Pseudoholomorphic quilts and Khovanov homology

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We further study the symplectic Khovanov homology of Seidel and Smith and its generalization to even tangles. We associate homomorphisms to elementary (as well as minimal) cobordisms between tangles.

We define the symplectic analogues $H^m_{ss}$ of Khovanov’s arc algebras and equip the invariant assigned to an $(m,n)$-tangle with the structure of an $(H^m_{ss}, H^m_{ss})$-bimodule. We show that $H^m_{ss}$ and Khovanov’s $H^m$ are isomorphic as algebras over $\mathbb{Z}/2$. We also obtain an exact triangle for the Seidel-Smith invariant similar to that of Khovanov homology.

1 Introduction

This paper is an addendum to [7] and its purpose is to prove some natural properties of the tangle invariant discussed therein. The tangle invariant is a generalization of the link invariant developed in [10] by Seidel and Smith as a candidate for a geometric model for Khovanov homology. It is called symplectic Khovanov homology and we denote it by $\mathcal{H}_{SS}$. Roughly speaking $\mathcal{H}_{SS}$ of a link $K$, given as the closure of a braid $\beta \in Br_m$, is defined as the Lagrangian Floer homology of a canonically defined Lagrangian $L$ and its image under a symplectomorphism associated to $\beta$. The Lagrangian $L$ lives in a symplectic manifold diffeomorphic to an $(m,m)$-type Springer fiber. One can hope that the geometric nature of $\mathcal{H}_{SS}$ can give more insight which is not available for Khovanov homology itself. (See [11] for a result in this direction.)

The invariant $\mathcal{H}_{SS}$ was generalized to even tangles in [7]. One motivation for this generalization was to make the proof of the equivalence of $\mathcal{H}_{SS}$ and Khovanov homology more feasible by decomposing a given link into elementary tangles.

In Section 5.1 we obtain (elementary) cobordism maps for the $\mathcal{H}_{SS}$ i.e. if two tangles $T, T'$ are related by an elementary cobordism $S$, we obtain a homomorphism

$$\mathcal{H}_{SS}(S): \mathcal{H}_{SS}(T) \to \mathcal{H}_{SS}(T').$$
These maps are given by counting pseudoholomorphic quilts. We do not attempt at
showing that our cobordism maps give a well-defined map when a general cobordism
is decomposed (into elementary ones) in different ways.

In the rest of Section 5 we use the cobordism maps (1) to show that \( \mathcal{H}_{SS} \) shares some
of the properties of Khovanov’s invariant. First of all we define a family of rings
\( \{H^m\}_{m \in \mathbb{N}} \) which are the symplectic analogues of Khovanov’s arc algebras \( H^m \) from
[3]. We define
\[
H^m_s := \mathcal{H}_{SS}(id_m)
\]
with product given by the maps associated to minimal cobordisms from Section 5.1
which in turn are given by counting pseudoholomorphic quilted triangles. We show
that the rings in these two families are isomorphic as algebras over \( \mathbb{Z}/2 \) (Theorem 5.5).

We equip the abelian group \( \mathcal{H}_{SS}(T) \), assigned to an \( (m,n) \)-tangle \( T \), with the structure
of an \( (H^m_s, H^n_s) \)-bimodule.

We provide further evidence for the equivalence of this invariant with Khovanov’s
combinatorially defined invariant by showing the equivalence for flat (crossingless)
tangles and elementary cobordisms between them (Propositions 5.11 and 5.12 ). In
5.3 we prove that for flat tangles Floer data can be chosen in such a way that Floer
differential vanishes.

At last in Section 5.5 we use the exact triangles for fibred Dehn twists from [13] to
prove an exact triangle for \( \mathcal{H}_{SS} \) (Corollary 5.19) which resembles that of Khovanov
homology after the collapse of bigrading. An argument similar to the one used by
Manolescu and Ozsvath [4] can then be used to show that the two invariants (i.e.
\( \mathcal{H}_{SS} \) and bigrading-collapsed Khovanov homology) agree on quasi-alternating links.
However we do not present a proof of this last claim.

We point out that in almost the same time that the first draft of this paper appeared
online, similar cobordism maps were introduced in an independent work by Jack
Waldron [12]. Waldron’s cobordism maps are defined in the original setting of the link
invariant of [10] and are given using the relative invariants associated to surfaces with
strip-like ends by Seidel [9]. But our maps are defined using the formalism of quilts
[16] and are defined after decomposing the involved tangles into elementary ones.
We expect that Waldron’s cobordism maps, when restricted to elementary cobordisms,
agree with our maps, when the latter is restricted to cobordisms of links.

**Remark** Symplectic Khovanov homology as defined by Seidel and Smith is an in-
variant of links given as closure of braids therefore comparison of this invariant and
Khovanov homology is nontrivial. See “symplectic Khovanov homology of crossing
diagrams” in Section 5.6 in [12]. However the the extension from [7] which we use here is assigned to tangle decompositions just as Khovanov’s invariant.

In the sections 2 to 4 we review, respectively, Khovanov’s invariant of tangles, Seidel-Smith invariant for links and the generalization of the latter to tangles. The notation we use here for the symplectic invariant is a bit different from that of [7]. In that paper we used the notation $\mathcal{K}_\text{sym}$ but here we use $\mathcal{HSS}$ (and $\mathcal{CSS}$ for the chain complex).

For the results of this paper to hold for Floer chain complexes with coefficients in $\mathbb{Z}$, one needs coherent orientations on the moduli spaces of pseudoholomorphic quilts used. This relies on work in progress by Wehrheim and Woodward [14].

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2 Khovanov homology of even tangles

Here we recall the basic facts about the Khovanov homology of even tangles.

2.1 Tangles

This subsection is a repetition of Section 4.1 in [7]. A tangle $T$ is defined to be a compact one-dimensional submanifold of (a diffeomorphic image of) $\mathbb{C} \times [0, 1]$ such that $i(T) := T \cap (\mathbb{C} \times \{0\}) \subset \mathbb{R} \times \{0\}$ and $t(T) := T \cap (\mathbb{C} \times \{1\}) \subset \mathbb{R} \times \{1\}$ and both sets are finite. The second assumption makes $i(T)$ and $t(T)$ ordered sets. In this thesis we deal only with tangles with an even number of initial points and end points. Such tangles are called even tangles. If $\#i(T) = 2m, \#(T) = 2n$ we say $T$ is an $(m, n)$-tangle and write $mTn$. We also allow $m$ and/or $n$ to be zero.

Definition 2.1 Two tangles $T, T'$ are called equivalent if there is a continuous family $T_t$ of tangles for $t \in [0, 1]$ such that $T_0 = T$ and $T_1 = T'$ and the order of $i(T_t)$ and of $t(T_t)$ is fixed.

Two tangles $T_1, T_2$ can be composed (concatenated) if $t(T_1) = i(T_2)$. Two equivalence classes $[T_1]$ and $[T_2]$ of tangles can be composed if $\#t(T_1) = \#i(T_2)$ and composition is done using the ordering on $t(T_1)$ and $i(T_2)$. Composition of tangles is denoted by juxtaposition. We will use the notation $id_m, \cap_{ijm}, \cup_{ijm}$ and $\sigma_{ijm}$ for the elementary
tangles in Figures 2 and 3 where \( m \) denotes the number of the strands. We might ignore \( m \) when there is no confusion. When we say a tangle \( T \) is equivalent to, say, \( \cap_{i\leq m} \), we implicitly have a one to one correspondence between \( i(T) \) and \( \{1,2,\ldots,2m-2\} \) and also between \( t(T) \) and \( \{1,2,\ldots,2m\} \) in mind.

A decomposition of \( T \) is a sequence of tangles

\[
n_0 T_1 n_1 T_2 \cdots n_{l-1} T_l n_l \quad n_0 = m, n_l = n
\]

such that \( T \) is equivalent to \( T_1 T_2 \cdots T_l \). A Morse-theoretic argument shows that any \( T \) can be expressed (not uniquely) as a composition of elementary tangles. Crossingless matchings (section 3.3) are a special class of \((0,n)\) or \((n,0)\)-tangles. Given a set of \( 2n \) points on the plane, a crossingless matching is a set of \( n \) non-intersecting curves each joining a pair of the given points. In the context of tangles a crossingless matching is regarded as a subset of \( \mathbb{C} \times [0,1] \).

**Definition 2.2** Let \( \mathcal{C}_n \) be the set of isotopy (in \( \mathbb{C} \)) classes of crossingless matchings between \( 2n \) points on the real line all of whose arcs lie in the upper half plane.

The cardinality of \( \mathcal{C}_n \) equals the \( n \)'th Catalan number.

One can define a category \( \text{Tang} \) whose objects are natural numbers and \( \text{hom}(m,n) \) consists of equivalence classes of \((m,n)\)-tangles. \( \text{Tang} \) has a monoidal structure given
by putting two tangles $kTl$ and $mTn$ “side-by-side” to obtain a $(k + m, l + n)$-tangle. We denote this by $T \oplus T'$. To each $(m, n)$-tangle $T$ there is assigned a “mirror image” $T'$ which is a $(n, m)$-tangle. There is a generators and relations description of $\text{Tang}$ due to Yetter [17] whose proof relies on Reidemeister’s description of plane diagram moves.

**Lemma 2.3** (Yetter [17]) The following are all the commutation relations between elementary tangles where “$\equiv$” means equivalence. If $|i - j| > 1$ we have:

\begin{align*}
(4) & \quad \sigma_i \sigma_j = \sigma_j \sigma_i \\
(5) & \quad \cap_i \cup_j = \cup_j \cap_i \\
(6) & \quad \cap_i \sigma_j = \sigma_j \cap_i \quad \cup_i \sigma_j = \sigma_j \cup_i,
\end{align*}

and for any $i$ we have:

\begin{align*}
(7) & \quad \sigma_j \cup_i = \cup_i \\
(8) & \quad \sigma_j \sigma_i' = id \\
(9) & \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\
(10) & \quad \cap_i \cup_{i+1} \cup_{i+2m} = id_{m-1} \\
(11) & \quad \sigma_i \cup_{i+1} = \sigma_{i+1} \cup_i \quad \sigma_i' \cup_{i+1} = \sigma_{i+1} \cup_i.
\end{align*}

The invariants that we consider here are invariants of oriented tangles. An oriented tangle comes with an orientation of each one of its components. Two example are shown in Figure 4.

## 2.2 The TQFT

Khovanov homology is based on a 1+1 dimensional TQFT $F$ whose definition we review here. 1+1 dimensional TQFTs are in one-to-one correspondence with Frobenius
algebras. Khovanov [2] defines the Frobenius algebra $\mathcal{V}$ to be equal to $H^*(S^2 \{ -1 \}$ (i.e. the cohomology of $S^2$ with its grading shifted down by one) as a ring. Let $\iota, x$ be degree $-1$ and degree 1 generators of $\mathcal{V}$ respectively. We define comultiplication by

$$\Delta(x) = x \otimes x \quad \Delta(\iota) = \iota \otimes x + x \otimes \iota.$$  

The unit map $\iota : \mathbb{Z} \to \mathcal{V}$ by $\iota(1) = \iota$. The trace map is defined by

$$\epsilon(x) = 1 \quad \epsilon(\iota) = 0.$$  

It is evident that multiplication is given by

$$m(1, x) = m(x, 1) = x \quad m(x, x) = 0 \quad m(1, 1) = 1.$$  

Definitions above are made by choosing a basis for $H^*(S^2)$. In section 5.2 we give a definition which does not need the choice of a basis. The TQFT $\mathcal{F}$ assigns to each closed one dimensional manifold (i.e. a circle) the vector space $\mathcal{V}$, to each cap the unit map $\iota : \mathbb{Z} \to \mathcal{V}$, to each cup the trace map $\epsilon : \mathcal{V} \to \mathbb{Z}$, to each pair of pants the multiplication $m : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$, and to each reverse pair of pants the comultiplication $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$.

### 2.3 Tangle cobordisms and the rings $H^m$

We denote the Cartesian coordinates on $\mathbb{C} \times [0, 1] \times [0, 1]$ by $(z, t, s)$. For a subset $A \subset (\mathbb{C} \times [0, 1] \times [0, 1])$ we set

$$\partial^v_i A = A \cap (\mathbb{C} \times [0, 1] \times \{i\})$$

and

$$\partial^h_i A = A \cap (\mathbb{C} \times \{i\} \times [0, 1]).$$

**Definition 2.4** Let $T_0, T_1$ be two $(m,n)$-tangles. A cobordism between $T_0$ and $T_1$ is a smoothly embedded surface $S$ in $\mathbb{C} \times [0, 1] \times [0, 1]$ s.t.

$$\partial^v_i S = T_i$$

for $i = 0, 1$. We also require $S$ to be the product of $\partial^h_i$ or $\partial^v_i$ with a small subinterval in a neighborhood of each face of $\mathbb{C} \times \partial([0, 1] \times [0, 1])$. 
The identity cobordism between $T$ and itself is denoted by $1_T$. Tangle cobordisms can be composed in two ways. First the vertical composition: if $S, S'$ are cobordisms between $T_0, T_1$ and $T_1, T_2$ then we get a cobordism

\[ S' \circ S = \frac{S' \cup S}{\partial_0 S' \sim \partial_1 S} \]

between $T_0$ and $T_1$. Secondly the horizontal composition: if $S$ is a cobordism between $kT_0$ and $mT_1$, and $S'$ is a cobordism between $lT'_0$ and $nT'_1$ then we get a cobordism

\[ S' \circ S = \frac{S' \cup S}{\partial_0 S' \sim \partial_1 S} \]

between $kT'_0 \circ T_0 L$ and $mT'_1 \circ T_1 N$. The last assumption in the definition of a cobordism ensures that compositions are smooth embedded surfaces.

For the purpose of this paper we just need to consider a special class of tangle cobordisms.

**Definition 2.5** For a crossingless matching $a \in c_m$, the minimal cobordism between $a^t$ and $id_m$ is the one which is given by merging the corresponding strands of $a^t$ and $a$ from the outermost one to the innermost one as depicted in Figure 5. We denote this minimal cobordism by $S_a$.

\[ a^H^m_b = F(a^t b)\{m\} \]

and

\[ H^m = \bigoplus_{a,b \in c_m} a^H^m_b. \]

Note that $a^t b$ is a disjoint union of circles so $a^H^m_b = q^\otimes k$ where $k$ is the number of the circles. Multiplication

\[ a^H^m_b \otimes c^H^m_d \rightarrow a^H^m_d \]
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is defined to be zero if \( b \neq c \). If \( b = c \), let \( S_b \) be the minimal cobordism between \( b'b \) and \( id_m \). The cobordism \( \mathcal{F}(1_a S_b 1_c) \) is a surface without corners so we get a map \( \mathcal{F}(1_a S_b 1_c) : aH^m_b \otimes bH^m_c \to aH^m_c \) which gives us multiplication.

We now recall a recursive decomposition of \( H^m \) from [7]. Denote by \( c'_m \) the subset of \( c_m \) consisting of elements which contain \( \cap_1 \), i.e. elements which contain an arc between points 1 and 2, and denote by \( c''_m \) its complement. (1 can be replaced with any \( 1 \leq i \leq 2m - 1 \).) \( c'_m \) is in one-to-one correspondence with \( c_{m-1} \). We have a map \( c''_m \to c_{m-1}, a \mapsto a' \), given by joining the two strands of \( a \) that stem from 1 and 2. Let \( cC'_1 \subset c''_m \times c''_m \) be the subset of all \((a, b)\) such that the arcs passing through points 1 and 2 in \( a'b \) form a single circle. If \((a, b)\) is in the complement of \( cC'_1 \), then the arcs passing through points 1 and 2 in \( a'b \) form two circles.

Let \( a, b \in c'_m \). If \( \bar{a} \) denotes \( a \) with the \( \cap_1 \) removed then we have

\[ aH^m_b = \bar{a}H^{m-1}_b \otimes \nu' \{1\}. \]

This contributes a summand of \( H^{m-1} \otimes \nu' \{1\} \) to \( H^m \). Set

\[ \bar{H}^m = \bigoplus_{a \in c'_m, b \in c'_m} \mathcal{F}(a', b) \{m\}. \]

The embedded circle \( C \) in \( a'b \) which passes through points 1 and 2 contributes a factor of \( \nu' \{1 + i\} \) to \( \bar{H}^m \) where \( i \) is the number of other pairs of points \( 2k - 1, 2k \) which \( C \) passes through. We can set

\[ \bar{H}^m = \bar{H}^m \otimes \nu' \{1\} \]

where \( \nu' \{1\} \) is the “local” contribution of the circle containing \( \cap_1 \) or \( \cup_1 \). This means that if \( a \in c'_m, b \in c''_m \) and we modify the strands of \( a'b \) passing through 1 and 2 only in a small neighborhood of the points 1 and 2 then we alter only the second factor in \( \bar{H}^m \otimes \nu' \{1\} \). Also denote by \( \bar{H}_1^m \) and \( \bar{H}_2^m \) the contribution of \( cC'_1 \) and its complement to \( H^m \). Again we can write \( \bar{H}_1^m = H_1^m \otimes \nu' \{1\} \) and \( \bar{H}_2^m = H_2^m \otimes \nu' \{1\} \otimes \nu' \{1\} \) where \( \nu' \{1\} \) resp. \( \nu' \{1\} \otimes \nu' \{1\} \) are “local” contributions from the single circle resp. the two circles formed by arcs passing through 1 and 2. Therefore we get

\[ (19) \quad H^m = \left( (H^{m-1} \oplus \bar{H}^m \oplus \bar{H}_1^m \oplus H_1^m) \otimes \nu' \{1\} \right) \bigoplus H_2^m \otimes \nu' \{1\} \otimes \nu' \{1\}. \]

as abelian groups.
2.4 The Khovanov invariant for flat tangles

**Definition 2.6** A flat tangle is a tangle which can be embedded into the plane i.e. a tangle without crossings.

For a flat \((m, n)\) tangle \(T\) we define

\[
\mathcal{Kh}(T) = \bigoplus_{a \in C_m, b \in C_n} \mathcal{F}(a'Tb)\{n\}.
\]

Obviously \(H_m = \mathcal{Kh}(id_m)\) as abelian groups. The abelian group \(\mathcal{Kh}(T)\) for an \((m, n)\)-tangle \(T\) has the structure of a \((H_m, H_n)\)-bimodule which is given by

\[
\mathcal{F}(1_a S_b 1_T 1_c) : a H_m^m \otimes \mathcal{F}(b'Tc) \to \mathcal{F}(a'Tc)
\]

for each \(a, b, c\) and zero map \(a H_m^m \otimes \mathcal{F}(c'Td) \to \mathcal{F}(a'Td)\) if \(b \neq c\).

For the unlinked union \(T \sqcup S^1\) we have

\[
\mathcal{Kh}(T \sqcup S^1) = \mathcal{Kh}(T) \otimes \psi' = \mathcal{Kh}(T)\{1\} \oplus \mathcal{Kh}(T)\{-1\}
\]

If \(S\) is a cobordism between two flat \((m, n)\)-tangles \(T_0, T_1\), it induces a bimodule map

\[
\mathcal{Kh}(S) : \mathcal{Kh}(T_0) \to \mathcal{Kh}(T_1)
\]

which is given on each component by

\[
\mathcal{F}(1_a' S_b 1_T 1_c) : \mathcal{F}(a'T_0b) \to \mathcal{F}(a'T_1b).
\]

The fact that \(\mathcal{Kh}(S)\) is independent of the isotopy class of \(S\) follows from the same property for the TQFT \(\mathcal{F}\). It is obvious that

\[
\mathcal{Kh}(S \circ S') = \mathcal{Kh}(S) \circ \mathcal{Kh}(S')
\]

**Lemma 2.7** ([3]) If \(lTm\) and \(mT'n\) are flat tangles then

\[
\mathcal{Kh}(T_0 \circ T_1) = \mathcal{Kh}(T_0) \otimes_{H^m} \mathcal{Kh}(T_1)
\]

as \((H^l, H^n)\)-bimodules. If \(T_0 S_0 T_0'\) and \(T_1 S_1 T_1'\) are minimal cobordisms then

\[
\mathcal{Kh}(S_0 S_1) = \mathcal{Kh}(S_0) \otimes_{H^n} \mathcal{Kh}(S_1).
\]
2.5 The Khovanov invariant in general

In this section we present Khovanov’s invariant for general tangles in a roundabout way which is shorter and suitable for our purpose. For a general tangle \( T \), \( \mathcal{H}(T) \) is a chain complex of graded bimodules over the rings \( H^m \) so it is doubly graded. For a flat tangle \( T \) the chain complex
\[
\cdots \to 0 \to \mathcal{H}(T) \to 0 \to \cdots
\]
with \( \mathcal{H}(T) \) in (first or homological) degree zero. We denote upward shift in first by \( \{ i \} \) and downward shift in second grading by \( [ i ] \). The only elementary braids which are not flat are the braids \( \sigma^+_m \) and \( \sigma^-_m \). Consider the chain complexes
\[
C^+_m \cdots \to 0 \to \mathcal{H}(id_m) \xrightarrow{\alpha} \mathcal{H}(\cup{id_m} \cap i_m) \{ -1 \} \to 0 \to \cdots
\]
\[
C^-_m \cdots \to 0 \to \mathcal{H}(\cup{i_m} \cap id_m) \xrightarrow{\beta} \mathcal{H}(id_m) \{ -1 \} \to 0 \to \cdots
\]
where the domain of maps \( \alpha \) and \( \beta \) are in (first or homological) degree zero. The map \( \alpha \) is \( \mathcal{H}(S_i) = \bigoplus_{a,b \in \mathbb{C}_m} \mathcal{F}(1^a S_i 1^b) \) where \( S_i \) is the minimal cobordism between \( \cup{i_m} \cap i_m \) and \( id_m \). The map \( \beta \) is obtained in the same way from \( S_t \) which is a cobordism between \( id_m \) and \( \cup{i_m} \cap i_m \). The \( -1 \) degree shift is to make the map \( \alpha \) of (the second or Jones) degree zero.

Let \( \sigma^+ = \sigma^+_m \) and \( \sigma^- = \sigma^-_m \) be as in the Figure 4. Khovanov defines
\[
\mathcal{H}(\sigma^+) = C^+_m \{ -1 \}
\]
\[
\mathcal{H}(\sigma^-) = C^-_m \{ 2 \}.
\]
Now let \( n_0 T_0 n_1 T_1 n_2 \cdots n_k T_k n_{k+1} \) be a decomposition of a tangle \( T \) into elementary tangles.

**Definition 2.8**

\[
\mathcal{H}(T) := \mathcal{H}(T_0) \otimes_{H^1} \mathcal{H}(T_1) \otimes_{H^2} \cdots \otimes_{H^k} \mathcal{H}(T_k)
\]

In [3], Khovanov defines his invariant using the cube of resolutions and obtains the above equation as a consequence. He also shows that \( \mathcal{H}(T) \) is independent of the decomposition and is invariant under isotopies of \( T \). If \( L \) is a link, the homology of \( \mathcal{H}(L) \) is the original Khovanov homology [2] of \( L \) with its first grading reversed. We set
\[
\mathcal{H}^j = \bigoplus_{j-k=i} H(\mathcal{H}(T))^j_k.
\]

Seidel and Smith [10] conjecture that their invariant \( \mathcal{H}_{SS} \) is equivalent to \( \mathcal{H} \).
Lemma 2.9  i) We have
\[ \mathcal{K}(\cup_{\text{im}} \cap_{\text{im}}) = \left( H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H_1^m \right) \otimes \mathcal{V} \{1\} \otimes \mathcal{V} \{2\}. \]

ii) On the first four direct summands, the map \( \alpha : \mathcal{F}(a^i \text{id}_m b) \to \mathcal{F}(a^i \cup_{\text{im}} \cap_{\text{im}} b) \) is given by the comultiplication \( \Delta : \mathcal{V} \{1\} \to \mathcal{V} \{1\} \otimes \mathcal{V} \) tensored with the identity map. On the last one it is given by the multiplication \( m : \mathcal{V} \{1\} \otimes \mathcal{V} \{1\} \to \mathcal{V} \) tensored with the identity.

iii) On the first four direct summands, the map \( \beta : \mathcal{F}(a \cup_{\text{im}} \cap_{\text{im}} b) \to \mathcal{F}(a^i \text{id}_m b) \) is given by the multiplication \( m : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \) tensored with the identity map. On the last one it is given by the comultiplication \( \Delta \) tensored with the identity.

iv) Since \( m : \mathcal{V} \to \mathcal{V} \) is surjective, we have
\[ H^1(C_{\text{im}}) = \frac{H_2^m \otimes \mathcal{V} \{1\} \otimes \mathcal{V} \{1\} \{\{-1\}}}{\text{Im } \Delta} = H_2^m \otimes \frac{\mathcal{V} \otimes \mathcal{V} \{1\}}{\text{Im } \Delta}. \]
\[ \mathcal{V} \otimes \mathcal{V} \{1\}/\text{Im } \Delta \text{ is isomorphic to } \mathbb{Z} < \{1 \otimes 1, 1 \otimes x - x \otimes 1\} > \}. \]
Therefore
\[ H^1(C_{\text{im}}) = H_2^m \{1\} \oplus H_2^m \{1\} \cong H_2^m \otimes \mathcal{V} \{1\}. \]

The map \( \Delta \) is injective and the kernel of \( m : \mathcal{V} \{1\} \otimes \mathcal{V} \to \mathcal{V} \{1\} \{\{-1\} \) equals \( \mathbb{Z} < x \otimes x, 1 \otimes x - x \otimes 1 \> \). Tensoring with \( x \otimes x \) has the effect of shifting degree by two, and tensoring with \( 1 \otimes x - x \otimes 1 \) does not shift the degree. Therefore
\[ H^0(C_{\text{im}}) = \left( H^{m-1} \{1\} \oplus \bar{H}^m \{1\} \oplus \bar{H}^m \{1\} \oplus H_1^m \{1\} \right) \otimes (\mathbb{Z} \{2\} \oplus \mathbb{Z}) \]
which is isomorphic to \( \left( H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H_1^m \right) \otimes \mathcal{V} \{2\} \).

Therefore
\[ \overline{\mathcal{K}}(\sigma_k) = (H^0(C_{\text{im}}) \oplus H^1(C_{\text{im}}) \{1\} \{+1\}) = (H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H_1^m) \otimes \mathcal{V} \{2\} \oplus H_2^m \otimes \mathcal{V} \{2\} \]

Proof  i) Follows easily by comparison to (19).

ii) This is because in the first four summands the cobordism merges two circles into one and in the last one it decomposes a circle into two.

iii) Similarly because in the first four summands the cobordism decomposes one circle into two and in the last one it merges two circles into one.

iv) Since \( m : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \) is surjective we have
\[ H^1(C_{\text{im}}) = \frac{H_2^m \otimes \mathcal{V} \{1\} \otimes \mathcal{V} \{1\} \{\{-1\}}}{\text{Im } \Delta} = H_2^m \otimes \frac{\mathcal{V} \otimes \mathcal{V} \{1\}}{\text{Im } \Delta}. \]
Now for $\sigma_i^+$ we have
\[
H^0(C_{km}^+) = H^m_2 \otimes \left( \ker m : \mathcal{V}\{1\} \otimes \mathcal{V}\{1\} \to \mathcal{V}\{2\}\{-1\} \right) \\
= H^m_2 \otimes \mathbb{Z} \langle x \otimes x, i \otimes x - x \otimes i \rangle > \{2\} \\
\cong H^m_2 \otimes \mathcal{V}\{3\}.
\]
\[
H^1(C_{km}^+) = \left( H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1 \right) \otimes \ker(\Delta : \mathcal{V}\{1\} \to \mathcal{V}\{1\} \otimes \mathcal{V}\{-1\}) \\
= \left( H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1 \right) \otimes \{i \otimes i, i \otimes x - x \otimes i\} \\
\cong \left( H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1 \right) \otimes \mathcal{V}\{-1\} \\
\cong \mathcal{K}h(\sigma_k^+) = \left( H^0(C_{km}^+) \oplus H^1(C_{km})\{1\}\{-1\} \right) \\
= \left( H^{m-1} \oplus \bar{H}^m \oplus \bar{H}^m \oplus H^m_1 \right) \otimes \mathcal{V}\{-1\} \oplus H^m_2 \otimes \mathcal{V}\{3 - 1\}
\]

3 The symplectic invariant of links

In this section we review the construction of Seidel and Smith [10]. This material appeared in [7] as well. Denote by Conf$_m$ the space of all unordered $m$-tuples of distinct complex numbers $(z_1, \ldots, z_m)$. Denote by Conf$_0^m$ the subset of Conf$_m$ consisting of $m$-tuples which add up to zero, i.e. $z_1 + \cdots + z_m = 0$. Let $I_{2 \times 2}$ and $0_{2 \times 2}$ denote the $2 \times 2$ identity matrices respectively. Let $S_m$ be the set of matrices in $\mathfrak{sl}_2$ of the form

\[
\begin{pmatrix}
  y_1 & I_{2 \times 2} \\
  y_2 & I_{2 \times 2} \\
  \vdots & \ddots \\
  y_{m-1} & I_{2 \times 2} \\
  y_m & 0_{2 \times 2}
\end{pmatrix}
\]

Here $y_1 \in \mathfrak{sl}_2$ and $y_i \in \mathfrak{gl}_2$ for $i > 1$. The set $S_m$ is in fact a homogeneous slice transverse to the orbit of $x$ ([10], Lemma 23). The map $\chi$ restricted to Conf$_{2m}^0$ is a differentiable fiber bundle ([10], Lemma 20). We denote the fiber of $\chi$ over $t$ by $\mathcal{Y}_{m,t}$, i.e.

$\mathcal{Y}_{m,t} = \chi^{-1}(t)$

If $t = (\mu_1, \ldots, \mu_{2m}) \notin \text{Conf}_0^m$, by $\mathcal{Y}_{m,t}$ we mean $\mathcal{Y}_{m,t'}$ where $t' = (\mu_1 - \sum \mu_i/2m, \ldots, \mu_{2m} - \sum \mu_i/2m)$. Each such fiber inherits a a Kähler structure from $S_m$ and this equips the
total space with a connection in a canonical way. Since these fibres are not compact
the existence of parallel transport maps is not guarantied. By modifying the Kähler
structure on $S_m$ Seidel and Smith assign rescaled parallel transport maps
\begin{equation}
\res_{\beta}^*: \mathcal{Y}_{m,\beta(0)} \to \mathcal{Y}_{m,\beta(1)}
\end{equation}
to each curve $\beta : [0, 1] \to \text{Conf}_{2m}$. The map $\res_{\beta}^*$ is a symplectomorphism defined on
arbitrarily large compact subsets of $\mathcal{Y}_{m,\beta(0)}$.

Let $E_\mu^\beta$ denote the $\mu$-eigenspace of $y$.

**Lemma 3.1** ([10], Lemmata 25 and 26) For any $y \in S_m$ and $\mu \in \mathbb{C}$ the projection
$\mathbb{C}^{2m} \to \mathbb{C}^2$ onto the first two coordinates gives an injective map $E_\mu^y \to \mathbb{C}^2$. Any
eigenspace of any element $y \in S_m$ has dimension at most two. Moreover the set of
elements of $S_m$ with 2 dimensional kernel can be canonically identified with $S_{m-1}$ and
this identification is compatible with $\chi$.

For a subset $A \subset \text{sl}_{2m}$, let $A_{\text{sub}, \lambda}^\mu$ (resp. $A_{\text{sub}^3, \lambda}^\mu$) be the subset of matrices in $A$
having eigenvalue $\lambda$ of multiplicity two (resp. three) and two Jordan blocks of size one
(resp. two Jordan blocks of sizes 1,2) corresponding to the eigenvalue $\lambda$ and no other
coincidences between the eigenvalues. Here are two results describing neighborhoods
of $S_{\text{sub}, \lambda}^m$ and $S_{\text{sub}^3, \lambda}^m$ in $S_m$.

**Lemma 3.2** ([10], Lemma 27) Let $D \subset \text{Conf}_0^{2m}$ be a disc consisting of the $2m$-
tuples $(\mu - \varepsilon, \mu - \varepsilon, \mu_3, \ldots, \mu_{2m})$ with $\varepsilon \in \mathbb{C}$ of small magnitude. Then there is a
neighborhood $U_\mu$ of $S_{\text{sub}, \mu}^m$ in $S_m \cap \chi^{-1}(D)$ and an isomorphism $\phi$ of $U_\mu$ with a
neighborhood of $S_{\text{sub}, \mu}^m$ in $S_{\text{sub}, \mu}^m \times \mathbb{C}^3$ such that $f \circ \phi = \chi$ on $S_m \cap \chi^{-1}(D)$ where
$f(x, a, b, c) = a^2 + b^2 + c^2$. Also if $N_y S_{\text{sub}, \mu}^m$ denotes the normal bundle to $S_{\text{sub}, \mu}^m$ at $y$
the we have
\begin{equation}
\phi(N_y S_{\text{sub}, \mu}^m) = \text{sl}(E_\mu^y) \oplus \zeta_y
\end{equation}
where $\zeta_y$ is the trace free part of $\{ \mathbb{C} \cdot 1 \subset \text{gl}(E_\mu^y) \} \oplus \text{gl}(E_\mu^y) \oplus \ldots \oplus \text{gl}(E_\mu^{2m})$.

Consider the line bundle $\mathcal{F}$ on $S_{\text{sub}^3, \mu}^m$ whose fiber at $y \in S_{\text{sub}^3, \mu}^m$ is $(y - m)E_{\chi_y}$ where
$\chi_y$ is the semisimple part of $y$. To $\mathcal{F}$ one associates a $\mathbb{C}^4$ bundle $\mathcal{L} = (\mathcal{F}\backslash \{0\}) \times _{\mathbb{C}^*} \mathbb{C}^4$
where $z \in \mathbb{C}^*$ acts on $\mathbb{C}^4$ by
\begin{equation}
(a, b, c, d) \to (a, z^{-2}b, z^2c, d).
\end{equation}
$\mathcal{L}$ decomposes as
\begin{equation}
\mathcal{L} \cong \mathbb{C} \oplus \mathcal{F}^{-2} \oplus \mathcal{F}^2 \oplus \mathbb{C}.
\end{equation}
Fibers of $\mathcal{L}$ should be regarded as transverse slices in $sl_3$. Upon choosing suitable coordinates on such a transverse slice (at the zero matrix) the function $\chi$ equals the function $p : sl_3 \rightarrow \mathbb{C}^2$ given by

$$p(a, b, c, d) = (d, a^3 - ad + bc).$$

$p$ is also well-defined as a function $\mathcal{L} \rightarrow \mathbb{C}^2$ because $b$ and $c$ are coordinates on line bundles which are inverses of each other. Denote by $\tau(d, z)$ the set of solutions of $
abla^3 - d\lambda + z = 0$.

**Lemma 3.3** ([10], Lemma 28) Let $P \subset \text{Conf}_{2m}^0$ be the set of $2m + 2$-tuples

$$\mu_1, \ldots, \mu_{i-1}, \tau(d, z), \mu_{i+3}, \ldots, \mu_{2m+2},$$

where $d$ and $z$ vary in a small disc in $\mathbb{C}$ containing the origin. There is a neighborhood $V$ of $S_m^{\text{sub}}$ in $S_m \cap \chi^{-1}(P)$ and an isomorphism $\phi'$ from $V$ to a neighborhood of zero section in $\mathcal{L}$ such that $p(\phi'(x)) = (d, z)$ if

$$\chi(x) = (\mu_1, \ldots, \mu_{i-1}, \tau(d, z), \mu_{i+3}, \ldots, \mu_{2m+2}).$$

If $\mu \in \mathbb{C}^{2m}/S_{2m}$ has only one element of multiplicity two or higher, which we denote by $\mu_1$, denote by $\mathcal{D}_{m, \mu}$ the set of singular elements of $(\chi^{-1}(\mu) \cap S_m)$ i.e.

$$\mathcal{D}_{m, \mu} = (\chi^{-1}(\mu) \cap S_m)^{\text{sub}, \mu_1}.$$

Let $\mathcal{D}_m$ be the union of all these $\mathcal{D}_{m, \tau}$ regarded as a subset of $S_m$. It inherits a Kähler metric from $S_m$. We have the map $\chi : \mathcal{D}_m \rightarrow \mathbb{C} \times \mathbb{C}^{2m-2}/S_{2m-2}$. By forgetting the first eigenvalue, the image of $\chi$ can be identified with $\text{Conf}_{2m-2}$.

### 3.1 Relative vanishing cycles

Let $X$ be a complex manifold and $K$ a compact submanifold. Let $g$ be a Kähler metric on $Y = X \times \mathbb{C}^3$ (not necessarily the product metric) and denote its imaginary part by $\Omega$. Consider the map $f : X \times \mathbb{C}^3 \rightarrow \mathbb{C}$ given by $f(x, a, b, c) = a^2 + b^2 + c^2$ and denote by $\phi_t$ the gradient flow of $-\text{Re} f$. Let $W$ be the set of points $y \in Y$ for which the trajectory $\phi_t(y)$ exists for all positive $t$. One can prove that $W$ is a manifold and the mapping $l : W \rightarrow X$ given by $l(y) = \lim_{t \to \infty} \phi_t(y)$ is well-defined and smooth. We have $\Omega|_W = l^*\Omega|_X$. The function $f$ restricted to $W$ is real and nonnegative.

Set $V_t(K) = \pi^{-1}(t) \cap l^{-1}(K) = l|_{\pi^{-1}(t)}^{-1}(K)$ which is a manifold for $t$ small. It follows from Morse-Bott lemma that $V_t(K)$ is a 2-sphere bundle on $K$ for $t$ small. To generalize the invariant to tangles we will need a slightly more general version of the
above construction in which \( K \) is noncompact and the metric equals the product metric outside a compact subset. (See subsection 4.1.) The resulting vanishing cycle equals (symplectically) the product bundle outside a compact subset.

### 3.2 Fibered \( A_2 \) singularities

Assume we have the same situation as in the Lemma 3.3, i.e. let \( \mathcal{F} \) be a holomorphic line bundle over a complex manifold \( X \) and define \( Y \) to be \( (\mathcal{F} \backslash 0) \times_{\mathbb{C}^*} \mathbb{C}^4 \) where the \( \mathbb{C}^* \) action is as in the formula (37). Let \( \Omega \) be an arbitrary Kähler form on \( Y \) and by regarding \( X \) as the zero section of \( Y \), \( \Omega \) restricts to a Kähler form on \( X \). Let \( (a, b, c, d) \) be the coordinates on fibers of \( Y \to X \) and \((d, z)\) coordinates on \( \mathbb{C}^2 \). Let the map \( p : Y \to \mathbb{C}^2 \) be as in Lemma 3.3. Let \( Y_d = p^{-1}(\mathbb{C} \times \{d\}) \) and \( p_d : Y_d \to \mathbb{C} \) be the restriction of \( p \). Set \( Y_{d, z} = p^{-1}(d, z) \). For \( d \neq 0 \) the critical values of \( p_d \) are \( \zeta_d^\pm = \pm 2\sqrt{d^3/27} \).

Let \( K \) be Lagrangian submanifold of \( X \). Using the relative vanishing cycle construction for the function \( p_d \) we can obtain a Lagrangian submanifold \( \Lambda_{d, z} \) of \( Y \) which is a sphere bundle over \( K \). (This construction works when \( Y \) is a nontrivial bundle over \( X \) as well.) There is another way of describing this Lagrangian as follows. Let \( Y \cong \mathbb{C}^4 \) be the fiber of \( Y \to X \) over some point of \( X \) and let \( p : Y \to \mathbb{C}^2 \) be as before. The restriction of the \( \mathbb{C}^* \) action to \( S^1 \) is a Hamiltonian action with the moment map \( \mu(a, b, c, d) = |c|^2 - |b|^2 \). Define

\[
C_{d, z, a} = \{(b, c) : \mu(b, c) = 0, a^3 - da - z = -bc\} \subset Y_{d, z}
\]

which is a point if \( a^3 - da - z = 0 \) and a circle otherwise. The three solutions of this equation correspond to the critical values of the projection \( q_{d, z} : Y_{d, z} \to \mathbb{C} \) to the \( a \) plane. In the situation of Lemma 3.3 they correspond to the three eigenvalues of \( Y \).

Let \( \alpha(r) \) be any embedded curve in \( \mathbb{C} \) which intersects these critical values (only) if \( r = 0, 1 \). Define

\[
\Lambda_\alpha = \bigcup_{r=0}^1 C_{d, z, \alpha(r)}
\]

which is a Lagrangian submanifold of \( Y_{d, z} \) (with Kähler form induced from \( \mathbb{C}^4 \)). Let \( \mathbf{c}, \mathbf{c}', \mathbf{c}'' \) be as in the Figure 6 where dots represent the critical values of \( q_{d, z} \). We can associate to \( K \) a Lagrangian submanifold \( \Lambda_{d, \alpha} \) of \( Y \) by defining \( \Lambda_{d, \alpha} = (Y|K) \times_{S^1} \Lambda_\alpha \). Seidel and Smith prove that these two procedures give the same result ([10], Lemma 40).
Lemma 3.4  If the Kähler form on $Y$ is obtained from a Kähler form on $X$, a Hermitian metric on $F$ and the standard form on $\mathbb{C}^4$ then $L_d = \Lambda_d e$.

![Figure 6](image)

3.3 Lagrangian submanifolds from crossingless matchings

Let $\mu \in \text{Conf}_{2m}$. A crossingless matching $D$ with endpoints in $\mu$ is a set of $m$ disjoint embedded curves $\delta_1, \ldots, \delta_m$ in $\mathbb{C}$ which have (only) elements of $\mu$ as endpoints. See Figure 7. To $D$ we associate a Lagrangian submanifold $L_D$ of $\mathcal{Y}_{m,\mu}$ as follows. Let $\{\mu_{2i-1}, \mu_{2i}\} \subset \mu$ be the endpoints of $\delta_k$ for each $k$. Let $\gamma$ be a curve in $\text{Conf}_m^0$ such that $\gamma(t) = \{\gamma_1(t), \gamma_2(t), \mu_3, \mu_4, \ldots, \mu_{2m}\}$, $\gamma(0) = \mu_i$, $i = 1, 2$ and as $s \to 1$, $\gamma_1(t), \gamma_2(t)$ approach each other on $\delta_1$ and collide. For example if $\delta_1(0) = \mu_1$, $\delta_1(1) = \mu_2$ the we can take $\gamma(t) = \{\delta(t/2), \delta(1 - t/2), \mu_3, \ldots, \mu_{2m}\}$. Set $\bar{\mu} = \mu \setminus \{\mu_1, \mu_2\}$, $\mu' = \gamma(1)$.

If $m = 1$ then relative vanishing cycle construction for $\chi: S_1 \to \mathbb{C}$ with the critical point over $\gamma(1) = 0$ gives us a Lagrangian sphere $L$ in $\mathcal{Y}_{1,\gamma(1-s)}$ for small $s$. Using reverse parallel transport along $\gamma$ we can move $L$ to $\mathcal{Y}_{1,\mu}$ to get our desired Lagrangian submanifold. Now for arbitrary $m$ assume by induction that we have obtained a Lagrangian $L_D \subset \mathcal{Y}_{m-1,\bar{\mu}}$ for $\bar{D}$ which is obtained from $D$ by deleting $\delta_1$. Now $\mathcal{Y}_{m-1,\bar{\mu}}$ can be identified with $D_{m,\tau}$ where $\tau = (0, 0, \mu_3 - (\mu_1 + \mu_2)/(2m - 2), \ldots, \mu_{2m} - (\mu_1 + \mu_2)/(2m - 2))$. Use parallel transport to move $L_{\bar{D}}$ to $D_{m,\gamma(1)}$. The later one is the set of singular points of $\mathcal{Y}_{m,\gamma(1-s)}$ so Lemma 3.2 tells us that we can use relative vanishing cycle construction for $L_D$ to obtain a Lagrangian in $\mathcal{Y}_{m,\gamma(1-s)}$ for small $s$. Parallel transporting it along $\gamma$ back to $\mathcal{Y}_{m,\mu}$ we obtain our desired Lagrangian which is topologically a trivial sphere bundle on $L_D$. We see that $L_D$ is diffeomorphic to a product of spheres. Different choices of the curve $\gamma$ result in Hamiltonian isotopic Lagrangians. The same holds if we isotope the curves in $D$ inside $\mathbb{C}\setminus\mu$.

Now we can define the Seidel-Smith invariant. Since each manifold $\mathcal{Y}_{m,\nu}$ is a submanifold of the affine space $S_m$ and has trivial normal bundle, its Chern classes are
zero. This together with the fact that $H_1(\mathcal{Y}_m) = 0$ implies that the canonical bundle of $\mathcal{Y}_{m,\nu}$ is trivial and so has a unique infinite Maslov cover. We start by choosing global sections $\eta_{S_m}$ and $\eta_{\eta/W}$. Then we choose trivializations for regular fibers of $\chi S_m$ characterized by $\eta_{S_m} \wedge \chi^* \eta_{\eta/W} = \eta_{S_m}$. If we choose a grading for $L \subset \mathcal{Y}_{m,t_0}$ and $\beta$ is a curve in $\text{Conf}_{2m}$ starting at $t_0$, one can continue the grading on $L$ uniquely to $h_{\beta|_{[0,t]}}(L)$ for any $s$. Therefore the grading of $L$ uniquely determines that of $h_{\beta}(L)$.

Let $\mathcal{D}_+$ be the crossingless matching at the left hand side of picture 7. If a link $K$ is obtained as closure of a braid $\beta \in \mathcal{B}_{r_m}$, let $\beta' \in \text{Conf}_{2m}$ be the braid obtained from $\beta$ by adjoining the identity braid $\text{id}_{m}$.

**Definition 3.5**

$$\mathcal{H}(S_S, \chi)(K) = HF^* + w(L_{\mathcal{D}_+}, h_{\beta'}^\text{reg}(L_{\mathcal{D}_+}))$$

Here $w$ is the writhe of the braid presentation, i.e. the number of positive crossings minus the number of the negative crossings in the presentation. Since the manifold is convex at infinity and the Lagrangians are exact, the above Floer cohomology is well-defined. Independence from choice of $\beta$ is proved in [10], section 5C.

## 4 Generalization to tangles

In this section we recall the generalization of the invariant of Seidel and Smith to tangles from [7].

### 4.1 The functor associated to a tangle

Let

$$n_0 T_1 n_1 T_2 \ldots n_{l-1} T_m l$$
be a decomposition of an oriented tangle $T$ into elementary tangles. Set $\nu_i = i(T_j)$ and $\nu_{i+1} = t(T_j)$ We have $\nu_i \in \text{Conf}_n$ for $i = 0, \ldots, l$. To each $T_i$ we want to associate a Lagrangian correspondence $L_{i,i+1} = L_{T_i}$ between $\mathcal{Y}_{n_i,\nu_i}$ and $\mathcal{Y}_{n_{i+1},\nu_{i+1}}$. In this way we can associate to $T$ a generalized Lagrangian correspondence

$$\Phi(T) = (L_{0,1}, L_{1,2}, \ldots, L_{m-1,m}) \{ -m - w \}$$

from $\mathcal{Y}_n$ to $\mathcal{Y}_m$. Here $m$ and $w$ are the number of cups and the writhe (number of positive crossings minus the number of negative ones) of the decomposition respectively.

If $T_k$ is an elementary braid in $Br_{2m}$, we set $L_{T_k}$ to be graph($h^{res}$) regardless of the orientation of the braid. Of course we can extend this definition to any braid. Let $V_i$ be the relative vanishing cycle for the map $f$ in Lemma 3.2 where $i$th and $(i+1)$th eigenvalues $(\mu_i, \mu_{i+1})$ of $\nu_k$ come together at some point $\mu$. Using a theorem of T. Perutz ([5], Theorem 2.19) one can describe monodromy maps around singularities of symplectic Morse-Bott fibrations as fibered Dehn twists. Therefore using the local picture of the Lemma 3.2 we see that if we have a subset $B \subset \mathcal{Y}_m$ for which the naive (non-rescaled) parallel transport map $h_{\sigma_i} |_B$ is well-defined then

$$h_{\sigma_i} \cong \tau_{V_i}.$$  

Let $V_{l,i}$ denote the $(S^2)$ fiber of $V_i$ over $x$. We grade $\tau_{V_i}$ in such a way that

$$\tau_{V_i} V_{l,i} = V_{l,i} \{ 1 \}$$

and the grading function vanishes outside a neighborhood of $V_i$. This grading is unique. (Lemma 5.6 in [8])

If $T_i = \bigcup_{j,m}$, we define a Lagrangian $L_{\bigcup_{j,m}}$, regardless of the orientation of $\bigcup_{k,m}$, as follows. The result depends on a real parameter $R > 0$. To simplify the notation we set $k = j, l = j + 1$. With $\nu_i$ as given above let $\nu = \nu_i = \{ \mu_1, \ldots, \mu_{2m} \}$. Let $\gamma$ be a curve in $\text{Conf}_{2m}^0$ such that $\gamma(0) = \nu_i$ and as $s \to 1$, $\mu_k$ and $\mu_l$ approach each other linearly and collide at a point $\mu'$. For example we can take

$$\gamma(t) = \{ \mu_1, \ldots, \mu_k + t(\mu_l - \mu_k)/2, \ldots, \mu_l - t(\mu_l - \mu_k)/2, \ldots, \mu_{2m} \}$$

provided that $\mu_k + t(\mu_l - \mu_k)/2$ does not intersect the other $\mu_i$. Set $\nu^{k,l} = \nu \setminus \{ \mu_k, \mu_l \}$, $\nu' = \gamma(1)$. We use Lemma 3.2 to identify a neighborhood of $S_{m,\mu'}^{\text{sub},\mu}$ in $S_m$ locally with $S_{m,\mu'}^{\text{sub},\mu} \times \mathbb{C}^3$. This induces a Kähler form and hence a metric on $S_m^{\text{sub},\mu} \times \mathbb{C}^3$. We perturb the complex structure outside a compact ball of radius $\rho$ (to be chosen below) so that outside that set the resulting metric equals the product metric. Now
we use the relative vanishing cycle construction for the whole $S_{m,\nu}$. It yields (after restriction) a sphere bundle $V = V_{(1-s)}(S_{m,\nu}) \subset Y_{m,\nu}$ for small $s$ with projection $\pi : V \to Y_{m,\nu} \cap S_{m,\nu}$. The relative vanishing cycle construction can be used because the metric equals the product metric outside a compact set.

We denote the image of $V$ under parallel transport map along $-\gamma$, i.e.
\[
h^{-1}_{\gamma(0,1)}(V) \subset Y_{m,\nu}
\]
by the same notation $V$. Composing $\pi$ with the parallel transport map $h^{-1}_{\gamma(0,1)}$ we obtain a projection $\pi : V \to Y_{m,\nu} \cap S_{m,\nu}$ which is a $S^2$ bundle. By Lemma 3.2, $Y_{m,\nu} \cap S_{m,\nu}$ can be identified with $D_{m,\nu}$ from (41). Let $\delta$ be a geodesic in $\text{Conf}_m$ joining $\nu$ to $p_{\nu}^{k,1}$. We can use parallel transport map (35) along the curve $\delta$ to map $D_{m-1,\nu'}$ to $D_{m-1,\nu''} \cup \{0,0\}$. The latter can be identified with $Y_{m-1,\nu''}$. So we obtain a fibration $\pi : V \to Y_{m-1,\nu''}$. We can use this map $\pi$ to pull $V$ back to $Y_{m-1,\nu''} \times Y_{m-1,\nu''}$. Let $\psi = \psi_1 + \psi_2$ be the plurisubharmonic function on $Y_{m,\nu} \times Y_{m-1,\nu''}$. We can choose $\mu$ in such a way that the inverse image of $\psi = R$ lies inside the ball of radius $\rho$. We have a projection $\pi : L_{\mu} \to \Delta \subset Y_{m-1,\nu''} \times Y_{m-1,\nu''}$.

As in the case of Lagrangians from crossingless matchings, replacing the curve $\gamma$ with another curve in the same homotopy class (with fixed endpoints) results in a new $L_{\mu}$ which is Lagrangian isotopic to the former one. Since the first homology group of this Lagrangian is zero, this isotopy is exact.

Lemma 4.1 below tells us that fibers of $L_{\mu}$ and $L_{\mu+1}$ over each point of the diagonal intersect transversely at only one point. We grade the $L_{\mu}$ in such a way that the absolute Maslov index of this intersection point (with regard to the two $S^2$ fibres) equals one. Construction for $\cap \mu$ is similar.

In order for $\Phi$ to define a functor, we must verify that the above correspondences satisfy the same commutation relations as the tangles they are associated to. The following is proven in [7]

**Lemma 4.1** If $|i - j| > 1$ we have

\[
L_{\sigma_i} L_{\sigma_j} \simeq L_{\sigma_j} L_{\sigma_i} \quad L_{\sigma_i} L_{\sigma_j} \simeq L_{\sigma_j} L_{\sigma_i},
\]

\[
L_{\sigma_i} L_{\sigma_j} \simeq L_{\sigma_j} L_{\sigma_i},
\]

\[
L_{\sigma_i} L_{\sigma_j} \simeq L_{\sigma_j} L_{\sigma_i}.
\]
For any $i$ we have

\begin{align}
L_{\sigma_i} L_{\sigma_{i+1}} L_{\sigma_i} &\simeq L_{\sigma_{i+1}} L_{\sigma_i} L_{\sigma_{i+1}} \\
L_{\gamma_i} L_{\sigma_i} &\simeq L_{\gamma_i} \{1\} \\
L_{\sigma_i} L_{U_{i+1}} &\simeq L_{\sigma_{i+1}} L_{U_i} \\
L_{\gamma_i} L_{U_{i+1}} &\simeq L_{\gamma_i} \{1\}.
\end{align}

Here “$\simeq$” means exact isotopy.

From 2.3 and the Lemma 4.1 we get the following.

**Theorem 4.2** The assignment $\Phi$ in (45) is a functor from the category of even tangles to the symplectic category.

### 4.2 The group valued invariant

We can obtain a group valued tangle invariant from the functor $\Phi$ as follows.

**Definition 4.3**

\begin{align}
\mathcal{HSS}(T) &= \bigoplus_{C \in C_m} HF(L^L, \Phi(T), L^C) \\
\mathcal{CSS}(T) &= \bigoplus_{C \in C_n} CF(L^L, \Phi(T), L^C)
\end{align}

We will, in subsection 5.3, put extra conditions on the chain complex (57) for $T$ a flat tangle. Each summand in the above direct sum is equal to the Floer cohomology of the Lagrangians

\begin{align}
\mathcal{L}_0 &= L_C \times L_{0,1} \times L_{1,2} \times \ldots \times L_{2k-1,2k} \\
\mathcal{L}_1 &= L_0 \times L_{1,2} \times \ldots \times L_{2k+1,2k} \times L^C
\end{align}

in $\mathcal{Y} = \mathcal{Y}_{n_0} \times \mathcal{Y}_{m_1} \times \ldots \times \mathcal{Y}_n$. Since these Lagrangians are not necessarily compact one has to make extra effort to prove that the above Floer homology is well-defined. This is done in [7] by truncating the Lagrangians near infinity using the Stein structure on the manifolds $\mathcal{Y}_m$.

**Theorem 4.4** For any tangle $T$, $\mathcal{HSS}(T)$ is well-defined and is independent of the decomposition of $T$ into elementary tangles.

It is clear that if $K$ is a $(0,0)$-tangle, i.e. a link, then the above invariant equals the original invariant of Seidel and Smith (3.5).
5 Results on the symplectic invariant

5.1 Maps induced by cobordisms

In this subsection we study the maps induced on $\mathcal{HSS}$ by cobordisms. Using elementary Morse theory one can decompose any cobordism $S$ between two tangles $T, T'$ into elementary cobordisms

$$ S = S_l \circ S_{l-1} \circ \cdots \circ S_1, $$

where each $S_i$ belongs to one of the three elementary types discussed below. We assign homomorphisms to each one of these elementary types and so get a homomorphism $\mathcal{HSS}(S_i)$ for each $S_i$. To define these elementary cobordism maps one needs to decompose the tangles involved into elementary ones. However cobordism maps do not depend on such decomposition of the tangle in an appropriate sense (Lemma 5.1). We then define $\mathcal{HSS}(S)$ to be the composition

$$ \mathcal{HSS}(S_l) \circ \mathcal{HSS}(S_{l-1}) \circ \cdots \circ \mathcal{HSS}(S_1). $$

Since such a decomposition is not unique, one can potentially get different maps from different decompositions. We do not address this problem here.

**Type I.** Cobordisms, equivalent to trivial cobordism, between equivalent tangles. The (iso-)morphism assigned to such a cobordism is given by the functoriality theorem. C.f. Theorem 4.4.

**Type II.** Birth or death of an unlinked circle:

$$ S_\circ : T \odot T' \rightarrow TT' $$

$$ S'_\circ : TT' \rightarrow T \odot T'. $$

We know from [7] that there is a canonical isomorphism

$$ \mathcal{HSS}(T \odot T') \cong \mathcal{HSS}(TT') \otimes \mathbb{Z} \nu. $$

We define the map

$$ \mathcal{HSS}(S_\circ) : \mathcal{HSS}(T \odot T') \rightarrow \mathcal{HSS}(TT') $$

induced by the cobordism $S_\circ$ to be $id \otimes \varepsilon$ and the map

$$ \mathcal{HSS}(S'_\circ) : \mathcal{HSS}(TT') \rightarrow \mathcal{HSS}(T \odot T') $$

to be $id \otimes \iota$. Here $\varepsilon$ and $\iota$ are the trace and unit maps from 2.2. Note that both maps can be defined as maps induced by quilts.
Type III. Saddle point cobordisms:

\( S_{\cap} : \quad T \cup_{\cap} T' \longrightarrow TT' \) \hspace{1cm} (62)

\( S'_{\cap} : \quad TT' \longrightarrow T \cup_{\cap} T'. \) \hspace{1cm} (63)

Let \( \Phi(T), \Phi(T') \) be the generalized Lagrangian correspondences associated to \( T \) and \( T' \). We define the following cobordism maps as follows.

\( CSS(S_{\cap}) : CSS(T \cup_{\cap} T') \rightarrow CSS(TT') \) \hspace{1cm} (64)

\( CSS(S'_{\cap}) : CSS(TT') \rightarrow CSS(T \cup_{\cap} T'). \) \hspace{1cm} (65)

The homomorphism (64) is defined to be the relative invariant associated to the quilt in Figure 5.1. The homomorphism (65) is the relative invariant of the transpose of this quilt.

Remember that we define the \( \mathcal{H}_{SS} \) of a tangle by using a decomposition of it into elementary tangles and using different decompositions result in isomorphic abelian groups. The following lemma shows that elementary cobordism maps are natural with respect to change of decomposition (or, in other words, isotoping the tangles involved).

**Lemma 5.1** Let \( T_0, T_1, T'_0, T'_1 \) be tangles such that \( T_i \) is equivalent to \( T'_i \) for \( i = 0, 1 \). Then we have the following commutative diagram

\[
\begin{array}{c}
\\
\mathcal{H}_{SS}(T_0 \cup_{\cap} T_1) \\
\downarrow
\\
\mathcal{H}_{SS}(T_0 T_1)
\end{array}
\begin{array}{c}
\\
\mathcal{H}_{SS}(T'_0 \cup_{\cap} T'_1) \\
\downarrow
\\
\mathcal{H}_{SS}(T'_0 T'_1)
\end{array}
\]
where the vertical maps are isomorphisms. One has a similar diagram for the type II cobordisms.

**Proof** The vertical isomorphisms were constructed by showing that Lagrangian correspondences assigned to elementary tangles satisfy the same commutation relations as the corresponding tangles. So the maps are given by Hamiltonian isotoping the corresponding Lagrangians and the functoriality theorem. For the first kind, the Hamiltonian isotopy induces a diffeomorphism between the corresponding moduli spaces of quilts. The second kind is an instance of “shrinking strips in quilted surfaces” and commutativity is given, in general settings, by Theorem 5.1 in [15]. Argument for type II cobordisms similar.

We next turn to the degree of saddle cobordism maps. We can use the local picture of the Lemma 3.2 to Hamiltonian isotope $L_{t_i}$ to $\gamma_{m-1} \times S^2$. Since the Lagrangians are now Cartesian products, $CSS(S_{\gamma_i})$ becomes the tensor product of the relative maps of the quilts in Figure 9 and we have the following commutative diagram.

$$
\begin{array}{c}
CSS(T \cup_{t_i} T') & \longrightarrow & CSS(T \cup_{t_i} T') \otimes CSS(\cup_{1} \cap_{1}) \\
| & & | \\
CSS(T) & \longrightarrow & CSS(T) \otimes CSS(id_1)
\end{array}
$$

The degree of the map on the right hand side is zero and therefore the degrees of the relative maps of the quilt $Q_0$ and $Q'_0$ is as well. See also Lemma 5.15 below. However because of the $-1$ degree shift coming from the extra cap in $T \cup_{t_i} T'$ we get

$$\deg CSS(S_{\gamma_i}) = 1 = \deg CSS(S'_{\gamma_i}).$$

**Minimal cobordisms** (c.f. Definition 2.5) are specific combinations of saddle cobordisms which deserve special attention. Let $a \in c_t, b \in c_m$ and $c \in c_n$. Recall from subsection 2.3 that we denote the minimal cobordism $b^0 \rightarrow id$ by $S_b$. From $S_b$ we get the cobordism $1_a 1_T S_b 1_T 1_c$ between $aTbT'c$ and $aTT'c$. We associated to this cobordism the quilt depicted in Figure 5.1 and we denote it by $Q_b$. The relative invariant associated to $Q_b$ gives a homomorphism of chain complexes

$$CF(Q_b) : CF(L^T_{a}, \Phi(T), L_b) \otimes Z CF(L^T_{b}, \Phi(T'), L_c) \longrightarrow CF(L^T_{a}, \Phi(T \circ T'), L_c),$$

as well as a homomorphism of graded groups.
Figure 9: Decomposing the quilt associated to a saddle cobordism

Figure 10: Quilt associated to a minimal cobordism
Pseudoholomorphic quilts and Khovanov homology

\[ HSS(S_a) : HF(Q_a) \to HF(L_a^b, \Phi(T), L_c) \to HF(L_a^b, L_c) \to HF(T, L_a^b, L_c) \]

Summing over all such \( a, b \) and \( c \) we get maps

\[ CSS(S_b) : CSS(T) \otimes Z \to CSS(T) \]

and

\[ HSS(S_b) : HSS(T) \otimes Z \to HSS(T') \]

If \( Q'_b \) is obtained from \( Q_b \) by reversing the incoming and outgoing ends we get

\[ HSS(S_t) : HSS(T) \otimes Z \to HSS(T) \]

It follows from the formula for the degree of the relative map of a quilt (Remark 2.4 in [15]) that

\[ \deg CSS(S_b) = 0 = \deg CSS(S'_b). \]

In the case that \( T = T' = id_1 \), \( Y_1 \) is diffeomorphic to \( T^*S^2 \) and has a symplectic form which is cohomologous and homotopic to the canonical symplectic structure on \( T^*S^2 \). So by Moser’s trick, applied to a compact neighborhood of the zero section, it is symplectomorphic to the standard one. Therefore we get maps

\[ m_{symp} = HSS(S_\cap) : HSS(\bigcirc) \otimes HSS(\bigcirc) \to HSS(\bigcirc) \]

and

\[ \Delta_{symp} = HSS(S'_\cap) : HSS(\bigcirc) \to HSS(\bigcirc) \otimes HSS(\bigcirc). \]

**Lemma 5.2** There is an isomorphism \( \phi : HSS(\bigcirc) \to \Psi \) under which \( \Delta_{symp} \) and \( m_{symp} \) correspond to \( \Delta \) and \( m \) respectively.

**Proof** Let \( f_0, f_1, f_2 \) be three Morse functions on a Riemannian manifold \( M \). Fukaya and Oh [1] prove that if we equip the cotangent bundle of \( M \) with the almost complex structure induced by the Levi-Civita connection on \( M \) then, for generic choice of the \( f_i \), there is an orientation preserving diffeomorphism between the moduli space of pseudoholomorphic triangles connecting intersection points of the \( df_i \) and the moduli space of pair of pants trajectories between the corresponding critical points of \( F = f_0 - f_1, G = f_1 - f_2, H = f_2 - f_0 \). The moduli space of pseudoholomorphic triangles in \( T^*S^2 \) with the almost complex structure from \( Y_1 \) is zero dimensional and cobordant
to the moduli corresponding to the almost complex structure induced by Levi-Civita connection. So the sum of the elements of the two are equal. Therefore after choosing (unique) homogeneous generators 1 and \( X \) for \( \mathfrak{g}SS(\bigcirc) \cong H^*(S^2) \), (73) corresponds to the wedge product on \( \nu' \) which is in turn equal to \( m \).

The same arguments show that \( \Delta_{\text{symp}} \) corresponds to the operation given by counting inverted Y’s in Morse homology of \( S^2 \). A direct computation shows that this operation is equal to \( \Delta \).

Note that the horizontal composition \( SS' \) of two cobordisms \( S \) and \( S' \) equals \( (S \text{id}) \circ (id \ S') \).

**Lemma 5.3** For two minimal cobordisms \( S_a \) and \( S_b \) we have

\[
\mathfrak{g}SS(S_a \text{id}) \circ \mathfrak{g}SS(id \ S_b) = \mathfrak{g}SS(id \ S_b) \circ \mathfrak{g}SS(S_a \text{id}).
\]

**Proof** From [16] we know if \( Q \) and \( Q' \) are two quilts which can be composed vertically (i.e. along the strip-like ends) then \( HF(Q \circ Q') = HF(Q) \circ HF(Q') \). We also know that \( HF(Q) \) is (at the cohomology level) invariant under the isotopy of the quilt \( Q \). The lemma follows from these two facts together with the isotopy in Figure 5.1.

![Isotopy between the composition of two quilts](image)

**5.2 \( H^n \) module structure**

We define the symplectic analogue of the rings \( H^n \) as

\[
H^n_{\text{symp}} = \mathfrak{g}SS(id_n) = \bigoplus_{a,b \in C_n} HF(L^i_a, L^i_b).
\]

The product map from \( HF(L^i_a, L^i_b) \otimes HF(L^i_c, L^i_d) \) to \( HF(L^i_a, L^i_d) \) is zero if \( b \neq d \) and is given by the map \( HF(Q_b) \) otherwise.
Lemma 5.4  For any \( a, b \in c_m \) we have \( HF(L_a^1, L_b) \cong H^*(S^2)^{\otimes k} \cong q^k \{ k \} \) where \( k \) is the number of circles in \( a'b \).

Proof  The Lagrangian \( L_a \) equals the composition of the Lagrangians associated to its arcs and similarly for \( L_b \) so \( \Phi(a') \# \Phi(b) = \Phi(a'b) \). Therefore

\[
HF(L_a^1, L_b) = HF(\Phi(a'), \Phi(b))\{m\} = HF(\Phi(a'b))\{m\} = \mathcal{HSS}(\Phi(k))\{m\} = H^*(S^2)^{\otimes k}.
\]

Theorem 5.5  For each \( n \) we have an isomorphism

\[
H^n_s \cong H^n
\]

as graded algebras over \( \mathbb{Z}/2 \).

Proof  Let \( a, b, c \in c_m \). Observe that \( HF(\Phi(a'), \Phi(b)) \otimes HF(\Phi(b'), \Phi(c)) \) is canonically isomorphic to \( HF(\Phi(a'b'b'c)) \). Label the arcs of \( b \) with numbers 1 to \( m \). The Lagrangian correspondence \( \Phi(a'b'b'c) \) is equivalent in the symplectic category to \( \Phi(C_1) \) where \( C_1 \) is an unlinked disjoint union of some \( k \) circles. Each of these circles has some marked points on it corresponding to the arcs of \( b \) and \( b' \) and these marked points are labelled with numbers which specify the cobordisms in \( S_b : a'b'b'c \rightarrow a'c \). So \( S_b \) is equivalent to the composition of cobordisms \( S_i : C_i \rightarrow C_{i+1} \), \( i = 1, \ldots, m \), where each \( C_i \) is an unlinked disjoint union of a number of circles and each \( S_i \) is either a saddle cobordism or it permutes two circles. We observe that each \( \Phi(C_i) \) is a generalized Lagrangian correspondence of the form

\[
(77) \quad pt \rightarrow Y_1 \rightarrow pt \rightarrow Y_1 \rightarrow pt \rightarrow \ldots \rightarrow pt \rightarrow Y_1 \rightarrow pt.
\]

Therefore we can use the Lemma 5.2 to conclude that we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{HSS}(C_i) & \xrightarrow{\mathcal{HSS}(S_i)} & \mathcal{HSS}(C_{i+1}) \\
\downarrow & & \downarrow \\
\mathcal{\Phi}(C_i) & \xrightarrow{\mathcal{\Phi}(S_i)} & \mathcal{\Phi}(C_{i+1})
\end{array}
\]

where (because of the special form of the Lagrangians (77)) the vertical arrows are canonical isomorphisms.

Now Lemma 5.1 ensures that the isomorphisms \( \mathcal{HSS}(a'b'b'c) \cong \mathcal{HSS}(C_1) \) and \( \mathcal{HSS}(a'c) \cong \mathcal{HSS}(C_m) \) intertwine the cobordism maps

\[
\mathcal{HSS}(S_b) : \mathcal{HSS}(a'b'b'c) \rightarrow \mathcal{HSS}(a'c)
\]
and

$$\mathcal{H}SS(S_m) \circ \cdots \circ \mathcal{H}SS(S_1) : \mathcal{H}SS(C_1) \to \mathcal{H}SS(C_m).$$

For an \((l, m)\)-tangle \(T\), \(\mathcal{H}SS(T)\) has a structure of a \((H^l, H^m)\) bimodule as follows. We have

$$\mathcal{H}SS(T) = \bigoplus_{b \in C_l, c \in C_m} HF(L_b, \Phi(T), L_c).$$

The part \(aH^l_b\) of \(H^l\) acts on \(HF(b, \Phi(T), c)\) from left by the map \(\mathcal{H}SS(S_b)\) (in (70)). So does \(cH^m_d\) from right by the map \(\mathcal{H}SS(S_c)\). We set the left action of \(aH^l_b\) on \(HF(b, \Phi(T), c)\) to be zero if \(b \neq b'\) and similarly for the right action. This way we obtain an \((H^l, H^m)\)-bimodule structure on \(\mathcal{H}SS(T)\).

**Remark** Note that since the cobordism maps \(CSS(S_b)\) are of degree zero, the chain complex \(CSS(T)\) can be regarded as a chain complex of \((H^l, H^m)\)-bimodules.

**Lemma 5.6** With the same notation as in (19) we have

$$\mathcal{H}SS(id_m) = H^m = \mathcal{H}\mathcal{H}(id_m)$$
$$\mathcal{H}SS(\cap_{i,m}) = \mathcal{H}\mathcal{H}(\cap_{i,m})$$
$$\mathcal{H}SS(\cup_{i,m}) = \mathcal{H}\mathcal{H}(\cup_{i,m})$$
$$\mathcal{H}SS(\sigma_{1,m}^{\pm}) = \mathcal{H}\mathcal{H}(\sigma_{1,m}^{\pm}) = (H^{m-1} \oplus H^{m'} \oplus H^{m'} \oplus H^1_1) \oplus \mathcal{V\{1 \mp 2\}} \bigoplus H^m_2 \otimes \mathcal{V\{2\}}$$

as \(H^m\) modules.

**Proof** The first three equations follow from the fact that \(\mathcal{H}\mathcal{H}\) and \(\mathcal{H}SS\) for disjoint union of \(k\) circles are equal to \(\mathcal{V} \otimes \mathcal{V}^k\). For the las one, the first equality was proved in [7] and the second one in the lemma 2.9.

$$\blacksquare$$

5.3 Vanishing of the differential for flat tangles

**Lemma 5.7** Let \(C_1, C_2 \in C_m\) be two crossingless matchings. Then we can choose Floer data in such a way that the Floer chain complex \(CF(L_{C_1}, L_{C_2})\) has differential equal to zero.
Proof We prove by induction on \( m \). If \( m = 1 \) then there is only one crossingless matching and the Floer chain complex equals \( CF(S^2, S^2) \) where \( S^2 \) is the zero section in \( \mathcal{Y}_m = T^*S^2 \). We can Hamiltonian isotope the zero section to a Lagrangian \( L \) s.t. \( L \) intersects the zero section at only two points. For example we can take \( L \) to be the graph of the one-form \( df \) where \( f \) is the height function on the zero section. In this case the Floer differential has to be zero because otherwise \( HF(S^2, L) \) will not be equal to \( H^*(S^2) \). This can also be seen by considering the Maslov indices of intersection points.

Now assume the statement holds for all crossingless matchings in \( C_k \) for \( k < m \). Let \( \alpha_1 \) be an arbitrary arc in \( C_1 \) and \( \mu_1, \mu_2 \) its endpoints. There are two cases. Either there is an arc \( \alpha_2 \) in \( C_2 \) joining \( p \) and \( q \) or there is no such arc. Proof for these two cases are similar to the proofs of the Kunneth formula and the Thom isomorphism for Floer homology [10]. In the first case let \( \tilde{C}_i \) be obtained from \( C_i \) by deleting \( \alpha_i \), \( i = 1, 2 \). Then we can use lemma 3.2 and then isotope the induced metric into the product metric. So \( L_{\tilde{C}_i} \) gets isotoped to \( L_{\tilde{C}_i} \times S^2 \). We choose a time dependent almost complex structure \( \tilde{J}_t \) on the base which is a compactly supported perturbation of its standard structure \( \tilde{J}_0 \). We choose the almost complex structure in a way similar to that of the first case above. Note that there are two possible configurations of the curves \( \alpha_i \).

In either case \( \Lambda_{\alpha_1} \) and \( \Lambda_{\alpha_2} \) intersect at only one point \( p \) corresponding to \( \mu_2 \). So we have

\[
CF(L_{C_1}, L_{C_2}) = CF(L_{\tilde{C}_1}, L_{\tilde{C}_2}) \otimes CF(S^2, S^2).
\]

So the claim follows from the induction hypothesis and the argument for the base case. In the second case let \( \alpha_2 \) be the unique arc in \( C_2 \) which has \( \mu_2 \) as an endpoint and let \( \mu_3 \) be its other end point. Now we can use lemma 3.3 to identify \( L_{\tilde{C}_i} \) with \( L_{\tilde{C}_i} \times S^1 \Lambda_{\alpha_i} \) where \( \Lambda_{\alpha} \) is the Lagrangian sphere associated to the curve \( \alpha \) as defined in subsection 3.2. We choose the almost complex structure in a way similar to that of the first case above. Note that there are two possible configurations of the curves \( \alpha_i \).

In either case \( \Lambda_{\alpha_1} \) and \( \Lambda_{\alpha_2} \) intersect at only one point \( p \) corresponding to \( \mu_2 \). So we have

\[
CF(L_{C_1}, L_{C_2}) = CF(L_{\tilde{C}_1}, L_{\tilde{C}_2}) \otimes \mathbb{Z} < p >.
\]

Let \( u \) be a holomorphic strip joining to intersection points of \( L_{C_1} \) and \( L_{C_2} \). So we have \( u = (u', u'') \) where \( u' \) is the projection to the first factor. By the induction hypothesis, \( u' \) is constant. Projection to the second factor is a holomorphic strip in \( \mathbb{C} \) which has its boundary on \( \alpha_1 \) and \( \alpha_2 \). Such a finite energy curve has to be constant by the
exponential convergence property of pseudoholomorphic strips. Therefore \( u'' \) is also constant so we get the desired result.

\[ \square \]

**Lemma 5.8** Let \( T \) be a flat \((m, n)\)-tangle. We can choose the Floer data in such a way that the Floer chain complex whose cohomology is \( \mathcal{H}_{SS}(T) \) has differential equal to zero.

**Proof** Let \( T = T_1 \cdots T_{k-1} \) be a decomposition of \( T \) and let \( T_0 \in \mathcal{C}_m \) and \( T_k \in \mathcal{C}_n \). Let \( L_{T_i} \) is a correspondence between \( \mathcal{Y}_{m_i} \) and \( \mathcal{Y}_{m_{i+1}} \). We use induction on \( m = \sum m_i \). The case \( m = 1 \) was treated in Lemma 5.7. If \( T_1 \) is the identity tangle then \( CF(L_{T_0}, L_{T_1}, \cdots, L_{T_k}) = CF(L_{T_0}, L_{T_1}, \cdots, L_{T_k}) \). So we can assume that \( T_1 \) is a cup. Therefore the both \( L_{T_0} \) and \( L_{T_1} \) are obtained by relative vanishing cycle construction from Lagrangians in \( \mathcal{Y}_{m_0-1} \) and \( \mathcal{Y}_{m_0-1} \times \mathcal{Y}_{m_1} \). Therefore we can use the same argument as in the proof of 5.7 for the induction step.

\[ \square \]

**Definition 5.9** For a flat \((m, n)\)-tangle \( T \) we require the chain complex

\[ \mathcal{C}_{SS}(T) = \bigoplus_{a \in \mathcal{C}_m, b \in \mathcal{C}_n} CF(L_a, \Phi(T), L_b) \]

to be given by Floer data in lemma 5.8.

### 5.4 Composition property of \( \mathcal{H}_{SS} \) for flat tangles

Let \( T \) and \( T' \) be \((l, m)\) and \((m, n)\) tangles respectively. Consider the map \( \psi \)

\[ \mathcal{H}_{SS}(T) \otimes_{\mathbb{Z}} \mathcal{H}_{SS}(T') \]

\[ \bigoplus_{a, b, b', c} HF(L'_a, \Phi(T), L_b) \otimes_{\mathbb{Z}} HF(L'_{b'}, \Phi(T'), L_c) \]

\[ \bigoplus_{a, c} HF(L'_a, \Phi(T' \circ T), L_c) \cong \mathcal{H}_{SS}(T' \circ T) \]

which is zero if \( b \neq b' \) and equals \( \mathcal{H}_{SS}(1_T S_b 1_T 1_c) \) otherwise. Here, as before, \( S_b \) is the minimal cobordism between \( bb' \) and \( id_m \). The abelian group \( \mathcal{H}_{SS}(T) \otimes_{\mathbb{Z}} \mathcal{H}_{SS}(T') \)
has the structure of a \((H^l, H^p)\)-bimodule and \(\psi\) is a \((H^l, H^p)\)-bimodule map. If \(x \in H^f(L_{a^l}, \Phi(T), L_{b^l})\), \(y \in H^f(L_{a^l}, \Phi(T'), L_{b^l})\) and \(\xi \in bH^m_{a^l}\) then

\[
\psi_\xi(x \otimes y) = \mathcal{HSS}(1_a T S_b l T c) \mathcal{HSS}(1_a T S_b l T c)(x, y)
\]

\[
\psi_\xi(x \otimes \xi y) = \mathcal{HSS}(1_a l T S_b l T c) \mathcal{HSS}(1_a l T S_b l T c)(x, y)
\]

It follows from (75) that these two are equal and so \(\psi_\xi\) factors through a map of bimodules \(\mathcal{HSS}(T) \otimes_{H^m} \mathcal{HSS}(T') \to \mathcal{HSS}(T \circ T')\) which we still denote by \(\psi_\xi\).

**Proposition 5.10** If \(T\) and \(T'\) are flat then \(\psi_\xi\) gives an isomorphism

\[
\psi_\xi : \mathcal{HSS}(T) \otimes_{H^m} \mathcal{HSS}(T') \cong \mathcal{HSS}(T \circ T').
\]

**Proof** Proof is exactly the same as that of Theorem 1 in [3]. The map \(\psi\) is the direct sum of the maps

\[
a^T \psi_c : \bigoplus_{b^l} H^f(L_{a^l}, \Phi(T), L_{b^l}) \bigotimes_{H^m} \bigoplus_{b^l} H^f(L_{a^l}, \Phi(T'), L_{b^l}) \rightarrow H^f(L_{a^l}, \Phi(T \circ T'), L_{c^l})
\]

We have \(\bigoplus_{b^l} H^f(L_{a^l}, \Phi(T), L_{b^l}) \cong \mathcal{HSS}(a^T T')\) and

\[
\bigoplus_{b^l} H^f(L_{a^l}, \Phi(T'), L_{b^l}) \cong \mathcal{HSS}(T' c^l)\{n\}
\]

as left and right \(H^m\)-modules respectively. We also have \(H^f(L_{a^l}, \Phi(T \circ T'), L_{c^l}) \cong \mathcal{HSS}(a^T T' c^l)\{n\}\). Therefore the argument is reduced to showing that

\[
\mathcal{HSS}(a^T T') \otimes_{H^m} \mathcal{HSS}(T' c^l) \cong \mathcal{HSS}(a^T T' c^l).
\]

Now \(a^T T'\) and \(T' c^l\) are \((0, m)\) and \((m, 0)\)-tangles respectively so \(a^T = a' \oplus i c\) and \(T' c^l = c' \oplus j c\) where \(a', b' \in c_m\) and \(i\) and \(j\) are the number of circles in \(a^T T'\) and \(T' c^l\) respectively. Thus we have \(\mathcal{HSS}(a^T) = \mathcal{HSS}(a') \otimes \mathcal{V}^l\), \(\mathcal{HSS}(T' c^l) = \mathcal{HSS}(c') \otimes \mathcal{V}^l\) and \(\mathcal{HSS}(a^T T' c^l) = \mathcal{HSS}(a' c') \otimes \mathcal{V}^{l+j}\). So we need to show that

\[
\mathcal{HSS}(a') \otimes_{H^m} \mathcal{HSS}(c') = \mathcal{HSS}(a' c').
\]

We have \(H^m \otimes_{H^m} H^m = H^m\) and if we multiply this identity with the idempotent \(1_{a'}\) from left and by \(1_{c'}\) from right we get the desired result. 

**Corollary 5.11** For any flat tangle \(T\) we have

\[
\mathcal{HSS}(T) = \tilde{\mathcal{K}}(T) = \mathcal{K}(T).
\]
Proposition 5.12  Let \( T, T' \) be flat \((m,n)\)-tangles and \( S \) a cobordism between \( T \) and \( T' \) which equals a composition of minimal cobordisms. Then, with coefficients in \( \mathbb{Z}/2 \), we have

\[
\mathcal{HSS}(S) = \mathcal{K}(S).
\]

Proof  By (75) we can assume that \( S \) consists of a single minimal cobordism. Therefore we have \( T = T_1 cc T_2 \) and \( T' = T_1 id T_2 \) for a crossingless matching \( c \) and \( S \) equals \( 1_{T_1}, S, 1_{T_2} \). For any \( a \in c_m \) and \( b \in c_n \), \( a' T_1 \) equals a crossingless matching \( a'_1 \in c_m \) disjoint union with some \( k \) circles. The same is true for \( T_2 b \) i.e. \( T_2 b \) equals \( b_2 \in c_n \) disjoint union with \( l \) circles. So, the problem is reduced to showing that the map

\[
\mathcal{HSS}(S) = HF(Q_c) : HF(L_{a_1}, L_c, L_{b_2}) \otimes q^{k+l} \longrightarrow HF(L_{a_1}, L_{b_2}) \otimes q^{k+l}
\]

equals \( \mathcal{K}(S) \). But

\[
HF(L_{a_1}, L_c, L_{b_2}) = HF(L'_{a_1}, L_c) \otimes HF(L'_c, L_{b_2}) = a_c H_c \otimes b H_{b_2}
\]

and \( HF(L'_{a_1}, L_{b_2}) = a_1 H_{b_2} \). Therefore the lemma follows from the isomorphism of the algebra structures on \( H^m \) and \( H^m_s \) over \( \mathbb{Z}/2 \) (Lemma 5.5).

5.5 Exact triangle for the symplectic invariant

In this subsection we prove an exact triangle for the Seidel-Smith invariant which is analogous to skein relations for knot polynomials. The tool we use is the exact triangle for Lagrangian Floer homology. This exact triangle was discovered by Seidel [9] for Dehn twists. We use a generalization of this triangle to fibred Dehn twists due to Wehrheim and Woodward [13]. Let \( M \) be a symplectic manifold and \( C \subset M \) a spherically fibred coisotropic fibering over a base \( B \). We denote the fibred Dehn twist along \( C \) by \( \tau_C \). The embedding \( (\iota \times \pi)C \) is a Lagrangian submanifold of \( M^- \times B \). By the abuse of notation we sometimes denote this submanifold by \( C \).

Let \( Q_0 \) be the quilt in the Figure 12. The exact triangle in [13] establishes a quasi-isomorphism between the Floer chain complex \( CF(L, \tau_C L') \) and the cone of the morphism \( f := CF(Q_0) \), i.e. the relative map associated with \( Q_0 \).

\[
f = CF(Q_0) : CF(L, (\pi \times \iota)C', (\iota \times \pi)C, L')\{-\frac{1}{2} \dim B\} \longrightarrow CF(L, L').
\]

More precisely we have the following.
If $C \subset M$ has codimension at least two and the triple $(L_0, L_1, C)$ is monotone and has Maslov index greater than or equal 3 then there is a quasi-isomorphism $(h\{1\}, k)$ from

$$\text{Cone}(f) = CF(L, C', C, L') \{ -\frac{1}{2} \dim B + 1 \} \bigoplus CF(L, L')$$

to $CF(L, \tau_C L')$.

Recall that a Lagrangian correspondence $L \subset M^- \times N$ between two symplectic manifolds is proper if for each $y \in N$ the set $\{ x \in M | (x, y) \in L \}$ is compact. (For Lagrangian submanifolds this is equivalent to compactness.) We call a Lagrangian submanifold $L$ of a Stein manifold $(M, \psi)$ allowable if the critical point set of $\psi|_L$ is compact. When doing Floer theory in a Stein manifold we use compatible almost complex structures which are invariant under the Liouville flow outside a compact subset.

Proposition 5.14 If $(M, \psi)$ is Stein, $C \subset M$ has codimension at least two and each one of $L_0, L_1, C$ is exact, proper and allowable then the conclusion of Theorem 5.13 holds.

Proof Properness implies that intersection in $(L, C', C, L')$ is compact. In [7, Lemma 3.3.2] it was shown that for any two allowable Lagrangian submanifolds $L_0, L_1$ of a Stein manifold $(M, \psi)$, any pseudoholomorphic curve with boundary on $L_0 \cup L_1$ lies in $M_{\psi \leq C}$ where $C$ is the maximum of $\psi$ on $L_0 \cap L_1$. This implies that the moduli spaces of pseudoholomorphic quilts used in the proof of 5.13 are compact in our case. The high Maslov index assumption in 5.13 is to rule out bubbling; in our case bubbling is ruled out by exactness of the Lagrangians.

\[\square\]
At the $A_\infty$ level one has the following exact triangle in $\text{DFuk}^\#(M)$.

$$
\begin{array}{ccc}
\text{graph } \tau_C & \rightarrow & \Delta_M \\
C'\#C \{-1/2 \dim B\} & \rightarrow & C \# \{−1/2 \dim B\}
\end{array}
$$

Here $\text{DFuk}^\#(M)$ is the generalized Fukaya category of a Stein manifold as described in [7, Section 4.3]. The objects of $\text{DFuk}^\#(M)$ are proper exact allowable generalized Lagrangian submanifolds of $M$. This category is somewhat similar to wrapped Fukaya category [?]. The difference is that in the wrapped Fukaya category one uses the Reeb Flow instead of the Liouville flow and also one takes the direct limit of the Floer chain complexes of the images of the Lagrangian submanifolds under this flow. In our case the more restrictive properness assumption frees us from taking direct sums.

If $L = (L_k, L_{k-1}, \ldots, L_1)$ is any generalized Lagrangian submanifold of $M$ then by applying the $A_\infty$ functor $\Phi^\#_L = \Phi^\#_{L_1} \circ \cdots \circ \Phi^\#_{L_1}$ to (5.19) we get the following exact triangle in $\text{DFuk}^\#(M)$.

$$
\begin{array}{ccc}
\text{graph } \tau_{C' L} & \rightarrow & L \\
C'\#C\#L \{-1/2 \dim B\} & \rightarrow & C'\#C\#L_1 \{-1/2 \dim B\}
\end{array}
$$

Therefore theorem 5.14 holds, without any change, if $L, L'$ are generalized Lagrangian submanifolds of $M$. One can prove this fact without using Fukaya categories.

With the same assumptions as in 5.14 let $M = M_1 \times M_2, B = B_1 \times M_2$, and $C$ be of the form $C_1 \times M_2$ where $C_1$ is a sphere bundle over $B_1$. Further assume that there are Lagrangian submanifolds $L_i \subset M_i$ and $L'_i \subset M'_i$ for $i = 1, 2$ such that $L = L_1 \times L_2$ and $L' = L'_1 \times L'_2$. Let $d = -\frac{1}{2} \dim B$ and $d_1 = -\frac{1}{2} \dim B_1$. Consider the map

(80) \quad $CF(\bar{Q}_0): CF(L_1, C'_1, C_1, L'_1) \{-1/2 \dim B_1\} \rightarrow CF(L_1, L'_1)$.

**Corollary 5.15**  Then $CF(L, \tau_C L')$ is quasi-isomorphic to

(81) \quad $\text{Cone}(\bar{Q}) \otimes CF(L_2, L'_2)$

and we have a commutative diagram

$$
\begin{array}{ccc}
CF(L, C', C, L') \{d\} & \xrightarrow{CF(\bar{Q}_0)} & CF(L, L') \\
\downarrow{\frac{1}{2} \dim M_2} & & \downarrow \text{identity} \\
CF(L_1, C_1, C_1, L'_1) \{d_1\} \otimes CF(L_2, L'_2) & \xrightarrow{CF(\bar{Q}_0)} & CF(L_1, L'_1) \otimes CF(L_2, L'_2)
\end{array}
$$


**Proof** We observe that \( \tau_C = \tau_{C_1} \times \text{id}_{M_2} \). The homomorphism \( CF(Q) \) is isomorphic to the tensor product of the maps induced by the two quilts in the Figure 13. The quilt on the left is \( Q \) and the quilt on the right induces the identity map. There is a grading shift \( CF(L_2, \Delta^1_{M_2}, \Delta_{M_2}, L'_2) = \frac{1}{2} \dim M_2 \) coming from the fact that the grading of \( \Delta^1_{M_2} \) equals \( \frac{1}{2} \dim M_2 \) minus the grading on \( \Delta_{M_2} \).

**Proposition 5.16** With the same assumptions as in Theorem 5.14, let \( L, L' \) be two generalized Lagrangian submanifolds of \( M \). Then \( CF(L', \text{graph}(\tau_C), L' \) is quasi-isomorphic to the cone of the map

\[
\text{(82)} \quad CF(L', C', C, L')\{ -\frac{1}{2} \dim B \} \longrightarrow CF(L', L')
\]

**Proof** Let \( L = (L_n, \ldots, L_k) \) and \( L' = (L'_{k-1}, \ldots, L_1) \) where \( L_i \subset M_{i+1} \times M_i \) and \( M_k = M \). We can assume, by adding identity Lagrangian correspondences if necessary, that

\[
\mathcal{L}_0 = L_n \times L_{n-2} \times \cdots \times L_k \times L_{k-1} \times \cdots \times L_2
\]

and

\[
\mathcal{L}_1 = L_{n-1} \times L_{n-3} \times \cdots \times \text{graph}(\tau_C) \times \cdots \times L_1.
\]

We have \( \mathcal{L}_1 = \tau_C(L_{n-1} \times L_{n-3} \times \cdots \times L_1) \) where \( C' = M_{n+1} \times \cdots \times M_{k+1} \times C \times M_{k-1} \times \cdots \times M_1 \) which fibers over \( M_{n-1} \times \cdots \times B \times \cdots \times M_1 \). The result follows from 5.15 by taking \( M_1 = M = M_k \) and \( M_2 \) to be the product of the rest of the manifolds \( M_i \).

**Corollary 5.17** \( CF(L, \tau_C^{-1}L') \) is quasi-isomorphic to the cone \( \text{Cone} \left( CF(Q'_0) \right) \{ -1 \} \).

**Proof** This is a standard argument. If \( l = \dim L \) then we have

\[
CF^*(L, \tau_C^{-1}L') = CF^*(\tau_C L, L') = CF^{l-\ast}(L', \tau_C L)^\vee = \text{Hom}(CF^{l-\ast}(L', \tau_C L), \mathbb{Z}).
\]

It follows from 5.14 that this is quasi-isomorphic to

\[
CF^{l-\ast}(L', (\nu \times \pi)C, (\pi \times \iota)C^l, L)\{ 1 \}^\vee \oplus CF^{l-\ast}(L', L)^\vee
\]

(with appropriate differential). This in turn equals

\[
CF(L, L') \oplus CF(L, (\nu \times \pi)C^l, (\pi \times \iota)C^l, L')\{ -1 \}.
\]
Now we use Theorem 5.14 (or more precisely 5.16) to obtain an exact triangle for the Seidel-Smith invariant. In the case under study $M = \gamma_l$ and the spherically fibred isotropic is $C = L \cap_i$. The coisotropic submanifold $C$ is a sphere bundle over $B = \gamma_{l-1}$. The fibred Dehn twist $\tau_C$ along $C$ equals the monodromy map $h_{\sigma_i}$ and so for any Lagrangian $L \subset M$ we have

$$(83) \quad L_{\sigma_i} \circ L \simeq \tau_C L.$$ 

Let $kTl$ and $lT'm$ be tangles, $\sigma_i^+, \sigma_i^- \in Br_2$ elementary braids and $T^\pm = T_{\sigma_i}^\pm T'$. We observe that if $a$ and $b$ are crossingless matchings and we take $L = (a', \Phi(T))$ and $L' = (\Phi(T'), b')$ then the map $f$ in (79) is the same as the cobordism map (64).

**Theorem 5.18** Let $e$ be the difference between the number of negative crossings in $T^\pm$ and $T \cup_i \cap_i T'$ (with the latter oriented arbitrarily). Then $CSS(T^-)$ is quasi-isomorphic to the cone of

$$(84) \quad CSS(T \cup_i \cap_i T')\{-2e\} \xrightarrow{CSS(1_{T_i}S_{\gamma_i}^1\gamma_i^1)} CSS(TT')$$

and $CSS(T^+)$ is quasi-isomorphic to the cone of

$$(85) \quad CSS(TT')\{-1\} \xrightarrow{CSS(1_T\gamma_i^1\gamma_i^1)} CSS(T \cup_i \cap_i T')\{-1 - 2e\}.$$ 

**Proof** We note that the sign conventions for positive braids and positive Dehn twists are opposites of each other. Since the degree of the map (64) equals 1, we apply the
degree shift \{-1\} to its target to obtain a map of degree zero. We have \(w(T^-) = w(T \cup_i \cap T') + 2e\) and \(w(T^-) = w(T T') - 1\) so we obtain (84).

In this case of \(T^+\) we have \(w(T^+) = w(T \cup_i \cap T') + 2e\), \(w(T^+) = w(T T') + 1\) and the cobordism map is of degree 1.

Corollary 5.19 We have the following exact triangles

\[
\begin{array}{c}
\text{CSS}(T^-) \\
\text{CSS}(T \cup_i \cap T') \{-2e\} \\
\text{CSS}(T T') \\
\text{CSS}(T^+) \\
\text{CSS}(T \cup_i \cap T') \{-1 - 2e\}
\end{array}
\]

which give the following cohomology exact sequences.

(86) \(\cdots \rightarrow \mathcal{H}SS^{i-1}(T^-) \rightarrow \mathcal{H}SS^{i-2e}(T \cup_i \cap T') \rightarrow \mathcal{H}SS^i(T T') \rightarrow \mathcal{H}SS^i(T^-) \rightarrow \cdots\)

(87) \(\cdots \rightarrow \mathcal{H}SS^{i-1}(T^+) \rightarrow \mathcal{H}SS^{i-1-2e}(T \cup_i \cap T') \rightarrow \mathcal{H}SS^i(T T') \rightarrow \mathcal{H}SS^i(T^+) \rightarrow \cdots\)

These two exact triangles are the similar to those for Khovanov homology ([6]) after the collapse of the bigrading. The same argument as in [4] can be used to deduce that the two invariants are equivalent for alternating (and more generally quasi-alternating) links.

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