CENTRALLY SYMMETRIC CONFIGURATIONS OF ORDER POLYTOPES

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Abstract. It is shown that the toric ideal of the centrally symmetric configuration of the order polytope of a finite partially ordered set possesses a squarefree quadratic initial ideal. It then follows that the convex polytope arising from the centrally symmetric configuration of an order polytope is a normal Gorenstein Fano polytope.

Introduction

The centrally symmetric configuration \( \mathbb{S} \) of an integer matrix supplies one of the powerful tools to construct normal Gorenstein Fano polytopes. The purpose of the present paper is to study the centrally symmetric configuration of the integer matrix associated with the order polytope of a finite partially ordered set.

Let \( \mathbb{Z}^{d \times n} \) denote the set of \( d \times n \) integer matrices. Given \( A \in \mathbb{Z}^{d \times n} \) for which no column vector is a zero vector, the centrally symmetric configuration of \( A \) is the \((d+1) \times (2n+1)\) integer matrix

\[
A^\pm = \begin{bmatrix}
0 & \vdots & A & -A \\
0 & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & 1 & \cdots & 1 \\
\end{bmatrix}.
\]

On the other hand, the centrally symmetric polytope arising from \( A \) is the convex polytope \( \mathcal{Q}_A^{(\text{sym})} \) which is the convex hull in \( \mathbb{R}^d \) of the column vectors of the matrix

\[
\begin{bmatrix}
0 & \vdots & A & -A \\
0 & \ddots & \ddots & \ddots \\
\end{bmatrix}.
\]

We focus our attention on the problem when \( \mathcal{Q}_A^{(\text{sym})} \) is a normal Gorenstein Fano polytope. In general, the origin is contained in the interior of \( \mathcal{Q}_A^{(\text{sym})} \subset \mathbb{R}^d \). Suppose that \( A \in \mathbb{Z}^{d \times n} \) satisfies \( \mathbb{Z}A = \mathbb{Z}^d \). Then a fundamental fact (Lemma \[\text{[1]}\]) on initial ideals guarantees that if the toric ideal \( I_{A^\pm} \) of \( A^\pm \) possesses a squarefree initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column \([0, \ldots, 0, 1]^t\) is smallest, then \( \mathcal{Q}_A^{(\text{sym})} \) is a normal Gorenstein Fano polytope.

In \( \mathbb{S} \) it is shown that if \( \text{rank}(A) = d \) and all nonzero maximal minors of \( A \) are \( \pm 1 \), then \( \mathcal{Q}_A^{(\text{sym})} \) is a normal Gorenstein Fano polytope.
Let \( P = [d] = \{1, \ldots, d\} \) be a finite partially ordered set and \( e_1, \ldots, e_d \) the unit coordinate vectors of \( \mathbb{R}^d \). Given a subset \( \alpha \subset P \), we write \( \rho(\alpha) \in \mathbb{R}^d \) for the vector \( \sum_{i \in \alpha} e_i \). A poset ideal of \( P \) is a subset \( \alpha \subset P \) such that if \( a \in \alpha \) and \( b \in P \) together with \( b \leq a \), then \( b \in \alpha \). In particular, the empty set as well as \( P \) itself is a poset ideal of \( P \). Let \( \mathcal{J}(P) \) denote the set of poset ideals of \( P \). The order polytope \( \mathcal{O}(P) \) is the \( d \)-dimensional \((0,1)\)-polytope which is the convex hull of \( \{ \rho(\alpha) \mid \alpha \in \mathcal{J}(P) \} \) in \( \mathbb{R}^d \). See [10]. We then write \( A_P \) for the integer matrix whose column vectors are those \( \rho(\alpha)^t \) with \( \alpha \in \mathcal{J}(P) \setminus \{\emptyset\} \).

**Theorem 0.1.** Let \( P \) be an arbitrary finite partially ordered set. Then the toric ideal \( I_{A_P}^\pm \) of \( A_P^\pm \) possesses a squarefree quadratic initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column \([0, \ldots, 0, 1]^t\) is smallest.

**Corollary 0.2.** The centrally symmetric polytope \( Q_{A_P}^{(sym)} \) arising from an arbitrary finite partially ordered set \( P \) is a normal Gorenstein Fano polytope.

The present paper is organized as follows. In Section 1 we recall basic materials on toric ideals of configurations as well as Fano polytopes. Theorem 0.1 together with Corollary 0.2 will be proved in Section 2. Finally, in Section 3, we compute the \( \delta \)-vectors of the centrally symmetric polytope \( Q_{A_P}^{(sym)} \), where \( P \) is an antichain.

1. **Centrally symmetric configurations and Fano polytopes**

We recall fundamental materials on centrally symmetric configurations and Fano polytopes. Let, as before, \( \mathbb{Z}^{d \times n} \) denote the set of \( d \times n \) integer matrices.

**a) Configuration**

A matrix \( A \in \mathbb{Z}^{d \times n} \) is called a configuration if there exists a hyperplane \( H \subset \mathbb{R}^d \) not passing the origin of \( \mathbb{R}^d \) such that each column vector of \( A \) lies on \( H \).

Let \( K \) be a field and \( K[T^{\pm 1}] = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}] \) the Laurent polynomial ring in \( d \) variables over \( K \). We associate each vector \( a = [a_1, \ldots, a_d] \in \mathbb{Z}^d \), with the Laurent monomial \( T^a = t_1^{a_1} \cdots t_d^{a_d} \in K[T^{\pm 1}] \). Given a configuration \( A \in \mathbb{Z}^{d \times n} \) with \( a_1, \ldots, a_n \) its column vectors, the toric ring \( K[A] \) of \( A \) is the monomial subalgebra of \( K[T^{\pm 1}] \) which is generated by \( T^{a_1}, \ldots, T^{a_n} \). Let \( K[X] = K[x_1, \ldots, x_n] \) denote the polynomial ring in \( n \) variables over \( K \). We define the surjective ring homomorphism \( \pi : K[X] \rightarrow K[A] \) by setting \( \pi(x_i) = T^{a_i} \) for \( i = 1, \ldots, n \). The kernel \( I_A \) of \( \pi \) is called the toric ideal of \( A \).

**b) Fano polytope**

A convex polytope \( \mathcal{P} \subset \mathbb{R}^d \) is called integral if each vertex of \( \mathcal{P} \) belongs to \( \mathbb{Z}^d \). We say that an integral convex polytope \( \mathcal{P} \subset \mathbb{R}^d \) is normal if, for each integer \( N > 0 \) and for each \( a \in N\mathcal{P} \cap \mathbb{Z}^d \), there exists \( a_1, \ldots, a_N \) belonging to \( \mathcal{P} \cap \mathbb{Z}^d \) such that \( a = a_1 + \cdots + a_N \).

Now, an integral convex polytope \( \mathcal{P} \subset \mathbb{R}^d \) is said to be a Fano polytope if the dimension of \( \mathcal{P} \) is \( d \) and if the origin of \( \mathbb{R}^d \) is a unique integer point belonging to the interior of \( \mathcal{P} \).
A Fano polytope \( P \subset \mathbb{R}^d \) is called Gorenstein if its dual polytope 
\[ P^\vee = \{ x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } y \in P \} \]
is integral, where \( \langle x, y \rangle \) is a canonical inner product of \( \mathbb{R}^d \).

We now come to a fundamental fact on normal Fano polytopes. We refer the reader to [12] and [5, Chapters 1 and 5] for basic information on initial ideals of toric ideals, regular and unimodular triangulations of integral convex polytopes.

**Lemma 1.1.** Let \( P \subset \mathbb{R}^d \) be an integral convex polytope such that the origin is contained in its interior. Let \( P \cap \mathbb{Z}^d = \{ a_1, \ldots, a_n \} \) and 
\[ A = \begin{bmatrix} a_1 & \cdots & a_n \\ 1 & \cdots & 1 \end{bmatrix} \in \mathbb{Z}^{(d+1)\times n}. \]
Suppose that \( \mathbb{Z}A = \mathbb{Z}^{d+1} \) and that there exists an ordering of the variables \( x_i_1 < \cdots < x_i_n \) for which \( a_i_1 = 0 \) such that the initial ideal \( \text{in}_{<} (I_A) \) of the toric ideal \( I_A \) with respect to the reverse lexicographic order \( < \) on \( K[X] \) induced by the ordering is squarefree. Then \( P \) is a normal Gorenstein Fano polytope.

**Proof.** The existence of an initial ideal as stated above guarantees that \( P \) possesses a unimodular triangulation \( \Delta \) for which each maximal face of \( \Delta \) contains the origin of \( \mathbb{R}^d \) as a vertex. It then follows easily that \( P \) is Fano and that the equation of the supporting hyperplane of each facet of \( P \) is of the form 
\[ a_1 z_1 + \cdots + a_d z_d = 1 \]
with each \( a_i \in \mathbb{Z} \). Hence the dual polytope of \( P \) is integral. Furthermore, in general, the existence of a unimodular triangulation of \( P \) says that \( P \) is normal. \( \square \)

c) Centrally symmetric configuration

In [8], the centrally symmetric configuration \( A^\pm \) of a matrix \( A \in \mathbb{Z}^{d\times n} \) is introduced:
\[ A^\pm = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ A \\ -A \end{bmatrix}. \]

A basic fact obtained in [8] is

**Proposition 1.2** ([8]). Let \( A \in \mathbb{Z}^{d\times n} \) be a matrix such that \( \mathbb{Z}A = \mathbb{Z}^d \) and whose nonzero maximal minors are \( \pm 1 \). Then the toric ideal \( I_{A^\pm} \) of \( A^\pm \) possesses a squarefree quadratic initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column \( [0, \ldots, 0, 1]^t \) is smallest. In particular, \( Q^{(\text{sym})}_A \) is a normal Gorenstein Fano polytope.

**Remark 1.3.** In general, \( A \in \mathbb{Z}^{d\times n} \) is called unimodular if \( \text{rank}(A) = d \) and all nonzero maximal minors of \( A \) have the same absolute value. If we assume \( \mathbb{Z}A = \mathbb{Z}^d \), then \( A \) is unimodular if and only if all nonzero maximal minors of \( A \) are \( \pm 1 \).
Example 1.4. Let $A$ be the following configuration:

$$A = \begin{bmatrix}
0 & 
\vdots 
& 
0 \\
1 & 
\cdots & 
1
\end{bmatrix} \quad \text{where} \quad A' = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.$$ 

Then we have $ZA = Z^0$ but $A$ is not unimodular. The initial ideal in $< (I_A)$ of $I_A$ with respect to any reverse lexicographic order $<$ is squarefree. Moreover, $I_A$ has a quadratic Gröbner basis with respect to some reverse lexicographic order such that the smallest variable is $x_1$. In addition, $K[A]$ is Gorenstein. On the other hand, $Q_A^{(\text{sym})}$ is not normal. We can check that $I_A^\pm$ is not generated by quadratic binomials and $Q_A^{(\text{sym})}$ is not Gorenstein by using the software package CoCoA [2].

2. Centrally symmetric configurations of order polytopes

In this section, we prove that, for any poset $P$,

- the toric ideal $I_{A_P}^\pm$ possesses a squarefree quadratic initial ideal with respect to a reverse lexicographic order such that the smallest variable corresponds to the origin; (Theorem [2.2]);
- $Q_A^{(\text{sym})}$ is a normal Gorenstein Fano polytope (Corollary [2.3]).

Let $P = [d] = \{1, \ldots, d\}$ be a poset and let $K[s, T^\pm] = K[s, t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}]$ be a Laurent polynomial ring in $d + 1$ variables over $K$. We set $S_P$ be the polynomial ring in variables $z, x_I (\emptyset \neq I \in \mathcal{J}(P))$, and $y_I (\emptyset \neq I \in \mathcal{J}(P))$ over $K$. We define the ring homomorphism $\pi : S_P \to K[s, T^\pm]$ by setting $\pi(z) = s$, $\pi(x_I) = s \prod_{i \in I} t_i$, and $\pi(y_I) = s \prod_{i \in I} t_i^{-1}$. Then the toric ideal $I_{A_P}^\pm$ is the kernel of $\pi$ and the toric ring $K[A_P^\pm]$ is the image of $\pi$. Let $<$ be a reverse lexicographic order on $S_P$ which satisfies $x_I < x_J$ and $y_I < y_J$ for all $I, J \in \mathcal{J}(P)$ with $I \subset J$. Here we set $x_\emptyset = y_\emptyset = z$ and hence $z$ is the smallest variable in $S_P$.

Example 2.1. Let $P = \{1, 2, 3, 4, 5\}$ be the poset with the partial order $1 < 3, 2 < 3, 2 < 4$ and $4 < 5$. In this case, $A_P$ is the following matrix

$$A_P = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4
\end{bmatrix}.$$ 

Then the initial ideal of $I_{A_P}^\pm$ with respect to a reverse lexicographic order $<$ is generated by 66 squarefree quadratic monomials.
Theorem 2.2. Work with the same notation as above. Let $G$ be the set of binomials consists of the following:

$$
\begin{align*}
x_I &- x_{I \cup J} x_{I \setminus J}, & y_I y_J &- y_{I \cup J} y_{I \setminus J} \quad (I \not\sim J) \\
x_I y_J &- x_{I \cap J} y_{I \setminus J} \quad (k \text{ is a maximal element of both } I \text{ and } J).
\end{align*}
$$

Here we set $x_0 = y_0 = z$. Then $G$ is a Gröbner bases of $I_{A_P^+}$ with respect to $<$ where the initial monomial of each binomial in $G$ is the first monomial.

Proof. Let $\text{in}(G) = \langle \text{in}_<(g) \mid g \in G \rangle$. Assume that $G$ is not a Gröbner bases of $I_{A_P^+}$. Then there exists a non-zero irreducible homogeneous binomial $f = u - v \in I_{A_P^+}$ such that neither $u$ nor $v$ belongs to $\text{in}(G)$. Let

$$
u = z^\alpha \prod_{i \in I} x_i^{p_i} \prod_{j \in J} y_j^{q_j}, \quad v = z^{\alpha'} \prod_{i' \in I'} x_{i'}^{p_{i'}} \prod_{j' \in J'} y_{j'}^{q_{j'}},
$$

where $0 < p_i, q_j, p_{i'}, q_{j'} \in \mathbb{Z}$ for all $I \in \mathcal{I}, J \in \mathcal{J}, I' \in \mathcal{I}'$ and $J' \in \mathcal{J}'$.

Since $x_I y_J \not\sim u, v$ for all $0 \not\in I \in \mathcal{J}(P)$, we have $\mathcal{I} \cap \mathcal{J} = \mathcal{I}' \cap \mathcal{J}' = \emptyset$. Moreover, since $x_I x_J, y_I y_J \not\sim u, v$ for all $I \not\sim J$, we can see that $\mathcal{I}, \mathcal{J}, \mathcal{I}'$ and $\mathcal{J}'$ are totally ordered subsets of $[d]$. In addition, we have $\mathcal{I} \cap \mathcal{I}' = \mathcal{J} \cap \mathcal{J}' = \emptyset$ since $f$ is irreducible.

Let $p = \sum_{I \in \mathcal{I}} p_I, q = \sum_{J \in \mathcal{J}} q_J, p' = \sum_{I' \in \mathcal{I}'} p_{i'}$ and $q' = \sum_{J' \in \mathcal{J}'} q_{j'}$. Then

$$
\pi(u) = s^{\alpha + p + q} \prod_{i \in \mathcal{I}} \left( \prod_{i \in \mathcal{I}} t_i \right)^{p_i} \prod_{j \in \mathcal{J}} \left( \prod_{j \in \mathcal{J}} t_j^{-1} \right)^{q_j},
$$

$$
\pi(v) = s^{\alpha' + p' + q'} \prod_{i' \in \mathcal{I}'} \left( \prod_{i' \in \mathcal{I}'} t_{i'} \right)^{p_{i'}} \prod_{j' \in \mathcal{J}'} \left( \prod_{j' \in \mathcal{J}'} t_{j'}^{-1} \right)^{q_{j'}}.
$$

Since $\pi(u) = \pi(v)$, we have $\pi(u') = \pi(v')$, where

$$
u' = z^{\alpha + 2q} \prod_{i \in \mathcal{I}} x_i^{p_i} \prod_{j' \in \mathcal{J}'} x_j^{q_{j'}}, \quad v' = z^{\alpha' + 2q'} \prod_{i' \in \mathcal{I}'} x_{i'}^{p_{i'}} \prod_{j \in \mathcal{J}} x_j^{q_j}.
$$

Hence $g = u' - v'$ does also belong to $I_{A_P^+}$. Suppose that $g \neq 0$ and $\text{in}_<(g) = u'$. Since $g \in K[z, x_I \mid 0 \not\in I \in \mathcal{J}(P)]$, we have $g \in \langle x_I x_J - x_{I \cup J} x_{I \setminus J} \mid I \not\sim J \rangle$ (see [3]). Then there exist $I$ and $J'$ with $I \not\sim J'$ such that $x_I x_{J'} \not\sim u'$. We may assume that $I \in \mathcal{I}$ and $J' \in \mathcal{J}'$ since both $\mathcal{I}$ and $\mathcal{J}'$ are totally ordered sets of $[d]$. In particular, $\mathcal{I}$ and $\mathcal{J}'$ are not empty.

Let $I_{\text{max}}, I_{\text{max}}', J_{\text{max}}$ and $J_{\text{max}}'$ be the maximal elements of $\mathcal{I}, \mathcal{I}', \mathcal{J}$ and $\mathcal{J}'$, respectively. (If $\mathcal{J} = \emptyset$, then let $J_{\text{max}} = \emptyset$.) Then $0 \not\in I_{\text{max}} \cup J_{\text{max}} = I_{\text{max}}' \cup J_{\text{max}}$ holds since $\pi(u') = \pi(v')$ and $\mathcal{I}, \mathcal{I}', \mathcal{J}$ and $\mathcal{J}'$ are totally ordered subsets of $[d]$. Thus, in particular, either $\mathcal{I}'$ or $\mathcal{J}$ is non-empty. We may assume that $\mathcal{I}' \neq \emptyset$. For $a \in P$, we define the poset ideal $\langle a \rangle = \{ b \in P \mid b \leq a \}$. If $a_1, \ldots, a_s$ are maximal elements of $I$, then we write $I = \bigcup_{i=1}^{s} \langle a_i \rangle$. Let $I_{\text{max}} \cup J_{\text{max}} = I_{\text{max}}' \cup J_{\text{max}} = \bigcup_{i=1}^{s} \langle e_i \rangle$.
and
\[
M_1 = \{ e_i | e_i \text{ is a maximal element of } I_{\text{max}} \},
\]
\[
M_2 = \{ e_i | e_i \text{ is a maximal element of } J'_{\text{max}} \},
\]
\[
M_3 = \{ e_i | e_i \text{ is a maximal element of } I'_{\text{max}} \},
\]
\[
M_4 = \{ e_i | e_i \text{ is a maximal element of } J_{\text{max}} \}.
\]
Then we have \( M_1 \cup M_2 = M_3 \cup M_4 = \{ e_1, \ldots, e_s \} \). Since \( x_I y_J \upharpoonright u, v \) for all pairs \( I = \bigcup_{i=1}^s (a_i) \) and \( J = \bigcup_{j=1}^t (b_j) \) with \( \{ a_1, \ldots, a_s \} \cap \{ b_1, \ldots, b_t \} \neq \emptyset \), we have \( M_1 \cap M_4 = M_2 \cap M_3 = \emptyset \). Hence it follows that \( M_1 = M_3 \) and \( M_2 = M_4 \).

Since \( \mathcal{I} \cap \mathcal{J}' = \emptyset \) we have \( I_{\text{max}} \not\subseteq I'_{\text{max}} \). Hence we may assume that \( I_{\text{max}} \not\subset I'_{\text{max}} \). Then there exists \( a_{\text{max}} \in I_{\text{max}} \) such that (i) \( a_{\text{max}} \) is a maximal element of \( I_{\text{max}} \), (ii) \( a_{\text{max}} \not\in I'_{\text{max}} \) and (iii) \( a_{\text{max}} < e_i \) in \( P \) for some \( e_i \in M_2 = M_4 \).

By \( \pi(u') = \pi(v') \),
\[
\sum_{I \in \mathcal{I}} p_I + \sum_{J \in \mathcal{J}} q_J = \sum_{I' \in \mathcal{I}'} p_{I'} + \sum_{J' \in \mathcal{J}'} q_{J'}
\]
holds for all \( a \in P \). Note that \( a_{\text{max}} \not\in I'_{\text{max}} \) and \( e_i \not\in I_{\text{max}} \cup I'_{\text{max}} \). Thus we have \( e_i \not\subset I \) for all \( I \in \mathcal{I} \cup \mathcal{I}' \) and \( a_{\text{max}} \not\subset I' \) for all \( I' \in \mathcal{I}' \). Hence we have
\[
\sum_{J \in \mathcal{J}} q_J = \sum_{J \in \mathcal{J}} q_J, \quad \sum_{I \in \mathcal{I}} p_I + \sum_{J \in \mathcal{J}} q_J = \sum_{J \in \mathcal{J}} q_J
\]
by \( \pi(u') = \pi(v') \). Note that if \( e_i \in J \in \mathcal{J}(P) \), then \( a_{\text{max}} \in J \). Thus we have
\[
0 < \sum_{I \in \mathcal{I}} p_I + \sum_{J \in \mathcal{J}} q_J = \sum_{J \in \mathcal{J}} q_J.
\]
Hence there exists \( J_{a_{\text{max}}} \in \mathcal{J} \) such that \( a_{\text{max}} \in J_{a_{\text{max}}} \) and \( e_i \not\in J_{a_{\text{max}}} \). Moreover, the following claim holds:

**Claim.** There exists a poset ideal \( J \in \mathcal{J} \) such that \( a_{\text{max}} \) is a maximal element of \( J \).

In fact, if \( a_{\text{max}} \) is not a maximal element of \( J_{a_{\text{max}}} \), then there exists \( b \in J_{a_{\text{max}}} \) such that \( a_{\text{max}} < b \) in \( P \). Note that \( b \not\in I_{\text{max}} \) since \( a_{\text{max}} \not\in I'_{\text{max}} \). In addition, \( b \not\in I_{\text{max}} \) since \( a_{\text{max}} \) is a maximal element of \( I_{\text{max}} \). Thus, by the above argument, there exists \( J \in \mathcal{J} \) such that \( a_{\text{max}} \in J \) and \( b \not\in J \). Then \( J_{a_{\text{max}}} \not\subset J \) since \( b \in J_{a_{\text{max}}} \) and \( J \) is a totally ordered subset of \([d]\). Therefore, we can get a desired poset ideal \( J \in \mathcal{J} \) by repeating this argument. Thus the claim holds.

Let \( J \) be a poset ideal as appeared in the above Claim. Then \( a_{\text{max}} \) is a maximal element of both \( I_{\text{max}} \) and \( J \). Hence \( x_{I_{\text{max}}} y_J \upharpoonright u \), but this is a contradiction. Therefore \( g = 0 \), that is, \( u' = v' \). Then we have \( (\mathcal{I} \cup \mathcal{I}') \cap (\mathcal{J} \cup \mathcal{J}') = \emptyset \) by \( \mathcal{I} \cap \mathcal{J}' = \mathcal{I}' \cap \mathcal{J} = \mathcal{I} \cap \mathcal{I}' = \mathcal{J} \cap \mathcal{J}' = \emptyset \). Therefore \( \prod_{I \in \mathcal{I}} x_I^p_I \prod_{J \in \mathcal{J}} x_J^q_J \) and \( \prod_{I' \in \mathcal{I}'} x_{I'}^{p_{I'}} \prod_{J' \in \mathcal{J}'} x_{J'}^{q_{J'}} \) have no common variables. Thus, we have \( \mathcal{I} = \mathcal{J} = \mathcal{I}' = \mathcal{J}' = \emptyset \) since \( u' = v' \). This is a contradiction. Therefore \( \mathcal{G} \) is a Gröbner basis of \( I_{A_{\mathbb{K}}} \). □
In Theorem 2.2, the initial ideal is squarefree and quadratic. By Lemma 1.1, we have the following:

**Corollary 2.3.** Let \( P \) be a poset. Then \( Q_{A_P}^{\text{sym}} \) is a normal Gorenstein Fano polytope.

Note that the matrix \( A_P \) of a poset \( P \) is not necessarily unimodular. A characterization of the unimodularity of \( A_P \) appeared in [7, Example 3.6] without proof. Here, we recall the characterization with a proof. Let \( \mathbb{N}^2 \) be the distributive lattice consisting of all pairs \( (i, j) \) of nonnegative integers with the partial order \( (i, j) \leq (k, \ell) \) if and only if \( i \leq k \) and \( j \leq \ell \). A distributive lattice \( D \) is said to be planar (see [11, p. 436]) if \( D \) is a finite sublattice of \( \mathbb{N}^2 \) with \( (0, 0) \in D \) which satisfies the following: for any \( (i, j), (k, \ell) \in D \) with \( (i, j) < (k, \ell) \), there exists a chain (totally ordered subset) of \( D \) of the form \( (i, j) = (i_0, j_0) < (i_1, j_1) < \cdots < (i_s, j_s) = (k, \ell) \) such that \( i_{k+1} + j_{k+1} = i_k + j_k + 1 \) for all \( k \).

**Proposition 2.4 ([7]).** Let \( J(P) \) be the distributive lattice associated with a poset \( P \). Then \( A_P \) is unimodular if and only if \( J(P) \) is planar.

**Proof.** Assume that \( J(P) \) is planar. Then \( K[A_P] \) is isomorphic to \( K[A] \) where \( A \) is the vertex-edge incidence matrix of a bipartite graph (see [6]). Since \( A \) is unimodular [9], \( A_P \) is also unimodular.

Conversely, assume that \( J(P) \) is not planar. Then there exist \( a, b, c \in P \) such that \( a \not< b, a \not< c \) and \( b \not< c \). Let \( I = \langle a \rangle \cup \langle b \rangle \cup \langle c \rangle \). Then the sublattice \( \{ J \in J(P) \mid I \setminus \{ a, b, c \} \subset J \subset I \} \) of \( J(P) \) is isomorphic to \( J(P') \), where \( P' = \{ a, b, c \} \) is an antichain. It is easy to see that

\[
A_{P'} = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

is not unimodular. Since \( K[A_{P'}] \) is a combinatorial pure subring (see [7]) of \( K[A_P] \), \( A_P \) is not unimodular.

### 3. The \( \delta \)-vector of \( Q_{A_P}^{\text{sym}} \) of an antichain poset \( P \)

In this section, we show that the \( \delta \)-vector of \( Q_{A_P}^{\text{sym}} \) of an antichain poset \( P \) is the Ehrhart number (see [11]).

Let \( \mathcal{P} \subset \mathbb{R}^d \) be an integral convex polytope of dimension \( d \). Given integers \( t = 1, 2, \ldots \), we set \( i(\mathcal{P}, t) = \#(t\mathcal{P} \cap \mathbb{Z}^d) \). Then \( i(\mathcal{P}, t) \) is a polynomial in \( t \) of degree \( d \) and called the Ehrhart polynomial of \( \mathcal{P} \). Its generating function satisfies

\[
1 + \sum_{t=1}^{\infty} i(\mathcal{P}, t) \lambda^t = \frac{\delta_{P}(\lambda)}{(1 - \lambda)^{d+1}}
\]

where \( \delta_{P}(\lambda) = \sum_{i=0}^{d} \delta_i \lambda^i \) is a polynomial in \( \lambda \) of degree \( \leq d \). The vector \( (\delta_0, \ldots, \delta_d) \) is called the \( \delta \)-vector of \( \mathcal{P} \). It is known that a Fano polytope \( \mathcal{P} \subset \mathbb{R}^d \) is Gorenstein if and only if \( \delta_i = \delta_{d-i} \) for all \( 0 \leq i \leq d \). See [4].
Theorem 3.1. Let $P = [d]$ be an antichain. Then we have

$$i(Q_{AP}^{(sym)}, t) = (t + 1)^{d+1} - t^{d+1},$$

$$1 + \sum_{t=1}^{\infty} i(Q_{AP}^{(sym)}, t)\lambda^t = \frac{\sum_{i=0}^{d} A(d+1, i)\lambda^i}{(1 - \lambda)^{d+1}},$$

where $A(d+1, i)$ is the Eulerian number.

Proof. Let $<$ be a reverse lexicographic order in Theorem 2.2. Since $Q_{AP}^{(sym)}$ is normal and $\mathbb{Z}A_P = \mathbb{Z}^{d+1}$, $i(Q_{AP}^{(sym)}, t)$ is equal to the number of standard monomials in $S_P$ of degree $t$ with respect to $\text{in}_< (I_{A_P^1})$. By Theorem 2.2 the set of all standard monomials of degree $t$ is

$$\text{SM}_t = \left\{ x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q} z^{t-p-q} \mid p, q \geq 0, p + q \leq t, I_1 \cap J_1 = \emptyset, I_1 \supset \cdots \supset I_p, J_1 \supset \cdots \supset J_q \right\}.$$ 

Let $V_t = \{ t_1^{m_1} \cdots t_d^{m_d} s^t \in K[s, T^{\pm 1}] \mid -t \leq m_i \leq t, \max_i(m_i) - \min_i(m_i) \leq t \}$. We can define the map $\pi|_{\text{SM}_t} : \text{SM}_t \rightarrow V_t$. On the other hand, we define a map $g : V_t \rightarrow \text{SM}_t$ by $g(t_1^{m_1} \cdots t_d^{m_d} s^t) = x_{i_1} \cdots x_{i_p} y_{j_1} \cdots y_{j_q} z^{t-p-q}$, where $p = \max_i(m_i)$, $q = -\min_i(m_i)$, $I_\alpha = \{ k \in [d] \mid m_k \geq \alpha \}$, and $J_\beta = \{ k \in [d] \mid m_k \leq -\beta \}$. Then $\pi|_{\text{SM}_t}$ is bijective since both $\pi|_{\text{SM}_t} \circ g$ and $g \circ \pi|_{\text{SM}_t}$ are identity maps. Hence $\# \text{SM}_t = \# V_t$.

For each $\gamma = 1, \ldots, t$, we can see that $\# \{ t_1^{m_1} \cdots t_d^{m_d} s^t \in V_t \mid \max_i(m_i) = \gamma \} = (t + 1)^d - t^d$. Moreover, $\# \{ t_1^{m_1} \cdots t_d^{m_d} s^t \in V_t \mid \max_i(m_i) \leq 0 \} = (t + 1)^d$. Therefore, we have

$$\# \text{SM}_t = \# V_t = t \{(t + 1)^d - t^d\} + (t + 1)^d = (t + 1)^{d+1} - t^{d+1}.$$

By a well-known identity

$$\sum_{i=0}^{\infty} t^d \lambda^t = \frac{\sum_{i=0}^{d} A(d, i)\lambda^{i+1}}{(1 - \lambda)^{d+1}},$$

for the Eulerian number, it is easy to show

$$1 + \sum_{t=1}^{\infty} ((t + 1)^{d+1} - t^{d+1})\lambda^t = \frac{\sum_{i=0}^{d} A(d+1, i)\lambda^i}{(1 - \lambda)^{d+1}}.$$

□

Remark 3.2. Let $P = [d]$ be an antichain poset. It is known that

$$1 + \sum_{t=1}^{\infty} i(O(P), t)\lambda^t = \frac{\sum_{i=0}^{d} A(d, i)\lambda^i}{(1 - \lambda)^{d+1}}.$$

Example 3.3. Let $P$ be the poset as appeared in Example 2.1. Then the $\delta$-vector of $Q_{AP}^{(sym)}$ is $(1, 15, 54, 54, 15, 1)$. Note that $O(P)$ is not Gorenstein since $P$ is not pure, and its $\delta$-vector is $(1, 5, 3)$. 

8
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