Revisiting the $U_A(1)$ problems

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Abstract

We survey various $U_A(1)$ problems and attempt to resolve the two puzzles related to the eta mesons that have experimental verification. Specifically, we first explore the Goldstone structure of the $\eta$ and $\eta'$ mesons in the context of $\eta-\eta'$ mixing using ideas based on QCD. Then we study the eta decays $\eta \rightarrow 3\pi^0$, $\eta' \rightarrow 3\pi^0$ and $\eta' \rightarrow \eta\pi\pi$. Finally we arrive at essentially the same picture in the dynamical scheme based on consistently coupled Schwinger-Dyson and Bethe-Salpeter integral equations. This chirally well-behaved bound-state approach clarifies the distinction between the usual axial-current decay constants and the $\gamma\gamma$ decay constants in the $\eta-\eta'$ complex. Allowing for the effects of the SU(3) flavor symmetry breaking in the quark–antiquark annihilation, leads to the improved $\eta-\eta'$ mass matrix.

11.30.Rd, 11.40.-q, 11.40.Ha, 11.10.St

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I. INTRODUCTION

Various statements of “UA(1) problems” related to the eta mesons and their possible resolutions have appeared in the literature now for almost three decades. Considerations of the eta UA(1) vacuum Ward identity were discussed by Glashow [1], Weinberg [2], Crewther [3] and collaborators. The UA(1) axial current and its anomalous addition were studied by Kogut and Susskind [4]. Semiclassical instantons with topological winding number were used by ’t Hooft [5]. Lastly, the large $N_c$ limit together with the $\theta$ vacuum were explored by Witten [6]. All of the above notions were invoked to resolve the UA(1) problem. These above UA(1) problems have recently been rekindled by ’t Hooft in his text [7].

We prefer to focus on two UA(1)-type problems that have empirical resolutions and which also have a theoretical basis:

1. Goldstone boson structure of the observed $\eta(547)$ and $\eta'(958)$ mesons via $\eta-\eta'$ mixing in the context of QCD.

2. Observed eta hadronic decay rates:
   (a) $\Gamma(\eta \rightarrow 3\pi^0) = 380 \pm 36$ eV [8] appears large since it should vanish by the Sutherland theorem [9], or be a factor of two smaller in the context of chiral perturbation theory [10].
   (b) $\Gamma(\eta' \rightarrow 3\pi^0) = 313 \pm 58$ eV [8] appears relatively suppressed because $\eta' \rightarrow 3\pi^0$ phase space is six times larger than for $\eta \rightarrow 3\pi^0$.
   (c) $\Gamma(\eta' \rightarrow \eta\pi\pi) = 131 \pm 8$ keV [8] is a strong decay, whereas the smaller $3\pi$ decays in 2a, 2b above change isospin by one unit and are non-strong decays proceeding through the quark mass difference $m_d - m_u$.
   (d) We invoke the $\Delta I = 1$ $u_3 = \bar{q}\lambda_3 q$ Coleman-Glashow (CG) [11] quark tadpole to support the current-current Sutherland [4] suppression of the $\eta \rightarrow 3\pi$ decay rates. The CG tadpole also explains all 13 hadron ($P, V, B, D$) SU(2) mass splittings [11,12]. Then we use PCAC Consistency [13] to compute the $\eta, \eta' \rightarrow 3\pi^0$ decay rates in 2a, 2b above.

The above problems are analyzed in Secs. II and III primarily on the basis of the input from meson phenomenology. However, the underlying notions of the quark model are also crucial in this analysis. Therefore, in Sec. IV we show the consistency of some of the results of Secs. II and III with a sophisticated quark model which has strong and clear connections with the fundamental theory – QCD. It is based on the so-called coupled Schwinger-Dyson (SD) and Bethe-Salpeter (BS) approach in which one, by solving the SD equation for dressed quark propagators of various flavors, explicitly constructs constituent quarks. They in turn build $q\bar{q}$ meson bound states which are solutions of the BS equation employing the dressed quark propagator obtained as the solution of the SD equation. If the SD and BS equations are so coupled in a consistent approximation, the light pseudoscalar mesons are simultaneously the $q\bar{q}$ bound states and the (quasi) Goldstone bosons of dynamical chiral symmetry breaking (D$\chi$SB). The resulting relativistically covariant constituent quark model (such as the variant of Ref. [14]) is consistent with current algebra because it incorporates the correct
chiral symmetry behavior thanks to $D\chi_{\text{SB}}$ obtained in an, essentially, Nambu–Jona-Lasinio fashion, but the former model interaction is less schematic. In Refs. [14–19] for example, it is combined nonperturbative and perturbative gluon exchange; the effective propagator function is the sum of the known perturbative QCD contribution and the modeled nonperturbative component. For details, we refer to Refs. [14–18], while here we just note that the momentum-dependent dynamically generated quark mass functions $M_f(q^2)$ (i.e., the quark propagator SD solutions for quark flavors $f$) illustrate well how the coupled SD-BS approach provides a modern constituent model which is consistent with perturbative and nonperturbative QCD. For example, the perturbative QCD part of the gluon propagator leads to the deep Euclidean behaviors of quark propagators (for all flavors) consistent with the asymptotic freedom of QCD [17]. However, what is important in the present paper, is the behavior of the same mass functions $M_f(q^2)$ for low momenta $[q^2 = 0$ to $-q^2 \approx (400 \text{ MeV})^2]$, where $M_f(q^2)$ (due to $D\chi_{\text{SB}}$) have values consistent with typical values of the constituent mass parameter in constituent quark models. For the (isosymmetric) $u$- and $d$-quarks, our concrete model choice [14] gives us $M_{u,d}(0) = 356$ MeV in the chiral limit (i.e., with vanishing $\tilde{m}_{u,d}$, the explicit chiral symmetry breaking bare mass term in the quark propagator SD equation, resulting in vanishing pion mass eigenvalue, $m_\pi = 0$, in the BS equation), and $M_{u,d}(0) = 375$ MeV [just 5% above $M_{u,d}(0)$ in the chiral limit] with the explicit chiral symmetry breaking bare mass $\tilde{m}_{u,d} = 3.1$ MeV, leading to a realistically light pion, $m_\pi = 140.4$ MeV. Similarly, for the $s$ quark, $M_s(0) = 610$ MeV. The simple-minded constituent mass parameters, denoted below by $\hat{m}$ in the case of the isosymmetric $u$ and $d$ quarks, and by $m_s$ in the case of the $s$ quarks, have therefore close analogues in the coupled SD-BS approach which explicitly incorporates some crucial features of QCD, notably $D\chi_{\text{SB}}$.

II. GOLDSTONE STRUCTURE OF ETA MESONS

To resolve $U_A(1)$ problem one, we invoke the $U(3)$ pseudoscalar nonet structure ($\pi, K, \eta, \eta'$) along with the Gell-Mann-Okubo mass formula $m_\pi^2 + 3m_\eta^2 = 4m_K^2$, requiring an octet eta mass $m_{\eta_8} \approx 567$ MeV. While this $\eta_8$ mass is presumed to vanish in the $SU(3) \times SU(3)$ chiral limit (CL), the companion singlet $\eta_0$ mass is not expected to vanish in the CL. Using the standard relation mixing $\eta, \eta'$ to $\eta_8, \eta_0$ away from the CL one knows

$$m_{\eta_8}^2 + m_{\eta_0}^2 = m_\eta^2 + m_{\eta'}^2 \approx 1.22 \text{ GeV}^2,$$

or $m_{\eta_0} \approx 947$ MeV

(1)

for masses $\eta(547), \eta'(958), \eta_8(567)$.

Here we assumed the standard, most traditional representation of the physical isoscalar pseudoscalars $\eta$ and $\eta'$ as the orthogonal mixture

$$|\eta\rangle = \cos \theta |\eta_8\rangle - \sin \theta |\eta_0\rangle,$$

(2a)

$$|\eta'\rangle = \sin \theta |\eta_8\rangle + \cos \theta |\eta_0\rangle,$$

(2b)

of the respective octet and singlet isospin zero states, $\eta_8$ and $\eta_0$. In the flavor $SU(3)$ quark model, they are defined through quark–antiquark ($q\bar{q}$) basis states $|f\bar{f}\rangle$ ($f = u, d, s$) as

$$|\eta_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle),$$

(3a)
\[ |\eta_0\rangle = \frac{1}{\sqrt{3}} (|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) . \quad (3b) \]

The exact SU(3) flavor symmetry \((m_u = m_d = m_s)\) is nevertheless badly broken. It is an excellent approximation to assume the exact isospin symmetry \((m_u = m_d)\), and a good approximation to take even the chiral symmetry limit \((m_u = m_d = 0)\) for \(u\) and \(d\)-quark, but for a realistic description, the strange quark mass \(m_s\) must be significantly heavier than \(m_u\) and \(m_d\). [In particular, the CL is obviously phenomenologically unrealistic in the strange sector, although it is qualitatively meaningful, and in fact useful as a theoretical limit in the discussions of the \(U_A(1)\) problem.] Thus, with \(|u\bar{u}\rangle\) and \(|d\bar{d}\rangle\) being practically chiral states as opposed to a significantly heavier \(|s\bar{s}\rangle\), Eqs. (3) do not define the octet and singlet states of the exact SU(3) flavor symmetry, but the effective octet and singlet states. Hence, as in Ref. [19] for example, only in the sense that the same \(q\bar{q}\) states \(|\bar{f}f\rangle\) \((f = u, d, s)\) appear in both Eq. (2a) and Eq. (2b) do these equations implicitly assume nonet symmetry (as pointed out by Gilman and Kauffman [20], following Chanowitz, their Ref. [8]). However, in order to avoid the \(U_A(1)\) problem, this symmetry must ultimately be broken at least at the level of the masses. In particular, it must be broken in such a way that \(\eta \rightarrow \eta_8\) becomes massless but \(\eta' \rightarrow \eta_0\) remains massive (as in Ref. [16]) when CL is taken for all three flavors, \(m_u, m_d, m_s \rightarrow 0\).

Alternatively, one can work in a nonstrange (NS)–strange (S) basis \(|\eta_{NS}\rangle\) and \(|\eta_S\rangle\), where

\[
|\eta_{NS}\rangle = \frac{1}{\sqrt{2}} (|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}} |\eta_8\rangle + \sqrt{\frac{2}{3}} |\eta_0\rangle , \quad (4a) \\
|\eta_S\rangle = |s\bar{s}\rangle = -\sqrt{\frac{2}{3}} |\eta_8\rangle + \frac{1}{\sqrt{3}} |\eta_0\rangle . \quad (4b)
\]

In analogy with Eq. (1), in this basis one finds

\[ m^2_{\eta_{NS}} + m^2_{\eta_S} = m^2_\eta + m^2_{\eta'} \approx 1.22 \text{ GeV}^2 , \quad (5) \]

since \(\langle \eta' | \eta \rangle = 0\) and since the NS–S mixing relations are

\[
|\eta\rangle = \cos \phi |\eta_{NS}\rangle - \sin \phi |\eta_S\rangle , \quad (6a) \\
|\eta'\rangle = \sin \phi |\eta_{NS}\rangle + \cos \phi |\eta_S\rangle . \quad (6b)
\]

The more familiar singlet-octet state mixing angle \(\theta\), defined by Eqs. (2), is geometrically related to the NS–S state mixing angle \(\phi\) above as [21]

\[ \theta = \phi - \arctan \sqrt{2} = \phi - 54.74^\circ . \quad (7) \]

Although mathematically equivalent to the \(\eta_8\)–\(\eta_0\) basis, the NS–S mixing basis is more suitable for some quark model considerations, being more natural in practice when the symmetry between the NS and S sectors is broken as described in the preceding passage. There is also another important reason to keep in mind the \(|\eta_{NS}\rangle$–\(|\eta_S\rangle$) state mixing angle \(\phi\). This is because it offers the quickest way to show the consistency of our procedures
and the corresponding results obtained using just one (θ or φ) state mixing angle, with the two-mixing-angle scheme considered in Refs. [22–28], which is defined with respect to the mixing of the decay constants. Namely, in the NS–S basis, the two decay-constant-mixing angles in the two-angle scheme (φ_q and φ_s) are very close to each other. This basis thus has the advantage that even in the case of the decay-constant-mixing, the reduction to just one mixing angle occurs in a good approximation and one can use just one mixing angle [24–26,28] around \( φ_q \approx \phi_s \). Moreover, phenomenology seems to justify the central assumption of Feldmann, Kroll and Stech (FKS) [26] that in the NS–S basis, the decay constants follow (in a good approximation) the pattern of particle state mixing, so that the NS–S state mixing angle \( \phi \approx \phi_q \approx \phi_s \). On the other hand, when we express our results in terms of \( \phi \) through the relation (7), which always holds since we use a state mixing angle, we find they are consistent with the FKS scheme. The application of the two-mixing-angle scheme is relegated to the Appendix, where we give our predictions for the \( \eta - \eta' \) decay constants, as well as \( \theta_8 \) and \( \theta_0 \), the two decay-constant-mixing angles (in the octet-singlet basis), in that scheme. Here, we simply note that our considerations will ultimately lead us to \( \phi \approx 42^\circ \), practically the same as the result of FKS scheme and theory (e.g., in Table 2 of Feldmann’s review [28]), and in agreement with data. World \( \eta - \eta' \) mixing angle data in 1989 led to

\[
\phi = 41^\circ \pm 2^\circ \quad \text{or} \quad \theta = -14^\circ \pm 2^\circ.
\]

(8)

A more recent detailed analysis [30] based on 1996 data for decays tensor to pseudoscalar \( T \to PP \), radiative vector to pseudoscalar (or vice versa) \( V \to P\gamma \), \( P \to V\gamma \), double radiative decays \( \eta \to \gamma\gamma \), \( \eta' \to \gamma\gamma \), and \( J/\psi \to VP \) decays (14 such decays) leads to the empirical \( \eta - \eta' \) mixing angles

\[
\phi = 43^\circ \pm 5^\circ \quad \text{or} \quad \phi = 42^\circ \pm 3^\circ
\]

(9)

found respectively from observed branching ratios, \( B(a_2 \to \eta\pi/K\overline{K}) = 2.96 \pm 0.53 \), \( B(a_2 \to \eta'\pi/\eta\pi) = 0.037 \pm 0.007 \), in complete agreement with (8). The \( \eta - \eta' \) mixing angles in (8) or (9) (for 4 of 14 determinations) depend on the constituent quark mass ratio \( m_s/\hat{m} \approx 1.45 \), as already found from baryon magnetic moments [31], meson charge radii [12] and \( K^* \to K\gamma \) decays [33]. (\( \hat{m} \) denotes the isosymmetric average mass \( m_{u,d} \).)

As for a theoretical determination of the \( \eta - \eta' \) mixing angle \( \phi \) or \( \theta = \phi - 54.74^\circ \), we follow the path of Refs. [21]. The contribution of the gluon axial anomaly to the singlet \( \eta_0 \) mass is essentially just parameterized and not really calculated, but some useful information can be obtained from the isoscalar \( q\bar{q} \) annihilation graphs of which the “diamond” one in Fig. 1 is just the simplest example. That is, we can take Fig. 1 in the nonperturbative sense, where the two-gluon intermediate “states” represent any even number of gluons when forming a \( C^+ \) pseudoscalar \( 7q \) meson [31], and where quarks, gluons and vertices can be dressed nonperturbatively, and possibly include gluon configurations such as instantons. Factorization of the quark propagators in Fig. 1 characterized by the ratio \( X \approx \hat{m}/m_s \) leads to the pseudoscalar mass matrix in the \( NS-S \) basis [21]

\[
\begin{pmatrix}
 m_\pi^2 + 2\beta & \sqrt{2}\beta X \\
 \sqrt{2}\beta X & 2m_K^2 - m_\pi^2 + \beta X^2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
 m_\eta^2 & 0 \\
 0 & m_{\eta'}^2
\end{pmatrix},
\]

(10)
where $\beta$ denotes the total annihilation strength of the pseudoscalar $q\bar{q}$ for the light flavors $f = u, d$, whereas it is assumed attenuated by a factor $X$ when a $s\bar{s}$ pseudoscalar appears. (The mass matrix in the $\eta_8$-$\eta_0$ basis reveals that in the $X \to 1$ limit, the CL-nonvanishing singlet $\eta_0$ mass is given by $3\beta$.) The two parameters on the left-hand-side (LHS) of (11), $\beta$ and $X$, are determined by the two diagonalized $\eta$ and $\eta'$ masses on the RHS of (10). The trace and determinant of the matrices in (11) then fix $\beta$ and $X$ to be [21]

$$\beta = \frac{(m_{\eta'}^2 - m_0^2)(m_0^2 - m_\pi^2)}{4(m_K^2 - m_\pi^2)} \approx 0.28 \text{ GeV}^2 \ , \ X \approx 0.78 \ , \ (11)$$

with the latter value suggesting a constituent quark mass ratio $X^{-1} = m_s/m_u \approx 1.3$, near the values in Refs. [29-33], $m_s/m_u \approx 1.45$.

This fitted nonperturbative scale of $\beta$ in (11) depends only on the gross features of QCD. If instead one treats the QCD graph of Fig. 1 in the perturbative sense of literally two gluons exchanged, then one obtains [33] only $\beta_{2g} \approx 0.09 \text{ GeV}^2$, which is about 1/3 of the needed scale of $\beta$ found in (11). (This indicates that just the perturbative “diamond” graph can hardly represent even the roughest approximation to the effect of the gluon axial anomaly operator $\epsilon a_\beta \epsilon \alpha \beta \epsilon a_{\mu \nu}$.) The above fitted quark annihilation (nonperturbative) scale $\beta$ in (11) can be converted to the NS–S $\eta$–$\eta'$ mixing angle $\phi$ in [3] from the alternative mixing relation

$$\phi = \arctan \left[ \frac{(m_{\eta'}^2 - 2m_K^2 + m_\pi^2)(m_0^2 - m_\pi^2)}{(2m_K^2 - m_\pi^2)(m_{\eta'}^2 - m_\pi^2)} \right]^{1/2} \approx 41.9^\circ \ . \ (12)$$

This kinematical QCD mixing angle (12) or $\theta = \phi - 54.74^\circ \approx -12.8^\circ$ has dynamical analogs [16], namely the coupled SD-BS approach mentioned in the Introduction and used in Sec. IV below. Since this predicted $\eta$–$\eta'$ mixing angle in (12) is compatible with the empirical values in (5) and (7), we use (12) in the mixing angle relations (6) to infer the nonstrange and strange $\eta$ masses,

$$m_{\eta NS}^2 = \cos^2 \phi \ m_\eta^2 + \sin^2 \phi \ m_{\eta'}^2 \approx (758 \text{ MeV})^2 \quad (13a)$$

$$m_{\eta S}^2 = \sin^2 \phi \ m_\eta^2 + \cos^2 \phi \ m_{\eta'}^2 \approx (801 \text{ MeV})^2 \ . \ (13b)$$

Thus it is clear that the true physical masses $\eta(547)$ and $\eta'(958)$ are respectively much closer to the Nambu-Goldstone (NG) octet $\eta_8(567)$ and the non-NG singlet $\eta_0(947)$ configurations than to the nonstrange $\eta_{NS}(758)$ and strange $\eta_{S}(801)$ configurations inferred in Eqs. (13). However, the mean $\eta$–$\eta'$ mass $(548 + 958)/2 \approx 753$ MeV is quite near the nonstrange $\eta_{NS}(758)$. But since $\eta_8(567)$ appears far from the NG massless limit we must ask: how close is $\eta_0(947)$ to the chiral-limiting nonvanishing singlet $\eta$ mass?

To answer this latter question, return to Fig. 1 and the quark annihilation strength $\beta \approx 0.28 \text{ GeV}^2$ in Eq. (11). These $\bar{q}q$ states presumably hadronize into the $U_A(1)$ singlet state $|\eta_0\rangle = |\bar{u}u + \bar{d}d + \bar{s}s\rangle/\sqrt{3}$, for effective squared mass in the SU(3) limit with $\beta$ remaining unchanged [34]:

$$m_{\eta_0}^2 = 3\beta \approx (917 \text{ MeV})^2 \ . \ (14)$$
This latter CL $\eta_0$ mass in (14) is only 3% shy of the exact chiral-broken $\eta_0(947)$ mass found in Eq. (1). (Such a 3% CL reduction also holds for the pion decay constant $f_\pi \approx 93$ MeV $\to 90$ MeV [33] and for $f_+(0) = 1 \to 0.97$ [36], the $K-\pi$ $K_{l3}$ form factor.)

Thus this $\eta-\eta'$ mixing resolution of the first $U_A(1)$ problem is that the physical $\eta(547)$ is 97% of the chiral-broken NG boson $\eta_8(567)$. Also the mixing-induced CL singlet mass of 917 MeV in (14) is 97% of the chiral-broken singlet $\eta_0(947)$ in (1), which in turn is 99% of the physical $\eta'$ mass $\eta'(958)$. This speaks to Weinberg’s question [2] as to why there is no isoscalar, pseudoscalar Goldstone boson (with mass less than about $\sqrt{3m_\pi} \sim 240$ MeV), associated with the spontaneous breakdown of the axial $U_A(1)$ symmetry.

III. HADRONIC ETA DECAYS AND THE $U_A(1)$ PROBLEM

As for the second $U_A(1)$ problem, Weinberg in [3] correctly identified the rapidly varying $\eta$ and $\pi^0$ poles for $\eta \to 3\pi^0$ decay. However, one must also fold in the PCAC consistency approach of Refs. [13,37] leading to the $\eta \to 3\pi^0$ amplitude magnitude with $f_\pi \approx 93$ MeV,

$$\left| \left\langle 3\pi^0 \right| H_{em} \left| \eta \right\rangle \right| = \left(3/2f_\pi^2\right) \left| \left\langle \pi^0 \right| H_{em} \left| \eta \right\rangle \right| + O(m_\pi^2/m_\eta^2) . \quad (15a)$$

Here the factor of $3/f_\pi^2$ on the RHS of (15a) corresponds to the three successive double pion PCAC reductions, while the factor of $1/2$ characterizes Weinberg’s [2] rapidly varying $\eta$ and $\pi^0$ pole terms. Also this $\Delta I = 1$ $\eta \to \pi$ non-strong transition in (15a) reduces to [12,13]

$$\left| \left\langle \pi^0 \right| H_{em} \left| \eta \right\rangle \right| = \cos \phi \left| \left\langle \pi^0 \right| u_3 \left| \eta_{NS} \right\rangle \right| = \cos 42^\circ (\Delta m_K^2 - \Delta m_{\pi}^2) \approx -3900$ MeV$^2$. \quad (15b)$$

In (15b) we have invoked the CG $u_3 = \bar{q}\lambda_3 q$ quark tadpole (which is known [11,12] to explain all $P, V, B, D$ hadron SU(2) electromagnetic (em) mass splittings) using the SU(3) form

$$\left| \left\langle \pi^0 \right| u_3 \left| \eta_{NS} \right\rangle \right| = \Delta m_K^2 - \Delta m_{\pi}^2 \approx -0.0052$ GeV$^2$, where $\Delta m_K^2 = m_{K^+}^2 - m_{K^0}^2$, etc. Also in (15b) we have again invoked the $\eta-\eta'$ mixing relations (16a) with mixing angle predicted by (12).

Substituting (15b) into (15a), one obtains the $\eta \to 3\pi^0$ amplitude

$$\left| \left\langle 3\pi^0 \right| H_{em} \left| \eta \right\rangle \right| = \left(3/2f_\pi^2\right) \left| \left\langle \pi^0 \right| H_{em} \left| \eta \right\rangle \right| \approx 0.68 . \quad (16a)$$

As for the experimental $\eta_{3\pi^0}$ decay amplitude, taking a constant matrix element (16a) integrated over the Dalitz plot, one predicts an $\eta \to 3\pi^0$ decay rate

$$\Gamma(\eta_{3\pi^0}) = (816$ eV) $\left| \left\langle 3\pi^0 \right| H_{em} \left| \eta \right\rangle \right|^2 \approx 377$ eV . \quad (16b)$$

The latter almost perfectly matches the 1998 PDG [4] rate of $380 \pm 36$ eV at the central value.

Alternatively we can extract the constant effective constant 3-body matrix elements $A_a, A_b, A_c$ from data [3]

$$\Gamma(\eta \to 3\pi^0) \approx 0.82 \ |A_a|^2$ keV $\approx 0.38$ keV, \quad (17a)$$

$$\Gamma(\eta' \to 3\pi^0) \approx 5.58 \ |A_b|^2$ keV $\approx 0.31$ keV, \quad (17b)$$

$$\Gamma(\eta' \to \eta\pi^0\pi^0) \approx 1.06 \ |A_c|^2$ keV $\approx 42$ keV, \quad (17c)$$
leading to the dimensionless 3-body amplitudes
\[
|A_u| \approx 0.68, \quad |A_b| \approx 0.24, \quad |A_c| \approx 6.3. \tag{17d}
\]
Note that the PCAC amplitude for \( \langle 3\pi^0 | H_{em} | \eta \rangle \) in (16a) recovers the observed \( \eta \to 3\pi^0 \) rate in (16) or equivalently the constant Dalitz plot amplitude forms in (17) give \( |A_u| \approx 0.68 \) which was earlier used to predict the \( \eta \to 3\pi^0 \) rate in Eqs. (16).

This consistency pattern can also be applied to \( \eta' \to 3\pi^0 \) decay, presumably dominated by \( \eta' \to \eta\pi^0\pi^0 \) followed by an em transition \( \langle \pi^0 | H_{em} | \eta' \rangle \):
\[
\left| \langle 3\pi^0 | H_{em} | \eta' \rangle \right| = 3 \left| \langle \pi^0 | H_{em} | \eta \rangle \langle \pi^0 \pi^0 | \eta' \rangle \right| (m_\eta^2 - m_\pi^2)^{-1} \\
\approx 3(3900 \text{ MeV}^2)(6.3)(281000 \text{ MeV}^2)^{-1} \approx 0.26. \tag{18a}
\]
In (18a) we have again used the em scale (15d) (three times), the \( \eta \) propagator on the \( \pi^0 \) mass shell and the constant amplitude \( |A_u| \approx 6.3 \) in (17d). The result 0.26 is near the constant amplitude \( |A_b| \approx 0.24 \) in (17d), or equivalently the \( \eta_0^* \) decay rate is predicted to be
\[
\Gamma(\eta' \to 3\pi^0) \approx 5.58 \left| \langle 3\pi^0 | H_{em} | \eta' \rangle \right|^2 \text{ keV} \approx 377 \text{ eV}, \tag{18b}
\]
near data (5) \( 313 \pm 58 \) eV.

Finally we consider the strong decays \( \eta' \to \eta\pi\pi \), with the charged to neutral pion branching ratio being (3) about 2, as expected via SU(2) symmetry. At first these decays were thought to be controlling the \( \eta-\eta' \) mixing angle. Now, however, one begins by assuming an \( \eta-\eta' \) mixing angle [such as \( \phi \approx 42^\circ \) or \( \theta \approx -13^\circ \) found earlier in Eqs. (8,12)] , and then attempts to explain the observed \( \eta' \to \eta\pi\pi \) rate given in Sec. I.

To this end Singh and Pasupathy in Ref. (33) studied the \( \delta = a_0(983) \) scalar meson pole amplitude in \( \eta' \to \delta\pi, \delta \to \eta\pi \). Later Deshpande and Truong in (39) also included a scalar meson \( \sigma \) pole in this analysis with \( \eta' \to \eta\sigma, \sigma \to \pi\pi \). These second authors in (33) justified introducing this latter \( \sigma \) in order to mask a soft-pion Adler zero which would drastically alter the \( \pi\pi \) phase space. In fact the \( \eta' \to \eta\pi^0\pi^0 \) data shows only a small deviation from phase space, with linear amplitude \( A(1 + \alpha y) \) now requiring (3) \( \alpha = -0.058 \pm 0.013 \), and \( \alpha = -0.08 \pm 0.03 \) for \( \eta' \to \eta\pi^+\pi^- \) decay.

Keeping only these two \( \delta \) and \( \sigma \) pole terms, we slightly modify Refs. (33) and write this combined \( \eta' \to \eta\pi^+\pi^- \) amplitude magnitude as
\[
A = |A(\eta' \to \eta\pi^+\pi^-)| \approx \left| \frac{G_\delta g_{\eta\pi\pi} g_{\eta'\pi\pi}}{m_\delta^2 - u - im_\delta \Gamma_\delta} + \frac{G_\sigma g_{\sigma\pi\pi} g_{\eta'\pi\pi}}{m_\sigma^2 - s - im_\sigma \Gamma_\sigma} \right|. \tag{19}
\]
Here the combined \( \delta \) and \( \sigma \) pole amplitudes have the same structure as in Ref. (33) except we always (rather than partially) keep the non-narrow widths (3) \( \Gamma_\delta \sim 100 \) MeV and \( \Gamma_\sigma \sim 700 \) MeV (3,37). Also to estimate the pole denominators in (19), we follow Ref. (33) and take \( m_\delta^2 - u \approx 2m_{\eta'}E_1 \approx 2m_{\eta'}(m_{\eta'} - m_\eta) \) in the \( \eta' \) rest frame with \( p_\pi \approx p_{\pi'} \approx 0 \) soft and \( s = [6.77 - 2.4y] m_\pi^2 \).

Finally we choose the nonstrange \( \sigma \) mass from the recent data analysis of Ref. (11):
\[
m_\sigma = 400 \text{ to } 900 \text{ MeV} \quad \text{mean mass } m_\sigma \approx 650 \text{ MeV}. \tag{20}
\]
This is near \( \varepsilon (700) \) used in [39] and is supported by the 1998 PDG tables [8]. Moreover a \( \sigma(650) \) is generated from linear \( \sigma \) model (LoSM) dynamics [42] with LoSM coupling constants using the mixing relations (\( g(\delta \eta \pi) \approx 1.56 \text{ GeV} \)):

\[
g(\delta \eta \pi) = \cos \phi g(\delta \eta_{NS}\pi) = \cos \phi \left( \frac{m_\eta^2 - m_{\eta_{NS}}^2}{2 f_\pi} \right) \approx 1.56 \text{ GeV},
\]

\[
g(\eta' \pi) = \sin \phi g(\eta'_{NS}\pi) = \sin \phi \left( \frac{m_\eta^2 - m_{\eta_{NS}}^2}{2 f_\pi} \right) \approx 1.40 \text{ GeV},
\]

\[
g(\sigma \pi \pi) = \frac{m_\sigma^2}{2 f_\pi} \approx 2.27 \text{ GeV},
\]

\[
g(\eta' \eta) = \cos \phi \sin \phi g(\sigma \pi \pi) \approx 1.13 \text{ GeV}.
\]

Note that \( g(\delta \eta_{NS}\pi) = g(\sigma \pi \pi) \) in the chiral limit and also that the \( \eta-\eta' \) mixing angle used (\( \phi = 41.9^\circ \)) is as found from Eq. (12).

Substituting the above numerical values back into (19) leads to the \( \eta' \to \eta \pi^+\pi^- \) amplitude magnitude

\[
|A| \approx \left| \frac{2.20}{0.79 - i0.10} + \frac{2.57}{0.29 - i0.46} \right| \approx 6.64.
\]

This LoSM prediction in (22) should be compared with the original estimates in [39] of \( |A| \approx 8.5, \alpha \approx -0.012 \). Also, \( |A| \approx 6.64 \) in (22) is near \( |A_c| \approx 6.3 \) in (17d) assuming a constant matrix element and isospin invariance. Lastly accounting for the \( \eta' \to \eta \pi^0 \pi^0 \) as well as the \( \eta' \to \eta \pi^+\pi^- \) amplitude, and folding in the slight Dalitz plot slope we predict the total decay rate (for the average slope \( \alpha \approx -0.07 \)):

\[
\Gamma(\eta' \to \eta \pi \pi) = 3 |A|^2 (1 + 0.24 \alpha + 0.27 \alpha^2) \text{ keV}
\]

\[
\approx 130 \text{ keV}.
\]

This prediction (23b) is in very good agreement with present data (131 ± 8 keV) as given in Sec. I.

We differ from Ref. [39] primarily in that we use the LoSM meson-meson couplings in Eqs. (21). An extraction of the \( \delta \eta \pi \) coupling from the width of \( \Gamma(\delta \eta \pi) \approx 100 \text{ MeV} \) [13] gives for \( q = 321 \text{ MeV} \):

\[
\Gamma(\delta \eta \pi) = \frac{q |2g(\delta \eta \pi)|^2}{8 \pi m_\delta^2} , \text{ or } |g(\delta \eta \pi)| \approx 1.38 \text{ GeV}.
\]

The latter coupling in (24) is reasonably near the LoSM coupling 1.56 GeV in (21a).

**IV. CONSISTENCY WITH DYNAMICAL CALCULATIONS**

As pointed out in the Introduction and Sec. II, there is a dynamical approach to the question of the Goldstone boson structure of the mixed \( \eta(547) \) and \( \eta'(958) \) mesons [10], namely the coupled SD-BS approach incorporating some crucial features of QCD, which leads to the similar conclusions on the mixing angle and masses as the analysis in Sec. II.
Before addressing its mass matrix, let us see what this approach tells us about the mixing angle that can be inferred from $\gamma\gamma$ decays. Since the SD-BS approach incorporates the correct chiral symmetry behavior thanks to $D\chi_{SB}$ and is consistent with current algebra, it reproduces (when care is taken to preserve the vector Ward-Takahashi identity of QED) the Abelian axial anomaly results, which are otherwise notoriously difficult to reproduce in bound-state approaches, as discussed in Ref. [17]. This gives particular weight to the constraints placed on the mixing angle $\theta$ by the SD-BS results on $\gamma\gamma$ decays of pseudoscalars.

A. $\gamma\gamma$ decays of the bound-state $\pi^0, \eta, \eta'$

We express the broken–SU(3) pseudoscalar states $\pi^0, \eta_8$ and $\eta_0$ through the quark basis states $|f\bar{f}\rangle$ by

$$|P\rangle = \sum_f \left( \frac{\lambda^P}{\sqrt{2}} \right)_{ff} |f\bar{f}\rangle,$$

(25)

where $P = \pi^0, \eta_8, \eta_0$ simultaneously have the meaning of the respective indices $j = 3, 8, 0$ on the SU(3) Gell-Mann matrices $\lambda^j$ ($j = 1, \ldots, 8$) and on $\lambda^0 \equiv (\sqrt{2}/3)1_3$. This picks out the diagonal $\lambda^3, \lambda^8, \lambda^0$ in Eq. (25). For future convenience we write the $P(p) \to \gamma(k)\gamma(k')$ amplitudes as

$$T_P(k^2, k'^2) = \sum_f \left( \frac{\lambda^P}{\sqrt{2}} \right)_{ff} Q_f^2 \bar{T}_{ff}(k^2, k'^2),$$

(26)

where $\bar{T}_{ff}(k^2, k'^2) \equiv T_{ff}(k^2, k'^2)/Q_f^2$ are the “reduced” two-photon amplitudes obtained by removing the squared charge factors $Q_f^2$ from $T_{ff}$, the $\gamma\gamma$ amplitude of the pseudoscalar quark-antiquark bound state of the hidden flavor $f\bar{f}$.

The decay amplitudes (into real photons, $k^2 = k'^2 = 0$) of the physical states $\eta$ and $\eta'$, are given in terms of the predicted $\gamma\gamma$ decay amplitudes of the SU(3) states $\eta_8$ and $\eta_0$ as

$$T_\eta(0, 0) = \cos \theta T_{\eta_8}(0, 0) - \sin \theta T_{\eta_0}(0, 0),$$

(27)

$$T_{\eta'}(0, 0) = \sin \theta T_{\eta_8}(0, 0) + \cos \theta T_{\eta_0}(0, 0).$$

(28)

The best fit to the experimental $\gamma\gamma$ decay amplitudes was found in Ref. [16] for $\theta = -12^\circ$ for the concrete SD-BS model and parameters [14] adopted there. In order to show that in the SD-BS approach $\gamma\gamma$ decays imply $\theta$ somewhere in that ballpark (i.e., less negative than values favored by $\chi$PT) regardless of any model choice, and to be able to compare with other theoretical approaches which usually try to express $P \to \gamma\gamma$ amplitudes in terms of the leptonic (axial-current) decay constants $f_P$, let us start with the light $u, d$ sector in the chiral (and soft) limit. There, the SD-BS approach yields analytically and exactly\footnote{The same holds [14,45] for the related process $\gamma \to \pi^+\pi^0\pi^-$.} and independently of the internal bound-state pion structure,
\[ \bar{T}_{u0}(0, 0) \equiv \bar{T}_{uu}(0, 0) = \bar{T}_{dd}(0, 0) = \frac{N_c}{2\sqrt{2}\pi^2 f_\pi}, \]  
(29)

\[ T_{\pi0}(0, 0) = \frac{N_c}{2\sqrt{2}\pi^2 f_\pi} \sum_f \left( \frac{\lambda^3}{\sqrt{2}} \right)_{ff} Q_f^2 = \frac{1}{4\pi^2 f_\pi}. \]  
(30)

Of course, the calculated \([14][18][16]\) value of \(f_\pi\) does depend on the (modeling of the) internal pion structure, but the empirically successful axial-anomaly chiral-limit relation (30) does not.

The \(\pi^0 \to \gamma\gamma\) decay amplitude for a possibly nonvanishing pion mass, can be used as a definition of pionic \(\gamma\gamma\)-decay constant \(\bar{f}_\pi\) by demanding that this amplitude be written in the form of the massless, CL amplitude (31), but with \(\bar{f}_\pi\) in place of \(f_\pi\): \(T_{\pi0}(0, 0) = 1/4\pi^2\bar{f}_\pi\). Obviously, \(\bar{f}_\pi = f_\pi\) in the CL, and \(\bar{f}_\pi\) is a convenient way to re-express the \(\gamma\gamma\) amplitude in the case of a nonvanishing pion mass, because the Veltman-Sutherland theorem, PCAC, and the empirical success of the chiral-limit anomaly result (30), guarantee that \(\bar{f}_\pi \approx f_\pi\) always holds for any realistic description of the light \(u, d\) sector. For simplicity of discussion, we therefore use \(\bar{f}_\pi = f_\pi\) in this subsection, as the Veltman-Sutherland theorem guarantees that this can be wrong only by several percent. Although the chiral limit formula (30) can be applied without reservations only to pions, it is customary to write the amplitudes for \(\eta_8, \eta_0 \to \gamma\gamma\) in the same form as (31), defining thereby the \(\gamma\gamma\)-decay constants \(\bar{f}_{\eta_8}\) and \(\bar{f}_{\eta_0}\):

\[ T_{\eta_8}(0, 0) \equiv \frac{N_c}{2\sqrt{2}\pi^2 f_{\eta_8}} \sum_f \left( \frac{\lambda^8}{\sqrt{2}} \right)_{ff} Q_f^2 = \frac{f_\pi}{f_{\eta_8}} \frac{T_{\pi0}(0, 0)}{\sqrt{3}}, \]  
(31)

\[ T_{\eta_0}(0, 0) \equiv \frac{N_c}{2\sqrt{2}\pi^2 f_{\eta_0}} \sum_f \left( \frac{\lambda^0}{\sqrt{2}} \right)_{ff} Q_f^2 = \frac{f_\pi}{f_{\eta_0}} \sqrt{3} T_{\pi0}(0, 0). \]  
(32)

As pointed out by \([14]\), \(\bar{f}_{\eta_8}\) and \(\bar{f}_{\eta_0}\) are not \textit{a priori} simply connected with the usual axial-current decay constants \(f_{\eta_8}\) and \(f_{\eta_0}\), in contrast to \(f_\pi \approx \bar{f}_\pi\). Expressing \(T_{\eta_8}(0, 0)\) and \(T_{\eta_0}(0, 0)\) through the \(\gamma\gamma\)-decay constants \(f_{\eta_8}\) and \(f_{\eta_0}\), yields the customary (see, e.g. \([17]\)) forms for the \(\eta\) and \(\eta'\) decay widths:

\[ \Gamma(\eta \to \gamma\gamma) = \frac{\alpha_{\text{em}}^2 m_\eta^3}{64\pi^3 3 f_\pi^2} \left[ \frac{f_\pi}{f_{\eta_8}} \cos \theta - \sqrt{8} \frac{f_\pi}{f_{\eta_0}} \sin \theta \right]^2, \]  
(33)

\[ \Gamma(\eta' \to \gamma\gamma) = \frac{\alpha_{\text{em}}^2 m_{\eta'}^3}{64\pi^3 3 f_\pi^2} \left[ \frac{f_\pi}{f_{\eta_8}} \sin \theta + \sqrt{8} \frac{f_\pi}{f_{\eta_0}} \cos \theta \right]^2. \]  
(34)

The even more customary version of (33) and (34) in which the axial-current decay constants \(f_{\eta_8}\) and \(f_{\eta_0}\) appear in place of \(\bar{f}_{\eta_8}\) and \(\bar{f}_{\eta_0}\) requires a derivation where PCAC and soft meson technique are applied to the \(\eta-\eta'\) complex \([17]\). For the indeed light pion, these assumptions are impeccable (leading to \(f_\pi = \bar{f}_\pi\)), but not for the \(\eta-\eta'\) complex. For such a heavy particle as \(\eta'\) they are quite dubious. However, we do not need and do not use these assumptions since we directly calculated the \(\eta_8\) and \(\eta_0\) decay amplitudes, i.e., \(\bar{f}_{\eta_8}\) and
just as the axial-current pseudoscalar decay constants $f_{\eta_8}$ and $f_{\eta_0}$ were calculated \[48\] independently of the $\gamma\gamma$ processes. In contrast to $f_\pi = f_{\pi}$, $f_{\eta_8}$ and $f_{\eta_8}$ cannot be equated, as the difference between them was found to be quite important \[49\].

The precise values of $f_{\eta_8}$ and $f_{\eta_0}$ are model dependent, but $f_{\eta_8} \approx f_{\pi}$ holds in this approach generally, i.e., independently of chosen model details, as long as the s-quark mass is realistically heavier than the $u,d$-quark masses. To see this, let us start by noting that $f_{\eta_8} \approx f_{\pi}$ is equivalent to $T_{\eta_8}(0,0) > T_{\pi}(0,0)/\sqrt{3}$, and since we can re-write Eq. (24) for $\eta_8$ as

$$T_{\eta_8}(0,0) = \frac{T_{\pi}(0,0)}{\sqrt{3}} + \frac{1}{9} \frac{2}{\sqrt{6}} \left[ \tilde{T}_{dd}(0,0) - \tilde{T}_{ss}(0,0) \right] ,$$

(35)

the inequality $f_{\eta_8} < f_{\pi}$ is in our approach simply the consequence of the fact that the ("reduced") $\gamma\gamma$-amplitude of the $s\bar{s}$-pseudoscalar bound state, $\tilde{T}_{ss}$, is smaller than the corresponding non-strange $\gamma\gamma$-amplitude $\tilde{T}_{dd}$ ($= \tilde{T}_{u\bar{u}} = \tilde{T}_{\pi}$ in the isosymmetric limit), for any realistic relationship between the non-strange and much larger strange quark masses. This is the reason why in this approach one cannot fit well the experimental $\eta, \eta' \rightarrow \gamma\gamma$ widths with the mixing angle as negative as in chiral perturbation theory descriptions ($\theta \approx -20^\circ$), but rather with $\theta \approx -12^\circ$. This is easily understood, for example, with the help of Fig. 1. of Ball et al. \[19\], where the values of $f_{\eta_8(0)}/f_{\pi}$ consistent with experiment are given as a function of the mixing angle $\theta$. Their curve shows that values $f_{\eta_8}/f_{\pi} < 1$ permit accurate reproduction of $\eta, \eta' \rightarrow \gamma\gamma$ widths only for $\theta$-values less negative than $-15^\circ$. [It does not matter that they in fact plotted $f_{\eta_8(0)}/f_{\pi}$ and not $f_{\eta_8}/f_{\pi}$. Namely, they used Eqs. (33)-(34) above for comparison with the experimental $\gamma\gamma$-widths, just with $f_{\eta_8(0)}/f_{\pi}$ instead of $f_{\eta_8(0)}/f_{\pi}$, so that the experimental constraints displayed in their Fig. 1 apply to whatever ratios are used in these expressions. One should also note that since in our approach $f_{\eta_8}, f_{\eta_0}$ and $f_{\pi}$ are not free parameters but predicted quantities, the two widths $\eta, \eta' \rightarrow \gamma\gamma$ cannot be fitted exactly by adjusting just one parameter, $\theta$. Rather, we fix $\theta$ by performing a $\chi^2$ fit to the widths.] On the other hand, the more negative values $\theta \lesssim -20^\circ$ give good $\eta, \eta' \rightarrow \gamma\gamma$ widths in conjunction with the ratio $f_{\eta_8}/f_{\pi} = f_{\eta_8}/f_{\pi} = 1.25$ obtained by \[13\] in $\chi$PT. However, the coupled SD-BS approach belongs among constituent quark approaches\[3\] and for them, considerably less negative angles, $\theta \approx -14^\circ \pm 2^\circ$, are natural.

Ref. \[10\] showed that these bounds and estimates are very robust under SD-BS model variations and can be taken as model independent. For example, for chiral $u,d$ quarks,

\[2\] This is different from chiral perturbation theory \[15\]. Nevertheless, for the axial-current decay constants our approach gives $f_{\eta_8} > f_{\pi}$ (see Appendix or Ref. \[14\]). Even the numerical value obtained in our concrete SD-BS calculation \[16\], $f_{\eta_8} = 1.31f_{\pi}$, is rather close to $f_{\eta_8} = 1.25f_{\pi}$ obtained in chiral perturbation theory \[13\].

\[3\] However, at least one effective–meson–Lagrangian approach, that of Benayoun et al. \[50\], yields results quite close to ours: $\theta = -11.59^\circ \pm 0.76^\circ$ and their Eq. (29), where their $(f_8, f_1)$ correspond to our $(f_{\eta_8}, f_{\eta_0})$, with $f_8/f_{\pi} = 0.82 \pm 0.02$ and $f_1/f_{\pi} = 1.15 \pm 0.02$. 

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leading to the bounds \( \frac{3}{5} f_\pi < \bar{f}_{\eta_s} < f_\pi \) and \( f_\pi < \bar{f}_{\eta_0} < \frac{6}{5} f_\pi \). Also, considerations based on the Goldberger–Treiman relation showed that \( \bar{T}_{ss}(0,0) < \bar{T}_{uu}(0,0) \) is simply due to \( f_{ss} \sim f_\pi + 2(f_{K+} - f_\pi) > f_\pi \) (where \( f_{ss} \) is the axial-current decay constant of the unphysical \( s\bar{s} \) pseudoscalar bound state), and that a good estimate of the \( \gamma\gamma \)-amplitude ratio is the inverse ratio of the pertinent constituent quark masses: \( \bar{T}_{ss}(0,0)/\bar{T}_{uu}(0,0) \approx \hat{m}/m_s \). Equations (36) then give the relations [reducing to \( \bar{f}_{\eta_s} = f_\pi \) and \( \bar{f}_{\eta_0} = f_\pi \) in the U(3) limit, just like Eqs. (30) themselves]

\[
\bar{f}_{\eta_s} \approx \frac{3 f_\pi}{5 - 2 \hat{m}/m_s} , \quad \bar{f}_{\eta_0} \approx \frac{6 f_\pi}{5 + \hat{m}/m_s} , \tag{37}
\]

obtained also by Ref. [51] using the simple quark loop model with constant constituent masses. These estimates are (for reasonable \( \hat{m}/m_s \)) close to what Ref. [16] calculated with a concrete SD-BS model choice [14], namely \( \bar{f}_{\eta_s}/f_\pi = 1.067 \) and \( \bar{f}_{\eta_0}/f_\pi = 0.797 \). For these concrete model values, \( \eta, \eta' \to \gamma\gamma \) widths (33)-(34) fit the data best for \( \theta = -12.0^\circ \).

### B. Introducing X into the SD-BS mass matrix

For the very predictive SD-BS approach to be consistent, the above mixing angle extracted from \( \eta, \eta' \to \gamma\gamma \) widths, should be close to the angle \( \theta \) predicted by diagonalizing the \( \eta-\eta' \) mass matrix. In this subsection, it is given in the quark \( f\bar{f} \) basis:

\[
M^2 = \text{diag}(M_{u\bar{u}}^2, M_{d\bar{d}}^2, M_{s\bar{s}}^2) + \beta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} . \tag{38}
\]

As in Sec. II, \( 3\beta \) (called \( \lambda_\eta \) in Ref. [16]) is the contribution of the gluon axial anomaly to \( m_{\eta_0}^2 \), the squared mass of \( \eta_0 \). We denote by \( M_{ff'} \) the masses obtained as eigenvalues of the BS equations for \( q\bar{q} \) pseudoscalars with the flavor content \( ff' \) (\( f, f' = u, d, s \)). However, since Ref. [16] had to employ a rainbow-ladder approximation (albeit the improved one of Ref. [14]), it could not calculate the gluon axial anomaly contribution \( 3\beta \). It could only avoid the \( U_A(1) \)-problem in the \( \eta-\eta' \) complex by parameterizing \( 3\beta \), namely that part of the \( \eta_0 \) mass squared which remains nonvanishing in the CL. Because of the rainbow-ladder approximation (which does not contain even the simplest annihilation graph – Fig. 1), the \( q\bar{q} \) pseudoscalar masses \( M_{ff'} \) do not contain any contribution from \( 3\beta \), unlike the nonstrange and strange \( \eta \) masses \( m_{\eta_{u,d,s}} \) [in Eq. (33a)] and \( m_{\eta_{s}} \) [in Eq. (33b)], which do, and which must not be confused with \( M_{u\bar{u}} = M_{d\bar{d}} = M_{s\bar{s}} \). Since the flavor singlet gluon anomaly contribution \( 3\beta \) does not influence the masses \( m_\pi \) and \( m_K \) of the non-singlet pion and kaon, the realistic rainbow-ladder modeling aims directly at reproducing the empirical values of these masses: \( M_{u\bar{u}} = M_{d\bar{d}} = m_\pi \) and \( M_{s\bar{s}} = m_K \). In contrast, the masses of the physical etas, \( m_\eta \) and \( m_{\eta'} \), must be obtained by diagonalizing the \( \eta_8-\eta_0 \) sub-matrix containing both \( M_{ff} \) and the gluon anomaly contribution to \( m_{\eta_0}^2 \).
bosons, whereas the chiral-limit-nonvanishing \( \eta \) for all three flavors, the SU(3) octet pseudoscalars including all \( \pi \) become massless Goldstone bosons, whereas the chiral-limit-nonvanishing \( \eta' \)-mass \( 3\beta \) is of order 1/\( N_c \) since it is purely due to the gluon anomaly. If one lets \( 3\beta \rightarrow 0 \) (as the gluon anomaly contribution behaves for \( N_c \rightarrow \infty \)), then for any quark masses and resulting \( M_{ff} \) masses, the “ideal” mixing (\( \theta = -54.74^\circ \)) takes place so that \( \eta \) consists of \( u, d \) quarks only and becomes degenerate with \( \pi \), whereas \( \eta' \) is the pure \( s\bar{s} \) pseudoscalar bound state with the mass \( M_{s\bar{s}} \).

In Ref. [16], numerical calculations of the mass matrix were performed for the realistic chiral and SU(3) symmetry breaking, with the finite quark masses (and thus also the finite BS \( q\bar{q} \) bound-state pseudoscalar masses \( M_{ff} \)) fixed by the fit [14] to static properties of many mesons but excluding the \( \eta-\eta' \) complex. The mixing angle which diagonalizes the \( \eta_{s}-\eta_0 \) mass matrix thus depended in Ref. [16] only on the value of the additionally introduced “gluon anomaly parameter” \( 3\beta \). Its preferred value turned out to be \( 3\beta = 1.165 \text{ GeV}^2 = (1079 \text{ MeV})^2 \), leading to the mixing angle \( \theta = -12.7^\circ \) [compatible with \( \phi = 41.9^\circ \) in Eq. (12)] and acceptable \( \eta \rightarrow \gamma\gamma \) and \( \eta' \rightarrow \gamma\gamma \) decay amplitudes. Also, the \( \eta \) mass was then fitted to its experimental value, but such a high value of \( 3\beta \) inevitably resulted in a too high \( \eta' \) mass, above 1 GeV. (Conversely, lowering \( 3\beta \) aimed to reduce \( m_{\eta'} \), would push \( \theta \) close to \(-20^\circ \), making predictions for \( \eta, \eta' \rightarrow \gamma\gamma \) intolerably bad.) However, unlike Eq. (10) in the present paper, it should be noted that Ref. [16] did not introduce into the mass matrix the “strangeness attenuation parameter” \( X \) which should suppress the nonperturbative quark \( f\bar{f} \rightarrow f'\bar{f}' \) annihilation amplitude (illustrated by the “diamond” graph in Fig. 1) when \( f \) or \( f' \) are strange.

On the other hand, the influence of this suppression should be substantial, since \( X \approx \hat{m}/m_s \) should be a reasonable estimate of it, and this nonstrange-to-strange \( \text{constituent mass ratio} \) in the considered variant of the SD-BS approach [16] is not far from \( X \) in Eq. (14) and from the mass ratios in Refs. [31–33], and is even closer to the mass ratios in the Refs. [31]. Namely, two of us found [16] it to be around \( M_u(0)/M_s(0) = 0.615 \) if the constituent mass was defined at the vanishing argument \( q^2 \) of the momentum-dependent SD mass function \( M_f(q^2) \).

We therefore introduce the suppression parameter \( X \) the same way as in the \( NS-S \) mass matrix [14], whereby the mass matrix in the \( f\bar{f} \) basis becomes

\[
M^2 = \text{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}}) + \beta \begin{bmatrix}
1 & 1 & X \\
1 & 1 & X \\
X & X & X^2
\end{bmatrix}.
\]

In a very good approximation, Eq. (33) recovers (in the \( \pi^0-NS-S \) basis) Eq. (10) for the \( 2 \times 2 \) \( \eta-\eta' \) subspace. This is because \( M^2_{s\bar{s}} \) differs from \( 2m^2_K - m^2_s \) only by a couple of percent, thanks to the good chiral behavior of the masses \( M_{ff} \) calculated in SD-BS approach. (These \( M^2_{ff} \) and the CL model values of \( f_\pi \) and quark condensate, satisfy Gell-Mann-Oakes-Renner relation to the first order in the explicit chiral symmetry breaking [14].) The SD-BS–predicted octet (quasi-)Goldstone masses \( M_{ff} \) are known to be empirically successful in our concrete model choice [14], but the question is whether the SD-BS approach can also give
some information on the $X$-parameter. If we treat both $3\beta$ and $X$ as free parameters, we can of course fit both the $\eta$ mass and the $\eta'$ mass to their experimental values. For the model parameters as in Ref. [14] (for these parameters our independent calculation gives $m_u = M_{u\bar{u}} = 140.4$ MeV and $M_{s\bar{s}} = 721.4$ MeV), this happens at $3\beta = 0.753$ GeV$^2 = (868$ MeV)$^2$ and $X = 0.835$. However, the mixing angle then comes out as $\theta = -17.9^\circ$, which is too negative to allow consistency of the empirically found two-photon decay amplitudes of $\eta$ and $\eta'$, with predictions of our SD-BS approach for the two-photon decay amplitudes of $\eta_8$ and $\eta_0$ [16].

Therefore, and also to avoid introducing another free parameter in addition to $3\beta$, we take the path where the dynamical information from our SD-BS approach is used to estimate $X$. Namely, our $\gamma\gamma$ decay amplitudes $T_{f\bar{f}}$ can be taken as a serious guide for estimating the $X$-parameter instead of allowing it to be free. We did point out in Sec. II that the attempted treatment [34] of the gluon anomaly contribution through just the “diamond diagram” contribution to $3\beta$, indicated that just this partial contribution is quite insufficient. This limits us to keeping $3\beta$ as a free parameter, but we can still suppose that this diagram can help us get the prediction of the strange-nonstrange ratio of the complete pertinent amplitudes $f\bar{f} \rightarrow f'\bar{f}'$ as follows. Our SD-BS modeling in Ref. [16] employs an infrared-enhanced gluon propagator [14,17] weighting the integrand strongly for low gluon momenta squared. Therefore, in analogy with Eq. (4.12) of Kogut and Susskind [4] (see also Refs. [53,54]), we can approximate the Fig. 1 amplitudes $f\bar{f} \rightarrow 2\text{gluons} \rightarrow f'\bar{f}'$, i.e., the contribution of the quark-gluon diamond graph to the element $ff'$ of the $3 \times 3$ mass matrix, by the factorized form

$$
\tilde{T}_{f\bar{f}}(0,0) \mathcal{C} \tilde{T}_{f'\bar{f}'}(0,0). \tag{40}
$$

In Eq. (40), the quantity $\mathcal{C}$ is given by the integral over two gluon propagators remaining after factoring out $\tilde{T}_{f\bar{f}}(0,0)$ and $\tilde{T}_{f'\bar{f}'}(0,0)$, the respective amplitudes for the transition of the $q\bar{q}$ pseudoscalar bound state for the quark flavor $f$ and $f'$ into two vector bosons, in this case into two gluons. The contribution of Fig. 1 is thereby expressed with the help of the (reduced) amplitudes $\tilde{T}_{f\bar{f}}(0,0)$ we calculated for the transition of $q\bar{q}$ pseudoscalars to two real photons ($k^2 = k'^2 = 0$). Although $\mathcal{C}$ is in principle computable, all this unfortunately does not amount to determining $\beta, \beta X$ and $\beta X^2$ in Eq. (39) since the higher (four-gluon, six-gluon, ..., etc.) contributions are clearly lacking. We therefore must keep the total (light-)quark annihilation strength $\beta$ as a free parameter. However, if we assume that the suppression of the diagrams with the strange quark in a loop is similar for all of them, Eq. (40) and the “diamond” diagram in Fig. 1 help us to at least estimate the parameter $X$ as $X \approx \tilde{T}_{ss}(0,0)/\tilde{T}_{u\bar{u}}(0,0)$. This is a natural way to build in the effects of the SU(3) flavor symmetry breaking in the $q\bar{q}$ annihilation graphs.

We get $X = 0.663$ from the two-photon amplitudes we obtained in the chosen SD-BS model [14]. This value of $X$ agrees well with the other way of estimating $X$, namely the nonstrange-to-strange constituent mass ratio of Refs. [22,23]. With $X = 0.663$, requiring that the $2 \times 2$ matrix trace, $m_\eta^2 + m_{\eta'}^2$, be fitted to its empirical value, fixes the chiral-limiting nonvanishing singlet mass squared to $3\beta = 0.832$ GeV$^2 = (912$ MeV)$^2$, just 0.5% below Eq. (44). The resulting mixing angle and $\eta, \eta'$ masses are

$$
\theta = -13.4^\circ, \quad m_\eta = 588 \text{ MeV}, \quad m_{\eta'} = 933 \text{ MeV}. \tag{41}
$$
The above results of the SD-BS approach are very satisfactory since they agree well with what was found in Sec. II by different methods. Let us close this section by exploring the stability of these results on model variations. Except the introduction of $3\beta = \lambda_\eta$, these SD-BS results were obtained without any other parameter fitting, with the model parameters resulting from the very broad previous fit [10], but actually giving us, in our independent calculation, a few percent too high results for $m_\pi$ and $m_K$. To possibly improve, and in any case check the robustness of the consistency with Sec. II (and subsection IV.A) on variations of our model description, we therefore perform a refitting in the sector of $u$, $d$ and $s$ quarks, to reproduce exactly the average isotriplet pion mass $m_\pi = M_{u\bar{u}} = 137.3$ MeV and isodoublet kaon mass $m_K = 495.7$ MeV. As Table I shows, the changes are small, and lead to $M_u(0)/M_s(0) = 0.622$ and $X = \tilde{T}_{s\bar{s}}(0,0)/\tilde{T}_{u\bar{u}}(0,0) = 0.673$. Using this $X$ to fit the sum of the squared $\eta$ and $\eta'$ masses to the empirical value, yields the column B in Table II, where we see a slight improvement in the $\eta$ and $\eta'$ masses with respect to the results [11], while the mixing angle is still acceptable, being less than $2^\circ$ away from the angle favored in Sec. II.

If we treat $X$ as the second free parameter (this procedure yields the column C of Table II) so that we are able to fit $m_\eta$ and $m_{\eta'}$ precisely to their experimental values, we get $X = 0.805$, along with the mixing angle $\theta = -14.9^\circ$ and the chiral-limit-nonvanishing singlet mass $3\beta = 0.801$ GeV$^2$=(895 MeV)$^2$. This is noticeably closer to $\theta$ and $3\beta$ resulting from other procedures (where $X$ is not a free parameter) than before the aforementioned $\pi^0 - K$ refitting to $m_\pi = 137.3$ MeV and $m_K = 495.7$ MeV.

Next, we note in the column D of Table II that the slightly improved fit to the masses also led to somewhat improved $\eta, \eta' \rightarrow \gamma\gamma$ widths when we extract from them $\theta = -12.8^\circ$, practically the same as Ref. [10] and the Sec. II result [12]. All the three possibilities B, C, and D, do not differ too much from each other, and agree reasonably with the experimental masses and $\gamma\gamma$ widths given in column E as well as with the corresponding results of Sec. II. This contrasts with column A, which also contains the results of the new fit but with $X = 1$. Column A shows that when $X = 1$, a good description of the masses requires a $\theta$ value too negative for a good description of the $\gamma\gamma$ widths in the SD-BS approach. Column A thus convinces us that it was precisely the lack of the strangeness attenuation factor $X$ that prevented Ref. [10] from satisfactorily reproducing the $\eta'$ mass when it successfully did so with the $\eta$ mass and $\gamma\gamma$ widths.

V. CONCLUSION

In Sec. II we studied the first $U_A(1)$ problem associated with the Goldstone structure of $\eta(547)$ and $\eta'(958)$ mesons. Following a QCD gluon-mediated approach to $\eta-\eta'$ particle mixing, we began by extracting an $\eta-\eta'$ mixing angle $\phi \approx 42^\circ$ in the $NS-S$ basis or $\theta \approx -13^\circ$ in the singlet-octet basis. This led to eta masses $\eta_0(567), \eta_0(947)$ with chiral-limiting (CL) $\eta_0(917)$. Then the physical eta mass $\eta(547)$ is 97% of $\eta_0(567)$, while $\eta'(958)$ is 104% of the CL $\eta_0(917)$. Such a 3–4% CL suppression is likewise found for the pion decay constant $f_\pi \approx 93$ MeV $\rightarrow$ 90 MeV and for the $K_{l3}$ form factor $f_+(0) = 1 \rightarrow 0.96-0.97$.

Then in Sec. III we studied the second $U_A(1)$ problem associated with eta meson hadronic decay rates. The $\eta, \eta' \rightarrow 3\pi^0$ ($\Delta I = 1$) decay rates of 377 eV followed from PCAC Consistency. Also a (strong) decay rate of 130 keV for $\eta' \rightarrow \eta\pi\pi$ was obtained from $\delta$ and $\sigma$ scalar meson poles combined with linear $\sigma$ model couplings. These three rates are com-
compatible with data finding \( \Gamma(\eta \to 3\pi^0) = 380 \pm 36 \text{ eV}, \Gamma(\eta' \to 3\pi^0) = 313 \pm 58 \text{ eV} \) and \( \Gamma(\eta' \to \eta\pi\pi) = 131 \pm 8 \text{ keV} \).

Finally, in Sec. IV we showed the consistency of the above results with those obtained in a chirally well-behaved quark model which was explicitly constructed through \( D\chi_{SB}, SD \) and \( BS \) equations. For example, described variations of our \( SD-BS \) approach lead to \( \theta \approx -13^\circ \pm 2^\circ \) and to the corresponding \( CL \) \( \eta_0 \) mass \( \sqrt{3\beta} = 912 \pm 18 \text{ MeV} \). Successful reproduction of the Abelian axial anomaly amplitudes in the \( CL \) in this bound-state approach, gives particular weight to our conclusion that so far away from the \( CL \) as in the case of the \( \eta-\eta' \) complex, \( \gamma\gamma \)-decay constants \( (\vec{f}_\eta,\vec{f}_\eta') \) differ significantly from the usual axial-current decay constants \( (f_\eta,f_\eta) \). By allowing for the effects of the \( SU(3) \) flavor symmetry breaking also in \( q\bar{q} \) annihilation graphs, we have improved the \( \eta-\eta' \) mass matrix with respect to the mass matrix in Ref. \( [16] \) (via the strangeness attenuation factor \( X = 0.663 \)).

The consistency of our approach and results with the two-mixing-angle scheme is shown in detail in the Appendix, where we also compute the mixing angles and axial decay constants in that scheme.

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**APPENDIX: CONNECTION WITH THE TWO–MIXING–ANGLES SCHEME**

In this appendix, we clarify the relationship of our approach with the two-mixing-angle scheme considered by Leutwyler and Kaiser \([22,23]\) as well as FKS \([24–27]\), and reviewed by Feldmann \([28]\).

The two-mixing-angle scheme is defined with respect to the mixing of the decay constants. As such, it is very suitable in studies where manipulating decay constants is crucial, e.g., when one expresses amplitudes through them with the help of PCAC. In other situations, the mixing of the states may be crucial. The \( SD-BS \) variant of our approach, where one explicitly solves for quark-antiquark bound states, and then uses them for direct calculation of amplitudes, is an especially clear example of that. (See Ref. \([24]\) for another recent example.) As FKS \([24]\) themselves state several lines below their Eq. (1.3), the appearance of the four parameters in the two-mixing-angle scheme, namely \( f_8, f_0, \theta_8 \) and \( \theta_0 \), raises anew the problem of their mutual relations and their connection with the mixing angle of the particle states, which is necessarily a single one. In our approach, it is convenient to utilize a mixing scheme defined with respect to a state basis corresponding to the broken \( SU(3) \) flavor symmetry, \( |\eta_{NS}\rangle \) and \( |\eta_8\rangle \) or, equivalently, the effective \( SU(3) \)-broken \( |\eta_8\rangle \) and \( |\eta_0\rangle \) [Eqs. (3)], i.e., the state mixing angle \( \phi \) of Eqs. (4) or (mathematically completely equivalently)

\[ \text{Note that in spite of the differences in notation, the effective octet and singlet states (3) of the broken SU(3) flavor, correspond to the effective octet and singlet states } \eta_8 \text{ and } \eta_0 \text{ given, e.g., by Eq. (85) in Feldmann’s review [28].} \]
the state mixing angle $\theta$ of Eqs. (2). Nevertheless, we show below that a) we can calculate quantities utilized in the two-mixing-angle scheme defined with respect to the mixing of the decay constants, and b) what we find for these quantities is close to what is quoted in Refs. [24, 28].

Independently of any specific approach, one can always define quite generally the axial-current decay constants $f_{\eta}^S$, $f_{\eta'}^S$, $f_{\eta}^0$, and $f_{\eta'}^0$ as the matrix elements

$$\langle 0 | A^{\mu}(x) | P(p) \rangle = if_P^a p^\mu e^{-ip \cdot x}, \quad a = 8, 0; \quad P = \eta, \eta'. \quad (A1)$$

These definitions are to be contrasted with the somewhat arbitrary definitions of the just two individual axial–current decay constants $f_{\eta}$ and $f_{\eta'}$ often used in this context, where $f_{\eta} \equiv \cos^2 \phi f_{NS} + \sin^2 \phi f_S$ and $f_{\eta'} \equiv \sin^2 \phi f_{NS} + \cos^2 \phi f_S$, where $f_{NS}$ and $f_S$ are defined below in Eqs. (A6). These $f_{\eta}$ and $f_{\eta'}$ stem from Eqs. (3) in conjunction with the rather arbitrary definitions $A_{\eta}^\mu(x) \equiv \cos \phi A_{NS}^\mu(x) - \sin \phi A_{NS}^\phi(x)$ and $A_{\eta'}^\mu(x) \equiv \sin \phi A_{NS}^\mu(x) + \cos \phi A_{NS}^\phi(x)$, with Eqs. (A3) below defining the nonstrange and strange axial currents, $A_{NS}^\mu(x)$ and $A_{NS}^\phi(x)$. In contrast to this, the four decay constants $f_{\eta}^S$, $f_{\eta'}^S$, $f_{\eta}^0$, and $f_{\eta'}^0$ are precisely defined by Eqs. (A4) and therefore can have unambiguous and process-independent meaning [28].

Following the convention of Leutwyler and Kaiser [22, 23], the four decay constants $f_{\eta}^S$, $f_{\eta'}^S$, $f_{\eta}^0$, and $f_{\eta'}^0$ are parametrized in terms of two decay constants $f_0$, $f_S$, and two angles $\theta_0$, $\theta_S$:

$$f_{\eta}^S = \cos \theta_S f_S, \quad (A2a)$$
$$f_{\eta'}^S = \sin \theta_S f_S, \quad (A2b)$$
$$f_{\eta}^0 = -\sin \theta_0 f_0, \quad (A2c)$$
$$f_{\eta'}^0 = \cos \theta_0 f_0. \quad (A2d)$$

We define the currents:

$$A_{NS}^\mu(x) = \frac{1}{\sqrt{3}} A^8 \mu(x) + \sqrt{\frac{2}{3}} A^0 \mu(x) = \frac{1}{2} \left( \bar{u}(x) \gamma^\mu \gamma_5 u(x) + \bar{d}(x) \gamma^\mu \gamma_5 d(x) \right), \quad (A3a)$$
$$A_{NS}^\phi(x) = -\sqrt{\frac{2}{3}} A^8 \mu(x) + \frac{1}{\sqrt{3}} A^0 \mu(x) = \frac{1}{\sqrt{2}} \bar{s}(x) \gamma^\mu \gamma_5 s(x). \quad (A3b)$$

The corresponding $NS-S$ decay constants (analogous to the constants $f_{\eta}^S$, $f_{\eta'}^S$, $f_{\eta}^0$, and $f_{\eta'}^0$ defined above) are defined as

$$\langle 0 | A^F(x) | P(p) \rangle = if_P^F p^\mu e^{-ip \cdot x}, \quad F = NS, S; \quad P = \eta, \eta'. \quad (A4)$$

The relations (A3) between the currents dictate that the relations between these two sets of decay constants are given, exactly and model independently, by

$$\begin{bmatrix} f_{\eta}^{NS} \\ f_{\eta'}^{NS} \\ f_{\eta}^{S} \\ f_{\eta'}^{S} \end{bmatrix} = \begin{bmatrix} f_{\eta}^S & f_{\eta}^0 \\ f_{\eta'}^S & f_{\eta'}^0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}, \quad (A5)$$

where we used matrix notation for compactness.
If we have well-defined nonstrange–strange states $|\eta_{NS}\rangle$ and $|\eta_S\rangle$ [as in our SD–BS approach] we can define the decay constants $f_{NS}$ and $f_S$ through

$$\langle 0 | A_{NS}^\mu(x) | \eta_{NS}(p) \rangle = i f_{NS} p^\mu e^{-ip^x},$$  \hspace{1cm} (A6a)

$$\langle 0 | A_S^\mu(x) | \eta_S(p) \rangle = i f_S p^\mu e^{-ip^x},$$  \hspace{1cm} (A6b)

$$\langle 0 | A_{NS}^\mu(x) | \eta_S(p) \rangle = 0 ,$$  \hspace{1cm} (A6c)

$$\langle 0 | A_{S}^\mu(x) | \eta_{NS}(p) \rangle = 0.$$  \hspace{1cm} (A6d)

Since the states $|\eta\rangle$ and $|\eta'\rangle$ are given by Eqs. (6) as the linear combinations of $|\eta_{NS}\rangle$ and $|\eta_S\rangle$, we can relate the constants $\{f_{\eta NS}, f_{\eta NS}', f_{\eta S}, f_{\eta S}'\}$ with $\{f_{NS}, f_S\}$:

$$\begin{bmatrix} f_{\eta NS} & f_{\eta S} \\ f_{\eta NS}' & f_{\eta S}' \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix}.$$  \hspace{1cm} (A7)

Using Eq. (A5), we can relate the decay constants $\{f_{8,\eta}, f_{8,\eta}', f_{0,\eta}, f_{0,\eta}'\}$ with $\{f_{NS}, f_S\}$:

$$\begin{bmatrix} f_{8,\eta} & f_{0,\eta} \\ f_{8,\eta}' & f_{0,\eta}' \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$  \hspace{1cm} (A8)

which is the same as in Feldmann et al. [26], who use the notation $f_q = f_{NS}, f_s = f_S$.

The above equations together with the definitions (A2) give the following solutions for $f_8, f_0, \theta_8,$ and $\theta_0$:

$$f_8 = \sqrt{\frac{1}{3} f_{NS}^2 + \frac{2}{3} f_S^2},$$  \hspace{1cm} (A9a)

$$\theta_8 = \phi - \arctan \left( \sqrt{2} f_{NS} / f_S \right),$$  \hspace{1cm} (A9b)

$$f_0 = \sqrt{\frac{2}{3} f_{NS}^2 + \frac{1}{3} f_S^2},$$  \hspace{1cm} (A9c)

$$\theta_0 = \phi - \arctan \left( \sqrt{2} f_{NS} / f_S \right).$$  \hspace{1cm} (A9d)

These relations (the same as already found by FKS [24 28]) are obtained in a general way, independently of specifics of any given approach; therefore, we can also apply them within our framework. In Sec. II, we have already pointed out the agreement of our results for our preferred state mixing angle $\phi \approx 42^\circ$ with the FKS results quoted in Refs. [24 28], but the values we find for $f_0, f_8, \theta_0,$ and $\theta_8$ are also similar to theirs. The important quantity here is $y = f_{NS}/f_S$, giving the extent of the SU(3) breaking. In our approach, this quantity is also present and plays a crucial role. The information it carries enables

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5Note that the last matrix in our Eq. (A8) is just $U^\dagger(\theta_{\text{ideal}})$ in the notation of Ref. [26], where $\theta_{\text{ideal}} \equiv \arctan \sqrt{2}$. 

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us to calculate what the decay-constant-mixing angles $\theta_8$ and $\theta_0$ would be in our approach (although we stress again that because of the way we calculate, their “rough average” $\theta$ must retain the central role because it has the meaning of the state-mixing angle). This is all because the $y$-ratio is essentially (as easily seen through the Goldberger–Treiman relation for constituent quarks) our parameter $X$, which can be (at least approximately) expressed as the ratio of the nonstrange-to-strange constituent quark mass: $X \approx \hat{m}/m_s$. In addition, in our very predictive coupled SD-BS approach, we can directly calculate all decay constants including $f_q = f_{NS}$ and $f_s = f_S$, and this again gives (in a good approximation) the same value for $X = y = f_{NS}/f_S$. While we have $f_{NS} = f_\pi$ (also assumed by FKS [26–28], in their theoretical analysis), our calculation yields $f_S = 1.4505f_\pi$, less than 3% more than the theoretical FKS prediction [26–28]. (Note that we used the symbol $f_{s\bar{s}}$ for $f_s = f_S$ [16].) Our chosen model therefore gives $y = f_{NS}/f_S = 0.6894$, leading to $f_8 = 1.318f_\pi$ and $f_0 = 1.170f_\pi$. (Interestingly, this is practically equal to our SD-BS model values $f_{\eta_8} = 1.31f_\pi$ and $f_{\eta_0} = 1.16f_\pi$ [16] for the octet and singlet axial-current decay constants $f_{\eta_8}$ and $f_{\eta_0}$, mentioned in Sec. IV. They are defined in the standard way through the matrix elements $\langle 0|A^{a\mu}\eta_a\rangle$, ($a = 8, 0$), so that the definitions (4) and (A3) imply that $f_{\eta_8}$ and $f_{\eta_0}$ are straightforwardly expressed through $f_{NS}$ and $f_S$ by the relations $f_{\eta_8} = \frac{1}{2}f_{NS} + \frac{2}{3}f_S$ and $f_{\eta_0} = \frac{2}{3}f_{NS} + \frac{1}{3}f_S$. We thus note that the quadratic relations (A9a) and (A9c) for differently defined octet and singlet constants $f_8$ and $f_0$, lead to similar values as the linear relations for $f_{\eta_8}$ and $f_{\eta_0}$.)

Using in Eqs. (A9) our preferred state mixing angle $\phi = 42^\circ$, our model value $y = f_{NS}/f_S = 0.6896$ also leads to the following decay-constant-mixing angles in the $\eta_8 - \eta_0$ basis: $\theta_8 = -22^\circ$ and $\theta_0 = -2.3^\circ$, close to the theoretical FKS results [26,27]. See also Table I in Ref. [28], line “FKS scheme & theory”, giving the $\theta_8 = -21.0^\circ$ and $\theta_0 = -2.7^\circ$, while the line “FKS scheme & phenomenology” in the same table has only somewhat more negative $\theta_0$ but larger $f_0/f_\pi$. The “FKS scheme & theory” then implies the state-mixing angle $\theta \approx -12^\circ$, in agreement with our results. This is as it should be, as we note that the mass matrix in Ref. [28], its Eq. (73), coincides with ours when the anomaly contribution $a^2$ is identified with the “Veneziano term” $\lambda_2^2/3(= \beta)$, as we do. Recalling that we have already pointed out the agreement of the $NS-S$ mixing angles obtained by us and by FKS [28], one can see that everything tallies.
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TABLE I. The first column displays results of refitting $\pi$, $K$ and $s\bar{s}$ masses ($m_{s\bar{s}} \equiv M_{s\bar{s}}$) obtained in the $q\bar{q}$ bound state SD-BS approach with the slightly changed explicit chiral symmetry breaking bare masses $\tilde{m}_{u,d} = 2.965$ MeV and $\tilde{m}_s = 69.25$ MeV. These $\pi$ and $s\bar{s}$-pseudoscalar masses are input parameters for $\eta-\eta'$ fit in Table II. The last column is the constituent quark mass $M_q(0)$ pertinent to the corresponding $q\bar{q}$ meson, namely $M_u(0) = M_d(0)$ for the pion and $M_s(0)$ for the unphysical $s\bar{s}$ pseudoscalar. The masses $m_P$ and $M_q(0)$ as well as the pseudoscalar axial-current decay constants $f_P$ are in units of GeV, while the $\gamma\gamma$ decay amplitudes $T_P(0,0)$ are in GeV$^{-1}$.

|   | $m_P$  | $f_P$  | $T_P(0,0)$ | $M_q(0)$ |
|---|--------|--------|------------|----------|
| $\pi$ | 0.1373 | 0.0931 | 0.257      | 0.374    |
| $K$   | 0.4957 | 0.113  |            |          |
| $s\bar{s}$ | 0.7007 | 0.135  | 0.0815     | 0.601    |

TABLE II. Various fits for the masses, mixing angle $\theta$, and $\gamma\gamma$ decay widths $\Gamma$ in the $\eta-\eta'$ complex. In columns A, B, and C, the free parameter in the mass matrix is $\beta$, whereas $X$ is free only in column C. Column A: results with $X = 1$ fixed by hand and $\beta$ fixed by fitting $m_\eta^2 + m_{\eta'}^2$. Column B: results with $X$ estimated from the ratio of the reduced strange and nonstrange $\gamma\gamma$ amplitudes, and $\beta$ fixed by fitting $m_\eta^2 + m_{\eta'}^2$. This column gives the best predictions, especially considering its only free parameter is $\beta$. Column C: results with $X$ treated as the second free parameter, making possible that $m_\eta$ and $m_{\eta'}$ are both fitted to their experimental values exactly. Column D: fitting the empirical $\gamma\gamma$ widths of $\eta$ and $\eta'$ with $\theta$ as the free parameter (and empirical $m_\eta$ and $m_{\eta'}$), independently of the masses and $\theta$ obtained from the mass matrix considerations. Column E: experimental values. Among the dimensionful quantities, $3\beta$ is in units of GeV$^2$, $m_\eta$ and $m_{\eta'}$ in GeV, while $\Gamma(\eta \to \gamma\gamma)$ and $\Gamma(\eta' \to \gamma\gamma)$ are in units of keV.

|   | A    | B    | C      | D       | E       |
|---|------|------|--------|---------|---------|
| $X$ | 1.0  | 0.673| 0.805  |         |         |
| $3\beta$ | 0.707 | 0.865| 0.801  |         |         |
| $\theta$ | $-19.5^\circ$ | $-11.1^\circ$ | $-14.9^\circ$ | $-12.8^\circ$ | -       |
| $m_\eta$ | 0.5048 | 0.5777| exp.   |         | 0.54730 |
| $m_{\eta'}$ | 0.9809 | 0.9398| exp.   |         | 0.95778 |
| $\Gamma(\eta \to \gamma\gamma)$ | 0.63 | 0.44 | 0.52   | 0.48    | 0.46 ± 0.04 |
| $\Gamma(\eta' \to \gamma\gamma)$ | 3.61 | 4.61 | 4.16   | 4.41    | 4.26 ± 0.19 |
APPENDIX: FIGURE CAPTIONS

Fig. 1: Nonperturbative QCD quark annihilation illustrated by the diagram with two-gluon exchange. It shows the transition of the $f \bar{f}$ pseudoscalar $P$ into the pseudoscalar $P'$ having the flavor content $f' \bar{f}'$. The dashed lines and full circles depict the $q \bar{q}$ bound-state pseudoscalars and vertices, respectively.
