Families of algebraic structures
Joint works with Loïc Foissy, Xing Gao and Yuanyuan Zhang

Dominique Manchon
CNRS-Université Clermont Auvergne

Higher structures emerging from renormalization,
Erwin Schrödinger Institut, October 12th 2020
1. Introduction: family Rota-Baxter algebras

2. Family dendriform algebras

3. Family pre-Lie algebras

4. Marcelo Aguiar’s results (2020)

5. Main results
Family Rota-Baxter algebras
Ebrahimi-Fard–Gracia-Bondía–Patras/Li Guo, 2007

- Rota-Baxter algebra : \((A, R)\) with

\[
R(a)R(b) = R(R(a)b + aR(b) + \lambda ab).
\]
Family Rota-Baxter algebras
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- Rota-Baxter algebra : \((A, R)\) with
  \[ R(a)R(b) = R(R(a)b + aR(b) + \lambda ab). \]

- **Family** Rota-Baxter algebra : \(\big( A, (R_\omega)_{\omega \in \Omega} \big)\) with \(\Omega\) semigroup and
  \[ R_\alpha(a)R_\beta(b) = R_{\alpha\beta}(R_\alpha(a)b + aR_\beta(b) + \lambda ab). \]
First instance (EGP), coming from the momentum scheme in renormalization (with $\Omega = \mathbb{N}$).
• **First instance** (EGP), coming from the momentum scheme in renormalization (with $\Omega = \mathbb{N}$).

• **Simplest example**, coming from minimal subtraction scheme (with $\Omega = \mathbb{Z}$) : Algebra of Laurent series $A = k[z^{-1}, z]]$. Rota-Baxter family algebra of weight $-1$, with $\Omega = (\mathbb{Z}, +)$. Here $R_\omega$ = projection onto the subspace $A_{<\omega}$ generated by $\{z^k, k < \omega\}$ parallel to the supplementary subspace $A_{\geq \omega}$ generated by $\{z^k, k \geq \omega\}$.
Another interesting example in weight zero: \( \Omega = (\mathbb{R}, +) \), and let \( A \) be the \( \mathbb{R} \)-algebra of continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \). For any \( \alpha \in \mathbb{R} \), define \( R_\alpha : A \rightarrow A \) by

\[
R_\alpha(f)(x) = e^{-\alpha a(x)} \int_0^x e^{\alpha a(t)} f(t) \, dt,
\]

where \( a \) is a fixed nonzero element of \( A \). Then \( (R_\alpha)_{\alpha \in \Omega} \) is a Rota-Baxter family of weight zero.
Family dendriform algebras
X. Gao - Y. Y. Zhang

- $\Omega$ semigroup,
- $(D, <_\omega, >_\omega)_{\omega \in \Omega}$ such that for $x, y, z \in D$ and $\alpha, \beta \in \Omega$,

\[
\begin{align*}
(x <_\alpha y) <_\beta z &= x <_{\alpha\beta} (y <_\beta z + y >_\alpha z), \\
(x >_\alpha y) <_\beta z &= x >_\alpha (y <_\beta z), \\
(x <_\beta y + x >_\alpha y) >_{\alpha\beta} z &= x >_\alpha (y >_\beta z).
\end{align*}
\]
The free $\Omega$-family dendriform algebra generated by a set $X$ can be described in terms of planar binary trees with internal nodes decorated by $X$ and edges typed by $\Omega$ (X. Gao - DM - Y. Y. Zhang).
The free $\Omega$-family dendriform (resp. tridendriform) algebra generated by a set $X$ can be described in terms of planar binary (resp. Schröder) trees with internal nodes (resp. internal node angles) decorated by $X$ and edges typed by $\Omega$ (X. Gao - DM - Y. Y. Zhang).
- The free $\Omega$-family dendriform (resp. tridendriform) algebra generated by a set $X$ can be described in terms of planar binary (resp. Schröder) trees with internal nodes (resp. internal node angles) decorated by $X$ and edges typed by $\Omega$ (X. Gao - DM - Y. Y. Zhang).

- The free $\Omega$-family Rota-Baxter algebra of weight $\lambda$ generated by a set $X$ can be described in terms of planar rooted trees with internal node angles decorated by $X$ and edges typed by $\Omega$ (X. Gao - DM - Y. Y. Zhang).
An $X$-decorated $\Omega$-typed PBT
Outline

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Family dendriform algebras
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Marcelo Aguiar’s results (2020)
Main results

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\alpha & \quad \beta \\
\quad \delta & \quad \gamma \\
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An angularly $X$-decorated $\Omega$-typed Schröder tree
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An $X$-decorated $\Omega$-typed PBT

An angularly $X$-decorated $\Omega$-typed Schröder tree

An angularly $X$-decorated $\Omega$-typed planar rooted tree
Any $\Omega$-family Rota-Baxter of weight zero (resp. one) is an $\Omega$-family dendriform (resp. tridendriform) algebra (family version a well-known result of M. Aguiar, resp. K. Ebrahimi-Fard).

The natural embedding of planar binary trees into planar rooted trees is the embedding of the free $\Omega$-family dendriform algebra into its enveloping Rota-Baxter algebra of weight zero.

The natural embedding of Schröder trees into planar rooted trees is the embedding of the free $\Omega$-family dendriform algebra into its enveloping Rota-Baxter algebra of weight one.
Family pre-Lie algebras
DM - Y. Y. Zhang

- Let $\Omega$ be a **commutative** semigroup.
- Left pre-Lie family algebra : \((A, (\triangleright_\omega)_{\omega \in \Omega})\) such that
  \[
  x \triangleright_\alpha (y \triangleright_\beta z) - (x \triangleright_\alpha y) \triangleright_{\alpha \beta} z = y \triangleright_\beta (x \triangleright_\alpha z) - (y \triangleright_\beta x) \triangleright_{\beta \alpha} z, \quad (1)
  \]
  where $x, y, z \in A$ and $\alpha, \beta \in \Omega$. 
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Left pre-Lie family algebra : $(A, (\triangleright_\omega)_{\omega \in \Omega})$ such that

$$x \triangleright_\alpha (y \triangleright_\beta z) - (x \triangleright_\alpha y) \triangleright_\alpha \beta z = y \triangleright_\beta (x \triangleright_\alpha z) - (y \triangleright_\beta x) \triangleright_\beta \alpha z,$$  \hspace{1cm} (1)

where $x, y, z \in A$ and $\alpha, \beta \in \Omega$.

If $A$ is an $\Omega$-family dendriform algebra with $\Omega$ commutative, it is an $\Omega$-family pre-Lie algebra with

$$x \triangleright_\omega y := x \succ_\omega y - y \prec_\omega x,$$  \hspace{1cm} for $\omega \in \Omega$.  

Description of the free $\Omega$-family pre-Lie algebra generated by $X$ in terms of $X$-decorated $\Omega$-typed non-planar rooted trees.
Description of the free $\Omega$-family pre-Lie algebra generated by $X$ in terms of $X$-decorated $\Omega$-typed non-planar rooted trees.
These examples call for a general approach.

What is a family $\mathcal{P}$-algebra for an operad $\mathcal{P}$?
Marcelo Aguiar’s approach (2020)

- **Principle**: A is an $\Omega$-family $\mathcal{P}$-algebra if and only if $A \otimes k\Omega$ is a graded $\mathcal{P}$-algebra.
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The family version of an operation of arity $n$ is parametrized by $\Omega^n$, where $\Omega$ is the semigroup at hand:

$$\alpha(a_1 \otimes \omega_1, \ldots, a_n \otimes \omega_n) = \alpha_{\omega_1, \ldots, \omega_n}(a_1, \ldots, a_n) \otimes \omega_1 \cdots \omega_n.$$
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In particular, the natural family version of a binary operation necessitates two parameters in $\Omega$. 

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In particular, the natural family version of a binary operation necessitates two parameters in $\Omega$.

The semigroup $\Omega$ must be commutative unless the operad is non-sigma, e.g. Assoc, Dup or Dend.
Example: family associative algebras.

\[ x \cdot_{\alpha,\beta} (y \cdot_{\beta,\gamma} z) = (x \cdot_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} z. \]
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\[ x \cdot_{\alpha,\beta,\gamma} (y \cdot_{\beta,\gamma} z) = (x \cdot_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} z. \]

The family associative algebra is commutative if moreover

\[ x \cdot_{\alpha,\beta} y = y \cdot_{\beta,\alpha} x. \]

This immediately yields the commutativity of the semigroup \( \Omega \).
Our approach
(L. Foissy - DM - Y. Y. Zhang)

- **Same Principle**: $A$ is an $\Omega$-family $\mathcal{P}$-algebra if and only if $A \otimes k\Omega$ is a graded $\mathcal{P}$-algebra.
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- **Same Principle**: $A$ is an $\Omega$-family $P$-algebra if and only if $A \otimes k\Omega$ is a graded $P$-algebra.

- **But**: $\Omega$ need not be a semigroup: it is just a set a priori.
Our approach
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- **Same Principle**: $A$ is an $\Omega$-family $P$-algebra if and only if $A \otimes k\Omega$ is a **graded** $P$-algebra.

- **But** $\Omega$ need not be a semigroup: it is just a set a priori.

- the starting (linear) operad $P$ **together with its presentation**

$$P = M_E/R = k.M_E/R$$

defines a $\mathfrak{P}$-algebra structure on $\Omega$, where $\mathfrak{P}$ is a set operad.
The set operad $\hat{\mathcal{P}}$ depends on $\mathcal{P}$ and its presentation:

$$\hat{\mathcal{P}} = \mathcal{M}_E / \mathcal{R},$$

where $\mathcal{R}$ is the set-operadic equivalence relation generated by $\mathcal{R}$. 
The set operad $\mathfrak{P}$ depends on $\mathcal{P}$ and its presentation:

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If $\mathcal{P}$ is (the linearization of) a set operad, then $\mathfrak{P} = \mathcal{P}$. 

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The set operad \( \mathcal{P} \) depends on \( \mathcal{P} \) and its presentation:

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where \( \mathcal{R} \) is the set-operadic equivalence relation generated by \( \mathcal{R} \).

If \( \mathcal{P} \) is (the linearization of) a set operad, then \( \mathcal{P} = \mathcal{P} \).

If \( \mathcal{P} \) is quadratic and if the Koszul dual \( \mathcal{P}^! \) of \( \mathcal{P} \) is a set operad, we have

\[
\mathcal{P} = \mathcal{P}^!.
\]
Upshot

Let $\mathcal{P} = \mathcal{M}_E/R$ be a finitely presented linear operad, and let $\mathcal{P}$ be the corresponding set operad.
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Let $\mathcal{P} = \mathcal{M}_E/R$ be a finitely presented linear operad, and let $\tilde{\mathcal{P}}$ be the corresponding set operad.

- The **color-mixing operad** $\tilde{\mathcal{P}}$ is a subquotient of the uniform $\Omega$-colored operad $\mathcal{P}^\Omega$. 
Upshot

Let $\mathcal{P} = \mathcal{M}_E / \mathcal{R}$ be a finitely presented linear operad, and let $\mathcal{P}$ be the corresponding set operad.

- The **color-mixing operad** $\tilde{\mathcal{P}}$ is a subquotient of the uniform $\Omega$-colored operad $\mathcal{P}^\Omega$.

- If $\Omega$ is a $\mathcal{P}$-algebra, an $\Omega$-family $\mathcal{P}$-algebra is an $\Omega$-graded algebra over $\tilde{\mathcal{P}}$, for which the underlying $\Omega$-graded object is uniform.
Upshot

Let $\mathcal{P} = \mathcal{M}_E/\mathcal{R}$ be a finitely presented linear operad, and let $\mathcal{P}$ be the corresponding set operad. The color-mixing operad $\tilde{\mathcal{P}}$ is a subquotient of the uniform $\Omega$-colored operad $\mathcal{P}^\Omega$.

If $\Omega$ is a $\mathcal{P}$-algebra, an $\Omega$-family $\mathcal{P}$-algebra is an $\Omega$-graded algebra over $\tilde{\mathcal{P}}$, for which the underlying $\Omega$-graded object is uniform.

In the color-mixing operad, the color of the output is obtained by combining the $n$ input colors by means of an operation of arity $n$ in $\tilde{\mathcal{P}}$.
Thank you for your attention!