NONABELIAN SUPERCONDUCTORS:
VORTICES AND CONFINEMENT
IN $\mathcal{N} = 2$ SQCD

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Abstract:

We study nonabelian vortices (flux tubes) in $SU(N)$ gauge theories, which are responsible for the confinement of (nonabelian) magnetic monopoles. In particular a detailed analysis is given of $\mathcal{N} = 2$ SQCD with gauge group $SU(3)$ deformed by a small adjoint chiral multiplet mass. Tuning the bare quark masses (which we take to be large) to a common value $m$, we consider a particular vacuum of this theory in which an $SU(2)$ subgroup of the gauge group remains unbroken. We consider $5 \geq N_f \geq 4$ flavors so that the $SU(2)$ sub-sector remains non asymptotically free: the vortices carrying nonabelian fluxes may be reliably studied in a semi-classical regime. We show that the vortices indeed acquire exact zero modes which generate global rotations of the flux in an $SU(2)_{C+F}$ group. We study an effective world sheet theory of these orientational zero modes which reduces to an $\mathcal{N} = 2$ $O(3)$ sigma model in (1+1) dimensions. Mirror symmetry then teaches us that the dual $SU(2)$ group is not dynamically broken.

July 2003
1. Introduction and Discussion

Some sort of nonabelian vortices are believed to be responsible for confinement in QCD. Although in string theory these objects appear naturally, they turn out to be somewhat elusive in four-dimensional field theories. The existing literature on the subject certainly provides an incomplete picture.

There are several reasons for this unsatisfactory situation. One of the reasons is that boundstates of vortices are not generally stable. An example is the case of an $SU(N)/\mathbb{Z}_N$ gauge theory (e.g., $SU(N)$ gauge theory with all fields in the adjoint representation) broken completely by a Higgs mechanism, where possible vortices represent nontrivial elements of the fundamental group

$$\pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N.$$  \hspace{1cm} (1.1)

$\mathbb{Z}_N$-charged objects cannot be BPS saturated \cite{1, 2}, and this fact, together with the unknown dependence of their properties on the form of the potential, number of the fields, etc., has obstructed investigations of such vortices.

Secondly, often these theories become strongly coupled at low energies and therefore an analytical study of the vortex configurations is very difficult. For instance, confinement in QCD may be due to the vortices of electric fields appearing in a dual (magnetic) $(SU(3), SU(2) \times U(1), \text{or } U(1)^2)$ theory. Unfortunately, neither the true nature of the effective magnetic degrees of freedom nor their form of interactions is known at the moment. 't Hooft’s suggestion that they be abelian monopoles of a gauge-fixed $U(1) \times U(1)$ theory \cite{3}, must still be verified. On the other hand, there is no experimental indication that the $SU(3)$ gauge group is dynamically broken to $U(1) \times U(1)$.

Finally, in the examples of classical solutions for “nonabelian vortices” discussed so far in the literature \cite{4} the vortex flux is actually always oriented in a fixed direction in the Cartan subalgebra, showing that they are basically abelian.

Useful hints come from the detailed study of a wide class of softly broken $\mathcal{N} = 2$ supersymmetric gauge theories where the dynamics appears particularly transparent. It was shown that, in fact, different types of confining vacua are realized in
these models \[5, 6, 7, 8, 9\]. It is possible that in some cases confinement is due to the condensation of monopoles associated with the maximally abelian subgroup (a dual Meissner effect), as in the \( \mathcal{N} = 1 \) vacua surviving the adjoint mass perturbation in the pure \( \mathcal{N} = 2 \) SYM \[5, 7, 8\]. These cases provided the first examples of four-dimensional gauge theory models in which the 't Hooft-Mandelstam mechanism of confinement \[3\] is realized and can be analysed quantitatively. A detailed study of these cases has shown however that dynamical abelianization takes place there, with a characteristically richer meson spectrum \[8, 1, 10, 11\]. Indeed the low-energy effective gauge group of the \( SU(N) \) theory is \( U(1)^{N-1} \) and the meson spectrum is classified according to the number of possible abelian strings via

\[
\pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1},
\]

(cfr. (1.1)). Thus vortices and therefore mesons come in infinite towers, a feature not expected in the real world QCD.

However, such is not the typical situation in softly broken \( \mathcal{N} = 2 \) theories with fundamental matter fields (quarks) \[12, 9\]. Confining vacua in \( SU(N) \), \( SO(N) \) and \( USp(2N) \) gauge theories with \( N_f \) quark flavor, are typically described by effective nonabelian dual gauge theories. For instance, in the so-called \( r \)-vacua of \( SU(N) \) gauge theory with \( N_f \) flavors and vanishing bare quark masses, the low-energy effective theory is a dual \( SU(r) \times U(1)^{N-r} \) theory. Addition of the adjoint chiral multiplet mass term \( \mu \text{Tr} \Phi^2 \) breaks supersymmetry to \( \mathcal{N} = 1 \), and the dual quarks in the \( r \) of \( SU(r) \) condense. These “dual quarks” have been recently identified \[13\] as the quantum Goddard-Nuyts-Olive-Weinberg monopoles \[14, 15\]. Their condensation is believed to give rise to nonabelian confinement via formation of nonabelian flux tubes.

In fact, the problem of nonabelian vortices is very closely related to (in a sense, it is one and the same problem as) that of the nonabelian monopoles on which they end. A key feature found in \[13\] is that the quantum behavior of the nonabelian monopoles, and in fact the vacuum properties themselves depend critically on the presence of massless flavors of matter. We shall find below that the existence of nonabelian vortices similarly requires the presence of massless flavors in the underlying theory.

Inspired by these developments, and based on a work by Marshakov and one of the authors (A.Y.) \[16\], we present in this paper a study of nonabelian superconductors, concentrating our attention on the properties of the vortices appearing in these systems. In a companion paper \[17\], we shall explore more extensively the properties of nonabelian monopoles themselves.
Our analyses are done in a context where the dynamics of the model is well understood and the transition from a theory with abelian vortices to one with nonabelian vortices can be studied in a weakly coupled semi-classical regime throughout. The model we consider is probably the simplest of such models, $\mathcal{N} = 2$ QCD with gauge group $SU(N)$ and $N_f$ hypermultiplets of fundamental matter (quarks). Upon deformation of this theory via a small mass term for the adjoint chiral multiplet, $\mu \text{Tr}\Phi^2$, the Coulomb branch of the theory shrinks to a number of isolated $\mathcal{N} = 1$ vacua.

Generically the vacuum expectation value (VEV) of the adjoint field breaks the $SU(N)$ gauge symmetry down to $U(1)^{N-1}$. However, it was shown in [12, 9] (see also [16]) that some of the $\mathcal{N} = 1$ vacua of $SU(N)\mathcal{N} = 2$ QCD preserve a nonabelian subgroup. These vacua are classified by an integer $r$. In a semiclassical regime, which is valid at large bare quark masses

$$m_A \gg \Lambda, \quad A = 1, \ldots N_f,$$

the adjoint scalar VEVs in those vacua take the form,

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \text{diag}(-m_1, -m_2, \ldots, -m_r, c, c, \ldots, c), \quad c = \frac{1}{N-r} \sum_{k=1}^{r} m_k,$$

where $r$ quark masses out of $N_f$ possible masses are chosen to satisfy the vacuum equations.

When the quark masses are tuned to a common value $m$, the pattern of the spontaneous breaking changes to

$$SU(N) \to SU(r) \times SU(N-r) \times U(1).$$

The $SU(N-r)$ sector is a pure $\mathcal{N} = 2$ Yang Mills theory \(^1\) and becomes strongly interacting at low energies and gets dynamically broken to $U(1)^{N-r-1}$. The $SU(r)$ sector, on the other hand, having $N_f$ massless flavors, remains weakly coupled as long as $r \leq \frac{N_f}{2}$.

Furthermore, in the presence of the aforementioned adjoint mass perturbation, the light squark fields acquire VEVs of color-flavor diagonal form (“Color-Flavor Locking”),

$$\langle q_i^a \rangle = \delta_i^a \sqrt{\mu m}, \quad i, a = 1, 2, \ldots r;$$

\(^1\)Recall that the quark masses come from the superpotentials $\tilde{Q}_i(\sqrt{2}\phi + m_i)Q_i$. 
which breaks the $SU(r) \times U(1)$ gauge group completely at scales far below the bare quark masses: $\sqrt{\mu m} \ll m$. The theory is now in the Higgs phase, and develops vortex configurations, representing nontrivial elements of

$$\pi_1\left(\frac{SU(r) \times U(1)^{N-r}}{\mathbb{Z}_r}\right) = \mathbb{Z}^{N-r}. \quad (1.7)$$

The key fact is that the system has an exact global $SU(r)_{C+F}$ symmetry, respected both by the interactions and by the scalar VEVS (1.4) and (1.6). A given vortex configuration however breaks this symmetry: it turns out that the symmetry is broken as $SU(r) \to SU(r-1) \times U(1)$ (see below.) As a result, exact orientation zero modes of $SU(r)/(SU(r-1) \times U(1)) \sim \mathbb{C}P^{r-1}$ are generated.

To work things out concretely, we analyse the case of the $r = 2$ vacua of the $SU(3)$ gauge theory ($N = 3, r = 2$ above) in detail in the main body of this paper. The $SU(2)$ subgroup, classically restored in the limit of equal quark masses, stays unbroken in the full quantum theory, as the relevant sector of the theory is infrared free if $N_f > 4$, or is conformal invariant if $N_f = 4$. On the other hand, of course, the underlying $SU(3)$ gauge theory is asymptotically free for $N_f \leq 5$, so we shall take $N_f$ to be either 4 or 5.

This is one of the important points of our analysis: by working in the regime in which the interactions remain weak at all scales, the continuous transition from the theory with abelian vortices (unequal quark masses) to the theory with nonabelian vortices which are qualitatively different, can be studied explicitly and reliably.

The unbroken gauge group $SU(2) \times U(1)/\mathbb{Z}_2$ is further broken at a much lower mass scale, yielding vortices representing the nontrivial elements of

$$\pi_1\left(\frac{SU(2) \times U(1)}{\mathbb{Z}_2}\right) = \mathbb{Z}. \quad (1.8)$$

Indeed, as the bare quark masses are tuned to a common value, $m_i \to m$, starting from unequal and generic values, the low-energy gauge group gets enhanced from $U(1) \times U(1)$ to $SU(2) \times U(1)$. The set of abelian vortices appearing in the unequal mass cases acquires a certain degeneracy and at the same time some orientation (in the color space) zero modes appear which relate the vortices of the same tension by global rotations. These zero modes are associated with the diagonal global $SU(2)_{C+F}$ subgroup of color $SU(2)_C$ crossed with the flavor $SU(2)_F \subset SU(4)_F$ which is an exact symmetry of the system. More precisely, the vortex zero modes parametrize
$SU(2)/U(1) \sim \mathbb{CP}^1 \sim S^2$ as each vortex solution breaks the exact $SU(2)_{C+F}$ symmetry to a $U(1)$ subgroup.

We then work out the effective world-sheet theory of the vortex zero modes, and show that it reduces to the $\mathcal{N} = 2$ $O(3)$ sigma model in (1+1) dimensions. Classically the $O(3)$ sigma model has spontaneous symmetry breaking and appears to yield massless Goldstone fields. In terms of strings in four dimensions this would mean that $SU(2)_{C+F}$ is spontaneously broken and the string flux is oriented in some particular direction inside the $SU(2)_C$ gauge subgroup.

However the quantum physics of the $\mathcal{N} = 2$ $O(3)$ sigma model in (1+1) is quite different. It is well understood using the mirror map [18], which relates it to a sine-Gordon theory. In particular it is known that the model has a mass gap and no spontaneous symmetry breaking. In terms of strings in 4D this means that the string is not oriented in any particular direction inside $SU(2)_C$ group. This ensures that our vortices are truly nonabelian. The sine-Gordon superpotential is generated dynamically in the effective (1+1)-dimensional worldsheet theory which produces exactly two vacua.

Our considerations can be straightforwardly generalized to the $r = N - 1$ vacua of the $SU(N)$ theory with $2N \geq N_f \geq 2(N - 1)$, with unbroken $SU(N - 1)$ group, although our analysis in these more general cases is less complete. In particular, in the case of an $SU(N)$ theory broken to $SU(N - 1) \times U(1)$ the zero modes of the vortex are described by a 2-dimensional $\mathbb{CP}^{N-2}$ sigma model whose mirror is an affine Toda theory with the desired $N - 1$ vacua.

The vortices studied in this paper, though stable in the low-energy theory, are strictly speaking metastable as the underlying gauge group (e.g., $SU(3)$) is simply connected. Their decay rates are however small, being exponentially suppressed by ratios of heavy monopole masses squared to the string tensions [19, 20].

Our result provides, albeit indirectly, a counterexample to the no-go theorem on the existence of monopoles with nonabelian charges discussed earlier [21]. These nonabelian monopoles do exist in our theory as stable solitons and act as the sources of the nonabelian vortices considered here, and are actually confined by them. We exhibit here explicitly the transformations among the vortices, which imply certain non-local transformations for their sources. We will see that the zero modes of the vortices are normalizable. To calculate the zero mode of a single monopole, which necessarily sources an infinite vortex, we must integrate that of the vortex along its
infinite length. Thus we find, as was seen in the flavorless cases of Refs. 21, that the zero mode of a single monopole is nonnormalizable. In a color-neutral configuration of monopoles the total length of the vortices may be taken to be finite and so the integral is finite, yielding normalizable zero modes which again generalize those known to exist in the flavorless case.

Throughout this paper we limit ourselves to cases with large bare quark masses where the original electric subgroup remains weakly coupled. When the bare quark masses are tuned to small values or even to zero, the low-energy system is weakly coupled when described in terms of the magnetic variables instead of the electric ones. The excitations which are quarks in the electric description at large quark masses become monopoles in the magnetic description at small quark masses 6, 22. The properties of the corresponding $r$- vacua have been studied in detail in 9, and in the case of a SCFT $r = 2$ vacua of $SU(3)$ theory, in 23. The properties of these corresponding vacua are closely related by holomorphy.

The organization of the paper is as follows. In Sect. 2 we review $\mathcal{N} = 2$ QCD with equal quark masses, work out its low-energy description, vacuum structure and the low-energy spectrum. In Sect. 3 we derive nonabelian Bogomolny equations, construct vortices and study their $SU(2)$ zero modes. We discuss the generalization to the more general case of $SU(N) \to SU(N - 1) \times U(1)$ breaking in Sect. 4. In Sect. 5 we work out the effective world sheet theory for orientational zero modes and discuss its physics.

While this work was in preparation Ref. 24 appeared which considers vortices in the very similar $\mathcal{N} = 2$ three-dimensional theory with an FI term. While these vortices are not strings but particles, the worldvolume theories appear to be related by dimensional reduction, and the vacuum structures and spectra appear to be the same. Thus many of our results as well as an extensive analysis of the relevant moduli spaces may be found there.

2. $\mathcal{N} = 2$ $SU(3)$ QCD

2.1. The Model

The field content of $\mathcal{N} = 2$ QCD with the gauge group $SU(3)$ and $N_f$ flavors of chiral multiplets is as follows. The $\mathcal{N} = 2$ vector multiplet consists of the gauge field $A_\mu$,
two Weyl fermions $\lambda^1_\alpha, \lambda^2_\alpha$ and the scalar field $\phi$, all in the adjoint representation of the gauge group. Here $\alpha = 1, 2$ is a spinor index while all adjoint fields are $3 \times 3$ matrices in the Lie algebra $SU(3)$.

The chiral multiplets of the $SU(3)$ theory consist of complex scalar squarks $q^{kA}$ and $\tilde{q}_{Ak}$ and Weyl fermion quarks $\psi^{kA}$ and $\tilde{\psi}_{Ak}$, all in the fundamental representation of the gauge group. Here $k = 1, 2, 3$ is a color index while $A$ is a flavor index, $A = 1, \ldots, N_f$.

This theory has a Coulomb branch on which the adjoint scalar acquires the vacuum expectation value (VEV)

$$\phi = \frac{1}{2} \begin{pmatrix} a_3 + \frac{a_8}{\sqrt{3}} & 0 & 0 \\ 0 & -a_3 + \frac{a_8}{\sqrt{3}} & 0 \\ 0 & 0 & -2\frac{a_8}{\sqrt{3}} \end{pmatrix} \equiv \lambda_3 a_3 + \lambda_8 a_8, \quad (2.1)$$

generically breaking the $SU(3)$ gauge group down to $U(1) \times U(1)$. Here $\lambda_3$ and $\lambda_8$ are the Gell-Mann matrices of the Cartan subalgebra.

In this paper we consider the special vacua for which

$$<a_3> = 0. \quad (2.2)$$

For these vacua the low-energy gauge group is $SU(2) \times U(1)$, at least classically.

We perturb the above theory by adding a small mass term for the adjoint matter via the superpotential

$$W = \mu \mathrm{Tr} \Phi^2. \quad (2.3)$$

Generally speaking, the superpotential breaks $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. The Coulomb branch shrinks to a number of isolated $\mathcal{N} = 1$ vacua \[12\]. In the limit $\mu \to 0$ these vacua correspond to special singular points on the Coulomb branch in which pairs of monopoles/dyons or quarks become massless. Three of these points are always at strong coupling. They correspond to $\mathcal{N} = 1$ vacua of the pure gauge theory. The massless quark points are at weak coupling if the quark masses $m_A$ are large, $m_A \gg \Lambda$. The vacua in which quarks become massless will be referred to as the quark vacua. We shall be mainly interested in these quark vacua.

It is important to note that $\mathcal{N} = 2$ supersymmetry is not broken to the leading order in the parameter $\mu$ in the effective theory \[11\]. In the effective low-energy theory the superpotential (2.3) gives rise to a superpotential linear in $a_8$ plus higher
order corrections. If only the linear term in a’s in the superpotential is kept and if we restrict our attention to the special vacua \(2.2\), then it reduces to a \(\mathcal{N} = 2\) Fayet-Iliopoulos term which does not break the \(\mathcal{N} = 2\) supersymmetry.

### 2.2. \(SU(2) \times U(1)\) symmetric low-energy theory

The \(SU(3)\) gauge group is broken down to \(U(1) \times U(1)\) by the VEV of the adjoint scalar \(2.1\) at generic values of quark masses. However, in the equal quark mass limit \((m_A = m)\) which we shall consider from now on, the VEV of the \(a_3\) field vanishes (see Sec. \(2.3.1\)), and the low-energy gauge group is \(SU(2) \times U(1)\). W-bosons which are charged with respect to both factors of the low-energy group acquire a large mass of order \(m\). The third color components of quarks also become heavy in this vacua with masses of order of \(m\).

Let us consider now the scales of order \(\sqrt{\mu m}\), which are well below W-boson masses (\(\mu\) is taken small, \(\mu \ll m\)). There the low-energy theory contains the following light fields of the \(\mathcal{N} = 2\) vector multiplet: four complex scalar light fields \(a_b\) and \(a_8\) where \(b = 1, 2, 3\) is the color \(SU(2)\) index, one \(SU(2)\) gauge field \(A^b_\mu\) and one \(U(1)\) gauge field \(A^{(8)}_\mu\) together with their fermionic superpartners. For example the gauge fields are defined as follows:

\[
A_\mu = \lambda_b A^b_\mu + \lambda_8 A^{(8)}_\mu
\]  

(2.4)

where our notation corresponds to expanding gauge and adjoint fields in the orthogonal basis of the Gell-Mann matrices, \(\lambda_a\) being the first three Gell-Mann matrices normalized as \(Tr(\lambda_a \lambda_b) = 1/2 \delta_{ab}\).

Light quark multiplets contain complex scalar \(SU(2)\)-doublets \(q^{kA}, \tilde{q}_{Ak}\) together with their fermionic superpartners, \(k = 1, 2\).

The bosonic part of the low-energy effective theory then acquires the form

\[
S_{eff} = \int d^4x \left[ \frac{1}{4g_2^2} (F^b_{\mu\nu})^2 + \frac{1}{4g_1^2} (F^{(8)}_{\mu\nu})^2 + \frac{1}{g_2^2} |D_\mu a_b|^2 + \frac{1}{g_1^2} |\partial_\mu a_8|^2 
+ |\nabla_\mu q^A|^2 + |\nabla_\mu \tilde{q}^A|^2 + V(q^A, \tilde{q}_A, a_b, a_8) \right].
\]  

(2.5)

Here \(D_\mu\) is the covariant derivative in the adjoint representation of \(SU(2)\) gauge subgroup, while

\[
\nabla_\mu = \partial_\mu - \frac{i}{2\sqrt{3}} A^8_\mu - i A^b_\mu \tau^b/2,
\]  

(2.6)
where we suppress the color $SU(2)$ indices and $\tau^b$ are $SU(2)$ Pauli matrices. The coupling constants $g_1$ and $g_2$ correspond to $U(1)$ and $SU(2)$ sectors respectively. The potential in the Lagrangian (2.5) is given by the D and F terms

$$V(q^A, \tilde{q}_A, a_b, a_8) = \frac{g_2^2}{8} \left( \tilde{q}_A \tau^b q^A - \tilde{q}_A \tau^b \tilde{q}^A \right)^2 + \frac{g_1^2}{24} \left( \tilde{q}_A q^A - \tilde{q}_A \tilde{q}^A \right)^2 + \
+ \frac{g_2^2}{2} \left| \tilde{q}_A \tau^b q^A \right|^2 + \frac{g_1^2}{6} \left| \tilde{q}_A q^A + \sqrt{6} \mu < a_8 > \right|^2 + \ldots,$$

(2.7)

where other D-terms involving the adjoint scalar fields $a_8$ and $a_b$ ($b = 1, 2, 3$) (which vanish at $< a_8 > \neq 0$ and $< a_b > = 0$) are left implicit. The term $\sqrt{6} \mu < a_8 >$ in the second line arises when we expand fields $a_8$ and $a_b$ in the superpotential (2.3) around their VEV’s and keep only terms linear in fluctuations of these fields. As we have already noted, this means that the theory in (2.5) is a bosonic part of a $N = 2$ supersymmetric theory. In particular this ensures that our theory has BPS vortices [1, 11, 16] (see also the seventh ref. in [4]).

The theory (2.5), (2.7) is an $SU(2) \times U(1)$ generalization of the low-energy theory for the $U(1) \times U(1)$ case studied in [16].

Below the scale $m$ the $SU(3)$ gauge group is broken and we have two coupling constants $g_1$ and $g_2$ which run according to the $U(1)$ and $SU(2)$ renormalization group flows respectively. Note that with a logarithmic accuracy we can neglect mixing of these two coupling constants. In the case with four flavors the $SU(2)$ coupling does not run ($SU(2)$ theory with $N_f = 4$ is conformal) and is given by its value at the scale $m$

$$\frac{8\pi^2}{g_2^2} = 2 \log \frac{m}{\Lambda} + \cdots. \quad (2.8)$$

Since at large $m$ the $SU(2)$ sector is weakly coupled, it remains so at low energies.

The $U(1)$ coupling undergoes an additional renormalization from the scale $m$ to the scale determined by the masses of light states in the low-energy theory (which are of the order of $\sqrt{\mu m}$, see next subsection). Thus we have

$$\frac{8\pi^2}{g_1^2} = 2 \log \frac{m}{\Lambda} + \frac{2}{3} \log \sqrt{\frac{m}{\mu}} + \cdots, \quad (2.9)$$

where we use the fact that the one loop coefficient of the $\beta$-function for $U(1)$ theory is $b = -2 n_e N_f$ and substitute $N_f = 4$ and the electric charge $n_e = 1/2\sqrt{3}$, see (2.6). Clearly, this coupling is even smaller than the one in the $SU(2)$ sector.
If the number of the quark flavors is taken to be five, the $SU(2)$ gauge coupling constant also runs to smaller values towards the infrared. In general therefore one has $g_1 \neq g_2$, both small, and we shall not need more details in the analyses below.

2.3. Vacuum structure and low-energy spectrum

In this subsection we review the vacuum structure and low-energy mass spectrum of $SU(3)\ N = 2$ QCD [12, 9] generalizing the analysis made in [16] to the case of the $SU(2) \times U(1)$ low-energy group. To find the vacua of the effective theory (2.5) we have to look for the zeros of the potential (2.7). At generic large values of quark masses solutions have the following structure [9, 16]. Besides the three strong coupling vacua which exist already in the pure $SU(3)$ gauge theory there are $2N_f$ $r = 1$ vacua and $N_f(N_f - 1)/2$ $r = 2$ vacua, were $r$ is the number of quark flavors which develop non-zero VEV’s.

Here we are mostly interested in $r = 2$ vacua, which have an $SU(2) \subset SU(3)$ unbroken gauge group which becomes exact in the case of equal quark masses. Clearly the minimal number of flavors for which we can have a $r = 2$ vacuum is $N_f = 2$. Let us consider this case first.

The adjoint scalar matrix is given by

$$\phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -2m \end{pmatrix}$$

where $m$ is the common mass of both flavors. In the above notation (2.10) reads

$$\langle a_3 \rangle = 0, \quad \langle a_8 \rangle = -\sqrt{6} m.$$ 

(2.11)

For real values of $m$ and $\mu$ we can use gauge rotations to make squark VEV’s real. We write the squark field as a $2 \times 2$ matrix $q^{kA}$ where $k = 1, 2$ is a color index and $A = 1, 2$ is a flavor one. Then the squark VEV’s are given by

$$\langle q^{kA} \rangle = \langle \bar{\tilde{q}}^{kA} \rangle = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(2.12)

where we have used color-flavor mixed matrix notation, and we have introduced

$$\xi = 6 \mu m.$$ 

(2.13)
which acts as the Fayet-Iliopoulos parameter of the $U(1)$. $\xi$ sets the scale of the low-energy theory \((2.5)\). Only the two upper color components and the first two flavors are shown in Eq.\((2.12)\): all other components have vanishing VEVs.

Now consider the spectrum of light fields in this vacuum. The $SU(2) \times U(1)$ low-energy gauge group is broken completely by squark VEV’s and all gauge bosons acquire masses. The mass matrix for the gauge fields $A_{\mu}^a$, $A_{\mu}^8$ can be read off of the kinetic terms for the quarks in \((2.5)\). It turns out that it is diagonal in the basis $A_{\mu}^a$, $A_{\mu}^8$. In particular, the mass of $A_{\mu}^8$ is given by

$$m_{A^8}^2 = \frac{1}{3} g_1^2 \xi, \quad (2.14)$$

while the mass of the $SU(2)$ W-boson is

$$m_{W}^2 = g_2^2 \xi. \quad (2.15)$$

The masses of the adjoint scalars $a_8$ and $a_6$ are identical to the ones in \((2.14)\) and \((2.15)\) as can be seen from \((2.7)\).

The mass matrix for squarks is now of size $16 \times 16$ including four real components of complex fields $q$ and $\tilde{q}$ for each color and flavor. It has four zero eigenvalues associated with the four states “eaten” by the Higgs mechanism for $U(1)$ and $SU(2)$ gauge factors and two non-zero eigenvalues coinciding with gauge boson masses \((2.14)\) and \((2.15)\). The eigenvalue \((2.14)\) corresponds to three squark eigenvectors while the one in \((2.15)\) corresponds to nine squark eigenvalues.

Altogether we have one long $\mathcal{N} = 2$ multiplet with mass \((2.14)\), containing eight bosonic states (3 states of the massive $A_{\mu}^8$ field plus 2 states of $a_8$ plus 3 squark states) and eight fermionic states. In addition we have three long $\mathcal{N} = 2$ multiplets with mass \((2.15)\) labeled by the color index $a = 1, 2, 3$ also containing eight bosonic and eight fermionic states each \(^2\). Note that no Nambu-Goldstone multiplets appear in this vacuum: all phases associated with broken symmetries are ”eaten” by Higgs mechanism.

Actually, in the theory with $N_f = 2$ discussed above, the $SU(2)$ gauge interactions become strong below the scale $m$, and the properties of the theory at low energies (at mass scales of order of $\sqrt{\mu m} \ll m$) cannot be determined from the Lagrangian \((2.5)\) only.

\(^2\)See \[11\] for a discussion of the emergence of $\mathcal{N} = 2$ long multiplets in Seiberg-Witten theory upon adjoint mass term deformation.
For this reason, we introduce more flavors into our theory and consider the $SU(3)$ theory with $N_f = 4$ or $N_f = 5$. The low-energy $SU(2) \times U(1)$ then remains in the weak coupling regime.

This theory has $\binom{N_f}{2}$ $r = 2$ vacua of the type described above, for unequal quark masses. Each of these vacua corresponds to choosing two flavors out of $N_f$ which develop VEV’s. This gives $N_f(N_f - 1)/2 = 6$ choices for $N_f = 4$. In the limit of equal masses all six vacua coalesce and a Higgs branch develops from the common root. The dimension of this Higgs branch is $8(N_f - 2)$ \cite{12,16}. To see this note that we have $8N_f$ real variables $q^{kA}$ subject to four $D$-term and eight $F$-term conditions in the potential (2.7). Also 3+1 gauge phases are eaten by the Higgs mechanism. Thus we have $8N_f - 12 - 4 = 8(N_f - 2)$ remaining degrees of freedom.

We consider below a special submanifold of the Higgs branch which admits BPS flux tubes (cf. \cite{1,12,16,26}). This base submanifold is compact and has the minimal value of the quark condensate $<|q^A|^2 >= <|\tilde{q}^A|^2 >= \xi$. One point on this submanifold which corresponds to non-zero VEV of the first flavor and non-zero VEV of the second flavor while all other components are zero is given in (2.12).

Other points on the base of the Higgs branch are given by a $SU(N_f)$ flavor rotation of (2.12). The dimension of the base submanifold of the Higgs branch is $4(N_f - 2)$ \cite{16}. To see this note that VEV’s of two flavors break $SU(N_f)$ symmetry down to $SU(N_f - 2)$. Thus the number of ”broken” generators is $\dim SU(N_f) - \dim SU(N_f - 2) = 4(N_f - 1)$ and also we have to subtract four phases “eaten” by the Higgs mechanism.

Other points on the $8(N_f - 2)$ dimensional Higgs branch correspond to non-zero VEV’s of massless moduli fields, and these points do not admit BPS strings. In particular, the ANO strings \cite{27} on the Higgs branch were studied in \cite{28,26}, they correspond to a limiting case of type I strings with the logarithmically thick tails associated with massless scalar fields. We shall not discuss here strings at generic points on the Higgs branch.

Before ending this subsection, we need to comment on the soliton sector. In the monopole sector, all solitonic states associated with the symmetry breaking (2.10) are massive. In particular, one finds an exactly degenerate doublet of BPS monopoles of minimum mass \cite{13}.

Apparently, a set of “monopole” states become massless as the bare quark masses
are tuned to a common value, \( m_i \to m \), at which point the low-energy gauge group gets enhanced from \( U(1) \times U(1) \) to \( SU(2) \times U(1) \). For instance, the BPS monopole carrying magnetic charge \((1,-1)\) with respect to two \( U(1) \) factors above has mass proportional to \( m_1 - m_2 \), and appears to become massless in the limit of \( SU(2) \) restoration. Classically this “state” becomes infinitely extended in space in such a limit, and at the same time the fields \( \phi, A_i \) degenerate into trivial vacuum configuration \( \phi(x) = A_i(x) = 0 \). More importantly, as the topological structure of the theory changes in the \( SU(2) \) restoration limit (from Eq. (3.6) to Eq. (3.13), see below) such a “massless monopole” is no longer topologically stable.

3. Non-abelian Vortices

We will now construct (BPS) vortex solutions in the theory described above and show that they possess exact zero modes.

3.1. Non-abelian Bogomolny Equations

As we have already anticipated, by restricting ourselves to a particular base submanifold of the Higgs branch of the theory with four flavors, we are able to deal with BPS strings throughout. By gauge and flavor rotations the squark VEVs can be taken to be of the form (2.12). Then classically only the two flavors which develop VEV’s will play a role in the vortex solution. Other flavors remain zero on the solution, and one can consider the squark fields \( q^{kA} \) to be \( 2 \times 2 \) matrices. Note however that the additional two flavors are crucial in the quantum theory, in keeping the \( SU(2) \) interactions weakly coupled.

Let us make an ansatz,

\[
q^{kA}(x) = \bar{q}^{kA}(x),
\]

and a convenient redefinition of the squark fields \( q^{kA} \to \frac{1}{\sqrt{2}}q^{kA} \). The low-energy action (2.5) then reduces (\( g_2 \) and \( g_1 \) stand for the \( SU(2) \) and \( U(1) \) coupling constants at the

\[3\]This is analogous to the fate of the ‘t Hooft - Polyakov monopole of the spontaneously broken \( SU(2) \to U(1) \) theory, in the limit \( v \to 0 \).

\[4\]In fact the additional flavors are important even classically. In the presence of additional flavors strings can turn into semilocal strings, see [29] for a review on semilocal strings. We shall not study this issue here.
scale $\xi$, respectively) to
\[
S = \int d^4x \left[ \frac{1}{4g_2^2} (F^a_{\mu\nu})^2 + \frac{1}{4g_1^2} (F^8_{\mu\nu})^2 + |\nabla_\mu q^A|^2 \right. \\
\left. + \frac{g_2^2}{8} (\bar{q}_A \tau^a q^A)^2 + \frac{g_1^2}{24} (\bar{q}_A q^A - 2\xi)^2 \right],
\]
where we have set the adjoint scalar fields to their VEVs (2.11). The string tension can be written à la Bogomolny \[30\]
\[
T = \int d^2x \left( \sum_{a=1}^3 \left[ \frac{1}{2g_2} F^{(a)}_{ij} \pm \frac{g_2}{4} (\bar{q}_A \tau^a q^A) \epsilon_{ij} \right]^2 \right. \\
\left. + \frac{1}{2g_1} F^{(8)}_{ij} \pm \frac{g_1}{4\sqrt{3}} (|q^A|^2 - 2\xi) \epsilon_{ij} \right]^2 \\
+ \frac{1}{2} |\nabla_i q^A \pm i\epsilon_{ij} \nabla_j q^A|^2 \pm \frac{\xi}{\sqrt{3}} \tilde{F}^{(8)} \right)
\]
where $\tilde{F}^{(8)} \equiv \frac{1}{2} \epsilon_{ij} F^{(8)}_{ij}$, leading to the following first order equations for strings
\[
\frac{1}{2g_2} F^{(a)}_{ij} \pm \frac{g_2}{4} (\bar{q}_A \tau^a q^A) \epsilon_{ij} = 0, \quad a = 1, 2, 3; \\
\frac{1}{2g_1} F^{(8)}_{ij} \pm \frac{g_1}{4\sqrt{3}} (|q^A|^2 - 2\xi) \epsilon_{ij} = 0; \\
\nabla_i q^A \pm i\epsilon_{ij} \nabla_j q^A = 0, \quad A = 1, 2, \ldots, N_f.
\]
(3.4)
Here $\epsilon = \pm$ is the sign of the total flux specified below.

The $U(1) \times U(1)$ string solutions found in the case of unequal quark masses \[16\] can be readily recognized as particular solutions of these equations. To construct them we further restrict the gauge field $A^a_\mu$ to the single color component $A^3_\mu$ (by setting $A^1_\mu = A^2_\mu = 0$), and consider only squark fields of the $2 \times 2$ color-flavor diagonal form:
\[
q^{kA}(x) = \bar{q}^{kA}(x) \neq 0, \quad \text{for} \quad k = A = 1, 2,
\]
by setting all other components to zero. For unequal masses the relevant topological classification was
\[
\pi_1\left(\frac{U(1) \times U(1)}{\mathbb{Z}_2}\right) = \mathbb{Z}^2
\]
(3.6)
and the allowed strings formed a lattice labeled by two integer winding numbers. In particular, assume that the first flavor winds $n$ times while the second flavor winds $k$.
times and look for solutions of (3.4) using the following ansatz
\[ q^k A(x) = \begin{pmatrix} e^{in}\varphi_1(r) & 0 \\ 0 & e^{ik}\varphi_2(r) \end{pmatrix}, \]
\[ A^3_i(x) = -\varepsilon\varepsilon_{ij} \frac{x_j}{r^2} \left( (n-k) - f_3(r) \right), \]
\[ A^8_i(x) = -\sqrt{3} \varepsilon\varepsilon_{ij} \frac{x_j}{r^2} \left( (n+k) - f_8(r) \right) \tag{3.7} \]
where \((r, \varphi)\) are polar coordinates in the (1,2) plane while the profile functions \(\varphi_1, \varphi_2\) for scalar fields and \(f_3, f_8\) for gauge fields depend only on \(r\).

With this ansatz the first-order equations (3.4) take the form
\[ r \frac{d}{dr} \varphi_1(r) - \frac{1}{2} \left( f_8(r) + f_3(r) \right) \varphi_1(r) = 0, \]
\[ r \frac{d}{dr} \varphi_2(r) - \frac{1}{2} \left( f_8(r) - f_3(r) \right) \varphi_2(r) = 0, \]
\[ -\frac{1}{r} \frac{d}{dr} f_8(r) + \frac{g_1^2}{6} \left( \varphi_1(r)^2 + \varphi_2(r)^2 - 2\xi \right) = 0, \]
\[ -\frac{1}{r} \frac{d}{dr} f_3(r) + \frac{g_2^2}{2} \left( \varphi_1(r)^2 - \varphi_2(r)^2 \right) = 0. \tag{3.8} \]
The profile functions in these equations are determined by the following boundary conditions
\[ f_3(0) = \varepsilon_{n,k} \left( n-k \right), \quad f_8(0) = \varepsilon_{n,k} \left( n+k \right), \]
\[ f_3(\infty) = 0, \quad f_8(\infty) = 0 \tag{3.9} \]
for the gauge fields, and the requirement that the squark fields be everywhere regular.

The behavior of the latter at \(r = \infty\),
\[ \varphi_1(\infty) = \sqrt{\xi}, \quad \varphi_2(\infty) = \sqrt{\xi} \tag{3.10} \]
and that at \(r = 0\) (e.g., \(\varphi_1(0) = 0\), if \(n \neq 0, k = 0\), follow from these requirements.

Here the sign of the string flux is
\[ \varepsilon = \varepsilon_{n,k} = \frac{n+k}{|n+k|} = \text{sign}(n+k) = \pm 1. \tag{3.11} \]

\[ \text{We use a notation slightly different from the one used in [16]: } \varphi_1(r) \text{ instead of } \varphi_u(r); \quad \varphi_2(r) \text{ instead of } \varphi_d(r). \]

The cylindrical coordinates are here denoted as \((z, r, \varphi)\), the vortex center extending along the \(z\) axis.
The tension of a \((n,k)\)-string for the case of equal quark masses is determined by the flux of the \(A^8_\mu\) gauge field alone and is given by

\[
T_{n,k} = 2\pi \xi |n + k|.
\]  

(3.12)

Note that \((1,0)\) and \((0,1)\)-strings are exactly degenerate.

Note also that \(\tilde{F}^{(3)}\) does not enter the central charge of the \(\mathcal{N} = 2\) algebra and so does not affect the string tension. The stability of the string in this case is due to the \(U(1)\) factor of the \(SU(2) \times U(1)\) low-energy group only.

The equations (3.4) represent a nonabelian generalization of the Bogomolny equations for the ANO string [30]. For a generic \((n,k)\)-string equations (3.8) do not reduce to the standard Bogomolny equations. For instance, for the \((1,1)\)-string these equations reduce to two Bogomolny equations while for the \((1,0)\) and \((0,1)\) strings they do not.

The numerical solution for the “elementary” \((1,0)\) string is shown in Fig. 1. Fig. 2. The \((0,1)\) string is obtained by the replacement, \(\phi_1 \leftrightarrow \phi_2; f_3 \leftrightarrow -f_3\).

![Figure 1: Vortex profile functions \(\phi_1(r)\) and \(\phi_2(r)\) of the \((1,0)\)-string. Note \(\phi_1(0) = 0\).](image)

The charges of \((n,k)\)-strings can be plotted on the Cartan plane of the \(SU(3)\) algebra. We shall use the convention of labeling the flux of a given string by the magnetic charge of the monopole which produces this flux and must be attached to each end. This is possible since both string fluxes and monopole charges are elements of the group \(\pi_1(U(1)^2) = \mathbb{Z}^2\). This convention is convenient because specifying the flux of a given string automatically fixes the charge of the monopole that it confines.

Our strings are formed by the condensation of squarks which have electric charges equal to the weights of \(SU(3)\) algebra. The Dirac quantization condition tells us that
Figure 2: The profile functions $f_3(r)$ (lower curve) and $f_8(r)$ (upper curve) for the $(1,0)$-string.

The lattice of $(n,k)$-strings is formed by roots of the $SU(3)$ algebra [16]. The lattice of $(n,k)$-strings is shown in Fig. 3. Two strings $(1,0)$ and $(0,1)$ are the “elementary” or “minimal” BPS strings. If we plot two lines along charges of these “elementary” strings (see Fig. 3) they divide the lattice into four sectors. It turns out [16] that the strings in the upper and lower sectors, which are labeled by black circles in Fig. 3, are BPS but they are marginally unstable at real quark mass ratios. Instead, strings in the right and left sectors, which are labeled in Fig. 3 by white circles, are bound states of the “elementary” ones but they are not BPS.

Figure 3: Lattice of $(n,k)$ vortices.
3.2. Minimal vortex of generic orientation: $S^2$ zero modes

Actually, the relevant homotopy group here is

$$\pi_1\left(\frac{SU(2) \times U(1)}{\mathbb{Z}_2}\right) = \mathbb{Z},$$

(3.13)

instead of (3.6), as we are working with the case of equal quark masses where the low-energy gauge group is $\frac{SU(2) \times U(1)}{\mathbb{Z}_2}$. The generator of the fundamental group is a loop which encircles the $U(1)/\mathbb{Z}_2$ once [17], and thus to calculate the tension of a string, or to determine whether it is stable, it suffices to simply count the winding number around this circle. This means that the lattice of $(n,k)$-strings reduces to a tower labeled by one integer $(n+k)$. For instance, the $(1,-1)$-string becomes completely unstable as it winds forward once and then backward once, and so there is no net winding and so no topological charge. On the restored $SU(2)$ group manifold it is also trivial, as it goes half way around the equator and then goes back. The $(2,0)$ string goes all of the way around the $SU(2)$ equator, making a contractible loop, but is stable because it wraps the $U(1)/\mathbb{Z}_2$ twice (it wraps the original $U(1)$ once).

![Figure 4: Reduced lattice of $\mathbb{Z}$ vortices.](image)

On the other hand, the $(1,0)$ and $(0,1)$ strings cannot be shrunk because they correspond to a half circle along the equator. They have the same tension (see (3.12)) for equal quark masses and thus apparently belong to doublet of an $SU(2)$.

In general non-BPS strings on the $(n,k)$-lattice (see Fig. 3) become unstable as they have tensions above their BPS bounds and we are left with $|n+k|+1$ BPS strings at each winding number $n+k$. The reduction of the string lattice is illustrated Fig. 4.

Most importantly, this suggests that there be a continuously infinite number of vortices of minimum winding and with the same tension,

$$T_1 = 2 \pi \xi$$

(3.14)
of which the (1, 0) and (0, 1) vortices discussed above are just two particular cases (Fig. 5). Below we show that this is indeed correct, by a continuous deformation of the (1, 0)-string solution transforming it into a (0, 1)-string. This deformation leaves the string tension unchanged and therefore corresponds to an orientational zero mode.

First let us separate physical variables from the gauge phases eaten by the Higgs mechanism in the quark fields. To do so we use the following parametrization of the $2 \times 2$ quark matrix

$$q^{kA} = U_{U(1)} U_{SU(2)} (q^0 + \tau^a q^a).$$

Here $U_{U(1)}$ and $U_{SU(2)}$ are matrices from the $U(1)$ and $SU(2)$ gauge factors respectively while $q^0(x)$ and $q^a(x)$ are real. The parametrization (3.15) represents eight real variables $q^{kA}$ in terms of $3+1=4$ gauge phases eaten by the Higgs mechanism and four physical variables $q^0$ and $q^a$. In particular, (2.12) corresponds to

$$<q^0> = \sqrt{\xi}, \quad <q^a> = 0.$$  

Now let us fix the unitary gauge (at least globally, which is enough for our purposes) by imposing the condition that squark VEV’s are given precisely by (3.16) and so all gauge phases are zero. Now transform the $(1, 0)$-string solution (3.7) into unitary gauge, which corresponds to the singular gauge in which the string flux comes from the singularity of the gauge potential at zero. In this gauge the solution (3.7)
for the (1, 0)-string takes the form

\[
q^{kA} = \begin{pmatrix}
\phi_1(r) & 0 \\
0 & \phi_2(r)
\end{pmatrix},
\]

\[
A_i^3(x) = \epsilon_{ij} \frac{x_j}{r^2} f_3(r), \quad A_i^8(x) = \sqrt{3} \epsilon_{ij} \frac{x_j}{r^2} f_8(r).
\]  \hspace{1cm} (3.17)

Note that a global diagonal subgroup in the product of gauge and flavor symmetries \(SU(2)_C \times SU(2)_F\) is not broken by the squark VEV. Namely,

\[
U < q > U^{-1} = < q >,
\]  \hspace{1cm} (3.18)

where \(U\) is a global rotation in \(SU(2)\) while the squark VEV matrix is given by (2.12). We call this unbroken group \(SU(2)_{C+F}\).

Now let us apply this global rotation to the (1, 0) string solution (3.17). We find

\[
q^{kA} = U \begin{pmatrix}
\phi_1(r) & 0 \\
0 & \phi_2(r)
\end{pmatrix} U^{-1},
\]

\[
A_i(x) = \frac{1}{2} U \tau^3 U^{-1} \epsilon_{ij} \frac{x_j}{r^2} f_3(r), \quad A_i^8(x) = \sqrt{3} \epsilon_{ij} \frac{x_j}{r^2} f_8(r),
\]  \hspace{1cm} (3.19)

where we use a matrix notation for the \(SU(2)\) gauge field, \(A_\mu = A_\mu^a \tau^a/2\). Using the representation

\[
U \tau^3 U^{-1} = n^a \tau^a,
\]  \hspace{1cm} (3.20)

where \(n^a\) is a unit vector on \(S^2\), \(n^2 = 1\), we can rewrite the \(SU(2)\) gauge potential of (3.19) in the form

\[
A_i(x) = \frac{1}{2} n^a \tau^a \epsilon_{ij} \frac{x_j}{r^2} f_3(r),
\]  \hspace{1cm} (3.21)

revealing that now the \(SU(2)\) flux of the string is directed along an arbitrary vector \(n^a\). It is easy to see that the rotated string (3.19) is a solution of nonabelian first order equations (3.4).

Since the \(SU(2)_{C+F}\) symmetry is not broken by squark VEV’s it is physical and does not correspond to any of the gauge rotations eaten by the Higgs mechanism. To see this explicitly let us rewrite the quark field of our solution (3.19) using the parametrization (3.15). We get

\[
U_{U(1)} = I, \quad U_{SU(2)} = I,
\]  

\[\text{Explicitly, if } n^a = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha), \text{ the rotation matrix is given by } U = \exp -i \beta \tau_3/2 \exp -ia \tau_2/2.\]
\[ q^0(x) = \frac{1}{2}(\phi_1 + \phi_2), \quad q^a(x) = n^a \frac{1}{2}(\phi_1 - \phi_2). \]  

We see that all gauge phases are zero while physical variables acquire an \( n \)-dependence. Clearly the solution \( 3.13 \) interpolates between \((1, 0)\) and \((0, 1)\) strings. In particular it gives a \((1, 0)\)-string for \( n = (0, 0, 1) \) and a \((0, 1)\)-string for \( n = (0, 0, -1) \).

The \( SU(2)_{C+F} \) symmetry is exact and the tension of the string solution \( 3.13 \) is independent of \( n^a \):

\[ T = 2 \, \pi \, \xi, \]  

see \( 3.12 \). However, an explicit vortex solution breaks the exact \( SU(2)_{C+F} \) as

\[ SU(2)_{C+F} \to U(1) : \]  

the two angles associated with vector \( n^a \) - two orientational bosonic zero modes of the string - parametrize the quotient space \( SU(2)/U(1) \sim \mathbb{C}P^1 \sim S^2 \).

In the regular gauge, the minimal nonabelian vortex of generic orientation \( 3.13 \) takes the form

\[ q^{kA} = U \left( e^{i \varphi \phi_1(r)} 0 \\ 0 \phi_2(r) \right) U^{-1} = e^{\frac{i}{2} \varphi \left( 1 + n^a \tau^a \right)} U \left( \phi_1(r) 0 \\ 0 \phi_2(r) \right) U^{-1}, \]

\[ A_i(x) = U \left[ -\frac{\tau_3}{2} \epsilon_{ij} \frac{x_j}{r^2} [1 - f_3(r)] \right] U^{-1} = -\frac{1}{2} n^a \tau^a \epsilon_{ij} \frac{x_j}{r^2} [1 - f_3(r)], \]

and

\[ A^8_i(x) = -\sqrt{3} \epsilon_{ij} \frac{x_j}{r^2} [1 - f_3(r)], \]  

(3.25)

where \( U \) is given by Eq. \( 3.20 \) and the profile functions are solutions of Eq.\( 3.8 \) for \((n, k) = (1, 0)\). In this gauge it is particularly clear that this solution smoothly interpolates between the \((1, 0)\) and \((0, 1)\) solutions: if \( n = (0, 0, 1) \) the first flavor squark winds at infinity while for \( n = (0, 0, -1) \) the second flavor does.

To further convince ourselves that the rotation considered above corresponds to physical zero modes we can construct a gauge invariant operator which has \( n^a \)-dependence on our solution. One example is

\[ O(x)_A^B = \bar{q}_B q^A(x), \]  

which is a matrix in flavor indices. Inserting the solution \( 3.25 \) this operator reads

\[ O(r) = \frac{1}{2} (\phi_1^2 + \phi_2^2)(r) + n^a \tau^a \frac{1}{2} (\phi_1^2 - \phi_2^2)(r). \]  

(3.27)
We see that $O(x)$ is a gauge invariant operator which has $n^a$-dependence localized near the string axis where $(\phi_1^2 - \phi_2^2)$ is non-zero.

As we have already mentioned the central charge of the $\mathcal{N} = 2$ algebra reduces to the $\tilde{F}^8$ component of the flux

$$\frac{1}{\sqrt{3}} \int d^2x \tilde{F}^8 = 2\pi. \quad (3.28)$$

If we define a gauge invariant flux $\int d^2x F^a \Phi^a$ it reduces to the one in (3.28) so the $SU(2)$ component of the flux does not enter\footnote{We can define an $SU(2)$ flux by constructing the operator $2 \int d^2x < \bar{q} > F^* < q > /\xi$ which is invariant under global gauge transformations. This flux is a matrix in flavor indices and on the string solution (3.25) reads $2\pi n^a(r^a)^A_B$.}. Still as we see from (3.27) there are gauge invariant quantities which acquire $n$-dependence.

3.3. Non-abelian monopoles as a multiplet of the unbroken dual group

In a sense, the result of the preceding subsection solves, albeit indirectly, the long-standing “existence problem” for the nonabelian monopoles discussed in the literature \cite{21}. In our model (with $m \gg \Lambda$), the monopoles generated by the symmetry breaking

$$SU(3) \Rightarrow SU(2) \times U(1) \xrightarrow{\mathbb{Z}_2} \quad (3.29)$$

are massive solitonlike states, which can appear as the sources of our vortices. The existence of the minimum vortices with generic orientation zero modes, which allows us to interpolate between the (1, 0) and (0, 1)-string solutions via $SU(2)$ rotations, implies the existence of the monopoles which behave truely as a doublet ((1, 0) and (0, 1)) of an $SU(2)$ group.

How has the “no-go” theorem of \cite{21} been avoided? First of all, these monopoles are non-local, finite-energy soliton states. The transformations among these configurations must be in the dual $SU(2)$ group, and not under the original, “electric” $SU(2)$ subgroup. Topological obstructions found in attempting to define globally the “electric” $SU(2)$ group in the monopole sector, do not apply to the dual group rotations, which are seen here indirectly as a consequence on the sources due to the global $SU(2)_{C+F}$ actions on the vortices.
Secondly, the existence of a massless flavors is fundamental to all of this. In fact, the orientation zero modes of the vortices are generated by the color-flavor diagonal $SU(2)_{C+F}$ which is an exact symmetry of the system. The fact that the dual of a gauge group involves the flavor group in some way, may appear surprising, but is not. In fact, it is one of the characteristic features of Seiberg’s duality in $\mathcal{N} = 1$ models. In MQCD it is yet less surprising as pairs of color and flavor branes fuse together in our vacua.

The fundamental importance of the massless flavors in generating truely non-abelian monopoles has already been noted in [13], where it was pointed out that because of renormalization effects only in a theory with a sufficient number of massless quark flavors does an unbroken (dual) gauge group remain exact at low energies. Otherwise, the semiclassical pattern of symmetry breaking has little to do with the true symmetry of the system. If the “unbroken group” is to be dynamically broken further at low energies, the degenerate multiplets of monopoles found in the semiclassical approximation mean simply the presence of an approximately degenerate set of monopoles. This is what occurs in a generic point of space of vacua in $\mathcal{N} = 2$ gauge theories.

Coming back to our model, the original local $SU(2) \times U(1)$ groups are completely broken by the squark VEVS at the scale $\xi = \sqrt{\mu m} \ll m$: the theory is in a Higgs phase. The dual $SU(2)$ theory must be in confinement phase. The (massive) doublet monopoles are confined. These conclusions are perfectly consistent with the result of the section 5 where we study the dynamics of the fluctuation of the $S^2$ zero modes.

4. Nonabelian Vortices in $SU(N)$ Gauge Theory

It is not difficult to generalize the whole discussion to the more general case in which the unbroken gauge group is

$$SU(K) \times U(1).$$

(4.1)

For instance, in the semiclassical vacuum of the $SU(N)$ theory where the adjoint scalar has a VEV of the form

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \text{diag} \left( -m, -m, \ldots, -m, (N - 1) m \right), \quad m \gg \Lambda,$$

(4.2)

the gauge symmetry is broken as

$$SU(N) \to SU(N - 1) \times U(1).$$

(4.3)
The unbroken gauge group remains weakly coupled at all scales if we take the number of flavors to be
\[ 2N > N_f \geq 2(N - 1) \] (4.4)
so that the semiclassical analysis is valid at all scales.

The construction of Sections 3.1. and 3.2. can be straightforwardly generalized to these more general cases with unbroken \( SU(K) \times U(1) \) group. For concreteness, the following equations will refer to the system Eq. (4.3), hence \( K = N - 1 \). With the ansatz (3.1) and after rescaling the squark fields as
\[ q^k A \rightarrow \frac{1}{\sqrt{2}} q^k A, \] (4.5)
the action (3.2) takes the form
\[ S = \int d^4x \left[ \frac{1}{4g^2} (F^{(a)}_{\mu\nu})^2 + \frac{1}{4e^2} (F^{(0)}_{\mu\nu})^2 + |\nabla_\mu q|^2 + \frac{g^2}{2} (\bar{q} t^a q)^2 + \frac{e^2}{4K(K + 1)} (\bar{q} q - K \xi)^2 \right], \] (4.6)
where the index \( a \) runs over \( 1, 2, \ldots, K^2 - 1 \), \( \xi = \text{const.} \sqrt{\mu m} \) and \( g \) and \( e \) are the \( SU(N - 1) \) and \( U(1) \) coupling constants, respectively. The covariant derivative is defined by
\[ \nabla_\mu = \partial_\mu - iA_\mu^a t^a - iA_\mu t^0, \quad t^0 = \frac{1}{\sqrt{2K(1 + K)}} \begin{pmatrix} 1_{K \times K} & 0 \\ 0 & -K \end{pmatrix}, \] (4.7)
t\( ^a \) being the generators of \( SU(K) \) in the fundamental representation.

In the sequel, we shall rewrite the abelian part in terms of
\[ \bar{e} \equiv \frac{e}{\sqrt{2K(1 + K)}}, \quad \bar{A}_i \equiv \frac{e}{\bar{e}} A_i \] (4.8)
(and subsequently drop the tildes) to simplify the equations somewhat. The net effect is a formal replacement \( \frac{e^2}{4K(K + 1)} \rightarrow \frac{e^2}{2} \), in Eq. (4.6).

The vortex tension has the form
\[ T = \int d^2x \left[ \sum_{a=1}^{K^2 - 1} \left( \frac{1}{2g} F^{(a)}_{ij} \pm \frac{g}{2} (\bar{q} A t^a q^A) \epsilon_{ij} \right)^2 \right. \]
\[ + \left[ \frac{1}{2e} F^{(0)}_{ij} \pm \frac{e}{2} (|q^A|^2 - K \xi) \epsilon_{ij} \right]^2 \]
\[ + \frac{1}{2} \left| \nabla_i q^A \pm i\epsilon_{ij} \nabla_j q^A \right|^2 \pm K \xi \tilde{F}^{(0)} \right). \] (4.9)
where $\tilde{F}^{(0)} := \frac{1}{2} \epsilon_{ij} F_{ij}^{(0)}$. A BPS vortex is a solution of the linear Bogomolny equations,

\[
\frac{1}{2g} F_{ij}^{(a)} + \frac{e}{2} \frac{g}{2} ( \tilde{q}_A e^a q^A ) \epsilon_{ij} = 0, \quad a = 1, 2, \ldots, K^2 - 1,
\]
\[
\frac{1}{2e} F_{ij}^{(0)} + \frac{e}{2} \left( |q_{ij}|^2 - K \xi \right) \epsilon_{ij} = 0;
\]
\[
\nabla_i q^A + i \epsilon_{ij} \nabla_j q^A = 0, \quad A = 1, 2, \ldots, N_f, \tag{4.10}
\]

where $\varepsilon = \pm$ is the sign of total flux.

### 4.1. Unbroken $SU(3)$

Let us now consider the specific case with $N = 4$ (unbroken $SU(3) \times U(1)$ group). Three particular solutions of these equations can be found by keeping $A_3^3, A_8^8$ and $A_0^{(0)}$, and by setting all other components to zero. The squark fields are labeled by three integers: $n, k, p$. These correspond to the squark winding numbers:

\[
q^{kA} = \begin{pmatrix}
  e^{i n \varphi} \phi_1(r) & 0 & 0 \\
  0 & e^{i k \varphi} \phi_2(r) & 0 \\
  0 & 0 & e^{i p \varphi} \phi_3(r)
\end{pmatrix}, \tag{4.11}
\]

with the conditions

\[
\phi_1, \phi_2, \phi_3 \rightarrow \sqrt{\xi}, \quad r \rightarrow \infty. \tag{4.12}
\]

As before, the only relevant color (vertical) and flavor (horizontal) components are shown above, all other components are set identically to zero in the vortex solution. At $\infty$ we have a pure gauge field $A_i \propto \epsilon_{ij} \frac{x_j}{r^2}$. We find the coefficients by imposing that the covariant derivatives go to zero. So we have:

\[
A_3^3(x) = -\epsilon_{ij} \frac{x_j}{r^2} \left( (n - k) - f_3(r) \right),
\]
\[
A_8^8(x) = -\frac{1}{\sqrt{3}} \epsilon_{ij} \frac{x_j}{r^2} \left( (n + k - 2p) - f_8(r) \right),
\]
\[
A_i(x) = -\frac{1}{3} \epsilon_{ij} \frac{x_j}{r^2} \left( (n + k + p) - f_0(r) \right). \tag{4.13}
\]

The profile functions should tend to zero at $r = \infty$, and their values at the origin (vortex center) are dictated by the regularity of the gauge fields ($f_3(0) = n - k$, etc). The first order equations for the profile functions are:

\[
r \frac{d}{dr} \phi_1(r) - \left( \frac{1}{2} f_3(r) + \frac{1}{6} f_8(r) + \frac{1}{3} f_0(r) \right) \phi_1(r) = 0,
\]

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The tension of this vortex is given by the $U(1)$ flux only,

\[ T_{n,k,p} = 2 \pi \xi |n + k + p|. \]

The $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$-strings all have the same tension. In the $A^3_i$ and $A^8_i$ plane, they form a equilateral triangle that corresponds to the antifundamental of $SU(3)$. It is possible to go from the $(1, 0, 0) \rightarrow (0, 1, 0)$ with the Weyl reflection:

\[ f_3 \rightarrow -f_3, \quad \phi_1 \leftrightarrow \phi_2, \]

other profile functions being left invariant. This corresponds to the global color-flavor $SU(3)_{C+F}$ rotation:

\[ U_{C+F} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \]

To do the transformation $(1, 0, 0) \rightarrow (0, 0, 1)$, we use the Weyl reflection:

\[ f_3 \rightarrow \frac{1}{2} f_3 - \frac{1}{2} f_8; \quad f_8 \rightarrow -\frac{3}{2} f_3 - \frac{1}{2} f_8, \quad \phi_1 \leftrightarrow \phi_3. \]

In the $A^3, A^8$ plane, this transformation is exatly the reflection:

\[ R_{Weyl} = \begin{pmatrix} \cos \frac{\pi}{3} & \sin \frac{\pi}{3} \\ -\sin \frac{\pi}{3} & -\cos \frac{\pi}{3} \end{pmatrix}. \]

The transformation corresponds to the colour-flavour rotation:

\[ U_{C+F} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \]
It is easy to see that with these transformations, equations Eq. (4.14) and the asymptotic conditions are left invariant.

We have found that these three solutions belong to the same set, and it is possible to continuously interpole between them with a gauge-flavor rotation. This set is parametrized by the coset $SU(3)_{C+F}/H$, where $H$ is the group left invariant by the vortex solution. Let us check this for the $(1,0,0)$ vortex. It is easily seen that it is possible to fix:

$$\phi_2 = \phi_3 = \phi, \quad f_3 = f_8 = f_{NA}$$

to reduce to four the number of profile functions satisfying:

$$r \frac{d}{dr} \phi_1(r) - \left( \frac{2}{3} f_{NA}(r) + \frac{1}{3} f(r) \right) \phi_1(r) = 0,$$

$$r \frac{d}{dr} \phi(r) - \left( - \frac{1}{3} f_{NA}(r) + \frac{1}{3} f(r) \right) \phi(r) = 0,$$

$$- \frac{1}{r} \frac{d}{dr} f_{NA}(r) + g^2 \left( \frac{1}{2} \phi_1(r)^2 - \frac{1}{2} \phi(r)^2 \right) = 0,$$

$$- \frac{1}{r} \frac{d}{dr} f(r) + 3e^2 \left( \phi_1(r)^2 + 2\phi(r)^2 - 3\xi \right) = 0. \quad (4.15)$$

Now it is possible to see that there is an unbroken subgroup $SU(2) \times U(1)$. For the $(0,0,1)$ vortex we can put:

$$\phi_1 = \phi_2 = \phi, \quad f_3 = 0, \quad f_8 = -2f_{NA}.$$

The four equations are the same as Eq. (4.15), with $\phi_1$ replaced by $\phi_3$.

To summarize, these vortices possess exact orientation zero modes, due to the fact that the system has an exact color-flavor diagonal symmetry, $SU(3)_{C+F}$. Since any particular vortex solution, like those found above, breaks this symmetry as

$$SU(3) \rightarrow SU(2) \times U(1), \quad (4.16)$$

there actually exist a continuous family of solutions of the same tension. The vortices of minimum tension of a generic orientation in $SU(3)$ are constructed starting from e.g., the $(1,0,0)$ solution, by $SU(3)_{C+F}$ transformations

$$q^{kA} = U \begin{pmatrix} e^{i\phi_1(r)} & 0 & 0 \\ 0 & \phi_2(r) & 0 \\ 0 & 0 & \phi_2(r) \end{pmatrix} U^\dagger, \quad (4.17)$$

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\[ A_i = U A_i^{(1,0,0)} U^\dagger, \quad (4.18) \]
where \( A_i^{(1,0,0)} \) stands for the gauge fields (4.13) with \((n, k, p) = (1, 0, 0)\). This family of vortices are labeled by the four real parameters of
\[ \frac{SU(3)}{SU(2) \times U(1)} \sim \mathbb{C}P^2. \quad (4.19) \]

4.2. Generalization to K-vacua

In the case with unbroken \( SU(K) \) symmetry, one apparently has \( 2K \) profile functions:
\[ \phi_1, \ldots, \phi_K, \quad f_3, \ldots, f_{K^2 - 1}, \quad f, \quad (4.20) \]
where \( f_{k^2-1} \)'s \((k = 2, 3, \ldots, K)\) correspond to \( K - 1 \) generators of the Cartan subalgebra. The ansatz is:
\[
d^kA = \begin{pmatrix} e^{in_1\alpha} \phi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{in_K\alpha}\phi_K \end{pmatrix},
\]
\[ A_i^3(x) = -\epsilon_{ij} x_j r^2 \left( (n_1 - n_2) - f_3 \right), \]
\[ A_i^{K^2-1}(x) = -\sqrt{\frac{2}{K(K-1)}} \epsilon_{ij} x_j r^2 \left( (n_1 + \ldots + n_{K-1} - (K-1)n_K) - f_{K^2-1} \right), \]
\[ A_i(x) = -\frac{1}{K} \epsilon_{ij} x_j r^2 \left( n_1 + \ldots + n_K - f \right). \quad (4.21) \]

Actually, the solution leaves an \( SU(K-1) \times U(1) \) symmetry invariant, as can be seen from the fact that they can be expressed in terms of four profile functions only, as in the \( SU(2) \) and \( SU(3) \) cases studied above. In fact, for the \((0, \ldots, 0, 1)\) vortex one can set:
\[ \phi_1 = \ldots = \phi_{K-1} = \phi, \]
\[ f_3 = \ldots = f_{(K-1)^2-1} = 0, \quad f_{K^2-1} = -(K-1)f_{NA} \quad (4.22) \]
reducing the linear equations to the set:
\[ \frac{d}{dr} \phi(r) - \left( -\frac{1}{K} f_{NA}(r) + \frac{1}{K} f(r) \right) \phi(r) = 0, \]
\[ r \frac{d}{dr} \phi_K(r) - \left( \frac{(K - 1)}{K} f_{NA}(r) + \frac{1}{K} f(r) \right) \phi_K(r) = 0, \]
\[ -\frac{1}{r} \frac{d}{dr} f_{NA}(r) + \frac{g^2}{2} \left( \phi_K(r)^2 - \phi(r)^2 \right) = 0, \]
\[ -\frac{1}{r} \frac{d}{dr} f(r) + K e^2 \left( (K - 1) \phi(r)^2 + \phi_K(r)^2 - K \xi \right) = 0. \] (4.23)

These equations reduce to Eq. (4.15) for \( K = 3 \).

Considering that the above system arises from the softly broken \( \mathcal{N} = 2 \) \( SU(N) \) theory with \( N_f \) flavors, \( 2N - 2 \leq N_f \leq 2N \), broken by the adjoint scalar VEVs as
\[ SU(N) \rightarrow SU(N - 1) \times U(1), \] (4.24)
the system has an exact \( SU(N - 1)_{C+F} \) symmetry, respected both by the adjoint and squark VEVs (\( K = N - 1 \) above). A vortex solution breaks this symmetry to
\[ SU(N - 1)_{C+F} \rightarrow SU(N - 2) \times U(1) \] (4.25)
and consequently a continuous \( 2(N - 2) \)-parameter family of degenerate vortices exist, representing the quotient space,
\[ \frac{SU(N - 1)}{SU(N - 2) \times U(1)} \sim \mathbb{C}P^{N-2}. \] (4.26)

5. The effective vortex world-sheet theory

We study now the effective low-energy theory for orientational collective coordinates on the string world sheet. We first restrict ourselves to the \( SU(3) \rightarrow SU(2) \times U(1) \) theory of Section 2 and Section 3, coming back to more general cases later. We shall study the bosonic collective coordinates \( n^a \) first and then use the unbroken supersymmetry to reconstruct the fermionic sector.

5.1. Kinetic term

Assume that the orientational collective coordinates \( n^a \) are slow varying functions of the string world sheet coordinates \( x_n, n = 0, 3 \). Then \( n^a \) become fields in a (1+1)-dimensional sigma model on the world sheet. Since the vector \( n^a \) parametrizes the string zero modes, there is no potential term in this sigma model. Let us work out the kinetic term.
To do so we substitute our solution (it is convenient to use it in the singular gauge (3.19)) into the action (3.2) assuming now that the fields acquire a dependence on coordinates \( x_n \) via \( n^a(x_n) \). However, before doing this we have to modify our solution. The point is that our solution was obtained as a \( SU(2)_{C+F} \) rotation of the \((1,0)\)-string. Now we make this transformation local (depending on \( x_n \)). Therefore, the \( n \)-components of the gauge potential are no longer zero. We assume the obvious ansatz for these components

\[
A_n = -i \partial_n U U^{-1} f(r),
\]

where we have introduced a new profile function \( f(r) \). It is determined by its own equation of motion which we will derive below. This function vanishes at infinity

\[
f(\infty) = 0,
\]

while the boundary condition at \( r = 0 \) will be determined shortly.

The kinetic term for \( n^a \) comes from gauge and quark kinetic terms in (3.2). Using (3.19) and (5.1) to calculate the \( SU(2) \) gauge field strength we find

\[
F_{ni} = \frac{1}{2} \partial_n n^a \tau^a \epsilon_{ij} x_j \frac{1}{r^2} f_3 [1 - f(r)] + i \partial_n U U^{-1} \frac{x_i}{r} \frac{d}{dr} f(r).
\]

We see that in order to have a finite contribution coming from \( Tr F_{ni}^2 \) we have to impose

\[
f(0) = 1.
\]

Now substituting the field strength (5.3) into the action (3.2) and taking into account also kinetic term for quarks we finally arrive at

\[
S^{(1+1)} = \beta \int d^2 x \frac{1}{2} (\partial n^a)^2,
\]

where the integration goes over world sheet coordinates \( x_n \) while the coupling constant \( \beta \) is given by

\[
\beta = \frac{2\pi}{g_2^2} \int_0^\infty r dr \left\{ \left( \frac{d}{dr} f(r) \right)^2 + \frac{1}{r^2} f_3^2 (1 - f)^2 + g_2^2 \left[ \frac{1}{2} f^2 (\phi_1^2 + \phi_2^2) + (1 - f)(\phi_1 - \phi_2)^2 \right] \right\}.
\]

We see that the effective world sheet theory for the string orientational zero mode is given by an \( O(3) \) sigma model. The symmetry group of this sigma model is nothing
but global $SU(2)_{C+F}$ whose 3-dimensional representation acts as the group of orientation preserving isometries on the target space, $\mathbb{CP}^1$. The coupling constant of this sigma model is determined by the minimum of action (5.6) for the function $f$. A numerical solution for the profile function $f(r)$ is given in Fig. 6. Note that the function $f$ satisfies a second order equation because, once we allow the dependence of $n^a$ on world sheet coordinates, the vortex is no longer BPS saturated. The emergence of new profile functions which determine the kinetic terms in the effective world volume theory of a string or domain wall was observed earlier in [20, 31].

Clearly (5.5) describes an effective low-energy theory. It has higher derivative corrections in powers of

$$\frac{\partial_n}{g_2\sqrt{\xi}},$$

where we use Eq. (2.15) to determine masses of gauge/quark multiplets in our $SU(2) \times U(1)$ low-energy theory. The sigma model (5.5) gives a good description at even lower scales, well below $g_2\sqrt{\xi}$ where higher derivative corrections are small. The scale $g_2\sqrt{\xi}$ determines also the inverse thickness of our string, in other words the effective sigma model (5.5) can be applied at scales below the inverse thickness of the string which plays a role of the UV cutoff for (5.5). It is quite natural that in the confinement phase the effective theory below the inverse thickness of a string becomes a two-dimensional sigma model on its world sheet.

Figure 6: The profile function $f(r)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{profile_function.png}
\caption{The profile function $f(r)$.}
\end{figure}
5.2. $\mathcal{N} = 2$ $O(3)$ sigma model in (1+1) dimensions

An $\mathcal{N} = 2$ supersymmetric theory in four dimensions has eight supercharges. The string solution of section 4 is 1/2 BPS. Thus we have four supercharges in the two dimensional sigma model on the string world sheet, which generate $\mathcal{N} = 2$ or more precisely (2,2) supersymmetry in 2 dimensions. We have seen that our effective theory (5.5) on the string world sheet is the bosonic part of the $\mathcal{N} = 2$ supersymmetric $O(3)$ sigma model in two dimensions. The physics of this theory is well understood using the mirror description [18]. Below we briefly review known results and interpret them in terms of strings in four dimensions.

The action of this model reads

$$S_{\sigma}^{(1+1)} = \frac{\beta}{2} \int d^2x \ d^2\theta \ d^2\bar{\theta} \ \log \left(1 + \bar{W}W\right), \quad (5.8)$$

where $W$ is a chiral superfield

$$W = w + \sqrt{2} \theta_\alpha \psi^\alpha + \theta^2 F. \quad (5.9)$$

Here $w$ is a complex bosonic field related to the vector $n^a$ by the stereographic projection

$$n^3 = \frac{1 - |w|^2}{1 + |w|^2}, \quad n^1 = \frac{Re \ w}{1 + |w|^2}, \quad n^2 = 2 \frac{Im \ w}{1 + |w|^2}, \quad (5.10)$$

while $\psi^\alpha$ is a complex fermion field, $\alpha = 1, 2$. The bosonic part of the action (5.8) has a standard form

$$S_{\sigma}^{(1+1)\text{bos}} = 2\beta \int d^2x \ \frac{\partial_n \bar{w} \ \partial_n w}{(1 + w \bar{w})^2}, \quad (5.11)$$

which is identical to the one in (5.5) upon substitution (5.10). Classicaly the $O(3)$ sigma model has a spontaneous breaking of the $SU(2)_{C+F}$ symmetry and two massless Goldstone bosons. This means that for a given string the vector $n^a$ is pointed towards a particular direction.

However, the quantum physics of the $\mathcal{N} = 2$ sigma model is quite different. The model is asymptotically free and runs into a strong coupling regime at low energies. The renormalized coupling constant as a function of the energy scale $E$ is given by

$$4\pi \beta = 2 \log \left(\frac{E}{\Lambda_\sigma}\right) + \cdots, \quad (5.12)$$

where $\Lambda_\sigma$ is the scale of the sigma model. This scale is determined by the condition that the sigma model coupling $\beta$ at the scale of the sigma model UV cut-off $g_2 \sqrt{\xi}$ is
given by the four dimensional low-energy coupling $g_2$ via (5.6). Thus,

$$\Lambda^2 \sim \xi e^{-\gamma \frac{m^2}{g^2}},$$  \hspace{1cm} (5.13)

where $\gamma$ is the value of the integral in the action (5.6).

The model has instantons which induce chiral symmetry breaking. Namely, there is a non-zero chiral fermion bilinear condensate

$$<\bar{\psi}(1 + \gamma_5)\psi> = \pm \text{const.}\Lambda_{\sigma},$$  \hspace{1cm} (5.14)

where $\gamma_5 = \tau_3$. The fact that there are two values of chiral condensate indicates that there are two vacua in the sigma model.

The physics of the model becomes more transparent in the mirror description. This is a description of the model in terms of the Coulomb gas of instantons, and is equivalent to a sine-Gordon theory. Explicitly, the model (5.8) is dual to the $\mathcal{N} = 2$ sine-Gordon theory

$$S^{(1+1)}_{\sigma} = \int d^2x \left[ d^2\theta d^2\bar{\theta} \frac{1}{\beta} \bar{Y} Y + \Lambda_{\sigma} d\theta^1 d\bar{\theta}_2 \cosh Y \right].$$  \hspace{1cm} (5.15)

Here the last term is a dual superpotential induced by instantons, while $Y$ is a twisted chiral superfield with the expansion

$$Y = y + \sqrt{2} \theta^1 \bar{\chi}_1 + \sqrt{2} \bar{\theta}_2 \chi_2 + \cdots.$$  \hspace{1cm} (5.16)

This theory has a mass gap of order of $\Lambda_{\sigma}$, indicating that there is no spontaneous breaking of $SU(2)_{C+F}$ and no Goldstone bosons.

### 5.3. $\mathcal{N} = 2$ CP$^{N-2}$ sigma model in (1+1) dimensions

An analogous conclusion follows in the more general case of an $SU(N)$ theory, Eq.(4.2), Eq.(4.3), Eq.(4.4). The low-energy action and its vacuum respect a global $SU(N - 1)_{C+F}$ symmetry, which is broken however by an individual vortex configuration to $SU(N - 2) \times U(1)$. See Eq.(4.26). We assume that a consideration analogous to the one given for the $SU(3)$ theory leads to an $\mathcal{N} = 2$, 

$$\frac{SU(N - 1)}{SU(N - 2) \times U(1)} \sim \text{CP}^{N-2}$$  \hspace{1cm} (5.17)

sigma model on the vortex world sheet. A study of such systems shows that the number of vacua in this sigma model is $N - 1$. 

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Acknowledgements

We are grateful to Hitoshi Murayama for discussions on the homotopy group properties of the nonabelian monopoles and Adam Ritz for useful conversations. A. Y. would like to thank Istituto Nazionale di Fisica Nucleare – Sezione di Pisa, for hospitality. The work of A.Y. was supported by Russian Foundation for Basic Research under the grant No 02-02-17115 and by INTAS grant No 00-00334. K. K. thanks Japan Society for the Promotion of Science (Fellow ID S-03034) and N. Sakai (Tokyo Institute of Technology) for hospitality.

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