Optical qudit-type entanglement creation at long distances by means of small cross-Kerr nonlinearities

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Entanglement represents an important resource for quantum information processing, but its generation itself requires physical resources that are limited. We propose a scheme for generating a wide class of entangled qudit-type states of optical field modes at sites separated by noisy medium when only weak optical nonlinearities are available at both sites. The protocol is also based on exploiting a weak probe field, transmitted between the sites and used for generation of quantum correlations between two spatially separated field modes. The idea of probabilistic entanglement enhancement by measurement is discussed, and corresponding scheme for measuring the probe field state with linear optics and photodetectors not resolving photon numbers is proposed. It is shown that the protocol is applicable in the case when decoherence, limited efficiency and dark counts of photodetectors, and uncertainty of nonlinear coupling constants are present.

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I. INTRODUCTION

One of the most intriguing features of quantum mechanical description of physical objects consists in presence of quantum superpositions and especially of entangled states [1]. At the very beginning of the quantum theory development such states were found to possess quite counterintuitive (from the classical point of view) properties [2-3]. The 20th century the attitude of physicists to such quantum states evolved from perceiving them as either evidences of theory incompleteness or interesting but quite useless features of the world [4-5] to understanding the opportunities provided by such quantum objects for solving numerous tasks of information processing [6-11]. It has been shown that quantum no-cloning theorem [12, 13] provides unconditional security of quantum cryptography protocols [6-8, 14-16], while using entangled states of quantum register can lead to essential speed-up of solving several classes of complicated computational tasks [7-11, 14-17, 18].

From this point of view entanglement represents an important resource for different information tasks. On the other hand, entanglement itself requires some physical resources for its generation and, thus, can be considered as an intermediate step on the way from physical devices to accomplishing tasks of information processing. Similarly to many other situations, the resources available nowadays for entanglement generation are limited, and one of the most important problems is to find methods for obtaining results as good as possible using as few resources as possible.

Among systems, being promising for efficient entanglement generation, optical field modes take their place due to possibility of long-distance transmission with relatively low decoherence and quite simple manipulation of the states with linear optics and photodetectors. Considering optical entanglement generation, one can divide necessary elements into two classes: quite simple elements, available "freely" (linear optical devices, photodetectors not resolving photon number, classical optical states), and resources — all other elements, being quite challenging for construction. It is impossible to generate entanglement with the first group of elements only (see e.g. Refs. [19-21]), and, therefore, some resources are necessarily required. The main resource, used for entanglement generation by optical methods, is nonlinearity, either measurement-induced [22, 23] or provided by interaction between the field modes in some medium. The corresponding physical resources are special detectors (e.g. resolving photon number [22, 23]) and nonlinear media respectively. Entangled states can be created also by means of linear optics from nonclassical states, which in their turn require nonlinearity for their generation.

Many quantum information processing and communication tasks (quantum cryptography, distributed quantum computation, teleportation of quantum states) require distribution of entanglement between parties, separated by noisy medium. For such tasks a high-quality quantum channel also becomes an important physical resource. So, the two main resources, required for generating entanglement between distant states with optical methods, are nonlinear interaction and a quantum channel. Therefore, efficient entanglement generation corresponds in this case to creating strongly entangled states with weak nonlinear interactions and noisy quantum channels (only such systems are available nowadays).

In present paper we consider weak local cross-Kerr interaction as a resource for creating nonlocal nonclassical states of spatially separated optical field modes. Certain progress has already been achieved in this field of research [24-31], especially in the case of creating qubit-like entangled states. However, it remained quite challenging to create more general classes of entangled states with such limited resources. We propose a protocol for creating wide class of qudit-type states (including entangled states) with arbitrary dimensionality in continuous variable (optical) system using weak cross-Kerr nonlin-
earty (as the main physical resources), linear beamsplitters, detectors not resolving photon numbers, and sources of coherent states. We show that entanglement of the states, created with our protocol, can be higher than unity (which is the limit for qubit-type states, that can be created by previously proposed methods) and, thus, our protocol provides more effective use of limited physical resources.

This paper is organized as follows. In the next section we discuss the main ideas of entanglement generation between distant sites when local nonlinear interaction and non-ideal quantum channel are available. Then the main operations of the proposed protocol and corresponding state transformations are presented. Section IV is devoted to discussion of the peculiarities of projecting the "raw" weakly entangled system state onto strongly entangled desired final state. It is shown that parameters of the protocol are determined in the unique way by the desired final state, and corresponding relations are found. In Section V we demonstrate several applications of the protocol to creation of nonclassical and entangled states of optical field modes. In the last section we prove applicability of the protocol for entanglement generation under realistic conditions by taking into account decoherence, limited efficiency and dark counts of photodetectors, and uncertainty of nonlinear coupling constants.

II. ENTANGLEMENT GENERATION WITH LOCAL CROSS-KERR NONLINEARITY

Cross-Kerr interaction itself can be used for generating entangled states starting from uncorrelated states of a pair of quantum objects. Suppose a field mode \( \hat{c} \) in a coherent state \( |\gamma\rangle_c \) interacts with another system (another optical field mode \( \hat{a} \) or an atomic system [24, 28, 34]). Due to the interaction the phase of the coherent state amplitude \( \gamma \) of the mode \( \hat{c} \) is shifted by the value, proportional to the number of excitations \( n \) of the system \( \hat{a} \):

\[
|n\rangle_a |\gamma\rangle_c \rightarrow |n\rangle_a |\gamma e^{i\chi n}\rangle_c,
\]

where \( \chi \) describes effective strength of the interaction. If the initial state of the object \( \hat{a} \) is a superposition of states with different excitation numbers (e.g. a coherent state \( |n\rangle_a \) in the case of field mode), the final state of the considered system will be an entangled state, composed by pairwise combinations of coherent states with different phases of mode \( \hat{c} \) and number states of the system \( \hat{a} \) (Fig. I(a)).

However, experimentally observed nonlinear interactions are quite weak [25, 35]: the effective nonlinearity strengths, predicted in the most promising 4-level atomic systems with electromagnetically induced transparency on the basis of theoretical calculations and experimental data, have the order of \( \chi \approx 10^{-3} \approx 10^{-2} [36, 42] \). The magnitude of phase space displacement caused by such cross-Kerr interaction is proportional to \( |\gamma\chi| \), and in general case the final state can be weakly entangled. One of the solutions of the problem consists in effective nonlinearity enhancement by using intense fields \( \hat{c} \): for \( |\gamma| \gg 1 \) the displacement magnitude can be large enough \(|\gamma|\chi \approx 1 \), while the amplitude of field \( \hat{a} \) or the number of excitations in the atomic system is also limited by losses in local storage. Therefore, this simple scheme is not applicable for generating strongly entangled states when the sites are separated by noisy medium. Only weak entanglement can be generated. On the other hand, if we take into account not only limited available nonlinearity, but also noisy medium between the sites (Fig. I(b)), then decoherence strongly limits maximal possible amplitudes of the transmitted field \( \hat{c} \) (\( |\gamma| \ll 1 \)), while the amplitude of field \( \hat{a} \) or the number of excitations in the atomic system is also limited by losses in local storage. Therefore, this simple scheme is not applicable for generating strongly entangled states when the sites are separated by noisy media. Hence, under realistic conditions a more sophisticated scheme, including some kind of entanglement enhancement, is required.

One way of obtaining strongly entangled quantum state is to implement entanglement distillation [12, 16]. This approach requires storage of quite a large number of initial weakly entangled states and can be quite chal-
lenging. Another solution of the problem can be based on probabilistic entanglement enhancement [27, 28, 30, 31]. For this purpose one can design certain measurement, carried out at Bob’s site, with successful outcome transforming initial weakly entangled state into a strongly entangled state. Direct measurement of the state of field mode \( \hat{c} \) makes this mode inaccessible for any further use.

Measurements, implemented with linear optics and photodetectors not resolving photon number on field modes, obtained by splitting the mode \( \hat{c} \), completely determine the state of the mode and destroy entanglement. Therefore, some kind of nonlinearity is required at Bob’s site, too. It is quite natural to suppose that this nonlinear interaction is the same as the one at Alice’s site (see e.g. Refs. [27, 28, 30, 31]).

Several schemes, based on probabilistic entanglement enhancement by a measurement at Bob’s site, have already been proposed for entangling distantly separated field modes, when effective strength of nonlinear interactions \( \chi \) is equal to \( \pi \) (strong nonlinear interaction) [27], and for entangling atomic qubits [28] as well as for creating qubit-type entangled states of optical field modes [30, 31], when only small nonlinearity is available.

In present paper we solve a more general task of designing the measurement scheme for creation of arbitrary qutrit-type state (including entangled states) from a wide class of possible states in continuous variable system using weak cross-Kerr nonlinearity, linear optical devices, detectors and sources of coherent states. The set of achievable final states of the field modes \( \hat{a} \) and \( \hat{b} \) has the form of a sum of phase-correlated pairs of coherent states of the modes:

\[
|\Psi_f\rangle_{ab} = \sum_n c_n |\alpha e^{i\chi_n} \rangle_{\hat{a}} |\beta e^{i\chi_n} \rangle_{\hat{b}},
\]

where coefficients \( c_n \) are arbitrary and can be fixed in an appropriate way for obtaining the state, most useful for certain practical applications.

### III. OPERATIONS OF THE PROTOCOL

We consider the following system (see Fig. 2): Alice and Bob possess local field modes \( \hat{a} \) and \( \hat{b} \) referred to below as main field modes; the probe beam (ancillary mode) is denoted as mode \( \hat{c} \); mode \( \hat{d} \) is a reference field, transmitted from Alice to Bob immediately before (or after) ancillary field for decreasing influence of dephasing in the quantum channel on the final states.

Main field modes, the ancillary and the reference fields are prepared in coherent states \( |\alpha\rangle_\hat{a} \) for Alice’s mode \( \hat{a} \), \( |\beta\rangle_\hat{b} \) for Bob’s mode \( \hat{b} \), \( |\gamma\rangle_\hat{c} \) for the probe field \( \hat{c} \) and \( |\tilde{\gamma}\rangle_\hat{d} \) for the reference field \( \hat{d} \). The initial state of the system is therefore uncorrelated. Local cross-Kerr interaction of the modes \( \hat{a} \) and \( \hat{c} \) (with effective strength \( \chi \) — the phase of a coherent state of mode \( \hat{a} \) increases by \( \chi \) radians per each photon in \( \hat{c} \)) leads to generation of correlations between the number of photons in the mode \( \hat{c} \) and the phase of the coherent state of the mode \( \hat{a} \). Then the field \( \hat{c} \) is transmitted to Bob’s site through the quantum channel. In this section we suppose for simplicity that all correlations are preserved by the channel.

After local interaction of the modes \( \hat{b} \) and \( \hat{c} \) (the effective strength of the interaction is supposed in this section to be also equal to \( \chi \)), taking place afterwards, the phases of the main modes \( \hat{a} \) and \( \hat{b} \) become correlated with the number of photons in the mode \( \hat{c} \):

\[
|\Psi_1\rangle_{abc} = \sum_n Q_n(\gamma) |\alpha e^{i\chi_n} \rangle_\hat{a} |\beta e^{i\chi_n} \rangle_\hat{b} |n\rangle_\hat{c},
\]

where \( Q_n(\gamma) = \frac{\gamma^n}{\sqrt{n!}} e^{-|\gamma|^2/2} \). However, the correlations, generated in the system, are weak in a realistic case. For small ancillary field amplitudes \( |\gamma|^2 \ll 1 \), required for decreasing decoherence in the quantum channel [27, 28, 31], the entanglement between the states of the mode \( \hat{a} \) possessed by Alice and the modes \( \hat{b} \) and \( \hat{c} \) possessed by Bob is much less than unity:

\[
E_1 \simeq h \left( \gamma^2 |\alpha|^2 |\gamma|^2 \right) \ll 1,
\]

where \( h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x) \).

As discussed in the previous section, the last stage of the protocol consists in detection of the probe beam state. This operation should lead (in the case of successful outcome) to transformation of the weakly correlated 3-modes state into a strongly correlated 2-modes state of the form (2).

The idea of obtaining the desired strongly correlated final state \( |\Psi_3\rangle_{abc} \) from the state (2) is based on the following decomposition:

\[
|\Psi_3\rangle_{abc} = |\Psi_f\rangle_{ab} \otimes |\varphi\rangle_\hat{c} + |\Psi_\perp\rangle_{abc},
\]

where \( |\Psi_\perp\rangle_{abc} \) denotes the part of the system state, orthogonal to \( |\varphi\rangle_\hat{c} \), and the vector \( |\varphi\rangle_\hat{c} \) is uniquely defined as

\[
|\varphi\rangle_\hat{c} = \text{const} \cdot \sum_n \left( c_n^* / Q_n(\gamma) \right) |n\rangle_\hat{c}.
\]

Thus, the transformation of the state (3) into the state (2) for arbitrary coefficients \( c_n \) can be realized by successful discrimination of the state \( |\varphi\rangle_\hat{c} \) from a set of ancillary mode states, containing \( |\varphi\rangle_\hat{c} \) and a complete system of states, orthogonal to \( |\varphi\rangle_\hat{c} \).

The most simple from theoretical point of view way of discriminating the state \( |\varphi\rangle_\hat{c} \) is implementing projective measurement described by operator \( \hat{P}_\varphi = |\varphi\rangle_\hat{cc} \langle \varphi| \) which satisfies the following relation:

\[
\hat{P}_\varphi \cdot |\Psi_3\rangle_{abc} = |\Psi_f\rangle_{ab} \otimes |\varphi\rangle_\hat{c}.
\]

However, implementation of such measurement without additional resources (e.g. nonlinear interaction) can be too complicated or impossible. Solution of this problem is discussed in the next section, and the discrimination technique, based on a special case of general class of POVM measurements — “elimination” measurements, — is proposed.
FIG. 2: Scheme of entanglement generation: Alice prepares entangled state of the field modes $\hat{a}$ and $\hat{\epsilon}$; then she sends the ancillary mode $\hat{c}$ and the reference mode $\hat{d}$ through the quantum channel to Bob; having correlated the states of the modes $\hat{b}$ and $\hat{c}$, Bob measures the final state of the ancillary field mode (with the help of the reference mode $\hat{d}$) and, in the case of successful outcome, announces Alice under the classical channel that entanglement was generated (otherwise the set of operations is repeated).

IV. DISCRIMINATION TECHNIQUE

A. "Elimination" measurements. General consideration

An important fact that can be used for designing required discrimination protocol is that coherent states represent a natural basis for linear optical devices (the set of coherent states is closed under linear optical transformations). From this point of view, it is convenient to base the discrimination scheme on comparing the state of the probe beam $\hat{c}$ with certain coherent states, obtained by splitting the reference mode $\hat{d}$. However, coherent states are nonorthogonal. It is this problem that leads to impossibility of implementing the discussed above projective measurement in a general case with linear optics. An important set of measurements, implementable with linear beamsplitters and photodetectors for coherent states, is the set of "elimination" measurements (see e.g. Refs. [28, 47] and Fig. 3(a) below) — the measurements with successful outcomes manifesting that the field mode state is not certain coherent state.

The concept of measurements with outcomes, manifesting that the input state is not certain fixed state, was proposed in Refs. [18, 54], as a part of unambiguous discrimination of two non-orthogonal states. The concept of "elimination" measurements was generalized in Ref. [47] for the case of discrimination between $N$ symmetric coherent states on a circle on phase plane with linear optical elements and photodetectors. In these papers the system was supposed to be prepared in one of the states from a fixed finite set (e.g. symmetric coherent states on a circle: $|\alpha e^{2\pi ik/N}\rangle$, $k = 0, ..., N - 1$). Discrimination of one of the states (e.g. $|\alpha\rangle$) is equivalent to elimination of all the remaining states from the fixed set (in the considered example $|\alpha e^{2\pi ik/N}\rangle$, $k = 1, ..., N - 1$).

Considering ancillary mode states in our protocol, one faces a more general situation. A complete set of independent states is infinite for a field mode, and the measurement does not necessarily coincide with certain basis vector, but can represent a superposition of basis vectors. Therefore, complete description of state transformations, occurring when elimination measurements are carried out, requires a more rigorous operator definition of the concept of elimination measurements.

We define measurement, eliminating state $|\psi\rangle$, as any POVM-type measurement (described by POVM $\{A_i\}$, $\sum_i [A_i]^{\dagger}A_i = 1$) with at least one of the outcomes (referred to in this paper as successful) being characterized by operator, denoted in further consideration as $A_{|\psi\rangle}$, with the property

$$\hat{A}_{|\psi\rangle}|\psi\rangle = 0.$$  \hspace{1cm} (8)

Having obtained this outcome, one can with certainty conclude that the measured state was not the state $|\psi\rangle$.

Considering a set of measurements eliminating states $\{|\psi_j\rangle\}$, we will require (simultaneously with Eq. 8 for all vectors $|\psi_j\rangle$) commutativity of operators, characterizing successful outcomes of the measurements:

$$[\hat{A}_{|\psi_j\rangle}, \hat{A}_{|\psi_i\rangle}] = 0.$$  \hspace{1cm} (9)

This condition means that successful elimination of the state $|\psi_i\rangle$ must not destroy the result of elimination of the state $|\psi_j\rangle$ for any pair of the states $|\psi_i\rangle$ and $|\psi_j\rangle$ from the considered set.

The conditions 8, 9 lead to the following important implication, useful for designing the required discrimination scheme. Let the set $\{|\psi_0\rangle, ..., |\psi_K\rangle\}$ be a (non-orthogonal) basis of finite-dimensional Hilbert space and operators $\hat{A}_{|\psi_0\rangle}$, ..., $\hat{A}_{|\psi_K\rangle}$ describe successful outcomes of corresponding elimination measurements. Any state $|\phi\rangle$ from the considered state space

$$|\phi\rangle = \phi_0|\psi_0\rangle + ... + \phi_K|\psi_K\rangle,$$  \hspace{1cm} (10)

subjected to successful elimination of the states $\{|\psi_1\rangle, ..., |\psi_K\rangle\}$, is transformed into the state

$$\hat{A}_{|\psi_K\rangle}...\hat{A}_{|\psi_1\rangle}|\phi\rangle = \phi_0\hat{A}_{|\psi_K\rangle}...\hat{A}_{|\psi_1\rangle}|\psi_0\rangle,$$  \hspace{1cm} (11)
where contribution of $|\psi_0\rangle$ only did not vanish. The final state of the system differs from the state $|\psi_0\rangle$, but it is a completely defined state and can be transformed unitarily into $|\psi_0\rangle$ (in our protocol the ancillary mode is finally discarded and this transformation is not necessary). Therefore, one can state that successful elimination of all the basis vectors except $|\psi_0\rangle$ leads to successful discrimination of the state $|\psi_0\rangle$.

Generalization of the definition to the case, when several operators $A_{\psi_j}^{(\mu)}$, $\mu=1, 2, ..., \mu\geq 1$, correspond to successful elimination of the state $|\psi_j\rangle$: $A_{\psi_j}^{(\mu)}|\psi_j\rangle = 0$, is straightforward: Eq. (10) retains its form and must be valid for all $A_{\psi_j}^{(\mu)}$, used instead of single operator $A_{\psi_j}$. The result, described by Eq. (11), also remains valid.

B. State discrimination with "elimination" measurements. From infinite-dimensional to finite-dimensional space

The described above technique can be used exactly for discrimination of the state $|\phi\rangle_c$ of the ancillary mode $c$ in a special case of finite-dimensional space of input states. Suppose that the final state of the whole system (modes $\hat{a}$, $\hat{b}$ and $c$) can be decomposed using finite number of ancillary field mode states, and these states together with the state $|\phi\rangle_c$ span $(K+1)$-dimensional space. Then we can choose $K$ such independent vectors $|\psi_j\rangle_c$, $j=1, ..., K$, orthogonal to $|\phi\rangle_c$, that the system $\{|\psi_1\rangle_c, ..., |\psi_K\rangle_c, |\phi\rangle_c\}$ is complete in corresponding subspace. As shown above, discrimination of the state $|\phi\rangle_c$ corresponds to successful elimination of the vectors $|\psi_j\rangle_c$, $j=1, ..., K$.

Such finite-dimensional case can be realized, for instance, when the nonlinearity strength is equal to $\chi = 2\pi/N$, where $N$ is an integer $\geq 27$. In these special case the states of the modes $\hat{a}$, $\hat{b}$ and $\hat{c}$ after the nonlinear interactions can be represented as a sum of $N$ terms consisting of coherent state of the form $|\gamma e^{\pm iNk}/N\rangle_c$, of the ancillary mode $\hat{c}$ and corresponding entangled state of modes $\hat{a}$ and $\hat{b}$. Discrimination of a fixed state $|\gamma e^{\pm iNk}/N\rangle_c$ (by elimination of $N-1$ coherent states $|\gamma e^{\pm iNk}/N\rangle_c$, $k \neq k_0$) maps initial weakly-entangled state onto strongly-entangled final state of the modes $\hat{a}$ and $\hat{b}$.

In general infinite-dimensional case, however, the state of the system cannot be decomposed in such a way and the described discrimination technique can be applied approximately. For the considered state $|\psi\rangle$ and for small ancillary field amplitudes $|\gamma| \ll 1$ high accuracy of exploiting the discrimination technique can be achieved by restricting consideration by a space of states with limited photon numbers. Suppose the expression (4) for the desired final state has $K+1$ nonzero terms, i.e. $c_n = 0$ for $n > K$. Then Eq. (10) for the state $|\phi\rangle_c$ also has $K+1$ nonzero terms and this state belongs to $(K+1)$-dimensional subspace of states with limited photons numbers $n \leq K$. According to Eq. (3), the probability of presence of more than $K$ photons in the mode $\hat{c}$ is proportional to $|\gamma|^{2(K+1)}$ and is small for the considered system. Therefore, the state $|\psi\rangle$ with probability close to unity belongs to the same $(K+1)$-dimensional subspace as the state $|\phi\rangle$:

$$\Psi_{abc} = \sum_{n=0}^{K} Q_{n}(\gamma) |\alpha e^{i\gamma n}\rangle_{a} |\beta e^{i\gamma n}\rangle_{b} |m\rangle_{c} +$$

$$+ |\delta\Psi_{1(K)}\rangle_{abc} = |\Psi_{1(K)}\rangle_{abc} + |\delta\Psi_{1(K)}\rangle_{abc},$$

where $||\delta\Psi_{1(K)}\rangle_{abc}|| = O \left( |\gamma|^{2(K+1)} \right)$. If we construct a measurement, which leads to correct discrimination of the state $|\phi\rangle_c$ for the subspace, spanned by the state vectors with photon numbers $n \leq K$, the distance between the obtained state (for the ideal system) and the desired final state $|\Psi_f\rangle$ will have the order not greater than

$$\frac{||\delta\Psi_{1(K)}\rangle_{abc}||^2}{\|A_{\psi_j}\|_{1(K)} - A_{\psi_j} |\Psi_{1(K)}\rangle_{abc}} = \frac{O \left( |\gamma|^{2(K+1)} \right)}{p_K}, \tag{13}$$

where $p_K$ is the probability of successful generation of the desired final state. Therefore, if the probability $p_K$ has the order at least $O \left( |\gamma|^{2K} \right)$, the error of exploiting the approximate discrimination technique will have the order $O \left( |\gamma|^2 \right)$ which is much smaller than the decrease of the final state fidelity caused by non-ideality of the quantum channel. Calculations below show that this condition is fulfilled for the considered system.

Thus, for generating the state $|\Psi_f\rangle_{abc}$ composed by the sum of $K+1$ phase correlated pairs of coherent states of modes $\hat{a}$ and $\hat{b}$, one needs to construct a scheme for elimination of $K$ vectors $|\psi_j\rangle_c$, $j=1, ..., K$.

C. Coherent states as the basis set for discrimination

The next step of designing the scheme for discrimination of the state $|\phi\rangle_c$ is construction of the set of independent states $\{|\psi_j\rangle_c\}$ in the form suitable for realizing elimination measurements with linear optical elements and photodetectors. Realization of such measurement for coherent states is known (see e.g. Ref. 47), therefore it is desirable that the basis states $|\psi_j\rangle_c$ be coherent states $|\gamma_j\rangle_c$ with amplitudes $\gamma_j$. In this case the orthogonality conditions have the form $\langle \phi | \gamma_j \rangle = \sum c_n \gamma_j^n / \left( Q_{n}(\gamma) \sqrt{n!} \right) = 0$. Then the coherent states amplitudes $\gamma_j$ can be found as $K$ roots of the $K$-th order equation

$$f(x) \equiv \sum_{n=0}^{K} c_n \left( \frac{x}{\gamma} \right)^n = 0 \tag{14}$$

and are uniquely defined for any given set $\{c_n\}$. These statements are correct in the nondegenerate case.
Presence of degenerate roots of Eq. (14) leads to linear dependence of the set \{\ket{\gamma_j}\}. In this case additional vectors must be added to the set to provide completeness. These vectors can be constructed in the following way. If the root \(\gamma_m\) has the multiplicity \(l_m > 1\), then for \(x = \gamma_m\) holds

\[
\frac{d^s f(x)}{dx^s} = 0, \quad 0 \leq s \leq l_m - 1.
\]

On the other hand,

\[
\frac{d^s f(\gamma_m)}{d\gamma_m^s} = \frac{d^s}{d\gamma_m^s} \left\{ \langle \varphi | \gamma_m \rangle_c e^{i\gamma_m^2/2} \right\}
= c \langle \varphi | (\hat{c}^\dagger)^s | \gamma_m \rangle_c e^{i\gamma_m^2/2}.
\]

Eqs. (15)–(16) prove orthogonality of the vectors \(\langle \gamma_m \rangle_c = (\hat{c}^\dagger)^s | \gamma_m \rangle_c\) to \(\langle \varphi \rangle_c\) for \(s \leq l_m - 1\). Instead of \(l_m\) copies of the vector \(\ket{\gamma_m}\) we obtain \(l_m\) vectors \(\ket{\gamma_m(s)}_c, s = 0, ..., l_m - 1\), that are (i) independent and (ii) orthogonal to \(\ket{\varphi}\).

Thus, in general case the set of vectors \(\ket{\gamma_m(s)}_c = (\hat{c}^\dagger)^s | \gamma_m \rangle_c\), where \(s = 0, ..., l_m - 1\) and index \(m\) enumerates distinct roots of Eq. (14), is the required set of \(K\) independent vectors, orthogonal to \(\ket{\varphi}\). This set of states can be constructed for any desired final state of the form \(\ket{\sum_{n \in \mathbb{N}} c_n | \psi_n \rangle}\) with finite number of terms and is uniquely defined for a fixed set of coefficients \(\{c_n\}\). Therefore, Eq. (14) provides the unique solution of the problem of generating any final state of the form \(\ket{\psi}\) with minimal exploited resources.

D. Implementation: nondegenerate case

In the nondegenerate case discrimination of the state \(\ket{\varphi}\) is based on quite a well known technique of eliminating coherent states (Fig. 3). For example, for eliminating a single coherent state \(\ket{\gamma_1}\), the measured state is displaced in phase space by the magnitude \(-\gamma_1\). The displacement is described by operator \(\hat{D}(\gamma_1) = \exp (-\gamma_1 \hat{c}^\dagger + \gamma_1 \hat{c})\). The displacement operator transforms coherent state \(\ket{\gamma_1}\) into the vacuum state, and detection of photons in the field mode after the displacement (Fig. 3(a)) manifests that the measured state was not \(\ket{\gamma_1}\). It should be noted that the considered detectors need not resolve photon numbers or be 100% efficient.

Elimination of a set of coherent states \(\{\ket{\gamma_1}\}_c\) can be carried out in a similar way by splitting the field mode \(\hat{c}\) into \(K\) modes (Fig. 3(b)). Each of the obtained modes is used for eliminating one of the coherent states \(\ket{\gamma_1}\) into \(K\) modes (Fig. 3(b)). In this case obtaining photocounts ("clicks") from all the detectors corresponds to successful outcomes of elimination of all the vectors \(\{\ket{\gamma_1}_j, j = 1, ..., K\}\) and, thus, to successful discrimination of \(\ket{\varphi}\).

The displacement operators can be effectively realized by mixing the field mode \(\hat{c}\) with additional reference modes \(\hat{d}_1, ..., \hat{d}_K\) in corresponding coherent states \(\ket{\gamma_1}_d, ..., \ket{\gamma_K}_d\) (prepared by splitting the reference mode \(\hat{d}\) and applying additional phase shifts) at linear beamsplitters (Fig. 3(c)). For example, for elimination of a single coherent state \(\ket{\gamma_1}\) one mixes the field \(\hat{c}\) with a single reference mode \(\hat{d}_1\) at linear beamsplitter \(BS_1\) with transmittance \(T_1 = \cos^2 \theta_1\) (Fig. 3(c)). If the amplitude of the reference mode coherent state equals \(\tilde{\gamma}_1 = -\gamma_1 \tan \theta_1\) and the measured state of the mode \(\hat{c}\) is \(\ket{\psi}_c\), after mixing at the beamsplitter the modes will be

\[
\hat{D}(\tilde{\gamma}_1) \hat{d}_1 \rightarrow \hat{c} \rightarrow \hat{d}_1.
\]

**FIG. 3:** (a) Elimination measurement for a single coherent state \(\ket{\gamma_1}\). \(\hat{D}(\gamma_1)\) is the operator of coherent displacement with the magnitude \(-\gamma_1\). (b) Scheme for elimination of a set of coherent states \(\ket{\gamma_2}\), \(j = 1, ..., K\). (c) Elimination measurement for a single coherent state \(\ket{\gamma_1}\) with the coherent displacement operator \(\hat{D}_c(\gamma_1)\) being effectively realized by mixing the mode \(\hat{c}\) with the reference mode \(\hat{d}_1\) in coherent state \(\ket{\gamma_1}_d\) at linear beamsplitter \(BS_1\). (d) Scheme for elimination of coherent states \(\ket{\gamma_2}\), \(j = 1, ..., K\), with linear beamsplitters (\(BS_1, ..., BS_{K-1}\); \(BS_1', ..., BS_{K-1}'\)) and photodetectors (\(D_1, ..., D_K\)); reference coherent states for implementing necessary displacements in phase space are obtained by splitting reference field \(\hat{d}\) and applying phase shifts \(\phi_1, ..., \phi_{K-1}\).
in the state $|\cos \theta_1, (\gamma_x + \gamma_1 \tan^2 \theta_1)\rangle_c |i \sin \theta_1, (\gamma_x - \gamma_1)\rangle_d_1$. The obtained state of the mode $\hat{c}$ can be used for some further operations, while the mode $d_1$ appears just in the state, required for implementing elimination measurement: the amplitude of the initial coherent state $|\gamma_x\rangle_c$ is displaced by magnitude $-\gamma_1$. Additional phase factor $i$ is irrelevant for detection of presence of photons in the mode, and decrease of the field intensity by the factor $\sin^2 \theta_1 = 1 - T_1$ influences only the probability of detecting photons and, thus, of obtaining successful elimination outcome. In an important case, when the mode $\hat{c}$ is discarded after the elimination measurement, the efficiency of the measurement can be improved by requiring $T_1 = \delta \ll 1$ (then $\sin^2 \theta_1 \approx 1$ and success probability is approximately the same as for the scheme in Fig. 3(a)).

The scheme suitable for implementing elimination measurements in general case of $K$ coherent states $\{|\gamma_j\rangle_c, j = 1, \ldots, K\}$ is shown in Fig. 3(d). If the measured state of the mode $\hat{c}$ is coherent state $|\gamma_1\rangle_c$, we require the states of the reference modes $\hat{d}_j$ after mixing with the mode $\hat{c}$ at beamsplitters to be coherent states $|iq(\gamma_x - \gamma_j)\rangle_{d_j}$, where $q$ is certain coefficient independent of the mode number $j$. These states correspond to displacement of the amplitude $\gamma_j$ of the measured coherent state by magnitudes $\gamma_j$ and can be used for carrying out corresponding elimination measurements.

Transmittances $T_j$ of the beamsplitters $BS_j$ in this case are defined in the unique way by the above requirement of obtaining correct coherent placements. As in the case of single coherent state $|\gamma_1\rangle_c$, we require the transmittance of the last beam splitter to be small: $T_K = \delta \ll 1$ (the mode $\hat{c}$ is discarded after implementing elimination measurements). The condition of dividing the amplitude $\gamma_j$ of measured coherent state into equal parts $q$ between the modes $\hat{d}_j$ leads to the following system of equations for the beamsplitter transmittances (we define parameters $\theta_j$ by $T_j = \cos^2 \theta_j$):

$$\begin{align*}
\sin \theta_1 &= q, \\
\sin \theta_2 \cos \theta_1 &= q, \\
&\cdots \\
\sin \theta_K \cos \theta_{K-1} \cdots \cos \theta_1 &= q.
\end{align*}$$

Solving these equations together with the requirement $T_K = \delta$, one finds the following expressions for the transmittances of the beamsplitters:

$$T_j = \frac{(K - j - 1)(1 - \delta) + 1}{(K - j)(1 - \delta) + 1} \approx \frac{K - j}{K + 1 - j},$$

and coefficient $q$:

$$q = \left( K + \frac{\delta}{1 - \delta} \right)^{-1/2} \approx 1/\sqrt{K}.$$

The amplitudes of the reference modes coherent states $\tilde{\gamma}_m$ are also defined in the unique way by the requirement of obtaining correct coherent displacements in implemented elimination measurements. The following recurrent system of equation can be obtained:

$$\begin{align*}
\tilde{\gamma}_j &= i \cos \theta_j + i\tilde{\gamma}_{j-1} \sin \theta_j = -i q \tilde{\gamma}_j, \\
\tilde{\gamma}_j &= \gamma_{j-1} \cos \theta_j + i\tilde{\gamma}_j \sin \theta_j, \\
\delta_{0} &= 0,
\end{align*}$$

where $\tilde{\gamma}_j$ is the amplitude of reference coherent state, mixed to the mode $\hat{c}$ by first $j$ beamsplitters. The solution of this system of equations is

$$\tilde{\gamma}_j = -\frac{i q}{\cos \theta_j} \{ \gamma_j + \sin^2 \theta_j (\gamma_1 + \ldots + \gamma_{j-1}) \}.$$

As stated above, coherent states of reference mode $\hat{d}_j$ with the required amplitudes $\tilde{\gamma}_j$ can be obtained by splitting coherent state $\tilde{\gamma}$ of the mode $\hat{d}$ (Fig. 3(d)). The transmittances $T_j' = \cos \theta_j'$ of the beamsplitters $BS_j'$ and the phase shifts $\phi_j$ are solutions of the following system of equations:

$$\begin{align*}
\tilde{\gamma}_j &= i \cos \theta_1' \cdots \cos \theta_{j-1}' \sin \theta_j' e^{i \phi_j} \tilde{\gamma}, \\
\gamma_{K} &= i \cos \theta_1' \cdots \cos \theta_{K-1}' \tilde{\gamma}.
\end{align*}$$

The system contains $K$ complex equations for $2K$ real variables $T_1', \ldots, T_{K-1}', \phi_1, \ldots, \phi_{K-1}, \Re \tilde{\gamma}, \Im \tilde{\gamma}$. The solution of the equations is quite cumbersome, and we will provide it in explicit form only for certain special cases, discussed below.

### E. Implementation: degenerate case

In the degenerate case we need to eliminate not only coherent states, but also the states of the form $|\gamma_m\rangle_c = (\hat{c}^+)^m|\gamma_0\rangle_c$, created from coherent states by adding fixed number of photons (photons-added coherent states — PACS). We will show that elimination of PACS can be done exactly in the same way as elimination of coherent states $|\gamma_1\rangle_c$.

For designing a scheme, suitable for elimination of the coherent state with one added photon, e.g. $\hat{c}^+|\gamma_1\rangle_c$, it is useful to notice that when a beam with a single-photon excitation is split into two parts by a beamsplitter, the excitation can be detected in one of the two beams, but not in both of the beams simultaneously. If the mode $\hat{c}$, the state of which is measured, is initially in the state $\hat{c}^+|\gamma_1\rangle_c$, after splitting the mode into two parts (e.g. modes $\hat{c}_1$ and $\hat{c}_2$) we obtain a superposition state $\left\{ \sqrt{\frac{1}{2}}(\hat{c}_1^+ + \hat{c}_2^+)|\gamma_1/\sqrt{2}\rangle_{c_1}|\gamma_1/\sqrt{2}\rangle_{c_2} + \sqrt{\frac{1}{2}}(\hat{c}_1^+ - \hat{c}_2^+)|\gamma_1/\sqrt{2}\rangle_{c_1}|\gamma_1/\sqrt{2}\rangle_{c_2} \right\}$, where the added photon can be found in one of the modes, but never in the two modes simultaneously. Therefore, for all the terms of the superposition at least one of the modes is in coherent state $|\gamma_1/\sqrt{2}\rangle$. If we implement measurements, eliminating the state $|\gamma_1/\sqrt{2}\rangle$, for both of the modes, two successful outcomes can never be obtained, if the measured state of the mode $\hat{c}$ was $\hat{c}^+|\gamma_1\rangle_c$. Therefore,
two simultaneously obtained successful outcomes of the elimination measurements correspond to elimination of the PACS with one added photon $\hat{c}^+|\gamma_1\rangle_c$. It should be noted that exactly the same scheme would be obtained, if we tried to eliminate two coherent states $|\gamma_1\rangle_c$ and $|\gamma_2\rangle_c$ with equal amplitudes $\gamma_1 = \gamma_2$ by the method, suitable for nondegenerate case.

Elimination of a PACS with $s$ added photons (e.g. $|\gamma^{(s)}\rangle_c$) can be carried out in a similar way, taking into account that when a mode with $s$ photons is split into $s + 1$ parts, it is impossible to detect a photon in each of the $s + 1$ modes simultaneously. Therefore, for elimination of the state $|\gamma^{(s)}\rangle_c$, one can splits the mode $\hat{c}$ into $s + 1$ modes. When $s$ photons, added to coherent state $|\gamma_1\rangle_c$ according to the definition of the state $|\gamma^{(s)}\rangle_c$, are distributed between $s + 1$ modes, at least one mode appear in coherent state $|\gamma_1/\sqrt{s + 1}\rangle$ without added photons. Therefore, $s + 1$ successful outcomes of elimination of coherent state $|\gamma_1/\sqrt{s + 1}\rangle$ for the $s + 1$ modes cannot be obtained simultaneously if the initial state of the mode $\hat{c}$ is the state $|\gamma^{(s)}\rangle_c$ (or any of the states $|\gamma_1\rangle_c$, $|\gamma^{(1)}\rangle_c$, ..., $|\gamma^{(s-1)}\rangle_c$ with lesser numbers of added photons). Thus, obtaining successful outcomes of the $s + 1$ measurements, eliminating coherent state $|\gamma_1/\sqrt{s + 1}\rangle$, corresponds to elimination of the state $|\gamma^{(s)}\rangle_c$, as well as of the states $|\gamma_1\rangle_c$, $|\gamma^{(1)}\rangle_c$, ..., $|\gamma^{(s-1)}\rangle_c$. As in the case, discussed in the previous paragraph, exactly the same scheme could be used for elimination of the set of coherent states $|\gamma_1\rangle_c$, ..., $|\gamma_s\rangle_c$ with equal amplitudes $\gamma_1 = \gamma_2 = ... = \gamma_s$ if the case were considered as nondegenerate.

The full algorithm of designing discrimination scheme for degenerate case can be summarized as follows. At first, $K$ roots $\gamma_j$ of Eq. (14) are found. Then the set of coherent states $|\gamma_j\rangle_c$ is constructed. If the root $\gamma_m$ has multiplicity $l_m > 1$, $l_m$ “copies” of the coherent state $|\gamma_m\rangle_c$ are replaced by $l_m$ independent states $|\gamma_m\rangle_c$, $|\gamma^{(1)}_m\rangle_c$, ..., $|\gamma^{(l_m-1)}_m\rangle_c$. In the obtained set of independent states single coherent states are eliminated by the method, discussed in the previous subsection. Sets of the states $|\gamma_m\rangle_c$, $|\gamma^{(1)}_m\rangle_c$, ..., $|\gamma^{(l_m-1)}_m\rangle_c$ with different numbers of photons, added to the same coherent state, are eliminated by carrying out measurements, eliminating coherent state $|\gamma_m\rangle$ (with amplitude, decreased by splitting, for $l_m$ modes, obtained after splitting the mode $\hat{c}$). Therefore, $l_m$ “copies” of the root $\gamma_m$, appearing in the list of roots of Eq. (14), correspond in the final discrimination scheme to $l_m$-fold elimination of the coherent state $|\gamma_m\rangle_c$. It means that one need not make any difference between degenerate and nondegenerate roots of Eq. (14), eliminating coherent states $|\gamma_j\rangle_c$ as many times, as they appear in the list of roots of Eq. (14). Thus, the scheme, designed for discrimination of the state $|\varphi\rangle_c$ in nondegenerate case (Fig. 5(d)), is also suitable for degenerate case.

F. Implementation: mathematical description

Quite interesting result of applicability of the same scheme for both nondegenerate and degenerate case can be given more rigorous mathematical proof on the basis of operator definition of elimination measurements (Eqs. (3), (9)).

As shown in Appendix A (see Eq. (A17)), transformation of the system density matrix in the case of successful outcomes of the $K$ measurements, eliminating coherent states $|\gamma_j\rangle_c$, (photocounts obtained from all the detectors $D_j$) has the form

$$
\rho_{abc}^{(\text{out})} = M \left\{ \sum_{n_j \geq 1} \hat{A}^{(n_j)}_{|\gamma_j\rangle_c} ... \hat{A}^{(n_1)}_{|\gamma_1\rangle_c} \rho_{abc} \right\} \otimes \left( \hat{A}^{(n_1)}_{|\gamma_1\rangle_c}^+ + ... \hat{A}^{(n_K)}_{|\gamma_K\rangle_c}^+ \right) \right\}
$$

where density matrices $\rho_{abc}^{(\text{in})}$ and $\rho_{abc}^{(\text{out})}$ describe the system state before and after implementing elimination measurements respectively; superoperator $M$ (Eq. (A9)) describes the part of system state transformation, which does not depend on measurement outcomes; the sets of operators $\{\hat{A}^{(n_j)}_{|\gamma_j\rangle_c} | n_j = 1, 2, ... \}$ correspond to successful elimination of coherent states $|\gamma_j\rangle_c$:

$$
\hat{A}^{(n_j)}_{|\gamma_j\rangle_c} | \gamma_j\rangle_c = 0, \quad n_j = 1, 2, ...
$$

and satisfy Eq. (9).

The scheme is apparently suitable for nondegenerate case, and one needs to shows that it also can be used when some roots of Eq. (14) are degenerate, i.e. that successful outcome of $\gamma_m$-fold elimination of the state $|\gamma_m\rangle_c$, corresponding to the root $\gamma_m$ with multiplicity $l_m$, or, in other words, successful elimination of $l_m$ “copies” of the state $|\gamma_m\rangle_c$ corresponds to elimination of the states $|\gamma_m\rangle_c$, $|\gamma^{(1)}_m\rangle_c$, ..., $|\gamma^{(l_m-1)}_m\rangle_c$. The successful result of $\gamma_m$-fold elimination of the state $|\gamma_m\rangle_c$ is described by operator

$$
\hat{A}^{(n_1)}_{|\gamma_m\rangle_c} ... \hat{A}^{(n_{l_m})}_{|\gamma_m\rangle_c} \propto (\hat{c} - \gamma_m)^r,
$$

where $r = n_1 + ... + n_{l_m} \geq l_m$. The expression

$$(\hat{c} - \gamma_m)^r (\hat{c}^+)^s = \left( \hat{c}^+ + \frac{\partial}{\partial \hat{c}} \right)^s (\hat{c} - \gamma_m)^r$$

contains powers of operator $(\hat{c} - \gamma_m)$ not less than $r - s \geq l_m - s \geq 1$ for $s = 0, ..., l_m - 1$. Therefore, operator (26) corresponds to elimination of the states $|\gamma_m\rangle_c$, $s = 0, ..., l_m - 1$:

$$
\left\{ \hat{A}^{(n_1)}_{|\gamma_m\rangle_c} ... \hat{A}^{(n_{l_m})}_{|\gamma_m\rangle_c} \right\} (\hat{c}^+)^s |\gamma_m\rangle_c \propto
$$

$$
(\hat{c} - \gamma_m)^{r-s} |\gamma_m\rangle_c = 0 \quad \text{for} \quad s = 0, ..., l_m - 1,
$$

which proves the conclusion made in the previous subsection.
G. Final state for successful and "semi-successful" results of discrimination

For obtaining the expression for the final state of the main field modes \( \hat{a} \) and \( \hat{b} \) after discrimination of the state \( |\phi\rangle_c \) by elimination measurements, it is convenient to introduce "phase-shifting" operator \( \hat{F}_{ab} = \exp \left( i \chi \left( \hat{a}^+ \hat{a} + \hat{b}^+ \hat{b} \right) \right) \) (it describes change of the main field modes state after cross-Kerr interaction with the mode \( \hat{c} \) possessing 1 photon) and to represent state \( |\Psi_1\rangle \) (described by Eq. (3)) using operator of coherent displacement with operator-type argument:

\[
|\Psi_1\rangle_{abc} = \hat{D}_c \left( \hat{F}_{ab} \gamma \right) |\alpha\rangle_a |\beta\rangle_b |0\rangle_c, \tag{28}
\]

with the following property:

\[
\hat{c} \hat{D}_c \left( \hat{F}_{ab} \gamma \right) = \hat{F}_{ab} \gamma \hat{D}_c \left( \hat{F}_{ab} \gamma \right), \tag{29}
\]

which leads to significant simplification of Eq. (23) for the final state density matrix. As shown in Appendix A (Eqs. (A19), (A20)), the final state of main field modes \( \hat{a} \) and \( \hat{b} \) after elimination of all the states \( \{ |\gamma\rangle_c \} \) and subsequent discarding the ancillary mode \( \hat{c} \) is described by density matrix

\[
\rho_{ab} = q^{2K} |\gamma|^2 \left| \Psi_f' \right\rangle_{ab} \langle \Psi_f' \right| + O \left( |\gamma|^{2K+2} \right), \tag{30}
\]

where

\[
|\Psi_f' \rangle_{ab} = \left( \hat{F}_{ab} - \frac{\gamma_n}{\gamma} \right) \cdots \left( \hat{F}_{ab} - \frac{\gamma_K}{\gamma} \right) |\alpha\rangle_a |\beta\rangle_b = \frac{1}{\epsilon_K} \sum_{n=0}^{K} c_n \hat{F}_{ab}^n |\alpha\rangle_a |\beta\rangle_b = \frac{1}{\epsilon_K} |\Psi_f \rangle_{ab}. \tag{31}
\]

The obtained expression means that the distance between the desired final state and the state \( \rho_{ab} \), generated by the scheme, has the order \( O \left( |\gamma|^2 \right) \) and is small enough to be neglected when nonideality of the system is taken into account.

It should be noted that in certain cases the final states, generated when "clicks" were obtained not from all the detectors, can also be useful ("semi-successful" results). Such states are described by expressions, similar to Eq. (31) but without multipliers, corresponding to the detectors (with numbers \( n_1, n_2, ... \)) that did not produce "clicks":

\[
|\Psi(n_1, n_2, ...)\rangle_{ab} = \prod_{m \neq n_1, n_2,...} \left( \hat{F}_{ab} - \frac{\gamma_m}{\gamma} \right) |\alpha\rangle_a |\beta\rangle_b. \tag{32}
\]

This expression can be decomposed in the form, similar to the desired final state (2) but with lower possible degree of entanglement:

\[
|\Psi(n_1, n_2, ...)\rangle_{ab} = \sum_n \tilde{c}_n (n_1, n_2, ...) |\alpha e^{i\chi n}\rangle_a |\beta e^{i\chi n}\rangle_b. \tag{33}
\]

E.g. for the case of absence of only one photocount the number of terms equals to \( K \) (instead of \( K + 1 \)) and the coefficients \( \tilde{c}_n \) can be found as

\[
\tilde{c}_n (n_1) = \sum_{m=0}^{n} c_m \epsilon_{K(n_1, n, m)} n + 1 - m, \tag{34}
\]

where \( n = 0, ..., K - 1 \).

In the next section we provide several examples of final states that can be generated by the protocol for successful and "semi-successful" discrimination outcomes.

V. EXAMPLES

A. Superpositions with correlated photon numbers

As the first example of possible applications of the protocol to nonclassical states generation we consider creation of a superposition of states of modes \( \hat{a} \) and \( \hat{b} \) with correlated photon numbers. We show that such superpositions arise quite naturally in our protocol and then use them to illustrate general formalism, developed in Section IV.

As discussed above (see Eq. (1)), cross-Kerr interaction correlates photon number of one of the interacting modes with the phase of coherent state of the other mode. The state \( |\Psi_1\rangle_{abc} \), obtained after cross-Kerr interaction of the main modes \( \hat{a} \) and \( \hat{b} \) with the ancillary mode \( \hat{c} \), can be considered either as a superposition, where phases of coherent states of the modes \( \hat{a} \) and \( \hat{b} \) are proportional to the number of photons in the mode \( \hat{c} \) (Eq. (3)), or alternatively as a superposition, where the phase of coherent state of the mode \( \hat{c} \) is proportional to the total number of photons in the modes \( \hat{a} \) and \( \hat{b} \). The latter interpretation of the state \( |\Psi_1\rangle_{abc} \) implies that discrimination of coherent state \( |\gamma e^{i\chi n}\rangle_c \) of the ancillary mode \( \hat{c} \) fixes the total number of photons in modes \( \hat{a} \) and \( \hat{b} \) to be equal to \( n \). The final state \( |\Psi_f\rangle_{ab} \) in this case is a superposition of Fock states of the modes \( \hat{a} \) and \( \hat{b} \) with the number of photons in each mode varying from 0 to \( n \) and the total number of photons being equal to \( n \) for each term.

Mathematically this statement can be proved in the following way. The state \( |\Psi_1\rangle_{abc} \) can be decomposed in the form:

\[
|\Psi_1\rangle_{abc} = \sum_{n=0}^{\infty} |\Phi(n)\rangle_{ab} |\gamma e^{i\chi n}\rangle_c, \tag{35}
\]

where

\[
|\Phi(n)\rangle_{ab} = \sum_{m=0}^{n} Q_m(\alpha) Q_{n-m}(\alpha) |m\rangle_a |n-m\rangle_b \tag{36}
\]

is a superposition of states of the modes \( \hat{a} \) and \( \hat{b} \) with fixed total number of photons (equal to \( n \)); function \( Q_m(\alpha) \)
is defined as $Q_m(\alpha) = \frac{a_m^* e^{-|\alpha|^2/2}}{\sqrt{m!}}$, $Q_m(\alpha) Q_{-m}(\alpha) = Q_n(\alpha) \sqrt{\frac{(m+n)!}{m!}}$; we assume for simplicity that $\alpha = \beta$.

Suppose that only $K$ terms are significant in the superposition $|\Phi(n, K)\rangle$: $|\psi_f\rangle = \sum_{n=0}^{K} \sum_{m=0}^{n} c_n e^{i\chi n} |m\rangle_n |n-m\rangle_b \times \{ 1 + O(Q_{K+1}(\alpha)/Q_n(\alpha)) \}$.

Then successful outcome of elimination of coherent states $\{ |\gamma_1\rangle, ..., |\gamma_n\rangle \}$ with subsequent discarding of the ancillary mode $\hat{c}$ transforms the state $|\psi_f\rangle_{ab}$ into the following state of the modes $\hat{a}$ and $\hat{b}$ with correlated photon numbers, described above:

$$|\psi_f\rangle_{ab} = 2^{-n/2} \sum_{n=0}^{K} \sum_{m=0}^{n} \left( \frac{m!}{n!} \right)^{1/2} c_n e^{i\chi n} \times \{ 1 + O(Q_{K+1}(\alpha)/Q_n(\alpha)) \}.$$  \hspace{1cm} (37)

Generation of states of the form $\{ |\gamma\rangle \}$ can be described by general formalism, developed in Section [ ] For this purpose we find coefficients $c_n$, for which the general final state $|\psi_f\rangle_{ab}$ (Eq. (3)) is equivalent for the desired final state $|\Phi(n, K)\rangle_{ab}$ (Eq. (38)). Then coherent states amplitudes $\chi_n$ can be found by solving Eq. (22), and expressions for the parameters of discrimination scheme can be derived.

The general expression Eq. (2) for the final state of the modes $\hat{a}$ and $\hat{b}$ can be transformed to the following form:

$$|\psi_f\rangle_{ab} = \sum_{n=0}^{K} \sum_{m=0}^{n} c_n e^{i\chi n} \left| \Phi(s) \right\rangle_{ab},$$  \hspace{1cm} (39)

by decomposing coherent states $|\alpha e^{i\chi n}\rangle_a$ and $|\beta e^{-i\chi n}\rangle_b$ in terms of Fock states, where states $|\Phi(s)\rangle_{ab}$ are defined by Eq. (35).

In order to obtain $|\psi_f\rangle_{ab} = |\Phi(s, K)\rangle_{ab}$, the coefficients $c_n$ must satisfy the following system of equations:

$$\sum_{n=0}^{K} c_n e^{i\chi n} = 0$$  \hspace{1cm} (40)

Before solving this system, it is useful to compare it with Eq. (13) for the amplitudes $\gamma_j$, and to notice, that if this coefficients $c_n$ satisfy Eq. (40), $K$ complex numbers $\gamma e^{i\chi s'}$, $s' = 0, 1, ..., s - 1, s + 1, ..., K$, apparently represent the $K$ roots of Eq. (13). Then, coefficients $c_n$ are defined in the unique way (except for overall normalization constant) by the complete system of roots and are equal to

$$c_0 = c_K \prod_{s'} e^{i\chi s'}, \hspace{1cm} c_{K-1} = c_K \sum_{s'} e^{i\chi s'}.$$  \hspace{1cm} (41)

For example, if the desired final state is the following one

$$|\psi_f\rangle_{ab} = |\Phi(2, 2)\rangle_{ab} =$$

$$= \frac{|0\rangle_a |2\rangle_b + \sqrt{2} |1\rangle_a |1\rangle_b + |2\rangle_a |0\rangle_b}{2\sqrt{2}} + O(|\alpha|),$$  \hspace{1cm} (42)

coefficients $c_n$ must be equal to $c_0 = e^{i\chi}, c_1 = -1 - e^{i\chi}$, $c_2 = 1$ (for unnormalized state). Amplitudes of the coherent states $|\gamma_j\rangle$, eliminated in discrimination scheme, are equal to $\gamma_1 = \gamma$ and $\gamma_2 = e^{i\chi}$ in this case. According to Eq. (18), transmittances of the beamsplitters $B_{S1}$ and $B_{S2}$ are equal approximately to $T_1 \approx 1/2$ (for $\delta \ll 1$) and $T_2 = \delta$. The amplitudes of the reference coherent states, defined by Eq. (21), are $\gamma_1 \approx -i\gamma_1$ and $\gamma_2 \approx -i(\gamma_1 + \gamma_2)/\sqrt{2\delta}$. Solving the system of equations (22), one finds $\phi_1 = -\chi/2, T_1 \approx 1 - \delta^2/2, \gamma \approx -i\gamma_2$.

For coefficients $c_n(\chi_1)$ (Eq. (41)), characterizing final state in the case of "semi-successful" outcomes of discrimination when the desired final state for successful outcome is described by Eq. (12), one obtains the following expressions: $c_0(1) = e^{i\chi}, c_1(1) = -1, c_2(1) = 1, c_2(2) = -1$. The final state, generated when "click" was obtained from detector $D_2$, is approximately a vacuum state:

$$|\psi(1)\rangle_{ab} = |0\rangle_a |0\rangle_b + O(|\alpha|^2),$$  \hspace{1cm} (43)

while the state, generated when "click" was obtained from detector $D_1$, belongs to the class of states, described by Eq. (35):

$$|\psi(2)\rangle_{ab} = \frac{|0\rangle_a |1\rangle_b + |1\rangle_a |0\rangle_b}{\sqrt{2}} + O(|\alpha|) \equiv |\Phi(1, 1)\rangle_{ab},$$  \hspace{1cm} (44)

and, therefore, can be useful for certain applications.

B. Maximally entangled states for protocols with fixed number of detectors

Another group of examples represents states of the form Eq. (2) with maximal entanglement, which is possible for a scheme with fixed number of photodetectors $K$ (and, therefore, with fixed number of terms in the expression (2) for the final state $|\psi_f\rangle_{ab}$).

In the most simple case of schemes with 1 detector ($K = 1$) the coefficients $c_0$ in Eq. (2), maximizing the final entanglement, can be found analytically as

$$c_0 = 1, \hspace{1cm} c_1 = -\exp(-i(\alpha^2 + |\beta|^2) \sin \chi),$$  \hspace{1cm} (45)

where for simplicity we consider unnormalized final state. Additional condition, required for maximization of entanglement in this case, is $|\alpha|^2 = |\beta|^2$ (for simplicity we will assume without loss of generality that $\alpha = \beta$).

The final state of the system is approximately a Bell state

$$|\psi^+\rangle_{ab} = (|+\rangle_a |-\rangle_b + |-\rangle_a |+\rangle_b)/\sqrt{2},$$  \hspace{1cm} (46)

where $|+\rangle_{a,b} = |+\rangle_{a,b}$ is the orthonormal basis for the states of the modes $\hat{a}$ and $\hat{b}$: $|\pm\rangle_{a,b} \sim (|\alpha\rangle_{a,b} \pm e^{-i|\alpha|^2 \sin \chi} |\alpha e^{i\chi}\rangle_{a,b})$. Therefore, the protocol can be used
for generating qubit-type quantum states with maximal entanglement (equal to 1), possible for qubit systems.

The set of states \( \{|\gamma_m\rangle\} \) consists of the only state \( |\gamma_1\rangle \) with the amplitude \( \gamma_1 = e^{i(|\alpha|^2 + |\beta|^2) \sin \chi} \), defined by Eq. (43). The transmittance of the beam splitter \( BS_1 \) (Fig. 3(c)) equals \( \delta \), and the amplitude of the reference coherent state is \( \gamma_1 = -i \gamma_1 (1 - \delta) / \delta \).

For the scheme with two detectors (\( K = 2 \)) the coefficients \( c_n \), providing maximal entanglement, can be found analytically for systems with \( \chi \ll 1 \) and \( |\alpha| = |\beta| \) in two limiting cases: \( |\alpha|^2 \chi^2 \ll 1 \) (low distinguishability of main field modes coherent states with and without phase shift equal to \( \chi \)): \( |\langle \alpha e^{i\chi} | \alpha \rangle| \approx 1 \) and \( |\alpha|^2 \chi^2 \gg 1 \) (high distinguishability: \( |\langle \alpha e^{i\chi} | \alpha \rangle| \ll 1 \)).

For \( |\alpha|^2 \chi^2 \ll 1 \) the final entanglement is maximal for

\[
\begin{align*}
c_0 &= 1, \\
c_1 &= -2 (1 - |\alpha|^2 \chi^2) e^{-2i|\alpha|^2 \chi}, \\
c_2 &= e^{-4i|\alpha|^2 \chi}. 
\end{align*}
\]

The final state of the system is

\[
|\Psi_f\rangle_{ab} = \frac{|u_1\rangle_a |u_3\rangle_b + \sqrt{2} |u_2\rangle_a |u_2\rangle_b + |u_3\rangle_a |u_1\rangle_b}{2},
\]

where

\[
\begin{align*}
|u_1\rangle_a &= q_1 (|\alpha\rangle_a e^{i|\alpha|^2 \chi} + |\alpha e^{i\chi}\rangle_a e^{-i|\alpha|^2 \chi} + 2|\beta\rangle_a |\alpha\rangle_a) + q_0 |\alpha e^{i\chi}\rangle_a, \\
|u_2\rangle_a &= q_2 (|\alpha\rangle_a e^{i|\alpha|^2 \chi} - |\alpha e^{i\chi}\rangle_a e^{-i|\alpha|^2 \chi} + 2|\beta\rangle_a |\alpha\rangle_a) - q_0 |\alpha e^{i\chi}\rangle_a, \\
|u_3\rangle_a &= q_3 (|\alpha\rangle_a e^{i|\alpha|^2 \chi} + |\alpha e^{i\chi}\rangle_a e^{-i|\alpha|^2 \chi} - 2|\beta\rangle_a |\alpha\rangle_a) + q_0 |\alpha e^{i\chi}\rangle_a,
\end{align*}
\]

are orthonormal basis vectors for the mode \( a \) and the basis vectors \( |u_j\rangle_b \) for the mode \( b \) are defined in a similar way (with \( \alpha \) being replaced by \( \beta \)); coefficients \( q_i \) are determined by the condition of orthonormality of the system of basis vectors. The final state (48) possesses entanglement \( E = 3/2 \), which is higher than the maximal value, achievable for a pair of qubits.

In this case the coherent states \( |\gamma_1\rangle_c \) and \( |\gamma_2\rangle_c \), exploited in the detection scheme, possess close amplitudes \( \gamma_{1,2} = (1 \pm i \sqrt{2} |\alpha| \chi - |\alpha|^2 \chi^2) e^{2i|\alpha|^2 \chi} \). Transmittances of the beamsplitters \( BS_1 \) and \( BS_2 \) are equal approximately to \( T_1 \approx 1/2 \) (for \( \delta \ll 1 \)) and \( T_2 = \delta \); amplitudes of the reference coherent states are \( \tilde{\gamma}_1 \approx -i \gamma_1 \) and \( \tilde{\gamma}_2 \approx -i (\gamma_1 + \gamma_2) / \sqrt{2} \). The parameters of discrimination scheme, defined by Eq. (22), are equal to \( \phi_1 = \sqrt{2} |\alpha| \chi, T'_1 \approx 1 - \delta^2 / 2, \tilde{\gamma} \approx -i \tilde{\gamma}_2 \).

In the opposite limiting case \( |\alpha|^2 \chi^2 \gg 1 \) (such condition is satisfied simultaneously with \( \chi \ll 1 \) in the systems with intense fields \( a \) and \( b \)) entanglement reaches the bound for 3-level system \( E = \log_2 3 \approx 1.58 \) when the

parameters are

\[
\begin{align*}
c_0 &= 1, \\
c_1 &= -e^{-2i|\alpha|^2 \chi}, \\
c_2 &= -e^{-4i|\alpha|^2 \chi}.
\end{align*}
\]

The final state in this case has the form

\[
|\Psi_f\rangle_{ab} = \frac{|u_1\rangle_a |u_3\rangle_b + |u_2\rangle_a |u_2\rangle_b + |u_3\rangle_a |u_1\rangle_b}{\sqrt{3}},
\]

where basis vectors \( |u_j\rangle_{a,b} \) are defined by Eq. (49) above. Amplitudes of coherent states \( |\gamma_1\rangle \) and \( |\gamma_2\rangle \) are equal to \( \gamma_{1,2} = (1 \pm i \sqrt{3}) e^{2i|\alpha|^2 \chi} / 2 \). The transmittances of the beamsplitters \( BS_1 \) and \( BS_2 \) and the amplitudes \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) of reference modes coherent states are the same functions of the amplitudes \( \gamma_1 \) and \( \gamma_2 \) in the previously discussed limiting case. Solving the system of equations (22), one can find \( \phi_1 = \pi / 3, T'_1 \approx 1 - \delta^2 / 2, \tilde{\gamma} \approx -i \tilde{\gamma}_2 \).

For intermediate values of distinguishability \( (|\alpha|^2 \chi^2 \sim 1) \), as well as for greater numbers of detectors (\( K \geq 3 \)), optimal coefficients \( c_n \) can be found numerically. The values of maximally possible entanglement for schemes with fixed number of detectors are shown in Fig. 3(b) (solid lines). One can see that the maximal possible value of entanglement grows with increase of the detectors number \( K \), and in certain cases it may be considered as more effective use of fixed resources (nonlinear interaction, quantum channel) than can be achieved in schemes with qubit-type entanglement.

Fig. 4 also illustrates entanglement of final states, generated by these schemes for ”semi-successful” outcomes (obtaining photocounts from lesser number of detectors; dashed, dotted and dot-dashed lines in Fig. 4). These final states possess non-zero entanglement and in certain cases can also be useful for solving information processing tasks. For example, the scheme with \( K = 3 \) detectors, optimized for obtaining maximal entanglement in the case of successful discrimination outcome, in the case of two absent photocounts can also produce maximally entangled state from the space of states \( |\Psi_f\rangle_{ab} \) with 2 non-zero terms (line 3f in Fig. 4).

VI. NONIDEAL SYSTEM

A. Considered types of nonideality

In previous sections we assumed for simplicity that the quantum channel and photodetectors are ideal. In order to prove realizability of the proposed entanglement generation method in realistic situations we discuss influence of system nonideality on the fidelity of obtaining final state.

In real system decoherence and dephasing accompany all the stages of the protocol: implementation of nonlinear interaction, transmission of the probe beam through the quantum channel, storage of main modes \( a \) and \( b \).
in local resonators. All these factors can be taken into account by solving corresponding master equations. However, for the system, considered in our paper, influence of some of the factors on the fidelity of final state generation is supposed to be small. Therefore, for the purpose of simplifying further description, we take into account only the following factors that can limit applicability of our protocol:

(i) decoherence of modes $\hat{a}$ and $\hat{b}$ during cross-Kerr interaction with probe beam $\hat{c}$ (decoherence of the mode $\hat{c}$ is assumed to have negligible effect due to much smaller amplitude of the probe beam: $|\gamma| \ll |\alpha|, |\beta|$); qualitatively, decoherence of the mode $\hat{c}$ during cross-Kerr interaction influences the final state in the same way as decoherence of this mode during transmission through the quantum channel, but is weaker);

(ii) inaccuracies in the nonlinearity values of the used Kerr media (we assume that effective strengths of nonlinear interactions carried out by Alice and Bob are equal to $\chi_{ac} = \chi + \Delta \chi_{ac}$ and $\chi_{bc} = \chi + \Delta \chi_{bc}$ respectively and differ from the value $\chi$ used in the scheme optimization);

(iii) decoherence caused by nonideality of local resonator at Alice’s site (Bob’s resonator is not used for storing part of an entangled state for a long time, and its nonideality is supposed to effect the final state fidelity negligibly; dephasing in the resonators is assumed to be small due to controllable laboratory conditions);

(iv) decoherence and dephasing in the quantum channel;

(v) limited efficiency (probability of detecting a photon, present in the field mode, equals $\lambda < 1$) and dark counts of photodetectors (obtaining photocount with probability $\zeta$ when the mode is in the vacuum state).

For describing nonideality of the system we consider four stages of the protocol separately and find (super)operators, describing difference between the states, obtained in ideal and nonideal systems. It should be noted that operator, transforming one fixed state into another fixed state (ideal state into nonideal one in the considered case), is not defined in the unique way: its action on states from orthogonal space can be arbitrary. In further consideration we try to choose operators, acting on the main modes only (but not on the ancillary one), from sets of equivalent operators, transforming ideal density matrix into the nonideal one. Then the final state after implementing all the stages of the protocol is expected to be presented in the form of certain superoperator, acting on the ideal final state $|\Psi_f\rangle_{ab}$ of the main modes $\hat{a}$ and $\hat{b}$.

B. Nonideal cross-Kerr interaction at Alice’s site

The first stage of the protocol is cross-Kerr interaction of modes $\hat{a}$ and $\hat{c}$. We describe this nonlinear interaction in nonideal case by the following master equation:

$$\frac{d}{dt}\rho = i\chi' \left[\hat{a}^+ \hat{a}^+ \hat{c}, \rho\right] + \kappa_1 \hat{L}(\hat{a})\rho,$$  \hspace{1cm} (52)

where $\hat{L}(\hat{X})\rho = \left[\hat{X}\rho, \hat{X}\right] + \left[\hat{X}, \rho\hat{X}\right]$. As discussed above (item (i) in the list of nonideality types), the only kind of nonideality taken into account by Eq. (52) is decoherence of the mode $\hat{a}$.

Assuming that the duration of interaction is $\Delta t_1$, one can characterize the nonlinear interaction by effective strength $\chi_{ac} = \chi' \Delta t_1$ and relative loses rate $A_1 = e^{2\kappa_1 \Delta t_1} - 1$ ($A_1 = (I_0 - I)/I$, where $I_0$ and $I$ are beam intensities before and after the interaction).

In the ideal case ($\kappa_1 = 0$) state transformation due to discussed cross-Kerr interaction is described by unitary operator

$$\hat{U}_{ac} = \exp(i\chi_{ac} \hat{a}^+ \hat{a}^+ \hat{c}^+ \hat{c}).$$  \hspace{1cm} (53)

The initial uncorrelated state $|\alpha\rangle_a |\beta\rangle_b |\gamma\rangle_c$ is transformed, therefore, into the superposition, where the phase of coherent state of the mode $\hat{a}$ is proportional to the number

![FIG. 4: Entanglement of the final states, generated by the protocol, versus coherent states distinguishability $|\alpha|^2$ $\chi^2$. Solid lines: maximal entanglement, possible for generation in the scheme with fixed number of photodetectors $K$: $K = 1$ (1), $K = 2$ (2), $K = 3$ (3). Dashed lines (2a, 2b): entanglement of the final state, generated by the scheme with $K = 2$ detectors with optimized coefficients $c_n$ in "semi-successful" case (one photocount instead of two); entanglement does not depend on the number of the detector, which "clicked". Dot-dashed lines (3a, 3b, 3c): final state entanglement for $K = 3$ and 1 absent photocount. Dotted lines (3d, 3e, 3f): line 3f coincides with the solid line 1): final entanglement for $K = 3$ and 2 absent photocounts.](image)
the end of the considered stage of the protocol: expressions for the quantities decomposition into Eq. (52) and solving the resulting sys-

is the initial uncorrelated density matrix (initial amplitudes \( \alpha_0, \beta_0, \gamma_0 \) of coherent states in the nonideal case must be larger that the amplitudes \( \alpha, \beta, \gamma \) that are expected to characterize final state).

The solution can be found by representing the density matrix of the modes \( \hat{a} \) and \( \hat{c} \) in the form

\[
\rho(t) = \sum_{n_1, n_2} \rho_{n_1 n_2}(t) \times \left| \alpha(t) e^{i \chi_n t} \right|^2_{\hat{a} a} \left| \beta(t) e^{i \chi_n t} \right|^2_{\hat{a} b} \left| \gamma(t) e^{i \chi_n t} \right|^2_{\hat{a} c} \left| n \right|^2_{\hat{c} c} \left| n' \right|^2_{\hat{c} c},
\]

which is preserved during evolution. Substituting this decomposition into Eq. (52) and solving the resulting system of differential equations, one can obtain the following expressions for the quantities \( \alpha(\Delta t_1) \) and \( \rho_{n_1 n_2}(\Delta t_1) \) at the end of the considered stage of the protocol:

\[
\alpha(\Delta t_1) = \alpha' \equiv \alpha(0) e^{-\kappa_1 \Delta t_1} \equiv \alpha(0) / \sqrt{\Lambda_1 + 1}, \tag{57}
\]

\[
\rho_{n_1 n_2}(\Delta t_1) \approx Q_{n_1}(\gamma(0)) Q_{n_2}(\gamma(0)) \exp \left\{ \left| \alpha(0) \right|^2 \times \Lambda_1 \left( i \chi_{ac}(n_1 - n_2) - \chi_{ac}^2 n_1 n_2 \right) \right\}, \tag{58}
\]

where smallness of nonideality is assumed for simplicity of derived expressions. The exponential factor in Eq. (58) describes influence of Kerr medium nonideality on the state, obtained after the interaction. For characterizing transition from the ideal state Eq. (51) to the nonideal one the following superoperator, acting in the state space of the mode \( \hat{a} \), can be chosen from the class of equivalent operators, describing this state transformation:

\[
\hat{M}_1(\hat{a}) : | \alpha(0) e^{i \chi_{ac} n_1} \rangle_{\hat{a} a} \langle \alpha(0) e^{i \chi_{ac} n_2} | \mapsto | \alpha' e^{i \chi_{ac} n_1} \rangle_{\hat{a} a} \langle \alpha' e^{i \chi_{ac} n_2} | + \exp \left( | \alpha(0) |^2 \times \Lambda_1 \left( i \chi_{ac}(n_1 - n_2) - \chi_{ac}^2 n_1 n_2 \right) \right),
\]

This superoperator adds small phase shift to the coefficients before coherent states of the mode \( \hat{a} \) and decreases non-diagonal elements of the density matrix.

Another factor, which influences fidelity of the final state generation but is not connected with nonideality of the Kerr medium itself, is deviation of the nonlinearity effective strength \( \chi_{ac} \) from its expected value \( \chi \), used during optimization of the discrimination scheme parameters. This factor can be accounted for by introducing superoperator

\[
\hat{M}_2(\hat{a}) : | \alpha(0) e^{i \chi_{ac} n_1} \rangle_{\hat{a} a} \langle \alpha(0) e^{i \chi_{ac} n_2} | \mapsto | \alpha'(0) e^{i \chi_{ac} n_1} \rangle_{\hat{a} a} \langle \alpha'(0) e^{i \chi_{ac} n_2} | \times \left( 1 + \frac{1}{2} \chi_{ac}^2 (n_1 - n_2)^2 \right),
\]

transforming the system state in the same way as \( \hat{M}_1(\hat{a}) \).

C. Storage of the mode \( \hat{a} \) in nonideal local resonator at Alice’s site

The second stage of the protocol consists in transmission of the ancillary field from Alice to Bob. At the same time the mode \( \hat{a} \), already correlated with the ancillary mode \( \hat{c} \), is stored at Alice’s site. These two modes interact with the environment independently, and corresponding kinds of nonideality are considered separately.

Decoherence of the mode \( \hat{a} \) in nonideal local resonator is described by the following master equation:

\[
\frac{d}{dt} \rho = \kappa_2 \hat{L}(\hat{a}) \rho. \tag{61}
\]

If duration of this stage is equal to \( \Delta t_2 \), relative losses rate equals to \( \Lambda_2 = e^{2 \kappa_2 \Delta t_2} - 1 \).

The solution of Eq. (61) can be found in the way, similar to the one used for the previous stage of the protocol. The amplitude of coherent states of the mode \( \hat{a} \) at time \( \Delta t_2 \) is \( \alpha'(0) e^{-\kappa_2 \Delta t_2} \) after this stage of the protocol must be equal to its final value \( \alpha(0) \).

The influence of decoherence in Alice’s local resonator on the state of the system can be described by the superoperator

\[
\hat{M}_2(\hat{a}) : | \alpha(0) e^{i \chi_{ac} n_1} \rangle_{\hat{a} a} \langle \alpha(0) e^{i \chi_{ac} n_2} | \mapsto | \alpha' e^{i \chi_{ac} n_1} \rangle_{\hat{a} a} \langle \alpha' e^{i \chi_{ac} n_2} | \times \left( 1 + \frac{1}{2} \chi_{ac}^2 (n_1 - n_2)^2 \right),
\]

transforming the system state in the same way as \( \hat{M}_1(\hat{a}) \).

D. Transmission of the ancillary field through nonideal quantum channel

Interaction with the environment of the ancillary mode \( \hat{c} \) during the second stage of the protocol is described by master equation

\[
\frac{d}{dt} \rho = \kappa \hat{L}(\hat{c}) \rho + \Gamma \hat{L}(\hat{c}^+ \hat{c}) \rho, \tag{63}
\]

where the first and the second terms describe decoherence and dephasing of the mode \( \hat{c} \) respectively (see item [iv] in the list of nonideality factors). For characterizing
this type of system nonideality one can introduce relative losses rate \( \Lambda = e^{2\Delta t_2} - 1 \) and mean phase error \( \Delta \phi = \sqrt{\Delta t_2} \).

Due to commutativity of \( \hat{L}(\hat{c}) \) and \( \hat{L}(\hat{c}^+ \hat{c}) \) (in the sense that \( \hat{L}(\hat{c}) \hat{L}(\hat{c}^+ \hat{c}) \rho = \hat{L}(\hat{c}^+ \hat{c}) \hat{L}(\hat{c}) \rho \)), the master equation Eq. (63) can be divided into two independent parts, describing decoherence and dephasing.

State transformation because of decoherence has the form:

\[
\rho \mapsto \sum_{n=0}^{\infty} \frac{(1 - e^{-2\Delta \phi^2})^n}{n!} e^{i\phi^2} \rho(\phi^+)^n. \tag{64}
\]

Dephasing of the mode \( \hat{c} \) transforms the system state as

\[
|n_1\rangle_{cc} \langle n_2| \mapsto |n_1\rangle_{cc} \langle n_2| e^{-\Delta \phi^2(n_1-n_2)^2}. \tag{65}
\]

As stated above, for certain simplification of further consideration it is useful to choose superoperators, acting on the main field modes only, from the set of equivalent superoperators, describing transition between ideal and nonideal case. For this purpose we take into account that in the expression [34] for the ideal system and nonideal case. For this purpose we take into account that in the expression (54) for the ideal system

\[
E. Nonideal cross-Kerr interaction at Bob’s site
\]

The third stage of the protocol is cross-Kerr interaction of modes \( \hat{b} \) and \( \hat{c} \), described by the following master equation:

\[
\frac{d}{dt}\rho = i\chi'' \left[ \hat{b}^+ \hat{b}^+ \hat{c}, \rho \right] + \kappa_3 \hat{L}(\hat{b}) \rho, \tag{68}
\]

and characterized by effective nonlinearity strength \( \chi_{bc} = \chi'' \Delta t_3 \) and relative losses rate \( \Lambda_3 = e^{2\Delta \phi^2} - 1 \) (we will assume for simplicity that \( \Lambda_3 = \Lambda_1 \)), where \( \Delta t_3 \) is the duration of the interaction.

In the ideal case this nonlinear interaction is described by operator

\[
\hat{U}_{bc} = \exp(i\chi_{bc} \hat{b}^+ \hat{c}^+ \hat{c}), \tag{69}
\]

with the obtained state \( \hat{U}_{bc} \hat{U}_{ac}(\alpha)_{bc} \hat{U}_{bc}^+ |\beta\rangle_c \) being equal to \( |\Psi_{1ab}\rangle_c \) (see Eq. (3)).

For nonideal system, master equation Eq. (68) can be solved exactly in the same way as Eq. (52). The superoperator, describing transition between ideal and nonideals cases, has the form

\[
\hat{M}_3(\hat{b}): |\beta(0)e^{i\chi_{bc}(n_1)}\rangle_{bb} \langle \beta(0)e^{i\chi_{bc}(n_2)}| \mapsto \nonumber
\]

\[
\mapsto |\beta e^{i\chi_{bc}(n_1)}\rangle_{bb} \langle \beta e^{i\chi_{bc}(n_2)}| \exp\left\{ |\beta(0)|^2 \times \nonumber \right. \]

\[
\left. \times \Lambda_3 \left[ \frac{(n_1-n_2)}{3} \right] e^{\chi_{bc}(n_1-n_2)^2} \right\}. \tag{70}
\]

and, similarly to \( \hat{M}_1(\hat{a}) \), adds small phase shift to the coefficients before coherent states of the mode \( \hat{b} \) and decreases non-diagonal elements of the density matrix.

Deviation of the nonlinearity effective strength \( \chi_{bc} \) from its expected value \( \chi \) can be accounted for by introducing the following superoperator, describing additional phase shift to coherent states of the mode \( \hat{b} \):

\[
\hat{M}_3(\hat{b}): |\beta(0)e^{i\chi_{bc}(n_1)}\rangle_{bb} \langle \beta(0)e^{i\chi_{bc}(n_2)}| \mapsto \nonumber
\]

\[
\mapsto |\beta(0)e^{i\chi_{bc}(n_1)}\rangle_{bb} \langle \beta(0)e^{i\chi_{bc}(n_2)}| \tag{71}
\]

F. Nonideality of discrimination scheme due to limited efficiency and dark counts of photodetectors

The last stage of the protocol is discrimination of the state \( |\varphi\rangle_c \) of the mode \( \hat{c} \) at Bob’s site. This stage includes operations on the ancillary mode \( \hat{c} \) only and does not influence directly modes \( \hat{a} \) and \( \hat{b} \) (their state is transformed due to previously generated correlations with the ancillary mode). All the superoperators \( \hat{M}_1(\hat{a}), \hat{M}_1(\hat{a}) \), \( \hat{M}_2(\hat{a}), \hat{M}_2(\hat{a}), \hat{M}_2(\hat{a}), \hat{M}_2(\hat{a}), \hat{M}_3(\hat{b}), \hat{M}_3(\hat{b}), \hat{M}_3(\hat{b}), \) introduced for describing nonideality of the three preceding stages of the protocol, act on the state spaces of the mode \( \hat{a} \) and \( \hat{b} \). Therefore, they must commute with any superoperators, characterizing the last stage of the protocol, and can be considered as acting after implementation of nonideal discrimination measurement. For such consideration the input state of the discrimination scheme is the state \( |\Psi_{1abc}\rangle \), defined by Eq. (3).

For the ideal system, obtaining successful outcome of all the elimination measurements, followed by discarding of the mode \( \hat{c} \), transforms the input weakly entangled state \( |\Psi_{1ab}\rangle \) into the desired final state \( |\Psi_{f}\rangle_{ab} \) (see Eqs. (29), (31)). Probability of this successful outcome equals \( p_{K(ideal)}(q|\gamma|^2)^K/|q|^2 \).

Limited efficiency of detectors (described by the probability \( \lambda \) of registering photons — see item (2) in the list of
types of system nonideality) leads to decrease of probability of obtaining successful discrimination outcome, effectively reducing fraction \(q\) of the coherent state amplitude of the ancillary mode, interacting with photodetectors, by factor \(\lambda; q \rightarrow \lambda q\). Then one-run success probability for nonideal detection scheme is equal to

\[
p_K = \left(\frac{q \lambda |\gamma|^2}{|K|^2}\right)^K \approx \left(\frac{|\gamma|^2}{K}\right)^K \frac{1}{|K|^2}. \tag{72}
\]

Dark counts of photodetectors lead to mixing density matrices, characteristic to "semi-successful" outcomes, to the final density matrix, corresponding to successful elimination of all the states \(|\gamma_j\rangle\). Then, according to Eqs. \(69 - 74\), successful discrimination of the state \(|\varphi\rangle\)

by the scheme with nonideal photodetectors transforms the state \(|\Psi_1\rangle_{abc}\) into the following mixed state:

\[
\rho_{ab} = |\Psi_f\rangle_{abab} \langle \Psi_f| + \frac{\zeta}{\lambda|\gamma|^2} [|\Psi(n_1)\rangle_{abab} \langle \Psi(n_1)| + \frac{\gamma^2}{|\gamma|^4} |\Psi(n_1, n_2)\rangle_{abab} \langle \Psi(n_1, n_2)| + \ldots,
\]

where \(j\)-th term corresponds to presence of \(j - 1\) dark counts.

**G. Final state in the nonideal case**

Summarizing the results concerning discussed types of system nonideality, we can express the density matrix of the final state of the modes \(\hat{a}\) and \(\hat{b}\) in the following form:

\[
\rho_{ab}^{\text{(final)}} = \hat{M}_3(\hat{b}) \hat{M}_4(\hat{b}) \hat{M}_2^{(2)}(\hat{a}) \hat{M}_2^{(2)}(\hat{a}) \hat{M}_2(\hat{a}) \times \hat{M}_1(\hat{a}) \hat{M}_3(\hat{a}) \rho_{ab}. \tag{74}
\]

This expression can be simplified by taking into account that coherent states of the modes \(\hat{a}\) and \(\hat{b}\) poses correlated phases in all terms of the expression for the density matrix \(\rho_{ab}\) (Eq. \(74\)) and appear only in groups of the form \(c_n|\alpha e^{i\chi_\alpha}\rangle_a|\beta e^{i\chi_\beta}\rangle_b\) or \(\tilde{c}_n(-)\rangle_a|\alpha e^{i\chi_\alpha}\rangle_a|\beta e^{i\chi_\beta}\rangle_b\). Therefore, superoperators \(\hat{M}_1(\hat{a})\), \(\hat{M}_2(\hat{a})\), \(\hat{M}_2^{(2)}(\hat{a})\), \(\hat{M}_3(\hat{b})\) act at the system state in the same way: they add small phase shifts to the coefficients \(c_n\) (or \(\tilde{c}_n(-)\)) and decreases non-diagonal elements of the density matrix. Due to commutativity of the superoperators, they can be collected in a single superoperator

\[
\hat{M}_0(\hat{a}, \hat{b}) = \hat{M}_3(\hat{b}) \hat{M}_4(\hat{b}) \hat{M}_2^{(2)}(\hat{a}) \hat{M}_2(\hat{a}) \hat{M}_1(\hat{a}), \tag{75}
\]

transforming pairs of coefficients \(c_n c_{n^*}\) of the state \(|\Psi_f\rangle\) (as well as pairs of coefficients \(\tilde{c}_n(n_1, n_2, ...)\) with the similar meaning, defined by Eq. \(54\)) in the following way:

\[
c_{n_1} c_{n_2}^* \rightarrow \tilde{c}_n(n_1, n_2), \tag{76}
\]

where

\[
\eta_1 = \frac{1}{2} \Lambda_1 (|\alpha|^2 \lambda_{ac} + |\beta|^2 \lambda_{bc}) + |\alpha|^2 \lambda_{ac} \Lambda_2 \tag{77}
\]

is a phase difference per photon and

\[
\eta_2 = \Delta \phi^2 + \frac{1}{3} \Lambda_1 (|\alpha|^2 \lambda_{ac} + |\beta|^2 \lambda_{bc}) + \frac{1}{2} |\alpha|^2 \lambda_{ac} \Lambda_2 \tag{78}
\]

describes decay of non-diagonal elements of density matrix.

The first term in the exponent of Eq. \(76\) corresponds to changing phase of coefficients \(c_n\) and can be compensated by corresponding changes in the detection scheme (by replacing \(\gamma_\eta e^{-\eta_1}\) in expressions for the scheme parameters). Therefore, only the second term of the exponent is essential for estimation of the deviation of the nonideal final state from the ideal one.

Finally, expression \(\rho_{ab}^{\text{(final)}}\) for the density matrix, obtained in for nonideal system, can be rewritten using the notations of Eq. \(75\) in the form

\[
\rho_{ab}^{\text{(final)}} = \hat{M}_0(\hat{a}, \hat{b}) \hat{M}_2^{(2)}(\hat{a}) \hat{M}_3(\hat{a}) \hat{M}_2(\hat{a}) \times \hat{M}_1(\hat{a}) \hat{M}_3(\hat{a}) \rho_{ab}, \tag{79}
\]

where \(\hat{M}_4(\hat{b})\) is a phase difference per photon and

\[
\eta_2 = \Delta \phi^2 + \frac{1}{3} \Lambda_1 (|\alpha|^2 \lambda_{ac} + |\beta|^2 \lambda_{bc}) + \frac{1}{2} |\alpha|^2 \lambda_{ac} \Lambda_2 \tag{78}
\]

describes decay of non-diagonal elements of density matrix.

In further consideration we assume that all types of nonideality, present in the system, are weak enough and the fidelity is close to unity (such systems are most useful from the practical point of view). Then the density matrix, defined by Eq. \(79\), is approximately equal to

\[
\rho_{ab}^{\text{(final)}} \approx \langle \Psi_f | \rho_{ab}^{\text{(final)}} | \Psi_f\rangle + \frac{\zeta}{\lambda |\gamma|^2} \sum_{n_1} |\Psi(n_1)\rangle_{abab} \langle \Psi(n_1)| + \ldots,
\]

where notation \(\Delta \hat{M}_i = \hat{I} - \hat{M}_i\) is introduced for the superoperators; only the leading order of small parameters, characterizing nonideality of the system, is taken into account.

**H. Fidelity of the final state generation**

According to the standard definition, the fidelity of generating the desired final state \(|\Psi_f\rangle_{ab}\) equals

\[
F = \frac{ab \langle \Psi_f | \rho_{ab}^{\text{(final)}} | \Psi_f\rangle_{ab}}{\text{Tr} \rho_{ab}^{\text{(final)}}}, \tag{81}
\]

where \(\rho_{ab}^{\text{(final)}}\) is the actual final state of the main field modes \(\hat{a}\) and \(\hat{b}\), defined by Eqs. \(75\), \(80\).

Using Eqs. \(80\), \(81\) and carrying out quite straightforward mathematical calculations, one can obtain the
following expression for the fidelity of final state generation:

\[ F \approx 1 - \sum_{(\Delta M_1)} \text{Tr} \left\{ P_\perp \Delta \hat{M}_i \rho_{ab}^{(\text{final})} \right\} - \frac{\zeta}{\lambda |\gamma|^2} |e_n|^2 \sum_{n_1} \text{Tr} \left\{ P_\perp |\Psi(n_1)\rangle_{abab} \langle \Psi(n_1)| \right\}, \]  

(82)

where \( P_\perp = 1 - |\Psi_f\rangle_{abab} \langle \Psi_f| \).

For further consideration it is convenient to introduce unnormalized states \( |\Psi_f(\alpha)\rangle_{ab} = \sum_n n^* c_n |\alpha e^{i\chi n}\rangle_a |\beta e^{i\chi n}\rangle_b \), which are useful, for example, when exponent is decomposed in Eq. (75). Then it is quite easy to show that

\[ \text{Tr} \left\{ P_\perp \Delta \hat{M}_0 (\hat{a}, \hat{b}) \rho_{ab}^{(\text{final})} \right\} = 2\eta_2 |\langle \Psi_f(\alpha) | \Psi_f(\alpha)\rangle_{ab}|^2, \]

(83)

where we suppose that the term proportional to \( \eta_1 \) vanishes due to correct phase compensation.

Representing states of the form \( |\alpha e^{i\chi n}\rangle_a |\beta e^{i\chi n}\rangle_b \) as

\[ |\alpha e^{i\chi n}\rangle_a = e^{i\Delta \chi_n \hat{a}^\dagger \hat{a}} |\alpha e^{i\chi n}\rangle_a \]

(84)

and decomposing exponents in power series in this expression, one can simplify the term in the sum in Eq. (82), corresponding to deviations of nonlinearity strengths from their expected value \( \chi \):

\[ \text{Tr} \left\{ P_\perp \Delta \hat{M}_i (\hat{a}, \hat{b}) \rho_{ab}^{(\text{final})} \right\} = |\langle \Psi_f^{(1)} | \left( \Delta \chi_{ac} \hat{a}^\dagger \hat{a} + \Delta \chi_{bc} \hat{b}^\dagger \hat{b} \right) | \langle \Psi_f^{(1)} \rangle|^2 \]

(85)

Analytical expression for the remaining term in the sum can be found in two limiting cases:

\[ \text{Tr} \left\{ P_\perp \Delta \hat{M}_2^{(1)} (\hat{a}) \rho_{ab}^{(\text{final})} \right\} \approx |\alpha|^2 \chi^2 (|\lambda|^2 + |\Delta|^2 |\gamma|^4) \]

(86)

for \( |\alpha|^2 \chi^2 \ll 1 \) and

\[ \text{Tr} \left\{ P_\perp \Delta \hat{M}_2^{(1)} (\hat{a}) \rho_{ab}^{(\text{final})} \right\} \approx \lambda |\gamma|^2 \]

(87)

for \( |\alpha|^2 \chi^2 \gg 1 \).

Eqs. (82), (83), (84) - (87) provide the expression for the final state generation fidelity for any desired state \( |\Psi_f\rangle_{ab} \) described by Eq. (2).

I. Estimation of the system parameters, required for protocol implementation

Further simplification of the derived above equations for the final state fidelity can be carried out in special cases, considered in Section [V]. We discuss the states with maximal entanglement, possible for a scheme with fixed number of detectors (\( K = 1 \) and \( K = 2 \)). On the basis of explicit expressions for the fidelity of such states generation we find conditions, that must be imposed on the system parameters in order to obtain sufficiently high fidelity of final state generation.

For most of physical systems, suitable for implementation of the protocol, assumptions that \( \chi \ll 1 \) and \( |\alpha|^2 \chi^2 \ll 1 \) are valid. In this paper we discuss analytical results obtained under these assumptions only (however, the opposite limiting case \( |\alpha|^2 \chi^2 \gg 1 \) can also be described analytically).

For maximally entangled state, generated by the scheme with \( K = 1 \) detector (Eqs. (15), (46)), we obtain

\[ \text{Tr} \left\{ P_\perp \Delta \hat{M}_0 (\hat{a}, \hat{b}) \rho_{ab}^{(\text{final})} \right\} = \frac{\eta_2}{|\alpha|^2 |\gamma|^2}, \]

(88)

\[ \text{Tr} \left\{ P_\perp \Delta \hat{M}_i (\hat{a}, \hat{b}) \rho_{ab}^{(\text{final})} \right\} = 2 |\alpha|^2 (\varepsilon_{ac} + \varepsilon_{be})^2 + (\varepsilon_{ac} - \varepsilon_{bc})^2 \]

(89)

and

\[ \frac{\zeta}{|\lambda| |\gamma|^2 |\alpha|^2 |\gamma|^2}, \]

\[ \text{Tr} \left\{ P_\perp |\Psi(n_1)\rangle_{abab} \langle \Psi(n_1)| \right\} = \frac{\zeta}{2 |\lambda| |\gamma|^2 |\alpha|^2 |\gamma|^2}, \]

where \( \varepsilon_{ac,be} = \Delta \chi_{ac,be}/\chi \) are relative inaccuracies of the nonlinear interaction strengths; we assumed for simplicity that \( |\alpha| = |\beta| \) and \( q = 1/\sqrt{K} \).

For the scheme with \( K = 2 \) detectors and coefficients \( c_n \) described by Eq. (17) the form of Eqs. (88) - (90) remains the same, but expressions in the right hand side of Eqs. (88) - (90) get numerical factor 2.

The finally obtained expression for decrease of the final state fidelity contains 6 distinct terms, corresponding to different types of processes in the system: Eq. (88) corresponds to 3 terms, proportional to \( \Delta \phi^2, \lambda_1 \) and \( \Lambda_2 \) (see Eq. (78) for \( \eta_2 \)); Eq. (89) and Eq. (90) provide expressions for the terms, describing nonlinearity strength inaccuracies and nonideality of photodetectors respectively; Eq. (85) corresponds to decoherence of the mode \( \hat{c} \) that can be described by effective discrete phase errors in the mode \( \hat{c} \). In order to estimate parameters values, suitable for final state generation with sufficient fidelity, we require each of the discussed 6 terms to be not greater than some small value \( \epsilon, \varepsilon \ll 1 \) (then the fidelity will be not less than \( F \geq 1 - 6\epsilon \)). The obtained 6 inequalities can be used for finding 6 independent system parameters.

For this purpose we divide parameters, describing the system, into four groups:

(i) parameters \( \zeta, \lambda, |\alpha| \) characterize exploited “local” equipment (photodetectors, maximal field intensities providing small decoherence during local op-
that entanglement generation at such distances requires parameters and the desired fidelity counts of photodetectors are present (see e.g. comments for optical methods of information processing when dark counts of photodetectors are present [51, 52]); we also assume that $|\alpha|^2 \sim 10$;

(ii) parameters $\chi, \Delta \chi_{ac}, \Delta \chi_{bc}$, $\Lambda_1$ characterize local cross-Kerr interaction; we use the inequalities to estimate these parameters values and to find out whether such nonlinear interaction can be realized experimentally;

(iii) parameters $\Lambda$, $\Delta \phi^2$, $\Lambda_2$ characterize properties of the quantum channel and local resonator and determine maximal distance of entanglement generation;

(iv) the ancillary field amplitude $|\gamma|$ can be changed and is chosen so as to provide maximal success probability for sufficiently high fidelity of the final state generation.

Therefore, the discussed inequalities, providing sufficient fidelity, can be expressed in the following way (for $K = 1$; for $K = 2$ the conditions are the same except for numerical factor 1/2 in conditions 2, 3, 4, 6):

$$\left\{ \begin{array}{l}
\Lambda < \frac{2\epsilon^2 \Lambda}{\zeta}, \\
\Lambda_2 < 2\epsilon, \\
\Delta \phi^2 < |\alpha|^2 \chi^2 \epsilon, \\
\Lambda_1 < \frac{3}{2} \epsilon, \\
|\gamma|^2 < \frac{\epsilon}{|\alpha|^2 \chi^2 \Lambda}, \\
\varepsilon_{ac}, \varepsilon_{bc} < \frac{\epsilon}{2|\alpha|^2}.
\end{array} \right. \quad (91)$$

The first of the conditions limits acceptable losses in the quantum channel. This limitation is fundamental for optical methods of information processing when dark counts of photodetectors are present (see e.g. comments in Ref. [10]). For instance, for the considered above parameters and the desired fidelity $F = 0.9$ the maximal acceptable attenuation of the channel is limited by values $(20 \div 28)$ dB, which for optical fiber with attenuation $0.20$ dB/km correspond to maximal distances about $L_{\text{max}} \approx (100 \div 140)$ km. However, it should be noted, that entanglement generation at such distances requires quite long storage of the field $\phi$ in Alice’s local resonator. The value of the resonator finesse, required for preserving sufficient state fidelity and defined on the basis of the second condition of Eq. (91), is about $10^{12}$ in this case. Even greater values have already been predicted theoretically for crystalline whispering gallery mode resonators [53]. Experimentally demonstrated high-quality resonators are characterized by values up to $10^9 \div 10^{11}$.

Therefore, we believe that the protocol will be more suitable for efficient entanglement generation when the best available quantum channel connecting Alice’s and Bob’s sites is lossier than optical fiber.

The third condition provides lower bound on the nonlinearity value. For $\Delta \phi \sim 10^{-3} \div 10^{-2}$ and the considered above parameters the minimal nonlinearity strength is $\chi_{\text{min}} \sim 10^{-3} \div 10^{-2}$. Such values has already been predicted in existing systems for the case of precise radiation focusing [55, 52]. It should be noted, that not only nonlinearity strength, but also acceptable signal attenuation during cross-Kerr interaction is limited (the fourth condition of Eq. (91) leads to requirement $\Lambda_1 < 0.025$).

The fifth condition of Eq. (91) limits maximal probe beam intensities and determines the maximal possible one-run success probability for the protocol, described by Eq. (72). For the protocol with $K = 1$ detector the success probability is sufficiently large for all losses values not exceeding the limit determined by the first condition of Eq. (91) (Fig. 5). For $K = 2$ generation of the desired final state can be implemented without too large number of ancillary field transmissions for losses not more than approximately $14$ dB (which correspond to the distances up to $70$ km in optical fiber).

The maximal acceptable relative inaccuracies of nonlinearity strengths $\varepsilon_{ac}, \varepsilon_{bc}$, defined by the last condition of Eq. (91), are equal to $0.09$ for $K = 1$ and to $0.06$ for $K = 2$.

Such parameters values can be achieved in real systems, and, therefore, our calculations prove applicability of the protocol for entanglement generation between sites, separated by lossy media, using contemporary experimental equipment.

VII. CONCLUSIONS

To summarize, in the present work we have proposed a protocol for creating a wide class of qudit-type states (including entangled states) with arbitrary dimensionality in continuous variable system using weak cross-Kerr nonlinearity, linear beamsplitters, detectors not resolving photon numbers, and sources of coherent states.

The method of entanglement generation is based on using an ancillary field mode, transmitted from Alice’s site to Bob’s one through lossy quantum channel. Weak nonlinear interaction of the mode with the main field modes possessing by Alice and Bob leads to creation of a weakly entangled 3-modes state. The main problem, solved in our work is designing a scheme for the ancillary mode state measuring leading to probabilistic entanglement enhancement and transforming the “raw” weakly correlated state into highly entangled final one. The found POVM measurement is shown to be implementable with linear optics and photodetectors, not resolving photon numbers, on the basis of elimination measurements. The equation, defining parameters of the detection scheme for a given desired final state in a unique way, is also
that the protocol can be used for creating quantum states with entanglement higher than unity and, therefore, in certain cases corresponds to more effective use of resources than can be achieved for protocols based on entangling qubit systems. The fidelity of final state generation $F = 0.9$ can be achieved when a quantum channel with losses rate up to $(20 \pm 28)$ dB is available (it corresponds to distances up to 140 km for optical fiber). Required cross-Kerr nonlinearity is $\chi \geq \chi_{\text{min}} \sim 10^{-3} \sim 10^{-2}$ and can be created using contemporary equipment.

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Appendix A: Elimination of coherent states as POVM measurements

Here we provide mathematical description of system state transformation when measurements, eliminating coherent states $\{|\gamma_j\rangle_c\}$, are implemented. The considered measurement scheme is shown in Fig. 3(d). The mode $\hat{c}$ is mixed at linear beamsplitters with $K$ reference modes $\hat{d}_j$, prepared in coherent states $|\gamma_j\rangle_{\hat{d}_j}$ by splitting the initial coherent state of the mode $\hat{d}$. Then photodetectors $D_j$ determine presence of photons in the modes $\hat{d}_j$.

Let the state of modes $\hat{a}$, $\hat{b}$ and $\hat{c}$ before elimination measurements be described by density matrix $\rho_{\hat{a}\hat{b}\hat{c}}^{(in)}$. Taking into account that coherent states represent a natural basis for describing linear optical transformations, it is convenient to decompose the input density matrix in terms of coherent states of the mode $\hat{c}$

$$\rho_{\hat{a}\hat{b}\hat{c}}^{(in)} = \int \hat{P}_{\hat{a}\hat{b}}(\gamma_x) |\gamma_x\rangle_{\hat{c}} \langle \gamma_x | d^2 \gamma_x,$$  \hspace{1cm} (A1)

where $\hat{P}_{\hat{a}\hat{b}}(\gamma_x)$ is an operator-valued function of variable $\gamma_x$ (acting as an operator on the modes $\hat{a}$ and $\hat{b}$), analogous to Glauber function of one-mode field.

The state of the expanded system, composed by the main modes $\hat{a}$ and $\hat{b}$, the ancillary mode $\hat{c}$ and reference modes $\hat{d}_j$, after mixing the ancillary mode $\hat{c}$ with reference modes $\hat{d}_j$ at beamsplitters is described by the following density matrix:

$$\rho_{\hat{a}\hat{b}\hat{c}\hat{d}_1...\hat{d}_K} = \hat{T}_{\hat{c}\hat{d}_1}...\hat{T}_{\hat{c}\hat{d}_K} \left( \rho_{\hat{a}\hat{b}\hat{c}}^{(in)} \otimes |\tilde{\gamma}_1\rangle_{\hat{d}_1} \langle \tilde{\gamma}_1| \otimes ... \otimes |\tilde{\gamma}_K\rangle_{\hat{d}_K} \langle \tilde{\gamma}_K| \right) \hat{T}_{\hat{c}\hat{d}_1}^+...\hat{T}_{\hat{c}\hat{d}_K}^+,$$  \hspace{1cm} (A2)

where $\hat{T}_{\hat{c}\hat{d}_j} = \exp \left\{ i \theta_j \left( \hat{c}^+ \hat{d}_j + \hat{c} \hat{d}_j^+ \right) \right\}$ are the unitary operators of field transformation by beamsplitters (the

FIG. 5: (a) Dependence of one-run success probability $p_K$ on the relative losses rate $\Lambda$ for the protocol with $K$ photodetectors for fixed fidelity of final state generation $F = 0.9$: $K = 1$ (1), $K = 2$ (2). Dotted line corresponds to the limit of entanglement generation because of dark counts of detectors. (b) Dependence of one-run success probability $p_K$ on the desired fidelity $F$ for the protocol with $K$ photodetectors and different losses rates $\Lambda$: $K = 1$, $\Lambda = 14$ dB (1), $K = 2$, $\Lambda = 14$ dB (2), $K = 2$, $\Lambda = 28$ dB (3), $K = 2$, $\Lambda = 28$ dB (4).
quantity \( \theta_j \) is related to transmittance as \( T_j = \cos^2 \theta_j \).

Using Eq. (A11) one can represent the density matrix (A2) in the form

\[
\rho_{abcd_1\ldots d_K} = \int \hat{P}_{ab}(\gamma_x) |\Psi(\gamma_x)\rangle_{cd_1\ldots d_K} \langle \Psi(\gamma_x)| d^2 \gamma_x, \tag{A3}
\]

where \( |\Psi(\gamma_x)\rangle_{cd_1\ldots d_K} = \hat{T}_{cd_1\ldots d_K} |\gamma_{l_1}\rangle_{d_1}\ldots |\gamma_{l_1}\rangle_{d_K} \).

The values of transmittances \( T_j \) of the beamsplitters \( BS_j \) (Eq. (13)) and amplitudes \( \gamma_j \) of the reference modes \( d_j \) (Eq. (21)) are chosen in such a way that, amplitude of coherent state \( |\gamma_x\rangle_c \) of the ancillary mode is split in equal parts between the modes \( d_j \), and each of the reference modes effectively undergoes coherent displacement \(-\gamma_j\):

\[
|\Psi(\gamma_x)\rangle_{cd_1\ldots d_K} = |\gamma_0(\gamma_x)\rangle_c \otimes \langle iq \cdot (\gamma_x - \gamma_1)|d_1\rangle \ldots \langle iq \cdot (\gamma_x - \gamma_K)|d_K\rangle, \tag{A4}
\]

where \( \gamma_0(\gamma_x) \) is a linear function of the amplitude \( \gamma_x \) of the ancillary mode coherent state, equal to

\[
\gamma_0(\gamma_x) = \sqrt{1 - Kq^2} \gamma_x + \sqrt{\frac{1 - \delta}{\delta}} q \sum_{j=1}^{K} \gamma_j. \tag{A5}
\]

Measuring presence of photons in the modes \( d_j \) by the photodetectors can be described by a set of pairs of projective operators

\[
\hat{P}_{-j} = |0\rangle\langle d_j|_j \quad \text{and} \quad \hat{P}_{+} = 1 - \hat{P}_{-j}, \tag{A6}
\]

describing absence and presence of photons in corresponding field mode respectively.

The final state of modes \( a, b \) and \( c \) after carrying out the measurements and discarding reference modes \( d_j \) is described by density matrix

\[
\rho_{abc}^{(out)} = \int \hat{P}_{ab}(\gamma_x) |\gamma_0(\gamma_x)\rangle_{cc} \langle \gamma_0(\gamma_x)| \prod_{j=1}^{K} \text{Tr}_{d_j} \left\{ \hat{P}_{\pm} |iq \cdot (\gamma_x - \gamma_j)\rangle_{d_j} \langle iq \cdot (\gamma_x - \gamma_j)|_{d_j} \right\} \cdot d^2 \gamma_x, \tag{A7}
\]

where the type of used projector ("-" or "-"+) depends on the obtained measurement outcome. This expression can be simplified using the following relation:

\[
\text{Tr}_{d_j} \left\{ \hat{P}_{\pm} |iq \cdot (\gamma_x - \gamma_j)\rangle_{d_j} \langle iq \cdot (\gamma_x - \gamma_j)|_{d_j} \right\} = \sum_{n_j \in \Omega_{\pm}} |d_j\rangle \langle n_j |i q \cdot (\gamma_x - \gamma_j)|_{d_j}\rangle = \sum_{n_j \in \Omega_{\pm}} \frac{q^{2n_j}}{n_j!} |\gamma_x - \gamma_j\rangle^{2n_j} e^{-q^2|\gamma_x - \gamma_j|^2}, \tag{A8}
\]

where \( \Omega_- = \{0\} \) and \( \Omega_+ = \{n \mid n \geq 1\} \) are the sets of photon numbers, corresponding to the measurement outcomes "-" (absence of photons detected) and "-" (photocount obtained) respectively.

Introducing superoperator, which describes measurement-invariant part of the ancillary mode state transformation by the definition

\[
\hat{M} : |\gamma_x\rangle_{cc} \langle \gamma_x| \mapsto |\gamma_0(\gamma_x)\rangle_{cc} \langle \gamma_0(\gamma_x)| \exp\left( -q^2 \sum_{j=1}^{K} |\gamma_x - \gamma_j|^2 \right), \tag{A9}
\]

one can transform Eq. (A7) for the density matrix \( \rho_{abc}^{(out)} \) to the form

\[
\rho_{abc}^{(out)} = \hat{M} \left\{ \int \hat{P}_{ab}(\gamma_x) |\gamma_x\rangle_{cc} \langle \gamma_x| \prod_{j=1}^{K} \sum_{n_j \in \Omega_{\pm}} \frac{q^{2n_j}}{n_j!} |\gamma_x - \gamma_j\rangle^{2n_j} \cdot d^2 \gamma_x \right\} , \tag{A10}
\]

Then one can define operators, which act as follows

\[
\hat{B}_{jn_j} = \frac{q^{n_j}}{n_j!} (\hat{c} - \gamma_j)^{n_j} : |\gamma_x\rangle \mapsto \frac{q^{n_j}}{n_j!} (\gamma_x - \gamma_j)^{n_j} |\gamma_x\rangle, \tag{A11}
\]

and transform the expression (A10) for the final state density matrix in the following way:

\[
\rho_{abc}^{(out)} = \hat{M} \left\{ \int \hat{P}_{ab}(\gamma_x) \cdot \sum_{n_j \in \Omega_{\pm}} \hat{B}_{Kn_K} \cdots \hat{B}_{1n_1} |\gamma_x\rangle_{cc} \langle \gamma_x| \hat{B}_{1n_1}^+ \cdots \hat{B}_{KKn_K}^+ \cdot d^2 \gamma_x \right\} = \hat{M} \left\{ \sum_{n_j \in \Omega_{\pm}} \hat{B}_{Kn_K} \cdots \hat{B}_{1n_1} \rho_{abc}^{(in)} \hat{B}_{1n_1}^+ \cdots \hat{B}_{KKn_K}^+ \right\} , \tag{A12}
\]
Eq. (A12) describes transformation of the system density matrix by elimination measurements in the form, similar to the one, corresponding to POVM measurements. It should be noted, however, that, the standard normalization condition is satisfied only for complete state transformation by the measuring setup (including action of superoperator $\hat{M}$ and K operators $\hat{B}_{ni}$, ..., $\hat{B}_{K\lambda K}$) rather than for single state elimination.

Operators $\hat{B}_{nj}$ for $n_j > 0$ correspond to the definition Eq. (3) of operators, describing successful elimination of the state $|\gamma_j \rangle_c$:

$$\hat{B}_{nj}|\gamma_j \rangle_c = \frac{q_{nj}}{\sqrt{n_j}}(\gamma_x - \gamma_j)^{n_j}|\gamma_x \rangle_c = 0, \ n_j = 1, 2, ... \ (A13)$$

Therefore, returning to the notations of Eqs. (8), (9), we can define operators $\hat{A}_{nj}(|\gamma_j \rangle_c)$, eliminating coherent state $|\gamma_j \rangle_c$, as

$$\hat{A}_{nj}^{(n_j)} = \hat{B}_{nj} \equiv \frac{q_{nj}}{\sqrt{n_j}}(\hat{c} - \gamma_j)^{n_j}, \ n_j = 1, 2, ... \ (A14)$$

These operators satisfy the conditions, provided by Eqs. (8), (9):

$$\hat{A}_{nj}^{(n_j)}|\gamma_j \rangle_c = 0, \ (A15)$$

$$\left[ \hat{A}_{nj}^{(n_j)}, \hat{A}_{nj'}^{(n_j')} \right] = 0 \text{ for all } n_i, n_j. \ (A16)$$

According to Eq. (A12), the final state of the system in the case of successful outcome of elimination of all the states $|\gamma_j \rangle_c$ is described by density matrix

$$\rho^{(\text{out})}_{abc} = \hat{M} \left\{ \sum_{n_j \geq 1} \hat{A}_{nj}^{(n_j)} ... \hat{A}_{nj}^{(n_j)} \rho_{abc} \left( \hat{A}_{nj}^{(n_j)} \right) + ... \left( \hat{A}_{nj}^{(n_j)} \right) \right\}, \ (A17)$$

If the input state is $|\Psi_1 \rangle_{abc} = \hat{D}_c \left( \hat{F}_{ab} \gamma \right) |\alpha \rangle_a |\beta \rangle_b |\gamma \rangle_c$ (see Eqs. (3), (28)), the expression for the final state can be rewritten as

$$\rho^{(\text{out})}_{abc} = \hat{M} \left\{ \sum_{n_j \geq 1} \frac{q^2 |\gamma|^2 n_1 + ... + 2n_K}{n_1! ... n_K!} \left( \hat{F}_{ab} - \frac{\gamma_K}{\gamma} \right)^{n_K} ... \left( \hat{F}_{ab} - \frac{n_1}{\gamma} \right)^{n_1} |\Psi_1 \rangle_{abc} \langle \Psi_1 | \left( \hat{F}_{ab}^{+} - \frac{n_1}{\gamma} \right)^{n_1} ... \left( \hat{F}_{ab}^{+} - \frac{\gamma_K}{\gamma} \right)^{n_K} \right\}, \ (A18)$$

where Eqs. (A14), (29) were taken into account. For small ancillary field amplitudes $|\gamma| \ll 1$ the main contribution to the final state density matrix is made by the term with $n_1 = ... = n_K = 1$ (the most probable case of successful elimination of the states $|\gamma_j \rangle$) corresponds to detection of exactly 1 photon by each of the detectors. After discarding mode $\hat{c}$, the final state of modes $\hat{a}$ and $\hat{b}$ is described by density matrix

$$\rho_{ab} = \text{Tr}_c \rho^{(\text{out})}_{abc} = q^{2K} |\gamma|^2 |\Psi_f \rangle_{ab} \langle \Psi_f | + O \left( |\gamma|^{2K+2} \right), \ (A19)$$

where

$$|\Psi_f \rangle_{ab} = \left( \hat{F}_{ab} - \frac{2K}{\gamma} \right) ... \left( \hat{F}_{ab} - \frac{n_1}{\gamma} \right) |\alpha \rangle_a |\beta \rangle_b = \frac{1}{cK} \sum_{n=0}^{K} c_n \hat{F}_{ab}^{n} |\alpha \rangle_a |\beta \rangle_b = \frac{1}{cK} |\Psi_f \rangle_{ab}; \ (A20)$$

we have taken into account that amplitudes $\{\gamma_j \}$ are roots of Eq. (14): $|\Psi_f \rangle_{ab}$ is the desired final state, described by Eq. (2).

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