MAPPING PROPERTIES FOR OPERATOR-VALUED
PSEUDODIFFERENTIAL OPERATORS ON TOROIDAL
BESOV SPACES

BIENVENIDO BARRAZA MARTÍNEZ, ROBERT DENK,
JAIRO HERNÁNDEZ MONZÓN, AND MAX NENDEL

Abstract. In this paper, we consider pseudodifferential operators on
the torus with operator-valued symbols and prove continuity properties
on vector-valued toroidal Besov spaces, without assumptions on the un-
derlying Banach spaces. The symbols are of limited smoothness with
respect to x and satisfy a finite number of estimates on the discrete
derivatives. The proof of the main result is based on a description of the
operator as a convolution operator with a kernel representation which
is related to the dyadic decomposition appearing in the definition of the
Besov space.

1. Introduction

In this note, we consider mapping properties of pseudodifferential op-
erators on the n-dimensional torus \( T^n = (\mathbb{R}/2\pi\mathbb{Z})^n \) in vector-valued Besov
spaces. Toroidal pseudodifferential operators are defined and investigated,
e.g., in the monograph [16] by Ruzhansky and Turunen. Here, the group
structure of \( T^n \) is used to define a global quantization with covariable \( k \in \mathbb{Z}^n \)
(Fourier series). This quantization is also the basis for the definition of the
Besov spaces on the torus by means of a dyadic decomposition of \( \mathbb{Z}^n \) (see
Definition 2.5 below). Compared to the other possible approach where \( T^n \) is
treated as a closed manifold, one has the advantage of a global quantization
without the necessity to introduce local coordinate charts. The theory of
pseudodifferential operators on the torus was developed by Agranovich [1],
McLean [13], Melo [14], Bu-Kim [6], [7] and others.

Mapping properties of toroidal pseudodifferential operators in \( L^p \)-spaces
were studied studied by Delgado [10], Molahajloo-Shahla-Wong [15], Wong
[19], Cardona [9] and others. In particular, in Cardona [9] mapping properties
in Besov and Hölder spaces are shown. The global quantization approach
mentioned above can be generalized to compact Lie groups, see Ruzhansky-
Turunen [17], Ruzhansky-Turunen-Wirth [18], Cardona [8] and references
therein.

The above references deal with the scalar-valued case. In the situation
where the considered functions have values in some Banach space \( E \), the sit-
uation depends on the geometric properties of \( E \). If \( E \) is a UMD space

\begin{itemize}
\item \textit{Date:} June 22, 2017.
\item 1991 \textit{Mathematics Subject Classification.} 35S05, 47D06, 35R20.
\item \textit{Key words and phrases.} Pseudodifferential operators, vector-valued Besov spaces, con-
volution kernels.
\item The authors would like to thank COLCIENCIAS (Project 121556933488) and DAAD
for the financial support.
\end{itemize}
Both the map-

tics (differences) $\Delta \varphi$ were studied, in the present note we investigate $x$-dependent vector-valued symbols with values in a general Banach space.

We consider pseudodifferential operators whose symbols have limited smoothness with respect to $x$ and satisfy a finite number of growth conditions in analogy to the conditions of Hörmander. The symbols have values in $L^p(E)$, the space of all bounded linear operators in $E$, where $E$ stands for an arbitrary Banach space. The main result (Theorem 3.3) states that the pseudodifferential operator $\text{op}[a]$ related to the symbol $a$ of order $m$ induces a bounded linear operator from $B^s_{pq}(T^n, E)$ to $B^s_{pq}(T^n, E)$, where the range of $s$ is in a natural way restricted by the smoothness of $a$ and where $p, q \in [1, \infty]$. One of the main steps in the proof consists of a description of the operators $\text{op}[a] \text{op}[\phi_j]$ and $\text{op}[\phi_j] \text{op}[a]$ as convolution operators (see Lemma 2.6). Here $(\phi_j)_{j \in \mathbb{N}_0}$ is a dyadic decomposition of $\mathbb{Z}^n$, and the kernels of these operators can be written in form of an infinite sum adapted to this dyadic decomposition. This allows to avoid oscillatory integrals and sum-integrals. We note that this approach gives a new proof of the Besov space continuity even in the $x$-independent case (cf. [4]), and therefore it may serve as a basis for future generalizations to locally compact abelian groups and to compact Lie groups (see also Remark 3.4 a)). Both the mapping properties and the convolution kernel description can be used to show generation of analytic semigroups for parabolic pseudodifferential operators on the torus. This will be the content of a subsequent paper.

2. Kernel Estimates for Toroidal Pseudodifferential Operators

In the following, let $E$ be a Banach space with norm $\| \cdot \|$. Throughout this paper, we fix $n \in \mathbb{N}$, $\rho \in \mathbb{N}$ with $\rho \geq n + 1$, $r \in [0, \infty)$ and $m \in \mathbb{R}$. We consider operator-valued pseudodifferential operators on the $n$-dimensional torus $T^n = (\mathbb{R} / (2\pi \mathbb{Z}))^n$, where we use $[-\pi, \pi]^n$ as a set of representatives. Note that in this case, the euclidian norm $|x|$ of a representative equals the distance of $x$ to 0 in the metric on $\mathbb{T}^n$. We use standard notation for smooth vector-valued functions $f \in C^\infty(T^n, E)$ and their Fourier series (discrete Fourier transform)

$$(\mathcal{F} f)(\mathbf{k}) := \hat{f}(\mathbf{k}) := \int_{T^n} e^{-i \mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x} \quad (\mathbf{k} \in \mathbb{Z}^n),$$

where $d\mathbf{x} := (2\pi)^{-n} d\mathbf{x}$. The Fourier transform is extended by duality to the space of vector-valued toroidal distributions $u \in \mathcal{D}'(T^n, E) := L(C^\infty(T^n), E)$, see [4], Section 2 for more details.

The symbol class on the torus is defined with help of the discrete derivatives (differences) $\Delta k$. For this, let $j \in \{1, \ldots, n\}$, and let $\delta_j := (\delta_{jk})_{k=1,\ldots,n}$ be the $j$-th unit vector in $\mathbb{R}^n$. For $a : \mathbb{Z}^n \to \mathbb{E}$ and $\alpha \in \mathbb{N}_0^n$, we set

$$\Delta_{k_j} a(\mathbf{k}) := a(\mathbf{k} + \delta_j) - a(\mathbf{k}) \quad (\mathbf{k} \in \mathbb{Z}^n),$$

$$\Delta_k^\alpha := \Delta_{k_1}^{\alpha_1} \ldots \Delta_{k_n}^{\alpha_n}.$$
We refer to [16], Sect. 3.3.1, for a more detailed discussion of the discrete analysis on the torus. In the following definition, we set \( \langle k \rangle := (1 + |k|^2)^{1/2} \) (\( k \in \mathbb{Z}^n \)).

**Definition 2.1.** a) Let \( S^{m,\rho,r} := S^{m,\rho,r}(\mathbb{T}^n \times \mathbb{Z}^n, L(E)) \) be the set of all functions \( a: \mathbb{T}^n \times \mathbb{Z}^n \to L(E) \) such that \( |x \mapsto a(x,k)| \in C^r(\mathbb{T}^n, L(E)) \) for all \( k \in \mathbb{Z}^n \), and \( \|a\|_{S^{m,\rho,r}} < \infty \). Here, in the case \( r \in \mathbb{N}_0 \) we define

\[
\|a\|_{S^{m,\rho,r}} := \max \left\{ \sup_{|\alpha| \leq \rho} |\mathbb{N}^{\alpha} a(x,k)| : x \in \mathbb{T}^n, k \in \mathbb{Z}^n, a(\mathbb{Z}^n) \right\},
\]

and in the case \( r \in (0, \infty) \setminus \mathbb{N} \) we define

\[
\|a\|_{S^{m,\rho,r}} := \max \left\{ \sup_{|\alpha| \leq \rho} |\mathbb{N}^{\alpha} a(x,k)| : x \in \mathbb{T}^n, k \in \mathbb{Z}^n, a(\mathbb{Z}^n) \right\}.
\]

b) For \( a \in S^{m,\rho,r} \) the pseudo-differential operator \( \text{op}[a] \) is defined by

\[
(\text{op}[a]f)(x) = \sum_{k \in \mathbb{Z}^n} e^{ik \cdot x} a(x,k)  \hat{f}(k) \quad (f \in C^{\infty}(\mathbb{T}^n, E), \quad x \in \mathbb{T}^n).
\]

**Remark 2.2.** a) It is easily seen that for \( f \in C^{\infty}(\mathbb{T}^n, E) \) we have

\[
\langle \hat{f}(k) \rangle_{k \in \mathbb{Z}^n} \in \mathcal{S}(\mathbb{Z}^n, E),
\]

where \( \mathcal{S}(\mathbb{Z}^n, E) \) stands for the Schwartz space of all functions \( \phi: \mathbb{Z}^n \to E \) with \( \sup_{k \in \mathbb{Z}^n} |\phi(k)| < \infty \) for all \( N \in \mathbb{N} \) (see, e.g., [4], Lemma 2.2). Therefore, the sum in (2–1) converges absolutely.

b) Inserting the definition of \( \hat{f}(k) \) into the right-hand side of (2–1), we formally get

\[
(\text{op}[a]f)(x) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{ik \cdot (x-y)} a(x,k) f(y) dy
\]

\[
= \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{ik \cdot y} a(x,k) f(x-y) dy.
\]

However, this sum-integral does not converge in general. To make such integrals convergent (and to change the order of integration and summation), one has to use either oscillatory sum-integrals (see [4], Remark 3.4) or use integration by parts (see [16], Remark 4.1.18). In the cases considered below, the symbols will be good enough to guarantee absolute convergence of the sum-integrals.

The definition of toroidal Besov spaces is based on a dyadic decomposition in the covariable space \( \mathbb{Z}^n \). We use the following definition.

**Definition 2.3.** A sequence \( (\varphi_j)_{j \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{Z}^n) \) is called a dyadic decomposition if the following conditions are satisfied.

1. We have \( \text{supp } \varphi_0 \subset \{k \in \mathbb{Z}^n : |k| \leq 2 \} \) and \( \text{supp } \varphi_j \subset \{k \in \mathbb{Z}^n : 2^{j-1} \leq |k| \leq 2^j - 1 \} \) for \( j \in \mathbb{N}_0 \).
2. For each \( k \in \mathbb{Z}^n \), we have \( 0 \leq \varphi_j(k) \leq 1 \) (\( j \in \mathbb{N}_0 \)) and \( \sum_{j \in \mathbb{N}_0} \varphi_j(k) = 1 \).
3. For each \( \alpha \in \mathbb{N}^n_0 \), exists a constant \( c_{\alpha} > 0 \) independent of \( j \) and \( k \) such that

\[
|\Delta^\alpha \varphi_j(k)| \leq c_{\alpha} \langle k \rangle^{-|\alpha|} \quad (j \in \mathbb{N}, \quad k \in \mathbb{Z}^n).
\]
Remark 2.4. A partition of unity on \( \mathbb{Z}^n \) can be obtained as a restriction of a partition of unity on \( \mathbb{R}^n \) in the sense of [4], Definition 3.5, or [2], Section 4. Here, the definition of a partition of unity \( (\tilde{\varphi}_j)_{j \in \mathbb{N}_0} \) on \( \mathbb{R}^n \) includes the condition

\[
|\partial^\alpha \tilde{\varphi}_j(\xi)| \leq c_n 2^{-j|\alpha|} \quad (\xi \in \mathbb{R}^n).
\]

Taking \( \varphi_j := \tilde{\varphi}_j|_{\mathbb{Z}^n} \), we obtain condition 2.3 (iii) by [16], proof of Theorem II.4.5.3, which states that for each \( k \in \mathbb{Z}^n \),

\[
\Delta_k^j \varphi_j(k) = \partial_{\xi}^j \tilde{\varphi}_j(\xi)|_{\xi = \tilde{\xi}}
\]

with some \( \tilde{\xi} \in [k_1, k_1 + \gamma_1] \times \ldots \times [k_n, k_n + \gamma_n] \). This implies

\[
|\Delta_k^j \varphi_j(k)| \leq C|k|^{-1} \quad (j \in \mathbb{N}, \ k \in \mathbb{Z}^n)
\]

using the conditions on the support of \( \varphi_j \).

Throughout the following, we will fix a dyadic decomposition \( (\varphi_j)_{j \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{Z}^n) \). We set \( \varphi_{-1} := 0 \) and define

\[
\chi_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1} \quad (j \in \mathbb{N}_0).
\]

Then \( \chi_j = 1 \) on \( \text{supp} \varphi_j \), i.e., we have \( \varphi_j \chi_j = \varphi_j \) for all \( j \in \mathbb{N}_0 \).

Definition 2.5. For \( p, q \in [1, \infty] \) and \( s \in \mathbb{R} \), the Besov space \( B^s_{pq}(\mathbb{T}^n, E) \) is defined as the space of all \( u \in \mathcal{S}(\mathbb{T}^n, E) \) with \( \|u\|_{B^s_{pq}(\mathbb{T}^n, E)} < \infty \), where

\[
\|u\|_{B^s_{pq}(\mathbb{T}^n, E)} := \left\| \left( 2^{js} \|\text{op}\varphi_j u\|_{L^p(\mathbb{T}^n, E)} \right)_{j \in \mathbb{N}_0} \right\|_{\ell^q(\mathbb{N}_0)}.
\]

For properties of vector-valued Besov spaces on the torus, we refer to [4], Remark 3.9. For the analog spaces in \( \mathbb{R}^n \), see [2], Section 5. The Besov space does not depend on the choice of the dyadic decomposition (in the sense of equivalent norms).

The estimates for pseudodifferential operators on toroidal Besov spaces below are based on their representation as integral operators and estimates for their kernels. We adapt this representation to the dyadic decomposition and obtain better convergence properties. In particular, there is no need to consider oscillatory sum-integrals.

Lemma 2.6. Let \( a \in S^{m, \rho, r} \), and let \( f \in C^\infty(\mathbb{T}^n, E) \).

a) We have

\[
\text{op}[a]f(x) = \sum_{\kappa \in \mathbb{N}_0} \text{op}[a_\kappa] \varphi_\kappa f(x) \quad (x \in \mathbb{T}^n).
\]

Here, the series on the right-hand side converges in \( C(\mathbb{T}^n, E) \) (i.e., uniformly in \( x \)).

b) For every \( x \in \mathbb{T}^n \) and \( j \in \mathbb{N}_0 \),

\[
\left( \text{op}[a] \text{op}[\varphi_j]f \right)(x) = \int_{\mathbb{T}^n} K_j(x, y) f(x - y) dy,
\]

where

\[
K_j(x, y) := \sum_{\kappa \in \mathbb{Z}^n} e^{i \kappa y} a(x, \kappa) \varphi_j(\kappa).
\]

(Note that this is a finite sum.)
c) For every \( x \in \mathbb{T}^n \) and \( j \in \mathbb{N}_0 \), \( a \)  

\[
(\text{op}[a] \text{op}[\varphi_j] f)(x) = \sum_{\kappa \in \mathbb{N}_0} \left[ \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K_j^{(1)}(x, y, z)(\text{op}[\chi_\kappa] f)(x - y - z) dy dz \right],
\]

where

\[
K_j^{(1)}(x, y, z) := \sum_{k, \ell \in \mathbb{Z}^n} e^{i\ell \cdot y} e^{i k \cdot z} \varphi_j(\ell) \varphi_\kappa(k) a(x, k).
\]

\[d) \text{For every } x \in \mathbb{T}^n \text{ and } j \in \mathbb{N}_0, \]

\[
(\text{op}[\varphi_j] \text{op}[a] f)(x) = \sum_{\kappa \in \mathbb{N}_0} \left[ \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K_j^{(2)}(x, y, z)(\text{op}[\chi_\kappa] f)(x - y - z) dy dz \right],
\]

where

\[
K_j^{(2)}(x, y, z) := \sum_{k, \ell \in \mathbb{Z}^n} e^{i\ell \cdot y} e^{i k \cdot z} \varphi_j(\ell) \varphi_\kappa(k) a(x - y, k).
\]

The series over \( \kappa \) in c) and d) converge in \( C(\mathbb{T}^n, E) \), the sums over \( k \) and \( \ell \) are finite.

\[\text{Proof. a) Because of } \sum_{\kappa \in \mathbb{N}_0} \varphi_\kappa = 1, \text{ we obtain}\]

\[
(\text{op}[a] f)(x) = \sum_{k \in \mathbb{Z}^n} e^{ix \cdot k} a(x, k) \hat{f}(k) = \sum_{k \in \mathbb{Z}^n} \left( \sum_{\kappa \in \mathbb{N}_0} e^{ix \cdot k} a(x, k) \varphi_\kappa(k) \hat{f}(k) \right). \tag{2–5}
\]

For every \( k \in \mathbb{Z}^n \), there are at most three \( \kappa \in \mathbb{N}_0 \) with \( \varphi_\kappa(k) \neq 0 \). This and \( \varphi_\kappa \leq 1 \) yield

\[
\sum_{k \in \mathbb{Z}^n} \| a(x, k) \varphi_\kappa(k) \hat{f}(k) \| \leq 3 \sum_{k \in \mathbb{Z}^n} \| a(x, k) \|_{L(E)} \| \hat{f}(k) \|_E \leq C \sum_{k \in \mathbb{Z}^n} \| k \|^m \| \hat{f}(k) \| < \infty.
\]

In the last step, we have used \( (\hat{f}(k))_{k \in \mathbb{Z}^n} \in \mathcal{S}(\mathbb{Z}^n, E) \). Therefore, the series in (2–5) converges in \( C(\mathbb{T}^n, E) \), and we may change the order of summation which yields a).

b) This follows from

\[
(\text{op}[a] \text{op}[\varphi_j] f)(x) = \int_{\mathbb{Z}^n} e^{i k \cdot x} a(x, k) \varphi_j(k) \hat{f}(k)
\]

\[= \sum_{k \in \mathbb{Z}^n} \left[ \int_{\mathbb{T}^n} e^{i k \cdot x} a(x, k) \varphi_j(k) e^{-i k \cdot z} f(z) dz \right]
\]

\[= \sum_{k \in \mathbb{Z}^n} \left[ \int_{\mathbb{T}^n} e^{i k \cdot y} a(x, k) \varphi_j(k) f(x - y) dy \right]
\]

\[= \int_{\mathbb{T}^n} K_j(x, y) f(x - y) dy.
\]

Note that the sum is finite, and therefore we may change the order of summation and integration.

c) We use \( \varphi_\kappa \chi_\kappa = \varphi_\kappa \) and \( \text{op}[\varphi_j] \text{op}[\varphi_\kappa] = \text{op}[\varphi_\kappa] \text{op}[\varphi_j] \) and apply a) to get

\[
\text{op}[a] \text{op}[\varphi_j] f = \sum_{\kappa \in \mathbb{N}_0} \text{op}[a] \text{op}[\varphi_\kappa] \text{op}[\varphi_j] \text{op}[\chi_\kappa] f.
\]
Here, the sum on the right-hand side converges in $C(T^n, E)$ due to a). Applying b), we see that
\[
(\op[a] \op[\varphi_k] \op[\varphi_j] \op[\chi_k] f)(x) = \int_{T^n} K_k(x, z) \big( \op[\varphi_j] \op[\chi_k] f \big)(x - z) dz
\]
with $K_k$ being defined in (2–4). Another application of b) with $a$ being replaced by the constant symbol $(x, k) \mapsto \id_E$ gives
\[
(\op[\varphi_j] \op[\chi_k] f)(x) = \int_{T^n} \tilde{K}_j(y) \big( \op[\chi_k] f \big)(x - y) dy
\]
with $\tilde{K}_j(y) := \sum_{\ell \in \mathbb{Z}^n} e^{i\ell y} \varphi_j(\ell)$. Altogether we obtain
\[
(\op[a] \op[\varphi_j] f)(x) = \sum_{\kappa \in \mathbb{N}_0} \int_{T^n} \int_{T^n} K^{(1)}_{j\kappa}(x, y, z)(\op[\chi_k] f)(x - y - z) dy dz
\]
with
\[
K^{(1)}_{j\kappa}(x, y, z) := K_k(x, z) \tilde{K}_j(y).
\]

(d) Similarly, we apply a) and twice b) to get
\[
(\op[\varphi_j] \op[a] f)(x) = \sum_{\kappa \in \mathbb{N}_0} \big( \op[\varphi_j] \op[a] \op[\varphi_k] \op[\chi_k] f \big)(x)
\]
\[
= \sum_{\kappa \in \mathbb{N}_0} \int_{T^n} \int_{T^n} K^{(2)}_{j\kappa}(x, y, z)(\op[\chi_k] f)(x - y - z) dy dz,
\]
with $K^{(2)}_{j\kappa}(x, y, z) := \tilde{K}_j(y) K_k(x - y, z)$ which shows the assertion in d). \qed

The following estimate on the kernel $K_j$ defined in Lemma 2.6 will be one key ingredient for the proof of Besov space continuity of toroidal pseudodifferential operators.

**Theorem 2.7.** Let $b \in S^{m, r, 0}$, and set
\[
K_j(x, y) := \sum_{k \in \mathbb{Z}^n} e^{i k y} \varphi_j(k) b(x, k) \quad (j \in \mathbb{N}_0).
\]

Then
\[
\|K_j(x, y)\|_{L(E)} \leq C 2^{jm} g_{j, \theta}(y) \|b\|_{m}^{(r, 0)} \quad (x, y \in T^n, j \in \mathbb{N}_0, \theta \in (0, 1)),
\]
where
\[
g_{j, \theta}(y) := \frac{(2j |y|)^\theta}{|y|^{n(1 + 2j |y|)}} \quad (y \in T^n).
\]

**Proof.** The proof follows the ideas from [4], proof of Lemma 4.8.

Note that $\varphi_j(k) = 0$ for $|k| > 2j + 1$ implies $\Delta^\gamma_k \varphi_j(k) = 0$ for $|k| > 2j + 1 + |\gamma|$. In the same way, $\varphi_j(k) = 0$ for $|k| < 2j - 1$ implies $\Delta^\gamma_k \varphi_j(k) = 0$ for $|k| < 2j - 1 - |\gamma|$.

Let $n_0$ be the smallest integer such that $2^{-n_0}(n + 1) \leq \frac{1}{4}$. Then
\[
2j + 1 + |\gamma| \leq 2j + 1 + (n + 1) \leq 2 \cdot 2j + 1 = 2j + 2
\]
and
\[
2j - 1 - |\gamma| \geq 2j - 1 - (n + 1) \geq \frac{1}{2} \cdot 2j - 1 = 2j - 2
\]
hold for all $j \geq n_0$ and all $|\gamma| \leq n + 1$.  

Condition 2.3 (iii) and the condition $a \in S^{m,\rho,0}$ imply with the discrete Leibniz formula that

$$\|\Delta^\gamma_k(\varphi_j(k)a(x, k))\|_{L(E)} \leq C \|a\|_{m}^{(\rho,0)} |k|^{m-|\gamma|}$$

for $(x, k) \in \mathbb{T}^n \times \mathbb{Z}^n$, $j \in \mathbb{N}_0$ and $|\gamma| \leq n+1$. Moreover, for each $x \in \mathbb{T}^n$ and $j \geq n$ we have

$$\Delta^\gamma_k(\varphi_j(k)a(x, k)) = 0 \quad (2-7)$$

if $|k| < 2^{j-2}$ or if $|k| > 2^{j+2}$.

Let $N \in \{n, n+1\}$, and set $(e^{i\eta} - 1)^\gamma := \prod_{k=1}^{n}(e^{i\eta_k} - 1)^{\gamma_k}$. Then we have (see [4], Remark 4.7)

$$|\eta|^N \leq C \sum_{|\gamma|=N} |(e^{-i\eta} - 1)^\gamma| \quad (\eta \in \mathbb{T}^n)$$

and

$$(e^{-i\eta} - 1)^\gamma K_j(x, \eta) = \sum_{k \in \mathbb{Z}^n} (e^{ik\cdot\eta} - 1)\Delta^\gamma_k(\varphi_j(k)a(x, k)).$$

In combination with the elementary inequality $|e^{ik\cdot\eta} - 1| \leq 2|k|\theta|\eta|^\theta$ which holds for all $\theta \in (0, 1)$, we get

$$|\eta|^N \|K_j(x, \eta)\|_{L(E)} \leq C \|a\|^{(\rho,0)}_{m} |\eta|^\theta \sum_{k \in B_j} |k|^{\theta} |k|^{m-N} \quad (x, \eta \in \mathbb{T}^n) \quad (2-8)$$

with $B_j := \{k \in \mathbb{Z}^n : 2^{j-2} \leq |k| \leq 2^{j+2}\}$. Due to [4], inequality (4-5), for all $\mu > 0$ the inequality

$$\sum_{\ell \in \mu^{-1} \mathbb{Z}^n \setminus \{0\}} |\ell|^{\theta-n} \mu^{-\ell} \leq C\theta$$

holds. Setting $\mu := 2^{j+2}$, we obtain

$$\sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{\theta-n} = \sum_{\ell \in \mu^{-1} \mathbb{Z}^n \setminus \{0\}} |\mu\ell|^{\theta-n} \leq \sum_{\ell \in \mu^{-1} \mathbb{Z}^n \setminus \{0\}} |\ell|^{\theta-n} \leq C\theta \mu^{\theta} = C2^{j\theta}. $$

Inserting this into $(2-8)$ with $N = n$ yields

$$\sum_{k \in B_j} |k|^{\theta} |k|^{m-N} \leq \left( \sup_{k \in B_j} \langle k \rangle^m \right) \sum_{k \in B_j} |k|^{\theta-n} \leq C \cdot 2^{j(m+\theta)}. \quad (2-9)$$

Note here that for $m \geq 0$ we used the estimate

$$\langle k \rangle^m \leq C \cdot 2^{(j+2)m} = C \cdot 2^{jm},$$

while for $m < 0$ we used

$$\langle k \rangle^m \leq C \cdot 2^{(j-2)m} = C \cdot 2^{jm}. $$

For $(2-8)$ with $N = n + 1$ we have in the same way

$$\sum_{k \in B_j} |k|^{\theta} |k|^{m-N-1} \leq C \sum_{k \in B_j} |k|^{\theta-n} |k|^{m-1} \leq C \cdot 2^{j(\theta+m-1)}. \quad (2-10)$$

Therefore, we obtain

$$|\eta|^n \|K_j(x, \eta)\|_{L(E)} \leq C \|a\|^{(\rho,0)}_{m} \cdot 2^{j(m+\theta)} |\eta|^\theta,$$
In particular, this yields that
\[ \|g\|_{L(E)} \leq C \|a\|_{m}^{(\rho,0)} \cdot 2^{j(m+\theta-1)}|\eta|^\theta. \]

Multiplying the second inequality by \(2^j\) and adding both inequalities yields
\[ \|K_j(x,\eta)\|_{L(E)} \leq C \|a\|_{m}^{(\rho,0)} \cdot 2^{jm} \frac{(2^j|\eta|)^\theta}{|\eta|^n(1+2^j|\eta|)} \quad (x, \eta \in \mathbb{T}^n, j \geq n_0). \]

\[ \square \]

3. Mapping properties in toroidal Besov spaces

In this section, we use the kernel estimates from above to show continuity of pseudodifferential operators in toroidal vector-valued Besov spaces.

**Lemma 3.1.** a) Let \(p \in [1, \infty]\), and let \(K: \mathbb{T}^n \times \mathbb{T}^n \to L(E)\) be measurable. Assume that there exists a function \(g \in L^1(\mathbb{T}^n)\) with
\[ \|K(x,y)\|_{L(E)} \leq g(y) \quad (x, y \in \mathbb{T}^n). \]
For \(x \in \mathbb{T}^n\) and \(f \in L^p(\mathbb{T}^n, E)\), define \(F(x) := \int_{\mathbb{T}^n} K(x,y)f(x-y)dy\). Then \(F(x)\) is well-defined for almost all \(x \in \mathbb{T}^n\) and
\[ \|F\|_{L^p(\mathbb{T}^n, E)} \leq \|g\|_{L^1(\mathbb{T}^n)}\|f\|_{L^p(\mathbb{T}^n, E)} \quad (f \in L^p(\mathbb{T}^n, E)). \]

b) Let \(p \in [1, \infty]\), let \(K: \mathbb{T}^n \times \mathbb{T}^n \to L(E)\) be measurable. Assume that there exist functions \(g, h \in L^1(\mathbb{T}^n)\) with
\[ \|K(x,y,z)\|_{L(E)} \leq g(y)h(z) \quad (x, y, z \in \mathbb{T}^n). \]
For \(x \in \mathbb{T}^n\), define \(F(x) := \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K(x,y,z)f(x-y-z)dxdz\). Then \(F(x)\) is well-defined for almost all \(s \in \mathbb{T}^n\) and
\[ \|F\|_{L^p(\mathbb{T}^n, E)} \leq \|g\|_{L^1(\mathbb{T}^n)}\|h\|_{L^1(\mathbb{T}^n)}\|f\|_{L^p(\mathbb{T}^n, E)} \quad (f \in L^p(\mathbb{T}^n, E)). \]

**Proof.** a) Let \(p \in [1, \infty]\). For \(x \in \mathbb{T}^n\), we have
\[ \|F(x)\| = \left\| \int_{\mathbb{T}^n} K(x,y)f(x-y)dy \right\| \leq \int_{\mathbb{T}^n} \|K(x,y)\|_{L(E)}\|f(x-y)\|dy \leq \int_{\mathbb{T}^n} g(y)\|f(x-y)\|dy = (g * \|f\|)(x). \]
Therefore,
\[ \|F\|_{L^p(\mathbb{T}^n, E)} \leq \|g\|_{L^1(\mathbb{T}^n)}\|f\|_{L^p(\mathbb{T}^n)} \leq \|g\|_{L^1(\mathbb{T}^n)}\|f\|_{L^p(\mathbb{T}^n, E)}. \]
In particular, this yields that \(F(x)\) is well-defined for almost all \(x \in \mathbb{T}^n\). The case \(p = \infty\) follows similarly.

b) This follows in the same way. By the assumption on \(K\), we can estimate
\[ \|F(x)\| \leq \int_{\mathbb{T}^n} \left[ \int_{\mathbb{T}^n} g(y)h(z)\|f(x-y-z)\|dy \right]dz = (h * (g * \|f\|))(x). \]
This yields the desired estimate on \(\|F\|_{L^p(\mathbb{T}^n, E)}\) and the fact that \(F(x)\) is well-defined for almost all \(x \in \mathbb{T}^n\). \(\square\)

**Lemma 3.2.** Let \(a \in S^{m,\rho,r}\) with \(r \in (0,1)\).

a) For all \(j \in \mathbb{N}_0\) and \(f \in C^\infty(\mathbb{T}^n, E)\),
\[ \|\text{op}[\alpha]\text{op}[\varphi_j]f\|_{L^p(\mathbb{T}^n, E)} \leq C\|a\|_{m}^{(\rho,r)m} \|\text{op}[\chi_j]f\|_{L^p(\mathbb{T}^n, E)}. \]
b) For all $j \in \mathbb{N}_0$ and $f \in C^\infty(\mathbb{T}^n, E)$,
\[
\| \text{op}[\varphi_j] \text{op}[a]f \|_{L^p(\mathbb{T}^n, E)} \leq C\|a\|^{(\rho, r)}_{m} (2^{jm}\| \text{op}[\chi_j]f \|_{L^p(\mathbb{T}^n, E)} + 2^{-jr}\| f \|_{B^m_{p,1}(\mathbb{T}^n, E)}) .
\]

Proof. a) By Lemma 2.6 b),
\[
(\text{op}[a] \text{op}[\varphi_j]f)(x) = \int_{\mathbb{T}^n} K_j(x, y) f(x - y)dy
\]
with $K_j$ being defined in (2.4). Due to Theorem 2.7, for arbitrary $\theta \in (0, 1)$,
\[
\|K_j(x, y)\|_{L(E)} \leq C 2^{jm}\|a\|^{(\rho, r)}_{m} g_j, \theta(y) \quad (x, y \in \mathbb{T}^n).
\]
Because of
\[
\|g_j, \theta\|_{L^1(\mathbb{T}^n)} = \int_{\mathbb{T}^n} \frac{(2^j|y|^\theta)}{|y|^n(1 + 2^j|y|)} dy = \int_{\mathbb{R}^n} \frac{|y|^\theta}{|z|^n(1 + |z|)} dz < \infty,
\]
we can apply Lemma 3.1 a) to obtain the assertion of a).

b) We consider the difference
\[
(\text{op}[\varphi_j] \text{op}[a] - \text{op}[a] \text{op}[\varphi_j]) f(x) = \sum_{\kappa \in \mathbb{N}_0} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K_{j, \kappa}(x, y, z)(\text{op}[\chi_{\kappa}]f)(x - y - z)dxdz
\]
with
\[
K_{j, \kappa}(x, y, z) := K_{j, \kappa}^{(2)}(x, y, z) - K_{j, \kappa}^{(1)}(x, y, z) = \sum_{\ell, \xi \in \mathbb{Z}^n} e^{i\ell \cdot y} e^{i\xi \cdot z} \varphi_j(\ell) \varphi_{\kappa}(\xi) (a(x - y, k) - a(x, k))
\]
\[
= \left( \sum_{\ell, \xi \in \mathbb{Z}^n} e^{i\ell \cdot y} \varphi_j(\ell) \right) \left( \sum_{\kappa \in \mathbb{Z}^n} e^{i\xi \cdot z} \varphi_{\kappa}(\xi) (a(x - y, k) - a(x, k)) \right)
\]
\[
=: \tilde{K}_j(y)K_{\kappa}'(x, y, z). \]

We apply Theorem 2.7 with $b(x, k) := a(x - y, k) - a(x, k)$ where $y \in \mathbb{T}^n$ is fixed. By the definition of $S^{m,p,r}$ we have
\[
\|b\|^{(\rho, 0)}_m = \max_{|\alpha| \leq \rho} \sup_{x \in \mathbb{T}^n} \sup_{k \in \mathbb{Z}^n} \|\Delta^\alpha_k(a(x - y, k) - a(x, k))\|_{L(E)} \leq |y|^r\|a\|^{(\rho, r)}_m.
\]
Note that $0 < r < 1$. From Theorem 2.7 we get
\[
\|K_{\kappa}'(x, y, z)\|_{L(E)} \leq C |y|^{r} 2^{knm}\|a\|^{(\rho, r)}_m g_{\kappa, \theta_1}(z) \quad (x, y, z \in \mathbb{T}^n)
\]
for arbitrary $\theta_1 \in (0, 1)$. Another application of Theorem 2.7 with constant symbol $b(x, k) = \text{id}_E$ yields
\[
\|\tilde{K}_j(y)\|_{L(E)} \leq C g_{j, \theta_2}(y) \quad (y \in \mathbb{T}^n)
\]
for all $\theta_2 \in (0, 1)$. Therefore,
\[
\|K_{j, \kappa}(x, y, z)\|_{L(E)} \leq C 2^{knm}\|a\|^{(\rho, r)}_m |y|^r g_{\kappa, \theta_1}(z) g_{j, \theta_2}(y).
\]
Because of $r \in (0, 1)$, we can choose $\theta_2 \in (0, 1 - r)$ and obtain for $\theta_0 := \theta_2 + r \in (0, 1)$
\[ |y|^{\tau} g_j,\theta_2(y) = |y|^{\tau} \frac{(2^j|y|)^{\theta_2}}{|y|^{\tau}(1 + 2^j|y|)} = 2^{-j} g_j,\theta_0(y). \]
Therefore,
\[ K_{j,\kappa}(x, y, z) \|_{L^1(E)} \leq C 2^{m\kappa} 2^{-j} \|a\|_m \| g_j,\theta_0(y) g_{\kappa,\theta_1}(z). \]
We have seen above that $\|g_j,\theta_0\|_{L^1(\mathbb{T}^n)} \leq C < \infty$ and $\|g_{\kappa,\theta_1}\|_{L^1(\mathbb{T}^n)} \leq C < \infty$. Therefore, we can apply Lemma 3.1 b) to get
\[ \|(\text{op}[\varphi_j] \text{op}[a] - \text{op}[a] \text{op}[\varphi_j]) f\|_{L^p(\mathbb{T}^n, E)} \leq C 2^{-j} \|a\|_m \| \text{op}[\chi_\kappa] f\|_{L^p(\mathbb{T}^n, E)}. \]
By the definition of $\chi_\kappa$,
\[ \sum_{\kappa \in \mathbb{N}_0} 2^{\kappa m} \| \text{op}[\chi_\kappa] f\|_{L^p(\mathbb{T}^n, E)} = \sum_{\kappa \in \mathbb{N}_0} 2^{\kappa m} \| \text{op}[\chi_\kappa - 1] + \text{op}[\varphi_\kappa] + \text{op}[\varphi_{\kappa + 1}] f\|_{L^p(\mathbb{T}^n, E)} \leq (2^{-m} + 1 + 2^m) \sum_{\kappa \in \mathbb{N}_0} 2^{\kappa m} \| \text{op}[\varphi_\kappa] f\|_{L^p(\mathbb{T}^n, E)} = C \| f\|_{B_{pq}^{s+m}(\mathbb{T}^n, E)}. \]
Therefore,
\[ \|(\text{op}[\varphi_j] \text{op}[a] - \text{op}[a] \text{op}[\varphi_j]) f\|_{L^p(\mathbb{T}^n, E)} \leq C 2^{-j} \|a\|_m \| f\|_{B_{pq}^{s+m}(\mathbb{T}^n, E)}. \]
Together with part a) this yields the assertion of b). \(\square\)

The last lemma is the essential step in the proof of Besov space continuity. The following theorem is the main result of the present paper.

**Theorem 3.3.** Let $m \in \mathbb{R}$, $\rho \in \mathbb{N}$ with $\rho \geq n + 1$, and $r \in (0, \infty)$, and let $a \in S^{m,\rho,r}$. Then for $s \in (0, r)$ and $p, q \in [1, \infty]$, the mapping
\[ \text{op}[a] : B_{pq}^{s+m}(\mathbb{T}^n, E) \rightarrow B_{pq}^s(\mathbb{T}^n, E) \]
is continuous. Moreover,
\[ (a \mapsto \text{op}[a]) \in L(S^{m,\rho,r}, L(B_{pq}^{s+m}(\mathbb{T}^n, E), B_{pq}^s(\mathbb{T}^n, E))). \]

**Proof.** (i) We first consider the case $r \in (0, 1)$. We start with showing that
\[ \text{op}[a] : B_{p1}^{s+m}(\mathbb{T}^n, E) \rightarrow B_{p1}^s(\mathbb{T}^n, E) \]
is continuous. For that we will use the density of $C^\infty(\mathbb{T}^n, E)$ in $B_{p1}^{s+m}(\mathbb{T}^n, E)$ (see [4], Theorem 3.15). Let $f \in C^\infty(\mathbb{T}^n, E)$. Then by Lemma 3.2 b) we obtain that
\[ \| \text{op}[a] f\|_{B_{p1}^s(\mathbb{T}^n, E)} = \sum_{j=0}^{\infty} 2^{js} \| \text{op}[\varphi_j] 2^j \| \text{op}[a] f\|_{L^p(\mathbb{T}^n, E)} \leq C \|a\|_m (\sum_{j \in \mathbb{N}_0} 2^{j(s+m)} \| \text{op}[\chi_j] f\|_{L^p(\mathbb{T}^n, E)} + \| f\|_{B_{p1}^{s+m}(\mathbb{T}^n, E)} \sum_{j \in \mathbb{N}_0} 2^{j(s-r)}) . \]
We have seen in the proof of Lemma 3.2 that the first sum can be estimated by \( C\|f\|_{B^{s+m}_{p,q}(T^n, E)} \). For the second term, we note that \( \sum_{j \in \mathbb{N}_0} 2^{j(s-r)} \) is finite because of \( r > s \) and use the continuous embedding \( B^{s+m}_{p,q}(T^n, E) \hookrightarrow B^m_{p,1}(T^n, E) \). Therefore,

\[
\| \text{op}[a]f \|_{B^m_{p,1}(T^n, E)} \leq C\|a\|^{(p,r)} \|f\|_{B^{s+m}_{p,q}(T^n, E)}
\]

which shows the continuity of \( \text{op}[a]: B^{s+m}_{p,q}(T^n, E) \rightarrow B^m_{p,1}(T^n, E) \) as well as the continuity of \( a \mapsto \text{op}[a] \) for \( q = 1 \).

For general \( q \in [1, \infty] \) we use real interpolation theory: For \( q \in [1, \infty] \), we choose some \( 0 < \varepsilon < 1 \) such that \( s - \varepsilon, s + \varepsilon \in (0, r) \). Then

\[
B^{s-\varepsilon}_{pq}(T^n, E) = \left( B^s_{p1}(T^n, E), B^s_{p1}(T^n, E) \right)_{1/2, q}
\]

for \( t \in \{s, s + m\} \).

Now the continuity of

\[
\text{op}[a]: B^{s-\varepsilon+\varepsilon}_{pq}(T^n, E) \rightarrow B^s_{p1}(T^n, E)
\]

and real interpolation immediately give the continuity of

\[
\text{op}[a]: B^{s+m}_{pq}(T^n, E) \rightarrow B^s_{pq}(T^n, E).
\]

In the same way, the continuity of the map \( a \mapsto \text{op}[a] \) follows.

(ii) Now let \( r \in [1, \infty) \), and let \( s \in (0, r) \). We first assume that \( s \notin \mathbb{N} \), i.e., \( s = s_0 + s_1 \) with \( s_0 \in \mathbb{N} \) and \( s_1 \in (0, 1) \). We choose \( r_1 := \tilde{r} - s_0 \in (0, 1) \). Then \( a \in S^{m,\tilde{r}} \) by the definition of the symbol class.

We make use of an equivalent norm in \( B^s_{pq}(\mathbb{T}^n, E) \). More precisely, there exist constants \( c_1, c_2 > 0 \) such that

\[
c_1 \sum_{|\alpha| \leq s_0} \| \partial^{\alpha}_x u \|_{B^s_{pq}(\mathbb{T}^n, E)} \leq \| u \|_{B^s_{pq}(\mathbb{T}^n, E)} \leq c_2 \sum_{|\alpha| \leq s_0} \| \partial^{\alpha}_x u \|_{B^s_{pq}(\mathbb{T}^n, E)}
\]

for all \( u \in B^s_{pq}(\mathbb{T}^n, E) \), see [2], (5.19), for the case of \( \mathbb{R}^n \), and [3], proof of Theorem 2.3, for the one-dimensional torus.

Let \( f \in C^\infty(\mathbb{T}^n, E) \), and let \( j \in \mathbb{N}_0 \) and \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq s_0 \). By Lemma 2.6 d) and the Leibniz rule, we have

\[
\text{op}[\varphi_j](\partial^\alpha_x \text{op}[a]f) = \partial^\alpha_x \text{op}[\varphi_j] \text{op}[a]f
\]

\[
= \partial^\alpha_x \left[ \sum_{\kappa \in \mathbb{N}_0} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K^{(2)}_{j\kappa}(x, y, z)(\text{op}[\chi_\kappa]f)(x-y-z)dydz \right]
\]

\[
= \sum_{\beta \leq \alpha} \sum_{\kappa \in \mathbb{N}_0} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} (\partial^\beta_x K^{(2)}_{j\kappa})(x, y, z)(\text{op}[\chi_\kappa] \partial^\alpha_{x-\beta} f)(x-y-z)dydz
\]

\[
= \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) (\text{op}[\varphi_j] \text{op}[a_\beta](\partial^\alpha_{x-\beta} f))(x)
\]

with the symbol \( a_\beta(x, k) := (\partial^\beta_x a)(x, k) \) \( (x \in \mathbb{T}^n, k \in \mathbb{Z}^n) \). Here we note that for all \( |\beta| \leq s_0 \), we have \( a_\beta \in S^{m,\tilde{r},\bar{r}_0} \) with \( \|a_\beta\|_{(\tilde{r},\bar{r}_0)} \leq C\|a\|_{(p,\tilde{r})} \leq C\|a\|_{(p,r)} \). In particular, the series over \( \kappa \) above are uniformly convergent with respect to \( x \) by Lemma 2.6 d) and we may change the order of differentiation and integration.
For $|\alpha| \leq s_0$ and $\beta \leq \alpha$, we can apply part (i) of the proof and obtain

$$\| \text{op}[a\beta] \partial_x^{\alpha-\beta} f \|_{B_{pq}^{s_0}(T^n,E)} \leq C\| a\beta \|_{m}^{(\rho,r)} \| \partial_x^{\alpha-\beta} f \|_{B_{pq}^{s_0+m}(T^n,E)} \leq \| a \|_{m}^{(\rho,r)} \| f \|_{B_{pq}^{s_0+m}(T^n,E)}.$$ 

Together with (3–1), this yields

$$\| \text{op}[a] f \|_{B_{pq}^{s}(T^n,E)} \leq c_2 \sum_{|\alpha| \leq s_0} \| \partial_x^\alpha \text{op}[a] f \|_{B_{pq}^{s_0}(T^n,E)} \leq C \| a \|_{m}^{(\rho,r)} \| f \|_{B_{pq}^{s_0+m}(T^n,E)}.$$

This shows the desired continuity in the case $s \in (0, \rho) \setminus N$. Finally, if $s \in N$, we choose $\varepsilon \in (0,1)$ with $0 < s - \varepsilon < s + \varepsilon < \rho$. As we have seen,

$$\text{op}[a] : B_{pq}^{s_0+m}(T^n,E) \to B_{pq}^{s_0+m}(T^n,E)$$

is continuous. Now the continuity of

$$\text{op}[a] : B_{pq}^{s_0+m}(T^n,E) \to B_{pq}^{s}(T^n,E)$$

again follows by real interpolation $(\ldots)_1/2,q$. So we have seen that the continuity of the operator $\text{op}[a]$ stated in the theorem as well as the continuity of $a \mapsto \text{op}[a]$ hold in all cases. □

Remark 3.4. a) As a particular case, we obtain the continuity of $\text{op}[a]$ in the case of $x$-independent symbols. In fact, this could more easily be obtained by the observation that $\text{op}[\varphi_j] \text{op}[a] = \text{op}[a] \text{op}[\varphi_j]$ holds in this case. Therefore, one can apply Lemma 2.6 b) and Lemma 3.2 a) and avoid double integrals.

The case of $x$-independent symbols was already shown in [4], Theorem 3.17. However, the proof in [4] was based on the connection between the symbols on $\mathbb{Z}^n$ and the symbols on $\mathbb{R}^n$. In fact, every symbol on $\mathbb{Z}^n$ can be extended to a symbol on $\mathbb{R}^n$ belonging to the same symbol class (see [16], Theorem II.4.5.3, and the transference principle in [11], Section 5.7). In the present paper, we formulated a proof which is independent of this fact. Therefore, the present proof might serve as a basis for generalizations to more general groups instead of $\mathbb{T}^n$.

b) As the symbols considered here are of restricted smoothness, we do not obtain continuity in the full scale of Besov spaces. That the range of continuity is restricted becomes obvious if we take a symbol $a(x,k) = b(x)$ independent of $k$, where $b \in C^\omega(T^n,L(E))$. In this case, $a \in S^{0,\rho,r}$ and

$$(\text{op}[a] f)(x) = \sum_{k \in \mathbb{Z}^n} \int_{T^n} e^{iky} b(x) f(x-y) dy = b(x) f(x) \quad (x \in T^n).$$

Taking $f(x)$ as a constant function, we see that in general $\text{op}[a] f \in C^\omega(T^n,E)$ cannot be improved.

References

[1] M. S. Agranovich. Spectral properties of elliptic pseudodifferential operators on a closed curve. Funktsional. Anal. i Prilozhen., 13(4):54–56, 1979.
[2] H. Amann. Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications. Math. Nachr., 186:5–56, 1997.
[3] W. Arendt and S. Bu. Operator-valued Fourier multipliers on periodic Besov spaces and applications. Proc. Edinb. Math. Soc. (2), 47(1):15–33, 2004.
[4] B. Barraza Martínez, R. Denk, J. Hernández Monzón, and T. Nau. Generation of semigroups for vector-valued pseudodifferential operators on the torus. *J. Fourier Anal. Appl.*, 22(4):823–853, 2016.

[5] B. Barraza Martínez, I. González Martínez, and J. Hernández Monzón. Operator-valued Fourier multipliers on toroidal Besov spaces. *Rev. Colombiana Mat.*, 50(1):109–137, 2016.

[6] S. Bu and J.-M. Kim. Operator-valued Fourier multiplier theorems on $L_p$-spaces on $\mathbb{T}^d$. *Arch. Math. (Basel)*, 82(5):404–414, 2004.

[7] S. Bu and J.-M. Kim. A note on operator-valued Fourier multipliers on Besov spaces. *Math. Nachr.*, 278(14):1659–1664, 2005.

[8] D. Cardona. Besov continuity of pseudo-differential operators on compact Lie groups revisited. *C. R. Math. Acad. Sci. Paris*, 355(5):533–537, 2017.

[9] D. Cardona. Hölder-Besov boundedness for periodic pseudo-differential operators. *J. Pseudo-Differ. Oper. Appl.*, 8(1):13–34, 2017.

[10] J. Delgado. $L^p$-bounds for pseudo-differential operators on the torus. In *Pseudo-differential operators, generalized functions and asymptotics*, volume 231 of *Oper. Theory Adv. Appl.* pages 103–116. Birkhäuser/Springer Basel AG, Basel, 2013.

[11] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2016.

[12] V. Keyantuo, C. Lizama, and V. Poblete. Periodic solutions of integro-differential equations in vector-valued function spaces. *J. Differential Equations*, 246(3):1007–1037, 2009.

[13] W. McLean. Local and global descriptions of periodic pseudodifferential operators. *Math. Nachr.*, 150:151–161, 1991.

[14] S. T. Melo. Characterizations of pseudodifferential operators on the circle. *Proc. Amer. Math. Soc.*, 125(5):1407–1412, 1997.

[15] S. Mohalahjloo and M. W. Wong. Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on $S^1$. *J. Pseudo-Differ. Oper. Appl.*, 1(2):183–205, 2010.

[16] M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries*, volume 2 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser Verlag, Basel, 2010. Background analysis and advanced topics.

[17] M. Ruzhansky and V. Turunen. Global quantization of pseudo-differential operators on compact Lie groups, SU(2), 3-sphere, and homogeneous spaces. *Int. Math. Res. Not. IMRN*, (11):2439–2496, 2013.

[18] M. Ruzhansky, V. Turunen, and J. Wirth. Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity. *J. Fourier Anal. Appl.*, 20(3):476–499, 2014.

[19] M. W. Wong. *Discrete Fourier analysis*, volume 5 of *Pseudo-Differential Operators. Theory and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.
B. Barraza Martínez, Universidad del Norte, Departamento de Matemáticas, Barranquilla (Colombia)
E-mail address: bbarraza@uninorte.edu.co

R. Denk, Universität Konstanz, Fachbereich für Mathematik und Statistik, Konstanz (Germany)
E-mail address: robert.denk@uni-konstanz.de

J. Hernández Monzón, Universidad del Norte, Departamento de Matemáticas, Barranquilla (Colombia)
E-mail address: jahernan@uninorte.edu.co

M. Nendel, Universität Konstanz, Fachbereich für Mathematik und Statistik, Konstanz (Germany)
E-mail address: max.nendel@uni-konstanz.de