Global asymptotic stability of predator-prey systems with stage structure and nonlinear birth rate

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Abstract. A predator-prey model with nonlinear birth rate and stage structure on prey species is revisited in this paper. By using the comparison theorem of differential equation and constructing some suitable Lyapunov function, we are able to show that the conditions which ensure the locally asymptotically stable of the equilibria is enough to ensure its globally asymptotically stable. Finally, the obtained results have substantially improved and extended the corresponding results of predecessors.

1. Introduction

Population dynamics is one of the main contents of theoretical ecology research. It takes human beings, insects and animals as the main research objects, and uses differential equations and dynamics methods to establish mathematical models to study the development and change rules of the population. In recent years, researchers have paid great attention to the dynamic behavior of the population model of stage structure, see article [1-8] and literatures. In [2], the persistence and extinction of the stage structure predator-prey population model is studied. A two-species May type cooperation model with stage structure is proposed by Chen et al. [4]. This shows that the phase structure is one of the essential cause of complex ecosystem dynamics behavior and it is one of the key factors to make the population extinct.

Huang [6] thinks that in real life, the number of individual populations is very small, and a more realistic model should take into account interspecific interaction. Therefore, they put forward the following predator-prey model with stage structure and nonlinear birth rate:

\[
\begin{align*}
\dot{x}_1(t) &= \frac{p}{q + x_1} x_2 - d_1 x_1 - \delta x_1 - r x_1 y, \\
\dot{x}_2(t) &= \delta x_1 - d_2 x_2, \\
\dot{y}(t) &= (k x_1 - d_3) y,
\end{align*}
\]

(1.1)

where \(x_1(t), x_2(t)\) denotes the population density of juvenile and adult populations at time \(t\). \(y(t)\) denotes the population density of predator populations at time \(t\). \(p, q, d_1, d_2, d_3, \delta, r, k\) are positive constants. As \(b = d_1 + \delta, \delta = c\), system (1.1) can be rewritten as
\[
\dot{x}_1(t) = \frac{p}{q + x_1} x_2 - bx_1 - rx_1 y, \\
\dot{x}_2(t) = cx_1 - d_3 x_2, \\
\dot{y}(t) = (kx_1 - d_1) y, 
\]

(1.2)

It is known by calculation that the system may have three nonnegative equilibrium points \( O(0, 0, 0) , E_1(\bar{x}_1, \bar{x}_2, 0) \) and \( E_2(\bar{x}_1^*, \bar{x}_2^*, y^*) \), where

\[
\bar{x}_1 = \frac{cp - bd_2 q}{bd_2} , \quad \bar{x}_2 = \frac{c(cp - bd_2 q)}{bd_2^2} 
\]

(1.3)

\[ x_1^* = \frac{d_3}{k} , \quad x_2^* = \frac{cd}{kd_2} , \quad y^* = \frac{ckp - bd_2 kq - bd_2 d_3}{d_2 r(k + d_3)} 
\]

(1.4)

The author of article [6] obtains the following two results which regard the stability of each equilibrium point of the system (1.2):

Theorem 1.1[6]: (a) when \( q < \frac{pc}{bd_2} \), the equilibrium point \( O(0, 0, 0) \) is unstable; when \( q > \frac{pc}{bd_2} \), the equilibrium point \( O(0, 0, 0) \) is locally asymptotically stable; (b) when \( q < \frac{pc}{bd_2} \), (i) if \( q < \frac{pc}{bd_2} - \frac{d_3}{k} \), then boundary equilibrium point \( E_1(\bar{x}_1, \bar{x}_2, 0) \) is unstable; (ii) if \( q > \frac{pc}{bd_2} - \frac{d_3}{k} \), then boundary equilibrium point \( E_1(\bar{x}_1, \bar{x}_2, 0) \) is globally asymptotically stable;

(c) when \( q < \frac{pc}{bd_2} - \frac{d_3}{k} \), \( E_2(\bar{x}_1^*, \bar{x}_2^*, y^*) \) is locally asymptotically stable.

Theorem 1.2[6]: When \( q > \frac{pc}{bd_2} - \frac{d_3}{k} \), \( E_2(\bar{x}_1^*, \bar{x}_2^*, y^*) \) is globally asymptotically stable.

Some interesting questions are raised: (1) the author discusses the global stability of the positive equilibrium, but the author does not discuss the global stability of the boundary equilibrium of the system, which is a very necessary research topic. With the over-exploitation of the natural world by human beings, more and more species have become endangered, the extinction of species has become a very important topic; (2) The author applies the Dulac test to prove theorem 1.2. As we all know, the Dulac test can only be applied to two-dimensional systems, but it cannot be directly applied to three-dimensional systems. Therefore, the author’s proof is not rigorous, and whether the conclusion of theorem 1.2 is valid remains to be discussed.

The purpose of this paper is to give a positive answer to the above two questions. In fact, we will give sufficient conditions and strict proof to guarantee the global asymptotic stability of the three equilibrium points of the system.

2. Main results
The main results of this article are described below.

Theorem 2.1: Considering system (1.2)

(1) when \( q > \frac{pc}{bd_2} \), the equilibrium point \( O(0, 0, 0) \) is globally asymptotically stable;

(2) when \( \frac{pc}{bd_2} - \frac{d_3}{k} < q < \frac{pc}{bd_2} \), boundary equilibrium point \( E_1(\bar{x}_1, \bar{x}_2, 0) \) is globally asymptotically stable;
(3) when $q < \frac{pc}{bd_2} - \frac{d_1}{k}$, $E_2(x_1^*, x_2^*, y^*)$ is globally asymptotically stable.

Before proving the theorem, we need the following lemma, which is a special example of theorem 2 in [5].

Lemma 2.1: Considering system

$$\dot{x}_1(t) = \frac{p}{q + x_1}x_2 - bx_1,$$

$$\dot{x}_2(t) = cx_1 - d_2x_2,$$

where $b > c$, the positive equilibrium point of the system $A(x_1, x_2)$ is globally asymptotically stable in the condition $q < \frac{pc}{bd_2}$, where

$$\bar{x}_1 = \frac{cp - bd_1q}{bd_2}, \quad \bar{x}_2 = \frac{c}{bd_2}\frac{cp - bd_1q}{bd_2}.$$

The following proof of theorem 2.1 is given.

Proof (1) According to the nonnegative property of the first two equations of system (1.2),

$$\dot{x}_1(t) \leq \frac{p}{q + x_1}x_2 - bx_1,$$

$$\dot{x}_2(t) \leq cx_1 - d_2x_2.$$

Considering system

$$\dot{u}_1(t) = \frac{p}{q + u_1}u_2 - bu_1,$$

$$\dot{u}_2(t) = cu_1 - d_2u_2,$$

we will prove that the equilibrium point $O(0, 0)$ of the system (2.4) is globally asymptotically stable when the condition $q > \frac{pc}{bd_2}$ is established. In fact, we need to construct a Lyapunov function

$$V_1(t) = \frac{c}{b}u_1 + u_2.$$

The derivative of $V_1(t)$ is calculated along the positive solution of the system (2.4)

$$D^+V_1(t) = \frac{c}{b}\left(\frac{p}{q + u_1}u_2 - bu_1 + (cu_1 - d_2u_2)\right) \leq \left(\frac{cp}{bq} - d_2\right)u_2.$$

Thus $q > \frac{pc}{bd_2}$, it can be known that $\frac{dV_1}{dt} \leq 0$. It takes the equal sign if and only if $u_1 = 0, u_2 = 0$, which shows that $O(0, 0)$ is globally asymptotically stable. That is to say, for any positive solution $(u_1(t), u_2(t))$ to system (2.4), there exists

$$\lim_{t \to \infty} u_1(t) = 0, \quad \lim_{t \to \infty} u_2(t) = 0.$$

For any positive solution $(x_1(t), x_2(t), y(t))$ of system (1.2), we suppose the initial value is $(x_1(0), x_2(0), y(0)) = (x_{10}, x_{20}, y_0)$ and $(u_1(t), u_2(t))$ is the solution which satisfy the initial value $(u_1(0), u_2(0)) = (x_{10}, x_{20})$ of system (2.4). Then from the comparison principle of differential equation, we know when $t \geq 0$,

$$x_i(t) \leq u_i(t).$$

According to the positivity of the solutions of system (1.2), and (2.7) and (2.8)
\begin{align}
0 \leq \lim \inf_{t \to +\infty} x_i(t) & \leq \lim \sup_{t \to +\infty} x_i(t) \leq \lim_{t \to +\infty} u_i(t) = 0, \quad i = 1, 2. \tag{2.9}
\end{align}

that is to say,
\begin{align}
\lim_{t \to +\infty} x_i(t) = 0, \quad i = 1, 2. \tag{2.10}
\end{align}

By (2.10), it follows that there is enough $T > 0$ and all $t > T$,
\begin{align}
x_i(t) & < \frac{d_i}{2k}. \tag{2.11}
\end{align}

thus when $t > T$, the third equation of system (1.2) is
\begin{align}
\frac{dy}{dt} & \leq -\frac{1}{2}d_3 y. \tag{2.12}
\end{align}

which shows that $t$ tends to infinity,
\begin{align}
y(t) & \leq y(T) \exp\left\{-\frac{1}{2}d_3 (t-T)\right\} \to 0. \tag{2.13}
\end{align}

(2.10) and (2.13) indicate that the boundary equilibrium $O(0, 0, 0)$ of the system (1.2) is globally asymptotically stable. The first part of the theorem is proved.

(2) According to the nonnegative property of the first two equations of system (1.2),
\begin{align}
\dot{x}_1(t) & \leq \frac{p}{q + x_1} x_2 - bx_1, \tag{2.14}
\end{align}
\begin{align}
\dot{x}_2(t) & \leq cx_1 - d_2 x_2.
\end{align}

Now considering the system
\begin{align}
\dot{u}_1(t) & = \frac{p}{q + u_1} u_2 - bu_1, \tag{2.15}
\end{align}
\begin{align}
\dot{u}_2(t) & = cu_1 - d_2 u_2.
\end{align}

according to lemma 2.1, when the condition $q < \frac{pc}{bd_2}$ is established, the equilibrium point $M(\bar{x}_1, \bar{x}_2)$ of system (2.15) is globally asymptotically stable, that is to say, any positive solution $(u_1(t), u_2(t))$ to system (2.15) is
\begin{align}
\lim_{t \to +\infty} u_1(t) = \bar{x}_1, \quad \lim_{t \to +\infty} u_2(t) = \bar{x}_2. \tag{2.16}
\end{align}

For any positive solution $(x_1(t), x_2(t), y(t))$ of system (1.2), its initial value may be set as
\begin{align}
(x_1(0), x_2(0), y(0)) = (x_{10}, x_{20}, y_0)
\end{align}

Let $(u_1(t), u_2(t))$ is the solution satisfying the initial value $(u_1(0), u_2(0)) = (x_{10}, x_{20})$ of the system (2.15), then the comparison principle of the differential equation tells us that when $t \geq 0$,
\begin{align}
x_i(t) & \leq u_i(t). \tag{2.17}
\end{align}

According to the positivity of solutions of system (1.2) and (2.16) and (2.17)
\begin{align}
\lim \sup_{t \to +\infty} x_i(t) & \leq \lim_{t \to +\infty} u_i(t) = \bar{x}_i, \quad i = 1, 2. \tag{2.18}
\end{align}

Theorem condition $\frac{pc}{bd_2} - \frac{d_3}{k} < q$ is equivalent to
\begin{align}
k\left(\frac{pc}{bd_2} - q\right) & < d_3. \tag{2.19}
\end{align}

In view of (2.19), for a positive number $\varepsilon_i > 0$ that is small enough, an inequality
\[
k \left( \frac{pc - bd_2 q}{bd_2} + \varepsilon_1 \right) + \varepsilon_1 < d_3.
\]

is true. So that is just

\[
k (\bar{x}_1 + \varepsilon_1) + \varepsilon_1 < d_3.
\]

Then by (2.18), there are enough \( T_1 > 0 \) for all \( t > T_1 \),

\[
x_1(t) < \bar{x}_1 + \varepsilon_1.
\]

thus when \( t > T_1 \), the third equation of system (1.2) has

\[
dy \leq \left[k(\bar{x}_1 + \varepsilon_1) - d_3\right]y.
\]

Because of these and (2.21), when \( t \) tends to infinity,

\[
y(t) \leq y(T_1) \exp[-\varepsilon_1 (t - T_1)] \to 0.
\]

so we know that the equality

\[
q < \frac{pc}{(b + re_2) d_2}
\]

is true for a positive number \( \varepsilon_2 \). That's small enough from the condition \( q < \frac{pc}{bd_2} \). For \( \varepsilon_2 \), we know from (2.24) that there is a large enough \( T_2 > T_1 \) that when \( t > T_2 \),

\[
y(t) < \varepsilon_2.
\]

By the first two equations of (2.26) and system (1.2), there exists \( t > T_2 \), such that

\[
\dot{x}_1(t) \geq \frac{p}{q + x_1} x_2 - bx_1 - re_2 x_1,
\]

\[
\dot{x}_2(t) \geq cx_1 - d_2 x_2.
\]

Considering the system

\[
\dot{v}_1(t) = \frac{p}{q + v_1} v_2 - (b + re_2) v_1,
\]

\[
\dot{v}_2(t) = cv_1 - d_2 v_2.
\]

according to (2.25) and lemma 2.1, the system (2.28) has a unique globally asymptotically stable positive equilibrium point \( E_2(\bar{v}_1, \bar{v}_2) \), where

\[
\bar{v}_1 = \frac{cp - (b + re_2) d_2 q}{(b + re_2) d_2^2}, \bar{v}_2 = \frac{c [cp - (b + re_2) d_2 q]}{(b + re_2) d_2^2}.
\]

In otherwords, for any positive solution \( (v_1(t), v_2(t)) \) to system (2.28), there has

\[
\lim_{t \to +\infty} v_1(t) = \bar{v}_1, \lim_{t \to +\infty} v_2(t) = \bar{v}_2.
\]

For any positive solution \( (x_1(t), x_2(t), y(t)) \) of system (1.2), its initial value may be set as \( (x_1(0), x_2(0), y(0)) = (x_{10}, x_{20}, y_0) \)

let \( (v_1(t), v_2(t)) \) is the solution satisfying the initial value \( (v_1(0), v_2(0)) = (x_{10}, x_{20}) \) of the system (2.28), then the comparison principle of the differential equation tells us that when \( t \geq T_2 \),

\[
x_i(t) \geq v_i(t)
\]

according to the positivity of the solutions of system (1.2) and (2.30) and (2.31),

\[
\liminf_{t \to +\infty} x_i(t) \geq \lim_{t \to +\infty} v_i(t) = \bar{v}_i, i = 1, 2.
\]

Then from (2.18) and (2.32), it can be shown that
\[ \bar{v}_i = \lim_{t \to \infty} v_i(t) \leq \lim \inf_{t \to \infty} x_i(t) \leq \lim \sup_{t \to \infty} x_i(t) \leq \lim_{t \to \infty} u_i(t) = \bar{x}_i, \quad i = 1, 2. \] (2.33)

it is noted that \( \varepsilon_2 \) is an arbitrarily small positive number and is easily known by (2.29) if \( \varepsilon_2 \to 0 \), \( \bar{v}_i \to \bar{x}_i \). It can be immediately available that

\[ \lim_{t \to \infty} x_i(t) = \bar{x}_i, \quad i = 1, 2. \] (2.34)

in (2.33), (2.24) and (2.34), which show that the boundary equilibrium \( E_i(\bar{x}_i, \bar{x}_2, 0) \) is globally asymptotically stable. The second part of the theorem is proved.

(3) We will prove this by constructing the appropriate Lyapunov function

\[ V_2(t) = K_1 \left( x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) + K_2 \left( x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right) + K_3 \left( y - y^* - y^* \ln \frac{y}{y^*} \right) \] (2.35)

where \( K_i, i = 1, 2, 3 \) is the normal number to be determined. It is noted that \( x_1^*, x_2^*, y^* \) satisfies this equation

\[ \frac{p}{q + x_1^*} x_2^* - bx_1^* - rx_1^* y^* = 0 \]
\[ cx_1^* - d_2 x_2^* = 0, \]
\[ k x_1^* - d_3 = 0. \] (2.36)

the derivative is calculated along the positive solution of the system (1.2) with the help of (2.36),

\[ D^+ V_2(t) = K_1 \frac{x_1 - x_1^*}{x_1} \left( \frac{p}{q + x_1} x_2 - bx_1 - rx_1 y \right) + K_2 \frac{x_2 - x_2^*}{x_2} \left( cx_1 - d_2 x_2 \right) + K_3 \frac{y - y^*}{y} \left( k x_1 - d_3 \right) y \]

\[ = K_1 \frac{x_1 - x_1^*}{x_1} \left( \frac{p}{q + x_1} x_2 - bx_1 - \frac{p}{q + x_1^*} x_2 - bx_1 - rx_1 y \right) + K_2 \frac{x_2 - x_2^*}{x_2} \left( \frac{p}{q + x_1^*} x_1^* x_2 - \frac{p}{q + x_1} x_2 - bx_1 - rx_1 y \right) \]
\[ + K_3 \frac{x_2 - x_2^*}{x_2} \left( \frac{c}{x_2} x_1 x_2^* - x_1 x_2 + x_1 x_2 - x_1^* x_2 \right) + K_3 \left( k x_1 - d_3 \right) y \]

\[ = - K_1 \frac{x_1 - x_1^*}{x_1} \left( \frac{p}{q + x_1^*} x_1^* x_2 - x_1 x_2 + x_1 x_2 - x_1^* x_2 \right) + K_3 \frac{x_2 - x_2^*}{x_2} \left( \frac{c}{x_2} x_1 x_2^* - x_1 x_2 + x_1 x_2 - x_1^* x_2 \right) + K_3 \left( k x_1 - d_3 \right) y \]

let \( K_1 = \frac{(q + x_1^*) x_2}{x_1^* x_2}, K_2 = 1, K_3 = \frac{x_1^*}{x_2^* p k} \), it follows that

\[ D^+ V_2(t) = - \frac{c x_2 x_1^*}{x_2^* x_1^* (q + x_1^*)} \left( x_1 - x_1^* \right)^2 - \frac{c}{x_2^*} \left( \frac{x_2}{x_1^*} \right)^2 \left( x_1 - x_1^* \right)^2 + \frac{c}{x_2^*} \left( \frac{x_2}{x_1^*} \right)^2 \left( x_2 - x_2^* \right)^2 \]

Consequently, for all any \( x, y > 0 \) and \( (x, y) \neq (x^*, y^*) \), such that \( D^+ V_2(t) < 0 \). There exists \( E_2(x_1^*, x_2^*, y^*) \) if and only if \( D^+ V_2(t) = 0 \). Then \( E_2(x_1^*, x_2^*, y^*) \) is globally asymptotically stable. The third part of the theorem is proved.
3. Conclusion
Huang[6] proposes the predator-prey model (1.1) with nonlinear birth rate of baiting, which became the system (1.2) after combining the coefficients. Huang[6] discusses the existence of each equilibrium point of the system, local stability and global stability of the positive equilibrium point. The global stability of positive equilibrium point is proved to be not enough strict, Dulac theorem cannot be directly applied to the three dimensional system. In this article, we rediscuss the global stability states of the three equilibrium points of (1.2). By using the comparison principle of differential equations and constructing appropriate Lyapunov functions, we know that the conditions to ensure the local stability of each equilibrium point of system (1.2) are sufficient to ensure that they are globally stable. In this way, we have greatly improved and promoted the main results of [6]. Our method is general, and it can be applied to other similar phase structure of the ecological system.

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