Quark-Antiquark System in Ultra-Intense Magnetic Field

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Abstract

We study the relativistic quark-antiquark system embedded in magnetic field (MF). The Hamiltonian containing confinement, color Coulomb and spin-spin interaction is derived. We analytically follow the evolution of the lowest neutral meson state as a function of MF strength. Calculating the color Coulomb energy \( \langle V_{\text{Coul}} \rangle \) we have observed the unbounded negative (at least in the limit of large \( N_c \)) contribution at large MF which makes the mass negative for \( eB > eB_{\text{crit}}^{QCD} \). We display the \( \pi^0 \) and \( \rho^0 \) masses as functions of MF in comparison with recent lattice data.

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1 Introduction

During the last years we have witnessed an impressive progress of the fundamental physics in ultra-intense magnetic field (MF) reaching the strength up
to $eB \sim 10^{18} G \sim m_{e}^{2}$ \[1\]. Until recently magnetars \[2\] were the only physical objects, where such, or somewhat weaker MF could be realized. Now MF of the above strength and even stronger is within reach in peripheral heavy ion collisions at RHIC and LHC \[3\]. High intensity lasers is another perspective tool to achieve MF beyond the Schwinger limit \[4\]. On the theoretical side a striking progress has been achieved along several lines. It is beyond our scope to discuss these works or even present a list of corresponding references. We mention only two lines of research which have a certain overlap with our work. The first one \[5, 6\] is the behavior of the hydrogen atom and positronium in very strong MF. The second one \[7\] is the conjecture of the vacuum reconstruction due to vector meson condensation in large MF. The relation between the above studies and our work will be clarified in what follows.

Our goal is to study from the first principles the spectrum of a meson composed of quark-antiquark embedded in MF. Use will be made of Fock-Feynman-Schwinger representation (see \[8\] for review and references) of the quark Green’s function with strong (QCD) interaction and MF included. An alternative approach could have been Bethe-Salpeter type formalism. However, for the confinement originating from the area law of the Wilson loop, the use of the gluon propagator is inadequate. Numerous attempts in this direction failed because of gauge dependence and the vector character of the gluon propagator, while confinement is scalar and gauge invariant. Therefore it is sensible to use the path integral technique for QCD+QED Green’s functions. This method in combination with the einbein technique (the method of effective masses) \[9\] enables one to construct explicit expressions for meson Hamiltonians without MF \[10\]. In this way spectra of light-light, light-heavy and heavy-heavy mesons were computed with a good accuracy, using the string tension $\sigma$, strong coupling constant $\alpha_{S}$ and quark current masses as an input \[11, 12\].

In what follows we expand this technique to incorporate the effects of MF on mesons. The latter contains: 1) direct influence of MF on quark and antiquark, and 2) the influence on gluonic fields, e.g., on $\alpha_{S}$, gluon propagator and on the gluon field correlators determining the string tension $\sigma$. However, since MF acts on charged objects, its influence on the gluonic degrees of freedom enters only via $(N_{c})^{-k}$, $k = 1, 2, ...$ In what follows the corrections of the second type will be neglected. 3) As will be discussed elsewhere, MF also changes quark condensate $\langle \bar{q}q \rangle$ and quark decay constants $f_{\pi}$ etc., and in this way strongly influences chiral dynamics.
The decisive step in our relativistic formalism is the implementation of the pseudomomentum notion and c.m. factorization in MF, suggested in the nonrelativistic case in [13] for neutral two particle systems.

The plan of the paper is the following. Section 2 contains a brief pedagogical reminder of how the two-body problem in MF is solved in quantum mechanics. The central point here is the integral of motion ("pseudomomentum") which allows the separation of the center of mass. Here we also show how to diagonalize the spin-dependent interaction. In section 3 we formulate the path integral for quark-antiquark system with QCD+QED interaction. Then from Green’s function the relativistic Hamiltonian is obtained. Section 4 is devoted to the treatment of confining and color Coulomb terms, we demonstrate the unboundedness of \( q\bar{q} \) spectrum due to the latter. This phenomenon can be called the “magnetic QCD collapse”, which occurs in the large \( N_c \) limit: \( n_f/N_c \to 0 \). Here we also present the derivation of the eigenvalue equations for the relativistic Coulomb problem. In section 5 we discuss the spectrum of the system focusing on the regime of ultra-strong MF. Section 6 contains the discussion of the results, comparison with lattice calculations, drawing further perspectives and intersections of our results with those of other authors [5, 6, 7].

2 Pseudomomentum and Wavefunction Factorization

The total momentum of \( N \) mutually interacting particles with translation invariant interaction is a constant of motion and the center of mass motions can be separated in Schrödinger equation. It was shown [13] that a system embedded in a constant MF also possesses a constant of motion – "pseudomomentum". As a result for the case of zero total electric charge \( Q = 0 \) the c.m. motion can be removed from the total Hamiltonian [13]. The simplest example is a two-particle system with equal masses \( m_1 = m_2 = m \) and electric charges \( e_1 = -e_2 = e \). We define

\[
R = \frac{r_1 + r_2}{2}, \quad \eta = r_1 - r_2, \quad P = p_1 + p_2.
\]

The case \( Q \neq 0 \) is more complicated and will be considered elsewhere.
Straightforward calculation in the London gauge $A = \frac{1}{2}(B \times r)$ yields

$$\hat{H} = \frac{1}{4m} \left( P - \frac{e}{2} (B \times \eta) \right)^2 + \frac{1}{m} \left( -i \frac{\partial}{\partial \eta} - \frac{e}{2} (B \times \mathbf{R}) \right)^2 + V(\eta).$$  \hspace{1cm} (2)$$

One can verify that the following “pseudomomentum” operator $\mathbf{F}$ commutes with the Hamiltonian $\hat{H}$

$$\hat{F} = \mathbf{F} = \mathbf{P} + \frac{e}{2} (B \times \eta).$$  \hspace{1cm} (3)$$

This immediately leads to the following factorization of the wave function (WF)

$$\Psi(\mathbf{R}, \eta) = \varphi(\eta) \exp \left\{ i \mathbf{PR} - \frac{e}{2} (B \times \eta) \mathbf{R} \right\}.$$  \hspace{1cm} (4)$$

For the oscillator-type potential $V(\eta)$ the problem reduces to a set of three oscillators, two of them are in a plane perpendicular to the magnetic field and their frequencies are degenerate, while the third one is connected solely with $V(\eta)$.

Next we briefly elucidate the spin interaction in presence of MF. The corresponding part of the Hamiltonian may be written as

$$\hat{H}_s = 4a_h f (\sigma_1 \sigma_2) - \mu B (\sigma_1 - \sigma_2),$$  \hspace{1cm} (5)$$

where $e_1 = -e_2 = e > 0$ and $\mu > 0$. Diagonalization of $\hat{H}_s$ yields the following four eigenvalues e.g. for $u\bar{u}$ system, comprising both $\rho$ and $\pi$ levels.

$$E_{1,2}^{(s)} = a_h f, \quad E_{3,4}^{(s)} = \pm a_h f \left( 2 \sqrt{1 + \left( \frac{\mu B}{a_h f} \right)^2} \mp 1 \right),$$  \hspace{1cm} (6)$$

where we assume that $B$ is aligned along the positive $z$-axis and $B = |\mathbf{B}|$. In a strong MF when $\mu B > a_h f$ spin-spin interaction becomes unimportant and $E_{3,4}^{(s)} \simeq \pm 2\mu B$. For the lowest level $E_4^{(s)}$ this corresponds to a configuration $|+ - \rangle$ when the spin of negatively charged particle is aligned antiparallel to $\mathbf{B}$, and the spin of the positively charged one – parallel to $\mathbf{B}$. This means that the spin (and isospin) are no more good quantum numbers and eigenvalues (6) correspond to the mixture of spin 1 and spin 0 states. As a result the $q\bar{q}$ state will split into 4 states (two of them coinciding $E_1^{(s)} = E_2^{(s)}$). Till now we treated a nonrelativistic system, to incorporate relativistic effects we shall exploit the path integral form of relativistic Green’s functions [8].
3 Relativistic $q\bar{q}$ Green’s function and effective Hamiltonian

The starting point is the Fock-Feynmann-Schwinger (world-line) representation of the quark Green’s function \cite{8}. The role of the “time” parameter along the path $z^{(i)}_\mu(s_i)$ of the $i$-th quark is played by the Fock-Schwinger proper time $s_i$, $i = 1, 2$. Consider a quark with a charge $e_i$ in a gluonic field $A_\mu$ and the electromagnetic vector potential $A_\mu^{(e)}$, corresponding to a constant magnetic field $B_i$. Then the quark propagator in the Euclidean space-time is

$$S_i(x, y) = \left( m_i + \hat{\partial} - ig\hat{A} - ie_i\hat{A}^{(e)} \right)^{-1}_{xy} \equiv \left( m_i + \hat{D}^{(i)} \right)^{-1}_{xy}. \quad (7)$$

The path-integral representation for $S_i$ is \cite{8}

$$S_i(x, y) = \left( m_i - \hat{D}^{(i)} \right) \int_0^\infty ds_i (Dz)_{xy} e^{-K_i \Phi^{(i)}_\sigma(x, y)} \equiv \left( m_i - \hat{D}^{(i)} \right) G_i(x, y), \quad (8)$$

where

$$K_i = m^2_i s_i + \frac{1}{4} \int_0^{s_i} d\tau_i \left( \frac{dz^{(i)}_\mu}{d\tau_i} \right)^2, \quad (9)$$

$$\Phi^{(i)}_\sigma(x, y) = P_A P_F \exp \left( ig \int_x^y A_\mu dz^{(i)}_\mu + ie_i \int_x^y A^{(e)}_\mu dz^{(i)}_\mu \right) \times \exp \left( \int_0^{s_i} d\tau_i \sigma_{\mu\nu}(g F_{\mu\nu} + e_i B_{\mu\nu}) \right). \quad (10)$$

Here $F_{\mu\nu}$ and $B_{\mu\nu}$ are correspondingly gluon and MF tensors, $P_A, P_F$ are ordering operators, $\sigma_{\mu\nu} = \frac{1}{4} (\gamma\mu\gamma\nu - \gamma\nu\gamma\mu)$. Eqs. (7-10) hold for the quark, $i = 1$, while for the antiquark one should reverse the signs of $e_i$ and $g$. In explicit form one writes

$$\sigma_{\mu\nu} F_{\mu\nu} = \begin{pmatrix} \sigma H & \sigma E \\ \sigma E & \sigma H \end{pmatrix}, \quad \sigma_{\mu\nu} B_{\mu\nu} = \begin{pmatrix} \sigma B & 0 \\ 0 & \sigma B \end{pmatrix}. \quad (11)$$

Next we consider $q_1\bar{q}_2$ system born at the point $x$ with the current $j_{\Gamma_1}(x) = \bar{q}_1(x)\Gamma_1 q_2(x)$ and annihilated at the point $y$ with the current $j_{\Gamma_2}(y)$. Here $x$ and $y$ denote the sets of initial and final coordinates of quark and antiquark. Using the nonabelian Stokes theorem and cluster expansion for the gluon field(see \cite{11} for reviews) and leaving the MF term intact, we can write

$$G_{q_1\bar{q}_2}(x, y) = \int_0^\infty ds_1 \int_0^\infty ds_2 (Dz^{(1)})_{xy} (Dz^{(2)})_{xy} (\hat{T}W_\sigma(A))_A \times$$
\[ \exp(i e_1 \int_y^x A_\mu^{(1)} dz_\mu^{(1)} - i e_2 \int_y^x A_\mu^{(2)} dz_\mu^{(2)}) + e_1 \int_0^{s_1} d\tau_1(\sigma \mathbf{B}) - e_2 \int_0^{s_2} d\tau_2(\sigma \mathbf{B})) \],

(12)

where

\[ \hat{T} = tr(\Gamma_1(m_1 - \dot{D}_1)\Gamma_2(m_2 - \dot{D}_2)) \],

(13)

and \( \Gamma_1 = \gamma_\mu, \Gamma_2 = \gamma_\nu \) for vector currents, while

\[ \langle W_\sigma(A) \rangle_A = \exp \left( -\frac{g^2}{2} \int d\pi_{\mu\nu}(1) d\pi_{\lambda\sigma}(2) \langle F_{\mu\nu}(1) F_{\lambda\sigma}(2) \rangle \right) \],

(14)

where \( d\pi_{\mu\nu} \equiv ds_{\mu\nu} + \sigma_{\mu\nu}^{(1)} d\tau_1 - \sigma_{\mu\nu}^{(2)} d\tau_2 \), and \( ds_{\mu\nu} \) is an area element of the minimal surface, which can be constructed using straight lines, connecting the points \( z^{(1)}(t) \) and \( z^{(2)}(t) \) on the paths of \( q_1 \) and \( \bar{q}_2 \) at the same time \( t \). Then the spin-independent part of the exponent reduces to the confinement term \( V_{\text{conf}}(|r|) \) plus color Coulomb potential \( V_{\text{Coul}} \), while spin-dependent part \( V_{SD} \) depends also on proper time variables \( \tau_1, \tau_2 \), (see [14] for derivation and discussion). For the case of zero quark orbital momenta with the minimal surface, discussed above, one obtains a simple answer for

\[ \langle W_\sigma(A) \rangle_A, \]

\[ \langle W_\sigma(A) \rangle_A = \exp \left( -\int_0^{t_E} dt_E \left[ \sigma |z^{(1)} - z^{(2)}| - \frac{4}{3} \frac{\alpha_s}{|z^{(1)} - z^{(2)}|} \right] \right), \]

(15)

containing \( V_{\text{conf}}(|r|) \) and \( V_{\text{Coul}}(|r|) \). Here \( \sigma \) is the string tension, \( \sigma = 0.2 \text{ GeV}^2 \) in our calculations.

At this point we introduce the method of einbein variables (effective masses) \( \omega_i \) defined via the connection between the proper time \( \tau_i \) and the real time \( t^E_1 = z_4(\tau_i) \)

\[ d\tau_i = \frac{dt^E_i}{2\omega_i}, \int ds_i(D^{(4)}z^{(i)})_{xy} = \text{const} \int D\omega_i(t)(D^{(3)}z^{(i)})_{xy}. \]

(16)

In this way the path integral in \( Dz^{(i)}_4 \) is replaced by \( D\omega^{(i)} \), and the latter can be denoted as: \( \int D\omega^{(1)} D\omega^{(2)} [... \equiv \langle [...] \rangle_\omega \), see [15] for the details.

First we need to find the Hamiltonian \( H_{q_i\bar{q}_2} \) of the system at \( t^E_1 = t^E_2 = t^E \). To this end we define the Euclidean Lagrangian \( L^E_{q_i\bar{q}_2} \). We write \( \frac{dz^{(i)}_k}{dt^E} = 2\omega_i \frac{dz^{(i)}_k}{dt^E} = 2\omega_i \dot{z}_k, k = 1, 2, 3. \) Then all terms in the exponents in [12], [14]
and (16) can be represented as \( \exp(-\int dt^E \mathcal{L}^E_{q_1 q_2}) \), and thus we arrive at the following action

\[
S^E_{q_1 q_2} = \int_0^{T_E} dt^E \left[ \frac{\omega_1 + \omega_2}{2} + \sum_i \left( \frac{\omega_i}{2} (\dot{z}_k^{(i)})^2 \right) - \right.
\]

\[
- i e_i A^{(e)}_k \dot{z}_k^{(i)} + \frac{m_1^2}{2\omega_1} + \frac{m_2^2}{2\omega_2} + e_1 \frac{\sigma_1 B}{2\omega_1} + e_2 \frac{\sigma_2 B}{2\omega_2} + \sigma |\mathbf{z}^{(1)} - \mathbf{z}^{(2)}| - \frac{4}{3} \frac{\alpha_s}{|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}|^2} \left. \right]\]

Here \( A^{(e)}_k \) is the \( k \)-th component of the QED vector potential, \( \sigma \) is the QCD string tension. The next step is the transition to the Minkowski metric. This is easy since confinement is already expressed in terms of string tension. We have \( \exp(-\int L^E dt_E) \rightarrow \exp(i \int L^M dt_M) \), \( t_E \rightarrow i t_M \), and

\[
H_{q_1 q_2} = \sum_i \dot{z}_k^{(i)} \dot{p}_k^{(i)} - L_M, \quad p_k^{(i)} = \frac{\partial L_M}{\partial \dot{z}_k^{(i)}} = \omega_i \dot{z}_k^{(i)} + e_i A^{(e)}_k. \quad (18)
\]

Next comes the key point of the Method of Effective Masses [9, 10]. It comprises the replacement of the path integral averaging over \( \omega_1, \omega_2 \) by the stationary point analysis. The applicability of this approximation may be justified by the following arguments. The \( q\bar{q} \) Green’s function (12) integrated over \( d^3(x - y) \) takes the “heat–kernel” form

\[
G_{q\bar{q}}(x, y) = \langle x | \hat{T} \exp(-H_{q_1 q_2} T) | y \rangle_{\omega_1, \omega_2} \quad (19)
\]

Integrating (12) over \( d^3(x - y) \), one obtains a simple expression:

\[
\int G^{(\mu\nu)}_{q_1 q_2}(x, y) d^3(x - y) = \left\langle \sum_{n, \lambda} \varepsilon^{(\lambda)}_\mu \varepsilon^{(\lambda)}_\nu \left( f_{n}^{(\lambda)} \right)^2 \frac{e^{-M_n^{(\lambda)} |x - y_1|}}{2M_n^{(\lambda)}} \right\rangle_{\omega_1, \omega_2} \quad (20)
\]

Here \( \varepsilon^{(\lambda)}_\mu \) is the polarization vector for the polarization state \( \lambda \), and \( M_n^{(\lambda)}, f_n^{(\lambda)} \) are correspondingly the Hamiltonian eigenvalue and quark decay constant, \( 2M_n^{(\lambda)} \) in the denominator stems from the normalization of the relativistic wave functions, \( n \) runs through all ordering numbers of the spectrum. All these quantities are functions of \( \omega_1, \omega_2 \). Therefore the integral (20) may be symbolically written as \( \langle K e^{-MT} \rangle_{\omega_1, \omega_2} = \int D\omega_1 D\omega_2 K(\omega_1, \omega_2) e^{-M(\omega_1, \omega_2) T} \) and it is essentially defined by the region of the stationary point of the exponent.

The effective masses \( \omega_i \) are to be found from the minimum of the total mass \( M(\omega_i) \), as it was suggested in [10]. To introduce the minimization
procedure and to check its accuracy we shall begin by the calculation of the eigenvalues of one and two quarks in MF, and the energy of the ground state of a relativistic charge in the atom in the next section, reproducing the known exact results.

We have the following equations defining $\omega_i$ from the total mass $M(\omega_i)$

$$\dot{H} \psi = M(\omega_i) \psi, \quad \frac{\partial M(\omega_i)}{\partial \omega_i} = 0. \quad (21)$$

For a single quark in MF the first of the above equations gives

$$M(\omega) = \frac{p_z^2 + m_q^2 + |eB|(2n + 1) - eB\sigma_z}{2\omega} + \frac{\omega}{2}. \quad (22)$$

Then the second equation yields the correct answer

$$\bar{M}_n = \left(\frac{p_z^2 + m_q^2 + |eB|(2n + 1) - eB\sigma_z}{2\omega}\right)^{1/2}. \quad (23)$$

Now we turn to the case of $q_1\bar{q}_2$ system and introduce the coordinates which are the generalization of (1)

$$\mathbf{R} = \frac{\omega_1 \mathbf{z}^{(1)} + \omega_2 \mathbf{z}^{(2)}}{\omega_1 + \omega_2}, \quad \eta = \mathbf{z}^{(1)} - \mathbf{z}^{(2)}, \quad (24)$$

$$\mathbf{P} = -i \frac{\partial}{\partial \mathbf{R}}, \quad \pi = -i \frac{\partial}{\partial \eta}. \quad (25)$$

It is convenient to introduce the following two additional parameters

$$\tilde{\omega} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}, \quad s = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}, \quad (26)$$

As before, for simplicity we consider only the neutral meson, so that $e_1 = -e_2 = e$. Then the total Hamiltonian may be written as

$$H_{q_1\bar{q}_2} = H_B + H_\sigma + W, \quad (27)$$

where

$$H_B = \frac{1}{2\omega_1} \left[ \frac{\tilde{\omega}}{\omega_2} \mathbf{P} + \pi - e \frac{1}{2} \mathbf{B} \times \left( \mathbf{R} + \frac{\tilde{\omega}}{\omega_1} \eta \right) \right]^2 +$$

$$+ \frac{1}{2\omega_2} \left[ \frac{\tilde{\omega}}{\omega_1} \mathbf{P} - \pi + e \frac{1}{2} \mathbf{B} \times \left( \mathbf{R} - \frac{\tilde{\omega}}{\omega_2} \eta \right) \right]^2 =$$
\[ \frac{1}{2\tilde{\omega}} \left( \pi - \frac{e}{2} B \times \mathbf{R} + \frac{s}{2} e B \times \eta \right)^2 + \frac{1}{2(\omega_1 + \omega_2)} \left( \mathbf{P} - \frac{e}{2} B \times \eta \right)^2. \] (28)

Equation (28) is an obvious generalization of (2). The two other terms in (27) read
\[ H_\sigma = \frac{m_1^2 + \omega_1^2 - e\sigma_1 B}{2\omega_1} + \frac{m_2^2 + \omega_2^2 + e\sigma_2 B}{2\omega_2}, \] (29)
\[ W = V_{\text{conf}} + V_{\text{Coul}} + \Delta W = \sigma|\eta| - \frac{4}{3} \frac{\alpha_s(\eta)}{\eta} + \Delta W, \] (30)
and \( \Delta W \) contains self–energy and spin–spin contributions. One can verify that the “pseudomomentum” operator in (3) introduced in Section 2 commutes with \( H_B \) and hence we can again separate the c.m. motion according to the ansatz (4). Then the problem reduces to the eigenvalue problem for \( \varphi(\eta) \) with the Hamiltonian \( H_B \) having the following form:
\[ H_B = \frac{1}{2\tilde{\omega}} \left( -i \frac{\partial}{\partial \eta} + s \frac{e}{2} B \times \eta \right)^2 + \frac{1}{2(\omega_1 + \omega_2)} \left( \mathbf{P} - eB \times \eta \right)^2 \] (31)

For \( \mathbf{P} \times B = 0 \) the system has a rotational symmetry and the c.m. is freely moving along the \( z \)-axis. Here we shall consider a state with zero orbital momentum \( (\mathbf{L}_\eta)_z = [\eta \times \frac{\partial}{\partial \eta}]_z = 0 \). As a result \( H_B \) is replaced by a purely internal space operator
\[ H_0 = \frac{1}{2\tilde{\omega}} \left( -\frac{\partial^2}{\partial \eta^2} + \frac{e^2}{4} (\mathbf{B} \times \eta)^2 \right), \] (32)

To test our method we put \( W = 0 \) and arrive at the equation
\[ (H_0 + h_\sigma)\varphi = M(\omega_1, \omega_2)\varphi. \] (33)

Consequent minimization of \( M(\omega_1, \omega_2) \) in \( \omega_1, \omega_2 \), as in (23), yields the expected answer for the two independent quarks,
\[ M = \sqrt{m_1^2 + eB(2n_1 + 1) - e\sigma_1 B} + \sqrt{m_2^2 + eB(2n_2 + 1) + e\sigma_2 B}. \] (34)

4 Treating confinement and color Coulomb terms. The magnetic QCD collapse

From (30), (32) it is clear, that inclusion of \( V_{\text{conf}} \) and \( V_{\text{Coul}} \) in \( H_0 + W \) leads to a differential equation in variables \( \eta_\perp, \eta_z \), which can be solved numerically.
However, in order to obtain a clear physical picture, we shall represent $V_{\text{conf}}$ in a quadratic form. This will allow to get an exact analytic solution in terms of oscillator functions with eigenvalue accuracy of the order of 5%. The color Coulomb contribution will be estimated as an average $\langle \varphi | V_{\text{Coul}} | \varphi \rangle$, thus yielding an upper limit for the total mass.

For $V_{\text{conf}}$ we choose the form

$$V_{\text{conf}} \rightarrow \tilde{V}_{\text{conf}} = \frac{\sigma}{2} \left( \frac{\eta^2}{\gamma} + \gamma \right) \quad (35)$$

Here $\gamma$ is a positive variational parameter; minimizing $\tilde{V}_{\text{conf}}$ w.r.t. $\gamma$, one returns to $V_{\text{conf}}$. We shall determine $M(\omega_1, \omega_2, \gamma)$ corresponding to $\tilde{V}_{\text{conf}}$, and to define $\gamma$ an additional condition

$$\left. \frac{\partial M(\omega_1, \omega_2, \gamma)}{\partial \gamma} \right|_{\gamma = \gamma_0} = 0 \quad (36)$$

will be added to (21). As a result $M(\omega_1^{(0)}, \omega_2^{(0)}, \gamma_0)$ will be the final answer for the mass of the system. The difference of the exact numerical solution from that obtained with the genuine potential $V_{\text{conf}}$ does not exceed 5%. The solution of the equation $(H_0 + \tilde{V}_{\text{conf}}) \varphi = M(\omega_1, \omega_2, \gamma_0) \varphi$ for the ground state is

$$\psi(\eta) = \frac{1}{\sqrt{\pi^{3/2} r_{\perp}^2 r_0}} \exp \left( -\frac{\eta_{\perp}^2}{2r_{\perp}^2} - \frac{\eta_z^2}{2r_0^2} \right), \quad (37)$$

where $r_{\perp} = \sqrt{\frac{2}{eB} \left( 1 + \frac{4\bar{\omega}}{e^2B^2} \right)^{-1/4}}$, $r_0 = \left( \frac{\gamma}{\sigma \bar{\omega}} \right)^{1/4}$. As we shall see below, for the lowest mass eigenvalue with $eB \gg \sigma$, one has $r_{\perp} \approx \frac{1}{\sqrt{eB}}$, $r_0 \approx \frac{1}{\sqrt{\sigma}}$ and the $(q_1, q_2)$ system acquires the form of an elongated ellipsoid. Similar quasi-one-dimensional picture was observed before for the hydrogen–like atoms in strong MF [5, 6]. In such geometrical configuration $V_{\text{Coul}}$ manifests itself in a peculiar way, again similar to what happens in hydrogen, or positronium atoms. However, as we shall see now, in QCD, at least in the large $N_c$ limit, the outcome is drastic.

We turn now to the color Coulomb term. As a starting point we present another check of our approach, namely we shall obtain the ground state energy of two relativistic particles with opposite charges without MF interacting via the Coulomb potential. The corresponding Hamiltonian reads
\[ H = H_0 + H_\sigma - \frac{a}{\eta}, \] then \( H\phi = M\phi \), and for \( eB = 0 \) we have

\[
M = -\frac{\tilde{\omega}\alpha^2}{2} + \frac{m_1^2 + \omega_1^2}{2\omega_1} + \frac{m_2^2 + \omega_2^2}{2\omega_2}.
\] (38)

Minimizing in \( \omega_1 \) in the limit \( m_2 \gg m_1 \) (the hydrogen atom), one obtains

\[
M = m_1\sqrt{1 - \alpha^2},
\] (39)

which coincides with the known eigenvalue of the Dirac equation.

In our \( (q_1\bar{q}_2) \) case one can calculate the expectation value of \( V_{\text{Coul}} = -\frac{4}{3} \frac{\alpha_s(\eta)}{\eta} \) with the asymptotic freedom and IR saturation behaviour in p-space (see [16] for a short review)

\[
\alpha_s(q) = \frac{4\pi}{\beta_0 \ln \left( \frac{q^2 + M_B^2}{\Lambda_{\text{QCD}}^2} \right)},
\] (40)

where \( M_B \) is proportional to \( \sqrt{\sigma} \), \( M_B \approx 1 \text{ GeV} \) [16]. With the wavefunction \( (37) \) the average value of \( V_{\text{Coul}} \) takes form

\[
\Delta M_{\text{Coul}} \equiv \int V_{\text{Coul}}(q)\tilde{\psi}^2(q)\frac{d^3q}{(2\pi)^3} = -\frac{4}{3\pi} \int_0^\infty \alpha_s(q)dqe^{-\frac{q^2r_0^2}{4}}I \left[ \frac{q^2(r_0^2 - r_\perp^2)}{4} \right],
\] (41)

where \( I(a^2) = \int_1^{+1} dx e^{-ax^2} \). Estimating the integral in (41), for \( eB \gg \sigma \), i.e. for \( r_0 \gg r_\perp \) one obtains for massless quarks

\[
\Delta M_{\text{Coul}} \approx -\frac{16\sqrt{\pi}}{3\beta_0^2} \ln \ln \frac{r_0^2}{r_\perp^2} \approx -\sqrt{\sigma} \ln \ln \frac{eB}{\sigma}.
\] (42)

With \( eB \) increasing the upper bound for the \( q\bar{q} \) mass is boundlessly decreasing. The exact eigenvalue should lie even lower. Surmising (as will be confirmed in the next section) that the contribution of the remaining part of Hamiltonian to the total mass is \( M_0 \approx 2\sqrt{\sigma} \), we can estimate the upper limit of \( eB \) compatible with the conditions \( M_0 + \Delta M_{\text{Coul}} \geq 0 \) namely

\[
(eB)_{\text{QCD}}^{\text{max}} \approx \sigma \exp \left( \exp \left( \frac{3\beta_0}{8\sqrt{\pi}} \right) \right) \approx 2.5 \cdot 10^{23} G \approx 2.8 \cdot 10^4 \text{ GeV}^2.
\] (43)

As we shall see, numerical calculations yield much smaller limit: \( (eB)_{\text{QCD}}^{\text{max}} \approx 6 \text{ GeV}^2 \). We note, that this upper limit is much smaller, than obtained in
QED from the positronium collapse [3]:

\[
(eB)^{QED}_{\text{max}} = \frac{m_e^2}{4} \exp\left(\frac{\pi^{3/2}}{\sqrt{\alpha}} + 2C_E\right) \approx 10^{40} G. \tag{44}
\]

One should note, that the QCD limit (43) is unaffected by higher order gluon loop corrections, since those contain only gluons, not sensitive to MF. However, the quark loop corrections to \( V_{\text{Coul}} \) are growing like \(|eB|\) and can possibly ensure the necessary screening. This is in line with QED, where such corrections produce screening and stabilization in the hydrogen case [6, 7]. Therefore the magnetic QCD collapse (42) refers only to the large \( N_c \) limit, when the quark loops contribution can be disregarded.

We shall not elaborate here more on this problem and its significance, leaving the topic to a dedicated paper.

5 Meson masses in magnetic field

Our next task is to calculate analytically the mass \( M_n(\omega_1, \omega_2, \gamma) \) of a \((q_1\bar{q}_2)\) meson. We have to solve the equation

\[
(H_0 + H_\sigma + W)\Psi_n(\eta) = M_n(\omega_1, \omega_2, \gamma)\Psi_n(\eta), \tag{45}
\]

where \( H_0, H_\sigma, W \) are given in (29-32). The result is

\[
M_n(\omega_1, \omega_2, \gamma) = \varepsilon_{n_1, n_2} + \frac{m_1^2 + \omega_1^2 - eB\sigma_1}{2\omega_1} + \frac{m_2^2 + \omega_2^2 + eB\sigma_2}{2\omega_2} + (\Delta M_{\text{Coul}}) + \Delta M_{\text{SE}}, \tag{46}
\]

where

\[
\varepsilon_{n_1, n_2} = \frac{1}{2\omega} \left[ \sqrt{\frac{e^2B^2}{\gamma} + \frac{4\sigma\omega}{\gamma}(2n_1 + 1)} + \sqrt{\frac{4\sigma\omega}{\gamma}} \left( n_2 + \frac{1}{2} \right) \right] + \frac{\gamma \sigma}{2}, \tag{47}
\]

\( \Delta M_{\text{Coul}} \) is given by (41), while according to [11, 12] \( V_{SS} \) and \( \Delta M_{\text{SE}} \) are given by

\[
V_{SS} = \frac{8\pi \alpha h_f |\varphi_n(0)|^2}{\omega_1 + \omega_2} (\sigma_1 \sigma_2), \quad \Delta M_{SE} = -\frac{2\sigma}{\pi \omega_1(0)} - \frac{2\sigma}{\pi \omega_2(0)}. \tag{48}
\]

We note that both \( V_{SS} \) and \( \Delta M_{SE} \) are to be considered as corrections and contain \( \omega_1^{(0)}, \omega_2^{(0)} \), obtained from minimization of the remaining part of
the Hamiltonian. The parameter $\gamma_0$ (see (35)) is defined from the condition

$$\left. \frac{\partial M}{\partial \gamma} \right|_{\gamma=\gamma_0} = \left. \frac{\partial \varepsilon}{\partial \gamma} \right|_{\gamma=\gamma_0} = 0,$$

(49)

In the lowest state $|\bar{u}\downarrow, u\uparrow\rangle$ we have $\sigma_1 B = -\sigma_2 B = B$, $\omega_1^{(0)} = \omega_2^{(0)} \equiv \omega^{(0)}$, and $\omega^{(0)}$ is obtained from $\frac{\partial M}{\partial \omega_i^{(0)}} = 0$, $i = 1, 2$.

![Figure 1: The mass of the system as a function of $\sqrt{eB}$. See the text for explanations.](image)

In Fig.1 we plot the mass of the system as a function of $\sqrt{eB}$. Calculations were performed according to (46) and the minimization procedure. The solid curve corresponds to the configuration $|\bar{u}\downarrow, u\uparrow\rangle$, the dashed one — $|\bar{u}\uparrow, u\uparrow\rangle$. The circles are from lattice calculations [17], the squares — from [18]. One can see that the mass is slowly decreasing and reaches zero at $(eB)_{\text{crit}}^{QCD}$ (note,
that the results plotted in Fig.1 were obtained for massless quarks). The behaviour is in agreement with the “magnetic QCD collapse” phenomena discussed above.

6 Discussion and conclusions

In our treatment of relativistic quark–aniquark system embedded in MF we relied on pseudomomentum factorization of the wave function and relativistic einbein technique. The Hamiltonian for neutral mesons in MF, containing confinement, colour Coulomb and spin interaction was derived. Using a suitable approximation for confining force we were able to calculate analytically meson masses as functions of the MF. Our eye was predominately on the lowest level with its mass decreasing with MF growing. This state is a mixture of the $\rho^0$ and $\pi^0$ as can be seen from its spin and isospin structure. Indeed, $u\bar{u}$ system under consideration is a mixture of isospin $I = 0$ and $I = 1$ states, and at large MF it has a spin structure $|u \uparrow, \bar{u} \downarrow\rangle$, which is a mixture of $S = 0$ and $S = 1$ states. With MF growing the mass of this state decreases (see Fig. 1), while the masses of all other states increase as $\sqrt{eB}$. A significant point is that these results are in line with recent lattice simulations \cite{17, 18} (see Fig.1). Calculating the Coulomb energy $\langle \Delta M_{\text{Coul}} \rangle$ we have obtained the unbounded negative contribution at large MF proportional to $\left(-\sqrt{\sigma} \ln \ln \frac{eB}{\sigma}\right)$ which makes the total mass negative for $eB > eB_{\text{crit}}^{QCD} \simeq 10 \text{ GeV}^2$.

Unlike the situation in hydrogen–like atoms, where loop corrections are able to produce saturation \cite{6, 7}, in QCD gluon loops are MF blind and uncapable to improve the results, while quark loops are suppressed in large $N_c$ limit. We call this problem “magnetic collapse in QCD” and plan to discuss it in detail in a separate paper where quark loops and gluon polarization operator will be considered, and possibly improve the situation, similarly to the hydrogen atom case.

In this paper to simplify things we started with $\rho^0$ meson states \cite{19} at $B = 0$ taking $\gamma_i$ in place of $\Gamma_1$ and $\Gamma_2$ in \cite{13}. In this way we essentially left aside the complicated problem of chiral dynamics and pseudo–Goldstone spectrum. As explained above in this oversimplified picture we can consider the lowest state as a mixture of $\rho^0$ and $\pi^0$. These two states are splitted by hyperfine interaction, this splitting is insignificant in the large MF limit. It

\footnote{In fact $u\bar{u}$ is a mock $\rho^0$}
is legitimate to compare the results of such treatment with the lattice data \cite{17,18} since in the latter the quark masses are not small and thus the chiral facet of $\pi^0$ is suppressed. As shown in Fig.1 our analytical results are in agreement with lattice calculations \cite{17,18} both for $\rho^0(\bar{u}u)$ states $|\uparrow\downarrow>$ and $|\uparrow\uparrow>$. The behavior of the total mass $M_0$ supports the conjecture of the “magnetic QCD collapse” existing in absence of quark loop corrections.

The methods used above can be generalized to the charged states thus shading the new light on the problem of charged vector boson condensation suggested in \cite{7}. As a preliminary foresight we note, that instability is an inherent property of elementary spin 1 bosons, while $\rho$–meson can not be considered as such an object, when MF is so strong that the Larmour radius is equal or smaller than its size. Another system which can be treated using the same technique is the neutral 3–body system, like neutron. The results might be important for the neutron stars physics.

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