RIBBON-MOVES FOR 2-KNOTS WITH 1-HANDLES ATTACHED
AND KOHVANOV-JACOBSSON NUMBERS

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Abstract. We prove that a crossing change along a double point circle on
a 2-knot is realized by ribbon-moves for a knotted torus obtained from the
2-knot by attaching a 1-handle. It follows that any 2-knots for which the
crossing change is an unknotting operation, such as ribbon 2-knots and twist-
spun knots, have trivial Khovanov-Jacobsson number.

A surface-knot or -link is a closed surface embedded in 4-space $\mathbb{R}^4$ locally flatly.
Throughout this note, we always assume that all surface-knots are oriented. A
ribbon-move (cf. [10]) is a local operation for (a diagram of) a surface-knot as
shown in Figure 1. We say that surface-knots $F$ and $F'$ are ribbon-move equivalent,
denoted by $F \sim F'$, if $F'$ is obtained from $F$ by a finite sequence of ribbon-moves.

The ribbon-move is a special case of the crossing change: Assume that a surface-
knot $F$ has a double point circle $L$ in a diagram such that (i) $L$ has no self-
intersection, and (ii) at every triple point on $L$, the sheet transverse to $L$ is either
top or bottom (not middle). The condition (i) means that $L$ does not go through
the same triple point twice. When $L$ satisfies these conditions, we can perform
a crossing change along $L$ by exchanging the roles of over- and under-sheets as
indicated in Figure 2 (cf. [10]). See [4] for details on diagrams of surface-knots.

For a 2-knot $K$ (a knotted sphere in $\mathbb{R}^4$), a crossing change is not necessarily
realized by ribbon-moves; indeed, a ribbon-move does not change the Farber-Levine
pairing of $K$ but a crossing change might (cf. [10]). On the other hand, when we
consider the $T^2$-knot (knotted torus in $\mathbb{R}^4$) $K + h$ obtained from $K$ by attaching a
1-handle $h$ on $K$, we obtain the following.

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change.
Theorem 1. Let $K$ and $K'$ be 2-knots such that $K'$ is obtained from $K$ by a crossing change. Then for any 1-handles $h$ and $h'$ on $K$ and $K'$, respectively, the $T^2$-knot $K + h$ is ribbon-move equivalent to $K' + h'$.

Proof. Along the double point circle $L$ for which we perform the crossing change, there is a neighborhood $N$ identified with $(B^3, t) \times S^1$, where $(B^3, t)$ is a tangle with two strings as shown in the left of Figure 3. In the figure, the orientations of tangles are induced from that of $K$, and all bands are attached in an orientation-compatible manner. For an interval $I$ in $S^1$, we take a 1-handle $h_1 = b_1 \times I$ on $K$, where $b_1$ is a band as indicated in the figure.

We observe that $K + h_1$ is ambient isotopic to $(K' \cup T) + h_2$ (cf. [12]), where $T = m \times S^1$ is a $T^2$-knot linking with $K'$, and the 1-handle $h_2 = b_2 \times I$ connects between $K'$ and $T$. See the center of Figure 3.

Consider a 1-handle $h_3 = b_3 \times I$ on $K' \cup T$. Since both of $h_2$ and $h_3$ connect between $K'$ and $T$, the $T^2$-knot $(K' \cup T) + h_2$ is ribbon-move equivalent to $(K' \cup T) + h_3$.

Finally we see that $(K' \cup T) + h_3$ is ambient isotopic to $K' + h_4$, where $h_4 = b_4 \times I$ is the 1-handle on $K'$ as shown in the right of the figure. Thus we obtain

$$K + h \sim K + h_1 = (K' \cup T) + h_2 \sim (K' \cup T) + h_3 = K' + h_4 \sim K' + h'.$$

This completes the proof.

We say that the crossing change is an unknotting operation for a surface-knot $F$ if the trivial surface-knot is obtained from $F$ by a finite sequence of crossing changes. It is still unknown whether the crossing change is an unknotting operation for any surface-knot.

Khovanov [8] introduced a categorification of the Jones polynomial, that is, a chain complex for a given classical knot diagram such that its graded Euler characteristic is the Jones polynomial. Khovanov [9] and Jacobsson [5] proved that...
it defines an invariant for cobordisms (relative to boundary diagrams). Specifically, a cobordism between two knot diagrams gives rise to a chain map (we call it a Khovanov-Jacobsson homomorphism) between corresponding chain complexes, that is invariant under equivalence of cobordisms of diagrams. See also [2]. In particular, a diagram of a $\mathbb{T}^2$-knot is a cobordism between empty diagrams, giving rise to a homomorphism $\mathbb{Z} \to \mathbb{Z}$ defined up to sign, a multiplication by a constant. We call this constant the Khovanov-Jacobsson number.

**Theorem 2.** Let $K$ be a 2-knot for which the crossing change is an unknotting operation. Then for any 1-handle $h$ on $K$, the $\mathbb{T}^2$-knot $K + h$ has the trivial Khovanov-Jacobsson number.

**Proof.** Let $K_0$ be the trivial 2-knot and $h_0$ the trivial 1-handle on $K_0$. By assumption and Theorem 1, the $\mathbb{T}^2$-knot $K + h$ is ribbon-move equivalent to $K_0 + h_0$, which is the trivial $\mathbb{T}^2$-knot.

Consider two movies as shown in Figure 4. It is seen from the definitions [2] [5] that the corresponding Khovanov-Jacobsson homomorphisms $H^*(\emptyset) \to H^*(\emptyset)$ are the same for these movies. This implies that a ribbon-move does not change the Khovanov-Jacobsson number. Hence the $\mathbb{T}^2$-knot $K + h$ has the same number as that of the trivial $\mathbb{T}^2$-knot $K_0 + h_0$. □

**Figure 4.**

By Theorem 2, if there is a 2-knot $K$ such that the Khovanov-Jacobsson number of $K + h$ is non-trivial, then the crossing change is not an unknotting operation for $K$. However, we have no such examples at present.

**Corollary 3.** Let $K$ be a ribbon 2-knot or twist-spun knot. Then for any 1-handle $h$ on $K$, the $\mathbb{T}^2$-knot $K + h$ has trivial Khovanov-Jacobsson number.

**Proof.** This follows from Theorem 2 and the fact that the crossing change is an unknotting operation for every ribbon 2-knot or twist-spun knot (cf. [1] [11]). □

We say that a surface-knot is pseudo-ribbon [7] if it has a diagram without triple points. The notions of ribbon and pseudo-ribbon 2-knots are the same [15] (see also [6]). On the other hand, for $\mathbb{T}^2$-knots, they are not coincident in the sense that the family of pseudo-ribbon $\mathbb{T}^2$-knots properly contains that of ribbon $\mathbb{T}^2$-knots.

**Proposition 4.** Any pseudo-ribbon $\mathbb{T}^2$-knot has trivial Khovanov-Jacobsson number.

**Proof.** By the results of Teragaito [14] and Shima [13], every pseudo-ribbon $\mathbb{T}^2$-knot $T$ is (i) a ribbon $\mathbb{T}^2$-knot, or (ii) a $\mathbb{T}^2$-knot obtained from a split union of a Boyle’s turned $\mathbb{T}^2$-knot $T'$ [3] and a trivial 2-link $U = U_1 \cup U_2 \cup \cdots \cup U_n$ by surgery along 1-handles $h_1, h_2, \ldots, h_n$ for some $n \geq 0$, where each $h_i$ connects between $T'$ and $h_i$ $(i = 1, 2, \ldots, n)$.
For the case (i), there is a ribbon 2-knot $K$ and a 1-handle $h$ such that $T = K + h$. Hence the conclusion follows from Corollary 3.

For the case (ii), we see that $T = (T' \cup U) + (\bigcup_{i=1}^n h_i)$ is ribbon-move equivalent to $T'$. We consider two movies for a classical knot diagram $D$ in a plane, one of which keep $D$ still and the other twists $D$ by a $2\pi$-rotation of the plane. Then it follows from the definitions \[2, 5, 9\] that the corresponding Khovanov-Jacobsson homomorphisms $H^*(D) \to H^*(D)$ are the same for these movies. This implies that $T'$ has the same Khovanov-Jacobsson number as that of a non-turned (that is, just spun) $T^2$-knot, which is ribbon. Hence this case reduces to (i). □

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References

[1] S. Asami and S. Satoh, An infinite family of non-invertible surfaces in 4-space, to appear in Bull. London Math. Soc.
[2] D. Bar-Natan, Khovanov’s homology for tangles and cobordisms, preprint available at: http://www.math.toronto.edu/~drorbn/papers/Cobordism/
[3] J. Boyle, The turned torus knot in $S^4$, J. Knot Theory Ramifications 2 (1993), 239–249.
[4] J.S. Carter and M. Saito, Knotted surfaces and their diagrams, Mathematical Surveys and Monographs, vol. 55, American Mathematical Society, Providence, RI. 1998.
[5] M. Jacobsson, An invariant of link cobordisms from Khovanov’s homology theory, preprint available at: http://xxx.lanl.gov/abs/math.GT/0206303
[6] T. Kanenobu and A. Shima, Two filtrations of ribbon 2-knots, Topology Appl. 121 (2002), 143–168.
[7] A. Kawauchi, On pseudo-ribbon surface-links, J. Knot Theory Ramifications 11 (2002), 1043–1062.
[8] M. Khovanov, A categorification of the Jones polynomial, Duke Math. J. 101(3) (1999), 359–426.
[9] An invariant of tangle cobordisms, preprint available at: http://xxx.lanl.gov/abs/math.GT/0207264
[10] E. Ogasu, Ribbon-moves of 2-knots: the Farber-Levine pairing and the Atiyah-Patodi-Singer-Casson-Gordon-Ruberman $\tilde{\eta}$-invariants of 2-knots, preprint available at: http://xxx.lanl.gov/abs/math.GT/0004007
[11] S. Satoh, Surface diagrams of twist-spun 2-knots, J. Knot Theory Ramifications 11 (2002), 413–430.
[12] A note on unknotting numbers of twist-spun knots, preprint.
[13] A. Shima, On simply knotted tori in $S^4$, II, Knots ’96 (Tokyo), 551–568, World Sci. Publishing, River Edge, NJ, 1997.
[14] M. Teragaito, Symmetry-spun tori in the four-sphere, Knots 90 (Osaka, 1990), 163–171, de Gruyter, Berlin, 1992.
[15] T. Yajima, On simply knotted spheres in $R^4$, Osaka J. Math. 1 (1964), 133–152.
[16] T. Yashiro, Deformations of surface diagrams, talk at First KOOK Seminar International Knot Theory and Related Topics, July 2004.
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