Effects of Scale-Free Disorder on the Anderson Metal-Insulator Transition

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We investigate the three-dimensional Anderson model of localization via a modified transfer-matrix method in the presence of scale-free diagonal disorder characterized by a disorder correlation function $g(r)$ decaying asymptotically as $r^{-\alpha}$. We study the dependence of the localization-length exponent $\nu$ on the correlation-strength exponent $\alpha$. For fixed disorder $W$, there is a critical $\alpha_c$, such that for $\alpha < \alpha_c$, $\nu = 2/\alpha$ and for $\alpha > \alpha_c$, $\nu$ remains that of the uncorrelated system in accordance with the extended Harris criterion. At the band center, $\nu$ is independent of $\alpha$ but equal to that of the uncorrelated system. The physical mechanisms leading to this different behavior are discussed.

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The successful analysis of the metal-insulator transition (MIT) in the Anderson model of localization has hitherto been limited to short-range or uncorrelated diagonal disorder. In this Letter we report the effects of long-range power-law correlated disorder — so called scale-free disorder — on the MIT. Scale-free disorder is omnipresent in nature. It is found in many diverse situations in biological and physical systems, in city growth patterns and in economics. The effects of scale-free disorder on the critical properties of physical systems have recently received much renewed attention.

For the Anderson model of localization previous investigations of scale-free disorder have concentrated on the one- (1D) and two-dimensional (2D) cases. For the 1D Anderson model, it has been shown that for energies close to the band edge the presence of scale-free diagonal disorder causes states to be strongly localized, while at the band center the states tend to have localization lengths $\xi$ comparable to the system size. In the 2D Anderson model with scale-free disorder an MIT of the Kosterlitz-Thouless transition type has been observed.

Of particular interest to us is the influence of scale-free disorder in the neighborhood of an MIT when the localization length $\xi$ becomes sufficiently large. Scale-free disorder can affect the character of the divergence as shown in Refs. for the classical percolation problem. The principle finding of Refs. is that the critical exponent $\nu$, governing the divergence of $\xi$, in a scale-free disordered classical system can change when the correlator $g(r - r')$ falls off with distance as a power law, i.e., $\propto |r - r'|^{-\alpha}$. The critical exponent of the classical percolation $\nu = 4/3$ crosses over to $\nu = 2/\alpha$ for $\alpha < 3/2$, i.e., when the decay of the correlator is slow enough.

The Harris criterion summaries the effects of short-range correlated disorder on a critical point. The criterion states that $\nu$ for the disordered system and the clean system are identical provided that $d\nu - 2 > 0$, with $d$ being the dimensionality of the system. The inequality is derived by demanding that the fluctuations of the random potential within a volume given by $\xi$ do not grow faster than their mean value as the transition is approached. For power-law correlated potentials, like the above scenario, there is an extension of this criterion originally suggested in Ref. and further studied in Refs. and . The behavior of $\nu$ in the presence of scale-free disorder is well described by the extended Harris criterion which can be stated formally as

$$
\nu = \begin{cases} 
2/\alpha & \text{if } \alpha < \alpha_c \\
\nu_0 & \text{if } \alpha > \alpha_c \end{cases}
$$

where $\nu_0$ is the critical exponent without correlations in the disorder. Eq. implies that there is a well-defined critical value $\alpha_c = 2/\nu_0$, below which correlations are relevant and above which correlations are irrelevant. Numerical studies for 2D classical percolation are indeed in good agreement with Eq. Quite recently, numerical investigations of long-range correlations in models of 2D quantum-Hall systems corroborated its validity for the quantum case. Therefore it appears to be possible that the criterion based on potential fluctuations can be applied also to quantum phase transitions such as the Anderson-type MIT, where $\nu$ is determined from the divergence of the quantum-localization length at the critical point. It is the purpose of this Letter to investigate this possibility.

We define $\nu_E$ and $\nu_W$ as critical exponents of the localization length $\xi \propto |E - E_c|^{-\nu_E}$ at fixed disorder strength $W$ and $\xi \propto |W - W_c|^{-\nu_W}$ at fixed energy $E$, respectively. We use the symbol $\nu$ to denote both exponents $\nu_E$ and $\nu_W$. For fixed $W$ and $\alpha < \alpha_c$, the critical exponent $\nu_E$ obeys the extended Harris criterion, as shown in Fig.
FIG. 1: The localization-length exponent $\nu_E(\alpha)$ as a function of the correlation-strength exponent $\alpha$ at $W = 12$. Error bars reflect one standard deviation. The horizontal line indicates the uncorrelated $\nu_0 = 1.66 \pm 0.06$, the vertical dotted line is $\alpha_c = 1.21$. The dashed line for $\alpha < \alpha_c$ gives the extended Harris criterion $\text{[4]}$. The grey areas denote error bounds of one confidence interval arising from the error in $\nu_0$. Deviations from the extended Harris criterion for small $\alpha < 0.5$ are due to finite-size effects.

At the band center $E = 0$, however, $\nu_W(\alpha)$ remains independent of $\alpha$ as can be seen in Fig. 2. This means that scale-free disorder increases the critical exponent $\nu_E$ for small $\alpha$ while leaving $\nu_W$ unchanged.

Our calculation is based upon the Anderson tight-binding Hamiltonian $\text{[28]}$ in site representation

$$\mathcal{H} = \sum_{\langle i,j \rangle} |i\rangle \langle j| + \sum_j \varepsilon_j |i\rangle \langle i|$$  \hspace{1cm} (2)

where $\langle i,j \rangle$ denotes a sum over nearest-neighbors and $|i\rangle$ is an atomic-like orbital at site $i$. The random on-site potentials $\varepsilon_i$ are chosen from a Gaussian distribution with zero mean and variance $W^2/12$. For uncorrelated Gaussian disorder, the dependence of $W_c(E)$ is known $\text{[29]}$, in particular $W_c(0) = 20.9 \pm 0.5$. We generate scale-free disorder by use of the modified Fourier-filtering method (FFM) as outlined in Refs. $\text{[30, 31]}$, so that the random on-site energies have an asymptotic correlation function $\langle \varepsilon_i \varepsilon_{i+r} \rangle \sim r^{-\alpha}$ in real space. The average is done over spatial positions and many disorder realizations. We note that large $\alpha$ corresponds to the nearly uncorrelated case which shall serve as our point of reference, while small $\alpha$ is the strongly correlated case.

In principle the usual iterative transfer-matrix method (TMM) $\text{[32, 33, 34, 35]}$ allows us to determine the localization length $\lambda$ of electronic states in a quasi-1D system with cross section $M \times M$ and length $L \gg M$, where typically a few million sites are needed for $L$ to achieve a reasonable accuracy for $\lambda$. However, the use of the non-iterative FFM procedure to generate scale-free disorder necessitates a complete storage of the on-site potentials and consequently, the iterative advantage of TMM is lost as computer memory requirements become rather large.

In order to circumvent this problem, we have modified the conventional TMM. We now perform the TMM on a system of fixed length $L_L$ of the quasi-1D bar. After the usual forward calculation with a global transfer matrix $T_L$, we add a backward calculation with transfer matrix $\bar{T}_L$. This forward-backward-multiplication procedure is repeated $K$ times. The effective total number of TMM multiplications is $L = 2KL_L$ and the global transfer-matrix $\tau_L$ is

$$\tau_L = (T_1^{b} \cdots T_n^{b} T_L^{b} T_{L=0}^{b} \cdots T_1^{b})^K$$

$$= (T_{L=0}^{b} T_L^{b})^K.$$  \hspace{1cm} (3)

As usual, we diagonalize the matrix

$$\Gamma_L \approx \lim_{K \to \infty} (\tau_L^\dagger \tau_L)^{1/4KL_L}.$$  \hspace{1cm} (4)

This modified TMM has previously been used for two interacting particles $\text{[37]}$.

After establishing Eq. 4, the calculation of $\lambda$ follows that for the conventional TMM $\text{[38]}$. The matrix $\Gamma_L$ is symplectic with $M^2$ paired eigenvalues $\exp(\pm \gamma)$ with Lyapunov exponents $\gamma$. Physically, $\gamma$ determines the increase or decrease of the envelope of the wave function at long distances. The localization length is defined as the inverse of the smallest Lyapunov exponent, $\lambda = 1/\gamma_{\text{min}}$. 

FIG. 2: The localization-length exponent $\nu_W(\alpha)$ as a function of the correlation-strength exponent $\alpha$ at $W = 3$. Error bars reflect one standard deviation. The horizontal line indicates the uncorrelated $\nu_0 = 1.49 \pm 0.03$, the vertical dotted line is $\alpha_c = 1.34$. The dashed line for $\alpha < \alpha_c$ gives the extended Harris criterion $\text{[4]}$. The grey areas denote error bounds of one confidence interval arising from the error in $\nu_0$.
For a reliable convergence check, only the accumulated changes to $\lambda$ after each complete forward-and-backward loop need to be taken into account, not all such changes in the bulk ($n = 2, 3, \ldots, L_0 - 1$) of the sample. Usually, convergence can be achieved after just a few $K$ multiplications.

The critical behavior can be determined by numerically establishing for the reduced-localization lengths $\Lambda = \lambda/M$ the one-parameter scaling hypothesis \cite{1, 3, 2, 4} as $\Lambda(x, \alpha) = F[\xi(x, \alpha)/M]$ in the vicinity of the MIT, where $\xi(x, \alpha)$ is the three-dimensional (3D) localization length in the thermodynamic limit. And accordingly where $\xi(x, \alpha)$ diverges when the tuning parameter $x$, which can be the disorder strength $W$ or the energy $E$, is tuned to its critical value $x_c$, i.e., $W_c$ or $E_c$ as

$$\xi(x) \sim |x - x_c|^{-\nu}. \quad (5)$$

The value of $\nu$ is estimated from the one-parameter scaling hypothesis \cite{1, 3} by finite-size scaling (FSS) \cite{42, 43, 44}. The FSS procedure performed here follows closely the approach in Refs. \cite{13, 42, 17, 18}. The basic idea \cite{3} is to construct a family of fit functions which include corrections to scaling \cite{3} due to an irrelevant scaling variable and due to non-linearities of the disorder dependence of the scaling variables \cite{12}.

Numerical results have been obtained for samples of lengths up to at least $L_0 = 1000$, system widths $M = 5, 7, 9, 11, 13$ and for various $\alpha$ values. For band center data as in Fig. 2, widths up to $M = 15$ have been used. For each sample, convergence was assumed when the value of $\gamma$ changed by less than $10^{-5}$ after a complete forward-and-backward loop. We emphasize that since no self-averaging is used, the more stringent standard TMM-convergence criterion \cite{3} is unnecessary. With at least 100 samples for each $\lambda(W, E, \alpha)$, we find a relative error of the sample-averaged localization lengths of $\lesssim 5\%$.

The phase diagram of localization, Fig. 3 shows the mobility-edge trajectories, which separate extended from localized states, for different values of $\alpha$. The symmetry with respect to $E = 0$ holds as in the uncorrelated case. $W_c(\alpha)$ at $E = 0$ increases monotonically as $\alpha \to 0$. In the absence of correlations, states in region I are extended whereas states in II are localized. The phase boundaries between the metallic states (region I) and the insulating states (region II) are modified in the presence of scale-free disorder. With uncorrelated disorder, states in region III are localized by quantum interference effects \cite{29, 50}. Correlation in the disorder potential will lead to a smoother disorder and a decrease of interference and thus increased localization lengths. Indeed, we find that for $\alpha < \infty$, states in region III become extended and the phase boundary is shifted to higher values of the disorder. The effect becomes most pronounced for strongly correlated disorder, i.e., small $\alpha$. For an initially insulating system with weakly-correlated disorder slightly larger than $W_c$, we find a transition to metallic behavior upon decreasing $\alpha$. In region IV, the phase boundary moves into the metallic phase I in contrast to region III. Thus states which are extended for weakly correlated disorder, become localized for strong correlation $\alpha \to 0$. As has been argued before \cite{24, 50}, the critical behavior near the mobility edges is governed by quantum interference as well as tunneling between potential wells. For small $W$ and large $|E|$ i.e., at the band edge the potential wells govern the localization behavior \cite{49}. Obviously, a disorder potential smoothened by correlations will also affect these tunneling processes, and away from the band center, this results in more localization. A similar effect has been seen for the 1D case \cite{23}.

The different influence of scale-free disorder on the states at the band center and at the band edge is also the origin of the difference in the behavior of $\nu_W(\alpha)$ and $\nu_E(\alpha)$ shown in Figs. 11 and 2. The extended Harris criterion essentially states that the critical behavior of the MIT remains unaffected if, as $E \to E_c$, the divergence of the localization length $\xi \sim |E - E_c|^{-\nu_E}$ is stronger than the divergence of a typical size $|E - E_c|^{-2/\alpha}$ associated with a correlated disorder potential fluctuation of energy $|E - E_c|$. Otherwise, $\nu_E = 2/\alpha$ \cite{24, 25}. Let us approximate $E_c(W)$ close to $W_c$ as

$$E_c(W) \approx |W - W_c|^{\beta}. \quad (6)$$

Then at $E = 0$

$$\xi \sim |E - E_c|^{-\nu_E} = |E_c|^{-\nu_E} \approx |W - W_c|^{-\beta \nu_E}, \quad (7)$$

so that the critical exponents should be related as

$$\nu_W \approx \beta \nu_E. \quad (8)$$

From Fig. 3 we see that $\beta \ll 1$ in the band center whereas $\beta = 1$ is possible for $|E| > 0$. Furthermore,
the above Harris-type argument is based on classical potential fluctuations, whereas at $E = 0$ the physics is dominated by quantum interference with an unchanged universal Anderson exponent $\nu_W \approx 1.6$.

In summary, we have shown that the extended Harris criterion is obeyed varying $E$ at fixed $W$. The resulting exponent $\nu_E$ agrees very well with the predictions of the extended Harris criterion. This is the first such demonstration for a fully quantum coherent situation in three dimensions to the best of our knowledge. Moreover, we show that the extended Harris criterion fails at the band center and trace this failure to the different mechanisms governing the MIT in the vicinity of the band center and outside. We emphasize that such different scaling behavior at the band center and at the band edges has indeed been speculatively discussed for a long time, although numerical studies of $\nu_E$ and $\nu_W$ for the uncorrelated case suggest a common value. Here we show that suitably long-ranged power-law correlations with $\alpha < \alpha_c$ give rise to a difference in $\nu_E$ and $\nu_W$.

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