On the structure and spectrum of classical two-dimensional clusters
with a logarithmic interaction potential

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Abstract

We present a numerical study of the effect of the repulsive logarithmic inter-particle interaction on the ground state configuration and the frequency spectrum of a confined classical two-dimensional cluster containing a finite number of particles. In the case of a hard wall confinement all particles form one ring situated at the boundary of the potential. For a general \( r^n \) confinement potential, also inner rings can form and we find that all frequencies lie below the frequency of a particular mode, namely the breathing-like mode. An interesting situation arises for the parabolic confined system (i.e. \( n = 2 \)). In this case the frequency of the breathing mode is independent of the number of particles leading to an upper bound for all frequencies. All results can be understood from Earnshaw’s theorem in two dimensions. In order to check the sensitivity of these results, the spectrum of vortices in a type II superconductor which, in the limit of large penetration depths, interact through a logarithmic potential, is investigated.

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I. INTRODUCTION

In recent years, classical clusters with a finite number of particles moving in a two-dimensional plane were studied for different kinds of confinement and interaction potentials. Bedanov et al. [1, 2] studied the classical system with a finite number of particles interacting through a repulsive $1/r$ potential and moving in a two-dimensional plane, confined by a parabolic potential. The $1/r$ potential is the Coulomb potential in a three-dimensional world. This theoretical system models experimental realizations, such as electrons on the surface of liquid helium [3], electrons in quantum dots in high magnetic fields [4], colloidal suspensions [5] and confined plasma crystals [6]. A detailed investigation of the structure [1] and the spectral properties [2] of these 2D clusters was carried out. Recently, ground state and metastable configurations and/or spectral properties of 2D clusters with a parabolic confinement potential but different interaction potentials were investigated (see Ref. [1, 2, 7, 8, 9, 10] for a $1/r$ potential, Ref. [10, 11] for a dipole interaction, Ref. [10, 11] for a logarithmic potential, Ref. [10, 12] for screened Coulomb interaction, ...) and with a $1/r$ Coulomb interaction potential but with different confinement potentials (see Ref. [13] for a Coulomb confinement potential, Ref. [14] for a hard wall potential).

Laughlin [15] has shown that for strongly interacting electrons in a magnetic field at certain fractional fillings ($1/3, 1/5, \ldots$) of the Landau levels, the maxima in the pair correlation function correspond to the equilibrium positions in a classical one dimensional plasma where the electrons interact via a repulsive logarithmic potential. The idea was extended to quantum dots [16] with few electrons as well as bosons and to rings [17] with a few electrons. It was shown that the electrons actually get localized at the positions where the pair correlation function peaks and the excitations of the system can be understood as rotational modes of the center of mass and vibrational modes. The eigenenergies of the system can be easily determined from the frequencies of the classical normal modes to a high degree of accuracy.

In this paper we consider systems with various confinement potentials containing particles interacting through a repulsive \textit{logarithmic} interaction. As the solution of the 2D Poisson equation yields a logarithmic potential, one can say that the Coulomb potential in a 2D world is logarithmic. Vortices in a film of liquid Helium interact through a logarithmic potential [18]. Also vortices in a type II superconducting 2D film, for a low concentration
of vortices, is expected to interact through a logarithmic potential \[19\]. The logarithmic interaction between vortices was used to study the stable vortex configurations in a disk shaped superconductor \[20\]. Since the vortices are infinitely long in the \( z \)-direction, the symmetry makes it effectively 2D. Hence the model may have some experimental realizations and it may be possible to verify experimentally the predictions made here.

In Section II we describe our model systems. The results for the ground state configurations are shown in Section III, and for the spectra in Section IV. Section V tests the sensitivity of the results by looking at the spectrum of vortices in a type II superconductor, which interact through a logarithmic potential for a large penetration depth. Our conclusions are given in Section V.

II. MODEL AND NUMERICAL APPROACH

The Hamiltonian of a 2D system of \( N \) charged particles in a \( r^n \) confinement potential and interacting through a repulsive logarithmic potential is given by

\[
H = \sum_{i=1}^{N} \frac{1}{2}m\omega_0^2 R^2 \left( \frac{r_i}{R} \right)^n + \sum_{i>j}^{N} V(|\vec{r}_i - \vec{r}_j|),
\]

where \( m \) is the mass of the particle, \( \omega_0 \) the radial confinement frequency, and \( \vec{r}_i = (x_i, y_i) \) the position of the \( i \)th particle with \( r_i \equiv |\vec{r}_i| \). A hard wall confinement is obtained for \( n \to \infty \) and in this case \( R \) equals the radius of the hard wall. The interaction potential is taken to be logarithmic:

\[
V(r) = -\beta \ln(r/R).
\]

We can write the Hamiltonian in a dimensionless form if we express the coordinates and energy in the following units: \( r' = \beta^{1/n} \alpha^{-1/n} R^{(n-2)/n}, E' = \beta \), with \( \alpha = \frac{1}{2}m\omega_0^2 \). The dimensionless Hamiltonian is given by

\[
H = \sum_{i=1}^{N} r'^n_i - \sum_{i>j}^{N} \ln |\vec{r}_i - \vec{r}_j|.
\]

In the limit of a hard wall confinement, the lengthunit becomes \( r' \to R \).

We also present results for the actual interaction potential for vortices in a type II superconductor for \( \lambda \to \infty \) as well as for smaller values of \( \lambda \) (with \( \lambda \) the penetration depth).
Vortices in a type II superconductor interact with a potential \[ V_s(r) = \beta K_0 \left( \frac{r}{\lambda} \right), \] (4)

with \( K_0 \) the zero-order Hankel function with imaginary argument, \( \beta = \Phi_0/(2\pi\lambda^2) \) and \( \Phi_0 = hc/2e \) the flux quantum. In the limit of \( \lambda \to \infty \), this potential becomes logarithmic:\[ V_s(r) \to -\beta \ln \lambda \text{ for } \lambda \to \infty \text{ (up to a constant)}. \] (5)

With the above units, the dimensionless form of this interaction potential is
\[ V_s(r) = K_0 \left( \frac{r}{\lambda} \right), \] (6)

where now also \( \lambda \) is expressed in the unit \( r' \).

The numerical method to obtain the ground state configuration is based on the Monte Carlo simulation technique supplemented with the Newton method in order to increase the accuracy of the energy. The latter technique is outlined and compared with the Monte Carlo technique in Ref. \[2\]. The eigenmode frequencies are obtained from the eigenvalues of the dynamical matrix
\[ E_{\alpha\beta,ij} = \frac{\partial^2 E}{\partial r_{\alpha,i} \partial r_{\beta,j}} \bigg|_{r_{\alpha,i} = r_{\alpha,i}^n}, \] (7)

where \( \{r_{\alpha,i}^n\} \) is the ground state configuration. The eigenvalues of the dynamical matrix are the squared eigenfrequencies of the system. The eigenfrequencies are expressed in the unit \( \omega' = \sqrt{E'/r'^2/m} \).

**III. GROUND STATE CONFIGURATIONS**

In this section we will discuss the numerically obtained ground state configurations for different powers of the confinement potential and show how they can be understood from Earnshaw’s theorem. Fig. [II] shows as an example the ground state configurations for 40 particles for (a) a hard wall confinement (i.e. \( n \to \infty \)), (b) a confinement with \( n = 3 \) and finally (c) a parabolic confinement (i.e. \( n = 2 \)). One can see that in the case of hard wall confinement all particles are situated on one ring at the boundary of the potential. This is true for any number of particles in the system, also for clusters with many particles. When the confinement differs from the hard wall confinement, particles can be situated in the central region. Notice that the density of particles for \( n = 3 \) is much larger at the edge than...
FIG. 1: Ground state configurations for 40 particles interacting through a logarithmic potential for (a) a hard wall confinement, (b) a confinement with \( n = 3 \) and (c) for a parabolic confinement. The scale is different in each figure, but the distance between the ticks is always one length unit.

FIG. 2: The eigenfrequencies for 2 up to 50 particles interacting through a logarithmic potential for (a) a \( n = 3 \) confinement potential and (b) a parabolic confinement potential.

in the central region, while for the parabolic confinement the density is uniform. These observations can be understood from Earnshaw’s theorem \[22\] in two dimensions. Since in the present case the inter-particle potential is logarithmic we can apply the 2D version of Gauss’s theorem

\[
\oint_C \mathbf{E} \cdot d\mathbf{r} \sim \int_S \rho(\mathbf{r}) dS, \tag{8}
\]

where \( S \) is the surface enclosed by \( C \), and \( \rho(\mathbf{r}) \) is the enclosed charge. Now it is easy to show that charged particles on a 2D surface can not have a stable static equilibrium when
they interact through a logarithmic potential (i.e. the 2D Coulomb potential). If we take a small circle anywhere in the 2D region with no enclosed charges, then, since the line integral of the field along the circle has to be zero, the number of outgoing lines of force are equal to the number of incoming ones. As long as there are outgoing lines of force, a charge left inside this circle can lower its energy by moving along it. Thus in case of hard wall confinement, particles can not be in a stable static equilibrium if they are not in contact with the hard wall potential. Our simulations indeed show that all particles are pushed on one ring situated at the boundary of the hard wall. This is different from the system with a hard wall confinement in which the particles interact through a $1/r$ potential. In Ref. [1] it was shown that stable multiple ring structures can be found if a sufficiently large number of particles are present in the 2D system.

As shown above the logarithmic interaction potential alone can not result in a stable multi-ring configuration. It is only possible if the hard wall confinement is changed into some soft wall $r^n$ confinement potential (as shown in Figs. 1(b) and (c)). In this case, when a particle in the central region is moved in the outward radial direction, the restoring force is given by the confinement potential alone. This must balance the nonrestoring forces provided by the Coulomb repulsion with the other particles. For the parabolic confinement case, this results in a uniform density as the restoring force is everywhere the same.

IV. THE EIGENMODE SPECTRUM

Also the eigenmode spectrum for systems with a logarithmic interaction potential differs from all the previously studied cases. Again the results can be understood from Earnshaw’s
theorem. The eigenfrequencies of the system with logarithmic interaction for a $n = 3$ confinement potential are, as an example for a soft wall confinement, shown in Fig. 2(a) for $N = 2$ up to 50. Notice that there is a pronounced highest frequency branch. We checked the corresponding eigenmodes and they all show the same behaviour: all outer particles move in the radial direction and have a large amplitude. An example of these breathing-like modes is shown in Fig. 3(a) for $N = 40$. It is interesting to look now at the eigenfrequencies in the case of a parabolic confinement potential, which is shown in Fig. 2(b). Again the mode corresponding to the highest frequency is the breathing mode (shown in Fig. 3(b) for $N = 40$). But, as the frequency of the breathing mode in the parabolic case is independent of the number of particles (and given by $\omega_{\text{breathing}} = \omega_{\text{max}} = 2^{[23]}$), there is an upper bound for the frequencies.

For confinement potentials with $n > 2$ the potential is steeper than the parabolic confinement at large distances from the center. So for the confinement with $n > 2$ a mode with larger velocities at the boundary than at the center overtakes the breathing mode to be the highest mode. But in all these modes there is a simultaneous contraction of a large number of particles so we can speak of breathing-like modes.

Normal modes are effectively a motion of free particles moving in a resultant potential created by the other particles and the confinement potential. So whenever we try to move a particle from its static equilibrium position, there are restoring forces that bring it back. Then, addition of extra particles complicates the effective potential in which the individual particles are moving and creates higher normal modes. This potential can become steeper and steeper as more particles are added. This results in higher and higher normal modes. But this argument breaks down for the logarithmic interaction for which Gauss’s theorem can be applied: as already shown in previous section, the confinement potential is the only restoring force in the outward radial direction and forms an upper bound to the restoring force. Addition of new particles does not create a more complicated effective potential but it can only result in a slight increase in the frequency of the breathing-like mode due to the increase of the nonrestoring Coulomb force on the outer particles.
V. VORTICES IN SUPERCONDUCTORS

As mentioned in the Introduction, the logarithmic potential is sometimes a good model for the interaction between vortices in a type II superconductor. In order to check how important small deviations from this logarithmic potential are, we also studied the spectrum of vortices in a type II superconductor for a more generally valid interaction potential. In Ref. [21] it
FIG. 6: The eigenmodes corresponding to highest frequency for $N = 20$ for a) $\lambda = 1$, b) $\lambda = 10$ and c) $\lambda = 50$, and corresponding to the highest but one frequency for d) $\lambda = 1$, e) $\lambda = 10$ and f) $\lambda = 50$.

is shown that the interaction potential between vortices in a type II superconductor is given by Eq. (4) if the penetration length $\lambda \gg$ the coherence length. In the limit of $\lambda \to \infty$, this potential reduces to the logarithmic potential studied in the previous sections.

The spectrum for the parabolically confined system is shown in Fig. 4 for penetration lengths $\lambda = 1, 10$ and 50. One can clearly see that for $\lambda = 1$ no upper bound frequency is found, while it is for $\lambda = 50$. That the logarithmic limit is not at all reached for $\lambda = 1$ is also seen in Fig. 6 which shows the eigenmode corresponding to the largest frequency for $N = 10$. As shown above, this eigenmode corresponds with the breathing mode in case of a logarithmic interaction. This is not at all the case for $\lambda = 1$ and 10 (see Figs. 5(a) and (b)), but for $\lambda = 50$ it does. Therefore, transition to the logarithmic-type of interaction takes place between $\lambda = 10$ and 50.

It is also interesting to look at the eigenmodes corresponding to the highest frequency for $N = 20$ (see Figs. 6(a-c)). Now the modes for $\lambda = 1, 10$ and 50 look all similar. However, they differ from the breathing mode: the middle and outer ring move in the opposite direction. From the spectrum one can observe that for $N > 10$ there are two highest frequencies which lie very close together. These highest but one frequencies are therefore plot in Fig. 6(d-f).
Notice that for \( \lambda = 10 \) and 50 this mode indeed corresponds with the breathing mode.

VI. CONCLUSIONS

We investigated the ground state structure and the spectral properties of a classical 2D cluster with a finite number of particles interacting through a repulsive logarithmic potential. The logarithmic potential is the Coulomb potential in a 2D world. It was shown that for a hard wall confinement all particles are situated on one ring at the boundary of the potential, as a consequence of Earnshaw’s theorem in 2D. Multi-ring configurations exist for soft confinement potentials in which case only the external confinement delivers the restoring force. The particles are uniformly distributed for a parabolic confinement.

We also found that, independently of the number of particles in the cluster, all eigen-frequencies lie below the frequency of a particular mode, namely the breathing-like mode. This is again a consequence of Earnshaw’s theorem because adding extra particles does not create extra minima in the potential nor makes it more complex. In the case of a parabolic confinement the frequency of this breathing mode is independent from \( N \), which results in an upper bound for the frequencies.

It is also shown that for vortices in a type II superconductor (for which the penetration length \( \lambda \gg \) coherence length) the transition to the logarithmic-type of interaction takes place between \( 10 < \lambda < 50 \). It is not always the highest mode anymore which corresponds with the breathing mode, it can also be the highest but one mode.

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