General analysis and numerical estimations of polarization observables in $\bar{N} + N \rightarrow \pi + e^+ + e^-$ reaction in an exclusive experimental setup

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Abstract

The dependence of the differential cross section and polarization observables in the $\bar{N} + N \rightarrow \pi + e^+ + e^-$ reaction on the polarizations of the proton target and antiproton beam (the produced electron may be unpolarized or longitudinally polarized) have been derived in a general form using hadron electromagnetic current conservation and $P-$ invariance of the hadron electromagnetic interaction. The analysis was done for the case of an exclusive experimental setup where the produced electron and pion are detected in coincidence. The explicit dependence of all polarization observables on two, from five, kinematic variables (the azimuthal angle $\phi$ and the virtual-photon parameter $\epsilon$), have been obtained assuming one-photon-exchange. The application to the particular case of a mechanism which contains information on time-like proton form factors in the unphysical region is considered in the Born approximation.
I. INTRODUCTION

Renewed interest in the reactions induced by antiprotons is related to the possibility to accelerate high intensity antiproton beams up to 15 GeV/c momentum at the FAIR facility [1]. The wide program foreseen at this facility by the PANDA collaboration [2] is focused on strong interaction and includes charmonium spectroscopy, hybrids, glueballs, and charm in nuclei (for a review, see [3]). Moreover, the selection in the final state of a pair of leptons accompanied by a pion will allow to study aspects of the electromagnetic interaction. In particular, it allows to access electromagnetic proton form factors in the so-called 'unphysical region'. Assuming the dominance of $t$ and $u$ channel exchange diagrams [4], the emission of a pion by the proton or the antiproton lowers the momentum squared ($q^2$) of a virtual photon which subsequently decays into a lepton pair. This mechanism allows to reach $q^2$ values under the $4M^2$ threshold ($M$ is the proton mass) in the time-like region of transferred momenta.

This idea was firstly suggested by M.P. Rekalo, using the crossing symmetry related reaction $\pi+N\rightarrow N+e^-+e^+ [5, 6]$. Pion scattering on nucleon and nuclei will be investigated in the resonance region by the HADES collaboration with the pion beam available at GSI (Darmstadt) in the next future [7]. The third reaction related by crossing symmetry to the previous ones is the pion electroproduction on proton, which has been widely investigated in semi-inclusive and exclusive polarized and unpolarized experiments, in different kinematical regions.

In this paper we extend our previous analysis [8] for the case of exclusive measurements, in an experimental setup where all final particles are detected. The five-fold differential cross section is derived in unpolarized and polarized case. Different polarization phenomena are considered with polarized or unpolarized beam and target, when the polarization of the scattered electron is measured. The expressions of the observables are given in terms of the six independent amplitudes (in general complex functions of three kinematical variables) which fully describe the considered reactions in the one-photon exchange approximation. A numerical application is done for the model suggested in Ref. [4].
II. GENERAL FORMALISM

A. General expression of the cross section

The general structure of the differential cross section for the reaction:

\[ \bar{p}(p_1) + p(p_2) \to \gamma^* + \pi^0 \to \ell^+(k_2) + \ell^-(k_1) + \pi^0(q_\pi), \quad \ell = e, \mu, \tau \]  

is considered in the frame of one-photon-exchange mechanism, and applied to the case of electrons, neglecting the lepton mass.

In this section, the formalism is based on the most general symmetry properties of the hadron electromagnetic interaction, such as gauge invariance (the conservation of the hadronic and leptonic electromagnetic currents), P-invariance (invariance with respect to space reflections) and does not depend on the details of the reaction mechanism.

Let us consider the reaction \( \bar{p} + p \to \gamma^* + \pi^0 \) where \( \gamma^* \) is a virtual photon. In the one-photon-exchange approximation, the matrix element of the reaction (1) in terms of the hadronic \( J_\mu \) and leptonic \( j_\mu \) currents can be written as

\[ M = \frac{4\pi\alpha}{q^2} J_\mu j^*_\mu, \]  

where \( q = k_1 + k_2 \) is the virtual photon four-momentum, \( J_\mu \) is the electromagnetic current describing the transition \( \bar{p} + p \to \pi^0 + \gamma^* \), \( j_\mu = \bar{u}(-k_2)\gamma_\mu u(k_1) \) describes the decay \( \gamma^*(q) \to e^+(k_2) + e^-(k_1) \).

The modulus squared of the matrix element can be written as the contraction of the hadronic \( H_{\mu\nu} \) and leptonic \( L_{\mu\nu} \) tensors:

\[ |M|^2 = \frac{16\pi^2\alpha^2}{q^4} H_{\mu\nu} L_{\mu\nu}, \]  

with

\[ H_{\mu\nu} = J_\mu J^*_\nu, \quad L_{\mu\nu} = j_\mu j^*_\nu. \]  

The most convenient system for the analysis of the spin structure of the matrix element is the antiproton-proton center-of-mass system (CMS).

To derive polarization observables it is necessary to define a particular reference frame. We choose a reference frame with the \( z \) axis directed along the momentum of the virtual photon \( \vec{k} \), the momentum of the antiproton beam \( \vec{p} \) lies in the \( xz \) plane. The \( y \) axis is normal
to the $\bar{p} + p \rightarrow \pi^0 + \gamma^*$ reaction plane and it is directed along the vector $\vec{k} \times \vec{p}$; $x$, $y$, and $z$ form a right-handed coordinate system.

In this system, the components of the particle four-momenta for the reaction $\bar{p} + p \rightarrow \pi^0 + \gamma^*$ are

$$p_1 = (E, \vec{p}), \quad p_2 = (E, -\vec{p}), \quad q = (k_0, \vec{k}), \quad q_\pi = (E_\pi, -\vec{k}).$$

The general expression for the differential cross section of the reaction considered has a standard form

$$d\sigma = \frac{\alpha^2}{(2\pi)^3} \frac{L_{\mu\nu} H_{\mu\nu}}{I^2} \frac{d^3 k_1}{2E_1} \frac{d^3 k_2}{2E_2} \frac{d^3 q_\pi}{2E_\pi} \delta^{(4)}(p_1 + p_2 - k_1 - k_2 - q_\pi),$$

where $I^2 = (p_1 \cdot p_2)^2 - M^4$, $M$ is the nucleon mass, $E_1(E_2)$ is the energy of the electron (positron) and $E_\pi$ is the pion energy.

Integrating the expression (6) over the positron variables and over the electron energy, with the help of $\delta$–function, we obtain the following form for the differential cross section (hereafter we neglect the electron mass)

$$\frac{d^3 \sigma}{dE_\pi d\Omega_\pi d\Omega_e} = \frac{\alpha^2}{64\pi^3} \frac{E_1 |\vec{k}|}{q^4} \frac{L_{\mu\nu} H_{\mu\nu}}{pW} \frac{1}{k_0 - |\vec{k}| \cos \theta_1},$$

where $W = 2E$ is the total energy of the final particles, $p = \sqrt{W^2 - 4M^2}/2$ is the modulus of the three-momentum of the proton or antiproton in the reaction CMS, $E_1$ is the energy of the electron, $\theta_1$ is the angle between the momenta of the virtual photon and of the detected electron, $k_0 = 2E - E_\pi$ and $|\vec{k}| = \sqrt{E_\pi^2 - m_\pi^2}$ are the energy and the momentum of the virtual photon, and $m_\pi$ is the pion mass. The electron energy in the reaction CMS is $E_1 = q^2/2(k_0 - |\vec{k}| \cos \theta_1)$, with $q^2 = 4E^2 - 4EE_\pi + m_\pi^2$.

The leptonic tensor, when the electron is unpolarized, is:

$$L_{\mu\nu}(0) = -2q^2 g_{\mu\nu} + 4(k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu}).$$

The longitudinal polarization of the electron induces a term in the leptonic tensor which has the form

$$L_{\mu\nu}(s) = 2i\lambda <\mu \nu q k_1 >,$$

where $<\mu \nu ab> = \varepsilon_{\mu \nu \rho \sigma} a_{\rho} b_{\sigma}$, and $\lambda$ is the degree of the longitudinal polarization of the electron. We use the convention $\varepsilon_{xyz0} = 1$. 

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Taking into account the conservation of the hadronic \( J_\mu \) and leptonic \( j_\mu \) currents, we can remove the time components of the hadronic and leptonic tensors and obtain:

\[
L_{\mu\nu}H_{\mu\nu} = \frac{1}{2}(L_{xx} + L_{yy})(H_{xx} + H_{yy}) + \frac{1}{2}(L_{xx} - L_{yy})(H_{xx} - H_{yy}) + \frac{q^4}{k_0^4}L_{zz}H_{zz} + \frac{1}{2}(L_{xy} + L_{yx})(H_{xy} + H_{yx}) + \frac{1}{2}(L_{xy} - L_{yx})(H_{xy} - H_{yx}) + \frac{q^2}{2k_0^2}(L_{zx} + L_{zx})(H_{zx} + H_{zx}) + \frac{q^2}{2k_0^2}(L_{zy} + L_{zy})(H_{zy} + H_{zy}) + \frac{q^2}{2k_0^2}(L_{yz} - L_{zy})(H_{yz} - H_{zy}).
\]

(10)

Let us express the components of the leptonic tensor in terms of measured quantities. The following relations, which can be derived from the energy and momentum conservation in our coordinate system, hold:

\[
k_{1x} = E_1 \sin \theta_1 \cos \varphi, \quad k_{1y} = E_1 \sin \theta_1 \sin \varphi, \quad k_{1z} = E_1 \cos \theta_1, \\
k_{2x} = -k_{1x}, \quad k_{2y} = -k_{1y}, \quad k_{2z} = |\vec{k}| - k_{1z},
\]

(11)

where \( \varphi \) is the azimuthal angle of the electron momentum, i.e., the angle between the \( xz \) plane and the electron-positron pair production plane which is determined by the momenta of the virtual photon and detected electron.

Then the general structure of the differential cross section for the reaction (11) for the case when produced electron and the pion are detected in coincidence and the detected electron is longitudinally polarized (the polarization states of the proton target and antiproton beam can be any) has the form

\[
\frac{d^3\sigma}{dE_d d\Omega_d d\Omega_e} = N\{H_{xx} + H_{yy} + \varepsilon \cos 2\varphi(H_{yy} - H_{xx}) + 2\varepsilon \frac{q^2}{k_0^2}H_{zz} - \varepsilon \sin 2\varphi(H_{xy} + H_{yx}) - \sqrt{\frac{q^2}{k_0}}\sqrt{2\varepsilon(1 - \varepsilon)}\eta \cos \varphi(H_{xx} + H_{xx}) - \sqrt{\frac{q^2}{k_0}}\sqrt{2\varepsilon(1 - \varepsilon)}\eta \sin \varphi(H_{yx} + H_{xz}) - i\lambda \sqrt{\frac{q^2}{k_0}}\sqrt{2\varepsilon(1 - \varepsilon)}\sin \varphi(H_{xx} - H_{xx}) + i\lambda \sqrt{\frac{q^2}{k_0}}\sqrt{2\varepsilon(1 - \varepsilon)}\cos \varphi(H_{yx} - H_{yz}) + i\lambda\sqrt{1 - \varepsilon^2}\eta(H_{xy} - H_{yx})\}.
\]

(12)

where \( \eta = \text{sign}(E_1 - E_2) \). Taking into account that \( E_1 - E_2 = |\vec{k}|(k_0 \cos \theta_1 - |\vec{k}|)/(k_0 - |\vec{k}| \cos \theta_1) \) and \( k_0 - |\vec{k}| \cos \theta_1 > 0 \) since \( k_0 > |\vec{k}| \) due to \( q^2 > 0 \), we have \( \eta = \text{sign}(k_0 \cos \theta_1 - |\vec{k}|) \).
We introduced the corresponding parameter \( \varepsilon \) (as it was done in the lepton–hadron scattering processes (see for example \[9\])

\[
\varepsilon^{-1} = \frac{q^2}{2E_1^2 \sin^2 \theta_1} - 1.
\]  

(13)

Let us do the following remarks on the expression (12) for the differential cross section:

- Its form is not related to a particular reaction mechanism. It is a consequence of a one-photon-exchange mechanism and of the P-invariance of the hadron electromagnetic interaction.

- Its general nature is due to the fact that its derivation requires only the hadron electromagnetic current conservation and the fact that the photon has spin one.

- As \( q^2 \gg 4m_e^2 \) (\( m_e \) is the electron mass), the components of the leptonic tensor are calculated in the limit of zero electron mass.

- It holds for the case of the longitudinally polarized electron and arbitrary polarization of the antiproton beam and proton target.

- The dependence of the differential cross section on two kinematical variables, which do not characterize the mechanism of the \( \bar{p} + p \to \pi^0 + \gamma^* \) reaction, \( \varphi \) (the angle between hadron reaction plane and lepton pair production plane) and parameter \( \varepsilon \) is explicitly singled out.

- All information about the \( \bar{p} + p \to \pi^0 + \gamma^* \) reaction mechanism is contained in the components of the hadronic tensor \( H_{ij} \).

B. The matrix element

The electromagnetic structure of hadrons, as probed by elastic and inelastic electron scattering, can be characterized by a set of structure functions. Each of these structure functions is determined by different combinations of the longitudinal and transverse components of the electromagnetic current \( J_\mu \), thus providing different pieces of information about the hadron structure and the possible mechanisms of the reaction under consideration. The formalism
of the structure functions is especially convenient for the investigation of polarization phenomena. Similar formalism can be applied to the annihilation reactions for the time-like region of the virtual photon.

Taking into account the conservation of the leptonic \( j_\mu \) and hadronic \( J_\mu \) electromagnetic currents, the matrix element of the \( \bar{p} + p \to e^+ + e^- + \pi^0 \) reaction can be written as

\[
\mathcal{M} = e e_\mu J_\mu = e \bar{e} \cdot \bar{J}, \quad e_\mu = \frac{e}{q^2} j_\mu, \quad \bar{e} = \frac{\bar{e} \cdot \bar{k}}{k_0^2} \bar{k} - \bar{e}.
\]  

(14)

The structure of this matrix element has to be the same as for the \( e^- + p \to e^- + p + \pi \) reaction (electroproduction of pions on nucleon) because it is the crossed channel of the reaction (1).

Let us use the approach, developed for the investigation of the polarization phenomena in the reaction of the electroproduction of pions on nucleon [9], for the case of the reaction (1).

Let us introduce, for convenience and to simplify the following calculations of the polarization observables, in the \( \bar{p}p \) CMS system, the orthogonal system of basic unit vectors \( \vec{q}, \vec{m} \) and \( \vec{n} \) which are built from the momenta of the particles participating in the reaction

\[
\vec{q} = \frac{k}{|k|}, \quad \vec{n} = \frac{k \times p}{|k \times p|}, \quad \vec{m} = \vec{n} \times \vec{q}.
\]  

(15)

The unit vectors \( \vec{q} \) and \( \vec{m} \) define the \( xz \)-reaction plane for the process \( \bar{p} + p \to \pi + \gamma^* \) and are directed along \( z \) and \( x \) axes, respectively. The unit vector \( \vec{n} \) is perpendicular to the reaction plane and it is directed along the \( y \) axis.

We get for the matrix element the following expression after converting from four- to two-component spinors:

\[
\mathcal{M} = e \varphi_1^\uparrow \varphi_2^\uparrow F \varphi_1,
\]  

(16)

where \( \varphi_1 (\varphi_2) \) is the proton (antiproton) spinor, respectively and \( F \) is the reaction amplitude which can be chosen in different but equivalent forms.

Being the crossed channel of the pion electroproduction on nucleon \( e + N \to e + N + \pi \), the amplitude \( F \), which describes the dynamics of the \( \bar{p} + p \to \gamma^* + \pi^0 \) reaction, is determined in the general case by six independent amplitudes for the P-conserving hadronic interaction. In the analysis of the polarization phenomena, it is convenient to use the following general form for the amplitude \( F \):

\[
F = \bar{e} \cdot \vec{m}(f_1 \vec{\sigma} \cdot \vec{m} + f_2 \vec{\sigma} \cdot \vec{q}) + \bar{e} \cdot \vec{n}(if_3 + f_4 \vec{\sigma} \cdot \vec{n}) + \bar{e} \cdot \vec{q}(f_5 \vec{\sigma} \cdot \vec{m} + f_6 \vec{\sigma} \cdot \vec{q}),
\]  

(17)
where \( f_i, i = 1 - 6 \) are the scalar amplitudes in the orthogonal basis which completely determine the reaction dynamics. These amplitudes contain information about the dynamics of the considered reaction and they depend on three independent kinematical variables, for example \( s = (p_1 + p_2)^2 \), \( q^2 \) and \( t = (p_1 - q)^2 \).

C. Hadronic tensor

Let us consider the general properties of the hadronic tensor. Taking into account that the hadronic and leptonic tensors satisfy the following conditions: \( H_{\mu\nu}q_\mu = H_{\mu\nu}q_\nu = 0 \), \( L_{\mu\nu}q_\mu = L_{\mu\nu}q_\nu = 0 \) (as a consequence of the hadronic and leptonic currents conservation) one can show that all polarization observables in the reaction \([1]\) are determined by the space components of the hadronic tensor only.

The hadronic tensor \( H_{ij} \) \((i, j = x, y, z)\) can be represented as a sum of terms which can be classified according to the polarization states of the proton-antiproton system, in the following way

\[
H_{ij} = H_{ij}(0) + H_{ij}(\xi_1) + H_{ij}(\xi_2) + H_{ij}(\xi_1, \xi_2),
\]

(18)

where the term \( H_{ij}(0) \) corresponds to the case of the unpolarized proton-antiproton system, the term \( H_{ij}(\xi_1)(H_{ij}(\xi_2)) \) corresponds to the case of polarized proton (antiproton) and the term \( H_{ij}(\xi_1, \xi_2) \) corresponds to the case when both initial particles are polarized.

The general structure of the first term of the hadronic tensor in Eq. \([18]\), which corresponds to unpolarized antiproton and proton, has the following form

\[
H_{ij}(0) = \alpha_1 q_i q_j + \alpha_2 n_i n_j + \alpha_3 m_i m_j + \alpha_4 (q_i m_j + q_j m_i) + i \alpha_5 (q_i m_j - q_j m_i).
\]

(19)

The real structure functions \( \alpha_i, i = 1-5 \), depend on three variables, for example \( s, q^2 \) and \( \cos \theta \) where \( \theta \) is the angle between the momenta of the antiproton and the virtual photon. Note that the structure function \( \alpha_5 \), for the crossed (scattering) channel \( e + N \rightarrow e + N + \pi \), is determined by the strong interaction effects of the final-state hadrons and vanishes for the pole diagrams contribution (the Born approximation) in all the kinematical range. In our case this structure function is non zero even in the Born approximation due to the fact that hadron form factors are complex here. This structure function gives contribution to the cross section only in the case of polarized leptons, where the lepton tensor contains an antisymmetric part. The structure functions are related to the reaction scalar amplitudes \( f_i \)
The general structure of the polarized hadronic tensor can be written as

\[ H_{ij}(\xi_1) = \vec{\xi}_1 \cdot \vec{n} \left( \beta_1 q_i q_j + \beta_2 m_i m_j + \beta_3 n_i n_j + \beta_4 \{q, m\}_{ij} + i \beta_5 [q, m]_{ij} \right) + \]
\[ + \vec{\xi}_1 \cdot \vec{q} \left( \beta_6 \{q, n\}_{ij} + \beta_7 \{m, n\}_{ij} + i \beta_8 [q, n]_{ij} + i \beta_9 [m, n]_{ij} \right) + \]
\[ + \vec{\xi}_1 \cdot \vec{m} \left( \beta_{10} \{q, n\}_{ij} + \beta_{11} \{m, n\}_{ij} + i \beta_{12} [q, n]_{ij} + i \beta_{13} [m, n]_{ij} \right), \]  

(20)

where \( \vec{\xi}_1 \) is the unit vector of the proton polarization and \( \{a, b\}_{ij} = a_i b_j + a_j b_i, [a, b]_{ij} = a_i b_j - a_j b_i \).

Let us denote as \( \vec{\beta}_i \) the structure functions that determine the hadronic tensor \( H_{ij}(\xi_2) \), where \( \vec{\xi}_2 \) is the antiproton polarization unit vector, corresponding to the case when the antiproton is polarized. The structure of this hadronic tensor is similar to the case of polarized proton and is given by Eq. (20), under replacement \( \beta_i \rightarrow \vec{\beta}_i \), and \( \xi_1 \rightarrow \xi_2 \).

Therefore, the dependence of the polarization observables on the nucleon (or antinucleon) polarization is determined by 13 structure functions. The expressions of these structure functions in terms of the reaction scalar amplitudes are also given in Appendix A.

Let us consider the case when both initial particles are polarized. The general structure of the hadronic tensor which describes the correlation of the nucleon and antinucleon polarizations can be written as

\[ H_{ij}(\xi_1, \xi_2) = \vec{\xi}_1 \cdot \vec{m} \vec{\xi}_2 \cdot \vec{n} \left( \gamma_1 q_i q_j + \gamma_2 m_i m_j + \gamma_3 n_i n_j + \gamma_4 \{q, m\}_{ij} + i \gamma_5 [q, m]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{m} \vec{\xi}_2 \cdot \vec{q} \left( \gamma_6 q_i q_j + \gamma_7 m_i m_j + \gamma_8 n_i n_j + \gamma_9 \{q, m\}_{ij} + i \gamma_{10} [q, m]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{q} \vec{\xi}_2 \cdot \vec{n} \left( \gamma_{11} q_i q_j + \gamma_{12} m_i m_j + \gamma_{13} n_i n_j + \gamma_{14} \{q, m\}_{ij} + i \gamma_{15} [q, m]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{n} \vec{\xi}_2 \cdot \vec{q} \left( \gamma_{16} q_i q_j + \gamma_{17} m_i m_j + \gamma_{18} n_i n_j + \gamma_{19} \{q, m\}_{ij} + i \gamma_{20} [q, m]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{q} \vec{\xi}_2 \cdot \vec{n} \left( \gamma_{21} q_i q_j + \gamma_{22} m_i m_j + \gamma_{23} n_i n_j + \gamma_{24} \{q, m\}_{ij} + i \gamma_{25} [q, m]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{n} \vec{\xi}_2 \cdot \vec{q} \left( \gamma_{26} \{q, n\}_{ij} + \gamma_{27} \{m, n\}_{ij} + i \gamma_{28} [q, n]_{ij} + i \gamma_{29} [m, n]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{n} \vec{\xi}_2 \cdot \vec{m} \left( \gamma_{30} \{q, n\}_{ij} + \gamma_{31} \{m, n\}_{ij} + i \gamma_{32} [q, n]_{ij} + i \gamma_{33} [m, n]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{q} \vec{\xi}_2 \cdot \vec{q} \left( \gamma_{34} \{q, n\}_{ij} + \gamma_{35} \{m, n\}_{ij} + i \gamma_{36} [q, n]_{ij} + i \gamma_{37} [m, n]_{ij} \right) + \]
\[ \vec{\xi}_1 \cdot \vec{q} \vec{\xi}_2 \cdot \vec{n} \left( \gamma_{38} \{q, n\}_{ij} + \gamma_{39} \{m, n\}_{ij} + i \gamma_{40} [q, n]_{ij} + i \gamma_{41} [m, n]_{ij} \right), \]  

(21)
where $\vec{\xi}_1(\vec{\xi}_2)$ is the nucleon (antinucleon) polarization unit vector. We see that the polarization observables, which are due to the correlation between the nucleon and antinucleon polarizations are determined by 41 structure functions $\gamma_i(i = 1 - 41)$. Their expressions in terms of the reaction scalar amplitudes $f_i(i = 1 - 6)$ are given in Appendix [3].

Let us stress that the results listed above have a general nature and are not related to a particular reaction mechanism. They are valid for the one-photon-exchange mechanism assuming P-invariance of the hadron electromagnetic interaction. The general nature of these results follows from the fact that their derivation requires only the hadron electromagnetic current conservation and the spin one nature of the virtual photon.

III. CROSS SECTION AND POLARIZATION OBSERVABLES

Let us calculate the five-fold differential cross section and various polarization observables for different types of experiments.

A. Unpolarized cross section

Let us calculate the differential cross section of the reaction $\bar{p}p \rightarrow e^+e^-\pi^0$ for the case when all particles are unpolarized. It is necessary to calculate the components of the hadronic tensor $H_{ij}(0)$ which general form is given in Eq. (19). One has

$$H_{xx}(0) \pm H_{yy}(0) = \alpha_3 \pm \alpha_2, \quad H_{zz}(0) = \alpha_1, \quad H_{xz}(0) + H_{zx}(0) = 2\alpha_4,$$

$$H_{xz}(0) - H_{zx}(0) = -2i\alpha_5. \quad(22)$$

The remaining components vanish.

The general form of the differential cross section for the reaction $\bar{p}p \rightarrow e^+e^-\pi^0$ in the case of unpolarized particles for the experimental conditions where the pion and the electron are detected in coincidence, can be written in terms of four independent contributions (the average over the spins of the initial particles is taken into account)

$$\frac{d^3\sigma_{un}}{dE_\pi d\Omega_\pi d\Omega_e} = N\Sigma, \quad \Sigma = \sigma_T + \epsilon \cos 2\varphi \sigma_P + \epsilon \sigma_L + \sqrt{2\epsilon(1 - \epsilon)} \cos \varphi \sigma_I,$$

$$\sigma_T = |f_1|^2 + |f_2|^2 + |f_3|^2 + |f_4|^2, \quad \sigma_L = 2\frac{q^2}{k_0^2}(|f_5|^2 + |f_6|^2),$$

$$\sigma_P = |f_3|^2 + |f_4|^2 - |f_1|^2 - |f_2|^2, \quad \sigma_I = -2\frac{\sqrt{q^2}}{k_0} \text{Re}(f_1f_5^* + f_2f_6^*)\eta. \quad(23)$$
Note that the scalar amplitudes $f_{i=1-4}$ are determined by the transverse components of the hadron electromagnetic current whereas $f_5$, $f_6$ are related to the longitudinal part. The subscripts $L$ and $T$ indicate that the corresponding contributions are determined by the longitudinal and transverse components of the electromagnetic current, respectively. The contribution $\sigma_T(\sigma_I)$ is determined by the interference of the transverse-transverse (transverse-longitudinal) components of this current. For the annihilation channel, Eq. (23) is similar to the corresponding expression in the scattering channel for the reactions of the following type $e + A \to e + h + B$ (for example, $e + N \to e + \pi + N$ or $e + d \to e + n + p$) for an experimental setup where the scattered electron and the produced hadron $h$ are detected in coincidence [9].

Since the differential cross section depends on the azimuthal angle $\varphi$ one can define the following asymmetry (the so-called left-right asymmetry in the case of the scattering channel)

$$A_{LT} = \frac{d\sigma(\varphi = 0^\circ) - d\sigma(\varphi = 180^\circ)}{d\sigma(\varphi = 0^\circ) + d\sigma(\varphi = 180^\circ)} = \frac{\sqrt{2\varepsilon(1 - \varepsilon)}\sigma_I}{\sigma_T + \varepsilon(\sigma_L + \sigma_P)}.$$  \hspace{1cm} (24)

This asymmetry is determined by the interference of the reaction amplitudes which characterize the emission of the virtual photon with nonzero longitudinal and transverse polarizations (note the subscript $LT$). In the annihilation channel, we show below that this asymmetry is proportional to $\sin \theta$.

**B. Longitudinally polarized electron**

The differential cross section for the case when the produced electron is longitudinally polarized can be written as

$$\frac{d^3\sigma}{dE_e d\Omega_e d\Omega_{\pi}} = \frac{1}{2}\frac{d^3\sigma_{un}}{dE_e d\Omega_e d\Omega_{\pi}}(1 + \lambda A'_{LT}),$$  \hspace{1cm} (25)

where the asymmetry $A'_{LT}$ is determined as follows

$$A'_{LT} = \frac{d\sigma(\lambda = 1) - d\sigma(\lambda = -1)}{d\sigma(\lambda = 1) + d\sigma(\lambda = -1)} = \sqrt{2\varepsilon(1 + \varepsilon)}\sigma'_{LT}\Sigma^{-1},$$

$$\sigma'_{LT} = 2\frac{\sqrt{q^2}}{k_0} Im (f_1 f_5^* + f_2 f_6^*).$$  \hspace{1cm} (26)

Note that, due to the specific $\varphi$-dependence, this asymmetry has to be measured in non-coplanar geometry (out-of-plane kinematics). One can show that this asymmetry is also proportional to $\sin \theta$.  

11
C. Longitudinally polarized electron and polarized proton target or polarized antiproton beam

Let us consider the production of the longitudinally polarized electron in the annihilation reaction \( p + p \to \gamma^* + \pi^0 \) where either the proton target or the antiproton beam are polarized.

The part of the differential cross section which depends on the polarization of the proton target can be written as:

\[
\frac{d^3\sigma}{dE_x d\Omega_x d\Omega_e} = \frac{1}{2} \frac{d^3\sigma_{un}}{dE_x d\Omega_x d\Omega_e} \left[ 1 + (A^p_x + \lambda T^p_x)\xi_{1x} + (A^p_y + \lambda T^p_y)\xi_{1y} + (A^p_z + \lambda T^p_z)\xi_{1z} \right], \tag{27}
\]

where \( A^p_i, i = x, y, z, \) are the analyzing powers (or the asymmetries) due to the polarization of the proton target, and \( T^p_i, i = x, y, z, \) are the coefficients of the polarization transfer from the polarized proton to the produced electron.

One can explicitly single out the dependence of the polarization observables \( A^p_i \) and \( T^p_i \) on the azimuthal angle \( \phi \) and parameter \( \varepsilon \) and write the expressions for these observables as functions of the contributions of the longitudinal (L) and transverse (T) components of the hadronic current in the \( p + p \to \gamma^* + \pi^0 \) reaction

\[
\begin{align*}
\Sigma A^p_x &= \sin \varphi \left[ \sqrt{2\varepsilon(1-\varepsilon)} A_x^{p(LT)} + \varepsilon \cos \varphi A_x^{p(TT)} \right], \\
\Sigma A^p_y &= A_y^{p(TT)} + \varepsilon A_y^{p(LL)} + \sqrt{2\varepsilon(1-\varepsilon)} \cos \varphi A_y^{p(LT)} + \varepsilon \cos 2\varphi \bar{A}_y^{p(TT)}, \\
\Sigma A^p_z &= \sin \varphi \left[ \sqrt{2\varepsilon(1-\varepsilon)} A_z^{p(LT)} + \varepsilon \cos \varphi A_z^{p(TT)} \right], \\
\Sigma T^p_x &= \sqrt{2\varepsilon(1+\varepsilon)} \cos \varphi T_x^{p(LT)} + \sqrt{1-\varepsilon^2} T_x^{p(TT)}, \\
\Sigma T^p_y &= \sqrt{2\varepsilon(1+\varepsilon)} \sin \varphi T_y^{p(LT)}, \\
\Sigma T^p_z &= \sqrt{2\varepsilon(1+\varepsilon)} \cos \varphi T_z^{p(LT)} + \sqrt{1-\varepsilon^2} T_z^{p(TT)}, \tag{28}
\end{align*}
\]

where the individual contributions to the above polarization observables in terms of the structure functions \( \beta_i \) are given by

\[
\begin{align*}
A_x^{p(TT)} &= -4\beta_{11}, \quad A_x^{p(LT)} = -4\beta_7, \quad A_x^{p(LL)} = -2\beta_{10}\eta \frac{\sqrt{q^2}}{k_0}, \quad A_x^{p(LT)} = -2\beta_6\eta \frac{\sqrt{q^2}}{k_0}, \\
A_y^{p(TT)} &= \beta_2 + \beta_3, \quad \bar{A}_y^{p(TT)} = \beta_3 - \beta_2, \quad A_y^{p(LL)} = 2\beta_1 \frac{q^2}{k_0}, \quad A_y^{p(LT)} = -2\beta_4 \eta \frac{\sqrt{q^2}}{k_0}, \\
T_x^{p(TT)} &= -2\beta_{13}\eta, \quad T_x^{p(LT)} = 2\beta_{12} \frac{\sqrt{q^2}}{k_0}, \quad T_y^{p(LT)} = -2\beta_5 \frac{\sqrt{q^2}}{k_0}, \\
T_z^{p(TT)} &= -2\beta_9\eta, \quad T_z^{p(LT)} = 2\beta_8 \frac{\sqrt{q^2}}{k_0}. \tag{29}
\end{align*}
\]
The structure functions $\beta_i$ describe the polarization phenomena for the case when the proton target is polarized. Similarly, the polarization effects for the case of polarized antiproton beam are described by the structure functions $\bar{\beta}_i$, $i = 1 - 13$. Their expressions in terms of the reaction scalar amplitudes are given in Appendix B.

The part of the differential cross section which depends on the polarization of the antiproton beam has the form given by Eq. (27), where it is necessary to do the following substitution: $\vec{\xi}_1 \rightarrow \vec{\xi}_2$, $A^p_i \rightarrow A^{\bar{p}}_i$, and $T^p_i \rightarrow T^{\bar{p}}_i$. The dependence of the polarization observables $A^{\bar{p}}_i$ and $T^{\bar{p}}_i$ on the azimuthal angle $\varphi$ and parameter $\varepsilon$ is the same as in Eq. (28). The expressions of the individual contributions $A^{\bar{p}(MN)}_i$ and $T^{\bar{p}(MN)}_i$, $MN = LL, LT, TT$, are given in Eq. (29), where it is necessary to do the substitution $\beta_i \rightarrow \bar{\beta}_i$.

D. Longitudinally polarized electron, polarized proton target and antiproton beam

The part of the differential cross section which depends on the product of the polarizations of the proton target and antiproton beam can be written as

$$\frac{d^3\sigma}{dE_\pi d\Omega_\pi d\Omega_e} = \frac{1}{2} \frac{d^3\sigma_{un}}{dE_\pi d\xi d\Omega_e} \left[ 1 + (C_{xx} + \lambda \bar{C}_{xx})\xi_1 x \xi_2 x + (C_{yy} + \lambda \bar{C}_{yy})\xi_1 y \xi_2 y + (C_{zz} + \lambda \bar{C}_{zz})\xi_1 z \xi_2 z \right. $$

$$
\left. + (C_{xy} + \lambda \bar{C}_{xy})\xi_1 x \xi_2 y + (C_{yx} + \lambda \bar{C}_{yx})\xi_1 y \xi_2 x + (C_{xz} + \lambda \bar{C}_{xz})\xi_1 x \xi_2 z + (C_{zx} + \lambda \bar{C}_{zx})\xi_1 z \xi_2 x + (C_{yz} + \lambda \bar{C}_{yz})\xi_1 y \xi_2 z + (C_{zy} + \lambda \bar{C}_{zy})\xi_1 z \xi_2 y \right],
$$

(30)

where $C_{ij}$, $i, j = x, y, z$, are the double spin correlation coefficients and $\bar{C}_{ij}$ $i, j = x, y, z$, are the triple spin coefficients describing the dependence of the longitudinal polarization of the electron on the polarization state of the polarized initial particles.

One can explicitly single out the dependence of the polarization observables $C_{ij}$ and $\bar{C}_{ij}$ on the azimuthal angle $\varphi$ and parameter $\varepsilon$ and write the expressions for these observables as functions of the contributions of the longitudinal (L) and transverse (T) components of
the hadronic current in the $\bar{p} + p \to \gamma^* + \pi^0$ reaction as:

\begin{align}
\Sigma C_{xx} &= C_{xx}^{(TT)} + \varepsilon C_{xx}^{(LL)} + \sqrt{2\varepsilon(1-\varepsilon)\cos \varphi C_{xx}^{(LT)}} + \varepsilon \cos 2\varphi \bar{C}_{xx}^{(TT)}, \\
\Sigma C_{yy} &= C_{yy}^{(TT)} + \varepsilon C_{yy}^{(LL)} + \sqrt{2\varepsilon(1-\varepsilon)\cos \varphi C_{yy}^{(LT)}} + \varepsilon \cos 2\varphi \bar{C}_{yy}^{(TT)}, \\
\Sigma C_{zz} &= C_{zz}^{(TT)} + \varepsilon C_{zz}^{(LL)} + \sqrt{2\varepsilon(1-\varepsilon)\cos \varphi C_{zz}^{(LT)}} + \varepsilon \cos 2\varphi \bar{C}_{zz}^{(TT)}, \\
\Sigma C_{xz} &= C_{xz}^{(TT)} + \varepsilon C_{xz}^{(LL)} + \sqrt{2\varepsilon(1-\varepsilon)\cos \varphi C_{xz}^{(LT)}} + \varepsilon \cos 2\varphi \bar{C}_{xz}^{(TT)}, \\
\Sigma C_{zx} &= C_{zx}^{(TT)} + \varepsilon C_{zx}^{(LL)} + \sqrt{2\varepsilon(1-\varepsilon)\cos \varphi C_{zx}^{(LT)}} + \varepsilon \cos 2\varphi \bar{C}_{zx}^{(TT)}, \\
\Sigma C_{xy} &= \sin \varphi \left[\varepsilon \cos \varphi C_{xy}^{(TT)} + \sqrt{2\varepsilon(1-\varepsilon)C_{xy}^{(LT)}}\right], \\
\Sigma C_{yx} &= \sin \varphi \left[\varepsilon \cos \varphi C_{yx}^{(TT)} + \sqrt{2\varepsilon(1-\varepsilon)C_{yx}^{(LT)}}\right], \\
\Sigma C_{yz} &= \sin \varphi \left[\varepsilon \cos \varphi C_{yz}^{(TT)} + \sqrt{2\varepsilon(1-\varepsilon)C_{yz}^{(LT)}}\right], \\
\Sigma C_{zy} &= \sin \varphi \left[\varepsilon \cos \varphi C_{zy}^{(TT)} + \sqrt{2\varepsilon(1-\varepsilon)C_{zy}^{(LT)}}\right], \\
\Sigma \bar{C}_{xx} &= \sqrt{2\varepsilon(1+\varepsilon)\sin \varphi \bar{C}_{xx}^{(LT)}}, \Sigma \bar{C}_{yy} = \sqrt{2\varepsilon(1+\varepsilon)\sin \varphi \bar{C}_{yy}^{(LT)}}, \\
\Sigma \bar{C}_{zz} &= \sqrt{2\varepsilon(1+\varepsilon)\sin \varphi \bar{C}_{zz}^{(LT)}}, \Sigma \bar{C}_{xy} = \sqrt{1-\varepsilon^2 C_{xy}^{(TT)} + \sqrt{2\varepsilon(1+\varepsilon)\cos \varphi C_{xy}^{(LT)}}, \\
\Sigma \bar{C}_{yx} &= \sqrt{1-\varepsilon^2 C_{yx}^{(TT)} + \sqrt{2\varepsilon(1+\varepsilon)\cos \varphi C_{yx}^{(LT)}}, \\
\Sigma \bar{C}_{yz} &= \sqrt{1-\varepsilon^2 C_{yz}^{(TT)} + \sqrt{2\varepsilon(1+\varepsilon)\cos \varphi C_{yz}^{(LT)}}, \\
\Sigma \bar{C}_{zy} &= \sqrt{1-\varepsilon^2 C_{zy}^{(TT)} + \sqrt{2\varepsilon(1+\varepsilon)\cos \varphi C_{zy}^{(LT)}}, \\
\Sigma \bar{C}_{xz} &= \sqrt{2\varepsilon(1+\varepsilon)\sin \varphi \bar{C}_{xz}^{(LT)}}, \Sigma \bar{C}_{zx} = \sqrt{2\varepsilon(1+\varepsilon)\sin \varphi \bar{C}_{zx}^{(LT)}}. \quad (31)
\end{align}

where the individual contributions to the above polarization observables in terms of the
structure functions $\gamma_i$ are given by

\[
C_{xx}^{(TT)} = 2(\gamma_2 + \gamma_3), \quad \bar{C}_{xx}^{(TT)} = 2(\gamma_3 - \gamma_2), \quad C_{xx}^{(LL)} = 4\gamma_1 \frac{q^2}{k_0^2}, \quad C_{xx}^{(LT)} = -4\gamma_4 \eta \frac{\sqrt{q^2}}{k_0},
\]

\[
C_{yy}^{(TT)} = 2(\gamma_{17} + \gamma_{18}), \quad \bar{C}_{yy}^{(TT)} = 2(\gamma_{18} - \gamma_{17}), \quad C_{yy}^{(LL)} = 4\gamma_{16} \frac{q^2}{k_0^2}, \quad C_{yy}^{(LT)} = -4\gamma_{19} \eta \frac{\sqrt{q^2}}{k_0},
\]

\[
C_{zz}^{(TT)} = 2(\gamma_{22} + \gamma_{23}), \quad \bar{C}_{zz}^{(TT)} = 2(\gamma_{23} - \gamma_{22}), \quad C_{zz}^{(LL)} = 4\gamma_{21} \frac{q^2}{k_0^2}, \quad C_{zz}^{(LT)} = -4\gamma_{24} \eta \frac{\sqrt{q^2}}{k_0},
\]

\[
C_{xx}^{(TT)} = 2(\gamma_7 + \gamma_8), \quad \bar{C}_{xx}^{(TT)} = 2(\gamma_8 - \gamma_7), \quad C_{xx}^{(LL)} = 4\gamma_6 \frac{q^2}{k_0^2}, \quad C_{xx}^{(LT)} = -4\gamma_9 \eta \frac{\sqrt{q^2}}{k_0},
\]

\[
C_{yy}^{(TT)} = 2(\gamma_{12} + \gamma_{13}), \quad \bar{C}_{yy}^{(TT)} = 2(\gamma_{13} - \gamma_{12}), \quad C_{yy}^{(LL)} = 4\gamma_{11} \frac{q^2}{k_0^2}, \quad C_{yy}^{(LT)} = -4\gamma_{14} \eta \frac{\sqrt{q^2}}{k_0},
\]

\[
C_{xy}^{(TT)} = -8\gamma_{27}, \quad C_{yx}^{(LT)} = -4\gamma_{26} \eta \frac{\sqrt{q^2}}{k_0}, \quad C_{xy}^{(TT)} = -8\gamma_{31}, \quad C_{yx}^{(LT)} = -4\gamma_{30} \eta \frac{\sqrt{q^2}}{k_0}, \quad C_{xz}^{(TT)} = -8\gamma_{35}, \quad C_{zx}^{(LT)} = -4\gamma_{34} \eta \frac{\sqrt{q^2}}{k_0}, \quad C_{xz}^{(TT)} = -8\gamma_{39}, \quad C_{zx}^{(LT)} = -4\gamma_{38} \eta \frac{\sqrt{q^2}}{k_0},
\]

\[
\bar{C}_{xx}^{(LL)} = -4\gamma_{5} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{yy}^{(LL)} = -4\gamma_{20} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{zz}^{(LL)} = -4\gamma_{25} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{xy}^{(LL)} = -4\gamma_{29} \eta, \quad \bar{C}_{yx}^{(LL)} = 4\gamma_{28} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{xy}^{(TT)} = -4\gamma_{33} \eta, \quad \bar{C}_{yx}^{(TT)} = 4\gamma_{32} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{yz}^{(LL)} = -4\gamma_{37} \eta, \quad \bar{C}_{zy}^{(LL)} = 4\gamma_{36} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{yz}^{(TT)} = -4\gamma_{41} \eta, \quad \bar{C}_{zy}^{(TT)} = 4\gamma_{40} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{zz}^{(TT)} = -4\gamma_{10} \frac{\sqrt{q^2}}{k_0}, \quad \bar{C}_{xx}^{(TT)} = -4\gamma_{15} \frac{\sqrt{q^2}}{k_0}.
\]

Note that the expressions for the structure functions $\gamma_i$ in terms of the reaction scalar amplitudes $f_i$ have a general nature, not depending on the reaction mechanism.

At this stage, the general model-independent analysis of the polarization observables in the reaction \( \Pi \) is completed. To proceed further in the calculation of the observables, one needs a model for the reaction mechanism.

### IV. BORN APPROXIMATION

The numerical estimation of the differential cross section and the polarization observables depends on the model that has been chosen to describe the $\bar{p} + p \rightarrow \pi + \gamma^*$ reaction. For the present calculation of the matrix element, let us use the approach of Ref. [4], which is based on the Compton-like Feynman amplitudes (the Born approximation). The Feynman diagrams for this reaction are shown in Figs. 1(a) and 1(b) for lepton pair emission from the
proton and from the antiproton, respectively. The interest of these diagrams is that they contain the $\gamma^*pp^{(*)}$ vertex, allowing to access nucleon form factors in the ‘unphysical region’ of the transferred momentum, below the physical threshold.

As previously discussed in Refs. [4, 6], one of the hadrons involved in the electromagnetic vertices is virtual and, rigorously speaking, the involved form factors should be modified by taking into account off-mass-shell effects. However, we use the standard expression for the nucleon electromagnetic current involving on-mass-shell nucleons, keeping in mind that the comparison with the future data will be meaningful in the kinematical region where the virtuality is small.

We use the following expression for the nucleon electromagnetic current which involves the Dirac $F_1(q^2)$ and Pauli $F_2(q^2)$ form factors

$$<N(p_2)|\Gamma_\mu(q^2)|N(p_1)> = \bar{u}(p_2)[F_1(q^2)\gamma_\mu + \frac{1}{4M}F_1(q^2)(\hat{q}\gamma_\mu - \gamma_\mu\hat{q})]u(p_1), \quad (33)$$

where $q$ is the four-momentum of the virtual photon. The nucleon form factors in the kinematical region of interest for the present work are largely unexplored complex functions.

Let us write the hadronic current corresponding to the two Feynman diagrams of Fig. 1 as follows [4]

$$J_\mu = \varphi_\pi^+(q_\pi)\bar{u}(-p_1)O_\mu u(p_2). \quad (34)$$

Here $\varphi_\pi$ is the pion wave function. Using the Feynman rules we can write

$$O_\mu = \frac{g}{d_1}\Gamma_\mu(q)(\hat{q} - \hat{p}_1 + M)\gamma_5 + \frac{g}{d_2}\gamma_5(\hat{p}_2 - \hat{q} + M)\Gamma_\mu(q), \quad (35)$$
where \( d_i = q^2 - 2q \cdot p_i, i = 1, 2, \) and \( g \) is the coupling constant describing the pion-nucleon vertex \( \pi NN^* \), \( (N^* \) is the off-mass-shell nucleon). The possible off-mass-shell effects of this coupling constant are neglected. Note that the hadronic current (34) is conserved, \( q \mu J^\mu = 0. \)

Let us write the matrix element in the proton-antiproton CMS. We obtain the following expression for the amplitude

\[
F = g_1 \varepsilon \cdot \vec{k} \sigma \cdot \vec{p} + g_2 \varepsilon \cdot \vec{p} \sigma \cdot \vec{k} + g_3 \varepsilon \cdot \vec{p} \sigma \cdot \vec{p} + g_4 \varepsilon \cdot \vec{p} \sigma \cdot \vec{p} + g_5 \varepsilon + i g_6 \varepsilon \cdot (\vec{k} \times \vec{p}),
\]

(36)

where \( g_i, i = 1 - 6, \) are the scalar amplitudes in a nonorthogonal basis. The previous structure of the matrix element (36) arises naturally in the transition from four- to two-component spinors and it is also a general expression that does not depend on the details of the reaction mechanism. In the case of the Born approximation, according to the Feynman diagrams in Fig. 1, the scalar amplitudes have the following form:

\[
\begin{align*}
g_1^B &= g \frac{p}{M} \left( \frac{1}{d_2} - \frac{1}{d_1} \right) F_2(q^2), \\
g_2^B &= g \frac{p}{M} \left( \frac{1}{d_1} - \frac{1}{d_2} \right) \left[ 2EF_1(q^2) + \left( k_0 + \frac{k}{M}p \cos \theta \right) F_2(q^2) \right], \\
g_3^B &= 2g \left( \frac{1}{d_1} + \frac{1}{d_2} \right) \left[ \frac{p}{M} F_2(q^2) + \frac{M}{p} G_M(q^2) \right], \\
g_4^B &= -2g \left( \frac{1}{d_1} + \frac{1}{d_2} \right) \left[ (2E - k_0)F_1(q^2) + \frac{k}{M}p \cos \theta F_2(q^2) \right], \\
g_5^B &= -2gMk \cos \theta \left( \frac{1}{d_1} + \frac{1}{d_2} \right) G_M(q^2), \\
g_6^B &= 2gE \frac{p}{M} \left( \frac{1}{d_1} + \frac{1}{d_2} \right) G_M(q^2),
\end{align*}
\]

(37)

where \( G_M(q^2) = F_1(q^2) + F_2(q^2), \) \( k \) \( (p) \) is the magnitude of the virtual photon (antiproton) three–momentum, and \( k_0 \) \( (E) \) is the energy of the virtual photon (antiproton). These quantities are expressed in the reaction CMS as

\[
E = \frac{\sqrt{s}}{2}, \quad k_0 = \frac{1}{2 \sqrt{s}} (s + q^2 - m^2_\pi), \quad p^2 = E^2 - M^2, \quad k^2 = k_0^2 - q^2,
\]

(38)

where \( m_\pi \) is the pion mass.

The scalar amplitudes in an orthogonal basis, \( f_i, \) which are used for the analysis of the polarization effects in the reaction (11) can be related to the scalar amplitudes \( g_i \) through the following relations

\[
\begin{align*}
f_1 &= p^2 \sin^2 \theta g_4 + g_5, \\
f_2 &= p \sin \theta (kg_3 + p \cos \theta g_4), \\
f_3 &= kp \sin \theta g_6, \\
f_4 &= g_5, \\
f_5 &= p \sin \theta (kg_2 + p \cos \theta g_4), \\
f_6 &= k^2 g_1 + g_5 + p \cos \theta (kg_2 + kg_3 + p \cos \theta g_4).
\end{align*}
\]

(39)
V. NUMERICAL RESULTS

In a high luminosity experiment and with a detector with good efficiency and acceptance, in particular at forward and backward angles, a precise angular distribution of the scattered electron can be measured.

Assuming the reaction mechanism suggested in Ref. [4], we illustrate in this section the angular dependence of different observables, for similar kinematical conditions as considered in the previous work [8]. The form factors parametrization is based on VMD model of Ref. [10], and follows Ref. [11].

For a value of the total energy squared $s=5.5$ GeV$^2$, three values of the momentum transfer squared are considered: $q^2 = 0.5, 2$ and $4$ GeV$^2$.

For these values of $s$ and $q^2$, in Fig. 2, the moduli squared of the amplitudes are shown as functions of $\cos \theta$. The amplitudes have a characteristic angular dependence and decrease when the transferred momentum increases. They are of the same order of magnitude, except $|f_5|^2$ which is ten times smaller. The amplitudes $|f_4|^2$, $|f_5|^2$ and $|f_6|^2$ vanish for $\cos \theta=0$.

For a simple illustration of the present results we chose to fix two more variables, $\theta_1 = 5^\circ$ and $\phi = 45^\circ$, and show the dependence of the variables as functions of $\cos \theta$.

The reduced cross section, $\Sigma$ and its four contributions, Eq. (23), are shown in Fig. 3. As expected, the largest contribution to the cross section is given by the transversal component, $\sigma_T$, which is the incoherent sum of four amplitudes. The reduced cross section takes its maximum value for backward and forward scattering, in the considered kinematics.

In Fig. 4, the angular asymmetry, Eq. (24), is illustrated as function of $\cos \theta$. It is an even function of $\cos \theta$, vanishing at forward and backward angles, and decreases when $Q^2$ increases. The single spin asymmetry, Eq. (26), vanishes in Born approximation.

The analyzing powers due to the polarization of the proton target $A^p_i$, $i = x, y, z$, and the coefficients of the polarization transfer from the polarized proton to the produced electron $T^p_i$, $i = x, y, z$, (Eq. (28)) are illustrated in Fig. 5 (same notations as Fig. 4).

The double spin correlation coefficients $C_{ij}$, $i, j = x, y, z$, , Eq. (31), when both initial particles are polarized, are shown in Fig. 6.

One can see from Figs. 4, 5 and 6 that polarization observables are generally large and show a characteristic dependence on angles and momentum transfer squared. They are generally larger at smaller $q^2$ values. The comparison with future experimental data will
allow to confirm or infirm the validity of the considered model.

VI. CONCLUSIONS

Model independent expressions for the five-fold cross section and polarization observables have been derived for the reaction $\bar{p} + p \rightarrow e^+ + e^- + \pi^0$, when the annihilation occurs through the one-photon exchange mechanism.

It has been shown that this reaction is completely described by a set of six amplitudes (in general complex) which are functions of three kinematical variables.

The expressions of the cross section, of the angular (out-of-plane) asymmetry and of single and double spin observables have been derived in terms of the relevant amplitudes, in
Fig. 3. (Color online) Reduced cross section $\Sigma$ (solid line, black) as function of $\cos \theta$ from Eq. (23) for the reaction $\bar{p} + p \rightarrow e^+ + e^- + \pi^0$, at $s = 5.5 \text{ GeV}^2$, $\theta_1 = 5^\circ$ and $\varphi = 45^\circ$ for three values of the momentum transfer squared: $q^2 = 0.5 \text{ GeV}^2$, $q^2 = 2 \text{ GeV}^2$ and $q^2 = 4 \text{ GeV}^2$ from top to bottom. The individual contributions to the reduced cross section are also reported: $\sigma_T$ (large dash triple dotted line, magenta), $\sigma_L$ (dashed line, red), $\sigma_P$ (dotted line, green), $\sigma_I$ (dash-dotted line, blue).

A general formalism, which holds for any model of the proton and for any reaction mechanism.

The numerical application has been done following the model of Ref. [4], which is based on $t$- and $u$-channel exchange diagrams, i.e. those diagrams which allow, in principle, to access hadron form factors in the time-like region under the physical threshold.

The results show large and measurable values of the observables, including an angular asymmetry, which is measurable with out-of-plane measurements in unpolarized particle reactions.
Fig. 4. (Color online) Angular asymmetry (Eq. 24) as function of $\cos \theta$ for the reaction $\bar{p} + p \rightarrow e^+ + e^- + \pi^0$, for $s=5.5$ GeV$^2$ and three values of the momentum transfer squared: $q^2 = 0.5$ GeV$^2$ (black solid line), $q^2 = 2$ GeV$^2$ (dashed red line) and $q^2 = 4$ GeV$^2$ (green dotted line).

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Appendix A: Helicity and scalar amplitudes

In this Appendix the helicity amplitudes and their relation with the reaction scalar amplitudes are given.

Let us define the helicity amplitudes as follows

$$M_{\lambda_1 \lambda_2}^\lambda = e \varphi_2^\lambda (\lambda_1) F(\lambda) \varphi_1(\lambda_2),$$  \hspace{1cm} (A.1)

where $\lambda$ and $\lambda_1(\lambda_2)$ are the helicities of the virtual photon and antiproton (proton). Using the expressions for the helicity states of the corresponding particles one obtains the following
relations between the helicity amplitudes and reaction scalar amplitudes

\begin{align*}
  h_1 &= M_{++}^0 = -ie\sqrt{q^2}k_0 (\sin \theta f_5 + \cos \theta f_6), \quad h_2 &= M_{+-}^0 = -ie\sqrt{q^2}k_0 (\cos \theta f_5 + \sin \theta f_6), \\
  h_3 &= M_{++}^+ = \frac{ie}{\sqrt{2}} (\sin \theta f_1 + \cos \theta f_2 + f_3), \quad h_4 &= M_{--}^+ = \frac{ie}{\sqrt{2}} (\sin \theta f_1 + \cos \theta f_2 - f_3), \\
  h_5 &= M_{+-}^+ = \frac{ie}{\sqrt{2}} (-\cos \theta f_1 + \sin \theta f_2 + f_4), \quad h_6 &= M_{-+}^+ = \frac{ie}{\sqrt{2}} (\cos \theta f_1 - \sin \theta f_2 + f_4),
\end{align*}

(A.2)

where \( \lambda = \pm, 0 \) means that the virtual photon helicity is \( \pm 1, 0 \), respectively, and index \( \lambda_1 = \pm (\lambda_2 = \pm) \) means that the antiproton (proton) helicity is \( \pm 1/2 (\pm 1/2) \). The remaining helicity amplitudes are not independent and they can be obtained from the relation

\[ M_{\lambda_1\lambda_2}^\lambda = (-1)^{\lambda_1 + \lambda_2 - \lambda} M_{-\lambda_1 - \lambda_2}^{-\lambda}. \]  

(A.3)
Fig. 6. (Color online) Same as in Fig. 4 for the following polarization observables from left to right, from top to bottom: $C_{xx}^{p}$, $C_{yy}^{p}$, $C_{zz}^{p}$, $C_{xz}^{p}$, $C_{zx}^{p}$, $C_{xy}^{p}$, $C_{yx}^{p}$, $C_{yz}^{p}$, $C_{zy}^{p}$.

We present also the inverse relation, i.e., the expressions of the reaction scalar amplitudes in terms of the helicity amplitudes. They are

$$
\begin{align*}
  f_1 &= -i e^{\sqrt{2}} \left[ \sin \theta (h_3 + h_4) - \cos \theta (h_5 - h_6) \right], \\
  f_2 &= -i e^{\sqrt{2}} \left[ \cos \theta (h_3 + h_4) + \sin \theta (h_5 - h_6) \right], \\
  f_3 &= -i e^{\sqrt{2}} (h_3 - h_4), \\
  f_4 &= -i e^{\sqrt{2}} (h_5 + h_6), \\
  f_5 &= \frac{i k_0}{e^{\sqrt{2} q^2}} (\sin \theta h_1 - \cos \theta h_2), \\
  f_6 &= \frac{i k_0}{e^{\sqrt{2} q^2}} (\cos \theta h_1 + \sin \theta h_2). 
\end{align*}
$$

\textbf{Appendix B: Structure functions and scalar amplitudes}

In this Appendix the explicit expressions for the structure functions $\alpha_i$ ($i = 1 - 5$), $\beta_i$ ($i = 1 - 13$) and $\gamma_i$ ($i = 1 - 41$) are given in terms of the orthogonal scalar amplitudes $f_i$. 
(i = 1 − 6) which determine the \( \bar{p} + p \to \pi + \gamma^* \) reaction.

The structure functions \( \alpha_i \ (i = 1 − 5) \), describing the hadronic tensor in the case of the unpolarized proton and antiproton, can be written as

\[
\alpha_1 = 2 \left( |f_5|^2 + |f_6|^2 \right), \quad \alpha_2 = 2 \left( |f_3|^2 + |f_4|^2 \right), \quad \alpha_3 = 2 \left( |f_1|^2 + |f_2|^2 \right),
\]
\[
\alpha_4 = 2 \text{Re}(f_1 f_5^* + f_2 f_6^*), \quad \alpha_5 = -2 \text{Im}(f_1 f_5^* + f_2 f_6^*). \quad (B.1)
\]

The expressions for the structure functions \( \beta_i (i = 1 − 13) \), describing the hadronic tensor in the case when only proton is polarized, are

\[
\beta_1 = -2 \text{Im} f_5 f_6^*, \quad \beta_2 = -2 \text{Im} f_1 f_2^*, \quad \beta_3 = -2 \text{Im} f_3 f_4^*, \quad \beta_4 = \text{Im}(f_2 f_5^* - f_1 f_6^*), \]
\[
\beta_5 = \text{Re}(f_2 f_5^* - f_1 f_6^*), \quad \beta_6 = -\text{Im}(f_3 f_6^* + f_4 f_5^*), \quad \beta_7 = \text{Im}(f_1 f_4^* + f_2 f_3^*), \]
\[
\beta_8 = -\text{Re}(f_3 f_6^* + f_4 f_5^*), \quad \beta_9 = -\text{Re}(f_1 f_4^* + f_2 f_3^*), \quad \beta_{10} = \text{Im}(f_4 f_6^* - f_3 f_5^*), \]
\[
\beta_{11} = \text{Im}(f_1 f_3^* - f_2 f_4^*), \quad \beta_{12} = \text{Re}(f_4 f_6^* - f_3 f_5^*), \quad \beta_{13} = \text{Re}(f_2 f_4^* - f_1 f_3^*). \quad (B.2)
\]

The expressions for the structure functions \( \gamma_i \ (i = 1 − 41) \), describing the hadronic tensor in the case when both the proton and antiproton are polarized (spin correlations), are

\[
\gamma_1 = -\frac{1}{2} \left( |f_5|^2 - |f_6|^2 \right), \quad \gamma_2 = -\frac{1}{2} \left( |f_3|^2 - |f_4|^2 \right), \quad \gamma_3 = -\frac{1}{2} \left( |f_3|^2 - |f_4|^2 \right),
\]
\[
\gamma_4 = -\frac{1}{2} \text{Re}(f_1 f_5^* - f_2 f_6^*), \quad \gamma_5 = -\frac{1}{2} \text{Im}(f_2 f_6^* - f_1 f_5^*), \quad \gamma_6 = -\text{Re} f_5 f_6^*, \quad \gamma_7 = -\text{Re} f_1 f_2^*, \]
\[
\gamma_8 = \text{Re} f_3 f_4^*, \quad \gamma_9 = -\frac{1}{2} \text{Re}(f_1 f_6^* + f_2 f_5^*), \quad \gamma_{10} = \frac{1}{2} \text{Im}(f_1 f_6^* + f_2 f_5^*),
\]
\[
\gamma_{11} = -\text{Re} f_5 f_6^*, \quad \gamma_{12} = -\text{Re} f_1 f_2^*, \quad \gamma_{13} = -\text{Re} f_3 f_4^*, \quad \gamma_{14} = -\frac{1}{2} \text{Re}(f_1 f_6^* + f_2 f_5^*),
\]
\[
\gamma_{15} = \frac{1}{2} \text{Im}(f_1 f_6^* + f_2 f_5^*), \quad \gamma_{16} = \frac{1}{2} \left( |f_3|^2 + |f_4|^2 \right), \quad \gamma_{17} = \frac{1}{2} \left( |f_3|^2 + |f_4|^2 \right),
\]
\[
\gamma_{18} = -\frac{1}{2} \left( |f_3|^2 + |f_4|^2 \right), \quad \gamma_{19} = \frac{1}{2} \text{Re}(f_1 f_5^* + f_2 f_6^*), \quad \gamma_{20} = \frac{1}{2} \text{Im}(f_5 f_6^* + f_4 f_2^*),
\]
\[
\gamma_{21} = \frac{1}{2} \left( |f_5|^2 - |f_6|^2 \right), \quad \gamma_{22} = \frac{1}{2} \left( |f_1|^2 - |f_2|^2 \right), \quad \gamma_{23} = -\frac{1}{2} \left( |f_3|^2 - |f_4|^2 \right),
\]
\[
\gamma_{24} = \frac{1}{2} \text{Re}(f_1 f_5^* - f_2 f_6^*), \quad \gamma_{25} = \frac{1}{2} \text{Im}(f_5 f_1^* - f_6 f_2^*), \quad \gamma_{26} = -\frac{1}{2} \text{Re}(f_4 f_5^* + f_3 f_6^*),
\]
\[
\gamma_{27} = \frac{1}{2} \text{Re}(f_1 f_4^* + f_2 f_3^*), \quad \gamma_{28} = \frac{1}{2} \text{Im}(f_4 f_5^* + f_3 f_6^*), \quad \gamma_{29} = -\frac{1}{2} \text{Im}(f_1 f_4^* + f_2 f_3^*),
\]
\[
\gamma_{30} = -\frac{1}{2} \text{Re}(f_4 f_5^* - f_3 f_6^*), \quad \gamma_{31} = -\frac{1}{2} \text{Re}(f_1 f_4^* - f_2 f_3^*), \quad \gamma_{32} = -\frac{1}{2} \text{Im}(f_3 f_6^* - f_4 f_5^*),
\]
\[
\gamma_{33} = -\frac{1}{2} \text{Im}(f_1 f_4^* - f_2 f_3^*), \quad \gamma_{34} = -\frac{1}{2} \text{Re}(f_4 f_5^* + f_3 f_6^*), \quad \gamma_{35} = -\frac{1}{2} \text{Re}(f_3 f_6^* + f_2 f_3^*),
\]
\[
\gamma_{36} = \frac{1}{2} \text{Im}(f_3 f_5^* + f_4 f_6^*), \quad \gamma_{37} = -\frac{1}{2} \text{Im}(f_1 f_5^* + f_2 f_4^*), \quad \gamma_{38} = -\frac{1}{2} \text{Re}(f_4 f_6^* - f_3 f_5^*),
\]
\[
\gamma_{39} = -\frac{1}{2} \text{Re}(f_2 f_4^* - f_1 f_3^*), \quad \gamma_{40} = -\frac{1}{2} \text{Im}(f_3 f_5^* - f_4 f_6^*), \quad \gamma_{41} = -\frac{1}{2} \text{Im}(f_2 f_4^* - f_1 f_3^*). \quad (B.3)
\]
The structure of the hadronic tensor describing the polarization of the antiproton is the same as for the case of polarized proton. Let us designate these structure functions as $\bar{\beta}_i$ and their expressions in terms of the scalar amplitudes are

\[
\begin{align*}
\bar{\beta}_1 &= -2Im f_5 f_6^*, \\
\bar{\beta}_2 &= -2Im f_1 f_2^*, \\
\bar{\beta}_3 &= 2Im f_3 f_4^*, \\
\bar{\beta}_4 &= Im(f_2 f_5^* - f_1 f_6^*), \\
\bar{\beta}_5 &= Re(f_2 f_5^* - f_1 f_6^*), \\
\bar{\beta}_6 &= Im(f_3 f_6^* - f_4 f_5^*), \\
\bar{\beta}_7 &= Im(f_1 f_4^* - f_2 f_3^*), \\
\bar{\beta}_8 &= Re(f_3 f_6^* - f_4 f_5^*), \\
\bar{\beta}_9 &= Re(f_2 f_3^* - f_1 f_4^*), \\
\bar{\beta}_{10} &= Im(f_4 f_6^* + f_3 f_5^*), \\
\bar{\beta}_{11} &= -Im(f_1 f_3^* + f_2 f_4^*), \\
\bar{\beta}_{12} &= Re(f_4 f_6^* + f_3 f_5^*), \\
\bar{\beta}_{13} &= Re(f_2 f_3^* + f_1 f_4^*),
\end{align*}
\]

which differ from Eq. [B.2] mainly by signs.

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