An autoregressive parametric method applied to the autowave process modelling

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Abstract. The authors of the paper research the topic of the autoregressive parametric spectral estimation method which is applied to oscillation (vibration). It is illustrated by the study of dynamically stable the combustion modes in a liquid-propellant engine. The scholars suppose a hypothesis that if a model parameter is selected correctly, the method offers a higher resolution. It does not require a dialogue-based weighted signal, and compensated parser/analyzer filter related to the accurate estimates of the oscillating system attributes. The survey proves that parametric spectral estimation amounts to solving an optimization problem with and seeking the autoregressive model value parameters of waveform shaping, in which the model would be as close as possible to the observed signal in reality. Based on the study, the authors design a digital model of the dynamic system that meets all the required properties.

1. Introduction
The investigation of the oscillating process represented by autoregressive models (AR models) has been actively observed recently by parametric methods of spectral estimation [1-3]. These methods assume an available procedure for generating an intelligence signal, namely the signal must be produced as a resonant response of a linear narrow-band system to a broad-band noise impact. For instance, according to the diagnostic model, it occurs on dynamically stable combustion modes in a liquid-propellant engine.

The simplest diagnostic model of a combustion chamber as a potentially self-oscillating system in the ‘noise’ modes of its operation [4] can represent a nonconservative dynamic second-order system, stimulated by random broad-band (white) noise:

\[
\frac{d^2Y_v}{dt^2} + 2\delta_v M \frac{dY_v}{dt} + \omega_{ov}^2 Y_v = \omega_{ov}^2 \xi(t), \quad \frac{\omega_{ov}}{\delta_v} \gg 1, \delta_v(M) = (\delta_{vd} - \delta_{vg}) > 0,
\]

(1)

where

- \(Y_v\) is the time realization of a narrow-band random process,
- \(\omega_{ov}\) is a radian frequency of natural/free vibrations (without damping),
\[ t \] is the time.

Accordingly, \( \delta_{\text{dd}}, \delta_{\text{dg}} \) are the coefficients of dissipation and generation of acoustic energy are the modes of normal vibrations, which are functions of \( M \) combustion chamber operating mode parameters, \( \xi(t) \) is a time-independent normal random broad-band impact (turbulent combustion noise).

Each of the equations (1) of the linear diagnostic model is similar to the equation of a simple \((R, L, C)\) resonant oscillatory circuit (resonator filter) stimulated by time-independent broad-band noise \([5, 6]\) and known from Radio engineering. The amplitude-frequency response (AFR) module of such a filter is defined by the corresponding expression (2)

\[
|W| = \left( \frac{\frac{1}{4r^2} - \frac{1}{1+r^2}}{1+\frac{1}{4r^2} + 4\delta^2\omega^2} \right)^{1/2} = \left( \frac{(2\pi f_0)^4}{16\pi^4(t_1^2 - \delta_1^2) + 16\pi^4t_0^2\delta^2} \right)^{1/2} .
\]

(2)

2. Materials and methods

The discrete digital simulator of the resonator filter (2) is a second-order autoregressive model (Yule model \([6]\)) declared by the equation

\[
Y_{i,n} = \alpha_1 Y_{i,n-1} + \alpha_2 Y_{i,n-2} + x_n ,
\]

where \( Y_{i,n}, \ldots \) is a sample at the output of i-filter, \( i = 1, 2, 3, \ldots, \ldots \),

\[
\alpha_1 = \frac{4r\cos2\pi f_1 h}{1+r},
\]

(4)

\[
\alpha_2 = -r,
\]

(5)

\[
r = \exp(-2\delta_d * h),
\]

(6)

\( f_1 \) is a resonant frequency of the filter,
\( h \) is a sample spacing, sec.,
\( \delta_d \) is a filtered damping coefficient, sec.\(^{-1}\).

\( Z \) is a rearrangement of the equation (3) can be represented as

\[
Y_n = \alpha_1 Y_n * Z^{-1} + \alpha_2 Y_n * Z^{-2} + x_n .
\]

(7)

Therefore there is an AFR filter

\[
W(Z) = \frac{1}{1-\alpha_1 Z^{-1} - \alpha_2 Z^{-2}} .
\]

(8)

Putting \( Z = e^{j2\pi f_t \Delta} \), where \( \Delta \) is the time step, one gets

\[
W(f) = \frac{1}{1-\alpha_1 e^{-j2\pi f_t \Delta} - \alpha_2 e^{-2j2\pi f_t \Delta}} = \frac{1}{1-\alpha_1 e^{-j\omega \Delta} - \alpha_2 e^{-2j\omega \Delta}} .
\]

(9)

AFR filter module

\[
|W(f)| = \left( \frac{1}{1+\alpha_1^2 + \alpha_2^2 - 2\alpha_1(1-\alpha_2)\cos(2\pi f_t \Delta) - 2\alpha_2\cos(4\pi f_t \Delta)} \right)^{1/2} .
\]

(10)

Comparison of the module results of the amplitude-versus-frequency response characteristics constructed in analogy with the expressions (2) and (10) stands for their identity property.

The digital simulator (3) is used as a waveform generator when testing the designed autoregressive method for spectral estimation of diagnostic value parameters.

It is shown in \([7]\) that equation (3) gives time-independent solutions for \( Y_n \), provided that \( \alpha_1 \) and \( \alpha_2 \) are within a triangular region.

\[ \alpha_1 + \alpha_2 < 1, \alpha_1 - \alpha_2 > -1, -1 < \alpha_2 < 1 \]

In this case, parametric spectral estimation amounts to solving an optimization problem, i.e., searching for the parameters of the autoregressive model of signal generation, whereby it would be as close as possible to the actually observed signal.
In the AR model of the assumed timing series, the discrete values \( x(n) \) are represented by a linear combination of the prior values and the size of error \( \varepsilon(n) \) [8, 9].

\[
x(n) = a_1 x(n-1) + a_2 x(n-2) + \ldots + a_K x(n-K) + a_P x(n-P) + \varepsilon(n) = \sum_{k=1}^{P} a_k x(n-k) + \varepsilon(n),
\]

where \( P \) is a model order.

The value \( \varepsilon(n) \) is the difference between the true/ideal value of \( x_K \) and value of \( \hat{x}_K \), forecasted by the AR model (prediction error).

It is assumed that \( \varepsilon(n) \) is white noise with a Gaussian probability density function of transient values and a uniform power spectrum \( N_\epsilon = const \). \( x(n) \) can be interpreted as the output of a recursive AR filter of \( P \) order, where white noise enters. AFR of such a filter is defined by the expression

\[
H(f) = \left( 1 - \sum_{k=1}^{P} a_k * e^{-iK\omega T} \right)^{-1},
\]

where \( T \) is the sampling period/time.

In accordance with it, the power spectral density of the AR model signal

\[
P(\omega) = \frac{\sigma^2_x(n)}{\left| 1 - \sum_{k=1}^{P} a_k * e^{-iK\omega T} \right|^2},
\]

where \( \sigma^2_x(n) \) is the white noise dispersion.

Thus, the AR method of spectral estimation amounts to estimating the coefficients \( a_k \) AR-модели of the AR model of \( P \) order, estimating the power \( \sigma^2_x \) of white noise, and calculating the power spectral density according to the formula (13).

The introduced parametric method for spectral estimation of resonant frequencies and vibration damping coefficients consists of three stages. At the first stage, the coefficients \( a_1 \ldots a_P \) of the autoregressive model (of \( P \) order) are estimated. The authors chose a modified covariance method out of the known methods for estimating these coefficients. At the second stage, the estimated values of the parameters \( (a_1, a_2, \ldots, a_P) \) are inserted into the abstract/theoretical expression for the AFR squared absolute value of the AR model (12).

At the third stage, in accordance with the diagnostic model, narrow-band rises of the squared absolute value of AFR module of the AR model are approximated by a theoretical expression for the squared absolute value of AFR module of a second-order linear resonant circuit, followed by an estimation of the oscillation damping coefficient \( \hat{\delta} \left( \hat{\delta} = \frac{\delta}{f_p} \right) \)

\[
|AFR|^2 \sim \frac{1}{(\omega - \omega_p)^2 + \delta^2},
\]

where \( \omega_p \) is the resonant circular frequency, and \( \delta \) is the dissipation loss coefficient.

Estimation of the required order of the autoregressive model is performed by comparing the obtained estimates of oscillation damping coefficients with the corresponding estimates obtained by regular methods (spectral, correlation).

In practice, the authors do not know the real correlation function of the signal explored, so they use the estimates of the autocorrelation function (ACF) to minimize the prediction error. There is a number of autoregressive estimation methods known, which differ mainly in the approach to processing edge/tip effects, i.e., in the method of involving into the calculation of those edge/tip samples of the signal for which there is no shifted twain when calculating the ACF.

All methods give virtually the same research results; differences begin to appear in case of short tones when reviewing long sequences of samples.

It is worth mentioning that the expression for the power spectral density of AR model signal is the same as the corresponding expression of the maximum entropy method, which is commonly used, for example, in time series mining/analysis in astrophysics [7]. Thus, if there is a relevant adoption of \( P \) order of the autoregressive process, an estimate of the spectral density that is supposed to be the best from the maximum entropy method can be obtained automatically.
The authors choose the last method among the widely accepted ones for estimating the coefficients of $a_k$ in the AR model (Yule-Walker, Berg, covariance, modified covariance). In this method, linear prediction parameters are estimated by minimizing the arithmetic mean-variance of fore and aft linear prediction errors (figure 1)

$$\sigma_{fb}^2 = \frac{1}{2}(\sigma_f^2 + \sigma_b^2),$$  \hspace{2cm} (15)

Where

$$\sigma_f^2 = \frac{1}{N-P} \sum_{n=P}^{N-1} x(n) + \sum_{K=1}^{P} a_K^2 x(n-K)$$

and

$$\sigma_b^2 = \frac{1}{N-P} \sum_{n=P}^{N-1} x(n) + \sum_{K=1}^{P} a_K^2 x(n-K).$$

Estimation of AR parameters using the maximum-likelihood method results in solving simultaneous equations

$$
\begin{pmatrix}
C_x(1,1) & C_x(1,2) & \cdots & C_x(1,P) \\
C_x(2,1) & C_x(2,2) & \cdots & C_x(2,P) \\
\vdots & \vdots & \ddots & \vdots \\
C_x(P,1) & C_x(P,2) & \cdots & C_x(P,P)
\end{pmatrix}
\begin{pmatrix}
a_{(1)}^1 \\ a_{(1)}^2 \\ \vdots \\ a_{(P)}^P
\end{pmatrix}
= 
\begin{pmatrix}
C_x(1,0) \\ C_x(2,0) \\ \vdots \\ C_x(P,0)
\end{pmatrix},
\hspace{2cm} (16)
$$

where the items of the covariance matrix are written as

$$C_{x(j,k)} = \frac{1}{2(N-P)} \left( \sum_{n=1}^{N-1} x(n-j) * x(n-k) + \sum_{n=1}^{N-1-P} x(n+j) * x(n+k) \right).$$  \hspace{2cm} (17)

White noise variance estimation

$$\sigma_e^2 = \sigma_{fb,\min}^2 = \frac{1}{2(N-P)} \left[ \sum_{n=P}^{N-1} x(n) + \sum_{K=1}^{P} a_K^2 x(n-K) \right] \cdot x(n) + \sum_{n=P}^{N-1-P} \left[ x(n) + \sum_{K=1}^{P} a_K^2 x(n-K) \right] x(n) ,$$

or

$$\sigma_e^2 = \sigma_{fb,\min}^2 = C_{x(0,0)} + \sum_{K=1}^{P} a_K \cdot C_{x(0,k)}.$$  \hspace{2cm} (18)

Several linear/serial algorithms for calculating the coefficients of the AR model are known (Yule-Walker, Levinson-Durbin, Berg, etc.) [8]. Combustion chamber noise modelling is designed in this study through a high-order AR process ($P=50 \ldots 300$). The modified covariance method (MCM) is chosen among the characteristics of algorithms for estimating the AR process parameters. It has a high-resolution capability. In particular, when it analyzes relatively short signal representations, it has no decoupling/division of spectral/wavelength peaks and some other advantages [3, 8, 9].

The authors use a sample of fast computation of the AR model coefficients proposed by Marple [3] as a pre-image of MCM algorithm, It shows a greater improvement of computer-based accuracy (for example, than solving simultaneous equations with decomposition according to Cholesky).

Figure 1. Flow chart of the linear prediction error filter fore and aft.
3. Method verification/checking
Model (1) makes it possible to produce a time implementation of linear narrow-band noise \( Y(t) \) with given statistical characteristics which are a prototype of acoustic noise of the combustion chamber at the frequency of \( v \)-normal mode.

To implement it, one can make a sample estimate of oscillation decrement and compare the obtained value with the given one. Using (1), the authors produce a time representation of linear narrow-band noise \( Y(t) \) with the values of the resonant frequency \( f = 1000 \) Hz and the value of the oscillation decrement \( d = 0.15 \). The processing results by Matlab \([10]\) are presented in table 1.

| \( f_p \) a, Hz | \( d \) b | \( \sigma \) c | \( \hat{\delta} \) d | \( \epsilon_d \) e, % | \( \epsilon_{dT} \) g, % | \( \epsilon_{\sigma} \) h, % |
|---|---|---|---|---|---|---|
| 1000 | 0.05 | 285.1 | 0.06 | 271.8 | 20 | 24.6 | 4.6 |
| 1000 | 0.10 | 219.4 | 0.09 | 222.7 | 10 | 17.4 | 1.5 |
| 1000 | 0.15 | 223.2 | 0.17 | 226.6 | 13 | 14.2 | 1.5 |
| 2000 | 0.15 | 214.4 | 0.16 | 213.8 | 6.6 | 10.0 | 0.2 |
| 4000 | 0.15 | 233.8 | 0.14 | 222.7 | 6.6 | 7.1 | 4.7 |

\( ^a f_p \) is a given decrement value;
\( ^b d \) - is a given decrement value;
\( ^c \sigma \) - is the root-mean-square value of the narrow-band signal, estimated according to the formula: \( \sigma = \left( \frac{1}{N} \sum_{i=1}^{N} y_i \right)^{1/2}, \ N=1000; \)
\( ^d \hat{\delta} \) is decrement estimation;
\( ^e \epsilon_d \) is the estimation of the root-mean-square value of the signal by integrating the spectrum of the AR model;
\( ^g \epsilon_{dT} \) is an abstract/theoretical statistical error;
\( ^h \epsilon_{\sigma} \) is the difference ratio between the estimates \( \sigma \) and \( \hat{\delta} \).

As can be seen, the estimation decrement error is not more than the theoretical statistical error defined by the expression\( \epsilon_{dT} = (\delta \star T_p)^{-1/2} \), where \( \delta \) is a given value of the damping coefficient; \( T_p \) is the implementation length.

The difference between the estimates of the mean-square values of the signal obtained straight according to \( Y(n) \) implementation samples and integrating the spectrum of the AR model is insignificant (no more than 5%).

4. Conclusion
Special test signals are usually used as a data sequence of finite length (representations) with the given characteristics to compare the capabilities of different methods of spectral estimation of dynamic system parameters (in the present case, resonant frequencies and vibration decrements). In particular, a representative of such signals is a model digital signal generated by a second-order linear dynamic system, which is supplied by white noise at the input. This model allows one to produce a time representation of linear narrow-band noise \( Y(n) \) with the given dynamic and statistic characteristics which is considered an equivalent of acoustic noise of the combustion chamber at the frequency of \( v \)-normal mode. When implementing it, one can make a sample estimation of the frequency and vibration decrement by one or another method and compare the obtained values with the given ones.

The values of frequencies and vibration decrements are a priori unknown while tested by the actual signal method (such as signals from pressure pulsation sensors). Therefore, these values, which are conventionally named ‘basic values’, are previously estimated using an alternative method (in this case, a correlation method according to the rate of decay of the autocorrelation function of the signal at the resonant frequency). Evidently, the estimates of oscillation decrements based on the AR model are close to the given values.
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