Analysis of Hartree equation with an interaction growing at the spatial infinity.

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Abstract
We consider nonlinear Schrödinger equation with Hartree-type nonlinearity. The case where an exponent describing a shape of nonlinearity is negative is studied. In such cases, the nonlinear potential grows at the spatial infinity. Under this situation, we prove the global well-posedness in an energy class. The key for proof is a transformation of the equation by using conservation of mass and conservation of momentum. Because of this respect, uniqueness holds under conservation of momentum. When the nonlinearity grows in the quadratic order, the solution is written explicitly and the uniqueness holds without conservation of momentum. By an explicit representation of the solution, it turns out that this kind of nonlinearity contains an effect like a linear potential.

1 Introduction
This article is devoted to the study of the Cauchy problem of Hartree equation

\[
\begin{cases}
  i\partial_t u + \frac{1}{2} \Delta u = \eta(|x|^{-\nu} * |u|^2)u, \\
  u(0) = u_0,
\end{cases}
\]  

(H)

where \((t,x) \in \mathbb{R} \times \mathbb{R}^d, d \geq 1,\) and \(\eta \in \mathbb{R}.\) The function space to which the initial data \(u_0\) belong will be specified later. We treat the case where the exponent \(\nu\) is negative. More specifically, let us consider \(\nu \in (-2, 0).\) To make notation clear, we introduce \(\gamma = -\nu \in (0, 2]\) and \(\lambda = -\eta,\) and consider

\[
\begin{cases}
  i\partial_t u + \frac{1}{2} \Delta u = -\lambda(|x|^\gamma * |u|^2)u, \\
  u(0) = u_0.
\end{cases}
\]  

(nH)

In what follows, we call (nH) as negative Hartree equation and distinguish it from (H) by assuming \(\nu, \gamma > 0.\)

The Hartree equation (H) and the negative Hartree equation (nH) are generalized models of Schrödinger-Poisson system

\[
\begin{cases}
  i\partial_t u + \frac{1}{2} \Delta u = V_P u, \\
  -\Delta V_P = |u|^2, \\
  u(0) = u_0.
\end{cases}
\]  

(SP)
When $d \geq 3$ the potential $V_P$ is given by $V_P(x) = c_d(|x|^{-(d-2)} * |u|^2)$, where $c_d$ is a positive constant. Hence (SP) corresponds to the special case of (H) such that $\eta = c_d > 0$ and $\nu = d - 2 > 0$. Hartree equation is extensively studied after [10] (see [5] and references therein). On the other hand, when $d \leq 2$ the potential $V_P$ has a different form:

$$V_P(x) = \begin{cases} -\frac{1}{2\pi} \log |x| * |u|^2, & (d = 2), \\ -\frac{1}{2} (|x| * |u|^2), & (d = 1). \end{cases}$$

One sees that that the nonlinearity grows at the spatial infinity. In particular, when $d = 1$ (SP) corresponds to (H) with $\eta = -1/2 < 0$ and $\nu = -1 < 0$, that is, to (nH) with $\lambda = 1/2 > 0$ and $\gamma = 1 > 0$. Then, the negative Hartree equation (nH) appears as a generalized model with respect to $d$ and $\gamma$. It turns out that the nonlinear interaction is defocusing (or repulsive) if $\lambda > 0$, and focusing (or attractive) if $\lambda < 0$.

This article is a consequence of [14] in which global well-posedness of (SP) for dimensions $d = 1$ and $d = 2$ is shown in an energy class (see also [6, 17, 18] for one dimensional case). The global well-posedness of (nH) for $\gamma \in (0, 1]$ follows by adapting the arguments in [14] (see Theorem A.1). Hence, in this article we concentrate on $\gamma \in (1, 2]$, in which case the growth rate of the nonlinear potential is higher than in the previous results. Another type of local existence result on (SP) for $d = 2$ is established in [13].

### 1.1 Main result 1 - the case $\gamma < 2$

We first state our result for $\gamma \in (1, 2)$. Throughout this article, we use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^d$ and denote by $\mathcal{F}$ the Fourier transform in $\mathbb{R}^d$:

$$\mathcal{F} f(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx.$$ 

For nonnegative $s$ and $r$, we define a function space $\Sigma^{s, r}$ by

$$\Sigma^{s, r} = \Sigma^{s, r}(\mathbb{R}^d) := \{ f \in H^s(\mathbb{R}^d) | \langle x \rangle^r f(x) \in L^2(\mathbb{R}) \}$$

with a norm $\| f \|^2_{\Sigma^{s, r}} := \| f \|^2_{H^s(\mathbb{R}^d)} + \| \langle x \rangle^r f \|^2_{L^2(\mathbb{R}^d)}$, where $H^s$ stands for the Sobolev space. Let us introduce an energy

$$E[u(t)] = \frac{1}{2} \| \nabla u(t) \|^2_{L^2} - \frac{\lambda}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^\gamma |u(t, x)|^2|u(t, y)|^2 dxdy,$$

a center of mass

$$X[u(t)] = \int_{\mathbb{R}^d} y |u(t, y)|^2 dy,$$

and a momentum

$$P[u(t)] = \text{Im} \int_{\mathbb{R}^d} \overline{u(t, y)} \nabla u(t, y) dy = \int_{\mathbb{R}^d} \xi |\mathcal{F} u(t, \xi)|^2 d\xi.$$ 

Notice that $E[u]$, $X[v]$, and $P[w]$ make sense for $u \in \Sigma^{1, \gamma/2}$, $v \in \Sigma^{0,1/2}$, and $w \in H^{1/2}$, respectively.
Theorem 1.1. Let \( d \geq 1 \), \( \gamma \in (1, 2) \), and \( \lambda \in \mathbb{R} \). Then, \( (nH) \) is globally well-posed. More precisely, for \( u_0 \in \Sigma^{1, \gamma/2} \), there exists a global solution \( u \in C(\mathbb{R}; \Sigma^{1, \gamma/2}) \cap C^1(\mathbb{R}; (\Sigma^{1, \gamma/2})') \) to \( (nH) \). The solution conserves the mass \( \|u(t)\|_{L^2} \), the energy \( E[u(t)] \), and the momentum \( P[u(t)] \). The solution is unique in the following class:

\[
\left\{ u \in C(\mathbb{R}; \Sigma^{1, \gamma/2}) \mid P[u(t)] = \text{const.} \right\}.
\]

Remark 1.2. 1. We say that uniqueness holds unconditionally if it holds under \( C(\mathbb{R}; \Sigma^{1, \gamma/2}) \). In this theorem, the uniqueness of \((nH)\) holds conditionally since the conservation of momentum is additionally required. For the unconditional uniqueness of nonlinear Schrödinger equation with power nonlinearity, we refer the reader to [7, 8, 11, 19].

2. Conservation of \( P[u(t)] \) is equivalent to \( X[u(t)] \equiv M(at+b) \), where \( M \), \( a \), and \( b \) are defined in (1.3) and (1.4) below.

3. We have the following bound on \( L^2 \)-norms of \( \nabla u(t) \) and \( \langle x \rangle^{\gamma/2} u(t) \) (see also Remark 1.6):

\[
\|\nabla u(t)\|_{L^2} \leq \begin{cases} C(t)^{\frac{2}{d-\gamma}} & \text{if } \lambda > 0, \\ C(t) & \text{if } \lambda < 0, \end{cases}
\]

4. We show this theorem in a general framework by replacing \( \lambda|\gamma| \) with an abstract potential in a class of divergent functions (Theorem 3.4).

Remark 1.3. A formula \( |x|^{-d+\alpha} = \Lambda(\alpha)(-\Delta)^{-\frac{\alpha}{2}} \) is known for \( 0 < \alpha < d \), where \( \Lambda(\alpha) = 2^{\alpha-d/2}\Gamma(\alpha/2)/\Gamma(d/2-\alpha/2) \) (see [16]). If \( \gamma \in (0, 2) \) and if \( \gamma - 2 > -d \) then we have

\[-\Delta(|\gamma|^* |u|^2) = -\gamma(\gamma - 2 + d)(|\gamma|^2 |u|^2) = \Lambda(\alpha)\Lambda(\alpha)(-\Delta)^{-\frac{\alpha}{2}} |u|^2.\]

In a sense, this implies \( |\gamma|^* = \Lambda(\alpha)\Lambda(\alpha)(-\Delta)^{-\frac{\alpha}{2}} \), which is an extension of the above formula to \( \alpha = d + \gamma > d \).

1.2 Main Result 2 - the case \( \gamma = 2 \)

We next consider the case \( \gamma = 2 \). There exists an explicit solution in this case. Moreover, uniqueness property holds without conservation of momentum.

Assume \( u_0 \in \Sigma^{1,1} \) and introduce the following vectors and numbers. We first let

\[
M = \|u_0\|_{L^2}^2
\]

and define constant vectors

\[
a := M^{-1} P[u_0], \quad b := M^{-1} X[u_0],
\]

which represents the (scaled) momentum and the center of mass, respectively. Let

\[
e := \|\nabla u_0\|_{L^2}, \quad d := \text{Im} \int u_0^* x \cdot \nabla u_0 dx, \quad e := \|xu_0\|_{L^2}^2.
\]
For \( \omega \in \mathbb{R} \), we set \( \mathcal{U}_\omega(t) = e^{it(\frac{\omega}{2} + \frac{|x|^2}{2})} \). An integral representation of \( \mathcal{U}_\omega(t) \) is known as Mehler’s formula. For a vector \( \mathbf{a} \) posed in \( \Sigma \), the unique solution of \( (nH) \) and \( \omega \) is given by \( \mathcal{U}_\omega(t) \).

**Theorem 1.4.** Let \( d \geq 1, \gamma = 2 \) and \( \lambda = \pm 1/2 \). Then, \( (nH) \) is globally well-posed in \( \Sigma^{1,1} \). Moreover, the uniqueness holds unconditionally. Furthermore, the unique solution of \( (nH) \) is given by

\[
\begin{aligned}
u(t) &= \begin{cases}
\exp \left( \frac{i|\mathbf{a}|^2}{2} t + i\psi_+(t) \right) \tau_{\mathbf{a}+b} \pi a \mathcal{U}_\omega(t) \pi_{-\mathbf{a}-b} u_0, & \text{if } \lambda = 1/2, \\
\exp \left( \frac{i|\mathbf{a}|^2}{2} t + i\psi_-(t) \right) \tau_{\mathbf{a}+b} \pi a \mathcal{U}_{-\omega}(t) \pi_{-\mathbf{a}-b} u_0, & \text{if } \lambda = -1/2,
\end{cases}
\end{aligned}
\]

where

\[
\psi_+(t) = \frac{e - M|\mathbf{a}|^2 + M(e - M|\mathbf{b}|^2)}{4M^{3/2}} \sinh(\sqrt{M}t) \cosh(\sqrt{M}t) + \frac{d - \mathbf{a} \cdot \mathbf{b}}{2M} \sin^2(\sqrt{M}t) + \frac{-e + M|\mathbf{a}|^2 + M(e - M|\mathbf{b}|^2)}{4M} t,
\]

and

\[
\psi_-(t) = \frac{e - M|\mathbf{a}|^2 - M(e - M|\mathbf{b}|^2)}{4M^{3/2}} \sin(\sqrt{M}t) \cos(\sqrt{M}t) + \frac{\mathbf{a} \cdot \mathbf{b} - d}{2M} \sin^2(\sqrt{M}t) + \frac{-e + M|\mathbf{a}|^2 - M(e - M|\mathbf{b}|^2)}{4M} t.
\]

By a scaling argument, the general \( \lambda > 0 \) case and \( \lambda < 0 \) case are reduced to the case \( \lambda = 1/2 \) and the case \( \lambda = -1/2 \), respectively.

**Remark 1.5.** From the explicit representation of the solution, we can deduce that the nonlinearity causes the following three effects on large time behavior of the solution. First is the nonlinear phase \( e^{i\psi_\pm(t)} \). It is known that if \( \nu > 1 \) (if \( \gamma < -1 \)) then the nonlinear dynamics is compared with free one but if \( \nu < 1 \) (if \( \gamma \geq -1 \)) then it is not and a phase correction must be taken into account (cf. long-range scattering [9]). It seems that \( e^{i\psi_\pm(t)} \) is a correction of this kind. Furthermore, the linear dynamics of which the nonlinear dynamics can be regarded as a perturbation changes from \( e^{it\Delta/2} \) into \( e^{it(\Delta + M|x|^2)/2} \). This is the second respect. It is important to remark that this modified linear dynamics depends on the mass of the solution. This phenomena occurs at least for \( \gamma > 0 \). When \( \gamma = 0 \), a solution of \( (nH) \) is given by \( u(t) = e^{-itM} e^{it\Delta/2} u_0 \), which is a free dynamics with a phase correction. Third is the translation in both Fourier and physical spaces, which depends only on the momentum \( \mathbf{a} \) and the center of mass \( \mathbf{b} \). This represents the motion of the center of mass and is involved at least for \( \gamma > 1 \).

**Remark 1.6.** Let \( \gamma = 2 \) and \( \lambda = 1/2 \). A calculation shows

\[
\|\nabla u\|^2_{L^2} = \left( \langle \cosh(\sqrt{M}t) \nabla + i\sqrt{M} \sinh(\sqrt{M}t)x \rangle u_0 \right)^2_{L^2} + M|\mathbf{a}|^2 - M \left| \cosh(Mt) \mathbf{a} + \sqrt{M} \sinh(\sqrt{M}t) \mathbf{b} \right|^2.
\]
and

\[
\left\| \sqrt{M} xu \right\|_{L^2}^2 = \left\| \left( \sinh(\sqrt{M}t) \nabla + i \sqrt{M} \cosh(\sqrt{M}t)x \right) u_0 \right\|_{L^2}^2 \\
+ M^2 |at + b|^2 - M \left\| \sinh(Mt)a + \sqrt{M} \cosh(\sqrt{M}t)b \right\|^2.
\]

Hence, \( \|u(t)\|_{L^2} = O(e^{t}) \) and \( \|xu(t)\|_{L^2} = O(e^{t}) \) as \( |t| \to \infty \) for a suitable data (for example, \( u_0(x) = e^{-|x|^2} \)). These growth rates are much faster than those for free solutions; \( \|e^{it\Delta/2}u_0\|_{L^2}^2 = O(1) \) and \( \|e^{it\Delta/2}u_0\|_{L^2} = O(|t|) \) as \( |t| \to \infty \). This is because the nonlinearity, which is regarded as a repulsive quadratic potential and a remainder, accelerates the dispersion. Carles studies effects of repulsive quadratic potentials in [2]. The above exponential growths for \( \gamma = 2 \) are, in a sense, equalities of (1.1) and (1.2) in the limit \( \gamma \to 2 \). If a similar acceleration occurred for \( \gamma < 2 \) and \( \lambda > 0 \), it seems reasonable that the time growths of \( \|\nabla u\|_{L^2} \) and \( \|ux\|_{L^2} \) are faster than those of free solutions as in (1.1) and (1.2).

### 1.3 Transformation of \((nH)\)

What is difficult when we solve \((nH)\) is the fact that the nonlinear potential \(|x|^\gamma \ast |u|^2\) grows at the spatial infinity. For this, it is hard to apply a usual perturbation argument to the corresponding integral equation. To overcome this respect, we introduce a transformation of \((nH)\). Let us now observe this with a formal computation.

We consider the case \(\gamma = 2\) as a model. The equation is then

\[
i\partial_t u + \frac{1}{2} \Delta u = -\lambda (|x|^2 \ast |u|^2) u.
\]

The right hand side is equal to

\[
-\lambda |x|^2 \|u(t)\|^2_{L^2} u(x) + 2\lambda x \cdot X[u] u(x) - \lambda \int_{\mathbb{R}^n} |y|^2 |u(y)|^2 dy u(x).
\]

As long as \( \lambda \in \mathbb{R} \), we can expect that \( \|u(t)\|_{L^2} \) is conserved. Hence the first term is regarded as \( -\lambda M |x|^2 u(x) \), where \( M \) is as in (1.3). Now, \( u \) solves

\[
i\partial_t u + \frac{1}{2} \Delta u + \lambda M |x|^2 u = 2\lambda x \cdot X[u] u - \lambda \int_{\mathbb{R}^n} |y|^2 |u(y)|^2 dy u.
\] (1.6)

Although the right hand side of this equation is still divergent, the main part of the nonlinearity is removed and so the growth rate is not \( O(|x|^2) \) any longer but \( O(|x|^1) \) as \( |x| \to \infty \). This argument is introduced in [14]. Now, let us go one step further. We next observe from \((nH)\) that

\[
\frac{d}{dt} X[u(t)] = P[u(t)]
\]

follows. Similarly, by a formal calculation, one verifies that \( \frac{d}{dt} P[u(t)] = 0 \). Thus, integrating twice gives us

\[
X[u(t)] = M(at + b),
\]
where \( a \) and \( b \) are defined in (1.4). Now, we introduce a new unknown

\[
\tilde{u}(t, x) = e^{-\frac{i|a|^2}{2t}(\pi_a t - b u)}(x).
\]

(1.7)

Namely, we work with the center of mass frame. Then, one verifies that \( \tilde{u} \) also solves (1.6) and \( X[\tilde{u}(t)] \equiv 0 \). These facts imply that \( \tilde{u} \) is a solution to

\[
i \partial_t \tilde{u} + \frac{1}{2} \Delta \tilde{u} + \lambda M |x|^2 \tilde{u} = -\lambda \int_{\mathbb{R}^d} |y|^2 |\tilde{u}(y)|^2 dy \tilde{u}.
\]

(1.8)

Now, the right hand side is bounded with respect to \( x \). Let us further set

\[
w(t, x) = \tilde{u}(t, x) \exp \left( -i \lambda \int_0^t \int_{\mathbb{R}^d} |y|^2 |\tilde{u}(s, y)|^2 dy ds \right).
\]

(1.9)

Then, \( w \) solves a linear Schrödinger equation \( i \partial_t w + \frac{1}{2} \Delta w + \lambda M |x|^2 w = 0. \) Applying inverses of (1.9) and (1.7), we obtain an explicit solution of (nH).

The argument in the case \( \gamma \in (1, 2) \) is similar. We introduce \( \tilde{u} \) as in (1.7) and try to solve a *modified Hartree equation*

\[
\left\{
\begin{aligned}
i \partial_t \tilde{u} &+ \frac{1}{2} \Delta \tilde{u} + \lambda M |x|^\gamma |\chi(|x|)| \tilde{u} \\
&= -\lambda \int_{\mathbb{R}^d} \left( |x - y|^\gamma |\chi(|x|) + \gamma \langle x, y \rangle \gamma^{-2} x \cdot y \right) \tilde{u}(y)^2 dy ds \\
\tilde{u}(0) &= \pi_a \tau_{-b} u_0
\end{aligned}
\right.
\]

(nH)

instead of (1.8), where \( \chi \) is a smooth non-decreasing function such that \( \chi(r) = 0 \) for \( r \leq 1 \) and \( \chi(r) = 1 \) for \( r \geq 2 \). It will turn out that (nH) can be solved in a standard way because the growth of the nonlinearity of (nH) is successfully removed by the transformation. It is important to note that (nH) is not a perturbation of free equation \( i \partial_t \psi + (1/2) \Delta \psi = 0 \) any more but of \( i \partial_t \psi + (1/2) \Delta \psi + \lambda M |x|^\gamma \chi(|x|) \psi = 0 \), which involves a linear potential.

Oh considered in [15] the Cauchy problem of nonlinear Schrödinger equation with a divergent potential and \( L^2 \)-subcritical power-type nonlinearity (see also [5]). In particular, the case where the potential is a quadratic polynomial is extensively studied. We refer the reader to [1, 2, 3, 4, 12, 20, 22].

The rest of this article is organized as follows: We prove Theorem 1.4 in the next Section, and Theorem 1.1 in Section 3.

### 2 Proof of Theorem 1.4

Let us prove our theorem for an equation with a harmonic potential

\[
i \partial_t u + \frac{1}{2} \Delta u + \frac{\eta}{2} |x|^2 u = -\frac{\zeta}{2}(|x|^2 + |u|^2) u, \quad u(0) = u_0 \in \Sigma^{1,1}(\mathbb{R}^d),
\]

(2.1)

where \( \eta \) and \( \zeta \) are real constants. For \( \omega \in \mathbb{R} \) and \( a, b \in \mathbb{R}^d \), we define an \( \mathbb{R}^d \)-valued function \( g_{\omega}(t) \) as

\[
g_{\omega}(t) = \begin{cases}
\frac{\sinh(\sqrt{\omega}t)}{\sqrt{\omega}} + b \cosh(\sqrt{\omega}t), & \omega > 0, \\
a t + b, & \omega = 0, \\
\frac{\sin(\sqrt{\omega}t)}{\sqrt{\omega}} + b \cos(\sqrt{\omega}t), & \omega < 0.
\end{cases}
\]

(2.2)
Notice that \( g_\omega(t) \) is a solution to \( g''_\omega(t) = \omega g_\omega(t) \) with \( g(0) = b \) and \( g'(0) = a \).

**Theorem 2.1.** 1. Let \( d \geq 1, \eta \in \mathbb{R}, \) and \( \zeta \in \mathbb{R} \). Then, (2.1) is globally well-posed in \( \Sigma^{1,1} \). The uniqueness holds unconditionally. Moreover, \( X[u(t)] = Mg_0(t) \) holds.

2. For a data \( u_0 \in \Sigma^{1,1}(\mathbb{R}^d) \), define \( M \) by (1.3) and set \( \omega = \eta + \zeta M \). Let \( g_\eta(t) \) be defined in (2.2) with a parameter \( \iota \in \mathbb{R} \) and the data \( a \) and \( b \) given by (1.4). Then, the unique solution to (2.1) is written as

\[
u(t,x) = e^{i\Psi_{\eta,\zeta}(t)}[\tau_{\eta_\omega(t)}\tau_{g_\omega(t)}(\tau_{-\eta_\omega(t)}\tau_{-g_\omega(t)}\iota(t))u_0](x)
\]

with

\[
\Psi_{\eta,\zeta}(t) = \frac{1}{2}(g_\eta(t) \cdot g'_\eta(t) - g_\omega(t) \cdot g'_\omega(t)) - \frac{\zeta M}{2} \int_0^t |g_\omega(s)|^2 ds + \frac{\zeta}{2} \psi_\omega(t),
\]

where \( \psi_\omega \) is defined with \( \iota, d, \) and \( e \) given by (1.5) as follows:

- If \( \omega > 0 \) then
  \[
  \psi_\omega(t) = \frac{c + \omega e}{2\omega^3/2} \sinh(\sqrt{\omega}t) \cosh(\sqrt{\omega}t) + \frac{d}{\omega} \sin^2(\sqrt{\omega}t) - \frac{c - \omega e}{2\omega} t;
  \]

- if \( \omega = 0 \) then
  \[
  \psi_0(t) = \frac{1}{3} ct^3 + dt^2 + et;
  \]

- if \( \omega < 0 \) then
  \[
  \psi_\omega(t) = -\frac{c + \omega e}{2|\omega|^{3/2}} \sin(\sqrt{|\omega|}t) \cos(\sqrt{|\omega|}t) + \frac{d}{|\omega|} \sin^2(\sqrt{|\omega|}t) + \frac{c - \omega e}{2|\omega|} t.
  \]

**Remark 2.2.** Thanks to Proposition 2.5 below, Theorem 1.4 immediately follows by taking \( \eta = 0 \) and \( \zeta = \pm 1 \) (and so \( \omega = \pm M \)).

**Remark 2.3.** In general, momentum of a solution is not conserved in the presence of a linear potential. Indeed, the solution of (2.1) given in this theorem satisfies \( \dot{P}^I[u(t)] = Mg'_\omega(t) \). This is not conserved unless \( \eta = 0 \) (or \( u_0 \) satisfies \( a = b = 0 \)).

**Remark 2.4.** Up to a translation in both physical and Fourier spaces and a nonlinear phase, the solution behaves as \( \mathcal{U}_\omega(t)u_0 \). In particular, we have \( \|u(t)\|_{L^p} = \|\mathcal{U}_\omega(t)u_0\|_{L^p} \) for all \( p \). It is worth pointing out that not \( \eta \) but the exponent \( \omega = \eta + \zeta M \) decides the linear profile of the solution. This means that, from the view point of change of the dispersive property, the nonlinearity has the same effect as by the linear potential. Recall that, however, the motion of the center of mass \( X[u(t)] \) is governed only by the linear potential. An interesting case would be \( \eta \zeta < 0 \). In this case, there exists a critical mass \( M_c = -\eta/\zeta \) such that the sign of \( \omega \) changes at this value. If \( \|u_0\|_{L^2}^2 = M_c \), then the effect of the linear potential is partially removed by the nonlinearity so that the solution of (2.1) is a solution of the free Schrödinger equation \( e^{i\Phi}u_0 \) up to a translation and a nonlinear phase.

**Proof.** Let us first consider the equation

\[
i\partial_t w + \frac{1}{2}\Delta w + \frac{\omega}{2}|w|^2w = 0, \quad w(0) = \pi_{-a\tau-b}u_0.
\]

(2.3)
Obviously, a solution is given by $w(t) = U_\omega(t)\pi-a\tau-b\omega$. One verifies that
\[ \frac{d}{dt}X[w(t)] = P[w(t)] \]
and
\[ \frac{d}{dt}P[w(t)] = \omega X[w(t)]. \]
Since $X[w(0)] = \int (y - b)u_0^2 dy = 0$ and $P[w(0)] = P[u_0] - Ma = 0$ hold, it follows that $X[w(t)] \equiv 0$. Set
\[ \tilde{u}(t, x) = w(t, x) \exp \left( i \frac{\zeta}{2} \int_0^t \int_{\mathbb{R}^d} |y|^2 |w(s, y)|^2 dy ds \right). \]  
(2.4)
Now, it is easy to see that $\tilde{u}$ solves
\[ i\partial_t \tilde{u} + \frac{1}{2} \Delta \tilde{u} + \frac{\omega}{2} |x|^2 \tilde{u} = -\frac{\zeta}{2} \int_{\mathbb{R}^d} |y|^2 |\tilde{u}(y)|^2 dy \tilde{u} \]  
(2.5)
and $\tilde{u}(0) = w(0) = \pi-a\tau-bu_0$. Since $|\tilde{u}(t, x)| = |w(t, x)|$, it also holds that $X[\tilde{u}(t)] \equiv X[w(t)] \equiv 0$. We introduce $g_\eta(t)$ by (2.2) with the data $a$ and $b$ given in (1.4). Recall that $g''_\eta = \eta g_\eta(t)$. Hence,
\[ i\partial_t \tilde{u} + \frac{1}{2} \Delta \tilde{u} + \frac{\eta}{2} |x+g_\eta(t)|^2 \tilde{u} - g''_\eta(t) \cdot x \tilde{u} \]
\[ = \zeta x \cdot X[\tilde{u}] \tilde{u} - \frac{\zeta}{2} \int_{\mathbb{R}^d} |y|^2 |\tilde{u}(y)|^2 dy \tilde{u}. \]  
(2.6)
Since $\omega = \eta + \zeta M = \eta + \zeta \|\tilde{u}(t)\|_{L^2}^2$, this equation is equivalent to
\[ i\partial_t \tilde{u} + \frac{1}{2} \Delta \tilde{u} + \frac{\eta}{2} |x+g_\eta(t)|^2 \tilde{u} - g''_\eta(t) \cdot x \tilde{u} = -\frac{\zeta}{2} \int_{\mathbb{R}^d} |y|^2 |\tilde{u}(y)|^2 dy \tilde{u}. \]
Hence, if we define $u$ by
\[ u(t, x) = e^{\frac{\zeta}{2} \int_0^t (|\eta g'_\eta(s)|^2 + |\eta g_\eta(s)|^2) ds} (\tau_{g_\eta(t)} \pi g''_\eta(t) \tilde{u})(x), \]  
(2.7)
then $u$ solves (2.1). It also holds that $u(0) = \tau_b \pi_a \tilde{u}(0) = \tau_b \pi_a \pi - a \tau - b u_0 = u_0$. Combining (2.7) and (2.4), we conclude that
\[ u(t, x) = e^{\frac{\zeta}{2} \int_0^t (|\eta g'_\eta(s)|^2 + |\eta g_\eta(s)|^2) ds} (\tau_{g_\eta(t)} \pi g''_\eta(t) \tilde{u})(x) \]
\[ = e^{\Psi(t)} (\tau_{g_\eta(t)} \pi g''_\eta(t) U_\omega(t) \pi - a \tau - b u_0)(x), \]  
(2.8)
where
\[ \Psi(t) = \frac{1}{2} \int_0^t (|\eta g'_\eta(s)|^2 + |\eta g_\eta(s)|^2) ds + \frac{\zeta}{2} \int_0^t \int_{\mathbb{R}^d} |y|^2 |w(s, y)|^2 dy ds \]
\[ = \frac{1}{2} (g_\eta(t) \cdot g''_\eta(t) - a \cdot b) + \frac{\zeta}{2} \int_0^t \int_{\mathbb{R}^d} |y|^2 |w(s, y)|^2 dy ds \]
Then, Propositions 2.5 and 2.6 show the stated representation of the solution.

Now, let us proceed to the proof of the uniqueness. Suppose that $u_1 \in C(\mathbb{R}; \Sigma^{1,1})$ is a solution of (2.1) in $(\Sigma^{1,1})'$. By the equation, we see that $u_1 \in C^1(\mathbb{R}; (\Sigma^{1,1})')$ and so that $\|u_1\|_{L^2}$ is conserved and $\frac{d}{dt}X[u_1(t)] = 8$.
By Proposition 2.7 below, we obtain $P[u_1(t)] \in C^1(\mathbb{R})$ and $\frac{d}{dt} P[u_1(t)] = \eta X[u_1(t)]$. Then, $X[u_1(t)] = Mg(t)$. Let us introduce

$$
\tilde{u}_1(t, x) = e^{-\frac{i}{2} \int_0^t (\tau_{g_0(t)\tau_{g_0(t)}} + \eta g_0(t)^2) ds} (\pi_{g_0(t)} \tau_{g_0(t)} u_1)(x),
$$

which is the inverse transform of (2.7). Then, $\tilde{u}_1$ solves (2.6). Since

$$
X[\tilde{u}_1(t)] = \int (y - g_0(t)) |u_1(t, y)|^2 dy = X[u_1(t)] - Mg(t) = 0,
$$

$\tilde{u}$ is also a solution to (2.5). Now, let us further introduce

$$
\omega(t, x) = \tilde{u}_1(t, x) \exp \left( -\frac{i}{2} \int_0^t \int_{\mathbb{R}^d} |g|^2 |\tilde{u}_1(s, y)|^2 dy ds \right).
$$

This is the inverse transform of (2.4) since $|\omega(t, x)| = |\tilde{u}_1(t, x)|$. Then, $\omega_1$ solves (2.3) and so $\omega_1(t, x) = \omega(x, t)$. Applying (2.7) and (2.4), we conclude that $u_1$ is identical to $\tilde{u}$.

**Proposition 2.5.** Let $\tau \in \mathbb{R}$ and $a, b \in \mathbb{R}^d$. Define $g_\tau(t)$ by (2.2). Then, it holds for all $t \in \mathbb{R}$ that

$$
\mathcal{U}_\tau(t) \pi_{g_\tau} \tau_{-b} = e^{\frac{i}{2} (g_\tau(t) \tau_{g_\tau(t)} - a \cdot b) \pi_{g_\tau(t)} \tau_{g_\tau(t)} \tau_{-b}} \mathcal{U}_\tau(t).
$$

**Proof.** Fix $\varphi \in \Sigma^{1,1}$. Let us set $w(t) = \mathcal{U}_\tau(t) \pi_{g_\tau} \tau_{-b} \varphi$. Then,

$$
i \partial_t w + \frac{1}{2} \Delta w + \frac{\kappa}{2} |x|^2 w = 0, \quad w(0) = \pi_{g_\tau} \tau_{-b} \varphi.
$$

Now, we introduce $\bar{w}(t, x) = e^{\frac{i}{2} (g_\tau(t) \tau_{g_\tau(t)} - a \cdot b) \pi_{g_\tau(t)} \tau_{g_\tau(t)} \tau_{-b}} \mathcal{U}_\tau(t) \pi_{g_\tau} \tau_{-b} \varphi$. One easily verifies that $i \partial_t \bar{w} = \mathcal{U}_\tau(t) \pi_{g_\tau} \tau_{-b} \varphi$. Hence,

$$
\mathcal{U}_\tau(t) \varphi = e^{\frac{i}{2} (g_\tau(t) \tau_{g_\tau(t)} - a \cdot b) \pi_{g_\tau(t)} \tau_{g_\tau(t)} \tau_{-b}} \mathcal{U}_\tau(t) \pi_{g_\tau} \tau_{-b} \varphi
$$

is valid for arbitrary $\varphi \in \Sigma^{1,1}$. Alternatively, let $\zeta = 0$ in (2.8).

**Proposition 2.6.** Let $w(t) = \mathcal{U}_\tau(t) \pi_{g_\tau} \tau_{-b} u_0$. Define $\psi_\omega(t)$ as in Theorem 2.1. Then,

$$
\int_0^t \int_{\mathbb{R}^d} |g|^2 |w(t)|^2 dy ds = \psi_\omega(t) - M \int_0^t \int_{\mathbb{R}^d} |g_\omega(s)|^2 dy ds.
$$

**Proof.** By the previous proposition, we obtain

$$
||xw(t)||_{L^2}^2 = \|x \tau_{-g_\omega(t)} u_0\|_{L^2}^2 = \|x \mathcal{U}_\tau(t) u_0\|_{L^2}^2 - M |g_\omega(t)|^2,
$$

where we have used $X[\mathcal{U}_\tau(t) u_0] = Mg_\omega(t)$. It therefore suffices to show that $\psi_\omega(t) = \int_0^t ||x \mathcal{U}_\tau(s) u_0||_{L^2}^2 ds$.

Assume $\omega > 0$. It is well known that $\mathcal{U}_\tau(t)$ is decomposed as $\mathcal{U}_\tau(t) = \mathcal{M}_\omega(t) \mathcal{D}_\omega(t) \mathcal{F} \mathcal{M}_\omega(t)$, where $\mathcal{M}_\omega(t)$ is a multiplication operator defined by

$$
\mathcal{M}_\omega(t) = \exp \left( \frac{\sqrt{\omega}}{2} \coth(\sqrt{\omega}|x|^2) \right)
$$

and $\mathcal{D}_\omega(t)$ and $\mathcal{F}$ are operators related to the Fourier transform.
and $\mathcal{D}_\omega(t)$ is a dilation operator defined by

$$(\mathcal{D}_\omega(t)f)(x) = \left(\frac{\sqrt{\omega}}{i\sinh(\sqrt{\omega}t)}\right)^\frac{d}{2} f \left(\frac{\sqrt{\omega}x}{\sinh(\sqrt{\omega}t)}\right).$$

Hence,

\[
\|x\mathcal{D}_\omega(t)u_0\|_{L^2}^2 = \left(\frac{\sinh(\sqrt{\omega}t)}{\sqrt{\omega}}\right)^2 \|x\mathcal{F}\mathcal{M}_\omega(t)u_0\|_{L^2}^2 = \left(\frac{\sinh(\sqrt{\omega}t)}{\sqrt{\omega}}\right)^2 \|\nabla (\mathcal{M}_\omega(t)u_0)\|_{L^2}^2 \]
\[
= \left(\frac{\sinh(\sqrt{\omega}t)}{\sqrt{\omega}}\right)^2 \|\nabla u_0 + i\sqrt{\omega} \coth(\sqrt{\omega}t)xu_0\|_{L^2}^2.
\]
\[
= \frac{c}{\omega} \sinh^2(\sqrt{\omega}t) + \frac{2d}{\sqrt{\omega}} \sinh(\sqrt{\omega}t) \cosh(\sqrt{\omega}t) + e \cosh^2(\sqrt{\omega}t)
\]
and so

$$\psi_\omega(t) = \left(\frac{c + \epsilon \omega}{2\omega^{1/2}}\right) \sinh(\sqrt{\omega}t) \cosh(\sqrt{\omega}t) + \frac{d}{\omega} \sinh^2(\sqrt{\omega}t) - \left(\frac{e - \epsilon \omega}{2\omega}\right)t,$$

where $c$, $d$, and $e$ are constants defined in (1.5). The proof for $\omega \leq 0$ is similar. We omit details. \hfill \square

**Proposition 2.7.** Let $d \geq 1$ and $\eta, \zeta \in \mathbb{R}$. Let $u_0 \in \Sigma^{1,1}$. Let $u \in C(\mathbb{R}; \Sigma^{1,1})$ solve (2.1) in $(\Sigma^{1,1})'$ sense. Then, $P[u(t)]$ is a continuously differentiable function of time and $\frac{d}{dt} P[u(t)] = \eta X[u(t)]$ holds.

**Proof.** Since $u \in C(\mathbb{R}; \Sigma^{1,1})$, we see that $X[u(t)] \in C(\mathbb{R})$. By (2.1), $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and $\frac{d}{dt} X[u(t)] = P[u(t)] \in C(\mathbb{R})$ hold. Set

$$v(t, x) = u(t, x) \exp \left(-\frac{i\zeta}{2} \int_0^t \int_{\mathbb{R}^d} |y|^2 |u(t, s)|^2 dy ds\right).$$

We have $|v(t, x)| = |u(t, x)|$ and so $X[v(t)] = X[u(t)]$ and $\|v(t)\|_{L^2} = \|u(t)\|_{L^2} = \|u_0\|_{L^2}$.

Hence, one sees that $v$ is a solution to

$$i\partial_t v + \frac{1}{2} \Delta v + \omega |x|^2 v = \lambda X[v(t)] \cdot xv, \quad v(0) = u_0,$$

where $\omega = \eta + \zeta M$. Let us introduce a function $H(t) \in C^3(\mathbb{R})$ as follows:

$$H(t) = -\zeta \int_0^t e^{\sqrt{\omega}(t-s)} \frac{e^{-\sqrt{\omega}(t-s)}}{2} \int_s^t X[u(\sigma)]d\sigma ds,$$

where $\frac{e^{\sqrt{\omega}} + e^{-\sqrt{\omega}}}{2} = \cos(\sqrt{\omega}t)$ if $\omega < 0$. Remark that $H$ is a solution of $H''(t) = \omega H(t) - \zeta X[u(t)]$ with $H(0) = H'(0) = 0$. Further set

$$v(t, x) = \exp \left(-\frac{i}{2} \int_0^t (|H'(s)|^2 + \omega |H(s)|^2) ds\right) [\pi_{\mathcal{H}(t)} \mathcal{T}_{H(t)} w(t)](x).$$
Then, $w$ solves $i\partial_t w + \frac{1}{2} \Delta w + \frac{\omega}{4}|x|^2 w = 0$ with $w(0) = u_0$, and so $X[w(t)] = Mg_\omega(t) \in C^\infty(\mathbb{R})$ follows. Hence, we conclude that

$$X[u(t)] = X[v(t)] = \int_{\mathbb{R}^d} (y + H(t))[w(t, y)]^2 dy = Mg_\omega(t) + MH(t) \in C^3(\mathbb{R}),$$

which gives us $P[u(t)] = \frac{d}{dt} X[u(t)] \in C^2(\mathbb{R})$. Moreover,

$$\frac{d}{dt} P[u(t)] = \frac{d^2}{dt^2} X[u(t)] = \frac{d^2}{dt^2} X[w(t)] + MH''(t) = Mg_\omega(t) + MH(t) - \zeta MX[u(t)] = (\omega - \zeta M) X[u(t)].$$

3 Proof of Theorem 1.1

We shall prove Theorem 1.1 in a general framework. Consider a generalized Hartree equation

$$\begin{cases}
i\partial_t u + \frac{1}{2} \Delta u = -((V + R) * |u|^2)u, \text{ in } \mathbb{R}^{1+d}, \\
u(0, x) = u_0,\end{cases}$$

(gH)

where $V$ and $R$ are real-valued functions of $x \in \mathbb{R}^d$.

A pair $(q, r)$ is admissible if $2 \leq q, r \leq \infty$ satisfy the relation $2/q = \delta(r) := d(1/2 - 1/r) \in [0, 1)$ (however, we exclude the case $(d, q, r) = (2, 2, \infty)$). For nonnegative numbers $p$, $q$ and $r$, define

$$W^{p, q}_{V, r} := \{ f \in L^r(\mathbb{R}^d); \langle \nabla \rangle^p f, (V(\cdot))^{q/2} f \in L^r(\mathbb{R}^d) \}$$

with a norm

$$\| f \|_{W^{p, q}_{V, r}} := \| \langle \nabla \rangle^p f \|_{L^r} + \| (V(\cdot))^{q/2} f \|_{L^r}.$$ 

Let $\Sigma^{p, q}_{V} := W^{p, q}_{V, 2}$.

Take a vector-valued function $W$ and set

$$K(x, y) = V(x - y) - V(x) + y \cdot W(x).$$

The assumptions on the potential $V$ and $R$ are the following.

**Assumption 3.1.** Suppose that $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth function satisfying the following properties:

(V1) $\partial^\alpha V(x) \in L^\infty(\mathbb{R}^d)$ for all index $\alpha$ with $|\alpha| \geq 2$;

(V2) There exist constants $C > 0$ and $\kappa \in [0, 1)$ such that $|\nabla V(x)| \leq C(V(x))^{\kappa/2}$;

The assumption on $R$ is the following.

**Assumption 3.2.** Suppose that $R : \mathbb{R}^d \to \mathbb{R}$ is a real-valued function satisfying the following properties:

(R1) $R(x) \in L^\infty(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$;

(R2) There exist constants $C > 0$ and $\kappa \in [0, 1)$ such that $|R(x)| \leq C|V(x)|^{\kappa/2}$;
(V3) There exists a vector-valued function $W$ such that $\partial^\alpha W(x) \in L^\infty(\mathbb{R}^d)$ for all $|\alpha| \geq 2$ and $K$ defined by (3.1) satisfies
\[
\sup_x |K(x, y)| + \sup_y |\nabla_y K(x, y)| \leq C (V(y));
\]

(V4) There exists a constant $C > 0$ such that $\langle x \rangle \leq C (V(x))$.

**Assumption 3.2.** Assume that $R$ satisfies the following.

(R1) $R \in L^s(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, where $s \in [1, \infty]$ and $s > d/4$.

(R2) $R^\circ := \max(0, R) \in L^\theta(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, where $\theta \in [1, \infty]$ and $\theta > d/2$.

**Remark 3.3.**

1. Roughly speaking, $V$ and $R$ denote a divergent part and a remainder of a (divergent) potential $V + R$, respectively. When we prove Theorem 1.1, we choose $V = \lambda|x|^{\gamma} \chi(x)$ and $R = \lambda|x|^{\gamma}(1 - \chi(x))$, where $\chi$ is a smooth radial function such that $\chi \equiv 1$ for $|x| \geq 2$ and $\chi \equiv 0$ for $|x| \leq 1$. Notice that $\lambda|x|^{\gamma}$ itself does not satisfy (V1).

2. $\Sigma_{V}^{1,1} \subset \Sigma_{V}^{1,1/2}$ as long as Assumption (V4) is satisfied, under which condition $X[u]$ is well-defined for $u \in \Sigma_{V}^{1,1}$.

3. If (R1) is satisfied then $(R * |u|^2) \in (H^1)'$ for $u \in H^1$. An example of $R$ satisfying (R1) is the $H^1$-subcritical Hartree-type nonlinearity; $\eta|x|^{-\nu}$ with $\eta \in \mathbb{R}$ and $0 < \nu < \min(4, d)$.

The main result of this section is the following.

**Theorem 3.4.** Suppose Assumptions (V1), (V2), (V3), (V4), (R1), and (R2) are satisfied. Then, (gH) is globally well-posed in $\Sigma_{V}^{1,1}$. More precisely, for $u_0 \in \Sigma_{V}^{1,1}$, there exists a global solution $u$ of (gH) in $C(\mathbb{R}; \Sigma_{V}^{1,1} \cap C^1(\mathbb{R}; (\Sigma_{V}^{1,1})'))$ which satisfies $u \in L^2(\mathbb{R}; W^{1,1}_{q,r})$ for all admissible pair $(q, r)$ and conserves the mass $\|u(t)\|_{L^2}$, the energy
\[
E[u(t)] = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 - \frac{1}{4} \int_{\mathbb{R}^{d+d}} (V(x - y) + R(x - y))|u(t, x)|^2|u(t, y)|^2dxdy,
\]
and the momentum $P[u(t)]$. The solution is unique in the class
\[
\left\{ u \in L^\infty(\mathbb{R}; \Sigma_{V}^{1,1}) \cap L^\infty_{\text{loc}}(\mathbb{R}; W^{1,1}_{q,r}) \mid P[u(t)] = \text{const} \right\}.
\]

Moreover, we have the following estimates: If $V \leq 0$ then
\[
\|\nabla u(t)\|_{L^2} \leq C, \quad \|\langle V(\cdot) \rangle^{1/2} u(t)\|_{L^2} \leq C (t)^{\nu};
\]
otherwise
\[
\|\nabla u(t)\|_{L^2} \leq C (t)^{\kappa}, \quad \|\langle V(\cdot) \rangle^{1/2} u(t)\|_{L^2} \leq C (t)^{\kappa},
\]
where $\kappa$ is the number defined in Assumption (V2).
We now apply the transform observed in the introduction. Then, the problem boils down to the Cauchy problem of the following modified version of the (gH):

\[
\begin{aligned}
&i\partial_t u + \frac{1}{2}\Delta u + MV(x)u = -u \int_{\mathbb{R}^d} K(\cdot, y)|u(y)|^2 dy - (R * |u|^2)u, \\
u(0) = u_0 \in \Sigma^{1,1}_V,
\end{aligned}
\]  

where \( M \) is as in (1.3) and \( K \) is defined by (3.1) with \( W \) given in Assumption (V3). If (V3) is satisfied then \( u \int K(\cdot, y)|u(y)|^2 dy \in \Sigma^{1,1}_V \) for \( u \in \Sigma^{1,1}_V \), where \( j = 0,1 \).

Denote \( A := \frac{1}{2}\Delta + MV(x) \). It is well known that \( \mathcal{A} \) is an essentially self-adjoint operator on \( L^2 \) as long as Assumption (V1) holds. Moreover, we have Strichartz’s estimate under this condition ([21]).

**Lemma 3.5 (Strichartz’s estimate).** Suppose (V1). For any fixed \( T > 0 \), the following properties hold:

- Suppose \( \varphi \in L^2(\mathbb{R}^2) \). For any admissible pair \((q,r)\), there exists a constant \( C = C(T,q,r) \) such that

\[
\|e^{itA}\varphi\|_{L^q(\{-T,T\};L^r)} \leq C \|\varphi\|_{L^2}.
\]

- Let \( I \subset (-T,T) \) be an interval and \( t_0 \in I \). For any admissible pairs \((q,r)\) and \((\gamma,\rho)\), there exists a constant \( C = C(t,q,r,\gamma,\rho) \) such that

\[
\left\| \int_{t_0}^t e^{i(t-s)A}F(s)ds \right\|_{L^q(I;L^r)} \leq C \|F\|_{L^{\gamma'}(I;L^\rho')}
\]

for every \( F \in L^{\gamma'}(I;L^\rho') \).

**Lemma 3.6.** Let \( A := \frac{1}{2}\Delta + MV(x) \). Suppose that Assumption (V1) holds. Denote by \( e^{itA} \) a one-parameter group generated by \( A \). Let \( F \) be an arbitrary weight function such that \( \nabla F \) and \( \Delta F \) are bounded. Then, for all \( f \in \Sigma^{1,1}_V \),

\[
e^{-itA}\nabla e^{itA}f = \nabla f + iM \int_0^t e^{-isA}(\nabla V)e^{isA}fds
\]

and

\[
e^{-itA}Fe^{itA}f = Ff + i \int_0^t e^{-isA}(\nabla F, \nabla + \frac{1}{2} \Delta F)e^{isA}fds.
\]

By means of this lemma, it immediately follows from (V1) and (V2) that

\[
\|e^{itA}\phi\|_{L^\infty((-T,T),\Sigma^{1,1}_V)} \leq \|\phi\|_{\Sigma^{1,1}_V} + CT \|e^{itA}\phi\|_{L^\infty((-T,T),\Sigma^{1,1}_V)} + CT \|\phi\|_{L^2}.
\]

This implies \( \|e^{itA}\phi\|_{\Sigma^{1,1}_V} \leq C(|t|,M)\|\phi\|_{\Sigma^{1,1}_V} \). A similar argument shows \( \|e^{itA}\phi\|_{\Sigma^{k,k}_V} \leq C(k,|t|,M)\|\phi\|_{\Sigma^{k,k}_V} \) for any positive integer \( k \). Further, as in [6, Lemma 2.11],

\( e^{itA}\phi \) is continuous in \( \Sigma^{k,k}_V \) with respect to \( M \) for each \( t \) and \( \phi \in \Sigma^{k,k}_V \).
3.1 Local well-posedness of $(mgH)$

We first give a unique local solution to $(mgH)$. Throughout this subsection and the next subsection we suppose that the constant $M$ in the operator $A$ is not necessarily equal to $||u_0||_{L^2}$. In what follows, we write $L^p((-T,T);X) = L^p_T X$, for short. The integral form of $(mgH)$ is

$$u(t) = Q[u] := e^{itA}u_0 + i \int_0^t e^{i(t-s)A}u(s) \int_{\mathbb{R}^d} K(x,y)u(s,y)^2dyds$$

$$+ i \int_0^t e^{i(t-s)A}((R*|u|^2)u)(s)ds.$$  \quad (3.3)

**Proposition 3.7.** Suppose that Assumptions (V1), (V2), (V3), and (R1) are satisfied. Let $u_0 \in \Sigma_V^{1,1}$. Define a Banach space

$$\mathcal{H}_{T,\delta} := \{ f \in L^\infty((-T,T);\Sigma_V^{1,1}) ; ||f||_{\mathcal{H}_T} \leq \delta \},$$

where

$$||f||_{\mathcal{H}_T} := ||f||_{L_{T,T}^\infty \Sigma_V^{1,1}} + ||f||_{L_{T,T}^2 W_{V,r}^{1,1}},$$

with $q = 8\zeta/d$ and $r = 4\zeta/(2\zeta - 1)$. Then, there exists $\delta_0$ depending only on $||u_0||_{\Sigma_V^{1,1}}$ such that for any $\delta \in [\delta_0, \infty)$, the operator $Q$ given in (3.3) is a contraction map from $\mathcal{H}_{T,\delta}$ to itself for suitable $T = T(\delta) > 0$.

**Proof.** We prove that $Q$ is a contraction map in two steps.

**Step 1**

Fix $\delta > 0$. We show that the existence of $T$ such that $Q[u] \in \mathcal{H}_{T,\delta}$ as long as $u \in \mathcal{H}_{T,\delta}$.

Let $u \in \mathcal{H}_{T,\delta}$. We first establish estimates on $R$. Set $R = R_1 + R_2$ with $R_1 \in L^\infty$ and $R_2 \in L^\infty$. One sees from Young’s inequality, Hölder’s inequality, and Sobolev’s embedding that

$$||A((R_1*|u|^2)u)||_{L_T^q L_r} \leq CT^{1-\gamma_0} ||R_1||_{L^\infty} ||u||_{L_T^2} ||\nabla u||_{L_T^2} ||u||_{L_T^2} ||Au||_{L_T^2}$$

$$\leq CT^{1-\delta_0} \delta^3$$

and

$$||A((R_2*|u|^2)u)||_{L_T^1 L_2} \leq CT ||R_2||_{L^\infty} ||u||_{L_T^2} ||Au||_{L_T^\infty L^2}$$

$$\leq CT \delta^3,$$

where $A = \text{Id}$, $\nabla$ or $(V(x))^{1/2}$. By Strichartz’s estimate, we have

$$||Q[u]||_{L_T^p L^q} + ||Q[u]||_{L_T^p L_{r'}} \leq C ||u_0||_{L^2} + C \left( u \int_{\mathbb{R}^d} K(\cdot,y)|u(y)|^2dy \right)_{L_T^1 L^2}$$

$$+ C \left( (R_1*|u|^2)u \right)_{L_T^q L_r'} + C \left( (R_2*|u|^2)u \right)_{L_T^1 L^2}.$$
Since \( \sup_x |K(x, y)| \leq C \langle V(y) \rangle \) holds by Assumption (V3), we have
\[
\left\| u \int_{\mathbb{R}^d} K(\cdot, y)|u(y)|^2dy \right\|_{L^2} \leq \left\| u(x)K(x, y)|u(y)|^2 \right\|_{L^2_x(L^2_y)} \leq \left\| u(x)K(x, y)|u(y)|^2 \right\|_{L^2_x(L^2_y)} \leq C \left\| u \right\|_{L^2} \left\| (V(\cdot))^{1/2} u \right\|_{L^2}^2.
\]
Take \( L^2_T \) norm to yield
\[
\left\| u \int_{\mathbb{R}^d} K(\cdot, y)|u(y)|^2dy \right\|_{L^2_T} \leq CT \left\| u \right\|_{L^2_T} \left\| u \right\|_{L^2}^2.
\]
Hence
\[
\left\| Q[u] \right\|_{L^2_T L^2} + \left\| Q[u] \right\|_{L^2_T L^r} \leq C \left\| u_0 \right\|_{L^2} + C(T + T^{1/2}) \delta^3. \quad (3.4)
\]
We next estimate \( \nabla Q[u] \). By Lemma 3.6, one sees that
\[
\nabla Q[u] = e^{itA} \nabla u_0 + i \int_0^t \frac{\partial}{\partial s} e^{i(t-s)A} \nabla \left( u(s) \int_{\mathbb{R}^d} K(x, y)|u(s, y)|^2dy \right) ds
\]
\[+ i \int_0^t \frac{\partial}{\partial s} e^{i(t-s)A} \nabla \left( (R_s * |u|^2)u \right) ds
\]
\[+ iM \int_0^t e^{i(t-s)A} (\nabla V) Q[u](s) ds.
\]
By Strichartz’s estimate,
\[
\left\| \nabla Q[u] \right\|_{L^2_T L^2} + \left\| \nabla Q[u] \right\|_{L^2_T L^r}
\leq C \left\| \nabla u_0 \right\|_{L^2} + C \left\| \nabla \left( u(s) \int_{\mathbb{R}^d} K(x, y)|u(s, y)|^2dy \right) \right\|_{L^2_T L^2}
\leq C \left\| \nabla (\nabla V) Q[u] \right\|_{L^2_T L^2}.
\]
Using Assumption (V3), we obtain
\[
\left\| \left( \nabla K(\cdot, y) \right)|u(y)|^2dy \right\|_{L^2_T L^2} \leq C \left\| (V(\cdot))^{1/2} u \right\|_{L^2_T L^2} \left\| u \right\|_{L^2_T L^2},
\]
\[
\left\| \nabla u \right\|_{L^2_T L^2} \leq \left\| (V(\cdot))^{1/2} u \right\|_{L^2_T L^2} \left\| \nabla u \right\|_{L^2_T L^2}.
\]
It follows from Assumption (V2) that
\[
\left\| (\nabla V) Q[u] \right\|_{L^2_T L^2} \leq C \left\| (V(\cdot))^{1/2} Q[u] \right\|_{L^2_T L^2}.
\]
We hence deduce that
\[
\left\| Q[u] \right\|_{L^2_T L^2} + \left\| \nabla Q[u] \right\|_{L^2_T L^r}
\leq C \left\| u_0 \right\|_{L^2} + C(T + T^{1/2}) \delta^3 + CMT \left\| (V(\cdot))^{1/2} Q[u] \right\|_{L^2_T L^2}. \quad (3.5)
\]

Let us proceed to the estimate of $\langle V(x) \rangle^{1/2} Q[u]$. Apply the second identity of Lemma 3.6 with $F = \langle V(x) \rangle^{1/2}$ to yield

$$
\langle V(x) \rangle^{1/2} Q[u] 
= e^{tA} \langle V(x) \rangle^{1/2} u_0 
+ i \int_0^t e^{i(t-s)A} \langle V(x) \rangle^{1/2} u(s) \int_{\mathbb{R}^d} K(x, y) |u(s, y)|^2 dy ds 
+ i \int_0^t e^{i(t-s)A} \langle V(x) \rangle^{1/2} (R * |u|^2) u ds 
+ i \int_0^t e^{i(t-s)A} \left( \nabla \langle V(x) \rangle^{1/2} \cdot \nabla + \frac{1}{2} \Delta \langle V(x) \rangle^{1/2} \right) Q[u](s) ds.
$$

Now, Assumptions (V1) and (V3) give us that

$$
\text{Proposition 3.8. Suppose that Assumptions (V1) and (V3) are satisfied.}
\int_0^t e^{i(t-s)A} \langle V(x) \rangle^{1/2} u(s) \int_{\mathbb{R}^d} K(x, y) |u(s, y)|^2 dy ds 
+ i \int_0^t e^{i(t-s)A} \langle V(x) \rangle^{1/2} (R * |u|^2) u ds 
+ i \int_0^t e^{i(t-s)A} \left( \nabla \langle V(x) \rangle^{1/2} \cdot \nabla + \frac{1}{2} \Delta \langle V(x) \rangle^{1/2} \right) Q[u](s) ds.
$$

Hence,

$$
\left\| \langle V(x) \rangle^{1/2} Q[u] \right\|_{L_T^\infty L^2} + \left\| \langle V(x) \rangle^{1/2} Q[u] \right\|_{L_T^1 L^r}
\leq C \left\| \langle V(x) \rangle^{1/2} u_0 \right\|_{L_T^1 L^2} + C(T + T^{1 - \frac{d}{2}}) \delta^3 
+ CT \left\| \nabla Q[u] \right\|_{L_T^\infty L^2} + \left\| Q[u] \right\|_{L_T^\infty L^2}.
$$

From (3.4), (3.5), and (3.6), we finally reach to

$$
\|Q[u]\|_{H_T} \leq C_1 \|u_0\|_{\Sigma_{V}^{1,1}} + C_2(T + T^{1 - \frac{d}{2}}) \delta^3 + C_4 T \|Q[u]\|_{H_T}.
$$

Letting $T$ so small that $C_3 T \leq 1/2$, we see $\|Q[u]\|_{H_T} \leq 2C_1 \|u_0\|_{\Sigma_{V}^{1,1}} + 2C_2(T + T^{1 - \frac{d}{2}}) \delta^3$. We now choose $\delta_0 := 3C_1 \|u_0\|_{\Sigma_{V}^{1,1}}$. Then, for any $\delta \geq \delta_0$, $Q$ maps $H_{T, \delta}$ to itself, provided $T$ is so small that $2C_2(T + T^{1 - \frac{d}{2}}) \delta^2 \leq 1/3$.

**Step 2**

We next show that $Q$ is a contraction map. In the same way as in Step 1, we obtain

$$
\|Q[u] - Q[v]\|_{H_T} \leq C_4(T + T^{1 - \frac{d}{2}}) \left( \|u\|^2_{H_T} + \|v\|^2_{H_T} \right) \|u - v\|_{H_T}
$$

for small $T$ and $u, v \in H_{T, \delta}$. Letting $T$ so small that $2C_4(T + T^{1 - \frac{d}{2}}) \delta^2 \leq 1/2$ if necessary, we conclude that $\|Q[u] - Q[v]\|_{H_T} \leq (1/2) \|u - v\|_{H_T}$.

**Proposition 3.8.** Suppose that Assumptions (V1), (V2), (V3), (V4), and (R1) are satisfied. Let $u_0 \in \Sigma_{V}^{1,2}$. Define a Banach space

$$
I_{T, \delta} := \{ f \in L^\infty((-T, T); \Sigma_{V}^{1,2}); \| f \|_{I_T} \leq \delta \},
$$

where $\| f \|_{I_T} := \| f \|_{L^\infty_H \Sigma_{V}^{1,2}} + \| f \|_{L^2_H W^{2,2}}$ with $q = 8C/d$ and $r = 4C/(2\zeta - 1)$. Then, there exists $\delta_0$ depending only on $u_0$ such that for any $\delta \in [\delta_0, \infty)$, the operator $Q$ given in (3.3) is a contraction map from $I_{T, \delta}$ to itself for suitable $T = T(\delta)$. 
The proof is done in a similar way.

**Theorem 3.9.** (1) Suppose that Assumptions (V1), (V2), (V3), and (R1) are satisfied. Then, for any \( u_0 \in \Sigma_{V,1}^1 \), there exists \( T = T(||u_0||_{\Sigma_{V,1}^1}) \) and a unique solution \( u \in C([-T,T], \Sigma_{V,1}^1) \cap L^{8c/d}([-T,T], W^{1,1}_{V,\Sigma_{V,1}^1}) \) of \((mgH)\). The solution belongs to \( C^1([-T,T], (\Sigma_{V,1}^1)') \) and \( L^q([-T,T], W^{1,1}_{V,\Sigma_{V,1}^1}) \) for all admissible pair \((q,r)\), and conserves mass \( \|u(t)\|_{L^2} \). Moreover, the solution depends continuously on the data.

(2) If Assumptions (V1), (V2), (V3), (V4), and (R1) are satisfied and if

\[
\|u_0\|_{L^2}^2 = M, \quad X[u_0] = P[u_0] = 0 \quad (3.8)
\]

are fulfilled, then the solution given in (1) conserves energy \( E[u(t)] \) defined in (3.2) and momentum \( P[u(t)] \). In other words, the Cauchy problem \((mgH)\) is locally well-posed in \( \Sigma_{V,1}^1 := \{u_0 \in \Sigma_{V,1}^1 | u_0 \) satisfies (3.8)\}. Moreover, the solution \( u \) solves \((gH)\).

**Remark 3.10.** The theorem holds even if we replace Assumption (V2) with a weaker one: (V2') \(|\nabla V(x)| \leq C |V(x)|^{1/2} \).

**Proof.** For \( u_0 \in \Sigma_{V,1}^1 \), we choose \( \delta \) and \( T \) so that \( Q[u] \) becomes a contraction map from \( \mathcal{H}_{T,\delta} \) to itself. Then, we obtain a unique solution \( u \in C([-T,T], \Sigma_{V,1}^1) \cap L^{8c/d}([-T,T], W^{1,1}_{V,\Sigma_{V,1}^1}) \) of integral version of \((mgH)\). From the equation \((mgH)\), we see that \( u \in C^1((-T,T),(\Sigma_{V,1}^1)') \). Strichartz’s estimate gives a bound in \( L^q([-T,T], W^{1,1}_{V,\Sigma_{V,1}^1}) \) for all admissible pair. As in (3.7), one gets

\[
\|u - v\|_{\mathcal{H}_T} \leq C \|u(0) - v(0)\|_{\Sigma_{V,1}^1} + C(T + T^{-\frac{d}{4}})(\|u\|_{\mathcal{H}_T}^2 + \|v\|_{\mathcal{H}_T}^2) \|u - v\|_{\mathcal{H}_T} \quad (3.9)
\]

for any two solutions \( u \) and \( v \) of \((mgH)\). From this estimate, we deduce continuous dependence of the solution on the data. Now, a use of \((mgH)\) shows that

\[
\frac{d}{dt} \|u(t)\|_{L^2}^2 = 2 \Re \langle u, \partial_t u \rangle_{\Sigma_{V,1}^1,(\Sigma_{V,1}^1)'} = 0,
\]

which completes the proof of the first statement of the theorem.

We now suppose that Assumption (V4) holds and \( u_0 \) satisfies (3.8). To justify the momentum conservation, we use a regularization argument. Let \( \{u_{0,n}\}_n \) be a sequence of functions in \( \Sigma_{V,2}^2 \) which satisfies \( \|u_{0,n}\|_{L^2}^2 = M \) and converges to \( u_0 \) in \( \Sigma_{V,1}^1 \) as \( n \to \infty \), and let \( u_n \in C([-\tilde{T}_n, \tilde{T}_n], \Sigma_{V,2}^2) \cap C^1((\tilde{T}_n - \tilde{T}_n, \tilde{T}_n), L^2) \) be corresponding solutions of \((mgH)\) with \( u_n(0) = u_{0,n} \). Notice that \( \tilde{T}_n \geq T/2 \) for large \( n \) since \( \|u_{0,n}\|_{\Sigma_{V,1}^1} \) converges to \( \|u_0\|_{\Sigma_{V,1}^1} \) as \( n \to \infty \). Thanks to (3.9), \( u_n \) converges to \( u \) in

\[
C([-T/2, T/2], \Sigma_{V,1}^1) \cap C^1((-T/2, T/2), (\Sigma_{V,1}^1)') \quad (3.10)
\]

It therefore holds for each \( n \) that

\[
\frac{d}{dt} P[u_n(t)] = \text{Im} \int_{\mathbb{R}^d} \frac{\partial_t u_n(t,y) \nabla u_n(t,y)}{y} dy + \text{Im} \int_{\mathbb{R}^d} \frac{u_n(t,y) \nabla \partial_t u_n(t,y)}{y} dy,
\]
which makes sense because $\nabla \partial_t u_n \in H^{-1}$. Integrating by parts and plugging 
\((mgH)\), we see that 
\[
\frac{d}{dt} P[u_n(t)] = \int_{\mathbb{R}^d} |u_n(t, x)|^2 \nabla (W(x) \cdot X[u_n(t)]) \, dx. \tag{3.11}
\]
Notice that the right hand side makes sense because $|\nabla W_j(x)| \leq C (V(x))^{\frac{1}{2}}$ is valid and $\Sigma_{V^{1,1}} \subset \Sigma_{V^{1,1/2}}$ holds by virtue of (V4), where $W_j$ denotes the $j$-th component of $W$. Indeed, take $y = e_j$ in (3.1) and differentiate with respect to $x$ to yield $\nabla W_j(x) = \nabla K(x, e_j) - \nabla V(x - e_j) + \nabla V(x)$. By Assumptions (V1), (V2), and (V3), one obtains $|\nabla W_j(x)| \leq C (V(x))^{1/2}$. We also have 
\[
\frac{d}{dt} X[u_n(t)] = P[u_n(t)]. \tag{3.12}
\]
Two estimates (3.11) and (3.12) give us 
\[
\frac{d}{dt} \left( |X[u_n(t)]|^2 + |P[u_n(t)]|^2 \right) \leq C \left( 1 + \|u_n(t)\|_{\Sigma_{V^{1,1}}}^2 \right) \left( |X[u_n(t)]|^2 + |P[u_n(t)]|^2 \right).
\]
Applying Gronwall’s lemma, we conclude that 
\[
|X[u_n(t)]|^2 + |P[u_n(t)]|^2 \leq C \delta^2 e^{T(1+\delta^2)} \|u_n - u_0\|_{\Sigma_{V^{1,1}}}^2, \tag{3.13}
\]
for $t \in (-T, T)$, where we have used the estimates 
\[
|X[u_n(0)]| = |X[u_{0,n}] - X[u_0]| = \left| \int_{\mathbb{R}^d} y(|u_{0,n}(y)|^2 - |u_0(y)|^2) dy \right| \leq C \|u_{0,n} - u_0\|_{\Sigma_{V^{1,1}}} \left( \|u_{0,n}\|_{\Sigma_{V^{1,1}}} + \|u_0\|_{\Sigma_{V^{1,1}}} \right)
\]
and 
\[
|P[u_n(0)]| = |P[u_{0,n}] - P[u_0]| = \left| \int_{\mathbb{R}^d} \xi (|F_{u_{0,n}}(\xi)|^2 - |F_{u_0}(\xi)|^2) d\xi \right| \leq \|u_{0,n} - u_0\|_{\Sigma_{V^{1,1}}} \left( \|u_{0,n}\|_{\Sigma_{V^{1,1}}} + \|u_0\|_{\Sigma_{V^{1,1}}} \right).
\]
Convergence of $u_n$ in the topology (3.10) allows us to claim $X[u(t)] \equiv P[u(t)] \equiv 0$ for $t \in (-T/2, T/2)$. By this fact and the mass conservation, the right hand side of $(mgH)$ becomes 
\[
- u \left( (V * |u|^2) - V(x) \|u_0\|_{L^2}^2 + W(x) \cdot X[u] \right) = -(V * |u|^2) u + MV(x) u - 0.
\]
Therefore, $u$ solves $(gH)$. Let us again consider the above sequence $\{u_{0,n}\}_n \in \Sigma_{V^{2,2}}$ approximating $u_0$ and the corresponding sequence of solutions $\{u_n\}_n$ to $(mgH)$. Conservation of mass allows us to rewrite $(mgH)$ into 
\[
\frac{i}{2} \partial_t u_n + \frac{1}{2} \Delta u_n = -((V + R) * |u_n|^2) u_n - W \cdot X[u_n] u_n.
\]
Multiplying this equality by $\partial_t u_n$ and integrating real parts of the resulting terms, we get
\[
\frac{d}{dt}E[u_n(t)] = \text{Re} \int W_n \partial_t u_n dx \cdot X[u_n].
\]
This calculation makes sense because $\partial_t u_n$ is a continuous $L^2$-valued function and $u_n$ satisfies (mgH) in $L^2$ sense. Since $\text{Re} \int W_j u_n \nabla W_j \cdot \nabla u_n dx$, one sees from (3.13) that
\[
|E[u_n(t)] - E[u_{0,n}]| \leq \frac{C_0^3}{1 + \delta^2} \langle 1 + \delta^2 \rangle u_n - u_0\|_{\Sigma^1_{\nu}} \to 0
\]
as $n \to \infty$ for $t \in (-T,T)$. By means of the convergence of $u_n$ in (3.10), $|E[u(t)] - E[u(0)]| \leq |E[u(t)] - E[u_n(t)]| + |E[u_n(t)] - E[u_{0,n}]| + |E[u_{0,n}] - E[u_0]| \to 0$ as $n \to \infty$ for $t \in (-T, T/2)$, which gives us the desired conservation law.

### 3.2 Global well-posedness of (mgH)

We next extend the above solution of (gH) to whole real line $\mathbb{R}$.

**Lemma 3.11.** Suppose Assumption (V2) is satisfied. Let $u \in C((-T,T); \Sigma^1_{\nu})$ be a solution to (gH) with data $u_0 \in \Sigma^1_{\nu}$. Then, it holds that
\[
\left\| (V(\cdot))^{1/2} u \right\|_{L^2} \leq C(t) \left( \|u_0\|_{\Sigma^1_{\nu}} \right) \left( 1 + \|u\|_{L^\infty_t L^2} \right)^{\frac{1}{2} + \kappa}
\]
for $t \in (-T,T)$, where $\kappa$ is the number defined in Assumption (V2).

**Proof.** Let us estimate $\|\sqrt{V} u\|_{L^2}$ because $\|u\|_{L^2}$ is conserved. For $r > 0$, we have
\[
\frac{d}{dt} \left\| 1_{\{|x| \leq r\}} \sqrt{V} u(t) \right\|_{L^2}^2 = 2 \text{Re} \left\langle 1_{\{|x| \leq r\}} |V| u, \partial_t u \right\rangle = \text{Im} \left\langle 1_{\{|x| \leq r\}} |V| u, \Delta u \right\rangle
\]
\[
= \text{Im} \int_{\{|x| \leq r\}} (\nabla u \cdot \nabla |V|) u dx,
\]
which implies
\[
\left| \frac{d}{dt} \left\| 1_{\{|x| \leq r\}} \sqrt{V} u(t) \right\|_{L^2}^2 \right| \leq \left\| 1_{\{|x| \leq r\}} \nabla V \right\|_{L^2} \left\| \nabla u(t) \right\|_{L^2}^2.
\]
Integrating in time and substituting $|\nabla V| \leq C|V|^\kappa/2$ give us
\[
\left\| 1_{\{|x| \leq r\}} \sqrt{V} u \right\|_{L^2}^2 \leq C \|u_0\|_{\Sigma^1_{\nu}}^2 + |t| \left\| \nabla u \right\|_{L^\infty_t L^2} \left\| 1_{\{|x| \leq r\}} |V|^{1/2} u \right\|_{L^\infty_t L^2} \left\| 1_{\{|x| \leq r\}} |V|^{1/2} u \right\|_{L^\infty_t L^2}^{1-\kappa}.
\]
Since $r$ is arbitrary, we pass to the limit $r \to \infty$ and reach to
\[
\left\| (V)^{1/2} u \right\|_{L^\infty_t L^2}^2 \leq C \|u_0\|_{\Sigma^1_{\nu}}^2 + C|t| \left\| \nabla u \right\|_{L^\infty_t L^2} \left\| u_0 \right\|_{L^2}^{1-\kappa} \left\| (V)^{1/2} u \right\|_{L^\infty_t L^2}^\kappa.
\]
(3.14)
It follows from Young’s inequality that

\[
C|t| \|\nabla u\|_{L^\infty_t L^2_x} \|u_0\|_{H^1}^{1/2} |u_0|^{1/2} \|u_0\|^\kappa \leq \frac{1}{2} \|\langle V \rangle \|_{L^\infty_t L^2_x}^2 + C_r (|t| \|\nabla u\|_{L^\infty_t L^2_x} \|u_0\|_{H^1}^{1/2} |u_0|^{1/2} \|u_0\|^\kappa)^{2/\kappa}.
\]

Plugging this estimate to (3.14), we conclude that

\[
\|\langle V \rangle \|_{L^\infty_t L^2_x}^2 \leq C \|u_0\|_{H^1}^2 + C \|u_0\|_{L^2}^{2\kappa - 1} \|\nabla u\|_{L^\infty_t L^2_x} \|u_0\|_{H^1}^{1/2} |u_0|^{1/2} \|u_0\|^\kappa.
\]

A blow-up criterion immediately follows from this lemma.

**Corollary 3.12 (Blow-up criterion).** Suppose Assumptions (V1), (V2), (V3), (V4), and (R1) and (3.8) are satisfied. Let \( u \in C((-T_{\min}, T_{\max}); \Sigma_{V_{\kappa}}^{1,1}) \) be a maximal solution to (gH). If \( T_{\max} < \infty \) (resp. \( T_{\min} < \infty \)) then \( \|\nabla u(t)\|_{L^2} \to \infty \) as \( t \uparrow T_{\max} \) (resp. as \( t \downarrow -T_{\min} \)).

Now, the following lemma shows that (mgH) is globally well-posed in \( \Sigma_{V_{\kappa}}^{1,1} \).

**Lemma 3.13 (Global \( H^1 \)-bound).** Suppose (V1), (V2), (V3), (V4), (R1), and (R2). Let \( u_0 \in \Sigma_{V_{\kappa}}^{1,1} \) and let \( u \in C((-T_{\min}, T_{\max}); \Sigma_{V_{\kappa}}^{1,1}) \) be a corresponding maximal solution to (mgH) which conserves energy \( E[u(t)] \) and momentum \( P[u(t)] \). Then, we have the following bounds for \( t \in (-T_{\min}, T_{\max}) \):

- If \( V \leq 0 \) then, \( \|\nabla u(t)\|_{L^2} \leq C \);
- otherwise, \( \|\nabla u(t)\|_{L^2} \leq C \langle t \rangle^{\kappa/2} \), where \( \kappa \) is the number defined in Assumption (V2).

**Proof.** If \( V \leq 0 \) then

\[
\|\nabla u(t)\|_{L^2}^2 \leq 2E[u(t)] + \frac{1}{2} \|R^+ + |u|^2\|_{L^1} \nabla u(t),
\]

\[
\leq 2E[u_0] + C \|R^+\|_{L^p} \|\nabla u(t)\|_{L^2} \|u_0\|_{L^2}^{4/\kappa - \frac{2}{\kappa}} + \frac{1}{2} \|R^+\|_{L^\infty} \|u_0\|_{L^2}^4.
\]

This gives \( \|\nabla u(t)\|_{L^2} \leq C \) since \( \theta > d/2 \). Otherwise, we have

\[
\|\nabla u(t)\|_{L^2}^2 \leq 2E[u_0] + \frac{1}{2} \int_{\mathbb{R}^{d+4}} V(x - y)|u(t, x)|^2|u(t, y)|^2dxdy
\]

\[
+ \frac{1}{2} \|R^+ + |u|^2\|_{L^1}.
\]
Since \( u_0 \in \tilde{\Sigma}^{1,1}_V \), we have \( X[u(t)] = 0 \). Therefore, Assumption (V3) yields
\[
\int_{\mathbb{R}^{d+1}} V(x-y)|u(t,x)|^2|u(t,y)|^2\,dxdy = \int_{\mathbb{R}^{d+1}} K(x,y)|u(t,x)|^2|u(t,y)|^2\,dxdy \]
\[
+ \int_{\mathbb{R}^d} V(x)|u(t,x)|^2\,dx \int_{\mathbb{R}^d} |u(t,y)|^2\,dy \]
\[
+ \int_{\mathbb{R}^d} W(x)|u(t,x)|^2\,dx \cdot X[u(t)] \]
\[
\leq C \|u_0\|_{L^2}^2 \|\langle V(\cdot)\rangle^{1/2} u(t)\|_{L^2}^2.
\]
Since \( u \) solves \((gH)\), plugging the estimate of Lemma 3.11, one sees that
\[
\|\nabla u\|_{L^\infty L^2} \leq C + C|t|^{\frac{2}{d-2}} \|\nabla u\|_{L^\infty L^2}^2 + C \|u(t)\|_{L^2}^\frac{2}{q}.
\]
By Young’s inequality,
\[
\|\nabla u\|_{L^\infty L^2}^2 \leq C + \left( \frac{1}{4} \|\nabla u\|_{L^\infty L^2}^2 + C_\kappa \left( \frac{C|t|^{\frac{2}{d-2}}}{\kappa} \right)^{\frac{d-2}{4}} \right) \]
\[
+ \left( \frac{1}{4} \|\nabla u\|_{L^\infty L^2}^2 + C_\theta \right).
\]
Thus, we conclude that \( \|\nabla u\|_{L^\infty L^2} \leq C \langle t \rangle^{\frac{1}{d-2}}. \]

### 3.3 Proof of Theorem 3.4

We have shown that \((mgH)\) is globally well-posed in \( \tilde{\Sigma}^{1,1}_V \). Let us next extend the global existence of solution to \((gH)\) for a general data, that is, for a data which does not necessarily satisfy (3.8).

**Proof of Theorem 3.4.** Take a nonzero \( u_0 \in \Sigma^{1,1}_V \) and define a positive constant \( M \) and \( d \)-dimensional vectors \( a \) and \( b \) as in (1.3) and (1.4), respectively. Set \( v_0(x) = (\pi_{-a} - b u_0)(x) \). One easily verifies that \( v_0 \) satisfies (3.8). Namely, \( v_0 \in \Sigma^{1,1}_V \). Then, applying Theorem 3.9 (2) and Lemma 3.13, we obtain a global solution \( \tilde{u} \) of \((gH)\) which conserves the mass, the energy, and the momentum. The solution depends continuously on \( v_0 \), and so on \( u_0 \) (Recall that \( e^{itA_\phi} \) is continuous with respect to the parameter \( M \) in \( A \)). We now define a function \( u \) by 
\[
\begin{align*}
\tilde{u}(t) &= \exp(itA_\phi) \tau_{a\pi t + b} \pi_{a} \tilde{u} \text{ as in (1.7). Then, } u \text{ belongs to the same class as } \tilde{u} \text{ and solves } (gH) \text{ with } u(0) = u_0. \text{ The solution } u \text{ conserves the mass because }
\end{align*}
\]
\[
\|u(t)\|_{L^2} = \|\tilde{u}(t)\|_{L^2} = \|v_0\|_{L^2} = \|u_0\|_{L^2}.
\]

Similarly,
\[
\|\nabla u(t)\|_{L^2}^2 = \|\nabla \tilde{u}(t) + i\tilde{u}(t)\|_{L^2}^2
\]
\[
= \|\nabla \tilde{u}(t)\|_{L^2}^2 - 2a \cdot P[\tilde{u}(t)] + |a|^2 \|\tilde{u}(t)\|_{L^2}^2
\]
and
\[ \int \int_{\mathbb{R}^{2d}} V(x-y)|u(t,x)|^2u(t,y)|^2\,dxdy = \int \int_{\mathbb{R}^{2d}} V(x-y)|\tilde{u}(t,x)|^2|\tilde{u}(t,y)|^2\,dxdy \]
hold. These give us the conservation of energy
\[ E[u(t)] = E[\tilde{u}(t)] - a \cdot P[\tilde{u}(t)] + \frac{1}{2}|a|^2 \|\tilde{u}(t)\|_{L^2}^2 \]
where
\[ \chi \]
and the conservation of momentum
\[ P[u(t)] = \text{Im} \int_{\mathbb{R}^{d}} \overline{\tilde{u}(t,y)}(\nabla \tilde{u}(t,y) + ia\tilde{u}(t,y))\,dy \]
\[ = P[\tilde{u}(t)] + a \|\tilde{u}(t)\|_{L^2}^2 = P[v_0] + aM = P[u_0]. \]
The estimate on \( \|\nabla u(t)\|_{L^2} \) is given in Lemma 3.13, and then the estimate on \( \|(V\cdot)^{1/2} u(t)\|_{L^2} \) follows from Lemma 3.11.

So far, we have shown all the statement except for the uniqueness. Let \( w \in L^\infty([-T,T],\Sigma_{V^{-1}}^{1}) \cap L^{8\zeta/d}([-T,T],W^{1,1}_{V^{-2\gamma}}(\mathbb{R},\mathbb{R}^{4\zeta})) \) be another solution of (gH) with \( u(0) = u_0 \) which conserves \( P[w(t)] \). Then, \( \partial_t w \in L^\infty((-T,T),(\Sigma_{V^{-1}}^{1})') \) holds from (gH). We define \( \tilde{v} \in L^\infty([-T,T],\Sigma_{V^{-1}}^{1}) \cap L^{8\zeta/d}_{lo}(\mathbb{R},W^{1,1}_{V^{-2\gamma}}(\mathbb{R},\mathbb{R}^{4\zeta})) \) as in (1.7). Then, \( \tilde{v} \) is also a solution of (gH) and conserves the momentum. Since \( X[\tilde{v}(0)] = P[\tilde{v}(0)] = 0 \), we can see that \( X[\tilde{v}(t)] = 0 \). Now, \( \partial_t \tilde{v} \in L^\infty((-T,T),(\Sigma_{V^{-1}}^{1})') \). Multiply (gH) by \( \overline{\tilde{v}} \) and integrate its imaginary part to yield \( \|\tilde{v}(t)\|_{L^2}^2 = \|\tilde{v}(0)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \). Then, we deduce that \( \tilde{v} \) solves (mgH). By the uniqueness of (mgH), we obtain \( \tilde{v} = v \). Back to the transform (1.7), this implies \( w = u \).

### 3.4 Proof of Theorem 1.1

Now, we are in a position to complete the proof of our main theorem. Set
\[ V(x) = \lambda|x|^\gamma \chi(x), \quad R(x) = \lambda|x|^\gamma(1 - \chi(x)), \]
where \( \chi \) is a smooth radial non-decreasing (with respect to \(|x|\)) function such that \( \chi \equiv 1 \) for \(|x| \geq 2 \) and \( \chi \equiv 0 \) for \(|x| \leq 1 \). One immediately sees that \( R \in L^\infty \) and Assumptions (R1) and (R2) are fulfilled with \( \zeta = \theta = \infty \). Thus, Theorem 1.1 follows from Theorem 3.4 if we prove that \( V \) satisfies Assumptions (V1), (V2), (V3), and (V4). We shall demonstrate merely (V3) since the others are trivial. Remark that (V2) holds with \( \kappa = \frac{2(\gamma - 1)}{\gamma} \) and that \( \kappa < 1 \) if and only if \( \gamma < 2 \).

**Lemma 3.14.** Let \( \gamma \in (1,2] \) and
\[ \tilde{K}(x,y) = |x-y|^{\gamma - |x|\gamma} + |x|\gamma^{-2}x \cdot y. \]
There exists a positive constant \( C \) depending only on \( \gamma \) such that
\[ \sup_{x \in \mathbb{R}^d} |\tilde{K}(x,y)| \leq C|y|^\gamma. \]
Proof. The case \( y = 0 \) is trivial. We hence fix \( \mathbb{R}^d \ni y \neq 0 \). It immediately follows that

\[
\sup_{x, |x| \leq |y|} |\tilde{K}(x, y)| \leq 3^\gamma |y|^\gamma + 2^\gamma |y|^\gamma + \gamma 2^{\gamma - 1} |y|^\gamma \leq C |y|^\gamma.
\]

We now consider the case \( |x| \geq 2|y| \). An elementary computation shows that \( K \) is written as

\[
\tilde{K}(x, y) = \int_0^1 \frac{\partial}{\partial a} |x - ay|^\gamma da + \gamma |x|^{-2} x \cdot y
\]

\[
= - \int_0^1 |x - ay|^{-2} y \cdot (x - ay) da + \gamma |x|^{-2} x \cdot y
\]

\[
= \gamma |y|^2 \int_0^1 a |x - ay|^{-2} da - \gamma \cdot y \int_0^1 \int_0^1 \frac{\partial}{\partial b} |x - bay|^{-2} db da
\]

\[
= \gamma |y|^2 \int_0^1 a |x - ay|^{-2} da
\]

\[
+ \gamma (\gamma - 4) x \cdot y \int_0^1 \int_0^1 |x - bay|^{-4} ay \cdot (x - bay) db da
\]

\[
= \tilde{K}_1(x, y) + \tilde{K}_2(x, y).
\]

For any integer \( m \geq 2 \), it holds that

\[
\sup_{x, |x| \geq 2|y|} |\tilde{K}_1(x, y)| \leq \gamma |y|^2 \int_0^1 a ((m - 1)|y|)^{-2} da = \frac{\gamma}{2(m - 1)^{2-\gamma}} |y|^\gamma.
\]

Therefore,

\[
\sup_{x, |x| \geq 2|y|} |\tilde{K}_1(x, y)| \leq \sup_{m \geq 2} \frac{\gamma}{2(m - 1)^{2-\gamma}} |y|^\gamma = \frac{\gamma}{2} |y|^\gamma.
\]

Similarly, we have

\[
\sup_{x, |x| \leq (m + 1)|y|} |\tilde{K}_2(x, y)|
\]

\[
\leq \gamma (4 - \gamma) (m + 1)|y|^2 \int_0^1 \int_0^1 ((m - 1)|y|)^{-4} a ((m + 1)|y|^2) db da
\]

\[
= \frac{\gamma (4 - \gamma) (m + 1)^2}{2(m - 1)^{4-\gamma}} |y|^\gamma.
\]

Since \( \sup_{m \geq 2} (m + 1)^2 (m - 1)^{4-\gamma} = 3^2 \), we conclude that

\[
\sup_{x, |x| \geq 2|y|} |\tilde{K}_2(x, y)| \leq \sup_{m \geq 2} \frac{\gamma (4 - \gamma) (m + 1)^2}{2(m - 1)^{4-\gamma}} |y|^\gamma = \frac{9\gamma (4 - \gamma)}{2} |y|^\gamma,
\]

which completes the proof. \( \square \)

**Proposition 3.15.** Let \( V \) be as in (3.15). If \( \gamma \in (1, 2) \) then \( V \) satisfies Assumption (V3) with \( W(x) = -\lambda \gamma x \langle x \rangle^{\gamma - 2} \). More precisely, if we put

\[
K(x, y) = \lambda \chi(|x - y|)|x - y|^{-\gamma} - \lambda \chi(|x|)|x|^\gamma + \lambda \gamma \langle x \rangle^{\gamma - 2} x \cdot y,
\]

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then there exists a positive constant $C$ depending only on $\gamma$ and $\lambda$ such that

$$\sup_{x \in \mathbb{R}^d} |K(x, y)| \leq C \langle y \rangle^{\gamma}, \quad \sup_{x \in \mathbb{R}^d} |\nabla_x K(x, y)| \leq C \langle y \rangle.$$

**Proof.** For simplicity, let $\lambda = 1$. Let $\tilde{K}$ be as in Lemma 3.14. We deduce that

$$\sup_{x \in \mathbb{R}^d} |\tilde{K}(x, y) - K(x, y)| \leq 2^{1+\gamma} + \sup_{x \in \mathbb{R}^d} |\gamma(|x|^{\gamma-2} - \langle x \rangle^{\gamma-2})x \cdot y|.$$ 

Let us estimate the second term of the right hand side. An elementary calculation shows

$$|x|^\nu - \langle x \rangle^\nu = -\int_0^1 \partial_n (a + |x|^2)^{\frac{\nu}{2}} da = -\frac{\nu}{2} \int_0^1 (a + |x|^2)^{\frac{\nu}{2} - 1} da$$ 

for any $\nu$, and so

$$\sup_{|x| \geq 1} |\gamma(|x|^{\gamma-2} - \langle x \rangle^{\gamma-2})x \cdot y| \leq \frac{\gamma(2 - \gamma)}{2} |y| \leq C \langle y \rangle.$$

It is obvious that

$$\sup_{|x| \geq 1} |\gamma(|x|^{\gamma-2} - \langle x \rangle^{\gamma-2})x \cdot y| \leq C \langle y \rangle.$$

Then, the first inequality follows from Lemma 3.14.

Let us proceed to the second inequality. Notice that

$$\nabla_x K(x, y) = \gamma|x - y|^\gamma - \gamma(x - y)\chi(|x - y|) + |x - y|^\gamma \nabla \chi(|x - y|)$$

$$- \gamma|x - y|^{\gamma-2}x \chi(|x|) - |x|^{\gamma-2} \nabla \chi(|x|)$$

$$+ \gamma \langle x \rangle^{\gamma-2}y + \gamma(\gamma - 2) \langle x \rangle^{\gamma-4} \langle x \cdot y \rangle x.$$ 

Since $\nabla \chi(|x|) = 0$ for $|x| \leq 1$ and $|x| \geq 2$, it holds that

$$\sup_{x \in \mathbb{R}^d} ||x - y|^{\gamma-2} \nabla \chi(|x - y|)| + ||x|^{\gamma-2} \nabla \chi(|x|)| \leq 2^{\gamma+1} \|
abla \chi\|_{L^\infty}.$$ 

We also deduce that

$$\sup_{x \in \mathbb{R}^d} |\langle x \rangle^{\gamma-2}y| + |\langle x \rangle^{\gamma-4} \langle x \cdot y \rangle x| \leq C|y|$$

for $\gamma \in (1, 2)$. Now, It holds that

$$|x - y|^{\gamma-2} (x - y)\chi(|x - y|) - |x|^{\gamma-2} x \chi(|x|)$$

$$= \int_0^1 \partial_n |x - ay|^{\gamma-2} (x - ay) \chi(|x - ay|) da$$

$$= (2 - \gamma) \int_0^1 |x - ay|^{\gamma-4} y \cdot (x - ay)(x - ay)\chi(|x - ay|) da$$

$$- \int_0^1 |x - ay|^{\gamma-2} y \chi(|x - ay|) da$$

$$- \int_0^1 |x - ay|^{\gamma-3} y \cdot (x - ay)\chi'(|x - ay|) da.$$ 

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Notice that \( \sup_{x \in \mathbb{R}^d} |x|^{\gamma - 2} \chi(|x|) \leq 1 \). Hence, the first term of the right hand side of the equality is estimated as

\[
\sup_{x \in \mathbb{R}^d} \int_0^1 |x - ay|^{\gamma - 4} (x - ay)(x - ay)\chi(|x - ay|) \, da 
\leq |y| \sup_{0 \leq a \leq 1} \sup_{x \in \mathbb{R}^d} |x - ay|^{\gamma - 2} \chi(|x - ay|) \leq |y|.
\]

One can obtain similar estimates for other terms. Thus, we conclude that

\[
\sup_{x \in \mathbb{R}^d} |x - y|^{\gamma - 2} \chi(|x - y|) - |x|^{\gamma - 2} \chi(|x|) \leq C|y|.
\]

\( \square \)

### A Global well-posedness of \((\text{nH})\) for \( \gamma \in (0, 1] \)

Here, we adapt the abstract theory established in Section 3 to the case \( \gamma \in (0, 1] \), which is characterized as the case where Assumption (V3) is satisfied with \( W = 0 \). In such a special setting, results in Section 3 become better. This is because we do not need the conservation of momentum any longer. As a result, Assumption (V4) can be removed and the uniqueness holds in the class in which the solution lies (without the conservation of momentum).

**Theorem A.1.** Let \( d \geq 1, \gamma \in (0, 1] \), and \( \lambda \in \mathbb{R} \). Then, \((\text{nH})\) is globally well-posed in \( \Sigma^{1,\gamma/2} \). Moreover, the uniqueness holds unconditionally.

This theorem is an immediate consequence of the following.

**Theorem A.2.** Let Assumptions (V1), (V2), (R1), and (R2) be satisfied. Also suppose that (V3) holds with \( W = 0 \). Then, \((\text{gH})\) is globally well-posed in \( \Sigma^{1,1}_{\lambda,\gamma} \).

**Remark A.3.** In Theorem A.2, the uniqueness holds in \( u \in C(\mathbb{R}, \Sigma^{1,1}_{\lambda,\gamma}) \cap L^{8/d}_{\text{loc}}(\mathbb{R}, W^{1,1}_{\frac{4}{\gamma - 1},\gamma}) \), where \( \zeta \) is the number defined in Assumption (R1). If we can choose \( \zeta = \infty \), then the uniqueness holds unconditionally.

**Proof of Theorem A.2.** We choose \( W = 0 \) and let \( u \) be the unique local solution of \((\text{mgH})\) given in Theorem 3.9 (1). By the conservation of mass, the right hand side of \((\text{mgH})\) becomes

\[
-u \left( (V * |u|^2) - V(x) \|u_0\|^2_{L^2} \right) = -(V * |u|^2)u + MV(x)u.
\]

Hence \( u \) solves \((\text{gH})\). Energy conservation follows from this fact as in the proof of Theorem 3.9 (2).

By Lemma 3.11 and energy conservation, we have

\[
\|\nabla u\|_{L^\infty \cap L^2} \leq C + C|t|^{\frac{2}{d - 4}} \|\nabla u\|_{L^\infty \cap L^2}^{\frac{2}{d - 4}} + C \|\nabla u\|_{L^\infty \cap L^2}^{\frac{d}{d - 4}}
\]

as in the proof of Lemma 3.13. This yields \( \|\nabla u\|_{L^\infty \cap L^2} \leq C \langle t \rangle^{\frac{2}{d - 4}} \). Hence, again by Lemma 3.11, one sees that \( \|u(t)\|_{\Sigma^{1,1}_{\lambda,\gamma}} \) never blows up in finite time.

We shall prove the uniqueness. Let \( v \in C(\mathbb{R}; \Sigma^{1,1}_{\lambda,\gamma}) \) be another solution of \((\text{gH})\). One verifies that \( v \) conserves mass and so that \( v \) solves \((\text{mgH})\). Thus, we conclude from the uniqueness of \((\text{mgH})\) (given in Theorem 3.9 (1)) that \( u = v \) follows. \( \square \)
Proof of Theorem A.1. Take a non-decreasing function $\chi$ so that $\chi(r) = 0$ for $r \leq 1$ and $\chi(r) = 1$ for $r \geq 2$. We put $V(x) = \lambda|x|^\gamma \chi(|x|)$ and $R(x) = \lambda|x|^\gamma (1 - \chi(|x|))$. It is obvious that $R \in L^\infty$ satisfies (R1) and (R2) with $\zeta = \theta = \infty$ and that $V$ satisfies Assumption (V1). We infer that (V2) is fulfilled with $\kappa = 0$. Furthermore, (V3) follows with $W = 0$ from the estimates $\|V(x - y) - |x - y|^\gamma\|_{L^\infty} \leq 2^\gamma$, $\|V(x) - |x|^\gamma\|_{L^\infty} \leq 2^\gamma$, and $\||x - y|\gamma - |x|^\gamma\|_{L^\infty} \leq |y|^\gamma$. Hence, Theorem A.1 follows from Theorem A.2. Uniqueness holds unconditionally since we can chose $\zeta = \infty$ (see Remark A.3).

We can also obtain results on
\[
\begin{cases}
  \imath \partial_t u + \frac{1}{2} \Delta u = -\lambda (\log |x| * |u|^2) u, \\
  u(0) = u_0,
\end{cases}
\]
(A.1)
where $(t,x) \in \mathbb{R}^{1+d}$.

**Theorem A.4.** Let $d \geq 1$ and $\lambda \in \mathbb{R}$. Then (A.1) is globally well-posed in $\{f \in H^1; \langle \log (x) \rangle^{1/2} f \in L^2\}$.

This reproduce [14, Theorem 1.1] when $d = 2$. The proof is similar to that of Theorem A.1. We choose $V(x) = \lambda \chi(|x|) \log |x|$ and $W(x) = \lambda (1 - \chi(|x|)) \log |x|$.

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**References**

[1] R. Carles, *Remarks on nonlinear Schrödinger equations with harmonic potential*, Ann. Henri Poincaré 3 (2002), no. 4, 757–772.

[2] ———, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal. 35 (2003), no. 4, 823–843 (electronic).

[3] ———, *Global existence results for nonlinear Schrödinger equations with quadratic potentials*, Discrete Contin. Dyn. Syst. 13 (2005), no. 2, 385–398.

[4] R. Carles, N. J. Mauser, and H. P. Stimming, *Semiclassical limit of the Hartree equation with harmonic potential*, SIAM J. Appl. Math 66 (2005), no. 1, 29–56.

[5] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.

[6] M. De Leo and D. Rial, *Well posedness and smoothing effect of Schrödinger-Poisson equation*, J. Math. Phys. 48 (2007), no. 9, 093509, 15.
[7] G. Furioli, F. Planchon, and E. Terraneo, *Unconditional well-posedness for semilinear Schrödinger and wave equations in $H^s$*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), Contemp. Math., vol. 320, Amer. Math. Soc., Providence, RI, 2003, pp. 147–156.

[8] G. Furioli and E. Terraneo, *Besov spaces and unconditional well-posedness for the nonlinear Schrödinger equation in $\dot{H}^s(\mathbb{R}^n)$*, Commun. Contemp. Math. 5 (2003), no. 3, 349–367.

[9] J. Ginibre and T. Ozawa, *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$*, Comm. Math. Phys. 151 (1993), no. 3, 619–645.

[10] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations with nonlocal interaction*, Math. Z. 170 (1980), no. 2, 109–136.

[11] T. Kato, *On nonlinear Schrödinger equations. II. $H^s$-solutions and unconditional well-posedness*, J. Anal. Math. 67 (1995), 281–306.

[12] R. Killip, M. Visan, and X. Zhang, *Energy-critical NLS with quadratic potentials*, Comm. Partial Differential Equations 34 (2009), no. 12, 1531–1565.

[13] S. Masaki, *Local existence and WKB approximation of solutions to Schrödinger-Poisson system in the two-dimensional whole space*, Comm. Partial Differential Equations, to appear.

[14] , *Energy solution to Schrödinger-Poisson system in the two-dimensional whole space*, archived as arXiv:1001.4308, 2010.

[15] Y.-G. Oh, *Cauchy problem and Ehrenfest’s law of nonlinear Schrödinger equations with potentials*, J. Differential Equations 81 (1989), no. 2, 255–274.

[16] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[17] H. Steinrück, *The one-dimensional Wigner-Poisson problem and its relation to the Schrödinger-Poisson problem*, SIAM J. Math. Anal. 22 (1991), no. 4, 957–972.

[18] H. P. Stimming, *The IVP for the Schrödinger-Poisson-Xo equation in one dimension*, Math. Models Methods Appl. Sci. 15 (2005), no. 8, 1169–1180.

[19] Y. Y. S. Win and Y. Tsutsumi, *Unconditional uniqueness of solution for the Cauchy problem of the nonlinear Schrödinger equation*, Hokkaido Math. J. 37 (2008), no. 4, 839–859.

[20] H. Wu and J. Zhang, *Energy-critical Hartree equation with harmonic potential for radial data*, Nonlinear Anal. 72 (2010), no. 6, 2821–2840.

[21] K. Yajima, *Smoothness and non-smoothness of the fundamental solution of time dependent Schrödinger equations*, Comm. Math. Phys. 181 (1996), no. 3, 605–629.
[22] X. Zhang, *Global wellposedness and scattering for 3D energy critical Schrödinger equation with repulsive potential and radial data*, Forum Math. 19 (2007), no. 4, 633–675.