LIOUVILLE THEOREMS FOR STABLE SOLUTIONS OF THE WEIGHTED LANE-EMDEN SYSTEM

HATEM HAJLAOUI
Institut Préparatoire aux Études d’Ingénieurs
Université de Kairouan, Tunisie

ABDELLAZIZ HARRABI
Institut Supérieur des Mathématiques Appliquées et de l’Informatique
Université de Kairouan, Tunisie

FOUED MTIRI
Institut Élie Cartan de Lorraine, IECL, UMR 7502
Université de Lorraine, France

(Communicated by Joachim Krieger)

Abstract. We examine the general weighted Lane-Emden system

\[-\Delta u = \rho(x)v^p, \quad -\Delta v = \rho(x)u^\theta, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N\]

where \(1 < p \leq \theta\) and \(\rho : \mathbb{R}^N \to \mathbb{R}\) is a radial continuous function satisfying \(\rho(x) \geq A(1 + |x|^2)^{\alpha/2}\) in \(\mathbb{R}^N\) for some \(\alpha \geq 0\) and \(A > 0\). We prove some Liouville type results for stable solution and improve the previous works [2, 9, 12]. In particular, we establish a new comparison property (see Proposition 1 below) which is crucial to handle the case \(1 < p \leq \frac{4}{3}\). Our results can be applied also to the weighted Lane-Emden equation \(-\Delta u = \rho(x)u^p\) in \(\mathbb{R}^N\).

1. Introduction. We consider the following weighted Lane-Emden system

\[-\Delta u = \rho(x)v^p, \quad -\Delta v = \rho(x)u^\theta, \quad u, v > 0 \quad \text{in} \quad \mathbb{R}^N\]  \hspace{1cm} (1)

where \(1 < p \leq \theta\) and \(\rho : \mathbb{R}^N \to \mathbb{R}\) is a radial continuous function satisfying the following assumption:

(*) There exists \(\alpha \geq 0\) and \(A > 0\) such that \(\rho(x) \geq A\rho_0(x)\) in \(\mathbb{R}^N\) where \(\rho_0(x) := (1 + |x|^2)^{\alpha/2}\).

Remark that under the scaling transformation \(u = \gamma^{\frac{1}{\alpha+\theta}} \tilde{u}, \quad v = \tilde{v}\) with \(\gamma > 0\), the following system

\[-\Delta \tilde{u} = \tilde{\rho}(x)\tilde{v}^p, \quad -\Delta \tilde{v} = \gamma \tilde{\rho}(x)\tilde{u}^\theta, \quad \tilde{u}, \tilde{v} > 0 \quad \text{in} \quad \mathbb{R}^N\]

is equivalent to (1) with \(\rho = \gamma^{\frac{1}{\alpha+\theta}} \tilde{\rho}\).
To define the notion of stability, let $\Omega$ be a subset of $\mathbb{R}^N$ and consider a general system given by
\[
-\Delta u = f(x, v), \quad -\Delta v = g(x, u), \quad \text{in } \Omega,
\]
with $f, g \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$. Following Montenegro [15] (see also [9, 2]), a smooth solution $(u, v)$ of (2) is said to be stable if the following eigenvalue problem
\[
-\Delta \xi = f_v(x, v) \zeta + \eta \xi, \quad -\Delta \zeta = g_u(x, u) \xi + \eta \zeta, \quad \text{in } \Omega,
\]
has a first positive eigenvalue $\eta > 0$, with corresponding positive smooth eigenfunction pair $(\xi, \zeta)$.

In this paper, we prove the following classification result:

**Theorem 1.1.** Suppose that $\rho$ satisfies (⋆) and let $x_0$ be the largest root of the polynomial
\[
H(x) = x^4 - \frac{16p\theta(p+1)(\theta+1)x^2}{(p\theta-1)^2} + \frac{16p\theta(p+1)(\theta+1)(p+\theta+2)}{(p\theta-1)^3} x - \frac{16p\theta(p+1)^2(\theta+1)^2}{(p\theta-1)^4}.
\]

i) If $\frac{4}{3} < p \leq \theta$ then (1) has no stable classical solution if $N < 2 + (2 + \alpha) x_0$. In particular, if $N \leq 10 + 4\alpha$, then (1) has no classical stable solution for all $\frac{4}{3} < p \leq \theta$.

ii) If $1 < p \leq \min\left(\frac{4}{3}, \theta\right)$, then (1) has no bounded classical stable solution, if
\[
N < 2 + \left[\frac{p}{2} + \frac{(2 - p)(p\theta - 1)}{(\theta + p - 2)(\theta + 1)}\right] (\alpha + 2) x_0.
\]

Therefore, if $N \leq 6 + 2\alpha$, the system (1) has no bounded classical stable solution for all $\theta \geq p > 1$.

When $\rho \equiv 1$, it is well known that if (1) admits an unbounded stable classical solution, then it admits a bounded stable classical solution (see [16, 19]). Therefore, we deduce from Theorem 1.1 the following result.

**Corollary 1.** Suppose that $\rho \equiv 1$ and let $x_0$ be the largest root of the polynomial $H(x)$ given by (3).

i) If $\frac{4}{3} < p \leq \theta$ then (1) has no stable classical solution if $N < 2 + 2(2 + \alpha) x_0$. In particular, if $N \leq 10 + 4\alpha$, then (1) has no classical stable solution for all $\frac{4}{3} < p \leq \theta$.

ii) If $1 < p \leq \min\left(\frac{4}{3}, \theta\right)$, then (1) has no classical stable solution for $N$ satisfying (4) with $\alpha = 0$.

Therefore, if $N \leq 6$, the system (1) has no classical stable solution for all $\theta \geq p > 1$.

As a consequence of Theorem 1.1, we obtain the following classification result for stable solution of the Lane-Emden equation
\[
-\Delta u = \rho(x) u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N.
\]

**Corollary 2.** Suppose that $\rho$ satisfies (⋆) and let $p > 1$.

i) If $\frac{4}{3} < p$ then (5) has no stable classical solution if
\[
N < 2 + \frac{2(2 + \alpha)}{p - 1} \left(p + \sqrt{p^2 - p}\right).
\]
In particular, if \( N \leq 10 + 4\alpha \), then (5) has no stable classical solution for all \( \frac{4}{3} < p \).

ii) If \( 1 < p \leq \frac{4}{3} \), (5) has no bounded stable classical solution for \( N \) verifying (6). Therefore, there is no bounded stable classical solution of (5) for all \( p > 1 \) if \( N \leq 10 + 4\alpha \).

Recalling that for the autonomous case, i.e. when \( \rho \equiv 1 \), the stable solutions of the corresponding Lane-Emden equation and system, or the biharmonic equation (corresponding to \( p = 1 \)) have been widely studied by many authors. See for instance [8, 19, 2, 11, 1, 6] and the references there in.

For the second order Lane-Emden equation (\( p > 1 \))
\[-\Delta u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \quad (7)\]
Farina classified completely in [8] all finite Morse index classical solutions for \( 1 < p < p_{JL} \), where \( p_{JL} \) stands for the Joseph-Lundgren exponent [13] (see also [10]). More precisely, the equation (7) admits nontrivial classical solutions with finite Morse index if and only if \( N \geq 3 \), \( p = \frac{N+2}{N-2} \) or \( N \geq 11 \) and \( p \geq p_{JL} \). For the biharmonic equation (\( p > 1 \)) \[\Delta^2 u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N, \quad (8)\]
Dávila-Dupaigne-Wang-Wei [6] recently gave a complete classification of finite Morse index solutions. They derived a monotonicity formula for the solutions of (8) and reduced the problem to the nonexistence of stable homogeneous solutions.

It is worthy to mention that Chen-Dupaigne-Chergu [1] proved an optimal Liouville type result for the radial stable solutions of (1) for \( \theta \geq p > 1 \) and \( \rho \equiv 1 \).

For the weighted equation or system with positive weights, the Liouville type results are less understood.

• Using Farina’s approach, Fazly proved the nonexistence of classical stable solutions of (5) for \( \rho = \rho_0 \), \( N \) satisfying (4) and \( p \geq 2 \). See Theorem 2.3 in [9].

• Using also Farina’s approach, Cowan-Fazly [4] established a Liouville type result for classical stable sub-solutions of (5) for \( N \) satisfying (6), \( p > 1 \) and \( \lim_{|x| \to \infty} \frac{\rho(x)}{\rho_0(x)} = C \in (0, \infty) \). See Theorem 1.3-(3) with \( \alpha = 0 \) in [4].

• Adopting the new approach in [3], Hu proved the following Liouville theorem for classical stable solutions of (1) for \( \rho = \rho_0 \) and \( \theta \geq p \geq 2 \) or \( \theta = p > \frac{4}{3} \), obtaining a direct extension of Theorem 1 in [2] for \( \rho \equiv 1 \). More precisely, let \( t_0^+ \) and \( t_0^- \) be the quantities used in [2]:
\[ t_0^\pm = \sqrt{\frac{p\theta(p+1)}{\theta+1}} \pm \sqrt{\frac{p\theta(p+1)}{\theta+1} - \frac{p\theta(p+1)}{\theta+1}}, \]
Hu proved in [12]:

**Theorem A.** Suppose that \( \rho = \rho_0 \) with \( \alpha \geq 0 \).

i) If \( 2t_0^- < p \leq \theta \) and \( N \) satisfies
\[ N < 2 + \frac{2(2 + \alpha)(\theta + 1)}{p\theta - 1} t_0^+, \]
then there is no classical stable solution of \((1)\). In particular there is no classical stable solution of \((1)\) for any \(2 \leq p \leq \theta\) and \(N \leq 10 + 4\alpha\).

ii) If \(p > \frac{4}{3}\) and \(N\) satisfies (6), then there is no classical stable solution of (5).

Remark 1. It is known that for \(1 < p \leq \theta\), there hold \(t_0^- < 1 < t_0^+\), \(t_0^-\) is decreasing and \(t_0^+\) is increasing in \(z := \frac{\theta(p+1)}{p+1}\). Moreover, \(\lim_{z \to \infty} t_0^- = \frac{1}{2}\) and \(\lim_{z \to \infty} t_0^+ = 1\).

Remark 2. We have \(2t_0^- < p\) if \(p > \frac{4}{3}\). Indeed, if \(p > \frac{4}{3}\) then \(\theta \geq p > \frac{4}{3}\) and \(z > \frac{16}{9}\). Since \(f(z) := \sqrt{z} - \sqrt{z - \sqrt{z}}\) is decreasing in \(z\), there holds \(2t_0^- = 2f(z) < 2f\left(\frac{16}{9}\right) = \frac{4}{3} < p\).

Using the above remark, we see that Theorem A (hence Theorem 1 in [2]) can be extended immediately for \(\frac{4}{3} < p \leq \theta\).

- We can show that \(2t_0^+ = \theta + 1 < x_0\) for any \(1 < p \leq \theta\) (see Lemma 6 below), where \(x_0\) is the largest root of the polynomial \(H\) given by (6). So Theorem 1.1 improves the bound given in Theorem A.
- In Theorem 1.1 and Corollary 2 we prove classification results for (1) and (5) with \(\rho\) satisfying (*), so without the restriction \(\rho = \rho_0\) in Theorem A or the condition (9) used in [4].
- Our approach permits to prove a Liouville type result for \(\theta \geq p > 1\). To the best of our knowledge, no general Liouville type result was known for stable solution of (1) with positive weight for \(1 < p \leq \frac{4}{3}\).

To prove Theorem 1.1 we will use the following Souplet type estimate [18]. Its proof is the same as for Lemma 2.3 in [12] where we replace just \(\rho_0\) by \(\rho\), so we omit the details.

**Lemma 1.** Let \(\theta \geq p > 1\) and \(\rho\) satisfy (*). Then any classical solution of (1) verifies
\[
u^{\theta+1} \leq \frac{\theta + 1}{p + 1} \nu^{p+1} \text{ in } \mathbb{R}^N.
\] (10)

However, to handle the case \(1 < p \leq \frac{4}{3}\), we need the following new comparison property between \(u\) and \(v\). It is somehow an inverse version of Souplet’s estimate [10], and has its own interest.

**Proposition 1.** Let \(\theta \geq p > 1\) and suppose that \(\rho\) satisfies (*). Let \((u, v)\) be a classical solution of (1) and assume that \(u\) is bounded, then
\[v \leq \|u\|_{\infty}^{\frac{\theta - p}{\theta + 1}} u.
\]

Our paper is organized as follows. In section 2, we prove some preliminary results, in particular we give the proof of Proposition 1. The proofs of Theorem 1.1 and Corollary 2 are given in section 3.

2. Preliminaries. In order to prove our results, we need some technical lemmas. In the following, \(C\) denotes always a generic positive constant independent on \((u, v)\), which could be changed from one line to another. The ball of center 0 and radius \(r > 0\) will be denoted by \(B_r\).
2.1. Comparison property. In this subsection, we give the proofs of Proposition 1. First, we can adapt the proof of Lemma 2.1 in [3] (which was inspired by the previous works [17, 14]), to obtain the following integral estimates for all classical solutions of (1).

Lemma 2. Let $p > 1$, $\theta > 1$ and suppose that $\rho$ satisfies (⋆). For any classical solution $(u, v)$ of (1) there exists $C > 0$ such that for any $R \geq 1$, holds

$$
\int_{B_R} \rho(x) v dx \leq CR^{N-\frac{2(p+1)p}{p\theta-1} - \frac{(p+1)\theta}{p\theta-1}}, \quad \int_{B_R} \rho(x) u^\theta dx \leq CR^{N-\frac{2(p+1)p}{p\theta-1} - \frac{(p+1)\theta}{p\theta-1}}.
$$

Proof. Let $\varphi_0 \in C_c^\infty(B_2)$ be a cut-off function verifying $0 \leq \varphi_0 \leq 1$, $\varphi_0 = 1$ for $|x| < 1$. Take $\psi := \varphi_0(R^{-1}x)$ for $R \geq 1$. Multiplying the equation $-\Delta u = \rho(x)v^p$ by $\psi^m$ and integrating by parts, there holds then

$$
\int_{\mathbb{R}^N} \rho(x)v^p \psi^m dx = - \int_{\mathbb{R}^N} u\Delta(\psi^m) dx \leq \frac{C}{R^2} \int_{B_{2R}\setminus B_R} u\psi^{m-2} dx.
$$

By H"{o}lder’s inequality, we get

$$
\int_{\mathbb{R}^N} \rho(x)v^p \psi^m dx \leq \frac{C}{R^2} \left( \int_{B_{2R}\setminus B_R} \rho(x)^{-\frac{m}{p}} dx \right)^\frac{1}{\theta} \left( \int_{B_{2R}\setminus B_R} \rho(x) u^\theta \psi^{m-2} dx \right)^\frac{1}{\theta},
$$

where $\frac{1}{\theta} + \frac{1}{m} = 1$. From (⋆) we deduce that for $R \geq 1$,

$$
\int_{\mathbb{R}^N} \rho(x)v^p \psi^m dx \leq CR^{N-\frac{\theta}{\theta-2} - \frac{2(p+1)p}{p\theta-1} - \frac{(p+1)\theta}{p\theta-1}} \left( \int_{B_{2R}\setminus B_R} \rho(x) u^\theta \psi^{(m-2)p} dx \right)^\frac{1}{\theta}.
$$

Similarly, using $-\Delta v = \rho(x)u^p$, we obtain, for $k \geq 2$,

$$
\int_{\mathbb{R}^N} \rho(x)u^\theta \psi^k dx \leq CR^{N-\frac{\theta}{\theta-2} - \frac{2(p+1)p}{p\theta-1} - \frac{(p+1)\theta}{p\theta-1}} \left( \int_{B_{2R}\setminus B_R} \rho(x) v^p \psi^{(k-2)p} dx \right)^\frac{1}{\theta},
$$

where $\frac{1}{\theta} + \frac{1}{p} = 1$. Take now $k$ and $m$ large verifying $m \leq (k-2)p$ and $k \leq (m-2)\theta$. Combining the two above inequalities, we get

$$
\int_{\mathbb{R}^N} \rho(x)v^p \psi^m dx \leq CR^{N-\frac{\theta}{\theta-2} - \frac{2(p+1)p}{p\theta-1} - \frac{(p+1)\theta}{p\theta-1}} \left( \int_{B_{2R}\setminus B_R} \rho(x) v^p \psi^{(k-2)p} dx \right)^\frac{1}{\theta} \leq CR^{N-\frac{\theta}{\theta-2} - \frac{2(p+1)p}{p\theta-1} - \frac{(p+1)\theta}{p\theta-1}} \left( \int_{\mathbb{R}^N} \rho(x) v^p \psi^m dx \right)^\frac{1}{\theta}.
$$

Hence

$$
\int_{B_R} \rho(x)v^p dx \leq \int_{\mathbb{R}^N} \rho(x)v^p \psi^m dx \leq CR^{N-\frac{2(p+1)p}{p\theta-1} - \frac{(p+1)\theta}{p\theta-1}}.
$$

Similarly, we obtain the estimate for $u$. \hfill \Box

Now we are in position to prove the inverse comparison property.

Proof of Proposition 7 Let $w = v - \lambda u$, where $\lambda = \|u\|_{L^\infty}^{-\frac{\theta}{p\theta-1}}$. We have, as $\theta \geq p$,

$$
\Delta w = \rho(x) (\lambda v^p - u^\theta) = \rho(x) \left[ \lambda v^p - \left( \frac{u}{\|u\|_\infty} \right)^\theta \left\| u \right\|_\infty^\theta \right]
$$
\[ \geq \rho(x) \left[ \lambda v^p - \left( \frac{u}{\|u\|_{\infty}} \right)^p \|u\|_{\infty}^{q-p} \right] \]
\[ = \rho(x)\|u\|_{\infty}^{q-p} \left( \frac{\lambda v^p}{\|u\|_{\infty}^{q-p}} - u^p \right) \]
\[ = \rho(x)\|u\|_{\infty}^{q-p} \left( \lambda^{p} v^p - u^p \right). \]

It follows that \( \Delta w \geq 0 \) in the set \( \{w \geq 0\} \). Consider \( w_+ := \max(w, 0) \). Next, we split the proof into two cases.

**Case 1.** \( p \geq 2 \). For any \( R > 0 \), there holds
\[ (p-1) \int_{B_R^+} w_+^{p-2} |\nabla w_+|^2 \, dx = - \int_{B_R^+} w_+^{p-1} \Delta w \, dx + \int_{\partial B_R^+} w_+^{p-1} \frac{\partial w}{\partial \nu} \, d\sigma \]
\[ \leq \int_{\partial B_R^+} w_+^{p-1} \frac{\partial w}{\partial \nu} \, d\sigma \]
\[ = \frac{R^{N-1}}{p} f'(R) \quad (11) \]

where
\[ f(R) := \int_{S^{N-1}} w_+^p (R\sigma) \, d\sigma \leq \int_{S^{N-1}} v^p (R\sigma) \, d\sigma =: g(R). \]

Hereafter, \( S^{N-1} \) denotes the unit sphere in \( \mathbb{R}^N \). By Lemma 2, we derive that
\[ \int_{0}^{R} r^{N-1} \rho(r) v^p(r) \, dr = \int_{0}^{R} r^{N-1} \rho(r) \int_{S^{N-1}} v^p(r\sigma) \, d\sigma \, dr \]
\[ \leq C R^{N-2} \frac{2(6+1)}{p+1} - \frac{2^{p+1}}{p+1} = o(R^N) \quad \text{as} \quad R \to \infty. \]

Using (*), there holds
\[ \int_{0}^{R} r^{N-1+\alpha} g(r) \, dr = o(R^N) \quad \text{as} \quad R \to \infty. \]

This implies that \( \liminf_{r \to \infty} g(r) = 0 \), hence \( \liminf_{r \to \infty} f(r) = 0 \). Consequently, there exist \( R_i \to \infty \) such that \( f'(R_i) \leq 0 \). Take \( \{1\} \) with \( R = R_i \) and let \( i \to \infty \), we conclude that \( w_+ \) is constant in \( \mathbb{R}^N \). If \( w \equiv C > 0 \) then \( v \geq C > 0 \) in \( \mathbb{R}^N \), which contradicts Lemma 2. Hence \( w_+ \equiv 0 \) in \( \mathbb{R}^N \), i.e. \( v - \lambda u \leq 0 \) in \( \mathbb{R}^N \).

**Case 2.** \( 1 < p < 2 \). For any \( R > 0 \) and \( \epsilon > 0 \), we have
\[ (p-1) \int_{B_R} (\epsilon + w_+)^{p-2} |\nabla w_+|^2 \, dx = - \int_{B_R} (\epsilon + w_+)^{p-1} \Delta w \, dx \]
\[ + \int_{\partial B_R} (\epsilon + w_+)^{p-1} \frac{\partial w}{\partial \nu} \, d\sigma \]
\[ \leq \int_{\partial B_R} (\epsilon + w_+)^{p-1} \frac{\partial w}{\partial \nu} \, d\sigma. \]

Letting \( \epsilon \to 0 \) (passing to limit in the l.h.s. via monotone convergence and use the dominated convergence on the r.h.s.), we get always the estimate \( \{1\} \), which will lead to the same conclusion: \( w_+ \equiv 0 \) in \( \mathbb{R}^N \).
2.2. Consequence of stability. With the ideas in [3, 7], we can proceed similarly as in the proof of Lemma 2.1 in [12] and claim

**Lemma 3.** If \((u, v)\) is a nonnegative classical stable solution of \((1)\), then

\[
\sqrt{p\theta} \int_{\mathbb{R}^N} \rho(x)u^{\frac{p-1}{2}}v^{\frac{p-1}{2}} \phi^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla\phi|^2 \, dx, \quad \forall \phi \in C^1_c(\mathbb{R}^N). \tag{12}
\]

The following Lemma is a consequence of the stability inequality \((12)\) and Proposition \((1)\). It plays a crucial role to handle the case \(1 < p \leq \frac{4}{3}\). Here we use also some ideas coming from \([10, 11]\).

**Lemma 4.** Let \((u, v)\) be a stable solution to \((1)\) with \(1 < p \leq \min\left(\frac{4}{3}, \theta\right)\). Assume that \(u\) is bounded and \(\rho\) satisfies (\(*\)), there holds

\[
\int_{B_R} \rho(x)v^2 \, dx \leq CR^N \frac{2(\rho+1)p}{\rho(p+1)} \left(\frac{2(\rho+1)(2-p)}{\rho+2}\right)^2, \quad \forall R > 0. \tag{13}
\]

**Proof.** Let \((u, v)\) be a stable solution of \((1)\), where \(u\) is bounded. Take \(\eta \in C^\infty_c(\mathbb{R}^N)\). Multiplying \(-\Delta v = \rho(x)u^\theta\) by \(v\eta^2\) and integrating by parts, there holds

\[
\int_{\mathbb{R}^N} |\nabla v|^2 \eta^2 \, dx = \int_{\mathbb{R}^N} \rho(x)u^\theta v^2 \eta^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} v^2 \Delta (\eta^2) \, dx.
\]

Using Lemma \([1]\) we get

\[
\int_{\mathbb{R}^N} |\nabla v|^2 \eta^2 \, dx \leq \sqrt{\frac{\theta + 1}{p + 1}} \int_{\mathbb{R}^N} \rho(x)u^{\frac{p-1}{2}}v^{\frac{p-1}{2}}v\eta^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} v^2 \Delta (\eta^2) \, dx.
\]

Set \(\phi = v\eta\) in \((12)\) and integrating by parts, we deduce that

\[
\sqrt{p\theta} \int_{\mathbb{R}^N} \rho(x)u^{\frac{p-1}{2}}v^{\frac{p-1}{2}} \phi^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla v|^2 \eta^2 \, dx + \int_{\mathbb{R}^N} v^2 |\nabla \eta|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} v^2 \Delta (\eta^2) \, dx.
\]

Combining the two last inequalities, we obtain

\[
\left(\sqrt{p\theta} - \sqrt{\frac{\theta + 1}{p + 1}}\right) \int_{\mathbb{R}^N} \rho(x)u^{\frac{p-1}{2}}v^{\frac{p-1}{2}} \eta^2 \, dx \leq \int_{\mathbb{R}^N} v^2 |\nabla \eta|^2 \, dx.
\]

Using Proposition \([1]\) there exists a positive constant \(C\) such that

\[
\int_{\mathbb{R}^N} \rho(x)u^{\frac{p+2}{2}} \eta^2 \, dx \leq C \int_{\mathbb{R}^N} v^2 |\nabla \eta|^2 \, dx.
\]

Take \(\varphi_0\) a cut-off function in \(C^\infty_c(B_2)\) verifying \(0 \leq \varphi_0 \leq 1\), \(\varphi_0 = 1\) for \(|x| < 1\). Let \(\eta = \varphi^m\) with \(\varphi := \varphi_0(R^{-1}x)\) for \(R > 0\), we arrive at

\[
\int_{\mathbb{R}^N} \rho(x)u^{\frac{p+2}{2}} \varphi^{2m} \, dx \leq \frac{C}{R^2} \int_{B_{2R}\setminus B_R} v^2 \varphi^{2m-2} \, dx.
\]

Using (\(*\)), there holds

\[
\int_{\mathbb{R}^N} \rho(x)u^{\frac{p+2}{2}} \varphi^{2m} \, dx \leq \frac{C}{R^{2+\alpha}} \int_{B_{2R}\setminus B_R} \rho(x)v^2 \varphi^{2m-2} \, dx \leq \frac{C}{R^{2+\alpha}} \int_{\mathbb{R}^N} \rho(x)v^2 \varphi^{2m-2} \, dx. \tag{14}
\]
Denote
\[ J_1 := \int_{\mathbb{R}^N} \rho(x) v^{\frac{q+2}{2}} \varphi^{2m} \, dx, \quad J_2 := \int_{\mathbb{R}^N} \rho(x) v^2 \varphi^{2m-2} \, dx. \]
Remark that \( p < 2 < \frac{\theta + p + 2}{2} \) for \( 1 < p \leq \frac{4}{3} \) and \( \theta \geq p \). A direct calculation yields
\[ 2 = p\lambda + \frac{\theta + p + 2}{2} (1 - \lambda) \quad \text{with} \quad \lambda = \frac{\theta + p - 2}{\theta + 2 - p} \in (0, 1). \]
Take \( m \) large such that \( m\lambda > 1 \). By Hölder’s inequality, Lemma 2 and (14), we get
\[
J_2 \leq J_1^{1 - \lambda} \left( \int_{\mathbb{R}^N} \rho(x) v^{p} \varphi^{2m\lambda - 2} \, dx \right)^{\lambda} \\
\leq \left( \frac{C J_2}{R^{2 + \alpha}} \right)^{1 - \lambda} \left( \int_{B_{2R}} \rho(x) v^{p} \, dx \right)^{\lambda} \\
\leq C' J_2^{1 - \lambda} R^{-2(\alpha + 1)(1 - \lambda)} \left( R^{N - \frac{2(\theta + p + 1)}{p\theta + 1} - \frac{(p + 1)\alpha}{p\theta + 1}} \right)^{\lambda},
\]
which implies
\[
J_2 \leq C R^{N - \frac{2(\theta + p + 1)}{p\theta + 1} - \frac{(p + 1)\alpha}{p\theta + 1}},
\]
so we are done. \( \square \)

3. Proofs of Theorem 1.1 and Corollary 2. The following lemma plays an important role in dealing with Theorems 1.1 and Corollary 2, where we use some ideas from [11]. Here and in the following, we define \( R_k = 2^k R \) for all \( R > 0 \) and integers \( k \geq 1 \).

**Lemma 5.** Suppose that \( \rho \) satisfies (\(*\)) and let \((u, v)\) be a stable solution of (1). Then for any \( s > \frac{p+1}{2} \) verifying \( L(s) < 0 \), there exists \( C < \infty \) such that
\[
\int_{B_{R}} \rho(x) u^s v^{s-1} \, dx \leq \frac{C}{R^2} \int_{B_{2R}} v^s \, dx, \quad \forall \ R > 0,
\]
where
\[
L(s) := s^4 - \frac{16p\theta(p+1)}{\theta + 1} s^2 + \frac{16p\theta(p+1)(p+\theta+2)}{(\theta+1)^2} s - \frac{16p\theta(p+1)^2}{(\theta+1)^2}. \quad (15)
\]

**Proof.** Take \( \phi \in C^2_{0}(\mathbb{R}^N) \). Let \((u, v)\) be a stable solution of (1), the integration by parts yields that
\[
\int_{\mathbb{R}^N} |\nabla u|^{q+1} \phi^2 \, dx = \frac{(q+1)^2}{4} \int_{\mathbb{R}^N} u^{q-1} |\nabla u|^2 \phi^2 \, dx \\
= \frac{(q+1)^2}{4q} \int_{\mathbb{R}^N} \phi^2 \nabla (u^q) \nabla u \, dx \quad (16) \\
= \frac{(q+1)^2}{4q} \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 \, dx + \frac{q+1}{4q} \int_{\mathbb{R}^N} u^{q+1} \Delta(\phi^2) \, dx, \]
and
\[
(q+1) \int_{\mathbb{R}^N} u^q \phi \nabla u \nabla \phi \, dx = \frac{1}{2} \int_{\mathbb{R}^N} \nabla (u^{q+1}) \nabla (\phi^2) \, dx = -\frac{1}{2} \int_{\mathbb{R}^N} u^{q+1} \Delta(\phi^2) \, dx. \quad (17)
\]
Take \( \varphi = u^{q+1} \phi \) with \( q > 0 \) into the stability inequality (12) and using (16)-(17), we obtain

\[
\sqrt{p} \int_{\mathbb{R}^N} \rho(x) u^{q+1} v^\frac{p-1}{2} u^{q+1} \phi^2 dx \\
\leq \int_{\mathbb{R}^N} \| \nabla \varphi \|^2 dx \\
\leq \frac{(q + 1)^2}{4q} \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 dx + C \int_{\mathbb{R}^N} u^{q+1} \left[ |\nabla \phi|^2 + \Delta(\phi^2) \right] dx,
\]

so we get

\[
a_1 \int_{\mathbb{R}^N} \rho(x) u^{q+1} v^\frac{p-1}{2} u^{q+1} \phi^2 dx \leq \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 dx + C \int_{\mathbb{R}^N} u^{q+1} \left[ |\nabla \phi|^2 + \Delta(\phi^2) \right] dx,
\]

with \( a_1 = \frac{4q \sqrt{p}}{(q + 1)^2} \). Choose now \( \varphi(x) = \varphi_0(R^{-1}x) \) where \( \varphi_0 \in C^\infty_0(B_2) \) such that \( \varphi_0 \equiv 1 \) in \( B_1 \), there holds then

\[
\int_{\mathbb{R}^N} \rho(x) u^{q+1} v^\frac{p-1}{2} u^{q+1} \phi^2 dx \leq \frac{1}{a_1} \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 dx + \frac{C}{R^2} \int_{B_{2R}} u^{q+1} dx. \tag{18}
\]

Similarly, applying the stability inequality (12) with \( \varphi = v^{r+1} \phi \), \( r > 0 \), we obtain

\[
\int_{\mathbb{R}^N} \rho(x) u^{q+1} v^\frac{p-1}{2} v^{r+1} \phi^2 dx \leq \frac{1}{a_2} \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 dx + \frac{C}{R^2} \int_{B_{2R}} v^{r+1} dx \tag{19}
\]

with \( a_2 = \frac{4r \sqrt{p}}{(r + 1)^2} \). Combining (18) and (19),

\[
I_1 + a_2 \frac{2(r+1)}{p+1} I_2 \\
= \int_{\mathbb{R}^N} \rho(x) u^{q+1} v^\frac{p-1}{2} u^{q+1} \phi^2 dx + a_2 \frac{2(r+1)}{p+1} \int_{\mathbb{R}^N} \rho(x) u^{q+1} v^\frac{p-1}{2} v^{r+1} \phi^2 dx \\
\leq \frac{1}{a_1} \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 dx + a_2 \frac{2(r+1)}{p+1} \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 dx \\
+ \frac{C}{R^2} \int_{B_{2R}} (u^{q+1} + v^{r+1}) \phi^2 dx.
\tag{20}
\]

Fix now

\[
q = \frac{(\theta + 1)x}{p+1} + \frac{\theta - p}{p+1}, \quad \text{or equivalently} \quad q + 1 = \frac{(\theta + 1)(r+1)}{p+1}.
\tag{21}
\]

Let \( r > \frac{p-1}{2} \), by Young’s inequality, there holds

\[
\frac{1}{a_1} \int_{\mathbb{R}^N} \rho(x) u^q v^p \phi^2 dx \\
= \frac{1}{a_1} \int_{\mathbb{R}^N} \rho(x) u^{q+1} v^{r+1} \frac{(\theta+1)r}{p+1} + \frac{\theta+1}{p+1} \frac{(1-p)}{p+1} v^{r+1} \phi^2 dx
\]
\[
= \frac{1}{a_1} \int_{\mathbb{R}^N} \rho(x) u^{\frac{\theta-1}{2}} v^{\frac{p-1}{2}} u^{(q+1)} \frac{2^q+1-p}{2(q+1)} v^{\frac{p+1}{2}} \phi^2 \, dx
\]
\[
\leq \frac{2r + 1 - p}{2(r + 1)} \int_{\mathbb{R}^N} \rho(x) u^{\frac{\theta-1}{2}} v^{\frac{p-1}{2}} u^{q+1} \phi^2 \, dx
\] 
\[+ \frac{p + 1}{2(r + 1)} a_1 \frac{2^q+1-p}{2(q+1)} \int_{\mathbb{R}^N} \rho(x) u^{\frac{\theta-1}{2}} v^{\frac{p-1}{2}} v^{r+1} \phi^2 \, dx
\]
\[= \frac{2r + 1 - p}{2(r + 1)} I_1 + \frac{p + 1}{2(r + 1)} a_1 \frac{2^q+1-p}{2(q+1)} I_2;
\]
and similarly we have
\[
a_2 \frac{2^q+1-p}{r+1} I_2 \leq \left[ \frac{2r + 1 - p}{2(r + 1)} \frac{a_2}{a_1} \frac{2^q+1-p}{r+1} + \frac{p + 1}{2(r + 1)} a_1 \right] I_2
\]
\[+ \frac{C}{R^2} \int_{B_{2R}} (u^{q+1} + v^{r+1}) \, dx,
\]
\[\text{hence}
\]
\[
\frac{p + 1}{2(r + 1)} \left[ (a_1 a_2)^{\frac{2^q+1-p}{r+1}} - 1 \right] I_2 \leq CR^{-2} a_1^{\frac{2^q+1-p}{r+1}} \int_{B_{2R}} (u^{q+1} + v^{r+1}) \, dx.
\]

Thus, if \(a_1 a_2 > 1\), by the choice of \(\phi\),
\[
\int_{B_R} \rho(x) u^{\frac{\theta-1}{2}} v^{\frac{p-1}{2}} v^{r+1} \, dx \leq I_2 \leq \frac{C}{R^2} \int_{B_{2R}} (u^{q+1} + v^{r+1}) \, dx.
\]

Using (21) and (11), there hold \(u^{q+1} \leq Cv^{r+1}\) and \(u^{\frac{\theta-1}{2}} v^{\frac{p-1}{2}} v^{r+1} \geq u^{\theta} v^r\). Denote \(s = r + 1\), we conclude that if \(a_1 a_2 > 1\) and \(s > \frac{p+1}{2}\),
\[
\int_{B_R} \rho(x) u^{\theta} v^{s-1} \, dx \leq \frac{C}{R^2} \int_{B_{2R}} v^{s} \, dx.
\]

Furthermore, we can check that \(a_1 a_2 > 1\) is equivalent to \(L(s) < 0\), the proof is completed.

Next, we give some properties of the polynomial \(L\) given by (15). Performing the change of variables \(x = \frac{s+1}{p} s\), a direct computation yields
\[
H(x) = \left( \frac{\theta + 1}{\rho \theta - 1} \right)^4 L(s)
\]
where \(H\) is given by (3). Hence \(H(x) < 0\) if and only if \(L(s) < 0\). In addition, we have

**Lemma 6.** Let \(1 < p \leq \theta\), then \(L(2) < 0\) and \(L\) has a unique root \(s_0\) in \((2, \infty)\) and \(2t^+_0 \leq s_0\). Moreover, if \(p > \frac{4}{3}\), then \(L(p) < 0\) and \(s_0\) is the unique root of \(L\) in \((p, \infty)\).
Proof. Using $1 < p \leq \theta$,

$$L(2) = 16 - \frac{64p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)(p+\theta+2)}{(\theta+1)^2} - \frac{16p\theta(p+1)^2}{(\theta+1)^2}$$

$$= 16 - \frac{64p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)^2}{(\theta+1)^2} - \frac{16p\theta(p+1)^2}{(\theta+1)^2}$$

$$= 16 - \frac{32p\theta(p+1)}{(\theta+1)} + \frac{16p\theta(p+1)}{(\theta+1)^2}$$

$$\leq 16 - \frac{32p\theta(p+1)}{(\theta+1)} + \frac{16p\theta(p+1)}{(\theta+1)}$$

$$= 16(1 - \frac{p^2}{\theta} + (1 - \theta p) < 0.$$

Very similarly, we can check that

$$L'(2) \leq 32 - \frac{32p\theta(p+1)}{(\theta+1)} < 0.$$

Furthermore, we have

$$L''(s) = 12s^2 - \frac{32p\theta(p+1)}{\theta+1},$$

then $L''$ can change at most once the sign from negative to positive for $s \geq 2$. As $\lim_{s \to \infty} L(s) = \infty$, it’s clear that $L$ admits a unique root in $(2, \infty)$. Moreover, we can check that

$$L(2t_0^+) = \frac{16p\theta(p+1)(\theta-p)}{(\theta+1)^2}(1 - 2t_0^+) \leq 0.$$

Hence, there holds $2t_0^+ \leq s_0$.

Now we consider $L(p)$. Rewrite

$$L(s) = s^4 - 16\frac{p\theta(p+1)}{\theta+1} \left( s^2 - \frac{p + \theta + 2}{\theta + 1} s + \frac{p + 1}{\theta + 1} \right).$$

For $s > 1$, we see that

$$\left( s^2 - \frac{p + \theta + 2}{\theta + 1} s + \frac{p + 1}{\theta + 1} \right)' = \frac{p + 1}{(\theta + 1)^2}(s - 1) > 0,$$

Then for $s > 1$, as $\theta \geq p > 1$, there holds

$$s^2 - \frac{p + \theta + 2}{\theta + 1} s + \frac{p + 1}{\theta + 1} > s^2 - 2s + 1 = (s - 1)^2 \quad \text{and} \quad \frac{p\theta(p+1)}{\theta+1} \geq p^2.$$

Finally, we get (for $p > 1$)

$$L(p) < p^4 - 16p^2(p-1)^2 = p^2(5p-4)(4-3p)$$

and

$$L'(p) = 4p^3 - 16\frac{p\theta(p+1)}{\theta+1} \left( 2p - \frac{p + \theta + 2}{\theta + 1} \right) < 4p^3 - 16p^2(2p-2) = 4p^2(8 - 7p),$$

We check readily that for $p > \frac{4}{3}$, $L(p) < 0$ and $L'(p) < 0$, so we can conclude as above.

We need also the following $L^1$ elliptic regularity result, see Lemma 5 in [2].
Lemma 7. For any $1 \leq \beta < \frac{N}{N-2}$, there exists $C > 0$ such that for any smooth non-negative function $w$, we have

$$
\left( \int_{B_{R}} w^{\beta}dx \right)^{\frac{1}{\beta}} \leq CR^{N\left(\frac{1}{\beta}-1\right)+2} \int_{B_{R+k+1}} |\Delta w|dx + CR^{N\left(\frac{1}{\beta}-1\right)} \int_{B_{R+k+1}} wdx.
$$

Applying the above two lemmas, we establish the following result which plays an essential role in iteration process.

Lemma 8. Suppose that $\rho$ satisfies $(\star)$ and let $(u, v)$ be a classical stable solution of \cite{1}, with $1 < p \leq \theta$. Then for any $1 \leq \lambda < \frac{N}{N-2}$, $2t_{0} < q < s_{0}$ and nonnegative integer $k \geq 1$, there holds

$$
\left( \int_{B_{R}} v^{q\lambda} dx \right)^{\frac{1}{q}} \leq CR^{N\left(\frac{1}{q}-1\right)+2} \int_{B_{R+k+1}} v^{q-2} |\nabla v|^{2}dx + CR^{N\left(\frac{1}{q}-1\right)+2} \int_{B_{R+k+1}} \rho(x)v^{q-1}u^{\theta}dx
$$

(23)

Proof. A simple calculation gives

$$
|\Delta(v^{q})| \leq q(q-1)v^{q-2} |\nabla v|^{2} + q\rho(x)v^{q-1}u^{\theta}.
$$

Using Lemma 7, we get

$$
\left( \int_{B_{R}} v^{q\lambda} dx \right)^{\frac{1}{q}} \leq CR^{N\left(\frac{1}{q}-1\right)+2} \int_{B_{R+k+1}} v^{q-2} |\nabla v|^{2}dx
$$

$$
+ CR^{N\left(\frac{1}{q}-1\right)+2} \int_{B_{R+k+1}} \rho(x)v^{q-1}u^{\theta}dx
$$

(23)

Now, take a cut-off function $\phi \in C_{0}^{2}(B_{R+k+2})$ verifying $\phi \equiv 1$ in $B_{R+k+1}$ and $|\nabla \phi| \leq \frac{C}{R}$. Multiplying $-\Delta v = \rho(x)u^{\theta}$ by $v^{q-1}\phi^{2}$ and integrating by parts, we have

$$
(q-1) \int_{\mathbb{R}^{N}} v^{q-2} |\nabla v|^{2} \phi^{2} dx = -2 \int_{\mathbb{R}^{N}} v^{q-1} \phi v \nabla \phi dx + \int_{\mathbb{R}^{N}} \rho(x)v^{q-1}u^{\theta}\phi^{2} dx.
$$

(24)

By Young’s inequality,

$$
2 \int_{\mathbb{R}^{N}} v^{q-1} |\nabla v| |\nabla \phi| \phi dx \leq \frac{q-1}{2} \int_{\mathbb{R}^{N}} v^{q-2} |\nabla v|^{2} \phi^{2} dx + C \int_{\mathbb{R}^{N}} v^{q} |\nabla \phi|^{2} dx.
$$

Inserting this into (24), using the properties of $\phi$, we obtain

$$
\int_{\mathbb{R}^{N}} v^{q-2} |\nabla v|^{2} \phi^{2} dx \leq C \int_{B_{R+k+2}} \rho(x)v^{q-1}u^{\theta} dx + \frac{C}{R^{2}} \int_{B_{R+k+2}} v^{q} dx.
$$

Substituting the above inequality into (23), there holds

$$
\left( \int_{B_{R}} v^{q\lambda} dx \right)^{\frac{1}{q}} \leq CR^{N\left(\frac{1}{q}-1\right)+2} \int_{B_{R+k+2}} \rho(x)v^{q-1}u^{\theta} dx + CR^{N\left(\frac{1}{q}-1\right)} \int_{B_{R+k+2}} v^{q} dx.
$$

Since $\rho$ satisfies $(\star)$, we can use Lemmas 5 to find (22).\qed

Now, we can follow exactly the iteration process as for Corollary 2 in \cite{2} (see also Proposition 3.1 in \cite{12}) to obtain
Corollary 3. Suppose that $1 < p \leq \theta$ and $\rho$ satisfies (\ast). Let $(u, v)$ be a classical stable solution of (1.1) and $q \in (2t_0^+, s_0)$, then for $q \leq \beta < \frac{N}{N-2}s_0$, there are $\ell \in \mathbb{N}$ and $C < \infty$ such that for any $R > 0$, 
\[
\left( \int_{B_R} v^q dx \right)^{\frac{1}{q}} \leq CR^N\left( \frac{1}{\beta} - \frac{1}{q} \right) \left( \int_{B_{R^\ell}} v^\theta dx \right)^{\frac{1}{\theta}}, \quad \text{with } R_\ell = 2^\ell R.
\]

Now we are in position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 completed. Let $(u, v)$ be a classical stable solution of (1.1) with $\rho$ satisfying (\ast). We split the proof into two cases.

Case 1. $p > \frac{4}{3}$. Let $p > q > 0$. Using Hölder’s inequality, there holds 
\[
\int_{B_R} v^q dx \leq \left( \int_{B_R} \rho^p dx \right)^{\frac{1}{p}} \left( \int_{B_R} \rho^{-\frac{p}{q}} dx \right)^{\frac{q}{p}}. \tag{25}
\]
Applying Lemma 2 from (\ast) we get 
\[
\int_{B_R} v^q dx \leq CR^N\left[ N - \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \right]^\frac{1}{q} \leq CR^N\left[ N - \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \right]^\frac{1}{q}. \tag{26}
\]

Note that 
\[
N \left( \frac{1}{\beta} - \frac{1}{q} \right) + \frac{1}{q} \left[ N - \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \right] < 0 \iff N < \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \beta.
\]

Suppose now 
\[
N < 2 + \left( \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \right) s_0.
\]
We can take $\beta$ small but close to $\frac{N}{N-2}s_0$ such that $N < \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \beta$. With a such $\beta$, tending $R \to \infty$ in (26), we get $\|v\|_{L^\theta(R^N)} = 0$, this is just impossible since $v$ is positive. In other words, the equation (1) has no classical stable solution if $N < 2 + (2 + \alpha)x_0$ where $x_0 = \frac{\theta + 1}{p\theta - 1} s_0$.

Moreover, adopting the proof of Remark 2 in [3], we can easily show that 
\[
2t_0^+ \left( \frac{\theta + 1}{p\theta - 1} \right) > 4, \quad \forall \theta \geq p > 1.
\]
By Lemma 6, $x_0 \geq 2t_0^+ \left( \frac{\theta + 1}{p\theta - 1} \right) > 4$. This means that if $N \leq 10 + 4\alpha$, (1) has no classical stable solution for any $\theta \geq p > \frac{4}{3}$.

Case 2. $1 < p \leq \frac{4}{3}$ and $u$ is bounded. Let now $2 > q > 0$, using (25), with $p$ is replaced by 2 and applying Lemma 2, it follows that for any $R > 1$, 
\[
\int_{B_R} v^q dx \leq CR^N\left[ N - \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \right]^\frac{1}{q} \leq CR^N\left[ N - \frac{(2 + \alpha)(\theta + 1)}{p\theta - 1} \right]^\frac{1}{q},
\]

Proceeding as above, we can apply Corollary 3 with $2t_0^- < q < 2$ and $q < \beta < \frac{N\sigma_0}{N-2}$ to complete the proof of Theorem 1.1.

**Proof of Corollary 2.** Let $u$ be a solution of the weighted Lane-Emden equation (5), then $v = u$ verify the system (1) with $p = \theta$. Remark that $u$ is stable for (5) means just the estimate (12) holds true with $v = u$ and $p = \theta$, which is the departure point of our study. Moreover, we have

$$t_0^\pm = p\pm \sqrt{p^2-p}$$

and

$$L(s) = s^4 - 16p^2s^2 + 32p^2s - 16p^2 = (s^2 + 4p(s-1))(s - 2t_0^-)(s - 2t_0^+).$$

Then, $2t_0^+$ is the largest root of $L$ as $t_0^+ > p > 1$. Therefore

$$x_0 = \frac{2p + 2\sqrt{p^2 - p}}{p - 1}$$

is the largest root of $H$, and we can check easily that $x_0 > 4$ for all $p > 1$. The result follows immediately by applying Theorem 1.1.

**Acknowledgments.** We would like to thank Professor Dong Ye for suggesting us this problem and for many helpful comments.

**REFERENCES**

[1] W. Chen, L. Dupaigne and M. Ghergu, A new critical curve for the Lane-Emden system Discrete Contin. Dyn. Syst., 34 (2014), 2469—2479.

[2] C. Cowan, Liouville theorems for stable Lane-Emden systems and biharmonic problems Nonlinearity, 26 (2013), 2357–2371.

[3] C. Cowan, Regularity of stable solutions of a Lane-Emden type system Methods Appl. Anal., 22 (2015), 301–311.

[4] C. Cowan and M. Fazly, On stable entire solutions of semilinear elliptic equations with weights Proc. Amer. Math. Soc., 140 (2012), 2003–2012.

[5] C. Cowan and N. Ghoussoub, Regularity of semi-stable solutions to fourth order nonlinear eigenvalue problems on general domains Calc. Var. PDE., 49 (2014), 291–305.

[6] J. Dávila, L. Dupaigne, K. Wang and J. Wei, A monotonicity formula and a Liouville-type theorem for a fourth order supercritical problem Adv. Math., 258 (2014), 240–285.

[7] L. Dupaigne, A. Farina and B. Sirakov, Regularity of the extremal solutions for the Liouville system, in: Geometric Partial Differential Equations, in: Publications of the Scuola Normale Superiore/CRM Series, 15 (2013), 139–144.

[8] A. Farina, On the classification of solutions of the Lane-Emden equation on unbounded domains of $\mathbb{R}^n$ J. Math. Pures Appl., 87 (2007), 537–561.

[9] M. Fazly, Liouville type theorems for stable solutions of certain elliptic systems Adv. Nonlinear Stud., 12 (2012), 1–17.

[10] C. Gui, W. Ni and X. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$ Comm. Pure Appl. Math., 45 (1992), 1153–1181.

[11] H. Hajlaoui, A. Harrabi and D. Ye, On stable solutions of the biharmonic problem with polynomial growth Pacific J. Math., 270 (2014), 79–93.

[12] L. Hu, Liouville type results for semi-stable solutions of the weighted Lane-Emden system J. Math. Anal. Appl., 432 (2015), 429–440.

[13] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rational Mech. Anal., 49 (1973), 241–269.

[14] E. Mitidieri and S. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, Tr. Mat. Inst. Steklova, 234 (2001), 1–384.

[15] M. Montenegro, Minimal solutions for a class of elliptic systems Bull. London Math. Soc., 37 (2005), 405–416.
P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic systems. Duke Math. J., 139 (2007), 555–579.

J. Serrin and H. Zou, Non-existence of positive solutions of Lane-Emden systems, Diff. Inte. Equations, 9 (1996), 635–653.

P. Souplet, The proof of the Lane-Emden conjecture in four space dimensions Adv. Math., 221 (2009), 1409–1427.

J. Wei and D. Ye, Liouville theorems for stable solutions of biharmonic problem Math. Ann., 356 (2013), 1599–1612.

Received February 2016; revised September 2016.

E-mail address: hajlaouihatem@gmail.com
E-mail address: abdelaziz.harrabi@yahoo.fr
E-mail address: mtirifoued@yahoo.fr