Gravity localization and effective Newtonian potential for Bent thick branes

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Abstract – In this letter, we first investigate the gravity localization and mass spectrum of gravity KK modes on de Sitter and anti-de Sitter thick branes. Then, the effective Newtonian gravitational potentials for these bent branes are discussed by two typical examples. The corrections of the Newtonian potential turn out to be ∆U(r) ∼ 1/r^2 at small r for both cases. These corrections are very different from that of the Randall-Sundrum brane model ∆U(r) ∼ 1/r^3.

Introduction. – The brane world models can supply new insights for solving gauge hierarchy and the cosmological constant problem [1–5]. The key idea of brane world is that gravity is free to propagate in all dimensions, while all matter fields are confined to a 3-brane with no contradiction with present gravitational experiments. In 1999, the brane world models with warped extra dimension were proposed by Randall and Sundrum (RS) [2]. In the famous RS brane-world model [2], the thickness of the brane is neglected. Subsequently, DeWolfe, Freedman, Gubser and Karch [6] proposed a more natural brane world model, the so-called thick-brane scenario, in which the brane is generated by a background scalar field coupled gravity. Since then, more and more authors have investigated the thick-brane scenario in higher-dimensional space-time [7–23]. In these models, the scalar fields do not play the role of bulk fields but provide the “material” of which the thick branes are made. For some comprehensive reviews about thick branes, please see ref. [24].

In brane world scenarios, there is an interesting and important issue, i.e., the localization of gravity. Generally, the gravity zero mode should be localized on the brane for the purpose that the effective four-dimensional gravity should be recovered. Moreover, from the phenomenological point of view, it is very important to seek configurations which allow for the existence of a mass gap in the spectrum of gravitational Kaluza-Klein (KK) excitations. The mass gap can decide the energy scale beyond which massive modes can be excited. From refs. [10,18] we know that the existence of a mass gap would in general make the localization of the massless graviton sure. Especially, the mass gap can result in different corrections of the Newton potential [22].

In this letter, we will investigate the localization and mass spectrum of gravity as well as the effective Newtonian potentials on de Sitter (dS) and anti-de Sitter (AdS) branes by some typical examples. For the case of dS branes, the potential of the corresponding Schrödinger equation for the gravity KK modes is a modified Fölsch-Teller potential, and we find that the correction term for the Newtonian potential at small r is ∼ 1/r^2, which is different from the correction caused by a volcano-like localized potential. For the case of AdS branes, the localized potential of the KK modes is an infinite deep potential, and all the KK modes are bound states. It is shown that the correction term of the gravitational potential at small distance for this case is also proportional to 1/r^2.

The organization of this letter is as follows. In next section, we review the localization of gravity. Then, we investigate the localization and mass spectrum of gravity on dS and AdS thick branes in the third and fourth section, respectively. The effective gravitational potentials on bent branes are studied in the fifth section. Finally, the conclusion is given in the last section.

The model. – We start with the following 5D action of bent thick branes, which are generated by a real scalar
field $\phi$ with a scalar potential $V(\phi)$,

$$S = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa_5^2} R - \frac{1}{2} \partial M^\phi \partial N^\phi - V(\phi) \right],$$

(1)

where $R$ is the 5D scalar curvature. The line element is assumed as

$$ds^2 = g_{MN} dx^M dx^N = e^{2A(z)} [d\tilde{\mu}_\nu(x) dx^\mu dx^\nu + dz^2],$$

(2)

where $e^{2A(z)}$ is the warp factor and $z$ stands for the extra coordinate. We suppose that $\tilde{g}_{\mu\nu}$ is some general 4D metric, and $\phi = \phi(z)$.

The field equations corresponding to the action (1) are read as

$$\kappa_5^2 \phi'^2 = 3A'^2 - 3A'' - A_4,$$

(3)

$$2\kappa_5^2 V(\phi) = 3 e^{-2A}(A_4 - 3A'^2 - A''),$$

(4)

$$\frac{dV(\phi)}{d\phi} = e^{-2A}(3A'\phi' + \phi'').$$

(5)

Here we set the 4D metric satisfies $\tilde{R}_{\mu\nu} = \Lambda_{\Phi} \tilde{g}_{\mu\nu}$ with $\Lambda_{\Phi}$ the 4D cosmology constant, i.e., the 4D space-time is a maximally symmetric 4D space-time. In the case of AdS$_4$ space, $\Lambda_{\Phi}$ is negative, while for dS$_4$ space the converse holds true. In the following sections, we will consider some explicit solutions of these equations.

Let us further consider the metric fluctuations $\delta g_{MN} = H_{MN} = e^{2A(z)} h_{MN}$ of (2). Under the axial gauge $H_{5M} = 0$, the total metric can be written as

$$ds^2 = e^{2A(z)} \left[[\tilde{g}_{\mu\nu}(x) + h_{\mu\nu}(x,z)] dx^\mu dx^\nu + dz^2 \right].$$

(6)

After imposing the transverse-traceless (TT) gauge condition $h_{\mu\nu}^\alpha = \nabla^\alpha h_{\mu\nu} = 0$, where $\nabla^\alpha$ denotes the covariant derivative with respect to $\tilde{g}_{\mu\nu}$, the equation for the perturbation $h_{\mu\nu}$ takes following form [20]:

$$\left( \partial_5^2 + 3(\partial A) \partial_5 + \tilde{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta - \frac{2}{3} A_4 \right) h_{\mu\nu}(x,z) = 0.$$

(7)

By making use of the KK decomposition $h_{\mu\nu}(x,z) = e^{-3A/2} \epsilon_{\mu\nu}(x) \Psi(z)$ with $\epsilon_{\mu\nu}(x)$ satisfying the TT condition, from eq. (7) we can get the following 4D equation:

$$\left( \tilde{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta - \frac{2}{3} A_4 \right) \epsilon_{\mu\nu}(x) = m^2 \epsilon_{\mu\nu}(x),$$

(8)

and the Schrödinger-like equation for the 5th dimensional sector

$$[-\partial_5^2 + V_{QM}(z)] \Psi(z) = m^2 \Psi(z),$$

(9)

where $m$ is the mass of the KK modes. The localizing potential is read as

$$V_{QM} = \frac{3}{2} \left[ \partial_5^2 A + \frac{3}{2}(\partial_5 A)^2 \right].$$

(10)

The zero mode is $\Psi_0(z) \propto e^{2A(z)}$ [20]. In order to localize the 4D graviton, $\Psi_0(z)$ should obey the normalization condition $\int dz |\Psi_0|^2 < \infty$. Generally, the character of the localization of massive KK modes depends on the specific expression of the potential $V_{QM}$ of the Schrödinger equation. In the following sections, we will investigate the mass spectrum and the corrections of the Newtonian potential on bent thick branes.

Localizations and mass spectrum of gravity on dS thick branes. – The general formulation of the metric for a dS thick brane embedded in AdS$_5$-like space-time is

$$ds^2 = e^{2A(z)} [-dt^2 + e^{2Ht}(dx_1^2 + dx_2^2 + dx_3^2 + dx_5^2)].$$

(11)

Here $H$ is the dS parameter and the 4D cosmological constant is $\Lambda_4 = 3H^2$.

**Example 1.** Firstly, we consider the following solution [10]:

$$A(z) = -\delta \ln[cosh z],$$

(12)

$$\phi(z) = \phi_0 \arcsin[tanh z],$$

(13)

$$V(\phi) = 3 \left[ 1 + 3\delta \right] H^2 \cos^{2(1-\delta)} \left( \frac{\phi}{\phi_0} \right).$$

(14)

where $z = H t / \delta$, $\delta$ is a constant which satisfies $0 < \delta < 1$, and $\phi_0 \equiv [3\delta (1 - \delta)]^{1/2}$. In this solution, the constant $\kappa_5^2$ is set as $\kappa_5^2 = 1$. Such a solution describes a dS thick brane located at $z = 0$, and the range of the fifth dimension is $-\infty < z < +\infty$.

From eq. (11), the potential of the Schrödinger-like equation can be expressed as

$$V_{QM}(z) = \frac{3H^2}{4\delta} \left[ 3\delta - (2 + 3\delta) sech^2 z \right].$$

(15)

Now we investigate the localization and mass spectrum of gravity KK modes. From eq. (14), we know that the potential has a minimum (negative value) $-3H^2/(2\delta)$ at $z = 0$ and a maximum (positive value) $9H^2/4$ at $z \to \pm \infty$. If we set $p = H/\delta$ and $q = 1 + 3\delta/2$, the Schrödinger equation (8) with the potential (14) becomes

$$[-\partial_5^2 - q(q-1)p^2 sech^2(pz)] \Psi_n = E_n \Psi_n,$$

(16)

where $E_n = m_n^2 - 9q^2/2$. The mass spectrum of the bound states is found to be [13,19]

$$m_n^2 = n(3\delta - n) H^2 / \delta^2,$$

(17)

where $n$ is an integer and satisfies $0 \leq n < 3\delta/2$. Note that the energy for $n = 0$ or $m_0 = 0$ always belongs to the spectrum of the potential (14) for $\delta > 0$. So it can be seen that, for $0 < \delta < 2$, there is only one bound state (ground state)

$$\Psi_0(z) \propto sech^{3\delta/2}(z)$$

(18)

with $m_0^2 = 0$, which is just the massless mode and stands for the localized 4D graviton on the dS brane. It is clear that there is no tachyonic graviton mode (with $m^2 < 0$).
The continuous spectrum starts at $m^2 = 9H^2/4$ and the corresponding KK modes asymptotically turn into plane waves, which represent delocalized KK massive gravitons. When $2/3 < \delta < 1$, there are two bound states, one is the ground state (17), the other is the first-excited state

$$\Psi_1(z) \propto \text{sech}^{3\delta/2}(z) \sinh \bar{z}$$

with mass $m_1^2 = (3\delta - 1)H^2/\delta$. The continuous spectrum also starts at $m^2 = 9H^2/4$.

The continuous massive states $\Psi_m(z)$ are expressed by a linear combination of the associated Legendre functions, and the normalized form is read as

$$\Psi_m(z) = \frac{1}{2}[\Gamma(1 + iM)P(3\delta/2, -iM, \tanh \bar{z}) + \Gamma(1 - iM)P(3\delta/2, iM, \tanh \bar{z})].$$

These KK modes would have contribution to the effective Newtonian potential, which will be considered later.

**Example 2.** In this subsection, we turn to another example. The solution is given by [21]

$$A(z) = -\frac{1}{2} \ln|s^2(1 + H^2) \cosh^2(kz)|,$$

$$\phi(z) = kz/b,$$

$$V(\phi) = \frac{9}{4}a^2(H^2 + 1)[H^2(s - 1) - 1] \cosh^2(b\phi) + 3a^2(H^2 + 1)^2,$$

where $k^2 = \frac{1+H^2}{s}$, $b = \pm \sqrt{\frac{2(1+H^2)}{3[1+(1-s)H^2]}}$, $s$ and $a$ are real parameters. In this example, the constant $\kappa^2_0$ is set as $\kappa^2_0 = 2$.

The potential of the KK modes (9) in the corresponding Schrödinger equation turns out to be

$$V_{QM} = (3k^2/4)[3 - 5 \text{sech}^2(kz)],$$

which is similar to (14). This potential supports two bound states. The first one is the ground state with $m^2_0 = 0$:

$$\Psi_0(z) = \sqrt{2k/\pi} \text{sech} \bar{z}(kz).$$

The second one is an excited state with $m^2_1 = 2k^2$:

$$\Psi_1(z) \propto \text{sech} \bar{z}(kz) \sinh(kz).$$

The continuous spectrum starts at $m^2 = 9k^2/4$. The continuous KK modes are similar to those in the first example and we do not give them here.

Note that, in the first solution, we cannot get a flat-brane solution by taking the limit $H \to 0$. While, the flat-brane scenario can be recovered from the second dS brane solution in the same limit. For both dS brane solutions, there exists a mass gap and the 4D graviton can be localized on the dS branes.

**Localization and mass spectrum of gravity on AdS thick branes.** For the case of AdS thick branes, the line element has the following form:

$$ds^2 = e^{2A(z)[e^{2Hx^2}(-dt^2 + dx_1^2 + dx_2^2) + dx_3^2 + dz^2]}.$$  \hspace{1cm} (26)

Here the 4D cosmological constant is $\Lambda_4 = -3H^2$. Next, we will use some examples to investigate the localization and mass spectrum of gravity on AdS branes.

**Example 1.** As the first example of AdS branes, we consider the following solution [10]:

$$A(z) = -\delta \ln \cos \bar{z},$$

$$\phi(z) = \phi_0 \arcsinh[\tan \bar{z}].$$

$$V(\phi) = \frac{3(1 + 3\delta)H^2}{2\delta} \cosh^{2(1-\delta)} \left( \frac{\phi}{\phi_0} \right),$$

where $\phi_0 \equiv \sqrt{3\delta(\delta - 1)}$ and the constant $\delta$ satisfies $\delta > 1$ or $\delta < 0$. In this example, the constant $\kappa^2_0$ is set as $\kappa^2_0 = 1$.

Such a solution describes an AdS brane located at $z = 0$, and the range of the fifth dimension is $-z_m < z < +z_m$ with $z_m = \frac{\delta}{2\delta}$.

The potential of the KK modes reads as

$$V_{QM} = \frac{3H^2}{4\delta} \left[ 2 + 3\delta \right] \sec^2 \bar{z} - 3\delta.$$

It is clear that $V_{QM}(z = 0) = 3H^2/(28)$. At the boundaries of the brane, the potential $V_{QM}$ tends to positive infinite and negative infinite for $\delta < -\frac{2}{3}$ or $\delta > 1$ and $-\frac{2}{3} < \delta < 0$, respectively. So a mass gap always exists in the spectrum of gravitational excitations of this system.

The case of $\delta > 1$ is not interesting because the zero mode does not exist. So we consider the case of $\delta < 0$, for which the solutions of the KK modes are found to be

$$\Psi_n(z) = c_n F_1 \left[ -n, n - 3\delta, (1 - 3\delta)/2, (1 - \sin \bar{z})/2 \right] \times \cos \frac{2\delta}{3}\bar{z} \quad (n = 0, 1, 2, \cdots).$$

All the KK modes are bound states and can be localized on the thick AdS brane. The mass spectrum of the KK modes is discrete:

$$m_n = |H/\delta| \sqrt{n + 3|\delta|}. $$

**Example 2.** Another solution of AdS branes can be obtained by making replacement $H^2 \to -H^2$ from (20)–(22) [21]. This solution describes an AdS thick brane located at $z = 0$, and the range of the fifth dimension is $-\infty < z < +\infty$. The gravitation localizing potential is the same as (23), so we do not need to repeat the analysis again.

**The effective Newtonian potentials.** In the brane world scenario, we need to obtain the 4D effective action from the 5D action (1), i.e.,

$$S_5 \supset M_5^2 \int d^5x \sqrt{-g}R_5 \supset M_5^2 \int d^4x \sqrt{-g}R_4.$$  \hspace{1cm} (33)
where $M_* = (2\kappa_5^2)^{-\frac{1}{2}}$ is the fundamental 5D Planck scale, and $M_{Pl} = (M_*^4 \int e^{3H(z)} dz)^{1/2}$ is the 4D Planck scale which determines the 4D gravitational coupling $G_N \sim M_{Pl}^2$. In other words, we can get the 4D effective theory of gravity. Generally, the localized zero mode will cause a 4D Newtonian interaction potential, and we also require that the other KK modes do not lead to unacceptable large corrections to the Newtonian gravitational potential in 4D effective theory [2,4,8,9,25]. In this section, we will study the effective Newtonian potentials on dS and AdS branes.

In the realistic braneworld scenario, the matter fields in the 4D theory on the brane would be smeared over the width of the brane in the transverse space, which is too complex to deal with. For simplicity, we just consider the gravitational potential between two point-like sources of mass $M_1$ and $M_2$ located at the location of the brane $z = 0$ [2,4,8,25]. This assumption is justified when the thickness of the brane is small compared with the bulk curvature. Now, we consider two examples for dS and AdS branes. For the brane is small compared with the bulk curvature. Now, we consider two examples for dS and AdS branes.

**dS branes.** Firstly, we study the first example for the dS brane given above. From eq. (16), we know that, there are one and two bound states for $0 < \delta < 2/3$ and $2/3 < \delta < 1$, respectively. Thus, we will discuss the two cases of $\delta$.

**Case I:** $0 < \delta < \frac{2}{3}$. In this case, the zero mode has been written as (17), and the continuum KK modes start at $m = 3H/2$. The effective potential between two point-like sources of mass $M_1$ and $M_2$ is from the contributions of the zero mode and the continuum KK modes, and can be expressed as [8]

$$U(r) = G_N \frac{M_1 M_2}{r} + \frac{M_1 M_2}{M_*^2} \int_{m_*}^\infty \frac{e^{-mr}}{r} |\Psi_m(0)|^2, \quad (34)$$

where $G_N = 1/(16\pi M_{Pl}^2)$ is the Newton’s gravitational constant. The standard Newtonian potential is from the contribution of the zero mode, and the second term is the correction of the Newtonian potential, which is from the contribution of the continuum KK modes (19). For $m = m_* = 3H/2$, we have $\Psi_{m_*}(0) = |P(3\delta/2, 0)|^2$. The curve of $|\Psi_m(0)|^2$ as a function of $m$ is plotted in fig. 1. It is shown that $|\Psi_m(0)|^2 \ll |\Psi_m(0)|^2$ and $|\Psi_m(0)|^2$ approaches to the constant 1 with the increase of the mass $m$, this is because the wave functions $\Psi_m(z)$ approach to plane waves as the eigenvalue $m$ becomes large.

In order to get the corrected Newtonian potential, we write the integrand as $I(m) = e^{-mr}/r |\Psi_m(0)|^2$. Rather than dealing with this complex integrand, we will consider other simple integrands, from which the integral can be achieved.

Firstly, we use the minimum $\Psi_{m_*}(0)$ to replace the function $\Psi_m(0)$, thus the integrand can be simply given by $I_1(m) = (e^{-mr}/r)|P(3\delta/2, 0)|^2$. The range of the parameter $\delta$ is $0 < \delta < 2/3$, the range of $P(3\delta/2, 0)$ is $0 < \Psi_{m_*}(0) = |P(3\delta/2, 0)|^2 < 1$. In this case, the correction of the Newtonian potential would be

$$\Delta U_1 = \left| P \left( \frac{3\delta}{2}, 0 \right) \right|^2 e^{-\frac{3\delta r}{2}} \frac{M_1 M_2}{M_*^2} \left( \frac{r}{r} \right)^2 = \eta_1 e^{-\frac{3\delta r}{2}} \frac{M_1 M_2}{M_*^2} \left( \frac{r}{r} \right)^2, \quad (35)$$

where $\eta_1 = |P(3\delta/2, 0)|^2$.

Secondly, we use the constant 1 to replace $|\Psi_m(0)|^2$, which refers to the limit $|\Psi_m(0)|^2 = 1$, and the integrand can be simply given by $I_2(m) = e^{-mr}/r$. Then the correction of the Newtonian potential would be

$$\Delta U_2 = e^{-\frac{3\delta r}{2}} \frac{M_1 M_2}{M_*^2}. \quad (36)$$

From fig. 1, we can see that $\eta_1 \leq |\Psi_m(0)|^2 < 1$, so the integrands satisfy the relation $I_1 \leq I_2 < I_m$. Thus, the correction of the Newtonian potential $\Delta U(r)$ must satisfy $\Delta U_1(r) \leq \Delta U(r) < \Delta U_2(r)$ and the effective Newtonian potential can be written as

$$U(r) = \frac{G_N}{r} \left( \frac{M_1 M_2}{r} + \frac{\eta_1 e^{-\frac{3\delta r}{2}} \frac{M_1 M_2}{M_*^2}}{r^2} \right) \quad (37)$$

with the constant $\eta$ satisfies $|P(3\delta/2, 0)|^2 < \eta_1 < 1$.

**Case II:** $\frac{2}{3} \leq \delta < 1$. In this case, the ground state $\Psi_0(z)$ (the zero mode) and the first-excited state $\Psi_1(z)$ are given by (17) and (18), respectively. The continuum KK modes also start at $m_* = 3H/2$. The effective potential between two point-like sources of mass $M_1$ and $M_2$ can be expressed as follows [8]:

$$U(r) = G_N \frac{M_1 M_2}{r} + \frac{M_1 M_2}{M_*^2} \int_{m_*}^\infty \frac{e^{-mr}}{r} |\Psi_m(0)|^2, \quad (38)$$

![Fig. 1: The shape of $|\Psi_m(0)|^2$ (black line) as a function of $m$. The black dashed line and the gray line represent $|\Psi(\frac{3\delta}{2}, 0)|^2$ and the constant 1, respectively. The parameters are set as $H = 1$, $\delta = 0.5$ (left) and $\delta = 0.9$ (right).](image-url)
where \( m_1 = \sqrt{(3\delta - 1)H^2/\delta} \). Because \( \Psi_1(z) \) is an odd function of \( z \), \( \Psi_1(0) = 0 \) and eq. (38) can be reduced to eq. (34). The continuous massive states \( \Psi_m(z) \) are also given by (19). The curve of \( |\Psi_m(0)|^2 \) as a function of \( m \) is plotted in fig. 1. We can see that \( 0 \leq |\Psi_m(0)|^2 < 1 \).

Following the same procedure for the first case, we can obtain the effective Newtonian potential,

\[
U(r) = G_N \frac{M_1 M_2}{r} + \eta_2 \frac{e^{-2\delta r}}{M_1^2} \frac{M_1 M_2}{r^2},
\]

(39)

where \( \eta_2 \) is a constant satisfying \( 0 < \eta_2 < 1 \).

From above discussion, it can be seen that, for the large distance \( r \) of two point-like sources, the correction of gravitational potential \( e^{-2\delta r}/r^2 \) is a high-order term compared with the term of the Newtonian potential \( 1 \) and hence can be neglected. However, when \( r \) is small, the correction \( e^{-2\delta r}/r^2 \sim 1 \) will be large and will be the main term of the effective Newtonian potential. The conclusions for the second examples for dS and AdS branes are similar to this one.

**AdS branes.** In this subsection, we investigate the first example for AdS branes with \( \delta < 0 \), for which all the KK modes including the zero mode are bound states. The localized zero mode will cause a 4D Newtonian potential, and all other KK modes will produce the correction to the potential. The effective potential of two point-like sources is read as [8]

\[
U(r) = G_N \frac{M_1 M_2}{r} + \sum_{n=1} \frac{M_1 M_2}{M_n^2} \frac{e^{-m_n r}}{r} |\Psi_n(0)|^2.
\]

(40)

Because all the odd-parity wave functions vanish at \( z = 0 \), we just only consider all even-parity wave functions. The even-parity wave functions are given by (31) but with \( n = 2i \), from which \( \Psi_n(0) = \Psi_{2i}(0) \) read as

\[
\Psi_{2i}(0) = c_{2i} \frac{\sqrt{\pi} \Gamma(\frac{1}{2} - \frac{33}{2})}{\Gamma(\frac{1}{2} - i) \Gamma(\frac{1}{2} + i + \frac{33}{2})}.
\]

(41)

We introduce a new function \( I(r) \) as follows:

\[
I(r) = \sum_{i=1} \frac{e^{-m_{2i} r}}{r} |\Psi_{2i}(0)|^2,
\]

(42)

where \( m_{2i} = |\frac{4}{\pi} \sqrt{2(2i + 3|\delta|)}| \). Then the correction of the Newtonian potential can be written as \( \Delta U(r) = (M_1 M_2/M_{2i}^2)I(r) \). Firstly, we need calculate the normalization constants \( c_{2i} \). The integrals are very complex, and we cannot give them. Although the analytic expressions for \( c_{2i} \) cannot be obtained, we can fix the parameters \( \delta = -1 \) and \( H = 1 \) and give the numerical results: \( |\Psi_{2i}(0)|^2 = 0.65625, 0.644531, 0.640869, 0.639267, 0.638426, 0.63793, 0.637614, 0.637399, 0.637248, 0.637136, 0.637052, 0.636987, 0.636935, 0.636894, 0.63686, \ldots \). We find that \( |\Psi_{2i}(0)|^2 \) are close to a constant for different values of the parameter \( i \). In the following calculation, we use the constant \( |\Psi_{2i}(0)|^2 \) to replace with all of \( |\Psi_{2i}(0)|^2 \).

Next, we need compare the approximate expression (43) with the exact result (42). We introduce a new function \( I_N(r) = \sum_{i=1} |\Psi_{2i}(0)|^2/M_{2i}^2 \) and the shapes of \( I_{\text{app}}(r) \) and the shapes of \( I_N(r) \) for different values of \( N \). It can be seen that with the increase of \( N \), \( I_N(r) \) would tend to the approximate expression \( I_{\text{app}}(r) \). Thus, our expression (43) is a good approximation. Then the effective Newtonian potential can be expressed as

\[
U(r) = G_N \frac{M_1 M_2}{r} + \frac{M_1 M_2}{M_{2i}^2} \frac{|\Psi_{2i}(0)|^2}{(e^2 |H|^2/\delta - 1)r}.
\]

(44)

So the effective potential \( U(r) \) can be reduced to the Newtonian potential for large distance \( r \). However, when the distance \( r \) is small, then the correction term for the gravitational potential \( \Delta U(r) \sim (e^2 |H|^2/\delta - 1)^{-1/2} \) will become the leading term.

From these two examples, we find that, when \( r \) is large enough, the correction of the gravitational potential can be ignored comparing with the Newtonian potential; however, when \( r \) is small, the correction term will be the main term of the potential. The corrections for the Newtonian potential for two different types of localization potentials are similar at short distance: \( \Delta U(r) \sim 1/r^2 \).

**Conclusions and discussions.** In this letter, the localization and mass spectrum of gravity, as well as the effective Newtonian potentials for dS and AdS thick branes have been investigated by two examples, respectively. We know that the dS/AdS space-time would become the flat
one if the ds/AdS parameter $H = 0$. Here, for the first examples of ds and AdS branes, the flat-brane solutions cannot be got in the limit of $H \to 0$. While, the ds and AdS branes would turn to flat ones for the second examples in the same limit. For all those bent-brane solutions, there exist mass gaps and the 4D graviton can be localized on the branes (for the first example of AdS brane, we need $\delta < 0$). The massive KK modes were obtained and the effective gravitational potentials on the ds and AdS branes were calculated. It was found that, for two different types of the potentials of the gravity KK modes (a modified Pöschl-Teller potential and an infinite deep potential), the corrections of the Newtonian potentials at short distance for two point-like sources are the same: $\Delta U(r) \approx \frac{\rho_1}{r}$, which is different from the correction in the RS brane model. The corrections can be neglected when the distance between the two sources is large.

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