On Optimal Weighted-Delay Scheduling in Input-Queued Switches

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Abstract
Motivated by relatively few delay-optimal scheduling results, in comparison to results on throughput optimality, we investigate an input-queued switch scheduling problem in which the objective is to minimize a linear function of the queue-length vector. Theoretical properties of variants of the well-known MaxWeight scheduling algorithm are established within this context, which includes showing that these algorithms exhibit optimal heavy-traffic queue-length scaling. For the case of $2 \times 2$ input-queued switches, we derive an optimal scheduling policy and establish its theoretical properties, demonstrating fundamental differences with the variants of MaxWeight scheduling. Our theoretical results are expected to be of interest more broadly than input-queued switches. Computational experiments demonstrate and quantify the benefits of our optimal scheduling policy.

1 Introduction
Input-queued switch architectures are widely used in modern computer and communication networks. The analysis and control of input-queued switches is critical for our understanding of design and performance issues related to internet routers, data-center switches and high-performance computing. There has been a large and rich literature around scheduling in such computer and communication systems.

The MaxWeight scheduling policy, first introduced in [6] for wireless networks and subsequently in [3] specifically for input-queued switches, is well-known for being throughput optimal. However, the issue of
delay-optimal scheduling for switches is less clear. In [3], the MaxWeight policy is shown to be asymptotically optimal in heavy traffic for an objective function of summation of the squares of the queue length with the assumption of complete resource pooling. In [2], MaxWeight scheduling is shown to be optimal in heavy traffic for an objective function of the summation of queue length, under the assumption that all the ports are saturated. These results are extended to the case of incompletely saturated in [1], though still for the summation of the average queue length. Nevertheless, the question of delay-optimal scheduling in input-queued switches remains open in general, as does the question of scheduling with more general objective functions.

In this paper, we seek to gain fundamental insights on delay-optimal scheduling in input-queued switches where a general linear cost function of queue length (delay) is associated with each queue. The objective of the corresponding stochastic control problem is to determine the scheduling policy that minimizes the discounted sum of these linear cost functions of the expected delays. Our derivation of an optimal solution involves the partitioning of the state space into different scheduling policy decision regions, where we derive an optimal policy and establish structural properties of the associated value function for these regions. In the $2 \times 2$ input-queued switch model, we completely identify three scheduling policy decision regions of interest, namely the trivial boundary, the critical boundary and the interior. We show that the optimal solution in the interior follows the $c\mu$ policy. On the other hand, for all but trivial regions of the boundary of the state space wherein the optimal solution is obvious, we establish that an optimal policy consists of a switching curve that takes into account the arrival processes. In some specific cases, we derive the optimal switching curve. More generally, we introduce an approach to approximate the optimal policy switching curve and we show that this renders an asymptotically optimal policy. We note that some of these results can also be extended to the case of general $n \times n$ input-queued switches. Meanwhile, our optimal weighted-delay scheduling analysis and results for the $2 \times 2$ switch are the first of a kind and the fundamental insights that we gain from such analysis and results will be valuable for our understanding of general systems.

Another important contribution of this paper to gain key insights on the fundamental properties of MaxWeight scheduling in the above setting for general $n \times n$ input-queued switches. In particular, we extend the results in [2] and prove that a weighted-variant of the MaxWeight policy has optimal scaling in heavy traffic for a general linear function of the average queue lengths. Although the heavy traffic analysis involves an objective function consisting of steady-state average of queue length, which can not be directly translated into the objective function of our discounted optimal control problem in general, the connections between these two frameworks are well understood; see, e.g., [4]. For the special case of unit cost functions, our results show that an optimal solution of the discounted control problem consists of choosing any admissible scheduling decision in the interior of the state space, following an appropriate switching curve in the nontrivial boundary of the state space, and controlling the trivial boundary with a work conserving policy. Since the input-queued switch system tends to spend all of its time in the interior of the state space asymptotically in the heavy-traffic regime, and the MaxWeight scheduling algorithm as an admissible policy is consistent with the optimal solution for the interior region derived herein, our results provide further understanding of the recent results in [2] showing MaxWeight to be delay-optimal asymptotically in the heavy-traffic regime for the unit cost function. Our results also shed light on the delay optimality of MaxWeight scheduling and its variants in general.

We also performed numerous computational experiments that compare our optimal scheduling policy with variants of MaxWeight scheduling. Under a symmetric scenario with Bernoulli arrivals, we derive the optimal switching curve which renders a policy that is optimal for both the total discounted cost and the long-run average cost. The relative optimality gap between our optimal policy and the MaxWeight scheduling policy is shown to be larger for heavier traffic intensities. More generally, computational experiments with our asymptotically optimal switching curve policy suggest rapid convergence to the optimal policy. Analogous to the symmetric case, the relative optimality gap between our optimal policy and the MaxWeight scheduling policy is once again shown to be larger for heavier traffic intensities.

We next present the details of our mathematical model and formulation for the general $n \times n$ input-queued switch model in Section 3. Our analysis and results for variants of MaxWeight scheduling is provided in Section 5. Our analysis and results for optimal scheduling and structural properties in $2 \times 2$ input-queued switches are presented in Section 4, followed by many of our proofs collected in Section 6. A representative sample of our computational experiments provided in Section 3 followed by concluding remarks. Additional proofs are presented in Appendix A and Appendix B.
2 Model and Formulation

Consider an input-queued switch with $n$ input ports and $n$ output ports. Each input port has a queue associated with every output port that stores packets waiting to be transmitted to the output port. Let $(i, j) \in \mathcal{I} := \{(i, j) : i, j \in [n]\}$, $[n] := \{1, \ldots, n\}$, index the queue associated with the $i$th input port and the $j$th output port. Packets arrive at queue $(i, j)$ from an exogenous stochastic process.

Time is slotted and denoted by a nonnegative integer $t \in \mathbb{Z}_+ := \{0, 1, \ldots\}$. At each time $t$, a scheduling policy selects a set of queues from which to simultaneously transmit packets under the constraints: (1) At most one packet can be transmitted from an input port; (2) At most one packet can be transmitted to an output port. We refer to a schedule as a subset of queues that satisfies these constraints.

A schedule is formally described by an $n^2$-dimensional binary vector $s = (s_{ij})_{(i, j) \in \mathcal{I}}$ such that $s_{ij} = 1$ if queue $(i, j)$ is in the schedule, and $s_{ij} = 0$ otherwise. Let $\mathcal{P}$ denote the set of all possible schedules:

$$\mathcal{P} = \left\{ s \in \{0, 1\}^{n^2} : \sum_{j \in [n]} s_{ij} = 1, \forall i \in [n] \right\},$$

and $\mathbf{S}^\pi(t) \in \mathcal{P}$ the schedule under policy $\pi$ for period $t$. Let $Q_{ij}^\pi(t) \in \mathbb{Z}_+$ denote the length of queue $(i, j)$ at time $t$ under policy $\pi$ and $A_{ij}(t) \in \mathbb{Z}_+$ the number of arrivals to queue $(i, j)$ during $[t, t+1)$. The queueing dynamics under policy $\pi$ then can be expressed as

$$Q_{ij}^\pi(t + 1) = Q_{ij}^\pi(t) + A_{ij}(t) - S_{ij}^\pi(t) - \mathbb{I}_{\{Q_{ij}^\pi(t) > 0\}},$$

where $\mathbb{I}_{\mathcal{A}}$ denotes an indicator function associated with event $\mathcal{A}$, returning 1 if $\mathcal{A}$ is true and 0 otherwise.

We assume that $\{A_{ij}(t) : t \in \mathbb{Z}_+, (i, j) \in \mathcal{I}\}$ are independent random variables and that, for fixed $(i, j) \in \mathcal{I}$, $\{A_{ij}(t) : t \in \mathbb{Z}_+\}$ are identically distributed with $\mathbb{E}[A_{ij}(t)] = \lambda_{ij}$. Define $Q^\pi(t) := (Q_{ij}^\pi(t))_{(i, j) \in \mathcal{I}}$, $A(t) := (A_{ij}(t))_{(i, j) \in \mathcal{I}}$, $\mathbf{S}^\pi(t) := (S_{ij}^\pi(t))_{(i, j) \in \mathcal{I}}$, and $\mathbf{U}^\pi(t) := (U_{ij}^\pi(t))_{(i, j) \in \mathcal{I}}$.

Our goal is to establish an optimal scheduling policy that minimizes the total discounted delay cost over an infinite horizon. Given the relationship between delays and queue lengths, we henceforth focus on cost as a function of queue length. Specifically, the cost under policy $\pi$ at time $t$ is a linear function of the queue lengths at time $t$:

$$c^\pi(t) = \sum_{(i, j) \in \mathcal{I}} c_{ij}Q_{ij}^\pi(t)$$

for the cost function constants $c_{ij}$. We are interested in the total discounted cost over an infinite horizon given by

$$J_\beta(q, \pi) := \sum_{t=0}^{\infty} \beta^t \mathbb{E}[c^\pi(t)], \quad Q^\pi(0) = q,$$

with discount factor $\beta \in (0, 1)$ and $Q^\pi(t)$ following \(\mathcal{I}\), or equivalently \(\mathcal{I}\).

Observe from \(\mathcal{I}\) that $Q^\pi(t + 1)$ is determined by $\mathbf{S}^\pi(t)$, which is under our control. A scheduling policy is admissible if the schedule $\mathbf{S}^\pi(t)$ at time $t$ is based solely on information revealed up to time $t$, such as $\mathbf{S}^\pi(s)$, $Q^\pi(s + 1)$, and $A(s)$ for all $s < t$. It follows from known results in Markov decision process theory \(\mathcal{I}\) that there exists an optimal stationary policy, on which $\mathbf{S}^\pi(t)$ depends only on $Q^\pi(t)$, and therefore we restrict our attention herein to stationary scheduling policies. Specifically, we seek to find a stationary scheduling policy with the following objective:

Minimize $J_\beta(q, \pi)$ over all stationary policies $\pi$. \(\mathcal{P}_\beta\)

3 Heavy Traffic Analysis of Weighted MaxWeight Algorithm

For the purposes of our heavy traffic analysis in this section, we alternatively express the queueing dynamics under the weighted MaxWeight algorithm by

$$Q_{ij}(t + 1) = Q_{ij}(t) + A_{ij}(t) - S_{ij}(t) + U_{ij}(t), \quad (2)$$
where $U_{ij}(t)$ denotes the unused service for queue $(i,j)$ at time $t$. We also start by defining a new inner product on $\mathbb{R}^{n^2}$ with respect to the vector $c$ as follows

$$
\langle x, y \rangle_c := \sum_{ij} c_{ij} x_{ij} y_{ij}.
$$

Hence, the corresponding norm of a vector $x \in \mathbb{R}^{n^2}$ is given by $\|x\|^2_c = \sum_{ij} c_{ij} x_{ij}^2$.

Consider the input-queued switch model of Section 2 under the $c$-weighted MaxWeight scheduling algorithm defined in Algorithm 1. Without loss of generality, we assume that the schedule selected is always a maximal schedule. If a non-maximal schedule is chosen, any links that can be added to make it maximal will have a zero queue length. Thus, we add those links to the schedule to make it maximal, which will simply result in an unused service on those links because there are no packets to serve.

**Algorithm 1 $c$-Weighted MaxWeight Scheduling Algorithm for Input-Queued Switch**

Let $c \in \mathbb{R}^n$ be a given positive weight vector, i.e., $c_{ij} \geq 0$, $\forall i,j$. Then, in every time slot $t$ under the $c$-weighted MaxWeight algorithm, each queue is assigned a weight $c_{ij}q_{ij}(t)$ and a schedule with the maximum weight is chosen, namely

$$
S(t) = \text{arg max}_{s \in \mathcal{P}} \sum_{ij} c_{ij}q_{ij}(t)s_{ij} = \text{arg max}_{s \in \mathcal{P}} \langle q(t), s \rangle_c.
$$

Ties are broken uniformly at random.

We study the switch system when the arrival rate vector $\lambda$ approaches a point on the boundary of the capacity region such that all the ports are saturated. In other words, we consider the arrival rate vector approaching the face $\mathcal{F}$ of the capacity region where

$$
\mathcal{F} = \left\{ \lambda \in \mathbb{R}_+^{n^2} : \sum_{i=1}^n \lambda_{ij} = 1, \sum_{j=1}^n \lambda_{ij} = 1, \quad \forall i,j \in [n] \right\},
$$

and where $e_c^{(i)} = \{ x \in \mathbb{R}^{n^2}, x_{ij} = \frac{1}{c_{ij}}, x_{i'j} = 0, \forall i' \neq i \}$ and $e_c^{(j)} = \{ x \in \mathbb{R}^{n^2}, x_{ij} = \frac{1}{c_{ij}}, x_{ij'} = 0, \forall j' \neq j \}$.

We will obtain an exact expression for the heavy traffic scaled weighted sum of queue lengths under the $c$-weighted MaxWeight algorithm in heavy traffic. The basic approach taken will be along the same lines as that in [1] but with the dot product redefined as in (3). To obtain the desired result for heavy traffic performance under the $c$-weighted MaxWeight algorithm, we first provide a simple universal lower bound on the average weighted queue length.

Throughout this section, we will consider a base family of switch systems having arrival processes $A^{(c)}(t)$ parameterized by $0 < \epsilon < 1$ such that the mean arrival rate vector is given by $\lambda^{(c)} = E[A^{(c)}(t)] = (1 - \epsilon)\nu$ for some $\nu \in \text{relint}(\mathcal{F})$ with $\nu_{\min} := \min_{ij} \nu_{ij} > 0$, and the arrival variance vector is given by $\text{Var}(A^{(c)}) = (\sigma^{(c)})^2 < \infty$.

### 3.1 Universal Lower Bound

This section presents a simple universal lower bound on the average weighted queue length.

**Proposition 3.1.** Consider the base family of switch systems and fix a scheduling policy under which the system is stable for any $0 < \epsilon < 1$. Suppose the queue length process $q^{(c)}$ converges in distribution to a steady state random vector $\overline{q}^{(c)}$. Then, for each of these systems, the average weighted queue length is lower
bounded by

\[ E \left[ \sum_{i,j} c_{ij} q_{ij}^{(e)} \right] \geq c_{\min} \left( \frac{\| \sigma^{(e)} \|^2}{2\epsilon} - \frac{n(1-\epsilon)}{2} \right), \]

and thus, in the heavy-traffic limit as \( \epsilon \downarrow 0 \), if \( (\sigma^{(e)})^2 \to \sigma^2 \), we have

\[ \liminf_{\epsilon \downarrow 0} c_{\epsilon} E \left[ \sum_{i,j} c_{ij} q_{ij}^{(e)} \right] \geq c_{\min} \left( \frac{\| \sigma \|^2}{2} \right), \]

where \( c_{\min} = \min c_{ij} \).

Proof. From Proposition 1 in [2], the average queue length of each switch system under a scheduling policy is bounded by

\[ E \left[ \sum_{i,j} q_{ij}^{(e)} \right] \geq \frac{\| \sigma^{(e)} \|^2}{2\epsilon} - \frac{n(1-\epsilon)}{2}. \]

We therefore obtain

\[ E \left[ \sum_{i,j} c_{ij} q_{ij}^{(e)} \right] \geq c_{\min} E \left[ \sum_{i,j} q_{ij}^{(e)} \right] \geq c_{\min} \left( \frac{\| \sigma^{(e)} \|^2}{2\epsilon} - \frac{n(1-\epsilon)}{2} \right). \]

Remark 3.1. The above bound is clearly loose in general and can be made tighter. However, since the above bound is sufficient for our purposes in this paper, we defer consideration of a tighter bound to future work.

3.2 State Space Collapse

In order to establish the desired state space collapse result, we first define the cone \( K_c \) to be the cone spanned by the vectors \( e^{(i)} \) and \( \tilde{e}^{(j)} \), namely

\[ K_c := \left\{ x \in \mathbb{R}^{n^2} : x_{ij} = \frac{w_i + \tilde{w}_j}{c_{ij}}, \quad w_i, \tilde{w}_j \in \mathbb{R}_+ \right\}. \]

For any \( x \in \mathbb{R}^{n^2} \), define \( x_{\parallel K_c} \) to be the projection of \( x \) onto the cone \( K_c \), i.e., \( x_{\parallel K_c} := \arg \min_{y \in K_c} \| x - y \|_c \). The error after projection is denoted by \( x_{\perp K_c} \), i.e., \( x_{\perp K_c} = x - x_{\parallel K_c} \). To simplify the notation throughout the paper, we will write \( x_{\parallel e} \) to mean \( x_{\parallel K_c} \) and write \( x_{\perp e} \) to mean \( x_{\perp K_c} \). Let \( S_c \) denote the space spanned by the cone \( K_c \), or more formally

\[ S_c := \left\{ x \in \mathbb{R}^{n^2} : x_{ij} = \frac{w_i + \tilde{w}_j}{c_{ij}}, \quad w_i, \tilde{w}_j \in \mathbb{R} \right\}. \]

The projection of \( x \in \mathbb{R}^{n^2} \) onto the space \( S_c \) is denoted by \( x_{\parallel S_c} \), with the error after projection denoted by \( x_{\perp S_c} \).

Now, consider the base family of switch systems under the \( e \)-weighted MaxWeight scheduling algorithm with the maximum possible arrivals in any queue denoted by \( A_{\text{max}} \). Let the variance of the arrival process be such that \( \| \sigma^{(e)} \|^2 \leq \tilde{\sigma}^2 \) for some \( \tilde{\sigma}^2 \) that is not dependent on \( \epsilon \). Let \( q^{(e)} \) denote the steady state random vector of the queue length process for each switch system parameterized by \( \epsilon \). We then have the following proposition.
Proposition 3.2. For each system defined above with $0 < \epsilon \leq \nu'_\text{min}$, the steady state queue length vector satisfies
\[ \mathbb{E} \left[ \| \mathbf{q}^{(\epsilon)} \|^{r} \right] \leq (M_r)^r, \quad \forall r \in \{1, 2, \ldots\}, \]
where $\nu'_\text{min}$ and $M_r$ are functions of $r, \tilde{\sigma}, \nu, A_{\text{max}}, \nu_{\text{min}}$ but independent of $\epsilon$.

The proof will be based on the study of a Lyapunov function of the form $V(\mathbf{q}) = \| \mathbf{q} \|^{2}$, and will be discussed in detail in Appendix A.1.

3.3 Weighted Sum of Queue Lengths in Heavy Traffic

We next exploit the above state space collapse result to obtain an exact expression for the heavy traffic scaled weighted sum of queue lengths in heavy traffic. Our main results are provided in the following theorem and subsequent corollary, with the proof of the former provided in Appendix A.2 and a general matrix solution approach for calculating the corresponding limit provided below.

Theorem 3.1. Consider the base family of switch systems under the $c$-weighted MaxWeight scheduling algorithm as in Proposition 3.2. Then, in the heavy traffic limit as $\epsilon \downarrow 0$, we have
\[
\lim_{\epsilon \to 0} \epsilon \mathbb{E} \left[ \sum_{ij} c_{ij} q_{ij}^{(\epsilon)} \right] = \frac{n}{2} (\sigma^2, \zeta)_c,
\]
where $\sigma^2 = (\sigma^2_{ij})_{ij}$ and the vector $\zeta$ is defined by
\[
\zeta_{ij} := \| (e_{ij})_{\|S_c\|} \|_c^2 \tag{5}
\]
with $e_{ij}$ being the matrix with a 1 in the $(i,j)^{th}$ position and 0 everywhere else.

Corollary 3.1. Suppose the size of the switch described in Theorem 3.2 is $n = 2$, i.e, we consider the base family of $2 \times 2$ switch systems under the $c$-weighted MaxWeight scheduling algorithm. Then, in the heavy traffic limit, we have
\[
\lim_{\epsilon \to 0} \epsilon \mathbb{E} \left[ \sum_{ij} c_{ij} q_{ij}^{(\epsilon)} \right] = \frac{1}{2} \sum_{ij} \sigma^2_{ij} c_{ij} \left( 1 - \frac{c^2_{ij}}{\sum_{i',j'} c^2_{i'j'}} \right).
\]

Proof. It can be confirmed that the following three vectors form an orthonormal basis for the subspace $S_c$:
\[
f_1 = \sqrt{\frac{c_{11} c_{22}}{c_{11} + c_{22}}} \begin{bmatrix} c_{11} & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix},
\]
\[
f_2 = \sqrt{\frac{c_{12} c_{21}}{c_{12} + c_{21}}} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix},
\]
\[
f_3 = \sqrt{\frac{(c_{11} + c_{22})(c_{12} + c_{21})}{c_{11} + c_{22} + c_{12} + c_{21}}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}.
\]
Calculating the norms of the projections $\| (e_{ij})_{\|S_c\|} \|_c^2$ using these basis vectors yields the desired result.

Unfortunately, there is no explicit expression for the limit in Theorem 3.1 for a general $n \times n$ switch analogous to the explicit expression we have in the above corollary for the $2 \times 2$ switch. Instead, we derive a general matrix solution approach by which the limit can be calculated through an alternative way to obtain $\| (e_{ij})_{\|S_c\|} \|_c^2$.  

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To start, we work with the following affine basis, for any $i,j \in [n-1]$ and general $n$,
\[
B_{ij} = \begin{pmatrix}
E_{ij} & -E_i \\
-E_j^t & 1
\end{pmatrix},
\]
where $E_{ij}$ denotes an $(n-1) \times (n-1)$ matrix with the $(i,j)$-th element being one and all other elements being zero, $E_i$ denotes an $(n-1)$-vector with the $i$-th element being one and all other elements being zero, and superscript $t$ denotes the transpose operator. This $(n-1)^2$ affine basis spans the $\perp_e$-space, whereas $e_1, \ldots, e_n, \tilde{e}_1, \ldots, \tilde{e}_{n-1}$ forms a basis for the $\|e\|$-space. Thus, we can use $\{g_{ij}\}$ to denote this basis where $g_{ij} = B_{ij}$ for $i,j \in [n-1]$, $g_{ni} = e_i$ for $i \in [n]$, and $g_{n} = \tilde{e}_i$ for $i \in [n-1]$.

Generally speaking, for an $n^2$-vector $v$, there is a unique coordinate in the above system, i.e., there exists a unique $n^2$-vector $x$ such that $v_j = \sum_i x_i g_{ij}$, or equivalently in matrix form $x^t G = v^t$, where each row of $G$ is one of the basis vectors. The limit can then be calculated through a general matrix solution approach such that, denoting by $\nu$ an $n^2$-vector with $v_{ij} = \sigma_{ij}^2$, we write
\[
\lim_{\epsilon \to 0} \epsilon E \left[ \sum_{ij} c_{ij} \pi_{ij}(\epsilon) \right] = (T(G^{-1})^t v)^T \Gamma(T(G^{-1})^t v),
\]
where the details of the matrices $T$, $G$ and $\Gamma$ are derived as follows.

Observe from the above setting that $G$ can be expressed as
\[
G = \begin{pmatrix}
I_{(n-1)^2} & B \\
C & D
\end{pmatrix},
\]
where $B$ is an $(n-1)^2 \times (2n-1)$ matrix, $C$ is a $(2n-1) \times (n-1)^2$ matrix, and $D$ is a $(2n-1) \times (2n-1)$ matrix. Among the composing matrices, $C$ and $D$ are the basis for the last $(2n-1)$-vector. Hence, for $i \in [n]$ and $j \in [n^2]$, we have
\[
g_{(n-1)^2+i,j} = \begin{cases}
\frac{1}{c_{ij}}, & j = i, i+1, \ldots, i+n-1, \\
0, & \text{otherwise},
\end{cases}
\]
and for $i \in [n-1]$ and $j \in [n^2]$, we obtain
\[
g_{(n-1)^2+n+i,j} = \begin{cases}
\frac{1}{c_{ij}}, & j = i, i+n, \ldots, i+n(n-2), \\
0, & \text{otherwise}.
\end{cases}
\]
Meanwhile, supposing the pair $(q,r)$ to be the quotient and the remainder when $i \in [(n-1)^2]$ is divided by $n$ with the Euclidean algorithm, we then have
\[
g_{i,(n-1)^2+j} = \begin{cases}
-1, & j = q+r+1, \\
1, & j = n^2, \\
0, & \text{otherwise}.
\end{cases}
\]
Upon taking its inverse we obtain
\[
G^{-1} = \begin{pmatrix}
I_{(n-1)^2} + B(D - CB)^{-1}C & -B(D - CB)^{-1} \\
-(D - CB)^{-1}C & (D - CB)^{-1}
\end{pmatrix},
\]
and thus we have $x^t = v^t G^{-1}$. From this we obtain $v_{i|e} = \sum_{i=1}^n x_{ni} g_{ni} + \sum_{j=1}^{n-1} x_{nj} g_{nj}$, or equivalently $v^t_{|e} = (Tx)^t G = (T(G^{-1})^t v)^t G$, where $T$ is the truncation matrix; namely, $T$ is an $n^2 \times n^2$ matrix whose first $(n-1)^2$ diagonal elements are one and all the other elements are zero.

It then follows that $||v||^2$ will have the quadratic form
\[
(T(G^{-1})^t v)^T \Gamma(T(G^{-1})^t v).
\]
where $\Gamma = (\gamma_{ij}) = \{ g_{q(i),r(i)+1}, g_{q(j),r(j)+1} \}_e$ with $q(i)$ and $r(i)$ denoting the quotient and remainder of $q$ divided by $n$, respectively. Furthermore, we know that, for any two pairs $(i, j)$ and $(k, \ell)$,

$$\langle g_{ij}, g_{k\ell} \rangle_e = \frac{\delta_{ij,k\ell}}{c_{ij}^2} + \frac{\delta_{i\ell,j}}{c_{in}^2} + \frac{\delta_{k\ell,j}}{c_{nk}^2} + \frac{1}{c_{nn}^2},$$

when $(i, j) \leq (n - 1, n - 1)$ with $\delta_{ij,k\ell} = 1$ only if $i = k$ and $j = \ell$, and zero otherwise; $\delta_{ij} = 1$ only if $i = j$, and zero otherwise. Moreover, for any $i, j \in [n]$,

$$\langle g_{ij}, g_{kn} \rangle_e = \langle g_{ij}, g_{kn} \rangle_e = 0,$$

$$\langle g_{nj}, g_{kn} \rangle_e = \left( \sum_{m=1}^{n} \frac{1}{c_{jm}} \right) \delta_{j\ell},$$

$$\langle g_{in}, g_{kn} \rangle_e = \left( \sum_{m=1}^{n} \frac{1}{c_{mi}} \right) \delta_{ik},$$

and

$$\langle g_{nj}, g_{kn} \rangle_e = \frac{1}{c_{jk}}.$$

### 4 Optimal Scheduling

Although some of our results and arguments hold for general $n$ and regions of the decision space, we focus our analysis and results on the case of $n = 2$, and thus $\mathcal{I} = \{ (1, 1), (1, 2), (2, 1), (2, 2) \}$. To derive an optimal policy and its structural properties, we first establish the equivalence of our stochastic optimal control problem with one based on a reward for each period in terms of the number of packets served. Then we derive an optimal scheduling policy for different regions of the decision space which we partition into the trivial boundary, the interior, and the critical boundary. Most of the proofs of our results are deferred until Section 5.

For notational convenience, we write $s \# s'$ when the two schedules $s, s' \in \mathcal{P}$ do not contain the same queue; e.g., $e_\mu \# e_\rho$ if $\mu \neq \rho$, $\mu, \rho \in \mathcal{I}$. We also write $\mu \# \nu$ when the two queues $\mu, \nu \in \mathcal{I}$ cannot be contained in any schedule; e.g., $(1, 1) \# (1, 2)$ and $(2, 2) \# (2, 1)$ in the $2 \times 2$ switch.

#### 4.1 Maximizing Service Rate

The cost for each time period in problem $[P_\beta]$ depends on the current queue lengths which involve both the arrival and service processes. We shall instead consider an equivalent problem that is based on a reward for maximizing the service rate, where the reward only depends on the current queue lengths and the service action. Specifically, upon choosing schedule $s \in \mathcal{P}$ with current queue length vector $q \in \mathbb{Z}_+^{[\mathcal{I}]}$, the reward function $r : \mathbb{Z}_+^{[\mathcal{I}]} \times \mathcal{P} \rightarrow \mathbb{R}_+$ is defined by

$$r(q, s) := \sum_{(i,j) \in \mathcal{I}} c_{ij} s_{ij} \cdot 1_{q_{ij} > 0}.$$

We associate a quantity with stationary policy $\pi$ by defining

$$\bar{J}_\beta(q, \pi) := \mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t r(Q^\pi(t), S^\pi(t)) \right],$$

where $Q^\pi(0) = q$ is the initial state. Then we can construct an alternative optimization problem as follows:

Maximize $\bar{J}_\beta(q, \pi)$ over all stationary policies $\pi$.

Next, we show that if there is an optimal (stationary) policy $\pi^*$ of $[P_\beta]$, then $\pi^*$ is an optimal policy of $[P_\beta]$. 

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Proposition 4.1. An optimal solution for problem \((\tilde{P}_\beta)\) is an optimal policy for problem \((P_\beta)\).

Proof From equation (1), we have
\[
J_\beta(q, \pi) = c\pi(t) + \sum_{(i,j) \in I} c_{ij} A_{ij}(t) - r\pi(t),
\]
where \(g = \sum_{t=0}^{\infty} \beta^t E[\sum_{(i,j) \in I} c_{ij} A_{ij}(t)]\) which does not depend on the policy \(\pi\). Hence, we obtain
\[
(1 - \beta) J_\beta(q, \pi) = c\pi(0) + \beta J_\beta(q, \pi) + g - \beta \tilde{J}_\beta(q, \pi),
\]
which implies the desired result.

Now let \(M\) be the set of all stationary policies. The maximum total discounted reward over \([0, n]\) is
\[
V_n(q) := \sup_{\pi \in M} \mathbb{E} \left[ \sum_{t=0}^{n} \beta^t r(Q\pi(t), S\pi(t)) \right],
\]
where \(V_0(q) = 0\) for all \(q \in Z_{+}^{\vert I\vert}\). We then have the following Bellman equation in the \((n + 1)th\) value iteration:
\[
V_{n+1}(q) = \max_{s \in P} \left\{ r(q, s) + \beta \mathbb{E}[V_n((q - s)^+ + A)] \right\},
\]
which is called the \((n + 1)th\) value function.

4.2 Trivial Boundary

For problem \((\tilde{P}_\beta)\), we first derive a work conserving optimal scheduling policy for a subset of the state space \(Z_{+}^{\vert I\vert}\).

Definition 4.1. A state \(q \in Z_{+}^{\vert I\vert}\) is in the trivial boundary if there exists \(s \in P\) such that
\[
q_{ij} = 0 \quad \text{if} \quad s_{ij} = 0.
\]
In other words, \(s\) is a schedule that can serve packets in all nonempty queues in \(q\).

Now our main result for the trivial boundary.

Theorem 4.1. An optimal policy in every value iteration for \(q\) in the trivial boundary is to choose a schedule \(s \in P\) that satisfies (7) at any time \(t\).

This theorem can be derived from the following proposition.

Proposition 4.2. Any value function \(V_n\) satisfies
\[
\beta V_n(q + e_{ij}) \leq \beta V_n(q) + c_{ij},
\]
for any \(q \in Z_{+}^{\vert I\vert}\) and \((i, j) \in I\).

4.3 Interior Region

We next investigate an optimal policy for an “interior” subset of the state space. A state \(q\) is in the interior if there exists a schedule that produces a maximal reward value
\[
r_{\text{max}} := \max_{q,s} \left\{ r(q, s) : q \in Z_{+}^{\vert I\vert}, s \in P \right\}
\]
with respect to \(q\).
Definition 4.2. A state $q$ is an interior point if

$$\max \{ r(q, s) : s \in P \} = r_{\text{max}}$$

and the interior region comprises the set of all interior points.

The following theorem identifies an optimal scheduling policy for the interior, rendering the $c\mu$ policy to be optimal.

Theorem 4.2. For $2 \times 2$ input-queued switches, an optimal schedule in any value iteration on an interior point $q$ is a schedule $s \in P$ such that $r(q, s) = r_{\text{max}}$.

For the $(n+1)$th value iteration, the above statement is true if $V_n$ satisfies the inequality of the next proposition.

Proposition 4.3. Let $q \in \mathbb{Z}_+^{|I|}$ be an interior point and $s \in P$ a schedule such that $r(q, s) = r_{\text{max}}$. Then, for any value function and any schedule $s' \in P$ with $s' \leq q$, we have

$$r(q, s) + \beta V_n(q - s) \geq r(q, s') + \beta V_n(q - s').$$

4.4 Critical Boundary: $c\mu$ is Optimal

In this section we start to consider the remaining region of the decision space, which we call the critical boundary, for the special case where only one buffer is empty and the $c\mu$ policy is optimal as in the interior region.

Throughout this section, let $I = \{\mu, \nu, \rho, \omega\}$ where $\mu\#\omega$ and $\mu\#\rho$. Further, we assume $c_{\mu} \leq c_{\rho} + c_{\omega} \leq c_{\mu} + c_{\nu}$ where $\mu\#\rho$ and $\nu\#\rho$. We then identify an optimal policy for state $q$ such that $q_{\nu} = 0$ and all other queues are nonempty.

Theorem 4.3. Let the state $q$ be such that $q_{\nu} = 0$, all other queues are nonempty in any value iteration, and $c$ is as assumed above. Then the optimal action on state $q$ is to serve packets in queues $\rho$ and $\omega$.

For the $(n+1)$th value iteration, the above statement is true if $V_n$ satisfies the inequality of the following proposition.

Proposition 4.4. For any value function, we have

$$c_{\rho} + c_{\omega} + \beta V_n(q + e_{\mu}) \geq c_{\mu} + \beta V_n(q + e_{\rho} + e_{\omega})$$

for any $q \in \mathbb{Z}_+^{|I|}$.

4.5 Critical Boundary: Switching Curve

In this section we consider the remainder of the critical boundary and show that an optimal policy of any value function has a switching curve structure.

Theorem 4.4. Fix a state $q \in \mathbb{Z}_+^{|I|}$. S1: In any value iteration, if the optimal action on $q$ is to serve queues $\mu$ and $\nu$, then these are optimal actions on $q + e_{\mu}$ and $q + e_{\nu}$. Furthermore, if the assumption of S1 holds, the optimal action on $q'$ is to serve queues $\mu$ and $\nu$ if $q'_{\mu} \geq q_{\mu}$, $q'_{\nu} \geq q_{\nu}$, and $q'_{\rho} \leq q_{\rho}$ for $\rho\#\mu$.

To establish the above theorem on a switching curve structure for the relevant portion of the critical boundary, we introduce in the next proposition inequalities that imply S1.

Proposition 4.5. For every $n \in \mathbb{Z}_+$, the $n$th value function satisfies the following inequalities: For any $q \in \mathbb{Z}_+^{|I|}$,

$$V_n(q + e_{\mu} + e_{\rho}) + V_n(q + e_{\mu}) \geq V_n(q + 2e_{\mu}) + V_n(q + e_{\rho}), \quad (12)$$
$$V_n(q + e_{\mu} + e_{\rho}) + V_n(q + e_{\mu} + e_{\nu}) \geq V_n(q + 2e_{\mu} + e_{\nu}) + V_n(q + e_{\rho}), \quad (13)$$
$$V_n(q + e_{\mu} + e_{\rho} + e_{\omega}) + V_n(q + e_{\mu}) \geq V_n(q + 2e_{\mu}) + V_n(q + e_{\rho} + e_{\omega}), \quad (14)$$

where $\mu, \rho, \omega \in I$, $\mu\#\rho$, $\mu\#\omega$, and $\rho \neq \omega$. 

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5  Proofs

5.1  Trivial Boundary

We first show that Theorem 4.1 follows from Proposition 4.2. Suppose that (8) holds for $V_n$. Let $q$ be a state in the trivial boundary and $s$ the schedule that satisfies (7). Then, for any schedule $s' \in P$, we have

$$(q - s')^+ = (q - s)^+ + \sum_{(i,j) \in I'} e_{ij},$$

$$r(q, s') = r(q, s) + \sum_{(i,j) \in I'} c_{ij},$$

where $(i,j) \in I'$ if and only if $q_{ij} \geq 1$ (so, $s_{ij} \geq 1$) and $s'_{ij} = 0$. Hence, we obtain

$$r(q, s') + \beta E[V_n((q - s')^+ + A)] = r(q, s') + \beta E\left[V_n((q - s)^+ + A + \sum_{(i,j) \in I'} e_{ij})\right]$$

$$\leq r(q, s') + \beta E[V_n((q - s)^+ + A)] + \sum_{(i,j) \in I'} c_{ij}$$

$$= r(q, s) + \beta E[V_n((q - s)^+ + A)],$$

where the second equation follows from Proposition 4.2. As a result, $s$ is the optimal schedule for $q$ in any value iteration.

Now we prove Proposition 4.2 by induction. Since $V_0(q) = 0$, (8) holds for $n = 0$. Next suppose that $V_k$ satisfies (8). Let $s \in P$ be a schedule.

i. If $(s_{ij} = 0)$ or $(s_{ij} = 1$ and $q_{ij} = 1$), we have

$$r(q + e_{ij}, s) = r(q, s),$$

$$\beta V_k((q + e_{ij} - s)^+ + A) = \beta V_k((q - s)^+ + A + e_{ij})$$

$$\leq \beta V_k((q - s)^+ + A) + c_{ij},$$

where the last equation follows from induction hypothesis.

ii. Otherwise (i.e., $s_{ij} = 1$ and $q_{ij} = 0$), we obtain

$$r(q + e_{ij}, s) = r(q, s) + c_{ij},$$

$$\beta V_k((q + e_{ij} - s)^+ + A) = \beta V_k((q - s)^+ + A).$$

From i. and ii., we derive

$$V_{k+1}(q + e_{ij}) = \max_{s \in P} \{r(q + e_{ij}, s) + \beta E[V_k((q + e_{ij} - s)^+ + A)]\}$$

$$\leq \max_{s \in P} \{r(q, s) + \beta E[V_k((q - s)^+ + A)]\} + c_{ij}$$

$$= V_{k+1}(q) + c_{ij},$$

which implies that $V_{k+1}$ satisfies (8) and by induction, the proof of Proposition 4.2 is complete.

5.2  Interior Region

We first assume that (10) holds for the $n$th value function $V_n$. Then, for any interior point $q$ with a schedule $s \in P$ such that $r(q, s) = r_{max}$ and any $s' \leq q$, we have

$$r(q, s) + \beta E[V_n(q - s + A)] = E[r(q + A, s) + \beta V_n(q + A - s)]$$

$$\geq E[r(q + A, s') + \beta V_n(q + A - s')]$$

$$= r(q, s') + \beta E[V_n(q + A - s')],$$

where the first and the last equations follow from $q + A \geq s, s'$ (which implies $r(q, s) = r(q, A, s)$), and the second equation follows from (10) for $V_n$. Hence, Theorem 4.2 holds for the $(n + 1)$th value iteration.
Now we prove Proposition 4.3 by induction on \( n \). Let \( q \) be an interior point, \( s \) a schedule with \( r(q, s) = r_{\text{max}} \), and \( s' \) a schedule such that \( s' \leq q \). First, for \( n = 0 \), (10) holds because \( V_0(q) = 0 \) and \( r(q, s) = r_{\text{max}} \geq r(q, s') \) for any \( s' \in \mathcal{P} \). Next, assume that \( V_k \) satisfies (10).

i. If \( s' \leq s \), then (10) for \( V_{k+1} \) immediately follows from Proposition 4.2.

ii. Otherwise, we have \( s' \# s \) in the \( 2 \times 2 \) input-queued switch.

Since \( q - s' \) is an interior point, by the above argument, we obtain

\[
V_{k+1}(q - s') = r(q - s', s) + \beta E[V_k(q - s - s' + A)]
\]

\[
= r(q, s) + \beta E[V_k(q - s - s' + A)]
\]

since \( s \# s' \), which implies \( q - s' \geq s \). On the other hand, from the definition of the value iteration, we also have

\[
V_{k+1}(q - s) \geq r(q - s, s') + \beta E[V_k(q - s - s' + A)]
\]

\[
= r(q, s') + \beta E[V_k(q - s - s' + A)].
\]

We therefore obtain

\[
V_{k+1}(q - s') - r(q, s) \leq V_{k+1}(q - s) - r(q, s'),
\]

which implies that (10) holds for \( n = k + 1 \).

The proof of Proposition 4.3 is complete by induction.

5.3 Critical Boundary: \( c\mu \) is Optimal.

We establish Proposition 4.4 by induction on \( n \). First, for \( n = 0 \), \( V_0 \) satisfies (11) because \( V_0 = 0 \) and \( c\mu \leq c_p + c_\omega \). Now, assume that (11) holds for \( V_k \).

i. Suppose that

\[
V_{k+1}(q + e_p + e_\omega) = c_{\mu} \cdot I_{\{q_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\mu + e_\nu)^+ + A + e_p + e_\omega)].
\]

If \( q_\mu \geq 1 \), we have

\[
V_{k+1}(q + e_p + e_\omega) = c_{\mu} + c_\nu \cdot I_{\{q_\nu > 0\}} + \beta E[V_n((q - e_\mu + e_\nu)^+ + A + e_p + e_\omega - e_\mu)]
\]

\[
\leq c_p + c_\omega + c_\nu \cdot I_{\{q_\nu > 0\}} + \beta E[V_n((q - e_\mu + e_\nu)^+ + A)]
\]

\[
\leq c_p + c_\omega + V_{k+1}(q + e_\mu) - c_\mu,
\]

where the second equation follows from the induction hypothesis and the last equation follows from the definition of the value iteration. Otherwise, if \( q_\mu = 0 \), we obtain

\[
V_{k+1}(q + e_p + e_\omega) = c_\nu \cdot I_{\{q_\nu > 0\}} + \beta E[V_n((q - e_\mu + e_\nu)^+ + A + e_p + e_\omega)]
\]

\[
\leq c_\nu + c_\omega + c_\nu \cdot I_{\{q_\nu > 0\}} + c_p + c_\omega + \beta E[V_n((q - e_\mu)^+ + A)]
\]

\[
\leq c_p + c_\omega + V_{k+1}(q + e_\mu) - c_\mu,
\]

where the second equation follows from Proposition 4.2 and the last follows from the definition of the value iteration.

ii. Next, suppose that

\[
V_{k+1}(q + e_p + e_\omega) = c_p + c_\omega + E[V_n(q + A)].
\]

If \( q_\nu \geq 1 \), we have

\[
V_{k+1}(q + e_p + e_\omega) = c_p + c_\omega + E[V_n(q + A)]
\]

\[
\leq c_p + c_\omega + c_\nu \cdot I_{\{q_\nu > 0\}} + E[V_n(q + A - e_\nu)]
\]

\[
\leq c_p + c_\omega + V_{k+1}(q + e_\nu) - c_\mu,
\]
where the second equation follows from Proposition 4.12 and the last follows from the definition of the value iteration. Otherwise, if \( q_\nu = 0 \), we obtain

\[
V_{k+1}(q + e_\rho + e_\omega) = c_\mu + c_\omega + E[V_n(q + A)] \\
\leq c_\rho + c_\omega + V_{k+1}(q + e_\mu) - c_\mu,
\]

where the second equation follows from the definition of the value iteration.

Hence, (11) holds for \( V_{k+1} \) and by induction, the proof of Proposition 4.4 is complete.

5.4 Critical Boundary: Switching Curve

We first show that the inequalities for \( V_n \) in Proposition 4.5 are sufficient conditions of \( S1 \) in the \( n \)th value iteration when state \( q \) is in the critical boundary. Then, by symmetry, \( q \) falls in one of the three conditions:

- **C1**: \( q_\mu \geq 1, q_\nu = 0, q_\rho \geq 1, \) and \( q_\omega = 0; \)
- **C2**: \( q_\mu \geq 1, q_\nu \geq 1, q_\rho \geq 1, \) and \( q_\omega = 0; \)
- **C3**: \( q_\mu \geq 1, q_\nu = 0, q_\rho \geq 1, \) and \( q_\omega \geq 1; \)

where \( \rho \neq \omega \) are queues that cannot be served with \( \mu \).

**Proof for C1.** From the assumption in \( S1 \), we have

\[
c_\mu + \beta E[V_n(q + A - e_\mu)] \geq c_\nu + \beta E[V_n(q + A - e_\rho)]. \tag{15}
\]

Substituting \( q + A - e_\mu - e_\rho \geq 0 \) for \( q \) in (12) yields

\[
V_n(q + A) + V_n(q + A - e_\rho) \geq V_n(q + A - e_\mu - e_\rho) + V_n(q + A - e_\mu).
\]

Taking expectation of the above equation for \( A \) and adding this to (15), we obtain

\[
c_\mu + \beta E[V_n(q + A)] \geq c_\nu + \beta E[V_n(q + A + e_\mu - e_\rho)],
\]

which implies that the optimal action on \( q + e_\mu \) is to serve queues \( \mu \) and \( \nu \).

On the other hand, for \( q + e_\nu \), we have

\[
c_\rho + \beta E[V_n(q + A + e_\nu - e_\rho)] \leq c_\rho + \beta E[V_n(q + A - e_\rho)] \\
\leq c_\mu + \beta E[V_n(q + A + e_\mu)],
\]

where the first equation follows from Proposition 4.2 and the second follows from (15). Hence, the optimal action on \( q + e_\mu \) is to serve queues \( \mu \) and \( \nu \).

**Proof for C2.** Assumption in \( S1 \) and condition C2 implies

\[
c_\mu + c_\nu + \beta E[V_n(q + A - e_\mu - e_\nu)] \geq c_\nu + \beta E[V_n(q + A - e_\mu)]. \tag{16}
\]

Substituting \( q + A - e_\mu - e_\nu - e_\rho \geq 0 \) for \( q \) in (13) yields

\[
V_n(q + A - e_\nu) + V_n(q + A - e_\rho) \geq V_n(q + A + e_\mu - e_\rho) + V_n(q + A - e_\mu - e_\nu).
\]

Taking expectation of the above equation for \( A \) and adding this to (16), we obtain

\[
c_\mu + c_\nu + \beta E[V_n(q + A - e_\nu)] \geq c_\nu + \beta E[V_n(q + A + e_\mu - e_\rho)],
\]

which implies that the optimal action on \( q + e_\mu \) is to serve packets in queues \( \mu \) and \( \nu \), and by symmetry this is also the optimal action on \( q + e_\nu \).

**Proof for C3.** From the assumption in \( S1 \), we have

\[
c_\mu + \beta E[V_n(q + A - e_\mu)] \geq c_\nu + c_\omega + \beta E[V_n(q + A - e_\rho - e_\omega)]. \tag{17}
\]

Substituting \( q + A - e_\rho - e_\omega \geq 0 \) for \( q \) in (14) yields

\[
V_n(q + A) + V_n(q + A - e_\rho - e_\omega) \geq V_n(q + A + e_\mu - e_\rho - e_\omega) + V_n(q + A - e_\mu).
\]
Taking expectation of the above equation for $A$ and adding this to (17), we obtain
\[ c_\mu + \beta E[V_n(q + A)] \geq c_\nu + c_\omega + \beta E[V_n(q + e_\mu + e_\rho - e_\omega)], \]
which implies that the optimal action on $q + e_\mu$ is to serve queues $\mu$ and $\nu$.

On the other hand, for $q + e_\nu$, we have
\[ c_\nu + c_\omega + \beta E[V_n(q + A + e_\nu + e_\rho - e_\omega)] \leq c_\nu + c_\mu + c_\omega + \beta E[V_n(q + A - e_\rho - e_\omega)] \]
\[ \leq c_\nu + c_\mu + \beta E[V_n(q + A - e_\mu)], \]
where the first equation follows from Proposition 4.2 and the second follows from (17). Hence, the optimal action on $q + e_\nu$ is to serve packets in queues $\mu$ and $\nu$.

In summary, statement $S1$ of Theorem 4.4 holds in the $n$th value iteration, if $V_n$ satisfies the inequalities in Proposition 4.5, the proof of which is presented in the next section.

5.5 Proposition 4.5

We prove Proposition 4.5 by induction on $n$, introducing new inequalities useful for proving inequalities (12)–(14):

\[ 2V_n(q + e_\mu) \geq V_n(q) + V_n(q + 2e_\mu), \quad (18) \]
\[ V_n(q + e_\mu + e_\nu) + V_n(q + e_\mu) \geq V_n(q + 2e_\mu + e_\nu) + V_n(q), \quad (19) \]
\[ V_n(q + e_\mu + e_\nu) + V_n(q) \geq V_n(q + e_\mu) + V_n(q + e_\nu), \quad (20) \]

where $\mu, \nu, \rho, \omega \in I$ such that $\rho \# \mu, \omega \# \mu, \rho \neq \omega, \mu \neq \nu$.

First, for $n = 0$, all equations (12)–(14) and (18)–(20) hold because $V_0(q) = 0$ for all $q \in \mathbb{Z}_+^{[2]}$. Next, assuming that the $k$th value function satisfies all of these equations, we prove $V_{k+1}$ satisfies the first three equations below and prove $V_{k+1}$ satisfies the remaining equations in Appendix B. Beyond the induction hypothesis on $V_k$, we need the following lemma, the proof of which is also provided in Appendix B.

Lemma 5.1. Suppose that $x, y, z, w \in \{0, 1\}^{[2]}$ satisfy: (a) $x \leq e_\mu + e_\nu$ and $y \leq e_\rho + e_\omega$, component-wise; (b) $x + y = z + w$. Then, any value function $V_n$ satisfies
\[ V_n(q + x) + V_n(q + y) \geq V_n(q + z) + V_n(q + w), \quad (21) \]
for any $q \in \mathbb{Z}_+^{[2]}$.

5.5.1 Proof of (12) for $V_{k+1}$

We prove (12) for $n = k + 1$ where $\mu \# \rho$. The right-hand side of this equation is involved with the optimal actions on $q + 2e_\mu$ and $q + e_\rho$ in the $(k + 1)$th value iteration, in which case statement $S1$ holds true because $V_k$ from the Bellman equation (6) satisfies (12)–(14). Hence, if the optimal action of the $(k + 1)$th value iteration on $q + 2e_\mu$ is to serve queues $\rho$ and $\omega$, then so is this the optimal action on $q + e_\rho$. Further, if the optimal action of the $(k + 1)$th value iteration on $q + e_\rho$ is to serve queues $\mu$ and $\nu$, then state $q + 2e_\mu$ has the same optimal action in the $(k + 1)$th value iteration. We therefore have, in the $(k + 1)$th iteration, the following three cases for optimal actions on those two states: (1) Both optimal actions are to serve queues $\mu$ and $\nu$; (2) Both optimal actions are to serve queues $\rho$ and $\omega$; (3) The optimal action on $(q + 2\mu)$ is to serve queues $\mu$ and $\nu$, and the optimal action on $q + \rho$ is to serve queues $\rho$ and $\omega$. We prove (12) for the $(k + 1)$th value function dealing with all three cases.

First, suppose that optimal actions on $q + 2e_\mu$ and $q + e_\rho$ are to transmit packets in queue $\mu$ and $\nu$ in the $(k + 1)$th value iteration. If $q_\mu \geq 1$, we obtain
\[ V_{k+1}(q + 2e_\mu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu \cdot I_{(q_\mu > 0)} + \beta E[V_k((q - e_\nu)^+ + A + e_\mu)] \]
\[ + c_\mu + c_\omega \cdot I_{(q_\omega > 0)} + \beta E[V_k((q - e_\omega)^+ + A - e_\mu + e_\rho)] \]
\[ \leq c_\mu + c_\nu \cdot I_{(q_\mu > 0)} + \beta E[V_k((q - e_\nu)^+ + A + e_\mu)] \]
\[ + c_\mu + c_\omega \cdot I_{(q_\omega > 0)} + \beta E[V_k((q - e_\omega)^+ + A + e_\rho)] \]
\[ \leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(q + e_\mu), \]
where the second equation follows from the induction hypothesis (substituting \((q - e_\nu)^+ + A - e_\mu\) for \(q\) in (12) for \(V_k\)) and the last equation follows from the definition of the value iteration. On the other hand, if \(q_\mu = 0\), we have

\[
V_{k+1}(q + 2e_\mu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\mu)] \\
+ c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\rho)] \\
\leq c_\mu + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A)] \\
+ c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\mu)] \\
\leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(q + e_\mu),
\]

where the second equation follows from Proposition 12 and the last follows from the definition of the value iteration.

Second, assume that optimal actions on \(q + 2e_\mu\) and \(q + e_\rho\) are to transmit packets in queue \(\rho\) and \(\omega\) in the \((k+1)\)th value iteration. If \(q_\rho \geq 1\), we obtain

\[
V_{k+1}(q + 2e_\mu) + V_{k+1}(q + e_\rho) \\
= c_\rho + c_\omega \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A + e_\rho + 2e_\mu)] \\
+ c_\rho + c_\omega \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A)] \\
\leq c_\rho + c_\omega \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A + e_\mu)] \\
+ c_\rho + c_\omega \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A - e_\rho + e_\mu)] \\
\leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(q + e_\mu),
\]

where the second equation follows from the induction hypothesis (substituting \((q - e_\omega)^+ + A - e_\rho\) for \(q\) in (12) for \(V_k\)) and the last equation follows from the definition of the value iteration. On the other hand, if \(q_\rho = 0\), we have

\[
V_{k+1}(q + 2e_\mu) + V_{k+1}(q + e_\rho) \\
= c_\rho + c_\omega \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A + 2e_\mu)] \\
+ c_\rho + c_\omega \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A)] \\
\leq c_\rho + c_\omega \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A + e_\mu)] \\
+ c_\rho \cdot 1_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A + e_\mu)] \\
\leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(q + e_\mu),
\]

where the second equation follows from the induction hypothesis (substituting \((q - e_\omega)^+ + A\) for \(q\) in (18) for \(V_k\)) and the last follows from the definition of the value iteration.

Finally, suppose that the optimal action on \(q + 2e_\mu\) and \(q + e_\rho\) is to serve packets in queues \(\mu\) and \(\nu\), and the optimal action on \(q + e_\rho\) is to transmit packets in queue \(\rho\) and \(\omega\) in the \((k+1)\)th value iteration. Then, we obtain

\[
V_{k+1}(q + 2e_\mu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\mu)] \\
+ c_\rho + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A)] \\
\leq c_\mu + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\mu)] \\
+ c_\rho + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\rho)] \\
\leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(Q + z) + V_k(Q + w),
\]

\[
V_{k+1}(q + e_\mu) + V_{k+1}(q + e_\mu + e_\rho) \geq c_\mu + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A)] \\
+ c_\rho + c_\nu \cdot 1_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\mu)] \\
\leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(Q + z) + V_k(Q + y),
\]

where \(Q := (q - e_\nu - e_\omega)^+ + A\), \(x := q - (q - e_\nu)^+ + e_\mu\), \(y := q - (q - e_\omega)^+\), \(z := q - (q - e_\omega)^+ + e_\mu\), \(w := q - (q - e_\nu)^+\). We also have \(x, y, z, w \in \{0, 1\}^{27}\), \(x \leq e_\mu + e_\nu\), \(y \leq e_\rho + e_\omega\), and \(x + y = z + w\).
Therefore, we obtain

\[
V_{k+1}(q + e_\mu) + V_{k+1}(q + e_\mu + e_\rho) \geq c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + c_\rho + c_\omega \cdot \mathbb{I}_{\{q_\omega > 0\}} + \beta \mathbb{E}[V_k(Q + x) + V_k(Q + y)]
\]

where the second equation follows from Lemma 5.1. Hence, (12) holds for the \((k+1)\)th value function \(V_{k+1}\).

5.5.2 Proof of (13) for \(V_{k+1}\)

We prove equation (13) for \(n = k+1\) where \(\mu \neq \rho, \mu \neq \omega, \nu \neq \rho, \nu \neq \omega, \rho \neq \omega, \text{ and } \mu \neq \nu\). The right-hand side of this equation is involved with the optimal actions on \(q + 2e_\mu + e_\nu\) and \(q + e_\rho\) in the \((k+1)\)th value iteration, in which case statement S1 holds true because \(V_k\) from the Bellman equation (6) satisfies (12)–(14). Hence, if the optimal action of the \((k+1)\)th value iteration on \(q + e_\mu\) is to serve queues \(\mu\) and \(\nu\), then so is this the optimal action on \(q + 2e_\mu + e_\nu\). Further, if the optimal action of the \((k+1)\)th value iteration on \(q + 2e_\mu + e_\nu\) is to serve queues \(\rho\) and \(\omega\), then state \(q + e_\rho\) has the same optimal action in the \((k+1)\)th value iteration. We therefore have, in the \((k+1)\)th iteration, the following three cases for optimal actions on those two states: (1) Both optimal actions are to serve queues \(\mu\) and \(\nu\); (2) Both optimal actions are to serve queues \(\rho\) and \(\omega\); (3) The optimal action on \(q + 2e_\mu + e_\nu\) is to serve queues \(\mu\) and \(\nu\) and the optimal action on \(q + e_\rho\) is to in queues \(\rho\) and \(\omega\). We prove (13) for the \((k+1)\)th value function dealing with all three cases.

First, suppose that both optimal actions are to transmit packets in queue \(\mu\) and \(\nu\) in \((k+1)\)th value iteration. If \(q_\mu \geq 1 \text{ and } q_\nu \geq 1\), we have

\[
V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A + e_\mu)] + c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A - e_\mu - e_\nu + e_\rho)]
\]

where the second equation follows from the induction hypothesis (substituting \(q + A - e_\mu - e_\nu\) for \(q\) in (13) for \(V_k\)) and the last equation follows from the definition of the value iteration. If \(q_\mu \geq 0 \text{ and } q_\nu = 0\), we obtain

\[
V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A + e_\mu)] + c_\mu + \beta \mathbb{E}[V_k(q + A - e_\mu + e_\rho)] + c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A)]
\]

where the second equation follows from the induction hypothesis (substituting \(q + A - e_\mu\) for \(q\) in (12) for \(V_k\)) and the last equation follows from the definition of the value iteration. Lastly, if \(q_\mu = 0\), we have

\[
V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A + e_\mu)] + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta \mathbb{E}[V_k((q - e_\nu)^+ + A + e_\rho)]
\]

where the second equation follows from the induction hypothesis (substituting \(q + A - e_\mu\) for \(q\) in (12) for \(V_k\)) and the last equation follows from the definition of the value iteration. Lastly, if \(q_\mu = 0\), we have
where the second equation follows from Proposition 4.2 and the last follows from the definition of the value iteration.

Second, assume that both optimal actions are to transmit packets in queue \( \rho \) and \( \omega \) in the \((k+1)\)th value iteration. If \( q_\rho \geq 1 \), we obtain

\[
V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q + e_\rho) = c_\rho + c_\omega \cdot \mathbb{I}_{\{q_\omega > 0\}} \\
+ \beta E[V_k((q - e_\omega)^+ + A - e_\rho + 2e_\mu + e_\nu)] \\
+ c_\rho + c_\omega \cdot \mathbb{I}_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A)] \\
\leq c_\rho + c_\omega \cdot \mathbb{I}_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A + e_\mu)] \\
+ c_\rho + c_\omega \cdot \mathbb{I}_{\{q_\omega > 0\}} + \beta E[V_k((q - e_\omega)^+ + A + e_\mu + e_\nu)] \\
\leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(q + e_\mu + e_\nu),
\]

where the second equation follows from the induction hypothesis (substituting \((q - e_\omega)^+ + A - e_\rho\) for \( q \) in (13) for \( V_k \)) and the last equation follows from the definition of the value iteration. On the other hand, if \( q_\rho = 0 \), we have

\[
V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu + \beta E[V_k(q + A + e_\mu)] \\
+ c_\rho + c_\omega + \beta E[V_k((q - e_\omega)^+ + A)] \\
=(c_\mu + c_\nu + c_\rho + c_\omega + \mathbb{I}_{\{q_\omega > 0\}}) \\
+ \beta E[V_k(Q + z) + V_k(Q + w)],
\]

where \( Q := (q - e_\omega)^+ + A \), \( x := e_\mu \), \( y := q - (q - e_\omega)^+ \), \( z := e_\mu + q - (q - e_\omega)^+ \), \( w := 0 \). We also have \( x, y, z, w \in \{0, 1\}^{|I|} \), \( x \leq e_\mu + e_\nu \), \( y \leq e_\rho + e_\omega \), and \( x + y = z + w \). Therefore, we obtain

\[
V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q + e_\rho) \\
\leq V_{k+1}(q + e_\mu + e_\rho) + V_{k+1}(q + e_\mu + e_\nu),
\]

where the second equation follows from the induction hypothesis (substituting \((q - e_\omega)^+ + A \) for \( q \) in (19) for \( V_k \)) and the last follows from the definition of the value iteration.

Finally, suppose that the optimal action on \( q + 2e_\mu + e_\nu \) is to serve packets in queues \( \mu \) and \( \nu \), and the optimal action on \( q + e_\rho \) is to transmit packets in queue \( \rho \) and \( \omega \) in \((k+1)\)th value iteration. Then, we obtain

\[
V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q + e_\rho) = c_\mu + c_\nu + \beta E[V_k(q + A + e_\mu)] \\
+ c_\rho + c_\omega + \beta E[V_k((q - e_\omega)^+ + A)] \\
=(c_\mu + c_\nu + c_\rho + c_\omega + \mathbb{I}_{\{q_\omega > 0\}}) \\
+ \beta E[V_k(Q + x) + V_k(Q + y)],
\]

which follows from Lemma 5.1.

Hence, (13) holds for the \((k+1)\)th value function \( V_{k+1} \).

### 5.5.3 Proof of (14) for \( V_{k+1} \)

We prove equation (14) for \( n = k + 1 \) where \( \mu \# \rho \), \( \mu \# \omega \), \( \nu \# \rho \), \( \nu \# \omega \), \( \rho \neq \omega \), and \( \mu \neq \nu \). The right-hand side of this equation is involved with the optimal actions on \( q + 2e_\mu \) and \( q + e_\rho + e_\omega \) in the \((k+1)\)th
value iteration, in which case statement S1 holds true because $V_k$ from the Bellman equation (6) satisfies (12)–(14). Hence, if the optimal action of the $(k+1)$th value iteration on $q + 2\varepsilon_\mu$ is to serve queues $\rho$ and $\omega$, then so is this the optimal action on $q + \varepsilon_\rho + \varepsilon_\omega$. Further, if the optimal action of the $(k+1)$th value iteration on $q + \varepsilon_\rho + \varepsilon_\omega$ is to serve queues $\mu$ and $\nu$, then state $q + 2\varepsilon_\mu$ has the same optimal action in the $(k+1)$th value iteration. We therefore have, in the $(k+1)$th iteration, the following three cases for optimal actions on those two states: (1) Both optimal actions are to serve queues $\mu$ and $\nu$; (2) Both optimal actions are to serve queues $\rho$ and $\omega$; (3) The optimal action on $q + 2\varepsilon_\mu$ is to serve queues $\mu$ and $\nu$ and the optimal action on $q + \varepsilon_\rho + \varepsilon_\omega$ is to serve queues $\rho$ and $\omega$. We prove (14) for the $(k+1)$th value function dealing with all three cases.

First, suppose that both optimal actions are to transmit packets in queue $\mu$ and $\nu$ in the $(k+1)$th value iteration. If $q_\mu \geq 1$, we have

$$V_{k+1}(q + 2\varepsilon_\mu) + V_{k+1}(q + \varepsilon_\rho + \varepsilon_\omega) = c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + \varepsilon_\mu)] + c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A - e_\mu + e_\rho + e_\omega)] \\ \leq c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + \varepsilon_\rho + e_\omega)] + c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A)] \\ \leq V_{k+1}(q + e_\mu + e_\rho + e_\omega) + V_{k+1}(q + e_\mu),$$

the second equation follows from the induction hypothesis (substituting $(q - e_\nu)^+ + A - e_\mu$ for $q$ in (14) for $V_k$) and the last equation follows from the definition of the value iteration. On the other hand, if $q_\mu = 0$, we obtain

$$V_{k+1}(q + 2\varepsilon_\mu) + V_{k+1}(q + \varepsilon_\rho + \varepsilon_\omega) = c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + \varepsilon_\mu)] + c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + e_\rho + e_\omega)] \\ \leq c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A + \varepsilon_\rho + e_\omega)] + c_\mu + c_\nu \cdot \mathbb{I}_{\{q_\nu > 0\}} + \beta E[V_k((q - e_\nu)^+ + A)] \\ \leq V_{k+1}(q + e_\mu + e_\rho + e_\omega) + V_{k+1}(q + e_\mu),$$

where the second equation follows from Proposition (12) and the last follows from the definition of the value iteration.

Second, assume that both optimal actions are to transmit packets in queue $\rho$ and $\omega$ in the $(k+1)$th value iteration. However, if $q_\rho = 0$ and $q_\omega = 0$, the optimal action cannot be optimal on $q + 2\varepsilon_\mu$, and thus one of $q_\rho$ and $q_\omega$ should not be zero. If $q_\rho \geq 1$ and $q_\omega \geq 1$, we have

$$V_{k+1}(q + 2\varepsilon_\mu) + V_{k+1}(q + \varepsilon_\rho + \varepsilon_\omega) = c_\rho + c_\omega + \beta E[V_k(q - \varepsilon_\rho - e_\omega + A + 2\varepsilon_\mu)] + c_\rho + c_\omega + \beta E[V_k(q + A)] \\ \leq c_\rho + c_\omega + \beta E[V_k(q + A + \varepsilon_\mu)] + c_\rho + c_\omega + \beta E[V_k(q - \varepsilon_\rho - e_\omega + A + e_\mu)] \\ \leq V_{k+1}(q + e_\mu + e_\rho + e_\omega) + V_{k+1}(q + e_\mu),$$

the second equation follows from the induction hypothesis (substituting $q - \varepsilon_\rho - e_\omega + A$ for $q$ in (14) for $V_k$) and the last equation follows from the definition of the value iteration. If only one of $q_\rho$ and $q_\omega$ is zero, without loss of generality, assume that $q_\rho \geq 1$ and $q_\omega = 0$. We then obtain

$$V_{k+1}(q + 2\varepsilon_\mu) + V_{k+1}(q + \varepsilon_\rho + \varepsilon_\omega) = c_\rho + c_\omega + \beta E[V_k(q - \varepsilon_\rho + A + 2\varepsilon_\mu)] + c_\rho + c_\omega + \beta E[V_k(q + A)] \\ \leq c_\rho + c_\omega + \beta E[V_k(q + A + \varepsilon_\mu)] + c_\rho + c_\omega + \beta E[V_k(q - \varepsilon_\rho + A + e_\mu)] \\ \leq V_{k+1}(q + e_\mu + e_\rho + e_\omega) + V_{k+1}(q + e_\mu),$$
where the second equation follows from the induction hypothesis (substituting \( q - e_\mu + A \) for \( q \) in (12) for \( V_k \)) and the last follows from the definition of the value iteration.

Finally, suppose that the optimal action on \( (q + 2e_\mu) \) is to serve packets in queues \( \mu \) and \( \nu \), and the optimal action on \( q + e_\mu + e_\omega \) is to transmit packets in queue \( \rho \) and \( \omega \) in the \((k+1)\)th value iteration. We then obtain

\[
V_{k+1}(q + 2e_\mu) + V_{k+1}(q + e_\mu + e_\omega) = c_\mu + c_\nu \cdot I_{\{q_\nu > 0\}} + \beta \mathbb{E}[V_k((q - e_\nu)^+ + A + e_\mu)] \\
+ c_\rho + c_\omega + \beta \mathbb{E}[V_k(q + A)] \\
= (c_\mu + c_\nu \cdot I_{\{q_\nu > 0\}} + c_\rho + c_\omega) \\
+ \beta \mathbb{E}[V_k((q - e_\nu)^+ + A + e_\mu) + V_k(q + A)] \\
+ \beta \mathbb{E}[V_k(q + A + e_\mu) + V_k((q - e_\nu)^+ + A)] \\
+ \beta \mathbb{E}[V_k(Q + z) + V_k(Q + w)],
\]

where \( Q = (q - e_\nu)^+ + A, x = e_\mu + q - (q - e_\nu)^+, y = 0, z = e_\mu, w = q - (q - e_\nu)^+. \) We also have \( x, y, z, w \in \{0, 1\}^{|I|}, x \leq e_\mu + e_\nu, y \leq e_\rho + e_\omega, \) and \( x + y = z + w. \) We therefore obtain

\[
V_{k+1}(q + e_\mu) + V_{k+1}(q + e_\mu + e_\rho + e_\omega) \leq V_{k+1}(q + 2e_\mu) + V_{k+1}(q + e_\rho + e_\omega),
\]

which follows from Lemma [5.1].

Hence, [14] holds for the \((k+1)\)th value function \( V_{k+1} \).

6 Computational Experiments

In this section, we present a representative sample of results from computational experiments on the performance of the (asymptotically) optimal solutions to the optimization problem \([P_B]\) in comparison with variants of the MaxWeight scheduling policy in the \(2 \times 2\) input-queued switch model. The case of symmetric arrival rates and unit costs across all queues is considered first, followed by consideration of the general case for the arrival-rate and cost vectors.

6.1 Symmetric Case

Our results above establish that an optimal policy follows the \( c_\mu \) rule in the interior region and in the trivial boundary while having a switching curve structure in the critical boundary. Hence, upon identifying the switching curve for the critical boundary, we have complete information about our optimal scheduling policy. An explicit characterization is possible in the case of unit costs (\( c_{ij} = 1 \)) and symmetric arrivals where we assume a Bernoulli arrival process with the same rate \( \lambda_{ij} = \lambda < 0.5 \), for all \((i, j) \in I\) and all \( t \).

For this symmetric case, the interior region comprises all states in which the queues \((1, 1)\) and \((2, 2)\) or the queues \((1, 2)\) and \((2, 1)\) are nonempty (i.e., the states in which the system can transmit two packets), whereas the trivial boundary comprises states with only one nonempty queue. The critical boundary consists of the states in which there are two nonempty queues but only one packet can be transmitted. We then have the following explicit characterization.

Claim 6.1. For the symmetric \(2 \times 2\) input-queued switch, an optimal scheduling policy to maximize discounted reward over the infinite time horizon is given by

i. Transmit two packets in the interior;

ii. Transmit a packet in the trivial boundary;

iii. Transmit a packet from the longest queue among the two nonempty queues when in the critical boundary.
Proof The results i. and ii. follow directly from Theorems 4.1 and 4.2. The result iii. follows upon applying the arguments in the proofs of Theorem 4.4 and Proposition 4.5 to the symmetric input-queued switch, which we omit. □

Since the policy in Claim 6.1 is also an optimal solution to the problem \( \mathcal{P}_\beta \) for any discounted factor \( \beta \in (0, 1) \), the policy minimizes the long-run average queue length. Note that this policy is the same as MaxWeight scheduling except for the actions in the interior region.

To investigate the performance between both scheduling policies in this symmetric case, we present in Figure 1 results from computational experiments for the long-run average queue length, together with 95% confidence intervals, under various arrival rates \( \lambda = 0.30, 0.35, 0.40, 0.45, 0.48 \) taken over 100 samples, for both our optimal policy and the MaxWeight policy, which we denote by MWS. We observe from these and related computational experiments that our optimal policy provides larger performance gaps over the MaxWeight policy in heavier traffic intensities (higher arrival rates). To quantify this gap, we present in Table 1 the relative optimality gap between MaxWeight scheduling and our optimal policy: \( \{(\text{MWS-OPT})/\text{OPT}\} \times 100 \).

![Figure 1: Long-run Average Queue Length](image)

| Arrival rate | Relative optimality gap |
|--------------|-------------------------|
| 0.30         | 1.31                    |
| 0.35         | 2.76                    |
| 0.40         | 5.28                    |
| 0.45         | 9.64                    |
| 0.48         | 15.01                   |

Note: For the highest arrival rate considered, our optimal policy renders a long-run average queue length 15% smaller than that of MaxWeight scheduling.

6.2 General Case

In contrast to the symmetric case, deriving an explicit switching curve is difficult in general because this structure depends on the discount factor \( \beta \), cost coefficients \( c_{ij} \), and arrival processes. Hence, instead of an explicit optimal solution, we investigate the performance of an approximate optimal policy based on value iterations, which we call the “Look-ahead policy” and show to be asymptotically optimal with respect to the look ahead. Throughout this section, let \( \mathcal{V} \) denote the set of bounded real-valued functions on the state space \( \mathbb{Z}_+^d \) with norm \( ||V|| := \sup\{|V(q)| : q \in \mathbb{Z}_+^d\}, \quad V \in \mathcal{V} \). We also define \( V^*_\beta, \tilde{V}^*_\beta \in \mathcal{V} \) by

\[
V^*_\beta(q) := \max\{J_\beta(q, \pi) : \pi \in \mathcal{M}\},
\]

\[
\tilde{V}^*_\beta(q) := \max\{\tilde{J}_\beta(q, \pi) : \pi \in \mathcal{M}\},
\]

recalling \( \mathcal{M} \) to be the set of all stationary policies.
6.2.1 Look-ahead Policy

As in Section 4.1, we consider value iteration on the optimization problem \( \{P_\beta\} \) starting with \( V_0 = 0 \). Then, we define the \( k \)th look-ahead policy \( \pi_k \) to be the policy that uses the \( k \)th value function as an approximation of an optimal solution. We therefore have

\[
\pi_k(q) := \arg \max \left\{ r(q, s) + \mathbb{E}\left[ V_k((q - s) + A) \right] : s \in P \right\}.
\]

This class of look-ahead policies has several benefits, two of which we briefly mention based on our theoretical results.

i. Our optimal results for the interior and trivial boundary reduce the computational burden of the look-ahead policy. Policy \( \pi_k \) is the same policy that generates the \( (k + 1) \)th value function. Since the optimal actions on states in the interior and the trivial boundary are known, we only need to find the actions for states in the critical boundary.

ii. For sufficiently large \( k \), policy \( \pi_k \) is a good approximation to an optimal solution of problem \( \{P_\beta\} \). Since \( \pi_k \) is based on value iterations to solve \( \{P_\beta\} \), \( J_\beta(q, \pi_k) \) is an approximation to \( \tilde{V}_\beta(q) \). Furthermore, we prove that \( J_\beta(q, \pi_k) \) converges to \( \tilde{V}_\beta(q) \) as \( k \) goes to infinity in the following theorem.

**Theorem 6.1.** Let \( V_0 = 0 \) and let \( \pi_k \) be the policy produced by the value iteration for \( k = 1, 2, \ldots \). Then, \( J_\beta(\cdot, \pi_k) \) converges to \( \tilde{V}_\beta(\cdot) \) as \( k \to \infty \). More precisely, if

\[
\|V^{k+1} - V^k\| < \frac{\varepsilon(1 - \beta)^2}{2\beta^2}, \tag{22}
\]

we then have \( \|J_\beta(\cdot, \pi) - \tilde{V}_\beta\| < \varepsilon \).

**Proof** From the proof of Proposition 4.1, we have

\[
(1 - \beta)J_\beta(q, \pi_k) = c^\pi(0) + g - \beta \tilde{J}_\beta(q, \pi_k),
\]

\[
(1 - \beta)V_\beta^* = c^\pi(0) + g - \beta \tilde{V}_\beta^*,
\]

where \( g = \sum_{t=0}^{\infty} \beta^{t+1} \mathbb{E}[\sum_{(i,j) \in E} c_{ij}A_{ij}(t)] \). Subtracting the second equation from the first yields

\[
\|J_\beta(\cdot, \pi_k) - V_\beta^*\| = \frac{\beta}{1 - \beta} \|\tilde{J}_\beta(\cdot, \pi_k) - \tilde{V}_\beta^*\|.
\]

On the other hand, from Theorem 6.3.1 in [4], we have that \( \tilde{J}(\cdot, \pi_k) \) converges to \( \tilde{V}_k \) and \( \|\tilde{J}(\cdot, \pi_k) - \tilde{V}_\beta^*\| < \frac{1 - \beta}{\beta} \varepsilon \) when (22) holds, which implies the desired results. \( \square \)

6.2.2 Performance of Look-ahead Policy

We compare the performance of our class of look-ahead policies, for different values of \( k \), with the performance of variants of MaxWeight scheduling. The standard MaxWeight policy, denoted by MWS, chooses a schedule that has larger total number of packets than the other schedule. The \( c \)-weighted MaxWeight policy, which we denote by C-MWS, chooses a schedule that has the larger weight than the other schedule where the weight is a linear function of the queue lengths and the cost coefficients; e.g., packets from queues \((1, 1)\) and \((2, 2)\) are transmitted when \( c_{11}q_{11} + c_{22}q_{21} > c_{12}q_{12} + c_{21}q_{21} \).

To investigate performance among the various scheduling policies, we calculate the expected total discounted queue length of the \( k \)th look-ahead policy (with look-ahead step size \( k \)) and compare these results with those for MWS and C-MWS. Figure 2 presents a representative sample of these computational experiment results, together with 95% confidence intervals, under arrival rates \( \lambda_{11} = 0.7, \lambda_{22} = 0.5, \lambda_{12} = 0.2 \) and \( \lambda_{21} = 0.29 \), cost vectors \( c_{11} = 2, c_{22} = 2, c_{12} = 10, c_{21} = 10 \), and discount factor \( \beta = 0.99 \), taken over 1000 samples. We observe from these and related experiments (across various distributions for arrivals, various arrival rates, and various cost coefficients) that the performance of the look-ahead policy is close to the optimal performance when the step size \( k \) is greater than or equal to 4. (Results for \( k = 6, \ldots, 10 \) are
nearly identical to those depicted for $k = 5$.) Table 2 presents relative optimality gaps between C-MWS and the look-ahead policies.

We further observe from these and related experiments that the look-ahead policies are good approximations to the optimal solution of problem (P $\beta$) even when the step size is relatively small, where the optimality gap varies from 7% to 16% depending on the experimental settings.

## 7 Conclusions

In this paper we investigated an input-queued switch scheduling problem in which the objective is to minimize a linear function of the queue-length vector. Within this context theoretical properties of variants of the well-known MaxWeight scheduling algorithm were established, which includes showing that these algorithms exhibit optimal heavy-traffic queue-length scaling. We derived an optimal scheduling policy and established its theoretical properties for $2 \times 2$ input-queued switches, demonstrating fundamental differences with the variants of MaxWeight scheduling. Computational experiments demonstrated and quantified the benefits of our optimal scheduling policy. We expect our results to be of interest more broadly than input-queued switches.

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[1] S. T. Maguluri, S. K. Burle, and R. Srikant. Optimal heavy-traffic queue length scaling in an incompletely saturated switch. Preprint, 2016.

[2] S. T. Maguluri and R. Srikant. Heavy traffic queue length behavior in a switch under the maxweight algorithm. Stoch. Syst., 6(1):211–250, 2016.
the new dot product by appropriately adapting the arguments in the proof of Lemma A.1. For all $\eta$ for positive numbers respectively, for all section. notation and clarify the presentation, we will omit the superscript $[6]$. L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling collapse in $[2]$, though with some important technical differences and details for the A. Additional Proofs for Heavy Traffic Analysis A.1 Proof of State Space Collapse

The proof of Proposition $[3,2]$ basically follows a similar logical approach to that in the proof of state space collapse in $[2]$, though with some important technical differences and details for the c-weighted MaxWeight algorithm due to the modified dot product, norms and projections with respect to the weight vector $c$. Here we will present the major arguments and point out the primary technical differences. To simplify the notation and clarify the presentation, we will omit the superscript $\epsilon$ associated with the variables of this section.

Our general approach consists of defining a Lyapunov function and its drift by

$$W_{\perp e}(q) := \|q_{\perp e}\|_c$$ and $$\Delta W_{\perp e}(q) := (W_{\perp e}(q(t + 1)) - W_{\perp e}(q(t)))1_{\{q(t) = q\}},$$

respectively, for all $q \in \mathbb{R}^{n^2}$. Then, upon proving that

$$\mathbb{E}[\Delta W_{\perp e}(q)|q(t) = q] \leq -\eta, \quad \forall q \text{ with } W_{\perp e}(q) \geq \kappa, \quad (23)$$

for positive numbers $\eta$, $\kappa$ and $D$ that depend on $\bar{\sigma}$, $\nu$, $A_{\text{max}}$ and $\nu_{\text{min}}$, but not on $c$, we can derive an expression for $M_\nu$ in Proposition $[3,2]$ and establish the desired result.

To prove $\text{(23)}$ we start with a version of Lemma 4 in $[2]$, which can be shown to hold more generally for the new dot product by appropriately adapting the arguments in the proof of Lemma 7 in $[7]$.

**Lemma A.1.** For all $q \in \mathbb{R}^{n^2}$, we have

$$\Delta W_{\perp e}(q) \leq \frac{1}{2\|q_{\perp e}\|_c}(\Delta V(q) - \Delta V_{\perp e}(q)). \quad (25)$$

Let us separately consider the two quantities $\Delta V(q)$ and $\Delta V_{\perp e}(q)$, recalling the queueing dynamics in $[2]$. For the first quantity, we obtain

$$\mathbb{E}[\Delta V(q)|q(t) = q] = \mathbb{E}[\|q(t) + A(t) - S(t)\|_c^2 + ||U(t)||_c^2 + 2\langle q(t + 1) - U(t), U(t) \rangle_c - \|q(t)\|_c^2|q(t) = q]$$

$$\leq \mathbb{E}[\|A(t) - S(t)\|_c^2 + 2\langle q(t), A(t) - S(t)\rangle_c|q(t) = q]$$

$$= \sum_{ij} c_{ij} a_{ij}^2(t) + c_{ij} s_{ij}(t) - 2c_{ij} A_{ij}(t) s_{ij}(t)|q(t) = q] + 2\langle q, \lambda - \mathbb{E}[S(t)|q(t) = q] \rangle_c$$

$$\leq \sum_{ij} c_{ij}(\lambda_{ij} + \sigma_{ij}^2) + \sum_{ij} c_{ij} s_{ij}(t) - 2\epsilon(q, \nu)_c + 2\min(q, \nu - r)_c,$$
where we take advantage of the facts that \( \langle q(t^+) \rangle, U(t) \rangle \) = 0 and that arrivals are independent of the queue length and service processes in each time slot, together with our definition of the \( c \)-weighted MaxWeight algorithm. The selection of \( r \) will be \( \nu + \nu_{\text{min}}^{c} |q_{\perp +} e| \), where \( \nu \) is an arrival rate vector that resides on the boundary of the capacity region with all input and output ports saturated and where \( \nu_{\text{min}}^{c} := \min \frac{\nu_{ij}}{\nu_{ij}} \). This selection of \( r \) guarantees that it is within the capacity region, which is readily verified by first observing that \( \nu_{ij} + \nu_{\text{min}}^{c} |q_{\perp +} e| \geq \nu_{ij} - \nu_{\text{min}} \geq 0 \) and then observing that \( \langle \nu + \nu_{\text{min}}^{c} |q_{\perp +} e|, e \rangle \leq 1 \) and \( \langle \nu + \nu_{\text{min}}^{c} |q_{\perp +} e|, e \rangle \leq 1 \). We therefore have
\[
E[\Delta V(q) | q(t) = q] \leq \sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}} - 2\langle q, \nu \rangle - 2 c_{\text{max}} ||q_{\perp +} e||. \]

Taking advantage of the fact that \( \langle q_{\perp +} e \rangle, q_{\perp +} e \rangle \leq 0 \). Turning to the second quantity, we obtain
\[
E[\Delta W_{\perp +} (q) | q(t) = q] \leq \sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}} - 2\langle q, \nu \rangle - 2 c_{\text{max}} ||q_{\perp +} e||. \]

And thus
\[
E[\Delta W_{\perp +} (q) | q(t) = q] \leq \frac{4(\sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}})}{\nu_{\text{min}}^{c}} - \frac{\epsilon}{\nu_{\text{min}}^{c}} - \frac{\epsilon}{\nu_{\text{min}}^{c}}. \]

Given
\[
\epsilon < \frac{\nu_{\text{min}}^{c}}{2 ||\nu||_{e}},
\]

Then on the set of
\[
W_{\perp +} (q) \geq \frac{4(\sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}})}{\nu_{\text{min}}^{c}},
\]

we obtain
\[
E[\Delta W_{\perp +} (q) | q(t) = q] \leq \frac{1}{2 ||q_{\perp +} e||} \left( \sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}} - 2\langle q, \nu \rangle - 2 c_{\text{max}} ||q_{\perp +} e|| + 2 \langle q_{\perp +} e, \nu \rangle \right)
\]
\[
\leq \frac{\sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}}}{2 ||q_{\perp +} e||} - \nu_{\text{min}}^{c} - \frac{\epsilon}{\nu_{\text{min}}^{c}} ||q_{\perp +} e||
\]
\[
\leq \frac{\sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}}}{2 ||q_{\perp +} e||} - \nu_{\text{min}}^{c} - \epsilon ||\nu||_{e}
\]
\[
\leq \frac{\sum_{ij} c_{ij} (\lambda_{ij} + \sigma_{ij}^2) + n c_{\text{max}}}{2 ||q_{\perp +} e||} - \frac{\nu_{\text{min}}^{c}}{4} \leq - \nu_{\text{min}}^{c}.\]
Hence, (23) holds with \( \eta = -\frac{\nu_{\min}}{\lambda} \).

To prove (24), we have

\[
|\Delta W_{l_e}(q)| \leq \|q_{l_e}(t + 1)\|_c - \|q_{l_e}(t)\|_c \leq \|q(t + 1) - q(t)\|_c
\]

\[
= \sqrt{\sum_{ij} c_{ij}(q_{ij}(t + 1) - q_{ij}(t))^2} \leq \sqrt{\sum_{ij} c_{ij} A^2_{ij}} \leq n\sqrt{c_{\max}} A_{\max},
\]

and thus (24) holds with \( D = n\sqrt{c_{\max}} A_{\max} \).

Now, from Lemma 3 in [2], we obtain

\[
\mathbb{E}\left[\|q_{l_e}^{(e)}\|^r\right] \leq (2\kappa)^4 + r\left(\frac{D + \eta}{\eta}\right)^r (4D)^r \leq (2\kappa)^r + \sqrt{\nu} \left(\frac{4D^r D + \eta}{\nu}\right)^r
\]

\[
\leq 2 \left( \max \left\{ 2\kappa, (\sqrt{\nu})^{1/r} 4D^{r/r} \frac{D + \eta}{\nu} \right\} \right)^r = (M_r)^r,
\]

where

\[
M_r = 2^{1/r} \max \left\{ 2\kappa, (\sqrt{\nu})^{1/r} 4D^{r/r} \frac{D + \eta}{\nu} \right\},
\]

which is a function of \( r, \bar{\sigma}, \nu, A_{\max} \) and \( \nu_{\min} \), but is independent of \( \epsilon \), hence completing the proof.

**Remark A.1.** For the special case of \( c = 1 \), we have the standard MaxWeight algorithm and our results reduce to the state space collapse in [3]. More generally, the capacity region and the maximal face \( F \) are not dependent upon the choice of the weight vector \( c \). However, for any positive weight vector, the state space collapses into the normal cone of the face \( F \) with respect to the dot product defined by the weight vector \( c \). This cone depends upon the choice of the weight vector, and thus the choice of the weight vector “tilts” the cone of collapse.

### A.2 Proof of Theorem 3.1

The proof basically follows a similar logical approach to that in [1], though with some important technical differences and details for the \( c \)-weighted MaxWeight algorithm due to the modified dot product, norms and projections with respect to the weight vector \( c \). Here we will present the major arguments and point out the primary technical differences. As in Appendix A.1, we will omit the superscript \( (e) \) associated with the variables of this section to simplify the notation and clarify the presentation.

Let \( A(t) \) denote the arrival vector in steady state, which is identically distributed to the random vector \( A(t) \) for any time \( t \in \mathbb{Z}_+ \). Further let \( S(q) \) and \( U(q) \) denote the steady state schedule and unused service vector, respectively, both of which depend on the queue length vector in steady state \( q \). Recalling the queueing dynamics in [1], define \( \bar{q}^+ := q + A - S(q) + U(q) \) to be the queue length vector at time \( t + 1 \), given the queue length vector at time \( t \) is \( q \). Clearly, \( q^+ \) and \( \bar{q}^+ \) have the same distribution.

The proof proceeds by setting the drift of the Lyapunov function \( V(q) = \|q\|_{S_e}^2 \) to zero in steady state, from which we obtain

\[
0 = \mathbb{E}[V(q^+) - V(q)]
\]

\[
= \mathbb{E}[\|A - S(q)\|_{S_e}^2] + 2\mathbb{E}[\langle q, A - S(q) \rangle_{S_e}] - \mathbb{E}[\|U|_{S_e}(q)\|_{S_e}^2] + 2\mathbb{E}[\langle q, U|_{S_e}(q) \rangle_{S_e}].
\]

This then yields an equation of the form

\[
2\mathbb{E}\left[\langle q, S(q) - A \rangle_{S_e} \right] = \mathbb{E}[\|A - S(q)\|_{S_e}^2] - \mathbb{E}[\|U|_{S_e}(q)\|_{S_e}^2] + 2\mathbb{E}\left[\langle q, U|_{S_e}(q) \rangle_{S_e} \right]. \quad (26)
\]
space collapse from Proposition 3.2, we have from the fact that the space spanned by the normal vectors of Cauchy-Schwartz inequality and making use of (28), we obtain where the second equation follows from the fact that (26) is upper bounded by thus yielding the left-hand side of the heavy-traffic limit in Theorem 3.1.

Now, we turn to the right-hand side of (26), for which we want to show convergence to \( \langle \sigma^2, \zeta \rangle_c \) where \( \zeta_{ij} := \| (e_{ij}) \|_{S_{c}}^2 \) as \( \epsilon \downarrow 0 \). Let us first investigate the expectation of the total unused service. Since \( \bar{q} \) and \( \bar{q}^+ \) have the same distribution, we obtain

\[
0 = \mathbb{E} \left[ \sum_{i,j} q_{ij}(t + 1) - \sum_{i,j} q_{ij}(t) \parallel q(t) = \bar{q} \right] = \mathbb{E} \left[ \sum_{i,j} A_{ij} - \sum_{i,j} S_{ij}(q) + \sum_{i,j} U_{ij}(q) \right]
\]

\[
= \sum_{i,j} \lambda_{ij} - n + \mathbb{E} \left[ \sum_{i,j} U_{ij}(q) \right] = (1 - \epsilon) \sum_{i,j} \nu_{ij} - n + \mathbb{E} \left[ \sum_{i,j} U_{ij}(q) \right] = -n \epsilon + \mathbb{E} \left[ \sum_{i,j} U_{ij}(q) \right],
\]

where the last equation follows from \( \nu \in \mathcal{F} \) and \( \sum_{i,j} \nu_{ij} = n \). We therefore have

\[
\mathbb{E} \left[ \sum_{i,j} U_{ij}(q) \right] = n \epsilon.
\]

Because of the non-expansive property of the projection and (28), the second term on the right-hand side of (26) is upper bounded by

\[
\mathbb{E} \left[ \| U_{\parallel S_c(q)} \|_c^2 \right] \leq \mathbb{E} \left[ \sum_{i,j} c_{ij} U_{ij}^2 \right] = \mathbb{E} \left[ \sum_{i,j} c_{ij} U_{ij} \right] \leq c_{\text{max}} n \epsilon,
\]

which implies that this second term converges to 0 as \( \epsilon \downarrow 0 \).

On the other hand, for the third term on the right-hand side of (26), we have

\[
2 \mathbb{E} \left[ \langle \bar{q} \| S_c(q) \rangle_c \right] - 2 \mathbb{E} \left[ \langle \bar{q}^+, U_{\parallel S_c(q)} \rangle_c \right] - 2 \mathbb{E} \left[ \langle \bar{q}^+ \| S_c(q) \rangle_c \right] = -2 \mathbb{E} \left[ \langle \bar{q}^+ \| S_c(q) \rangle_c \right],
\]

where the second equation follows from the fact that \( \bar{q}^+_{ij} = 0 \) whenever \( U_{ij}(\bar{q}) = 1 \). Upon employing the Cauchy-Schwartz inequality and making use of (28), we obtain

\[
-2 \sqrt{\mathbb{E} \left[ \| \bar{q}^+ \|_c^2 \right] \mathbb{E} \left[ \| \bar{q}^+ \|_c^2 \right]} \leq 2 \mathbb{E} \left[ \langle \bar{q}^+ \| S_c(q) \rangle_c \right] \leq 2 \sqrt{\mathbb{E} \left[ \| \bar{q}^+ \|_c^2 \right] \mathbb{E} \left[ \| U_{\parallel S_c(q)} \|_c^2 \right]},
\]

\[
\Rightarrow -2 M_2 \sqrt{\mathbb{E} \left[ \| \bar{q}^+ \|_c^2 \right]} \leq 2 \mathbb{E} \left[ \langle \bar{q}^+ \| S_c(q) \rangle_c \right] \leq 2 M_2 \sqrt{\mathbb{E} \left[ \| U_{\parallel S_c(q)} \|_c^2 \right]},
\]

\[
\Rightarrow -2 M_2 \sqrt{2n \epsilon} \leq 2 \mathbb{E} \left[ \langle \bar{q}^+ \| S_c(q) \rangle_c \right] \leq 2 M_2 \sqrt{2n \epsilon},
\]
where \( M_2 \) is the constant in Proposition 3.2. This then implies that the third term also converges to 0 as \( \epsilon \downarrow 0 \).

Finally, turning to investigate the first term, the dimension of the underlying space is \( 2n - 1 \), and let us assume that \( f_1, f_2, \ldots, f_{2n-1} \) is orthonormal for this space. Hence, from basic properties of the space, we know that there exist \( v_{\ell,i} \) and \( \tilde{v}_{\ell,j} \) such that \( f_{\ell ij} - v_{\ell,i} - \tilde{v}_{\ell,j} \). We then can derive

\[
\mathbb{E}[\| (A - S(\bar{q})) \|_{\phi_{\epsilon}}^2] \\
= \sum_{\ell=1}^{2n-1} (A - S(\bar{q}), f_\ell)^2 \\
= \sum_{\ell=1}^{2n-1} \left( \sum_{ij} (A_{ij} - S_{ij}) \left( \frac{v_{\ell ij} + \tilde{v}_{\ell ij}}{c_{ij}} \right) c_{ij} \right)^2 \\
= \sum_{\ell=1}^{2n-1} \left[ \sum_i v_{\ell i} \left( \sum_j A_{ij} - S_{ij} \right)^2 + \sum_j \tilde{v}_{\ell j} \left( \sum_i (A_{ij} - S_{ij}) \right)^2 \right] \\
= \sum_{\ell=1}^{2n-1} \text{Var} \left[ \sum_i v_{\ell i} \sum_j A_{ij} + \sum_j \tilde{v}_{\ell j} \sum_i A_{ij} \right] \\
= \sum_{\ell=1}^{2n-1} \left[ \sum_i v_{\ell i}^2 \sum_j \sigma_{ij}^2 + \sum_j \tilde{v}_{\ell j}^2 \sum_i \sigma_{ij}^2 + 2 \sum_{ij} v_{\ell i} \tilde{v}_{\ell j} \sigma_{ij}^2 \right] \\
= \sum_{ij} c_{ij} \sigma_{ij} \sum_{\ell=1}^{2n-1} \left( \frac{v_{\ell i} + \tilde{v}_{\ell j}}{c_{ij}} \right)^2 c_{ij} = \sum_{ij} c_{ij} \sigma_{ij} \sum_{\ell=1}^{2n-1} (f_\ell, e_{ij})^2 = \sum_{ij} c_{ij} \sigma_{ij} \| (e_{ij}) \|_{\phi_{\epsilon}}^2 = \| \sigma^2 \|_{\phi_{\epsilon}}^2,
\]

which completes the proof of Theorem 3.1.

### B Additional Proofs for Optimal Policy

#### B.1 Proof of (18) for \( V_{k+1} \)

We prove that equation (18) holds in each case depending on the three optimal actions of the \((k+1)\)th value iteration on \( q \) and \( q + 2e_\mu \). In particular, from Theorem 4.4 for the \((k+1)\)th value iteration, serving queues \( \mu \) and \( \nu \) on \( q \) and serving queues \( \rho \) and \( \omega \) on \( q + 2e_\mu \) cannot be optimal at the same time.

First, assume that both optimal actions are serving queues \( \mu \) and \( \nu \). If \( q_\mu > 0 \), we have

\[
V_{k+1}(q) + V_{k+1}(q + 2e_\mu) = c_\mu + c_\nu + \mathbb{E}[V_k((q - e_\nu) + A - e_\mu)] \\
+ c_\mu + c_\nu + \mathbb{E}[V_k((q - e_\nu) + A + e_\mu)] \\
\leq c_\mu + c_\nu + \mathbb{E}[V_k((q - e_\nu) + A)] \\
+ c_\mu + c_\nu + \mathbb{E}[V_k((q - e_\nu) + A)] \\
\leq 2V_{k+1}(q + e_\mu),
\]

where the second equation follows from the induction hypothesis (substituting \( (q - e_\nu)^+ + A - e_\mu \) for \( q \) in (18) for \( V_k \)) and the last equation follows from the definition of the value iteration. On the other hand, if
where the second equation follows from Proposition 4.2.

Second, suppose that the optimal actions are serving queues $\rho$ and $\omega$. Then, we have

\[ V_{k+1}(q) + V_{k+1}(q + 2e_\mu) = 2(c_\rho \cdot I_{\{q > 0\}} + c_\omega \cdot I_{\{q > 0\}}) + E[V_k((q - e_\rho - e_\omega)^+ + A)] + E[V_k((q - e_\rho - e_\omega)^+ + A + 2e_\mu)] \]

where the second equation follows from the induction hypothesis (substituting $\rho$ and $\omega$ for $q$ in (18) for $V_k$) and the last equation follows from the definition of the value iteration.

Finally, assume that the optimal action on $q$ is to serve queues $\rho$ and $\omega$, and the optimal action on $q + 2e_\mu$ is to serve queues $\mu$ and $\nu$. We then obtain

\[ V_{k+1}(q + 2e_\mu) + V_{k+1}(q) = c_\rho \cdot I_{\{q \geq \rho\geq 0\}} + c_\omega \cdot I_{\{q \geq \omega\geq 0\}} + \beta E[V_k((q - e_\rho - e_\omega)^+ + A + 2e_\mu)] \]

where $Q := (q - e_\rho - e_\omega)^+ + A$, $x := e_\mu + q - (q - e_\rho - e_\omega)^+$, $y := q - (q - e_\rho - e_\omega)^+$, $z := e_\mu + q - (q - e_\rho - e_\omega)^+$, $w := q - (q - e_\rho - e_\omega)^+$. We also have $x, y, z, w \in \{0, 1\}^{E_2}$. We therefore obtain

\[ 2V_{k+1}(q + e_\mu) \geq c_\mu \cdot I_{\{q \geq \mu\geq 0\}} + \beta E[V_k((q - e_\rho - e_\omega)^+ + A)] \]

which follows from Lemma 5.1.

Hence, (18) holds for the $(k+1)$th value function $V_{k+1}$.

**B.2 Proof of (19) for $V_{k+1}$**

We prove that equation (19) for $n = k + 1$, where $\mu$ and $\nu$ can be served simultaneously, holds in each case depending on the three optimal actions of the $(k+1)$th value iteration on $q$ and $q + 2e_\mu + e_\nu$. In particular, from Theorem 4.4 for the $(k+1)$th value iteration, serving queues $\mu$ and $\nu$ on $q$ and serving queues $\rho$ and $\omega$ on $q + 2e_\mu + e_\nu$ cannot be optimal at the same time.
First, assume that both optimal actions are serving queues $\mu$ and $\nu$. If $q_\mu > 0$ and $q_\nu > 0$, we obtain

$$V_{k+1}(q) + V_{k+1}(q + 2e_\mu + e_\nu) = c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + 2e_\mu + e_\nu)]$$
$$+ c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + 2e_\mu + e_\nu)]$$
$$\leq c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A)]$$
$$+ c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A) - e_\nu]$$
$$\leq V_n(q + e_\mu + e_\nu) + V_n(q + e_\mu),$$

where the second equation follows from the induction hypothesis (substituting $q + A - e_\mu - e_\nu$ for $q$ in (19) for $V_k$) and the last equation follows from the definition of the value iteration. On the other hand, if $q_\mu > 0$ and $q_\nu = 0$, we have

$$V_{k+1}(q) + V_{k+1}(q + 2e_\mu + e_\nu) = c_\mu + \beta \mathbb{E}[V_k(q + A - e_\mu)]$$
$$+ c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A - e_\mu)]$$
$$\leq c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A)]$$
$$+ c_\mu + c_\nu + \beta \mathbb{E}[V_k(q + A)]$$
$$\leq V_n(q + e_\mu) + V_n(q + e_\mu + e_\nu),$$

where the second equation follows from the induction hypothesis (substituting $q + A - e_\mu$ for $q$ in (18) for $V_k$) and the last equation follows from the definition of the value iteration. Lastly, if $q_\mu = 0$, we obtain

$$V_{k+1}(q) + V_{k+1}(q + 2e_\mu + e_\nu) = c_\nu \cdot 1_{(q_\nu > 0)} + \mathbb{E}[V_k((q - e_\nu)^+ + A)]$$
$$+ c_\mu + c_\nu + \mathbb{E}[V_k(q + A + e_\mu)]$$
$$\leq c_\nu + c_\nu \cdot 1_{(q_\nu > 0)} + \mathbb{E}[V_k((q - e_\nu)^+ + A)]$$
$$+ c_\mu + c_\nu + \mathbb{E}[V_k(q + A)]$$
$$\leq V_n(q + e_\mu) + V_n(q + e_\mu + e_\nu),$$

where the second equation follows from Proposition 4.2.

Second, suppose that both optimal actions are serving queues $\rho$ and $\omega$. Then, we have

$$V_{k+1}(q) + V_{k+1}(q + 2e_\mu + e_\nu) = c_\rho \cdot 1_{(q_\rho > 0)} + c_\omega \cdot 1_{(q_\omega > 0)}$$
$$+ \beta \mathbb{E}[V_k((q - e_\rho - e_\omega)^+ + A)]$$
$$+ \beta \mathbb{E}[V_k((q - e_\rho - e_\omega)^+ + A + 2e_\mu + e_\nu)]$$
$$\leq 2(c_\rho \cdot 1_{(q_\rho > 0)} + c_\omega \cdot 1_{(q_\omega > 0)})$$
$$+ \beta \mathbb{E}[V_k((q - e_\rho - e_\omega)^+ + A + e_\mu + e_\nu)]$$
$$+ \beta \mathbb{E}[V_k((q - e_\rho - e_\omega)^+ + A + e_\mu)]$$
$$\leq V_n(q + e_\mu + e_\nu) + V_n(q + e_\mu),$$

where the second equation follows from the induction hypothesis (substituting $(q - e_\rho - e_\omega)^+ + A$ for $q$ in (19) for $V_k$) and the last equation follows from the definition of the value iteration.

Finally, assume that the optimal action on $q$ is to serve queues $\rho$ and $\omega$, and the optimal action on
where $Q:= (q - e_p - e_\omega)^+ + A$, $x:= e_\mu$, $y:= q - (q - e_p - e_\omega)^+$, $z:= e_\mu + q - (q - e_p - e_\omega)^+$, $w:= 0$. We also have $x, y, z, w \in \{0, 1\}^{I_2}$, $x \leq e_\mu + e_\nu$, $y \leq e_p + e_\omega$, and $x + y = z + w$. Therefore, we obtain

$$V_n(q + e_\mu + e_\nu) + V_n(q + e_\mu) \geq V_{k+1}(q + 2e_\mu + e_\nu) + V_{k+1}(q)$$

which follows from Lemma 13.

Hence, (20) holds for all value functions.

### B.3 Proof of (20) for $V_{k+1}$

We prove that equation (20) for $n = k + 1$, where $\mu$ and $\nu$ can be served simultaneously, holds in each case depending on the three optimal actions of the $(k + 1)$th value iteration on $q + e_\mu$ and $q + e_\nu$.

First, suppose that the optimal actions of $(k + 1)$th value iteration on $q + e_\mu$ and $q + e_\nu$ are to serve packets in queues $\rho$ and $\omega$. Then, we have

$$V_{k+1}(q + e_\mu) + V_{k+1}(q + e_\nu) = c_\rho \cdot I_{\{|q_\rho| > 0\}} + c_\omega \cdot I_{\{|q_\omega| > 0\}} + \beta V_k((q - e_p - e_\omega)^+ + A + e_\mu) + \beta V_k((q - e_p - e_\omega)^+ + A + e_\nu) \leq c_\mu (1 + I_{\{|q_\mu| > 0\}}) + c_\nu (1 + I_{\{|q_\nu| > 0\}}) + \beta V_k(Q + z) + V_k(Q + w),$$

where the second equation follows from the induction hypothesis (substituting $(q - e_p - e_\omega)^+ + A$ for $q$ in (20) for $V_k$) and the last equation follows from the definition of the value iteration.

Second, assume that both optimal actions are to serve packets in queues $\mu$ and $\nu$. We then obtain

$$V_{k+1}(q + e_\mu) + V_{k+1}(q + e_\nu) = c_\mu (1 + I_{\{|q_\mu| > 0\}}) + c_\nu (1 + I_{\{|q_\nu| > 0\}}) + \beta V_k((q - e_\mu)^+ + A) + \beta V_k((q - e_\mu)^+ + A) \leq c_\mu (1 + I_{\{|q_\mu| > 0\}}) + c_\nu (1 + I_{\{|q_\nu| > 0\}}) + \beta V_k(Q + z) + V_k(Q + w),$$

$$V_{k+1}(q + e_\mu + e_\nu) + V_{k+1}(q) \geq c_\mu (1 + I_{\{|q_\mu| > 0\}}) + c_\nu (1 + I_{\{|q_\nu| > 0\}}) + \beta V_k(q + A) + \beta V_k((q - e_\mu - e_\nu)^+ + A) \leq c_\mu (1 + I_{\{|q_\mu| > 0\}}) + c_\nu (1 + I_{\{|q_\nu| > 0\}}) + \beta V_k(Q + z) + V_k(Q + w).$$
where \( Q := (q - \mu - \nu)^+ + A \), \( x := q - (q - \mu - \nu)^+ \), \( y := 0 \), \( z := q - (q - \mu)^+ \), \( w := q - (q - \nu)^+ \). We also have \( x, y, z, w \in \{0, 1\}^{Z_1} \), \( x \leq \mu + \nu \), \( y \leq \mu + \omega \), and \( x + y = z + w \). Therefore, we obtain

\[
V_{k+1}(q + \mu + \nu) + V_{k+1}(q) \geq V_{k+1}(q + \mu) + V_{k+1}(q + \nu),
\]

which follows from Lemma 5.1

Finally, suppose that the two optimal actions are different from each other. Then, by symmetry, we can assume that the optimal action on \( q + \nu \) is to serve packets in queues \( \mu \) and \( \nu \), and the optimal action on \( q + \mu \) is to serve packets in queues \( \rho \) and \( \omega \). Thus, at least one of \( q_\rho \) and \( q_\omega \) is nonzero. If \( q_\rho = 0 \) and \( q_\omega = 0 \), we have

\[
V_{k+1}(q + \mu) + V_{k+1}(q + \nu) = c_{\nu} + \beta E[V_k(q + A)] + c_\mu + c_\nu + \beta E[V_k((q - e_\mu - e_\nu)^+ + A + e_\mu)]
\]

\[
\leq c_\mu + c_\nu + \beta E[V_k(q + A)] + c_\mu + c_\nu + \beta E[V_k(q + A)] + c_\mu + c_\nu + \beta E[V_k(q + A)] + V_{k+1}(q),
\]

where the second equation follows from Proposition 4.2. On the other hand, we now assume that \( q_\mu \geq 1 \). If \( q_\rho \geq 1 \) and \( q_\omega \geq 1 \), we obtain

\[
V_{k+1}(q + \mu) + V_{k+1}(q + \nu) = c_\rho + c_\omega + \beta E[V_k(q + A + e_\mu - e_\rho - e_\omega)]
\]

\[
+ c_\mu + c_\nu + \beta E[V_k(q + A - e_\mu)] + c_\mu + c_\nu + \beta E[V_k(q + A - e_\mu)] + c_\mu + c_\nu + \beta E[V_k(q + A - e_\mu)] + V_{k+1}(q),
\]

where the second equation follows from the induction hypothesis (substituting \( q + A - e_\mu - e_\rho - e_\omega \) for \( q \) in (14) for \( V_k \)) and the last equation follows from the definition of the value iteration. If \( q_\rho = 1 \) and \( q_\omega = 0 \), we have

\[
V_{k+1}(q + \mu) + V_{k+1}(q + \nu) = c_\mu + c_\nu + \beta E[V_k(q + A - e_\mu)]
\]

\[
+ c_\mu + c_\nu + \beta E[V_k(q + A - e_\mu)] + c_\mu + c_\nu + \beta E[V_k(q + A - e_\rho)] + V_{k+1}(q),
\]

where the second equation follows from the induction hypothesis (substituting \( q + A - e_\mu - e_\rho \) for \( q \) in (12) for \( V_k \)) and the last equation follows from the definition of the value iteration.

Since (20) holds for \( V_{k+1} \), it holds for all value functions.

**B.4 Proof of Lemma 5.1**

We prove Lemma 5.1 by induction. The proposition is true for \( V_0 \) because \( V_0(q) = 0 \) for all \( q \in \mathbb{Z}_1^{Z_1} \). Now, suppose that the proposition holds for the \( k \)th value function \( V_k \) and that \( x, y, z, w \in \{0, 1\}^{Z_1} \) satisfies the assumptions in Lemma 5.1. We then show that (21) holds for the \((k+1)\)th value function \( V_{k+1} \) in each case depending on the optimal actions of \( k \)th value iteration on \( q + z \) and \( q + w \) as follows.

I. Both optimal actions are same.

With out loss of generality, assume that both optimal actions are serving queues \( \mu \) and \( \nu \). Then, the
right-hand side of (21) becomes

\[
V_{k+1}(q + z) + V_{k+1}(q + w) \\
= c_\mu \cdot I_{\{q_\mu + z_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + z_\nu > 0\}} \\
+ \beta E[V_k((q + z - e_\mu - e_\nu)^+ + A)] \\
+ c_\mu \cdot I_{\{q_\mu + w_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + w_\nu > 0\}} \\
+ \beta E[V_k((q + w - e_\mu - e_\nu)^+ + A)]
\]

where

\[
\begin{align*}
Q &= (q - e_\mu - e_\nu)^+ + A, \\
z' &= (q + z - e_\mu - e_\nu)^+ - (q - e_\mu - e_\nu)^+, \\
w' &= (q + w - e_\mu - e_\nu)^+ - (q - e_\mu - e_\nu)^+.
\end{align*}
\]

On the other hand, by the definition of value iteration, we obtain

\[
V_{k+1}(q + x) + V_{k+1}(q + y) \\
\geq c_\mu \cdot I_{\{q_\mu + x_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + x_\nu > 0\}} \\
+ \beta E[V_k((q + x - e_\mu - e_\nu)^+ + A)] \\
+ c_\mu \cdot I_{\{q_\mu + y_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + y_\nu > 0\}} \\
+ \beta E[V_k((q + y - e_\mu - e_\nu)^+ + A)]
\]

where

\[
\begin{align*}
x' &= (q + x - e_\mu - e_\nu)^+ - (q - e_\mu - e_\nu)^+, \\
y' &= (q + y - e_\mu - e_\nu)^+ - (q - e_\mu - e_\nu)^+. \\
x' + y' &= z' + w',
\end{align*}
\]

and thus the first two equations in (29) and (30) are the same.

For the last part, it is readily verified that

\[
\begin{align*}
x', y', z', w' \in \{0, 1\}^{2I}, \\
x' \leq e_\mu + e_\nu, y' \leq e_\rho + e_\omega, \\
x' + y' &= z' + w',
\end{align*}
\]

which implies that

\[
V_k(Q + z') + V_k(Q + w') \leq V_k(Q + x') + V_k(Q + y')
\]

because $V_k$ satisfies Lemma [5.1] (induction hypothesis).

Hence, (21) holds for $V_{k+1}$ in this case.

**II.** Both optimal actions are different from each other.

Without loss of generality, assume that the optimal action on $(q + z)$ are serving queues $\mu$ and $\nu$. Then, as before, the right-hand side and left-hand side of (21) become

\[
V_{k+1}(q + z) + V_{n+1}(q + w) \\
= c_\mu \cdot I_{\{q_\mu + z_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + z_\nu > 0\}} \\
+ c_\mu \cdot I_{\{q_\mu + w_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + w_\nu > 0\}} \\
+ \beta E[V_k(Q + z') + V_k(Q + w')]
\]

\[
V_{k+1}(q + x) + V_{n+1}(q + y) \\
\geq c_\mu \cdot I_{\{q_\mu + x_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + x_\nu > 0\}} \\
+ c_\mu \cdot I_{\{q_\mu + y_\mu > 0\}} + c_\nu \cdot I_{\{q_\nu + y_\nu > 0\}} \\
+ \beta E[V_n(Q + x') + V_n(Q + y')]
\]

because $V_k$ satisfies Lemma [5.1] ( induction hypothesis).
where $Q = (q - e_\mu - e_\nu - e_\rho - e_\omega)^+ + A$, $x' = (q + x - e_\mu - e_\nu)^+ - (q - e_\mu - e_\nu)^+$, $y' = (q + y - e_\rho - e_\omega)^+ - (q - e_\rho - e_\omega)^+$, $z' = (q + z - e_\mu - e_\nu)^+ - (q - e_\mu - e_\nu)^+$, $w' = (q + w - e_\rho - e_\omega)^+ - (q - e_\rho - e_\omega)^+$.

Now, we calculate the terms involving $\mu$ in (31) and (32) for each case:

i. If $x_\mu = z_\mu$, then $w_\mu = 0$ because $y_\mu = 0$, and thus we obtain

$$c_\mu \cdot I(q_\mu + x_\mu > 0) = c_\mu \cdot I(q_\mu + z_\mu > 0),$$

$$x'_\mu + y'_\mu = z'_\mu + w'_\mu.$$  

ii. Suppose that $x_\mu > z_\mu$ and $q_\mu > 0$. Then, since $x_\mu + y_\mu = z_\mu + w_\mu$ and $y_\mu = 0$, we have $w_\mu = 1$, and therefore we obtain

$$c_\mu \cdot I(q_\mu + x_\mu > 0) = c_\mu \cdot I(q_\mu + z_\mu > 0) = c_\mu$$

$$x'_\mu + y'_\mu = z'_\mu + w'_\mu,$$

where the second line follows from $x'_\mu = z'_\mu + 1$ and $y'_\mu = w'_\mu$.

iii. Lastly, assume that $x_\mu = 0$, $z_\mu = 0$, $w_\mu = 1$, and $q_\mu = 0$. We then have

$$c_\mu \cdot I(q_\mu + x_\mu > 0) = c_\mu \cdot I(q_\mu + z_\mu > 0) = 0,$$

$$x'_\mu + y'_\mu = z'_\mu = 0, \quad w'_\mu = 1.$$  

Define $w'' = w' - e_\mu$. Then, we have $x'_\mu + y'_\mu = z'_\mu + w''$, and from the trivial boundary Proposition we obtain $V_k(Q + w') \leq c_\mu + V_k(Q + w'')$.

In summary, upon letting

$$w'' := \begin{cases} 
  w' & \text{if } x_\mu = z_\mu \\
  w' & \text{if } x_\mu > z_\mu \text{ and } q_\mu = 1 \\
  w' - e_\mu & \text{if } x_\mu > z_\mu \text{ and } q_\mu = 0
\end{cases},$$

we have

$$V_{k+1}(q + z) + V_{k+1}(q + w) \leq c_\mu \cdot I(q_\mu + x_\mu) + c_\nu \cdot I(q_\nu + x_\nu) + c_\nu \cdot I(q_\mu + w_\rho) + c_\omega \cdot I(q_\omega + w_\omega) + \beta E[V_k(Q + z') + V_k(Q + w')]$$

and

$$x' + y' = z' + w''.$$  

Repeating these steps for other components, yields $\widehat{w}, \widehat{z} \in \{0, 1\}^{2\mathbb{Z}}$ such that $x' + y' = \widehat{z} + \widehat{w}$ and

$$V_{k+1}(q + z) + V_{k+1}(q + w) \leq c_\mu \cdot I(q_\mu + x_\mu) + c_\nu \cdot I(q_\nu + x_\nu) + c_\nu \cdot I(q_\mu + y_\rho) + c_\omega \cdot I(y_\omega + w_\omega) + \beta E[V_k(Q + x') + V_k(Q + y')],$$

where the second equation follows from the induction hypothesis and the third equation follows from (32).

Hence, the $(k + 1)$th value function $V_{k+1}$ satisfies (21) and Lemma [5.1] holds.