A two-dimensional interacting system obeying Fractional Exclusion Statistics

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We consider \( N \) fermions in a two-dimensional harmonic oscillator potential interacting with a very short-range repulsive pair-wise potential. The ground-state energy of this system is obtained by performing a Thomas-Fermi as well as a self-consistent Hartree-Fock calculation. The two results are shown to agree even for a small number of particles. We next use the finite temperature Thomas-Fermi method to demonstrate that in the local density approximation, these interacting fermions are equivalent to a system of noninteracting particles obeying the Haldane-Wu fractional exclusion statistics. It is also shown that mapping onto a system of \( N \) noninteracting quasi-particles enables us to predict the energies of the excited states of the \( N \)-body system.

I. INTRODUCTION

Haldane \cite{1} first proposed the generalised exclusion statistics in the context of excitations in antiferromagnetic spin-chains. In this generalised scheme, one considers a finite dimensional single-particle Hilbert space with \( N \) identical particles. Increasing the particle number by one results in the blocking off of \( \alpha \) single-particle states for the others, the special cases \( \alpha = 1 \) and \( \alpha = 0 \) corresponding to fermions and bosons respectively. It is to be noted that the statistical parameter \( \alpha \) is \( N \)-independent, and is determined solely by the interaction strength between the particles. An exact realization of this statistics is found in the Calogero-Sutherland-Moser (CSM) one-dimensional model \cite{2}, in which the pair interaction potential falls off as the inverse square of the relative straight-line distance between the particles, which are either on an infinite line, or on a circle. Both the ground-state as well as the excited state energies of this interacting CSM-system may be exactly reproduced by a set of \( N \) quasi-particles obeying Haldane statistics \cite{3}. The finite temperature distribution function for the generalised exclusion statistics is known \cite{4}. For the CSM system, it has been shown \cite{3} that the noninteracting quasiparticles have such occupancies. In principle, generalised exclusion statistics may be obeyed in higher dimensions also.

The aim of this paper is to present a many-body Hamiltonian in two dimensions that obeys fractional exclusion statistics. This model was first proposed in \cite{5}, and consisted of \( N \) fermions in a two-dimensional confinement potential, and interacting with a very short-range repulsive two-body force. For the one-dimensional CSM case,
the ground and the excited state energies of the many-body system are analytically known, so one may test the mapping to the quasi-particle spectrum. The difficulty in the two-dimensional case is that the energy spectrum of the proposed many-body system is not known - in fact not even the ground state energy is known, so the model cannot be tested unless elaborate numerical calculations are performed. In ref [5], only the Thomas-Fermi (TF) expression for the spatial density had suggested that the system may obey fractional exclusion statistics. No attempt was made in that paper to construct a quasi-particle scheme to map the energy spectrum. In this paper, we perform a self-consistent Hartree-Fock (HF) calculation to estimate the ground-state energy of the two-dimensional system. It is found that the simple TF calculation for the energy agrees surprisingly well with HF result. We then use the finite temperature TF method to show that in the local-density approximation, the interacting fermions are equivalent to a system of noninteracting particles obeying the Haldane-Wu fractional exclusion statistics. Finally, we present a quasi-particle spectrum that enables us to calculate the ground as well as the excited state energies of the system. Although not expected to be exact like the CSM in the one-dimensional case, the quasi-particle method yields analogous finite-N corrections that lowers the energy. Our model may be useful for the description of interacting electrons in a quantum dot, since it shows that the short-range component of the effective interaction in a two-dimensional system may induce fractional exclusion statistics. The mapping to a system of non-interacting quasi-particles further enables us to easily calculate the excited states of a system of electrons in a quantum dot, albeit approximately. This, in the basis of electrons, would be a daunting task indeed.

II. THE MANY-BODY PROBLEM IN TWO-DIMENSIONS

A. The Hamiltonian

We consider a system of $N$ fermions in two dimensions, confined in a one-body potential $V_1$, and interacting pair-wise through a short-range repulsive two-body interaction $V_2$. The Hamiltonian is given by

$$
H = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} V_1(r_i) + \sum_{j<k} V_2(|r_j - r_k|).
$$

(1)

As in [6], the spins are taken to be unpolarized. The mean-field expression for the energy at zero temperature is given by

$$
E[\rho(r)] = \int d^2r \left[ \tau(r) + V_1(r)\rho(r) + \frac{1}{2} \rho(r) \int d^2r' \rho(r')V_2(|r - r'|) - \frac{1}{2} \int d^2r' \rho(r, r')^2V_2(|r - r'|) \right],
$$

(2)
where \( \tau(r) \), \( \rho(r) \) are the spatial single-particle kinetic energy and number density, and \( \rho(r,r') \) is the density-matrix. As shown in [3], the density-matrix expansion following Skyrme [6], together with angle-averaging, yields the form

\[
E[\rho(r)] = \int d^2r \left[ \tau(r) + V_1(r)\rho(r) + \frac{1}{4} \rho^2(r)M_0 + \frac{1}{8} \{\pi \rho^3(r) - [\nabla \rho(r)]^2\}M_2 + \ldots \right],
\]

(3)

where \( M_n = \int d^2r \, r^n V_2(r) \) is the \( n \)th moment of the two-body potential. If \( V_2(r) \) has short range, the higher moments fall off rapidly. In this paper we only retain the lowest moment \( M_0 \), and drop the terms involving \( M_2, M_3 \) etc. This is tantamount to taking an effective interaction of zero range. In this approximation, we show that the interacting system may be mapped onto \( N \) noninteracting quasi-particles obeying fractional exclusion statistics. We may therefore consider an effective Hamiltonian

\[
\hat{H} = \frac{1}{2m} \sum_{i=1}^N p_i^2 + \sum_{i=1}^N V_1(r_i) + \frac{2\pi \hbar^2}{m}(\alpha - 1) \sum_{j<k} \delta(|r_j - r_k|).
\]

(4)

where \( \alpha = (1 + \frac{mM_0}{\pi \hbar^2}) \) is dimensionless, and will play the role of the statistical parameter. Note from the above equation that \( \alpha = 1 \) describes a system of noninteracting fermions.

**B. Thomas-Fermi (TF) approximation**

Consider, within the TF formalism, the above system at zero and finite temperatures. We show that this system of interacting fermions is equivalent to a set of noninteracting particles in \( V_1(r) \) obeying fractional exclusion statistics locally. The usage of the term locally will be explained shortly. Let us first recapitulate the \( T = 0 \) case. In the TF approximation, \( \tau(r) \) in Eq.(3) is replaced by \( \frac{\hbar^2}{2m} \pi \rho^2(r) \). For the zero-range interaction for \( V_2 \) given by (4), only the lowest moment \( M_0 \) in Eq.(3) is nonzero. For this case, (with a spin-degeneracy factor of 2 included), the TF energy density functional is given by

\[
E[\rho(r)] = \int d^2r \left[ \frac{\hbar^2}{2m} \pi \rho^2(r) + V_1(r)\rho(r) + \frac{\pi \hbar^2}{2m}(\alpha - 1)\rho(r) \right].
\]

(5)

The ground state energy \( E_0^{TF} \) and the spatial density \( \rho_0^{TF}(r) \) are now determined by the variation \( \delta(E - \mu(0)N) = 0 \), where \( \mu(0) \) is the chemical potential at \( T = 0 \), such that \( \int d^2r \, \rho_0^{TF}(r) = N \). On performing the variation, we get

\[
\rho_0^{TF}(r) = \frac{m}{\pi \hbar^2 \alpha} [\mu(0) - V_1(r)], \quad r \leq r_0,
\]

(6)

\[
= 0, \quad r > r_0,
\]

(7)
where \( r_0 \) is the classical turning point, defined by \( \mu(0) = V_1(r_0) \). The form of Eq.(7) for the spatial density prompted us to suggest in [5] that it may also be generated by a set of noninteracting particles, but with a zero-temperature occupancy factor \( 1/\alpha \) instead of 1. Indeed, the same feature was earlier found [7] for CSM, where the ground-state energy was also reproduced correctly to leading order in \( N \). The chemical potential itself is determined by the requirement that the spatial density, integrated over space, should give the particle number \( N \). For the special choice \( V_1(r) = m\omega^2 r^2/2 \) which we will explore in some detail, this gives the relation

\[
\mu(0) = \hbar \omega \sqrt{\alpha N}. \tag{8}
\]

For this case, so far as the chemical potential is concerned, the energy scale has been stretched by a factor \( \sqrt{\alpha} \), an important clue for the construction of the quasi-particle spectrum later. On performing the \( r-\)integration in (5) with the \( \rho_0(r) \) given by (7), and making use of the above expression for \( \mu \), we get the ground state energy

\[
E_{0}^{TF} = \frac{2}{3} \alpha^{1/2} N^{3/2} \hbar \omega. \tag{9}
\]

Our considerations here are all based on the TF model. It is important to test the accuracy of the model, specially for small \( N \), by using a more elaborate method. To this end, we present in the next section the self-consistent Hartree-Fock results.

Before doing this, however, we demonstrate that this system approximately obeys, within the TF framework, fractional exclusion statistics even at a finite temperature \( T \). A system of noninteracting particles at a finite temperature obeying fractional exclusion statistics has the following distribution function at equilibrium temperature \( T \)

\[
n_\alpha(\epsilon, T) = \frac{1}{w(\epsilon, T) + \alpha}, \tag{10}\]

where \( w \) is the solution of the equation \( w^\alpha[1 + w]^{1-\alpha} = e^{(\epsilon - \mu)/T} \). For the special case of a two-dimensional translationally invariant system (with constant spatial density \( \rho \) and constant single-particle density of states), Wu [4] had shown that

\[
\mu(T) = \frac{\alpha \pi \hbar^2 \rho}{m} + T \ln[1 - \exp(-\pi \hbar^2 \rho/mT)] \tag{11}\]

In [5], we applied finite temperature TF for a system of fermions in a potential \( V_1(r) \), and interacting with a zero-range force. In this case it is easily shown that

\[
[\mu(T) - V_1(r)] = \frac{\alpha \pi \hbar^2 \rho(r, T)}{m} + T \ln[1 - \exp(-\pi \hbar^2 \rho(r, T)/mT)] \tag{12}\]

Here \( \rho(r, T) \) is given by [5].
\[ \rho(r, T) = \frac{mT}{\pi \hbar^2} \ln \left[ 1 + \exp \left\{ (\mu(T) - V(r, T))/T \right\} \right]. \quad (13) \]

This is the local version of Eq. (11) above, where \( \mu(T) \) and a constant \( \rho \) are replaced by their local values \( \mu(T) - V_1(r) \) and \( \rho(r, T) \) respectively. So, in the presence of a fixed external potential and a small number \( N \) of electrons, as in a quantum dot, for example, fractional exclusion statistics may be obeyed in the local-density approximation. The latter term means that at any point \( r \), the local fermi-energy and spatial density obey the same equations as their counterparts in the translationally invariant system. This generally is true in the interior of the system, but deviations may occur at the edges if the spatial density changes rapidly in a scale compared to the size of the system. The local-density approximation has been successfully used in sophisticated nuclear many-body calculations even with a relatively small number of nucleons [8], where a large fraction of them may reside in the surface region.

In case the single-particle density of states is not a constant, the Wu-equation (11) is generalised to the following form [7]:

\[ \mu(T) = \alpha \nu + T \ln[1 - \exp(-\nu/T)] , \quad (14) \]

where

\[ \nu(T) = \int_0^\infty n_\alpha(\epsilon, T) d\epsilon . \quad (15) \]

We emphasize that Eq. (14) is quite general, and does not make any restrictive assumption about the density of states. We now show that our interacting fermionic system also obeys a local version of Eq. (14) in the TF framework. To demonstrate this, we define

\[ \nu(r, T) = \int_0^\infty d\epsilon n_F(r, p, T) , \quad (16) \]

where

\[ n_F(r, p, T) = \frac{1}{\exp[(p^2/2m + V(r) - \mu)/T] + 1} . \quad (17) \]

For the system given by the effective Hamiltonian [9],

\[ V(r, T) = V_1(r) + \frac{\pi \hbar^2}{m} (\alpha - 1) \rho(r, T) . \quad (18) \]

Replacing \( d\epsilon \) in Eq. (16) by \( dp/m \) at a fixed point \( r \), and performing the p-integration, we then obtain

\[ \nu(r, T) = \frac{\pi \hbar^2}{m} \rho(r, T) . \quad (19) \]

Using Eqs (13) and (12), both based on the interacting fermion system, we can then write

\[ [\mu(T) - V_1(r)] = \alpha \nu(r, T) + T \ln[1 - \exp(-\nu(r, T)/T)] . \quad (20) \]
This is the local version of Eq. (14) that we set out to show, and emphasizes that the local density approximation is applicable even when the single-particle density of states is not a constant.

We now go on to check the accuracy of the TF approximation for the ground state by performing a self-consistent HF calculation in the next section.

C. The Hartree-Fock (HF) calculation

The advantage of performing a HF calculation with $\tilde{H}$ given by (4) is that the Fock term is local. A little algebra shows that the HF single-particle orbitals $\psi_i(r)$ with energies $e_i$ obey the one-body equation

$$\left[ \hat{h}_0 + \frac{\pi \hbar^2}{m} (\alpha - 1) \rho(r) \right] \psi_i(r) = e_i \psi_i(r), \quad (21)$$

where $\hat{h}_0 = \hat{p}^2/2m + m\omega^2 r^2/2$, and $\rho = \sum_i \psi^* \psi$, the prime indicating that the sum is over the occupied orbitals only. Note that the one-body potential in (21) is the confining harmonic oscillator in $\hat{h}_0$, plus the contribution coming from the direct and the exchange matrix-elements of the two-body interaction. Let us denote the latter by

$$U(r) = \frac{\pi \hbar^2}{m} (\alpha - 1) \rho(r). \quad (22)$$

The potential $U$, proportional to $\rho$, is determined by the occupied single-particle orbitals. Therefore the equations have to be solved iteratively, starting with a trial $\rho$, until self-consistency is achieved to a specified accuracy. To this end, the orbitals $\psi_i(r)$ are expressed in the basis set generated by the unpertubed harmonic oscillator. The basis set, of course, has to be truncated in practice. For numerical accuracy, we find that if $N$ shells are occupied in the unperturbed configuration, then at least $2N$ shells should be included in the basis set. For $N = 12$ particles, where $N = 3$, we have tested that the self-consistent results remain essentially unaltered when the dimension of the basis set is increased from shells 6 to 10. The HF single-particle Hamiltonian, given by the left-hand side of Eq. (21) (with a trial $\rho$), is represented by a matrix in this basis. It is diagonalsied to obtain the orbitals and their eigenvalues. The density $\rho$ and the corresponding HF Hamiltonian are recalculated, and the procedure repeated for self-consistency. Typically, for moderate strength of the interaction ($\alpha = 3/2$), about 40 iterations or more are needed to obtain self-consistent occupied orbitals $\psi_i$. The HF ground-state energy is given by

$$E_0^{HF} = \sum_i \left( e_i - \frac{1}{2} <\psi_i|U|\psi_i> \right), \quad (23)$$
where \( U \) is defined by Eq.\((22)\), and the sum over \( i \) is over occupied states only. The self-consistent results for 3 different systems \( (N = 12, 20 \) and 42 fermions), each for 2 values of \( \alpha \), are presented in Table 1. These \( N \)-values are closed shells configurations for \( h_0 \). For non-closed shell systems, we found that numerical convergence for self-consistency is harder to achieve. Interestingly, as is shown in Table 1, \( E_{0}^{HF} \) is very close to \( E_{0}^{TF} \) calculated from Eq.\((9)\). In Fig.1 we display, for \( N = 20 \) and \( \alpha = 3/2 \), the shape of the self-consistent HF one-body potential, \( (m\omega^2r^2/2 + U(r)) \) as a function of the dimensionless variable \( R = (m\omega/\bar{\hbar})^{1/2}r \), and the corresponding low-lying single-particle energies \( e_i \) in units of \( \hbar\omega \). For comparison, the unperturbed harmonic oscillator and its energy eigenvalues are also shown in the same diagram.

Two features are to be noted: (a) the relatively large upward shift at the center in the HF one-body potential, resulting in an overall shift in the single-particle spectrum, and (b) the almost equal but reduced spacing between the levels in the HF spectrum, with the degeneracy between the \((2s,1d)\) states being intact to 4 significant figures, and the splitting in \((2p,1f)\) and \((3s,2d,1g)\) sub-shells remaining negligible. These characteristics suggest that, like the CSM, the ground and the excited states of the \( N \)-particle system are shifted by the same overall constant whose magnitude is determined by the strength of the interaction. (For systems in which the last filled shell is only partially occupied, preliminary results indicate that the splittings between the states within a shell are larger.) We also note that the states \((n,l)\) and \((n,-l)\) remain strictly degenerate. We also found that in every case, the spacing between the HF shells are almost equal, with an effective oscillator constant given by \( \omega/\sqrt{\alpha} \). This behaviour of the HF \( e_i \)'s is very different from the quasi-particle spectrum \( \epsilon_i \)'s presented in the next section, the latter actually spreading out by a factor \( \sqrt{\alpha} \). We may express the HF single-particle energies in the form

\[
e_n = \Delta(\alpha,N_0) + (n+1)\omega/\sqrt{\alpha} , n = 0, 1, .. \quad (24)
\]

where \( N_0 \) is number of filled shells in the system. The gap \( \Delta \) is proportional to \( (N_0 + 1) \), and the constant of proportionality increases with \( \alpha \).

As mentioned in the previous section, the TF result \( \| \) had suggested that the ground state energy of the interacting system may be estimated by filling up each level of the corresponding noninteracting system by a fraction \( 1/\alpha \), but preserving the total particle number. We call this the “global” fractional-statistics rule. It is useful only for the ground state, in contrast to the quasi-particle formalism given in the next section. In Table 1, the numerical values for \( E^{G}(\text{global}) \) are also given for comparison.

**III. THE QUASI-PARTICLE SPECTRUM**

We wish to express the energy of an interacting \( N \)-particle system as a sum of noninteracting quasi-particle
energies, both for the ground and the excited states. This can be done exactly for CSM, which behaves as an ideal gas obeying fractional exclusion statistics. For the two-dimensional model presented here, we cannot prove that such is the case. Nevertheless, we show that when the degeneracies of the levels are taken into account, a consistent scheme for the quasi-particle spectrum may be formulated, obeying the same rules as given for CSM [3, 9]. This procedure reproduces the large-N TF results for the ground state, and predicts the finite-N corrections. Following the steps of [3], we then show that such a mapping leads naturally to the thermodynamic distribution function formulated in [4].

In what follows, we take the two-dimensional harmonic oscillator as the confining potential, and a spin-degeneracy of two. At the outset, we clarify certain definitions that are used in this presentation. The unperturbed harmonic oscillator spectrum is specified by shells, containing degenerate states in the shell. For example, the \( j \)th shell is at energy \( j \omega \) and has \( \eta_j = 2j \) states in it, where we have included the spin degeneracy of two in \( \eta_j \).

The quasi-particle spectrum has exactly the same structure of these shells and states, except that the energy of a quasi-particle is determined by the occupancies of the states below it, as specified by the rules below. In this mapping, the unperturbed energy \( \epsilon_k = k \omega \) of a particle is mapped onto the quasi-particle energy \( \epsilon_A(k, \alpha) \). The analogy with CSM leads us to suggest the following

\[
\epsilon_A(k, \alpha) = \epsilon_k - \omega(1 - \sqrt{\alpha})N(\epsilon_k, 0),
\]

where \( N(\epsilon_k, 0) \) is the sum of the occupancy fractions of the shells with energies \( \epsilon < \epsilon_k \). This is given by

\[
N(\epsilon_k) = \sum_{j=1}^{\infty} \Theta(\epsilon_k - \epsilon_j) \frac{1}{\eta_j} \sum_{i=1}^{\eta_j} n_{j,i},
\]

where \( \Theta(x) = 0 \) for \( x \leq 0 \) and \( 1 \) for \( x > 0 \), and \( n_{j,i} \) is the occupancy (0 or 1) of the \( i \)th state of the \( j \)th shell. An example is illustrated in Fig. 2, where (a) is an unperturbed harmonic oscillator excited state configuration in which one particle from the \((2s, 1d)\) shell has been excited to the next \((2p, 1f)\) shell. In (b), the corresponding mapping to the quasi-particle spectrum is shown. We use the notation \( \tilde{\omega} = \sqrt{\alpha} \omega \). Using Eq. (26), we note that \( N(\epsilon_1) = 0, N(\epsilon_2) = 1, N(\epsilon_3) = 2, N(\epsilon_4) = 17/6, N(\epsilon_5) = 71/24 \). Now we may use Eq. (25) to deduce the quasi-particle energies as shown in Fig. 2(b).

For the ground state of a closed-shell configuration, all the shells are fully filled up to the \( N_0 \)th shell, and are unoccupied thereafter. The above rules then lead to

\[
N(\epsilon_j) = j - 1, \quad 1 \leq j \leq N_0;
\]

\[
= N_0, \quad j > N_0.
\]

The energy levels \( \epsilon_A(k, \alpha) \) in the ground-state configuration are given by \( \omega, \omega + \tilde{\omega}, \omega + 2\tilde{\omega}, ..., \omega + (N_0 - 1)\tilde{\omega}, 2\omega + (N_0 - 1)\tilde{\omega}, 3\omega + (N_0 - 1)\tilde{\omega}, ... \). The ground state energy is then given by
\[ E_0^{(qp)} = \sum_{k=1}^{N_0} \epsilon_A(k, \alpha) \eta_k , \]
\[ = 2 \sum_{k=1}^{N_0} k \epsilon_k - 2 \omega (1 - \sqrt{\alpha}) \sum_{k=1}^{N_0} (k - 1) , \]
\[ = \omega \sqrt{\alpha} N_0 (N_0 + 1) (2N_0 + 1)/3 - \omega (\sqrt{\alpha} - 1) N_0 (N_0 + 1) . \quad (28) \]

where \( N_0 \) is the last filled level. Expressing \( N_0 \) in terms of \( N \),
\[ N = 2 \sum_{k=1}^{N_0} k , \quad (29) \]
the ground state energy is given by
\[ E_0^{(qp)} = \frac{2}{3} \omega \sqrt{\alpha} N^{3/2} \left[ 1 + \frac{1}{8N} + \ldots \right] - (\sqrt{\alpha} - 1) \omega N . \quad (30) \]

The term proportional to \( N^{3/2} \) is the same as the TF expression \( [4] \), while the others are the finite-\( N \) corrections arising from the microscopic approach taken here.

The values obtained for \( E_0^{(qp)} \) by Eq. (30) are displayed in Table 1. Note that these are always lower than the corresponding HF values, as is to be expected. Note also that for \( \alpha = 0 \), Eq. (30) gives the bosonic limit with all particles in the lowest energy state at \( \omega \). Although the expression (30) is only valid for a closed-shell system, the ground state energy of an “open shell” configuration may also be calculated easily using this formalism.

The excited state energies for any configuration \( \{ n_{k,i} \} \) may now be written using Eqs. (25, 26):
\[ E[\{n_{k,i}\}] = \sum_{k=1}^{\infty} \epsilon_A(k, \alpha) \sum_{i=1}^{n_k} n_{k,i} , \quad (31) \]
\[ = \sum_{k=1}^{\infty} \epsilon_k \sum_{i=1}^{n_k} n_{k,i} - \omega (1 - \sqrt{\alpha}) \sum_{k'(\leq k)=1}^{\infty} \sum_{1}^{\infty} \sum_{i=1}^{n_k} \sum_{i'=1}^{n_{k'}} n_{k,i} n_{k',i'} . \quad (32) \]

Note that the energy levels \( \epsilon_A(k, \alpha) \) depend on the configuration \( \{ n_{k,i} \} \). Like the CSM, the excitation energies of the system are given only by the first term in Eq. (32).

The second term depends only on the total number of particles. The correspondence with CSM and the agreement of the leading \( N \)-dependent term of \( E_0^{(qp)} \) with the TF and the HF results suggests that (32) may give a good description for the excited states. As an aside we remark that if the effective interaction between \( N \) electrons in a quantum dot is dominated by a short range interaction, then the above expression for the total energy as a sum of the quasiparticle energies immediately yields not only the approximate ground state but also its excited states. A direct calculation of these excited states is in general extremely difficult.

We now consider the distribution function for the quasi-particles. First consider the ground state \( (T = 0) \).
So far as the energy spacing between the occupied quasi-particle levels are concerned, it is given by \( \tilde{\omega} = \sqrt{\alpha \omega} \). The number of particles \( \Delta N_A \) in an energy interval between \( \epsilon_A \) and \( \epsilon_A + \Delta \epsilon_A \) is given by

\[
\Delta N_A = \frac{1}{\tilde{\omega}^2} \epsilon_A \Delta \epsilon_A ,
\]

\[
= \frac{1}{\alpha \omega^2} \epsilon_A \Delta \epsilon_A .
\]

Thus the occupancy factor for the filled quasi-particle states in the energy scale of \( \omega \) is \( 1/\alpha \) instead of 1. This is precisely the \( T = 0 \) distribution function for the occupied states in fractional exclusion statistics, irrespective of dimensionality.

In conclusion, we have shown that for fermions in a two-dimensional harmonic potential interacting with a very short-range repulsive two-body force, the TF calculation for the ground state energy agrees well with the corresponding HF result. The HF self-consistent single-particle states get almost a constant upward shift, while still retaining equal spacings in a slightly shallower potential. Gaining confidence with this success of the TF method, we apply it for the finite temperature problem in a confinement potential \( V_1(r) \). We find that in this approximation, the interacting fermions obey the same equations locally as to be expected from a system of non-interacting particles obeying fractional exclusion statistics. This is true for any confinement potential, so long as the finite-range effects of the two-body interaction is neglected. It means that for such a system, the interacting fermions in the bulk obey fractional exclusion statistics, at least so far as energies are concerned. For the special case of harmonic confinement, a quasi-particle spectrum is constructed to predict the ground- as well as the excited state energies of the system. This is analogous to the CSM case in the one-dimensional problem.

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TABLE I. Ground state energy of N-particle (closed shell) systems for various $N$'s (column 1) and interaction strengths (column 2). Column 3 displays the sum of the occupied orbital energies, and the next the HF ground state energy (see Eq. (23)). Column 5 is the TF energy given by Eq. (9). Column 6, with the heading $E^G$, is the “global” approximation as explained at the end of the HF section. The last column is calculated from the “microscopic” quasi-particle approach, as given by Eq. (30).

| $N$ | $\alpha$ | $\sum \epsilon_i$ | $E^{HF}$ | $E^{TF}$ | $E^G$ | $E^{qp}$ |
|-----|---------|------------------|----------|----------|-------|---------|
| 12  | 5/4     | 34.424           | 31.301   | 30.984   | 32.000 | 29.890  |
| 12  | 3/2     | 39.969           | 34.278   | 33.941   | 34.667 | 31.598  |
| 20  | 5/4     | 73.781           | 67.079   | 66.667   | 68.000 | 64.723  |
| 20  | 3/2     | 85.694           | 73.472   | 73.030   | 73.333 | 68.991  |
| 42  | 5/4     | 223.818          | 203.478  | 202.879  | 204.400| 198.526 |
| 42  | 3/2     | 260.002          | 222.886  | 222.243  | 224.000| 213.465 |

FIG. 1. Plot of the HF self-consistent potential ($m\omega^2r^2/2 + U(r)$) and the corresponding single-particle energies $\epsilon_i$ (in bold), for $N = 20$ and $\alpha = 3/2$. For comparison, $V_1(r) = m\omega^2r^2/2$ and its energy levels (in light) are also shown. The distance $R$ is in units of $\sqrt{\hbar/m\omega}$, and the energy scale is in units of $\hbar\omega$.

FIG. 2. The quasi-particle spectrum for a one-particle one-hole excited state configuration for $N = 12$. The crosses denote the particles, and the circle the hole. On the left, the configuration is shown for the harmonic potential (unperturbed), to the right is the quasi-particle spectrum to which the interacting system has been mapped.
$N=12$

\[
\begin{align*}
\text{UNPERTURBED} & \quad \epsilon_i^{(0)} \\
\omega & \quad \omega \\
2\omega & \quad \omega \\
3\omega & \quad \omega \\
4\omega & \quad \omega \\
5\omega & \quad \omega \\
\end{align*}
\]

\[
\begin{align*}
\text{QUASIPARTICLE} & \quad \epsilon_i \\
\omega & \quad \omega \\
\omega + \omega & \quad \omega \\
\omega + 2\omega & \quad \omega \\
\omega + \frac{7}{6}\omega + \frac{17}{6}\omega & \quad \frac{49}{24}\omega + \frac{71}{24}\omega \\
\end{align*}
\]