The Casimir effect and mass renormalization for a massive Bosonic string in background B-field

Y. Koohsarian\(^1\) A. Shirzad\(^2\)

Department of Physics, Isfahan University of Technology  
P.O.Box 84156-83111, Isfahan, IRAN,  
School of Physics, Institute for Research in Fundamental Sciences (IPM)  
P.O.Box 19395-5531, Tehran, IRAN.

Abstract

We study the Casimir effect for a massive bosonic string terminating on D-brains, and living in a flat space with an antisymmetric background B-field. We find the Casimir energy and Casimir force as functions of the mass and length of the string and show the force does not depend on B-field. We also find the divergent part of the vacuum state energy, as a function of the mass and length of the string and offer an idea for renormalization of the mass of the string.

Keywords: Casimir force, Background B-field, Bosonic strings, Mass renormalization

1 Introduction

Imposing definite boundary conditions on a quantum field changes the spectrum of the quantum states and leads in particular to changing the vacuum energy of the system. These zero-point fluctuations results in some observable quantum effects such as the well known Casimir force \([1]\). As we know, this force depends on the features of the space-time manifold and on the boundary conditions imposed on the field.

A variety of theoretical models with different boundary conditions have been considered in the literature in which the casimir effect is analyzed and for some configurations the Casimir force is observed or measured experimentally (see, e.g., \([2,3]\) as a review). The essential point for each model is finding the Hamiltonian of the system as a combinations of different physical modes (mostly harmonic oscillators) which acquire positive excitation energies above the vacuum state. Then turning off all of the excitations, one finds the vacuum state energy of the system. The Casimier force emerges as the change in the vacuum energy due to a small displacement of the boundaries.

In this paper we consider the model of a massive bosonic string in a background B-field introduced initially in \([4]\). This model is a generalization of the massless case which is a famous model in the context of the string theory, specially because of exhibiting noncommutative coordinates on the brains attached to the endpoints of the string\([5]\). In a previous paper \([6]\), considering the boundary conditions as Dirac constraints and imposing them on

\(^1\)koohsarian_ramian@yahoo.com  
\(^2\)shirzad@ipm.ir
the Fourier expansions of the fields, we found the physical modes of the system as an infinite set of harmonic oscillators. This enabled us to write down the canonical Hamiltonian as a summation over Hamiltonians of simple harmonic oscillators with definite frequencies. Hence, we can read out the zero point energy as the summation over vacuum energy of individual oscillators and regularize it to find out the Casimir energy in terms of the length of the string. We will apply the well known Abel-Plana formula for regularization of the vacuum energy. Then by differentiating the Casimir energy with respect to the string length, we will find the associated Casimir force as an interaction between the D-brains (section 2).

Finally, we calculate the divergent part of the vacuum state energy of the string in section 3. It’s reasonable to interpret the divergencies of the vacuum state energy as the renormalization of some classical parameters of the system [7]. Here, we make a connection between the divergent part of the vacuum state energy and renormalization of the mass of the bosonic string.

## 2 The Casimir force for the massive Bosonic string

Suppose an even number of fields, $X_i$, among Bosonic fields $X^\mu$ living in a flat target space, are coupled to an antisymmetric external tensor B-field. In the simplest case the subspace of $X_i$’s is a two dimensional Euclidian space and the constant B-field is exhibited by

$$B_{ij} \equiv \begin{pmatrix} 0 & \tilde{B} \\ -\tilde{B} & 0 \end{pmatrix}. \quad (1)$$

Thus, neglecting those components of $X^\mu$ which does not couple to the B-field, the simplified Lagrangian is given as [4]

$$L = \frac{1}{2} \int_0^l d\sigma \left[ \ddot{X}^2 - X'^2 - m^2 X^2 + 2B_{ij} \dot{X}_i X'_j \right], \quad (2)$$

where "dot" and "prime" represent differentiation with respect to $\tau$ and $\sigma$ respectively. Consistency of the variational principal is achieved by considering the boundary conditions $X'_i + B_{ij} \dot{X}_j = 0$ at the end-points $\sigma = 0$ and $\sigma = l$. In the canonical formulation the Hamiltonian reads

$$H = \frac{1}{2} \int_0^l d\sigma \left[ (P - BX')^2 + X'^2 + m^2 X^2 \right], \quad (3)$$

where $P_i = \dot{X}_i + B_{ij} X'_j$ are conjugate momentum fields. Hence, the boundary conditions can be considered as vanishing of the primary constraint $\Phi_i(\sigma, \tau) = M_{ij} \partial_\sigma X_j(\sigma, \tau) + B_{ij} P_j(\sigma, \tau)$ at the end-points $\sigma = 0$ and $\sigma = l$ where $M = 1 - B^2$. As shown in details in [6], the consistency of primary constraints, in the language of constrained systems, gives the following two infinite sets of constraints at the end-points

$$(\partial^2_\sigma - m^2)^n \left[ M_{ij} \partial_\sigma X_j(\sigma, \tau) + B_{ij} P_j(\sigma, \tau) \right] = 0,$$

$$(\partial^2_\sigma - m^2)^n \left[ \partial_\sigma P_i(\sigma, \tau) - m^2 B_{ij} X_j(\sigma, \tau) \right] = 0, \quad (4)$$

where $n = 0, 1, 2, \ldots$. Imposing the above constraints on the most general Fourier expansions of the fields $X(\sigma, \tau)$ and $P(\sigma, \tau)$, gives their expansions in terms of an enumerable set of
physical modes $a_n$ and $c_n$ as follow

$$ X(\sigma, \tau) = \frac{1}{\sqrt{l}} \left[ a_0(\tau) \cosh k_0(\sigma - \frac{l}{2}) - \frac{1}{k_0} M^{-1} B a_0(\tau) \sinh k_0(\sigma - \frac{l}{2}) \right]$$

$$ + \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[ a_n(\tau) \cos \frac{n\pi}{l} \sigma - \frac{l}{n\pi} M^{-1} B c_n(\tau) \sin \frac{n\pi}{l} \sigma \right], $$

$$ P(\sigma, \tau) = \frac{1}{\sqrt{l}} \left[ c_0(\tau) \cosh k_0(\sigma - \frac{l}{2}) - \frac{1}{k_0} M^{-1} B c_0(\tau) \sinh k_0(\sigma - \frac{l}{2}) \right]$$

$$ + \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[ c_n(\tau) \cos \frac{n\pi}{l} \sigma - \frac{l}{n\pi} M^{-1} B a_n(\tau) \sin \frac{n\pi}{l} \sigma \right]. \quad (5) $$

Using the symplectic approach gives finally the classical brackets of the physical modes as

$$ [a_n, c_s] = N_{n-1} \delta_{ns}, \quad (6) $$

where

$$ N_0 \equiv \frac{\sinh k_0 l}{k_0 l}, \quad N_n \equiv 1 + \frac{k_0^2 l^2}{n^2 \pi^2} \quad n \neq 0. \quad (7) $$

Inserting the expansions (5) of the fields in (3) gives the Hamiltonian in terms of physical modes as

$$ H = \frac{1}{2} \sum_{n=0}^{\infty} N_n (M^{-1} c_n^2 + M \omega_n^2 a_n^2), \quad (8) $$

where

$$ \omega_0^2 = m^2 M, \quad \omega_n^2 = m^2 + \frac{n^2 \pi^2}{l^2} \quad n \neq 0. \quad (9) $$

The Hamiltonian (8) is, obviously, a superposition of infinite number of independent harmonic oscillators with $a$’s as positions and $c$’s as momenta.

Now, we can use these results to study the Casimir effect for the current problem. From Eqs. (9) the zero-point energy of the system is

$$ E_0(l, m) = \frac{1}{2} \left( \omega_0 + \sum_{n=1}^{\infty} \omega_n \right) = \frac{1}{2} \left( m \sqrt{1 + B^2} + \sum_{n=1}^{\infty} \sqrt{m^2 + \frac{n^2 \pi^2}{l^2}} \right), \quad (10) $$

where we have used the Planck units in which $\hbar = 1$ and $c = 1$. The sum (10) is obviously infinite, as usual in quantum field theory in assigning the ground state energy of a system. In order to regularize (10), we use a generalized form of the known Abel-Plana formula [7] as follow

$$ \sum_{n=0}^{\infty} G_A(n) - \int_{0}^{\infty} dk G_A(k) = \frac{1}{2} G_A(0) - 2 \int_{A}^{\infty} \frac{dk}{\exp(2\pi k) - 1} (k^2 - A^2)^{\frac{1}{2}}, \quad (11) $$

where $k$ is a continuous variable corresponding to $n$ and $G_A(k) = \sqrt{A^2 + k^2}$. To find the convergent part of Eq. (10) we just have to take $G_m(n) = \frac{1}{2} \sqrt{m^2 + \frac{n^2 \pi^2}{l^2}}$. After some simplifications we have

$$ \frac{1}{2} \sum_{n=0}^{\infty} \sqrt{m^2 + \frac{n^2 \pi^2}{l^2}} - \frac{1}{2} \int_{0}^{\infty} dk \sqrt{m^2 + k^2} = -\frac{m}{4} - \frac{1}{4\pi l} \int_{\mu}^{\infty} \frac{dy}{\exp(y) - 1} \sqrt{y^2 - \mu^2}, \quad (12) $$

3
where \( y = 2\pi k \) and \( \mu = 2ml \). So, comparing Eqs. (12) and (10) we obtain the Casimir energy as
\[
E_c(l, m) = \left( \sqrt{1 + \tilde{B}^2} - \frac{1}{2} \right) \frac{m}{2} - \frac{1}{4\pi l} \int_{\mu}^{\infty} dy \frac{\exp(y) - 1}{\sqrt{y^2 - \mu^2}}.
\] (13)

Note that the result (13) has no dependence on the choice of the regularization method.

Here, it seems necessary to attend to a physical point. If we reasonably consider the parameter "\( m \)" as the mass of the bosonic string, we anticipate the Casimir energy should vanish by taking the limit \( m \to \infty \). The reason is, for large values of \( m \), we expect the zero-point fluctuations of the vacuum state of the string tend to zero. This requirement can be generally considered as a universal condition for the Casimir energy of a massive system [7]. As a result of this condition, the first term in Eq. (13) seems to be problematic. However, this term may be absorbed as a renormalization term for the mass of the string, as we will see in the next section.

Now differentiating Eq. (13) with respect to \( l \) gives the Casimir force as follow
\[
F_c(l, m) = -\frac{1}{4\pi l^2} \int_{\mu}^{\infty} dy \left( \frac{\mu^2}{\exp(y) - 1} \sqrt{y^2 - \mu^2} + \frac{\sqrt{y^2 - \mu^2}}{\exp(y) - 1} \right).
\] (14)

For the massless bosonic string the corresponding results can be obtained simply, by taking the limit \( m \to 0 \) in Eqs. (13) and (14) as
\[
E_c(l) = -\frac{\pi}{24}\frac{l}{l^2}.
\]
\[
F_c(l) = -\frac{\pi}{24}\frac{l}{l^2}.
\] (15)

These results can also be obtained using the well known Zeta function regularization. Obviously the Casimir force (15) associated with the massless string as well as that of massive one (14), has no dependence on B-field. Hence, we conclude that the background B-field does not play any role in the Casimir effect for massive or massless bosonic string, as we see in above plot, the Casimir force, in terms of Planck force \( (F_p \approx 1.2 \times 10^{44} N) \), decreases when the string mass, in terms of Planck mass \( (m_p \approx 2.2 \times 10^{-8} kg) \), or the string length, in terms of Planck length \( (l_p \approx 1.6 \times 10^{-35} m) \) increases, as expected.
3 Mass Renormalization

In this section we want to show that the divergent part of the zero-point energy should not be considered just as a redundant part which should be canceled. As we will see, it may have, in fact, physical interpretation.

The Abel-plana formula in the previous section just gave us the convergent part of the vacuum energy. To find the divergent part of the zero-point energy of the string, we initially write

\[ E_0(s) = \frac{\mu^{2s}}{2} \sum_{j=0}^{\infty} \omega_j^{1-2s}, \]

in which \( \mu \) is a constant with the dimension of mass, included for compensating the dimensions in opposite sides of Eq. (16) and \( j \) is the number of mode. Apparently \( E_0(s) \) in the limit \( s \to 0 \) tends to vacuum state energy. From the well-known definition of Gamma function we can write

\[ \omega_j^{1-2s} = \int_0^\infty \frac{dt}{t} \frac{t^{s-\frac{1}{2}}}{\Gamma(s - \frac{1}{2})} \exp(-t\omega_j^2). \]

For the massive bosonic string, using Eq. (9) we have

\[ E_0(s) = \frac{\mu^{2s}}{2} \int_0^\infty \frac{dt}{t} \frac{t^{s-\frac{1}{2}}}{\Gamma(s - \frac{1}{2})} \exp(-tm^2) \left( \sum_{n=1}^{\infty} \exp(-n^2\pi^2l^2) \right) + \frac{\mu^{2s}}{2} \omega_0. \]

As is seen, the integrand in Eq. (18) is well-behaved for large \( t \). For small \( t \), however, using the known Poisson summation formula we obtain an asymptotic expansion as

\[ \sum_{n=0}^{\infty} \exp(-n^2\pi^2l^2) \approx \frac{l}{\sqrt{4\pi t}}. \]

Hence, Eqs. (18) and (19) result in

\[ E_0(s) \approx \frac{\mu^{2s}}{2} \frac{l}{\sqrt{4\pi}} \int_0^\infty \frac{dt}{t} \frac{t^{s-1}}{\Gamma(s - \frac{1}{2})} \exp(-tm^2) + \frac{\mu^{2s}}{2} \omega_0 \]

\[ = \frac{\mu^{2s}}{2} \frac{lm^2}{\sqrt{4\pi}} \frac{\Gamma(s - 1)}{\Gamma(s - \frac{1}{2})} + \frac{\mu^{2s}}{2} \omega_0. \]

Here, an allusion to the theme of the familiar expansion of heat kernel may seem interesting. It is well-known that the heat kernel \( K(t) \), has an asymptotic expansion for \( t \approx 0 \) as follows

\[ K(t) \approx \frac{1}{\sqrt{(4\pi t)^{d}}} \sum_{n=0}^{\infty} a_n t^n, \]

in which, \( d \) is the dimension of the space where the dynamical fields live in, and \( a_n \)'s are recognized as the heat kernel coefficients. Comparing Eqs. (19) and (21), we deduce that the heat kernel asymptotic expansion for the bosonic string can be derived, simply, if we take \( d = 1 \), \( a_0 = l \) and \( a_i \approx 0 \), for \( i \geq 1 \). In other word, since \( a_0 \) is often considered as the volume of the underlying space, the bosonic string, in the framework of heat kernel
expansion, can be seen as a one dimensional system with volume \( l \). Now turning back to Eq. (20), we have for \( s \to 0 \)

\[
E_0(s) \approx \frac{lm^2}{8\pi} \frac{1}{s} + \frac{\omega_0}{2},
\]

(22)

where the asymptotic value \( \Gamma(s-1) \approx -\frac{1}{s} \) for \( s \approx 0 \), has been employed and also we have inserted \( \Gamma(-\frac{1}{2}) = -2\sqrt{\pi} \). So the divergent part of the vacuum state energy for the massive bosonic string is

\[
E_0^{\text{div}}(s) = \frac{lm^2}{8\pi} \frac{1}{s}, \quad s \to 0.
\]

(23)

Note that \( E_0^{\text{div}}(s) \) in Eq. (23) is proportional to the mass squared as well as the length of the string; hence, it may have some physical significance. Using the idea given in Ref. [7], it would be more clear if we reasonably, add the vacuum state energy to the classical ground state energy \( E_{0,\text{class}} \), to evaluate the total ground state energy, \( E_{0,\text{total}} \), of a system, as follows

\[
E_{0,\text{total}} = E_{0,\text{class}} + E_0(s) = E_{0,\text{class}} + E_0^{\text{div}}(s) + E_0(s) - E_0^{\text{div}}(s)
\]

\[
= E_{0,\text{class}} + E_{0,\text{qu}}.
\]

(24)

Here we have subtracted the divergent part from vacuum state energy and appended it to the classical ground state energy, as a reasonable interpretation for the renormalization of the classical ground state energy of the system.

For the relativistic massive string, it is reasonable to consider the mass of the string, as the classical ground state energy, \( E_{0,\text{class}} = m \). Hence we can write

\[
E_{0,\text{class}}^{\text{ren}} = m + \frac{lm^2}{8\pi} \frac{1}{s}, \quad s \to 0.
\]

(25)

Eventually, we can think of Eq. (25), as renormalization of the mass of bosonic string as

\[
m \to m + \frac{lm^2}{8\pi} \frac{1}{s}, \quad s \to 0,
\]

(26)

As we said in the previous section, we should impose the physical requirement \( E_{0,\text{qu}} \to 0 \) for \( m \to \infty \). To fulfill this condition the first term in Eq. (13) can be considered as another mass renormalization term. Hence, the final expression for renormalization of the string mass can be written as follow

\[
m \to m + \left( \sqrt{1 + \tilde{B}^2} - 1 \right) \frac{m}{2} + \frac{lm^2}{8\pi} \frac{1}{s}, \quad s \to 0.
\]

(27)

Note, the last two terms in Eq. (27) are resulted from quantization of the system, and vanish in the classical limit \( \hbar \to 0 \). We see that the background B-field plays a role in the mass renormalization, although it has no influence on the Casimir effect for the Bosonic string, as we previously realized.

References

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet 51, 793 (1948).
[2] V. M. Mostepanenko and N.N. Turnov, *The Casimir effect and its applications*, (Oxford University Press, Oxford, 1997).

[3] M. Bordag, *The Casimir Effect. 50 years latre* (World Scientific, Singapore, 1999).

[4] C.-s. Cho and P.-M. Ho, *Non-commutative D-brane and open string in pp-wave background with B-field* Nucl. Phys. B **636**, 141-158, (2002).

[5] C.-s. Cho and P.-M. Ho, *Noncommutative open string and D-brane* Nucl. Phys. B **550**, 151-168, (1999), hep-th/9812219.

[6] A. Shirzad, A. Bakhshi and Y. Koohsarian, hep-th 1112.5781

[7] M. Bordag, G. L. Klimchitskaya, U. Mohideen and V. M. Mostepanenko, *Advances in Casimir Effect*, (Oxford Science Publications 2008).