HOW TO USE THE FUNCTIONAL EMPirical PROCESS FOR DERIVING
ASYMPTOTIC LAWS FOR FUNCTIONS OF THE SAMPLE

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Abstract. The functional empirical process is a very powerful tool for deriving asymptotic laws for almost any kind of statistics whenever we know how to express them into functions of the sample. Since this method seems to be applied more and more in the very recent future, this paper is intended to provide a complete but short description and justification of the method and to illustrate it with a non trivial example using bivariate data. It may also serve for citation without repeating the arguments.

1. Introduction

The idea of this paper came out from master and PhD courses on weak convergence theory of the author in many universities. We aimed at giving the attendance a powerful tool for finding the asymptotic normality law of almost any kind of statistics, whenever the latter exists and the expressions of the statistics in the sample are known. This tool is the functional empirical process. Usually, this theory is not taught in normal courses, but is usually part of dissertation projects. The reason is the theory is new and it is implicitly assumed that one has to deal with Donsker classes, Vapnik-Chervonenkis theory (see Vapnik and Chervonenkis (1971)) or entropy numbers, in brief that one has to deal with convergence of stochastic processes. Here, we do not need the big theory in van der vaart and Wellner van der Vaart and Wellner (1996), Gaenssler Gaenssler (1983) or in Pollard Pollard (1984) or in books like them.

But at the finite-distribution level, functional empirical processes can be reasonably easy to use and to justify while keeping its intrinsic power. We decided to write a paper that explains all about it at the theoretical level. And next to give a non trivial example. We choose to deal with the liner correlation that has two interests. On one hand, we deal with samples of couples of real random variables, to show that the method goes far beyond real random variables. On another hand, the computations are indeed somewhat heavy but are also reasonable to allow to serve as an example for any other case.

It turns now that the paper is not only for master students but for researchers in Probability and Statistics, in Economics, and other areas using stochastic methods. As an example, a new theory of arbitrary Jarque-Berra’s type normality law (see Jarque and Bera

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(1987)) is entirely based on this tool in Lo et al. (2015).

We are sure that readers of the paper will find it very useful in their everyday research works.

The remainder of the paper is organized as follows. In Section 2, we describe the parameter we are estimating, here the linear correlation coefficient, and its estimator and next establish the asymptotic normality. In Section 3, we describe our tool, the empirical process and provide all of its needed features. The proof of the results will be given finally in Section 4. We end the paper with a conclusion and recommendations in Section 5.

2. Weak convergence of the empirical linear correlation coefficient

As said already, we are going to illustrate our tool on the plug-in estimator of the linear correlation coefficient of two random variable \((X, Y)\), with neither of \(X\) and \(Y\) is degenerated, defined as follows

\[
\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}
\]

where

\[
\mu_x = \int x \, dP_X(x), \quad \mu_y = \int x \, dP_X(x), \quad \sigma_{xy} = \int (x - \mu_x)(y - \mu_y) \, dP_{(X,Y)}(x, y).
\]

\[
\sigma_x^2 = \int (x - \mu_x)^2 \, dP_X(x), \quad \sigma_y^2 = \int (x - \mu_x)^2 \, dP_X(y).
\]

We also dismiss the case the case where \(|\rho| = 1\), for which one of \(X\) and \(Y\) is an affine function of the other, for example \(X = aY + b\). It is clear that centering the variables \(X\) and \(Y\) and normalizing them by their standard deviations \(\sigma_x\) and \(\sigma_y\) does not change the correlation coefficients \(\rho\). So we may and do center \(X\) and \(Y\) at their expectations and normalize them so that we can and do assume that

\[
\mu_x = \mu_y = 0, \quad \sigma_x = \sigma_y = 1.
\]

However, we will let these coefficient appear with their names and we only use their particular values at the conclusion stage.

Let us consider the plug-in estimator of \(\rho\). To this end, let \((X_1, Y_1), (X_2, Y_2), ...\) be a sequence independent observations of \((X, Y)\). For each \(n \geq 1\), the plug-in estimator is the following

\[
\rho_n = \left\{ \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})(Y_i - \overline{Y}) \right\}^{1/2} \left\{ \frac{1}{n^2} \sum_{i=1}^{n} (X_i - \overline{X})^2 \times \sum_{i=1}^{n} (Y_i - \overline{Y})^2 \right\}^{-1/2}
\]

We are going to give the asymptotic theory of \(\rho_n\) as an estimator of \(\rho\). Introduce the notation

\[
\mu_{(p,x),(q,y)} = E((X - \mu_x)^p(Y - \mu_y)^q), \quad \mu_{4,x} = E(X - \mu_x)^4, \quad \mu_{4,y} = E(Y - \mu_y)^4
\]
Here is our main Theorem

**Theorem 1.** Suppose that neither of $X$ and $Y$ is degenerated and both have finite fourth moments and that $X^3Y$ and $XY^3$ have finite expectations. Then, as $n \to \infty$,

$$
\sqrt{n}(\rho_n - \rho) \rightsquigarrow N(0, \sigma^2),
$$

where

$$
\sigma^2 = \sigma_x^{-2}\sigma_y^{-2}(1 + \rho^2/2)\mu_{(2,x),(2,y)} + \rho^2(\sigma_x^{-4}\mu_{4,x} + \sigma_y^{-4}\mu_{4,y})/4 - \rho(\sigma_x^{-3}\sigma_y^{-1}\mu_{3,x,(1,y)} + \sigma_x^{-1}\sigma_y^{-3}\mu_{(1,x),(3,y)})
$$

This result enables to test independence between $X$ and $Y$, or to test non linear correlation in the following sense.

**Theorem 2.** Suppose that the assumptions of Theorem 1 hold. Then

(1) If $X$ and $Y$ are not linearly correlated, that is $\rho = 0$, we have

$$
\sqrt{n}\rho_n \rightsquigarrow N(0, \sigma_1^2),
$$

where

$$
\sigma_1^2 = \sigma_x^{-2}\sigma_y^{-2}\mu_{(2,x),(2,y)}.
$$

(2) If $X$ and $Y$ are independent, then $\rho = 0$, and

$$
\sqrt{n}\rho_n \rightsquigarrow N(0, 1)
$$

3. **The Functional Empirical Process and Other Tools**

Let $Z_1, Z_2, \ldots$ be a sequence of independent copies of a random variable $Z$ defined on the same probability space with values on some metric space $(S, d)$. Define for each $n \geq 1$, the functional empirical process by

$$
\mathbb{G}_n(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f(Z_i) - \mathbb{E}f(Z_i)),
$$

where $f$ is a real and measurable function defined on $\mathbb{R}$ such that

(3.1) \hspace{1cm} \mathbb{V}_Z(f) = \int (f(x) - \mathbb{P}_Z(f))^2 \, dP_Z(x) \leq \infty,

which entails

(3.2) \hspace{1cm} \mathbb{P}_Z(|f|) = \int |f(x)| \, dP_Z(x) \leq \infty.
Denote by $F(S)$ - $F$ for short - the class of real-valued measurable functions that are defined on $S$ such that (3.1) holds. The space $F$, when endowed with the addition and the external multiplication by real scalars, is a linear space. Next, it remarkable that $G_n$ is linear on $F$, that is for $f$ and $g$ in $F$ and for $(a, b) \in \mathbb{R}^2$, we have

$$aG_n(f) + bG_n(g) = G_n(af + bg).$$

We have this result

**Lemma 1.** Given the notation above, then for any finite number of elements $f_1, ..., f_k$ of $S, k \geq 1$, we have

$$t^t(G_n(f_1), ..., G_n(f_k)) \rightsquigarrow N_k(0, \Gamma(f_i, f_j)_{1 \leq i, j \leq k}),$$

where

$$\Gamma(f_i, f_j) = \int (f_i - P_Z(f_i))(f_j - P_Z(f_j)) dP_Z(x), 1 \leq j \leq k.$$

**Proof.** It is enough to use the Cramér-Wold Criterion (see for example Billingsley (1968), page 45), that is to show that for any $a = (a_1, ..., a_k) \in \mathbb{R}^k$, by denoting $T_n = t^t(G_n(f_1), ..., G_n(f_k))$, we have $<a, T_n> \rightsquigarrow <a, T>$ where $T$ follows the $N_k(0, \Gamma(f_i, f_j)_{1 \leq i, j \leq k})$ law and $<\cdot, \cdot>$ stands for the usual product scalar in $\mathbb{R}^k$. But, by the standard central limit theorem in $\mathbb{R}$, we have

$$<a, T_n> = G_n\left(\sum_{i=1}^k a_i f_i\right) \rightsquigarrow N(0, \sigma^2_\infty),$$

where, for $g = \sum_{1 \leq i \leq k} a_i f_i$,

$$\sigma^2_\infty = \int (g(x) - P_Z(g))^2 dP_Z(x)$$

and this easily gives

$$\sigma^2_\infty = \sum_{1 \leq i, j \leq k} a_i a_j \Gamma(f_i, f_j),$$

so that $N(0, \sigma^2_\infty)$ is the law of $<a, T>$. The proof is finished.
3.1. **How to use the tool.** We usually work with usual asymptotic statistics on $\mathbb{R}^k$. Once we have our sample $Z_1, Z_2, \ldots$ as random variables defined in the same probability space with values in $\mathbb{R}^k$, the studied statistics, say $T_n$, is usually a combinations of expressions of the form

$$H_n = \frac{1}{n} \sum_{i=1}^k H(Z_i)$$

for $H \in \mathcal{F}$. We use this simple expansion, for $\mu(H) = \mathbb{E}H(Z)$,

$$H_n = \mu(H) + n^{-1/2} G_n(H).$$

We have that $G_n(H)$ is asymptotically bounded in probability since $G_n(H)$ weakly converges to, say $M(H)$ and then by the continuous mapping theorem $\|G_n(H)\| \Rightarrow \|M(H)\|$. Since all the $G_n(H)$ are defined on the same probability space, we get for all $\lambda > 0$, by the assertion of the Portmanteau Theorem for concerning open sets,

$$\lim_{n \to \infty} \sup_{\lambda \to \infty} P(\|G_n(H)\| > \lambda) \leq P(\|M(H)\| > \lambda)$$

and then

$$\lim_{\lambda \to \infty} \lim_{n \to \infty} \sup_{\lambda \to \infty} P(\|G_n(H)\| > \lambda) \leq \lim_{n \to \infty} P(\|M(H)\| > \lambda) = 0.$$

From this, we use the big $O_p$ notation, that is $G_n(H) = O_p(1)$. Formula (3.3) becomes

$$H_n = \mu(H) + n^{-1/2} G_n(H) = \mu(H) + O_p(n^{-1/2})$$

and we will be able to use the delta method. Indeed, let $g : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable on a neighborhood of $\mu(H)$. The mean value theorem leads to

$$g(H_n) = g(\mu(H)) + g'(\mu_n(H)) n^{-1/2} G_n(H)$$

where

$$\mu_n(H) \in [(\mu(H) + n^{-1/2} G_n(H)) \land \mu(H), (\mu(H) + n^{-1/2} G_n(H)) \lor \mu(H)]$$

so that

$$|\mu_n(H) - \mu(H)| \leq n^{-1/2} G_n(H) = O_p(n^{-1/2}).$$

Then $\mu_n(H)$ converges to $\mu_n(H)$ in probability (denoted $\mu_n(H) \Rightarrow \mu_n(H)$). But the convergence in probability to a constant is equivalent to the weak convergence. Then $\mu_n(H) \Rightarrow \mu(H)$. Using again the continuous mapping theorem, $g'(\mu_n(H)) \Rightarrow g'(\mu(H))$ which in turn yields $g'(\mu_n(H)) \Rightarrow g'(\mu(H))$ by the characterization of the weak convergence to a constant. Now (3.4) becomes

$$g(H_n) = g(\mu(H)) + (g'(\mu(H)) + o_P(1)) n^{-1/2} G_n(H)$$

$$= g(\mu(H)) + g'(\mu(H)) n^{-1/2} G_n(H) + o_P(1) n^{-1/2} G_n(H)$$

$$= g(\mu(H)) + n^{-1/2} G_n(g'(\mu(H)) H) + o_P(n^{-1/2})$$
We arrive at the final expansion
\begin{equation}
(3.5) \quad g(H_n) = g(\mu(H)) + n^{-1/2} \mathbb{G}_n(g'(\mu(H)H) + o_P(n^{-1/2}).
\end{equation}

The method consists in using the expansion (3.5) as many times as needed and next to do some algebra on these expansions. By using the same techniques as above, we have these three formulas

**Lemma 2.** Let \((A_n)\) and \((B_n)\) be two sequences of real valued random variables defined on the same probability space holding the sequence \(Z_1, Z_2, \ldots\) Let \(A\) and \(B\) be two real numbers and let \(L(z)\) and \(H(z)\) be two real-valued functions of \(z \in S\). Suppose that \(A_n = A + n^{-1/2} \mathbb{G}_n(L) + o_P(n^{-1/2})\) and \(B_n = B + n^{-1/2} \mathbb{G}_n(H) + o_P(n^{-1/2}).\) Then
\begin{align*}
A_n + B_n &= A + B + n^{-1/2} \mathbb{G}_n(L + H) + o_P(n^{-1/2}), \\
A_nB_n &= AB + n^{-1/2} \mathbb{G}_n(BL + AH) + o_P(n^{-1/2}).
\end{align*}

and if \(B \neq 0,\)
\begin{equation}
\frac{A_n}{B_n} = \frac{A}{B} + n^{-1/2} \mathbb{G}_n \left( \frac{1}{B}L - \frac{A}{B^2}H \right) + o_P(n^{-1/2}).
\end{equation}

By putting together all the described steps in a smart way, the methodology will lead us to a final result of the form
\begin{equation}
T_n = t + n^{-1/2} \mathbb{G}_n(h) + o_P(n^{-1/2})
\end{equation}
which entails the weak convergence
\begin{equation}
\sqrt{n}(T_n - t) = \mathbb{G}_n(h) + o_P(1) \rightsquigarrow N(0, \Gamma(h, h)).
\end{equation}

We are now in position to apply right here the methodology on the empirical linear correlation coefficient.

### 4. Proofs of the results

We are going to use the function empirical process based on the observations \((X_i, Y_i), i = 1, 2, \ldots\) that are independent copies of \((X, Y)\). Write
\begin{equation}
\rho_n^2 = \frac{\frac{1}{n} \sum_{i=1}^{n} X_i Y_i - \overline{X} \overline{Y}}{\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \overline{X}^2 \right\}^{1/2} \left\{ \frac{1}{n} \sum_{i=1}^{n} Y_i^2 - \overline{Y}^2 \right\}^{1/2}} = \frac{A_n}{B_n}.
\end{equation}

Let us say for once that all the functions of \((X, Y)\) that will appear below are measurable and have finite second moments. Let us handle separately the numerator and denominator. To treat \(A_n\), using the empirical process implies that
\[
\begin{align*}
\frac{1}{n} \sum_{i=1}^{n} X_i Y_i &= \mu_{xy} + n^{-1/2} G_n(p), \\
\bar{X} &= \mu_x + n^{-1/2} G_n(\pi_1), \\
\bar{Y} &= \mu_y + n^{-1/2} G_n(\pi_2),
\end{align*}
\]

where \( p(x, y) = xy, \pi_1(x, y) = x, \pi_2(x, y) = y. \) From there we use the fact that \( G_n(g) = O_P(1) \)
for \( E(g(X, Y)^2) < +\infty \) and get
\[
A_n = \mu_{xy} + n^{-1/2} G_n(p) - (\mu_x + n^{-1/2} G_n(\pi_1))(\mu_y + n^{-1/2} G_n(\pi_2)).
\]
This leads to
\[
A_n = \sigma_{xy} + n^{-1/2} G_n(H_1) + o_P(n^{-1/2})
\]
with
\[
H_1(x, y) = p(x, y) - \mu_x \pi_2 - \mu_y \pi_1.
\]

Next, we have to handle \( B_n. \) Since the roles of \( \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}^2 \right\}^{1/2} \) and of \( \left\{ \frac{1}{n} \sum_{i=1}^{n} Y_i^2 - \bar{Y}^2 \right\}^{1/2} \) are symmetrical, we treat one of them and extend the results to the other. Consider \( \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}^2 \right\}^{1/2}. \) From (4.1), use the delta method to get
\[
\bar{X}^2 = \left( \mu_x + n^{-1/2} G_n(\pi_1) \right)^2 = \mu_x^2 + 2\mu_x n^{-1/2} G_n(\pi_1) + o_P(n^{-1/2})
\]
that is
\[
\bar{X}^2 = \left( \mu_x + n^{-1/2} G_n(\pi_1) \right)^2 = \mu_x^2 + n^{-1/2} G_n(2\mu_x \pi_1) + o_P(n^{-1/2}).
\]

From there, we get
\[
\frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}^2 = m_{2,x} + n^{-1/2} G_n(\pi_1^2) - \bar{X}^2
\]
\[
= m_{2,x} - \mu_x^2 + n^{-1/2} G_n(\pi_1^2 - 2\mu_x \pi_1) + o_P(n^{-1/2})
\]
\[
= \sigma_x^2 + n^{-1/2} G_n(\pi_1^2 - 2\mu_x \pi_1) + o_P(n^{-1/2}).
\]

Using the Delta-method once again leads to
\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}^2 \right\}^{1/2} = \sigma_x + n^{-1/2} G_n(\frac{1}{2\sigma_x} \{ \pi_1^2 - 2\mu_x \pi_1 \}) + o_P(n^{-1/2}).
\]

In a similar way, we get
\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} Y_i^2 - \bar{Y}^2 \right\}^{1/2} = \sigma_y + n^{-1/2} G_n(\frac{1}{2\sigma_y} \{ \pi_2^2 - 2\mu_y \pi_2 \}) + o_P(n^{-1/2}).
\]
We arrive at
\[ B_n = \left\{ \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \overline{X}^2 \right\}^{1/2} \left\{ \frac{1}{n} \sum_{i=1}^{n} Y_i^2 - \overline{Y}^2 \right\}^{1/2} = \sigma_x \sigma_y + n^{-1/2}G_n \left( \frac{\sigma_y}{2\sigma_x} \left\{ \sigma_x^2 - 2 \mu_x \pi_1 \} + \frac{\sigma_x}{2\sigma_y} \left\{ \sigma_y^2 - 2 \mu_y \pi_2 \right\} \right) \]

By setting
\[ H_2(x, y) = \frac{\sigma_y}{2\sigma_x} \left\{ \sigma_x^2 - 2 \mu_x \pi_1 \} + \frac{\sigma_x}{2\sigma_y} \left\{ \sigma_y^2 - 2 \mu_y \pi_2 \right\} \]
we have
(4.3) \[ B_n = \sigma_x \sigma_y + n^{-1/2}G_n(H_2) + n^{-1/2}. \]

Now, combining (4.2) and (4.3) and using Lemma 1 yield
\[ \sqrt{n}(\rho_n^2 - \rho^2) = n^{-1/2}G_n \left( \frac{1}{\sigma_x \sigma_y} H_1 - \frac{\sigma_{xy}}{\sigma_x^2 \sigma_y^2} H_2 \right) + o_P(1). \]

Put
\[ H = \frac{1}{\sigma_x \sigma_y} (p(x, y) - \mu_x \pi_2 - \mu_y \pi_1) - \frac{\rho}{\sigma_x \sigma_y} \left\{ \frac{1}{2\sigma_x} \left\{ \sigma_y^2 - 2 \mu_y \pi_2 \right\} \right\} + \frac{1}{2\sigma_y} \left\{ \sigma_x^2 - 2 \mu_x \pi_1 \right\} \]

Now we continue with the centered and normalized case to get
\[ H(x, y) = p(x, y) - \frac{\rho}{2}(\pi_1^2 + \pi_2^2) \]
and
\[ H(X, Y) = XY - \frac{\rho}{2}(X^2 + Y^2). \]

Denote
\[ \mu_{(p,x),(q,y)} = E((X - \mu_x)^p(Y - \mu_y)^q). \]

We have
\[ EH(X, Y) = \sigma_{xy} - \rho = 0 \]
and \( \text{var}H(X, Y) \) is equal to
\[ \mu_{(2,x),(2,y)} + \rho^2(\mu_{4,x} + \mu_{4,y})/4 - \rho(\mu_{3,x},(1,y) + \mu_{1,x},(3,y)) + \rho^2(\mu_{2,x},(2,y))/2 \]
and finally \( \text{var}H(X, Y) = \sigma_0^2 \) with
\[ \sigma_0^2 = \frac{1}{1 + \rho^2/2}\mu_{(2,x),(2,y)} + \rho^2(\mu_{4,x} + \mu_{4,y})/4 - \rho(\mu_{3,x},(1,y) + \mu_{1,x},(3,y)). \]

This gives the conclusion that for centered and normalized \( X \) and \( Y, \)
\[ \sqrt{n}(\rho_n - \rho) \sim N(0, \sigma_0^2). \]
Next, if we use the normalizing coefficients in $\sigma_0$, we get

$$ \sigma^2 = \sigma^2_x \sigma^2_y (1 + \rho^2 / 2) \mu_{(2,x),(2,y)} + \rho^2 (\sigma^4_x \mu_{4,x} + \sigma^4_y \mu_{4,y}) / 4 $$

and we conclude in the general case that

$$ \sqrt{n}(\rho_n - \rho) \xrightarrow{d} N(0, \sigma^2) $$

The proof of Theorem 2 follows by easy computations under the particular conditions of $\rho$ and under independence.

5. Conclusion and recommendations

This paper was intended to learn how to use the functional empirical process for deriving asymptotic normality laws. We saw that the method is powerful and sometimes straightforward. But the computations may be huge. Fortunately, in each case, we might be able to use specific properties of studied statistics. In the example we treated, we use the property that the correlation coefficient remains the same under location and scale parameter changes. This allowed to reduce the computations to centered and normalized random variables.

This works has sparked in our mind the ideas of a handbook in which as much as possible applications of this method in any area of Probability theory and Statistics will be gathered. We will surely go into it in the future

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