APPROXIMATION OF RANDOM INVARIANT MANIFOLDS FOR A STOCHASTIC SWIFT-HOHENBERG EQUATION

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ABSTRACT. Random invariant manifolds are considered for a stochastic Swift-Hohenberg equation with multiplicative noise in the Stratonovich sense. Using a stochastic transformation and a technique of cut-off function, existence of random invariant manifolds and attracting property of the corresponding random dynamical system are obtained by Lyaponov-Perron method. Then in the sense of large probability, an approximation of invariant manifolds has been investigated and this is further used to describe the geometric shape of the invariant manifolds.

1. Introduction. The Swift-Hohenberg equation was originally derived by Swift and Hohenberg[27] as a model for convective instability in Rayleigh-Bénard convection. There has been some interesting works for the local Swift-Hohenberg equation [14, 18, 20, 21, 17]. However, when the distance from the change of stability is sufficiently small, or Rayleigh number is near thermal equilibrium, the influence of small noise or molecular noise is detected in various convection experiments [22, 23, 25]. It is difficult to stabilize the control parameters (e.g. temperature in Rayleigh-Bénard convection) to the precision of the noise strength, which is extremely small in the case of thermal fluctuations. The effects of thermal fluctuations (i.e. additive noise) on the onset of convective motion needs to be taken into account, and thus Swift and Hohenberg’s original model is actually a stochastic partial differential equation (SPDE)

\[ u_t = -(1 + \partial_{xx})^2 u + \mu u - u^3 + \sigma \xi, \]

where \( u(x, t) \) is the unknown function, \( \sigma \) is a positive parameter, \( \mu \) is a real parameter, and \( \xi = \xi(t) \) is a noisy process.

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Furthermore, it is appropriate to consider the effects of small noise on the parameter \( \mu \), and in this case, a stochastic Swift-Hohenberg equation with multiplicative noise arises [2]

\[
    u_t = -(1 + \partial_{xx})^2 u + \mu u - u^3 + \sigma u \circ \xi, \tag{1}
\]

where the symbol \( \circ \) indicates stochastic differential in the Stratonovich sense. In this paper, the noisy process \( \xi = \frac{dW}{dt} \) is the generalized derivative of a standard scalar Brownian motion \( W(t) \), which represents Gaussian white noise [13].

The purpose of the present paper is to study random invariant manifolds for the stochastic Swift-Hohenberg equation (1), together with their approximations and geometric shapes.

Stochastic partial differential equations (SPDEs) play important roles in modeling, analyzing, simulating and predicting complex phenomena under random fluctuation in various fields [19, 13, 31]. They have been extensively investigated recently as appropriate mathematical models in the context of random dynamical systems [8, 9, 11, 12, 13, 15, 10, 26, 30]. Invariant manifolds are special invariant sets which provide geometric structures for understanding stochastic dynamics. Although the existence of random invariant manifolds for a class of SPDEs is known [11, 12, 13], the geometric shapes of these invariants manifolds are much less clear [3, 5, 6].

This paper is organized as follows. After recalling relevant definitions and known results concerning random dynamical systems and invariant manifolds in section 2, we devote section 3 to the study of the existence of invariant manifolds and attracting properties for the corresponding random dynamical system for the stochastic Swift-Hohenberg equation (1). To this end, we employ a stochastic transformation, a technique of cut-off function, and the Lyaponov-Perron method. In section 4, we consider the approximation of the invariant manifolds, in the sense of large deviation, a technique of cut-off function, and the Lyaponov-Perron method. In section 4, we consider the approximation of the invariant manifolds, in the sense of large deviation, a technique of cut-off function, and the Lyaponov-Perron method.

2. Preliminaries. Rewrite the stochastic Swift-Hohenberg equation (1) as follows

\[
    u_t = -Au + \mu u + F(u) + \sigma u \circ \dot{W}, \quad u(0, x) = u_0(x), \tag{2}
\]

where \( u \) is subject to zero Dirichlet boundary condition on \( D = (0, \pi) \), \( A = (1+\partial_{xx})^2 \) a closed self-adjoint linear operator with domain \( Dom(A) \) in \( H = L^2(D) \), and the nonlinear function \( F(u) = -u^3 \). In Hilbert space \( H = L^2(D) \), we use the usual norm \( \| \cdot \| \) and scalar product \( \langle \cdot , \cdot \rangle \).

The eigenvalues and eigenfunctions for \( A \) are \( \lambda_n = (1 - n^2)^2 \) and \( e_n(x) = \sin nx, n = 1, 2, \ldots \). Here, the kernel space of \( A \) is \( H_c = \text{span}\{e_1(x)\} \). We denote \( \lambda_c = \lambda_1 = 0 \) and \( \lambda_s = \lambda_2 = 9 \). Except for \( \lambda_c = 0 \), all other eigenvalues for \( A \) are positive and denote \( H_s \) as the eigenspace spanned by eigenfunctions corresponding to these positive eigenvalues. Note that \( H_c \oplus H_s = H \).

Denote the orthogonal projection from \( H \) to \( H_c \) by \( P_c \) and \( A_c = P_cA \). In addition, denote \( P_s = I - P_c \) and \( A_s = P_sA \). In the following, we use the subscript “c” for projection onto \( H_c \) and “s” for projection onto \( H_s \). We know that \( P_c \) is a continuous projection commuting with \( A \).

Recall that the interpolation space \( H^\alpha, \alpha > 0 \), is defined as the domain of \( A^\alpha \) endowed with scalar product \( \langle u, v \rangle_\alpha = \langle A^\frac{\alpha}{2}u, A^\frac{\alpha}{2}v \rangle \) and the norm \( \| \cdot \|_\alpha \). In addition, we introduce \( H^{-\alpha} \) as the dual of \( H^\alpha \) with respect to the scalar product in \( H \). The nonlinear function \( F(u), \) is a mapping \( F : H \times H \times H \rightarrow H^{-\alpha} \). Also...
note that \( H^\alpha = H_\alpha \oplus H_\alpha^\alpha \), where \( H_\alpha^\alpha = H^\alpha \cap H_\alpha \) with \( \alpha \in [0, 1) \). By Theorem 6.13 in [24], there exists \( M > 0 \) such that

\[
\|e^{-A_t \pi}\|_{L(H^\alpha, H^\alpha)} \leq Me^{-\lambda_\alpha t}, \quad t \geq 0, \\
\|e^{-A_t \pi}\|_{L(H^\alpha, H)} \leq \frac{M}{e^{\alpha}} e^{-\lambda_\alpha t}, \quad t \geq 0.
\]

Some basic concepts related to one of dynamics behaviors for stochastic dynamical systems[1, 7, 13] are recalled here. Let \((X, \| \cdot \|_X)\) be a separable Hilbert space with Borel \( \sigma \)-algebra \( B(X) \), and \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) be an ergodic metric dynamical system.

**Definition 2.1.** (See[1, 7, 13]) Let \( t \in \mathbb{R} \) and \( \omega \in \Omega \). A continuous random dynamical system on \( X \) over \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) is \((\mathcal{B}(\mathbb{R}_+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))\) measurable mapping

\[
\varphi : \mathbb{R}_+ \times \Omega \times X \to X, \quad (t, \omega, u) \mapsto \varphi(t, \omega, u)
\]
such that following conditions

1. \( \varphi(0, \omega, \cdot) \) is the identity on \( X \);
2. \( \varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \varphi(s, \omega, \cdot)) \) for all \( s, t \geq 0 \);
3. \( \varphi(t, \omega, \cdot) : X \to X \) is continuous for all \( t \geq 0 \).

We also need a random transformation to convert an SPDE to a partial differential equation random coefficients. To this end, we consider the following scalar Langevin equation

\[
dz + zd\tau = \sigma dW.
\]

It is known that this equation has a unique stationary solution \( z(\theta_t \omega) \), which is called the stationary Ornstein-Uhlenbeck (O-U) process. From [13], for the solution of (5), the following properties hold in a \( \theta_t \), invariant set \( \Omega \subset \Omega^* \) of full probability

\[
\lim_{t \to \pm \infty} \frac{z(\theta_t \omega)}{t} = 0, \quad \lim_{t \to \pm \infty} \frac{\int_0^t z(\theta_\tau \omega) d\tau}{t} = 0, \forall \omega \in \Omega^*.
\]

Setting \( z(t) = z(\theta_t \omega) \), \( z(0) = z(\omega) \) holds. Introduce a random transformation by \( v = e^{-z(\theta_t \omega)} u(t) = e^{-z(t)} u(t) \). Then the original SPDE (2) becomes a random partial differential equation

\[
\frac{dv}{dt} = -Av + (\mu + z)v + e^{-z(\theta_t \omega)} F(ve^{z(\theta_t \omega)}),
\]

\[
v(x, 0) = e^{-z(\omega)} u_0, \quad x \in [0, \pi],
\]

\[
v(0, t) = v(\pi, t) = \frac{\partial v}{\partial x}(0, t) = \frac{\partial v}{\partial x}(\pi, t) = 0.
\]

In order to consider dynamic behaviors of for this random system in a neighborhood of the fixed point \( v = 0 \), we use a cut-off technique such that the nonlinear term will then satisfy a global Lipschitz condition. Define a cut-off function

\[
\chi(v) = \begin{cases} 
1, & \text{if } \|v\|_\alpha \leq 1, \\
0, & \text{if } \|v\|_\alpha > 2.
\end{cases}
\]

Then for any positive parameter \( R \), we define \( \chi_R(v) = \chi(\frac{v}{R}) \) for all \( v \in H^\alpha \). Therefore,

\[
\chi_R(v) = \chi\left(\frac{v}{R}\right) = \begin{cases} 
1, & \text{if } \|v\|_\alpha \leq R, \\
0, & \text{if } \|v\|_\alpha > 2R.
\end{cases}
\]
In addition, define \( F(R)(v) = \chi_R(v)F(v) \) for convenience. Then for every positive constant \( L > 0 \) and for every \( \omega \in \Omega \), there exists a positive random variable \( R > 0 \) such that the local Lipschitz condition holds
\[
\|F(R)(v) - F(R)(\tilde{v})\| \leq L\|v - \tilde{v}\|.
\] (10)

Then the truncated system obtained by the cut-off technique
\[
u_t = -Au + \mu u + F(R)(u) + \sigma u \circ \dot{W}, \quad u(0, x) = u_0(x)
\] (11)
will be considered. That is, we will now consider the following random system
\[
\frac{dv}{dt} = -Av + (\mu + z)v + e^{-z(\theta_t \omega)}F(R)(ve^{z(\theta_t \omega)}),
\] (12)
\[
v(x, 0) = e^{-z(\omega)}u_0, \quad x \in [0, \pi],
\] (13)
\[
v(0, t) = v(\pi, t) = \frac{\partial v}{\partial x}(0, t) = \frac{\partial v}{\partial x}(\pi, t) = 0.
\] (14)

The system (12)-(14) has a unique solution for every \( \omega \in \Omega \) and generates a continuous random dynamical system \( \varphi(t, \omega) \) on \( H \), i.e., \( \varphi(t, \omega)(v_0) \) is defined to be the solution \( v(t, \omega, v_0) \) (with initial value \( v_0 \)). We consider a random mapping
\[
T(\omega, v_0) = v_0 e^{-z(\omega)}
\] (15)
and its inverse transform
\[
T^{-1}(\omega, v_0) = v_0 e^{z(\omega)}
\] (16)
for all \( v_0 \in H^\alpha \) and \( \omega \in \Omega \).

Note that \( T^{-1}(\theta_t \omega, v(t, \omega; T(\omega, x))) \) is just \( u(t, \omega; x) \), which is the random dynamical system generated by the SPDE (2). Thus, the mapping \( T \) maps random dynamical system \( v \) to \( u \), and vice versa.

Now recall a definition for a local random invariant manifold are given.

**Definition 2.2.** (See [3]) The system (7)-(9) is said to have a local random invariant manifold with radius \( R \), if there is a random set \( \mathcal{M}^R(\omega) \), which is defined by the graph of a random continuous function \( \psi(\omega, \cdot) : B_R(0) \cap \mathcal{X}_1 \rightarrow \mathcal{X}_2 \), such that for all bounded sets \( B \) in \( B_R(0) \subset \mathcal{X}_1 \) we have the local dynamical system \( \varphi \) generated by (7)-(9) satisfies
\[
\varphi(t, \omega) \left[ \mathcal{M}^R(\omega) \cap B \right] \subset \mathcal{M}^R(\theta_t \omega)
\] (17)
for all \( t \in (0, \tau_0(\omega)) \) with
\[
\tau_0(\omega) = \tau_0(\omega, B) = \inf\{t \geq 0 : \varphi(t, \omega) \left[ \mathcal{M}^R(\omega) \cap B \right] \not\subset B_R(0)\}.
\] (18)

The system (12)-(14) is the truncated version of the system (7)-(9) with a cut-off radius \( R \). It is clear that \( \mathcal{M}^R(\omega) = \mathcal{M}(\omega) \cap B_R(0) \) is a local invariant manifold for the system (7)-(9). Therefore, from now on, we only consider the invariant manifold \( \mathcal{M}(\omega) \) of the truncated system (12)-(14).

**3. Existence of a random invariant manifold and an attracting property.**

First, we construct the local random invariant manifold of (12)-(14) for sufficiently small \( R > 0 \) and prove its exponentially attracting property, by using the method
in [11]. In this end, we consider the following two projected random equations on $H_e$ and $H_x$, respectively,

\begin{align}
\partial_t v_c &= (\mu + z) v_c + P_e e^{-z} F^{(R)}(ve^z), \quad v_c(0) = P_x u_0 e^{-z(0)} = \xi, \\
\partial_t v_x &= -A_x v_x + (\mu + z) v_x + P_x e^{-z} F^{(R)}(ve^z), \quad v_x(0) = P_x u_0 e^{-z(0)}.
\end{align}

The Lyapunov-Perron method will be used as in [11] on the random space with random norm depending on $\omega$, which is defined as follows.

**Definition 3.1.** For $-\mu < \eta < \lambda_x - \mu$, define the Banach space

$$C_\eta^- = \left\{ v \in C\left((-\infty, 0]; H^\alpha \right): \sup_{t \leq 0} \left\{ e^{\eta t - f_0^r z(r) dr} \|v\|_\alpha \right\} < \infty \right\}$$

with norm

$$\|v\|_{C_\eta^-} = \sup_{t \leq 0} \left\{ e^{\eta t - f_0^r z(r) dr} \|v\|_\alpha \right\}.$$  

Notice that

\begin{align}
\|e^{-A_x + \mu} (t-r)+f_0^r z(r) dr \|_{C_\eta^-} &\leq M e^{-\lambda_x + \mu} (t-r)+f_0^r z(r) dr \|v_x(r)\|_\alpha \\
&\leq M \|v_x\|_{C_\eta^-} e^{-\lambda_x + \mu} (t-r)+f_0^r z(r) dr \rightarrow 0, \quad \text{as} \ r \rightarrow -\infty.
\end{align}

We introduce the following mapping and investigate its uniform contraction property as in [11].

**Definition 3.2.** Define the nonlinear operator $J$ on $C_\eta^-$ for given $\xi \in H^\alpha_e$ and $\omega \in \Omega_0$ as

$$J(v, \xi)(t) = e^{\mu t + f_0^r z(r) dr} \xi + \int_0^t e^{\mu (t-\tau) + f_0^r z(r) dr} e^{-z(r)} P_x F^{(R)}(v(\tau) e(z(\tau))) d\tau$$

$$+ \int_{-\infty}^t e^{-(A_x + \mu) (t-\tau) + f_0^r z(r) dr} e^{-z(r)} P_x F^{(R)}(e(z(\tau)) v(\tau)) d\tau.$$  

According to the Lipschitz property of $F^{(R)}(u)$, it easy to obtain that for any $\xi \in H^\alpha_e$ and $\omega \in \Omega_0$, there is a mapping $J(\cdot, \xi)(t): C_\eta^- \rightarrow C_\eta^-$; refer to[10]. Further, we can show it is a Lipschitz continuous map by some estimates.

**Theorem 3.3.** Suppose $-\mu < \eta < \lambda_x - \mu$ and condition (26). Then the operator $J$ has a unique fixed point $v^* = v^*(\omega, \xi) \in C_\eta^-$. 

**Proof.** For any $v, \bar{v} \in C_\eta^-$, consider

$$\|J(v, \xi) - J(\bar{v}, \xi)\|_{C_\eta^-} \leq L_R \|v - \bar{v}\|_{C_\eta^-} \sup_{t \leq 0} \int_0^t e^{(\mu + \eta) (t-\tau)} d\tau + ML_R \|v - \bar{v}\|_{C_\eta^-} \sup_{t \leq 0} \int_{-\infty}^t e^{(\mu + \eta) (t-\tau)} d\tau$$

$$\leq \left[ \frac{L_R}{\mu + \eta} + ML_R \frac{\Gamma(1 - \alpha)}{(\lambda_x - \eta - \mu)^{1-\alpha}} \right] \|v - \bar{v}\|_{C_\eta^-},$$

where $Lip(J) = \frac{L_R}{\mu + \eta} + ML_R \frac{\Gamma(1 - \alpha)}{(\lambda_x - \eta - \mu)^{1-\alpha}}$ is actually independent of $\xi$ and $\omega$. Therefore, if for sufficient small $R > 0$ such that

$$Lip(J) = \frac{L_R}{\mu + \eta} + ML_R \frac{\Gamma(1 - \alpha)}{(\lambda_x - \eta - \mu)^{1-\alpha}} < 1,$$

the mapping $J(\cdot, \xi)(t): C_\eta^- \rightarrow C_\eta^-$ is a contraction. So according to the uniform contraction mapping principle we can obtain the result. \qed
For the fixed point \( v^* = v^*(\omega; \xi) \in C^\alpha_T \), define \( h(\omega, \xi) = P_s v^*(0, \omega; \xi) \). Thus, we have
\[
h(\omega, \xi) = \int_{-\infty}^0 e^{(A_\omega - \mu)\tau + \int_0^\tau z(\tau) d\tau} e^{-z(\tau)} P_s \mathcal{F}^R(\tau, \omega) e^{z(\tau)} d\tau.
\] (27)

Then by the similar discussion in [16], we have a local random Lipschitz invariant manifold for the random dynamical system \( \varphi^R(t, \omega) \) of the system (2) as follows
\[
M^R_{\text{cut}}(\omega) = \{ (\xi, e^{z(\omega)} h(\omega, e^{-z(\omega)} \xi)) : \xi \in H^\alpha_c \},
\] (28)
which is the graph of \( e^{z(\omega)} h(\xi, e^{-z(\omega)} \xi) \). In addition, define a mapping
\[
\psi(\omega, \cdot) : H_c \bigcap B_R(0) \rightarrow H^\alpha_s, \quad \xi \rightarrow \psi(\omega, \xi) = e^{z(\omega)} h(\xi, e^{-z(\omega)} \xi).
\] (29)

So, it is easy to obtain that
\[
\mathcal{M}^R(\omega) = \text{graph} (\psi(\omega, \cdot)) = \mathcal{M}^R_{\text{cut}}(\omega) \bigcap B_R(0),
\] (30)
is a local Lipschitz random invariant manifold of the random dynamical system \( \varphi(t, \omega) \) for the system (2).

**Theorem 3.4.** Suppose \(-\mu < \eta < \lambda_s - \mu\) and condition (26). Then the random dynamical system \( \varphi(t, \omega) \) for the system (2) has a local Lipschitz random invariant manifold \( \mathcal{M}^R(\omega) \) for sufficiently small \( R > 0 \), which is defined as the graph of a random Lipschitz mapping \( h(\omega, \cdot) : H_c \rightarrow H^\alpha_s \)
\[
M^R_{\text{cut}}(\omega) = \{ (\xi, e^{z(\omega)} h(\omega, e^{-z(\omega)} \xi)) : \xi \in H^\alpha_c \}.
\] (31)

Next, we prove the attracting property of the random invariant manifold \( M^R_{\text{cut}}(\omega) \) for the system (11). According to the approach in [29], it is enough to obtain the cone invariant property in the following.

**Lemma 3.5.** Let \( v, \bar{v} \) be two solutions of the system (12)-(14). Fix \( \delta > 0 \) and define the cone
\[
K_\delta = \{ u \in H^\alpha : \| u \|_\alpha \leq \delta \| u_c \|_\alpha \}.
\] (32)
Suppose that \( R \) is sufficiently small such that
\[
\lambda_s \geq 2\lambda_s^{-1} L^2_\delta (1 + \frac{1}{2})^2 + 4R(1 + \delta),
\] (33)
and
\[
\lambda_s > 4\mu + 2\lambda_s^{-1} L^2_\delta (1 + \frac{1}{2})^2.
\] (34)
If \( v(t_0) - \bar{v}(t_0) \in K_\delta \), then \( v(t) - \bar{v}(t) \in K_\delta \) for all \( t \geq t_0 \).
Moreover, if \( v(t_0) - \bar{v}(t_0) \) is outside \( K_\delta \) at some time \( t_0 \), then
\[
\| v_s(t, \omega) - \bar{v}_s(t, \omega) \|^2 \leq \| u_0 - \bar{u}_0 \|^2 e^{-\frac{1}{2} \lambda_s t - 2z(\omega) + 2 \int_0^t \theta_s(\omega) d\tau}
\] (35)
for all \( t \in [0, t_0] \).

**Proof.** Firstly, let \( p = v_c - \bar{v}_c \) and \( q = v_s - \bar{v}_s \). Then we consider two systems
\[
\partial_t p = (\mu + z)p + P_s e^{-z} \mathcal{F}^R(e^z v) - P_s e^{-z} \mathcal{F}^R(e^z \bar{v}),
\] (36)
\[
p(0) = e^{z(\omega)} P_s (u_0 - \bar{u}_0) = e^{z(\omega)} (u_0c - \bar{u}_0c).
\] (37)
and
\[
\partial_t q = -A_s q + (\mu + z)q + P_s e^{-z} \mathcal{F}^R(e^z v) - P_s e^{-z} \mathcal{F}^R(e^z \bar{v}),
\] (38)
\[
q(0) = e^{z(\omega)} P_s (u_0 - \bar{u}_0) = e^{z(\omega)} (u_{0s} - \bar{u}_{0s}).
\] (39)
Taking inner with $A^* p$, we can obtain
\[
\frac{1}{2} \frac{d}{dt} \|p\|^2 \geq (\mu + z)\|p\|^2 - L_R(\|p\|^2 + \|p\|\|q\|). \tag{40}
\]

Taking inner with $A^* q$, we have
\[
\frac{1}{2} \frac{d}{dt} \|q\|^2 \leq -\|q\|^2 + (\mu + z)\|q\|^2 + \lambda_s \frac{1}{2} L_R(\|p\| + \|q\|)\|q\| + \delta^2 L_R(\|p\| + \|p\|\|q\|). \tag{41}
\]

We use the approach of proof of contradiction. Suppose that $v - \bar{v}$ is outside of $\mathcal{K}_\delta$ for some $t$, that is we have $\|q\|^2 = \delta^2 \|p\|^2$ for some $t$. Then from above inequalities, using Young's inequality, we can obtain
\[
\frac{1}{2} \frac{d}{dt} (\|q\|^2 - \delta^2 \|p\|^2) \leq -\lambda_s (\|q\|^2),
\]

which yields the desired cone invariance.

Secondly, let $p + q$ is outside the cone at time $t_0$, that is $\|q(t_0)\|_\alpha \geq \|p(t_0)\|_\alpha$. Then by the first result we can obtain $\|q(t)\|_\alpha > \|p(t)\|_\alpha$ for all $t \in [0, t_0]$. Then from (41), we yield
\[
\frac{1}{2} \frac{d}{dt} \|q\|^2 \leq -\|q\|^2 + (\mu + z)\|q\|^2 + \lambda_s \frac{1}{2} L_R(\|p\| + \|q\|)\|q\| + \delta^2 L_R(\|p\| + \|p\|\|q\|).
\]

Then due to condition (33) and by Poincaré inequality, we have
\[
\frac{d}{dt} (\|q\|^2 - \delta^2 \|p\|^2) \leq -\lambda_s (\|q\|^2), \tag{43}
\]

Then due to condition (34), we can obtain that
\[
\frac{d}{dt} \|q\|^2 \leq \frac{\lambda_s}{2} \|q\|^2 + (2\mu + z) + \frac{L_R^2(1 + \frac{1}{\delta})^2}{\lambda_s} - \lambda_s (\|q\|^2).
\]

Then according to comparison principle we can obtain for almost all $\omega$,
\[
\|q(t, \omega)\|^2 \leq e^{-\frac{\lambda_s}{2} + 2z(\theta, \omega)\|q(0, \omega)\|^2}
\]
\[
\leq e^{-\frac{\lambda_s}{2} + 2z(\theta, \omega)\|u(0, \omega) - \bar{u}(0, \omega)\|^2} e^{-2z(\omega)},
\]

where $\|q(0, \omega)\| = \|v_\delta(0, \omega) - \bar{v}_\delta(0, \omega)\| \leq \|u(0, \omega) - \bar{u}(0, \omega)\| e^{-z(\omega)}$.

The cone invariant property obtained in above Lemma coincides to the existence of random attractor in state space. In order to obtain our main results, we need to give the following lemma, which is similarly obtained as in [11].

**Lemma 3.6.** For any given $T > 0$, the following initial value problem
\[
\partial_t v_c = (\mu + z)v_c + P_c e^{-z} F(R)((v_c + v_s)e^z), \quad v_c(0) = \xi \in H_c; \tag{46}
\]
\[
\partial_t v_s = (-A_s + (\mu + z)v_s + P_s e^{-z} F(R)((v_c + v_s)e^z), \quad v_s(0) = \kappa^c(v_c(0)) \tag{47}
\]
has a unique solution $(v_c(t), v_s(t)) \in C(0, T; H_c \times H_s^0)$, which lies on the manifold $\mathcal{M}^R_{\text{cut}}(\theta, \omega)$ a.a..
Furthermore, according to the contradicting discussion in [29], the attracting property of the random invariant manifold for corresponding random dynamical system can be proved as follows.

**Theorem 3.7.** Assume that conditions (26), (33) and (34) hold. For any solution $u(t, \omega)$ of truncated system (11), there is one orbit $U(t, \omega)$ on $M^{R}_{cut}(\theta_{t}\omega)$ with $P_{c}U(t, \omega)$ solves the equation

$$
\partial_{t}u_c = (\mu + z)u_c + P_{c}F^{(R)}(u_c + e^{z(\theta_{t}\omega)}h(\theta_{t}\omega, z^{-z(\theta_{t}\omega)}u_c)) + \sigma u_c \circ \dot{W}(t)
$$

such that

$$
\|u(t, \omega) - U(t, \omega)\|_{\alpha} \leq D(t, \omega)\|u(0, \omega) - U(0, \omega)\|_{\alpha}e^{-\lambda_{c}t}
$$

where $D(t, \omega)$ is a tempered increasing process defined by

$$
D(t, \omega) = e^{-z(\theta_{t}\omega)} + \int_{0}^{t} z(\theta_{s}\omega)ds.
$$

Therefore, from the proceeding theorem, due to our transformation process and random invariant manifolds obtained former, the main result is directly obtained in this section.

**Theorem 3.8.** Assume that conditions (26), (33) and (34) hold, then the local Lipschitz random invariant manifold $M^{R}(\omega)$ is locally exponentially attracting almost surely in the small ball $B_{R}(0)$. That is for any $\|u_{0}\|_{\alpha} < R$, the distance

$$
dist(\varphi(t, \omega)u_{0}, M^{R}(\theta_{t}\omega)) \leq 2RD(t, \omega)e^{-\lambda_{c}t}
$$

for all $t < \tau_{0}(\omega) = \inf\{t > 0 : \varphi(t, \omega)u_{0} \notin B_{R}(0)\}$ with $D(t, \omega)$ is a tempered increasing process.

In addition, since the results above are obtained under the condition of a cut-off function for corresponding local system about (2), then we can easily and directly obtain the corresponding result for original system.

**Corollary 1.** There is a local random invariant manifold $M(\omega)$ for system (2) in a small ball $B(0, R)$ with $R$ and $\mu$ satisfy conditions (26), (33) and (34). Moreover for $\|u_{0}\|_{\alpha} < R$ we have

$$
dist(\varphi(t, \omega)u_{0}, M(\omega)) \leq 2RD(t, \omega)e^{-\lambda_{c}t}
$$

for all $t \in [0, \tau_{0})$ where $\tau_{0} = \inf\{t > 0 : \varphi(t, \omega)u_{0} \notin B_{R}(0)\}$.

**Remark 1.** Here $\mu$ is indeed to satisfy the conditions (26), (33) and (34), because it is taken in $0 < \mu \leq 4$, which is not optimization according to above Corollary. But it does not impact our discussion of the main results.

### 4. Approximation of the random invariant manifold

In this section, we will consider the approximation and local geometric shape of the random invariant manifold $M^{R}(\omega) = \{(\xi, e^{z(\omega)}h(\omega, e^{-z(\omega)}\xi)) : \xi \in H^{\alpha} \cap B_{R}(0)\}$. If $v^{*}$ in $C_{\eta}$, then from Section 3, it is known that

$$
v^{*}_{s}(t, \omega; \xi) = \int_{-\infty}^{t} e^{(-A_{s} + \mu)(t-\tau) + \int_{\tau}^{t} z(\tau)d\tau}e^{-z(\tau)}P_{s}F^{(R)}(e^{z(\tau)}v^{*}(\tau))d\tau,
$$

and $v^{*}_{s}(0, \omega; \xi) = h(\omega, \xi)$. First, some properties of $v^{*}(\xi)$ can be obtained.
Lemma 4.1. Suppose $\lambda_s > \mu + \eta > 0$. Then there is a constant $C > 0$ such that for all $\xi \in H^s_\infty$

\[
\|v^*(\xi)\|_{C^s} \leq C\|\xi\|_\alpha; \|v^*_c(\xi)\|_{C^s} \leq CR^2\|\xi\|_\alpha; \tag{53}
\]

\[
\|v^*_c(\xi)\|_{C^s} \leq C\|\xi\|_\alpha; \|v^*(\xi_1) - v^*(\xi_2)\|_{C^s} \leq C\|\xi_1 - \xi_2\|_\alpha; \tag{54}
\]

Proof. Notice that

\[
\|J(0)(\xi)\|_{C^s} = \sup_{t \leq 0} e^{nt-f_0^t(z(r))dr}\|J(0)(\xi)\|_\alpha \leq \sup_{t \leq 0} e^{nt-f_0^t(z(r))dr} e^{\mu t+\int_0^t z(r)dr}\|\xi\|_\alpha \tag{55}
\]

It is obtained that the $J(0)(\xi)$ is bounded. Due to the contraction of $J$, there is

\[
\|v^*\|_{C^s} \leq \|J(v^*) - J(0)\|_{C^s} + \|J(0)\|_{C^s} \leq Lip(J)\|v^*\|_{C^s} + \|\xi\|_\alpha. \tag{56}
\]

So the first claim yields

\[
\|v^*\|_{C^s} \leq \frac{1}{1 - Lip(J)}\|\xi\|_\alpha. \tag{57}
\]

For the second bounded result, using semigroup estimates properties, it is obtained that

\[
\|v^*_c\|_\alpha = \|\int_{-\infty}^t e^{(-A_s+\mu)(t-\tau)+\int_0^\tau z(r)d\tau} e^{-\mu t+\int_0^t z(r)dr}P_s F^R(e^{z(\tau)}v^*_c(\tau))d\tau\|_\alpha \tag{58}
\]

Then there is

\[
\|v^*_c\|_{C^s} \leq C\|v^*(\tau)\|_{C^s} \leq C\|\xi\|_\alpha; \tag{59}
\]

\[
\|v^*_c\|_{C^s} = \|v - v^*_c\|_{C^s} \leq \|v\|_{C^s} + \|v^*_c\|_{C^s} \leq C\|\xi\|_\alpha. \tag{60}
\]

In addition, using the similar estimates, it is obtained that

\[
\|v^*(\xi_1) - v^*(\xi_2)\|_{C^s} \leq C\|\xi_1 - \xi_2\|_\alpha, \tag{61}
\]

\[
\|h^*(\xi_1) - h^*(\xi_2)\|_{C^s} \leq C\|\xi_1 - \xi_2\|_\alpha. \tag{62}
\]

Now, the main estimates about random invariant manifold will be given step by step as follows.

Firstly, define

\[
g_1(t) = \int_{-\infty}^t e^{(-A_s+\mu)(t-\tau)+\int_0^\tau z(r)d\tau} e^{-\mu t+\int_0^t z(r)dr} P_s F^R(e^{z(\tau)}v^*_c(\tau))d\tau. \tag{63}
\]

Lemma 4.2. Suppose $\lambda_s > \mu + \eta > 0$. Then there exists a positive constant $C$ such that

\[
\|v^*_s(\xi)(t) - g_1(t)\|_{C^s} \leq CLR\|\xi\|_\alpha. \tag{64}
\]
Proof. From Lemma 4.1, and semigroup estimates properties (3)-(4), there is
\[ \|v^*_s(t) - g_1(t)\|_{C^2_\alpha} \]
\[ = \| \int_{-\infty}^{t} e^{(\mu - \lambda_s)(t-\tau) - z(\tau) + f^*_v z(\tau) d\tau} [P_s F(R)(e^{z(\tau)} \nu_\tau^*(\tau)) - P_s F(R)(e^{z(\tau)} \nu_\tau^*(\tau))] d\tau \|_{C^2_\alpha} \]
\[ \leq M_{LR} \sup_{t \leq 0} e^{\eta t - \int_{-\infty}^{t} z(\tau) d\tau} \int_{-\infty}^{t} \frac{1}{(t - \tau)^{\alpha}} e^{(\mu - \lambda_s)(t-\tau) + f^*_v z(\tau) d\tau} \| v^*_s(\tau) - v^*_s(\tau) \|_\alpha d\tau \]
\[ \leq M_{LR} \sup_{t \leq 0} \| v^*(\tau) - v^*_s(\tau) \|_{C^2_\alpha} \int_{-\infty}^{t} \frac{1}{(t - \tau)^{\alpha}} e^{(\lambda_s + \mu + \eta)(t-\tau) d\tau} \]
\[ \leq C_{LR} \Gamma(1 - \alpha) \| v^*_s(\tau) \|_{C^2_\alpha} \leq C_{LR} \| \xi \|_\alpha. \]
Then the result of this Lemma is obtained. \( \square \)

Therefore, from Lemma 4.2, it is easily obtained that
\[ \| v^*_s(0) - g_1(0)\| \leq C_{LR} \| \xi \|_\alpha. \] (65)

Secondly, define
\[ g_2(t) = \int_{-\infty}^{t} e^{(\lambda_s - \mu)(t-\tau) + f^*_v z(\tau) d\tau} e^{-z(\tau)} P_s F(R)(e^{z(\tau)} e^{\mu \tau + f^*_v z(\tau) d\tau} \xi) d\tau. \] (67)
Then there is
\[ g_2(0) = \int_{-\infty}^{0} e^{(\lambda_s - \mu)\tau + f^*_v z(\tau) d\tau} e^{-z(\tau)} P_s F(R)(e^{z(\tau)} e^{\mu \tau + f^*_v z(\tau) d\tau} \xi) d\tau. \] (68)

From \( v^*(t) \), the following estimate is obtained
\[ \| v^*_s(\tau) - \xi e^{\mu \tau + f^*_v z(\tau) d\tau} \|_{C^2_\alpha} \leq C_{LR} \| v^*_s(\tau) \|_{C^2_\alpha} \int_{-\infty}^{0} e^{(\mu + \eta)(t-\tau) d\tau} \]
\[ \leq C_{LR} \| v^*_s(\tau) \|_{C^2_\alpha} \leq C_{LR} \| \xi \|_\alpha. \] (69)
which will be used in the following estimates.

**Lemma 4.3.** Suppose \( \lambda_s > \mu + \eta > 0 \). Then there exists a positive constant \( C \) such that
\[ \| g_1(0) - g_2(0)\| \leq C \| \xi \|_\alpha. \] (70)

**Proof.** Notice that
\[ \| g_1(t) - g_2(t)\|_{C^2_\alpha} \]
\[ = \| \int_{-\infty}^{t} e^{(\mu - \lambda_s)(t-\tau) - z(\tau) + f^*_v z(\tau) d\tau} P_s F(R)(e^{z(\tau)} v^*_s(\tau) - \xi e^{z(\tau)} \mu \tau + f^*_v z(\tau) d\tau) d\tau \|_{C^2_\alpha} \]
\[ \leq M_{LR} \sup_{t \leq 0} e^{\eta t - \int_{-\infty}^{t} z(\tau) d\tau} \int_{-\infty}^{t} \frac{1}{(t - \tau)^{\alpha}} e^{(\mu - \lambda_s)(t-\tau) + f^*_v z(\tau) d\tau} \]
\[ \cdot \| v^*_s(\tau) - \xi e^{\mu \tau + f^*_v z(\tau) d\tau} d\tau \|_\alpha d\tau \]
\[ \leq M_{LR} \sup_{t \leq 0} \| v^*_s(\tau) - \xi e^{\mu \tau + f^*_v z(\tau) d\tau} d\tau \|_{C^2_\alpha} \int_{-\infty}^{t} \frac{1}{(t - \tau)^{\alpha}} e^{(\lambda_s + \mu + \eta)(t-\tau) d\tau} \]
\[ \leq C_{LR} \Gamma(1 - \alpha) \| v^*_s(\tau) - \xi e^{\mu \tau + f^*_v z(\tau) d\tau} d\tau \|_{C^2_\alpha} \leq C_{LR} \| \xi \|_\alpha. \] (71)
Then from above inequality, there is
\[ \|g_1(0) - g_2(0)\| \leq C\|\xi\|_\alpha. \]  
(72)
Then the result holds. \(\square\)

Thirdly, define
\[ g_3(t) = \int_{-\infty}^{t} e^{-A_\tau t + \int_0^\tau z(r)dr} e^{-z(t)} P_\tau F(e^{z(t)} e^\mu + \int_0^\tau z(r)dr) d\tau \]  
(73)
So, there is
\[ g_3(0) = \int_{-\infty}^{0} e^{A_\tau t + \int_0^\tau z(r)dr} e^{-z(t)} P_\tau F(e^{z(t)} e^\mu + \int_0^\tau z(r)dr) d\tau \]
\[ = - \int_{-\infty}^{0} e^{A_\tau t + \int_0^\tau z(r)dr} e^{-z(t)} e^{3z(t)} e^{3\mu + 3 \int_0^\tau z(r)dr} \xi \]  
\[ = - e^{2z(0)} \xi \int_{-\infty}^{0} e^{A_\tau t + 2\mu + 2 \int_0^\tau z(r)dr} \xi d\tau. \]  
(74)

Now, the key Lemma is given, which is a main result property of a stochastic transform process used in present paper.

**Lemma 4.4.** (See [3]) There is a random variable \(K_1(\omega)\) such that \(K_1(\omega) - 1\) has a standard exponential distribution and
\[ \int_0^t z(t) dt + z(t) = z(0) + \sigma \omega(t) \leq \sigma (K_1(\omega) + |t|), \]  
for all \(t \leq 0\), \(z\) satisfies equation (5). Also,
\[ |\omega(t)| \leq \max\{|\omega(t)|, -\omega(t)|\} \leq K_1^\pm(\omega) + |t|, \]  
for all \(t \leq 0\), \(K_1^\pm(\omega) = K_1(\omega) + K_1(\omega)\) and \(K_1(\omega)\) has the same law as \(K_1(\omega)\). Furthermore, a similar estimate is true for \(|z(0)|\).

Define
\[ K_2(\omega) = \sup_{\tau \leq 0} \left| \frac{1 - e^\mu + \sigma \omega(\tau)}{e^{\delta |\tau|}} \right|, \]  
(77)
where \(\gamma\) and \(\delta\) are positive constants satisfying \(\gamma \geq \max\{\mu, \sigma\}\) and \(\delta > \mu + \sigma\).

Using \(|1 - e^{st}| \leq |s||t|\), one gets
\[ K_2(\omega) = \sup_{\tau \leq 0} \left| \frac{1 - e^{\mu} \sigma \omega(\tau)}{e^\delta |\tau|} \right| \leq \sup_{\tau \leq 0} \left| \frac{\mu |\tau| + |\sigma \omega(\tau)| e^{\mu |\tau| + |\sigma \omega(\tau)| - \delta |\tau|}}{\gamma} \right| \]
\[ \leq C e^{\sigma K_1^\pm(\omega)} \sup_{\tau \leq 0} |\tau| + K_1^\pm(\omega)) e^{(\mu + \sigma - \delta) |\tau|} \leq C e^{\sigma K_1^\pm(\omega)} (1 + K_1^\pm(\omega)), \]  
(78)
because the term of the right side of above inequality have an exponentially decreasing function. Now, the following estimates are given.

**Lemma 4.5.** Let \(0 < \sigma < \frac{\lambda + \mu}{3}\) and \(e^{z(0)}\|\xi\|_{\alpha} \leq R\). There exists a positive constant \(C\) such that
\[ \|g_2(0) - g_3(0)\| \leq C e^{-z(0)} 2^{\sigma K_1(\omega)} K_2(\omega) \|\xi\|_{\alpha}^2, \]  
(79)
Let \( \sigma < \lambda \). Similar to the process of the former, as proof.

Then there is a constant \( \sigma \).

Proof. Then the inequality (81) is obtained.

Then, define

\[
g_4(t) = e^{2z(0)} \int_{-\infty}^{t} e^{-A_s(t-\tau)-2\mu(t-\tau)} d\tau (-\xi_3^s). \tag{83}
\]

Then there is

\[
g_4(0) = e^{2z(0)} \int_{-\infty}^{0} e^{A_s \tau + 2\mu \tau} d\tau \xi_3^s = e^{2z(0)} (A_s + 2\mu)^{-1} (-\xi_3^s). \tag{84}
\]

Lemma 4.7. Let \( 0 < \sigma < \frac{\lambda + 2\mu}{2} \) and \( e^{z(0)} \| \xi \|_\alpha \leq R \). There exists a positive constant \( C \) such that

\[
\| g_3(0) - g_4(0) \| \leq C K_3(\omega) \| \xi \|_\alpha. \tag{85}
\]
Lemma 4.9. Let $g_4(0)$ and $g_5(0)$, it is obtained that
\[
\|g_3(0) - g_4(0)\|
\leq \|e^{2z(0)} \int_{-\infty}^{0} e^{A_s\tau + 2\mu\tau} \xi_3^3 d\tau - e^{2z(0)} \int_{-\infty}^{0} e^{A_s\tau + 2\mu\tau} d\tau \xi_3^3\|
=e^{2z(0)} \int_{-\infty}^{0} e^{A_s\tau + 2\mu\tau} [e^{2\sigma\omega(\tau)} - 1] \xi_3^3 d\tau
\leq M\gamma_1 L_R K_3(\omega)\|\xi\| \alpha \int_{-\infty}^{0} \frac{1}{(-\tau)^\alpha} e^{(\lambda_2 + 2\mu - 2\delta_1)\tau} \leq C K_3(\omega)\|\xi\| \alpha, \tag{86}
\]
where
\[
K_3(\omega) = \sup_{\tau \leq 0} \left| \frac{1 - e^{2\sigma\omega(\tau)}}{\gamma_1 e^{2\delta_1|\tau|}} \right| \leq C e^{2\sigma K^2(\omega)} (1 + K^2(\omega)), \tag{87}
\]
here $\gamma_1 > \sigma, \delta_1 > \sigma$.

Then there is
\[
ge_5(t) = e^{2z(t)} \int_{-\infty}^{t} e^{-A_s(\tau - \tau)} d\tau (-\xi_3^3). \tag{88}
\]
Finally, define
\[
ge_5(0) = e^{2z(0)} \int_{-\infty}^{0} e^{A_s\tau} d\tau (-\xi_3^3) = e^{2z(0)} A_s^{-1}(\xi_3^3). \tag{89}
\]

Lemma 4.8. Let $\lambda_2 - 2\mu > 0$ and $e^{z(0)}\|\xi\| \alpha \leq R$. There exists a positive constant $C$ such that
\[
\|g_4(0) - g_5(0)\| \leq C\|\xi\| \alpha. \tag{90}
\]

Proof. From the representation of $g_4(0)$ and $g_5(0)$, similar to former, it is obtained that
\[
\|g_4(0) - g_5(0)\|
\leq \|e^{2z(0)} \int_{-\infty}^{0} e^{A_s\tau + 2\mu\tau} d\tau \xi_3^3 - e^{2z(0)} \int_{-\infty}^{0} e^{A_s\tau + 2\mu\tau} d\tau \xi_3^3\|
\leq \|e^{2z(0)} \int_{-\infty}^{0} e^{A_s\tau} [e^{2\sigma\omega(\tau)} - 1] \xi_3^3 d\tau\|
\leq M L_R \|\xi\| \alpha \int_{-\infty}^{0} \frac{1}{(-\tau)^\alpha} e^{\lambda_2(1 - e^{2\mu\tau})} d\tau
\leq C\|\xi\| \alpha. \tag{91}
\]
The proof is completed.

Lemma 4.9. Let $\lambda_2 > \mu + \eta > 0$, $0 < \sigma < \min\{\frac{\lambda_2 + \mu}{4}, \frac{\lambda_2 + 2\mu}{2}\}$, $\lambda_2 - 2\mu > 0$ and $e^{z(0)}\|\xi\| \alpha \leq R$. There exists a positive constant $C$ such that
\[
\|h(\omega, \xi) - g_5(0)\| \leq C [1 + K_3(\omega) + e^{z(0)} e^{2\sigma K_1(\omega)} K_2(\omega)\|\xi\| \alpha] \|\xi\| \alpha. \tag{92}
\]
Therefore, according the inverse transform $T^{-1}$, the main result is immediately obtained.
Theorem 4.10. (Shape of random invariant manifold) Let \( \lambda_3 > \mu + \eta > 0 \), \( 0 < \sigma < \min\{\frac{\lambda_3 + \mu}{3}, \frac{\lambda_3 + 2\mu}{2}\} \), \( \lambda_3 - 2\mu > 0 \) and \( e^{\zeta(\omega)}\|\xi\|_\alpha \leq R \). There exists a positive constant \( C \) such that

\[
\|e^{\zeta(\omega)}h(\omega, e^{-\zeta(\omega)}\xi)) - A_3^{-1}(e_3^3)\| \leq Ce^{\zeta(\omega)}[1 + \|\xi\|_\alpha]\|\xi\|_\alpha
\]

holds with probability larger than \( 1 - Ce^{-\frac{1}{2}} \). Therefore in a neighborhood of zero for equation (2), the graph \( (\xi, e^{\zeta(\omega)}h(\omega, e^{-\zeta(\omega)}\xi)) \) of the invariant manifold \( M_R^R(\omega) \) is approximately given by \( (\xi, A_3^{-1}(e_3^3)) \) with probability larger than \( 1 - Ce^{-\frac{1}{2}} \).

Proof. Due to \( \Omega_K = \{\omega \in \Omega | K^\pm(\omega) > \frac{1}{\sigma}\} \). By the Lemma this set has probability less than \( Ce^{-\frac{1}{2}} \). So, there exists a positive \( C \) on the complement \( \Omega_K^c \) such that \( K_1(\omega), K_3(\omega), K_3(\omega) \) are bounded. Then, from Lemma 4.8 we can obtain the theorem.

In present paper, it is enough to give the small positive \( 0 < \mu < 4.5 \) in a special case taking \( \lambda_3 = 9 \) for the approximation of random invariant manifolds because of \( H = \text{span}\{e_1(x) = \sin x\} \). If we write \( \xi = r \cdot e_1 \), then the invariant manifold \( M_R^R(\omega) \) is locally given in high probability as the graph of

\[
A_3^{-1}(e_3^3) = r^3A_3^{-1}(e_3^3) = r^3v_s \in H_s.
\]

Therefore, \( M_R^R(\omega) \) lies approximately in the plane spanned by \( e_1 \) and \( v_s \), and it can be described by a cubic curve, unless \( v_s = 0 \). Then we can obtain the approximation manifolds above, and the graph of it can be given in the plane in this special case.

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