Massive gravitational waves in Chern-Simons modified gravity

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Abstract. We consider the nondynamical Chern-Simons (nCS) modified gravity, which is regarded as a parity-odd theory of massive gravity in four dimensions. We first find polarization modes of gravitational waves for $\theta = x/\mu$ in nCS modified gravity by using the Newman-Penrose formalism where the null complex tetrad is necessary to specify gravitational waves. We show that in the Newman-Penrose formalism, the number of polarization modes is one in addition to an unspecified $\Psi_4$, implying three degrees of freedom for $\theta = x/\mu$. This compares with two for a canonical embedding of $\theta = t/\mu$. Also, if one introduces the Ricci tensor formalism to describe a massive graviton arising from the nCS modified gravity, one finds one massive mode after making second-order wave equations, which is compared to five found from the parity-even Einstein-Weyl gravity.

Keywords: modified gravity, gravity

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1 Introduction

Topologically massive gravity (TMG) including a gravitational Chern-Simons term (gCS) is a three-dimensional gravity theory [1, 2] with a massive propagating degree of freedom (DOF). Since the gCS term is odd under parity, the theory shows a single DOF of a given helicity, whereas the other helicity mode remains massless. Li, Song, and Strominger [3] have shown that the third-order Einstein equation (the first-order equation for the linearized Einstein tensor $\delta G_{\mu\nu}$) of cosmological TMG is changed into the same first-order equation for a massive graviton $h_{\mu\nu}$ when one chooses the transverse-traceless gauge. This shows clearly that the cosmological TMG is regarded as the first-order gravity theory. Later on, Bergshoeff, Hohm, and Townsend have proposed new massive gravity (NMG) by adding a quadratic curvature term to the Einstein-Hilbert action [4]. Although this term was designed to reproduce the Fierz-Pauli action for a massive graviton, there is no way to avoid ghost states in higher dimensions ($D \geq 4$), because of the fourth-order theory in the metric tensor formalism [5]. Since the NMG preserves parity unlike the TMG, a massive graviton acquires the same mass for both helicity states, showing two DOF. However, we did not know what are polarization modes of gravitational waves (GWs) in TMG and NMG. Since two theories belong to higher-order gravity in three dimensions, we need to introduce the Newman-Penrose (NP) formalism [6] where the null real tetrad is necessary to specify a few polarization modes of GWs. It turned out that one mode is $\Phi_{12}$ for the TMG, whereas two modes are $\Phi_{22}$ and $\Phi_{12}$ for the NMG [7]. This shows a perfect matching between the Einstein tensor [4] and NP formalisms because in three dimensions, only the massive graviton exists and the Weyl tensor vanishes.

On the other hand, there is no consensus on the number of graviton DOF in nondynamical Chern-Simons (nCS) [8–10] and dynamical Chern-Simons (dCS) modified gravity [11–14] in four dimensions, because it depends on the (non)dynamical CS as well as the choice of $\theta$. The number of graviton DOF propagating around a Minkowski background was two ($h^{TT}_{ij}$) as general relativity for the canonical CS coupling of $v_{\mu} = \partial_{\mu} \theta = (1/\mu, 0)$ [8]. Choosing a class of exact solutions describing plane-fronted gravitational waves ($pp$-waves) along $+z$ axis, the graviton is also described by two DOF ($\Psi_4 = -R_{n\bar{m}n\bar{m}} = \hat{h}_+ - i\hat{h}_{\times}$) for the nCS gravity [9], while it has three ($\Psi_4$ and $\Phi_{22} = -R_{nmn\bar{m}} = \frac{1}{2}\hat{h}_{mn\bar{m}}$) for the dCS gravity [15]. Here $h_{+,\times}$ are the plus/crossing polarizations of the waveform and the overhead dot (˙) denotes the differentiation with respect to the retarded null coordinate $u = t - z$. However, in the weak
amplitude regime of $\theta^2 \simeq 0$, the effect of $\Phi_{22}$ is negligible, which establishes $\Psi_4$ as in general relativity.

At this stage, we should mention the difference between the canonical embedding $\theta = t/\mu$ in Minkowski background [8] and a spacelike choice $\theta = x/\mu$ in Minkowski background [16] and AdS$_4$ background [10]. As was emphasized previously, there is no additional degrees of freedom for the canonical embedding because this does not induce time derivative but space derivative additionally. However, a spacelike choice of either $\theta = x/\mu$ or $\theta = y/\mu$ provides a time derivative in addition to second-order time derivative. The latter mimics the TMG in three dimensions. Recently, it was shown that six and ten initial conditions are needed to specify the time evolution of physical perturbations for $\theta = \mu t + \Psi(r)$ nCS and dCS theories in the Schwarzschild black hole spacetime [17]. Explicitly, noting that the Einstein-Hilbert term provides four, the dynamical CS scalar $\theta$ has two, and the CS term brings four, we have ten in the dCS gravity. However, one has six for the nCS gravity because the Pontryagin constraint $^*RR = 0$ kills two in eight. Actually, this corresponds to three and five DOF for the nCS and dCS gravity, respectively. This supports that the number of DOF is three for the nCS gravity when one uses the metric tensor formalism as well as $\theta = \mu t + \Psi(r)$ with $\Psi(r) \neq 0$. Actually, the number of propagating DOF is independent of a given background.

Even though a quadratic gravity required null complex tetrad to specify six independent polarization modes of $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ in the NP formalism [18, 19], the Einstein-Weyl (EW) gravity

$$S_{EW} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{\mu^2} \left( R_{\mu\nu}R^{\mu\nu} - \frac{1}{3} R^2 \right) \right]$$

(1.1)

has seven modes $\{\Psi_3, \Psi_4, \Psi_4, \Phi_{22}\}$ with $\Psi_4$ the massive ghost modes because the EW gravity is a fourth-order theory in the metric formalism [20]. Here $\Psi_3, \Psi_4$, and $\Psi_4$ are complex, providing six DOF. Analyzing the rotational behavior of the former set shows the respective helicity values $s = \{0, \pm 1, \pm 2, 0\}$. However, $\Psi_4$ always remains unspecified here because it must be determined by the Riemann tensor. This implies that the NP formalism is not appropriate for describing both massless and massive modes with seven DOF totally. To account for a massive mode only, it would be better to introduce the Ricci tensor formalism without ghost problem, instead of the metric tensor formalism with ghost problem.

Accordingly, it is urgent to clarify how many DOF are in nCS modified gravity (3.1) when the nCS modified gravity with $\theta = x/\mu$ is considered as a massive gravity. In order to find it, we first propose the similarity and difference between 3D and 4D massive gravities based on the metric tensor formalism:

| dimensions | parity-odd gravity | parity-even gravity | remarks |
|------------|-------------------|-------------------|---------|
| 3D         | TMG (1DOF)        | NMG (2)           | massless and massive modes |
| 4D         | nCS gravity (3)   | EW gravity (7)    |         |

We emphasize that the $\theta = x/\mu$ nCS modified gravity was considered as a 4D extension of the TMG, whereas the dCS modified gravity has been formulated by treating the CS scalar $\theta$ as a dynamical field. Hence, in this work, we investigate polarization modes of gravitational waves in the nCS modified gravity by using the NP formalism where the null complex tetrad is necessary to specify gravitational waves. The linearized Einstein equation corresponds to the first-order equation for the linearized Ricci tensor which will be used to kill the independent polarization modes propagating on the Minkowski spacetime. It turns out that the number of polarization modes is three for the nCS case by taking into account
an unspecified $\Psi_4$ which corresponds to three in the metric tensor formalism. If one uses the Ricci tensor formalism in the nCS modified gravity, there is no ghost problem because it lowers the third-order equation to the first-order one. Using the linearized TMG and NMG [3, 4], we could develop the linearized Einstein equation which is a second-order wave equation. This might describe a tensor wave propagating along $+z$ axis. In this case, we obtain one DOF for a massive graviton propagating on the Minkowski background.

2 The Newman-Penrose formalism

Let us first consider GWs propagating in the $+z$ direction for simplicity. In this case, all waves are functions of $t$ and $z$. At any point $P$, the null complex tetrad vectors $\{k, n, m, \tilde{m}\}$ are related to the Cartesian tetrad vectors $\{e_t, e_x, e_y, e_z\}$ in four dimensions with metric signature $(-,+,+,+)$ as

$$\frac{k}{n} = \frac{1}{\sqrt{2}}(e_t \pm e_z), \quad \frac{m}{\tilde{m}} = \frac{1}{\sqrt{2}}(e_x \pm ie_y),$$

where they satisfy the relations

$$-k \cdot n = m \cdot \tilde{m} = 1, \quad k \cdot \{m, \tilde{m}\} = n \cdot \{m, \tilde{m}\} = k^2 = n^2 = 0.$$  \hfill (2.2)

We note that a tensor $T$ can be written as

$$T_{abc...} = T_{\mu\nu\rho...}a^\mu b^\nu c^\rho...,$$

where Latin indices $(a,b,c,...)$ run over $(k, n, m, \tilde{m})$, while Greek indices $(\mu, \nu, \rho,...)$ run over $(t,x,y,z)$ because we are working with Cartesian coordinates to specify the Minkowski background. It is well-known that the Weyl tensor has ten components in four dimensions. Therefore, the Riemann tensor with twenty components can be decomposed into the Ricci tensor with ten and Weyl tensor with ten as

$$R_{\rho\sigma\mu\nu} = (g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) - \frac{1}{3}Rg_{\rho[\mu}g_{\nu]\sigma} + C_{\rho\sigma\mu\nu}.$$ \hfill (2.4)

Then, the NP quantities of five complex Weyl scalars $\Psi$’s $[C_{\rho\sigma\mu\nu}]$, nine $\Phi$’s $[R_{\mu\nu}]$ and $\Lambda[R]$ represent irreducible parts of the Riemann tensor $R_{\rho\sigma\mu\nu}$.

Choosing nearly plane waves propagating $+z$ direction reduces $R_{\rho\sigma\mu\nu}$ to six nonvanishing components representing by a set of $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ in the generic metric theory [18]. The first and last ones are real scalars, while the second and third are complex scalars. Figure 1 shows the six polarization modes of weak, plane, null GW permitted in any metric theory of gravity. Using the tetrad basis, the NP quantities are represented by the Riemann tensor and Ricci tensor as

$$\Psi_2 = -\frac{1}{6}R_{nknk} = -\frac{1}{6}R_{nk},$$
$$\Psi_3 = -\frac{1}{2}R_{nkn\tilde{m}} = -\frac{1}{2}R_{n\tilde{m}},$$
$$\Psi_4 = -R_{n\tilde{m}n\tilde{m}},$$
$$\Phi_{22} = -R_{n\tilde{m}n\tilde{m}} = -\frac{1}{2}R_{nn}.$$ \hfill (2.5)
Figure 1. Six polarization modes of weak, plane GW permitted in four-dimensional gravity, which indicate (a) : $\text{Re}\{\Psi_4\}$, (b) : $\text{Im}\{\Psi_4\}$, (c) : $\Phi_{22}$, (d) : $\Psi_2$, (e) : $\text{Re}\{\Psi_3\}$, (f) : $\text{Im}\{\Psi_3\}$. All modes (a) $\sim$ (f) are propagating in the +z direction. The solid displacement shows that each mode induces on a sphere of test particles, while the dashed displacement indicates that each mode induces on a sphere of test particles after half-period.

We note that $\Psi_4$ could not be represented by the Ricci tensor and thus, it remains unconstrained in any metric theory of gravity. Another relation is

$$R_{nknm} = R_{nm}, \quad (2.6)$$

while a relation for the Ricci scalar is given by

$$R = -2R_{nk} = -2R_{nknk} = 12\Psi_2, \quad (2.7)$$

which implies that the non-propagation of the Ricci scalar ($R = 0$) indicates $\Psi_2 = 0$.

The naive $E(2)$ classification of polarization waves under Lorentz transformation [18] is useful to find gravitational wave polarizations found in nCS modified gravity. It is given by

- Class $\Pi_6$: $\Psi_2 \neq 0$;
- Class $\Pi_5$: $\Psi_2 = 0$, $\Psi_3 \neq 0$;
- Class $N_3$: $\Psi_2 = \Psi_3 = 0$, $\Psi_4 \neq 0$, $\Phi_{22} \neq 0$;
- Class $N_2$: $\Psi_2 = \Psi_3 = \Phi_{22} = 0$, $\Psi_4 \neq 0$;
- Class $O_1$: $\Psi_2 = \Psi_3 = \Psi_4 = 0$, $\Phi_{22} \neq 0$;
- Class $O_0$: $\Psi_2 = \Psi_3 = \Psi_4 = \Phi_{22} = 0$.

The Einstein gravity of $R$ (equivalently, its equation $R_{\mu\nu} = 0$ and $R = 0$) is of class $N_2$, while the EW gravity of $R + \gamma (R^2_{\mu\nu} - R^2/3)$ may be classified by $\Pi_5$ because of the non-propagation of Ricci scalar ($R = 0$). A quadratic gravity of $R + \alpha R^2 + \gamma R^2_{\mu\nu}$ may be categorized by class $\Pi_6$. However, we note that the number of DOF is seven for the EW gravity and eight for the
quadratic gravity in the metric tensor formalism because of the presence of massive ghost modes.

The Brans-Dicke theory belongs to class $N_3$ because the Brans-Dicke scalar provides $\Phi_{22}$ [21]. We note that $\Phi_{22}$ and $\Psi_2$ correspond to a perpendicular scalar mode (breathing mode) and to a longitudinal scalar mode. They are arising from either the massive Brans-Dicke theory or the metric $f(R)$-gravity, in addition to an unspecified $\Psi_4$. These theories are of class $I_6$ even though they have four polarization modes with $\Psi_3 = 0$.

The $pp$-wave approach in nCS modified gravity leads to class $N_2$ [15]. If a theory is of class $I_6$ or $I_{15}$, the corresponding amplitudes cannot be identified with massless particle fields like $\Psi_4$ and $\Phi_{22}$. Furthermore, the nCS modified gravity is a constraint theory regardless of any choice $\theta$. All solutions must satisfy the Pontryagin constraint which translates into a reality condition of the quadratic invariant for the Weyl spinor $I$ on the $E(2)$ classification [9]

$$\Im(I) = 3\Im[\Psi_2^2] = 0 \rightarrow \Psi_2 = 0. \quad (2.8)$$

It implies that any spacetime of Petrov types III, N, and O automatically satisfies the Pontryagin constraint, while spacetimes of Petrov type II could violate it. We propose that if the nCS gravity is a massive gravity theory, it belongs to $I_{15}$ but the number of DOF is between five and two. Hence, we have to determine what is the $E(2)$ class of nCS modified gravity in the Minkowski background really.

### 3 nCS modified gravity

Let us first consider the nCS modified gravity in four dimensions whose action is given by

$$S_{CS} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{\theta}{4} *RR \right]. \quad (3.1)$$

where $\kappa^2 = 8\pi G$, $\theta$ is a nondynamical field, and $*RR = *R^{\eta}_{\xi\mu\nu} R^\eta_{\eta\mu\nu}$ is the Pontryagin density with

$$*R^{\eta}_{\xi\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R^\eta_{\xi\rho\sigma}. \quad (3.2)$$

Varying for $g_{\mu\nu}$ on the action (3.1), we find the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + C_{\mu\nu} = 0. \quad (3.3)$$

Here, the four-dimensional Cotton tensor $C_{\mu\nu}$ is written by

$$C_{\mu\nu} = \nabla_\gamma \theta \epsilon^{\gamma\rho\sigma} (\rho \nabla_{(\sigma} R_{\nu)\rho}) + \frac{1}{2} \nabla_\gamma \nabla_\rho \theta \epsilon^{\gamma\sigma\delta} R^\rho_{(\mu)\sigma\delta}. \quad (3.4)$$

where the Cotton tensor is a traceless and symmetric tensor. We stress that even though some works [13, 14, 22] have focussed on the latter term in (3.4) for technical reasons, the former term with $\theta = x/\mu$ shows really a 4D extension of the TMG in three dimensions (see eq. (3.10)). Hence, we keep the former term in a sense that the nCS gravity is considered as a parity-odd theory of massive gravity in four dimensions.
It is important to note that applying $\nabla^\mu$ to (3.3) leads to a term of $\nabla^\mu C_{\mu\nu} = 0$ which can be rewritten as

$$-\frac{\partial_\nu \theta}{8} * RR = 0,$$

being the Pontryagin constraint of $* RR = 0$ for non-constant $\theta$.\(^1\) Note that this is an outcome of the Bianchi identity. In this work, we shall consider the Minkowski background metric and scalar ansatz:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2,$$

$$\bar{\theta} = \frac{1}{\mu} x,$$

where $\mu$ is a constant related to a mass parameter. The scalar ansatz is necessary to obtain a massive graviton propagating on the Minkowski background.

We would like to mention a few things about the particular choice (3.7) of $\bar{\theta}$. The first is that $1/\mu$ plays a role of constant coupling, which will be shown in eq. (3.8). The second is that it is needed to describe the plane GWs propagating in the $+z$ direction, implying that all quantities depend on the null retarded time of $(t - z)$ only. The other choice of $\bar{\theta} = y/\mu$ is also possible, but it does not make a significant difference when one compares with (3.7). The canonical embedding of $\bar{\theta} = t/\mu$, even though it is useful for describing a massless graviton with 2 DOF in the metric tensor formalism [8], is not suitable for developing all modes in the nCS gravity. This is because there are no time derivatives in the linearized Cotton tensor for the canonical embedding.

Now we introduce the metric perturbation as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then the linearized equation (3.3) can be cast into the form

$$R_L^{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R^L + \frac{1}{\mu} \bar{\epsilon}^{x\rho\sigma} (\mu \partial_{[\sigma]} R^L_{\nu]\rho} = 0,$$

where $R^L_{\mu\nu}$ and $R^L$ are the linearized Ricci tensor\(^2\) and scalar which consist of the components of the first-order metric perturbation $h_{\mu\nu}$. The Levi-Civita symbol $\bar{\epsilon}^{\mu\nu\rho\sigma}$ is defined by $\bar{\epsilon}^{xyz} = 1$. On the other hand, taking the trace of eq. (3.8) leads to

$$R^L = 0,$$

which implies the non-propagation of Ricci scalar. Imposing $R^L = 0$, the linearized equation (3.8) leads to the first-order time equation for the Ricci tensor

$$R^L_{\mu\nu} + \frac{1}{\mu} \bar{\epsilon}^{x\rho\sigma} (\mu \partial_{[\sigma]} R^L_{\nu]\rho} = 0,$$

which is our main equation for the linearized nCS modified gravity around the Minkowski background. As far as we know, this is a newly derived equation which is similar to the

\(^1\)For a constant $\theta$, eq. (3.5) is automatically satisfied. However, in this case, the Einstein equation (3.3) reduces to that of general relativity with $C_{\mu\nu} = 0$.

\(^2\)We note that a quantity of $R^L_{\mu\nu}$ given in the Ricci tensor formalism is referred to a boosted-up tensor of the massive graviton [5, 23], because it can be written as $R^L_{\mu\nu} = -\frac{1}{2} \square h^{TT}_{\mu\nu}$ under the transverse-traceless gauge.
linearized equation given in cosmological TMG [3]. It is easy to check that the linearized Bianchi identity is satisfied as

\[ \partial \mu R^L_{\mu \nu} = 0 \] (3.11)

because of \( \partial \mu C^L_{\mu \nu} = 0 \). Also, its traceless equation is trivial: \( R^L = C^L = 0 \).

On the other hand, for the canonical embedding of \( \tilde{\theta} = t/\mu \), its linearized equation takes the form

\[ R^L_{\mu \nu} + \frac{1}{\mu} \varepsilon^{\mu \rho \sigma} (\mu | \partial_{[\sigma} R^L_{\nu] \rho]} = 0, \] (3.12)

which is a genuine second-order time equation for the metric tensor \( h_{\mu \nu} \) (a first-order space equation for the Ricci tensor \( R^L_{\mu \nu} \)) because of \( \sigma \neq t \).

4 Polarization modes of GWs

As was introduced in section 2, the NP formalism determines the number of polarization states of GWs in four-dimensional gravity models. Now we wish to find the number of GWs modes \( \theta = x/\mu \) in nCS modified gravity by using the NP formalism. To this end, we first recall the null complex tetrad \( \{ k, n, m, \bar{m} \} \) (2.1) and the GWs set \( \{ \Psi_2, \Psi_3, \Psi_4, \Phi_{22} \} \) (2.5).

We point out that the vanishing of the linearized Ricci scalar (3.9) implies \( \Psi_2 = 0 \) which can be obtained by the relation (2.7). It is worth noting that the condition of \( \Psi_2 = 0 \) also satisfies the Pontryagin constraint \( * R R = 0 \), which translates into a reality condition on a quadratic invariant of the Weyl spinor \( I \) [9]

\[ \Im(I) = \Im(3\Psi_2^2) = 0. \] (4.1)

Here, we used \( \Psi_0 = \Psi_1 = 0 \) because the two Newman-Penrose scalars \( \Psi_0 \) and \( \Psi_1 \) vanish for plane waves propagating along \( +z \) axis [18]. On the other hand, it is noted that the GW mode \( \Psi_2 = 0 \) also satisfies the Pontryagin constraint \( * R R = 0 \), which translates into a reality condition on a quadratic invariant of the Weyl spinor \( I \) [9]

\[ \Im(I) = \Im(3\Psi_2^2) = 0. \] (4.1)

We are now in a position to check whether the remaining modes \( \Psi_3 \) and \( \Phi_{22} \) (more explicitly, \( R^L_{nm}, R^L_{n\bar{m}}, R^L_{nn} \)) are truly independent components. These are modes of helicity \( \pm 1 \) and 0. To this end, we realize that the components of the linearized Ricci tensor in eq. (3.10) are coupled to other components due to the Levi-Civita tensor. It turns out that the \( (t, t), (t, z), \) and \( (z, z) \) components of eq. (3.10) yield one relation

\[ R^L_{tt} + 2R^L_{tz} + R^L_{zz} = -(\partial_t + \partial_z) \left\{ \frac{1}{\mu} (R^L_{zt} + R^L_{yz}) \right\}, \] (4.2)

which implies that the l.h.s. of eq. (4.2) vanishes, because acting an operation \((\partial_t + \partial_z \propto \partial_v)\) on the linearized Ricci tensor with functions of \((t - z \propto u)\) leads to zero. This shows clearly a non-propagation of \( \Phi_{22} \) (\( \Phi_{22} = 0, R^L_{nn} = 0 \)) by observing the relations

\[ \Phi_{22} = -\frac{1}{2} R^L_{nm}, \quad R^L_{nn} = R^L_{tt} + 2R^L_{tz} + R^L_{zz}. \] (4.3)

After some manipulations, all remaining components of eq. (3.10) together with the linearized Bianchi identity (3.11) provide the other relation

\[ \left\{ 1 + \frac{1}{4\mu^2}(\partial_t^2 - \partial_z^2) \right\} \partial_t (R^L_{xt} + R^L_{xz}) = 0, \] (4.4)
which includes a case of the non-propagation of $R_{xt}^L + R_{xz}^L$ (that is, $R_{xt}^L + R_{xz}^L = 0$) at the solution level. At this stage, introducing $\Psi_3 = \text{Re}[\Psi_3] + i\text{Im}[\Psi_3]$ expressed in terms of linearized Ricci tensor

$$\text{Re}[\Psi_3] = R_{nm}^L + R_{n\tilde{m}}^L = -(R_{xt}^L + R_{xz}^L), \quad (4.5)$$

$$\text{Im}[\Psi_3] = -i(R_{nm}^L - R_{n\tilde{m}}^L) = R_{yt}^L + R_{yz}^L, \quad (4.6)$$

(4.2) and (4.4) indicate clearly that we have only one independent component $\text{Im}[\Psi_3]$. This is reminiscent of the fact that the TMG shows one DOF of a given helicity since the gCS term is odd under parity, while the NMG has two DOF because it belongs to a parity-even theory.

It seems that in the $\theta = x/\mu$ nCS modified gravity, there exist three independent modes of GWs:

$$\left( \Psi_2 = 0, \quad \Phi_{22} = 0, \quad \text{Re}[\Psi_3] = 0 \right)$$

$$\text{Re}[\Psi_4] \neq 0, \quad \text{Im}[\Psi_4] \neq 0, \quad \text{Im}[\Psi_3] \neq 0, \quad (4.7)$$

which correspond to the polarization modes (a), (b), and (f) in figure 1, respectively. Thus, the nCS modified gravity might belong to class III$^5$ with three DOF because $\Psi_2 = 0$ and $\Psi_3 \neq 0$. On the other hand, the other choice of $\tilde{\theta} = y/\mu$ might lead to the class III$^5$ with three DOF since $\Psi_2 = 0$ and $\Psi_3 \neq 0$ ($\text{Re}[\Psi_3] \neq 0, \text{Im}[\Psi_3] = 0$) which indicates polarization modes (a), (b), and (e) in figure 1.

However, we would like to mention a weak point of the NP formalism to represent full modes propagating on the Minkowski background in the EW gravity. In the NP formalism, there is no constraint on $\Psi_4$ in any metric theory of gravity because it is determined by Riemann tensor $\Psi_4 = -R_{n\tilde{m}n\tilde{m}}$ even for $R_{\mu\nu}^L = 0$ of Einstein gravity. Accordingly, two of seven GWs in the EW gravity are given by $\Psi_4$, which describes a massive tensor mode. The other five are given by $\Psi_4$, $\Psi_3$ and $\Phi_{22}$ in the NP formalism. On the other hand, in the Ricci tensor formalism (see section 5 for details), the linearized Ricci tensor equation (5.6) indicates that there exist five ($10 - 1 - 4 = 5$) DOF in the EW gravity, which consist of massive GWs only. Here, 5 constraints are obtained by requiring both the traceless condition $R_{\mu\nu}^L = 0$ and the transverse condition of $\partial_{\mu}R_{\mu\nu}^L = 0$. Therefore, in the next section, we use the Ricci tensor formalism to count the number of DOF of propagating massive graviton in the $\theta = x/\mu$ nCS modified gravity.

5 Massive gravity equation

In this section, we explore an explicit wave equation by employing the Ricci tensor formalism. Thereby, we count the number of massive modes in the nCS modified gravity.

Before going further, we comment briefly on TMG and NMG. Let us first introduce a operator $D^\mu_r(m) = \delta^\mu_r + \frac{1}{m} \epsilon_{\alpha\beta} \partial_{\alpha}[4]$ in three dimensions. Using the operator $D$, the linearized Einstein equation of the TMG takes the form $[D(m)\delta R]_{\mu\nu} = 0$ with $\delta R = 0$. Its second-order equation is given by

$$[D(-m)D(m)\delta R]_{\mu\nu} = 0 \rightarrow (\Box - m^2)\delta R_{\mu\nu} = 0, \quad (5.1)$$

which is the linearized equation obtained from the NMG with $\delta R = 0$ and $\partial^\mu \delta R_{\mu\nu} = 0 [4]$. This indicates that the second-order equation (5.1) with a single mass-squared $m^2$ could be
obtained from the first-order equation naturally for the TMG. More schematically, we have

$$\left[ D(-m_-)D(m_+)\delta R \right]_{\mu\nu} = \begin{cases} 
  m_+ = m_- = m & \Rightarrow \square R_{\mu\nu} - m^2 \delta R_{\mu\nu} = 0 \\
  m_+ = m, m_- \to \infty & \text{or} \\
  m_- = -m, m_+ \to \infty & \Rightarrow \delta R_{\mu\nu} + \frac{1}{m} \epsilon^{\rho\sigma} \mu \partial_{\rho} \delta R_{\sigma\nu} = 0 
\end{cases} \quad (5.2)$$

In this sense, the parity-odd TMG with one DOF, which corresponds to a propagating mode with a mass \(m_+\) or \(m_-\), is considered as a “square-root” of the parity-even NMG with two DOF.

Now we are in a position to develop a second-order wave equation in four dimensions, which can describe a massive graviton propagation explicitly. For this purpose, we first express the first-order equation (3.10) arisen from the nCS modified gravity by introducing \(D\) operators:

$$D_{\mu
u}^{\lambda\nu}(\mu) \equiv \delta_{\mu}^{\lambda} \delta_{\nu}^{\nu} + \frac{1}{\mu} \delta_{\mu}^{\lambda} \epsilon_{\nu}^{\sigma} \partial_{\sigma},$$

where \((A, B)\) denotes the symmetrization of \((A, B) = (AB + BA)/2\). Using the \(D(\mu)\) operator, eq. (3.10) can be written compactly as

$$D_{\mu
u}^{\lambda\nu}(\mu) R_{\lambda\nu}^{L} \equiv \left[ D(\mu) R^{L} \right]_{\mu\nu} = 0. \quad (5.4)$$

Acting \(\left[ D(-\mu)D(\mu) \right]^{\lambda\nu} \) on \(R_{\lambda\nu}^{L}\), we obtain a complicated second-order equation

$$\left[ D(-\mu)D(\mu) R^{L} \right]_{\mu\nu} = R_{\mu\nu}^{L} - \frac{1}{\mu^2} \left[ \square R_{\mu\nu}^{L} + \frac{1}{2} \eta_{\mu\nu} \square R_{xx}^{L} - \frac{1}{2} \partial_{\rho} \partial_{\nu} R_{xx}^{L} - \frac{3}{2} \delta_{(\mu}^{\rho} \square R_{\nu)x}^{L} \right] = 0. \quad (5.5)$$

This is not obviously a tensor wave equation that describes a massive graviton propagating in the Minkowski background

$$\square R_{\mu\nu}^{L} - \mu^2 R_{\mu\nu}^{L} = 0, \quad (5.6)$$

which was obtained from the EW gravity (1.1) together with \(R^{L} = 0\), and \(\partial^{\mu} R_{\mu\nu}^{L} = 0 [19]\). Eq. (5.6) is a standard wave equation which describes five DOF of a massive graviton propagating on the Minkowski background in the Ricci tensor formalism. However, it is important to note that the \((x, x)\) component of eq. (5.5) leads simply to \(R_{xx}^{L} = 0\), which kills one DOF. Therefore, we have four DOF \((5 - 1 = 4)\) totally. It turns out that finally, the tensor wave equation (5.5) can be classified into two types:

$$\left[ D(-\mu)D(\mu) R^{L} \right]_{\mu\nu} = \begin{cases} 
  \square R_{\mu\nu}^{L} - \mu^2 R_{\mu\nu}^{L} = 0, & \text{[mode 1]} \\
  \square R_{\mu\nu}^{L} - 4 \mu^2 R_{\mu\nu}^{L} = 0, & \text{[mode 2]}
\end{cases} \quad (5.7)$$

We would like to mention a few things observed from eqs. (3.10) and (5.7). Firstly, the components of linearized Ricci tensor in mode 1 do not couple to those in mode 2. Secondly, for example, one can check from the equation (3.10) that \(R_{xy}^{L}\) (one component of mode 1 in eq. (5.7)) can be expressed in terms of components \(R_{xz}^{L}\) and \(R_{yz}^{L}\):

$$R_{xy}^{L} = \frac{1}{\mu} (\partial_{\rho} R_{xz}^{L} - \partial_{z} R_{xy}^{L}).$$
we have one independent component of the Ricci tensor coming from mode 1 when one requires five constraints from (3.10) for $(R^L_{1tL}, R^L_{yL}, R^L_{zL}, R^L_{yy}, R^L_{yz})$. The mode 2 is ruled out because its mass-squared is not $\mu^2$ but $4\mu^2$. Thirdly, $\text{Im}[\Psi_3] = R^L_{yL} + R^L_{yz}$ belongs to the mode 1, while $\text{Re}[\Psi_3] = -(R^L_{ztL} + R^L_{zz})$ belongs to the mode 2.

By analogy of the TMG and NMG (5.2), we propose that the first second-order equation in (5.7) is related to the first-order equation (3.10) as follows:

\[
\left[D(-m_1)D(m_2)R^L_{\mu\nu}\right]_{\mu\nu} = \begin{cases} 
  m_1 = m_2 = \mu 
  \Rightarrow \Box R^L_{\mu\nu} - \mu^2 R^L_{\mu\nu} = 0 \\
  m_1 = -\mu, \ m_2 \rightarrow \infty \\
  \text{or} \\
  m_2 = \mu, \ m_1 \rightarrow \infty 
\end{cases} 
\]

which implies that when the $\theta = x/\mu$ nCS modified gravity is considered as the parity-odd theory of massive gravity in four dimensions, we have one DOF which corresponds to a massive propagating mode with mass $\mu$.

6 Discussions

In this work, we have considered the nCS modified gravity with $\theta = x/\mu$ as a parity-odd model of massive gravity in four dimensions like as the TMG [2] in three dimensions. This means that in this case, the nCS gravity is not a modified gravity which mimics Einstein gravity of describing two massless modes $\Psi_4$. Here we have found one additional polarization mode of $\text{Im}[\Psi_3]$ by using the NP formalism. The presence of $\text{Im}[\Psi_3]$ implies that the nCS gravity provides three DOF (two for massless mode and one for massive mode). This was compared to the nCS gravity with the canonical embedding $\theta = t/\mu$ where two massless modes are found only.

However, we have to point out an insufficiency of the NP formalism to represent full modes in the EW gravity. In the NP formalism, there is no constraint on $\Psi_4$ in any metric theory of gravity because it must be determined by the Riemann tensor $\Psi_4 = -R_{\alpha\beta\gamma\delta}$, which corresponds to the massless mode. In the case of the EW gravity, we have two modes of $\tilde{\Psi}_4$ the massive ghost mode, in addition to five of $\Psi_3$, $\Phi_{22}$, and $\Psi_4$ [19]. It shows that the whole propagating modes are seven in the EW gravity.

As were previously shown in three-dimensional massive gravity models [3, 4], the third- and fourth-order gravity theories become a first- and second-order gravity theories when one introduces either the linearized Ricci tensor or Einstein tensor (cosmological TMG and NMG) instead of metric tensor. A usage of Ricci and Einstein tensors may be a known way to avoid ghost problem in higher-derivative gravity theories. Similarly, we have introduced the Ricci tensor formalism to extract a massive mode because we have regarded the $\theta = x/\mu$ nCS gravity as a massive gravity. As a byproduct, we do not worry about the appearance of the massive ghost states.

If the Ricci tensor was employed to describe the linearized Einstein equation, the latter became the first-order equation (3.10) which could be used to kill non-propagating modes in the nCS gravity. In order to count the number of independent Ricci tensor modes, we have to find the second-order wave equation (5.6) together with the transverse-traceless condition found in the EW gravity. As a result, we have found the tensor wave equation (5.7) for one independent mode which is compared to five from the EW gravity [19]. Taking into account
full modes in the metric tensor formalism, the nCS gravity provides three, while the EW gives us seven. In this sense, we may regard the nCS gravity (parity-odd) as a square-root of the EW gravity (parity-even), which is a four-dimensional version of “TMG as a square-root of NMG”. In summary, comparing with the metric formalism, both the NP and Ricci tensor formalisms are not suitable for counting number of seven DOF in the EW gravity, while the Ricci tensor formalism is not appropriate for finding three DOF in the $\theta = x/\mu$ nCS gravity:

| gravity models | formalisms | metric tensor | Ricci tensor | NP |
|----------------|------------|---------------|--------------|----|
| EW gravity     | 7 DOF (2:massless+5:massive) | 5 (0+5) | 5 (2+3) |
| $\theta = x/\mu$ nCS gravity | 3 (2+1) | 1 (0+1) | 3 (2+1) |

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