CODES, VERTEX OPERATORS AND TOPOLOGICAL MODULAR FORMS

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Abstract. We describe a new link between the theory of topological modular forms and representations of vertex operator algebras obtained by certain lattices. The construction is motivated by the arithmetic Whitehead tower of the orthogonal groups. The tower discloses the role of codes in representation theory.

1. Introduction

During the past decades there have been several attempts in various directions to relate the theory of topological modular forms $tmf$ to conformal field theories. Gourbonov, Malikov and Schechtman [GMS00] among others provided an interpretation of the Witten genus in terms of chiral conformal field theories. More precisely, they constructed a sheaf of differential vertex operator algebras, known as the chiral de Rham complex, from which the elliptic genus or the Witten genus can be recovered. This object might present the Euler class in $tmf$ but a full geometric description of the cohomology theory itself is still missing (cf. [Seg88] [ST04]).

This survey article has been written by algebraic topologists who are guided by the purpose to establish a geometric interpretation of the $tmf$-cocycles in terms of sheaves of vertex operator algebras. Its new perspective comes from coding theory which relates the representation theory of very specific lattice algebras to the arithmetic Whitehead tower of the orthogonal groups. This connection from cocycle candidates to the string groups differs from the usual one, which only uses index theory and the Witten genus.

Fortunately, the representation theory of vertex operator algebras associated to even lattices is well understood. In this respect, vertex operator algebras are at an advantage to the more modern concept of factorization algebras, which is not part of this work, though. It is well known that the representations are related to modular forms via the partition function.

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The paper contains a new result, Theorem 5.4, establishing a one to one correspondence between representations of the mentioned lattice algebras and the coefficients of tmf\((p)\), topological modular forms with \(\Gamma(p)\)-structure. This result can be interpreted as higher analogue of Atiyah-Bott-Shapiro’s result which relates representations of Clifford algebras to the coefficients of \(K\)-theory.

Codes already play an important role in \(K\)-theory. This was explored in the work of Jay A. Wood [Woo89]. Phenomena like periodicity and triality of the spin groups can be best understood using coding theory. We use the \((8,4)\)-Hamming code to develop the symmetries and the representations of Clifford groups. These objects come together in the arithmetic Whitehead tower of the orthogonal groups. While climbing up the tower, we explain how vertex operator algebras become part of the picture and how codes govern their representation theory.

The article starts with a short introduction to coding theory and \(\theta\)-series. The details can be found in [Ebe13]. Section 3 does not contain new results but formulates the van der Geer-Hirzebruch result for nonlinear codes. This will be important when it comes to partition functions of vertex operator algebra representations. Section 4 deals with the arithmetic Whitehead tower. It is shown how codes can be used to construct the interesting representations of the Clifford groups. Afterwards, string extensions are studied in the complex case and the role of lattices \(L\) is demonstrated. In Section 5 we proceed to give a model of the loop space extension of the torus \(BL\) in terms of vertex operator algebras and specify its representations. The main Theorem 5.4 gives a one-to-one correspondence between the representation ring and the ring of Hilbert modular forms. Finally, Section 6 describes the relation of the theorem to topological modular forms with level structures. This last section surveys the ideas to describe the cocycles by sheaves of vertex algebra modules. Corollary 6.1 is the promised analogue of the Atiyah-Bott-Shapiro theorem.

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2. Codes and Lattices

Let \(F\) be a field. A code of length \(n\) over \(F\) is a subset of \(F^n\). Let \(C\) and \(C'\) be codes over \(F\) with the same length \(n\). A (strict) morphism from \(C\) to \(C'\) is a monomial transformation \(g\) such that \(g(C) \subset C'\). This means that \(g\) is a linear endomorphism of \(F^n\) which in terms of the standard basis takes
the form
\[ g : e_i \mapsto c_i e_{\sigma^{-1}(i)} \]
for some \( c_i \in F^\times \) and some \( \sigma \in \Sigma_n \). In particular, a strict automorphism of \( C \) is an element in
\[ (F^\times)^n \rtimes \Sigma_n. \]
A code is called **linear** if it is a linear subspace. A linear code \( C \) is **self-orthogonal** if \( C \subseteq C^\perp \), and **self-dual** if \( C = C^\perp \) where orthogonality is with respect to the standard inner product.

2.1. The (8,4)-Hamming Code. The (8,4)-Hamming code \( H_8 \) is a doubly even self-dual simply error correcting binary code, and it is the smallest code with these properties. As we shall see in Section 4.2 below, these features are responsible for a particularly nice structure of the real Clifford algebra \( Cl_8 \), which is at the heart of real Bott periodicity, the \( \hat{A} \)-genus and the triality symmetries of the group \( Spin(8) \). To recall the construction of \( H_8 \), we introduce the Fano Plane
\[ \mathcal{F} = F^2(F_2), \]
viewed as a finite geometry, with lines given by the 2-dimensional subspaces of \( F_2^3 \). Its power set \( \mathcal{P}(\mathcal{F}) \) endowed with the symmetric difference operation is a seven dimensional \( F_2 \)-vector space. Inside it, we have the three dimensional linear subspace \( C \) of line complements. For any vector \( S \in \mathcal{P}(\mathcal{F}) \), the set-theoretic complement is
\[ \mathcal{F} \setminus S = \mathcal{F} + S. \]
The \((7,4)\)-Hamming code is the subspace
\[ \mathcal{H}_7 = C \oplus \langle \mathcal{F} \rangle \]
of \( \mathcal{P}(\mathcal{F}) \). More precisely, any choice of numbering of the elements of the Fano plane defines an identification \( F_2^3 \cong \mathcal{P}(\mathcal{F}) \) with different choices giving strictly isomorphic codes. We give two such numberings in Figure 1. Independent of the numbering, the incidence vector of the full set \( \mathcal{F} \) is given by
\[ \mathbf{1} := (1, 1, 1, 1, 1, 1, 1)^T. \]
The parity check isomorphism
\[ F_2^7 \xrightarrow{\cong} (F_2^8)^{ev} \]
\[ v \mapsto (|v|, v) \]
transports \( \mathcal{H}_7 \) into a doubly even code \( \mathcal{H}_8 \) of length eight. This is the (8,4)-Hamming Code. The Fano plane is the smallest interesting example of a finite projective geometry, and as such it is dually isomorphic to itself, meaning that there is a bijection between points and lines preserving the incidence relation. We can realize this duality in a manner that decomposes \((F_2^3)^{ev}\) as the sum of two (strictly isomorphic) copies of \( C \). To do so, we number the points of \( \mathcal{F} \) in two different ways, as depicted in Figure 1. The dual isomorphism is then obtained by numbering the lines on either side as in Figure 2. Writing \( c_i \) for the complement of Line \( i \) in the right-hand
Figure 1. Two numberings of the Fano plane

Figure 2. Incidence table relating points and lines in Figure 1. The dual isomorphism is reflected in the diagonal symmetry.

picture and $b_i$ for the complement of Line $i$ in the left-hand picture, we have

$$c_i + b_i = \{i\} + 1.$$

As a result, we obtain the decompositions

$$(\mathbb{F}_2^7)^{ev} = \mathcal{B} \oplus \mathcal{C}$$
and
\[(F_3^8)^{ev} \cong F_2^7 = B \oplus \langle 1 \rangle \oplus C,\]
where $B$ and $C$ are the subspaces of line complements consisting of the $b_i$ (respectively $c_i$) and the zero vector. A further feature of point-line duality is that lines on the right govern addition in $B$ and vice versa: three distinct numbers $i$, $j$ and $k$ form a line in the right-hand picture, if and only if
\[b_i + b_j + b_k = 0.\]
Similarly,
\[c_i + c_j + c_k = 0\]
in $C$ if and only if $i$, $j$ and $k$ form a line on the left. This will become important in our explicit calculations of spinor representations.

2.2. The Standard Ternary Code. A code is called ternary if $F = F_3$. A famous ternary code is
\[T = \{(s, a, a + s, a + 2s) \mid a, s, \in F_3\}.\]
It is a linear code of dimension 2 in $F_3^4$ which is self-dual. Its automorphism group is $GL_2(F_3)$ as one easily verifies. This group coincides with the automorphism group of the Lubin-Tate curve
\[C : y^2 + y = x^3\]
over $F_4 = \{0, 1, \omega, \bar{\omega}\}$. This curve plays an important role in the construction of the spectrum $tmf$. It will be reconsidered in Section 6.

2.3. The ternary Golay Code. The ternary Golay code is a self-dual linear code $C$ in $F_3^{12}$ with 729 words. Its automorphism group is the Mathieu group $2.M_{12}$. This code seems to be related to the triality of spin groups as considered in 4.3, but the authors have not yet been able to make this connection precise.

3. Lattices over integers of number fields

For details the reader is referred to Chapters 5.1 to 5.4 of [Ebe13]. Let $p$ be an odd prime. Let $F$ be the number field $\mathbb{Q}(\zeta)$ for some primitive $p$th root of unity. This is a field extension over $\mathbb{Q}$ of degree $p - 1$. For $r$ ranging from 1 to $p - 1$ there are embeddings $\sigma_r$ of $F$ in $\mathbb{C}$ given by $\sigma_r(\zeta) = \zeta^r$. The trace of an element $\alpha \in F$ is defined by
\[\text{Tr}(\alpha) = \sum_{r=1}^{p-1} \sigma_r(\alpha).\]
We write $\mathcal{O}$ for the ring of integers in $F$. The underlying additive group of $\mathcal{O}$ is free abelian of rank $p - 1$ generated by $\zeta^0, \ldots, \zeta^{p-2}$. Let $\mathcal{P}$ be the principal ideal of $\mathcal{O}$ generated by $1 - \zeta$. Then $\mathcal{P}$ is the kernel of the map $\rho : \mathcal{O} \to \mathbb{Z}/p\mathbb{Z}$ given by
\[a_0 + a_1\zeta + \cdots + a_{p-2}\zeta^{p-2} \mapsto (a_0 + a_1 + \cdots + a_{p-2}).\]
The pairing
\[ \langle x, y \rangle = \text{Tr} \left( \frac{xy}{p} \right) \]
defines a bilinear form on \( O \) with values in the rationals. It makes \( P \) into an even lattice, that is, \( \langle x, y \rangle \in \mathbb{Z} \) and \( \langle x, x \rangle \in 2\mathbb{Z} \). It is isomorphic to the root lattice \( A_{p-1} \).

3.1. Lattices from Codes. More generally, let \( \rho : \mathbb{Q}^n \to \mathbb{F}_p^n \) be the reduction map modulo the principal ideal \( P \) in each coordinate. For a code \( C \) set
\[ \Gamma_C = \rho^{-1}(C) \subset \mathbb{O}^n. \]
Assume that \( C \) is a self-orthogonal linear code. Then the symmetric bilinear form
\[ \langle x, y \rangle = \sum_{i=1}^n \text{Tr} \left( \frac{x_i \bar{y}_i}{p} \right). \]
turns \( \Gamma_C \) to an even lattice of rank \( n(p-1) \) with discriminant \( p^{n-2m} \). The dual lattice
\[ \Gamma^\vee = \text{Hom}(\Gamma, \mathbb{Z}) = \{ x \in F^n | \langle x, y \rangle \in \mathbb{Z} \} \]
satisfies
\[ \Gamma^\vee \cong \Gamma_C^\perp \]
(see f.i. [Ebe13, Proposition 5.2 and Lemme 5.5] for details). In particular, if \( C \) is self dual, then \( \Gamma_C \) is isomorphic to its dual lattice. Self-dual lattices are known as unimodular lattices.

Example 3.1. The standard ternary code gives the root lattice \( E_8 \). The ternary Golay code produces an even unimodular lattice of rank 24 which contains \( 12A_2 \).

3.2. Theta series. Lattices come with theta series, and these are key to elliptic genera. The usual theta series of the lattice \( \Gamma_C \) is
\[ \vartheta_C(z) = \sum_{x \in \Gamma_C} e^{\pi i z \text{Tr}(\frac{x}{p})} = \sum_{x \in \Gamma_C} e^{2\pi i z \text{Tr}_k(\frac{x}{p})}. \]
where \( k = \mathbb{Q}(\zeta + \zeta^{-1}) \) is the real subfield of \( F \). There is an refinement of \( \vartheta_C \) which is a holomorphic function on \( r = (p-1)/2 \)-copies of the upper half-plane \( \mathbb{H} \): set
\[ \theta_C(z) = \sum_{x \in \Gamma_C} e^{2\pi i z \text{Tr}_k(\frac{x}{p})} \text{ with } \text{Tr}_k \left( \frac{zx}{p} \right) = \sum_{l=1}^r \frac{\sigma_l(x \bar{x})}{p}. \]
Set \( \theta_j = \theta_{x+j} \) for \( j \in \mathbb{F}_p \). These functions are symmetric Hilbert modular forms of weight \( m = 1 \) with respect to the congruence group
\[ \Gamma(\mathfrak{p}) = \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{O}) | A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ mod } \mathfrak{p} \}. \]
where $p = \mathbb{P} \cap k$. In detail, this means they are holomorphic functions $f$ which transform as

$$f(\sigma_1(A)z_1, \ldots, \sigma_r(A)z_r) = f(z)\prod_{l=1}^r(\sigma_l(c)z_l + \sigma_l(d))^m.$$  

The symmetry means that they are invariant under the action of the Galois group $\text{Gal}(k, \mathbb{Q})$ which permutes the coordinates of $H_r$. It turns out that every function $\theta_C$ is a sum of monomials in the $\theta_j$s and hence is a symmetric Hilbert modular form. In order to formulate this result more precisely, recall that the weight enumerator of a code $C$ is the polynomial

$$W_C(x_0, x_1, \ldots, x_n) = \sum_{w \in C} x_0^{l_0(w)} x_1^{l_1(w)} \cdots x_r^{l_r(w)}$$

where $l_j(w)$ is the number of zeroes in $w$ and $l_j(w)$ is the number of $\pm j$. The following result is the Theorem of Alpbach by van der Geer-Hirzebruch. It is formulated there for linear self-orthogonal codes in [Ebe13][Theorem 5.3]. We repeat the proof below because the assumption is unnecessary.

**Theorem 3.2.** Let $C$ be arbitrary (not necessary linear). Then

$$\theta_C = W_C(\theta_0, \theta_1, \ldots, \theta_r)$$

In particular, each code of length $n$ gives a symmetric $\Gamma(p)$-Hilbert modular form of weight $n$.

**Proof.** For $c \in C$ calculate

$$\sum_{x \in p^{-1}(c)} e^{2\pi i \text{Tr}_k\left(z \frac{x+c}{p}\right)} = \sum_{x_1 \in p^{-1}(c_1)} e^{2\pi i \text{Tr}_k\left(z \frac{x_1+c_1}{p}\right)} \cdots \sum_{x_n \in p^{-1}(c_n)} e^{2\pi i \text{Tr}_k\left(z \frac{x_n+c_n}{p}\right)}$$

$$= \theta_0^{l_0(c)} \theta_1^{l_1(c)} \cdots \theta_r^{l_r(c)}(z)$$

since

$$\theta_j(z) = \sum_{x \in \mathbb{F}+j} e^{2\pi i \text{Tr}_k\left(z \frac{x}{p}\right)} = \sum_{x \in \mathbb{F}-j} e^{2\pi i \text{Tr}_k\left(z \frac{x}{p}\right)}.$$  

The claim follows after summing over all code words. \qed

**Corollary 3.3.** The usual theta series $\vartheta_C$ of $C$ is a modular form for $\Gamma(p)$ of weight $nr$. Self dual codes give $SL_2(\mathbb{Z})$-invariant modular forms.

**Proof.** The first statement follows from the theorem after taking the diagonal map from $H$ to $H_r$. The second statement uses [3] and the well known fact that $\theta$-series of self dual lattices are invariant under the full modular group. It can also be obtained from the theorem by analyzing the action of $SL_2(\mathbb{F}_p)$ (see [Ebe13 Section 5] for $p = 3, 5$). \qed

**Remark 3.4.** Doi-Naganuma have constructed a map from the group of modular forms for the congruence group $\Gamma_0(p)$ to Hilbert modular forms for $\Gamma(p)$ which preserves the weight and Hecke eigenforms. It would be
interesting to analyze those symmetric Hilbert modular forms which are in the image of the DN lift in terms of code words.

4. Arithmetic Whitehead towers

The organizing principle for the different cohomology theories considered in this paper comes from the Whitehead towers of the orthogonal and the unitary groups. These are the towers

\[ O(n) \leftarrow SO(n) \leftarrow \mathbb{Z}/2 \leftarrow Spin(n) \leftarrow PU(H) \leftarrow String(n) \leftarrow \ldots \]

and

\[ U(n) \leftarrow SU(n) \leftarrow PU(H) \leftarrow String(n) \leftarrow \ldots \]

defined for large \( n \). The step to the right is obtained by killing off the lowest homotopy group. For instance, \( SO(n) \) is a connected component of \( O(n) \) and \( Spin(n) \) is its simply connected cover, which turns out to be 2-connected.

For \( n \geq 3 \) the smallest non-trivial homotopy group of \( Spin(n) \) is

\[ \pi_3(Spin(n)) \cong \begin{cases} \mathbb{Z} \cong \langle Spin(3) \rangle & \text{for } n \neq 4 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 4 \end{cases} \]

[MT91] Chapter VI, Theorem 4.17.

The next step in the tower, \( String(n) \), is classified by a generator of the integral cohomology \( H^4(BSpin(n)) \). To be specific, we recall the low degree cohomology groups:

(i) Let \( n \geq 3 \). Then \( H^k(BSpin(n)) \) vanishes for \( k \leq 3 \) and is cyclic of infinite order for \( k = 4 \) except for \( n = 4 \) where it is two copies of the integers.

(ii) For \( n \geq 4 \), one generator is \( p_1/2 \) where \( p_1 \) is the first Pontryagin class of the canonical bundle associated to the adjoint representation.

(iii) For \( n = 3 \), the Euler class of the spinor representation generates. It coincides with \( p_1/4 \).

An explicit model realizing \( String(n) \) as topological group can be found in [ST04 Section 5]. This method can be continued, but for the purposes of the paper at hand, we are only interested in the first steps of the tower. For \( n \geq 4 \), \( String(n) \) fibers over \( Spin(n) \) with fibre \( PU(H) = K(\mathbb{Z}, 2) \), the projective unitary group of an infinite dimensional separable Hilbert space. There is also a realization of \( String_n(\mathbb{R}) \) as compact 2-group with fibre \( \mathbb{B}U(1) \), whose geometric realization gives the topological group, see for instance [SP11].

4.1. The Clifford groups and codes. We revisit the (8, 4)-Hamming Code in the context of Whitehead tower. If we make the identifications

\[ \mathbb{F}_2^8 \cong O(1)^8 \cong O(8) \cap \text{Diag} \]
then the Fano line complements can be realized as the three-fold tensor products of the diagonal Pauli matrices

\[ \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

More precisely, in the second picture in Figure 1, the point \([a:b:c] \in F\) has number \(4a + 2b + c\). The resulting isomorphism

\[ \mathcal{P}(F) \cong F_2^7 \cong (F_2^8)^{ev} \]

identifies the linear map

\[ C: F_2^3 \rightarrow \mathcal{P}(F) \quad \quad p \mapsto F \setminus p^\perp \]

with the map

\[ F_2^3 \rightarrow O(1)^8 \quad \quad (a, b, c) \mapsto \sigma_3^a \otimes \sigma_3^b \otimes \sigma_3^c. \]

If \(p\) is a point in \(F\) (a line in \(F_2^3\)), then the orthogonal complement \(p^\perp\) is a line in \(F\) (a plane in \(F_2^3\)), and \(F \setminus p^\perp\) is a line complement. So, the image of \(C\) is the subspace of line complements. The vector \(1\) corresponds to \(-\text{id}\), so the full Hamming code \(H_8\) is identified with the subgroup

\[ H = \left\{ \pm \sigma_3^a \otimes \sigma_3^b \otimes \sigma_3^c \mid a, b, c \in \{0, 1\} \right\}. \]

The various properties making \(H_8\) suitable for coding theory are, to a topologists, statements about the Whitehead tower. The pre-image of \(O(1)^n\) inside \(Pin(n)\) is the extraspecial 2-group \(F_n\) of Clifford words. Writing \(-1 \in F_n\) for the central element, this group consists of words in the symbols \(e_0, \ldots, e_{n-1}\) subject to the relations

\[ e_i^2 = -1 \quad \text{and} \quad e_i e_j = -e_j e_i. \]

The code \(H_8\) is even, meaning

\[ H \subset SO(8). \]

Moreover, \(H_8\) is self dual, so its spin-cover

\[ \tilde{H} \subset Spin(8) \]

is abelian. The fact that \(H_8\) is doubly even makes \(\tilde{H}\) elementary abelian,

\[ \tilde{H} \cong (\pm 1) \times H, \]

meaning that \(H\) lifts to a subgroup of \(Spin(8)\). Indeed, \(\tilde{H}\) is a maximal abelian subgroup of the group \(F_8^{ev}\) of even Clifford words. The fact that \(H_8\) is simply error correcting means that the Clifford symbols \(e_0, \ldots, e_7\) form a system of coset representatives for \(F_8^{ev}/\tilde{H}\). The orthogonal decomposition

\[ F_8^{ev} = B \oplus H_8 \]
discussed in Section 2.1 allows us to write $F_{ev}^8$ as semi-direct product of two elementary abelian groups,

$$F_{ev}^8 \cong B \ltimes \tilde{H}.$$ 

Here $B$ acts on $\tilde{H}$ by

$$b : \tilde{h} \mapsto (-1)^{(b,h)} \tilde{h},$$

where $\tilde{h} \in \tilde{H}$ is a lift of $h \in H$.

4.2. Spinors. The twisted group algebra of $F_n$ is the Clifford algebra $Cl_n$ of $\mathbb{R}^n$ with its standard quadratic form

$$Cl_n = \mathbb{R}[F_n]/(-1)_{\mathbb{R}} \sim (-1)_{F_n}.$$ 

In particular, Clifford modules coincide with centre-faithful representations $\rho$ of $F_n$. Because of the inclusion

$$Spin(n) \subset Cl_{ev}^8 \cong \mathbb{R}^{tw}[F_{ev}^8]$$

[ABS64], it follows that representations of $F_{ev}^8$ give rise to representations of the spin groups. Let now $n = 8$. The spinor representations $\Delta^+$ and $\Delta^-$ are the two irreducible modules over the Clifford algebra $Cl_8$. Each has dimension 8. Following Pressley and Segal [PS86, Section 12.6], the spinor representations are constructed by inducing up centre-faithful representations of a maximal abelian subgroup. This is the point where it becomes essential that our maximal abelian subgroup is, in fact, elementary abelian, i.e., that $H_8$ is a Type II code. Let $\chi^+$ and $\chi^-$ be the characters

$$\chi^+ : (\pm 1) \times H_8 \rightarrow O(1)$$

$$-1 \mapsto -1$$

$$h \mapsto 1$$

and

$$\chi^- : (\pm 1) \times H_8 \rightarrow O(1)$$

$$-1 \mapsto -1$$

$$h \mapsto h_0.$$ 

Then the eight dimensional spinor representations $\Delta^\pm$ are given as

$$\Delta^\pm = \text{ind}_{\tilde{H}}^{F_{ev}^8} \chi^{\pm}.$$ 

Due to the semi-direct product decomposition $F_{ev}^8 = B \ltimes \tilde{H}$, elements of $B$ act as permutation matrices of order two, which turn out to be three-fold tensor products of the $2 \times 2$ permutation matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
More precisely, $\mathcal{B}$ acts by addition on itself, which is governed by $\mathcal{C}$ and given by the doubly even permutations

$$
\begin{align*}
b_1 &\mapsto (01) (23) (45) (67) = \sigma_0 \otimes \sigma_0 \otimes \sigma_1, \\
b_2 &\mapsto (02) (13) (46) (57) = \sigma_0 \otimes \sigma_1 \otimes \sigma_0, \\
b_3 &\mapsto (03) (12) (47) (56) = \sigma_0 \otimes \sigma_1 \otimes \sigma_1, \\
b_4 &\mapsto (04) (15) (26) (37) = \sigma_1 \otimes \sigma_0 \otimes \sigma_0, \\
b_5 &\mapsto (05) (14) (27) (36) = \sigma_1 \otimes \sigma_0 \otimes \sigma_1, \\
b_6 &\mapsto (06) (17) (24) (35) = \sigma_1 \otimes \sigma_1 \otimes \sigma_0, \\
b_7 &\mapsto (07) (16) (25) (34) = \sigma_1 \otimes \sigma_1 \otimes \sigma_1.
\end{align*}
$$

The subgroup $\mathcal{H}$ is fixed by $\Delta^+$ and acted upon by $\chi^- \otimes \text{id}$ for $\Delta^-$.  

4.3. **Triality.** The relationship between spinor groups and codes was already explored in the work of Wood [Woo89], who also writes about the triality symmetry. Together with the fundamental representation

$$
\pi : \text{Spin}(8) \rightarrow \text{SO}(8),
$$

the spinor representations $\Delta^\pm$ form the eight dimensional irreducible representations of $\text{Spin}(8)$. All three are double covers, their respective kernels generated by the three non-trivial center elements:

$$
\ker(\Delta^+ ) = \langle \omega \rangle, \quad \ker(\Delta^- ) = \langle -\omega \rangle, \quad \ker(\pi ) = \langle -1 \rangle,
$$

where $\omega = e_0 \cdots e_7$ is the second generator of the center of $\text{Spin}(8)$. The spin covers of the spinor representations $\Delta^\pm$ are the extraordinary outer automorphisms of $\text{Spin}(8)$ depicted in Figure 3.

4.4. **Bott periodicity.** The same objects that define triality are also responsible for real Bott periodicity. The Bott element is given by the full spinor representation

$$
\Delta = \text{ind}_{\mathcal{H}}^{\text{Spin}(8)} \chi^+ = \Delta^+ \oplus e_0 \Delta^+,
$$

combining $\Delta^+$ and $\Delta^-$ in the sense that

$$
\text{res}_{\mathcal{F}_{8^v}} \Delta \cong \Delta^+ \oplus \Delta^-.
$$

The Pauli matrix point of view has the advantage that it gives an explicit description of the Bott element, which is more direct than the approach in the standard literature [ABS64] [LM89]. One checks directly that the
Figure 3. Triality symmetry of the Dynkin diagram $D_4$.

The extraordinary automorphisms $\tau^\pm$ are the lifts of the spinor representations, while $\sigma$ is conjugation with $e_0$. The action of the symmetric group $S_3$ on the center is indicated in red.

matrices

\[
E_1 = \begin{bmatrix}
1 & -1 \\
1 & -1 & 1 \\
-1 & 1 \\
-1 & 1
\end{bmatrix}, \quad E_2 = \begin{bmatrix}
1 & -1 \\
1 & -1 & 1 \\
-1 & 1 \\
-1 & 1
\end{bmatrix}, \quad E_3 = \begin{bmatrix}
1 & -1 \\
-1 & 1 & -1 \\
1 & -1 & 1 \\
1 & -1 & 1
\end{bmatrix}, \quad E_4 = \begin{bmatrix}
1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & 1 & 1
\end{bmatrix}
\]
satisfy the Clifford relations. Hence
\[ \Delta^\pm (e_0 e_i) = \pm E_i \]
defines two modules over the even Clifford algebra, which are readily checked to have the desired effect on centers. Bott periodicity is the statement that the representation
\[ \Delta : Cl_8 \rightarrow Mat_{16 \times 16} \]
is an isomorphism. This is now an easy consequence of the fact that the real Pauli matrices
\[ \sigma_0, \; \sigma_1, \; \sigma_1 \sigma_0, \; \text{and} \; \sigma_3, \]
form a basis of the $2 \times 2$-matrices, taking into account that $\Delta$ sends the 256 positive Clifford words to the 256 possible 4-fold tensor products of the real Pauli matrices. Indeed, the 64 three-fold tensor products are the possible products of the matrices $E_1, \ldots, E_7$ above, which are easily read off from the Pauli matrix realizations of $H$ and $B$. Combined with $\omega \mapsto \sigma_3 \otimes \text{id} \otimes \text{id} \otimes \text{id}$ and $e_1 \mapsto \sigma_1 \otimes \text{id} \otimes \text{id} \otimes \text{id}$, these determine all of $\Delta$.

If we allow ourselves to work over the complex numbers, the doubly even condition can be dropped, and an even self-dual code will suffice, since unlike $O(1)$, the group $U(1)$ can accommodate elements of order 4. This is the reason why complex Bott periodicity already occurs at $n = 2$, where we have the first even self-dual code.

4.5. The integral spin and string groups. There is a version of the Whitehead tower for finite groups, where the successive killing of lower degree group cohomology has the meaning of forming first the commutator subgroup then the universal central extension (Schur extension), then the categorical Schur extension. This process requires the commutator subgroup of the original group to be perfect, so we will limit ourselves to considering
only such groups. For details we refer the reader to [EG17]. As before, a tower of topological groups can be obtained by taking classifying spaces of the fibers, so that the categorification interpretation is really a matter of taste.

We saw above how codes fit into the Whitehead tower for $O(n) = O_n(\mathbb{R})$. Since codes, however, are finite objects, a more natural home is the arithmetic Whitehead tower for $O_n(\mathbb{Z})$, the group of orthogonal matrices with integral entries. These groups are known as the hyper-octahedral groups. They are generated by permutation matrices and diagonal matrices with $\pm 1$. It is not hard to verify the isomorphism

$$O_n(\mathbb{Z}) \cong \Sigma_n \ltimes F_n^2.$$  

The spinor representations $\Delta^\pm$ discussed above map the group $F^e_8$ to $SO_8(\mathbb{Z})$, or rather its spin double cover, which we will denote $\text{Spin}_8(\mathbb{Z})$.

In the stable range, the arithmetic Whitehead tower of the symmetric groups is governed by the homotopy groups of spheres [EG17], while that of $GL_n(\mathbb{Z})$ is governed by the $K$-theory of the integers, see Section 4.6 below. The inclusions

$$\Sigma_n \subset O_n(\mathbb{Z}) \subset GL_n(\mathbb{Z}).$$

suggest to think of the Whitehead tower of the hyper-octahedral groups as interpolating between these two.

**Theorem 4.1.** For large enough $n$, the arithmetic Whitehead tower of the hyper-octahedral groups starts with the terms

$$O_n(\mathbb{Z}) \leftarrow A_n \ltimes (F^e_n)^{(\mathbb{Z}/2)^2} \leftarrow \widetilde{A}_n \ltimes F^e_n \leftarrow \ldots$$

**Proof.** For $n \geq 2$, the abelisation of $O_n(\mathbb{Z})$ is a Klein 4-group. This is discussed, for instance, in [Ste92]. The index 2 subgroups of the hyper-octahedral group consist of

(i) the group $SO(\mathbb{Z})$ of elements with even determinant,

(ii) the group $A_n \ltimes F_0^e$ of signed even permutations and

(iii) the group $D_n = \Sigma_n \ltimes (F^e_n)^{ev}$ of evenly signed permutations.

The latter is a Weyl group of type $D$. The intersection of these three groups is the group

$$H = A_n \ltimes (F^e_2)^{ev}$$

of evenly signed even permutations. This is the commutator subgroup of $O_n(\mathbb{Z})$. We shall see below that it is perfect and hence possesses a universal central extension. Write $E$ for the elementary 2-group $(F^e_2)^{ev}$. Recall that the universal coefficient theorem identifies the Pontryagin dual of the integral homology with cohomology with circle coefficients

$$\widehat{H}_i(G) \cong H^i(G; U(1)),$$

and we will switch between these two as convenient for calculating the Schur multiplier of $H$. Consider the Lyndon-Hochschild-Serre spectral sequence for
the $U(1)$-cohomology of the extension

\[ 0 \to E \to H \to A_n \to 0. \]

We have

\[ E^{0,1}_2 = (E^*)_A = 0 \]

for $n \geq 3$, while

\[ E^{0,2}_2((\Lambda^2 E)_A^*) = (\Lambda^2 E^*)_A \]

has two elements. The non-trivial element is the bilinear form

\[ \beta = \sum_{1 \leq i < j \leq n} x_i \wedge x_j, \]

where $(x_i)_i$ is the dual basis to $(e_i)_i$. Here we have used the results in [Rom05] to identify the invariants under the alternating group. Note that $\beta$ is a cocycle for $F_{ev}^n$. This is checked on the Clifford words of length 2, which generate $F_{ev}^n$. On the other axis, we have $E^{1,0}_2 = 0$ for $n \geq 5$, since the alternating groups become perfect in this range. Further, for $n \geq 8$, the Schur multiplier of the alternating group, $E^{2,0}_2$, has order 2. The non-trivial element classifies the spin extension $\tilde{A}_n$. We now turn our attention to the entry $E^{1,1}_2$, which we claim to be zero. Then the proof will follow from Corollary 4.3 below.

**Lemma 4.2.** Let $f$ be a crossed homomorphism from $A_n$ to $E^*$ with respect to the action permuting the basis elements of $E$. Then $f$ is principal.

*Proof.* The alternating group is generated by 3-cycles. So, if a crossed morphism is zero on all 3-cycles, it is zero. Without loss of generality, we therefore assume that $f(123) \neq 0$. We will continue to identify $E^*$ with the degree one polynomials in the variables $x_i$ with $\mathbb{F}_2$-coefficients, modulo the elementary symmetric polynomial $x_1 + \cdots + x_n$. In this notation, $f(123)$ and $f(132)$ each have two summands, with indices from the set $\{1, 2, 3\}$. Of course, there is a degree of freedom here, we could add $x_1 + \cdots + x_n$ to obtain the other representative; we choose the former representation. To be precise, if $f(123) = x_i + x_{i+1}$ (indices modulo 3), then $f(132) = (32)(x_i + x_{i+1}) = x_{i+1} + x_{i+2}$. We will define a polynomial $p = \sum_{i=1}^n \alpha_i x_i$ such that $f$ agrees with the principal crossed homomorphism associated to $p$. We set $\alpha_i = 0 = \alpha_{i+2}$ and $\alpha_{i+1} = 1$. If $n = 3$, we are done. Otherwise, for any $k > 3$, we consider $f((12)(3k))$. This either contains both $x_3$ and $x_k$ as summands, or neither. In the first case, we set $\alpha_k = \alpha_3 = 1$, in the second case, we set $\alpha_k = \alpha_3$. This is a generating set on which $f$ now agrees with the principal crossed homomorphism associated to $\sum_{i=1}^n \alpha_i x_i$. \hfill $\square$

**Corollary 4.3.** For $n \geq 5$, the group $H$ is perfect. For $n \geq 8$, the Schur extension of $H$ is given by the semi-direct product

\[ \tilde{A}_n \times F_{ev}^n. \]
Proof. The spectral sequence calculation above implies that $H$ is perfect and that its Schur multiplier has order 4. Comparing to the Schur multiplier of the hyper-octahedral group $S_n \ltimes \mathbb{F}_2^n$, which was identified by Stembridge in [Ste92] as an elementary abelian 2-group of order 8, we conclude that $H_2(H)$ is in fact a Klein 4-group. To be precise, this follows from the spectral sequence for the extension

$$1 \rightarrow H \rightarrow S_n \ltimes \mathbb{F}_2^n \rightarrow \{\pm 1\}^2 \rightarrow 1,$$

taking into account that $H_2(\{\pm 1\}^2)$ has order 2. Write $\tilde{H}$ for the Schur extension of $H$. By the universal property of Schur extension, we obtain a map of central extensions

$$
\begin{array}{cccccc}
0 & \rightarrow & H_2(H) & \rightarrow & \tilde{H} & \rightarrow & H & \rightarrow & 0 \\
0 & \rightarrow & \{\pm 1\}^2 & \rightarrow & \tilde{A}_n \ltimes F_{n}^{ev} & \rightarrow & A_n \ltimes (\mathbb{F}_2^n)^{ev} & \rightarrow & 0
\end{array}
$$

The arrow between the central subgroups must be an isomorphism, since the bottom extension cannot be reduced to an extension by $\{\pm 1\}$. It follows that the middle arrow is an isomorphism, as well. □

As the Schur extension of the perfect group $A_n \ltimes E$, the group $\tilde{A}_n \ltimes F_{n}^{ev}$ is super perfect and hence possesses a categorical Schur extension, giving the next step in the arithmetic Whitehead tower of the hyper-octahedral groups (for $n \geq 8$). In the stable range, we have maps

$$\pi_3(S^0) \rightarrow \tilde{A}_n \ltimes F_{n}^{ev} \rightarrow \pi_3(K\mathbb{Z}).$$

It is therefore natural to conjecture that the categorical Schur multiplier of $\tilde{A}_n \ltimes F_{n}^{ev}$ also stabilizes for large enough $n$ and that its order is a multiple of 24.

4.6. Arithmetic Whitehead tower for $GL_n(\mathbb{Z})$. Mason and Stothers [MS74, Theorem 5.1] have shown that for $n \geq 3$ the subgroup $E_n(\mathbb{Z})$ of $SL_n(\mathbb{Z})$ generated by the elementary matrices is perfect and coincides with the commutator subgroup of $GL_n(\mathbb{Z})$. We have

$$SO_n(\mathbb{Z}) \subset E_n(\mathbb{Z}).$$

The elementary matrices satisfy the following relations

(5) $e_{ij}(a)e_{ij}(b) = e_{ij}(a + b)$

(6) $e_{ij}(a)e_{kl}(b) = e_{kl}(b)e_{ij}(a), \ j \neq k$ and $i \neq l$

(7) $e_{ij}(a)e_{jk}(b)e_{ij}(a)^{-1}e_{jk}(b)^{-1} = e_{ik}(ab); \ i, j, k$ distinct

(8) $e_{ij}(a)e_{ki}(b)e_{ij}^{-1}(a)e_{ki}(b)^{-1} = e_{kl}(-ba); \ i, j, k$ distinct
The universal central extension of $E_n(\mathbb{Z})$ is known as Steinberg group $St_n(\mathbb{Z})$. It is defined by symbols $x_{ij}(a)$ with $i \neq j, 1 \leq i, j \leq n, a \in \mathbb{Z}$ subject to the relations (5)(6)(7)(8). The kernel of the map $St_n(\mathbb{Z}) \rightarrow E_n(\mathbb{Z}); x_{ij}(a) \mapsto e_{ij}(a)$ is the algebraic K homology group $K_2(\mathbb{Z})$. In particular, the first two homology groups of $St_n(\mathbb{Z})$ vanish and the Schur multiplier is $H_2(E_n(\mathbb{Z})) \cong K_2(\mathbb{Z}) \cong \mathbb{Z}/2$.

Using standard techniques from algebraic K-theory [Ros94, 5.2.7] it is not hard to see that $H_3(St_n(\mathbb{Z})) \cong K_3(\mathbb{Z}) \cong \mathbb{Z}/48$.

So, we have the tower

$$GL_n(\mathbb{Z}) \leftarrow E_n(\mathbb{Z}) (\mathbb{Z}/2)^2 \xrightarrow{\mathbb{Z}/48} St_n(\mathbb{Z}) \xrightarrow{\mathbb{Z}/48} St(\mathbb{Z}) \leftarrow \ldots$$

The group $K_3(\mathbb{Z})$ plays a key role in the computation of the fourth cohomology of the Conway group $H^4(\text{Co}1; \mathbb{Z}) \cong \mathbb{Z}/24$ by Johnson-Freyd and Treumann [JFT20].

4.7. **Integers in cyclotomic fields and string extensions.** We have seen that good codes lead to interesting representations of the integral spin groups. We like to see similar phenomena for the integral string groups. For this purpose, we consider the complex Whitehead tower and restrict it to integers in cyclotomic fields.

Let $U_n(\mathbb{C})$ be the group of unitary matrices, that is, complex isometries of $\mathbb{C}^n$. Since the fourth cohomology of $BSU_n(\mathbb{C})$ is generated by the second Chern class we obtain a $PU(H)$-extension $\widetilde{SU}_n(\mathbb{C})$ in the same way as in the real case. Let $\mathcal{O}$ be the ring of integers in the number field $K = \mathbb{Q}(\zeta)$ considered in Section 3. The integral unitary group $U_n(\mathcal{O}) = U_n(\mathbb{C}) \cap Gl_n(\mathcal{O})$ coincides with the semi direct product of diagonal with permutation matrices

$$U_n(\mathcal{O}) = \{ \pm \zeta^i | i = 0, \ldots, p - 1 \}^n \rtimes \Sigma_n.$$ 

A code of length $n$ can be considered as subset of $U_n(\mathcal{O})$ by identifying $\mathbb{F}_p$ with the multiplicative group of roots $\zeta^i$.

Consider the pullback diagram

$$
\begin{array}{cccc}
\mathbb{F}_p^{\det=1} & \longrightarrow & \widetilde{SU}_n(\mathcal{O}) & \longrightarrow & \widetilde{SU}_n(\mathbb{C}) \\
PU(H) & \downarrow & PU(H) & \downarrow & PU(H) \\
\mathbb{F}_p^{\det=1} & \longrightarrow & SU_n(\mathcal{O}) & \longrightarrow & SU_n(\mathbb{C}).
\end{array}
$$
There are several ways to deal with representations of \( PU(H) \)-extensions of topological groups \( G \). One way is to take based loops \( \Omega \) and observe that \( \Omega PU(H) \cong S^1 \). Hence, \( \Omega G \) is equivalent to an extension of \( \Omega G \) by a compact Lie group. Then one looks for representations which mimic the structure of the loop space of a representation of \( \tilde{G} \).

This approach does not work in the arithmetic setting since the base spaces are discrete. In our case, however, we may employ the short exact sequence \( P \to O \to F_p \) of Section 3 to obtain the fibre sequence

\[
\mathbb{F}_p^n \to B \mathbb{P}^n \to B \mathbb{O}^n.
\]

Explicitly, the topological group \( B \mathbb{P} \) is the \((p-1)\)-dimensional torus

\[
\mathbb{T}_P = (\mathbb{O} \otimes \mathbb{R})/(\mathbb{P} \otimes \mathbb{Z})
\]

and \( \mathbb{F}_p \) sits in \( \mathbb{T}_P \) as \( \mathbb{O} \otimes \{1\}/\mathbb{P} \otimes \{1\} \). The inclusion of \( \mathbb{F}_p \) in \( U_1(\mathbb{C}) \cong \mathbb{R}/\mathbb{Z} \) factorizes through \( \mathbb{T}_P \) by the map

\[
(a_0 + a_1 \zeta + \cdots + a_{p-2} \zeta^{p-2}) \mapsto (a_0 + a_1 + \cdots + a_{p-2})/p.
\]

with \( a_i \in \mathbb{R} \). If one chooses the basis

\[
1 - \zeta, \zeta - \zeta^2, \ldots, \zeta^{p-2} - \zeta^{p-1}
\]

of \( \mathbb{P} \) then the induced map

\[
(\mathbb{R}/\mathbb{Z})^{p-1} \cong \mathbb{R}/\mathbb{Z} \otimes \mathbb{P} \xrightarrow{\rho} \mathbb{R}/\mathbb{Z}
\]

is the projection to the last factor.

**Lemma 4.4.** The extension of \((\mathbb{F}_p)^n_{\text{det}=1}\) defined by the quadratic form of the lattice \( \mathbb{P}^n \) coincides with \((\mathbb{F}_p)^n_{\text{det}=1}\) which was defined by the second Chern class.

**Proof.** The quadratic form of the lattice \( \mathbb{P} \) with respect to the basis (10) has the form

\[
q(x) = \sum_{i=0}^{p-1} x_i^2 + x_i x_{i+1}.
\]

As a 4-dimensional cohomology class it restricts to a generator \( z^2 \) in \( H^4 B \mathbb{F}_p \) by what we said before. The computation

\[
\sum_i z_i^2 = \sigma_1^2 - 2\sigma_2
\]

in the elementary symmetric polynomials \( \sigma_i \) implies that the quadratic form of orthogonal sum \( \mathbb{P}^n \) corresponds to \( c_2 = \sigma_2 \) up to the unit \(-2 \mod p \) once \( c_1 \) vanishes. \( \square \)

In the pull back diagram above, one may hence replace the group \( \mathbb{F}_p^n \) with the torus \( \mathbb{T}_P^n \) and look for loop representation of the extension \( \Omega \mathbb{T}_P^n \) with the center \( S^1 \) acting non trivially. This will be done in the next section in the infinitesimal setting. The role of codes in \( \mathbb{F}_p^n \) will then be disclosed.
5. Lattice vertex operator algebras and their representations

The mathematical notion of a vertex algebra goes back to the work of Borcherds in 1986 [Bor86] who axiomatized the relations amongst lattice vertex operators. The definition appears rather technical but will become more explicit when we look at the example $\Omega^T_L$ where $L$ is a lattice.

5.1. Vertex operator algebras and modules. A vertex algebra consists of the following data:

(i) a $\mathbb{Z}_+$-graded vector space $V$
(ii) a vacuum vector $1 \in V_0$, $1 \neq 0$
(iii) a linear map $D : V \to V$ of degree one, $D1 = 0$
(iv) a linear map $Y(\cdot, z) : V \to \text{End}(V)[z, z^{-1}]$, $Y(v, z) = \sum v_n z^{-n-1}$

The endomorphisms $v_n$ are called modes of $v$. These data should satisfy the following properties for all $v, w \in V$ (see [Fre02] [MT10] for details):

(i) there is an integer $N = N(w)$ with $v_n(w) = 0$ for all $n > N$
(ii) $Y(v, z) \in V[z, z^{-1}][z^{-1}]$ and there is a $k \geq 0$ with $(z_1 - z_2)^k [Y(v_1, z_1), Y(v_2, z_2)] = 0$.

This property is called locality and one writes $Y(v, z) \sim Y(w, z)$ if a power of $(z_1 - z_2)$ annihilates the bracket.

(iii) $Y(v, z)1 = v + O(z)$
(iv) $[D, Y(v, z)] = \partial Y(u, z)$ where $\partial$ is the formal derivative with respect to $z$

Note that the operator $D$ is entirely determined by the state-field correspondence $Y$: property (iv) implies $Dv = v_{-2}1$.

A vertex operator algebra is a vertex algebra which is equipped with a distinguished state $\omega \in V$. Its modes $L_n = \omega_{n+1}$ of $\omega$ should satisfy

(i) $V_n = \{ v \in V \mid L_0 v = nv \}$
(ii) $Y(L_{-1} v, z) = \partial Y(v, z)$
(iii) $[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m,-n} c id_V$ for some number $c$.

In addition, one requires $\dim V_n < \infty$, $V_n = 0$ for $n \ll 0$. The state $\omega$ also carries the name conformal vector. (It is responsible for the action of the Virasoro algebra, that is, the reparametrization action of loops.)

A module over a vertex operator algebra is a vector space $M$ together with a linear map

$Y_M : V \to \text{End}(M)[z, z^{-1}]$, $v \mapsto Y_M(v, z) = \sum v_n^M z^{-n-1}$

with the properties:

(i) $Y_M(1, z) = id_M$
(ii) $Y_M(v, z) \sim Y_M(w, z)$
(iii) for \( k > 0 \) associativity holds, that is,

\[(z_1 + z_2)^k Y_M(v, z_1 + z_2) Y_M(w, z_2) = (z_1 + z_2)^k Y_M(Y_M(v, z_1) w, z_2).\]

Usually one requires another grading \( M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda} \) such that \( \dim M_{\lambda} < \infty \), \( M_{\lambda + n} = 0 \) for \( n \ll 0 \) and \( L_0 m = \lambda m \) for \( m \in M_{\lambda} \). It turns out that irreducible modules over tame vertex operator algebras have this property. Here, a \( V \)-module is irreducible if it contains no proper, nonzero submodule (invariant under all modes \( v_n^M \)).

Note that if \( M \) is irreducible and if \( M_{\lambda + n} \neq 0 \) for some \( n \in \mathbb{Z} \) then \( M = \bigoplus_{n \in \mathbb{Z}} M_{\lambda + n} \) because the right module is invariant under all modes. Set \( h = \lambda + n \) with the smallest \( n \) such that \( M_{\lambda + n} \neq 0 \). This number turns out to be rational and it is called the conformal weight of \( M \).

The partition function of a representation \( M \) is the function on the upper half plane given by the formula

\[ Z(z) = tr_M q^{L_0^M - c/24} = q^{h - c/24} \sum_{n \geq 0} \dim(M_{h + n}) q^n \]

where \( q = e^{2\pi i z} \).

5.2. Lattice algebras. We now specify to the vertex algebra associated to a \( d \)-dimensional lattice \( L \) with a non degenerate symmetric bilinear form \( q \). We will construct an infinitesimal model for \( \widetilde{\Omega T_L} \) where \( T_L = L \otimes S^1 \).

Set \( \mathfrak{h} = \mathbb{C}^d \) and think of \( \mathfrak{h} \) as the abelian complexified Lie algebra of \( T_L \). Define the Heisenberg algebra

\[ \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} K \]

with central element \( K \) and brackets

\[ [v \otimes t^m, w \otimes t^n] = m \langle v, w \rangle \delta_{m, -n} K. \]

The first summand in \( \hat{\mathfrak{h}} \) corresponds to loops in \( T_L \) and the second one gives a central extension by \( S^1 \). This already is a Lie algebra model for \( \widetilde{\Omega T_L} \). However, the loop space of a group and its extension has more structure than just being a group. For example, one can reparameterize loops. This action should be incorporated into the model.

The Heisenberg algebra comes with a triangular decomposition given by

\[ \hat{\mathfrak{h}}^\pm = \oplus_{\pm n > 0} \mathfrak{h} \otimes t^n; \quad \hat{\mathfrak{h}}^0 = \mathfrak{h} \oplus \mathbb{C} K. \]

Let \( M \) be a \( \mathfrak{h} \)-module \( X \). Let \( l \) be a scalar. Extend the action of \( \mathfrak{h} \) to \( \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^0 \) by letting \( \hat{\mathfrak{h}}^+ \) annihilate \( M \) and by letting \( K \) act by multiplication with the level \( l \). Let \( V_\mathfrak{h}(l, M) \) be the induced \( \mathfrak{h} \)-module. Explicitly,

\[ V_\mathfrak{h}(l, M) = U(\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^0) \otimes M \cong U(\hat{\mathfrak{h}}^-) \otimes M \cong S(\hat{\mathfrak{h}}^-) \otimes M \]
where \( \mathcal{U} \) denotes the universal enveloping algebra and \( S \) is the symmetric algebra. Set
\[
Y(v, z) = \sum_{n} v(n) z^{-n-1}
\]
and \( v(n) \) is the action of \( v \otimes t^n \) on \( V_{h}(l, M) \). This state-field correspondence \( Y \) makes
\[
M_{d}^{0} \overset{\text{def}}{=} V_{h}(1, \mathbb{C}1)
\]
into a vertex algebra. The conformal vector is
\[
\omega = \frac{1}{2} \sum_{i=1}^{d} (v_{i})((-1)^{i}(v^{i})(-1)) \mathbf{1}
\]
and central charge is \( d \). Here, \( \{v_{i}\} \) is a basis of \( \mathbb{C}^{d} \) and \( \{v^{i}\} \) is the dual basis with respect to \( Q \). Moreover, \( V_{h}(1, M) \) is a module over \( M_{d}^{0} \) (see [MT10] for details).

Now assume that \( L \) is an even lattice in \( \mathfrak{h} = \mathbb{R}^{d} \) with respect to a positive definite real quadratic form \( q_{\mathfrak{h}} \) and let \( \mathfrak{h} \) and \( q \) be the complexifications. The irreducible representations \( M_{\alpha} \) of \( \mathfrak{h} \) are one-dimensional and indexed by elements of the dual space \( \alpha \in \mathfrak{h}^{\ast} \). Set
\[
V_{L} = \bigoplus_{\alpha \in L} V_{h}(1, M_{\alpha^{\vee}}) \cong S(\hat{\mathfrak{h}}^{-}) \otimes \mathbb{C}[L]
\]
where \( \alpha^{\vee} \) denotes the dual of \( \alpha \). The construction makes \( V_{L} \) into a module over the Heisenberg vertex algebra \( M_{d}^{0} \).

With some effort, the vector space \( V_{L} \) can itself be given the structure of a vertex operator algebra: for \( e^{\alpha} \in M_{\alpha^{\vee}} \) set
\[
Y(e^{\alpha}, z) = \exp(\sum_{n>0} \frac{\alpha(-n)}{n} z^{n}) \exp(\sum_{n<0} \frac{\alpha(-n)}{n} z^{n}) e^{\alpha} z^{\alpha}.
\]
Here, \( \alpha_n \) are the modes of \( Y(\alpha, z) \) in \( M_{d}^{0} \), \( z^{\alpha} : v \otimes e^{\beta} \mapsto z^{q(\alpha, \beta)} v \otimes e^{\beta} \) for \( v \in \mathfrak{h}^{-} \) and \( e^{\alpha} : v \otimes e^{\beta} \mapsto \epsilon(\alpha, \beta) v \otimes e^{\alpha+\beta} \) for a certain bilinear 2-cocycle \( \epsilon : L \otimes L \to \{\pm 1\} \). This imposes the structure of a vertex algebra on \( V_{L} \). See the textbook [Kac98] or [PLM88] for details.

**Proposition 5.1.** (see [MT10], Equation (75)) The partition function of \( V_{L} \) is
\[
Z_{V_{L}}(z) = \eta(q)^{-d} \sum_{\alpha \in L} q^{\langle \alpha, \alpha \rangle/2} = \eta(q)^{-d} \vartheta_{L}(q)
\]
where \( \eta \) is the Dedekind \( \eta \)-function
\[
\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^{n}).
\]

The irreducible modules of \( V_{L} \) were computed by Dong. They are indexed by elements of the set \( C_{f} = L^{\vee}/L \) where \( L^{\vee} \) denotes the dual lattice
\[
L^{\vee} = \{ \beta \in \mathbb{R}^{d} | Q(\alpha, \beta) \in \mathbb{Z} \text{ for all } \alpha \in L \}. 
\]
For $\lambda \in C_f$ the module takes the form
\[ V_{L+\lambda} = \bigoplus_{\alpha \in L} V_{\mathbb{h}}(1, M_{(\alpha+\lambda)^+}) \cong S(\mathbb{h}^-) \otimes \mathbb{C}[L + \lambda] \]
and its partition function is
\[ Z_{V_{L+\lambda}}(z) = \eta(q)^{-d} \sum_{\alpha \in L} q^{(\alpha+\lambda,\alpha+\lambda)/2}. \]

**Theorem 5.2** ([Don93]). The $V_L$-modules $\{V_{L+\lambda}\}$ for $\lambda \in C$ are all distinct and they provide a complete list for the isomorphism classes of irreducible $V_L$-modules.

5.3. **The Eisenstein vertex algebra and its representations.** Let $\mathbb{O}$ be the ring of integers of the cyclotomic field $\mathbb{Q}[\zeta]$ as considered before. Let $C$ be a linear code in $\mathbb{F}_p^n$ with $C \subset C^\perp$. In section 3 we defined the even lattice $\Gamma_C \subset \mathbb{O}^n$ with bilinear form
\[ \langle x, y \rangle = \sum_{i=1}^n \text{Tr}(x_i \bar{y}_i / p). \]
Its dual is the lattice $\Gamma_{C^\perp}$. In particular, for $L = \Gamma_0 = \mathbb{F}_p^n$ this means
\[ (\mathbb{F}_p^n)^\vee = \mathbb{O}^n. \]

In the following, the associated vertex operator algebra $V_n = V_{\mathbb{F}_p^n}$ will be called **Eisenstein vertex algebra**. Theorem 5.2 tells us that the isomorphism classes of irreducible representations of $V_n$ are indexed by words $w$ of the full code $C_f = \mathbb{O}^n/\mathbb{F}_p^n \cong \mathbb{F}_p^n$.

Let $\mathcal{R}ep(V_n)$ be the free group generated by these isomorphism classes. Lattice algebras associated to positive-definite even lattices are rational, that is, any module is a direct sum of irreducible modules (see [DM04]). These sums are necessary finite because of the restrictions on the dimensions. We conclude that $\mathcal{R}ep(V_n)$ coincides with the Grothendiek group of the category of $V_d$-modules. Set
\[ \mathcal{R}ep(V) = \sum_{n \geq 0} \mathcal{R}ep(V_n) \]
with $\mathcal{R}ep(V_0) = \mathbb{Z}$. It has a ring structure since
\[ \mathcal{R}ep(V_{n+n'}) \cong \mathcal{R}ep(V_n) \otimes \mathcal{R}ep(V_{n'}). \]
The automorphism group $\text{Aut}(C_f) = (\mathbb{F}_p^\times)^n \rtimes \Sigma_n$ acts on $\mathcal{R}ep(V_n)$ in the obvious way. It contains the subgroup
\[ \text{Aut}_\mathbb{R}(C_f) = \{ \pm 1 \}^n \rtimes \Sigma_n \]
of real automorphisms.

Before formulating the main result we need to introduce some notation. For a ring $R$ let $\mathcal{R}ep_R(V)$ be the product $\mathcal{R}ep(V) \otimes R$. It is the free $R$-module generated by the irreducibles. Let $mf(p)$ be the ring of modular forms for the congruence group $\Gamma(p)$ with $q$-expansion at $\infty$ in $\mathbb{O}[[\mathbb{F}_p]][q^{1/p}]$. 
Similarly, write $\text{hmf}^{(p)}_{\Sigma} \sum(p)$ for the $O[\frac{1}{p}]$-algebra of symmetric Hilbert modular forms for $\Gamma(p)$ generated by $\theta_j$ for $j = 0, \ldots, r$.

**Example 5.3.** For $p = 3$, the ring $mf(3)_C$ is freely generated by $\theta_0, \theta_1$ (see f.i. [Ebe13 Theorem 5.4]). They have $q$-expansions at $\infty$

\[
\begin{align*}
\theta_0 &= \sum_{(x,y) \in \mathbb{Z}^2} q^{x^2 - xy + y^2} = 1 + 6(q + q^2 + q^4 + 2q^7 + \ldots) \\
\theta_1 &= q^{\frac{1}{3}} \sum_{(x,y) \in \mathbb{Z}^2} q^{x^2 - xy + y^2 + x - y} = 3q^{\frac{1}{3}}(1 + q + 2q^2 + 2q^4 + \ldots).
\end{align*}
\]

The first two coefficients of $\theta_0$ and $\theta_1$ imply that $mf(3)$ itself is freely generated by $\theta_0, \theta_1$ and hence $\text{hmf}^{(p)}_{\Sigma}(3) = \text{hmf}(3) = O[\frac{1}{3}][\theta_0, \theta_1]$.

\(12\) For $p = 5$, Hirzebruch has proved in [Hir77] that the ring of complex symmetric Hilbert modular forms is a polynomial ring in $\theta_0, \theta_1, \theta_2$. This implies

\(13\) $\text{hmf}^{(p)}_{\Sigma}(5) \cong O[\frac{1}{5}][\theta_0, \theta_1, \theta_2]$.

**Theorem 5.4.**

(i) The partition function induces a well defined ring map

\[
Z : \text{Rep}_O[\frac{1}{p}](V) / \text{Aut}_R(C_f) \rightarrow mf(p) \quad [M] \mapsto \eta^{n(p-1)} Z_M
\]

(ii) The map $Z$ is the composite of a map of graded rings

\[
\tilde{Z} : \text{Rep}_O[\frac{1}{p}](V) / \text{Aut}_R(C_f) \rightarrow \text{hmf}^{(p)}_{\Sigma}
\]

with the map induced by the diagonal.

(iii) The map $\tilde{Z}$ is an isomorphism for $p = 3$ and $p = 5$.

**Proof.** Irreducible representations of $V_n$ are indexed by words $w \in \mathbb{F}_p^n$ and have the partition function $\eta^{-n(p-1)}d(w)$ by Proposition 5.1. This expression coincides with the diagonal of

\[
\theta_{\{w\}} = \theta_0^{d_0(c)} \theta_1^{d_1(c)} \cdots \theta_r^{d_r(c)}(c)
\]

by Theorem 3.2 and hence is invariant under the action of $\text{Aut}_R(C_f)$. Since each $\theta_j$ lies in $\text{hmf}^{(p)}_{\Sigma}$ we obtain a well defined map $\tilde{Z}$ and the first two assertions are shown.

For the last claim, observe that the target is a polynomial ring in the $\theta_j$s by Equations (12)(13) and the map sends a representation indexed by symmetrized word to the corresponding monomial.

**Corollary 5.5.** For a code $C \subset \mathbb{F}_p^n$ define the $V_n$-module $M_C = \sum_{w \in C} V_p^{\dim w}$. Suppose $C$ is self dual.

(i) The partition function $Z$ of $M_C$ is a $SL_2(\mathbb{Z})$-invariant modular form.
(ii) The refined partition function $\tilde{Z}$ is a $SL_2(\mathbb{Q})$-invariant Hilbert modular form for $p = 5$.

Proof. The first statement follows from Theorem 5.4 and Corollary 3.3. The second statement for $p = 5$ follows from Theorem 5.4 and a result of Gleason-Pierce and Sloane (see [MMS72],[Ebe13, Corollary 5.4,5.5]). □

Remark 5.6. For $p \geq 5$ the refined partition function $\tilde{Z}$ should have a more intrinsic description in terms of traces of Eisenstein vertex operators. Also, it would be worth to know the representations which allow the Doi-Naganuma lift (c. Remark 3.4).

Remark 5.7. There is an action of $SL_2(\mathbb{Z})$ on the ring of integral $\Gamma(3)$-modular forms: if $\theta$ is a modular form of weight $n$ and level 3 then

$$\theta \left( \frac{az + b}{cz + d} \right) (cz + d)^{-n}$$

is again a modular form of weight $n$ and level 3. Since the subgroup $\Gamma(3)$ acts trivially it is an action of

$$G_1 = SL_2(\mathbb{F}_3) \cong SL_2(\mathbb{Z})/\Gamma(3).$$

Explicitly, the generators $S$ and $T$ act on $\theta_0$ by

$$\theta_0 \left( \frac{-1}{z} \right) = \frac{z(-1 - 2\zeta)}{3} (\theta_0(z) + 2\theta_1(z))$$

$$\theta_0(z + 1) = \theta_0(z)$$

and on $\theta_1$ by

$$\theta_1 \left( \frac{-1}{z} \right) = \frac{z(-1 - 2\zeta)}{3} (\theta_0(z) - \theta_1(z))$$

$$\theta_1(z + 1) = \zeta \theta_1(z).$$

The fix point ring

$$mf(3)^{G_1} = mf_{0[3]}$$

coincides with the ring of integral modular forms for the full group $SL_2(\mathbb{Z})$.

6. Topological modular forms with level structure

Atiyah-Bott-Shapiro have discovered the role of Clifford algebra representations in $K$-theory [ABS64]. We have seen in Section 4 that these correspond to representations of the finite group $F_n$ in the arithmetic Whitehead tower and are best understood using coding theory. In Section 5, we investigated the representations of the string extension of the corresponding group in the complex arithmetic Whitehead tower. Theorem 5.4 gives a correspondence of its representation ring with modular forms for the congruence subgroup $\Gamma(N)$. In this section, we like to interpret this result as
a higher version of the Atiyah-Bott-Shapiro construction. The higher version of $K$-theory will be $TMF(N)$, the cohomology theory of topological modular forms with full level $N$ structure.

6.1. The spectrum $TMF(N)$. Elliptic cohomology has its origin in the study of elliptic genera. F. Hirzebruch [Hir88] generalized the earlier notion of S. Ochanine from level 2 to higher levels and constructed ring maps

$$\chi_* : MU_* \to mf(N)_*$$

where $MU_*$ is the bordism ring of stably almost complex manifolds. These elliptic genera come along with rigidity results for the equivariant index of the Dirac operator on manifolds with circle actions. E. Witten related them to index theory on loop spaces of manifolds with string structures. The Witten genus is a ring map

$$W : MString_* \to mf_*$$

where $MString$ denotes the cobordism theory of manifolds with string structures.

In the late 80s, Landweber-Ravenel-Stong [LRS95] introduced a cohomology theory, nowadays denoted by $TMF_1(2)$, for the congruence group $\Gamma_1(2)$. It was generalized to other levels by A. Baker [Bak94], J.L. Brylinski [Bry90] (with 2 inverted) and by the second author with the help of a result by J. Franke [Fra92]. The method given in [Lau99] shows that the Hirzbruch genera define periodic homology and cohomology theories given by

$$TMF(N)_*(X) = MU_*(X) \otimes _* mf(N)_*[\Delta^{-1}].$$

The corresponding ring spectra come with ring maps

$$\chi : MU \to TTMF(N)$$

but these complex orientations are not compatible amongst each other along the maps $TMF(N) \to TTMF(NM)$. A family version of the Witten genus

$$W : MString \to tmf$$

was achieved by M. Hopkins et al. [Hop02, DPHH14]. Its target is the spectrum of topological modular forms $tmf$ (or $Tmf$ for the periodic version). Its coefficient ring maps to the ring $mf_*$ of integral modular forms. This map is neither surjective nor injective but rationally it is an isomorphism.

There are basically two different constructions of the spectrum $tmf$ as a highly structured ring spectrum. One construction is based on its localizations with respect to Morava $K$-theories (see [DFHH14, Chapter 11]). It uses the fact that its $K(2)$-localization is the Lubin-Tate theory $E_2$. This theory originates from deformations of the elliptic curve mentioned in Section 2.2. Its $K(1)$-localization is related to the theory of $p$-adic modular forms (see [Lau04]). The other construction uses derived algebraic geometry, that is, algebraic geometry with ordinary rings replaced by highly structured ring spectra. The reader is referenced to the survey article [Lur09]. For a construction of the connective theories $tmf(N)$ along these lines see [HL16].
There is an equivalence of homotopy fixed point spectra
\[ T \text{MF}[1/p] \to T \text{MF}(p)^{hG_{2}(\mathbb{F}_{p})} \]
which should be regarded as a homotopy version of Equation (14). The spectral sequences
\[ H^{s}(G_{2}(\mathbb{F}_{p}), \pi_{2l}T \text{MF}(p)) \Rightarrow \pi_{2l-s}T \text{MF}[1/p] \]
where computed by Mahowald-Rezk in [MR09] for \( p = 3 \) and by Behrens-Omsky in [BO16] for \( p = 5 \).

For \( p = 3 \), the proposed generalization of the Atiyah-Bott-Shapiro theorem is the following:

**Corollary 6.1.**

(i) The partition function \( Z \) gives an isomorphism
\[ \mathcal{R}ep_{\mathbb{Z}[\mathbb{F}_{3}]}(V)/\text{Aut}_{\mathbb{R}}(C_{f}) \overset{\cong}{\longrightarrow} \text{tmf}^{0}(S^{2n}). \]

(ii) If \( C \) is a self dual ternary code of length \( n \) then there is an element of \( \text{tmf}^{0}(S^{2n}) \) whose image in \( \text{tmf}^{0}(S^{2n}) \) is \( Z(M_{C}) \).

**Proof.** The first claim only is a reformulation of Theorem 5.4 for the case \( p = 3 \). The second claim follows from Equation (3) and [Hop02, Proposition 5.12]. \( \Box \)

H. Tamanoi argued in [Tam99] that the equivariant index of the formal Dirac operator on loop spaces is a virtual module over some vertex operator algebra. Of course, this is a consequence of Corollary 6.1 for \( p = 3 \), since the index coincides with the elliptic genus and hence takes values in the ring of \( \Gamma(3) \)-modular forms. This suggests that there is a geometric construction of the cohomology theory \( \text{tmf}(3) \) along these lines.

It turns out that bundles whose fibers are representations of the Eisenstein vertex algebra lead to a cohomology theory \( K_{V} \) by the usual construction: one takes the Grothendieck group of isomorphism classes and inverts the representation which is attached to the discriminant \( \Delta \) to get a periodic theory. The corollary then says that \( K_{V} \) has the same coefficients as \( T \text{MF}(3) \). However, using results from Dong-Liu-Ma-Zhou [DLMZ04], it is not hard to see that one only gets a form of \( K \)-theory
\[ K_{V}(X) = K(X) \otimes mf(3)_{*}[\Delta^{-1}] \]
since the associated genus is equivalent to the Todd genus. Bundles of representations hence do not lead to the correct elliptic cocycles. On the other hand, Equation (15) implies that a cohomology theory with the same coefficients and the correct Euler classes has to coincide with \( T \text{MF}(3) \).

### 6.2. Sheaves of vertex operator algebras and Euler classes.

Bundles correspond to quasi-coherent sheaves. Gorbounov-Malikov-Schechtman-Vaintrob [GMS00] constructed a more general sheaf \( MSV(X) \) of vertex operator algebras for each complex manifold \( X \). It comes with a multiplication map
\[ \mathcal{O}(X) \otimes MSV(X) \rightarrow MSV(X) \]
by functions on $X$ which, however, is not associative. There is a filtration of $\text{MSV}(X)$ compatible with the multiplication such that the associated graded is the coherent sheaf (cf. [BL00])

$$\text{ell}_X = \Lambda_{y^{-1}} T^* X \otimes \bigotimes_{n=1}^{\infty} \Lambda^{y n} T^* X \otimes \bigotimes_{n=1}^{\infty} S_q^n T^* X \otimes \bigotimes_{n=1}^{\infty} S_q^n T^* X.$$

Take $y = -\zeta$ and consider $\text{ell}_X$ as an element of $K_{\text{Tate}}(N)(X) = K(X)_{[1/N]}(q^{1/N})$.

**Proposition 6.2.** Up to a normalization factor, $\text{ell}_X$ coincides with the Euler class of $T X$ in $K_{\text{Tate}}(N)$ once $K_{\text{Tate}}(N)$ is equipped with the orientation coming from the level $N$-elliptic character [Lau99] [Lau00]

$$\lambda : \text{TMF}(N) \to K_{\text{Tate}}(N).$$

**Proof.** This can easily be verified with a Chern character computation (see [HBJ92], p.117ff)).

Proposition 6.2 suggests that $\text{MSV}(X)$ should rather be considered as an element of $\text{TMF}^{2n}(X)$, the elliptic Euler class, which maps under $\lambda$ to $\text{ell}_X$. Up to a normalization factor, the elliptic character hence is the map which takes the associated graded object.

The normalization factor also plays a role in Corollary 6.1 where the considered objects have the same spirit. In fact, the construction of $\text{MSV}(X)$ given in [LL07] resembles Equation (15) where the module $M$ is replaced by a quasi-coherent sheaf. Lian-Linshaw even provided equivariant versions of $\text{MSV}(X)$ in the same article for $X$ which are equipped with an action of a compact Lie group. Needless to say that such geometric interpretations in terms of vertex operator algebras would tremendously enhance the theory of topological modular forms and conformal field theories.

**References**

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology 3 (1964), no. suppl., suppl. 1, 3–38. MR 167985

[Bak94] Andrew Baker, *Elliptic genera of level N and elliptic cohomology*, J. London Math. Soc. (2) 49 (1994), no. 3, 581–593. MR 1271552

[BL00] Lev A. Borisov and Anatoly Libgober, *Elliptic genera of toric varieties and applications to mirror symmetry*, Invent. Math. 140 (2000), no. 2, 453–485. MR 1757003

[BO16] Mark Behrens and Kyle Ormsby, *On the homotopy of Q(3) and Q(5) at the prime 2*, Algebr. Geom. Topol. 16 (2016), no. 5, 2459–2534. MR 3572338

[Bor86] Richard E. Borcherds, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), no. 10, 3068–3071. MR 843307

[Bry90] Jean-Luc Brylinski, *Representations of loop groups, Dirac operators on loop space, and modular forms*, Topology 29 (1990), no. 4, 461–480. MR 1071369

[DFHH14] Christopher L. Douglas, John Francis, André G. Henriques, and Michael A. Hill (eds.), *Topological modular forms*, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014. MR 3223024
[DLMZ04] Chongying Dong, Kefeng Liu, Xiaonan Ma, and Jian Zhou, *K*-theory associated to vertex operator algebras, Math. Res. Lett. **11** (2004), no. 5-6, 629–647. MR 2106231

[DM04] Chongying Dong and Geoffrey Mason, Rational vertex operator algebras and the effective central charge, Int. Math. Res. Not. (2004), no. 56, 2989–3008. MR 2097833

[Don93] Chongying Dong, Vertex algebras associated with even lattices, J. Algebra **161** (1993), no. 1, 245–265. MR 1245855

[Ebe13] Wolfgang Ebeling, Lattices and codes, third ed., Advanced Lectures in Mathematics, Springer Spektrum, Wiesbaden, 2013, A course partially based on lectures by Friedrich Hirzebruch. MR 2977354

[EG17] Narthana Epa and Nora Ganter, Platonic and alternating 2-groups, High. Struct. **1** (2017), no. 1, 122–146. MR 3912053

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meurman, Vertex operator algebras and the Monster, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988. MR 996026

[Fra92] Jens Franke, On the construction of elliptic cohomology, Math. Nachr. **158** (1992), 43–65. MR 1235295

[Fre02] Edward Frenkel, Vertex algebras and algebraic curves, vol. 276, 2002, S´ eminaire Bourbaki, Vol. 1999/2000, pp. 299–339. MR 1886764

[GMS00] Vassily Gorbounov, Fyodor Malikov, and Vadim Schechtman, Gerbes of chiral differential operators, Math. Res. Lett. **7** (2000), no. 1, 55–66. MR 1748287

[HBJ92] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung, Manifolds and modular forms, Aspects of Mathematics, E20, Friedr. Vieweg & Sohn, Braunschweig, 1992, With appendices by Nils-Peter Skoruppa and by Paul Baum. MR 1189136

[Hir77] F. Hirzebruch, The ring of Hilbert modular forms for real quadratic fields in small discriminant, Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 1977, pp. 287–323. Lecture Notes in Math., Vol. 627. MR 0480355

[Hir88] Friedrich Hirzebruch, Elliptic genera of level $N$ for complex manifolds, Differential geometrical methods in theoretical physics (Como, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 250, Kluwer Acad. Publ., Dordrecht, 1988, pp. 37–63. MR 981372

[HL16] Michael Hill and Tyler Lawson, Topological modular forms with level structure, Invent. Math. **203** (2016), no. 2, 359–416. MR 3455154

[Hop02] M. J. Hopkins, Algebraic topology and modular forms, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 291–317. MR 1989190

[JFT20] Theo Johnson-Freyd and David Treumann, $H^3(Co_0; \mathbb{Z}) = \mathbb{Z}/24$, Int. Math. Res. Not. IMRN (2020), no. 21, 7873–7907. MR 4176841

[Kac98] Victor Kac, Vertex algebras for beginners, second ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998. MR 1651389

[Lau99] Gerd Laures, The topological $q$-expansion principle, Topology **38** (1999), no. 2, 387–425. MR 1660325

[Lau00] Gerd Laures, On cobordism of manifolds with corners, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5667–5688. MR 1781277

[Lau04] Gerd Laures, $K(1)$-local topological modular forms, Invent. Math. **157** (2004), no. 2, 371–403. MR 2076927

[LL07] Bong H. Lian and Andrew R. Linshaw, Chiral equivariant cohomology. I, Adv. Math. **209** (2007), no. 1, 99–161. MR 2294219
[LM89] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR 1031992

[LRS95] Peter S. Landweber, Douglas C. Ravenel, and Robert E. Stong, *Periodic cohomology theories defined by elliptic curves*, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 317–337. MR 1320998

[Lur09] J. Lurie, *A survey of elliptic cohomology*, Algebraic topology, Abel Symp., vol. 4, Springer, Berlin, 2009, pp. 219–277. MR 2597740

[MMS72] F. Jessie MacWilliams, Colin L. Mallows, and Neil J. A. Sloane, *Generalizations of Gleason’s theorem on weight enumerators of self-dual codes*, IEEE Trans. Inform. Theory IT-18 (1972), 794–805. MR 398664

[MR09] Mark Mahowald and Charles Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Q. 5 (2009), no. 2, Special Issue: In honor of Friedrich Hirzebruch. Part 1, 853–872. MR 2508904

[MS74] A. W. Mason and W. W. Stothers, *On subgroups of GL(n, A) which are generated by commutators*, Invent. Math. 23 (1974), 327–346. MR 338209

[MT91] Mamoru Mimura and Hirosi Toda, *Topology of Lie groups. I, II*, Translations of Mathematical Monographs, vol. 91, American Mathematical Society, Providence, RI, 1991, Translated from the 1978 Japanese edition by the authors. MR 1122592

[MT10] Geoffrey Mason and Michael Tuite, *Vertex operators and modular forms*, A window into zeta and modular physics, Math. Sci. Res. Inst. Publ., vol. 57, Cambridge Univ. Press, Cambridge, 2010, pp. 183–346. MR 2648364

[PS86] Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1986, Oxford Science Publications. MR 900587

[Rom05] Matthieu Romagny, *The fundamental theorem of alternating functions*, [https://perso.univ-rennes1.fr/matthieu.romagny/notes/FTAF.pdf](https://perso.univ-rennes1.fr/matthieu.romagny/notes/FTAF.pdf) September 2005.

[Ros94] Jonathan Rosenberg, *Algebraic K-theory and its applications*, Graduate Texts in Mathematics, vol. 147, Springer-Verlag, New York, 1994. MR 1282290

[Seg88] G. B. Segal, *The definition of conformal field theory*, Differential geometrical methods in theoretical physics (Como, 1987), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 250, Kluwer Acad. Publ., Dordrecht, 1988, pp. 165–171. MR 981378

[SP11] Christopher J. Schommer-Pries, *Central extensions of smooth 2-groups and a finite-dimensional string 2-group*, Geom. Topol. 15 (2011), no. 2, 609–676. MR 2803631

[ST04] Stephan Stolz and Peter Teichner, *What is an elliptic object?*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 247–343. MR 2079378

[Ste92] John R. Stembridge, *The projective representations of the hyperoctahedral group*, J. Algebra 145 (1992), no. 2, 396–453. MR 1144940

[Tam99] Hirotaka Tamanoi, *Elliptic genera and vertex operator super-algebras*, Lecture Notes in Mathematics, vol. 1704, Springer-Verlag, Berlin, 1999. MR 1716689

[Woo89] Jay A. Wood, *Maximal abelian subgroups of spinor groups and error-correcting codes*, Algebraic topology (Evanston, IL, 1988), Contemp. Math., vol. 96, Amer. Math. Soc., Providence, RI, 1989, pp. 333–350. MR 1022693
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