Euclidean Rings of $S$-integers in Complex Quadratic Fields

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Abstract. We give an elementary approach to studying whether rings of $S$-integers in complex quadratic fields are Euclidean with respect to the $S$-norm.

1. INTRODUCTION. In a first course on elementary number theory, one establishes the Fundamental Theorem of Arithmetic, which says that every positive integer factors uniquely as product of primes. In the standard proof, one needs to know that for any prime $p$, if $p$ divides $ab$, then $p$ divides $a$ or $p$ divides $b$. This seemingly obvious fact about primes is usually derived from the Euclidean algorithm. The following crucial property of $\mathbb{Z}$ allows one to show that the Euclidean algorithm terminates in a finite number of steps: For all $a,b \in \mathbb{Z}$, $b \neq 0$, there exists $q,r \in \mathbb{Z}$ such that $a = qb + r$ and $|r| < |b|$. Equivalently, this “Euclidean property” reads: For all $x \in \mathbb{Q}$ there exists $c \in \mathbb{Z}$ such that $|x - c| < 1$.

Algebraic number fields are one of the main objects of study in algebraic number theory. The simplest number fields other than the rational numbers $\mathbb{Q}$ are the complex quadratic fields which take the form $K = \mathbb{Q}(\sqrt{-d}) = \{x + y\sqrt{-d} \mid x, y \in \mathbb{Q}\}$, where $d > 0$ is squarefree. We view $K$ as a subset of the complex numbers $\mathbb{C}$. Let $N : K \to \mathbb{Q}$ denote the norm function (which multiplies an element by its conjugate)

$$N(x + y\sqrt{-d}) = (x + y\sqrt{-d})(x - y\sqrt{-d}) = x^2 + dy^2.$$

One has $N(\xi \eta) = N(\xi)N(\eta)$ for all $\xi, \eta \in K$, as is verified by direct computation. Let $\mathcal{O}$ denote the ring of integers in $K$, which consists of those elements that satisfy a monic polynomial with integer coefficients. One can show $\mathcal{O} = \mathbb{Z}[w] = \{a + bw \mid a, b \in \mathbb{Z}\}$ with

$$w = \begin{cases} \sqrt{-d} & -d \equiv 2, 3 \pmod{4} \\ \frac{1 + \sqrt{-d}}{2} & -d \equiv 1 \pmod{4} \end{cases}.$$

The ring $\mathcal{O}$ in $K$ is the analogue of $\mathbb{Z}$ in $\mathbb{Q}$, but properties enjoyed by $\mathbb{Z}$ (such as unique factorization) do not always hold in $\mathcal{O}$.

We now give the classical generalization of the Euclidean algorithm. We call $K$ norm-Euclidean if for every $\xi \in K$ there exists $\gamma \in \mathcal{O}$ such that $N(\xi - \gamma) < 1$; this is equivalent to the condition that $\mathcal{O}$ is a Euclidean ring with respect to the function $N$. It is known that $K$ is norm-Euclidean exactly when $d = -1, -2, -3, -7, -11$. This goes back to Dedekind’s supplement to Dirichlet’s book [4].

Let $S \subseteq \mathbb{Z}^+$ be a finite (possibly empty) set of primes, and let $T \subseteq \mathbb{Z}^+$ be the set of all finite products of elements in $S$ (where $1 \in T$ by convention). In this paper, we are
interested in the so-called ring of $S$-integers

$$\mathcal{O}_S = \left\{ \frac{a + bw}{c} : a, b, c \in \mathbb{Z}, c \in T \right\} \subseteq K.$$  

This is an example of the ring-of-fractions construction one might encounter in a first course on ring theory.

We define the $S$-norm of an element $\xi \in K$, denoted by $N_S(\xi)$, by deleting the primes in $S$ from the prime factorization of the numerator and denominator of the rational number $N(\xi)$. For example, when $K = \mathbb{Q}(\sqrt{-5})$, $S = \{2\}$, $\xi = (3 + 3\sqrt{-5})/7$, one has $N(\xi) = 54/49$ and hence $N_S(\xi) = 27/49$. One can express $N_S$ in terms of the $p$-adic valuation. Given a nonzero $x \in \mathbb{Q}$, we write $v_p(x)$ for the power of $p$ (which could be positive, negative, or zero) showing up in the prime factorization of $x$. Then one has $N_S(\xi) = N(\xi) \prod_{p \in S} p^{-v_p(N(\xi))}$. It follows from the multiplicative property of $N$ that the function $N_S$ is also multiplicative.

We call $K$ $S$-norm-Euclidean if for every $\xi \in K$ there exists $\gamma \in \mathcal{O}_S$ such that $N_S(\xi - \gamma) < 1$; this is equivalent to the statement that $\mathcal{O}_S$ is a Euclidean ring with respect to the function $N_S$. One would like to know: When is $K$ $S$-norm-Euclidean? Even in our setting, where $K$ is a complex quadratic field, this question is, in general, unsolved. To give a simple example, if $K = \mathbb{Q}(\sqrt{-5})$ and $S = \{2\}$, then $\mathcal{O}_S = \mathbb{Z}[\sqrt{-5}, 1/2]$; in this case $K$ is $S$-norm-Euclidean, although $K$ is not norm-Euclidean.

In this paper we give an elementary approach that allows us to prove a few results about $S$-norm-Euclidean fields in the complex quadratic setting. Our first result furnishes a number of examples.

**Theorem 1.** The complex quadratic field $K = \mathbb{Q}(\sqrt{-d})$ is $S$-norm-Euclidean for the following choices of $d$ and $S$:

| $S$     | Values of $d$ |
|---------|---------------|
| $\emptyset$ | $1, 2, 3, 7, 11$ |
| $\{2\}$   | $1, 2, 3, 5, 6, 7, 11, 15, 19, 23$ |
| $\{2, 3\}$ | $1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 23, 31, 35, 39, 43, 47, 51, 55, 59, 67, 71$ |
| $\{2, 3, 5\}$ | $1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 39, 43, 47, 51, 55, 59, 67, 71, 79, 83, 87, 91, 95, 103, 107, 111, 115, 119, 123, 127, 131, 139, 143$ |

Note the previous theorem is not claiming these lists are complete or even finite, although both are true when $S = \emptyset$. As discussed, when $S = \emptyset$, this result is classical. When $S = \{2\}$ these examples appear in [8, 14]. However, we have not seen any examples with $S = \{2, 3\}$ explicitly given in the literature.

A natural question arises: Does there always exist a set $S$ so that $K$ is $S$-norm-Euclidean? The answer to this question is yes, and we give a quantitative version of this claim. Before stating this result, we require an additional piece of terminology and notation. We write $D$ to denote the absolute value of the discriminant of $K$. By definition, the discriminant is $\text{det}([1, w; 1, \overline{w}])^2$ where $\overline{w}$ denotes the complex conjugate of $w$; therefore the discriminant equals $-D$ where

$$D = \begin{cases} 4d & -d \equiv 2, 3 \pmod{4} \\ d & -d \equiv 1 \pmod{4}. \end{cases}$$

Given $x \in \mathbb{R}$ we write $\lceil x \rceil$ to denote the ceiling of $x$; i.e., $\lceil x \rceil$ is the smallest integer greater than or equal to $x$. 

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Theorem 2. Let $K$ be a complex quadratic field of discriminant $-D$. If $S$ contains all primes less than $\lceil \sqrt{D}/\sqrt{3} \rceil$, then $K$ is $S$-norm-Euclidean.

In a first course in algebraic number theory, one shows that the class group is generated by the set of all prime ideals whose norm is less than the Minkowski bound; it turns out that Theorem 2 implies this result in our setting. However our proof of Theorem 2 does not use the Minkowski bound, the class group, or even the notion of an ideal! For those familiar with the rudiments of algebraic number theory, further discussion is given in an Appendix. We should note that Theorem 2 appears in [11] but that our proof is different.

To give another example, consider $K = \mathbb{Q}(\sqrt{-163})$ and $S = \{2, 3, 5, 7\}$, where

$$O_S = \mathbb{Z} \left[ \frac{1 + \sqrt{-163}}{2}, \frac{1}{120} \right] = \mathbb{Z} \left[ \sqrt{-163}, \frac{1}{120} \right].$$

Theorem 2 implies that $K$ is $S$-norm-Euclidean; however, $K$ is not norm-Euclidean (even though $O = \mathbb{Z} \left[ (1 + \sqrt{-163})/2 \right]$ is a PID and hence a UFD).

On the other hand, one might consider a fixed set $S$ and ask which complex quadratic fields $K$ are $S$-norm-Euclidean. When $S$ contains several elements, this seems to be a difficult problem, so we restrict ourselves to the case where $S$ contains a single prime $p \in \mathbb{Z}$. Even in this simplest case, we will further restrict which $(d, p)$ pairs we consider in order to obtain a complete classification. In what follows $(n/p)$ denotes the usual Legendre symbol; namely $(n/p) = 1$ if $x^2 \equiv n \pmod{p}$ has a solution with $x \not\equiv 0 \pmod{p}$, $(n/p) = 0$ if $p$ divides $n$, and $(n/p) = -1$ otherwise.

Theorem 3. Let $K = \mathbb{Q}(\sqrt{-d})$ with $d > 0$ squarefree and $S = \{p\}$.

1. Suppose $p \geq 11$ and $(-d/p) \neq 1$. Then $K$ is $S$-norm-Euclidean only if $d \in \{1, 2, 3, 7, 11\}$.
2. Suppose $p = 2$ and $-d \not\equiv 1 \pmod{8}$. Then $K$ is $S$-norm-Euclidean if and only if $d \in \{1, 2, 3, 5, 6, 10, 11, 19\}$.
3. Suppose $p \in \{3, 5, 7\}$ and $(-d/p) \neq 1$. Then $K$ is $S$-norm-Euclidean if and only if one of the following holds:
   
   $p = 3$, $d \in \{1, 3, 7, 11, 15\}$,
   
   $p = 5$, $d \in \{2, 3, 7, 15, 35\}$,
   
   $p = 7$, $d \in \{1, 2, 7, 11, 35\}$.

We note that the previous theorem follows from Theorem 0.19 of [14]; however, our approach requires very little background knowledge and our proof is relatively short. In comparing Theorem 1 with Theorem 3, the reader may have noticed the addition of $d = 10$ to the list when $S = \{2\}$. It turns out that the technique used to prove Theorem 1 does not quite apply in the case of $(d, p) = (10, 2)$, but we find a way to treat this case and four other "exceptional cases" in Section 5.

We view the following question as a natural one. In light of Theorem 3 one can restrict attention to those $(d, p)$ pairs such that $(-d/p) = 1$.

Problem. Are there infinitely many $d$ for which there exists $p$ such that $K$ is $S$-norm-Euclidean when $S = \{p\}$?

Associated to any number field $K$ is a finite abelian group, called the class group of $K$. Roughly speaking, this group governs the complexity of factorizations of elements
of $\mathcal{O}$ into irreducible elements, and this group is trivial if and only if unique factorization holds. It turns out that if $K$ is {$p$}-norm-Euclidean, then the class group of $K$ is cyclic.

According to the Cohen–Lenstra heuristics (see [3]), one expects that the latter event should happen infinitely often, but this is currently unproven. (See Appendix for additional discussion.) If one were to resolve the problem above in the affirmative, then one would prove that class group of $K$ is cyclic infinitely often, resolving a major open problem. (Of course, this likely means that producing such a proof may be difficult.) On the other hand, if the resolution of this problem is in the negative, then proving such a result would be interesting in its own right. We note that Stark has already suggested looking at Euclidean rings of $S$-integers in connection with class number problems (see [13]).

Finally, since we are posing problems anyway, here is another:

**Problem.** Are there infinitely many $d$ for which $K$ is {$2, 3$}-norm-Euclidean?

### 2. MAIN IDEA.

In this section, we develop the main idea of this paper, which provides sufficient conditions for $K$ to be $S$-norm-Euclidean. We will see that Theorems 1 and 2 will then immediately follow. We will give a small amount of additional background material along the way as necessary. Let $K$ denote a complex quadratic field and adopt all the notation given in Section 1. In particular, we write $K = \mathbb{Q}(\sqrt{-d})$ with $d > 0$ squarefree and discriminant $-D$.

The norm function $N : K \to \mathbb{Q}$ also maps $\mathcal{O} \to \mathbb{Z}$. In the case $-d \equiv 2, 3 \pmod{4}$ this is evident from our definition; in the case where $-d \equiv 1 \pmod{4}$, observe that

$$N(a + bw) = \left(a + \frac{b}{2}\right)^2 + d \left(\frac{b}{2}\right)^2 = a^2 + ab + \left(\frac{d + 1}{4}\right)b^2,$$

where $(d + 1)/4 \in \mathbb{Z}$. It follows from this that $N_S(\alpha) \leq N(\alpha)$ for all $\alpha \in \mathcal{O}$. The following lemma gives an inequality that involves elements of $K$ that are not integral.

**Lemma 4.** Suppose

$$\xi = \frac{x + yw}{z} \in K, \quad a = \frac{a + bw}{c} \in \mathcal{O}_S$$

with $x, y, z \in \mathbb{Z}$, $z$ not divisible by any prime in $S$, $a, b \in \mathbb{Z}$, $c \in T$. Then

$$N_S(\xi - \alpha) \leq c^2 N(\xi - \alpha).$$

**Proof.** Observe

$$N(\xi - \alpha) = N_S(c\xi - \alpha) = \frac{N(c\xi - cz\alpha)}{z^2c^2}$$

and $cz^2 - cz\alpha \in \mathcal{O}$ which implies

$$N_S(\xi - \alpha) \leq \frac{N(c\xi - cz\alpha)}{z^2} = c^2 N(\xi - \alpha).$$

We define the fundamental domain for $K$ as

$$\mathcal{F} = \{x + yw \mid 0 \leq x, y \leq 1\} \subseteq \mathbb{C}.$$
This will assist us in visualizing the norm-Euclidean property graphically. This parallelogram $F$ is a rectangle when $-d \equiv 1 \pmod{4}$. Notice that any $z \in \mathbb{C}$ can be written as $z = z' + \alpha$ where $z' \in F$ and $\alpha \in \mathcal{O}$. Indeed, translations of $F$ give a tessellation of $\mathbb{C}$. Hence, when studying the norm-Euclidean property, we can focus our attention on the points in $F$. Indeed, observe that $K$ is norm-Euclidean if and only if for all $\xi \in K \cap F$ there exists $\gamma \in \mathcal{O}$ such that $|\xi - \gamma| < 1$. In particular, $N(\xi - \gamma) = |\xi - \gamma|^2$ where $|\cdot|$ denotes the complex modulus.

**Definition.** For $\alpha \in \mathcal{O}_S$ let $\text{denom}_S(\alpha)$ denote the minimal $c$ over all representations $\alpha = (a + bw)/c$ with $a, b \in \mathbb{Z}, c \in T$.

**Lemma 5.** We have $K$ is $S$-norm-Euclidean if for all $\xi \in K \cap F$ there exists $\alpha \in \mathcal{O}_S$ such that $|\xi - \alpha| < 1/\text{denom}_S(\alpha)$.

**Proof.** Let $\xi \in K$ be arbitrary. We must show that there exists $\alpha \in \mathcal{O}_S$ such that $N_S(\xi - \alpha) < 1$. Without loss of generality, we may assume that $\xi = (x + y\omega)/z$ where $z$ is not divisible by any primes in $S$, by multiplying $\xi$ by an element of $T$ if necessary; indeed, if $u \in T$, then $N_S(u\xi - \alpha) = N_S(u)N_S(\xi - u^{-1}\alpha)$ and $N_S(u) = 1$. Moreover, we may assume $\xi \in F$ by subtracting an appropriate element of $\mathcal{O}$. Noting that $N(\xi - \alpha) = |\xi - \alpha|^2$, Lemma 4 gives the result.

**Example.** Consider the case of $K = \mathbb{Q}(\sqrt{-5})$ whose fundamental domain is depicted in Figure 1. As one can see, we cannot cover $F$ by circles of radius 1 centered at the points of $\mathcal{O}$. However, in light of Lemma 5, if we choose $S = \{2\}$, we can additionally consider circles of radius $1/2$ centered at points in $\mathcal{O}_S$ of the form $(a + bw)/2$. In this case, we can cover $F$ as depicted in Figure 1. This allows us to conclude that $K$ is $\{2\}$-norm-Euclidean.

![Figure 1. $K = \mathbb{Q}(\sqrt{-5})$.](image)

We now generalize the idea presented in the previous example. Consider the collection of all $S$-integers in the fundamental domain (i.e., elements of $\mathcal{O}_S \cap F$). This collection is precisely the set of all $\alpha_{i,j}^k = (i + j\omega)/k$ for $k \in T$ and $0 \leq i, j \leq k$. In light of Lemmas 4 and 5, we know that the condition $F \subseteq \bigcup_{i,j,k} B_{1/k}(\alpha_{i,j}^k)$ is sufficient for $K$ to be $S$-Euclidean. (Here $B_r(\alpha)$ denotes the open ball centered at $\alpha$ of radius $r$.)
Moreover, we will only need to consider the $B_{1/k}(\alpha_{i,j}^k)$ where $\gcd(j, k) = 1$. To each “row of circles” $\bigcup_{i=0}^{k} B_{1/k}(\alpha_{i,j}^k)$, we can associate a horizontal strip

$$R_j^k = \left\{ x + iy \in F : \left| y - \frac{j \Im(w)}{k} \right| < \frac{\sqrt{3}}{2k} \right\}.$$

Note that the $R_j^k$ are maximal with the property that $R_j^k \subseteq F \cap \bigcup_{i,j} B_{1/k}(\alpha_{i,j}^k)$. The $R_j^k$ give rise to intervals via the projection map $\pi(x + iy) = y/\Im(w)$. We define

$$I_j^k = \left( \frac{j}{k} - \frac{\sqrt{3}}{2k \Im(\omega)}, \frac{j}{k} + \frac{\sqrt{3}}{2k \Im(\omega)} \right) = \left( \frac{j - \sqrt{3/D}}{k}, \frac{j + \sqrt{3/D}}{k} \right).$$

Notice that $\Im(w) = \sqrt{D}/2$, which allowed us to rewrite the intervals in terms of $D$ rather than $\Im(w)$. The following lemma gives sufficient conditions for $K$ to be $S$-norm-Euclidean.

**Lemma 6.** Notation as above. If $[0, 1] \cap Q \subseteq \bigcup_{j,k} I_j^k$, then $K$ is $S$-norm-Euclidean.

**Proof.** Suppose $[0, 1] \cap Q \subseteq \bigcup_{j,k} I_j^k$. Taking $\pi^{-1}$ of both sides allows us to observe that

$$K \cap F \subseteq \bigcup_{j,k} \pi^{-1}(I_j^k) \subseteq \bigcup_{i,j,k} B_{1/k}(\alpha_{i,j}^k).$$

In light of the preceding discussion, we conclude that $K$ is $S$-norm-Euclidean.

**Example.** Consider the field $K = Q(\sqrt{-67})$ and choose $\{2, 3\}$. We write down the intervals $I_j^k$ for $0 \leq j \leq k \leq 4$ and $\gcd(j, k) = 1$, and check that these cover the unit interval. Thus $K$ is $S$-norm-Euclidean. Figure 2 gives the intervals $I_j^k$ and the corresponding covering of $F$. We have sorted the intervals to make it easier to “see” them in the covering.

We rephrase Lemma 6 in a slightly different form, that will be convenient in the sequel. In what follows, we write $\{x\} = x - \lfloor x \rfloor$ to denote the fractional part of a real number $x$.

**Corollary 7.** Suppose that for every rational $y \in [0, 1]$ there exists $c \in T$ such that

$$\{cy\} < \frac{\sqrt{3}}{\sqrt{D}} \quad \text{or} \quad \{cy\} > 1 - \frac{\sqrt{3}}{\sqrt{D}}.$$

Then $K$ is $S$-norm-Euclidean.

**Remark.** Let $q$ be the smallest prime such that $q \notin S$. Since $\{c/q\} \geq 1/q$ for all $c \in T$, the condition in Corollary 7 can only be satisfied when $D \leq 3q^2$.

As illustrated in the previous example, our results thus far lead to the following simple procedure:

1. Let $q$ be the smallest prime not in $S$ and set $X = 3q^2$.
2. Enumerate the intervals $I_j^k$ for $0 \leq j \leq k \leq X$ with $\gcd(j, k) = 1$. 
3. If $[0, 1] \subseteq \bigcup_{j,k} I^k_j$, then $K$ is $S$-norm-Euclidean.

Notice that this procedure will never prove that $K$ is not $S$-norm-Euclidean. However, as was stated in Theorem 1 it allows us to show a number of complex quadratic fields $K$ are, in fact, $S$-norm-Euclidean.

3. PROOF OF THEOREMS 1 AND 2.

Proof of Theorem 1. The procedure described in the previous section leads to the proof of Theorem 1. Implementing this procedure on a computer gives the result immediately, but this could be done by hand as the number of intervals necessary is not large. Below we give the value of $X$ necessary and the number of intervals produced.

| $S$ | $X$ | #$I^k_j$ |
|-----|-----|---------|
| $\emptyset$ | 1   | 2       |
| {2}  | 2   | 3       |
| {2, 3} | 4   | 7       |
| {2, 3, 5} | 6   | 13      |

Proof of Theorem 2. The proof is a simple application of Corollary 7. Let $y \in [0, 1] \cap \mathbb{Q}$. Set $X = \sqrt{D}/\sqrt{3}$. Suppose that $S$ contains all primes $\leq \lceil X \rceil$ so that $T$ contains all positive integers $\leq \lceil X \rceil$. We will follow the proof of Dirichlet’s approximation theorem to show that there exists a $c \in T$ such that $\{cy\} < 1/X$ or $\{cy\} > 1 - 1/X$. Subdivide the unit interval as

$$[0, 1) = \left[ 0, \frac{1}{X} \right) \cup \left[ \frac{1}{X}, \frac{2}{X} \right) \cup \cdots \cup \left[ \frac{k - 1}{X}, 1 \right)$$
Section 5.

Proof of Theorem 3 (p is odd). Let \( d > 0 \) be squarefree and \( S = \{p\} \) with \( p \) odd and \( (-d/p) \neq 1 \). Set \( \xi_0 = (1 + w)/2 \). Let \( \alpha \in \mathcal{O}_S \) be arbitrary and write \( \alpha = (a + bw)/p^n \) where \( a, b, n \in \mathbb{Z} \), \( n \geq 0 \), with \( a \) and \( b \) not both divisible by \( p \). Then we have

\[
\alpha - \xi_0 = \frac{(2a - p^n) + (2b - p^n)w}{2 \cdot p^n} = \frac{p^n(A + Bw)}{2 \cdot p^n},
\]

where we have rewritten \( 2a - p^n = p^mA \) and \( 2b - p^n = p^nB \) with \( m \geq 0 \) maximal. Note that \( A, B \neq 0 \) and both of \( A, B \) cannot be divisible by \( p \).

First consider the case where \(-d \equiv 2, 3 \pmod{4}\). We have

\[
N(\alpha - \xi_0) = \frac{p^{2m}(A^2 + dB^2)}{2^2 \cdot p^{2n}}.
\]  

Suppose \((-d/p) = -1\). We claim that \( A^2 + dB^2 \) is not divisible by \( p \). Indeed if, \( p \) divides \( A^2 + dB^2 \), then we must have \( A, B \equiv 0 \pmod{p} \), lest both \( A \) and \( B \) be divisible by \( p \); in this case, one can solve \( A^2 + dB^2 \equiv 0 \pmod{p} \) to show that \(-d\) is a square mod \( p \), a contradiction. Therefore \( N_S(\alpha - \xi_0) \geq (1 + d)/4 \geq 1 \) when \( d \geq 3 \).

We turn to the case where \((-d/p) = 0\). This is the same except that \( A^2 + dB^2 \) is possibly divisible by one copy of \( p \). If \( A + dB^2 \) is not divisible by \( p \), the result follows as before, so we may assume \( p \) divides \( A^2 + dB^2 \). In this case, \( A \) must be divisible by \( p \) as well, which implies that \( A^2 + dB^2 \geq p^2 + d \) and hence \( N_S(\alpha - \xi_0) \geq (p^2 + d)/(4p) \geq 1 \) when \( d \geq 3 \). Consequently, \( K \) is not \( S \)-norm-Euclidean when \(-d \equiv 2, 3 \pmod{4}\) and \( d \geq 3 \).

Next we turn to the case \(-d \equiv 1 \pmod{4}\). We have

\[
N(\alpha - \xi_0) = \frac{p^{2m}(2A + B) + 4dB}{2^2 \cdot p^{2n}} = \frac{p^{2m}((2A + B)^2 + dB^2)}{16 \cdot p^{2n}}.
\]

If \((-d/p) = -1\), then \( (2A + B)^2 + dB^2 \) is not divisible by \( p \), and it follows that then \( N_S(\alpha - \xi_0) \geq (1 + d)/16 \geq 1 \) when \( d \geq 15 \). If \((-d/p) = 0\) then \( (2A + B)^2 + dB^2 \) is divisible by at most one copy of \( p \). As before, we may assume \( p \) divides \( (2A + B)^2 + dB^2 \). In this case, we find that \( 2A + B \) is divisible by \( p \) and thus \( N_S(\alpha - \xi) \geq (p^2 + d)/(16p) \). This gives \( N_S(\alpha - \xi_0) \geq 1 \) when \( d \geq 15 \) except when \((d, p)\) equals one of the pairs \((15, 3)\), \((15, 5)\), \((35, 5)\), \((35, 7)\). We deal with the remaining cases in Section 5.

Proof of Theorem 3 (p = 2). Let \( d > 0 \) be squarefree and \( S = \{2\} \). Set \( \xi_0 = (1 + w)/3 \). Let \( \alpha \in \mathcal{O}_S \) be arbitrary and write \( \alpha = (a + bw)/2^n \) where \( a, b, n \in \mathbb{Z} \), \( n \geq 0 \), with \( a \) and \( b \) not both divisible by \( 2 \); in this case, one can solve \( a + bw \equiv 0 \pmod{2^n} \) to show that \(-d\) is a square mod \( 2 \), a contradiction. Therefore \( N_S(\alpha - \xi_0) \geq (1 + d)/8 \geq 1 \) when \( d \geq 7 \).

We turn to the case where \((-d/p) = 0\). This is the same except that \( A^2 + dB^2 \) is possibly divisible by one copy of \( p \). If \( A + dB^2 \) is not divisible by \( p \), the result follows as before, so we may assume \( p \) divides \( A^2 + dB^2 \). In this case, \( A \) must be divisible by \( p \) as well, which implies that \( A^2 + dB^2 \geq p^2 + d \) and hence \( N_S(\alpha - \xi_0) \geq (p^2 + d)/(8p) \geq 1 \) when \( d \geq 7 \). Consequently, \( K \) is not \( S \)-norm-Euclidean

4. PROOF OF THEOREM 3. Since Theorem 3 gives a classification of sorts, the proof will require some effort. The first part concerns the case where \( S = \{p\} \) with \( p \) odd, and the second part concerns the case where \( S = \{2\} \). In each part we must treat the cases of \(-d \equiv 2, 3 \pmod{4}\) and \(-d \equiv 1 \pmod{4}\) separately. At first pass, the reader may wish to skip one or more of these subproofs.
We aim to show that \( N_S(\alpha - \xi_0) \geq 1 \). Observe that
\[
\alpha - \xi_0 = \frac{(3a - 2^n) + (3b - 2^n)w}{3 \cdot 2^n}.
\]
We rewrite \( 3a - 2^n = 2^m A \) and \( 3b - 2^n = 2^m B \) where \( A, B \in \mathbb{Z} \) and \( m \geq 0 \) is maximal; note that \( A, B \neq 0 \) and that \( A \) and \( B \) cannot both be even.

First consider the case where \( -d \equiv 2, 3 \pmod{4} \). We have
\[
N(\alpha - \xi_0) = \frac{2^{2m}(A^2 + dB^2)}{3^2 \cdot 2^{2n}}.
\]
We claim that \( A^2 + dB^2 \) is divisible by at most one power of 2; indeed, the assumption \( A^2 + dB^2 \equiv 0 \pmod{4} \) together with \( -d \equiv 2, 3 \pmod{4} \) leads to \( A, B \equiv 0 \pmod{2} \), a contradiction. Whence
\[
N_S(\alpha - \xi_0) \geq \frac{A^2 + dB^2}{2 \cdot 3^2} \geq \frac{1 + d}{18},
\]
which implies \( N_S(\alpha - \xi_0) \geq 1 \) when \( d \geq 17 \). Since \( \alpha \in O_S \) was arbitrary, this proves that \( K \) is not \( S \)-norm-Euclidean when \( -d \equiv 2, 3 \pmod{4} \) and \( d \geq 17 \).

This leaves open the cases of \( d = 10, 13, 14 \). However, if \( d \) is even then the assumption that 2 divides \( A^2 + dB^2 \) leads to 2 divides \( A \) and therefore \( N_S(\alpha - \xi_0) \geq (2^2 + d)/18 \geq 1 \) when \( d \geq 14 \). We will return to the particular cases of \( d = 10, 13 \) later.

Next, consider the case where \( -d \equiv 1 \pmod{4} \). We assume, in addition, that \( -d \not\equiv 1 \pmod{8} \) so that \( (1 + d)/4 \) is an odd integer. We have
\[
N(\alpha - \xi_0) = \frac{2^{2m}(A^2 + AB + (\frac{1+d}{4})B^2)}{3^2 \cdot 2^{2n}}.
\]
We claim that \( A^2 + AB + ((1 + d)/4)B^2 \) is not divisible by 2; indeed, if this were the case, then \( A^2 + AB + B^2 \equiv 0 \pmod{2} \) which leads to \( A, B \equiv 0 \pmod{2} \), a contradiction. Therefore
\[
N_S(\alpha - \xi_0) \geq \frac{A^2 + AB + (\frac{1+d}{4})B^2}{3^2} = \frac{(2A + B)^2 + dB^2}{4 \cdot 3^2} \geq \frac{1 + d}{36}.
\]
Hence \( N_S(\alpha - \xi_0) \geq 1 \) when \( d \geq 35 \).

Now we return to the case of \( d = 13 \). We choose \( \xi_0 = w/3 \). In a similar manner as before, we write
\[
N(\alpha - \xi_0) = \frac{2^{2m}(A^2 + 13B^2)}{2^{2n}6^2},
\]
where \( 2^m A = 3a, 2^m B = 3b - 2^n, A, B \) not both even, and \( A^2 + 13B^2 \) is divisible by at most one power of 2. Notice that \( B \neq 0 \) but that \( A \) could equal 0. If 2 does not divide \( A^2 + 13B^2 \), then \( N_S(\alpha - \xi_0) \geq 13/9 \geq 1 \). Hence we may assume 2 divides \( A^2 + 13B^2 \). In this case, it follows that \( A \) and \( B \) are both odd, and since \( A \) is a multiple of 3, we have \( N_S(\alpha - \xi_0) \geq (3^2 + 13)/18 \geq 1 \).

It turns out the case of \( (d, p) = (10, 2) \) will require a little more care. We return to this case in the next section.

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It may be useful to remark that we have now completely dealt with the “only if” portion of Theorem 3. The “if” portion of the theorem follows from Theorem 1 except for the five \((d, p)\) pairs which we will now consider.

5. EXCEPTIONAL CASES. Here we deal with the cases where \((d, p)\) is one of the following pairs:

\[(10, 2), (15, 3), (15, 5), (35, 5), (35, 7).\]

It turns out that our criterion from Section 2 just barely fails for the cases \(d = 10\) and \(d = 15\). In particular, the countable union of intervals \(\bigcup_{j,k} I^k_j\) fail to cover \([0, 1]\) by just finitely many points. The case of \(d = 35\) is a little more difficult to deal with.

Proposition 8. \(\mathbb{Q}(\sqrt{-10})\) is \(2\)-norm-Euclidean.

Proof. For the first part of the proof, we will show that

\[
\bigcup_{j,k} I^k_j \supseteq [0, 1] \setminus \left\{ \frac{1}{3}, \frac{2}{3} \right\}.
\]

To establish this, we show that the condition in Corollary 7 holds for all \(y \in [0, 1]\) except when \(y \in \{1/3, 2/3\}\). First, observe that \(y \in \{1/3, 2/3\}\) implies the fractional part \(2^n y\) belongs to \([1/3, 2/3]\) for all \(n \geq 0\), and hence the condition clearly fails for these two values of \(y\). Note that \(\sqrt{3}/40 \approx 0.274 < 1/3\).

Let \(y \in [0, 1]\) be arbitrary. If there exists \(n \geq 0\) such that \(2^n y \in [1/3, 2/3] \cup [3/4, 1]\), then we are done, since \(1/4 < \sqrt{3}/40\). By way of contradiction, suppose that \(2^n y \in [1/4, 1/2]\) for all \(n \geq 0\). We have \(y \in J_0 := [1/4, 1/2]\) or \(y \in J'_0 := [1/2, 3/4]\). For sake of concreteness, suppose \(y \in J_0\), but the case of \(y \in J'_0\) is treated in a similar manner. Since \(y \in [1/4, 1/2]\), we have \(2y \in [1/2, 1]\) but by hypothesis it must be that \(2y \in J_1 := [1/2, 3/4]\). Similarly, \(4y \in J_2 := [5/4, 3/2]\). We inductively define a sequence of intervals \(J_0, J_1, J_2, \ldots\) in this manner, all of length \(1/4\), having the property that \(2^n y \in J_n\). Therefore \(y \in \bigcap_n 2^{-n} J_n\), and since this intersection contains at most one element, we arrive at a contradiction, except for one possible value of \(y\). Similarly, if \(y \in J'_0\), we can inductively define \(J'_0, J'_1, J'_2, \ldots\) and obtain a contradiction, except for one possible value of \(y\). Hence we have proven that the condition in Corollary 7 holds with at most two possible exceptions. But since we already know \(1/3\) and \(2/3\) are exceptions, this proves (6).

It suffices to show that given \(\xi = r/s + w/3 \in \mathcal{F}\) with \(s\) odd, there exists \(\alpha \in \mathcal{O}_S\) such that \(N_S(\xi - \alpha) < 1\). (If \(\xi\) is of the form \(r/s + 2w/3\), then multiply by 2 and translate back into \(\mathcal{F}\) using an element of \(\mathcal{O}\).) Notice that \(0 \leq r/s \leq 1\) since \(\xi \in \mathcal{F}\). First set \(\alpha = w/2\). We find

\[
\alpha - \xi_0 = \frac{6r + sw}{6s}, \quad N(\alpha - \xi_0) = \frac{18r^2 + 5s^2}{2 \cdot 3^2 s^2}.
\]

Therefore \(N_S(\alpha - \xi_0) < 2(r/s)^2 + 5/9 < 1\) when \(r/s < \sqrt{2}/3 \approx 0.47\). If \(\alpha = (2 + w)/2\), then we find \(N_S(\alpha - \xi_0) < 2(1 - r/s)^2 + 5/9 < 1\) when \(r/s > 1 - \sqrt{2}/3 \approx 0.53\). If \(\alpha = (2 + w)/4\), then

\[
N_S(\alpha - \xi_0) \leq 4(1/2 - r/s)^2 + 5/9 < 1
\]

when \(1/6 \leq r/s \leq 5/6\). This completes the proof.

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Proposition 9. $\mathbb{Q}(\sqrt{-15})$ is $\{3\}$-norm-Euclidean and $\{5\}$-norm-Euclidean.

Proof. Suppose $S = \{p\}$ with $p = 3$ or $p = 5$. We proceed in a similar manner as in Proposition 8. First we claim that $\bigcup_{j,k} E_j \supseteq [0,1) \setminus \{1/2\}$. For sake of concreteness, we prove this when $p = 3$. Let $y \in [0,1)$ be arbitrary. Suppose $\{3^ny\} \in [1/3, 2/3]$ for all $n \geq 0$. (If this condition does not hold, then we are done since $1/3 < \sqrt{3/15}$.) This means that $y \in J_0 := [1/3, 2/3]$, $3y \in J_1 := [4/3, 5/3]$, $9y \in J_2 := [13/3, 14/3]$, and so on. Inductively, we find

$$3^ny \in J_n := \left[\frac{3^n}{2} - \frac{1}{6}, \frac{3^n}{2} + \frac{1}{6}\right].$$

Therefore $y \in \bigcap_n 3^{-n} J_n = \{1/2\}$. This proves the claim when $p = 3$. When $p = 5$, the proof is similar. Noting that $2/5 < \sqrt{3/15}$, we can assume that $\{5^ny\} \in [2/5, 3/5]$ for all $n \geq 0$. This leads to $J_n := \{5^n/2 - 1/10, 5^n/2 + 1/10\}$ and the result follows as before.

Let $\xi = r/s + w/2 \in \mathcal{F}$ with $(p,s) = 1$. We may assume $0 \leq r/s < 1$. If $\alpha = w$, then $N_S(\xi - \alpha) \leq (r/s - 1/4)^2 + 15/16 < 1$ when $0 < r/s < 1/2$. Similarly, if $\alpha = 1$, then $N_S(\xi - \alpha) \leq (r/s - 3/4)^2 + 15/16 < 1$ when $1/2 < r/s < 1$. This leaves only the points $\xi = w/2$ and $\xi = (1 + w)/2$. When $\xi = w/2$, we compute $N(\xi) = 15/16$ so that $N_S(\xi) < 1$ in either case. Consider $\xi = (1 + w)/2$. We have $N(\xi) = 3/2$ so that $N_S(\xi) = 1/2 < 1$ in the case $p = 3$. For the case $p = 5$, we compute $N_S(\xi - 2) = 5/2$ so that $N_S(\xi - 2) = 1/2 < 1$. This completes the proof.

It appears that the strategy employed for the previous two propositions does not work when $d = 35$. Indeed, although it turns out that $K$ is $S$-norm-Euclidean in the remaining exceptional cases, Lemma 4 may not be sufficient to prove this. We require a new idea. The following lemma essentially says that in some cases, we can increase the radii of our circles.

Lemma 10. Notation as in Lemma 4. Assume $p \in S$ and $2 \not\in S$. Assume $-d \equiv 1 \pmod{4}$ and $p \mid d$. If the following holds,

$$c \equiv 0 \pmod{p}, \ b \not\equiv 0 \pmod{p}, \ 2a + b \equiv 0 \pmod{p} \quad (7)$$

then

$$N_S(\xi - \alpha) \leq \frac{c^2}{p} N(\xi - \alpha).$$

In particular, in Lemma 5, it suffices to find a value of $\alpha$ satisfying (7) such that

$$|\xi - \alpha| < \frac{\sqrt{p}}{c}.$$

Proof. We have $\xi - \alpha = (A + Bw)/(cz)$ where $A = cx - az$ and $B = cy - bz$ so that

$$N(\xi - \alpha) = \frac{(2A + B)^2 + dB^2}{4c^2z^2}.$$ 

As in the proof of Lemma 4 we have

$$N_S(\xi - \alpha) \leq \frac{(2A + B)^2 + dB^2}{4z^2} = \frac{c^2}{p} N(\xi - \alpha).$$

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Table 1. Circles \(|\xi - \alpha| < r\) that cover \(F\) when \(p = 5\).

| \(r\) | \(\alpha\) |
| --- | --- |
| 1 | 0, 1, \(w\), \(1 + w\) |
| \(1/5\) | \(1 + 2w\), \(3/5\), \(3 + 3w\), \(4 + 3w\) |
| \(\sqrt{5}/5\) | \(2 + w\), \(-1 + 2w\), \(6 + 3w\), \(3 + 4w\), \(4 + 2w\), \(1 + 3w\) |

If, in addition, the conditions (7) hold, then \(2A + B \equiv 0 \pmod{p}\) which implies the numerator is divisible by \(p\), and therefore \(N_S(\xi - \alpha) \leq p + c^2 N(\xi - \alpha)\).

**Proposition 11.** \(\mathbb{Q}(\sqrt{-35})\) is \(\{5\}\)-norm-Euclidean and \(\{7\}\)-norm-Euclidean.

**Proof.** Set \(S = \{p\}\). We first consider \(p = 5\). Making use of Lemmas 5 and 10 we obtain a list of 14 circles of various radii that suffice to cover \(F\). See Table 1 for the centers of the chosen circles (as elements \(\alpha \in \mathcal{O}_S\)) and the radius afforded by our lemmas. In principle, one could verify by hand that the 14 provided circles cover \(F\). We believe the picture in Figure 3 that displays the covering is fairly convincing. If one is not convinced by the picture, one could partition the fundamental domain into \(5^6\) parallelograms of equal area in the obvious way, and use a computer to check that each parallelogram lies entirely inside one of the given circles. When \(p = 7\), the argument is similar. We give a list of 18 circles (see Table 2) that carry out the covering (also depicted in Figure 3).

**Appendix.** We finish with some discussion that requires a modicum of algebraic number theory. (One possible reference is [6].) Let \(K\) be a number field with ring of integers \(\mathcal{O}\), and let \(N\) denote the absolute value of the norm map. In the quadratic setting, it is known that there are only finitely many norm-Euclidean fields. When \(K\) is complex quadratic, this result is classical. When \(K\) is real quadratic, the classification was completed by Chatland and Davenport in 1950 (see [2]). It turns out that \(K = \mathbb{Q}(\sqrt{d})\) with \(d > 0\) squarefree is norm-Euclidean if and only if \(d = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\). Barnes and Swinnerton-Dyer corrected an error from earlier work (see [1]); for a time, it was thought that
Figure 3. $K = \mathbb{Q}(\sqrt{-35})$.

the field $\mathbb{Q}(\sqrt{97})$ was norm-Euclidean. See [5] for a self-contained exposition of this result. Davenport generalized the result of [2] to show that there are only finitely many norm-Euclidean fields with rank($\mathcal{O}^\times$) = 1. Very little is known about norm-Euclidean fields when rank($\mathcal{O}^\times$) > 1. See [9, 10] for an introduction to the topic of Euclidean number fields, or [7] for a survey of the field.

Recall that we can associate to every prime $v$ of $K$ (finite or infinite) an absolute value $|·|_v : K \to \mathbb{R}_{\geq 0}$. Let $S$ be a finite set of primes of $K$ containing the set of infinite primes $S_{\infty}$. We define the $S$-integers of $K$ and the $S$-norm as

$$\mathcal{O}_S = \{ \xi \in K \mid v(\xi) \geq 0 \text{ for all } v \notin S \}, \quad N_S(\xi) = \prod_{v \in S} |\xi|_v.$$ 

A theorem of O’Meara (see [12]) says that for any number field $K$ there exists a finite set of primes $S$ such that $K$ is $S$-norm-Euclidean. It would be desirable to have a quantitative form of this theorem. In principle, the techniques developed by Lenstra in [8] would allow one to prove such a result (see the last paragraph of Section 1 of [8]). However, to our knowledge this has not been carried out anywhere in the literature. On the other hand, O’Meara’s Theorem is also proved in [11]; in principle this approach could be made quantitative as well, but the author only explicitly states such a result in the complex quadratic case.

The determination of all $S$-norm-Euclidean number fields (and function fields) with $\#S = 2$ was completed by van der Linden (see [14]). The analogue of Dirichlet’s Unit Theorem says that rank($\mathcal{O}_S^\times$) = $\#S - 1$ and consequently, $\#S = 2$ is precisely the situation where $\mathcal{O}_S^\times$ has rank one. Again, not much is known when rank($\mathcal{O}_S^\times$) > 1.

We now explain how the setting in which we have been working throughout the paper fits into this picture. Let $K$ be a complex quadratic field and $S$ be a finite set of rational primes as in Section 1. Let $S_K$ denote the set all prime ideals of $K$ lying above primes in $S$ together with the set of all infinite primes $S_{\infty}$. In this setting one could write $\mathcal{O}_S$ instead of the more cumbersome but accurate $\mathcal{O}_{S_K}$, and $N_S$ instead of $N_{S_K}$. It turns out that $\mathcal{O}_S$ is then the same ring we introduced in Section 1, and $N_S$ is the same
function. It can now be seen that our proof of Theorem 2 is an elementary proof of a quantitative version of O’Meara’s theorem in the complex quadratic case.

Considering the case of \( S = \{ p \} \) we find that \( \#S_K = 2 \) when \( p \) is inert or ramified, and \( \#S_K = 3 \) if \( p \) splits. Because the splitting behavior of \( p \) can be detected using the Legendre symbol, the conditions in Theorem 3 were secretly insisting that \( \text{rank}(\mathcal{O}_S^\times) = 1 \). Therefore this theorem is subsumed by van der Linden’s classification. We still believe it is worthwhile to put down, as our proof is elementary in nature. Additionally, Theorem 1 contains a number of examples in which \( \text{rank}(\mathcal{O}_S^\times) > 1 \).

We now give the connection with the class group of \( K \), denoted by \( \text{Cl}(K) \). Recall that \( \text{Cl}(K) \) is trivial if and only if \( \mathcal{O} \) is a UFD. Suppose \( S \) be a set of primes in \( K \) containing \( S_\infty \). The analogously defined \( S \)-class group \( \text{Cl}_S(K) \) is isomorphic to \( \text{Cl}(K) \) modulo the subgroup generated by the finite primes in \( S \). In particular, if \( K \) is \( S \)-norm-Euclidean, then \( \text{Cl}_S(K) \) is trivial, and hence the primes (or even just the non-principal primes) in \( S \) generate \( \text{Cl}(K) \). This explains the comment immediately following Theorem 2; indeed, in the complex quadratic setting, the Minkowski constant is \( 2/\pi > 1/\sqrt{3} \).

Finally, consider the situation of Theorem 3 and the remark that follows. If \( p \) is inert, then \( \text{Cl}(K) \cong \text{Cl}_S(K) \) and hence \( K \) being \( S \)-norm-Euclidean implies \( \text{Cl}(K) \) is trivial. A famous theorem of Baker–Heegner–Stark states that this only happens for \( d = 1, 2, 3, 7, 11, 19, 43, 67, 163 \). If \( p \) is ramified then \( (p) = p^2 \) and \( \text{Cl}_S(K) \) is isomorphic to \( \text{Cl}(K)/\langle p \rangle \) where \( p^2 \) is trivial in \( \text{Cl}(K) \); consequently, \( K \) being \( S \)-norm-Euclidean implies that \( \text{Cl}(K) \) has order 1 or 2. Similarly, complex quadratic fields with class number 2 have been classified, so the list of possibilities is finite. (Keep in mind that the solutions to these class number problems are deep results and that our proof of Theorem 3 did not make use of these results in any way.) If \( p \) is split then \( (p) = p\overline{p} \), and since \( p, \overline{p} \) are inverses in \( \text{Cl}(K) \), it follows again that \( \text{Cl}_S(K) \cong \text{Cl}(K)/\langle p \rangle \); consequently, \( K \) being \( S \)-norm-Euclidean implies the class group is cyclic generated by \( p \). This event should happen infinitely often, but currently this is unproven. This explains the discussion following Theorem 3.

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