On the Brauer Groups of Symmetries of Abelian Dijkgraaf–Witten Theories

Jürgen Fuchs$^1$, Jan Priel$^2$, Christoph Schweigert$^2$, Alessandro Valentino$^2$

$^1$ Teoretisk fysik, Karlstads Universitet Universitetsgatan 21, 651 88 Karlstad, Sweden
$^2$ Fachbereich Mathematik, Universität Hamburg, Bereich Algebra und Zahlentheorie Bundesstraße 55, 20 146 Hamburg, Germany. E-mail: christoph.schweigert@uni-hamburg.de

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Abstract: Symmetries of three-dimensional topological field theories are naturally defined in terms of invertible topological surface defects. Symmetry groups are thus Brauer–Picard groups. We present a gauge theoretic realization of all symmetries of abelian Dijkgraaf–Witten theories. The symmetry group for a Dijkgraaf–Witten theory with gauge group a finite abelian group $A$, and with vanishing 3-cocycle, is generated by group automorphisms of $A$, by automorphisms of the trivial Chern–Simons 2-gerbe on the stack of $A$-bundles, and by partial e-m dualities. We show that transmission functors naturally extracted from extended topological field theories with surface defects give a physical realization of the bijection between invertible bimodule categories of a fusion category $\mathcal{A}$ and braided auto-equivalences of its Drinfeld center $\mathcal{Z}(\mathcal{A})$. The latter provides the labels for bulk Wilson lines; it follows that a symmetry is completely characterized by its action on bulk Wilson lines.

1. Symmetries of Abelian Dijkgraaf–Witten Theories

Dijkgraaf–Witten theories are extended topological field theories that have a mathematically precise gauge theoretic formulation with finite gauge group. In that setting, the fields of the Dijkgraaf–Witten theory with gauge group $G$ are obtained by first considering $G$-bundles, to which, in a second step, a linearization procedure is applied (see [Mo] for a recent description). In the present note we investigate the notion of symmetries of three-dimensional Dijkgraaf–Witten theories, regarded as extended 1-2-3-dimensional topological field theories. To keep the presentation simple, we restrict ourselves to the case in which the gauge group is an abelian group, which we denote by $A$.

Braided auto-equivalences of bulk Wilson lines. The task of understanding symmetries in Dijkgraaf–Witten theories can be approached from two different angles, either algebraically or gauge theoretically. From a purely algebraic point of view, one would consider the modular category of bulk Wilson lines, which is the representation category...
\(\mathcal{D}(A)\)-mod of the Drinfeld double of \(A\). Symmetries should then in particular induce braided auto-equivalences of \(\mathcal{D}(A)\)-mod.

The group of braided auto-equivalences (up to monoidal natural equivalence) can be described as follows. Denote by \(A^*\) the group of complex characters of \(A\). The group \(A \oplus A^*\) comes with a natural quadratic form \(q: A \oplus A^* \to \mathbb{C}^\times\), given by \(q(g+\chi) = \chi(g)\) for \(g+\chi \in A \oplus A^*\). The automorphism group of \(A \oplus A^*\) then has a subgroup, denoted by \(O_q(A \oplus A^*)\), consisting of those group automorphisms \(\varphi\) that preserve this form, i.e., satisfy \(q(\varphi(z)) = q(z)\) for all \(z \in A \oplus A^*\). Now the group of braided auto-equivalences is isomorphic to this group \(O_q(A \oplus A^*)\) [ENOM]. Simple objects of \(\mathcal{D}(A)\)-mod, and thus simple labels for bulk Wilson lines of the Dijkgraaf–Witten theory with gauge group \(A\), are in bijection with elements of \(A \oplus A^*\); a braided auto-equivalence induces the natural action of the corresponding element of \(O_q(A \oplus A^*)\) on the group \(A \oplus A^*\).

In this approach the auto-equivalences of \(\mathcal{D}(A)\)-mod are not intrinsically realized in the Dijkgraaf–Witten theory as a gauge theory. It is therefore not clear whether every braided auto-equivalence of the category of bulk Wilson lines preserves all aspects of the three-dimensional topological field theory so that it can indeed be regarded as a full-fledged symmetry of the theory. It is not clear either whether a braided auto-equivalence would then describe a symmetry uniquely. There might be several different realizations, or also none at all, of the auto-equivalences on other field theoretic quantities, such as boundary conditions.

**Universal kinematical symmetries.** It is thus important to find a field theoretic realization of the auto-equivalences, relating to the fact that Dijkgraaf–Witten theories can be formulated as gauge theories. At the same time we then get additional insight into the structure of the group \(O_q(A \oplus A^*)\). From a gauge theoretic point of view it is natural to expect that the symmetries of the stack \(\text{Bun}(G)\) of \(G\)-bundles are symmetries of both classical and quantum Dijkgraaf–Witten theories.\(^1\) One might call these symmetries universal kinematical symmetries—kinematical, because they are symmetries of the kinematical setting, i.e., \(G\)-bundles; and universal, because the manifold on which the \(G\)-bundles are defined does not enter.

The symmetries of \(\text{Bun}(G)\) form the 2-group \(\text{AUT}(G)\), i.e. the category whose objects are group automorphisms \(\varphi: G \to G\) and whose morphisms \(\varphi_1 \to \varphi_2\) are given by group elements \(h \in G\) that satisfy \(\varphi_2(g) = h \varphi_1(g) h^{-1}\) for all \(g \in G\). Since in the case of our interest the group \(A\) is abelian, we can safely ignore the morphisms in the category \(\text{AUT}(A)\) and work with the ordinary automorphism group \(\text{Aut}(A)\) of the group \(A\).

The group \(\text{Aut}(A)\) of symmetries of \(\text{Bun}(A)\) can be identified in a natural way with a subgroup of the group \(O_q(A \oplus A^*)\) of braided auto-equivalences. Indeed, for any \(\alpha \in \text{Aut}(A)\), the automorphism \(\alpha \oplus (\alpha^{-1})^*\) of \(A \oplus A^*\) belongs to \(O_q(A \oplus A^*)\), where \((\alpha)^* \in \text{Aut}(A^*)\) is defined by \([\alpha^* \chi](a) := \chi(\alpha(a))\) for all \(\chi \in A^*\) and all \(a \in A\). But this argument is purely group theoretical, and it is not clear at this point whether the embedding has any physical relevance and relates symmetries of bundles to braided auto-equivalences of bulk Wilson lines.

**Universal dynamical symmetries.** The realization of Dijkgraaf–Witten theories as gauge theories leads to even more symmetries. Apart from a finite group \(G\), a three-cocycle \(\omega \in Z^3(G, U(1))\) is another ingredient of a Dijkgraaf–Witten theory. Geometrically this

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\(^1\) Actually, a general Dijkgraaf–Witten theory involves a 3-cocycle \(\omega \in Z^3(G, \mathbb{C}^\times)\). Here we only consider the case of trivial \(\omega\), and hence do not expect any compatibility relations between the automorphism and \(\omega\).