Cosmic magnetism, curvature and the expansion dynamics

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Abstract

We discuss how a cosmological magnetic field could affect the expansion of the universe, through its interaction with the spacetime geometry. The tension of the field lines means that the magneto-curvature coupling tends to accelerate positively curved regions and decelerate those with negative curvature. Depending on the equation of state of the matter, the effect varies from mimicking that of a cosmological constant to resembling a time-decaying quintessence. Most interestingly, the coupling between magnetism and geometry implies that even weak fields have a significant impact if the curvature contribution is strong. This leads to kinematical complications that can inhibit the onset of the accelerated phase in spatially open inflationary models. We employ a simple cosmological model to illustrate these effects and examine the conditions necessary for them to have a appreciable impact.

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1 Introduction

Current observations provide strong evidence for the widespread presence of magnetic fields in the universe. Magnetic fields appear to be a common property of the intracluster medium of galaxy clusters, extending well beyond the core regions. Strengths of ordered magnetic fields in the intracluster medium of cooling-flow clusters exceed those typically associated with the interstellar medium of the Milky Way, suggesting that galaxy formation and even cluster dynamics are, at least in some cases, influenced by magnetic forces. Furthermore, reports of Faraday rotation associated with high redshift Lyman-α absorption systems seem to imply that dynamically significant magnetic fields may be present in condensations at high redshift. In summary, the more we look for extragalactic magnetic fields, the more ubiquitous we find them to be.

The origin of cosmic magnetism remains a mystery and is still a matter of debate. Over the years, a number of possible solutions has been proposed, ranging from eddies and density fluctuations in the early plasma to cosmological phase-transitions, inflationary and superstring

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inspired scenarios \[3\]. Historically, studies of magnetogenesis were motivated by the need to explain the origin of large-scale galactic fields. Typical spiral galaxies have magnetic fields of the order of a few \(\mu G\) coherent over the plane of their disc. Such fields could arise from a relatively large primordial seed field, adiabatically amplified by the collapse of the protogalaxy, or by a much weaker one that has been strengthened by the galactic dynamo. Provided that this mechanism is efficient, the seed can be as low as \(\sim 10^{-23} G\) at present. However, in the absence of nonlinear dynamo amplification, seeds of the order of \(10^{-12} G\) or even \(10^{-8} G\) are required \[4\].

Magnetic fields introduce new ingredients into the standard, but nevertheless uncertain, picture of the early universe. A fundamental and unique property of magnetic fields is their vectorial nature, which couples the field to the spacetime geometry via the Ricci identity (see Eq. \(4\)). An additional, also unique, characteristic is the tension (i.e. the negative pressure) exerted along the field’s lines of force. This means that every small magnetic flux tube behaves like an infinitely elastic rubber band \[5\]. Intuitively, what the magneto-curvature coupling does, is to inject these elastic properties of the field into space itself. The implications of such an interaction are kinematical as well as dynamical with quite unexpected results. Kinematically speaking, the magneto-curvature effect tends to accelerate positively curved perturbed regions, while it decelerates regions with negative local curvature \[6\]. Dynamically, the most important magneto-curvature effect is that it can reverse the pure magnetic effect on density perturbations. To be precise, in the absence of curvature, the field is found to slow down the growth of density gradients \[6\]. However, when curvature is taken into account, the inhibiting magnetic effect is reduced and, in the case of ‘maximum’ spatial curvature contribution, even reversed \[7\]. Here, we focus upon the kinematics and provide an example of how a cosmological magnetic field can modify, through its coupling to geometry, the expansion rate of an almost-FRW universe.

We assume a spacetime filled with a perfectly conducting barotropic fluid and permeated by a weak primordial magnetic field. The energy density and the anisotropic pressure of the field are treated as first-order perturbations upon the FRW background. The vectorial nature of the field results in a magneto-geometrical term in the Raychaudhuri equation, which depends on the curvature of the unperturbed model. The negative pressure, that is the tension, carried by the magnetic force-lines makes the implications of this term unique. Qualitatively speaking the effect depends on the curvature sign of the background spacelike sections. When the unperturbed universe is spatially open, the magneto-geometrical term adds to the decelerating effect of ordinary matter. For a spatially closed background, however, the magneto-curvature contribution tends to accelerate the expansion. In both cases the magnetic tension brings the expansion rate closer to that of a flat FRW model. Quantitatively, the effect depends on the relative strength of the field and on the type of matter that fills the universe. In particular, the magneto-geometrical term remains constant throughout an epoch of stiff-matter domination. During this period the field acts as an effective (positive or negative) cosmological constant. As the universe progresses into the radiation and subsequently the dust era, the magneto-curvature term progressively decreases mimicking a time-decaying quintessence. Under normal circumstances these effects are relatively weak, although subtle enough to make an open FRW universe look less open and a closed one look less closed. On the other hand, the magneto-curvature effect on the expansion can be dramatic, if the field or the curvature are strong. Moreover, even weak magnetic fields have a significant overall impact in a strongly curved universe. In fact, the mere magnetic presence in spatially open inflationary models leads to kinematical complications that
can suppress the onset of the accelerated phase. In a particular example, weakly magnetised FRW universes with $p = -\rho$ cannot enter a period of de Sitter inflation as long as $\Omega < 0.5$.

In what follows, we employ the covariant perturbation formalism \cite{8, 9} (applied to magnetised cosmologies in \cite{6, 7}) to illustrate the kinematical implications of cosmological magnetic fields. The reader is referred to the aforementioned articles for further discussion and details.

2 Magneto-curvature effects on the expansion

We consider a perturbed, slightly inhomogeneous and anisotropic, FRW universe filled with a single perfectly conducting barotropic medium with energy density $\rho$. We also allow for a magnetic field ($B_a$), which is weak relative to the dominant matter component (i.e. $B^2 = B_a B_a \ll \rho$). The magnetic field is assumed to be a test-field on the FRW background. The energy density ($\rho_{\text{mag}} = \frac{1}{2} B^2$), the isotropic pressure ($p_{\text{mag}} = \frac{1}{6} B^2$) and the anisotropic stresses ($\pi_{ab} = -B_{(a} B_{b)}$) of the field will be treated as first-order perturbations.

Note that the spatial hypersurfaces of the unperturbed model may be closed or open as well as flat. Thus, the zero-order Friedmann equation is given by

$$\frac{3k}{a^2} = 8\pi G \rho_0 - 3H^2 + \Lambda,$$

where $k = 0, \pm 1$ is the curvature index of the spatial hypersurfaces, $\rho_0$ is the background matter density, $H = \dot{a}/a$ is the Hubble parameter ($a$ is the scale factor) and $\Lambda$ the cosmological constant.

In the background $H = \frac{1}{3} \Theta$, where $\Theta$ describes the rate of the (volume) expansion and obeys Raychaudhuri's formula. In its non-linear form, the latter reads

$$\dot{\Theta} + \frac{1}{3} \Theta^2 + 4\pi G \left( \rho + 3p + B^2 \right) - D^a A_a - A_a A^a + 2 \left( \sigma^2 - \omega^2 \right) - \Lambda = 0,$$

where $p$ is the fluid pressure, $A_a$ is the 4-acceleration and $\sigma^2, \omega^2$ are respectively the shear and vorticity magnitudes. Note that $D_a$ is the covariant derivative operator projected orthogonally to the fluid flow. The crucial magnetic effects propagate through the fluid acceleration, which satisfies the non-linear Euler equation

$$\left( \rho + p + \frac{2}{3} B^2 \right) A_a + c_s^2 D_a \rho + \varepsilon_{abc} B^b \text{curl} B^c + A^b \pi_{ba} = 0,$$

where $c_s^2$ is the sound speed of the barotropic fluid. Linearising the divergence $D^a A_a$ and using the commutation law between the projected gradients of spacelike vectors, namely

$$D_{[a} D_{b]} B_c = \frac{1}{2} R_{dcba} B^d - \varepsilon_{abcd} \omega^d \hat{B}_{(c)},$$

we can calculate the first-order magnetic contribution to Eq. \cite{3}. Note that formula \cite{4}, also known as the 3-Ricci identity, is the source of the magneto-curvature coupling discussed here. It illustrates the vectorial nature of the field and leads inevitably to curvature-dependent terms every time the gradients of the magnetic vector commute. Here, it is written in its exact, fully

\footnote{Angled brackets denote the projected, symmetric, trace-free part of tensors and the orthogonal projections of vectors. The same notation is also used for the orthogonally projected time derivatives.}
non-linear, form. Thus, $\omega_a$ and $\mathcal{R}_{abcd}$ are respectively the vorticity vector and the ‘spatial’ Riemann tensor of the real spacetime.

Expressed in terms of the deceleration parameter and given the weakness of the field, the linearised Raychaudhuri equation becomes

$$\frac{1}{3} \dot{\Theta}^2 q = 4\pi G \rho (1 + 3w) - \frac{2k c_s^2}{(1 + w)a^2} + \frac{c_s^2 \Delta}{(1 + w)a^2} + \frac{c_s^2 B}{2(1 + w)a^2} - \frac{2}{\rho (1 + w)} \left[ (D_{(a} B_{b)})^2 - (D_{[a} B_{b]})^2 \right] - \Lambda,$$

where $q = -1 - 3\dot{\Theta}/\Theta^2$ is the deceleration parameter, $w = p/\rho$ and $c_s^2 = B^2/\rho$ is the Alfvén speed. In deriving Eq. (5) we have used the fact that $D^a B_a = 0$, and employed the zero-order expressions $\mathcal{R}_{ab} = \frac{1}{3} \mathcal{R} h_{ab}$ and $\mathcal{R} = 6k/a^2$ for the spatial Ricci tensor and Ricci scalar respectively (recall that $\mathcal{R}_{ab} = \mathcal{R}^{c}_{\quad ab}$). Also, the scalars $\Delta = (a^2/\rho) D^2 \rho$ and $\mathcal{B} = (a^2/B^2) D^2 B^2$ represent fluctuations in the matter and the magnetic energy densities respectively.

Clearly, the sign of the right-hand side of Eq. (5) determines the state of the expansion. Negative terms accelerate the universe, while positive ones slow the expansion down. Thus, the coupling between the magnetic field and the background spatial curvature affects the average deceleration of a perturbed magnetised FRW universe. Qualitatively, the effect depends on the geometry of the spacelike hypersurfaces. In particular, when $k = -1$, the magneto-curvature term in Eq. (5) simply adds to the gravitational pull of the (ordinary) matter component.

Locally, the first-order scalars $\Delta$ and $\mathcal{B}$ are either positive or negative, depending on whether the perturbed region is respectively over-dense or under-dense. On average, however, one expects that $\Delta = 0 = \mathcal{B}$. On the other hand, the mean $\rho$ and $c_s^2$ are always positive. As a result, the spatial average of Eq. (5) gives

$$\frac{1}{3} \dot{\Theta}^2 q = 4\pi G \rho (1 + 3w) - \frac{2k c_s^2}{(1 + w)a^2} - \frac{c_s^2 B}{2(1 + w)a^2} - \Lambda,$$

with $\dot{\rho} = -(1 + w)\Theta \rho$ and $(c_s^2)^* = -\frac{1}{2}(1 - 3w)\Theta c_s^2$, given that $(B^2)^* = -\frac{4}{3} \Theta B^2$ to first order. Thus, the coupling between the magnetic field and the background spatial curvature affects the average deceleration of a perturbed magnetised FRW universe. Qualitatively, the effect depends on the geometry of the spacelike hypersurfaces. In particular, when $k = -1$, the magneto-curvature term in Eq. (5) simply adds to the gravitational pull of the (ordinary) matter component. On the other hand, for a spatially closed background the magneto-curvature coupling tends to accelerate the expansion, thus opposing the matter effect. In both cases the overall result of the
field presence is to bring the expansion rate closer to that of a flat universe. This unconventional behaviour is caused by the negative pressure experienced along the magnetic lines of force, that is by the field’s tension. The latter tends to smooth out the kinematic effects of curvature, imprinted in Eq. (1), by modifying the expansion rate of the universe accordingly. Intuitively, one might argue that the magneto-curvature coupling has transferred the elastic properties of the field into space itself [6, 7]. Note that for $k = +1$ the field will reverse the fluid effect on the expansion, if the magneto-geometrical term in Eq. (7) is stronger than the matter term. This is possible when the field is strong or when the spatial regions are strongly curved. Even a moderate magneto-curvature contribution, however, can boost the expansion rate of a closed FRW universe and make it look less closed, or slow down an open one to make it look less open.

How important the above effects are and what period in the lifetime of the universe they affect most, depends on the type of the matter that fills the universe. As we shall see next, for cosmological models with ordinary matter (i.e. $0 \leq w \leq 1$), the most intriguing magneto-curvature effects occur in a $k = +1$ model. On the other hand, if exotic matter dominates (e.g. $-1 \leq w \leq -\frac{1}{3}$), the magneto-curvature coupling is crucial when $k = -1$. This case also offers an example of how a relatively weak magnetic field can have a strong impact.

3 The magnetic field as an effective cosmological constant

Equation (7) raises the interesting question as to whether the magneto-curvature term can mimic a cosmological constant. The answer is positive, depending on the matter component of the universe. To be precise, the magneto-geometrical term in Eq. (5) is time-independent if $c_a^2 \propto a^2$. Given that $c_a^2 \propto a^{3w-1}$ this happens when $w = 1$. Thus, provided that stiff matter dominates, the magnetic field introduces an effective $\Lambda$-term through its coupling to the background curvature. Such an effective cosmological constant is positive when the unperturbed FRW universe is positively curved and negative if $k = -1$. Here, we will focus upon the $k = +1$ case because then the magnetic effects on the expansion oppose those of the matter. We set $\Lambda = 0$ and consider an early period with $p = \rho$. Then Eq. (7) becomes

$$\frac{1}{2} \Theta^2 q = 16\pi G \rho - \frac{c_a^2}{a^2},$$

(8)

where $c_a^2 \propto a^2$ since $\rho \propto a^{-6}$. Note how the magneto-curvature term acts as a positive cosmological constant, having effectively replaced $\Lambda$. This term leads to exponential expansion as long as it dominates the right-hand side of Eq. (8). On using expression (7), Eq. (8) gives

$$\frac{1}{3} \Theta^2 q = 6H^2 \Omega \left[ 1 - \frac{(\Omega_0 - 1) c_a^2}{6\Omega} \right],$$

(9)

where $\Omega \equiv \rho/\rho_c$, $\Omega_0 \equiv \rho_0/\rho_c$ are respectively the average and background density parameters, with $\rho_c \equiv 3H^2/8\pi G$ representing the background critical density. For a weak magnetic field $\rho \simeq \rho_0$ on average, which means that $\Omega \simeq \Omega_0$. On these grounds, we will no longer distinguish between $\Omega$ and $\Omega_0$ but use them interchangeably. Hence, the expansion is accelerated if

$$c_a^2 > \frac{6\Omega}{\Omega - 1},$$

(10)
This implies that marginally closed universes, with $0 < \Omega - 1 \ll 1$, require very strong magnetic fields to accelerate. When $\Omega - 1 \sim 1$, however, a cosmological field with energy density comparable to that of the stiff matter could trigger a period of accelerated expansion. Stiff-matter FRW models are encountered in the so-called pre-Big-Bang scenarios as the dual counterparts of the string-theory inspired dilaton cosmologies. They also correspond to scalar-field models dominated by the field’s kinetic energy [11].

One should keep in mind that as the magnetic field gets stronger the almost-FRW treatment given here becomes less reliable. In this respect, condition (10) should only be taken as indicative. Having said that, studies of perturbed magnetised Bianchi I models have shown that, qualitatively speaking, the magnetic effects on average scalars (such as $\Theta$) remain very close to those predicted by the FRW treatments (see Eq. (68) in [f]).

In Eq. (8) the magneto-geometrical term drops slower than the matter term, which means that it can accelerate the expansion later in the stiff-matter era. Using the evolution law $\rho = \rho_*(1 + z)^6$, we find that the acceleration starts at redshift

$$z \simeq -1 + \sqrt{\frac{(c_a^2)_*}{6 \Omega_*} (\Omega_* - 1)}$$

where for convenience $z_* = 0$ at the end of the stiff-matter era rather than today.

4 The magnetic field as quintessence

Let us now consider a radiation dominated universe. When $w = \frac{1}{3}$ Eq. (9) becomes

$$\frac{1}{3} \Theta^2 q = 8\pi G \rho - \frac{3 c_a^2}{2a^2},$$

with $\rho \propto a^{-4}$ and $c_a^2 = \text{constant}$. The magneto-curvature term is no longer constant but decreases as $a^{-2}$, slower than the matter term. Here, the magneto-geometrical effects resemble those attributed to a time-decaying quintessence [12]. As before, Eqs. (11) and (12) imply that the accelerated phase commences if $c_a^2 > 2\Omega / (\Omega - 1)$ at redshift $z \simeq -1 + \sqrt{(c_a^2)_0 (\Omega_0 - 1) / 2\Omega_0}$, the latter measured at the time of matter-radiation equality. Although magnetic fields of such strength are not allowed at nucleosynthesis, [13], they are not a priori excluded earlier in the radiation era. Neutrino damping means that relatively strong magnetic fields in the early radiation era can efficiently dissipate their energy to satisfy the nucleosynthesis limits [14].

Qualitatively speaking, the picture does not change in the dust era. The difference now is that the magneto-curvature term drops as fast as the fluid term (i.e. $\propto a^{-3}$). Similarly to the radiation era, the field mimics a time-decaying quintessence and leads to accelerated expansion if $c_a^2 > 3\Omega / 4 (\Omega - 1)$. Such magnetic fields, however, are beyond the limits set by current observations [15]. Nevertheless, even weak fields can slightly accelerate the expansion and thus make a magnetised closed FRW universe look less closed than it actually is. Note that when $k = -1$ only the sign of the effects discussed so far changes. In spatially open models the magneto-curvature effects on the expansion are complementary to those of the ordinary matter.
5 Strong effects from weak magnetic fields

So far we have restricted ourselves to magnetised cosmologies filled with ordinary matter (i.e. $0 \leq w \leq 1$). In these environments the magneto-curvature effects, subtle though they may be, remain secondary unless the field is relatively strong. However, strong magnetic fields are not always necessary for the magneto-curvature effect to be significant. In fact, the aforementioned interaction between magnetism and geometry may also challenge the widespread perception that magnetic fields are relatively unimportant for cosmology. This belief is based on current observations, which point towards a weak magnetic presence at nucleosynthesis and recombination. However, the magneto-curvature coupling could make the field into a key player irrespective of the magnetic strength. In principle, even weak magnetic fields can lead to appreciable effects, provided that there is a strong curvature contribution. To illustrate how this can happen, we turn to spatially open cosmological models containing matter with negative pressure. For $k = -1$ Eq. (7) becomes

$$\frac{1}{3} \Theta^2 q = 4\pi G \rho (1 + 3w) + \frac{2c_s^2}{(1 + w)a^2},$$

(13)

where we have set $\Lambda = 0$. We begin with a simple qualitative argument. In Eq. (13) the magneto-curvature term evolves as $c_s^2/a^2 \propto a^{-3(1-w)}$ and the matter term obeys the standard evolution law $\rho \propto a^{-3(1+w)}$. Thus, $(c_s^2/a^2)/\rho \propto a^{6w}$, which implies that the magneto-geometrical effects dominate the right hand side of Eq. (13) at sufficiently early times if $w < 0$. Note that the Alfvén speed, which measures the relative strength of the field, behaves as $c_s^2 \propto a^{-1+3w}$. This means that, as we go back in time, the ratio $(c_s^2/a^2)/\rho$ grows faster than the Alfvén speed provided that $w < -\frac{1}{3}$. Therefore, when $-1 < w < -\frac{1}{3}$, the magneto-geometrical effects can dominate the dynamics of the early expansion while the field is still relatively weak. In these cases the phase of accelerated expansion, which otherwise would have been inevitable, may not happen. Instead, the universe goes through a period of decelerated expansion. This is an interesting possibility that puts a question mark on the efficiency of inflationary models in the presence of primordial magnetism. Recall that an initial curvature era was never considered as a problem for inflation, given the smoothing power of the accelerated phase. However, this may not be the case when a primordial magnetic field is present, no matter how weak the latter is.

There is a plethora of scenarios, which utilise out-of-equilibrium epochs in the early universe to generate primeval magnetic fields [1]. The energy scales involved vary from $\sim 100$ MeV at the QCD phase transition, to $\sim 100$ GeV in the case of electroweak (EW) physics and closer to the Planck energy scale for inflation or string cosmology. The viability of the proposed mechanisms depends primarily on the field’s subsequent evolution, in view of the current observational constraints. Crucially for our purposes, only a weak (seed) magnetic field is required. As we shall see next, when curvature dominates, it is the mere presence of the field that is important and not its relative strength. Note that in the current section we assume a magnetic presence at epochs earlier than the EW phase transition, when the $SU(2) \times U(1)_Y$ symmetry is restored. If primordial magnetic fields were to be present at these high temperatures, they should correspond to the $U(1)_Y$ hypercharge rather than $U(1)_{EM}$. Such hyper-electromagnetic fields have the same stress-tensor of their EM counterparts and, in an infinitely conducting medium, the remaining hyper-magnetic field obeys an induction equation and satisfies a vanishing-divergence
law analogous to the standard Maxwell equations (see e.g. [16]). Thus, the formulae derived in Sec. 2 are also compatible with epochs prior to the EW symmetry-breaking.

Equation (13) can also apply to weakly magnetised, almost-FRW, scalar-field dominated cosmologies. Indeed, given the absence of a background magnetic field (see Sec. 2), consider a FRW unperturbed model containing a self-interacting complex scalar field $\phi$. Relative to a timelike 4-velocity $u_a$, the stress tensor associated with $\phi$ has the perfect-fluid form $T_{ab} = \rho u_a u_b + p h_{ab}$, where $\rho = \frac{1}{2} \dot{\phi} \dot{\phi}^* + V(\phi \phi^*)$ and $p = \frac{1}{2} \dot{\phi} \dot{\phi}^* - V(\phi \phi^*)$ [17]. On the other hand, the magnetic field always behaves as an imperfect fluid with $T_{ab} = \frac{1}{2} B^2 u_a u_b + \frac{1}{6} B^2 h_{ab} - B_{(a} B_{b)}$. Thus, the total energy momentum tensor reads

$$T_{ab} = (\rho + \frac{1}{2} B^2) u_a u_b + (p + \frac{1}{6} B^2) h_{ab} - B_{(a} B_{b)} ,$$

(14)

assuming that, to leading order, the coupling between $\phi$ and magnetism does not affect the perfect-fluid behaviour of the scalar field. This is a reasonable approximation given the weakness of the magnetic field. Moreover, the pure geometrical nature of the magneto-curvature interaction means that imperfections in the fluid description of $\phi$ are of minor importance for our purposes. The effects we are examining here are triggered solely by the spacetime geometry and by the tension of the magnetic force-lines. In addition to, their impact on the expansion dynamics depends almost entirely on the strength on the background curvature. On these grounds, one may substitute expression (14) into the conservation law $\nabla^b T_{ab} = 0$, obtain the formulae of Sec. 2 and eventually recover Eq. (13), this time for a spatially open scalar-field dominated universe.

Let us now consider the implications of the magnetic presence for a simple inflationary model. The initial conditions at the onset of inflation are rather unclear and subject to debate. Usually, the universe enters the inflationary regime from the Planck era or after a highly relativistic epoch. The accelerated expansion is driven by the dominating inflaton field $\phi$, with an equation of state that satisfies the condition $-\rho \leq p < -\frac{1}{3} \rho$. Here, we also allow for a weak primeval magnetic field. To begin with, recall our qualitative argument that as long as the index $w = p/\rho$ varies within $(-1, -\frac{1}{3})$, there is always an early period when the expansion is dominated by the magneto-curvature effects. The latter could suppress the accelerated phase. To refine and quantify this statement we employ Eq. (1) and then (see Sec. 3) rewrite Eq. (13) as

$$\frac{1}{3} \Theta^2 q = \frac{2}{7} H^2 \left[ (1 + 3w) \Omega + \frac{4 c_a^2 (1 - \Omega)}{3(1 + w)} \right] ,$$

(16)

which reduces to the standard non-magnetised expression when the field term is dropped [16]. Thus, the condition for suppressing acceleration (i.e. for $q > 0$) reads

$$c_a^2 > -\frac{3(1 + w)(1 + 3w) \Omega}{4(1 - \Omega)} .$$

(17)

$^2$Following scalar electrodynamics, a typical Lagrangian coupling $\phi$ to the electromagnetic vector potential $A_a$ has the form

$$\mathcal{L} = -\frac{1}{4} D_a \phi (D^a \phi)^* - V(\phi \phi^*) - \frac{1}{4} F_{ab} F^{ab} .$$

(15)

In the above $F_{ab} = 2 \nabla_{[a} A_{b]}$ is the Faraday tensor, $D_a = \nabla_a - i e A_a$ is the gauge covariant derivative ($e$ is the electromagnetic coupling) and $V = V(\phi \phi^*)$ is the potential that describes the self-interaction of $\phi$. 8
In spatially open inflationary cosmologies, the density parameter diverges from the boundary point \((a = 0, \Omega = 0)\) towards the \(\Omega = 1\) limit \([18]\). Such models go through a curvature dominated early stage characterised by \(\Omega \ll 1\). During this period, a relatively weak magnetic field (with \(c_a^2 \ll 1\)) is capable of slowing the expansion down, as condition \((17)\) shows. For example, when \(w = -\frac{2}{3}\) condition \((17)\) reduces to \(4c_a^2 > \Omega (1 - \Omega)\), \((18)\)

which is satisfied by a weak magnetic field provided \(\Omega \ll 1\). This strong curvature requirement is considerably relaxed as \(w \to -1, -\frac{1}{3}\), since then the right-hand side of \((17)\) becomes arbitrarily small. Note that the limit \(w = -1\) corresponds to standard slow-roll inflation with \(\frac{1}{2} \dot{\phi} \dot{\phi}^* \ll V\).

On the other hand, at \(w = -\frac{1}{3}\) we have, what one might call, a ‘minimal inflation’ scenario. In both cases condition \((17)\) is reduced to the requirement that \(c_a^2 > 0\). This, in turn, suggests that the mere magnetic presence will suppress the accelerated phase in any open universe with \(w = -1, -\frac{1}{3}\), irrespective of how strong its spatial curvature is. This is not surprising given that at \(w = -1\) the magneto-curvature impact on the expansion maximises, while at \(w = -\frac{1}{3}\) the gravitational effects take their minimum value (see Eq. \((13)\)).

One can refine these results by recalling the, in the magnetic presence, the total gravitational mass of the universe is \(\rho + 3p + B^2\) instead of \(\rho + 3p\) and the total energy density is \(\rho + p + \frac{2}{3} B^2\) rather than \(\rho + p\). Given the weakness of the field, this correction is of very little importance when \(-1 < w < -\frac{1}{3}\). When \(w = -\frac{1}{3}\), however, the total gravitational mass is \(B^2\) and still positive, which explains why the model decelerates. Of particular interest is the behaviour near the \(w = -\frac{1}{3}\) limit. In this case one should replace \(1 + w\) in condition \((17)\) with \(2c_a^2\) rather than zero. Then, one obtains a refined, curvature depended, requirement \(\Omega < 0.5\) for the suppression of the accelerated phase. Therefore, moderately open universes dominated by a slowly rolling scalar field cannot enter a period of de Sitter inflation as there is an arbitrarily weak magnetic field present.

In the absence of cosmic magnetism, our model goes through an accelerated phase until a more conventional equation of state is restored, as \(\phi\) rolls towards the minimum of its potential. In the process, the curvature of the space is smoothed out and the universe ends up arbitrarily flat. This apparent solution of the flatness problem has long been considered a major point in favour of the inflationary paradigm. However, by introducing even a weak primordial magnetic field, one could drastically change this picture. The results presented so far indicate that, in the presence of primeval magnetism, inflation may not be able to cope with negative curvature effectively.

In principle, one could still argue that our model might eventually enter a late accelerated phase. Clearly, whether this could happen or not depends on how strong the initial curvature is and how long the inflaton-dominated regime lasts. The details rest with the particular model that one might have in mind. However, it seems plausible that the stronger the initial magneto-curvature effects are, the longer the inflaton domination must be if the universe is ever to accelerate. Thus, provided that the effective index \(w\) is trapped within \((-1, -\frac{1}{3})\) long enough, the aforementioned magneto-curvature effects may eventually become too weak to suppress

\[The\ value\ w = -\frac{4}{3}\ also\ corresponds\ to\ the\ effective\ equation\ of\ state\ associated\ with\ network\ of\ infinite\ planar\ domain\ walls.\ Similarly,\ w = -\frac{1}{3}\ also\ represents\ a\ network\ of\ infinitely\ extended\ cosmic\ strings\ [19].\]
inflation. On the other hand, if the universe remains inflaton-dominated for a brief spell only, the accelerated phase will never have the chance to begin.

Let us emphasise that the magneto-curvature effects discussed above should be seen as the field’s kinematic reaction to the geometry of the spatial sections rather, than as a direct attempt to suppress inflation. In an open universe the tension of the magnetic force-lines slows down the expansion rate to bring it closer to that of a flat FRW model (see discussion in Sec. 2). The stronger the curvature is the more dramatic the effect. In this respect, the magnetic presence does not generically target inflation although it might seem so at first. In fact, as one can immediately see through Eqs. (7) and (13), the field would have assisted the inflationary expansion if the space had been closed (i.e. if \( k = +1 \)) instead of open.

6 Discussion

Despite their established widespread presence, research on cosmological magnetic fields remains rather marginal. The reasons could be the perceived weakness of the field effects or the lack, as yet, of a consistent theory explaining the origin of cosmic magnetism. The fact that magnetic fields further complicate the picture of the early universe may be an additional factor. However, magnetic fields have been observed everywhere where modern technology has made their detection possible. Thus, we feel justified to argue that a magnetic-free picture of the early universe is not a complete picture. Moreover, we believe that the potential of some unique magnetic characteristics, such as their vectorial nature and tension properties, has not been fully appreciated. The magneto-geometrical interaction discussed here is a consequence of these properties and of the general relativistic geometrical interpretation of gravity. Intuitively, what the magneto-curvature coupling does, is to inject the elastic properties of the field into space itself. The resulting effects are unexpected and potentially very important. The main task of this paper was to draw attention to these issues.

Our examples illustrate the impact of this coupling between magnetism and geometry on the evolution of the universe. We have discussed how, depending on the circumstances, the magneto-curvature effects mimic those of a positive cosmological constant, or those attributed to a time-decaying quintessence. For a spatially closed background, the overall magnetic impact varies from weakly opposing deceleration to accelerated expansion depending on the field’s strength. The stronger the field is, the more dramatic the effect. On the other hand, when the background is open, the magnetic contribution to the expansion is simply complementary to that of ordinary matter. Even then, however, the field’s role is subtle, making the universe look less open than it actually is.

The most intriguing result, however, is that even weak magnetic fields can become, through their coupling to geometry, key players in the evolution of the universe. We argue that, when the curvature is strong, the mere presence of a magnetic field leads to effects that can alter the picture of the universe in unexpected ways. In particular, we have demonstrated how spatially open cosmological models containing matter with negative pressure are not guaranteed a period of early accelerated expansion if a magnetic field is present. In fact, for \( p = -\rho \) the mere presence of the field can suppress the inflationary phase even in moderately curved spaces with \( \Omega < 0.5 \). Strong curvature is important when \(-\rho < p < -\frac{1}{3}\rho\), if appreciable deceleration is to
be achieved. Still, this effect can be triggered by arbitrarily weak fields. The magnetic strength is not the issue any more. Once the vectorial nature of the field has brought geometry into play, the overall magnetic impact no longer depends on the field alone. It is the presence of the magnetic field that is important and not its relative strength.

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