Abstract. Let $B$ be an unknown linear evolution process on $\mathbb{C}^d \simeq \ell^2(\mathbb{Z}_d)$ driving an unknown initial state $x$ and producing the states \{${B^\ell x, \ell = 0, 1, \ldots}$\} at different time levels. The problem under consideration in this paper is to find as much information as possible about $B$ and $x$ from the measurements $Y = \{x(i), Bx(i), \ldots, B^\ell i x(i) : i \in \Omega \subset \mathbb{Z}^d\}$. If $B$ is a “low-pass” convolution operator, we show that we can recover both $B$ and $x$, almost surely, as long as we double the amount of temporal samples needed in [4] to recover the signal propagated by a known operator $B$. For a general operator $B$, we can recover parts or even all of its spectrum from $Y$. As a special case of our method, we derive the centuries old Prony’s method [5, 7, 12] which recovers a vector with an $s$-sparse Fourier transform from $2s$ of its consecutive components.

1. Introduction.

Sampling of physical processes is done by sensors or measurement devices that are placed at various locations and can be activated at different times. Dynamical sampling involves studying the time-space patterns formed by the locations of the measurement devices and the times of their activation [4, 3, 2, 11, 13, 8]. One of the goals in dynamical sampling is to identify the patterns that allow one to deduce the desired information about the evolution process. In [4, 3] we considered the problem of spatiotemporal sampling in which an initial state of an evolution process is to be recovered from a set of samples at different time levels. There it is assumed that the evolution process is driven by a known well-behaved filter arising from a well-studied physical process such as diffusion. In [1, 2, 6], the problem of signal recovery was studied for more general processes but the operator governing the evolution was still assumed to be known. In this paper, we shift the emphasis to recovering the spectrum of the evolution operator, which is no longer assumed to be known. In the case of [4], the operator is defined by a well-behaved filter and, thus, is completely determined by its spectrum.

In [4, 3] we have shown that it is possible to trade off spatial samples for time samples at essentially a one-to-one ratio without any loss of information. In this paper we show that if the number of time samples in an invariant process is
doubled, we can still solve the problem even if the filter propagating the signal is not known \textit{a priori}. For more general processes, we may not be able to recover the evolution operator completely, but we are still able to recover part or all of its spectrum.

Dynamical sampling setup is in many ways similar to that of the Slepian-Wolf distributed source coding problem \cite{14} and the distributed sampling problem in \cite{9}. Our setting, however, is fundamentally different from the above in the nature of the processes we study. Distributed sampling problem typically deals with two signals correlated by a transmission channel. We, on the other hand, can observe an evolution process at several instances and over longer periods of time. In the invariant case, we gain access to a number of signals (observations) correlated via the same filter. We then employ a reincarnation of a well-known Prony’s method \cite{5, 7, 12} that uses these observations first for the almost sure recovery of the correlating filter and next for the recovery of the initial state of the process as in \cite{4}. Intuitively, one can think of recovering of the shape of a wave by observing its amplitude at a single location over a long period of time as opposed to acquiring all of the amplitudes at once. For more general processes, we develop more sophisticated Krylov subspace methods to recover (a part of) the spectrum of an evolution operator.

Let us introduce some of the relevant notation and describe our problem in more detail. In this paper we limit our attention to the finite dimensional case and expect to address the infinite dimensional analog elsewhere.

As in \cite{4}, the signal $x$ here is represented by a vector in $\ell^2(\mathbb{Z}_d) \cong \mathbb{C}^d$, where $\mathbb{Z}_d$ is the cyclic group of order $d \in \mathbb{N}$. The evolution operator $B \in B(\ell^2)$ is represented by a $d \times d$ matrix. We call the evolution process \textit{invariant} if $B$ is a self-adjoint circular matrix.

The \textit{sampling operator} or \textit{sampler} $A \in B(\ell^2)$ is another $d \times d$ matrix. An \textit{ideal sampler} $A = S_\Omega$ is the diagonal projection

$$S_\Omega x = \sum_{j \in \Omega} \langle x, e_j \rangle e_j, \quad x \in \mathbb{C}^d,$$

where $\{e_j, j = 1, \ldots, d\}$ is the standard orthonormal basis in $\mathbb{C}^d$. If $m$ divides $d$ and $\Omega = \Omega_m = \{0, m, 2m, \ldots\}$, we shall write $S_m$ instead of $S_{\Omega_m}$ and call it a \textit{uniform} (ideal) sampler.

By the \textit{dynamical samples} of a signal $x \in \mathbb{C}^d$ we mean the collection of vectors

$$y_\ell = AB^\ell x, \quad \ell = 0, 1, \ldots$$

The general goal of the research in this paper is to recover as much information about the evolution operator $B$ as possible from the dynamical samples $y_\ell$. Given sufficiently many dynamical samples, we will provide a method to recover all the eigenvalues of $B$ that can possibly be recovered from these samples. Moreover, in the case of an invariant process and a uniform sampler, we describe an algorithm
to recover the operator $B$ completely from $2m$ dynamical samples for almost every (unknown) signal $x \in \mathbb{C}^d$. The vector $x$ can then be recovered as in [4].

The paper is organized as follows. In Section 2, we introduce the theory that connects the minimal (annihilating) polynomial of a matrix $B$ to several other types of annihilating polynomials such as the $A$-altered minimal polynomial of $B$ and annihilators of certain altered Krylov subspaces. This connection together with an algorithm for finding the annihilator of an $A$-altered Krylov subspace will allow us, in Section 3, to find (a part of) the spectrum of the operator $B$ from measurements obtained by sampling $B^\ell x$, $\ell = 0, \ldots, \ell_i$ at locations $i \in \Omega \subset \mathbb{Z}^d$, where $x$ is an unknown vector in $\mathbb{C}^d$. Section 4 is devoted to the special case when the operator $A$ is a convolution operator, as is common in applications.

2. Krylov subspaces and annihilating polynomials

Around 1930 the engineer and scientist Alexei Nikolaevich Krylov used Krylov subspaces to compute the coefficients of a minimal polynomial, the roots of which are eigenvalues of a matrix [10]. In this section we show that Krylov’s idea can still be used in the case of altered (or preconditioned) Krylov subspaces that we introduce. The definitions and the general theory developed here will be used in subsequent sections. As before, $A$ and $B$ are $d \times d$ complex matrices.

Definition 2.1. A Krylov subspace of order $r$ generated by the matrix $B$ and a vector $x \in \mathbb{C}^d$ is

$$K_r(B,x) = \text{span}\{x, Bx, \ldots, B^{r-1}x\}.$$ 

Definition 2.2. An altered Krylov subspace of order $r$ generated by matrices $A, B$ and a vector $x \in \mathbb{C}^d$ is

$$AK_r(A; B, x) = AK_r(B, x) = \text{span}\{Ax, ABx, \ldots, AB^{r-1}x\}.$$ 

We consider the following minimal annihilating polynomials with regard to Krylov and altered Krylov subspaces.

Definition 2.3.

1. The minimal polynomial of $B$, denoted by $p_B$, is the monic polynomial of smallest degree among all the polynomials $p$ such that $p(B) = 0$. We will denote the degree of $p_B$ by $r_B$.

2. The $A$-altered minimal polynomial of $B$, denoted by $p_A^B$, is the monic polynomial of smallest degree among all the polynomials $p$ such that $Ap(B) = 0$. We let $r_A^B$ denote the degree of $p_A^B$.

3. The $(A, B, x)$-annihilator, denoted by $p_{A,x}^B$, is the monic polynomial of smallest degree among all the polynomials $p$ such that $Ap(B)K_{r_B}^B(B, x) = \{0\}$, $x \in \mathbb{C}^d$. We let $r_{A,x}^B = \text{deg} p_{A,x}^B$.

Observe that the degrees of the minimal polynomials correspond to the orders of the respective maximal Krylov and altered Krylov subspaces. We also note
that the above definition implies that if \( x \in \mathbb{C}^d \) is such that \( \mathcal{A} \mathcal{K}_{r_A}(B, x) = \{0\} \) then the \((A, B, x)\)-annihilator is trivial, i.e. \( p_{A,x}^B \equiv 1 \).

**Proposition 2.1.** The polynomial \( p_A^B \) always divides \( p^B \).

**Proof.** Since \( Ap^B(B) = 0 \), we have \( r^B \geq r_A^B \). Then \( p^B = p_A^B g + h \) where \( h \) is a polynomial of degree less than \( r_A^B \), and \( 0 = Ap^B(B) = Ap_A^B(B)g(B) + Ah(B) \) implies \( Ah(B) = 0 \). But then \( h = 0 \) due to the minimality of \( p_A^B \).

**Corollary 2.2.** The roots of \( p_A^B \) are eigenvalues of \( B \).

The proposition above can be restated in terms of ideals generated by minimal polynomials, so that we have the following inclusion \( \langle p^B \rangle \subseteq \langle p_A^B \rangle \). The next lemma is key to the rest of this section and states that the set of all the polynomials \( p \) such that \( Ap(B)K_{r_A}^B(B, x) = \{0\} \) is an ideal generated by \( p_{A,x}^B \). Note that the polynomials \( p \) such that \( Ap(B)K_L(B, x) = \{0\} \) for \( L < r_A^B \) do not necessarily form an ideal.

**Lemma 2.3.** Assume that a polynomial \( q \) satisfies \( Ap(B)K_{r_A}^B(B, x) = \{0\} \). Then for any \( \ell \in \mathbb{N} \) we have \( Ap(B)K_{\ell}^B(B, x) = \{0\} \). Equivalently, for any polynomial \( s \) we have \( Ap(B)s(B)x = 0 \).

**Proof.** Let \( s \) be a polynomial. If \( \deg s < r_A^B \), then the conclusion is contained in the assumption. Otherwise, we represent \( s \) as \( s = p_A^B g + h \) where \( g \) and \( h \) are polynomials such that \( \deg h < \deg p_A^B = r_A^B \). Then \( Ap(B)s(B)x = Ap(B)(p_A^B g(B) + h(B))x = Ap_A^B(B)g(B)x + Ap(B)h(B)x = 0 \).

As a consequence we get the following inclusion of ideals \( \langle p_A^B \rangle \subseteq \langle p_{A,x}^B \rangle \). **Proposition 2.4.** The \((A, B, x)\)-annihilator \( p_{A,x}^B \) divides \( p_A^B \), and, hence, the roots of \( p_{A,x}^B \) are eigenvalues of \( B \).

**Proof.** We only need to prove something if \( r_{A,x}^B > 0 \). Directly from Definition 2.3 we have \( r_{A,x}^B \leq r_A^B \). Hence, we can write \( p_A^B = p_{A,x}^B g + h \) where \( g \) and \( h \) are polynomials and \( \deg h < r_{A,x}^B \). Using Lemma 2.3, we have

\[
\{0\} = Ap_A^B(B)K_{r_A}^B(B, x) = Ap_{A,x}^B(B)g(B) + A(h(B))K_{r_A}^B(B, x) = Ah(B)K_{r_A}^B(B, x)
\]

and the minimality of \( p_{A,x}^B \) implies \( h = 0 \).

The next proposition is important for the development of the rest of this paper, and it states that the ideals \( \langle p_A^B \rangle \) and \( \langle p_{A,x}^B \rangle \) are equal for almost all \( x \).

**Proposition 2.5.** The set \( \Upsilon = \{ x \in \mathbb{C}^d : p_{A,x}^B \neq p_A^B \} \) has Lebesgue measure 0.

**Proof.** Let \( q \) be a monic polynomial of degree less than \( r_A^B \) that divides \( p_A^B \). There are finitely many such polynomials. The set \( \Upsilon_q = \{ x \in \mathbb{C}^d : p_{A,x}^B = q \} \) is a subspace of \( \mathbb{C}^d \) and \( \Upsilon_q \cap \mathbb{C}^d \) since \( p_A^B \) is minimal. Hence, \( \Upsilon_q \) has measure 0. So does the set \( \Upsilon = \bigcup_q \Upsilon_q \), which is a finite union of null sets.

\( \square \)
The above results show that the spectral identification in dynamical sampling hinges on the ability to compute the \((A, B, x)\)-annihilator. The following result identifies the number of dynamical samples that is sufficient for the computation.

**Proposition 2.6.** Assume \(r_{A,x}^B > 0\). Then the \((A, B, x)\)-annihilator can be recovered from \(2r_{A,x}^B\) dynamical samples \(y_\ell = AB^\ell x, \ell = 0, 1, \ldots, 2r_{A,x}^B - 1\).

**Proof.** For \(r_x \in \mathbb{N}\), consider the system
\[
AB^{rx} B^k x + \sum_{\ell=0}^{r_x-1} \alpha_\ell AB^\ell B^k x = 0, \quad k = 0, \ldots, r_{A,x}^B - 1,
\]
of linear equations in \(\alpha = (\alpha_0, \ldots, \alpha_{r_x-1})\). Clearly, the only \(r_x\) for which the above system has a unique solution equals \(r_{A,x}^B\) and \(\alpha\) is the vector of the coefficients of the \((A, B, x)\)-annihilator. Since \(r_{A,x}^B \leq r_{A,x}^B\), the samples \(y_\ell = AB^\ell x, \ell = 0, 1, \ldots, 2r_{A,x}^B - 1\), provide enough information to explicitly write the above system when \(r_x = r_{A,x}^B\).

**Example 2.1.** In this example we let the evolution operator \(B\) be the circular shift, \((Bx)(n) = x(n+1), x \in \ell^2(\mathbb{Z}_d)\), and the sampler \(A\) be equal to \(S_{\{j\}}\) for some \(j \in \mathbb{Z}_d\). Observe that \(\sigma(B) = \{e^{2\pi i n/d} : n \in \mathbb{Z}_d\}\). We shall assume that the signal \(x\) is such that its Fourier transform \(\hat{x}\) is \(s\)-sparse, i.e. the cardinality of \(\text{supp } \hat{x}\) is \(s < \frac{d}{2}\). Let \(V\) be the subspace \(V = \{y \in \mathbb{C}^d : \text{supp } \hat{y} \subseteq \text{supp } \hat{x}\}\). Since \(V\) is invariant under \(B\), we see that \(r_{S_{\{j\}},x}^B \leq r_{A,x}^B\), and the system of \(2s\) equations as in the proof of Proposition 2.6 gives the coefficients of the annihilator \(p_{S_{\{j\}},x}^B\) as its solution. The roots of the annihilator are the \(d\)-th roots of unity and, moreover, \(e^{2\pi i n/d}\) is a root if and only if \(n \in \text{supp } \hat{x}\). These roots constitute the spectrum \(\sigma(B|_V)\) of the restriction \(B|_V\) of \(B\) to \(V\). Thus, for this case, the spectral recovery in dynamical sampling is equivalent to the well-known Prony’s method [5, 7] for the recovery of a vector with an \(s\)-sparse Fourier transform from \(2s\) of its consecutive samples. In fact, working out the details, one sees that our algorithm is the same as that of Prony (see Example 4.1 below). A closely related point of view on the Prony’s method and its generalizations is presented in [12].

**Remark 2.1.** Let us emphasize that in the above example, the vector \(x\) belongs to the set \(\Upsilon\) defined in Proposition 2.5.

**Remark 2.2.** In numerical linear algebra the modern Krylov subspace methods are typically based on the Lanczos or Arnoldi processes [15] and references therein. In our future research we plan to investigate if any of these methods can be modified to solve the problem of spectral identification in dynamical sampling.

3. Spectral recovery for diagonalizable matrices

To simplify the exposition, in the remainder of this paper we consider only diagonalizable evolution operators. Thus, we write \(B = UDU^{-1}\), where \(D\) is a
diagonal matrix and $U$ is an invertible matrix. The columns of the matrix $U^*$ will be denoted by $u_i = U^*e_i$, $i = 1, \ldots, d$, where, as usually, $\{e_j, j = 1, \ldots, d\}$ is the standard orthonormal basis in $\mathbb{C}^d$. We let $\sigma(B) = \sigma(D) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of the matrix $B$, which we seek to recover from the dynamical samples. We write the spectral decomposition of the matrix $D$ as $D = \sum_{j=1}^n \lambda_j P_j$.

**Definition 3.1.** Let $\Omega$ be a subset of $\{1, \ldots, d\}$. An eigenvalue $\lambda_j \in \sigma(B)$ is called $\Omega$-observable if $S_\Omega U P_j \neq 0$. The set of all $\Omega$-observable eigenvalues will be denoted by $\sigma_\Omega(B)$.

In this section, we show that, for almost every vector $x \in \mathbb{C}^d$, an eigenvalue $\lambda_j$ can be recovered from the dynamical samples if and only if it is $\Omega$-observable.

We begin by considering an important special case which occurs when the sampler $A = S_{(i)}$ is the rank 1 orthogonal projection onto the $i$-th basis element. Recall that we have defined $u_i = U^*e_i$.

**Lemma 3.1.** The $(I, D^*, u_i)$-annihilator $p_{I,u_i}^{D^*}$ coincides with the polynomial $\tilde{p}_{S_{(i)}}^B$ whose coefficients are complex conjugates of the coefficients of the $S_{(i)}$-altered minimal polynomial $p_{S_{(i)}}^B$.

**Proof.** The claim follows from the equalities

$$
\langle S_{(i)} B^k x, e_i \rangle = \langle B^k x, e_i \rangle = \langle UD^k U^{-1} x, e_i \rangle = \langle U^{-1} x, (D^*)^k u_i \rangle,
$$

$k \in \mathbb{N}$, $x \in \mathbb{C}^d$, and the fact that $\langle S_{(i)} B^k x, e_j \rangle = 0$ for all $j \neq i$. \hfill \Box

**Lemma 3.2.** Let $R(p_{I,u_i}^{D^*}) \subseteq \mathbb{C}$ be the set of all roots of the $(I, D^*, u_i)$-annihilator $p_{I,u_i}^{D^*}$. Then $\sigma_{(i)}(B) = \overline{R(p_{I,u_i}^{D^*})}$, where the bar denotes complex conjugation.

**Proof.** To simplify the notation, in this proof we will write $q$ instead of $p_{I,u_i}^{D^*}$. First, observe that for $j = 1, \ldots, n$ we have $\lambda_j \in \sigma_{(i)}(B)$ if and only if $P_j U^* S_{(i)} \neq 0$, and the latter inequality is equivalent to $P_j u_i = P_j U^* e_i = P_j U^* S_{(i)} e_i \neq 0$.

Secondly, observe that for any polynomial $p$ we have

$$
p(D^*) u_i = \sum_{j=1}^n p(\overline{\lambda_j}) P_j u_i,
$$

and the collection of vectors $\{P_j u_i : P_j u_i \neq 0\}$ is linearly independent. In particular, if $p = q$, then $0 = \sum_j q(\overline{\lambda_j}) P_j u_i = 0$ implies $q(\overline{\lambda_j}) = 0$ for all $\lambda_j \in \sigma_{(i)}(B)$.

Hence, $\sigma_{(i)}(B) \subseteq \overline{R(q)}$.

Finally, the polynomial $h$ defined by

$$
h(\lambda) = \prod_{j : P_j u_i \neq 0} (\lambda - \overline{\lambda_j}) = \prod_{j \in \sigma_{(i)}(B)} (\lambda - \overline{\lambda_j})
$$

satisfies $h(D^*) u_i = 0$. Hence, since $\sigma_{(i)}(B) \subseteq \overline{R(q)}$ and $q$ is minimal, we must have $h = q$. \hfill \Box
As a corollary of Lemmas 3.1 and 3.2 we immediately get

**Corollary 3.3.** We have \( r_{S_{\lambda(i)}}^B = r_{P_{\lambda(i)}}^{D^*} = |\sigma_{\lambda(i)}(B)| \), where \( |\sigma_{\lambda(i)}(B)| \) is the cardinality of the set.

The following result is an immediate extension of the above observations to all ideal samplers.

**Proposition 3.4.** Let \( p_{\Omega}^{D^*} \) denote the least common multiple of the \( (I, D^*, u_i) \)-annihilators \( p_{I,u_i}^{D^*} \), \( i \in \Omega \). Then the coefficients of \( p_{\Omega}^{D^*} \) coincide with the complex conjugates of the coefficients of the \( S_\Omega \)-altered minimal polynomial \( p_{\Omega}^B \).

**Proof.** This follows directly from the fact that \( S_\Omega = \sum_{i \in \Omega} S_{\lambda(i)} \) and the minimality of \( p_{\Omega}^B \). \( \square \)

**Corollary 3.5.** We have \( r_{S_\Omega}^B = |\sigma_\Omega(B)| \).

Combining Proposition 2.5 and corollary 3.3 with Proposition 3.3 we get

**Theorem 3.6.** For almost every \( x \in \mathbb{C}^d \) the set \( \sigma_{\lambda(i)}(B) \) can be recovered from the measurements \( \{(B^kx)(i) : k = 0, \ldots, 2r_i - 1\} \), where \( r_i = |\sigma_{\lambda(i)}(B)| \). Consequently, for any \( \Omega \subseteq \mathbb{Z}_d \), the set \( \sigma_\Omega(B) \) can be recovered from the dynamical samples \( \{(B^kx)(i) : i \in \Omega, k = 0, \ldots, 2r_i - 1\} \) for almost every \( x \in \mathbb{C}^d \).

**Remark 3.1.** From the proof of Lemma 3.2 we know that \( \lambda_j \in \sigma(B) \) is \( \Omega \)-observable if and only if there exists \( i \in \Omega \) such that \( P_j u_i \neq 0 \), where \( u_i = U^*e_i \). If \( \Omega \) is a set that allows the reconstruction of any vector \( x \in \mathbb{C}^d \) from samples \( \{(B^kx)(i) : i \in \Omega, k = 0, \ldots, 2r_i - 1\} \), then, from the results in [2], \( \{P_j u_i : i \in \Omega\} \) is a frame for the range of \( P_j \). Thus, it follows that if \( \Omega \) is a set that allows the reconstruction of any \( x \in \mathbb{C}^d \) from samples \( \{(B^kx)(i) : i \in \Omega, k = 0, \ldots, 2r_i - 1\} \), then \( \sigma_\Omega(B) = \sigma(B) \).

We conclude the section with an estimate on the number of time samples needed for the recovery of the spectrum under an additional condition.

**Theorem 3.7.** Let \( \{u_i : i \in \Omega\} \) be the column vectors of the matrix \( U \) corresponding to \( \Omega \) and let \( L \) be a fixed integer. Assume that \( \{(D^*)^L u_i : i \in \Omega\} \subset \text{span}\{u_i, D^* u_i, \ldots, (D^*)^{L-1} u_i : i \in \Omega\} \). Then the set \( \sigma_\Omega(B) \) can be recovered from \( \{(B^k)x(i) : i \in \Omega, k = 0, \ldots, (|\Omega| + 1)L - 1\} \) for almost every \( x \in \mathbb{C}^d \).

**Proof.** The assumption of the theorem implies that there exist numbers \( \alpha_{\ell j}(i) \), \( i, j \in \Omega, \ell = 0, \ldots, L - 1 \), such that

\[
(D^*)^L u_i = \sum_{j \in \Omega} \sum_{\ell=0}^{L-1} \alpha_{\ell j}(i)(D^*)^\ell u_j, \quad i \in \Omega.
\]
From (3.1) it follows that for almost every $x \in \mathbb{C}^d$ we have

$$(3.2) \quad (B^{L+k}x)(i) = \sum_{j \in \Omega} \sum_{\ell = 0}^{L-1} \alpha_{\ell j}(i)(B^{\ell+k}x)(j), \quad i \in \Omega, \ k \in \mathbb{Z}.$$ 

Observe that for $k = 0, \ldots, |\Omega|L - 1$ and $i \in \Omega$ the equations in (3.2) form a square system of linear equations which has a solution by the assumption of the theorem. Computing the coefficients $\alpha_{\ell j}(i), \ i, j \in \Omega, \ \ell = 0, \ldots, L - 1,$ from these square systems and plugging them back into (3.2) we obtain $(B^kx)(i)$ for any $k \in \mathbb{Z}$ and $i \in \Omega$. It remains to invoke Theorem 3.6. □

**Remark 3.2.** Note that in many cases we have that $|\Omega|L$ is approximately $d$. Thus, a typical number of measurements needed to obtain the spectrum via Theorem 3.7 is approximately $(|\Omega| + 1)d$.

**Remark 3.3.** Theorems 3.6 and 3.7 are also valid for general matrices $B$ with some minor modifications. In particular, Theorem 3.7 remains valid if the Jordan decomposition $B = UJU^{-1}$ is used, and the matrix $D$ is replaced by the matrix $J$ in its statement.

4. RECOVERY OF THE EVOLUTION OPERATOR IN THE CASE OF A REGULAR INVARIANT DYNAMICAL SAMPLING PROBLEM.

An important question for spectral recovery is determining the actual number of dynamical samples needed to find the spectrum of $B$. Proposition 2.6 provides essentially the best answer one could have in the general case. From the algorithmic point of view, however, this answer is not very satisfactory. Indeed, to determine the number $r_B^A$ used in Proposition 2.6, one needs to have some prior knowledge about $B$. In this section we address the case of an invariant evolution process with a uniform ideal sampler. This turns out to be enough prior information to get a good upper bound on the sufficient number of dynamical samples.

Recall from the introduction, that in the case under consideration the sampler $A$ is given by

$$(4.1) \quad A = S_m : \ell^2(\mathbb{Z}_d) \to \ell^2(\mathbb{Z}_d), \quad (S_m x)(n) = \delta_{(n \mod m),0} x(n),$$

where $m$ is an odd integer that divides an odd integer $d$ (oddness is assumed for the sake of computational simplicity). The evolution operator $B$ in this case is a convolution operator: $Bx = a * x, \ a \in \ell^2(\mathbb{Z}_d) \simeq \mathbb{C}^d$. Since the matrix of $B$ is circular, it is diagonalized by the $d$-dimensional discrete Fourier transform (DFT) $F_d$ defined by

$$\hat{x}(k) = (F_d x)(k) = \sum_{\ell = 0}^{d-1} x(\ell) e^{-\frac{2\pi i \ell k}{d}}, \quad x \in \mathbb{C}^d, \ k = 1, \ldots, d.$$
Thus, we have
\begin{equation}
B = F_d^{-1}DF_d = \frac{1}{d}F_d^*DF_d,
\end{equation}
where $D$ is the diagonal matrix defined by $\hat{a}$ – the DFT of the filter $a$.

In this section we show that for almost every $x \in \mathbb{C}^d$ the knowledge of $2m$ dynamical samples
\begin{equation}
y_\ell = S_mB^\ell x, \; \ell = 0, \ldots, 2m - 1,
\end{equation}
is sufficient to reconstruct the spectrum of $B$, that is the set of entries of $\hat{a}$.

Observe that knowing the order of entries in $\hat{a}$ would then allow one to recover the operator $B$ completely via (4.2). In particular, this can be done if one assumes that $\hat{a}$ is real, symmetric, and decreasing on $\{0, 1, \ldots, \frac{d-1}{2}\}$, which is a natural assumption for diffusion-type processes. Once the filter $a$ is known, one can use the results in [4] to recover $x$.

**Theorem 4.1.** Assume that $B$ is an unknown convolution operator on $\ell^2(\mathbb{Z}_d)$. Then, for almost all $x \in \mathbb{C}^d$, the spectrum of $B$ can be recovered from the dynamical samples $y_\ell$, $\ell = 0, \ldots, 2m - 1$, defined in (4.3).

**Proof.** We use the DFT and the Poisson summation formula to rewrite (4.3) in the following way:
\[
\hat{y}_\ell = F_dS_mF_d^{-1}F_d^*B^\ell F_d^{-1}\hat{x} = \sum_{j=1}^{J} E_jD^\ell \hat{x},
\]
where each $E_j$, $j = 1, \ldots, J$, is a rank-1 projection given by
\begin{equation}
(E_jz)(k) = \begin{cases}
\frac{1}{m} \sum_{i=0}^{m-1} z(k + Ji), & k = j \text{ mod } J; \\
0, & \text{otherwise};
\end{cases}
z \in \mathbb{C}^d, \; k = 1, \ldots, d.
\end{equation}
Observe that $E_jE_k = \delta_{jk}E_k$, where $\delta_{jk}$ is the usual Kronecker delta. For $j = 1, \ldots, J$, let $\Omega(j) = \{k \in \mathbb{Z}_d : k = j \text{ mod } J\}$. Since the $k$-th row of $E_j$ is zero for any $k \notin \Omega(j)$ and $D$ is diagonal, the polynomials
\[
p_j(\lambda) = \prod_{k \in \Omega(j)} (\lambda - \hat{a}(k))
\]
satisfy $E_j p_j(D) = 0$, $j = 1, \ldots, J$. Moreover, the set $R(j)$ of all roots of the minimal polynomial $p_j^D$ coincides with the set of all roots of the polynomial $p_j$ (cf. the proof of Lemma 3.2). It immediately follows that
\begin{equation}
\sigma(B) = \bigcup_{j=1}^{J} R(j)
\end{equation}
and $r_{E_j}^D \leq m$ for each $j = 1, \ldots, J$. Hence, we can apply Proposition 2.6 to recover the $(E_j, D, \hat{x})$-annihilators $p_{E_j,\hat{x}}^D$, $j = 1, \ldots, J$. Due to Proposition 2.5,
these annihilators will coincide with the minimal polynomials $p_{E_j}^D$ for almost every $x \in \mathbb{C}^d$ and (4.5) means that the proof is complete. \hfill \Box

4.1. **Algorithm.** The proof of Proposition 2.6 essentially provides an algorithm for the recovery of the spectrum $\sigma(B)$. For the case in Theorem 4.1, the algorithm in Proposition 2.6 is unnecessarily complicated and can be simplified as follows.

Our goal is to find for each $j = 1, \ldots, J$ the set of roots $R(j)$ of the $(E_j, D, \hat{x})$-annihilator $p_{E_j,\hat{x}}^D$. According to the proof of Proposition 2.6, the coefficients $\alpha(j) = (\alpha_0(j), \ldots, \alpha_{r_j-1}(j))$ of the polynomial $p_{E_j,\hat{x}}^D$ satisfy the system of linear equations

\[
E_j D^{r_j} D^k \hat{x} + \sum_{\ell=0}^{r_j-1} \alpha_\ell(j) E_j D^\ell D^k \hat{x} = 0, \quad k = 0, \ldots, r_j^D - 1,
\]

where $r_j$ is the degree of $p_{E_j,\hat{x}}^D$, i.e. the minimal integer for which the system has a (unique) solution.

Since $r_j^D \leq m$, and in view of Lemma 2.3, we can let $k$ vary from 0 to $m - 1$ without altering the solution of the system (4.6). Since $E_j$ has rank one, for each fixed $k$, the system of $d = mJ$ equations can be replaced by a single equation

\[
\sum_{i=0}^{m-1} \hat{a}^{k+r_j}(j + iJ) \hat{x}(j + iJ) + \sum_{\ell=0}^{r_j-1} \alpha_\ell(j) \sum_{i=0}^{m-1} \hat{a}^{k+\ell}(j + iJ) \hat{x}(j + iJ) = 0.
\]

Recalling that $\hat{y}_\ell(j) = \frac{1}{m} \sum_{i=0}^{m-1} \hat{a}^\ell(j + iJ) \hat{x}(j + iJ), \ j \in \mathbb{Z}^d, \ \ell = 0, 1, \ldots$, we deduce that the system (4.6) is equivalent to

\[
\hat{y}_{k+r_j}(j) + \sum_{\ell=0}^{r_j-1} \alpha_\ell(j) \hat{y}_{k+\ell}(j) = 0, \quad k = 0, \ldots, m - 1.
\]

For each $j$, this last system of equations can be set from the dynamical samples that are available to us by assumption. Thus, we have the following crude algorithm for the recovery of $\sigma(B)$.

**Step I.** For each $j = 1, \ldots, J$

1. find the minimal integer $r_j$ for which the system (4.7) has a solution $\alpha(j)$ and find that solution;
2. let $p_j(\lambda) = \lambda^{r_j} + \sum_{\ell=0}^{r_j-1} \alpha_\ell(j) \lambda^\ell$ and find the set $R(j)$ of all roots of $p_j$.

**Step II.** Recover the spectrum $\sigma(B)$ from (4.5).

**Example 4.1.** Prony’s method for finding vectors with sparse Fourier transform includes solving a system of linear equations that is a special case of the one in our algorithm. As in Example 2.1, we let $(Bx)(n) = x(n+1)$ and $x \in l^2(\mathbb{Z}^d)$ be $s$ sparse. We also let the subsampling factor $m$ be equal to $d$. Then the system of
equations (4.7) with \( r_j = s \) will coincide with the system of equations for finding the supp \( \hat{x} \) in Prony’s method.

**Remark 4.1.** In [4] we noted that the filter \( a \) is typically such that \( \hat{a} \) is real, symmetric, and decreasing for \( \ell = 0, 1, \ldots, d-1 \). In this case, it easily follows that \( r_j = r^D_{E_j} = m \) for \( j = 1, \ldots, J-1 \), and \( r_J = r^D_{E_J} = \frac{m+1}{2} \) for almost every \( x \in \mathbb{C}^d \). Moreover, the monotonicity condition allows us to properly order the roots in \( \sigma(B) \) and completely recover \( \hat{a} \) and, hence, the evolution matrix \( B \). We also observe that in this case, the imaginary part of the system (4.7) can be ignored and the real part of the system is given by a self-adjoint Hankel matrix. This allows one to use special methods to solve it.

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**References**

[1] R. Aceska and S. Tang, *Dynamical sampling in hybrid shift invariant spaces*, Contemp. Math., Amer. Math. Soc., Providence, RI, 2014. To appear.

[2] A. Aldroubi, C. Cabrelli, U. Molter, and S. Tang, *Dynamical sampling*. ArXiv:1409.8333.

[3] A. Aldroubi, J. Davis, and I. Krishtal, *Exact reconstruction of spatially undersampled signals in evolutionary systems*, J. Fourier Anal. Appl. DOI: 10.1007/s00041-014-9359-9. ArXiv:1312.3203.

[4] ———, *Dynamical sampling: time-space trade-off*, Appl. Comput. Harmon. Anal., 34 (2013), pp. 495–503.

[5] T. Blu, P.-L. Dragotti, M. Vetterli, P. Marziliano, and L. Coulot, *Sparse sampling of signal innovations*, Signal Processing Magazine, IEEE, 25 (2008), pp. 31–40.

[6] J. Davis, *Dynamical sampling with a forcing term*, Contemp. Math., Amer. Math. Soc., Providence, RI, 2014. To appear.
[7] G. C. F. M. R. de Prony, *Essai expérimental et analytique: sur les lois de la dilatabilité de fluides élastiques et sur les celles de la force expansive de la vapeur de l’eau et de la vapeur de l’alkool, à différentes températures.*, J. de l’École Polytechnique Floréal et Plairial, An. III (1795), pp. 24–76.

[8] K. Gröchenig, J. L. Romero, J. Unnikrishnan, and M. Vetterli, *On minimal trajectories for mobile sampling of bandlimited fields*, arXiv:1312.7794v2, (2014).

[9] A. Hormati, O. Roy, Y. Lu, and M. Vetterli, *Distributed sampling of signals linked by sparse filtering: Theory and applications*, Signal Processing, IEEE Transactions on, 58 (2010), pp. 1095–1109.

[10] A. Krylov, *On the numerical solution of the equation by which in technical questions frequencies of small oscillations of material systems are determined*, Izv. Akad. Nauk SSSR. Otd. Mat. Estest., Ser. VII (1931), pp. 491–539.

[11] Y. Lu and M. Vetterli, *Spatial super-resolution of a diffusion field by temporal oversampling in sensor networks*, in Acoustics, Speech and Signal Processing, 2009. ICASSP 2009. IEEE International Conference on, april 2009, pp. 2249–2252.

[12] T. Peter and G. Plonka, *A generalized Prony method for reconstruction of sparse sums of eigenfunctions of linear operators*, Inverse Problems, 29 (2013). 025001.

[13] G. Reise, G. Matz, and K. Grochenig, *Distributed field reconstruction in wireless sensor networks based on hybrid shift-invariant spaces*, Signal Processing, IEEE Transactions on, 60 (2012), pp. 5426–5439.

[14] D. Slepian and J. Wolf, *Noiseless coding of correlated information sources*, Information Theory, IEEE Transactions on, 19 (1973), pp. 471 – 480.

[15] D. S. Watkins, *The matrix eigenvalue problem. GR and Krylov subspace methods*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

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