On algebraic values of function
\[ \exp (2\pi i \, x + \log \log y) \]

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Abstract

It is proved that for all but a finite set of square-free integers \( d \) the value of transcendental function \( \exp (2\pi i \, x + \log \log y) \) is an algebraic number for the algebraic arguments \( x \) and \( y \) in the real quadratic field of discriminant \( d \). Such a value generates the Hilbert class field of imaginary quadratic field of discriminant \( -d \).

Key words and phrases: real multiplication; Sklyanin algebra

MSC: 11J81 (transcendence theory); 46L85 (noncommutative topology)

1 Introduction

It is an old problem to determine if given irrational value of a transcendental function is algebraic or transcendental for certain algebraic arguments; the algebraic values are particularly remarkable and worthy of thorough investigation, see [Hilbert 1902] [1], p. 456. Only few general results are known, see e.g. [Baker 1975] [1]. We shall mention the famous Gelfond-Schneider Theorem saying that \( e^{\beta \log \alpha} \) is a transcendental number, whenever \( \alpha \notin \{0, 1\} \) is an algebraic and \( \beta \) an irrational algebraic number. In contrast, Klein’s invariant \( j(\tau) \) is known to take algebraic values whenever \( \tau \in \mathbb{H} := \{x+iy \in \mathbb{C} \mid y > 0\} \) is an imaginary quadratic number.

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The aim of our note is a result on the algebraic values of transcendental function
\[ J(x, y) := \{ e^{2\pi i x} + \log \log y \mid -\infty < x < \infty, \ 1 < y < \infty \} \]  
(1)
for the arguments \( x \) and \( y \) in a real quadratic field; the function \( J(x, y) \) can be viewed as an analog of Klein’s invariant \( j(\tau) \), hence the notation. Namely, let \( \mathfrak{t} = \mathbb{Q}(\sqrt{d}) \) be a real quadratic field and \( \mathfrak{A}_i = \mathbb{Z} + j\mathfrak{t} \) be an order of conductor \( j \geq 1 \) in the field \( \mathfrak{t} \); let \( h = |\text{Cl} (\mathfrak{A}_i)| \) be the class number of \( \mathfrak{A}_i \) and denote by \( \{ \mathbb{Z} + j\mathfrak{t} \mid 1 \leq 1 \leq h \} \) the set of pairwise non-isomorphic pseudo-lattices in \( \mathfrak{t} \) having the same endomorphism ring \( \mathfrak{A}_i \), see [Manin 2004] [5], Lemma 1.1.1.

Finally, let \( \varepsilon \) be the fundamental unit of \( \mathfrak{A}_i \) and let \( f \geq 1 \) be the least integer satisfying equation \( |\text{Cl} (R_f)| = |\text{Cl} (\mathfrak{A}_i)| \), where \( R_f = \mathbb{Z} + f\mathfrak{O}_k \) is an order of conductor \( f \) in the imaginary quadratic field \( k = \mathbb{Q}(\sqrt{-d}) \). Our main result can be formulated as follows.

**Theorem 1** For each square-free positive integer \( d \notin \{1, 2, 3, 7, 11, 19, 43, 67, 163\} \) the values \( \{ J(\theta_i, \varepsilon) \mid 1 \leq i \leq h \} \) of transcendental function \( J(x, y) \) are algebraically conjugate numbers generating the Hilbert class field \( H(k) \) of the imaginary quadratic field \( k = \mathbb{Q}(\sqrt{-d}) \) modulo conductor \( f \).

**Remark 1** Since \( H(k) \cong \mathbb{Q}(j(\tau)) \cong \mathbb{Q}(f\sqrt{-d}, j(\tau)) \) with \( \tau \in R_f \), one gets an inclusion \( J(\theta_i, \varepsilon)) \in \mathbb{Q}(f\sqrt{-d}, j(\tau)) \).

**Remark 2** Note that even though the absolute value \( |z| = \sqrt{z\bar{z}} \) of an algebraic \( z \) is an algebraic number, the absolute value of \( J(\theta_i, \varepsilon) \) is transcendental. It happens because \( |z| \) belongs to a quadratic extension of the real field \( \mathbb{Q}(z\bar{z}) \) which may have no real embeddings at all. (Compare with the CM-field, i.e. a totally imaginary quadratic extension of the totally real number field.)

The structure of article is as follows. Some preliminary facts can be found in Section 2. Theorem 1 is proved in Section 3 and Section 4 contains an example illustrating the theorem.

## 2 Preliminaries

The reader can find basics of the \( C^* \)-algebras in [Murphy 1990] [6] and their \( K \)-theory in [Blackadar 1986] [2]. The noncommutative tori are covered in
2.1 Noncommutative tori

By a noncommutative torus $\mathcal{A}_\theta$ one understands the universal $C^*$-algebra generated by the unitary operators $u$ and $v$ acting on a Hilbert space $\mathcal{H}$ and satisfying the commutation relation $vu = e^{2\pi i \theta} uv$, where $\theta$ is a real number.

Remark 3  Note that $\mathcal{A}_\theta$ is isomorphic to a free $\mathbb{C}$-algebra on four generators $u, u^*, v, v^*$ and six quadratic relations:

$$
\begin{align*}
vu &= e^{2\pi i \theta} uv, \\
v^*u^* &= e^{2\pi i \theta} u^*v^*, \\
v^*u &= e^{-2\pi i \theta} uv^*, \\
v^*v &= e^{-2\pi i \theta} u^*v, \\
u^*u &= uu^* = e, \\
v^*v &= vv^* = e.
\end{align*}
$$

Indeed, the first and the last two relations in system (2) are obvious from the definition of $\mathcal{A}_\theta$. By way of example, let us demonstrate that relations $vu = e^{2\pi i \theta} uv$ and $u^*u = uu^* = v^*v = vv^* = e$ imply the relation $v^*u = e^{-2\pi i \theta} uv^*$ in system (2). Indeed, it follows from $uu^* = e$ and $vv^* = e$ that $uu^*vv^* = e$. Since $uu^* = u^*u$ we can bring the last equation to the form $u^*uvu^*$ = e and multiply the both sides by the constant $e^{2\pi i \theta}$; thus one gets the equation $u^* (e^{2\pi i \theta} u) v = e^{2\pi i \theta}$. But $e^{2\pi i \theta} uv = vu$ and our main equation takes the form $u^*vvu^* = e^{2\pi i \theta}$. We can multiply on the left the both sides of the equation by the element $u$ and thus get the equation $uu^*vvu^* = e^{2\pi i \theta} u$; since $uu^* = e$ one arrives at the equation $vu^*v = e^{2\pi i \theta} u$. Again one can multiply on the left the both sides by the element $v^*$ and thus get the equation $v^*vu^*v = e^{2\pi i \theta} v^*u$; since $v^*v = e$ one gets $v^*v = e^{2\pi i \theta} v^*u$ and the required identity $v^*u = e^{-2\pi i \theta} uv^*$. The remaining two relations in (2) are proved likewise; we leave it to the reader as an exercise in non-commutative algebra.

Recall that the algebra $\mathcal{A}_\theta$ is said to be stably isomorphic (Morita equivalent) to $\mathcal{A}_{\theta'}$, whenever $\mathcal{A}_\theta \otimes \mathcal{K} \cong \mathcal{A}_{\theta'} \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of all compact operators on $\mathcal{H}$; the $\mathcal{A}_\theta$ is stably isomorphic to $\mathcal{A}_{\theta'}$ if and only if

$$
\theta' = \frac{a\theta + b}{c\theta + d} \quad \text{for some matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).
$$

(3)
The $K$-theory of $A_{\theta}$ is two-periodic and $K_0(A_{\theta}) \cong K_1(A_{\theta}) \cong \mathbb{Z}^2$ so that the Grothendieck semigroup $K^+_0(A_{\theta})$ corresponds to positive reals of the set $\mathbb{Z} + \mathbb{Z}\theta \subset \mathbb{R}$ called a pseudo-lattice. The torus $A_{\theta}$ is said to have real multiplication, if $\theta$ is a quadratic irrationality, i.e. irrational root of a quadratic polynomial with integer coefficients. The real multiplication says that the endomorphism ring of pseudo-lattice $\mathbb{Z} + \mathbb{Z}\theta$ exceeds the ring $\mathbb{Z}$ corresponding to multiplication by $m$ endomorphisms; similar to complex multiplication, it means that the endomorphism ring is isomorphic to an order $R_f = \mathbb{Z} + \mathfrak{O}_k$ of conductor $f \geq 1$ in the real quadratic field $k = \mathbb{Q}(\theta)$ – hence the name, see [Manin 2004] [5]. If $d > 0$ is the discriminant of $k$, then by $A^{(d,f)}_{RM}$ we denote a noncommutative torus with real multiplication by the order $\mathfrak{R}_f$.

2.2 Elliptic curves

For the sake of clarity, let us recall some well-known facts. An elliptic curve is the subset of the complex projective plane of the form $E(\mathbb{C}) = \{(x, y, z) \in \mathbb{C}P^2 \mid y^2z = 4x^3 + axz^2 + bz^3\}$, where $a$ and $b$ are some constant complex numbers. Recall that one can embed $E(\mathbb{C})$ into the complex projective space $\mathbb{C}P^3$ as the set of points of intersection of two quadric surfaces given by the system of homogeneous equations

$$\begin{cases} u^2 + v^2 + w^2 + z^2 &= 0, \\ Av^2 + Bw^2 + z^2 &= 0, \end{cases}$$

(4)

where $A$ and $B$ are some constant complex numbers and $(u, v, w, z) \in \mathbb{C}P^3$; the system (4) is called the Jacobi form of elliptic curve $E(\mathbb{C})$. Denote by $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$ the Lobachevsky half-plane; whenever $\tau \in \mathbb{H}$, one gets a complex torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Each complex torus is isomorphic to a non-singular elliptic curve; the isomorphism is realized by the Weierstrass $\wp$ function and we shall write $E_{\tau}$ to denote the corresponding elliptic curve. Two elliptic curves $E_\tau$ and $E_{\tau'}$ are isomorphic if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d} \text{ for some matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

(5)

If $\tau$ is an imaginary quadratic number, elliptic curve $E_\tau$ is said to have complex multiplication; in this case lattice $\mathbb{Z} + \mathbb{Z}\tau$ admits non-trivial endomorphisms realized as multiplication of points of the lattice by the imaginary quadratic numbers, hence the name. We shall write $E^{(-d,f)}_{CM}$ to denote elliptic curve with
complex multiplication by an order $R_f = \mathbb{Z} + fO_k$ of conductor $f \geq 1$ in the imaginary quadratic field $k = \mathbb{Q}(\sqrt{-d})$.

### 2.3 Sklyanin algebras

By the **Sklyanin algebra** $S_{\alpha,\beta,\gamma}(\mathbb{C})$ one understands a free $\mathbb{C}$-algebra on four generators and six relations:

$$
\begin{align*}
    x_1 x_2 - x_2 x_1 &= \alpha (x_3 x_4 + x_4 x_3), \\
    x_1 x_2 + x_2 x_1 &= x_3 x_4 - x_4 x_3, \\
    x_1 x_3 - x_3 x_1 &= \beta (x_4 x_2 + x_2 x_4), \\
    x_1 x_3 + x_3 x_1 &= x_4 x_2 - x_2 x_4, \\
    x_1 x_4 - x_4 x_1 &= \gamma (x_2 x_3 + x_3 x_2), \\
    x_1 x_4 + x_4 x_1 &= x_2 x_3 - x_3 x_2,
\end{align*}
$$

where $\alpha + \beta + \gamma + \alpha \beta \gamma = 0$. The algebra $S_{\alpha,\beta,\gamma}(\mathbb{C})$ represents a twisted homogeneous coordinate ring of an elliptic curve $E_{\alpha,\beta,\gamma}(\mathbb{C})$ given in its Jacobi form

$$
\begin{align*}
    u^2 + v^2 + w^2 + z^2 &= 0, \\
    \frac{1-\alpha}{1+\beta} v^2 + \frac{1+\alpha}{1-\gamma} w^2 + z^2 &= 0,
\end{align*}
$$

see [Smith & Stafford 1993] [10], p.267 and [Stafford & van den Bergh 2001] [11], Example 8.5. The latter means that algebra $S_{\alpha,\beta,\gamma}(\mathbb{C})$ satisfies an isomorphism $\text{Mod} (S_{\alpha,\beta,\gamma}(\mathbb{C}))/\text{Tors} \cong \text{Coh} (E_{\alpha,\beta,\gamma}(\mathbb{C}))$, where $\text{Coh}$ is the category of quasi-coherent sheaves on $E_{\alpha,\beta,\gamma}(\mathbb{C})$, $\text{Mod}$ the category of graded left modules over the graded ring $S_{\alpha,\beta,\gamma}(\mathbb{C})$ and $\text{Tors}$ the full sub-category of $\text{Mod}$ consisting of the torsion modules, see [Stafford & van den Bergh 2001] [11], p.173. The algebra $S_{\alpha,\beta,\gamma}(\mathbb{C})$ defines a natural automorphism $\sigma$ of elliptic curve $E_{\alpha,\beta,\gamma}(\mathbb{C})$, *ibid.*

### 3 Proof of theorem 1

For the sake of clarity, let us outline main ideas. The proof is based on a categorical correspondence (a covariant functor) between elliptic curves $E_{\tau}$ and noncommutative tori $A_{\theta}$ taken with their “scaled units” $\frac{1}{\mu} e$. Namely, we prove that for $\sigma^4 = \text{Id}$ the norm-closure of a self-adjoint representation of the Sklyanin algebra $S_{\alpha,\beta,\gamma}(\mathbb{C})$ by the linear operators $u = x_1, u^* = x_2, v = x_3, v^* = x_4$ on a Hilbert space $\mathcal{H}$ is isomorphic to the $C^*$-algebra $A_{\theta}$ so that
its unit $e$ is scaled by a positive real $\mu$, see lemma 2 because $S_{\alpha,\beta,\gamma}(\mathbb{C})$ is a coordinate ring of elliptic curve $E_{\alpha,\beta,\gamma}(\mathbb{C})$ so will be the algebra $A_{\theta}$ modulo the unit $\frac{1}{\mu}e$. Moreover, our construction entails that a coefficient $q$ of elliptic curve $E_{\alpha,\beta,\gamma}(\mathbb{C})$ is linked to the constants $\theta$ and $\mu$ by the formula $q = \mu e^{2\pi i \theta}$, see lemma 1. Suppose that our elliptic curve has complex multiplication, i.e. $E_{\alpha,\beta,\gamma}(\mathbb{C})$ has CM, see lemma 1. Therefore one gets an inclusion

$$\mu e^{2\pi i \theta} \in H(k),$$

where $\theta \in \mathbb{Q}(\sqrt{d})$ and $\mu = \log \varepsilon$. (Of course, our argument is valid only when $q \not\in \mathbb{R}$, i.e. when $|Cl(R_f)| \geq 2$; but there are only a finite number of discriminants $d$ with $|Cl(R_f)| = 1$.)

Let us pass to a detailed argument.

**Lemma 1** If $\sigma^4 = Id$, then the Sklyanin algebra $S_{\alpha,\beta,\gamma}(\mathbb{C})$ endowed with the involution $x_1^3 = x_2$ and $x_3^4 = x_4$ is isomorphic to a free algebra $\mathbb{C}(x_1, x_2, x_3, x_4)$ modulo an ideal generated by six quadratic relations

$$\begin{align*}
    x_3 x_1 &= \mu e^{2\pi i \theta} x_1 x_3, \\
    x_4 x_2 &= \frac{1}{\mu} e^{2\pi i \theta} x_2 x_4, \\
    x_4 x_1 &= \mu e^{-2\pi i \theta} x_1 x_4, \\
    x_3 x_2 &= \frac{1}{\mu} e^{-2\pi i \theta} x_2 x_3, \\
    x_2 x_1 &= x_1 x_2, \\
    x_4 x_3 &= x_3 x_4,
\end{align*}$$

where $\theta = \text{Arg} (q)$ and $\mu = |q|$ for a complex number $q \in \mathbb{C} \setminus \{0\}$.

**Proof.** (i) Since $\sigma^4 = Id$, the Sklyanin algebra $S_{\alpha,\beta,\gamma}(\mathbb{C})$ is isomorphic to a free algebra $\mathbb{C}(x_1, x_2, x_3, x_4)$ modulo an ideal generated by the skew-symmetric relations

$$\begin{align*}
    x_3 x_1 &= q_{13} x_1 x_3, \\
    x_4 x_2 &= q_{24} x_2 x_4, \\
    x_4 x_1 &= q_{14} x_1 x_4, \\
    x_3 x_2 &= q_{23} x_2 x_3, \\
    x_2 x_1 &= q_{12} x_1 x_2, \\
    x_4 x_3 &= q_{34} x_3 x_4,
\end{align*}$$

where $q_{ij} = q_{ji}$ and $q_{ij} = 0$ for $i < j$.
where \( q_{ij} \in \mathbb{C} \setminus \{0\} \), see [Feigin & Odesskii 1989] \(^3\), Remark 1.

(ii) It is verified directly, that relations (10) are invariant of the involution \( x_1^* = x_2 \) and \( x_3^* = x_4 \), if and only if

\[
\begin{align*}
q_{13} &= (\bar{q}_{24})^{-1}, \\
q_{24} &= (\bar{q}_{13})^{-1}, \\
q_{14} &= (\bar{q}_{23})^{-1}, \\
q_{23} &= (\bar{q}_{14})^{-1}, \\
q_{12} &= \bar{q}_{12}, \\
q_{34} &= \bar{q}_{34},
\end{align*}
\]

(11)

where \( \bar{q}_{ij} \) means the complex conjugate of \( q_{ij} \in \mathbb{C} \setminus \{0\} \).

Remark 4 The invariant relations (11) define an involution on the Sklyanin algebra; we shall refer to such as a Sklyanin \( * \)-algebra.

(iii) Consider a one-parameter family \( S(q_{13}) \) of the Sklyanin \( * \)-algebras defined by the following additional constraints

\[
\begin{align*}
q_{13} &= \bar{q}_{14}, \\
q_{12} &= q_{34} = 1.
\end{align*}
\]

(12)

It is not hard to see, that the \( * \)-algebras \( S(q_{13}) \) are pairwise non-isomorphic for different values of complex parameter \( q_{13} \); therefore the family \( S(q_{13}) \) is a normal form of the Sklyanin \( * \)-algebra \( S_{\alpha,\beta,\gamma}(\mathbb{C}) \) with \( \sigma^4 = Id \). It remains to notice, that one can write complex parameter \( q := q_{13} \) in the polar form \( q = \mu e^{2\pi i \theta} \), where \( \theta = \text{Arg} (q) \) and \( \mu = |q| \). Lemma \(^4\) follows. \( \square \)

Lemma 2 (basic isomorphism) The system of relations (2) for noncommutative torus \( \mathcal{A}_\theta \) with \( u = x_1, u^* = x_2, v = x_3, v^* = x_4 \), i.e.

\[
\begin{align*}
x_3 x_1 &= e^{2\pi i \theta} x_1 x_3, \\
x_4 x_2 &= e^{2\pi i \theta} x_2 x_4, \\
x_4 x_1 &= e^{-2\pi i \theta} x_1 x_4, \\
x_3 x_2 &= e^{-2\pi i \theta} x_2 x_3, \\
x_2 x_1 &= x_1 x_2 = e, \\
x_4 x_3 &= x_3 x_4 = e,
\end{align*}
\]

(13)
is equivalent to the system of relations (9) for the Sklyanin *-algebra, i.e.

$$
\begin{align*}
\begin{cases}
x_3x_1 &= \mu e^{2\pi i \theta} x_1 x_3, \\
x_4x_2 &= \frac{1}{\mu} e^{2\pi i \theta} x_2 x_4, \\
x_4x_1 &= \mu e^{-2\pi i \theta} x_1 x_4, \\
x_3x_2 &= \frac{1}{\mu} e^{-2\pi i \theta} x_2 x_3, \\
x_2x_1 &= x_1 x_2, \\
x_4x_3 &= x_3 x_4,
\end{cases}
\end{align*}
$$

(modulo the following “scaled unit relation”

$$
x_1 x_2 = x_3 x_4 = \frac{1}{\mu} e. \tag{15}\n$$

Proof. (i) Using the last two relations, one can bring the noncommutative torus relations (13) to the form

$$
\begin{align*}
\begin{cases}
x_3 x_1 x_4 &= e^{2\pi i \theta} x_1, \\
x_4 &= e^{2\pi i \theta} x_2 x_4 x_1, \\
x_4 x_1 x_3 &= e^{-2\pi i \theta} x_1, \\
x_2 &= e^{-2\pi i \theta} x_4 x_2 x_3, \\
x_1 x_2 &= x_2 x_1 = e, \\
x_3 x_4 &= x_4 x_3 = e.
\end{cases}
\end{align*}
$$

(ii) The system of relations (14) for the Sklyanin *-algebra complemented by the scaled unit relation (15), i.e.

$$
\begin{align*}
\begin{cases}
x_3 x_1 &= \mu e^{2\pi i \theta} x_1 x_3, \\
x_4 x_2 &= \frac{1}{\mu} e^{2\pi i \theta} x_2 x_4, \\
x_4 x_1 &= \mu e^{-2\pi i \theta} x_1 x_4, \\
x_3 x_2 &= \frac{1}{\mu} e^{-2\pi i \theta} x_2 x_3, \\
x_2 x_1 &= x_1 x_2 = \frac{1}{\mu} e, \\
x_4 x_3 &= x_3 x_4 = \frac{1}{\mu} e,
\end{cases}
\end{align*}
$$

is equivalent to the system

$$
\begin{align*}
\begin{cases}
x_3 x_1 x_4 &= e^{2\pi i \theta} x_1, \\
x_4 &= e^{2\pi i \theta} x_2 x_4 x_1, \\
x_4 x_1 x_3 &= e^{-2\pi i \theta} x_1, \\
x_2 &= e^{-2\pi i \theta} x_4 x_2 x_3, \\
x_2 x_1 &= x_1 x_2 = \frac{1}{\mu} e, \\
x_4 x_3 &= x_3 x_4 = \frac{1}{\mu} e.
\end{cases}
\end{align*}
$$

(14)
by using multiplication and cancellation involving the last two equations.

(iii) For each \( \mu \in (0, \infty) \) consider a scaled unit \( e' := \frac{1}{\mu} e \) of the Sklyanin \( * \)-algebra \( S(q) \) and the two-sided ideal \( I_\mu \subset S(q) \) generated by the relations \( x_1 x_2 = x_3 x_4 = e' \). Comparing the defining relations \( (13) \) for \( S(q) \) with relation \( (13) \) for the noncommutative torus \( A_{\theta} \), one gets an isomorphism

\[
S(q) / I_\mu \cong A_{\theta}.
\]

The isomorphism maps generators \( x_1, \ldots, x_4 \) of the Sklyanin \( * \)-algebra \( S(q) \) to such of the \( C^* \)-algebra \( A_{\theta} \) and the scaled unit \( e' \in S(q) \) to the ordinary unit \( e \in A_{\theta} \). Lemma 2 follows. □

**Remark 5** It follows from \( (19) \) that noncommutative torus \( A_{\theta} \) with the unit \( \frac{1}{\mu} e \) is a coordinate ring of elliptic curve \( E_\tau \). Moreover, such a correspondence is a covariant functor which maps isomorphic elliptic curves to the stably isomorphic (Morita equivalent) noncommutative tori; the latter fact follows from an observation that isomorphisms in category \( \text{Mod} \) correspond to stable isomorphisms in the category of underlying algebras. Such a functor explains the same (modular) transformation law in formulas \( (3) \) and \( (5) \).

**Lemma 3** The coordinate ring of elliptic curve \( E_{CM}^{(-d,f)} \) is isomorphic to the noncommutative torus \( A_{RM}^{(d,f)} \) with the unit \( \frac{1}{\log \varepsilon} e \), where \( f \) is the least integer satisfying equation \( |\text{Cl}(R_f)| = |\text{Cl}(R_f)| \) and \( \varepsilon \) is the fundamental unit of order \( R_f \).

**Proof.** The fact that \( A_{RM}^{(d,f)} \) is a coordinate ring of elliptic curve \( E_{CM}^{(-d,f)} \) was proved in [Nikolaev 2014] [7]. We shall focus on the second part of lemma 3 saying that the scaling constant \( \mu = \log \varepsilon \). To express \( \mu \) in terms of intrinsic invariants of pseudo-lattice \( K^+_0(A_{RM}^{(d,f)}) \cong \mathbb{Z} + \mathbb{Z}\theta \), recall that \( \mathfrak{R}_f \) is the ring of endomorphisms of \( \mathbb{Z} + \mathbb{Z}\theta \); we shall write \( \mathfrak{R}^+_f \) to denote the multiplicative group of units (i.e. invertible elements) of \( \mathfrak{R}_f \). Since \( \mu \) is an additive functional on the pseudo-lattice \( \Lambda = \mathbb{Z} + \mathbb{Z}\theta \), for each \( \varepsilon, \varepsilon' \in \mathfrak{R}^+_f \) it must hold \( \mu(\varepsilon\varepsilon'\Lambda) = \mu(\varepsilon\varepsilon')\Lambda = \mu(\varepsilon)\Lambda + \mu(\varepsilon')\Lambda \). Eliminating \( \Lambda \) in the last equation, one gets

\[
\mu(\varepsilon\varepsilon') = \mu(\varepsilon) + \mu(\varepsilon'), \quad \forall \varepsilon, \varepsilon' \in \mathfrak{R}^+_f.
\]

The only real-valued function on \( \mathfrak{R}^+_f \) with such a property is the logarithmic function (a regulator of \( \mathfrak{R}^+_f \)); thus \( \mu(\varepsilon) = \log \varepsilon \), where \( \varepsilon \) is the fundamental unit of \( \mathfrak{R}_f \). Lemma 3 is proved. □
Remark 6 (Second proof of lemma 3) The formula \( \mu = \log \varepsilon \) can be derived using a purely measure-theoretic argument. Indeed, if \( h_x : \mathbb{R} \to \mathbb{R} \) is a “stretch-out” automorphism of real line \( \mathbb{R} \) given by the formula \( t \mapsto tx, \forall t \in \mathbb{R} \), then the only \( h_x \)-invariant measure \( \mu \) on \( \mathbb{R} \) is the “scale-back” measure \( d\mu = \frac{1}{t} dt \). Taking the antiderivative and integrating between \( t_0 = 1 \) and \( t_1 = x \), one gets
\[
\mu = \log x. \tag{21}
\]
It remains to notice that for pseudo-lattice \( K_0^+ (\mathcal{A}_{RM}^{(d,f)}) \cong \mathbb{Z} + \mathbb{Z} \theta \), the automorphism \( h_x \) corresponds to \( x = \varepsilon \), where \( \varepsilon > 1 \) is the fundamental unit of order \( \mathfrak{R} \). Lemma 3 follows. □.

One can prove theorem 1 in the following steps.

(i) Let \( d \not\in \{1, 2, 3, 7, 11, 19, 43, 67, 163\} \) be a positive square-free integer. In this case \( h = |Cl (R_f)| \geq 2 \) and \( \mathcal{E}_{CM}^{(-d,f)} \not\sim \mathcal{E} (\mathbb{Q}) \).

(ii) Let \( \{\mathcal{E}_1, \ldots, \mathcal{E}_h\} \) be pairwise non-isomorphic elliptic curves having the same endomorphism ring \( R_f \). From \( |Cl (R_f)| = |Cl (\mathfrak{R})| \) and lemma 3 one gets \( \{\mathcal{A}_1, \ldots, \mathcal{A}_h\} \) pairwise stably non-isomorphic noncommutative tori; the corresponding pseudo-lattices \( K_0^+ (\mathcal{A}_i) = \mathbb{Z} + \mathbb{Z} \theta_i \) will have the same endomorphism ring \( \mathfrak{R} \). Thus for each \( 1 \leq i \leq h \) one gets an inclusion
\[
(\log \varepsilon) e^{2\pi i \theta_i} \in H(k), \tag{22}
\]
where \( H(k) \) is the Hilbert class field of quadratic field \( k = \mathbb{Q}(\sqrt{-d}) \) modulo conductor \( f \). Since \( (\log \varepsilon) \exp(2\pi i \theta_i) = \exp(2\pi i \theta_i + \log \log \varepsilon) := \mathcal{J}(\theta_i, \varepsilon) \), one concludes that \( \mathcal{J}(\theta_i, \varepsilon) \in H(k) \).

(iii) Finally, because \( Gal (H(k)|k) \cong Cl (R_f) \cong Cl (\mathfrak{R}) \), it is easy to see that the set \( \{\mathcal{J}(\theta_i, \varepsilon) \mid 1 \leq i \leq h\} \) is invariant of the action of group \( Gal (H(k)|k) \) on \( H(k) \); in other words, numbers \( \mathcal{J}(\theta_i, \varepsilon) \) are algebraically conjugate.

Theorem 1 is proved. □
4 Example

In this section we shall use remark 1 to estimate $J(\theta, \varepsilon)$ for special values of the discriminant $d$; the reader is encouraged to construct examples of his own.

Example 1 Let $d = 15$ and $f = 1$. It is well known that the class number of order $R_f \cong O_k$ of the field $k = \mathbb{Q}(\sqrt{-15})$ is equal to 2. Because the class number of the field $\mathfrak{r} = \mathbb{Q}(\sqrt{15})$ is also 2, one concludes from equation $|Cl (\mathfrak{r})| = |Cl (R_f)|$ that conductor $j = 1$. Let $\tau \in O_k$; it is well known that in this case $j(\tau) \in \mathbb{Q}(\sqrt{5})$, see e.g. [Silverman 1994] Example 6.2.2. In view of remark 1 one gets an inclusion $J(\theta_i, \varepsilon) \in \mathbb{Q}(\sqrt{-15}, \sqrt{5})$. Since one of $\theta_i$ is equal to $\sqrt{15}$ and the fundamental unit $\varepsilon$ of the field $\mathfrak{r} = \mathbb{Q}(\sqrt{15})$ is equal to $4 + \sqrt{15}$, one gets the following inclusion

$$J(\sqrt{15},\ 4 + \sqrt{15}) := e^{2\pi i \sqrt{15} + \log\log(4 + \sqrt{15})} \in \mathbb{Q}\left(\sqrt{-15}, \sqrt{5}\right). \quad (23)$$

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