GENERAL CYCLING OPERATIONS IN GARSIDE GROUPS

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Abstract. In this article, we introduce the notion of cycling operations of arbitrary order in Garside groups, which is a full generalization of the cycling and decycling operations. Theoretically, this notion together with other related concepts provides a context in which various definitions and arguments concerning Garside groups are unified and simplified as well as improved. Practically, it yields a new algorithm which has a considerably improved performance on solving the conjugacy problem of reducible braids.

Key words. Garside groups, braid groups, conjugacy problem, cycling operation, summit set.

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1. Introduction

The solution of the conjugacy problem in braid groups backdated to Garside [9] who established the first algorithm to solve the problem by means of calculating a conjugacy invariant of braids, the so called summit set. In the past decade, with many efforts (for example, [5, 2, 3, 8]) put on a refined version of the summit set, the super summit set, the algorithm was improved in various aspects. The algorithm and its improvements were also applied to a large family of groups, known as the Garside groups or small Gaussian groups [7].

Recent progress on this issue was addressed to [10], in which the super summit set was refined again to the ultra summit set by posing the cycling-recurrence condition. Remarkably, the algorithm resulted is so efficient that it makes practically possible to solve the conjugacy problem of generic braids (pseudo-Anosov braids) with large number of strands and word length.

Nevertheless, in contrast with such success, when confined to a specific class of braids (but still generic in practical sense), the reducible braids, even the best algorithm due to [10] practically fails. We will justify this point in Section 6 by giving examples and experimental data. To sum up, in the case of reducible braids, the cycling-recurrence condition loses its control on the components, so the performance of ultra summit set degenerates to the level of super summit set.

To remedy this deficiency, a natural way is to further refine the ultra summit set by posing the cycling-recurrence condition on the components of a reducible braid. At first sight, applying cycling operation on the components requires knowledge of the reduction system of a reducible braid. However, this is not the...
case. The refinement is easily implemented by introducing the notion of cycling operations of arbitrary order in Garside groups, which is a full generalization of the cycling and decycling operations. With a slight modification to the algorithms for computing super summit set and ultra summit set, one is able to compute the fully refined summit set effectively and achieve great performance improvement on solving the conjugacy problem of reducible braids.

Apart from practical significance, the notion of the general cycling operations turns out to be a very fundamental concept. Together with the concepts of pushforward and pullback along general cycling operations, it provides a context in which various definitions and arguments concerning Garside groups are unified and simplified, hence sheds light on these aspects (see Section 7 and the end of Section 2). From the theoretical point of view, these new concepts provide a very convenient and powerful tool for future study of Garside groups.

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## 2. Notations and basic facts

The notion of Garside group [7] is a natural generalization of braid group and, more generally, Artin group of finite type. In this section, we give a brief review of Garside groups and state some basic facts and know results for later use or comparison. Readers are referred to [2, 7, 14, 6, 8, 10] for more details.

Let $M$ be a monoid. We say $x \in M$ is an atom if $x \neq 1$ and $x = yz$ implies either $y = 1$ or $z = 1$. $M$ is said to be atomic if it is generated by its atoms and for every $x \in M$ there exists a finite number $\|x\|$, called the norm of $x$, such that $x$ is a product of at most $\|x\|$ atoms.

A cancellative, atomic monoid $M$ is said to be Gaussian if every two elements of $M$ have both a left (and right) greatest common divisor and a left (and right) least common multiple.
A Garside monoid $M$ is a Gaussian monoid which admits a Garside element. The Garside element is an element $\Delta \in M$ such that its left divisors coincide with its right divisors, they forming a finite set and generating $M$. The divisors of the Garside element are called simple elements.

Every Garside monoid admits a group of fractions. A group $G$ is called a Garside group if it is the group of fractions of a Garside monoid.

The braid groups are main examples of Garside groups. If we write the $n$-strand braid group in Artin presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i - j| \geq 2, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \quad |i - j| = 1 \right\rangle,$$

then the monoid given by the same presentation is a Garside monoid with Garside element (the half twist)

$$\Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1.$$  

This Garside structure of $B_n$ is referred to as the classical structure.

An alternative Garside structure of $B_n$ was given in [2], referred to as the dual structure or BKL structure. Since this structure will not be used in this article we omit the precise description here.

Throughout this article, let $G$ denote a Garside group associated with Garside monoid $M$ and Garside element $\Delta$. Let $S$ and $A$ denote the finite sets of simple elements and atoms of $G$, respectively.

For $x, y \in G$ we denote by $x \prec y$ the relation that $x$ is a left divisor of $y$, i.e. $x^{-1}y \in M$, by $x \wedge y$ and $x \vee y$ the left greatest common divisor and left least common multiple of $x, y$ respectively. The conjugation $u^{-1}xu$ is denoted as $x^u$ and the specific conjugation $\Delta^{-1}x\Delta$ is also denoted as $\tau(x)$.

A fundamental fact about a Garside group $G$ is that, for every $x \in G$, there is a unique decomposition $x = \Delta^{p_1}x_1 \cdots x_l$, called the (left) normal form of $x$, satisfying the conditions $x \wedge \Delta^{p+i} = \Delta^{p}x_1 \cdots x_i$ and $x_i \in S \setminus \{1, \Delta\}$. The infimum, supremum and canonical length of $x$ are defined to be $\inf x = p$, $\sup x = p + l$ and $\text{len } x = l$, respectively.

The following basic facts will be repeatedly used in the article without explanation.

(1) The relation $\prec$ is a partial order.
(2) $\tau(S) = S$. So $\tau^e = \text{id}$ for some $e > 0$ and $\Delta^e$ lies in the center of $G$.
(3) $\tau(M) = M$. So $x \prec y$ if and only if $\tau(x) \prec \tau(y)$. Moreover, we have $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$, $\tau(x \vee y) = \tau(x) \vee \tau(y)$ and $x\Delta \wedge y\Delta = (x \wedge y)\Delta$, $x\Delta \vee y\Delta = (x \vee y)\Delta$.
(4) $\Delta^{\inf x} \prec x \prec \Delta^{\sup x}$. So, $1 \prec x^n\Delta^{-n}\inf x$ and $1 \prec x^{-n}\Delta^{n}\sup x$ for $n \geq 0$.
(5) Each set $\{x \mid p_1 \leq \inf x, \sup x \leq p_2\}$ is finite. So, a sequence of sufficient length in it has repetitions.

The approach found by Garside to solve the conjugacy problem in a Garside group $G$ is to associate to each element $x \in G$ a computable, nonempty subset
\[ \tilde{C}(x) \subset C(x) \] which is only dependent on the conjugacy class \( C(x) \) of \( x \). Then given two elements \( x, y \in G \), one just computes and compares \( \tilde{C}(x) \) and \( \tilde{C}(y) \) to see whether \( x \) and \( y \) belong to the same conjugacy class. The summit set and its refinements, the super summit set and the ultra summit set, are such type of conjugacy invariants.

The **summit infimum**, **summit supremum** and **summit length** of the conjugacy class \( C(x) \) of \( x \) are

\[
\inf_s x = \max\{ \inf y \mid y \in C(x) \}, \\
\sup_s x = \min\{ \sup y \mid y \in C(x) \}, \\
\text{len}_s x = \sup_s x - \inf_s x.
\]

The **cycling** and **decycling** operations on \( x = \Delta^p x_1 \cdots x_l \) in normal form are the conjugations

\[
c(x) = x^{\Delta^p x_1 \Delta^- p} = \Delta^p x_2 \cdots x_l \tau^{-p}(x_1), \\
d(x) = x^{\Delta^p x_1 \cdots x_{l-1}} = \Delta^p \tau^p(x_l) x_1 \cdots x_{l-1}.
\]

Note that both operations neither decrease the infimum nor increase the supremum.

With these notations, one defines the **super summit set**

\[ C^s(x) = \{ y \in C(x) \mid \inf y = \inf_s x, \ \sup y = \sup_s x \}, \]

and the **ultra summit set**

\[ C^u(x) = \{ y \in C^s(x) \mid c^N(y) = y \text{ for some } N > 0 \}. \]

The finiteness of both sets are clear. The nonemptiness and the computability of these conjugacy invariants can be derived from the following theorems (see the references linked).

**Theorem 2.1** ([5, 2, 14]). If \( \inf x < \inf_s x \) then \( \inf c^N(x) > \inf x \) for some \( N > 0 \). Similarly, if \( \sup x > \sup_s x \) then \( \sup d^N(x) < \sup x \) for some \( N > 0 \).

**Theorem 2.2** ([8]). If \( x^u, x^v \in C^s(x) \) then \( x^{u \wedge v} \in C^s(x) \).

**Theorem 2.3** ([10]). If \( x^u, x^v \in C^u(x) \) then \( x^{u \wedge v} \in C^u(x) \).

As an evidence of the powerfulness of our new machinery, all these theorems will appear as easy corollaries in the next section.

### 3. New Definitions and Main Results

The **cycling operation of order** \( q \) on \( x \) is the conjugation

\[ c_q(x) = x^{x \Delta^q}. \]

We have the \( c_q \)-recurrence set

\[ G_q = \{ x \in G \mid c^N_q(x) = x \text{ for some } N > 0 \}. \]
The following properties are immediate from definition. In particular, the last one says that each sequence \( x, c_q(x), c_q^2(x), \ldots \) eventually runs into a closed orbit, so \( C(x) \cap G_q \) is always nonempty.

**Lemma 3.1. Properties of cycling operations.**

1. \( \tau c_q(x) = c_q \tau(x) \). So, \( \tau(G_q) = G_q \).
2. For \( x = \Delta^p x_1 \cdots x_l \) in normal form, we have
   \[
   c_q(x) = \begin{cases} 
   \tau^q(x), & q \leq \inf x, \\
   x_{q-p+1} \cdots x_l \Delta^p x_1 \cdots x_{q-p}, & \inf x < q < \sup x, \\
   x, & q \geq \sup x.
   \end{cases}
   \]
3. \( x \in G_q \) for \( q \leq \inf x \) or \( q \geq \sup x \).
4. \( \inf x \leq \inf c_q(x) \) and \( \sup c_q(x) \leq \sup x \).

These new cycling operations are indeed natural generalizations of the cycling and decycling operations. Note that
\[
c(x) = \tau^{-\inf x} c_{\inf x+1}(x), \quad d(x) = c_{\sup x-1}(x).
\]

In the next section we derive the following theorem. Since for each \( x \) the \( c_q \) orbit of \( x \) eventually runs into \( G_q \), it follows from the theorem that if \( \inf x < q \leq \inf x \) then \( c_q^N(x) \geq q \) holds for sufficient large \( N \) and a similar statement for supremum. In particular, the specific case \( q = \inf x + 1 \) or \( q = \sup x - 1 \) gives rise to Theorem 2.1.

**Theorem 3.2.** We have \( x \not\in G_q \) for \( \inf x < q \leq \inf s \) or \( \sup s \leq q < \sup x \).

In the sequel, the super summit set and the ultra summit set are nothing but
\[
C^s(x) = C(x) \cap \bigcap_{q \in \{\inf x, \sup x\}} G_q,
\]
\[
C^u(x) = C(x) \cap \bigcap_{q \in \{\inf x, \inf x+1, \sup x\}} G_q.
\]

The fully refined summit set we define here is
\[
C^s(x) = C(x) \cap \bigcap_{q \in \mathbb{Z}} G_q = C(x) \cap \bigcap_{\inf x \leq q \leq \sup x} G_q.
\]

Remark the obvious inclusions
\[ C^s(x) \subset C^u(x) \subset C^s(x). \]

The following theorem will also be proved in the next section. As an immediate consequence, we conclude that \( x^{u \wedge v} \in C^s(x) \) (resp. \( C^s(x), C^u(x) \)) provided \( x^u, x^v \in C^s(x) \) (resp. \( C^s(x), C^u(x) \)). So, we reach an alternative proof of Theorem 2.2 and Theorem 2.3.

**Theorem 3.3.** If \( x^u, x^v \in G_q \) then \( x^{u \wedge v} \in G_q \). In particular, \( c_q(G_p) \subset G_p \) for all \( p, q \in \mathbb{Z} \).
From Lemma 3.1(3) and the inclusion \( c_q(G_p) \subset G_p \), we have the following algorithm. In particular, the set \( C^*(x) \) is always nonempty.

**Algorithm 3.4.** Given an element \( x \) of \( G \), the following algorithm computes an element of \( C^*(x) \).

1. Set \( q = \inf x + 1 \).
2. **while** \( q < \sup x \) do
   1. Compute \( x, c_q(x), c^2_q(x), \ldots, c^N_q(x) \) until repetition encountered.
   2. Set \( x = c^N_q(x) \).
   3. Set \( q = q + 1 \).
3. **end while**
4. **return** \( x \)

Now we proceed to present an algorithm for computing the whole \( C^*(x) \).

Define the **full cycling trajectory** of \( x \in G \)

\[
T(x) = \{ c_{q_k} \cdots c_{q_2} c_{q_1}(x) \mid q_i \in \mathbb{Z}, k \geq 0 \}.
\]

The validity of the next algorithm follows from Lemma 3.1(1),(2).

**Algorithm 3.5.** Given an element \( x \) of \( G \), the following algorithm computes the full cycling trajectory \( T(x) \).

1. Set \( T = \{ x, \tau(x), \ldots, \tau^{e-1}(x) \} \) where \( e > 0 \) satisfies \( \tau^e = \text{id} \).
2. **for** \( y \in T \) do
   1. Set \( T = T \cup \{ c_q(y) \mid \inf y < q < \sup y \} \).
3. **end for**
4. **return** \( T \)

Let \( A(x) \) denote the set of \( \prec \)-minimal elements in \( \{ u \in S \backslash \{1\} \mid x^u \in C^*(x) \} \).

The following theorems are proved in the next section. Thanks to them we have Algorithm 3.8 for computing \( C^*(x) \).

**Theorem 3.6.** For each pair \( x_1, x_2 \in C^*(x) \) there exists a sequence

\[
y_1 = x_1, \ y_2, \ y_3, \ldots, \ y_k = x_2 \in C^*(x)
\]

such that \( y_{i+1} = y_{i}^{u_i} \) for some \( u_i \in A(y_i) \).

**Theorem 3.7.** For each pair \( x_1, x_2 \in C^*(x) \) with \( T(x_1) = T(x_2) \) and for each \( u_1 \in A(x_1) \), there exists \( u_2 \in A(x_2) \) such that \( T(x_1^{u_1}) = T(x_2^{u_2}) \).

**Algorithm 3.8.** Given an element \( x \) of \( G \), the following algorithm computes \( C^*(x) \).

1. Compute \( \tilde{x} \in C^*(x) \) and set \( \mathcal{T} = \{ T(\tilde{x}) \} \).
2. **for** \( T \in \mathcal{T} \) do
   1. Choose \( y \in T \) and set \( \mathcal{T} = \mathcal{T} \cup \{ T(y^u) \mid u \in A(y) \} \).
3. **end for**
4. **return** \( \bigcup_{T \in \mathcal{T}} T \)
Note that Algorithm 3.8 involves a computation of the set \( \{ T(y^u) \mid u \in A(y) \} \), which we will work out in Section 5 along the lines of [8, 10]. Although we can alternatively compute the superset \( \{ T(y^u) \mid u \in S \setminus \{1\}, y^u \in C^*(x) \} \) in the algorithm, which is much easier to be implemented, as argued in [8] this may decrease the performance considerably, because a Garside group may have a large number of simple elements while only a few atoms. For example, the braid group \( B_n \) endowed with the classical Garside structure has \( n! \) simple elements but only \( n - 1 \) atoms. So a delicate implementation of Algorithm 3.8 is necessary for practical use.

4. Pushforward and pullback I

The notions of pushforward and pullback were introduced in [10] (pushforward was called transport instead) where they were used to keep track of the cycling orbits of various conjugations of an element \( x \) and were proved to be very powerful in the study of ultra summit set.

These notions are also applicable for general cycling operations. Inspiringly, in this new setting they can be defined in a very concise form. The pushforward \( \phi_{x,q}(u) \) and pullback \( \pi_{x,q}(u) \) of \( u \) along the cycling operation \( x \to c_q(x) \) are defined as

\[
\phi_{x,q}(u) = x'' u \wedge x'^{-1} \Delta^q \tau^q(u),
\]

\[
\pi_{x,q}(u) = \Delta^\inf u \lor x''^{-1} u \lor x' \Delta^{-q} \tau^{-q}(u),
\]

respectively, where \( x' = x \wedge \Delta^q \) and \( x'' = x'^{-1} x \). We clarify these definitions by a pair of lemmas.

In what follows, if the context is clear we omit the subscripts of \( \phi, \pi \).

**Lemma 4.1.** Properties of pushforward.

1. \( (x \wedge \Delta^q) \phi(u) = u(x^u \wedge \Delta^q) \), so \( c_q(x)^{\phi(u)} = c_q(x^u) \). See the diagram below.

\[
\begin{array}{ccc}
x^u & \xrightarrow{x^u \wedge \Delta^q} & c_q(x^u) \\
\uparrow & & \uparrow \phi(u) \\
x & \xrightarrow{x \wedge \Delta^q} & c_q(x)
\end{array}
\]

2. \( \phi(\Delta^p) = \Delta^p \).
3. If \( u \prec v \) then \( \phi(u) \prec \phi(v) \).
4. \( \inf u \leq \inf \phi(u) \) and \( \sup \phi(u) \leq \inf u \).
5. \( \phi(u \wedge v) = \phi(u) \wedge \phi(v) \).
6. If \( x^v = x^u \) and \( \phi(u) = \phi(v) \) then \( u = v \).

**Proof.**

1. \( (x \wedge \Delta^q) \phi(u) = x' (x'' u \wedge x'^{-1} u \Delta^q) = x u \wedge u \Delta^q = u(x^u \wedge \Delta^q) \).
2. \( \phi(\Delta^p) = x'' \Delta^p \wedge x'^{-1} \Delta^q \Delta^p = x'^{-1} (x \wedge \Delta^q) \Delta^p = \Delta^p \).
3. If \( u \prec v \) then \( \phi(u) = x'' u \wedge x'^{-1} \Delta^q \tau^q(u) \prec x'' v \wedge x'^{-1} \Delta^q \tau^q(v) = \phi(v) \).
Properties of pullback.

(1) By the definition of pushforward, we have $\Delta^u \preceq u \preceq \Delta^v$, from (2) and (3) we have $\Delta^u = \phi(\Delta^u) \preceq \phi(u) \preceq \phi(\Delta^v) = \Delta^v$. Hence $\inf u \leq \inf \phi(u)$ and $\sup \phi(u) \leq \sup u$.

(5) $\phi(u \land v) = x''(u \land v) \land x''(u \land v) = x''(u \land x''v \land x''\Delta^q(u) \land x''\Delta^q(v) = \phi(u) \land \phi(v)$.

(6) By (1) we have $u = (x \land \Delta^q)\phi(u)(x^u \land \Delta^q)^{-1}$ and $v = (x \land \Delta^q)\phi(v)(x^v \land \Delta^q)^{-1}$. So $x^u = x^v$ and $\phi(u) = \phi(v)$ imply $u = v$.

Lemma 4.2. Properties of pullback.

(1) $\pi(u) \prec v$ if and only if $\inf u \leq \inf v$ and $u \prec \phi(v)$.

(2) $\pi(\Delta^p) = \Delta^p$.

(3) If $u \prec v$ then $\pi(u) \prec \pi(v)$.

(4) $\inf u \leq \inf \pi(u)$ and $\sup \pi(u) \leq \inf u$.

(5) $u \prec \phi\pi(u)$.

(6) If $\inf \phi(v) = \inf v$ then $\pi\phi(v) \prec v$.

Proof. (1) By the definition of pushforward, we have $u \prec \phi(v) \iff u \prec x''v$ and $u \prec x''\Delta^q(v) \iff x''u \prec v$ and $x''\Delta^q(u) \prec v \iff x''u \lor x''\Delta^q(u) \prec v$. Therefore, $\pi(u) \prec v$ if and only if $\Delta^u \prec v$ and $u \prec \phi(v)$.

(2) $\pi(\Delta^p) = \Delta^p \lor x''\Delta^p \lor x''\Delta^p = (1 \lor x''\Delta^p \lor x''\Delta^p) \Delta^p = \Delta^p$.

(3) If $u \prec v$ then $\pi(u) = \Delta^u \lor x''u \lor x''\Delta^q(u) \prec v \lor x''\Delta^q(v) = \pi(v)$.

(4) Follows from (2) and (3).

(5),(6) Apply (1) for $v = \pi(u)$ and $u = \phi(v)$ respectively.

For specific values of $q$, we have several more properties of the pushforward.

Lemma 4.3. For $q \leq \inf x$ we have

(1) $\tau^{-q}c_q(x) = x$ and $\tau^{-q}\phi(u) = u(x^u \land \Delta^q)\Delta^{-q}$, so $x^{-q}\phi(u) = \tau^{-q}c_q(x^u)$;

\[ \begin{array}{cccc}
  x^u & \xrightarrow{x^u \land \Delta^q} & c_q(x^u) & \xrightarrow{\Delta^{-q}} & \tau^{-q}c_q(x^u) \\
  u & \uparrow & \phi(u) & \uparrow & \tau^{-q}\phi(u) \\
  x & \xrightarrow{\Delta^q} & c_q(x) & \xrightarrow{\Delta^{-q}} & x
  \end{array} \]

(2) $\tau^{-q}\phi(u) \prec u$ with equality holds if and only if $\inf x^u \geq q$.

Similarly, for $q \geq \sup x$ we have

(1) $c_q(x) = x$ and $\phi(u) = u(x^u \land \Delta^q)$, so $\phi(u) = c_q(x^u)$;

(2) $\phi(u) \prec u$ with equality holds if and only if $\sup x^u \leq q$.

Proof. For $q \leq \inf x$ we have $x \land \Delta^q = \Delta^q$. Then (1) is a special case of Lemma 4.1(1). Moreover, from $\tau^{-q}\phi(u) = u(x^u \land \Delta^q)\Delta^{-q}$ we have $\tau^{-q}\phi(u) \prec u\Delta^q\Delta^{-q} = u$ with equality holds if and only if $\Delta^q \prec x^u$. Hence (2) holds.

The supremum part is proved similarly.
Combining pushforwards and pullbacks along single cycling operations, one has the pushforward and pullback along an arbitrary cycling orbit
\[ x \rightarrow c_{q_1}(x) \rightarrow c_{q_2}c_{q_1}(x) \rightarrow \cdots \rightarrow c_{q_k} \cdots c_{q_1}(x). \]
For example, the pushforward \( \phi_{x,q}^{(n)}(u) \) and pullback \( \pi_{x,q}^{(n)}(u) \) of \( u \) along the cycling orbit \( x \rightarrow c_{q}(x) \rightarrow c_{q}^2(x) \rightarrow \cdots \rightarrow c_{q}^n(x) \) are defined by induction as
\[ \phi_{x,q}^{(0)}(u) = u, \quad \phi_{x,q}^{(n)}(u) = \phi_{c_q^{-1}(x),q}^{(n-1)}(u), \quad \pi_{x,q}^{(0)}(u) = u, \quad \pi_{x,q}^{(n)}(u) = \pi_{x,q}^{(n-1)} \pi_{c_q(q(x),q)}^{(n-1)}(u). \]
Now suppose \( x \in G_q \) and let \( L \) be the \( c_q \)-orbit length of \( x \), i.e. the minimal positive integer such that \( c_q^L(x) = x \). The pushforward and pullback of \( u \) along the cycling orbit \( x \rightarrow c_q(x) \rightarrow c_q^2(x) \rightarrow \cdots \rightarrow c_q^L(x) \) will be denoted as
\[ \tilde{\phi}_{x,q}(u) = \phi_{x,q}^{(L)}(u), \quad \tilde{\pi}_{x,q}(u) = \pi_{x,q}^{(L)}(u). \]

The following proposition plays a crucial role in this article. Thanks to it, all theorems we claimed in the previous section are derived readily.

**Proposition 4.4.** Suppose \( x \in G_q \). Then \( x^u \in G_q \) if and only if \( \tilde{\phi}^N(u) = u \) for some \( N > 0 \).

**Proof.** Let \( L \) be the \( c_q \)-orbit length of \( x \). If \( \tilde{\phi}^N(u) = u \) then by Lemma 4.1(1)
\[ c_q^{NL}(x^u) = c_q^{NL}(x)^{\tilde{\phi}^N(u)} = x^u \text{ thus } x^u \in G_q. \]

Conversely, suppose \( x^u \in G_q \). By Lemma 4.1(4) the equality \( \tilde{\phi}^{N_1}(u) = \tilde{\phi}^{N_2}(u) \) holds for some \( 0 \leq N_1 < N_2 \). Then from Lemma 4.1(1) we have
\[ c_q^{NL}(x^u) = x^{\tilde{\phi}^{N_1L}(u)} = x^{\tilde{\phi}^{N_2L}(u)} = c_q^{N_2L}(x^u). \]
Therefore, \( c_q^{N_1L-i}(x^u) = c_q^{N_2L-i}(x^u) \) for \( i = 0, 1, \ldots, N_1L \), since \( x^u \in G_q \). By Lemma 4.1(1) again
\[ c_q^{N_1L-i}(x^u)^{\tilde{\phi}^{(N_1L-i)}(u)} = c_q^{N_1L-i}(x^u) = c_q^{N_2L-i}(x^u)^{\tilde{\phi}^{(N_2L-i)}(u)} = c_q^{N_1L-i}(x)^{\tilde{\phi}^{(N_2L-i)}(u)} \]
for \( i = 0, 1, \ldots, N_1L \). Note that in the last equality we used \( c_q^L(x) = x \). Finally, starting from \( \tilde{\phi}^{(N_1L)}(u) = \tilde{\phi}^{(N_2L)}(u) \) and applying Lemma 4.1(6) \( N_1L \) times we get \( u = \tilde{\phi}^{(N_2L-N_1L)}(u) \), i.e. \( u = \tilde{\phi}^{N_2-N_1}(u) \).

**Proof of Theorem 3.2.** Suppose \( x \in G_q \) for some inf \( x < q \leq \inf_q x \). Choose \( y \), \( u \) such that inf \( y \geq q \) and \( x = y^u \). By Lemma 4.3(2) we have
\[ (\tau^{-q} \phi_{y,q})^N(u) < (\tau^{-q} \phi_{y,q})^{N-1}(u) < \cdots < (\tau^{-q} \phi_{y,q})^{N-1}(u) < u. \]
Since both \( y, y^u \in G_q \), using the same argument in the proof of Proposition 4.4 we can show \( (\tau^{-q} \phi_{y,q})^N(u) = u \) for some \( N > 0 \). It follows that \( \tau^{-q} \phi_{y,q}(u) = u \).

By Lemma 4.3(2) we have inf \( y^u \geq q \), which contradicts the assumption inf \( y^u = \inf x < q \).

The other case of the theorem is proved similarly.
Proof of Theorem 3.3. For \( x \in G_q \), if \( x^u, x^v \in G_q \) then by Proposition 1.2 there exists \( N > 0 \) such that \( \tilde{\phi}^N(u) = u \) and \( \tilde{\phi}^N(v) = v \). So by Lemma 1.1(5) \( \tilde{\phi}^N(u \wedge v) = \tilde{\phi}^N(u) \wedge \tilde{\phi}^N(v) = u \wedge v \). Applying Proposition 1.2 again yields \( x^u \wedge v \in G_q \).

For general \( x \in G \), suppose \( x^w \in G_q \). Then \( (x^w)^{-1}u, (x^w)^{-1}v \in G_q \). Therefore \( x^u \wedge v = (x^w)^{-1}u \wedge v \in G_q \).

As to the inclusion \( c_q(G_p) \subset G_p \), one notices that if \( x \in G_p \) then \( x^u, x^v \in G_p \) hence \( c_q(x) = x(x^u \wedge v) \in G_p \).

□

Corollary 4.5. For \( x \in G_q, \phi_{x,q} \) restricts to a bijection
\[
\phi_{x,q} : \{ u \mid x^u \in C^*(x) \} \to \{ u \mid c_q(x)^u \in C^*(x) \}.
\]

Proof. For \( x^u \in C^*(x) \), it follows from Lemma 1.1(1) and the inclusion \( c_q(G_p) \subset G_p \) that \( c_q(x)^{\phi_{x,q}(u)} = c_q(x)^u \in C^*(x) \). Therefore, \( \phi_{x,q} \) does restrict to above map. By Proposition 1.2 the map is invertible.

□

Proof of Theorem 3.7. Suppose \( x_2 = x_1^u \). Multiplying a power of \( \Delta \) if necessary, we assume \( u \in M \). The theorem is proved by induction on the norm \( \|u\| \). First, set \( y_1 = x_1 \). If \( \|u\| = 0 \) then \( x_1 = x_2 \) and we have nothing to do. Otherwise, since \( x_1^u, x_1^\Delta \in C^*(x) \), by Theorem 3.3 \( x_1^u \wedge \Delta \in C^*(x) \) hence we can choose \( u_1 \in A(x_1) \) such that \( u_1 \preceq u \wedge \Delta \). Set \( y_2 = y_1^{u_1} \). Then \( y_2^{u_1} = x_2 \) but \( \|u_1\| < \|u\| \). By inductive hypothesis, there exists a sequence \( y_2, y_3, \ldots, y_k = x_2 \) such that \( y_{i+1} = y_i^{u_i} \) for some \( u_i \in A(y_i) \).

□

Proof of Theorem 3.7. Suppose \( x_1, x_2 \) are connected by a cycling orbit
\[
x_1 \to c_{q_1}(x_1) \to c_{q_2}c_{q_1}(x_1) \to \cdots \to c_{q_k} \cdots c_{q_2}c_{q_1}(x_1) = x_2.
\]

By Corollary 1.3 the pushforward \( \psi \) along the orbit restricts to a bijection
\[
\psi : \{ u \mid x_1^u \in C^*(x) \} \to \{ u \mid x_2^u \in C^*(x) \}
\]

By Lemma 1.1(2), (3), \( \psi \) further restricts to a bijection \( \psi : A(x_1) \to A(x_2) \). From Lemma 1.1(1), we have \( x_2^\psi(u_1) = c_{q_k} \cdots c_{q_2}c_{q_1}(x_1^{u_1}) \) thus \( T(x_2^\psi(u_1)) = T(x_1^{u_1}) \) for each \( u_1 \in A(x_1) \).

□

Till now, we have not made any use of the notion of pullback. We conclude this section by preparing the following proposition. Roughly speaking, \( \pi \)-recurrency guarantees lower bounds for \( \phi \)-orbits (see also Lemma 5.3(1)).

Proposition 4.6. Suppose \( x \in G_q \) and \( \pi^{N_1}(u) = \pi^{N_2}(u) \) for some \( 0 \leq N_1 < N_2 \). Then for every \( v \) satisfying \( \pi^{N_1}(u) \prec v \) there exists arbitrarily large \( N \) such that \( u \prec \pi^N(v) \).

Proof. From Lemma 1.2(5) and the hypotheses \( \pi^{N_1}(u) = \pi^{N_2}(u) \), \( \pi^{N_1}(u) \prec v \), we have
\[
u \prec \pi^{N_1+k(N_2-N_1)} \pi^{N_1+k(N_2-N_1)}(u) = \pi^{N_1+k(N_2-N_1)} \pi^{N_1}(u) \prec \pi^{N_1+k(N_2-N_1)}(v)
\]
for any \( k \geq 0 \).
5. Pushforward and Pullback II

In this section we derive a delicate implementation of Algorithm 3.8.

For \( x \in \cap_{q \in \mathbb{Z}} G_q \), let \( \Phi_x \) be the free group generated by \( \{ \tilde{\phi}_{x,q} \mid q \in \mathbb{Z} \} \) which, by Corollary 4.5, acts on the set \( \{ u \mid x^u \in C^*(x) \} \).

**Lemma 5.1.** Properties of \( \psi \).

1. \( T(x^{\psi(u)}) = T(x^u) \) for every \( \psi \in \Phi_x \).
2. \( \Phi_x \) preserves the partial order \( \prec \).
3. \( \inf u = \inf \psi(u) \) and \( \sup \psi(u) = \sup u \) for every \( \psi \in \Phi_x \).
4. If \( u \prec v \) and \( \Phi_x u = \Phi_x v \) then \( u = v \).

**Proof.** (1),(2),(3) Follows from Lemma 4.1(1),(3),(4) respectively, together with Lemma 5.3.

(4) Suppose \( v = \psi(u) \) for some \( \psi \in \Phi_x \). From (2) we have \( u \prec \psi(u) \prec \psi^2(u) \prec \cdots \). But by (3) \( \psi^N(u) = u \) holds for some \( N > 0 \). It follows that \( u = \psi(u) \), hence \( u = v \).

**Proposition 5.2.** The relation \( \prec \) defined on the set of \( \Phi_x \)-orbits by

\[
\Phi_x u \prec \Phi_x v \text{ if } \psi_1(u) \prec \psi_2(v) \text{ for some } \psi_1, \psi_2 \in \Phi_x
\]

is a partial order.

**Proof.** Reflexivity and transitivity are clear. Suppose \( \Phi_x u \prec \Phi_x v \prec \Phi_x u \), i.e. \( \psi_1(u) \prec \psi_2(v) \) and \( \psi_2'(v) \prec \psi_2''(u) \) for some \( \psi_1, \psi_2, \psi_1', \psi_2' \in \Phi_x \). Then by Lemma 5.1(2), \( u \prec \psi_1^{-1}\psi_2(v) \prec \psi_1^{-1}\psi_2\psi_2'\psi_2'^{-1}\psi_2''(u) \) and by Lemma 5.1(4), \( u = \psi_1^{-1}\psi_2\psi_2'\psi_2'^{-1}\psi_2''(u) \). Hence \( u = \psi_1^{-1}\psi_2(v) \) and \( \Phi_x u = \Phi_x v \), i.e. the relation is symmetric.

Thanks to Theorem 3.8 we have a well defined map

\[
\mu_x : G \to \{ v \mid x^v \in C^*(x) \}, \quad u \mapsto \wedge\{ v \mid u \prec v, x^v \in C^*(x) \}.
\]

That is, \( \mu_x(u) \) is the \( \prec \)-minimal element satisfying \( u \prec \mu_x(u) \) and \( x^{\mu_x(u)} \in C^*(x) \). Slightly abusing notation, we denote by \( \Phi_x u \) the \( \Phi_x \)-orbit of \( \mu_x(u) \) for arbitrary \( u \in G \).

By definition, \( \mu_x(u) \prec \mu_x(v) \) and \( \Phi_x u \prec \Phi_x v \) hold for \( u \prec v \). This fact will be used repeatedly in the remainder of this section.

**Lemma 5.3.** For \( x \in \cap_{q \in \mathbb{Z}} G_q \) we have the followings.

1. \( \Phi_x \tilde{\phi}_{x,q}(u) \prec \Phi_x u \).
2. \( \Phi_x \tilde{\pi}_{x,q}(u) \prec \Phi_x u \).

**Proof.** (1) \( \Phi_x \tilde{\phi}_{x,q}(u) \prec \Phi_x \tilde{\phi}_{x,q}\mu_x(u) = \Phi_x u \).

(2) By Lemma 5.1(3), \( \inf \tilde{\phi}_{x,q}\tilde{\phi}_{x,q}^{-1}\mu_x(u) = \inf \tilde{\phi}_{x,q}^{-1}\mu_x(u) \). Applying Lemma 4.2(6) yields \( \tilde{\pi}_{x,q}\tilde{\phi}_{x,q}\tilde{\phi}_{x,q}^{-1}\mu_x(u) \prec \tilde{\phi}_{x,q}^{-1}\mu_x(u) \). Therefore,

\[
\Phi_x \tilde{\pi}_{x,q}(u) \prec \Phi_x \tilde{\pi}_{x,q}\mu_x(u) = \Phi_x \tilde{\pi}_{x,q}\tilde{\phi}_{x,q}\tilde{\phi}_{x,q}^{-1}\mu_x(u) \prec \Phi_x \tilde{\phi}_{x,q}\mu_x(u) = \Phi_x u.
\]
Algorithm 5.4. Given \( x \in \cap_{q \in \mathbb{Z}} G_q \) and \( u \in G \), the following algorithm computes \( \mu_x(u) \).

Set \( l = \text{len} x \) and choose a permutation \( q_0, q_1, \ldots, q_l \) of the integers from \( \inf x \) to \( \sup x \).

Set \( u_0 = u \).

\[
\text{for } i = 0 \text{ to } l \text{ with step } +1 \text{ do }
\]

\[
\text{Compute } u_i, \pi_{x,q_i}(u_i), \pi_{x,q_i}^2(u_i), \ldots, \pi_{x,q_i}^N(u_i) \text{ until repetition encountered.}
\]

Set \( u_{i+1} = \pi_{x,q_i}^N(u_i) \).

\[
\text{end for}
\]

Set \( v_l = u_{l+1} \).

\[
\text{for } i = l \text{ to } 0 \text{ with step } -1 \text{ do }
\]

\[
\text{Compute } v_{i+1}, \tilde{\phi}_{x,q_i}(v_{i+1}), \tilde{\phi}_{x,q_i}^2(v_{i+1}), \ldots, \tilde{\phi}_{x,q_i}^N(v_{i+1}) \text{ until repetition encountered and } u_i < \tilde{\phi}_{x,q_i}^N(v_{i+1}).
\]

Set \( v_i = \tilde{\phi}_{x,q_i}^N(v_{i+1}) \).

\[
\text{end for}
\]

\text{return } v_0

Proof. First, remark that \( u_i < v_i \). By Proposition 4.4, there exists arbitrarily large \( N \) such that \( u_{i-1} < \tilde{\phi}_{x,q_i-1}^N(v_i) \), so the algorithm stops in finite steps. Moreover, we have \( \Phi_x u_i < \tilde{\phi}_{x,q_i}^N(v_i) \). Also notice that, by Lemma 5.3, \( \Phi_x v_0 < \Phi_x v_1 < \cdots \cdots < \Phi_x v_{l+1} = \Phi_x u_{l+1} < \cdots < \Phi_x u_1 < \Phi_x u_0 \). Summarizing, we have \( u_0 < v_0 \) and \( \Phi_x v_0 = \Phi_x v_0 \). Therefore, by Lemma 5.1(4), \( \mu_x(v_0) = \mu_x(u_0) \).

Further, remark that \( x^{v_i} \in G_{q_i} \) by Proposition 4.4. By Lemma 4.1(1) and the latter claim of Theorem 3.3, \( x^{v_i} \in G_q \) for \( \inf x \leq q \leq \sup x \). Since \( \inf x = \inf_s x \) and \( \sup_s x = \sup x \), it follows from the definition of \( C^*(x) \) that \( x^{v_i} \in C^*(x) \).

Hence \( v_0 = \mu_x(v_0) \).

Finally, we conclude that\( v_0 = \mu_x(v_0) = \mu_x(u_0) = \mu_x(u) \). \( \square \)

Remark the inclusion \( A(y) \subset \{ \mu_y(a) \mid a \in A \} \). With above algorithm one may implement Algorithm 3.8 by computing the superset \( \{ T(y^{\mu_y(a)}) \mid a \in A \} \) instead of \( \{ T(y^u) \mid u \in A(y) \} \), both having a cardinality not greater than the number of atoms of \( G \).

Moreover, as in [8], short-cuts can be used to increase the efficiency. Actually, the set computed by the following algorithm suffices for implementing Algorithm 3.8.

Algorithm 5.5. Given \( x \in \cap_{q \in \mathbb{Z}} G_q \), the following algorithm computes a set \( \mathcal{T} \) satisfying \( \{ T(x^u) \mid u \in A(x) \} \subset \mathcal{T} \subset \{ T(x^{\mu_x(a)}) \mid a \in A \} \).

Set \( Q = A \) and \( \mathcal{T} = \emptyset \).

\[
\text{for } a \in A \text{ do }
\]

\[
\text{Compute } \mu_x(a) \text{ by using Algorithm 5.4}
\]

\[
\text{if } \text{ meanwhile } a' < \pi_{x,q_i}(u_i) \text{ or } a' < \tilde{\phi}_{x,q_i}^N(v_i) \text{ for some } a' \in Q \setminus \{ a \} \text{ then}
\]

\[
\text{Set } Q = Q \setminus \{ a \}.
\]
else
    Set $\mathcal{T} = \mathcal{T} \cup \{T(\chi_{\mu(z)}(a))\}$.
end if
return $\mathcal{T}$

Proof. It is clear that $\mathcal{T} \subset \{T(\chi_{\mu(z)}(a)) \mid a \in A\}$. Suppose an atom $a$ is excluded from $Q$ by another atom, say $a_1$. Consider the sequence of atoms $a_1, a_2, \ldots, a_k$ in which $a_i$ is excluded from $Q$ by $a_{i+1}$ and $a_k$ survives in $Q$ when the algorithm stops. By Lemma 5.3, $\Phi \in \mathcal{T}$ is excluded from $Q$ if $a_{i+1} < \Phi \cdot a_i$ in which $\Phi \in \mathcal{T}$ from $Q$. It follows.

Proposition 5.6. Given $x = \Delta^p x_1 \cdots x_l$ in normal form, $0 \leq k \leq l$ and $u \in S \setminus \{\Delta\}$, define

$$
\begin{align*}
  u_0 &= \tau^p(u), \\
  u_i &= \Delta \wedge x_i^{-1}u_{i-1}\Delta \quad \text{for } i = 1, \ldots, k, \\
  u_{l+1} &= u, \\
  v_k &= u, \\
  v_{i-1} &= 1 \vee x_i v_i \Delta^{-1} \quad \text{for } i = k, \ldots, 1, \\
  v_{k+1} &= u, \\
  v_i &= 1 \vee x_i^{-1}v_i \quad \text{for } i = k + 1, \ldots, l.
\end{align*}
$$

Then $\phi_{x,p+k}(u) = u_k \wedge u_{k+1}$ and $\pi_{x,p+k}(u) = \tau^{-p}(v_0) \vee v_{l+1}$.

Proof. By induction one verifies the followings

$$
\begin{align*}
  u_i &= \Delta \wedge x_i^{-1} \cdots x_1^{-1}\tau^p(u)\Delta^i \quad \text{for } i = 0, \ldots, k, \\
  u_i &= \Delta \wedge x_i \cdots x_l u \quad \text{for } i = l + 1, \ldots, k + 1, \\
  v_i &= 1 \vee x_i+1 \cdots x_k u \Delta^{i-k} \quad \text{for } i = k, \ldots, 0, \\
  v_i &= 1 \vee x_i^{-1} \cdots x_{k+1} u \quad \text{for } i = k + 1, \ldots, l + 1.
\end{align*}
$$

Let $x' = \Delta^p x_1 \cdots x_k$ and $x'' = x_{k+1} \cdots x_l$. Note that $\phi_{x,p+k}(u) \in S$ and $\inf u = 0$. So

$$
\begin{align*}
  u_k \wedge u_{k+1} &= \left(\Delta \wedge x_k^{-1} \cdots x_1^{-1}\tau^p(u)\Delta^k\right) \wedge \left(\Delta \wedge x_{k+1} \cdots x_l u\right) \\
  &= \Delta \wedge x' u \Delta^{p+k} \wedge x'' u = \phi_{x,p+k}(u) \\
  \tau^{-p}(v_0) \vee v_{l+1} &= \tau^{-p}(1 \vee x_1 \cdots x_k u \Delta^{-k}) \vee (1 \vee x_l^{-1} \cdots x_{k+1}^{-1}) \\
  &= 1 \vee x' u \Delta^{-p-k} \wedge x''^{-1} u = \pi_{x,p+k}(u)
\end{align*}
$$

\[\square\]

6. Revisiting braid groups

In this section braid groups are supposed to be endowed with the classical Garside structure.
First, we argue that the algorithm by computing the ultra summit set practically fails for solving conjugacy problem of reducible braids. Let us consider a simple example. For any braid $\beta \in B_{n-1}$ with $\inf \beta > 0$, appending one additional trivial strand yields a reducible braid $\beta' \in B_n$. Note that the cycling operation on $\beta'$ is essentially trivial (merely the conjugation by the Garside element of $B_{n-1}$ on the subbraid $\beta$), so the cycling-recurrence condition is always satisfied. In the sequel, whenever $\beta$ lies in its super summit set, so does $\beta'$ in its ultra summit set. Therefore, the ultra summit set of $\beta'$ is at least as large as the super summit set of $\beta$. As argued in [10], computation of super summit set has been practically inaccessible for those braids with moderate number of strands and word length, thus so is the computation of ultra summit sets for such reducible braids.

However, one notices that the cycling operation on the components of a reducible braid can be achieved by applying general cycling operations on the total braid. In above example, to apply the cycling operation on $\beta'$ it suffices to apply $c_{\inf \beta + 1}$ on the total braid $\beta'$. Then, with the cycling-recurrence condition posed on the subbraid $\beta$, the cardinality $|C^*(\beta')|$ is comparable to $|C^u(\beta)|$, contrasting with the fact that $|C^u(\beta')|$ is not smaller than $|C^s(\beta)|$. Therefore, the set $|C^*(\beta')|$ can still be effectively computed and the conjugacy problem can be practically solved.

In the remainder of this section, we present some experimental data to compare the performance of the ultra summit set $C^u$ with the new summit set $C^*$ on solving conjugacy problem in braid groups. In all tables, each entry involves a computation of 5,000 random braids.

**Test 1.** This test compares the performance of $C^u$ and $C^*$ on the reducible braids described in above example. For several values of $n$ and $l$, we choose at random positive braids $\beta \in B_{n-1}$ with $\sup_s \beta = l$. Then, for each of them we append one additional trivial strand to make it into a reducible braid in $B_n$ and compute the summit sets $C^u$ and $C^*$ of the braid resulted. See Table 1.

Random braids are generated as follows. Choose independent random simple elements $x_1, x_2, \ldots$ until $\sup(x_1 \cdots x_k) = l$. Set $\beta = x_1 \cdots x_k$. Repeat this process until $\beta$ satisfies $\sup_s \beta = l$.

**Test 2.** This test compares the performance on a type of nested braids. For several values of $n$ and $l$ with $n$ a multiple of three, we choose at random positive braids $\beta \in B_3$ with $\sup_s \beta = l$ in the same way as previous test. Then, for each $\beta$ we choose independent random simple elements $x_{i1}, x_{i2}, \ldots, x_{il} \in B_{n/3}$ for $i = 1, 2, 3$ and replace each strand of $\beta$ by the braid $x_{i1}x_{i2} \cdots x_{il}$ to produce a nested braid of $n$ strands, then compute its summit sets. See Table 2.

**Test 3.** This test compares the performance on generic braids. For several values of $n$ and $l$, we choose at random positive braids $\beta \in B_n$ with $\text{len}_s \beta = l$ and compute the summit sets $C^u$ and $C^*$. See Table 3.
Table 1. Experimental data for Test 1. Average/maximal sizes of $C_u, C^*$ and average/maximal times $T_u, T^*$ spent on computing them. $1K=1,000$. Times are given in ms, unless stated otherwise.

| $n$ | $l$ | 3   | 5   | 10  | 20  | 30  | 40  |
|-----|-----|-----|-----|-----|-----|-----|-----|
|     | $|C_u|$ | 21.6/48 | 81.4/168 | 599/3000 | 2345/64K | 3760/239K | 4938/191K |
|     | $|C^*|$ | 11.9/32 | 15.9/80 | 25.2/160 | 43.0/220 | 60.9/216 | 81.0/172 |
|     | $T_u$ | 1/54 | 2/54 | 26/164 | 166/4560 | 364/23s | 603/23s |
|     | $T^*$ | 1/54 | 1/54 | 2/54 | 17/109 | 68/164 | 186/384 |

| $n$ | $l$ | 7   |
|-----|-----|-----|
|     | $|C_u|$ | 245/1824 | 7228/119K |
|     | $|C^*|$ | 31.3/288 | 27.2/612 |
|     | $T_u$ | 11/109 | 443/7472 |
|     | $T^*$ | 1/54 | 1/54 | 5/109 | 43/164 | 184/439 | 529/769 |

| $n$ | $l$ | 9   |
|-----|-----|-----|
|     | $|C_u|$ | 3676/188K |
|     | $|C^*|$ | 41.3/1320 | 29.4/528 |
|     | $T_u$ | 314/17s |
|     | $T^*$ | 1/54 | 2/54 | 10/54 | 79/274 | 317/769 | 883/1318 |

Random braids are generated in the same way as [10]. Choose at random an integer $p \in \{0, 1\}$ and choose independent random simple elements $x_1, x_2, \ldots$ until $\text{len}(x_1 \cdots x_k) = l$. Set $\beta = \Delta^p x_1 \cdots x_k$. Repeat this process until $\beta$ satisfies $\text{len}_n \beta = l$.

From above example and experimental data, we conclude that, with a slight loss of efficiency for generic braids (in the worst case, running time is prolonged approximately $\text{len}_n \beta$ times for a braid $\beta$), a considerable improvement is achieved on solving conjugacy problem of reducible braids by computing the new summit set $C^*$ instead of the ultra summit set $C_u$.

**Remark 6.1.** Although the new summit set $C^*$ is very likely bounded above by a polynomial function of word length for fixed number of strands (see also [2, 8] for the conjectures on the bound of super summit set), it is exponential in the number of strands. For example, fix a braid $\beta \in B_3$ with $|C^*(\beta)| > 1$, then the new summit set $C^*$ of the reducible braid of $3n$ strands yielded by juxtaposing $n$ copies of $\beta$ has a cardinality not smaller than $|C^*(\beta)|^n$ which is obviously exponential in $n$. 
Table 2. Experimental data for Test 2. Average/maximal sizes of $C^u, C^*$ and average/maximal times $T^u, T^*$ spent on computing them. 1K=1,000. Times are given in ms, unless stated otherwise.

| $n$ | $l$ | 2 | 3 | 5 | 10 | 15 | 20 |
|-----|-----|---|---|---|----|----|----|
| $|C^u|$ | 192/2740 | 267/6640 | 2681/197K | – | – | – |
| $|C^*|$ | 192/2740 | 66.4/2740 | 123/2880 | 416/14K | 1070/38K | 2770/125K |
| $T^u$ | 10/164 | 17/604 | 225/24s | – | – | – |
| $T^*$ | 4/109 | 1/54 | 3/109 | 29/879 | 160/11s | 913/231s |

| $n$ | $l$ | 12 |
|-----|-----|----|
| $|C^u|$ | 6064/355K | – |
| $|C^*|$ | 6064/355K | 445/77K | 614/20K | 3124/161K | 4121/80K | 18K/440K |
| $T^u$ | 641/38s | – | – | – | – | – |
| $T^*$ | 252/15s | 18/3021 | 29/824 | 406/26s | 1346/98s | 14s/979s |

Table 3. Experimental data for Test 3. Average/maximal sizes of $C^u, C^*$ and average/maximal times $T^u, T^*$ spent on computing them. Times are given in ms, unless stated otherwise.

| $n$ | $l$ | 20 |
|-----|-----|----|
| $|C^u|$ | 12.1/110 | 20.2/80 | 40.0/40 | 60.0/60 | 80.0/80 | 100.0/100 |
| $|C^*|$ | 12.1/110 | 20.2/80 | 40.0/40 | 60.0/60 | 80.0/80 | 100.0/100 |
| $T^u$ | 5/54 | 11/164 | 31/109 | 66/274 | 113/439 | 176/604 |
| $T^*$ | 5/109 | 24/164 | 172/659 | 542/1538 | 1593/5109 | 3447/9780 |

| $n$ | $l$ | 50 |
|-----|-----|----|
| $|C^u|$ | 10.0/20 | 20.0/20 | 40.0/40 | 60.0/60 | 80.0/80 | 100.0/100 |
| $|C^*|$ | 10.0/20 | 20.0/20 | 40.0/40 | 60.0/60 | 80.0/80 | 100.0/100 |
| $T^u$ | 24/109 | 38/109 | 106/329 | 208/604 | 351/1428 | 526/2307 |
| $T^*$ | 19/54 | 78/219 | 518/1703 | 1544/4505 | 4461/12s | 9609/23s |

So, along the lines of solving conjugacy problem in braid groups by computing some type of summit set, a polynomial algorithm both in number of strands and word length will inevitably involve a reduction process of reducible braids. Reader is referred to [1] for recent work on this direction. See also [4, 11] for efforts on the relation between the reduction systems and the super/ultra summit sets of reducible braids.
7. Other Applications

In this section we present several more applications of the machinery developed in the previous sections. More precisely, we give simple proves, but in strong forms, to several results in \[3, 12, 1\].

7.1. Summit infimum and supremum. The following main theorem of \[3\] gives rise to a bound for computing summit infimum and supremum of a braid conjugacy class by applying cycling and decycling operations. Indeed the argument involved there is applicable to all Garside groups. Recall that \(\|\Delta\|\) denotes the norm of the Garside element, which only depends on the Garside structure of a Garside group.

**Theorem 7.1** (\[3, Theorem 1\]). If \(\inf x < \inf y\) then \(\inf c^{\|\Delta\|-1}(x) > \inf x\). Similarly, if \(\sup x > \sup y\) then \(\sup d^{\|\Delta\|-1}(x) < \sup x\).

Below we give a simple proof to a stronger version of above theorem. (Note that the case \(q = \inf x + 1\) or \(q = \sup x - 1\) recovers above theorem.)

**Theorem 7.2.** If \(\inf x < q \leq \inf y\) then \(\inf c_q^{\|\Delta\|-1}(x) \geq q\). Similarly, if \(\sup x \leq q < \sup y\) then \(\sup c_q^{\|\Delta\|-1}(x) \leq q\).

**Proof.** We prove the first claim of the theorem and the latter claim can be proved in the same way. Suppose \(\inf x < q \leq \inf y\). By Theorem 3.2 there exists \(N > 0\) such that \(\inf c_N^q(x) \geq q\). Let \(N\) be the minimum in possible. We have to show that \(N < \|\Delta\|\).

Set \(y = (\tau^{-q}c_q)^N(x) = \tau^{-Nq}c_N^q(x)\) and \(\psi = \tau^{-q}\phi_{y,q}\). Then \(\inf y \geq q\). Choose \(u\) such that \(x = y^u\) and set \(u' = \psi^N(u)^{-1}u\). Applying Lemma 4.3(1) on the following diagram we obtain \(x = y^{u'}\) and \(\psi^n(u') = \psi^N(u)^{-1}\psi^n(u)\).

\[
\begin{array}{cccccc}
x & \longrightarrow & \tau^{-q}c_q(x) & \longrightarrow & \cdots & \longrightarrow & (\tau^{-q}c_q)^N(x) \\
\uparrow u & & \uparrow \psi(u) & & \cdots & & \uparrow \psi^N(u) \\
(\tau^{-q}c_q)^N(x) & \longrightarrow & (\tau^{-q}c_q)^N(x) & \longrightarrow & \cdots & \longrightarrow & (\tau^{-q}c_q)^N(x) \\
\uparrow \psi^N(u)^{-1} & & \uparrow \psi^N(u)^{-1} & & \cdots & & \uparrow \psi^N(u)^{-1} \\
(\tau^{-q}c_q)^N(x) & \longrightarrow & (\tau^{-q}c_q)^N(x) & \longrightarrow & \cdots & \longrightarrow & (\tau^{-q}c_q)^N(x)
\end{array}
\]

Moreover, by Lemma 4.3(2) and the minimality of \(N\) we have
\[1 = \psi^N(u') \preceq \psi^{N-1}(u') \preceq \cdots \preceq \psi(u') \preceq u'.\]

Note that by Lemma 4.4(2),(5) we have \(\psi^n(u' \wedge \Delta) = \psi^n(u' \wedge \Delta)\). We claim that \(\inf y^\psi^n(u' \wedge \Delta) < q\) for \(0 \leq n < N\). Otherwise, by Lemma 4.3(2)
\[1 = \psi^N(u' \wedge \Delta) = \psi^{N-1}(u' \wedge \Delta) = \cdots = \psi^n(u' \wedge \Delta).
\]
Hence we derive \(\psi^n(u') \wedge \Delta = 1\), i.e. \(\psi^n(u') = 1\), a contradiction.
Finally, applying Lemma 4.3(2) again yields
\[ 1 = \psi^N(u' \wedge \Delta) \preceq \psi^{N-1}(u' \wedge \Delta) \preceq \cdots \preceq \psi(u' \wedge \Delta) \preceq u' \wedge \Delta \succeq \Delta \]
which implies \( N < \|\Delta\|. \)

7.2. Stable summit set. Very recently, the behavior of the cycling and decycling operations on the powers of an element attracted much attention in the study of Garside groups. See, for example, \([1, 12, 13]\). From this point of view, one perhaps is willing to introduce the cycling operation of order \((p, q)\)
\[ c_{p,q}(x) = x^{xp} \wedge \Delta^q \]
and the \(c_{p,q}\)-recurrence set
\[ G_{p,q} = \{ x \in G \mid c_{p,q}^N(x) = x \text{ for some } N > 0 \}. \]
Notice that \( c_q(x^p) = (c_{p,q}(x))^p \), so applying a \( c_q \) operation on \( x^p \) is equivalent to applying a \( c_{p,q} \) operation on \( x \). In particular, \( x^p \in G_q \) if and only if \( x \in G_{p,q} \).

Most arguments and results concerning the single order cycling operations in this article can be generalized to the double order version straightforwardly. The pushforward and pullback along the cycling operation \( x \to c_{p,q}(x) \) are defined as
\[
\phi_{x,p,q}(u) = x''u \wedge x'^{-1}\Delta^q\tau^q(u), \\
\pi_{x,p,q}(u) = \Delta^{\inf}u \vee x'^{-1}u \vee x'\Delta^{-q}\tau^{-q}(u),
\]
respectively, where \( x' = x^p \wedge \Delta^q \) and \( x'' = x'^{-1}x^p \).

With a suitable modification, the algorithms for computing \( C^*(x) \) can be used to compute the set
\[ C^{[m,n],*}(x) = C(x) \cap \bigcap_{m \leq p \leq n, q \in \mathbb{Z}} G_{p,q} \]
which is hence nonempty.

However, it should be pointed out that the knowledge of present article does not lead to an algorithm to compute the refined summit set subject to all \( c_{p,q} \)-recurrence conditions
\[ C^{*,*}(x) = C(x) \cap \bigcap_{p,q \in \mathbb{Z}} G_{p,q}, \]
because we do not know how to bound the order \( p \). Nevertheless we have the following theorem.

**Theorem 7.3.** The set \( C^{*,*}(x) \) is nonempty.

**Proof.** Note that \( C^{*,*}(x) \) is the intersection of the descending sequence of finite, nonempty sets
\[ C^{[-1,1],*}(x) \supset C^{[-2,2],*}(x) \supset C^{[-3,3],*}(x) \supset \cdots. \]
Since a descending sequence of finite, nonempty sets always has nonempty intersection (we leave it to the reader as an easy exercise), the theorem follows. □

In the sequel, as supersets of $C^\ast(x)$ the stable super summit set

$$C(x) \cap \bigcap_{p \geq 1, q \in \{\inf x^p, \sup x^p\}} G_{p,q}$$

and the stable ultra summit set

$$C(x) \cap \bigcap_{p \geq 1, q \in \{\inf x^p, \inf x^p+1, \sup x^p\}} G_{p,q}$$

are both nonempty.

7.3. Rigid elements. Rigid elements became of interest in the conjugacy problem in Garside groups because, on the one hand, these elements have many nice properties and, on the other hand, these elements are generic enough, for example, it is shown in [1] that for every pseudo-Anosov braid $x$ some power of $x$ is rigid up to conjugacy.

In [1], an element $x = \Delta^px_1 \cdots x_l$ in normal form is said to be rigid if $l > 0$ and

$$\Delta^px_1 \cdots x_l \tau^{-p}(x_1)$$

is also in normal form. Actually the second condition is equivalent to say

$$x^2 = \Delta^p \tau^p(x_1) \cdots \tau^p(x_l) x_1 \cdots x_l$$

is in normal form. So the condition in the definitions can be stated more intrinsically as $\text{len } x > 0$ and

$$x^2 \wedge \Delta^{\inf x + \sup x} = x \Delta^{\inf x}.$$

As mentioned above, rigid elements have many nice properties. For example, a nontrivial power of a rigid element is also rigid. Another example is the behavior of the cycling operations on them is very simple. Indeed, the action of each $c_{p,q}$ on a rigid element $x = \Delta^px_1 \cdots x_l$ in normal form is merely a cyclic permutation together with some possible $\tau$ actions on the $x_i$’s. It follows that the rigid elements of a Garside group $G$ are contained in $\cap_{p,q \in \mathbb{Z}} G_{p,q}$

The following theorem is one of the main result in [1].

**Theorem 7.4** ([1] Theorem 3.22 and Theorem 3.34). (1) If $\text{len } x > 1$ and $x$ is rigid then the ultra summit set $C^u(x)$ is precisely the rigid conjugates of $x$.

(2) If $\text{len } x > 1$ and a power $x^N$ is conjugate to a rigid element, then $N$ can be chosen so that $0 < N < \|\Delta\|^2$.

Below we give an alternative proof to above theorem, with the boring condition $\text{len } x > 1$ dropped.
Theorem 7.5. (1) If \( x \) is rigid then the set \( C^*(x) \) is precisely the rigid conjugates of \( x \).

(2) If a power \( x^N \) is conjugate to a rigid element, then \( N \) can be chosen so that \( 0 < N < \|\Delta\|^2 \).

The proof depends on several lemmas.

Lemma 7.6. Suppose \( x \) is rigid and \( p = \inf x \), \( q = \inf x + \sup x \). Then for \( x^u \in C^s(x) \) we have \( u < \tau^{-p}\phi_{x,2,q}(u) \) with equality holds if and only if \( x^u \) is rigid.

Proof. By the definition of rigidity, \( x^2 \wedge \Delta = x^p \). A straightforward calculation shows that \( \tau^{-p}\phi_{x,2,q}(u) = x^{-1}u((x^u)^2 \wedge \Delta^q)\Delta^{-p} \) (see the diagram below).

\[
\begin{array}{c}
\xymatrix{
x^u \ar[r]^{(x^u)^2 \wedge \Delta} & c_{2,q}(x^u) \ar[r]^{\Delta^{-p}} & \tau^{-p}c_{2,q}(x^u) \\
x \ar[u] \ar[r]^{x \Delta^p} & c_{2,q}(x) \ar[u]_{\phi_{x,2,q}(u)} \ar[r]^{\Delta^{-p}} & x \\
}
\end{array}
\]

Since both \( x, x^u \in C^s(x) \), we have \( \text{len} x^u = \text{len} x > 0 \) and \( \inf x^u = p, \sup x^u = q - p \). Therefore,

\[
\inf x^u \leq u((x^u)^2 \wedge \Delta^q)\Delta^{-p} = x^{-1}u((x^u)^2 \wedge \Delta^q)\Delta^{-p} = \tau^{-q}\phi_{x,2,q}(u)
\]

with equality holds if and only if \( x^u \Delta^{-p} \wedge (x^u)^{-1} \Delta^q \Delta^{-p} = 1 \), i.e., \( (x^u)^2 \wedge \Delta^q = x^u\Delta^p \), that is, \( x^u \) is rigid. \( \square \)

Lemma 7.7. If \( x^m (m > 0) \) is rigid and \( \inf x^m = m \inf x, \sup x^m = m \sup x \) then \( x \) is also rigid.

Proof. Clearly \( \text{len} x = \text{len} x^m/m > 0 \). Let \( p = \inf x \), \( q = \inf x + \sup x \). Then \( \inf x^m = mp, \sup x^m = mq - mp \) and the rigidity of \( x^m \) says \( x^{2m} \wedge \Delta^{mq} = x^m\Delta^{mp} \). Therefore,

\[
x\Delta^p < x^2 \wedge \Delta^q < x^{(m+1)}\Delta^{-(m-1)p} \wedge x^{-(m-1)}\Delta^{-(m-1)p+mq}
\]

\[
= x^{-(m-1)}(x^{2m} \wedge \Delta^{mq})\Delta^{-(m-1)p} = x^{-(m-1)}(x^m\Delta^{mp})\Delta^{-(m-1)p} = x\Delta^p,
\]

which implies \( x^2 \wedge \Delta^q = x\Delta^p \). So, \( x \) is rigid. \( \square \)

Lemma 7.8. For every \( x \) there exist integers \( 0 < N_1, N_2 \leq \|\Delta\| \) such that \( \inf_s x^{nN_1} = n \inf_s x^{N_1} \) and \( \sup_s x^{nN_2} = n \sup_s x^{N_2} \) for all \( n > 0 \).

Proof. Let \( N_1, N_2 \) be the denominators of the rational numbers

\[
\max\{\inf_s x^n/n \mid n = 1, 2, \ldots, \|\Delta\|\}
\]

and

\[
\min\{\sup_s x^n/n \mid n = 1, 2, \ldots, \|\Delta\|\}
\]

respectively. Clearly, \( 0 < N_1, N_2 \leq \|\Delta\| \). Then \cite{13} Lemma 2.4(ii), Theorem 3.2 and Theorem 5.1(i)] says \( N_1, N_2 \) are exactly what we want. \( \square \)
Proof of Theorem 7.5. (1) Let \( p = \inf x, q = \inf x + \sup x \). Since the rigid elements of \( G \) are contained in \( \cap_{p,q \in \mathbb{Z}} G_{p,q} \), the rigid conjugates of \( x \) belong to \( C^{*,*}(x) \). Moreover, we have the obvious inclusion \( C^{*,*}(x) \subseteq C^*(x) \cap G_{2,q} \). So, it suffices to show that all elements of \( C^*(x) \cap G_{2,q} \) are rigid.

Suppose \( x^u \in C^*(x) \cap G_{2,q} \). By Lemma 7.6,
\[
\tau^{-p} \phi_{x,2,q}(u) \prec \cdots \prec (\tau^{-p} \phi_{x,2,q})^N(u).
\]
Since both \( x, x^u \in G_{2,q} \), by the same argument in the proof of Proposition 4.4 we have \( (\tau^{-p} \phi_{x,2,q})^N(u) = u \) for some \( N > 0 \). Therefore, \( u = \tau^{-p} \phi_{x,2,q}(u) \), and Lemma 7.6 says that \( x^u \) is rigid.

(2) Suppose a power of \( x \), say \( x^m \) with \( m > 0 \), is conjugate to a rigid element. By Theorem 7.3 we can choose \( y \in C^{*,*}(x) \). Then \( y^n \in C^{*,*}(x^n) \) hence \( \inf y^n = \inf y \) and \( \sup y^n = \sup y \) for all \( n \in \mathbb{Z} \).

Let \( N_1, N_2 \) be as in Lemma 7.8 and let \( N \) be the least common multiple of \( N_1, N_2 \). Note that \( N < \|\Delta\|^2 \); otherwise, we must have \( \|\Delta\| = 1 \), which implies \( G \) is the infinite cyclic group generated by \( \Delta \) hence has no rigid element.

Since \( x^{mN} \) is also conjugate to a rigid element, \( y^{mN} \in C^{*,*}(x^{mN}) \) is rigid by the conclusion of (1). Moreover, from the choice of \( N \) we have \( \inf y^{mN} = m \inf y \) and \( \sup y^{mN} = m \sup y \). By Lemma 7.6 \( y \) is rigid. This completes the proof of the theorem.

From the proof we have the follow algorithm for deciding whether some power of \( x \) is conjugate to a rigid element and computing the possible rigid element. First, compute the integers \( 0 < N_1, N_2 \leq \|\Delta\| \) as in Lemma 7.8 (which indeed can be done in polynomial time in the case of braid groups). Then, compute an element \( \tilde{x} \) of \( C^*(x^N) \cap G_{2,\inf x^N + \sup x^N} \) where \( N \) is the least common multiple of \( N_1, N_2 \). Then some power of \( x \) is conjugate to a rigid element if and only if \( \tilde{x} \) is rigid.

Moreover, the proof says for every rigid element \( x \) we have
\[
C^{*,*}(x) = C^*(x) \cap G_{2,\inf x + \sup x}
\]
which is precisely the rigid conjugates of \( x \).

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