GENERIC EXPONENTIAL SUMS ASSOCIATED TO LAURENT POLYNOMIALS IN ONE VARIABLE

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Abstract. The generic Newton polygons for $L$-functions of exponential sums associated to Laurent polynomials in one variable are determined when $p$ is large. The corresponding Hasse polynomials are also determined.

Key words: exponential sum, $L$-function, generic Newton polygon

MSC2000: 11L07, 14F30

1. Introduction

We shall determine the generic Newton polygon of $L$-functions of exponential sums associated to Laurent polynomials in one variable.

Throughout this paper, $p$ denotes a prime number, and $q$ denotes a power of $p$. Write $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Let $\bar{\mathbb{F}}_p$ be a fixed algebraic closure of the finite field $\mathbb{F}_p$, and $\mathbb{F}_q$ the finite field with $q$ elements in $\bar{\mathbb{F}}_p$.

Let $f$ be a Laurent polynomial over $\mathbb{F}_q$. We assume that the leading exponents of $f$ are prime to $p$. One associates to $f$ the Artin-Schreier curve

$$C_f : y^q - y = f(x).$$

Let $N_k$ be the number of $\mathbb{F}_{q^k}$-rational points including the infinities on $C_f$. The zeta function of $C_f$ is defined by

$$Z(t, C_f, \mathbb{F}_q) = \exp(\sum_{k=1}^{+\infty} N_k t^k).$$

Let $\mathcal{Q}$ be a fixed algebraic closure of $\mathbb{Q}$. Let $\psi$ denote any nontrivial character of $\mathbb{F}_p$ into $\mathcal{Q}^\times$. Let $V_f$ be the affine line $\mathbb{A}$ over $\mathbb{F}_q$ if $f$ is a polynomial, and let $V_f$ be the one-dimensional torus $\mathbb{T}$ over $\mathbb{F}_q$ if $f$ is not a polynomial. We have

$$N_k = q^k + 1 + \sum_{\alpha \in \mathbb{F}_q^\times} S(k, \alpha f, \mathbb{F}_q),$$

where the exponential sum $S(k, f, \mathbb{F}_q)$ is defined by

$$S(k, f, \mathbb{F}_q) = \sum_{x \in V_f(\mathbb{F}_{q^k})} \psi(\text{Tr}_{\mathbb{F}_q^k/\mathbb{F}_p}(f(x))).$$

So we have

$$(1 - t)(1 - qt)Z(t, C_f, \mathbb{F}_q) = \prod_{\alpha \in \mathbb{F}_q^\times} L(t, \alpha f, \mathbb{F}_q),$$

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where the $L$-function $L(t, f, \mathbb{F}_q)$ is defined by

$$L(t, f, \mathbb{F}_q) = \exp\left(\sum_{k=1}^{+\infty} S(k, f, \mathbb{F}_q) t^k \right).$$

It is well-known that the function $L(t, f, \mathbb{F}_q)$ is a polynomial in $t$ with coefficients in $\mathbb{Q}$.

Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, and $\mathbb{Q}_p$ the field of $p$-adic numbers. Fix an embedding of $\mathbb{Q}$ into $\mathbb{Q}_p$. Let $\text{ord}_p(\cdot)$ be the $p$-adic order function of $\mathbb{Q}_p$, and define the $q$-adic order function as $\text{ord}_q(\cdot) = \frac{1}{\text{ord}_p(q)} \cdot \text{ord}_p(\cdot)$. As $L(t, f, \mathbb{F}_q)$ has coefficients in $\mathbb{Q}$, one can talk about the $p$-adic absolute values of its reciprocal roots. These $p$-adic absolute values are completely determined by the Newton polygon of $L(t, f, \mathbb{F}_q)$ defined as follows.

**Definition 1.1.** Let $g(t) = 1 + \sum_{i=1}^{n} c_i t^i$ be a polynomial in $t$ with coefficients $c_i \in \mathbb{Q}_p$. The $q$-adic Newton polygon of $g$ is the lower convex closure of the points $(0, 0), (n, \text{ord}_q(c_n)), n = 1, \cdots, u$.

It is very hard to determine the Newton polygon of $L(t, f, \mathbb{F}_q)$ in general. However, it is easier to give a good lower bound. The simplest one is the Hodge polygon defined as follows.

**Definition 1.2.** Let $d$ be a positive integer. The Hodge polygon of the interval $[0, d]$ is the polygon whose vertices are $(n, \frac{n(n+1)}{2d}), n = 0, 1, 2, \cdots, d-1$.

**Definition 1.3.** Let $d$ and $e$ be positive integers. The Hodge polygon of $[-e, d]$ is the polygon with initial point $(0, 0)$, end point $(d+e, d+e) = (d, d)$, and the vertices $(m+n+1, \frac{m(n+1)}{2e} + \frac{n(n+1)}{2d})$ with $(m,n)$ running over pairs satisfying $-\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}, 0 \leq m < e, 0 \leq n < d$.

Let $\Delta(f)$ be the smallest closed interval of the real line containing 0 and the exponents of the monomials of $f$. So $\Delta(f) = [0, d]$ if $f$ is a polynomial of degree $d$, and $\Delta(f) = [-e, d]$ if $f$ is a Laurent polynomial with leading term $a_{-e} x^{-e} + a_d x^d$. The well-known Hodge bound is stated as the following theorem.

**Theorem 1.4** (Hodge bound). The $q$-adic Newton polygon of $L(t, f, \mathbb{F}_q)$ lies above the Hodge polygon of $\Delta(f)$. Moreover, both polygons have the same initial point and the same end point.

By Grothendieck’s specialization lemma (Confer [K76] and [W04]), the $q$-adic Newton polygon of $L(t, f, \mathbb{F}_q)$ is constant for a generic $f$ with fixed $\Delta(f) = \Delta$. That constant polygon is called the generic Newton polygon of $\Delta$. Let $d > 0$ and $e \geq 0$ be integers. Let $D = d$ if $e = 0$, and let $D$ be the least common multiple of $d$ and $e$ if $e > 0$. We assume that $D$ is prime to $p$. The following theorem says that in nice situations the generic polygon coincides with the Hodge polygon.

**Theorem 1.5** (Stickelberger’s theorem [W93]). The generic Newton polygon of $[-e, d]$ coincides with its Hodge polygon if and only if $p \equiv 1 \pmod{D}$.
As a special case of a conjecture of Wan [W04], the following theorem says that the Hodge bound is approximately the best.

**Theorem 1.6** (Zhu [Zh03, Zh04, Zhu04]). *The generic Newton polygon of \([-e, d]\) goes to its Hodge polygon as \(p\) goes to infinity.*

In proving the above theorem in the case \(e = 0\), Zhu used Dwork’s \(p\)-adic theory, a kind of Diéonné-Manin diagonalization, and some force computations to produce a list of polynomials she denoted as \(f_i\)’s. She then used a kind of maximal-monomial-locating technique to prove that one of these \(f_i\)’s does not vanish. Blache-Férard [BF] discovered that Zhu’s maximal-monomial-locating technique can prove the nonvanishing of \(f_0\). This enabled them to get the following theorem.

**Theorem 1.7** (Blache-Férard). *If \(p \geq 3D\), the generic Newton polygon of \([0, d]\) is the polygon with vertices*

\[
(n, \frac{1}{p-1} \sum_{i=1}^{n} \left| \frac{pi - n}{d} \right|), \quad n = 0, 1, \ldots, d-1.
\]

The condition \(p \geq 3D\) in the theorem is very clean. To achieve that clean condition Blache-Férard abolished Zhu’s Diéonné-Manin diagonalization technique, and made recourse to Dwork’s original method.

It should be mentioned that Yang [Ya] computed the Newton polygons for \(L\)-function of exponential sums associated to polynomials of the form \(x^d + \lambda x\), and Hong [H01, H02] computed the Newton polygons for \(L\)-function of exponential sums associated to polynomials of degree 4 and 6.

From now on we assume that \(e > 0\). We shall determine the generic Newton polygon of \([-e, d]\).

Write

\[
p_{[0,d]}(n) = \frac{1}{p-1} \sum_{i=1}^{n} \left| \frac{pi - n}{d} \right|, \quad n = 0, 1, \ldots, d.
\]

And write

\[
p_{[-e,d]}(0) = 0, \quad p_{[-e,d]}(d + e) = \frac{d + e}{2},
\]

\[
p_{[-e,d]}(k) = \min_{(m,n) \in I_k} \{p_{[0,e]}(m) + p_{[0,d]}(n)\}, \quad k = 1, \ldots, d + e - 1,
\]

where

\[
I_k = \{(m,n) \mid m + n + 1 = k, -\frac{1}{e} \leq \frac{m}{e} - \frac{n}{d} \leq \frac{1}{d}, 0 \leq m < e, 0 \leq n < d\}.
\]

**Definition 1.8.** *The arithmetic polygon of \([-e, d]\) is defined to be the graph of the function on \([0, d + e]\) which is linear between consecutive integers and takes on the value \(p_{[-e,d]}(k)\) at integers \(k = 0, 1, \ldots, d + e\).*

We shall prove the following theorem.

**Theorem 1.9.** *The generic Newton polygon of \([-e, d]\) coincides with its arithmetic polygon if \(p \geq 3D\).*
It would be interesting if one can extend the result to twisted exponential sums and to exponential sums associated to functions studied in [Zhu04] and [BFZ].

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2. THE ARITHMETIC POLYGON

Recall that, for $k=1,\cdots,d+e-1$,

$$I_k = \{(m,n) \mid m+n+1 = k, -\frac{1}{e} \leq \frac{m}{e} - \frac{n}{d} \leq \frac{1}{d}, 0 \leq m < e, 0 \leq n < d\}.$$ 

Let $V_k$ be the subset of $I_k$ consisting pairs at which the function

$$(m,n) \mapsto p_{[0,e]}(m) + p_{[0,d]}(n)$$

takes on the minimal value. In this section we shall prove the following theorem.

**Theorem 2.1.** Let $p > 3D$. Then the arithmetic polygon of $[-e,d]$ is convex. Moreover, $(k,p_{[-e,d]}(k))$ $(0 < k < d+e)$ is a vertex if and only if $V_k$ contains only one pair.

We begin with the following lemma.

**Lemma 2.2.** The set $I_k$ contains one or two pairs. If $I_k = \{(m,n)\}$, then

$$-\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}.$$ 

If $I_k$ contains exactly two pairs, then it is of form $\{(m,n), (m+1,n-1)\}$ with

$$\frac{m+1}{e} = \frac{n}{d}.$$ 

**Proof.** Define a degree function on $\mathbb{Z}$ by

$$\deg(i) = \begin{cases} i/d, & i \geq 0, \\ -i/e, & i \leq 0. \end{cases}$$

There is a positive integer $u$ such that

$$k = \#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\},$$

or

$$\#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\} < k < \#\{i \in \mathbb{Z} \mid \deg(i) \leq (u+1)/D\}.$$ 

If $k = \#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\}$, then $I_k$ is of form $\{(m,n)\}$ with

$$-\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}.$$ 

If $\#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\} < k < \#\{i \in \mathbb{Z} \mid \deg(i) \leq (u+1)/D\}$, then $I_k$ is of form $\{(m,n), (m+1,n-1)\}$ with

$$\frac{m+1}{e} = \frac{n}{d}.$$ 

The lemma is proved.

It is easy to see that Theorem 2.1 follows from the following three theorems.
Theorem 2.3. Let $k = 1, 2, \cdots, d + e - 1$. If $V_k$ contains two pairs, then

$$2p_{[-e,d]}(k) = p_{[-e,d]}(k - 1) + p_{[-e,d]}(k + 1).$$

Theorem 2.4. Let $k = 1, 2, \cdots, d + e - 1$. If $I_k$ contains two pairs but $V_k$ contains only one pair, then

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k - 1) + p_{[-e,d]}(k + 1).$$

Theorem 2.5. Let $p > 3D$. Let $k = 1, 2, \cdots, d + e - 1$. If $I_k$ contains only one pair, then

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k - 1) + p_{[-e,d]}(k + 1).$$

Proof of Theorem 2.3. Suppose that $V_k$ contains two pairs. Then so does $I_k$. Assume that $I_k = \{(m,n), (m+1,n-1)\}$. Then $\frac{m+1}{e} = \frac{n}{d}$. It follows that $I_{k-1} = \{(m,n-1)\}$ and $I_{k+1} = \{(m+1,n)\}$. Note that

$$p_{[-e,d]}(k) = p_{[0,e]}(m) + p_{[0,d]}(n) = p_{[0,e]}(m + 1) + p_{[0,d]}(n - 1).$$

It follows that

$$2p_{[-e,d]}(k) = p_{[0,e]}(m) + p_{[0,d]}(n) + p_{[0,e]}(m + 1) + p_{[0,d]}(n - 1) = p_{[-e,d]}(k - 1) + p_{[-e,d]}(k + 1).$$

Theorem 2.3 is proved.

Proof of Theorem 2.4. Assume that $I_k = \{(m,n), (m+1,n-1)\}$. Then $\frac{m+1}{e} = \frac{n}{d}$. It follows that $I_{k-1} = \{(m,n-1)\}$ and $I_{k+1} = \{(m+1,n)\}$. Without loss of generality, we assume that $V_k = \{(m,n)\}$. Then

$$p_{[0,e]}(m) + p_{[0,d]}(n) < p_{[0,e]}(m + 1) + p_{[0,d]}(n - 1).$$

It follows that

$$2p_{[0,e]}(m) + 2p_{[0,d]}(n) < p_{[0,e]}(m) + p_{[0,d]}(n) + p_{[0,e]}(m + 1) + p_{[0,d]}(n - 1).$$

That is,

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k - 1) + p_{[-e,d]}(k + 1).$$

Theorem 2.4 is proved.

Proof of Theorem 2.5. Assume that $I_k = \{(m,n)\}$. Then

$$\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}.$$

Let $(m_1,n_1) \in V_{k-1}$. Then $m_1 = m$ or $n_1 = n$. Without loss of generality, we assume that $m_1 = m$. Then $n_1 = n - 1$. Let $(m_2,n_2) \in V_{k+1}$. Then $m_2 = m$ or $n_2 = n$.

First, we assume that $m_2 = m$. Then $n_2 = n + 1$. Note that

$$p_{[0,d]}(n + 1) - p_{[0,d]}(n) \geq \frac{1}{p - 1}([p - 1]\frac{n + 1}{d} - 1),$$

and

$$p_{[0,d]}(n) - p_{[0,d]}(n - 1) \leq \frac{1}{p - 1}([p - 1]\frac{n}{d}],$$

and

$$[p - 1]\frac{n + 1}{d} < [p - 1]\frac{n}{d} - 1.$$

It follows that

$$2p_{[0,d]}(n) < p_{[0,d]}(n + 1) + p_{[0,d]}(n - 1).$$
Therefore
\[ 2p_{[0,e]}(m) + 2p_{[0,d]}(n) = p_{[0,e]}(m) + p_{[0,d]}(n + 1) + p_{[0,e]}(m) + p_{[0,d]}(n - 1). \]
That is,
\[ 2p_{[-e,d]}(k) < p_{[-e,d]}(k - 1) + p_{[-e,d]}(k + 1). \]
Secondly, we assume that \( n_2 = n \). Then \( m_2 = m + 1 \). Note that
\[ p_{[0,e]}(m + 1) - p_{[0,e]}(m) \geq \frac{1}{p - 1} \left( \left\lceil \frac{m + 1}{e} \right\rceil - 1 \right), \]
\[ p_{[0,d]}(n) - p_{[0,d]}(n - 1) \leq \frac{1}{p - 1} \left( \left\lceil \frac{n}{d} \right\rceil \right), \]
and
\[ \left\lceil \frac{(p - 1)n}{d} \right\rceil < \left\lceil \frac{(p - 1)(m + 1)}{e} \right\rceil - 1. \]
It follows that
\[ p_{[0,e]}(m) + p_{[0,d]}(n) < p_{[0,e]}(m + 1) + p_{[0,d]}(n - 1). \]
Therefore
\[ 2p_{[0,e]}(m) + 2p_{[0,d]}(n) < p_{[0,e]}(m + 1) + p_{[0,d]}(n) + p_{[0,e]}(m) + p_{[0,d]}(n - 1). \]
That is,
\[ 2p_{[-e,d]}(k) < p_{[-e,d]}(k - 1) + p_{[-e,d]}(k + 1). \]
Theorem 2.5 is proved.

3. HASSE POLYNOMIAL

For \( \vec{a} = (a_{-e}, \cdots, a_d) \in \mathbb{F}_q^{d+e+1} \), we write
\[ f_{\vec{a}}(x) = \sum_{i=-e}^{d} a_i x^i. \]
It is easy to see that the Newton polygon of \( L(t, f_{\vec{a}}, \mathbb{F}_q) \) is independent of \( a_0 \). So one can take \( a_0 \) to be any preferred number. We take \( a_0 = 1 \) so that Lemmas 5.2 and 5.3 are expressed in a simpler form.

In this section we define a polynomial \( H \) such that the Newton polygon of \( L(t, f_{\vec{a}}, \mathbb{F}_q) \) coincides with the generic Newton polygon of \([-e,d]\) if and only if \( H(\vec{a}) \neq 0 \).

**Definition 3.1.** Let \( k = 1, 2, \cdots, d + e - 1 \) be such that \( V_k = \{(m, n)\} \). We define \( S_k \) to be the set of permutations \( \tau \) of \( \{-m, -m + 1, \cdots, n\} \) such that
\[
\tau(i) = \begin{cases} 
\geq n - d \left\{ -\frac{e - n}{d} \right\}, & \text{if } i > 0, \\
= 0, & \text{if } i = 0, \\
\leq -m + e \left\{ \frac{d + m}{e} \right\}, & \text{if } i < 0.
\end{cases}
\]
Let 
\[ E(t) = \exp\left( \sum_{i=0}^{\infty} \frac{t^p}{p^i} \right). \]

It is a power series in \( \mathbb{Z}_p[[t]] \), and we call it the Artin-Hasse exponential series. We write 
\[ E(t) = \sum_{n=0}^{+\infty} \lambda_n t^n. \]

**Definition 3.2.** Let \( k = 1, 2, \ldots, d + e - 1 \) be such that \( V_k = \{(m, n)\} \). We write \( r_i = \begin{cases} n - d\left\lfloor \frac{p_i - n}{d} \right\rfloor + d, & 1 \leq i \leq n, \\ m - e\left\lfloor \frac{p_i + m}{e} \right\rfloor + e, & -m \leq i \leq -1. \end{cases} \)

We define a polynomial \( H_k \) in the variables \( x_{-e}, \ldots, x_d \), by 
\[ H_k(x) = \sum_{\tau \in S_k} u_\tau \prod_{i=-m}^{-1} x_{-r_i - \tau(i)} \prod_{i=1}^{n} x_{r_i - \tau(i)}, \]
where 
\[ u_\tau = \text{sgn}(\tau) \left( \prod_{i=1}^{n} \lambda_{\left\lfloor \frac{p_i - \tau(i)}{d} \right\rfloor} \lambda_{\left\lfloor \frac{p_i + \tau(i)}{e} \right\rfloor} \right) \prod_{i=-m}^{-1} \lambda_{\left\lfloor \frac{-p_i + \tau(i)}{e} \right\rfloor} \lambda_{\left\lfloor \frac{-p_i - \tau(i)}{d} \right\rfloor} \in \mathbb{Z}_p^\times. \]

**Definition 3.3.** The Hasse polynomial \( H \) of \([-e, d]\) is defined by 
\[ H = x_d x_{-e} \prod_{V_k=1} \hat{H}_k, \]
where \( \hat{H}_k \) is the reduction of \( H_k \) modulo \( p \).

We shall prove the following theorem.

**Theorem 3.4.** The Hasse polynomial \( H \) of \([-e, d]\) is non-zero.

Let \( k = 1, 2, \ldots, d + e - 1 \) be such that \( V_k = \{(m, n)\} \). It is easy to see that Theorem 3.4 follows from the following one.

**Theorem 3.5.** Among the monomials 
\[ \prod_{i=-m}^{-1} x_{-r_i - \tau(i)} \prod_{i=1}^{n} x_{r_i - \tau(i)}, \tau \in S_k, \]
there is a monomial which appears exactly once.

That theorem plays a crucial role in the determination of the generic Newton polygon of \([-e, d]\). In the case \( e = 0 \), Blache-Férand [BF] used Zhu’s maximal-monomial-locating technique to prove the theorem. In the case \( e > 0 \), the maximal-monomial-locating technique no longer works. Fortunately, a minimal-monomial-locating technique will play the role.

Set \( x_1 < x_2 < \cdots < x_d \) and \( x_{-1} < x_{-2} < \cdots < x_{-e} \). Define \( \prod_{i \in I} x_i > \prod_{j \in J} x_j \) and \( \prod_{i \in I} x_i > \prod_{j \in J} x_j \) if \( I \) and \( J \) are finite subsets of positive integers and there is an \( i \in I \) which is greater than all \( j \in J \). Define \( g_1 g_3 \geq g_2 g_4 \) if \( g_1, g_2, g_3, g_4 \) are monomials such that \( g_1 \geq g_2 \) and \( g_3 \geq g_4 \).

It is easy to see that Theorem 3.5 follows from the following one.
Theorem 3.6. Among the monomials
\[ \prod_{i=-m}^{-1} x_{-r_i - \tau(i)} \prod_{i=1}^{n} x_{r_i - \tau(i)}, \quad \tau \in S_k, \]
the minimal monomial appears exactly once.

Proof. Note that \( r_i \neq r_j \) and \( r_i \neq r_j \) if \( i \) and \( j \) are distinct positive integers. So we can order them such that
\[
\begin{align*}
    r_{i_1} > r_{i_2} > \cdots > r_{i_n}, \quad i_j > 0,
\end{align*}
\]
and
\[
\begin{align*}
    r_{t_1} > r_{t_2} > \cdots > r_{t_m}, \quad t_j < 0.
\end{align*}
\]
Note that \( r_{i_j} \leq n + d \) and \( r_{t_j} \leq m + e \). So we have
\[
\begin{align*}
    r_{i_j} &\leq n + d + 1 - j, \quad \text{and} \quad r_{t_j} \leq m + e + 1 - j.
\end{align*}
\]
Recall that \( \tau \in S_k \) if and only if \( \tau(i) \geq r_i - d \) if \( i > 0 \), and \( \tau(i) \leq -r_i + e \) if \( i < 0 \). Hence, if we define \( \tau_0 \) by
\[
\begin{align*}
    \tau_0(i_j) &= n + 1 - j, \quad \text{and} \quad \tau_0(t_j) = -(m + 1 - j),
\end{align*}
\]
then \( \tau_0 \in S_k \).

We claim that, for any \( \tau \in S_k \),
\[
\prod_{j=1}^{n} x_{r_{i_j} - \tau(i_j)} \geq \prod_{j=1}^{n} x_{r_{i_j} - (n+1-j)}
\]
with equality holding if and only if \( \tau(i_j) = n + 1 - j \) for all \( 1 \leq j \leq n \). Suppose that \( \tau(i_j) \neq n + 1 - j \) for some \( 1 \leq j \leq n \). Let \( j_0 \) be the least one with this property. Then
\[
\tau(i_{j_0}) < n + 1 - j_0.
\]
Hence
\[
\begin{align*}
    r_{i_{j_0}} - \tau(i_{j_0}) > r_{i_{j_0}} - (n + 1 - j_0) \geq r_{i_j} - (n + 1 - j), \quad \text{for all} \quad j \geq j_0,
\end{align*}
\]
Therefore
\[
\prod_{j=1}^{n} x_{r_{i_j} - \tau(i_j)} > \prod_{j=1}^{n} x_{r_{i_j} - (n+1-j)}
\]
as claimed.

Similarly, we can prove that, for any \( \tau \in S_k \),
\[
\prod_{j=1}^{m} x_{-r_{t_j} - \tau(t_j)} \geq \prod_{j=1}^{m} x_{-r_{t_j} + (m+1-j)}
\]
with equality holding if and only if \( \tau(t_j) = -(m + 1 - j) \) for all \( 1 \leq j \leq m \). It follows that the monomial
\[
\prod_{j=1}^{n} x_{r_{i_j} - (n+1-j)} \prod_{j=1}^{m} x_{-r_{t_j} + (m+1-j)}
\]
is minimal and occurs in the monomials
\[
\prod_{i=-m}^{-1} x_{-r_i - \tau(i)} \prod_{i=1}^{n} x_{r_i - \tau(i)}, \quad \tau \in S_k.
\]
The theorem is proved.

4. DWORK’S $p$-ADIC ANALYTIC METHOD

In this section we give a brief survey on Dwork’s $p$-adic analytic method. Proofs of theorems in this section may be omitted. Interested readers may consult [Dw62, Dw64] and [AS87, AS89] for detailed proofs.

Write $\mathbb{Z}_q := \mathbb{Z}_p[\mu_{q-1}]$ and $\mathbb{Q}_q := \mathbb{Q}_p(\mu_{q-1})$, where $\mu_n$ is the group of $n$-th roots of unity.

Recall that

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^p^i}{p^i}\right) = \sum_{n=0}^{+\infty} \lambda_n t^n \in \mathbb{Z}_p[[t]]$$

is the Artin-Hasse exponential series. Choose $\pi \in \mathbb{Q}_p(\mu_p)$ such that $E(\pi) = \psi(1)$. We have $\text{ord}_p(\pi) = \frac{1}{p-1}$ and $\sum_{i=0}^{\infty} \frac{\pi^p^i}{p^i} = 0$.

Let $L$ be the Banach space over $\mathbb{Q}_q[\pi^{1/D}]$ with formal basis $\pi^{\deg(i)}x^i$, $i \in \mathbb{Z}$. That is, $L = L_0 \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$ with $L_0 = \{ \sum_{i \in \mathbb{Z}} c_i \pi^{\deg(i)}x^i : c_i \in \mathbb{Z}_q[\pi^{1/D}] \}$.

The space is closed under multiplication. So it is an algebra.

For $\vec{a} = (a_{-e}, \cdots, a_d) \in \mathbb{F}_q^{d+e+1}$, we write

$$E_{\vec{a}}(x) := \prod_{i = -e}^{d} E(\pi a_i x^i),$$

where $\hat{a}_i$ is the Teichmüler lifting of $a_i$. As each $E(\pi a_i x^i)$ lies in $L$, so does $E_{\vec{a}}$.

The Galois group $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ acts on $L$ but keeps $\pi^{1/D}$ and $x$ fixed. Let $\sigma$ be the Frobenius element of that Galois group. Write

$$\hat{E}_{\vec{a}}(x) = \prod_{j=0}^{+\infty} E^{\sigma j}_{\vec{a}}(x^{p^j}).$$

Define an operator $\partial : L \to L$ by

$$\partial(g) = xg'(x) + g(x)x \frac{d \log \hat{E}_{\vec{a}}(x)}{dx}.$$ 

It is easy to see that $L_0$ is stable under $\partial$.

Define an operator $\psi_p : L \to L$ by

$$\psi_p \left( \sum_{i \in \mathbb{Z}} c_i x^i \right) = \sum_{i \in \mathbb{Z}} c_{pi} x^i.$$ 

And write

$$\Psi_p := \sigma^{-1} \circ \psi_p \circ E_{\vec{a}}.$$ 

That is,

$$\Psi_p (g) = \sigma^{-1}(\psi_p (gE_{\vec{a}})).$$

Note that $\Psi_p$ is $\mathbb{Q}_p[\pi^{1/D}]$-linear, but $\mathbb{Q}_q[\pi^{1/D}]$-semi-linear.
Define $\Psi_p^n = \Psi_p^q$. So $\Psi_q^n = \Psi_q^q$. It is easy to check that $\Psi_q$ is $\mathbb{Z}_q[\pi^{1/D}]$-linear. Moreover, we have

$$q\partial \Psi_q = \Psi_q \partial.$$

Let $\tilde{\Psi}_p$ be the induced operator of $\Psi_p$ on $L/(\partial L)$. We have the following three theorems.

**Theorem 4.1.** We have

$$L(s, f, \mathbb{F}_q) = \det(1 - s\tilde{\Psi}_q | L/(\partial L) \text{ over } \mathbb{Q}_q(\pi^{1/D})).$$

**Theorem 4.2.** The $q$-adic Newton polygons of $\det(1 - s^b \tilde{\Psi}_q | L/(\partial L) \text{ over } \mathbb{Q}_q(\pi^{1/D}))$ and $\det(1 - s \tilde{\Psi}_p | L/(\partial L) \text{ over } \mathbb{Q}_p(\pi^{1/D}))$ coincide.

**Theorem 4.3.** Over $\mathbb{Z}_q[\pi^{1/D}]$, the lattice $L_0/(\partial L_0)$ has a basis represented by

$$\pi^{\deg(i)} x^i, -e \leq i \leq d - 1.$$

5. Elementary estimates

In this section we give some elementary estimates on the matrix coefficients of the operator $\tilde{\Psi}_p$ on $L/(\partial L)$.

Write

$$E_{\tilde{a}}(x) = \sum_{i \in \mathbb{Z}} \gamma_i x^i.$$

We have

$$\gamma_i = \sum_{\sum_{j=-e}^d n_j = i} \prod_{j=-e}^d \lambda_j \tilde{a}_j^{n_j}.$$

**Definition 5.1.** We write $\alpha = O(\pi^t)$ to mean that $\text{ord}_\pi(\alpha) \geq t$, where $\text{ord}_\pi(\cdot) = \frac{1}{\text{ord}_p(\pi) \text{ord}_p(\cdot)}$.

**Lemma 5.2.** If $i \geq 0$,

$$\gamma_i = \pi^\left\lfloor \frac{i}{d} \right\rfloor \lambda_{\left\lfloor \frac{i}{d} \right\rfloor} \lambda_{\left\{ \frac{i}{d} \right\}} \hat{a}_d^{\left\lfloor \frac{i}{d} \right\rfloor} \hat{a}_{\left\{ \frac{i}{d} \right\}} + O(\pi^\left\lceil \frac{i}{d} \right\rceil + 1).$$

**Proof.** If $\sum_{j=-e}^d j n_j = i$ ($n_j \geq 0$), then $\sum_{j=-e}^d n_j \geq \left\lfloor \frac{i}{d} \right\rfloor$ with equality holding if and only if

$$n_j = \begin{cases} \left\lceil \frac{i}{d} \right\rceil, & j = d \\ \left\{ \frac{i}{d} \right\}, & j = d \left\{ \frac{i}{d} \right\} \\ 0, & \text{otherwise}. \end{cases}$$

The lemma now follows.

Similarly, we have the following lemma.

**Lemma 5.3.** If $i < 0$,

$$\gamma_i = \pi^\left\lceil \frac{-i}{d} \right\rceil \lambda_{\left\lceil \frac{-i}{d} \right\rceil} \lambda_{\left\{ \frac{-i}{d} \right\}} \hat{a}_d^{\left\lceil \frac{-i}{d} \right\rceil} \hat{a}_{\left\{ \frac{-i}{d} \right\}} + O(\pi^\left\lfloor \frac{-i}{d} \right\rfloor + 1).$$

From the last two lemmas we infer the following corollary.
Corollary 5.4. We have
\[
\gamma_i = O(\pi^{\deg(i)}).
\]

Let \( F = (F_{ij})_{-e \leq i,j \leq d-1} \) be the matrix defined by
\[
\psi_p \circ E_{\mathbf{a}}(x^j) \equiv \sum_{i=-e}^{d-1} F_{ij} x^i \pmod{\partial L}.
\]

Lemma 5.5. Let \( p \geq 3D \), and \(-e \leq i,j \leq d-1\). We have
\[
F_{ij} = \gamma_{pi-j} + \begin{cases} 
O(\pi^{\deg(p_i)+2}), & i \neq -e \\
O(\pi^p), & i = -e.
\end{cases}
\]

Proof. We have
\[
\psi_p \circ E_{\mathbf{a}}(x^j) = \sum_{i_0 \in \mathbb{Z}} \gamma_{pi_0-j} x^{i_0} = \sum_{i_0=-e}^{d-1} \gamma_{pi_0-j} x^{i_0} + \sum_{i_0 \not\in \{-e, \cdots, d-1\}} \gamma_{pi_0-j} x^{i_0}.
\]

For \( i_0 \not\in \{-e, \cdots, d-1\} \), we write
\[
\pi^{\deg(i_0)} x^{i_0} = \sum_{i=-e}^{d-1} c_{i_0} \pi^{\deg(i)} x^i \pmod{\partial L}, \quad c_{i_0} \in \mathbb{Z}_q[\pi^{1/(de)}].
\]

Then
\[
\psi_p \circ E_{\mathbf{a}}(x^j) = \sum_{i=-e}^{d-1} x^i (\gamma_{pi-j} + \sum_{i_0 \not\in \{-e, \cdots, d-1\}} c_{i_0} \pi^{\deg(i)-\deg(i_0)} \gamma_{pi_0-j}) \pmod{\partial L}.
\]

It follows that
\[
F_{ij} = \gamma_{pi-j} + \sum_{i_0 \not\in \{-e, \cdots, d-1\}} c_{i_0} \pi^{\deg(i)-\deg(i_0)} \gamma_{pi_0-j}.
\]

If \( i_0 \not\in \{-e, -(e-1), \cdots, d-1\} \), and \( i \neq -e \), we have
\[
\deg(i) - \deg(i_0) + \ord_{\pi}(\gamma_{pi_0-j}) \geq \deg(i) - \deg(i_0) + \deg(p_i) - 1
\]
\[
\geq \deg(p_i) + (p-1)(\deg(i_0) - \deg(i)) - 1
\]
\[
\geq \lceil \deg(p_i) \rceil + \frac{p-1}{D} - 1 \geq \lceil \deg(p_i) \rceil + 2.
\]

If \( i_0 \not\in \{-e, -(e-1), \cdots, d-1, d\} \), and \( i = -e \), we also have
\[
\deg(i) - \deg(i_0) + \ord_{\pi}(\gamma_{pi_0-j}) \geq \lceil \deg(p_i) \rceil + 2.
\]

If \( i_0 = d \), and \( i = -e \), we have
\[
\deg(i) - \deg(i_0) + \ord_{\pi}(\gamma_{pi_0-j}) \geq p.
\]

Therefore
\[
F_{ij} = \gamma_{pi-j} + \begin{cases} 
O(\pi^{\deg(p_i)+2}), & i \neq -e \\
O(\pi^p), & i = -e.
\end{cases}
\]

The lemma is proved.
6. Generic polygon

In this section we prove Theorem 1.9. It follows immediately from the following theorem.

**Theorem 6.1.** Let \( p \geq 3D \). Then the \( q \)-adic Newton polygon of \( L(t, f_{\vec{a}}, \mathbb{F}_q) \) coincides with the arithmetic polygon of \([-e, d]\) if and only if \( H(\vec{a}) \neq 0 \).

Write
\[
\det(1 - s\bar{\Psi}_p \mid L/(\partial L) \text{ over } \mathbb{Q}_p(\pi^{1/D})) = \sum_{i=0}^{b(d+e)} (-1)^i c_i s^i.
\]

By Theorems 4.2 and 2.1, Theorem 6.1 follows from the following two theorems.

**Theorem 6.2.** Let \( p > 3D \). Let \( k = 1, 2, \cdots, d + e - 1 \) be such that \( V_k \) contains two pairs. Then
\[
\ord_q(c_{bk}) \geq p([-e, d]) (k).
\]

**Theorem 6.3.** Let \( p > 3D \). Let \( k = 1, 2, \cdots, d + e - 1 \) be such that \( V_k \) contains exactly one pair. Then
\[
\ord_q(c_{bk}) \geq p([-e, d]) (k)
\]
with equality holding if and only if \( \bar{H}_k(\vec{a}) \neq 0 \).

From now on, we suppose that \( q = p^b \), and let \( \zeta \) be a primitive \((q-1)\)-th roots of unity.

**Definition 6.4.** We define the matrix \( G = (G(i,u),(j,w))_{-e \leq i,j < d-1, 0 \leq u,w \leq b-1} \) by
\[
(\zeta^w)^{\sigma-1} F_{ij}^{\sigma-1} = \sum_{u=0}^{b-1} G(i,u),(j,w) \zeta^u.
\]

**Lemma 6.5.** We have
\[
\Psi_p(\zeta^w x^j) \equiv \sum_{i=-e}^{d-1} \sum_{u=0}^{b-1} G(i,u),(j,w) \zeta^u x^i \pmod{\partial L}.
\]

That is, \( G \) is the matrix of \( \bar{\Psi}_p \) with respect to the basis over \( \mathbb{Q}_p(\pi^{1/D}) \) represented by \( \zeta^u x^i, \ -e \leq i \leq d-1, 0 \leq u \leq b-1 \).

**Proof.** Recall that
\[
\psi_p \circ E_{\vec{a}}(x^j) \equiv \sum_{i=-e}^{d-1} F_{ij} x^i \pmod{\partial L}.
\]

So
\[
\Psi_p(\zeta^w x^j) \equiv (\zeta^w)^{\sigma-1} \sum_{i=-e}^{d-1} F_{ij}^{\sigma-1} x^i \pmod{\partial L}.
\]

By definition,
\[
(\zeta^w)^{\sigma-1} F_{ij}^{\sigma-1} = \sum_{u=0}^{b-1} G(i,u),(j,w) \zeta^u.
\]

The lemma now follows.
Corollary 6.6. We have
\[ \det(1 - s\tilde{\Psi}_p \mid L/(\partial L) \text{ over } \mathbb{Q}_p(\pi^{1/D})) = \det(1 - sG). \]

In particular,
\[ c_{bk} = \sum_T \det((G_{i,u},(j,w))(i,u),(j,w) \in T), \]
where \( T \) runs over subsets of
\[ \{(i, u) \mid -e \leq i \leq d - 1, 0 \leq u \leq b - 1\} \]
with cardinality \( bk \).

Lemma 6.7. Let \( T_1 \) and \( T_2 \) be two finite sets with equal cardinality. Let \( g_1 \) and \( g_2 \) be real-valued functions on \( T_1 \) and \( T_2 \) respectively. Suppose that \( g_1 \) and \( g_2 \) agree on \( T_1 \cap T_2 \), and that \( g_2(t_2) \geq g_1(t_1) \) for \( t_2 \in T_2 \setminus T_1 \) and \( t_1 \in T_1 \setminus T_2 \). Then
\[ \sum_{i \in T_2} g_2(t) \geq \sum_{i \in T_1} g_1(t). \]

Moreover, if \( g_2(t_2) > g_1(t_1) \) for \( t_2 \in T_2 \setminus T_1 \) and \( t_1 \in T_1 \setminus T_2 \), then the equality holds if and only if \( T_1 = T_2 \).

Proof. Obvious.

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. It suffices to show that, for any subset \( T \) of
\[ \{(i, u) \mid -e \leq i \leq d - 1, 0 \leq u \leq b - 1\} \]
with cardinality \( bk \), and any permutation \( \tau \) of \( T \), we have
\[ \text{ord}_\pi\left( \prod_{(i,u) \in T} G_{i,u,\tau(i,u)} \right) \geq b(p-1)p_{[-e,d]}(k). \]

Let \( V_k = \{(m-1,n+1),(m,n)\} \). Then \( \frac{n+1}{d} = \frac{m}{e} \). Moreover, the cardinality of the set \( \{1 \leq i \leq m-1 \mid pi \equiv m \pmod{e}\} \) is equal to that of \( \{1 \leq i \leq n \mid pi \equiv n + 1 \pmod{d}\} \). Without loss of generality, we assume that both of them are of cardinality 1. Then
\[ (p-1)p_{[-e,d]}(k) = \sum_{i=1}^{n} \left\lfloor \frac{pi - n}{d} \right\rfloor + \sum_{i=1}^{m-1} \left\lfloor \frac{pi - m + 1}{e} \right\rfloor + \left\lfloor \frac{(p-1)m}{e} \right\rfloor - 1. \]

Note that
\[ \text{ord}_\pi(G_{i,u,\tau(i,u)}) = \text{ord}_\pi(F_{i,\tau(i)}). \]
So, if \( i > 0 \), then
\[ \text{ord}_\pi(G_{i,u,\tau(i,u)}) \geq \begin{cases} \left\lfloor \frac{pi-n}{d} \right\rfloor, & \tau(i) \leq n, \\ \left\lfloor \frac{pi-n}{d} \right\rfloor - 1, & \tau(i) > n, \end{cases} \]
\[ \begin{cases} \left\lfloor \frac{pi-m+1}{e} \right\rfloor, & i > n + 1. \end{cases} \]

Similarly, if \( i < 0 \), then
\[ \text{ord}_\pi(G_{i,u,\tau(i,u)}) \geq \begin{cases} \left\lfloor \frac{-pi+1}{d} \right\rfloor, & \tau(i) \geq -m + 1, \\ \left\lfloor \frac{-pi-m+1}{e} \right\rfloor - 1, & \tau(i) \leq -m, \\ \frac{pm}{e} + 1, & i < -m. \end{cases} \]
Therefore
\[
\text{ord}_\pi \left( \prod_{(i,u) \in T} G_{(i,u), \tau((i,u))} \right) \geq \sum_{(i,u) \in T: 1 \leq i \leq n} \left\lfloor \frac{pi - n}{d} \right\rfloor + \sum_{(-i,u) \in T: 1 \leq i \leq m-1} \left\lfloor \frac{pi - m + 1}{e} \right\rfloor
+ \left\lfloor \frac{(p-1)m}{e} \right\rfloor \sum_{(i,u) \in T: i=n+1 \text{ or } i=-m} 1 + \sum_{(i,u) \in T: i>n+1 \text{ or } i<-m} \left\lfloor \frac{pm}{e} \right\rfloor
+ \left( \left\lfloor \frac{(p-1)m}{e} \right\rfloor - 1 \right) \sum_{(i,u) \in T: i=n+1 \text{ or } i=-m} 1 + \sum_{(i,u) \in T: i>n+1 \text{ or } i<-m} \left\lfloor \frac{pm}{e} \right\rfloor.
\]

By Lemma 6.7, we have
\[
\text{ord}_\pi \left( \prod_{(i,u) \in T} G_{(i,u), \tau((i,u))} \right) \geq b \left( \sum_{i=1}^{n} \left\lfloor \frac{pi - n}{d} \right\rfloor + \sum_{i=1}^{m-1} \left\lfloor \frac{pi - m + 1}{e} \right\rfloor + \left\lfloor \frac{(p-1)m}{e} \right\rfloor - 1 \right).
\]

The proof is completed.

It remains to prove Theorem 3.3

**Lemma 6.8.** Let \( p > 3D \). Let \( k = 1, 2, \ldots, d + e - 1 \) be such that \( V_k = \{(m, n)\} \). Then
\[
c_{bk} = \det((G_{(i,u), (j,w)})_{-m \leq i, j, 0 \leq u, w \leq b-1}) + O(\pi^{\min(1)\cdot\pi^{e\cdot\pi^{d}(k)+\frac{1}{2}}}).
\]

**Proof.** It suffices to show that, for any subset \( T \) of
\[
\{(i, u) \mid -e \leq i \leq d - 1, 0 \leq u \leq b - 1\}
\]
with cardinality \( bk \) which is different from \( \{-m, \ldots, n\} \times \{0, \ldots, b-1\} \), and any permutation \( \tau \) of \( T \), we have
\[
\text{ord}_\pi \left( \prod_{(i,u) \in T} G_{(i,u), \tau((i,u))} \right) > b(p - 1)\pi^{(p|e\cdot\pi^{d}(k)}(k).
\]

First we suppose that \( I_k = \{(m, n)\} \). Note that, if \( i > 0 \), then
\[
\text{ord}_\pi(G_{(i,u), \tau((i,u))}) \geq \begin{cases} 
\left\lfloor \frac{pi - n}{d} \right\rfloor, & \tau(i) \leq n, \\
\left\lfloor \frac{pi - n}{d} \right\rfloor - 1, & \tau(i) = n, \\
\left\lfloor \frac{pm}{e} \right\rfloor + \frac{e}{d} - 2, & i > n.
\end{cases}
\]

Similarly, if \( i < 0 \), then
\[
\text{ord}_\pi(G_{(i,u), \tau((i,u))}) \geq \begin{cases} 
\left\lfloor \frac{-pi - m}{e} \right\rfloor, & \tau(i) \geq -m, \\
\left\lfloor \frac{-pi - m}{e} \right\rfloor - 1, & \tau(i) = -m, \\
\left\lfloor \frac{pm}{e} \right\rfloor + \frac{e}{d} - 2, & i < -m.
\end{cases}
\]

So
\[
\text{ord}_\pi \left( \prod_{(i,u) \in T} G_{(i,u), \tau((i,u))} \right) \geq \sum_{(i,u) \in T: 1 \leq i \leq n} \left\lfloor \frac{pi - n}{d} \right\rfloor + \sum_{(-i,u) \in T: 1 \leq i \leq m} \left\lfloor \frac{pi - m}{e} \right\rfloor
\]
\[
\begin{align*}
&\sum_{(i,u)\in T: i>n} \left\lfloor \frac{pm}{d} \right\rfloor + \sum_{(i,u)\in T: i>m} \left\lfloor \frac{pm}{e} \right\rfloor \\
&+ \sum_{(i,u)\in T: i>n \text{ or } i<-m} \left( \frac{p}{D} - 2 \right) - \sum_{(i,u)\in T: \tau(i)>n \text{ or } \tau(i)<-m} 1 \\
&> \sum_{(i,u)\in T: 1\leq i\leq n} \left\lfloor \frac{pi - n}{d} \right\rfloor + \sum_{(i,u)\in T: 1\leq i\leq m} \left\lfloor \frac{pi - m}{e} \right\rfloor \\
&+ \sum_{(i,u)\in T: i>n} \left\lfloor \frac{pm}{d} \right\rfloor + \sum_{(i,u)\in T: i>m} \left\lfloor \frac{pm}{e} \right\rfloor
\end{align*}
\]

By Lemma 6.7, we have

\[
\text{ord}_n \left( \prod_{(i,u)\in T} G_{(i,u),\tau(i,u)} \right) > b(p - 1)p_{[-e,d]}(k).
\]

Secondly, we suppose that \( I_k \) contains two pairs. Without loss of generality, we may assume that \( I_k = \{(m,n), (m+1,n-1)\} \). Then \( \frac{m+1}{e} = \frac{n}{d} \),

\[
pi \equiv m + 1 (\text{ mod } e), \quad 1 \leq i \leq m,
\]

and there is exactly one \( 1 \leq i \leq n - 1 \) such that

\[
pi \equiv n (\text{ mod } d).
\]

So

\[
(p - 1)p_{[-e,d]}(k) = \sum_{i=1}^{n-1} \left\lfloor \frac{pi - n + 1}{d} \right\rfloor + \sum_{i=1}^{m} \left\lfloor \frac{pi - m - 1}{e} \right\rfloor + \left\lfloor \frac{(p - 1)n}{d} \right\rfloor - 1.
\]

Note that, if \( i > 0 \), then

\[
\text{ord}_n(G_{(i,u),\tau(i,u)}) \geq \left\{ \begin{array}{ll}
\left\lfloor \frac{pi-n+1}{d} \right\rfloor, & \tau(i) \leq n - 1, \\
\left\lfloor \frac{pi-n+1}{d} \right\rfloor - 1, & \tau(i) \geq n, \\
\left\lfloor \frac{pm}{d} \right\rfloor + \frac{p}{d} - 2, & i > n.
\end{array} \right.
\]

Similarly, if \( i < 0 \), then

\[
\text{ord}_n(G_{(i,u),\tau(i,u)}) \geq \left\{ \begin{array}{ll}
\left\lfloor \frac{-pi-m-1}{e} \right\rfloor, & \tau(i) \geq -m - 1, \\
\left\lfloor \frac{-pi-m-1}{e} \right\rfloor - 1, & \tau(i) \leq -m - 1, \\
\left\lfloor \frac{pm}{e} \right\rfloor + \frac{p}{e} - 2, & i < -m - 1.
\end{array} \right.
\]

So

\[
\text{ord}_n \left( \prod_{(i,u)\in T} G_{(i,u),\tau(i,u)} \right) \geq \sum_{(i,u)\in T: 1\leq i\leq n} \left\lfloor \frac{pi - n + 1}{d} \right\rfloor + \sum_{(i,u)\in T: 1\leq i\leq m} \left\lfloor \frac{pi - m - 1}{e} \right\rfloor \\
+ \left\lfloor \frac{(p - 1)n}{d} \right\rfloor + \sum_{(i,u)\in T: i=n \text{ or } i=-m-1} 1 + \sum_{(i,u)\in T: \tau(i)>n \text{ or } \tau(i)<-m-1} \left\lfloor \frac{pm}{d} \right\rfloor \\
+ \sum_{(i,u)\in T: i>n \text{ or } i<-m-1} \left( \frac{p}{D} - 2 \right) - \sum_{(i,u)\in T: \tau(i)>n \text{ or } \tau(i)<-m-1} 1.
\]
If \( \{(i, u) \in T : i > n \text{ or } i < -m - 1\} \neq \emptyset \), then

\[
\text{ord}_\pi \left( \prod_{(i, u) \in T} G_{(i,u), \tau(i,u)} \right) > \sum_{(i, u) \in T : 1 \leq i < n} \left\lfloor \frac{pi - n + 1}{d} \right\rfloor + \sum_{(-i, u) \in T : 1 \leq i \leq m} \left\lfloor \frac{pi - m - 1}{e} \right\rfloor \\
+ \left\lfloor \frac{(p - 1)n}{d} \right\rfloor - 1 \sum_{(i, u) \in T : i = n} 1 + \left\lfloor \frac{(p - 1)m}{d} \right\rfloor - 1.
\]

By Lemma 6.7, we have

\[
\text{ord}_\pi \left( \prod_{(i, u) \in T} G_{(i,u), \tau(i,u)} \right) \geq \sum_{(i, u) \in T : 1 \leq i < n} \left\lfloor \frac{pi - n + 1}{d} \right\rfloor + \sum_{(-i, u) \in T : 1 \leq i \leq m} \left\lfloor \frac{pi - m - 1}{e} \right\rfloor \\
+ \left\lfloor \frac{(p - 1)n}{d} \right\rfloor - 1 \sum_{(i, u) \in T : i = n} 1 + \left\lfloor \frac{(p - 1)m}{d} \right\rfloor - 1.
\]

By Lemma 6.7, we also have

\[
\text{ord}_\pi \left( \prod_{(i, u) \in T} G_{(i,u), \tau(i,u)} \right) > \sum_{i=1}^{n-1} \left\lfloor \frac{pi - n + 1}{d} \right\rfloor + \sum_{i=1}^{m} \left\lfloor \frac{pi - m - 1}{e} \right\rfloor + \left\lfloor \frac{(p - 1)n}{d} \right\rfloor - 1.
\]

The proof is completed.

**Definition 6.9.** We write \( \alpha \sim \beta \) to mean that \( \alpha = u\beta \) for some \( p \)-adic unit \( u \).

**Theorem 6.10.** Let \( p > 3D \). Let \( k = 1, 2, \ldots, d + e - 1 \) be such that \( V_k = \{(m, n)\} \). Then

\[
c_{bb} \sim \det((F_{ij})_{-m \leq i,j \leq n})^b + O(b^{(p-1)[p-\varepsilon,d(k)+\frac{1}{2}]})
\]

**Proof.** It suffices to show that

\[
\det((G_{(i,u),(j,w)})_{-m \leq i,j \leq n,0 \leq u,w \leq b-1}) \sim \det((F_{ij})_{-m \leq i,j \leq n})^b.
\]

Let \( V = \oplus_{i=-m}^{n-1} Q_q(\pi_1/D)e_i \) be a \( k \)-dimensional vector space over \( Q_q(\pi_1/D) \) with standard basis \( e_{-m}, \ldots, e_n \). Let \( F = (F_{ij})_{-m \leq i,j \leq n} \) act on it in the standard way, and let \( \sigma \) act on it coordinate-wise. Then

\[
\sigma^{-1} \circ F(\zeta^we_i) = (\zeta^w)^{\sigma^{-1}} \sum_{i=-m}^{n} F_i^{-1}e_i.
\]

Therefore, \( G \) is the matrix of \( \sigma^{-1} \circ F \) on \( V \) with respect to the basis over \( Q_p(\pi_1/D) \):

\[
\zeta^we_i, \quad -m \leq i \leq n, 0 \leq u \leq b - 1.
\]

As \( \sigma \) is just a re-ordering of the basis, we have

\[
\det((G_{(i,u),(j,w)})_{-m \leq i,j \leq n,0 \leq u,w \leq b-1}) \sim \det((F_{ij})_{-m \leq i,j \leq n})^b.
\]

The theorem is proved.
Lemma 6.11. Let \( p > 3D \). Let \( k = 1, 2, \ldots, d + e - 1 \) be such that \( V_k = \{(m, n)\} \). Then
\[
\det((F_{ij})_{-m \leq i,j \leq n}) = \sum_{\tau \in S_k} \text{sgn}(\tau) \prod_{i=-m}^{n} F_{i,\tau(i)} + O(\pi^{(p-1)p^{(e,d)(k+1/D)}}).
\]

Proof. For \( j \leq n \), we have
\[
\left\lfloor \frac{p_i - j}{d} \right\rfloor = \left\lfloor \frac{p_i - n + (n-j)}{d} \right\rfloor \geq \left\lfloor \frac{p_i - n}{d} \right\rfloor
\]
with equality holding if and only if
\[
j \geq n - d\left\{ \frac{p_i - n}{d} \right\}.
\]
Similarly, for \( j \geq -m \), we have
\[
\left\lfloor \frac{-p_i + j}{e} \right\rfloor = \left\lfloor \frac{-p_i - m + (m+j)}{e} \right\rfloor \geq \left\lfloor \frac{-p_i - m}{e} \right\rfloor
\]
with equality holding if and only if
\[
j \leq -m + e\left\{ \frac{p_i + m}{e} \right\}.
\]
So, if \( \tau \notin S_k \) is a permutation of \( \{-m, -(m-1), \ldots, n\} \), then
\[
\text{ord}_\pi\left( \prod_{i=-m}^{n} F_{i,\tau(i)} \right) \geq \sum_{i=-m}^{n} \left\lfloor \text{deg}(p_i - \tau(i)) \right\rfloor
\]
\[
\geq 1 + \sum_{i=1}^{n} \left\lfloor \frac{p_i - n}{d} \right\rfloor + \sum_{i=1}^{m} \left\lfloor \frac{p_i - m}{e} \right\rfloor.
\]
Hence
\[
\det((F_{ij})_{-m \leq i,j \leq n}) = \sum_{\tau \in S_k} \text{sgn}(\tau) \prod_{i=-m}^{n} F_{i,\tau(i)} + O(\pi^{(p-1)p^{(e,d)(k+1/D)}}).
\]
The lemma is proved.

We are now ready to prove Theorem 6.3. By the above lemmas, it suffices to prove the following theorem.

Theorem 6.12. Let \( p > 3D \). Let \( k = 1, 2, \ldots, d + e - 1 \) be such that \( V_k = \{(m, n)\} \). Then
\[
\det((F_{ij})_{-m \leq i,j \leq n}) = \pi^{(p-1)p^{(e,d)(k)}} \hat{a}_d^u \hat{a}_e^v H_k(\tilde{a}) + O(\pi^{(p-1)p^{(e,d)(k+1/D)}}),
\]
where \( \tilde{a} = (\hat{a}_e, \ldots, \hat{a}_d) \), and \( u, v \) are integers depending on \( k \).

Proof. By Lemmas 6.11 and 5.3, we have
\[
\det((F_{ij})_{-m \leq i,j \leq n}) = \sum_{\tau \in S_k} \text{sgn}(\tau) \prod_{i=-m}^{n} \gamma_{p_i,\tau(i)} + O(\pi^{(p-1)p^{(e,d)(k+1/D)}}).
\]
By Lemmas 5.2 and 5.3, we have
\[
\gamma_{p_i,\tau(i)} = \begin{cases} 
\pi^{\left\lfloor \frac{p_i - n}{d} \right\rfloor} \lambda_{\left\lfloor \frac{p_i - n}{d} \right\rfloor} \hat{a}_d^{\left\lfloor \frac{p_i - n}{d} \right\rfloor - 1} a_{r_i,\tau(i)} + O(\pi^{\left\lfloor \frac{p_i - n}{d} \right\rfloor}), & i > 0 \\
\pi^{\left\lfloor \frac{-p_i - m}{e} \right\rfloor} \lambda_{\left\lfloor \frac{-p_i - m}{e} \right\rfloor} \hat{a}_e^{\left\lfloor \frac{-p_i - m}{e} \right\rfloor - 1} a_{-e,\tau(i)} + O(\pi^{\left\lfloor \frac{-p_i - m}{e} \right\rfloor + 1}), & i < 0.
\end{cases}
\]
The theorem now follows.

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