REGULARIZATION FOR FRACTIONAL INTEGRAL. APPLICATION TO NONLINEAR EQUATIONS WITH SINGULARITIES

MIRJANA STOJANOVIĆ

Abstract. We give the regularization for fractional integral by delta sequence and apply it to obtain existence-uniqueness theorems in Colombeau algebras for nonlinear equations with singularities: nonlinear system of integral equations with polar kernel and nonlinear parabolic equations (of ordinary type, with nonlinear conservative term and with Schrödinger kernel) with strongly singular initial data and non-Lipschitz nonlinearities. In a case of nonlinear parabolic equations we do in fact regularization of heat semigroup with delta sequence with respect to the time variable \( t \). We do the same for linear Schrödinger equation.

1. Introduction

Fractional time derivatives have been considered in many problems in mechanics with application to ordinary fractional differential equations in mechanics (cf. [17]).

In recent years considerable interest to fractional calculus has been stimulated by various applications of that calculus in numerical analysis and different areas of physics and engineering possibly including fractal phenomena (cf. [12]).

The purpose of this paper is study of effects of fractional derivative term in system of nonlinear integral equations, nonlinear parabolic equations and linear Schrödinger equation.

We find a fractional derivative term in system of nonlinear integral equations with polar kernel and in heat kernel in nonlinear parabolic equations as well as in Schrödinger semigroup in linear Schrödinger equation. We regularize it by delta sequence with respect to \( t \) to obtain existence-uniqueness theorems in Colombeau vector type spaces.

The content of the paper is the following.

We give the regularization for fractional integral by delta sequence and obtain the logarithmic boundedness of fractional derivative of delta distribution. Then, we apply it to the system of nonlinear Volterra type integral equations with polar kernel and singularities for free term to obtain the existence-uniqueness theorems in Colombeau space \((G(I))^n\), where \( I \) is an interval around zero.

1Department of Mathematics and Informatics, Faculty of science, University of Novi Sad, Trg D.Obradovića 4, 21 000 Novi Sad, Serbia and Montenegro
stojanovic@im.ns.ac.yu

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We give the regularization for heat kernel by delta sequence with respect to the time variable \( t \) to nonlinear parabolic equations of ordinary type, parabolic equations with nonlinear conservative term and nonlinear parabolic equations with Schrödinger kernel with strongly singular initial data and non-Lipschitz nonlinearities. We apply the results for regularized fractional integral to obtain the existence-uniqueness theorems in Colombeau vector type spaces \( \mathcal{G}_{C^{1}([L^{p}, L^{q}])}([0, T], \mathbb{R}^{n}) \), \( 1 \leq p, q \leq \infty \), for those choices of \( p, q \) and the dimension space \( n \) for which corresponding Colombeau space is an algebra with multiplication. For example, it holds for \( p = q = 2 \), \( n \leq 3 \) due to Sobolev imbedding theorems.

We do the same for linear Schrödinger equation with strongly singular potential and the initial data.

Let us mention that the pioneering work concerning the regularization of semi-groups of nonlinear PDEs with respect to the time variable \( t \) in Colombeau algebra was done in [6].

2. Colombeau algebras

For general theory of Colombeau generalized functions and for the algebra \( \mathcal{G}(\Omega) \), where \( \Omega \) is an open set, cf. [7], [8], [2], [15] and recently [9], [10]. We recall the basic definitions from [9].

Suppose that \( \Omega \) is an open set. Define \( \mathcal{E}(\Omega) = (\mathcal{C}^{\infty}(\Omega))^{I} \),
\[
\mathcal{E}_{M}(\Omega) = \{(u_{\varepsilon})_{\varepsilon} \in \mathcal{E}(\Omega)|\forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_{0}^{m} \exists N \in \mathbb{N} \ \text{with} \ \sup_{x \in K} |\partial^{\alpha}u_{\varepsilon}(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},
\]
\[
\mathcal{N}(\Omega) = \{(u_{\varepsilon})_{\varepsilon} \in \mathcal{E}(\Omega)|\forall K \subset \subset \Omega \forall \alpha \in \mathbb{N}_{0}^{m} \forall s \in \mathbb{N} : \sup_{x \in K} |\partial^{\alpha}u_{\varepsilon}(x)| = O(\varepsilon^{s}) \text{ as } \varepsilon \to 0\}.
\]

Elements of \( \mathcal{E}_{M}(\Omega) \) and \( \mathcal{N}(\Omega) \) are called moderate, resp. negligible functions. The Colombeau algebra on \( \Omega \) is defined as \( \mathcal{G}(\Omega) = \mathcal{E}_{M}(\Omega)/\mathcal{N}(\Omega) \).

Recall the construction of the \( \mathcal{G}_{p,q}(\Omega) \) algebras from [3].

Let \( \Omega \subset \mathbb{R}^{n} \) be open, \( m \in \mathbb{Z}, 1 \leq p \leq \infty \). Denote by \( W^{m,p}(\Omega) \) usual Sobolev space whose all derivatives up to the order \( m \) are finite in corresponding norm. Then,
\[
W^{\infty,p}(\Omega) = \cap_{m} W^{m,p}(\Omega), \ W^{-\infty,p}(\Omega) = \cup_{m} W^{-m,p}(\Omega).
\]

Let \( 1 \leq p, q \leq \infty \). Define
\[
\mathcal{E}(\Omega) = \{u|(0, \infty) \times \Omega \to \mathbb{R} \text{ such that } u_{\varepsilon}(x) \text{ is } C^{\infty} \text{ in } x \in \Omega, \ \forall \varepsilon > 0\}
\]
\[
\mathcal{E}_{p}(\Omega) = \{u \in \mathcal{E}(\Omega), \text{ such that } u_{\varepsilon} \in W^{\infty,p}(\Omega), \ \forall \varepsilon > 0\}
\]
\[
\mathcal{E}_{M,p}(\Omega) = \{u \in \mathcal{E}_{p}, \text{ such that } \forall \alpha \in \mathbb{N}_{0}^{m} \exists N \in \mathbb{N} \text{ such that } ||\partial^{\alpha}u_{\varepsilon}(\cdot)||_{p} = O(\varepsilon^{-N}), \varepsilon \to 0\}
\]
\[
\mathcal{N}_{p,q}(\Omega) = \{u \in \mathcal{E}_{M,p}(\Omega) \cap \mathcal{E}_{q}(\Omega) \text{ such that } \forall \alpha \in \mathbb{N}_{0}^{m} \forall s \in \mathbb{N}, ||\partial^{\alpha}u_{\varepsilon}(\cdot)||_{q} = O(\varepsilon^{s}), \varepsilon \to 0\}
\]
where \( ||\cdot||_{p} \) denotes \( L^{p} \)-norm. For properties of these spaces see [3]. Colombeau space \( \mathcal{G}_{p,q}([0, T], \mathbb{R}^{n}) \), \( 1 \leq p, q \leq \infty \) is the factor set
\[
\mathcal{G}_{p,q}([0, T], \mathbb{R}^{n}) = \mathcal{E}_{M,p}([0, T], \mathbb{R}^{n})/\mathcal{N}_{p,q}([0, T], \mathbb{R}^{n}).
\]

We shall give the construction of Colombeau vector type spaces \( \mathcal{G}_{C^{1}([L^{p}, L^{q}])}([0, T], \mathbb{R}^{n}) \), \( 1 \leq p, q \leq \infty \). All of these spaces are not algebra with multiplication. For \( n \leq 3, p = q = 2 \) due to the Sobolev imbedding \( L^{2}(\mathbb{R}^{n}) \subset L^{\infty}(\mathbb{R}^{n}) \), the Colombeau vector
type space $\mathcal{G}_{C^1, (L^2, L^2)}([0, T), \mathbb{R}^n)$ is an algebra with multiplication. For the proof cf. [14].

Let $\Omega$ be an open set in $\mathbb{R}^n$, $1 \leq p, q \leq \infty$.

Space $\mathcal{E}_{C^1, (L^p, L^q)}([0, T), \mathbb{R}^n)$ is a vector space of nets $(G_\varepsilon)_\varepsilon$ such that

$$G_\varepsilon \in C^0([0, T), \mathbb{R}^n) \cap C^1([0, T), \mathbb{R}^n), \varepsilon < 1,$$

with the following property: $\forall T_1 \in (0, T) \exists N \in \mathbb{N}$ such that

$$\max \left\{ \sup_{t \in [0, T]} \|G_\varepsilon(t)\|_{L^p}, \sup_{t \in [T_1, T]} \|\partial_t G_\varepsilon(t)\|_{L^p} \right\} = O(\varepsilon^{-N}), \varepsilon \to 0. \quad (1)$$

If

$$\max \left\{ \sup_{t \in [0, T]} \|G_\varepsilon(t)\|_{L^q}, \sup_{t \in [T_1, T]} \|\partial_t G_\varepsilon(t)\|_{L^q} \right\} = O(\varepsilon^s), \varepsilon \to 0,$$

holds for $\forall s \in \mathbb{N}$ we obtain null space $\mathcal{N}_{C^1, (L^p, L^q)}([0, T), \mathbb{R}^n)$. Colombeau vector type space is defined as the quotient space

$$\mathcal{G}_{C^1, (L^p, L^q)}([0, T), \mathbb{R}^n) = \mathcal{E}_{C^1, L^p}([0, T), \mathbb{R}^n)/\mathcal{N}_{C^1, (L^p, L^q)}([0, T), \mathbb{R}^n).$$

3. Basic facts from fractional calculus

Fractional integrals are defined as a generalization of Cauchy formula for a repeated indefinite integral

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \ t >, \ n \in \mathbb{N},$$

where $J f$ is a primitive function of $f$:

$$(Jf)(t) := \int_0^t f(\tau) d\tau,$$

and $J^n$ denotes the $n^{th}$ power of $J$.

For an arbitrary real number $\alpha > 0$ and sufficiently regular function $f$ on $\mathbb{R}$ we define fractional integral of order $\alpha > 0$

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ t >, \ \alpha \in \mathbb{R}^+. \quad (2)$$

The equation (2) can be expressed in terms of distributions for arbitrary complex number $\alpha$ (cf. [11]), as

$$\Phi_\alpha(t) := \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, \\ \frac{1}{D^{(n)}} \Phi_{\alpha+n}, & \alpha \leq 0, \ \alpha + n > 0, \ n \in \mathbb{N}, \end{cases} \quad (3)$$

(where $D^{(n)}$ is the $n^{th}$ distributional derivative) in the form $J^\alpha f := \Phi_\alpha * f_+$, where $f_+(t) := H(t) f(t)$, $H(t)$ is the Heaviside function, convolution is defined in distributional sense. The distribution $t^{\alpha-1}_+$ for $\alpha > 0$ is defined by

$$t^{\alpha-1}_+ = \begin{cases} 0 & t < 0, \\ t^{\alpha-1} & t > 0. \end{cases}$$

For other values of $\alpha$ this distribution is defined by analytic continuation or by regularization of divergent integrals (cf. [11]). We have

$$J^\alpha f(t) = \Phi_\alpha * f(t), \ \alpha > 0, \ \text{and} \ D\Phi_\alpha = \Phi_{\alpha-1},$$
and semigroup property holds

\[ \Phi_\alpha(t) * \Phi_\beta(t) = \Phi_{\alpha+\beta}(t), \quad \alpha, \beta > 0. \]

With interpretation of the quotient as a limit if \( t = 0 \) (cf. [12])

\[ \Phi_{-n} := \frac{t_n}{\Gamma(-n)} = \delta^{(n)}(t), \quad n \in \mathbb{N}_0, \]

where \( \delta^{(n)}(t) \) is the generalized derivative of order \( n \) of the Dirac delta distribution.

Then,

\[ \frac{d^n}{dt^n} f(t) = f^{(n)}(t) = \Phi_{-n}(t) * f(t) = \int_{0^-}^{t^+} f(\tau)\delta^{(n)}(t-\tau)d\tau, \quad t > 0, \]

since

\[ \int_{0^-}^{t^+} f(\tau)\delta^{(n)}(\tau-t)d\tau = (-1)^n f^n(t), \quad \delta^{(n)}(t-\tau) = (-1)^n\delta^{(n)}(\tau-t). \]

The formal definition of fractional derivative could be

\[ \Phi_{-\alpha}(t) * f(t) = \frac{1}{\Gamma(-\alpha)} \int_{0^-}^{t^+} \frac{f(\tau)}{(t-\tau)^{\alpha+1}}d\tau, \quad \alpha \in \mathbb{R}^+. \quad (4) \]

In general, \( \Phi_{-\alpha}(t) \) is not locally absolutely integrable and integral is divergent.

In order to obtain the definition that is still valid for classical functions we regularize the divergent integral by delta sequence. Instead of the function \( f \) in (4) we shall use the mollifier \( \phi_\varepsilon(t) = |ln\varepsilon|\phi(x \cdot |ln\varepsilon|) \) (resp. using \( (ln|ln\varepsilon|) \) where \( \phi(t) \in C_0^\infty(\mathbb{R}) \), \( \phi(t) \geq 0, \int \phi(t)dt = 1 \).

We shall consider three different cases with various values of \( \alpha \).

1. Consider (4). For the elements of the set \( n \in \mathbb{N}_0, \Phi_{-n}(t) \ast \phi_\varepsilon(t) = \delta^{(n)}(t) \ast \phi_\varepsilon(t) \).

We shall find \( L^1 \)-norm of the above convolution for later use. We have

\[ \|\Phi_{-n}(t) \ast \phi_\varepsilon(t)\|_{L^1} = \|\delta^{(n)}(t) \ast \phi_\varepsilon(t)\|_{L^1} = \|\delta(t) \ast D^{(n)}\phi_\varepsilon(t)\|_{L^1} = \|D^{(n)}\phi_\varepsilon(t)\|_{L^1} \leq C|ln\varepsilon|^n \]

where \( D^{(n)} \) denotes distributional derivative.

2. Let \( \alpha > 0 \). Then, \( \Phi_\alpha(t) := \frac{t^\alpha}{\Gamma(\alpha)}, \alpha > 0 \), and fractional integral is given by (2).

Then, \( J^{\alpha}f(t) = \Phi_\alpha(t) \ast f(t), \quad \alpha > 0 \). Then, \( \|J^{\alpha}\phi_\varepsilon(t)\|_{L^1} \leq \|\Phi_\alpha(\cdot)\|_{L^1}\|\phi_\varepsilon(\cdot)\|_{L^1} \leq C \).

By this, we cover the case \( 0 < \alpha < 1 \).

3. Consider the case \( \alpha < 0 \). By classical definition (3) of the function \( \Phi_\alpha(t) \) for \( \bar{\alpha} \in \mathbb{R}_+, \alpha = -\bar{\alpha}, \alpha > 0 \), we have \( \Phi_{-\bar{\alpha}}(t) = D^{(n)}\Phi_{-\bar{\alpha}+n}(t), \bar{\alpha} \geq 0, -\bar{\alpha}+n > 0, n \in \mathbb{N}, \bar{\alpha} < n \). Then,

\[ J^{-\bar{\alpha}}\phi_\varepsilon(t) = \frac{1}{\Gamma(-\bar{\alpha})} \int_{0^-}^{t^+} (t-\tau)^{-\bar{\alpha}-1}\phi_\varepsilon(\tau)d\tau, \quad \bar{\alpha} \in \mathbb{R}_+, \quad t > 0, \]

and

\[ J^{-\bar{\alpha}}\phi_\varepsilon(t) = \Phi_{-\bar{\alpha}} \ast \phi_\varepsilon(t) = D^{n}\Phi_{-\bar{\alpha}+n}(t) \ast \phi_\varepsilon(t) = \Phi_{-\bar{\alpha}+n}(t) \ast D^{n}\phi_\varepsilon(t), \quad -\bar{\alpha} + n > 0. \]

By case 2,

\[ \|J^{-\bar{\alpha}}\phi_\varepsilon(t)\|_{L^1} \leq \|\Phi_{-\bar{\alpha}+n}(t)\|_{L^1}\|D^{n}\phi_\varepsilon(t)\|_{L^1} \leq C|ln\varepsilon|^n. \]
Thus, we have for each $\alpha \in \mathbb{R}$
\[
|||x|^{|\alpha-1} \ast \phi_\epsilon(x)|||_{L^1} \leq \left\{
\begin{array}{ll}
C & \text{when } \alpha > 0, \\
C|\ln \epsilon|^m & \text{when } \alpha \leq 0, \; m > \bar{\alpha}, \; \alpha = -\bar{\alpha}, \; \bar{\alpha} > 0.
\end{array}
\right.
\] (5)

We have just proved the following Lemma.

**Lemma 1.** Fractional integral of the delta sequence

\[
J^\alpha \phi_\epsilon(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \phi_\epsilon(\tau) d\tau, \; t > 0, \; \alpha \in \mathbb{R},
\]

where $\phi(x) \in C_0^\infty(\mathbb{R})$, $\phi(x) \geq 0$, $\int \phi(x) dx = 1$, and $\phi_\epsilon(x) = |\ln \epsilon| \phi(x \cdot |\ln \epsilon|)$ (resp. using $(\ln|\ln \epsilon|)$ as instead of $|\ln \epsilon|$) has the following bounds in $L^1$-norm:
\[
||J^\alpha (\phi_\epsilon(t))||_{L^1} = |||t|^{|\alpha-1} \ast \phi_\epsilon(t)|||_{L^1} \leq \left\{
\begin{array}{ll}
C & \text{when } \alpha > 0, \\
C|\ln \epsilon|^m & \text{(resp. } (\ln|\ln \epsilon|)^m) \text{ when } \alpha \leq 0, \; m > \bar{\alpha}, \; \alpha = -\bar{\alpha}, \; \bar{\alpha} > 0.
\end{array}
\right.
\]

**Corollary 1.** When $\alpha = 0$ we have $||v_p^\frac{1}{|x|+\epsilon} \ast \phi_\epsilon(x)||_{L^1} \leq C|\ln \epsilon|^m$, $m > 0$.

4. Application to nonlinear Volterra integral equation

Consider a system of nonlinear Volterra integral equations with polar kernel
\[
f^i(x) = g^i(x) + \int_0^x \frac{K^i(x, y, f(y))}{|x-y|^\alpha + 1} dy, \; x \in I, \; i = 1, .., n, \; \alpha \in \mathbb{R},
\] (6)

where $I$ is an interval around zero, $g^i(x) \in D'(I)$, $K^i \in L^\infty_{loc}(I \times I \times \mathbb{R}^2)$ and it is not of Lipschitz class, $i = 1, 2, ..., n$.

**Theorem 1.** Nonlinear system of integral equations (6) with three type of singularities: $g(x) \in (D'(I))^n$, $K(x, y, f(y)) \in (L^\infty_{loc}(I \times I \times \mathbb{R}^2))^n$ is nonlinear and non-Lipschitz, and integral is divergent on the set $\{y | y = x\}$ has a unique solution in the space $[f_\epsilon(x)] \in (G(I))^n$.

**Proof.** The function $K^i$ is substituted by $K^i_\epsilon$ and $g^i$ by $g^i_\epsilon$, $i = 1, ..., n$, $\epsilon \in (0, 1]$, in order to avoid non-Lipschitz nonlinearity of $K^i$ and singularities of $g^i$, $i = 1, ..., n$.

For given $g = (g^1, ..., g^n) \in (D'(I))^n$, we put $g^i_\epsilon = (g^i \kappa_\epsilon) \ast \phi_\epsilon$, $i = 1, ..., n$, where $\kappa_\epsilon \in C_0^\infty(I)$,
\[
\kappa_\epsilon = \left\{
\begin{array}{ll}
1 & \text{on } I_{2\epsilon} \\
0 & \text{on } I \setminus I_{\epsilon},
\end{array}
\right.
\]

$I = (-a, a)$, $I_{2\epsilon} = (-a + j \epsilon, a - j \epsilon)$, $j = 1, 2$ and $\phi_\epsilon(x) = |\ln \epsilon| \phi(x \cdot |\ln \epsilon|)$, $x \in \mathbb{R}$, is mollifier with the properties: $\phi \in S(\mathbb{R})$, $\int \phi(x) dx = 1$, $\int x^\beta \phi(x) dx = 0$, for all $\beta \in N_0$, if $a = -\infty$, or $b = \infty$ then $a + j \epsilon = -\infty$, or $b - j \epsilon = \infty$.

By cut-off method (cf. [16]) we have
\[
|\nabla K_\epsilon(x, y, f_\epsilon(y))| \leq C|\ln \epsilon|^b, \; (\text{resp. } \leq C(\ln|\ln \epsilon|)),
\]

where $b$ will be determined to handle the problem under consideration. Then, the integral is singular only due to $|x - y|^\alpha$.

Consider the convolution form of the above equation
\[
f_\epsilon(x) = g_\epsilon(x) + K_\epsilon(x, y, f_\epsilon(y)) \ast |x|^{|\alpha-1} \ast \phi_\epsilon(x),
\] (7)
where we regularized fractional part of the integrand by delta sequence to avoid the divergence of the integral along the diagonal \( x = y \). In integral form we have

\[
f_\varepsilon(x) = g_\varepsilon(x) + \int_{0}^{x} \int_{\mathbb{R}} \frac{K_\varepsilon(x, y, f_\varepsilon(y))}{|x - y|^{-\alpha + 1}} \phi_\varepsilon(x - y - s)dsdy.
\]  

(8)

By Hölder inequality

\[
|f_\varepsilon(x)| \leq C|\ln\varepsilon|^{1} + \int_{0}^{x} ||K_\varepsilon(x, y, f_\varepsilon(y)||_{L^\infty}||\phi_\varepsilon(x - y)||_{L^1}dy.
\]

Let \( \alpha \leq 0 \). By (5)

\[
f_\varepsilon(x) = g_\varepsilon(x) + \int_{0}^{x} ||\nabla K_\varepsilon(x, y, \theta f_\varepsilon(y))||_{L^\infty} \sup_{y} |f_\varepsilon(y)||\ln\varepsilon|^{m}dy, -\bar{\alpha} + m > 0,
\]

where \( \alpha = -\bar{\alpha}, \bar{\alpha} \geq 0 \). Gronwall inequality yields

\[
|f_\varepsilon(x)| \leq C|\ln\varepsilon|^{1} \exp(C|\ln\varepsilon|^{b+m}) \leq C\varepsilon^{-N}, \ \exists N > 0,
\]

where \( m \geq \bar{\alpha} \), i.e. \( b + \bar{\alpha} < 1 \). For \( \alpha > 0 \) due to (5) we have only the condition \( b < 1 \).

Consider \( \beta \)-th derivative, \( \beta \in N_0, \beta \geq 1 \). We have in (8)

\[
D^\beta f_\varepsilon(x) = D^\beta g_\varepsilon(x) + \int_{0}^{x} \int_{\mathbb{R}} K_\varepsilon(x, y, f_\varepsilon(y)) \frac{\partial^\beta \phi_\varepsilon(x - y - s)}{|x - y|^{-\alpha + 1}}dsdy.
\]

By Hölder inequality we obtain

\[
D^\beta f_\varepsilon(x) = D^\beta g_\varepsilon(x) + \int_{0}^{x} \sup_{y} ||\nabla K_\varepsilon(x, y, f_\varepsilon(y))||_{L^\infty} \sup_{y} |f_\varepsilon(y)||\ln\varepsilon|^{m}dy, -\bar{\alpha} + m > 0
\]

(9)

We shall calculate the first

\[
||\partial^\beta \phi_\varepsilon(x) \ast |x|_+^{\alpha - 1}||_{L^1} \leq \int_{\mathbb{R}} ||\phi_\varepsilon(x) \ast \partial^\beta |x|_+^{\alpha - 1}||_{L^1}dx
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} ||(\alpha - 1)(\alpha - 2)...(\alpha - \beta - 1)||\phi_\varepsilon(s)||x - s|_+^{\alpha - 1 - \beta}dsdx.
\]

By Lemma 1 we have

\[
||\phi_\varepsilon(x) \ast \partial^\beta |x|_+^{\alpha - 1}||_{L^1} \leq \begin{cases} C & \text{when } \alpha > 0, \alpha > \beta, \\ C|\ln\varepsilon|^{m}, & \alpha > 0, \alpha < \beta, m > -\beta + \alpha \\ C|\ln\varepsilon|^{m} & \alpha < 0, \alpha < \beta < 0, m > -\bar{\alpha} - \beta, \alpha = -\bar{\alpha}, \bar{\alpha} > 0 \end{cases}
\]

(10)

Setting this in (9) we obtain for \( \alpha \leq 0 \)

\[
D^\beta f_\varepsilon(x) = D^\beta g_\varepsilon(x) + \int_{0}^{x} ||\ln\varepsilon|| \sup_{y} |f_\varepsilon(y)||\ln\varepsilon|^{m}dy, m > -\bar{\alpha} - \beta, \bar{\alpha} < \beta.
\]

Since \( g(x) \in D'(I) \) we shall give a regularization as in ([16]) \( ||\kappa_\varepsilon f(x) \ast \phi_\varepsilon(x)||_{L^\infty} \leq C|\ln\varepsilon| \). Then Gronwall inequality yields

\[
|D^\beta f_\varepsilon(x)| \leq C|\ln\varepsilon|^{\beta + 1} + \int_{0}^{x} \sup_{y} |f_\varepsilon(y)||\ln\varepsilon|^{b+m}dy, m > -\bar{\alpha} - \beta, \bar{\alpha} < \beta, \bar{\alpha} > 0
\]

and

\[
|D^\beta f_\varepsilon(x)| \leq C|\ln\varepsilon|^{1+\beta} \exp(C|\ln\varepsilon|^{b+m}) \leq C\varepsilon^{-N}, \ \exists N > 0, x \in \mathbb{R}, \varepsilon < \varepsilon_0,
\]

where \( m > -\bar{\alpha} - \beta, b + m < 1, b < 1 + \bar{\alpha} + \beta \). This condition is included in the condition imposed at the first step \( m > \bar{\alpha}, b < 1 - \bar{\alpha} \).
If $\alpha > 0$ we have by (10)

$$D^\beta f_\varepsilon(x) = D^\beta g_\varepsilon(x) + \int_0^x C \left\{ \begin{array}{ll} C & \alpha > 0, \alpha > \beta \\ C[\ln\varepsilon]^m, & \alpha > 0, \alpha < \beta, m > -\beta + \alpha \\ \end{array} \right. |\ln\varepsilon|^b \sup_y |f_\varepsilon(y)|dy.$$ 

Gronwall inequality yields

$$|D^\beta f_\varepsilon(x)| \leq C|\ln\varepsilon|^{1+\beta} \exp(C \left\{ \begin{array}{ll} C & \alpha > 0, \alpha > \beta \\ C[\ln\varepsilon]^m, & \alpha > 0, \alpha < \beta, m > -\beta + \alpha \\ \end{array} \right. |\ln\varepsilon|^b) \leq C\varepsilon^{-N},$$

$\exists N > 0$, $x \in \mathbb{R}$, $\varepsilon < \varepsilon_0$, under the condition $b < 1$, when $\alpha > \beta$, and $b + m < 1$, $b < 1 + \beta - \alpha$, $\alpha < \beta$, what is included in the first step condition $b < 1$. Thus, we have the moderateness of the solution $f_\varepsilon(x) \in E_M(I)$.

Let us prove the uniqueness. Suppose that $f_{1\varepsilon}(x)$ and $f_{2\varepsilon}(x)$ are two solutions to the equation (8) and denote their difference by $F_\varepsilon(x)$. Then, we have

$$|F_\varepsilon(x)| \leq \int_0^x |K_\varepsilon(x, y, f_{1\varepsilon}(y)) - K_\varepsilon(x, y, f_{2\varepsilon}(y))|\frac{\phi_\varepsilon(x - y - s)}{|x - y|^{1-\alpha}} dsdy + d_\varepsilon(x),$$

(11)

where $d_\varepsilon(x) \in \mathcal{N}(I)$. By mean value theorem

$$|K_\varepsilon(x, y, f_{1\varepsilon}(y)) - K_\varepsilon(x, y, f_{2\varepsilon}(y))| \leq C|f_{1\varepsilon}(y) - f_{2\varepsilon}(y)|$$

$$\int_0^1 |\nabla K_\varepsilon(x, y, \theta f_{1\varepsilon}(y) + (1 - \theta)f_{2\varepsilon}(y))|d\theta \leq C|F_\varepsilon(y)||\ln\varepsilon|^b.$$

Setting this in (11) we obtain

$$|F_\varepsilon(y)| \leq C \int_0^x \sup_y |F_\varepsilon(y)||\ln\varepsilon|^b\phi_\varepsilon(x - y) + |x - y|^\alpha_1 - 1||_{L^1} dy + d_\varepsilon(x).$$

Applying Lemma 1 we obtain for $\alpha = -\tilde{\alpha} < 0$, where $\tilde{\alpha} > 0$

$$|F_\varepsilon(x)| \leq C \int_0^x \sup_y |F_\varepsilon(y)||\ln\varepsilon|^b + d_\varepsilon(x), m > \tilde{\alpha}.$$ 

Gronwall inequality yields

$$|F_\varepsilon(x)| \leq C d_\varepsilon(x) \exp(C|\ln\varepsilon|^b + m) \leq C\varepsilon^s, \forall s \in \mathbb{N}, x \in \mathbb{R}, \varepsilon < 1,$$

when $b + m < 1$, $m > \tilde{\alpha}$, i.e. $b + \tilde{\alpha} < 1$.

If $\alpha > 0$ by Lemma 1 we have

$$|F_\varepsilon(x)| \leq C \int_0^x \sup_y |F_\varepsilon(y)||\ln\varepsilon|^b dy + d_\varepsilon(x).$$

Gronwall inequality yields

$$|F_\varepsilon(x)| \leq C d_\varepsilon(x) \exp(C|\ln\varepsilon|^b) \leq C\varepsilon^s, \forall s \in \mathbb{N}, x \in \mathbb{R}, \varepsilon < \varepsilon_0,$$

when $b < 1$.

Consider $\beta^{th}$-derivative, $\beta \in \mathbb{N}_0$, $\beta \geq 1$, applied on $F_\varepsilon(x)$ in the equation (9) to obtain

$$|D^\beta F_\varepsilon(x)| \leq \int_0^x |\int_0^1 \nabla K_\varepsilon(x, y, \theta f_{1\varepsilon} + (1 - \theta)f_{2\varepsilon})|d\theta|$$

$$\sup_y |F_\varepsilon(y)||\phi_\varepsilon(x - y) + \partial_\varepsilon^\beta |x - y|^\alpha_1 - 1||_{L^1} dy + d_\varepsilon(x), 0 < \theta < 1.$$
By (10) we have
\[
|F_\varepsilon^\beta(x)| \leq C \int_0^x |\ln \varepsilon|^b \sup_y |F_\varepsilon(y)| \left\{ \begin{array}{ll}
C & \text{when } \alpha > \beta, \alpha > 0 \\
C|\ln \varepsilon|^m, & \text{when } \alpha > 0, \alpha - \beta < 0, m > \alpha - \beta \\
C|\ln \varepsilon|^m, & \alpha < 0, \alpha - \beta < 0, m > -\bar{\alpha} - \beta
\end{array} \right.
\]
dy + d_\varepsilon(x).

Using the null properties of $F_\varepsilon(x)$ from the first step of the induction we obtain
\[
|D^\beta F_\varepsilon(x)| \leq Cd_\varepsilon(x) + \left\{ \begin{array}{ll}
C & \text{when } \alpha > \beta, \alpha > 0 \\
C|\ln \varepsilon|^m, & \text{when } \alpha > 0, \alpha - \beta < 0, m > \alpha - \beta \\
C|\ln \varepsilon|^m, & \alpha < 0, \alpha - \beta < 0, m > -\bar{\alpha} - \beta
\end{array} \right.
\]
\forall s \in \mathbf{N}, \varepsilon < \varepsilon_0, x \in \mathbf{R}, \text{ under the condition}
\[
\left\{ \begin{array}{ll}
b < 1, & \alpha - \beta > 0, \alpha > 0 \\
b + m < 1, & m > \alpha - \beta, \alpha > 0, \alpha - \beta < 0 \\
b + m, & m > -\bar{\alpha} - \beta, \alpha < 0, \alpha - \beta < 0
\end{array} \right.
\]
which is included in the condition imposed at the first step
\[
\left\{ \begin{array}{ll}
b < 1, & \alpha > 0 \\
b < 1 - \bar{\alpha}, & \alpha < 0.
\end{array} \right.
\]

Then, $F_\varepsilon(x) \in \mathcal{N}(I)$. Follows $f_{1\varepsilon}(x) \approx f_{2\varepsilon}(x)$, the solution is unique.

Thus, there exists a unique solution in the space $[f_\varepsilon] \in (\mathcal{G}(I))^n$. □

**Remark 1.** The condition imposed on parameters gives a link between the singularity of the fractional integral and non-Lipschitz nonlinearity of the kernel $K$. The singularity of the fractional integral is reflected by the growth of the mollifiers. The singularity of the free term is reflected by $|\ln \varepsilon|$-boundedness which is necessary for moderateness in applying Gronwall inequality. When we consider $|\ln \varepsilon|$-boundedness for both $K$ and fractional part, the slower growth of mollifiers contributes to the moderateness.

5. **Application to nonlinear parabolic equations with strongly singular initial data**

Consider nonlinear parabolic equations with strongly singular initial data and non-Lipschitz nonlinearity for free term for ordinary nonlinear parabolic equations (cf. [4]), equations with nonlinear conservative term (cf. [5]) and parabolic equation with Schrödinger kernel (cf. [14]).

We shall prove the existence-uniqueness theorems in the Colombeau vector type spaces $\mathcal{G}^{1, \beta}_{C, (L^p, L^q)}([0, T), \mathbf{R}^n), 1 \leq p, q \leq \infty$, using the regularization for fractional part of the heat kernel by delta sequence with respect to the time variable $t$.

We shall consider:

1. Cauchy problem for nonlinear parabolic equation
\[
\partial_t u = \triangle u + g(u), \ t > 0, \ x \in \mathbf{R}^n, \ u(0, x) = \mu(x),
\]
where $g(u) \in L^\infty_{\text{loc}}([0, T), \mathbf{R}^n)$ and does not satisfy Lipschitz condition, $\mu(x) \in \mathcal{M}^k(\mathbf{R}^n) \subset \mathcal{D}'(\mathbf{R}^n), k \in \mathbf{Z}_+$, where $\mathcal{M}^k(\mathbf{R}^n)$ is the strong dual of Banach space $C^k_b(\mathbf{R}^n)$ of all $C^k(\mathbf{R}^n)$ functions with bounded derivatives up to the order $k$. For example, strongly singular initial data are given in the form of regularization by smooth mollifiers of sum
of derivatives of Dirac measures. We put some conditions on mollifiers regularizing the leading term of the initial data:

\[ \phi_{\varepsilon} \in C_0^\infty (\mathbb{R}^n), \quad \int_{\mathbb{R}^n} \phi_{\varepsilon} (x) dx = 1, \quad \lim_{\varepsilon \to 0} \phi_{\varepsilon} (x) = \delta (x) \]

in \( \mathcal{D}'(\mathbb{R}^n) \) where \( \phi_{\varepsilon} (x) = |ln\varepsilon|^{an} \phi (x \cdot |ln\varepsilon|), \varepsilon > 0, \) and \( \delta (x) \) stands for the Dirac measure massed at the point zero (cf. [13]):

\[ \lim_{\varepsilon \to 0} u_{0\varepsilon} (x) = u_0 (x) := \sum_{i=1}^{n} \sum_{j=0}^{k} a_{i,j} \delta^{(j)} (x - \xi_i), \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \]

2. Cauchy problem for nonlinear parabolic equation with conservative nonlinear term

\[ \partial_t u - \Delta u + \partial_x \cdot \vec{g} (u) = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \quad u(0, x) = u_0 (x) = D^k \psi (x), \]

where \( u = (u_1, \ldots, u_m), \vec{g} (u) = (g(u), \ldots, g_n (u)) \in L^\infty_{loc} ([0, T), \mathbb{R}^n), \partial_x \cdot \vec{g} (u) = \vec{g} (u) \cdot \nabla u = \sum_{j=1}^{n} g_{j} (x) \partial_{x_j} u, k \geq 0, \) where the initial data are the following:

\[ \mu (x) = |D|^k \psi, \quad \psi \in L^p (\mathbb{R}^n), \quad k > 0, \]

\[ 1 \leq p \leq \infty, \quad D = (-\Delta)^{1/2}, \quad \psi \in L^p (\mathbb{R}^n), \quad 1 \leq p \leq \infty. \]

We use the following regularization for the initial data:

\[ \mu_{\varepsilon} (x) = D^k \psi (x) \ast \phi_{\varepsilon} (x) = \psi (x) \ast D^k \phi_{\varepsilon} (x), \psi \in L^p (\mathbb{R}^n), \phi_{\varepsilon} (x) = |ln\varepsilon|^{an} \phi (x \cdot |ln\varepsilon|), \]

and then

\[ ||\mu_{\varepsilon} (\cdot)||_{L^p} \leq C ||D^k \phi_{\varepsilon} (\cdot)||_{L^1} \leq C |ln\varepsilon|^{n(a-1)+k}. \]

As the initial data we can consider the powers of delta distribution \( u_0 (x) = \delta^{(\beta)} (x), \beta \in (0, \infty), x \in \mathbb{R}^n. \) Without loss of generality suppose that \( u_0 (x) = \delta (x), \delta_\varepsilon (x) = |ln\varepsilon|^{an} \phi (x \cdot |ln\varepsilon|), \phi (x) \in C_0^\infty (\mathbb{R}^n), \int \phi (x) dx = 1, \phi (x) \geq 0. \]

3. Nonlinear parabolic equation with Schrödinger kernel, delta as the potential

delta as the initial data:

\[ (\partial_t - \Delta) u + V (x) u + g (u) = 0, \quad u(0, x) = u_0 (x), \quad x \in \mathbb{R}^n, \]

where \( V (x) \) and \( u_0 (x) \) are the singular distributions, say \( \delta \)-distributions, \( g(u) \in L^\infty_{loc} ([0, T), \mathbb{R}^n), \) is nonlinear, non-Lipschitz. The initial data and potential could be sum of derivatives and powers of \( \delta \)-distribution. Without loss of generality we shall consider the case when \( u_0 (x) = V (x) = \delta (x). \)

5.1. Nonlinear parabolic equation (12). Consider regularized equation (12)

\[ \partial_t u_{\varepsilon} (t, x) = \Delta u_{\varepsilon} (t, x) + g_{\varepsilon} (u_{\varepsilon} (t, x)), \quad u_{0\varepsilon} (x) = \delta_{\varepsilon} (x), \]

where

\[ \delta_{\varepsilon} (x) = |ln\varepsilon|^{an} \phi (x \cdot |ln\varepsilon|), \quad |\nabla g_{\varepsilon} (\theta u_{\varepsilon})| \leq C |ln\varepsilon|^{b}, \]

\[ \phi (x) \in C_0^\infty (\mathbb{R}^n), \phi \geq 0, \int \phi (x) dx = 1, \quad g(u) \text{ is regularized by cut-off to avoid non-Lipschitz nonlinearity and the heat semigroup is regularized as follows.} \]

We shall regularize heat kernel in \( L^1 \)-norm. We have

\[ E_{\varepsilon \varepsilon} (t, x - y) = E_n (t, x - y) \ast \phi_{\varepsilon} (t) = \int_{\mathbb{R}} E_n (t - \tau, x - y) \phi_{\varepsilon} (\tau) d\tau \]
where
\[ E_n(t - \tau, x - y) = (4\pi(t - \tau))^{-n/2} \exp \left(-\frac{|x - y|^2}{4(t - \tau)}\right). \]

Then,
\[
||E_n\varepsilon(t, x - \cdot)||_{L^1} = \int_{\mathbb{R}^n} \int_{\mathbb{R}} |(4\pi(t - \tau))^{-n/2}| \exp \left(-\frac{|x - y|^2}{4(4\pi(t - \tau))}\right) |\phi_\varepsilon(\tau)| d\tau dy,
\]

and
\[
\partial^2_x E_n\varepsilon(t, x - y) = \partial^2_x E_n(t, x - y) \ast \phi_\varepsilon(t) = \int_{\mathbb{R}} \partial^2_x E_n(t - \tau, x - y) \phi_\varepsilon(\tau) d\tau
\]
\[
\leq C \int_{\mathbb{R}} (4\pi(t - \tau))^{-n/2}(t - \tau)^{-\beta} \exp \left(-\frac{|x - y|^2}{4(4\pi(t - \tau))}\right) \phi_\varepsilon(\tau) d\tau.
\]

Since from [5] we have
\[
|t^{k/2+n/2(1-1/r)} \partial^2_x E_n(t, \cdot)||_{L^r} \leq \infty, \quad |\beta| \leq k, \quad 1 \leq r \leq \infty,
\]
by Fubini theorem we obtain
\[
||\partial^2_x E_n(t, x - \cdot)||_{L^1} \leq C \int_{\mathbb{R}} \int_{\mathbb{R}^n} |(t - \tau)^{-n/2-\beta}||\phi_\varepsilon(\tau)|| \exp \left(-\frac{|x - y|^2}{4(4\pi(t - \tau))}\right) dy d\tau,
\]
and by (17) we continue
\[
\leq C \int_{\mathbb{R}} (t - \tau)^{-\beta/2}||(t - \tau)^{\beta/2} \partial^2_x E_n(t - \tau, x - \cdot)||_{L^1} d\tau.
\]

Since $$||(t - \tau)^{\beta/2} \partial^2_x E_n(t - \tau, x - \cdot)||_{L^1} \leq C$$ we obtain
\[
||\partial^2_x E_n\varepsilon(t - \tau, x - \cdot)||_{L^1} \leq \left\{
\begin{array}{ll}
C & \text{when } \beta < 2 \\
C(\ln|\ln\varepsilon|)^m, & \text{when } \beta \geq 2, \quad m > \beta/2 - 1,
\end{array}
\right.
\]
where we used the (\ln|\ln\varepsilon|)-boundedness of the heat semigroup to accomplish the moderateness in corresponding Colombeau space.

**Theorem 2.** The regularized equation (15) to the nonlinear parabolic equation (12) where \(u_0(x) = \delta(x), \ g(u) \in L_\infty([0, T), \mathbb{R}^n)\) is nonlinear and non-Lipschitz has a unique solution in Colombeau vector type spaces \(\mathcal{G}_{C^1(L^p, L^q)}([0, T), \mathbb{R}^n)\) for those choices \(1 \leq p, q \leq \infty\) for which corresponding Colombeau space is an algebra with multiplication. For \(p = q = 2\) the space \(\mathcal{G}_{C^1(L^2, L^2)}([0, T), \mathbb{R}^n)\) is an algebra with multiplication for \(n \leq 3\).

**Proof.** Consider regularized integral form of the nonlinear parabolic equation (12)
\[
u_\varepsilon(t, x) = \int_{\mathbb{R}^n} E_{n\varepsilon}(t, x - y) u_0\varepsilon(y) dy + \int_0^t \int_{\mathbb{R}^n} E_{n\varepsilon}(t - \tau, x - y) g_\varepsilon(u_\varepsilon(\tau, y)) dy d\tau,
\]
\(t \in [0, T), \ x \in \mathbb{R}^n\). We have in \(L^p\)-norm
\[
||u_\varepsilon(t, \cdot)||_{L^p} \leq ||E_{n\varepsilon}(t, x - \cdot)||_{L^1} ||u_0\varepsilon(\cdot)||_{L^p}
\]
\[
+ \int_0^t ||E_{n\varepsilon}(t - \tau, x - \cdot)||_{L^1} ||\nabla g_\varepsilon(\theta u_\varepsilon)||_{L^\infty} ||u_\varepsilon(\tau, \cdot)||_{L^p} d\tau.
\]

Then, by (18)
\[
||u_\varepsilon(t, \cdot)||_{L^p} \leq C ||\ln\varepsilon||^{n(1-1/p)} + \int_0^t C ||\ln\varepsilon||^p ||u_\varepsilon(\tau, \cdot)||_{L^p} d\tau.
\]
Gronwall inequality yields 
\[ \|u_\varepsilon(t, \cdot)\|_{L^p} \leq C|\ln \varepsilon|^{n(1-1/p)} \exp (CT|\ln \varepsilon|^b) \leq C\varepsilon^{-N}, \]
\[ \exists N > 0, \ t \in [0, T), \ T > 0, \ x \in \mathbb{R}^n, \ b < 1. \] The same holds for the first derivative.

Consider \( \beta^{th} \)-derivative, \( \beta \geq 2, \)
\[
\partial^\beta_x u_\varepsilon(t, x) = \int_{\mathbb{R}^n} \partial^\beta_x E_{\varepsilon}(t, x-y)u_0\varepsilon(y)dy + \int_0^t \int_{\mathbb{R}^n} \partial^\beta_x E_{\varepsilon}(t-\tau, x-y)g_\varepsilon(u_\varepsilon(\tau, y))dyd\tau.
\]
We have,
\[
\|\partial^\beta_x u_\varepsilon(t, \cdot)\|_{L^p} \leq \|\partial^\beta_x E_{\varepsilon}(t, x-\cdot)\|_{L^1}\|u_0\varepsilon(\cdot)\|_{L^p} \\
+ \int_0^t \|\partial^\beta_x E_{\varepsilon}(t-\tau, x-\cdot)\|_{L^1}\|\nabla g_\varepsilon(\theta u_\varepsilon)(\cdot)\|_{L^\infty}\|u_\varepsilon(\tau, \cdot)\|_{L^p}d\tau.
\]
Then,
\[
\|\partial^\beta_x u_\varepsilon(t, \cdot)\|_{L^p} \leq C(\ln |\ln \varepsilon|)^m|\ln \varepsilon|^{n(1-1/p)} + \int_0^t C(\ln |\ln \varepsilon|)^m|\ln \varepsilon|^b|u_\varepsilon(\tau, \cdot)\|_{L^p}d\tau,
\]
where \( m > \beta/2 - 1. \) Using the moderateness of \( u_\varepsilon(t, x) \) from the first step we obtain
\[
\|\partial^\beta_x u_\varepsilon(t, \cdot)\|_{L^p} \leq C(\ln |\ln \varepsilon|)^m|\ln \varepsilon|^{n(1-1/p)} + (CT(\ln |\ln \varepsilon|)^m|\ln \varepsilon|^b\varepsilon^{-N}) \leq C\varepsilon^{-N},
\]
\[ \exists N > 0, \ t \in [0, T), \ x \in \mathbb{R}^n, \ \varepsilon < \varepsilon_0, \] where \( b < 1, \ m > \beta/2 - 1, \ \beta \geq 2. \)
Thus, \( u_\varepsilon(t, x) \in \mathcal{E}_{M,p}([0, T), \mathbb{R}^n). \)

Consider the uniqueness. If we suppose that \( W_\varepsilon(t, x) = u_{1\varepsilon}(t, x) - u_{2\varepsilon}(t, x) \) where \( u_{1\varepsilon}(t, x) \) and \( u_{2\varepsilon}(t, x) \) are two solutions to the equation (15), then, we should to solve the equation
\[
\partial_t W_\varepsilon(t, x) = \Delta W_\varepsilon(t, x) + w_\varepsilon(t, x)W_\varepsilon(t, x) + N_\varepsilon(t, x)
\]
\[
u_0\varepsilon(x) = N_0\varepsilon(x) \in \mathcal{N}_{p,q}(\mathbb{R}^n), \ N_\varepsilon(t, x) \in \mathcal{N}_{C^1,1(L^p,L^q)}([0, T), \mathbb{R}^n),
\]
\[
w_\varepsilon(t, x) = \int_0^t \nabla g_\varepsilon(\theta u_{1\varepsilon}(t, x) + (1-\theta)u_{2\varepsilon}(t, x))d\theta.
\]
In integral form we have
\[
W_\varepsilon(t, x) = \int_{\mathbb{R}^n} E_{\varepsilon}(t, x-y)N_\varepsilon(y)dy + \int_0^t \int_{\mathbb{R}^n} E_{\varepsilon}(t-\tau, x-y)w_\varepsilon(\tau, y)W_\varepsilon(\tau, y)dyd\tau
\]
\[ + \int_0^t \int_{\mathbb{R}^n} E_{\varepsilon}(t-\tau, x-y)N_\varepsilon(\tau, y)dyd\tau.
\]
In \( L^q \)-norm, where \( 1 \leq q \leq \infty \) we obtain
\[
||W_\varepsilon(t, \cdot)||_{L^q} \leq \|E_{\varepsilon}(t, x-\cdot)||_{L^1}\|N_\varepsilon(\cdot)||_{L^q} + \int_0^t \|E_{\varepsilon}(t-\tau, x-\cdot)||_{L^1}\|w_\varepsilon(\tau, \cdot)||_{L^\infty}d\tau
\]
\[ + \int_0^t \|E_{\varepsilon}(t-\tau, x-\cdot)||_{L^1}\|N_\varepsilon(\tau, \cdot)||_{L^q}d\tau.
\]
Then,
\[
||W_\varepsilon(t, \cdot)||_{L^q} \leq C\varepsilon^s + \int_0^t \|\ln \varepsilon|^b\||W_\varepsilon(\tau, \cdot)||_{L^q}d\tau + \int_0^t C\varepsilon^s d\tau.
\]
Employ Gronwall inequality to obtain
\[
||W_\varepsilon(t, \cdot)||_{L^q} \leq C\varepsilon^s \exp (CT|\ln \varepsilon|^b) \leq C\varepsilon^s,
\]
\[ \forall s \in \mathbb{N}^n, \ x \in \mathbb{R}^n, \ \varepsilon < \varepsilon_0, \] where \( b < 1. \) The same holds for the first derivative.
For $\beta^{th}$-derivative, $\beta \geq 2$, we obtain
\[
\frac{\partial^3}{\partial x^3} W_\varepsilon(t, x) = \int_{\mathbb{R}^n} \frac{\partial^3}{\partial x^3} E_{ne}(t, x - y) N_\varepsilon(y)dy + \int_0^t \int_{\mathbb{R}^n} \frac{\partial^3}{\partial x^3} E_{ne}(t - \tau, x - y) w_\varepsilon(\tau, y)dyd\tau.
\]

In $L^q$-norm we have
\[
\|\frac{\partial^3}{\partial x^3} W_\varepsilon(t, \cdot)\|_{L^q} \leq C \|\frac{\partial^3}{\partial x^3} E_{ne}(t, x - \cdot)\|_{L^1} ||N_\varepsilon(\cdot)||_{L^q} + \int_0^t \|\frac{\partial^3}{\partial x^3} E_{ne}(t - \tau, x - \cdot)\|_{L^1} ||\partial^2_x E_{ne}(t - \tau, x - \cdot)\|_{L^1} d\tau,
\]
for $m > \beta/2 - 1$. By the first step and (18)
\[
\|\frac{\partial^3}{\partial x^3} W_\varepsilon(t, \cdot)\|_{L^q} \leq C (\ln|\ln \varepsilon|)^m \varepsilon^s + \int_0^t C (\ln|\ln \varepsilon|)^m |\ln \varepsilon|^b \varepsilon^s d\tau + \int_0^t C (\ln|\ln \varepsilon|)^m \varepsilon^s d\tau
\]
\[
\leq C \varepsilon^s, \ \forall s \in \mathbb{N}, \ t \in [0, T), \ x \in \mathbb{R}^n, \ \varepsilon < \varepsilon_0, \ under \ the \ condition \ b < 1, \ where \ m > \beta/2 - 1.
\]

Finally, $W_\varepsilon(t, x) \in \mathcal{N}_{C^1, \beta}([0, T), \mathbb{R}^n)$, $1 \leq p, q \leq \infty$. Thus, the solution is unique in the Colombeau vector space $\mathcal{G}_{C^1, \beta}([0, T), \mathbb{R}^n)$ for those choices $1 \leq p, q \leq \infty$ for which corresponding Colombeau vector space is an algebra with multiplication. This is the consequence of the Sobolev imbedding theorems, cf. [1].

\[
\square
\]

5.2. Parabolic equation with nonlinear conservative term (13). The equation (13) has the following regularized integral form
\[
u_\varepsilon(t, x) = E_{ne}(t, x) * \mu_\varepsilon(x) + \int_0^t \nabla E_{ne}(t - \tau, \cdot) * \tilde{g}_\varepsilon(u_\varepsilon(\tau, \cdot))d\tau, \ \mu_\varepsilon(x) = \delta_\varepsilon(x),
\]
where the regularization for each term with subscript $\varepsilon$ except for the heat kernel, are given in (16).

We shall regularize the gradient of the heat kernel as follows:
\[
\nabla E_{ne}(t, x) = \nabla E_n(t, x) * \phi_\varepsilon(t).
\]
In $L^1$-norm, we have
\[
||\nabla E_{ne}(t - \tau, x - \cdot)||_{L^1} = \int_{\mathbb{R}^n} |\nabla E_n(t - \tau, x - y) \phi_\varepsilon(\tau)| d\tau dy
\]
\[
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(t - \tau)^{-n/2 - 1}\exp(-|x - y|^2/(4(t - \tau)))| \phi_\varepsilon(\tau)| d\tau dy.
\]
Then,
\[
||\frac{\partial^3}{\partial x^3} \nabla E_{ne}(t - \tau, x - \cdot)||_{L^1} = \int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} \frac{\partial^3}{\partial x^3} \nabla E_n(t - \tau, x - y) \phi_\varepsilon(\tau) d\tau| dy
\]
\[
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(t - \tau)^{-n/2 - 1 - \beta}| \exp(-|x - y|^2/(4(t - \tau)))| \phi_\varepsilon(\tau)| d\tau dy.
\]
Apply Fubini theorem to obtain
\[
||\frac{\partial^3}{\partial x^3} \nabla E_{ne}(t - \tau, x - \cdot)||_{L^1} \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(t - \tau)^{-n/2 - 1 - \beta}| \exp(-|x - y|^2/(4(t - \tau)))| d\tau dy.
\]
Then, we continue
\[
\leq C \int_{\mathbb{R}} |(t - \tau)^{-\beta/2 - 1/2}||(t - \tau)^{(\beta + 1)/2} \partial_\tau^\beta \nabla E_n(t - \tau, x - \cdot)||_{L^1} d\tau
\]
\[
\leq C \int_{\mathbb{R}} (t - \tau)^{-\beta/2 - 1/2} \phi_\varepsilon(\tau) d\tau.
\]

Apply Lemma 1 to obtain
\[
\|\partial_x^\beta \nabla E_n(t - \tau, x - \cdot)||_{L^1} \leq \begin{cases} 
C & \text{for } \beta < 1 \\
C(\ln|\ln\varepsilon|)^m & \text{for } m > (\beta - 1)/2, \beta \geq 1.
\end{cases}
\] (20)

The initial data are regularized by delta sequence, nonlinear term \( g(u) \in L^\infty_{\text{loc}}([0, T), \mathbb{R}^n) \) which is nonlinear and non-Lipschitz, it is regularized by cut-off and the fractional part of the heat kernel is regularized as described in Lemma 1. We have the following theorem.

**Theorem 3.** The regularized equation (19) have a unique solution \( [u_\varepsilon] \in \mathcal{G}_{C^0, (L^p, L^q)} ([0, T), \mathbb{R}^n) \) for those choices \( 1 \leq p, q \leq \infty \) for which the corresponding Colombeau space is an algebra with multiplication. In a limiting case, when \( \varepsilon \to 0 \), we obtain the solution to the equation (13).

**Proof.** Taking the \( L^p \)-norm, \( 1 \leq p \leq \infty \) of (19) we obtain
\[
\|u_\varepsilon(t, \cdot)||_{L^p} \leq \|E_n(t, \cdot)||_{L^1} \|\mu_\varepsilon(\cdot)||_{L^p}
\]
\[
+ \int_0^t \|\nabla E_n(t - \tau, x - \cdot)||_{L^1} \|\nabla g_\varepsilon(\theta u_\varepsilon)||_{L^\infty} \|u_\varepsilon(\tau, \cdot)||_{L^p} d\tau.
\]

Since \( \mu_\varepsilon(x) = \delta_\varepsilon(x) = |\ln\varepsilon|^{an}\phi(x \cdot |\ln\varepsilon|) \) and by \( (|\ln|\cdot|) \)-boundedness of gradient we have
\[
\|u_\varepsilon(t, \cdot)||_{L^p} \leq C|\ln\varepsilon|^{n(a-1/p)} + \int_0^t C|\ln|^{b}\|u_\varepsilon(\tau, \cdot)||_{L^p} d\tau.
\]

By Gronwall inequality
\[
\|u_\varepsilon(t, \cdot)||_{L^p} \leq C|\ln\varepsilon|^{n(a-1/p)} \exp (CT|\ln|^{b}) \leq C\varepsilon^{-N},
\]
\( \exists N > 0, t \in [0, T), x \in \mathbb{R}^n, b < 1, a > 0. \)

**Remark 2.** If we use \( (|\ln|\cdot|) \)-boundedness for \( |\nabla g_\varepsilon| \) and for the heat semigroup then, there is no condition on parameters. Slower growth of mollifiers contributes to the moderateness.

Suppose that \( \beta \in \mathbb{N}_0^0, \beta > 0 \). Then, we have from (19)
\[
\partial_\tau^\beta u_\varepsilon(t, x) = \partial_\tau^\beta E_n(t, x) * \mu_\varepsilon(x) + \int_0^t \int_{\mathbb{R}^n} \partial_\tau^\beta \nabla E_n(t - \tau, x - y) g_\varepsilon(u_\varepsilon(\tau, y)) dy d\tau.
\]

By (20)
\[
\|\partial_\tau^\beta u_\varepsilon(t, \cdot)||_{L^1} \leq C(\ln|\ln\varepsilon|)^m|\ln\varepsilon|^{n(1-1/p)} + \int_0^t C(\ln|\ln\varepsilon|)^m \|\nabla g_\varepsilon(\theta u_\varepsilon)||_{L^\infty} \|u_\varepsilon(\tau, \cdot)||_{L^p} d\tau.
\]

Using the moderateness of \( u_\varepsilon(t, x) \) to obtain
\[
\|\partial_\tau^\beta u_\varepsilon(t, \cdot)||_{L^p} \leq C(\ln|\ln\varepsilon|)^m|\ln\varepsilon|^{n(1-1/p)} + CT(\ln|\ln\varepsilon|)^m|\ln\varepsilon|^b \varepsilon^{-N} \leq C\varepsilon^{-N},
\]
\( \exists N > 0, t \in [0, T), x \in \mathbb{R}^n, \varepsilon < \varepsilon_0, m > (\beta - 1)/2 \) under the condition \( b < 1 \) what is included in the condition imposed at the first step.
Thus, we have the moderateness in the space \( u_\varepsilon(t, x) \in \mathcal{E}_{M,p}([0, T), \mathbb{R}^n) \).
Concerning the uniqueness we should to solve the equation
\[
\partial_t W_\varepsilon(t, x) = \Delta W_\varepsilon(t, x) + w_\varepsilon(t, x)W_\varepsilon(t, x) + N_\varepsilon(t, x)
\]
\[
W_{\varepsilon_0}(0, x) = N_{\varepsilon}(x) \in \mathcal{N}_{p,q}(\mathbb{R}^n)
\]
where \( 1 \leq p, q \leq \infty, N_\varepsilon(t, x) \in \mathcal{N}^{C^1}_{C^1,L_p,P,L_0}([0, T), \mathbb{R}^n) \), \( W_\varepsilon(t, x) = u_\varepsilon(t, x) - u_\varepsilon(t, x) \)
where \( u_\varepsilon(t, x) \) and \( u_\varepsilon(t, x) \) are two solutions to the regularized equation to (13) and \( w_\varepsilon(t, x) = \int_0^t \nabla g_\varepsilon(t, \theta u_\varepsilon(t, x) + (1 - \theta) u_\varepsilon(t, x))d\theta, \ 0 < \theta < 1 \). In integral form we have
\[
W_\varepsilon(t, x) = E_{v_\varepsilon}(t, x) * N_\varepsilon(x) + \int_0^t \int_{\mathbb{R}^n} \nabla E_{v_\varepsilon}(t - \tau, x - y)w_\varepsilon(\tau, y)W_\varepsilon(\tau, y)dyd\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} \nabla E_{v_\varepsilon}(t - \tau, x - y)N_\varepsilon(\tau, y)dyd\tau.
\]
In \( L^q \)-norm where \( 1 \leq q \leq \infty \), we have
\[
\|W_\varepsilon(t, \cdot)\|_{L^q} \leq \|E_{v_\varepsilon}(t, \cdot)\|_{L^1}\|N_\varepsilon(\cdot)\|_{L^q} + \int_0^t \|\nabla E_{v_\varepsilon}(t - \tau, x - \cdot)\|_{L^1}\|w_\varepsilon(\tau, \cdot)\|_{L^\infty}
\]
\[
\|W_\varepsilon(\tau, \cdot)\|_{L^q}d\tau + \int_0^t \|\nabla E_{v_\varepsilon}(t - \tau, x - \cdot)\|_{L^1}\|N_\varepsilon(\tau, \cdot)\|_{L^q}d\tau.
\]
Then, we obtain
\[
\|W_\varepsilon(t, \cdot)\|_{L^q} \leq C\varepsilon^s + \int_0^t C|\ln\varepsilon|^{b}\|W_\varepsilon(\tau, \cdot)\|_{L^q}d\tau.
\]
Employ Gronwall inequality to obtain
\[
\|W_\varepsilon(t, \cdot)\|_{L^q} \leq C\varepsilon^s \exp\left(CT|\ln\varepsilon|^{b}\right) \leq C\varepsilon^s,
\]
\( \forall s \in \mathbb{N}, \ t \in [0, T), \ x \in \mathbb{R}^n, \ \varepsilon < \varepsilon_0, \) where \( b < 1 \).
For \( \beta^{th} \)-derivative we have for \( \beta > 0 \)
\[
\partial_\beta^t W_\varepsilon(t, x) = \partial_\beta^t E_{v_\varepsilon}(t, x) * N_\varepsilon(x) + \int_0^t \int_{\mathbb{R}^n} \partial_\beta^t \nabla E_{v_\varepsilon}(t - \tau, x - y)w_\varepsilon(\tau, y)W_\varepsilon(\tau, y)dyd\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} \partial_\beta^t \nabla E_{v_\varepsilon}(t - \tau, x - y)N_\varepsilon(\tau, y)dyd\tau.
\]
In \( L^q \)-norm we have
\[
\|\partial_\beta^t W_\varepsilon(t, \cdot)\|_{L^q} \leq \|\partial_\beta^t E_{v_\varepsilon}(t, \cdot)\|_{L^1}\|N_\varepsilon(\cdot)\|_{L^q} + \int_0^t \|\partial_\beta^t \nabla E_{v_\varepsilon}(t - \tau, x - \cdot)\|_{L^1}
\]
\[
\|w_\varepsilon(\tau, \cdot)\|_{L^q} = \|W_\varepsilon(\tau, \cdot)\|_{L^q}d\tau + \int_0^t \|\partial_\beta^t \nabla E_{v_\varepsilon}(t - \tau, x - \cdot)\|_{L^1}\|N_\varepsilon(\tau, \cdot)\|_{L^q}d\tau,
\]
and for \( m > (\beta - 1)/2 \) due to null property of \( W_\varepsilon(t, x) \) obtained at the first step of induction we obtain
\[
\|\partial_\beta^t W_\varepsilon(\tau, \cdot)\|_{L^q} \leq C(\ln|\ln\varepsilon|)^m\varepsilon^s + \int_0^t C(\ln|\ln\varepsilon|)^m|\ln\varepsilon|^{b}\varepsilon^s d\tau + \int_0^t C(\ln|\ln\varepsilon|)^m\varepsilon^s d\tau
\]
\[
\leq C\varepsilon^s, \ \forall s \in \mathbb{N}, \ t \in [0, T), \ x \in \mathbb{R}^n, \ \varepsilon < \varepsilon_0, \) under the condition imposed at the first step, \( b < 1 \).
Thus, \( W_\varepsilon(t, x) \in \mathcal{N}^{C^1}_{C^1,L_p,P,L_0}([0, T), \mathbb{R}^n) \), \( 1 \leq p, q \leq \infty \). Follows, there exists the unique solution to the equation (13) in the space \( \mathcal{G}^{C^1}_{C^1,L_p,P,L_0}([0, T), \mathbb{R}^n) \), \( 1 \leq p, q \leq \infty \), for those choices of \( p, q \) for which corresponding Colombeau space is an algebra with
multiplication. If \( p = q = 2 \) due to the Sobolev imbedding theorem for \( n \leq 3 \), 
\( L^2(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n) \) and the space \( \mathcal{G}_{C^1(L^2(L^2))}([0, T), \mathbb{R}^n) \) is an algebra with multiplication (cf. [14]). □

5.3. **Nonlinear parabolic equation with Schrödinger kernel (14).** Consider nonlinear parabolic equation with Schrödinger kernel (14) where \( u_0(x) = \delta(x), V(x) = \delta(x), g(u) \in L^2_{\text{loc}}([0, T), \mathbb{R}^n) \) is nonlinear, non-Lipschitz and it is regularized as in [16] to obtain
\[
\| \nabla g_\varepsilon(\theta u_\varepsilon) \| \leq C|\ln\varepsilon|^b, \quad (\text{resp. } (\ln|\ln\varepsilon|)).
\]

Initial data and potential are regularized as follows:
\[
u_{0\varepsilon}(x) = |\ln\varepsilon|^{cn}\phi(x \cdot |\ln\varepsilon|), \quad V_\varepsilon(x) = |\ln\varepsilon|^{cn}\phi(x \cdot |\ln\varepsilon|),
\]
and the heat semigroup has the regularization described in Section 5.1.

We have in integral form
\[
u_\varepsilon(t, x) = E_{n\varepsilon}(t, x) * \mu_\varepsilon(x) + \int_0^t \int_{\mathbb{R}^n} E_{n\varepsilon}(t - \tau, x - y)V_\varepsilon(y)u_\varepsilon(\tau, y)dyd\tau
\]
\[+(\int_0^t \int_{\mathbb{R}^n} E_{n\varepsilon}(t - \tau, x - y)g_\varepsilon(x \cdot |\ln\varepsilon|)dyd\tau].
\]

Then, in \( L^p \)-norm, \( 1 \leq p \leq \infty \), we have
\[
\|u_\varepsilon(t, \cdot)\|_{L^p} \leq \|E_{n\varepsilon}(t, \cdot)\|_{L^1}||\mu_\varepsilon(\cdot)||_{L^p} + \int_0^t \|E_{n\varepsilon}(t - \tau, \cdot)\|_{L^1}\|V_\varepsilon(\cdot)\|_{L^\infty}\|u_\varepsilon(\tau, \cdot)\|_{L^p}d\tau \\
+ \int_0^t \|E_{n\varepsilon}(t - \tau, x - \cdot)\|_{L^1}\|\nabla g_\varepsilon(\theta u_\varepsilon)\|_{L^\infty}\|u_\varepsilon(\tau, \cdot)\|_{L^p}d\tau.
\]

By regularization (18) given for heat kernel we have for \( \beta < 1 \)
\[
\|u_\varepsilon(t, \cdot)\|_{L^p} \leq C|\ln\varepsilon|^{\alpha(1 - 1/p)} + \int_0^t C|\ln\varepsilon|^{cn}\|u_\varepsilon(\tau, \cdot)\|_{L^p}d\tau \\
+ \int_0^t C|\ln\varepsilon|^b\|u_\varepsilon(\tau, \cdot)\|_{L^p}d\tau.
\]

Employ Gronwall inequality to obtain
\[
\|u_\varepsilon(t, \cdot)\|_{L^p} \leq C|\ln\varepsilon|^{\alpha(1 - 1/p)} \exp(Ct(|\ln\varepsilon|^{cn} + |\ln\varepsilon|^b)) \leq C\varepsilon^{-N},
\]
\( \exists N > 0, \varepsilon < \varepsilon_0, t \in [0, T], x \in \mathbb{R}^n, \ cn < 1, \ b < 1. \) The same holds for the first derivative.

**Remark 3.** The condition \( \begin{cases} \text{cn < 1} \\ \text{b < 1} \end{cases} \) gives the link between the singularity of the heat semigroup, nonlinearity of \( g(u) \) and the singularity of the potential. Singularity of the initial data is given by \( n(1 - 1/p) > 0 \) what holds and it is necessary that this part is logarithmically bounded.

Consider \( \beta^{th} \)-derivative, \( \beta \geq 2 \). We have in (21)
\[
\partial^\beta_x u_\varepsilon(t, x) = \partial^\beta_x E_{n\varepsilon}(t, x) * \mu_\varepsilon(x) + \int_0^t \int_{\mathbb{R}^n} \partial^\beta_x E_{n\varepsilon}(t - \tau, x - y)V_\varepsilon(y)u_\varepsilon(\tau, y)dyd\tau \\
+ \int_0^t \int_{\mathbb{R}^n} \partial^\beta_x E_{n\varepsilon}(t - \tau, x - y)g_\varepsilon(x \cdot |\ln\varepsilon|)dyd\tau.
\]
In $L^p$-norm, where $1 \leq p \leq \infty$, we obtain
\[
\|\partial^2_x u_\varepsilon(t, \cdot)\|_{L^p} \leq \|\partial^2_x E_{n\varepsilon}(t, \cdot)\|_{L^1}\|\mu_\varepsilon(\cdot)\|_{L^1} + \int_0^t \|\partial^2_x E_{n\varepsilon}(t - \tau, x - \cdot)\|_{L^1} \|V_\varepsilon(\cdot)\|_{L^\infty} + \int_0^t \|\partial^2_x E_{n\varepsilon}(t - \tau, x - \cdot)\|_{L^1} \|\nabla g_\varepsilon(\theta u_\varepsilon)\|_{L^\infty} \|u_\varepsilon(\tau, \cdot)\|_{L^p} d\tau,
\]
and by the regularization we have for $m > \beta/2 - 1$
\[
\|\partial^2_x u_\varepsilon(t, \cdot)\|_{L^p} \leq C (\ln |\ln \varepsilon|)^m |\ln \varepsilon|^{n(1 - 1/p)} + \int_0^t C (\ln |\ln \varepsilon|)^b |\ln \varepsilon|^{m} \varepsilon^{-N} d\tau + \int_0^t C (\ln |\ln \varepsilon|)^m |\ln \varepsilon|^{b} \varepsilon^{-N} d\tau.
\]
We used the moderateness of $u_\varepsilon(t, x)$ obtained at the first step. Thus,
\[
\|\partial^2_x (t, \cdot)\|_{L^p} \leq C \varepsilon^{-N}, \exists N > 0, t \in [0, T), x \in \mathbb{R}^n, \varepsilon < \varepsilon_0,
\]
under the conditions $b < 1, cn < 1$, imposed at the first step, and $m > \beta/2 - 1$.

Concerning the uniqueness we should to solve the equation
\[
\partial_t W_\varepsilon(t, x) = \Delta W_\varepsilon(t, x) + V_\varepsilon(x) W_\varepsilon(t, x) + w_\varepsilon(t, x) W_\varepsilon(t, x) + N_\varepsilon(t, x)
\]
where $N_\varepsilon(t, x) \in N_{C^1([0, T), \mathbb{R}^n)}$, $w_\varepsilon(t, x) = u_{1\varepsilon}(t, x) - u_{2\varepsilon}(t, x)$ where $u_{1\varepsilon}(t, x)$ and $u_{2\varepsilon}(t, x)$ are two solutions to the regularization for the equation (14), $w_\varepsilon(t, x) = \int_0^1 \nabla g_\varepsilon(\theta u_{1\varepsilon}(t, x)) + (1 - \theta) u_{2\varepsilon}(t, x)) d\theta$.

In integral form we have
\[
W_\varepsilon(t, x) = E_{n\varepsilon}(t, x) * N_{0\varepsilon}(x) + \int_0^t \int_{\mathbb{R}^n} E_{n\varepsilon}(t - \tau, x - y) V_\varepsilon(y) W_\varepsilon(\tau, y) dy d\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} E_{n\varepsilon}(t - \tau, x - y) g_\varepsilon(u_\varepsilon(\tau, y)) dy d\tau + \int_0^t \int_{\mathbb{R}^n} E_{n\varepsilon}(t - \tau, x - y) N_\varepsilon(\tau, y) dy d\tau.
\]
In $L^q$-norm, $1 \leq q \leq \infty$, we obtain
\[
\|W_\varepsilon(t, \cdot)\|_{L^q} \leq C \|E_{n\varepsilon}(t, \cdot)\|_{L^1} |\ln \varepsilon(\cdot)|_{L^q} + \int_0^t \|E_{n\varepsilon}(t - \tau, x - \cdot)\|_{L^1} \|V_\varepsilon(\cdot)\|_{L^\infty} d\tau + \int_0^t \|E_{n\varepsilon}(t - \tau, x - \cdot)\|_{L^1} \|w_\varepsilon(\tau, \cdot)\|_{L^\infty} d\tau + \int_0^t \|E_{n\varepsilon}(t - \tau, x - \cdot)\|_{L^1} \|N_\varepsilon(\tau, \cdot)\|_{L^q} d\tau.
\]
Employ the regularizations to obtain
\[
\|W_\varepsilon(t, \cdot)\|_{L^q} \leq C \varepsilon^s + \int_0^t C |\ln \varepsilon|^{cn} \|W_\varepsilon(\tau, \cdot)\|_{L^q} d\tau + \int_0^t C |\ln \varepsilon|^{b} \|W_\varepsilon(\tau, \cdot)\|_{L^q} d\tau.
\]
By Gronwall inequality
\[
\|W_\varepsilon(t, \cdot)\|_{L^q} \leq C \varepsilon^s \exp (CT(\ln \varepsilon|^{cn} + |\ln \varepsilon|^{b})) \leq C \varepsilon^s,
\]
$\forall s \in \mathbb{N}, x \in \mathbb{R}^n, t \in [0, T), \varepsilon < \varepsilon_0$, where the condition $cn < 1, b < 1$ is imposed at the first step in the proof of moderateness. The same holds for the first derivative due to (18).
For $\beta^{th}$-derivative, $\beta \geq 2$, we have in (22)

$$
\partial^\beta_x W_\varepsilon(t, x) = \partial^\beta_x E_{n\varepsilon}(t, x) * N_{0\varepsilon}(x) + \int_0^t \int_{\mathbb{R}^n} \partial^\beta_x E_{n\varepsilon}(t - \tau, x - y)V_\varepsilon(y)W_\varepsilon(\tau, y)dyd\tau \\
+ \int_0^t \int_{\mathbb{R}^n} \partial^\beta_x E_{n\varepsilon}(t - \tau, x - y)w_\varepsilon(\tau, y)W_\varepsilon(\tau, y)dyd\tau + \int_0^t \int_{\mathbb{R}^n} \partial^\beta_x E_{n\varepsilon}(\tau, y)N_\varepsilon(\tau, y)dyd\tau.
$$

Then, in $L^q$-norm, $1 \leq q \leq \infty$, we have

$$
||\partial^\beta_x W_\varepsilon(t, \cdot)||_{L^q} \leq ||\partial^\beta_x E_{n\varepsilon}(t, \cdot)||_{L^q} ||N_{0\varepsilon}(\cdot)||_{L^q} + \int_0^t ||\partial^\beta_x E_{n\varepsilon}(t - \tau, \cdot)||_{L^q} ||w_\varepsilon(\tau, \cdot)||_{L^p} ||W_\varepsilon(\tau, \cdot)||_{L^q} d\tau
\\
+ \int_0^t ||\partial^\beta_x E_{n\varepsilon}(\tau, \cdot)||_{L^q} ||N_\varepsilon(\tau, \cdot)||_{L^q} d\tau.
$$

Setting the regularization (18) and due to the null property of $W_\varepsilon(t, x)$ we obtain

$$
||\partial^\beta_x W_\varepsilon(t, \cdot)||_{L^q} \leq C(ln|ln\varepsilon|)^m \varepsilon^s + \int_0^t C(ln|ln\varepsilon|)^m |ln\varepsilon|^{cn\varepsilon} \varepsilon^s d\tau
\\
+ \int_0^t C(ln|ln\varepsilon|)^m |ln\varepsilon|^{b\varepsilon} \varepsilon^s d\tau \leq C\varepsilon^s,
$$

$\forall s \in \mathbb{N}$, $\varepsilon < \varepsilon_0$, $t \in [0, T)$, $x \in \mathbb{R}^n$, $m > \beta/2 - 1$, under the condition $b < 1$, $cn < 1$, imposed at the first step of induction.

Thus, $W_\varepsilon(t, x) \in N_{C^1([0, T), \mathbb{R}^n)}$, $1 \leq p, q \leq \infty$.

Follows, the solution is unique in the spaces $\mathcal{G}_{C^1([0, T), \mathbb{R}^n)}$, $1 \leq p, q \leq \infty$, for those choices of $p, q$ for which the corresponding Colombeau space is an algebra with multiplication. That is the consequence of Sobolev imbedding theorems. □

**Remark 4.** We can remain the heat semigroup to stay $|ln\varepsilon|$-bounded and set the potential and $|\nabla g|_x$ to be $(ln|ln\varepsilon|)$-bounded to obtain existence-uniqueness theorems in corresponding Colombeau algebras with appropriate conditions on parameters what leads to the balance between the singularities of the initial data, heat semigroup, potential and the nonlinearity of $g$.

6. Application to linear Schrödinger equation with singular potential and initial data

Consider the linear Schrödinger equation with strongly singular potential and initial data

$$
\frac{1}{i} \partial_t u(t, x) = (\Delta - V(x))u(t, x), \ u(0, x) = u_0(x) = \delta(x), \ V(x) = \delta(x).
$$

(23)

The following regularization for delta distribution will be used:

$$
u_{0\varepsilon}(x) = |ln\varepsilon|^m \phi(x \cdot |ln\varepsilon|), \ V_\varepsilon(x) = |ln\varepsilon|^m \phi(x \cdot |ln\varepsilon|), \ (\text{resp. using } (ln|ln\varepsilon|)),
$$

where $\phi(x) \in C^\infty_0(\mathbb{R}^n)$, $\phi(x) \geq 0$, $\int \phi(x)dx = 1$.

We shall give the regularization for Schrödinger semigroup with respect to $t$ by delta sequence to handle the existence-uniqueness result in the vector space $\mathcal{G}_{C^1([0, T], \mathbb{R}^n)}$.
We apply this to obtain norm in $L$, where the injection is continuous and each space is dense in succeeding one. Then, $||S_{nε}(t, x)|| \leq \sup \int_{0}^{t} S_{n}(t, y) \phi_{ε}(y) dy \leq C$, since $\sup_{x} |exp(i|x|^{2}/(4t)|) \leq 1$. This is the fractional derivative of $δ$-sequence and by Lemma 1 it follows

$$\sup_{x} |S_{nε}(t, x)| \leq \begin{cases} C & \text{for } n < 2 \\ C(ln|ln|ε|)^{m} & \text{for } m > n/2 - 1, \ n \geq 2. \end{cases} \tag{24}$$

Consider the $β^{th}$-derivative, $β \in N_{0}^{0}$, of Schrödinger semigroup. We have $∂_{x}^{β} S_{nε}(t, x) = ∂_{x}^{β} S_{n}(t, x) * φ_{ε}(t)$. Since $\sup_{x} |∂_{x}^{β} S_{nε}(t, x)| \leq C(t - τ)^{-n/2 - β}$ we obtain

$$\sup_{x} |∂_{x}^{β} S_{nε}(t, x)| \leq C \int_{0}^{t} (t - τ)^{-n/2 - β} φ_{ε}(τ) dy. \tag{25}$$

By Lemma 1

$$\sup_{x} |∂_{x}^{β} S_{nε}(t, x)| \leq \begin{cases} C & \text{for } n = 1, \ β = 0 \\ C(ln|ln|ε|)^{m} & \text{for } m > β + n/2 - 1, \ β \geq 0, \ n \geq 2. \end{cases} \tag{26}$$

We indicate interpolation spaces for noninteger $m$, from [18]. Suppose that $X, Y$ are two Hilbert space, $X \subset Y$, $X$ is dense in $Y$ and the injection is continuous. The interpolation between the spaces provides a family of Hilbert spaces $[X, Y]_{θ}$, $0 ≤ θ ≤ 1$, with the following properties: $[X, Y]_{0} = X$, $[X, Y]_{1} = Y$ and $X \subset [X, Y]_{θ} \subset Y$, where the injection is continuous and each space is dense in succeeding one. Then,

$$||f||_{[X, Y]_{θ}} \leq c(θ)||f||_{X}^{1-θ}||f||_{Y}^{θ}, \ \forall f \in X, \ ∀θ, \ 1 ≤ θ ≤ 1. \tag{27}$$

We apply this to obtain norm in $L^{p/(p+1)}$-space. Since $p/(p+1) = 1 - 1/p$, then this space is settled between $L^{2}$ and $L^{1}$. We have $L^{2} \subset [L^{p}, L^{1}]_{θ} \subset L^{1}, \ 1 ≤ θ ≤ 1$. Then,

$$||u_{0ε}(\cdot)||_{L^{p/(p+1)}} \leq c(θ)||u_{0ε}(\cdot)||_{L^{2}}^{1-θ}||u_{0ε}(\cdot)||_{L^{1}}^{θ}, \ 0 ≤ θ ≤ 1,\tag{28}$$

and

$$||u_{0ε}(\cdot)||_{L^{p/(p+1)}} \leq c(θ)||ln|ε|^{n(a-1)(1-θ)+n(a-1/2)θ}, \ 1 ≤ θ ≤ 1.\tag{29}$$

Thus,

$$||u_{0ε}(\cdot)||_{L^{p/(p+1)}} \leq c(θ)||ln|ε|^{n(a-1/2)}, \ 0 ≤ θ ≤ 1. \tag{30}$$

Now we can prove the following theorem.

**Theorem 4.** Regularized equation to Schrödinger equation (23) with strongly singular initial data and potential where the regularizations for potential and initial data are given by (16) and the regularization for Schrödinger semigroup is described above, has a unique solution in the space $G_{C^{1}([0, T), R^{n})}^{p, q}((0, T), R^{n}), \ 1 ≤ p, q ≤ ∞$, for those choices $p, q$ for which corresponding Colombeau vector space is an algebra with multiplication (for example, for $p = q = 2, n ≤ 3$, this holds).
**Proof.** The regularized equation for (23) has the following integral form
\[
u_ε(t, x) = S_{ne}(t, x) * u_{0ε}(x) + \int_0^t \int_{\mathbb{R}^n} S_{ne}(t - \tau, x - y)V_ε(y)u_ε(\tau, y)dydτ.\]
(27)

In $L^p$-norm we have
\[
\|u_ε(t, \cdot)\|_{L^p} \leq \|S_{ne}(t, \cdot)\|_{L^\infty}\|u_{0ε}(\cdot)\|_{L^{p/(p+1)}}
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} \|S_{ne}(t - \tau, x - \cdot)\|_{L^\infty}\|V_ε(\cdot)\|_{L^1}\|u_ε(\tau, \cdot)\|_{L^p}dτ.
\]
Since $\|V_ε(\cdot)\|_{L^1} \leq C|lnε|^n(e-1)$ and by (26) and (24) we obtain
\[
\|u_ε(t, \cdot)\|_{L^p} \leq \left\{ \begin{array}{ll} \frac{C}{C(ln|lnε|)^m}, & n > 2, m > n/2 - 1 \|lnε\|^{n(a-1/θ/2)} \\
\frac{C}{C(ln|lnε|)^m}, & n > 2, m > n/2 - 1 \|lnε\|^{n(e-1)}\|u_ε(τ, \cdot)\|_{L^p}dτ.
\end{array} \right.
\]
By Gronwall inequality
\[
\|u_ε(t, \cdot)\|_{L^p} \leq \left\{ \begin{array}{ll} \frac{C}{C(ln|lnε|)^m}, & n > 2, m > n/2 - 1 \|lnε\|^{n(e-1)}\|u_ε(τ, \cdot)\|_{L^p}dτ.
\end{array} \right.
\]
Thus,
\[
\|u_ε(t, \cdot)\|_{L^p} \leq Cε^{-N}, \exists N > 0, x \in \mathbb{R}^n, t \in [0, T], ε < ε_0, 1 < p < \infty, c < 1 + 1/n.
\]
Consider $β^{th}$-derivative, $β \in \mathbb{N}^n$, $β \geq 1$, of (27)
\[
\partial^β_xu_ε(t, x) = \partial^β_xS_{ne}(t, x) * u_{0ε}(x) + \int_0^t \int_{\mathbb{R}^n} \partial^β_xE_{ne}(t - \tau, x - y)V_ε(y)u_ε(\tau, y)dydτ.
\]
Then, we have
\[
\|\partial^β_xu_ε(t, \cdot)\|_{L^p} \leq \|\partial^β_xS_{ne}(t, \cdot)\|_{L^\infty}\|u_{0ε}(\cdot)\|_{L^{p/(p+1)}}
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} \|\partial^β_xS_{ne}(t - \tau, x - \cdot)\|_{L^\infty}\|V_ε(\cdot)\|_{L^1}\|u_ε(\tau, \cdot)\|_{L^p}dτ.
\]
Setting the regularization we obtain
\[
\|\partial^β_xu_ε(t, \cdot)\|_{L^p} \leq \left\{ \begin{array}{ll} \frac{C}{C(ln|lnε|)^m}, & m > n/2 + β - 1, β \geq 0, n \geq 2 \|lnε\|^{n(a-1/θ/2)} \\
\frac{C}{C(ln|lnε|)^m}, & m > n/2 + β - 1, β \geq 0, n \geq 2 \|lnε\|^{n(e-1)}\|u_ε(τ, \cdot)\|_{L^p}dτ.
\end{array} \right.
\]
The moderateness of $u_ε(t, x)$ from the first step yields
\[
\|\partial^β_xu_ε(t, \cdot)\|_{L^p} \leq \left\{ \begin{array}{ll} \frac{C}{C(ln|lnε|)^m}, & m > n/2 + β - 1, β \geq 0, n \geq 2 \|lnε\|^{n(a-1/θ/2)} \\
\frac{C}{C(ln|lnε|)^m}, & m > n/2 + β - 1, β \geq 0, n \geq 2 \|lnε\|^{n(e-1)}\|u_ε(τ, \cdot)\|_{L^p}dτ.
\end{array} \right.
\]
\[
\exists N > 0, x \in \mathbb{R}^n, t \in [0, T], 1 < θ < 1, c < 1 + 1/n. Thus, u_ε \in \mathbb{E}_C^{1, L^p}([0, T), \mathbb{R}^n), 1 \leq p \leq \infty.
\]
Concerning the uniqueness we should to solve the equation

\[
\frac{1}{t} \partial_t W_\varepsilon(t, x) = (\triangle - V_\varepsilon(x)) W_\varepsilon(t, x) + N_\varepsilon(t, x)
\]

\(W_\varepsilon(0, x) = N_{0 \varepsilon}(x) \in \mathcal{N}_{p,q}(\mathbb{R}^n), \quad N_\varepsilon(t, x) \in \mathcal{N}_{C^1(L_p, L_q)}([0, T), \mathbb{R}^n), \quad 1 \leq p, q \leq \infty, \)

\(W_\varepsilon(t, x) = u_{1 \varepsilon}(t, x) - u_{2 \varepsilon}(t, x)\) where \(u_{1 \varepsilon}(t, x)\) and \(u_{2 \varepsilon}(t, x)\) are two regularized solutions to the equation (23). In integral form we have

\[
W_\varepsilon(t, x) = S_{n \varepsilon}(t, x) * N_{0 \varepsilon}(x) + \int_0^t \int_{\mathbb{R}^n} S_{n \varepsilon}(t, x - y)V_\varepsilon(y) W_\varepsilon(\tau, y) dy d\tau
\]

\[+ \int_0^t \int_{\mathbb{R}^n} S_{n \varepsilon}(t - \tau, x - y) N_\varepsilon(\tau, y) dy d\tau.
\]

Then,

\[
||W_\varepsilon(t, \cdot)||_{L^q} \leq ||S_{n \varepsilon}(t, \cdot)||_{L^\infty} ||N_{0 \varepsilon}(\cdot)||_{L^{q/(q+1)}} + \int_0^t \int_{\mathbb{R}^n} ||S_{n \varepsilon}(t - \tau, x - \cdot)||_{L^\infty} ||V_\varepsilon(\cdot)||_{L^1} d\tau.
\]

Setting the regularization to obtain

\[
||W_\varepsilon(t, \cdot)||_{L^q} \leq \left\{ \begin{array}{ll}
C, & n < 2 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 - 1, \quad n \geq 2 \varepsilon^s
\end{array} \right.
\]

\[+ \int_0^t \left\{ \begin{array}{ll}
C, & n < 2 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 - 1, \quad n \geq 2 \varepsilon^s
\end{array} \right. |\ln\varepsilon|^{n(c-1)} ||W_\varepsilon(\tau, \cdot)||_{L^q} d\tau.
\]

Gronwall inequality yields

\[
||W_\varepsilon(t, \cdot)||_{L^q} \leq \left\{ \begin{array}{ll}
C, & n < 2 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 - 1, \quad n \geq 2 \varepsilon^s
\end{array} \right.
\]

\[
\exp(C(T - \varepsilon)) \left\{ \begin{array}{ll}
C, & n < 2 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 - 1, \quad n \geq 2 \varepsilon^s
\end{array} \right. |\ln\varepsilon|^{n(c-1)} \leq C\varepsilon^s
\]

\(\forall s \in \mathbb{N}, \quad x \in \mathbb{R}^n, \quad t \in [0, T], \quad \varepsilon < \varepsilon_0, \quad c < 1 + 1/n.
\)

For the \(\beta^{th}\)-derivative, \(\beta \in \mathbb{N}_0, \beta \geq 1,\) we have

\[
\partial_x^\beta u_\varepsilon(t, x) = \partial_x^\beta S_{n \varepsilon}(t, x) * N_{0 \varepsilon}(x) + \int_0^t \int_{\mathbb{R}^n} \partial_x^\beta S_{n \varepsilon}(t - \tau, x - y)V_\varepsilon(y) u_\varepsilon(\tau, y) dy d\tau
\]

\[+ \int_0^t \int_{\mathbb{R}^n} \partial_x^\beta S_{n \varepsilon}(t - \tau, x - y) N_\varepsilon(\tau, y) dy d\tau,
\]

and

\[
||\partial_x^\beta W_\varepsilon(t, \cdot)||_{L^q} \leq ||\partial_x^\beta S_{n \varepsilon}(t, \cdot)||_{L^\infty} ||N_{0 \varepsilon}(\cdot)||_{L^{q/(q+1)}} + \int_0^t ||\partial_x^\beta S_{n \varepsilon}(t - \tau, x - \cdot)||_{L^\infty} d\tau
\]

\[||V_\varepsilon(\cdot)||_{L^1} ||W_\varepsilon(\tau, \cdot)||_{L^q} d\tau + \int_0^t ||\partial_x^\beta S_{n \varepsilon}(t - \tau, x - \cdot)||_{L^\infty} ||N_\varepsilon(\tau, \cdot)||_{L^{q/(q+1)}} d\tau.
\]

By (25) we have

\[
||\partial_x^\beta W_\varepsilon(t, \cdot)||_{L^q} \leq \left\{ \begin{array}{ll}
C, & \beta = 0, \quad n = 1 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 + \beta - 1, \quad \beta \geq 0, \quad n \geq 2 \varepsilon^s
\end{array} \right.
\]

\[+ \int_0^t \left\{ \begin{array}{ll}
C, & \beta = 0, \quad n = 1 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 + \beta - 1, \quad \beta \geq 0, \quad n \geq 2 \varepsilon^s
\end{array} \right. |\ln\varepsilon|^{n(c-1)} ||W_\varepsilon(\tau, \cdot)||_{L^q} d\tau.
\]
The first step of the induction yields

\[ ||\partial^2_x W_\varepsilon(t,.)||_{L^q} \leq \begin{cases} 
C, & \beta = 0, n = 1 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 + \beta - 1, \beta \geq 0, n \geq 2 \varepsilon^s
\end{cases} \]

\[ + (CT \begin{cases} 
C, & \beta = 0, n = 1 \\
C(\ln|\ln\varepsilon|)^m, & m > n/2 + \beta - 1, \beta \geq 0, n \geq 2 \varepsilon^s|\ln\varepsilon|^{n(c-1)} \end{cases} \leq C\varepsilon^s \]

\[ \forall s \in \mathbb{N}, x \in \mathbb{R}^n, t \in [0,T), \varepsilon < \varepsilon_0, \text{under the condition } c < 1 + 1/n \text{ where we supposed the } (\ln|\ln\varepsilon|)-\text{boundedness of the heat semigroup.} \]

**Remark 5.** Another possibility is to set (\ln|\ln\varepsilon|)-boundedness of potential \( V_\varepsilon(x) \) and (|\ln\varepsilon|)-boundedness of Schrödinger semigroup under the appropriate condition on the growth of the mollifier, or put the both to be (\ln|\ln\varepsilon|)-bounded. Everything leads to the same conclusion: existence-uniqueness results in Colombeau vector type spaces \( \mathcal{G}_{C^1(L^p, L^q)}([0,T), \mathbb{R}^n) \) for those \( 1 \leq p, q \leq \infty \) for which corresponding space is an algebra with multiplication.

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