SYMMETRIC TENSOR REPRESENTATIONS, QUASIMODULAR FORMS, AND WEAK JACOBI FORMS

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Abstract. We establish a correspondence between vector-valued modular forms with respect to a symmetric tensor representation and quasimodular forms. This is carried out by first obtaining an explicit isomorphism between the space of vector-valued modular forms with respect to a symmetric tensor representation and the space of finite sequences of modular forms of certain type. This isomorphism uses Rankin-Cohen brackets and extends a result of Kuga and Shimura, who considered the case of vector-valued modular forms of weight two. We also obtain a correspondence between such vector-valued modular forms and weak Jacobi forms.

1. Introduction and statement of main results

It is well-known that the integrals of certain vector-valued differential forms attached to automorphic forms with respect to a Fuchsian group play an important role in the arithmetic theory of modular correspondences, and the study of such integrals was first introduced by Eichler [4] and Shimura [13].

In [9] Kuga and Shimura studied a correspondence between vector-valued differential forms, or equivalently vector-valued modular forms of weight two, and certain finite sequences of scalar-valued modular forms. One of our goals in this paper is to extend such a correspondence to the case of vector-valued modular forms of arbitrary weight. We then use this correspondence to obtain an isomorphism between the space of vector-valued modular forms with respect to a symmetric tensor representation and that of quasimodular forms. Quasimodular forms generalize classical modular forms and were first introduced by Kaneko and Zagier in [8]. It appears naturally in various places (see [6, 7, 10], for instance). In fact, an explicit isomorphism can be obtained between the space of finite sequences of modular forms of certain type and the space of vector valued modular form with respect to a symmetric tensor representation. This isomorphism is defined by using Rankin-Cohen brackets for vector-valued modular forms. On the other hand, finite sequences of modular forms of the same type are known to correspond to certain quasimodular forms (see [12]) and also to weak Jacobi forms (cf. [5]).

In this paper we establish mutual correspondences among vector-valued modular forms with respect to a symmetric tensor representation, finite sequences of modular forms, quasimodular forms, and weak Jacobi forms.

Given a discrete subgroup Γ of SL(2, R) and integers k and n with n ≥ 0, let $M_k(Γ)$ and $\tilde{M}_k^{n+1}(Γ, ρ_n)$ be the space of modular forms of weight k for Γ and the
space of vector-valued modular forms of weight \( k \) for \( \Gamma \) with respect to the \( n \)-th symmetric tensor representation \( \rho_n \), respectively. We also denote by \( QP^m_k(\Gamma) \) and \( QM^m_k(\Gamma) \) with \( m \geq 0 \) the spaces of quasimodular polynomials of degree at most \( m \) and quasimodular forms depth at most \( m \), respectively, of weight \( k \) for \( \Gamma \) (see Section 1). Our main results involve establishing certain isomorphisms contained in the following theorems, which will be proved in Section 7.

**Theorem 1.1.** (i) Given an integer \( \ell \) with \( 0 \leq \ell \leq n \), the formula
\[
V_{k,n,\ell}(g_{\ell}) = [g_{\ell}, \hat{v}_n]_{n-\ell}^{(k-n+2\ell,-n)}
\]
for \( g_{\ell} \in M_{k-n+2\ell}(\Gamma) \) determines a complex linear map
\[
V_{k,n,\ell} : M_{k-n+2\ell}(\Gamma) \to \widehat{M}_{n+1}^n(\Gamma, \rho_n),
\]
where \( \hat{v}_n(z) = f(z^n, \ldots, z, 1) \in \mathbb{C}^{n+1} \), and \([g_{\ell}, \hat{v}_n]_{n-\ell}^{(k-n+2\ell,-n)}\) denotes the \((n-\ell)\)-th Rankin-Cohen bracket of \( g_{\ell} \) and \( \hat{v}_n \) given by (2.5). (ii) If \( \hat{G}_{k-n+2\ell}(\Gamma, \rho_n) = \text{Im} (V_{k,n,\ell}) \) denotes the image of the map (1.2) for \( 0 \leq \ell \leq n \), then \( \widehat{M}_{n+1}^n(\Gamma, \rho_n) \) has a direct sum decomposition of the form
\[
\widehat{M}_{n+1}^n(\Gamma, \rho_n) = \bigoplus_{\ell=0}^n \hat{G}_{k-n+2\ell}(\Gamma, \rho_n).
\]
(iii) The map (1.2) determines an isomorphism
\[
M_{k-n+2\ell}(\Gamma) \cong \hat{G}_{k-n+2\ell}(\Gamma, \rho_n)
\]
for each \( \ell \in \{0, 1, \ldots, n\} \); hence we obtain a canonical isomorphism
\[
\widehat{M}_{n+1}^n(\Gamma, \rho_n) \cong \bigoplus_{\ell=0}^n M_{k-n+2\ell}(\Gamma)
\]
between vector-valued modular forms with respect to \( \rho_n \) and finite sequences of modular forms.

We can also consider an explicit linear map from vector-valued modular forms to scalar-valued modular forms in (1.2) by using Rankin-Cohen brackets for vector-valued modular forms as in the next theorem.

**Theorem 1.2.** Given integers \( k \) and \( n \) with \( n \geq 0 \), the formula
\[
W_{k,n}(F) = (\hat{u}_n, F)]_{0}^{(-n,k)}, (\hat{u}_n, F)]_{1}^{(-n,k)}, \ldots, (\hat{u}_n, F)]_{n}^{(-n,k)}
\]
for \( F \in \widehat{M}_{n+1}^n(\Gamma, \rho_n) \) determines a complex linear map
\[
W_{k,n} : \widehat{M}_{n+1}^n(\Gamma, \rho_n) \to \bigoplus_{\ell=0}^n M_{k-n+2\ell}(\Gamma),
\]
where the brackets are the Rankin-Cohen brackets in (2.3) and \( \hat{u}_n = f(1, (-z), \ldots, (-z)^n) \).

Given an element \( (f_0, f_1, \ldots, f_n) \in \bigoplus_{\ell=0}^n M_{k-n+2\ell}(\Gamma) \), the corresponding polynomial \( \sum_{r=0}^n f_r(z)X^r \) belongs to \( MP^n_{k-n}(\Gamma) \), and this correspondence determines an isomorphism
\[
\bigoplus_{\ell=0}^n M_{k-n+2\ell}(\Gamma) \cong MP^n_{k-n}(\Gamma).
\]
On the other hand, it is also known that

\[ QP^n_j(\Gamma) \cong MP^n_{j-2n}(\Gamma) \]

for \( j > 2n \) (see Section 4). From these isomorphisms and \( (1.3) \) we obtain the following result.

**Theorem 1.3.** There is a canonical isomorphism.

(1.5) \[ \hat{M}^{n+1}_k(\Gamma, \rho_n) \cong QP^n_{k+n}(\Gamma) \]

between vector-valued modular forms and quasimodular polynomials for \( k > n \).

The next theorem gives an expression of the formula for the vector-valued function corresponding to a quasimodular form via the isomorphism \( (1.5) \) in terms of derivatives of \( \hat{v}_n \) and the coefficients of the given quasimodular form.

**Theorem 1.4.** Let \( F \) be the complex algebra of holomorphic functions on the Poincaré upper half plane \( \mathcal{H} \). If \( F(z,X) \) is a polynomial of degree at most \( n \) over the ring of holomorphic functions on \( \mathcal{H} \) given by

\[ F(z,X) = \sum_{r=0}^{n} f_r(z)X^r, \]

we set

(1.6) \[ \mathcal{U}_n(F(z,X)) = \sum_{\ell=0}^{n} (-1)^n(n-\ell)!D^\ell(\hat{v}_n)f_\ell, \]

where \( \hat{v}_n \) is as in Theorem 1.1 and \( D = d/dz \). Then this formula determines an isomorphism

\[ \mathcal{U}_n : QP^n_{k+n}(\Gamma) \approx \hat{M}^{n+1}_k(\Gamma, \rho_n) \]

of complex vector spaces for each \( k > n \).

**Example 1.5.** We consider the weight 12 cusp form

(1.7) \[ \Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} \]

for \( \Gamma(1) = SL(2,\mathbb{Z}) \) with \( q = e^{2\pi i z} \). Using \( (1.1) \) for \( n = 2, k = 14, \ell = 0 \) and \( g_0 = \Delta \), we see that

(1.8) \[ \mathcal{V}_{14,2,0}(\Delta)(z) = \Delta''(z) \left( \frac{z^2}{i} \right) + 13\Delta'(z) \left( \frac{2a}{i} \right) + 78\Delta(z) \left( \frac{b}{i} \right) \in \hat{M}^{3}_{14}(\Gamma(1), \rho_2). \]

On the other hand, for \( n = 2 \) the formula \( (1.6) \) determines a vector-valued modular form

(1.9) \[ \mathcal{U}_2(F(z,X)) = 2\tilde{v}_2(z)f_0(z) + (D\tilde{v}_2)(z)f_1(z) + (D^2\tilde{v}_2)(z)f_2(z) \in \hat{M}^{3}_{14}(\Gamma(1), \rho_2) \]

corresponding to a quasimodular polynomial \( F(z,X) \in QP^2_{16}(\Gamma) \). By comparing \( (1.8) \) and \( (1.9) \) we obtain the quasimodular polynomial

\[ (\mathcal{U}_2^{-1} \circ \mathcal{V}_{14,2,0})(\Delta)(z,X) = \frac{1}{2}\Delta''(z) + 13\Delta'(z)X + 78\Delta(z)X^2 \in QP^2_{16}(\Gamma). \]
2. Vector-valued modular forms

In this section we review the definition of vector-valued modular forms and introduce Rankin-Cohen brackets for such modular forms.

Let \( \mathcal{H} \) be the Poincaré upper half plane on which the group \( SL(2, \mathbb{R}) \) acts as usual by linear fractional transformations. Thus, if \( z \in \mathcal{H} \) and \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL(2, \mathbb{R}) \), we have

\[
\gamma z = \frac{az + b}{cz + d}.
\]

For such \( z \) and \( \gamma \), we set

\[
J(\gamma, z) = cz + d, \quad K(\gamma, z) = \frac{c}{cz + d}.
\]

Then the resulting maps \( J, K : SL(2, \mathbb{R}) \times \mathcal{H} \to \mathbb{C} \) satisfy

\[
J(\gamma \gamma', z) = J(\gamma, \gamma' z) J(\gamma', z),
\]
\[
K(\gamma, \gamma' z) = J(\gamma', z)^2 (\overline{K(\gamma \gamma', z)} - K(\gamma', z))
\]
for all \( z \in \mathcal{H} \) and \( \gamma, \gamma' \in SL(2, \mathbb{R}) \).

Let \( F \) be the ring of holomorphic functions on \( \mathcal{H} \), and let \( \hat{F}^n \) with \( n > 0 \) be the space of \( \mathbb{C}^n \)-valued holomorphic functions on \( \mathcal{H} \). Throughout this paper, we shall use \( t(\cdot) \) to denote the transpose of the matrix \( (\cdot) \). In particular, if \( x_1, \ldots, x_\ell \in \mathbb{C} \), the corresponding column vector belonging to \( \mathbb{C}^\ell \) will be denoted by \( t(x_1, \ldots, x_\ell) \). Let \( \rho : \Gamma \to GL(n, \mathbb{R}) \) be a representation of a discrete subgroup \( \Gamma \) of \( SL(2, \mathbb{R}) \), and let \( \rho^* : \Gamma \to GL(n, \mathbb{R}) \) be the contragredient, or the inverse transpose, of \( \rho \), so that

\[
\rho^*(\gamma) = t(\rho(\gamma))^{-1}
\]
for all \( \gamma \in SL(2, \mathbb{R}) \). If \( f \in F \), \( \hat{f} \in \hat{F}^n \), \( \gamma \in SL(2, \mathbb{R}) \) and \( \ell \in \mathbb{Z} \), then we set

\[
(f |_\ell \gamma)(z) = J(\gamma, z)^{-\ell} f(\gamma z), \quad (\hat{f} |_\ell \gamma)(z) = J(\gamma, z)^{-\ell} \hat{f}(\gamma z)
\]
for all \( z \in \mathcal{H} \). We now modify the usual definition of modular forms by suppressing the growth condition at the cusps.

**Definition 2.1.** (i) Given an integer \( k \), a **modular form of weight** \( k \) **for** \( \Gamma \) is an element \( f \in F \) satisfying

\[
 f |_k \gamma = f
\]
for all \( \gamma \in \Gamma \).

(ii) An element \( \hat{f} \in \hat{F}^n \) is a **vector-valued modular form of weight** \( k \) **for** \( \Gamma \) **with respect to** \( \rho \) if it satisfies

\[
\hat{f} |_k \gamma = \rho(\gamma) \hat{f}
\]
for all \( \gamma \in \Gamma \).

We shall denote by \( M_k(\Gamma) \) and \( \hat{M}_k(\Gamma, \rho) \) the spaces of modular forms for \( \Gamma \) and vector-valued modular forms for \( \Gamma \) with respect to \( \rho \), respectively, of weight \( k \).

Rankin-Cohen brackets are bilinear maps of scalar-valued modular forms (see e.g. [3]), and they can be extended to the vector-valued case as described in the following lemma.
Lemma 2.2. (i) Given nonnegative integers \( \lambda_1, \lambda_2, w \) and vector-valued modular forms \( \phi_1 \in \tilde{M}^{n_1}_{\lambda_1}(\Gamma, \rho_1) \) and \( \phi_2 \in \tilde{M}^{n_2}_{\lambda_2}(\Gamma, \rho_2) \), we set

\[
[\phi_1, \phi_2]_{w}^{(\lambda_1, \lambda_2)}(z) = \sum_{r=0}^{w} (-1)^r \binom{\lambda_1 + w - 1}{w-r} \binom{\lambda_2 + w - 1}{r} \phi_1^{(r)}(z) \otimes \phi_2^{(w-r)}(z)
\]

for all \( z \in \mathcal{H} \). Then \( [\phi_1, \phi_2]_{w}^{(\lambda_1, \lambda_2)} \) is a vector-valued modular form belonging to \( \tilde{M}^{n_1 + n_2}_{\lambda_1 + \lambda_2 + 2w}(\Gamma, \rho_1 \otimes \rho_2) \).

(ii) Given vector-valued modular forms \( \phi \in \tilde{M}^n_\alpha(\Gamma, \rho) \) and \( \psi \in \tilde{M}^n_\beta(\Gamma, \rho^*) \) with \( \alpha, \beta \geq 0 \) and a nonnegative integer \( w \), we set

\[
[[\phi, \psi]]_{w}^{(\alpha, \beta)}(z) = \sum_{r=0}^{w} (-1)^r \binom{\alpha + w - 1}{w-r} \binom{\beta + w - 1}{r} t(\phi^{(r)}(z)) \psi^{(w-r)}(z)
\]

for all \( z \in \mathcal{H} \), where \( \rho^* \) is the contragredient of \( \rho \). Then \( [[\phi, \psi]]_{w}^{(k,f)} \) is a modular form belonging to \( M_{k+f+2w}(\Gamma) \).

Proof. See [11].

From Lemma 2.2 we obtain the bilinear maps

\[
[ \ , \ ]_{w}^{(\lambda_1, \lambda_2)} : \tilde{M}^{n_1}_{\lambda_1}(\Gamma, \rho_1) \times \tilde{M}^{n_2}_{\lambda_2}(\Gamma, \rho_2) \to M^{n_1 + n_2}_{\lambda_1 + \lambda_2 + 2w}(\Gamma, \rho_1 \otimes \rho_2),
\]

which may be regarded as Rankin-Cohen brackets for vector-valued modular forms. We can consider the bracket given by (2.2) when one of the modular forms is scalar valued by considering it as a 1-dimensional vector-valued modular form with respect to the trivial representation. Thus, for example, we have a bilinear map of the form

\[
[ \ , \ ]_{w}^{(\lambda, \mu)} : M_{\lambda}(\Gamma) \times \tilde{M}^{n}_{\mu}(\Gamma, \rho) \to M^{n}_{\lambda+\mu+2w}(\Gamma, \rho),
\]

given by

\[
[f, \phi]_{w}^{(\lambda, \mu)} = \sum_{r=0}^{w} (-1)^r \binom{\lambda + w - 1}{w-r} \binom{\mu + w - 1}{r} f^{(r)}(w-r)
\]

for \( f \in M_{\lambda}(\Gamma) \) and \( \phi \in \tilde{M}^{n}_{\mu}(\Gamma, \rho) \).

Remark 2.3. Although the proof of Lemma 2.2 in [11] was given for vector-valued modular forms of nonnegative weight, we shall use the same formulas to consider brackets for modular forms of negative weight by using

\[
\binom{k}{r} = \frac{k(k-1) \cdots (k-r+1)}{r!}
\]

for \( k, r \in \mathbb{Z} \) with \( r > 0 \). In particular, the bracket in the statement of Theorem 1.1 is the extended version of the Rankin-Cohen bracket in (2.3). Similarly, the bracket in (2.3) can also be extended to the negative weight case, and such brackets are used in Theorem 1.2.
3. Symmetric tensor representations

In this section we introduce certain vector-valued modular forms with respect to symmetric tensor representations and discuss some of their properties.

We fix a positive integer $n$, and denote by

$$\rho_n : SL(2, \mathbb{R}) \to GL(n + 1, \mathbb{R})$$

the $n$-th symmetric tensor representation of $SL(2, \mathbb{R})$ given by

$$\rho_n(\gamma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}^n = \left( \gamma \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^n$$

for all $\gamma \in SL(2, \mathbb{R})$, where

$$\left( \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^n = t(z_1^n, z_1^{n-1}z_2, \ldots, z_1z_2^{n-1}, z_2^n) \in \mathbb{C}^{n+1}$$

with $(z_1, z_2) \in \mathbb{C}^2$.

**Proposition 3.1.** Let $\Gamma$ be a discrete subgroup of $SL(2, \mathbb{R})$, and let $\hat{v}_n, \hat{u}_n \in \hat{F}^n$ be vector-valued functions on $\mathcal{H}$ given by

$$\hat{v}_n(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}^n, \quad \hat{u}_n(z) = \begin{pmatrix} 1 \\ -z \end{pmatrix}^n$$

for all $z \in \mathbb{C}$. Then $\hat{v}_n$ and $\hat{u}_n$ are vector-valued modular forms with

$$\hat{v}_n \in \hat{M}^{n+1}_n(\Gamma, \rho_n), \quad \hat{u}_n \in \hat{M}^{n+1}_n(\Gamma, \rho^*_n),$$

where $\rho^*_n : SL(2, \mathbb{R}) \to GL(n + 1, \mathbb{R})$ is the contragredient of $\rho_n$.

**Proof.** For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, using (3.1), we see that

$$\rho_n(\gamma) \hat{v}_n(z) = \rho_n(\gamma) \begin{pmatrix} z \\ 1 \end{pmatrix}^n = \left( \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} \right)^n = \begin{pmatrix} az + b \\ cz + d \end{pmatrix}^n = (cz + d)^n \begin{pmatrix} \gamma z \\ 1 \end{pmatrix} = \mathfrak{J}(\gamma, z)^n \hat{v}_n(\gamma z).$$

for all $z \in \mathcal{H}$; hence we obtain $\hat{v}_n \in \hat{M}^{n+1}_n(\Gamma, \rho_n)$. On the other hand, we have

$$\rho^*_n(\gamma) \hat{u}_n(z) = \rho_n(t^{-1} \gamma^{-1}) \hat{u}_n(z) = \left( t^{-1} \gamma^{-1} \begin{pmatrix} 1 \\ -z \end{pmatrix} \right)^n$$

$$= \left( \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ -z \end{pmatrix} \right)^n = \left( -cz + d \right)^n = \mathfrak{J}(\gamma, z)^n \hat{u}_n(\gamma z),$$

which show that $\hat{u}_n \in \hat{M}^{n+1}_n(\Gamma, \rho^*_n)$. □
We now introduce the matrix-valued function

\[ L_n : \mathcal{H} \to GL(n + 1, \mathbb{C}) \]

defined by

\[ L_n(z) = \rho_n \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \]

for all \( z \in \mathcal{H} \). Then for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \) it is known (see p.266 in [9]) that

\[ L_n(\gamma z)^{-1} \rho_n(\gamma)L_n(z) = \rho_n \begin{pmatrix} \tilde{J} & 0 \\ c & \tilde{J} \end{pmatrix} = J^n A, \]

where \( A \) is the \((n + 1) \times (n + 1)\) lower triangular matrix whose \((r + 1)\)-th row is of the form

\[ (c^r \tilde{J}^{n-r}, (\tilde{r}) c^r-1 \tilde{J}^{n-r-1}, \ldots, (r-1) c^0 \tilde{J}^{n-r}, \tilde{J}^{n-r}, 0, \ldots, 0) \]

for \( 0 \leq r \leq n \) with \( \tilde{J} = \tilde{J}(\gamma, z) \) in (2.1).

**Lemma 3.2.** Let \( \omega \in \widetilde{M}^{n+1}_k(\Gamma, \rho_n) \) be a vector-valued modular form of the form

\[ \omega(z) = L_n(z) \cdot (f_0(z), f_1(z), \ldots, f_n(z)) \]

for all \( z \in \mathcal{H} \), where \( f_0, \ldots, f_n \in F \). If \( r \) is an integer with \( 0 \leq r \leq n \) such that \( f_\ell \equiv 0 \) for all \( \ell < r \), then \( f_r \) is a modular form belonging to \( M_{k-n+2r}(\Gamma) \).

**Proof.** Given \( \omega \in \widetilde{M}^{n+1}_k(\Gamma, \rho_n) \) as in (3.5) and an element \( \gamma \in \Gamma \), using (3.4) and the relation \( \omega|_k \gamma = \rho(\gamma) \omega \), we see that

\[ \tilde{J}(\gamma, z)^{-k} L_n(\gamma z) \cdot (f_0(\gamma z), f_1(\gamma z), \ldots, f_n(\gamma z)) = \rho_n(\gamma)L_n(z) \cdot (f_0(z), f_1(z), \ldots, f_n(z)) = L_n(\gamma z) \tilde{J}(\gamma, z)^n A \cdot (f_0(z), f_1(z), \ldots, f_n(z)); \]

hence we have the identity

\[ \tilde{J}(\gamma, z)^{-k} \cdot (f_0(\gamma z), f_1(\gamma z), \ldots, f_n(\gamma z)) = \tilde{J}(\gamma, z)^n A \cdot (f_0(z), f_1(z), \ldots, f_n(z)); \]

of vectors in \( \mathbb{C}^{n+1} \). Thus by comparing the \((r + 1)\)-th entries we obtain

\[ \tilde{J}(\gamma, z)^{-k} f_r(\gamma z) = \tilde{J}(\gamma, z)^{n-2r} f_r(z), \]

and therefore the lemma follows. \( \square \)

If \( \lambda, w, r \in \mathbb{Z} \), we set

\[ \alpha_{w,r}^\lambda = (-1)^r \binom{\lambda + w - 1}{w - r} \binom{w - n - 1}{r}. \]

Then from (2.5) we see that

\[ [f, \hat{v}_n]^{(\lambda, -n)} = \sum_{r=0}^w \alpha_{w,r}^\lambda f^{(r)} \hat{v}_n^{(w-r)} \]

for \( f \in M_\lambda(\Gamma) \).
Lemma 3.3. If \( g_j \in M_{k-n+2j}(\Gamma) \) with \( 0 \leq j \leq 2n \), we have
\[
[g_j(z), \hat{v}_n(z)]_{n-j}^{(k-n+2j,-n)} = L_n(z) \cdot ^t(g_0^L(z), g_1^L(z), \ldots, g_n^L(z))
\]
for all \( z \in \mathcal{H} \), where
\[
 g^L_\ell = \begin{cases} 
 0 & \text{for } 0 \leq \ell \leq j - 1; \\
 (n - \ell)! \alpha_{n-j,\ell-j}^k D^{\ell-j} g_j & \text{for } j \leq \ell \leq n
\end{cases}
\]
with \( D = d/dz \) and \( \alpha_{k-n+2j,\ell-j}^k \) is as in (3.6).

Proof. A direct computation shows that
\[
(D^n \hat{v}_n, D^{n-1} \hat{v}_n, \ldots, \hat{v}_n) = L_n(z) \cdot \diag [n!, (n - 1)!], \ldots, 1],
\]
where \( \diag [\cdots] \) denotes the \( (n + 1) \times (n + 1) \) diagonal matrix having \( [\cdots] \) as its diagonal entries. Thus we see that
\[
L_n(z) \cdot ^t(g_0^L(z), g_1^L(z), \ldots, g_n^L(z))
\]
\[
= (D^n \hat{v}_n, D^{n-1} \hat{v}_n, \ldots, \hat{v}_n) \cdot \diag [n!, (n - 1)!], \ldots, 1]^{-1}
\]
\[
\times ^t(g_0^L(z), g_1^L(z), \ldots, g_n^L(z))
\]
\[
= \sum_{i=0}^n \frac{1}{(n - i)!} (D^{n-i} \hat{v}_n(z)) g_i^L(z)
\]
\[
= \sum_{i=j}^n \alpha_{n-j,\ell-j}^k (D^{n-i} \hat{v}_n(z)) (D^{\ell-j} g_j)(z)
\]
\[
= \sum_{\ell=0}^{n-j} \alpha_{n-j,\ell}^k (D^{n-\ell-j} \hat{v}_n(z)) (D^\ell g_j)(z)
\]
for all \( z \in \mathcal{H} \). On the other hand, from (3.7) we see that
\[
[g_j, \hat{v}_n]_{n-j}^{(k-n+2j,-n)} = \sum_{\ell=0}^{n-j} \alpha_{n-j,\ell}^k (D^\ell g_j)(D^{n-\ell-j} \hat{v}_n);
\]
hence the lemma follows. \( \square \)

4. Quasimodular and modular polynomials

In this section we describe a correspondence between quasimodular and modular polynomials as well as the identification of quasimodular forms with quasimodular polynomials. By combining these isomorphisms with the main theorems stated in Section 1, we establish a correspondence between vector-valued modular forms and quasimodular forms.

We fix a nonnegative integer \( m \), and denote by \( \mathcal{F}_m[X] \) the space of polynomials over the ring \( \mathcal{F} \) of holomorphic functions on \( \mathcal{H} \) of degree at most \( m \). Given \( \lambda \in \mathbb{Z} \) and a polynomial
\[
\Phi(z, X) = \sum_{r=0}^m \phi_r(z) X^r \in \mathcal{F}_m[X],
\]
we set

\[(4.1) \quad (\Phi |^X \gamma)(z, X) = \sum_{r=0}^{m}(\phi_r |^{\lambda+2r} \gamma)(z)X^r,\]

\[(\Phi \parallel \lambda \gamma)(z, X) = \mathcal{J}(\gamma, z)^{-\lambda}\Phi(\gamma z, \mathcal{J}(\gamma, z)^2(X - \mathcal{R}(\gamma, z)))\]

for all \(z \in \mathcal{H}\) and \(\gamma \in SL(2, \mathbb{R})\). Then it can be shown that the two operations \(|^X\) and \(\parallel\) determine right actions of \(SL(2, \mathbb{R})\) on \(\mathcal{F}_m[X]\).

**Definition 4.1.** (i) An element \(F(z, X) \in \mathcal{F}_m[X]\) is a modular polynomial for \(\Gamma\) of weight \(\lambda\) and degree at most \(m\) if it satisfies

\[F |^X \gamma = F\]

for all \(\gamma \in \Gamma\).

(ii) An element \(\Phi(z, X) \in \mathcal{F}_m[X]\) is a quasimodular polynomial for \(\Gamma\) of weight \(\lambda\) and degree at most \(m\) if it satisfies

\[\Phi \parallel \lambda \gamma = \Phi\]

for all \(\gamma \in \Gamma\).

(iii) An element \(f \in \mathcal{F}\) is a quasimodular form for \(\Gamma\) of weight \(\lambda\) and depth at most \(m\) if there are functions \(f_0, \ldots, f_m \in \mathcal{F}\) such that

\[(4.2) \quad (f |^X \gamma)(z) = \sum_{r=0}^{m} f_r(z)\mathcal{R}(\gamma, z)^r\]

for all \(z \in \mathcal{H}\) and \(\gamma \in \Gamma\).

We denote by \(MP^m_\lambda(\Gamma)\) and \(QP^m_\lambda(\Gamma)\) the spaces of modular and quasimodular, respectively, polynomials for \(\Gamma\) of weight \(\lambda\) and degree at most \(m\). We also denote by \(QM^m_\lambda(\Gamma)\) the space of quasimodular forms for \(\Gamma\) of weight \(\lambda\) and depth at most \(m\).

**Remark 4.2.** (i) It follows from (4.1) and Definition 4.1(i) that a polynomial \(F(z, X) = \sum_{r=0}^{m} f_r(z)X^r\) is a modular polynomial belonging to \(MP^m_\lambda(\Gamma)\) if and only if \(f_r(z) \in M^{\lambda+2r}_{\lambda}(\Gamma)\) for each \(r \in \{0, 1, \ldots, m\}\).

(ii) If \(\gamma \in \Gamma\) is the identity matrix in (4.2), then \(\mathcal{R}(\gamma, z) = 0\). Thus, if \(f \in QM^m_\lambda(\Gamma)\) satisfies (4.2), we see that

\[(4.3) \quad f = f_0.\]

On the other hand, if \(m = 0\), the relation (4.2) can be written in the form

\[f |^X \gamma = f_0 = f;\]

hence \(QM^0_\lambda(\Gamma)\) coincides with the space \(M_\lambda(\Gamma)\) of modular forms of weight \(\lambda\) for \(\Gamma\).

(iii) If (4.2) is satisfied for another set of functions \(\hat{f}_0, \ldots, \hat{f}_m \in \mathcal{F}\), then we have

\[\sum_{r=0}^{m} (\hat{f}_r(z) - f_r(z))\mathcal{R}(\gamma, z)^r = 0\]

for all \(\gamma\) belonging to the infinite set \(\Gamma\); hence it follows that \(\hat{f}_r = f_r\) for each \(r\). Thus we see that the quasimodular form \(f\) determines the associated functions \(f_0, \ldots, f_m \in \mathcal{F}\) uniquely.
Given $\lambda, m \in \mathbb{Z}$ with $\lambda > 2m \geq 0$ and a polynomial $F(z, X) = \sum_{r=0}^{m} f_r(z)X^r \in \mathcal{F}_m[X]$, we set

\begin{align}
(\Lambda^m_\lambda F)(z, X) &= \sum_{r=0}^{m} f^\Lambda_r(z)X^r, \\
(\Xi^m_\lambda F)(z, X) &= \sum_{r=0}^{m} f^\Xi_r(z)X^r,
\end{align}

where

\begin{align*}
f^\Lambda_k &= \frac{1}{k!} \sum_{r=0}^{m-k} \frac{1}{r!(\lambda - 2k - r - 1)!} f^{(r)}_{m-k-r}, \\
f^\Xi_k &= (\lambda + 2k - 2m - 1) \sum_{r=0}^{k} \frac{(-1)^r}{r!} (m - k + r)(2k + \lambda - 2m - r - 2)! f^{(r)}_{m-k+r}
\end{align*}

for $0 \leq k \leq m$.

**Theorem 4.3.** The complex linear endomorphisms $\Lambda^m_\lambda$ and $\Xi^m_\lambda$ of $\mathcal{F}_m[X]$ defined by (4.4) and (4.5) induce isomorphisms

\begin{align}
\Lambda^m_\lambda : MP^m_{\lambda-2m}(\Gamma) &\rightarrow QP^m_\lambda(\Gamma), \\
\Xi^m_\lambda : QP^m_\lambda(\Gamma) &\rightarrow MP^m_{\lambda-2m}(\Gamma)
\end{align}

with

\begin{align*}
(\Xi^m_\lambda)^{-1} &= \Lambda^m_\lambda
\end{align*}

for each $\lambda > 2m$.

**Proof.** See [12].

If $f(z) \in QM^m_\lambda(\Gamma)$ is a quasimodular form satisfying (4.2), we define the corresponding polynomial $(Q^m_\lambda f)(z, X) \in \mathcal{F}_m[X]$ by

\begin{align}
(Q^m_\lambda f)(z, X) &= \sum_{r=0}^{m} f_r(z)X^r
\end{align}

for all $z \in \mathcal{H}$, which is well-defined by Remark 4.2(iii). Then it is known (see [12]) that the resulting complex linear map

\begin{align}
Q^m_\lambda : QM^m_\lambda(\Gamma) &\rightarrow QP^m_\lambda(\Gamma)
\end{align}

is an isomorphism whose inverse is given by

\begin{align*}
(Q^m_\lambda)^{-1} F(z, X) &= F(z, 0)
\end{align*}

for $F(z, X) \in QP^m_\lambda(\Gamma)$. 

5. Weak Jacobi forms

Weak Jacobi forms generalize usual Jacobi forms (cf. [5]), and they appear, for example, in the theory of elliptic genera and quantum field theory (see e.g. [14]). In this section we describe connections between vector-valued modular forms and weak Jacobi forms for the modular group $\Gamma(1) = SL(2, \mathbb{Z})$.

Definition 5.1. Given integers $k$ and $\ell$, a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is a weak modular form of weight $k$ and index $\ell$ for $\Gamma(1)$ if it satisfies
\[
\phi(\gamma z, \bar{\gamma} \bar{z}^{-1} w) = \bar{\gamma} \bar{z} \phi(z, w),
\]
\[
\phi(z, w + \mu z + \nu) = (-1)^{2\ell} \phi(z, w)
\]
for all $(z, w) \in \mathcal{H} \times \mathbb{C}$, $\gamma \in \Gamma$ and $\mu, \nu \in \mathbb{Z}$, and it has a Fourier expansion of the form
\[
\phi(z, \zeta) = \sum_{m \geq 0} \sum_{r \in \ell + \mathbb{Z}} c(m, r) q^m \zeta^r
\]
with $q = e^{2\pi iz}$ and $\zeta = e^{2\pi i w}$. We denote by $\tilde{J}_{k, \ell}(\Gamma(1))$ the space of weak modular forms of weight $k$ and index $\ell$ for $\Gamma(1)$.

In [5] Eichler and Zagier discussed various properties of the two Jacobi forms
\[
\phi_{10,1}(z, w) = \frac{1}{144} (E_6(z) E_{4,1}(z, w) - E_4(z) E_{6,1}(z, w)),
\]
\[
\phi_{12,1}(z, w) = \frac{1}{144} (E_4^2(z) E_{6,1}(z, w) - E_6(z) E_{6,1}(z, w))
\]
of index 1 of weight 10 and 12. Here $E_4, E_6$ are the usual elliptic Eisenstein series given by
\[
E_4(z) := 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n, \quad E_6(z) := 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n
\]
for $z \in \mathcal{H}$ (see [15]), and $E_{4,1}, E_{6,1}$ are Jacobi Eisenstein series of weights 4 and 6, respectively, with index 1 of the form
\[
E_{4,1}(z, w) = 1 + (\zeta + 56 \zeta + 126 + 56 \zeta^{-1} + \zeta^{-2}) q + (126 \zeta^2 + \ldots + 126 \zeta^{-2}) q^2 + \ldots,
\]
\[
E_{6,1}(z, w) = 1 + (\zeta^2 - 88 \zeta - 330 - 88 \zeta^{-1} + \zeta^{-2}) q + (-330 \zeta^2 - \ldots - 330 \zeta^{-2}) q + \ldots
\]
for $(z, w) \in \mathcal{H} \times \mathbb{C}$ (cf. [5] p. 23). If we set
\[
(5.1) \quad M_* = M_*(\Gamma(1)) = \bigoplus_{k \geq 0} M_k(\Gamma(1)) \cong \mathbb{C}[E_4, E_6],
\]
the functions
\[
\tilde{\phi}_{-2,1}(z, w) = \frac{\phi_{10,1}(z, w)}{\Delta(z)}, \quad \tilde{\phi}_{0,1} = \frac{\phi_{12,1}(z, w)}{\Delta(z)}
\]
with $\Delta(z)$ as in (3.7) satisfy the following property.

Theorem 5.2. The ring $\tilde{J}_{ev,4}(\Gamma(1))$ of weak Jacobi forms of even weight is a polynomial algebra over the ring $M_*$ on two generators $\tilde{\phi}_{-2,1}$ and $\tilde{\phi}_{0,1}$.

Proof. See [5]. \(\square\)

The same functions can be used to obtain a correspondence between vector-valued modular forms and weak modular forms as is stated below.

\[\text{11}\]
Theorem 5.3. The map \( \Psi_{n,k} : \hat{M}^{n+1}_k(\Gamma(1), \rho_n) \to \tilde{J}_{k-n,n}(\Gamma(1)) \) defined by
\[
(5.2) \quad \Psi_{n,k}(F) = \sum_{\ell=0}^{n} \nu_{k,n,\ell}^{-1}(F) \cdot \tilde{\phi}^\ell_{-2,1} \cdot \tilde{\phi}^{n-\ell}_{0,1}
\]
for any \( F \in \hat{M}^{n+1}_k(\Gamma(1), \rho_n) \) is an isomorphism. Here, \( \nu_{k,n,\ell}^{-1} \) is the inverse of the isomorphism in (1.1).

Proof. It is known that the map \( P_{k-n}^n : \bigoplus_{\ell=0}^{n} M_{k-n+2\ell}(\Gamma(1)) \to \tilde{J}_{k-n,n}(\Gamma(1)) \) given by
\[
P_{k-n}^n(f_0, f_1, \ldots, f_n) = \sum_{\ell=0}^{n} f_\ell \tilde{\phi}^\ell_{-2,1} \tilde{\phi}^{n-\ell}_{0,1}
\]
with \( f_\ell \in M_{k-n+2\ell}(\Gamma(1)) \) for \( 0 \leq \ell \leq n \) is an isomorphism (see [5, p.108]). Thus the theorem follows from this and the isomorphism (1.3) in Theorem 1.1. \( \square \)

The above results suggest the existence of certain isomorphisms included in the next theorem.

Theorem 5.4. There are canonical isomorphisms of the form
\[
\bigoplus_{k \equiv n \, (\text{mod } 2)} \hat{M}^{n+1}_k(\Gamma(1), \rho_n) \cong \bigoplus_{k \equiv n \, (\text{mod } 2)} \bigoplus_{\ell=0}^{n} M_{k-n+2\ell}(\Gamma(1)) \cong \bigoplus_{k \equiv n \, (\text{mod } 2)} \tilde{J}_{k-n,n}(\Gamma(1)) \cong M^*[\tilde{\phi}_{0,1}, \tilde{\phi}_{2,1}],
\]
where \( M^* \) is as in (5.1).

6. Cohen-Kuznetsov liftings

We denote by \( \mathcal{F}[[X]] \) the complex algebra of formal power series in \( X \) with coefficients in \( \mathcal{F} \). We fix a positive integer \( n \) and denote by \( \hat{\mathcal{F}}^{n+1} \) the space of \( \mathbb{C}^{n+1} \)-valued holomorphic functions on \( \mathcal{H} \). Then the space \( \hat{\mathcal{F}}^{n+1}[[X]] \) of formal power series in \( X \) with coefficients in \( \hat{\mathcal{F}}^{n+1} \) has the structure of a module over \( \mathcal{F}[[X]] \). Note that, unlike \( \mathcal{F}[[X]], \hat{\mathcal{F}}^{n+1}[[X]] \) does not have a ring structure. Let \( \Gamma \) be a discrete subgroup of \( SL(2, \mathcal{F}) \), and let \( \rho : \Gamma \to GL(n+1, \mathbb{C}) \) be a representation of \( \Gamma \) in \( \mathbb{C}^{n+1} \). Then the associated action of \( \Gamma \) on the coefficients induces an action of \( \Gamma \) on \( \hat{\mathcal{F}}^{n+1}[[X]] \).

Definition 6.1. (i) Given an integer \( \lambda \), a Jacobi-like form of weight \( \lambda \) for \( \Gamma \) is a formal power series \( \Phi(z,X) \in \mathcal{F}[[X]] \) satisfying
\[
\Phi(\gamma z, \bar{\mathfrak{J}}(\gamma, z)^{-2} X) = \bar{\mathfrak{J}}(\gamma, z)^{\lambda} e^{\bar{\mathfrak{J}}(\gamma, z)} \Phi(z,X)
\]
for all \( z \in \mathcal{H} \) and \( \gamma \in \Gamma \).
(ii) A vector-valued Jacobi-like form of weight $\lambda$ for $\Gamma$ with respect to $\rho$ is a formal power series $\tilde{\Phi}(z, X) \in \tilde{J}^{n+1}[X]$ satisfying
\[ (6.1) \quad \tilde{\Phi}(\gamma z, \tilde{J}(\gamma, z)^{-2}X) = \tilde{J}(\gamma, z)^{\lambda}e^{\tilde{R}(\gamma, z)X} \rho(\gamma) \tilde{\Phi}(z, X) \]
for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

We denote by $\mathcal{J}_\lambda(\Gamma)$ and $\mathcal{J}_\lambda(\Gamma, \rho)$ the spaces of Jacobi-like forms and vector-valued Jacobi-like forms with respect to $\rho$, respectively, of weight $\lambda$ for $\Gamma$.

**Lemma 6.2.** If $\tilde{\nu}_n \in \tilde{M}_n^{n+1}(\Gamma, \rho_n)$ is as in \[\text{(6.2)}\], the associated formal power series given by
\[ (6.2) \quad \tilde{\Phi}_{\tilde{\nu}_n}(z, X) = \sum_{j=0}^{\infty} \frac{(-1)^j(n-j)!D^j(\tilde{\nu}_n)}{j!} X^j \]
is a vector-valued Jacobi-like form belonging to $\tilde{J}_{-n}(\Gamma, \rho_n)$.

**Proof.** Given $\alpha \in SL(2, \mathbb{R})$, it can be shown by induction that
\[ (6.3) \quad ((D^\nu(\tilde{\nu}_n)) |_{-n+2\nu} \alpha)(z) = \sum_{\ell=0}^{\nu} \frac{(-1)^{\nu-\ell}\nu!(n-\ell)!}{\ell!(\nu-\ell)!(n-\nu)!} \mathcal{R}(\gamma, z)^{\nu-\ell}D^\ell(\tilde{\nu}_n |_{-n} \alpha)(z) \]
for each $\nu \geq 0$ and $z \in \mathcal{H}$. If $\gamma \in \Gamma$ and if $\tilde{\Phi}_{\tilde{\nu}_n}(z, X)$ is the formal power series given by \[\text{(6.2)},\] we have
\begin{align*}
\tilde{\Phi}_{\tilde{\nu}_n}(\gamma z, \tilde{J}(\gamma, z)^{-2}X) &= \sum_{j=0}^{\infty} \frac{(-1)^j(n-j)!}{j!} (D^j\tilde{\nu}_n)(\gamma z) \tilde{J}(\gamma, z)^{-2j}X^j \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j(n-j)!}{j!} ((D^j\tilde{\nu}_n) |_{-n+2j} \gamma)(z) \tilde{J}(\gamma, z)^{-n}X^j \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j(n-j)!}{j!} \tilde{J}(\gamma, z)^{-n}X^j \\
&\quad \times \sum_{\ell=0}^{j} \frac{(-1)^{j-\ell}\ell!(n-\ell)!}{(j-\ell)!(n-j)!} \mathcal{R}(\gamma, z)^{j-\ell}D^\ell(\tilde{\nu}_n)(z) \\
&= \tilde{J}(\gamma, z)^{-n} \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} \frac{(-1)^\ell(n-\ell)!}{\ell!(j-\ell)!} \mathcal{R}(\gamma, z)^{j-\ell}(D^\ell\tilde{\nu}_n)(z)X^j,
\end{align*}
where we used the relation $\tilde{\nu}_n |_{-n} \gamma = \tilde{\nu}_n$. On the other hand, we see that
\begin{align*}
\tilde{J}(\gamma, z)^{-n}e^{\tilde{R}(\gamma, z)X} \tilde{\Phi}_{\tilde{\nu}_n}(z, X) &= \tilde{J}(\gamma, z)^{-n} \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j(n-j)!}{j!\ell!} \mathcal{R}(\gamma, z)^{j+\ell}X^{j+\ell} \\
&= \tilde{J}(\gamma, z)^{-n} \sum_{p=0}^{\infty} \sum_{j=0}^{p} \frac{(-1)^j(n-j)!}{j!(p-j)!} \mathcal{R}(\gamma, z)^{p-j}X^p.
\end{align*}
Hence we obtain
\[ (6.4) \quad \tilde{\Phi}_{\tilde{\nu}_n}(\gamma z, \tilde{J}(\gamma, z)^{-2}X) = \tilde{J}(\gamma, z)^{-n}e^{\tilde{R}(\gamma, z)X} \rho_n(\gamma) \tilde{\Phi}_{\tilde{\nu}_n}(z, X), \]
and therefore the lemma follows. □

The vector-valued Jacobi-like form \( \tilde{\Phi}_v(z, X) \in \tilde{J}_{-n}(\Gamma, \rho_n) \) may be regarded as the vector-valued version of the Cohen-Kuznetsov lifting (cf. \([3, 15]\)) of the vector-valued modular form \( \hat{v}_n \in \widehat{M}_{n+1}^n(\Gamma, \rho_n) \).

7. Proof of main theorems

Proof of Theorem 1.1 We consider an element

\[
\hat{g} = (g_0, g_1, \ldots, g_n) \in \bigoplus_{\ell=0}^n M_{k-n+2\ell}(\Gamma)
\]

with \( g_\ell \in M_{k-n+2\ell} \) for each \( \ell \in \{0, 1, \ldots, n\} \). Then the Cohen-Kuznetsov lifting of the modular form \( g_\ell \) given by

\[
(7.1) \quad \tilde{g}_\ell(z, X) = \sum_{j=0}^{\infty} \frac{g_\ell^{(j)}(z)}{j!(j+k-n+2\ell-1)!} X^j
\]

is a Jacobi-like form belonging to \( J_{k-n+2\ell}(\Gamma) \) and therefore satisfies

\[
(7.2) \quad \tilde{g}_\ell(\gamma z, \tilde{J}(\gamma, z)^{-2} X) = \tilde{J}(\gamma, z)^{k-r+n+2\ell} e^{\tilde{R}(\gamma, z)X} \tilde{g}_\ell(z, X)
\]

for all \( \gamma \in \Gamma \) (see e.g. \([3, 15]\)). Similarly, by Lemma 6.2 the vector-valued version of the Cohen-Kuznetsov lifting of \( \hat{v}_n \) given by

\[
(7.3) \quad \tilde{\Phi}_v(z, X) = \sum_{j=0}^{\infty} (-1)^j(n-j)! \hat{v}_n^{(j)}(z) X^j,
\]

satisfies

\[
(7.4) \quad \tilde{\Phi}_v(\gamma z, \tilde{J}(\gamma, z)^{-2} X) = \tilde{J}(\gamma, z)^{k-r+n} e^{\tilde{R}(\gamma, z)X} \rho_n(\gamma) \tilde{\Phi}_v(z, X)
\]

for all \( \gamma \in \Gamma \). We now set

\[
\hat{F}(z, X) = \tilde{g}_\ell(z, -X) \tilde{\Phi}_v(z, X).
\]

Then, using (7.2) and (7.4), we have

\[
(7.5) \quad \hat{F}(\gamma z, \tilde{J}(\gamma, z)^{-2} X) = \tilde{J}(\gamma, z)^{k-r+n+2\ell} e^{\tilde{R}(\gamma, z)X} \tilde{g}_\ell(z, -X) \times \tilde{J}(\gamma, z)^{-n} e^{\tilde{R}(\gamma, z)X} \rho_n(\gamma) \tilde{\Phi}_v(z, X)
\]

\[
\quad = \tilde{J}(\gamma, z)^{k-2n+2\ell} \rho_n(\gamma) \hat{F}(z, X)
\]

for all \( \gamma \in \Gamma \). Hence, if we write

\[
\hat{F}(z, X) = \sum_{j=0}^{\infty} \hat{\eta}^{\hat{g}_\ell}(z) X^j,
\]

from (7.5) we see that

\[
(7.6) \quad \hat{\eta}^{\hat{g}_\ell} \in \widehat{M}_{k-2n+2\ell+2j}(\Gamma, \rho_n)
\]
for each \( j \geq 0 \). On the other hand, using (7.1) and (7.3), we obtain
\[
\hat{F}(z, X) = \hat{g}_t(z, -X)\hat{\Phi}_{\nu_0}(z, X)
\]
\[
= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{r+s} (n - s)! g_t^{(r)}(z) \hat{\nu}_n^{(s)}(z)}{r! s!(r + k - n + 2\ell - 1)!} X^{r+s}
\]
\[
= \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{(-1)^j (n - j + r)! g_t^{(r)}(z) \hat{\nu}_n^{(j-r)}(z)}{r!(j-r)!(r + k - n + 2\ell - 1)!} X^j;
\]
hence we have
\[
\hat{\eta}_j^{g_t} = \sum_{r=0}^{j} \frac{(-1)^j (n - j + r)! g_t^{(r)} \hat{\nu}_n^{(j-r)}}{r!(j-r)!(r + k - n + 2\ell - 1)!}
\]
for all \( j \geq 0 \). Furthermore, using (2.5), for \( 0 \leq \ell \leq n \) and \( j \geq 0 \) we have
\[
[g_t, \hat{\nu}_n]^{(k-n+2\ell,-n)}_{j} = \sum_{r=0}^{j} (-1)^r \binom{k-n+2\ell+j-1}{j-r} \binom{-n+j-1}{r} g_t^{(r)} \hat{\nu}_n^{(j-r)}
\]
\[
= \sum_{r=0}^{j} (-1)^r \frac{(k-n+2\ell+j-1)!}{(j-r)!(k-n+2\ell+r-1)!} \cdot \frac{(-n+j-1) \cdots (-n+j-r)!}{r!} g_t^{(r)} \hat{\nu}_n^{(j-r)}
\]
\[
= \sum_{r=0}^{j} \frac{(k-n+2\ell+j-1)!(n+r-j)!}{(j-r)!(k-n+2\ell+r-1)!r!(n-j)!} \hat{\eta}_j^{g_t}
\]
\[
= (-1)^j \frac{(k-n+2\ell+j-1)!}{(n-j)!} \hat{\eta}_j^{g_t},
\]
which belongs to \( \hat{M}_{k-2n+2\ell+2j}^{n+1}(\Gamma, \rho_n) \) by (7.6). Hence it follows that
\[
[g_t, \hat{\nu}_n]^{(k-n+2\ell,-n)}_{n-\ell} \in \hat{M}_k^{n+1}(\Gamma, \rho_n),
\]
which proves (i). We now consider a vector-valued modular form \( F \in \hat{M}_k^n(\Gamma, \rho_n) \), and assume that
\[
L_n(z)^{-1} F(z) = (t(f_0(z), f_1(z), \ldots, f_n(z))
\]
for all \( z \in \mathcal{H} \) with \( f_0, \ldots, f_n \in F \). Let \( t \) be the first positive integer such that \( f_t \) is not identically zero. Then by Lemma 3.2 the function \( f_t \) is a modular form belonging to \( M_{k-n+2\ell}(\Gamma) \). Here, we note that \( k - n + 2\ell > 0 \) because otherwise \( M_{k-n+2\ell}(\Gamma) = \{0\} \). Using Lemma 3.3 we see that the first nonzero entry of the vector
\[
L_n(z)^{-1} \mathcal{V}_{k,n,t}(f_t)(z) = L_n(z)^{-1}[f_t(z), \hat{\nu}_n(z)]^{(k-n+2\ell,-n)}_{n-\ell}
\]
is the \( t \)-th component which is equal to
\[
(n-t)! \alpha_{k-n+2\ell,0} f_t(z).
\]
Thus, if we set
\[
\xi_{0,t} = \frac{1}{(n-t)! \alpha_{k-n+2\ell,0}} \mathcal{V}_{k,n,t}(f_t) \in \text{Im} \mathcal{V}_{k,n,t} = \hat{G}_{k-n+2\ell}^{\alpha}(\Gamma, \rho_n),
\]
the first \(t + 1\) entries of the vector
\[
L_n(z)^{-1}(F(z) - \xi_0, t(z))
\]
are zero with \(F - \xi_0, t \in \hat{M}_k^{n+1}(\Gamma, \rho_n)\). Applying the same argument to the vector-valued modular form \(F - \xi_0, t\), we can find an element \(\xi_{1, t+1} \in \hat{G}_{k-n+2t+2}(\Gamma, \rho_n)\) such that the first \(t + 1\) components of the vector
\[
L_n(z)^{-1}(F(z) - \xi_0, t(z) - \xi_{1, t+1}(z))
\]
are all zero. By repeating this process we obtain the expression
\[
F = \sum_{\ell=0}^{n+2-t} \xi_{\ell, t+\ell} \in \bigoplus_{\ell=0}^{n+2-t} \hat{G}_{k-n+2t+2\ell}(\Gamma, \rho_n),
\]
which implies (ii). In order to prove (iii), given \(\ell \in \{0, 1, \ldots, n\}\), we assume that \(V_{k,n,\ell} : M_{k-n-2\ell}(\Gamma) \to \hat{G}_{k-n+1}(\Gamma, \rho_n)\). Then from (1.1) and Lemma 3.3 we see that
\[
0 = [g - h, \hat{v}_n]_{k-n-2\ell} = L_n(z), \{(f_0(z), f_1(z), \ldots, f_n(z))
\]
for all \(z \in \mathcal{H}\), where
\[
f_j = \begin{cases} 
0 & \text{for } 0 \leq j \leq \ell - 1; \\
(n - j)! \alpha_{k-n-2\ell, j-\ell} D^{j-\ell} (g - h) & \text{for } \ell \leq j \leq n.
\end{cases}
\]
Thus we have
\[
0 = f_\ell = (n - \ell)! \alpha_{k-n-2\ell, 0} (g - h),
\]
and therefore \(g = h\), which show that \(V_{k,n,\ell}\) is injective; hence it follows that the map
\[
V_{k,n,\ell} : M_{k-n-2\ell}(\Gamma) \to \hat{G}_{k-n+2\ell}(\Gamma, \rho_n) = \text{Im} \,(V_{k,n,\ell})
\]
is an isomorphism verifying (iii).

**Proof of Theorem 1.2**  Given integers \(k\) and \(n\) with \(n \geq 0\), using the bilinear map (2.4) and the fact that \(\hat{u}_n \in \hat{M}_k^{n+1}(\Gamma, \rho_n^*)\) (see Proposition 3.1), we see easily that the map \(W_{k,n}\) is well-defined; hence the theorem follows.

**Proof of Theorem 1.3**  Given \(k \in \mathbb{Z}\), by Theorem 1.1(iii) there is a canonical isomorphism
\[
\hat{M}_k^n(\Gamma, \rho_n) \cong \bigoplus_{\ell=0}^n M_{k-n-2\ell}(\Gamma).
\]
However, using Remark 4.2(i), we can identify the direct sum \(\bigoplus_{\ell=0}^n M_{k-n-2\ell}(\Gamma)\) with the space \(MP^n_{k-n}(\Gamma)\) of modular polynomials. Since
\[
MP^n_{k-n}(\Gamma) \cong QP^n_{k+n}(\Gamma)
\]
by (4.6), we obtain the desired isomorphism. By combining (1.3) with the isomorphism (4.8) we also obtain a correspondence
\[
\hat{M}^{n+1}_k(\Gamma, \rho_n) \cong QM^n_{k+n}(\Gamma)
\]
between vector-valued modular forms and quasimodular forms.
Proof of Theorem 1.4. Given \( k \in \mathbb{Z} \), we consider a quasimodular polynomial \( F(z, X) \in QP_{k+n}^{n}(\Gamma) \) of the form

\[
F(z, X) = \sum_{r=0}^{n} f_r(z)X^r.
\]

Since we already have the isomorphism (1.5), it suffices to show that \( V_n(F(z, X)) \) belongs to \( \hat{M}_{k}^{n+1}(\Gamma, \rho_n) \). First, we note that \( F(z, X) \) can be lifted to a Jacobi-like form

\[
\Phi_F(z, X) = \sum_{\ell=0}^{\infty} \phi_{\ell}(z)X^\ell \in J_{k-n}(\Gamma)
\]

with

\[
f_r = \frac{1}{r!}\phi_{n-r}
\]

for \( 0 \leq r \leq n \) satisfying

\[
(7.7) \quad \Phi_F(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) = \mathfrak{J}(\gamma, z)^{k-n}e^{\mathfrak{J}(\gamma, z)X}\Phi_F(z, X)
\]

for all \( \gamma \in \Gamma \) (cf. [2]). If \( \tilde{\Phi}_{v_n}(z, X) \in \hat{J}_{-n}(\Gamma, \rho_n) \) is as in (6.2), we set

\[
\Psi(z, X) = \Phi_F(z, -X)\tilde{\Phi}_{v_n}(z, X)
\]

\[
= \sum_{j=0}^{\infty} \sum_{\ell=0}^{n} \frac{(-1)^{j+\ell}(n-j)!}{j!}D^j(\tilde{\upsilon}_n)(z)\phi_{\ell}(z)X^{j+\ell}
\]

\[
= \sum_{r=0}^{\infty} \sum_{\ell=0}^{r} \frac{(-1)^{r}(n-r+\ell)!}{(r-\ell)!}D^{r-\ell}(\tilde{\upsilon}_n)(z)\phi_{\ell}(z)X^r,
\]

assuming that \( \phi_{\ell} = 0 \) for \( \ell > n \). Thus we may write

\[
(7.8) \quad \Psi(z) = \sum_{r=0}^{\infty} \psi_r(z)X^r,
\]

where

\[
\psi_r = \sum_{\ell=0}^{r} \frac{(-1)^{r}(n-r+\ell)!}{(r-\ell)!}D^{r-\ell}(\tilde{\upsilon}_n)\phi_{\ell}
\]

\[
= \sum_{\ell=0}^{r} \frac{(-1)^{r}(n-r+\ell)!(n-\ell)!}{(r-\ell)!}D^{r-\ell}(\tilde{\upsilon}_n)f_{n-\ell}
\]

\[
= \sum_{\ell=0}^{r} \frac{(-1)^{r}(2n-r-\ell)!\ell!}{(r-n+\ell)!}D^{r-n+\ell}(\tilde{\upsilon}_n)f_{\ell}
\]

for \( r \geq 0 \). Using (6.4) and (7.7), we see that

\[
\Psi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) = \mathfrak{J}(\gamma, z)^{k-2n}\rho_n(\gamma)\Psi(z, X).
\]

for \( \gamma \in \Gamma \). From this and (7.8) it follows that

\[
\psi_r \in \hat{M}_{k-2n+2r}^{-1}(\Gamma, \rho_n)
\]
for each $r \geq 0$. In particular, we obtain
\[\psi_n = \sum_{\ell=0}^{n} (-1)^n(n-\ell)!D^\ell(\hat{v}_n)f_\ell \in \hat{M}^{n+1}_k(\Gamma, \rho_n),\]
and therefore the theorem follows from this and the identity $\psi_n(z) = \mathcal{U}_n(F(z, X))$.

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