Approximations of the Images and Integral Funnels of the $L_p$ Balls under a Urysohn-Type Integral Operator

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Received December 26, 2021; in final form, June 2, 2022; accepted June 10, 2022

ABSTRACT. Approximations of the image and integral funnel of a closed ball of the space $L_p$, $p > 1$, under a Urysohn-type integral operator are considered. A closed ball of the space $L_p$, $p > 1$, is replaced by a set consisting of a finite number of piecewise constant functions, and it is proved that, for appropriate discretization parameters, the images of these piecewise constant functions form an internal approximation of the image of the closed ball. This result is applied to approximate the integral funnel of a closed ball of the space $L_p$, $p > 1$, under a Urysohn-type integral operator by a set consisting of a finite number of points.

KEY WORDS: Urysohn integral operator, image of $L_p$ ball, integral funnel, approximation, input-output system.

DOI: 10.1134/S0016266322040050

1. Introduction

Nonlinear integral operators arise in mathematical models of various physical, mechanical, economic, and biological phenomena. Note that integral models have certain advantages over differential ones. For example, outputs for such systems can be defined as continuous, even $p$-integrable, functions (see, e.g., [1]–[4]). In particular, the mathematical models of various input-output systems are based on integral operators of Urysohn type. Therefore, the construction of the set of images and integral funnel of the input functions under a given integral operator is very important from the point of view of applications.

Two of the important constructions of the theory of control systems described by ordinary differential equations are those of an attainable set and an integral funnel. An attainable set of a system is defined as the set of the points in the phase space to which the trajectories of the system at a given instant of time arrive. The integral funnel of a system is defined in the extended phase space as the set consisting of the graphs of trajectories generated by all admissible control functions and is a generalization of the integral curve notion from the theory of differential equations (see, e.g., [5]–[7]). The attainable sets and the integral funnel of a system include the complete information about this system and often make it possible to construct trajectories with prescribed properties (see, e.g., [8]). Various topological properties and approximate construction methods of the attainable sets and the integral funnel of a given control system are the topics of a vast number of investigations (see, e.g., [9]–[12] and references therein). For a linear control system, the attainable sets can be described as the image of the set of control functions under appropriate Volterra, Fredholm, or Hilbert–Schmidt integral operators.

In this paper internal approximations of the image and integral funnel of a closed ball of the space $L_p(\Omega; \mathbb{R}^m)$, $p > 1$, with radius $r$ centered at the origin under a Urysohn-type integral operator are studied. The integral funnel is defined as the set of graphs of the images of all functions from the $L_p$ ball under consideration. The closed ball is replaced by its subset consisting of a finite number of piecewise constant functions. Using the Steklov average of an integrable function and introducing $\Delta$-partitions of a compact set, we approximate the image of the closed ball by the images of these piecewise constant functions. The obtained result allows us to approximate the integral funnel by a set consisting of a finite number of points.
The presented results can be applied to approximate the set of outputs of an input-output system described by a Urysohn-type integral operator, where the inputs are chosen from a closed ball of the space $L_p(\Omega; \mathbb{R}^m)$, $p > 1$. Such inputs in general characterize consumption, e.g., of energy, fuel, finance, food, etc. (see, e.g., [13]–[15]).

The paper is organized as follows. In Section 2 the basic conditions and propositions which are used in the further arguments are given. In Section 3, step by step, a closed ball of the space $L_p$ is replaced by a set consisting of a finite number of piecewise constant functions. It is proved that the set of images of these piecewise constant functions is an internal approximation of the image of the closed ball under the integral operator under consideration (Theorem 3.1). An adequate approximation for the integral funnel is also given by the same theorem.

2. Preliminaries

Consider the Urysohn-type integral operator

$$U(x(\cdot))(\xi) = \int_{\Omega} K(\xi, s, x(s)) ds,$$

(2.1)

where $\xi \in E$, $s \in \Omega$, $E \subset \mathbb{R}^b$ and $\Omega \subset \mathbb{R}^k$ are compact sets, $x(\cdot) \in V_{p,r}$,

$$V_{p,r} = \{ x(\cdot) \in L_p(\Omega; \mathbb{R}^m) : \|x(\cdot)\|_p \leq r \},$$

(2.2)

$p > 1$, $L_p(\Omega; \mathbb{R}^m)$ is the space of Lebesgue measurable functions $x(\cdot): \Omega \to \mathbb{R}^m$ such that $\|x(\cdot)\|_p < +\infty$, $\|x(\cdot)\|_p = \left( \int_{\Omega} \|x(s)\|^p ds \right)^{1/p}$, and $\| \cdot \|$ denotes the Euclidean norm. It is assumed that the function $K(\cdot)$ satisfies the following conditions.

A. The function $K(\cdot): E \times \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.

B. There exists an $l_0 > 0$ such that

$$\|K(\xi, s, x_1) - K(\xi, s, x_2)\| \leq l_0 \|x_1 - x_2\|$$

for any $(\xi, s, x_1) \in E \times \Omega \times \mathbb{R}^m$ and $(\xi, s, x_2) \in E \times \Omega \times \mathbb{R}^m$.

C. There exist functions $\omega(\cdot, \cdot): \Omega \times \mathbb{R}^m \to [0, +\infty)$ and $\varphi(\cdot): [0, +\infty) \to [0, +\infty)$ and numbers $\beta_0 \geq 0$ and $\beta_1 \geq 0$ such that

$$\omega(s, x) \leq \beta_0 \|x\| + \beta_1, \quad \varphi(0) = 0, \quad \varphi(\tau) \to 0^+ \text{ as } \tau \to 0^+$$

for every $(s, x) \in \Omega \times \mathbb{R}^m$ and

$$\|K(\xi_1, s, x) - K(\xi_2, s, x)\| \leq \omega(s, x) \cdot \varphi(\|\xi_1 - \xi_2\|)$$

for any $(\xi_1, s, x) \in E \times \Omega \times \mathbb{R}^m$ and $(\xi_2, s, x) \in E \times \Omega \times \mathbb{R}^m$.

We denote

$$U_{p,r} = \{ U(x(\cdot))(\cdot) : x(\cdot) \in V_{p,r} \},$$

(2.3)

$$U_{p,r}(\xi) = \{ y(\xi) \in \mathbb{R}^n : y(\cdot) \in U_{p,r} \}, \quad \xi \in E,$$

(2.4)

$$\mathcal{F}_{p,r} = \{(\xi, y(\xi)) \in E \times \mathbb{R}^n : y(\cdot) \in U_{p,r} \}.$$  

(2.5)

It is obvious that the set $U_{p,r}$ is the image of the closed ball $V_{p,r}$ under the Urysohn integral operator (2.1) and the set $\mathcal{F}_{p,r}$ consists of the graphs of the functions in $U_{p,r}$. The set $\mathcal{F}_{p,r}$ is called the integral funnel of the set $V_{p,r}$ under the Urysohn integral operator (2.1).
Conditions A and B imply that, for each \( x( \cdot ) \in V_{p,r} \), the image \( U(x( \cdot ))(\cdot ) \) is a continuous function and the set \( U_{p,r} \) is a bounded subset of the space \( C(E; \mathbb{R}^n) \), where \( C(E; \mathbb{R}^n) \) is the space of continuous functions \( w(\cdot): E \rightarrow \mathbb{R}^n \) with norm \( \| w(\cdot) \|_C = \max \{ \| w(\xi) \|: \xi \in E \} \). We set

\[
B_C(1) = \{ y(\cdot) \in C(E; \mathbb{R}^n): \| y(\cdot) \|_C \leq 1 \},
\]

\[
\alpha_* = M_0 \mu(\Omega) + l_0 r [\mu(\Omega)]^{(p-1)/p},
\]

\[
\beta_* = \beta_1 \mu(\Omega) + \beta_0 r [\mu(\Omega)]^{(p-1)/p},
\]

where \( \mu(\Omega) \) denotes the Lebesgue measure of the set \( \Omega \), the numbers \( l_0, \beta_0, \) and \( \beta_1 \) are defined in Conditions B and C, and \( M_0 = \max\{\| K(\xi, s, 0) \|: (\xi, s) \in E \times \Omega\} \).

The Hausdorff distance between sets \( G \subset \mathbb{R}^n \) and \( Q \subset \mathbb{R}^n \) is denoted by the symbol \( H_n(G, Q) \), and the Hausdorff distance between sets \( W \subset C(E; \mathbb{R}^n) \) and \( Y \subset C(E; \mathbb{R}^n) \) is denoted by the symbol \( H_C(W, Y) \). Conditions A–C imply the following propositions.

**Proposition 2.1.** The inequality

\[
\| y(\cdot) \|_C \leq \alpha_*,
\]

holds for every \( y(\cdot) \in U_{p,r} \), where \( \alpha_* \) is defined by (2.7).

**Proposition 2.2.** For any \( y(\cdot) \in U_{p,r} \), \( \xi_1 \in E \), and \( \xi_2 \in E \), the inequality

\[
\| y(\xi_1) - y(\xi_2) \| \leq \beta_* \cdot \varphi(\| \xi_1 - \xi_2 \|)
\]

holds, and hence

\[
H_n(U_{p,r}(\xi_1), U_{p,r}(\xi_2)) \leq \beta_* \cdot \varphi(\| \xi_1 - \xi_2 \|)
\]

for any \( \xi_1 \in E \) and \( \xi_2 \in E \), where \( \beta_* \) is defined by (2.8).

Proposition 2.2 implies the convergence

\[
H_n(U_{p,r}(\xi), U_{p,r}(\xi_*)) \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \xi_*,
\]

for each fixed \( \xi_* \in E \).

The Arzelà–Ascoli theorem and Propositions 2.1 and 2.2 imply the precompactness of the set \( U_{p,r} \).

**Proposition 2.3.** The set \( U_{p,r} \) is a precompact subset of the space \( C(E; \mathbb{R}^n) \).

Now let us give the definition of a finite \( \Delta \)-partition of a set \( Q \subset \mathbb{R}^{n*} \), where \( \Delta > 0 \) is a given number.

**Definition 2.1.** Let \( Q \subset \mathbb{R}^{n*} \) be a given set, and let \( \Delta > 0 \) be a given number. A finite system of sets \( \mathcal{T} = \{ Q_1, \ldots, Q_T \} \) is said to be a finite \( \Delta \)-partition of \( Q \) if

\( (d_1) \) \( Q_i \subset Q \) and \( Q_i \) is Lebesgue measurable for every \( i = 1, \ldots, T; \)

\( (d_2) \) \( Q_i \cap Q_j = \emptyset \) for every \( i \neq j \), where \( i = 1, \ldots, T \) and \( j = 1, \ldots, T; \)

\( (d_3) \) \( Q = \bigcup_{i=1}^{T} Q_i; \)

\( (d_4) \) \( \text{diam}(Q_i) \leq \Delta \) for every \( i = 1, \ldots, T \), where \( \text{diam}(Q_i) = \sup \{ \| x - y \| : x \in Q_i, y \in Q_i \} \).

Since \( \mu(Q_i) \rightarrow 0^+ \) as \( \text{diam}(Q_i) \rightarrow 0^+ \), we can assume without loss of generality that, for the partition \( \mathcal{T} = \{ Q_1, \ldots, Q_T \} \), the inequality \( \mu(Q_i) \leq \Delta \) is also satisfied for every \( i = 1, \ldots, T \).

**Proposition 2.4.** Let \( Q \subset \mathbb{R}^{n*} \) be a compact set. Then, for every \( \Delta > 0 \), it has a finite \( \Delta \)-partition.

The following proposition will be used in the further arguments.
Proposition 2.5. Let \((X, d)\) be a metric space, and let \(P \subset X\) be a precompact set. Suppose that \(P_i \subset P_{i+1} \subset P\) for every \(i = 1, 2, \ldots\) and 
\[
H_X \left( P, \bigcup_{i=1}^{\infty} P_i \right) \leq \theta_*. 
\]
Then, for every \(\varepsilon > 0\), there exists an \(i(\varepsilon) > 0\) such that, for each \(i \geq i(\varepsilon)\), the inequality 
\[
H_X(P, P_i) \leq \theta_* + \varepsilon
\]
is satisfied, where \(H_X(\cdot, \cdot)\) stands for the Hausdorff distance between subsets of the metric space \((X, d)\).

3. Approximation

Let \(\gamma > 0\), and let \(\Lambda = \{0 = w_0 < w_1 < \ldots < w_q = \gamma\}\) be a uniform partition of the closed interval \([0, \gamma]\) with \(\delta = w_{\lambda+1} - w_\lambda\) for \(\lambda = 0, 1, \ldots, q - 1\). Since \(\Omega \subset \mathbb{R}^k\) and \(E \subset \mathbb{R}^b\) are compact sets, it follows from Proposition 2.4 that, for every \(\Delta > 0\), they have finite \(\Delta\)-partitions \(\gamma_1 = \{\Omega_1, \ldots, \Omega_M\}\) and \(\gamma_2 = \{E_1, \ldots, E_N\}\), respectively.

Let \(S = \{x \in \mathbb{R}^m : \|x\| = 1\}\), and, for given \(\sigma > 0\), let \(S_\sigma = \{b_1, \ldots, b_g\}\) be a finite \(\sigma\)-net of \(S\). An algorithm for specifying a finite \(\sigma\)-net of \(S\) is given in [10]. We set 
\[
V_{p,r}^\gamma,\gamma_1,\Lambda,\sigma = \{x(\cdot) \in V_{p,r} : x(s) = w_\lambda b_i, \text{ for every } s \in \Omega_j, \quad j = 1, \ldots, M, w_\lambda, b_i \in S_\sigma\} \quad (3.1)
\]
and let 
\[
U_{p,r}^\gamma,\gamma_1,\Lambda,\sigma = \{U(x(\cdot))(\cdot) : x(\cdot) \in V_{p,r}^\gamma,\gamma_1,\Lambda,\sigma\}, \quad (3.2)
\]
\[
U_{p,r}^\gamma,\gamma_1,\Lambda,\sigma(\xi) = \{y(\xi) : y(\cdot) \in U_{p,r}^\gamma,\gamma_1,\Lambda,\sigma\}, \quad \xi \in E. \quad (3.3)
\]

Now, for each \(i = 1, \ldots, N\), we choose an arbitrary \(\xi_i \in E_i\) and denote 
\[
F_{p,r}^\gamma,\gamma_1,\Lambda,\sigma,\xi_2 = \bigcup_{i=1}^{N}(\xi_i, U_{p,r}^\gamma,\gamma_1,\Lambda,\sigma(\xi_i)). \quad (3.4)
\]

Note that the set \(V_{p,r}^\gamma,\gamma_1,\Lambda,\sigma\) defined by (3.1) can be redefined as 
\[
V_{p,r}^\gamma,\gamma_1,\Lambda,\sigma = \left\{x(\cdot) \in L_p(\Omega; \mathbb{R}^m) : x(s) = w_\lambda b_i, \text{ for every } s \in \Omega_j, \quad j = 1, \ldots, M, w_\lambda, b_i \in S_\sigma, \sum_{j=1}^{M} \mu(\Omega_j)w_\lambda^p b_i^p \leq r^p \right\}.
\]

It is obvious that the set \(V_{p,r}^\gamma,\gamma_1,\Lambda,\sigma\) consists of a finite number of piecewise constant functions, the set \(U_{p,r}^\gamma,\gamma_1,\Lambda,\sigma\) consists of those continuous functions which are the images of the functions in the set \(V_{p,r}^\gamma,\gamma_1,\Lambda,\sigma\) under the operator (2.1), and the set \(F_{p,r}^\gamma,\gamma_1,\Lambda,\sigma,\xi_2\subset \mathbb{R}^{k+n}\) is a finite union of sets consisting of a finite number of points.

Theorem 3.1. For every \(\varepsilon > 0\), there exists \(a_\gamma(\varepsilon) > 0\), \(a_\Delta(\varepsilon) > 0\), \(a_\delta(\varepsilon) > 0\), and \(a_{\sigma(\varepsilon)} = \sigma(\varepsilon, \gamma_\varepsilon(\varepsilon)) > 0\) such that, for any \(\Delta \in (0, \Delta_\varepsilon(\varepsilon)]\), \(\delta \in (0, \delta_\varepsilon(\varepsilon)]\), and \(\sigma \in (0, \sigma_\varepsilon(\varepsilon)]\), the inequalities 
\[
H_C(U_{p,r}, U_{p,r}^\gamma(\varepsilon), \gamma_1, \Lambda, \sigma) < \varepsilon, \quad (3.5)
\]
\[
H_{k+n}(F_{p,r}, F_{p,r}^\gamma(\varepsilon), \gamma_1, \Lambda, \sigma, \xi_2) < \varepsilon \quad (3.6)
\]
are satisfied, where the set $\mathcal{F}_{p,r}$ is defined by (2.5), the set $\mathcal{F}_{p,r}^\gamma \subset \mathcal{F}_{p,r}$ is defined by (3.4), $\mathcal{Y}_1$ is a finite $\Delta$-partition of the compact set $\Omega$, $\mathcal{Y}_2$ is a finite $\Delta$-partition of the compact set $E$, $A$ is a uniform partition of the closed interval $[0, \gamma_\varepsilon \gamma]$ and $\delta$ is its diameter.

**Proof.** The proof of the theorem will be carried out in seven steps. At first, let us prove inequality (3.5).

**Step 1.** For given $\gamma > 0$, we denote

$$\gamma V_{p,r} = \{x(\cdot) \in V_{p,r} : \|x(s)\| \leq \gamma \text{ for every } s \in \Omega\},$$

(3.7)

$$\gamma U_{p,r} = \{U(x(\cdot))(\cdot) : x(\cdot) \in V_{p,r}\}$$

(3.8)

and let

$$\kappa_* = 2l_0 r_p.$$

(3.9)

Let us prove the inequality

$$H_C(\gamma U_{p,r}, \gamma U_{p,r}^\gamma) \leq \frac{\kappa_*}{\gamma^{p-1}},$$

(3.10)

where the sets $\gamma U_{p,r}$ and $\gamma U_{p,r}^\gamma$ are defined by (2.3) and (3.8), respectively.

Choose an arbitrary $y(\cdot) \in \gamma U_{p,r}$. It is the image of some $x(\cdot) \in V_{p,r}$ under the operator (2.1), where $V_{p,r}$ is defined by (2.2). We define a new function $x_*(\cdot) : \Omega \to \mathbb{R}^n$ by setting

$$x_*(s) = \begin{cases} 
\frac{x(s)}{\|x(s)\|} & \text{if } \|x(s)\| > \gamma, \\
2 \|x(s)\| & \text{if } \|x(s)\| \leq \gamma.
\end{cases}$$

(3.11)

It is not difficult to show that $x_*(\cdot) \in \gamma V_{p,r}$. Let a function $y_*(\cdot) : E \to \mathbb{R}^n$ be the image of the function $x_*(\cdot) \in \gamma V_{p,r}$, where the set $\gamma V_{p,r}$ is defined by (3.7). It is obvious that $y_*(\cdot) \in \gamma U_{p,r}$. We denote $\Omega_* = \{s \in \Omega : \|x(s)\| > \gamma\}$. Since $x(\cdot) \in V_{p,r}$, it follows from Chebyshev’s inequality (see [16, p. 82]) that

$$\mu(\Omega_*) \leq \frac{r_p}{\gamma^{p-1}}.$$ 

(3.12)

From (2.1), (3.9), (3.11), (3.12), Condition B, and Hölder’s inequality we obtain

$$\|y(\xi) - y_*(\xi)\| \leq \int_{\Omega_*} l_0 \|x(s) - x_*(s)\| ds \leq 2r_0 l_0 [\mu(\Omega_*)]^{(p-1)/p} \leq \frac{2l_0 r_p^{p-1}}{\gamma^{p-1}} = \frac{\kappa_*}{\gamma^{p-1}}$$

for every $\xi \in E$ and, consequently,

$$\|y(\cdot) - y_*(\cdot)\|_{C} \leq \frac{\kappa_*}{\gamma^{p-1}}.$$ 

(3.13)

Since $y(\cdot) \in \gamma U_{p,r}$ is chosen arbitrarily, it follows from (3.13) that

$$\gamma U_{p,r} \subset \gamma U_{p,r}^\gamma + \frac{\kappa_*}{\gamma^{p-1}} B_{C}(1),$$

(3.14)

where $B_{C}(1)$ is defined by (2.6). The inclusion $\gamma U_{p,r}^\gamma \subset \gamma U_{p,r}$ and (3.14) yield inequality (3.10).

Let

$$\gamma_\varepsilon = \left(\frac{10k_*}{\varepsilon}\right)^{1/(p-1)},$$

(3.15)

where $k_*$ is defined by (3.9). Relations (3.10) and (3.15) imply

$$H_C(\gamma U_{p,r}, \gamma U_{p,r}^{\gamma_\varepsilon}) \leq \frac{\varepsilon}{10}.$$ 

(3.16)
Step 2. For given $\gamma_s(\varepsilon) > 0$, we denote
\[
V_{p,r}^{\gamma_s(\varepsilon),\text{Lip}} = \{ x(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon)} : x(\cdot): \Omega \to \mathbb{R}^m \text{ is Lipschitz continuous} \} \tag{3.17}
\]
and let
\[
U_{p,r}^{\gamma_s(\varepsilon),\text{Lip}} = \{ U(x(\cdot))(\cdot) : x(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon),\text{Lip}} \}. \tag{3.18}
\]
At this step it will be shown that
\[
h_C(U_{p,r}^{\gamma_s(\varepsilon)}; U_{p,r}^{\gamma_s(\varepsilon),\text{Lip}}) = 0, \tag{3.19}
\]
where the sets $U_{p,r}^{\gamma_s(\varepsilon)}$ and $U_{p,r}^{\gamma_s(\varepsilon),\text{Lip}}$ are defined by (3.8) and (3.18), respectively.

For $w \in \mathbb{R}^k$ and $\alpha > 0$, we denote $B_k(w, \alpha) = \{ y \in \mathbb{R}^k : ||y - w|| < \alpha \}$ and $\overline{B}_k(w, \alpha) = \{ y \in \mathbb{R}^k : ||y - w|| \leq \alpha \}$.

Let us choose an arbitrary $x(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon)}$, and let $h \in (0, 1)$ be fixed. Now we define a function $x_h(\cdot) : \Omega \to \mathbb{R}^m$ by setting
\[
x_h(s) = \frac{1}{v_h} \int_{B_h(s,h)} x(\nu) d\nu, \quad s \in \Omega, \tag{3.20}
\]
where $v_h$ is the Lebesgue measure of the ball centered at the origin with radius $h$ in the space $\mathbb{R}^k$, i.e., $v_h = \mu(B_k(0,h))$. If $\nu \notin \Omega$, then we set $x(\nu) = 0$ in (3.20). It is known that
\[
v_h = c_s \cdot h^k, \tag{3.21}
\]
where $c_s = \pi^{k/2}/\Gamma(k/2 + 1)$ and $\Gamma(\cdot)$ is the Euler function.

The function $x_h(\cdot)$ is called the Steklov average of the function $x(\cdot)$. In [4; Chap. 9, Sec. 1, Lemma 1] it was proved that if $x(\cdot) \in L_p(\Omega; \mathbb{R}^m)$, then $x_h(\cdot) \in C(\Omega; \mathbb{R}^m)$. Here, following the proof scheme in [4], we prove that, for $x(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon)}$, we have $x_h(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon),\text{Lip}}$, where $V_{p,r}^{\gamma_s(\varepsilon),\text{Lip}}$ is defined by (3.17).

Since $x(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon)}$, it follows from (3.20) that $||x_h(s)|| \leq \gamma_s(\varepsilon)$ for every $s \in \Omega$. Applying Hölder’s inequality and taking into consideration the inequality $||x(\cdot)||_p \leq r$, it is not difficult to verify that the inequality $||x_h(\cdot)||_p \leq r$ holds. Thus, we have $x_h(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon)}$. Now, let us prove that the function $x_h(\cdot) : \Omega \to \mathbb{R}^k$ is Lipschitz continuous. Choose arbitrary $s_1 \in \Omega$ and $s_2 \in \Omega$.

For fixed $h \in (0, 1)$ and chosen $s_1 \in \Omega$ and $s_2 \in \Omega$, two cases are possible:

Case 1: $||s_2 - s_1|| \geq 2h$. From the membership $x(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon)}$, Hölder’s inequality, (3.20), and (3.21) it follows that
\[
||x_h(s_2) - x_h(s_1)|| \leq \frac{r}{c_s h^{k+p}} \cdot ||s_2 - s_1||, \tag{3.22}
\]
where $c_s$ is defined in (3.21).

Case 2: $||s_2 - s_1|| < 2h$. It can be shown that (3.20), (3.21), and $x(\cdot) \in V_{p,r}^{\gamma_s(\varepsilon)}$ imply the inequality
\[
||x_h(s_2) - x_h(s_1)|| \leq \frac{k \gamma_s(\varepsilon)}{h} ||s_2 - s_1||. \tag{3.23}
\]

We denote
\[
\chi(\gamma_s(\varepsilon), h) = \max \left\{ \frac{r}{c_s h^{k+p} \cdot \frac{k \gamma_s(\varepsilon)}{h}} \right\}. \tag{3.24}
\]

By virtue of (3.22), (3.23), and (3.24), for any $s_1 \in \Omega$ and $s_2 \in \Omega$, we have
\[
||x_h(s_2) - x_h(s_1)|| \leq \chi(\gamma_s(\varepsilon), h) \cdot ||s_2 - s_1||. \tag{3.25}
\]
This means that, for fixed \( \varepsilon > 0 \) and \( h \in (0,1) \), the function \( x_h(\cdot) : \Omega \to \mathbb{R}^m \) is Lipschitz continuous. Since \( x_h(\cdot) \in V^\gamma_{p,r} \), it follows from (3.17) and (3.25) that \( x_h(\cdot) \in V^\gamma_{p,r}. \)

Let \( \rho > 0 \) be an arbitrarily chosen number. We will prove that

\[
H_C(U^\gamma_{p,r}, U^\gamma_{p,r}^{\text{Lip}}) \leq \rho,
\]

where \( U^\gamma_{p,r} \) is defined by (3.18).

Choose an arbitrary \( \tilde{y}(\cdot) \in U^\gamma_{p,r} \). It is the image of \( \tilde{x}(\cdot) \in V^\gamma_{p,r} \) under the operator (2.1). Choose a sequence \( \{h_j\}_{j=1}^\infty \) such that \( h_j \in (0,1) \) for every \( j = 1, 2, \ldots \) and \( h_j \to 0^+ \) as \( j \to \infty \). We define a new function \( x_j(\cdot) : \Omega \to \mathbb{R}^m \) by setting

\[
x_j(s) = \frac{1}{v_{h_j}} \int_{B_h(s,h_j)} \tilde{x}(\nu) d\nu, \quad s \in \Omega,
\]

which is the Steklov average of the function \( \tilde{x}(\cdot) \) for \( h_j \in (0,1) \). Then we have \( x_j(\cdot) \in V^\gamma_{p,r} \) for every \( j = 1, 2, \ldots \). According to Lemma 4 of [4; Chap. 9, Sec. 1], we have \( \|x_j(\cdot) - \tilde{x}(\cdot)\|_p \to 0 \) as \( j \to +\infty \), and hence, for \( \rho > 0 \), there exists a \( j_* > 0 \) such that

\[
\|x_{j_*}(\cdot) - \tilde{x}(\cdot)\|_p \leq \frac{\rho}{l_0[\mu(\Omega)]^{(p-1)/p}},
\]

where \( l_0 \) is defined in Condition B.

Let \( \tilde{y}_*(\cdot) : E \to \mathbb{R}^n \) be the image of \( x_{j_*}(\cdot) \in V^\gamma_{p,r} \) under the operator (2.1). Then \( \tilde{y}_*(\cdot) \in U^\gamma_{p,r}^{\text{Lip}} \), and (2.1), (3.27), Condition B, and Hölder’s inequality imply

\[
\|\tilde{y}(\xi) - \tilde{y}_*(\xi)\| \leq l_0 \int_{\Omega} \|\tilde{x}(s) - x_{j_*}(s)\| ds \leq l_0[\mu(\Omega)]^{(p-1)/p}\|\tilde{x}(\cdot) - x_{j_*}(\cdot)\|_p \leq \rho
\]

for every \( \xi \in E \), and hence \( \|\tilde{y}(\cdot) - \tilde{y}_*(\cdot)\|_C \leq \rho \). This means that

\[
U^\gamma_{p,r} \subset U^\gamma_{p,r}^{\text{Lip}} + \rho B_C(1).
\]

Since \( U^\gamma_{p,r} \subset U^\gamma_{p,r}^{\text{Lip}} \), it follows from (3.28) that inequality (3.26) holds. Finally, since \( \rho > 0 \) is chosen arbitrarily, (3.26) yields (3.19).

**Step 3.** For given \( \gamma(\varepsilon) > 0 \) and an integer \( R > 0 \), we denote

\[
V^\gamma_{p,r}^{\text{Lip},R} = \{x(\cdot) \in V^\gamma_{p,r}^{\text{Lip}} : \text{Lipschitz constant of } x(\cdot) \text{ is not greater than } R\},
\]

\[
U^\gamma_{p,r}^{\text{Lip},R} = \{U(x(\cdot))((\cdot)) : x(\cdot) \in V^\gamma_{p,r}^{\text{Lip},R}\}.
\]

It is not difficult to verify that \( V^\gamma_{p,r}^{\text{Lip},R} \subset C(\Omega; \mathbb{R}^m) \) and \( U^\gamma_{p,r}^{\text{Lip},R} \subset C(E; \mathbb{R}^n) \) are compact sets. Moreover, one can show that \( V^\gamma_{p,r}^{\text{Lip}} = \bigcup_{R=1}^{+\infty} V^\gamma_{p,r}^{\text{Lip},R} \) and hence

\[
U^\gamma_{p,r}^{\text{Lip}} = \bigcup_{R=1}^{+\infty} U^\gamma_{p,r}^{\text{Lip},R},
\]

where \( V^\gamma_{p,r}^{\text{Lip},R} \) and \( U^\gamma_{p,r}^{\text{Lip},R} \) are defined by (3.29) and (3.30), respectively.

Now, from (3.16), (3.19), and (3.31) it follows that

\[
H_C\left(U_{p,R}, \bigcup_{R=1}^{+\infty} U^\gamma_{p,r}^{\text{Lip},R}\right) \leq \frac{\varepsilon}{10}.
\]
According to Proposition 2.1, the set \( \mathcal{U}_{p,r} \subset C(E; \mathbb{R}^n) \) is precompact and \( \mathcal{U}_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R} \subset \mathcal{U}_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R+1} \subset \mathcal{U}_{p,r} \) for every \( R = 1, 2, \ldots \). Thus, by virtue of (3.32) and Proposition 2.4, for \( \varepsilon/10 \), there exists an integer \( R_s(\varepsilon) > 0 \) such that

\[
H_C(\mathcal{U}_{p,r},\mathcal{U}_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R}) \leq \frac{\varepsilon}{10} + \frac{\varepsilon}{10} = \frac{\varepsilon}{5}
\]

for every \( R \geq R_s(\varepsilon) \), and consequently

\[
H_C(\mathcal{U}_{p,r},\mathcal{U}_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R_s(\varepsilon)}) \leq \frac{\varepsilon}{5}.
\] (3.33)

**Step 4.** For a given \( \Delta \)-partition \( \Upsilon_1 = \{ \Omega_1, \ldots, \Omega_M \} \) of the compact set \( \Omega \), we set

\[
V_{p,r}^{\gamma_{s}(\varepsilon),\Upsilon_1} = \{ x(\cdot) \in V_{p,r}^{\gamma_{s}(\varepsilon)} : x(s) = x_j \text{ for every } s \in \Omega_j, j = 1, \ldots, M \}
\] (3.34)

and let

\[
\mathcal{U}_{p,r}^{\gamma_{s}(\varepsilon),\Upsilon_1} = \{ U(x(\cdot))(\cdot) : x(\cdot) \in V_{p,r}^{\gamma_{s}(\varepsilon),\Upsilon_1} \},
\]

where \( \gamma_{s}(\varepsilon) \) is defined by (3.15).

Now let us choose an arbitrary \( \hat{y}(\cdot) \in \mathcal{U}_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R_s(\varepsilon)} \). It is the image of \( \hat{x}(\cdot) \in V_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R_s(\varepsilon)} \), where \( V_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R_s(\varepsilon)} \) is defined by (3.29). This means that

\[
\| \hat{x}(\cdot) \|_p \leq r, \quad \| \hat{x}(s) \| \leq \gamma_{s}(\varepsilon) \quad \text{for every } s \in \Omega,
\]

\[
\| \hat{x}(s) - \hat{x}(s^*) \| \leq R_s(\varepsilon)\|s - s^*\| \quad \text{for any } s, s^* \in \Omega.
\] (3.35)

We define the function \( \hat{x}_{s}(\cdot) : \Omega \to \mathbb{R}^m \) by setting

\[
\hat{x}_{s}(s) = \frac{1}{\mu(\Omega_j)} \int_{\Omega_j} \hat{x}(\nu) \, d\nu, \quad s \in \Omega_j, j = 1, \ldots, M.
\] (3.36)

From (3.35) and (3.36) it follows that \( \| \hat{x}_{s}(s) \| \leq \gamma_{s}(\varepsilon) \) for every \( s \in \Omega \) and

\[
\int_{\Omega_j} \| \hat{x}_{s}(\nu) \|^p \, d\nu \leq \int_{\Omega_j} \| \hat{x}(\nu) \|^p \, d\nu
\]

for every \( j = 1, \ldots, M \). The last inequality implies \( \| \hat{x}_{s}(\cdot) \|_p \leq \| \hat{x}(\cdot) \|_p \leq r \), and hence \( \hat{x}_{s}(\cdot) \in V_{p,r}^{\gamma_{s}(\varepsilon),\Upsilon_1} \).

Let us choose an arbitrary \( s \in \Omega \) and fix it. Since \( \Upsilon_1 = \{ \Omega_1, \ldots, \Omega_M \} \) is a finite \( \Delta \)-partition of \( \Omega \), it follows from Definition 2.1 that there exists a \( j_s \in \{ 1, \ldots, M \} \) such that \( s \in \Omega_{j_s} \), where \( \text{diam}(\Omega_{j_s}) \leq \Delta \). The relation \( \hat{x}(\cdot) \in V_{p,r}^{\gamma_{s}(\varepsilon),\text{Lip},R_s(\varepsilon)} \) and (3.36) imply

\[
\| \hat{x}(s) - \hat{x}_{s}(s) \| = \frac{1}{\mu(\Omega_{j_s})} \int_{\Omega_{j_s}} \| \hat{x}(s) - \hat{x}_{s}(\nu) \| \, d\nu
\]

\[
\leq \frac{1}{\mu(\Omega_{j_s})} R_s(\varepsilon) \int_{\Omega_{j_s}} \| s - \nu \| \, d\nu \leq R_s(\varepsilon)\Delta.
\] (3.37)

Now let \( \hat{y}_{s}(\cdot) \) be the image of \( \hat{x}_{s}(\cdot) \) under the operator (2.1). Then \( \hat{y}_{s}(\cdot) \in \mathcal{U}_{p,r}^{\gamma_{s}(\varepsilon),\Upsilon_1} \), and Condition B and (3.37) imply

\[
\| \hat{y}(\xi) - \hat{y}_{s}(\xi) \| \leq \int_{\Omega} l_0 \| \hat{x}(\nu) - \hat{x}_{s}(\nu) \| \, d\nu \leq l_0 \mu(\Omega) R_s(\varepsilon)\Delta
\]
for every \( \xi \in E \) and, consequently,

\[
\| \hat{y}(\cdot) - \tilde{y}_*(\cdot) \|_C \leq \ell_0 \mu(\Omega) R_* \varepsilon \Delta.
\]

Since \( \hat{y}(\cdot) \in U^{\gamma_*(\varepsilon), \text{Lip}, R_* \varepsilon} \) is chosen arbitrarily and \( \hat{y}_*(\cdot) \in U^{\gamma_*(\varepsilon), \mathcal{T}_1} \), it follows from the last inequality that

\[
U^{\gamma_*(\varepsilon), \text{Lip}, R_* \varepsilon} \subset U^{\gamma_*(\varepsilon), \mathcal{T}_1} + \ell_0 \mu(\Omega) R_* \varepsilon \Delta B_C(1),
\]

where \( B_C(1) \) is defined by (2.6).

Since \( \varphi(\tau) \to 0^+ \) as \( \tau \to 0^+ \), there exists a \( \Delta_1(\varepsilon) > 0 \) such that

\[
\varphi(\Delta) \leq \frac{\varepsilon}{10 \beta_*}.
\]

for every \( \Delta \in (0, \Delta_1(\varepsilon)] \), where \( \varphi(\cdot) \) is given in Condition C and \( \beta_* \) is defined by (2.8). We denote

\[
\Delta_*(\varepsilon) = \min \left\{ \frac{\varepsilon}{10 \ell_0 \mu(\Omega) R_* \varepsilon}, \frac{\varepsilon}{10}, \Delta_1(\varepsilon) \right\}.
\]

From (3.33), (3.38), and (3.40) it follows that, for every \( \Delta \in (0, \Delta_*(\varepsilon)] \), we have

\[
U_{p,r} \subset U_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1} + \frac{3 \varepsilon}{10} B_C(1).
\]

Since \( U_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1} \subset U_{p,r} \), it follows from the last inclusion that, for every \( \Delta \in (0, \Delta_*(\varepsilon]) \),

\[
H_C(U_{p,r}, U_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1}) \leq \frac{3 \varepsilon}{10}.
\]

**Step 5.** Given \( \gamma_*(\varepsilon) > 0 \), a \( \Delta \)-partition \( \mathcal{T}_1 = \{ \Omega_1, \ldots, \Omega_M \} \) of the set \( \Omega \), and a uniform \( \delta \)-partition \( \Lambda = \{ 0 = w_0 < w_1 < \cdots < w_q = \gamma_*(\varepsilon) \} \) of the closed interval \([0, \gamma_*(\varepsilon)]\), we set

\[
V_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1, \Lambda} = \{ x(\cdot) \in V_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1} : x(s) = x_j \text{ for every } s \in \Omega_j, \quad \|x_j\| \in \Lambda, \ j = 1, \ldots, M \},
\]

where \( \delta = w_{\lambda+1} - w_\lambda \) for \( \lambda = 0, 1, \ldots, q-1 \) and \( \gamma_*(\varepsilon) \) is defined by (3.15), and let

\[
U_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1, \Lambda} = \{ U(x(\cdot))(\cdot) : x(\cdot) \in V_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1, \Lambda} \}.
\]

Let us choose an arbitrary \( y_0(\cdot) \in U_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1} \). It is the image of \( x_0(\cdot) \in V_{p,r}^{\gamma_*(\varepsilon), \mathcal{T}_1} \). By virtue of (3.34) we have

\[
x_0(s) = x_j, \quad \|x_j\| \leq \gamma_*(\varepsilon) \text{ for any } s \in \Omega_j \text{ and } j = 1, \ldots, M,
\]

\[
\sum_{j=1}^M \mu(\Omega_j) \|x_j\|^p \leq r^p.
\]

The inequality \( 0 \leq \|x_j\| \leq \gamma_*(\varepsilon) \) implies that if \( \|x_j\| < \gamma_*(\varepsilon) \), then there exists a \( w_{\lambda_j} \in \Lambda \) such that

\[
\|x_j\| \in [w_{\lambda_j}, w_{\lambda_j+1}).
\]

We define a function \( \bar{x}_0(\cdot) : \Omega \to \mathbb{R}^m \) by setting

\[
\bar{x}_0(s) = \begin{cases} \frac{x_j}{\|x_j\|}w_{\lambda_j} & \text{if } 0 < \|x_j\| < \gamma_*(\varepsilon), \\ x_j & \text{if } \|x_j\| = 0 \text{ or } \|x_j\| = \gamma_*(\varepsilon), \end{cases}
\]

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where \( s \in \Omega \) and \( w_{\lambda_j} \in \Lambda \) is defined in (3.44), \( j = 1, \ldots, M \). From (3.44) and (3.45) it follows that

\[
\|x_0(s) - \tilde{x}_0(s)\| \leq \delta
\]  

(3.46)

for every \( s \in \Omega \).

Relations (3.43), (3.44), and (3.45) yield \( \|\tilde{x}_0(s)\| \leq \|x_0(s)\| \) for every \( s \in \Omega \) and, consequently, \( \|\tilde{x}_0(s)\| \leq \gamma_*(\varepsilon) \) for every \( s \in \Omega \) and \( \|\tilde{x}_0(\cdot)\|_p \leq \|x_0(\cdot)\|_p \leq r \). Thus, we obtain \( \tilde{x}_0(\cdot) \in V_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \).

Let \( \tilde{y}_0(\cdot) : E \to \mathbb{R}^n \) be the image of the function \( \tilde{x}_0(\cdot) \) under the operator (2.1). Then \( \tilde{y}_0(\cdot) \in \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \), and Condition B, (2.1), and (3.46) imply

\[
\|y_0(\xi) - \tilde{y}_0(\xi)\| \leq l_0 \int_{\Omega} \|x_0(s) - \tilde{x}_0(s)\|\,ds \leq l_0 \mu(\Omega) \delta
\]

for every \( \xi \in E \), whence

\[
\|y_0(\cdot) - \tilde{y}_0(\cdot)\|_C \leq l_0 \mu(\Omega) \delta.
\]  

(3.47)

Since \( y_0(\cdot) \in \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1} \) is an arbitrarily chosen function and \( \tilde{y}_0(\cdot) \in \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \), it follows from (3.47) that

\[
\mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \subset \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1} + l_0 \mu(\Omega) \delta \cdot B_C(1).
\]

This inclusion, together with \( \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \subset \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1} \),

implies

\[
H_C(\mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda}, \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1}) \leq l_0 \mu(\Omega) \delta. 
\]  

(3.48)

We denote

\[
\delta_*(\varepsilon) = \frac{\varepsilon}{10l_0 \mu(\Omega)}. 
\]  

(3.49)

From (3.48) and (3.49) it follows that, for every partition \( \Lambda \) such that \( \varepsilon \in (0, \delta_*(\varepsilon)] \), the inequality

\[
H_C(\mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda}, \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1}) \leq \frac{\varepsilon}{10} 
\]  

(3.50)

is satisfied.

Note that inequality (3.50) holds true for every finite \( \Delta \)-partition \( \gamma_1 \) of the compact set \( \Omega \). Finally, from (3.41) and (3.50) we obtain that, for any finite \( \Delta \)-partition \( \gamma_1 \) of the compact set \( \Omega \) and uniform \( \delta \)-partition \( \Lambda \) of the closed interval \( [0, \gamma_*(\varepsilon)] \) such that \( \Delta \in (0, \Delta_*(\varepsilon)] \) and \( \delta \in (0, \delta_*(\varepsilon)] \), the inequality

\[
H_C(\mathcal{U}_{p,r}^{\gamma_*(\varepsilon)}, \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda}) \leq \frac{2\varepsilon}{5} 
\]  

(3.51)

is satisfied, where \( \Delta_*(\varepsilon) \) is defined by (3.40).

**Step 6.** Suppose given \( \gamma_*(\varepsilon) > 0 \), a finite \( \Delta \)-partition \( \gamma_1 = \{\Omega_1, \ldots, \Omega_M\} \) of the set \( \Omega \), and a uniform \( \delta \)-partition \( \Lambda = \{0 = w_0 < w_1 < \cdots < w_q = \gamma_*(\varepsilon)\} \) of the closed interval \( [0, \gamma_*(\varepsilon)] \), where \( \delta = w_{\lambda+1} - w_\lambda \) for \( \lambda = 0, 1, \ldots, q - 1 \). Let us show that, for any \( \sigma > 0 \), we have

\[
\mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \subset \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda,\sigma} + l_0 \mu(\Omega) \gamma_*(\varepsilon) \sigma B_C(1), 
\]  

(3.52)

where the set \( \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda,\sigma} \) is defined by (3.2).

Choose an arbitrary \( \overline{y}(\cdot) \in \mathcal{U}_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \). It is the image of some \( \overline{x}(\cdot) \in V_{p,r}^{\gamma_*(\varepsilon),\gamma_1,\Lambda} \). In view of (3.42) we have

\[
\overline{x}(s) = w_{\lambda_j} a_j, \quad s \in \Omega_j, 
\]  

(3.53)
where \( w_{\lambda j} \in A \) and \( a_j \in S = \{ x \in \mathbb{R}^m : \| x \| = 1 \} \) for \( j = 1, \ldots, M \) and \( \sum_{j=1}^{M} \mu(\Omega_j)w_{\lambda j}^p \leq r^p \). Since \( S_\sigma \) is a finite \( \sigma \)-net of \( S \), it follows that, for each \( a_j \in S \), there exists a \( b_{\lambda j} \in S_\sigma \) such that \( \| a_j - b_{\lambda j} \| \leq \varepsilon \). We define a new function \( \varphi_s(\cdot) : \Omega \to \mathbb{R}^m \) by setting
\[
\varphi_s(s) = w_{\lambda j}b_{\lambda j}, \quad s \in \Omega_j,
\]
where \( j = 1, \ldots, M \). From (3.1), (3.53), and (3.54) it follows that \( \varphi_s(\cdot) \in V_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma} \) and
\[
\| \varphi(s) - \varphi_s(s) \| \leq \gamma_\varepsilon(\varepsilon)\sigma
\]
for every \( s \in \Omega \).

Let \( \overline{\varphi}_s(\cdot) : E \to \mathbb{R}^n \) be the image of the function \( \varphi_s(\cdot) \in V_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma} \) defined by (3.54). Then \( \overline{\varphi}_s(\cdot) \in \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma} \) and Condition B, (2.1), and (3.55) imply
\[
\| \overline{\varphi}(\xi) - \overline{\varphi}_s(\xi) \| \leq \int_{\Omega} \int_0^1 \| \varphi(s) - \varphi_s(s) \| ds \leq l_0\mu(\Omega)\gamma_\varepsilon(\varepsilon)\sigma
\]
for every \( \xi \in E \), and hence
\[
\| \overline{\varphi}(\cdot) - \overline{\varphi}_s(\cdot) \|_{C} \leq l_0\mu(\Omega)\gamma_\varepsilon(\varepsilon)\sigma.
\]

Since \( \overline{\varphi}(\cdot) \in \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda} \) is an arbitrarily chosen function and \( \overline{\varphi}_s(\cdot) \in \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma} \), we obtain inclusion (3.52) from the last inequality. From (3.52) and the inclusion \( \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma} \subset \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda} \), it follows that
\[
H_C(\mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda}, \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma}) \leq l_0\mu(\Omega)\gamma_\varepsilon(\varepsilon)\sigma.
\]

Let
\[
\sigma_\varepsilon(\varepsilon) = \sigma_\varepsilon(\varepsilon, \gamma_\varepsilon(\varepsilon)) = \frac{\varepsilon}{10l_0\mu(\Omega)\gamma_\varepsilon(\varepsilon)}.
\]

Relations (3.56) and (3.57) imply that, for every \( \sigma \in (0, \sigma_\varepsilon(\varepsilon)] \), the inequality
\[
H_C(\mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda}, \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma}) \leq \frac{\varepsilon}{10}
\]
holds. From (3.51) and (3.58) we conclude that, for every finite \( \Delta \)-partition \( Y_1 \) of the compact set \( \Omega \), every uniform \( \delta \)-partition \( \Lambda \) of the closed interval \( [0, \gamma_\varepsilon(\varepsilon)] \) such that \( \Delta \in (0, \Delta_\varepsilon(\varepsilon)] \) and \( \delta \in (0, \delta_\varepsilon(\varepsilon)] \), and every \( \sigma \in (0, \sigma_\varepsilon(\varepsilon)] \), the inequality
\[
H_C(\mathcal{U}_{p,r}, \mathcal{U}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma}) \leq \frac{\varepsilon}{2}
\]
holds. Thereby, inequality (3.5) is proved.

**Step 7.** Now, at the last step, inequality (3.6) will be proved.

It is obvious that, for any finite \( \Delta \)-partitions \( Y_1 = \{ \Omega_1, \ldots, \Omega_M \} \) of the compact set \( \Omega \subset \mathbb{R}^k \) and \( Y_2 = \{ E_1, \ldots, E_N \} \) of the compact set \( E \subset \mathbb{R}^b \), any uniform \( \delta \)-partition \( \Lambda = \{ 0 = w_0 < w_1 < \cdots < w_\delta = \gamma_\varepsilon(\varepsilon) \} \) of the closed interval \( [0, \gamma_\varepsilon(\varepsilon)] \), where \( \delta = w_{\lambda+1} - w_{\lambda} \) for \( \lambda = 0, 1, \ldots, q - 1 \), and any \( \sigma > 0 \), the inclusion
\[
\mathcal{F}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma,Y_2} \subset \mathcal{F}_{p,r}
\]
holds, where \( \gamma_\varepsilon(\varepsilon) > 0 \) is defined by (3.15), \( \mathcal{F}_{p,r} \) is the integral funnel of the closed ball \( V_{p,r} \subset L_p(\Omega; \mathbb{R}^m) \) under the operator (2.1) defined by (2.5), and the set \( \mathcal{F}_{p,r}^{\gamma_\varepsilon(\varepsilon),Y_1,\Lambda,\sigma,Y_2} \) consists of a finite number of points and is defined by (3.4).

Now let \( \gamma_\varepsilon(\varepsilon) > 0 \) be defined by (3.15), finite \( \Delta \)-partitions \( Y_1 = \{ \Omega_1, \ldots, \Omega_M \} \) of the compact set \( \Omega \) and \( Y_2 = \{ E_1, \ldots, E_N \} \) of the compact set \( E \subset \mathbb{R}^b \) be such that \( \Delta \in (0, \Delta_\varepsilon(\varepsilon)] \), a uniform \( \delta \)-partition \( \Lambda = \{ 0 = w_0 < w_1 < \cdots < w_\delta = \gamma_\varepsilon(\varepsilon) \} \) of the closed interval \( [0, \gamma_\varepsilon(\varepsilon)] \) be such that
δ ∈ (0, δ_0(ε)], and σ be a number in (0, σ_0(ε)], where δ = w_{λ+1} - w_λ for λ = 0, 1, ..., q - 1 and Δ_0(ε) > 0, δ_0(ε) > 0, and σ_0(ε) > 0 are defined by (3.40), (3.49), and (3.57), respectively. By virtue of (3.59) we have

\[ H_n(U_{p,r}(ξ), U_{p,r}^{γ_ε}(e), 1, Λ, σ(ξ)) ≤ \frac{ε}{2} \]

(3.61)

for every ξ ∈ E, where the sets U_{p,r}(ξ) and U_{p,r}^{γ_ε}(e), 1, Λ, σ(ξ) are defined by (2.4) and (3.3), respectively.

Now, let us prove that

\[ \mathcal{F}_{p,r} ⊂ \mathcal{F}_{p,r}^{γ_ε}(e), 1, Λ, σ, 2 + \frac{5ε}{6} Φ_{k+n}(1), \]

(3.62)

where \( Φ_{k+n}(1) = \{z ∈ \mathbb{R}^{k+n} : \|z\| ≤ 1\} \).

Choose an arbitrary \((ξ_*, z_*) \in \mathcal{F}_{p,r} \). Then we have \( z_* ∈ U_{p,r}(ξ_*) \). By virtue of (3.61) there exists a \( w_* ∈ U_{p,r}^{γ_ε}(e), 1, Λ, σ(ξ_*) \) such that

\[ \|z_* - w_*\| < \frac{3ε}{5}. \]

(3.63)

Since \( Υ_2 = \{E_1, ..., E_N\} \) is a finite Δ-partition of E, it follows from Definition 2.1 that there exists an \( i_* \) such that \( ξ_* ∈ E_{i_*} \) and \( \|ξ_* - ξ_{i_*}\| ≤ Δ \). By analogy with Proposition 2.2 we can show that

\[ H_n(U_{p,r}^{γ_ε}(e), 1, Λ, σ(ξ_*), U_{p,r}^{γ_ε}(e), 1, Λ, σ(ξ_{i_*})) ≤ β_ε · φ(\|ξ_* - ξ_{i_*}\|). \]

(3.64)

Since \( Δ ∈ (0, Δ_0(ε)] \), it follows from (3.39), (3.40), and (3.64) that, for \( w_* ∈ U_{p,r}^{γ_ε}(e), 1, Λ, σ(ξ_*) \), there exists an \( f_* ∈ U_{p,r}^{γ_ε}(e), 1, Λ, σ(ξ_{i_*}) \) such that

\[ \|w_* - f_*\| < \frac{ε}{9}. \]

(3.65)

It is obvious that \((ξ_*, f_*) ∈ \mathcal{F}_{p,r}^{γ_ε}(e), 1, Λ, σ, 2 \). Since \( \|ξ_* - ξ_{i_*}\| ≤ Δ < Δ_0(ε), \) from (3.40), (3.63) and (3.65) we obtain

\[ \|(ξ_*, z_*) - (ξ_*, f_*)\| ≤ \|ξ_* - ξ_{i_*}\| + \|z_* - f_*\| \]

\[ ≤ \|ξ_* - ξ_{i_*}\| + \|z_* - w_*\| + \|w_* - f_*\| < \frac{5ε}{6}. \]

(3.66)

Thus, we have proved that, for an arbitrary \((ξ_*, z_*) ∈ \mathcal{F}_{p,r} \), there exists a \((ξ_{i_*}, f_*) ∈ \mathcal{F}_{p,r}^{γ_ε}(e), 1, Λ, σ, 2 \) such that inequality (3.66) holds. This proves inclusion (3.62).

Finally, (3.60) and (3.62) imply inequality (3.6). The proof of the theorem is completed. □

**Remark 3.1.** Note that the given method can be applied to approximate the image and integral funnel of the closed ball

\[ V_{p,r}(x_*(\cdot)) = \{x(\cdot) ∈ L_p(Ω; \mathbb{R}^m) : \|x(\cdot) - x_*(\cdot)\|_p ≤ r\} \]

and of any set of the form

\[ G_{p,r_1, ..., r_n}(x_1(\cdot), ..., x_n(\cdot)) = \bigcup_{i=1}^{n_0} V_{p,r_i}(x_i(\cdot)), \]

where \( x_*(\cdot) ∈ L_p(Ω; \mathbb{R}^m), x_1(\cdot) ∈ L_p(Ω; \mathbb{R}^m), ..., x_{n_0}(\cdot) ∈ L_p(Ω; \mathbb{R}^m) \) are given functions.
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