Symmetries and linearization of ordinary difference equations

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Abstract

The connection between symmetries and linearizations of discrete-time dynamical systems is being investigated. It is shown, that existence of semigroup structures related to the vector field and having linear representations enables reduction of linearization problem to a system of first order partial differential equations. By means of inverse of the Poincaré map one can relate symmetries in such linearizable systems to continuous and discrete ones of the corresponding differential equations.

1 Introduction

The principal objective behind the concept of (exact) linearization is to express an apparently involved evolution law in simple terms, facilitating further analysis. The purpose is also to bring a system to some 'canonical' form enabling comparison among studied dynamical systems in terms of e.g. minimal embedding dimension, which can be regarded as a complexity measure. To that end one seeks a projection map (2) from a suitable linear space \( \mathcal{E} \) onto the phase space \( \mathcal{M} \), which defines a homomorphism of appropriate (semi)group actions. The desired feature is of course finiteness of the embedding space \( \mathcal{E} \), otherwise the linearization remains formal. On the other hand, it is worthwhile to recall the applications of linearization in infinite-dimensional spaces. An example are the methods of Hilbert spaces in spectral theory (1).

Since the works of Lie, the notion of a symmetry as an automorphism in the space of solutions plays central role in the analysis of differential equations (6 7 8). In particular, it is well-known that the knowledge of an infinitesimal Lie-point and generalized symmetries, allows to reduce the system of ordinary differential equations (ODEs). An analogous result for ordinary difference equations (O\(\Delta\)Es) was obtained by Maeda (2), and expanded by Quispel et al. (2 4 5). Namely, they showed that determination of a Lie-point or generalized symmetry, enables reduction of the dimension of an O\(\Delta\)E, and in particular cases its linearization (2 4). The approach taken up by Maeda and Quispel et al. relies on finding an infinitesimal time-translation symmetry, and transformation into normal coordinates. It should be noted, that determination of such particular symmetry may be in general highly nontrivial task.

In this work, we introduce a method for the study of difference equations, based upon observation that finite symmetries of the evolution generator form a structured set of relations, identified as a semigroup of similarities. If at least one of its closed subsets is known, then that structure can be used to form a linear semigroup of isomorphic structure, and obtain differential system of constraints for the desired projection map \( p \). This method is illustrated by two examples: the ‘classical’ Riccati equation related to the symmetry group \( \text{SL}(2) \), and a thorough analysis of a logistic equation, with emphasis on the role played by similarities in classification of obtained continuous-time solutions (the inverse Poincaré maps).

2 Linearization and symmetries

Let us begin with a brief account of the problem of linearization of recurrences, and its relation to the symmetries of evolution generators.

Let \( f : \mathcal{M} \to \mathcal{M} \) be a function defined on a manifold \( \mathcal{M} \). Consider the autonomous system of difference equations

\[ x_{n+1} = f(x_n), \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \quad (1) \]

Iteration of the function \( f \) generates an associated semigroup with multiplication defined by composition of functions, and unity (monoid)

\[ G := \{ f^n \mid f^0 = \text{id}, \ f^{n+1} = f^n \circ f, \ n \in \mathbb{N}_0 \}. \]

The linearization of Eq. (1) is a pair \((L, p)\) comprising a linear map \( L : \mathcal{E} \to \mathcal{E} \) acting on a vector space \( \mathcal{E} \), and projection \( p : \mathcal{E} \to \mathcal{M} \), such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{L^n} & \mathcal{E} \\
p \downarrow & & \downarrow p \\
\mathcal{M} & \xrightarrow{f^n} & \mathcal{M}
\end{array}
\]

for all \( n \in \mathbb{N}_0 \). Equivalently, one can write

\[ f^n \circ p = p \circ L^n. \quad (2) \]

Iterations of the linear map \( L \) form another semigroup

\[ G_{\text{lin}} := \{ L^n \mid n \in \mathbb{N}_0 \}, \]

which, according to (2) is homomorphic to \( G \). Therefore, the linearization involves (i) fibration \( \mathcal{E} \xrightarrow{p} \mathcal{M}, \) (ii) linear function \( L \), and (iii) semigroup homomorphism.

Further, the kernel of \( p \) is a free monoid acting along the fibre:

\[ \ker p = \{ \kappa : \mathcal{E} \to \mathcal{E} \mid p \circ \kappa = p \} \quad (3) \]
By definition, the elements of this kernel satisfy
\[ \forall \kappa \exists \kappa': \quad \kappa \circ L = L \circ \kappa' \]  
where \( \kappa, \kappa' \in \ker p \). Notice, that the sections \( \{ \sigma \} \in p^{-1} \) and consequently the kernel elements too, are typically multivalued. Whenever they appear in some expression, it is understood that the equality holds true only for particular subset of these maps. For instance, the inverse to Eq. (4)
\[ \sigma \circ f^n = \kappa \circ L^n \circ \sigma, \]
means that for every \( \sigma \in p^{-1} \) there exists \( \kappa \in \ker p \) (or vice versa) satisfying this relation.

The iteration of function \( f \) is often a result of temporal discretization, and interpreted as the Poincaré map of some underlying, continuous-time system. The reciprocal operation is what we call the inverse Poincaré map (IPM), namely an extension \( f^n, \mathcal{M} \mapsto (f^t, \mathcal{M}) \), such that \( \mathcal{M} \supset \mathcal{M} \), \( f^t : \mathcal{M} \to \mathcal{M} \) for \( t \in \mathbb{R}_+ \), and
\[
\begin{align*}
(1) & \quad f^1 \equiv f, \\
(2) & \quad \lim_{t \to 0} f^t = \text{id}, \\
(3) & \quad \forall t, s \in \mathbb{R}_+: \quad f^t \circ f^s = f^{t+s},
\end{align*}
\]
In other words, the inverse Poincaré map is a one-parameter continuous monoid \( \{ f^t \} \) congruent to \( \{ f^n \} \) at \( t \in \mathbb{N}_0 \). The linearization \( (L, p) \) of Eq. (1), provides a suitable framework for obtaining IPMs. It must, however, be borne in mind that unlike \( L^n \), the power \( L^t \), \( t \in \mathbb{R}_+ \) is multivalued what makes characterization of the semigroup \( G_\text{lin} \) by its generator \( L \) imprecise. One should instead use a particular branch of its logarithm \( E \), that is
\[ L^t = \exp(tE), \quad L^t : \hat{\mathcal{E}} \to \hat{\mathcal{E}}, \quad \sigma \in \hat{\mathcal{E}}, \]
which is well-defined. If the kernel of \( p \) is linear, then \( E \circ \kappa = E \circ \kappa' \).

Further, the notion of inverse Poincaré map allows to discriminate between apparently different linearizations, namely two linearizations \( (p_1, E_1), (p_2, E_2) \) are regarded as equivalent if the corresponding IPMs
\[
\begin{align*}
f^t_1 &= p_1 \circ e^{tE_1} \circ \sigma_1, \\
f^t_2 &= p_2 \circ e^{tE_2} \circ \sigma_2,
\end{align*}
\]
are homomorphic, i.e.,
\[ \exists \phi : \hat{\mathcal{M}}_1 \Rightarrow \hat{\mathcal{M}}_2 : \quad \phi \circ f^t_1 \cong f^t_2 \circ \phi, \]
where the symbol ‘\( \cong \)’ indicates multivalence, and ‘\( \cong \)’ is used instead of ‘\( \equiv \)’, to emphasize that the branches of map \( \phi \) on both sides may be different.

According to the above definition, equivalence implies that, besides \( \phi \), there must exist a map \( U : \hat{\mathcal{E}}_1 \to \hat{\mathcal{E}}_2 \), such that
\[
\begin{align*}
p_2 \circ U &= \phi \circ p_1, \\
k_2 \circ e^{tE_2} \circ U &= U \circ e^{tE_1} \circ k_1.
\end{align*}
\]

For linear kernels the condition (5a) reduces to ‘similarity’ between the logarithms \( k_2 \circ e^{tE_2} \circ U = U \circ e^{tE_1} \circ k_1 \). Whenever the two flows are equal, \( f^t_1 = f^t_2 \), the solution of eqs. (5) is of the form
\[ U_{\text{eq}} = \sigma_2 \circ p_1, \]
and \( \phi = \text{id} \). Indeed, this reduces eq. (5a) to an identity, while (5b) can be verified by direct calculus:
\[
\begin{align*}
k_2 \circ e^{tE_2} \circ U_{\text{eq}} &= \sigma_2 \circ p_2 \circ e^{tE_2} \circ \sigma_2 \circ p_1 \\
&= \sigma_2 \circ f^t_2 \circ p_1 \\
&= \sigma_2 \circ f^t_1 \circ p_1 \\
&= \sigma_2 \circ p_1 \circ k_1 \circ e^{tE_1} \\
&= U_{\text{eq}} \circ e^{tE_1} \circ k_1.
\end{align*}
\]

If \( f^t_1 \neq f^t_2 \) and one can find an alternative solution of (5b), that is \( U \neq U_{\text{eq}} \), then
\[ \phi = p_2 \circ U \circ \sigma_1, \quad \phi : \mathcal{M} \Rightarrow \hat{\mathcal{M}}_2, \]
is one of the desired homomorphism maps, establishing equivalence between these two IPMs.

At the same time \( \phi \), becomes a symmetry of the function \( f \) itself, or of its extension defined on \( \hat{\mathcal{M}}_2 \), that is
\[ \phi \circ f \cong f \circ \phi. \]

Conversely, any such discrete symmetry provides similarity transformation between two IPMs. If the image of \( \phi \) is \( \mathcal{M} \), then it is a proper symmetry of the original function \( f \). Equation (7) directly suggests another way of finding symmetries by means of the linearization \( (p, L) \).

Any linear mapping \( M : \hat{\mathcal{E}} \to \hat{\mathcal{E}} \) whose commutant with \( L \) is an element of the kernel of projection map
\[ L \circ M \circ L^{-1} \circ M^{-1} \in \ker p, \]
gives rise to a symmetry \( \phi \) defined by
\[ \phi := p \circ M \circ \sigma. \]

The case, \( M = L^t \) for fixed \( t \in \mathbb{R}_+ \) leads to the ‘trivial’ symmetry coinciding with the time-translation one \( \phi = f^t \).

The condition (8), can often be rewritten in alternative form as a solution of certain implicit system
\[ S(x, \phi(x)) = 0, \]
such that the following implication holds
\[ S(x, x') = 0 \quad \Rightarrow \quad S(f(x), f(x')) = 0. \]
Accordingly, the solutions of (1) are invariant subsets of this relation \( S(x_n, x_{n+1}) \equiv 0 \), i.e. the symmetry maps solutions into solutions.

The symmetries \( \phi(x) \), similarly to linearizing transformations \( p \), can be expanded locally into power series in \( x \) in the neighbourhood of fixed points (FPs) of the generating map \( f \). Moreover, the same reasons which cause nonexistence of (local) linear normal forms, also prevent unambiguous symmetry expansion whenever \( f \) has resonant spectra at at FP \( x_* \). If we denote by \( x_i^t \), \( i = 1, \ldots, \dim \mathcal{M} \) the local coordinates of a point \( x \in \mathcal{M} \), and the spectra of the first derivatives \( f'_{i,j} \equiv \partial f' / \partial x^j \) by

\[
\text{sp } f'_{i,j}(x_*) = \{ \lambda_{(1)}, \ldots, \lambda_{(m)} \}, \quad m = \dim \mathcal{M},
\]

then this resonance condition takes the form

\[
\exists \bar{k} \in \mathbb{N}^m : \prod_{i=1}^{\dim \mathcal{M}} (\lambda_{(i)} - k_i) \in \text{sp } f'_{i,j}(x_*).
\]

If the condition (3) is to be satisfied by a single branch of \( \phi \) near FP \( x_* \), then one finds \( \phi(x_*) = x_* \). The expansion coefficients in the series

\[
\phi(x_* + \delta x)^n = x_* + \phi_i^j \delta x^j + \frac{1}{2} \phi_i^j k \delta x^j \delta x^k + \ldots, \quad (12)
\]

where summation over repeated indices is understood, are recursively defined functions of \( f'_{i,j}(x_*) \) and the free parameter tensor \( \phi_i^j(x_*) \). In particular, choosing

\[
\phi_i^j(x_*) = (f^n)'_{i,j}(x_*), \quad n \in \mathbb{Z},
\]

leads to the trivial symmetries \( \phi = f^n \), \( n \in \mathbb{N}_0 \). Nevertheless this does not imply that \( \phi \) coincides with the time translation symmetry \( f^t \) for all \( t \in \mathbb{R}_+ \), since in general \( x_* \) is not a fixed point of \( f^t \). Other special choices include \( \phi_i^j(x_*) = 0 \Rightarrow \phi \equiv x_* = \text{const} \), \( \phi_i^j(x_*) = \delta_i^j \Rightarrow \phi = \text{id} \), and \( \phi_i^j(x_*) = -\delta_i^j \) yields self-inverse symmetries, i.e. \( \phi^{-1} = \phi \). In the second example provided in this article, we use local expansion (12) together with the property (11), to investigate numerically the symmetries of logistic map. In the same example, the many-valued symmetries do not satisfy \( \phi(x_*) = x_* \) for all fixed-points \( x_* \).

Composition of symmetries form semigroups with identity (groups, if all elements are invertible), which can be in general uncountable. The whole set of symmetries of a given map \( f \) is always non-empty, containing at least the semigroup \( G \) itself. In particular, if a subset \( \mathcal{F} := \{ \phi^s \} \) admits continuous parametrization \( s \in \mathbb{R} \), satisfying \( \lim_{s \to 0} \phi^s = \text{id} \), \( \phi^s \circ \phi^t = \phi^{s+t} \), then one can construct a piece-wise autonomous (i.e. autonomous on open intervals of the evolution parameter) vector field

\[
\frac{d}{ds} \phi^s(x) \bigg|_{s=0} = \Phi(x), \quad \dot{\Phi} := \Phi_i^j \partial_i + \Phi_i^j \partial_i,
\]

where barred indices refer to complex-conjugated variables if \( \mathcal{M} \) is a complex manifold – real case is obtained by restriction to vanishing imaginary parts. Differentiation of the symmetry condition

\[
\phi^s \circ f \equiv f \circ \phi^s
\]

leads to the tangent map (push-forward) \( \Phi|_f = f_\ast \dot{\Phi} \), which in local coordinates has the form

\[
\Phi^i \circ f = f'_{i,j} \Phi^j,
\]

provided \( f \) is holomorphic. As pointed out by Maeda [2] and Quispel et al. [3, 5], existence of the vector field \( \phi \) satisfying (13), can be used to reduce dimensionality of Eq. (1) in the same way as it is done in the theory of differential equations. More precisely, if there exists an IPM \( f^t \) such that the symmetry condition holds true for all \( t \in \mathbb{R}_+ \), then the corresponding vector field \( \dot{X} \) defined by

\[
\frac{d}{dt} f^t(x) \bigg|_{t=0} = X(x), \quad \dot{X} := X_i^j \partial_i + \dot{X}_i \partial_i,
\]

satisfies the Lie point-symmetry condition

\[
[\dot{X}, \Phi] = 0.
\]

### 3 One-parameter family of similarities

Instead of constructing symmetries and similarities upon linearizations, we now consider a situation when one knows at least some of them in advance. The question arises as to whether this information can be exploited to determine the linearizations, and in this paragraph we investigate this possibility.

The generating function \( f \) together with its symmetries form a semigroup with respect to composition, equipped with unity. That the function in concern \( f \) may be part of a structured set of maps, is a key element, which significantly simplifies the search for a homomorphism with linear representation of the same structure, that is – a linearization.

Similarity between two maps generalizes the notion of a symmetry. Analogously, a set of similarities form a semigroup with respect to composition

\[
H := \{ h(\alpha) : \mathcal{M} \rightarrow \mathcal{M} | \forall \alpha, \beta : h(\alpha) \circ h(\beta) \equiv h(\gamma) \circ h(\alpha) \},
\]

In particular, if all elements of \( H \) are invertible then this set becomes an inner automorphism group with composition \( h(\alpha) : h(\beta) \mapsto h(\alpha) \circ h(\beta) \circ h^{-1}(\alpha) = h(\gamma) \).

Suppose, that \( H \) is continuously parametrized, and contains the function \( f \). The structure of this set can be used to determine (formally) isomorphic semigroup \( H_{\text{lin}} = \{ L(\alpha) \} \) of linear maps acting on some vector space \( \mathcal{E} \). Assuming there exists a homomorphism between \( H \) and \( H_{\text{lin}} \) the problem of linearization reduces to finding a map \( p : \mathcal{E} \rightarrow \mathcal{M} \) such that

\[
\forall \alpha : p \circ L(\alpha) = h(\alpha) \circ p.
\]
Let the explicit, coordinate form of \( L(\alpha) \) be
\[
L^n(y; \alpha) = L^n_b(\alpha)y^b, \quad a, b = 1, \ldots, m, \\
L^n_b(\alpha) \in \mathbb{C}.
\]

Differentiating with respect to \( y \) and \( \alpha \) gives two independent relations
\[
p^{i, \alpha}_a(L\tilde{y})L^n_b = h^i_j(p(y))p^{j, \alpha}_b(y) \\
p^{i, \alpha}_a(L\tilde{y})\partial_\alpha L^n_b y^b = \partial_\alpha h^i_j(p(y))
\]
where \( \tilde{y} = \{y^a\} \) is the coordinate vector of the point \( y \), \( \partial_\alpha \equiv \partial/\partial_\alpha \). Comparing the above equations we get
\[
h^i_j p^{j, \alpha}_b E^b_c y^c = \partial_\alpha h^i_j.
\]
which where all omitted arguments of \( p \) are \( y \), and as before \( L(\alpha) = e^{E(\alpha)} \), that is \( \partial_\alpha E^b_c = [L^{-1}]^b_d \partial_\alpha L^d_c \). Therefore, the existence of continuous 1-parameter semigroup of similarities leads to the first order system of partial differential equations (PDEs) for the mapping \( p \). Its solution, together with the linear map \( L(\alpha) \) such that \( h(\alpha) = f \), provides the desired linearization of the system \( \mathbf{1} \).

We stress, that the crucial point in the above construction is the formal isomorphism between structured sets of nonlinear and linear mappings, and not their particular composition rule – cf. example \( \mathbf{11} \) below.

### 3.1 Example: homographic map

For an easy illustration of the procedure described above, we first analyse the case of Riccati equation. Let \( x \in U = \mathbb{C} \), and the dynamical system \( \mathbf{1} \) be the equation generated by a homography \( f : U \to U \)
\[
x_{n+1} = f(x_n) := \frac{ax_n + b}{\gamma x_n + \delta}, \quad \alpha \delta \neq \gamma \beta.
\]
(16)

Recall, that the set of all homographies is a Kleinian group, and under the above constraint imposed on the parameters, which exclude uninteresting constant case, it becomes isomorphic with \( \text{SL}(2, \mathbb{C}) \). A local parametrization can be given by
\[
H = \left\{ h(x) = \frac{\alpha (\delta + \gamma x) - 1}{\gamma (\delta + \gamma x)}, \quad \delta \neq 0 \right\}
\]
(17a)
\[
H_{\text{lin}} = \left\{ L = \frac{\alpha}{\gamma} \frac{\alpha \delta - 1}{\delta}, \quad \delta \neq 0 \right\}
\]
(17b)

Clearly \( f \in H \).

Out of the three continuous parameters \( (\alpha, \gamma, \delta) \) it is sufficient to take the one specific – choose \( \alpha \) for concreteness. The differential constraint \( \mathbf{15} \) becomes
\[
\frac{\partial h}{\partial x} \frac{\partial p}{\partial y} [L^{-1}]^b_d \partial_\alpha L^d_c y^d = \frac{\partial h}{\partial \alpha}.
\]
Inserting \( \mathbf{17} \), one gets single PDE of the form
\[
\frac{\partial p}{\partial y^1} \frac{\delta}{(\delta + \gamma y^1)^2} - \frac{\partial p}{\partial y^2} \frac{\gamma}{(\delta + \gamma y^2)^2} = \frac{1}{\gamma y^1 + \delta y^2}.
\]

The solution of this equation has two integration constants \( C \neq 0 \) and \( D \): \( p(y) = \frac{y^1 + \delta CD}{y^2 - \gamma CD} \).

Taking into account \( \mathbf{14}, \mathbf{17} \) and \( \mathbf{18} \), one finds that the only possibility left to satisfy \( C \neq 0 \) is to set \( D = 0 \). Therefore
\[
p(y) = \frac{y^1}{y^2},
\]
which is the desired linearizing map.

Having found the linearization, it is straightforward to provide the inverse Poincaré maps \( p'(x) \) of the Riccati equation \( \mathbf{16} \). These maps are also homographies with \( t \)-dependent coefficients, and in general, for arbitrary initial position \( x \in U \) they go to infinity in finite time. This implies, that the motion is better described in complex projective space \( \mathcal{M} = \mathbb{C} \cup \{ \infty \} = \mathbb{C}P^1 \), instead of a single chart \( U \). Taking the second local chart as \( U' = \mathbb{C} \), with transition functions \( g : U \to U' \), \( x' = g(x) := 1/x \), one has \( f' = g \circ f \circ g^{-1} \), that is
\[
x_{n+1}' = f'(x_n') = \frac{\gamma (\delta x_n' + \gamma)}{(\alpha \delta - 1)x_n' + \alpha \gamma}.
\]

The projection map in these coordinates \( p' = g \circ p \), is simply \( p'(y) = y^2/y^1 \). Both \( p \) and \( p' \) define local inhomogeneous coordinates of the projective space \( \mathcal{M} = \mathbb{C}P^1 \).

### 3.2 Example: the logistic equation

Consider a more involved example of the logistic equation
\[
x_{n+1} = f(x_n) := 4x_n(1 - x_n).
\]
(19)

It is well-known that it is linearizable – we now recover this linearization following the approach described earlier. We first observe, that \( f \) is a member of the following set\(^2\)
\[
H = \{ h(x; \alpha) = \sin^2[\pi \alpha + \arccos(1 - 2x)] \mid \alpha \in \mathbb{R} \}
\]
(20)

for \( \alpha \in \mathbb{Z} \). The compositions
\[
h(\alpha) \circ h(\beta) \cong h(\gamma) \circ h(\alpha),
\]

satisfy either
\[
\gamma = 2\beta - \alpha \quad \text{or} \quad \gamma = 3\alpha - 2\beta.
\]

\(^2\)The branches of infinitely multivalued map \( \arccos : [-1, 1] \to \mathbb{R} \), can be expressed explicitly as \( \arccos_k(x) = (-1)^k \arccos(x) + \pi |k - ((-1)^k - 1)/2| \), where \( \arccos : [-1, 1] \to [0, \pi] \) is the principal branch.
The following linear operators $L_{\pm} = L_{\pm}(\alpha)$

$$L_+ = \begin{bmatrix} 2 & \alpha \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad L_- = \begin{bmatrix} -2 & \alpha \\ 0 & 1 \end{bmatrix}.$$ 

define two representations $H_{\text{lin}}$ of $L$, that is

$$L_+(\alpha)L_+(\beta) = L_+(2\beta - \alpha)L_+(\alpha),$$

$$L_-(\alpha)L_-(\beta) = L_-(3\alpha - 2\beta)L_-(\alpha).$$

The equations (15), in each case reduce to a single PDE

$$-\frac{\partial p}{\partial t} = \pm \frac{2\pi}{y^2} \sqrt{p(1-p)},$$

where the ‘+’ signs correspond to $L_+$ and the ‘−’ signs correspond to $L_−$ respectively. Their solutions however, are formally identical

$$p(y) = \frac{1}{2} - \frac{1}{2} \cos \left( \pi \frac{y^2}{y^2} + C_\pm \right), \quad C_\pm = C_\pm(y^2).$$

Substituting $p(y)$ into Eq. (13) $p(L_{\pm} y) = h(p(x))$, we find the functions $C_\pm$ to be constants, and for each label ‘+’, two further possibilities arise, namely we have

$$p_{+,1}(y) = \frac{1}{2} - \frac{1}{2} \cos \left( \pi \frac{y^2}{y^2} - \alpha \right),$$

$$p_{+,2}(y) = \frac{1}{2} - \frac{1}{2} \cos \left( \pi \frac{y^2}{y^2} + 3\alpha \right),$$

$$p_{-,1}(y) = \frac{1}{2} - \frac{1}{2} \cos \left( \pi \frac{y^2}{y^2} - \frac{3}{2} \alpha \right),$$

$$p_{-,2}(y) = \frac{1}{2} - \frac{1}{2} \cos \left( \pi \frac{y^2}{y^2} + \alpha \right).$$

The corresponding inverses are

$$\sigma_{+,1}(x) = [y^2 \left( \frac{1}{2} \arccos(1-2x) + \alpha \right), y^2],$$

$$\sigma_{+,2}(x) = [y^2 \left( \frac{1}{2} \arccos(1-2x) - 3\alpha \right), y^2],$$

$$\sigma_{-,1}(x) = [y^2 \left( \frac{1}{2} \arccos(1-2x) + \frac{1}{2} \alpha \right), y^2],$$

$$\sigma_{-,2}(x) = [y^2 \left( \frac{1}{2} \arccos(1-2x) - \alpha \right), y^2].$$

The nonlinear IPMs can be computed from

$$f_{\pm}^i = p \circ L_{\pm} \circ \sigma$$

where the powers of $L_\pm$ are\(^3\)

$$L^i_+ = \begin{bmatrix} 2^i & \alpha(2^i - 1) \\ 0 & 1 \end{bmatrix},$$

$$L^i_- = \begin{bmatrix} (-2)^i & \alpha(1 - (-2)^i) \\ 0 & 1 \end{bmatrix}.$$ 

Now, the composition (21) yields

$$f_{+,1}^i(x) = \sin^2 \left( 2^i - 1 \arccos(1-2x) + \pi \alpha(2^i - 1) \right),$$

$$f_{+,2}^i(x) = \sin^2 \left( 2^i - 1 \arccos(1-2x) - \pi \alpha(2^i - 1) \right),$$

$$f_{-,1}^i(x) = \sin^2 \left( (-2)^i - 1 \arccos(1-2x) + \pi \alpha(1 - (-2)^{i}) \right)$$

$$+ \frac{2\pi}{y^2} \alpha(1 - (-2)^i),$$

$$f_{-,2}^i(x) = \sin^2 \left( (-2)^i - 1 \arccos(1-2x) \right).$$

Clearly, $f_{+,1}^i$ and $f_{-,2}^i$ represent the same family of maps, and for the second pair, $f_{-,2}^i$ is a subset of $f_{-,1}^i$. Therefore, there are actually two distinct sets of (multivalued) continuous trajectories $f_{+,1}^i$ and $f_{-,2}^i$, however only in the first one there exists a real-valued branch of $2^i$, and hence $f_{+,1}^i(x) \in \mathbb{R}$, provided $x \in [0, 1]$. All remaining trajectories are complex.

Our purpose now, is to discuss the symmetries constructed from obtained linearizations. Since $L_{+,1} = L_{+,2}$, therefore we use $U = 1$ as a solution to (55). The first pair composed according to Eq. (7), that is

$$p_{+,1} \circ \sigma_{+,2} \cong p_{+,2} \circ \sigma_{+,1}$$

$$\cong \sin^2 \left[ \pi \alpha + \frac{1}{2} \arccos(1-2x) \right],$$

becomes an identity for $\alpha \in \mathbb{Z}$, which is a trivial symmetry. But the second pair

$$p_{-,1} \circ \sigma_{-,2} \cong p_{-,2} \circ \sigma_{-,1}$$

$$\cong \sin^2 \left[ \frac{2}{3} \pi \alpha + \frac{1}{2} \arccos(1-2x) \right]$$

$$=: \phi_\epsilon(x)$$

yield a nontrivial one at $\alpha = 1, 2, 3k, k \in \mathbb{Z}$. After reduction this is

$$\phi_\epsilon(x) = \frac{1}{2} \left( 3 - 2x \pm 2\sqrt{3x(1-x)} \right),$$

namely, the solution $x' = \phi_\epsilon(x)$ to an equation of ellipse inscribed into the square $[0, 1] \times [0, 1]$:

$$S_\epsilon(x, x') := (x + x' - 1)^2 + \frac{1}{2} (x - x')^2 - \frac{1}{4} = 0. \quad (22b)$$

Compositions of $\phi_\epsilon$ with $f_{\pm}^i$ gives uncountable number of symmetries out of this one. Nevertheless, this set is not exhaustive; other symmetries can for instance be constructed according to Eq. (10). In order to illustrate, consider the following linear maps

$$M_+ := \begin{bmatrix} i & (i - 1) \alpha \\ 0 & 1 \end{bmatrix}, \quad M_- := \begin{bmatrix} i & \frac{1}{3}(1 - (i-1)\alpha) \\ 0 & 1 \end{bmatrix},$$

trivially satisfying the condition (9), by $[M_+, L_\pm] = 0$. Restricting to the case of integer parameter $\alpha$, the compositions $p_\pm \circ M_\pm \circ \sigma_\pm := \phi_{\pm,h}$ take the form

$$\phi_{+,h} = - \sinh^2 \left[ \pi \alpha + \frac{1}{2} \arccos(1-2x) \right],$$

$$\phi_{-,h} = - \sinh^2 \left[ \frac{2}{3} \pi \alpha + \frac{1}{2} \arccos(1-2x) \right].$$

We point out that, neither of these maps the interval $[0, 1]$ into itself, as $\phi_\epsilon$ does, therefore they are not proper symmetries of the logistic equation in the sense of definition given earlier. In particular, both the above sets have one common element for $\alpha = 0$. It is rather special one for being real-valued and self-inverse $\phi_{\epsilon}^{-1} = \phi_{\epsilon}$.

$$\phi_{\epsilon} : \mathbb{R} \implies \mathbb{R},$$

\(^3\)In the complex domain both $2^i$ and $(-2)^i$ are countably many-valued: A particular branch $E_{\pm} = \ln L_{\pm}$ is selected by fixing the integer $k$ in $(\pm 2)^k = \exp[i \pi 2 + \pi i (2k + (1 \mp 1)/2)].$
\[
\phi_h : x \mapsto \begin{cases}
\sin^2 \left[ \frac{1}{2} \arccosh(1 - 2x) \right] & x \in (-\infty, 0], \\
-\sinh^2 \left[ \frac{1}{2} \arccos(1 - 2x) \right] & x \in [0, 1], \\
\cosh^2 \left[ \frac{1}{2} \arccos(2x - 1) \right] & x \in [1, \infty)
\end{cases}
\]

This relation \(\phi_h\), together with \(\phi_e\), are illustrated in Fig. 1b).

**Figure 1** Examples of the symmetry curves of general logistic equation \(f(x) = \mu x(1 - x)\), in several cases of its control parameter \(\mu\). The two cases \(\mu = 3.83187\) (close to periodic window) and \(\mu = 3.5625\) (just before onset of chaos), were found numerically through procedure described in the text. The curves \(\phi_e\) on b), c) seem to be fractal, for not being closed and filling densely subsets of \(\mathbb{R}^2\).

We have also investigated symmetries of the logistic equation \(x_{n+1} = \mu x_n(1 - x_n)\) numerically, in other cases of the parameter \(\mu\). The applied procedure directly uses the symmetry definition (11). Namely, assuming \(\phi(x_*) = x_*\), one can develop \(\phi\) into power series in some neighbourhood of the fixed point \(x_*\), as described in Sect. 2. We confine ourselves to the self-inverse symmetries \(\phi^{-1} = \phi\), by fixing the free expansion parameter to \(\delta_x \phi(x_*) = -1\). Starting from \(x_0 := x_* + \delta x\), \(|\delta x| \ll 1\), the expansion can then be used to approximate \(x'_0 := \phi(x_0)\) to an arbitrary precision. By iteration

\[
\{x_n, x'_n\} \mapsto \{x_{n+1}, x'_{n+1}\} = \{f(x_n), f(x'_n)\},
\]

a finite number of points belonging to the symmetry relation is obtained. Many repetitions of this algorithm, with various initial displacements \(\delta x\) yielded sufficient data to draw reliable pictures of \(\phi\) in number of cases \(\mu \in [2, 4]\) — two particular are shown on Fig. 1b), c).

Both \(\phi_e\) and \(\phi_h\) exist for all checked values of the control parameter \(\mu\), except \(\mu = 2\) where \(\phi_e\) degenerates to a single point at \((x_*, x_*) = (\frac{1}{2}, \frac{1}{2})\). At this extreme the second curve \(\phi_h\) can be given closed analytical form, thanks to the well-known linearization

\[
2x(1 - x) = p(2\sigma(x)), \quad p(y) = \frac{1}{2} (1 - e^{2y}), \quad \sigma(x) = \frac{1}{2} \ln(1 - 2x).
\]

Again, taking \(M = -1\) in Eq. (10) gives (cf. Fig. 1d))

\[
\phi_h(x) = p(-\sigma(x)) = \frac{x}{2x - 1},
\]

or, in the equivalent form (cf. Eqs. (22), (22b)), the solution \(x' = \phi_h(x)\) to

\[
S_h(x, x') = xx' - \frac{1}{2}(x + x') = 0.
\]
It should be noted that, in contrast to all other cases of \( \mu > 2 \), here \( \phi_h \) is an ordinary (i.e. single-valued) function.

We also remark, that by construction each of the fixed points \( x_* = 0, 1 - \mu^{-1} \) of the vector field \( f \) is also a fixed point of one of the two symmetries \( \phi_h \) and \( \phi_e \) respectively:

\[
\begin{align*}
0 & \notin \text{img} \phi_e(0), & 1 - \mu^{-1} & \notin \text{img} \phi_e(1 - \mu^{-1}),
0 & \in \text{img} \phi_h(0), & 1 - \mu^{-1} & \in \text{img} \phi_h(1 - \mu^{-1})
\end{align*}
\]

This is not generally the case for the time-translation symmetries \( f^\pm \). It is therefore plausible to conjecture that, apart from special degenerate cases, presence of non-trivial discrete symmetries associated with equilibrium points of the vector field \( f : \mathcal{M} \rightarrow \mathcal{M} \) is a generic feature in difference equations. In this sense, \( \phi_e \) and \( \phi_h \) can be considered typical of logistic map \( f(x) = \mu x(1 - x) \).

## 4 Conclusions

In this paper, we have investigated the relationships among symmetries, similarities, linearizations and inverse Poincaré maps of ordinary difference equations. The general conclusion is, that the existence of additional, continuously parametrized family of similarities, admissible by the system, plays an important role for the analytic solution of ODE and/or its linearization. Existence of such structure can be regarded as an analog of Lie algebra in the case of Lie-type ordinary differential systems. This structure generalizes the notion of a symmetry for it does not necessarily have to be a group, but may instead form a semigroup without unity. The important point is, that if a linear representation of this structure exists, then the linearization problem of original ODE can be reduced to a solution of a finite partial differential system.

Finally, as suggested by the results presented here, the number of non-trivial discrete symmetries appears to be directly related to the cardinality of the fixed subset of phase space \( \mathcal{M} \). Interestingly, these relations need not to be simultaneously symmetries of the inverse Poincaré maps \( f^\dagger \), but instead provide similarity transformations among them.

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