CR and Holomorphic Embeddings and Pseudo-conformally Flat Metrics

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1 Introduction

In Several Complex Variables, understanding when a CR manifold can be embedded into a sphere is a subtle problem. Forstneric [F86] and Faran [Fa88] proved the existence of real analytic strictly pseudoconvex hypersurfaces in \( \mathbb{C}^{n+1} \) which do not admit any germ of non-constant holomorphic map taking \( M \) into sphere \( \partial B^{N+1} \) for any positive integer \( N \). Zaitsev constructed explicit examples for the Forstneric-Faran phenomenon [Z08]. Meanwhile, there have been much work done to prove the uniqueness of such embeddings up to the action of automorphisms. For instance, a well-known rigidity theorem says that that if \( M^{2n+1} \) is a CR spherical immersion inside \( \partial B^{N+1} \) with \( N \leq 2n - 1 \), then \( M \) must be totally geodesic (i.e., \( M \) is the image of \( \partial B^{n+1} \) by a linear fractional CR map). Ebenfelt, Huang and Zaitsev ([EHZ04], Theorem 1.2) proved that if \( d < \frac{n}{2} \), any smooth CR-immersion \( f : M \rightarrow \partial B^{n+d+1} \), where \( M \) is a smooth CR hypersurface of dimension \( 2n + 1 \), is rigid. Oh in [Oh] obtained a very interesting result on the non-embeddability for real hyperboloids into spheres of low codimension. Kim and Oh [KO06] found a necessary and sufficient condition for the local holomorphic embeddability into a sphere of a generic strictly pseudoconvex pseudo-Hermitian CR manifold in terms of its Chern-Moser curvatures. Along these lines, we mention recent studies in the papers of Huang-Zhang [HZ], Ebenfelt-Sun [ES] and Huang-Zaitsev [HZ]. We also refer the reader to a recent survey paper [HJ07] by the first two authors and many references therein. Our fist goal in this paper is to study the non-embeddability property for a class of hypersurfaces, called real hypersurfaces of involution type, in the low codimensional case, by making use of property of a naturally related Gauss curvature. We mention also the paper by Kolar-Lambel where degenerate revolution hypersurfaces in \( \mathbb{C}^2 \) were studied.

Consider a real hypersurface of revolution type defined by
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\[ M = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} \mid r = 0\} \]
\[ r = p(z, \overline{z}) + q(w, \overline{w}), \quad q(w, \overline{w}) = \overline{q(w, \overline{w})}, \quad d(q)|_{\{q=0\}} \neq 0, \]
\[ p(z, \overline{z}) = \sum_{1 \leq \alpha, \beta \leq n} h_{\alpha\beta} z^{\alpha} \overline{z}^{\beta}. \]  

(1)

Here \((h_{\alpha\beta})\) is a positive definite Hermitian matrix. Such a real hypersurface apparently admits a \(U(n)\)-action and was studied by Webster in [W02]. Associated with such a real hypersurface is a domain \(D_0\) in \(\mathbb{C}\) defined by \(D_0 := \{w \in \mathbb{C} : q(w, \overline{w}) < 0\}\). Assume that \(M\) is strongly pseudoconvex in a certain neighborhood \(U_0\) of \(w_0 \in D_0\), Webster observed that then \(h := -(\log q)_{w\overline{w}} > 0\) in \(U_0\) and thus we have a well-defined Hermitian metric \(ds^2 = h dw d\overline{w}\). Write the Gauss curvature of such a metric as \(K\). Define the Gauss curvature of this metric by \(K = -\frac{1}{h} \frac{\partial^2}{\partial \overline{w} \partial w} \log h\). Write \(M_0 \subset M\) for an open piece of \(M\) whose projection to the \(w\)-space in \(U_0\). We first prove the following result, which reveals the connection between the hermitian geometry over \(D_0\) and the local smooth CR embeddability of \(M\) into a sphere with lower codimension:

**Theorem 1.1** Let \(M\) be a strongly pseudoconvex real hypersurface of revolution in \(\mathbb{C}^{n+1}\) defined as in (1) with \(2 \leq n \leq N \leq 2n - 2\). Let \(D_0, U_0, K\) and \(M_0\) be just defined as above. Suppose the Gauss curvature \(K \geq -2\) over \(U_0\) and there is a non-constant smooth CR map from \(M_0\) into \(\partial \mathbb{B}^{N+1}\). Then \(K \equiv -2\) over \(U_0\) and the embedding image of \(M\) in \(\partial \mathbb{B}^{N+1}\) is totally geodesic, namely, a CR transversal intersection of an affine complex subspace of dimension \((n+1)\) with \(\partial \mathbb{B}^{N+1}\).

**Example 1.2** Let \(q = |w|^2 + \varepsilon |w|^4 - 1\) and \((h_{\alpha\beta}) = I_{n \times n}\) in (1). Then, for \(\varepsilon > 0\), \(M\) admits a non-totally geodesic holomorphic embedding into the unit sphere in \(\mathbb{C}^{n+2}\) through the map: \((z, w) \mapsto (z, w, \sqrt{\varepsilon} w^2)\). However, for \(\varepsilon < 0\), the Gauss curvature \(K\) of \(ds^2 = -(\log q)_{w\overline{w}} dw \otimes d\overline{w}\) is given by \(K = -2 - 4 \varepsilon + o(1) > -2\) near a neighborhood of \(w = 0\). (See Example 7.1.) Thus, by our theorem and the algebraicity theorem of the first author [Hu94], \(M\) in this setting can not be locally embedded into \(\partial \mathbb{B}^{N+1}\) with \(N \leq 2n - 2\). Hence the curvature assumption is needed in Theorem 1.1. Similarly, let \(q = |w|^2 + \varepsilon |w|^4 + |w|^6 - 1\) with \(\varepsilon < 0, |\varepsilon| << 1\). Then \(M\) defined by \(r = |z|^2 + |w|^2 + \varepsilon |w|^4 + |w|^6 - 1\) is now compact and strongly pseudoconvex. Since the Gauss curvature \(K\) defined above now is greater than \(-2\) in a neighborhood of \(0\) in \(D_0\), combing Theorem 1.1 with the algebraicity theorem of the first author in [Hu94], we also see that any open piece of \(M\) can not be smoothly CR embedded into \(\partial \mathbb{B}^{N+1}\) with \(N \leq 2n - 2\). However, we do not know if the assumption \(N \leq 2n - 2\) can be dropped.
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Our proof of Theorem 1.1 is based on the framework established in [EHZ04], computations of Pseudo-Hermitian curvature tensor in [We02] and the following rigidity lemma obtained by the first author:

**Regidity Lemma** [Hu99]: Let $g_1, \ldots, g_k, f_1, \ldots, f_k$ be holomorphic functions in $z \in \mathbb{C}^n$ near 0. Assume $g_j(0) = f_j(0) = 0$ for all $j$. Let $A(z, \overline{z})$ be real-analytic near the origin such that

$$
\sum_{j=1}^{k} g_j(z) f_j(z) = |z|^2 A(z, \overline{z}).
$$

(2)

If $k \leq n - 1$, then $A(z, \overline{z}) \equiv 0$ and $\sum_{j=1}^{k} g_j(z) f_j(z) \equiv 0$.

This rigidity lemma has also played an important role in understanding many other problems in CR geometry. For instance, the proof of the third gap theorem [HJY12] is obtained by repeatedly applying this lemma in subtle ways. In [EHZ04], a different formulation of the above lemma was formulated. A new formulation of this rigidity lemma is presented in Lemma 2.1 of §2, and will be used in this paper.

Along the same lines of applying the above rigidity lemma, we also study rigidity problems for conformal maps between a class of Kähler manifolds with pseudo-conformally flat metrics. More precisely, we prove the following:

**Theorem 1.3** Let $f : (X, \omega) \to (Y, \sigma)$ be a holomorphic conformal embedding, where $(X, \omega)$ and $(Y, \sigma)$ are Kähler manifolds with $\dim \mathbb{C} X = n$ and $\dim \mathbb{C} Y = N$. Suppose $2 \leq n \leq N \leq 2n - 1$ and that the curvature tensors of $(X, \omega)$ and $(Y, \sigma)$ are pseudo-conformally flat. Then $f(X)$ is a totally geodesic submanifold of $Y$.

Here we mention that a holomorphic map $f : (M, \omega) \to (N, \sigma)$ between Hermitian manifolds $M$ and $N$ is called conformal if $f^* \sigma = k \omega$ holds for some positive constant $k$ on $M$. A tensor $T_{\alpha\overline{\beta}\mu\nu}$ over a complex manifold is called pseudo-conformally flat (cf. [EHZ04]) if in any holomorphic chart, we have

$$
T_{\alpha\overline{\beta}\mu\nu} = H_{\alpha\overline{\beta}} g_{\mu\nu} + \hat{H}_{\mu\overline{\nu}} g_{\alpha\overline{\beta}} + \hat{H}^*_{\alpha\sigma} g_{\mu\overline{\nu}} + \hat{H}^*_{\mu\sigma} g_{\alpha\overline{\beta}}
$$

(3)

where $(H_{\alpha\overline{\beta}}), (\hat{H}_{\alpha\overline{\beta}}), (H^*_{\alpha\sigma})$ and $(\hat{H}^*_{\mu\sigma})$ are smoothly varied Hermitian matrices, and $(g_{\alpha\overline{\beta}})$ is the smoothly varied Hermitian metric, over the chart.

Basic examples for Hermitian manifolds with pseudo-conformally flat curvature tensors are the complex space forms: $\mathbb{C}^n$ with Euclidean metric, $\mathbb{CP}^n$ with the Fubini-Study metric and $\mathbb{B}^n$ with Poincaré metric (see §2). Other more complicated examples contain the Bochner-Kahler manifolds [Br01].
Concerning the dimension condition \( N \leq 2n - 1 \) in Theorem 1.2, we recall some related results on global holomorphic immersions. For \( \mathbb{CP}^n \), Feder proved in 1965 [Fed65] that any holomorphic immersion \( f : \mathbb{CP}^n \to \mathbb{CP}^N \) with \( N \leq 2n - 1 \) has totally geodesic image (realizing \( \mathbb{CP}^n \) as a linear subvariety). For \( X = \mathbb{B}^n / \Gamma \), Cao and Mok proved in 1990 [CM90] that if \( f : X \to Y \) is a holomorphic immersion where \( X \) and \( Y \) are complex hyperbolic space forms of complex dimension \( n \) and \( N \) respectively, such that \( X \) is compact and \( N \leq 2n - 1 \), then \( f \) has totally geodesic image. In CR geometry, we have the rigidity theorem [Hu99]: if \( F : \partial \mathbb{B}^{n+1} \to \partial \mathbb{B}^{N+1} \) is a CR map which is \( C^2 \)-smooth with \( 1 \leq n \leq N \leq 2n - 1 \), then \( F \) must be linear fractional. Also, Mok had constructed an example [Mok02] of a non-totally geodesic holomorphic isometric embedding from the disc \( \Delta \) into \( \Delta^p \). For other related rigidity results, we refer the reader to the papers by Calabi [Ca53], Mok-NG [MN], Mok [Mok], Yuan-Zhang [YZ12] and many references therein.

2 A tensor version of the rigidity lemma

We first reformulate the rigidity lemma mentioned in (2) into the following version: (See also related formulations in [EHZ04])

**Lemma 2.1** Let \( A_{\alpha\beta} \) and \( B_{\alpha\beta} \) be complex numbers where \( 1 \leq \alpha, \beta \leq n, n+1 \leq a \leq N \). Let \((g_{\alpha\beta})\) and \((G_{ab})\) be Hermitian matrices with \((g_{\alpha\beta})\) positive definite. Let \((H^{(l)}_{\alpha\beta}), (\hat{H}^{(l)}_{\alpha\beta}), (H^{*(l)}_{\alpha\beta}), (\tilde{H}^{(l)}_{\alpha\beta})\) be Hermitian matrices where \( 1 \leq l \leq k \). Suppose that \( N - n \leq n - 1 \) and that

\[
\sum_{a,b=n+1}^{N} G_{ab} A_{\alpha\beta} B_{\mu\nu} X^{\alpha} X^{\beta} X^{\mu} X^{\nu} = \sum_{l=1}^{k} (H^{(l)}_{\alpha\beta} g_{\mu\nu} + \hat{H}^{(l)}_{\alpha\beta} g_{\mu\nu} + H^{*(l)}_{\alpha\beta} g_{\mu\nu} + \tilde{H}^{(l)}_{\alpha\beta} g_{\mu\nu}) X^{\alpha} \overline{X^{\beta}} X^{\mu} X^{\nu}
\]

holds for any \( X = (X^\alpha) = (X^\beta) = (X^\mu) = (X^\nu) \in \mathbb{C}^n \). Then

\[
\sum_{a,b=n+1}^{N} G_{ab} A_{\alpha\beta} X^{\alpha} X^{\beta} B_{\mu\nu} X^{\mu} X^{\nu} \equiv 0, \quad \forall X \in \mathbb{C}^n.
\]
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Proof: The right-hand-side of (4) is equal to
\[
\sum_{l=1}^{k} (H^{(l)}_{\alpha\beta}g_{\alpha\beta} + \tilde{H}^{(l)}_{\mu\nu}g_{\alpha\beta} + \tilde{H}^{(l)}_{\alpha\beta}X^{\alpha}X^{\mu}X^{\nu}) = |X|^2 \sum_{l=1}^{k} \left( H^{(l)}_{\alpha\beta}X^{\alpha}X^{\beta}X^{\mu}X^{\nu} + \tilde{H}^{(l)}_{\mu\nu}X^{\mu}X^{\nu} \right) = |X|^2 A(X, X)
\]
where \( A(X, X) \) is some real analytic function of \( X \). Then the left hand side of (4) is equal to
\[
\sum_{a,b=n+1}^{N} G_{ab} A_{a,\beta} X^{\alpha}X^{\beta}X^{\mu}X^{\nu} = \sum_{a=n+1}^{N} g_a(X) h_a(X)
\]
where \( g_a(X) = \sum_{\alpha,\beta} A_{a,\beta} X^{\alpha}X^{\beta} \) and \( h_a(X) = \sum_{b=n+1}^{N} \sum_{\alpha,\beta} G_{ab} B_{a,\beta} X^{\alpha}X^{\beta} \) are holomorphic functions. Namely, we have
\[
\sum_{a=n+1}^{N} g_a(X) h_a(X) = |X|^2 A(X, X), \quad \forall X \in \mathbb{C}^n.
\]
By the hypothesis: \( N - n < n \), it concludes from (2) that \( A(X, X) \equiv 0 \), and thus (5) holds. \( \Box \)

3 Pseudo-Hermitian geometry

CR submanifold of hypersurface type Let \( M \) be a smooth strictly pseudoconvex \((2n+1)\)-dimensional CR submanifold in \( \mathbb{C}^{n+1} \). We have the complexified tangent bundle \( CTM \) which admits the decomposition \( CTM = T^{(1,0)}M \oplus T^{(0,1)}M \). A non-zero real smooth 1-form \( \theta \) along \( M \) is said to be a contact of \( M \) is \( \theta|_p \) annihilates \( T^{(1,0)}_p M \oplus T^{(0,1)}_p M \) for any \( p \in M \). Let \( r \) be a local defining function of \( M \). Then \( \theta = i\partial_r \) is a contact form of \( M \) and any other contact form is a multiple of \( \theta \): \( k\theta \) with \( k \neq 0 \) a smooth function along \( M \).

Now, fix a contact form \( \theta \). Then there is a unique smooth vector field \( T \), called the Reeb vector field such that: (i) \( \theta(T) \equiv 1 \), (ii) \( d\theta(T, X) \equiv 0 \) for any smooth tangent vector field \( X \) over \( M \). The Levi-form \( L_{\theta} \) with respect to \( \theta \) at \( p \in M \) is defined by
\[
L_{\theta}(u, v) := -i\theta(u \wedge \overline{v}) = i\theta([u, \overline{v}]), \quad \forall u, v \in T^{1,0}_p(M), \quad \forall p \in M.
\]
Recall that we say \((M, \theta)\) to be \textit{strictly pseudoconvex} if the Levi-form \(L_\theta\) is positive definite for all \(z \in M\).

Let \(T' M\) be the annihilator bundle of \(\mathcal{V} := T^{(0,1)} M\) which is a rank \(n + 1\) subbundle of \(\mathbb{C} T^* M\).

\textbf{Admissible coframe} If we choose a local basis \(L_\alpha, \alpha = 1, \ldots, n\), of \((1, 0)\) vector fields (i.e. sections of \(\mathcal{V} = T^{1,0}_M\)), so that \((T, L_\alpha, L_\alpha^\ast)\) is a frame for \(\mathbb{C} T M := \mathbb{C} \otimes T M\) where \(L_\alpha^\ast = \overline{L_\alpha}\). Then the equation in (ii) above is equivalent to

\[d\theta = ig_\alpha^\beta \theta^\alpha \wedge \theta^\beta.\] 

(9)

Here \(\theta^\beta = \overline{\theta^\beta}\) and \((g_\alpha^\beta)\) is the (hemitian) Levi form matrix and \((\theta, \theta^\alpha, \theta^\beta)\) is the coframe dual to \((T, L_\alpha, L_\alpha^\ast)\). (For brevity, we shall say that \((\theta, \theta^\alpha)\) is the coframe dual to \((T, L_\alpha)\)). Note that \(\theta\) and \(T\) are real whereas \(\theta^\alpha\) and \(L_\alpha\) always have non-trivial real and imaginary parts.

Without mentioning \(T\), we can complete \(\theta\) to a coframe \((\theta, \theta^\alpha)\) by adding \((1, 0)\)-cotangent vectors (the cotangent vectors that annihilate \(V\) \(\theta^\alpha\)). The coframe is called \textit{admissible} if \(\langle \theta^\alpha, T \rangle = 0, \) for \(\alpha = 1, \ldots, n\). As other equivalent definitions, \((\theta, \theta^\alpha)\) is admissible if (9) holds.

\textbf{Pseudo-Hermitian geometry on} \(M\) \hspace{1cm} Observe that (by the uniqueness of the Reeb vector field) for a given contact form \(\theta\) on \(M\), the admissible coframes are determined up to transformations

\[\tilde{\theta}^\alpha = u_\beta^\alpha \theta^\beta, \hspace{0.5cm} (u_\beta^\alpha) \in GL(\mathbb{C}^n).\]

Every choice of a contact form \(\theta\) on \(M\) is called \textit{pseudo-Hermitian structure} and defines a hermitian metric on \(\mathcal{V}\) (and on \(\overline{\mathcal{V}}\)) via the (positive-definite) Levi form (see (8)). For every such \(\theta\), Tanaka [T75] and Webster [W78] defined a \textit{pseudo-Hermitian connection} \(\nabla\) on \(\mathcal{V}\) (and also on \(\mathbb{C} T M\)) which is expressed relative to an admissible coframe \((\theta, \theta^\alpha)\) by

\[\nabla L_\alpha = \omega^\beta_\alpha \otimes L_\beta\]

where the 1-forms \(\omega^\alpha_\beta\) on \(M\) are uniquely determined by the conditions

\[d\theta^\beta = \theta^\alpha \wedge \omega^\beta_\alpha \text{ mod } (\theta \wedge \theta^\alpha), \hspace{0.5cm} dg_\alpha^\beta = \omega^\beta_\alpha + \omega^\beta_\alpha^\gamma \wedge \omega^\gamma_\alpha.\] 

(10)

We may rewrite the first condition in (10) as

\[d\theta^\beta = \theta^\alpha \wedge \omega^\beta_\alpha + \theta \wedge \tau^\beta, \hspace{0.5cm} \tau^\beta = A^\beta_\alpha \theta^\alpha, \hspace{0.5cm} A^\alpha_\beta = A^\beta_\alpha \]

(11)

for a suitably determined torsion matrix \((A^\beta_\alpha)\), where the last symmetry relation holds automatically (see [W78]).

The \textit{pseudo-Hermitian curvature} \(R^\beta_\alpha_\mu_\nu\) and \(W^\beta_\alpha_\mu\) of the pseudoHermitian connection is given, in view of [W78, (1.27), (1.41)], by

\[d\omega^\beta_\alpha - \omega^\beta_\gamma \wedge \omega^\gamma_\alpha = R^\beta_\alpha_\mu_\nu \theta^\mu \wedge \theta^\nu + W^\beta_\alpha_\mu \theta^\mu \wedge \theta - W^\beta_\alpha_\mu_\nu \theta^\nu \wedge \theta + i\theta^\alpha \wedge \tau^\beta - i\tau^\alpha \wedge \theta^\beta.\]

(12)
4 Local CR embeddings

Coframes on $f : M \to \hat{M}$ Let $f : M \to \hat{M}$ be a local CR embedding where $M$ is a strictly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$ and $\hat{M}$ is a strictly pseudoconvex hypersurface in $\mathbb{C}^{\hat{n}+1}$. We use a $\hat{}$ to denote objects associated to $\hat{M}$. We shall also omit the $\hat{}$ over frames and coframes if there is no ambiguity. It will be clear from the context if a form is pulled back to $M$ or not. Under the above assumptions, we identify $M$ with the submanifold $f(M)$ and write $M \subset \hat{M}$. Capital Latin indices $A, B, \ldots$ will run over the set $\{1, \ldots, \hat{n}\}$. Greek indices $\alpha, \beta, \ldots$ will run over $\{1, \ldots, n\}$; Small Latin indices $a, b, \ldots$ will run over the complementary set $\{n + 1, \ldots, \hat{n}\}$.

Let $(\theta, \theta^\alpha)$ and $(\hat{\theta}, \hat{\theta}^A)$ be coframes on $M$ and $\hat{M}$ respectively, and recall that $f$ is a CR mapping if

$$f^*(\hat{\theta}) = a \theta, \quad f^*(\hat{\theta}^A) = E^A_{\alpha} \theta^\alpha + E^A \theta,$$

where $a$ is a real-valued function and $E^A_{\alpha}, E^A$ are complex-valued functions. Applying $f^*$ to the equation

We identify $M$ with the submanifold $f(M)$ of $\hat{M}$ and write $M \subset \hat{M}$. Then the CR bundle $\mathcal{V} = T^{0,1}M$ is a rank $n$ subbundle of $\mathcal{V} = T^{0,1} \hat{M}$ along $M$. Then there is a rank $(\hat{n} - n)$ subbundle $N'M$ consisting of $1$-forms on $\hat{M}$ whose pullbacks to $M$ by $f$ vanish. The subbundle $N'M$ is called the holomorphic conormal bundle of $M$ in $\hat{M}$.

We write $i^*$ for the standard pull back map and $i_*$ for the push-forward map. Notice that our consideration is purely local. We let $p \in M$ and fix a local admissible coframe $\{\theta, \theta^\alpha\}$ for $M$. Let $T$ be the Reeb vector field associated with $\theta$. Assume that $\hat{M}$ is a small neighborhood of $0$ in $\mathbb{R}^\hat{m}$, $p = 0$ and $M$ is defined near $0$ by $x_j = 0$ with $j = m + 1, \ldots, \hat{m}$. First, we can extend $\theta$ to a contact form of $\hat{M}$ in a neighborhood of $0$. Write $x' = (x_1, \ldots, x_m)$. Define $\hat{\theta} = u \theta$, with $u(x', 0) \equiv 1$. Then $d\hat{\theta} = du \wedge \theta + u d\theta$. We want $d\hat{\theta} \wedge T = 0$ along $M$. For this, we write $u d\theta \wedge T = \sum_{j=m+1}^{\hat{m}} d_j(x', 0) dx_j$. Then, we need to have, along $M$: $du = \sum_{j=1}^{m} d_j(x', 0) dx_j$. Since $T$ is the Reeb vector field for $\theta$ along $M$, we have $d_j(x', 0) = 0$ for $j \leq m$. Thus, choose $u = 1 + \sum_{j=m+1}^{\hat{m}} d_j(x', 0) x_j$. Then we have $d\hat{\theta} \wedge T = 0$ along $M$. Now, by the uniqueness of the Reeb vector field, we see that Reeb vector field $\hat{T}$ of $\hat{\theta}$, when restricted to $M$, coincides with $T$. Extend $\theta^\alpha$ to a neighborhood of $0$ in $\hat{M}$ to get $\hat{\theta}^\alpha$, and add $\theta^\alpha$ so that $\{\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^\alpha\}$ forms a basis for $T'\hat{M}$ near $0$. Apparently, after a linear change for the forms $\{\hat{\theta}^\alpha, \hat{\theta}^\alpha\}$, we can assume that the pull-back of $\hat{\theta}^\alpha$ to $M$ is zero for each $\alpha = n + 1, \ldots, \hat{n}$, the pull back of $\hat{\theta}^\alpha$ to $M$ is $\theta^\alpha$ for $\alpha = 1, \ldots, n$, $\theta$ remains the same, and $\{\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^\alpha\}$ is an admissible coframe along $\hat{M}$ near $0$.

Next, suppose that $d\theta = \sqrt{-1}g_{\alpha\beta} \theta^\alpha \wedge \theta^\beta$ with $g_{\alpha\beta} = \delta_{\alpha\beta}$ along $M$. We can even make the Levi form of $\hat{M}$ with respect to the co-frame $\{\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^\alpha\}$ also the identical matrix along $M$. 


Indeed, let \( \{T, L_\alpha\} \) be the dual frame of \( \{\theta, \theta^\alpha\} \) along \( M \). Extend \( L_\alpha \) to a vector field of type \((1, 0)\) in a neighborhood of \( 0 \) in \( \tilde{M} \). Find \( \{\tilde{L}_\alpha\} \) so that \( \{\tilde{L}_\alpha, \tilde{L}_\alpha\} \) forms a base of vector fields of type \((1, 0)\) over \( \tilde{M} \) with its Levi form along \( \tilde{M} \) near \( 0 \) the identical matrix. Let \( \{\tilde{\theta}, \tilde{\theta}^\alpha, \tilde{\theta}^{\alpha}\} \) be the dual co-frame of \( \{\tilde{T}, \tilde{L}_\alpha\} \). Then along \( M \), \( i^*(\tilde{\theta}^\alpha, L_\alpha) = <\tilde{\theta}^\alpha, \tilde{L}_\alpha > |_M = 0; \)
\( i^*(\tilde{\theta}^{\alpha}), T > = <\tilde{\theta}^\alpha, \tilde{T}|_M > = 0 \). Hence the pull back of \( \tilde{\theta}^\alpha \) to \( M \) is zero. Clearly, the pull-back to \( \tilde{\theta}^{\alpha} \) to \( M \) is \( \theta^{\alpha} \) and \( i^*(\hat{\theta}) = \theta \). Assume that
\[
d\hat{\theta} = \sqrt{-1}g_{AB}\hat{\theta}^A \wedge \hat{\theta}^B + \sum_{A=1}^{\hat{n}} (e_A(x)\hat{\theta}^A + \overline{e_A(x)}\overline{\hat{\theta}^A}) \wedge \hat{\theta}.
\]
Contracting along \( \hat{T} \), we see that \( e_A \equiv 0 \). Hence, we see that \( \{\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^{\alpha}\} \) is an admissible co-frame. Now, the Levi form of \( \tilde{M} \) along \( M \) is the identity with respect to such a frame.

We say that the pseudo-Hermitian structure \((\tilde{M}, \hat{\theta})\) is admissible for the pair \((M, \tilde{M})\) if the Reeb vector field \( \hat{T} \) for \( \hat{\theta} \) is tangent to \( M \). With the just obtained co-frame \( \{\hat{\theta}, \hat{\theta}^A\} \) on \( \tilde{M} \) where \( A = 1, 2, \ldots, \hat{n} \), the holomorphic conormal bundle \( N'M \) is spanned by the linear combinations of the \( \hat{\theta}^a \). Summarizing the above, we see the following basic fact from \([EHZ04]\):

**Proposition 4.1 ([EHZ04], Corollary 4.2)** Let \( M \) and \( \tilde{M} \) be strictly pseudoconvex CR-manifolds of dimensions \( 2n+1 \) and \( 2\hat{n}+1 \) respectively. Let \( f : M \to \tilde{M} \) be a CR embedding. If \((\theta, \theta^\alpha)\) is any admissible coframe on \( M \), then in a neighborhood of any point \( \hat{p} \in f(M) \) in \( \tilde{M} \) there exists an admissible coframe \((\hat{\theta}, \hat{\theta}^A)\) on \( \tilde{M} \) with \( f^*(\hat{\theta}, \hat{\theta}^A, \hat{\theta}^{\alpha}) = (\theta, \theta^\alpha, 0) \). In particular, \( \hat{\theta} \) is admissible for the pair \((f(M), \tilde{M})\), i.e., the Reeb vector field \( \hat{T} \) is tangent to \( f(M) \). Also, when the Levi form of \( M \) with respect to the co-frame \((\theta, \theta^\alpha)\) is the identical matrix, then we can also choose \((\hat{\theta}, \hat{\theta}^A)\) such that the Levi form of \( \tilde{M} \) with respect to \((\hat{\theta}, \hat{\theta}^A)\) is also the identical matrix.

If we fix an admissible co-frame \((\theta, \theta^\alpha)\) on \( M \) and let \((\hat{\theta}, \hat{\theta}^A)\) be an admissible co-frame on \( \tilde{M} \) near a point \( \hat{p} \in f(M) \), we shall say \((\hat{\theta}, \hat{\theta}^A)\) is adapted to \((\theta, \theta^\alpha)\) on \( M \) if it satisfies the conclusions of the above Proposition above. We also normalize the Levi-forms with these frame such that they are identical.

**Second fundamental form** Equation (11) implies that when \((\theta, \theta^A)\) is adapted to \( M \), if the pseudoconformal connection matrix of \((\tilde{M}, \hat{\theta})\) is \( \hat{\omega}_B^\alpha \), then that of \((M, \theta)\) is the pullback of \( \hat{\omega}_B^\alpha \). The pulled back torsion \( \hat{\tau}^\alpha \) is \( \tau^\alpha \), so omitting the \(^*\) over these pullbacks will not cause any ambiguity and we shall do that from now on. By the normalization of the Levi form, the second equation in (10) reduces to
\[
\omega_B^\alpha + \omega_{B\tilde{A}} = 0,
\]
5 THE PSEUDO-CONFORMAL GEOMETRY

where as before $\omega_{AB} = \overline{\omega_{BA}}$.

The matrix of 1-forms $(\omega^b_a)$ pulled back to $M$ defines the second fundamental form of the embedding $f : M \to \hat{M}$. Since $\theta^b = 0$ on $M$, equation (11) implies that on $M$,

$$\omega^b_a \wedge \theta^a + \tau^b \wedge \theta = 0,$$

and this implies that

$$\omega^b_a = \omega^b_{a \beta} \theta^\beta, \quad \omega^b_{a \beta} = \omega^b_{\beta a}, \quad \tau^b = 0.$$

Following [EHZ04], we identify the CR-normal space $T^{1,0}_p \hat{M} / T^{1,0}_p M$, also denoted by $N^{1,0}_p \hat{M}$ with $\mathbb{C}^{n-n}$ by choosing the equivalence classes of $L_\alpha$ as a basis. Therefore for fixed $\alpha, \beta = 1, ..., n$, we view the component vector $(\omega^a_{\alpha \beta})_{a=n+1, ..., \hat{n}}$ as an element of $\mathbb{C}^{n-n}$. Also view the second fundamental form as a section over $M$ of the bundle $T^{1,0}M \otimes N^{1,0}_p \hat{M} \otimes T^{1,0}M$.

5 The Pseudo-conformal geometry

Pseudo-conformal geometry We will need the pseudo-conformal connection and structure equations introduced by Chern and Moser in [CM74]. Let $Y$ be the bundle of coframes $(\omega, \omega^\alpha, \omega^\beta, \phi)$ on the real ray bundle $\pi_E : E \to M$ of all contact forms defining the same orientation of $M$, such that $d\omega = ig_{\alpha \beta} \omega^\alpha \wedge \omega^\beta + \omega \wedge \phi$ where $\omega^\alpha \in \pi_E^*(T' M)$ and $\omega$ is the canonical 1-form on $E$. In [CM74] it was shown that these forms can be completed to a full set of invariants on $Y$ given by the coframe of 1-forms

$$(\omega, \omega^\alpha, \omega^\beta, \phi, \phi^\alpha_{\beta}, \phi^\beta, \psi)$$

which define the pseudo-conformal connection on $Y$.

$$\begin{align*}
\phi_\alpha + \phi^\alpha_\beta &= d\phi^\beta, \\
d\omega &= i\omega^\mu \wedge \omega_\mu + \omega \wedge \phi, \\
d\omega^\alpha &= \omega^\mu \wedge \phi^\alpha_\mu + \omega \wedge \phi^\alpha, \\
d\phi &= i\omega^\gamma \wedge \phi_\gamma + i\phi_\sigma \wedge \phi_\sigma + \omega \wedge \psi, \\
d\phi^\alpha_{\beta} &= \phi^\mu_{\beta} \wedge \phi^\alpha_\mu + i\omega_{\beta} \wedge \phi^\alpha - i\phi_{\beta} \wedge \omega^\alpha - i\delta_{\beta}\phi^\alpha_\mu \wedge \omega^\mu - \frac{\delta_{\beta}\phi^\alpha_{\mu}}{2} \psi \wedge \omega + \Phi^\alpha_{\beta}, \\
d\phi^\alpha &= \phi \wedge \phi^\alpha + \phi^\mu \wedge \phi^\alpha_\mu - \frac{1}{2} \psi \wedge \omega^\alpha + \Phi^\alpha, \\
d\psi &= \phi \wedge \psi + 2i\omega^\mu \wedge \phi_\mu + \Psi,
\end{align*}$$

(17)
where the curvature 2-forms $\Phi^\beta_\alpha$, $\Phi^\alpha_\beta$ and $\Psi$ are decomposed as

$$
\begin{align*}
\Phi^\beta_\alpha &= S^\beta_\alpha \mu \nu \omega^\mu \wedge \omega^\nu + V^\beta_\alpha \mu \omega^\mu \wedge \omega + V^\beta_\alpha \nu \omega^\nu \wedge \omega, \\
\Phi^\alpha_\beta &= V^\alpha_\beta \mu \nu \omega^\mu \wedge \omega^\nu + P^\alpha_\beta \mu \omega^\mu \wedge \omega + Q^\alpha_\beta \nu \omega^\nu \wedge \omega, \\
\Psi &= -2iP^\mu_\alpha \nu \omega^\mu \wedge \omega + R^\mu_\nu \omega^\mu \wedge \omega + R^\nu_\mu \omega^\nu \wedge \omega.
\end{align*}
$$

(18)

where the functions $S^\alpha_\beta \mu \nu$, $V^\alpha_\beta \mu$, $P^\alpha_\beta$, $Q^\alpha_\beta$ together represent the pseudo-conformal curvature of $M$.  

As in [CM74] we restrict our attention here to coframes $(\theta, \theta^\alpha)$ for which the Levi form $(g_\alpha^\beta)$ is constant. The 1-forms $\phi^\alpha$, $\phi^\alpha$, $\phi^\beta_\alpha$, $\psi$ are uniquely determined by requiring the coefficients in (18) to satisfy certain symmetry and trace conditions (see [CM74] and the appendix), e.g.

$$
S^\beta_\alpha \mu \nu = S^\beta_\alpha \mu \nu \alpha \beta = S^\beta_\mu \mu \alpha \beta = S^\mu_\alpha \mu \beta = 0
$$

Let us fix a contact form $\theta$ that defines a section $M \rightarrow E$. Then any admissible coframe $(\theta, \theta^\alpha)$ for $T^{1,0}M$ defines a unique section $M \rightarrow Y$ for which the pullbacks of $(\omega, \omega^\alpha)$ coincide with $(\theta, \theta^\alpha)$ and the pullback of $\phi$ vanishes. As in [W78], we shall use the same notation for the pulled back forms on $M$ (that now depend on the choice of the admissible coframe). With this convention, we have

$$
\theta = \omega, \theta^\alpha = \omega^\alpha, \phi = 0
$$

(19)

on $M$.

Relationship between pseudo-conformal geometry and pseudo-Hermitian geometry  

In view of Webster [W78, (3.8)], the pulled back tangential pseudoconformal curvature tensor $S^\alpha_\beta \mu \nu$ can be obtained from the tangential pseudo-Hermitian curvature tensor $R^\alpha_\beta \mu \nu$ in (12) by

$$
S^\alpha_\beta \mu \nu = R^\alpha_\beta \mu \nu - \frac{R^\alpha_\beta \mu \nu \alpha \beta + R^\alpha_\beta \mu \beta \alpha + R^\alpha_\beta \mu \alpha \gamma g^\gamma_\beta + R^\mu_\beta \alpha \gamma g^\gamma_\beta}{n+2} + \frac{R(g_\alpha^\gamma g^\gamma_\beta + g_\alpha^\beta g^\beta_\gamma)}{(n+1)(n+2)}
$$

(20)

where

$$
R^\alpha_\beta := R^\mu_\alpha \beta \mu \text{ and } R = R^\mu_\mu
$$

are respectively the pseudo-Hermitian Ricci and scalar curvature of $(M, \theta)$.

---

1The indices of $S^\alpha_\beta \mu \nu$ here are interchanged comparing to [CM74] to make them consistent with indices of $R^\alpha_\beta \mu \nu$ in (12).
Traceless component  Following the terminology in [EHZ04], we call a tensor $T_{\alpha_1...\alpha_t,\beta_1...\beta_s}$ pseudo-conformally equivalent to 0 or pseudo-conformally flat if it is a linear combination of $g_{\alpha_i\beta_j}$ for $i = 1, 2, ..., t$ and $j = 1, 2, ..., s$. Two tensors $T_{\alpha_1,\beta_1} - T_{\alpha_2,\beta_2}$ are called conformally equivalent if $R_{\alpha_3,\beta_3}$ are pseudo-conformally flat. For any tensor $R_{\alpha\beta\mu\nu}$, its traceless component is the unique tensor that is trace zero and that is conformally equivalent to $R_{\alpha\beta\mu\nu}$. We denote the traceless component by $[R_{\alpha\beta\mu\nu}]$. Formula (20) expresses the fact that $S_{\alpha\beta\mu\nu}$ is the “traceless component” of $R_{\alpha\beta\mu\nu}$ (cf. [EHZ04], (5.5)): 

$$S_{\alpha\beta\mu\nu} = [R_{\alpha\beta\mu\nu}].$$  

(21)

6  Real Hypersurface of Revolution

Real hypersurfaces of revolution  Let $M = \{(z,w) \mid r = 0\}$ be a real hypersurface of revolution in $\mathbb{C}^{n+1}$ with $n \geq 2$ where

$$r = p(z,\bar{z}) + q(w,\bar{w}), \quad q = \bar{q} \text{ and } p(z,\bar{z}) = h_{\alpha\beta}z^\alpha\bar{z}^\beta.$$  

(22)

where $(g_{\alpha\beta})$ is a positive definite Hermitian matrix. Also $d(q) \neq 0$ when $q = 0$.

Define $D := \{(z,w) \mid r < 0\}$. As the auxiliary curve and domain in $\mathbb{C}$, we define $M_0 := \{w \mid q(w,\bar{w}) = 0\}$ and $D_0 := \{w \mid q(w,\bar{w}) < 0\}$. $M$ is strictly pseudoconvex if and only if on $D_0 := \{q < 0\}$, $h := -(\log q)_{w\bar{w}} = q_{w\bar{w}}/q^2 > 0$. Assume that $M$ is strictly pseudoconvex. Then $D_0$ admits a Hermitian metric $ds^2 = hdwd\bar{w}$. We denote by $K$ its Gaussian curvature on $D_0$. It was proved in [W02] that for $w \in D_0$ and $(z,w) \in M$ with $n \geq 2$ and $dq \neq 0$, the fourth order Chern-Moser tensor $S(z,w) = 0$ if and only if $K(w) = -2$.

The pseudo-Hermitian curvature of $M$  By Webster, at the point where $d(q) \neq 0$, the pseudo-Hermitian curvature of $M$ is calculated as

$$R_{\beta\gamma\rho\sigma} = -A(g_{\beta\alpha}g_{\rho\sigma} + g_{\beta\rho}g_{\alpha\sigma}) - Bp_{\beta}p_{\alpha}p_{\rho}p_{\sigma}$$  

(23)

where

$$A = -\frac{Q}{1 - Qq}, \quad g_{\alpha\beta} = h_{\alpha\beta} + Qp_{\alpha}p_{\beta}, \quad \theta = -i\partial r, \quad \theta^\alpha = dz^\alpha - i\eta^\alpha \theta, \quad \eta^\alpha = g^{\alpha\beta}\eta_{\beta}, \quad \eta_{\alpha} = -Qp_{\alpha};$$  

(24)

and

$$B = \frac{Q_{w\bar{w}}}{q_wq_{\bar{w}}} + 2Q\left(\frac{Q_w}{q_w} + \frac{Q_{\bar{w}}}{q_{\bar{w}}}\right) + 3Q^3 + \frac{q(Q_{w\bar{w}}/q_{w\bar{w}}) + Q^2|q_{w\bar{w}}|}{1 - Qq}$$  

(25)
where \( Q = \frac{q w q w}{q w q w} \). Notice that the formulas above were slightly modified from those in [We02], since we need \((g_{\alpha \beta})\) to be positive definite to apply the Gauss-Codazzi equation here.

Here \( B \) can also be calculated as

\[
B = \frac{(K + 2)k^2}{q^3(q w q w)^2}
\]

where \( k = q w q w - q w q w \). We notice that \( B \) is a real-valued function and \( B \leq 0 \) if and only if \( K + 2 \geq 0 \).

**Umbilic points of the fourth order Chern-Moser tensor** \( S \)

Let \( S \) be the fourth order Chern-Moser tensor when \( n \geq 2 \). (For \( n = 1 \), it is replaced by the Cartan invariant). A point \((z, w) \in M\) is called a **umbilic point** if \( S(z, w) = 0 \).

It was proved by Webster [W02] that let \( w \in D_0 \) and \((z, w) \in M\). Then at points where \( dq \neq 0 \),

\[
S(z, w) = 0 \quad \text{if and only if} \quad K(w) = -2.
\]

If \( B \equiv 0 \), it implies \( K \equiv -2 \) by (26).

### 7 Proof of Theorem 1.1

Let \( M_0 \) be a connected open piece of \( M = \{(z, w) \mid r = 0\} \), that is strongly pseudoconvex in \( \mathbb{C}^{n+1} \) with \( n \geq 2 \). Here \( M \) is as in (22). Assume that \( M_0 \) project down to an open subset \( U_0 \) of \( D_0 \). Suppose that there is a non-constant CR map \( F : M_0 \to \partial \mathbb{B}^{N+1} \). By the Hopf lemma and shrinking \( M_0 \), we can assume that \( F \) is a CR embedding. Under the assumption as in Theorem 1.1, we then need to prove that \( F(M) \) must be the CR transversal intersection of an affine subspace with the sphere. After shrinking \( M_0 \) and thus \( U_0 \), we can assume that \( q w \neq 0 \) over \( U_0 \).

We take an admissible coframe \((\theta, \theta^\alpha)\) on \( M \) as mentioned before with \( \theta := -i \partial_z r \) as the contact form. Fixing any point \( p \in M_0 \), by Proposition 4.1, there exists a neighborhood \( \hat{U} \) of \( \hat{p} := F(p) \) in \( \partial \mathbb{B}^{N+1} \) and an admissible coframe \((\hat{\theta}, \hat{\theta}^\alpha)\) on \( \hat{U} \) such that \( F^*(\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^\alpha) = (\theta, \theta^\alpha, 0) \) on \( U \), where \( U \) is a neighborhood of \( p \) in \( M_0 \) such that \( F(U) = \hat{U} \).

Consider the pseudo-conformal Gauss equation (cf. (5.9) in [EHZ04])

\[
[\hat{S}(X, X, X, X)] = S(X, X, X, X) + [(II(X, X), II(X, X))], \quad \forall X \in T^1_0 F(M),
\]

where \( S \) is the pseudo-conformal curvature of \( F(M) \), \( \hat{S} \) is the restriction of the pseudo-conformal curvature of \( \partial \mathbb{B}^{N+1} \) on \( F(M) \), and \( II(X, X) \) is the second fundamental form of
\( F(M) \subset \partial B_{N+1} \). Here the notation \([\ ]\) in (21) is used and we can regard \( X \) as a vector in \( \mathbb{C}^n \). Locally it can be written as

\[
[S_{\alpha \beta \mu \nu}] = S_{\alpha \beta \mu \nu} + [g_{ab} \omega^a_{\alpha} \mu \omega^b_{\beta \nu}]
\]

where \((\omega^a_{\alpha})\) is the second fundamental form of \( F(M) \) and \( \omega^a_{\alpha} = \omega^b_{\alpha} \theta^a \), and \((g_{ab})\) is the (Levi) positive definite Hermitian matrix. Here \( \omega^a_{\alpha} \theta^a \) are functions satisfying \( \omega^a_{\alpha} \theta^a = \omega^a_{\alpha} \beta \theta^\beta \) (cf. [EHZ04], (4.3) and (5.6)). Recall the facts that the pseudo-conformal curvature of a sphere vanishes and that we have

\[
S_{\alpha \beta \mu \nu} = [R_{\alpha \beta \mu \nu}]
\]

where \( R_{\alpha \beta \mu \nu} \) is the pseudo-Hermitian curvature induced by the pseudo-Hermitian metric on \( F(M) \). Then (29) becomes

\[
0 = [R_{\alpha \beta \mu \nu} + g_{ab} \omega^a_{\alpha} \mu \omega^b_{\beta \nu}] \quad \text{(30)}
\]

Since \( F \) is a local CR embedding, we can identify the pseudo-Hermitian structure \((M, \theta)\) with \((F(M), (F^{-1})^* \theta)\). In other words, we can identify the pseudo-Hermitian curvature \( R_{\alpha \beta \mu \nu} \) on \( F(M) \) as the pseudo-Hermitian curvature over \( M \). Then from (7.1), we have

\[
p_{\beta} = h_{\beta \beta} z^\beta, \quad p_{\mu} = h_{\alpha \alpha} z^\alpha
\]

and thus

\[
\sum_{\alpha, \beta, \mu, \nu} p_{\alpha} p_{\beta} p_{\mu} p_{\nu} = \sum_{\alpha, \beta, \mu, \alpha', \beta', \mu'} h_{\alpha \alpha'} h_{\beta \beta'} h_{\mu \mu'} h_{\nu \nu'} h_{\beta \beta} z^\beta h_{\alpha \alpha} z^\alpha
\]

\[
= \left| \sum_{\beta, \mu, \beta', \mu'} h_{\beta \beta} z^\beta h_{\mu \mu} z^\mu \right|^2 \quad \text{(31)}
\]

Now, as in the proof of lemma 2.1, we have the following computation:

\[
A_{\alpha \beta} g_{\alpha \beta} X^\alpha \overline{X}^\beta X^\mu \overline{X}^\nu = B(X, \overline{X}) |X|^2,
\]

where \( |X|^2 = g_{ab} X^a \overline{X}^b \) and \( B(X, \overline{X}) = A_{ab} X^a \overline{X}^b \). We substitute (7.1) and (31) into (30) to obtain

\[
0 = |X|^2 E(X, \overline{X})
\]

\[
- B \left| \sum_{\beta, \mu, \beta', \mu'} h_{\beta \beta} z^\beta h_{\mu \mu} z^\mu \overline{X}^\beta X^\nu \right|^2
\]

\[
+ \sum_{n+1 \leq a, b \leq N} g_{ab} \omega^a_{\alpha} X^\alpha X^\mu \omega^b_{\beta \mu} \overline{X}^\beta X^\nu \quad \forall X \in \mathbb{C}^n, \text{ at } \hat{p} \in \hat{U}
\]
for some real analytic function $E(X, \overline{X})$. Since $(N - n) + 1 \leq (2n - 2) - n + 1 = n - 1$, we apply Lemma 2.1 to yield that

$$-B \left( \sum_{\beta,\nu,\beta',\nu'} h_{\beta,\beta'} z^\beta h_{\nu,\nu'} z^{\nu'} X^\beta X^{\nu'} \right) \left( \sum_{\alpha,\mu,\alpha',\mu'} \overline{h_{\alpha,\alpha'}} z^{\alpha'} h_{\mu,\mu'} z^{\mu'} X^\alpha X^\mu \right) + \sum_{a,b=n+1}^N g_{ab} \omega^a_{\alpha} X^\alpha X^\mu \omega_b^{\overline{\beta}} \overline{X}^{\beta} X^{\nu} = 0, \quad \forall X \in \mathbb{C}^n. \tag{33}$$

When $B \leq 0$, then both terms in the left hand side of the above equation is nonnegative. Hence, we get that $B \equiv 0$ over $U_0$ and

$$\sum_{a,b=n+1}^N g_{ab} \omega^a_{\alpha} X^\alpha X^\mu \omega_b^{\overline{\beta}} \overline{X}^{\beta} X^{\nu} \equiv 0, \quad \forall X \in \mathbb{C}^n.$$

Since $(g_{ab})$ is Hermitian and positive definite, it implies $\omega^a_{\alpha} \equiv 0$, $\forall a, \alpha, \mu$ so that the second fundamental form of $F(M)$ is zero.

Then either by the result of Webster in (27) or by the result in [JY10], $F(M)$ and $M$ must be spherical. Thus $F(M)$ is in the image $G(\partial \mathbb{B}^{n+1})$ for some linear fractional map $G : \partial \mathbb{B}^{n+1} \rightarrow M \subset \partial \mathbb{B}^{N+1}$, by the well-known rigidity result in [Hu99]. The proof of Theorem 1.1 is complete. \qed

**Example 7.1** Let $q = |w|^2 + \epsilon|w|^4 + \phi(w, \overline{w}) - 1$ with $\epsilon \in \mathbb{R}$ and $\phi = o(|w|^4)$ being smoothly real-valued. Now $D_0 = \{w \in \mathbb{C} : q < 0\}$. $ds^2 = -(\log q)_{w\overline{w}} dw \otimes d\overline{w}$ defines a Hermitian metric in a neighborhood of $0 \in D_0$. The formula for its Gauss curvature was derived in [(15), We02]:

$$K = -2 + q^3 k^{-3} \left( k q_{ww\overline{w}} + q |q_{ww\overline{w}}|^2 - 2 \Re(q_{ww\overline{w}}q_{w\overline{w}}) + q_{w\overline{w}} |q_{ww}|^2 \right)$$

with $k = q_w q_{\overline{w}} - q q_{\overline{w}}$. By a direct computation, one sees that $K = -2 - 4\epsilon + o(|w|)$. Hence, for $\epsilon < 0$, we have $K > 2$ in a small neighborhood of $0$ in $D_0$.
8 Examples of pseudo-conformally flat Kähler manifolds

Complex space forms A Kähler manifold of constant holomorphic sectional curvature is called a complex space form. The universal complex space forms are $\mathbb{C}^n$, $\mathbb{C}P^n$ and $\mathbb{B}^n$ equipped with the Kahler metric

$$ h_{ij} = \frac{\delta_{ij}}{1 + \kappa |z|^2} - \frac{\kappa z_i \overline{z}_j}{(1 + \kappa |z|^2)^2} $$

with $\kappa = 0, 1$ and $-1$ respectively. Also, $z \in \mathbb{C}^n$ in the $\mathbb{C}^n$ and $\mathbb{P}^n$ (locally chart in this setting) case; and $|z| < 1$ in the hyperbolic space case. The curvature tensor is given by

$$ \Theta_{ij} = \kappa \left( \sum_{k,l=1}^{n} h_{k\overline{l}} dz_k \wedge d\overline{z}_l \right) \delta_{ij} - \kappa \sum_{l=1}^{n} h_{i\overline{l}} d\overline{z}_l \wedge dz_j $$

and

$$ R_{i\overline{j}k\overline{l}} = \kappa (h_{i\overline{j}} h_{k\overline{l}} + h_{i\overline{l}} h_{k\overline{j}}), $$

Complex space forms are certainly pseudo-conformally flat.

Bochner-Kähler manifolds Let $(M, \omega)$ be a Kähler manifold. Write $\omega = \sum_{i\overline{j}} g_{i\overline{j}} dz_i \otimes d\overline{z}_j$ in a local holomorphic chart. The Bochner curvature tensor of $(M, \omega)$ is defined as the following tensor:

$$ B_{\beta\alpha\gamma\delta} = R_{\beta\alpha\gamma\delta} - \frac{g_{i\overline{j}} R_{i\overline{j}\gamma\delta} + g_{j\overline{i}} R_{j\overline{i}\gamma\delta} + g_{i\overline{j}} R_{j\overline{i}\gamma\delta} + g_{j\overline{i}} R_{i\overline{j}\gamma\delta}}{n+2} + \frac{R(g_{\beta\gamma} g_{\alpha\delta} + g_{\beta\delta} g_{\alpha\gamma})}{(n+1)(n+2)} $$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor of $(M, \omega)$, $R_{\alpha\beta}$ is the Ricci tensor and $R$ is the scalar curvature of $(M, \omega)$. $(M, \omega)$ is called a Bochner-Kähler manifold if its Bochner curvature tensor is identically zero. There have been extensive studies on Bochner-Kähler manifolds in the literature, for which we refer the reader to the paper of Bryant ([Br01]). Bochner-Kähler manifolds are apparently pseudo-conformally flat in our definition.

9 The proof of the Theorem 1.2

To prove Theorem 1.2, for any point $u_0 \in X$, let $z = (z_1, ..., z_n)$ be a holomorphic coordinate system of $f(X)$ at $z_0 = f(u_0)$, and $\tilde{z} = (z_1, ..., z_n, z_{n+1}, ..., z_N)$ an extension of $(z_1, ..., z_n)$.
to a coordinate system of $Y$ at $z_0$. We shall fix the following convention for indices: $1 \leq i, j, \ldots, \leq N$, $1 \leq \alpha, \beta, \mu, \nu, \gamma, \delta \ldots \leq n$, $n + 1 \leq a, b, A, B, \ldots \leq N$.

Let us denote by $\tilde{g}_{ij}$ the Hermitian metric of $(Y, \sigma)$ and $\tilde{R}_{\alpha\beta\gamma\delta}$ the curvature tensor of this metric on $Y$. Let us denote by $g_{\alpha\beta}$ the restriction metric of the metric $\tilde{g}_{ij}$ on $f(X)$ and and $R_{\alpha\beta\gamma\delta}$ the curvature tensor of this reduced metric $g_{ij}$ on $f(X)$.

By the Gauss equation, we have the following equation of tensors:

$$\tilde{R}_{\alpha\beta\gamma\delta}|_{f(X)} - R_{\alpha\beta\gamma\delta} = h^A_{\alpha\gamma} \tilde{h}^B_{\beta\delta} \tilde{g}_{AB}$$  \hspace{1cm} (34)

where $h^A_{\alpha\gamma} = \tilde{g}^{A\beta} \frac{\partial \tilde{g}_{\alpha\gamma}}{\partial z^\beta}$ is the second fundamental form of $f(X)$ in $Y$.

Since $(Y, \sigma)$ is pseudo-conformally flat, the restriction of the curvature also satisfies

$$\tilde{R}_{\alpha\beta\gamma\delta}|_{f(X)} = (G_{\alpha\beta} \tilde{g}_{\mu\nu} + \tilde{G}_{\mu\beta} \tilde{g}_{\alpha\nu} + G^*_{\alpha\nu} \tilde{g}_{\mu\beta} + \tilde{G}_{\mu\sigma} \tilde{g}_{\alpha\beta})|_{f(X)}$$  \hspace{1cm} (35)

where $G_{\alpha\beta}, \tilde{G}_{\alpha\beta}, G^*_{\alpha\beta}, \tilde{G}_{\mu\sigma}$ are some Hermitian matrices on $f(X)$.

Since $(X, \omega)$ is pseudo-conformally flat, so is $(f(X), (f^{-1})^*(\omega))$. Since $f$ is holomorphic conformal, we have $(f^{-1})^*\omega = k \sigma|_{f(X)}$ for a positive constant $k > 0$. By the assumption that $(X, \sigma)$ is pseudo-conformally flat, we conclude that $(f(X), \sigma|_{f(X)})$ is also pseudo-conformally flat, and hence the curvature tensor $R_{\alpha\beta\gamma\delta}$ is conformally flat on $f(X)$ and it can be written as

$$R_{\alpha\beta\gamma\delta} = H_{\alpha\beta} \tilde{g}_{\mu\nu} + \tilde{H}_{\mu\beta} \tilde{g}_{\alpha\nu} + H^*_{\alpha\nu} \tilde{g}_{\mu\beta} + \tilde{H}_{\mu\sigma} \tilde{g}_{\alpha\beta}$$  \hspace{1cm} (36)

where $H_{\alpha\beta}, \tilde{H}_{\mu\beta}, H^*_{\alpha\nu}, \tilde{H}_{\mu\sigma}$ are some Hermitian matrices on $f(X)$.

By (34)(35) and (36), we have

$$\begin{align*}
(G_{\alpha\beta} \tilde{g}_{\mu\nu} &+ \tilde{G}_{\mu\beta} \tilde{g}_{\alpha\nu} + G^*_{\alpha\nu} \tilde{g}_{\mu\beta} + \tilde{G}_{\mu\sigma} \tilde{g}_{\alpha\beta})(z_0) X^\alpha \overline{X}^\beta X^\mu \overline{X}^\nu \\
- (H_{\alpha\beta} \tilde{g}_{\nu\mu} &+ \tilde{H}_{\mu\beta} \tilde{g}_{\alpha\nu} + H^*_{\alpha\nu} \tilde{g}_{\mu\beta} + \tilde{H}_{\mu\sigma} \tilde{g}_{\alpha\beta})(z_0) X^\alpha \overline{X}^\beta X^\mu \overline{X}^\nu \\
= (h^A_{\alpha\mu} \tilde{h}^B_{\beta\nu} X^\alpha \overline{X}^\beta X^\mu \overline{X}^\nu, \tilde{g}_{AB})(z_0) &
\end{align*}$$  \hspace{1cm} (37)

for any $X = (X^\alpha) = (X^\beta) = (X^\mu) = (X^\nu) \in \mathbb{C}^n$.

By the same calculation as in (6), the left hand side of (37) is equal to $|X|^2 A(X, \overline{X})$. Since $N - n \leq 2n - 1 - n = n - 1$, we can apply Lemma 3.1 to conclude

$$\sum_{A,B=\alpha+1}^{\alpha+n} h^A_{\alpha\mu}(z_0) X^\alpha \overline{X}^\beta \tilde{h}^B_{\beta\nu}(z_0) X^\mu \overline{X}^\nu \tilde{g}_{AB}(z_0) = 0, \hspace{0.5cm} \forall X \in \mathbb{C}^n.$$
Since the Hermitian metric $(\hat{g}_{A\overline{B}}(z_0))$ is positive definite, $h^A_{\alpha\mu}(z_0) = 0$ for all $\alpha, \mu$, and $A$. Since this holds for any point $z$ in $X$, we have proved that the second fundamental form of $f(X)$ is identically zero, and hence $f(X)$ is totally geodesic in $Y$, proving the Theorem 1.2. □

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