Quantum Systems at The Brink
Existence and Decay Rates of Bound States at Thresholds; Critical Potentials and dimensionality

Dirk Hundertmark
Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany

Michal Jex
Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Břehová 7, 11519 Prague, Czech Republic

M. Lange
SISSA, Mathematics Area, Via Bonomea 265, 34136 Trieste, Italy
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One of the crucial properties of a quantum system is the existence of bound states. In this paper we present a necessary and sufficient condition for the Schrödinger operator to have a zero energy bound state. In particular we show that the asymptotic behaviour of the potential is the crucial ingredient. We derive the necessary and sufficient conditions for existence and absence with respect to the dimension. Our results are sharp and show high dependence on dimension.

I. INTRODUCTION

Since the dawn of quantum physics, the existence of bound states plays a crucial role for the properties of quantum systems. Of special importance is the ground state, i.e., the eigenfunction corresponding to the lowest eigenvalue of the Hamiltonian describing the system. In this paper we consider a Schrödinger operator of the form

\[ H = -\Delta + V \]  

on \( \mathbb{R}^d \) where \( V \in K^0_{d}(\mathbb{R}^d) \) is a real potential within Kato class. We say that \( V \in K_d(\mathbb{R}^d) \) if

\[
\begin{align*}
    d &\geq 3, \quad \lim_{\alpha \downarrow 0} \sup_{|x| \leq R} \int_{|x-y| \leq \alpha} |x-y|^{-(d-2)} |V(y)| dy = 0, \quad \forall R > 0, \\
    d &\geq 2, \quad \lim_{\alpha \downarrow 0} \sup_{|x| \leq R} \int_{|x-y| \leq \alpha} -\ln(|x-y|) |V(y)| dy = 0, \quad \forall R > 0, \\
    d &\geq 1, \quad \sup_{x} \int_{|x-y| < 1} |V(y)| dy < \infty.
\end{align*}
\]

We say that \( V \in K^0_{d}(\mathbb{R}^d) \) if \( V I_{B_R(0)} \in K_d(\mathbb{R}^d) \) for all \( R \). The condition on the potential is chosen in such a way that the operator can be easily defined via quadratic forms and associated eigenfunctions are then continuous [1, Theorem C.1.1].

We are interested in the special case when the ground state energy is at the threshold of the essential spectrum. For simplicity we therefore assume that the spectrum of \( H \) is contained in \([0, \infty)\) and investigate for which potentials \( V \) a zero-energy eigenstate exists. One of the first result of this kind was shown for Helium type atoms in [2]. They proved the existence of zero energy bound state for an atom with scaled repulsion between the electrons. A more general version of this claim was proved for a general long-range Coulomb repulsion in [3]. Contrary to this the absence result was shown in [4]. The authors showed that any continuous potential decaying faster than \( |V|_+ \leq \frac{1}{|x|^2} \left( \frac{4}{3} + \frac{1}{\ln(|x|)} \right) \) in \( \mathbb{R}^3 \) do not produce zero energy eigenstates and that the constants \( 3/4 \) and \( 1 \) are optimal. A further improvement was given by Gridnev and Garcia in [5] for the case \( \mathbb{R}^3 \). They reproved that any potential smaller then \( \frac{1}{|x|^2} \) can not support zero energy bound state. Complementary to this they also showed that for any potential decaying slower then \( \frac{1}{|x|^2} \) the bound states approaching the threshold do not spread and eventually become bound states at the threshold. In short, long-range repulsion can in fact stabilize quantum systems by prohibiting eigenstates to

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* Also at Department of Mathematics, Altgeld Hall, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, USA
† Also at CEREMADE, Dauphine University, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France
tunnel to infinity. This leads to a localization effect. In this paper we extend the result in [3] to arbitrary dimension and arbitrary precision by applying higher order corrections. In particular we show that the dimension $d = 4$ is critical in the sense that every short-range potential $V$ in dimension $d \leq 4$ does not support a zero-energy bound state. Where as for $d \geq 5$ this changes and we provide a sharp condition when a critical potential has a zero energy bound state. By a critical potential we mean a potential $V \in L^1_{loc}(\mathbb{R}^d)$ such that

$$-(1 - \delta)\Delta + V$$

has a negative bound state and essential spectrum equal to $[0, \infty)$ for every $\delta > 0$. Note that $K^d_{loc} \subset L^1_{loc}(\mathbb{R}^d)$.

Let $\ln_1(x) := \ln(|x|)$ and $\ln_{j+1}(x) := \ln(\ln_j(x))$ and let $H$ be as above. Our main results can be summarized as follows

- **Absence of a zero energy bound state**
  Any potential $V$ satisfying
  $$V(x) \leq \frac{d(4 - d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{n} \ln^{-1}_j(x)$$
  for a given $n \in \mathbb{N}$ and all $|x| \gg 1$ can not have a zero energy ground state.

- **Existence of a zero energy bound state**
  Any critical potential $V$ satisfying
  $$V(x) \geq \frac{d(4 - d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{n} \ln^{-1}_j(x) + \varepsilon \prod_{k=1}^{n} \ln^{-1}_k(x)$$
  for a given $n \in \mathbb{N}$, $\varepsilon > 0$ and all $|x| \gg 1$ has a zero energy bound state.

**Remark 1.** We want to stress that equations (2) and (3) clearly show that the behaviour changes drastically for $d > 4$.

**Remark 2.** In [3] the authors studied the behaviour of resonances and eigenstates at the threshold in dimension 3. Furthermore based on the remark of a referee they note that there can not be any resonances for $d = 4$ but their approach is not applicable for dimension 4. A detailed analysis of eigenstates and resonances at the threshold for the case of nonlocal operators was carried out in [4]. Compactly supported zero energy $L^2$ solutions were constructed in [4] for potentials in $L^p(\mathbb{R}^d)$ for $p < d/2$ and $d \geq 3$ or $L^1(\mathbb{R}^2)$. Such potentials are not in Kato class.

In Section II we present all the necessary definitions and formulate the main results. Section III is devoted to their proofs. We note that it will be necessary to have a positive ground state for the non-existence proof. Thus in general we cannot prove the absence of ground state for fermionic particle statistics. However the existence proof works without this requirement.

## II. DEFINITIONS AND MAIN RESULTS

The operator $H$ is a self-adjoint realization of the differential expression $P = -\Delta + V$ in $L^2(\mathbb{R}^d)$. We consider solutions of the Schrödinger equation

$$-\Delta u(x) + V(x)u(x) = 0, \quad \text{in } \Omega$$

where $-\Delta u$ is the Laplacian on a set $\Omega \subseteq \mathbb{R}^d$ and $V$ is a real function in $K^d_{loc}(\Omega)$. We define solutions, supersolutions and subsolutions of (4) in the following sense:

**Definition 1.**
- **$u$ is a solution of (4)** if $u$ is a function in $H^1_{loc}(\Omega)$ with $Vu \in L^1_{loc}(\Omega)$ such that
  $$\int_{\Omega} (\nabla u \cdot \nabla \phi + Vu\phi)dx = 0$$
  for every $\phi \in C^0_0(\Omega)$.

- **$u$ is a supersolution of (4)** if $u$ is a function in $H^1_{loc}(\Omega)$ with $Vu \in L^1_{loc}(\Omega)$ such that
  $$\int_{\Omega} (\nabla u \cdot \nabla \phi + Vu\phi)dx \geq 0$$
  for every non-negative $\phi \in C^0_0(\Omega)$. 

...
\* u is a subsolution of (4) if u is a function in \( H^1_{\text{loc}}(\Omega) \) with \( Vu \in L^1_{\text{loc}}(\Omega) \) such that

\[
\int_{\Omega} (\nabla u \cdot \nabla \phi + Vu) \, dx \leq 0
\]

for every non-negative \( \phi \in C_0^{\infty}(\Omega) \).

We assume that the spectrum of \( H \) satisfies

\[
\sigma(H) \subseteq [0, \infty)
\]

and note that this for example holds if \( \lim_{|x| \to \infty} V = 0 \). Recall that we call a potential \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \) critical if for any \( \delta > 0 \) the operator

\[
-(1-\delta)\Delta + V
\]

has essential spectrum equal to \([0, \infty)\) and admits a negative bound state. We note that there are analogous alternative definitions for critical potentials. One can check that \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \) is a critical if and only if \( H = -\Delta + V \geq 0 \) but, for any nonnegative function \( W \neq 0 \), the perturbed operator \( H - W \) has a negative bound state. This is closely related to virtual levels, i.e. either resonances or eigenfunctions which lie at the threshold of the essential spectrum and can be transformed to negative eigenvalues by a small perturbation maintaining the essential spectrum \([9, 10]\).

We recall the shorthand notation introduced in the Introduction

\[
\ln_0(x) := |x|,
\ln_{j+1}(x) := \ln(\ln_j(x))
\]

for all \( j \in \mathbb{N}_0 \).

**Theorem 1** *(Non-existence result).* Let \( H \) be as in (1). Then \( H \) does not have a ground state with eigenvalue 0 if there exists \( R_0 > 0 \) such that for all \( |x| \geq R_0 \)

\[
V(x) \leq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{n} \prod_{k=1}^{j} \ln^{-1}_k(x) + \frac{1}{4|x|^2} \left( \sum_{j=1}^{n} \prod_{k=1}^{j} \ln^{-1}_k(x) \right)^2 + \frac{1}{2|x|^2} \sum_{j=1}^{n} \sum_{l=1}^{j} \prod_{k=1}^{j} \ln^{-1}_k(x) \ln^{-1}_m(x)
\]

for arbitrary \( n \in \mathbb{N}_0 \).

The inequality (2) is a simple consequence of (6). It is obtained simply by omitting positive terms of higher order within (6). Complementary to the absence result we have

**Theorem 2** *(Existence result).* Let \( H \) be as in (1) with a critical potential \( V \). Then \( H \) has a ground state with eigenvalue 0 if there exists \( R_0 > 0 \) such that for all \( |x| \geq R_0 \)

\[
V(x) \geq \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{n} \prod_{k=1}^{j} \ln^{-1}_k(x) + \frac{1}{4|x|^2} \left( \sum_{j=1}^{n} \prod_{k=1}^{j} \ln^{-1}_k(x) \right)^2 + \frac{1}{2|x|^2} \sum_{j=1}^{n} \sum_{l=1}^{j} \prod_{k=1}^{j} \ln^{-1}_k(x) \ln^{-1}_m(x)
\]

\[
+ \frac{\varepsilon^2}{4|x|^2} \prod_{j=1}^{n} \ln^{-2}_j(x) + \frac{\varepsilon}{|x|^2} \prod_{j=1}^{n} \ln^{-1}_j(x) + \frac{\varepsilon}{|x|^2} \sum_{k=1}^{n} \prod_{j=1}^{k} \prod_{m=1}^{n} \ln^{-1}_j(x) \ln^{-1}_m(x)
\]

for arbitrary \( n \in \mathbb{N}_0 \) and \( \varepsilon > 0 \).

Again the expressions within (7) can be simplified into (3). This is achieved by controlling all the higher order terms by the leading term with \( \varepsilon \) and changing the value of \( \varepsilon \) in front of it. 

**Remark 3.** We can think of the expressions (6) and (7) as a sort of expansions. Considering \( n = 0 \) we get

\[
V(x) \leq \frac{d(4-d)}{4|x|^2}
\]

for the absence and

\[
V(x) \geq \frac{d(4-d)+\varepsilon}{4|x|^2}
\]
for the existence with $\epsilon > 0$. For $d = 3$ this agrees with the result proved in [5]. However, our result is not restricted to $d = 3$ and provide a much simpler proof by avoiding Green functions. By utilizing the higher order corrections from Eqs. (6) and (7) we obtain a sharp distinction between existence and non-existence in the case of a critical potential. For example, for $n = 1, 2$ we obtain

\[
V(x) \leq \frac{d(4 - d)}{4|x|^2} + \frac{1}{|x|^2 \ln |x|}
\]

for the absence and

\[
V(x) \geq \frac{d(4 - d)}{4|x|^2} + \frac{1 + \epsilon}{|x|^2 \ln |x|} + \frac{1}{|x|^2 \ln |x| \ln (\ln |x|)}
\]

for the existence, where $\epsilon > 0$ are arbitrary. Moreover, we can also deduce that for any given $n$ it is enough to enlarge the bound on the potential which guarantees the absence by

\[
\frac{\epsilon}{|x|^2} \prod_{j=1}^{n} \ln^{-1}(x)
\]

to obtain the existence condition for arbitrary $\epsilon > 0$.

Each potential having the zero energy ground state has to be critical. This can be summarized in the following Lemma.

Lemma 1. Let $H$ be as in (1). Moreover let $H$ have a ground state with the eigenvalue $E_0 = 0$. Then $V$ must be a critical potential.

Proof. Due to the assumptions on $H$ it is straightforward to see that the ground state $\psi$ is strictly positive. Then for any $W \geq 0$ which is not identically 0 we have

\[
\langle \psi, (H - W)\psi \rangle = \langle \psi, (-W)\psi \rangle = -\langle \psi, W\psi \rangle < 0
\]

where we used that $H\psi = 0$.

The converse to Lemma 1 does not hold. A simple counter-example is given by the potential $-\frac{1}{4|x|^2}$ in $\mathbb{R}^3$. To see this consider the well-known Hardy inequality

\[
\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{4|x|^2} \, dx \leq \int_{\mathbb{R}^3} |\nabla \psi(x)|^2 \, dx.
\]

where $\psi \in H^1_0(\mathbb{R}^3)$. It is important to note that the Hardy inequality is sharp. It implies that $-\Delta - \frac{1}{4|x|^2}$ is positive and critical but $-\Delta - \frac{1}{4|x|^2}$ does not have a ground state with eigenvalue 0 by Theorem 1.

III. PROOFS OF MAIN THEOREMS

Our proofs rely on Comparison Lemma [11, 12]. We use the version presented in [13, Theorem 2.7].

Theorem 3. Let $w$ be a positive and continuous supersolution of the equation $(P - \lambda)u = 0$ in a neighborhood of infinity $\Omega_R = \mathbb{R}^d \setminus B_R(0)$. Let $v$ be a continuous subsolution of the same equation in $\Omega_R$. Suppose that

\[
\lim_{N \to \infty} \left( \frac{1}{N^2} \int_{N \leq |x| \leq aN} |v|^2 \, dx \right) = 0
\]

(8)

for some $\alpha > 1$. Then there exists a positive constant $C$ such that

\[
v(x) \leq Cw(x)
\]

in $\Omega_{R+1}$. 
Remark 4. We note that the condition (8) is trivially satisfied for every \( v \in L^2(\mathbb{R}^d) \).

The general idea of the proofs is to find a suitable functions to use in Comparison Lemma as either lower or upper bound. A suitable function is one which is at the edge of \( L^2 \)-integrability outside of a bounded region. We start by defining functions \( \phi'_n, \phi''_n \in C(\mathbb{R}^d \setminus B_R(0)) \) for \( R > 0 \)

\[
\phi'_n(x) := \frac{1}{|x|^\frac{d}{2} \ln(|\ln(|x|)\ldots \ln(\ldots \ln(|x|)\ldots))|^{\frac{n}{2}}}
\]

and for \( \varepsilon > 0 \)

\[
\phi''_n(x) := \phi'_n(x) \ln^{\frac{1}{2}}(\ln\ldots \ln(|x|)\ldots)|^{\frac{2}{n}}.
\]

It is not hard to see that for each \( n \in \mathbb{N}_0 \) the function \( \phi'_n \) is barely not \( L^2 \)-integrable outside of a bounded region whereas the function \( \phi''_n \) is barely \( L^2 \)-integrable outside of a bounded region. Using the notation \( (a) \) we have

\[
\phi'_n(x) = \frac{1}{|x|^\frac{d}{2}} \prod_{j=1}^{n} \ln^{-\frac{1}{2}}(|x|),
\]

\[
\phi''_n(x) = \frac{1}{|x|^\frac{d}{2}} \left( \prod_{j=1}^{n} \ln^{-\frac{1}{2}}(|x|) \right) \ln^{-\frac{2}{n}}(x).
\]

By a direct calculation we can check that the functions \( \phi'_n \) and \( \phi''_n \) satisfy (9).

Lemma 2. Let \( H \) be as in (1). Let \( \psi \) be a normalized eigenfunction associated to the eigenvalue \( E \). Then the functions \( |\text{Re}\psi| \) and \( |\text{Im}\psi| \) are subsolutions of \((H - E)v\).

Proof. We start by showing this for a real eigenfunction. Using [13, Lemma 2.9] on a real eigenfunction \( \phi \), we obtain that a function \( \phi_+ := \max\{\phi, 0\} \) is a subsolution of \((H - E)v = 0\). Using the same Lemma for \(-\phi\) we obtain that \( \phi_- := \max\{-\phi, 0\} \) is a subsolution of \((H - E)v = 0\). Due to linearity we have that \( \phi_+ + \phi_- = |\phi| \) is a subsolution which completes the proof for a real eigenfunction.

Now we extend this to a general eigenfunction \( \psi \). Due to the form of \((H - E)\) it is straightforward to see that \( \overline{\psi} \) is also an eigenfunction associated to \( E \). Using linearity of the eigenproblem we get that \( 2\text{Re}(\psi) = \psi + \overline{\psi} \) and \( 2\text{Im}(\psi) = -i(\psi - \overline{\psi}) \) are also eigenfunctions corresponding to the eigenvalue \( E \). We can therefore apply the first part of the proof to \( \text{Re}(\psi) \) and \( \text{Im}(\psi) \) which completes the proof.

Lemma 3. Let \( H \) be as in (1). Then any eigenfunction \( \psi \) associated to \( E \) is uniformly bounded

\[
\|\psi\|_\infty \leq e^{tE}C(t)\|\psi\|_2.
\]

for every \( t \in \mathbb{R} \) where \( C(t) = \|\exp(-tH)\|_{2 \to \infty}. \)

Proof. Using the eigenfunction equation \( H\psi = E\psi \) we can write

\[
e^{-tH}\psi = e^{-tE}\psi.
\]

Using [1, Lemma B.1.1] we get that the operator \( e^{-tH} \) is a bounded operator between \( L^2 \) and \( L^\infty \). This implies

\[
\|e^{-tH}\psi\|_\infty = e^{-tE}\|\psi\|_\infty
\]

which can be rewritten as

\[
\|\psi\|_\infty = e^{tE}\|e^{-tH}\psi\|_\infty \leq e^{tE}\|e^{-tH}\|_{2 \to \infty}\|\psi\|_2
\]

which completes the proof.
A. Proof of Theorem 1

The main idea is to construct a potential $V$ in such a way that the function $\phi_n^l$ is a (sub)solution of $P$ and then use Theorem 2. To do this we express $V$ with the help of the equation

$$-\Delta \phi_n^l + V \phi_n^l \leq 0$$

as

$$V \leq \frac{\Delta \phi_n^l}{\phi_n^l}.$$ 

Due to the fact that $\phi_n^l$ depends only on the radial variable we express $\Delta \phi_n^l$ as

$$\Delta \phi_n^l = \frac{\partial^2 \phi_n^l}{\partial r^2} + \frac{d - 1}{r} \frac{\partial \phi_n^l}{\partial r}.$$ 

By a direct calculation we get

$$\frac{\partial \phi_n^l}{\partial r} = -\phi_n^l \left( \frac{d}{2|x|} + \frac{1}{2|x|} \sum_{j=1}^{n} j \ln^{-1}(x) \right),$$

$$\frac{\partial^2 \phi_n^l}{\partial r^2} = \phi_n^l \left( \frac{d}{2|x|} + \frac{1}{2|x|} \sum_{j=1}^{n} j \ln^{-1}(x) \right)^2 + \phi_n^l \left( \frac{d}{2|x|^2} + \frac{1}{2|x|^2} \sum_{j=1}^{n} \sum_{k=1}^{j} \ln^{-1}(x) \ln^{-1}(x) \right)$$

where we have used

$$\frac{\partial \ln_1(r)}{\partial r} = \frac{1}{r}, \quad \frac{1}{\ln n_1(r) \ln n_2(r) \cdots \ln n_l(r) r}.$$ 

This implies

$$\frac{\Delta \phi_n^l}{\phi_n^l} = \frac{d(4-d)}{4|x|^2} + \frac{1}{|x|^2} \sum_{j=1}^{n} \prod_{k=1}^{j} \ln^{-1}(x) + \frac{1}{4|x|^2} \left( \sum_{j=1}^{n} \prod_{k=1}^{j} \ln^{-1}(x) \right)^2 + \frac{1}{2|x|^2} \sum_{j=1}^{n} \sum_{k=1}^{j} \prod_{m=1}^{j} \ln^{-1}(x) \ln^{-1}(x) \ln^{-1}(x).$$

In other words for each potential satisfying the equation (6) the function $\phi_n^l$ is a subsolution of (4). Now we are ready to complete the proof via contradiction. Assume that $\psi$ is a ground state of $H$, i.e., $\psi \in L^2(\Omega)$ and it satisfies the equation (4). This means that $\psi$ is also a supersolution of (4). Then by Theorem 3 we have $0 < \phi_n^l \leq C\psi$ for some $C > 0$. This implies that $\psi$ is not $L^2$-integrable which is a contradiction completing the proof.

Remark 5. There is also another alternative approach how to obtain the expression $\frac{\Delta \phi_n^l}{\phi_n^l}$ based on induction. We use an identity which holds for radially symmetric functions. For any $f(|x|)$ and $g(|x|)$ we have

$$\Delta(gf) = f \Delta g + 2fg' + (\Delta f)g.$$ (12)

Using the identity (12), the equality $\phi_{n+1}^l(x) = \phi_n^l(x) \ln^{-\frac{1}{2}}_{n+1}(x)$ along with the notation $V_n := \frac{\Delta \phi_n^l}{\phi_n^l}$ we obtain

$$V_{n+1} = V_n + \frac{\Delta \ln^{-\frac{1}{2}}_{n+1}(x)}{\ln^{-\frac{1}{2}}_{n+1}(x)} + 2 \frac{\partial \phi_n^l}{\partial r} \frac{\partial \ln^{-\frac{1}{2}}_{n+1}(x)}{\ln^{-\frac{1}{2}}_{n+1}(x)}.$$ 

A straightforward calculation yields

$$\frac{\Delta \ln^{-\frac{1}{2}}_{n+1}(x)}{\ln^{-\frac{1}{2}}_{n+1}(x)} = \frac{d}{4|x|^2} \prod_{k=1}^{n+1} \ln^{-2}(x) + \frac{d}{2|x|^2} \sum_{k=1}^{n+1} \prod_{j=1}^{k} \ln^{-1}(x) + \frac{1}{2|x|^2} \sum_{j=1}^{n+1} \sum_{k=1}^{j} \prod_{m=1}^{j} \ln^{-1}(x) \ln^{-1}(x),$$

$$2 \frac{\partial \phi_n^l}{\partial r} \frac{\partial \ln^{-\frac{1}{2}}_{n+1}(x)}{\ln^{-\frac{1}{2}}_{n+1}(x)} = \left( \frac{d}{|x|} + \frac{1}{|x|} \sum_{j=1}^{n} \prod_{k=1}^{j} \ln^{-1}(x) \right) \left( \frac{1}{2|x|^2} \prod_{j=1}^{n+1} \ln^{-1}(x) \right),$$

$$V_{n+1} - V_n = \frac{3}{4|x|^2} \prod_{k=1}^{n+1} \ln^{-2}(x) + \frac{1}{|x|^2} \sum_{j=1}^{n+1} \ln^{-1}(x) + \frac{1}{|x|^2} \sum_{j=1}^{n+1} \sum_{k=1}^{j} \prod_{m=1}^{j} \ln^{-1}(x) \ln^{-1}(x).$$
Calculating

\[ V_0 = |x|^{\frac{d}{2}} \Delta |x|^{-\frac{d}{2}} = \frac{d \cdot d+2}{2} |x|^{-2} - (d-1) \frac{d}{2} |x|^{-2} = \frac{4d-d^2}{4|x|^2} \]

completes this alternative approach.

**B. Proof of Theorem**

The general idea remains the same also for the existence result. We just switch the roles of the eigenfunction and the function \( \phi_n^\mu \). We construct \( V \) in such a way that the function \( \phi_n^\mu \) is a (super)solution, i.e.,

\[ -\Delta \phi_n^\mu + V \phi_n^\mu \geq 0 \Rightarrow V \geq \frac{\Delta \phi_n^\mu}{\phi_n^\mu}. \]

To calculate the expression \( \frac{\Delta \phi_n^\mu}{\phi_n^\mu} \) we can use the previous calculations. We write

\[ \phi_n^\mu = \phi_n^\mu \ln \frac{x^2}{\nu} (x) \]

and use (12). Using the notation \( W_n := \frac{\Delta \phi_n^\mu}{\phi_n^\mu} \) we get

\[
W_{n+1} = \frac{\Delta \ln \frac{x^2}{\nu}(x)}{\ln \frac{x^2}{\nu}(x)} + 2 \frac{\partial_r \ln \frac{x^2}{\nu}(x)}{\ln \frac{x^2}{\nu}(x)} \varphi_{n+1} \nonumber \]

\[
= \frac{\epsilon^2 \prod_{k=1}^{n+1} \ln_k^{-2}(x) + \epsilon \prod_{k=1}^{n+1} \ln_k^{-1}(x) + \epsilon \prod_{j=1}^{n+1} \prod_{m=1}^{n+1} \ln_k^{-1}(x) \ln_m^{-1}(x)}{\prod_{k=1}^{n+1} \ln_k^{-2}(x) + \prod_{k=1}^{n+1} \ln_k^{-1}(x) + \prod_{j=1}^{n+1} \prod_{m=1}^{n+1} \ln_k^{-1}(x) \ln_m^{-1}(x)}
\]

Under the assumption that an eigenstate exists at the threshold we obtain

\[ |\psi| \leq C \phi_n^\mu \]

for a given \( C > 0 \) where we used Lemma(2) and \( |\psi| \leq |\text{Re} \psi| + |\text{Im} \psi| \). To show the existence of an eigenfunction at the threshold we need to construct a weakly convergent sequence of eigenfunctions. We construct such a sequence in the following way. We take a sequence of operators

\[ H_n = -\Delta + V - \frac{1}{n} \chi_{\Omega^c} (0) \]

where \( n \in \mathbb{N} \). Due to the criticality of \( V \) each \( H_n \) has an eigenfunction \( \psi_n \) with an eigenvalue \( E_n < 0 \) s.t. \( \lim_{n \to \infty} E_n = 0 \). The sequence \( \{ \psi_n \}_{n \in \mathbb{N}} \) converges weakly. We need to show that it converges strongly. We show this using Tightness (14). A weakly converging sequence is strongly converging provided that

\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} |\psi_n(x)|^2 \, dx = 0,
\]

\[
\lim_{L \to \infty} \limsup_{n \to \infty} \int_{|k| > L} |\psi_n(k)|^2 \, dk = 0
\]

where \( \hat{\psi} \) is the Fourier transform of \( \psi \). The second equation is implied by the finiteness of energy and its relation to Sobolev norm. Details of this argument can be found in (15). The first equation can be shown using bounds similar to (13). We have

\[
(\Delta - V) |\psi_n| \leq (\Delta - V - E_n)|\psi_n| = 0
\]

\[
(\Delta - V - E_n) |\phi_n^\mu| \leq (\Delta - V) |\phi_n^\mu| \geq 0
\]

in \( \Omega_R \) where we used Lemma(2) and the fact that \( -E \) is in \( \Omega_R \). This is due to the fact that \( \chi_{\Omega_R} (0) \) is supported outside of \( \Omega_R \). This implies

\[ |\psi_n| \leq C_n |\phi_n^\mu| \]
which is analogous to our bound \[13\]. Now we need to check that the constants \(C_n\) are uniformly bounded as \(n\) goes to infinity. Using Lemma 3 we get that each \(\psi_n\) is bounded outside of a compact set by

\[
|\psi_n(x)| \leq e^{tE} C(t) \|\psi\|_2.
\]

where the constant \(C(t)\) is

\[
C(t) = \|\exp(-tH)\|_{2\to\infty}.
\]

This implies that the constants \(C_n\) can be chosen to be bounded uniformly in \(n\). This proves that \(\psi_n\) converges strongly which completes the proof.

\[\square\]

**Remark 6.** We note that several parts in the proof are more difficult than they need to be. We intentionally avoided the use of strict positivity of the eigenfunction. This allows the existence result to be usable also for fermionic systems and higher eigenstates (provided the necessary modifications).

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