S–POLYREGULAR BARGMANN SPACES

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ABSTRACT. We introduce two classes of right quaternionic Hilbert spaces in the context of slice polyregular functions, generalizing the so-called slice and full hyperholomorphic Bargmann spaces. Their basic properties are discussed and the explicit formulae of their reproducing kernels are given and associated Segal-Bargmann transforms are also introduced and studied. The spectral description as special subspaces of $L^2$-eigenspaces of a second order differential operator involving the slice derivative is also investigated.

1 Introduction

The classical Bargmann functional space $F^2$ is defined as the phase space on the complex plane consisting of all $e^{-|z|^2}dxdy$–square integrable entire functions. It is known to be unitarily isomorphic to the quantum mechanical configuration space $L^2(\mathbb{R}; dx)$ by means of the classical Segal-Bargmann transform (see for examples [7, 15, 25]). As special generalization in the context of polyanalytic functions are the generalized Bargmann spaces $F^2_n$ of level $n = 0, 1, 2, \ldots$, (see for example [24, 2, 3]), so that $F^2_0 = F^2$. The corresponding theory has found remarkable applications in time-frequency analysis, analysis of the higher Landau levels and in the multiplexing of signals, see for instance [3] and the references therein.

A quaternionic analogue of $F^2$ was introduced in [5] for slice regular functions $\mathcal{SR}$ on the quaternion algebra $\mathbb{H}$, i.e., the space of $\mathbb{H}$–valued real differentiable functions $f$ on $\mathbb{H} \equiv \mathbb{R}^4$ such that

$$\overline{\partial}_I f(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f|_{I}(x + yI)$$

vanishes identically on $\mathbb{H}$ for every $I \in \mathbb{S} = \{ q \in \mathbb{H}; q^2 = -1 \}$. More precisely, JMP

$$F^2_{\text{slice}} = \left\{ f(q) = \sum_{j=0}^{+\infty} q^j c_j \mathbb{H}, \quad \sum_{j=0}^{+\infty} j! |c_j|^2 < +\infty \right\} \subset \mathcal{SR} \cap L^2(\mathbb{C}_I; e^{-|q|^2} d\lambda_I),$$

(1.2)

where $\mathbb{C}_I := \mathbb{R} + \mathbb{R}I$ is a slice in $\mathbb{H}$ and $I \in \mathbb{S} = \{ q \in \mathbb{H}; q^2 = -1 \}$. It is shown there that $F^2_{\text{slice}}$ is independent of $I$ and is a reproducing kernel quaternionic Hilbert space. The related quaternionic Segal-Bargmann transform is considered in [13]. It connects $F^2_{\text{slice}}$ to the $L^2$–Hilbert space of quaternionic–valued functions on the real line.

Motivated by the works [6, 24, 16, 2, 3] studying and characterizing the polyanalyticity in the complex setting as well as by Brackx’ works [9, 10] studying the $k$–monogenic functions with respect to the Füter operator, we devote the present paper to possible generalizations of $F^2_{\text{slice}}$ in the context of the slice polyregular ($\mathcal{SR}_n$)

The research work of A.G. was partially supported by a grant from the Simons Foundation.
functions with respect to the slice derivative. They are natural analogs of the classical polyanalytic functions (see Definition 2.1) and appear as special subspaces of the Hilbert space

$$\mathcal{SR}_n^2 := \mathcal{SR}_n \cap L^2(\mathbb{C}_I, e^{-|q|^2} d\lambda_I),$$

the space of all S–polyregular functions $F : \mathbb{H} \rightarrow \mathbb{H}$ subject to the norm boundedness $\|f\|_{\mathcal{C}_I} < +\infty$. The norm here is the one associated to the inner product

$$\langle f, g \rangle_{\mathcal{C}_I} = \int_{\mathbb{C}_I} f(q) \overline{g(q)} e^{-|q|^2} d\lambda_I(q).$$

These spaces will be called here S–polyregular Bargmann space of level $n$ of first and second kind and we will denote them by $\mathcal{SR}_{1,n}^2$ and $\mathcal{SR}_{2,n}^2$, respectively.

Both $\mathcal{SR}_{1,n}^2$ and $\mathcal{SR}_{2,n}^2$ are associated to some system of partial differential equations involving the operators

$$\nabla_i^{j,k} = \bar{\partial}_i^{j,l} \partial_i^{l,k},$$

where $\bar{\partial}_i$ is the complex conjugate of $\partial_i$ in (1.1) and $\partial_i^{j,l}$ is the usual derivative with respect to the $I$–variable, $I \in \mathbb{S}$ (see Definitions 3.2 and 3.3). It should be noted here that $\mathcal{SR}_{1,0}^2 = \mathcal{SR}_{2,0}^2$ reduces further to $\mathcal{F}^2_{\text{slice}}$ in (1.2) (see Section 3).

The concrete description of these spaces is somehow the quaternionic Hermite polynomials

$$H_{m,n}^Q(q, \overline{q}) = m!n! \sum_{j=0}^{\min(m,n)} \frac{(-1)^j}{j!} \frac{q^{m-j} \overline{q}^{n-j}}{(m-j)!(n-j)!}.$$

We refer to [14] for an accurate systematic study of them.

Our main aim is to give a concrete description of these spaces. We prove that $\mathcal{SR}_{1,n}^2$ and $\mathcal{SR}_{2,n}^2$ are reproducing kernel quaternionic Hilbert spaces whose reproducing kernels are given explicitly in terms of the Laguerre polynomials (see Theorem 3.6). The proof is based essentially on a weak version of the Identity Principal for S–polyregular functions that we establish in Subsection 2.2 (Proposition 2.7) and on a natural extension of the left star product for S–polyregular functions. Moreover, a hilbertian decomposition of $L^2(\mathbb{C}_I, e^{-|q|^2} d\lambda_I)$ in terms of $\mathcal{SR}_{2,n}^2$ is also given (Theorem 3.9).

Associated Segal–Bargmann transforms $B_{\ell,n}$; $\ell = 1, 2$, are then introduced and studied in some details (see Theorems 4.1 and 4.4). They are defined on $L^2_{\mathbb{Q}}(\mathbb{R}; dt)$, the $L^2$-Hilbert space of left-sided quaternionic-valued functions on the real line and their kernel functions involve the Hermite polynomials extended to the quaternion.

It should be noted here that for $n = 0$, the transform $B_{\ell,0}$ is equal to the one considered in [13].

Another task of the present paper is to show that the constructed spaces are closely connected to the concrete $L^2$–spectral analysis of the semi–elliptic (sliced) second-order differential operator

$$\Box_q = -\partial_q \overline{s} + \overline{q} \partial_s,$$

where $\overline{s}$ denotes the conjugate of the left slice–derivative $\partial_s$ defined by

$$\partial_s f(q) = \begin{cases} \frac{1}{2} \left( \frac{\partial f |_{x_q}}{\partial x} - I_q \frac{\partial f |_{y_q}}{\partial y} \right)(q), & \text{if } q = x_q + y_q I_q \in \Omega \setminus \mathbb{R}; \\ \frac{df}{dx}(x_q), & \text{if } q = x_q \in \Omega \cap \mathbb{R}, \end{cases}$$

(1.6)
In fact, such spaces are realized as special subspaces of the $L^2$–eigenspaces
\[ \mathcal{F}_n^2 = \left\{ f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda); \Box_q f = nf \right\}. \tag{1.8} \]

Here $d\lambda$ denotes the Lebesgue measure on $\mathbb{H} \simeq \mathbb{R}^4$. The $L^2$–spectral description of $\Box_q f = nf$ was possible by dealing first with the $C^\infty$ right–eigenvalue problem $\Box_q f = f\mu$ on $\mathbb{H} := \mathbb{H} \setminus \mathbb{R}$ and by extending appropriately the obtained explicit solutions to the whole $\mathbb{H}$ (Theorem 5.1). Thereby, by manipulating the asymptotic behaviour of such eigenfunctions, we show that the spectrum of $\Box_q$ is purely discrete and consists of the eigenvalues $\mu = n$ which occur with infinite degeneracy (see Theorem 5.7).

The spaces $S\mathcal{R}_{\ell,n}^2$; $\ell = 1, 2$, appear then as specific subspaces of $\mathcal{F}_n^2$ described by Theorem 5.7. This becomes clear in the discussion provided in the last section.

The rest of this paper is structured as follows
- S–polyregular functions.
- S–polyregular Bargmann spaces of first and second kinds.
- Explicit formulas for the reproducing kernels.
- Associated Segal-Bargmann transforms.
- Spectral realization of the S–polyregular Bargmann spaces.
- Concluding Remarks and full S–polyregular Bargmann spaces

## 2 S–POLYREGULAR FUNCTIONS

### 2.1 Notations. The elements of the divisor algebra of quaternions $\mathbb{H}$ are 4-component extended complex numbers of the form $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the imaginary components $i, j, k$ satisfy the Hamiltonian computation rules $i^2 = j^2 = k^2 = ijk = -1$. $ki = -ik = j$. According to this algebraic representation, the quaternionic conjugate is defined to be $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$, so that $pq = \overline{p\overline{q}}$ for $p, q \in \mathbb{H}$. The modulus of $q$ is defined to be $|q| = \sqrt{q\overline{q}}$. The polar representation is given by $q = re^{i\theta}$, where $r = |q| \geq 0$, $\theta \in [0, 2\pi]$, and $I$ belongs to the set of imaginary units $\mathbb{S} = \{ q \in \mathbb{H}; q^2 = -1 \}$, which can be identified with the unit sphere $S^2 = \{ q \in \mathbb{H}; |\mathbb{S}(q)| = 1 \}$ in $\mathbb{S}\mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. The representation $q = re^{i\theta}$ is not unique unless $q$ is not real. Another interesting representation of $q \in \mathbb{H}$ is given by $q = x + iy$ for some real numbers $x$ and $y$ and imaginary unit $I \in \mathbb{S}$. It is unique for any $q \in \mathbb{H} = \mathbb{H} \setminus \mathbb{R}$ with $y > 0$. Thus, $\mathbb{H}$ can be seen as the infinite union of the slices $\mathbb{C}_I := \mathbb{R} + \mathbb{R}I$. The last representation was been crucial in developing the theory of quaternionic slice hyperholomorphic functions that has been introduced by Gentili and Struppa in the seminal work [17]. Since then, they have been object of intensive research and the corresponding hyper-complex analysis has been developed. It has found many interesting applications in operator theory, quantum physics, Schur analysis [11] 4 [19].

### 2.2 S–polyregular functions and first properties. The solution of the Cauchy–Riemann equation $\overline{\partial} f = 0$ involving the derivative in (1.1) leads to the theory of slice regular functions. A natural generalization is the S–polyregular functions.

**Definition 2.1.** A quaternionic–valued function $f$ on a domain $\Omega \subset \mathbb{H}$ is said to be (left) slice polyregular (S-polyregular) of level $n$ (order $n + 1$), if it is a real differentiable in $\Omega$ and its restriction $f_I$ is polyanalytic in $\Omega_I := \Omega \cap \mathbb{C}_I$ for every
$I \in S$, in the sense that the function $\overline{\partial}_l f : \Omega_l \rightarrow \mathbb{H}$ vanishes identically on $\Omega_l$. We denote by $\mathcal{SR}_n(\Omega)$ the corresponding right quaternionic vector space.

**Remark 2.2.** For $n = 0$, we recover the standard space $\mathcal{SR}_0(\Omega) = \mathcal{SR}(\Omega)$ of slice regular functions [17] [11] [19] [12], whose elements are left power series

$$\varphi(q) = \sum_{j=0}^{+\infty} q^j \alpha_j(I)$$

where $\alpha_j$ are seen as functions $\alpha_j : I \mapsto \alpha_j(I)$ on $S$ with values in $\mathbb{H}$.

Topologically, the space $\mathcal{SR}_n(\Omega)$ is endowed with the natural topology of uniform convergence on compact sets in $\Omega$, so that it turns out to be a right vector space over the noncommutative field $\mathbb{H}$. We review below some of their basic properties that we need to develop the rest of the paper for the case $\Omega = \mathbb{H}$. Thus, one can easily prove the following characterization for the elements in $\mathcal{SR}_n := \mathcal{SR}_n(\mathbb{H})$.

**Proposition 2.3.** For every $f \in \mathcal{SR}_n$ there exist some $\varphi_k \in \mathcal{SR}$; $k = 0, 1, \cdots, n$, such that

$$f(q, \overline{q}) = \sum_{k=0}^{n} q^k \varphi_k(q).$$

Thanks to this decomposition many interesting analytic properties of S–polyregular functions can be derived from their analogs for the slice regular functions. However, one must be careful since $\mathcal{SR}_n$ enjoy some properties that are different from the ones of $\mathcal{SR}$. For example, the S–polyregular functions even vanish on an accumulation set. This is the case of $1 - q\overline{q}$ which is a nonzero S–polyregular on $\mathbb{H}$ but vanishes on the closed set $\{q \in \mathbb{H}, |q| = 1\}$.

Analogically to the complex setting, the operator $\partial_s - \overline{q}$ will play a crucial role in this theory. It is the adjoint operator of $\overline{\partial}_l$ when acting on the Hilbert space $L^2(\mathbb{H}, e^{-|q|^2} d\lambda)$ (see [12]). Using the fact that $(\partial_l - \overline{q})F$ is orthogonal to $F$ when $F$ is slice regular, we can show by induction that $(\partial_l - \overline{q})^n F; n = 0, 1, 2, \cdots$, are S–polyregulars and form an orthogonal system in $L^2(\mathbb{C}, e^{-|s|^2} d\lambda)$. This procedure gives rise to the quaternionic Hermite polynomials $H_Q^{m,n}$ by specifying $F(q) = F_m(q) = q^n$, to wit,

$$H_{m,n}^Q(q, \overline{q}) = (\partial_l - \overline{q})^n(F_m)(q) = (-1)^m e^{\overline{q}|q|^2} \partial_s^m(e^{-|q|^2} q^m).$$

Accordingly, one considers the differential transformation

$$[\mathcal{H}_n(F)](q) := (-1)^n e^{\overline{q}|q|^2} \partial_s^m(e^{-|q|^2} F)$$

so that for $[\mathcal{H}_n(F_m)](q) = H_{m,n}^Q(q, \overline{q})$. Therefore, we claim that

**Proposition 2.4.** A function $f$ belongs to $\mathcal{SR}_n$ if and only if there exists some $F \in \mathcal{SR}$ such that $f = \mathcal{H}_n(F)$. Moreover, we have

$$\mathcal{SR}_n = \sum_{k=0}^{n} \mathcal{H}_k(\mathcal{SR})$$

and is spanned by the polynomials $H_{j,n}^Q; j = 0, 1, 2, \cdots$.

The following result is a Splitting Lemma for the S–polyregular functions generalizing the standard one for the slice regular functions.
Proposition 2.5 (Splitting lemma for S–polyregular functions). If \( f \) is a S–polyregular function, then for every \( I \in \mathbb{S} \), and every \( J \in \mathbb{S} \) perpendicular to \( I \), there are two polyanalytic functions \( F, G : \mathbb{C}_I \rightarrow \mathbb{C}_I \) such that for any \( q = x + Iy \)
\[
f_I(q) = F(q) + G(q)J.
\]

Proof. The proof follows by means of Proposition 2.3 combined with the standard Splitting Lemma for the slice regular functions. \( \square \)

An analogue of the identity principal for the S–polyregular functions can also be obtained. To this end, we begin by recalling the standard one for the slice regular functions

Lemma 2.6 ([19, 12]). Let \( f : U \rightarrow \mathbb{H} \) be a slice regular function on a slice domain \( U \). Denote by \( Z_f = \{ q \in U; f(q) = 0 \} \) the zero set of \( f \). If there exists \( I \in \mathbb{S} \) such that \( \mathbb{C}_I \cap Z_f \) has an accumulation point, then \( f \equiv 0 \) on \( U \).

This principle is no longer valid for S–polyregular functions as shown by the counterexample \( 1 - q \). However, we can provide a weak version of such uniqueness theorem.

Proposition 2.7 (Identity Principle for S–polyregular functions). Let \( f \) be a S–polyregular function in \( SR_n \) such that \( f \) is identically zero on a subregion \( \Omega \subset \mathbb{C}_I \) for some \( I \in \mathbb{S} \). Then \( f \) is identically zero on the whole \( \mathbb{H} \).

Proof. According to Proposition 2.3, we can write \( f \in SR_n \) as \( f(q) = \sum_{k=0}^{n} q^k \varphi_k(q) \) with \( \varphi_k \in SR \). Now, by the assumption that \( f|_\Omega \equiv 0 \) with \( \Omega \) is a subregion of some slice \( \mathbb{C}_I \), we obtain
\[
n!|_\Omega = \partial_I^n \left( \sum_{k=0}^{n} (x - Iy)^k \varphi_k(x + Iy) \right) \equiv 0.
\]
Repeating this procedure, we conclude that \( \varphi_k|_\Omega \equiv 0 \) for every \( k = n, n-1, \cdots, 1, 0 \). Therefore, \( \varphi_k \equiv 0 \) on the whole \( \mathbb{H} \) by Lemma 2.6. This implies that \( f \equiv 0 \) on \( \mathbb{H} \). \( \square \)

Remark 2.8. Other powerful uniqueness theorems for S–polyregular functions can be obtained. Furthermore, additional properties for the S–polyregular functions will be the subject of a forthcoming investigation of S–polyregular functions.

We conclude this section by introducing the star product for S–polyregular functions. To this end, recall that the left \( \ast^L_s \)–product for left slice regular functions is defined by
\[
f \ast^L_s g(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{k=0}^{n} a_k b_{n-k} \right)
\]
for given \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) and \( f(q) = \sum_{n=0}^{\infty} q^n b_n \) in \( SR \). The involved series is convergent and is a slice regular function [18]. This product is introduced to overcome the fact that the product of left slice regular functions is not necessarily a left slice regular function. To surmount analogue problem in the context of left S–polyregular functions, a natural extension of the \( \ast^L_s \)–product can be defined by considering
\[
(f \ast^L_{sp} g)(q, \bar{q}) = \sum_{j,k=0}^{m,n} \bar{q}^{j+k} \varphi_j \ast^L_s \psi_k(q)
\]
for given \( f(q, \overline{q}) = \sum_{j=0}^{m} \overline{q}^j \varphi_j(q) \in \mathcal{SR}_m \) and \( g(q, \overline{q}) = \sum_{k=0}^{n} \overline{q}^k \psi_k(q) \in \mathcal{SR}_n \). We define in a similar way the right star product for right S–polyregular functions \( f(q, \overline{q}) = \sum_{j=0}^{m} \varphi_j(q) \overline{q}^j \) and \( g(q, \overline{q}) = \sum_{k=0}^{n} \psi_k(q) \overline{q}^k \) as follows

\[
(f \star^R g)(q, \overline{q}) = \sum_{j,k=0}^{m,n} \varphi_j \star^L_k \psi_k(q) \overline{q}^{j+k}.
\]

(2.1)

Thus, one can easily check the following

**Lemma 2.9.** For every \( f \in \mathcal{SR}_m \) and \( g \in \mathcal{SR}_n \), we have

1. \( f \star^L g = \overline{g} \star^R f \).
2. \( f \star^L g = g \star^R f \) if the coefficients of any components slice regular functions \( \varphi_j \) and \( \psi_k \) commute.

As basic example of computation with such \( \star^L \)–product, we explicit the one of the following function

\[
S_k(\overline{p}, p; q, \overline{q}) := \left( |p-q|^2 \right)^{k \star^L_{sp}}
\]

where \( |p-q|^2 := (p-q) \star^L_{sp} (p-q) = h_q(p) \star^L_{sp} h_q(p) \), where we have set \( h_q(p) = p-q \).

For \( k = 1, 2 \) we have

\[
S_1(\overline{p}, p; q, \overline{q}) = \overline{p}(p-q) - (p-q)\overline{q} = \overline{p}h_q(p) - h_q(p)\overline{q}
\]

and

\[
S_2(\overline{p}, p; q, \overline{q}) = \overline{p}^2 h_{q^2}(p) - 2\overline{p} h_{q^2}(p)\overline{q} + h_{q^2}(p)\overline{q}^2.
\]

More generally, we claim

**Lemma 2.10.** Let \( h_q(p) = p - q \). Then, for every \( k = 1, 2, \ldots \) and \( p, q \in \mathbb{H} \), we have

\[
S_k(\overline{p}, p; q, \overline{q}) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \overline{p}^{k-j} h_{q^2}^{k \star^L_{sp}} (p)\overline{q}^j.
\]

Whose the proof can be handled by induction. Moreover, it is clear that the following assertions hold true.

1. The function \( p \rightarrow S_k(\overline{p}, p; q, \overline{q}) \) is left S–polyregular for every fixed \( q \).
2. The function \( q \rightarrow S_k(\overline{p}, p; q, \overline{q}) \) is right S–polyregular for every fixed \( p \).
3. We have \( S_k(\overline{p}, p; q, \overline{q}) = S_k(q, \overline{q}; p, p) \) for every \( p, q \in \mathbb{H} \).

Therefore, we assert the following result which is needed in the establishment of the explicit expression of the reproducing kernels for the S–polyregular Bargmann spaces we deal with in the next section.

**Lemma 2.11.** Let \( L_n^{(\gamma)}(x) \) denote the generalized Laguerre polynomial of degree \( n \) and argument \( \gamma > -1 \). Then, the function

\[
(p, q) \rightarrow L_n^{(\gamma)} \left( |p-q|^2 \right)
\]

satisfies the properties (i), (ii) and (iii) above.
Proof. This readily follows since the generalized Laguerre polynomial \( L_n(\gamma)(x) \) is a finite linear expansion of the functions \( S_k(p, q; \gamma) \) with real coefficients. \( \square \)

In the next sections, we introduce appropriately two classes of infinite dimensional right quaternionic reproducing kernels Hilbert spaces that will be considered as the analog of polyanalytic Bargmann spaces.

\section{S–polyregular Bargmann Spaces}

Let \( \mathcal{SR}_{1,n}^2 \) denote the space of all \( f \) belonging to the right \( \mathbb{H} \)-vector space \( \mathcal{SR}_n^2 := \mathcal{SR}_n \cap L^2(\mathbb{C}_{I_0}, e^{-|\xi|^2} d\lambda) \), for some \( I_0 \in \mathbb{S} \), and such that

\[ \nabla^{(j, k)}_{I,0} f(q, \overline{q}) = \partial^j_I f(q, \overline{q}) |_{q=0} = 0, \]

for every \( j = 0, 1, 2, \cdots \) and \( k = 0, 1, 2, \cdots, n \). The condition (3.1) is equivalent to say that the coefficients \( \alpha_{j,k} : \mathbb{S} \rightarrow \mathbb{H} \) in the expansion of each component slice regular function \( \varphi_k \) (Remark 2.2) are constants on \( \mathbb{S} \).

The particular case of \( n = 0 \) gives rise to the slice Bargmann space \( \mathcal{F}_{\text{slice}}^2 \) considered in [5], for which the monomials \( e_{m,j}(q) := q^m \) constitute an orthogonal basis. In contrast to what one can think, the monomials \( e_{j,k}(q, \overline{q}) := q^j \overline{q}^k \) does not form an orthogonal system in \( \mathcal{SR}_n^2 \) as showed by \( \langle e_{j,0}, e_{j+k,k} \rangle_{\mathcal{C}_I} = \|e_{j+k}\|_{\mathcal{C}_I} = \pi (j + k)! \). Thus, motivated by Proposition 2.4, we will make use of the univariate quaternionic Hermite polynomials \( H_{j,k}^Q \) instead of monomials \( e_{j,k} \), to describe \( \mathcal{SR}_n^2 \).

\begin{proposition}
A function \( f \) belongs to \( \mathcal{SR}_{1,n}^2 \) if and only it can be expanded as follows

\[ f(q, \overline{q}) = \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} H_{j,k}^Q(q, \overline{q}) \alpha_{j,k} \]

for some quaternionic constants \( \alpha_{j,k} \) satisfying the growth condition

\[ \sum_{j=0}^{+\infty} j! |\alpha_{j,k}|^2 < +\infty \]

for every \( k = 0, 1, \cdots, n \).
\end{proposition}

\textbf{Proof.} The direct implication follows making use of [14, Proposition 3.8], expressing the monomials \( \overline{q}^k q^l \) in terms of \( H_{r,s}^Q \),

\[ q^m \overline{q}^n = m!n! \sum_{k=0}^{m \wedge n} \frac{H_{m-k,n-k}^Q(q, \overline{q})}{k!(m-k)!(n-k)!}. \]

The orthogonality

\[ \langle H_{m,n}^Q, H_{j,k}^Q \rangle_{\mathcal{C}_I} = \pi m!n! \delta_{m,j} \delta_{n,k} \]

of \( H_{r,s}^Q \) shows that the condition \( \|f\|_{\mathcal{C}_I} < +\infty \) becomes equivalent to

\[ \|f\|^2_{\mathcal{C}_I} = \sum_{k,k' = 0}^{+\infty} \sum_{j,j' = 0}^{+\infty} \overline{\alpha}_{j,k} \langle H_{j,k}^Q, H_{j',k'}^Q \rangle_{\mathcal{C}_I} \alpha_{j',k'} = \sum_{k=0}^{+\infty} \sum_{j=0}^{+\infty} |\alpha_{j,k}|^2 \|H_{j,k}^Q\|^2. \]

The argument for obtaining the inverse implication is Proposition 2.4. \( \square \)

\begin{definition}
The right vector space \( \mathcal{SR}_{1,n}^2 \), generalizing the slice hyperholomorphic Bargmann space \( \mathcal{F}_{\text{slice}}^2 \), is called first kind \( S \)-polyregular Bargmann space of level \( n \).
\end{definition}
Another interesting subspace to deal with is the following
\[
\mathcal{SR}_{2,k}^2 := \left\{ \sum_{j=0}^{+\infty} H_{j,k}^Q(q,\overline{q})c_j; c_j \in \mathbb{H}, \quad \sum_{j=0}^{+\infty} j|c_j|^2 < +\infty \right\}.
\]

**Definition 3.3.** The right vector space \(\mathcal{SR}_{2,k}^2\) is called here \(S\)-polyregular Bargmann space of (exact) level \(k\) of second kind.

**Theorem 3.4.** The spaces \(\mathcal{SR}_{1,n}^2\) and \(\mathcal{SR}_{2,k}^2\) are Hilbert spaces with orthogonal basis \(\{H_{j,k}^Q; k = 0, 1, \ldots, n; j = 0, 1, \ldots\}\) and \(\{H_{j,k}; j = 0, 1, \ldots\}\), respectively. Moreover, we have
\[
\mathcal{SR}_{1,n}^2 = \bigoplus_{k=0}^n \mathcal{SR}_{2,k}^2. 
\]

**Proof.** As for \(n = 0\), it is not difficult to see that the considered spaces are close subspaces of the Hilbert space \(L^2(\mathbb{C}, e^{-|q|^2}d\lambda_1)\), and therefore they are right quaternionic Hilbert spaces. Now, for fixed nonnegative integer \(k\), the polynomials \(H_{j,k}^Q; j = 0, 1, 2, \ldots\), form an orthogonal system with respect to the gaussian measure and generate \(\mathcal{SR}_{2,k}^2\). Their linear independence is equivalent to their completion. In fact, for a given \(g = \sum_{j=0}^{+\infty} H_{j,k}^Qc_j \in \mathcal{SR}_{2,k}^2\), the condition that \(\langle f, H_{j,k}^Q \rangle = 0\), for every \(\ell = 0, 1, 2, \ldots\), implies that \(c_\ell = 0\) and therefore \(g\) is identically zero on \(\mathbb{H}\), for \(\langle f, H_{j,k}^Q \rangle_{\mathcal{C}_l} = \overline{\pi_j}H_{j,k}^Q\). Thus, \(\{H_{j,k}; j = 0, 1, \ldots\}\) is orthogonal basis of \(\mathcal{SR}_{2,k}^2\). The assertion that \(\{H_{j,k}^Q; k = 0, 1, \ldots, n; j = 0, 1, \ldots\}\) form an orthogonal basis of \(\mathcal{SR}_{1,n}^2\) follows in a similar way. It is also an immediate consequence of (3.4). The decomposition (3.4) readily follows since for given \(f \in \mathcal{SR}_{1,n}^2\), we have
\[
f = \sum_{k=0}^{+\infty} \sum_{j=0}^{n} H_{j,k}^Qc_j = \sum_{k=0}^{n} g_k, \quad g_k := \sum_{j=0}^{+\infty} H_{j,k}^Qc_j.
\]
Then, it is clear that \(g_k \in \mathcal{SR}_{2,k}^2\). In addition, the \(\{g_k; k = 0, 1, \ldots, n\}\) is orthogonal, since for \(k \neq k'\), we have
\[
\langle g_k, g_{k'} \rangle_{\mathcal{C}_l} = \left( \sum_{j,j'=0}^{+\infty} \alpha_{j,k} \alpha_{j',k'} \delta_{j,j'} \right) \left| H_{j,k}^Q \right|^2_{\mathcal{C}_l} = 0.
\]
Accordingly, we have \(\mathcal{SR}_{1,n}^2 = \bigoplus_{k=0}^n \mathcal{SR}_{2,k}^2\). Moreover,
\[
\|f\|_{\mathcal{C}_l}^2 = \sum_{k=0}^{n} \|g_k\|_{\mathcal{C}_l}^2 = \pi \sum_{k=0}^{+\infty} j!|\alpha_{j,k}|^2. \tag{3.5}
\]

In order to show that the considered Hilbert spaces \(\mathcal{SR}_{1,n}^2\) and \(\mathcal{SR}_{2,k}^2\) possess reproducing kernels, we need to the following.

**Lemma 3.5.** For every fixed \(q \in \mathbb{H}\), the evaluation map \(\delta_q f = f(q, \overline{q})\) is a continuous linear form on the Hilbert spaces \(\mathcal{SR}_{1,n}^2\) and \(\mathcal{SR}_{2,k}^2\). Moreover, we have
\[
|f(q, \overline{q})| \leq \frac{1}{\sqrt{\pi}} \|f\|_{\mathcal{C}_l}. \tag{3.6}
\]
Proof. Let $g \in \mathcal{SR}^2_{2,k}$ such that $g = \sum_{j=0}^{+\infty} H_{j,k}^Q c_j$. Using the Cauchy-Schwartz inequality and the expression of the square norm of $g$, $\|g\|_{C_I}^2 = \pi k! \sum_{j=0}^{+\infty} j! |c_j|^2$, we get
\[
|g(q, \overline{q})| \leq \left( \sum_{j=0}^{+\infty} \frac{|H_{j,k}^Q(q, \overline{q})|^2}{\pi j! k!} \right)^{\frac{1}{2}} \|g\|_{C_I}.
\] (3.7)

The series in the right-hand side of (3.7) is absolutely convergent on $B(0, r_0)$ for every fixed $r_0$ and is independent of $g$. This follows readily making use of the following upper bound (see [14 Corollary 4.3]):
\[
|H_{n+k,n}^Q(q, \overline{q})| \leq \frac{(n+k)!}{k!} |q|^k e^{\frac{|q|^2}{2}}.
\] (3.8)

More explicitly, we have
\[
\sum_{j=0}^{+\infty} \frac{|H_{j,k}^Q(q, \overline{q})|^2}{\pi j! k!} = e^{|q|^2}.
\] (3.9)

by means of [3 Eq. 29]. This proves that
\[
|g(q, \overline{q})| \leq \frac{1}{\sqrt{\pi}} e^{\frac{|q|^2}{2}} \|g\|_{C_I}.
\] (3.10)

Now, for $f \in \mathcal{SR}_{1,n}^2$, we have $f = \sum_{k=0}^{n} g_k$ with $g_k \in \mathcal{SR}_{2,k}^2$, and therefore, we obtain
\[
|f(q, \overline{q})|^2 \leq \sum_{k=0}^{n} |g_k(q, \overline{q})|^2 \leq \sum_{k=0}^{n} \frac{1}{\pi} e^{\frac{|q|^2}{2}} \|g_k\|^2 \leq \frac{1}{\pi} e^{\frac{|q|^2}{2}} \|f\|_{C_I}^2
\] by means of (3.5) and (3.10). This completes the proof. \qed

The previous Lemma combined with the quaternionic version of the Riesz representation theorem [22 Theorem 1] ensures the existence of the reproducing kernels of $\mathcal{SR}_{1,n}^2$ and $\mathcal{SR}_{2,k}^2$. The next result gives their explicit expressions in terms of the Laguerre polynomial $L_n^{(q)}(x)$ and the special convergent series
\[
e_{n,b} := \sum_{k=0}^{+\infty} \frac{a^k b^k}{k!}.
\]

**Theorem 3.6.** The reproducing kernels of $\mathcal{SR}_{1,n}^2$ and $\mathcal{SR}_{2,k}^2$ are given respectively by
\[
\mathcal{K}_{1,n}(p, q) = \frac{1}{\pi} e^{\frac{|p-q|}{2}} \theta_{sp} L_n^{(1)}(|p-q|^2)_{sp},
\] (3.11)
and
\[
\mathcal{K}_{2,k}(p, q) = \frac{1}{\pi} e^{\frac{|p-q|}{2}} \theta_{sp} L_k(|p-q|^2)_{sp}.
\] (3.12)

 Succinctly, we have
\[
f(p, \overline{p}) = \langle \mathcal{K}_{1,n}(p, \cdot), f \rangle_{C_I} \quad \text{and} \quad g(p, \overline{p}) = \langle \mathcal{K}_{2,k}(p, \cdot), g \rangle_{C_I}
\] for every $f \in \mathcal{SR}_{1,n}^2$ and $g \in \mathcal{SR}_{2,k}^2$. \[\]
Proof. For \( \mathcal{S}\mathcal{R}^2_{2,k} \), the computation of \( \mathcal{K}_{2,k}(p, q) \) can be done by performing

\[
\mathcal{K}_{2,k}(p, q) = \frac{1}{\pi k!} \sum_{j=0}^{\infty} \frac{H^Q_{j,k}(q, \eta)H^Q_{k,j}(p, p)}{j!},
\]

since \( \{H^Q_{j,k}; j = 0, 1, \cdots \} \) is an orthogonal basis of \( \mathcal{S}\mathcal{R}^2_{2,k} \) (see Theorem 3.4) and \( \overline{H^Q_{k,j}} = H^Q_{j,k} \). For real \( q = x \) or for \( p, q \) belonging to the same slice \( \mathbb{C}_I \), the result follows by means of

\[
\frac{1}{\pi} \sum_{j=0}^{\infty} \frac{H^Q_{j,k}(z, \overline{z})H^Q_{k,j}(w, \overline{w})}{k!} = \frac{(-1)^j}{\pi} e^{\pi w} H^Q_{j,j}(z - w, \overline{z} - \overline{w}) = \frac{1}{\pi} e^{\pi w} L_j(|z - w|^2)
\]

which is readily an immediate consequence of Theorem 3.1 in [8], to wit

\[
\sum_{j=0}^{\infty} \frac{H^Q_{j,k}(z, \overline{z})H^Q_{k,j}(w, \overline{w})}{j!} = t^j H^Q_{k,k}(z - tw, \overline{z} - \overline{w}) e^{\pi w},
\]

valid for every \( |t| = 1 \) and \( z, w \in \mathbb{C} \), combined with \( H^Q_{k,k}(\xi, \overline{\xi}) = (-1)^k k! L_k(|\xi|^2) \).

Now, for given fixed nonreal \( q \), let \( I_q \in \mathbb{S} \) such that \( q \in \mathbb{C}_{I_q} \). The functions

\[
\varphi : p \mapsto \mathcal{K}_{2,k}(p, q) \quad \text{and} \quad \psi : p \mapsto \frac{1}{\pi} e^{[p, q]} L_{p} L_{k}(|p - q|^2_{L})
\]

are clearly \( S \)-polyregular thanks to the definiton (2.1) of the \( \star_{sp} \)-product and Lemma 2.11. Moreover, they coincide on the slice \( \mathbb{C}_{I_q} \) by means of (3.13). Thus, by invoking the identity principle for \( S \)-polyregular functions (Proposition 2.7), we conclude that \( \phi \equiv \psi \) on the whole \( \mathbb{H} \). This holds for arbitrary \( q \in \mathbb{H} \). Therefore, we have

\[
\mathcal{K}_{2,k}(p, q) = \frac{1}{\pi} e^{[p, q]} L_{p} L_{k}(|p - q|^2_{L})
\]

for \( p, q \in \mathbb{H} \). This completes our check for (3.12).

To conclude for Theorem 3.6, it suffices to observe that since \( \mathcal{S}\mathcal{R}^2_{1,n} = \bigoplus_{k=0}^{n} \mathcal{S}\mathcal{R}^2_{2,k} \), we have

\[
\mathcal{K}_{1,n}(p, q) = \sum_{k=0}^{n} \mathcal{K}_{2,k}(p, q).
\]

Hence, by virtue of \( \sum_{k=0}^{n} L_k^{(\gamma)}(x) = L_n^{(\gamma+1)}(x) \) (see [20]), we get

\[
\mathcal{K}_{1,n}(p, q) = \sum_{k=0}^{n} \frac{1}{\pi} e^{[p, q]} L_{p} L_{k}(|p - q|^2_{L}) = \frac{1}{\pi} e^{[p, q]} L_{p} L_{n}^{(1)}(|p - q|^2_{L}).
\]

\[\square\]

Remark 3.7. The expression of \( \mathcal{K}_{1,n}(p, q) \) can be rewritten in the equivalent form

\[
\mathcal{K}_{1,n}(p, q) = \frac{1}{\pi} L_{n}^{(1)}(|p - q|^2_{L}) e^{[p, q]} L_{p} L_{n}^{(1)}(|p - q|^2_{L}) \quad (3.15)
\]

thanks to (ii) of Lemma 2.9. The same observation holds for \( \mathcal{K}_{2,k}(p, q) \).
**Remark 3.8.** The operator \( f \mapsto P_k f \) given by \( P_k f(p, \overline{p}) = \langle K_{2,k}(p, \cdot), f \rangle_{C_t} \) defines the orthogonal projection of \( L^2(C_t; e^{-|q|^2} d\lambda_I) \) onto \( \mathcal{SR}_{2,k}^2 \). More explicitly, it reads

\[
P_k f(p, \overline{p}) = \frac{1}{\pi} \int_{C_t} e^{\frac{(|p|^2 - |q|^2)}{L_{sp}}} L_k(|p - q|^2 e^{-|q|^2} d\lambda_I(q)}
\]

which we can rewrite also as

\[
P_k f(p, \overline{p}) = \frac{1}{\pi} \int_{C_t} L_k(|p - q|^2 e^{-|q|^2} d\lambda_I(q)}
\]

by means of (ii) in Lemma 2.9.

We conclude this section with the following result giving an orthogonal Hilbertian decomposition of the Hilbert space \( L^2(C_t; e^{-|q|^2} d\lambda_I) \).

**Theorem 3.9.** We have the following hilbertian decomposition

\[
L^2(C_t; e^{-|q|^2} d\lambda_I) = \bigoplus_{k=0}^{+\infty} \mathcal{SR}_{2,k}^2.
\]

**Proof.** Notice first that such decomposition is equivalent to that the orthogonal complement of \( \bigoplus_{k=0}^{+\infty} \mathcal{SR}_{2,k}^2 \) in \( L^2(C_t; e^{-|q|^2} d\lambda_I) \) is \( \{0\} \). To this end, we claim that

\[
T(t|q) := \int_{C_t} \frac{1}{(1-t)} e^{\frac{|\xi|^2}{L_{sp}}} L_k(\sum_{k=0}^{+\infty} |L_k(w-q|^2 e^{-|q|^2} d\lambda_I(q)) = 0
\]

holds for every \( f \in \bigoplus_{k=0}^{+\infty} \mathcal{SR}_{2,k}^2 \), every \( t \in [0,1[ \) and every fixed \( q \in \mathbb{H} \). In fact, this follows readily making use of the generating function for the Laguerre polynomials ([20 Eq. (14), p. 135])

\[
\sum_{k=0}^{+\infty} t^k L_k^{(\alpha)}(\xi) = \frac{1}{(1-t)^{1+1}} \exp \left( \frac{t \xi}{1-t} \right).
\]

Indeed,

\[
T(t|q) = \int_{C_t} e^{\frac{|\xi|^2}{L_{sp}}} L_k(\sum_{k=0}^{+\infty} t^k L_k(|w-q|^2 e^{-|q|^2} d\lambda_I(q))
\]

\[
= \sum_{k=0}^{+\infty} t^k \int_{C_t} e^{\frac{|\xi|^2}{L_{sp}}} L_k(\sum_{k=0}^{+\infty} |L_k(w-q|^2 e^{-|q|^2} d\lambda_I(q)
\]

\[
= 0.
\]

The limit \( t \rightarrow 1^- \) in (3.18) yields an integral involving the Dirac \( \delta \)-function at the point \( q \in \mathbb{H} \). From that, the left-hand side of (3.18) reduces further to \( e^{\frac{|\xi|^2}{L_{sp}}} f(\xi) e^{-|q|^2} \). Therefore, \( f(q) = 0 \) for every \( q \in \mathbb{H} \). \( \square \)

4 SEGAL-BARGMANN TRANSFORMS FOR S–POLYREGULAR BARGMANN SPACES

In this section, we introduce a family of appropriate transforms of Bargmann type defined on the right quaternionic Hilbert space \( L^2_{\mathbb{H}}(\mathbb{R}; dt) \), consisting of all square integrable \( \mathbb{H} \)-valued functions with respect to the inner product

\[
\langle f, g \rangle_{\mathbb{H}} := \int_{\mathbb{R}} f(t) \overline{g(t)} dt.
\]
Their images will be the $S$-polyregular Bargmann spaces defined and studied in the previous section. To this end, we define the kernel functions $B_{\ell,n}(x; q); \ell = 1, 2,$ on $\mathbb{R} \times \mathbb{H}$ to be the bilinear generating function

$$
B_{2,k}(x; q) = \sum_{j=0}^{\infty} \frac{h_j(t)H_{j,k}(q, q)}{\|h_j\|_{\mathbb{R}}\|H_{j,k}\|_{\mathbb{C}^I}}.
$$

(4.1)

and

$$
B_{1,n}(x; q) = \sum_{k=0}^{n} B_{2,k}(x; q).
$$

(4.2)

Here, $h_j(t)$ denotes the $j^{th}$ real Hermite function,

$$
h_j(t) = (-1)^j e^{\frac{-t^2}{2}} \frac{d^j}{dt^j} (e^{-t^2}),
$$

(4.3)

that form an orthogonal basis of $L^2_{\mathbb{H}}(\mathbb{R}; dt)$, with square norm given by

$$
\|h_j\|^2_{\mathbb{R}} = 2^j j! \sqrt{\pi}.
$$

(4.4)

Thus, we have

**Theorem 4.1.** For every $t \in \mathbb{R}$ and $q \in \mathbb{H}$, we have

$$
B_{2,k}(t; q) = \exp \left( -\frac{\ell^2+\pi^2}{2} + \sqrt{2\pi} q t \right) \sqrt{2\pi} H_k \left( \frac{q + q}{\sqrt{2}} - t \right).
$$

Subsequently, the function

$$
B_{2,k,q} : t \rightarrow B_{2,k}(t; q) := \left( \frac{1}{\pi} \right)^\frac{1}{4} \frac{1}{\sqrt{2^k k!}} e^{-\frac{\ell^2+\pi^2}{2} + \sqrt{2\pi} q t} H_k \left( \frac{q + q}{\sqrt{2}} - t \right)
$$

belongs to $L^2_{\mathbb{H}}(\mathbb{R}; dt)$ for every fixed $q \in \mathbb{H}$. Moreover, we have

$$
\|B_{2,k,q}\|_{\mathbb{R}} = \frac{1}{\sqrt{\pi}} e^{\frac{\ell q^2}{2}}.
$$

(4.5)

**Proof.** The explicit expression of the kernel function can be obtained by [14, Theorem 5.7]. For the second assertion, fix $q = x + iy$ in $\mathbb{H}$ and write the modulus of the kernel function $B_{2,k}(t; q)$ as

$$
|B_{2,k}(t; q)|^2 = \left( \frac{1}{\pi} \right)^\frac{1}{4} \frac{1}{2^k k!} \left| e^{-\frac{\ell^2+\pi^2}{2} + \frac{x^2+y^2+1+\sqrt{2\pi} y q}{2}} \right|^2 \left| H_k(\sqrt{2x} - t) \right|^2
$$

$$
= \left( \frac{1}{\pi} \right)^\frac{1}{4} \frac{1}{2^k k!} e^{-\ell^2-x^2+y^2+2\sqrt{2\pi} y t} \left| H_k(\sqrt{2x} - t) \right|^2.
$$
Therefore, it follows that
\[
\|B_{2,k,q}\|_R^2 = \left(\frac{1}{\pi}\right)^\frac{3}{2} \frac{1}{2^{k+1}k!} e^{x^2+y^2} \int_R e^{-(t-\sqrt{2}x)^2} |H_k(t - \sqrt{2}x)|^2 dt
= \left(\frac{1}{\pi}\right)^\frac{3}{2} \frac{1}{2^{k+1}k!} e^{q^2} \int_R e^{-u^2} |H_k(u)|^2 du
= \left(\frac{1}{\pi}\right)^\frac{3}{2} \frac{1}{2^{k+1}k!} e^{q^2} \int_R |h_k(u)|^2 du
= \left(\frac{1}{\pi}\right)^\frac{3}{2} \frac{1}{2^{k+1}k!} e^{q^2} \|h_k\|_R^2
= \frac{1}{\pi} e^{q^2}.
\]
□

**Remark 4.2.** By comparing (4.5) to (3.9), we conclude that \(\|B_{2,k,q}\|_R = \sqrt{K_k(q,q)}\) for every \(q \in \mathbb{H}\).

Associated to the kernel function \(B_{2,k}\) given through (4.1), we consider the integral transform defined by \([B_k \phi](q) := \langle B_{2,k}(\cdot; q), \phi \rangle_R\) which reads explicitly
\[
[B_k \phi](q) := \left(\frac{1}{\pi}\right)^\frac{3}{2} \frac{1}{\sqrt{2^{k+1}k!}} \int_R e^{-\frac{t^2+q^2}{\sqrt{2}} + \sqrt{2}qt} H_k \left(\frac{q + \sqrt{2}t}{\sqrt{2}} - x\right) \phi(t) dt = \langle B_{2,k}(\cdot; q), \phi \rangle_R
\]
for a given function \(\phi : \mathbb{R} \to \mathbb{H}\), provided that the integral exists. The following result shows that \(B_k\) is well-defined on \(L^2_H(\mathbb{H}; dt)\). Namely, we have

**Lemma 4.3.** For every quaternion \(q \in \mathbb{H}\) and every \(\phi \in L^2_H(\mathbb{R}; dt)\), we have
\[
|B_n \phi(q, \bar{q})| \leq \frac{1}{\sqrt{\pi}} e^{\frac{|q|^2}{4}} \|\phi\|_R.
\]

**Proof.** The proof readily follows by applying the Cauchy-Schwartz inequality, we obtain
\[
|B_n \phi(q, \bar{q})| \leq \int \|B_{2,k}(t; q)|\phi(t)| dt \leq \|B_{2,k,q}\|_R \|\phi\|_R. \tag{4.6}
\]
In view of (4.5), the inequality (4.6) reduces further to
\[
|B_n \phi(q, \bar{q})| \leq \frac{e^{\frac{|q|^2}{4}}}{\sqrt{\pi}} \|\phi\|_R.
\]
□

**Theorem 4.4.** The transform \(B_k : L^2_H(\mathbb{R}; dt) \rightarrow \mathcal{S}R^2_{2,k}\) defines a Hilbert space isomorphism.

**Remark 4.5.** The Segal-Bargmann transform \(B_k\) maps the orthogonal basis \(h_j\) of \(L^2_H(\mathbb{R}; dt)\) to the orthogonal basis \(H^Q_{2,k}(q, \bar{q})\) of the \(S\)-polyregular Bargmann space \(\mathcal{S}R^2_{2,k}\). More exactly, we have
\[
[B_k(h_j)](q, \bar{q}) = \left(\frac{1}{\pi}\right)^\frac{3}{2} \frac{1}{\sqrt{2^j k!}} H^Q_{2,k}(q, \bar{q}).
\]
5 Spectral realization of the S–polyregular Bargmann spaces

5.1 Discussion. We conclude this paper by showing that the S–polyregular Bargmann space \( S^2_{2,n} \) (and therefore \( S^2_{1,n} \)) is closely connected to the concrete \( L^2 \)–spectral analysis of the slice differential operator \( \Box_q \). To this end, we begin by considering the \( C^\infty \)–spectral properties of \( \Box_q \) which requires the surmount of two problems. The first one is connected to the uniqueness problem of the polar representation \( q = re^{i\theta} \) and the slice representation \( q = x + iy \), of given \( q \in \mathbb{H} \). This can be resolved by restricting \( q \) to belong to \( \mathbb{H} = \mathbb{H} \setminus \mathbb{R} \) and next extend, in some how, the obtained results to the whole \( \mathbb{H} \). The second problem is related to the notion of the slice derivative given by (1.7) which remains the differential operator not necessarily elliptic.

To see this, notice that \( \partial_s \) can be rewritten in the following unified form

\[
\frac{\partial}{\partial q} = \frac{1}{2} \left( (1 + \chi_R(q)) \frac{\partial}{\partial x} - (1 - \chi_R(q)) I_q \frac{\partial}{\partial y} \right),
\]

so that the operator \( \Box_q \) reads

\[
\Box_q = \frac{1}{4} \left\{ \frac{(1 + \chi_R(q))^2}{\partial x^2} + \frac{(1 - \chi_R(q))^2}{\partial y^2} \right\} + \frac{1}{2} \left( 1 + \chi_R(q) \right) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + \frac{I_q}{2} \left( 1 - \chi_R(q) \right) \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right).
\]

It can be seen as a family of second order differential operators on \( \mathbb{R}^2 \) labeled by \( S \). Accordingly, for every fixed \( I_q \in S \), the operator \( \Box_q \) is not elliptic nor uniform elliptic. However, it is semi–elliptic since the eigenvalues of the corresponding matrix

\[
\begin{pmatrix}
-\frac{1}{4} (1 + \chi_R(q))^2 & 0 \\
0 & (1 - \chi_R(q))^2
\end{pmatrix}
\]

are clearly non–negatives (but not necessary positives). Therefore, to do so, we emphasize to begin by studying the eigenvalue problem of \( \Box_q \) in (5.2) when acting on both the \( C^\infty \)– and the \( L^2 \)–quaternionic–valued functions on \( \mathbb{H} \), extend the results to the real line. It should be noted here that \( \mathbb{R} \) is a negligible Borel measurable set with respect to the gaussian measure on \( \mathbb{H} \), and therefore

\[
\int_{\mathbb{H}} f(q)e^{-|q|^2} \, d\lambda(q) = \int_{\mathbb{R}^+} f(r) e^{-r^2} \, dr \, d\sigma(I_q),
\]

where \( dr \) (resp. \( d\theta \)) denotes the Lebesgue measure on positive real line (the unit circle) and \( d\sigma(I) \) stands for the standard area element on \( S \). This observation will be used systematically when discussing square integrability of the appropriate extension on the whole \( \mathbb{H} \).

5.2 \( C^\infty \)–right–eigenvalue problem. Let \( \mu \) be a fixed quaternionic number and consider the right eigenvalue problem \( \Box_q f = f \mu \) for \( \Box_q \) acting on the right quaternionic vector space \( C^\infty(\mathbb{H}) \) of all quaternionic–valued functions that are \( C^\infty \) on the whole \( \mathbb{H} \simeq \mathbb{R}^4 \). Thus, associated to \( \mu \), we perform the \( C^\infty \)–eigenspace

\[
\mathcal{E}_\mu^\infty(\mathbb{H}, \Box_q) := \{ f \in C^\infty(\mathbb{H}) ; \Box_q f = f \mu \}.
\]

Notice for instance that \( \mathcal{E}_\mu^\infty(\mathbb{H}, \Box_q) \) is not necessarily a quaternionic right vector space. However, it is a \( C_\mu \)–right vector space, where \( C_\mu := \{ p \in \mathbb{H} , p\mu = \mu p \} \) is the
for every integer and its action on the functions

For the ansatz

is given by the following

Its expression in the polar coordinates

\( \Delta q \) in the right–eigenvalue problem

\( \Delta \mathbb{H} \setminus \mathbb{R} \), we consider \( \Delta_q \tilde{f} = \tilde{f}_\mu \), where \( \Delta_q \) denotes the restriction of the sliced differential operator \( \Box_q \) in (1.6) to \( \mathbb{H} \). It takes the form

\[
\Delta_q = -\frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + \frac{I}{2} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right). 
\]

Its expression in the polar coordinates \( q = re^{i\theta} \), with \( r > 0, 0 \leq \theta \leq 2\pi \) and \( I \in \mathbb{S} \), is given by the following

and its action on the functions \( e^{i\theta a^f_j} \) is given by

\[
\Delta_q(e^{i\theta a^f_j}(r)) = -\frac{e^{i\theta a^f_j}}{4r} \left[ r^2 \frac{\partial^2}{\partial r^2} + (1 - 2r^2) r \frac{\partial}{\partial r} + (2jr^2 - j^2) \right] a^f_j(r). 
\]

Now, by smooth regularity, any \( \tilde{f} \in C^\infty(\mathbb{H}) \) can be expanded as

where the functions \( (r, I) \mapsto a^f_j(r) \) are \( C^\infty \) on \([0, +\infty) \times \mathbb{S} \). Therefore, by inserting (5.9) in the right–eigenvalue problem \( \Delta_q \tilde{f} = \tilde{f}_\mu \) and making use of (5.8), we see

\[
\left[ r^2 \frac{\partial^2}{\partial r^2} + (1 - 2r^2) r \frac{\partial}{\partial r} + (2jr^2 - j^2) \right] a^f_j(r) = -4r^2 a^f_j(r) \mu 
\]

for every integer \( j \) and fixed \( r \) and \( I \). By the changes of variable \( t = r^2 > 0 \) and of function \( a^f_j(r) = t^\alpha h^f_j(t, I) \), we get

\[
\alpha h^f_j(\alpha) + (2\alpha + 1 - t) h^f_j(\alpha + \frac{j}{2}) + \frac{1}{\alpha} \left( \alpha - \frac{j}{2} \right) \left( \alpha + \frac{j}{2} - t \right) h^f_j(\alpha + \frac{j}{2} - t) h^f_j(\alpha, I) = -h^f_j(\alpha, I) \mu. 
\]

For the ansatz \( \alpha = |j|/2 \), we recognize the left–quaternionic version of the confluent hypergeometric differential equation

\[
th^f(\cdot, I) + (c - t) h^f(\cdot, I) = h^f(\cdot, I) a 
\]
satisfied by \( h_j(\cdot, I) \) on \([0, +\infty[\), with \( c = |j| + 1 \) and \( a = -\mu - j \chi_{\mathbb{R}^-}(j) = -\mu - (1 - \text{sgn}(j)) \frac{j}{2} \in \mathbb{H} \). Here \( \text{sgn} \) is the signum function. Its first solution is given by the Kummer’s function \( M\left( \frac{a}{c} \left| t \right. \right) \). For \( c = |j| + 1 \) being a positive integer, the second (linearly independent) solution is given by the Tricomi’s logarithmic function \cite[p. 21]{23} (see also \cite[p.504]{1})

\[
U \left( \frac{a}{|j| + 1} \left| t \right. \right) := \frac{(|j| - 1)!}{\Gamma(a)} S^a_j(t) + \frac{(-1)^{|j|+1}}{|j|! \Gamma(a - |j|)} \left\{ M\left( \frac{a}{|j| + 1} \left| t \right. \right) \ln t + \sum_{k=0}^{+\infty} \frac{(a)_k}{(|j| + 1)_k} (\psi(a + k) - \psi(1 + k) - \psi(|j| + 1 + k)) \frac{t^k}{k!} \right\},
\]

where \( \psi(x) \) denotes the logarithmic derivative of the gamma function, \( \psi(x) = \Gamma'(x)/\Gamma(x) \), and \( S^a_j(t) \) is the finite sum given by

\[
S^a_j(t) := \sum_{k=0}^{+\infty} \frac{(a - |j|)_k t^{k-|j|}}{(1 - |j|)_k k!}
\]

and interpreted as 0 when \( j = 0 \). Thus, the only solution of (5.12) that can be extended at \( t = 0 \) is given by

\[
h_j(t, I) = M\left( \frac{-\mu - (1 - \text{sgn}(j)) \frac{j}{2}}{|j| + 1} \left| t \right. \right) \gamma^I_{\mu,j}
\]

for some quaternionic constants \( \gamma^I_{\mu,j} \in \mathcal{C}_\mu \) viewed as functions on \( \mathbb{S} \). Therefore, the corresponding \( \mathcal{C}^{\infty} \) on \( \mathbb{H} \) and solutions of the right–eigenvalue problem \( \Delta_q f = f \mu \) on \( \mathbb{H} \) is given by

\[
f(re^{i\theta}) = \sum_{j \in \mathbb{Z}} r^{j} e^{j\theta} M\left( \frac{-\mu - (1 - \text{sgn}(j)) \frac{j}{2}}{|j| + 1} \left| r \right. \right) \gamma^I_{\mu,j}
\]

that we can rewrite as in (5.6). Such expression is well–defined as a \( \mathcal{C}^{\infty} \) function on the whole \( \mathbb{H} \). Its restriction to \( \mathbb{R} \) is exactly the solution of (5.11). \( \square \)

**Remark 5.2.** The extension of the solution of differential equation (5.12) at the regular singular point 0 corresponds to the extension of the solution of the right–eigenvalue problem \( \Delta_q f = f \mu \) on \( \mathbb{H} \) to the whole \( \mathbb{H} \).

**Remark 5.3.** The quaternionic \( C_{\mu} \)–right–vector space \( \mathcal{E}_{\mu}^{\infty}(\mathbb{H}, \Box_q) \) is generated by the functions

\[
\psi_{\mu,j}(q) := q^{(1 + \text{sgn}(j)) \frac{|j|}{2}} q^{(1 - \text{sgn}(j)) \frac{|j|}{2}} M\left( \frac{-\mu - (1 - \text{sgn}(j)) \frac{j}{2}}{|j| + 1} \left| q \right. \right). \quad (5.13)
\]

The expansion of any \( f \in \mathcal{E}_{\mu}^{\infty}(\mathbb{H}, \Box_q) \) in terms of \( \psi_{\mu,j}(q) \) involves sliced right coefficients \( \gamma^I_{\mu,j} \) in \( \mathcal{C}_\mu \).

5.3 \( L^2 \)–right–eigenvalue problem. In the sequel, we are interested in giving a concrete description of \( L^2 \)–eigenspaces of the right–eigenvalue problem \( \Box_q f = f \mu \). To this end, we perform

\[
\mathcal{F}^2_{\mu} := \left\{ f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda); \Box_q f = f \mu \right\}, \quad (5.14)
\]

as well as

\[
\overline{\mathcal{F}}^2_{\mu} := \left\{ f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda); \Delta_q f = f \mu \right\}, \quad (5.15)
\]
where $L^2(\mathbb{H}; e^{-|q|^2}d\lambda)$ denotes the right Hilbert space of all quaternionic–valued square integrable functions on $\mathbb{H}$ with respect to the inner product
\[
\langle f, g \rangle_{\mathbb{H}} := \int_{\mathbb{H}} f(q)g(q)e^{-|q|^2}d\lambda(q)
\]  
with $d\lambda(q) = dx_0dx_1dx_2dx_3$ being the Lebesgue measure on $\mathbb{H} \simeq \mathbb{R}^4$. We define in a similar way $L^2(\mathbb{H}; e^{-|q|^2}d\lambda)$ and $\langle \tilde{f}, \tilde{g} \rangle_{\mathbb{H}}$. Thus, the following lemmas are fundamental for our investigation of the $L^2$–eigenspaces $F^2_\mu$.

**Lemma 5.4.** Keep notations as above.

(i) We have
\[
\text{Spectre}_{L^2(\mathbb{H}; e^{-|q|^2}d\lambda)}(\Box_q) \subset \text{Spectre}_{L^2(\mathbb{H}; e^{-|q|^2}d\lambda)}(\Delta_q).
\]

(ii) The space $F^2_\mu$ is a $L^2$–subspace of the $C^\infty$–eigenspace $E^\infty_\mu(\mathbb{H}, \Delta_q)$ and we have
\[
F^2_\mu \subset F^2_\mu = L^2(\mathbb{H}; e^{-|q|^2}d\lambda) \cap E^\infty_\mu(\mathbb{H}, \Delta_q).
\]

**Proof.** The first assertion follows readily since for every $f \in L^2(\mathbb{H}; e^{-|q|^2}d\lambda)$, we have $\tilde{f} \in L^2(\mathbb{H}; e^{-|q|^2}d\lambda)$ with $\|f\|_{\mathbb{H}} = \|\tilde{f}\|_{\mathbb{H}}$. The second assertion is an immediate consequence of the ellipticity of $\Delta_q$ seen as a second order differential operator on $\mathbb{R} \times \mathbb{R}^*$. \hfill \Box

The second key lemma concerns the elementary functions
\[
\varphi_{\mu,j}(q) := \psi_{\mu,j}(q)\alpha_j^f
\]
associated to given $\alpha_j^f \in C_\mu$, where $q = x + iy \in \mathbb{H}$ and $\psi_{\mu,j}$ are as in (5.13).

**Lemma 5.5.** We assert the following

(i) The functions $\varphi_{\mu,j}$ are pairwise orthogonal in the sense that $\langle \varphi_{\mu,j}, \varphi_{\mu,k} \rangle = 0$ whenever $j \neq k$.

(ii) The functions $\varphi_{\mu,j}$ belong to $L^2(\mathbb{H}; e^{-|q|^2}d\lambda)$ if and only if $\mu_j = \mu + j$ is a nonnegative integer.

(iii) Let $\mu_j = 0, 1, 2, \cdots$. Then, the square norm of $\varphi_{\mu,j}$ in $L^2(\mathbb{H}; e^{-|q|^2}d\lambda)$ is given by
\[
\|\varphi_{\mu,j}\|_{\mathbb{H}}^2 = \pi \frac{\mu_j!}{(\mu_j + j)!} \int_{\mathbb{S}} |\alpha_j^f|^2d\sigma(I).
\]

**Proof.** The first assertion follows by direct computation using the polar coordinates $q = re^{i\theta}$. Indeed, in these coordinates, the Lebesgue measure $d\lambda$ becomes the product of the standard Lebesgue measures $rdr$ on $\mathbb{R}^+$ and $d\theta$ on the unit circle times the standard area element $d\sigma(I)$ on $\mathbb{S}$, the two–dimensional sphere of imaginary units in $\mathbb{H}$. Therefore, for every $\alpha_j^f \in \mathbb{H}$, we have
\[
\langle \varphi_{\mu,j}, \varphi_{\mu,k} \rangle = \int_{\mathbb{H}} \psi_{\mu,j}(q)\alpha_j^f\psi_{\mu,k}(q)\alpha_k^f e^{-|q|^2}d\lambda(q) = \int_{\mathbb{S}} \int_0^\infty r^{j+k+1} \int_{\mathbb{S}} \alpha_j^f R_{j,k}(I)\alpha_k^f e^{-r^2}d\sigma(I)dr,
\]
where $R_{j,k}(I)$ stands for
\[
R_{j,k}(I) := M\left(\frac{-\mu_j}{|j|+1} r^2\right) \left(\int_0^{2\pi} e^{(k-j)\theta} d\theta\right) M\left(\frac{-\mu_k}{|k|+1} r^2\right).
\]
The use of the well–known fact $\int_0^{2\pi} e^{(k-j)\theta} d\theta = 2\pi \delta_{j,k}$ and the change of variable $t = r^2$ complete our check of (i). Indeed, we obtain

$$\langle \varphi_{\mu,j}, \varphi_{\mu,k} \rangle = \pi \left( \int_S |\alpha_j|^2 d\sigma(I) \right) \left( \int_0^\infty t^j \left| M \left( \frac{-\mu_j}{|j| + 1}, t \right) \right|^2 e^{-t} dt \right) \delta_{j,k}. \quad (5.20)$$

To prove the second assertion, we make use of the asymptotic behavior

$$M \left( \frac{a}{c}, t \right) \sim \frac{e^{t^{a-c}}}{\Gamma(a)} \frac{t^a}{e^t}$$

for $t$ large enough and $a \neq 0, -1, -2, \cdots$, that follows from the Poincaré-type expansion [21, Section 7.2]

$${t}^{a-c} \frac{t^a}{e^t} \sim \frac{e^{t^{a-c}}}{\Gamma(a)} \frac{t^a}{e^t} \frac{1}{k!} \sum_{k=0}^\infty (1-a)_k (c-a)_k t^{-k}.$$ 

Indeed, if $\mu_j \neq 0, 1, 2, \cdots$, then the nature of the integral involved in the right–hand side of (5.20) is equivalent to

$$\frac{1}{|\Gamma(-\mu_j)|^2} \int_0^\infty t^{-(2\Re(\mu_j) + |j| + 2)} e^t dt$$

which is clearly divergent. Thus, we necessarily have $\mu_j = 0, 1, 2, \cdots$. In this case, the involved Kummer’s function is the generalized Laguerre polynomial ([20, Eq. (1), p. 200])

$$M \left( \frac{-\mu_j}{|j| + 1}, t \right) = \frac{\mu_j!}{(|n| + 1)_{\mu_j}} L^{(j)}_{\mu_j}(t) \quad (5.21)$$

which satisfies the orthogonality property [20, Eq. (4), p. 205 - Eq. (7), p. 206]

$$\int_{\mathbb{R}^+} L^{(\alpha)}_j(t) L^{(\alpha)}_k(t) t^\alpha e^{-t} dt = \frac{\Gamma(\alpha + j + 1)}{\Gamma(j + 1)} \delta_{j,k}. \quad (5.22)$$

More precisely, starting from (5.20), the explicit computation yields

$$\| \varphi_{\mu,j} \|_H^2 = \pi \left( \frac{\mu_j!}{(|j| + 1)_{\mu_j}} \right)^2 \left( \int_0^\infty (L^{(j)}_{\mu_j}(t))^2 t^j e^{-t} dt \right) \left( \int_S |\alpha_j|^2 d\sigma(I) \right) \times \left( \int_S |\alpha_j|^2 d\sigma(I) \right)$$

$$= \pi \left( \frac{\mu_j!}{(|j| + 1)_{\mu_j}} \right)^2 \left( \int_S |\alpha_j|^2 d\sigma(I) \right).$$

This completes our check of (ii) and (iii). \hfill \Box

**Remark 5.6.** If $\mu$ is a fixed nonnegative integer $\mu = n$, then $\psi_{\mu,j}\alpha_j^I$ belongs to $L^2(\mathbb{H}; e^{-\tau^2} d\lambda)$ if and only if $j \geq -n$, unless the corresponding $\alpha_j^I$ is zero. In this case, the square norm of $\psi_{n,j}$ (in (5.13)) is given by

$$\| \psi_{n,j} \|_H^2 = \pi \frac{n!(j!)^2}{(n+j)!} \text{Area}(\mathbb{S}). \quad (5.23)$$

The next result shows in particular that the spectrum of $\Box_q$ acting $L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ is purely discrete and reduces to the quantized eigenvalues known as Landau levels.
**Theorem 5.7.** The space $F^2_{\mu}$ is nontrivial if and only if $\mu = n = 0, 1, 2, \cdots$. In this case, a nonzero quaternionic--valued function $f$ belongs to $F^2_{n}(\mathbb{H})$ if and only if it can be expanded as

$$f(q) = \sum_{j=-n}^{+\infty} q^j M \left( \frac{-n}{|j| + 1} |q|^2 \right) C_j(I). \quad (5.24)$$

where the quaternionic constants $C_j(I)$ satisfy the growth condition

$$\|f\|^2_{\mathbb{H}} = \pi \sum_{j=-n}^{+\infty} \frac{n!(j!)^2}{(n+j)!} \left( \int_{\mathbb{S}} |C_j(I)|^2 d\sigma(I) \right) < +\infty. \quad (5.25)$$

**Proof.** Fix $\mu \in \mathbb{H}$ and assume that there is a nonzero function $f \in L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$ solution of $\square_q f = \mu f$. Then, the realization (5.17) and the proof of Theorem 5.1 show that $\tilde{f} := f|_{\mathbb{H}}$ admits the expansion

$$\tilde{f}(|q| e^{i\theta}) = \sum_{j \in \mathbb{Z}} \psi_{\mu,j}(q) \gamma^j_{\mu,j}$$

The orthogonality of the $(\psi_{\mu,j})_j$ (see (i) of Lemma 5.5) infers that

$$\|\tilde{f}\|^2_{\mathbb{H}} = \sum_{j \in \mathbb{Z}} \|\psi_{\mu,j} \gamma^j_{\mu,j}\|^2_{\mathbb{H}} = \frac{\pi}{\text{Area}(\mathbb{S})} \sum_{j \in \mathbb{Z}} \left( \int_{\mathbb{S}} |\gamma^j_{\mu,j}|^2 d\sigma(I) \right) \|\psi_{\mu,j}\|^2_{\mathbb{H}}.$$

Therefore, since the nonzero function $f$ belongs to $F^2_{\mu}$, we have necessarily $\|\psi_{\mu,j}\|^2_{\mathbb{H}}$ is finite for every $j$ such that

$$\int_{\mathbb{S}} |\gamma^j_{\mu,j}|^2 d\sigma(I) \neq 0.$$

Now, (ii) of Lemma 5.5 readily implies that $\mu$ is necessary of the form $\mu = n = 0, 1, 2, \cdots$ and $j \geq -n$. In such case, the $\gamma^j_{\mu,j} = C_j(I)$ are arbitrary in $\mathbb{H} = \mathcal{C}_\mu$ for $\mu$ being real. Moreover, we have

$$\|f\|^2_{\mathbb{H}} = \pi \sum_{j=-n}^{+\infty} \frac{n!(j!)^2}{(n+j)!} \int_{\mathbb{S}} |C_j(I)|^2 d\sigma(I).$$

This yields the growth condition (5.25) and the proof is completed. \hfill $\square$

The following result describes the fact that the elements of $F^2_{n}$ can be expanded as series of the quaternionic Hermite polynomials $H^Q_{j,n}(q, \overline{q})$.

**Corollary 5.8.** The space $F^2_{n}$ contains the quaternionic Hermite polynomials defined by (1.5). Moreover, every element $f$ belonging to $F^2_{n}(\mathbb{H})$ can be expanded as

$$f(q) = \sum_{j=-n}^{+\infty} \frac{(-1)^j j!}{(n+j)!} H^Q_{n+j,n}(q, \overline{q}) C_j(I) \quad (5.26)$$

for some sliced quaternionic constants $C_j(I)$ displaying the growth condition (5.25).
Proof. This lies in the fact that the involved confluent hypergeometric function is connected to the quaternionic Hermite polynomials given through [14]. In fact, we have

\[ q^jM \left( \frac{-n}{|j| + 1} \left| q \right|^2 \right) = \frac{(-1)^n j^j}{(n + j)!} H^{Q}_{n+j,n}(q, \bar{q}). \]  

(5.27)

Therefore, the expression of \( f(q) \) given through (5.24) reduces further to (5.26) with the same growth condition (5.25).

\[ \square \]

5.4 Connection to S–polyregular Bargmann spaces of first kind. By Corollary 5.8, the space \( F^2_{n} \) can be reexpressed as follows

\[
F^2_{n} = \left\{ f(q) = \sum_{j=0}^{+\infty} \frac{(-1)^n j^j}{j!} H^{Q}_{j,n}(q, \eta) C_j(I), \pi \sum_{j=-n}^{+\infty} \frac{n!(j)!}{(n+j)!} \left( \int_S |C_j(I)|^2 d\sigma(I) \right) < +\infty \right\}.
\]

It reduces further to \( SR^2_{2,n} \) when the \( C_j(I) \) are assumed to be constants functions on \( S \), \( C_j(I) = C_j \). This is to say that \( SR^2_{2,n} \) are realized as

\[
SR^2_{2,n} = F^2_{n} \cap \left( \bigcap_{j=0}^{\infty} \text{Ker}(\nabla^{j,n}_{I,0}) \right)
\]

where \( \nabla^{j,n}_{I,0} \) acts as in (3.1). More particularly, by taking \( n = 0 \), the previous growth condition reads simply

\[
\left( \sum_{j=0}^{+\infty} j! |C_j|^2 \right) \text{Vol}(S) < +\infty.
\]

Comparing this to the sequential characterization of the slice hyperholomorphic Bargmann space \( F^{2}_{\text{slice}} \) given by Proposition 3.11 in [5], we see that \( F^{2}_{\text{slice}} = F^2_{n} \cap \left( \bigcap_{j=0}^{\infty} \text{Ker}(\nabla^{j,0}_{I,0}) \right) \subset F^2_{n} \).

6 Concluding Remarks: Full S–polyregular Bargmann spaces

Motivated by Theorem 4.2 in [14] asserting that the quaternionic Hermite polynomials \( (H^{Q}_{j,k})_{j,k} \) form a sliced basis of the Hilbert space \( L^2(\mathbb{H}; e^{-|q|^2} d\lambda) \), equipped with the scalar product

\[
(f, g)_{\mathbb{H}} = \int_{\mathbb{H}} f(q)g(q)e^{-|q|^2} d\lambda(q),
\]

(6.1)

we define \( SR^2_{n,\text{full}} \) to be the space of S–polyregular functions spanned by the quaternionic Hermite polynomials \( H^{Q}_{j,n} \), for varying \( j = 0, 1, 2, \ldots \), and belonging to \( L^2(\mathbb{H}; e^{-|q|^2} d\lambda) \). Then, we have

\[
(f, g)_{\mathbb{H}} = \int_S \langle f, g \rangle_{C_I} d\sigma(I)
\]

and subsequently, the space \( SR^2_{n,\text{full}} \) can be described as

\[
SR^2_{n,\text{full}} = \left\{ \sum_{j=0}^{+\infty} H^{Q}_{j,n}(q, \eta) C_j(I_q); C_j : S \rightarrow \mathbb{H}, \pi n! \sum_{j=0}^{\infty} \int_S |C_j(I)|^2 d\sigma(I) < +\infty \right\}
\]

(6.2)
which is exactly the sequential characterization of $L^2$-eigenspace $F^2_n$. The particular case of $n = 0$ corresponds to the full hyperholomorphic Bargmann space

$$\mathfrak{B}_{f_{\text{full}}}^2 := \mathcal{S}\mathcal{R} \cap L^2(\mathbb{H}; e^{-|q|^2} d\lambda)$$

(6.3)

defined as the right quaternionic Hilbert space of all slice regular functions that are $e^{-|q|^2} d\lambda$-square integrable on $\mathbb{H}$. This lies on the fact $F^2_{f_{\text{full}}}$ can decribed as the space of functions $f(q) = \sum_{j=0}^{\infty} q^j C_j(I)$ satisfying

$$\|f\|_H^2 = \pi \sum_{j=0}^{+\infty} j! \left( \int_{\mathbb{S}} |C_j(I)|^2 d\sigma(I) \right) < +\infty.$$ 

Thus, it is not difficult to prove that the space $\mathcal{S}\mathcal{R}_{n, f_{\text{full}}}^2$ are right quaternionic Hilbert space. We call it here the full S–polyregular Bargmann spaces of second kind. The quaternionic Hermite polynomials $H^{Q}_{j,n}$, for varying $j = 0, 1, 2, \ldots$, constitute an orthogonal 'sliced' basis of it.

Acknowledgements. This work was initialized with the second–named author in 2016 and a part of it figure her Ph.D thesis (January 2017). Another part of this investigation was completed during the third–named author’s visit to departimenti di Mathematica di Politecnico di Milano May - June 2017 (see arXiv:1707.01674). He would like to express his gratitude to Professor I.M. Sabadini for hospitality and many interesting discussions. The present version is an ameliorated version that we have finalized with the Ph.D. Student Abdelhadi Benahmadi. It was the subject of a recent talk in 'International Conference on Advances in Applied Mathematics ICAAM - 2018, Sousse, Tunisia'.

The research work of A.G. was partially supported by a grant from the Simons Foundation.

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