$B^0 - \bar{B}^0$ mixing in unquenched lattice QCD

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(Dated: March 25, 2002)

We present an unquenched lattice calculation for the $B^0 - \bar{B}^0$ transition amplitude. The calculation, carried out at an inverse lattice spacing $1/a = 2.22(4)$ GeV, incorporates two flavors of dynamical quarks described by the fully $O(a)$-improved Wilson fermion action and heavy quarks described by NRQCD. A particular attention is paid to the uncertainty that arises from the chiral extrapolation, especially the effect of pion loops, for light quarks, which we find could be sizable for the leptonic decay of the $B_d$ meson, whereas it is small for the $B_s$ meson and the $B$ parameters. We obtain $f_{B_d} = 191(10)(^{+12}_{-11})$ MeV, $f_{B_s}/f_{B_d} = 1.13(3)(^{+13}_{-12})$, $B_{d}(m_s) = 0.836(27)(^{+56}_{-62})$, $B_{s}/B_{d} = 1.017(16)(^{+56}_{-57})$, $\xi = 1.14(3)(^{+3}_{-4})$, where the first error is statistical, and the second is systematic, including uncertainties due to chiral extrapolation, finite lattice spacing, heavy quark expansion and perturbative operator matching.

PACS numbers: 12.38.Gc, 12.39.Fe, 12.39.Hg, 13.20.He, 14.40.Nd

The unitarity test for the Cabibbo-Kobayashi-Maskawa (CKM) matrix entered a new era with the BaBar and Belle measurements of the angle $\phi_1$. The test requires the determination of the other angles and the sides of the unitarity triangle, the precision of the latter being limited by uncertainties in hadronic matrix elements. Lattice QCD in principle offers a model-independent calculation of such matrix elements. Those matrix elements provided by lattice calculations to date, however, are based on the quenched approximation, blindly hoping that quenching does not introduce large errors.

Simulations including creation and annihilation of a quark anti-quark pair in the vacuum have become feasible only recently. In this letter we present an unquenched lattice calculation of the hadronic matrix elements appearing in the $B^0 - \bar{B}^0$ mixing amplitude, which is needed in the determination of the CKM matrix element $|V_{td}|$ from the mass difference $\Delta M_d$. The matrix element is parametrized as $\langle B^0 \mid b \gamma_{\mu}(1 - \gamma_5)q \bar{b} \gamma_{\mu}(1 - \gamma_5)q \mid B^0 \rangle \equiv \frac{\sqrt{\pi}}{7} f_{B_q} B_{B_q} M_{B_q}^2$, where $f_{B_q}$ is the $B$ meson decay constant and $B_{B_q}$ denotes the $B$ parameter ($q$ represents $d$ or $s$ quark). Our prime interest is to calculate $B_{B_q}$, as unquenched calculations of $f_{B_q}$ are already available.

We include, however, the calculation of $f_{B_q}$ to provide a consistent set of the matrix element for $B^0 - \bar{B}^0$ mixing.

With current lattice calculations, systematic uncertainties due to the discretization error and the perturbative matching between continuum and lattice operators amount to 10–20%. One may improve the accuracy of $|V_{td}|$ by studying the ratio $\Delta M_s/\Delta M_d$ if $B_s - \bar{B}_s$ mixing is measured. The relevant quantity is $\xi \equiv (f_{B_s}/\sqrt{B_{B_s}})/(f_{B_d}/\sqrt{B_{B_d}})$, where many of the systematic uncertainties cancel in the lattice calculation.

A remaining major uncertainty arises from the chiral extrapolation of the lattice simulation which is made with relatively heavy dynamical quarks. One may resort to chiral perturbation theory (ChPT) as a theoretical guide for the extrapolation. The problem is that the currently available lattice data do not show the logarithmic behavior expected from long-distance pion loops in ChPT. It is in the scope of the present work to discuss the uncertainty in the the matrix elements associated from the chiral extrapolation in the absence of the observable logarithmic behavior.

The calculation is carried out on the unquenched gauge configurations generated at $\beta = 5.2$ on a $20^3 \times 48$ lattice. Two flavors of dynamical quarks for the $u$ and $d$ quarks are simulated at five values of quark mass in the range $(0.7 - 2.9)m_s$ with $m_s$ the physical strange quark mass. This corresponds to the pseudoscalar to vector mass ratio of 0.6–0.8. The hopping parameter chosen is $K_{sea} = 0.1340, 0.1343, 0.1346, 0.1350,$ and 0.1355. For each sea quark mass, 1,200 configurations are accumulated for measurements from 12,000 HMC trajectories separated by 10 trajectories. The lattice spacing we adopt is determined from $\rho$ meson mass and equals $2.22(4)$ GeV after the extrapolation to the chiral limit. This value is consistent with $2.19(^{+3}_{-2})$ GeV from the Sommer scale $r_0$ assuming the physical value of $0.49$ fm, and $2.25(5)$ GeV...
from $f_K$ (with an additional $O(5\%)$ error from the perturbative matching). This suggests that the large width of $\rho$ may not seriously affect the chiral extrapolation of the $\rho$ meson mass. Other details of the simulation are described in [8]. We adopt the lattice NRQCD formalism [9,10] for heavy quarks. The non-perturbatively improved Wilson quark action [11] is employed for both valence and sea light quarks.

We take five values of heavy quark mass $m_Q$ ($am_Q = 1.3, 2.1, 3.0, 5.0$ and $10.0$) to cover $m_Q = 3$–20 GeV for the NRQCD action that contains all corrections of the order $1/m_Q$. We take the valence light quark mass set equal to the sea quark mass and then extrapolate to the physical $u$- and $d$-quark masses, unlike the ‘partially quenched analysis’ often adopted in the literature. The strange quark is treated in the quenched approximation. There is the uncertainty in the determination of the strange quark mass depending upon which strange hadron is used as input. We take three values for the hopping parameter $K_s = 0.13465, 0.13468$ and $0.13491$ and interpolate to $K_s(K) = 0.13486(3)$ (for the $K$ meson as input) and $K_s(\phi) = 0.13471(4)$ (for the $\phi$ meson as input) [9]. The method to calculate $f_B$ and the $B$ parameter follows our previous studies in the quenched approximation [12,13].

Figure 1 shows the chiral extrapolation of the decay constants $f_{B_s}$ expressed in terms of $\Phi_{f_{B_s}} \equiv f_{B_s}/\sqrt{M_{B_s}}$ as a function of the pion mass squared. For $f_{B_s}$, the result is shown at $K_s = 0.13465$. In order to absorb the change of effective lattice scale that varies with $K_{sea}$ at a fixed bare coupling $\beta$, both axes are normalized with $r_0$ determined from the heavy quark potential at each sea quark mass. The heavy quark mass is interpolated to the $b$ quark using the lattice data.

Open triangles show quenched results obtained at a similar lattice spacing $1/a = 1.83(2)$ GeV ($\beta = 6.0$) with the non-perturbatively improved Wilson quark action. Our observation that they lie close to the unquenched data (filled circles) implies that the $B$ meson decay constant takes a similar value in quenched and two-flavor QCD if the scale is normalized by $r_0$. With the more conventional normalization of using the $\rho$ mass, however, the unquenched values are higher by about $20\%$ (see, e.g., [4]), as is seen by the fact that $r_0m_\rho = 1.91(2)$ for $N_f = 2$ while it is $2.20(3)$ on the quenched lattice [8]. This is understood as systematic errors of the quenched approximation, with which the determination of the lattice scale depends on which physical quantity is the input. With the dynamical quarks these errors are significantly reduced, leading to a convergent determination of the lattice scale.

The solid line represents a linear plus quadratic fit in $(r_0m_\pi)^2$, which describes the lattice data well. This fit, however, does not contain the chiral logarithmic term which is predicted by ChPT [12]:

$$\Phi_{f_{B_s}}^{(0)} = 1 - \frac{3(1 + 3\gamma^2)}{4} \ln \frac{m_\pi^2}{\mu^2} + \cdots,$$

where terms regular in $m_\pi^2$ are omitted, and the coupling $g$ is the $B^* B \pi$ interaction in ChPT. Lattice calculations [13,14] give a value consistent with the empirical one measured for $D^* \rightarrow D \pi$ decay, $g = 0.59 \pm 0.01 \pm 0.07$ [15]. Although there is an uncertainty in translating the value at the $D^*$ meson to that at the infinitely heavy quark mass (where the heavy-light ChPT is formulated), we take $g = 0.6$ to estimate the effect of the chiral logarithm.

Let us here consider a simpler case. For the pion decay constant the one-loop logarithmic term is controlled by the number of dynamically active quark flavors $N_f$ as

$$-\frac{N_f}{2} \ln \frac{m_\pi^2}{\mu^2} + \frac{m_\pi^2}{\mu^2} \ln \frac{m_\pi^2}{\mu^2} + \cdots$$

no uncertain parameters such as $g$ are involved. Thus, the test for the presence of the chiral logarithm is less ambiguous. Our high statistics unquenched data, shown in Figure 2, exhibit quite a linear behavior for $(r_0m_\pi)^2 > 2$, i.e., $m_\pi > 500$ MeV; no appreciable curvature characteristic of the chiral logarithm is observed.

One may suspect that pions in the simulation are too heavy to validate the use of ChPT. Another possibility may be the effect of explicit chiral symmetry breaking of the Wilson quark action at finite lattice spacings, as was discussed recently in the context of ChPT by [10] (see also [17,18]). Here we explore the more naturally looking, former possibility that the pion loop is suppressed for heavy pions and that the chiral logarithm manifests itself only for sufficiently small sea quark masses. The authors of [13] proposed a model that incorporate such a behavior by introducing a hard cutoff regularization of the one-loop ChPT calculation. This amounts to the replacement, $m_\pi^2 \ln m_\pi^2/\mu^2 \rightarrow m_\pi^2 \ln m_\pi^2/(m_\pi^2 + \mu^2)$, where $\mu$
The effect of the chiral logarithm is small for $f_{B_s}$, since the particle circulating the loop is kaon or eta. The formula in the partially quenched QCD is given in \[21\]. The chiral extrapolation is shown in Figure 1 with the lines for two extreme cases $\mu = 0$ and $\infty$ MeV. The difference between the two is only 1%.

To quote our results we take the central value from the polynomial fit and include the variation in the presence of the chiral logarithm as an error. We obtain

$$f_{B_s} = 191(10)^{+6}_{-10}/(12)(-) \text{ MeV},$$  

$$f_{B_d} = 215(9)^{+6}_{-0}/(13)(+6) \text{ MeV},$$  

$$\frac{f_{B_s}}{f_{B_d}} = 1.13(3)^{+12}_{-0}/(2)^{+3},$$

where the first error is statistical, the second is the uncertainty from the chiral extrapolation and the other two are systematic errors explained in what follows. The error from the chiral extrapolation is one sided, since the polynomial fit is taken as our central value. The systematic error given in the third parenthesis is those arising from the finite lattice spacing (truncation of the actions and currents, and their perturbative matching) and the truncation of terms higher order in $1/m_Q$ in the NRQCD action. An order estimate of the truncation errors shows that the most important contributions are $O(A_0^2/m_Q^2) \sim 4\%$ and $O(a_s^2) \sim 4\%$. We add these errors by quadratures together with other possible errors. In [10], it is shown for the quenched lattice that such estimates correctly describe the error of finite lattice spacing (see also [11] for $B_B$). The errors we obtained is consistent with those in the quenched case at the comparable lattice spacing. The errors in the last parenthesis represent the ambiguity in the determination of the strange quark mass, for which we adopt the value with the $K$ mass as the central value.

For the $B$ parameter, ChPT predicts $-(1-3g^2)/2$ for the coefficient of the chiral log term instead of $3(1+3g^2)/4$ in [11, 21]. Therefore, the effect of the chiral logarithm is negligible in practice. For $B_B$, there is no chiral logarithm as a function of sea quark mass.

Figure 3 shows the chiral extrapolation of $B_{B_s}(\mu_b)$ at $\mu_b = m_b = 4.8$ GeV and the fits without the chiral logarithm. The triangles show the quenched results. The sea quark effect is small for this quantity.

Our unquenched results obtained with a linear chiral extrapolation are

$$B_{B_s}(m_b) = 0.836(27)^{+0}_{-27}/(56)(-),$$  

$$B_{B_s}(m_b) = 0.850(22)^{+0}_{-27}/(57)(+5),$$  

$$\frac{B_{B_s}}{B_{B_d}} = 1.017(16)^{+5}_{-0}/(17)(+6).$$

The meaning of errors is the same as for $f_B$, except for the second one, i.e., those associated with the chiral extrapolation: we take the central value from the linear fit

![Diagram](image-url)
and put the difference from the linear plus quadratic fit as the systematic error.

The amplitude of neutral $B$ meson mixing is proportional to $f_B^2 B_B$. Using the conventionally adopted renormalization-scale independent definition $\hat{B}_B$, which is related to $B_B(m_b)$ as $\hat{B}_B = 1.528 B_B(m_b)$ for $\Lambda^{(5)}_{\overline{MS}} = 225$ MeV, we find

$$f_{B_d} \hat{B}_{B_d} = 215(11)(^{+6}_{-2})(15)(-) \text{ MeV}, \quad (8)$$

$$f_{B_s} \hat{B}_{B_s} = 245(10)(^{+3}_{-2})(17)(^{+7}_{-5}) \text{ MeV}, \quad (9)$$

and for the SU(3) breaking ratio $\xi$,

$$\xi = 1.14(3)(^{+1.3}_{-0.7})(^{2}_{-1}) \text{ MeV}, \quad (10)$$

The chiral extrapolation gives the largest entry of systematic errors for $\xi$, as also suggested in [22]. Compared to the commonly assumed number in the phenomenological analysis, $f_{B_d} \hat{B}_{B_d} = 230(28)(28)$ MeV (e.g., [23]), where the second error is the quenching uncertainty, our central value of (8) is slightly lower and the quenching error is eliminated.

In conclusion we have obtained unquenched lattice estimate of the $B^0 - \bar{B}^0$ mixing matrix elements, including the decay constant, in a consistent set of simulations. Although the simulation is made at a relatively large mass of dynamical quarks, we explored the range of errors associated with the chiral extrapolation: we expect that the true values of the matrix elements are within the range of indicated errors, even if the chiral logarithm would become manifest at a small quark mass.

This work is supported by Large Scale Simulation Program No. 79 (FY2002) of High Energy Accelerator Research Organization (KEK), and also in part by the Grants-in-Aid of the Ministry of Education (Nos. 12740133, 13135204, 13640259, 13640260, 14046202, 14540289, 14740173, 15204015, 15540251, 15540279).

[1] B. Aubert et al. [BABAR Collaboration], Phys. Rev. Lett. 89, 201802 (2002) [arXiv:hep-ex/0207042].
[2] K. Abe et al. [Belle Collaboration], Phys. Rev. D 66, 071102 (2002) [arXiv:hep-ex/0208025].
[3] K. Hagihara et al. [Particle Data Group Collaboration], Phys. Rev. D 66, 010001 (2002).
[4] N. Yamada, a review talk at 20th International Symposium on Lattice Field Theory (LATTICE 2002), Boston, Massachusetts, 24-29 Jun 2002; arXiv:hep-lat/0210035.
[5] S. Hashimoto et al. [JLQCD Collaboration], at 20th International Symposium on Lattice Field Theory (LATTICE 2002), Boston, Massachusetts, 24-29 Jun 2002; arXiv:hep-lat/0209091.
[6] S. Aoki et al. [JLQCD Collaboration], arXiv:hep-lat/0212039.
[7] B. A. Thacker and G. P. Lepage, Phys. Rev. D 43, 196 (1991).
[8] G. P. Lepage, L. Magnea, C. Nakhle, U. Magnea and K. Hornbostel, Phys. Rev. D 46, 4052 (1992) [arXiv:hep-lat/9205007].
[9] K. Jansen and R. Sommer [ALPHA collaboration], Nucl. Phys. B 530, 185 (1998) [Erratum-ibid. B 643, 517 (2002)] [arXiv:hep-lat/9803017].
[10] K. I. Ishikawa et al. [JLQCD Collaboration], Phys. Rev. D 61, 074501 (2000) [arXiv:hep-lat/9905036].
[11] S. Aoki et al. [JLQCD Collaboration], Phys. Rev. D 67, 014506 (2003) [arXiv:hep-lat/0208038].
[12] B. Grinstein, E. Jenkins, A. V. Manohar, M. J. Savage and M. B. Wise, Nucl. Phys. B 380, 369 (1992) [arXiv:hep-ph/9204207].
[13] G. M. de Divitiis, L. Del Debbio, M. Di Pierro, J. M. Flynn, C. Michael and J. Peisa [UKQCD Collaboration], JHEP 9810, 010 (1998) [arXiv:hep-lat/9807032].
[14] A. Abada et al., Phys. Rev. D 66, 074504 (2002) [arXiv:hep-ph/0206237].
[15] A. Anastassov et al. [CLEO Collaboration], Phys. Rev. D 65, 032003 (2002) [arXiv:hep-ex/0108043].
[16] S. Aoki, arXiv:hep-lat/0306027.
[17] G. Rupak and N. Shoresh, Phys. Rev. D 66, 054503 (2002) [arXiv:hep-lat/0201019].
[18] O. Baer, G. Rupak and N. Shoresh, arXiv:hep-lat/0306021.
[19] W. Detmold et al., Phys. Rev. Lett. 87, 172001 (2001) [arXiv:hep-lat/0103006].
[20] J. J. Sanz-Cillero, J. F. Donoghue and A. Ross, arXiv:hep-ph/0305181.
[21] S. R. Sharpe and Y. Zhang, Phys. Rev. D 53, 5125 (1996) [arXiv:hep-lat/9510037].
[22] A. S. Kronfeld and S. M. Ryan, Phys. Lett. B 543, 59 (2002) [arXiv:hep-ph/0206058].
[23] A. Hocker, H. Lacker, S. Laplace and F. Le Diberder, Eur. Phys. J. C 21, 225 (2001) [arXiv:hep-ph/0104062].