The purpose of this paper is to show that one can find an analog to the complex Klein-Gordon equation in curved space-time by considering matter which is moving in an higher dimensional space-time. This is motivated on the one hand by the fact that there are alternative approaches to quantum mechanics. Early such attempts have been discussed within the so called Bohmian mechanics [1, 2]. Similarities between a higher dimensional wave equation and the non-relativistic quantum theory have been pointed out in [3]. It was also suggested that quantum field theory might emerge from a chaotic classical theory with friction [4, 5]. For an overview see [6]. On the other hand a further motivation comes from the existing ideas on an additional time dimension. The consequences of more than one time dimensions were studied in [7, 8, 9, 10, 11]. Further, an additional time dimension was used to interpret geodesics in four dimensions as null paths in the higher dimensions [12]. There is also an intersection of both motivations. Some papers relate a quantum field theory on the horizon of a black hole to a corresponding classical theory in the higher dimensional spacetime [13, 14]. Similar effects are found in the context of supergravity and string theory involving a holographic principle and extra time dimensions [15, 16, 17]. In a purely conceptual discussion an extra time dimension is suggested as an alternative to quantization [18]. In a stochastic approach [19] the classical Langevin equation for an auxiliary time coordinate \( \tilde{t} \) is used in order to obtain Euclidian quantum field theory in the limit \( \tilde{t} \to \infty \). All those ideas can be seen as motivation for this paper, where the structure of the complex Klein-Gordon equation is found from classical differential geometry with one additional time dimension.

The motivation of our approach is the idea that the movement of a classical particle with respect to an additional but unobservable time dimension \( \tilde{t} \) could produce effects which are usually described by quantum mechanics. As first application of this idea we will derive the structure of the relativistic complex Klein-Gordon equation from a classical theory with one additional dimension.

I. THE COMPLEX KLEIN-GORDON EQUATION IN CURVED SPACETIME

The relativistic Klein-Gordon equation in curved spacetime [20] is

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi(x,t) \right) = \frac{m^2}{\hbar^2} \Phi(x,t) ,
\]

where \( g \) stands for the determinant of the metric \( \hat{g}_{\mu\nu} \). This equation for the complex wave function \( \Phi \) can be expressed in terms of the real function \( S_Q(x,t) \) and the real function \( \rho(x,t) \) by defining \( \Phi(x,t) = \sqrt{\rho} \exp(i S_Q/\hbar) \). This gives two coupled differential equations

\[
\Box \sqrt{\rho} = \frac{\sqrt{\rho}}{\hbar^2} \left( (\partial^\mu S_Q)(\partial_\mu S_Q) - m^2 \right) ,
\]

\[
0 = \partial_\mu \left( \sqrt{-g} \hat{g}^{\mu\nu} (\partial_\nu S_Q) \right) .
\]

In the first equation we used the abbreviation \( \Box \sqrt{\rho} \equiv \partial_\mu \left( \sqrt{-g} \hat{g}^{\mu\nu} \partial_\nu \sqrt{\rho} \right) / \sqrt{-g} \) for the covariant d’Alembert operator.

The redefinition of the complex wave function in terms of the real functions \( S_Q \) and \( \rho \) does not change the meaning or the interpretation of Eq. (I). The function \( S_Q \) is usually understood as the phase of the quantum wave function.

II. GENERAL RELATIVITY WITH ONE ADDITIONAL TIME DIMENSION

In this section we will consider the geometric structure of Einsteins equations with one additional coordinate \( \tilde{t} \). A simple ansatz for the metric in \( 2 + 3 \) dimensions will be made. We then show that it is possible to choose the energy-momentum tensor \( T_{AB} \) in such a way that the classical equations of motion in the higher dimensional theory (after the additional coordinate \( \tilde{t} \) is integrated out) correspond to the relativistic quantum equation for a spinless particle. We use the coordinate notation \( x_A = (\tilde{t}, x_\mu) = (\tilde{t}, t, x) \), where capital Latin indices \( A \) run from 0 to 4 and Greek indices run from 1 to 3.
As starting point we take the Einstein equations in (2+3) dimensions
\[ R_{AB} - \frac{1}{2}g_{AB}R = GD_{TAB} - \frac{1}{2}g_{AB}\Lambda, \] (4)
where \( R_{AB} \) is the higher dimensional Ricci tensor, \( R \) is its contraction, and \( GD \) is the higher dimensional gravitational coupling constant. The term \( \Lambda \) is directly connected to the cosmological constant \( \Lambda_4 \). Our ansatz for the higher dimensional metric is
\[ g_{AB}dx^A dx^B = \alpha^2(\bar{t})\rho(t, \vec{x})d\bar{t}^2 + g_{\mu\nu}(\bar{t}, \vec{x}, \alpha)dx^\mu dx^\nu. \] (5)

Later, the function \( \alpha \) will be separated into a constant plus a small \( \bar{t} \) dependent part
\[ \alpha(\bar{t}) = \alpha_0 + \epsilon_0\tilde{\alpha}(\bar{t}). \] (6)

Note that this metric is similar to the Kaluza-Klein metric for a vanishing electromagnetic field \( A_\mu = 0 \) \[ \[21, 22, 23]. \] However, in contrast to the Kaluza-Klein approach the energy momentum tensor \( T_{AB} \) and \( \Lambda \) are not assumed to be identically zero and all the functions have real values. An other difference is that the functions in the metric (\( \alpha^2\rho \) and \( g_{\mu\nu} \)) and the energy momentum tensor \( T_{AB} \) are allowed to have an explicit \( \bar{t} \) dependence. With the metric (5) the twenty-five coupled differential equations (4) read
\[ \left( -\alpha^2\sqrt{\rho} \Box \sqrt{\rho} - \frac{(\alpha^2\rho_{,\bar{t}})_{,\bar{t}}}{2} + \frac{\dot{\alpha}\rho}{2\alpha} \right) = GD \left( T_{00} - \frac{\alpha^2\rho T_A}{3} + \frac{\alpha^2\rho}{3} \Lambda \right), \] (7)

\[ \left( \frac{g_{\lambda\beta}(\dot{g}_{\lambda\beta\rho,\beta} - \rho_{,\beta}\dot{g}_{\lambda\beta})}{4\rho} - \frac{\partial_4(g_{\lambda\beta}\rho_{,\beta})}{2} \right) = GD T_{\delta\beta}, \] (8)

\[ \left[ R_{\delta\beta} - \frac{\sqrt{\rho}(\dot{\rho})_{,\delta\beta}}{\sqrt{\rho}} + \frac{1}{2\alpha^2\rho} \left( \frac{\dot{\alpha}\rho_{,\beta}}{\alpha} - \ddot{g}_{\beta\delta} + \frac{g^{\lambda\gamma}g^{\beta\delta,\gamma}g_{\mu\beta,\lambda}}{4} + \frac{g^{\beta\delta}g_{\mu\beta,\lambda}}{4} \right) \right] = GD \left( T_{\delta\beta} - \frac{\alpha^2\rho T_A}{3} + \frac{\alpha^2\rho}{3} \Lambda \right). \] (9)

Here, we denoted a derivative with respect to \( \bar{t} \) as \( \partial_\bar{t}X = \dot{X} \), the covariant derivative in the four dimensional subspace as \( \nabla_\mu X = X_\mu \), and \( \Box \sqrt{\rho} = (\sqrt{\rho})^{,\bar{t}}_{,\bar{t}}. \) \( R_{\delta\beta} \) is the normal four dimensional form of the Ricci tensor which only contains derivatives of \( g_{\mu\nu} \) with respect to \( x_\mu \). The Hamilton-Jakobi definition of the energy momentum tensor of a free particle in curved spacetime is
\[ T_A^B = (\partial^A S_H)(\partial_B S_H), \] (10)
\[ T_A^A = ((\partial^A S_H)(\partial_B S_H) + (\partial^\mu S_H)(\partial_\mu S_H)) \] (11)

where \( S_H \) is the density of Hamilton’s principal function \[ \[24, 25]. \] A priori, the \( \bar{t} \) dependence of \( S_H \) is not known. Therefore, we make a separation ansatz where \( S_H \) can be decomposed into a classical part \( \bar{S} \) which only depends on the observable coordinates \( \bar{t}, \vec{x} \) and a part \( B(\bar{t})\sqrt{\rho} \) which depends on all coordinates \( \bar{t}, \vec{x} \):
\[ S_H(\bar{t}, \vec{x}) = \epsilon_1 \sqrt{\rho} B(\bar{t}) + \bar{S}(\bar{t}, \vec{x}), \] (12)

Using the definitions (5, 10), and after a multiplication with \(-\sqrt{\rho}/2\) the scalar equation (12) reads
\[ \Box \sqrt{\rho} \int d\bar{t} = \sqrt{\rho} \left[ \frac{GD}{3} \int d\bar{t} (\partial^\mu S_H)(\partial_\mu S_H) \right. \] (13)

\[ -\left( \frac{5}{6} \Lambda \int d\bar{t} - \frac{1}{2\sqrt{\rho}} \int d\bar{t} \left( \frac{\rho^\beta}{\sqrt{\rho}} \frac{\rho^\beta}{\sqrt{\rho}} \int d\bar{t} \right) \right] \] (14)

The source term on the right hand side contains the tensor \( P_{\mu\nu} = (\dot{g}_{\mu\nu} - g_{\mu\nu}g_{\alpha\beta}(\dot{g}_{\alpha\beta})/2 \). Apart from the last two terms this equation has always the same \( \rho \) dependence as Eq. (6), which justifies the ansatz (11). Those last terms have the form of a source, which indicates that a \( \tilde{t} \) dependence of \( g_{\mu\nu} \) can lead to particle production. This result is not too surprising because it is known that already a normal time dependence of the metric can lead to particle production \[ \[26\]. \] However, we are primarily interested in problems with a seemingly constant (with respect to \( \tilde{t} \)) four dimensional spacetime. Therefore, we decompose the four dimensional submetric \( g_{\mu\nu} \) into a \( \tilde{t} \) independent part \( g_{\mu\nu} \), and a traceless \( \bar{t} \)-variable part \( \gamma_{\mu\nu}(\bar{t}, x^\alpha) \)
\[ g_{\mu\nu}(\bar{t}, x^\alpha) = g_{\mu\nu} + \epsilon_2^2 \gamma_{\mu\nu}(\bar{t}, x^\alpha). \] (15)

Now we assume that perturbations due to \( \tilde{t} \) are small and expand Eq. (15) to lowest order in \( \epsilon_2 \leq 1 \) \( (\tilde{t} = 0, 1, 2) \). With Eq. (11) one finds up to order \( O(\epsilon_2) \)
The second to last term of Eq. (13) vanished, since it is proportional to the trace of $\gamma_{\mu\nu}$, and the last term vanished because it is of higher order in $\epsilon_i$. A comparison of Eq. (15) with the complex Klein-Gordon equation shows that both equations are the same if two identifications are made. The first identification relates the Hamilton principal function $\tilde{S}$ to the quantum phase $S_Q$ by

$$S_Q \equiv \frac{\hbar}{\sqrt{G_D}} \tilde{S} \quad .$$

The second identification relates the mass $m$ to the functions $\hat{R}_\nu^\mu$ and $\Lambda$ by

$$m^2 = \frac{\hbar^2}{6} \left( 5\Lambda - 3\hat{R}_\nu^\mu \right) \quad .$$

One can already see that such an identification makes only sense if the scalar curvature in the four dimensional subspace $\hat{R}_\nu^\mu$ does not depend on $x_\rho$. (This is for instance the case in de-Sitter spacetimes. One also has to check that a solution of Eq. (15) allows to solve all the twenty-five equations (7) in the $\epsilon_i$ expansion. The first check is to compare Eq. (15) with Eq. (7). It turns out that both equations are only consistent if

$$\hat{R}_\nu^\mu = \Lambda \quad .$$

Consequently, the mass identification simplifies to

$$m^2 = \frac{\hbar^2}{3} \Lambda \quad .$$

This shows that a mass term can only be defined if the four dimensional background is curved due to a non zero cosmological constant $\Lambda$. A similar result was previously obtained for massive gravitons in four dimensional spacetime. In the case of an exactly flat Minkowski background the mass term of this toy model is zero. Apart from the vanishing mass, an exactly flat Minkowski background is a solution of the toy model. The remaining twenty-four checks will be postponed to the subtopic “Is this theory consistent with all Einstein- and conservation equations?” in the discussion. One also sees that for matching several different masses ($m_j$) one has to impose several constants ($\Lambda_j$).

Since Eq. (2) was found, the next step is to derive the second Klein-Gordon equation. For this we consider the covariant conservation law for the energy-momentum tensor $T_{AB}$

$$0 = \nabla_B \left( T^B_A \right) = \partial_B T^B_A - \Gamma^D_{AB} T^B_D + \Gamma^B_{BD} T^D_A \quad .$$

Now we take the 0 component of this equation and apply the definitions

$$0 = \left[ \frac{\epsilon_1}{\sqrt{\rho}} \partial_\mu \left( \rho \partial^\mu (\tilde{S} + \epsilon_1 \rho \beta) \right) \right] \beta \quad (21)$$

$$+ \frac{\epsilon_2^2}{2} g^{\mu\nu} \left( \partial_\mu S_H \right) \left( \partial^\nu S_H \right) + \frac{\epsilon_2^2}{2} g^{\gamma\beta} \gamma_{\gamma\beta} \beta^2 + \epsilon_2 \partial_\mu \left( \frac{\beta^2}{\alpha_5} \right) + \frac{\epsilon_2}{2} g^{\gamma\beta} \gamma_{\alpha\beta} \sqrt{\rho} \left( \partial^\nu S_H \right) \beta + O(\epsilon_i^3) \quad .$$

Only the first and the last term survive in a first order $\epsilon_i$ expansion and the continuity equation simplifies to

$$0 = \partial_\mu \left( \sqrt{-g} \rho (\tilde{S}_\nu^\mu \partial_\nu \tilde{S}) \right) + O(\epsilon_i) \quad .$$

With the identification, this is exactly the second Klein-Gordon equation. Up to now we have used the classical but $t$ dependent higher dimensional Einstein equations and the covariant continuity equation and expanded them for small perturbations around the $t$-independent solution. From this we have found two matching rules such that the complex Klein-Gordon equations in a curved spacetime can be exactly identified with the classical equations.

## III. CRITICAL POINTS

We will now try to point out some checks, criticism, and limitations of the idea and the derivation presented here.

### Is this quantum mechanics?

No, finding a dual to the complex Klein-Gordon equation does not mean that one has automatically derived quantum mechanics. To make such a statement one would have to construct a self-consistent philosophical and mathematical theory like the pilot wave theory of Bohm. Such a construction goes beyond the scope of this paper.

### Why is the second time not visible?

The easiest way to explain this, is with a compactification of the hidden time such that $\tilde{t} = t + \tilde{T}$. As long the “radius” is just short enough, this would only lead to violations of Lorentz invariance on the small scale $\tilde{T}$. A similar construction for an extra but compact time variable (imaginary or not) can be found in [10, 28]. Thus there are several viable theoretical models that allow and have extra time dimensions. Within those models there exist however experimental bounds on the size of the additional time dimension $\tilde{T}$.

### What is the interpretation of the higher dimensional metric?

The strongest assumption for the higher dimensional metric $\tilde{g}_{\mu\nu}$ is that $g_{00} \sim \rho$. This can be justified by a classical probability argument: The idea of this new approach becomes most clear for a system that does not
change in the observable time direction \( t \). In a deterministic system, the position \( \vec{x} \) and velocity \( \vec{v} = d\vec{x}/dt \) of the particle are known as soon as the initial position \( \vec{x}_0 \), the initial velocity \( \vec{v}_0 \), and the propagation time \( \tau \) are known. Not knowing the initial conditions one has to deal with a probability density \( f(\vec{x}_0, \vec{v}_0, \tau) \). Without loss of generality we choose some fixed starting point which leaves a probability density \( g(\vec{v}_0, \tau) \). In a deterministic system the initial velocity can be calculated as a function of the actual velocity and the time \( \vec{v}_0 = \vec{v}_0(\vec{v}(\tau), \tau) \). Therefore, the whole system can also be described by a statistical probability density \( h(\vec{v}_0(\vec{v}(\tau), \tau)) \). The probability \( \rho (\vec{x}) \) of finding the particle at a point \( \vec{x} \) will than be obtained by an integration over the proper time variable

\[
\rho (\vec{x}) = \int d\tau h(\vec{v}_0(\vec{v}(\tau), \tau)) \quad .
\]

The key observation, which relates a probability to the extra time variable, is: The faster a particle moves at a certain point \( \vec{x} \) the smaller is the probability of finding the particle at this point \( h(\vec{v}_0(\vec{v}(\tau), \tau)) \sim 1/\vec{v}_0(\vec{v}(\tau), \tau) \) and hence

\[
\rho (x) \sim \int d\tau \frac{1}{\vec{v}_0(\vec{v}(\tau), \tau)} = \int d\tau \frac{1}{\vec{v}_0(\vec{v}(\tau), \tau)} \quad .
\]

This probability is directly related to the 00 component of the metric \( g_{\mu\nu} \). We already assumed that the particle moves mostly in parallel to the additional time coordinate \( (dt/d\tau \gg dx/d\tau > dt/dx) \). In this limit the differential of the proper time \( \tau \) can be approximated by the differential of the additional time variable \( d\tau \approx dt/\sqrt{g_{00}} \).

Under the assumption that some part of From the integral in Eq. (24) one finds

\[
\rho (x) \sim \int \frac{dt}{\sqrt{g_{00}}} \quad .
\]

This shows that a statistical interpretation of an unobservable additional time leads naturally to the metric ansatz (2).

**Is this theory consistent with all Einstein- and conservation equations?**

Since we only worked with a subset of the Einstein equations we have to check whether this can lead to inconsistencies with the full set. From comparing Eq. (15) with Eq. (7) we already found that \( \tilde{R}_\mu^\nu = \tilde{\Lambda} \). The same condition is found when comparing Eq. (15) with the trace of Eq. (9). Because of this condition one can always decompose the four dimensional Ricci tensor as

\[
\tilde{R}_\mu^\nu = n_1 \tilde{g}_\mu^\nu + G_D p_\mu p_\nu \quad ,
\]

where the four vector \( p_\mu \) and the number \( n_1 \) do not depend on \( x_\mu \). Please note that we do not know the solution to Eq. (20) but we assume that it exists. In the \( \epsilon_i \) expansion the eight equations (8) do not create any new conditions, since the LHS is of order \( \epsilon_i^2 \) and the RHS is composed of a vanishing \( t \) boundary term plus a term which is also of higher order in \( \epsilon_i \). In the same scheme and after using Eqs. (18) (20) the remaining nine independent equations in (9) read

\[
\int \frac{dt}{\sqrt{g_{00}}} \delta^{\mu\nu} = - \int \frac{dt}{\sqrt{g_{00}}} \left[ G_D \left( \partial_\mu \tilde{S} \partial_\nu \tilde{S} - p_\mu p_\nu \right) \right] + \tilde{g}_{\mu\nu} \left( G_D \partial_\mu \tilde{S} \partial_\nu \tilde{S} - \tilde{\Lambda} \right)
\]

It is hard to prove in general that those equations and the Klein-Gordon equations can always be consistently solved. However, we can show that this is the case for the most simple solution of the Klein-Gordon equation. The most simple solution of the first Klein-Gordon equation (15) is: \( \rho = \text{const}, \tilde{S}_\mu = p_\mu, \) and \( p^\mu p_\mu = \Lambda/G_D \). For this solution the nine independent equations (29) are obviously fulfilled since both sides of the equation vanish. A comparison with Eq. (13) further shows that \( n_1 = 0 \).

Thus, we have shown that, at least in the discussed special case, all twenty-five Einstein equations can be fulfilled consistently in our theory. As last consistency check we have to study the four remaining continuity equations from Eq. (20)

\[
\nabla_\nu \left( T^\mu_\nu \right) = \partial_\mu \left( g^{\nu\sigma} \tilde{S}_\mu \tilde{S}_\nu \right) - \Gamma^{\mu\nu}_{\rho\sigma} \left( g^{\gamma\nu} \tilde{S}_\rho \tilde{S}_\sigma \right)
\]

For the above solution of the Klein-Gordon equation this is the conservation law for the four momentum \( p_\mu \sim \tilde{S}_\mu \). Therefore, we have checked the consistency of an exemplary solution of our theory with all its thirty equations (7) (9) (20). Because of the condition (20) the Klein-Gordon equation on a flat Minkowski background can only result from our theory in an asymptotic region of spacetime. Since energy always curves spacetime, this what one would expect also from the standard interpretation of quantum mechanics and general relativity.

**What is the classical limit?**

The construction with its higher dimensional energy momentum tensor \( T_{\mu\nu} \) and the higher dimensional metric \( g_{\mu\nu} \) should (in the limit of large distancescales and large timescales) be consistent with the usual four dimensional classical general relativity. In this limit the \( \rho \) dependence and the terms proportional to \( \epsilon_i \) can be omitted. However, a different normalization in the metric expansion (13) is useful such that \( \tilde{g}_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}/2 \). Now Eq. (29) simplifies to

\[
\tilde{R}_{\mu\nu} = G_D \left( \tilde{S}_\mu \tilde{S}_\nu - \tilde{g}_{\mu\nu} \tilde{S}_\alpha \tilde{S}_\alpha \right) + \frac{\tilde{g}_{\mu\nu} \tilde{\Lambda}}{2} \quad ,
\]

For an infinitely small normalizing volume at the position of the particle trajectory we denote the density of the energy momentum tensor as \( \tilde{S}_\mu \tilde{S}_\nu \rightarrow \tilde{p}_\mu \tilde{p}_\nu \delta^3(\vec{x} - \vec{x}_n) / E \). Thus, we found the four dimensional Einstein field equations for a free particle (31)

\[
\tilde{R}_{\mu\nu} = G_D \left( \tilde{p}_\mu \tilde{p}_\nu + \frac{\tilde{g}_{\mu\nu} \tilde{p}^2}{2} \right) \delta^3(\vec{x} - \vec{x}_n) + \frac{\tilde{g}_{\mu\nu} \tilde{\Lambda}}{2} \quad .
\]
Please note that in contrast to [33, 34] the gravitating matter is free to propagate into all space and time dimensions and therefore the higher dimensional gravitational coupling $G_D$ is equal to the Newton constant $G_N$. This shows that classical general relativity can be obtained from the ansatz (11) after integrating out $i$ and taking the limit of large time and distance scales.

**Isn’t the cosmological constant very small?**

The higher dimensional cosmological term $\Lambda$ is on the one hand related to the mass in the quantum equation and on the other hand to the four dimensional cosmological constant $\Lambda_4$. The observed cosmological constant $\Lambda_4$ is a very small number [35] and therefore inconsistent with a typically much bigger particle mass in Eq. (19). A possible way out of the mass-cosmological constant problem might be found by considering higher orders in the exponential or by taking a possible $\bar{\alpha}$ into account. Even if the issue remains in our toy model, this might be a good sign because also most of the conventional quantum theories have a problem in getting the cosmological constant right [36].

**Which parameters had to be engineered?**

The presented mechanism only worked after demanding Eqs. (16) (19) to be true. The first equation (16) relates the classical principal functional $\tilde{S}$ to the phase of the quantum wave function $S_Q$. The second equation (19) relates the constant $\Lambda$ to the mass of the quantum particle.

**What about spinors?**

One of the great successes of quantum mechanics was the prediction and explanation of a particle with a Landé factor ($g = 2$) due to the Dirac equation. It will be a challenging task to find the analog of the Dirac equation in our picture. However, it has been shown by [37] that the $g = 2$ factor is also present in a Kerr-Newman spacetime. Later, [38] found out how to relate the electromagnetic field of a Dirac electron to the surrounding field of the Kerr-Newman solution. Those progresses indicate that there could be a way to explain spin 1/2 with the help of classical general relativity.

**Quantum field theory?**

Questions concerning the formulation of the second quantization in quantum field theory, interacting field theory, or Yang Mills theories in this picture are hopefully subject to future studies.

### IV. SUMMARY

In this paper we study the higher dimensional Einstein equations (11) and the higher dimensional continuity equation (21) with one additional time dimension $i$. For the higher dimensional metric we make the ansatz (4), and for the higher dimensional principal function we make the ansatz (11). Then we make an expansion in the parameter $\epsilon_i$ to ensure a small $i$ dependence. After defining the quantum phase (16) and the quantum mass (19) we find the structure of the Klein-Gordon equation in curved spacetime

\[ \Box \sqrt{\rho} = \frac{\sqrt{\rho}}{h^2} \left( (\partial \mu S_Q) (\partial \mu S_Q) - m^2 \right) + \mathcal{O}(\epsilon_i), \]

\[ 0 = \partial_i \left( \sqrt{-g} g^{\nu \rho} (\partial_i S_Q) \right) + \mathcal{O}(\epsilon_i). \]

Thus, by constructing this toy model we explicitly showed that it is possible to find a gravitational dual to the complex Klein-Gordon equation in curved spacetime. This achievement of the model is accompanied by a number of possible issues which have to be further understood. First, the weak $i$ dependence of the functions $S_H$ and $\bar{\alpha}$ is just an ansatz and not a necessity. Second, except of the case with $m = 0$ we could not find a solution of all twentyfive Einstein equations. Third, having a gravitational dual to the complex Klein-Gordon equation does not mean that one also has a dual for relativistic quantum mechanics. However, the toy model is one of several (very different) theories [15, 16, 17, 18, 19] that make a formal connection between the quantum structure of nature and an introduction of an additional time dimension. Our conclusion is that this connection could be more than a pure mathematical peculiarity, it might actually reflect the structure of spacetime.

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