On the Extension of the Gaussian Moat Problem

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Abstract

In this paper, we have developed an algorithm for the prime searching in $\mathbb{R}^3$. This problem was proposed by M. Das [Arxiv.2019]. This paper is an extension of her work. As we know the distribution of primes will get more irregular as we are going to infinity and going to the higher dimensions. We have also shown that why it is not possible to extend the Gaussian Moat problem for the higher dimensions (more than four dimensional plane).

1 Introduction

In number theory, the Gaussian moat problem asks whether it is possible to find an infinite sequence of distinct Gaussian prime numbers such that the difference between consecutive numbers in the sequence is bounded. The problem was first posed in 1962 by Basil Gordon [7] (although it has sometimes been erroneously attributed to Paul Erdős) and it remains unsolved. If we think that one is walking through the Gaussian primes and taking the steps at Gaussian primes then the problem is to find the moat to get the distribution of Gaussian primes. If we see the same problem on the real number line then it is more simple than the Gaussian Moat problem. It can be said directly that for any $n$, the $n-1$ consecutive numbers $n! + 2, n! + 3, \ldots, n! + n$ are all composite. The result for the real number line also follows from the Prime Number Theorem.

The Gaussian Moat problem says about the prime distribution for the $\mathbb{R}^2$ plane. The distribution of the primes in the $xy$ plane is not so regular and there is no direct connection with the natural prime distribution. So it is really hard to prove, especially for the large prime values. There are many several result on this problem [8,10,12,13,15]. The work was done by Genther et.al [9] helped to calculate the moat up to $\sqrt{26}$. But there is no bound for the Gaussian prime which can help to find the moats for all values of primes. Recently M. Das [1] has developed an algorithm to find the Gaussian primes, as well one can calculate the moat from it for the large values of prime also.

In her paper, she has proposed the same problem for the three-dimensional plane. Which is same as saying that: One can walk through the infinity taking the bounded length steps on the three-dimensional plane? Clearly, this answer is No. It is hard to prove because for the two-dimensional plane it is hard to find the prime distribution. Then for the three-dimensional plane, it will be harder.

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In this paper, we have extended Das’s algorithm to find the primes in a three-dimensional plane. This algorithm shows that the primes are getting rare as we are approaching infinity. For the Gaussian prime problem, we know that there exist infinitely many Gaussian primes \[11\]. In this article, we have defined the primes for the three-dimensional plane and also proved that there exist infinitely many three-dimensional primes using the Dirichlet’s theorem. As we have already said this paper is an extension of the Das’s algorithm, we have also proved that the algorithm for the three-dimensional prime also covers all the three-dimensional primes up to \(10^A\) for the sufficiently large value of \(A\) but finite.

Last but not the list, we have discussed about the same problem for higher dimensions. Well, the Waring problem assures us that we can not generalize this problem for higher values. According to the to the Lagrange’s four-square theorem we can extend the problem for \(\mathbb{R}^4\) but that may not be so interesting.

2 Background

In this section, we are going to say about the few details concerning the prime distribution on three dimensional plane. This problem was posed by M. Das \[1\]. We know about the natural prime numbers. Now the question arises that what is the definition of prime for higher dimensions? Well, for two dimensional plane it is a well known result call Gaussian Prime.

**Definition 1. Gaussian Primes:** Gaussian primes \[2\] are Gaussian integers \(z = a + bi\) satisfying one of the following properties.

- If both \(a\) and \(b\) are non zero then, \(a + bi\) is a Gaussian prime iff \(a^2 + b^2\) is an ordinary prime.
- If \(a = 0\) or \(b = 0\), then \(b\) or \(a\) is a Gaussian prime iff \(|b|\) or \(|a|\) is an ordinary prime and \(|b|\) or \(|a|\) \(\equiv\) \((\mod 4)\).

For the natural prime distribution we have the Prime Number Theorem \[3\] with a controllable error term. If we see the same problem for complex plane then it leads to the famous unsolved problem in Number theory which we call the Gaussian moat problem. In 2019, M. Das has generalized this idea for the three dimensional plane and posed the problem that: “**In the plane** \(\mathbb{R}^3\), is it possible to walk to infinity in the higher lattices taking the steps on the primes with the bounded length?” In this paper, we are going to construct an algorithm with the help of M. Das’s idea about the distribution of prime numbers in the three dimensional plane.

Now one can ask that what is the problem for the higher dimensions. Before answer this question let us look at an important invention of number theory, posed by Edward Waring \[4\] in 1770.

**Waring problem:** Whether each natural number \(k\) has an associated positive integer \(s\) such that every natural number is the sum of at most \(s\) natural numbers to the power of \(k\).

It’s an interesting an well known problem to find the bound for the function \(g(k)\) where for every \(k\), let \(g(k)\) denote the minimum number \(s\) of \(k\)-th powers of naturals needed to represent all positive integers. Every positive integer is the sum of one first power, itself, so \(g(1) = 1\). In this paper, this is not our motivation. From this result we just want to say that we can define the primes only up to four dimensional plane as M. Das has said in her paper.
Now let us see enlisted some important result which will help us to define the prime in three dimensional plane.

**Legendre’s three-square theorem:** A natural number can be represented as the sum of three squares of integers

\[
n = a^2 + b^2 + c^2
\]

if and only if \( n \) is not of the form \( n = 4^x(8y + 7) \) for integers \( x \) and \( y \). So, if \( x > 1 \) then \( n \) can not be a prime. We need to concern about the numbers \( 8y + 7 \) only. From this result we can define the primes in the three dimensional plane.

**Definition 2.** Three dimensional primes are three dimensional integer lattices i.e., \( a\hat{i} + b\hat{j} + c\hat{k} \) satisfying the following properties:

- If \( a, b, c \) all are non zero then \( a\hat{i} + b\hat{j} + c\hat{k} \) is a three dimensional prime iff \( |a^2 + b^2 + c^2| \) is an ordinary prime not of the form congruent to 7 modulo 8.
- If \( a \) or \( b \) or \( c \), one of them is equal to zero then it leads to a Gaussian prime if \( |a^2 + b^2| \) or \( |b^2 + c^2| \) or, \( |c^2 + a^2| \) is an ordinary prime of the form congruent to 1 or 5 modulo 3.
- If two of \( a, b, c \) equal to zero then it leads to a natural prime number number with \( |a| \) or, \( |b| \), or \( |c| \) is congruent to 7 modulo 8.

In this paper, we are going to develop an algorithm (which is an extension of the Das’s result) concerning the distribution of three dimensional prime.

Now let us explain that why it is not possible to generalize the idea for the higher dimensions. We have already seen the Waring problem and it is clear from that we do not know that it is possible to write an number of the form as a sum of \( n \) number squares or not. For four dimensional plane, we have Lagrange’s four-square theorem.

**Lagrange’s four-square theorem:** Every natural number can be represented as the sum of four integer squares.

\[
p = a_0^2 + a_1^2 + a_2^2 + a_3^2
\]

where the four numbers \( a_0, a_1, a_2, a_3 \) are integers.

From this result we can say directly that any natural prime number can be written as sum of four squares. So, any natural prime can be a four dimensional prime. We can also extend Das’s result for four dimensional plane and it is very easy to do. In this paper, our main goal is to extend her result for three dimensions and at the end of the paper, we will discuss this for four dimensions also but the result for four dimensions may not be so interesting. Now let us start our main discussion.

### 3 Overview of the Algorithm

In this section, we are going to give an overview of the main algorithm. We are going to give an overview of the prime searching algorithm for \( \mathbb{R}^3 \). We are going to analyse the three-dimensional prime (see definition) distribution for each line \( y = c \) (for \( c \geq 1 \)) on the \( xyz \) plane. For one
of $x, y, z = 0$ leads to the two-dimensional plane. We have to take care of the primes which are distributed on the planes, not on the axis. We know from the definition that if one of the $a, b$ or $c$ equal to zero then it leads to a Gaussian prime and we do not need to take care of that. We don’t have any asymptotic formula for the primes of the form $7$ modulo $8$. From the Chebyshev’s bias, we can get an approximation that among all the primes in an interval how many of them are of the form $7$ modulo $8$. But there is an error term. So we don’t need to think about it. One most important point that why we are discussing this three-dimensional case separately. Well, if we see the definition of three-dimensional prime then it is clear that they are the primes not of the form $7$ modulo $8$. For the four dimensions onwards it doesn’t hold.

Before starting this discussion let us see another important result which we are going to use to find the radius of the sphere we will consider for this algorithm. This result is formulated by the Swedish mathematician Harald Cramér in 1936,

Cramér’s conjecture: The gaps between consecutive primes are always small, and the conjecture quantifies asymptotically just how small they must be, i.e.,

$$p_{n+1} - p_n = O \left( (\log p_n)^2 \right),$$

where $p_n$ denotes the $n$th prime number. This supports the stronger statement

$$\limsup_{n \to \infty} \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1.$$ 

Now let’s start the main discussion. If we see the three-dimensional plane then it can be thought as three two dimensional planes making an angle with each other. So, we will analyze the prime distribution for one of these plane and the other planes follow similarly (see figure 1). Then let’s label a plane say plane 1 denoted by $P_1$. If we see the geometry of the three dimensional plane we will say a point $p$ is in $P_1$, if for the point $p = (a, b, c)$ the triplet satisfy the condition that $a, b > c$. One more thing to note that we are avoiding the negative axes. Now if we are standing on a three dimensional prime on the plane $P_1$ then Cramèr’s conjecture is a very helpful result to find the next prime on the line $y = c$ but it will be more easy to calculate if we can get a prime inside of a sphere with suitable radius. So we will consider a sphere with a suitable choice of its radius and this is the most important thing. Why is that? Because this choice of the radius can reduce a lot of computation and it is also important to the area of the sphere, in which we are going to check the primes. This choice is also important to reduce computation.

There is an important question arises that: Is it possible to cover all the three-dimensional prime using this algorithm? The answer is Yes, it is possible to cover all the three-dimensional prime using this algorithm. We are just slicing the plane in such fashion that we can get the three-dimensional primes easily, with avoiding computations. The way of slicing the plane is the main trick of our algorithm. One more thing to note that is, one can not walk to infinity using the bounded length step and stepping on the three-dimensional prime as the analytic number theorists say. We agree with this statement and also this algorithm supports that the length will increase as one is approaching to infinity. Later, we will develop the mathematics and the geometrical view of this algorithm. Before that, we will construct the lines which are slicing the plane.

We need to prove an important result before writing the main algorithm that is there exist infinitely many primes, not of the form congruent to $7$ modulo $8$. To prove this result we need to know Dirichlet’s theorem on an arithmetic progression.
Figure 1: The plane $x = y = z$ has divided the plane $P_1$ into two symmetric planes.

**Dirichlet’s theorem:** For any two positive co-prime integers $a$ and $d$, there are infinitely many primes of the form $a + nd$, where $n$ is a non-negative integer. In other words, there are infinitely many primes that are congruent to $a$ modulo $d$.

**Theorem 1.** There exist infinitely many three-dimensional primes.

**Proof:** The statement of this theorem follows directly from the Dirichlet’s theorem. There are primes of the form $8k + 1, 8k + 3, 8k + 5, 8k + 7$. It is clear that $\gcd(8, 1) = \gcd(8, 3) = \gcd(8, 5) = 1$. So, there exist infinitely many three dimensional primes.

One more interesting thing about the three-dimensional prime is how many three dimensional primes are there corresponding to each natural prime number. Well, consider the prime 3, then we can write $29 = \pm(4)^2 + \pm(2)^2 + \pm(2)^2$. We have analyzed only for the positive values. So for the positive choice of integers, there are $3! = 6$ many primes corresponding to each natural prime number. So for each prime triplet $(a, b, c)$ the possibilities are $(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)$.

This counting will help us to understand the geometry for a three-dimensional integer lattice. Actually, this counting result is true for all the three-dimensional integer lattice.

Now let us get back to the discussion about the distribution of three-dimensional primes in details.

4 Discussion of the Algorithm

In this section, we are going to discuss the main algorithm. The logic for this algorithm, mathematical description and it’s geometrical interpretation. So, let’s start the main description of this algorithm.

The algorithm which we are going to develop for three-dimensional prime, we will consider only one side of the three-dimensional plane and analyze the prime distribution on it. For the
other sides, it will follow similarly. Why is it so? Let’s discuss about its geometry. If we see the three-dimensional plane then it can be thought as three two dimensional planes making an angle with each other. So, we will analyze the prime distribution for one of these plane and the other planes follow similarly (see figure 3). For the betterment of understanding let us label the planes. If \(x, y \neq 0\) and \(z = 0\) then it is called plane 1 and denoted by \(P_1\). Similarly, if \(y, z \) or, \(z, x \neq 0\) and \(x = 0\) or, \(y = 0\) then it is called plane 2 and plane 3 and denoted by \(P_2, P_3\) respectively.

Now consider that we are on the plane \(P_1\), and let’s see the geometrical view of a three-dimensional integer lattice which lies on the \(P_1\). Let \(p = (a, b, c)\) be a three-dimensional integer lattice with \(a, b > c\), this inequality is the condition for a three-dimensional integer lattice lies on the plane \(P_1\). If we inspect this plane then we can see that the plane \(x = y = z\) is dividing the plane \(P_1\) symmetrically. The points with the inequality \(a > b > c\) lies on the lower side of \(P_1\) and the points with the inequality \(b > a > c\) lies on the upper side of \(P_1\). We can define the function \(f\) as follows: \(f((a, b, c)) = (b, a, c)\) with the inequality \(a > b > c\). It is clear that the two sides of the plane \(x = y = z\) are symmetric. So, it is enough to analyze the prime distribution for the one side of this plane. We know for each prime triplet \((a, b, c)\) there exist six different three-dimensional primes if we have considered \(a, b, c > 0\).

Now focus on the geometry of the integer lattice \(p = (a, b, c)\) with \(a > b > c\). The algorithm we are going to develop is nothing but an extension of the Das’s [1] algorithm for the Gaussian Primes. We have used the same idea for this algorithm. If we are standing on a three-dimensional integer lattice then there are twelve possibilities to move another point. If we are trying to move forward then there are four possibilities to move (see figure 2).

Now we will construct the planes and paths to develop our main algorithm. Before that let us
write the mathematical expression, which is,

\[ A = \{ p : p = a^2 + b^2 + c^2, p \leq R, a, b, c \neq 0 \} \]
\[ A = \{ p : c^2 - p = a^2 + b^2, p \leq R, a, b, c \neq 0 \} \]
\[ R - p = a^2 + b^2, p \leq R. \]
\[ R = a^2 + b^2 + p, R \approx 10^A. \]

We are going to develop an algorithm to search the primes in the three-dimensional plane and it will be easy and more efficient technique if we can reduce computation. If we have got a set of primes then we will connect it by a path and then we will construct the planes with respect to these paths which will help us to find the other paths and to cover all the three-dimensional primes up to the \( 10^A \) for some sufficiently large \( A \).

[Note that, we have considered a sphere with the radius \( 10^A \), for some sufficiently large \( A \) and trying to find the distribution primes inside this sphere.]

Previously, we said that we will consider the plane \( x = y = z \) and will start the prime searching from it. It is clear that 3 is the only prime on the plane \( x = y = z \) (see remark \([1]\)). First, we will find the primes which are near to the plane \( x = y = z \) and then we will connect those primes and construct a path. After constructing that path we will start to find the primes which lie near to that path and then we will connect them again and continue the process like this. But the paths are not so smooth and we want to reduce computation. That’s why we will construct the planes which make easy to get the next path.

Before constructing the planes we need to define the paths.

**Definition 3.** A path \( P_{(n, q_1)} \) is a set of all three dimensional primes which holds the property:

\[ P_{(n, q_1)} = \{ p_{(n, c, q_1)} : \text{all the primes } p \text{ lies on the plane } q_1 \text{ and in the line } z = c \text{ and close to the plane } x = y = z \}. \]

where the value of \( c \) varies.

[Note: For the \( n \)-th prime on the plane \( q_1 \) lies on the line \( z = c \) and in the path \( P_{(n, q_1)} \) is denoted by \( p_{(n, c, q_1)} \).]

By definition \([3]\) it is clear that path is nothing but a set of primes in the line \( y = c \), on the plane \( q_1 \), where the value of \( c \) varies. As we know the path is not so smooth so we will find the primes which are the furthest primes from the plane \( x = y = z \) but lies on the path \( P_1 \) and join them. For the starting of this plane construction, we will start from the origin. So, it is clear that we can find the equation of that plane easily from the high school algebra because we know the co-ordinate of that point and starting from the origin. Likewise, we will continue the construction process. The motivation of this plane construction is to find the paths easily. According to the construction method, we can say that the initial plane is the plane \( x = y = z \) with respect to which we are constructing our path \( P_1 \) and we can call it by \( Q_0 \). Similarly, we will construct the plane \( Q_1 \) by talking the primes from the path \( P_1 \) which are the furthest from the plane \( Q_0 \). Likewise, we will get our second path \( P_2 \) with respect to the line \( Q_1 \) and then we construct the plane \( Q_2 \) with respect to the path \( P_2 \) and continue the process.

After continuing this process in this manner we will have a sequence of planes \( Q_0, Q_1, \ldots, Q_n \) for some \( n \in \mathbb{N} \). We continue until we can cover all the primes up to the end of the big sphere with
radius $10^A$. Clearly, $A$ is large but finite so the value of $n$ is also finite. We have finished the plane construction, now let us describe the main algorithm and its logic.

We start from the point $(1, 1, 1)$ because it is the smallest three-dimensional prime. Now we take the sphere with the radius $O((\log p_{(n,c,p_1)})^2)$ (where $n$ and $c$ equal to 1 and we are on the plane $P_1$ and in the line $z = 1$). This is the first sphere in the line $z = c$ on the plane $P_1$, it is denoted by $S_{(1,c,P_1)}$. So the $n$-th sphere in the line $y = c$ on the plane $P_1$ is denoted by $S_{(n,c,P_1)}$. We choose the radius of the sphere from the Cramér’s estimate for the difference between consecutive prime numbers.

Our aim is to move forward. So, we will consider the rightmost quadrant of the sphere. The plane $Q_0$ will cut this sphere, we will not check the primes in the upper side of that cut. Because it is on the upper side of the plane $x = y = z$, so we need not think about it. We have already said that why it is enough to analyze for the lower half portion of the plane $x = y = z$. We have the area in the sphere bounded by the plane $x = y = z$ and the arc of the sphere.

We will check the primes inside this area. If there are primes then choose the nearest prime and move to it. If there is no prime then choose a tube with the same radius of the sphere $S_{(n,c,P_1)}$ and check the primes inside the upper right area of the tube bounded by the plane $x = y = z$ and the arc of the tube. If there are primes then choose the nearest one and move to it. If there are no primes then choose another tube. Note that when we have got a prime and moved to it then we will choose another sphere $S_{(n+1,c',P_1)}$ (say) (where $c' > c$ for some $c$ and $c'$) and continue the process.

Then we check the primes inside that considered area of the sphere and choose the nearest prime from that. Then we move to that prime and continue the process. Continuing this process we will
be able to cover all the primes inside the sphere \( x^2 + y^2 + z^2 = R^2 \) where \( R = 10^A \) for sufficiently large \( A \).

**Theorem 2.** We can cover all the three-dimensional prime inside the sphere \( x^2 + y^2 + z^2 = R^2 \) where \( R = 10^A \) for sufficiently large \( A \).

**Proof:** We have started from the prime \( p_{(1,c,\mathcal{P}_1)} \) and proceed by considering the sphere \( S_{(n,c,\mathcal{P}_1)} \) and continuing the process as described in the algorithm. We have got the three-dimensional primes near to the plane \( x = y = z \) and will get the path \( P_1 \) by connecting them. We have described the plane construction with respect to the path \( P_1 \). With the help of this plane, we will get our second path i.e., \( P_2 \).

Likewise, we will get a sequence of paths, \( \{ P_1, P_2, \ldots, P_n \} \) with respect to the planes \( \{ Q_0, Q_1, \ldots, Q_{n-1} \} \). As we have said that we don’t need to calculate the area of the sphere which is on the upper side of the plane \( x = y = z \) for the path \( P_1 \). Similarly, we do not need to consider the area of the sphere which is in the upper side of the plane \( Q_{n-1} \) for the path \( P_n \) (for all \( n \geq 0 \)). The primes on the plane \( Q_n \) has already been calculated and we will continue the process until we rich the end of the sphere \( x^2 + y^2 + z^2 = R^2 \). At last, we will take the union of all the paths. It is clear that we have covered all the primes inside the sphere \( x^2 + y^2 + z^2 = R^2 \) where \( R = 10^A \) for sufficiently large \( A \).

**Remark 1.** If a point is on the plane \( x = y = z \) then we can write the point is \( (a, a, a) \) for some values of \( a \). Then \( |a^2 + a^2 + a^2| = 3a^2 \), clearly which is not a prime except the case that \( a = 1 \). So, 3 is the only prime on the plane \( x = y = z \).

## 5 The Algorithm

In this section, we will write the algorithm. In the previous section, we have described the algorithm in details. Also mentioned that this algorithm is nothing but an extension of the Gaussian Prime Algorithm done by M. Das [1]. Let’s write the main algorithm for three-dimensional prime.

**Algorithm 1:** Algorithm for the path \( P_1 \) in three dimensions

1. Select the starting point \( p_{(1,c,\mathcal{P}_1)} \) (say) (where \( c \in \mathbb{N} \)) and \( p_{(1,c,\mathcal{P}_1)} \) lies in the line \( z = c \) on the plane \( \mathcal{P}_1 \).
2. Let \( |p_{(1,c,\mathcal{P}_1)}| = |a^2 + b^2 + c^2| \). Take the length \( O \left( (\log p_{(1,c,\mathcal{P}_1)})^2 \right) \).
3. Consider the sphere \( S_{(1,c,\mathcal{P}_1)} \) with the radius \( r_{(1,c,\mathcal{P}_1)} = O \left( (\log p_{(1,c,\mathcal{P}_1)})^2 \right) \).
4. Consider the upper-right one quartet part of the sphere.
5. Select the considered area below the plane \( Q_0 \) (i.e., \( x = y = z \)).
6. Check the primes inside the selected area.
7. if there are primes then
8. select the nearest prime from the point \( p_{(1,c,\mathcal{P}_1)} \);
9. else
10. Take the tube with the inner radius \( r_{(1,c,\mathcal{P}_1)} \) and check the primes inside it;
11. end
12. Choose the nearest prime.
13. Continue the process.
Algorithm 2: Algorithm for the path $P_k$, for all $k \geq 2$ in three dimensions

1. Select the starting point $p(k,c,P_1)$ (say) (where $c \in \mathbb{N}$) and $p(1,c,P_1)$ lies in the line $z = c$ on the plane $P_1$.
2. Let $|p(k,c,P_1)| = |a^2 + b^2 + c'^2|$. Take the length $O((\log p(k,c,P_1))^2)$.
3. Consider the sphere $S(k,c,P_1)$ with the radius $r(k,c,P_1) = O((\log p(k,c,P_1))^2)$.
4. Consider the upper-right one quarter part of the sphere.
5. Select the considered area below the plane $Q_{k-1}$ (i.e., $x = y = z$).
6. Check the primes inside the selected area.
7. if there are primes then
   8. select the nearest prime from the point $p(k,c,P_1)$;
   9. else
      10. Take the tube with the inner radius $r(k,c,P_1)$ and check the primes inside it;
11. end
12. Choose the nearest prime.
13. Continue the process.

6 Generalization

We have stated the Waring problem, it says that every natural number is the sum of at most 4 squares, 9 cubes, or 19 fourth powers. From this statement, we have the Lagrange’s four-square theorem which says that every natural number can be written as the sum of four squares. These results ensure that it is not possible to generalize this idea for the higher dimensions.

For the four-dimensional plane, we can write all the primes as the sum of four squares. So, this problem may not be so interesting but we can extend this algorithm for four dimensions. Instead of taking sphere we will take the four-dimensional balls and instead of taking two-dimensional planes we will take the three-dimensional planes. The distribution will be more irregular than the three-dimensional plane but this problem may not be so interesting.

7 Conclusion

In this paper, we have developed the algorithm to search the three-dimensional prime. We have shown that there exist infinitely many three dimensional primes using Dirichlet’s theorem.

We have used the properties of a three-dimensional integer lattice to develop the algorithm. Also, we have shown that we can cover all the three-dimensional primes inside the sphere with radius $10^A$ for the sufficiently large value of $A$. This algorithm is an extension of the work done by M. Das [1].

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11