Predictability, Distinguishability and Entanglement

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Recent times have seen a spurt of research activity focused on “completing” certain wave-particle duality relations using entanglement or polarization. These studies use a duality relation involving path-predictability, and not path-distinguishability. Quantum origins of these results are explored here, in the more general framework of multipath quantum interference. Multipath interference with a path-detector is theoretically analyzed to find the connection between predictability and distinguishability. It is shown that entanglement is what quantitatively connects distinguishability with predictability. Thus, a duality relation between distinguishability and coherence, can also be viewed as a triality between predictability, entanglement and coherence. There exist two different kind of duality relations in the literature, which pertain to two different kinds of interference experiments, with or without a path-detector. Results of this study show that the two duality relations are quantitatively connected via entanglement. The roots of the new results in the classical optical domain, including the polarization coherence theorem, can be understood in the light of this work. Additionally, the triality relations obtained can quantify wave-particle duality in the interesting case of a quanton with an internal degree of freedom. The relations can also be employed to experimentally determine the degree of bipartite entanglement.

Introduction.— Wave-particle duality is an old subject in quantum physics [1], and has seen a sustained interest, which has only increased with time. Two-path interference became a testing ground for all such ideas. Greenberger and Yasin looked at the issue of wave-particle duality from a simplified perspective where the “particleness” is inferred only if the quanton is more likely to follow one path than the other. For such a scenario, they derived the inequality [2, 3]

\[ P^2 + V^2 \leq 1, \quad (1) \]

where \( P \) is path-predictability, or just predictability, and \( V \) is the interference visibility, a measure of the wave nature of the quanton. In a two-path experiment, the states corresponding to the quanton passing through different paths, are necessarily orthogonal, and they can be used as basis states of an effectively 2-dimensional Hilbert space. The density matrix of the quanton can be written in this basis, and the diagonal elements \( \rho_{11}, \rho_{22} \) represent the probabilities of the quanton to pass through path 1 and 2, respectively. Predictability is defined as \[ P = |\rho_{11} - \rho_{22}| = \sqrt{1 - 4\rho_{11}\rho_{22}}. \quad (2) \]

It is obvious that if the two beams are of same intensity, \( P = 0 \). The visibility is defined simply as the standard fringe contrast

\[ V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}, \quad (3) \]

where \( I_{\text{max}}, I_{\text{min}} \) are the maximum and minimum intensities, respectively. The inequality (1) becomes an equality if the quanton state is pure, but if the quanton state is not pure due a variety of reasons, the inequality holds. On the other hand, if one wants to experimentally know which of the two paths the quanton followed, even if it is equally likely to go through any of the two, one has to have a path-detector in place, to make the two paths distinguishable [4]. Englert derived the following duality relation for such a situation [5]

\[ D^2 + V^2 \leq 1, \quad (4) \]

where \( D \) is called path-distinguishability, or just distinguishability. Here too, the inequality becomes an equality if the quanton and path-detector state is pure. Wave-particle duality has also been widely explored in classical optics. It should be emphasized here that in the classical optical scenario, the duality relation of the kind (1) is meaningful. Since a question like, which of the two paths did the quanton actually follow, are not meaningful in the classical optical realm, formulation of a duality relation of the kind (4) is not needed there. However, any experimental test of (4) will also hold in the classical optical domain.

Recently it was shown that, in the realm of classical optics, including polarization in the two-slit interference experiment, the following “triality” relation can be derived [6]

\[ D^2 + V^2 + C^2 = 1, \quad (5) \]

where \( D \) is what the authors call “distinguishability,” but should actually be identified with the predictability \( P \), \( V \) is the interference visibility given by (3) and \( C \) is concurrence, a measure of entanglement. The authors interpret this equality as “completing the duality relation” given by (1), which is an inequality. This work was built upon their earlier work recognizing entanglement like non-separability involving polarization or a generalized polarization, which has important consequences in classical optics [7, 8], and also upon the recognition of the role of entanglement in complementarity relations.
Photon
Path-detector
Multi path

FIG. 1. Schematic representation of a n-path interference experiment with a path-detector.

in a bipartite system of two qubits [9]. The relationship between polarization, indistinguishability, and entanglement has also been probed earlier [10]. This work was followed by a flurry of papers on similar theme [11–16]. These results point to the interesting role of entanglement in wave-particle duality. Extending these ideas to the level of single photons should throw more light on the quantum origin of these results. That is the aim of the present investigation. In particular, we consider multipath quantum interference in the presence of a path detector.

**Multipath interference.**— Let us consider a quanton passing through a multipath arrangement, and there is a path-detector present whose precise nature we need not specify for the present purpose (see FIG. 1). We start by specifying a general pure state of a quanton passing through a n-path interferometer. If \(|\psi_k\rangle\) represent the state corresponding to the quanton taking the k’th path, the initial state, before the quanton interacts with the path-detector, is given by

\[
|\Psi_0\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots + c_n|\psi_n\rangle,
\]  

where \(|c_k|^2\) represents the probability of the quanton taking the k’th path. The states \(|\psi_k\rangle\} can be assumed to form an orthonormal set, without loss of generality. The combined state of the quanton and the path-detector, as the quanton emerges from the multipath arrangement, can be represented as

\[
|\Psi\rangle = c_1|\psi_1\rangle|d_1\rangle + c_2|\psi_2\rangle|d_2\rangle + \cdots + c_n|\psi_n\rangle|d_n\rangle,
\]  

where \(|d_i\rangle\} represent certain normalized states of the path-detector which may not necessarily be orthogonal to each other. This entanglement is a fundamental requirement of the process of measurement, as laid down by von Neumann [17].

In the absence of the path-detector, one can derive a duality relation [18]

\[
P^2 + C^2 = 1.
\]  

where \(P\) is a generalized predictability

\[
P \equiv \sqrt{1 - \frac{1}{n-1} \sum_{j \neq k} \sqrt{\rho_{jj} \rho_{kk}}}.
\]  

\(\rho\) being the density operator for the pure quanton state \(|\Psi_0\rangle\rangle\), and the basis states are taken to be \(|\psi_i\rangle\}. For \(n = 2\), (9) reduces to (2). The wave-nature is quantified by the recently introduced measure of coherence \([19, 20]\)

\[
C \equiv \frac{1}{n-1} \sum_{j \neq k} |\rho_{jk}|,
\]  

where \(\rho\) is the density operator of the quanton, and the basis is chosen to be \(|\psi_i\rangle\}. This duality relation (8) is the generalization of Greenberger and Yasin’s relation (1) to n paths. For \(n = 2\), the \(P\) and \(C\) of (8) reduced to the \(P\) and \(V\) of (1), respectively. If the quanton experiences some incoherence because of some reasons, (8) becomes an inequality.

Interestingly, a simpler form of predictability can be formulated, as has been shown earlier in the classical case [21]. This simpler predictability has the following form

\[
P_Q \equiv 1 - \frac{1}{n-1} \sum_{j \neq k} \sqrt{\rho_{jj} \rho_{kk}}.
\]

This yields a simple and interesting duality relation

\[
P_Q + C = 1.
\]  

We wish to emphasize here that the coherence \(C\), given by (14) is a good measure of the wave-nature. Just like conventional visibility, it can be measured in an interference experiment [20, 28]. Using these definitions, a tight
multopath wave-particle duality relation has been derived before [23]

\[ D_Q + C = 1. \]  \hspace{1cm} (15)

This relation is an equality even though the quanton is entangled with the path-detector. It can also be cast in a quadratic form similar to that of (4), by defining a different distinguishability, as \( D = \sqrt{D_Q(2 - D_Q)} \). With this new distinguishability, one can write the tight duality relation [29]

\[ D^2 + C^2 = 1, \]  \hspace{1cm} (16)

which should be viewed as the multipath generalization of (4). It should be mentioned here that for \( n = 2 \), this \( D \) reduces to the \( D \) of (4), and (16) reduces to (4).

**Distinguishability and Predictability.**—Relations (8) and (12) involve predictability, and pertain to an experiment without any path-detector. Relations (16) and (15) involve distinguishability, and pertain to interference experiments where there is a path-detector. Could there be a connection between the two? From (11) and (13), path-distinguishability may be written as

\[ D_Q = 1 - \frac{1}{n-1} \sum_{i \neq j} \sqrt{\rho_{ii}\rho_{jj}} \]

\[ + \frac{1}{n-1} \sum_{i \neq j} (\sqrt{\rho_{ii}\rho_{jj}} - \sqrt{\rho_{ii}\rho_{jj}}|d_i|d_j|) \]

\[ = P_Q + E_Q. \]  \hspace{1cm} (17)

Now, the quantity \( E_Q \) is interesting, and can be written as follows

\[ E_Q = \frac{1}{n-1} \sum_{i \neq j} (\sqrt{\rho_{ii}\rho_{jj}} - \sqrt{\rho_{ii}\rho_{jj}}|d_i|d_j|) \]

\[ = \frac{1}{n-1} \sum_{i \neq j} (\sqrt{\rho_{ii}\rho_{jj}} - |\rho_{ij}|), \]  \hspace{1cm} (18)

where \( \rho_{ij} \) is the reduced density operator for the state \( |\Psi\rangle \), given by (7). Notice that if all \( |d_i| \) are identical, i.e., the state \( |\Psi\rangle \) is disentangled, \( E_Q = 0 \). If all \( |d_i| \) are mutually orthogonal, and \( |c_i| = 1/\sqrt{n} \), for all \( i \), which means the state is maximally entangled, \( E_Q = 1 \). Thus \( E_Q \) can be considered a measure of entanglement. It is interesting to compare \( E_Q \) to a well known entanglement measure, *I-concurrence* defined as [30, 31]

\[ E^2_c = 2 \sum_{i \neq j} (\rho_{ii}\rho_{jj} - |\rho_{ij}|^2) . \]  \hspace{1cm} (19)

One can see that \( E_Q \) is a measure of purity of \( \rho_{ij} \) in the same way as \( E_c \). It is expected to show the same monotonicity as \( E_c \). Thus we can consider \( E_Q \) as a good measure of entanglement. So, the relation

\[ D_Q = P_Q + E_Q \]  \hspace{1cm} (20)

shows that distinguishability is connected to predictability in a precise way through entanglement. The duality relation (15) can now be written as

\[ P_Q + C + E_Q = 1, \]  \hspace{1cm} (21)

which is a triality relation between predictability, quantum coherence and entanglement.

A similar analysis can be carried out for the quadratic form of the duality relation (16). Here we write \( D \) in terms of \( P \) using (9):

\[ D^2 = D_Q(2 - D_Q) = 1 - \left( \frac{1}{n-1} \sum_{i \neq j} \sqrt{\rho_{ii}\rho_{jj}}|d_i|d_j| \right)^2 \]

\[ = 1 - \left( \frac{1}{n-1} \sum_{i \neq j} \sqrt{\rho_{ii}\rho_{jj}} \right)^2 \]

\[ + \frac{1}{(n-1)^2} \left[ \left( \sum_{i \neq j} \sqrt{\rho_{ii}\rho_{jj}} \right)^2 - \left( \sum_{i \neq j} \sqrt{\rho_{ii}\rho_{jj}}|d_i|d_j| \right)^2 \right] \]

\[ = P^2 + E^2, \]  \hspace{1cm} (22)

where

\[ E^2 = \frac{1}{(n-1)^2} \left[ \left( \sum_{i \neq j} \sqrt{\rho_{ii}\rho_{jj}} \right)^2 - \left( \sum_{i \neq j} \sqrt{\rho_{ii}\rho_{jj}}|d_i|d_j| \right)^2 \right] \]

(23)

can also be considered a measure of entanglement. For \( n = 2 \), \( E \) reduces to the I-concurrence \( E_c \) given by (19). For \( n > 2 \), \( E \) is bounded between 0 and 1, whereas \( E_c \) is not. Thus we arrive at another triality relation

\[ P^2 + C^2 + E^2 = 1, \]  \hspace{1cm} (24)

between predictability, coherence and entanglement. This relation may be looked upon as a multipath, quantum version of (5). It should rather be looked upon as the root from which relations like (5) arise, because the derivation is fully quantum. The entanglement in it is also genuinely quantum. In the classical optical regime, it is not genuine entanglement, but non-separability of two degrees of freedom of the light field [32]. Similar ideas have recently been explored in a somewhat different way [33].

**Entanglement and wave-particle duality.**—This result is profound as it tells us that entanglement is what connects distinguishability to predictability, something that was not recognized earlier. It tells us that the pairs of relations (1) and (4), (8) and (16), and (12) and (15), are not unconnected duality relations. One is connected to the other through entanglement. For example, if in a multipath interference, the entanglement measure (between the quanton and path detector) goes to zero, one smoothly goes from (say) (16) to (8).

Connecting distinguishability to predictability is just one aspect of this analysis. The role of entanglement is more than that. If the system, the quanton is entangled with, is part of the path-detecting device, then of
course entanglement is useful in relating distinguishability to predictability. But what if the quanton is entangled to a system which is not part of the path-detecting apparatus? For example, in a two-path neutron interference experiment, if the spin state of the neutron is different in each path, the triality relations (21) and (24) will tightly quantify the relation between the predictability, visibility (or coherence) and the measure of entanglement between the neutron paths and the neutron spin. In this respect, entanglement can be considered an integral part of wave-particle duality.

Several years back an interesting question was addressed in the context of wave-particle duality. If the quanton is equipped with an internal degree of freedom, e.g., spin, and the interaction with the environment leads to some path information being deposited in the environment, can one infer the amount this information from the interference visibility [34]? A duality relation was formulated for such a situation, using a much involved “generalized visibility” [34]. It is easy to see that the triality relations (21) and (24) can accomplish this task in a much simpler way. If one has knowledge of the intensities of the two beams, and hence the predictability, measuring coherence from the interference will yield information about how much entanglement has been generated between the quanton and the environment. These relations are also applicable to the already studied problem of duality relation for a quanton in the presence of a quantum memory [35].

These triality relations may also be used to measure bipartite entanglement in some experiments. If one wants to quantify the entanglement between a quanton and an ancillary system, knowing the relative intensity of the two beams, and measuring coherence from the interference will yield the measure of entanglement.

Polarization coherence theorem.— A recently introduced notion, in classical optics, was to connect the wave-particle duality relation of the kind (1) to a generalized version of polarization. The result was the polarization coherence theorem [8]

\[ P_P^2 = D^2 + V^2 \]  

where \( P_P \) is the “degree of polarization.” It could represent polarization due to the spin, or a “mode polarization” related different spatial modes. The quantity \( D \) should be identified with predictability. The triality relation (24) can be suggestively written as

\[ 1 - \mathcal{E}^2 = \mathcal{P}^2 + \mathcal{C}^2, \]

where \( \sqrt{1 - \mathcal{E}^2} \) plays the role of the “generalized polarization” \( P_P \). In fact, for \( n = 2 \)

\[ 1 - \mathcal{E}^2 = 1 - 4 \left[ \rho_{r11} \rho_{r22} - |\rho_{r12}|^2 \right] \]

is closely similar to the expression of \( P_P^2 \) in eqn. (9) of Ref. [8]. This shows that the polarization coherence theorem has origins in the entanglement between the quanton paths and some degrees of freedom or “modes.”

Conclusions.— We have looked at multipath interference of a quanton in the presence of a pathdetector. We find that the path-distinguishability is quantitatively connected to path-predictability via entanglement. The well known duality relations between path-distinguishability and coherence can then be viewed as triality relations between path-predictability, coherence and entanglement. We believe that these quantum mechanical results lie at the root of a lot of recent interesting results in the realm of classical optics. The works on “completing” wave-particle duality [6], and “turning off” duality [14], are interestingly connected to the relation between distinguishability and predictability. The results obtained here also connect with the recently formulated polarization coherence theorem [8]. The triality relations obtained here, have several interesting uses. They show that the two different kinds of duality relations, that have independently existed in the literature, one based on predictability, and the other based on distinguishability, are quantitatively connected to each other via entanglement. This is something that was not recognized earlier. The triality relations provide a simpler solution to the problem of wave-particle duality of a quanton with an internal degree of freedom [34], or that of a quanton in the presence of quantum memory [35]. With some ingenuity, the relations can be employed for measuring the degree of bipartite entanglement by looking at the interference of the quanton. We believe that this new picture in terms of the triality between predictability, coherence, entanglement, opens the door to a wide variety of interesting possibilities.

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