Hunting for Reduced Polytopes

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Abstract We show that there exist reduced polytopes in three-dimensional Euclidean space. This partially answers the question posed by Lassak (Israel J Math 70(3):365–379, 1990) on the existence of reduced polytopes in d-dimensional Euclidean space for d ≥ 3.

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1 Introduction

Constant width bodies, i.e., convex bodies for which parallel supporting hyperplanes have constant distance, have a long and rich history in mathematics [3]. Due to Meissner [18], constant width bodies in Euclidean space can be characterized by diametrical completeness, that is, the property of not being properly contained in a set of the same diameter. Constant width bodies also belong to a related class of reduced convex bodies introduced by Heil [6]. This means that constant width bodies do not properly contain a convex body of same minimum width. Remarkably, the classes of reduced bodies and constant width bodies do not coincide, as a regular triangle in the Euclidean plane shows.

Reduced bodies are extremal in remarkable inequalities for prescribed minimum width, as in Steinhagen’s inequality [3] or Pál’s problem [19], so far only solved in the planar case. In full generality, any non-decreasing inclusion functional of convex bodies with prescribed minimum width attains its minimum at some reduced body.

Reduced bodies in the Euclidean space have been extensively studied in [9,10,14], and the concept of reducedness has been translated to finite-dimensional normed spaces [12,13,15]. In reference to the existence of reduced polygons in the Euclidean plane, Lassak [9] posed the question whether there exist reduced polytopes in Euclidean $d$-space for $d \geq 3$. Several authors addressed the search for reduced polytopes in finite-dimensional normed spaces [1,2,11,16,17]. For Euclidean space starting from dimension 3 several polytopes were proved to be not reduced, such as pyramids with polytopal base [1, Thm. 1] (cf. also [16,17]), symmetric polytopes [13, Cl. 2], or simple polytopes [2, Cor. 8]. The purpose of this article is to present a reduced polytope in $\mathbb{R}^3$ in Sect. 3. The validity of our example can be checked using the algorithm provided in Sect. 4.

2 Notation and Basic Results

Throughout this paper, we work in $d$-dimensional Euclidean space, that is, the vector space $\mathbb{R}^d$ equipped with the inner product $\langle x \mid y \rangle := \sum_{i=1}^{d} x_i y_i$ and the norm $\|x\| := \sqrt{\langle x \mid x \rangle}$, where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ denote two points in $\mathbb{R}^d$. A subset $K \subset \mathbb{R}^d$ is said to be convex if the line segment

$$[x, y] := \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$$

is contained in $K$ for all choices of $x, y \in K$. Convex compact subsets of $\mathbb{R}^d$ having non-empty interior are called convex bodies. The smallest convex superset of $K \subset \mathbb{R}^d$ is called its convex hull $\text{co}(K)$, whereas the smallest affine subspace of $\mathbb{R}^d$ containing $K$ is denoted by $\text{aff}(K)$, the affine hull of $K$. The affine dimension $\text{dim}(K)$ of $K$ is the dimension of its affine hull. The support function $h(K, \cdot) : \mathbb{R}^d \to \mathbb{R}$ of $K$ is defined by
\[ h(K, u) := \sup \{ \langle u \mid x \rangle : x \in K \} . \]

For \( u \in \mathbb{R}^d \setminus \{0\} \), the hyperplane \( H(K, u) := \{ x \in \mathbb{R}^d : \langle u \mid x \rangle = h(K, u) \} \) is a supporting hyperplane of \( K \). The width of \( K \) in direction \( u \in \mathbb{R}^d \), defined by

\[ w(K, u) := h(K, -u) + h(K, u), \]

equals the distance of the supporting hyperplanes \( H(K, \pm u) \) multiplied by \( \|u\| \). The minimum width of \( K \) is \( \omega(K) := \inf \{ w(K, u) : \|u\| = 1 \} \). A polytope is the convex hull of finitely many points. The boundary of a polytope consists of faces, i.e., intersections of the polytope with its supporting hyperplanes. We shall refer to faces of affine dimension 0, 1, and \( d - 1 \) as vertices, edges, and facets, respectively. Faces of polytopes are lower-dimensional polytopes and shall be denoted by the list of their vertices. By definition, attainment of the minimal width of a polytope \( P \) is related to a binary relation on faces of \( P \) called strict antipodality, see [1].

**Definition 2.1** Let \( P \subset \mathbb{R}^d \) be a polytope. Distinct faces \( F_1, F_2 \) of \( P \) are said to be strictly antipodal if there exists a direction \( u \in \mathbb{R}^d, \|u\| = 1 \), such that \( H(P, u) \cap P = F_1 \) and \( H(P, -u) \cap P = F_2 \).

Gritzmann and Klee [5, (1.9)] formulated a necessary condition on strictly antipodal pairs whose distance equals the minimum width. Here, \( F_1 + F_2 = \{ x + y : x \in F_1, y \in F_2 \} \) denotes the Minkowski sum of sets \( F_1, F_2 \subset \mathbb{R}^d \).

**Theorem 2.2** Suppose that \( P \subset \mathbb{R}^d \) is a polytope with non-empty interior, and that \( F_1 \) and \( F_2 \) form a strictly antipodal pair of faces of \( P \) whose distance is equal to \( \omega(P) \). Then

\[ \dim(F_1 + F_2) = d - 1, \] \( (2.1) \)

with \( \dim(F_1) = \dim(F_2) = d - 1 \) when \( P \) is centrally symmetric.

For arbitrary subsets \( A, B \subset \mathbb{R}^d \), we shall denote by

\[ \rho(A, B) = \inf \{ \|x - y\| : x \in \text{aff}(A), y \in \text{aff}(B) \} \]

the minimal distance between points of \( \text{aff}(A) \) and \( \text{aff}(B) \). In the situation of Theorem 2.2, \( \rho(F_1, F_2) \) is then said to be the distance between the respective parallel supporting hyperplanes of \( P \).

The following definition by Heil [6] is central to the present investigation.

**Definition 2.3** A convex body \( K \) is reduced if we have \( \omega(K') < \omega(K) \) for all convex bodies \( K' \subset K \).

Reduced polytopes can be characterized using vertex-facet distances, see [2, Thm. 4] and [11, Thm. 1] for the following result (cf. also [1, Lem. 2]).

**Theorem 2.4** A polytope \( P \subset \mathbb{R}^d \) is reduced if and only if for every vertex \( v \) of \( P \), there exists a strictly antipodal facet \( F \) of \( P \) such that the distance between \( v \) and \( \text{aff}(F) \) equals \( \omega(P) \).
3 A Reduced Polytope

In contrast to the various classes of polytopes which are shown to be non-reduced in the literature, we present a reduced polytope $P$ now. Consider the points

$$
\begin{align*}
    v_1 &:= (r, 0, -t), & v_2 &:= (-r, 0, -t), & v_3 &:= (0, r, t), \\
    v_4 &:= (0, -r, t), & v_5 &:= (h, x, s), & v_6 &:= (-h, x, s), \\
    v_7 &:= (h, -x, s), & v_8 &:= (-h, -x, s), & v_9 &:= (x, h, -s), \\
    v_{10} &:= (x, -h, -s), & v_{11} &:= (-x, h, -s), & v_{12} &:= (-x, -h, -s).
\end{align*}
$$

For properly chosen parameters $t, x, s, h, r > 0$ the points $v_1, \ldots, v_{12}$ are the vertices of our polytope $P$. The combinatorial structure of this polytope is shown in Fig. 1.

The polytope $P$ possesses the same symmetry as the Zalgaller–Johnson solid $J_{84}$ (however, not the same combinatorial structure, cf. [7,20]). Hence, it is sufficient to
control few facet–vertex and edge–edge distances. In fact, we are going to solve the equations

\[
\begin{align*}
\rho(v_1, v_3 v_1 v_12 v_4) &= 1, & \rho(v_1 v_2, v_3 v_4) &= \delta_1, & \rho(v_1 v_9, v_4 v_8) &= \delta_3, \\
\rho(v_5, v_2 v_8 v_12) &= 1, & \rho(v_1 v_5, v_4 v_8) &= \delta_2,
\end{align*}
\]

with respect to \( t, x, s, h, r \). Here, \( \delta_1, \delta_2, \delta_3 \geq 1 \) are suitably chosen. By introducing the normal vectors

\[
\begin{align*}
n_1 &:= (v_{11} - v_3) \times (v_{12} - v_3), & n_4 &:= (v_1 - v_5) \times (v_4 - v_8), \\
n_2 &:= (v_8 - v_2) \times (v_{12} - v_2), & n_5 &:= (v_1 - v_9) \times (v_4 - v_8), \\
n_3 &:= (v_1 - v_2) \times (v_3 - v_4) = (0, 0, 4 r^2),
\end{align*}
\]

where \( u \times w \) denotes the usual cross product of the vectors \( u, w \in \mathbb{R}^3 \), these equations can be rewritten as

\[
\begin{align*}
\langle n_1 | v_1 - v_3 \rangle^2 - \|n_1\|^2 &= 0, \\
\langle n_3 | v_3 - v_1 \rangle^2 - \delta_1^2 \|n_3\|^2 &= 0, \\
\langle n_5 | v_1 - v_4 \rangle^2 - \delta_2^2 \|n_5\|^2 &= 0, \\
\langle n_2 | v_5 - v_2 \rangle^2 - \|n_2\|^2 &= 0, \\
\langle n_4 | v_1 - v_4 \rangle^2 - \delta_3^2 \|n_4\|^2 &= 0.
\end{align*}
\]

Now, it is easy to see that the third equation is equivalent to \( 2 t = \delta_1 \). Moreover, it is tedious to check that we can factor out \( h^2 \) in the first equation and \( (h + r - x)^2 \) in the fifth. Hence, we are going solve the four equations

\[
\begin{align*}
\begin{cases}
\langle n_1 | v_1 - v_3 \rangle^2 - \|n_1\|^2 = 0, \\
\langle n_2 | v_5 - v_2 \rangle^2 - \|n_2\|^2 = 0, \\
(h + r - x)^{-2} \left( \langle n_5 | v_1 - v_4 \rangle^2 - \delta_3^2 \|n_5\|^2 \right) = 0, \\
\langle n_4 | v_1 - v_4 \rangle^2 - \delta_2^2 \|n_4\|^2 = 0,
\end{cases}
\end{align*}
\]

under \( t = \delta_1/2 \) with respect to the remaining variables \( \langle x, s, h, r \rangle \). Note that each left-hand side of the four equations in (3.1) are multivariate polynomials of degree at most 6 in the four unknowns \( \langle x, s, h, r \rangle \).

Numerically, we use \( \delta_1 = 1.1, \delta_2 = 1.003 \) and \( \delta_3 = 1.004 \) and solve equations (3.1) by Newton’s method starting with \( (x_0, s_0, h_0, r_0) = (0.62, 0.13, 0.09, 0.35) \). After 5 iterations, this results in

\[
\begin{align*}
t &= 0.55, & x &\approx 0.6176490959799, & s &\approx 0.1351384931026, \\
h &\approx 0.0984300252409, & r &\approx 0.3547183586709
\end{align*}
\]
and the numerical residuum in the four equations is below $10^{-15}$. In order to prove that the polynomial system (3.1) has an exact root in the neighborhood of our numerical approximation, we employ Kantorovich’s theorem, see [8, Thm. XVIII.1.6].

**Theorem 3.1** Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuously differentiable and let $z \in \mathbb{R}^n$ and a neighborhood $U$ of $z$ be given such that $F'(z)$ is invertible and such that the conditions

\[ \| F'(z_1) - F'(z_2) \| \leq L \| z_1 - z_2 \| \quad \forall z_1, z_2 \in U, \]

\[ B(z, a) \subset U, \quad \text{where } a := 2 \| F'(z) \|^{-1} \| F(z) \|, \]

\[ L \| F'(z)^{-1} \|^2 \| F(z) \| < \frac{1}{2} \]

are satisfied, where $B(z, a) = \{ x \in \mathbb{R}^d : \| x - z \| \leq a \}$ and $L > 0$ is a constant. Then, the iterates of Newton’s method starting in $z$ converge to a point $\hat{z} \in B(z, a)$ with $F(\hat{z}) = 0$.

Since $F$ given by (3.1) is a polynomial and since the initial residual is already very small, it is straightforward to verify the applicability of Theorem 3.1 with $z$ as given in (3.2). Indeed, the interested reader may verify the validity of the hypothesis of Theorem 3.1 with $z = (x, s, h, r)$ given in (3.2), which results in $\| F(z) \| \leq 2 \cdot 10^{-13}$, $\| F'(z)^{-1} \| \leq 10$, $L := 50$, and $U := B(z, \hat{a})$ with $a \leq \hat{a} := 4 \cdot 10^{-12}$. Hence, successive iterations of Newton’s method with this initial condition converge to an exact solution of (3.1) in a small neighborhood (namely $B(z, a)$) of our numerical approximation (3.2). Moreover, the residual drops below $10^{-15}$ after one iteration and this yields that the 13 digits provided in (3.2) are all the right ones of the exact solution.

Using these parameters, we can check that the remaining distances are

\[ \rho(v_1 v_9 v_{10}, \ v_{11} v_{12}) \approx 1.0433929735637, \quad \rho(v_5 v_9, \ v_8 v_{12}) \approx 1.0126888049628. \]

Thus, the width of our polytope using these parameters is really 1.0, see Theorem 2.2. Consequently, our polytope is reduced by Theorem 2.4.

Since the Jacobian of the (left-hand sides of) equations (3.1) with respect to $(x, s, h, r)$ is invertible at our point of interest, it follows from the implicit function theorem that we also obtain a solution for small changes of the parameters $\delta_1$, $\delta_2$, and $\delta_3$. Hence, we obtain a whole family of reduced polytopes possessing three degrees of freedom.

### 4 Evaluating Your Catches

It is quite a delicate and tedious procedure to check the reducedness of a given polytope $P \subset \mathbb{R}^3$. Hence, we present an algorithm based on Theorems 2.2 and 2.4. It consists of two steps:

1. Compute the width of $P$, compare Theorem 2.2.
2. Check whether each vertex has a strictly antipodal facet at distance $\omega(P)$, compare Theorem 2.4.
An implementation in pseudocode is given in Algorithm 4.1. In step 4 of the algorithm, we denoted by $e_1 \times e_2$ a vector normal to the skew edges $e_1$ and $e_2$. Using Theorems 2.2 and 2.4, it is easy to check its correctness. A Matlab implementation is provided at zenodo, see [4].

Algorithm 4.1 Algorithm for checking reducedness of polytopes in $\mathbb{R}^3$

1: input: polytope $P \subset \mathbb{R}^3$
2: set $w \leftarrow +\infty$
3: for all skew pairs of edges $e_1$ and $e_2$ of $P$ do
4: set $w \leftarrow \min\{w, w(e_1 \times e_2/\|e_1 \times e_2\|, P)\}$
5: end for
6: unmark all vertices of $P$
7: for all facets $F$ of $P$ do
8: compute the strictly antipodal face $\hat{F}$
9: set $\hat{w} \leftarrow \rho(F, \hat{F})$
10: if $\hat{w} < w$ then
11: unmark all vertices of $P$
12: set $w \leftarrow \hat{w}$
13: end if
14: if $\hat{F}$ consists of a single vertex $v$ and $\hat{w} = w$ then
15: mark vertex $v$
16: end if
17: end for
18: return Are all vertices of $P$ marked?

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