Reconstruction of permutations distorted by single transposition errors

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Abstract— The reconstruction problem for permutations on \( n \) elements from their erroneous patterns which are distorted by transpositions is presented in this paper. It is shown that for any \( n \geq 3 \) an unknown permutation is uniquely reconstructible from 4 distinct permutations at transposition distance at most one from the unknown permutation. The transposition distance between two permutations is defined as the least number of transpositions needed to transform one into the other. The proposed approach is based on the investigation of structural properties of a corresponding Cayley graph. In the case of at most two transposition errors it is shown that \( N(x,y) \geq n \) erroneous patterns are required in order to reconstruct an unknown permutation. Similar results are obtained for two particular cases when permutations are distorted by given transpositions. These results confirm some bounds for regular graphs which are also presented in this paper.

I. INTRODUCTION

Efficient reconstruction of arbitrary sequences was introduced and investigated by Levenshtein for combinatorial channels with errors of interest in coding theory such as substitutions, transpositions, deletions and insertions of symbols [1], [2]. Sequences are considered as elements of a vertex set \( V \) of a graph \( = (V,E) \) where an edge \( xy \in E \) is viewed as the single error transforming \( x \) into \( y \). One of the metric problems which arises here is the problem of reconstructing an unknown vertex \( x \) from a minimum number of vertices in the metric ball \( B_r(x) \) of radius \( r \) centered at the vertex \( x \). It is reduced to finding the value

\[
N(x,y) = \max_{x,y \in V} |B_r(x) \setminus B_r(y)| \geq 1
\]

where \( N(x,y) \geq 1 \) is the least number of distinct vertices in the ball \( B_r(x) \) around the unknown vertex \( x \) which are sufficient to reconstruct \( x \) subject to the condition that at most \( r \) single errors have happened. As one can see, this problem is based on considering metric balls in a graph but it differs from traditional packing and covering problems in various ways. It is motivated by a transmission model where information is realized in the presence of noise without encoding or redundancy, and where the ability to reconstruct a message (vertex) uniquely depends on having a sufficiently large number of erroneously patterns of this message.

The value \( N(x,y) \) was studied for the Hamming and Johnson graphs [2]. Both graphs are distance-regular and the first is a Cayley graphs. The problem of finding the value \( N(x,y) \) is much more complicated for graphs which are not distance-regular. Cayley graphs of this kind arise for instance on the symmetric group and the signed permutation group, when the reconstruction of permutations and signed permutations is considered for distortions by single reversal errors [3], [4].

In this paper we continue these investigations and consider the reconstruction problem for permutations distorted by single transposition errors which consist of swapping 1) any two elements of a permutation; 2) any two neighboring elements of a permutation; and 3) the first and any other element of a permutation. The corresponding graphs are the transposition Cayley graph, the bubble-sort Cayley graph and the star Cayley graph. They are regular but not distance-regular. We investigate the combinatorial properties of these graphs and present the values \( N(x,y) \) when \( r = 1,2 \) in each case. Some bounds on \( N(x,y) \) for regular graphs are also considered. It is shown that the bubble-sort and star Cayley graphs are examples for which these bounds are attained.

II. DEFINITIONS, NOTATION, GENERAL RESULTS

Let \( G \) be a finite group and let \( S \) be a set of generators of \( G \) such that the identity element \( e \) of \( G \) does not belong to \( S \) and such that \( S = S^{-1} \); where \( S^i = f^s \) is \( : s \in S \). In the \emph{Cayley graph} \( = Cay(G,S) = (V,E) \) vertices correspond to the elements of the group, i.e. \( V = G \), and edges correspond to multiplication on the right by generators, i.e. \( E = \{ f g | g \in S \} \). Denote by \( d(x,y) \) the \emph{path distance} between the vertices \( x \) and \( y \) in \( G \); and by \( d(\cdot) = m \max_{x \neq y} d(x,y) \) the \emph{diameter} of \( G \). In other words, in a Cayley graph the diameter is the maximum, over \( g \in G \), of the length of a shortest expression for \( g \) as a product of generators. For the vertex \( x \) let \( S_r(x) = f y \in V : d(x,y) = r \) and \( B_r(x) = f y \in V : d(x,y) \leq r \); \( B_r(x) \) is the \emph{ sphere} and the \emph{ball} of radius \( r \) centered at \( x \); respectively. The vertices \( y \in B_r(x) \) are \( r \)-neighbors of the vertex \( x \).

As mentioned in the Introduction, the value \( N(x,y) \) was investigated initially for distance-regular graphs such as the Hamming and Johnson graphs. Let us recall that a simple connected graph is \emph{distance-regular} if there are integers \( b_i \) for \( i \geq 0 \) such that for any two vertices \( x \) and \( y \) at distance \( d(x,y) = i \) there are precisely \( c_i \) neighbors of \( x \) at distance \( j \).
y in $S_{i+1}(x)$ and $b_2$ neighbors of $y$ in $S_{i+1}(x)$; Evidently is regular of valency $k = b_2$; or $k$-regular. A $k$-regular simple graph is strongly regular if there exist integers and such that every adjacent pair of vertices has common neighbors, and every nonadjacent pair of vertices has common neighbors.

The Hamming space $F_q^n$ consists of the $q^n$ vectors of length $n$ over the alphabet $\{0,1,\ldots,q\}$ and $q = 2$. It is endowed with the Hamming distance $d$ where $d(x,y)$ is the number of coordinate positions in which $x$ and $y$ differ. It can be viewed as a graph $L_n(q)$ with vertex set given by the vector space $F_q^n$ (where $F_q$ is the field of $q$ elements) where $x \cdot y$ is an edge of $L_n(q)$ iff $d(x,y) = 1$. This Hamming graph is the Cayley graph on the additive group $F_q^n$ when we take the generator set $S = \{x \in F_q : x \cdot 2 \neq 0 \}$. It was shown in [1], [2] that for any $n$ and $q$ and $r \geq 1$,

$$N(L_n(q); r) = q \binom{n}{i}$$

for the particular case $n = 2$ the Hamming graph $L_2(q)$ is the lattice graph over $F_q$. This graph is strongly regular with parameters $\nu = q^2$; $k = 2(\nu - 1)$; $\delta = q$; $\rho = 2$; and from [2] we get $N(L_2(q); 1) = q$ and $N(L_2(q); 2) = q^2$:

The Johnson graph $J_q^n$ is defined on the subset $V = J_q^n$ consisting of all $0q$-vectors with exactly $e$ entries equal to 1 for a fixed $e$ and $n$. On $J_q^n$ the Johnson distance is defined as half the (even) Hamming distance, and two vertices $x, y$ are joined by an edge iff they are at Johnson distance 1 from each other. In particular, $J_q^n$ is not a Cayley graph although the notion of errors being represented by edges makes sense all the same. In particular, two vertices are at distance 1 from each other iff one is obtained from the other by the interchange of two coordinate positions. In [1], [2] it was shown that

$$N(J_q^n; r) = q \binom{n}{i} \binom{n-1}{i-1}$$

for any $n$, $2r \leq 1$, $r \geq 1$. In the particular case $e = 2$ and $n = 4$ the Johnson graph $J_4^3$ is the triangular graph $T(4)$. As vertices it has the 2-element subsets of an $n$-element set and two vertices are adjacent iff they are not disjoint. This graph is strongly regular with parameters $\nu = q^2 - 1$; $k = 2n - 2$; $\delta = 4$; and from [3] we obtain $N(T(4); 1) = n$ and $N(T(4); 2) = 2n^2 - 12$:

These two results were the first analytic formulas for the reconstruction problem we are interested in. Their uniformity depends on the fact that these graphs are distance-regular. What then are the general results for simple graphs, regular graphs and Cayley graphs? We start with a few observations from [3] for any connected simple graphs $G(V,E)$. In the spirit of distance regularity we put $k_i(x) = |\beta_i(x)|$ and define numbers $c_i(x,y)$; $b_2(x,y)$ and $a_i(x,y)$ for any two vertices $x \in V$ and $y \in S_i(x)$ such that

$$c_i(x,y) = \{z \in S_{i-1}(x) : d(z,y) = 1\}$$

$$b_2(x,y) = \{z \in S_{i-1}(x) : d(z,y) = 1\}$$

$$a_i(x,y) = \{z \in S_i(x) : d(z,y) = 1\}$$

From this $a_i(x,y) = a_i(y,x)$ is the number of triangles over the edge $d(x,y)$ and $c_0(x,y)$ is the number of common neighbors of $x \in V$ and $y \in S_0(x)$:

$$= ( \frac{m}{x \cdot y} a_i(x,y)$$

$$= ( \frac{m}{x \cdot y} a_i(x,y)$$

Since $B(x) \setminus B(y) > 0$ for $x \neq y$ if and only if $d(x,y) > 2$ we have

$$N(x,r) = \frac{m}{x \cdot y} N_s(x,y)$$

where $N_s(x,y) = m \frac{B(x) \setminus B(y) > 0}{x \cdot y}$.

In particular, $N_1(1) = 2$ and $N_2(1) = 2$ so that

$$N(x,1) = m \frac{(x + 2)}{2}$$

One can easily check that using this formula for the lattice graph $L_2(q)$ and the triangular graph $T(n)$ we obtain again the earlier formulas [2] and [3]. Indeed, since $n = 2$ and $4$ for $T(n)$; $n = 4$ we have $N(T(n); 1) = n$ from [7]. By the same reason we have $N(L_2(q); 1) = q$ since $q = 2$ and $2$ in this case.

We have no general results for $N(x,r)$ when $x$ is a regular graph. The numbers $c_0(x,y)$ and $b_2(x,y)$ usually depend on $y \in S_i(x)$ and this causes difficulties when searching for general estimates of $N(x,r)$. However, some bounds on $N(x,1)$ and $N(x,2)$ were obtained in [5]. Here it is assumed that is connected, $k$-regular of diameter $d(x) = 2$ with $v$ vertices and parameters $0 < k < 2$;

$$N(x,1) = \frac{1}{2}(v + 1)$$

This theorem is proved by checking that $+ 2 \frac{1}{2}(v + 1)$ and $\frac{1}{2}(v + 1)$: The first inequality takes place since $k < 2$ and $k < 2$. Moreover, there is equality only if $v = 4$ and $k = 2$. The second inequality is true since counting edges between $S_1(x)$ and $S_2(x)$ for any $x \in V$ we have $|x \cdot S_1(x)| = 1 a_i(x,y) = 1 v \cdot 2$ and the fact that $k_2(x) \wedge x \cdot k = 1$ we get $k \cdot (v + 1)$ with equality if and only if $x$ is strongly regular. Let us note here that the equality $k = 1$ is well-known for strongly regular graphs. From this and the fact that $k = 2$ we have $1 < k < 2$ and hence $k < 2(v + 1$) is valid for any regular graph. By taking into account these two inequalities and we get [8] from [7]. Moreover, [8] is attained on the strongly regular $t$-partite graph $K^{(t)}_k$ with $t(k)$ vertices partitioned into $t$ parts, where $t(k) = \frac{2k}{k}$ is an integer, and with edges connecting any two vertices of different parts.
Theorem 2: For any k-regular graph we have
\[
N_2 ( ; 2) = \frac{3}{4} (\ N ( ; 1) \ N ( ; 1)) + 2 (9)
\]

In proving the linear programming problem arises for the vertex subset \( U = \bigcup_{i=1}^{n} B_i (x_i) \ n \times B(y) \) where \( (x_i, y) \) is a vertex at distance 1 from both \( x \) and \( y \). The task is to minimize \( j = h \) \( u_h \) for nonnegative numbers \( u_h \) satisfying the following conditions
\[
\begin{align*}
X & = \sum_{h=1}^{n} u_h h^2 \ 
\left( k \ 1 \right) ; \\
X & = \sum_{h=1}^{n} u_h h \ 
\left( 2 \ 2 \right) \ \Omega \left( G ; 1 \right) ; \\
\end{align*}
\]

where \( u_h = \left\{ \begin{array}{ll}
0 & \text{if} \ h \text{ is not distance–transitive} \\
1 & \text{if} \ h \text{ is distance–transitive}
\end{array} \right. \) and \( U (1) \) is the set of vertices in \( U \) belonging to \( h \) sets \( B_i (x_i) \), \( i = 1, \ldots, n \).

The details of the proofs for Theorems 1 and 2 can be found in [5]. From the last theorem one can immediately get the following corollaries.

Corollary 1: For a k-regular graph \( \; ; \) in \( (1) ; \) \( \) then \( N_2 ( ; 2) = k + 1 \); \n(ii) if \( \) \( N \); \( \) \( \) \( ; 2 \) \( \) \( \) \( 2k \); \n(iii) if \( \) \( N \); \( \) \( \) \( ; 1 \) \( \) \( \) \( 3 \); \nthen \( N \); \( \) \( \) \( ; 2 \) : \n\[
\begin{align*}
N_2 ( ; 2) & = N_1 ( ; 2) ; \\
N_2 ( ; 2) & = 2k \ ( \ 2 \ ) \ \Omega \left( 1 \right) \ 
\left( 3 \ 4 \right) \ (1) = 0 ;
\end{align*}
\]

and finally we obtain (10).

III. The reconstruction of permutations in Cayley graphs generated by transpositions

Let \( \text{Sym}_n \) be the symmetric group on \( n \) symbols. We write a permutation in one–line notation as \( \sigma = [ \ i \ j \ : \ : \ : \ : \ n \ ] \) where \( i = (i) \) for every \( i \ 2 \ \Omega \ 1 \). For the transposition Cayley graph \( \text{Sym}_n ( \Gamma ) \) on \( \text{Sym}_n \); the generator set consists of all transpositions \( T = \{ t_{ij} \} \ 2 \ \text{Sym}_n \); \( i \ 1 \ 2 \ j \) \( \Omega \ j \ = \ n \); \( t_{ij} \) interchanges positions \( i \) and \( j \) when \( i \neq j \) or \( i = j \) and \( t_{ij} \Omega \ j \ = \ x \). As any \( k \)-cycle can be written as a product of \( k \) transpositions (but no fewer), the diameter of \( \text{Sym}_n ( \Gamma ) \) is \( \Omega (1) \). The graph is bipartite since any edge joins an even permutation to an odd permutation. The symmetry properties of \( \text{Sym}_n ( \Gamma ) \) have been discussed in [6]. The graph is edge–transitive but not distance–regular and hence not distance–transitive. All these properties and other basic facts are collected in the following statements.

Lemma 2: The transposition Cayley graph \( \text{Sym}_n ( \Gamma ) ; \ n \ 3 \) \( \) is a connected bipartite \( n \ 2 \)-regular graph of order \( n ! \) and diameter \( \Omega (1) \); \n(iii) it is not distance–regular and hence not distance–transitive; \n(iii) it does not contain subgraphs isomorphic to \( K_{2,4} \) and each of its vertices belongs to \( n \ 3 \) subgraphs isomorphic to \( K_{3,2} \).

(Here \( K_{p,q} \) is the complete bipartite graph with \( p \) and \( q \) vertices in the two parts, respectively.)

Theorem 3: For any \( n \ 3 \) we have \( \Omega ( \text{Sym}_n ( \Gamma ) ; 1) = 3 \);

This means that any unknown permutation is uniquely reconstructible from 4 distinct permutations at transposition distance at most one from the unknown permutation. The proof of these statements is based on considering a permutation \( \sigma = [ \ i \ j \ : \ : \ : \ : \ n \ ] \) in cycle notation, with cycle type \( \sigma = [ \ i \ 1 \ 2 \ : \ : \ : \ h_i \ ] \), where \( h_i \) \( = \Omega \) is the number of cycles of length \( i \). In particular \( \sigma = [ \ i \ ] \Omega \) \( h_i = n \). The permutation can be also presented as a product of a least number of transpositions. Each such product represents a shortest path in \( \text{Sym}_n ( \Gamma ) \) from \( e \) to \( \sigma \). The number of such paths was obtained in [7]. This result is based on Ore’s theorem on the number of trees with \( n \) labeled vertices and presented by the following theorem.

Theorem 4: [7] Let \( 2 \ < \ \Omega \ n \ 3 \) have cycle type \( \sigma = [ \ i \ 1 \ 2 \ : \ : \ : \ h_i \ ] \), consisting of \( n \ 1 \ \Omega \ l_i = \ n \ 1 \) cycles where \( l_i \ 1 \ \Omega \ n \ 1 \) Then the number of distinct ways to express as a product of transpositions is equal to

\[
\begin{align*}
\Omega (n) & = \prod_{i=1}^{n} \sum_{j=1}^{h_i} \binom{h_i}{j} \Omega (j-1) ! \Omega (j) !, \\
\end{align*}
\]

According to the above theorem, the following lemma gives us formulas for the numbers \( c_i ( \sigma ) = c_i ( \sigma ; e ) ; b_i ( \sigma ) = b_i ( \sigma ; e ) ; a_i ( \sigma ) = a_i ( \sigma ; e ) \) for small \( \sigma \) in Cayley graphs on the symmetric group \( \text{Sym}_n \) will be presented in the next section when the generator set \( S \) consists of transpositions.
Lemma 3: In the transposition graph $Sym_n(\Gamma)$ the sets $S_i = S_i(e); 1 \leq i \leq n$; are the permutations consisting of $(n) i$ disjoint cycles, counting also 1-cycles. For any $2 S_i$ with cycle type $\alpha = 1^n 2^h_1 \cdots 2^h_i$ we have $a_1(\alpha) = 0$ and

$$c_1(\alpha) = \frac{1}{2} h^2_1 n^2; a_2(\alpha) = \frac{1}{2} h^2_2 n^2; \cdots; a_i(\alpha) = \frac{1}{2} h^2_i n^2.\quad (10)$$

In particular, since $a_1(\alpha) = 0$ for any $1 \leq i \leq n$; then from this lemma and by (4) we have $\mathcal{S}(Sym_n(\Gamma)) = 0$ Moreover, it is well-known that two permutations are conjugate by an element of $Sym_n$ if and only if they have the same cycle type. If $1^n 2^h_1 \cdots 2^h_i$ denotes the conjugacy class of an element of cycle type $1^n 2^h_1 \cdots 2^h_i$ then it is shown in [5] that $S_i; 1 \leq i \leq n$ is the disjoint union

$$S_i = \{ 1^n 2^h_1 \cdots 2^h_i \}^{2^h_i}; \quad (11)$$

where

$$j^n 1^n 2^h_1 \cdots 2^h_i = \frac{n!}{h!_1 h!_2 \cdots h!_n}.\quad (12)$$

Hence, from (11) we have $S_2 = \{ 1^n 3 1^2 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2$ and then by Lemma 3 we get $c_2(\alpha) = 3$ if $\alpha = 1^n 3 1^2$; and $c_2(\alpha) = 2$ if $\alpha = 1^n 4 2^2$. From these and Lemma 5 we have $\mathcal{S}(Sym_n(\Gamma)) = 3$; and therefore, by (7) we get Theorem 5 Moreover, there are no subgraphs isomorphic to $K_{2n}$ in $Sym_n(\Gamma)$ since $\mathcal{S}(Sym_n(\Gamma)) = 3$. The number $n^3$ of subgraphs isomorphic to $K_{3,3}$ and having e as one of its vertices is obtained from (12) for any $2 (1^n \cdots 3^2)$. By vertex–transitivity the same holds for any vertex in $Sym_n(\Gamma)$ (see condition (iii) in Lemma 2).

So, any unknown permutation is uniquely reconstructible from 4 distinct permutations at transposition distance at most 1 from the unknown permutation. As the following shows, in the case of at most two transposition errors the reconstruction of the permutation requires many more distinct 2-neighbors of $\Gamma$.

Theorem 5: For $n \geq 3$ we have

$$N(\mathcal{S}(Sym_n(\Gamma))) = \frac{3}{2} (n^2 + 1).\quad (13)$$

This follows from the fact that the normalizer of $\Gamma$ is $\mathcal{G} = Sym_n$ itself and from the following lemma.

Lemma 4: For any $2 S_i; 1 \leq i \leq n$ the number of vertices in $(1^n 2^h_1 \cdots 2^h_i)\{\Gamma\}$ at a given distance from depends only on the conjugacy class to which belongs.

To prove Theorem 5 it is therefore sufficient to consider the numbers of vertices in all subsets of $B_2(e)$ at minimal distance at most 2 from a given vertex $2 S_i; 1 \leq i \leq 4$. By (11) we have $S_1 = \{ 1^n 2^2 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2, S_2 = \{ 1^n 3 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2, S_3 = \{ 1^n 4 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2, S_4 = \{ 1^n 5 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2, S_5 = \{ 1^n 6 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2, S_6 = \{ 1^n 7 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2, S_7 = \{ 1^n 8 \} \cdots \frac{1^n}{2} 1^2 4^2 \cdots 2^2$. It is shown that $N(\mathcal{S}(Sym_n(\Gamma))) = 20$ for $n = 5; \quad N_3(\mathcal{S}(Sym_n(\Gamma))) = 12$ for $n = 4; \quad N_2(\mathcal{S}(Sym_n(\Gamma))) = \frac{n}{2} (n + 1) \cdots (n + 1) + 1$ for all $n \geq 3$; From these and by (5) one can conclude (13).

The statements of Theorem 5 and Corollary 2 are generalized in the following conjecture.

Conjecture 1: For any $2 (1^n \cdots 3^i)\Gamma$; for any $r \geq 1$ and $n \geq 2r + 1$ we have

$$N(\mathcal{S}(Sym_n(\Gamma))) = \begin{cases} n & \text{for } r = 1; \\ 2^n & \text{for } r > 1. \end{cases}\quad (14)$$

Now let us consider the bubble–sort graph $Sym_n(\Gamma)$ This is the Cayley graph on $Sym_n$ for the generator set $\mathcal{T} = \{ t_{ij+1}, t_{ij+1} \in S_i; e \}$ then necessarily $t_{ij+1} t_{ij+1} = t_{ij+1} t_{ij+1} t_{ij+1} t_{ij+1}$ and $t_{ij+1} t_{ij+1} t_{ij+1} t_{ij+1}$ is disjoint. It suffices to verify this for permutations on 4 letters. Hence there are at most two such neighbors and so $\mathcal{S}(Sym_n(\Gamma)) = 2$. It can be also verified that we have $N(\mathcal{S}(Sym_n(\Gamma))) = 4$ for $n = 5; \quad N(\mathcal{S}(Sym_n(\Gamma))) = 2$ for $n = 4; \quad N(\mathcal{S}(Sym_n(\Gamma))) = 2$ for $n = 3$; From all these and by (5) and (7) we get the following theorem.

Theorem 6: For any $n \geq 3$ we have

$$N(\mathcal{S}(Sym_n(\Gamma))) = 2 \quad \text{and} \quad N(\mathcal{S}(Sym_n(\Gamma))) = 2.\quad (15)$$

Almost the same results appear for the star Cayley graph $Sym_n(st)$ generated by the set of prefix–transpositions $st = \{ s_{ij}, \Gamma \}$. $\Gamma$ is connected $\Gamma$ is it is one of the most investigated graphs in the theory of interconnected networks since many parallel algorithms are efficiently mapped on the star Cayley graph.

Lemma 6: [8] The star Cayley graph $Sym_n(st)$ is a connected bipartite $(n + 1)$-regular graph of order $n$ with diameter $d = \frac{3(n - 1)}{2}$.

The star Cayley graph $Sym_n(st)$ is not distance–regular for $n \geq 4$ [6] and has no cycles of lengths 3, 4, 5 or 7. Hence $\mathcal{S}(Sym_n(st)) = 0$ and $\mathcal{S}(Sym_n(st)) = 1$. Moreover, it is easy to verify that $N(\mathcal{S}(Sym_n(st))) = 4$ for $n = 5; \quad N(\mathcal{S}(Sym_n(st))) = 4$ for $n = 4; \quad N(\mathcal{S}(Sym_n(st))) = 2$ for $n = 3$; From these properties and by (5) and (7) we get the following theorem.
**Theorem 7:** For any \( n \geq 4 \) we have

\[ N(\text{Sym}_n(\text{st}); 1) = 2 \quad \text{and} \quad N(\text{Sym}_n(\text{st}); 2) = 2(n-1). \]

Thus, in the bubble–sort and star Cayley graphs any unknown permutation is uniquely reconstructible from 3 distinct 1–neighbors of. Similarly, for the unique reconstruction of from neighbors at distance at most 2 we see that any \( 2n-1 \) distinct 2–neighbors of are sufficient. These two graphs are examples for which the inequality \((2)\) in Corollary (1) is attained.

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