DRINFELD FUNCTOR AND FINITE-DIMENSIONAL REPRESENTATIONS OF YANGIAN

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Abstract. We extend the results of Drinfeld on the Drinfeld functor to the case \( \ell \geq n \). We present the character of finite-dimensional representations of the Yangian \( Y(\mathfrak{sl}_n) \) in terms of the Kazhdan-Lusztig polynomials as a consequence.

Introduction

In this article we study the representations of the Yangian \( Y(\mathfrak{sl}_n) \). The Yangian is a quantum group introduced by V. G. Drinfeld ([D1]). The parameterization of the simple finite-dimensional representations of \( Y(\mathfrak{sl}_n) \) was obtained in [D3] by the sequences of monic polynomials \( Q(u) = (Q_1(u), \ldots, Q_{n-1}(u)) \) called the Drinfeld polynomials. Furthermore, he has constructed in [D2] a functor from the category \( \mathcal{C}_H_\ell \) of finite-dimensional representations of the degenerate affine Hecke algebra \( H_\ell \) to the category \( \mathcal{C}_Y(\mathfrak{sl}_n) \) of finite-dimensional representations of \( Y(\mathfrak{sl}_n) \). This functor is called the Drinfeld functor. It was stated in [D2] that as well as the classical Frobenius-Schur duality, the Drinfeld functor gives the categorical equivalence between \( \mathcal{C}_H_\ell \) and the certain subcategory of \( \mathcal{C}_Y(\mathfrak{sl}_n) \) in the case \( \ell < n \). Chari-Pressley generalized this duality to the case between the affine Hecke algebra and the quantum affine algebra. They proved that the categorical equivalence holds in this case as well provided that \( \ell < n \) ([CP2]).

However, due to the restriction \( \ell < n \), the above categorical equivalence does not describe all the finite-dimensional representations of the Yangian \( Y(\mathfrak{sl}_n) \). In particular, even the characters of finite-dimensional representations of \( Y(\mathfrak{sl}_n) \) have not been known, except for the case \( n = 2 \) ([CP3]) and the special class of the representations called tame ([NT1]).

The main purpose of this article is to extend the Drinfeld’s results to the case \( \ell \geq n \). To be more precise, we first show the followings without restriction \( \ell < n \):

1. The Drinfeld functor sends the standard modules of \( H_\ell \) to zero or the highest modules of \( Y(\mathfrak{sl}_n) \) (Theorem 8).
2. The Drinfeld functor sends the simple modules of \( H_\ell \) to zero or the simple modules of \( Y(\mathfrak{sl}_n) \) (Theorem 10).

Here the standard modules are certain induced \( H_\ell \)-modules which have unique simple quotients (see subsection 1.4). We also determine the explicit images of the standard modules. It turns out that the highest weight modules obtained as the images of the standard modules are those tensor product modules of the evaluation representations studied in [AK]. We note that any simple \( Y(\mathfrak{sl}_n) \)-module is isomorphic to the image of a simple \( H_\ell \)-module for some \( \ell \).
Further, combining the above results with that of the representation theory of $\mathcal{H}_t$, we state the following:

3. The multiplicity formula of $Y(\mathfrak{sl}_n)$ expressed in terms of the Kazhdan-Lusztig polynomials (Theorem [3]).

This is the result of considering the composition $D_\ell \circ F_\lambda$ of the two exact functors, where $F_\lambda$ is the functor from the Bernstein-Gelfand-Gelfand category $\mathcal{O}_r$ of the complex Lie algebra $\mathfrak{gl}_r$ to the category $\mathcal{C}_\mathcal{H}_t$, obtained by Suzuki and the author in [AS].

1. Preliminaries

1.1. Yangian. Let $n$ be a positive integer. First we review some fundamental facts about the algebra structure of the Yangian $Y(\mathfrak{sl}_n)$. Our main references are [D1], [D3], [MNO] and we basically follow the notation of [MNO].

Let
\[ R(u) = 1 - \frac{P}{u} \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n), \]
where $P$ is the permutation operator in $\mathbb{C}^n \otimes \mathbb{C}^n$ and $u$ is a parameter. Let $E_{ij} \in \mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$ denote the usual matrix operator on $\mathbb{C}^n$. The Yangian $Y(\mathfrak{gl}_n)$ is the universal associative algebra over $\mathbb{C}$ with generators $t_{ij}^{(k)} (1 \leq i, j \leq n, k = 0, 1, 2, \ldots)$ and the defining relations
\[ R_{12}(u-v)t_{1}(u)t_{2}(v) = t_{2}(v)t_{1}(u)R_{12}(u-v), \quad (1.1.1) \]
where
\[ t(u) = \sum_{i,j} t_{ij}(u) \in Y(\mathfrak{gl}_n) \otimes \text{End}(\mathbb{C}^n), \]
\[ t_{ij}(u) = \sum_{d=0}^{\infty} u^{d-1}t_{ij}^{(d)} \in Y(\mathfrak{gl}_n)([[u^{-1}]]). \]
Here we put $t_{ij}^{(-1)} = \delta_{ij}id$ and both sides of (1.1.1) are regarded as elements of $Y(\mathfrak{gl}_n)([[u^{-1}]]) \otimes \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$ and the subindexes of $t(u)$ and $R(u)$ indicate to which copy of $\text{End}(\mathbb{C}^n)$ these matrices correspond.

The defining relations (1.1.1) are equivalent to the following relations:
\[ [t_{ij}^{(r)}, t_{kl}^{(s)}] - t_{kl}^{(r)}t_{ij}^{(s)} - t_{ij}^{(r)}t_{kl}^{(s)} = t_{kj}^{(r-s)}t_{il}^{(s-r)} - t_{kj}^{(s-r)}t_{il}^{(r-s)} \quad (1.1.2) \]
\[ (1 \leq i, j, k, l \leq n, r, s \in \mathbb{Z}_{\geq 0}) \quad (\text{[MNO]}). \]

The algebra $Y(\mathfrak{gl}_n)$ is a Hopf algebra with coproduct
\[ \Delta : t_{ij}(u) \mapsto \sum_{a=1}^{n} t_{ia}(u) \otimes t_{aj}(u), \quad (1.1.3) \]
antipode $S(t(u)) = t(u)^{-1}$ and counit $\varepsilon(t(u)) = 1$.

Let $U(\mathfrak{gl}_n)$ denote the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n$. The algebra $U(\mathfrak{gl}_n)$ is considered as a subalgebra of $Y(\mathfrak{gl}_n)$ by the inclusion homomorphism defined by
\[ U(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_n) \]
\[ E_{ij} \mapsto t_{ij}^{(0)}. \]

\[ ^1 \text{After completing this article, the author was notified that E. Vasserot obtained the similar formula in terms of intersection cohomologies in the case of the quantum affine algebra by geometrical method ([preprint, math.QA/9803024]).} \]
On the other hand, for \( a \in \mathbb{C} \), the map

\[
ev_a : Y(\mathfrak{gl}_n) \rightarrow U(\mathfrak{gl}_n)
\]

\[
t_{ij}(u) \mapsto \delta_{ij} + \frac{E_{ij}}{u - a}
\]

(1.1.4)

defines an algebra homomorphism. For a \( \mathfrak{gl}_n \)-module \( V \), let \( \ev_a(V) \) denote the \( Y(\mathfrak{gl}_n) \)-module obtained by pulling \( V \) back by (1.1.4).

The quantum determinant \( \text{qdet} t(u) \) is defined as

\[
\text{qdet} t(u) = \sum_{w \in W_n} (-1)^{\text{sgn}(w)} t_{w(1),1}(u)t_{w(2),2}(u - 1)\cdots t_{w(n),n}(u - n + 1),
\]

where \( W_n \) denotes the symmetric group \( \mathfrak{S}_n \). The coefficients of the quantum determinant are algebraically independent and generate the center \( Z(Y(\mathfrak{gl}_n)) \) of \( Y(\mathfrak{gl}_n) \).

For a formal series \( f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]] \), the multiplication

\[
t(u) \mapsto f(u) t(u)
\]

(1.1.5)
defines an automorphism of \( Y(\mathfrak{gl}_n) \). It is known that the Yangian \( Y(\mathfrak{sl}_n) \) can be defined as the subalgebra of \( Y(\mathfrak{gl}_n) \) consisting of elements fixed by all automorphisms of the form (1.1.3) ([MNO]). One has a tensor product decomposition

\[
Y(\mathfrak{gl}_n) \cong Z(Y(\mathfrak{gl}_n)) \otimes Y(\mathfrak{sl}_n).
\]

(1.1.6)

Hence any \( Y(\mathfrak{gl}_n) \)-module can be considered as a \( Y(\mathfrak{sl}_n) \)-module.

Let

\[
F_i Y(\mathfrak{gl}_n) := \sum_{i \in \mathbb{Z}_{\geq 0}} \sum_{r_1, \ldots, r_k} \mathcal{C} t_{i+r_1,j_1}^{(i_1, j_1)} \cdots t_{i+r_k,j_k}^{(i_k, j_k)} \quad (i \in \mathbb{Z}_{\geq 0}).
\]

(1.1.7)

Then, by (1.1.2), \( F_i Y(\mathfrak{gl}_n) \cdot F_j Y(\mathfrak{gl}_n) \subset F_{i+j} Y(\mathfrak{gl}_n) \), and (1.1.7) defines a filtration on \( Y(\mathfrak{gl}_n) \). Let \( \text{gr} Y(\mathfrak{gl}_n) \) denote the corresponding the graded algebra;

\[
\text{gr} Y(\mathfrak{gl}_n) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} F_i Y(\mathfrak{gl}_n)/F_{i-1} Y(\mathfrak{gl}_n) \quad (F_{-1} Y(\mathfrak{gl}_n) = 0).
\]

Then, \( \text{gr} Y(\mathfrak{gl}_n) \) is isomorphic to \( U(\mathfrak{gl}_n[t]) \), where \( U(\mathfrak{gl}_n[t]) \) is the universal enveloping algebra of the polynomial current algebra \( \mathfrak{gl}_n[t] := \mathfrak{gl}_n \otimes \mathbb{C}[t] \) with \( \mathbb{Z}_{\geq 0} \)-grading such that the degree of the element \( X \otimes t^r \) (\( X \in \mathfrak{gl}_n \)) equals \( r \) ([MNO]).

1.2. Drinfeld polynomials. In this subsection we review the classification theory of finite-dimensional simple \( Y(\mathfrak{sl}_n) \)-modules studied by Drinfeld ([D3], see also [CP1, MO]).

A representation \( V \) of \( Y(\mathfrak{gl}_n) \) is called highest weight if there exists a cyclic vector \( v \) such that \( t_{ij}(u) \cdot v = 0 \) (\( 1 \leq i < j \leq n \)) and \( t_{ii}(u - i) \cdot v = \zeta_i(u)v \) (\( 1 \leq i \leq n \)) for some formal series \( \zeta_i(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]] \). The vector \( v \) is called the highest weight vector of \( V \) and the sequence \( \zeta(u) = (\zeta_1(u), \ldots, \zeta_n(u)) \) is called the highest weight of \( V \). The central element \( \text{qdet} t(u) \) acts as a constant \( \zeta_1(u) \zeta_2(u - 1) \cdots \zeta_n(u - n + 1) \) on a highest weight module \( V \). As in the classical Lie algebra theory, any highest weight \( Y(\mathfrak{gl}_n) \)-module has a unique simple quotient, in which the image of its highest weight vector is nonzero.

It is known by Drinfeld that a simple highest weight module of \( Y(\mathfrak{gl}_n) \) is finite-dimensional if and only if there exists a sequence of monic polynomials \( Q(u) = \)
\( (Q_1(u), \ldots, Q_{n-1}(u)) \) such that
\[
\frac{\zeta_k(u)}{\zeta_{k+1}(u + 1)} = \frac{Q_k(u + 1)}{Q_k(u)}
\]
for \( k = 1, \ldots, n - 1 \). A theorem of Drinfeld states that there is a one-to-one correspondence between the finite-dimensional simple \( Y(\mathfrak{g}_n) \)-modules and the sequences of monic polynomials \( Q(u) = (Q_1(u), \ldots, Q_{n-1}(u)) \) defined by (1.2.1) (D3). The \( Q(u) \) are called Drinfeld polynomials.

**Remark 1.** The standard symbol for the Drinfeld polynomials is \( P(u) \). However, this symbol for a polynomial is reserved for the Kazhdan-Lusztig polynomials in this article.

### 1.3. Degenerate Affine Hecke Algebra

Let \( \ell \) be a positive integer. Let \( \mathfrak{h}_\ell \) be the Cartan subalgebra of \( \mathfrak{gl}_\ell \), which consists of the diagonal matrices. Define a basis \( \{ \epsilon_i \}_{i=1}^\ell \) of \( \mathfrak{h}_\ell \) by putting \( \epsilon_i = E_{ii} \). The dual space \( \mathfrak{h}_\ell^* \) is identified with \( \mathfrak{h}_\ell \) via the inner product \( \langle \epsilon_i, \epsilon_j \rangle = \delta_{ij} \).

Let \( R_\ell \) be the root system of \( \mathfrak{gl}_\ell \):
\[
R_\ell = \{ \alpha_{ij} = \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq \ell \},
R_\ell^+ = \{ \alpha_{ij} \in R_\ell \mid i < j \},
\Pi_\ell = \{ \alpha_i = \alpha_{ii+1} \mid i = 1, \ldots, \ell - 1 \},
\]
where \( R_\ell^+ \) is the set of positive roots and \( \Pi_\ell \) is the set of simple roots. Let \( \rho = \frac{1}{\ell} \sum_{\alpha \in R_\ell^+} \alpha \). We identify the coroots with the roots throughout this article.

Let \( s_\alpha \in W_\ell \) denote the reflection corresponding to \( \alpha \in R_\ell \);
\[
s_\alpha \cdot \lambda = \lambda - \lambda(\alpha)\alpha \quad (\lambda \in \mathfrak{h}_\ell^*).
\]

Put \( s_{ij} = s_{\alpha_{ij}} \) (\( \alpha_{ij} \in R_\ell \)) and \( s_i = s_{\alpha_i} \) (\( \alpha_i \in \Pi_\ell \)).

Let \( S(\mathfrak{h}_\ell) \) be the symmetric algebra of \( \mathfrak{h}_\ell \), which is isomorphic to the polynomial ring over \( \mathfrak{h}_\ell^* \).

The *degenerate affine Hecke algebra* \( \mathcal{H}_\ell \) of \( GL_\ell \) (D2) is the associative algebra over \( \mathbb{C} \) such that
\[
\mathcal{H}_\ell \cong \mathbb{C}[W_\ell] \otimes S(\mathfrak{h}_\ell)
\]
as a vector space, the subspaces \( \mathbb{C}[W_\ell] \otimes \mathbb{C} \) and \( \mathbb{C} \otimes S(\mathfrak{h}_\ell) \) are subalgebras of \( \mathcal{H}_\ell \), and the following relations hold in it:
\[
s_\alpha \cdot \xi - s_\alpha(\xi) \cdot s_\alpha = -\alpha(\xi) \quad (\alpha \in \Pi_\ell; \xi \in \mathfrak{h}_\ell),
\]
where the elements \( \xi \in \mathfrak{h}_\ell \) and \( w \in W_\ell \) are identified with \( 1 \otimes \xi \in \mathcal{H}_\ell \) and \( w \otimes 1 \in \mathcal{H}_\ell \) respectively. One has
\[
\mathcal{H}_\ell = \mathbb{C}[W_\ell] \cdot S(\mathfrak{h}_\ell) = S(\mathfrak{h}_\ell) \cdot \mathbb{C}[W_\ell].
\]

We put \( \mathcal{H}_0 = \mathbb{C} \) for convenience.

The center \( Z(\mathcal{H}_\ell) \) of this algebra equals the \( W_\ell \)-invariant polynomials \( S(\mathfrak{h}_\ell)^{W_\ell} = \mathbb{C}[\epsilon_1, \ldots, \epsilon_\ell]^{W_\ell} \) (Lus).

Define elements \( y_i \in \mathcal{H}_\ell \) (\( i = 1, \ldots, \ell \)) by
\[
y_i := s_{1i} \cdot \epsilon_i \cdot s_{1i} = \epsilon_i - \sum_{j < i} s_{ji}.
\]
Then one can see by direct calculations that
\[ w \cdot y_i = y_{w(i)} \cdot w \quad (i = 1, \ldots, \ell, \ w \in W_\ell), \]  
(1.3.3)
\[ [y_i, y_j] = -(y_i - y_j)s_{ij} \quad (1 \leq i, j \leq \ell). \]  
(1.3.4)
The algebra $H_\ell$ is isomorphic to the $\mathbb{C}$-algebra with generators $w \in W_\ell$ and $y_i$  
($i = 1, \ldots, \ell$) with the defining relations (1.3.3), (1.3.4) and the Coxeter relations of $w$'s in $W_\ell$.

Let \( F_1 H_\ell := \sum_{w \in W_\ell} \sum_{d_1 + \cdots + d_i \leq i} \mathbb{C}y_1^{d_1}y_2^{d_2} \cdots y_\ell^{d_\ell}w \quad (i \in \mathbb{Z}_{\geq 0}). \]  
(1.3.5)
Then, by (1.3.3), $F_i H_\ell \cdot F_j H_\ell \subset F_{i+j} H_\ell$, and (1.3.5) defines a filtration on $H_\ell$.

Let $\text{gr} \ H_\ell$ denote the corresponding graded algebra;

\[ \text{gr} \ H_\ell = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} F_i H_\ell/F_{i-1} H_\ell \quad (F_{-1} H_\ell = 0). \]

Let $\bar{y}_i$ denote the image of $y_i$ in $\text{gr} \ H_\ell$. Then $\text{gr} \ H_\ell$ is isomorphic to the graded $\mathbb{C}$-algebra generated by $\mathbb{C}[W_\ell]$ and the polynomial ring $\mathbb{C}[\bar{y}_1, \ldots, \bar{y}_\ell]$ with the relations  
\[ w \cdot \bar{y}_i = \bar{y}_{w(i)} \cdot w \quad (i = 1, \ldots, \ell, \ w \in W_\ell), \]  
whose grading is given by $\text{deg}(\bar{y}_i) = 1$ and $\text{deg}(w) = 0$. In particular, $\text{gr} \ H_\ell \cong \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_\ell] \otimes \mathbb{C}[W_\ell]$ as a $\mathbb{C}$-vector space.

### 1.4. Representations of degenerate affine Hecke algebra

In this subsection we review the theory of the representations of $H_\ell$ studied in [Zel, Ro, Gr] (see also [CG]), along the line introduced in [AS] and developed in [S]. Let $\mathcal{C}_H$ denote the category of finite-dimensional representations of $H_\ell$. Let $r$ be a nonnegative integer. The representation theory of $H_\ell$ is well-described by the language of the Bernstein-Gelfand-Gelfand category $\mathcal{O}_r$ of $\mathfrak{gl}_r$, via the functors $F_\lambda : \mathcal{O}_r \to \mathcal{C}_H$ constructed in [AS, S].

Let

\[ P_r^+ := \left\{ \lambda \in \mathfrak{h}_r^* \mid \lambda(\alpha) \notin -1, -2, \ldots \text{ for all } \alpha \in R_r^+ \right\}, \]
\[ P_{r, \mathbb{Z}}^+ := \bigoplus_{i=1}^r \mathbb{Z}e_i \subset P_r^+. \]

An element of $P_r^+$ (resp. $P_{r, \mathbb{Z}}^+$) is called the dominant (resp. integral dominant) weight.

For $\lambda \in \mathfrak{h}_r^*$, there is a functor form $\mathcal{O}_r$ to $\mathcal{C}_H$ defined by

\[ F_\lambda(X) := H_0(\mathfrak{n}_r^-, X \otimes (\mathbb{C}^r)^{\otimes \ell})_{\lambda-\rho} \quad (X \in \mathcal{O}_r) \]  
(1.4.1)
\[ = [X \otimes (\mathbb{C}^r)^{\otimes \ell}/\mathfrak{n}_r^-(X \otimes (\mathbb{C}^r)^{\otimes \ell})]_{\lambda-\rho}, \]

where $\mathfrak{n}_r^-$ denotes the nilpotent subalgebra of $\mathfrak{gl}_r$ generated by the lower triangular matrices $\{E_{ij} | i - j > 0\}$ and $X_{\lambda}$ denotes the weight space of weight $\lambda$ of a $\mathfrak{gl}_r$-module $X$. The action of $H_\ell$ on the space $F_\lambda(X)$ is given by

\[ e_i \mapsto \sum_{j=0}^i \Omega_{ji} + \frac{r-1}{2} \quad (1 \leq i \leq \ell) \]
\[ s_i \mapsto \Omega_{ii+1} \quad (1 \leq i \leq \ell - 1), \]

where $\Omega_{ij}$ denotes an endomorphism of $X \otimes (\mathbb{C}^r)^{\otimes \ell}$ which acts as the Casimir $\Omega = \sum_{rs} E_{rs} \otimes E_{sr}$ on the tensor product of $i$-th and $j$-th factors and by identity
on all the other factors. Here the 0-th factor corresponds to \( X \in \mathcal{O}_r \). The functor \( F_\lambda \) is exact if \( \lambda \in P^+_r \).

For complex numbers \( a, b \) such that \( b - a + 1 = \ell \), let \( \mathbb{C}_{[a, b]} = \mathbb{C}1_{[a, b]} \) denote the one-dimensional representation of \( \mathcal{H}_\ell \), defined by

\[
\begin{align*}
s_i \cdot 1_{[a, b]} &= 1_{[a, b]} & (i = 1, \ldots, \ell - 1), \\
\epsilon_i \cdot 1_{[a, b]} &= (a + i - 1)1_{[a, b]} & (i = 1, \ldots, \ell).
\end{align*}
\]

(1.4.2)  (1.4.3)

Let \( \text{Wt}(X) \) denote the space of weights of a \( \mathfrak{gl}_r \)-module \( X \). Then, clearly

\[
\text{Wt}((\mathbb{C}^\ell) \otimes \lambda) = \left\{ \sum_{i=1}^r n_i \epsilon_i \mid n_i \in \{0, 1, \ldots, \ell\}, \sum_{i=1}^r n_i = \ell \right\}.
\]

For \( \lambda \in \mathfrak{h}^*_r \), let

\[
S(\lambda; \ell) := \left\{ \mu \in \mathfrak{h}^*_r \mid \lambda - \mu \in \text{Wt}((\mathbb{C}^\ell) \otimes \lambda) \right\}.
\]

(1.4.4)

For \( \lambda \in \mathfrak{h}^*_r \) and \( \mu \in S(\lambda; \ell) \), define an \( \mathcal{H}_\ell \)-module

\[
\mathcal{K}(\lambda, \mu) := \mathcal{H}_\ell \otimes (\mathcal{H}_\ell \otimes \cdots \otimes \mathcal{H}_\ell) (\mathbb{C}_{[\mu_1, \lambda_1 - 1]} \otimes \cdots \otimes \mathbb{C}_{[\mu_r, \lambda_r - 1]}),
\]

where \( \lambda_i = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_\ell) \). Here \( \mathcal{H}_\ell \otimes \mathcal{H}_\ell \otimes \cdots \otimes \mathcal{H}_\ell \) is regarded as a subalgebra of \( \mathcal{H}_\ell \) via the embeddings \( \mathcal{H}_{\ell_h} \hookrightarrow \mathcal{H}_\ell \) defined by \( \epsilon_a \mapsto \epsilon_a + \sum_{j=1}^{i-1} \ell_j \) and \( s_a \mapsto s_a + \sum_{j=1}^{i-1} \ell_j \).

Let

\[
1_{\lambda, \mu} := 1_{[\mu_1, \lambda_1 - 1]} \otimes \cdots \otimes 1_{[\mu_r, \lambda_r - 1]} \in \mathcal{K}(\lambda, \mu).
\]

Then the correspondence \( 1_{\lambda, \mu} \mapsto 1 \) defines an isomorphism of \( \mathcal{H}_\ell \)-modules

\[
\mathcal{K}(\lambda, \mu) \cong \mathbb{C}[W_\ell/W_{\ell_1} \times \cdots \times W_{\ell_\ell}] \quad (1.4.6)
\]

For a partition \( \nu \) of \( \ell \), let \( U(\nu) \) denote the simple \( W_\ell \)-module associated with \( \nu \). For \( \lambda \in \mathfrak{h}^*_r \) and \( \mu \in S(\lambda; \ell) \), let \( \nu_{\lambda, \mu} \) denote the partition of \( \ell \) obtained by forgetting the order of \( (\ell_1, \ldots, \ell_\ell) \), where \( \ell_i = (\lambda_1 - \mu_1)(\epsilon_i) \). Then by (1.4.6), \( \mathcal{K}(\lambda, \mu) \) decomposes as

\[
\mathcal{K}(\lambda, \mu) \cong U(\nu_{\lambda, \mu}) \oplus \bigoplus_{\nu > \nu_{\lambda, \mu}} U(\nu)^{\otimes c_{\nu}}.
\]

(1.4.7)

as a \( W_\ell \)-module, where \( > \) is the dominance order in the set of the partitions and \( c_{\nu} \) is some nonnegative integer (see [14], for example). It is known that if \( \lambda \in P^+_r \), the \( W_\ell \)-simple component \( U(\nu_{\lambda, \mu}) \) generates \( \mathcal{K}(\lambda, \mu) \) over \( \mathcal{H}_\ell \), hence it has an unique simple quotient \( \mathcal{L}(\lambda, \mu) \) which contains \( U(\nu_{\lambda, \mu}) \) with multiplicity one (Zelevinsky, see also [3]). The module \( \mathcal{K}(\lambda, \mu) \) with \( \lambda \in P_r^+ \) and \( \mu \in S(\lambda; \ell) \) is called a standard module of \( \mathcal{H}_\ell \).

Let \( W_\lambda \subset W_\ell \) denote the stabilizer of \( \lambda \in \mathfrak{h}^*_r \). Notice that if \( \mu \in S(\lambda; \ell) \), then \( w \cdot \mu \in S(\lambda; \ell) \) for all \( w \in W_\lambda \). One has

\[
\begin{align*}
\mathcal{K}(\lambda, \mu) \cong \mathcal{K}(\lambda, \mu') & \iff \mathcal{L}(\lambda, \mu) \cong \mathcal{L}(\lambda, \mu') \\
& \iff \mu' = w \cdot \mu \text{ for some } w \in W_\lambda
\end{align*}
\]

(1.4.8)

for \( \lambda \in P^+_r \) and \( \mu, \mu' \in S(\lambda; \ell) \). It is known that any simple \( \mathcal{H}_\ell \)-module is isomorphic to \( \mathcal{L}(\lambda, w \cdot \mu) \) for some \( \lambda, \mu \in P^+_r \) and \( w \in W_\lambda \setminus W_\ell/W_\mu \) such that \( w \cdot \mu \in S(\lambda; \ell) \) for some \( r \in \mathbb{N} \). (Zelevinsky, Ro.)
Let \( M(\lambda) \) be the Verma module of \( \mathfrak{gl}_n \) with highest weight \( \lambda - \rho \) and let \( L(\lambda) \) denote its unique simple quotient. Let \( \lambda, \mu \in P^+ \) and \( w \in W_\lambda \backslash W_\mu / W_\mu \) such that \( w \cdot \mu \in S(\lambda; t) \). Then one of the main results in \cite{AS} is stated as follows:

\[
F_\lambda(M(w \cdot \mu)) \cong K(\lambda, w \cdot \mu),
\]

\[
F_\lambda(L(w \cdot \mu)) \cong \begin{cases} \mathcal{L}(\lambda, w \cdot \mu) & \text{if } w \cdot \mu = w_{LR} \cdot \mu, \\ 0 & \text{otherwise}, \end{cases}
\]

(1.4.9) (1.4.10)

where \( w_{LR} \) denotes the longest length representative of \( w \) in the double coset \( W_\lambda w W_\mu \).

2. Drinfeld Functor

2.1. Drinfeld Functor. For a left \( \mathcal{H}_t \)-module \( M \), consider an \( \mathcal{H}_t \otimes U(\mathfrak{gl}_n) \)-module \( M \otimes (\mathbb{C}^n)^{\otimes t} \). Here we regard \( \mathbb{C}^n \) as the vector representation of \( \mathfrak{gl}_n \). For \( x \in \mathfrak{gl}_n \) and \( i = 1, \ldots, t \), let \( \tau_i(x) \) denote the endomorphism of \( M \otimes (\mathbb{C}^n)^{\otimes t} \) which acts as \( x \in \mathfrak{gl}_n \) on the \( i \)-th factor of the tensor product \( (\mathbb{C}^n)^{\otimes t} \) and by identity on all the other factors.

Define an action of the Yangian \( Y(\mathfrak{gl}_n) \) on \( M \otimes (\mathbb{C}^n)^{\otimes t} \) by

\[
\pi : t(u) \mapsto T_1(u - \epsilon_1)T_2(u - \epsilon_2) \cdots T_t(u - \epsilon_t),
\]

(2.1.1)

where

\[
T_i(u - \epsilon_i) = 1 + \frac{1}{u - \epsilon_i} I_i
\]

and

\[
I_i = \sum_{1 \leq a, b \leq n} \tau_i(E_{ab}) \otimes E_{ab} \in \text{End}(M \otimes (\mathbb{C}^n)^{\otimes t}) \otimes \text{End}(\mathbb{C}^n).
\]

By the fact that \( S(\mathfrak{h}_t) \) is commutative, (2.1.1) gives a well-defined action of \( Y(\mathfrak{gl}_n) \) (recall (1.1.3) and (1.1.4)).

The symmetric group \( W_t \) naturally acts on \( M \otimes (\mathbb{C}^n)^{\otimes t} \) by \( s_{ij} \mapsto K_{ij} P_{ij} \), where \( K_{ij} \) denotes its action on \( M \) and \( P_{ij} \) denotes its action on \( (\mathbb{C}^n)^{\otimes t} \) by permutation.

Now define

\[
D_t(M) := (M \otimes (\mathbb{C}^n)^{\otimes t}) / \sum_{i=1}^t \text{Im}(s_i + 1),
\]

(2.1.2)

where \( \text{Im}(s_i + 1) \) denotes the subspace \((s_i + 1)(M \otimes (\mathbb{C}^n)^{\otimes t})\). Let \([m \otimes u]\) denote the equivalence class of \( m \otimes u \in M \otimes (\mathbb{C}^n)^{\otimes t} \) in \( D_t(M) \).

The following proposition is due to Drinfeld (\cite{D2}, see also \cite{BGHP, CP2}).

**Proposition 2.** Let \( M \) be an \( \mathcal{H}_t \)-module. Then, \( \pi \) induces an action of \( Y(\mathfrak{gl}_n) \) on the space \( D_t(M) \).

**Proof.** It is enough to show that \( \left( \prod_{i=1}^t (u - \epsilon_i) \right) \pi(t(u)) \) preserves the denominator space of (2.1.2) since \( \left( \prod_{i=1}^t (u - \epsilon_i) \right) \in Z(\mathcal{H}_t)[u] \). This follows from the formula

\[
(u - \epsilon_i + I_i)(u - \epsilon_{i+1} + I_{i+1})s_i \equiv s_i(u - \epsilon_i + I_i)(u - \epsilon_{i+1} + I_{i+1}) + (s_i + 1) [I_{i+1}, I_i]
\]

\((s_i = K_{i+1}P_{i+1})\), which can be proven by direct calculations using the defining relations (1.3.1) and the commutation relations \([P_i, I_i] = [I_{i+1}, I_i]\). \(\square\)
The action of $Y(\mathfrak{g} l_n)$ on the space $D_\ell (M)$ will be denoted by the same symbol $\pi$.

Let $\mathcal{C}_Y (\mathfrak{g} l_n)$ and $\mathcal{C}_Y (\mathfrak{s} l_n)$ denote the category of finite-dimensional representations of $Y(\mathfrak{g} l_n)$ and $Y(\mathfrak{s} l_n)$ respectively. Then $D_\ell$ defines an exact functor from $\mathcal{C}_\mathcal{H}_\ell$ to $\mathcal{C}_Y (\mathfrak{g} l_n)$ or $\mathcal{C}_Y (\mathfrak{s} l_n)$. The functor $D_\ell$ is called the Drinfeld Functor ([D2]). Note that our definition of the Drinfeld functor slightly differs from that of [D2]. A theorem of Drinfeld states that if $\ell < n$, $D_\ell$ gives a categorical equivalence between $\mathcal{C}_\mathcal{H}_\ell$ and the certain subcategory of $\mathcal{C}_Y (\mathfrak{s} l_n)$ ([D2]). One can deduce its unpublished proof from the paper of Chari-Pressly [CP2], in which the categorical equivalence was generalized to the case between the affine Hecke algebra and the quantum affine algebra. However, the method in [CP2] does not apply to the case $\ell \geq n$.

2.2. The following proposition follows from the Frobenius-Schur duality.

**Proposition 3.** Let $M$ be an $\mathcal{H}_\ell$-module. Let $M = \bigoplus_\nu \mathbb{C} U(\nu)^{\oplus c_\nu}$ ($c_\nu \in \mathbb{Z}_{\geq 0}$) be a decomposition of $M$ as a $W_\ell$-module. Then,

$$D_\ell (M) \cong \bigoplus_{\nu, (\ell, \nu) \leq n} L(\nu')^{\oplus c_{\nu'}}$$

as a $\mathfrak{g} l_n$-module, where $\nu'$ is the transpose of a partition $\nu$ identified with the dominant integral weight of $\mathfrak{g} l_n$.

See [CP2] for the proof of the following proposition.

**Proposition 4.** Let $M_1$ and $M_2$ be representations of $\mathcal{H}_{\ell_1}$ and $\mathcal{H}_{\ell_2}$ respectively. Then,

$$D_{\ell_1} (M_1) \otimes D_{\ell_2} (M_2) \cong D_{\ell_1 + \ell_2} \left( \mathcal{H}_{\ell_1 + \ell_2} \otimes (\mathcal{H}_{\ell_1} \otimes \mathcal{H}_{\ell_2}) (M_1 \otimes M_2) \right)$$

as a $Y(\mathfrak{g} l_n)$ and $Y(\mathfrak{s} l_n)$-module.

The following formula was stated in [BGFP] as a conjecture.

**Proposition 5.** Let $M$ be an $\mathcal{H}_\ell$-module. Then,

$$\pi (t_{ab}(u)) \equiv \delta_{ab} + \sum_{i=1}^\ell \frac{1}{u - y_i} \otimes \tau_i (E_{ab}). \quad (2.2.1)$$

on the space $D_\ell (M)$. In particular, $\pi (t_{ab}^{(d)}(u))$ acts as $\sum_{i=1}^\ell y_i^d \otimes \tau_i (E_{ab})$.

**Proof.** We prove by induction on $k$ that

$$\left(1 + \frac{I_1}{u - \epsilon_1}\right) \left(1 + \frac{I_2}{u - \epsilon_2}\right) \ldots \left(1 + \frac{I_k}{u - \epsilon_k}\right) \equiv \left(1 + \sum_{i=1}^k \frac{1}{u - y_i} I_i\right)$$

on $D_\ell (M)$.

There is nothing to prove for $k = 1$. Let $k > 1$. By induction hypothesis,

$$\left(1 + \frac{I_1}{u - \epsilon_1}\right) \left(1 + \frac{I_2}{u - \epsilon_2}\right) \ldots \left(1 + \frac{I_k}{u - \epsilon_k}\right)
\equiv 1 + \left(\sum_{i=1}^{k-1} \frac{1}{u - y_i} I_i\right) + \frac{1}{u - \epsilon_k} I_k + \left(\sum_{i=1}^{k-1} \frac{1}{u - y_i}\right) \cdot \frac{1}{u - \epsilon_k} I_k I_k$$
Since \( I_i \cdot I_k = P_{ik} \cdot I_k \) and \( P_{ik} \equiv -K_{ik} \) on \( D_{\ell}(M) \),
\[
\frac{1}{u - \epsilon_k} I_k + \left( \sum_{i=1}^{k-1} \frac{1}{u - y_i} \right) \frac{1}{u - \epsilon_k} I_i I_k = \frac{1}{u - y_k} \left( u - y_k \sum_{i=1}^{k-1} K_{ik} \right) I_k = \frac{1}{u - y_k} \cdot (u - \epsilon_k) \cdot \frac{1}{u - \epsilon_k} I_k = \frac{1}{u - y_k} I_k.
\]

Let \( \Lambda_k = \sum_{i=1}^{k} \epsilon_i \in \mathbb{P}_n^+ \) for \( k = 0, \ldots, n \) and let \( \nu_{\Lambda_k} \) denote the highest weight vector of the simple \( \mathfrak{gl}_n \)-module \( L(\Lambda_k + \rho) \). Then we can identify \( \nu_{\Lambda_k} \) with the highest weight vector of the simple \( Y(\mathfrak{gl}_n) \)-module \( \psi_a(L(\Lambda_k + \rho)) \) \((a \in \mathbb{C})\). It can be checked directly that its weight \((\zeta_1(u), \ldots, \zeta_n(u))\) is given by
\[
\zeta_i(u) = \begin{cases} 1 + \frac{1}{u - i - a} & \text{if } 1 \leq i \leq k \\ 1 & \text{otherwise.} \end{cases} \quad (2.2.2)
\]

The following proposition, which is easily follows from Proposition 5, is due to Chari-Pressley \([\text{CP2}]\).

**Proposition 6.** Let \( a, b \) be complex numbers such that \( b - a + 1 = \ell \). Then, as a \( Y(\mathfrak{gl}_n) \)-module,
\[
D_{\ell}(\mathbb{C}[a, b]) \cong \begin{cases} \psi_a(L(\Lambda_\ell + \rho)) & \text{if } \ell \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2.3)
\]

### 3. Main results

#### 3.1. For a subspace \( M' \) of an \( \mathcal{H}_N \)-module \( M \), let \( D_{\ell}(M') \) denote the image of \( M' \) by the Drinfeld functor in \( D_{\ell}(M) \). The proof of the following proposition is in Section 4.

**Proposition 7.** Let \( M \) be an \( \mathcal{H}_N \)-module such that \( M \) is generated by some simple \( W_\ell \)-submodule \( U \) of \( M \). Suppose that \( D_{\ell}(U) \) is nonzero. Then, \( D_{\ell}(U) \) generates \( D_{\ell}(M) \) over \( Y(\mathfrak{gl}_n) \).

Now let \( r \in \mathbb{N} \). For \( \lambda \in \mathfrak{h}_r^\ast \), let
\[
S^{(n)}(\lambda) = \{ \mu \in \mathfrak{h}_r \mid (\lambda - \mu)(\epsilon_i) \in \{0, 1, \ldots, n\} \mbox{ for } i = 1, \ldots, r\}.
\]

For \( \lambda \in \mathfrak{h}_r^\ast \) and \( \mu \in S^{(n)}(\lambda) \), define a tensor product module \( M(\lambda, \mu) \) of \( Y(\mathfrak{gl}_n) \) by
\[
M(\lambda, \mu) := \psi_{\nu_{\mu_1}}(L(\Lambda_{\ell_1} + \rho)) \otimes \cdots \otimes \psi_{\nu_{\mu_r}}(L(\Lambda_{\ell_r} + \rho)), \quad (3.1.1)
\]

where \( \mu_i = \mu(\epsilon_i) \), \( \ell_i = (\lambda - \mu)(\epsilon_i) \). Here \( Y(\mathfrak{gl}_n) \) acts via the coproduct \((1.1.3)\). Let \( v_{\lambda, \mu} := v_{\lambda_{\ell_1}} \otimes \cdots \otimes v_{\lambda_{\ell_r}} \in M(\lambda, \mu) \). Then by \((2.2.2)\), \( t_i(u) \cdot v_{\lambda, \mu} = \zeta_{\lambda, \mu; i}(u)v_{\lambda, \mu} \) for
Let $S^{(n)}(\lambda; \ell) = S(\lambda; \ell) \cap S^{(n)}(\lambda)$. Notice that for $\mu \in S(\lambda; \ell)$, the condition $\mu \in S^{(n)}(\lambda; \ell)$ is equivalent to $\nu_{\lambda, \mu}(\epsilon_1) \leq n$, where the partition $\nu_{\lambda, \mu}$ is identified with a dominant integral weight (recall subsection 1.4).

**Theorem 8.**

(1) The Drinfeld functor sends a standard module of $\mathbb{H}_\ell$ to zero or a highest weight module of $Y(\mathfrak{gl}_n)$.

(2) More precisely, let $\lambda \in P^+_\ell$ and $\mu \in S(\lambda; \ell)$. Then,

$$D_\ell(K(\lambda, \mu)) \cong \begin{cases} 
M(\lambda, \mu) & \text{if } \mu \in S^{(n)}(\lambda; \ell), \\
0 & \text{otherwise}. 
\end{cases}$$

(3.1.3)

In particular, $M(\lambda, \mu)$ is highest weight with the highest weight vector $\nu_{\lambda, \mu}$ and the highest weight $\zeta_{\lambda, \mu}(u)$.

**Proof.** (1) Let $M$ be a standard module and suppose that $D_\ell(M) \neq 0$. Since $M$ is a standard module, $M \cong U(\nu) \oplus \bigoplus_{\gamma \supset \nu} U(\gamma)^{\otimes c_{\gamma}}$ and $M = \mathbb{H}_\ell \cdot U(\nu)$ for some partition $\nu$ of $\ell$ such that $\nu(\epsilon_1) \leq n$. By Proposition 3,

$$D_\ell(M) \cong L(\nu' + \rho) \oplus \bigoplus_{\gamma' < \nu', \gamma(\epsilon_1) \leq n} L(\gamma' + \rho)^{\otimes c_{\gamma}}$$

But by Proposition 3, the highest weight vector of $L(\nu' + \rho)$ generates $D_\ell(M)$ over $Y(\mathfrak{gl}_n)$ in this decomposition. Since the other $\mathfrak{gl}_n$-weights appearing in $D_\ell(M)$ is smaller than $\nu'$ with respect to the dominance order, it follows that $D_\ell(M)$ is a highest weight module whose highest weight vector is the $\mathfrak{gl}_n$-highest weight vector of $L(\nu' + \rho) \subset D_\ell(M)$. (2) The isomorphism (3.1.3) follows from Proposition 4 and Proposition 3. In fact, (3.1.3) holds without restriction $\lambda \in P^+_\ell$. The rest of the statement follows from (1). \hfill \square

**Remark 9.** The fact that $M(\lambda, \mu)$ is highest weight for $\lambda \in P^+_\ell$ was proved by Akasaka-Kashiwara (AK) in the case of the quantum affine algebra and by Nazarov-Tarasov (NT2) in the case of the Yangian. The above thorem provides another proof of it.

Let us call those $Y(\mathfrak{gl}_n)$-modules $M(\lambda, \mu)$ with $\lambda \in P^+_\ell$ and $\mu \in S^{(n)}(\lambda; \ell)$ standard tensor product modules of $Y(\mathfrak{gl}_n)$. By Theorem 8 (2), a standard tensor product module $M(\lambda, \mu)$ has a unique simple quotient, which is denoted by $V(\lambda, \mu)$. Then by (3.1.2), its Drinfeld polynomials $Q_{\lambda, \mu}(u) = (Q_{\lambda, \mu; 1}(u), \ldots, Q_{\lambda, \mu; n-1}(u))$ are calculated as

$$Q_{\lambda, \mu; k}(u) = \prod_{i=1, \ldots, r}^{\lambda_i - \mu_i = k} (u - \lambda_i),$$

where $\lambda_i = \lambda(\epsilon_i)$ and $\mu_i = \mu(\epsilon_i)$ (AK, CP2, NT2).
Theorem 10.
(1) The Drinfeld functor sends a simple \( \mathcal{H}_\ell \)-module to zero or a simple \( Y(\mathfrak{gl}_n) \)-module.
(2) More precisely, let \( \lambda \in P_\tau^+ \) and \( \mu \in S(\lambda; \ell) \). Then,
\[
D_\ell(\mathcal{L}(\lambda, \mu)) \cong \begin{cases} 
V(\lambda, \mu) & \text{if } \mu \in S^{(n)}(\lambda; \ell), \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. (1) Let \( \mathcal{L} \) be a simple \( \mathcal{H}_\ell \)-module. Suppose that \( D_\ell(\mathcal{L}) \neq 0 \). Let \( V \) be a proper \( Y(\mathfrak{gl}_n) \)-submodule of \( D_\ell(\mathcal{L}) \). We suppose that \( V \neq 0 \) and deduce a contradiction. Let \( L \) be a simple \( \mathfrak{gl}_n \)-submodule of \( V \). Then \( L = D_\ell(U) \) for some simple \( W_\ell \)-submodule \( U \) of \( M \). But since \( M \) is simple, \( M = \mathcal{H}_\ell \cdot U \). Thus by Proposition 3.2, \( D_\ell(M) = Y(\mathfrak{gl}_n) \cdot L \subset Y(\mathfrak{gl}_n) \cdot V \), which contradicts the assumption that \( V \) is proper. (2) follows from (1), Theorem \( \mathfrak{8} \) (2) and and the fact that \( \mathcal{L}(\lambda, \mu) \) contains the simple \( W_\ell \)-module \( U(\nu_\lambda, \mu) \) with multiplicity one.

Remark 11. By (3.1.4), it is easy to see that every simple \( Y(\mathfrak{sl}_n) \)-module appears as the image of a simple \( \mathcal{H}_\ell \)-module by the Drinfeld functor for some \( \ell \).

Remark 12. When \( \lambda - \rho \) and \( \mu - \rho \) are both dominant weights, \( V(\lambda, \mu) \) belongs to the class of representations called tame \( (\text{NTT}) \). Conversely, any simple tame module is isomorphic to \( V(\lambda, \mu) \) with \( \lambda - \rho, \mu - \rho \in P_\tau^+ \) for some \( \ell \).

3.2. Multiplicity Formula. For \( \lambda, \mu \in P_\tau^+ \), define
\[
W^{(n)}(\lambda, \mu) = \{ w \in W_\ell \mid w \cdot \mu \in S^{(n)}(\lambda; \ell) \} \subset W_\ell.
\]
Then \( S^{(n)}(\lambda; \ell) = \bigsqcup_{\mu \in P_\tau^+} \bigsqcup_{w \in W^{(n)}(\lambda, \mu)} \{ w \cdot \mu \} \). Notice that if \( \mu \in S^{(n)}(\lambda; \ell) \), then \( w \cdot \mu \in S^{(n)}(\lambda; \ell) \) for all \( w \in W_\lambda \).

Lemma 13. Let \( \lambda, \mu \in P_\tau^+ \) and \( w, w' \in W^{(n)}(\lambda, \mu) \). Then \( Q_{\lambda,w,\mu}(u) = Q_{\lambda,w',\mu}(u) \) if and only if \( w \equiv w' \) in the double coset \( W_\lambda \backslash W_\ell / W_\mu \).

Proof. First notice that the condition \( w \equiv w' \) in the double coset \( W_\lambda \backslash W_\ell / W_\mu \) is equivalent to the condition that the following sets of pairs of complex numbers are equal:
\[
\{(\lambda(\epsilon_i), w \cdot \mu(\epsilon_i)) \mid i = 1, \ldots, r \}, \quad \{(\lambda(\epsilon_i), w' \cdot \mu(\epsilon_i)) \mid i = 1, \ldots, r \}.
\]
Hence the direction \( \Leftarrow \) is easy to see. \( \Rightarrow \). Let
\[
\{(a_1, b_1), \ldots, (a_k, b_k)\}, \quad \{(a_1', b_1'), \ldots, (a_k', b_k')\}
\]
be the result of removing the common pairs from (3.2.1) so that \( a_i \geq a_j \) if \( a_i - a_j \in \mathbb{Z} \) for \( 1 \leq i < j \leq k \). We suppose that \( k \geq 1 \) and deduce a contradiction. By the assumption \( Q_{\lambda,w,\mu}(u) = Q_{\lambda,w',\mu}(u) \), the differences \( a_i - b_i \) and \( a_i - b_i' \) are all 0 or \( n \).

Now by the assumption, \( b_1 \neq b_1' \). We can assume \( a_1 = b_1 + n = b_1' \). Since \( \{b_i\}_{i=1}^k = \{b_i'\}_{i=1}^k \), there exists \( p \) such that \( b_p = b'_p \). Then \( a_p = a'_p \) since \( a_p = b_p \) or \( b_p + n \) and \( a_1 \geq a_p \). Hence \( (a_1, b_1) = (a_p, b_p) \), which contradicts the assumption that there is no common pair between the two sets in (3.2.2). \( \square \)
Proposition 14. Let $\lambda, \mu \in P^+_r$ and $w, w' \in W_r(n(\lambda, \mu))$. Then the following conditions are equivalent:

1. $M(\lambda, w \cdot \mu) \cong M(\lambda, w' \cdot \mu)$.
2. $V(\lambda, w \cdot \mu) \cong V(\lambda, w' \cdot \mu)$.
3. $w \equiv w'$ in the double coset $W_\lambda \backslash W_r / W_\mu$.

Proof. (1) $\Rightarrow$ (2) follows from Theorem 8 and the fact that a simple quotient of a highest weight module is unique. (2) $\Rightarrow$ (3) follows from Lemma 13 and (3) $\Rightarrow$ (1) follows from (1.4.8).

Let $W_r(n)(\lambda, \mu)$ denote the image of $W_r(n)(\lambda, \mu)$ in the double coset $W_\lambda \backslash W_r / W_\mu$. By Proposition 14, each correspondence

\[ w \mapsto V(\lambda, w \cdot \mu), \quad w \mapsto M(\lambda, w \cdot \mu) \]

defines an injective map from the set $W_r(n)(\lambda, \mu)$ to the set of equivalence classes of finite-dimensional $Y(\mathfrak{sl}_n)$-modules.

Let $\leq$ denote the Bruhat ordering in $W_r$ and let $P_w(z)$ denote the Kazhdan-Lusztig polynomial associated with the Weyl group $W_r$. In the following theorem we state the multiplicity formula of $Y(\mathfrak{sl}_n)$. For simplicity, we only consider the essential cases when the roots of Drinfeld polynomials are integers.

Theorem 15. Let $\lambda, \mu \in P^+_r$ and $w \in W_r(n)(\lambda, \mu)$.

1. The family \(\left\{V(\lambda, x \cdot \mu) \mid x \in W_r(n)(\lambda, \mu), xLR \geq wLR\right\}\) is exactly the set of all simple $Y(\mathfrak{sl}_n)$-modules which appear as the composition factors of the standard tensor product module $M(\lambda, w \cdot \mu)$. Moreover, their multiplicities are expressed as

\[ [M(\lambda, w \cdot \mu), V(\lambda, x \cdot \mu)] = P_{wLR,xLR}(1) \]

for $x \in W_r(n)(\lambda, \mu)$.

2. Conversely, the simple $Y(\mathfrak{sl}_n)$-module $V(\lambda, w \cdot \mu)$ is expressed as

\[ [V(\lambda, w \cdot \mu)] = \sum_{x \in W_r(n)(\lambda, \mu)} (-1)^{\ell(wLRw_0) - \ell(xw_0)} P_{xw_0,wLRw_0}(1) \cdot [M(\lambda, x \cdot \mu)]. \]

in the Grothendieck group of $C_{Y(\mathfrak{sl}_n)}$, where $w_0$ denotes the longest element of $W_r$.

Proof. Due to the well-known Kazhdan-Lusztig conjecture ([BR, BK]) and the translation principle ([Fad]), one has

\[ [M(\lambda, w \cdot \mu)] = \sum_{x \in W/W_\mu} P_{wR,xR}(1) \cdot [L(x \cdot \mu)] \]

(3.2.3)

\[ [L(\lambda, \mu)] = \sum_{x \in W_r} (-1)^{\ell(wLRw_0) - \ell(xw_0)} P_{xw_0,wRw_0}(1) \cdot [M(x \cdot \mu)], \]

(3.2.4)

for $\mu \in P^+_r$ and $w \in W/W_\mu$ in the Grothendieck group of the category $O_r$ of $\mathfrak{gl}_r$, where $w_R$ denote the longest length representative in the coset $wW_\mu$. 
Then by \((1.4.3), (1.4.10),\) Theorem 8 and Theorem 10, applying the exact functor \(D_t \circ F_\lambda\), one has
\[
[M(\lambda, w \cdot \mu)] = \sum_{x \in W_{t, \mu}^1(\lambda, \mu) \setminus W_L} P_{w_R, x \cdot L}(1) [V(\lambda, x \cdot \mu)] \tag{3.2.5}
\]
and
\[
[V(\lambda, w \cdot \mu)] = \sum_{x \in W_{t, \mu}^1(\lambda, \mu) \setminus W_L} (-1)^{\ell(w_R w \cdot 0) - \ell(x \cdot 0)} P_{w_R, x \cdot L w \cdot 0}(1) [M(\lambda, x \cdot \mu)] \tag{3.2.6}
\]
in the Grothendieck group of \(C_{\nu}(\mathfrak{sl}_n)\). Hence (2) is proved and (1) follows form Proposition 14 and (3.2.5).

Remark 16. If \(x_R \geq w\) (resp. \(x_L \geq w\)), then \(x_R \geq w_R\) and \(P_{w, x_R}(q) = P_{w_R, x_R}(q)\) (resp. \(x_L \geq w_L\) and \(P_{w, x_L}(q) = P_{w_L, x_L}(q)\)), where \(w_L\) denotes the longest length representative in the coset \(W_\lambda w\) (see [Hum, Corollary 7.14], for example). Hence it follows that \(P_{w, x_L}(q) = P_{w_R, x_L}(q) = P_{w_R, x_L}(q)\).

Remark 17. Let \(\lambda - \rho\) and \(\mu - \rho\) are both dominant integral weights, one can obtain the resolution of \(V(\lambda, \mu)\) by applying the exact functor \(D_t \circ F_\lambda\). This provides an alternative proof of the resolutions of the simple elementary module \((\text{NT}2)\) constructed by Cherednik \((\text{Che})\). A generalization of such resolutions is possible to some extent by using the generalized BGG resolution obtained in \(\text{[S]}\) (see \(\text{[S]}\) for the corresponding statement in the case of the degenerate affine Hecke algebra).

4. Proof of Proposition 7

Let \(M\) be an \(H_t\)-module and \(U \subset M\) be a simple \(W_t\)-module such that \(D_t(U) \neq 0\) as in the proposition. The proof is divided into 4 parts.

4.1. Let \(\tilde{C}_{H_t}\) be the category of \(H_t\)-modules who decompose into (possibly infinite) direct sum of finite-dimensional \(W_t\)-modules. Let \(\tilde{C}_{\nu}(\mathfrak{gl}_n)\) be the category of \(Y(\mathfrak{gl}_n)\)-modules who decompose into (possibly infinite) direct sum of finite-dimensional \(\mathfrak{gl}_n\)-modules. Extend the Drinfeld functor \(D_t\) to the functor from the category \(\tilde{C}_{H_t}\) to the category \(\tilde{C}_{\nu}(\mathfrak{gl}_n)\). It is easy to see that the extended functor \(D_t\) is still an exact functor.

Let \(\nu = \sum_{i=1}^n \nu_i \epsilon_i\) be the partition such that \(U \cong U(\nu')\), where \(\nu'\) denotes the transpose of \(\nu\) as before. Define an \(W_t\)-module
\[
J_W(\nu) := \mathbb{C}[W_t] \otimes (\mathbb{C}[W_{\nu_1}] \otimes \ldots \otimes \mathbb{C}[W_{\nu_n}]) (\mathbb{C}1_{\text{sign}, \nu_1} \otimes \ldots \otimes \mathbb{C}1_{\text{sign}, \nu_n}),
\]
where \(\mathbb{C}1_{\text{sign}, \nu_i}\) is one-dimensional representation of \(\mathbb{C}[W_{\nu_i}]\) such that \(s_i \cdot 1_{\text{sign}, \nu_i} = -1_{\text{sign}, \nu_i}\). It is well-known that there exists a surjective homomorphism \(\varphi_W : J_W(\nu) \to U(\nu')\) of \(W_t\)-modules. Let \(J(\nu)\) be an \(H_t\)-module defined by
\[
J(\nu) := H_t \otimes_{\mathbb{C}[W_t]} J_W(\nu).
\]
Then \(J(\nu)\) is an object of \(\tilde{C}_{H_t}\). Let \(\varphi : J(\nu) \to M\) be the surjective \(H_t\)-homomorphism induced by \(\varphi_W\). Then, the \(Y(\mathfrak{gl}_n)\)-homomorphism
\[
D_t(\varphi) : D_t(J(\nu)) \longrightarrow D_t(M)
\]
is surjective and \(D_t(\varphi)(J_W(\nu)) = D_t(U)\). Hence it is suffice to prove that
\[
D_t(J(\nu)) = Y(\mathfrak{gl}_n) \cdot D_t(J_W(\nu)). \tag{4.1.1}
\]
4.2. Recall the filtrations on $\mathcal{H}_\ell$ and $Y(\mathfrak{gl}_n)$. For a gr $\mathcal{H}_\ell$-module $\bar{M}$, let $\bar{D}_\ell(\bar{M})$ denote the gr $Y(\mathfrak{gl}_n)$-module $\bar{D}_\ell(\bar{M}) := (\bar{M} \otimes (\mathbb{C}^n)^{\otimes \ell})/\sum \text{Im}(s_i + 1)$, in which $U(\mathfrak{gl}_n[t]) \cong \text{gr} Y(\mathfrak{gl}_n)$ acts as

$$E_{ij} \otimes t^r \mapsto \sum_{u=1}^{\ell} \bar{y}_u \otimes \tau_u(E_{ij}).$$

Then $\bar{D}_\ell$ defines an exact functor from the category of gr $\mathcal{H}_\ell$-modules which decomposes into direct sum of finite-dimensional $W_\ell$-modules to the category of $U(\mathfrak{gl}_n[t])$-modules decomposes into direct sum of finite-dimensional $\mathfrak{gl}_n$-modules.

Now introduce a filtration on $J(\nu)$ by the followings;

$$F_{-1}J(\nu) = 0, F_0J(\nu) = J_W(\nu), F_iJ(\nu) = (F_i\mathcal{H}_\ell) \cdot (F_0J(\nu)) \quad (i \in \mathbb{N}).$$

Let $\bar{J}(\nu)$ denote the the corresponding graded module of $J(\nu)$. Then, as a gr $\mathcal{H}_\ell$-module,

$$\bar{J}(\nu) = \text{gr} \mathcal{H}_\ell \cdot J_W(\nu) \cong (\mathbb{C}[\bar{y}_1, \ldots, \bar{y}_\ell] \otimes \mathbb{C}[W_\ell]) \otimes_{\mathbb{C}[W_\ell]} J_W(\nu).$$

In particular, $\bar{J}(\nu) \cong \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_\ell] \otimes_{\mathbb{C}} J_W(\nu)$ as a $\mathbb{C}$-vector space.

Introduce a filtration on $D_\ell(J(\nu))$ induced from that of $J(\nu)$;

$$F_{-1}D_\ell(J(\nu)) := D_\ell(F_0J(\nu)). \quad (4.2.1)$$

By Proposition 18, one can easily check that $F_iY(\mathfrak{gl}_n) \cdot F_iD_\ell(J(\nu)) \subset F_{i+j}D_\ell(J(\nu))$.

Let $\text{gr} D_\ell(J(\nu))$ denote the corresponding graded module of $D_\ell(J(\nu))$. Then, by Proposition 18 and the fact that $\bar{D}_\ell$ is exact, one has

$$\text{gr} D_\ell(J(\nu)) \cong \bar{D}_\ell(J(\nu))$$

as a gr $Y(\mathfrak{gl}_n) \cong U(\mathfrak{gl}_n[t])$-module. Now (4.1.1) is reduced to the following proposition.

**Proposition 18.** As a $U(\mathfrak{gl}_n[t])$-module,

$$\bar{D}_\ell(J(\nu)) = U(\mathfrak{gl}_n[t]) \cdot \bar{D}_\ell(J_W(\nu)),$$

where $\bar{D}_\ell(J_W(\nu))$ denotes the image of $J_W(\nu) \subset J(\nu)$ in $\bar{D}_\ell(J(\nu))$.

4.3. As a preparation for the proof of Proposition 18, we shall first consider the simplest case when $\nu = \ell \epsilon_i$ for some $i$. Put $\bar{J}(\ell) = \bar{J}(\ell \epsilon_i)$. Then as a gr $\mathcal{H}_\ell$-module,

$$\bar{J}(\ell) \cong \mathbb{C}[\bar{y}_1, \ldots, \bar{y}_\ell]1_{\text{sign}},$$

where $W_\ell$ act on the right-hand-side by $w \cdot (f 1_{\text{sign}}) = (-1)^{\ell(w)}w(f)1_{\text{sign}}$.

Let $\mathbf{i} = (i_1, \ldots, i_n)$ be a permutation of $(1, 2, \ldots, n)$. For $\gamma = \sum_{i=1}^n \gamma_i \epsilon_i \in \text{Wt} ((\mathbb{C}^n)^{\otimes \ell})$, let

$$u_i(\gamma) := u_{i_1}^{\otimes \gamma_i} \otimes u_{i_2}^{\otimes \gamma_i} \otimes \cdots \otimes u_{i_n}^{\otimes \gamma_i} \in (\mathbb{C}^n)^{\otimes \ell}.$$

**Lemma 19.** For a fixed permutation $\mathbf{i} = (i_1, \ldots, i_n)$ of $(1, 2, \ldots, n)$, the set

$$\left\{ \left[ y_1^{d_1} y_2^{d_2} \cdots y_\ell^{d_\ell} 1_{\text{sign}} \otimes u_i(\gamma) \right] \mid \gamma = \sum_{i=1}^n \gamma_i \epsilon_i \in \text{Wt} ((\mathbb{C}^n)^{\otimes \ell}), d_m \geq d_{m+1} \text{ if } \sum_{j=1}^{m-1} \gamma_i < m < \sum_{j=1}^{m} \gamma_i \right\}$$

for some $a$.

forms a $\mathbb{C}$-basis of $\bar{D}_\ell(J(\ell))$. (4.3.1)
Proof. Clearly
\[ \tilde{D}_t (\tilde{J}(t)) \cong \bigoplus_{\gamma \in \text{Wt}(\mathbb{C}^n)^{\otimes t}} (\tilde{J}(t) \otimes \left[(\mathbb{C}^n)^{\otimes t}\right]_{\gamma})/\sum \text{Im}(s_i + 1) \]
as a \mathbb{C}-vector space. In the case when \( \gamma = \ell \epsilon_i \) for some \( i \), it is well-known that
\[ (\tilde{J}(t) \otimes \left[(\mathbb{C}^n)^{\otimes t}\right]_{\ell \epsilon_i})/\sum \text{Im}(s_i + 1) = \bigoplus_{d_1 \geq \cdots \geq d_t} \mathbb{C}[y_1^{d_1} \cdots y_t^{d_t} 1_{\text{sign}} \otimes u_t^{\otimes t}] \quad (4.3.2) \]
For a general weight \( \gamma \in \text{Wt}(\mathbb{C}^n)^{\otimes t} \), notice that the correspondence \( w \mapsto w \cdot u_t(\gamma) \) defines an isomorphism \( \mathbb{C}[W_{\ell} / W_{\gamma_1} \times \cdots \times W_{\gamma_n}] \cong \left[(\mathbb{C}^n)^{\otimes t}\right]_{\gamma} \). On the other hand, \( \tilde{J}(t) \cong \tilde{J}(\gamma_1) \otimes \cdots \otimes \tilde{J}(\gamma_n) \) as a \( W_{\gamma_1} \times \cdots \times W_{\gamma_n} \)-module. Hence, by using the same argument as in the Proposition 18, the statement reduces to (4.3.2).

For \( k \in \{1, \ldots, n\} \), let \( U(\mathfrak{g}_{n}[t])_k \) denote the subalgebra of \( U(\mathfrak{g}_{n}[t]) \) generated by the elements \( E_{ik} \otimes t^r \) \( (i = 1, \ldots, n, \ r \in \mathbb{Z}_{\geq 0}) \).

Lemma 20. For any \( k \in \{1, \ldots, n\} \),
\[ \tilde{D}_t (\tilde{J}(t)) \cong U(\mathfrak{g}_{n}[t])_k \cdot [1_{\text{sign}} \otimes u_k^{\otimes t}] \quad (4.3.3) \]
Proof. Let \( i = (i_1, \ldots, i_n) = (1, 2, \ldots, k-1, k+1, k+2, \ldots, n, k) \). For \( m = 0, 1, \ldots, \ell \), let \( \tilde{D}_t (\tilde{J}(t))_{k,m} \) be the subspace of \( \tilde{D}_t (\tilde{J}(t)) \) spanned by the vectors of the form \( \tilde{J}(1) \) such that \( d_j = 0 \) for all \( j > m \). We prove by induction on \( m \) that \( \tilde{D}_t (\tilde{J}(t))_{k,m} \subset U(\mathfrak{g}_{n}[t])_k \cdot [1_{\text{sign}} \otimes u_k^{\otimes t}] \).

Let \( m = 0 \). Then for any \( \gamma = \sum_{i=1}^n \gamma_i \epsilon_i \in \text{Wt}(\mathbb{C}^n)^{\otimes t} \),
\[ (E_{i_1,k})^{\gamma_1} (E_{i_2,k})^{\gamma_2} \cdots (E_{i_{m-1},k})^{\gamma_{m-1}} \cdot [1_{\text{sign}} \otimes u_k^{\otimes t}] \]
where \( E_{ij} = E_{ij} \otimes 1 \in \mathfrak{g}_{n}[t] \). Hence \( \tilde{D}_t (\tilde{J}(t))_{k,0} \subset U(\mathfrak{g}_{n}[t])_k \cdot [1_{\text{sign}} \otimes u_k^{\otimes t}] \).

Next let \( m > 0 \) and suppose that \( \tilde{D}_t (\tilde{J}(t))_{k, m-1} \subset U(\mathfrak{g}_{n}[t])_k \cdot [1_{\text{sign}} \otimes u_k^{\otimes t}] \).

Let \( a \) be the integer such that \( \sum_{j=1}^a \gamma_j < m \leq \sum_{j=1}^a \gamma_j \). Then, one has
\[ (E_{i_a,k} \otimes t^m) \cdot [g_{y_{1}^{d_1} y_{2}^{d_2} \cdots y_{m-1}^{d_{m-1}} 1_{\text{sign}}}] \otimes u_k(\gamma - \epsilon_{i_a} + \epsilon_k) \]
for \( d_m > 0 \). Here the equality holds modulo \( \tilde{D}_t (\tilde{J}(t))_{k, m-1} \) if \( a = n \) (i.e. \( i_a = k \)). Hence by induction hypothesis, \( \tilde{D}_t (\tilde{J}(t))_{k, m} \subset U(\mathfrak{g}_{n}[t])_k \cdot [1_{\text{sign}} \otimes u_k^{\otimes t}] \). \( \square \)

4.4. Let turn ourselves back to the proof of Proposition 18. The following lemma is an analogue of Proposition 4.

Lemma 21. As a \( \mathfrak{g}_{n}[t] \)-module,
\[ \tilde{D}_t (\tilde{J}(\nu)) \cong \tilde{D}_{\nu_1} (\tilde{J}(\nu_1)) \otimes \tilde{D}_{\nu_2} (\tilde{J}(\nu_2)) \otimes \cdots \otimes \tilde{D}_{\nu_n} (\tilde{J}(\nu_n)) \]

Now let us complete the proof of Proposition 18. Let \( v_i = [1_{\text{sign}} \otimes u_k^{\otimes t}] \) and \( v_{[k]} := v_k \otimes \cdots \otimes v_n \). Notice that \( U(\mathfrak{g}_{n}[t])_k \cdot v_a \subset \mathbb{C} v_a \) if \( a \neq k \). Hence one can show by induction on \( k \) that
\[ U(\mathfrak{g}_{n}[t])_1 \cdot v_1 \otimes U(\mathfrak{g}_{n}[t])_2 \cdot v_2 \otimes \cdots \otimes U(\mathfrak{g}_{n}[t])_k \cdot v_k \otimes \mathbb{C} v_{[k+1]} \subset U(\mathfrak{g}_{n}[t]) \cdot v_{[1]} \].

Now Proposition 18 follows from Lemma 20.
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