Path Integrals in a Multiply-Connected Configuration Space (50 Years After)

Amaury Mouchet

Received: 22 January 2021 / Accepted: 2 September 2021 / Published online: 3 November 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract
The proposal made 50 years ago by Schulman (Phys Rev 176(5):1558–1569, 1968), Laidlaw and Morette-DeWitt (Phys Rev D 3(9):1375–1378, 1971) and Dowker (J Phys A 5:936–943, 1972) to decompose the propagator according to the homotopy classes of paths was a major breakthrough: it showed how Feynman functional integrals opened a direct window on quantum properties of topological origin in the configuration space. This paper casts a critical look at the arguments brought by this series of papers and its numerous followers in an attempt to clarify the reason why the emergence of the unitary linear representation of the first homotopy group is not only sufficient but also necessary.

Keywords Path integrals and topology · Path integrals in multiply-connected space · Topological phases · Anyons

We must neglect our models and study our capabilities.

Edgar Allan Poe ([1], p. 122)

This article comes back to the 50-years-old following statement: when the quantum propagator in configuration space is split into homotopy classes of paths according to

\[
K(q_f, t_f, q_i, t_i) = \sum_{c \in \pi_1(q_i, q_f)} E(c) \int_{c \in \mathbb{C}} e^{iS[c]} d[c],
\]

(1)

then the coefficients \(E(c)\) are necessarily given by the images of a unitary representation of the first homotopy group of the configuration space. After having presented the context and the high stakes of such decomposition, Sect. 2 will briefly recall

\[\text{Amaury Mouchet}
\text{mouchet@phys.univ-tours.fr}
\]

1 Institut Denis Poisson de Mathématiques et de Physique Théorique, Université de Tours — CNRS (UMR 7013), Parc de Grandmont, 37200 Tours, France
the basic concepts while setting up the notations. The core of this work will be the object of Sect. 3 where a proof of this statement will be proposed. This will provide us a vantage point from which we will be able to cast, in Sect. 4, a critical eye on the arguments advocated by Schulman [2–4], Laidlaw and Morette-DeWitt [5] and Dowker [6] and their numerous followers. Despite its fundamental importance, to my knowledge, almost all the attempts of justifying the decomposition (1) concern the fact that a unitary representation is sufficient to get a consistent model for the quantum evolution. The only exceptions being the works of Laidlaw and Morette-DeWitt [5] and Schulman ([7], § 23.3), to which the literature on the subject seems to always refer eventually. While underlying their major contributions, I will, in the same time, try to explain why these rely on unsatisfactory weak points and therefore are, in my opinion, incomplete and require to be rebuilt. Sect. 4 will illustrate some of the previously discussed points in the more concrete, but still general, models whose non trivial topology is induced by periodic boundary conditions. Surprisingly, after the pioneer article on the subject done by Schulman [3], only caricature models seem to have been retained in the literature whereas the generality and the simplicity of the decomposition (1) for spatially periodic models would deserve a better attention. The concluding Sect. 5 will emphasize the difference that may be put to experimental tests between the unitary representation of the first homotopy group and of another topological group like the first homology group.

1 Context, Stakes and Starting Concepts

Topology shares a long history with physics since the xixth century [8, 9] and was introduced into the quantum arena by Dirac’s [10] seminal work on the magnetic monopole.1 Within the Schrödinger formalism, the topology of the configuration space mainly appears through the boundary conditions imposed to the wavefunctions that constitute the Hilbert space of states. However, in this context, untangling the global and, by definition, robust topological properties from the local (differential) ones is not straightforward; all the more so than, when dealing with a curved manifold, the definition of the momentum operator and, more generally, the set up of the quantum canonical formalism is actually far from being canonical. The long history—initiated also by Dirac [11] himself—and the abundant literature, not free of tough controversies, about the quantum operator associated with an angle variable (in phase as well as in configuration space) reflects these concerns.2 By offering a direct connection between the quantum evolution and the paths in configuration space, Feynman’s formulation in its original form [14, 15], hides better but

1 Not referring explicitly to topology does not mean, of course, that it is absent, all the more so when the mathematical concepts were not stabilized: even before Poincaré works at the turn of the xix–xxth centuries, topological arguments irrigated fluid dynamics and electromagnetism thought the works of Helmholtz and of the Anglo-Irish-Scottish school including Stokes. In his paper, Dirac follows repeatedly a topological reasoning.

2 For an historical survey on the quantum phase operator see [12] and for a compilation of papers see [13].
does not get rid of the ordering-operator ambiguities, nor does it make disappear the issues that emerge when trying to associate a quantum transformation to a (non linear) change of variables required when covering a manifold (different from $\mathbb{R}^L$) with several patches of curvilinear coordinates (for instance see [16], chap. 7; [17]; [18]; [19]; [20], for a review; [21] for phase-space path integrals; [22] for configuration-space path integrals; [7, chap. 24]; a more recent presentation on these matters can be found in [23], chap. 2). However, the great advantage of writing the propagator as the result of a collective interference from a bunch of paths in configuration space allows to clearly separate the local properties, that are encapsulated in the Lagrangian, from the topological ones, that are encapsulated in the global properties of paths on which the integral is computed. It is worth mentioning that the appealing formulation of path integrals in terms of phase-space paths, despite its many assets over the path integrals in configuration space, seems to be of poor interest when dealing with topological properties: the reason is mainly that the phase-space paths that mainly contribute to the propagator, though continuous in position, are discontinuous in momenta ([15], Eq. (50); [24], Fig. 7-1) and by definition, discontinuity is ruled out from topological considerations.

The freedom of choosing the coefficients $E$ as the images a unitary representation of the fundamental group of the configuration space provides extra resources when combined to the freedom of choosing a Lagrangian alone. It offers a way to include some features and probe some properties that are insensitive to out-of-control perturbations. It unifies in a coherent and common scheme the quantum treatment of gauge models, including the original Dirac monopole but also the Ehrenberg–Siday–Aharonov–Bohm effect [25, 26]. As far as non-relativistic particles are concerned, not only it provides a new understanding on the fundamental dichotomy between bosons in fermions through the two possible unitary scalar representations of the permutation group, the trivial one and the signature [5], but it opens the doors, in effective two dimensions, to the intermediate behaviour of anyons (27–29 for instance and other papers in Chaps. 5 and 6 of Shapere & Wilczek’s collection) whose existence has been proven recently by a direct experimental evidence [30]. These two examples, gauge theory and the statistics of identical non relativistic particles, not to speak about solitons in field theory, are sufficient by themselves to understand the importance of the possibilities brought by the decomposition (1).

All along this work, I have tried to keep the notations to be self-explaining and standard enough. The reader who has already some acquaintance with the subject may skip directly to Sect. 3, possibly coming back to the following for clarification. In the remaining of the present Sect. 2, all the definitions and notations that will be used are specified. The reader is supposed to be familiar with the elementary notions of homotopy theory that are sketchily provided for the sake of self-containedness. For more complete constructions, examples and proofs see the chapters 1 in the remarkable books of Hatcher [31] or Fomenko and Fuchs [32]. After Schulman [2–4], Dowker [6] was the first to emphasize that a neat justification of (1) requires an auxiliary space called the universal covering space of the configuration space $Q$,
which we will denote by $Q$. Whereas its definition and its properties have now
became overspread in the physics literature on the subject we deal with, it seems
that its general and systematic construction (therefore the proof of its existence)
remains confined to some algebraic topology textbooks (31§ 1.3 from p. 63 and up
or [32], § 6.12). Therefore, to understand why $Q$ does not come out of the blue, a
special, somehow extended, place is devoted below to this construction.

**Paths and concatenation.** To be more specific, the configuration space of the sys-
them with t. degrees of freedom is supposed to be a real manifold $Q$ equipped with a
sufficiently smooth differential structure so that a Schrödinger equation (resp. a
Lagrangian) can be defined to model the quantum (resp. classical) evolution. A
path $c$ will be a continuous map $t \in [t_1, t_f] \mapsto q(t) \in Q$; in addition to its geometri-
ical image, made of 1d-continuous subset of $Q$, it is important to keep in mind that
the dynamical course, through the parametrisation in time $t$, is an essential charac-
teristic of $c$: even though they share the same image, two paths having a different
velocity $\dot{q}$ at the same point will be considered as distinct.\(^3\) To any path $c$ join-
ing $q_1 = q(t_1)$ to $q_f = q(t_f)$ is associated a unique inverse denoted by $c^{-1}$ obtained
by reversing the course of time through the reparametrisation $\hat{q}(t) = q(t_f + t_1 - t)$. A
path $c'$ joining $q_1' = q(t_1')$ to $q_f' = q(t_f')$ can be concatenated to any other path $c$ join-
ing $q_1 = q(t_1)$ to $q_f = q(t_f)$ provided that $t_f \in [t_1, t_f']$ and $q_f = q_{f'}$; the result is a path
defined by $t \in [t_1, t_f]$ such that $\hat{q}(t) = q(t)$ when $t \in [t_1, t_f]$ and $\hat{q}(t) = q(t')$
when $t \in [t_f, t_f']$ and will be denoted by $c \cdot c'$ (when $t_f > t_f$, note that the chronologi-
cal ordering is chosen to go from left to right). The concatenation is associative:
$(c \cdot c') \cdot c'' = c \cdot (c' \cdot c'')$ whenever the concatenation of the three paths is possi-
ble. A loop is a path such that $q(t_f) = q(t_1)$. As far as I know, all the relevant configu-
ration spaces in physics are pathwise-connected (any pairs of points are the end-
points of a path) and so will be $Q$.

**Homotopy.** Two paths $q(t)$ and $q'(t)$ are said to be homotopic if they are defined on
the same time interval $[t_1, t_f]$, if they share the same endpoints, $q'(t_f) = q(t_f)$,
$q'(t_f') = q(t_f')$ and if they can be continuously deformed one into the other. This equiva-
ience relation allows to classify the paths within homotopy classes that will be denoted
by lower-case Gothic letters. When there is a risk of ambiguity, the endpoints will be
specified as in $c_{q_1, q_f}$. The set of all classes sharing the same endpoints will be denoted
by $\pi_1(q_1, q_f)$. The concatenation law is transferred to the set of classes: $c_{q_1, q_f} \cdot c'_{q_1', q_f'}$ is the
common homotopic class of every path $c \cdot c'$ obtained for any $c \in c_{q_1, q_f}$ and
any $c' \in c'_{q_1', q_f'}$. This classification erases the time parametrisation since two different
paths $t \mapsto q(t)$ and $s \mapsto q(t(s))$ are homotopic whenever $t(s)$ is a continuous bijection.
We will denote by $c_{q_1, q_f}^{-1}$ the homotopy class of $c^{-1}$ whatever is $c \in c_{q_0, q_f}$. When
restricted to the set $\pi_1(q_0, q_0)$ of the classes of loops $I_{q_0, q_0}$, the concatenation becomes
an internal law, having a neutral element, the class $c_{q_0}$ of all the loops starting and end-
ing at the basepoint $q_0$ that are homotopic to the constant path $q(t) = q_0$ for all $t$.

\(^3\) Whereas, in differential geometry, (oriented) paths or curves are usually defined up to a (monotonically increasing) bijective continuous reparametrisation that can always be taken to be, say, $s \in [0, 1]$. 

Springer
Then, $I_q^{-1}_{q_0}$ is precisely the inverse of $I_{q_0,q_0}$ for the concatenation law. Endowed with the latter, $\pi_1(q_0,q_0)$ is a group for every choice of the basepoint $q_0$ and all these groups are isomorphic one to the other through a left and right concatenation by a class and its inverse connecting the two basepoints—this is a particular case of equation (2) below when $q_f = q_i$ (Fig. 1a)—therefore, they can be upgraded to an abstract group $\pi_1(Q)$, independent of $q_0$, called the fundamental group of $Q$ which constitutes a topological invariant of $Q$ (i.e. preserved by any continuous deformation of $Q$). As shown in Fig. 3, this group is not necessarily Abelian. The configuration space $Q$ is said to be simply-connected when all its loops can be continuously deformed into one point, in other words when $\pi_1(Q) = \{e\}$. Otherwise, $Q$ is said to be multiply-connected.

When $q_f \neq q_i$, $\pi_1(q_i,q_f)$ is not a group (because concatenation between two of its elements is not possible) but can be constructed from $\pi_1(Q)$ in the following way: for any choice of $q_0 \in Q$, $c_{q_0,q_i} \in \pi_1(q_0,q_i)$ and $c'_{q_0,q_f} \in \pi_1(q_0,q_f)$, each element $c_{q_i,q_f} \in \pi_1(q_i,q_f)$ has a unique decomposition of the form (Fig. 1b, c)

$$c_{q_i,q_f} = c^{-1}_{q_f,q_0} \cdot I_{q_f,q_0} \cdot c'_{q_0,q_f}$$

(2)

where $I_{q_0,q_0} \in \pi_1(q_0,q_0)$: trivially, $I_{q_0,q_0}$ is uniquely given by $c_{q_0,q_i} \cdot c_{q_i,q_f} \cdot c'^{-1}_{q_f,q_0}$. We will take advantage of this bijective map between $\pi_1(q_i,q_f)$ and $\pi_1(Q)$ to label the elements of the former with the elements of the latter.

Construction of the universal covering space. (Fig. 2a) Once a basepoint $q_0$ is chosen in $Q$ then the universal covering space can be obtained as

$$\tilde{Q}_{q_0} \overset{\text{def}}{=} \bigcup_{q \in Q} \pi_1(q_0,q)$$

(3)

which is a disjoint union, that is, for every $\tilde{q} \in \tilde{Q}_{q_0}$ there exists a unique $q = \Pi(\tilde{q}) \in Q$ such that $\tilde{q} \in \pi_1(q_0,q)$. Because of the bijective map between any two $\pi_1(q_i,q_f)$ obtained from (2), two different choices of basepoint will provide two bijectively related $\tilde{Q}_{q_0}$’s and all these sets can be abstracted into a basepoint-independent set $\tilde{Q}$. When $Q$ is simply-connected, all the $\pi_1(q_0,q)$ have just one element that can be identified with the endpoint $q$ itself and therefore $\tilde{Q} = Q$. When $Q$ is multiply-connected, $\tilde{Q}$ is a patchwork made of several copies of $Q$, each being labelled by the elements of $\pi_1(Q)$: more precisely, each $q$ is in correspondence with several elements in $\tilde{Q}$, namely the elements of $\pi_1(q_0,q)$ which are themselves, as we
have seen, bijectively related to $\pi_1(Q)$. To avoid multivaluedness this correspondence is rather described by its inverse, the projection $\Pi$ from $\bar{Q}$ to $Q$ defined above, which associates to each element $\bar{q} = \epsilon_{q_0,q} \in \bar{Q}$ the final point $q$ of any of the paths in $\epsilon_{q_0,q}^{-1}$. In other words $\Pi^{-1}(q) = \pi_1(q_0,q)$.

From any $\bar{q} = \epsilon_{q_0,q} \in \bar{Q}$ and any $I \in \pi_1(q_0,q_0)$ isomorphically associated to $g \in \pi_1(Q)$ the class $I \cdot \epsilon_{q_0,q}$ still remains in $\pi_1(q_0,q)$ and therefore corresponds to a class $\bar{q}' = \epsilon_{q_0,q}'$. Then, to each element $g \in \pi_1(Q)$ we have a map $T_g : \bar{q} = \epsilon_{q_0,q} \mapsto \bar{q}' = I \cdot \epsilon_{q_0,q}'$ that transforms an element of $\Pi^{-1}(q)$ into another element of $\Pi^{-1}(q)$. For convenience we will use the lighter notation $g\bar{q} = T_g(\bar{q})$. 

Fig. 2 a Construction of the universal covering space $\bar{Q}$ from $Q$ whose multi-connectedness comes from a forbidden region (hatched region). b The differential structure of $\bar{Q}$ is obtained through the inverse of the projection $\Pi$ (represented by vertical downward arrows) which allows to lift the coordinate patches $U_q$ of the differential manifold $Q$ (Color figure line).
Because of its associativity, the concatenation is isomorphically transferred to the composition of the $T$’s: $T_{q',q} T_{q,q'} = T_{q'',q}$. The $T$’s define an action of the fundamental group of $Q$ on its universal covering space $\tilde{Q}$. Because $1 \cdot e_{q_0,q_0} = e_{q_0,q_0}$ if and only if $I = e_{q_0}$, the group action is free that is, by definition, for every $\tilde{q}$ in $\tilde{Q}$, $g\tilde{q} = \tilde{q}$ if and only if $g$ is the neutral element $e$ of $\pi_1(Q)$.

Conversely, from any pair $\tilde{q}' = e'_{q_0,q_0}$ and $\tilde{q} = e_{q_0,q_0}$ there exists a unique $g \in \pi_1(Q)$ —the one associated to the loop $e'_{q_0,q_0}, e^{-1}_{q_0,q_0}$ in $\pi_1(Q,q_0)$—such that $g\tilde{q} = \tilde{q}'$. One can then adopt the more common inverse perspective and recover $Q$ from $\tilde{Q}$: it is the set of the orbits in $\tilde{Q}$ under the action of $\pi_1(Q)$ or, in other words, we have $Q = \tilde{Q}/\pi_1(Q)$ the set of equivalence classes in $\tilde{Q}$ where two elements $\tilde{q}'$ and $\tilde{q}$ are defined to be equivalent if there is a $g \in \pi_1(Q)$ such that $\tilde{q}' = g\tilde{q}$.

The differential structure of the universal covering space. (Fig. 2b) The differential structure of the manifold $Q$ can be lifted to $\tilde{Q}$ for the main reason that the open sets that cover $Q$, from which the charts are defined, can be chosen to be simply-connected. For any $\tilde{q} = e_{q_0,q_0} \in \tilde{Q}$, there exists a simply-connected open set $U_q$ in $Q$ containing $q$, isomorphic to an open set in $\mathbb{R}^1$. It can be lifted into $\tilde{U}_q$ defined to be the subset of $\tilde{Q}_{q_0}$ made of all the classes $e_{q_0,q_0}$ such that there exists a path between $q$ and $q'$ entirely included in $U_q$ or, in other words $\tilde{q}' = e'_{q_0,q_0}$ will be in $\tilde{U}_q$ if and only if the class $e^{-1}_{q_0,q_0}$ contains a path included in $U_q$. This requires of course that $q' \in U_q$. Clearly $\tilde{q} \in \tilde{U}_q$ because the constant path equal to $q$ is in $U_q$ but $g\tilde{q} \notin \tilde{U}_q$ for all $g \neq e$. Indeed, suppose $\tilde{q}' = g\tilde{q}$ belongs to $\tilde{U}_q$ then $e^{-1}_{q_0,q_0}$ is a class of loops (because $q' = q$) and this class would be $e_q$ (because it contains a loop included in $U_q$ which is simply-connected). Therefore, we would have $e'_{q_0,q_0} = e_{q_0,q_0}$ that is $g\tilde{q} = \tilde{q}$ and hence, as we have seen above, $g = e$. Then, all the $\tilde{U}_{g\tilde{q}}$ that can be constructed in the same way are pairwise disjoint.

Moreover, being a differential manifold, $Q$ is also locally pathwise-connected (every neighbourhood of every point contains a pathwise-connected neighbourhood). Then, for every $q'$ belonging to $U_q$ there exists a path in $U_q$ connecting $q$ and $q'$. Its class is uniquely defined because $U_q$ is simply-connected and therefore there exists a unique $\tilde{q}'$ in $\pi_1(Q_0,q')$, given by $e_{q_0,q_0} \circ e_{q_0,q_0}$, belonging to $\tilde{U}_q$.

Therefore $\Pi^{-1}(U_q) = \bigcup_{g \in \pi_1(Q,q_0)} \tilde{U}_{g\tilde{q}}$ appears to be a disjoint union and each $\tilde{U}_{g\tilde{q}}$ is bijectively related to $U_q$ through the restriction $\Pi_{g\tilde{q}}$ which happens to be a homeomorphism since every neighbourhood of $q$ included in $U_q$ can be lifted in an analogous way and can be used to define a basis of open sets in $\tilde{Q}$.

The composition of these homeomorphisms transfer the charts covering $Q$ in charts covering $\tilde{Q}$ which eventually inherits of all the differential structure of $Q$.

The class $\tilde{q}_0 \overset{def}{=} e_{q_0,q_0}$ is a privileged element of $Q_{q_0}$ and we can safely identify $\tilde{U}_{\tilde{q}_0}$ with $U_{q_0}$ by considering $\Pi_{\tilde{q}_0}$ as a trivial inclusion map.

Lifted paths. Every path $\gamma$ in $Q$ given by $t \in [t_i,t_f] \mapsto q(t)$ connecting $q_i$ to $q_f$, once a $\tilde{q}_i$ is chosen in $\Pi^{-1}(q_i)$, is lifted into a unique path $\gamma'$ in $\tilde{Q}$ given by $t \in [t_i,t_f] \mapsto \tilde{q}(t)$ where $\tilde{q}(t) \in \Pi^{-1}(q(t))$ is uniquely defined by covering $\gamma'$ with simply-connected patches on which the restriction of $\Pi^{-1}$ is bijective. Two non-homotopic paths sharing the same endpoints in $Q$ will be lifted in $\tilde{Q}$ into two paths ending to different $\tilde{q}_f = \tilde{q}(t_f)$ if they both start at $\tilde{q}_i$.
Simply-connectedness of the universal covering space. So to speak, \( \tilde{Q} \) is obtained by unfolding \( Q \) in order to get a simply connected space. If we consider a loop \( \mathcal{L} \) in \( \tilde{Q} \) given by \( \tilde{q}(t) = \epsilon_{q_0,q(t)} \) such that \( \tilde{q}(t_i) = \tilde{q}(t_f) \), then its projection by \( \Pi \) in \( Q \), namely \( q(t) \), is a loop \( \mathcal{L} \) in \( e_q \) precisely because \( \epsilon_{q_0,q(t_i)} = \epsilon_{q_0,q(t_f)} \). Then, there exists a continuous deformation that contracts \( \mathcal{L} \) into the constant path equal to \( \tilde{q}(t) = \tilde{q}(t_i) \) for all \( t \), therefore \( \tilde{Q} \) is indeed simply connected.

2 Emergence of the Unitary Representation of \( \pi_1(Q) \)

2.1 General Characteristics of the Propagator

All the quantum evolution operators \( \hat{U}(t_f, t_i) \) share the following characteristic properties: for any times \( (t_i, t, t_f) \), we have the composition law

\[
\hat{U}(t_f, t) \hat{U}(t, t_i) = \hat{U}(t_f, t_i),
\]

endowed with the neutral element representing a non-evolution

\[
\hat{U}(t_j, t_i) = 1,
\]

which makes the \( \hat{U} \)'s unitary provided the exchange of time arguments corresponds to Hermitian conjugation

\[
(\hat{U}(t_f, t_i))^* = \hat{U}(t_i, t_f).
\]

The chronological ordering is completely free, in particular one cannot impose systematically \( t \in [t_i, t_f] \) because conditions (4) do not allow to conclude that \( (\hat{U}(t_f, t_i))^* = (\hat{U}(t_i, t_f))^* \) if \( \hat{U}(t, t) \) cannot be decomposed into \( \hat{U}(t, t_i) \hat{U}(t_i, t) \) for any \( t_f \): it is required to use the composition law between operators whose arguments change their chronological order one from the other. We note also that (4b) is not a consequence of (4a) (by taking \( t_f = t_i \) for instance) if we refrain posing a priori that the \( \hat{U} \)'s are invertible. The propagators \( K(q_f, t_f, q_i, t_i) \) in configuration space \( Q \) are (generalised) functions of two points \( (q_f, q_i) \) in \( Q \) that can be thought has the matrix elements \( \langle q_f|\hat{U}(t_f, t_i)|q_i \rangle \) where \( \{|q_i\} \) denotes a non-normalisable basis of the Hilbert space on which the \( \hat{U} \)'s are defined. The properties (4) have their exact translation in terms of propagators ([16], Eqs. 30.6, 7, 8)

\[
K(q_f, t_f, q_i, t_i) = \int_Q K(q_f, t_f, q, t)K(q, t, q_i, t_i)\,dq;
\]

\[
K(q_f, t_i, q_i, t_i) = \delta(q_f - q_i);
\]
We denote by \( d^q q \) a given measure on \( Q \) (including a non-homogeneous Jacobian when curvilinear coordinates are used) associated with the Dirac function \( \delta \) such that
\[
\int_Q f(q) \delta(q' - q) \, d^q q = f(q')
\]
for any test function \( f \) defined on \( Q \). It is important to note that the integral that constitutes the right-hand side of (5a) covers the whole \( Q \); this is the reason why, when dividing the evolution into a sequence of infinitesimal-time slices, to build up, by their composition, the path integral
\[
K(q_f, t_f, q_i, t_i) = \int e^{i S[\mathcal{C}]/\hbar} \, d[\mathcal{C}],
\]
the integration domain includes all the paths \( \mathcal{C} \) on \( Q \) such that \( q(t_i) = q_i \) and \( q(t_f) = q_f \). Since we want to emphasize the topological properties, we ought not to choose an explicit expression of the action \( S \) and actually we will not. We will retain only its additivity by concatenation of paths:
\[
S[\mathcal{C}_2 \cdot \mathcal{C}_1] = S[\mathcal{C}_2] + S[\mathcal{C}_1]; \quad S[\mathcal{C}^{-1}] = -S[\mathcal{C}]
\]
(the latter does not assume any time-reversal invariance, which is in fact not satisfied as soon as a magnetic field is present; rather, it may provide a definition of the transformed Lagrangian under \( t \mapsto t_f + t_i - t \)). We will neither use a precise definition of the path integrals. We shall assume that actually the right-hand side of (6) satisfies (5) which is not an easy statement to prove or, conversely, that must be included in any constructivist approach. An important departure from the original construction proposed by Feynman and Hibbs ([24], Eq. (4-28)) is that their propagator is defined to be zero for \( t_f < t_i \) or, equivalently, they consider the matrix elements of \( \hat{U}(t_f, t_i) \) multiplied by the Heavside step function \( \Theta(t_f - t_i) \). We will rather not to because we want to preserve the property (5c) which is essential to the group property of the \( \hat{U} \)'s; we will keep working with a function \( K \) that fulfills the same evolution equation as a normalisable state, namely the time dependent Schrödinger equation, without any supplementary \( \delta(t - t_i) \) terms turning it into a retarded Green function. For the same reason, we will not allow to use the Wick-substitution “mantra” that leads to an irreversible evolution governed by a semi-group. The difficulty of defining mathematically an oscillatory path integral is the sign that using an imaginary time is not harmless from the physical point of view; as soon as we suppress, by construction, the central notion of quantum interferences, one is expected to miss a lot of physics including the topological phases as they appear in (1). We will show how the latter are directly connected to the unitarity of the evolution.

For \( t_f \neq t_i \), one expects the function \((q_f, q_i) \mapsto K(q_f, t_f, q_i, t_i)\) to be smooth on the configuration space if the Schrödinger time-dependent equation satisfied by \( K \) involves potentials that are regular enough. In particular, for a model invariant under time translations, from a stationary orthonormal eigenbasis \( \{ |\phi_n>\} \) labelled by the quantum numbers \( n \) and corresponding to an energy spectrum \( \{ \epsilon_n \} \), we have the following spectral decomposition of \( K \) in terms of the corresponding wavefunctions \( \phi_n(q) \equiv <q|\phi_n> \) defined on \( Q \).
\[ K(q_f, t_f, q_i, t_i) = \sum_{\nu} \phi_\nu(q_f)\phi_\nu^*(q_i) e^{-\frac{i}{\hbar}(q_f - q_i)\nu} \]  

(8)

and the smoothness of \( K \) is at least the same as the smoothness of the stationary wavefunctions.

### 2.2 Propagators in the Universal Covering Space

Once the domain of integration in the right-hand side of (6) has been split in homotopy classes \( c_{q_i, q_f} \), the weights \( E(c_{q_i, q_f}) \) must be chosen in order for the new \( K \) given by (1) to still satisfy (5) and, then, to keep its interpretation of being the density of probability amplitude for reaching the configuration \( q_f \) at time \( t_f \) assuming that the system is in the configuration \( q_i \) at \( t_i \). On the other hand, the partial path integrals defined by

\[ k_\epsilon(q_f, t_f, q_i, t_i) \overset{\text{def}}{=} \int_{\mathcal{C}} e^{\frac{i}{\hbar}S[\epsilon]} d[\epsilon] \]  

(9)

are not expected to satisfy (5) and, notably, one should expect that

\[ k_\epsilon(q_f, t_f, q_i, t_i) \neq \delta(q_f - q_i) \]  

(10)

because to obtain the right-hand side requires a path integration with no restriction on the homotopy classes as in (6). In fact, one expects the function \((q_f, q_i) \mapsto k_\epsilon(q_f, t_f, q_i, t_i)\) to have discontinuities in \( Q \) because there is no way of deforming continuously, say, a class of paths \( c_{q_i, q(s)} \) along a non-trivial loop \( q(s) \) without ending with another class of paths. For instance, let us choose \( s \in [0, 1] \mapsto q(s) \) to be a loop \( \mathcal{L} \) in \( Q \) with \( q(0) = q(1) = q_f \) which is not continuously deformable into the constant path \( q_f \). Then, we can try to maintain the continuity of \( s \mapsto k_\epsilon(q(s), t_f, q_i, t_i) \) by transforming continuously the domain of integration of (9) in the following way: to define \( c_{q_i, q(s+ds)} \), all the paths in \( c_{q_i, q(s)} \) are concatenated with the portion of \( \mathcal{L} \) between \( q(s) \) and \( q(s + ds) \) while changing the time parametrisation to keep the paths always defined for \([t_f, t_f]\). Nevertheless, at the end of this process, when the final point \( s = 1 \) is reached, all the paths have been concatenated with \( \mathcal{L} \) modulo a time reparametrisation and therefore belong to the class \( c'_{q_i, q_f} = c_{q_i, q_f} \cdot I_{q_f, q_f} \); the latter is different from the class we started with at \( s = 0 \) if and only if the class \( I_{q_f, q_f} \) of \( \mathcal{L} \) is different from the neutral class \( \epsilon_{q_f} \). Having two different integration domains, \( k_\epsilon \) and \( k_{\epsilon'} \) are a priori different and therefore, even though the limits \( s \to 0^+ \) and \( s \to 1^- \) lead to the same value \( q(0) = q(1) = q_f \), they give two different values for \( k_{\epsilon_{c_{q_i, q(s)}}, q(s), t_f, q_i, t_i} \). Since \( \mathcal{L} \) is arbitrary with at least one point of discontinuity on it, we expect to have in \( Q \) at least one hypersurface of codimension 1 of points of discontinuity for each \( k \).

These singularities of the \( k \)'s in \( Q \) do not contradict the regularity of \((q_f, q_i) \mapsto K(q_f, t_f, q_i, t_i) \) for \( t_f \neq t_i \); within the sum (1), when following the loop \( \mathcal{L} \) from \( s = 0^+ \) to \( s = 1^- \) a permutation occurs where all terms are swapped one with an other maintaining the continuity of the whole sum. One can then even
redefine a continuous family of functions on $Q$ with the help of a Heaviside $\Theta$ functions splitting the two sides of the hypersurfaces of discontinuities of the partial path integrals\(^4\) but, as we have seen, these functions cannot be associated with the same homotopy class on the two sides of these surfaces.

To keep following the same homotopy class without dealing with cuts, the price to be paid is to somehow establish a distinction between paths in $Q$ (sharing the same endpoints) is transferred to a difference in the endpoints in the homotopic distinction between paths in $Q$ (sharing the same endpoints) is transferred to a difference in the endpoints in the homotopy class on the two sides of these surfaces.

Then, pick up one $\tilde{q}_i \in \Pi^{-1}(q_i)$, set $q_0 = q_f$ and define $\tilde{q}_f = \epsilon_{q_f}$. Then each path $\gamma$ in $Q$ connecting $q_i$ to $q_f$ is lifted into a unique path $\tilde{\gamma}$ in $\tilde{Q}$ connecting $\tilde{q}_i$ to $g\tilde{q}_f$ for $g$ in $\pi_1(\tilde{Q})$ associated to $I$. When restricted to simply-connected patches on $\tilde{Q}$, all the differential structure of $Q$ can be lifted to $\tilde{Q}$, in particular the coordinates charts, the action functional and the measure on paths; the definition of which are part of the translation of the partial path integral into the universal covering space according to

\[
K(q_f, t_f, q_i, t_i) = \sum_{\gamma \in \pi_1(q_i, q_f)} \mathcal{E}(\epsilon_0 \cdot I) \int_{\gamma \in \epsilon_0 \cdot I} e^{\frac{i}{\hbar} \int_{\gamma} \mathcal{A}} d[\gamma]. \tag{11}
\]

Then, pick up one $\tilde{q}_i \in \Pi^{-1}(q_i)$, set $q_0 = q_f$ and define $\tilde{q}_f = \epsilon_{q_f}$. Then each path $\gamma$ in $Q$ connecting $q_i$ to $q_f$ is lifted into a unique path $\tilde{\gamma}$ in $\tilde{Q}$ connecting $\tilde{q}_i$ to $g\tilde{q}_f$ for $g$ in $\pi_1(\tilde{Q})$ associated to $I$. When restricted to simply-connected patches on $\tilde{Q}$, all the differential structure of $Q$ can be lifted to $\tilde{Q}$, in particular the coordinates charts, the action functional and the measure on paths; the definition of which are part of the translation of the partial path integral into the universal covering space according to

\[
k_{\epsilon_0 \cdot I}(q_f, t_f, q_i, t_i) \overset{\text{def}}{=} \int_{\gamma \in \epsilon_0 \cdot I} e^{\frac{i}{\hbar} \int_{\gamma} \mathcal{A}} d[\gamma] = \int_{\tilde{\gamma} \in \tilde{\epsilon}_{\tilde{q}_i, \tilde{q}_f}} e^{\frac{i}{\hbar} \int_{\tilde{\gamma}} \mathcal{A}} d[\tilde{\gamma}] \overset{\text{def}}{=} \tilde{K}(g\tilde{q}_f, t_f, \tilde{q}_i, t_i) \tag{12}\]

where the integration domain is now the homotopy class $\tilde{\epsilon}_{\tilde{q}_i, \tilde{q}_f}$ of the paths $\tilde{q}(t)$ in $\tilde{Q}$ such that $\tilde{q}(t_i) = \tilde{q}_i$ and $\tilde{q}(t_f) = g\tilde{q}_f$. In working in the universal covering space, we have expressed each partial path integrals $k$ into a plain Feynman integral $\tilde{K}(g\tilde{q}_f, t_f, \tilde{q}_i, t_i)$ where all the paths connecting two points are considered with no restriction on their homotopy classes since, by construction, $\tilde{Q}$ is simply connected. The equality (12) concerns the values of $k$ and $\tilde{K}$ and not the functions themselves whose arguments are defined in different spaces and whose smoothness properties are not the same.

\(^4\) If $k_{\sigma}^n(s_0^0) = k_{\sigma^{-1}}(s_0^0)$ for a permutation $\sigma$ of the discrete labels $n$, then the function $k_{\sigma^{-1}}(s_0(s)) = k_{\sigma^{-1}}(s_0(s)) = k_{\sigma^{-1}}(s_0(s))$ is continuous at $s_0$ and one can express the discontinuous functions $k_n$ with the continuous functions $k_n$. Let $k_n(s) = \tilde{k}_n(s)\Theta(s - s_0) + \tilde{k}_n(s)\Theta(s_0 - s)$.

\(^5\) The same line of thought leads to the construction of the Riemann surfaces from the complex plane. The latter is unfolded into $n$ connected sheets to avoid the line cuts of the function $z \mapsto z^{1/n}$.
2.3 Linear Independence of the $k$’s

As any plain Feynman integral, $\bar{K}$ satisfies (5b),

$$\bar{K}(g\bar{q}_f, t_f, \bar{q}_i, t_i) = \delta(g\bar{q}_f - \bar{q}_i),$$

(13)
in contrast with (10). Then, if for one reason or another, a linear combination of the form $\sum_{g \in \pi_1(Q)} A(g) \bar{K}(g\bar{q}_f, t_f, \bar{q}_i, t_i)$ vanishes identically for all times, by taking $t_f = t_i$, this implies that the coefficients $A(g)$ are zero because $g\bar{q}_f \neq \bar{q}_f$ as soon as $g \neq e$. From (12), this linear independence of the $\delta(g\bar{q}_f - \bar{q}_i)$ for $g \in \pi_1(Q)$ is directly transmitted to the $k$’s:

$$\sum_{l \in \pi_1(q_f, q_I)} A(c_0 \cdot I) k_{c_0,1}(q_f, t_f, q_I, t_I) = 0 \iff \forall l \in \pi_1(q_f, q_I), A(c_0 \cdot I) = 0.$$  

(14)

In other words, we have established that the decomposition (1) of a given propagator $K$ is necessarily unique.

2.4 Composition

For the decomposition (1) to be consistent with (5a), we must have

$$\sum_{c \in \pi_1(q_I, q_f)} E(c) \int_{c \in c} e^{\frac{i}{\hbar} S[c]} d[c]$$

$$= \int_Q d^c q_I \sum_{c_1 \in \pi_1(q_I, q_f)} \sum_{c_2 \in \pi_1(q_f, q_I)} E(c_2) E(c_1) \int_{c_1 \in c_1} e^{\frac{i}{\hbar} (S[c_2] + S[c_1])} d[c_2] d[c_1].$$

(15)

Take $t_i < t_f$ choose $t_f \in ]t_i, t_f[$. Every path $C$ involved in the integral of the left-hand side, connecting $(q_I, t_i)$ to $(q_f, t_f)$, is uniquely obtained by concatenation of one path $C_2$, connecting $(q_I, t_i)$ to $(q_f, t_f)$, to one path $C_1$, connecting $(q_f, t_f)$ to $(q_I, t_f)$ where $q_I$ given by $q(t_f)$; the class $c$ of $C$ is then uniquely decomposed into $c_1 \cdot c_2$ where $c_1$ (resp. $c_2$) denotes the homotopy class of $C_1$ (resp. $C_2$). Therefore each path of the left-hand side appears once and only once among the paths in the right-hand side.

Conversely, every path involved in the right-hand side is obtained by concatenation of a path connecting $(q_I, t_i)$ to $(q_f, t_f)$ to a path connecting $(q_I, t_f)$ to $(q_f, t_i)$ for a given $q_I$ and then appears once and only once in the left-hand side.

Moreover, because of the additivity property (7), we can collect the paths of the right-hand side according to
\[
\int_Q \sum_{c_1 \in \pi_1(\mathcal{q}_i, \mathcal{q}_f)} \sum_{c_2 \in \pi_1(\mathcal{q}_f, \mathcal{q}_j)} E(c_2)E(c_1) \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} e^{\frac{i}{\hbar}(\mathcal{S}[\mathcal{C}_2]+\mathcal{S}[\mathcal{C}_1])} d[\mathcal{C}_2]d[\mathcal{C}_1]
\]

\[
= \sum_{c \in \pi_1(\mathcal{q}_i, \mathcal{q}_j)} E(c)k(c, \mathcal{q}_f, \mathcal{q}_i, \mathcal{q}_j, \mathcal{q}_i, \mathcal{q}_f)
= \sum_{c \in \pi_1(\mathcal{q}_i, \mathcal{q}_j)} E(c_2)E(c_1)k(c, \mathcal{q}_f, \mathcal{q}_i, \mathcal{q}_j, \mathcal{q}_f)
\]

The identification with the left-hand side (15) reads

\[
E(c_1 \cdot c_2) = E(c_2)E(c_1)
\]

because of the linear independence of the \(k_i\)'s. Then, the generalisation (1) preserves the original Feynman’s interpretation: the probability amplitude brought by the path \(\mathcal{C} = \mathcal{C}_1 \cdot \mathcal{C}_2\) to the propagator \(K\) remains equal to the product of the amplitudes brought by \(\mathcal{C}_1\) and \(\mathcal{C}_2\); since the integral involves all the possible paths, the two concatenated pieces are considered to be independent as soon as the continuity of \(\mathcal{C}\) is maintained. This multiplication of the amplitudes reads

\[
E(c_1 \cdot c_2)e^{\frac{i}{\hbar}\mathcal{S}[\mathcal{C}]} = E(c_2)E(c_1)e^{\frac{i}{\hbar}(\mathcal{S}[\mathcal{C}_2]+\mathcal{S}[\mathcal{C}_1])}
\]

which is guaranteed both by (18) and by the additivity of the action with respect to concatenation.

As an immediate consequence of (18) by taking for \(c_2\) any neutral class \(\mathcal{C}\) and \(c_1 \in \pi_1(\mathcal{q}_i, \mathcal{q}_j)\) we get

\[
E(\mathcal{C}) = 1
\]

and by choosing \(c_2 = \mathcal{C}^{-1}\),

\[
E(\mathcal{C}^{-1}) = (E(\mathcal{C}))^{-1}
\]

### 2.5 Conjugation

The third and last characteristic property of a propagator is the Hermitian conjugation rule (5c). Then we must have

\[
\sum_{c \in \pi_1(\mathcal{q}_i, \mathcal{q}_j)} E(c)\int_{\mathcal{C} \in c} e^{\frac{i}{\hbar}\mathcal{S}[\mathcal{C}]}d[\mathcal{C}] = \sum_{c^{-1} \in \pi_1(\mathcal{q}_j, \mathcal{q}_i)} (E(\mathcal{C}^{-1}))^{-1} \int_{\mathcal{C}^{-1}} e^{-\frac{i}{\hbar}\mathcal{S}[\mathcal{C}]}d[\mathcal{C}]
\]

where all the classes \(c^{-1}\) involved in the right-hand side are made of paths connecting \((\mathcal{q}_f, \mathcal{q}_j)\) to \((\mathcal{q}_i, \mathcal{q}_i)\). Yet, to each of these paths is associated a unique inverse \(c^{-1}\) connecting \((\mathcal{q}_j, \mathcal{q}_i)\) to \((\mathcal{q}_f, \mathcal{q}_j)\) whose action is opposite by virtue of (7): \(\mathcal{S}[\mathcal{C}^{-1}] = -\mathcal{S}[\mathcal{C}]\).
On the right-hand side, the sum on the path in $c^{-1}$ can be obtained by a sum on the opposite paths in the classes $c$:

$$\int_{\mathcal{C} \in c^{-1}} e^{-\frac{i}{\hbar}S[\mathcal{C}]} d[\mathcal{C}] = \int_{\mathcal{C} \in c} e^{\frac{i}{\hbar}S[\mathcal{C}]} d[\mathcal{C}]$$  \hspace{1cm} (22)

(if the path integral is defined as a limit of a discretisation, the Jacobian of such a transformation equals to one because it just consists in a permutation of the discrete coordinates; in a constructivist perspective this Jacobian is defined to be one). Therefore we obtain

$$\sum_{c} E(c) k_{c}(q_{f}, t_{f}, q_{i}, t_{i}) = \sum_{c} \left( E(c^{-1}) \right)^{*} k_{c}(q_{f}, t_{f}, q_{i}, t_{i})$$  \hspace{1cm} (23)

that is

$$E(c^{-1}) = \left( E(c) \right)^{*},$$  \hspace{1cm} (24)

by using the linear independence of the $k_{c}$'s again. Combined with (20) we obtain

$$\left( E(c) \right)^{*} = \left( E(c) \right)^{-1}.$$  \hspace{1cm} (25)

### 2.6 Unitary Representation

To turn $E$ into a morphism of groups, besides (18), one must restrict its arguments to the class of loops $\pi_{1}(Q)$. But this can be done by picking up one $q_{0} \in Q$ and two classes $c_{f} \in \pi_{q_{0}, q_{f}}, c_{i} \in \pi_{q_{0}, q_{i}}$, and use the loops in $\pi_{1}(q_{0}, q_{0})$ to label the paths $c$ in the sum (1):

$$K(q_{f}, t_{f}, q_{i}, t_{i}) = \sum_{c_{f} \in \pi_{1}(q_{0}, q_{f})} E(c_{i}^{-1} \cdot p_{c} \cdot c_{f}) \int_{\mathcal{C} \in c_{i}^{-1} \cdot c_{f}} e^{\frac{i}{\hbar}S[\mathcal{C}]} d[\mathcal{C}];$$  \hspace{1cm} (26a)

$$= E(c_{f}) \sum_{c \in \pi_{1}(q_{0}, q_{0})} E(1) \int_{\mathcal{C} \in c_{f}^{-1} \cdot c_{f}} e^{\frac{i}{\hbar}S[\mathcal{C}]} d[\mathcal{C}] \left( E(c_{i}) \right)^{-1};$$  \hspace{1cm} (26b)

where now, together with (18) and (25), the coefficients $E(1)$ are the images of a unitary representation of $\pi_{1}(Q)$. The pre- and post-factors $E(c_{f})$ and $\left( E(c_{i}) \right)^{-1}$ warrant that the composition law is satisfied.

By the way, in all of the above, we had no need to work with scalar propagators exclusively ([33], § 5). Where some discrete quantum numbers $\alpha$ label the components of the wavefunctions to take into account the spin or some internal degree of freedom of a bounded system, the propagators $K$ are implicitly labelled by two such labels, having a matrix-like structure explicitly given by $K_{\alpha' \alpha}$. The action functional a priori depends on these numbers and so the endpoints and the position kets $|q, \alpha\rangle$ come with such multiplicity. The right-hand side of (5b) implicitly
contains a Kronecker symbol $\delta_{a'a}$ and so on. The coefficients $E$ are then given by a matrix whose entries are explicitly $(E_{a'a})$. The relation (18) is to be understood as a matrix product and in (25) a Hermitian conjugation is involved. This offers a direct bridge leading to non-Abelian gauge theories [34, 35].

When dealing with scalar wavefunctions and propagators, from (25) we deduce that each $E(e)$ is a pure phase factor and even though $\pi_1(Q)$ is not commutative, its $U(1)$-representations are. As noted by ([33], Eq. (2.5)), the group which is then represented, obtained by quotienting by the non-commutative part of $\pi_1(Q)$, appears to be the first holonomy group (a coarser topological invariant of $Q$). In that case, the difference of phases associated with $E(e_j)(E(e_i))^{-1}$ can be absorbed by adding a total derivative in the action $S$.

3 Critical Discussion on Previous Arguments

The decomposition (1) was first proposed by Schulman [2] for $Q$ being the configuration space of a rotating solid (in 2 and 3 dimensions). Whereas this article focuses on the path integral approach, as its title highlights it, two subsequent articles [3, 4] explore further some decomposition of the propagator to other systems but without recourse to path integrals. The role of the universal covering space is put forward specially in ([2], § 3 and fig. 1; [4], sec. I, fig. 3) and occupies a central place in the unified treatment proposed by Dowker [6]. However, all these works, which involve only scalar wavefunctions, suppose a priori that the coefficients $E$ are of unit modulus; this is also taken for granted—and even so (18) occasionally—in succeeding articles until recently ([36], Eq. (1.1); [37], Eq. (1.1); [38], Eq. (1); [33], Eq. (2.1); [39], Eq. (3); [40], before (2.8); [41], just after eq. (22); [42].)

The origin of this hypothesis is easily understood if one thinks that the covering space has a genuine physical meaning, that is, on which wavefunctions have the usual quantum interpretation. Actually, in all the models presented in these series of papers, and in our Sect. 4 as well, $\hat{Q}$ is the primary physical configuration space from which the multiple-connected base space $Q$ is built by imposing some boundary conditions (periodicity, forbidden region). Together with this folding of $\hat{Q}$ taken to be $\mathbb{R}^1$, the wavefunctions are folded as well by identifying $\hat{\phi}(g\hat{q})$ with $\hat{\phi}(\hat{q})$ up to a phase $E(g) = e^{i\chi(g)}$ because the latter is unobservable in $\hat{Q}$:

$$\hat{\phi}(g\hat{q}) = e^{i\chi(g)} \hat{\phi}(\hat{q}).$$

Then, since $Q$ appears somehow secondary or at last artificially introduced, one has no qualms about violating the very principles of quantum theory by considering multivalued wavefunctions or propagators in $Q$; $\hat{Q}$ never ceases to be the genuine physical space where wavefunctions and propagators remain monovalued. The point of view adopted in the present paper is quite the opposite and the ambiguity inherent to some multivalued quantities has never been introduced neither in $\hat{Q}$ nor, of course, in $\hat{Q}$. By laying our foundations on the Feynman path integral, we keep

---

6 In the context of the Ehrenberg–Siday–Aharonov–Bohm effect, see Berry [43]’s fair denunciation of the use of multivalued wavefunctions.
the possibility of considering the multi-connected space $Q$ as our primary physical space whereas $\bar{Q}$ is therefore constructed as an auxiliary space to establish the linear independence (14) of the partial path integrals. This is not an undue theoretical issue to consider models where $\bar{Q}$ cannot pretend to have a physical meaning. In a Young interference configuration with charged particles, for instance, two magnetic impenetrable tori—like the one used in Tonomura [44]’s famous experiment on the Ehrenberg–Siday–Aharonov–Bohm effect—the non-commutativity of the first homotopy group (Fig. 3) gives to $\bar{Q}$ the structure of an infinite tree-like manifold; such an “unnatural” covering space can also be obtained in lower dimension by considering 8-shaped wire. Clearly, in such situations, the physical presence must be given to $Q$ over $\bar{Q}$ and multivalued quantities in $Q$ cannot be supported. In any case, banishing multivalued functions preserves the flexibility of interpreting $Q$ or $\bar{Q}$ as the primary physical space.

The first attempt to prove that, in the scalar case, the $E$’s not only can but must be obtained from a $U(1)$-representation of the first homotopy group $\pi_1(Q)$ was proposed by Laidlaw and Morette-DeWitt [5] in the first part of their article. There, the linear independence of the partial path integrals together with their behaviour at $t_f \to t_i$ was already understood to be key in determining the weight factors $E$. Unfortunately, their arguments suffer from several flaws coming from the ubiquitous confusion between $Q$ and $\bar{Q}$.

To prove (14), it seems to be appealing to avoid passing by the universal covering space but, as far as I know, this challenge remains to be met if ever it makes sense; in the introductory Sect. 2, I have recalled through (3) the construction of $\bar{Q}$ to show how inseparable it is from the analysis of the topology of paths in $Q$. In a subsequent review of which Morette-DeWitt is also a co-author ([46], p. 295), it is still written that “There are two equivalent ways of giving meaning to [eq. (1)]. We give here the one which does not require auxiliary concepts; the other one [6] proceeds via the universal covering.” and the proof of (14) is referred to [5]. In a more recent mathematical synthesis, Cartier and DeWitt-Morette (47, § II-4, p. 2268) eventually adopt Dowker [6]’s approach and work starting with the universal covering space together with the hypothesis (27).

Coming back to the arguments used in [5], their key step II concerning the short-time behaviour of $k_r$ relies on the debatable assumption that the action is an increasing function of the length of the (not necessarily classical) path for short-time intervals. This may be true for a quasi-free (in the absence of vector/scalar potential), stationary, short-length path but not in general for the non-infinitesimally short paths they also consider. The neighbourhood of hard wall boundaries, where some diffractive non classical paths may minimise or maximise the action seem to fall out the scope of their analysis. In fact, if we want to go on with their semiclassical arguments, we must take care of the non-commutative limits $\hbar \to 0$ and $t_f - t_i \to 0^+$.

---

7 Supposedly, this motivated Dowker ([6], Introduction) “to present a somewhat neater and more attractive derivation of [the result (1)].”

8 To reuse their notations, they write p. 1376 that if a path is given by the concatenation $q(a, a') = q(a, b)q(b, a')$ hence $S[q(a, a')] > S[q(a, b)]$ or, translated into the notations of the present article, $\mathcal{C} = \mathcal{C}_1 \cdot \mathcal{C}_2 \Rightarrow S[\mathcal{C}] > S[\mathcal{C}_1]$. 
and, when $q_f \neq q_i$, the behaviour of $k_r(q_f, t_f, q_i, t_i)$ is given by one or more oscillatory integral for each homotopy class whose prefactor must be tamed and within this “battle of exponentials” between the semiclassical contributions, it is not simple to identify a winner if there is any. In any case, deducing that $|E| = 1$ exactly rather that approximately without, say, any real exponential prefactor is, in my opinion, the privilege of a too restrictive class of models. More generally, as argued above, the details of the differential structure of the action, should not be relevant in a topological analysis.

Another objection may be raised when considering [5]'s definition of linear independence. They use a much stronger condition than $\sum_{\epsilon \in \pi_1} A(\epsilon \cdot 1)k_{\epsilon \cdot 1} \equiv 0$; they require that this cancellation should occur for any alternative choice of $\epsilon_0$ while not affecting the coefficients $A$, in other words, to transcript their condition (p. 1376, top right column): $\sum_{\epsilon \in \pi_1} A(\epsilon' \cdot 1)k_{\epsilon' \cdot 1} \equiv 0$ for all $\epsilon'$ while $A(\epsilon' \cdot 1) = A(\epsilon_0 \cdot 1)$. But the latter condition is not generally fulfilled precisely because a phase factor is allowed to appear when passing from $A(\epsilon' \cdot 1)$ to $A(\epsilon \cdot 1)$ when $\epsilon \cdot 1 \neq \epsilon_0$.

Another criticism, which has no serious repercussion on their argument but is crucial in Schulman ([7], § 23.3)’s justification, can be brought when they work with a propagator $K$ whose value depends, up to a phase, on a choice of “mesh” to label the classes connecting $q_f$ to $q_i$ with the loops which correspond to our $\epsilon_f$ and $\epsilon_i$ in (26). The complete propagator $K$ cannot depend on the purely conventional choice of $\epsilon_f$ and $\epsilon_i$ and, therefore, a change of the latter cannot have any impact on $K$, even by simply changing its global phase. If this were the case, we would be led to the spurious multivalued propagator and then to the not less spurious multivalued wavefunctions $\phi(q, t_f) = \int Q K(q, t_f, q_i, t_i)\phi(q_i, t_i)\, dL$. From our starting expression (1) where no choice of $(\epsilon_f, \epsilon_i)$ is required, or by a direct elementary computation of the right-hand side of (26), $K(q, t_f, q_i, t_i)$ remains completely insensitive to the choice of $(\epsilon_f, \epsilon_i)$.

From a birds-eye view, all the proofs are suspicious that do not use, in one way or another, the characteristic property (4c), or, in other words, that do not fully use the unitary character of the quantum evolution; indeed, when the first homotopy group has an infinite number of elements $\pi_1(Q)$ labelled by some integers $n$, an exponential $E_n = e^{i\theta}$ with $\text{Re}\ \theta \neq 0$ should be considered (their exponentially increase when $n \to \pm\infty$ may be dominated by the exponential decrease of the corresponding oscillatory path integral and the sum (1) could remain convergent).

### 4 Crystalline Systems

Let us illustrate some of the points raised above in the case of crystals. We will consider a quantum system made, for simplicity, of one particle (this restriction is not essential) whose dynamics is governed by a time-independent Hamiltonian expressed in terms of canonical Hermitian operators $\hat{H} = H(\hat{p}, \hat{r})$ which is spatially

---

9 When $\pi_1(Q)$ is finite, (18) is sufficient for the $E$’s to be given by the roots of unity.
periodic on a Bravais lattice $\mathcal{R}$. Then $L = D$ the dimension of the direct space identified with $\mathbb{R}^D$. We will denote by $\mathbf{R}$ the vectors with integer components that constitute $\mathcal{R}$. The unitary operator $\hat{T}(\mathbf{R}) = e^{i\mathbf{p}\cdot \mathbf{R}}/\hbar$ represents the spatial translation by $\mathbf{R}$. All the $\hat{T}$’s commute one with the other and with $\hat{H}$. We can diagonalise them in the same orthonormal basis \[48\]

where $\sigma$ denotes a set of discrete quantum numbers and $k$ denotes $D$ continuous quantum numbers in the reciprocal space defined modulo a translation of the reciprocal lattice $\mathcal{R}$. To obtain the complete spectrum and the associated eigenbasis, it is necessary and sufficient for $k$ to run through an elementary cell that we will choose, for instance, to be the first Brillouin zone $\mathcal{C}$. Bloch theorem ([49], chap. 8 for instance) essentially says that the Hilbert space $\mathcal{H}$ of the states of the system can be decomposed in a direct sum of subspaces $(\mathcal{H}_k)_{k \in \mathcal{C}}$ where, for a given $k$, being the unitary translation operator by $-\hbar k$ in the reciprocal space, the discrete eigenvalues of

\[30\] are precisely labelled by $\sigma$ and allow to reconstitute the whole spectrum $E_\sigma(k)$. The associated eigenvectors of $\hat{H}_k$ are given by the Bloch states

\[31\]

that is

\[32a\] and are strictly $\mathcal{R}$-periodic,

\[32b\] in contrast with (28b). The corresponding wavefunctions $\phi_{\sigma,k}(r) = \langle r | \phi_{\sigma}(k) \rangle$ and the associated Bloch functions $u_{\sigma,k}(r) = \langle r | u_\sigma(k) \rangle$ are related by

\[33\]

whereas (32b) reads, for any $\mathbf{R} \in \mathcal{R}$,

\[34\]

Consider the propagator given by (8) where the sum is restricted to $\mathcal{H}_k$.\[ Springer\]
\[ K_k(r_f, t_f, r_i, t_i) = \sum_{\sigma} \phi_{\sigma,k}(r_f) \phi_{\sigma,k}^*(r_i) e^{-\frac{i}{\hbar} (t_f - t_i) E_{\sigma}(k)}; \quad (35) \]

\[ = \sum_{\sigma} u_{\sigma,k}(r_f) u_{\sigma,k}^*(r_i) e^{ik(r_f - r_i - t_f + t_i) E_{\sigma}(k)} . \quad (36) \]

Now the \( \tilde{R} \)-periodicity of the bands \( k \mapsto E_{\sigma}(k) \) and \( k \mapsto \phi_{\sigma,k} \) allows to expand \( K_k \) into Fourier series according to:

\[ K_k(r_f, t_f, r_i, t_i) = \sum_{R \in \mathcal{R}} e^{-ik\cdot R} K_R(r_f, t_f, r_i, t_i) \quad (37) \]

with (\( \tilde{v} \) stands for the volume of \( \tilde{C} \))

\[ K_R(r_f, t_f, r_i, t_i) = \frac{1}{\tilde{v}} \int_{\tilde{C}} K_k(r_f, t_f, r_i, t_i) e^{ik\cdot R} d^0k; \quad (38a) \]

\[ = \frac{1}{\tilde{v}} \int_{\tilde{C}} \sum_{\sigma} u_{\sigma,k}(r_f) u_{\sigma,k}^*(r_i) e^{ik(r_f - r_i - R) - \frac{i}{\hbar} (t_f - t_i) E_{\sigma}(k)} d^0k; \quad (38b) \]

\[ = \frac{1}{\tilde{v}} \int_{\tilde{C}} \sum_{\sigma} \phi_{\sigma,k}(r_f + R) \phi_{\sigma,k}^*(r_i) e^{-\frac{i}{\hbar} (t_f - t_i) E_{\sigma}(k)} d^0k; \quad (38c) \]

\[ = \tilde{K}(r_f + R, t_f, r_i, t_i). \quad (38d) \]

In the antepenultimate expression we have recognised the full propagator \( \tilde{K} \), \textit{i.e.} built with the complete spectrum of \( \tilde{H} \), for the particle going from \( \tilde{q}_i = r_i \) to \( \tilde{q}_f = r_f + R \). In fact, \( K_R(r_f, t_f, r_i, t_i) \) or \( \tilde{K}(r_f, t_f, r_i, t_i) \) are generally not \( \mathcal{R} \)-periodic, even up to a phase, neither in \( R \), neither in \( r_f \), neither in \( r_i \) but rather, for all \( R' \in \mathcal{R} \), from (34),

\[ K_R(r_f + R', t_f, r_i, t_i) = K_R(r_f, t_f, r_i - R', t_i) = K_{R+R'}(r_f, t_f, r_i, t_i) \quad (39) \]

whereas \( K_k \) inherits of the boundary conditions satisfied by \( \phi_{\sigma,k} \):

\[ K_k(r_f + R, t_f, r_i, t_i) = e^{ik\cdot R} K_k(r_f, t_f, r_i, t_i); \quad (40a) \]

\[ K_k(r_f, t_f, r_i + R, t_i) = e^{-ik\cdot R} K_k(r_f, t_f, r_i, t_i). \quad (40b) \]

In terms of path integrals, \( \tilde{K}(r_f + R, t_f, r_i, t_i) \) involves all the paths connecting \( (r_i, t_i) \) to \( (r_f + R, t_f) \) and, like \( K_R(r_f, t_f, r_i, t_i) \), is defined in the whole (simply-connected) \( \tilde{Q} = \mathbb{R}^0 \). In the equality (38d), we recover the identity (12) where \( \tilde{Q} \) is a primary cell \( C \) of the crystal, \( g \) is associated to a spatial translation by \( R \) in the Bravais lattice, and the partial propagator \( k_R(r_f, t_f, r_i, t_i) \) defined to be the restriction
of \((r_f, r_i) \mapsto K_R(r_f, t_f, r_i, t_i)\) to \(C\). As explained in the introduction, one can identify an open set of \(Q\) with an open set of its universal covering space \(\tilde{Q}\). This is done naturally when restricting \(\tilde{q} = q = r\) to the interior of \(C\); yet, as soon as we add \(R \neq 0\) to such an \(r\) we get outside this identification zone. The configuration space \(Q\) is obtained by identifying in \(\tilde{Q}\) every two points \((r, r')\) if and only if \(r' - r \in \mathcal{R}\). Then, \(Q\) reduces to the primary cell \(\tilde{C}\) with its opposite boundary edges identified; it is obtained by quotienting \(\tilde{Q} = \mathbb{R}^D\) by the commutative \(\mathcal{R}\)-translation group which is then interpreted as the first homotopy group \(\mathbb{Z}^D\) of the \(D\)-torus thus obtained. The relations (39) when one of the \((r_f, r_i)\) lies on one edge of the boundary of \(C\) illustrate what we have generally established, namely the discontinuity of the \(k\)’s. Taking \(t_f \rightarrow t_i\) we have, as an illustration of (13),

\[
\tilde{K}(r_f + R, t_i, r_i, t_i) = \delta(r_f + R - r_i)
\]  

(41)
a well-defined distribution in \(\tilde{Q}\) but that becomes problematic when tried to be restricted to \(C\), see (10), because it is not \(\mathcal{R}\)-periodic even up to a phase unlike

\[
K_k(r_f, t_i, r_i, t_i) = \sum_{R \in \mathcal{R}} e^{-ik \cdot R} \delta(r_f + R - r_i)
\]  

(42)
whose restriction to the interior of \(C\) coincides with \(\delta(r_f - r_i)\) in agreement with (5b). Now, when \((r_f, r_i) \in \mathcal{C}^2\), and when folding each path \(\tilde{C}\) in \(\tilde{Q} = \mathbb{R}^D\) to a path \(C\) in \(Q\), the Fourier series (37) is exactly an expansion of the form (1). The Bloch angles \(k\) label the \(U(1)\)-representation of \(\pi_1(Q) = \mathbb{Z}^D\)

\[
E(R) = e^{-ik \cdot R}.
\]  

(43)
Schulman ([3], Eqs. (2.2) & (2.6)) has proposed the decomposition (37) but did not explicitly interpreted \(K_R\) beyond of being a simple Fourier coefficient, all the more that he is sticking to Green functions rather than matrix elements of the evolution operator (without a time Heaviside function). However, surprisingly, to my knowledge, in all texts that try to introduce the path-integral formulation in a multi-connected space, including ([7], § 23.1), this general Bloch framework is abandoned to exemplify (1) for the sole free motion on the circle (\(D = 1\)) (in mathematical physics see (and its references [42]) that deal exclusively with the Laplace-Beltrami operator), heavily reinforced with Poisson summation formulae or Jacobi functions. Again, the choice of a particular Lagrangian as well as any differential structure, can only reduce the perspective and mask the fact that we deal with topology.

5 Homotopy Versus Homology

There is another family of groups that provides topological invariants of the configuration space, namely the homology/cohomology groups ([31], chap. 2 for instance). Among those is the first homology group, traditionally denoted by \(H_1(Q)\), which is made of one-dimensional cycles (loops with a moveable basepoint) that are not boundaries of a two-dimensional surface included in \(Q\); two chains being equivalent if they define the boundary of a two-dimensional surface in \(Q\). A non-unit element
of $H_1(Q)$ is typically the equivalence class of a chain around a “hole” in $Q$ and therefore both the groups $\pi_1(Q)$ and $H_1$ probe the “holes” in $Q$. However, they are generally different since unlike the first one, $H_1$ is always commutative ([31], § 2A).

With the use of Stokes theorem, one physically associates $H_1(Q)$ with the magnetic flux through the hole obtained by computing the circulation of a vector potential $A$ (a one-form) along a cycle $c$ in $H_1$: for a unit electric charge,

$$\Phi = \int_c A \, dx$$

(44)

and, obviously, the position of a base point chosen to compute the integral along a loop is irrelevant. This is the kind of topological phase that plays a key role in Dirac’s work on magnetic monopole and in the Ehrenberg–Siday–Aharonov–Bohm effect both mentioned in § 2. As long as we work with factors $E$ that are in $U(1)$, the physical properties coming from (1) cannot discriminate between $H_1(Q)$ and $\pi_1(Q)$: the commutativity of the phases is not able to reflect the non-commutativity of $\pi_1(Q)$ and if we want a finer signature of this non-commutativity we must consider systems whose $E$ are unitary matrices of dimension at least 2. One can also understand this requirement in Fig. (3): (b) and (c) are homotopically different (you cannot move the base point) but homologically identical (when you can move the base point) since the magnetic flux through them is the same, namely the magnetic flux carried by the left torus only.

Even in the case of anyons, one is unable to say if the topological properties at stakes are the ones of $\pi_1$ rather that $H_1$. Anyons can be interpreted as scalar particles moving in a two-dimensional surface, each of them carrying an individual magnetic flux $\Phi$ perpendicular to the surface. As they classically evolve with an interaction that prevents them from being at the same place at the same time, the trajectories of each anyon accumulate an Ehrenberg–Siday–Aharonov–Bohm phase while wrapping one around each other in a braid-like structure. Actually, the braid group with $N$ strands is precisely the $\pi_1$ of the configuration space of $N$ anyons and it is not commutative as soon as $N \geq 3$. If $b_n$ stands for the generator of the braid group where particles $n$ and $n+1$ are exchanged, we actually have the Artin-Yang-Baxter relation

$$b_n b_{n+1} b_n = b_{n+1} b_n b_{n+1}.$$  

(45)

For a set of identical anyons, the phase of each generator of the braid group $E(b_n)$ is independent of $n$ for (45) to be satisfied and can be taken to be $e^{i\Phi/(2\hbar)}$ but, again, the non-commutativity of the braid group is lost. To recover it, one should not only work with a model of particles with an internal degree of freedom such that their propagator involves not pure topological phases but unitary matrices $E_n = E(b_n)$ but also accept to deal with $E_n$ that depends on $n$ which is hardly sustainable for identical particles. However, when different species of anyons are present, one may recover some properties that emerge from the non-Abelian character of their intertwining (see [50] for a pedagogical review of these so called non-Abelian anyons).

Coming back to a configuration space with two holes, each of them being associated with one generator of $\pi_1(Q)$: we can easily conceive an experimental set up where this is relevant, with two superconducting tori or two Mach-Zehnder interferometers can be
used. One can even think of a representation of $\pi_1$ still keeping its non-commutativity but having a finite number of images provided by $E$. The smallest non-commutative finite group is the permutation group of 3 elements and the smallest dimension of a unitary non-commutative linear representation of it is 3 (as in the 1D-case, the constraints imposed by $r^2 = 1$ for any transposition $\tau$ that generates the group force the 2D unitary matrices to be $\pm 1$ and therefore commutative). For instance, up to any global rotation, to represent two transpositions one can choose

$$E_1 \overset{\text{def}}{=} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}; \quad E_2 \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{(46)}$$

as the two non-commutative generators. The third transposition is represented by

$$E_3 \overset{\text{def}}{=} E_2 E_1 E_2 = E_1 E_2 E_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{(47)}$$

and the circular permutations are represented by

$$E_+ \overset{\text{def}}{=} E_1 E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad E_- \overset{\text{def}}{=} E_2 E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{(48)}$$

On a three dimensional Euclidean vectors all the $E$’s belong to $SO(3)$; the $E_1, E_2, E_3$ are rotations of angle $\pi$ around the axis $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ respectively, and $E_\pm$ are rotations of angle $\pm 2\pi/3$ around the axis $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

$$E_1^2 = E_2^2 = E_3^2 = 1; \quad E_\pm^2 = E_\mp; \quad E_3^\pm = 1 \quad \text{(49)}$$

Having two independent non commuting generators, say $I_1$ and $I_2$ as in Fig. 3, $\pi_1(Q)$ is isomorphic to the set of the double infinite sequences of integers $(n_1, m_1, \ldots, n_i, m_i, \ldots)$ but any element $I_1^{n_1} \cdot I_2^{m_1} \cdots I_1^{n_i} \cdot I_2^{m_i} \cdots$ can be unitarily represented by one of the six rotations defined above $\{1, E_1, E_2, E_3, E_+, E_-\}$ if we choose $E(I_1) = E_1$ and $E(I_2) = E_2$. One may conceive that such rotations may be physically implemented on a spin-1 particle, for instance a fictitious spin obtained by working with cold atoms where only a bunch of 3-(sub)levels is relevant to describe their interaction with light.

**Acknowledgements** It is a pleasure to acknowledge my deep gratitude to Alain Comtet of the Laboratoire de Physique Théorique et de Modèles Statistiques de l’Université Paris Saclay for his precious advices on this work; all the more that he somehow initiated it some decades ago while guiding my first steps in physics research. Many thanks also to Dominique Delande for his continuous support and hospitality at the Laboratoire Kastler-Brossel.
References

1. Poe, E.A.: Marginal notes ii. a sequel to the “marginalia” of the “democratic review”. Godey’s magazine and Lady’s Book, vol. XXXI, pp. 120–123 (1845)
2. Schulman, L.S.: A path integral for spin. Phys. Rev. 176(5), 1558–1569 (1968)
3. Schulman, L.S.: Green’s function for an electron in a lattice. Phys. Rev. 188(3), 1139–1142 (1969)
4. Schulman, L.S.: Approximate topologies. J. Math. Phys. 12(2), 304–308 (1971)
5. Laidlaw, M.G.G., Morette-DeWitt, C.: Feynman functional integrals for systems of indistinguishable particles. Phys. Rev. D 3(9), 1375–1378 (1971)
6. Dowker, J.S.: Quantum theory on multiply connected spaces. J. Phys. A 5, 936–943 (1972)
7. Schulman, L.S.: Techniques and Applications of Path Integration. Wiley, New York (1981)
8. Nash, C.: Topology and physics—a historical essay. In [55], pp. 359–415, Chap. 12. See also arxiv: hep-th/9709135
9. Mouchet, A.: Drowning by numbers: topology and physics in fluid dynamics. In [54], pp. 249–266. See also arxiv:1706.09454
10. Dirac, P.A.M.: Quantised singularities in the electromagnetic field. Proc. R. Soc. Lond. Ser. A, 133, 60–72 (1931). reproduced in §2.1 of [59].
11. Dirac, P.A.M.: The quantum theory of the emission and absorption of radiation. Proc. R. Soc. Lond. Ser. A, 114, 243–265 (1927). reproduced in [57] ( paper. 1 p. 1) and in [13] (paper. 1 p. 7)
12. Nieto, M.M.: Quantum phase and quantum phase operators: some physics and some history. Phys. Scripta T48, 5–12 (1993)
13. Barnett, S.M., Vaccaro, J.A.: The Quantum Phase Operator. A Review. Series in Optics and Optoelectronics. Taylor and Francis, New York (2007)
14. Feynman, R.P.: The principle of least action in quantum mechanics (1942), reproduced in [51], pp. 1–69.
15. Feynman, R.P.: Space-time approach to non-relativistic quantum mechanics. Rev. Modern Phys. 20(2), 367–387 (1948). reproduced in [57], p. 321 and in [51], p. 71.
16. Pauli, W.: Selected topics in field quantization, volume 6 of Pauli Lectures on Physics: Selected Topics in Field Quantization. MIT press, Cambridge, 1957/73. In: Enz, C.P., and translated by S. Margulies and H.R. Lewis from the German notes Ausgewählte Kapitel aus der Feldquantisierung (Zürich, Verlag des Vereins der Mathematiker und Physiker an der ETH, 1957) prepared by U. Hochstrasser und M. R. Schafroth from a course given in ETH in 1950–51
17. DeWitt, B.S.: Dynamical theory in curved spaces. i. A review of the classical and quantum action principles. Rev. Modern Phys. 29(3), 377–397 (1957)
18. Edwards, S.F., Gulyaev, Y.V.: Path integrals in polar co-ordinates. Proc. R. Soc. London Ser. A 279(1377), 229–235 (1964)
19. McLaughlin, D.W., Schulman, L.S.: Path integrals in curved spaces. J. Math. Phys. 12(12), 2520–2524 (1971)
20. Dowker, J.S.: Covariant Feynman derivation of Schrodinger’s equation in a Riemannian space. J. Phys. A 7, 1256–1265 (1974)
21. Fanelli, R.: Canonical transformations and phase space path integrals. J. Math. Phys. 17(4), 490–493 (1976)
22. Gervais, J.-L., Jevicki, A.: Point canonical transformations in the path integral. Nuclear Phys. A 110, 93–112 (1976)
23. Prokhorov, L.V., Shabanov, S.V.: Hamiltonian Mechanics of Gauge Systems. Cambridge Monographs on Mathmatica Physics. Cambridge University Press, Cambridge (2011)
24. Feynman, R.P., Hibbs, A.R.: Quantum mechanics and path integrals. International series in pure and applied physics. McGraw-Hill Publishing Company, New York, 1965. Emended edition by Daniel F. Styer (Dover, 2005)
25. Ehrenberg, W., Siday, R.E.: The refractive index in electron optics and the principles of dynamics. Proc. Phys. Soc. B 62(1), 8–21 (1949)
26. Aharonov, Y., Bohm, D.: Significance of electromagnetic potentials in the quantum theory. Phys. Rev. 115(3), 485–491 (1959)
27. Leinaas, J.M., Myrheim, J.: On the theory of identical particles. Nuovo Cimento B 37(1), 1–23 (1977)
28. Wilczek, F.: Quantum mechanics of fractional-spin particles. Phys. Rev. Lett. 49(14), 957–959 (1982)
29. Arovas, D.P.: Topics in fractional statistics. In [58], pp. 284–322.
30. Bartolomei, H., Kumar, M., Bisognin, R., Marguerite, A., Berroir, J.-M., Bocquillon, E., Plaçais, B., Cavanna, A., Dong, Q., Gennser, U., Jin, Y., Fève, G.: Fractional statistics in anyon collisions. Science 368(6487), 173–177 (2020)
31. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge, 2002. ISBN 0-521-79160-X, freely available from pi.math.cornell.edu/hatcher
32. Fomenko, A., Fuchs, D.: Homotopical Topology, volume 273 of Graduate Texts in Mathematics. Springer, 2016. ISBN 9783319234885. Second edition
33. Horváthy, P., Morandi, G., Sudarshan, E.C.G.: Inequivalent quantizations in multiply connected spaces. Nuovo Cimento 11D(1–2), 201–228 (1989)
34. Oh, C.H., Soo, C.P., Lai, C.H.: The propagator in the generalized Aharonov–Bohm effect. J. Math. Phys. 29(5), 1154–1157 (1988)
35. Balachandran, A.P.: Classical topology and quantum phases: quantum mechanics. In [52], pp. 1–28.
36. Berg, H.P.: Feynman path integrals on manifolds and geometric methods. Nuovo Cimento 66A(4), 441–449 (1981)
37. Tarski, J: Path integrals over manifolds. In [53], pp. 229–239.
38. Anderson, A.: Changing topology and nontrivial homotopy. Phys. Lett. B 212(3), 334–338 (1988)
39. Ho, V.B., Morgan, M.J.: Quantum mechanics in multiply-connected spaces. J. Phys. A 29, 1497–1510 (1996)
40. Tanimura, S., Tsutsui, I.: Inequivalent quantizations and holonomy factor from the path-integral approach. Ann. Phys. 258, 137–156 (1997)
41. Forte, S: Spin in quantum field theory. In [56], pp. 66–94 and also arxiv:hep-th/0507291
42. Kocábová, P., Št’ovíček, P.: Generalized Bloch analysis and propagators on Riemannian manifolds with a discrete symmetry. J. Math. Phys. 49(033518), 1–15 (2008)
43. Berry, M.V.: Exact Aharonov–Bohm wavefunction obtained by applying Dirac’s magnetic phase factor. Eur. J. Phys. 1, 240–244 (1980)
44. Tononura, A.: Direct observation of thitherto unobservable quantum phenomena by using electrons. Proc. Natl. Acad. Sci. USA 102(42), 14952–14959 (2005)
45. Webb, R.A., Washburn, S., Umbach, C.P., Laibowitz, R.B.: Observation of $h/e$ Aharonov–Bohm oscillations in normal-metal rings. Phys. Rev. Lett. 54(25), 2696–2699 (1985)
46. DeWitt-Morette, C., Maheshwari, A., Nelson, B.: Path integration in non-relativistic quantum mechanics. Phys. Rep. 50(5), 255–372 (1979)
47. Cartier, P., DeWitt-Morette, C.: A new perspective on functional integration. J. Math. Phys. 36(5), 2237–2312 (1995)
48. Zak, J.: Finite translations in solid-state physics. Phys. Rev. Lett. 19(24), 1385–1387 (1967)
49. Ashcroft, N.W., Mermin, N.D.: Solid State Physics. Saunders College, Philadelphia (1976)
50. Nayak, C., Simon, S.H., Stern, A., Freedman, M., Sarma, S.D.: Non-abelian anyons and topological quantum computation. Rev. Modern Phys. 80(3), 1083–1159 (2008)
51. Brown, L.M.: Feynman’s Thesis: A New Approach to Quantum Theory. World Scientific, New Jersey, 2005. ISBN 9812563660
52. De Filippo, S., Marinaro, M., Marmo, G., Vilasi, G., editors. Geometrical and Algebraic Aspects of Nonlinear Field Theory, Amsterdam, 1989. In: Proceedings of the meeting on Geometrical and Algebraic Aspects of Nonlinear Field Theory, Amalfi, Italy, May 23–28, 1988, North-Holland. ISBN 0-444-87359-7
53. Doebner, H.D., Andersson, S.I., Prety, H.R. editors. Differential Geometric Methods in Mathematical Physics, volume 905 of Lecture notes in Mathematics, Berlin, 2007. Schladming Winter School in Theoretical Physics, Springer. ISBN 3-540-38590-8
54. Schrödinger, J. (ed.): Selected Papers on Quantum Electrodyanamics. Dover, New York (1958)
55. Shapere, A., Wilczeck, F. (eds.): Geometric Phases in Physics, volume 5 of Advanced Series in Mathematical Physics. World Scientific, Singapore, 1989. ISBN 9971-50-599-1
59. Thouless, D.J.: Topological Quantum Numbers in Nonrelativistic Physics. World Scientific, Singapore (1998)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.