Transitions in quantum computational power

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APCWQIS 2014, NCKU, Tainan

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Outline

I. Introduction
   - quantum computation by local measurement
   - notion of transitions in quantum computational power

II. Example Hamiltonians: Building blocks

III. 2D/3D structure
   - Phase diagrams for quantum computational power

IV. Summary
2D cluster state

- Created by applying CZ gates to each pair with edge

\[ |C\rangle \equiv \bigotimes_{\langle i,j \rangle} CZ_{i,j} \left( |+\rangle |+\rangle \cdots |+\rangle \right) \]

[Raussendorf&Briegel '01]

- Ground state of 5-body Hamiltonian

\[ H = - \sum_v K_v = - \sum_v \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix} \]

(except boundary spins)

\[ X, Y, Z \text{ Pauli matrices} \]
Resource for universal quantum computation

- Carve out entanglement structure by local Z measurement

[Raussendorf&Briegel ’01]

(1) Measurement along each wire simulates one-qubit evolution (gates)

(2) Measurement along each bridge simulates two-qubit gate (CNOT)

Universal measurement-based quantum computation (MBQC)
Other states for universal QC?

- The first known resource state is the 2D cluster state
  - Other 2D graph/cluster states* on regular lattices: triangular, honeycomb, kagome, etc. [Van den Nest et al. '06]
  - A few other states from tensor network construction and TriCluster state [Gross & Eisert '07, '10]
    [Chen et al. '09]
  - Family of 2D Affleck-Kennedy-Lieb-Tasaki states [Wei, Affleck & Raussendorf '11&12; Miyake '11; Wei '13, Wei, Hagnagadar & Raussendorf '14]

 опасно

So far still no complete characterization for resource states
Resource states from ground states?

- Unique ground states of certain gapped Hamiltonians?
  \[ H | \Psi_{\text{resource}} \rangle = E_0 | \Psi_{\text{resource}} \rangle \]
  
  - If so, create resources by cooling!
  
  - Desire simple and short-ranged (nearest nbr) 2-body Hamiltonians

  - Cluster states require few-body (e.g. 5-body) interactions! \[ \text{[Nielsen '06]} \]
  
  - AKLT states are ground state of two-body interacting Hamiltonians (possibly gapped) \[ \text{[AKLT '87,88]} \text{[Garcia-Saez,Murg,Wei '12]} \]

- What about thermal states for quantum computation?
  \[ \rho_{\text{resource}} = e^{-\beta H} / \text{Tr}(e^{-\beta H}) \] useful?
  
  \[ \text{[Li,Browne,Kwek,Raussendorf & Wei '11]} \]
Main motivations here

- Universal resource states from certain phases of matter?
  - it’s a difficult question in general, but some examples:
    - Browne et al. --- percolation
    - Bartlett et al. --- (1) Cluster in B field; (2) deformed AKLT
    - Murao et al. --- interacting cluster at finite T
    - Li et al. --- thermal states for QC

- Can we characterize regions in the phase diagram by the quantum computational power of the equilibrium states?

- System parameter vs. temperature “phase diagram” in terms of universal quantum computation?
Some examples why transitions in quantum computational power make sense...
Cluster state and percolation

- Cluster-state at faulty square lattice:

  \[ p_{\text{occupy}} : \text{system parameter} \Rightarrow \text{QC possible if } p_{\text{occupy}} > p_{\text{perco.threshold}} \]

  \[ \Rightarrow \text{Transition in quantum computational power} \]

\[ \text{[Browne, Elliot, Flammia, Merkel, Miyake, Short '08]} \]
\[ \text{cf [Gross, Eisert, Schuch, Perez-Garcia rowne, '07]} \]
Cluster phase?

- Doherty and Bartlett: Cluster Hamiltonian in B field

\[ H = - \sum_{v} K_v - B \sum_{v} X_v \]  \[\text{[Doherty, Bartlett '09]}\]

\[ H = -\sum_{\square} \hat{Z} \hat{Z} - B \sum_{\mu} \hat{X}_{\mu} \]

- They argue that the phase $|B|<1$ is characterized by fidelities of universal gates

- transition in quantum computational power

- Question: what about finite temperature $T$?
Deforming AKLT state

- Spin-3/2 AKLT state on honeycomb lattice is universal
  - [Wei, Affleck & Raussendorf '11; Miyake '11]

- Deformed AKLT also universal (in a range of deformation)
  \[ |\psi(a)\rangle \propto \left(D(a)^{-1}\right)^{\otimes N} |\psi_{\text{AKLT}}\rangle \]

  - Transition to non-universal coincides with transition to Neel order
  - Q: finite T?

\[
D(a) = \frac{\sqrt{3}}{a} (|\frac{3}{2}\rangle\langle\frac{3}{2}| + |\frac{-3}{2}\rangle\langle\frac{-3}{2}|)
+ |\frac{1}{2}\rangle\langle\frac{1}{2}| + |\frac{-1}{2}\rangle\langle\frac{-1}{2}|
\]

\[
H = \sum_{\langle i,j \rangle} D(a)_i \otimes D(a)_j h_{i,j}^{[\text{AKLT}]} D(a)_i \otimes D(a)_j
\]
Interacting cluster Hamiltonian at finite $T$

$$H = - \sum_v K_v = - \sum_v \begin{array}{c} Z \cr \otimes \cr Z \end{array}$$

$\Rightarrow$ $H' = - \sum_{\langle u,v \rangle} K_u K_v \rightarrow \sum_{\langle u,v \rangle} X_u X_v$ (mapped to classical Ising by CZ)

[Fujii, Nakata, Ohzeki, Murao ’13]

- Gate fidelity shows a transition at classical Ising transition $T_c$

$\Rightarrow$ Q: can we add local field to vary Hamiltonian?

$$H'' = - \sum_{\langle u,v \rangle} K_u K_v - B \sum_v X_v$$
Thermal states as resource

- 2D & 3D spin Hamiltonians

\[ V_{\text{line}} = \Delta (S_c^{x} A_b^x + S_c^{y} A_b^y + S_c^{z} A_b^z) \]
\[ V_{\text{dash}} = \Delta (S_c^{x} B_b^x + S_c^{y} B_b^y + S_c^{z} B_b^z) \]

- Thermal states can be used for universal QC, if \( T/\Delta \) is such that error below threshold

Q: can we vary some system parameter or local field?
Goal: Phase diagram with temperature and other system parameter

- Models illustrated:
  - Percolation: vary probability only
  - Cluster with B field and deformed AKLT: only at zero T
    \[ H = - \sum_v K_v - B \sum_v X_v \]
    \[ H = \sum_{\langle i,j \rangle} D(a)_i \otimes D(a)_j h[^{\text{AKLT}}]_{i,j} D(a)_i \otimes D(a)_j \]
  - Interacting cluster at finite T: mapped to classical Ising
    \[ H' = - \sum_{\langle u,v \rangle} K_u K_v \to - \sum_{\langle u,v \rangle} X_u X_v \]
  - Thermal states for universal QC: Hamiltonian Heisenberg-like

- Varying both T & some system parameter?
  \[ \Rightarrow \text{difficult problem: Hamiltonian not solvable in general} \]
Goal: Phase diagram of quantum computational power

- Another question: Does transition in quantum computational power necessarily coincide with transition in the phase of matter?

- Will construct two models to investigate these
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First: Toy model

- 3 spin-1/2 at sides and 1 spin-3/2 at center $u$

Mathematically project joint state of $(1'2'3')$ to their symmetric subspace (i.e. virtual to physical)

\[
\Pi_{S}^{[1'2'3'\rightarrow u]} = \left| +\frac{3}{2}\right\rangle \left\langle 000\right| + \left| -\frac{3}{2}\right\rangle \left\langle 111\right| \\
+ \left| +\frac{1}{2}\right\rangle \left\langle W\right| + \left| -\frac{1}{2}\right\rangle \left\langle \overline{W}\right|
\]

$|\overline{W}\rangle \equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left| -\frac{1}{2}\right\rangle$

$|W\rangle \equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left| \frac{1}{2}\right\rangle$
Spin 3/2 and three virtual qubits

- Addition of angular momenta of 3 spin-1/2’s

\[
\begin{align*}
\frac{1}{2} \otimes \frac{1}{2} &= 0 \oplus 1 \\
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}
\end{align*}
\]

- The four basis states in the symmetric subspace

\[
\begin{align*}
|111\rangle &\equiv |↓↓↓\rangle \leftrightarrow \left| -\frac{3}{2} \right\rangle & |\overline{W}\rangle &\equiv \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle) \leftrightarrow \left| -\frac{1}{2} \right\rangle \\
|000\rangle &\equiv |↑↑↑\rangle \leftrightarrow \left| \frac{3}{2} \right\rangle & |W\rangle &\equiv \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \leftrightarrow \left| \frac{1}{2} \right\rangle
\end{align*}
\]
Toy model: building block

- 3 spin-1/2 at sides and 1 spin-3/2 at center \( u \)

\[
|\text{Singlets}\rangle \equiv (|01\rangle - |10\rangle)_{1'1} (|01\rangle - |10\rangle)_{22'} (|01\rangle - |10\rangle)_{33'}
\]

\[
\Pi_{S[^{1'2'3'}\to u]} = |\frac{3}{2}\rangle\langle 000| - |\frac{3}{2}\rangle\langle 111| + |\frac{1}{2}\rangle\langle W| - |\frac{1}{2}\rangle\langle W|
\]

- Form an AKLT-like state:

\[
|\psi_{\text{AKLT}}\rangle \equiv \Pi_{S[^{1'2'3'}\to u]}|\text{Singlets}\rangle
\]

\[
= |000\rangle_{123} \otimes |\frac{3}{2}\rangle_u - |111\rangle_{123} \otimes |\frac{3}{2}\rangle_u - |W\rangle_{123} \otimes |\frac{1}{2}\rangle_u + |\overline{W}\rangle_{123} \otimes |\frac{1}{2}\rangle_u
\]

- Unique ground state of \( H \), projector to total \( S=2 \) joint subspace of spins \( i \) and \( u \)

\[
H = \sum_{i=1}^{3} P_{i,u}^{[S=2]} = \sum_{i} \left[ \frac{1}{2} \mathbf{s}_u \cdot \mathbf{s}_i + \frac{5}{8} \right]
\]
Toy model Hamiltonians

- Exactly diagonalizable
  - Finite energy gap = 1
- Next: Allow the Hamiltonians to vary

\[
H = \sum_{i=1,2,3} \vec{S}_u \cdot \vec{s}_i
\]

\[
H_1 = \sum_{i=1,2,3} \left[ S_u^x s_i^x + S_u^y s_i^y + (1 + \delta) S_u^z s_i^z \right]
\]

\[
H_2 = \sum_{i=1,2,3} \left( \vec{S}_u \cdot \vec{s}_i \right) - d_z \left( S_u^z \right)^2
\]
Toy model is for 4 spins.

Q: how do we scale to 2D or 3D structure for universal quantum computation?

Ans. Use toy model as a building block

→ Patch up 2D or 3D structure
2D or 3D spin models

- Use this as a building block to construct spin systems with a spectral gap

- Regard two bond qubits as a spin-3/2

- The spectral property is inherited from toy model
2D or 3D structure

- Use this as a building block to construct spin systems with a spectral gap

\[
H_1 = \sum_{i=1,2,3} [S_u^x s_i^x + S_u^y s_i^y + (1 + \delta)S_u^z s_i^z]
\]

\[
H_2 = \sum_{i=1,2,3} (\vec{S}_u \cdot \vec{s}_i) - d_z \left( S_u^z \right)^2
\]
Two spin models

\[ H_1 = \sum_{i=1,2,3} \left[ S_u^x s_i^x + S_u^y s_i^y + (1 + \delta) S_u^z s_i^z \right] \]

\[ H_2 = \sum_{i=1,2,3} \left( \vec{S}_u \cdot \vec{s}_i \right) - d_z \left( S_u^z \right)^2 \]

- No transitions at finite T (free energy analytic)

- \( f = -T \log \left[ \text{Tr}(e^{-\beta H}) \right] \)
Two spin models

\[ H_1 = \sum_{i=1,2,3} \left[ S_u^x s_i^x + S_u^y s_i^y + (1 + \delta) S_u^z s_i^z \right] \]

\[ H_2 = \sum_{i=1,2,3} \left( \vec{S}_u \cdot \vec{s}_i \right) - d_z \left( S_u^z \right)^2 \]

1\textsuperscript{st}

What about transitions in quantum computational power?
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How to understand the computational phase in $\delta (d_z)$ –vs.- $T$ plane?

- Model 1: Useful for QC
- Model 2: Not useful for QC
Zero temperature

\[ H_1 = \sum_{i=1,2,3} \left[ S_{u}^x s_i^x + S_{u}^y s_i^y + (1 + \delta) S_{u}^z s_i^z \right] \]

\[ H_2 = \sum_{i=1,2,3} \left( \vec{S}_{u} \cdot \vec{s}_i \right) - d_z \left( S_{u}^z \right)^2 \]

Wavefunction:
\[ |\Psi(\delta)\rangle = -(|3/2, 111\rangle - | -3/2, 000\rangle) + \frac{-2\delta + \sqrt{9 + 4\delta^2}}{3} (|1/2, \bar{W}\rangle - | -1/2, W\rangle) \]

\[ \Rightarrow \text{For } \delta, d_z \geq -\sqrt{3}/2, \text{ can filter out a GHZ } |\uparrow \downarrow \downarrow \uparrow \rangle - | \downarrow \uparrow \uparrow \uparrow \rangle \]

with probability=1 via generalize measurement (next slides)
Distill a four-spin GHZ state (@ δ=δz=0)

\[ H_1 = \sum_{i=1,2,3} \left[ S_u^x s_i^x + S_u^y s_i^y + (1 + \delta) S_u^z s_i^z \right] \]

\[ H_2 = \sum_{i=1,2,3} \left( \vec{S}_u \cdot \vec{s}_i \right) - d_z \left( S_u^z \right)^2 \]

- Central spin has 4 levels: how to reduce to 2 levels?
  - Use projection? 
    \[ P_z = \left| \frac{3}{2} \right>_u \left\langle \frac{3}{2} \right| + \left| -\frac{3}{2} \right>_u \left\langle -\frac{3}{2} \right| \]

\[ |000\rangle_{123} \otimes \left| -\frac{3}{2} \right>_u - \frac{3}{2} \right>_u - |111\rangle_{123} \otimes \left| \frac{3}{2} \right>_u - |W\rangle_{123} \otimes - \left| \frac{1}{2} \right>_u + |\bar{W}\rangle_{123} \otimes \left| \frac{1}{2} \right>_u \]

- What if the projection does not succeed?
Distill a four-spin GHZ state (@ \( \delta = d_z = 0 \))

\[
H_1 = \sum_{i=1,2,3} [S_{u}^{x} s_i^{x} + S_{u}^{y} s_i^{y} + (1 + \delta) S_{u}^{z} s_i^{z}]
\]

\[
H_2 = \sum_{i=1,2,3} (\vec{S}_u \cdot \vec{s}_i) - d_z (S_{u}^{z})^2
\]

- Central spin has 4 levels: how to reduce to 2 levels?

\[
|\psi\rangle_{123u} = |000\rangle_{123} \otimes |0\rangle_u - \frac{3}{2} |1\rangle_u - |111\rangle_{123} \otimes |\frac{3}{2}\rangle_u - |W\rangle_{123} \otimes |-\frac{1}{2}\rangle_u + |\overline{W}\rangle_{123} \otimes |\frac{1}{2}\rangle_u
\]

- Use projection?

\[
P_z = |\frac{3}{2}\rangle_u \langle \frac{3}{2}| - |\frac{3}{2}\rangle_u \langle \frac{1}{2}| - |\frac{1}{2}\rangle_u \langle \frac{3}{2}|
\]

\[
|\psi\rangle_{123u} \xrightarrow{P_z} |000\rangle_{123} \otimes |\frac{3}{2}\rangle_u - |111\rangle_{123} \otimes |\frac{3}{2}\rangle_u
\]

- What if the projection does not succeed?
Generalized measurement (POVM)

- \( x \) and \( y \) axes are also good:

\[
F_{u,z} = \frac{\sqrt{2}}{3} \left( \left| \frac{3}{2} \right\rangle_{a} \left\langle \frac{3}{2} \right|_{z} + \left| -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_{z} \right)
\]

\[
F_{u,x} = \frac{\sqrt{2}}{3} \left( \left| \frac{3}{2} \right\rangle_{a} \left\langle \frac{3}{2} \right|_{x} + \left| -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_{x} \right)
\]

\[
F_{u,y} = \frac{\sqrt{2}}{3} \left( \left| \frac{3}{2} \right\rangle_{a} \left\langle \frac{3}{2} \right|_{y} + \left| -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_{y} \right)
\]

- Three elements satisfy:

\[
F_{u,x} F_{u,x} + F_{u,y} F_{u,y} + F_{u,z} F_{u,z} = I_u
\]

- POVM outcome \( a_u \) = \{ \( x \), \( y \), or \( z \) \} is random

- effective 2-level system

\[
\left| \frac{3}{2} \right\rangle_{a_u} \leftrightarrow \left| 000 \right\rangle_{a_u}, \left| -\frac{3}{2} \right\rangle_{a_u} \leftrightarrow \left| 111 \right\rangle_{a_u}
\]

- \( a_u \): new quantization axis

\[
\bar{Z} \equiv \left| \frac{3}{2} \right\rangle_{a_u} \left\langle \frac{3}{2} \right|_{a_u} - \left| -\frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_{a_u}, \quad \bar{X} \equiv \left| \frac{3}{2} \right\rangle \left\langle -\frac{3}{2} \right|_{a_u} + \left| -\frac{3}{2} \right\rangle \left\langle +\frac{3}{2} \right|_{a_u}
\]

- state becomes \( \left| \Phi \right\rangle \rightarrow F_{u,a_u} \left| \Phi \right\rangle \)
Distill a four-spin GHZ state (@ $\delta=d_z=0$)

$H_1 = \sum_{i=1,2,3} [S_u^x s_i^x + S_u^y s_i^y + (1 + \delta)S_u^z s_i^z]$  
$H_2 = \sum_{i=1,2,3} (S_u \cdot \vec{s}_i) - d_z (S_u^z)^2$

- POVM outcome $a_u=\{x,y,z\}$ indicates a randomly chosen quantization axis and the state becomes

$$F_{u,a_u} |\psi_{AKLT}\rangle = |(+3/2)_u (111)_{123}\rangle_{a_u} - |(-3/2)_u (000)_{123}\rangle_{a_u}$$

- Re-label states @ site u:  $|(+3/2)_u\rangle \rightarrow |1_u\rangle$, $-|(-3/2)_u\rangle \rightarrow |0_u\rangle$

⇒ Post-POVM state is a GHZ state:  $|(0)_u (000)_{123}\rangle + |(1)_u (111)_{123}\rangle$
Distill a four-spin GHZ state (@ δ=d_z≠0)

\[ |\Psi(\delta)\rangle = -(|3/2, 111\rangle - | -3/2, 000\rangle) + \frac{-2\delta + \sqrt{9 + 4\delta^2}}{3}(|1/2, W\rangle - | -1/2, W\rangle) \]

\[ D(a) = | + 3/2\rangle\langle +3/2 | + | -3/2\rangle\langle -3/2 | + a(|1/2\rangle\langle 1/2 | + | -1/2\rangle\langle -1/2 |) \]

- Use combination of filtering D(a) and POVM F’s

\[ F'_{x}(a) = \frac{1}{a} F_{x} D(a) \]
\[ F'_{z}(a) = \sqrt{\frac{3a^2 - 1}{2a^2}} F_{z} D(a) \]
\[ F'_{y}(a) = \frac{1}{a} F_{y} D(a) \]

⇒ Distill GHZ state with certainty for \[ a \geq \frac{1}{\sqrt{3}} \quad (\delta, d_z \geq -\sqrt{3}/2) \]

- For \[ a < \frac{1}{\sqrt{3}} \], need a different POVM but succeeds with

\[ p = \frac{1 - 3a^2}{1 + a^2} \]

⇒ Percolation consideration for cluster state
GHZ network + CZ

→ Measurement of two virtual qubits enacts Controlled-Z gate between neighboring center particles

| ↑↓↓↓⟩ - | ↓↑↑↑⟩
Cluster state (universal for QC) is created (for $\delta, d_z \geq -\sqrt{3}/2$) on honeycomb lattice with unit probability.
Zero temperature ($\delta, d_z$ near -1)

For $-1.288 < \delta, d_z < -\sqrt{3}/2$, can filter out a GHZ with probability > percolation threshold

Connected 2D cluster state on faulty honeycomb lattice
Finite T diagram (T=0 understood)

- Model exactly solvable $\Rightarrow$ GHZ fraction at finite T is known
- Use techniques from fault-tolerant quantum computation
  $\Rightarrow$ to locate temperature where error rate = FT threshold
  $\Rightarrow$ transition temperature
3D is more robust

- Model 1: Useful for QC
- Model 2: Not useful for QC

- Can create 3D cluster state ➔ topological protection for QC (Measurement-based version of so-called surface code QC)

[Raussendorf, Harrington, Goyal ‘06]
[Raussendorf, Harrington, PRL (2007)]
Summary

- Introduce notion of “computational phase” via resource states in measurement-based quantum computation
- Explicitly construct two model spin-3/2 Hamiltonians \( \Rightarrow \) QC phase diagram of \( T \text{-vs. } \delta \) (or \( d_z \))

- Transitions in quantum computational power need NOT coincide with transitions in phases of matter

Main Refs.: Li, Browne, Kwek, Raussendorf & Wei, PRL 107,060501 (2011)

Wei, Li & Kwek, PRA 89,0502315 (2014)

Related Refs.: Wei, Affleck, Raussendorf, PRL 106,070501 (2011)

Wei, PRA 88,062307 (2013); Wei et al, PRA 90, 042333 (2014)

Garcia-Saez, Murg, Wei, PRB 88, 245118 (2013)
Supplementary slides
One-qubit gate

\[ U = e^{-i \frac{\xi}{2} \sigma_x} e^{-i \frac{\eta}{2} \sigma_z} e^{-i \frac{\xi}{2} \sigma_x} \]

input \[ |\psi_{in}\rangle \]

output \[ |\psi_{out}\rangle \sim U |\psi_{in}\rangle \]

Measurement pattern:

Observables:

\[ \sigma_x \]

\[ \cos(\xi) \sigma_x \pm \sin(\xi) \sigma_y \]
CNOT gate

\[
CNOT = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

\[|\psi_{\text{in}}\rangle \rightarrow |\psi_{\text{out}}\rangle \sim \text{CNOT}|\psi_{\text{in}}\rangle\]

\(\Rightarrow\) simulates CNOT (via entanglement between wires)
Generating a cluster state

- Example: 2D cluster state on square lattice
Creating a cluster state

- After POVM on center particles, each block is an effective 4-qubit GHZ state

\[ \rightarrow |0000\rangle + |1111\rangle \]

- Perform measurement on the bond particles

  ➔ Effective joint measurement on the two virtual qubits (e.g. Bell-state measurement or in 2-qubit cluster state basis)
  
  ➔ Induce control-phase gate between two center qubits (up to Z gates)
  
  ➔ Give rise to a cluster state on a hexagonal (honeycomb) lattice
Error analysis

Table 1: Consequence of qubit errors (from 0-3 and 0′-3′ of block C in Fig. b on logical cluster-state qubits. The logical error $X_C$ is equivalent to $Z_U Z_L Z_D Z_R$ due to the cluster-state stabilizer operator $X_C Z_U Z_L Z_D Z_R$. 

\[
\begin{array}{|c|c|}
\hline
\text{errors on 0-3 or 0′-3′} & \text{errors on C, U, L, D, and R} \\
\hline
X_0 & I \text{ (no error)} \\
X_0' & X_C \equiv Z_U Z_L Z_D Z_R \\
X_1 \text{ or } X_1' & Z_U Z_L \\
X_2 & Z_U \\
X_2' & Z_D \\
X_3 & Z_L \\
X_3' & Z_R \\
Z_i \text{ (for all } i) & Z_C \\
\hline
\end{array}
\]
Error analysis

As goal is to investigate intrinsic property of quantum computational power, assume error caused by finite $T$ (i.e. assume perfect measurement)

$$\rho_T = \frac{e^{-H/T}}{\text{Tr} e^{-H/T}}$$

$$\rho_{GHZ} = U_y F_x \rho_T F^\dagger_x U_y^\dagger + U_x F_y \rho_T F^\dagger_y U_x^\dagger + F_z \rho_T F^\dagger_z$$

$$\rho'_{GHZ} = \prod_{K \in \{K\}} \frac{1}{2} ([I] + [K]) \rho_{GHZ} = \sum_{\sigma \in \{\sigma\}} p_\sigma [\sigma] |GHZ\rangle \langle GHZ|$$

$$p_z \simeq 2(p_{Z_0} + 2p_{X_1} + p_{X_2} + p_{X_3} + 3p_{Z_0}X_1 + 2p_{Z_0}X_2 + 2p_{Z_0}X_3)$$

- **2D**: transition at
  $$p'_z \approx 10^{-7} \Rightarrow p_z \approx \frac{1}{3} 10^{-7} k(p_l)$$

- **3D**: $$\frac{p_l}{24.9\%} + \frac{p_z}{2.93\%} \approx 1$$

[Wei, Li, Kwek, PRA ‘14]
Computational phases

Model 1

Model 2

2D:

3D:
Fault tolerance at 3D

- Builds upon Raussendorf-Harrington-Goyal scheme on 3D cluster state

\[ \text{Error threshold: 1.4\% for depolarizing error and 0.11 \% (later improved to 0.75\%) on preparation-, gate-, storage-, and measurement errors} \]

[Ann of Phys 321, 2242 (2006)]

[Raussendorf & Harrington, PRL (2007)]