The refractive index of relic gravitons

Massimo Giovannini

Department of Physics, Theory Division, CERN, 1211 Geneva 23, Switzerland INFN, section of Milan-Bicocca, I-20126 Milan, Italy

E-mail: massimo.giovannini@cern.ch

Received 3 December 2015, revised 22 March 2016
Accepted for publication 30 March 2016
Published 11 May 2016

Abstract

The dynamical evolution of the refractive index of the tensor modes of the geometry produces a specific class of power spectra characterized by a blue (i.e. slightly increasing) slope which is directly determined by the competition of the slow-roll parameter and of the rate of variation of the refractive index. Throughout the conventional stages of the inflationary and post-inflationary evolution, the microwave background anisotropies measurements, the pulsar timing limits and the big-bang nucleosynthesis constraints set stringent bounds on the refractive index and on its rate of variation. Within the physically allowed region of the parameter space the cosmic background of relic gravitons leads to a potentially large signal for the ground-based detectors (in their advanced version) and for the proposed space-borne interferometers. Conversely, the lack of direct detection of the signal will set a qualitatively new bound on the dynamical variation of the refractive index.

Keywords: background radiations, cosmology, gravitational radiation

1. Introduction

It was speculated long ago that gravitational waves might acquire an effective refractive index when they evolve in curved space-times [1]. As electromagnetic waves develop a refractive index when they travel in globally neutral (but intrinsically charged) media, a similar possibility can also be envisaged in the case of linearized gravity. In this investigation it is suggested that the consistent variation of the refractive index throughout the conventional stages of the cosmological evolution leads to the production of a stochastic background of relic gravitons with blue spectral slopes.

Relic gravitons are known to have been produced in the early Universe thanks to the pumping action of the gravitational field [2]. This phenomenon occurs in a variety of different scenarios and, in particular, in the case of conventional inflationary models (see e.g. [3–10] for an incomplete but potentially interesting list of time-ordered references). While
inflationary models typically predict decreasing slopes, in the conventional lore blue spectral indices of the relic graviton backgrounds can arise when a long phase (dominated by stiff sources) takes place after inflation but prior to the dominance of radiation [5]. Other less conventional possibilities include gravity theories which are not of Einstein–Hilbert type (see e.g. the last paper of [3]) and the violation of the dominant energy condition in the early Universe [11]. In this paper we are going to argue that blue spectral slopes may arise from a comparatively more mundane possibility, namely the temporal variation of the refractive index of the tensor modes while the evolution of the background geometry follows exactly the same patterns of the concordance paradigm.

The stochastic backgrounds of relic gravitons are subjected to three complementary classes of constraints. The first class of direct limits stems from the temperature and polarization anisotropies of the cosmic microwave background [12–14] and it is customarily expressed in terms of a bound on the tensor-to-scalar ratio $r_{\ell}(k_p)$ at a conventional pivot wavenumber $k_p$, where the large-scale power spectra are assigned. The pulsar timing measurements [15, 16] impose instead an upper bound on the cosmic graviton background at a typical frequency roughly corresponding to the inverse of the observation time along which the pulsars’ timing has been monitored. The big-bang nucleosynthesis limits [17] set an indirect constraint on the extra-relativistic species (and, among others, on the relic gravitons) at the time when light nuclei have been firstly formed. There is finally the possibility of using the energy balance of the plasma at the time of matter-radiation equality of photon decoupling to constrain the integrated energy density of the relic gravitons [18]. This bound is conceptually very similar to the one of big-bang nucleosynthesis but applicable over smaller frequencies or, equivalently, smaller wavenumbers.

In this paper we intend to compute the cosmic background of the relic gravitons induced by the consistent variation of the refractive index during the early stages of the evolution of the geometry. The appropriately constrained spectra shall be compared with the frequency window of the ground-based interferometers such as Ligo/Virgo [19, 20], Geo600 [21] and the recently proposed Kagra [22] (ideal prosecution of the Tama300 experiment [23]). There also exist daring projects of wide-band detectors in space like the Lisa interferometer [24] (in one of its different incarnations) or the Bbo/Decigo [25] project1.

The variation of the refractive index along the different stages of the evolution of the background must be continuous and differentiable at least once. This basic requirement stems directly from the evolution equations of the tensor modes of the geometry. In the case of a conventional inflationary and post-inflationary evolution the scale factor approximately evolves as

$$a_{\text{inf}}(\tau) = \left(-\frac{\tau}{\tau_{1}}\right)^{\beta}, \quad \tau \leq -\tau_{1}. \tag{1.1}$$

$$a_{\text{p}}(\tau) = \frac{\beta\tau + (\beta + 1)\tau_{1}}{\tau_{1}}, \quad -\tau_{1} < \tau \leq \tau_{2}. \tag{1.2}$$

1 The acronyms appearing in this and in the previous sentences refer to the corresponding projects: Lisa (Laser Interferometer Space Antenna), Bbo (Big Bang Observer), Decigo (Deci-hertz Interferometer Gravitational Wave Observatory) and Kagra (Kamioka Gravitational Wave Detector).

2 We are assuming here a conformally flat Friedmann–Robertson–Walker background metric $\gamma_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$ where $a(\tau)$ is the scale factor, $\tau$ denotes the conformal time coordinate and $\eta_{\mu\nu}$ is the Minkowski metric. This is the simplest way of complying with the concordance scenario where the extrinsic curvature is always much larger than the intrinsic (spatial) curvature.
where \( \tau_1 \) coincides with the end of the inflationary phase and \( \tau_2 \) coincides with the time of matter-radiation equality; note that \( \beta \rightarrow 1 \) in the case of a pure de Sitter phase and \( \beta = 1 - \mathcal{O}(\epsilon) \) in the quasi-de Sitter case. The transition to the domination of the dark energy will be discussed later on since, in practice, it does not affect the slope and it has a mild effect on the amplitude of the spectrum. Equations (1.1)–(1.3) are all continuous with their first derivatives at the transition points.3

Since relic gravitons are produced because of the pumping action of the background curvature (containing second derivatives of the scale factor), the continuity of equations (1.1)–(1.3) at the transition points ensures that the evolution equations of the relic gravitons will not have singularities in \( \tau_1 \) and \( \tau_2 \) but, at most, jump discontinuities. To guarantee a continuous variation of the refractive index without imposing further conditions, we are led to the following parametrization4:

\[
\alpha(n) = n_1 a^\alpha(n), \quad \alpha > 0,
\]

where \( \alpha \) measures, in practice, the rate of variation of the refractive index in units of the Hubble rate. Equation (1.4) implies that \( n(\tau) \) is automatically continuous and differentiable in \( \tau_1 \) and \( \tau_2 \) provided the scale factor shares the same properties at the transition points. The parametrization (1.4) is minimal insofar as it contains only two arbitrary parameters, namely \( n_1 \) and \( \alpha \). Furthermore the value of \( n_1 \) is not totally arbitrary5 since at the onset of the inflationary phase the refractive index must be larger than (or equal to) 1 to avoid superluminal phase and group velocities. Note that when the background is ever expanding, the positivity of \( \alpha \) guarantees that this condition is preserved throughout the evolution of the geometry.

Before concluding this introduction it is appropriate to stress the main purpose of the present paper is not to endorse a specific scenario but rather to point out what we regard as an interesting possibility. In the bulk of the paper we shall consider the following action for the relic gravitons,

\[
S = \frac{1}{8\ell_p^2} \int d^3x \int d\tau \, a^2(\tau) \left[ \partial_\tau h_{ij} \partial_\tau h_{ij} - \frac{1}{n^2(\tau)} \partial_k h_{ij} \partial_k h_{ij} \right],
\]

where \( h_{ij} \) is the (transverse and traceless) tensor amplitude, \( \ell_p \) is the Planck length, \( a(\tau) \) is the scale factor and \( n(\tau) \) is the refractive index already introduced in equation (1.4). The form of equation (1.5) is rather obvious: in the limit \( n(\tau) \rightarrow 1 \) we get back the standard action for the relic gravitons whereas in the case \( a(\tau) \rightarrow 1 \) we have the action for the gravitational waves in the presence of a refractive index in flat space-time. Equation (1.5) can also be written in a slightly different form, namely

3 This means, more specifically, that for \( \tau = -\tau_1 \) we have that \( a_i(-\tau_1) = a_i(-\tau) \) and \( a'_i(-\tau_1) = a'_i(-\tau) \). Similarly at the second transition point \( a_i(\tau_2) = a_i(\tau) \) and \( a'_i(\tau_2) = a'_i(\tau) \).

4 We stress that \( \alpha \) is the rate of variation of the refractive index in units of the Hubble rate. This practice is widely used to bound the cosmological variation of a specific quantity like, for instance, the size of the internal dimensions, the time variation of the Newton constant, the time variation of the fine structure constant and so on and so forth.

5 For practical reasons the bounds on the amplitude of the spectral index can be more simply expressed in terms of \( n_1 = n(\tau) \) where \( \tau_1 \) coincides with beginning of the inflationary phase. Since the evolution during inflation is known \( n_1 \) can be easily related to \( n_\tau \) by a simple redshift factor.
\[ S = \frac{1}{8\ell_p^2} \int \! d^4x \int \! d\tau [A(\tau) \partial_\tau H_j \partial_\tau H_j - B(\tau) \partial_\tau H_j \partial_\tau H_j + C(\tau) H_j H_j]. \]  

(1.6)

Equations (1.5) and (1.6) are related by field redefinitions are therefore equivalent up to total time derivatives. More specific technical details can be found in appendix A.

A refractive index varying on cosmological scales is implicitly contained in the work of Szekeres [1] where the aim was the formulation of a macroscopic theory of gravitation. Equations (1.5) and (1.6) may also arise in modified theories of gravity violating the equivalence principle. In these theories the variation of the refractive index could be connected with the variation of a homogeneous field which does not necessarily dominate the background. These fields are often called spectator fields. A simple example of this dynamical situation is given by

\[ S_{\varphi} = -\frac{1}{2\ell_p^2} \int \! d^4x \sqrt{-g} \left[ R - \frac{1}{M^2} \partial^\mu \varphi \partial_\mu \varphi R_{\mu\nu} \right], \]  

(1.7)

where \( M \) is a given mass scale and \( \varphi \) could be a spectator field which has a homogenous background without being dominant. If \( \varphi \) is a space-time constant equation (1.7) reduces to the standard Einstein–Hilbert action. During the inflationary phase \( \varphi \) might even become dominant and coincide with the primary inflaton field, under certain circumstances. The action \( S_{\varphi} \) can be then perturbed to second-order in the amplitude of the tensor fluctuations (i.e. \( S_{\varphi} \rightarrow \delta^{(2)} S_{\varphi} \)). After some algebra (swiftly reported in appendix A) the second-order (tensor) fluctuation of equation (1.7) falls into the equivalence class of equations (1.5) and (1.6). Another example of similar nature is provided by the following action:

\[ S_0 = -\frac{1}{2\ell_p^2} \int \! d^4x \sqrt{-g} \left[ R - \frac{1}{M^2} q(\varphi) R^2_{GB} \right], \quad R^2_{GB} = R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R_{\mu\nu}R_{\mu\nu} + R^2, \]  

(1.8)

where \( R^2_{GB} \) denotes the Gauss–Bonnet combination. The second-order fluctuations of the action (1.8) will produce, again, an action of the type of equation (1.6). In equation (1.8) the Gauss–Bonnet combination is essential since it guarantees that the evolution equations of the gravitational waves will not contain higher-order time derivatives. In this respect it is also essential that \( \partial_\tau q(\varphi) \approx 0 \); if \( \partial_\tau q(\varphi) = 0 \) (i.e. if \( q(\varphi) \) is a space-time constant) the whole second term appearing inside the square brackets in equation (1.8) corresponds to a total derivative and does not contribute to the equations of motion. Thanks to the preceding observation we can also infer the general form of \( A(\tau) \), \( B(\tau) \) and \( G(\tau) \). For instance, \( A(\tau) = 1 + a_1 \partial_\tau q / M^2 + a_2 \partial^2_\tau q / M^2 + a_3 (\partial_\tau q)^2 / M^2 \); i.e. it must contain all terms required by power counting and it must go to one in the limit \( \partial_\tau q \rightarrow 0 \). From this type of analysis, it is also clear that \( A(\tau) \rightarrow B(\tau) \rightarrow C(\tau) \).

Equations (1.7) and (1.8) describe possible violations of the equivalence principle leading to equations (1.5) and (1.6). There are also other frameworks where the propagation of gravitational waves in fully inhomogeneous universes (such as Swiss-cheese models) may produce similar effects. While we leave all these potentially interesting themes for future analyses we want to stress that the main inspiration of this paper is more modest: we just want to show that a time-dependent refractive index can affect the spectra of relic gravitons and to demonstrate this point we have decided to deal with a minimal and consistent set-up which is the one described earlier in this introduction.

6 We shall use the standard notations where \( \mathcal{H} = \partial_\tau \ln a \); furthermore \( \mathcal{H} = aH \) where \( H \) is the standard Hubble rate defined in the cosmic time parametrization.
The plan of this paper is the following. In section 2 the most relevant technical aspects of the analysis are derived. The power spectra and the spectral energy density of the relic gravitons are computed in section 3 and in the framework of the conventional cosmological evolution. In section 4 the spectra of the relic gravitons are confronted with all the available constraints. In the final part of section 4 the prospects for the wide-band detectors of gravitational waves are illustrated. Section 5 contains our concluding remarks. Some useful but lengthy results are collected in appendices A–C.

2. Relic gravitons with refractive index

2.1. Basic definitions

The two polarizations of the gravitational wave are defined as

\[ e^{(\hat{\ell})}_{ij}(\hat{k}) = (\hat{m}_i \hat{m}_j - \hat{q}_i \hat{q}_j), \quad e^{(\hat{\ell})}_{ij}(\hat{k}) = (\hat{m}_i \hat{q}_j + \hat{q}_i \hat{m}_j), \]  

(2.1)

where \( \hat{k}_i = k_i / |\hat{k}| \), \( \hat{m}_i = m_i / |\hat{m}| \) and \( \hat{q} = q_i / |\hat{q}| \) are three mutually orthogonal directions and \( \hat{k} \) is oriented along the direction of propagation of the wave. It follows directly from equation (2.1) that

\[ \sum_l e^{(\lambda)}_{ij}(\hat{k}) e^{(\lambda)}_{lm}(\hat{k}) = 2 \delta_{ij} \]  

while the sum over the polarizations gives:

\[ \sum_{ij} e^{(\lambda)}_{ij}(\hat{k}) e^{(\lambda)}_{ij}(\hat{k}) = [p_{mn}(\hat{k})p_{ij}(\hat{k}) + p_{ij}(\hat{k})p_{mn}(\hat{k}) - p_{ij}(\hat{k})p_{jm}(\hat{k})], \]  

(2.2)

where \( p_{ij}(\hat{k}) = (\delta_{ij} - \hat{k}_i \hat{k}_j) \). Defining the Fourier transform of \( h_{ij}(\vec{x}, \tau) \) as

\[ h_{ij}(\vec{x}, \tau) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda} \int d^3k \ h_{ij}(\vec{k}, \tau) e^{-i\vec{k} \cdot \vec{x}}, \]  

(2.3)

the tensor power spectrum \( P_T(k, \tau) \) determines the two-point function at equal times:

\[ \langle h_{ij}(\vec{k}, \tau) h_{mn}(\vec{p}, \tau) \rangle = \frac{2\pi^2}{k^3} P_T(k, \tau) S_{\lambda \mu \nu}(\vec{k}) \delta^{(3)}(\vec{k} + \vec{p}), \]  

(2.4)

where \( S_{\lambda \mu \nu}(\vec{k}) = \sum_{ij} e^{(\lambda)}_{ij}(\hat{k}) e^{(\lambda)}_{ij}(\hat{k}) / 4 \). The analog of equation (2.4) for \( \hat{h}_{ij}(\hat{k}, \tau) \) is given by:

\[ \langle h'_{ij}(\vec{k}, \tau) h'_{mn}(\vec{p}, \tau) \rangle = \frac{2\pi^2}{k^3} Q_T(k, \tau) S_{\lambda \mu \nu}(\vec{k}) \delta^{(3)}(\vec{k} + \vec{p}), \]  

(2.5)

where \( Q_T(k, \tau) \) is the corresponding power spectrum and where the prime denotes a derivation with respect to the conformal time coordinate \( \tau \). Note that equation (2.4) follows the same conventions used when deriving the spectrum of curvature perturbations on comoving orthogonal hypersurfaces (customarily denoted by \( \mathcal{R}(\vec{x}, \tau) \))

\[ \langle \mathcal{R}(\vec{k}, \tau) \mathcal{R}(\vec{p}, \tau) \rangle = \frac{2\pi^2}{k^3} P_R(k, \tau) \delta^{(3)}(\vec{k} + \vec{p}), \]  

(2.6)

which is exactly the quantity employed to set the initial conditions for the evolution of the temperature and polarization anisotropies of the cosmic microwave background [12–14]. We also recall for future convenience that, according to the standard convention, the scalar power spectrum is assigned as:

\[ P_R(k) = A_R \left( \frac{k}{k_p} \right)^{n_s - 1}, \quad k_p = 0.002 \text{ Mpc}^{-1}, \]  

(2.7)

where \( k_p \) is called the pivot scale, \( n_s \) is the scalar spectral index and \( A_R \) is the amplitude of the scalar power spectrum at the pivot scale.
2.2. Power spectra and spectral energy density

The equation obeyed by \( h_{ij}(x, \tau) \) follows from the second order action:

\[
S = \frac{1}{8\ell_p^2} \int \! d^3x \int \! d\tau \, a^2(\tau) \left[ \partial_\tau h_{ij} \partial_\tau h_{ij} - \frac{1}{n^2(\tau)} \partial_k h_{ij} \partial_k h_{ij} \right],
\]

which reduces to the conventional action \([10, 26]\) in the limit \( n(\tau) \to 1 \). Note that in equation (2.8) \( \ell_p = \sqrt{8\pi G} = 8\pi/M_p \) and \( M_p = 1.22 \times 10^{19} \) GeV. From equation (2.8) the equations of motion for \( h_i^\mu \) are:

\[
h_i^{\mu \nu} + 2\mathcal{H}h_i^{\nu} - \frac{\nabla^2 h_i^{\nu}}{n^2(\tau)} = 0,
\]

where \( \mathcal{H} = (\ln a)' = a H \) and \( H \) is the conventional Hubble rate. In equation (2.9) the contribution of the (transverse and traceless) anisotropic stress has been neglected. At low frequencies and in the concordance paradigm the contribution to the anisotropic stress is due to the presence of (effectively massless) neutrinos \([30]\). At high frequencies the anisotropic stress induced by waterfall fields may lead to an enhancement of the spectral energy density (see third paper in \([10]\)). Both effects will be neglected in what follows for two independent reasons. We shall neglect neutrinos because they are known to suppress the energy density of the relic gravitons at intermediate frequencies but their numerical relevance is not strictly essential for the present considerations. The waterfall field, on the contrary, may lead to large effects which are, however, model dependent, insofar as they arise in a given and specific class of inflationary scenarios.

In the absence of anisotropic stress \( h_{ij}(x, \tau) \) can be quantized and the corresponding field operator is:

\[
\hat{h}_{ij}(x, \tau) = \frac{\sqrt{2}\ell_p}{(2\pi)^{3/2}} \sum_{\lambda} \int \! d^3k \, e^{i(k \cdot \hat{x})} \left[ F_{\ell,\lambda}(\tau) \hat{a}_{\ell,\lambda} e^{-ik \cdot \hat{x}} + F_{\ell,\lambda}^\dagger(\tau) \hat{a}_{\ell,\lambda}^\dagger e^{ik \cdot \hat{x}} \right],
\]

where \( F_{\ell,\lambda}(\tau) \) is the (complex) mode function obeying equation (2.9) and the sum is performed over the two physical polarizations of equation (2.1); note that \( [\hat{a}_{\ell,\lambda}, \hat{a}_{\ell',\lambda'}^\dagger] = \delta_{\ell,\ell'} \delta^{(3)}(\vec{k} - \vec{\ell}) \). The same expansion of equation (2.10) can be obtained for the derivative of the amplitude

\[
\hat{h}_{ij}(x, \tau) = \frac{\sqrt{2}\ell_p}{(2\pi)^{3/2}} \sum_{\lambda} \int \! d^3k \, e^{i(k \cdot \hat{x})} \left[ G_{\ell,\lambda}(\tau) \hat{a}_{\ell,\lambda} e^{-ik \cdot \hat{x}} + G_{\ell,\lambda}^\dagger(\tau) \hat{a}_{\ell,\lambda}^\dagger e^{ik \cdot \hat{x}} \right],
\]

where, this time, \( G_{\ell} = F_{\ell}' \). The power spectra introduced in equations (2.4) and (2.5) become, in this specific case:

\[
\mathcal{P}_T(k, \tau) = \frac{4\ell_p^2}{n^2} \frac{k^3}{|F_{\ell}(\tau)|^2},
\]

\[
\mathcal{Q}_T(k, \tau) = \frac{4\ell_p^2}{n^2} \frac{k^3}{|G_{\ell}(\tau)|^2}.
\]

The mode functions \( F_{\ell}(\tau) \) obey the following equation which is the Fourier space analog of equation (2.9):

\[
F_{\ell}'' + \frac{2a'}{a} F_{\ell}' + \frac{k^2}{n^2(\tau)} F_{\ell} = 0,
\]
that can also be written as

\[ f''_k + \left[ \frac{k^2}{n^2(\tau)} - \frac{a''}{a} \right] f_k = 0. \]  

(2.15)

Following Ford and Parker [26] (see also [27–29] for complementary approaches) the energy density of the relic gravitons can be written as

\[ \rho_{gw} = \frac{1}{8\ell_P^4 a^2} \left[ \partial_i h_{ij} \partial_i h_{ij} + \frac{1}{n^2(\tau)} \partial_i h_{ij} \partial_i h_{ij} \right]. \]  

(2.16)

Within the established notations\(^7\) the energy density per logarithmic interval of wavenumber becomes

\[ \frac{d\rho_{gw}}{d\ln k} = \frac{1}{8\ell_P^4 a^2} \left[ \frac{k^2}{n^2(\tau)} \mathcal{P}_T(k, \tau) + \mathcal{Q}_T(k, \tau) \right] \rightarrow \frac{k^2}{4\ell_P^4 a^2(\tau) n^2(\tau)} \mathcal{P}_T(k, \tau), \]  

(2.17)

where the final result holds when the modes are inside the Hubble radius since, in this case, \( k^2 \mathcal{P}_T(k, \tau)/n^2(\tau) \rightarrow \mathcal{Q}_T(k, \tau). \) In the opposite limit we have instead that \( \mathcal{Q}_T(k, \tau) \rightarrow \mathcal{H}^2 \mathcal{P}_T(k, \tau). \) When discussing the graviton spectra over various orders of magnitude in frequency it is more practical to deal with the spectral energy density of the relic gravitons in critical units per logarithmic interval of wavenumber:

\[ \Omega_{gw}(k, \tau) = \frac{1}{\rho_{crit}} \frac{d\rho_{gw}}{d\ln k}, \quad \rho_{crit} = 3H^2 / \ell_P^2. \]  

(2.18)

The energy density of the relic gravitons per logarithmic interval of comoving wavenumber (or logarithmic interval of comoving frequency) introduced in equations (2.17) and (2.18) will be occasionally called the spectral energy density of the cosmic graviton background.

**2.3. Practical time parametrizations**

We conclude this section with few remarks involving the time parametrizations. As we saw the evolution of the mode functions can be perfectly well discussed in the conformal time parametrization. However, for an explicit solution of the equations, it is convenient to use the \( \eta \)-time parametrization. Indeed, the action can be expressed in a simpler form by introducing a different time coordinate defined by \( d\tau = n(\eta)d\eta. \) In this case the action of equation (2.8) can be expressed as:

\[ S = \frac{1}{8\ell_P^2} \int d^3x \int d\eta \ b^2(\eta) \left[ \partial_i h_{ij} \partial_i h_{ij} - \partial_i h_{ij} \partial_i h_{ij} \right], \quad b(\eta) = \frac{a(\eta)}{\sqrt{n(\eta)}}. \]  

(2.19)

The mode expansion is analogous to equation (2.10) and it is given by:

\[ h_{ij}(\tilde{x}, \eta) = \frac{\sqrt{2} \ell_P}{(2\pi)^{3/2}} \sum \int d^3k \ e^{ik\xi} (\tilde{k}) \left[ \mathcal{F}_{ij,\lambda}(\eta) \hat{a}_{\ell,\lambda} e^{-ik\xi} + \mathcal{F}_{ij,\lambda}^\dagger(\eta) \hat{a}_{\ell,\lambda}^\dagger e^{ik\xi} \right], \]  

(2.20)

\(^7\) We take the opportunity for an elementary observation which is however rather crucial to avoid potential confusions: in this paper the natural logarithms will be denoted by ‘ln’ while the common logarithms will be denoted by ‘log’.
where, however, the evolution equation obeyed by $F_k(\eta)$ differs from equation (2.14) and it is given by
\[
\frac{\partial^2 F_k}{\partial \eta^2} + \frac{2}{b} \left( \frac{\partial b}{\partial \eta} \right) \frac{\partial F_k}{\partial \eta} + k^2 F_k = 0.
\] (2.21)

The evolution of the mode function rescaled through $b(\eta)$ will then read
\[
\frac{\partial^2 \tilde{F}_k}{\partial \eta^2} + \left[ k^2 - \frac{1}{b} \left( \frac{\partial^2 b}{\partial \eta^2} \right) \right] \tilde{F}_k = 0, \quad \tilde{F}_k = b(\eta) F_k(\eta).
\] (2.22)

The parametrization of equation (1.4) implies that power-law behaviours in the $\tau$-parametrization translate into power-laws in the $\eta$-parametrization. This is always true except for the case when the relation between $\eta$ and $\tau$ is logarithmic. This happens, for instance, when $a = 1$ and the scale factor evolves during the radiation-dominated phase (i.e. equation (1.2)). The same thing happens when $\alpha = 1/2$ and the scale factor is the one of dusty matter (as in equation (1.3)).

### 3. Power spectra in the different phases

The evolution of the mode functions of equations (2.14) and (2.15) must be solved by taking into account the evolution of the refractive index (see equation (1.4)) in each of the different stages defined, respectively, by equations (1.1)–(1.3). At the practical level the strategy is to pass from the $\tau$-parametrization to the $\eta$-parametrization and then transform back the obtained result in the conformal time coordinate. Since this procedure is algebraically lengthy but completely straightforward, we shall simply present the final result for the correctly normalized mode function and avoid pedantic details. We finally mention that it is useful to introduce, in some of the forthcoming equations, the obvious notation
\[
\omega_{\text{inf}}(\tau) = \frac{k}{n_1 a_{\text{inf}}^{\alpha}(\tau)}, \quad \omega_r(\tau) = \frac{k}{n_1 a_{\mu}^{\alpha}(\tau)}, \quad \omega_m(\tau) = \frac{k}{n_1 a_{m}^{\alpha}(\tau)},
\] (3.1)
where the scale factors in the different epochs are parametrized as in equation (1.1)–(1.3).

#### 3.1. Power spectrum during inflation

When the scale factor and the refractive index are given, respectively, by equations (1.1) and (1.4), the normalized solution of equation (2.15) is given by:
\[
f_k(\tau) = \frac{D}{\sqrt{2 \omega_{\text{inf}}(\tau)}} \sqrt{-\omega_{\text{inf}}(\tau) \tau} H^{(1)}_{\mu}[g(\tau)], \quad \mu = \frac{3 - \epsilon}{2(1 - \epsilon)} \left[ 1 + \alpha \beta \right],
\] (3.2)
where $H^{(1)}_{\mu}[g(\tau)]$ is the Hankel function of the first kind [31, 32] with argument $g(\tau)$ and index $\mu$. Equation (3.2) has been derived in the case where the slow-roll parameter $\epsilon = -\dot{H}/H^2$ is constant in time. In this case it turns out that $\beta = 1/(1 - \epsilon)$ since $aH = -1/(1 - \epsilon) \tau$. In equation (3.2) the normalization $|D| = \sqrt{\pi/(2[1 + \alpha \beta])}$
guarantees that, up to an irrelevant phase, equation (3.2) coincides with a plane wave in the large argument limit of Hankel functions. Since $\omega_{\text{inf}}(\tau)$ depends on $\tau$ it is practical to introduce a single argument $g_\tau(\tau)$ as

$$g_\tau(\tau) = \frac{-\tau \omega_{\text{inf}}(\tau)}{1 + \alpha}\frac{1}{1 + \alpha\beta} n_1 \left( \frac{\tau}{\tau_1} \right)^{1+\alpha\beta}.$$ (3.3)

Inserting equation (3.2) into equation (2.12) we obtain, after some algebra, the explicit expression of the inflationary power spectrum:

$$\mathcal{P}_T(k, \tau) = 8 \left( \frac{H_r}{M_p} \right)^2 \frac{|k n_1|^2}{(1 + \alpha\beta)} \left( \frac{\tau}{\tau_1} \right)^{1+2\alpha\beta} |H_\mu^{(1)}[g_\tau(\tau)]|^2,$$ (3.4)

where $\mu$ can also be expressed as $\mu = (3 - \epsilon)/(2(1 + \epsilon + \alpha))$. Using equation (3.3) and considering the modes that are larger than the Hubble radius, equation (3.4) becomes:

$$\mathcal{P}_T(k, k_{\text{max}}) = \mathcal{C}(\epsilon, \alpha) n_1^{3-n_T(\epsilon, \alpha)} \left( \frac{H_r}{M_p} \right)^2 \left( \frac{k}{k_{\text{max}}} \right)^{n_T(\epsilon, \alpha)},$$

$$\mathcal{C}(\epsilon, \alpha) = \frac{2^6 - n_T(\epsilon, \alpha)}{\pi^2} \left( \frac{3}{2} - n_T(\epsilon, \alpha) \right) \left[ 1 + \frac{\alpha}{1 - \epsilon} \right]^{2-n_T(\epsilon, \alpha)}.$$ (3.5)

Denoting with $\Omega_{80}$ the present value of the critical fraction of radiative species (in the concordance paradigm photons and neutrinos) and with $A_R$ the amplitude of the scalar power spectrum at the pivot scale (see equation (2.7)) the value of $k_{\text{max}}$ can be expressed, for instance, in Mpc$^{-1}$ units:

$$\left( \frac{k_{\text{max}}}{\text{Mpc}^{-1}} \right) = 2.247 \times 10^{23} \left( \frac{H_r}{H_0} \right)^{\gamma-1/2} \left( \frac{\epsilon}{0.01} \right)^{1/4} \left( \frac{A_R}{2.41 \times 10^{-9}} \right)^{1/4} \left( \frac{h_0^2 \Omega_{80}}{4.15 \times 10^{-5}} \right)^{1/4}.$$ (3.6)

In equation (3.6) $\gamma$ accounts for the possibility of a delayed reheating terminating at a Hubble scale $H_r$ smaller than the Hubble rate during inflation.

In what follows, as already mentioned in the introduction, we shall rather stick to the conventional case where the reheating is sudden and $\gamma = 1/2$ (or $H_0 = H_r$ since the end of the inflationary phase coincides with the beginning of the radiation epoch). In more general terms, however, $H_r$ can be as low as $10^{-44}M_p$ (but not smaller) corresponding to a reheating scale occurring just prior to the formation of the light nuclei.

In the limit $\alpha \to 0$ and $n_1 \to 1$, equation (3.5) leads to the standard result, namely:

$$\lim_{\alpha \to 0, n_1 \to 1} \mathcal{P}_T(k, \tau) \to \frac{16}{\pi} \left( \frac{H_r}{M_p} \right)^2 \left( \frac{k}{k_{\text{max}}} \right)^{-2\epsilon},$$ (3.7)

which implies $H_r/M_p = \pi r_T A_R/16$ with $r_T = 16\epsilon$ and $n_T = -r_T/8$. This kind of consistency relation (stipulating that the tensor–scalar ratio exactly equals $16\epsilon$) will not be valid anymore in the present context and the specific form of the tensor-to-scalar ratio will be used in section 4 to constrain the possible values of $\alpha$.

---

8 Since $\alpha \geq 0$ and $\beta = 1/(1 - \epsilon)$ the absolute value is pleonastic. During the radiation and matter epochs the analog factors are not necessarily positive definite. To keep a homogeneous notation the absolute values have been always included even when not mandatory.
We observe that whenever $\alpha > 0$ the spectral index can increase since it can be naively larger than the slow-roll parameter. More specifically expanding $n_T(\epsilon, \alpha)$ in the limit $\epsilon < 1$ we will have that

$$n_T = \frac{3\alpha}{1 + \alpha} + \frac{(\alpha - 2)}{(1 + \alpha)^2} \epsilon + \mathcal{O}(\epsilon^2).$$

(3.8)

Two possible situations can be envisaged. If $\alpha > 1$ the spectral slope is always violet (i.e. sharply increasing); in the limiting case $\alpha \gg 1$ we have, according to equation (3.8), that $n_T \to 3$. This possibility is strongly constrained by backreaction effects as we shall specifically see in section 4. If $0 < \alpha < 1$ the spectra are blue (i.e. slightly increasing) provided $\alpha > 2\epsilon/(3 + 5\epsilon)$; in the opposite case (i.e. $\alpha < 2\epsilon/(3 + 5\epsilon)$) the conventional limit is recovered and $n_T \simeq -2\epsilon$.

The physical region of the parameters corresponds to the situation where at the onset of inflation the refractive index is larger than (or equal to) 1. If this is the case a superluminal phase velocity is avoided. Indeed, denoting with $\tau_i$ the initial time of the evolution we shall have that $\frac{\alpha}{\alpha_0} \gg 1.

$$n_i = n_i\left(\frac{a_i}{a_0}\right)^\alpha = n_i e^{-\alpha N_i}, \quad n_i \geq 1. \quad (3.9)$$

If $n_i = 1$ we shall have that $n_i \to \exp(\alpha N_i)$ where $N_i$ is the total number of inflationary e-folds. As we shall see the detailed discussion of section 5 implies that $n_i$ must indeed be $\mathcal{O}(1)$ even if not strictly equal to 1.

### 3.2. Power spectrum during the radiation epoch

Following the conventions established in equations (3.2) and (3.3) the expression of the mode function during the radiation epoch shall be written as:

$$f_k(\tau) = \frac{D_\Omega}{\sqrt{2\omega_r(\tau)}} \sqrt{\omega_r(\tau)} y(\tau, \tau_i) \{ c_+(k, \tau_i) H_\rho^{(2)}[g_r(\tau)] + c_-(k, \tau_i) H_\rho^{(1)}[g_r(\tau)] \}, \quad (3.10)$$

where

$$y(\tau, \tau_i) = \left( \tau + \frac{\beta + 1}{\beta} \tau_i \right), \quad \rho = \frac{1}{2|1-\alpha|}, \quad |D_\Omega| = \frac{\pi}{\sqrt{2|1-\alpha|}}. \quad (3.11)$$

In terms of $y(\tau, \tau_i)$ the argument of the Hankel functions $g_r(\tau)$ is defined as

$$g_r(\tau) = \frac{\omega_r(\tau)}{|1-\alpha|} y(\tau, \tau_i) = \frac{k}{m|1-\alpha|} \left( \frac{\tau_i}{\beta} \right)^\beta \left[ \tau + \frac{\beta + 1}{\beta} \tau_i \right]^{1-\alpha}. \quad (3.12)$$

The coefficients $c_{\pm}(k, \tau_i)$ are complex and they obey $|c_+(k, \tau_i)|^2 - |c_-(k, \tau_i)|^2 = 1$. The exact expression of the two mixing coefficients is reported in equations (B.1) and (B.2) of appendix B and can be determined by matching continuously the inflationary mode function (i.e. equation (3.2)) with the one of equation (3.10) in $\tau = -\tau_i$; in formulae the following pair of conditions must be imposed:

---

9 We recall that the parametrization of the scale factors given in equations (1.1)-(1.3) stipulates that $a_0 = a(-\tau_i) = 1$. There are some who prefer to set $a_0 = 1$ (where $a_0$ is the present value of the scale factor) but this is not the convention adopted in the present paper.
\[ f_k^{(\text{inf})}(-\tau) = f_k^{(\text{rad})}(-\tau), \quad \frac{\partial f_k^{(\text{inf})}}{\partial \tau} \bigg|_{\tau=-\tau} = \frac{\partial f_k^{(\text{rad})}}{\partial \tau} \bigg|_{\tau=-\tau}. \] (3.13)

The requirements of equation (3.13) follow directly from the continuity of the scale factors and of the extrinsic curvature. We recall that the continuity of the scale factor guarantees, in the present approach, the continuity of the refractive index. The continuity of the extrinsic curvature (related to the conformal time derivative of the scale factor) guarantees that \(a^\rho/a = \mathcal{H}^2 + \mathcal{H}'\) will have, at most, jump discontinuities. The exact expression of the mixing coefficients determined in the present situation reproduces the conventional results when \(\alpha \to 0\) (see equation (B.8) of appendix B).

The exact results of equations (B.6) and (B.7) can be expanded in powers of \(g_r \simeq |g| \ll 1\):

\[ c_\ast(k, \tau) + c_\ast(k, \tau) = \mathcal{O}(\mathbb{G}_r) + \mathcal{O}(\mathbb{G}_r), \]

\[ c_\ast(k, \tau) + c_\ast(k, \tau) = 2 c_\ast(k, \tau) = \frac{2}{g_r^\mu} g_r^\mu [i \xi(\alpha, \epsilon) + \mathcal{O}(\mathbb{G}_r) + \mathcal{O}(\mathbb{G}_r)], \] (3.14)

where \(\mathbb{G}_r = g_r(\tau)\) and \(\mathbb{G}_r = g_r(\tau)\) (see also equation (B.3) of appendix B). For practical reasons, the following combination

\[ \mathcal{E}(\alpha, \epsilon) = \frac{2^{\nu+\mu} \Gamma(\mu) \Gamma(\nu)}{8 \sqrt{\beta} (1 - \alpha) \pi} \left\{ \frac{[1 - \alpha]}{\sqrt{1 + \alpha \beta}} \left[ \beta [1 - 2(1 - \alpha) \rho - 2 \mu \alpha + 1 - 2 \mu] \right] \right\}, \]

has been introduced in equation (3.14); note that \(\mathcal{E}(\alpha, \epsilon)\) only depends on \(\alpha\) and \(\epsilon\) since all the other auxiliary variables (i.e. \(\rho\) and \(\mu\)) are independent functions of \(\alpha\) and \(\epsilon\). The result of equation (3.14) can be made more explicit by using the expressions of \(\mathbb{G}_r\) and \(\mathbb{G}_r\); to lowest order in \(x_1 = k \tau_0\), the approximate expression of \(|c_{\ast}(\xi)|^2\) is given by

\[ |c_{\ast}(\xi)|^2 = \mathcal{E}^2(\alpha, \epsilon) \beta^{2 \nu} \frac{[1 - \alpha]}{\sqrt{1 + \alpha \beta}} \left[ \frac{1}{\beta} [1 - 2(1 - \alpha) \rho - 2 \mu \alpha + 1 - 2 \mu] \right]^{2(\mu + \rho)}. \] (3.16)

Since equation (3.14) we have that \(c_\ast(k, \tau) \simeq c_\ast(k, \tau)\), for \(\tau > -\tau_0\) the radiation power spectrum is therefore given by:

\[ \mathcal{P}_\tau(k, \tau) = \frac{4 k^2 \mathcal{E}^2}{\pi [1 - \alpha a^2(\tau)]^2} y(\tau, \tau)|c_{\ast}(k, \tau)|^2 J_\nu^2[g_r(\tau)], \] (3.17)

where \(J_\nu[g_r(\tau)] = \left[H^\nu_1(g_r) + H^\nu_2(g_r)\right]/2\). The argument of \(J_\nu[g_r(\tau)]\) is \(g_r\) (not \(\mathbb{G}_r\)) so that deep in the radiation epoch the power spectrum can be obtained in the limit \(g_r \ll 1\). In this case, using the standard limits of the Bessel functions, the power spectrum becomes:

\[ \mathcal{P}_\tau(k, \tau) = \frac{64}{\pi} \left[ \frac{H_1}{M_p} \right]^2 \left( \frac{a_1}{a} \right)^2 n_r(\tau)|k\tau_0|^2 |c_{\ast}(k, \tau)|^2 \cos^2[g_r(\tau)]; \]

(3.18)

where the large argument limit of \(J_\nu[g_r(\tau)]\) has been used. In equation (3.18) we can replace \(\cos^2[g_r(\tau)] \to 1/2\) as it is customary in this kind of analysis. Thus, from equation (3.18) we can also deduce the energy density and express it in critical units:

\[ \Omega_{gw}(k, \tau) = \frac{8}{3 \pi} \left( \frac{H_1}{M_p} \right)^2 \left| k\tau_0 \right|^2 |n_r(\tau)| |c_{\ast}(k, \tau)|^2. \] (3.19)

### 3.3. Power spectrum during the matter epoch

In the matter-dominated epoch, using equation (1.3) into equation (1.4), the normalized solution of equation (2.15) for \(\tau \geq \tau_2\) is given by:
\[ f_k (\tau) = \frac{D_m}{\sqrt{2\omega_m (\tau)}} \sqrt{\omega_m (\tau)} z (\tau, \tau_1, \tau_2) \{ d_+ (k, \tau_1, \tau_2) H^{(2)}_\sigma [g_m (\tau)] + d_- (k, \tau_1, \tau_2) H^{(1)}_\sigma [g_m (\tau)] \}, \]

where

\[ z (\tau, \tau_1, \tau_2) = \left( \tau + \tau_2 + \frac{2(\beta + 1)}{\beta} \right), \quad \sigma = \frac{3}{2 |1 - 2\alpha|}, \]

\[ |D_m| = \frac{\pi}{2|1 - 2\alpha|}. \]  

(3.20)

Notice finally that \( g_m (\tau) \) in equation (3.20) is defined as

\[ g_m (\tau) = \frac{\omega_m (\tau)}{|1 - 2\alpha|} \left( \tau + \tau_2 + \frac{2(\beta + 1)}{\beta} \right)^3 \]

\[ = \frac{k}{n|1 - 2\alpha|} \left( \frac{4\eta_1 [\beta \tau_2 + (\beta + 1)\tau_1]}{\beta^2} \right)^\alpha \left( \tau + \tau_2 + \frac{2(\beta + 1)}{\beta} \right)^{1 - 2\alpha}. \]  

(3.21)

As in the case of \( c_\pm (k, \eta) \) also the mixing coefficients \( d_\pm (k, \tau_1, \tau_2) \) can be determined by continuous matching of the relevant mode functions across \( \tau_2 \). In this case we shall then impose

\[ f_k^{(\text{rad})} (\tau_2) = f_k^{(\text{mat})} (\tau_2), \quad \frac{\partial f_k^{(\text{rad})}}{\partial \tau} \bigg|_{\tau = \tau_2} = \frac{\partial f_k^{(\text{mat})}}{\partial \tau} \bigg|_{\tau = \tau_2}. \]  

(3.22)

The explicit form of \( d_\pm (k, \tau_1, \tau_2) \) is reported in equations (C.1) and (C.2) of appendix C together with a specific discussion of some relevant physical limits.

As in the case of radiation the power spectrum can be easily obtained in all the interesting regions. More specifically, recalling equations (C.1) and (C.2) we have that \( d_\pm (k, \tau_1, \tau_2) \equiv d_\pm (k, \tau_1, \tau_2) \) in the relevant physical limit. From equations (3.13) and (3.23) the power spectrum during the matter phase is therefore given by:

\[ \mathcal{P}_T (k, \tau) = \frac{4k^2 \gamma^2}{\pi |1 - 2\alpha| a^2 (\tau)} z (\tau, \tau_1, \tau_2) |d_- (k, \tau_1, \tau_2)|^2 J^2_\sigma [g_m (\tau)], \]  

(3.23)

where \( J_\sigma (g_m) = [H^{(1)}_\sigma (g_m) + H^{(2)}_\sigma (g_m)]/2. \) Deep in the matter epoch the power spectrum can be obtained in the limit \( g_m \gg 1 \). In this case, using the standard limits of the Bessel functions, the power spectrum becomes:

\[ \mathcal{P}_T (k, \tau) = \frac{64}{\pi} \left( \frac{H_\sigma}{M_P} \right)^2 \left( \frac{a_1}{a} \right)^2 n_m (\tau) |k\tau|^2 |d_- (k, \tau_1, \tau_2)|^2 \cos^2 [g_m (\tau)]. \]  

(3.24)

In equation (3.25) we can replace \( \cos^2 [g_m (\tau)] \rightarrow 1/2 \) and we can also deduce the energy density and express it in critical units:

\[ \Omega_{gw} (k, \tau) = \frac{8}{3\pi} \left( \frac{H_\sigma}{M_P} \right)^2 \left( \frac{H_0}{H_{eq}} \right)^{2/3} \left[ \frac{|k\tau|^4}{n_m (\tau)} \right] |d_- (k, \tau_1, \tau_2)|^2. \]  

(3.25)

Using the same expansions discussed in the radiation case we can obtain, for instance, the leading order expression for \( |d_- (k, \tau_1, \tau_2)|^2 \) for \( |k\tau| \ll 1 \) and \( |k\tau_2| \ll 1 \).
\[ |d(k, \tau_1, \tau_2)|^2 = M(\alpha, \epsilon) \left( \frac{\tau_1}{\beta \tau_2} \right)^{2(\mu+\sigma)} n_1^{2(\mu+\sigma)} |k\gamma|^2 |k\gamma|^2 (\rho-\sigma), \quad (3.27) \]

where \( M(\alpha, \epsilon) \) is given by:

\[
M(\alpha, \epsilon) = \frac{2^{2\mu-9}\beta^4 \pi^2 \Gamma(\mu) \Gamma(\mu)}{\pi^2 \rho^2 \left[ 1 - \alpha^2 \right]^2 \left[ 1 - 2\alpha \right]^{2\mu} \left[ 1 + \alpha \beta \right]^{2\mu}} \times \left[ \frac{\beta + 1}{2\beta} - (1 - \alpha) \rho - \mu \left( \frac{1 + \alpha \beta}{\beta} \right)^2 \right], \quad (3.28)\]

and it is only a function of \( \alpha \) and \( \epsilon \) because the other parameters (i.e. \( \mu, \rho \) and \( \sigma \)) are all independent functions of \( \alpha \) and \( \epsilon \) and they have been defined, respectively, in equations (3.2), (3.11) and (3.21).

An interesting limit of equation (3.27) is the one \( \beta \to 1 \) and \( \alpha \to 0; \) in this limit \( \mu \to 3/2, \rho \to 1/2 \) and \( \sigma \to 3/2. \) In this case (setting also \( n_1 \to 1 \)) we have that \( |d| \to (9/64)|k\gamma|^4 |k\gamma|^2. \) This is the standard result for the mixing coefficients in the case of a transition from a pure de Sitter phase to the matter-dominated epoch passing through the conventional radiation dominance.\(^{10}\)

It is finally possible to obtain a general expression encompassing the radiation and matter-dominated phases for the energy density of the relic gravitons in critical units. The expression applies for modes inside the Hubble radius at the present time and it is given by:

\[
\Omega_{gw}(k, \tau_0) = \mathcal{N}(\alpha, \epsilon) \left( \frac{H_0}{H_{eq}} \right)^{2/3 + \alpha[1/(\rho-\sigma)-1]} \left( \frac{H_1}{M_p} \right)^{2-\alpha[2(\rho-\sigma)+1]-1/2} \left( \frac{H_0}{M_p} \right)^{\alpha[2(\rho-\sigma)+1]-1/2} \times \left( \frac{H_1}{M_p} \right)^{2(\mu+\sigma)-\alpha} \mathcal{N}_{\alpha}[2(\mu+\sigma)-\alpha] \left( \frac{k}{k_{max}} \right)^{4-2(\mu+\rho)} T(k, k_{eq}),
\]

\[
T^2(k, k_{eq}) = 1 + b_1 \left( \frac{k}{k_{eq}} \right)^{2(\rho-\sigma)} + c_1 \left( \frac{k}{k_{eq}} \right)^{4(\rho-\sigma)},
\]

\[
\mathcal{N}(\alpha, \epsilon) = \frac{2^{2\alpha+3}}{3\pi} \left( 1 + 2(\rho-\sigma) \right) \mathcal{M}(\alpha, \epsilon),
\quad (3.29)\]

where \( b_1 \) and \( c_1 \) are numerical constants of \( \mathcal{O}(1); \) in equation (3.29) we have already expressed \( n_1 \) in terms of \( n_i, \) i.e. the initial value of the spectral index. The accurate value of \( b_1 \) and \( c_1 \) can be also obtained numerically by computing the transfer function of the energy density of the relic gravitons introduced in e.g. \( [10]. \) Alternatively one can compute the transfer function for the power spectrum and then compute the energy density \( [4]. \) In both cases the idea is to integrate numerically the background and the mode functions across the matter–radiation transition. For the present ends what matters, however, is equation (3.29) in the limit \( k \gg k_{eq}. \) When \( \alpha \to 0 \) and \( n_i \to 1 \) and for \( k \gg k_{eq} \) equation (3.29) reproduces the standard result

\[
\Omega_{gw}(k, \tau_0) = \frac{3}{8\pi} \left( \frac{H_0}{H_{eq}} \right)^{2/3} \left( \frac{H_1}{M_p} \right)^{2} \left( \frac{k}{k_{max}} \right)^{-2\epsilon}. \quad (3.30)
\]

\(^{10}\) The limit of the exact expressions in this specific case (see equations (C.6) and (C.7)) coincides with the limit of the general expression obtained above: this is a useful check of the whole algebraic consistency.
As in the standard case, when \( k \gg k_{\text{max}} \) we have that \( \Omega_{gw} \) is exponentially suppressed as \( \exp[-\delta k/k_{\text{max}}] \) where \( \delta \) is a numerical factor that can be estimated in a specific model of smooth transition (see e.g. [10]).

In equation (3.29) the ratio \( H_0/M_\text{P} \) is raised to an \( \alpha \)-dependent power that disappears in the conventional limit of equation (3.30) (i.e. \( \alpha \rightarrow 0 \)). The explicit values of \( H_0/M_\text{P} \) and \( H_{eq}/H_0 \) can be explicitly written as

\[
\frac{H_0}{M_\text{P}} = 1.228 \times 10^{-61} \left( \frac{h_0}{0.7} \right), \\
\frac{H_{eq}}{H_0} = 10^{6.29} \left( \frac{h_0^2 \Omega_{M0}}{0.1364} \right)^{3/2} \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{-3/2},
\]

where \( h_0 \) is the indetermination on the present value of the Hubble rate, \( \Omega_{M0} \) is the critical fraction of matter density and \( \Omega_{R0} \) is the critical fraction of radiation energy density.

It is finally useful to remark that the spectral slope of \( \Omega_{gw} \) in the high-frequency branch (i.e. for \( k \gg k_{eq} \)) is simply given by \( 4 - 2(\mu + \rho(\alpha)) \) as it can be immediately verified from the explicit expression of equation (3.29). Recalling equations (3.2) and (3.10) the high-frequency slope can be written more explicitly:

\[
4 - 2[\mu(\epsilon, \alpha) + \rho(\alpha)] = 3 + \frac{1}{\alpha - 1} + \frac{\alpha - 2}{1 + \alpha - \epsilon} \\
\simeq 2(\alpha - \epsilon) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\alpha^2),
\]

where the second equality follows by expanding the exact expression first in powers of \( \epsilon \) and then in powers of \( \alpha \). This limit captures an important corner of the parameter space (see the discussion of section 4). Equation (3.32) implies that in the pure de Sitter limit without variation of the spectral index (i.e. \( \alpha = 0 \) and \( \epsilon = 0 \)) \( \Omega_{gw} \) is constant in frequency with amplitude given by equation (3.30) in the limit \( k \gg k_{eq} \).

### 3.4. Typical frequencies

We shall always use wavenumbers\(^{11}\) expressed either in units of \( \text{Mpc}^{-1} \) or in units of Hz. The reason for this potential ambiguity is that the discussion mixes constraints arising over large length-scales (where the wavenumbers are typically measured in \( \text{Mpc}^{-1} \)) and other limits coming from comparatively much shorter scales (where the wavenumbers are typically assigned in Hz) [10]. For instance, in what follows we shall be dealing with the big-bang nucleosynthesis wavenumber \( k_{\text{bbn}} \)

\[
k_{\text{bbn}} = 1.47 \times 10^{-10} \left( \frac{g_\text{p}}{10.75} \right)^{1/4} \left( \frac{T_{\text{bbn}}}{10^3 \text{MeV}} \right)^{1/2} \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{1/4} \text{Hz},
\]

where \( g_\text{p} \) denotes the effective number of relativistic degrees of freedom entering the total energy density of the plasma and \( T_{\text{bbn}} \) is the big-bang nucleosynthesis temperature determining the size of the Hubble radius at the corresponding epoch. The typical value of the frequency corresponding to equation (3.33) is \( \nu_{\text{bbn}} = k_{\text{bbn}}/2\pi = 2.3 \times 10^{-11} \text{Hz} \). Similar observations can be made in all the other cases. For future convenience \( k_{\text{max}} \) and \( k_{eq} \) can also be expressed in Hz:

\(^{11}\) We shall often measure comoving wavenumbers in Hz and refer to typical comoving frequencies. Note that, in natural units, \( k = 2\pi \nu \). Frequencies and wavenumbers are not exactly coincident even if it is useful, at a practical level, to measure wavenumbers in Hz.
\[ k_{\text{max}} = 2.183 \left( \frac{H}{H_0} \right)^{-1/2} \left( \frac{\epsilon}{0.01} \right)^{1/4} \left( \frac{A_R}{2.41 \times 10^{-9}} \right)^{1/4} \left( \frac{h_0^2 \Omega_{R0}}{4.15 \times 10^{-5}} \right)^{1/4} \text{ GHz}, \quad (3.34) \]

\[ k_{\text{eq}} = 9.69 \times 10^{-17} \left( \frac{h_0^2 \Omega_{R0}}{0.1364} \right)^{-1/2} \text{ Hz.} \quad (3.35) \]

The frequencies corresponding to the fiducial values of the parameters given in equations (3.34) and (3.35) are given, respectively, by \( \nu_{\text{max}} = 0.34 \text{ GHz} \) and by \( \nu_{\text{eq}} = 1.54 \times 10^{-17} \text{ Hz} \).

### 3.5. Secondary effects

In the present analysis we neglected, for the sake of simplicity, a number of secondary effects that may interfere with the variation of the refractive index. For \( k < k_{\text{bbn}} \) the power spectra and the energy density of the gravitons are suppressed due to the neutrino free streaming. The effective energy-momentum tensor acquires, to first-order in the amplitude of the plasma fluctuations, an anisotropic stress (see e.g. [30] and references therein). The overall effect of collisionless particles is a reduction of the spectral energy density of the relic gravitons.\(^{12}\)

The second effect leading to a further suppression of the energy density is the late dominance of the dark energy. The redshift of \( \Lambda \)-dominance is given by \( \Omega_{\Lambda 0}/\Omega_{\text{M0}} \). In principle there should be a break in the spectrum for the modes reentering the Hubble radius after \( \tau_{\Lambda} \). This tiny modification of the slope is practically irrelevant and it occurs anyway for \( k < k_{\text{eq}} \). However, the adiabatic damping of the tensor mode function across the \( \tau_{\Lambda} \)-boundary reduces the amplitude of the spectral energy density by a factor \( (\Omega_{\text{M0}}/\Omega_{\Lambda 0})^2 \simeq 0.10 \). This figure is comparable with the suppression due to the neutrino free streaming. These effects have been discussed in the past (see [7, 10] and references therein). Further effects leading to similar reductions of \( \Omega_{\text{ev}} \) are related to the evolution of the relativistic species.

The effects mentioned in the two previous paragraphs are secondary since they can be easily reabsorbed by the variation of one of the other unknown parameters of the cosmic graviton background. At the same time they become truly essential if the absolute normalization of the graviton spectrum is known (see last paper of [10]). In the present case the inclusion of these secondary effects is unimportant for the final conclusions, as we explicitly checked.

### 4. Phenomenological considerations

The limits on the variation of the refractive index over various scales will now be derived. There are four qualitatively different sets of bounds to be examined and they involve, respectively, (i) the backreaction constraints during inflation, (ii) the limits stemming from the tensor-to-scalar ratio obtained from the temperature and polarization anisotropies of the cosmic microwave background, (iii) the bounds arising from the millisecond pulsar timing measurements and finally (iv) the so-called big-bang nucleosynthesis constraints. At the end of the section the impact of the derived limits on the prospects for the wide-band interferometers shall be addressed.

\(^{12}\) Assuming that the only collisionless species in the thermal history of the Universe are the neutrinos, the amount of suppression can be parametrized by the function \( F(R_e) = 1 - 0.539R_e + 0.134R_e^2 \), where \( R_e \) is the fraction of neutrinos in the radiation plasma. In the case \( N_e = 3, R_e = 0.405 \) and the suppression of the spectral energy density is proportional to \( F^2(0.405) = 0.645 \). This suppression will be effective for relatively small frequencies which are larger than \( k_{\text{eq}} \) and smaller than \( k_{\text{bbn}} \).
4.1. Limits from backreaction effects

The considerations of section 3 suggesting an upper limit on \( \alpha \) can be made more concrete by computing the total energy density of the gravitational waves and by comparing it with the critical energy density during inflation. From equations (2.17) and (2.18) the total energy density of the produced gravitons in critical units is given by:

\[
\frac{\rho_{gw}(a, \alpha, \epsilon)}{\rho_{\text{crit}}} = \frac{1}{24H_i^2 a^2} \int_{1/\eta}^{1/\gamma_i} \frac{dk}{k} \left[ \frac{k^2}{n^2(\tau)} P_{T}(k, \tau) + Q_{T}(k, \tau) \right],
\]

where the integration is extended from modes exiting the horizon at the onset of inflation up to those reentering exactly at the onset of the radiation phase. Inserting equations (3.2)–(3.4) into equation (4.1) and performing the indicated integrals we can easily obtain the following result:\(^{13}\)

\[
\frac{\rho_{gw}(a, \alpha, \epsilon)}{\rho_{\text{crit}}} = S(\alpha, \epsilon) \left( \frac{H_i}{M_p} \right)^2 \left( \frac{a}{a_i} \right)^{2\alpha} \left( \frac{a_i}{a} \right)^{(3-2\beta)/\beta} \left[ 1 + \frac{(3 - 2\mu)}{(5 - 2\mu) n_i^2} \left( \frac{a}{a_i} \right)^{-2\alpha} \left( \frac{a_i}{a} \right)^{2/\beta} \right] \]

\[
S(\alpha, \epsilon) = \frac{2^{2\mu} \Gamma^2(\mu)}{3 \pi^2 (3 - 2\mu) n_i^{-1}}.
\]

(4.2)

The function \( S(\alpha, \epsilon) \) appearing in equation (4.2) only depends on \( \alpha \) and \( \epsilon \) since \( \beta = \beta(\epsilon) \) and \( \mu = \mu(\alpha, \epsilon) \); moreover, the dependence on the scale factor in equation (4.2) can be traded for the total number of inflationary e-folds \( N_t \).

It is not necessary to analyze the independent variation of \( N_t, \alpha \) and \( \epsilon \); the upper bound on \( \alpha \) is anyway less constraining than the ones to be examined later on. In fact \( \alpha \) cannot exceed 0.1 when the remaining parameters are fixed to their fiducial values: from equation (4.2) with \( a = a_f = a_i \) and \( n_i = \mathcal{O}(1) \) we have that \( |\rho_{gw}/\rho_{\text{crit}}| < 1 \) provided

\[
\alpha < -\frac{\ln(\pi \epsilon \mathcal{A}_R)}{3 N_t},
\]

(4.3)

where, as usual, \( \mathcal{A}_R \) is the amplitude of the scalar power spectrum at the pivot scale and has been introduced in equation (2.7). The total number of e-folds \( N_t \) appearing in equation (4.5) must be larger than (or equal to) \( N_{\text{max}} \):

\[
N_{\text{max}} = 61.43 + \frac{1}{4} \ln \left( \frac{h_0^2 \Omega_{\text{R0}}}{4.15 \times 10^{-5}} \right) - \ln \left( \frac{h_0}{0.7} \right) + \frac{1}{4} \ln \left( \frac{A_R}{2.41 \times 10^{-9}} \right) + \frac{1}{4} \ln \left( \frac{\epsilon}{0.01} \right),
\]

(4.4)

which is the maximal number of e-folds presently accessible to large-scale observations\(^{14}\). In the case \( (\epsilon, N_t, A_R) = (0.01, 65, 2.41 \times 10^{-9}) \) equation (4.4)

\(^{13}\) The cases \( \mu = 3/2 \) and \( \mu = 5/2 \) are singular: this simply means that the corresponding integrals must be separately computed and lead to a logarithmic contribution which is only present, strictly speaking, in the case \( \epsilon \to 0 \) and \( \alpha \to 0 \) (i.e. pure de Sitter evolution).

\(^{14}\) In practice \( N_{\text{max}} \) is determined by redshifting the inflationary event horizon at the present time and by identifying the obtained results with the current value of the Hubble radius.
implies, for instance, \( \alpha < 0.11 \) when \( n_i = \mathcal{O}(1) \). Similar results can be obtained from slightly different choices of parameters.

### 4.2. Limits from the tensor-to-scalar ratio

The long wavelength gravitons induce direct temperature and polarization. Technically they can affect the TT power spectra (i.e. the temperature autocorrelations), the EE power spectra (i.e. the polarization autocorrelations) and the TE power spectra (i.e. the cross-correlation between temperature and polarization). These power spectra can interfere with temperature and polarization power spectra induced by the scalar mode and this is why the upper bounds on the tensor-to-scalar ratio can be derived from the accurate determinations of the temperature anisotropies and polarization anisotropies \([12, 13]\). Depending on the combined data sets the WMAP 5-year data provided bounds on \( r_T \) in the framework of the concordance model with values ranging from \( r_T < 0.58 \) to \( r_T < 0.2 \). Similar bounds have been obtained from the WMAP 7-year data. The 9-year WMAP data release gave a limit \( r_T < 0.38 \) always in the light of the concordance model in the presence of the tensor. In the last three years there have been more direct determinations of the B-mode polarization of the cosmic microwave background. The first detection of a B-mode polarization, not caused by relic gravitons but coming from the lensing of the E-mode polarization, has been published by the South Pole Telescope \([34]\). The Bicep2 experiment \([35]\) claimed the observation of a primordial B-mode compatible with \( r_T = 0.2^{+0.07}_{-0.05} \) which turned out to be induced, at least predominantly, by a polarized foreground. The present Planck data imply \( r_T < 0.1 \) \([14]\) with a sensitivity to the B-mode polarization comparable with that of the Bicep2 collaboration.

While tensor contribution to the cosmic microwave background observables can be cleanly ruled out (or ruled in) by direct observations of the B-mode polarization (as attempted by Bicep2 and by other previous experiments directly sensitive to polarization (see e.g. \([36]\)) for the present ends what matters is not the specific value of the bound but the generic order of magnitude that \( r_T \) should not exceed at a conventional pivot scale\(^{15}\) \( k_p = 0.002 \text{ Mpc}^{-1} \). From equation (3.5) the tensor-to-scalar ratio reads

\[
r_T(k_p, \alpha, n_i, \epsilon, N_i) = \pi \epsilon C(\epsilon, \alpha) n_i^{3-n_T(\epsilon, \alpha)} \epsilon^{2\alpha(3-n_T(\epsilon, \alpha))N_i} \left( \frac{k_p}{k_{\text{max}}} \right)^{n_T(\epsilon, \alpha)}.
\]

To get a superficial idea of the orders of magnitude involved we can first consider the case \( N_i = 65 \) and \( \epsilon = \mathcal{O}(10^{-2}) \). In this case we have that \( r_T < 0.1 \) provided \( 0 < \alpha < 0.13 \) for \( n_i = 1 \). A slight increase of \( n_i \) or of the total number of e-folds strengthen the limit on \( \alpha \). For instance if \( n_i = 1.5 \) we will have that \( 0 < \alpha < 0.04 \) (for \( N_i = 65 \)) and \( 0 < \alpha < 0.02 \) (for \( N_i = 70 \)). A more detailed discussion is summarized figures 1 and 2. In figure 1 all the parameters are fixed except \( n_i \) and \( \alpha \). Along each of the curves \( \log r_T \) is constant and the labels refer to the value of the common logarithm (i.e. to base 10) of \( r_T \) computed from equation (4.5). For illustration the arrows indicate the curve \( \log r_T = -1 \); the physical region, compatible with the current constraints, demands that \( \log r_T < -1 \).

In figure 1 (plot on the left) the total number of e-folds is \( N_i = 60 \), while in the plot on the right \( N_i = 65 \). As the number of e-folds increases the value of \( \alpha \) is pushed towards 0.1.

The same trend is observed in figure 2 (plot on the left) where \( r_T \) is illustrated in the \((\alpha, N_i)\) plane. As the total number of e-folds increases beyond 65, \( \alpha \) is driven towards 0.

\(^{15}\) The WMAP collaboration consistently chooses \( k_p = 0.002 \text{ Mpc}^{-1} \). The Bicep2 collaboration used \( k_p = 0.05 \text{ Mpc}^{-1} \). The first data release of the Planck collaboration assigned the scalar power spectra of curvature perturbations \( P_k \) at \( k_p = 0.05 \text{ Mpc}^{-1} \) while the tensor-to-scalar ratio \( r_T \) is assigned at \( k_p = 0.002 \text{ Mpc}^{-1} \).
Whenever $n_i$ gets smaller than 1 the parameter space in the $(\alpha, n_i)$ plane gets larger, depending on the value of $n_i$. This region has been excluded since it would lead to a superluminal phase velocity which coincides, in this case, with the group velocity. In specific situations where the group velocity does not coincide with the phase velocity, the regions $0 < n_i < 1$ might become phenomenologically viable. However in the present context we just want to focus on the most conservative situation.

Figure 1. The value of $r_T$ computed from equation (4.5) is illustrated in the $(\alpha, n_i)$ plane.

Figure 2. The value of $r_T$ is illustrated in the $(\alpha, N_i)$ plane and in the $(\alpha, \log \epsilon)$ plane.
4.3. Limits from the pulsar timing bound

The bounds on $r_T$ are derived in the hypothesis that the consistency relations between the tensor amplitude and the tensor spectral index are verified. This is not necessarily true in the present context. Thus the bounds stemming from $r_T$ might be even less stringent than the ones we just analyzed. Ultimately this is not an important limitation since the most constraining bounds, in the present situation, come from higher wavenumbers (or higher frequencies). Indeed, the pulsar timing constraint demands
where $k_{\text{pulsar}}$ roughly corresponds to the inverse of the observation time along which the pulsars’ timing has been monitored. The same strategy discussed in the previous subsection can now be applied to equation (4.6). In figures 3 and 4 we used exactly the same range of parameters already employed in figures 1 and 2.

In figures 3 and 4 we illustrate the common logarithm of $W_g$ computed from equation (3.29). Comparing figures 3 with figure 1 the values of $\alpha$ allowed by the pulsar bound are much smaller than 0.1 for the same range of variation of $n_r$. The same conclusion follows from the comparison of figures 2 and 4. The pulsar bound is systematically more constraining because when the refractive index is dynamical the energy density of the relic gravitons is increasing (rather than decreasing) for $k \gg k_{\text{eq}}$.

In figures 3 and 4 the arrow indicates, approximately, the curve where the pulsar bound is saturated.

### 4.4. Limits from the big-bang nucleosynthesis

A conclusion compatible with the pulsar bound can be drawn in the case of the big-bang nucleosynthesis constraint (BBN in what follows) stipulating that the bound on the extra-relativistic species at the time of big-bang nucleosynthesis can be translated into a bound on the cosmic graviton backgrounds. This constraint is customarily expressed in terms of $\Delta N$, representing the contribution of supplementary neutrino species but the extra-relativistic species do not need to be fermionic. If the additional species are relic gravitons we have:

$$
\Omega(k_{\text{pulsar}}, \tau_0) < 1.9 \times 10^{-8}, \quad k_{\text{pulsar}} \simeq 10^{-8} \text{ Hz},
$$

(4.6)

where $k_{\text{pulsar}}$ roughly corresponds to the inverse of the observation time along which the pulsars’ timing has been monitored. The same strategy discussed in the previous subsection can now be applied to equation (4.6). In figures 3 and 4 we used exactly the same range of parameters already employed in figures 1 and 2.

In figures 3 and 4 we illustrate the common logarithm of $\Omega_{gw}$ computed from equation (3.29). Comparing figures 3 with figure 1 the values of $\alpha$ allowed by the pulsar bound are much smaller than 0.1 for the same range of variation of $n_r$. The same conclusion follows from the comparison of figures 2 and 4. The pulsar bound is systematically more constraining because when the refractive index is dynamical the energy density of the relic gravitons is increasing (rather than decreasing) for $k \gg k_{\text{eq}}$.

In figures 3 and 4 the arrow indicates, approximately, the curve where the pulsar bound is saturated.

$$
\Omega(k_{\text{pulsar}}, \tau_0) < 1.9 \times 10^{-8}, \quad k_{\text{pulsar}} \simeq 10^{-8} \text{ Hz},
$$

(4.6)

where $h_{\text{bbn}}$ and $h_{\text{max}}$ have been computed, respectively, in equations (3.33) and (3.34); note that in equation (4.7) $\Omega_{0,0}$ denotes the present critical fraction of energy density coming just from photons. The bounds on $\Delta N$, range from $\Delta N \lesssim 0.2$ to $\Delta N \lesssim 1$. The bounds stemming from equation (4.7) can be easily inferred from figures 5 and 6. In both cases we illustrate the
common logarithm of the left-hand side of equation (4.7) computed in the case of a dynamical refractive index from equation (3.29). By comparing figures 3 and 5 we see that the pulsar and the BBN bound are largely compatible. Conversely by comparing figures 1 and 5 the big-bang nucleosynthesis bound is always more constraining than the bounds stemming from $r_f(k_p)$.

The same conclusion is reached if we compare the variation of a different set of parameters, like in figures 2, 4 and 6. Note finally that in figures 5 and 6 the arrow has been used to guide the eye towards the curve where the big-bang nucleosynthesis bound is approximately saturated.

4.5. Limits from radiation energy density at decoupling

The same logic employed for the derivation of the BBN bound can be applied at another typical epoch of the life of the Universe, namely the decoupling of matter and radiation. The main physical difference between the two approaches can be understood from equation (3.33). While the typical frequency of BBN is $\mathcal{O}(10^{-10})$ the typical frequencies of matter radiation equality is $\mathcal{O}(10^{-16})$ (see equation (3.35)). Since the decoupling between matter and radiation occurs after equality according to [18] we have that

$$h_0^2 \int_{k_{dec}}^{\overline{k_{max}}} \Omega_{GW}(k, \tau_0) \ln k \leq 8.7 \times 10^{-6}.$$ (4.8)

While the bound itself is numerically similar to the one of equation (4.7) it is certainly more constraining since $k_{dec} \ll k_{bbn}$. The plots already discussed in BBN case must therefore be reexamined in the light of equation (4.8). If we compute the same integral and compare it with the bound of equation (4.8) we obtain the results of figure 7. Figure 7 must be compared with figure 5 since the two figures share the same parameters. The dashed lines highlighted by the arrow correspond, in figure 5, to the common logarithm of $5.61 \times 10^{-6}$. The dashed lines highlighted by the arrow in figure 7 correspond instead to the common logarithm of $8.7 \times 10^{-6}$. We could have expected a bigger difference in the contour plots between the bounds of equations (4.7) and (4.8). The rationale for the obtained results can be understood.
in simple terms The frequency difference between \( k_{\text{dec}} \) and \( k_{\text{bbn}} \) affects the integral itself. But the spectral slope, for the physical range of \( \alpha \), increases. Therefore the only difference between the two bounds stems actually from the numerical upper limits of the two integrals. The same exercise performed in the case of figure 5 could be repeated for the parameters of figure 6 with analog results. For the sake of conciseness we shall not report the corresponding plots.

4.6. Prospects for wide-band detectors

The bounds examined in the previous subsections suggest that for the standard fiducial values of \((N_\text{t}, \epsilon)\) the values of \( n_1 \) and \( \alpha \) are constrained to be within the following window:

\[
1 < n_1 < 10, \quad 0 < \alpha < 0.07.
\] (4.9)

The sensitivity of a given pair of wide-band detectors to a stochastic background of relic gravitons depends upon the relative orientation of the instruments (see e.g. \([37, 38]\)) and a specific analysis of the signal-to-noise ratio is beyond the scope of this paper. While the advanced version of the wide-band interferometers is still a matter of debate (and the published results do not seem conclusive at the moment) the frequency window of the detectors will always be between few Hz (where the seismic noise dominates) and 10 kHz (where the shot noise eventually dominates). The wideness of the band (important for the correlation among different instruments) is not as large as 10 kHz but much narrower. There are projects of wide-band detectors in space like the Lisa and the Bbo/Decigo. The common feature of these three projects is that they are all space-borne missions; the Lisa interferometer should operate between \(10^{-4}\) and 0.1 Hz. Nominally the Decigo project will be instead sensitive to frequencies between 0.1 and 10 Hz.

Recalling the results obtained so far we can safely say that growing spectra can arise in the following range:
where the lower bound of equation (4.10) has been derived after equation (3.8) while the upper bound follows from equation (4.3). Since the upper limit of equation (4.10) is larger than that of equation (4.9) we can conclude that growing spectra arise in practice in the whole range of variation of $\alpha$. Using the illustrative example discussed before we have that for $k_{\text{L}i\alpha} = \mathcal{O}(10^{-3})$ Hz (i.e. compatible with the Lisa window) we would have $\Omega_{\text{gw}}(\text{mHz}) \simeq 10^{-8.21}$ for $\alpha = 0.06$, $N_{i} = 65$, $\epsilon = 0.001$ and $n_{i} = 1$. Larger values of $n_{i}$ and $N_{i}$ make the signal smaller. For the Ligo/Virgo frequencies and for the same parameters chosen in the Lisa case we would have instead $\Omega_{\text{gw}}(0.1 \text{ kHz}) \simeq 10^{-7.69}$ where $k_{\text{L}i\alpha} = 0.1 \text{ kHz}$.

This trend is confirmed by the results illustrated in figure 8 where we report the common logarithm of $\Omega_{\text{gw}}$ in the Lisa window (plot on the left) and in the Ligo/Virgo window (plot on the right). The dashed lines in both plots corresponds to the common logarithm of the big-bang nucleosynthesis bound in the $\Delta N_{i} = 1$ case (i.e. $\log(5.61 \times 10^{-6}) = -5.25$). The allowed region of the parameter space must be, in both plots of figure 8, below the dashed lines. While the noise power spectra of Lisa are still rather hypothetical, the advanced version of terrestrial interferometers might get down to $10^{-10}$ in $\Omega_{\text{gw}}$ after an appropriate integration time. The shaded area in figure 8 illustrates the allowed region in the parameter space where the relic gravitons are potentially detectable. As already mentioned we shall not dwell here on the detectability prospects in this paper. The value $\Omega_{\text{gw}} = \mathcal{O}(10^{-10})$ has been mainly quoted to guide the eye and whether this sensitivity will be in fact reached by terrestrial of space-borne detectors is an entirely different issue. If this value will be indeed reached by cross-correlation of two instruments more detailed discussions could be necessary (see e.g. [37, 38] and discussions therein).

Figure 8. The energy density of the relic gravitons is illustrated in the $(\alpha, n_{i})$ plane in the Lisa window (plot on the left) and in the Ligo/Virgo window (plot on the right).
It is finally appropriate to mention that the maximal signal due to the variation of the refractive index occurs in a frequency region between the MHz and the GHz. In this range of frequencies microwave cavities or even wave guides \[^{39-41}\] can be used as detectors of gravitational waves.

5. Concluding remarks

The continuous and differentiable evolution of the refractive index leads to power spectra and spectral energy densities of the relic gravitons that are slightly increasing as a function of the comoving wavenumber (or of the comoving frequency). While the rate of variation of the refractive index can be stringently bounded, the derived limits do not exclude the potential relevance of the cosmic graviton background for either ground-based or space-borne interferometers aimed at a direct detection of gravitational waves. The spectral slopes are determined by the competition of the slow-roll parameter against \(\alpha\) which measures the rate of variation of the refractive index in units of the Hubble rate. The phenomenologically allowed region implies that \(1 \leq n_i < 10\) and \(0 < \alpha < 0.07\), where \(n_i\) denotes the value of the refractive index at the onset of the inflationary expansion.

The sensitivity of wide-band detectors\(^\text{17}\) to the relic graviton backgrounds is customarily expressed in terms of the minimal detectable spectral energy density of the relic gravitons in critical units. The operating windows of the ground-based and space-borne interferometers are complementary: the typical frequency range of space-borne interferometers extends between \(\nu = -8.74\) Hz and 10 kHz \(\approx 10^4\) Hz, and a few Hz. Conversely, the window of ground-based detectors extends between a few Hz (where seismic noise dominates) and 10 kHz \(= 10^4\) Hz (where shot noise dominates). While the time scales for the realization of space-borne interferometers

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\(\nu\) & \(\log[\Omega_{gw}]\) & \(\log[S_hHz]\) & \(\log[\sqrt{S_hHz}]\) \\
\hline
10^{-4}\ Hz & -8.15 & -32.56 & -16.28 \\
10^{-3}\ Hz & -7.94 & -38.35 & -19.17 \\
1 Hz & -7.74 & -44.14 & -22.07 \\
10^2 Hz & -7.53 & -49.93 & -24.96 \\
10^3 Hz & -7.32 & -55.73 & -27.86 \\
10^4 Hz & -7.11 & -61.52 & -30.76 \\
10^5 Hz & -6.90 & -67.31 & -33.65 \\
10^6 Hz & -6.69 & -73.10 & -22.07 \\
\hline
\end{tabular}
\caption{Typical values of the spectral energy density and of the strain amplitude in the operating windows of ground-based and space-borne interferometers. Consistently with the discussion of section 4 the parameters \((n_i, \alpha, \epsilon, N)\) have been chosen to be \((1, 0.06, 0.001, 65)\).}
\end{table}

It is finally appropriate to mention that the maximal signal due to the variation of the refractive index occurs in a frequency region between the MHz and the GHz. In this range of frequencies microwave cavities or even wave guides \[^{39-41}\] can be used as detectors of gravitational waves.

\[^{17}\text{We recall, for the sake of precision, that the expression of the signal-to-noise ratio in the context of optimal processing required for the detection of stochastic backgrounds \[^{37, 38}\] depends on an integral over the frequency band (between few Hz and 10 kHz in the case of the ground-based interferometers of Ligo-type). The numerator of the integrand contains }\left[\dot{\Omega}_{gw}\right]^{\nu} \text{ while in the denominator we have the sixth power of the frequency multiplied by the noise power spectra of each of the two correlated detectors. The signal-to-noise ratio depends also on the relative orientation of the interferometers and on the total observation time which is crucial to increase the sensitivity. Naively, if the minimal detectable signal (by one detector) is }h_0^2\Omega_{gw}, \text{ then the cross-correlation of two identical instruments might increase the sensitivity by a factor }1/\sqrt{\Delta\nu T} \text{ where }\Delta\nu \text{ is the bandwidth and }T, \text{ as already mentioned, is the observation time. Therefore if a single instrument detects }h_0^2\Omega_{gw} \approx 10^{-6} \text{ the correlation may detect }h_0^2\Omega_{gw} \approx 10^{-10} \text{ provided }\Delta\nu \approx 100 \text{ Hz and }T \approx O(1yr) \[10].\]
interferometers are still vague it is useful to illustrate our findings by keeping ground-based detectors and space-borne interferometers on equal footing. In table 1 the frequencies encompass the operating ranges of space-borne and ground-based detectors. In the first column we illustrate the common logarithm of $\Omega_{gw}$. In the two remaining columns we illustrate the common logarithms of the strain amplitude $\Sigma$ and of its square root. In table 1 the range of frequencies has been extended well beyond the window of ground-based interferometers. The last frequency, i.e. 10 GHz, is even larger than $\nu_{\text{max}} = k_{\text{max}}/(2\pi) = O(\text{GHz})$ and we just included it to show that the spectral energy density is always well below the constraints previously discussed.

If the correlation of (advanced) wide-band interferometers will eventually reach sensitivities $O(10^{-10})$ in $\Omega_{gw}$, the present considerations might become relevant since the signal due to a dynamical refractive index with $\alpha = O(0.06)$ and $n_i = O(1)$ can even be $O(10^{-7.53})$ for the phenomenologically allowed region of the parameter space and for a typical frequency $O(0.1 \text{ kHz})$. Even if the noise power spectra and the specific features of the hypothetical space-borne interferometers are still unclear, a potential sensitivity $O(10^{-8.05})$ in $\Omega_{gw}$ cannot be excluded for a typical frequency $O(\text{mHz})$. The lack of detection of a stochastic background of relic gravitons either by ground-based detectors or by space-borne interferometers may therefore provide a further (and potentially much more stringent) bound on the rate of variation of the refractive index.

**Appendix A. Second order actions**

It is useful to start this discussion by noticing that equations (1.5) and (1.6) are indeed related by simple field redefinitions and up to total time derivatives. More precisely, let us consider, for the sake of concreteness, equation (1.5), and let us redefine the tensor amplitude as $H_{ij} = a h_{ij}$. After inserting this field redefinition in the action we obtain the following expression:

$$ S = \frac{1}{8\ell_p^2} \int d^3x \int d\tau \left[ \partial_i H_{ij} \partial^i H_{ij} - 2 \mathcal{H} H_{ij} \partial_i H_{ij} + \mathcal{H}^2 H_{ij} H_{ij} - \frac{1}{n^2(\tau)} \partial_i H_{ij} \partial^i H_{ij} \right] $$ (A.1)

Clearly the term $-2 \mathcal{H} H_{ij} \partial_i H_{ij}$ can be integrated by parts so that the result of equation (A.1) will be that of equation (1.6) with $A(\tau) = 1$, $B(\tau) = 1/n^2(\tau)$ and $C(\tau) = (\mathcal{H}^2 + \partial_i \mathcal{H})$. Another plausible field redefinition could be $h_{ij} = n(\tau) H_{ij}$. Repeating the same analysis we shall obtain, again, an action falling into the equivalence class defined by equation (1.6).

The second-order actions for the relic gravitons can take different forms depending on the field redefinitions which also produce terms that can be integrated by parts. Having established this point we now want to perturb to second order in the amplitude of the tensor modes the action (1.7) and show that the obtained action has the same form of equations (1.5) and (1.6), up to field redefinition. We recall that the metric fluctuations to first and second order are given by

$$ \delta^{(1)} g_{ij} = -a^2 h_{ij}, \quad h^i = \partial_i h^j = 0, \quad (A.2) $$

18 The sensitivity is often expressed by means of $S_0$ or in terms of its square root. The precise relation between $S_0$ and $\Omega_{gw}$ is given by $S_0(\nu; \tau_0) = 7.981 \times 10^{-11} (100 \text{ Hz}/\nu)^3 h_0^2 \Omega_{gw}(\nu; \tau_0) \text{Hz}^{-1}$. Note that $S_0$ is measured in $1/\text{Hz} = \text{sec}$. 

25
\[ \delta^{(1)} g^{ij} = \frac{\hbar^i}{a^2}, \quad \delta^{(2)} g^{ij} = -\frac{h_i^j h^i}{a^2}, \quad (A.3) \]

where \( \delta^{(1)} \) and \( \delta^{(2)} \) denote, respectively, the first and second order (tensor) fluctuation of the corresponding quantity. The tensor modes discussed here correspond, obviously, to the divergenceless and traceless rank-two tensors in three-dimensions \( h_{ij} \). Since the fluctuations of the Christoffel connections to first and second order are:

\[
\begin{align*}
\delta^{(1)} \Gamma^0_{ij} &= \frac{1}{2} \left[ \partial_j h_{ij} + 2 \partial(h_{ij}) \right], \\
\delta^{(1)} \Gamma^k_{ij} &= \frac{1}{2} \left[ \partial_j h^k_{ij} + \partial_i h^k_{ij} - \partial^k h_{ij} \right], \\
\delta^{(2)} \Gamma^0_{ij} &= -\frac{1}{2} h^k \partial_j h_{ij}, \\
\delta^{(2)} \Gamma^k_{ij} &= \frac{1}{2} h^k \left[ \partial_j h_{ijk} - \partial_k h_{ij} - \partial_i h_{jk} \right], \quad (A.4)
\end{align*}
\]

it is straightforward but rather lengthy to compute \( \delta^{(2)} S_a \). For this purpose let us first remark that equation \( (1.7) \) can also be written as

\[
\delta^{(2)} S_a = \frac{1}{2 \ell^2} \int \! d^4x \sqrt{-g} \left[ R - p \ u^\mu u^\nu R_{\mu\nu} \right], \quad u^\mu = \frac{\partial \varphi}{M \sqrt{p}}, \quad p = \frac{1}{M^2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi, \quad (A.5)
\]

where \( g^{\alpha\mu} u_\mu u_\nu = 1 \). Since, by definition, \( \varphi \) has a homogeneous field (and not necessarily dominant) the second order action becomes

\[
\delta^{(2)} S_a = \frac{1}{8 \ell^2} \int \! d^4x \{ (1 + p) a^2 \partial_j h_{ij} \partial_j h_{ij} - a^2 \partial_j h_{ij} \partial_k h_{ij} \\
+ \frac{pa^2}{4} \left[ 2 \mathcal{H}^2 + 3 \mathcal{H} \frac{\partial p}{p} + 2 \left( \frac{\partial^2 p}{p^2} \right) - \partial^2 h_{ij} \partial_j h_{ij} \right]. \quad (A.6)
\]

To obtain equation \( (A.6) \) it is easier to perturb separately the two terms of equation \( (A.5) \). The first term is standard and can be immediately expressed in a form where the known surface term is already absent, namely,

\[
\int \! d^4x \sqrt{-g} R = \int \! d^4x \sqrt{-g} g^{\alpha\beta} \left( \Gamma^\mu_{\alpha\beta} \Gamma^\nu_{\mu \nu} - \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} \right). \quad (A.7)
\]

The second order fluctuation of equation \( (A.7) \) can be formally given by:

\[
\begin{align*}
\int \! d^4x \sqrt{-g} & \left[ \delta^{(2)} g^{\alpha\beta} (\Gamma^\mu_{\alpha\beta}) \Gamma^\nu_{\mu \nu} - \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} \right] \\
+ g^{\alpha\beta} \delta^{(1)} \Gamma^\mu_{\alpha\beta} \Gamma^\nu_{\mu \nu} + \Gamma^\mu_{\alpha \nu} \delta^{(1)} \Gamma^\nu_{\mu \beta} - \delta^{(1)} \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} - \Gamma^\mu_{\alpha \nu} \delta^{(1)} \Gamma^\nu_{\mu \beta} \\
+ \delta^{(1)} \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} + \Gamma^\mu_{\alpha \nu} \delta^{(1)} \Gamma^\nu_{\mu \beta} - \delta^{(1)} \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} - \Gamma^\mu_{\alpha \nu} \delta^{(1)} \Gamma^\nu_{\mu \beta} \\
+ \delta^{(2)} \sqrt{-g} \left[ g^{\alpha\beta} (\Gamma^\mu_{\alpha\beta}) \Gamma^\nu_{\mu \nu} - \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} \right] \\
+ \delta^{(1)} \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} - \delta^{(1)} \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} \\
+ \delta^{(1)} \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} - \delta^{(1)} \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\mu \beta} \right], \quad (A.8)
\end{align*}
\]

where \( g^{\alpha\beta} \) and \( \Gamma^\mu_{\alpha \beta} \) are, respectively, the background values of the metric and of the Christoffel connections. Using now equations \( (A.2) \) and \( (A.4) \) inside equation \( (A.8) \) the
explicit expression of equation (A.6) follows, after simple algebra, by recalling that
\[
\delta_{(2)}^{(2)}R_{\mu\nu} = \frac{1}{4} \partial_{\alpha} h_{\beta\gamma} \partial_{\mu} h_{\nu} + \frac{H}{2} h_{\beta\gamma} \partial_{\mu} h_{\nu} + \frac{1}{2} \partial_{(1)}^{2} h_{\mu\nu} h_{\alpha\beta}.
\]  
(A.9)

Indeed the only second order fluctuation of the four-dimensional Ricci tensor contributing to the second term inside the square bracket in equation (1.7) is exactly given by equation (A.9). As discussed in the bulk of the paper and at the beginning of this appendix the tensor modes can be freely redefined with the purpose of canonically normalizing the term containing the two time derivatives of the tensor amplitude of equation (A.6). The same procedure outlined in this discussion can be used to perturb the action of equation (1.8) containing the Gauss–Bonnet contribution.

Appendix B. Mixing coefficients during the radiation phase

The explicit expression of the mixing coefficients appearing in equation (3.10) can be written as
\[
c_{+}(k, \tau) = -\frac{\pi i}{4(1 - \alpha)} \left[ \sqrt{\beta} \ q H_{\mu}^{(1)}(\epsilon_{\gamma}) A_{\mu}^{(1)}(\epsilon_{\rho}) - \frac{q}{\sqrt{\beta}} H_{\mu}^{(2)}(\epsilon_{\gamma}) B_{\mu}^{(1)}(\epsilon_{\rho}) \right],
\]  
(B.1)
\[
c_{-}(k, \tau) = \frac{\pi i}{4(1 - \alpha)} \left[ \sqrt{\beta} \ q H_{\mu}^{(1)}(\epsilon_{\gamma}) A_{\mu}^{(2)}(\epsilon_{\rho}) + \frac{q}{\sqrt{\beta}} H_{\mu}^{(2)}(\epsilon_{\gamma}) B_{\mu}^{(1)}(\epsilon_{\rho}) \right],
\]  
(B.2)

where \( q = \sqrt{|1 - \alpha|/|1 + \alpha \beta|} \). In equations (B.1) and (B.2) the following notation have been employed:
\[
g_{\rho}(-\gamma) = \epsilon_{\rho} = \frac{k \gamma}{n[1 + \alpha \beta]}, \quad g_{\rho}(-\gamma) = \epsilon_{\rho} = \frac{k \gamma}{n[1 - \alpha \beta]}.
\]  
(B.3)

Furthermore, in equations (B.1) and (B.2) \( A_{\mu}^{(1)}(\epsilon_{\rho}) \) and \( B_{\mu}^{(2)}(\epsilon_{\rho}) \) are two auxiliary expressions defined as:
\[
A_{\mu}^{(1)}(\epsilon_{\rho}) = H_{\mu}^{(1)}(\epsilon_{\rho}) \left[ (1 - \alpha) \rho + \frac{1}{2} \right] - \epsilon_{\rho} (1 - \alpha) H_{\mu+1}^{(1)}(\epsilon_{\rho}),
\]  
(B.4)
\[
B_{\mu}^{(2)}(\epsilon_{\rho}) = H_{\mu}^{(1)}(\epsilon_{\rho}) \left[ (1 + \alpha \beta) \mu + \frac{1}{2} \right] - \epsilon_{\rho} (1 + \alpha \beta) H_{\mu+1}^{(1)}(\epsilon_{\rho}),
\]  
(B.5)

where, following the properties of the Hankel functions under complex conjugation, we shall have \( A_{\mu}^{(1)}(\epsilon_{\rho}) = A_{\mu}^{(1)}(\epsilon_{\rho}) \) and \( B_{\mu}^{(2)}(\epsilon_{\rho}) = A_{\mu}^{(1)}(\epsilon_{\rho}) \). In more explicit terms equations (B.1) and (B.2) can be written as:
\[
c_{+}(k, \tau) = -\frac{k \pi \beta}{4(1 - \alpha)} \left[ \frac{|1 - \alpha|}{|1 + \alpha \beta|} \left\{ H_{\mu}^{(1)}(\epsilon_{\rho}) H_{\mu}^{(1)}(\epsilon_{\rho}) \right\} \right.
\times \left[ \frac{\beta + 1}{2} + (1 - \alpha) \rho + \frac{\mu(1 + \alpha \beta)}{\beta} \right]
\left. - (1 - \alpha) \epsilon_{\rho} H_{\mu}^{(1)}(\epsilon_{\rho}) H_{\mu+1}^{(1)}(\epsilon_{\rho}) - \frac{(1 + \alpha \beta)}{\beta} \epsilon_{\rho} H_{\mu}^{(1)}(\epsilon_{\rho}) H_{\mu+1}^{(1)}(\epsilon_{\rho}) \right].
\]  
(B.6)
As it can be explicitly verified from equations (B.6) and (B.7), $|c_+ (k, \tau_1)|^2 = |c_-(k, \tau_1)|^2 = 1$.

It is useful to mention that equations (B.6) and (B.7) reproduce exactly the standard results in the limit $\alpha \to 0$, $n_i \to 1$ and $\beta \to 1$. More specifically this limit refers to the situation where there is a transition from an exact de Sitter phase (i.e. $\epsilon \to 0$ and $\mu \to 3/2$) to a conventional radiation-dominated phase where $\rho \to 1/2$. In the standard limit we also have that $\bar{g}_s \to k_\tau$ and $\bar{g}_r \to k_\tau$ and, from the explicit form of equations (B.6) and (B.7),

$$c_+(x_1) = \frac{i}{2 \pi_x} [2 \eta (x_1 + i) - 1] e^{2i \alpha}, \quad c_-(x_1) = -\frac{i}{2 \pi_x}; \quad (B.8)$$

in this case the mixing coefficients depend on the single argument $x_1 = k_\tau$.

### Appendix C. Mixing coefficients during the matter phase

The explicit expression of the mixing coefficients appearing in equation (3.20) can be written as

$$d_+(k, \tau_1, \tau_2) = -\frac{i \pi}{4 \sqrt{2} (1 - 2 \alpha)} \sqrt{\frac{1 - 2 \alpha}{1 - \alpha}} \left[ c_+(k, \tau_1) \times [H^{(2)}_{\rho}(\bar{g}_s) \mathcal{F}^{(1)}_{\sigma}(\bar{g}_m) - 2 H^{(1)}_{\sigma}(\bar{g}_m) \mathcal{G}^{(2)}_{\rho}(\bar{g}_s)] + c_-(k, \tau_1) [H^{(2)}_{\rho}(\bar{g}_s) \mathcal{F}^{(1)}_{\sigma}(\bar{g}_m) - 2 H^{(1)}_{\sigma}(\bar{g}_m) \mathcal{G}^{(2)}_{\rho}(\bar{g}_s)] \right], \quad (C.1)$$

$$d_-(k, \tau_1, \tau_2) = -\frac{i \pi}{4 \sqrt{2} (1 - 2 \alpha)} \sqrt{\frac{1 - 2 \alpha}{1 - \alpha}} \left[ c_+(k, \tau_1) [H^{(2)}_{\rho}(\bar{g}_s) \mathcal{F}^{(1)}_{\sigma}(\bar{g}_m) - 2 H^{(1)}_{\sigma}(\bar{g}_m) \mathcal{G}^{(2)}_{\rho}(\bar{g}_s)] + c_-(k, \tau_1) [H^{(2)}_{\rho}(\bar{g}_s) \mathcal{F}^{(1)}_{\sigma}(\bar{g}_m) - 2 H^{(1)}_{\sigma}(\bar{g}_m) \mathcal{G}^{(2)}_{\rho}(\bar{g}_s)] \right], \quad (C.2)$$

where $\bar{g}_m$ and $\bar{g}_r$ are defined as:

$$g_m(\tau_2) = \bar{g}_m = \frac{2 k_\tau_2}{n_1 (1 - 2 \alpha)} \left( \frac{\tau_1}{\beta_\tau_2} \right)^{\nu} \left[ 1 + \frac{\beta + 1}{\beta} \left( \frac{\tau_1}{\tau_2} \right) \right]^{1 - \alpha}, \quad g_r(\tau_2) = \bar{g}_r = \frac{1 - 2 \alpha}{2 (1 - \alpha)} \bar{g}_m \quad (C.3)$$

In equations (C.1) and (C.2) the following auxiliary functions have been introduced:

$$\mathcal{F}^{(1)}_{\sigma}(\bar{g}_m) = \left[ \frac{1}{2} + \sigma (1 - 2 \alpha) \right] H^{(1)}_{\sigma}(\bar{g}_m) - (1 - 2 \alpha) \bar{g}_m H^{(1)}_{\rho + 1}(\bar{g}_m), \quad (C.4)$$

$$\mathcal{G}^{(2)}_{\rho}(\bar{g}_s) = \left[ \frac{1}{2} + \rho (1 - \alpha) \right] H^{(1)}_{\rho}(\bar{g}_s) - (1 - \alpha) \bar{g}_s H^{(1)}_{\rho + 1}(\bar{g}_s), \quad (C.5)$$

and furthermore, following the standard notations, we have that $\mathcal{F}^{(1)}_{\sigma}(\bar{g}_m) = \mathcal{F}^{(1)}_{\sigma}(\bar{g}_s)$ and $\mathcal{G}^{(2)}_{\rho}(\bar{g}_s) = \mathcal{G}^{(2)}_{\rho}(\bar{g}_r)$. Using equations (C.4) and (C.5) the relation $|d_+|^2 - |d_-|^2 = 1$ can be explicitly verified. As in appendix B it is useful to investigate the specific limit
\[ \alpha \to 0, \beta \to 1. \] In this case, defining \( x_1 = k_\eta \) and \( x_2 = k_\gamma \) the exact form of \( d_{\alpha}(x_1, x_2) \) can be written as:

\[ d_{\alpha}(x_1, x_2) = \frac{e^{i\frac{x_1}{x_1^2}}}{16x_1^4} \left[ e^{-2ix_2} + e^{2i\eta}[8x_2^2 + 4ix_2 - 1][2\eta(x_1 + i) - 1}\right], \quad (C.6) \]

\[ d_{\alpha}(x_1, x_2) = \frac{e^{-3ix_2}}{16x_1^4} \left[ e^{2i\eta}[8x_2^2 - 4ix_2 - 1] + e^{2i\eta}[1 - 2\eta(i + x_1)]\right]. \quad (C.7) \]

The expressions of equations (C.6) and (C.7) can be expanded in order to derive approximate expressions of the transfer function of the relic graviton spectrum across the radiation–matter transition. The exact results for the mixing coefficients have been used as a systematic cross-check: the power spectra and the energy density of the gravitons in the case of a dynamical refractive index are complicated functions of \( \alpha \) which must anyway reduce to known expressions in the \( \alpha \to 0 \) limit. This is not only true for the exact expressions but also for the corresponding approximated mixing coefficients computed in the limits \( |k_\eta| < 1 \) and \( |k_\gamma| < 1 \).

References

[1] Szekeres P 1971 Ann. Phys. 64 599
Peters P C 1974 Phys. Rev. D 9 2207

[2] Grishchuk L P 1975 Sov. Phys. JETP 40 409
Grishchuk L P 1974 Zh. Eksp. Teor. Fiz. 67 825
Grishchuk L P 1977 Ann. N. Y. Acad. Sci. 302 439

[3] Rubakov V A, Sazhin M V and Veryaskin A V 1982 Phys. Lett. B 115 189
Allen B 1988 Phys. Rev. D 37 2078
Sahni V 1990 Phys. Rev. D 42 453
Grishchuk L P and Solokhin M 1991 Phys. Rev. D 43 2566
Gasperini M and Giovannini M 1992 Phys. Lett. B 282 36
Gasperini M and Giovannini M 1993 Phys. Rev. D 47 1519

[4] Turner M S, White M J and Lidsey J E 1993 Phys. Rev. D 48 4613
Brustein R, Gasperini M, Giovannini M and Veneziano G 1995 Phys. Lett. B 361 45

[5] Giovannini M 1998 Phys. Rev. D 58 083504
Giovannini M 1999 Phys. Rev. D 60 123511
Giovannini M 1999 Class. Quantum Grav. 16 2905
Babusci D and Giovannini M 1999 Phys. Rev. D 60 083511

[6] Boyle L A, Steinhardt P J and Turok N 2004 Phys. Rev. D 69 127302
Zhao W and Zhang Y 2006 Phys. Rev. D 74 043503
Zhang Y, Zhao W, Xia T and Yuan Y 2006 Phys. Rev. D 74 083006

[7] Watanabe Y and Komatsu E 2006 Phys. Rev. D 73 123515
Chongchitnan S and Efstathiou G 2006 Phys. Rev. D 73 083511
Chongchitnan S and Efstathiou G 2006 Prog. Theor. Phys. Suppl. 163 204

[8] Giovannini M 2009 Class. Quantum Grav. 26 045004
Giovannini M 2008 Phys. Lett. B 668 44
Giovannini M 2010 Phys. Rev. D 82 083523
Giovannini M 2014 Class. Quantum Grav. 31 225002

[9] Giovannini M 1999 Phys. Rev. D 59 121301
Cataldo M and Mella P 2006 Phys. Lett. B 642 5

[10] Spergel D N et al 2003 Astrophys. J. Suppl. 148 175
Spergel D N et al 2007 Astrophys. J. Suppl. 170 377
Page L et al 2007 Astrophys. J. Suppl. 170 335

[11] Gold B et al 2011 Astrophys. J. Suppl. 192 15
Larson D et al 2011 Astrophys. J. Suppl. 192 16
Bennett C L et al 2011 Astrophys. J. Suppl. 192 17
Hinshaw G et al 2013 Astrophys. J. Suppl. 208 19
Bennett C L et al 2013 Astrophys. J. Suppl. 208 20

[14] Ade P A R et al (Planck Collaboration) 2014 Astron. Astrophys. 571 A22
Ade P A R et al (Planck Collaboration) 2014 Astron. Astrophys. 571 A16
Ade P A R et al (Planck Collaboration) 2015 arXiv:1502.02114 [astro-ph.CO]
Ade P A R et al (Planck Collaboration) 2015 arXiv:1502.01594 [astro-ph.CO]

[15] Kaspi V M, Taylor J H and Ryba M F 1994 Astrophys. J. 428 713

[16] Jenet F A et al 2006 Astrophys. J. 653 1571
Demorest P B et al 2013 Astrophys. J. 762 94

[17] Schwartzmann V F 1969 JETP Lett. 9 184
Giovannini M, Kurki-Suonio H and Sihvola E 2002 Phys. Rev. D 66 043504
Cyburt R H, Fields B D, Olive K A and Skillman E 2005 Astropart. Phys. 23 313

[18] Smith T L, Pierpaoli E and Kamionkowski M 2006 Phys. Rev. Lett. 97 021301
Sendra I and Smith T L 2012 Phys. Rev. D 85 123002

[19] Abbott B et al (LIGO Collaboration) 2007 Astrophys. J. 659 918
Abbott B et al (ALLEGRO Collaboration and LIGO Scientific Collaboration) 2007 Phys. Rev. D 76 022001

Cella G, Colacino C N, Cuoco E, Di Virgilio A, Regimbau T, Robinson E L and Whelan J T 2007 Class. Quantum Grav. 24 S639

[20] Abbott B P et al (LIGO Scientific and VIRGO Collaborations) 2009 Nature 460 990
Abadie J et al (LIGO Scientific and VIRGO Collaborations) 2012 Phys. Rev. D 85 122001
Aasi J et al (LIGO Scientific and VIRGO Collaborations) 2014 arXiv:1406.4556 [gr-qc]

[21] Lück H et al 1997 Class. Quantum Grav. 14 1471
Vahlbruch H, Khalaidovski A, Lastzka N, Graf C, Danzmann K and Schnabel R 2010 Class. Quantum Grav. 27 084027

[22] Somiya K and (KAGRA Collaboration) 2012 Class. Quantum Grav. 29 124007
Aso Y et al (KAGRA Collaboration) 2013 Phys. Rev. D 88 043007

[23] Ando M et al 2001 Phys. Rev. Lett. 86 3950

[24] Hughes S A 2002 Mon. Not. R. Astron. Soc 331 805
Hughes S A 2007 arXiv:0711.0188 [gr-qc]

[25] Corbin V and Cornish N J 2006 Class. Quantum Grav. 23 2435
Harry G M, Fritschel P, Shaddock D A, Folkner W and Phinney E S 2006 Class. Quantum Grav. 23 4887
Harry G M, Fritschel P, Shaddock D A, Folkner W and Phinney E S 2006 Class. Quantum Grav. 23 7361 (erratum)

Kawamura S et al 2008 J. Phys. Conf. Ser. 120 032004

[26] Ford L H and Parker L 1977 Phys. Rev. D 16 1601
Ford L H and Parker L 1977 Phys. Rev. D 16 245
Hu B L and Parker L 1977 Phys. Lett. A 63 217

[27] Landau L D and Lifshitz E M 1971 The Classical Theory of Fields (New York: Pergamon)

Isaacson R 1968 Phys. Rev. 166 1263
Isaacson R 1968 Phys. Rev. 166 1272

[29] Abramov R L 1999 Phys. Rev. D 60 064004
Babak S V and Grishchuk L P 2000 Phys. Rev. D 61 024038
Giovannini M 2006 Phys. Rev. D 73 083505
Su D and Zhang Y 2012 Phys. Rev. D 85 104012
Giovannini M 2015 Phys. Rev. D 91 023521

[30] Weinberg S 2004 Phys. Rev. D 69 023503
Dicus D A and Repko W W 2005 Phys. Rev. D 72 088302
Boyle L A and Steinhardt P J 2008 Phys. Rev. D 77 063504
Watanabe Y and Komatsu E 2006 Phys. Rev. D 73 123515

[31] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions (New York: Dover)
[32] Erdelyi A, Magnus W, Oberhettinger F and Tricomi F 1953 Higher Transcendental Functions (New York: McGraw-Hill)
[33] Liddle A R and Leach S M 2003 Phys. Rev. D 68 103503
[34] Hanson D et al (SPTpol Collaboration) 2013 Phys. Rev. Lett. 111 141301
[35] Ade P A R et al (BICEP2 Collaboration) 2014 Phys. Rev. Lett. 112 241101
[36] Ade P et al (QUaD Collaboration) 2008 Astrophys. J. 674 22
Pryke C et al (QUaD Collaboration) 2009 Astrophys. J. 692 1247
Bischoff C et al (QUaD Collaboration) 2011 Astrophys. J. 741 111
Araujo D et al (QUaD Collaboration) 2012 Astrophys. J. 760 145

[37] Babusci D and Giovannini M 2000 Class. Quantum Grav. 17 2621
Babusci D and Giovannini M 2001 Int. J. Mod. Phys. D 10 477

[38] Saulson P R 1994 Fundamentals of Interferometric Gravitational Wave Detectors (Singapore: World Scientific)

[39] Pegoraro F, Radicati L A, Bernard P and Picasso E 1978 Phys. Lett. A 68 165
Reece C E, Reiner P J and Melissinos A C 1986 Nucl. Instrum. Methods A 245 299
Reece C E, Reiner P J and Melissinos A C 1984 Phys. Lett. A 104 341
Bernard P, Gemme G, Parodi R and Picasso E 2001 Rev. Sci. Instrum. 72 2428
Ballantini R, Bernard P, Chincarini A, Gemme G, Parodi R and Picasso E 2004 Class. Quantum Grav. 21 S1241

[40] Cruise A M 2000 Class. Quantum Grav. 17 2525
Cruise A M and Ingley R M 2005 Class. Quantum Grav. 22 S479
Cruise A M and Ingley R M 2006 Class. Quantum Grav. 23 6185

[41] Li F Y, Tang M X and Shi D P 2003 Phys. Rev. D 67 104008
Li F Y, Wu Z H and Zhang Y 2003 Chin. Phys. Lett. 20 1917
Nishizawa A et al 2008 Phys. Rev. D 77 022002