SECOND ORDER ESTIMATES FOR BOUNDARY BLOW-UP SOLUTIONS OF ELLIPTIC EQUATIONS

CLAUDIA ANEDDA AND GIOVANNI PORRU
Dipartimento di Matematica e Informatica
Via Ospedale 72
09124 Cagliari, Italy

ABSTRACT. We investigate blow-up solutions of the equation $\Delta u = f(u)$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Under appropriate growth conditions on $f(t)$ as $t$ goes to infinity we show how the mean curvature of the boundary $\partial \Omega$ appears in the second order term of the asymptotic expansion of the solution $u(x)$ as $x$ goes to $\partial \Omega$.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, and let $f(t)$ be a smooth function, increasing for $t \geq 0$, which satisfies $f(0) = 0$ and the Keller-Osserman condition

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} < \infty, \quad F'(t) = f(t).$$

It is well known [14], [17] that under these conditions the Dirichlet problem

$$\Delta u = f(u) \text{ in } \Omega, \quad u(x) \to \infty \text{ as } x \to \partial \Omega,$$

has a classical solution called a boundary blow-up (explosive, large) solution. Moreover, the one dimensional problem

$$\phi'' = f(\phi), \quad \phi(s) > 0, \quad \lim_{s \to 0} \phi(s) = \infty,$$

has a solution satisfying

$$\int_{\phi(s)}^\infty \frac{dt}{\sqrt{2F(t)}} = s, \quad F(t) = \int_0^t f(\tau) d\tau.$$

Under some additional condition on $f$, it is possible to show the estimate [7]

$$\lim_{x \to \partial \Omega} \frac{u(x)}{\phi(\delta(x))} = 1,$$

where $\delta(x)$ denotes the distance of $x$ from $\partial \Omega$. This means that the main part of the asymptotic behaviour of the solution $u(x)$ near $\partial \Omega$ is independent of the geometry of the domain. The behaviour of boundary blow-up solutions near the boundary has been investigated by many researchers, see [1], [2], [3], [4], [5], [7], [8], [9], [13], [15]. Let us recall a result of C. Bandle and M. Marcus, taken from Theorem 4 of

2000 Mathematics Subject Classification. Primary: 35J25; Secondary: 35B05, 35B40.

Key words and phrases. Elliptic equations, Blow-up solutions, Second order boundary estimates.
Let $f(t) > 0$ and $F(t)$ as in (2); moreover, if $G(t) = \int_0^t \sqrt{F(\tau)} d\tau$, suppose there exist $a$, $b$, with $1 < a < b$ such that

$$a \frac{F(t)}{f(t)} \leq G(t) \leq b \frac{F(t)}{f(t)}$$

(3)

for large $t$. Note that (3) implies the Keller-Osserman condition and that $F(t)/t^2$ is increasing for $t$ large. Under condition (3), in [8] it is proved that

$$C \frac{(\delta(x))^2 \phi'(\delta(x))}{\phi(\delta(x))} \leq \frac{u(x)}{\phi(\delta(x))} - 1 \leq C \delta(x),$$

(4)

where $\phi$ is defined as in (2) and $C$ is a suitable constant.

A function which satisfies (3) is $f(t) = t^p$, $p > 1$. In this case we have

$$\phi(s) = (a_p s^\frac{2}{p^2}, \quad a_p = \frac{p - 1}{\sqrt{2(1 + p)}}.$$

For this special case C. Bandle [4] has improved the estimates (4) proving the expansion

$$u(x) = (a_p \delta(x))^\frac{2}{p^2} \left[1 + \frac{(N - 1)K(\delta)}{p + 3} \delta(x) + o(\delta(x))\right],$$

where $K(\delta)$ denotes the mean curvature of $\partial \Omega$ at the point $\delta$ nearest to $x$, and $o(\delta)$ has the usual meaning.

The object of the present paper is to find a similar expansion for a suitable class of functions $f$. We suppose that

$$\frac{f'(t)F(t)}{(f(t))^2} = \frac{p}{p + 1} + O(1)t^{-\beta}, \quad F(t) = \int_0^t f(\tau) d\tau,$$

(5)

where $p > 1$, $\beta > 0$, and $O(1)$ denotes a bounded quantity. In case of $1 < p \leq 3$ we also use the following additional condition:

$$\exists \theta_0 < 1, \exists a > 1 : \forall \theta \in (\theta_0, 1), \forall t > t_0, \theta f(t) > f(\theta t).$$

(6)

Furthermore, suppose there is a constant $M$ such that for all $\theta \in (1/2, 2)$ and for $t$ large we have

$$\frac{|f''(\theta t)|t^2}{f(t)} \leq M.$$  

(7)

Then we find the estimate

$$u(x) = \phi(\delta) \left[1 + \frac{N - 1}{p + 3} K(x) \delta + O(1)\delta^\sigma\right],$$

where $\phi$ is defined as in (2), $K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = constant\}$ and $\sigma > 1$ depends on $\beta$ and $p$. A typical example which satisfies all the conditions in above is $f(t) = t^p + P(t)$, where $P(t)$ has a polynomial growth $q$ with $q < p$. Note that this special case has been discussed in [3] in a different context. Indeed, in [3] the basic function $\phi$ used to give the expansion of the solution $u(x)$ was related to the principal part $t^p$ only. Now we take $\phi$ defined as in (2), which makes a strong difference when $q$ is close to $p$.

Results of existence for singular equations in presence of a gradient term are also been discussed, see for example [5], [10]. Also the cases of weighted quasilinear equations as well as $p$-Laplace equations have been investigated, see [10], [12] and references therein.
2. **Main result.** Let \( f(t) : [0, +\infty) \rightarrow \mathbb{R} \) be smooth, increasing in \([0, \infty)\) with \( f(0) = 0 \). Suppose that (5) holds, and let us write this equation as

\[
(F(t))^{\frac{1}{p+1}} \left( \frac{(F(t))^{\frac{1}{p+1}}}{f(t)} \right) + O(1) t^{-\beta} = 0.
\]

Integration by parts on \((1, t)\) yields

\[
\frac{F(t)}{tf(t)} = \frac{1}{p+1} + g(t),
\]

where

\[
|g(t)| \leq \begin{cases} 
C t^{-\beta} & \text{if } 0 < \beta < 1, \\
C t^{-1} \log t & \text{if } \beta = 1, \\
C t^{-1} & \text{if } \beta > 1.
\end{cases}
\]

Here and in what follows, \( C \) is a suitable constant. By (5) and (8) we find

\[
tf'(t) f(t) = p + g(t),
\]

where \( g(t) \) is not necessarily the same as in (8), but it again satisfies estimate (9).

By (10) we find, for some constant \( C > 1 \) and \( t \) large

\[
\frac{1}{C} t^p < f(t) < Ct^p, \quad \frac{1}{C} t^{p+1} < F(t) < Ct^{p+1}.
\]

If \( \phi \) is defined as in (2), by using the last estimates, for \( s \) small we get

\[
\frac{1}{C} s^{\frac{2}{p+1}} < (\phi(s))^{-1} < Cs^{\frac{2}{p+1}}.
\]

**Lemma 1.** If (5) holds and if \( \phi = \phi(s) \) is defined as in (2) then we have

\[
\lim_{s \to 0} \frac{\phi(s)}{s^2 f(\phi(s))} = \frac{(p-1)^2}{2(p+1)}.
\]

**Proof.** Let us write

\[
\frac{\phi(s)}{s^2 f(\phi(s))} = \left( \frac{\phi(\frac{1}{2})(f(\phi))^{-\frac{1}{2}}}{\int_{\phi}^{\infty} (2F(\tau))^{-\frac{1}{2}} d\tau} \right)^2.
\]

Recall that (5) implies (8). Putting \( \phi(s) = t \), and using de l’Hôpital rule, (5) and (8) we find

\[
\lim_{s \to 0} \frac{\phi(\frac{1}{2})(f(\phi))^{-\frac{1}{2}}}{\int_{\phi}^{\infty} (2F(\tau))^{-\frac{1}{2}} d\tau} = \lim_{t \to \infty} \frac{t^\frac{1}{2}(f(t))^{-\frac{1}{2}}}{\int_t^{\infty} (2F(\tau))^{-\frac{1}{2}} d\tau} = \frac{1}{2} \lim_{t \to \infty} \frac{t^\frac{1}{2}(f(t))^{-\frac{1}{2}} - t^\frac{1}{2}(f(t))^{-\frac{1}{2}} f'(t)}{-(2F(t))^{-\frac{1}{2}}}
\]

\[
= \frac{1}{2} \lim_{t \to \infty} \left[ -\left( \frac{tf(t)}{2F(t)} \right)^{-\frac{1}{2}} + \left( \frac{tf(t)}{2F(t)} \right)^{\frac{1}{2}} f'(t) F(t) \right] F(t)^{\frac{1}{2}}
\]

\[
= \frac{1}{2} \left[ -\left( \frac{p+1}{2} \right)^{-\frac{1}{2}} + \left( \frac{p+1}{2} \right)^{\frac{1}{2}} \frac{2p}{p+1} \right] = \left( \frac{p+1}{2} \right)^{\frac{1}{2}} p - 1.
\]

This estimate and (14) yield (13). The lemma is proved.

\(\square\)
Lemma 2. If (5) holds and if $\phi = \phi(s)$ is defined as in (2) then we have
\[
-\frac{\phi'}{f(\phi)} = \frac{p-1}{p+1}s + O(1)s\phi^{-\beta},
\]
where $O(1)$ denotes a bounded quantity.

Proof. By the relation
\[
-1 + 2\left(\frac{p}{p+1} + O(1)t^{-\beta}\right) = \frac{p-1}{p+1} + O(1)t^{-\beta},
\]
using (5) we have
\[
-1 + 2F(t)f'(t)(f(t))^{-2} = \frac{p-1}{p+1} + O(1)t^{-\beta}.
\]
Multiplying by $(2F(t))^{-\frac{1}{2}}$ we find
\[
-(2F(t))^{-\frac{1}{2}} + (2F(t))^{\frac{1}{2}}f'(t)(f(t))^{-2} = \frac{p-1}{p+1}(2F(t))^{-\frac{1}{2}} + O(1)(2F(t))^{-\frac{1}{2}}t^{-\beta},
\]
and
\[
-((2F(t))^{\frac{1}{2}}(f(t))^{-1})' = \frac{p-1}{p+1}(2F(t))^{-\frac{1}{2}} + O(1)(2F(t))^{-\frac{1}{2}}t^{-\beta}.
\]
Using (11) we observe that $(2F(t))^{\frac{1}{2}}(f(t))^{-1} \to 0$ as $t \to \infty$. Therefore, integrating on $(t, \infty)$ we get
\[
(2F(t))^{\frac{1}{2}}(f(t))^{-1} = \frac{p-1}{p+1}\int_{t}^{\infty} (2F(\tau))^{-\frac{1}{2}}d\tau + O(1)\int_{t}^{\infty} (2F(\tau))^{-\frac{1}{2}}\tau^{-\beta}d\tau.
\]
Using de l'Hôpital rule and (8) we find
\[
\lim_{t \to \infty} t^{-\beta}f(t)^{\frac{1}{2}}(f(t))^{-1}d\tau = 1 + \beta \lim_{t \to \infty} \int_{t}^{\infty} (2F(\tau))^{-\frac{1}{2}}d\tau = 1 + \beta \lim_{t \to \infty} \frac{1}{1 - t(2F(t))^{-1}} = 1 + \frac{2\beta}{p - 1}.
\]
In view of the last estimate, equation (16) can be rewritten as
\[
\frac{(2F(t))^{\frac{1}{2}}}{f(t)} = \frac{p-1}{p+1}\int_{t}^{\infty} (2F(\tau))^{-\frac{1}{2}}d\tau + O(1)t^{-\beta}\int_{t}^{\infty} (2F(\tau))^{-\frac{1}{2}}d\tau.
\]
Putting $t = \phi(s)$ and recalling that $-\phi'(s) = (2F(\phi(s)))^{\frac{1}{2}}$, the lemma follows. $\square$

Theorem 1. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with a smooth boundary $\partial\Omega$, let $p > 1$, $\beta > 0$ be real numbers, and let $f(t)$ be smooth, increasing in $[0, \infty)$ with $f(0) = 0$, satisfying (5) and (7); in case of $1 < p \leq 3$ we also let condition (6) holds. If $u(x)$ is a solution to problem (1) then
\[
u(x) = \phi(\delta)[1 + \frac{N - 1}{p + 3}K\delta + O(1)\delta^\sigma],
\]
where $\phi$ is defined as in (2), $\delta = \delta(x)$ denotes the distance of $x$ from $\partial\Omega$, $K = K(x)$ is the mean curvature of the surface \{x \in \Omega : \delta(x) = \text{constant}\}, 1 < \sigma \leq 2$, with $\sigma < \min\left[\frac{2\beta}{p - 1} + 1, \frac{2}{p - 1} + 1\right]$. 
Proof. We look for a super-solution of the kind
\[ w(x) = \phi(\delta) + A\phi(\delta)\delta + \alpha\phi(\delta)\delta^2, \]
where
\[ A = \frac{H}{p+3}, \quad H = (N-1)K \] (18)
and \( \alpha \) is a positive constant to be determined. Denoting by \( \cdot \) differentiation with respect to \( \delta \), we have
\[ w_{x_i} = \phi'\delta_{x_i} + A_{x_i}\phi\delta + A(\phi\delta)'\delta_{x_i} + \alpha(\phi\delta^2)'\delta_{x_i}. \]
Since \[ 11 \]
we find
\[ \Delta w = \phi'' - \phi'H + \Delta A\phi\delta + 2\nabla A \cdot \nabla \delta(\phi\delta)' + A(\phi\delta)'' - A(\phi\delta)'H + \alpha\left( (\phi\delta')'' - (\phi\delta')'H \right) \]
\[ = \phi'' - \phi'(H - 2A) + \Delta A\phi\delta + 2\nabla A \cdot \nabla \delta(\phi'\delta + \phi) + A\phi''\delta - A(\phi'\delta + \phi)H + \alpha\left( \phi''\delta^2 + 2\sigma(\phi'\delta\delta^{-1} + \sigma(\sigma - 1))\phi\delta\delta^{-2} - \phi'\delta\delta^{-1}H - \sigma\phi\delta\delta^{-1}H \right). \] (19)
Recall that
\[ \phi'' = f(\phi). \]
Furthermore, by (13) with \( s = \delta \) we have
\[ \frac{\phi}{f(\phi)} = \frac{(p-1)^2}{2(p+1)}\delta^2 + o(1)\delta^2, \]
where \( o(1) \) denotes a quantity which tends to zero as \( \delta \) tends to zero. Using the last equations as well as (15) with \( s = \delta \), by (19) we find
\[ \Delta w = f(\phi)\left[ 1 + \left( \frac{p-1}{p+1}(H - 2A) + A \right)\delta + O(1)\phi^{-\beta}\delta + O(1)\delta^2 \right. 
\[ + \alpha\delta^2 \left( 1 - 2\sigma\frac{p-1}{p+1} + \sigma(\sigma - 1)\frac{(p-1)^2}{2(p+1)} + o(1) \right). \] (20)
Since \( O(1) \) is a bounded quantity, we can find suitable constants \( C_i \) such that
\[ \Delta w < f(\phi)\left[ 1 + \left( \frac{p-1}{p+1}(H - 2A) + A \right)\delta + C_1\phi^{-\beta}\delta + C_2\delta^2 \right. 
\[ + \alpha\delta^2 \left( 1 - 2\sigma\frac{p-1}{p+1} + \sigma(\sigma - 1)\frac{(p-1)^2}{2(p+1)} + o(1) \right). \] (21)
On the other side, using Taylor’s expansion we have
\[ f(w) = f(\phi)\left[ 1 + \frac{f'(\phi)}{f(\phi)}\phi(\delta + \alpha\delta^2) + \frac{f''(\phi)}{2f(\phi)}(A\delta + \alpha\delta^2)^2 \right], \] (22)
with \( \delta \) between \( \phi \) and \( \phi(1 + A\delta + \alpha\delta^2) \). After \( \alpha \) is fixed, we consider only points \( x \in \Omega \) such that
\[ -\frac{1}{2} < A\delta + \alpha\delta^2 < 1. \] (23)
This means that $1/2 < 1 + A\delta + \alpha \delta^\sigma < 2$. Therefore, the term $\bar{\phi}$ which appears in (22) satisfies $\bar{\phi} = \theta \phi$ with $1/2 < \theta < 2$, and we can use (7). Using (10) and (7), by (22) we find

$$f(w) = f(\phi)[1 + pA\delta + \alpha p\delta^\sigma + O(1)g(\phi)\delta + g(\phi)\alpha \delta^\sigma + O(1)\delta^2 + O(1)(\alpha \delta^\sigma)^2]. \quad (24)$$

By (24), we can take suitable constants $C_i$ such that

$$f(w) > f(\phi)[1 + pA\delta + \alpha p\delta^\sigma - C_3|g(\phi)|\delta - |g(\phi)|\alpha \delta^\sigma - C_4\delta^2 - C_5(\alpha \delta^\sigma)^2]. \quad (25)$$

Since by (18)

$$\frac{p - 1}{p + 1}(H - 2A) + A = pA,$$

by (21) and (25) we have

$$\Delta w < f(w) \quad (26)$$

provided

$$C_1\phi^{-\beta} \delta + C_2\delta^2 + \alpha \delta^\sigma \left(1 - 2\sigma\frac{p - 1}{p + 1} + \sigma(\sigma - 1)\frac{(p - 1)^2}{2(p + 1)} + o(1)\right)$$

$$< \alpha \delta^\sigma - C_3|g(\phi)|\delta - |g(\phi)|\alpha \delta^\sigma - C_4\delta^2 - C_5(\alpha \delta^\sigma)^2.$$

Rearranging and using (9) we find

$$C_1\phi^{-\beta} \delta^{1-\sigma} + C_3|g(\phi)|\delta^{1-\sigma} + (C_2 + C_4)\delta^{2-\sigma}$$

$$< \alpha\left(p - 1 + 2\sigma\frac{p - 1}{p + 1} - \sigma(\sigma - 1)\frac{(p - 1)^2}{2(p + 1)} + o(1) - C_5\alpha \delta^\sigma\right). \quad (27)$$

Let us discuss inequality (27). By using (12) we find

$$\phi^{-\beta} \delta^{1-\sigma} < C\delta^{-\frac{2\beta}{p - 1} + 1-\sigma}. $$

If $0 < \beta \leq 1$, by assumption we have $\alpha < \frac{2\beta}{p - 1} + 1$, and, therefore, $\frac{2\beta}{p - 1} + 1 - \sigma > 0$; if $\beta > 1$ we have $\alpha < \frac{2\beta}{p - 1} + 1$, and, therefore, $\frac{2\beta}{p - 1} + 1 - \sigma > 0$. Therefore, in both cases we have $\phi^{-\beta} \delta^{1-\sigma} = 0$ as $\delta \to 0$. 

If $0 < \beta < 1$, by (9) we have $|g(\phi)| < C\phi^{-\beta}$. Therefore, by the previous argument we have $|g(\phi)|\delta^{1-\sigma} \to 0$ as $\delta \to 0$. If $\beta = 1$ then $|g(\phi)| < C\phi^{-1} \log \phi$, and, using (12) we find $|g(\phi)|\delta^{1-\sigma} < C\delta^{-\frac{2\beta}{p - 1} + 1-\sigma} \log(\delta^{-1})$. Since $\sigma < \frac{2\beta}{p - 1} + 1$, it follows that $|g(\phi)|\delta^{1-\sigma} \to 0$ as $\delta \to 0$ also for $\beta = 1$. If $\beta > 1$ then $|g(\phi)| < C\phi^{-1}$, and using (12) again we find

$$|g(\phi)|\delta^{1-\sigma} < C\delta^{-\frac{2\beta}{p - 1} + 1-\sigma}. $$

Since now $\sigma < \frac{2\beta}{p - 1} + 1$, $|g(\phi)|\delta^{1-\sigma} \to 0$ also in this situation. Clearly, since $\sigma \leq 2$, $\delta^{2-\sigma}$ is bounded as $\delta \to 0$. Moreover, we have

$$p - 1 + 2\sigma\frac{p - 1}{p + 1} - \sigma(\sigma - 1)\frac{(p - 1)^2}{2(p + 1)} = \frac{(p - 1)^2}{2(p + 1)}(\sigma + 1)\left(2\frac{p + 1}{p - 1} - \sigma\right) > 0.$$

Hence, we can take $\alpha_0$ large and $\delta_0$ small so that (23) and (27) hold for $\sigma \geq \alpha_0$, $\delta \leq \delta_0$ with $\alpha \delta^\sigma \leq \alpha_0 \delta^\sigma_0$.

Let us show now that we can choose $\alpha$ and $\delta_1$ so that $w(x) \geq u(x)$ for $\delta(x) = \delta_1$.

If $G(t) = \int_0^t \sqrt{F(\tau)}d\tau$, using de l’Hôpital rule and (5), we find

$$G(t) = \int_0^t \sqrt{F(\tau)}d\tau,$$
\[ \lim_{t \to \infty} \frac{G(t) f(t)}{G'(t) F(t)} = \lim_{t \to \infty} \frac{G(t)}{(F(t))^{\frac{2}{p}} (f(t))^{-1}} = \lim_{t \to \infty} \frac{1}{3/2 - (f(t))^{-2} F(t) F'(t)} = \frac{2(p + 1)}{p + 3} > 1. \]

Therefore (3) holds and we can use the estimates (4). In our situation, the left hand side of (4) can be simplified. Indeed, we claim that

\[ -C\delta(x) \leq \frac{u(x)}{\phi(\delta(x))} - 1 \leq C\delta(x). \]

Recall that (5) implies (8). Putting \( \phi(s) = t \), using de l’Hôpital rule and (8) we find

\[ \lim_{\theta \to 0} \frac{\phi(s)}{s} = \lim_{t \to \infty} \frac{t(2F(t))^{-\frac{1}{2}}}{\int_{t}^{\infty} (2F(\tau))^{-\frac{1}{2}} d\tau} = \lim_{t \to \infty} \frac{(2F(t))^{-\frac{1}{2}} - t(2F(t))^{-\frac{3}{2}} f(t)}{-(2F(t))^{-\frac{3}{2}}} = -1 + \lim_{t \to \infty} \frac{tf(t)}{2F(t)} = -1 + \frac{p + 1}{2} = \frac{p - 1}{2}. \]

Inequality (28) follows from (4) and the above estimate.

From (28) we find

\[ \lim_{x \to \partial \Omega} \frac{\phi(\delta(x))}{u(x)} = 1. \tag{29} \]

Let \( \rho = \alpha \delta^\sigma \), where \( \alpha \) and \( \delta \) are such that (23) and (27) hold. By (29) we can decrease \( \delta \) (increasing \( \alpha \) according to \( \alpha \delta^\sigma = \rho \)) until

\[ \frac{\phi(\delta(x))}{u(x)} > \frac{2}{2 + \rho} \]

for \( \delta(x) \leq \delta_1 \). As a consequence, multiplying by \( (1 + A\delta + \alpha \delta^\sigma) \) we have

\[ \frac{u(x)}{u(x)} > \frac{2}{2 + \rho} (1 + A\delta + \alpha \delta^\sigma). \]

Decrease \( \delta_1 \) again (and increase \( \alpha \)) in order to have \( A\delta_1 > -\rho/2 \) and \( \alpha \delta_1^\sigma = \rho \). Then \( u(x) > u(x) \) for \( \delta(x) = \delta_1 \).

If \( p > 3 \), by (12) we have \( \delta \phi(\delta) \to 0 \) as \( \delta \to 0 \); hence, by (28) we find easily that \( w(x) - u(x) \to 0 \) as \( x \to \partial \Omega \). By (26), (1) and the comparison principle ([11], Theorem 10.1) it follows that \( w(x) \geq u(x) \) on \( \{ x \in \Omega : \delta(x) < \delta_1 \} \).

If \( 1 < p \leq 3 \), let \( t_0 \) and \( \theta_0 \) be the constants of condition (6). Decrease \( \delta_1 \) (and increase \( \alpha \) according to \( A\delta_1^\sigma = \rho \)) in order to have \( u(x) > t_0 \) for \( \delta(x) < \delta_1 \). For \( \theta \in (\theta_0, 1) \) we have (trivially) \( w(x) > \theta u(x) \) on \( \{ x \in \Omega : \delta(x) = \delta_1 \} \). On the other side, since \( \phi(\delta(x))/u(x) \to 1 \) as \( x \to \partial \Omega \), we find that \( w(x) > \theta u(x) \) near \( \partial \Omega \). Moreover, using (6) we find

\[ \Delta(\theta u) = \theta f(u) > f(\theta u). \]

By the latter inequality, (26) and the comparison principle it follows that \( w(x) \geq \theta u(x) \) on \( \{ x \in \Omega : \delta(x) < \delta_1 \} \). As \( \theta \to 1 \) we find that \( w(x) \geq u(x) \) on the same set.

We look for a sub-solution of the kind

\[ v(x) = \phi(\delta) + A\phi(\delta) \delta - \alpha \phi(\delta) \delta^\sigma, \]
where \( A \) is the same as before and \( \alpha \) is a positive constant to be determined. Instead of (20) now we have

\[
\Delta v = f(\phi) \left[ 1 + \left( \frac{p-1}{p+1}(H-2A) + A \right) \delta + O(1) \phi^{-\beta} \delta + O(1) \delta^2 \right]
\]

\[-\alpha \delta^\sigma \left( 1 - 2\sigma \frac{p-1}{p+1} + \sigma(\sigma-1) \frac{(p-1)^2}{2(p+1)} + o(1) \right) \].

This means that we can find suitable constants \( C_i \) (not necessarily the same as before) such that

\[
\Delta v > f(\phi) \left[ 1 + \left( \frac{p-1}{p+1}(H-2A) + A \right) \delta - C_1 \phi^{-\beta} \delta - C_2 \delta^2 \right]
\]

\[-\alpha \delta^\sigma \left( 1 - 2\sigma \frac{p-1}{p+1} + \sigma(\sigma-1) \frac{(p-1)^2}{2(p+1)} + o(1) \right) \]. \tag{30}

After \( \alpha \) is fixed, we consider only points \( x \in \Omega \) such that

\[-\frac{1}{2} < \frac{1}{A} \leq \alpha \delta^\sigma < 1. \tag{31}\]

Using Taylor’s expansion, (9) and (7) we find

\[
f(v) < f(\phi) \left[ 1 + p\alpha \delta - \alpha \delta^\sigma + C_3 |g(\phi)| \delta + |g(\phi)| \alpha \delta^\sigma + C_4 \delta^2 + C_5 (\alpha \delta^\sigma)^2 \right]. \tag{32}\]

Recalling that

\[
\frac{p-1}{p+1}(H-2A) + A = pA,
\]

by (30) and (32) we have

\[
\Delta v > f(v) \tag{33}
\]

when

\[
-C_1 \phi^{-\beta} \delta - C_2 \delta^2 - \alpha \delta^\sigma \left( 1 - 2\sigma \frac{p-1}{p+1} + \sigma(\sigma-1) \frac{(p-1)^2}{2(p+1)} + o(1) \right)
\]

\[> - \alpha \delta^\sigma + C_3 |g(\phi)| \delta + |g(\phi)| \alpha \delta^\sigma + C_4 \delta^2 + C_5 (\alpha \delta^\sigma)^2.
\]

Rearranging and using (9) we find

\[
C_1 \phi^{-\beta} \delta^{1-\sigma} + C_3 |g(\phi)| \delta^{1-\sigma} + (C_2 + C_4) \delta^{2-\sigma} \]

\[< \alpha \left( p - 1 + 2\sigma \frac{p-1}{p+1} - \sigma(\sigma-1) \frac{(p-1)^2}{2(p+1)} + o(1) - C_5 \alpha \delta^\sigma \right),
\]

which looks like (27) possibly with different values of the constants \( C_i \). Therefore we can take \( \delta_0 \) small and \( \alpha_0 \) large in order to satisfy this inequality as well as (31) for \( \delta < \delta_0 \) and \( \alpha > \alpha_0 \) with \( \alpha \delta^\sigma \leq \alpha_0 \delta_0^\sigma \). Take \( \alpha \) and \( \delta \) as in above and put \( \rho = \alpha \delta^\sigma \).

By (29), we can decrease \( \delta \) (increasing \( \alpha \) according to \( \alpha \delta^\sigma = \rho \)) until

\[
\frac{\phi(\delta(x))}{u(x)} < \frac{2}{2 - \rho}
\]

for \( \delta(x) \leq \delta_1 \). Multiplying by \( 1 + A\delta - \alpha \delta^\sigma \) we have

\[
\frac{v(x)}{u(x)} < \frac{2}{2 - \rho} \left( 1 + A\delta - \alpha \delta^\sigma \right).
\]

Decrease \( \delta_1 \) again (and increase \( \alpha \)) in order to have \( A\delta_1 < \rho/2 \) and \( \alpha \delta_1^\sigma = \rho \). Then \( v(x) < u(x) \) for \( \delta(x) = \delta_1 \).
As in the previous case, if \( p > 3 \), by (28) we find easily that \( v(x) - u(x) \to 0 \) as \( x \to \partial \Omega \). By (33) and (1) it follows that \( v(x) \leq u(x) \) for \( \{ x \in \Omega : \delta(x) < \delta_1 \} \).

Let \( 1 < p \leq 3 \), and rewrite condition (6) as

\[
\exists \theta_0 < 1, \, \exists t_0 > 1 : \, \forall t \in (\theta_0, 1), \, \forall t > t_0, \, \frac{1}{\theta} f(t) < f\left(\frac{1}{\theta} t\right).
\]

If \( t_0 \) and \( \theta_0 \) are the constants in above, decrease \( \delta_1 \) (and increase \( \alpha \) according to \( \alpha \delta_1^\theta = \rho \)) in order to have \( u(x) > t_0 \) for \( \delta(x) < \delta_1 \). For \( \theta \in (\theta_0, 1) \) we have (trivially) \( v(x) < \frac{1}{\theta} u(x) \) on \( \{ x \in \Omega : \delta(x) = \delta_1 \} \). On the other side, since \( \phi(\delta(x))/u(x) \to 1 \) as \( x \to \partial \Omega \), we find that \( v(x) < \frac{1}{\theta} u(x) \) near \( \partial \Omega \). Moreover, using the above assumption we find

\[
\Delta \left(\frac{1}{\theta} u\right) = \frac{1}{\theta} f(u) < f\left(\frac{1}{\theta} u\right).
\]

By the latter inequality and (33) it follows that \( v(x) \leq \frac{1}{\theta} u(x) \) on \( \{ x \in \Omega : \delta(x) < \delta_1 \} \).

As \( \theta \to 1 \) we find that \( v(x) \leq u(x) \) on the same set. The theorem follows.

3. Concluding remarks. If the function \( f(t) \) has an exponential growth instead of a polynomial growth then the behaviour of the second order term of the solution to problem (1) is quite different. For example, in case of \( f(t) = e^t \), C. Bandle [4] has found the expansion

\[
u(x) = \log \frac{2}{\delta^2} + (N - 1) K(\pi) \delta + o(\delta),
\]

where \( \pi \) is the point of \( \partial \Omega \) nearest to \( x \), and \( o(\delta) \) has the usual meaning.

More generally, assume \( f(t) : \mathbb{R} \to \mathbb{R} \) smooth and such that

\[
f(t) > 0, \quad f'(t) \geq 0, \quad \int_{-\infty}^{0} f(\tau)d\tau < \infty.
\]

Suppose there is \( \beta > 0 \) such that, for \( t > 0 \),

\[
\frac{F(t)f'(t)}{(f(t))^2} = 1 + O(1)t^{-\beta}, \quad F(t) = \int_{-\infty}^{t} f(\tau)d\tau,
\]

where \( O(1) \) is a bounded quantity. Furthermore suppose that, again for \( t > 0 \),

\[
\frac{F(t)}{f(t)} = \frac{1}{\beta} t^{1-\beta} (1 + O(1)t^{-\beta}).
\]

Finally, we suppose that for some \( m > 2 \) there are \( \epsilon > 0 \) and \( M, t_0 \) large such that

\[
\frac{|f''(\theta t)|}{f(t)} \leq Mt^{2\beta-2}(F(t))^{\frac{\delta}{\beta}}, \quad 1 - \epsilon < t < 1 + \epsilon,
\]

for all \( t > t_0 \). Under these conditions one proves that

\[
u(x) = \Phi(\delta) \left[1 + \beta^{-1}(N-1)K(x)(\Phi(\delta))^{-\beta} \delta + O(1)(\Phi(\delta))^{-2\beta} \delta \right],
\]

where \( \Phi(s) \) is defined as follows:

\[
\int_{s}^{\infty} \frac{dt}{\sqrt{2F(t)}} = s.
\]

Note that all these conditions hold for \( f(t) = e^{t\left|t^{\beta-1}\right|P(t)} \), where \( P(t) > 0 \) has a polynomial growth. The proof of this result will appear in a forthcoming paper.
REFERENCES

[1] L. Andersson and P. T. Chruściel, Solutions of the constraint equation in general relativity satisfying “hyperbolic conditions”, Dissertationes Mathematicae, 355 (1996), 1–100.

[2] C. Anedda, A. Buttu and G. Porru, Second order estimates for boundary blow-up solutions of special elliptic equations, Boundary Value Problems, Article ID, 45859 (2006), 1–12.

[3] C. Anedda and G. Porru, Higher order boundary estimates for blow-up solutions of elliptic equations, Differential and Integral Equations, 19 (2006), 345–360.

[4] C. Bandle, Asymptotic behaviour of large solutions of quasilinear elliptic problems, ZAMP, 54 (2003), 1–8.

[5] C. Bandle and E. Giarrusso, Boundary blow up for semilinear elliptic equations with non-linear gradient terms, Advances in Differential Equations, 1 (1996), 133–150.

[6] C. Bandle and M. Marcus, Dependence of blowup rate of large solutions of semilinear elliptic equations on the curvature of the boundary, Complex Var. Theory Appl., 49 (2004), 555–570.

[7] C. Bandle and M. Marcus, Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, J. d’Anal. Math., 58 (1992), 9–24.

[8] C. Bandle and M. Marcus, On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems, Differential and Integral Equations, 11 (1998), 23–34.

[9] S. Berhanu and G. Porru, Qualitative and quantitative estimates for large solutions to semilinear equations, Communications in Applied Analysis, 4 (2000), 121–131.

[10] M. Ghergu and V. Radulescu, Multiparameter bifurcation and asymptotics for the singular Lane-Emden-Fowler equation with a convection term, Proc. Roy. Soc. Edinburgh Sect. A, 135 (2005), 61–84.

[11] D. Gilbarg and N. S. Trudinger, “Elliptic Partial Differential Equations of Second Order,” Springer Verlag, Berlin, 1977.

[12] F. Gladiali, Boundary behaviour of solutions to quasilinear elliptic singular problems, Int. J. Appl. Math., 1 (1999), 489–498.

[13] A. Greco and G. Porru, Asymptotic estimates and convexity of solutions of semilinear elliptic equations, Differential and Integral Equations, 10 (1997), 219–229.

[14] J. B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math., 10 (1957), 503–510.

[15] A. C. Lazer and P. J. McKenna, Asymptotic behaviour of solutions of boundary blow-up problems, Differential and Integral Equations, 7 (1994), 1001–1019.

[16] A. Mohammed, Existence and asymptotic behavior of blow-up solutions to weighted quasilinear equations, J. Math. Anal. Appl., 298 (2004), 621–637.

[17] R. Osserman, On the inequality $\Delta u \geq f(u)$, Pacific J. Math., 7 (1957), 1641–1647.

Received July 2006; revised March 2007.

E-mail address: canedda@unica.it
E-mail address: porru@unica.it