Full counting statistics and fluctuation–dissipation relation for periodically driven systems

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We derive the fluctuation theorem for a stochastic and periodically driven system coupled to two reservoirs with the aid of a master equation. We write down the cumulant generating functions for both the current and entropy production in closed compact forms so as to treat the adiabatic and nonadiabatic contributions systematically. We derive the fluctuation theorem by taking into account the property that the instantaneous currents flowing into the left and the right reservoir are not equal. It is found that the fluctuation–dissipation relation derived from the fluctuation theorem involves an expansion with respect to the time derivative of the affinity in addition to the standard contribution.

I. INTRODUCTION

The fluctuation theorem is known as one of the most fundamental relations in nonequilibrium physics [1–5]. It reduces to the standard linear response theory near equilibrium and holds even when the system is far from equilibrium. Using the formal representation of the fluctuation theorem, we can derive a series of nontrivial relations such as the fluctuation–dissipation relation and Onsager’s reciprocal relations.

When the system coupled to several reservoirs is driven periodically, it can be viewed as a heat engine and properties such as the efficiency can be considered. Although the second law of thermodynamics gives a universal upper bound on the efficiency of heat engines, the bound is not tight for systems operated at finite speed. Since dissipative effects are inevitable in such systems, it is interesting to consider the problem from a perspective of nonequilibrium statistical mechanics [4].

The study of this problem is motivated by not only fundamental nonequilibrium physics but also experimentally-accessible phenomena. If the system is driven periodically, it causes nontrivial dynamical phenomena such as Thouless pumping [5] in which there exists a finite current even in the absence of net bias. The current has a geometrical interpretation and is described by the Berry phase [7] even in classical stochastic systems [8]. We also find many applications from a viewpoint of engineering. In quantum systems, a periodic protocol induces nontrivial effective interactions and is used as a method of controlling the system and of finding nontrivial states [9,10].

In this paper, we treat classical stochastic systems driven by periodic modulations. Some previous studies of Thouless pumping processes indicated that the presence of the geometrical current leads to the violation of the fluctuation theorem, which originates from the absence of the Levitov–Lesovik–Gallavotti–Cohen (LLGC) symmetry in the cumulant generating function of full counting statistics [2,11]. They discussed that the current distribution function exhibits a non-Gaussian property [12,16]. Their studies are based on the adiabatic approximation and it is not clear how the nonadiabatic effect affects the results. We note that the word “adiabatic” used in this paper is equivalent to quasistatic. The range of applicability of the standard proof of the fluctuation theorem is a subtle problem and we need a careful treatment to establish the theorem.

The main aim of the present work is to study nonadiabatic effects on the fluctuation theorem and relations derived from it. For this purpose, we only focus on a two-level system whose master equation involves transition rates depending periodically on time. We use the method of full counting statistics to study distributions of the current and entropy production [11,17–20]. The present study is a natural extension of our previous work on nonadiabatic effects of the average current [21]. The cumulant generating function treated in the full counting statistics reflects the underlying symmetry of the system and is useful in deriving the fluctuation theorem.

The paper is organized as follows. In Sec. II we describe the model, and summarize previous results. In Sec. III we introduce the full counting statistics and define the cumulant generating function. It is analyzed by using the dynamical invariant in Sec. V and the obtained compact form of the function is studied in detail in Sec. V. We prove the fluctuation theorem in Sec. VI and derive the fluctuation–dissipation relation in Sec. VII. The last section VIII is devoted to conclusions.

II. SYSTEM

We consider a classical stochastic system which couples to a left and right reservoir. The system has two possible states, “empty” and “filled”, and the dynamics is described by the two-level classical master equation

\[ \frac{d}{dt} \langle p(t) \rangle = W(t)p(t). \]
$|p(t)\rangle$ is a two-component vector and $W(t)$ is a transition-rate matrix. The first (second) component of $|p(t)\rangle$ represents the probability that the state is empty (filled). $W(t)$ is decomposed into two parts $W^{(L)}(t) + W^{(R)}(t)$ and each part is parametrized as

$$W^{(\mu)}(t) = \begin{pmatrix} -k^{(\mu)}_{\text{in}}(t) & k^{(\mu)}_{\text{out}}(t) \\ k^{(\mu)}_{\text{in}}(t) & -k^{(\mu)}_{\text{out}}(t) \end{pmatrix},$$

where $\mu$ represents L or R. Here, $k^{(\mu)}_{\text{in}}(t)$ is the incoming rate from reservoir $\mu$ and $k^{(\mu)}_{\text{out}}(t)$ is the corresponding outgoing rate. They are both assumed to be positive. We use the following probability and the above form is the most general one in the present two-level system. Although we basically assume a periodic system with period $T_0 = 2\pi/\omega$, the following analysis is general and can be applied to arbitrary dynamical systems.

The matrix form of $W(t)$ is restricted by the conservation of probability and the above form is the most general one in the present two-level system. Although we basically assume a periodic system with period $T_0 = 2\pi/\omega$, the following analysis is general and can be applied to arbitrary dynamical systems.

The two-level master equation is solved exactly in Ref. [21]. With the initial condition $|p(0)\rangle = (p_0, 1 - p_0)^T$, the solution is given by

$$|p(t)\rangle = \left( \begin{array}{c} p_{\text{out}}(t) + \delta(t) \\ 1 - p_{\text{out}}(t) - \delta(t) \end{array} \right) + (p_0 - p_{\text{out}}(0)) e^{-\int_0^t dt' k(t')} \left( \begin{array}{c} 1 \\ -1 \end{array} \right),$$

where $p_{\text{out}}(t) = k_{\text{out}}(t)/k(t)$ and

$$\delta(t) = - \int_0^t dt' p_{\text{out}}(t') e^{-\int_0^{t'} dt'' k(t'')}.$$

The dot denotes time derivative. Since the second term on the right hand side of Eq. (7) decays as $t$ increases, the first term gives the state in the long-time limit. Note that $\delta(t)$ in principle depends on the whole history of the system and gives rise to nonadiabatic effects.

The exact form of the probability distribution can be used to calculate the average current as

$$\langle \dot{J}^{(\mu)} \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_T^{T+T_0} dt \, J_1^{(\mu)}(t),$$

where

$$J_1^{(\mu)}(t) = k^{(\mu)}_{\text{in}}(t)p_2(t) - k^{(\mu)}_{\text{out}}(t)p_1(t),$$

We assume in Eq. (9) that the system reaches a periodic state whose period is equal to the modulation period $T_0$. $p_1$ ($p_2$) denotes the first (second) component of $|p(t)\rangle$. Using the solution of the master equation, we can write

$$J_1^{(L)}(t) = J_d(t) + \frac{k^{(L)}_{\text{in}}(t)}{k(t)} \left( \dot{p}_{\text{out}}(t) + \dot{\delta}(t) \right),$$

$$J_1^{(R)}(t) = -J_d(t) + \frac{k^{(R)}_{\text{in}}(t)}{k(t)} \left( \dot{p}_{\text{out}}(t) + \dot{\delta}(t) \right),$$

where $J_d(t)$ represents the dynamical (“classical”) part of the current as

$$J_d(t) = \frac{k^{(L)}_{\text{in}}(t)k^{(R)}_{\text{out}}(t) - k^{(L)}_{\text{out}}(t)k^{(R)}_{\text{in}}(t)}{k(t)}.$$

The second term on the right hand side of Eqs. (11) and (12) represents the geometric part of the current. To find current correlations systematically, we need to introduce the generating function for the current distribution. This is the main aim of this study.

III. CUMulant generating function

To calculate statistical quantities, we introduce a set of counting fields $\chi(t) = (\chi^{(L)}(t), \chi^{(R)}(t))$, where $\chi^{(L)}(t)$ is for the left coupling and $\chi^{(R)}(t)$ is for the right. The transition-rate matrix is modified as $W(t) \to W(t, \chi(t))$ where the diagonal part is unchanged and the off-diagonal part is changed as

$$W_{12}(t) \to k^{(L)}_{\text{out}}(t) := k^{(L)}_{\text{out}}(t)e^{\chi^{(L)}(t)} + k^{(R)}_{\text{out}}(t)e^{\chi^{(R)}(t)},$$

$$W_{21}(t) \to k^{(L)}_{\text{in}}(t) := k^{(L)}_{\text{in}}(t)e^{-\chi^{(L)}(t)} + k^{(R)}_{\text{in}}(t)e^{-\chi^{(R)}(t)}.$$

The explicit form of the counting fields depends on the quantity we wish to calculate; several examples will be specified. Here, we only assume that they are real functions of $t$. Since all the variables can be taken to be real, we do not use the standard choice $e^{\alpha(t)}$. It should be noted that most of the previous studies on Thouless pumping processes use only $\chi^{(L)}$ or $\chi^{(R)}$ [12–16]. However, it is important to introduce two counting fields for both couplings to discuss symmetry relations such as the fluctuation theorem.

Let us solve the modified master equation

$$\frac{d}{dt} |p^1(t)\rangle = W(t, \chi(t)) |p^1(t)\rangle,$$

to calculate the cumulant generating function, which is a functional of $\chi(t)$, as

$$g[\chi] = \lim_{T \to \infty} \frac{1}{T} \ln \langle |p^1(T)\rangle \rangle,$$

where $\langle 1 \rangle = (1, 1)$. Since $|p^1(t)\rangle \to |p(t)\rangle$ and $\langle 1 | p^1(t)\rangle \to 1$ for $\chi \to 0$, we have $g[0] = 0$. The current correlation functions can be calculated by differentiating $g[\chi]$ with respect to $\chi(t)$.

The definition of the cumulant generating function in Eq. (17) is a standard choice for systems with time-independent parameters. We can calculate the long time average of statistical quantities by using $g[\chi]$. In the present case with periodic modulations, we are interested in average statistical quantities over one cycle as in Eq. (9). We show in the next section that Eq. (17) can be used to calculate the average quantities when the periodically modulated system approaches a periodic state.

In this study, we consider the following choices of the counting field.
1. Average current:

\[ \chi(t) = \chi = (\chi^{(L)}, \chi^{(R)}), \]  

(18)

where \( \chi^{(L)} \) and \( \chi^{(R)} \) are independent of time. The cumulants of current from the system to the left (right) reservoir is obtained by differentiation of \( g = g(\chi) \) with respect to \( \chi^{(L)}(\chi^{(R)}) \). For example, we have

\[ \frac{\partial}{\partial \chi^{(a)}(\mu)} g(\chi) \bigg|_{\chi=0} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, J^{(a)}(t). \]  

(19)

This is equal to \( \langle \dot{J}^{(a)} \rangle \) in Eq. (9) when the system approaches a periodic state.

2. Entropy production:

\[ \chi(t) = \left( \chi \ln \left( \frac{k_{\text{in}}^{(L)}(t)}{k_{\text{in}}^{(R)}(t)} \right), \chi \ln \left( \frac{k_{\text{out}}^{(L)}(t)}{k_{\text{out}}^{(R)}(t)} \right) \right), \]  

(20)

where \( \chi \) is independent of time. The entropy production is obtained by differentiation of \( g \) with respect to \( \chi \). This choice of the counting field is based on the fact that the entropy production in the reservoir \( \mu \) for a process \( j \to i \) is given by \( \sigma^{(a)}_{ij} = \ln(W_{ij}^{(a)}/W_{ji}^{(a)}) \).

3. Excess entropy production:

\[ \chi(t) = \left( \chi \ln \left( \frac{k_{\text{out}}^{(L)}(t)}{k_{\text{out}}^{(R)}(t)} \right), \chi \ln \left( \frac{k_{\text{out}}^{(L)}(t)}{k_{\text{out}}^{(R)}(t)} \right) \right). \]  

(21)

\( \sigma^{(a)}_{ij} \) is decomposed by using the steady-state distribution \( p^{(s)} \) as

\[ \sigma^{(a)}_{ij} = \ln \left( \frac{W_{ij}^{(a)} p_{ij}^{(s)}}{W_{ji}^{(a)} p_{ji}^{(s)}} \right) + \ln \left( \frac{p_i^{(s)}}{p_j^{(s)}} \right). \]  

(22)

The second term on the right hand side represents the excess entropy production [22]. In the present two-state case, the detailed balance condition leads to

\[ \sigma^{ex}_{ij} = \ln \left( \frac{p_i^{(s)}}{p_j^{(s)}} \right) = \ln \left( \frac{W_{ij}}{W_{ji}} \right), \]  

(23)

which suggests the choice of the counting field as shown in Eq. (21). We note that this definition of the excess entropy production is different from that in Ref. 20.

IV. DYNAMICAL INVARIANT

Our task is to calculate the cumulant generating function for a given transition rate matrix with \( \chi(t) \). The introduction of the counting field spoils the property \( \langle 1 \rangle W = 0 \), which makes difficult to solve Eq. (10) in a straightforward manner [21].

The main idea of this paper is to use the dynamical invariant originally introduced for quantum harmonic oscillator systems [23]. It is recognized as the fundamental quantity in the method of shortcuts to adiabaticity [24, 25]. The extension of the concept to the present system is straightforward and we summarize the relevant results below.

A matrix \( F^x(t) \) is called a dynamical invariant when it satisfies the relation

\[ \frac{dF^x(t)}{dt} = W(t, \chi(t))F^x(t) - F^x(t)W(t, \chi(t)). \]  

(24)

This quantity has good properties under the only assumption that \( F^x \) and \( W \) are diagonalizable. First, the instantaneous eigenvalues of \( F^x(t) \) are independent of time. Second, the solution of the master equation is expressed in the most convenient way by using the instantaneous eigenstates of \( F^x(t) \). We can generally write

\[ |p^x(t)\rangle = \sum_{n=1}^2 C_n e^{\int_0^t d\tau \langle \phi_0^{(s)}(\tau) | W(\tau, \chi(\tau)) | \phi_0^{(s)}(\tau) \rangle} \times e^{-\int_0^t d\tau \langle \phi_1^{(s)}(\tau) | W(\tau, \chi(\tau)) | \phi_1^{(s)}(\tau) \rangle} |\phi_n^{(s)}(t)\rangle, \]  

(25)

where \( |\phi_n^{(s)}(t)\rangle \) represents a right eigenstate of \( F^x(t) \) and \( \langle \phi_0^{(s)}(t) | \phi_1^{(s)}(t) \rangle \) represents the corresponding left eigenstate. \( C_n \) is shown to be time independent, which means that the solution is given by the adiabatic state with respect to \( F^x \). The exponential factor in the second line of Eq. (25) represents the geometric “phase”. It makes the state invariant under the transformation

\[ |\phi_0^{(s)}(t)\rangle \to R_n(t)|\phi_1^{(s)}(t)\rangle, \]  

(26)

\[ \langle \phi_0^{(s)}(t) | \to \langle \phi_1^{(s)}(t) | R_n^{-1}(t), \]  

(27)

where \( R_n(t) \) represents an arbitrary real function with \( R_n(0) = 1 \).

Since \( \text{Tr} F^x(t) \) is independent of \( t \) and the equation is unchanged under the multiplicative change \( F^x(t) \to r F^x(t) \), we can take \( F^x(t) \) to be traceless without loss of generality. Thus, the eigenvalues are found to be \( \pm 1 \). To solve Eq. (24), we parametrize \( F^x(t) \) as

\[ F^x(t) = \begin{pmatrix} z^x(t) & (1 + z^x(t))^{1/2} \frac{1}{s^x(t)} \\ (1 - z^x(t))s^x(t) & -z^x(t) \end{pmatrix}. \]  

(28)

Then, \( s^x(t) \) and \( z^x(t) \) are obtained by solving

\[ s^x = k_{\text{out}}^x (s^x - s^x_0)(s^x - s^x), \]  

(29)

\[ z^x = \left( k_{\text{in}}^x s^x + k_{\text{out}}^x \right) z^x + k_{\text{out}}^x s^x - \frac{k_{\text{in}}^x}{s^x}, \]  

(30)

where

\[ s^x_0 = \frac{k_{\text{out}} - k_{\text{in}} \mp \sqrt{(k_{\text{out}} - k_{\text{in}})^2 + 4k_{\text{out}}k_{\text{in}}^x}}{2k_{\text{out}}}. \]  

(31)

The parametrization in Eq. (28) is convenient because Eq. (29) for \( s^x \) is independent of \( z^x \) and Eq. (30) for \( z^x \) is formally
asymptotically identical behavior to evolution depending on the initial condition, a fixed point and plot the solutions for several choices of the initial condition. For the initial condition \( s'(0) < s^r(0) \), \( s'(t) \) has asymptotically identical behavior to \( s'(t) \sim s^r(t) \). When \( s' \) diverges for \( s'(0) > s^r(0) \), we can switch to the variable \( r^v = 1/s' \), which results in the same asymptotic behavior \( r^v \sim r^v_0 \).

 solutions. Using the solutions of \( s^v \) and \( z^v \), we can write the left and right eigenstates of \( F^v(t) \) as

\[
\langle \phi^v_n | \rangle = \left\{ \begin{array}{ll}
1 & \frac{1 + z^v}{1 - z^v s^v} \\
1 & \frac{1 + z^v}{1 - z^v s^v} \end{array} \right. \nonumber,
\]

\[
| \phi^v_n \rangle = \left\{ \begin{array}{ll}
\frac{1}{2} \left( 1 - z^v s^v \right) & \frac{1 - z^v s^v}{1 + z^v} \\
\frac{1}{2} \left( 1 - z^v s^v \right) & \frac{1 - z^v s^v}{1 + z^v} \end{array} \right. .
\]

This set of states satisfies the orthogonality relations \( \langle \phi^v_m | \phi^v_n \rangle = \delta_{mn} \) and the resolution of unity \( \sum_n | \phi^v_n \rangle \langle \phi^v_n | = 1 \).

Equation (29) for \( s^v \) has two fixed points \( s^v \). Since \( k^v_{out} \) is a nonnegative quantity, \( s^v \) is unstable with respect to deviations while \( s^v \) is stable. Then, we expect that \( s^v \) approaches the stable fixed point \( s^v \) as time goes on (See Fig. 1). After transient evolution depending on the initial condition, \( s^v \) relaxes into a state similar to \( s^v \). The solution of Eq. (29) at large \( t \) takes negative values and the corresponding solution of \( z^v \) diverges exponentially as we see from Eq. (30). We note that the fixed point of \( z^v \) given by

\[
z^v_0 = \frac{k^v_{out} s^v - k^v_{in}}{k^v_{out} s^v + k^v_{in}} = -\frac{k^v_{out} - k^v_{in}}{\sqrt{(k^v_{out} - k^v_{in})^2 + 4k^v_{out} k^v_{in}}}.
\]

describes the adiabatic solution. The eigenstates of \( F^v(t) \) in Eqs. (32) and (33) become those of \( W(t, \chi(t)) \) by substituting \( s^v = s^v_0 \) and \( z^v = z^v_0 \).

To obtain a closed form of the cumulant generating function, we evaluate \( |p^v(t)⟩ \) in Eq. (25). We have

\[
\langle \phi^v_n | W | \phi^v_n \rangle - \langle \phi^v_n | \phi^v_n \rangle = \left\{ \begin{array}{ll}
\frac{1}{2} & (1 - z^v)(1 - s^v) n = 1 \\
\frac{1 - z^v}{1 + z^v} & n = 2
\end{array} \right. . \quad (36)
\]

where the limit \( z^v \to \infty \) is taken in Eq. (35). Taking the logarithm of Eq. (36) and using the solution of \( z^v \) as

\[
z^v(T) \sim \exp \left\{ -\int_0^T dt \left( k^v_{out} s^v + \frac{k^v_{in}}{s^v} \right) \right\}, \quad (37)
\]

we obtain

\[
\lim_{T \to \infty} \frac{1}{T} \ln \langle | \phi^v_n (T) \rangle \rangle = \left\{ \begin{array}{ll}
-\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \left( k^v_{out} s^v(t) + k^v_{in}(t) \right) \nonumber,
\end{array} \right. \quad (38)
\]

Combining everything together, we obtain a compact form

\[
g[\chi] = \frac{1}{T} \int_0^T dt \left( k^v_{out}(t) s^v(t) + k^v_{in}(t) \right), \quad (39)
\]

\( s^v(t) \) is obtained by solving the first-order differential equation in Eq. (29). We note that both contributions \( n = 1 \) and \( n = 2 \) give the same expression. The property that two independent modes contribute to the result is guaranteed by the fact that the equation has two possible solutions.

As we see from Fig. 1, under periodic modulation, the system approaches a periodic state. Then, we can rewrite Eq. (39) as

\[
g[\chi] = -\lim_{T \to \infty} \frac{1}{T} \int_0^{T+T_0} dt \left( k^v_{out}(t) s^v(t) + k^v_{in}(t) \right), \quad (40)
\]

which shows that we can calculate statistical quantities averaged over one cycle by using \( g[\chi] \). The integral is taken over a finite interval and we can find a geometrical expression as we show in the next section.

V. PROPERTIES OF THE CUMULANT GENERATING FUNCTION

A. Decomposition

The cumulant generating function is decomposed into several parts. We rewrite Eq. (29) for \( s^v \) as

\[
s^v = s^v_0 - \frac{1}{k^v_{out}} \frac{s^v}{s^v_0 - s^v}, \quad (41)
\]
and substitute this into Eq. (39). Then, \( g \) is decomposed as \( g = g_d + g_g \). The first part is obtained by substituting \( \delta^k \) into \( \delta^v \) as
\[
g_d = \frac{1}{T_0} \int_0^{T_0} dt \frac{\sqrt{(k_{\text{out}} - k_{\text{in}})^2 + 4k_{\text{out}}k_{\text{in}}}}{2},
\]
where we consider a periodic system with the period \( T_0 \). This represents the dynamical part and is obtained from the dynamical \( \langle \phi i | W | \phi in \rangle \) in the standard adiabatic treatment.

The other part represents the geometric part:
\[
g_g = \lim_{T \to \infty} \frac{1}{T_0} \int_T^{T+T_0} \frac{\partial \chi}{\partial \chi_{\mu}} \frac{\delta^v}{s^+ - s^-}.
\]

The adiabatic part is obtained by substituting the adiabatic solution \( \delta^k = \delta^v \) into \( g_g \) as
\[
g_{\text{ad}} = \frac{1}{T_0} \int_0^{T_0} dt \frac{\delta^v}{s^+ - s^-}.
\]

This part is obtained from the geometric phase (\( \langle \phi i | \delta^k | \phi in \rangle \)) in the adiabatic treatment. The difference between \( g_g \) and \( g_{\text{ad}} \) represents nonadiabatic effects. The form of \( g_{\text{ad}} \) leads to a geometric interpretation as can be seen from the representation
\[
g_{\text{ad}} = \frac{1}{T_0} \int_0^{T_0} d\lambda \frac{1}{s^+ - s^-} \frac{\partial}{\partial \lambda} \delta^v,
\]
where we assume that the time dependence is controlled by periodic time-dependent parameters \( \lambda(t) \). The adiabatic part has a purely geometric interpretation and its behavior is characterized by a closed trajectory in the parameter space. A similar interpretation is possible for the whole geometrical part by extending the parameter space to include a dynamically generated parameter \( \chi(t) \).

When the parameters and counting fields are periodic functions with period \( T_0 = 2\pi/\omega \), \( g \) is expanded with respect to \( \omega \) as
\[
g[\chi] = \sum_{k=0}^{\infty} \varepsilon^k g^{(k)}[\chi],
\]
where \( \varepsilon \) is a nondimensionalization of \( \omega \) by a proper scale (\( k_0 \) in an example of Sec. [VI.B] in \( W(t) \)). The dynamical part is given by the zeroth order term \( g_d = g^{(0)} \), the adiabatic part is of first order \( g_{\text{ad}} = \varepsilon g^{(1)} \), and the rest represents nonadiabatic contributions. Each part can be found by solving Eq. (41) iteratively:
\[
\delta^k = s^+ - \frac{\delta^v}{k_{\text{out}}(s^+ - s^-)} = s^- - \frac{\delta^v}{k_{\text{out}}(s^+ - s^-)} + \cdots.
\]

B. Average current

The first order term of the cumulant generating function gives the first moment, i.e., the mean of the statistical quantity. We derive the average current to confirm that the present formulation is consistent with previous research [21].

We choose \( \chi(t) \) as in Eq. (18). The dynamical part of the current is calculated from Eq. (42). We easily find that the expansion to first order in \( \chi \) gives \( J_d(t) \) in Eq. (13). The geometrical part is obtained from
\[
\frac{\partial g_\chi}{\partial \chi_{\mu}} \bigg|_{\chi=0} = \lim_{T \to \infty} \frac{1}{T_0} \int_T^{T+T_0} dt \frac{1}{s^+ - s^0} \frac{\partial \chi^v}{\partial \chi_{\mu}} \bigg|_{\chi=0}.
\]

At \( \chi = 0, s^0 = 1 \), and Eq. (29) can be solved as
\[
s^0(t) = 1 - \frac{1}{p_{\text{out}}(t) - p_{\text{out}}(0) + \delta(t) + \frac{1}{T-\delta(t)}},
\]
where \( \delta(t) \) is given in Eq. (5). We also have
\[
\frac{\partial s^v}{\partial \chi_{\mu}} \bigg|_{\chi=0} = -\frac{k^{(\mu)}(t)}{k(t)},
\]
As a result, we obtain
\[
\frac{\partial g_{\chi}}{\partial \chi_{\mu}} \bigg|_{\chi=0} = \lim_{T \to \infty} \frac{1}{T_0} \int_T^{T+T_0} dt \frac{k^{(\mu)}(t)}{k(t)} \left( \hat{p}_{\text{out}}(t) + \delta(t) \right),
\]
where we use the property that \( \delta(t) \) becomes periodic for long times. Thus, we conclude that the average current in the present formulation gives the known result shown in Eqs. (11) and (12).

VI. FLUCTUATION THEOREM

A. Formal considerations

As mentioned in the introduction, some studies discussed extended fluctuation relations that do not satisfy the conventional fluctuation theorem [12–16]. Nevertheless, the fluctuation theorem is fundamentally a symmetry relation, and as such, it should be recoverable when highly symmetrical situations are considered.

In the standard analysis of the full counting statistics for systems with time-independent parameters, it is sufficient to introduce the counting field only for the left, or right, coupling, because the average currents must satisfy the relation \( \langle J^L(t) \rangle = -\langle J^R(t) \rangle \). Although this current conservation holds even in the present time-dependent systems under one cycle, the instantaneous currents \( J^{(\mu)}(t) \) in Eqs. (11) and (12) show that \( J^{(L)}(t) \neq -J^{(R)}(t) \). To treat the instantaneous currents for both couplings, we introduce two counting fields, which allows us to discuss whether the fluctuation theorem exists in highly symmetric situations.

We investigate the LLGC symmetry of the system. Under the transformation
\[
\chi^{(\mu)}(t) \to \tilde{\chi}^{(\mu)}(t) := -\chi^{(\mu)}(t) - A^{(\mu)}(t),
\]
where \( A^{(\mu)}(t) \) denotes the affinity
\[
A^{(\mu)}(t) = \ln \left( \frac{k^{(\mu)}_{\text{out}}(t)}{k^{(\mu)}_{\text{in}}(t)} \right),
\]

we find that the transition rate matrix is transformed to the transposed matrix as
\[ W(t,\chi(t)) \rightarrow W(t,\tilde{\chi}(t)) = W^T(t,\chi(t)), \]
Equation (54). When we use the counting fields for the entropy production and the excess entropy production in Eqs. (20) and (21), the transformation is achieved by
\[ \chi \rightarrow \tilde{\chi} = -\chi - 1. \]
(55)

To find the corresponding symmetry of the cumulant generating function, we write
\[ g[\chi] = \lim_{T \to \infty} \frac{1}{T} \ln \left( \text{exp}_{\leftarrow} \left( \int_0^T dt W(t,\chi(t)) \right) | p(0) \right), \]
where \text{exp}_{\leftarrow} denotes the time-ordered exponential. This expression can be rewritten as
\[ g[\chi] = \lim_{T \to \infty} \frac{1}{T} \ln \langle \text{exp} \left( \int_0^T dt W(t,\tilde{\chi}(t)) \right) | 1 \rangle. \]
(57)

When the time evolution reaches a steady state, the result must be insensitive to the choice of the initial condition. Comparison of Eq. (58) with Eq. (56) gives
\[ g[\chi] = g^*[\tilde{\chi}], \]
(59)
where \( g^* \) represents the cumulant generating function for the time-reversed matrix \( W(\tilde{t},\chi(\tilde{t})) \). To find this symmetric relation, it is crucial to apply the time-reversal operation \( t \rightarrow -t \) as well as the operation \( \chi \rightarrow \tilde{\chi} \), which is different from the previous analysis [12–16].

Equation (59) is a simple consequence of the symmetry of \( W(t,\chi(t)) \) in Eq. (54). The cumulant generating function is transformed into the current distribution function by the Fourier transformation as
\[ P[J] = \int [d\chi(t)] e^{T g[\chi]} \exp \left[ -i \sum_{\mu=L,R} \int_0^T dt J^{(\mu)}(t) \chi^{(\mu)}(t) \right], \]
(60)
where \( J(t) = (J^{(L)}(t), J^{(R)}(t)) \) and the functional integral is taken over all possible counting fields \( \chi(t) = (\chi^{(L)}(t), \chi^{(R)}(t)) \). Using Eq. (59), we find the fluctuation theorem for the current:
\[ \lim_{T \to \infty} \frac{1}{T} \ln \frac{P[J]}{P^*[−J]} = \lim_{T \to \infty} \frac{1}{T} \sum_{\mu=L,R} \int_0^T dt J^{(\mu)}(t) A^{(\mu)}(t), \]
(61)
Similarly, we can derive a relation for the distribution of the entropy production. Writing the cumulant generating function as \( g(\chi) \), the distribution function defined as
\[ P(\sigma) = \int \frac{d\chi}{2\pi} e^{T(\chi^* - i\chi/\sigma)}, \]
(62)
satisfies the relation
\[ \ln \frac{P(\sigma)}{P(−\sigma)} = \sigma. \]
(63)

Thus, by careful application of the time-dependent counting fields, we confirm the existence of standard fluctuation theorem even in periodically driven systems such as the Thouless pumping process. We note that the present discussion is applicable to any multi-level system provided we can find a suitable transformation \( \chi(t) \rightarrow \tilde{\chi}(t) \).

The present result is different from those in Refs. [12–16]. The previous works studied a symmetry between \( g[\chi^{(L)}] \) and \( g[\chi^{(R)}] \), which leads to a non-Gaussian relation between \( P[J] \) and \( P[−J] \). On the other hand, Eq. (61) is the relation between \( P[J] \) and \( P^*[−J] \).

**B. Detailed properties**

The cumulant generating function \( g \) can be expanded in powers of frequency as in Eq. (47), where each term satisfies Eq. (59). We easily confirm from the explicit form in Eq. (42) that \( g_0 \) satisfies Eq. (59). In Appendix A we confirm that \( g_0 \) satisfies Eq. (59). We also find the relation for each of two operations as
\[ g^{(k)}[\chi] = g^{(k)}[\tilde{\chi}] = (−1)^k g^{(k)}[\chi]. \]
(64)
If \( k \) is odd, \( g^{(k)} \) introduced in Eq. (47) changes the sign under the time-reversal operation.

The above relations hold for any type of counting field. When we treat the entropy production and the excess entropy production, we have additional relations. In Appendix A we show that \( g_0 = g_0 = 0 \) for the excess entropy production. Since the cumulant generating function only has a nonadiabatic part, the use of the adiabatic approximation does not make sense when investigating excess entropy production.

We plot the cumulant generating functions in Figs. 2–5, where we use the parametrization
\[ k^{(L)}_m(t) = k_0 (1 + R_1 \cos \omega t), \]
(65)
\[ k^{(R)}_m(t) = k_0 (1 + R_2 \sin \omega t), \]
(66)
\[ k^{(L)}_m(t) = k_0, \]
(67)
\[ k^{(R)}_m(t) = k_0, \]
(68)
with \( R_1 = 0.6 \) and \( R_2 = 0.4 \). We take \( (\chi^{(L)}(t), \chi^{(R)}(t)) = (\chi, 0) \) for the counting field of the current.

In the case of the current, the dynamical part is a dominant contribution and the geometric part takes rather small values.
as we see in Figs. 2 and 3. When the frequency is small, the difference between \( g \) and \( g_{\text{ad}} \) is small, which means that the adiabatic approximation gives accurate results. The difference becomes large for large frequencies. We note that the slope at \( \chi = 0 \) represents the average current \( \langle \hat{J}(L) \rangle \). The current is significantly suppressed for large values of \( \omega \). This result is consistent with that in Ref. [21]. We also confirm that Eq. (59) holds.

The cumulant generating functions for the entropy production and the excess entropy production are plotted in Figs. 4 and 5. We find that \( g(0) = g(-1) = 0 \) should be satisfied. We also see that \( g(\chi) \) grows rapidly as the frequency increases in both cases.

VII. FLUCTUATION–DISSIPATION RELATIONS

A. Formal considerations

It is important to extract physically relevant results from the fluctuation theorem in Eq. (59). In this section, we derive the fluctuation–dissipation relation to examine nonadiabatic effects.

First, we derive a formal expression of the relation. Generally, the cumulant generating function for the current is expanded as

\[
g[\chi] = \frac{1}{T} \int_0^T \text{d} t \chi^{(\mu)}(t) J_1^{(\mu)}(t) + \frac{1}{2!T} \int_0^T \text{d} t_1 \text{d} t_2 \chi^{(\mu)}(t_1) J_2^{(\mu)}(t_1, t_2) \chi^{(\nu)}(t_2) + \frac{1}{3!T} \int_0^T \text{d} t_1 \text{d} t_2 \text{d} t_3 \chi^{(\mu)}(t_1) \chi^{(\nu)}(t_2) \chi^{(\lambda)}(t_3) J_3^{(\mu \nu \lambda)}(t_1, t_2, t_3) + \cdots \tag{69}
\]

where the summation is taken over repeated indices. \( \{ J_1^{(\mu_1, \ldots, \mu_k)}(t_1, \ldots, t_k) \} \) denotes a set of local current correlators to be obtained from the cumulant generating function. This expression is compared to Eq. (39) where the cumulant generating function is represented by an integral over a single variable. When we treat the dynamical part only, we see
from Eq. (42) that each term is written as
\[ J_{k}^{(g_{i}^{(u)})(t_{1}, \ldots, t_{k}) \chi_{L}(t_{1}) \chi_{R}(t_{1}) \chi_{L}(t_{1}) \chi_{R}(t_{1}) \cdots \delta(t_{1} - t_{2}) \cdots \delta(t_{1} - t_{k})}. \] (70)
This means that the correlations in the dynamical part are local and survive only if some type of symmetry exists between the left and right currents.

The geometric part includes nonlocal correlations and they can be treated by using the derivative expansion. With the aid of Appendix B, we obtain the expression
\[ g[\chi] = \frac{1}{T} \int_{0}^{T} dt \left( J_{1}^{(-\chi)} + J_{1}^{(+\chi)} \right)(t) + \frac{1}{2T} \int_{0}^{T} dt \left( J_{20}^{(-\chi)} + J_{20}^{(+\chi)} \right)^{2}(t) + \frac{1}{\omega} J_{21}^{(-\chi)} \chi_{L}(t) + \frac{1}{\omega^{2}} J_{22}^{(+\chi)} \chi_{R}(t) \chi_{L}(t) \cdots , \] (71)
where \( \chi_{L}(t) = \chi_{L}(t) + \chi_{R}(t) \). Explicit expressions of \( J_{1}^{(\pm \chi)} \) in the first term are given by
\[ J_{1}^{(\pm \chi)}(t) = \pm \left( J_{1}^{(\chi)}(t) + J_{1}^{(-\chi)}(t) \right) \] (72)
The definitions of the other terms such as \( J_{20}^{(-\chi)} \) are given in Appendix B.

Imposing Eq. (59), we can obtain a series of nontrivial relations for the coefficient functions. At the zeroth order in the counting field, \( g[-A] = g^{*}[0] = 0 \) and we have the relation
\[ -\frac{1}{T} \int_{0}^{T} dt \left( J_{1}^{(-\chi)} + J_{1}^{(+\chi)} \right)(t) + \frac{1}{2T} \int_{0}^{T} dt \left( J_{20}^{(-\chi)} + J_{20}^{(+\chi)} \right)^{2}(t) + \frac{1}{\omega} J_{21}^{(-\chi)} \chi_{L}(t) + \frac{1}{\omega^{2}} J_{22}^{(+\chi)} \chi_{R}(t) \chi_{L}(t) \cdots = 0 , \] (73)
where \( A^{(\pm)} = A^{(L)} \pm A^{(R)} \).

We are mostly interested in the first-order term in \( \chi \) because it gives the fluctuation–dissipation relation. After integrating the result over \( t \), we obtain two relations:
\[ \langle \hat{J}^{(L)} \rangle + \langle \hat{J}^{(R)} \rangle^{*} = \frac{1}{T} \int_{0}^{T} dt \left( J_{20}^{(-\chi)} + J_{20}^{(+\chi)} + \frac{1}{\omega} J_{21}^{(-\chi)} \right)(t) + \cdots , \] (74)
\[ 0 = \frac{1}{T} \int_{0}^{T} dt \left( J_{20}^{(+\chi)} + J_{20}^{(+\chi)} \right)(t) + \cdots . \] (75)
\( \langle \hat{J}^{(L)} \rangle^{*} \) denotes the current of the time-reversed protocol. We have used \( \int dt J_{i}^{(\pm \chi)}(t) = 0 \) to derive the second relation. The physical meaning of this result is discussed below. Similarly, we can obtain nontrivial relations for higher-order terms.

### B. Detailed properties

If we only keep the zeroth order term in \( \omega \), the contribution is described by the dynamical part. We see from Eq. (70) that among two-point correlators only \( J_{20}^{(-\chi)}(t) \) has a nonzero contribution. We obtain from Eq. (74)
\[ \frac{2}{T_{0}} \int_{0}^{T} dt J_{d}(t) = \frac{1}{T_{0}} \int_{0}^{T} dt \left( J_{2d}(t) \chi_{L}(t) + O((A^{(\pm \chi)}(t))^{2}) \right) . \] (76)
This is the standard form of the fluctuation–dissipation relation for the dynamical current. We also see that Eq. (75) is trivially satisfied.

Since the current with the time-reversed protocol is given by the original current with the replacement \( \omega \rightarrow -\omega \), the left hand side of Eq. (74) includes even powers of \( \omega \) only. This means that the adiabatic part of the current does not contribute to this relation. From the present analysis, it is not clear whether the adiabatic part obeys a similar relation. We note that extended fluctuation–dissipation relations for the adiabatic current were discussed in Refs. [14, 16]. In order to find nontrivial corrections to Eq. (76), we need to go beyond the adiabatic approximation. Conventionally, due to the left–right symmetry, the fluctuation–dissipation relation is obtained by
using $A^{(-)}(t)$ only. This is not the case for the present time-dependent system where $\dot{A}^{(+)}(t)$ appears in the expansion. The property that the time derivative of the parameters appears in the expansion is expected from the general consideration of the adiabatic response \[26–28\]. Our formulation is self-contained and naturally leads to these results.

Examination of the explicit form of $e^2g^{(2)}$, given by the expression

$$e^2g^{(2)}(\chi) = -\frac{1}{T_0} \int_0^{T_0} dt \frac{s^Y_{\chi} s^X_\chi}{k^a(s^Y_{\chi} - s^X_\chi)},$$

shows that $\tilde{J}^{(L)}_{20}(t) = 0$ and $\tilde{J}^{(L)}_{21}(t) = 0$. Then, up to second order in $\omega$, we can write the fluctuation–dissipation relation as

$$\langle \tilde{J}^{(L)} \rangle + (\tilde{J}^{(L)})^* = \frac{1}{T_0} \int_0^{T_0} dt \left( \tilde{J}^{(L)}_{20}(\cdot) + \frac{1}{\omega} J_{21} \dot{A}^{(+)}(\cdot) \right)(t).$$

The neglected part involves terms such as $(\cdot)^2$, $(\cdot)^2$, and higher powers. The explicit forms of $\tilde{J}^{(L)}_{20}(t)$ and $\tilde{J}^{(L)}_{21}(t)$ can be found in Appendix \[B\]. The important point is that the adiabatic response is described by the correlator $J_{21}(t)$ written as

$$\frac{1}{\omega} J_{21}(t) = -\int_{-\infty}^{\infty} dt \tau \tilde{f}^{(LR)}_2 (t + \frac{\tau}{2} t - \frac{\tau}{2}).$$

This gives the correlation between the left and right coupling. The leading contribution of the adiabatic response comes from $g^{(2)}$. When the parameters are functions of $\theta = \omega t$, Eq. \((78)\) can be written as

$$\langle \tilde{J}^{(L)} \rangle + (\tilde{J}^{(L)})^* \approx \int_0^{\tau^2} \frac{d\theta}{2\pi} f_{2d}(\theta) A^{(-)}(\theta)$$

$$+ e^2 \int_0^{\tau^2} \frac{d\theta}{2\pi} \left( a(\theta) A^{(-)}(\theta) + b(\theta) \frac{dA^{(+)}(\theta)}{d\sigma} \right) + O(e^4),$$

where we rewrite $A^{(\theta)}(t)$ as $A^{(\theta)}(\theta)$, $J_{2d}(t)$ as $J_{2d}(\theta)$. The functions $a(\theta)$ and $b(\theta)$ are obtained from $g^{(2)}(\chi)$ in Eq. \((77)\).

VIII. CONCLUSION

In conclusion, we obtained the cumulant generating function for two-level stochastic systems and derived a series of fluctuation relations applicable to periodically driven systems converging to a periodic state. Our findings are summarized as follows.

- All of our results are derived from a compact form of the cumulant generating function shown in Eq. \((59)\). We derive the expression by using the properties of the dynamical invariant. Although we need to solve the differential equation in Eq. \((29)\), the equation is simple and a systematic treatment is possible. In fact, we can extract the adiabatic part and the nonadiabatic part from the expression and study how each part contributes to the result.

- The cumulant generating function is useful not only for calculating the current distribution but also for finding the underlying symmetry. To derive the fluctuation theorem, Eq. \((59)\), we stress that introducing the instantaneous counting field for coupling to each reservoir is important. It allows us to consider the symmetry under the transformation in Eq. \((52)\).

- We also stress that the time-reversal operation is crucial in deriving the fluctuation theorem. Previous studies on the violation of the fluctuation theorem paid attention to a relation between $P[J]$ and $P[-J]$, rather than $P[J]$ and $P'[\cdot]$. We showed that, even within the adiabatic approximation, the effect of the time-reversal operation is important.

- Equation \((59)\) gives a series of nontrivial relations including the fluctuation–dissipation relation in Eq. \((78)\). The result is expanded with respect to $\dot{A}^{(+)}(t)$ as well as $A^{(-)}(t)$. We need to go beyond the adiabatic approximation to find a nonzero result for $\dot{A}^{(+)}(t)$.

- Equation \((78)\) does not include the adiabatic part. An extended fluctuation–dissipation relation of the adiabatic current was discussed in previous studies \[14,16\]. There is no corresponding result in the present analysis.

The most important result of this paper is a systematic method of dealing with the nonadiabatic effects with desired accuracy. We have discussed the formal structure of the cumulant generating function in this paper. Our method is applicable to a broad range of periodically driven open systems. Our future goal is to obtain useful results for different systems using this method.

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Appendix A: On the adiabatic part of the cumulant generating function

1. General properties

We consider the adiabatic part of the cumulant generating function

$$g_{\text{ad}}(\chi) = \frac{1}{T_0} \int_0^{T_0} dt \frac{\dot{s}^Y_{\chi} s^X_\chi}{s^Y_\chi - s^X_\chi} = \frac{1}{2T_0} \int_0^{T_0} dt \frac{\dot{s}^Y_{\chi} + \dot{s}^X_{\chi}}{s^Y_\chi - s^X_\chi}. \quad (A1)$$
Using Eq. (A1), we rewrite Eq. (A1) as

\[ g_{ad}[\chi] = \frac{1}{2T_0} \int_T^0 \, dt \left( \frac{k_{\text{out}} - k_{\text{in}}}{\sqrt{(k_{\text{out}} - k_{\text{in}})^2 + 4k_{\text{out}}^k k_{\text{in}}^k}} \right) \left( \frac{\dot{k}_{\text{out}}^k - k_{\text{in}}^k}{\dot{k}_{\text{out}}^k} \right) \right]. \tag{A2} \]

To show that this function satisfies the relation in Eq. (59), we apply the time-reversal operation \( g \to g^* \) and the transformation of the counting field \( \chi \to \tilde{\chi} \). We have

\[ g_{ad}^*[\tilde{\chi}] = -\frac{1}{2T_0} \int_T^0 \, dt \left( \frac{k_{\text{out}} - k_{\text{in}}}{\sqrt{(k_{\text{out}} - k_{\text{in}})^2 + 4k_{\text{out}}^k k_{\text{in}}^k}} \right) \left( \frac{k_{\text{out}}^k - k_{\text{in}}^k}{k_{\text{out}}^k} \right) \right]. \tag{A3} \]

We note that the combination \( k_{\text{out}}^k k_{\text{in}}^k \) is invariant under the replacement \( \chi \to \tilde{\chi} \). Then, taking the difference between \( g_{ad}[\chi] \) and \( g_{ad}^*[\tilde{\chi}] \), we obtain

\[ g_{ad}[\chi] - g_{ad}^*[\tilde{\chi}] = \frac{1}{T_0} \int_T^0 \, dt \left( \frac{k_{\text{out}} - k_{\text{in}}}{\sqrt{(k_{\text{out}} - k_{\text{in}})^2 + 4k_{\text{out}}^k k_{\text{in}}^k}} \right) \left( \frac{k_{\text{out}}^k - k_{\text{in}}^k}{k_{\text{out}}^k} \right) \right) \right). \tag{A4} \]

We change the variables according to

\[ k_{\text{out}} - k_{\text{in}} = r \cos \theta, \tag{A5} \]

\[ 4k_{\text{out}}^k k_{\text{in}}^k = r \sin \theta, \tag{A6} \]

to find

\[ g_{ad}[\chi] - g_{ad}^*[\tilde{\chi}] = \frac{1}{T_0} \int_T^0 \, dt \frac{\dot{\theta}}{\sin \theta} = 0. \tag{A7} \]

We note that \( 0 < \theta < \pi \) in the present parametrization with \( k_{\text{out}}^k k_{\text{in}}^k > 0 \) and the integral is not divergent.

2. Cumulant generating function for the excess entropy production

When we consider the excess entropy production, we choose the counting field as shown in Eq. (21). The cumulant generating function in that case is written as

\[ g_{ad}[\chi] = \frac{1}{2T_0} \int_T^0 \, dt \left( k_{\text{out}} k_{\text{in}} - k_{\text{out}} k_{\text{in}}^k \right) \left( k_{\text{in}} - \chi(k_{\text{out}} - k_{\text{in}}) \right) \frac{k_{\text{out}} k_{\text{in}}}{k_{\text{out}} k_{\text{in}}^k}. \tag{A8} \]

Then, by changing variables according to \( k_{\text{out}} = r \cos \theta, k_{\text{in}} = r \sin \theta \), we find that the integrand depends only on \( \theta \) as in Eq. (A7). Then, the adiabatic part of the cumulant generating function is identically zero.

Appendix B: Derivative expansion of the cumulant generating function

We consider the derivative expansion of Eq. (69). The two-point correlator is written as

\[ \int \! dr_1 dr_2 \chi^{(\mu)(t_1)} J_2^{(\nu)}(t_1, t_2) \chi^{(\nu)}(t_2) = \int \! dt d\tau \chi^{(\mu)}(t + \tau/2) J_2^{(\nu)}(t; \tau) \chi^{(\nu)}(t - \tau/2). \tag{B1} \]

where we write \( J_2^{(\nu)}(t + \tau/2, t - \tau/2) \) as \( J_2^{(\nu)}(t; \tau) \). We expect from the result for the dynamical part in Eq. (70) that, for slow driving, the dominant contributions of the current come from the domain \( \tau \approx 0 \). We expand the counting field as

\[ \chi(t \pm \tau/2) = \chi(t) \pm \frac{\tau}{2} \chi'(t) + \frac{\tau^2}{8} \chi''(t) + \cdots. \tag{B2} \]

To calculate the cumulant generating function to second order in \( \omega, e^2 g^{(2)}[\chi] \), we keep the derivative expansion up to second order. Then, we obtain

\[ \int \! dt_1 dt_2 \chi^{(\mu)(t_1)} J_2^{(\nu)}(t_1, t_2) \chi^{(\nu)}(t_2) \]

\[ \simeq \int \! dt \left[ J_{20}^{(\nu)}(t) \chi^{(\nu)}(t) \right]
\]

\[ + \frac{1}{\omega^2} J_{22}^{(\nu)}(t) \left( \chi^{(\mu)}(t) \chi^{(\nu)}(t) - \chi^{(\mu)}(t) \chi^{(\nu)}(t) \right) \]

\[ + \frac{1}{\omega^2} J_{21}^{(\nu)}(t) \left( \chi^{(\mu)}(t) \chi^{(\nu)}(t) - 2 \chi^{(\mu)}(t) \chi^{(\nu)}(t) + \chi^{(\mu)}(t) \chi^{(\nu)}(t) \right) \]

\[ = \int \! dt \left[ J_{20}^{(\nu)}(t) \chi^{(\nu)}(t) \right]
\]

\[ + \frac{1}{\omega^2} J_{22}^{(\nu)}(t) \left( \chi^{(\mu)}(t) \chi^{(\nu)}(t) - \chi^{(\mu)}(t) \chi^{(\nu)}(t) \right) \]

\[ + \frac{1}{\omega^2} J_{21}^{(\nu)}(t) \left( \chi^{(\mu)}(t) \chi^{(\nu)}(t) + J_{21}^{(\nu)}(t) \chi^{(\mu)}(t) \chi^{(\nu)}(t) \right) \tag{B3} \]

where

\[ J_{20}^{(\nu)}(t) = \int \! dt J^{(\nu)}(t; \tau), \tag{B4} \]

\[ \frac{1}{\omega} J_{21}^{(\nu)}(t) = \int \! dt \frac{\tau}{2} J^{(\nu)}(t; \tau), \tag{B5} \]

\[ \frac{1}{\omega^2} J_{22}^{(\nu)}(t) = \int \! dt \frac{\tau^2}{8} J^{(\nu)}(t; \tau), \tag{B6} \]
where integration by parts was used in the second equality. Using these expressions, we obtain Eq. (71) and

\begin{align}
J_{1}^{(+)} &= \frac{1}{2} \left( J_{1}^{(L)} \pm J_{1}^{(R)} \right), \\
J_{20}^{(+)} &= \frac{1}{4} \left( J_{20}^{(LL)} + J_{20}^{(RR)} \pm J_{20}^{(LR)} \pm J_{20}^{(RL)} \right) + \frac{1}{4\omega_2} \left( J_{22}^{(LL)} + J_{22}^{(RR)} \pm J_{22}^{(LR)} \pm J_{22}^{(RL)} \right), \\
J_{20}^{(-)} &= \frac{1}{2} \left( J_{20}^{(L)} - J_{20}^{(R)} \right) - \frac{1}{4\omega_2} \left( J_{21}^{(LR)} - J_{21}^{(RL)} \right) + \frac{1}{2\omega_2} \left( J_{22}^{(LL)} - J_{22}^{(RR)} \right), \\
J_{21} &= -2J_{21}^{(L)}, \\
J_{22}^{(+)} &= \left( J_{22}^{(LL)} + J_{22}^{(RR)} \pm J_{22}^{(LR)} \pm J_{22}^{(RL)} \right), \\
J_{22}^{(-)} &= -2\left( J_{22}^{(L)} - J_{22}^{(R)} \right).
\end{align}

[B7] [B8] [B9] [B10] [B11] [B12]

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