The alternative model of the spherical oscillator
Levon Mardoyan
Yerevan State University
Alex Manougian str., 1, 0025 Yerevan, Armenia
E-mail: mardoyan@ysu.am

Abstract
The quasiradial wave functions and energy spectra of the alternative model of spherical oscillator on the $D$-dimensional sphere and two-sheeted hyperboloid are found.

Keywords: Spherical oscillator, sphere, two-sheeted hyperboloid.

1 Introduction
The spherical oscillator was suggested by Higgs [1, 2]. The $D$-dimensional spherical oscillator is defined by the potential

$$V_{SD} = \frac{\omega^2}{2} \frac{x_{\mu} x_{\mu}}{x_0^2}, \quad \mu = 1, 2, \ldots, D,$$

(1)

where $x_0$, $x_{\mu}$ are the Euclidean coordinates of the ambient space $\mathbb{R}^{D+1}$: $x_0^2 + x_{\mu} x_{\mu} = r_0^2$ for $D$-dimensional sphere and $x_0^2 - x_{\mu} x_{\mu} = r_0^2$ for $D$-dimensional two-sheeted hyperboloid. (We use a system of units in which the reduced mass $m$ and Planck constant $\hbar$ satisfy $m = \hbar = 1$.) The spherical oscillator (1) on the $D$-dimensional sphere and two-sheeted hyperboloid is considered in [3] in detail.

The oscillator problem on spheres and pseudospheres was discussed from many point of view in [4, 5, 6, 7, 8, 9, 10].

The alternative model of spherical oscillator, which was suggested in our previous papers [11, 12], is defined by the potential

$$V_{SD}^D = 2\omega^2 r_0^2 \frac{x_0^2 - x_0}{r_0 + x_0}$$

(2)

on the $D$-dimensional sphere, and

$$V_{DH}^D = 2\omega^2 r_0^2 \frac{x_0^2 - r_0}{x_0 + r_0}$$

(3)

on the $D$-dimensional two-sheeted hyperboloid.

The two-dimensional case of the oscillator potentials (2) and (3) was considered in [13, 14].

2 Quasiradial function on $D$-sphere
The Schrödinger equation describing the nonrelativistic quantum motion in the $D$-dimensional curved space has the following form:

$$\hat{H}\Psi = \left[ -\frac{1}{2} \Delta_{LB} + V(\vec{x}) \right] \Psi = E\Psi,$$

(4)

where the Laplace-Beltrami operator in arbitrary curvilinear coordinates $\xi_{\mu}$ is

$$\Delta_{LB} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi_{\mu}} \left( g^{\mu\nu} \sqrt{g} \frac{\partial}{\partial \xi_{\nu}} \right), \quad g = \text{det} g_{\mu\nu}, \quad g_{\alpha\beta} g^{\mu\beta} = \delta^{\mu}_{\alpha}.$$

In the hyperspherical coordinates

$$x_0 = r_0 \cos \chi,$$

$$x_1 = r_0 \sin \chi \cos \theta_1,$$

$$x_2 = r_0 \sin \chi \sin \theta_1 \cos \theta_2,$$

$$\vdots$$

$$x_{D-1} = r_0 \sin \chi \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{D-2} \cos \varphi,$$

$$x_D = r_0 \sin \chi \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{D-2} \sin \varphi,$$
where \( \chi, \theta_1, \ldots, \theta_{D-2} \in [0, \pi] \), \( \varphi \in [0, 2\pi] \), the oscillator potential \( V_S^D \) reads
\[
V_S^D = 2\omega^2 r_0^2 \tan^2 \frac{\chi}{2}.
\]

The Schrödinger equation \( \text{(1)} \) for the potential \( \text{(5)} \) may be solved by searching for a wave function in the form
\[
\Psi(\chi, \theta_1, \ldots, \theta_{D-2}, \varphi) = R(\chi) Y_{L_l l_2 \ldots l_{D-2}}(\theta_1, \ldots, \theta_{D-2}, \varphi),
\]
where \( L \) is the total angular momentum, \( l_i \) are the angular hypermomenta and \( Y_{L_l l_2 \ldots l_{D-2}}(\theta_1, \ldots, \theta_{D-2}, \varphi) \) is the solution of the Laplace-Beltrami eigenvalue equation on the \((D-1)\)-dimensional sphere. After the separation of variables in \( \text{(4)} \) we obtain the quasiradial equation
\[
\frac{1}{(\sin \chi)^{D-1}} \frac{\partial}{\partial \chi} \left[ (\sin \chi)^{D-1} \frac{\partial R}{\partial \chi} \right] + \left[ 2r_0^2 E - \frac{L(L+D-2)}{\sin^2 \chi} - 4\omega^2 r_0^4 \tan^2 \frac{\chi}{2} \right] R = 0.
\]

Using the substitution
\[
R(\chi) = (\sin \chi)^{-\frac{D-1}{2}} Z(\chi)
\]
we find the Pöschl-Teller type equation
\[
d^2 Z \quad \frac{d^2 Z}{d\xi^2} \left[ \epsilon - \nu^2 \left( \frac{1}{4} - \frac{1}{\cos^2 \xi} - \frac{(D-2)^2}{2} \right) \right] Z = 0,
\]
where \( \xi = \frac{\chi}{2} \in \left[ 0, \frac{\pi}{2} \right] \), and
\[
\epsilon = 8r_0^2 E + (D-1)^2 + 16\omega^2 r_0^4, \quad \nu = \sqrt{\left( L + \frac{D-2}{2} \right)^2 + 16\omega^2 r_0^4}.
\]

The solution of Eq. \( \text{(6)} \) regular for \( \xi \in \left[ 0, \frac{\pi}{2} \right] \) and expressed in terms of the hypergeometric function is \( \text{(15)} \)
\[
R^D_{n_r \nu \nu}(\chi) = C^D_{n_r \nu \nu} \left( \sin \frac{\chi}{2} \right)^L \left( \cos \frac{\chi}{2} \right)^{\nu - \frac{D}{2} + 1} \times
\]
\[
\times _2 F_1 \left( -n_r, n_r + L + \nu + \frac{D}{2}; L + \frac{D}{2}; \sin^2 \frac{\chi}{2} \right),
\]
and the \( \epsilon \) is quantized as
\[
\epsilon = \left( 2n_r + L + \nu + \frac{D}{2} \right)^2,
\]
where \( n_r = 0, 1, 2, \ldots \) is a "quasiradial" quantum number. The eigenvalues \( E \) are given by
\[
E^D_N = \frac{1}{8r_0^2} \left[ (N + 1)(N + D) + (2\nu - 1) \left( N + \frac{D}{2} \right) + L(L + D - 2) - \frac{D}{2}(D-1) \right],
\]
where \( N = 2n_r + L = 0, 1, 2, \ldots \) is the principal quantum number.

For the quasiradial wave function \( R^D_{n_r \nu \nu}(\chi) \) we choose the normalization condition
\[
\int_0^{\infty} r_0^2 \left| R^D_{n_r \nu \nu}(\chi) \right|^2 (\sin \chi)^{D-1} d\chi = 1
\]
and find:
\[
C^D_{n_r \nu \nu} = \sqrt{\frac{(2n_r + L + \nu + \frac{D}{2}) \Gamma(n_r + L + \nu + \frac{D}{2}) \Gamma(n_r + L + \frac{D}{2})}{2^{D-1} r_0^D (n_r)! \Gamma(n_r + \nu + 1) \left[ \Gamma\left( L + \frac{D}{2} \right) \right]^2}}.
\]
In the limit \( r_0 \to \infty, \chi \to 0 \) and \( \chi r_0 \sim r \) fixed and \( \nu = 4\omega r_0^2 \), we see that

\[
\lim_{r_0 \to \infty} E_N^D = \omega \left( N + \frac{D}{2} \right)
\]

and

\[
\lim_{r_0 \to \infty} R_{N,L\nu}^D(\chi) = \frac{\omega^{\frac{D}{2} + \frac{1}{2}}}{\Gamma \left( L + \frac{D}{2} \right)} \sqrt{\frac{2\Gamma \left( \frac{D}{2} + \frac{1}{2} \right)}{(\frac{D}{2} - L)!}} e^{\frac{-\chi^2}{2}} F \left( -\frac{N - L}{2}; L + \frac{D}{2}, \omega \chi^2 \right),
\]

where \( F (a; c; x) \) is the confluent hypergeometric function. Formula \( (11) \) coincides with the known formula for \( D \)-dimensional flat radial wave functions \( [16] \).

### 3 Oscillator on the \( D \)-dimensional hyperboloid

The pseudospherical coordinates on the \( D \)-dimensional two-sheeted hyperboloid: \( x_0^2 - x_1^2 - x_2^2 - x_D^2 = r_0^2 \), \( x_0 \geq r_0 \), are

\[
\begin{align*}
ex_0 &= r_0 \cosh \tau, \\
x_1 &= r_0 \sinh \tau \cos \theta_1, \\
x_2 &= r_0 \sinh \tau \sin \theta_1 \cos \theta_2, \\
&\vdots \\
x_{D-1} &= r_0 \sinh \tau \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{D-2} \cos \varphi, \\
x_D &= r_0 \sinh \tau \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{D-2} \sin \varphi,
\end{align*}
\]

where \( \tau \in [0, \infty) \). Variables in the Schrödinger equation \( (4) \) may be separated for oscillator potential \( (3) \) which in the pseudospherical coordinates has the form

\[
V_H^D = 2\omega^2 r_0^2 \tanh \frac{\tau}{2},
\]

by the ansatz

\[
\Psi (\tau, \theta_1, \ldots, \theta_{D-2}, \varphi) = R(\tau) Y_{L,\ell_1,\ldots,\ell_{D-2}} (\theta_1, \ldots, \theta_{D-2}, \varphi),
\]

where, as in the previous case \( \ell_1 \), are the angular hypermomenta and \( L \) is the total angular momentum, and the hyperspherical function \( Y_{L,\ell_1,\ldots,\ell_{D-2}} (\theta_1, \ldots, \theta_{D-2}, \varphi) \) is the solution of the Laplace-Beltrami eigenvalue equation on the \((D-1)\)-dimensional sphere. After separation of variables in \( (4) \) we find the quasiradial equation

\[
\frac{1}{(\sinh \tau)^{D-1}} \frac{\partial}{\partial \tau} \left[ (\sinh \tau)^{D-1} \frac{\partial R}{\partial \tau} \right] + \left[ 2r_0^2 E - \frac{L(L + D - 2)}{\sinh^2 \tau} - 4\omega^2 r_0^4 \tanh^2 \frac{\tau}{2} \right] R = 0.
\]

Using now the substitution

\[
R(\tau) = (\sinh \tau)^{-\frac{D-1}{2}} Z(\tau)
\]

we come to the equation

\[
\frac{d^2 Z}{d\rho^2} + \left[ \epsilon - \frac{\nu^2 - \frac{1}{4}}{\cosh^2 \rho} - \frac{(L + \frac{D-2}{2})^2}{\sinh^2 \rho} - \frac{1}{4} \right] Z = 0,
\]

where \( \rho = \frac{\tau}{2} \in [0, \infty) \), and \( \epsilon = 8r_0^2 - (D - 1)^2 - 16\omega^2 r_0^4 \).

Thus, the oscillator problem on the two-sheeted hyperboloid is described by the modified Pöschl-Teller equation and, unlike the oscillator equation on the sphere which has only a discrete spectrum, equation \( (12) \) possesses both bound and unbound states.
The discrete quasiradial wave function regular on the line \( \tau \in [0, \infty) \) and normalized by the condition
\[
\int_0^\infty |R_{n_r,L^\nu}^{D}\left(\tau\right)|^2 \left(\sinh \tau\right)^{D-1} \, d\tau = 1
\]
has the form
\[
R_{n_r,L^\nu}^{D}\left(\tau\right) = \frac{1}{\Gamma\left(\frac{L + \nu + D}{2}\right)} \sqrt{\left(\nu - 2n_r - L - \frac{D}{2}\right) \Gamma\left(n_r + L + \frac{D}{2}\right)} \times (13)
\]
\[
\times \left(\sinh \frac{\tau}{2}\right)^L \left(\cosh \frac{\tau}{2}\right)^{2n_r-\nu-\frac{D}{2}+1} \times _2F_1\left(-n_r, -n_r + \nu; L + \frac{D}{2}; \tanh^2 \frac{\tau}{2}\right),
\]
with the "quasiradial" quantum number \( n_r = 0, 1, 2, \ldots, \left[\frac{1}{2} \left(\nu + L - \frac{D}{2}\right)\right]\). The \( \epsilon \) is quantized by
\[
\epsilon = -\left(2n_r + L - \nu + \frac{D}{2}\right)^2,
\]
and the energy spectrum for the alternative model of quantum spherical oscillator on the \( D \)-dimensional two-sheeted hyperboloid takes the value
\[
E_N^D = \frac{1}{8r_0^2} \left[(2\nu - 1) \left(N + \frac{D}{2}\right) - N(N + D - 1) - L(L + D - 2) + \frac{D}{2}(D - 1)\right]. (14)
\]
Here \( N = 2n_r + L \) is the principal quantum number and the bound state solution is possible only for
\[
0 \leq N \leq \left[\nu - \frac{D}{2}\right].
\]
In the contractio limit \( r_0 \to \infty, \tau \sim r/r_0 \) and \( \nu \sim 4\omega r_0^2 \), we see that the continuous spectrum vanishes while the discrete spectrum is infinite, and it is easy to reproduce the oscillator energy spectrum and wave function [11].

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