A comparative example of cyclostationary description of a non-stationary random process

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Abstract. The paper deals with cyclostationarity as a natural extension of stationarity as the key property in designing the widely-used models of random processes. The comparative example of two processes, one is wide-sense stationary and the other is wide-sense cyclostationary, is given in the paper and reveals the lack of the conventional stationary description based on one-dimensional autocorrelation functions. It is shown that two significantly different random processes appear to be characterized by exactly the same autocorrelation function while their two-dimensional autocorrelation functions provide outlook where the difference between processes of two above-mentioned classes becomes much clearer. More concise representation by expanding the two-dimensional autocorrelation function to its Fourier series where the cyclic frequency appears as the transform parameter is illustrated. The closed-form expression for the components of the cyclic autocorrelation function is also given for the random process which is an infinite pulse train made of rectangular pulses with randomly varying amplitudes.

1. Introduction
The stationary model for description of random processes could not be the best choice in the case of process containing a structural periodicity which is impossible to be reliably identified by the conventional Fourier analysis or extracted by a direct subtraction. The vital alternative for the description of these processes could be the global probabilistic model with the hidden periodicity taking into account the regular change of their statistical properties. The cyclostationarity (CS), a term originally introduced by W. Bennet \cite{1} in the 1950s primarily to describe digitally modulated pulses, provides the formal description for the concept of the hidden periodicity including but not limited to the prominent case of periodically correlated time series and continuous signals. The cyclostationarity, which is a natural yet not evident expansion of the stationarity, has been developing by W. Gardner \cite{2}, A. Napolitano \cite{3}, H. Hurd \cite{4}, I. Javorsky \cite{5} and others \cite{6}, although the main results obtained over few decades are still not familiar to the significant majority of modern researchers in various fields.

In general, the process $x(t)$ will be named cyclostationary (of order $K$) if there exists its transformation of the $k$-th degree such that its expectation (via ensemble averaging) is a periodic in time function. Considering a simpler case of a wide-sense cyclostationary process, one could obtain its characterization through the varying yet periodic mean $m_k(t)$, which is the first-order transformation:

$$m_k(t) = E[x(t)] = E[x(t + T)] = m_k(t + T), \quad (1)$$
and two-dimensional autocorrelation function \( R_x(t, \tau) \), which is a possible second-order transformation:

\[
R_x(t, \tau) = E\left[ x(t + 0.5\tau)x^*(t - 0.5\tau) \right], \quad R_x(t, \tau) = R_x(t + T, \tau).
\]  

(2)

The CS approach extends the statistical description based on the habitual characteristics of the well-known stationary model by taking into account periodic change in time of the chosen characteristics, e.g., the mean (1) and the auto-correlation function (2). One of the advantages of the CS approach is that the characteristics of the stationary model become the essential and additive parts of the corresponding characteristics of the more general CS model.

The paper is organized as follows. A comparative example of two processes sharing the same autocorrelation function is introduced in section 2. The importance of two-dimensional correlation function is shown in section 3 together with its link to conventional one. In section 4, the cyclic autocorrelation is described for the strictly periodic two-dimensional correlation function and gives a concise explanation for the feature of cyclostationarity. The paper ends with the conclusion.

2. Insufficiency of stationary description

One possible and relatively easy understandable example showing the clear difference between wide-sense stationary (WSS) and wide-sense cyclostationary processes (WSCS) are presented below. At first, let us define a random process \( x(t) \) which is generated as

\[
x(t) = \sum_{n=-\infty}^{\infty} AX_n g(t - nT),
\]

(3)

where \( T \) is a period, \( A \) is an amplitude multiplier, \( \{X_n\} \) is a set of independent identically distributed (i.i.d.) random numbers defined on the same probability space and \( g(t) \) is a deterministic signal of finite energy. For further simplification, the distribution for \( \{X_n\} \) is chosen to be Bernoulli taking values from a set \( \{-1, 1\} \) with equal probabilities. Similarly, the waveform of \( h(t) \) is chosen to be a symmetric rectangle of the width \( \Delta \), such as \( \Delta \leq T \):

\[
g(t) = \text{rect}(t\Delta^{-1}).
\]

(4)

A typical realization of \( x(t) \) is shown in figure 1, where one can see the time sequence of the same width rectangular pulses with altering amplitudes.

![Figure 1. A typical realization of the process belonging to the cyclostationary class.](image)

The second process being introduced as \( y(t) \) has a different origin. Its typical realization is shown in figure 2. It is generated as the output of the linear time-invariant system with impulse response \( h(t) = g(t)/\Delta \), where \( g(t) \) is defined by equation (4), when it is fed with a white Gaussian noise \( n(t) \) with uniform two-sided power spectral density \( N_0 = P\Delta \), where \( P = A^2\Delta/T \) is the average power of the process \( x(t) \). The assumption about \( N_0 \) has been made to make the average powers of both processes equal.

![Figure 2. A typical realization of the stationary process.](image)
The conventional description of a random process could be obtained via its autocorrelation function or power spectral density as the Fourier-counterpart in frequency domain. The autocorrelation function $R_y(\tau)$ for the process $y(t)$ could be obtained using well-known convolution relation:

$$R_y(\tau) = R_x(\tau) \ast R_h(\tau), \quad (5)$$

where $R_\delta(\tau) = N_0 \delta(\tau)$ is the autocorrelation function of white noise and $R_h(\tau)$ is the autocorrelation function of the impulse response $h(t)$. The autocorrelation function of $y(t)$ is the triangle waveform shown in figure 3 together with its power spectral density $S_y(f) = P \Delta \text{sinc}^2(\pi f \Delta)$ plotted in figure 4.

The correlation function $R_x(\tau)$ of the process $x(t)$ can be derived via its probability density as the expectation of the product of two observation $x^*(t)$ and $x(t+\tau)$, here superscript '*' denotes a complex conjugation. After $R_x(\tau)$ is obtained one will come to the fact, that if the process $x(t)$ is assumed to be stationary – it is not indeed but is thought in the vast majority cases – both processes will share exactly the same autocorrelation function as well as the same power spectral density.

![Figure 3. Autocorrelation function of both random processes.](image1)

![Figure 4. Power spectral density of both random processes.](image2)

A simple look at two realization in figures 1 and 2 can reveal the dissimilarity of those two processes despite the fact they have the same correlation functions. However, if those processes had been superimposed with some noise, the distinction would not remain so noticeable any more. Moreover, it is no doubt impossible to rely on such an argumentation as visible structure during an automated or machine processing of signals. The cyclostationary theory is expected to be a perfect tool for solving this and other related problems.

3. Two-dimensional correlation function

3.1. Function of two instant time arguments

The first characteristic one should consider when they move to the cyclostationary description of some process $s(t)$ is its two-dimensional autocorrelation function define via probabilistic expectation as follows:

$$R_s(t_1, t_2) = E[s(t_1)s^*(t_2)], \quad (6)$$

where $t_1$ and $t_2$ are two instants taken at two moments along the observation time.

The two-dimensional autocorrelation functions of processes $x(t)$ and $y(t)$ are shown in figures 5 and 6 correspondingly by means of the planar color diagrams which third dimension (values of the correlation) is drawn by the colour intensity. So the higher intensity is, the larger value of correlation function it expresses. The process $x(t)$ exhibits a periodical correlation pattern that is a sequence of identical prisms along the diagonal $t_2 = t_1$ depicted as squares with sharp edges due to the rectangular waveform of $g(t)$ given by equation (4).

The two-dimensional ACF of $y(t)$ is different. It could basically be describe with the lines $t_1-t_2=\tau$ where the correlation remains the same and equal to $R_y(\tau)$ along. The diagram in figure 6 displays the linear gradient decay from the main diagonal if one moves in the perpendicular direction.

3.2. Symmetric form of ACF

Another form of two-dimensional ACF could be obtained by the coordinate transformation, where each point is defined by $(t, \tau)$ rather than by pair $(t_1, t_2)$. In the former pair, $\tau$ stands for a difference, or
time shift, between moments of the observations while $t$ is the current time that could be chosen in different ways. The choice enforcing useful symmetry means that $t$ relates to the centre of the time interval $(t_2, t_1)$:

$$\tau = t_1 - t_2, \quad t = 0.5(t_1 + t_2).$$

(7)

In fact, this transformation leads to the function which is generally periodic only in one dimension denoted by $t$. Thus, the ACF $R_x(t, \tau)$ of the process $x(t)$, which is shown in figure 7, exhibits periodicity in $t$ with the same period as in its model given by equation (1):

$$R_x(t, \tau) = R_x(t + T, \tau).$$

(8)

The squares from figure 5 have changed their form to diamonds according to the transformation given by equation (7). In contrast, for the process $y(t)$, its ACF $R_y(t, \tau)$ remains the same while $t$ is varying.

Figure 5. Two-dimensional autocorrelation function of process $x(t)$.

Figure 6. Two-dimensional autocorrelation function of process $y(t)$.

Figure 7. Symmetric $(t, \tau)$ autocorrelation function of process $x(t)$.

Figure 8. Symmetric $(t, \tau)$ autocorrelation function of process $y(t)$.

3.3. Averaging and stationarization

The link from two-dimensional ACF described above to the conventional one-dimensional ACF can be established using the averaging over $t$:

$$R_x(\tau) = \left< R_x(t, \tau) \right> = \lim_{B \to \infty} \frac{1}{B} \int_{-B/2}^{B/2} R_x(t, \tau) dt.$$

(9)

The averaging performed on $R_x(t, \tau)$, which is a constant over $t$ for each $\tau$, just keeps its triangle profile (see figures 8 and 3) having integrated the variable $t$ out: $R_x(t, \tau) = R_x(\tau)$

The infinite-time averaging (7) applied to the strictly periodic function $R_x(t, \tau)$ can be simplified to the averaging over one its period:
\[ R_x(\tau) = \{R_x(t, \tau)\}_t = \frac{1}{T} \int_{-T/2}^{T/2} R_x(t, \tau)dt. \]  

(10)

The figure 7 can be easily used to conduct this averaging. Thus, for any \( \tau \) such that \(|\tau|<\Delta\) the cross-section of ACF is a rectangular function of \( t \) with width \( \Delta-|\tau| \). If, for instance, the value \( \tau=0 \) is taken, than \( R_x(0) = A^2 \Delta/T = P \). Again, if \( \tau=\pm \Delta/2 \), \( R_x = A^2 \Delta(2/T) = P/2 \) since the width of the underlying rectangle turns to be twice shorter. Thus, the conventional ACF \( R_x(\tau) \) comes as a function of time shift \( \tau \) only

There is a concise interpretation of time averaging named a stationarization. The latter means that a random delay \( \theta \), which may be modelled as a random number distributed uniformly over the interval \([0; T] \), affects each single observation of the process \( x(t) \). Making ensemble averaging over \( \theta \) one will come to the stationarized process made of the original \( x(t) \). In turns, the stationarized process is well-described by \( R_x(\tau) \), but this function is not enough to characterization of \( x(t) \) itself.

4. Cyclostationary description

Since the two-dimensional ACF \( R_x(t, \tau) \) is a periodic function of \( t \) as well as a constant over \( t \), it could be represented as its Fourier series

\[ R_x(t, \tau) = \sum_{\alpha \in A} R_x^\alpha(\tau) \exp\left(j2\pi\alpha t\right), \]  

(11)

where \( \alpha \) is called cyclic frequency and the set of cyclic frequencies \( A = \{\alpha : \alpha = m/T, m \in \mathbb{Z}\} \) consisting of multiples of the fundamental frequency \( 1/T \); \( j \) is the imaginary unit here, i.e. \( j^2 = -1 \).

The terms standing before the complex exponentials are functions of the time shift \( \tau \). Basically, they can be obtained as the coefficient of Fourier series:

\[ R_x^\alpha(\tau) = \{R_x(t, \tau)\exp\left(-j2\pi\alpha t\right)\}_t = \frac{1}{T} \int_{-T/2}^{T/2} R_x(t, \tau)\exp\left(-j2\pi\alpha t\right)dt. \]  

(12)

The set of all non-zero functions \( \{R_x^\alpha(\tau)\} \) represents the cyclic autocorrelation function (CACF) of random process \( s(t) \). This process will be called wide-sense stationary (WSS) if and only if its cyclostationary representation consists of \( R_x^0(\tau) \) only, otherwise it can be called wide-sense cyclostationary (WSCS). From that point of view, it is easy to conclude that WSS is a subclass of WSCS. As a matter of fact, all components belonging to cyclic frequencies other than zero, \( \alpha \neq 0 \), describe the varying part of \( R_x(t, \tau) \) and relate to the non-stationarity in the process.

Returning to the pulse amplitude-modulated [7] process \( x(t) \), its CACF could be expressed [8], in a closed form for each integer number \( m \):

\[ R_x^{m/T}(\tau) = \begin{cases} P\left(1-|\tau|\Delta^{-1}\right)\text{sinc}\left[\pi m(\Delta-|\tau|)T^{-1}\right], & |\tau|<\Delta; \\ 0, & \text{otherwise}. \end{cases} \]  

(13)

where \( \text{sinc}(x) \) defines \( \sin(x)/x \) with value 1 at \( x=0 \). The components of cyclic correlation associated with zero cyclic frequency \( \alpha=0 \) \( (m=0) \) and \( m=\pm 1, \pm 2, \pm 3 \) are shown by means of three-dimensional plot in figure 9. The rich presence of cyclic components reveal the fact that \( x(t) \) belongs to WSCS class rather than WSS.

On the contrary, the WSS process \( y(t) \) is described by a single function associated with \( m=0 \) only:

\[ R_y^0(\tau) = \begin{cases} P\left(1-|\tau|\Delta^{-1}\right), & |\tau|<\Delta; \\ 0, & \text{otherwise}. \end{cases} \]  

(14)

The cyclic decomposition of \( y(t) \) shown in figure 10 is a single triangle at \( \alpha=0 \). The dashed lines are shown only for the comparison with \( R_x^m(\tau) \) since no guess about period could be possible for \( y(t) \).
5. Conclusion

The comparative example of wide-sense cyclostationary process considered in the paper can give one
a simple introduction to cyclostationarity with revealing the reasonable distinction between two types
of processes – wide-sense stationary and wide-sense cyclostationary. The essential feature standing
the WSCS processes out against the stationary is the behaviour of its two-dimensional autocorrelation
function. For WSCS, it basically exhibits a periodic pattern depending on both time variables rather
than a one-dimensional profile extended in the cross-dimension holding for WSS.

Actually, the two-dimensional autocorrelation function could be expanded into a Fourier series
with respect to the current-time variable in the case of its strict periodicity. A countable set of
functional coefficients forms the representation of cyclic autocorrelation function. Its necessary part
associated with zero cyclic frequency is just a conventional autocorrelation function while the other
parts, if they present, are in charge of the non-stationary behaviour.

6. Acknowledgement

This work was supported by state assignment of the Ministry of Education and Science of the Russian
Federation (project 8.8502.2017/BP).

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