DIFFERENTIAL CALCULUS ON QUANTUM COMPLEX
GRASSMANN MANIFOLDS I: CONSTRUCTION

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ABSTRACT. Covariant first order differential calculus over quantum complex Grassmann manifolds is considered. It is shown by a Pusz–Woronowicz type argument that under restriction to calculi close to classical Kähler differentials there exist exactly two such calculi for the homogeneous coordinate ring. Complexification and localization procedures are used to induce covariant first order differential calculi over quantum Grassmann manifolds. It is shown that these differential calculi behave in many respects as their classical counterparts. As an example the $q$-deformed Chern character of the tautological bundle is constructed.

Covariant first order differential calculus is a concept first introduced by S. L. Woronowicz [Wor87], [Wor89] to generalize the notion of differential form from commutative algebra to quantum groups and quantum spaces. The task to find well behaved analogues of differential forms for the noncommutative deformed coordinate algebras appearing in the framework of quantum groups is still of considerable interest [Her98], [Sch99], [SV98]. In [PW89] W. Pusz and S.L. Woronowicz proved that for the quantum vector space of dimension $\geq 3$ there exist exactly two differential calculi freely generated by the differentials of the generators. In [Pod92] P. Podleś classified differential structures on the quantum 2-sphere $S^2_{qc}$ which have certain properties similar to classical differential forms. It turned out, that only in the so called quantum subgroup case $c = 0$ such a differential calculus exists and is then uniquely determined.

Podleś quantum 2-sphere is an example of a $q$-deformed Grassmann manifold. The undeformed Grassmann manifold $\text{Gr}(r,N)$ of $r$-dimensional subspaces in $\mathbb{C}^N$ is a projective algebraic variety. It is well known [CP94] that its homogeneous coordinate ring $\mathcal{O}(\mathcal{V}_r)$ can be $q$-deformed to a quantum space $\mathcal{O}_q(\mathcal{V}_r)$. On the other hand M. Noumi, M. S. Dijkhuizen and T. Sugitani introduced a large class of quantum Grassmannians [NDS97]. Here only the quantum subgroup case is considered and will be denoted by $\mathcal{O}_q(\text{Gr}(r,N))$. The aim of this paper is to construct a canonical covariant first order differential calculus over $\mathcal{O}_q(\text{Gr}(r,N))$ and investigate its properties.

The $q$-deformed coordinate rings $\mathcal{O}_q(\mathcal{V}_r)$ and $\mathcal{O}_q(\text{Gr}(r,N))$ are closely related. More explicitly we consider the complexification of $\mathcal{O}_q(\mathcal{V}_r)$, an algebra obtained by adding complex conjugate elements. This complexification has a distinguished

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invariant element, and \( \mathcal{O}_q(\text{Gr}(r,N)) \) can be obtained as a subalgebra of the localization with respect to this element. It is the main observation of our work that this construction allows an analogue on the level of differential calculus. To achieve this the notions of complexification and localization of first order differential calculus are introduced. In generalization of the above mentioned classification result of \( \text{PW89} \) differential calculi over \( \mathcal{O}_q(V_r) \) are classified. For this purpose the \( U(\mathfrak{sl}_N) \)-module structure of classical Kähler differentials over \( \mathcal{O}(V_r) \) is analyzed. Under suitable additional assumptions all covariant first order differential calculi with the same \( U_q(\mathfrak{sl}_N) \)-module structure are classified and it is seen that exactly two such calculi exist. Complexification and localization then lead to a calculus \( \Gamma^1_q(\text{Gr}(r,N)) \) over \( \mathcal{O}_q(\text{Gr}(r,N)) \) which in the case of \( S^2_{q0} \) is seen to be isomorphic to the calculus constructed by Podleś. Thus a relationship between the calculus of Pusz and Woronowicz on the quantum plane and Podleś’ calculus on \( S^2_{q0} \) is established and generalized to arbitrary Grassmann manifolds.

The construction of \( \Gamma^1_q(\text{Gr}(r,N)) \) given here allows to perform explicit computations. As an example it is shown that as a left module \( \Gamma^1_q(\text{Gr}(r,N)) \) is generated by the differentials of the generators of \( \mathcal{O}_q(\text{Gr}(r,N)) \). The dimension of \( \Gamma^1_q(\text{Gr}(r,N)) \) in the sense of \( \text{Her} \) is estimated.

Considering \( \mathcal{O}_q(\text{Gr}(r,N)) \) in terms of the appropriate set of generators and relations it is straightforward to construct a \( q \)-deformed analogue of the module of sections of the tautological bundle over \( \text{Gr}(r,N) \). This allows to introduce the \( q \)-deformed Chern character of the tautological bundle. It is a sum of closed differential forms in the universal higher order differential calculus of \( \Gamma^1_q(\text{Gr}(r,N)) \) which are seen to be central. Thus it is to be expected that the cohomology ring is independent of the deformation parameter.

The ordering of the paper is as follows. Section 1 serves to fix notations. In Section 2 quantum Grassmann manifolds are recalled. The notion of complexification and localization are applied to the homogeneous coordinate ring \( \mathcal{O}_q(V_r) \) establishing the relation between \( \mathcal{O}_q(V_r) \) and \( \mathcal{O}_q(\text{Gr}(r,N)) \). This part owes much to the detailed analysis of \( \text{Sto98} \). Explicit relations for different sets of generators of \( \mathcal{O}_q(\text{Gr}(r,N)) \) are established. In a slight digression these relations are used to calculate the kernel of the quantum analogue of the canonical map dual to the inclusion \( \text{Gr}(r-1,N-1) \hookrightarrow \text{Gr}(r,N) \). The explicit relations between different sets of generators involve a considerable number of \( R \)-matrices. To simplify notations the graphical calculus of \( \text{Tur94} \) is used. In Appendix A this calculus is recalled and useful simplifications of morphisms often met in our framework are provided.

The explicit construction of the canonical covariant first order differential calculus is performed in Section 3. First the notations of localization and complexification for differential calculus on \( q \)-spaces are introduced. Then under suitable additional assumptions all differential calculi over the homogeneous coordinate ring \( \mathcal{O}_q(V_r) \) are classified. They are factored by their torsion submodules and then turn out to be given by relations of \( R \)-matrix type. Localization and complexification is applied to induce the desired calculus over \( \mathcal{O}_q(\text{Gr}(r,N)) \).

The last section is devoted to the construction of the \( q \)-deformed Chern character for the tautological bundle over \( \text{Gr}(r,N) \).
1. Notations and conventions

Unless stated otherwise all notations and conventions coincide with those used in [KS97]. Throughout this work $q^k \neq 1$ for all $k \in \mathbb{N}$ is assumed.

The $q$-deformed universal enveloping algebra $U_q(\mathfrak{sl}_N)$ is the complex algebra generated by elements $E_i, F_i, K_i$ and $K_i^{-1}, i = 1, \ldots, N-1$ and relations

$$
K_i K_i^{-1} = K_i^{-1} K_i = 1 \\
K_i K_j = K_j K_i \\
K_i E_j = q^{\alpha_{ij}} E_j K_i \\
K_i F_j = q^{-\alpha_{ij}} F_j K_i
$$

$$
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}
$$

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k}_q E_i^{1-a_{ij}-k} E_j E_i^k = 0, \quad i \neq j
$$

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1 - a_{ij}}{k}_q F_i^{1-a_{ij}-k} F_j F_i^k = 0, \quad i \neq j.
$$

Here $a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{else} \end{cases}$ denotes the Cartan matrix of $\mathfrak{sl}_N$ and the $q$-deformed binomial coefficient is defined by

$$
\binom{n}{k}_q = \frac{[n][n-1] \ldots [n-k+1]}{[k][k-1] \ldots [1]}
$$

where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$. The algebra $U_q(\mathfrak{sl}_N)$ obtains a Hopf algebra structure by

$$
\Delta K_i = K_i \otimes K_i \\
\Delta E_i = E_i \otimes K_i + 1 \otimes E_i \\
\Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i \\
\epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0 \\
S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i.
$$

Among the irreducible finite dimensional representations of $U_q(\mathfrak{sl}_N)$ are the so called type 1 representations $V(\lambda)$ which are uniquely determined by a highest weight $\lambda$. More explicitly if $\omega_i, i = 1, \ldots, N-1$ denote the fundamental weights and if $\lambda = \sum_i \lambda_i \omega_i \in \sum_i \mathbb{N} \omega_i$ is a dominant integral weight there exists $v \in V(\lambda)$ such that

$$
F_i v = 0, \quad K_i^{-1} v = q^{\lambda_i} v.
$$

Dually the $q$-deformed algebra of regular functions $O_q(SL(N))$ is the subalgebra of the dual Hopf algebra of $U_q(\mathfrak{sl}_N)$ generated by the matrix coefficients of the irreducible type 1 representations. It is generated by the matrix coefficients $u^i_j$,
i, j = 1, ..., \(N\) of the vector representation. By construction the type 1 representations of \(U_q(\mathfrak{sl}_N)\) coincide with the right \(\mathcal{O}_q(\text{SL}(N))\)-comodules. If \(q \in \mathbb{R}\) the Hopf algebra \(\mathcal{O}_q(\text{SL}(N))\) can be endowed with a *-structure by \((u^*_i)^* = S(u^*_i)\) and is then denoted by \(\mathcal{O}_q(\text{SU}(N))\). The category of type 1 representations of \(U_q(\mathfrak{sl}_N)\) is a braided monoidal category. There exists a universal r-form
\[
    r : \mathcal{O}_q(\text{SL}(N)) \otimes \mathcal{O}_q(\text{SL}(N)) \to \mathbb{C}
\]
such that the braiding \(\hat{R}_{VW} : V \otimes W \to W \otimes V\) can be written as
\[
v \otimes w \mapsto w_{(0)} \otimes v_{(0)} r(v_{(1)}, w_{(1)})
\]
where Sweedler notation is used. The convolution inverse of \(r\) is denoted by \(\bar{r}\). For any \(a, b \in \mathcal{O}_q(\text{SL}(N))\) the relations
\[
    r(a_{(1)}, b_{(1)})a_{(2)}b_{(2)} = b_{(1)}a_{(1)}r(a_{(2)}, b_{(2)}) \quad (1.2)
\]
\[
    \bar{r}(a_{(1)}, b_{(1)})b_{(2)}a_{(2)} = a_{(1)}b_{(1)}\bar{r}(a_{(2)}, b_{(2)}) \quad (1.3)
\]
hold. In terms of the solution
\[
    \hat{R}_{ij}^{kl} = \begin{cases} 
        q & \text{if } i = j = k = l, \\
        1 & \text{if } i = l \neq j = k, \\
        q - q^{-1} & \text{if } k = i < j = l, \\
        0 & \text{else}
    \end{cases} \quad (1.4)
\]
of the braid relation the universal r-form is determined by
\[
    r(u^i_j, u^k_l) = p \hat{R}_{ij}^{kl} \quad (1.5)
\]
where \(p \in \mathbb{C}\) satisfies \(p^N = q^{-1}\). This universal r-form is real, i.e.
\[
    r(a^*, b^*) = \overline{r(b, a)}.
\]
Introduce variants of the \(\hat{R}\)-matrix by
\[
    (\hat{R})_i^{jk} = (\hat{R})_{ik}^{ji} \quad (\hat{R})_i^{jk} = (\hat{R})_{ij}^{kj} \quad (\hat{R})_i^{jk} = q^{2j-2k}(\hat{R})_{ki}^{lj}.
\]
The meaning of these variants becomes clear if one relates them to the universal r-form
\[
    (\hat{R})_i^{jk} = p^{-1} r(S(u^i_j), S(u^k_l)) \quad (\hat{R})_i^{jk} = p\bar{r}(S(u^i_j), S(u^k_l)) \quad (\hat{R})_i^{jk} = p^{-1} r(S(u^i_j), u^k_l) \quad (\hat{R})_i^{jk} = pr(S(u^i_j), u^k_l) \quad (\hat{R})_i^{jk} = p^{-1} r(S(u^i_j), u^l_j) \quad (\hat{R})_i^{jk} = pr(u^i_j, S(u^l_j)).
\]
In terms of the matrix coefficients \(u^i_j\) of the vector representation \(V(\omega_1)\) the matrix coefficients of the fundamental representation \(V(\omega_s)\) are given by s-minors. More explicitly let \(I = \{i_1 < \ldots < i_s\}\) and \(J = \{j_1 < \ldots < j_s\}\) denote subsets of \(\{1, \ldots, N\}\). Define
\[
    \mathcal{D}_j^I := \sum_{\sigma \in S_s} (-q)^{\ell(\sigma)} u_{i_{\sigma(1)}}^{i_{\sigma(1)}} \ldots u_{i_{\sigma(s)}}^{i_{\sigma(s)}}
\]
where \(S_s\) denotes the symmetric group in \(s\) elements and \(\ell(\sigma)\) is the length of the permutation \(\sigma \in S_s\). Then the matrix coefficients
\[
    \{x_I := \mathcal{D}_j^{\{r+1, \ldots, N\}} \mid I = \{i_1 < \ldots < i_s\} \subset \{1, \ldots, N\}\}
\]
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form a basis of a left $U_q(\mathfrak{sl}_N)$-module isomorphic to $V(\omega_s)$ with corresponding right \( \mathcal{O}_q(\text{SL}(N)) \)-comodule structure

$$\Delta(x_I) = x_I \otimes D_I^f.$$ 

As for the vector representation $[1.4]$ a solution $(\hat{R}^{KL}_{ij})$ of the braid relation on $V(\omega_s) \otimes V(\omega_s)$ is given by

$$(1.9) \quad r(D_i^L, D_j^K) = p^{s^2} \hat{R}^{KL}_{ij}.$$ 

In analogy to $(1.10)-(1.11)$ define

$$(1.11) \quad \hat{R}^{KL}_{ij} = p^{-s^2} \pi(S(D_i^K), S(D_j^K)) \quad (\hat{R}^{-})^{KL}_{ij} = p^{s^2} r(S(D_i^K), D_j^K)$$ 

and

$$(1.12) \quad \hat{R}^{KL}_{ij} = p^{-s^2} \pi(S(D_i^K), D_j^K) \quad (\hat{R}^{-})^{KL}_{ij} = p^{s^2} r(D_i^K, S(D_j^K)).$$ 

For further details on $q$-deformed universal enveloping algebras $U_q(\mathfrak{g})$ and coordinate algebras $\mathcal{O}_q(G)$ consult $[KS97]$. 

For any subset $S$ of a complex vector space $V$ the symbol $\text{Lin}_C$ will denote the complex linear span of $S$. For any linear map $A : V^\otimes n \to V^\otimes m$ and some tensor power $V^\otimes p$, $p \geq k+n$ the map $\text{Id}^\otimes k \otimes A \otimes \text{Id}^\otimes p-k-n$ on $V^\otimes p$ will be denoted by the symbol $A_{k+1,\ldots,k+n}$.

2. Quantum Grassmann Manifolds

2.1. Localization and Complexification. Let $\mathcal{X}$ denote an $U_q(\mathfrak{g})$-module algebra without zero divisors and let $1 \in S \subset \mathcal{X}$ denote a left and right Ore subset, i.e. $S$ is multiplicatively closed and satisfies

$$\forall x \in \mathcal{X}, s \in S \quad \exists x_L, x_R \in \mathcal{X}, s_L, s_R \in S \quad s_Lx = x_Ls, xs_R = s_Rx.$$ 

Then the localization $\mathcal{X}(S)$ is defined $[MR88]$. It is possible to give criteria when the $U_q(\mathfrak{g})$-module structure of $\mathcal{X}$ induces a $U_q(\mathfrak{g})$-module structure on $\mathcal{X}(S)$ $[LR97]$. In the example considered here the existence of such an action of $U_q(\mathfrak{g})$ will be straightforward.

Let now $q \in \mathbb{R}$ and consider the compact real form of some quantum group $\mathcal{O}_q(G)$ with real universal $r$-form $\mathfrak{r}$. Assume $\mathcal{X} = \oplus_\lambda \mathcal{X}_\lambda$ where $\mathcal{X}_\lambda$ are irreducible type 1 representations of $U_q(\mathfrak{g})$, i.e. $\mathcal{X}$ is also an $\mathcal{O}_q(G)$-comodule algebra. Let $\overline{\mathcal{X}}$ denote the complex conjugate vector space with the opposite multiplication $x \cdot y := yx$. Then $\overline{\mathcal{X}}$ is also an $\mathcal{O}_q(G)$-comodule algebra with comultiplication

$$\overline{\Delta}_\mathcal{X} : x \mapsto x(0) \otimes x^*_1 \in \overline{\mathcal{X}} \otimes \mathcal{O}_q(G).$$ 

Assume further that $\mathcal{X}$ is given by generators and homogeneous relations. For any homogeneous $x \in \mathcal{X}$ (resp. $y \in \overline{\mathcal{X}}$) let $\deg(x)$ (resp. $\deg(y)$) denote the degree, i.e. the number of generators occurring in each summand of $x$ (resp. $y$). Then for each $\lambda \in \mathbb{C}$ the tensor product $\mathcal{X}_\lambda^\overline{\mathcal{X}} := \mathcal{X} \otimes \overline{\mathcal{X}}$ obtains an algebra structure by

$$(2.1) \quad (x \otimes y) \cdot (x' \otimes y') = x^{\deg(y) \deg(x')} (x(0) \otimes (y(1) y') \mathfrak{r}(y(0), x'(1)))$$ 

and turns into an $\mathcal{O}_q(G)$-comodule algebra $[KS97$, Lemma 10.31]. The algebra $\mathcal{X}_\lambda^\overline{\mathcal{X}}$ will be called the complexification of $\mathcal{X}$. If $\lambda \in \mathbb{R}$ then $\mathcal{X}_\lambda^\overline{\mathcal{X}}$ is a *-algebra with $(x \otimes y)^* = y \otimes x$. 

2.2. The homogeneous coordinate ring. The Grassmann manifold $\Gr(r,N)$ of $r$-dimensional subspaces of $\mathbb{C}^N$ is the $\SL(N)$-orbit of a highest weight vector $v \in V(\omega_r)$ in the projective space $\mathbb{P}(V(\omega_r))$. Its homogeneous coordinate ring $O(V_r)$ is generated by the set of functions
\begin{equation}
\{f_\ell := \ell(\cdot,v) | \ell \in V(\omega_r)^*\}
\end{equation}
in $O(\SL(N))$. This approach allows a well known analogue in the $q$-deformed setting for arbitrary flag manifolds [CP94], [LR92], [TT91], [Soi92]. We restrict ourselves to the special case of Grassmannians. More explicitly, if $v \in V(\omega_r)$ is a highest weight vector then $O_q(V_r)$ is defined to be the subalgebra generated by the matrix coefficients (2.2) in $O_q(\SL(N))$.

**Proposition 2.1.** ([Soi92], Prop. 1) As a $U_q(\mathfrak{sl}_N)$-module algebra $O_q(V_r)$ is isomorphic to the direct sum $\oplus_{k \geq 0} V(k\omega_r)^*$ endowed with the Cartan multiplication.

Recall that the Cartan multiplication is given on homogeneous components by the projection
\[ V(k\omega_r)^* \otimes V(m\omega_r)^* \to V((k+m)\omega_r)^*. \]
The $q$-deformed coordinate algebra $O_q(V_r)$ can further be described in terms of generators $V(\omega_r)^* = V(\omega_s)$, $s = N-r$, and relations by $V(\omega_s) \otimes V(\omega_r) \supset V(\lambda) = 0$ if $\lambda \neq 2\omega_s$ [Bra94], [TT91]. The notation $s = N-r$ will be used throughout this paper.

The eigenvalue of the braiding $\hat{R}_{\omega_s,\omega_r}$ induced by the universal $r$-form on $V(2\omega_s) \subset V(\omega_s) \otimes V(\omega_s)$ is given by $q^{(N-s)/N}$, compare [KS97] 8.4.3, Prop. 22. Thus the eigenvalue of the rescaled braiding $(\hat{R}_{Kj}^{KL})$ on $V(2\omega_s) \subset V(\omega_s) \otimes V(\omega_s)$ equals $q^s$.

This eigenvalue occurs on no other irreducible subspace of $V(\omega_s) \otimes V(\omega_s)$. Therefore the $q$-deformed homogeneous coordinate ring $O_q(V_r)$ can also be defined by generators $x_I$ and relations
\[ x_I x_J = q^{-s} x_K x_L \hat{R}_{IJ}^{KL}. \]
The complexification $O_q(V_r)^{\mathbb{C}}$ can also be given explicitly in terms of generators and relations.

**Lemma 2.1.** Let $(x_I)$ denote the basis of $V(\omega_s)$ from above and let $(y_I)$ denote the dual basis of $V(\omega_s)^* = V(\omega_r)$. Then the complexification $O_q(V_r)^{\mathbb{C}}$ is isomorphic to
\[ O_q(V_r) \otimes O_q(\mathfrak{g}_s) \]
and a complete list of relations is given by
\begin{align}
2.3 & \quad x_I x_J = q^{-s} x_K x_L \hat{R}_{IJ}^{KL} \\
2.4 & \quad y_I y_J = q^s y_K y_L (\hat{R}^-)^{KL}_{IJ} \\
2.5 & \quad y_I x_J = p^{-s^2} \lambda x_K y_L (\hat{R}^-)^{KL}_{IJ}. 
\end{align}
If $\lambda = p^{r^2} q^s$ then any invariant element $c \in V(\omega_s) \otimes V(\omega_r) \subset O_q(V_r)^{\mathbb{C}}$ is central.

**Proof.** To prove (2.3) note that $\Delta x_I = x_J \otimes D_I^j$ implies $\Delta y_I = y_J \otimes S(D_I^j)$ and by definition
\[ y_I x_J = \lambda x_K y_L r(S(D_I^j), D_J^K) = p^{-s^2} \lambda x_K y_L (\hat{R}^-)^{KL}_{IJ}. \]
Decomposition of the tensor product $V(\omega_s) \otimes V(\omega_r)$ yields that any invariant element $c \in V(\omega_s) \otimes V(\omega_r)$ is a complex multiple of $\sum_i x_i y_i$. Now the last statement follows from the relations (1.9)–(1.12) and the properties of the universal $r$-form.

In what follows we will only consider the case $\lambda = p^s q^r$ and drop the index $\lambda$.

2.3. $q$-Grassmann Manifolds. The complex Grassmann manifold $Gr(r, N)$ has the structure of a homogeneous space $Gr(r, N) \cong G/K$ with $G = SU(N)$ and $K = S(U(r) \times U(N-r))$. The harmonic analysis of the square integrable functions on $Gr(r, N)$ endowed with the Haar measure is given by [Hel84], V Thm. 4.3,

$$L^2(Gr(r, N)) = \bigoplus_{\lambda \in P^+_K} V(\lambda).$$

Here $P^+_K$ denotes the set of $K$-spherical dominant weights. In the case of $Gr(r, N)$ the set $P^+_K$ consists of all weights of the form [Hel84], V Thm. 4.1, [Sto98], Thm. 4.2.

$$\lambda = \sum_{i=1}^{\min(r,s)} n_i (\omega_i + \omega_{N-i}), \quad n_i \in \mathbb{N}_0$$

where as before $s = N-r$. The direct sum

$$\mathcal{O}(Gr(r, N)) = \bigoplus_{\lambda \in P^+_K} V(\lambda)$$

is multiplicatively closed and will be called the coordinate algebra of $Gr(r, N)$. As $Gr(r, N)$ is a projective variety this notion deviates from the classical formalism of algebraic geometry.

Consider the generators $V(\omega_s) \subset \mathcal{O}(V_r)$ of the homogeneous coordinate ring of $Gr(r, N)$. To any $f \in V(\omega_s)$ associate a complex conjugate function $f^*$ defined by

$$f^*(p) = \overline{f(p)}.$$

The vector space $\overline{V(\omega_s)}$ of complex conjugate functions is endowed with the complex conjugate scalar multiplication, i.e. $(\lambda \cdot f^*)(p) = \overline{\lambda f(p)}$. The $SU(N)$-invariant scalar product on $V(\omega_s)$ induces an isomorphism of representations

$$\overline{V(\omega_s)} \rightarrow V(\omega_s)^* = V(\omega_r)$$

$$f^* \rightarrow \langle \cdot, f \rangle$$

which extends to an isomorphism of algebras

$$\overline{\mathcal{O}(V_r)} \rightarrow \mathcal{O}(V_s).$$

Denote the algebra of functions on the affine cone $V_r$ generated by $V(\omega_s)$ and $\overline{V(\omega_s)}$ by $\mathcal{O}(V_r)_C$. The multiplication $\mathcal{O}(V_r) \otimes \mathcal{O}(V_s) \rightarrow \mathcal{O}(V_r)_C$ can be seen to be an isomorphism. Let $c \in V(\omega_s) \otimes V(\omega_r) \subset \mathcal{O}(V_r)_C$ denote a nonzero invariant element (which is uniquely determined up to a scalar factor) and consider the algebra of functions on $V_r$ generated by

$$V = \text{Lin}_C \left\{ \frac{x y}{c} \mid x \in V(\omega_s) \subset \mathcal{O}(V_r), y \in V(\omega_r) \subset \mathcal{O}(V_s) \right\}.$$
The representation \( V \) is isomorphic to
\[
V \cong \bigoplus_{k=0}^{\min(s,r)} V(\omega_k + \omega_{N-k})
\]
where by abuse of notation \( V(\omega_0 + \omega_{N-0}) \) denotes the trivial representation. The elements of \( V \) are homogeneous of degree 0 and therefore induce functions on the Grassmann manifold. As the localization \( \mathcal{O}(V_r)_C(c) \) has no zero divisors the elements of \( V \) generate the algebra \( \mathcal{O}(\text{Gr}(r,N)) \). Thus classically \( \mathcal{O}(\text{Gr}(r,N)) \) is isomorphic to the subalgebra of \( \mathcal{O}(V_r)_C(c) \) generated by \( V \).

This construction of \( \mathcal{O}(\text{Gr}(r,N)) \) allows a straightforward analogue in the \( q \)-deformed setting, \( q \in \mathbb{R} \setminus \{-1,0,1\} \). Namely consider the complexification
\[
\mathcal{O}_q(V_r)_C \cong \mathcal{O}_q(V_r) \otimes \mathcal{O}_q(V_r)
\]
of \( \mathcal{O}_q(V_r) \) and let \( c \in V(\omega_s) \otimes V(\omega_r) \subset \mathcal{O}_q(V_r)_C \) denote a nonzero invariant element. By Lemma 2.1 the element \( c \) is central in \( \mathcal{O}_q(V_r)_C \). The subalgebra of the localization \( \mathcal{O}_q(V_r)_C(c) \) generated by
\[
V = \mathrm{Lin}_C \left\{ \frac{xy}{c} \bigg| x \in V(\omega_s) \subset \mathcal{O}_q(V_r), y \in V(\omega_r) \subset \mathcal{O}_q(V_r) \right\}
\]
will be denoted by \( \mathcal{O}_q(\text{Gr}(r,N)) \) and will be called the \( q \)-deformed coordinate algebra of the Grassmann manifold. This definition of \( \mathcal{O}_q(\text{Gr}(r,N)) \) will prove useful for our purpose as one can use the FODC on \( \mathcal{O}_q(V_r) \) to induce a FODC on \( \mathcal{O}_q(V_r)_C(c) \) and therefore on the \( q \)-deformed Grassmann manifold. The aim of the next theorem is to make contact with the definition of \( \mathcal{O}_q(\text{Gr}(r,N)) \) in the literature.

By construction there is an inclusion
\[
\mathcal{O}_q(V_r) \hookrightarrow \mathcal{O}_q(\text{SL}(N))
\]
\[
x_I \mapsto D_I^{(r+1,\ldots,N)}
\]
which induces a covariant \( * \)-algebra homomorphism
\[
i_C : \mathcal{O}_q(V_r)_C \to \mathcal{O}_q(\text{SU}(N)).
\]
Indeed, map \( y_I \) to \( (D_I^{(r+1,\ldots,N)})^* = S(D_I^{(r+1,\ldots,N)}) \). It follows from (1.2) that for \( A = \{r+1,\ldots,N\} \)
\[
S(D_A^I)D_J^I = \sum_{M,N,K,L} \hat{R}_{MN}^{AA}D_K^M S(D_N^K)(\hat{R}^-)^{KL}_{IJ}.
\]
Now \( (\hat{R})_{MN}^{AA} = q^*\delta_{MA}\delta_{NA} \) and therefore \( i_C(x_I) \) and \( i_C(y_J) \) satisfy the relation (2.3). Note that \( i_C(c) = 1 \). Therefore \( i_C \) induces a map denoted by the same symbol
\[
i_C : \mathcal{O}_q(\text{Gr}(r,N)) \to \mathcal{O}_q(\text{SL}(N)).
\]
Define \( K \subset U_q(\mathfrak{sl}_N) \) to be the subalgebra generated by
\[
\{E_i,F_i,K_j \mid i \neq r, j = 1,\ldots,N-1\}
\]
and set \( K^+ = \{k \in K \mid \epsilon(k) = 0\} \). The following theorem implies that \( i_C \) is an isomorphism of \( \mathcal{O}_q(\text{Gr}(r,N)) \) onto the \( q \)-deformed coordinate algebra
\[
\mathcal{O}_q(U/K) = \{ b \in \mathcal{O}_q(\text{SL}(N)) \mid \langle k, b(1) \rangle b(2) = 0 \quad \text{for all} \quad k \in K^+ \}
\]
of the Grassmann manifold defined for instance in [DS97] (5.1). Here \( \langle \cdot, \cdot \rangle \) denotes the pairing \( U_q(\mathfrak{sl}_N) \otimes \mathcal{O}_q(\text{SL}(N)) \to \mathbb{C} \) given in [KS97], 9.4.
Theorem 2.2. $O_q(Gr(r, N)) \cong \bigoplus_{\lambda \in P^+} V(\lambda)$.

Proof. Note first that for any $k \in \mathbb{N}$
\[ V(k^r) \otimes V(k^r) \cong V((k-1)^r) \otimes V((k-1)^r) \bigoplus (2.8) \]
\[ \bigoplus_{(n_1, n_2, \ldots)} V \left( \sum_{i=1}^{\min(r, s)} n_i (\omega_i + \omega_{N-i}) \right) \]
where the summation is over all min$(r, s)$-tuples $(n_1, n_2, \ldots)$ such that $\sum n_i = k$.
As a vector space $O_q(Gr(r, N)) \subset O_q(V_r)_{\mathbb{C}}(c)$ is generated by subspaces
\[ W_k := \frac{1}{C^k} V(k^r) \otimes V(k^r) \subset O_q(V_r)_{\mathbb{C}}(c) \]
consisting of all $k$-fold products of elements of $V$. In $O_q(Gr(r, N))$ the subspace
$W_{k,k-1} \subset W_k$ isomorphic to
\[ V((k-1)^r) \otimes V((k-1)^r) \]
is identified with $W_{k-1}$. Indeed, if $v_i \in O_q(V_r)_{\mathbb{C}}$ denotes a highest weight vector of
\[ V(\omega_i + \omega_{N-i}) \subset V(\omega_i) \otimes V(\omega_r) \subset O_q(V_r)_{\mathbb{C}} \]
the element $c^s v_i^r \cdots v_i^r \sum n_i + \ell = k$ is up to a scalar multiple the unique highest weight vector in $V(k^r) \otimes V(k^r) \subset O_q(V_r)_{\mathbb{C}}$ of weight $\sum n_i (\omega_i + \omega_{N-i})$. Dividing by $c^k$ implies the claimed identification of $W_{k,k-1}$ with $W_{k-1}$ in $O_q(Gr(r, N))$. Now the claim of the theorem follows from [2].

Note the inclusion $i : O_q(Gr(r, N)) \subset O_q(U/K)$. In fact it is known that $\text{Im}(i)$ and $O_q(U/K)$ are isomorphic and that the decomposition of $\text{Im}(i)$ into irreducible modules is the same as the decomposition of $O_q(Gr(r, N))$ in Theorem 2.2 [Sto98]. This implies the following Corollary.

Corollary 2.3. The map $i : O_q(Gr(r, N)) \rightarrow O_q(U/K)$ is an isomorphism of left $U_q(s\mathfrak{n})$-module algebras.

2.4. Generators and relations. To present $O_q(Gr(r, N))$ in terms of generators and relations consider the standard basis $x_i \in \Lambda^r(\mathbb{C}^N) = V(\omega_i)$ with multi-indices
$I = (i_1 < \cdots < i_s) \subset \{1, \ldots, N\}$. If $e_1, \ldots, e_N$ denotes the standard basis of $V(\omega_1) = \mathbb{C}^N$ then
\[ x_I = \sum_{\sigma \in S_s} (-1)^{t(\sigma)} e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_s)} \in V(\omega_i) \subset V(\omega_1)^{\otimes s}. \]

Let further $e_1^*, \ldots, e_N^*$ denote the dual basis of $V(\omega_{N-1}) = (\mathbb{C}^N)^*$ and define
\[ y_I = \sum_{\sigma \in S_s} (-q)^{t(\sigma)} e_{\sigma(i_1)}^* \otimes \cdots \otimes e_{\sigma(i_s)}^* \in V(\omega_i) \subset V(\omega_{N-1})^{\otimes s}. \]

Evaluation
\[ e_{i_1}^* \otimes \cdots \otimes e_{i_s}^* (e_{j_1} \otimes \cdots \otimes e_{j_k}) = \delta_{i_1,j_1} \cdots \delta_{i_s,j_k} \]
leads to
\[ y_I(x_J) = \left( \sum_{\sigma \in S_s} q^{2t(\sigma)} \right) \delta_{IJ} = q^{s(s-1)/2} s! \delta_{IJ}. \]
Thus up to normalization \((y_I)\) is the dual basis of \((x_I)\). The element
\[
(2.11) \quad c = \sum_I x_I \otimes y_I \in V(\omega_s) \otimes V(\omega_r) \subset \mathcal{O}_q(\text{Gr}(r, N))
\]
is invariant and the generators \(z_{IJ} = x_I y_J / c \in \mathcal{O}_q(\text{Gr}(r, N))\) fulfill the relations
\[
(2.12) \quad \text{tr}(z) := \sum_I z_{II} = 1
\]
\[
(2.13) \quad \sum_J q^{-s} z_{IM} z_{NK} R_{jj}^{ MN} = z_{IK}
\]
\[
(2.14) \quad z_{IR} z_{SL} R_{JK}^{RS} P_{MK}^{LK} = 0
\]
\[
(2.15) \quad z_{IR} z_{SL} R_{JK}^{RS} T_{MN}^{IL} = 0.
\]
Here \(P\) (resp. \(\overline{P}\)) denotes any projector onto a subspace \(V(\mu) \subset V(\omega_s) \otimes V(\omega_r)\) such that \(\mu \neq 2\omega_s\) (resp. \(V(\nu) \subset V(\omega_r) \otimes V(\omega_r)\) such that \(\nu \neq 2\omega_r\)). By construction of \(\mathcal{O}_q(\text{Gr}(r, N))\) as a subalgebra of \(\mathcal{O}_q(\text{Gr}(r, N))_{*}(c)\) it is clear that the above list of relations is complete.

To gain geometric understanding of \(\mathcal{O}_q(\text{Gr}(r, N))\) it is useful to consider another set of generators. Let \(z_{ij}\) denote the basis element of
\[
V(0) \oplus V(\omega_1 + \omega_{N-1}) \subset V
\]
defined as the image in \(\mathcal{O}_q(\text{Gr}(r, N))_{*}(c)\) of the basis vector \(e_i \otimes e_j^* \in V(\omega_1) \otimes V(\omega_{N-1})\) by the map
\[
(2.16) \quad q^{s(3-s)/2} \frac{[s][s]}{c} \begin{array}{c} x \end{array} \begin{array}{l} \downarrow \tilde{\rho} \end{array} \begin{array}{c} y \end{array}
\]
where the graphical calculus from Appendix A is applied.

The generator \(z_{ij}\) can be written more explicitly
\[
z_{ij} = q^{s(3-s)/2}[s][s]^{-1} \sum_{\substack{i_1, \ldots, i_s \atop k_1, \ldots, k_s \atop i_1, \ldots, i_s}} e_{k_1} \cdot \cdot \cdot e_{k_s} A_{i_1, i_2, \ldots, i_s}^{k_1, \ldots, k_s} e_{i_1}^* \cdot \cdot \cdot e_{i_s}^* A_{i_1, i_2, \ldots, i_s}^{k_1, \ldots, k_s}
\]
where \(A\) and \(\hat{A}\) denote the projectors onto the invariant subspaces \(V(\omega_s)\) and \(V(\omega_r)\) respectively. Note that Lemma A.3 implies for \(m_1 < m_2 < \cdots < m_s\) and arbitrary \(i_1, \ldots, i_s\)
\[
\hat{A}_{i_1, i_2, \ldots, i_s}^{m_1, m_2, \ldots, m_s} = \hat{A}_{i_1, i_2, \ldots, i_s}^{m_2, m_3, \ldots, m_s} = \frac{q^{-s(s-1)/2}}{[s]!} \sum_{\sigma \in S_s} (-q)^{\ell(\sigma)} \delta_{i_1, \sigma(m_1)} \cdots \delta_{i_s, \sigma(m_s)}.
\]
Therefore
\[
z_{ij} = \frac{q^{-s(s+1)/2}}{[s-1]!} \sum_{i_2, \ldots, i_s} (-q)^{\ell(i, i_2, \ldots, i_s) + \ell(j, i_2, \ldots, i_s)} x_{\{i, i_2, \ldots, i_s\}} y_{\{j, i_2, \ldots, i_s\}}
\]
where \(x_{\{i, i_2, \ldots, i_s\}}\) and \(y_{\{j, i_2, \ldots, i_s\}}\) are zero if two indices coincide and \(\ell(i, i_2, \ldots, i_s)\) denotes the length of the permutation which transforms \(i, i_2, \ldots, i_s\) into an increasing sequence. As noted in the last section there is a \(*\)-algebra homomorphism
The generators $z_{ij}$ will be called the little generators of $O_q(\text{Gr}(r, N))$ while the generators $z_{IJ}$ will be called the big generators. Given a $U_q(\mathfrak{sl}_N)$-module algebra $\mathcal{O}$ and any two morphisms $z_1, z_2 \in \text{Hom}(V(\omega_1) \otimes V(\omega_{N-1}), \mathcal{O})$ define their product $z_1 \bullet z_2 \in \text{Hom}(V(\omega_1) \otimes V(\omega_{N-1}), \mathcal{O})$ by

$$(z_1 \bullet z_2)_{ik} = \sum_n (z_1)_{i\alpha} (z_2)_{\alpha k} R_{\alpha n}^{ab} = q^{2N+1} \sum_j q^{-2j} (z_1)_{ij} (z_2)_{jk}.$$ 

Here as above $(z_1)_{i\alpha}$ resp. $(z_1 \bullet z_2)_{ik}$ denotes the image of $e_i \otimes e_\alpha$. Graphically $z_1 \bullet z_2$ is represented by

Instead of $(z_1 \bullet z_2)_{ik}$ we will also write $\sum_j (z_1)_{ij} \bullet (z_2)_{jk}$ or just $(z_1)_{ij} \bullet (z_2)_{jk}$ which indicates that $\bullet$ is a $q$-deformed matrix multiplication.

**Proposition 2.2.** In $O_q(\text{Gr}(r, N))$ the following relations hold

1. $\sum_i z_{ii} = q^{-s}[s], \quad \sum_i z_{i\alpha} \bullet z_{\alpha j} = z_{jk},$
2. $q^{2l-2j} z_{ij} z_{\alpha k} \hat{R}^{\alpha l}_{j l} \hat{R}^{kl}_{cd} = q^{2l-2k} z_{ci} z_{jk} \hat{R}^{jl}_{cd} \hat{R}^{ab}_{kl}$, (Reflection Equation)
3. $z_{IJ} = (-1)^{s(s-1)/2} q^{s^2 ([s]!)^{-1}}.$

**Proof.** Recall the following relations

\begin{align*}
\hat{R}uu &= uu\hat{R} \\
\hat{R}uu^e &= u^e u\hat{R} \\
\hat{R}u^e u &= u^e u\hat{R}
\end{align*}

(2.19)
which follow from the properties (1.2), (1.3) of the universal r-form. By Corollary 2.3 it suffices to verify the above relations for the elements $i_C(z_{ij}) \in \mathcal{O}_q(\text{SL}(N))$. Property 1. is immediately checked. To verify the reflection equation note first that
\[
J_{kl} = \begin{cases} 
2^k - 2^N - 1 & \text{if } k = l > r, \\
0 & \text{else}
\end{cases}
\]
is a solution of the reflection equation
\[
(2.20) \quad J_{ij} J_{ka} \tilde{R}_{lb} \tilde{R}_{cd} = J_{ci} J_{jk} \tilde{R}_{il} \tilde{R}_{ba}.
\]
Then the relations (2.19) imply that $z_{ij} = u^m_i S(u^n_j) J_{mn}$ is also a solution of (2.20) concluding the proof of 2.

The last property will be verified by explicit calculation using the graphical calculus in Appendix A.

By the above proposition $\mathcal{O}_q(\text{Gr}(r, N))$ can be considered as a $q$-deformed version of the coordinate algebra of the affine variety of projectors onto $s$-dimensional subspaces of $\mathbb{C}^N$.

2.5. Inclusion of quantum Grassmann manifolds. Consider the surjective map of Hopf algebras
\[
\mathcal{O}_q(\text{SU}(N)) \to \mathcal{O}_q(\text{SU}(N - 1)),
\]
\[
\begin{cases} 
u^n_i & \text{if } i = 1 \text{ or } j = 1, \\
u^{i-1}_j & \text{else.}
\end{cases}
\]
As the little generators $z_{ij} \in \mathcal{O}_q(\text{Gr}(r, N)) \subset \mathcal{O}_q(\text{SU}(N))$, $i,j \geq 2$ map onto the little generators $z_{i-1,j-1} \in \mathcal{O}_q(\text{Gr}(r-1, N-1)) \subset \mathcal{O}_q(\text{SU}(N-1))$ one obtains a surjection
\[
i^*_r : \mathcal{O}_q(\text{Gr}(r, N)) \to \mathcal{O}_q(\text{Gr}(r-1, N-1)).
\]
Classically this surjection corresponds to the inclusion
\[
\text{Gr}(r-1, N-1) \to \text{Gr}(r, N).
\]

Proposition 2.3. Let $\mathcal{I} \subset \mathcal{O}_q(\text{Gr}(r, N))$, $r \geq 1$ denote the ideal generated by the set
\[
\{z_{1k}, z_{l1} | k, l = 1, \ldots, N\}.
\]
Then $\text{ker}(i^*_r) = \mathcal{I}$.

Proof. Let $\mathcal{L} \subset \mathcal{O}_q(\text{Gr}(r, N))$ denote the ideal generated by
\[
\{z_{1j} | 1 \in I \text{ or } 1 \in J\}.
\]
It follows from Proposition 2.2.3 and (2.17) that $\mathcal{L} \subset \mathcal{I} \subset \text{ker}(i^*_r)$. Therefore it suffices to show that the induced surjection
\[
\mathcal{O}_q(\text{Gr}(r, N))/\mathcal{L} \to \mathcal{O}_q(\text{Gr}(r-1, N-1))
\]
is also injective.
For any multi-index $K = (k_1 < \cdots < k_s)$, $k_i \in \{2, \ldots, N\}$ let $K'$ denote the multi-index $(k_1-1, \ldots, k_s-1)$. Note that

\begin{align}
(2.21) & \quad R^{KL}_{IJ} = \hat{R}^{KL}_{IJ}' \\
(2.22) & \quad \hat{R}^{KL}_{IJ} = \hat{R}^{KL'}_{IJ'}
\end{align}

if non of the occurring multi-indices $I, J, K, L$ contains 1.

Recall that (2.12)-(2.15) form a complete set of relations for $O_q(\text{Gr}(r,N))$. Let $\hat{z}_{IJ} \in O_q(\text{Gr}(r,N))$ denote the canonical preimage of $z_{IJ} \in O_q(\text{Gr}(r-1,N-1))$ obtained by raising all components of the multi-indices $I$ and $J$ by one. It remains to verify that modulo $\mathcal{L}$ the elements $\hat{z}_{IJ} \in O_q(\text{Gr}(r,N))$ satisfy the defining relations (2.12)-(2.13) of $O_q(\text{Gr}(r-1,N-1))$. This clearly holds for (2.12). As to (2.13) note that (1 $\in I$ or 1 $\in J$) and $\hat{R}^{KL}_{IJ} \neq 0$ imply (1 $\in K$ or 1 $\in L$). Thus modulo $\mathcal{L}$ summation has only to be taken over all $J \neq 1$. Now (2.21) implies that the preimages $\hat{z}_{IJ}$ satisfy the defining relation (2.13) of $O_q(\text{Gr}(r-1,N-1))$ modulo $\mathcal{L}$. The desired property concerning (2.14) follows from (2.21), (2.22) and the fact that $P$ can be replaced by $R-q^a$, similarly for (2.13).

3. Construction of Differential Calculus

3.1. Covariant Differential Calculus. Let $\mathcal{X}$ denote a $\mathbb{C}$-algebra. A first order differential calculus (FODC) over $\mathcal{X}$ is a $\mathcal{X}$-bimodule $\Gamma$ together with a $\mathcal{X}$-linear map

$$d : \mathcal{X} \to \Gamma$$

such that $\Gamma = \text{Lin}_\mathbb{C}\{a db c | a,b,c \in \mathcal{X}\}$ and $d$ satisfies the Leibniz rule

$$d(ab) = a db + da b.$$ 

Let in addition $\mathcal{A}$ denote a Hopf algebra and $\Delta_\mathcal{X} : \mathcal{X} \to \mathcal{X} \otimes \mathcal{A}$ a right $\mathcal{A}$-comodule algebra structure on $\mathcal{X}$. If $\Gamma$ possesses the structure of a right $\mathcal{A}$-comodule

$$\Delta_\mathcal{A} : \Gamma \to \Gamma \otimes \mathcal{A}$$

such that

$$\Delta_\mathcal{A}(a db c) = (\Delta_\mathcal{X}a)((d \otimes \text{Id})\Delta_\mathcal{X}b)(\Delta_\mathcal{X}c)$$

then $\Gamma$ is called covariant. A FODC $\Gamma$ over a $*$-algebra $\mathcal{X}$ is called a $*$-calculus if there exists an involution $* : \Gamma \to \Gamma$ such that $*a db c = c^* d(b^*) a^*$. For further details of first order differential calculi consult [KS97].

3.2. Localization and Complexification. Let $\Gamma$ denote an $\mathcal{X}$-bimodule and let $1 \in S \subset \mathcal{X}$ denote an Ore-subset as in section 2.1. Then

$$\Gamma(S) = \mathcal{X}(S) \otimes_\mathcal{X} \Gamma \otimes_\mathcal{X} \mathcal{X}(S)$$

is called the localization of $\Gamma$ with respect to $S$. By the Leibniz rule the localization of a covariant FODC $d : \mathcal{X} \to \Gamma$ allows a uniquely determined differential $d_S : \mathcal{X}(S) \to \Gamma(S)$ such that $d_S|_\mathcal{X} = i_S \circ d$ where $i_S$ denotes the canonical map $i_S : \Gamma \to \Gamma(S)$.

To define the complexification of a FODC assume as in section 2.1 that $q \in \mathbb{R}$ and consider the compact real form of some quantum group $O_q(G)$. The complex conjugate vector space $\overline{\Gamma}$ of $\Gamma$ endowed with the opposite $\overline{\mathcal{X}}$-bimodule structure $a \cdot \gamma := \gamma a$, $\gamma \cdot b := b \gamma$, $\gamma \in \overline{\Gamma}$, $a,b \in \overline{\mathcal{X}}$ obtains the structure of a covariant FODC by $\overline{d} = d$ and $\overline{\Delta}_\Gamma : \gamma \mapsto \gamma(0) \otimes \gamma^*(1)$. Assume again $\mathcal{X}$ to be generated by elements
of an irreducible type 1 representation $V$ and homogeneous relations. Consider the $X^\lambda_C$-bimodules

$\Gamma^X := (X \otimes \overline{X}) \otimes_X \Gamma \otimes_X (X \otimes \overline{X})$

and define $\Gamma^\lambda := \Gamma^X \oplus \Gamma^\overline{X}/\sim$, where $\sim$ denotes the equivalence relation

(3.1) $y \otimes dx + dy \otimes x \sim \lambda^{\deg(y) \deg(x)} \left( x(0) \otimes dy(0) + dx(0) \otimes y(0) \right) r(y(1), x(1))$

obtained by differentiation of (2.1) using the Leibniz rule. Assume that $\Gamma$ is a graded $X$-bimodule and $\deg(dx) = \deg(x) = 1$ for $x \in V$. For $x \in V \subset X$, $y \in \overline{V} \subset \overline{X}$ impose the additional relation

(3.2) $y \otimes dx = \lambda dx(0) \otimes y(0) r(y(1), x(1))$.

**Lemma 3.1.** Let $\Gamma^\lambda_C$ denote the quotient of $\Gamma^\lambda$ by (3.2), then

$\Gamma^\lambda_C \cong (\overline{X} \otimes \Gamma) \oplus (X \otimes \Gamma)$.

**Proof.** By (3.1) the vector space $\Gamma^\lambda_C$ is equal to the quotient of $\Gamma^X \oplus \Gamma^\overline{X}$ by (3.2) and

(3.3) $dy \otimes x = \lambda x(0) \otimes dy(0) r(y(1), x(1))$

Thus the claim follows from

$\Gamma^X/(3.2) \cong \overline{X} \otimes \Gamma$, \hspace{1cm} $\Gamma^\overline{X}/(3.3) \cong X \otimes \Gamma$.

The covariant FODC $\Gamma^\lambda_C$ will be called the complexification of $\Gamma$. If $\lambda \in \mathbb{R}$ then $\Gamma^\lambda_C$ is a $\ast$-calculus and $(\Gamma^X)^\ast = \Gamma^\overline{X}$.

3.3. Differential calculus over $\mathcal{O}_q(V_r)$. As a guiding principle in the construction of FODC over $q$-spaces one demands that dimensions should coincide with dimensions in the classical situation. Recall that the classical Kähler differential 1-forms over the homogeneous coordinate ring $\mathcal{O}(V_r)$ are given by

$\Omega^1 = (\mathcal{O}(V_r) \otimes V(\omega_s))/R$

where $R$ denotes the $\mathcal{O}(V_r)$-submodule generated by the irreducible components $V(\lambda) \subset \text{Sym}^2(V(\omega_s)), \lambda \neq 2\omega_s$. As the defining relations of $\mathcal{O}(V_r)$ and $\Omega^1$ are homogeneous $\Omega^1$ is a graded $\mathcal{O}(V_r)$-module. An element is homogeneous of degree $k$ if it can be written as $\sum_j p_j \otimes dx^j$ where $p_j$ are homogeneous polynomials of degree $k-1$ in the generators $x^j$ of $\mathcal{O}(V_r)$. Denote the elements of degree $k$ by $\Omega^1_k$.

**Lemma 3.2.** The homogeneous components of the Kähler differential 1-forms over $\mathcal{O}(V_r)$ are given by

(3.4) $\Omega^1_1 \cong V(\omega_s)$

(3.5) $\Omega^1_2 \cong V(2\omega_s) \oplus \bigoplus_{k=0}^t V(\omega_s-(2k+1) + \omega_s+(2k+1))$

(3.6) $\Omega^1_{k>2} \cong V(k\omega_s) \oplus V(\omega_{s-1} + (k-2)\omega_s + \omega_{s+1})$

where $t = \min\left(\frac{s-1}{2}, \frac{s-3}{2}\right)$ in (3.5).
Proof. By construction the $U(\mathfrak{sl}_N)$-module structure of $\Omega_1^1$ and $\Omega_2^1$ is given by (3.4) and (3.5). Indeed

$$\text{Sym}^2(V(\omega_s)) = \bigoplus_{k=0}^\min(|\omega_s|,|\hat{\omega}_s|) V(\omega_{s-2k} + \omega_{s+2k}).$$

The irreducible components isomorphic to $V(k\omega_s)$ and $V((k-1)\omega_s)$ in $V((k-1)\omega_s) \otimes V(\omega_s) \subset \mathcal{O}_q(V_r) \otimes V(\omega_s)$ are nonzero in $\Omega_{k>2}^1$ because they don’t occur in $V((k-2)\omega_s) \otimes \text{Sym}^2(V(\omega_s))$.

Thus it remains to check, that the other components in $V((k-1)\omega_s) \otimes V(\omega_s)$ can be written in terms of the relations. This is achieved by direct computation. □

The aim of this subsection is to construct a graded covariant first order differential calculus $\Gamma$ over $\mathcal{O}_q(V_r)$ with the following properties.

1. As a left $\mathcal{O}_q(V_r)$-module structure $\Gamma$ is generated by the differentials of the generators of $\mathcal{O}_q(V_r)$.
2. The FODC $\Gamma$ has the same $U_q(\mathfrak{sl}_N)$-module structure as its classical counterpart $\Omega^1$.

In the case $r = N/2$ to obtain uniqueness we demand instead the stronger conditions

1a. $\text{Lin}_C\{u \, dv| u, v \in V(\omega_s) \subset \mathcal{O}_q(V_r)\} = \text{Lin}_C\{dv| u, v \in V(\omega_s) \subset \mathcal{O}_q(V_r)\}$.
2a. Every homogeneous component of $\Gamma$ has the same $U_q(\mathfrak{sl}_N)$-module structure as its classical counterpart $\Omega^1$.

Conditions 1., 2., and 2a. imply that one has to define $\Gamma_k \subset \mathcal{O}_q(V_r) \otimes V(\omega_s)$ by the right hand side of the expression for $\Omega^1_k$ in Lemma 3.2. It remains to show that $\Gamma = \bigoplus_k \Gamma_k$ can be endowed with a right $\mathcal{O}_q(V_r)$-module structure and a differential $d: \mathcal{O}_q(V_r) \to \Gamma$ such that it obtains the structure of a covariant FODC and without factoring by a nontrivial submodule.

To define the right module structure of $\Gamma$ note that the Leibniz rule, the covariance, condition 1a. and the defining relations of $\mathcal{O}_q(V_r)$ imply that for any $\lambda \neq 2\omega_s$ and any $\sum x_i \otimes y_i \in V(\lambda) \subset V(\omega_s) \otimes V(\omega_s)$

$$\sum_i dx_i y_i = - \sum_i x_i dy_i.$$

Thus the right module structure of $\Gamma$ is uniquely determined by a complex parameter $c_T$ such that

$$dt_1 t_1 = c_T t_1 dt_1$$

for any highest weight vector $t_1$ of $V(\omega_s)$. From the structure of $U_q(\mathfrak{sl}_N)$ one obtains that $t_1 \otimes t_2 - q t_2 \otimes t_1$ is a highest weight vector of $V(\omega_{s-1} + \omega_{s+1})$ if $t_2 = E_s(t_1)$. On the other hand applying $E_s$ twice to (3.8) yields

$$q^{-1} dt_2 t_1 + dt_1 t_2 = c_T (q^{-1} t_2 dt_1 + t_1 dt_2)$$

$$dt_2 t_2 = c_T t_2 dt_2.$$

Rewrite these equations as

$$dt_j t_j = c_T t_j dt_j \quad \text{for} \quad j = 1, 2$$

$$dt_2 t_1 = \frac{c_T + 1}{q^{-1} + q} t_1 dt_2 + \frac{c_T q^{-1} - q}{q^{-1} + q} t_2 dt_1$$

$$dt_1 t_2 = \frac{c_T q - q^{-1}}{q^{-1} + q} t_1 dt_2 + \frac{c_T + 1}{q^{-1} + q} t_2 dt_1.$$
Using (3.3)–(3.11) to calculate $dt_1(t_1t_2 - qt_2t_1)$ one obtains

$$0 = (c_1q - q^{-1})(c_1q^{-1} - q)[t_1^2dt_2 - qt_1t_2dt_1] .$$

As by assumption the expression in the square brackets does not vanish, this implies $c_1 = q^2$. Thus it is proved that there exist at most two covariant FODC satisfying the conditions 1. and 2. above. It follows from the covariance and the choice of $c$ that the right module structure of $\Gamma$ given by (3.7) and (3.8) is indeed well defined. The main result is summarized in the following theorem.

**Theorem 3.3.** Let $r \in \{1, \ldots, N-1\}$. Then there exist exactly two covariant FODC over $O_q(V_r)$ satisfying conditions 1. and 2. (resp. 1a. and 2a. in the case $r = N/2$) above.

The above theorem generalizes the classification result of [PW89] for covariant FODC on quantum vector spaces to all homogeneous coordinate rings of quantum Grassmann manifolds. Pusz' and Woronowicz’ differential calculus on the $q$-deformed vector space is obtained by application of the above construction to $O_q(V_{N-1})$. In the case $N = 2, r = 1$ there exist two families of FODC labelled by a complex parameter satisfying conditions 1. and 2. [PW89]. Thus to obtain uniqueness in this case additional requirements like 1a. and 2a. are indeed necessary.

An interesting feature of the above differential calculi over $O_q(V_r)$ is that the commutation relations between generators and their differentials are in general no longer given by multiplication by the universal $R$-matrix. Indeed, the commutation relations are given by a covariant map $A : V(\omega_s) \otimes^2 \to V(\omega_s) \otimes^2$ with eigenvalue $-1$ on $V(\omega_s) \otimes^2 \setminus \text{Sym}^2 V(\omega_s)$.

This apparent deficit can be dealt with considering the $O_q(V_r)$-torsion in $\Gamma$.

**Lemma 3.4.** Let $\Gamma_{\text{tor}}$ denote the $O_q(V_r)$-torsion submodule of $\Gamma$. Then $\Gamma_{\text{tor}}$ is concentrated in degree $k = 2$ and

$$(3.12) \quad \Gamma_{\text{tor}} = \bigoplus_\lambda V(\lambda)$$

where summation is taken over all $\lambda = \omega_{s-m} + \omega_{s+m}$ with $1 < m \leq \min(s, r)$ odd.

**Proof.** Indeed, it follows from (3.6) and the covariance of the multiplication that the elements of $\Gamma_{\text{tor}}$ defined by (3.12) are annihilated by any $x \in V(\omega_s) \subset O_q(V_r)$.

Assume on the other hand that $x \in O_q(V_r), \gamma \in \Gamma$ such that $x\gamma = 0$. Without loss of generality assume $x$ and $\gamma$ to be homogeneous, i.e.

$$x \in V(k\omega_s) \subset O_q(V_r) \quad \gamma = \gamma_1 + \gamma_2 \in V(l\omega_s) \oplus V(\omega_{s-1} + (l-2)\omega_s + \omega_{s+1}) \subset \Gamma.$$ If $x$ and $\gamma$ are linear combinations of weight vectors the summands of maximal weight with respect to the lexicographic order of the weights are also torsion. Thus we can even assume $x$ and $\gamma$ to be weight vectors. Choose bases $(x_i)_{i \in I_x}$ of $V(k\omega_s)$ and $(\gamma_i^1)_{i \in I_1}$ of $V(l\omega_s)$ and $(\gamma_i^2)_{i \in I_2}$ of $V(\omega_{s-1} + (l-2)\omega_s + \omega_{s+1})$ consisting of weight vectors such that $x = x_{i_1}$, $\gamma_1 = \gamma_{i_1}^1$ and $\gamma_2 = \gamma_{i_2}^2$ for some $i_1 \in I_x, i_1 \in I_1$ and $i_2 \in I_2$. Denote by $(D^1x)_j, (D^1y)_j, (D^2)_j \in O_q(SL(N))$ the corresponding matrix
coefficients, i.e.
\[
\Delta x_j = x_i \otimes (D^x)^{i}_j \\
\Delta \gamma^1_j = \gamma^1_i \otimes (D^1)^{i}_j \\
\Delta \gamma^2_j = \gamma^2_i \otimes (D^2)^{i}_j.
\]

Now \(x\gamma = 0\) implies
\[
(3.13) \quad 0 = \Delta(x\gamma) = x_i \gamma^1_j \otimes (D^x)^{i}_j (D^1)^{j}_i + x_i \gamma^2_j \otimes (D^x)^{i}_j (D^2)^{j}_i.
\]

As \(O_q(SL(N))\) is an integral domain and the weight \((k+l)\omega_s\) only appears in the product of matrix coefficients of the first summand \(\gamma^1 \neq 0\) implies \(x_{\max} \gamma^1_{\max} = 0\) for the highest weight vectors \(x_{\max} \in V(k\omega_s) \subseteq O_q(V_r)\) and \(\gamma^1_{\max} \in V(l\omega_s) \subseteq \Gamma\).

As \(x_{\max} \gamma^1_{\max}\) is a highest weight vector of \(V((k+l)\omega_s) \subseteq \Gamma\) this is a contradiction. Thus \(\gamma^1 = 0\), \(\gamma^2 \neq 0\) and \(3.13\) would imply \(x_{\max} \gamma^2_{\max} = 0\) where \(\gamma^2_{\max} \in V(\omega_{s-1} + (l-2)\omega_s + \omega_{s+1}) \subseteq \Gamma\) denotes a highest weight vector. This again leads to a contradiction. Hence \(\gamma = 0\).

In the construction of a covariant differential calculus on \(q\)-deformed Grassmann manifolds in subsection [3.4] a localization with respect to an invariant element \(c \in V(\omega_{s}) \otimes V(\omega_{s})^*\) of the complexification of \(O_q(V_r)\) will be considered. In this localization all \(\Gamma_{\text{tor}}\) vanishes. Therefore it makes sense to divide by \(\Gamma_{\text{tor}}\) and consider
\[
\Gamma^{\text{red}} := \Gamma/\Gamma_{\text{tor}} = \bigoplus_k \Gamma_k
\]

where now \(\Gamma_k\) denotes the homogeneous torsion free component of degree \(k\), i.e.
\[
\Gamma_1 \cong V(\omega_s), \quad \Gamma_{k \geq 2} \cong V(k\omega_s) \otimes V(\omega_{s-1} + (k-2)\omega_s + \omega_{s+1}).
\]

Note that in the case of quantum vector spaces there is no torsion and therefore \(\Gamma^{\text{red}} = \Gamma\) is still the calculus of [FW89].

Consider the complexification \(\Gamma^{\text{red}}_{c^{r}}\) of the calculus \(\Gamma^{\text{red}}\) determined by the constant \(c_\Gamma = q^2\). By Lemma [21] and the general construction a complete set of relations of \(\Gamma^{\text{red}}_{c^{r}}\) is given by
\[
(3.14) \quad x_I x_J = q^{-s} x_K x_L \hat{R}_{i j}^{KL}
\]
\[
(3.15) \quad dx_I x_J = q^{2s} x_K dx_L \hat{R}_{i j}^{KL}
\]
\[
(3.16) \quad y_I y_J = q^s y_K y_L (\hat{R}^-)^{KL}_{i j}
\]
\[
(3.17) \quad dy_I y_J = q^{-2} y_K dy_L (\hat{R}^-)^{KL}_{i j}
\]
\[
(3.18) \quad y_I x_J = q^s x_K y_L (\hat{R}^-)^{KL}_{i j}
\]
\[
(3.19) \quad dx_I y_J = q^{-s} y_K dx_L \hat{R}_{i j}^{KL}
\]
\[
(3.20) \quad dy_I x_J = q^s x_K dy_L \hat{R}_{i j}^{KL}.
\]

To obtain the relations in the case \(c_\Gamma = q^{-2}\) replace \(3.13\) and \(3.17\) by
\[
(3.21) \quad dx_I x_J = q^{2s} x_K dx_L (\hat{R}^-)^{KL}_{i j}
\]
\[
(3.22) \quad dy_I y_J = q^{-2} y_K dy_L \hat{R}_{i j}^{KL}.
\]
3.4. Differential calculus over $O_q(\text{Gr}(r,N))$. Classical Kähler differentials over a commutative algebra $A$ are given by $\Gamma^1(A) = I/I^2$ where $I$ denotes the kernel of the multiplication $m_A : A \otimes A \to A$. Thus if $A$ is a subalgebra of $B$ the kernel of the induced map $\Gamma^1(A) \to \Gamma^1(B)$ is given by
\[(\ker m_A \cap (\ker m_B)^2)/(\ker m_A)^2.\]

It can be shown that in the case $A = O(\text{Gr}(r,N))$ and $B = O(\mathcal{V}_r)_C(c)$ this quotient vanishes. Thus classically Kähler differentials over $O(\text{Gr}(r,N)) \subset O(\mathcal{V}_r)_C(c)$ coincide with the $O(\text{Gr}(r,N))$-submodule of $\Gamma^1(O(\mathcal{V}_r)_C(c))$ generated by $d_z J$. This observation and the construction of section 3.3 allow to introduce a $q$-deformed version of classical Kähler differentials over $O(\text{Gr}(r,N))$. To do so consider the covariant FODC $\Gamma^+$ over $\mathcal{X} = O_q(\mathcal{V}_r)$ constructed in section 3.3 uniquely determined by $c_T = q^2$. The complex conjugate FODC $\Gamma^+_{\overline{\mathcal{X}}}$ for $\overline{\mathcal{X}}$ is seen to belong to $\Gamma^+_{\overline{\mathcal{V}}} = q^{-2}$. The complexification $\Gamma^+_{\mathcal{X}}$ of $\Gamma^+$ is a covariant FODC over $\mathcal{X}_C$. Localize this FODC with respect to the invariant element $c \in \mathcal{X}$ and denote the resulting calculus by $\Gamma^+_C(c)$. The canonical FODC over $O_q(\text{Gr}(r,N))$ is defined to be the subcalculus of $\Gamma^+_C(c)$ generated by $O_q(\text{Gr}(r,N))$ and is denoted by $\Gamma^+_{q}(\text{Gr}(r,N))$. Note that $\Gamma^+_{q}(\text{Gr}(r,N))$ is a $*$-calculus as $\Gamma^+_{\mathcal{X}}$ is a $*$-calculus. To perform calculations in $\Gamma^+_{q}(\text{Gr}(r,N))$ it is useful to point out some further relations in the localization $\Gamma^+_C(c)$. Introduce partial differentiations
\[
(3.23) \quad \partial := d \otimes \text{Id} : O_q(\mathcal{V}_r) \otimes O_q(\mathcal{V}_r) \to \Gamma^+_{\mathcal{X}}
\]
\[
(3.24) \quad \overline{\partial} := \text{Id} \otimes d : O_q(\mathcal{V}_r) \otimes O_q(\mathcal{V}_r) \to \Gamma^+_{\overline{\mathcal{X}}}.\]

Then in particular $\partial c = dz J y I$ and $\overline{\partial} c = x J dy I$. The following Lemma can be proved by direct calculation using (3.14)–(3.22).

**Lemma 3.5.** In $\Gamma^+_C(c)$ the following relations hold
\[
\begin{align*}
\partial c x I &= q^2 x I \partial c \quad &\text{and} & &\overline{\partial} c x I &= x I \overline{\partial} c \\
\partial c y I &= y I \partial c \quad & &\overline{\partial} c y I &= q^{-2} y I \overline{\partial} c \\
\partial c e^n &= q^{2n} c e^n \\n\partial c z I J &= z I J \partial c \\
\overline{\partial} c z I J &= z I J \overline{\partial} c \\
d x I c &= c d x I + (q^2 - 1) x I \partial c \\
d x I e^{-1} &= e^{-1} d x I + (q^{-2} - 1) e^{-1} x I \partial c \\
(3.25) \quad \partial z I J &= e^{-1} d x I y J - z I J e^{-1} \partial c \\
\overline{\partial} z I J &= c^{-1} x I d y J - z I J c^{-1} \overline{\partial} c \\
(3.26) \quad \partial z I J z K L &= q^2 z M N \partial z O P T_{I J K L}^{M N O P} \\
\overline{\partial} z I J z K L &= q^{-2} z M N \overline{\partial} z O P T_{I J K L}^{M N O P} \\
\end{align*}
\]

where $T = \tilde{R}_{23} \tilde{R}_{12} \tilde{R}_{34} \tilde{R}_{43}$.

**Corollary 3.6.** As a left $O_q(\text{Gr}(r,N))$-module $\Gamma^+_{q}(\text{Gr}(r,N))$ is generated by the differentials $dz J$. More explicitly
\[
\begin{align*}
d z I J K L &= (q^2 + q^{-2}) z M N d z O P T_{I J K L}^{M N O P} - z M N z O P d z R S T_{A B D L}^{O P R S} T_{I J K C}^{M N A B} C C D \\
\end{align*}
\]
where $C C D = \sum_J q^{-s} \tilde{R}_{I J}^{C D}$.
Proof. All indices will be dropped in the calculations. Note first that by (3.26)
\[ zd = (q^{-2} \partial z \bar{z} + q^{2} \partial \bar{z} z) T^{-1} = (q^{2} d \bar{z} z - (q^{2} - q^{-2}) \partial \bar{z} z) T^{-1} \]
\[ z \bar{z} d \bar{z} = (q^{-4} \partial \bar{z} z z + q^{4} \partial \bar{z} z z) T_{1234} T^{-1}_{3456} = (q^{4} d \bar{z} z z - (q^{4} - q^{-4}) \partial \bar{z} z z) T_{1234} T^{-1}_{3456} \]
which implies by (2.13)
\[ zdT = q^{2} d \bar{z} z - (q^{2} - q^{-2}) \partial \bar{z} z \]
(3.27)
\[ z \bar{z} d \bar{z}T = q^{4} d \bar{z} z - (q^{4} - q^{-4}) \partial \bar{z} z \]
(3.28)
Taking differences yields the desired relation. 

Note that (3.27) implies \( \partial z \in \Gamma_{1}^{q}(\text{Gr}(r,N)) \) and thus also \( \partial \bar{z} \in \Gamma_{1}^{q}(\text{Gr}(r,N)) \).
Therefore \( \Gamma_{q}^{q}(\text{Gr}(r,N)) \) can be written as a direct sum
\[ \Gamma_{q}^{1}(\text{Gr}(r,N)) = \Gamma_{+} \oplus \Gamma_{-} \]
where \( \Gamma_{+} \) (resp. \( \Gamma_{-} \)) denotes the \( \mathcal{O}_{q}(\text{Gr}(r,N)) \)-subbimodule generated by the set \( \{ \partial z_{I,J} | I,J \} \) (resp. \( \{ \partial \bar{z} z_{I,J} | I,J \} \)). The differential \( \partial \) (resp. \( \partial \bar{z} \)) endows \( \Gamma_{+} \) (resp. \( \Gamma_{-} \)) with the structure of a covariant FODC. Relation (3.26) implies that \( \Gamma_{+} \) (resp. \( \Gamma_{-} \)) is even generated by \( \{ \partial z_{I,J} | I,J \} \) (resp. \( \{ \partial \bar{z} z_{I,J} | I,J \} \)) as a left \( \mathcal{O}_{q}(\text{Gr}(r,N)) \)-module. Writing (3.26) graphically
\[ \partial z z = q^{2} \]
(3.29)
\[ \partial \bar{z} z = q^{-2} \]
one sees that as a left and as a right \( \mathcal{O}_{q}(\text{Gr}(r,N)) \)-module \( \Gamma_{+} \) (resp. \( \Gamma_{-} \)) is even generated by the differentials of the little generators \( \{ \partial z_{i,j} | i,j \} \) (resp. \( \{ \partial \bar{z} z_{i,j} | i,j \} \)).

Lemma 3.7. In terms of the little generators of \( \mathcal{O}_{q}(\text{Gr}(r,N)) \) the following relations hold in \( \Gamma_{q}^{1}(\text{Gr}(r,N)) \)
\[ z \bullet \partial z = 0 \]
(3.30)
\[ \partial z \bullet z = \partial z \]
\[ z \bullet \partial \bar{z} z = \partial \bar{z} z \]
(3.31)
\[ \partial \bar{z} z \bullet z = 0. \]

Proof. To verify (3.30) note that the second identity follows from the first one, the projector property \( z \bullet z = z \) and the Leibniz rule. The proof of \( z \bullet \partial z = 0 \) is performed.
graphically. If \( \mu = q^{(s-3)/2}[s][s] \) then by definition of the little generators

\[
z_{ij} \cdot \partial z_{jk} = z_{ij} \cdot \left( \mu c^{-1} \frac{\partial x}{x} \right) = \mu c^{-1} z_{jk} \partial c
\]

The middle crossing can be resolved by means of Lemma A.2.

\[
= \mu^2 \frac{c^2}{s^2}
\]

The conjugate relations (3.31) follow from (3.30) by application of \( * \) in the \( * \)-calculus \( \Gamma_C^+ \).

Let \( A \) denote a Hopf algebra and \( B \subset A \) a right comodule subalgebra. Assume that \( A \) is faithfully flat as \( B \)-module. Up to translation from right to left it was shown in [Her] that there exists a one to one correspondence between \( A \)-covariant FODC \( \Gamma \) over \( B \) and certain left ideals \( L \subset B^+ \), where \( B^+ = \ker(\epsilon|_B) \). The dimension of a covariant FODC \( \Gamma \) over \( B \) is defined by

\[
(3.32) \quad \dim(\Gamma) = \dim_C(\Gamma/\Gamma B^+) = \dim_C(B^+/L).
\]

It has been proven in [MS99] that \( \mathcal{O}_q(SU(N)) \) is a faithfully flat \( \mathcal{O}_q(Gr(r,N)) \)-module.

**Proposition 3.1.** The differential calculi \( \Gamma_+ \) and \( \Gamma_- \) are nonisomorphic. Their dimensions can be estimated by \( 0 < \dim(\Gamma_+) \leq r(N-r) \).
Proof. Note first that by construction of $\Gamma^1_q(\text{Gr}(r,N))$ the differentials $\partial z_{ij}$ do not vanish and therefore $\Gamma_+ \neq 0$. Thus there exist $i,j$ such that $\partial z_{ij} \neq 0$. Lemma 3.7 then implies that $\Gamma_+$ and $\Gamma_-$ are not isomorphic.

To prove the second property note that for any differential calculus $\Gamma$ with corresponding left ideal $L$ the relation $\dim(\Gamma) = 0$ implies $L = B^+$. By the general construction of differential calculi in terms of left ideals in [Her] this is equivalent to $\Gamma = 0$.

The vector space $\Gamma_-/\Gamma_-B^+$ is generated by $N^2$ elements $\bar{\partial} z_{ij}$, $i,j = 1, \ldots, N$. Thus to verify the upper bound it suffices to show that $\bar{\partial} z_{ij} \in \Gamma_- B^+$ if $j > r$ or $i \leq r$. This follows from Lemma 3.7 which implies for $j > r$

$$\bar{\partial} z_{ij} = q^{2N-2j+1} \epsilon(z_{jj}) \bar{\partial} z_{ij} - \bar{\partial} z_{ik} \bullet z_{kj} = -\bar{\partial} z_{ik} \bullet z_{kj}^+ \in \Gamma_- B^+.$$ 

For $i \leq r$ consider the relation $\bar{\partial} z_{ij} = z_{ik} \bullet \bar{\partial} z_{kj}$. Proposition 2.2.3 implies $z_{IJ} \in B^+$ if one of the multi-indices $I$ or $J$ contains an index $\leq r$. Thus the explicit form of the occurring $R$-matrices in (3.29) implies that $z_{IJ} \bar{\partial} z_{KL}$ if $I$ contains an index $\leq r$. As $\bar{\partial} z_{ij} = z_{ik} \bullet \bar{\partial} z_{kj}$ for $i \leq r$ can be written as a linear combination of such expressions $z_{IJ} \bar{\partial} z_{KL}$, the proof for $\Gamma_-$ is completed.

The estimate concerning $\Gamma_+$ is obtained similarly. Note first that $\partial z_{ij} = \partial z_{ik} \bullet z_{kj}$ implies $\partial z_{ij} \in \Gamma_+ B^+$ if $j \leq r$. On the other hand applying (3.29) to $z_{ik} \bullet \partial z_{kj} = 0$ one obtains the relation

$$\partial z_{ij} = 0$$

for any $i,j$. Assume $i > r$ and apply $\epsilon$ to the right $z$ factor. It follows from the explicit form of the occurring $R$-matrices that one obtains only terms of the following types:

- Multiples of $\partial z_{kk}$, $k < r$
- Multiples of $\partial z_{kj}$, $r < k < i$
- $\partial z_{ij}$ with coefficient

$$q^{-1} \epsilon(z_{ii}) + (q^{-1} - q) \sum_{r < i' < i} \epsilon(z_{i'i'}) = q^{2r-2N}.$$ 

Thus by induction on $i$ one gets $\partial z_{ij} \in \Gamma_+ B^+$ for all $i > r$. 

For the construction of $\Gamma^1_q(\text{Gr}(r,N))$ one could have also started with the FODC $\Gamma^\ominus$ over $\mathcal{X} = \mathcal{O}_q(V_r)$ which is uniquely determined by $c_\Gamma = q^{-2}$. Indeed, consider $\Gamma_\mathcal{C}$ and localize with respect to the invariant element $c$. Define complex and complex conjugate differentiations $\partial$ and $\bar{\partial}$ as in (3.23) and (3.24) with $\Gamma_\mathcal{C}^+$ replaced by $\Gamma_\mathcal{C}^\ominus$. In analogy to Lemma 3.5 one obtains
Lemma 3.8. In $\Gamma^{-}_C(c)$ the following relations hold
\[
\begin{align*}
\partial c x_I &= x_I \partial c + (q^{-2} - 1) c dx_I \\
\partial c y_I &= y_I \partial c \\
\partial c e^n &= q^{-2n} e^n \partial c \\
\partial c z_{IJ} &= z_{IJ} \partial c + (q^{-2} - 1) c \partial z_{IJ} \\
\partial c y_I &= y_I \partial c + (q^{-2} - 1) c dy_I \\
\partial c e^n &= q^{-2n} e^n \partial c \\
\partial c z_{IJ} &= z_{IJ} \partial c + (q^{-2} - 1) c \partial z_{IJ} \\
\partial c z_{I,J} &= q^{-2} c^{-1} dx_I y_J - q^{2} z_{IJ} c^{-1} \partial c \\
\partial z_{I,J} e^n &= q^{-2n} e^n \partial z_{I,J} \\
\partial z_{I,J} z_{K,L} &= q^{2} z_{MN} \partial z_{OP} T_{IJKL}^{MNOP} \\
\partial z_{I,J} y_I &= q^{2} z_{MN} \partial z_{OP} T_{IJKL}^{MNOP}
\end{align*}
\]
(3.33)
(3.34)

where $T = R_{23} R_{12} R_{34} R_{23}$.

Corollary 3.9. The $O_q(\text{Gr}(r,N))$-submodule of $\Gamma^+_C(c)$ generated by $\{\partial z_{IJ} \mid I,J\}$ (resp. $\{\partial z_{IJ} \mid I,J\}$, $\{y_I \mid I,J\}$) is isomorphic to the submodule of $\Gamma^-_C(c)$ generated by $\{\partial z_{IJ} \mid I,J\}$ (resp. $\{\partial z_{IJ} \mid I,J\}$, $\{y_I \mid I,J\}$).

Proof. Note first that the $O_q(\text{V}_r)_C(c)$ submodules of $\Gamma^+_C(c)$ and $\Gamma^-_C(c)$ generated by $\{\partial z_I \mid I\}$ are isomorphic as left $O_q(\text{V}_r)_C(c)$-modules. By (3.22) and (3.33) there exists a left $O_q(\text{V}_r)_C(c)$-module isomorphism of these submodules mapping $\partial z_{IJ} \in \Gamma^+_C(c)$ to $\partial z_{IJ} \in \Gamma^-_C(c)$. Thus the left $O_q(\text{Gr}(r,N))$-submodules of $\Gamma^+_C(c)$ and $\Gamma^-_C(c)$ generated by $\{\partial z_{IJ} \mid I,J\}$ which by (3.24) and (3.31) coincide with the $O_q(\text{Gr}(r,N))$-submodules generated by $\{\partial z_{IJ} \mid I,J\}$ are isomorphic as left modules. But (3.24) and (3.31) also imply that the right module structures coincide. The claims concerning $\partial$ and $d$ are obtained analogously.

The above isomorphisms of $O_q(\text{Gr}(r,N))$-bimodules preserve the differentials. Therefore a construction of $\Gamma^+_C(\text{Gr}(r,N))$ starting from $\Gamma^-$ leads to the same differential calculus.

In the case $N = 2$, $r = 1$ of Podleś’ quantum 2-sphere $O_q(\text{S}^2) = O_q(\text{Gr}(1,2))$ Proposition 3.1 implies that $\dim(\Gamma^+_q(\text{Gr}(1,2))) = 2$. It has been shown in [Her] that there exists exactly one covariant first order $\ast$-calculus over $O_q(\text{S}^2)$ of dimension 2 and that this calculus coincides with the calculus of [Pod92].

4. Chern Classes

Let $O(M)$ denote the coordinate algebra of an affine algebraic variety $M$. Algebraic vector bundles over $M$ are in one to one correspondence to projective modules over $O(M)$. A projective module $V$ can be uniquely determined by a projector $p \in \text{Mat}_k(O(M))$, i.e. by a surjective map
\[
p : O(M)^k \rightarrow V \subset O(M)^k, \quad p^2 = p.
\]
By the general theory [Kar87] to any such projector is associated the curvature 2-form
\[
R = pdpdp \in \text{Mat}_k(\Omega^2(M)).
\]
The differential forms \( c_i \in \Omega^2(M) \) defined by
\[
\det(1 + R) = 1 + c_1 + c_2 + \cdots + c_k
\]
are closed. The corresponding cohomology classes are the Chern classes of the vector bundle. The differential forms \( c_1, \ldots, c_k \) can be expressed in terms of the closed forms
\[
(4.1) \quad ch(V) = \frac{1}{n!} \text{tr}(R^t).
\]
Consult the first chapter of \([Kar87]\) for further details.

In the example of the Grassmann manifold \( \text{Gr}(r,N) \) resp. the homotope affine algebraic variety with coordinate ring \( \mathbb{C}[z_{ij} \mid i, j = 1, \ldots, N]/(z^2 = z, \text{tr}(z) = N-r) \) the matrix \( z \) with entries \( z_{ij} \) can be considered as a projector describing the tautological bundle over \( \text{Gr}(r,N) \). The Chern classes of the tautological bundle over \( \text{Gr}(r,N) \) generate the cohomology ring \( H^*(\text{Gr}(r,N)) \).

The aim of this section is to construct a \( q \)-deformed analogue of the differential forms \( ch(V(\text{Gr}(r,N))) \) associated with the module of sections \( V(\text{Gr}(r,N)) \) of the tautological bundle. Consider the universal higher order differential calculus \( \Gamma_q^*(\text{Gr}(r,N)) \) over \( \mathcal{O}_q(\text{Gr}(r,N)) \) with first order calculus \( \Gamma^1_q(\text{Gr}(r,N)) \). The matrix \( z = (z_{ij}) \in \text{Mat}_N(\mathcal{O}_q(\text{Gr}(r,N))) \) of little generators can be considered as a \( q \)-deformed analogue of the projector which describes the tautological bundle. Define \( q \)-deformed analogues of the differential forms \( (4.1) \) by

\[
ch_q,t(\text{Gr}(r,N)) = \frac{1}{[t]!} \text{tr}(z \cdot d z \cdot d z \cdot \cdots \cdot d z) = \frac{1}{[t]!} \text{tr}(R \cdot R \cdot \cdots \cdot R)
\]

**Proposition 4.1.** The differential forms \( ch_q,t \) are closed in \( \Gamma_q^*(\text{Gr}(r,N)) \).

The main step of the proof is achieved by the following Lemma.

**Lemma 4.1.**

\[
\text{tr}(z \cdot \bar{z} \cdot d z \cdot \bar{z} \cdot \cdots \cdot \bar{z} \cdot \partial z \cdot \bar{z}) = 0
\]
\[
\text{tr}(\partial z \cdot \bar{z} \cdot \partial z \cdot \bar{z} \cdot \cdots \cdot \bar{z} \cdot \partial z \cdot z) = 0.
\]

**Proof.** The proof is performed graphically. By the commutation relations \( (3.26) \) resp. \( (3.29) \) one has

\[
\text{tr}(z \cdot \bar{z} \cdot \partial z \cdot \cdots \cdot \bar{z} \cdot \partial z \cdot \bar{z}) = q^2
\]
\[ q^2 \text{tr}(\partial z \cdot \partial z \cdot \ldots \cdot \partial z \cdot \partial z \cdot \partial z \cdot z) = 0 \]

where the last equation follows from (3.31). The second relation is verified analogously. □

**Proof of Prop. 4.1.** Note that in \( \Gamma^*_q(\text{Gr}(r,N)) \)

\[ z \cdot d z \cdot d z = d z \cdot d z - d z \cdot z \cdot d z = d z \cdot d z \cdot z \]

and by Lemma 3.7

\[ z \cdot d z \cdot d z \cdot \ldots \cdot d z = z \cdot \overline{\partial z} \cdot \partial z \cdot \ldots \cdot \overline{\partial z} \]

\[ d z \cdot d z \cdot \ldots \cdot d z \cdot z = \partial z \cdot \overline{\partial z} \cdot \partial z \cdot \ldots \cdot \overline{\partial z} \cdot z. \]

Lemma 4.1 and the relations (4.2) - (4.4) imply

[\text{d(ch}_{q,l}(\text{Gr}(r,N)))] = \frac{1}{[l]!} \text{tr}(d z \cdot \ldots \cdot d z)

\[ = \frac{1}{[l]!} \left( \text{tr}(z \cdot d z \cdot \ldots \cdot d z) + \text{tr}(d z \cdot z \cdot d z \cdot \ldots \cdot d z) \right) \]

\[ = \frac{1}{[l]!} \left( \text{tr}(z \cdot \overline{\partial z} \cdot \partial z \cdot \ldots \cdot \overline{\partial z}) + \text{tr}(\partial z \cdot \overline{\partial z} \cdot \partial z \cdot \ldots \cdot \overline{\partial z} \cdot z) \right) \]

\[ = 0. \]

□

**Corollary 4.2.** The differential forms \( \text{ch}_{q,l}(\text{Gr}(r,N)) \) are central in \( \Gamma^*(\text{Gr}(r,N)) \).

**Proof.** It follows from (3.29) that the differential forms \( \text{ch}_{q,l}(\text{Gr}(r,N)) \) commute with the big generators \( z_{I,J} \). Differentiation implies the claim. □

**Appendix A. Graphical calculus**

The finite dimensional type 1 representations of the Hopf algebra \( U_q(\mathfrak{g}) \) form a ribbon category \( U_q(\mathfrak{g})-\text{Rep} \) (KS97, 8 Prop. 19, 21, resp. Tur94 for the notion of ribbon category). Thus there exists a functor \( \mathcal{F}_{U_q(\mathfrak{g})-\text{Rep}} \) from the category of \( U_q(\mathfrak{g})-\text{Rep} \) coloured ribbon graphs to \( U_q(\mathfrak{g})_{\text{rep}} \) defined in Tur94, Thm. 2.5, which allows a graphical calculus for morphisms in \( U_q(\mathfrak{g})-\text{Rep} \). Restrict to the case \( \mathfrak{g} = \mathfrak{sl}_N \). On the second tensor power of the vector representation the braiding of \( U_q(\mathfrak{g})-\text{Rep} \) is given by multiplication by the \( R \)-Matrix \( p\hat{R} \) (4.3). To suppress
the rational exponent $p = q^{-1/N}$ in this action a rescaled version of the graphical calculus is considered.

More explicitly we restrict ourselves to tensor powers $V_1 \otimes V_2 \otimes \ldots \otimes V_k$ where $V_i$, $i = 1, \ldots, k$, denotes either the vector representation $V$ or its dual $V^*$. With respect to the standard basis $e_1, \ldots, e_N$ of $V$ (resp. $e_1^*, \ldots, e_N^*$ of $V^*$) crossings of lines correspond to multiplication with rescaled $R$-matrices as follows:

$\hat{R}$

$\hat{R}^{-}$

and $\bigcup V^*$ denotes the canonical inclusion $\mathbb{C} \rightarrow V \otimes V^*$.

Projections onto sub-representations will be denoted by boxes containing the highest weight of the sub-representation. Thus:

(A.1)

\[ \begin{array}{c}
\vdots \\
\omega_i \\
\vdots
\end{array} \]

denotes the projection onto the alternating sub-representations $V(\omega_s) \subset V^s$. If a sub-representation occurs with some multiplicity the box denotes the projection onto the whole isotypic component. Boxes labelled by $\sum n_i \omega_i$ will be considered as projectors of $V^k \otimes \Sigma n_i$ or $(V^*)^k \otimes \Sigma (N-1)n_i$. Thus in- and outgoing lines and lines between boxes are often dropped. Lines labelled by $s \in \mathbb{N}$ represent $s$ lines. With respect to the standard basis of $V$ (resp. $V^*$) diagrams with ingoing lines labelled by indices are identified with the image of the corresponding basis vectors. Similarly for morphisms on $V(\omega_s)$ (resp. $V(\omega_s)^*$) and the standard basis $(x_I)$ (resp. $(y_I)$) defined by (2.9) (resp. (2.10)).

Note that different from [Tur94] here diagrams are considered as morphisms from the bottom to the top. This convention is motivated by the choice to consider right comodule algebras, i.e. generators with lower indices.

Let $\mathcal{O}$ denote a $U_q(\mathfrak{sl}_N)$-module algebra with generators $(\bar{x}_I)$. Assume that the generators $(\bar{x}_I)$ form the basis of a finite dimensional representation $\overline{V}$ of $U_q(\mathfrak{sl}_N)$. Assume in addition that there is a given isomorphism between $\overline{V}$ and some sub-representation of $V^\otimes n \otimes (V^*)^\otimes m$ for some $m, n$. Writing elements $\bar{x}$ into upper boxes with $n$ ingoing lines labelled by $V$ and $m$ ingoing lines labelled by $V^*$ we will also consider the corresponding morphism as a map to the algebra $\mathcal{O}$.

The following technical lemmata give a number of useful simplification results for morphisms in $U_q(\mathfrak{sl}_N)-Rep$ which finally lead to a proof of Proposition 2.3.3.
Lemma A.1.

where \( n_{\lambda,\mu} = i - j \) if the Young tableau of \( V(\lambda) \) is obtained from the Young tableau of \( V(\mu) \) by adding a box in the \((j, i)\)-position, or in terms of fundamental weights \( \lambda = \mu + \omega_j - \omega_{j-1} \).

Proof. Express \( \mu = \sum_k \mu_k \omega_k, \ \mu_k \in \mathbb{N}_0 \) in terms of the fundamental weights. By [KS97], Prop. 8.22, the exponent \( n_{\lambda,\mu} \) is given by

\[
2 \left| \mu \right| - (\mu, \mu + 2\rho) - (\omega_1, \omega_1 + 2\rho) + (\mu + \omega_j - \omega_{j-1}, \mu + \omega_j - \omega_{j-1} + 2\rho),
\]

where \( |\mu| = \sum_k k\mu_k \) denotes the number of boxes of the Young diagram of \( V(\mu) \). This expression can be simplified using the relations

\[
(\rho, \omega_i) = \frac{1}{2}i(N - i) \quad \text{and} \quad (\omega_i, \omega_j) = \frac{1}{N}i(N - j)
\]

which hold for \( j \geq i \). □

Lemma A.2. For \( k = 0, 1, \ldots, s \) define morphisms by

\[
\phi_k := \begin{array}{ccc}
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\end{array} \quad \text{and} \quad \varphi_k := \begin{array}{ccc}
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\end{array}
\]

Then

\[
\phi_k = q^{k(s-k+1) \choose k} \varphi_0 \quad \text{and} \quad \varphi_k = q^{-k(s-k+1) \choose k} \varphi_0.
\]

Proof. For \( k = 0, 1, \ldots, s \) and \( m = 0, 1, \ldots, k \) define morphisms by

\[
\lambda_k := \begin{array}{ccc}
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\end{array} \quad \text{and} \quad \lambda_{km} := \begin{array}{ccc}
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\omega_s & \omega_s & \omega_s \\
\end{array}
\]

The relation \( \hat{R}^{-1} = \hat{R} - \hat{q} \text{Id} \) implies for \( m < k \)

\[
\lambda_{km} = \lambda_{k,m+1} - \hat{q}^{-m} \phi_{k-1}
\]
and therefore
\[(A.2) \quad \lambda_k = \lambda_{k0} = \lambda_{kk} - \hat{q} \left( \sum_{m=0}^{k-1} q^{-2m} \right) \phi_{k-1} = \phi_k - \hat{q}q^{-k}[k] \phi_{k-1}.\]

On the other hand by Lemma \[A.1\] one has
\[\varphi_k = q^{2(s-k)}\]
\[\varphi_k = q^{2(s-k+1)}\]
and therefore
\[(A.3) \quad \phi_k = q^{2(s-k+1)} \lambda_k.\]

The relations \[(A.2)\] and \[(A.3)\] imply
\[\phi_k = \frac{\hat{q}q^{-1-k}[k]}{1 - q^{-2(s-k+1)}} \phi_{k-1} = q^{2(1-k)+s[k]} \phi_{k-1} = \prod_{l=1}^{k} \left( q^{s-l} [l] \right) \phi_0 = q^{k(s-k+1)} \phi_0.\]

The claim concerning \(\varphi_k\) can be obtained from the result for \(\phi_k\) as follows.
\[\varphi_k = q^{-2k(s-k)}\]
\[\varphi_k = q^{-2k(s-k)+1}\]
where in the first and in the third equation \[KS97, \text{Prop. 8.22}\] is used again. \(\square\)
Let $A_s$ denote the antisymmetrizer, i.e. the projector onto $V(\omega_s) \subset V^{\otimes s}$ given graphically by (A.4).

**Lemma A.3.**

(A.4) $A_{s+1} = \frac{1}{[s+1]} \left( q^s(A_s)_{1,...,s} - [s](A_s)_{1,...,s} \hat{R}_{s,s+1}(A_s)_{1,...,s} \right)$

(A.5) $= \frac{1}{[s+1]} \left( q^s(A_s)_{2,...,s+1} - [s](A_s)_{2,...,s+1} \hat{R}_{1,2}(A_s)_{2,...,s+1} \right)$, where $(A_s)_{1,...,s}$ denotes the action of $A_s$ on the first $s$ tensor components of $V^{\otimes s+1}$, similarly $(A_s)_{2,...,s+1}$, $\hat{R}_{s,s+1}$ and $\hat{R}_{1,2}$.

Proof. If $P_{s+1}$ is defined by the right hand side of (A.4) then

$$P_{s+1}|_{V(\omega_{s+1})} = \frac{1}{[s+1]}(q^s + q^{-1}[s]) = 1.$$

On the other hand it follows from Lemma A.2 for $k = 1$ that $P_{s+1}$ restricted to $V(\omega_s + \omega_1) \subset V(\omega_s) \otimes V \subset V^{\otimes (s+1)}$ does indeed vanish. This proves (A.4) and the second expression is obtained similarly.

For $k = 0, 1, \ldots, s$ and $t = 0, 1, \ldots, s$ consider the morphisms defined by

$$\Psi_{kl} := \omega_{s-1} \quad \Psi_k := \frac{1}{[s-k+1]} \begin{array}{ccc} \omega_{k+1} & \omega_k \end{array}$$

and

$$\Psi_{k+1,k} := \begin{array}{ccc} \omega_{k+1} & \omega_k \end{array}$$

such that $\Psi_{kl} := \Psi_{kl} \Psi_k$.

**Lemma A.4.** The coefficients $\psi_{kl}$ satisfy the following relations:

1. $\psi_{kl} = \psi_{lk}$,
2. $\psi_{kl} = \psi_{kj} \psi_{jl}$ for all $k > j > l$,
3. $\psi_{kk} = 1$,
4. $\psi_{k+1,k} = \frac{[s-k]}{[k+1][s]}$.

Proof. Define $W \subset V^{\otimes (s+t)}$ to be the isotypical component of $V(\omega_s + \omega_t)$ in $V^{\otimes (s+t)}$. Define morphisms $K = (A_{s-k} \otimes A_{t+k})|_W$ and $L = (A_{s-t} \otimes A_{t+k})|_W$. Then the relation

$$\psi_{kl} \psi_{kl} K = \psi_{kl} K L K L = \psi_{kl} \Psi_k K$$

proves the first property. Relations 2. and 3. follow immediately from the definitions. To verify 4. define coefficients $\rho_k \in \mathbb{C}$, $k = 0, 1, \ldots, s$ by

$$\begin{array}{ccc} \omega_{k-1} & \omega_k & \omega_{k+1} \\ \omega_{k-1} & \omega_k & \omega_{k+1} \\ \omega_{k-1} & \omega_k & \omega_{k+1} \\
\end{array} = \rho_k \Psi_k.$$
It follows from (A.4) with \( s \) replaced by \( s-k \) and from (A.5) with \( s \) replaced by \( t+k \) that
\[
\rho_k = \frac{q^{s-k}}{[s-k]} \psi_{k,k-1} = \frac{q^{t+k}}{[t+k]} \psi_{k,k+1}.
\]
This relation and the symmetry property \( \psi_{k+1,k} = \psi_{k,k+1} \) imply the recursion formula
\[
\psi_{k+1,k} = \frac{[t+k][s-k+1]}{[t+k+1][s-k]} \psi_{k,k-1} + \frac{[s-t-2k]}{[t+k+1][s-k]}.
\]
Now property 4. follows by induction. ∎

The above Lemma can be used to calculate all coefficients \( \psi_{kl} \). In particular
\[
(A.6) \quad \psi_{00} = \prod_{j=0}^{t-1} \psi_{j+1,j} = \prod_{j=0}^{t-1} \frac{[s-t-j][j+1]}{[t+j+1][s-j]} = \frac{(s)}{[t]\cdot(q)}.
\]

**Lemma A.5.**
\[
\begin{array}{c}
\omega_{s-1} \quad \omega_s \quad \omega_{s+1} \quad \cdots \quad \omega_{s-1} \\
\omega_s \quad \omega_{s+1} \quad \omega_s \quad \cdots \quad \omega_s \\
\omega_{s-1} \quad \omega_{s+1} \quad \omega_{s-1} \quad \cdots \quad \omega_{s-1} \\
(s-1)\omega_s
\end{array} = [s]^s \prod_{j=0}^{(s)} \frac{(s)}{q}^{-2}.
\]

**Proof.** Note first that the irreducible representation \( V((s-1)\omega_s) \) occurs in each of the tensor powers \( V(\omega_{s-1})^{\otimes s} \) and \( V(\omega_s)^{\otimes s-1} \) with multiplicity one. Therefore there exists a well defined coefficient \( \lambda_s \) such that the morphism on the left hand side of the above equation is equal to \( \lambda_s \) times the morphism on the right hand side. Lemma A.4 for \( l = 0 \) and \( k = s-t-1 \) can be applied successively to simplify the diagram on the left hand side. This yields
\[
\lambda_s = \prod_{t=0}^{s-1} \psi_{0,s-t-1} = \prod_{t=0}^{s-1} \frac{[s]}{([t] \cdot (s)}{q}^{-2} = [s]^s \prod_{j=0}^{(s)} \frac{(s)}{q}^{-2}.
\]

**Lemma A.6.** Set \( \mu_s = q^{(s-1)(-(s-1)(s-2))/6} \) then
\[
\begin{array}{c}
\omega_1 \quad \cdots \quad \omega_s \quad \omega_1 \quad \cdots \quad \omega_s \\
\omega_1 \quad \cdots \quad \omega_s \quad \omega_1 \quad \cdots \quad \omega_s \\
\omega_1 \quad \cdots \quad \omega_s \quad \omega_1 \quad \cdots \quad \omega_s \\
8s\omega_s
\end{array} = \mu_s \prod_{j=0}^{(s)} \frac{(s)}{q}^{-1}.
\]
where $V(\omega_s)$ is considered as a sub-representation of $V^{\otimes s}$, and

$$= \mu_{s-1} \prod_{j=0}^{s-1} \left( \frac{s}{j} \right) q$$

where $V(\omega_r)$ is considered as a sub-representation of $(V^*)^{\otimes s}$.

Proof. To prove the first relation note that Lemma A.2 implies

$$= q^{t(s-t+1)} \frac{t}{s} \frac{s}{t} \gamma$$

Applying this successively from the right to the left hand side of the first relation yields the desired coefficient. The second relation is verified similarly dualizing.

Now the proof of the last property of Proposition 2.4 can be performed graphically. Abbreviate $\gamma = (-q)^{-\frac{s(s+1)}{2}} (q^{s-1} \frac{1}{s} \frac{s}{s-1} [s] [s]^{s-1}) e^{-s}$, then

$$= \gamma$$
By Lemma A.5 it is possible to insert \( s - 1 \) projectors onto \( V(\omega_s) \). This leads to

\[
= \gamma[s]^{-s} \prod_{j=1}^{s} \left( \frac{s}{j} \right)^2_{q}.
\]

Now the left and the right half of the above diagram can be simplified using Lemma A.6.

\[
= \gamma[s]^{-s}.
\]

\[
= (-1)^{(s-1)/2} q^{-s^2} [s]! \mu_{J, J}
\]

which proves the last property of Proposition 2.2.

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