GRAVITON PRODUCTION IN ELLIPTICAL AND HYPERBOLIC UNIVERSES

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Abstract

The problem of cosmological graviton creation for homogeneous and isotropic universes with elliptical ($\varepsilon = +1$) and hyperbolic ($\varepsilon = -1$) geometries is addressed. The gravitational wave equation is established for a self-gravitating fluid satisfying the barotropic equation of state $p = (\gamma - 1)\rho$, which is the source of the Einstein’s equations plus a cosmological $\Lambda$-term. The time dependent part of this equation is exactly solved in terms of hypergeometric functions for any value of $\gamma$ and spatial curvature $\varepsilon$. An expression representing an adiabatic vacuum state is then obtained in terms of associated Legendre functions whenever $\gamma \neq \frac{2}{3} \frac{(2n+1)}{(2n-1)}$, where $n$ is an integer. This includes most cases of physical interest such as $\gamma = 0, 4/3, 1$. The mechanism of graviton creation is reviewed and the Bogoliubov coefficients related to transitions between arbitrary cosmic eras are also explicitly evaluated.

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1 Introduction

After the pioneer works of Grishchuk [1], the formation of a gravitational wave (GW) background in the course of the cosmological expansion has been extensively studied by several authors (see, for example, [2] and references therein). These ‘Grishchuk gravitons’ [3] are created because the definition of particle states at early and late times is not the same and hence an initial vacuum state seen later becomes a multi-particle state [2],[4]–[6]. The direct detection of these waves would provide us with a new view of the very early phases of the universe since any graviton should have decoupled from matter and from other fields as early as the Planck time ($10^{-43}s$) [7]–[10]. In principle, even without detecting them directly, we may obtain useful information and constrain important parameters of any cosmological models by analyzing their indirect effects. In fact, besides probing the current view of the very early universe, the presence of a GW background can, for example, cause deviations from the Hubble flow for the velocities of galaxies and clusters of galaxies [3], affect the timing of orbital motions [3] and pulsars [11], influence the results of primordial nucleosynthesis [2, 4], and, through the Sachs-Wolfe effect [12], may also contribute to the anisotropies of the Cosmic Microwave Background (CMB) [13, 14].

In spite of the importance of the subject, the large majority of the studies on graviton production found in the literature deal only with the case of spatially flat ($\varepsilon = 0$) Friedmann-Robertson-Walker (FRW) universes. In fact, to the best of our knowledge, only in two articles the time dependent part of the GW equation is solved explicitly for non-spatially flat FRW cosmologies, namely: in the original work of Lifshitz [15], where the solution is given for radiation- and matter-dominated universes with hyperbolic geometries ($\varepsilon = -1$), and in the recent paper by Allen, Caldwell and Koranda [16] where the authors have considered an elliptical ($\varepsilon = +1$) FRW universe and a sequence of a de Sitter (inflationary) phase, followed by a radiation-dominated era and a matter-dominated (dust) phase. In ref. [17] the GW equation for $\Lambda = 0$ and any $\varepsilon$ was established, but not solved for $\varepsilon \neq 0$, whereas in [18] the problem of long GW’s in a homogeneous closed universe (type G3 IX in the Bianchi classification) was addressed (see also [19]).

In the present paper we generalize previous results by studying the creation of gravitons in both elliptical ($\varepsilon = +1$) and hyperbolic ($\varepsilon = -1$) FRW cosmologies. For these values of $\varepsilon$, we derive the general solution of the GW equation in a model where the density $\rho$ and pressure $p$ of the cosmic fluid obey the equation of state $p = (\gamma - 1)\rho$, $\gamma \leq 2$. We find the solution representing an adiabatic vacuum state
for a large class of values of \( \gamma (\gamma \neq \frac{2}{3} \frac{(2n+1)}{(2n-1)}, n = 0, \pm 1, \pm 2, \ldots) \). By considering an arbitrary sequence of cosmic eras with \( \gamma \) in this class, we evaluate the Bogoliubov coefficients which are necessary to obtain the graviton spectrum \([2, 4]\) and to give an accurate determination of their contribution to the anisotropy of the CMB \([13]\). This exact result implies that it is possible to perform a direct comparison among the stochastic background of GW’s produced in different FRW geometries (\( \varepsilon = 0, \pm 1 \)), and potentially it may also be used to impose constraints on several cosmological parameters (the case \( \varepsilon = 0 \) was treated in detail in \([2, 4]\)).

The paper is organized as follows: In Sec. 2 we establish the wave equation for any value of \( \varepsilon \). In Sec. 3 the cosmological model considered is described and the general solution for the scale factor in terms of the conformal time, \( \eta \), is given. In Sec. 4 we derive the general solution for the GW equation in terms of hypergeometric functions, whereas in Sec. 5 we obtain the solutions representing an adiabatic vacuum as a combination of associated Legendre functions. The mechanism of graviton creation is reviewed in Sec. 6 and the Bogoliubov coefficients are evaluated for the model under consideration. A brief summary of our results is presented in a conclusion section.

Unless otherwise stated, the units used are such that \( \hbar = c = k_B = 1 \). Greek indices run from 0 to 3 (spacetime indices) and latin indices from 1 to 3 (spatial indices).

## 2 The wave equation

The study of primordial GW’s is done through the use of a perturbation formalism developed by Lifshitz \([15]\). This consists in making an expansion in the deviations from an unperturbed, non-radiative configuration, and linearizing the Einstein equations about that configuration \([21]\). The spacetime metric is

\[
ds^2 = g^{(T)}_{\mu\nu} dx^\mu dx^\nu ,
\]

with

\[
g^{(T)}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} ,
\]

\[
h_{\mu\nu} \ll g_{\mu\nu} ,
\]

where \( h_{\mu\nu} \) are small perturbations to the background metric \( g_{\mu\nu} \) which can be decomposed into scalar, vector, and transverse-traceless tensor modes. The first two modes represent density and rotational perturbations, respectively. The latter transforms as a 3-tensor under spatial coordinate transformations, and corresponds to sourceless,
weak GW’s. We, therefore, consider only perturbations of the Einstein equations such that the perturbed value of the energy-momentum tensor, $T_{\mu\nu}$, is set equal to zero (see Eq. (2.20) below).

For a homogeneous and isotropic universe, and in the co-moving coordinate system, the background line element takes the FRW form

$$ds^2 = dt^2 - a^2(t) d\ell^2 = a^2(\eta) (d\eta^2 - d\ell^2) ,$$

(2.4)

where $t$ and $\eta$ are, respectively, the cosmic and conformal times, related by

$$dt = a \, d\eta .$$

(2.5)

The 3-space metric is

$$dl^2 = \delta_{ij} dx^i dx^j$$

(2.6)

$$= \frac{1}{1 - \varepsilon r^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2)$$

(2.7)

$$= d\chi^2 + F^2(\chi) (d\theta^2 + \sin^2 \theta \, d\varphi^2) ,$$

(2.8)

$$F(\chi) = \frac{\sin(\sqrt{\varepsilon} \chi)}{\sqrt{\varepsilon}}$$

$$= \sin \chi , \quad \varepsilon = +1$$

$$= \chi , \quad \varepsilon = 0$$

$$= \sinh \chi , \quad \varepsilon = -1 .$$

(2.9)

We shall use the form of $dl^2$ in terms of $\chi , \theta , \varphi$ with $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$, $0 \leq \chi \leq \pi$ for $\varepsilon = +1$, and $0 \leq \chi < \infty$ for $\varepsilon = 0 , -1$. Note that

$$g_{ij} = -a^2(t) \delta_{ij} .$$

(2.10)

The Einstein equations read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} ,$$

(2.11)

with $R_{\mu\nu}$, $R$, and $\Lambda$ being the Ricci tensor, the Ricci scalar, and the cosmological constant, respectively. For a perfect fluid the energy-momentum tensor is

$$T_{\mu\nu} = (\rho + p) U_{\mu} U_{\nu} - p g_{\mu\nu} ,$$

(2.12)
and $U^\mu$ is the four-velocity of the fluid. The field equations take the form

$$8\pi G \frac{\rho}{3} + \Lambda \frac{\dot{a}^2}{3} = \frac{\ddot{a}}{a^2} + \frac{\varepsilon}{a^2},$$  \hfill (2.13)

$$8\pi G p - \Lambda = -2 \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\varepsilon}{a^2},$$  \hfill (2.14)

where a dot means derivative with respect to the cosmic time $t$. By assuming that the equation of state is

$$p = (\gamma - 1) \rho, \quad \gamma \leq 2,$$  \hfill (2.15)

we get from (2.13) and (2.14)

$$\rho = \rho_0 \left( \frac{a_0}{a} \right)^{3\gamma},$$  \hfill (2.16)

$$a\ddot{a} + \frac{3\gamma - 2}{2} \left( \dot{a}^2 + \varepsilon \right) - \frac{1}{2} \gamma \Lambda \dot{a}^2 = 0.$$  \hfill (2.17)

As one may check, the above equation has the first integral

$$\dot{a}^2 = \left( a_0^2 H_0^2 + \varepsilon + \frac{1}{3} \Lambda a_0^2 \right) \left( \frac{a_0}{a} \right)^{3\gamma - 2} - \varepsilon + \frac{1}{3} \Lambda a^2,$$  \hfill (2.18)

where the Hubble parameter is defined by

$$H(t) = \frac{\dot{a}}{a} = \frac{a'}{a^2},$$  \hfill (2.19)

the primes represent derivatives with respect to $\eta$, and $\rho_0 \equiv \rho(t_0)$, etc.

To obtain the GW equation we perturb the Einstein equations to first order and set

$$\delta \rho = \delta p = \delta U^\mu = 0,$$  \hfill (2.20)

$$U^\mu h_{\mu\nu} = 0.$$  \hfill (2.21)

We also impose the gauge conditions

$$\nabla_{\nu} h^{\mu\nu} = 0,$$  \hfill (2.22)

where $\nabla_{\nu}$ indicates the covariant derivative. Eqs. (2.21) and (2.22) lead to

$$h_{\mu\nu} = 0.$$  \hfill (2.23)
(When $a(t) = \text{constant}$, (2.22) constitutes only 3 independent conditions and (2.23) must be imposed separately [17].) In the coordinate system of (2.4) $U^\mu = \delta^\mu_0$, and Eq. (2.21) gives

$$h_{0\mu} = 0 , \quad \text{(2.24)}$$

and from (2.22)

$$\hat{\nabla}_j h^{ij} = 0 . \quad \text{(2.25)}$$

($\hat{\nabla}_j$ represents the covariant derivative with respect to the spatial metric $\hat{g}_{ij}$.)

We are then left with only two independent components of $h^{\mu\nu}$, corresponding to the two polarizations of a GW.

In the gauge used we obtain [17]

$$\frac{d^2 h_{ij}^j}{dt^2} + 3 H \frac{dh_{ij}^j}{dt} + \frac{2\varepsilon}{a^2} h_{ij}^j = \frac{1}{a^2} \hat{\nabla}^2 h_{ij}^j , \quad \text{(2.26)}$$

where

$$\hat{\nabla}^2 h_{ij}^j \equiv \hat{g}^{lm} \hat{\nabla}_l \hat{\nabla}_m h_{ij}^j = \frac{1}{\sqrt{\hat{g}}} \frac{\partial}{\partial x^l} \left( \sqrt{\hat{g}} \hat{g}^{lm} \frac{\partial h_{ij}^j}{\partial x^m} \right) , \quad \text{(2.27)}$$

and $\hat{g}$ is the determinant of $\hat{g}_{ij}$. In terms of the conformal time, Eq. (2.26) can be recast as

$$(h_{ij}^j)'' + 2 \frac{a'}{a} (h_{ij}^j)' + 2\varepsilon h_{ij}^j = \hat{\nabla}^2 h_{ij}^j . \quad \text{(2.28)}$$

If we write

$$\vec{x} = (\chi, \theta, \varphi) , \quad \text{(2.29)}$$

then it is possible to find mode solutions of (2.26) (or (2.28)), labeled by $\tilde{k}$, in the form

$$h_{ij}^j(\tilde{k}, t, \vec{x}) = \Psi(k, t) G_i^j(\tilde{k}, \vec{x}) , \quad \text{(2.30)}$$

where the functions $G_i^j(\tilde{k}, \vec{x})$ and $\Psi(k, t)$ satisfy

$$\hat{\nabla}^2 G_i^j(\tilde{k}, \vec{x}) = -(k^2 - 3\varepsilon) G_i^j(\tilde{k}, \vec{x}) , \quad \text{(2.31)}$$
$$\hat{\nabla}_j G_i^j(\tilde{k}, \vec{x}) = 0 , \quad \text{(2.32)}$$
$$G_i^i(\tilde{k}, \vec{x}) = 0 , \quad \text{(2.33)}$$

$$\ddot{\Psi}(k, t) + 3 \frac{a'}{a} \dot{\Psi}(k, t) + (k^2 - \varepsilon) \frac{\Psi(k, t)}{a^2} = 0 , \quad \text{(2.34)}$$

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or still, in the conformal time

$$\Psi''(k, \eta) + 2 \frac{a'}{a} \Psi'(k, \eta) + (k^2 - \varepsilon) \Psi(k, \eta) = 0 . \quad (2.35)$$

In the above equations, $k$ is the co-moving wave number, related to the physical wavelength $\lambda$ and frequency $\omega$ by

$$k = \frac{2\pi a}{\lambda} = \omega a , \quad (2.36)$$

and $[15]$

$$0 < k < \infty \quad , \quad \varepsilon = 0, -1 , \quad (2.37)$$

$$k = 3, 4, \ldots \quad , \quad \varepsilon = +1 . \quad (2.38)$$

For $\varepsilon = 0$, the symbol $\tilde{k}$ stands for

$$\tilde{k} \equiv (K; \vec{k}) , \quad (2.39)$$

where $K$ represents one of the two possible polarization states of the GW’s, i. e., $K = K_a$ or $K = K_b$; $\vec{k}$ is the co-moving wave vector,

$$\vec{k} = (k_1, k_2, k_3) \quad , \quad -\infty < k_j < \infty , \quad (2.40)$$

$$k = |\vec{k}| = \left( \sum_{i=1}^{3} k_i^2 \right)^{1/2} . \quad (2.41)$$

For $\varepsilon = \pm 1$,

$$\tilde{k} \equiv (K; k, J, M) , \quad (2.42)$$

$$M = -J, -J+1, \ldots, J , \quad (2.43)$$

$$J = 2, 3, \ldots, k-1 \quad , \quad \varepsilon = +1$$

$$= 2, 3, \ldots \quad , \quad \varepsilon = -1 . \quad (2.44)$$

(Note that ‘our’ $k^2$ coincides with the $\beta^2$ of [19], but is $3\varepsilon$ plus the $k^2$ of ref. [17], and that ‘our’ $k$ is 1 plus the $L$ of ref. [16].)

It is convenient to express the temporal dependence of the wave in terms of the function $\mu(k, t)$ defined by

$$\Psi(k, t) = \frac{\mu(k, t)}{a(t)} . \quad (2.45)$$
Eqs. (2.34) and (2.35) lead, respectively, to

\[
\ddot{\mu}(k,t) + \frac{\dot{a}}{a} \dot{\mu}(k,t) + \left[\frac{(k^2 - \varepsilon)}{a^2} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a}\right] \mu(k,t) = 0 ,
\]

(2.46)

\[
\mu''(k,\eta) + \left( k^2 - \varepsilon - \frac{a''}{a} \right) \mu(k,\eta) = 0 .
\]

(2.47)

We shall take Eq. (2.47) as the basic wave equation to be solved.

The general solution representing the tensor perturbations can therefore be written as

\[
h_{\mu^\nu}(\eta,\vec{x}) = \sum k \left[ A(\tilde{k}) \Psi(k,\eta) G_{\mu^\nu}(\tilde{k},\vec{x}) + \text{H.C.} \right] ,
\]

(2.48)

where H.C. represents the Hermitian conjugate of the previous term, and \(\sum\) refers to summation over discrete values and integration over continuous ones, i.e.,

\[
\sum_{k} = \sum_{K} \int d^3k = \sum_{K} \int_{0}^{\infty} k^2 dk , \quad \varepsilon = 0
\]

(2.49)

\[
= \sum_{K} \sum_{k,J,M} , \quad \varepsilon = +1
\]

\[
= \sum_{K} \int_{0}^{\infty} dk \sum_{J,M} , \quad \varepsilon = -1 ,
\]

(2.50)

\[
\sum_{K} = \sum_{K=K_a}^{K_b} ,
\]

(2.51)

\[
\sum_{k,J,M} = \sum_{k=3}^{\infty} \sum_{J=2}^{J} \sum_{M=-J}^{J} , \quad \varepsilon = +1 ,
\]

(2.52)

\[
\sum_{J,M} = \sum_{J=2}^{\infty} \sum_{M=-J}^{J} , \quad \varepsilon = -1 .
\]

In the coordinates used, the spatial functions are specified by Eqs. (2.31)–(2.33) and by

\[
G_{\mu}^{0} = 0 .
\]

(2.53)
We shall impose the normalization condition
\[
\int G_{\mu}^{\nu}(\tilde{k}, \vec{x}) G_{\mu}^{*\nu}(\tilde{k}', \vec{x}) \sqrt{g} d^3 x = 16 \pi G \delta(\tilde{k}, \tilde{k}') \quad (2.54)
\]
where the asterisk indicates the complex conjugate and \( \delta(\tilde{k}, \tilde{k}') \) is the delta function with respect to \( \sum_{\tilde{k}} \) [22], i.e.,
\[
\sum_{\tilde{k}} g(\tilde{k}') \delta(\tilde{k}, \tilde{k}') = g(\tilde{k}) \quad (2.55)
\]
It proves convenient to choose the phases of the \( G_{\mu}^{\nu}(\tilde{k}, \vec{x}) \) in such a way that
\[
G_{\mu}^{*\nu}(\tilde{k}, \vec{x}) = G_{\mu}^{\nu}(-\tilde{k}, \vec{x}) \quad (2.56)
\]
where
\[
-\tilde{k} = -(K; \tilde{k}) = (K; -\tilde{k}) \quad , \quad \varepsilon = 0 \quad (2.57)
-\tilde{k} = -(K; k, J, M) = (K; k, J, -M) \quad , \quad \varepsilon = \pm 1 \quad (2.58)
\]
(See [22] for details.)

In the classical theory \( A(\tilde{k}) \) and \( A^\dagger(\tilde{k}) \) in Eq. (2.48) are just complex constants (Fourier-Bessel coefficients), whereas in the quantized theory they become annihilation and creation operators, respectively. As usual, these operators are required to satisfy the commutation relations
\[
[A(\tilde{k}), A^\dagger(\tilde{k}')] = \delta(\tilde{k}, \tilde{k}') \quad , \quad [A(\tilde{k}), A(\tilde{k}')] = 0 \quad (2.59)
\]
We further impose the following normalization condition on the Wronskian of the time-dependent part of the unified solution
\[
\Psi(k, \eta) \Psi'(k, \eta) - \Psi^*(k, \eta) \Psi'(k, \eta) = \frac{i}{a^2} \quad (2.61)
\]
or, equivalently,
\[
\mu(k, \eta) \mu'(k, \eta) - \mu^*(k, \eta) \mu'(k, \eta) = i \quad (2.62)
\]
3 The solution of the field equations

3.1 General equations

The problem of determining the spectrum of primordial GW’s in a consistent way may be summarized in four basic steps:

1. First of all one finds the $a(\eta)$ for the cosmological model under consideration by solving (2.17) or, equivalently, (2.18). This is required in order to obtain the explicit form of the wave equation (2.47). Particular solutions of the field equations have usually been considered in the literature. In this work a general treatment will be presented.

2. The solution of (2.47) should then be found. As remarked earlier, some approximate methods have been used in the literature (see, for example, [23]).

3. By studying the behavior of the solutions in the high-frequency limit, the particular solution of (2.47) representing an adiabatic vacuum state [20] is identified.

4. Finally, the Bogoliubov coefficients are evaluated using, for instance, the general procedure described in [3].

In this section we deal with the first step by writing the solutions in a convenient way for our purposes. Following the procedure developed in ref. [24], we initially recast (2.17) using the conformal time [25]

$$\frac{a''}{a} + \frac{(3\gamma - 4)}{2} \left(\frac{a'}{a}\right)^2 + \frac{(3\gamma - 2)}{2} \varepsilon - \frac{1}{2} \gamma \Lambda a^2 = 0 .$$

(3.1)

For $\gamma = 2/3$ we make the substitution

$$b = \ln a$$

(3.2)

to get

$$b'' - \frac{1}{3} \Lambda e^{2b} = 0 ,$$

(3.3)

which can be easily solved for $\Lambda = 0$. The scale factor is then

$$a(\eta) = a_0 e^{\frac{2}{3} H_0 (\eta - \eta_0)} , \quad \gamma = \frac{2}{3} , \quad \Lambda = 0 .$$

(3.4)
For $\gamma \neq 2/3$, by replacing

\[ b = a^{1/q}, \quad (3.5) \]

\[ q \equiv \frac{2}{3\gamma - 2}, \quad (3.6) \]

Eq. (3.1) takes the following form

\[ b'' + \frac{\varepsilon}{q^2} b - \frac{\gamma \Lambda}{2q} b^{2q+1} = 0, \quad (3.7) \]

which formally is the nonlinear equation satisfied by a forced harmonic oscillator.

From now on, we restrict our analysis to models with vanishing cosmological constant. In this case, (3.7) reduces to the equation of a simple harmonic oscillator, whose solution is

\[ b = c_1 \sin \left( \sqrt{\frac{\varepsilon}{q^2}} \eta + c_2 \right), \quad \gamma \neq \frac{2}{3}, \quad \Lambda = 0, \quad (3.8) \]

and from (3.3), we write the unified expression for the scale factor as

\[ a(\eta) = a_0 \left( \frac{\sin \Theta}{\sin \Theta_0} \right)^q, \quad \gamma \neq \frac{2}{3}, \quad \Lambda = 0, \quad (3.9) \]

where,

\[ \Theta \equiv \frac{\sqrt{\varepsilon}}{|q|} (\eta - \eta_0) + \Theta_0, \quad (3.10) \]

\[ \Theta_0 \equiv \Theta(\eta_0) = \arctan \left( \frac{q a_0 H_0}{|q| \sqrt{\varepsilon}} \right), \quad (3.11) \]

and by definition

\[ a_0 \equiv a(\eta_0), \quad (3.12) \]

\[ H_0 \equiv H(\eta_0). \quad (3.13) \]

Note also that the unified expression for the Hubble parameter is given by

\[ H(\eta) = \frac{q \sqrt{\varepsilon} \cot \Theta}{|q| a}. \quad (3.14) \]
\[ \rho_c = \frac{3H^2}{8\pi G}, \]  
(3.15)

the density parameter is then
\[ \Omega \equiv \frac{\rho}{\rho_c} = 1 + \frac{\varepsilon a^2 H^2}{(\cos \Theta)^2}. \]
(3.16)

In the limit \( \varepsilon \to 0 \), (3.9) reduces to the general form of the solution for the spatially flat case (Eq. (28) of ref. [4]). Since the graviton spectrum for \( \varepsilon = 0 \) was studied in detail in [2, 4], in what follows we shall treat only the cases \( \varepsilon = \pm 1 \).

### 3.2 Elliptical models (\( \varepsilon = +1 \))

In these models, \( \Theta \) and \( \Theta_0 \) are real and, for \( q > 0 \) (\( \gamma > 2/3 \), non-inflationary scenarios), \( \Theta_0 = \arctan(a_0H_0) \), whereas for \( q < 0 \) (\( \gamma < 2/3 \), inflationary scenarios), \( \Theta_0 = \pi - \arctan(a_0H_0) \). Therefore, for any \( q \), \( 0 < \Theta_0 < \pi \) and \( \sin \Theta_0 > 0 \). For \( a(\eta) \) to be positive we must then require \( 0 < \Theta < \pi \).

For \( q > 0 \) (\( \dot{a} < 0 \)) there is an initial singularity at \( \Theta = 0 \) (\( \eta_s = \eta_0 - q\Theta_0 \)). An expansion phase occurs for \( 0 < \Theta < \pi/2 \) until \( a(\eta) \) reaches the maximum value \( a_{\text{max}} = a_0/(\sin \Theta_0)^q \) at \( \Theta = \pi/2 \) (\( \eta_{\text{max}} = \eta_0 + q\left(\frac{\pi}{2} - \Theta_0\right) \)). This is followed by a contraction stage for \((\pi/2) < \Theta < \pi \), and finally by a ‘big-crunch’ at \( \Theta = \pi \) (\( \eta_s = \eta_0 + q(\pi - \Theta_0) \)).

For \( q < 0 \) (\( \ddot{a} > 0 \)), \( a \) is arbitrarily large as \( \Theta \to 0 \) (\( \eta \to \eta_{\infty 1} = \eta_0 - |q|\Theta_0 \)), decreases for \( 0 < \Theta < \pi/2 \), reaches the minimum \( a_{\text{min}} = a_0(\sin \Theta_0)^{|q|} \) at \( \Theta = \pi/2 \) (\( \eta_{\text{min}} = \eta_0 + |q|\left(\frac{\pi}{2} - \Theta_0\right) \)), grows for \((\pi/2) < \Theta < \pi \), and becomes arbitrarily large as \( \Theta \to \pi \) (\( \eta \to \eta_{\infty 2} = \eta_0 + |q|(\pi - \Theta_0) \)).

Note that \( \cos \Theta \) is positive for \( 0 < \Theta < \frac{\pi}{2} \), \( q > 0 \), \( \left(\frac{\pi}{2} < \Theta < \pi \right., q < 0 \), and negative otherwise.

### 3.3 Hyperbolical models (\( \varepsilon = -1 \))

For \( \varepsilon = -1 \) we define

\[ \Phi \equiv \frac{\Theta}{i} = \frac{1}{|q|} [(\eta - \eta_0) + q \Phi_0], \]
(3.17)

\[ \Phi_0 \equiv \frac{|q| \Theta_0}{q i} = \text{arcth} \left( a_0H_0 \right), \]
(3.18)
where $|a_0H_0| > 1$. Hence,

$$a(\eta) = a_0 \left[ \frac{\sinh \Phi}{\sinh (q/|q| \Phi_0)} \right]^q = a_0 \left[ \frac{|q| \sinh \Phi}{q \sinh \Phi_0} \right]^q,$$

(3.19)

$$H = \frac{q \coth \Phi}{|q| a},$$

(3.20)

$$\Omega = \frac{1}{(\cosh \Phi)^2}.$$

(3.21)

For an expanding universe we must have

$$a_0H_0 > 1.$$

(3.22)

Consequently, $\Phi_0 > 0$, $\sinh \Phi_0 > 0$. The condition $a > 0$ then requires $\Phi > 0$ ($\eta > \eta_0 - q \Phi_0$) for $q > 0$, and $\Phi < 0$ ($\eta < \eta_0 + |q| \Phi_0$) for $q < 0$.

For $q > 0$ there is a singularity at $\Phi = 0$ ($\eta_s = \eta_0 - q \Phi_0$) and $a \to \infty$ as $\Phi \to \infty$ ($\eta \to \infty$). For $q < 0$, $a \to 0$ as $\Phi \to -\infty$ ($\eta \to -\infty$) and $a \to \infty$ as $\Phi \to 0$ ($\eta \to \eta_0 - q \Phi_0$).

### 4 General solution for the wave equation and the effective potential

For the cosmological model described in Sec. 3, the wave equation (2.47) reads

$$\mu''(k, \eta) + (k^2 - \varepsilon - a_0^2H_0^2) \mu(k, \eta) = 0, \quad \gamma = \frac{2}{3},$$

(4.1)

$$\mu''(k, \eta) + \left[ k^2 + \frac{(1-q)}{q} \frac{\varepsilon}{(\sin \Theta)^2} \right] \mu(k, \eta) = 0, \quad \gamma \neq \frac{2}{3}. $$

(4.2)

The solution of (4.1) is trivial and we shall concentrate only in the case $\gamma \neq \frac{2}{3}$.

Note that, as it happens in the spatially flat geometry, the effective potential

$$V_{eff} = \left| \frac{(1-q) \varepsilon}{q (\sin \Theta)^2} \right|$$

(4.3)
approaches zero as $\gamma \to 4/3$. If we define the Hubble co-moving wave number by

$$k_H \equiv \frac{2\pi a}{\lambda_H} = \frac{2\pi a}{H^{-1}},$$

(4.4)

and

$$S \equiv \frac{V_{\text{eff}}}{k_H^2},$$

(4.5)

we see that the ratio $S$ is time dependent, i.e.,

$$S(\Theta(\eta)) = \frac{1}{4\pi^2} \left| \frac{1 - q}{q} \right| \frac{1}{(\cos \Theta)^2}.$$  

(4.6)

Therefore, the amplification condition \cite{1, 2, 3, 20}

$$k^2 \ll V_{\text{eff}}$$

(4.7)

does not necessarily represent modes outside the Hubble radius, $\lambda \gg \lambda_H$ ($k^2 \ll k_H^2$).

In fact, this will happen only for those time intervals where $S(\Theta(\eta)) < 1$. In elliptical geometries, however, we may have $S(\Theta(\eta)) > 1$ as long as

$$-\frac{1}{2\pi} \sqrt{\left| \frac{1 - q}{q} \right|} < \cos \Theta < \frac{1}{2\pi} \sqrt{\left| \frac{1 - q}{q} \right|},$$

(4.8)

that is, close to $a_{\text{max}}$ for $q > 0$ and close to $a_{\text{min}}$ for $q < 0$. As $|\cos \Theta| \leq 1$, (4.8) is restricted to hold for $q$ in one of the following intervals: $(1 + 4\pi^2)^{-1} \leq q \leq 1$, $q > 1$, or $q \leq (1 - 4\pi^2)^{-1}$.

In the hyperbolic geometries, $S(\Theta(\eta)) > 1$ if

$$\cosh \Phi < \frac{1}{2\pi} \sqrt{\left| \frac{1 - q}{q} \right|},$$

(4.9)

i.e., near the singularity when $q > 0$, or as $a \to \infty$ for $q < 0$. This can happen only for $0 < q < (1 + 4\pi^2)^{-1}$ or $(1 - 4\pi^2)^{-1} < q < 0$ (as $\cosh \Phi \geq 1$).

In order to solve (4.2) we first make the substitution

$$Z = (\sin \Theta)^2$$

(4.10)
and write
\[
\mu(k, \eta) = \left( \frac{Z}{\varepsilon} \right)^{q/2} f(Z). \tag{4.11}
\]

It is easily seen that the function \( \mu \) defined by (4.11) is a solution of the wave equation (4.2) as long as \( f \) satisfies
\[
Z (1 - Z) \frac{d^2 f}{dZ^2} + \left[ \left( q + \frac{1}{2} \right) - (q + 1) Z \right] \frac{df}{dZ} + \frac{q^2}{4 \varepsilon} (k^2 - \varepsilon) f = 0, \tag{4.12}
\]
which is a hypergeometric equation [27] with parameters
\[
A = \frac{q}{2} \left( 1 + \frac{k}{\sqrt{\varepsilon}} \right), \tag{4.13}
\]
\[
B = \frac{q}{2} \left( 1 - \frac{k}{\sqrt{\varepsilon}} \right), \tag{4.14}
\]
\[
C = q + \frac{1}{2}. \tag{4.15}
\]

The general solution of the gravitational wave equation is then given by (4.11) with \( f(Z) \) being the general solution of the hypergeometric equation (4.12). The explicit form of this solution depends on several relations involving the parameters \( A, B, C \), and may be found in [27]–[29].

5 Adiabatic vacuum solutions

The next step is to find the particular solution which represents a ‘physical’ vacuum state. As it is usually done in the literature, we adopt the adiabatic approach for defining particle states. In the high-frequency limit the adiabatic definition reduces to the usual positive-frequency Minkowski modes (\( \sim e^{-ik\eta} \)). A detailed discussion of this prescription can be found in [20] and, in our model, it can be easily accomplished whenever
\[
C \neq 0, \pm 1, \pm 2, \ldots , \tag{5.1}
\]
which is equivalent to
\[
q \neq n - \frac{1}{2} \quad , \quad n = 0, \pm 1, \pm 2, \ldots , \tag{5.2}
\]
or still
\[ \gamma \neq \frac{2}{3} \left( \frac{2n + 1}{2n - 1} \right), \quad n = 0, \pm 1, \pm 2, \ldots . \quad (5.3) \]

Under this restriction (which covers interesting cases such as \( \gamma = 0, \gamma = 4/3, \gamma = 1 \)), the general solution of Eq. (4.12) can be written as [27]–[29]

\[ f(Z) = C_1 F(A, B, C; Z) + C_2 Z^{1-c} F(A', B', C'; Z), \quad (5.4) \]

where \( F \) stands for hypergeometric functions, \( C_1 \) and \( C_2 \) are arbitrary constants, and

\[ A' = A - C + 1, \quad (5.5) \]
\[ B' = B - C + 1, \quad (5.6) \]
\[ C' = 2 - C. \quad (5.7) \]

The solution for the wave equation can then be expressed as

\[ \mu(k, \eta) = C \left( \frac{Z}{\varepsilon} \right)^{\eta/2} \left[ F(A, B, C; Z) + C_0 Z^{1-c} F(A', B', C'; Z) \right]. \quad (5.8) \]

The constant \( C_0 \) must be found by imposing the adiabatic constraints, whereas \( C \) is determined up to an irrelevant phase by the normalization condition (2.62).

The fact that for both values of \( \varepsilon \),

\[ C = A + B + \frac{1}{2}, \quad (5.9) \]
\[ C' = A' + B' + \frac{1}{2}, \quad (5.10) \]

enables us to write the hypergeometric functions appearing in (5.8) in terms of associated Legendre functions [27]–[29].

We anticipate the final result derived below and write the adiabatic solution for the GW equation in the unified form

\[ \mu(k, \eta) = e^{i\psi} e^{-i \frac{1}{2} \pi (\frac{1}{2} + \frac{q}{\varepsilon})} \Gamma(m + 1) \left[ \frac{|q| \Gamma(-m)}{4 \Gamma(m + 1)} \right]^{1/2} (\sin \Theta)^{1/2} \mathcal{L}(X), \quad (5.11) \]
where:

\[ \mathcal{L}(X) \equiv P_{\nu}^{-m}(X) - e^{\pm i m \pi \left(\frac{1+\varepsilon}{2}\right)} \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_{\nu}^{m}(X), \quad (5.12) \]

\[ X \equiv |\cos \Theta|, \quad (5.13) \]

\[ m \equiv q - \frac{1}{2} = \frac{3}{2} \left(\frac{2 - \gamma}{3\gamma - 2}\right), \quad (5.14) \]

\[ \nu \equiv \sqrt{\varepsilon} |q| k - \frac{1}{2}, \quad (5.15) \]

\[ I \equiv \text{Im} \left\{ e^{-i m \pi \left(\frac{1+\varepsilon}{2}\right)} \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} \right\}, \quad (5.16) \]

\( \text{Im}\{z\} \) stands for the imaginary part of \( z \), \( \psi \) is a constant phase (whose value is irrelevant for the evaluation of the Bogoliubov coefficients), \( \Gamma \) is the gamma function, and \( P_{\nu}^{m} \) represents the associated Legendre function of the first kind, of order \( m \) and degree \( \nu \). The upper (lower) sign applies if \( \cos \Theta > 0 \) (\( < 0 \)).

Note that for \( \varepsilon = -1 \), \( \cos \Theta = \cosh \Phi \geq 1 \), but for \( \varepsilon = +1 \), \( \cos \Theta > 0 \) during an expansion phase if \( q > 0 \) and during a contraction phase if \( q < 0 \). We also have

\[ 0 < Z \leq 1 \quad 0 < X < 1 \quad (\varepsilon = +1) \quad (5.17) \]

and

\[ -\infty < Z = -(\sinh \Phi)^2 < 0 \quad 1 < X = \cosh \Phi < \infty \quad (\varepsilon = -1) \quad (5.18) \]

In ref. \[28\] the ‘slanted’ symbols \( P_{\nu}^{m}(X), Q_{\nu}^{m}(X) \) are used to indicate the associated Legendre functions of the first and second kind defined for \( \text{Re}\{X\} > 1 \). The ‘straight’ symbols \( P_{\nu}^{m}(X), Q_{\nu}^{m}(X) \) are used for \( |X| < 1 \). We adhere to this notation whenever we are dealing explicitly with the cases \( \varepsilon = -1 \) and \( \varepsilon = +1 \), respectively.

The calligraphic symbols \( P_{\nu}^{m} \) and \( Q_{\nu}^{m} \) will be used only in unified expressions like (5.11) that are valid for both values of \( \varepsilon \). (Note that, as it was remarked in \[17\], there is a misprint at page 999 of ref. \[28\]: the first paragraph should refer to ‘straight’ \( P \) and the second paragraph to ‘slanted’ \( P \).)

Some functional relations involving the associated Legendre functions (relations with the hypergeometric functions, asymptotic behavior, derivatives, etc.), which are needed to derive the adiabatic solution and the Bogoliubov coefficients, are distinct for \( P_{\nu}^{m}(X) \) and \( P_{\nu}^{m}(X) \) \[28\]. We must therefore treat the two cases separately.
For the elliptical models, and due to equations (5.9) and (5.10), we can use Eq. (15.4.13) of ref. [27] to write \( F(A, B, C; Z) \) and \( F(A', B', C'; Z) \) in terms of \( P^{-m}_\nu(X) \) and \( P^m_\nu(X) \), respectively. If we require \( \mu(k, \eta) \) to have the desired asymptotic behavior for large \( k \), the constant \( C_0 \) may be determined. This can be done through the use of Eq. (8.10.7) of [27]. In order to evaluate \( C \) we apply the condition (2.62) on the Wronskian of the solutions by using Eq. (8.741.1) of [28]. Whenever we are forced to deal with \( P^{-m}_\nu(-\cos \Theta) \), we use Eq. (8.737.2) of [28]. The final result is then given by (5.11)–(5.16) with \( \epsilon = +1 \) and \( \nu \rightarrow - \nu \).

For the hyperbolical geometries, equations (5.9) and (5.10) enable us to use Eq. (15.4.12) of [27] to write \( F(A, B, C; Z) \) and \( F(A', B', C'; Z) \) in terms of \( P^{-m}_\nu(cosh \Phi) \) and \( P^m_\nu(cosh \Phi) \), respectively. To analyze the asymptotic behavior for large \( k \) we make successive use of equations (8.723.1) of [28] and (15.7.1) of [27]. The constant \( C_0 \) is then determined. For the Wronskian condition, we first write \( P^{-m}_\nu(X) \) as a combination of \( P^m_\nu(X) \) and \( Q^m_\nu(X) \) using Eq. (8.736.1) of [28] and then apply (8.1.8) of [27]. We finally get (5.11)–(5.16) with \( \epsilon = -1 \) and \( \nu \rightarrow - \nu \).

6 The Bogoliubov coefficients and the graviton spectrum

6.1 Review of the graviton creation mechanism

The general procedure to calculate the graviton spectrum was explained in detail in [2]. We shall give here only a brief resume of this method and then evaluate the Bogoliubov coefficients for our present model.

In realistic cosmological models the equation of state (2.13) will change its form during the cosmic evolution. Prior to a time \( \eta_r \) the equation reads

\[
p_{(n-1)}(\eta) = (\gamma_{r-1} - 1) \rho_{(n-1)}(\eta) \quad (\eta < \eta_r),
\]

whereas for \( \eta > \eta_r \)

\[
p_{(r)}(\eta) = (\gamma_r - 1) \rho_{(r)}(\eta) \quad (\eta > \eta_r).
\]

We are working in the sudden transition approximation where the change in \( \gamma \) occurs instantaneously at the transition time \( \eta_r \). (See [2] for the conditions of applicability of this approximation and other details.) It is possible to impose the continuity conditions,

\[
a_{(n-1)}(\eta_r) = a_{(r)}(\eta_r),
\]

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\( a'_{(r-1)}(\eta_r) = a'_{(r)}(\eta_r) \). \hspace{1cm} (6.4)

\( g_r(\eta) \) indicates the appropriate form for the quantity \( g \) in the interval \([\eta_r, \eta_{r+1}]\). The transition times are \( \eta_r, \eta_{r+1}, \) etc. For the first phase we take \( \eta_0 \) to be any instant in that phase. The notation \( g_r \) indicates \( g_r(\eta_r) \), etc.

We write

\[ Y_{(r-1)}(k, \eta) = \frac{\mu_{(r-1)}(k, \eta)}{a_{(r-1)}(\eta)} , \quad \eta < \eta_r , \]

\[ = \frac{1}{a_{(r)}(\eta)} \left[ \alpha_r(k) \mu_{(r)}(k, \eta) + \beta_r(k) \mu_{(r)}^*(k, \eta) \right] , \quad \eta > \eta_r , \] \hspace{1cm} (6.5)

and

\[ Y_{(r)}(k, \eta) = \frac{1}{a_{(r)}(\eta)} \left[ \alpha_r^*(k) \mu_{(r-1)}^*(k, \eta) - \beta_r(k) \mu_{(r-1)}(k, \eta) \right] , \quad \eta < \eta_r \]

\[ = \frac{\mu_{(r)}(k, \eta)}{a_{(r)}(\eta)} , \quad \eta > \eta_r , \] \hspace{1cm} (6.6)

where \( \mu_{(r-1)} \) and \( \mu_{(r)} \) are solutions to the wave equation describing an adiabatic vacuum in their respective regions. The two modes \( Y_{(r-1)} \) and \( Y_{(r)} \) define two quantizations of the field associated with two different Fock spaces, but the associated operators in each case represent physical particle observables only inside their respective eras. It is easily seen that

\[ Y_{(r-1)}(k, \eta) = \alpha_r(k) Y_{(r)}(k, \eta) + \beta_r(k) Y_{(r)}^*(k, \eta) . \] \hspace{1cm} (6.7)

Due to (2.62), the Bogoliubov coefficients \( \alpha_r(k), \beta_r(k) \) satisfy

\[ |\alpha_r(k)|^2 - |\beta_r(k)|^2 = 1 \], \hspace{1cm} (6.8)

and the operators in the two epochs are related by a Bogoliubov transformation

\[ A_{(r)}(\bar{k}) = \alpha_r(k) A_{(r-1)}(\bar{k}) + \beta_r^*(k) A_{(r-1)}^\dagger(\bar{k}) , \] \hspace{1cm} (6.9)

\[ A_{(r-1)}(\bar{k}) = \alpha_r^*(k) A_{(r)}(\bar{k}) - \beta_r(k) A_{(r)}^\dagger(\bar{k}) . \] \hspace{1cm} (6.10)

Consequently, the vacuum state at the region \((r-1)\), labelled \( |0_{(r-1)}> \), is annihilated by \( A_{(r-1)} \) but not by \( A_{(r)} \), and, if \( N_{(r)}(\bar{k}) \) is the number operator for the mode \( \bar{k} \) at stage \((r)\),

\[ N_{(r)}(\bar{k}) \equiv A_{(r)}^\dagger(\bar{k}) A_{(r)}(\bar{k}) \], \hspace{1cm} (6.11)
\[ < N(\tilde{k}) >_r \equiv 0_{(r-1)}|N(r)(\tilde{k})|0_{(r-1)} > = |\beta_r(k)|^2. \]  

(6.12)

Hence the vacuum state \( |0_{(r-1)} > \) is a multi-particle state when we use the definition of particles appropriate for the epoch \( \eta > \eta_r \). This is interpreted by saying that particles have been created by the expansion dynamics.

In a multi-stage model we need to relate the operators \( A \) and \( A^\dagger \) of stages separated by several transitions. This can be done recursively by

\[
\alpha_{Tr}(k) = \alpha_r(k) \alpha_{T_{r-1}}(k) + \beta_{Tr}^*(k) \beta_{T_{r-1}}(k),
\]

(6.13)

\[
\beta_{Tr}(k) = \beta_r(k) \alpha_{T_{r-1}}(k) + \alpha_r^*(k) \beta_{T_{r-1}}(k).
\]

(6.14)

The Bogoliubov coefficients can be found if we impose the continuity of \( \gamma_{(r-1)} \) and \( \gamma_{(r)} \) and their first derivatives at \( \eta_r \). The result is

\[
\alpha_r(k) = i \left[ \mu_r(k; \eta_r) \mu_{(r-1)}'(k; \eta) - \mu_{(r-1)}(k; \eta_r) \mu_{(r)}'(k; \eta) \right],
\]

(6.15)

\[
\beta_r(k) = i \left[ \mu_{(r-1)}(k; \eta_r) \mu_r'(k; \eta) - \mu_r(k; \eta_r) \mu_{(r-1)}'(k; \eta) \right].
\]

(6.16)

The power spectrum of the graviton background can be described by the quantity \( P_g(\omega) \), defined in such way that \( P_g(\omega) d\omega \) represents the energy per unit volume between the frequencies \( \omega \) and \( \omega + d\omega \). In the units used in this paper

\[
P_g(\omega) = \frac{\omega^3}{\pi^2} < N_\omega(k) >.
\]

(6.17)

The above expression is valid for waves with \( \lambda \leq \lambda_H \). Modes outside the Hubble radius \( \lambda > \lambda_H = H^{-1} \) cannot contribute to the energy density since this is a locally-defined quantity (see [2, 4, 8, 9, 23, 31] for details).

Note that equations (6.15)-(6.17) are independent of the cosmological model considered. Eq. (6.12) assumes that no particles were initially present. If the initial state is not the vacuum, then ‘stimulated’ graviton creation may also occur [2, 20, 32].

### 6.2 The Bogoliubov coefficients

In our model, the expressions for the Bogoliubov coefficients evaluated from (5.11), (5.13), (5.16) may be simplified if we use the continuity conditions (5.3), (6.4), which read

\[
\sin \Theta_{(r-1)}(\eta_r) = \sin \Theta_{r-1} \left( \frac{a_r}{a_{r-1}} \right)^{1/\eta_{r-1}},
\]

(6.18)
\[
cot \Theta_{(r-1)}(\eta_r) = \frac{|q_{r-1}|}{q_{r-1}} \frac{q_r}{|q_r|} \cot \Theta_r = \frac{|q_{r-1}|}{q_{r-1}} \frac{a_r H_r}{\sqrt{\varepsilon}}.
\] (6.19)

For \(\varepsilon = -1\), (6.18) and (5.19) reduce to

\[
\sinh \Phi_{(r-1)}(\eta_r) = \frac{q_{r-1}}{|q_{r-1}|} \sinh \Phi_{r-1} \left( \frac{a_r}{a_{r-1}} \right)^{\frac{1}{q_{r-1}}},
\] (6.20)

\[
\coth \Phi_{(r-1)}(\eta_r) = \frac{|q_{r-1}|}{q_{r-1}} \coth \Phi_r = \frac{|q_{r-1}|}{q_{r-1}} a_r H_r.
\] (6.21)

For \(\varepsilon = +1\) the final result is

\[
\alpha_r(k) = i e^{i(\psi_{r-1}-\psi_r)} \left| \frac{q_{r-1}}{q_r} \right|^{1/2} \frac{B_{r-1} B_r}{\pi} \sin \Theta_r \left( \frac{Y_r}{X_r} \right)^{1/2} D_{\alpha_r},
\] (6.22)

\[
\beta_r(k) = i e^{i(\psi_{r-1}+\psi_r)} \left| \frac{q_{r-1}}{q_r} \right|^{1/2} \frac{B_{r-1} B_r}{\pi} \sin \Theta_r \left( \frac{Y_r}{X_r} \right)^{1/2} D_{\beta_r},
\] (6.23)

\[
X_r = |\cos \Theta_r| = \sin \Theta_r |a_r H_r| = \Omega_r^{-1/2},
\] (6.24)

\[
Y_r \equiv |\cos \Theta_{(r-1)}(\eta_r)| = \frac{\sin \Theta_{r-1}}{\sin \Theta_r} \left( \frac{a_r}{a_{r-1}} \right)^{\frac{1}{q_{r-1}}} X_r,
\] (6.25)

\[
B_r \equiv \frac{\Gamma(-m_r) \Gamma(\nu_r - m_r + 1)}{\Gamma(-m_r) \Gamma(\nu_r + m_r + 1)} \left[ \frac{-\sin(m_r \pi)}{I_r} \right]^{1/2},
\] (6.26)

\[
D_{\alpha_r} \equiv \pm D_r^{(1)} + \frac{i}{2} D_r^{(2)} + \frac{\pi}{2} q_{r-1} \frac{q_r}{|q_r|} \left[ \pm \frac{\pi}{2} D_r^{(3)} - i D_r^{(4)} \right],
\] (6.27)

\[
D_{\beta_r} \equiv \mp D_r^{(1)} + \frac{i}{2} D_r^{(2)} + \frac{\pi}{2} q_{r-1} \frac{q_r}{|q_r|} \left[ \pm \frac{\pi}{2} D_r^{(3)} + i D_r^{(4)} \right],
\] (6.28)

\[
D_r^{(1)} \equiv \frac{q_r}{q_{r-1}} Q^{(l_r)+1}_{n_r}(Y_r) Q^{(m_r)-1}_{\nu_r}(X_r) - Q^{(l_r)}_{n_r}(Y_r) Q^{(m_r)+1}_{\nu_r}(X_r),
\] (6.29)

\[
D_r^{(2)} \equiv \frac{q_r}{q_{r-1}} Q^{(l_r)+1}_{n_r}(Y_r) P^{(m_r)}_{\nu_r}(X_r) - Q^{(l_r)}_{n_r}(Y_r) P^{(m_r)+1}_{\nu_r}(X_r),
\] (6.30)

\[
D_r^{(3)} \equiv \frac{q_r}{q_{r-1}} P^{(l_r)+1}_{n_r}(Y_r) P^{(m_r)}_{\nu_r}(X_r) - P^{(l_r)}_{n_r}(Y_r) P^{(m_r)+1}_{\nu_r}(X_r),
\] (6.31)

\[
D_r^{(4)} \equiv \frac{q_r}{q_{r-1}} P^{(l_r)+1}_{n_r}(Y_r) Q^{(m_r)}_{\nu_r}(X_r) - P^{(l_r)}_{n_r}(Y_r) Q^{(m_r)+1}_{\nu_r}(X_r),
\] (6.32)
\[ l_r \equiv m_{r-1}, \quad (6.33) \]
\[ n_r \equiv \nu_{r-1}. \quad (6.34) \]

The quantities \( \Theta_r, \psi_r, q_r, m_r, \nu_r, I_r \) are those defined by equations (3.11), (5.11)–(5.16) appropriate to the epoch \( \eta_r < \eta < \eta_{r+1} \) (and similarly for \( \Theta_{r-1} \), etc.). The upper (lower) sign applies if \( \cos \Theta_r > 0 \) \( ( < 0) \).

For \( \varepsilon = -1 \) the coefficients can be written as
\[ \alpha_r(k) = i e^{(\psi_{r-1} - \psi_r)} e^{-i(l_r + m_r)\pi} \left( \frac{q_{r-1}}{q_r} \right)^{1/2} B_{r-1} B_r \frac{\sinh \Phi_r \left( Y_r \Phi_r \right)}{\pi} \frac{1}{2} D_{\alpha_r}, \quad (6.35) \]
\[ \beta_r(k) = i e^{(\psi_{r-1} + \psi_r)} e^{-i(l_r + m_r)\pi} \left( \frac{q_{r-1}}{q_r} \right)^{1/2} B_{r-1} B_r \frac{\sinh \Phi_r \left( Y_r \Phi_r \right)}{\pi} \frac{1}{2} D_{\beta_r}, \quad (6.36) \]
\[ X_r \equiv \cosh \Phi_r, \quad (6.37) \]
\[ Y_r \equiv \cosh \Phi_{(r-1)}(\eta_r) = \frac{\sinh \Phi_{r-1}}{\sinh \Phi_r} \left( \frac{a_r}{a_{r-1}} \right)^{\frac{1}{q_{r-1}}} X_r, \quad (6.38) \]
\[ D_{\alpha_r} \equiv D_r^{(1)} + E_r D_r^{(2)}, \quad (6.39) \]
\[ D_{\beta_r} \equiv -D_r^{(1)}, \quad (6.40) \]
\[ E_r \equiv i \pi \frac{e^{im_r \pi}}{\sin(m_r \pi)} \frac{\text{Im}\{\Gamma^*(\nu_r - m_r + 1) \Gamma(\nu_r + m_r + 1)\}}{\Gamma(\nu_r - m_r + 1) \Gamma* (\nu_r + m_r + 1)}, \quad (6.41) \]
\[ D_r^{(1)} \equiv \frac{q_r}{q_{r-1}} O_{n_r}^{(l_r+1)}(Y_r) Q_{m_r}^{m_r}(X_r) - Q_{n_r}^{l_r}(Y_r) Q_{m_r+1}^{(m_r+1)}(X_r), \quad (6.42) \]
\[ D_r^{(2)} \equiv \frac{q_r}{q_{r-1}} O_{n_r}^{(l_r+1)}(Y_r) P_{m_r}^{m_r}(X_r) - Q_{n_r}^{l_r}(Y_r) P_{m_r+1}^{(m_r+1)}(X_r). \quad (6.43) \]

7 Concluding remarks

We have addressed the problem of graviton creation in elliptical (\( \varepsilon = +1 \)) and hyperbolic (\( \varepsilon = -1 \)) Friedmann-Robertson-Walker cosmologies with a gamma-law type equation of state. After establishing the general solution for the scale factor \( a(\eta) \) we have obtained the general solution for the gravitational wave equation. We have then found those modes representing an adiabatic vacuum state for \( \gamma \neq \frac{2}{3} \left( \frac{2n+1}{2n-1} \right) \), \( n = 0, \pm 1, \pm 2, \ldots \). This includes most cases of physical interest, such as a de Sitter
phase ($\gamma = 0$), a radiation-dominated era ($\gamma = 4/3$), and a dust phase ($\gamma = 1$). By considering an arbitrary sequence of epochs with $\gamma$ in this class, we have derived the Bogoliubov coefficients associated with graviton creation in this model. These coefficients are necessary to obtain the gravitational wave power spectrum, $P_g(\omega)$, and to give an accurate determination of the graviton contribution to the anisotropy of the CMB [13].

The application of our results to a definite model, that is, to a definite sequence of cosmic eras corresponding to particular values of $\gamma$, can, in principle, be used to compare the behavior of the relic gravitons in the three possible geometries ($\varepsilon = 0, \pm 1$). In fact, by imposing the observational constraints associated with the anisotropy of the CMB, with the nucleosynthesis of $^4$He, and with the limits on the present value of $\Omega$, it may be possible to obtain theoretical constraints involving several cosmological parameters, such as the value of $\gamma$ in an initial inflationary period, the value of $H$ at the end of inflation, and $\Omega$ itself.

Some other interesting questions also arise. For example, in elliptical models, what happens to graviton creation as the expansion maximum is reached? And, for models with a big-bounce ($q < 0$), what can be said about this production near the expansion minimum? Some of these issues will be addressed in a forthcoming communication [34].

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