THE 3-FORM MULTIPLE IN SUPERGRAVITY

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Abstract

We derive the couplings of the 3-form supermultiplet to the general supergravity-matter-Yang-Mills system. Based on the methods of superspace geometry, we identify component fields, establish their supergravity transformations and construct invariant component field actions. Two specific applications are addressed: the appearance of fundamental 3-forms in the context of strong-weak duality and the use of the 3-form supermultiplets to describe effective degrees of freedom relevant to the mechanism of gaugino condensation.

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The classical equivalence between a rank-two antisymmetric tensor and a pseudoscalar is based on the Hodge duality between the field strength of a 2-form gauge field and the derivative of a 0-form. A similar duality relation exists between a rank-3 antisymmetric tensor and a constant scalar field. Such a relation was in particular considered some time ago in connection with the cosmological constant problem [1, 2, 3].

The supersymmetric generalisation of this duality is particularly interesting. In the first case, the rank-two antisymmetric tensor is incorporated
into the linear supermultiplet \[\mathcal{H}\] which includes also a real scalar and a Majorana spinor field. The duality transformation relates this supermultiplet to a chiral supermultiplet \[\mathcal{H}\] whose content includes the original scalar field as well as the pseudoscalar dual to the antisymmetric tensor.

Such a connection is often used in the context of superstring theories. Indeed, the massless string modes include a dilaton and an antisymmetric tensor which, together with a dilatino spinor field, form a linear multiplet which plays an important role in the effective field theory. A duality transformation is often performed which turns these fields into a dilaton-axion (or rather a field with axion-like couplings) system, central in all discussions of the behavior of the theory under conformal and chiral transformations.

The role of supersymmetry is even more striking when one considers the rank-three antisymmetric tensor. Whereas in the non-supersymmetric case such a field does not correspond to any physical degree of freedom (through its equation of motion, its field strength is a constant 4-form), supersymmetry couples it with propagating fields. Indeed, the 3-form supermultiplet \[\mathcal{H}\] can be described by a chiral superfield \(Y\) and an antichiral field \(\bar{Y}\):

\[
\bar{D}{}^\dot{\alpha} Y = \mathbf{0}, \quad D_a Y = \mathbf{0}, \tag{1.1}
\]

which are subject to the additional constraint:

\[
D^2 Y - \bar{D}^2 \bar{Y} = \frac{8i}{3} \epsilon^{klmn} \Sigma_{klmn}, \tag{1.2}
\]

where \(\Sigma_{klmn}\) is the gauge-invariant field strength of the rank-three gauge potential superfield which we will denote by \(C_{klm}\):

\[
\Sigma_{klmn} = \partial_k C_{lmn} - \partial_l C_{mnk} + \partial_m C_{nkl} - \partial_n C_{klm}. \tag{1.3}
\]

Such a structure is obtained by solving the following constraint on the super-4-form field strength:

\[
\Sigma_{\alpha \beta A} = \mathbf{0}, \tag{1.4}
\]

where \(\alpha\) is a (dotted or undotted) spinorial index whereas \(A\) is a general superspace index. Such a constraint is reminiscent of what is encountered in the case of supersymmetric Chern-Simons forms.
The component fields of the (anti)chiral superfield $Y$ and $\bar{Y}$ are propagating. Therefore, supersymmetry couples the rank three antisymmetric tensor with dynamical degrees of freedom, while respecting the gauge invariance associated with the 3-form.

Let us note that $Y$ is not a general chiral superfield since it must obey the constraint (1.2). Indeed, such a constraint is possible only if $Y$ derives from a prepotential $\Omega$ which is real:

$$Y = -4\bar{D}^2\Omega, \quad \bar{Y} = -4D^2\Omega.$$  \hspace{1cm} (1.5)

All the preceding formulas find a generalisation in supergravity theories [7, 8] which will be the framework of the present paper.

Rank-three antisymmetric tensors might play an important role in several problems of interest, connected with string theories. One of them is the breaking of supersymmetry through gaugino condensation. As the formalism of the super-3-form is modelled along the lines of the Yang-Mills Chern-Simons superforms, this should come as no surprise. Indeed, in supersymmetric theories where the Yang-Mills fields are coupled to a dilaton described by a linear multiplet – such as effective superstring theories –, the effective theory below the scale of condensation is described by a chiral superfield subject to the constraint (1.2); its scalar component is the gaugino condensate itself. This chiral superfield derives from a vector superfield such as in (1.5), which is interpreted (see the first reference [5]) as a “fossil” Chern-Simons superfield [9].

Another interesting appearance of the 3-form supermultiplet occurs in the context of strong-weak coupling duality. More precisely, the dual formulation of ten-dimensional supergravity [10] appears as an effective field theory of some dual formulation of string models, such as five-branes [11]. The Yang-Mills field strength which is a 7-form in ten dimensions may precisely yield in four dimensions a 4-form field strength. The corresponding 3-form may then play an important role in such a key issue as the cosmological constant problem [1].

In the next section, we present the 3-form supermultiplet in the context
of supergravity. In particular, we give the explicit solutions of the constraints (1.4) and present the supersymmetry transformations. In section 3, we derive the form of the action involving this supermultiplet coupled with chiral supermultiplets. In section 4, we comment on two types of applications where our analysis might apply: composite 3-form describing effective degrees of freedom below the gaugino condensation scale and fundamental 3-form arising from the compactification of the dual formulation of ten-dimensional supergravity.

2 THE 3-FORM AND SUPERGRAVITY

2.1 General definitions

The superspace geometry of the 3-form multiplet has been known for some time \([1]\). Its coupling to the general supergravity-matter system is most conveniently described by generalizing the approach of \([1]\) to the framework of \(U_K(1)\) superspace \([2]\). This means simply that we have to deal with a 3-form gauge potential

\[
B^3 = \frac{1}{3!} E^A E^B E^C B_{CBA}^3, \quad (2.1)
\]

where now \(E^A\) denotes the frame of the full \(U_K(1)\) superspace. The 3-form gauge potential is subject to the gauge transformations

\[
B^3 \mapsto \Lambda^B = B^3 + d\Lambda, \quad (2.2)
\]

with parameters given as a superspace 2-form,

\[
\Lambda = \frac{1}{2} E^A E^B \Lambda_{BA}. \quad (2.3)
\]

The invariant field strength

\[
\Sigma = dB^3, \quad (2.4)
\]

is a 4-form in superspace,

\[
\Sigma = \frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA}, \quad (2.5)
\]
with coefficients
\[
\frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA} = \frac{1}{4!} E^A E^B E^C E^D \left( 4 \mathcal{D}_B B^D_{CBA} + 6 T_{DC}^E B^D_{FBA} \right).
\]
(2.6)

Here, the full $U_K(1)$ superspace covariant derivatives and torsions appear. Likewise, the Bianchi identity,
\[
d\Sigma = 0,
\]
(2.7)
is a $U_K(1)$ superspace five-form with coefficients
\[
\frac{1}{5!} E^A E^B E^C E^D E^E \left( 5 \mathcal{D}_E \Sigma_{DCBA} + 10 T_{ED}^F \Sigma_{FBCA} \right) = 0.
\]
(2.8)

In these formulas we have kept the covariant differentials in order to keep track of the graded tensorial structure of the coefficients.

### 2.2 Constraints and Bianchi identities

The multiplet containing the 3-form gauge potential is obtained after imposing constraints on the covariant field-strength coefficients. Following [6] we require
\[
\Sigma_{\dot{\alpha} \gamma \beta A} = 0,
\]
(2.9)
where $\alpha \sim \dot{\alpha}, \dot{\alpha}$ and $A \sim a, \alpha, \dot{\alpha}$. The consequences of these constraints can be studied by analyzing consecutively the Bianchi identities, from lower to higher canonical dimensions (i.e. a spinor index contributes one-half while a vector index contributes one in suitable units). The tensorial structures of the coefficients of $\Sigma$ at higher canonical dimensions are then subject to restrictions due to the constraints. In addition, covariant superfield conditions involving spinorial derivatives will emerge. The contraints serve to reduce the number of independent component fields but do not imply any dynamical equations.

As a result of this analysis, all the coefficients of the 4-form field strength $\Sigma$ can be expressed in terms of the two superfields $\nabla$ and $Y$, which are
identified as follows in the tensorial decomposition:

\[
\Sigma_{\delta\gamma}{}^{ba} = \frac{1}{2}(\sigma_{ba} \epsilon)_{\delta\gamma} \Upsilon, \\
\Sigma^{\delta\gamma}{}^{ba} = \frac{1}{2}(\sigma_{ba} \epsilon)_{\delta\gamma} \Upsilon.
\]  

(2.10) \hspace{1cm} (2.11)

As we are working in \( U_K(1) \) superspace these identifications also allow to read off the \( U_K(1) \) weights of \( \Upsilon \) and \( \bar{\Upsilon} \), which are

\[
\kappa(\Upsilon) = +2, \quad \kappa(\bar{\Upsilon}) = -2.
\]

(2.12)

resulting in covariant (exterior) derivatives

\[
D_{\alpha} \Upsilon = d\Upsilon + 2A_{\alpha} \Upsilon, \quad D_{\dot{\alpha}} \bar{\Upsilon} = d\bar{\Upsilon} - 2\bar{A}_{\dot{\alpha}} \bar{\Upsilon},
\]

(2.13)

with \( A = E^M A_M \) the \( U_K(1) \) gauge potential. On the other hand, the Weyl weights are determined to be

\[
\omega(\Upsilon) = \omega(\bar{\Upsilon}) = +3.
\]

(2.14)

By a special choice of conventional constraints (i.e. a covariant redefinition of \( B_{cba}^3 \)), it is possible to impose

\[
\Sigma_{\delta\gamma}{}^{ba} = 0.
\]

(2.15)

The one spinor-three vector components of \( \Sigma \) are given as

\[
\Sigma_{\delta}{}^{cb} = -\frac{1}{16} \sigma_{\delta\dot{\delta}} \epsilon^{dcb} D_{\dot{\delta}} \bar{\Upsilon}, \\
\Sigma^{\delta}{}^{cb} = +\frac{1}{16} \sigma^{d\delta} \epsilon^{dcb} D_{\delta} \Upsilon.
\]

(2.16) \hspace{1cm} (2.17)

At the same time one finds that the superfields \( \bar{\Upsilon} \) and \( \Upsilon \) are subject to the chirality conditions

\[
D_{\alpha} \bar{\Upsilon} = 0, \quad D_{\dot{\alpha}} \Upsilon = 0.
\]

(2.18)

Moreover they are constrained by the relation

\[
\frac{8i}{3} \epsilon^{dcb} \Sigma_{dcb} = (D^\alpha D_{\alpha} - 24R^1) \Upsilon - (D_{\dot{\alpha}} D^{\dot{\alpha}} - 24R) \bar{\Upsilon}.
\]

(2.19)
This equation involving double spinorial derivatives is a nontrivial restriction besides the chirality constraints, because $\Sigma_{dcb\alpha}$ contains (among other terms) the curl of the purely vectorial coefficient of the 3-form. As a consequence, its lowest superfield component is not an independent field but is expressed in terms of other components, as will be explained in detail in the next subsection.

In conclusion, we have seen that all the coefficients of the superspace 4-form $\Sigma$, subject to the constraints, are given in terms of the superfields $Y$ and $\bar{Y}$ and their spinorial derivatives. It is a matter of straightforward computation to show that all the remaining Bianchi identities do not contain any new information.

2.3 Explicit solution of constraints : the unconstrained prepotential

The analysis of the constraints via the Bianchi identities showed how the 3-form superspace geometry is described in terms of the superfields $Y$ and $\bar{Y}$, themselves subject to the constraints (2.18, 2.19). On the other hand, the direct solution of the constraints for the 3-form gauge potential allows to identify an unconstrained prepotential. As a result, $Y$ and $\bar{Y}$ given in terms of this unconstrained prepotential automatically satisfy (2.18, 2.19).

We shall give here a brief sketch of this and refer to [8] for a more detailed account of the explicit solution of the constraints in the case coupled to supergravity.

As an illustration of the method consider the constraints

\[ \Sigma^{\delta \gamma \beta A} = 0, \quad \Sigma^{\delta \dot{\gamma} \dot{\beta} A} = 0, \]  

(2.20)

which are solved in terms of prepotentials $U_{\beta A}$ and $V_{\dot{\beta} A}$ such that, respectively,

\[ B^{\dot{\gamma} \beta A} = D_{A} U_{\gamma \beta} + \oint_{\gamma \beta} \left( D_{\gamma} U_{\beta A} + T_{\gamma} F_{\beta} U_{F} \right), \]  

(2.21)

and

\[ B^{3 \dot{\gamma} \dot{\beta} A} = D_{A} V_{\dot{\gamma} \dot{\beta}} + \oint_{\dot{\gamma} \dot{\beta}} \left( D^{\dot{\gamma}} V_{\dot{\beta} A} + T_{\dot{\gamma}} F_{\dot{\beta}} V_{F} \right). \]  

(2.22)
Since the prepotentials $U_{\beta A}$ and $V^{\dot{\beta} A}$ should reproduce the gauge transformations of the gauge potentials $B^{3}_{\gamma \beta A}$ and $B^{3\dot{\gamma} \dot{\beta} A}$ they are assigned gauge transformations

$$U_{\beta A} \mapsto U_{\beta A} + \Lambda_{\beta A}, \quad V^{\dot{\beta} A} \mapsto V^{\dot{\beta} A} + \Lambda^{\dot{\beta}} A. \quad (2.23)$$

On the other hand, the prepotentials are still allowed to change under pregauge transformations which leave the gauge potentials themselves unchanged. In the case at hand the pregauge transformations are

$$U_{\beta A} \mapsto U_{\beta A} + D_{\beta} \chi^{A} - \left(-\right)^{a} \mathcal{D}_{A} \chi^{\beta} + T^{A}_{\beta A} F \chi, \quad (2.24)$$

and

$$V^{\dot{\beta} A} \mapsto V^{\dot{\beta} A} + D^{\dot{\beta}} \psi^{A} - \left(-\right)^{a} \mathcal{D}_{A} \psi^{\dot{\beta}} + T^{A}_{\dot{\beta} A} F \psi. \quad (2.25)$$

Playing around with these transformations it is quite straightforward to convince oneself that the non-trivial unconstrained prepotential $\Omega$ is identified in

$$B^{3}_{\gamma \dot{\beta} a} = -2i(\sigma_{a} \epsilon)_{\gamma}^{\beta} \Omega, \quad (2.26)$$

up to certain field dependent gauge transformations which we have neglected here (for a more elaborate description see [8]). Moreover, for the other non-vanishing components of the 3-form gauge potential one finds

$$B^{3}_{\gamma ba} = 2(\sigma_{ba})_{\gamma}^{\phi} \mathcal{D}_{\phi} \Omega, \quad B^{3\dot{\gamma} ba} = 2(\sigma_{ba})_{\dot{\gamma}}^{\phi} \mathcal{D}^{\dot{\phi}} \Omega, \quad (2.27)$$

and, for the purely vectorial part,

$$(\left[D_{\alpha}, D_{\dot{\alpha}}\right] - 4G_{\alpha \dot{\alpha}}) \Omega = \frac{1}{3} \sigma_{d \alpha \dot{\alpha}} \epsilon^{d c b a} B^{3}_{cba}. \quad (2.28)$$

Explicit substitution of these expressions for the 3-form gauge potentials in the field strength gives rise to

$$\bar{Y} = -4(D^{\alpha} \mathcal{D}_{\alpha} - 8R^{\dagger}) \Omega, \quad (2.29)$$

$$Y = -4(D_{\dot{\alpha}} \mathcal{D}^{\dot{\alpha}} - 8R) \Omega. \quad (2.30)$$
As already indicated above, these last three equations are the explicit solution of the constraint equations (2.18,2.19). The explicit expressions for $Y$ and $\bar{Y}$ illustrate also the fact that the prepotential remains undetermined up to a linear superfield, i.e. its pregauge-transformations

$$\Omega \mapsto \Omega' = \Omega + \lambda,$$  \hspace{1cm} (2.31)

are parametrized in terms of a linear superfield $\lambda$ which satisfies

$$(D^2 - 8 R^\dagger) \lambda = 0, \quad (\bar{D}^2 - 8 R) \lambda = 0.$$  \hspace{1cm} (2.32)

### 2.4 Component fields and supergravity transformations

We define the component fields as the lowest components of some superfields. First of all, the three-index component field is identified as

$$B^3_{klm}|_{\theta = \bar{\theta} = 0} = C_{klm}(x).$$  \hspace{1cm} (2.33)

As to the components of $Y$ and $\bar{Y}$ we define

$$Y|_{\theta = \bar{\theta} = 0} = Y(x), \quad D_{\alpha}Y|_{\theta = \bar{\theta} = 0} = \sqrt{2} \eta_{\alpha}(x),$$  \hspace{1cm} (2.34)

and

$$\bar{Y}|_{\theta = \bar{\theta} = 0} = \bar{Y}(x), \quad \bar{D}^{\dot{\alpha}}\bar{Y}|_{\theta = \bar{\theta} = 0} = \sqrt{2} \bar{\eta}^{\dot{\alpha}}(x).$$  \hspace{1cm} (2.35)

At the level of two covariant spinor derivatives we define the component $H(x)$ as

$$D^2 Y|_{\theta = \bar{\theta} = 0} + \bar{D}^2 \bar{Y}|_{\theta = \bar{\theta} = 0} = -8 H(x).$$  \hspace{1cm} (2.36)

The orthogonal combination however is not an independent component field; a look at our superspace geometry shows that it is given as

$$D^2 Y|_{\theta = \bar{\theta} = 0} - \bar{D}^2 \bar{Y}|_{\theta = \bar{\theta} = 0} = -\frac{32i}{3} e^{klmn} \partial_k C_{lmn}$$

$$+ 2 \sqrt{2} i (\bar{\psi}_m \sigma^m)^\alpha \eta_{\alpha} - 2 \sqrt{2} i (\psi_m \sigma^m)^{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}}$$

$$- 4 (\bar{\psi}_m \sigma^{mn} \bar{\psi}_n) Y + 4 (M + \psi_m \sigma^{mn} \bar{\psi}_n) \bar{Y}. $$  \hspace{1cm} (2.37)

---

3 Observe that for the special gauge choice $\Omega = 1$ one obtains the identifications $\bar{Y} = 32 R^\dagger$, $Y = 32 R$ and $-12 G^a = e^{abcd} B^{cba}$. In this case eq. (2.19) becomes simply $D^2 R - \bar{D}^2 R^\dagger = 4i D_{\alpha} G^{\alpha}$, one of the supergravity equations (see for instance (B.88) in [12]).
This expression illustrates also how the superspace approach takes care of the modifications which arise from the coupling to supergravity, here the appearance of the Rarita-Schwinger field and the supergravity auxiliary field, in the particular combination $M \bar{\nabla} - \bar{M} Y$.

The component fields in the other sectors, i.e. supergravity, matter and Yang-Mills are defined as usual [12]. Some new aspects arise in the treatment of the field dependent $U_K(1)$ prepotential due to the presence of the fields $Y$ and $\bar{Y}$, carrying non-vanishing $U_K(1)$ weights. It is for this reason that we refrain from calling $K$ a Kähler potential, we rather shall refer to the field dependent $U_K(1)$ prepotential as kinetic prepotential.

Before turning to the derivation of the supergravity transformations we shall shortly digress on the properties of the composite $U_K(1)$ connection arising from the kinetic prepotential

$$K(\phi, Y, \bar{\phi}, \bar{Y})$$

subject to Kähler transformations

$$K(\phi, Y, \bar{\phi}, \bar{Y}) \mapsto K(\phi, Y, \bar{\phi}, \bar{Y}) + F(\phi) + \bar{F}(\bar{\phi}).$$

Because of the non-zero $U_K(1)$ weight of the fields $Y$ and $\bar{Y}$, invariance of the Kähler potential itself under $U_K(1)$ imposes the condition

$$YK_Y = \bar{Y}K_{\bar{Y}}.$$ 

(2.38)

We will make systematic use of this relation in what follows. An example of a non-trivial Kähler potential which satisfies this condition is

$$K(Y, \bar{Y}) = \ln(1 + Y\bar{Y}),$$

(2.39)

or, if we want to include some dependence on the matter fields $\phi, \bar{\phi}$

$$K(Y, \bar{Y}) = \ln \left( X(\phi, \bar{\phi}) + Z(\phi, \bar{\phi}) Y\bar{Y} \right),$$

(2.40)

where $X$ and $Z$ are two functions of the matter fields.
Now recall first of all that the $U_K(1)$ connection component field is defined as the projection to the lowest component of the superfield $A_m$,

$$A_m(x) = A_m| = -\frac{1}{2}\bar{\sigma}_m^\dot{\alpha} A_{\alpha\dot{\alpha}} + \frac{1}{2} \bar{\psi}_m^\alpha A_\alpha + \frac{1}{2} \bar{\psi}_m A^\dot{\alpha},$$

with

$$A_\alpha = \frac{1}{4} E_\alpha^M \partial_M K,$$
$$A^\dot{\alpha} = -\frac{1}{4} \bar{E}_\dot{\alpha}^M \partial_M K,$$

and

$$A_{a\dot{a}} = -\frac{i}{8} [\mathcal{D}_a, \mathcal{D}_{\dot{a}}] K + \frac{\bar{\psi}}{2} G_{a\dot{a}}.$$

Exploiting the (super)field dependence of the kinetic prepotential one finds for the commutator term

$$[\mathcal{D}_a, \mathcal{D}_{\dot{a}}] K = 2 i K_k \mathcal{D}_a \phi^k - 2 i K_{\dot{k}} \mathcal{D}_{\dot{a}} \bar{\phi}^{\dot{k}} + 2 i K_Y \mathcal{D}_a Y - 2 i K_{\bar{Y}} \mathcal{D}_{\dot{a}} \bar{Y} + 2 K_{A\dot{A}} \mathcal{D}_a \Psi^A \mathcal{D}_{\dot{a}} \bar{\Psi}^\dot{A} + 6 (Y K_Y + \bar{Y} K_{\bar{Y}}) G_{a\dot{a}},$$

where $\Psi^A$ a short hand notation for $\phi^k, Y$, and $\bar{\Psi}^\dot{A}$ for $\bar{\phi}^{\dot{k}}, \bar{Y}$. The important point is that on the right hand the $U_K(1)$ connection appears in the covariant derivatives of $Y$ and $\bar{Y}$ due to the non-vanishing $U_K(1)$ weights. Explicitly one has

$$\mathcal{D}_a Y|_{\theta=\bar{\theta}=0} = \sigma_{a\dot{a}}^m \left( \partial_m Y - 2 A_m \bar{Y} - \frac{1}{\sqrt{2}} \bar{\psi}_m A_\alpha \bar{\eta}^\dot{\alpha} \right),$$

$$\mathcal{D}_{\dot{a}} \bar{Y}|_{\theta=\bar{\theta}=0} = \sigma_{a\dot{a}}^m \left( \partial_m Y + 2 A_m Y - \frac{1}{\sqrt{2}} \bar{\psi}_m A^\dot{\alpha} \eta_\alpha \right).$$

Substituting in the defining equation for $A_m$ and factorizing gives then rise to

$$A_m(x) + \frac{i}{2} e_m^a b_a = \frac{1}{4} \frac{1}{1 - Y K_Y} \left( K_k \mathcal{D}_m A^k - K_{\dot{k}} \mathcal{D}_m \bar{A}^{\dot{k}} + K_Y \partial_m Y - K_{\bar{Y}} \partial_m \bar{Y} + i \bar{\sigma}_m^\alpha K_{A\dot{A}} \Psi^A \bar{\Psi}^\dot{A} \right).$$

Again, we have introduced a short hand notation: $\Psi^A_\alpha$ stands for $\chi^k_\alpha$ or $\eta_\alpha$, while $\bar{\Psi}^\dot{A}_{\dot{\alpha}}$ stands for $\bar{\chi}^{\dot{k}}_{\dot{\alpha}}$ or $\bar{\eta}_{\dot{\alpha}}$. As is easily verified by an explicit calculation, $A_m$ defined this way transforms as it should under the $U_K(1)$ transformations given above, i.e.

$$A_m \mapsto A_m + \frac{1}{4} \partial_m (F - \bar{F}).$$
Observe that the denominator accounts for the non-trivial $U_K(1)$ phase transformations

$$Y \mapsto Ye^{-\frac{1}{2}(F - \bar{F})}, \quad Y' \mapsto Ye^{\frac{1}{2}(F - \bar{F})}$$

of the 3-form scalar fields. In the following we shall frequently use the particular combination

$$v_m(x) = A_m(x) + \frac{i}{2} e_m^a b_a. \quad (2.43)$$

We come now back to the issue of the supergravity transformations of the component fields of the 3-form multiplet as defined above. In general, in the spirit of [13], supergravity transformations are defined as combinations of superspace diffeomorphisms (i.e. Lie-derivatives in superspace as defined in some detail in ref. [14]) and field dependent gauge transformations. In the case of the 3-form one has

$$\delta B^3 = (i\xi d + d\xi)B^3 + d\Lambda = i\xi \Sigma + d\left(\Lambda + i\xi B^3\right), \quad (2.44)$$

and the corresponding supergravity transformation is defined as a diffeomorphism of parameter $\xi^A = i\xi E^A$ together with a compensating infinitesimal 2-form gauge transformation of parameter

$$\Lambda = -i\xi B^3,$$

giving rise to

$$\delta B^3 = i\xi \Sigma = \frac{1}{3!} E^A E^B E^C \xi^D \Sigma_{DCBA}. \quad (2.45)$$

The supergravity transformation of the component 3-form gauge field $C^{klm}$ is then simply obtained from the double projection [14] (simultaneously to lowest superfield components and to space-time differential forms) as

$$\delta B^3 = \frac{1}{3!} e^A e^B e^C \xi^D \Sigma_{DCBA}.$$

Taking into account the definition $e^A = E^{A||}$ and the particular form of the coefficients of $\Sigma$ one obtains

$$\delta C^{mlk} = \sqrt{\frac{3}{2}} \left(\bar{\zeta} \bar{\sigma}^n \eta - \xi \sigma^n \bar{\eta}\right) \varepsilon_{nmlk} + \frac{1}{2} \int_{mlk} \left[ (\psi_m \sigma_{lk} \zeta) \bar{Y} + (\bar{\psi}_m \bar{\sigma}_{lk} \bar{\zeta}) Y \right]. \quad (2.47)$$
Let us turn now to the transformations of the remaining components. To start, note that at the superfield level one has

$$
\delta Y = \iota \xi dY = \iota \xi DY - 2 \iota \xi AY, \quad (2.48)
$$

$$
\delta Y = \iota \xi dY = \iota \xi D\bar{Y} + 2 \iota \xi A\bar{Y}. \quad (2.49)
$$

Taking into account the explicit form of the field-dependent factor \( \iota \xi A \) = \( \zeta^\alpha A_{\alpha} \) one finds

$$
\delta Y = \sqrt{2} \zeta^\alpha \left\{ (1 - \frac{1}{2} Y K_Y) \eta_\alpha - \frac{1}{2} Y K_k \chi^k_\alpha \right\} + \frac{1}{\sqrt{2}} \tilde{\zeta}\dot{\zeta} Y \\
\left\{ K_m \bar{\eta}^\dot{\alpha} + K_{\dot{\alpha}} \chi^k \right\}, \quad (2.50)
$$

$$
\delta Y = \sqrt{2} \zeta^\alpha \left\{ (1 - \frac{1}{2} Y K_Y) \bar{\eta}^\dot{\alpha} - \frac{1}{2} \bar{\nabla} K_k \bar{\chi}^\dot{k} \right\}
+ \frac{1}{\sqrt{2}} \zeta^\alpha \bar{Y} \left\{ Y \eta_\alpha + K_k \chi^k_\alpha \right\}, \quad (2.51)
$$

It is more convenient to use a notation where one keeps the combination

$$
\Xi = \zeta^\alpha A_{\alpha} = \frac{1}{2 \sqrt{2}} \zeta^\alpha \left( K_k \chi^k_\alpha + K_Y \eta_\alpha \right) - \frac{1}{2 \sqrt{2}} \tilde{\zeta}\dot{\zeta} \left( K_{\dot{\alpha}} \chi^k \right), \quad (2.52)
$$

giving rise to a compact form of the supersymmetry transformations :

$$
\delta Y = \sqrt{2} \zeta^\alpha \eta_\alpha - 2 \Xi Y, \quad \delta Y = \sqrt{2} \zeta^\alpha \bar{\eta}^\dot{\alpha} + 2 \Xi \bar{Y}. \quad (2.53)
$$

The transformation law for the 3-”forminos” comes out as

$$
\delta \eta_\alpha = \sqrt{2} \zeta^\alpha H + \frac{4 i \sqrt{2}}{\sqrt{3}} \zeta^\alpha \bar{c}_{klmn} \partial_k C_{lmn} + i \sqrt{2} (\bar{c} \sigma^m e)_\alpha \bar{Y} - \Xi \eta_\alpha \\
- \frac{i}{2} \bar{c}^\alpha \left( \bar{\psi}_m \bar{\sigma}^m \eta - \psi_m \sigma^m \bar{\eta} \right) - i \sqrt{2} (\bar{c} \sigma^m e)_\alpha \psi_m \sigma^m \eta \phi \\
+ \frac{1}{\sqrt{2}} \zeta^\alpha \left\{ \left( \bar{M} + \bar{\psi}_m \bar{\sigma}^m \bar{\psi}_n \right) Y - \left( M + \psi_m \sigma^m \psi_n \right) \bar{Y} \right\}, \quad (2.54)
$$

and

$$
\delta \bar{\eta}^\dot{\alpha} = \sqrt{2} \tilde{\zeta} \bar{\eta}^\dot{\alpha} H + i \sqrt{2} (\bar{c} \sigma^m \bar{e})^\alpha \bar{D}_m \bar{Y} - \frac{4 i \sqrt{2}}{\sqrt{3}} \tilde{\zeta} \bar{c}_{klmn} \partial_k C_{lmn} + \Xi \bar{\eta}^\dot{\alpha} \\
+ \frac{i}{2} \tilde{c}^\dot{\alpha} \left( \bar{\psi}_m \bar{\sigma}^m \eta - \psi_m \sigma^m \bar{\eta} \right) - i (\bar{c} \sigma^m \bar{e})_\dot{\alpha} \psi_m \sigma^m \bar{\eta} \\
- \frac{1}{\sqrt{2}} \bar{\zeta}^\dot{\alpha} \left\{ \left( \bar{M} + \bar{\psi}_m \bar{\sigma}^m \bar{\psi}_n \right) Y - \left( M + \psi_m \sigma^m \psi_n \right) \bar{Y} \right\}. \quad (2.55)
$$

4 The covariant derivative \( D_m \) differs from \( \bar{D}_m \) in that it is given in terms of the particular combination \( v_m = A_m + \frac{1}{2} e_m a \) i.e. the covariant derivative \( D_m \) does not contain the field \( b_a \)
Finally, the supergravity transformation of $H$ is given as

$$\delta H = \frac{1}{\sqrt{2}}(\bar{\zeta} \bar{\sigma}^m) a D_m \eta a + \frac{i}{2}(\bar{\zeta} \bar{\sigma}^m \gamma^m \bar{\psi}_m)(D_n Y - \frac{1}{\sqrt{2}} \psi_m \gamma^m \phi)$$

$$+ \frac{1}{\sqrt{2}}(\zeta^m) a D_m \pi^a + \frac{i}{2}(\zeta^m \gamma^m \psi_m)(D_n \bar{\gamma}^a - \frac{1}{\sqrt{2}} \bar{\psi}_m \gamma^m \pi^a)$$

$$+ \frac{1}{3\sqrt{2}} M \zeta^a \eta a + \frac{1}{3\sqrt{2}} M \bar{\zeta}_a \bar{\eta} a + \frac{1}{3\sqrt{2}}(\bar{\zeta} \gamma^a \eta + \zeta \gamma^a \bar{\eta}) b a$$

$$+ Y \bar{\zeta}_a X^a |_{\theta = \bar{\theta} = 0} + \bar{\gamma}^a \zeta^a X^a |_{\theta = \bar{\theta} = 0} - \frac{1}{\sqrt{2}}(\bar{\zeta} \gamma^m \psi_m + \zeta \gamma^m \bar{\psi}_m) H$$

$$+ \frac{2}{\sqrt{2}}(\bar{\zeta} \gamma^a \psi_a - \zeta \gamma^a \bar{\psi}_a) \bar{\psi}_{k m n} \partial_k C_{l m n}$$

$$- \frac{i}{4\sqrt{2}}(\bar{\zeta} \gamma^m \psi_m - \zeta \gamma^m \bar{\psi}_m)(M + \bar{\psi}_m \gamma^m \bar{\psi}_n) Y - (M + \psi_m \gamma^m \psi_n) \bar{\gamma}$$

$$- \frac{i}{4\sqrt{2}}(\bar{\zeta} \gamma^m \psi_m - \zeta \gamma^m \bar{\psi}_m)(\bar{\psi}_m \gamma^m \eta - \psi_m \gamma^m \bar{\eta}). \quad (2.56)$$

In this transformation law appear the lowest components of the superfields $X^a$ and $X^{\bar{a}}$. These superfields which play a key role in the construction of invariant actions, are defined as follows:

$$X^a = -\frac{1}{8} \left( \mathcal{D}^2 - 8 R \right) D_a K, \quad \bar{X}^{\bar{a}} = -\frac{1}{8} \left( \mathcal{D}^2 - 8 R^\dagger \right) \bar{D}^{\bar{a}} K. \quad (2.57)$$

One may now successively apply the spinorial derivatives to the kinetic potential to evaluate the explicit form of these superfields. Alternatively one may use the expression

$$A = \frac{1}{4} K_A D \Psi^A \frac{1}{4} K_A D \bar{\Psi}^\dagger + \frac{i}{4} E^a \bar{\sigma}^{\dagger a} K_A D_a \Psi^A D_a \bar{\Psi}^\dagger$$

$$\quad + \frac{3i}{4} E^a G_a \left( 1 - \frac{1}{2} (Y K_Y + \bar{Y} K_{\bar{Y}}) \right), \quad (2.58)$$

for the composite $U_K(1)$ connection, take the exterior derivative $dA = F$ and identify $\bar{X}^{\bar{a}}$ and $X^a$ in the 2-form coefficients

$$F^\beta_{\alpha} = + \frac{i}{2} \sigma^\alpha_{\beta \beta} \bar{X}^{\bar{\beta}} + \frac{3i}{2} \bar{D}_\beta G_a, \quad F^\beta_{\alpha} = - \frac{i}{2} \bar{\sigma}^{\beta \bar{\alpha}} X^\alpha + \frac{3i}{2} \bar{D}^{\bar{\beta}} G_a. \quad (2.59)$$

A straightforward calculation then yields the component field expression\(^5\)

$$\bar{X}^{\bar{a}}(1 - \bar{Y} K_{\bar{Y}})|_{\theta = \bar{\theta} = 0} = - \frac{i}{\sqrt{2}} K_A \bar{\Psi}^A \bar{\sigma}^{\dagger m} \eta a \left( D_m \bar{\Psi}^\dagger - \frac{1}{\sqrt{2}} \bar{\psi}_m \gamma^m \bar{\Psi}^\dagger \right)$$

\(^5\) We make use, in the Yang-Mills sector, of the suggestive notations

$$K_k (\lambda^a A)^{\bar{a}} = \bar{\lambda}^a (K_k T_{\bar{a}}^k A^{\bar{a}}), \quad K_k (\lambda^a \cdot A)^{\bar{a}} = \lambda^a (K_k T_{\bar{a}}^k A^\dagger).$$
by definition does not depend on the superfield $G$. Note here that we are using the space-time covariant derivative $D_{\alpha\dot{\alpha}}$. As pointed out in a previous subsection, the $U_K(1)$ gaugino superfield $X_{\alpha}$ is given as

$$2i X_{\alpha} (1 - Y K_{\gamma}) =$$

$$K_{\dot{A}A} \bar{D}^{\dot{A}} \bar{\Psi}^{\dot{A}} D_{\dot{\alpha}a} \Psi_{a}^{\alpha} - \frac{i}{4} K_{\dot{A}A} D_{\alpha} \Psi^{A} \bar{D}^{2} \bar{\Psi}^{A}$$

$$- \frac{i}{4} K_{\dot{A}B} \bar{D}_{\dot{\alpha}} \bar{\Psi}^{\dot{A}} \bar{D}^{\dot{A}} \bar{\Psi}^{\dot{B}} D_{\alpha} \Psi^{A} - 2i K_{k} (\mathcal{W}_{\alpha} \cdot \phi)^{k}. \quad (2.62)$$

Note here that we are using the space-time covariant derivative $D_{\alpha\dot{\alpha}}$, which by definition does not depend on the superfield $G_{\alpha\dot{\alpha}}$. In full detail:

$$D_{\alpha\dot{\alpha}} Y = D_{\alpha\dot{\alpha}} Y - 3i G_{\alpha\dot{\alpha}} Y, \quad D_{\alpha\dot{\alpha}} Y = D_{\alpha\dot{\alpha}} Y + 3i G_{\alpha\dot{\alpha}} Y, \quad (2.63)$$

These are the component field expressions which are to be used in the transformation law of $H$ (2.56). The same expressions will be needed later on in the construction of the invariant action.

### 2.5 The matter $D$-term superfield

For later convenience, we discuss now shortly the form of the $D$-term superfield $D_{\alpha} X_{\alpha}$ in the presence of the 3-form multiplet. As pointed out in a previous subsection, the $U_K(1)$ gaugino superfield $X_{\alpha}$ is given as

$$-\frac{\sqrt{2}}{8} D_{\alpha} \phi^{k} \left| K_{\dot{A}A} \bar{D}^{\dot{A}} \bar{\Psi}^{\dot{A}} + \frac{1}{\sqrt{2}} H K_{\gamma} \bar{\Psi}^{\dot{A}} + \frac{4i}{3\sqrt{2}} \bar{\Psi}^{\dot{A}} K_{\dot{A}Y} \varepsilon^{klmn} \partial_{k} C_{lmn} ight.$$
and

\[ D_{\alpha\dot{\alpha}} D_\beta Y = D_{\alpha\dot{\alpha}} D_\beta Y - \frac{3i}{2} G_{\alpha\dot{\alpha}} D_\beta Y, \quad (2.64) \]

\[ D_{\alpha\dot{\alpha}} D_\beta Y = D_{\alpha\dot{\alpha}} D_\beta Y + \frac{3i}{2} G_{\alpha\dot{\alpha}} D_\beta Y. \quad (2.65) \]

In deriving the explicit expression for \( D^\alpha X_\alpha \) we shall make systematic use of this derivative, which somewhat simplifies the calculations and will be useful anyway when passing to the component field expression later on. In applying the spinorial derivative to (2.62) it is convenient to make use of the following relations

\[ D_\alpha \bar{D}_\alpha Y = -2i D_{\alpha\dot{\alpha}} Y, \quad (2.66) \]

\[ D_\alpha \bar{D}_\alpha^2 Y = -4i D_{\alpha\dot{\alpha}} \bar{D}_\beta Y + 2 G_{\alpha\dot{\alpha}} D_\beta Y - 8 X_\alpha Y, \quad (2.67) \]

\[ D_\alpha \bar{D}_\alpha^2 \bar{\phi}^k = -4i D_{\alpha\dot{\alpha}} \bar{D}_\dot{\alpha} \bar{\phi}^k + 2 G_{\alpha\dot{\alpha}} \bar{D}_\dot{\alpha} \bar{\phi}^k + 8 (\bar{\psi}_\alpha \cdot \bar{\phi})^k. \quad (2.68) \]

In order to obtain a compact form for \( D^\alpha X_\alpha \), we introduce \( K^{\dot{A}A} \) as the inverse of \( K_{A\dot{A}} \) and we define

\[ -4 F^A = D^\alpha D_\alpha \Psi^A + \Gamma_{BC}^A D^\alpha \Psi^B D_\alpha \Psi^C, \quad (2.69) \]

\[ -4 \bar{F}^{\dot{A}} = \bar{D}_\dot{\alpha} \bar{D}_\dot{\alpha} \bar{\psi}^{\dot{A}} + \bar{\Gamma}_{BC\dot{A}} \bar{D}_\dot{\alpha} \bar{\psi}^B \bar{D}_\dot{\alpha} \bar{\psi}^C, \quad (2.70) \]

with

\[ \Gamma_{BC}^A = K^{A\dot{A}} K_{\dot{A}B\dot{C}}, \quad \bar{\Gamma}_{BC\dot{A}} = K^{A\dot{A}} K_{A\dot{B}\dot{C}}. \quad (2.71) \]

Moreover we define the new covariant derivatives

\[ \nabla_{\alpha\dot{\alpha}} D^\alpha \Psi^A = D_{\alpha\dot{\alpha}} D^\alpha \Psi^A + \Gamma_{BC}^A D_{\alpha\dot{\alpha}} \Psi^B D_\alpha \Psi^C, \quad (2.72) \]

\[ \nabla_{\alpha\dot{\alpha}} \bar{D}_\dot{\alpha} \bar{\psi}^{\dot{A}} = D_{\alpha\dot{\alpha}} \bar{D}_\dot{\alpha} \bar{\psi}^{\dot{A}} + \bar{\Gamma}_{BC\dot{A}} D_{\alpha\dot{\alpha}} \bar{\psi}^B \bar{D}_\dot{\alpha} \bar{\psi}^{\dot{C}}, \quad (2.73) \]

With these definitions the superfield expression of \( D^\alpha X_\alpha \) becomes simply

\[
2i D^\alpha X_\alpha \left( 1 - \bar{\nabla} K_{\bar{\alpha} \bar{\beta}} \right) = 4i \bar{\nabla} K_{A\bar{A}} X^\alpha D_\alpha \Psi^A + 4i Y K_{V\bar{A}} \bar{X}_\alpha \bar{D}_\alpha \bar{\psi}^\bar{A}
- 2i K_{A\bar{A}} D^{\alpha\dot{\alpha}} \bar{\psi}^{\dot{A}} D_{\alpha\dot{\alpha}} \Psi^A - 4i K_{A\bar{A}} F^{\dot{A}} F_\bar{A}
- K_{A\bar{A}} \bar{D}_\dot{\alpha} \bar{\psi}^{\dot{A}} \nabla_{\alpha\dot{\alpha}} D^\alpha \Psi^A - K_{A\bar{A}} D^{\alpha\dot{A}} \Psi^A \nabla_{\alpha\dot{\alpha}} \bar{D}_\dot{\alpha} \bar{\psi}^\bar{A}
\]

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This looks indeed very similar to the usual case. One of the differences however is that the $F$-terms and their complex conjugates for the superfields $Y$ and $\bar{Y}$ have special forms. We will come back to this point later on in the discussion of the full component field action.

### 2.6 The superpotential superfield

The superpotential superfield is given as the combination

$$P = e^{K/2}W(\phi, y),$$

with $y$ defined as a holomorphic section

$$y = e^{-K/2}Y.$$

The superfield $P$ is covariantly chiral, $D\bar{\alpha} P = 0$, and carries $U_K(1)$ weight $\kappa(P) = +2$.

We parametrize the covariant spinorial derivatives of $P$ such that

$$D_\alpha P = \Sigma_A D_\alpha \Psi^A,$$

and

$$D^2 P = -4 \Sigma_A F^A + \Sigma_{AB} D^a \Psi^A D_\alpha \Psi^B.$$

The various components of the coefficients $\Sigma_A$ and $\Sigma_{AB}$ are given as

$$\Sigma_k = e^{K/2}(W_k + K_k W) - YW_y K_k,$$

$$\Sigma_Y = e^{K/2}WK_Y + W_y(1 - YK_Y)$$

and

$$\Sigma_{kl} = (e^{K/2}W - YW_y)(K_{kl} + K_k K_l).$$
\[-Y(W_{ky}K_l + W_{ly}K_k) + e^{K/2}(W_{kl} + W_{lk}K_l + W_lK_k)\]
\[+ e^{-K/2}Y^2 K_kK_lW_{yy} - \Sigma_A \Gamma^A_{kl}, \quad (2.81)\]

\[\Sigma_{kY} = (e^{K/2}W - YW_y)(K_{kY} + K_kK_Y)\]
\[+ W_{ky} (1 - YK_Y) + e^{K/2}W_kK_Y\]
\[- e^{-K/2} YK_kW_{yy} (1 - YK_Y) - \Sigma_A \Gamma^A_{kY}, \quad (2.82)\]

\[\Sigma_{YY} = (e^{K/2}W - YW_y)(K_{YY} + K_YK_Y)\]
\[+ e^{-K/2}W_{yy} (1 - YK_Y)^2 - \Sigma_A \Gamma^A_{YY}. \quad (2.83)\]

Complex conjugate expressions are obtained from
\[\bar{P} = e^{K/2}\bar{W}(\bar{\phi}, \bar{y}), \quad (2.84)\]
with \(\bar{y} = e^{-K/2}\bar{y}\).

\section{THE COMPONENT FIELD ACTION}

\subsection{General action terms}

Our starting point for the construction of supersymmetric and \(U_K(1)\) invariant component field actions is the generic expression

\[\mathcal{L}(r, \bar{r}) = -\frac{1}{4}e\left(\bar{D}^2 - 24\bar{R}\right)\bar{r} - \frac{1}{4}e\left(D^2 - 24R\right)r\]
\[+ \frac{i}{2}e(\bar{\psi}_m\bar{\sigma}^m)^\alpha D_\alpha r| + \frac{i}{2}e(\psi_m\sigma^m)^\dot{\alpha} \bar{D}^\dot{\alpha}\bar{r}|\]
\[= -e(\bar{\psi}_m\sigma^{mn}\bar{\psi}_n) r| - e(\psi_m\sigma^{mn}\psi_n) \bar{r}|, \quad (3.1)\]

with \(r\) and \(\bar{r}\) chiral resp. antichiral superfields of \(U_K(1)\) weights
\[\kappa(r) = +2, \quad \kappa(\bar{r}) = -2. \quad (3.2)\]

Particular component field actions are then obtained by choosing \(r\) and \(\bar{r}\) appropriately. The complete action we are going to consider here will consist of three separately supersymmetric pieces,

\[\mathcal{L}_{total} = \mathcal{L}_{S+M} + \mathcal{L}_{POT} + \mathcal{L}_{YM}, \quad (3.3)\]
with
\[ \mathcal{L}_{S+M} = \mathcal{L} \left( -3R, -3R^\dagger \right), \quad (3.4) \]
the combination of the supergravity action and the kinetic terms of the matter sector, including the 3-form multiplet,
\[ \mathcal{L}_{\text{POT}} = \mathcal{L} \left( e^{K/2} W, e^{K/2} \bar{W} \right), \quad (3.5) \]
the 3-form dependent superpotential and
\[ \mathcal{L}_{\text{YM}} = \mathcal{L} \left( \frac{1}{4} f_{rs} W^r \sigma^s W^s, \frac{1}{4} \tilde{f}_{rs} \bar{W}^r \bar{W}^s \bar{\sigma}^s \right), \quad (3.6) \]
the Yang-Mills kinetic terms with 3-form independent gauge coupling functions. In the following we shall discuss one by one the three individual contributions to the total Lagrangian.

The relevant superfield relations for the supergravity plus matter kinetic actions are
\[ -3 \mathcal{D}_a R = X_a + 2 (T_{cb} \sigma^{cb} e)_{\alpha}, \quad -3 \mathcal{D}^{\dot{\alpha}} R^\dagger = \mathcal{X}^{\dot{\alpha}} + 2 (T_{cb} \bar{\sigma}^{cb} e)^{\dot{\alpha}} \quad (3.7) \]
and
\[ \frac{3}{4} \left( \mathcal{D}^2 - 24 R^\dagger \right) R + \frac{3}{4} \left( \mathcal{D}^\dagger - 24 R \right) R^\dagger = -\frac{1}{2} R b^a b_a + 3 G^a G_a - 12 R^\dagger R - \frac{1}{2} \mathcal{D}^2 X_a. \quad (3.8) \]
It is then convenient to decompose the supergravity plus matter action such that
\[ \mathcal{L}_{S+M} = \mathcal{L}_{\text{supergravity}} + e \mathcal{D}_M, \quad (3.9) \]
with the D-term matter component field \( \mathcal{D}_M \) defined as
\[ \mathcal{D}_M = -\frac{1}{2} D^a X_a \sigma^a \sigma^{\dot{a}} + \frac{i}{2} \psi^a \sigma^a m^{a \dot{a}} X_{\dot{a}} \sigma^{\dot{a}} + \frac{i}{2} \bar{\psi}^{a \dot{a}} \sigma^{a \dot{a}} X_a. \quad (3.10) \]
The pure supergravity part is given by the usual expression, i.e.
\[ \frac{1}{e} \mathcal{L}_{\text{supergravity}} = -\frac{1}{2} \mathcal{R} + \frac{1}{2} e^{k mn} (\bar{\psi}_k \sigma_l D_m \psi_n - \psi_k \sigma_l D_m \bar{\psi}_n) - \frac{1}{3} \mathcal{M} M + \frac{1}{3} b^a b_a. \quad (3.11) \]
except that the $U_K(1)$ covariant derivatives of the Rarita-Schwinger field contain now the new composite $U_K(1)$ connection as defined above. For the matter part one obtains

$$(1 - YK_Y) D_m = -\sqrt{2} X^\alpha | \Psi^A_\alpha \overline{\nabla} K_{A\overline{\alpha}}^A - \sqrt{2} \overline{X}_\alpha | \overline{\Psi}^{\overline{\alpha}} Y \overline{K}_{\overline{A}}$$

$$- g^{mn} K_{A\overline{A}} D_n \Psi^A + K_{A\overline{A}} F^A \overline{F}^\overline{A}$$

$$- i K_{A\overline{A}} \overline{\Psi}^{\overline{A}} \sigma_{\alpha\overline{\alpha}} \nabla^\alpha \Psi^A$$

$$+ \frac{1}{2} R_{AB} \Psi^A \overline{\Psi}^B - \frac{1}{2} K_{A\overline{A}} \Psi^A \overline{\Psi}^{\overline{A}}$$

$$- \frac{1}{\sqrt{2}} (\overline{\psi}_m \overline{\sigma}^n \sigma^m \overline{\Psi}^A) K_{A\overline{A}} D_n \Psi^A - \frac{1}{\sqrt{2}} (\overline{\psi}_m \sigma^m \overline{\Psi}^A) K_{A\overline{A}} D_n \overline{\Psi}$$

$$- (\overline{\psi}_m \sigma^m \overline{\Psi}^A) K_{A\overline{A}} (\overline{\psi}_n \overline{\Psi}^A)$$

$$- \frac{1}{2} K_{A\overline{A}} \overline{g}^{mn} (\psi_m \Psi^A)(\overline{\psi}_n \overline{\Psi}^A)$$

$$- (D \cdot \overline{A}) \overline{k} - i \sqrt{2} K_{\overline{A}k} (\overline{\lambda} \cdot \overline{A})^k + i \sqrt{2} K_{\overline{A}k} \Psi^A (\lambda \cdot \overline{A})^k$$

$$- \frac{1}{2} (\overline{\psi}_m \sigma^m)^\alpha K_k (\lambda \cdot A)^k + \frac{1}{2} (\psi_m \sigma^m) \overline{k} (\overline{\lambda} \cdot \overline{A})^k,$$  \hspace{1cm} (3.12)

with the terms in the first line given as

$$- \sqrt{2} X^\alpha | \Psi^A_\alpha \overline{\nabla} K_{A\overline{\alpha}}^A - \sqrt{2} \overline{X}_\alpha | \overline{\Psi}^{\overline{\alpha}} Y \overline{K}_{\overline{A}} =$$  \hspace{1cm} (3.13)

$$\left[ \frac{1}{1 - YK_Y} \left[ +i \overline{\nabla} K_{\overline{B}} \Psi^A_{\overline{A}} \overline{\Psi}^{\overline{B}} \sigma_{\overline{\alpha}}^m \left( D_m \Psi^A - \frac{1}{\sqrt{2}} \overline{\psi}_m \bar{\sigma} \Psi^A \right) + i YK_{\overline{B}} K_{A\overline{A}} \overline{\Psi}^{\overline{B}} \Psi^A \sigma_{\overline{\alpha}}^m \left( D_m \Psi^A - \frac{1}{\sqrt{2}} \psi_m \overline{\sigma} \overline{\Psi}^{\overline{A}} \right) - YK_{\overline{B}} K_{A\overline{A}} \overline{\Psi}^{\overline{B}} \overline{\Psi} \overline{F}^A - YK_{\overline{B}} K_{A\overline{A}} \Psi^A \overline{B} \overline{F}^\overline{A} - i \sqrt{2} YK_{\overline{A}} \Psi^A K_k (\lambda \cdot A)^k + i \sqrt{2} YK_{\overline{A}} \Psi^A K_k (\overline{\lambda} \cdot \overline{A})^k \right] \right].$$

Making use of the superpotential superfield and the corresponding definitions given at the end of the previous section one derives easily the component field expression

$$\frac{1}{e^2} \mathcal{L}_{POT} = \Sigma_A F^A - \frac{1}{2} \Sigma_{AB} \Psi^A \Psi^B + \frac{i}{\sqrt{2}} \Sigma_A \left( \overline{\psi}_m \overline{\sigma}^m \Psi^A \right)$$

$$- e^2 W \left( \overline{M} + \overline{\psi}_m \overline{\sigma}^m \overline{\psi}_n \right) + \text{h.c.}, \hspace{1cm} (3.14)$$
where $\Sigma_A$ and $\Sigma_{AB}$ are defined in (2.79-2.83).

Finally the Yang-Mills component field action reads

$$\frac{1}{4}L_{YM} = -\frac{1}{4}f_{rs} \left[ f_{rn} f_{sm} + 2i \lambda^r \sigma^m D_m \bar{\lambda}^s + 2i \bar{\lambda}^s \sigma^m D_m \lambda^r \right]$$

$$-2D^s D^s + \frac{i}{2} \varepsilon^{klmn} f_{kl} f^s_{mn} - 2(\lambda^r \sigma^a \bar{\lambda}^s) b_a$$

$$-\frac{1}{4} \frac{\partial f_{rs}}{\partial A^i} \left[ \sqrt{2} (\chi^i \sigma^{mn} \lambda^r) f^s_{mn} - \sqrt{2} (\chi^i \lambda^r) D^s + (\lambda^r \lambda^s) F^i \right]$$

$$-\frac{1}{4} \frac{\partial \bar{f}_{rs}}{\partial \bar{A}^i} \left[ \sqrt{2} (\bar{\chi}^i \sigma^{mn} \bar{\lambda}^r) f^s_{mn} - \sqrt{2} (\bar{\chi}^i \bar{\lambda}^r) D^s + (\bar{\lambda}^r \bar{\lambda}^s) \bar{F}^i \right]$$

$$+ \frac{1}{8} \left( \frac{\partial^2 f_{rs}}{\partial A^k \partial A^l} - \frac{\partial f_{rs}}{\partial A^k} \Gamma^i_{kl} \right) (\chi^i \chi^j)(\lambda^r \lambda^s)$$

$$+ \frac{1}{8} \left( \frac{\partial^2 \bar{f}_{rs}}{\partial \bar{A}^k \partial \bar{A}^l} - \frac{\partial \bar{f}_{rs}}{\partial \bar{A}^k} \bar{\Gamma}^i_{kl} \right) (\bar{\chi}^i \bar{\chi}^j)(\bar{\lambda}^r \bar{\lambda}^s)$$

plus $\psi_m$, $\bar{\psi}_m$ dependent terms, \hspace{1cm} (3.15)

with the Yang-Mills field strength tensor

$$f^r_{mn} = \partial_m a^r_n - \partial_n a^r_m + a^s_m a^t_n c^r_{st}, \hspace{1cm} (3.16)$$

and covariant derivatives of the gauginos

$$D_m \lambda^r_\alpha = \partial_m \lambda^r_\alpha - \omega_{m \alpha} \phi^r_\phi + v_m \lambda^r_\alpha - a^t_m \lambda^r_\alpha c^r_{st}, \hspace{1cm} (3.17)$$

$$D_m \bar{\lambda}^{r \dot{\alpha}} = \partial_m \bar{\lambda}^{r \dot{\alpha}} - \omega_{m \dot{\alpha}} \phi^{r \dot{\alpha}} + v_m \bar{\lambda}^{r \dot{\alpha}} - a^t_m \bar{\lambda}^{r \dot{\alpha}} c^r_{st}. \hspace{1cm} (3.18)$$

### 3.2 Solving for the auxiliary fields

In the different pieces of the whole Lagrangian, we isolate the contributions containing auxiliary fields and proceed sector by sector as much as possible.

Diagonalization in $b_a$ makes use of the terms

$$\Lambda_b = \frac{1}{3} b^a b_a - \frac{1}{2} M_{\AA\bar{\AA}} \left( \Psi^A \sigma^a \bar{\Psi}^A \right) b_a + \frac{1}{2} f_{rs} (\lambda^r \sigma^a \bar{\lambda}^s) b_a, \hspace{1cm} (3.19)$$

with

$$M_{\AA\bar{\AA}} = \frac{1}{1 - Y_Y K_{\AA\bar{\AA}}}, \hspace{1cm} (3.20)$$
whereas the relevant terms for the Yang-Mills auxiliary sector are

$$
\Lambda_D = \frac{1}{2} f_{rs} D^r D^s + \frac{1}{1 - Y K_Y} D^s \left( K_Y T_8 \bar{A}^I \right) + \sqrt{2} D^s \left( \frac{\partial f_{rs}}{\partial A_k} \bar{\lambda}^k \lambda + \frac{\partial f_{rs}}{\partial A_k} \bar{\lambda}^k \lambda \right).
$$

The $F$-terms of chiral matter and the 3-form appear in the general form

$$
\Lambda_{F,\mathcal{F}} = F^A M_{\bar{A}A} \mathcal{F}_{\bar{A}} + F^A P_A + \mathcal{F}_A \mathcal{F}_{\bar{A}},
$$

with the definitions

$$
P_k = \Sigma_k - \frac{1}{4} \frac{\partial f_{rs}}{\partial A_k} (\lambda^r \lambda^s) - Y M_{YB} M_{k \bar{A}} \bar{\Psi}_\alpha \overline{\Psi}_\alpha^{i\bar{A}},
$$

$$
P_Y = \Sigma_Y - Y M_{YB} M_{Y \bar{A}} \bar{\Psi}_\alpha \overline{\Psi}_\alpha^{i\bar{A}}.
$$

We write this expression as

$$
\Lambda_{F,\mathcal{F}} = \mathcal{F}^k M_{kk} \mathcal{F}_{\bar{A}} - \mathcal{F}_{\bar{A}} M_{\bar{A}A} P_A + \mathcal{F}_Y \frac{1}{M_{YY}} \mathcal{F}_{\bar{Y}},
$$

where $M_{\bar{A}A}$ is the inverse of $M_{A\bar{A}}$ and in particular

$$
\frac{1}{M_{YY}} = M_{Y\bar{Y}} - M_{Yk} \mathfrak{M}^{kk} M_{k\bar{Y}},
$$

with $\mathfrak{M}^{kk}$ the inverse of $M_{kk}$. Moreover

$$
\mathcal{F}^k = \mathcal{F} + \left( \mathcal{P}_{\bar{k}} + \mathcal{F} Y M_{Y\bar{k}} \right) \mathfrak{M}^{kk},
$$

$$
\mathcal{F}_{\bar{k}} = \mathcal{F}_{\bar{k}} + \mathfrak{M}^{kk} \left( P_k + M_{k\bar{Y}} \mathcal{F}_{\bar{Y}} \right),
$$

and

$$
\mathcal{F}^Y = \mathcal{F} + \mathcal{P}_{\bar{A}} M_{\bar{A}Y}, \quad \mathcal{F}_{\bar{Y}} = \mathcal{F}_{\bar{Y}} + M_{\bar{A}A} P_A.
$$

We use now the particular structure of the 3-form multiplet to further specify these $F$-terms. Using (2.36), (2.37), (2.69), (2.70) we parametrise

$$
\mathcal{F}^Y = H + i \left( \Delta + \frac{M_Y - M_{\bar{Y}}}{2i} \right) + f^Y,
$$

$$
\mathcal{F}_{\bar{Y}} = H - i \left( \Delta + \frac{M_Y - M_{\bar{Y}}}{2i} \right) + f_{\bar{Y}}.
$$
with

\[ f^Y = -\frac{1}{4} \Gamma^Y_{BC} D^\alpha \Psi^B D_\alpha \Psi^C + \bar{\mathcal{F}}_A M^A \bar{Y}, \]

\[ \bar{f}^Y = -\frac{1}{4} \bar{\Gamma}^Y_{BC} D^\alpha \bar{\Psi}^B \bar{D}_\alpha \bar{\Psi}^C + M^A \bar{Y} P_A, \]

as well as

\[ \Delta = \frac{4}{3} \epsilon^{klmn} \partial_k C_{lmn} - \frac{1}{2\sqrt{2}} \left( \bar{\psi}_m \sigma^m \eta - \psi_m \sigma^m \bar{\eta} \right) + \frac{1}{2i} \left[ \left( \bar{\psi}_m \sigma^mn \bar{\psi}_n \right) Y - \left( \psi_m \sigma^mn \psi_n \right) \bar{Y} \right]. \]

In terms of these notations the last term in (3.25) takes then the form

\[ \mathcal{F}^Y \frac{1}{M^Y} \bar{\mathcal{F}}^Y = \frac{1}{M^Y} \left( H + \frac{f^Y + \bar{f}^Y}{2} \right)^2 + \frac{1}{M^Y} \left( \Delta + \frac{M^Y - M\bar{Y}}{2i} + \frac{f^Y - \bar{f}^Y}{2i} \right)^2. \]

In this equation the last term makes a contribution to the sector \( M, \bar{M} \) and the 3-form we consider next. Except for this term the sum of \( \Lambda_b, \Lambda_D, \Lambda_F, \bar{F} \) will give rise to the following diagonalised expression:

\[ \frac{1}{e} \mathcal{L}(F^k, F^k, b_a, D^r, H) = \frac{1}{3} \hat{b}_a \hat{b}^a + \frac{1}{2} \bar{D}^r f_{rs} \bar{D}^s + \mathcal{F}^k M_{kk} \bar{F}^k \]

\[ + \frac{1}{M^Y} \left( H + \frac{f^Y + \bar{f}^Y}{2} \right)^2 - \frac{3}{16} \mathbb{B}_a \mathbb{B}^a \]

\[ - \frac{1}{2} \bar{D}_r (f^{-1})^{rs} \bar{D}_s - \bar{P}_A M^A P_A, \]

where \( \hat{b}_a = b_a + \mathbb{B}_a \) with

\[ \mathbb{B}^a = -M_{A\bar{A}} \left( \Psi^A \sigma^a \bar{\Psi}^\bar{A} \right) + f_{rs} \left( \bar{\chi}^r \sigma^a \bar{\chi}^s \right), \]

and \( \bar{D}^r = D^r + (f^{-1})^{rs} \bar{D}_s \) with

\[ \bar{D}_r = -\frac{1}{1 - Y K_Y} (K_k T_{rA})^k \]

\[ + \frac{\sqrt{2}}{4} \left( \frac{\partial f_{rs}}{\partial A^k} (\bar{\chi}^k \bar{\chi}^s) + \frac{\partial f_{rs}}{\partial \bar{A}^k} (\bar{\chi}^k \bar{\chi}^s) \right). \]
Use of the equations of motion simply sets to zero the first four terms leaving for the Lagrangian of (3.36)

\[
\frac{1}{e} \mathcal{L} = -\frac{3}{16} B_a B^a - \frac{1}{2} D_r (f^{-1})^{rs} D_s - \frac{1}{M Y Y} P_Y
- \left( \frac{F_k - F_Y M Y}{M Y Y} \right) M^{kk} \left( P_k - \frac{M Y Y}{M Y Y} P_Y \right),
\]

(3.39)

where we have block diagonalised \( M^{AA} \).

As to the \( M, \overline{M} \) dependent terms of the full action we observe that they are intricately entangled with the field strength tensor of the 3-form, a novel structure compared to the usual supergravity-matter couplings. The relevant terms for this sector are identified to be

\[
\Lambda_{M, \overline{M}} = 3e^K |W|^2 - \frac{1}{3} |M| + 3e^{K/2} W^2
+ \frac{1}{M Y Y} \left[ \Delta - \frac{1}{2i} (MY - \overline{MY}) + \frac{1}{2i} (f Y - \overline{f Y}) \right]^2.
\]

(3.40)

One recognises in the first two terms the usual superpotential contributions whereas the last term is new and comes from (3.35). This expression contains all the terms of the full action which depend on \( M, \overline{M} \) or the 3-form \( C_{klm} \).

The question we have to answer is as to how far the \( M, \overline{M} \) sector and the 3-form sector can be disentangled, if at all. Clearly, the dynamical consequences of this structure deserve careful investigation.

The 3-form contribution is not algebraic, so one cannot use the solution of its equation of motion (\( e.o.m. \)) in the Lagrangian \( \mathcal{L} \). One way out is to derive the \( e.o.m. \)'s and look for an equivalent Lagrangian giving rise to the same \( e.o.m. \)'s. Explicitly one obtains for the 3-form:

\[
\partial_k \left\{ \frac{1}{M Y Y} \left[ \Delta - \frac{1}{2i} (MY - \overline{MY}) + \frac{1}{2i} (f Y - \overline{f Y}) \right] \right\} = 0,
\]

(3.41)

solved by setting

\[
\frac{1}{M Y Y} \left[ \Delta - \frac{1}{2i} (MY - \overline{MY}) + \frac{1}{2i} (f Y - \overline{f Y}) \right] = c,
\]

(3.42)
where $c$ is a real constant. Then the e.o.m.’s for $M$ and $\overline{M}$ read

$$M + 3e^{K/2}W = -3icY \quad ; \quad \overline{M} + 3e^{K/2}\overline{W} = 3ic\overline{Y}. \quad (3.43)$$

At last, one considers the e.o.m. for e.g. $\overline{Y}$, in which we denote by $L(\overline{Y})$ the many contributions of $\overline{Y}$ to the Lagrangian, except for $\Lambda_{MM}$,

$$\partial_m \frac{\delta L(\overline{Y})}{\delta \partial_m \overline{Y}} - \frac{\delta L(\overline{Y})}{\delta \overline{Y}} - \frac{\delta \Lambda_{MM}}{\delta \overline{Y}} = 0. \quad (3.44)$$

Using (3.42) and (3.43) the last term assumes the form

$$\frac{\delta \Lambda_{MM}}{\delta \overline{Y}} = \frac{\delta}{\delta \overline{Y}} \left\{ 3e^K |W + icy|^2 - c^2 \overline{M}^2 Y - ic(f^Y - f^\overline{Y}) \right. \left. - ic \left[ (\overline{\psi}_m \sigma^{mn} \psi_n) Y - (\psi_m \sigma^{mn} \overline{\psi}_n) \overline{Y} \right] \right\}. \quad (3.45)$$

This suggests that the equations of motion can be derived from an equivalent Lagrangian obtained by dropping the 3-form contribution and shifting the superpotential $W$ to $W + icy$. This can be seen more clearly by restricting our attention to the scalar degrees of freedom as in the next section.

### 3.3 The scalar potential

The analysis presented above allows to obtain the scalar potential of the theory:

$$V = \left( \Sigma_k - (\Sigma_\overline{Y} - ic) \frac{M_{Yk}}{M_{YY}} \right) M^{kk} \left( \Sigma_k - \frac{M_{YY}}{M_{YY}} (\Sigma_Y + ic) \right) + (\Sigma_\overline{Y} - ic) \frac{1}{M_{YY}} (\Sigma_Y + ic) - 3e^K |W + icy|^2 + \frac{1}{2} \frac{1}{1 - YK_Y} K_k (T_r \overline{A})^k (f^{-1})^{rs} \frac{1}{1 - YK_Y} K_k (T_s \cdot A)^k. \quad (3.46)$$

We note that the shift $W \mapsto W + icy$ induces $\Sigma_k \mapsto \Sigma_k$ and $\Sigma_Y \mapsto \Sigma_Y + ic$, which are precisely the combinations which appear in (3.46).

In fact (3.46) is nothing but the scalar potential of some matter fields $\phi^k$ of Kähler weight 0 plus a field $Y = ye^{K/2}$ of Kähler weight 2 with a
superpotential $W + icy$ in the usual formulation of supergravity. In order to show this, let us consider $y$ and $\overline{y}$ as our new field variables and define

$$K(Y, \phi, \overline{Y}, \bar{\phi}) = K(y, \phi, \overline{y}, \bar{\phi}),$$

(3.47)

Taking as an example the Kähler potential in (2.39), one finds

$$y = \frac{Y}{(1 + Y \overline{Y})^{1/2}}, \quad \overline{y} = \frac{\overline{Y}}{(1 + Y \overline{Y})^{1/2}},$$

$$K(y, \overline{y}) = -\ln(1 - y \overline{y}).$$

(3.48)

which is a typical Kähler potential with $SU(1, 1)$ noncompact symmetry.

We can express the matrix $M_{A\overline{A}}$ and its inverse $M_{\overline{A}A}$ in terms of the derivatives of $K$, namely $K_{A\overline{A}}$ and of its inverse $K_{\overline{A}A}$ ($A$ denotes $k, y$ as well as $k, Y$ depending on the context). Then it appears that the expression of the scalar potential becomes very simple as we use the relevant relations collected in the appendix. Indeed, if we use the following definitions

$$\hat{W} = W + icy,$$

$$D_A \hat{W} = \hat{W}_A + K_{A\overline{A}} \hat{W},$$

(3.49)

then

$$V = e^K \left( D_{\overline{A}} \hat{W} K^{A\overline{A}} D_A \hat{W} - 3|\hat{W}|^2 \right)$$

$$+ \frac{1}{2} K_k \left( T_{r.\overline{A}} \right)^k \left( T_{s.A} \right)^{rs} K_{k} \left( T_{s.A} \right)^{k},$$

(3.50)

which is the familiar expression of the scalar potential of the scalar fields $\phi_k$ and $y$ in the standard formulation of supergravity.

4 APPLICATIONS

4.1 Fundamental 3-form

Fundamental 3-forms naturally appear in the context of strong-weak coupling duality. This can be seen most easily using the language of five-branes

\footnote{The new fields $y$ and $\overline{y}$ are chiral when using the derivatives covariant with respect to the new Kähler potential $K(y, \overline{y})$}
In the critical spacetime dimension $d = 10$, five-brane theories are conjectured to be dual to string theories in the sense that a weakly coupled five-brane is a dual representation of a strongly coupled heterotic string. After compactification to four dimensions this may lead to a string/string duality.

The effective field theory corresponding to the five-brane scenario would necessarily be described by the formulation of supergravity in $d = 10$ dimensions with a seven-form field strength \[ g^{10} \]. Under compactification, this would naturally yield a 4-form field strength, i.e. the field strength of a fundamental 3-form.

In what follows, we will use a simple dimensional reduction \[ 15, 16 \] to infer some of the couplings of this fundamental 3-form as they arise from compactification. In 10 dimensions the kinetic term for the six-form (dual of the 2-form found among the massless string modes in the standard formulation) involves the seven-form field strength we just referred to. We note this field strength $K_{M_1 M_2 M_3 M_4 M_5 M_6 M_7}$ ($M_i = 1 \cdots 10$) and the corresponding action term reads \[ 10, 11 \]:

\[
S^{(10)} = \int d^{10}x \sqrt{-g^{(10)}} e^{\phi} K_{M_1 M_2 M_3 M_4 M_5 M_6 M_7} K_{M_1 M_2 M_3 M_4 M_5 M_6 M_7}, \tag{4.1}
\]

where $\phi$ is the dilaton field and the upper indices are related to the lower ones through the ten-dimensional metric tensor $g^{MN}$.

Under compactification to four dimensions, we recover our 4-form field strength through the components:

\[
\Sigma_{klmn} = K_{klmnIJK}, \tag{4.2}
\]

where the indices $I, J, K$ refer to the compact manifold. In our simple compactification scheme \[ 15 \],

\[
g^{(10)}_{IJ} = e^\sigma \delta_{IJ}, \quad g^{(10)}_{mn} = e^{-3\sigma} g_{mn}, \tag{4.3}
\]

where $g_{mn}$ is the 4-dimensional metric and $\sigma$ is the “breathing mode” of the compact manifold.
The effective theory can be written in terms of the scalar fields

\[ s = e^{-\phi/2}e^{3\sigma}, \quad t = e^{\phi/2}e^{\sigma}, \quad (4.4) \]

which are the dilaton and modulus fields. In the present formulation they are parts of respectively a chiral and a linear supermultiplet. The effective theory is described by the Kähler potential

\[ K = -\ln s - 3\ln t \quad (4.5) \]

The action (4.1) yields the following term in the 4-dimensional action

\[ S^4 \sim \int d^4x \sqrt{-g}st^3\Sigma^{klmn}\Sigma_{klmn} = \int d^4x \sqrt{-g}e^{-K}\Sigma^{klmn}\Sigma_{klmn}. \quad (4.6) \]

This should be compared with the corresponding term in (3.40) (where \( \Delta \) contains a term proportional to \( \Sigma^{klmn}\Sigma_{klmn} \)). It remains to be seen which field could be interpreted as the chiral field \( Y \) appearing in the 3-form supermultiplet.

The field dependence of this kinetic term is given by \( 1/M^{FY} \), whose explicit form in terms of the fields \( \phi^k \) and \( y \) is given by (see appendix)

\[ 1/M^{FY} = \frac{e^{-K}}{(1 + yK)^2} \frac{1}{K\gamma^f + \beta_kK\gamma^k + \beta_k\beta_k\gamma^k}, \quad (4.7) \]

where \( \beta_k \) and \( \beta_k \) are defined in the appendix. One recognises precisely the \( e^{-K} \) dependence.

One might wonder which field of superstring models plays the role of the field \( y \) accompanying the 3-form in the supergravity multiplet. In compactification schemes such as Calabi-Yau manifolds where there is only one independent 3-form, there should be a single \( y \) field. We are working in the dual formulation of supergravity where Kähler moduli are in linear multiplets whereas the dilaton is in a chiral multiplet. Therefore a natural candidate for \( y \) is the dilaton. Since the dilat Kähler potential has a \( SU(1,1) \) invariance, we would, under this hypothesis, readily make the following identification:

\[ y = \frac{1 - S}{1 + S} \quad (4.8) \]
with a Kähler potential given by the example presented above for illustrative purpose in (3.48). One may worry that the terms other than \( e^{-K} \) in (4.7) would induce an extra dependence in \( y \) and thus in \( S \). But with a Kähler potential (3.48), one obtains
\[
\frac{1}{(1 + yK_y)^2} \frac{1}{K_y} = 1
\] (4.9)

Finally, Kähler transformations \( y \rightarrow ye^{-F} \) are related in this case to \( SU(1, 1) \) transformations on \( S \) in a straightforward manner:
\[
S \rightarrow \frac{Scosh \frac{F}{2} + sinh \frac{F}{2}}{Ssinh \frac{F}{2} + cosh \frac{F}{2}}.
\] (4.10)

It is also interesting to perform at this level a duality transformation in order to see the content of the theory in the usual formulation. This transformation reads at the level of the scalar fields:
\[
st^3 \Sigma_{kmln} \sim \epsilon_{kmln} c,
\] (4.11)

where \( c \) is a scalar field (constant through its equation of motion). The corresponding action term then reads:
\[
S_{(4)}' = \int d^4x \sqrt{-g} e^K c^2.
\] (4.12)

A term of this exact form was actually proposed in this context \[17\] as a remnant, in the 4-dimensional theory, of the field strength of the 2-form: \( H_{IJK} \) (\( I, J, K \) compact indices). It is known to break supersymmetry spontaneously \[18\].

### 4.2 Composite 3-form: gaugino condensates.

Another aspect of supersymmetry breaking where 3-forms play a role is gaugino condensation. This is not completely surprising since the constraints (2.9) on the 4-form field strength \( \Sigma_{ABCD} \) superfield in supergravity are imposed by analogy with the case of a product of two Yang-Mills 2-forms \( F_{AB}F_{CD} \). The corresponding 3-form is then the Chern-Simons form.
This appears most clearly in formulations of gaugino condensation which involve a dilaton field, such as in superstring models: the dilaton field is then incorporated into a linear multiplet $L$ (see also [19]) in the fundamental theory. The composite degrees of freedom are described, in the effective theory below the scale of condensation, by a vector superfield $V$ which incorporates also the components of the fundamental linear multiplet $L$. The chiral superfield

$$U = -(\mathcal{D}_\alpha \mathcal{D}^\alpha - 8R)V$$

(4.13)

has the same quantum numbers (in particular the same Kähler weight) as the superfield $W^\alpha W_\alpha$. Its scalar component, for instance, is interpreted as the gaugino condensate.

Alternatively, the vector superfield is interpreted as a “fossil” Chern-Simons field which includes the fundamental degrees of freedom of the dilaton supermultiplet. It can be considered as a prepotential for the chiral superfield $U$: as such, its reality imposes the constraint (2.13) with $U = Y = Y$.

K form of the superpotential is dictated by the anomaly structure of the underlying theory [20] and is expressed as in (2.76) through the variable $u = U \exp(-K/2)$ which can be understood in terms of the ratio of the infrared cut-off ($U^{1/3}$) and the effective ultraviolet cut-off ($\exp(K/6)$). It reads simply [9]:

$$W(u) = u \ln u$$

(4.14)

and its component form can be read off (3.14).

Certainly, the two applications just described deserve further study. In this paper, we have restricted our attention to the derivation of the couplings of a 3-form supermultiplet to supergravity and we have tried to be general enough in order to be able to describe the different physical situations where such a supermultiplet might play a relevant role, somewhat neglected until now.

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A Appendix

Here we gather some relations obtained as we use $y, \overline{y}$ as variables. Let

$$K(Y, \overline{Y}) = K(y, \overline{y}),$$

and the conjugate, then defining

$$\alpha = \frac{1}{(1 + yK_y)}, \quad \beta_k = \frac{yK_k}{(1 + yK_y)},$$

we obtain

$$M_{Y Y} = e^{-K/2} \alpha^2 K_y \overline{y},$$

$$M_{Y k} = e^{-K/2} \alpha \left( y K_y y + \beta_k K_{y k} \right),$$

$$M_{k \overline{k}} = K_{k \overline{k}} - \beta_k K_{y k} - \beta_k K_{k y} + \beta_k \beta_k K_{y y}.$$ (A.17)

This allows to compute the inverse of $M$ in terms of the inverse of $K$,

$$M_{Y Y} = e^K \alpha^{-1} \left( K_{y y} + \beta_k K_{y k} + \beta_k K_{k y} + \beta_k \beta_k K_{y y} \right),$$

$$M_{Y k} = e^K \alpha^{-1} \left( K_{y k} + \beta_k K_{k k} \right),$$

$$M_{k \overline{k}} = K_{k k}.$$ (A.18)

The $\Sigma$'s appearing in the potential part also take simple forms

$$\Sigma_Y = \frac{D_y \overline{W}}{(1 + yK_y)}; \quad \Sigma_k = D_k \overline{W} - \beta_k D_y \overline{W}.$$ (A.19)

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