Hydrodynamic impacts of short laser pulses on plasmas

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Abstract

We determine conditions allowing to simplify the description of the impact of a short and arbitrarily intense laser pulse onto a cold plasma at rest. If both the initial plasma density and pulse profile have plane symmetry, then suitable matched upper bounds on the maximum and the relative variations of the initial density, as well as the intensity and duration of the pulse, ensure a strictly hydrodynamic evolution of the electron fluid (without wave-breaking or vacuum-heating) during its whole interaction with the pulse, while ions can be regarded as immobile. We use a recently developed fully relativistic plane model whereby the system of the (Lorentz-Maxwell and continuity) PDEs is reduced into a family of highly nonlinear but decoupled systems of non-autonomous Hamilton equations with one degree of freedom, with the light-like coordinate $\xi = ct - z$ instead of time $t$ as an independent variable, and new apriori estimates (eased by use of a Liapunov function) of the solutions in terms of the input data (initial density and pulse profile). If the laser spot radius $R$ is finite but not too small the same conclusions hold for the part of the plasma close to the axis $\vec{z}$ of cylindrical symmetry. These results may help in drastically simplifying the study of extreme acceleration mechanisms of electrons.

1 Introduction and preliminaries

Laser-plasma interactions induced by ultra-intense laser pulses lead to a variety of very interesting phenomena \cite{1,2,3,4,5}, notably plasma compression for inertial fusion \cite{6}, laser wakefield acceleration (LWFA) \cite{7,8,9} and other extremely compact acceleration mechanisms (e.g. hybrid laser-driven and particle-driven plasma wakefield acceleration \cite{10}) of charged particles, which hopefully will be at the basis of a generation of new, table-top accelerators. This is paramount because accelerators have extremely important applications

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in particle physics, materials science, medicine, industry, environmental remediation, etc; therefore huge investments are being devoted all over the world to the development of such accelerators. Similar extreme conditions (huge electromagnetic fields and huge accelerations of charged particles in plasmas) occur also in a number of violent astrophysical processes, see e.g. [9] and references therein. In general, these phenomena are ruled by the equations of a kinetic theory coupled to Maxwell equations, which can be solved only numerically via particle-in-cell (PIC) techniques. Unfortunately, PIC codes involve huge and expensive computations for each choice of the free parameters; even with the presently ever increasing computational power, exploring the parameter space blindly to single out interesting regions remains prohibitive. Sometimes, good predictions can be obtained also by treating the plasma as a multicomponent (electron and ions) fluid and by numerically solving, via multifluid codes (such as QFluid [14]) or hybrid kinetic/fluid codes, the (simpler) associated hydrodynamic equations; but in general it is not known a priori in which conditions, or spacetime regions, this is possible. Therefore all analytical insights that can simplify the work, at least in special cases or in a limited space-time region, are welcome.

This applies in particular to studying the impact of a very short (and possibly very intense) laser pulse perpendicularly onto a cold diluted plasma at rest (or onto matter which is locally ionized into a plasma by the front of the pulse itself). As it is well-known, electrons start oscillating orthogonally to the direction $\vec{z}$ of propagation of the pulse and drifting in the positive $z$-direction, respectively pushed by the electric and magnetic parts of the Lorentz force induced by the pulse; thereafter, electrons start oscillating also longitudinally (i.e. in $\vec{z}$-direction), pushed by the restoring electric force induced by charge separation. It turns out that the initial dynamics is simpler if the pulse is essentially short. We shall say (see Definition [1]) that the pulse is essentially short if it overcomes each electron before the $z$-displacement $\Delta$ of the latter reaches a negative minimum for the first time; that an essentially short pulse is strictly short if it overcomes each electron before $\Delta$ becomes negative for the first time. In other words, we regard a pulse as strictly short (resp. essentially short) if it overcomes each electron before it finishes the first $1/2$ (resp. $3/4$) longitudinal oscillation. In the nonrelativistic (NR) regime a pulse which is symmetric under inversion about its center is strictly short, essentially short if its duration $l/c$ does not respectively exceed $1/2$, $1$ times the NR plasma oscillation period $t_{\nu}^{nr} \equiv \sqrt{\pi m/n_b e^2}$ associated to the maximum $n_b$ of the initial electron density, i.e. if

$$G_b := \sqrt{\frac{n_b e^2}{\pi m c^2}} \leq \frac{1}{2}$$  \hspace{1cm} (1)

(see Proposition [1]; here $-e, m$ are the electron charge and mass, and $c$ is the speed of light. The relativistic plasma oscillation period is not independent of the oscillation amplitude, but grows with the latter, which in turn grows with the pulse intensity. Correspondingly, eq. (16) can be fulfilled also with a larger $G_b$; in addition, it is compatible with maximizing the oscillation amplitude, and thus also the energy transfer from the pulse to the plasma wave,

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1. We just mention the EU-funded project Eupraxia [11, 12, 13].
2. The reader can recognize such initial longitudinal motions e.g. from fig. 1c and from the electron worldlines reported in fig.s 8, 7.
3. If the pulse is a slowly modulated monochromatic wave $58$ with wavelength $\lambda = 2\pi/k$ this implies a fortiori $\frac{\pi^2}{mc^2} n_b \lambda^2 \ll 1$, so that the plasma is underdense.
because for given \( n_b \) and pulse energy such a maximization can be achieved \([\ref{3}, \ref{24}]\) through a suitable

\[
l \sim \tilde{\xi}_2; \quad \text{(i.e. when } G_b \sim 1/2, \text{ in the nonrelativistic regime).} \tag{2}\]

We believe that such impacts require a deeper understanding because, among other things, they may generate: i) a plasma wave (PW) \([\ref{15}, \ref{16}]\), or even a ion bubble \([\ref{17}, \ref{18}, \ref{19}, \ref{20}, \ref{21}, \ref{22}, \ref{23}]\), producing the LWFA, i.e. accelerating a small bunch of (socalled witness) electrons trailing the pulse to very high energy, in the forward direction; ii) the slingshot effect \([\ref{24}, \ref{25}, ?]\), i.e. the backward acceleration and expulsion of energetic electrons from the vacuum-plasma interface, during or just after the impact. The present work is one out of a few papers \([\ref{26}, \ref{27}]\) arguing that, with the help of the plane, fully relativistic Lagrangian model of Ref. \([\ref{28}, \ref{29}]\) and very little computational power, we can obtain important information about such an impact, in particular the formation of a PW, its persistence before wave-breaking (WB), the features of the latter. As known, a small WB is not necessarily undesired; it may be used to produce and inject the mentioned witness electrons in the PW (self-injection).

The plane model is as follows. One assumes that the plasma is initially neutral, unmagnetized and at rest with zero densities in the region \( z < 0 \). More precisely, the \( t = 0 \) initial conditions for the electron fluid Eulerian density \( n_e \) and velocity \( v_e \) are of the type

\[
v_e(0, x) = 0, \quad n_e(0, x) = \tilde{n}_0(z), \tag{3}\]

where the initial electron (as well as proton) density \( \tilde{n}_0(z) \) fulfills

\[
\tilde{n}_0(z) = 0 \text{ if } z \leq 0, \quad 0 < \tilde{n}_0(z) \leq n_b \text{ if } z > 0 \tag{4}\]

for some \( n_b > 0 \) (a few examples are reported in fig. 2). One assumes that before the impact the laser pulse is a free plane transverse wave travelling in the \( z \)-direction, i.e. the electric and magnetic fields \( E, B \) are of the form

\[
E(t, x) = E^\perp(t, x) = \epsilon^\perp(\epsilon t - z), \quad B = B^\perp = k \times E^\perp \quad \text{if } t \leq 0 \tag{5}\]

(the superscript \( \perp \) denotes vector components orthogonal to \( k \equiv \nabla z \)), where \( \epsilon^\perp(\xi) \) has a bounded support with \( \xi = 0 \) as the left extreme (i.e. the pulse reaches the plasma at \( t = 0 \)). The input data of a specific problem are the functions \( \tilde{n}_0(z), \epsilon^\perp(\xi) \); it is convenient to define also the related functions

\[
\alpha^\perp(\xi) := -\int_{-\infty}^{\xi} d\zeta \, \epsilon^\perp(\zeta), \quad v(\xi) := \left[ \frac{\epsilon \alpha^\perp(\xi)}{mc^2} \right]^2, \tag{6}\]

\[
\tilde{N}(Z) := \int_{0}^{Z} d\zeta \, \tilde{n}_0(\zeta), \quad U(\Delta; Z) := K \int_{0}^{\Delta} d\zeta \, (\Delta - \zeta) \, \tilde{n}_0(Z + \zeta), \tag{7}\]

where \( K := \frac{4\pi e^2}{mc^2} \). By definition \( v \) is dimensionless and nonnegative, and \( \tilde{N}(Z) \) strictly grows with \( Z \). One describes the plasma as a fully relativistic collisionless fluid of electrons and a static fluid of ions (as usual, in the short time lapse of interest here the motion of the

\[\text{Namely, a region containing only ions, because all electrons have been expelled out of it.}\]
much heavier ions is negligible), with $E$, $B$ and the plasma dynamical variables fulfilling the Lorentz-Maxwell and continuity equations. Since at the impact time $t = 0$ the plasma is made of two static fluids, by continuity such a hydrodynamical description (HD) is justified and one can neglect the depletion of the pulse at least for small $t > 0$; the specific time lapse is determined a posteriori, by self-consistency. This allows us to reduce (see [28, 29, or 30, 31, 32, 33] for shorter presentations) the system of Lorentz-Maxwell and continuity partial differential equations (PDEs) into ordinary ones, more precisely into the following continuous family of decoupled Hamilton equations for systems with one degree of freedom. Each system determines the complete Lagrangian (in the sense of non-Eulerian) description of the motion of the electrons having a same initial longitudinal coordinate $Z > 0$ (the $Z$-electrons, for brevity), and reads

$$\Delta'(\xi, Z) = \frac{1 + v(\xi)}{2s^2(\xi, Z)} - \frac{1}{2}, \quad s'(\xi, Z) = K\{\tilde{N}[Z + \Delta(\xi, Z)] - \tilde{N}(Z)\}; \quad (8)$$

it is equipped with the initial conditions

$$\Delta(0, Z) = 0, \quad s(0, Z) = 1. \quad (9)$$

Here the unknown basic dynamical variables $\Delta(\xi, Z)$, $s(\xi, Z)$ are respectively the longitudinal displacement and s-factor of the Z-electrons expressed as functions of $\xi, Z$, while $z_e(\xi, Z) := Z + \Delta(\xi, Z)$ is the present longitudinal coordinate of the Z-electrons; we consider all dynamical variables $f$ (in the Lagrangian description) as functions of $\xi, Z$ instead of $t, Z$; $f'$ stands for the total derivative $df/d\xi := \partial f/\partial \xi + s'\partial f/\partial s + \Delta'\partial f/\partial \Delta; Z$ plays the role of the family parameter. All the other electron dynamical variables can be expressed in terms of $\Delta, s$ and the initial coordinates $X \equiv (X, Y, Z)$ of the generic electron fluid element. In particular, the dimensionless variable $u^\perp := p^{\perp}/mc$, i.e. the electrons’ transverse momentum in $mc$ units, in the basic approximation is given by $u^\perp = \frac{v^\perp}{mc};$ hence $v = u^\perp$. The light-like coordinate $\xi = ct - z$ in Minkowski spacetime can be adopted instead of time $t$ as an independent variable because all particles must travel at a speed lower than $c$; at the end, to express the solution as a function of $t$ one just needs to replace everywhere $\xi$ by the inverses $\xi(t, Z)$ of the strictly increasing (in $\xi$) functions $t(\xi, Z) := (\xi + z_e(\xi, Z))/c$. Eq. (8) are Hamilton equations with $\xi, \Delta, -s$ playing the role of the usual $t, q, p$ and (dimensionless) Hamiltonian

$$H(\Delta, s, \xi; Z) \equiv \frac{s^2 + 1 + v(\xi)}{2s} + U(\Delta; Z); \quad (10)$$

the first term gives the kinetic + rest mass energy, while $U$ plays the role of a potential energy due to the electric charges’ mutual interaction. Consequently, along the solutions of $H'$ = $\partial H/\partial \xi = v'/2s$. Integrating the latter identity by parts, and using the definition of $H$ in $H'$ of $H$, we find

$$\frac{(s-1)^2}{2s} + U(\Delta; Z) = H(\xi, Z) - 1 - \frac{v(\xi)}{2s(\xi, Z)} = \int_0^\xi d\eta \frac{v_s'(\eta, Z)}{2s^2(\eta, Z)} = \nu(\xi, Z). \quad (11)$$

5Namely, $s$ is the light-like component of the 4-velocity of the Z electrons, or equivalently is related to their 4-momentum $p$ by $p^\perp - cp^\perp \equiv mc^2 s$; it is positive-definite. In the NR regime $|s-1| \ll 1$; in the present fully relativistic regime it needs only satisfy the inequality $s > 0$.

6In the cited papers [28, 29, 30, 31, 32, 33] the two dependences are denoted as $f(\xi, Z)$ and $f(t, Z)$ respectively; since here we use only the former, we denote it simply as $f(\xi, Z)$ (without the $^*$).
The Hamilton eqs (8) are non-autonomous for $0 < \xi < l$, where $[0, l]$ is the smallest closed interval containing the support of $\varepsilon^\perp$; ultra-intense pulses are characterized by $\max_{\xi \in [0, l]} \{ v(\xi) \} \gg 1$ and induce ultra-relativistic electron motions. For $\xi \geq l$ (8) can be solved also by quadrature, using the energy integrals of motion $H(\xi, Z) = H(l, Z) =: h(Z) = \text{const.}$

Solving (8-9) yields the motions of the Z-electrons’ fluid elements, which are fully represented through their worldlines in Minkowski space. In fig. 7, 8 we have displayed the projections onto the $z, ct$ plane of these worldlines for two specific sets of input data; as we can see, the PW emerges from them as a collective effect. Mathematically, the PW features can be derived passing to the Eulerian description of the electron fluid; the resulting flow is laminar, with $xy$ plane symmetry. The Jacobian of the transformation $X \mapsto x_e \equiv (x_e, y_e, z_e)$ from the Lagrangian to the Eulerian coordinates reduces to $J_{\xi, Z} = \partial z_e(\xi, Z)/\partial Z$, because $x_e^+ - X^+$ does not depend on $X^+$.

The HD breaks where worldlines intersect, leading to WB of the PW. No WB occurs as long as $J$ remains positive. If the initial density is uniform, $\tilde{n}_0(Z) = n_0 = \text{const}$, both the equations (8) and the initial conditions (9) become $Z$-independent, because (8b) takes the form $s' = M\Delta$, where $M := Kn$. Consequently, also their solutions become $Z$-independent, and $J \equiv 1$ at all $\xi$. Otherwise, WB occurs after a sufficiently long time [34].

Our main goal here is to determine manageable sufficient conditions on $\tilde{n}_0(z), \varepsilon^+(\xi)$ guaranteeing that $J(\xi, Z) > 0$ for all $Z > 0$ and $\xi \in [0, l]$, without solving the Cauchy problems (8-9). This will ensure that there is no wave-breaking during the laser-plasma interaction (WBPLI), i.e. while the Hamilton equations (8) are non-autonomous (due to the dependence of $v$ on $\xi$). We reach this goal by determining upper and lower bounds first on $\Delta, s, H$ (section 2), then on $J$ and $\partial s/\partial Z$ (section 3), also with the help of a suitable Liapunov function. These bounds provide also useful approximations of these dynamical variables in the interval $0 \leq \xi \leq l$. As said, the NR short-pulse conditions (1) are generalized by the ones (16) in the present, fully relativistic regime. Inequalities (35), (36) are respectively sufficient conditions for (16a), (16b). Instead of (35) one may first check the stronger, but also more easily verifiable, condition (40), or even the simplest (and strongest) one $Mn l^2 \leq 2$. In case (36) is satisfied, we can exclude WBPLI: in the NR regime, if also (48) is fulfilled; in the general case, if one of the three conditions of Proposition 1 is fulfilled (if $Q_0 < 1$, the strongest but easiest to compute, is not fulfilled, then one may check $Q_1 < 1$, or $Q_2 < 1$), namely if initially the plasma is sufficiently diluted and/or the local relative variations of its density are sufficiently small. In section 4 we compare the dynamics of $s, \Delta, J, \sigma$ induced by the same pulse on five representative $\tilde{n}_0(z)$ having the same upper bound and asymptotic value $n_0$, both by numerically solving the equations and by applying the mentioned inequalities.

In particular, we learn that the density profile at the very edge of the plasma is critical; for instance, if $\tilde{n}_0(z) \sim z$ as $z \to 0^+$ then WB occurs earlier (albeit electrons collide with very small relative velocities) than if $\tilde{n}_0(z) \sim z^2$, or if $\lim_{z \to 0^+} \tilde{n}_0(z) > 0$ (discontinuous density at $z = 0$). To produce LWFA one usually shoots the laser pulse orthogonally to a supersonic gas (e.g. hydrogen or helium) jet; since outside the jet nozzle it is $\tilde{n}_0(z) \sim z^2$, our results imply that under rather broad conditions such an impact occurs in the hydrodynamic regime.

For $\xi \geq l$, using the conservation of energy, one can show [27] that, while $\Delta$ and $s$ are periodic with a suitable period $\xi_u, J$ and $\sigma$ are \textit{linearly quasiperiodic}, namely of the form

$$ f(\xi) = a(\xi) + \xi b(\xi), \quad \xi \geq l, $$

(12)
where \(a, b\) are periodic in \(\xi\) with period \(\xi_H\), and \(b(\xi)\) oscillates between positive and negative values, and so does the second term, which dominates as \(\xi \to \infty\), with \(\xi\) acting as a modulating amplitude. Therefore the occurrence of WB after the laser-plasma interaction is best investigated studying the dependence \([12, 27]\).

The spacetime region where the present plane hydrodynamic model (predicting a laminar and \(xy\)-symmetric flow) is self-consistent is determined (section 4) by the conditions \(J > 0\) (no collisions) and \((62)\) (undepleted pulse approximation); for typical LWFA experiments the length and time sizes allowed by \((62)\) are respectively of the order of several hundreds of microns, femtoseconds. The spacetime region where the predictions of the model can be trusted is further reduced (section 4) by the finite transverse size \(R\) of the laser pulse.

According to our model, other phenomena characteristic of plasma physics, like turbulent flows, diffusion, heating, heat exchange, as well the very motion of ions, can be excluded inside the latter region, but of course can and will occur outside.

Finally, we point out that recent advances in multi-timescale analysis permitted also (semi)analytical studies of laser-plasma interaction (including some aspects of WB) in the ultrarelativistic regime using Vlasov description \([35]\).

### 2 Apriori estimates of \(\Delta, s, H\) for small \(\xi > 0\)

The Cauchy problem \((8-9)\) is equivalent to the following integral one:

\[
\Delta(\xi, Z) = \int_0^\xi \frac{d\eta}{2s^2(\eta; Z)} \left[ \frac{\xi}{2} - s(\xi, Z) - 1 \right] = \int_0^\xi \int_Z dZ' K\tilde{n}_0(Z').
\]

If \(\tilde{n}_0(Z) = n_0 = \text{const}\), then \(U(\Delta) = M\Delta^2/2, s' = M\Delta, \) and \((13)\) amount to

\[
s(\xi) = 1 + \frac{M}{2} \left[ -\frac{\xi^2}{2} + \int_0^\xi d\eta (s' - \eta) \frac{1 + v(\eta)}{s^2(\eta)} \right],
\]

where \(M := Kn_0\); once \((14)\) is solved, one obtains \(\Delta\) from \(\Delta = s'/M\). In fig. 1 we plot an example of a monochromatic laser pulse slowly modulated by a Gaussian and the corresponding solution \((s, \Delta)\) in a constant density plasma; the qualitative behaviour of the solution remains the same also if \(\tilde{n}_0(z) \neq \text{const}\). In the NR regime \(v \ll 1\), whence \(|\Delta/l| \ll 1, |\delta| \ll 1\), where \(\delta := s - 1\); at lowest order in \(\delta, \Delta, \) \((8-9)\) reduce to the equations \(\delta' = M\Delta, \Delta' = v/2 - \delta\) of a forced NR harmonic oscillator with trivial initial conditions. The solution is

\[
\Delta(\xi) = \int_0^\xi \frac{d\eta}{2} \sin \left[ \sqrt{M}(\xi - \eta) \right], \quad \delta(\xi) = \int_0^\xi \frac{d\eta}{2} \sin \left[ \sqrt{M}(\xi - \eta) \right].
\]

By \((8)\), the zeroes of \(\Delta(\cdot, Z)\) are extrema of \(s(\cdot, Z)\), and vice versa, because \(\tilde{N}(Z)\) grows with \(Z\). Let us recall how \(\Delta, s\) start evolving from their initial values \((9)\). As said, for \(\xi > 0\) all electrons reached by the pulse start to oscillate transversely and drift forward; in fact, \(v(\xi)\) becomes positive, implying in turn that so does the right-hand side (rhs) of \((8a)\) and
Figure 1: a) Normalized amplitude of a linearly polarized \( \psi = 0 \) monochromatic laser pulse slowly modulated by a Gaussian with full width at half maximum \( l' \) and peak amplitude \( a_0 \equiv \lambda e E^\perp_{M} / mc^2 = 1.3 \); this yields a moderately relativistic electron dynamics, and \( \Delta \equiv \Delta^0 (0) \approx 0.45 l' \). If \( l' = 7.5 \mu m \) (corresponding to a pulse duration of \( \tau' = l'/c \approx 2.5 \times 10^{-14} s \)), and the wavelength is \( \lambda = 0.8 \mu m \), then the corresponding peak intensity must be \( I = 7.25 \times 10^{18} W/cm^2 \); these are typical values obtainable by Ti:Sapphire lasers in LWFA experiments.  

b) Corresponding forcing term \( v(\xi) \), and average-over-a-cycle \( v_a(\xi) \) of the latter.  
c) Corresponding solution of (8-9), or equivalently of (14), if \( Kn_{0} l^2 \approx 4 \); this value is obtained if \( l' = 7.5 \mu m \) and \( \tilde{n}_0(Z) \equiv n_0 = 2 \times 10^{18} \text{cm}^{-3} \) (a typical value of the electron density used in LWFA experiments). As expected, \( s \) is insensitive to the rapid oscillations of \( \epsilon^\perp \) for \( \xi \in [0, l] \), while for \( \xi \geq l \) the energy \( H \) is conserved, and the solution is periodic. The length \( l \) is determined on physical grounds; if e.g. the plasma is created locally by the impact of the front of the pulse itself on a gas (e.g. hydrogen or helium), then \( [0, l] \) has to contain all points \( \xi \) where the pulse intensity is sufficient to ionize the gas. Here for simplicity and conventionally we have fixed it to be \( l = 4l' \) [the possible inaccuracy of such a cut is very small, because \( \epsilon(0^+) = \epsilon(l^-) \) is \( 2^{-16} \) times the maximum \( \epsilon(l/2) \) of the modulation, i.e. practically zero] what makes \( G_b \equiv \sqrt{Kn_0 l} \approx 8 \).
\( \Delta \): the \( Z=0 \) electrons leave behind themselves a layer of ions of finite thickness completely evacuated of electrons. If the density vanished (\( \tilde{n}_0 \equiv 0 \)) then we would obtain

\[
s \equiv 1, \quad \Delta(\xi, Z) = \int_0^\xi d\eta \frac{v(\eta)}{2} =: \Delta^0(\xi);
\]

\( \Delta^0(\xi) \) grows with \( \xi \) and is almost constant for \( \xi > l \) if \( v(\xi) \simeq 0 \) for \( \xi > l \) (what occurs if the pulse is slowly modulated (58)). On the contrary, since the density is positive, the growth of \( \Delta \) implies also the growth of the rhs of (8a) (because the latter grows with \( \Delta \)), and of \( s(\xi, Z) \). \( \Delta(\xi, Z) \) keeps growing as long as \( 1 + v(\xi) > s^2(\xi, Z) \), reaches a maximum at \( \bar{\xi}_1(Z) \equiv \) the smallest \( \xi > 0 \) such that the rhs (8b) vanishes. \( s(\xi, Z) \) keeps growing as long as \( \Delta(\xi, Z) \geq 0 \), reaches a maximum at the first zero \( \bar{\xi}_2 \) of \( \Delta(\xi, Z) \) and decreases for \( \xi > \bar{\xi}_2 \), while \( \Delta(\xi, Z) \) is negative. \( \Delta(\xi, Z) \) reaches a negative minimum at \( \bar{\xi}_3(Z) = \) the smallest \( \xi > \bar{\xi}_2 \) such that the rhs (8b) vanishes again. We also denote by \( \bar{\xi}_3(Z) \) the smallest \( \xi > \bar{\xi}_3 \) such that \( s(\xi, Z) = 1 \), and \( \mathcal{I} := [0, \bar{\xi}_3] \). We encourage the reader to single out \( \bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \bar{\xi}_3 \) for the solution considered in fig. 1 from the graphs of 1. If \( \epsilon^* \) is slowly modulated then \( v(l) \simeq 0 \) (see section 5.2, appendix 5.4 in \( \cite{29} \) for details); then \( \bar{\xi}_3 \simeq \bar{\xi}_3 \) if in addition \( l < \bar{\xi}_3 \).

**Definition 1** A pulse is strictly short, essentially short w.r.t. \( \tilde{n}_0 \) if it respectively fulfills

\[
\begin{cases}
\Delta(\xi, Z) \geq 0, & \forall \xi \in [0, l], Z \geq 0 \quad \Leftrightarrow \quad l \leq \bar{\xi}_2(Z) \\
s(\xi, Z) \geq 1, & \forall \xi \in [0, l], Z \geq 0 \quad \Leftrightarrow \quad \bar{\xi}_3(Z) \quad \forall Z \geq 0.
\end{cases}
\]

In the NR regime if \( \tilde{n}_0(Z) \equiv n_0 = \text{const} \) these respectively amount to requiring that the corresponding solution (15) fulfills \( \Delta(l) \geq 0, \delta(l) \geq 0 \); if \( \tilde{n}_0(Z) \neq \text{const} \) it is sufficient to replace \( n_0 \) by \( n_b \) to obtain sufficient conditions for the fulfillment of (16); they are

\[
\begin{align*}
2\Delta(\xi) &= \int_0^\xi d\eta v(\eta) \cos \left[ \sqrt{Kn_b}(\xi - \eta) \right] \geq 0, & \forall \xi \in [0, l], \\
2\delta(\xi) &= \int_0^\xi d\eta v(\eta) \sin \left[ \sqrt{Kn_b}(\xi - \eta) \right] \geq 0.
\end{align*}
\]

**Proposition 1** If \( v \) is symmetric about \( \xi = l/2 \), i.e. \( v(\xi) = v(l-\xi) \), then (17a) amounts to \( G_b \leq 1/2 \); if \( v \) in addition has a unique maximum in \( \xi = l/2 \), then (17b) amounts to \( G_b \leq 1 \).

The proof is in the appendix (section 5.1). The assumption that \( v(\xi) \) be symmetric is satisfied with very good approximation if the pulse is a slowly modulated one (58) with a symmetric modulation \( \epsilon(\xi) \) about \( \xi = l/2 \) (as in fig. 1), by (59).

The following estimates hold for \( \xi \in \mathcal{I} \equiv [0, \bar{\xi}_3] \). First, \( s \geq 1 \) and (13a) imply the bound

\[
\Delta(\xi, Z) \leq \Delta^0(\xi).
\]
2.1 Constant density case

If \( \tilde{n}_0(Z) \equiv n_0 > 0 \), for \( \xi \in \mathcal{I} \) we find by (18) \( s(\xi) - 1 = M \int_0^\xi d\eta \Delta(\eta) \leq M \int_0^\xi d\eta \Delta_0(\eta) \), i.e.

\[
s(\xi) \leq 1 + \frac{M}{2} \int_0^\xi d\eta (\xi - \eta) v(\eta) =: s^{(1)}(\xi).
\]

(19)

\( s^{(1)}(\xi) \) grows strictly with \( \xi \) and is convex. Eq. (19) and (13) in turn imply

\[
\Delta(\xi) \geq \int_0^\xi d\eta \frac{1 + v(\eta)}{[s^{(1)}(\eta)]^2} - \frac{\xi}{2} =: \Delta^{(1)}(\xi).
\]

(20)

\( \Delta^{(1)}(\xi) \) vanishes at \( \xi = 0 \), grows with \( \xi \) for small \( \xi > 0 \) until its (unique) maximum point; for larger \( \xi \) it decreases and becomes negative. Hence, a lower bound \( \tilde{\xi}_2^{(1)} \) for \( \tilde{\xi}_2 \) is the smallest \( \xi > 0 \) such that \( \Delta^{(1)}(\xi) = 0 \). Therefore \( \Delta^{(1)}(l) \geq 0 \) ensures that \( \tilde{\xi}_2 \geq \tilde{\xi}_2^{(1)} \geq l \).

Eq. (13b) also implies \( s(\xi) - 1 = M \int_0^\xi d\eta \Delta(\eta) \geq M \int_0^\xi d\eta \Delta^{(1)}(\eta) \), namely

\[
s(\xi) \geq 1 + \frac{M}{2} f(\xi) =: s^{(2)}(\xi), \quad f(\xi) := \int_0^\xi d\eta (\xi - \eta) [1 + v(\eta)] =: \frac{\xi^2}{2}.
\]

At least for small \( \xi \), this is a more stringent lower bound for \( s \) in \( \mathcal{I} \) than \( s \geq 1 \): \( f(\xi) \) vanishes at \( \xi = 0 \), grows with \( \xi \) for small \( \xi > 0 \) until its (unique) maximum point; for larger \( \xi \) it decreases and becomes negative. Hence, a lower bound \( \tilde{\xi}_3^{(1)} \) for \( \tilde{\xi}_3 \) is the smallest \( \xi > 0 \) such that \( f(\xi) = 0 \). Therefore \( f(l) \geq 0 \) ensures that \( \tilde{\xi}_3 \geq \tilde{\xi}_3^{(1)} \geq l \).

2.2 Generic density case

Let \( \tilde{n}(\xi, Z) := \tilde{n}_0[z_e(\xi, Z)] \), \( \tilde{\xi}_2' := \min\{\tilde{\xi}_2, l\} \), \( n_u, n_d, n_d > 0 \) be some upper, lower bounds on \( \tilde{n} \)

\[
n_d(Z) \leq \tilde{n}(\xi, Z) \leq n_u(Z)
\]

(22)

for \( 0 \leq \xi \leq \tilde{\xi}_2' \). If \( \tilde{n}_0(Z) \equiv n_0 \) then \( \tilde{n} \equiv n_0 \), and we can set \( n_u = n_d = n_0 \). In general, a \( Z \)-independent choice of \( n_u \) is \( n_u = n_b \), see [4]. More accurately, given \( \Delta_u > 0 \) such that

\[
\Delta(\xi, Z) \leq \Delta_u \quad \forall \xi \in [0, \tilde{\xi}_2'],
\]

(23)

then \( 0 \leq \Delta(\xi, Z) \leq \Delta_u(Z) \) for all \( \xi \in [0, \tilde{\xi}_2'] \) (by the definition of \( \tilde{\xi}_2' \)), and (22) holds choosing

\[
n_u(Z) = \max_{Z' \in [Z, Z + \Delta_u]} \{\tilde{n}_0(Z')\}, \quad n_d(Z) = \min_{Z' \in [Z, Z + \Delta_u]} \{\tilde{n}_0(Z')\};
\]

(24)

\( n_u(Z) \) is a lower (and therefore better) upper bound than \( n_u = n_b \). We also abbreviate \( M_u(Z) := Kn_u(Z), \quad M_d(Z) := Kn_d(Z) \). By [18], we can adopt the simple choice \( \Delta_u := \Delta_0(l) \).

---

7Actually, if \( v(l) \approx 0 \), as it occurs if the pulse is a slowly modulated one (58), then \( \Delta_0(\xi) \approx \Delta_0(l) \) if \( \xi > l \), and (24) holds with \( \Delta_u = \Delta_0(l) \) for all \( 0 \leq \xi \leq \tilde{\xi}_2' \), even if \( \tilde{\xi}_2' > l \).
Lemma 1 For all $\xi \in [0, \tilde{\xi}_3]$ the rhs $\nu$ of (11) can be bound by

$$\nu(\xi, Z) \leq \frac{M_u(Z)}{2} [\Delta^{(0)}(\xi)]^2$$

(25)

Proof For $\xi \in [0, \tilde{\xi}_2]$ the inequality is proved as follows:

$$\nu(\xi) = \int_0^{\xi} d\eta \frac{v(\eta)}{2s^2(\eta)} \leq \int_0^{\xi} d\eta \frac{v(\eta)}{2s^2(\eta)} \int_{\xi}^{\xi+\Delta(\eta)} dz M_u = M_u \int_0^{\xi} d\eta \frac{v(\eta)}{2s^2(\eta)} \leq M_u \int_0^{\xi} d\eta \frac{v(\xi)}{2} = \nu_u(\xi)$$

where we have used $\Delta^{(0)} = v/2$ and for brevity we have not displayed the $Z$ argument. The inequality holds also in $[\tilde{\xi}_2, \tilde{\xi}_3]$ because there $\nu$ decreases, whereas $\nu_u$ grows.

The maximum of $\nu(\xi, Z)$ in $[0, \tilde{\xi}_3]$ is in $\xi = \tilde{\xi}_2$ because $s' > 0$ in $[0, \tilde{\xi}_2]$ and $s' < 0$ in $[\tilde{\xi}_2, \tilde{\xi}_3]$. To obtain upper, lower bounds $s_u, s_d$ for $s(\xi, Z)$ and a lower bound $\Delta_d(Z)$ for $\Delta(\xi, Z)$ in the longer interval $0 \leq \xi \leq \xi'_3 := \min\{l, \xi_3\}$ we use (25) to majorize

$$\nu(\xi, Z) \leq \nu(\xi_2, Z) \leq \frac{M_u}{2} [\Delta^{(0)}(\xi_2)]^2 \leq \frac{M_u}{2} \Delta^2_u.$$  

This, replaced in (11), yields $(s-1)^2/2s \leq M_u \Delta^2_u/2$ and $\mathcal{U} \leq M_u \Delta^2_u/2$, whence

$$s_d \leq s(\xi, Z) \leq s_u, \quad \frac{s_u}{s_d} := 1 + \frac{M_u}{2} \Delta^2_u \pm \sqrt{\left(1 + \frac{M_u}{2} \Delta^2_u\right)^2 - 1}, \quad \Delta(\xi, Z) \geq \Delta_d(Z),$$

(26)

where $\Delta_d$ is the negative solution of the equation $\mathcal{U}(\Delta; Z) = M_u(Z) \Delta^2_u/2$ (as a first estimate, $\Delta_d = -\Delta_u$). Clearly, $1/s_d = s_u > 1$.

Proposition 2 For all $\xi \in [0, \xi'_3]$ the dynamical variables $\Delta, s$ are bounded as follows:

$$\Delta^{(0)}(\xi, Z) \geq \Delta(\xi, Z) \geq \Delta^{(1)}(\xi, Z),$$

$$s^{(1)}(\xi, Z) \geq s(\xi, Z) \geq s^{(2)}(\xi, Z),$$

(27)

where $\Delta^{(0)}(\xi) := \int_0^{\xi} d\eta \nu(\eta)/2$, and

$$s^{(1)}(\xi, Z) := \min \{s_u, 1 + g(\xi, Z)\}, \quad g(\xi, Z) := \frac{M_u}{2} \int_0^{\xi} d\eta (\xi-\eta) \nu(\eta),$$

$$\Delta^{(1)}(\xi, Z) := \max \{\Delta_d, d(\xi, Z)\}, \quad d(\xi, Z) := \int_0^{\xi} d\eta \frac{1+\nu(\eta)}{2 [s^{(1)}(\eta, Z)]^2} - \frac{\xi}{2},$$

$$s^{(2)}(\xi, Z) := \begin{cases} 1 + \frac{M_d}{2} f(\xi, Z) & 0 \leq \xi \leq \tilde{\xi}_2^{(1)} \\ \max \left\{s_d, 1 + \left[\frac{M_d}{2} - \frac{M_u}{2}\right] f\left(\tilde{\xi}_2^{(1)}, Z\right) + \frac{M_u}{2} f(\xi, Z)\right\} & \tilde{\xi}_2^{(1)} < \xi \leq \tilde{\xi}_3, \end{cases}$$

$$f(\xi, Z) := \int_0^{\xi} d\eta (\xi-\eta) \left\{\frac{1+\nu(\eta)}{[s^{(1)}(\eta, Z)]^2} - 1\right\},$$

(28)

\footnote{Again, if $\nu(l) \simeq 0$, as it occurs if the pulse is a slowly modulated one, then $\Delta^{(0)}(\xi) \simeq \Delta^{(0)}(l)$ if $\xi > l$, and these results remain valid if we replace $\tilde{\xi}_3$ by $\tilde{\xi}_3$, even if $\tilde{\xi}_3 > l$.}
where \( \xi_2(Z) < \bar{\xi}_2 \) is the zero of \( d(\xi, Z) \) as well as the maximum point of \( f(\xi, Z) \), and \( M'_u/K = n' := \max_{Z' \in [Z + \Delta_d, Z]} \{ \bar{n}_0(Z') \} \). Moreover, the value of the Hamiltonian is bounded by

\[
H_d := 1 + \frac{v}{2s^{(1)}} + \nu_d \leq H \leq 1 + \frac{v}{2s^{(2)}} + \frac{M_u}{2} [\Delta^{(0)}]^2 =: H_u,
\]

where, dubbing the maximum of \( d(\xi, Z) \) by \( \bar{\xi}_1(Z) \), we have defined

\[
u_d(\xi, Z) := \begin{cases} 
\frac{M_d(Z)}{2} [\Delta^{(1)}(\xi, Z)]^2 & \xi \in [0, \bar{\xi}_1], \\
\frac{M_d(Z)}{2} [\Delta^{(1)}(\bar{\xi}_1)]^2 & \xi \in [\bar{\xi}_1, \bar{\xi}_1], \\
\frac{M_d(Z)}{2} [\Delta^{(1)}(\bar{\xi}_1, Z)]^2 + M'_u(Z) \int_{\bar{\xi}_1}^\xi d\eta \frac{v\Delta^{(0)}}{2}(\eta, Z) & \xi \in [\bar{\xi}_1, \bar{\xi}_2].
\end{cases}
\]

[Note that \( \bar{\xi}_1(Z) < \bar{\xi}_1 \)]. Eq. (27-28) reduce to (18-21) if \( \bar{n}_0(Z) \equiv n_0 = \text{const.} \)

**Proof** The left inequality in (27a) is the already proven (18). Eq. (13b) by (24), (18) implies \( s(\xi, Z) - 1 = \int_0^\xi d\eta M_u \Delta(\eta, Z) \leq M_u \int_0^\xi d\eta \Delta^0(\eta) \), what, together with (26a), implies the left inequality in (27b); the latter, together with (13a), (26b) in turn imply the right inequality in (27a). \( d(\xi, Z) = f'(\xi, Z) \) vanishes at \( \xi = 0 \), grows with \( \xi \) for small \( \xi > 0 \), as far as it reaches a maximum for sufficiently large \( \xi \); then decreases to negative values. Hence \( \tilde{\xi}_2 \), i.e. the smallest \( \xi > 0 \) such that \( \Delta^{(1)}(\xi, Z) = 0 \), is indeed a lower bound for \( \bar{\xi}_2 \); \( \tilde{\xi}_2 \) is also the maximum point of \( f \) and \( s^{(2)} \). Eq. (13b) also implies for all \( \xi \in [0, \tilde{\xi}_2] \)

\[
s(\xi, Z) - 1 \geq M_d \int_0^\xi d\eta \Delta(\eta, Z) \geq M_d \int_0^\xi d\eta d(\eta, Z) = \frac{M_d}{2} f(\xi, Z).
\]

If \( \xi \in [\tilde{\xi}_2, \tilde{\xi}_3] \), integrating \( \xi \) over \( [\tilde{\xi}_2, \tilde{\xi}_3] \) and recalling that \( \Delta_d \leq \Delta < 0 \) there, we find

\[
s(\xi, Z) - s(\tilde{\xi}_2, Z) = -K \int_{\tilde{\xi}_2}^\xi d\eta \int_{Z + \Delta(\eta, Z)} Z dZ' \bar{n}_0(Z') \geq M'_u \int_{\tilde{\xi}_2}^\xi d\eta \Delta(\eta, Z) \geq M'_u \int_{\tilde{\xi}_2}^\xi d\eta \Delta^{(1)}(\eta, Z)
\]

\[
> M'_u \int_{\tilde{\xi}_2}^\xi d\eta \Delta^{(1)}(\eta, Z) \geq \frac{M'_u}{2} \left[ f(\xi, Z) - f \left( \tilde{\xi}_2, Z \right) \right]
\]

(the first inequality in the last line holds because \( \Delta^{(1)}(\eta, Z) < 0 \) if \( \eta \in [\tilde{\xi}_2, \tilde{\xi}_3] \)); since \( s \) has its maximum in \( \tilde{\xi}_2 \), \( [31] \) implies in particular \( s(\tilde{\xi}_2, Z) \geq s(\tilde{\xi}_2, Z) \geq 1 + \frac{M_d}{2} f(\tilde{\xi}_2, Z) \), which replaced in \( [32] \) gives

\[
s(\xi, Z) \geq 1 + \frac{M_d}{2} f(\tilde{\xi}_2, Z) + \frac{M'_u}{2} \left[ f(\xi, Z) - f \left( \tilde{\xi}_2, Z \right) \right];
\]

the latter inequality and \( [31] \) amount to the right inequality in (27b), which holds together with \( s(\xi, Z) \geq 1 \). The right inequality in (29) follows from \( [11] \) and \( [25] \). From \( 2\Delta^{(1)} = (1 + v)/s^{(2)} - 1 \leq v/s^{(2)} \) it follows for \( \xi \in [0, \tilde{\xi}_2] \)

\[
\nu(\xi, Z) \geq M_d \int_0^\xi d\eta \frac{v\Delta}{2s^2}(\eta) \geq M_d \int_0^\xi d\eta \frac{v\Delta^{(1)}}{2s^{(2)}}(\eta) \geq M_d \int_0^\xi d\eta \frac{\Delta^{(1)}(\eta)\Delta^{(1)}(\eta)}{2}\left( \eta^2 \right) = \frac{M_d}{2} \left[ \Delta^{(1)}(\xi) \right]^2,
\]

\[
11
\]
where for brevity again we have not displayed the Z argument; since the rhs has its maximum in \( \xi = \tilde{\xi}_1^{(1)} \), whereas \( \nu(\xi, Z) \) still grows in \([\tilde{\xi}_1^{(1)}, \tilde{\xi}_2] \), we obtain

\[
\nu(\xi, Z) \geq \begin{cases} 
\frac{M_d}{2} \left[ \Delta^{(1)}(\xi) \right]^2 & \xi \in [0, \tilde{\xi}_1^{(1)}] \\
\frac{M_d}{2} \left[ \Delta^{(1)}(\tilde{\xi}_1^{(1)}) \right]^2 & \xi \in [\tilde{\xi}_1^{(1)}, \tilde{\xi}_2] 
\end{cases} \quad (33)
\]

If \( \xi \in [\tilde{\xi}_2, \tilde{\xi}_3] \) then \( s', \Delta < 0 \) in \([\tilde{\xi}_2, \xi] \), and

\[
\int_{\xi}^{\xi} \frac{d\eta}{2s^2}(\eta) = -\int_{\xi}^{\xi} \frac{Kv}{2s^2}(\eta) d\eta n_0(\xi) \geq M_d \int_{\xi}^{\xi} d\eta \frac{v\Delta}{2} (\eta) \geq M_d' \int_{\xi}^{\xi} d\eta \frac{v\Delta^{(1)}}{2} (\eta) > M_d' \int_{\xi}^{\xi} d\eta \frac{v\Delta^{(1)}}{2} (\eta)
\]

(the last inequality holds because \( \Delta^{(1)} < 0 \) in \([\tilde{\xi}_2^{(1)}, \tilde{\xi}_2] \)); summing this inequality and \( \nu(\tilde{\xi}_2, Z) \geq M_d \left[ \Delta^{(1)}(\tilde{\xi}_1^{(1)}) \right]^2 / 2 \) we obtain the one \( \nu(\xi, Z) \geq M_d \left[ \Delta^{(1)}(\tilde{\xi}_1^{(1)}) \right]^2 / 2 + M_d' \int_{\xi}^{\xi} d\eta v\Delta^{(1)}(\eta)/2 \), which actually holds for all \( \xi \in [\tilde{\xi}_2^{(1)}, \tilde{\xi}_3] \), by (33) and because the second term is negative if \( \xi \in [\tilde{\xi}_2^{(1)}, \tilde{\xi}_2] \). Summing up, we find \( \nu(\xi, Z) \geq \nu_d(\xi, Z) \) in all the interval \([0, \tilde{\xi}_3] \); this, together with (11), implies also the left inequality (29). \( \square \)

By (26), inequalities (22) hold for all \( \xi \in [0, \tilde{\xi}_3] \) if, rather than by (24), we define \( n_u, n_d \) by

\[
n_u(Z) = \max_{Z' \in [Z + \Delta_d, Z + \Delta_u]} \{ n_0(Z') \}, \quad n_d(Z) = \min_{Z' \in [Z + \Delta_d, Z + \Delta_u]} \{ n_0(Z') \}. \quad (34)
\]

As said, \( \Delta^{(1)}(\xi, Z) = d(\xi, Z) = f'(\xi, Z) \) vanishes at \( \xi = 0 \), grows up to its unique positive maximum at \( \tilde{\xi}_1^{(1)} \), then decreases to negative values; \( \tilde{\xi}_2^{(1)} \) is the unique \( \xi > \tilde{\xi}_1^{(1)} \) such that \( \Delta^{(1)}(\xi, Z) = 0 \). Hence, \( \tilde{\xi}_2^{(1)} \) is a lower bound for \( \xi_2 \). Therefore the condition

\[
\Delta^{(1)}(l, Z) \geq 0 \quad (35)
\]

ensures that \( \tilde{\xi}_2(Z) \geq \tilde{\xi}_2^{(1)}(Z) \geq l \), namely that the pulse is strictly short. Similarly, \( s^{(2)} - 1 \) vanishes at \( \xi = 0 \), grows up to its unique positive positive maximum at \( \tilde{\xi}_3^{(1)} \), then decreases to negative values. Hence, a lower bound \( \tilde{\xi}_3^{(1)} \) for \( \tilde{\xi}_3 \) is the unique \( \xi > \tilde{\xi}_2^{(1)} \) such that \( s^{(2)}(\xi, Z) = 1 \), and the condition

\[
s^{(2)}(l, Z) \geq 1 \quad (36)
\]

ensures that \( \tilde{\xi}_3(Z) \geq \tilde{\xi}_3^{(1)}(Z) \geq l \equiv \tilde{\xi}_3^{(1)} \), namely that the pulse is essentially short.

Moreover, under this assumption we find by (29) the following upper and lower bounds on the final Z-electron energy \( h(Z) \) after their interaction with the pulse:

\[
H_u(l, Z) \leq h(Z) \leq H_u(l, Z). \quad (37)
\]

In fig. 4 we plot \( s(\cdot, Z), \Delta(\cdot, Z) \) for two values of \( Z \) and the associated upper and lower bounds corresponding to the densities of fig. 2 which have the same asymptotic value \( n_0 \). As we can see, the bounds agree well. Moreover, a useful lower bound for \( \Delta^{(1)} \) is provided by

**Lemma 2**

\[
\Delta^{(1)}(\xi, Z) \geq \Delta^{(0)}(\xi) \left[ 1 - \frac{M_d(Z)}{2} \xi^2 - M_d(Z) \xi \Delta^{(0)}(\xi) \right]. \quad (38)
\]
Proof \( g \geq 0 \) implies \( \frac{1}{1+g} \geq 1-g, \frac{1}{1+g}^2 \geq (1-g)^2 \geq 1-2g \), so that

\[
2\Delta^{(1)}(\xi, Z) \geq \int_0^\xi d\eta \left\{ \left[ 1 + v(\eta) \right] \left[ 1 - 2g(\eta, Z) \right] - 1 \right\} = 2\Delta^{(0)}(\xi) - \int_0^\xi d\eta 2g(\eta, Z)\left[ 1 + v(\eta) \right].
\] (39)

The definitions of \( g, \Delta^{(0)} \) immediately imply the inequality \( g(\xi, Z) \leq M_u \xi \Delta^{(0)}(\xi) \), whence

\[
\int_0^\xi d\eta 2g(\eta, Z) \leq M_u \int_0^\xi d\eta 2\Delta^{(0)}(\eta) \leq M_u \int_0^\xi d\eta 2\Delta^{(0)}(\xi) \leq M_u \xi^2 \Delta^{(0)}(\xi),
\]

\[
\int_0^\xi d\eta 2g(\eta, Z)v(\eta) \leq M_u \int_0^\xi d\eta 2\Delta^{(0)}(\eta)v(\eta) \leq M_u \int_0^\xi d\eta \xi^4 \left[ \Delta^{(0)}(\eta) \right] (\eta) = 2M_u \xi \Delta^{(0)}(\xi)^2.
\]

Replacing the latter in (39) we obtain (38). □

As a consequence, if the square bracket at the rhs(38) is nonnegative, then so is \( \Delta^{(1)}(\xi, Z) \), and therefore another condition ensuring that \( \tilde{\xi}_2 > l \) (i.e. that the pulse is strictly short) is

\[
M_u(Z)l^2 \left[ 1 + 2\Delta_u \right] \leq 2,
\] (40)

which is more easily computable, but also more difficult to satisfy, than (35).

One could find more stringent (but also less easily computable) bounds than (27) by replacing them in (13) and reiterating the previous arguments, but for the scope of the present work we content ourselves with these basic relations.

3 Bounds on the Jacobian for small \( \xi > 0 \)

Differentiating (8) we find that the dimensionless variables

\[
\varepsilon(\xi, Z) := J(\xi, Z) - 1 = \frac{\partial s(\xi, Z)}{\partial Z}, \quad \sigma(\xi, Z) := l \frac{\partial s(\xi, Z)}{\partial Z}
\] (41)

fulfill the Cauchy problem

\[
\varepsilon' = -\kappa \sigma, \quad \sigma' = Kl \left( \tilde{n} - \tilde{n}_0 + \tilde{n} \varepsilon \right),
\]

\[
\varepsilon(0, Z) = 0, \quad \sigma(0, Z) = 0,
\] (42)

where we have abbreviated \( \kappa := \frac{1+\nu}{l s^2} \geq 0 \). From \( \sigma \) one can immediately obtain \( \partial u_z/\partial Z \) via

\[
l \frac{\partial u_z}{\partial Z} = -\frac{1 + v}{2s^2} \sigma.
\] (43)

(this is dimensionless, as well). To bound \( \varepsilon, \sigma \) for small \( \xi \) we introduce the Liapunov function

\[
V := \varepsilon^2 + \beta \sigma^2,
\] (44)

\footnote{The first step is the new bound \( \Delta(\xi, Z) \leq \Delta^{(2)}(\xi, Z) := \int_0^\xi d\eta \frac{1+\nu(\eta)}{2s^2(\eta, Z)} - \frac{\xi}{2} \), the second is setting \( \Delta_u := \max_{\xi \in [0,l]} \{ \Delta^{(2)}(\xi) \} \) instead of \( \Delta_u := \Delta^{(0)}(l) \) in (24), and so on.}
where $\beta(Z) \in \mathbb{R}^+$ is specified below. Clearly $V(0, Z) = 0$. Eq. (42) implies

$$V' = \varepsilon \sigma 2 (\beta K l \bar{n} - \kappa) + \sigma 2 \beta K l (\bar{n} - \bar{n}_0)$$  (45)

and, since $2|\varepsilon \sigma| \leq V/\sqrt{\beta}$, $|\sigma| \leq \sqrt{V/\beta}$, we obtain

$$V' \leq 2A\sqrt{V} + 2BV ~ \Rightarrow ~ \left(\sqrt{V}\right)' \leq A + B\sqrt{V},$$

where we have abbreviated $B(\xi, Z) := |\beta K l \bar{n} - \kappa|/2\sqrt{\beta}$ and introduced some $A(Z)$ such that $A \geq Kl\sqrt{\beta} \max\{|\bar{n} - \bar{n}_0|\}$. By the comparison principle $[36]$, $\sqrt{V} \leq R$, where $R(\xi, Z)$ is the solution of the Cauchy problem $R' = A + BR$, $R(0, Z) = 0$, what implies

$$|\varepsilon(\xi, Z)| \leq \sqrt{V(\xi, Z)} \leq R(\xi, Z) = A(Z) \int_0^\xi d\eta \exp \left[ \int_\eta^\xi d\zeta B(\zeta, Z) \right]$$  (46)

and $\sqrt{\delta}|\sigma| \leq R$. Since $R(\xi, Z)$ grows with $\xi$, if $R(l, Z) < 1$ no WBDLPI may involve the $Z$-electrons. Choosing $\beta = 1/M_u l^2$, we find for all $\xi \in [0, \xi_3]$

$$2lB \beta = \left| \frac{1+v}{s^3} - \frac{\bar{n}}{n_u} \right| \leq \left| \frac{1+v}{s^3} - 1 \right| + \left| 1 - \frac{\bar{n}}{n_u} \right| \leq |D| + \delta,$$

where $D(\xi, Z) := \frac{1+v(\xi)}{[s(\xi, Z)]^3} - 1$, $\delta(Z) := 1 - \frac{n_d(Z)}{n_u(Z)}$.  (47)

In the NR regime, which is characterized by $v \ll 1$, we have $s \simeq 1$, $\kappa \simeq 1/l$, $\Delta_u/l \ll 1$, $D \simeq 0$; setting $D = 0$ leads by a straightforward computation to

$$R(l, Z) \simeq R_{nr}(Z) := f[r(Z)], \quad f(r) := 2 \left( e^{r/2} - 1 \right), \quad r(Z) := \delta(Z) \sqrt{M_u(Z)} l.$$

$f(r)$ grows with $r \geq 0$ and reaches the value 1 for $r \simeq 0.81$; therefore the condition

$$r(Z) = \delta(Z) \sqrt{M_u(Z)} l < 0.81$$  (48)

is sufficient to ensure that the $Z$-electrons are not involved in WBDLPI. This is automatically satisfied if $\sqrt{M_u(Z)} l < 0.81$, because by definition $\delta \leq 1$, otherwise it is a very mild condition on the relative variation $\delta$ of the initial electron density across an interval of length $\Delta_u \ll l$; in fact, to violate (48) one needs a discontinuous, or a continuous but very steep, $\bar{n}_0(Z)$ with large relative variations around $Z$, see section 4.

Now consider the general case. In the interval $[0, \xi_3]$ the inequalities $s^{(1)} \geq s \geq s^{(2)} \geq 1$ imply

$$|D(\xi, Z)| \leq D(\xi, Z) := \max\left\{ \frac{1+v(\xi)}{[s^{(2)}(\xi, Z)]^3} - 1, 1 - \frac{1+v(\xi)}{[s^{(1)}(\xi, Z)]^3} \right\} \leq \tilde{v}(\xi) := \max\{v(\xi), 1\} \leq \max\{v_M, 1\} =: \tilde{v}_M,$$  (49)
where \( v_M, \tilde{v}_M \) are the maxima of \( v, \tilde{v} \). Hence we obtain the bounds

\[
\int_{\eta}^{\xi} d\zeta B(\zeta) \leq \frac{\sqrt{M_u}}{2} \left[ (\xi-\eta) \delta + \int_{\eta}^{\xi} d\zeta D(\zeta) \right]
\leq \frac{\sqrt{M_u}}{2} \left[ (\xi-\eta) \delta + \int_{\eta}^{\xi} d\zeta \tilde{v}(\zeta) \right]
\leq \frac{\sqrt{M_u}}{2} (\tilde{v}_M + \delta) (\xi-\eta),
\]

which replaced in (46) by choosing \( A = \sqrt{M_u} \delta \) respectively imply

\[
R(\xi,Z) \leq \delta(Z)\sqrt{M_u(Z)} \int_{0}^{\xi} d\eta \exp\left\{ \frac{\sqrt{M_u(Z)}}{2} \left[ (\xi-\eta) \delta(Z) + \int_{\eta}^{\xi} d\zeta D(\zeta,Z) \right] \right\} =: Q(\xi,Z) \quad (50)
\]

\[
Q(l,Z) \leq \delta(Z)\sqrt{M_u(Z)} \int_{0}^{l} d\eta \exp\left\{ \frac{\sqrt{M_u(Z)}}{2} \left[ (l-\eta) \delta(Z) + \int_{\eta}^{l} d\zeta \tilde{v}(\zeta) \right] \right\} =: Q_1(Z) \quad (51)
\]

\[
\leq \frac{2\delta(Z)}{\tilde{v}_M + \delta(Z)} \left\{ \exp\left[ \frac{\tilde{v}_M + \delta(Z)}{2} \sqrt{M_u(Z)} l \right] - 1 \right\} =: Q_0(Z) \quad (52)
\]

(here we have redisplayed the Z-argument). \( Q_2(Z) := Q(l,Z) \) is the most difficult to compute, \( Q_0(Z) \) is the easiest. We thus arrive at

**Theorem 1** Assume that condition (36) is fulfilled. Then no WBDLPI involves the Z-electrons if, in addition, \( Q_0(Z) < 1 \), or at least \( Q_1(Z) < 1 \), or at least \( Q_2(Z) < 1 \). If one of these conditions is fulfilled for all \( Z \), then WBDLPI occurs nowhere.

Consequently, for any fixed pump there is no WBDLPI if \( n_b \) is sufficiently small. A simple sufficient condition is given by

**Corollary 1** For any pulse (5) there is no WBDLPI if \( Kn_b l^2 < 4 \left[ \log 2/(1+\tilde{v}_M) \right]^2 \) and (36) are fulfilled. In particular, it suffices that \( Kn_b l^2 < \min \{ 4 \left[ \log 2/(1+\tilde{v}_M) \right]^2, 2/(1+2\Delta_u/l) \} \).

**Proof** From \( \delta \leq 1 \leq \tilde{v}_M \) it follows \( \delta l \sqrt{M_u} \leq L := \frac{\tilde{v}_M + \delta}{2} \sqrt{M_u} \). Hence \( L < \log 2 \) implies \( e^L < 2 \), whence

\[
Q_0 = \delta l \sqrt{M_u} \frac{e^{L}-1}{L} \leq e^L - 1 < 1,
\]

which together with (36) implies the first claim. The second follows as (40) implies (36). \( \square \)

### 4 Discussion and conclusions

As we have seen, if the inequality (35) is fulfilled, then \( \tilde{\xi}_2 > l \), i.e. the pulse is strictly short (namely, it completely overcomes the Z-electrons while their longitudinal displacement is
still nonnegative). If at least the inequality (36) is fulfilled, then \( \xi_3 > l \), so that the pulse is essentially short (namely, the pulse completely overcomes the \( Z \)-electrons before their longitudinal displacement reaches its first negative minimum), and the inequalities (27), (29), (50-52) apply. If in addition one of the conditions of Proposition 1 is satisfied, then we can indeed exclude WBDLP1.

As seen, the more easily computable - but also more difficult to satisfy - condition (40) implies (55) and also the inequality \( M_u(Z)^2 \leq \frac{2}{1+2\Delta_u/l} \), which after a substitution in (50-52) simplifies the computation of their rhs; in particular (52) becomes

\[
M \implies (35) \text{ and also the inequality } \quad Q_0(Z) \leq \frac{2\delta(Z)}{\bar{v}_m + \delta(Z)} (e^C - 1) =: \bar{Q}_0(Z), \quad C := \frac{\bar{v}_m + \delta(Z)}{\sqrt{2[1+2\Delta_u/l]}}.
\] (53)

Therefore (40) and \( \bar{Q}_0(Z) < 1 \) provide a sufficient condition to exclude WBDLP1, as well.

Note that, in the above conditions, several dimensionless numbers characterizing the input data, viz. \( \bar{v}_m, \Delta_u/l, C_0^2 = M_0^2, M_1^2, M_d^2, \delta \), and possibly also \( s_u, M_u^2 \), play a key role in the main inequalities of the present paper; therefore their computation represents the first step to check whether/where such conditions are fulfilled or violated.

In the NR regime (48) is equivalent to either inequality\(^{10}\)

\[
\frac{n_d}{p} > \frac{n_u}{p} - \sqrt{\frac{2n_u}{p}} \iff \frac{n_u}{p} < 1 + \frac{n_d}{p} + \sqrt{1 + \frac{2n_d}{p}}, \quad p := \frac{(0.81)^2}{2Kl^2}.
\] (54)

If \( \tilde{n}_0 \) grows in \([0, \tilde{Z} + \Delta_u]\), also \( q(z) := \tilde{n}_0(z)/p \) does, and for all \( Z \in [0, \tilde{Z}] \) the previous conditions become

\[
q(Z) > q(Z + \Delta_u) = \sqrt{2q(Z + \Delta_u)} \iff q(Z + \Delta_u) < 1 + q(Z) + \sqrt{1 + 2q(Z)}.
\] (55)

This is fulfilled e.g. if in \([0, \tilde{Z} + \Delta_u]\) \( \tilde{n}_0 \) is continuous (without excluding \( \tilde{n}_0(0^+) > 0 \)), at least piecewise \( C \), and \( 0 \leq \frac{d\tilde{n}(z)}{dz} \Delta_u < 1 + \sqrt{1 + 2q(Z)} \) for all \( Z \in [0, \tilde{Z}] \) and \( z \in [Z, \tilde{Z} + \Delta_u] \).

Now we impose that \( \tilde{n}_0 \) is continuous in \( -\infty, \tilde{Z} + \Delta_u \) and reaches a given value \( \check{n} > 0 \) at \( Z = \tilde{Z} \) while respecting (55). We compare the minimum \( Z \) for a linear and a quadratic \( \tilde{n}_0 \); note that \( dq/dZ \) for the former violates the above bound at \( Z = 0 \). We find\(^{11}\)

\[
\tilde{n}_0(Z) = n_1(Z) := \theta(Z)\frac{n Z}{Z} \quad \text{fulfills (55) if } \quad \frac{Z}{\Delta_u} > \frac{n}{2p} = \frac{K\check{n}l^2}{(0.81)^2} =: \frac{\check{Z}_1}{\Delta_u} \quad (56)
\]

\[
\tilde{n}_0(Z) = n_2(Z) := \theta(Z)\frac{n Z^2}{Z^2} \quad \text{fulfills (55) if } \quad \frac{Z}{\Delta_u} > \sqrt{\frac{n}{2p} + \sqrt{\frac{n}{2p} + \frac{1}{4}}} - =: \frac{\check{Z}_2}{\Delta_u}.
\] (57)

\(^{10}\)Ineq. (48) amounts to \( n_u - n_d < \sqrt{2p} n_u \), i.e. (54); taking the square one obtains the equivalent inequality \( n_u^2 - 2n_u(n_d + p) + n_d > 0 \), which is fulfilled if \( n_u < n_+ \), where \( n_+ := n_d + p + \sqrt{(n_d + p)^2 - n_d^2} = n_d + p + \sqrt{p^2 + 2pn_d} \). Solve the equation \( x^2 - 2x(n_u + p) + n_d = 0 \) in the unknown \( z \); the left inequality is automatically satisfied because \( n_u \geq n_d > n_d^2/n_+ = n_+ \). Dividing the inequality \( n_u < n_+ \) by \( p \) we obtain (54).

\(^{11}\)In fact, if \( \tilde{n}_0 = n_1 \) then (55) becomes \( \tilde{n}\Delta_u/p\tilde{Z} < 1 + \sqrt{1 + 2\tilde{n}\tilde{Z}/p\tilde{Z}} \) for all \( Z \); the rhs is lowest for \( Z = 0 \), whereby the inequality becomes (56), as claimed. If \( \tilde{n}_0 = n_2 \) then (55) becomes the condition

\[
F(Z) := \sqrt{\frac{2p}{\tilde{n}}(Z + \Delta_u) - \Delta_u^2} - 2\Delta_u Z > 0;
\]

this is of first degree in \( Z \), hence is fulfilled for all \( Z \in [0, \tilde{Z}] \) if it is for \( Z = 0, \tilde{Z} \); the quadratic polynomial \( F(\tilde{Z}) \) in \( \tilde{Z} \) is positive if \( \tilde{Z} > \tilde{Z}_2 \), as claimed, because \( \tilde{Z}_2 \) is the positive solution of the equation \( F(z) = 0 \) in the unknown \( z \), and \( \tilde{Z} > \tilde{Z}_2 \) automatically makes also \( F(0) > 0 \).
(here $\theta$ is the Heavisde step function). If $\sqrt{K\bar{n}}l$ is considerably larger than 1, then $\bar{Z}_1$ is considerably larger than $Z_2$; in particular, assuming $\bar{n}_0 = \bar{n}$, i.e. $G_b \sim \pi$, yields $Z_1 \sim 15.04\Delta_u$, $Z_2 \sim 6.78\Delta_u$. Therefore choosing $\bar{Z} \in [Z_2, Z_1]$ we can exclude WBDLPI adopting $\bar{n}_0(Z) = n_2(Z)$, but not $\bar{n}_0(Z) = n_1(Z)$. Such a result is relevant for LWFA experiments, which usually fulfill \cite{2}. From the physical viewpoint, it allows one to exclude WBDLPI because: i) the density $\bar{n}_0(Z)$ obtained just outside the nozzle of a supersonic gas jet (orthogonal to the $\vec{z}$) typically is $C^1(\mathbb{R})$ with $\bar{n}_0(0) = 0 = d\bar{n}_0(0)$ (see e.g. fig. 2 in \cite{2}, or fig. 5 in \cite{37}), and therefore is closer to type $n_2(Z)$ than to type $n_1(Z)$; ii) by causality, the effects of a pulse with a finite spot radius near its symmetry axis $\vec{z}$ are the same as with a plane wave ($R = \infty$), at least for small $\xi$. From the viewpoint of mathematical modeling, it suggests that it makes a big difference to describe the edge of the plasma by $n_1$ or by $n_2$: in the first case we can correctly predict the plasma evolution only by kinetic theory and PIC codes, while in the second we can do also by a hydrodynamic description and (less computationally demanding) multifluid codes.

If we allow a discontinuous (in $Z = 0$) linear Ansatz $\bar{n}_0(Z) = \theta(Z)\bar{n}[a + (1-a)Z/\bar{Z}]$ ($0 < a \leq 1$), then (55) is fulfilled if $\bar{Z} > \bar{n}(1-a)\Delta_u/p[1+\sqrt{1+2a\bar{n}/p}]$, which is again smaller than $Z_1$.

Although our results apply to all $\epsilon^\perp$ with support contained in $[0, l]$, regardless of their Fourier analysis, in most applications one deals with a modulated monochromatic wave\cite{17}

$$
\epsilon^\perp(\xi) = \epsilon(\xi) \{ i \cos \psi \sin(k\xi + \varphi_1) + j \sin \psi \sin(k\xi + \varphi_2) \},
$$

(58)

modulation carrier wave $\epsilon^\parallel(\xi)$

where $i = \nabla x$, $j = \nabla y$. If, as it follows from (1), $Kn_bA^2 \ll 1$ ($\lambda = 2\pi/k$ is the wavelength), i.e. the plasma is underdense, then the relative variations of $\Delta(\xi, Z)$ (and $z_\epsilon(\xi, Z)$) in a $\xi$-interval of length $\leq \lambda$ are much smaller than those of $x^\perp(\xi)$, and those $s(\xi)$ even smaller; in fact, as $s > 0$, $v \geq 0$, the integral in (13a) averages the fast variations of $v$ to yield much smaller relative variations of $\Delta$, and the first integral in (13b) averages the residual small variations of $\bar{N}[z_\epsilon(\xi)]$ to yield an essentially smooth $s(\xi)$, see e.g. fig. 1. On the contrary, $\alpha^\perp(\xi)$, $x^\perp(\xi)$ vary fast as $\epsilon^\perp(\xi)$. Under rather general assumptions (see the appendix)\cite{20}

$$
\alpha^\perp(\xi) = -\frac{\epsilon(\xi)}{k} \epsilon^\parallel(\xi) + O\left(\frac{1}{k^2}\right) \simeq -\frac{\epsilon(\xi)}{k} \epsilon^\parallel(\xi),
$$

(59)

where $\epsilon^\parallel(\xi) := \epsilon(\xi + \lambda/4)$, and similarly for other integrals with modulated integrands; in the appendix we recall upper bounds for the remainders $O(1/k^2)$. If $|\epsilon^\perp| \ll |k\epsilon|$ (slow modulations) the right estimate is very good, and $v$ can be approximated very well by $v \simeq \epsilon^\parallel(\epsilon^\perp/e/kmc^2)^2$. For the reasons mentioned above, replacing $v$ by its (approximated) average over a cycle,

$$
v_a(\xi) := \frac{1}{2} \left( \frac{\epsilon(\xi)}{kmc^2} \right)^2,
$$

(60)

has only a small effect on $\Delta$ and almost no effect on $s$, $V$, and similarly on the functions $\Delta^{(0)}$, $s^{(1)}$, ... introduced in sections $2, 3$ to bound $\Delta, s, V$; but it simplifies their computation a

\footnote{The elliptic polarization in (58) is ruled by $\psi, \varphi_1, \varphi_2$; it reduces to a linear one in the direction of $\vec{a} := i \cos \varphi + j \sin \varphi$ if $\varphi_1 = \varphi_2$, to a circular one if $|\cos \psi| = |\sin \psi| = 1/\sqrt{2}$ and $\varphi_1 = \varphi_2 \pm \pi/2$.}
Figure 2: Plots of the ratios $\tilde{n}_0/n_b$ for the following initial densities:

0) $\tilde{n}_0(z) = n_b \theta(z)$.
1) $\tilde{n}_0(z) = \frac{1}{\sqrt{\pi}} n_b \theta(z) [1 + \theta(l' - z) z/l' + \theta(z - l')]$.
2) $\tilde{n}_0(z) = n_b [\theta(z) \theta(l' - z) z/l' + \theta(z - l')]$.
3) $\tilde{n}_0(z) = n_b \left\{ \frac{\bar{z}}{2} \frac{f(z)}{f(\bar{z})} \theta(z - \bar{z}) + \theta(z - \bar{z}) \frac{f(z)}{f(\bar{z})} \right\}$, where $f(z) := (0.1 z/l')^2 + (0.2 z/l')^4$ and $\bar{z} = 6.5l'$; this grows as $z$ for $z \leq \bar{z}$ and coincides with the next one for $z > \bar{z}$.
4) $\tilde{n}_0(z) = n_b \theta(z) \frac{f(z)}{f(\bar{z})}$.

lot. As a consequence, the bounds (27), (29) and (37), as well as the short pulse conditions (35), (36) and the no-WB conditions of proposition 1 remain essentially valid also if in computing the bounds we replace $v$ by $v_a$.

We illustrate the results obtained so far considering a pulse (58) with a linear polarization (e.g. $\psi = 0$) and a modulation of Gaussian type, except that it is cut-off outside the support $0 \leq \xi \leq l$:

$$\frac{e}{kmc^2} \epsilon(\xi) = a_0 \exp \left[ \frac{(\xi - l/2)^2}{l'^2} - 2 \log 2 \right] \theta(\xi) \theta(l - \xi); \quad (61)$$

$l'$ is the full width at half maximum of the intensity $I$ of the electromagnetic (EM) field. More precisely, we adopt the pulse plotted in fig. 1a, which has the maximum at $\xi = l/2$ and $a_0 = 1.3$; this yields a moderately relativistic electron dynamics and $\Delta_u \equiv \Delta^{(0)}(l) \simeq 0.45l'$. In fig. 1b we have plot the associated $v$ and $v_a$. We perform all computations and plots running our specifically designed programs using an “off the shelf,” general-purpose numerical package on a common notebook for a time lapse between several seconds and several minutes. Let us compare the impact of such a pulse on the density profiles plotted in fig. 2.

The upper bound $n_b$ plays also the role of asymptotic value. Here we choose $n_b = 2 \times 10^{18}$ cm$^{-3}$, which is the same as the $n_0$ of fig. 1 what yields $Kn_b l'^2 \simeq 4$; but the results for the dimensionless variables remain the same if we change $n_b, \lambda, l'$ keeping $\lambda/l'$ and $Kn_b l'^2$ constant. As already said, in the case of the step-shaped density 0), if $\xi_2 > l$, i.e. if the pulse is strictly short [a sufficient condition for that is (35)], then $n_u = n_d = n_b, \delta = 0$, and by (50,52) there is no WBDLPI; more directly, this is a consequence of $J(\xi, Z) \equiv 1$ for $0 \leq \xi \leq l$, which follows from the $Z$-independence of $\Delta$ in such an interval. As for the other profiles, we have respectively plotted:

- In fig. 3 and fig. 5 the solutions $J, \sigma$ of eq. (42) and, by (43), the associated function $l' \partial u^z/\partial Z$ for $0 \leq \xi \leq 6l'$ and a few values of $Z$, assuming the initial electron density profiles 1), 2) and 3), 4) respectively.
• In fig. 4 and fig. 6 the solutions $s, \Delta$ of (8-9), their upper and lower bounds $s^{(1)}, s^{(2)}, \Delta^{(1)}, \Delta^{(2)}$ [eq. (28)] and the function $Q$ of eq. (50), for $0 \leq \xi \leq 6l'$ and a few values of $Z$, assuming the initial electron density profiles 1) - 2) and 3) - 4), respectively.

• In fig. 7 and fig. 8 the corresponding worldlines of the $Z$-electrons, for $0 \leq ct \leq 20l'$ and $Z = n l'/20$, $n = 0, 1, ..., 200$, associated to the initial electron density profiles 3), 4). The support of the EM pulse is coloured pink (the red part is the more intense part); the laser-plasma interaction takes place in the spacetime region which has nonempty intersection with worldlines of electrons or protons.

We can compare the results for the densities 1) - 2) and those for the densities 3) - 4) side-by-side. In case 1) WBDLPI is avoided assuming that $\tilde{n}_0(0) > 0$, but worldlines intersect and WB takes place not far from the laser-plasma interaction region; in case 2) WBDLPI takes place for $Z \simeq 0$, due to the steep growth of $\tilde{n}_0(Z) = Z/l' n_b$ from the value $\tilde{n}_0(0) = 0$. In case 3), albeit the growth $\tilde{n}_0(Z) \propto Z$ is much less steep, again worldlines intersect and WB takes place not very far from the laser-plasma interaction region, whereas in case 4) this occurs quite far from the latter, consistently with the results $|J - 1| \ll 1$, $Q_2 < 1$. Thus we note that, although such dynamics are moderately relativistic, rather than nonrelativistic, switching from profile 1) to profile 2), or from profile 3) to profile 4) has the same qualitative effect of avoiding (or distancing from) WBDLPI. We also note that in case 1), 3) the $Z \simeq 0$ worldlines first intersect with very small angles, or equivalently, that when the corresponding electrons collide their longitudinal momenta differ by a very small amount: by formula (43) $l \partial u^z / \partial Z$ is very small not only because $\sigma$ is, but also because $\nu \simeq 0$ and $s > 1$. Hence we can expect that these collisions will lead only to a very small momentum spreading.

Essential the same results are reached choosing a different pulse polarization, because $v_a$ will be of the same type. In the case of circular polarization ($\varphi_1 - \varphi_2 = \pm \pi/2, \cos \psi = \pm \sin \psi$) and Gaussian modulation $\nu$ itself will essentially coincide with $v_a$, thus displaying a single maximum (cf. fig. 1b).

As mentioned in the introduction, the equations of motion for the $Z$-electrons can be reduced to the form (8), and thus are decoupled from those of all $Z'$-electrons, $Z' \neq Z$, only in the idealization where the laser pulse is "undepleted", i.e. not affected by its interaction with the plasma. The latter is expected to be an acceptable approximation only for small $t, z > 0$. Actually, for a slowly modulated monochromatic wave one can show by self-consistency [27] that this is a good approximation in the spacetime region that is the intersection of the ‘laser-plasma interaction’ stripe $0 \leq ct - z \leq l$ with the orthogonal stripe

$$0 \leq \frac{e^2 n_b \lambda}{2 mc^2} (ct + z) \ll 1.$$  

In view of the inequalities $\lambda \ll l$ and [11] or [16] we see that, to our satisfaction, this region is much longer than $l$ in the $ct + z$ direction. Thus, if $n_b = 2 \times 10^{18} \text{cm}^{-3}$, and the pulse is as in fig. 1 we can consider the latter as undepleted, and the electrons’ motion determined above as accurate, for time intervals $[0, t_d]$, where $t_d$ is at least a few $10^{-13}$s.

The above predictions are based on idealizing the initial laser pulse as a plane EM wave (5). In a more realistic picture the $t = 0$ laser pulse is cylindrically symmetric around
Figure 3: The initial electron densities 1), 2) of fig. 2 (first line; respectively left, right) and, below, the corresponding plots of $J, \sigma, \partial u^z / \partial z$ vs. $\xi$ during the interaction with the pulse of fig. 1 for a few sample values of $Z$. As we see, the right $J$ keeps positive for $\xi < l$ and all $Z$, while the left $J$ becomes negative for very small $Z$ and $\xi \lesssim l$; correspondingly, the right worldlines do not intersect, while the right ones do (see the down $z_e$-graphs).
Figure 4: The initial electron densities 1), 2) of fig. 2 (first line; respectively left, right); below, assuming \( n_b = 2 \times 10^{18} \text{cm}^{-3} \), we plot the corresponding \( s, \Delta \), their upper and lower bounds \( s^{(1)}, s^{(2)}, \Delta^{(1)}, \Delta^{(2)} \) and the function \( Q \), vs. \( \xi \) during the interaction with the pulse of fig. 1, for the same sample values \( Z = l'/20 \) and \( Z = l'/2 \) of \( Z \). The values \( Q_2(Z) := Q(l,Z) \) can be read off the plots. As we can see, the bounds are much better for the density 1); the values \( Q_2(Z) \lesssim 1 \) are consistent with all worldlines intersecting rather far from the laser-plasma interaction spacetime region. Whereas the large value of \( Q_2(Z) \) for the density 2) is an indication that some worldlines intersect within, or not far from, the laser-plasma interaction spacetime region. Our computations lead also to \( Q_0(l'/20) = 10.64 \), \( Q_0(l'/2) = 88.53 \), \( Q_0(l'/20) = 32.35 \) with the density 2).
Figure 5: The initial electron densities 3), 4) of fig. 2 (respectively left, right) with \( n_b = 2 \times 10^{18}\text{cm}^{-3} \), and below the corresponding plots of \( J, \sigma, l' \partial u^z / \partial z \) during the interaction with the pulse of fig. 1 for a few sample values of \( Z \). As we can see, \( J \) keeps positive at least for all \( \xi \in [0, 2l] \) if the density is of type 4) (which grows as \( Z^2 \) for \( Z \sim 0 \)), whereas it becomes negative for \( \xi \sim 6.5l' \) and small \( Z \) if the density is of type 3), (which grows as \( Z \) for \( Z \sim 0 \)); correspondingly, the right worldlines do not intersect, while the left ones do (see the down \( z_e \)-graphs).
Figure 6: The initial electron densities 3), 4) of fig. 2 (respectively left, right) with \( n_b = 2 \times 10^{18} \text{cm}^{-3} \), and corresponding plots of \( s, \Delta \), their upper and lower bounds \( s^{(1)}, s^{(2)}, \Delta^{(1)}, \Delta^{(2)} \), and the function \( Q(\xi, Z) \), vs. \( \xi \) for the same sample values \( Z = l'/20 \) and \( Z = l'/2 \) of \( Z \). The values \( Q_2(Z) := Q(l',Z) \) can be read off the plots. As we can see, the bounds are much better for the density 4); the values \( Q_2(Z) \leq 1 \) are consistent with all worldlines intersecting rather far from the laser-plasma interaction spacetime region. Whereas the large value of \( Q_2(Z) \) for the density 3) is an indication that some worldlines intersect not far from the laser-plasma interaction spacetime region. Our computations lead also to \( Q_0(l'/20) = 4.62, Q_0(l'/2) = 3.08 \) with the density 3), \( Q_0(l'/20) = 0.49, Q_0(l'/2) = 0.85 \) with the density 4).
the \( \vec{z} \)-axis and has a *finite* spot radius \( R \), namely the \( t = 0 \) EM fields are of the form 
\[
E = \varepsilon^i(-z) \chi(\rho), \quad B = k \times E,
\]
where \( \rho^2 = x^2 + y^2 \) and \( \chi(\rho) \geq 0 \) is 1 for \( \rho \leq 1 \) and rapidly goes to zero for \( \rho > R \). By causality, the motion of the electrons remains \( \rho > R \) strictly the same in the future Cauchy development\(^\text{13}\) \( D^+(D) \) of \( D = D_1 \cup D_2 \), where \( D_1 := \{(ct, x) = (0, 0, 0, z > 0)\} \) and \( D_2 := \{(0, x) | \rho \leq R\} \), and almost the same in a neighbourhood of \( D^+(D) \); therefore the conditions described above remain sufficient to exclude WBDLP\(I\) at least in such a region.

Finally, the conditions of Proposition \( \text{I} \) are very general, in that they apply also to discontinuous \( \tilde{n}_0 \), or non-monotone \( \tilde{n}_0 \), but if \( \tilde{n}_0 \) has a bounded derivative it turns out that they are unnecessarily too strong for ensuring that no WBDLP\(I\) occurs. Weaker no-WBDLP\(I\) conditions under the latter assumptions will be treated in \( \text{[27]} \).

5 Appendix

5.1 Proof of Proposition \( \text{I} \)

We abbreviate \( \Omega := \sqrt{K} \tilde{n}_0 \), \( \zeta := l - \eta \). Eq. \( (17) \) yields
\[
2\Delta(l) = \int_0^{l/2} d\eta v(\eta) \cos [\Omega(l - \eta)] + \int_0^l d\eta v(l - \eta) \cos [\Omega(l - \eta)]
= \int_0^{l/2} d\eta v(\eta) \cos [\Omega(l - \eta)] - \int_{l/2}^0 d\zeta v(\zeta) \cos (\Omega\zeta)
= \int_0^{l/2} d\eta v(\eta) \cos [\Omega(l - \eta)] + \int_0^{l/2} d\eta v(\eta) \cos (\Omega\eta)
= [1 + \cos (\Omega\eta)] \int_0^{l/2} d\eta v(\eta) \cos (\Omega\eta) + \sin (\Omega\eta) \int_0^{l/2} d\eta v(\eta) \sin (\Omega\eta)
\tag{63}
\]
\[
2\delta(l) = \int_0^{l/2} d\eta v(\eta) \sin [\Omega(l - \eta)] + \int_0^l d\eta v(l - \eta) \sin [\Omega(l - \eta)]
= \int_0^{l/2} d\eta v(\eta) \sin [\Omega(l - \eta)] - \int_{l/2}^0 d\zeta v(\zeta) \sin (\Omega\zeta)
= [1 - \cos (\Omega\eta)] \int_0^{l/2} d\eta v(\eta) \sin (\Omega\eta) + \sin (\Omega\eta) \int_0^{l/2} d\eta v(\eta) \cos (\Omega\eta)
\tag{64}
\]

If \( \Omega l \leq \pi \) both integrands in \( \text{(63)} \) are nonnegative, and so are the factors of both integrals; moreover the latter and \( \Delta(l) \) itself are positive if \( \Omega l < \pi \), zero if \( \Omega l = \pi \). In either case \( \Delta(\xi) > 0 \) for all \( \xi \in]0, l[, \) because \( \Delta'(l) = v(l) - \delta(l) < 0 \) (since \( v(l) \simeq 0, \delta(l) > 0 \)). Moreover, if \( \varepsilon := \Omega l - \pi > 0 \) is sufficiently small, then both integrals and \( 1 + \cos (\Omega l) = 1 - \cos \varepsilon \simeq \varepsilon^2 / 2 \) are still positive, whereas sin (\( \Omega l \)) = -\sin \varepsilon \simeq -\varepsilon; the second negative term will dominate and make \( \text{(63)} \) negative as well. Therefore \( \text{(17a)} \) will be satisfied iff \( \Omega l \leq \pi \) i.e. if \( G_b \leq 1 / 2 \).

Similarly, if \( \Omega l = 2\pi \) the factors of both integrals in \( \text{(64)} \) vanish, and \( \delta l = 0 \). If \( \varepsilon := \Omega l - 2\pi \neq 0 \) is sufficiently small, then \( 1 - \cos (\Omega l) = 1 - \cos \varepsilon \simeq \varepsilon^2 / 2 \), whereas sin (\( \Omega l \)) = sin \varepsilon \simeq \varepsilon.\(^\text{13}\)

\(^\text{13}\)We recall that the *future Cauchy development* \( D^+(D) \) of a region \( D \) in Minkowski spacetime \( M^4 \) is defined as the set of all points \( x \in M^4 \) for which every past-directed causal (i.e. non-spacelike) line through \( x \) intersects \( D \).
and the second term dominates over the first. Under our assumptions the second integral will be negative, because \( v(\eta) \) is larger where \( \cos(\Omega \eta) < 0 \). Hence (64) will be negative if \( \varepsilon > 0 \) (and sufficiently small); if \( \varepsilon < 0 \) (and sufficiently small), then (64) will be positive, and \( \delta(\xi) > 0 \) also for \( 0 < \xi < l \), because \( \delta'(l) = M \Delta(l) < 0 \) (since in this case \( \Delta(l) < 0 \). Therefore (17b) will be satisfied iff \( \Omega l \leq 2\pi \) i.e. if \( G_0 \leq 1 \).

### 5.2 Estimates of oscillatory integrals

Here we recall some useful estimates [29] of oscillatory integrals, such as (6a) in the case (58). Given a function \( f \in C^2(\mathbb{R}) \), integrating by parts we find for all \( n \in \mathbb{N} \)

\[
\int_{-\infty}^{\xi} d\zeta f(\zeta)e^{ik\zeta} = -\frac{i}{k}f(\xi)e^{ik\xi} + R_1^f(\xi),
\]

(65)

\[
R_1^f(\xi) := \frac{i}{k} \int_{-\infty}^{\xi} d\zeta f'(\zeta) e^{ik\zeta} = \left( \frac{i}{k} \right)^2 \left[ -f'(\xi) e^{ik\xi} + \int_{-\infty}^{\xi} d\zeta f''(\zeta) e^{ik\zeta} \right].
\]

(66)

Hence we find the following upper bounds for the remainder \( R_1^f \):

\[
\left| R_1^f(\xi) \right| \leq \frac{1}{|k|^2} \left[ |f'(\xi)| + \int_{-\infty}^{\xi} |f''(\zeta)| \right] \leq \frac{\|f'\|_{\infty} + \|f''\|_1}{|k|^2},
\]

(67)

It follows \( R_1^f = O(1/k^2) \). All inequalities in (67) are useful: the left inequalities are more stringent, while the right ones are \( \xi \)-independent.

Equations (65), (67) and \( R_1^f = O(1/k^2) \) hold also if \( f \in W^{2,1}(\mathbb{R}) \) (a Sobolev space), in particular if \( f \in C^2(\mathbb{R}) \) and \( f, f', f'' \in L^1(\mathbb{R}) \), because the previous steps can be done also under such assumptions. Equations (65) will hold with a remainder \( R_1^f = O(1/k^2) \) also under weaker assumptions, e.g. if \( f' \) is bounded and piecewise continuous and \( f, f', f'' \in L^1(\mathbb{R}) \), but \( R_1^f \) will be a sum of contributions like (66) for every interval in which \( f' \) is continuous.

Letting \( \xi \to \infty \) in (65), (67) we find for the Fourier transform \( \hat{f}(k) = \int_{-\infty}^{\xi} d\zeta f(\zeta)e^{-iky} \) of \( f(\xi) \)

\[
|\hat{f}(k)| \leq \frac{\|f''\|_{\infty} + \|f''\|_1}{|k|^2},
\]

(68)

hence \( \hat{f}(k) = O(1/k^2) \) as well. Actually, if \( f \in S(\mathbb{R}) \) then \( \hat{f}(k) \) decays much faster as \( |k| \to \infty \), since \( \hat{f} \in S(\mathbb{R}) \) as well. For instance, if \( f(\xi) = \exp[-\xi^2/2\sigma] \) then \( \hat{f}(k) = \sqrt{\pi\sigma} \exp[-k^2\sigma/2] \).

To prove approximation [59] now we just need to choose \( f = \epsilon \) and note that every component of \( \alpha^\perp \) will be a combination of (65) and (65) \( k \to -k \).

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Figure 7: Down: the initial electron density 3) of fig. 2. Up: The worldlines of $Z$-electrons interacting with the pulse of fig. 1 for 200 equidistant values of $Z$; the support $0 \leq ct-z \leq l$ and the 'effective support' $(l-l')/2 \leq ct-z \leq (l+l')/2$ of the pulse are pink, red; the spacetime region of the pure-ion layer is yellow. Horizontal arrows pinpoint where particular subsets of worldlines first intersect.
Figure 8: Down: the initial electron density 4) of fig. 2. Up: The worldlines of $Z$-electrons interacting with the pulse of fig. 1 for 200 equidistant values of $Z$; the support $0 \leq ct-z \leq l$ and the 'effective support' $(l-l')/2 \leq ct-z \leq (l+l')/2$ of the pulse are pink, red; the spacetime region of the pure-ion layer is yellow. Horizontal arrows pinpoint where particular subsets of worldlines first intersect; as we can see, small $Z$ worldlines first intersect quite farther from the laser-plasma interaction spacetime region (in pink) than in the linear homogenous case 3).