Universal algebraic growth of entanglement entropy in many-body localized systems with power-law interactions

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Power-law interactions play a key role in a large variety of physical systems. In the presence of disorder, these systems may undergo many-body localization for a sufficiently large disorder. Within the many-body localized phase the system presents in time an algebraic growth of entanglement entropy, \( S_{vN}(t) \propto t^{\gamma} \). Whereas the critical disorder for many-body localization depends on the system parameters, we find by extensive numerical calculations that the exponent \( \gamma \) acquires a universal value \( \gamma_c \approx 0.33 \) at the many-body localization transition, for different lattice models and power laws. Moreover, our results suggest an intriguing relation between \( \gamma_c \) and the critical minimal decay power of interactions necessary for the observation of many-body localization.

The interplay between disorder and interactions plays a key role in the understanding of transport and thermalization in many-body quantum systems. Whereas quantum interference leads to Anderson localization in non-interacting disordered systems [1], localization may occur even in highly excited states in the presence of interactions [2]. Many-body localization (MBL) is of fundamental relevance in quantum statistical mechanics, being the only known robust mechanism that may prevent thermalization in an isolated system. As a result, MBL has attracted a huge attention in recent years [3–7], including breakthrough experiments [8–14].

Whereas MBL research has mostly focused on local interactions, recent works are unveiling the intriguing thermalization and MBL physics in disordered systems with power-law interactions [15–32]. One on hand, this is justified by the possible relevance of long-range interacting systems for the understanding of MBL in dimensions larger than one [23, 33]. On the other hand, power-law interactions (van der Waals, dipolar, Coulomb, or even of variable power) are fundamentally relevant for a large variety of physical systems, including nuclear spins [35], nitrogen vacancy centers in diamonds [34], polar molecules [36], magnetic atoms [37, 38], Rydberg gases [39, 40], atoms at photonic crystals [41], and trapped ions [42, 43]. One-dimensional XXZ spin models, with both Ising and exchange interactions decaying with the interparticle distance \( r \) as \( 1/r^a \), have been predicted to present MBL for a sufficiently large disorder for \( a > a_c = 2 \) [15, 17, 18]. For XY models, with just spin exchange, MBL has been predicted for \( a > a_c = 3/2 \) due to emerging Ising interactions [19], although recent numerical calculations indicate that \( a_c \) may be smaller [26, 24, 32].

Entanglement dynamics within the MBL phase presents intriguing features. In particular, whereas the entanglement entropy, \( S_{vN} \), saturates in the Anderson localized case to a system-size independent value, in MBL systems entanglement propagates due dephasing even in the absence of energy or particle transport. In particular, for local interactions, \( S_{vN} \) grows logarithmically in time until reaching a volume-law value [44–47]. This entropy growth results from the interaction between exponentially localized local integrals of motion (LIOMs) [48, 49] adiabatically connected with the single-particle states. In contrast, in power-law interacting systems single-particle localization is rather algebraic [50, 51], and a similar LIOM logic predicts algebraic entropy growth [27], as observed numerically [16, 26, 27].

In this Letter, we investigate the MBL phase of hard-core bosons with power-law hops and interactions, or equivalently spin models with power-law exchange and Ising terms. Using exact diagonalization, we determine the onset of MBL from level spacing statistics for different models of experimental relevance: XY model, XXZ model with equal decay power for Ising and exchange interactions, and extended-Hubbard model (EHM) with nearest-neighbor (NN) hops and power-law interactions. Using exact evolution and Krylov techniques, we analyze the entanglement dynamics, and in particular the algebraic growth \( S_{vN}(t) \propto t^{\gamma} \). In contrast to previous studies, which concentrated in a narrow range of \( a \) values and a single disorder strength \( W \) well within the MBL regime [26, 27], we analyze in detail the dependence on \( W \), showing that the algebraic growth presents a remarkable universality at the critical disorder \( W_c(a) \) that marks the onset of MBL. At criticality, \( \gamma = \gamma_c \approx 0.33 \), for both the XY and XXZ models, irrespective of the decay power. Interestingly, our results suggest a surprising relation between \( \gamma_c \) and the critical \( a_c \) for MBL.

Model.— We consider hard-core bosons in a disordered 1D lattice, which present both power-law hopping and interactions. The system is described by the Hamiltonian:

\[
\hat{H} = -J \sum_{i,j \neq i} \frac{1}{|r_i - r_j|^a} (\hat{b}_i^\dagger \hat{b}_j + \text{h.c.}) + V \sum_{i,j \neq i} \frac{1}{|r_i - r_j|^a} \hat{n}_i \hat{n}_j + \sum_j \epsilon_j \hat{n}_j, \tag{1}
\]

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where $\hat{b}_j$ are bosonic operators at site $j$ $(\langle \hat{b}_j^2 \rangle = 0)$, $\hat{n}_j = \hat{b}_j^\dagger \hat{b}_j$, $J = 1$ and $V$ are, respectively, the hopping amplitude and interaction strength to nearest-neighbors (NN), and $\epsilon_j$ is a random on-site energy uniformly distributed between $-W$ and $W$. Hamiltonian (1) is interesting for a large variety of physical problems. It may be mapped to a spin-1/2 Hamiltonian with power-law exchange and Ising terms, and random on-site magnetic field. In particular, when $V = 0$ (or equivalently $b = 0$, due to number conservation), Eq. (1) reduces to an XY Hamiltonian, as that already realized, in absence of disorder, in polar molecules with two available rotational states [38], Rydberg atoms [40], and trapped ions [42, 43]. The case $a = b$, which reduces to a power-law XXZ model, is directly relevant for two-component magnetic atoms [37] and polar molecules in the presence of an external electric field. Finally, for $a = \infty$ the model reduces to the extended Hubbard model (EHM) with NN hopping and power-law interactions, already realized in magnetic atoms polarized in the maximally stretched state [39]. Below, we focus on the localization properties and dynamics of these experimentally relevant cases.

Many-body localization.— We first establish, for the different cases, the critical disorder strength to achieve MBL. We consider a lattice of $L$ sites with open-boundary conditions. By means of exact diagonalization for up to $L = 18$ sites, we determine the eigenstates for $L/2$ hard-core bosons. We study the level-spacing statistics, characterized by $r_n = \min(\delta_n, \delta_{n-1})/\max(\delta_n, \delta_{n-1})$, where $\delta_n \equiv E_{n+1} - E_n$, and $E_n$ denotes the eigenenergies in growing order. We obtain $\langle r \rangle$ after averaging $r_n$ over all states with $-W < E_n < W$ (in order to avoid spurious effects given by states at the spectral edges) and over up to 1000 disorder realizations. We determine $\langle r \rangle$ for different system sizes $L = 14$, 16, and 18. In the thermodynamic limit, it is expected that integrable or MBL systems are characterized by a Poissonian level spacing distribution, characterized by $\langle r \rangle \approx 0.386$, whereas ergodic systems present a Wigner-Dyson distribution in the Gaussian Orthogonal Ensemble, which results in $\langle r \rangle \approx 0.529$. The critical disorder strength $W_c$ marking the onset of MBL is then given by the crossing point of the $\langle r \rangle$ curves for different $L$, which is hence stationary under scaling of the system size. In order to determine properly $W_c$, we perform a finite-size scaling analysis, expressing $\langle r \rangle$ as a function of $(W - W_c)^{1/\nu}$, such that curves for different $L$ collapse, as illustrated in Fig. 1(a) for $b = 0$ and $a = 3$.

Phase diagrams.— Figure 1(b) depicts the phase diagram as a function of $W/J$ and $a$ for the XY model ($b = 0$ or $V = 0$). For $a < a_c \approx 3/2$, we observe, for the available system sizes in our numerics, no clear crossing point in the finite-size scaling of $\langle r \rangle$, and hence no trace of MBL. This is in agreement with the predictions of Ref. [19], although we cannot rule out that the critical power may be slightly lower [26, 29, 32]. Figure 1(c) shows the XXZ case ($a = b$) with $V = 1$. In agreement with Ref. [15, 18], we observe no MBL for $a < a_c \approx 2$. Finally, Fig. 1(d) depicts the EHM with NN hopping, for which MBL occurs at any $b$ value, being enhanced for $b < 2$ [29].

Entanglement entropy.— We are particularly interested in the entanglement dynamics within the MBL region. In the following, we consider that the system is initially prepared in a half-filled density wave state $\ldots 101010 \ldots$. This choice is inspired by experiments [8], but other choices should not alter the results. The entanglement dynamics is monitored by means of the entanglement entropy, $S_{\nu,N}(t) = -\text{Tr}[\hat{\rho}_A \ln \hat{\rho}_A]$, with $\hat{\rho}_A = \text{Tr}_B[\hat{\rho}]$ the reduced density matrix of the left half of the system $(A)$ when tracing over the other half $(B)$.

For systems up to $L = 18$ sites, we determine the dynamics at any time $t > 0$ using exact evolution. Krylov subspace techniques allow us calculations with larger system sizes (up to $L = 22$) but they are limited to moderate time scales. We have checked that for a given disorder realization and up to $L = 18$ the Krylov and the exact calculation provide the same result within $10^{-6}$ relative error in the determination of $S_{\nu,N}(t)$. A large number of disorder realizations is crucial to achieve good statistics and converging results for $\gamma$, since anomalous regions of

![FIG. 1: (a) $\langle r \rangle$ for $b = 0$, $a = 3$ as function of $W$ for $L = 14$, 16, and 18. Phase diagrams, evaluated using the level spacing statistics, as a function of $W$ and the decay power $a$ for (b) XY model ($b = 0$), (c) XXZ model ($a = b$ and $V = 1$), and (d) EHM model ($a = \infty$, $V = 1$). We indicate the value of the power $\gamma$ of the growth of $S_{\nu,N}(t)$ at the MBL transition.](image)
small disorder increase the value of \( \gamma \). For \( L = 14, 16, 18 \) we choose up to 2000 samples, and for \( L = 20 \) up to 1000 samples, which lead to converging results \([52]\).

Figure 2(a) shows our results for the XY model \((b = 0)\) with \( a = 3 \) and \( W = 8 \), which illustrate a typical entropy growth in our calculations. Initially, \( S_{cN}(t) = 0 \) since we start with a Fock state. At short times local dynamics leads to an entropy growth that is independent of the power \( \gamma \), and indeed it is shared by the \( a = \infty \) (NN) case, which presents Anderson localization. After this initial dynamics, for finite \( a \), the entropy grows algebraically, \( S_{cN}(t) \propto t^\gamma \), until saturating at a value that depends on the system size. We note that the onset of the algebraic growth is delayed to longer times when \( a \) grows, resulting in an entropy plateau shared with the NN case. The onset time diverges when \( a \to \infty \).

As shown in Fig. 2(a), the slope of the log-log curves converges within various decades for \( L = 14, 16, 18 \) and 20 (the latter obtained with the Krylov method, and hence limited to shorter times), allowing us to exclude finite-size effects in the determination of \( \gamma \) by fitting only within the converged time window.

**Universal entropy growth in the XY model.** The value of the exponent \( \gamma \) depends on both the disorder strength \( W \) and the powers \( a \) and \( b \). Figure 2(b) depicts our results for the XY model with \( a = 3 \) and \( L = 20 \) sites, for \( W \) around the critical \( W_c \simeq 3.5 \). Both for the MBL and the extended phase we have an algebraic growth. Although our focus is on the MBL regime, we note that the entropy growth in the extended regime is far from linear as one would expect for an ergodic system \([52]\), indicating possible non-ergodicity of the extended regime, at least in the vicinity of the MBL phase. The power \( \gamma \) decreases with growing \( W \), with a critical power \( \gamma_c \simeq 0.33 \) at \( W = W_c \). As we discuss in the following, this critical entropy growth turns out to be universal. This universality constitutes the main result of this paper.

In Fig. 3(a1) we depict \( \gamma \) for different \( a \) values. Although the level-spacing analysis, which provides \( W_c(a) \), and the study of the dynamics, which provides \( \gamma \), are independent from each other, we observe that all \( \gamma(a, W) \) curves converge at criticality, \( W = W_c(a) \), at a value that within our numerical accuracy is approximately \( \gamma_c \simeq 0.33 \pm 0.02 \) \([54]\). Hence, remarkably, the critical algebraic entropy growth at the MBL on-set is independent of \( a \) (see also Fig. 2(a2)). Moreover, for \( W > W_c(a) \) in the vicinity of the MBL boundary, \( \gamma \) becomes to a good approximation a universal function of \((a - 3/2)(W - W_c(a))\) (Fig. 3a2).

For large \( W \), we may expect that the LIOMs can be approximated by the population of single-particle states, which remain algebraically localized at lattice sites with the same power \( a \) of the XY exchange \([50, 51]\). Hence, the interaction between LIOMs placed at a distance \( r \) (resulting from the hard-core constraint) should decay as \( 1/r^2a \). As a result, we would expect for large \( W \), \( \gamma = \gamma_\infty(a) = 1/2a \) at large disorder. As shown in Fig. 3(a1), this is approximately the case (dashed lines indicate \( \gamma_\infty(a) \)). Calculations with large system sizes would be however necessary to establish the asymptotic \( \gamma_\infty(a) \) dependence more precisely, since due to the low saturation entropy we cannot perform reliable fits of \( \gamma \) for \( W > 12 \) for the system sizes we can evaluate. Interestingly, if \( \gamma = \gamma_c \) holds for the MBL transition all the way till \( a = a_c \), and since \( W_c(a_c) \) diverges, then we would...
expect $\gamma_c = \gamma_\infty(a_c)$. Note that this is indeed fulfilled for
$\gamma_\infty(a) \simeq 1/2a$, $a_c \simeq 3/2$ and $\gamma_c \simeq 1/3$.

Universal entropy growth in other models.— Interestingly,
a similar analysis for the XXZ model with $a = b$ reveals that the algebraic
growth of $S_{cN}(t)$ is also universal at the onset of MBL with the same
exponent $\gamma_c \simeq 0.33$ (see Fig. 3(b1)). Moreover, within the MBL
in the vicinity of $W_c(a)$ $\gamma$ is a universal function of
$(W - W_c(a))(a - 2)$ (see Fig. 3(b2)). For large $W$, follow-
ing the arguments of Ref. 10, we may expect a de-
pendence $\gamma_\infty(a) \simeq 1/(a + 1)$. Our numerical calculations,
which as for the XY model are limited to $W < 12$, suggest
that this is approximately the case. Similar as above, we
would expect $\gamma_c = \gamma_\infty(a_c)$. Note that the latter would
be fulfilled for $\gamma_\infty(a) \simeq 1/(a + 1)$, $a_c \simeq 2$, and $\gamma_c \simeq 1/3$.
The results for the XXZ and XY models hence suggest
that there is an intriguing relation between the universal
growth of $S_{cN}(t)$ at the MBL transition, given by $\gamma_c$, and
the critical power $a_c$ for observation of MBL.

Finally, we have analyzed as well the dynamics in the
EHM ($a = \infty$), see Fig. 1(d) 58. We obtain a critical
$\gamma_c \simeq 0.33$ for $b > 2$, but the critical power signifi-
cantly deviates from the universal behavior for $b < 2$, indic-
ating that finite power-law hopping may be necessary to
retrieve universal entropy growth at criticality for the
case of long-range Ising-like interactions.

Conclusions.— Hard-core bosons with power-law-
decaying hops and interactions in disordered 1D lattices,
or equivalently spin models with power-law-decaying ex-
change and Ising terms present MBL for a sufficiently
large disorder. By means of level spacing statistics, we have determined the MBL regime for three models of par-
ticular experimental relevance: XY model, XXZ model,
and EHM model with power-law interactions. Due to
algebraic localization of LIOMs, the entanglement dy-
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[1] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[2] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Ann. Phys. 321, 1126 (2006).
[3] R. Nandkishore and D.A. Huse, Annu. Rev. Condens. Matter Phys. 6, 15 (2015).
[4] E. Altman and R. Vosk, Annu. Rev. Condens. Matter Phys. 6, 383 (2015).
[5] D.A. Abanin and Z. Papić, Ann. Phys. 529, 1700169 (2017).
[6] F. Alet and N. Laflorencie, C. R. Physique 19, 498 (2018).
[7] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Rev. Mod. Phys. 91, 021001 (2019).
[8] M. Schreiber et al., Science 349, 842 (2015).
[9] J. Choi et al., Science 352, 1547 (2016).
[10] P. Bordia et al., Phys. Rev. Lett. 116, 140401 (2016).
[11] J. Smith et al., Nat. Phys. 12, 907 (2016).
[12] H. P. Lüschen et al., Phys. Rev. Lett. 119, 260401 (2017).
[13] A. Lukin et al., Science 364, 256 (2019).
[14] M. Rispoli et al., Nature 573, 385 (2019).
[15] A. L. Burin, arXiv:0611387.
[16] M. Pino, Phys. Rev. B 90, 174204 (2014).
[17] N. Yao et al., Phys. Rev. Lett. 113, 243002 (2014).
[18] A. L. Burin, Phys. Rev. B 91, 094202 (2015).
[19] A. L. Burin, Phys. Rev. B 92, 104428 (2015).
[20] P. Hauke and M. Heyl, Phys. Rev. B 92, 134204 (2015).
[21] H. Li, J. Wang, X.-J. Liu, and H. Hu, Phys. Rev. A 94, 063625 (2016).
[22] D. B. Gutman et al., Phys. Rev. B 93, 245427 (2016).
[23] R. Singh, R. Moessner, and D. Roy, Phys. Rev. B 95, 094205 (2017).
[24] R. M. Nandkishore and S. L. Sondhi, Phys. Rev. X 7, 041021 (2017).
[25] K. S. Tikhonov and A. D. Mirlin, Phys. Rev. B 97, 214205 (2018).
[26] A. Safavi-Naini et al., Phys. Rev. A 99, 033610 (2019).
[27] G. De Tomasi, Phys. Rev. B 99, 054204 (2019).
[28] S. Nag and A. Arg, Phys. Rev. B 99, 224203 (2019).
[29] S. Roy and D. E. Logan, SciPost Phys. 7, 042 (2019).
[30] T. Botzung et al., Phys. Rev. B 100, 155136 (2019).
[31] J. Choi et al., Phys. Rev. Lett. 122, 043603 (2019).
[32] S. Schiffer, J. Wang, X.-J. Liu, and H. Hu, arXiv:1908.04031.
[33] B. Kloss and Y. Bar Lev, arXiv:1911.07587.
[34] G. Waldherr et al., Nature 506, 204 (2014).
[35] G. A. Álvarez, D. Suter, and R. Kaiser, Science 349, 846
[36] B. Yan et al., Nature **501**, 521 (2013).
[37] A. de Paz et al., Phys. Rev. Lett. **111**, 185305 (2013).
[38] S. Baier et al., Science **352**, 201 (2016).
[39] J. Zeiher et al., Phys. Rev. X **7**, 041063 (2017).
[40] S. de Léséleuc et al., Science **365**, 775 (2019).
[41] C.-L. Hung, A. González-Tadela, J. I. Cirac, and H. J. Kimble, PNAS **113** E4946 (2016).
[42] P. Richerme et al., Nature **511**, 198 (2014).
[43] P. Jurcevic et al., Nature **511**, 202 (2014).
[44] M. Znidaric, T. Prosen, and P. Prelovsek, Phys. Rev. B **77**, 064426 (2008).
[45] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. **109**, 017202 (2012).
[46] M. Serbyn, Z. Papic, and D. A. Abanin, Phys. Rev. Lett. **110**, 260601 (2013).
[47] R. Vosk and E. Altman, Phys. Rev. Lett. **110**, 067204 (2013).
[48] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. **111**, 127201 (2013).
[49] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B **90**, 174202 (2014).
[50] X. Deng, B. L. Altshuler, G. V. Shlyapnikov, and L. Santos, Phys. Rev. Lett. **117**, 020401 (2016).
[51] X. Deng, V. E. Kravtsov, G. V. Shlyapnikov, and L. Santos, Phys. Rev. Lett. **120**, 110602 (2018).
[52] Time-dependent density-matrix renormalization group (t-DMRG) calculations allow for larger sizes [26], but they are necessarily limited to short time scales (compromising the fitting discussed in the main text), and too expensive to afford for the necessary large number of disorder realizations.
[53] H. Kim and D. A. Huse, Phys. Rev. Lett. **111**, 127205 (2013).
[54] The error bar is determined by the accuracy of the linear fitting of the algebraic growth region in log-log plots such as Fig. 2(a). Error bars are typically smaller than the symbols depicted in Figs. 3.
[55] Algebraic entanglement growth in this model was discussed in Ref. [16] for $V \ll 1$ and a fixed large disorder $W = 8$ for up to $L = 16$. We have checked that our $\gamma$ coincides with that of that reference for $L = 16$, $b = 1$ and $V = 0.1$.
[56] We note that for the XY model, for the particular $W$ value and window of $a$ values discussed in Ref. [26], we obtain approximately the same exponent $\gamma$. 