A Diagrammatic Interpretation of the Boltzmann Equation

M.E. Carrington\textsuperscript{a,b}, Hou Defu\textsuperscript{a,b,c} and R. Kobes\textsuperscript{b,d}

\textsuperscript{a} Department of Physics, Brandon University, Brandon, Manitoba, R7A 6A9 Canada
\textsuperscript{b} Winnipeg Institute for Theoretical Physics, Winnipeg, Manitoba
\textsuperscript{c} Institute of Particle Physics, Huazhong Normal University, 430070 Wuhan, China
\textsuperscript{d} University of Winnipeg, Winnipeg, Manitoba, R3B 2E9 Canada

We study nonlinear response in weakly coupled nonequilibrium $\phi^4$ theory in the context of both classical transport theory and real time quantum field theory, based on a generalized Kubo formula which we derive. A novel connection between these two approaches is established which provides a diagrammatic interpretation of the Boltzmann equation.

Fluctuations occur in a system perturbed slightly away from equilibrium. The responses to these fluctuations are described by transport coefficients. The investigation of transport coefficients in high temperature gauge theories is important in cosmological applications such as electroweak baryogenesis and in the context of heavy ion collisions. There are two basic methods to calculate transport coefficients: transport theory and linear response theory. Using the transport theory method one starts from a local equilibrium form for the distribution function and performs an expansion in the gradient of the four-velocity field. The coefficients of this expansion are determined from the classical Boltzmann equation. In the response theory approach one divides the Hamiltonian into a bare piece and a perturbative piece that is linear in the gradient of the four-velocity field. One uses standard perturbation theory to obtain the Kubo formula for the viscosity in terms of retarded Green functions, which are then evaluated using equilibrium quantum field theory. As is typical in finite temperature field theory, it is not sufficient to calculate perturbatively in the coupling constant: there are certain infinite sets of diagrams that contribute at the same order in perturbation theory and have to be resummed.

To date, most calculations of transport coefficients have been done to the order of linear response. In many physical situations however nonlinear response can be important. In this Letter, we study nonlinear response using transport theory and using quantum field theory, and explain the connection between these approaches. We perform a Chapman-Enskog expansion of the Boltzmann equation keeping up to quadratic contributions. We obtain a generalized nonlinear Kubo formula, and a set of integral equations which resum ladder and extended ladder diagrams. We show that these two equations have exactly the same structure, and thus provide a diagrammatic interpretation of the Chapman-Enskog expansion of the Boltzmann equation, up to quadratic order.

We start from the definition of shear viscosity. In a system that is out of equilibrium, the existence of gradients in thermodynamic parameters like the temperature and the four dimensional velocity field give rise to thermodynamic forces which lead to deviations from the equilibrium expectation value of the viscous shear stress:

$$\delta \langle \pi_{\mu \nu} \rangle = \eta^{(1)} H_{\mu \nu} + \eta^{(2)} H_{\mu \nu}^{T2} + \cdots$$

where $\eta^{(1)}$ and $\eta^{(2)}$ are the coefficients of the terms that are linear and quadratic respectively in the gradient of the four-velocity. The first coefficient is the usual shear viscosity. The second has not been widely discussed in the literature – we will call it the quadratic shear viscous response.

The Boltzmann equation can be used to calculate transport properties for weak coupling $\lambda \phi^4$ theory with zero chemical potential. We introduce a phase space distribution function $f(x,k)$ (the underlined momenta are on shell). The form of $f(x,k)$ in local equilibrium is,

$$f^{(0)} = \frac{1}{e^{\beta(x)u_{\mu}(x)k^\mu} - 1} := n_k; \quad N_k := 1 + 2n_k.$$  

We expand $f$ around $f^0$ using a gradient expansion in the local rest frame where $\bar{u}(x) = 0$. We keep only linear terms that contain one power of $H_{\mu\nu}$ and quadratic terms that contain two powers of $H_{\mu\nu}$, since these are the only terms that contribute to the viscosity coefficients we are trying to calculate. We write,

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \cdots$$

with,

$$f^{(1)} \sim \bar{u}_{\mu} \partial^{\mu} f^{(0)}; \quad f^{(2)} \sim \bar{u}_{\mu} \partial^{\mu} f^{(1)}.$$  

Using we obtain,
The collision term is the potential transition rate for particles of momentum \( k_i \) and \( k_f \) given by the rate of change of the distribution function \( f(k) \) with \( \beta \) being the inverse temperature.

\[
f^{(1)} := -n_k (1 + n_k) \phi_k; \quad \phi_k = \beta^2 \frac{1}{2} B_{ij}(k) H_{ij} \quad (5)
\]

\[
f^{(2)} := n_k (1 + n_k) N_k \theta_k; \quad \theta_k := \beta^2 \frac{1}{4} C_{ijkm}(k) H_{ij} H_{km} \quad .
\]

We write

\[
B(k)_{ij} = \hat{I}_{ijm}(k) B(k); \quad C_{ijkm}(k) = \hat{I}_{ijm}(k) \hat{I}_{kmn}(k) C(k)
\]

\[
\hat{I}_{ijm}(k) = \langle k_i k_j - \frac{1}{3} \delta_{ij} k^2 \rangle;
\]

The viscous shear stress tensor is given by

\[
\langle \pi_{ij} \rangle = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} f(k_i k_j - \frac{1}{3} \delta_{ij} k^2) .
\]

Using the expansion, we have,

\[
\delta \langle \pi_{ij} \rangle = -\frac{\beta}{15} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} n_k (1 + n_k) k^2 B(k) H_{ij}
\]

\[
+ \frac{2\beta^2}{105} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} [n_k (1 + n_k) N_k] k^2 C(k) H_{ij}^T .
\]

Comparing with \( \pi \), we have,

\[
\eta^{(1)} = -\frac{\beta}{15} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} n_k (1 + n_k) k^2 B(k)
\]

\[
\eta^{(2)} = \frac{2\beta^2}{105} \int \frac{d^3 k}{(2\pi)^3 2\omega_k} [n_k (1 + n_k) N_k] k^2 C(k) .
\]

The first order equation is

\[
\mathbb{K}_i \partial^\mu f(x, k) = C[f] .
\]

The collision term is \( C[f] := \frac{1}{2} \int \Gamma_{12+3k} d\Gamma_{2+3} f_{12} (1 + f_3) (1 + f_4) f_{12} f_{34} f_{12} f_{34} \) with \( f_i : = f(x, p_i) \), \( f_k : = f(x, k) \). The symbol \( \Gamma_{12+3k} \) represents the differential transition rate for particles of momentum \( P_1 \) and \( P_2 \) to scatter into momenta \( P_3 \) and \( K \).

The first order equation is

\[
\mathbb{K}_i \partial^\mu f^{(1)}(x, k) = C[f^{(0)}; f^{(1)}] .
\]

where we keep terms linear in \( f^{(1)} \) on the right hand side. Using (3) and comparing the coefficients of \( H_{ij} \) on both sides of (11) we obtain (13).

\[
I_{ij}(k) = \frac{1}{2} \int \Gamma_{12+3k} d\gamma_n \{ B_{ij}(p_1) 
\]

\[
+ B_{ij}(p_2) - B_{ij}(k) - B_{ij}(p_3) \} \]

where \( \gamma_n = (1 + n_1) (1 + n_2) n_3 / (1 + n_k) \).

The second order contribution to (4) is

\[
\mathbb{K}_i \partial^\mu f^{(2)}(x, k) = C[f^{(0)}; f^{(1)}; f^{(2)}] .
\]
in the interaction. After a lengthy calculation we find that the result can be written as a retarded three-point correlator \( \delta \langle \pi_{\mu\nu}(x, t) \rangle = \eta^{(2)} H_{\mu\nu}^T \)

\[
\eta^{(2)} = \frac{3}{70 dq_0 dq_0^*} \text{Re} \left[ \lim_{q 	o 0} G_{R1}(-Q - Q', Q, Q') \right] |_{q_0 = q_0^* = 0}
\]

with

\[
G_{R1}(x, y, z) = \theta(t_x - t_y)\theta(t_y - t_z)\left[ (\pi(x), \pi(y), \pi(z)) + \theta(t_x - t_y)\theta(t_z - t_y)\left[ (\pi(x), \pi(z), \pi(y)) \right] \right].
\]

This is an inter. We have obtained a type of nonlinear Kubo formula that allows us to obtain the quadratic shear viscous response from a retarded three-point function using equilibrium quantum field theory.

Next we obtain a perturbative expansion for the correlation functions of composite operators \( D_R(x, y) \) and \( G_{R1}(x, y, z) \) which work in the Keldysh representation. We define the vertices \( \Gamma_{ij} \) and \( M_{ijklm} \) by truncating external legs from the following connected vertices:

\[
\Gamma_{ij} = \langle T_c \pi_{ij}(x)\phi(y)\phi(z) \rangle
\]

\[
M_{ijklm} = \langle T_c \pi_{ij}(x)\pi_{klm}(y)\phi(z)\phi(w) \rangle
\]

where \( \pi_{ij}(x) = \partial_x \phi(x)\partial_y \phi(x) - \frac{1}{2} \delta_{ij}(\partial_x \phi(x))(\partial_y \phi(x)) \), and \( T_c \) is the time ordering operator on the CTP contour. These definitions allow us to write the two- and three-point correlators of the form depicted in Fig. [1]: (a) Two-point function for shear viscosity from linear response; (b) Three-point function for quadratic shear viscous response. The dashed external line represents the composite operator \( \pi_{ij} \).

The square box is the four-point function \( M \) and the round blob is the three-point vertex \( \Gamma \).

After performing the sum over Keldysh indices using the Mathematica program described in [17] we obtain,

\[
D_R(Q) = i \int dK (N_{k+q} - N_k) \Gamma_{ij}(K, Q, -K - Q)
\]

\[
D_A(K)D_R(K + Q)I_{ji}(k)
\]

\[
G_{R1}(-Q - Q', Q, Q') = -4 \int dK (\bar{M}_F)_{kkj}(K, Q, Q')
\]

\[
D_R(K)D_A(K + Q + Q')I_{ji}(k)
\]

where \( \bar{M}_F = M_F + N_1 M_{R1} + N_3 M_{R1} \) is a particular combination of four-point vertices [18]. Rotational invariance leads to \( M_{ijklm} := I_{ij}I_{klm}M; \Gamma_{ij} := I_{ij}\Gamma \). We regulate the pinching singularity with the imaginary part of the HTL self energy \( \Sigma_k \) and obtain [14],

\[
D_R(K)D_A(K + Q) \rightarrow -\frac{\rho_k}{2\text{Im}\Sigma_k}
\]

with \( \rho_k = i(D_R(K) - D_A(K)) \). Now we expand in \( q_0 \) and \( q_0^* \). In [20] and [21] we keep terms proportional to \( q_0 \) and \( q_0^* \) respectively, since these terms are the only ones that contribute to [17] and [18]. Substituting (20) and (21) into (17) and (18) we obtain,

\[
\eta^{(1)} = \frac{\beta}{15} \int dK k^2 \rho_k n_k (1 + n_k) \left[ \frac{\text{Re} R_{2}(K)}{\text{Im} \Sigma_k} \right]
\]

\[
\eta^{(2)} = -\frac{2\beta^2}{105} \int dK k^2 \rho_k n_k (1 + n_k) n_k \left[ \frac{\text{Re} R_{1}(K)}{\text{Im} \Sigma_k} \right].
\]

Comparing with [8] and [9] we see that the results are identical if we identify

\[
B(k) = \frac{\text{Re} R_{2}(K)}{\text{Im} \Sigma_k}, \quad C(k) = -\frac{\text{Re} R_{1}(K)}{\text{Im} \Sigma_k}
\]

with the momentum \( K \) on the shifted mass shell: \( \delta(K^2 - m_{th}^2) \) where \( m_{th}^2 = m^2 + 2\text{Re} \Sigma_k \).

It is well known that ladder diagrams give the dominate contributions to the vertex \( \Gamma_{ij} \). They contribute to the viscosity to the same order in perturbation theory as the bare one loop graph and thus need to be included in a resummation. The integral equation that one obtains from resumming ladder contributions to the three-point vertex (see Fig. [2]) has exactly the same form as the equation obtained from the linearized Boltzmann equation [19] with thermal excitat

![Fig. [2]: Integral equation for the ladder resum-mation.](image)

Following the pinch effect argument [3][1], one can show that an infinite set of ladder graphs and some other contributions which we call extended ladder graph contribute to the same order to vertex \( M_{ijklm} \) as the tree diagram. Therefore, for consistent calculation, we consider an integr

![Fig. [3]: Integral equation for an extended-ladder resum-mation.](image)
We keep only the pinching contributions, using (22) to regulate, and expand in $q_0$ and $q_0\phi$, keeping only the term proportional to $q_0\phi_0$, since that is the only term that will contribute to the quadratic shear viscous response coefficient. We obtain [13]:

$$N_k M_{RI}^{ijlm}(K) = -N_k I_{ij} \frac{\Gamma_{R2}^{ijlm}(K)}{\text{Im}\Sigma_k}$$

(26)

$$\frac{\lambda^2}{4} \int dP dR d_n \rho_R \rho_R \left[ \frac{-N_P M_{RI}^{ijlm} (P)}{\text{Im}\Sigma_P} \right] + \frac{1}{2} \frac{\Gamma_{R2}^{ij}(P)}{\text{Im}\Sigma_P} \frac{\Gamma_{R2}^{ij}(R)}{\text{Im}\Sigma_P} \tilde{N}_{rp} + \frac{1}{2} \frac{\Gamma_{R2}^{ij}(P)}{\text{Im}\Sigma_P} \frac{\Gamma_{R2}^{ij}(K)}{\text{Im}\Sigma_P} \tilde{N}_{pq} \right].$$

We introduce the symmetric notation: $P := P_3; P' := P_2; R := P_3$ and rewrite the equation above after symmetrizing on the integration variables. We obtain,

$$N_k I_{ij} \frac{\Gamma_{R2}^{ijlm}(K)}{\text{Im}\Sigma_k} = \frac{\lambda^2}{12} \int dP_1 dP_2 dP_3 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - K)$$

$$d_n \rho_1 \rho_2 \rho_3 \left[ \frac{N_P M_{RI}^{ijlm}(P)}{\text{Im}\Sigma_p} \right] + \frac{N_P M_{RI}^{ijlm}(K)}{\text{Im}\Sigma_k}$$

$$\tilde{N}_{rp} + \frac{1}{2} \left( N_{12} \frac{\Gamma_{R2}^{ij}(P)}{\text{Im}\Sigma_p} \frac{\Gamma_{R2}^{ij}(P)}{\text{Im}\Sigma_p} \frac{\Gamma_{R2}^{ij}(P)}{\text{Im}\Sigma_p} \frac{\Gamma_{R2}^{ij}(P)}{\text{Im}\Sigma_p} \right)$$

(27)

Note that once again we have obtained an integral equation that is decoupled: it only involves $M_{R1}$ and $\Gamma_{R2}$. With $\Gamma_{R2}$ determined from the integral equation for the ladder resummation, (27) can be solved to obtain $M_{R1}$. Finally, comparing (24) and (27) with [3] and [14] we see that calculating the quadratic shear viscous response using transport theory describing effective thermal excitations and keeping terms that are quadratic in the gradient of the four-velocity field in the expansion of the Boltzmann equation is equivalent to calculating the same response coefficient from quantum field theory at finite temperature using the next-to-linear response Kubo formula with a vertex given by a specific integral equation. This integral equation shows that the complete set of diagrams that need to be resummed includes the standard ladder graphs, and an additional set of extended ladder graphs. Some of the diagrams that contribute to the viscosity are shown.

Fig. [4]: Some of the ladder and extended ladder diagrams that contribute to quadratic shear viscous response.

This result provides a diagrammatic interpretation of the Chapman-Enskog expansion of Boltzmann equation, up to quadratic order.

There are several directions for future work. It has been shown that the Boltzmann equation can be derived from the Kadanoff-Baym equations by using a gradient expansion and keeping only linear terms [13]. The connection between this result and the work discussed in this paper can probably be understood by studying the dual roles of the gradient expansion and the quasiparticle approximation. In addition, it would be interesting to generalize this work to gauge field theories.

[1] S.R. de Groot, W.A. van Leeuwen, and Ch.G. van Weert, *Relativistic Kinetic Theory*. (North-Holland Publishing, 1980).
[2] See, for example, D. Teaney and E. V. Shuryak, Phys. Rev. Lett. 83, 4951 (1999).
[3] G. Baym, et al., Phys. Rev. Lett. 64, 1867 (1990).
[4] D.N. Zubarev, *Nonequilibrium Statistical Thermodynamics*, (Plenum, New York, 1974).
[5] A. Hosoya, M. Sakagami, and M. Takao, Ann. of Phys. (NY) 154, 229 (1984), and references therein.
[6] S. Jeon, Phys. Rev. D52, 3591 (1995); S. Jeon and L. Yaffe, Phys. Rev. D 53, 5790 (1996).
[7] R.D. Pisarski, Phys. Rev. Lett. 63, 1129 (1989); E. Braaten and R.D. Pisarski, Nucl. Phys. B 337, 569 (1990).
[8] V.V. Lebedev and A.V. Smilga, Physica A 181, 187 (1992); M.E. Carrington, Phys. Rev. D48, 3836 (1993).
[9] D. Bodeker, Phys. Lett. B 426, 351 (1998); D. F. Litim and C. Manuel, Phys. Rev. Lett. 82, 4981 (1999).
[10] R. Jackiw and V.P. Nair, Phys. Rev. Lett. 64, 4991 (1990).
[11] J-P Blaizot and E. Iancu, hep-ph/0010119.
[12] P. Arnold, D. T. Son and L. G. Yaffe, Phys. Rev. D59, 105020 (1999).
[13] M.E. Carrington, Hou Defu and R. Kobes, hep-ph/0102256.
[14] M.E. Carrington, Hou Defu and R. Kobes, Phys. Rev. D62, 025010 (2000).
[15] L.V. Keldysh, Zh. Eksp. Teor. Fiz. 47, 1515 (1964).
[16] K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. 118, 1 (1985).
[17] M.E. Carrington, Hou Defu, A. Hachkowski, D. Pickering and J. C. Sowiak, Phys. Rev. D61, 25011 (2000).
[18] M.E. Carrington, Hou Defu and J.C. Sowiak, Phys. Rev. D62, 065003, (2000).
[19] S. Leupold, Nucl. Phys. A 672, 475, (2000).