RENORMALIZED ENERGY FOR DISLOCATIONS IN QUASI-CRYSTALS

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Abstract. Anti-plane shear deformations of a hexagonal quasi-crystal with multiple screw dislocations are considered. Using a variational formulation, the elastic equilibrium is characterized via limit of minimizers of a core-regularized energy functional. A sharp estimate of the asymptotic energy when the core radius tends to zero is obtained using higher-order Γ-convergence. Also, the interaction between dislocations and the Peach-Köhler force at each dislocation are analyzed.

Keywords: dislocation; renormalized energy; Γ-convergence.

1. Introduction

1.1. Problem Settings. Quasi-crystals were introduced in 1982 by Shechtman (see [17]) as a kind of non-crystalline condensed matter state. In contrast with crystals with periodic atomic arrangement, quasi-crystals only exhibit quasi-periodicity, i.e. they have perfect long-range order (like mirror symmetry) but no three-dimensional periodicity.

Unlike many other amorphous solids, quasi-crystals have similar elastic properties to these of crystals. More importantly, based on the Landau density wave theory (see [9]), quasi-crystals can be described as a projection of higher-dimensional crystals into a lower-dimensional space. This requires two displacement fields $\vec{u}$ and $\vec{w}$ defined in the physical domain of the quasi-crystal, where $\vec{u}$ is a phonon field which is similar to the displacement field in crystals and $\vec{w}$ is an extra phase field. Also, we may define the strain and stress tensors in phonon space and phase space.

To be precise, we consider anti-plane shear deformations of a one-dimensional hexagonal quasi-crystal (see [5], [6], [7], [8], [9], [14]). Given an elastic body $\Xi = \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is simply-connected, bounded and open, with Lipschitz $\partial \Omega$, we denote the phonon deformation as

$$\Phi : (x, y, z) \rightarrow (x, y, z + u(x, y)),$$

and the phase deformation as

$$\Psi : (x, y, z) \rightarrow (x, y, z + w(x, y)),$$

for some functions $u, w : \Omega \rightarrow \mathbb{R}$. This allows us to reduce the three-dimensional problem to a two-dimensional setting. Hence, the phonon strain tensor is defined as

$$\dot{U} := 0 = \nabla (0, 0, u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \end{pmatrix},$$

which can be symmetrized as

$$\hat{U} := \frac{U + U^T}{2} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \frac{\partial u}{\partial x} \\ 0 & 0 & \frac{1}{2} \frac{\partial u}{\partial y} \\ \frac{1}{2} \frac{\partial u}{\partial x} & \frac{1}{2} \frac{\partial u}{\partial y} & 0 \end{pmatrix}.$$
and the non-symmetric phase strain tensor is defined as

\[
W := \nabla(0, 0, w) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\partial w/\partial x & \partial w/\partial y & 0 
\end{pmatrix}.
\] (1.2)

The relations (1.1) and (1.2) hold for a quasi-crystal when dislocations are absent. If dislocations are taken into consideration, then the strain tensor is singular at the site of the dislocations, and in particular it is a line singularity for a screw dislocation. Dislocations are one-dimensional defects in a crystalline-type material, whose presence may greatly affect the elastic and other properties (see [11] and [15]). Dislocation lines of quasi-crystals were observed in experiments soon after Shechtman's discovery (see [1], [12], [13], [14]).

In a quasi-crystal undergoing a shear deformation, a screw dislocation may be described by a position \((x, y) \in \Omega\) and a Burgers vector \(\vec{b} = be_z\). Here \(\vec{e}_z\) denotes the unit vector in the \(z\) direction and \(b\), the Burgers modulus, represents the magnitude of the dislocation. The presence of dislocation yields a singularity at position \((x, y)\) and thus strain tensors fail to be the gradients of smooth displacement fields, i.e. (1.1) and (1.2) do not hold any more.

To be precise, consider \(N\) dislocations at \(d_i = (x_i, y_i)\) for \(i = 1, 2, \ldots, N\), with Burger's vector for the phonon field given by \(\vec{b}_u^i = b_u^i \vec{e}_z\) and for the phase field given by \(\vec{b}_w^i = b_w^i \vec{e}_z\). The strain tensors \(U\) and \(W\) now satisfy

\[
(\nabla \times U) \cdot \vec{e}_z = \sum_{i=1}^{N} \vec{b}_u^i \delta d_i, \quad (\nabla \times W) \cdot \vec{e}_z = \sum_{i=1}^{N} \vec{b}_w^i \delta d_i,
\]

which is equivalent to

\[
b_u^i = \int_{\ell_i} U \cdot t ds, \quad b_w^i = \int_{\ell_i} W \cdot t ds,
\]

where \(\ell_i\) is any counterclockwise loop that surrounds \(d_i\) and no other dislocation points, \(t\) is the tangent of \(\ell_i\) and \(ds\) is the line differential. Similarly, we can still define the symmetrized phonon strain tensor \(\tilde{U} = U + U^T\).

Denote the phonon stress tensor as \(\sigma\) and the phase stress tensor as \(\rho\), which are \(3 \times 3\) matrices in principle. For the convenience of computation, we may straighten \(\sigma, \rho, U\) and \(W\) to column vectors with 9 components. Then the generalized Hooke’s law (see [9]) reads as

\[
\begin{pmatrix}
\sigma \\
\rho
\end{pmatrix} = \begin{pmatrix}
\mathcal{C} & \mathcal{R} \\
\mathcal{R}^T & \mathcal{K}
\end{pmatrix} \begin{pmatrix}
\tilde{U} \\
W
\end{pmatrix},
\]

where \(\mathcal{C}, \mathcal{R}, \mathcal{K}\) are \(9 \times 9\) matrices such that \(\begin{pmatrix}
\mathcal{C} & \mathcal{R} \\
\mathcal{R}^T & \mathcal{K}
\end{pmatrix}\) is positive definite and depends on the species of the quasi-crystal. The equilibrium equations are

\[
\nabla \cdot \sigma = 0, \quad \nabla \cdot \rho = 0,
\]

where the divergence is performed row by row. Here we use straightened vectors and matrices interchangeably. The free energy is

\[
J[U, W] := \int_{\Xi} \widehat{\mathcal{F}}[U, W] dx dy dz,
\]

where the energy density \(\widehat{\mathcal{F}}\) is given by

\[
\widehat{\mathcal{F}}[U, W] := \frac{1}{2} \begin{pmatrix}
\tilde{U}^T & W^T
\end{pmatrix} \begin{pmatrix}
\mathcal{C} & \mathcal{R} \\
\mathcal{R}^T & \mathcal{K}
\end{pmatrix} \begin{pmatrix}
\tilde{U} \\
W
\end{pmatrix}.
\]

We intend to study the structure of the energy associated with this system.
1.2. Problem Simplification. Since $U$ and $W$ are sparse matrices, we can reduce the 18-variable problem to a 4-variable problem (see [9]). In particular, for $N$ dislocation points at $\vec{d}_i, i = 1, 2, \ldots, N$, with Burger’s vectors for the phonon field given by $\vec{b}_u^i$ and for the phase field given by $\vec{b}_w^i$, it suffices to consider $\mathcal{U} = (\mathcal{U}_x, \mathcal{U}_y)$ and $\mathcal{W} = (\mathcal{W}_x, \mathcal{W}_y)$ satisfying

$$
\begin{bmatrix}
\sigma \\
\rho
\end{bmatrix} = \begin{pmatrix}
C & R \\
R^T & K
\end{pmatrix} \begin{bmatrix}
\mathcal{U} \\
\mathcal{W}
\end{bmatrix},
\nabla \times \mathcal{U} = \sum_{i=1}^{N} b_u^i \delta_{\vec{d}_i}, \nabla \times \mathcal{W} = \sum_{i=1}^{N} b_w^i \delta_{\vec{d}_i},
\nabla \cdot \mathcal{U} = 0, \nabla \cdot \mathcal{W} = 0,
$$

(1.3)

where $\sigma = (\sigma_x, \sigma_y), \rho = (\rho_x, \rho_y)$ are vectors with 2 components, and $C, R, K$ are $2 \times 2$ matrices, $C$ and $K$ are symmetric and positive definite, $\nabla \cdot \vec{f} := \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}$ and $\nabla \times \vec{f} := \frac{\partial f_y}{\partial x} - \frac{\partial f_y}{\partial y}$. Roughly speaking, $\mathcal{U}$ plays the role of $(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$ and $\mathcal{W}$ plays the role of $(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y})$. Here we omit the symmetrization procedure of $\mathcal{U}$ since it can be directly incorporated into Hooke’s law, and we do not change the notation for $\sigma, \rho, C, R, K$. The free energy is

$$J[\mathcal{U}, \mathcal{W}] := \int_{\Omega} \mathcal{F}[\mathcal{U}, \mathcal{W}] dxdy,$$

with density

$$\mathcal{F}[\mathcal{U}, \mathcal{W}] := \frac{1}{2} \begin{pmatrix}
\mathcal{U}^T & \mathcal{W}^T
\end{pmatrix} \begin{pmatrix}
C & R \\
R^T & K
\end{pmatrix} \begin{pmatrix}
\mathcal{U} \\
\mathcal{W}
\end{pmatrix}.$$

In a hexagonal quasi-crystal (see [9]), we may further simplify the Hooke’s law as

$$C = \begin{pmatrix}
C & 0 \\
0 & C
\end{pmatrix}, \quad R = R^T = \begin{pmatrix}
R & 0 \\
0 & R
\end{pmatrix}, \quad K = \begin{pmatrix}
K & 0 \\
0 & K
\end{pmatrix},$$

for some constants $C, R, K$ with

$$C, K > 0 \text{ and } CK > R^2,$$

(1.4)

i.e. the matrix $\begin{pmatrix}
C & R \\
R^T & K
\end{pmatrix}$ is positive definite. Also, the free energy density reduces to

$$\mathcal{F}[\mathcal{U}, \mathcal{W}] = \frac{1}{2} \left( C \|\mathcal{U}\|^2 + K \|\mathcal{W}\|^2 + 2R(\mathcal{U} \cdot \mathcal{W}) \right).$$

(1.5)

1.3. Core Regularization. It is well-known that in a neighborhood of a dislocation point, the free energy blows up (see [7] and [8]). Similar to the techniques in [6] and [8] for crystals, we consider a variational formulation by removing a core $B_\varepsilon(\vec{d}_i) = \{ \vec{d} = (x, y) : |\vec{d} - \vec{d}_i| \leq \varepsilon \}$ around each dislocation, and we consider the minimization problem

$$\min_{(\mathcal{U}, \mathcal{W}) \in H_0^r} \int_{\Omega_\varepsilon} \mathcal{F}[\mathcal{U}, \mathcal{W}] dxdy,$$

(1.6)

where $\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^{N} B_\varepsilon(\vec{d}_i)$ and the admissible set is defined by

$$H_0^r := \left\{ (\mathcal{U}, \mathcal{W}) \in L^2(\Omega_\varepsilon), \nabla \times \mathcal{U} = 0, \nabla \times \mathcal{W} = 0 \quad \text{in} \quad \Omega_\varepsilon, \right\}$$

$$\begin{align*}
\int_{\partial B_\varepsilon(\vec{d}_i)} \mathcal{U} \cdot tds &= b_u^i, \\
\int_{\partial B_\varepsilon(\vec{d}_i)} \mathcal{W} \cdot tds &= b_w^i, \quad i = 1, 2, \ldots, N,
\end{align*}$$

where $t$ is the unit tangent vector at $\partial B_\varepsilon(\vec{d}_i)$. Here $\mathcal{U} \cdot t$ and $\mathcal{W} \cdot t$ are the tangential traces of $\mathcal{U}$ and $\mathcal{W}$, which are well-defined in the $L^2$ curl-free space (see [7] and [8]).
Assume that the solution to the above minimization problem admits a unique solution as \((\mathcal{V}_\epsilon, \mathcal{W}_\epsilon)\). Our goal is to study the behavior of \((\mathcal{V}_\epsilon, \mathcal{W}_\epsilon)\) and of the free energy

\[
J_\epsilon[\mathcal{V}_\epsilon, \mathcal{W}_\epsilon] := \int_{\Omega} \mathfrak{F} [\mathcal{V}_\epsilon, \mathcal{W}_\epsilon] \, dx dy,
\]
as \(\epsilon \to 0\).

1.4. **Main Theorem.** We intend to use \(\Gamma\)-convergence to analyze the minimizer and energy structure. Define the functional \(J_\epsilon^{(0)} : L^2(\Omega) \times L^2(\Omega) \to [0, \infty]\) by

\[
J_\epsilon^{(0)}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] := \begin{cases}
\int_{\Omega} \frac{1}{2} \left( C |\mathcal{U}_\epsilon|^2 + K |\mathcal{W}_\epsilon|^2 + 2R(\mathcal{U}_\epsilon \cdot \mathcal{W}_\epsilon) \right) \, dx dy \\
\infty \quad \text{otherwise in } L^2(\Omega) \times L^2(\Omega).
\end{cases}
\]

**Theorem 1.1.** (Compactness) *(see Section 3.1)* Assume that (1.4) holds and \((\mathcal{U}_\epsilon, \mathcal{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)\) satisfy

\[
sup_{\epsilon \to 0} J_\epsilon^{(0)}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \leq C_0.
\]

Then there exists \(v_u, v_w \in H^1(\Omega)\) such that up to the extraction of subsequence (non-relabelled),

\[
(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon) \rightharpoonup (\nabla v_u, \nabla v_w) \quad \text{in weak-}L^2 \text{ as } \epsilon \to 0.
\]

With compactness theorem in hand, we can show the zeroth-order \(\Gamma\)-convergence.

**Theorem 1.2.** (0th-Order \(\Gamma\)-Convergence) *(see Section 3.2)* Assume that (1.4) holds. Define the functional \(J_0^{(0)} : L^2(\Omega) \times L^2(\Omega) \to [0, \infty]\) as

\[
J_0^{(0)}[\mathcal{U}, \mathcal{W}] := \begin{cases}
\int_{\Omega} \frac{1}{2} \left( C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) + \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \\
\infty \quad \text{otherwise in } L^2(\Omega) \times L^2(\Omega).
\end{cases}
\]

Then

1. For any sequence of pairs \((\mathcal{U}_\epsilon, \mathcal{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)\) such that \((\mathcal{U}_\epsilon, \mathcal{W}_\epsilon) \rightharpoonup (\mathcal{U}, \mathcal{W})\) in weak-\(L^2(\Omega)\), we have \(\lim \inf_{\epsilon \to 0} J_\epsilon^{(0)}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \geq J_0^{(0)}[\nabla v_u, \nabla v_w]\).
2. There exists a sequence of pairs \((\mathcal{U}_\epsilon, \mathcal{W}_\epsilon) \in L^2(\Omega) \times L^2(\Omega)\) such that \((\mathcal{U}_\epsilon, \mathcal{W}_\epsilon) \rightharpoonup (\mathcal{U}, \mathcal{W})\) in weak-\(L^2(\Omega)\), we have \(\lim \sup_{\epsilon \to 0} J_\epsilon^{(0)}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \leq J_0^{(0)}[\nabla v_u, \nabla v_w]\),

which means

\[
J_\epsilon^{(0)}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \to J_0^{(0)}[\mathcal{U}, \mathcal{W}],
\]
in the sense of \(\Gamma\)-convergence in weak-\(L^2(\Omega)\).

\(\Gamma\)-convergence naturally yields the convergence of minimum of energy functionals.

**Corollary 1.3.** (Core Energy) *(see Section 3.2)* Assume that (1.4) holds. We have

\[
\inf_{\mathcal{U}, \mathcal{W}} J_0^{(0)}[\mathcal{U}, \mathcal{W}] = \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}.
\]

Assume \((\mathcal{W}_\epsilon', \mathcal{W}_e')\) is the minimizer of \(J_\epsilon^{(0)}\), then we have

\[
J_\epsilon^{(0)}[\mathcal{W}_\epsilon', \mathcal{W}_e'] = E_0 + o(1),
\]

where the rescaled leading-order energy

\[
E_0 = \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}.
\]
The zeroth-order \( \Gamma \)-convergence result tells us the asymptotic behavior of leading-order free energy. However, the rescaling in \( J^{(0)} \) suppress \( O(1) \) term in the energy. As \([3]\) revealed, more detailed information can be discovered when we get rid of the rescaling and go to first-order \( \Gamma \)-convergence. Define the functional \( J^{(1)}: L^2(\Omega) \times L^2(\Omega) \to [0, \infty) \) as

\[
J^{(1)}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) := \begin{cases} 
\int_{\Omega} \frac{1}{2} \left( C |\tilde{\mathbf{u}}|_e^2 + K |\mathbf{w}_e|_e^2 + 2 R(\tilde{\mathbf{u}}_e \cdot \mathbf{w}_e) \right) dx dy - |\ln(\epsilon)| \inf_{\tilde{\mathbf{u}}, \tilde{\mathbf{w}}} J^{(0)}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) 
\quad \text{if } (\tilde{\mathbf{u}}_e, \mathbf{w}_e) \in H'_0(\Omega), \\
\infty \text{ otherwise in } L^2(\Omega) \times L^2(\Omega).
\end{cases}
\]

\textbf{Theorem 1.4.} \((1^{\text{st}}-\text{Order } \Gamma-\text{Convergence})\) (see Section 3.3) Assume that (1.4) holds. Define the functional \( J^{(1)}: L^2(\Omega) \times L^2(\Omega) \to [0, \infty) \) as

\[
J^{(1)}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) := \begin{cases} 
E_{\text{self}} + E_{\text{int}} + E_{\text{elastic}} \text{ if } (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = \left( \nabla v_u + \sum_{i=1}^N \mathbf{w}_i, \nabla v_w + \sum_{i=1}^N \mathbf{w}_i \right), \\
\infty \text{ otherwise in } L^2(\Omega) \times L^2(\Omega),
\end{cases}
\]

where

\[
E_{\text{self}} := \sum_{i=1}^N \int_{\Omega \setminus B_r(\tilde{\mathbf{d}})} \frac{1}{2} \left( C |\mathbf{w}_i|_e^2 + K |\mathbf{w}_i|_e^2 + 2 R(\mathbf{w}_i \cdot \mathbf{w}_i) \right) dx dy + \sum_{i=1}^N \left( \frac{C(b_{i,e}^2) + K(b_{i,w}^2)}{4\pi} + \frac{2 R(b_{i,e}^2) b_{i,w}^2}{4\pi} \right) ln(r),
\]

\[
E_{\text{int}} := \sum_{i=1}^{N-1} \sum_{j=1}^N \int_\Omega \left( C(\mathbf{w}_i \cdot \mathbf{w}_j) + K(\mathbf{w}_i \cdot \mathbf{w}_j) + R(\mathbf{w}_i \cdot \mathbf{w}_j) + R(\mathbf{w}_i \cdot \mathbf{w}_j) \right),
\]

\[
E_{\text{elastic}} := J[\nabla v_u, \nabla v_w] + \sum_{i=1}^N \int_{\partial \Omega} \left( v_u (C \mathbf{w}_i + R \mathbf{w}_i) + v_w (K \mathbf{w}_i + R \mathbf{w}_i) \right) \cdot nds.
\]

Then

1. For any sequence of pairs \( (\tilde{\mathbf{u}}_e, \tilde{\mathbf{w}}_e) \in L^2(\Omega) \times L^2(\Omega) \) such that \( (\tilde{\mathbf{u}}_e, \tilde{\mathbf{w}}_e) \to (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \) in weak-\( L^2(\Omega) \), we have \( \liminf_{\epsilon \to 0} J^{(1)}(\tilde{\mathbf{u}}_\epsilon, \tilde{\mathbf{w}}_\epsilon) \geq J^{(1)}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) \).
2. There exists a sequence of pairs \( (\mathbf{u}_\epsilon, \mathbf{w}_\epsilon) \in L^2(\Omega) \times L^2(\Omega) \) such that \( (\mathbf{u}_\epsilon, \mathbf{w}_\epsilon) \to (\mathbf{u}, \mathbf{w}) \) in weak-\( L^2(\Omega) \), we have \( \limsup_{\epsilon \to 0} J^{(1)}(\mathbf{u}_\epsilon, \mathbf{w}_\epsilon) \leq J^{(1)}(\mathbf{u}, \mathbf{w}) \),

which means

\[
J^{(1)}(\tilde{\mathbf{u}}_\epsilon, \tilde{\mathbf{w}}_\epsilon) \to J^{(1)}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}),
\]

in the sense of \( \Gamma \)-convergence in weak-\( L^2(\Omega) \).

Similarly, we have a better approximation of energy functionals.

\textbf{Corollary 1.5.} \((\text{Renormalized Energy})\) (see Section 3.3) Assume that (1.4) holds. We have

\[
\inf_{\tilde{\mathbf{u}}, \tilde{\mathbf{w}}} J^{(1)}(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}},
\]
where
\[ F_{\text{self}} : = \sum_{i=1}^{N} \int_{\Omega} \left( \frac{1}{2} \left( C|\mathcal{U}|^2 + K|\mathcal{V}|^2 \right) + 2R(\mathcal{U}, \mathcal{V}) \right) \, dx \, dy \]
\[ + \sum_{i=1}^{N} \frac{\left( C(\delta_u^i)^2 + K(\delta_v^i)^2 + 2R(\delta_u^i, \delta_v^i) \right)}{4\pi} \ln(r), \]
\[ F_{\text{int}} : = \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\Omega} \left( C(\mathcal{U}, \mathcal{V}) + K(\mathcal{V}, \mathcal{V}) + R(\mathcal{U}, \mathcal{V}) + R(\mathcal{V}, \mathcal{V}) \right), \]
\[ F_{\text{elastic}} : = J[\nabla u_0, \nabla w_0] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( u_0(C\mathcal{U}_i + R\mathcal{V}_i) + w_0(K\mathcal{V}_i + R\mathcal{U}_i) \right) \cdot nds, \]
in which \((u_0, w_0)\) is the minimizer of
\[ I[u, w] = J[\nabla u, \nabla w] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( v_u(C\mathcal{U}_i + R\mathcal{V}_i) + v_w(K\mathcal{V}_i + R\mathcal{U}_i) \right) \cdot nds. \]
Assume \((\mathcal{U}'_0, \mathcal{V}'_0) \in H_0^1\) is the minimizer of \(J^{(1)}_e\), then we have
\[ J^{(1)}_e[\mathcal{U}'_0, \mathcal{V}'_0] = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}} + o(1). \]

As corollaries, we can now state a characterization of the structure of minimizer \((\mathcal{U}_\epsilon, \mathcal{V}_\epsilon)\) and energy \(J_e[\mathcal{U}_\epsilon, \mathcal{V}_\epsilon] \) in (1.6).

**Theorem 1.6.** (Minimizer Structure) (see Section 3.4) Assume that (1.4) holds. The problem (1.6) admits a unique solution
\[ \mathcal{U}_\epsilon = \sum_{i=1}^{N} \mathcal{U}_i + \nabla u_\epsilon, \quad \mathcal{V}_\epsilon = \sum_{i=1}^{N} \mathcal{V}_i + \nabla w_\epsilon, \]
where
\[ \mathcal{U}_i = \frac{\delta_u^i}{2\pi} \left( -\frac{1}{(x - x_i)^2 + (y - y_i)^2} \right), \]
\[ \mathcal{V}_i = \frac{\delta_v^i}{2\pi} \left( -\frac{1}{(x - x_i)^2 + (y - y_i)^2} \right), \]
and \((u_\epsilon, w_\epsilon)\) is the unique minimizer of
\[ I_e[u_\epsilon, w_\epsilon] : = J_e[\nabla u_\epsilon, \nabla w_\epsilon] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( u_\epsilon(C\mathcal{U}_i + R\mathcal{V}_i) + w_\epsilon(K\mathcal{V}_i + R\mathcal{U}_i) \right) \cdot nds \]
\[ - \sum_{i=1}^{N} \sum_{j \neq i} \int_{\partial B(x_i, y_i)} \left( u_\epsilon(C\mathcal{U}_j + R\mathcal{V}_j) + w_\epsilon(K\mathcal{V}_j + R\mathcal{U}_j) \right) \cdot nds, \]
subject to \(\int_B u_\epsilon \, dx \, dy = 0\) and \(\int_B w_\epsilon \, dx \, dy = 0\) for some ball \(B \subset \Omega_\epsilon\), with \(n\) the outward unit normal vector on \(\partial \Omega\).

Furthermore, \((\mathcal{U}_\epsilon, \mathcal{V}_\epsilon)\) converges in weak-\(L^2(\Omega)\) as \(\epsilon \to 0\) to \((\mathcal{U}_0, \mathcal{V}_0)\) where
\[ \mathcal{U}_0 = \sum_{i=1}^{N} \mathcal{U}_i + \nabla u_0, \quad \mathcal{V}_0 = \sum_{i=1}^{N} \mathcal{V}_i + \nabla w_0, \]
and \([u_0, w_0]\) is the unique minimizer of
\[ I_0[u_0, w_0] = J[\nabla u_0, \nabla w_0] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( u_0(C\mathcal{U}_i + R\mathcal{V}_i) + w_0(K\mathcal{V}_i + R\mathcal{U}_i) \right) \cdot nds, \]
subject to \( \int_B u_0 \, dx \, dy = 0 \) and \( \int_B w_0 \, dx \, dy = 0 \) for some ball \( B \subset \Omega_\epsilon \).

**Theorem 1.7.** (Energy Structure) (see Section 3.4) Assume that (1.4) holds. We have

\[
J_\epsilon [\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = \int_{\Omega_\epsilon} \mathcal{F}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \, dx \, dy = E_0 \ln \left( \frac{1}{\epsilon} \right) + F + o(1),
\]

where the core energy \( E_0 \) is defined in (1.7) and the renormalized energy \( F = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}} \) is defined in (1.8).

**Remark 1.8.** The core energy is a leading singular term of \( O(\ln(1/\epsilon)) \), which confirms that the free energy is not finite when dislocations are present. The \( O(1) \) term is usually called the renormalized energy and is physically meaningful. This type of asymptotic expansion was first derived for Ginzburg-Landau vortices in [4], and extended to the context of dislocation in [8]. The techniques to prove \( \Gamma \)-convergence results were first introduced in the study of the Ginzburg-Landau vortices (see [2] and [16]).

Note that the renormalized energy is independent of the radius \( \epsilon \) and thus fully characterizes the energy structure around dislocations.

As an application of the energy structure, we prove that the interaction energy \( F_{\text{int}} \) obeys the inverse logarithmical law of the distance between two dislocations.

**Theorem 1.9.** (Interaction Energy) (see Section 4.1) Assume that (1.4) holds. We have

\[
F_{\text{int}} = \sum_{i=1}^{N-1} \sum_{j=i}^N \frac{C b_i^k b_j^k + K b_i^k b_j^k + R b_i^k b_j^k + R b_i^k b_j^k}{2\pi} \ln \left( \frac{1}{|d_i^k - d_j^k|} \right) + O(1).
\]

When multiple dislocations are present, defects interact with themselves by means of the so-called Peach-Köhler force, which is defined as the negative gradient of renormalized energy \( F \) at the dislocation points (see [10]).

**Theorem 1.10.** (Peach-Köhler force) (see Section 4.2) Assume that (1.4) holds. The Peach-Köhler force acting at \( \vec{d}_k \) is given by

\[
\nabla_{\vec{d}_k} F = -\int_{\partial B_r(\vec{d}_k)} \left( \mathcal{F}[\mathcal{U}_0, \mathcal{W}_0] \mathbf{1} - (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{W}_0) \right) \cdot \mathbf{n} \, ds,
\]

for \( r < \frac{1}{2} \min_k \left( \text{dist}(\vec{d}_k, \partial \Omega) \right) \).

**Remark 1.11.** The integrand in Theorem 1.10

\[
E = -\left( \mathcal{F}[\mathcal{U}_0, \mathcal{W}_0] \mathbf{1} - (C \mathcal{U}_0 \otimes \mathcal{U}_0 + K \mathcal{W}_0 \otimes \mathcal{W}_0 + R \mathcal{U}_0 \otimes \mathcal{W}_0 + R \mathcal{W}_0 \otimes \mathcal{W}_0) \right)
\]

is usually called the Eshelby stress tensor.

Our paper is organized as follows: in Section 2 we present some preliminary results on the minimization problem (1.6) of \( J_\epsilon \) for fixed \( \epsilon \); in Section 3 we derive the zeroth-order and first-order \( \Gamma \)-convergence of the free energy when \( \epsilon \to 0 \) and study the structure of minimizer and energy; Finally, in Section 4 we introduce two applications of the renormalized energy: the interaction between dislocations and the Peach-Köhler force.

### 2. Preliminaries

In this section, we consider the minimization problem (1.6) of \( J_\epsilon \) for fixed \( \epsilon \).
2.1. **Euler-Lagrange Equation.** We start with the equations that minimizer of $J_\epsilon$ should satisfy and the uniqueness of minimizer.

**Lemma 2.1.** Assume that (1.4) holds and $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ is the minimizer of $J_\epsilon$ in $H^1_0(\Omega)$. Then it satisfies the equations

$$
\begin{align*}
\nabla \cdot (C\mathcal{U}_\epsilon + R\mathcal{W}_\epsilon) &= \nabla \cdot (K\mathcal{W}_\epsilon + R^T\mathcal{U}_\epsilon) = 0 \quad \text{in } \Omega, \\
(C\mathcal{U}_\epsilon + R\mathcal{W}_\epsilon) \cdot n &= (K\mathcal{W}_\epsilon + R^T\mathcal{U}_\epsilon) \cdot n = 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

(2.1)

where $n$ is the outward normal vector to $\partial \Omega$. Moreover, the solution to (2.1) is unique.

**Proof.** The free energy density in $\Omega$, is given by

$$
\mathcal{F}[\mathcal{U}, \mathcal{W}] = \frac{1}{2} \begin{pmatrix} \mathcal{U}^T & \mathcal{W}^T \end{pmatrix} \begin{pmatrix} C & R \\ R^T & K \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{W} \end{pmatrix}.
$$

For any $(\mathcal{U}, \mathcal{W})$ and $(\bar{\mathcal{U}}, \bar{\mathcal{W}})$ in $H^1_0$, we must have $\mathcal{U} - \bar{\mathcal{U}} = \nabla P$ and $\mathcal{W} - \bar{\mathcal{W}} = \nabla Q$ for some $P, Q \in H^1(\Omega)$ due to curl-free condition. Hence, the first-order variation is

$$
\delta J_\epsilon[\mathcal{U}, \mathcal{W}](p, q) = \lim_{\theta \to 0} \frac{J_\epsilon[\mathcal{U} + \theta \nabla p, \mathcal{W} + \theta \nabla q] - J_\epsilon[\mathcal{U}, \mathcal{W}]}{\theta} = - \int_{\Omega} \left(p \nabla \cdot (C\mathcal{U} + R\mathcal{W}) + q \nabla \cdot (K\mathcal{W} + R^T\mathcal{U}) \right) d\mathbf{x}d\mathbf{y} + \int_{\partial \Omega} \left(p(C\mathcal{U} + R\mathcal{W}) \cdot n + q(K\mathcal{W} + R^T\mathcal{U}) \cdot n \right) ds.
$$

Thus, setting $\delta J_\epsilon[\mathcal{U}, \mathcal{W}](p, q) = 0$ for any $p, q \in H^1(\Omega)$, we can deduce that the minimizer $(\mathcal{U}_\epsilon, \mathcal{W}_\epsilon)$ is a weak solution of the Euler-Lagrange equations (2.1).

To prove uniqueness, assume that $(\mathcal{U}_1, \mathcal{W}_1)$ and $(\mathcal{U}_2, \mathcal{W}_2)$ are two solutions to (2.1). The difference $(f, g) = (\mathcal{U}_2 - \mathcal{U}_1, \mathcal{W}_2 - \mathcal{W}_1)$ must be curl-free and has zero loop integral around $\partial B_\epsilon(\bar{d}_0)$. Therefore, we must have $(f, g) = (\nabla F, \nabla G)$ for some $F, G \in H^1(\Omega)$. Since $F$ and $G$ satisfy the equation

$$
\int_{\Omega} \left( (p \nabla F + \nabla G) + (q \nabla F + \nabla G) \right) d\mathbf{x}d\mathbf{y} = 0,
$$

for any $p, q \in H^1(\Omega)$, taking $p = F$ and $q = G$, considering $\begin{pmatrix} C & R \\ R^T & K \end{pmatrix}$ is positive definite, we must have $\nabla F = \nabla G = 0$, and the uniqueness follows. \hfill \square

2.2. **Estimate and Energy for Single Dislocation.** In this section, we further restrict the discussion to the case in which $\Omega = B_\epsilon(\bar{d}_0)$ for constant $r >> \epsilon$, with only one dislocation at $\bar{d}_0 = (x_0, y_0)$ with Burger’s vector of phonon field as $\bar{b}_u$ and of phase field as $\bar{b}_w$. Solving the above Euler-Lagrange equations (2.1), by a linear combination, we get

$$
\begin{align*}
\nabla \cdot \mathcal{U}_\epsilon &= \nabla \cdot \mathcal{W}_\epsilon = 0 \quad \text{in } \Omega, \\
\mathcal{U}_\epsilon \cdot n &= \mathcal{W}_\epsilon \cdot n = 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

in $H^1_0(\Omega)$. Hence, there exists potential functions $U_\epsilon(x, y)$ and $W_\epsilon(x, y)$ such that $\nabla U_\epsilon = \mathcal{U}_\epsilon$, $\nabla W_\epsilon = \mathcal{W}_\epsilon$ and

$$
\Delta U_\epsilon = \Delta W_\epsilon = 0 \quad \text{in } \Omega.
$$

Therefore, we are lead to solving Laplace’s equations in an annulus with Neumann boundary $\frac{\partial U_\epsilon}{\partial n} = \frac{\partial W_\epsilon}{\partial n} = 0$. This system has a unique solution subject to the normalization conditions $\int_{\partial B_\epsilon(\bar{d}_0)} dU_\epsilon = b_u$.
and \(\int_{\partial B, (\tilde{d}_e)} dW_e = b_w\), and we obtain the explicit solution as

\[
U_e = \frac{b_u}{2\pi} \arctan \left( \frac{y - y_0}{x - x_0} \right), \quad W_e = \frac{b_w}{2\pi} \arctan \left( \frac{y - y_0}{x - x_0} \right)
\]

for \((x, y) \in \Omega_e\).

Hence, we have

\[
\mathcal{U}_e = \frac{b_u}{2\pi} \frac{1}{(x - x_0)^2 + (y - y_0)^2} (- (y - y_0), (x - x_0)),
\]

\[
\mathcal{W}_e = \frac{b_w}{2\pi} \frac{1}{(x - x_0)^2 + (y - y_0)^2} (- (y - y_0), (x - x_0))
\]

for \((x, y) \in \Omega_e\), and we note that these are independent of \(\epsilon\) and \(r\). Therefore, the minimum free energy can be obtained explicitly as

\[
J_e = \int_{\Omega_e} \mathbb{F} [\mathcal{U}_e, \mathcal{W}_e] dx dy = (C b_u^2 + K b_w^2 + 2 R b_u b_w) \frac{1}{4\pi} \ln \left( \frac{r}{\epsilon} \right).
\]

### 2.3. Estimate and Energy for Multiple Dislocations

Now we consider the case with multiple dislocations in general domains. For fixed \(\tilde{d}_e = (x_i, y_i)\), assume that the single-dislocation solution is \((\mathcal{U}_i, \mathcal{W}_i)\). Based on analysis in Lemma 2.1, we must have

\[
\mathcal{U}_e := \sum_{i=1}^{N} \mathcal{U}_i + \nabla u_e, \quad \mathcal{W}_e := \sum_{i=1}^{N} \mathcal{W}_i + \nabla w_e.
\]

for some \(u_e, w_e \in H^1(\Omega_e)\). We deduce

\[
J_e[\mathcal{U}_e, \mathcal{W}_e] = I_e[u_e, w_e] + \sum_{i=1}^{N} J_e[\mathcal{U}_i, \mathcal{W}_i]
\]

\[
+ \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \int_{\Omega_e} \left( C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_i \cdot \mathcal{U}_j) \right),
\]

where

\[
I_e[u_e, w_e] := J_e[\nabla u_e, \nabla w_e] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( u_e(C \mathcal{U}_i + R \mathcal{W}_i) + w_e(K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot nds
\]

\[
- \sum_{i=1}^{N} \sum_{j \neq i} \int_{\partial B_e(x_i, y_i)} \left( u_e(C \mathcal{U}_j + R \mathcal{W}_j) + w_e(K \mathcal{W}_j + R \mathcal{U}_j) \right) \cdot nds.
\]

Therefore, in order to minimize \(J_e\), it suffices to consider the problem:

\((M_e)\): Minimize \(I_e[u, w]\) for \(u, w \in H^1(\Omega_e)\) subject to \(\int_B u dx dy = 0\) and \(\int_B w dx dy = 0\) for some ball \(B \subset \Omega_e\), i.e. find the solution of

\[
\min_{u, w \in H^1(\Omega_e)} I_e[u, w]. \tag{2.4}
\]

This normalization is for the convenience of coercivity and will not affect the minimizing process since adding a constant to \(u\) or \(w\) will not affect the value of \(I_e[u, w]\).
Lemma 2.2. Assume that (1.4) holds and \((u_\epsilon, w_\epsilon)\) is the solution of the minimization problem (2.4) for \(I_\epsilon\). Then it satisfies the equations

\[
\begin{align*}
\nabla \cdot (\mathcal{C} \nabla u_\epsilon + \mathcal{R} \nabla w_\epsilon) &= \nabla \cdot (\mathcal{K} \nabla w_\epsilon + \mathcal{R}^T \nabla w_\epsilon) = 0 \quad \text{in} \quad \Omega_\epsilon, \\
\left( \mathcal{C} \sum_{k=1}^{N} \mathcal{U}_i + \nabla u_\epsilon + \mathcal{R} \left( \sum_{k=1}^{N} \mathcal{U}_i + \nabla u_\epsilon \right) \right) \cdot n &= 0 \quad \text{on} \quad \partial \Omega, \\
\left( \mathcal{K} \sum_{k=1}^{N} \mathcal{U}_i + \nabla w_\epsilon + \mathcal{R}^T \left( \sum_{k=1}^{N} \mathcal{U}_i + \nabla u_\epsilon \right) \right) \cdot n &= 0 \quad \text{on} \quad \partial \Omega, \\
\mathcal{C} \left( \sum_{j \neq i} \mathcal{U}_i + \nabla u_\epsilon + \mathcal{R} \left( \sum_{j \neq i} \mathcal{U}_i + \nabla u_\epsilon \right) \right) \cdot n &= 0 \quad \text{on} \quad \partial B_i(\tilde{d}_i), \\
\mathcal{K} \left( \sum_{j \neq i} \mathcal{U}_i + \nabla w_\epsilon + \mathcal{R}^T \left( \sum_{j \neq i} \mathcal{U}_i + \nabla u_\epsilon \right) \right) \cdot n &= 0 \quad \text{on} \quad \partial B_i(\tilde{d}_i).
\end{align*}
\]

Moreover, the solution to (2.5) is unique.

Proof. This follows a standard argument via first-order variation. Letting

\[
\delta I_\epsilon[u, w](p, q) = \lim_{\theta \to 0} \frac{I_\epsilon[u + \theta p, w + \theta q] - I_\epsilon[u, w]}{\theta}
\]

\[
= - \int_{\Omega_\epsilon} \left( \mathbf{p} \cdot (\mathcal{C} \nabla u + \mathcal{R} \nabla w) + \mathbf{q} \cdot (\mathcal{K} \nabla w + \mathcal{R}^T \nabla u) \right) dx dy \\
+ \int_{\partial \Omega} \mathbf{p} \left( \mathcal{C} \left( \sum_{k=1}^{N} \mathcal{U}_i + \nabla u \right) + \mathcal{R} \left( \sum_{k=1}^{N} \mathcal{U}_i + \nabla w \right) \right) \cdot n ds \\
+ \int_{\partial \Omega} \mathbf{q} \left( \mathcal{K} \left( \sum_{k=1}^{N} \mathcal{U}_i + \nabla w \right) + \mathcal{R}^T \left( \sum_{k=1}^{N} \mathcal{U}_i + \nabla u \right) \right) \cdot n ds \\
- \int_{\partial B_i(\tilde{d}_i)} \mathbf{p} \left( \mathcal{C} \left( \sum_{j \neq i} \mathcal{U}_i + \nabla u \right) + \mathcal{R} \left( \sum_{j \neq i} \mathcal{U}_i + \nabla w \right) \right) \cdot n ds \\
- \int_{\partial B_i(\tilde{d}_i)} \mathbf{q} \left( \mathcal{K} \left( \sum_{j \neq i} \mathcal{U}_i + \nabla w \right) + \mathcal{R}^T \left( \sum_{j \neq i} \mathcal{U}_i + \nabla u \right) \right) \cdot n ds.
\]

If \(\delta I_\epsilon[u, w](p, q) = 0\) for any \(p, q \in H^1(\Omega_\epsilon)\), then the system (2.5) is satisfied. The uniqueness follows from a standard argument as in the proof of Lemma 2.1.

\]

2.4. Minimization of the Energy.

Lemma 2.3. Assume that (1.4) holds. There exist constants \(C_1, C_2 > 0\) independent of \(\epsilon\) such that

\[
I_\epsilon[u, w] \geq C_1 \left( ||u||^2_{H^1(\Omega_\epsilon)} + ||w||^2_{H^1(\Omega_\epsilon)} \right) - C_2 \left( ||u||_{H^1(\Omega_\epsilon)} + ||w||_{H^1(\Omega_\epsilon)} \right),
\]

for all \(u, w \in H^1(\Omega_\epsilon)\) subject to the normalization condition \(\int_B u dx dy = 0\) and \(\int_B w dx dy = 0\) for some ball \(B \subset \Omega_\epsilon\). Moreover, the minimization problem (2.4) for \(I_\epsilon\) admits a unique solution \((u_\epsilon, w_\epsilon) \in H^1(\Omega_\epsilon)\) satisfying

\[
||u||^2_{H^1(\Omega_\epsilon)} + ||w||^2_{H^1(\Omega_\epsilon)} \leq M,
\]

for some constant \(M > 0\) independent of \(\epsilon\).
Proof. Recall that
\[
I_\epsilon[u, w] : = J_\epsilon[\nabla u, \nabla w] + \sum_{i=1}^{N} \int_{\partial\Omega} \left( u(C\mathcal{U}_i + R\mathcal{W}_i) + w(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot nds
\]
\[
- \sum_{i=1}^{N} \sum_{j \neq i} \int_{\partial B_{e}(x_i, y_i)} \left( u(C\mathcal{U}_j + R\mathcal{W}_j) + w(K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot nds.
\]
Since \( I \) is positive definite, we directly estimate
\[
I_\epsilon[u, w] \geq C \int_{\Omega_e} \left( |\nabla u|^2 + |\nabla w|^2 \right) dx dy - C' \int_{\partial\Omega} \left( |u| + |w| \right) ds - C' \int_{\partial B_{e}(x_i, y_i)} \left( |u| + |w| \right) ds.
\]
By Poincaré's inequality (see [8]), we have for \( C_1 > 0 \) independent of \( \epsilon \),
\[
\int_{\Omega_e} |\nabla u|^2 dx dy \geq C_1 \| u \|^2_{H^1(\Omega_e)},
\]
\[
\int_{\Omega_e} |\nabla w|^2 dx dy \geq C_1 \| w \|^2_{H^1(\Omega_e)}.
\]
In these two estimates, the normalization condition is essential. Also, we have for \( C_2 > 0 \) independent of \( \epsilon \),
\[
\int_{\partial\Omega} \left( |u| + |w| \right) ds \leq C_2 \left( \| u \|^2_{H^1(\Omega_e)} + \| w \|^2_{H^1(\Omega_e)} \right), \tag{2.6}
\]
\[
\int_{\partial B_{e}(x_i, y_i)} \left( |u| + |w| \right) ds \leq C_2 \left( \| u \|^2_{H^1(\Omega_e)} + \| w \|^2_{H^1(\Omega_e)} \right). \tag{2.7}
\]
Hence, the coercivity is naturally valid, i.e.
\[
I_\epsilon[u, w] \geq C_1 \left( \| u \|^2_{H^1(\Omega_e)} + \| w \|^2_{H^1(\Omega_e)} \right) - C_2 \left( \| u \|^2_{H^1(\Omega_e)} + \| w \|^2_{H^1(\Omega_e)} \right).
\]
Since \( I_\epsilon \) is strictly convex (see [8]) and \( I_\epsilon[0, 0] = 0 \), the existence and uniqueness follow. \( \square \)

We have established the following result.

**Theorem 2.4.** Assume that (1.4) holds. The problem (1.6) admits a unique solution
\[
\mathcal{U}_\epsilon = \sum_{i=1}^{N} \mathcal{U}_i + \nabla u_\epsilon, \quad \mathcal{W}_\epsilon = \sum_{i=1}^{N} \mathcal{W}_i + \nabla w_\epsilon,
\]
where
\[
\mathcal{U}_i = \frac{b_i}{2\pi} \frac{1}{(x-x_i)^2 + (y-y_i)^2} \left( - (y-y_i), (x-x_i) \right),
\]
\[
\mathcal{W}_i = \frac{b_i}{2\pi} \frac{1}{(x-x_i)^2 + (y-y_i)^2} \left( - (y-y_i), (x-x_i) \right),
\]
and \((u_\epsilon, w_\epsilon)\) is the minimizer of
\[
I_\epsilon[u_\epsilon, w_\epsilon] : = J_\epsilon[\nabla u_\epsilon, \nabla w_\epsilon] + \sum_{i=1}^{N} \int_{\partial\Omega} \left( u_\epsilon(C\mathcal{U}_i + R\mathcal{W}_i) + w_\epsilon(K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot nds
\]
\[
- \sum_{i=1}^{N} \sum_{j \neq i} \int_{\partial B_{e}(x_i, y_i)} \left( u_\epsilon(C\mathcal{U}_j + R\mathcal{W}_j) + w_\epsilon(K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot nds.
\]
subject to \( \int_B u_\epsilon dx dy = 0 \) and \( \int_B w_\epsilon dx dy = 0 \) for some ball \( B \subset \Omega_\epsilon \), with \( n \) the outward unit normal vector on \( \partial\Omega \).

This theorem tells us the existence and uniqueness of minimizer in (1.6). The asymptotic behaviors of minimizer and energy as \( \epsilon \to 0 \) are left open at this stage.
3. Γ-Convergence

In this section, we use higher-order Γ-convergence to dig more information into the structure of minimizer and energy.

3.1. Weak-$L^2$ Compactness. Notice that for any $(\tilde{\mathbf{U}}, \tilde{\mathbf{W}}) \in H^\varepsilon_0$, using (1.4), we have

$$
\int_{\Omega^\varepsilon} \left( C |\tilde{\mathbf{U}}|^2 + K |\tilde{\mathbf{W}}|^2 + 2R(\tilde{\mathbf{U}} \cdot \tilde{\mathbf{W}}) \right) dxdy
\geq \sum_{i=1}^N \int_{B_r(d_i) \setminus B_r(\tilde{d}_i)} \left( C |\tilde{\mathbf{U}}|^2 + K |\tilde{\mathbf{W}}|^2 + 2R(\tilde{\mathbf{U}} \cdot \tilde{\mathbf{W}}) \right) dxdy
\geq C_0 \sum_{i=1}^N \int_r \int_{\partial \Omega} \left( |\tilde{\mathbf{U}}|^2 + |\tilde{\mathbf{W}}|^2 \right) dxdy = C_0 \sum_{i=1}^N \int_r \int_{\partial \Omega} \left( |\tilde{\mathbf{U}}|^2 + |\tilde{\mathbf{W}}|^2 \right) dxdy
\geq C_0 \sum_{i=1}^N \int_r \frac{1}{2p} \int_{B_r(d_i)} \left( (\tilde{\mathbf{U}} \cdot t + \tilde{\mathbf{W}} \cdot t) ds \right)^2 d\rho
= C_0 \sum_{i=1}^N \frac{(b_u^i + b_w^i)^2}{2\pi} \ln \left( \frac{r}{\varepsilon} \right).
$$

Therefore, we know the energy blows up when $\varepsilon \to 0$. We need a proper scaling in order to show compactness. For the minimizer $(\mathcal{U}_\varepsilon, \mathcal{W}_\varepsilon)$, we may directly estimate

$$
\int_{\Omega^\varepsilon} \left( C |\mathcal{U}_\varepsilon|^2 + K |\mathcal{W}_\varepsilon|^2 + 2R(\mathcal{U}_\varepsilon \cdot \mathcal{W}_\varepsilon) \right) dxdy
= \sum_{i=1}^N \int_{B_r(d_i) \setminus B_r(\tilde{d}_i)} \left( C |\mathcal{U}_\varepsilon|^2 + K |\mathcal{W}_\varepsilon|^2 + 2R(\mathcal{U}_\varepsilon \cdot \mathcal{W}_\varepsilon) \right) dxdy
+ \int_{\Omega^\varepsilon} \left( C |\mathcal{U}_\varepsilon|^2 + K |\mathcal{W}_\varepsilon|^2 + 2R(\mathcal{U}_\varepsilon \cdot \mathcal{W}_\varepsilon) \right) dxdy
\leq C_0 \sum_{i=1}^N \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi} \ln \left( \frac{r}{\varepsilon} \right).
$$

Therefore, we need to consider the scaling $\frac{1}{\ln(\varepsilon)^{1/2}}$.

Define the functional $J^{(0)} : L^2(\Omega) \times L^2(\Omega) \to [0, \infty]$ by

$$
J^{(0)}[\mathcal{U}_\varepsilon, \mathcal{W}_\varepsilon] := \begin{cases} 
\int_{\Omega^\varepsilon} \frac{1}{2} \left( C |\mathcal{U}_\varepsilon|^2 + K |\mathcal{W}_\varepsilon|^2 + 2R(\mathcal{U}_\varepsilon \cdot \mathcal{W}_\varepsilon) \right) dxdy & \text{if } (\mathcal{U}_\varepsilon, \mathcal{W}_\varepsilon) = \left( \frac{\tilde{\mathbf{U}}}{\ln(\varepsilon)^{1/2}}, \frac{\tilde{\mathbf{W}}}{\ln(\varepsilon)^{1/2}} \right) \text{ for some } (\tilde{\mathbf{U}}, \tilde{\mathbf{W}}) \in H^\varepsilon_0, \\
\infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega).
\end{cases}
$$

Theorem 3.1. (Compactness) Assume that (1.4) holds and $(\mathcal{U}_\varepsilon, \mathcal{W}_\varepsilon) \in L^2(\Omega) \times L^2(\Omega)$ satisfy

$$
\sup_{\varepsilon > 0} J^{(0)}[\mathcal{U}_\varepsilon, \mathcal{W}_\varepsilon] \leq C_0.
$$

Then there exists $v_u, v_w \in H^1(\Omega)$ such that up to the extraction of subsequence (non-relabeled),

$$(\mathbf{1}_\Omega, \mathcal{U}_\varepsilon, \mathbf{1}_\Omega, \mathcal{W}_\varepsilon) \rightharpoonup (\nabla v_u, \nabla v_w) \text{ in weak } - L^2 \text{ as } \varepsilon \to 0.$$
In turn, by Poincaré's inequality, we have
\[
\nabla \times \left( \hat{\mathbf{u}}_c - \sum_{i=1}^{N} \mathcal{U}_i \right) = \nabla \times \left( \hat{\mathbf{w}}_c - \sum_{i=1}^{N} \mathcal{W}_i \right) = 0.
\]
\[
\int_{\partial B_i(\tilde{u}_i)} \left( \hat{\mathbf{u}}_c - \sum_{i=1}^{N} \mathcal{U}_i \right) \cdot ds = \int_{\partial B_i(\tilde{u}_i)} \left( \hat{\mathbf{w}}_c - \sum_{i=1}^{N} \mathcal{W}_i \right) \cdot ds = 0.
\]
Therefore, using the analysis of Lemma 2.1, we obtain
\[
\hat{\mathbf{u}}_c - \sum_{i=1}^{N} \mathcal{U}_i = \nabla u_c,
\]
\[
\hat{\mathbf{w}}_c - \sum_{i=1}^{N} \mathcal{W}_i = \nabla w_c,
\]
for some \( u_c, w_c \in H^1(\Omega_c) \). Also, because of (3.1) and
\[
\int_{\Omega_c} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \leq C_0 |\ln(\epsilon)|,
\]
we know that
\[
\int_{\Omega_c} \left( C |\nabla u_c|^2 + K |\nabla w_c|^2 + 2R(\nabla u_c \cdot \nabla w_c) \right) dx dy \leq C_0 |\ln(\epsilon)|.
\]
In turn, by Poincaré’s inequality, we have
\[
\|u_c\|_{H^1(\Omega_c)} + \|w_c\|_{H^1(\Omega_c)} \leq C |\ln(\epsilon)|.
\]
We can define a natural extension (see [8]) of \((u_c, w_c)\) from \(\Omega_c\) to \(\Omega\) as \((\hat{u}_c, \hat{w}_c)\) such that
\[
\|\hat{u}_c\|_{H^1(\Omega)} + \|\hat{w}_c\|_{H^1(\Omega)} \leq C |\ln(\epsilon)|.
\]
It is easy to see that up to extracting a subsequence,
\[
\left( \frac{\hat{u}_c}{|\ln(\epsilon)|^{1/2}}, \frac{\hat{w}_c}{|\ln(\epsilon)|^{1/2}} \right) \to (v_u, v_w),
\]
in weak-\(H^1(\Omega)\) for some \((v_u, v_w)\) in \(H^1(\Omega)\). On the other hand, note that \(\mathcal{U}_i, \mathcal{W}_i \notin L^2(\Omega)\), but \(\mathcal{U}_i, \mathcal{W}_i \in L^p(\Omega)\) for any \(1 \leq p < 2\), and also
\[
\int_{\Omega_c} \left( |\mathcal{U}_i|^2 + |\mathcal{W}_i|^2 \right) dx dy \leq C |\ln(\epsilon)|.
\]
Hence, we know that up to extracting a subsequence
\[
\left( \frac{1_{\Omega_c}}{|\ln(\epsilon)|^{1/2}}, \frac{1_{\Omega_c}}{|\ln(\epsilon)|^{1/2}} \right) \to (U^i, W^i),
\]
in weak-\(L^2(\Omega)\), for some \(U^i, W^i \in L^2(\Omega)\). Taking \(\phi, \psi \in C_0^\infty(\Omega)\), we have
\[
\int_{\Omega_c} \frac{\mathcal{U}_i \phi + \mathcal{W}_i \psi}{|\ln(\epsilon)|^{1/2}} dx dy \leq \frac{||\mathcal{U}_i||_{L^\infty}}{|\ln(\epsilon)|^{1/2}} + \frac{||\mathcal{W}_i||_{L^\infty}}{|\ln(\epsilon)|^{1/2}} \leq \frac{C}{|\ln(\epsilon)|^{1/2}} \to 0 \quad \text{as} \quad \epsilon \to 0.
\]
Therefore, we must have \(U^i = W^i = 0\), i.e.
\[
\left( \frac{1_{\Omega_c}}{|\ln(\epsilon)|^{1/2}}, \frac{1_{\Omega_c}}{|\ln(\epsilon)|^{1/2}} \right) \to (0, 0).
Thus define
\[ \hat{U}_\epsilon := \sum_{i=1}^{N} U_i + \nabla \hat{u}_\epsilon, \]
\[ \hat{M}_\epsilon := \sum_{i=1}^{N} W_i + \nabla \hat{w}_\epsilon. \]

such that \( \hat{U}_\epsilon = \hat{U}_\epsilon \) and \( \hat{M}_\epsilon = \hat{M}_\epsilon \) in \( \Omega \). In summary, we have shown that
\[ \left( \frac{1_{\Omega}, \hat{U}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{1_{\Omega}, \hat{M}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \rightarrow \left( \frac{1_{\Omega}, \nabla \hat{u}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{1_{\Omega}, \nabla \hat{w}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right) \quad \text{in weak-L}^2(\Omega). \]

\[ \Box \]

3.2. Zeroth-Order \( \Gamma \)-Convergence.

**Theorem 3.2.** (Zeroth-Order \( \Gamma \)-Convergence) Assume that (1.4) holds. Define the functional \( J^{(0)}_0 : L^2(\Omega) \times L^2(\Omega) \to [0, \infty) \) as
\[ J^{(0)}_0[\mathcal{U}, \mathcal{M}] := \begin{cases} \int \frac{1}{2} \left( C |\nabla u|^2 + K |\nabla v|^2 + R(\nabla u \cdot \nabla v) \right) + \sum_{i=1}^{N} \frac{C(b_i^u)^2 + K(b_i^v)^2 + 2R(b_i^u)(b_i^v)}{4\pi} \right) & \text{if } (\mathcal{U}, \mathcal{M}) = (\nabla u, \nabla v) \quad \text{for some } u, v \in H^1(\Omega), \\ \infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases} \]

Then
\begin{enumerate}
\item For any sequence of pairs \( (\mathcal{U}_n, \mathcal{M}_n) \in L^2(\Omega) \times L^2(\Omega) \) such that \( (\mathcal{U}_n, \mathcal{M}_n) \rightharpoonup (\mathcal{U}, \mathcal{M}) \) in weak-L\( ^2(\Omega) \),
\item There exists a sequence of pairs \( (\mathcal{U}_n, \mathcal{M}_n) \in L^2(\Omega) \times L^2(\Omega) \) such that \( (\mathcal{U}_n, \mathcal{M}_n) \rightharpoonup (\mathcal{U}, \mathcal{M}) \) in weak-L\( ^2(\Omega) \),
\end{enumerate}

which means
\[ J^{(0)}_0[\mathcal{U}_n, \mathcal{M}_n] \to J^{(0)}_0[\mathcal{U}, \mathcal{M}], \]
in the sense of \( \Gamma \)-convergence in weak-L\( ^2(\Omega) \).

**Proof.** We divide the proof into two steps:

Step 1: lim inf.

Assume that \( (\mathcal{U}_n, \mathcal{M}_n) \in H^1_0, \quad \left( \frac{\mathcal{U}_n}{|\ln(\epsilon)|^{1/2}}, \frac{\mathcal{M}_n}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (\mathcal{U}, \mathcal{M}) \) and \( J^{(0)}_0[\nabla u, \nabla v] \) is finite. Then due to weak convergence in \( L^2 \) and quadratic \( \mathcal{F} \), we know \( J^{(0)}_0[\mathcal{U}_n, \mathcal{M}_n] \leq C_0 |\ln(\epsilon)| \). Based on compactness and Theorem 3.1, we must have
\[ \left( \frac{1_{\Omega}, \mathcal{U}_n}{|\ln(\epsilon)|^{1/2}}, \frac{1_{\Omega}, \mathcal{M}_n}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (\nabla u, \nabla v), \]
for some \( u, v \in H^1(\Omega) \), i.e., we must have
\[ (\mathcal{U}, \mathcal{M}) = (\nabla u, \nabla v). \]

Based on
\[ \tilde{U}_\epsilon = \sum_{i=1}^{N} U_i + \nabla u, \quad \tilde{M}_\epsilon = \sum_{i=1}^{N} W_i + \nabla v, \]
and the fact that
\[ \left( \frac{U_i}{|\ln(\epsilon)|^{1/2}}, \frac{W_i}{|\ln(\epsilon)|^{1/2}} \right) \rightharpoonup (0, 0) \text{ in weak } - L^2(\Omega), \]
we deduce that
\[
\left( \frac{\nabla u_\varepsilon}{|\ln(\varepsilon)|^{1/2}}, \frac{\nabla w_\varepsilon}{|\ln(\varepsilon)|^{1/2}} \right) \to (\nabla v_u, \nabla v_w) \text{ in weak } L^2(\Omega).
\]

Hence, we obtain
\[
\left( \frac{u_\varepsilon}{|\ln(\varepsilon)|^{1/2}}, \frac{w_\varepsilon}{|\ln(\varepsilon)|^{1/2}} \right) \to (v_u, v_w) \text{ in weak } H^1(\Omega).
\]

For \( r > \varepsilon \), we write
\[
\frac{1}{|\ln(\varepsilon)|} \int_{\Omega_r} \frac{1}{2} \left( C |\varepsilon| + 2K |\varepsilon| + 2R(\varepsilon \cdot \nabla \varepsilon) \right) dx dy
\]
\[
= \frac{1}{|\ln(\varepsilon)|} \int_{\Omega_r} \frac{1}{2} \left( C |\varepsilon| + 2K |\varepsilon| + 2R(\varepsilon \cdot \nabla \varepsilon) \right) dx dy
\]
\[
+ \frac{1}{|\ln(\varepsilon)|} \sum_{i=1}^{N} \int_{B_r(d_i) \setminus B_r(d_i)} \frac{1}{2} \left( C |\varepsilon| + 2K |\varepsilon| + 2R(\varepsilon \cdot \nabla \varepsilon) \right) dx dy
\]
\[
+ \frac{1}{|\ln(\varepsilon)|} \sum_{i \neq j} \int_{B_r(d_i) \setminus B_r(d_j)} \frac{1}{2} \left( C |\varepsilon| + 2K |\varepsilon| + 2R(\varepsilon \cdot \nabla \varepsilon) \right) dx dy
\]
\[
:= I + II + III + IV + V + VI.
\]

By weak lower semi-continuity, we always have
\[
\liminf_{\varepsilon \to 0} I \geq \int_{\Omega_r} \frac{1}{2} \left( C |\nabla v_u| + 2K |\nabla v_w| + 2R(\nabla v_u \cdot \nabla v_w) \right) dx dy
\]
\[
\to \int_{\Omega} \frac{1}{2} \left( C |\nabla v_u| + 2K |\nabla v_w| + 2R(\nabla v_u \cdot \nabla v_w) \right) dx dy,
\]
as \( r \to 0 \). On the other hand, a direct computation based on explicit formula (2.2) and (2.3) reveals
\[
\lim_{\varepsilon \to 0} II = \sum_{i=1}^{N} \frac{C(b_{i}^u)^2 + 2K(b_{i}^w)^2 + 2R(b_{i}^u)(b_{i}^w)}{4\pi}.
\]

It is easy to see \( III \geq 0 \), which means
\[
\liminf_{\varepsilon \to 0} III \geq 0.
\]

Since \( i \neq j \) in \( IV \), then in the integral, at most one of \( \Psi_i \) or \( \Psi_j \) can contribute \( |\ln(\varepsilon)|^{1/2} \). A similar argument holds for \( \Psi_i \) and \( \Psi_j \). Hence, we have
\[
\liminf_{\varepsilon \to 0} IV = 0,
\]
and
\[
\liminf_{\varepsilon \to 0} V = 0.
\]

Since
\[
\begin{align*}
\nabla \cdot \Psi_i &= \nabla \cdot \Psi_i = 0 \quad \text{in } B_r(d_i) \setminus B_r(d_i), \\
\Psi_i \cdot n &= \Psi_i \cdot n = 0 \quad \text{on } \partial B_r(d_i),
\end{align*}
\]
we may integrate by parts to get
\[ \lim_{\epsilon \to 0} V I = 0. \]
We have shown that
\[ \liminf_{\epsilon \to 0} J^{(0)}_{\epsilon} [\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \geq J^{(0)}_0 [\nabla v_u, \nabla v_w]. \]
Similarly, the compactness and Theorem 3.1 imply that when \( J^{(0)}_0 [\nabla v_u, \nabla v_w] = \infty \), we must have \( J^{(0)}_{\epsilon_0} [\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \to \infty \).

Step 2: \( \limsup \).
The \( J^{(0)}_0 [\nabla v_u, \nabla v_w] = \infty \) case is trivial, we only consider the case when \( J^{(0)}_0 [\nabla v_u, \nabla v_w] \) is finite. Define
\[ (\hat{\mathcal{U}}_\epsilon, \hat{\mathcal{W}}_\epsilon) := \left( \frac{|\ln(\epsilon)|^{1/2} \nabla v_u + \sum_{i=1}^{N} \mathcal{Y}_i}{|\ln(\epsilon)|^{1/2}}, \frac{|\ln(\epsilon)|^{1/2} \nabla v_w + \sum_{i=1}^{N} \mathcal{Z}_i}{|\ln(\epsilon)|^{1/2}} \right). \]
We have
\[ \left[ \frac{1_{\Omega} \hat{\mathcal{U}}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{1_{\Omega} \hat{\mathcal{W}}_\epsilon}{|\ln(\epsilon)|^{1/2}} \right] \to [\nabla v_u, \nabla v_w] \text{ in weak } L^2(\Omega), \]
and
\[ \frac{1}{|\ln(\epsilon)|} \int_{\Omega} \frac{1}{2} \left( C |\hat{\mathcal{U}}_\epsilon|^2 + K |\hat{\mathcal{W}}_\epsilon|^2 + 2R(\hat{\mathcal{U}}_\epsilon \cdot \hat{\mathcal{W}}_\epsilon) \right) \, dx \, dy \]
\[ = \int_{\Omega} \frac{1}{2} \left( C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) \, dx \, dy \]
\[ + \frac{1}{|\ln(\epsilon)|} \sum_{i=1}^{N} \int_{\Omega} \frac{1}{2} \left( C |\mathcal{Y}_i|^2 + K |\mathcal{Z}_i|^2 + 2R(\mathcal{Y}_i \cdot \mathcal{Z}_i) \right) \, dx \, dy \]
\[ + \frac{1}{|\ln(\epsilon)|} \sum_{i \neq j} \int_{\Omega} \frac{1}{2} \left( C(\mathcal{Y}_i \cdot \mathcal{Z}_j) + K(\mathcal{Z}_i \cdot \mathcal{Y}_j) + 2R(\mathcal{Z}_i \cdot \mathcal{Y}_j) \right) \, dx \, dy \]
\[ + \frac{1}{|\ln(\epsilon)|^{1/2}} \sum_{i=1}^{N} \int_{\Omega} \frac{1}{2} \left( C(\nabla v_u \cdot \mathcal{Y}_i) + K(\nabla v_w \cdot \mathcal{Z}_i) + 2R(\nabla v_u \cdot \mathcal{Z}_i) + 2R(\nabla v_w \cdot \mathcal{Y}_i) \right) \, dx \, dy \]
\[ := I + II + III + IV. \]
Estimating it term by term, and using the techniques similar to those in Step 1, we have
\[ \limsup_{\epsilon \to 0} I \leq \int_{\Omega} \frac{1}{2} \left( C |\nabla v_u|^2 + K |\nabla v_w|^2 + 2R(\nabla v_u \cdot \nabla v_w) \right) \, dx \, dy, \]
\[ \limsup_{\epsilon \to 0} II \leq \sum_{i=1}^{N} \frac{C(b_u^i)^2 + K(b_w^i)^2 + 2R(b_u^i)(b_w^i)}{4\pi}, \]
\[ \lim_{\epsilon \to 0} III = 0, \]
\[ \lim_{\epsilon \to 0} IV = 0, \]
and conclude that
\[ \limsup_{\epsilon \to 0} J^{(0)}_{\epsilon} [\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] \leq J^{(0)}_0 [\mathcal{U}, \mathcal{W}]. \]

By Theorem 2.4 and the basis properties of \( \Gamma \)-convergence, we can naturally obtain an approximation of energy.
Corollary 3.3. Assume that (1.4) holds. We have

\[
\inf_{\mathcal{U}, \mathcal{W}} J_0^{(0)}[\mathcal{U}, \mathcal{W}] = \sum_{i=1}^{N} \frac{C(b_i^j)^2 + K(b_i^w)^2 + 2R(b_i^j)(b_i^w)}{4\pi}.
\]

Assume \((\mathcal{W}', \mathcal{W}'')\) is the minimizer of \(J_0^{(0)}\), then we have

\[
J_0^{(0)}[\mathcal{W}', \mathcal{W}''] = E_0 + o(1),
\]

where the rescaled leading-order energy is \(E_0 = \sum_{i=1}^{N} \frac{C(b_i^j)^2 + K(b_i^w)^2 + 2R(b_i^j)(b_i^w)}{4\pi}\).

3.3. First-Order \(\Gamma\)-Convergence. Since the leading order energy \(E_0\) only concerns with magnitude of the Burger's vectors and loses information about the dislocation position, we need more detailed analysis of convergence and selection process, which leads us to considering the first-order \(\Gamma\)-convergence.

Now we get rid of the rescaling \(\frac{1}{|\ln(\epsilon)|^{1/2}}\). Define the functional \(J_0^{(1)} : L^2(\Omega) \times L^2(\Omega) \to [0, \infty]\) as

\[
J_0^{(1)}[\mathcal{U}, \mathcal{W}] := \begin{cases} E_{\text{self}} + E_{\text{int}} + E_{\text{elastic}} & \text{if } (\tilde{\mathcal{U}}, \tilde{\mathcal{W}}) = \left(\nabla v_u + \sum_{i=1}^{N} \mathcal{Z}_i, \nabla v_w + \sum_{i=1}^{N} \mathcal{W}_i\right), \\
\infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega),
\end{cases}
\]

where

\[
E_{\text{self}} := \sum_{i=1}^{N} \int_{B_r(d_i)} \frac{1}{2} \left( C |\mathcal{Z}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{Z}_i \cdot \mathcal{W}_i) \right) \, dx dy
\]

\[
+ \sum_{i=1}^{N} \frac{(C(b_i^j)^2 + K(b_i^w)^2 + 2R(b_i^j)(b_i^w)) \ln(r)}{4\pi},
\]

\[
E_{\text{int}} := \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \int_{\Omega} \left( C(\mathcal{Z}_i \cdot \mathcal{Z}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{Z}_i \cdot \mathcal{W}_j) + R(\mathcal{Z}_j \cdot \mathcal{W}_i) \right),
\]

\[
E_{\text{elastic}} := J[\nabla v_u, \nabla v_w] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( v_u(C\mathcal{Z}_i + R\mathcal{W}_i) + v_w(K\mathcal{W}_i + R\mathcal{Z}_i) \right) \cdot n ds.
\]

Then

1. For any sequence of pairs \((\tilde{\mathcal{U}}, \tilde{\mathcal{W}})\) in \(L^2(\Omega) \times L^2(\Omega)\) such that \((\tilde{\mathcal{U}}, \tilde{\mathcal{W}}) \rightharpoonup (\mathcal{U}, \mathcal{W})\) in weak-\(L^2(\Omega)\), we have \(\liminf_{\epsilon \to 0} J_0^{(1)}[\mathcal{U}, \mathcal{W}] \geq J_0^{(1)}[\nabla v_u, \nabla v_w]\).

2. There exists a sequence of pairs \((\tilde{\mathcal{U}}, \tilde{\mathcal{W}})\) in \(L^2(\Omega) \times L^2(\Omega)\) such that \((\tilde{\mathcal{U}}, \tilde{\mathcal{W}}) \rightharpoonup (\mathcal{U}, \mathcal{W})\) in weak-\(L^2(\Omega)\), we have \(\limsup_{\epsilon \to 0} J_0^{(1)}[\mathcal{U}, \mathcal{W}] \leq J_0^{(1)}[\nabla v_u, \nabla v_w]\),

which means

\[
J_0^{(1)}[\mathcal{U}, \mathcal{W}] \to J_0^{(1)}[\nabla v_u, \nabla v_w],
\]

in the sense of \(\Gamma\)-convergence in weak-\(L^2(\Omega)\).
Proof. We naturally have
\[ J_{\epsilon}^{(1)} \left[ \tilde{u}_\epsilon, \tilde{w}_\epsilon \right] = \begin{cases} \int_{\Omega} \frac{1}{2} \left( C |\tilde{u}_\epsilon|^2 + K |\tilde{w}_\epsilon|^2 + 2R(\tilde{u}_\epsilon \cdot \tilde{w}_\epsilon) \right) dx dy - |\ln(\epsilon)| \sum_{i=1}^{N} \frac{C(b_i^\epsilon)^2 + K(b_i^\omega)^2 + 2R(b_i^\epsilon)(b_i^\omega)}{4\pi} & \text{if } (\tilde{u}_\epsilon, \tilde{w}_\epsilon) \in H_0^1(\Omega) \\ \infty & \text{otherwise in } L^2(\Omega) \times L^2(\Omega). \end{cases} \]

We first prove the lim inf part. Consider weakly convergent sequence
\[ (\tilde{u}_\epsilon, \tilde{w}_\epsilon) = 1_{\Omega} \left( \nabla u + \sum_{i=1}^{N} \mathcal{U}_i, \nabla w + \sum_{i=1}^{N} \mathcal{W}_i \right) \rightarrow (\tilde{u}, \tilde{w}) = \left( \nabla v_u + \sum_{i=1}^{N} \mathcal{U}_i, \nabla v_w + \sum_{i=1}^{N} \mathcal{W}_i \right). \]

Direct computation using (2.2) and (2.3) yields
\[ 1_{\Omega} \left( \sum_{i=1}^{N} \mathcal{U}_i, \sum_{i=1}^{N} \mathcal{W}_i \right) \rightarrow \left( \sum_{i=1}^{N} \mathcal{U}_i, \sum_{i=1}^{N} \mathcal{W}_i \right). \]

Naturally, we have
\[ (\nabla u, \nabla w) \rightarrow (\nabla v_u, \nabla v_w). \]

Hence, weak convergence yields boundedness \( \|\nabla u\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \leq C' \) for some constant \( C' \) independent of \( \epsilon \). We may decompose
\[
\begin{align*}
\int_{\Omega} \frac{1}{2} \left( C |\tilde{u}_\epsilon|^2 + K |\tilde{w}_\epsilon|^2 + 2R(\tilde{u}_\epsilon \cdot \tilde{w}_\epsilon) \right) dx dy - |\ln(\epsilon)| \sum_{i=1}^{N} \frac{C(b_i^\epsilon)^2 + K(b_i^\omega)^2 + 2R(b_i^\epsilon)(b_i^\omega)}{4\pi} \\
= \left( \sum_{i=1}^{N} \int_{\Omega} \frac{1}{2} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy - |\ln(\epsilon)| \sum_{i=1}^{N} \frac{C(b_i^\omega)^2 + 2R(b_i^\epsilon)(b_i^\omega)}{4\pi} \right) \\
+ \sum_{i \neq j} \int_{\Omega} \frac{1}{2} \left( C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_j \cdot \mathcal{W}_i) \right) dx dy \\
+ \int_{\Omega} \frac{1}{2} \left( C(\nabla u \cdot \nabla w) + 2R(\nabla u \cdot \nabla w) \right) dx dy \\
+ \sum_{i=1}^{N} \int_{\Omega} \frac{1}{2} \left( C(\nabla u \cdot \mathcal{U}_i) + K(\nabla w \cdot \mathcal{W}_i) + R(\nabla u \cdot \mathcal{W}_i) + R(\nabla w \cdot \mathcal{U}_i) \right) dx dy \\
:= I + II + III + IV.
\end{align*}
\]

Here the argument is similar to that in the proof of 0th-order \( \Gamma \)-convergence, so we only describe the main strategy. For \( I \), decompose \( \Omega_\epsilon = \Omega_r \cup (\Omega_r \setminus \Omega_r) \) for some \( r > \epsilon \), i.e.
\[
I = \sum_{i=1}^{N} \int_{\Omega_r \setminus \Omega_r} \frac{1}{2} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy + \sum_{i=1}^{N} \int_{\Omega_r \setminus \Omega_r} \frac{1}{2} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy
\]

- \( |\ln(\epsilon)| \sum_{i=1}^{N} \frac{C(b_i^\omega)^2 + 2R(b_i^\epsilon)(b_i^\omega)}{4\pi} \)

Direct computation using (2.2) and (2.3) reveals that
\[
\lim_{\epsilon \to 0} \left( \sum_{i=1}^{N} \int_{\Omega_r \setminus \Omega_r} \frac{1}{2} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) dx dy \right) = |\ln(\epsilon)| \sum_{i=1}^{N} \frac{C(b_i^\omega)^2 + 2R(b_i^\epsilon)(b_i^\omega)}{4\pi} + \sum_{i=1}^{N} \frac{C(b_i^\omega)^2 + 2R(b_i^\epsilon)(b_i^\omega)}{4\pi} \ln(r).
\]

Hence, we know
\[
\lim_{\epsilon \to 0} I = E_{\text{self}}.
\]
Similarly, a direct computation using (2.2) and (2.3) shows that

$$\lim_{\epsilon \to 0} II = E_{\text{int}}.$$ 

Based on weak convergence \((\nabla u_\epsilon, \nabla w_\epsilon) \to (\nabla v_u, \nabla v_w)\) and weak lower semi-continuity, we know that

$$\liminf_{\epsilon \to 0} III \geq J[\nabla v_u, \nabla v_w].$$

Finally, after integrating by parts, by weak convergence and the equations (2.1) satisfied by \((\mathcal{U}_i, \mathcal{W}_i)\), we know that

$$\lim_{\epsilon \to 0} IV = \sum_{i=1}^{N} \int_{\partial \Omega} \left( v_u (C \mathcal{U}_i + R \mathcal{W}_i) + v_w (K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot nds.$$

Therefore,

$$\liminf_{\epsilon \to 0} (III + IV) \geq E_{\text{elastic}}.$$ 

To summarize, this concludes the proof of the lim inf part.

For the lim sup part, consider the sequence

\[
(\tilde{\mathcal{U}}_\epsilon, \tilde{\mathcal{W}}_\epsilon) = \mathbf{1}_{\Omega_0} \left( \nabla v_u + \sum_{i=1}^{N} \mathcal{U}_i, \nabla v_w + \sum_{i=1}^{N} \mathcal{W}_i \right),
\]

and we have

\[
(\tilde{\mathcal{U}}_\epsilon, \tilde{\mathcal{W}}_\epsilon) \to \left( \nabla v_u + \sum_{i=1}^{N} \mathcal{U}_i, \nabla v_w + \sum_{i=1}^{N} \mathcal{W}_i \right).
\]

Therefore, a direct computation using explicit formula (2.2) and (2.3) justifies the result, and thus the \(\Gamma\)-convergence holds.\(\square\)

Similar to the analysis of Corollary 3.3, Theorem 2.4 and the basic property of \(\Gamma\)-convergence justify a more detailed energy approximation.

**Corollary 3.5.** Assume that (1.4) holds. We have

$$\inf_{\tilde{\mathcal{U}}, \tilde{\mathcal{W}}} J^{(1)}_0[\tilde{\mathcal{U}}, \tilde{\mathcal{W}}] = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}},$$

where

\[
F_{\text{self}} : = \sum_{i=1}^{N} \int_{\Omega \setminus B_r(d_i)} \frac{1}{2} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2 R (\mathcal{U}_i \cdot \mathcal{W}_i) \right) \, dx \, dy \tag{3.3}
\]

\[
+ \sum_{i=1}^{N} \left( C (b_{w_i})^2 + K (b_{w_i})^2 + 2 R (b_{w_i}^i b_{w_i}^j) \right) \ln(r),
\]

\[
F_{\text{int}} : = \sum_{i=1}^{N-1} \sum_{j=1}^{N} \int_{\Omega} \left( C (\mathcal{U}_i \cdot \mathcal{U}_j) + K (\mathcal{W}_i \cdot \mathcal{W}_j) + R (\mathcal{U}_i \cdot \mathcal{W}_j) + R (\mathcal{W}_i \cdot \mathcal{W}_j) \right),
\]

\[
F_{\text{elastic}} : = J[\nabla u_0, \nabla w_0] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( v_u (C \mathcal{U}_i + R \mathcal{W}_i) + v_w (K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot nds,
\]

in which \((u_0, w_0)\) is the minimizer of

\[
I[v_u, v_w] = J[\nabla v_u, \nabla v_w] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( v_u (C \mathcal{U}_i + R \mathcal{W}_i) + v_w (K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot nds.
\]

Assume \((\mathcal{U}_\epsilon', \mathcal{W}_\epsilon') \in H^1_0\) is the minimizer of \(J^{(1)}_\epsilon\), then we have

$$J^{(1)}_\epsilon[\mathcal{U}_\epsilon', \mathcal{W}_\epsilon'] = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}} + o(1).$$
Remark 3.6. The existence and uniqueness of minimizer \((u_0, w_0)\) can be proved using a similar argument as in Section 2.3 and 2.4.

Remark 3.7. We can show that \(F_{\text{self}}\) is independent of the choice of \(r\). Assume \(r' < r\), say \(r' < r\), then we have

\[
\sum_{i=1}^{N} \int_{\Omega \setminus B_{r'}(d_i)} \frac{1}{2} \left( C |\mathcal{W}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) \, dx \, dy \\
+ \sum_{i=1}^{N} \int_{B_{r'}(d_i) \setminus B_{r'}(d_i)} \frac{1}{4\pi} \ln \left( \frac{r}{r'} \right) \\
= \sum_{i=1}^{N} \int_{\Omega \setminus B_{r'}(d_i)} \frac{1}{2} \left( C |\mathcal{W}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) \, dx \, dy \\
+ \sum_{i=1}^{N} \int_{B_{r'}(d_i) \setminus B_{r'}(d_i)} \frac{1}{4\pi} \ln \left( \frac{r'}{r} \right) \\
+ \sum_{i=1}^{N} \left( C(b_u)^2 + K(b_w)^2 + 2R(b_u')(b_w') \right) \frac{1}{4\pi} \ln (r') \\
= \sum_{i=1}^{N} \int_{\Omega \setminus B_{r'}(d_i)} \frac{1}{2} \left( C |\mathcal{W}_i|^2 + K |\mathcal{W}_i|^2 + 2R(\mathcal{U}_i \cdot \mathcal{W}_i) \right) \, dx \, dy \\
+ \sum_{i=1}^{N} \left( C(b_u)^2 + K(b_w)^2 + 2R(b_u')(b_w') \right) \frac{1}{4\pi} \ln (r). 
\]

Hence, choosing \(r'\) or \(r\) gives exactly the same \(F_{\text{self}}\).

3.4. Minimizer and Energy Structure. Combining Corollary 3.3 and Corollary 3.5, we can describe the structure of minimizer and energy.

Theorem 3.8. Assume that (1.4) holds. The problem (1.6) admits a unique solution

\[
\mathcal{U}_e = \sum_{i=1}^{N} \mathcal{U}_i + \nabla u_e, \quad \mathcal{W}_e = \sum_{i=1}^{N} \mathcal{W}_i + \nabla w_e, 
\]

where

\[
\mathcal{U}_i = \frac{b_u}{2\pi} \frac{1}{(x-x_i)^2 + (y-y_i)^2} \left( - (y-y_i), (x-x_i) \right), \\
\mathcal{W}_i = \frac{b_u}{2\pi} \frac{1}{(x-x_i)^2 + (y-y_i)^2} \left( - (y-y_i), (x-x_i) \right), 
\]

and \((u_e, w_e)\) is the unique minimizer of

\[
I_e[u_e, w_e] : = J_e[\nabla u_e, \nabla w_e] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( u_e (C\mathcal{U}_i + R\mathcal{W}_i) + w_e (K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot nds \\
- \sum_{i=1}^{N} \sum_{j \neq i} \int_{\partial B_{r}(x_i, y_i)} \left( u_e (C\mathcal{U}_j + R\mathcal{W}_j) + w_e (K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot nds, 
\]

\[
I_e[u_e, w_e] : = J_e[\nabla u_e, \nabla w_e] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( u_e (C\mathcal{U}_i + R\mathcal{W}_i) + w_e (K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot nds \\
- \sum_{i=1}^{N} \sum_{j \neq i} \int_{\partial B_{r}(x_i, y_i)} \left( u_e (C\mathcal{U}_j + R\mathcal{W}_j) + w_e (K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot nds, 
\]

\[
I_e[u_e, w_e] : = J_e[\nabla u_e, \nabla w_e] + \sum_{i=1}^{N} \int_{\partial \Omega} \left( u_e (C\mathcal{U}_i + R\mathcal{W}_i) + w_e (K\mathcal{W}_i + R\mathcal{U}_i) \right) \cdot nds \\
- \sum_{i=1}^{N} \sum_{j \neq i} \int_{\partial B_{r}(x_i, y_i)} \left( u_e (C\mathcal{U}_j + R\mathcal{W}_j) + w_e (K\mathcal{W}_j + R\mathcal{U}_j) \right) \cdot nds, 
\]
Theorem 3.9. Assume that (1.4) holds. We have

$$J_0[u_0, w_0] = J[\nabla u_0, \nabla w_0] + \sum_{i=1}^N \int_{\partial \Omega} \left(u_0(C\mathcal{U}_i + R\mathcal{W}_i) + w_0(K\mathcal{U}_i + R\mathcal{W}_i)\right) \cdot n ds,$$

subject to $$\int_B u_0 dx dy = 0$$ and $$\int_B w_0 dx dy = 0$$ for some ball $$B \subset \Omega, \text{ with } n$$ the outward unit normal vector on $$\partial \Omega$$.

Furthermore, $$(\mathcal{U}, \mathcal{W})$$ converges in weak-$$L^2(\Omega)$$ as $$\epsilon \to 0$$ to $$(\mathcal{U}_0, \mathcal{W}_0)$$ where

$$\mathcal{U}_0 = \sum_{i=1}^N \mathcal{U}_i + \nabla u_0, \quad \mathcal{W}_0 = \sum_{i=1}^N \mathcal{W}_i + \nabla w_0.$$

and $$[u_0, w_0]$$ is the unique minimizer of

$$I_0[u_0, w_0] = J[\nabla u_0, \nabla w_0] + \sum_{i=1}^N \int_{\partial \Omega} \left(u_0(C\mathcal{U}_i + R\mathcal{W}_i) + w_0(K\mathcal{U}_i + R\mathcal{W}_i)\right) \cdot n ds,$$

subject to $$\int_B u_0 dx dy = 0$$ and $$\int_B w_0 dx dy = 0$$ for some ball $$B \subset \Omega$$.

Proof. The existence and uniqueness of minimizer have been shown in Theorem 2.4. $$\Gamma$$-convergence naturally yields that minimizer of $$J_0^{(1)}$$ goes to minimizer of $$J^{(1)}$$. Hence, this result is obvious. \(\square\)

**Theorem 3.9.** Assume that (1.4) holds. We have

$$J_{\epsilon}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = \int_{\Omega_\epsilon} \tilde{J}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] dx dy = E_0 \ln \left(\frac{1}{\epsilon}\right) + F + o(1),$$

where the core energy $$E_0$$ is defined in (3.2) and the renormalized energy $$F = F_{\text{self}} + F_{\text{int}} + F_{\text{elastic}}$$ is defined in (3.3).

Proof. We directly compute

$$J_{\epsilon}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = |\ln(\epsilon)| J_{\epsilon}^{(0)} \left[\frac{\mathcal{U}_\epsilon}{|\ln(\epsilon)|^{1/2}}, \frac{\mathcal{W}_\epsilon}{|\ln(\epsilon)|^{1/2}}\right] = E_0 \ln \left(\frac{1}{\epsilon}\right) + J_{\epsilon}^{(1)}[\mathcal{U}_\epsilon, \mathcal{W}_\epsilon] = E_0 \ln \left(\frac{1}{\epsilon}\right) + F + o(1).$$

\(\square\)

4. Application of Renormalized Energy

4.1. Interaction between Dislocations. In this section, we will prove that the energy related to interaction between dislocation $$F_{\text{int}}$$ obeys the inverse logarithmic law of the distance between two dislocations.

**Theorem 4.1.** Assume that (1.4) holds. We have

$$F_{\text{int}} = \sum_{i=1}^{N-1} \sum_{j=i+1}^N \frac{C b_i^j b_i^j + K b_i^j b_i^j + R b_i^j b_i^j + R b_i^j b_i^j}{2\pi} \ln \left(\frac{1}{|\vec{d}_i - \vec{d}_j|}\right) + O(1).$$

Proof. Since

$$F_{\text{int}} = \sum_{i=1}^{N-1} \sum_{j=i+1}^N \int_{\Omega} \left(C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_j \cdot \mathcal{W}_i)\right),$$

let $$\vec{d}_i, \vec{d}_j \in \Omega$$ and let $$\gamma$$ be a segment of line that connects $$\vec{d}_j$$ to $$\partial \Omega$$ and is parallel to $$\vec{d}_i - \vec{d}_j$$. We rewrite

$$\gamma = \{\vec{d} \in \Omega : \vec{d} = \vec{d}_j + s(\vec{d}_j - \vec{d}_i) \text{ for } s \in [0, \bar{s}]\}$$

where $$\bar{s}$$ depends on the distance between $$\vec{d}_i, \vec{d}_j$$ and $$\partial \Omega$$. Let

$$\vec{m} = \left(\frac{\vec{d}_j - \vec{d}_i}{|\vec{d}_j - \vec{d}_i|}\right)^\perp$$
Lemma 4.2. Lemmas proved in [8]. In this section, we will show its relation with the renormalized energy. Here we first present three

\[ \int_{\Omega} \left( C(\mathcal{U}_i \cdot \mathcal{V}_j) + K(\mathcal{V}_i \cdot \mathcal{V}_j) + R(\mathcal{U}_i \cdot \mathcal{V}_j) + R(\mathcal{V}_i \cdot \mathcal{V}_j) \right) \]
\[ = \int_{\Omega} \left( C(\mathcal{U}_i \cdot \nabla U) + K(\mathcal{V}_i \cdot \nabla W) + R(\mathcal{U}_i \cdot \nabla W) + R(\mathcal{V}_i \cdot \nabla U) \right) \]
\[ = \int_{\partial \Omega} \left( C\mathcal{U}_i(U \cdot n) + K\mathcal{V}_i(W \cdot n) + R\mathcal{U}_i(W \cdot n) + R\mathcal{V}_i(U \cdot n) \right) ds \]
\[ - \int_{\gamma} \left( C\mathcal{U}_i[U] + K\mathcal{V}_i[W] + R\mathcal{U}_i[W] + R\mathcal{V}_i[U] \right) \cdot \vec{m} ds. \]

The first integral is bounded since all quantities are uniformly bounded on \( \partial \Omega \). For the second integral, we estimate

\[ - \int_{\gamma} \left( C\mathcal{U}_i[U] + K\mathcal{V}_i[W] + R\mathcal{U}_i[W] + R\mathcal{V}_i[U] \right) \cdot \vec{m} ds \]
\[ = \int_{\gamma} \left( C\mathcal{U}_i b^i_u + K\mathcal{V}_i b^i_w + R\mathcal{U}_i b^i_w + R\mathcal{V}_i b^i_u \right) \cdot \vec{m} ds. \]

By explicit formula (2.2) and (2.3), we know

\[ \mathcal{U}_i(\hat{d}) = -\frac{b^i_u}{2\pi} \frac{\vec{m}}{|\hat{d} - \hat{d}_i|}, \quad \mathcal{V}_i(\hat{d}) = -\frac{b^i_w}{2\pi} \frac{\vec{m}}{|\hat{d} - \hat{d}_i|}. \]

Hence, we have

\[ \int_{\gamma} \left( C\mathcal{U}_i b^i_u + K\mathcal{V}_i b^i_w + R\mathcal{U}_i b^i_w + R\mathcal{V}_i b^i_u \right) \cdot \vec{m} ds \]
\[ = \int_{\gamma} \frac{Cb^i_u b^i_u + Kb^i_w b^i_w + Rb^i_w b^i_w + Rb^i_u b^i_u}{2\pi} \frac{1}{|\hat{d} - \hat{d}_i|} ds \]
\[ = \frac{Cb^i_u b^i_u + Kb^i_w b^i_w + Rb^i_w b^i_w + Rb^i_u b^i_u}{2\pi} \int_0^s \frac{1}{|\hat{d}_i - \hat{d}_j| + s} ds \]
\[ = \frac{Cb^i_u b^i_u + Kb^i_w b^i_w + Rb^i_w b^i_w + Rb^i_u b^i_u}{2\pi} \left( \ln \left( \frac{1}{|\hat{d}_i - \hat{d}_j|} \right) + \ln \left( |\hat{d}_i - \hat{d}_j| + s \right) \right). \]

The result follows since we always have \( \bar{s} > 0 \). \( \Box \)

4.2. Peach-Köhler Force. The Peach-Köhler Force acting on the dislocation \( \vec{d}_k \) is given by \( \nabla_{\vec{d}_k} F \) (see [10]). In this section, we will show its relation with the renormalized energy. Here we first present three lemmas proved in [8].

Lemma 4.2. Define

\[ D_k^\theta f(\vec{d}) = \left. \frac{d}{d\theta} f(\vec{d}; \vec{d}_1, \vec{d}_2, \cdots, \vec{d}_k + \theta \vec{V}, \cdots, \vec{d}_N) \right|_{\theta = 0}. \]
Then we have
\[
D_k^\nu \mathcal{U}_k = 0 \quad \text{for} \quad k \neq i, \\
D_k^\nu \mathcal{W}_k = 0 \quad \text{for} \quad k \neq i, \\
D_k^\nu \mathcal{U}_k = -D_k^\nu \mathcal{W}_k \cdot \vec{V} = -\nabla (\mathcal{U}_k \cdot \vec{V}), \\
D_k^\nu \mathcal{W}_k = -D_k^\nu \mathcal{U}_k \cdot \vec{V} = -\nabla (\mathcal{W}_k \cdot \vec{V}), \\
D_k^\nu \mathcal{U}_0 = \nabla U = \nabla (D_k^\nu u_0 - \mathcal{U}_k \cdot \vec{V}), \\
D_k^\nu \mathcal{W}_0 = \nabla W = \nabla (D_k^\nu w_0 - \mathcal{W}_k \cdot \vec{V})
\]
where \( D \) is the derivative with respect to \( \vec{d} \).

**Lemma 4.3.** We have
\[
\frac{d}{d\theta} \int_{B_r(\vec{d}_0 + \theta \vec{V})} f(\vec{d}, \theta) dxdy \bigg|_{\theta = 0} = \int_{B_r(\vec{d}_0)} D_\theta f(\vec{d}, 0) dxdy
\]
\[
= \int_{B_r(\vec{d}_0)} \partial_\theta f(\vec{d}, 0) dxdy + \int_{\partial B_r(\vec{d}_0)} f(\vec{d}, 0) \vec{V} \cdot nds,
\]
\[
\frac{d}{d\theta} \int_{\partial B_r(\vec{d}_0 + \theta \vec{V})} g(\vec{d}, \theta) ds \bigg|_{\theta = 0} = \int_{\partial B_r(\vec{d}_0)} D_\theta g(\vec{d}, 0) ds,
\]
\[
\frac{d}{d\theta} \int_{\Omega \setminus B_r(\vec{d}_0 + \theta \vec{V})} r(\vec{d}, \theta) dxdy \bigg|_{\theta = 0} = \int_{\Omega \setminus B_r(\vec{d}_0)} \partial_\theta r(\vec{d}, 0) dxdy - \int_{\partial B_r(\vec{d}_0)} r(\vec{d}, 0) \vec{V} \cdot nds,
\]
where \( D_\theta = \partial_\theta + \vec{V} \cdot \nabla \).

**Lemma 4.4.** We have
\[
D_\theta \mathcal{U}_i(\vec{d}; \vec{d}_i + \theta \vec{V}) = 0,
\]
for any \( \vec{V} \).

Now we can prove the main result.

**Theorem 4.5.** Assume that (1.4) holds. The Peach-Köhler force acting at \( \vec{d}_k \) is given by
\[
\nabla \mathcal{F}_k = -\int_{\partial B_r(\vec{d}_k)} \left( \hat{S}[\mathcal{U}_0, \mathcal{W}_0] - (C |\mathcal{U}_0| + K |\mathcal{W}_0| + R|\mathcal{U}_0| \otimes \mathcal{W}_0 + R|\mathcal{W}_0| \otimes \mathcal{U}_0) \right) \cdot nds,
\]
for \( r < \frac{1}{2} \min_k (\text{dist}(\vec{d}_k, \partial \Omega)) \).

**Proof.** We decompose the renormalized energy
\[
F(\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_N) = G(\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_N) + H(\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_N),
\]
where
\[
G(\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_N) := \int_{\Omega} \frac{1}{2} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R|\mathcal{U}_i| \otimes \mathcal{W}_i \right) dxdy,
\]
\[
H(\vec{d}_1, \vec{d}_2, \ldots, \vec{d}_N) := \sum_{i=1}^{N} \sum_{m \neq i} \int_{B_r(\vec{d}_m)} \frac{1}{2} \left( C |\mathcal{U}_i|^2 + K |\mathcal{W}_i|^2 + 2R|\mathcal{U}_i| \otimes \mathcal{W}_i \right) dxdy
\]
\[
+ \sum_{m=1}^{N} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \int_{B_r(\vec{d}_m)} \left( C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_i \cdot \mathcal{U}_j) \right) dxdy
\]
\[
+ \sum_{m=1}^{N} \int_{B_r(\vec{d}_m)} \frac{1}{2} \left( C |\nabla u_0|^2 + K |\nabla w_0|^2 + 2R|\nabla u_0| \cdot \nabla w_0 \right) dxdy
\]
\[
+ \sum_{m=1}^{N} \sum_{i=1}^{N} \int_{\partial B_r(\vec{d}_m)} \left( u_0(C \mathcal{U}_i + R \mathcal{W}_i) + w_0(K \mathcal{W}_i + R \mathcal{U}_i) \right) \cdot nds,
\]
with
\[ D^V_k F = D^V_k G + D^V_k H. \]

We divide the proof into several steps:

Step 1: Estimate of \( D^V_k G \).

We write
\[
I : = D^V_k \left( \int \frac{1}{2} \left( C \| \mathcal{V}_0 \|^2 + K \| \mathcal{W}_0 \|^2 + 2R(\mathcal{V}_0 \cdot \mathcal{W}_0) \right) \, dx \, dy \right)
\]
\[
= \int_{\Omega} \left( C \mathcal{V}_0 \cdot D^V_k \mathcal{V}_0 + K \mathcal{W}_0 \cdot D^V_k \mathcal{W}_0 + R \mathcal{V}_0 \cdot D^V_k \mathcal{W}_0 + R \mathcal{W}_0 \cdot D^V_k \mathcal{V}_0 \right) \, dx \, dy
\]
\[
- \int_{\partial B_i(d_i)} \frac{1}{2} \left( C \| \mathcal{V}_0 \|^2 + K \| \mathcal{W}_0 \|^2 + 2R(\mathcal{V}_0 \cdot \mathcal{W}_0) \right) \, \mathcal{V} \cdot nds.
\]

Hence, by the equations (2.1), we have
\[
\int_{\Omega} \left( C \mathcal{V}_0 \cdot D^V_k \mathcal{V}_0 + K \mathcal{W}_0 \cdot D^V_k \mathcal{W}_0 + R \mathcal{V}_0 \cdot D^V_k \mathcal{W}_0 + R \mathcal{W}_0 \cdot D^V_k \mathcal{V}_0 \right) \, dx \, dy
\]
\[
= \int_{\Omega} \left( C \mathcal{V}_0 \cdot \nabla(D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) + K \mathcal{W}_0 \cdot \nabla(D^V_k w_0 - \mathcal{W}_k \cdot \mathcal{V}) + R \mathcal{V}_0 \cdot \nabla(D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) \right) \, dx \, dy
\]
\[
= - \sum_{j=1}^{N} \int_{\partial B_i(d_i)} \left( C \mathcal{V}_0 \cdot (D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) \cdot n + K \mathcal{W}_0 \cdot (D^V_k w_0 - \mathcal{W}_k \cdot \mathcal{V}) \cdot n + R \mathcal{V}_0 \cdot (D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) \cdot n \right) \, ds
\]
\[
- \int_{\partial B_i(d_i)} \frac{1}{2} \left( C \| \mathcal{V}_0 \|^2 + K \| \mathcal{W}_0 \|^2 + 2R(\mathcal{V}_0 \cdot \mathcal{W}_0) \right) \, \mathcal{V} \cdot nds
\]
\[
= - \sum_{j=1}^{N} \int_{\partial B_i(d_i)} \left( C \mathcal{V}_0 \cdot (D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) \cdot n + K \mathcal{W}_0 \cdot (D^V_k w_0 - \mathcal{W}_k \cdot \mathcal{V}) \cdot n + R \mathcal{V}_0 \cdot (D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) \cdot n \right) \, ds
\]
\[
- \int_{\partial B_i(d_i)} \left( C \mathcal{V}_0 \cdot (D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) \cdot n + K \mathcal{W}_0 \cdot (D^V_k w_0 - \mathcal{W}_k \cdot \mathcal{V}) \cdot n + R \mathcal{V}_0 \cdot (D^V_k u_0 - \mathcal{V}_k \cdot \mathcal{V}) \cdot n \right) \, ds
\]
\[
= I_1 + I_2 + I_3.
\]
In above estimates, $I_1$ is the desired term, so we only focus on $I_2$ and $I_3$. We need to cancel

$$I_2 = - \sum_{j \neq k} \int_{\partial B_{\epsilon}(d_j)} \left( C \| \nabla \varphi \| \cdot (D_k^V u_0 - \nabla_k \cdot \vec{V}) \cdot n + K \| \nabla \varphi \| - (D_k^V u_0 - \nabla_k \cdot \vec{V}) \right) \cdot n
+ R \| \nabla \varphi \| \cdot (D_k^V u_0 - \nabla_k \cdot \vec{V}) \cdot n ds,$$

and

$$I_3 = - \int_{\partial B_{\epsilon}(d_k)} \left( C \| \nabla \phi \| \cdot (D_0 u_0 + \sum_{j \neq k} \nabla_j \cdot \vec{V}) \cdot n + K \| \nabla \phi \| \cdot (D_0 u_0 + \sum_{j \neq k} \nabla_j \cdot \vec{V}) \cdot n
+ R \| \nabla \phi \| \cdot (D_0 u_0 + \sum_{j \neq k} \nabla_j \cdot \vec{V}) \cdot n
+ R \| \nabla \phi \| \cdot (D_0 u_0 + \sum_{j \neq k} \nabla_j \cdot \vec{V}) \cdot n ds.$$

**Step 2: Estimate of $D_k^V H$ - First Term.**

We directly write

$$II : = D_k^V \left( \sum_{i=1}^{N} \sum_{m \neq i} \int_{B_{\epsilon}(d_m)} \frac{1}{2} \left( C \| \nabla \iota \| + K \| \nabla \iota \| + 2R(\nabla \iota \cdot \nabla \iota) \right) dx dy \right)
= D_k^V \left( \sum_{m \neq k} \int_{B_{\epsilon}(d_k)} \frac{1}{2} \left( C \| \nabla \iota \| + K \| \nabla \iota \| + 2R(\nabla \iota \cdot \nabla \iota) \right) dx dy \right)
+ D_k^V \left( \sum_{m \neq k} \sum_{m \neq l} \int_{B_{\epsilon}(d_m)} \frac{1}{2} \left( C \| \nabla \iota \| + K \| \nabla \iota \| + 2R(\nabla \iota \cdot \nabla \iota) \right) dx dy \right)
= II_1 + II_2.$$

In $II_1$, we know each $D_0 \nabla \iota = D_0 \iota = 0$ since $m \neq k$, then we have

$$II_1 = \sum_{m \neq k} \int_{\partial B_{\epsilon}(d_k)} \left( C \| \nabla \iota \| \cdot \nabla(\nabla \iota \cdot \vec{V}) + K \| \iota \| \cdot \nabla(\iota \cdot \vec{V})
+ R \| \nabla \iota \| \cdot \nabla(\iota \cdot \vec{V}) + R \| \iota \| \cdot \nabla(\iota \cdot \vec{V}) \right) dx dy
= \sum_{m \neq k} \int_{\partial B_{\epsilon}(d_k)} \left( C \| \nabla \iota \| \cdot \iota(\nabla \iota \cdot \vec{V}) + K \| \iota \| \cdot \iota(\iota \cdot \vec{V})
+ R \| \nabla \iota \| \cdot \iota(\iota \cdot \vec{V}) + R \| \iota \| \cdot \iota(\iota \cdot \vec{V}) \right) ds.$$
Also, since the domain and functions do not move for $i \neq k$, we have

$$II_2 = D_k^V \left( \sum_{m \neq k, i \neq j} \int_{B_r(\delta_m)} \frac{1}{2} \left( C|\mathcal{W}_k|^2 + K|\mathcal{W}_k|^2 + 2R(\mathcal{W}_k \cdot \mathcal{W}_k) \right) dx dy \right)$$

$$= - \sum_{m \neq k} \int_{\partial B_r(\delta_m)} \left( C\mathcal{W}_k \cdot \nabla(\mathcal{W}_k \cdot \mathcal{V}) + K\mathcal{W}_k \cdot \nabla(\mathcal{W}_k \cdot \mathcal{V}) \right) ds$$

Step 3: Estimate of $D_k^V H$ - Second Term.

We directly decompose

$$III : = D_k^V \left( \sum_{i < j} \int_{B_r(\delta_k)} \left( C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_j \cdot \mathcal{U}_i) \right) dx dy \right)$$

$$+ D_k^V \left( \sum \sum \sum_{m \neq k, i < j} \int_{B_r(\delta_m)} \left( C(\mathcal{U}_i \cdot \mathcal{U}_j) + K(\mathcal{W}_i \cdot \mathcal{W}_j) + R(\mathcal{U}_i \cdot \mathcal{W}_j) + R(\mathcal{W}_j \cdot \mathcal{U}_i) \right) dx dy \right)$$

$$= III_1 + III_2.$$

Then we have

$$III_1 = \sum \sum \sum_{i \neq k, j \neq i} \int_{B_r(\delta_k)} \left( C\mathcal{U}_i \cdot \nabla(\mathcal{U}_j \cdot \mathcal{V}) + K\mathcal{W}_i \cdot \nabla(\mathcal{W}_j \cdot \mathcal{V}) \right) dx dy$$

$$+ R\mathcal{U}_i \cdot \nabla(\mathcal{W}_j \cdot \mathcal{V}) + R\mathcal{W}_i \cdot \nabla(\mathcal{U}_j \cdot \mathcal{V}) ds.$$

$$III_2 = - \sum \sum \sum_{m \neq k, i \neq j} \int_{B_r(\delta_m)} \left( C\mathcal{U}_i \cdot \nabla(\mathcal{U}_k \cdot \mathcal{V}) + K\mathcal{W}_i \cdot \nabla(\mathcal{W}_k \cdot \mathcal{V}) \right) dx dy$$

$$+ R\mathcal{U}_i \cdot \nabla(\mathcal{W}_k \cdot \mathcal{V}) + R\mathcal{W}_i \cdot \nabla(\mathcal{U}_k \cdot \mathcal{V}) ds.$$

Step 4: Estimate of $D_k^V H$ - Third Term.
We directly decompose

\[ IV : = D^V_k \left( \int_{B_0(\bar{d}_0)} \frac{1}{2} \left( C |\nabla u_0|^2 + K |\nabla w_0|^2 + 2R(\nabla u_0 \cdot \nabla w_0) \right) dxdy \right) + D^V_k \left( \sum_{m \neq k} \int_{B_0(\bar{d}_m)} \frac{1}{2} \left( C |\nabla u_0|^2 + K |\nabla w_0|^2 + 2R(\nabla u_0 \cdot \nabla w_0) \right) dxdy \right) = IV_1 + IV_2. \]

By integrating by parts, we know

\[ IV_1 = \int_{\partial B_0(\bar{d}_0)} \left( C \nabla u_0 \cdot n(D^V_k u_0 + \nabla w_0 \cdot \nabla \hat{V}) + K \nabla w_0 \cdot n(D^V_k w_0 + \nabla w_0 \cdot \nabla \hat{V}) + R \nabla w_0 \cdot n(D^V_k w_0 + \nabla w_0 \cdot \nabla \hat{V}) + R \nabla w_0 \cdot n(D^V_k u_0 + \nabla u_0 \cdot \nabla \hat{V}) \right) ds. \]

Similarly, we have

\[ IV_2 = \sum_{m \neq k} \int_{\partial B_0(\bar{d}_m)} \left( CD^V_k u_0(n \cdot n) + KD^V_k w_0(n \cdot n) + RD^V_k w_0(n \cdot n) + RD^V_k u_0(n \cdot n) \right) ds. \]

Step 5: Estimate of $D^V_k H$ - Fourth Term.

We directly decompose

\[ V : = D^V_k \left( \sum_{i=1}^N \int_{\partial B_i(\bar{d}_i)} \left( u_0(C \mathcal{W}_i + R \mathcal{W}_i) + w_0(K \mathcal{W}_i + R \mathcal{W}_i) \right) ds \right) + D^V_k \left( \sum_{m \neq k} \sum_{i=1}^N \int_{\partial B_i(\bar{d}_m)} \left( u_0(C \mathcal{W}_i + R \mathcal{W}_i) + w_0(K \mathcal{W}_i + R \mathcal{W}_i) \right) ds \right) = V_1 + V_2. \]

Similarly to previous steps, we have

\[ V_1 = \sum_{i=1}^N \int_{\partial B_i(\bar{d}_i)} \left( D_0 u_0(C \mathcal{W}_i + R \mathcal{W}_i) + D_0 w_0(K \mathcal{W}_i + R \mathcal{W}_i) \right) ds + \sum_{i \neq k} \int_{\partial B_i(\bar{d}_k)} \left( \nabla u_0(C \mathcal{W}_i + R \mathcal{W}_i) \cdot \hat{V} + \nabla w_0(K \mathcal{W}_i + R \mathcal{W}_i) \cdot \hat{V} \right) ds. \]

Also, we have

\[ V_2 = \sum_{m \neq k} \sum_{i=1}^N \int_{\partial B_i(\bar{d}_m)} \left( D^V_k u_0(C \mathcal{W}_i + R \mathcal{W}_i) + D^V_k w_0(K \mathcal{W}_i + R \mathcal{W}_i) \right) ds - \sum_{m \neq k} \sum_{i=1}^N \int_{\partial B_i(\bar{d}_m)} \left( \nabla u_0(C \mathcal{W}_i + R \mathcal{W}_i) \cdot \hat{V} + \nabla w_0(K \mathcal{W}_i + R \mathcal{W}_i) \cdot \hat{V} \right) ds. \]

Step 6: Synthesis.

Collecting all above terms, we have

\[ II_1 + III_1 + IV_1 + V_1 = \int_{\partial B_0(\bar{d}_0)} \left( C \mathcal{W}_0 \cdot (D_0 u_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \hat{V}) \cdot n + K \mathcal{W}_0 \cdot (D_0 w_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \hat{V}) \cdot n \right. \]

\[ + R \mathcal{W}_0 \cdot (D_0 w_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \hat{V}) \cdot n + R \mathcal{W}_0 \cdot (D_0 u_0 + \sum_{j \neq k} \mathcal{W}_j \cdot \hat{V}) \cdot n \]

\[ = - I_3. \]
and
\[ II_2 + III_2 + IV_2 + V_2 = \sum_{j \neq k} \int_{\partial B_r(\delta_j)} \left( C^k \mathcal{V}_0 \cdot (D^V_k u_0 - \mathcal{V} \cdot \mathcal{V}) \cdot n + K^0 \cdot (D^V_k u_0 - \mathcal{V} \cdot \mathcal{V}) \cdot n \ight. \\
+ R^0 \cdot (D^V_k u_0 - \mathcal{V} \cdot \mathcal{V}) \cdot n + R^0 \cdot (D^V_k u_0 - \mathcal{V} \cdot \mathcal{V}) \cdot n \right) ds \\
= -I_2. \]

Summarizing all above, we obtain
\[ I + II + III + IV + V = I_1 \]
\[ = -\int_{\partial B_r(\delta_0)} \left( C^0 \mathcal{V}_0 \otimes \mathcal{V}_0 + K^0 \otimes \mathcal{V}_0 + R^0 \otimes \mathcal{V}_0 + R^0 \otimes \mathcal{V}_0 \right) \mathcal{V} \cdot nds. \]

Then our result naturally follows. \qed

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