Optomechanical force sensor in a non-Markovian regime

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Abstract

An optomechanical force sensor for a mechanical oscillator in a non-Markovian environment is presented. By performing homodyne detection, we obtain a general expression for the output signal. It is shown that the weak force detection is sensitive to the non-Markovian environment. The additional noise can be reduced and the mechanical sensitivity can be obviously amplified compared to the Markovian condition even in resolved sideband regimes without using assistant systems or squeezing. Our results provide a promising platform for improving the sensitivity of weak-force ultrasensitive detection.

1. Introduction

Optomechanical systems provide a platform for high-precision measurements, including ultrasensitive force detection [1], small quantities of adsorbed mass detection [2], and low-reflectivity object detection [3]. Such systems exploit the huge susceptibility around the resonance frequency of oscillators with excellent mechanical quality factor $Q_m$, combined with high-sensitivity interferometric measurements [1, 4]. The photon shot noise in the optomechanical systems will broaden the optical response spectrum and finally affect the sensitivity of detection during the frequency measurement [5, 6], which means that the shot noise should be reduced. However, reduced shot noise would increase quantum back-action noise force due to the opposite scalings with the optical field intensity [7]. Many schemes have been proposed to optimally compromise between photon shot noise and quantum back-action [8], which leads to the standard quantum limit (SQL) in weak force sensing [9, 10]. Various approaches to beyond-SQL measurements have been proposed [11–15], including optical squeezing in the optomechanical system [5, 14], atomic assistance in a separate cavity [13], mechanical modification by light [15], and so on. Up to now, most of the measurement schemes have been based on the Born–Markov approximation. The noise effect from a structured bath for optomechanical measurement still has not been discussed. On the other hand, improving detecting precision with a structured bath is also unresolved. Thus, investigation of measurement noise under a structured environment is a practical requirement for the further development of high-precision measurements.

Generally speaking, the quantal consideration of thermal noise of the optomechanical measurement system can be adequately described as a movable mirror undergoing quantum Brownian motion with coupling through the reservoir momentum [16]. The dynamics of this system are a non-Markovian process essentially. Since the non-Markovian environment exhibits a memory effect [17–21] that can be used to store quantum information [17], generate and protect entanglement [18, 22], and enhance the side-band cooling effect [19, 23], it might be of benefit for high-precision measurements due to the same requirements of quantum behavior protection. Most recently, a kind of non-Markovian environment for mechanical oscillators has been designed in which spectrum density of the environment was detected [24], making it possible to detect weak force under a structured environment. With this consideration in mind, we investigate detection property based on an elementary optomechanical system where the mechanical oscillator is coupled to a non-Markovian reservoir, while the bath of the cavity is a Markovian environment so as to output the oscillator signal through the cavity.

In this paper, we introduce a non-Markovian environment for a mechanical oscillator and obtain the solution of output signal under homodyne detection. Then we study the sensibility and additional noise of an optomechanical weak-force detection system with different spectrum densities $J(\omega)$, including that of the...
Markovian condition. We find that some environments with super-Ohmic or experimental cut-off spectra [24] do have obvious enhanced sensibility compared with that under Markovian conditions. Furthermore, we greatly reduce additional noise, even in the unsolved sideband regime.

2. Model

We consider a typical optomechanical system where the frequency of the cavity and mechanical resonator are \( \omega_c \) and \( \omega_m \), respectively. The weak force is sensed by the mechanical oscillator, and the environment noise simultaneously exerts a stochastic force to the oscillator. In order to detect the signal force, we assume that the mechanical oscillator is coupled to a non-Markovian reservoir so as to decrease stochastic force. Considering the feasibility, a Markovian environment for the optical mode easily outputs the optical signal to perform homodyne detection in the experiment. Therefore, we consider the optical mode in a Markovian regime. As shown in figure 1, the output signal can be processed in the standard homodyne detection. The Hamiltonian of the system can be described as

\[
H = H_S + H_E
\]

where \( H_S \) describes the cavity mode driven by a laser coupled to the mechanical resonator via radiation pressure with the coefficient \( g_0 = (\omega_c/L) \sqrt{\hbar/2m\omega_m} \), and \( \omega_d \) is the angular frequency of the laser. \( E \) is the cavity driving strength given by \( E = 2\sqrt{P\kappa_c/\hbar\omega_d} \), with \( P \) being the input power of the laser and \( \kappa_c \) being the input rate of the cavity. The first term of \( H_E \) is the energy of the mechanical reservoir for the \( k \)th environmental oscillator with frequency \( \omega_k \). The second term of \( H_E \) describes the coupling between the mechanical oscillator and the reservoir with the coupling strength \( \omega_k \gamma_k \) for the \( k \)th environmental mode. For convenience, we take \( \hbar = 1 \) throughout the paper. In the rotating frame at the driving laser frequency \( \omega_d \), the time evolution of the system and reservoir operators in the Heisenberg picture are

\[
a = -\left( i\Delta_c + \frac{\kappa}{2} \right) a + ig_0 a q_m + E + \sqrt{\kappa} a_m,
\]

\[
q_m = \omega_m p_m,
\]

\[
p_m = -\omega_m q_m + g_0 a^\dagger a - \sum_k \omega_k \gamma_k q_k,
\]

\[
q_k = \omega_k p_k,
\]
\[
\dot{p}_m = -\omega_m q_m - \omega_q \gamma_l q_m \gamma_l q_m ,
\]

(2c)

where \( \Delta_c = \omega_c - \omega_d \), and \( \kappa \) and \( \alpha_m \) denote the dissipation rate and noise operator of the cavity, respectively.

The autocorrelation function of the vacuum noise is \( \langle a_m(t) a_m^\dagger(\tau) \rangle = \delta(t - \tau) \). Solving equations (2d) and (2c), we have

\[
q_m(t) = q_m(0) \cos(\omega_q t) + p_m(0) \sin(\omega_q t) - \omega_q \gamma_l \int_0^t d\tau q_m(\tau) \sin[\omega_q (t - \tau)].
\]

(3)

Substituting it into equation (2c),

\[
\dot{p}_m = -\omega_m q_m + \gamma_m a + \int_0^t d\tau \{ f(t - \tau) q_m(\tau) + G^a a + G^d + F_m \},
\]

where \( f(t) = \sum \omega^2 \gamma_c^2 \sin(\omega_q t) \int \frac{d\omega}{2\pi} j(\omega) \sin(\omega t) \) and \( \xi(t) = -\sum \omega \gamma_c q_m(0) \cos(\omega_q t) + p_m(0) \sin(\omega_q t) \) is the input noise of the oscillator, which depends on the initial states of the reservoir. In the Markovian regime, this term is usually written as \( \sqrt{\gamma_m} F_m \), where \( \gamma_m \) is the dissipation rate of the mechanics and \( F_m \) is the noise operator. \( F_m \) is the external forces to be measured \([13, 14]\), which can be an accelerated mass \([25]\), magnetostrictive material \([26]\), atomic force \([27]\), or gravitational waves \([28]\).

Currently, most experimental realizations of cavity optomechanics are still in the single-photon, weak coupling with strong driving condition \([29–32]\). Under this condition, we can linearize the equations of motion around the steady state with \( p_{m0} \to p_m \to p_0 \), \( q_{m0} \to q_m \to q_0 \), \( a \to \alpha + a \), where \( p_0 \equiv \langle p_m \rangle \), \( q_0 \equiv \langle q_m \rangle \), and \( \alpha \equiv \langle a \rangle \). Neglecting the nonlinear terms, the linearized equation of motion can be written as

\[
\dot{a} = -\left(i \Delta_c^l + \frac{\kappa}{2} \right) a + i G q_m + \sqrt{\kappa} a_m ,
\]

\[
\dot{q}_m = \omega_m p_m ,
\]

\[
\dot{p}_m = -\omega_m q_m + \int_0^t d\tau \{ f(t - \tau) q_m(\tau) + G^a a + G^d + F_m \},
\]

(4)

where \( G = \alpha g_0 \) is the linearized coupling strength and \( \Delta_c^l = \Delta_c - g_0 q_0 \) denotes the effective detuning of the cavity. In order to solve the dynamics of the system and find the input noise sources, we now switch into the frequency domain by introducing the Fourier transform operator \( O(\omega) = \frac{1}{\sqrt{2\pi}} \int dt O(t)e^{i\omega t} \) and obtain

\[
a(\omega) = \chi_c(\omega) [i G q_m(\omega) + \sqrt{\kappa} a_m(\omega)] ,
\]

\[
q_m(\omega) = \chi_m(\omega) [G^a a(\omega) + G^d (-\omega) + F_m(\omega)] ,
\]

(5)

where \( \chi_c \equiv [\kappa / 2 - i(\omega - \Delta_c)]^{-1} \) and \( \chi_m \equiv -\omega_m/(\omega^2 - \omega_m^2) + \omega_m \Sigma(\omega) \) are susceptibilities of cavity and mechanical oscillator with \( \Sigma(\omega) = \int d\omega' \frac{\omega' j(\omega')}{(\omega^2 - \omega'^2)^{1/2}} + \frac{i\pi}{\omega(\omega - \Theta(\omega - \omega))} \) being the Laplace transform of the self-energy correction \([18, 33]\), where \( \Theta(\omega) \) is a step function. Here, \( \chi_m \) denotes the effect of the mechanical bath which depends on the spectrum density \( j(\omega) \). The commonly used Ohmic-type spectral density of the form \( j(\omega) = \eta \omega \) is a real number that determines the \( \omega \) dependence of \( j(\omega) \) in the low-frequency region. The baths with \( 0 < s < 1 \), \( s = 1 \), and \( s > 1 \) are termed as 'sub-Ohmic', 'Ohmic' and 'super-Ohmic' baths, respectively. In the Markovian condition, \( \chi_m = -\omega_m / [\omega^2 - \omega_m^2] + \gamma_m \) \( \omega_m \), where \( \gamma_m \) is the damping rate of the mechanical oscillator. Similarly, in the non-Markovian regime, we can also define the equivalent dissipation rate \( \gamma_{\text{eff}} \) which depends on the spectrum density \( j(\omega) \). Solving equation (3), we have

\[
q_m(\omega) = \frac{G^a \chi_c \sqrt{\kappa} a_m(\omega) + G \chi_m \sqrt{\kappa} a_m^\dagger(-\omega) + F_m(\omega)}{\chi_m^{-1} - iG[\chi_c - \chi_m]} ,
\]

(6)

where \( \chi_c' = [\kappa / 2 - i(\omega + \Delta_c)]^{-1} \). In equation (6), the coordinate of mechanical operator in the frequency domain is composed of two parts. The one term is proportional to the input field of the cavity through the radiation pressure coupling with the coefficient \( G \). The other term \( F_m(\omega) \) results from the bath of the oscillator and external force. If we neglect the effect from the cavity, we can rewrite equation (6) as \( q_m(\omega) = \chi_m(\omega) F_m \).

There is an obvious positive correlation between the external force and the position spectrum of the oscillator. For weak-force detection, we need a large susceptibility \( \chi_m(\omega) \) to magnify the weak signal \( F_m \). On the other hand, the thermal noise from the environment should be reduced because the noise can be coequally amplified with the detecting signal by the system. In the Markovian regime, the two requirements will demand a high mechanical quality factor and low bath temperature \([1]\). But it is more complex in that the non-Markovian condition, \( \chi_m(\omega) \) totally depends on the self-energy correction \( \Sigma(\omega) \), which is a frequency-dependent parameter up to the structure of the bath. We will specifically discuss this in the following section.

It is hard for us to directly detect the oscillator experimentally, but the signal from the external force can be obtained and enhanced by the cavity through the optomechanical interaction. Usually, we use the output photon from the optomechanical cavity as an indirect information carrier. Under the Markovian regime for the optical field, we can use the standard input–output relation \( O_{\text{out}} = \sqrt{\kappa} O - O_m \). Considering a homodyne
measurement shown in figure 1, we have the signal
\[ M_{\text{out}}(\omega) = i[a_{\text{out}}^+(\omega) e^{-i\theta} - a_{\text{out}}(\omega) e^{i\theta}] = A(\omega) a_{\text{in}}(\omega) + B(\omega) a_{\text{in}}^+(\omega) + C(\omega) F_{\text{in}}, \]
where
\[ A(\omega) = \frac{4e^{-i\theta} \kappa G^2 \chi_m}{\sqrt{2} [\Delta'_c^2 + (\kappa - 2\omega_i^2)]}, \]
\[ B(\omega) = \frac{4e^{-i\theta} \kappa G^2 \chi_m + i D^2 (4G^2 \chi_m + D^2) + 4\omega_i^2]}{\sqrt{2} [\Delta'_c^2 + (\kappa - 2\omega_i^2)]}, \]
\[ C(\omega) = \frac{2i \kappa G^2 \chi_m e^{-i\theta} D^2 (2\omega) - e^{-i\theta} G(2\omega)}{\sqrt{2} [\Delta'_c^2 + (\kappa - 2\omega_i^2)]}, \]
with \( D = 2\Delta'_c + i\kappa \), and the phase \( \theta \) is introduced and can be optimized to enhance the sensitivity of the weak-force detection [14].

To obtain the relationship between the detecting force and the output signal, we can rewrite equation (7) as
\[ \frac{M_{\text{out}}(\omega)}{C(\omega)} = \frac{A(\omega)}{C(\omega)} a_{\text{in}}(\omega) + \frac{B(\omega)}{C(\omega)} a_{\text{in}}^+(\omega) + F_{\text{in}}. \]

Considering \( F_{\text{in}} = F_{\text{ext}} + \xi(\omega) \) and defining \( M_{\text{out}}(\omega) / C(\omega) = F_{\text{ext}} + F_{\text{add}}(\omega) \), then we have
\[ F_{\text{add}} = \xi(\omega) + \frac{A(\omega)}{C(\omega)} a_{\text{in}}(\omega) + \frac{B(\omega)}{C(\omega)} a_{\text{in}}^+(\omega), \]
where \( F_{\text{add}} \) is the additional noise of the detecting force. The first term denotes the thermal noise operator of the mechanical environment, and the second and third terms denote the input noise of the cavity. From the general definition of the noise spectrum, we have
\[ S_{\text{add}}(\omega) = \frac{1}{2}[S_{FF}(\omega) + S_{FF}(-\omega)], \]
where \( S_{FF}(\omega) = \int d\omega' \langle F_{\text{add}}(\omega) F_{\text{add}}(\omega') \rangle \). Here, we assume that any two parts initially have no correlation. The vacuum radiation input noise \( a_{\text{in}} \) satisfies the \( \delta \)-correlation function. The additional noise spectrum density becomes
\[ S_{\text{add}}(\omega) = S_{\xi\xi}(\omega) + \frac{|A(\omega)|^2 + |B(\omega)|^2}{2|C(\omega)|^2}, \]
where \( S_{\xi\xi}(\omega) \) is thermal noise with the structured bath, which does not depend only on the bath temperature but also on the spectrum density \( \mathcal{J}(\omega) \). For simplicity, we choose \( \theta = 0 \), and then equation (12) can be rewritten as
\[ S_{\text{add}}(\omega) = S_{\xi\xi}(\omega) + \frac{|P(\omega)|^2 + |Q(\omega)|^2}{8\kappa G^2 D^2 - G^2 D^2 - 2\omega_i^2 G^2 + G^2}. \]

where
\[ P(\omega) = 4(\kappa G^2 - iD|G|^2) + \frac{i(D^2 - 4\omega_i^2)}{\chi_m}, \]
\[ Q(\omega) = 4(\kappa G^2 + iD^2|G|^2) - \frac{i(D^2 - 4\omega_i^2)}{\chi_m}. \]

3. Mechanical susceptibility and thermal correlation with a structured environment

Before we investigate the additional noise spectrum, we first analyze the effect of the ability of amplification \( \chi_m(\omega) \) and thermal noise spectrum \( S_{\xi\xi}(\omega) \) under non-Markovian environment. As we have mentioned in the last section, the sensibility of the mechanical oscillator for the weak force ultrasensitive detection in the optomechanical system is determined by the quantity \( \chi_m(\omega) \) and has been widely discussed in the Markovian regime [36, 37]. In the non-Markovian regime, \( \chi_m(\omega) \) is a spectrum-dependent parameter. According to [6], considering the effect of cavity, we now present the character of mechanical susceptibility by plotting \( \chi_m(\omega)/\chi_{x0} \) as a function of \( \omega \) with the commonly used Ohmic-type spectrum in figures 2(a) and (b), where \( \chi_{x0} = \chi_{x0}^{-1} - i|G|^2(\chi_c - \chi_s) \) denotes the mechanical sensitivity, \( \chi_{x0} = -\gamma_{\text{eff}} - i|G|^2[\chi_c(\omega_m) - \chi_s(\omega_m)] \) is
the optimal mechanical susceptibility in the Markovian regime. As shown in figure 2(a), it is clearly seen that the maximal sensitive frequency area is around the oscillator frequency $\omega_m$ in the Markovian regime, which is consistent with the general results in weak-force detection in [1]. For Ohmic-type spectrum, a super-Ohmic environment could provide an obvious amplification for susceptibility of detection. We also notice that a structured bath will cause a frequency displacement of the maximal susceptibility because $\omega_m$ is substituted by the effective frequency $\omega_{eff}$.

In figure 2(b), we plot the maximal ratio $\chi_{mm}(\omega)/\chi_{mm}$ of the susceptibility with a different environment as a function of the equivalent damping rate $\gamma_{eff}$. Here, $\gamma_{eff}$ is a spectrum-dependent parameter which describes the dissipation strength of the structured bath; through the inverse Laplace transform of $\Sigma(\omega)$, we have $\gamma_{eff} \approx \pi \int \mathcal{J}(\omega) \, d\omega$ [38]. Under this condition, $\gamma_{eff}$ is proportional to the system--bath coupling strength $\eta$. It is shown that the maximal ratio of $\chi_{mm}(\omega)/\chi_{mm}$ exhibits vibration behavior with the increase of an effective dissipation rate $\gamma_{eff}$. The maximal sensitivity ratio of the system will reach the peak value at some specific effective dissipation rate $\gamma_{eff}$ and does not require the system–environment coupling factor $\eta$, which is too strong. Thus, we can enhance the sensitivity with a structured environment, and the corresponding detection frequency should also be modulated.

The thermal noise $S_{\xi}(\omega)$ as background noise negatively affects the weak-force detection. In order to improve the precision of the weak-force detection, we should reduce the effect result from the thermal noise of the bath of the oscillator. We consider a movable mirror undergoing quantum Brownian motion reservoir. The thermal-noise spectral density is defined as $S_{\xi}(\omega) \equiv \int_{-\infty}^{\infty} dt \, e^{i \omega t} \langle [\xi(t), \xi(0)] \rangle$. Considering the structure of the environment, we have

$$S_{\xi}(\omega) = \int_{-\infty}^{\infty} dt \, e^{i \omega t} \int_{0}^{\infty} d\omega' \mathcal{J}(\omega') \left[ n_{ib}(\omega') \cos(\omega't) + \frac{1}{2} e^{-\omega't} \right],$$

where $n_{ib}(\omega) = (e^{\omega h} - 1)^{-1}$ is the phononic distribution function of the reservoir, $\mathcal{J}(\omega)$ describes the character of the reservoir. For Born–Markov approximation where the system–reservoir coupling rate is weak and the interaction time is short enough, the environment can be described as a flat spectrum and the integral for $\omega$ is a delta function of time; therefore, the environment presents no memory effect for the system, i.e.

$S_{\xi} = \gamma_{m} n_{ib}(e^{\omega h} - 1)$.

As shown in figure 2(c), we plot the thermal noise spectral density for Markovian and non-Markovian environments as a function of $\omega$. For the exponent $s = 0.5, 1, 2$ the corresponding coupling strength of system--bath for sub-Ohmic, Ohmic, and super-Ohmic are $\eta_{0.5} = 5.5 \times 10^{-3}, \eta_1 = 1.2 \times 10^{-2}, \eta_2 = 6.1 \times 10^{-2}$, respectively, when we choose the same equivalent damping rate $\gamma_{eff} = \pi \times 10^{-3} \omega_m$ as that for Markovian condition. For a fair comparison, the other parameters are also selected the same for different structured baths. From figure 2(c), we see that different structures of baths will cause different distributions of thermal excitation. But, around frequency $\omega_m$, $S_{\xi}$ is below 0.025 around $\omega_m$ in $\omega_{eff}$ and just shifts slightly, we can reasonably ignore the thermal noise $S_{\xi}$ because one can observe that $S_{\xi}$ is below 0.025 around $\omega_m$.  

Figure 2. (a) Ratio of mechanical susceptibility $\chi_{mm}/\chi_{mm}$ as a function of $\omega$ with Markovian condition and Ohmic-type spectrum. $s = 0.5, 1, 2$ for three kinds of Ohmic-type spectrum, respectively, and the equivalent damping rate $\gamma_{eff}/\omega_m = \pi \times 10^{-3}$. (b) The maximal ratio of mechanical susceptibility $\chi_{mm}/\chi_{mm}$ as a function of the equivalent damping rate $\gamma_{eff}$ for a different spectrum. (c) Thermal noise with three kinds of Ohmic-type spectrum densities. In the Markovian regime, the noise exists only at the frequency $\omega_m$. The directrix is plotted by the black dashed line, while $s = 0.5$ for the sub-Ohmic spectrum, $s = 1$ for the Ohmic spectrum, and $s = 2$ for the super-Ohmic spectrum. The equivalent dissipation rate $\gamma_{eff} = \pi \times 10^{-3} \omega_m$. Other parameters are the oscillator frequency $\omega_m = 10^5$ Hz, bath temperature $T = 1$ mK, and cut-off frequency $\omega_0/\omega_m = 10$. 

\begin{align}
S_{\xi}(\omega) &= \int_{-\infty}^{\infty} dt \, e^{i \omega t} \int_{0}^{\infty} d\omega' \mathcal{J}(\omega') \left[ n_{ib}(\omega') \cos(\omega't) + \frac{1}{2} e^{-\omega't} \right], \\
\end{align}
\( \omega_m \), Under Markovian reservoirs, according to equation (15), for the commonly used detection frequency area \( \omega_m \), the noise \( S_{\text{opt}} \approx \gamma_{\text{eff}} \frac{k_B T}{\hbar \omega_m} = k_B T / (\hbar Q_{\text{eff}}) \). Thus we can reduce the thermal noise by cooling down the system [19] or improve the effective mechanical quality factor \( Q_{\text{eff}} \) directly. According to the experiment parameters in nano-mechanical systems [12], where \( \omega_m = 2 \pi \times 1.04 \) MHz, mechanical quality factor \( Q_m = 6.2 \times 10^3 \), and environment temperature \( T = 77 \) mK, the thermal noise \( S_{\text{opt}} / \omega_m \ll 1 \), which means we can ignore the thermal noise for weak-force detection under the current experimental conditions [6].

4. Additional noise with a structured environment

For weak-force detection, in addition to a high sensitivity, good linearity, and high response speed, we expect to reduce additional noise, which is also widely used as a detection waveband such as weak-force detection through optomechanical induced transparency [39], microwave quantum illumination by optomechanical systems [3], or gravitational-wave detectors with unstable optomechanical filters [28]. Now we show that, in certain environments, we can obtain high sensitivity and reduced additional noise.

Recently, a spectral density of a mechanical environment was detected experimentally through the emitted light of a micro-optomechanical system [24]. The demonstration device consists of a thick layer of SiNx with a high-reflectivity mirror pad in its center as a mechanically moving end mirror in a Fabry–Pérot cavity where the spectral density is described by \( \mathcal{J}(\omega) = C \omega^k \), with \( C > 0 \) and \( k = -2.30 \pm 1.05 \). The region of \( \omega \) satisfies \( \omega \in [\omega_{\text{min}}, \omega_{\text{max}}] \) centered around mechanical resonance frequency \( \omega_m = 914 \) kHz. Here, \( \omega_{\text{min}} = 885 \) kHz, \( \omega_{\text{max}} = 945 \) kHz, and the corresponding bandwidth \( \Gamma = \approx 0.07 \omega_m \). We employ this cut-off experimental spectral density \( \mathcal{J}(\omega) = C \omega^k \), where \( C = \mathcal{J}(\omega_m) / \omega_m^k \), and we choose the bandwidth \( \Gamma_m = 0.2 \omega_m \), exponent \( k = -2 \).

We plot the additional noise and the susceptibility for the different types of environments of the mechanical oscillator in figure 3, where we reasonably ignore thermal noise \( S_{\text{opt}} \) around \( \omega_m \) [6] in equation (13), according to the conclusion in Section 3. As shown in figure 3(a), we plot the optimal additional noise \( S_{\text{add}} \) as a function of linearized coupling rate \( G \). It is obvious that, for different spectrum, there are minimum values of \( S_{\text{add}} \) at certain value of \( G \). In addition to the super-Ohmic spectrum, the evolution trend of the curve with the coupling rate \( G \) is almost the same. When the coupling rate \( G / \omega_m \) is less than 0.017, the additional noise of the super-Ohmic spectrum is larger than that of other ones. On the contrary, i.e., \( G / \omega_m > 0.017 \), the additional noise of the super-Ohmic spectrum will be less than that of other spectrums. In the large \( G \) scale, the additional noise is independent of the structure of the environment. Under this region, the additional noise is mainly governed by the vacuum fluctuations of the cavity through optomechanical interaction, and the noise from the mechanical environment can be ignored. Thus, in order to reduce the additional noise, the driving strength of the cavity should not be too strong.
In figure 3(b), we plot the optimal additional noise $S_{\text{add}}$ as a function of damping rate $\kappa$, which can be adjusted by Q-technology in the measurement [40]. As shown in figure 3(b), evolution curves of the additional noise with the dissipation rate $\kappa$ tend to be consistent, except for the super-Ohmic spectrum. There is a peak value of the additional noise at the specific dissipation rate $\kappa$. The additional noise decreases first and then increases as the dissipation rate $\kappa$ increases. For the super-Ohmic spectrum, when the dissipation rate $\kappa/\omega_m$ is less than a specific value 0.16, the additional noise is much smaller than that of other structures. With the increase of the dissipation rate, the additional noise of the super-Ohmic spectrum will be larger than that of other ones. In the resolved sideband regime, the additional noise of the system for different spectral structures can be maintained at a low level $10^{-3}$. In other words, the small value of damping rate can help us reduce the additional noise for the super-Ohmic spectrum. However, there is no obvious effect on reducing the additional noise for other spectral structures under the same effective dissipation $\kappa$.

Employing the optimized parameters based on figures 3(a) and (b), we plot the additional detection noise and susceptibility in the frequency region shown in figures 3(c) and (d). The additional noise $S_{\text{add}}/\omega_m$ can be reduced to near $10^{-3}$ at the effective frequency $\omega_m$. The mechanical sensitivity of the system has been significantly improved for the Ohmic-type spectrum. Especially for the super-Ohmic spectrum, the sensitivity is about $10^2$ times of the Markovian condition. Comparing figures 3(c) and (d), one can observe that the frequency region with minimum detection noise is exactly the frequency region with optimal sensitivity. As shown in the numerator of the second term of equations (13) and (14), the maximal sensitivity $|\chi_m|$ results in the optimal $S_{\text{add}}$ (Because $D$ is independent of frequency $\omega$). Thus, in our scheme, we can detect the weak force with maximal sensitivity and minimum additional noise.

Considering the feasibility, we can easily adjust the rate $G/\omega_m$ by controlling the driving power of the cavity so that $G/\omega_m < 0.03$ (see figure 3(a)). Since the cut-off spectrum [24] has been realized in experiment, we can employ it to reduce the additional noise even in the unsolved sideband regime, which shows in figure 3(b). In order to maintain the coherence, the low loss rate of the mechanical oscillator is still needed.

5. Force sensing in general environment

In order to show the advantages of the detection under non-Markovian environment, we provide an example of measuring the mass of the human chromosome-1. The mass of one chromosome-1 molecule is about $2.7 \times 10^{-13}$ g [41]. The external accretion mass will introduce an additional frequency responded to by the mechanical resonator. The mass response of the mechanical resonator can be defined as $R = \partial \omega/\partial m$, with the typical value $R = 10^{22}$ Hz \cdot g^{-1} [42, 43]. Then we deposit a few chromosomes onto the surface of the mechanical resonator and observe the output signal of the system. As shown in figure 4, we plot the output signal $S_{\text{out}}(\omega)$ (see Appendix for details) of the optomechanical mass sensor with differently structured environment. For fair comparison, we choose the same effective dissipation rates as well as optomechanical cavity parameters for different environments. From figures 4(a) to (e), we can see that the energy of the output spectrum of the
resonance frequency increases as the number of adsorbed chromosomes increases. That is to say, we can detect the number of chromosomes by measuring the strength of the resonant output energy \( I_{\text{out}} = S_{\text{out}}(\omega_{\text{eff}}) \). By comparing the Markovian condition Ohmic-type spectra and experimental cut-off spectrum in figure 5(a), we find that the detection energy response is enhanced while the noise is reduced (the bandwidth is narrowed) for the Ohmic-type and cut-off spectra. Especially for super-Ohmic spectrum, the energy response of mass detection is almost \( 5 \times 10^4 \) times that of the Markovian conditions. This conclusion is similar to what we discussed in the previous section: we can detect the weak force with maximal sensibility and minimum additional noise in specific non-Markovian environments.

As shown in figure 5(a), there is a significant linear relationship between the resonance response energy \( I_{\text{out}} \) and the number of the chromosomes \( N \). Therefore, our scheme is totally consistent with the basic requirements of weak-force detection. As shown in figure 5(b), by comparing two different conditions, we can see that the output energy of the detection can be concentrated in a small area around the effective frequency \( \omega_{\text{eff}} \) due to the specific structure of the super-Ohmic spectrum. The sensitivity or the response energy of the sensor to the input signal in a super-Ohmic environment is much higher than that in the Markovian condition, which can be seen in the subgraph of figure 5(b). As we have chosen the same effective dissipation rate of the oscillator and other parameters of the optomechanical system, the noise energy from the cavity and mechanical environment are the same. Therefore, the signal–noise ratio of the mass sensor in the super-Ohmic environment is larger than that in the Markovian condition while the bandwidth of the output spectrum is narrowed. This process is similar to 'squeezing' the response energy of the input signal. We can understand the mechanism of the optimized weak-force detection in the non-Markovian regime by analyzing the response of the output field to the non-Markovian environment. The mechanical oscillator is a sensor of weak force while the external force can be regarded as a part of the mechanical environment undoubtedly. The environment of the sensor affects the weak-force detection, which can be seen in figure 4. Thus, the response of the weak-force detection system depended on the coupling effect between the mechanical oscillator and its environment. The only difference in the comparison between the Markovian and non-Markovian condition is the characters of the environment structure. In Born–Markov approximation, the environment is equivalent to a flat spectrum. The effective response of the oscillator to the environment is the combination of the average coupling effect of all bath modes and frequency detuning between the mechanical mode and bath mode, which can be seen in the subgraph of figure 5(b), where the response coefficient \( \chi_{\text{xm}} \) is a symmetrical distribution around \( \omega_{\text{m}} \). However, the system–bath response is dependent on the environment spectrum \( f(\omega) \) for structured bath.

In order to understand the reason why the super-Ohmic are superior to others, we plot figure 5(c). As shown in figure 5(c), the \( f(\omega) \) distribution of the super-Ohmic spectrum mainly concentrates in the low-frequency region (detection region), and the contribution of the high-frequency mode of the environment can even be ignored. But the distribution for sub-Ohmic and Ohmic spectra are more gentle than the super-Ohmic spectrum. Therefore, we can safely say that the super-Ohmic and experimental cut-off spectra are more far away from the flat spectrum of the Markovian environment than that for the sub-Ohmic and Ohmic spectra. As we all know, the Markovian environment only contributes a stochastic force. Therefore, it is reasonable that the non-Markovian back action can reduce the noise. Since the super-Ohmic spectrum is the most different from the Markovian flat spectrum, it can be the best for decreasing noise force. The subgraph in figure 5(b) clearly shows that the response coefficient \( \chi_{\text{xm}} \) of the super-Ohmic is much higher than that for the other spectrum in

\[ I_{\text{out}} / \omega_m = \text{constant} \]

\[ S_{\text{out}}(\omega_{\text{eff}}) \]

\[ f(\omega) \]

\[ \chi_{\text{xm}} \]
the detection frequency region. In addition, the superiority of the super-Ohmic spectra over the Ohmic and sub-Ohmic spectra in decreasing noise [44] had also been observed in cooling the mechanical oscillator [19]. Therefore, in our scheme, the output signal of the weak force detection could exhibit high response and high sharpness spectrum in the non-Markovian regime, which can be implemented to improve the detection accuracy for both the energy [45, 46] and frequency response sensors [41, 47].

In the preceding discussion, the environment of the optical cavity is Markovian. If we would like to introduce a non-Markovian environment for the cavity field, the additional term \( H_{CE} = \sum_k \{ \hbar \kappa_k b_k^\dagger b_k + i \hbar \kappa_k (b_k a_k^\dagger - h.c.) \} \) should be added in the Hamiltonian \( H \). Instead, equation (2a) should be substituted by another two equations, and the output relation also should be renewed, so the problem becomes very complicated. It is hard for us to directly foresee the function of the non-Markovian environment of the cavity field. We will finish it elsewhere.

6. Conclusion

We investigate the weak-force detection of an optomechanical system in a non-Markovian regime. By solving the exact dynamics of the optomechanical system, we obtain a general analytical result of the output signal. We have shown that: (i) The thermal noise for weak-force detection can be ignored even under non-Markovian environments, while the susceptibility is efficiently amplified in the effective frequency region \( \omega_{\text{eff}} \). (ii) The additional noise can be significantly reduced in the structured bath, especially for super-Ohmic spectra. The additional noise can be maintained at a quite low level and the quantum effect can be better protected with a structured bath by comparison with the Markovian condition in a resolved sideband regime. (iii) Employing super-Ohmic environments to reduce the additional noise and amplification detection signal does not require a high-quality cavity. Meanwhile, optimized \( G / \omega_{\text{in}} \) is demanded. Furthermore, we provide an example by comparing the Markovian and non-Markovian conditions to measure the mass of the human chromosome-1, and then we analyze the mechanism of the optimization detection in the non-Markovian regime. Instead of introducing squeezing and improving the experimental conditions, such low bath temperature and high mechanical quality factor, our results provide another effective way for reducing the additional noise by utilizing the engineered non-Markovian reservoir in ultrasensitive detection.

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Appendix Output signal of the mass sensor

By using the general definition of the noise spectrum, according to equation (7), we can obtain the output signal of the weak force detection system

\[
S_{\text{out}}(\omega) = \frac{1}{2} \left[ |A(\omega)|^2 + |B(\omega)|^2 + S_{\text{in}}(\omega) |C(\omega)|^2 \right],
\]

where \( S_{\text{in}}(\omega) \) denotes the input signal from the external force \( F_{\text{ext}} \) after ignoring the thermal noise of the mechanical oscillator. Here, we intend to measure the mass of the human chromosome-1 as example, where the external accretion mass will introduce an additional frequency responded to by the mechanical resonator with mass responsivity \( R = 10^{11} \text{ Hz} \cdot \text{g}^{-1} \). Then we deposit a few chromosomes onto the surface of the mechanical resonator. The additional energy of the input signal can be described as \( S_{\text{in}} = N m R \), where \( m = 2.7 \times 10^{-13} \text{ g} \) is the mass of one chromosome-1 molecule, \( N \) is the number of the deposit chromosomes. In the non-Markovian regime, \( \omega_{\text{in}} \) is replaced by the effective frequency \( \omega_{\text{eff}} \), and the optimal detection area should be on resonance with this effective frequency. We define the strength of the frequency resonance spectrum \( I_{\text{out}} = S_{\text{out}}(\omega_{\text{eff}}) \), where \( \omega_{\text{eff}} = \omega_{\text{in}} \) in the Markovian regime. The energy of \( I_{\text{out}} \) will linearly increase as the number of adsorbed chromosomes increases. This allows us to detect the number of the chromosomes by measuring the strength of the resonant output energy.

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