The optimality of word lengths. Theoretical foundations and an empirical study.

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Zipf’s law of abbreviation, namely the tendency of more frequent words to be shorter, has been viewed as a manifestation of compression, i.e. the minimization of the length of forms – a universal principle of natural communication. Although the claim that languages are optimized has become trendy, attempts to measure the degree of optimization of languages have been rather scarce. Here we present two optimality scores that are dually normalized, namely, they are normalized with respect to both the minimum and the random baseline. We analyze the theoretical and statistical pros and cons of these and other scores. Harnessing the best score, we quantify for the first time the degree of optimality of word lengths in languages. This indicates that languages are optimized to 62 or 67 percent on average (depending on the source) when word lengths are measured in characters, and to 65 percent on average when word lengths are measured in time. In general, spoken word durations are more optimized than written word lengths in characters. Our work paves the way to measure the degree of optimality of the vocalizations or gestures of other species, and to compare them against written, spoken, or signed human languages.
1. Introduction

Language universals – properties that apply to all languages on Earth – are “vanishingly few”, as exceptions can often be found [1]. Some of the promising candidates are linguistic laws [2,3], though these have taken a rather secondary role in mainstream linguistics since Zipf’s pioneering research. One of the most robust patterns is Zipf’s law of abbreviation: more frequent words tend to be shorter [2,4]. It has been claimed that there is an even stronger law which relates the length of a word with its context of occurrence [5], but further research has demonstrated that this new law is weaker and supported in fewer languages than expected [6–8]. Therefore, Zipf’s law of abbreviation is, at present, one of the strongest laws of language in terms of theoretical understanding [9–11], and in terms of empirical support both across languages [4] and in other species [3]. Here we aim to shift the main focus from the surface of linguistic behavior (e.g. linguistic laws), and the all-or-nothing logics of absolute universals [1], to the principles underlying a general theory of natural communication [3,12,13]. In this setting, the principles that govern communication across species are the true universals which manifest themselves in production (potentially with exceptions), and may require specific experimental conditions to appear [14,15]. This shift of focus requires a split between principles on one hand, and their manifestations on the other, as well as a theoretical understanding of when and why the manifestations of a principle reach the surface of the observable world. Since Zipf’s pioneering research, the law of abbreviation has been argued to be a manifestation of word length minimization [2,10,16], currently known as the principle of compression [3,16]. In its simplest form, the principle of compression can be formulated as the minimization of the average length of tokens from a repertoire of \( n \) types, that is defined as

\[
L = \sum_{i=1}^{n} p_i l_i, \tag{1.1}
\]

where \( p_i \) and \( l_i \) are, respectively, the probability and the length of the \( i \)-th type. In practical applications, \( L \) is calculated replacing \( p_i \) by the relative frequency of a type, that is

\[
p_i = f_i / T,
\]

where \( f_i \) is the absolute frequency of a type and \( T \) is the total number of tokens, i.e.

\[
T = \sum_{i=1}^{n} f_i.
\]

This leads to a definition of \( L \) that is

\[
L = \frac{1}{T} \sum_{i=1}^{n} f_i l_i.
\]

Although the claim that languages are optimized has become trendy [17–19], attempts to measure the degree of optimization of languages have been rather scarce [20–23]. Here we aim to quantify the degree of optimality of \( L \) as a window to the strength of compression in languages.

The degree of optimality of syntactic dependency distances – reflecting the manifestation of the dependency distance minimization principle – has been investigated for two decades [22,24–28]. In contrast, the degree of optimality of word lengths has been investigated only recently [20,29,30]. This degree of optimization has been measured with the score [20,31]

\[
\eta = \frac{L_{\text{min}}}{L}, \tag{1.2}
\]

where \( L \) is the average length of tokens and \( L_{\text{min}} \) is the minimum baseline, namely the minimum value that \( L \) can achieve. The analyses using \( \eta \) indicate that languages are 30% optimal on average under non-singular coding and 40% optimal on average under unique decodability [20].
Here we will investigate the degree of optimality of languages with two new scores that we develop from recent research on the optimality of dependency distances [22]. One is

$$\Psi = \frac{L_r - L}{L_r - L_{min}},$$  

(1.3)

where $L_r$ is the random baseline for $L$. The other is

$$\Omega = \frac{\tau}{\tau_{min}},$$  

(1.4)

where $\tau$ is the Kendall correlation coefficient between $p_i$ and $l_i$, and $\tau_{min}$ is the minimum baseline for $\tau$.

The remainder of the article is organized as follows. In section 2, we introduce the definition of $L_{min}$, the minimum baseline, and the definition of $L_r$, the random baseline, that are the building blocks of $\Psi$ and $\Omega$. In section 3, we analyze the mathematical properties of $L$, $\eta$, $\Psi$ and $\Omega$ and other optimality scores, showing the technical superiority of $\Psi$ and $\Omega$. In section 4 and section 5, we present, respectively, the materials and methods that will be used to investigate these scores in real languages. In section 6, we show that $\Psi$ is the best optimality score according to a wide range of criteria. $\Psi$ indicates that languages are optimized to a 62% or 67% on average (depending on the source) when word length is measured in characters and to a 65% when word length is measured in time. We also find that word lengths in time are more optimized than word lengths in characters in sufficiently large samples. Moreover, Chinese and Japanese are more optimized when word lengths are measured in characters rather than in strokes (or in Latin characters after romanization). Finally, in section 7, we discuss the findings examining a wide range of research questions borrowed from a general research program on the optimality of languages. That program was introduced in the context of syntactic dependency distances [22]. We also make proposals for future research.

2. The baselines

2.1 The random baseline

Our null hypothesis is a random mapping of word probabilities into word lengths, that leads to a random baseline that is defined as [32]

$$L_r = \frac{1}{n} \sum_{i=1}^{n} l_i.$$  

(2.1)

Note that $L_r$ actually corresponds to the mean length of word types, whereas the mean word length $L$ corresponds to the mean length of word tokens.

2.2 The minimum baseline

$L_{min}$ is the minimum value that $L$ can achieve making certain assumptions. If the only assumptions are $l_i \geq 0$ and

$$\sum_{i=1}^{n} p_i = 1,$$

then $L_{min} = 0$ [10] and $\eta$ becomes constant ($\eta = 0$ according to Equation 1.2). If the $l_i$’s stem from the lengths of strings, and strings of length zero are not considered valid types, then $l_i \geq 1$ and $L_{min} = 1$.

In standard information theory and its extensions, the $p_i$’s are assumed to be given and optimization is assumed to operate only on the $l_i$’s [10,20]. To calculate $\eta$, $L_{min}$ is computed here making two kinds of further assumptions borrowed from standard information theory and its extensions [10,20]:
• Non-singular coding (NS), namely two word types should not be assigned the same string. We use $L_{NS}^{min}$ to refer to the specific value of $L_{min}$ in this setting.

• Unique decodability (UD), namely the assigned strings, in addition to being non-singular, should have a unique segmentation when concatenated forming a sequence of strings without blanks or other sorts of separators. We use $L_{UD}^{min}$ to refer to the specific value of $L_{min}$ in this setting.

Each of these assumptions is known as coding scheme in the language of information theory [33]. See [10] for further details on these two ways of defining the minimum baseline.

Table 1: Matrix indicating the frequency and length of three types. The mean type length is $L = \frac{235}{25} = 1.88$.

|    | $f_i$  | $l_i$ |
|----|--------|-------|
| 1  | 100    | 2     |
| 2  | 20     | 1     |
| 3  | 5      | 3     |

Previous research on the optimality of word lengths has used $L_{NS}^{min}$ and $L_{UD}^{min}$ [20,29]. Here we focus on a third way of computing $L_{min}$, that we call rank ordering (RO) [29] and that is called “Zipfian coding” in related work [30]. We use $L_{RO}^{min}$ to refer to this specific value of $L_{min}$. This method follows from the generalized theory of compression [10] and has already been used in previous research on the optimality of word lengths [29,30].

$L_{RO}^{min}$ is obtained when the current values of $l_i$ are reassigned to frequencies (or equivalently probabilities) so as to minimize $L$. $L$ is minimized when probabilities are sorted decreasingly and lengths are sorted increasingly [10]. Then the $i$-th most frequent type gets the $i$-th shortest length, hence the name rank ordering.

To see this, consider a matrix with two columns, $f_i$ and $l_i$, that are used to compute $L$. The matrix in Table 1 gives $L = \frac{235}{25} = 1.88$. Suppose that we shuffle the column of $l_i$ in Table 1. We use $l'_i$ to refer to the new value of $l_i$ after the shuffling and $L'$ to refer to the new value of $L$, that is

$$L' = \frac{1}{T} \sum_{i=1}^{n} f_i l'_i.$$ 

Table 2 shows all the possible shufflings (permutations) of the $l_i$ column that can be produced and the corresponding values of $L'$ for the matrix in Table 1.

Then $L_{RO}^{min}$ is the value of $L'$ of the permutation of column $l_i$ that minimizes the value of $L'$. According to Table 1, $L_{RO}^{min} = \frac{155}{25} = 1.24$. By symmetry, $L_{RO}^{min}$ is also the value of $L'$ of the permutation of column $f_i$ that minimizes the value of $L'$, where now

$$L' = \frac{1}{T} \sum_{i=1}^{n} f'_i l_i,$$

and $f'_i$ is the new value of $f_i$ after the shuffling.

The rationale of rank ordering is that it is sensitive to imperceptibility and distinguishability factors as well as to the phonotactic constraints shaping the length of words in a language [10, Section 2.2]. In contrast, optimal non-singular coding and optimal uniquely decodable encoding completely neglect constraints that are language specific or specific to humans (e.g. their cognition, their anatomy, etc.), and abstract away from the cultural and biological evolution processes that lead to more efficient languages [9]. Rank ordering is conservative with respect to
Table 2: All the $3! = 6$ permutations of the column $l_i$ in Table 1 that can be produced. Each permutation is indicated with letters from A to F. A is the one minimizing $L$. $L'$, the mean length of tokens in a given permutation, is shown at the bottom for each permutation.

| i | $f_i$ | $l'_i$ | $l'_i$ | $l'_i$ | $l'_i$ | $l'_i$ | $l'_i$ |
|---|---|---|---|---|---|---|---|
| 1 | 100 | 1 | 1 | 2 | 2 | 3 | 3 |
| 2 | 20 | 2 | 3 | 1 | 3 | 1 | 2 |
| 3 | 5 | 3 | 2 | 3 | 1 | 2 | 1 |

$L' = \frac{155}{125} = 1.24, \frac{170}{125} = 1.36, \frac{235}{125} = 1.88, \frac{265}{125} = 2.12, \frac{330}{125} = 2.64, \frac{345}{125} = 2.76$

optimality because it assumes that a language cannot produce word lengths that are better than the ones that are being observed. In contrast, non-singular coding and uniquely decodable coding point towards the theoretical limits of optimal coding in languages.

The Kendall $\tau$ correlation can be defined as

$$\tau = \frac{n_c - n_d}{\binom{n}{2}},$$

where $n_c$ is the number of concordant pairs and $n_d$ is the number of discordant pairs. $\tau_{RO}^{\text{min}}$ is the smallest value that $\tau$ can achieve by shuffling the column of the $p_i$'s or the column of the $l_i$'s in the matrix. $\tau_{RO}^{\text{min}}$ coincides with the ordering of the $l_i$'s in the matrix such that $L$ is minimized, namely, $L = L_{RO}^{\text{min}}$ [10]. $\tau_{RO}^{\text{min}} = -1$ if there are no ties both in the $p_i$'s and the $l_i$'s; otherwise (the common situation), $\tau_{RO}^{\text{min}} > -1$ (Appendix A, Property A.3).

$L_{RO}^{\text{min}}$ and $\tau_{RO}^{\text{min}}$ are unaffected if the lengths are replaced by new “lengths” that result from applying a monotonically increasing function of $l$ as the following property states. That property will be used to analyze the mathematical properties of distinct scores in section 3.

Property 2.1. Let $L_{RO}^{\text{min}}$ be the minimum value of $L$ under rank ordering, defined as

$$L_{RO}^{\text{min}} = \sum_{i=1}^{n} p_i l_{i, \text{min}}.$$  

Let $h(l)$ be a strictly monotonically increasing function of $l$. We use a prime mark, $'$, to indicate the new value of a function after applying $h(l)$ to the $l_i$'s. Accordingly,

$$L' = \sum_{i=1}^{n} p_i h(l_i)$$

and

$$\tau' = \tau(p, h(l)).$$

Then the minimum value that $L'$ can achieve is

$$L_{RO}^{\text{min}} = \sum_{i=1}^{n} p_i h(l_{i, \text{min}})$$

while the minimum of $\tau$ remains unaltered, i.e.

$$\tau_{RO}^{\text{min}} = \tau_{RO}^{\text{min}}.$$  

Proof: Recall that $L_{RO}^{\text{min}}$ is computed by sorting the lengths increasingly and the probabilities decreasingly [10] in the matrix that is used to calculate $L$. Applying the same method to calculate
it turns out that, if the strictly monotonic transformation is increasing, the sorting by the new lengths that are obtained after replacing each original length $l$ by $h(l)$ will preserve the position of the lengths in the original sorting, giving Equation 2.3. For the same reason, $\tau RO_{\text{min}} = \tau_{\text{min}}$, because the ordering of the $l_i$'s that minimizes $L$ (yielding $L_{\text{RO}}^{\text{min}}$) coincides with the ordering of the $h(l_i)$'s that minimizes $L_{\text{RO}}^\prime$ since $h(l)$ is strictly increasing.

In this article, we will investigate the degree of optimality of languages focusing on rank ordering as a lower bound of the true degree of optimality of word lengths in languages. The choice of rank ordering is for the sake of simplicity and to ensure that the optimality scores are properly normalized as explained in Appendix A. Given that focus, hereafter, we use $L_{\text{min}}$ and $\tau_{\text{min}}$ to refer to $L_{\text{RO}}^{\text{min}}$ and $\tau_{\text{RO}}^{\text{min}}$, respectively.

### 2.3 The properties of the baselines

Here we investigate optimality scores, i.e. $\eta$, $\Psi$ and $\Omega$, that are defined on top of these baselines. The minimum and the random baseline share various statistical properties with the original source:

(i) The distribution of word frequencies and the distribution of word probabilities.
(ii) The number of tokens and the number of types.
(iii) The alphabet size in the case of written language (and the repertoire of phonemes and larger constructs in the case of spoken language).

Therefore, all the scores reviewed in section 3 assume that all these statistical characteristics are fixed; only the mapping of word probabilities into word lengths can be changed. For comparison, the optimality scores for dependency distances discussed in [22,29] assume that the dependency structure is fixed and that only the order of words can be changed. Besides, the minimum baselines of standard information theory and its extensions (optimal non-singular coding and optimal uniquely decodable encoding; section 2.2) do not preserve at least properties item (i) and item (ii) above.

### 3. The optimality scores

#### 3.1 The design of a new optimality score

By adapting the new score for the optimality of dependency distances [22] to word lengths, we obtain Equation 1.3. Importantly, it can be shown that $\Psi$ is proportional to the Pearson correlation $r$ between frequency and length (section A.1), i.e.

$$\Psi = -ar,$$

where

$$a = \frac{(n-1)s_ps_l}{L_r - L_{\text{min}}}$$

is a constant determined by the distribution of the $l_i$'s and the distribution of the $p_i$'s; $s_p$ and $s_l$ are the standard deviations of word probability and word length, respectively. Traditionally, $r$ is seen as a measure of linear association between two variables ([35] and references therein). Therefore, a potential limitation of $\Psi$ is that it may be unable to capture the actual non-linear association between frequency and length [36,37].

This takes us to a second attempt to define an optimality score for word lengths, that is based on Kendall $\tau$ correlation. Applying the same template that we used for the design of $\Psi$, we get
the score
\[ \Omega = \frac{\tau - \tau_r}{\tau_r - \tau_{min}}, \]  
(3.1)

where \( \tau \) is the actual Kendall \( \tau \) correlation between frequency and length, \( \tau_r \) is the expected value of \( \tau \) under our null hypothesis [32] and \( \tau_{min} \) is defined by rank ordering as before. As \( \tau_r = E[\tau] = 0 \) [35], \( \Omega \) becomes simply
\[ \Omega = \frac{\tau}{\tau_{min}}. \]

Since \( L = L_{min} \) implies \( n_c = 0 \) [10], the definition of Kendall \( \tau \) (Equation 2.2) yields
\[ \Omega = \frac{n_d - n_c}{\binom{n}{2}} = \frac{n_d - n_c}{n_{d,min}}, \]

where \( n_{d,min} \) is the number of discordant pairs when \( L = L_{min} \).

Having said that, notice that one cannot expect \( \Psi \) to be unable to capture non-linear associations. It has been shown that there are non-linear associations that are better captured by Pearson correlation, against common belief [35]. Therefore, \( \Psi \) and \( \Omega \) should be seen, for the time being, as distinct approaches to deal with monotonic associations.

### 3.2 The significance of an optimality score

According to the mathematical analysis in section A.1, testing if \( \eta \) or \( \Psi \) are significantly large (or \( L \) is significantly small), as expected by the principle of compression, is equivalent to testing if Pearson \( r \) is significantly small by means of a left-sided correlation test. In contrast, testing if \( \Omega \) is significantly large is equivalent to testing if \( \tau \) is significantly small. Therefore, testing if \( \Omega \) is significantly large is also equivalent to testing the law of abbreviation by means of a left-sided Kendall \( \tau \) correlation test. Similarly, testing if \( \Psi \) (or \( \eta \)) is significantly large is also equivalent to testing the law of abbreviation by means of a left-sided Pearson \( r \) correlation test. In light of these equivalences, previous research on the law of abbreviation by means of Pearson \( r \) or Kendall \( \tau \) [4,32,38,39] has implicitly tested if \( \eta \), \( \Psi \) or \( \Omega \) are significantly large.

### 3.3 Overview

Here we examine the theoretical properties of distinct scores, including correlation coefficients between word frequency and word length (Pearson \( r \) and Kendall \( \tau \)). Firstly, \( L \) is the only score that is not normalized. However, while \( \eta \) is singly normalized (it is normalized only with respect to the minimum baseline, \( L_{min} \)), \( \Psi \) and \( \Omega \) are dualy normalized because they are normalized with respect to the minimum baseline (\( L_{min} \) or \( \tau_{min} \)) and the random baseline (\( L_r \) or \( \tau_r = 0 \)). \( \eta \), \( \Psi \) and \( \Omega \) exhibit constancy under optimal coding (namely \( \Omega = \Psi = \eta = 1 \) when \( L = L_{min} \)). \( \Psi \) and \( \Omega \), as the correlation scores, exhibit stability under a random mapping of probabilities into length – namely their expected value in that random mapping is zero – while that of \( \eta \) is moving (depending on certain parameters of the communication system). Hence \( \eta \) lacks this property. Constancy under optimality and stability under the null hypothesis together make a critical difference. The scores that lack one of them or both, i.e. \( L \), \( \eta \) and the correlation scores (\( r \) and \( \tau \)) are less informative about the actual distance to optimality than \( \Psi \) and \( \Omega \) when comparing distinct systems, e.g. distinct languages [20] or the same language at different evolutionary stages [40].

From a mathematical standpoint, we evaluate the scores considering the following desirable properties:

- **Stability under the null hypothesis.** A score exhibits stability under the null hypothesis if its expectation takes a constant value under the null hypothesis [22]. The optimality
Table 3: Summary of the mathematical properties of each of the word length scores: mean word length ($L$), the Pearson and the Kendall correlation between frequency and length ($r$ and $\tau$, respectively) and the optimality scores $\eta$, $\Psi$ and $\Omega$. Concerning invariance, we assume for simplicity that the transformation is applied to word length only.

| Properties                                      | $L$ | $\eta$ | $r$ | $\Psi$ | $\tau$ | $\Omega$ |
|------------------------------------------------|-----|--------|-----|--------|--------|---------|
| Constancy under optimal coding                  | ✓   | ✓      | ✓   | ✓      | ✓      | ✓       |
| Stability under the null hypothesis             | ✓   | ✓      | ✓   | ✓      | ✓      | ✓       |
| Invariance under translation                    | ✓   | ✓      | ✓   | ✓      | ✓      | ✓       |
| Invariance under proportional change of scale   | ✓   | ✓      | ✓   | ✓      | ✓      | ✓       |
| Invariance under increasing linear transformation| ✓   | ✓      | ✓   | ✓      | ✓      | ✓       |
| Invariance under strictly increasing non-linear transformation | ✓   | ✓      | ✓   | ✓      | ✓      | ✓       |

scores $\Psi$ and $\Omega$ as well as the correlation coefficients $r$ and $\tau$, are stable under the null hypothesis, namely their expectation is zero when the mapping of word probabilities into word lengths is random [32]. In contrast, $\eta$ lacks that property.

- **Invariance under linear transformation.** A score exhibits invariance under linear transformation when its value does not change if a linear transformation is applied to the target variable [22]. The Pearson correlation between two variables is invariant under linear transformation only if (a) the linear transformation that is applied to only one of the variables is increasing or (b) the transformations that are applied to each of the variables are both increasing or both decreasing; the same applies to Kendall $\tau$ correlation (Appendix A). For simplicity, here we focus on increasing linear transformations of word length.

$\Psi$ and $\Omega$ are invariant (do not change) if length is transformed linearly, namely, each length $l$ is replaced by a linear function of it, i.e. $g(l) = al + b$, where $a$ and $b$ are some constants, with $a > 0$. In contrast, $\eta$, is not invariant under a linear transformation but invariant under a proportional change of scale, which is obtained when $b = 0$.

- **Invariance under monotonic transformation.** A score exhibits invariance under strictly increasing transformation if its value does not change if a transformation of this sort is applied to the target variable. The Kendall $\tau$ correlation between two variables is invariant under strictly non-linear monotonic transformation only if (a) the transformation that is applied to only one of the variables is increasing or (b) the transformations that are applied to each of the variables are both increasing or both decreasing (Appendix A). In contrast, that invariance is only warranted for Pearson correlation when the transformations are linear.

For simplicity, here we focus on increasing non-linear transformations of word length. $\Omega$ is invariant (does not change) if each length $l$ is replaced by a strictly increasing function of it, $h(l)$. Notice that $h$ can be non-linear. In contrast, $\Psi$ and $\eta$ are not invariant under that transformation.

Table 3 summarizes the mathematical properties of these scores.

### 3.4 Problems in communication systems

Borrowing the setting of previous research [10], we reduce a communication system to a mapping from types to codes, that may not be discrete. The key of the mapping is the code lengths, that may be real numbers as when one measures the duration of a vocalization or a gesture [3]. Then suppose that we have two communication systems, A and B.

**The weak recoding problem.** In this recoding problem, B is just a transformation of A assigning a new code, e.g. a new string, to each type of A (hence preserving the frequency of every type
from A in B while changing their length). Then suppose that \( s(X) \) is the score of a communication system \( X \). The question is, under what conditions \( s(A) = s(B) \)? That question cannot be answered unless we clarify what we consider to be relevant or not for \( s(A) = s(B) \). Invariance under increasing linear transformation is an answer to that question: we will have \( s(A) = s(B) \) when the score satisfies that invariance and the new lengths in B are an increasing linear transformation of those of B. Similarly, another condition for \( s(A) = s(B) \) is provided by invariance under strictly increasing transformation.

In this article, we will deal with three instances of the weak recoding problem:

1. **Length in characters versus duration in the same language.** In this instance, the lengths in characters in one of the communication system, say \( A \), have been replaced by the corresponding durations in time to produce the other, say \( B \).
2. **Length in Chinese/Japanese characters versus other discrete lengths (romanizations and strokes).** Here lengths in Chinese/Japanese characters are replaced by lengths in strokes or in romanizations, namely Latin script conversions (via Pinyin for Chinese and Romaji for Japanese). Certainly, such replacement leads to an increase in \( L \) (Table B1), but a key question is if the degree of optimality will also change. We will show that the degree of optimality reduces in line with the increase in \( L \). This suggests that romanizations or strokes are a poor proxy to investigate the degree of optimality of these languages with respect to languages written in other scripts.
3. **Removal of vowels in Latin script languages or romanizations.** The motivation for this recoding is to check if the scores are robust to varying decisions with regard to the design of a writing system. For instance, writing systems differ in how they code for vowels. Catalan – as all Romance languages – uses a Latin script, where both consonants and vowels are represented. In contrast, the primary writing system of Arabic is an abjad, where only consonants are represented (unless fatah diacritics are used to indicate vowels). Certainly, removing vowels in Romance languages leads to a decrease in \( L \), but a key question is if their degree of optimality will also change. Interestingly, we will show that the degree of optimality of word lengths decreases only slightly after removing vowels, despite mean word length being reduced considerably. Hence, it is important to use powerful optimality scores.

In these instances, we can only predict \( s(A) = s(B) \) if the actual transformation satisfies the assumptions of its invariance conditions (Table 3).

In these instances, the fact that length is measured in characters implies that invariance under increasing linear transformation cannot be achieved by any linear transformation. Truncation or rounding of the non-integer values when recoding may break the requirement that the transformation must be strictly increasing. Then we have essentially two options. First, not caring about the choice of the score. Second, choosing a score whose theoretical properties are only achieved under ideal conditions. By ideal conditions we mean conditions that may be difficult, if not impossible, to satisfy in a study with real data. Here our choice is the second, hoping that scores that exhibit powerful properties under ideal conditions will behave robustly or yield a reasonable approximation even when these ideal conditions are not met.

Suppose now that coding in A is non-singular, namely no two distinct types are assigned the same code. After replacing the strings in A by new strings it may turn out that B ceases to be non-singular. That could happen because two types in A end up having the same code or one type is assigned the empty string. This motivates the need of defining a strong recoding problem where non-singularity is preserved.

**Strong recoding problem.** In this recoding problem, B is initially obtained from A as in the weak recoding problem, but additionally, all types in B that are assigned the same string are merged into the same type, and types that are assigned the empty string are dropped. Notice
that strong recoding may change the distribution of word frequencies, a feature that the weak recoding problem preserves. Notice that if \( A \) is non-singular then \( B \) will also be non-singular.

All the instances of the weak recoding problem presented above can be investigated under the framework of the strong recoding problem. However, in this article, we investigate directly only the weak recoding problem for the sake of simplicity and due to pressure for space.

The free comparison problem. Above, we have introduced problems where \( B \) stems from \( A \).

The free comparison problem appears when \( A \) and \( B \) are a priori two different communication systems, hence the number of types and their distribution differs. This setting raises the question of whether the optimality scores of two languages with distinct writing systems or distinct evolutionary histories are comparable. This is a big research challenge for analyses of the degree of optimality in written languages, given the disparity of writing systems of the world [41], which is to some degree also reflected in the dataset in the present study [32, Table 4 and 5]. Some specific questions are:

(i) Can we compare Romance languages against Arabic, although the former code for vowels and the latter essentially does not? We will show that removing vowels in Romance languages does not have a big impact in the value of the score, suggesting that the original scores for Romance languages are comparable to those of Arabic.

(ii) Can we compare Romance languages – which use a script that essentially codes for phonemes – against syllabic writing systems, namely systems that use a character for every syllable? Examples of syllabic writing systems are Chinese Hanzi, where every character stands for a syllable, or abugida writing systems.

We wish to design scores that abstract away from cultural differences while providing an insight into the actual degree of optimality of word lengths. Ultimately, this may be impossible to accomplish, but we hope to find scores that can at least give reliable results under ideal conditions.

In spite of these considerations, one may argue that the relationship between the old and the new word length does not need to be linear. Crucially, only \( \Omega \) is invariant under a strictly monotonic non-linear transformation of lengths. Further details about these mathematical properties are given in the coming subsections. The last subsection ends with additional desirable properties that will turn out to be crucial to choose the best score.

3.5 Other desirable properties

The score with more checks in Table 3 is not necessarily the best score. Other properties are desirable.

First, our baselines, and therefore the scores that are defined on top of them, assume that the number of tokens (\( T \)), the number of types (\( n \)) and the alphabet size (\( A \)) are fixed and that only the mapping of probabilities into words can be changed (section 2.3). In this setting, a desirable property of a score is that it is not influenced by these characteristics. If a score was heavily determined by any of these parameters when examining various languages, it could be argued that, rather than observing differences between the degree of optimality of languages, we are observing differences in text length (in tokens) or in the size of the alphabet or the vocabulary. A critical dependency would be on the number of tokens, that would be particularly worrying when using non-parallel corpora, where higher variation in text length is expected with respect to parallel texts.

Second, an important question is whether a score will converge to a concrete value in a sufficiently large sample of the same language and how fast it will converge. This approaches the problem of dependency on the number of tokens within a given language; the equivalent problem across languages is tackled by the first point above.

Third, another important and related question is replaceability. To what extent a score could be replaced by a simpler one, in spite of all the theoretical arguments summarized in Table 3?
the extreme case, could one of the best scores, say \( \Psi \), be replaced simply by \( L \) because the former is strongly correlated with the latter?

All these questions are difficult to investigate theoretically and will be tackled empirically at the beginning of section 6 so as to help choose a primary score for reporting results. We will show that \( \Psi \) is the most powerful score in terms of independence of these parameters (text length and size of the alphabet or size of the vocabulary) as well as in terms of convergence; \( \Psi \) cannot be replaced by neither \( L \) nor \( \eta \) in parallel texts.

4. Material

We borrow a dataset with information about word length and word frequency from recent research [32], available in the repository of the article [42]. The dataset has been extracted from two collections of texts: Common Voice Forced Alignments, hereafter CV, and Parallel Universal Dependencies, hereafter PUD. PUD comprises 20 distinct languages from 7 linguistic families and 8 scripts. CV comprises 46 languages from 14 linguistic families (we include ‘Conlang’, i.e. ‘constructed languages’, as a family for Esperanto and Interlingua) and 10 scripts. The typological information (language family) is obtained from Glottolog 4.6. The writing systems are determined according to ISO-15924 codes. See [32] for further details about the dataset.

4.1 The datasets

The dataset provides the length of a word in characters (in PUD and CV) and its duration (in CV only) in two variants, median duration and mean duration (median is used by default but mean is used for control in certain cases). It also provides the length in strokes and in romanizations for Japanese and Chinese (Romaji and Pinyin, respectively) in PUD. The traditional writing systems of Japanese and Chinese yield very short word lengths in characters, while the number of distinct characters is very large, especially compared to Western languages with mostly alphabetic writing systems [40,43]. We wish to test whether these differences are reflected in optimality scores.

Common Voice Forced Alignments

Notice that Abkhazian, Panjabi, and Vietnamese have a critically low number of tokens (less than 1000 tokens according to Table 5 in [32]). However, we decided to include them in the analyses so as to understand their limitations.

Parallel Universal Dependencies

Notice that three Japanese words that are hapax legomena could not be romanized and thus the number of tokens and types varies slightly with respect to the original Japanese characters (Table 4 in [32]). Thus Japanese in strokes follows the setting of the weak recoding problem approximately.

5. Methodology

All the code used to produce the results is available in the repository of the article [42]. The bulk of the methods are borrowed from a preceding article, including the unsupervised method to filter words with unusual or “foreign” characters [32]. Next we simply highlight specific methods or specific variants in this article.

5.1 Weak recoding problem

When measuring word length in Latin script compared to strokes in Japanese or Chinese, we simply preserve the original character types but recompute their length, in agreement with the

1https://glottolog.org/

2See Table 4 in [32] for alphabet size and Table B1 for average word length in Chinese and Japanese.
definition of the weak recoding problem we introduced in section 3.4. We do not recompute types according to distinct strings in Latin script (that would be the strong recoding problem, which is not the target of this article). It is possible and expected that distinct character types are assigned the same string in Latin script. For instance, 書, 格, 角, 援, 核, 画, 各, 佳, 確, 斯く, 昇く, 殻, 隙, 隙 are all written as ‘kaku’ in Romaji. Hence Japanese romanizations do not comply with the setting of the strong recoding problem.

5.2 Immediate constituents in writing systems

When measuring word length in written languages, we are mainly using immediate constituents of written words. In Romance languages, we are measuring word length in letters of the alphabet, the immediate constituents in the written form, which are a proxy for phonemes. For syllabic writing systems (as Chinese in our dataset), the immediate constituents of words are characters that correspond to syllables. In addition, for Chinese and Japanese, we are considering two other possible word length units: strokes and letters in Latin script romanizations. In the hierarchy from words to other units, only the original characters are immediate constituents. This means that, for each of these languages, words are unfolded into three systems, one for each unit of encoding (original characters, strokes, romanized letters/characters). For simplicity and to avoid that these two languages are hence over-represented in statistical analyses on the correlation between scores and certain parameters at the level of languages (section 3.5), we will use only their immediate constituents (namely original characters). The results based on original characters are hence compared with those for the other languages.

5.3 Statistical testing

When measuring the association between two variables, we use both Pearson correlation and Kendall correlation [34]. Note that the traditional view of Pearson being a measure of linear association and Kendall correlation being a measure of monotonic non-linear association has been challenged [35].

Given the long computational time required for Kendall \( \tau \) correlation, the naive default algorithm \( O(n^2) \) for computing \( \tau \) was replaced by one that runs in \( O(n \log n) \) time [44] and is available in \( \text{R}^3 \).

We applied a Holm-Bonferroni correction to \( p \)-values\(^4\) in correlograms, where a correlation test was applied to all pairs of random variables from a specific set.

6. Results

In section 1, we highlighted the importance of distinguishing between principles and their manifestations. In section 3, we have addressed the problem of how to measure the degree of compression of word lengths through the relationship between the frequency of a type and its length, examining distinct scores for the optimality of word lengths. In section 3, we have shown that \( \eta, \Psi, \text{ and } \Omega \) are indices of the intensity of compression because they always reach one when the system is fully optimized according to the minimum baseline (rank ordering) in section 2. In contrast, Pearson \( r \) and Kendall \( \tau \) correlations fail to give a constant value when the system is fully optimized. Therefore, \( r \) and \( \tau \) are suitable to investigate the manifestation of compression, i.e. the law of abbreviation [32], but are a poor reflection of the actual degree of optimality with respect to the minimum baseline. First, we provide some intuitive understanding of key optimality scores and establish that \( \Psi \) is the best score in terms of the best combination of mathematical properties (Table 3) and other desirable properties (section 3.5). Second, we choose \( \Psi \) accordingly to measure the degree of optimality of languages. We also compare \( \Psi \) against the other optimality scores (excluding the correlation scores, \( r \) and \( \tau \), for the reasons mentioned above).

3 https://www.rdocumentation.org/packages/pcaPP/versions/2.0-1/topics/cor.fk
4 https://stat.ethz.ch/R-manual/R-devel/library/stats/html/p.adjust.html
6.1 Understanding the optimality scores

Here we focus on a qualitative understanding of the optimality scores $\eta$, $\Psi$ and $\Omega$ as a preparation for the results that are presented below. All these scores are ratios of the form $y/x$ (Equation 1.2, Equation 1.3 and Equation 1.4) and give a percentage of word length optimization with respect to some baseline(s). Figure 1, Figure 2 and Figure 3 show $y$ as a function of $x$ for each of them. By definition, all these scores have to lay below the identity line ($y = x$) because they are designed to satisfy $y \leq x$ (the maximum value of these scores is 1). In case of optimal coding (minimum word length), languages would lay on the identity line, because these scores exhibit constancy under optimal coding. Figure 1, Figure 2 and Figure 3 indicate that languages are not coding optimally. The expected behavior of the scores in other conditions depends on the score. In the absence of any word length effect (for or against compression), languages are expected to be on the abscissa axis (the line $y = 0$) in case of $\Psi$ and $\Omega$, because $\Psi$ and $\Omega$ exhibit stability under the null hypothesis; in case of $\eta$, languages are going to be distributed according to $x = L = L_r$ because $\eta$ is not stable under the null hypothesis. In case of some degree compression of word lengths, languages must be over the line $y = 0$ in case of $\Psi$ and $\Omega$, which is what these figures show; in case of $\eta$, the lack of stability under the null hypothesis requires a comparison of the actual value of $\eta$ against its expected value under the null hypothesis, that is $L_{\text{min}}/L_r$. Figure 2 and Figure 3 reveal the presence of some degree of compression; in fact points are in general closer to the diagonal than to the abscissa axis.

A central score in the analyses to come is $\Psi$. The percentage $\Psi$ yields is with respect to $L_r - L_{\text{min}}$, that is the gap between the minimum possible mean word length (obtained by assigning the $i$-th most frequent word the $i$-th shortest length in a language and recomputing mean word length) and the random mean word length (obtained by shuffling at random word lengths or word probabilities in a language). In more detail, $\Psi$ is the percentage of that gap ($L_r - L_{\text{min}}$) covered by the gap $L_r - L$, i.e. the gap between the actual word lengths and the random mean word length. In Chinese (when the units are Han characters), $\Psi = 72.3\%$ is the percentage of the gap $x = L_r - L_{\text{min}} = 2.18 - 1.47 = 0.71$ covered by the gap $y = L_r - L = 2.18 - 1.67 = 0.41$ (Figure 1 and Table B1).

Finally, notice that similar scores or their components have already appeared implicitly in previous research on the optimality of word lengths [30]. In Figure 4 of [30], one finds the gap $L_r - L$, the numerator of $\Psi$ (Equation 1.3), in the $x$-axis of the left panel as well as $1/\eta$ in the $y$-axis of the right panel (but we used rank ordering to define $L_{\text{min}}$ for $\eta$; Equation 1.2).
Figure 1: The ingredients of the ratio $\eta$: $L_{min}$ (the minimum baseline) versus $L$ (the actual word length). The purple line is the identity function (slope of 1 and zero intercept). Red points indicate the points that are expected under the null hypothesis (where $L = L_r$ is expected). (a) PUD collection. (b) CV collection with length measured in characters. (c) CV collection with length measured in duration.
Figure 2: The ingredients of the ratio $\Psi$: the gap $L_r - L$ versus the gap $L_r - L_{\text{min}}$. The purple line is the identity function (slope of 1 and zero intercept). Red points indicate the points that are expected under the null hypothesis (where $L = L_r$ is expected). (a) PUD collection. (b) CV collection with length measured in characters. (c) CV collection with length measured in duration.
Figure 3: The ingredients of the ratio $\Omega$: negative Kendall $\tau$ correlation between frequency and length, versus the negative $\tau_{\text{min}}$, the minimum baseline for $\tau$. Here we use the definition of $\Omega$ Equation 3.1 with $\tau_r = 0$, hence the minus sign in the correlations displayed by each axis. Red points indicate the points that are expected under the null hypothesis (where $\tau = \tau_r = 0$ is expected). The purple line is the identity function (slope of 1 and zero intercept). (a) PUD collection. (b) CV collection with length measured in characters. (c) CV collection with length measured in duration.
6.2 Desirable properties of the scores

In order to identify the best score for cross-linguistic research on the degree of optimality, we take into account what we consider to be desirable properties for an optimality score, in addition to those that can be established by a purely mathematical analysis (Table 3).

First, a score should not be related to the basic language parameters – namely observed alphabet size (number of distinct characters, $A$), observed vocabulary size (number of types, $n$), and sample size (number of tokens, $T$) – which are assumed to be constant in the definition of the baselines. The only score consistently satisfying these properties across all collections and length definitions is $\Psi$, while both $\eta$ and $\Omega$ show some degree of significant association with $A$ and $T$ when data is more heterogeneous, i.e. in the CV collection (Figure C1).

Second, the optimality score of a given language should not depend too strongly on the size of its available sample, meaning that it should converge, and ideally it should converge fast. Again, the score which gets closer to this ideal behaviour is $\Psi$, which varies within a rather small range, and appears to approach convergence in some languages. In contrast, both $\eta$ and $\Omega$ keep their decreasing trend even for large sample sizes, yet with different speeds (Figure C2, Figure C3 and Figure C4).

Third, a complex score should not be replaceable by a simpler score, namely it should not be significantly and strongly associated to it. When using parallel data, none is replaceable by $L$, the simplest score; we only find that $\Omega$ could be replaced by $\eta$ (Figure C5). Thus $\Psi$ is the best complex score for parallel data. Solid conclusions for non-parallel data are beyond the scope of this article.

See Appendix C for further details on the analysis of the desirable properties.

6.3 The distribution of the optimality of word lengths

Here we review the distribution of the values of each of the scores. For the scores on word length in characters, we excluded strokes and romanizations for Chinese and Japanese for the sake of homogeneity (then we use only immediate written word constituents). In Figure 4, we show the density plots for both collections, color-coded by definition of length. All scores show a rather narrow distribution, both in PUD and in CV, despite its heterogeneity and large size in terms of languages. The bulk of the values is far from zero, the value expected under the null hypothesis, specially for $\eta$ and $\Psi$. In Table 4, we report the main summary statistics. In all cases, $\Psi$ and $\Omega$ take positive values suggesting some degree of optimization in every language. Indeed, the correlation tests on the same data support [32] that $\Psi$ or $\Omega$ are significantly large in each individual language. The magnitude strongly depends on the score used. Concerning $\Psi$, our preferred score for cross-linguistic analysis, the mean values computed in the three considered scenarios (in a range from 0 to 100%) are close to each other: 67% in PUD with length measured in characters, 62% in CV when considering characters, and 66% when considering duration.

In Table B1, Table B2 and Table B3, we detail the values of all scores and those of their ingredients for each language and collection.
Table 4: Summary statistics of optimality scores $\eta$, $\Psi$ and $\Omega$ for every collection and definition of length. For length in characters, we only use immediate word constituents for the sake of homogeneity. Accordingly, scores in strokes or in romanizations for Chinese and Japanese in PUD are excluded. Refer to Table B1, Table B2 and Table B3 for values on individual languages.

| score | collection     | Min. | 1st Qu. | Median | Mean  | 3rd Qu. | Max.  | sd  |
|-------|----------------|------|---------|--------|-------|---------|-------|-----|
| $\eta$ | PUD-characters | 0.69 | 0.78    | 0.81   | 0.81  | 0.85    | 0.88  | 0.05|
|       | CV-characters   | 0.52 | 0.69    | 0.75   | 0.73  | 0.80    | 0.96  | 0.10|
|       | CV-duration     | 0.46 | 0.72    | 0.76   | 0.76  | 0.81    | 0.92  | 0.09|
| $\Psi$ | PUD-characters | 0.42 | 0.64    | 0.70   | 0.67  | 0.72    | 0.78  | 0.08|
|       | CV-characters   | 0.33 | 0.57    | 0.63   | 0.62  | 0.68    | 0.79  | 0.09|
|       | CV-duration     | 0.33 | 0.60    | 0.69   | 0.66  | 0.73    | 0.83  | 0.11|
| $\Omega$ | PUD-characters | 0.13 | 0.23    | 0.29   | 0.29  | 0.35    | 0.44  | 0.08|
|       | CV-characters   | 0.04 | 0.19    | 0.24   | 0.26  | 0.30    | 0.68  | 0.12|
|       | CV-duration     | 0.09 | 0.22    | 0.25   | 0.26  | 0.31    | 0.45  | 0.07|

Figure 4: Density distribution of the scores $\eta$, $\Psi$ and $\Omega$ for length in characters or duration over languages in the PUD and CV collections. For length in characters, we only use immediate word constituents for the sake of homogeneity. Accordingly, scores in strokes or in romanizations for Chinese and Japanese in PUD are excluded. Refer to Table B1, Table B2 and Table B3 for values on individual languages. The vertical dashed lines show mean values (the values of the means are shown in Table 4).

6.4 Sorting languages by their degree of optimality

Here we focus on PUD, as it is the only parallel corpus, and on $\Psi$ for the sake of simplicity given its mathematical and statistical properties. Sorting languages in CV might lead to misinterpretations,
as its intrinsic heterogeneity hampers valid cross-linguistic comparisons (see Appendix C for problems and risks of drawing conclusions from CV).

In Figure 5 (a), we show the languages in the PUD collection sorted decreasingly by $\Psi$. In Figure 5 (b), we display a breakdown of the composition of $\Psi$ intended to help understand its definition as a ratio of two differences.

Tables showing the concrete values of the scores and further details on both collections can be found in section B.1.

Figure 5: Optimality of word lengths in PUD according to $\Psi$ with length measured in characters.
(a) Absolute values of $\Psi$. (b) Visual depiction of the composition of $\Psi = (L_r - L)/(L_r - L_{\text{min}})$ using bars that consist of two segments, a dark blue one followed by a light blue one. The bar starts at $L_{\text{min}}$, changes color at $L$, and ends at $L_r$. Then the dark blue segment of the bar is the difference $L - L_{\text{min}}$. This quantity is positive because $L_{\text{min}} \leq L$ by definition. The light blue segment is the difference $L_r - L$. This quantity is positive because $L < L_r$ due to compression [32], hence the length of the whole bar is the difference $L_r - L_{\text{min}}$, and the length of the light blue segment over the whole represents $\Psi$.

6.5 The weak recoding problem

Length in characters versus duration in the same language. To understand the relation between the optimality of a language in written and in oral form, we compare the values of $\Psi$ when length is measured in duration against its values when length is measured in characters (Figure 6). This comparison is possible only in the CV collection. In Figure 6, we also show the outcome of fitting a linear model. Both when considering length in median and in mean duration, we find that the scores are generally preserved as supported by a significant and positive Pearson correlation over 0.74, and a linear regression that yields a slope near 1, as well as a small positive intercept. Interestingly, the great majority of languages show more optimization in oral than in written form (dots above the purple identity line). The intensity of this phenomenon seems to be related with sample size, in the sense that the languages with the smallest samples tend to be
below the identity line. Recall, however, that we found no significant correlation between $\Psi$ and $T$ (section 3.5).

![Figure 6: $\Psi$ measured in duration ($\Psi_{\text{dur}}$) versus $\Psi$ measured in characters ($\Psi_{\text{chars}}$) in the CV collection. Languages are color-coded by number of tokens in logarithmic scale to indicate orders of magnitude in sample size. We fit a robust linear model by Theil-Sen regression (blue line), which is then compared to the identity function, $\Psi_{\text{dur}} = \Psi_{\text{chars}}$ (purple line). 95% confidence intervals for the regression line are shown as a gray band. The choice of robust linear regression is motivated by the presence of undersampled languages whose scores deviate from the remainder of languages in this collection. For written language, only immediate word constituents are used (Japanese and Chinese are not included in CV). We report the parameters of the linear model (slope and intercept), Pearson correlation ($r$) and the standard error of the regression ($S$). (a) Word length defined as median duration. $y = 0.066 + 0.966 \times x$, $r = 0.794$ with $p$-value $4.5 \times 10^{-11}$ and $S = 0.227$. (b) Word length defined as mean duration. $y = 0.022 + 1.03x$, $r = 0.745$ with $p$-value $3 \times 10^{-9}$ and $S = 0.239.

Length in Chinese/Japanese characters versus other discrete lengths (Latin scripts characters and strokes). Here we analyze the impact of replacing word length in Chinese and Japanese characters, the immediate word constituents in written form, by word length in strokes or Latin script characters after romanization. We find that the value of $\Psi$ reduces in all cases (Figure 5). However, the value of the optimality score is still significantly large according to the corresponding correlation test, even after the replacement [32, Figure 1 (a, c)]. In particular, romanization affects both languages in a comparable way (scores decrease by 15% in Chinese, and 13% in Japanese), while measuring length in strokes penalizes Chinese (18% decrease), and has a smaller effect on Japanese (9% decrease).

Removal of vowels in Latin script languages or romanizations. We also investigate the impact of removing vowels on the 15 languages using Latin scripts (including Chinese Pinyin and Japanese Romaji) in the PUD collection, comparing the original value of the score against its new value after this particular transformation. Figure 7 shows the new scores as a function of the original ones, alongside the results of a linear regression. For both $\Psi$ and $\Omega$, the best fit of the linear model gives a slope very close to 1 and a positive offset of almost 0. In contrast, $\eta$ yields a slope of 1.4 and an intercept of -0.4. Moreover, the standard error $S$ of its linear regression is more than twice the values found for $\Psi$ and $\Omega$. Undoubtedly, $\eta$ is not robust to the removal of
vowels. Among all the scores, $\Omega$ has the highest Pearson correlation and the lowest standard error, suggesting it is the score that is the least sensitive to vowel removal.
Figure 7: The value of a score after removing vowels, indicated with the subindex \( \text{rem} \) (e.g. \( \Psi_{\text{rem}} \)), as a function of the original value with all characters (e.g. \( \Psi_{\text{chars}} \)) in languages using the Latin script in the PUD collection. For a given language, the new value of the score is obtained after removing vowels that may include different accents or tones. The purple line is the identity function (slope of 1 and zero intercept), while the blue line is the least squares linear regression. 95% confidence intervals for the regression line are shown as a gray band. For each score, we indicate the parameters of the best fit of a linear model (slope and intercept), as well as the Pearson correlation coefficient (\( r \)) and the standard error of the regression (\( S \)).

(a) \( \eta \) score. \( y = -0.401 + 1.42x \), \( r = 0.913 \) with \( p \)-value \( 2.1 \times 10^{-6} \) and \( S = 0.177 \). (b) \( \Psi \) score. \( y = 0.005 + 0.92x \), \( r = 0.967 \) with \( p \)-value \( 4.0 \times 10^{-9} \) and \( S = 0.067 \). (c) \( \Omega \) score. \( y = 0.006 + 0.919x \), \( r = 0.951 \) with \( p \)-value \( 5.3 \times 10^{-8} \) and \( S = 0.083 \).
6.6 Impact of disabling the filter of words that contain “foreign” characters

All results presented in this section have been obtained after applying the new method to filter out highly unusual characters and words described in [32]. If the filter is disabled, we obtain some slight changes in the values, but the qualitative results remain the same.

7. Discussion

The degree to which languages are optimized is receiving increasing attention [20–23]. Quite generally, linguistic research is subject to a tension between acknowledging the astonishing uniformity of languages at higher levels of abstraction, and the empirical diversity which languages exhibit in a myriad of structural features from phonology to syntax. In the domain of coding efficiency, it has been argued that distinct languages exhibit similar degrees of efficiency [21]. In this article, we contribute to each of these complementary views. Questions 1-2 put emphasis on potential homogeneity whereas Question 3 and Question 4 shed some light on diversity.

Question 1. What are the best optimality scores for cross-linguistic research?

We measured optimality with three scores: $\eta$, $\Psi$, and $\Omega$. As shown in Table 3, the two latter scores are endowed with better statistical and mathematical properties, and are thus more reliable than $\eta$. Concerning the comparison between the remaining two, we argue that $\Psi$ is endowed with some additional desirable properties which make it better suited for the scope of cross-linguist research.

Desirable properties. First, an ideal score should not be sensitive to the basic language parameters that are assumed constant in the definition of the baselines, such as alphabet size, vocabulary size, and sample size. To check whether this requirement holds for the considered scores, we computed the Kendall $\tau$ (and Pearson $r$) correlation between them and the mentioned basic parameters (Figure C1). We want to stress that PUD is the only parallel corpus in the analysis, and the correlations computed in CV might suffer from confounding effects.

On the one hand, no significant linear or non-linear association could be detected for $\Psi$, consistently across collections and length definitions. On the other hand, both $\eta$ and $\Omega$ show some significant negative correlations with observed alphabet size and sample size in the CV collection, where these two parameters are however also strongly associated between them. Indeed, the observed alphabet size depends on the amount of seen data (Figure C1 (b,c,e,f)), and the correlation of $\eta$ and $\Omega$ with $A$ could potentially just be a side-effect of the relationships above, by transitivity. Nevertheless, the most critical correlation is the one observed with sample size $T$, as it implies that these two scores tend to give larger values in under-sampled languages, which we do not consider a desirable property for a score. This can also be observed when looking at the convergence speed of the scores. As shown in Figure C2 for PUD, Figure C3 for CV when length is measured in characters, and Figure C4 for CV when length is measured in duration, both $\eta$ and $\Omega$ keep their decreasing trend even for large sample sizes. In contrast, $\Psi$ generally varies within a smaller range, and importantly, it seems to reach a quite stable value in many languages, especially in CV where larger sample sizes are available. While the convergence analysis allows one to capture the evolution of the scores for different sample sizes within a language, it also gives an understanding of the behaviour of scores across languages with different sample sizes. In fact, since $\eta$ and $\Omega$ tend to decrease, they will be likely to have higher values in languages with a smaller sample, thus leading to potentially misleading conclusions in non parallel data, or data with excessive heterogeneity in terms of sample size. Moreover, $\Psi$ and $\Omega$ have a very similar shape up to around $10^2$ tokens, after which the latter generally drops down, suggesting the importance not only of parallel corpora, but also of large enough samples.
Thus, even if the $\Psi$ scores obtained on non-parallel data might be a mirror of the level of optimization of a language in a certain context rather than of the language itself, they are less likely to be influenced by the heterogeneity in terms of text sizes, a feature that arises naturally even in parallel corpora but to a lower degree. For this reason, despite its relation with Pearson correlation, which poses a potential challenge on the ability of $\Psi$ to grasp a non-linear relation between length and frequency, this score turns out to be more suited for non-parallel data. Besides, $\Omega$ could theoretically have more power in detecting a non-linear association, but its observed association with the number of tokens in CV (characters) raises serious concerns about its reliability in non-parallel data. Moreover, the steep decreasing trend observed in the convergence analysis suggests that, even when dealing with texts of a comparable size, $\Omega$ will be more strictly related to it.

Complementary arguments about the best score in the context of the recoding problem are given in Question 5.

**Possible alternatives** In our derivation of the new optimality scores for word lengths, namely $\Psi$ and $\Omega$, we followed the template to design a new score for dependency distances [22]. Equivalent scores may be obtained by other means. We cannot exclude the possibility that there are simpler scores or existing scores that have the same statistical properties as our $\Psi$ score for word lengths. A promising candidate for future research is the $\gamma$ index, a variant of Kendall $\tau$ correlation [45], whose estimator is

$$\gamma = \frac{n_c - n_d}{n_c + n_d}.$$  

The inverted sign $\gamma$, i.e., $-\gamma$, seems to have the same mathematical properties as $\Omega$ but may satisfy the desirable properties of $\Psi$: $\gamma$ is able to achieve $-1$ when the number of concordant pairs is $n_c = 0$ whereas $\tau$ only achieves that when ties are missing [34].

**Question 2. Is compression a universal optimization principle? Or, are there languages showing no optimization at all?**

Despite the different definitions and properties, each of the three considered optimality scores reaches 1 when a language is fully optimized, while a value of 0 when there is no optimization at all is only expected for $\Psi$ and $\Omega$. The distribution of the values of the scores (Figure 4) and, with greater detail, the summary statistics (Table 4), indicate that $\Psi$ and $\Omega$ always take values that are larger than 0, with magnitudes depending on how optimization is measured (the score and the units of length). This finding indicates that, globally, languages are shaped by compression (it is unlikely that all languages yield a positive value just by chance). However, to ensure that compression has shaped each of the languages, a test of the significance of the score is required. We will focus on $\Psi$ given our discussion above on the best score for cross-linguistic research. We have shown that testing if $\Psi$ is significantly high is equivalent to testing the law of abbreviation using a one-sided Pearson correlation test, whose outcomes have already been shown [32, Figure 1 (c,d)].

All languages in the sample show some significant degree of optimization in written form, independently of the units chosen to measure the latter. The test of $\Psi$ yields significance at a 99% confidence level for all the languages in PUD. Concerning oral form (duration), the only exception is Panjabi in the CV collection, for which we could not find a significantly small correlation. Thus its $\Psi$ score is not significantly large despite being greater than 0. This seems likely to be related with its severe under-sampling issue, as it sample contains only 98 tokens. Indeed, the only languages which are significant at the 95% (rather than 99%) confidence level are Abkhazian, Dhivehi, and Vietnamese, all having very small samples compared to the other languages of the CV collection. Therefore, we conclude that optimization is a universal principle, that manifests exceptionlessly in our ensemble of languages when a large enough sample of a language is available.
Question 3. What is the degree of optimization of languages?

It has been argued that language production is not optimal but “good enough” [46]. A necessary condition for the frequency-length association to be “good enough” is that the degree of optimality is statistically significant, which is the case [32, Figure 1 (c,d)]. A further step is quantifying how “good” the mapping is, an issue addressed by inspecting the absolute value of $\Psi$ in a scale from 0 to 100% as its values are significantly large.

By looking at the summary statistics in Table 4 and at the probability density plots in Figure 4, we can observe how the distribution of $\Psi$ is similar across the three scenarios (PUD, CV considering characters, CV considering duration). Indeed, their median, mean, and standard deviation values are comparable across collections and definitions of length. In particular, the mean value of $\Psi$ in parallel data is 67%, while for CV it ranges between 62% (with length measured in characters) and 66% (with length measured in duration). Moreover, half of the languages in PUD show an optimization score larger than 70%, while this value changes slightly to 69% in CV when length measured in duration, and drops to 63% when length measured in characters.

This robustness suggests that the values of $\Psi$ that we observe are a reasonable approximation of the real degree of optimization of natural languages, at least within the scope of the families and scripts considered in the analysis.

In sum, the degree of optimization of word lengths in languages ranges between 60% and 70% for the majority of the languages, both in parallel and non parallel data. These findings are in line with the claim that languages exhibit a similar degree of coding efficiency in spite of their diversity [21]. Interestingly, our measurements of word length are on a scale from 0 to 100% as a result of dual normalization, showing a high concentration of values within a narrow range.

Finally, although we have excluded $\Omega$ to inspect the degree of optimization of languages for the sake of simplicity, we would like to make a point about the possible origin of the low values (compared to $\Psi$) that $\Omega$ exhibits. We are certain that they are not due to the choice of $\tau$ as opposed to Spearman $\rho$ correlation in the definition of $\Omega$ (Equation 1.4). See section B.2 for further details and a further discussion.

Question 4. What are the most and the least optimized languages of the world?

By means of $\Psi$, we are able to quantify the level of optimization of languages taking advantage of the fact that its is not correlated with the basic parameters ($A$, $n$, $T$) of each language. However, a truly fair comparison of the values of $\Psi$ in languages might be non-trivial to establish even in the parallel collection. Indeed, text sizes in PUD are not large enough for $\Psi$ to reach convergence, meaning that the scores computed on our sample are likely to be not fully representative of the real ones (Figure C2). This still has to be kept in mind when comparing languages by optimization degree even knowing that $\Psi$ decays slowly once the text sample is sufficiently large. Besides, large samples are available for CV but parallelism is lacking and sample sizes are very heterogeneous, making cross-linguistic comparisons problematic. For these reasons, here we focus on PUD and make emphasis on the relative order of the languages when sorted by $\Psi$ rather than on the concrete values, assuming that the ranks of languages by $\Psi$ are more stable than their absolute value of $\Psi$ when sample size is increased.

Figure 5 shows the ranking of languages in PUD obtained by sorting their $\Psi$ values in decreasing order. We assume for simplicity that every difference in the scores of two languages is significant for the purposes of the ranking, which might not necessarily be the case. Future research should address the problem of the statistical significance of the differences between the optimality scores of two languages [22].
Having said this, the languages vary in optimization between 42.8% in Indonesian and 77.6% in Japanese according to immediate word constituents (Figure 5). Japanese is the language showing the highest degree of optimization but only when word lengths are measured in characters (the immediate constituents of its written word forms), with $\Psi = 77.6\%$. The writing systems used in Japanese, Chinese, and Korean clearly lead to lower values of $L_{\text{min}}$, $L_r$, and $L_c$ (when length is measured in characters) compared to other scripts, and are contained in a rather small interval (Figure 5 (b)). However, these three languages have a very different position in the ranking, meaning that the reduced variability in word length is not a priori an advantage or a disadvantage. Together with Japanese, at the top of the ranking we find Chinese and a cluster of Romance languages: Portuguese, Italian, Spanish, and French. Romance languages show a degree of optimality that ranges between 71–75% (above English, that has 66%) and also show a very similar score composition (Figure 5 (b)).

Japanese Kanji are borrowed from traditional Chinese Hanzi but Japanese adds Hiragana and Katakana. The finding that written Japanese is more optimized than written Chinese when using the same unit (characters, romanizations and strokes) suggests that syllabaries along with Japanese Kanji result in a higher degree of optimization of written language than the single use of Hanzi in Chinese, according to our metrics. However, this issue deserves further investigation as there is a hidden parameter that we are not controlling for: the level of granularity in the definition of a word in Chinese and Japanese (by level granularity we mean the criteria to segment a sequence of characters into words). The current differences in optimality may reduce if the same level of granularity was used.

Question 5. What is the optimality of languages after recoding?

We investigated the effect of recoding on optimization level by assessing the degree to which scores or ranks are preserved when the recoding transformation is applied. In particular, we addressed the issue from three different viewpoints.

**Length in characters versus duration in the same language.** By means of the CV corpus, we were able to explore the effect on optimality of measuring a word length in duration rather than in characters. Most languages are above the identity line, indicating that they show more optimization in word durations (Figure 6). Undersampled languages deviate from that trend. Especially when median duration is used (Figure 6 (a)) the slope is very close to 1 (0.966), and the intercept is small (0.066), meaning that scores are actually preserved to a large degree but shifted up. However, there is a positive relation between the number of tokens and the extent to which a language is more optimized in oral rather than written form, as measured by the ratio $\Psi_{\text{dur}}/\Psi_{\text{chars}}$.

Notice that not only the undersampled languages are below the diagonal, but also the larger the sample the higher the ratio $\Psi_{\text{dur}}/\Psi_{\text{chars}}$ (the Pearson correlation between $T$ and the ratio $\Psi_{\text{dur}}/\Psi_{\text{chars}}$ is $r = 0.389$; $p$-value 0.008). Therefore, the larger the sample, the stronger the evidence for word durations being more optimized than word lengths in characters. Critically, the opposite finding in certain languages (lower optimality for word durations) is likely to be due to undersampling.

These considerations notwithstanding, it has not escaped our attention that French, a language whose alphabetic writing system is among the farthest with respect to actual phonetic form among Romance languages (namely it is one of the less transparent writing systems among Romance languages), exhibits one of the highest contrasts between its lower optimality in written form and its higher optimality in oral form. However, we cannot exclude that this finding is related to the larger sample size of French with respect to other Romance languages (Table 4 in [32]).

The range is based on immediate constituents of characters, this is why we are neglecting the minimum $\Psi$ achieved by Chinese characters in strokes, that is an even lower value than that of Indonesian.
Length in Chinese/Japanese characters versus other discrete lengths (Latin script characters and strokes). While for all the main analyses we consider the immediate constituents of words in the original language, we re-code Chinese and Japanese using strokes and romanizations to investigate the effect on optimality of measuring lengths in different units. It is obvious that recoding increases the length of Chinese/Japanese words. Hence, the conclusion that word lengths are more optimized in Chinese/Japanese characters than in romanizations or strokes based on plain word lengths would be straightforward but would sweep under the carpet that, after recoding, the random and minimum baselines have also changed. For this reason, optimality scores that take into account these baselines are required and the conclusions obtained with them are non-trivial a priori.

We find a drop in the optimality scores for both alternative discrete lengths in both languages (Figure 5), yet the degree of optimization is still significant [32, Figure 1 (c)]. While romanization has a comparable effect in decreasing the scores in the two languages (the optimality of Japanese drops from 78% to 67% and that of Chinese drops from 72% to 61%), the impact of using strokes as a unit leads to a larger effect in Chinese (Japanese drops from 78% to 71% while Chinese drops from 72% to 60%), as shown in Figure 5.

The reduction of optimality when not using immediate constituents suggests that compression operates on characters rather than on strokes and that characters (or components within characters) form units where the number of strokes is less relevant. Finally, notice that the Latin-like scripts that are commonly used to type characters on digital devices (cell phones or computers) in Chinese or Japanese, do not reach the same level of optimality as the original characters. A direct translation of Chinese/Japanese characters into a Latin alphabet plus some additional diacritics hence results in a reduction of the optimality of these languages in written form. Here our emphasis in the comparison is on reading written characters. In case of writing characters, bear in mind that Chinese and Japanese speakers cannot, in general, type a single Hanzi or Kanji character by touching just one key. For that reason, in the context of writing, the comparison would be fair if they were always able to type a single character by touching just one key. All these considerations together suggest that the pathways Western and Eastern writing systems took are not arbitrary, but (to some extent) guided by coding efficiency at the level of immediate constituents of words.

Removal of vowels in Latin script languages or romanizations. We finally consider a scenario in which recoding implies applying a decreasing transformation to word lengths, namely removing vowels. Given the invariance properties of $\Psi$ and $\Omega$ (that $\eta$ lacks; Table 3), we would expect them to yield approximately the same values for the original and the new scores computed by excluding vowels (while $\eta$ be less robust to that transformation). Consistently, the linear regression slopes for both $\Psi$ and $\Omega$ are close to 1, while that of $\eta$ is about 1.4. Furthermore, $\eta$ has both the largest standard regression error and the largest intercept (in absolute value).

All the scores experience a decrease after the removal of values, whose magnitude seems to be negatively associated with the original score. Importantly, $\Psi$ and $\Omega$ show a greater stability, but the latter seems to be the most capable of preserving the scores under weak recoding, having the lowest standard error, the highest Pearson correlation, and the smallest intercept. This is reasonable, given that $\Omega$ is endowed with invariance also under non-linear transformations, and there is no reason to assume that vowels would reduce word lengths in a linear way. Indeed, while on average a syllable contains one vowel, syllables can have different lengths, and thus be affected in different ways by the vowels removal. In conclusion, $\Psi$ and $\Omega$ are robust to vowel removal but $\Omega$ turns out to be more powerful than $\Psi$ in this respect.

A fundamental conclusion is that we have demonstrated the existence of scores that have the potential to abstract from certain characteristics in the design of a writing system, e.g. how vowels are coded (abjads as in Standard Arabic, where only consonants are code, or the Latin script in Romance languages, which codes for both consonants and vowels). Crucially, the choice of the optimality score should be determined by the characteristics of writing systems we want to
Question 6. Why do languages deviate from optimality?

We found that word lengths in languages are not 100% optimal. In previous research on the optimality of word lengths, we have argued that perceptibility and distinguishability factors as well as phonotactic constraints are likely to prevent languages from reaching theoretical optimality [10]. Along these lines, it has been demonstrated that the gap between average word length and the optimal one vanishes after controlling for morphological and graphotactical constraints [30]. However, while these constraints explain the actual level of compression of word lengths in languages, it is yet to be explained

(i) Why languages with shorter syntactic dependencies between words also tend to have shorter words [13]. That suggests that the online memory pressures leading to syntactic dependency distance minimization are responsible for compression of word lengths.

(ii) The relationship between word length and frequency rank matches the theoretical prediction that results from either (a) optimal non-singular coding or (b) a combination of Zipf’s rank-frequency law and optimal uniquely decodable encoding [47].

In addition to these factors that concern individuals (a speaker or a listener), the nature of the cultural process by which languages are optimized may also divert them from full optimality [9]. Languages are distributed and conventional communication systems. Reaching full optimality would imply coordinated changes in a system that may not be possible in a real community of speakers. In this article, we have considered a minimum baseline that preserves the strings that are already being used by a language and simply attempts to reorder their corresponding lengths so as to minimize average word length. Once a communication system has been established, such reordering is a serious challenge in a conventional system like language, as this implies changing the meaning of the strings. Languages can only increase their optimality gradually. Two speakers can change locally their conventions to increase their optimality but the challenge is to propagate those changes over the entire population. Among the gradual changes, some look easier than others for a cognitive system. Optimizing a system implies changing the lengths of the strings that are being assigned to each type. The lower the resemblance between the new string and the old string, the higher the cognitive effort. Thus language users will have difficulties to interpret a string with its new meaning or to change their mental mapping of meanings to strings. We actually see every-day examples of word length reduction of the cognitively easy kind: acronyms and abbreviations for frequently used expressions are used to communicate more efficiently (e.g. ‘org.’ to say ‘organization’, ‘asap’ to say ‘as soon as possible’). That is the case of certain areas of specialization (e.g. ‘CPU’ to say ‘central processing unit’ in Computer Science, ‘ACL’ to say ‘anterior cruciate ligament’ in Anatomy), communication on social media (e.g. ‘idk’ to say ‘I don’t know’, ‘brb’ to say ‘be right back’), or when taking personal notes.

However, there is also preliminary evidence for the opposite trend: word lengths seem to have been increasing over historical time in Chinese and Arabic [40,48]. This is relevant to our study for three reasons. First, it indicates a potential diachronic pattern of word lengths at odds with the law of abbreviation, which is a synchronic observation. Second, it suggests that the contribution to the cost of communication stemming from word lengths has been increasing over time in these languages. Third, it highlights the necessity for scores which are comparable over time. In fact, we have defined above word length scores which are less affected by the scarcity of texts samples. These are hence amenable to diachronic studies where textual material from earlier time periods is often sparse.

In sum, all the arguments above raise the question if full optimality could be reached just by gradual, local, distributed and cognitively easy changes. We thus have to ponder if reaching full optimality is actually necessary for a language. The actual degree of optimality of word lengths...
in languages (more than 50% on scale from 0 to 100% according to $\Psi$) may just be “good enough” in language production [46].

Future research
In this article, we have introduced two new scores that require further theoretical and empirical research. Although we have focused on $\Psi$ for the sake of simplicity, further attention to $\Omega$ and its variants is necessary. We have made a significant contribution to the problem of the best score for cross-linguistic research, but the problem should be the subject of further research.

For simplicity, we have investigated only the weak recoding problem. The next step is initiating research on the strong recoding problem, which may provide further arguments for the choice of the best score.

We have approached the degree of optimality of word lengths only synchronically. However, we have established some foundations for research on the evolution of communication. In particular, we hope that $\Psi$ contributes to revise the finding that word lengths have been increasing over centuries in Chinese and Arabic based on $L$, namely simple average word lengths [40,48], and extending the analysis to more languages. At present, these findings suggest that languages may have been evolving against the principle of compression.

Finally, our investigation of the degree of optimality of word durations has paved the way for exploring the degree of optimality of the durations of vocalizations or gestures of other species [3].

Data accessibility
Data and code for this research work are stored in GitHub: [https://github.com/IQL-course/IQL-Research-Project-21-22](https://github.com/IQL-course/IQL-Research-Project-21-22) [42].

Authors’ contributions
SP: Conceptualization, Formal Analysis, Investigation, Software, Supervision, Validation, Visualization, Writing-original draft, Writing-review & editing; ACM: Data curation, Resources, Software; JCM: Writing-review & editing; MW: Writing-review & editing, Software, Visualization; CB: Conceptualization, Writing-review & editing; RFC: Conceptualization, Formal analysis, Funding acquisition, Methodology, Project Administration, Supervision, Writing-original draft, Writing-review & editing.

Conflict of interest declaration
We declare we have no competing interests.

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References

1. Evans N, Levinson SC. 2009 The myth of language universals: language diversity and its importance for cognitive science. Behavioral and Brain Sciences 32, 429–492.
2. Zipf GK. 1949 Human behaviour and the principle of least effort. Cambridge (MA), USA: Addison-Wesley.
3. Semple S, i Cancho RF, Gustison M. 2022 Linguistic laws in biology. Trends in Ecology and Evolution 37, 53–66.
4. Bentz C, Ferrer-i-Cancho R. 2016 Zipf’s law of abbreviation as a language universal. In Bentz C, Jäger G, Yanovich I, editors, Proceedings of the Leiden Workshop on Capturing Phylogenetic Algorithms for Linguistics. University of Tübingen.
5. Piantadosi ST, Tily H, Gibson E. 2011 Word lengths are optimized for efficient communication. Proceedings of the National Academy of Sciences 108, 3526–3529.
6. Meylan SC, Griffiths TL. 2021 The challenges of large-scale, web-based language datasets: word length and predictability revisited. Cognitive Science 45, e12983.
7. Levshina N. 2022 Frequency, informativity and word length: insights from typologically diverse corpora. Entropy 24.
8. Koplenig A, Kupietz M, Wolfer S. 2022 Testing the relationship between word length, frequency, and predictability based on the German reference corpus. Cognitive Science 46, e13090.
9. Kanwal J, Smith K, Culbertson J, Kirby S. 2017 Zipf’s law of abbreviation and the principle of least effort: language users optimise a miniature lexicon for efficient communication. Cognition 165, 45–52.
10. Ferrer-i-Cancho R, Bentz C, Seguin C. 2019 Optimal coding and the origins of Zipfian laws. Journal of Quantitative Linguistics 29, 165–194.
11. Ferrer-i-Cancho R, Debowski L, Moscoso del Prado Martín F. 2013 Constant conditional entropy and related hypotheses. Journal of Statistical Mechanics p. L07001.
12. Ferrer-i-Cancho R. 2018 Optimization models of natural communication. Journal of Quantitative Linguistics 25, 207–237.
13. Ferrer-i-Cancho R, Gómez-Rodríguez C. 2021 Dependency distance minimization predicts compression. In Proceedings of the Second Workshop on Quantitative Syntax (Quasy, SyntaxFest 2021) pp. 45–57 Sofia, Bulgaria. Association for Computational Linguistics.
14. Ferrer-i-Cancho R, Hernández-Fernández A. 2013 The failure of the law of brevity in two New World primates. Statistical caveats. Glottotheory 4.
15. Ferrer-i-Cancho R, Gómez-Rodríguez C. 2021 Anti dependency distance minimization in short sequences. A graph theoretic approach. Journal of Quantitative Linguistics 28, 50 – 76.
16. Ferrer-i-Cancho R, Hernández-Fernández A, Lusseau D, Agoramoorthy G, Hsu MJ, Semple S. 2013 Compression as a universal principle of animal behavior. Cognitive Science 37, 1565–1578.
17. Liu H, Xu C, Liang J. 2017 Dependency distance: a new perspective on syntactic patterns in natural languages. Physics of Life Reviews 21, 171–193.
18. Gibson E, Futrell R, Piantadosi ST, Dautriche I, Mahowald K, Bergen L, Levy R. 2019 How efficiency shapes human language. Trends in Cognitive Sciences 23, 389–407.
19. Levshina N. 2022 Communicative efficiency: language structure and use. Cambridge: Cambridge University Press.
20. Ferrer-i-Cancho R, Bentz C. 2018 The evolution of optimized language in the light of standard information theory. In Cuskley C, Flaherty M, Little H, McCrohon L, Ravignani A, Verhoeef T, editors, The Evolution of Language: Proceedings of the 12th International Conference (EVOLANG XII). NCU Press.
21. Coupé C, Oh YM, Dediu D, Pellegrino F. 2019 Different languages, similar encoding efficiency: Comparable information rates across the human communicative niche. Science Advances 5, eaaw2594.
22. Ferrer-i-Cancho R, Gómez-Rodríguez C, Esteban JL, Alemany-Puig L. 2022 Optimality of syntactic dependency distances. *Physical Review E* **105**, 014308.

23. Koplenig A. 2021 Quantifying the efficiency of written language. *Linguistics Vanguard* **7**, 20190057.

24. Hawkins JA. 1998 Some issues in a performance theory of word order. In Siewierska A, editor, *Constituent order in the languages of Europe*. Berlin: Mouton de Gruyter.

25. Ferrer-i-Cancho R. 2004 Euclidean distance between syntactically linked words. *Physical Review E* **70**, 056135.

26. Tily HJ. 2010 The role of processing complexity in word order variation and change. PhD thesis Stanford University. Chapter 3: Dependency lengths.

27. Gulordava K, Merlo P. 2015 Diachronic trends in word order freedom and dependency length in dependency-annotated corpora of Latin and ancient Greek. In *Proceedings of the Third International Conference on Dependency Linguistics (Depling 2015)* pp. 121–130 Uppsala, Sweden. Uppsala University.

28. Gulordava K, Merlo P. 2016 Multi-lingual dependency parsing evaluation: a large-scale analysis of word order properties using artificial data. *Transactions of the Association for Computational Linguistics* **4**, 343–356.

29. Moreno Fernández J. 2021 The optimality of word lengths. Master’s thesis Barcelona School of Informatics Barcelona.

30. Pimentel T, Nikkarinen I, Mahowald K, Cotterell R, Blasi D. 2021 How (non-)optimal is the lexicon?. In *North American Chapter of the Association for Computational Linguistics*. 2021.

31. Borda M. 2011 *Fundamentals in information theory and coding*. Berlin: Springer.

32. Petrini S, Casas-i-Muñoz A, Cluet-i-Martinell J, Wang M, Bentz C, Ferrer-i-Cancho R. 2022 Direct and indirect evidence of compression of word lengths in languages. Zip’s law of abbreviation revisited. [https://arxiv.org/abs/2303.10128](https://arxiv.org/abs/2303.10128).

33. Cover TM, Thomas JA. 2006 *Elements of information theory*. New York: Wiley. 2nd edition.

34. Conover WJ. 1999 *Practical nonparametric statistics*. New York: Wiley. 3rd edition.

35. van den Heuvel E, Zhan Z. 2022 Myths about linear and monotonic associations: Pearson’s *r*, Spearman’s *ρ*, and Kendall’s *τ*. *The American Statistician* **76**, 44–52.

36. Sigurd TM, Thomas JA. 2006 *Elements of information theory*. New York: Wiley. 2nd edition.

37. Strauss U, Grzybek P, Altmann G. 2007 Word length and word frequency. In *Grzybek P, editor, Contributions to the science of text and language*, pp. 277–294. Dordrecht: Springer.

38. Semple S, Hsu MJ, Agoramoorthy G. 2010 Efficiency of coding in macaque vocal communication. *Biology Letters* **6**, 469–471.

39. Ferrer-i-Cancho R, Lusseau D. 2009 Efficient coding in dolphin surface behavioral patterns. *Complexity* **14**, 23–25.

40. Chen H, Liang J, Liu H. 2015 How does word length evolve in written Chinese?. *PLOS ONE* **10**, 1–12.

41. Daniels PT. 1990 Fundamentals of grammatology. *Journal of the American Oriental Society* **110**, 727–731.

42. Petrini S, Casas-i-Muñoz A, Cluet-i-Martinell J, Wang M, Bentz C, Ferrer-i-Cancho R. 2022 The optimality of word lengths. Theoretical foundations and an empirical study. [Github](https://github.com).

43. Joyce T, Hodošček B, Nishina K. 2012 Orthographic representation and variation within the Japanese writing system: some corpus-based observations. *Written Language and Literacy* **15**, 254–278.

44. Christensen D. 2005 Fast algorithms for the calculation of Kendall’s *τ*. *Computational Statistics* **20**, 51–62.

45. Goodman LA, Kruskal WH. 1963 Measures of association for cross classifications III: Approximate sampling theory. *Journal of the American Statistical Association* **58**, 310–364.

46. Goldberg AE, Ferreira F. 2022 Good-enough language production. *Trends in Cognitive Sciences* **26**, 300–311.

47. Hernández-Fernández A, G. Torres I, Garrido JM, Lacasa L. 2019 *Linguistic Laws in Speech: The Case of Catalan and Spanish*. Entropy **21**.

48. Milička J. 2018 Average word length from the diachronic perspective: the case of Arabic. *Linguistic Frontiers* **1**, 81–89.

49. Gujarati D, Porter D. 2008 *Basic econometrics*. McGraw-Hill Education.

50. DeGroot MH, Schervish MJ. 2002 *Probability and statistics*. Boston: Wiley 3rd edition.
Appendices

A. Theory

A.1 The relationship between $L$, $\eta$, $\Psi$ and Pearson correlation

First we show the relationship between $\Psi$ and covariance or Pearson correlation through their estimators.

Given a sample of $n$ points, $\{(x_1, y_1), ..., (x_i, y_i), ..., (x_n, y_n)\}$, the sample covariance is defined as

$$ s_{xy} = \frac{1}{n-1} \left( \sum_{i=1}^{n} x_i y_i - n \bar{x} \bar{y} \right), $$

where $\bar{x}$ is the sample mean of $x$ and $\bar{y}$ is the sample mean for $y$, i.e.

$$ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, $$
$$ \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i. $$

Now we replace the random variables $x$ and $y$ by $p$ (the probability of a type) and $l$ (the length/duration of a type). Accordingly, the sample of $n$ points becomes $\{(p_1, l_1), ..., (p_i, l_i), ..., (p_n, l_n)\}$, one point per type. Then the covariance between $p$ and $l$ is

$$ s_{pl} = \frac{1}{n-1} \left( \sum_{i=1}^{n} p_i l_i - n \bar{p} \bar{l} \right). $$

Recalling the definition of $L$ (Equation 1.1) and noting that $\bar{p} = \frac{1}{n}$ and $\bar{l} = L_r$ (Equation 2.1), we finally obtain

$$ s_{pl} = \frac{1}{n-1} (L - L_r). \quad (A.1) $$

Then $\Psi$ turns out to be proportional to the sample covariance, i.e.

$$ \Psi = \frac{L_x - L}{L_r - L_{min}} = -\frac{(n-1)}{L_r - L_{min}} s_{pl} \quad (A.2) $$

but with an opposite sign.

The sample Pearson correlation is

$$ r = \frac{s_{xy}}{s_x s_y}, \quad (A.3) $$

where $s_x$ and $s_y$ are the sample standard deviation of $x$ and $y$, i.e.

$$ s_x = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} $$
$$ s_y = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2}. $$

Combining Equation A.1 and Equation A.3, we find that

$$ r = \frac{L - L_r}{(n-1)s_{pl}}. $$
Combining Equation A.2 and Equation A.3, we find that $\Psi$ is negatively proportional to the sample Person correlation, i.e.

$$\Psi = -\frac{(n-1)s_ps_l}{L_r - L_{\min} r}.$$ 

Similarly, it is easy to see that $\eta$ and $L$ are linear functions of $r$ or $s_{pl}$. For instance,

$$\eta = \frac{1}{L_{\min}} [(n-1)s_{pl}r + L_r].$$

Other linear relationships are left as an exercise.

A.2 Invariance of Pearson correlation under linear transformation

The Pearson correlation between two random variables $X$ and $Y$ is the covariance normalized by the product of the standard deviations, i.e.

$$r(X,Y) = \frac{COV(X,Y)}{\sigma(X)\sigma(Y)}.$$ 

It is often stated that Pearson correlation $r(X,Y)$ between two random variables $X$ and $Y$ (or its estimator) is invariant under linear transformation, i.e. $r(X,Y)$ does not change if $X$ is replaced by $f(X) = aX + b$ and $y$ is replaced by $g(Y) = cY + d$ [49, Question 3.11]. However, the statement is not accurate enough as the following property clarifies.

**Property A.1.** $r(X,Y)$ is invariant under linear transformation.

- In a strict or strong sense, i.e. $r(aX + b, cY + d) = r(X,Y)$, if and only if $\text{sgn}(a) = \text{sgn}(c)$ and $r(aX + b, cY + d) = -r(X,Y)$, if and only if $\text{sgn}(a) \neq \text{sgn}(c)$.

- In a weak sense, i.e. $|r(aX + b, cY + d)| = |r(X,Y)|$, in any circumstance (namely independently of $a$, $b$, $c$ and $d$).

**Proof.** Knowing that [50, Sections 4.3 and 4.6]

$$COV(aX + b, cY + d) = acCOV(X,Y)$$

$$\sigma(aX + b) = |a|\sigma(X)$$

$$\sigma(cY + d) = |c|\sigma(Y),$$

it follows that

$$r(aX + b, cY + d) = \frac{ac}{|a||c|} r(X,Y).$$

Hence the statements of this property. \qed
A.3 Invariance of Kendall $\tau$ correlation under strictly monotonic transformation

The Kendall $\tau$ correlation between to random variables $X$ and $Y$ is defined on the probability that two pairs of values $(X_1, Y_1)$ and $(X_2, Y_2)$ are concordant, i.e.

$$Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\},$$

and on the probability that they are discordant, i.e.

$$Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\},$$

as [35]

$$\tau = \tau(X, Y) = Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\}.$$

Kendall $\tau$ correlation $\tau(X, Y)$ is invariant under non-linear transformation but under certain conditions. If $X$ is replaced by applying some function $h_x$ and $Y$ is replaced by some function $h_y$, then $\tau$ is invariant when both $h_x(X)$ and $h_y(Y)$ are strictly monotonically increasing or both $h_x(X)$ and $h_y(Y)$ are strictly monotonically decreasing ([35] and references therein). If only one variable is replaced, say $X$, then $h(X)$ must be strictly monotonically increasing. The argument is detailed in the following property and its proof.

**Property A.2.** $\tau(X, Y)$ is invariant strictly monotonic linear transformation

- In a strict or strong sense, i.e.
  
  $$\tau(h_x(X), h_y(Y)) = \tau(X, Y)$$

  if the trend of $h_x(X)$ and that of $h_y(Y)$ is the same (both increasing or both decreasing) and

  $$\tau(h_x(X), h_y(Y)) = -\tau(X, Y)$$

  if the trends differ (one is increasing and the other is decreasing) and

- In a weak sense, i.e.
  
  $$|\tau(h_x(X), h_y(Y))| = |\tau(X, Y)|,$$

  in any circumstance (namely independently of the trend of $h_x(X)$ and $h_y(Y)$).

**Proof.** Notice that

- $\tau$ is symmetric, namely, $\tau(X, Y) = \tau(Y, X)$.
- When $h_x(X)$ and $h_y(Y)$ are strictly monotonic (increasing or decreasing; the direction of $h_x(X)$ and $h_y(Y)$ does not need to be the same),

  $$Pr\{(h_x(X_1) - h_x(X_2))(h_y(Y_1) - h_y(Y_2)) = 0\} = Pr\{(X_1 - X_2)(Y_1 - Y_2) = 0\}.$$

  If we apply a strictly monotonically increasing ($i$) function $h_i$ to one of the variables, say $X$, we get that $\tau(h_i(X), Y) = \tau(X, Y)$. To see it, notice that

  $$Pr\{(h_i(X_1) - h_i(X_2))(Y_1 - Y_2) > 0\} = Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\}$$

  and then

  $$Pr\{(h_i(X_1) - h_i(X_2))(Y_1 - Y_2) < 0\} = 1 - Pr\{(h_i(X_1) - h_i(X_2))(Y_1 - Y_2) > 0\} - Pr\{(h_i(X_1) - h_i(X_2))(Y_1 - Y_2) = 0\} = 1 - Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\} - Pr\{(X_1 - X_2)(Y_1 - Y_2) = 0\} = Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\}.$$
By symmetry, we also have that $\tau(X, h_i(Y)) = \tau(X, Y)$. If we apply a strictly monotonically decreasing (d) function $h_d$ to one of the variables, say $X$, we get that $\tau(h_d(X), Y) = -\tau(X, Y)$. To see it, recall that

$$\Pr\{(h_d(X_1) - h_d(X_2))(Y_1 - Y_2) = 0\} = \Pr\{(X_1 - X_2)(Y_1 - Y_2) = 0\}$$

and notice that

$$\Pr\{(h_d(X_1) - h_d(X_2))(Y_1 - Y_2) > 0\} = \Pr\{(X_1 - X_2)(Y_1 - Y_2) < 0\}$$

and

$$\Pr\{(h_d(X_1) - h_d(X_2))(Y_1 - Y_2) < 0\} = \Pr\{(X_1 - X_2)(Y_1 - Y_2) > 0\}.$$ 

Hence $\tau(h_d(X), Y) = -\tau(X, Y)$. By symmetry, we also have that $\tau(X, h_d(Y)) = -\tau(X, Y)$.

Now suppose that we apply a strictly monotonic function $h_x$ to $X$ and a strictly monotonic function $h_y$ to $Y$.

- If both functions are increasing, by iterating on the arguments above, we get

$$\tau(h_x(X), h_y(Y)) = \tau(X, h_y(Y)) = \tau(X, Y).$$

- If both functions are decreasing, by iterating on the arguments above, we get

$$\tau(h_x(X), h_y(Y)) = -\tau(X, h_y(Y)) = -\tau(X, Y).$$

- If one function is decreasing and the other is increasing, say $h_x$ is increasing and $h_y$ is decreasing,

$$\tau(h_x(X), h_y(Y)) = \tau(X, h_y(Y)) = -\tau(X, Y).$$

If $h_x$ is decreasing and $h_y$ is increasing,

$$\tau(h_x(X), h_y(Y)) = -\tau(X, Y)$$

again by the symmetry of $\tau$.

Hence the statements of this property.

A.4 Constancy under optimal coding

A score exhibits constancy under optimality if it takes a constant value in case the system under consideration is organized optimally [22]. $\eta, \Psi$ and $\Omega$ exhibit constancy under optimality, namely they take a value of 1 when word lengths are minimum and actually their value never exceeds one. In case of optimal coding, $\tau \leq 0$ [10]. Although $r$ and $\tau$ are bounded below by -1, they do not necessary reach -1 when lengths are minimum [10]. Obviously, $L$ lacks any of these properties since $L \geq L_{min}$ and $L_{min}$ does not need to be one. In the language of mathematics,

Property A.3.

(i) $\Psi, \Omega, \eta \leq 1$, with equality if and only if $L_{min} = L$.

(ii) $-1 \leq r, \tau \leq 1$, but $L_{min} = L$ does not imply $r = -1$ or $\tau = -1$.

Proof.
(i) Since \( L_{\text{min}} \leq L \) by definition,
\[
\begin{align*}
\Omega &= \frac{\tau}{\tau_{\text{min}}} \leq 1 \\
\eta &= \frac{L_{\text{min}}}{L} \leq 1.
\end{align*}
\]
For the same reason,
\[-L \leq -L_{\text{min}}
\]
and adding \( L_r \) to both sides of the equation, one obtains
\[L_r - L \leq L_r - L_{\text{min}}.
\]
Dividing by \( L_r - L_{\text{min}} \) on both sides of the inequality, one gets
\[\frac{L_r - L}{L_r - L_{\text{min}}} \leq \frac{L_r - L_{\text{min}}}{L_r - L_{\text{min}}}
\]
because \( L_r \geq L_{\text{min}} \) and hence \( \Psi \leq 1 \).
Finally, notice that \( \tau_{\text{min}} \leq \tau \) by definition and \( \tau_{\text{min}} \) is negative [10]. Then, dividing by \( \tau_{\text{min}} \) on both sides of \( \tau \geq \tau_{\text{min}} \), we obtain
\[\Omega = \frac{\tau}{\tau_{\text{min}}} \leq \frac{\tau_{\text{min}}}{\tau_{\text{min}}} = 1.
\]
(ii) It is well-known that \(-1 \leq r, \tau \leq 1 \) [34, Section 5.4]. When \( L_{\text{min}} = L_r = -1 \) is only possible if the relationship between \( p \) and \( l \) is perfectly linear in the optimal shuffling of value \( l_i \)'s in the matrix; \( \tau = -1 \) can only be achieved if there are not ties neither within the \( p_i \)'s nor within the \( l_i \)'s [10].

\[\square\]

A.5 Technical remarks on normalization
\( \eta \) was designed to be normalized, namely, \( 0 \leq \eta \leq 1 \) [10]. However, a theoretical challenge in the calculation of \( \eta \) with the two versions of \( L_{\text{min}} \) from information theory (optimal non-singular coding and optimal uniquely decodable encoding in section 2.2), namely \( L_{\text{min}}^{\text{NS}} \) and \( L_{\text{min}}^{\text{NS}} \), is to ensure that the assumptions of each coding scheme hold. If a language satisfies them, then \( \eta \leq 1 \) because \( L \leq L_{\text{min}} \). If not, then \( \eta > 1 \) may be possible. The same problem may concern \( \Psi \) or \( \Omega \) depending on the choice of the minimum baseline. This is another reason why we choose a minimum baseline based on rank ordering, i.e. \( L_{\text{min}}^{\text{RO}} \), so that the random baseline and the minimum baseline are perfectly aligned. Thus, our choice of the baselines warrants that \( \eta, \Psi, \Omega \leq 1 \) for any communication system.

A.6 Stability under the null hypothesis

Property A.4.
\[\mathbb{E}[\Psi] = \mathbb{E}[\Omega] = \mathbb{E}[r] = \mathbb{E}[\tau] = 0
\]
while
\[\mathbb{E}[\eta] \geq \frac{L_{\text{min}}}{L_r}.
\]
Proof. It is well-known that $E[r] = E[\tau] = 0$ (see [35] and references therein). On the one hand,

$$E[\Omega] = E[\tau_{min}] = 1$$

and

$$E[\Psi] = E\left[\frac{L_r - L}{L_r - L_{min}}\right]$$

$$= \frac{1}{L_r - L_{min}} (L_r - E[L])$$

$$= \frac{1}{L_r - L_{min}} (L_r - L_r)$$

$$= 0.$$

On the other hand,

$$E[\eta] = E\left[\frac{L_{min}}{L}\right]$$

$$= L_{min} E\left[\frac{1}{L}\right].$$

$\frac{1}{L}$ is a convex function of $L$ because $L$ is positive by definition. Jensen’s inequality gives

$$E\left[\frac{1}{L}\right] \geq \frac{1}{E[L]}.$$ 

Finally

$$E[\eta] \geq \frac{L_{min}}{L_r}.$$

Obviously, $L$ lacks stability under the null hypothesis since $E[L] = L_r$ and that is the mean length of types.

Property A.4 only indicates a lower bound for $E[\eta]$. Then the true $E[\eta]$ may be stable under the null hypothesis in the sense that $E[\eta] = k$, where $k$ is some constant. The fact that $L_{min}, L_r > 0$ and Equation A.4 imply $k > 0$. In contrast, $E[\Psi] = E[\Omega] = k$ with $k = 0$. In section B.3, we demonstrate that $\eta$ is not stable under the null hypothesis with real data, indeed $E[\eta] \approx \frac{L_{min}}{L_r}$, confirm that $\Psi$ and $\Omega$ are stable under the null hypothesis while $L$ is not.

A.7 Invariance under linear transformation

Property A.5. We use a prime, $'$, to indicate the new value of a score after applying a linear transformation $g(l) = al + b$ to $l$ with $a > 0$. Let $\eta''$ be the value of $\eta$ after applying a proportional change of scale to $l$, namely $g(l)$ with $b = 0$. Then

$$\Psi = \Psi' \quad \Omega = \Omega'$$

$$r = r' \quad \tau = \tau'$$

$$\eta = \eta''$$

while $L = L'$ and $\eta = \eta'$ do not generally hold.
Proof. $r$ and $\tau$ are invariant under that linear transformation (section A.2 and section A.3), hence $r = r'$ and $\tau = \tau'$.

Knowing that

$$L = \sum_i p_i l_i,$$

it is easy to show that, after applying a linear transformation to word lengths ($g(l)$ can be increasing or decreasing), one obtains

$$L' = aL + b$$

$$L'_r = aL_r + b.$$ 

The condition $a > 0$ and Property 2.1 give

$$L'_\min = aL_{\min} + b.$$  \hspace{1cm} (A.5)

Hence

$$\psi' = \frac{L'_\tau - L'}{L'_r - L'_\min}$$

$$= \frac{aL_r + b - aL - b}{aL_r + b - aL_{\min} - b}$$

$$= \frac{a(L_r - L)}{a(L_r - L_{\min})}$$

$$= \psi.$$ 

In contrast,

$$\eta' = \frac{L_{\min}}{L'}$$

$$= \frac{aL_{\min} + b}{aL + b}.$$ 

When $b = 0$,

$$\eta' = \eta'' = \eta.$$

Concerning $\Omega$, recall that $\tau$ is invariant under strictly increasing transformation (section A.3). When $a > 0$, the linear transformation is strictly increasing, hence the invariance of $\tau$ under that transformation gives

$$\tau' = \tau(p, g(l)) = \tau(p, l) = \tau.$$ 

Besides, $a > 0$ and Property 2.1 give $\tau'_\min = \tau_{\min}$. \hfill \Box

A.8 Invariance under non-linear monotonic transformation

In the language of mathematics,

**Property A.6.** We use a prime, $'$, to indicate the new value of a score after applying a strictly increasing non-linear transformation $h(l)$ to $l$. Then

$$\tau = \tau'$$

$$\Omega = \Omega'$$

while

$$L = L'$$

$$r = r'$$

$$\psi = \psi'$$

$$\eta = \eta'.$$
do not generally hold.

Proof. Since \( h(l) \) is strictly increasing, then \( \tau \) is invariant (section A.2 and section A.3), i.e.

\[
\tau(p, l) = \tau(p, h(l)).
\]

As \( h(l) \) is monotonically increasing, Property 2.1 gives

\[
\tau_{\min}' = \tau_{\min}(p, h(l)) = \tau_{\min}(p, l).
\]

Finally, combining the results above

\[
\Omega' = \frac{\tau(p, h(l))}{\tau_{\min}(p, h(l))} = \frac{\tau(p, l)}{\tau_{\min}(p, l)} = \Omega.
\]

\( L' = L \) is trivially unwarranted. To see that \( r' = r \) is not warranted, suppose that \( r(p, l) = 1 \), namely the association between both variables is perfectly linear and increasing. If we apply a strictly increasing non-linear transformation \( h(l) \), then \( r(p, h(l)) < 1 \) because the association between \( p, h(l) \) is not linear. Similarly, \( \Psi' = \Psi \) is unwarranted. A counterexample suffices. Consider configuration \( C \) in Table 1. \( L_r = 2, L = 1.88 \) and \( L_{\min} = 1.14 \) give \( \Psi \approx 0.158 \). Now suppose that we apply \( h(l) = 2^l \) to \( C \). Then \( L_r' = \frac{14}{12}, L' = \frac{32}{25} \), and \( L_{\min}' = \frac{64}{75} \) give \( \Psi' = 0.8 \). Hence \( \Psi \neq \Psi' \). The finding is not surprising given that \( \Psi \) is a linear function of the Pearson correlation coefficient (section A.1). \( \eta \) cannot be invariant under that transformation because it is not even invariant if the transformation is strictly increasing but linear with non-zero slope (Property A.6).

\[\square\]

B. Analysis

Here we present complementary analyses, tables and plots.

B.1 Optimality scores

In Table B1, Table B2 and Table B3, we show the values of the optimality scores and their ingredients for PUD and for CV when length is measured in characters and also in duration, respectively.
Table B1: Word lengths and their optimality in PUD. Word length is measured in number of characters. Han script is used in its traditional variant.

| Language | Family     | Script | $L_{min}$ | $L$ | $L_r$ | $\tau$ | $\tau_{min}$ | $\eta$ | $\Psi$ | $\Omega$ |
|----------|------------|--------|-----------|-----|-------|--------|--------------|-------|-------|---------|
| Arabic   | Afro-Asiatic| Arabic | 3.40      | 4.16| 5.56  | -0.13  | -0.73        | 0.82  | 0.65  | 0.18    |
| Indonesian| Austronesian| Latin  | 4.14      | 5.98| 7.36  | -0.11  | -0.80        | 0.69  | 0.43  | 0.14    |
| Russian  | Indo-European| Cyrillic| 5.18     | 6.04| 8.08  | -0.19  | -0.64        | 0.86  | 0.71  | 0.30    |
| Hindi    | Indo-European| Devanagari| 3.11   | 4.09| 5.85  | -0.19  | -0.78        | 0.76  | 0.64  | 0.24    |
| Czech    | Indo-European| Latin  | 4.69      | 5.44| 7.27  | -0.22  | -0.67        | 0.86  | 0.71  | 0.34    |
| English  | Indo-European| Latin  | 3.77      | 4.86| 6.99  | -0.20  | -0.76        | 0.78  | 0.66  | 0.26    |
| French   | Indo-European| Latin  | 3.66      | 4.78| 7.55  | -0.17  | -0.76        | 0.77  | 0.71  | 0.22    |
| German   | Indo-European| Latin  | 4.51      | 5.73| 8.55  | -0.23  | -0.68        | 0.79  | 0.70  | 0.34    |
| Icelandic| Indo-European| Latin  | 4.44      | 5.32| 7.91  | -0.26  | -0.66        | 0.84  | 0.75  | 0.40    |
| Italian  | Indo-European| Latin  | 3.83      | 4.84| 7.74  | -0.18  | -0.76        | 0.79  | 0.74  | 0.23    |
| Polish   | Indo-European| Latin  | 5.17      | 6.06| 7.99  | -0.19  | -0.64        | 0.85  | 0.69  | 0.29    |
| Portuguese|           | Latin  | 3.56      | 4.52| 7.48  | -0.18  | -0.74        | 0.79  | 0.75  | 0.25    |
| Spanish  | Indo-European| Latin  | 3.73      | 4.63| 7.59  | -0.16  | -0.74        | 0.77  | 0.72  | 0.22    |
| Swedish  | Indo-European| Latin  | 4.30      | 5.41| 7.99  | -0.23  | -0.68        | 0.80  | 0.70  | 0.34    |
| Japanese | Japonic     | Japanese| 1.45    | 1.73| 2.71  | -0.21  | -0.60        | 0.84  | 0.78  | 0.35    |
| Japanese-strokes | Japonic | Japanese| 5.15 | 7.70| 13.91 | -0.09 | -0.78        | 0.67  | 0.71  | 0.12    |
| Japanese-romaji | Japonic | Latin  | 2.75     | 3.83| 6.07  | -0.15  | -0.80        | 0.72  | 0.67  | 0.19    |
| Korean   | Koreanic    | Hangul  | 2.42     | 2.75| 3.24  | -0.24  | -0.64        | 0.88  | 0.60  | 0.38    |
| Thai     | Kra-Dai     | Thai   | 3.16     | 4.33| 6.33  | -0.23  | -0.82        | 0.73  | 0.63  | 0.28    |
| Chinese  | Sino-Tibetan| Han     | 1.47     | 1.67| 2.18  | -0.24  | -0.55        | 0.88  | 0.72  | 0.44    |
| Chinese-strokes | Sino-Tibetan| Han     | 10.47    | 15.00| 21.59 | -0.18 | -0.77        | 0.70  | 0.59  | 0.24    |
| Chinese-pinyin | Sino-Tibetan| Latin  | 3.85     | 4.88| 6.53  | -0.18  | -0.77        | 0.79  | 0.61  | 0.23    |
| Turkish  | Turkic      | Latin  | 5.33     | 6.41| 7.93  | -0.24  | -0.66        | 0.83  | 0.58  | 0.36    |
| Finnish  | Uralic      | Latin  | 6.55     | 7.51| 9.31  | -0.23  | -0.60        | 0.87  | 0.65  | 0.39    |
Table B2: Word lengths and their optimality in CV. Word length is measured in number of characters. ‘Conlang’ stands for ‘constructed language’, that is an artificially created language.

This is not a family in the proper sense, and Conlang languages are not related in the common family sense.

| Language | family | script | $L_{min}$ | $L$ | $L_{opt}$ | $\tau_{MIN}$ | $\eta$ | $\Psi$ | $\Omega$ |
|----------|--------|--------|-----------|-----|----------|-------------|----|-----|-----|
| Arabic   | Afro-Asiatic | Arabic | 3.26 | 4.10 | 5.06 | -0.14 | -0.87 | 0.80 | 0.53 | 0.16 |
| Malay    | Afro-Asiatic | Latin | 3.72 | 5.07 | 7.35 | -0.20 | -0.81 | 0.73 | 0.63 | 0.24 |
| Vietnamese | Austroasiatic | Latin | 2.75 | 3.24 | 3.47 | -0.19 | -0.75 | 0.85 | 0.53 | 0.26 |
| Indonesian | Austroasian | Latin | 3.70 | 5.37 | 7.24 | -0.20 | -0.91 | 0.69 | 0.53 | 0.22 |
| Esperanto | Conlang | Latin | 3.23 | 4.83 | 7.73 | -0.18 | -0.92 | 0.67 | 0.65 | 0.19 |
| Interlingua | Conlang | Latin | 3.36 | 4.43 | 7.43 | -0.24 | -0.79 | 0.76 | 0.74 | 0.30 |
| Tamil    | Dravidian | Tamil | 4.65 | 5.68 | 7.08 | -0.28 | -0.88 | 0.82 | 0.58 | 0.32 |
| Persian  | Indo-European | Arabic | 2.69 | 3.80 | 5.49 | -0.21 | -0.92 | 0.71 | 0.60 | 0.22 |
| Assamese | Indo-European | Assamese | 4.10 | 4.57 | 5.36 | -0.31 | -0.68 | 0.90 | 0.62 | 0.46 |
| Russian  | Indo-European | Cyrillic | 4.05 | 6.31 | 9.00 | -0.13 | -0.94 | 0.64 | 0.54 | 0.14 |
| Ukrainian | Indo-European | Cyrillic | 4.52 | 5.52 | 7.67 | -0.16 | -0.86 | 0.82 | 0.68 | 0.19 |
| Panjabi  | Indo-European | Devanagari | 3.55 | 3.68 | 3.88 | -0.32 | -0.48 | 0.96 | 0.60 | 0.68 |
| Modern Greek | Indo-European | Greek | 3.61 | 4.85 | 7.64 | -0.24 | -0.85 | 0.75 | 0.69 | 0.29 |
| Breton   | Indo-European | Latin | 2.90 | 3.97 | 6.31 | -0.24 | -0.90 | 0.73 | 0.69 | 0.26 |
| Catalan  | Indo-European | Latin | 2.99 | 4.90 | 8.58 | -0.15 | -0.92 | 0.61 | 0.66 | 0.17 |
| Czech    | Indo-European | Latin | 3.81 | 4.83 | 7.17 | -0.22 | -0.92 | 0.79 | 0.69 | 0.24 |
| Dutch    | Indo-European | Latin | 3.24 | 4.72 | 8.26 | -0.28 | -0.95 | 0.69 | 0.71 | 0.29 |
| English  | Indo-European | Latin | 2.39 | 4.61 | 7.79 | -0.07 | -0.83 | 0.52 | 0.59 | 0.09 |
| French   | Indo-European | Latin | 2.67 | 5.04 | 8.13 | -0.04 | -0.85 | 0.53 | 0.57 | 0.04 |
| German   | Indo-European | Latin | 3.00 | 5.73 | 10.30 | -0.12 | -0.87 | 0.52 | 0.63 | 0.13 |
| Irish    | Indo-European | Latin | 3.01 | 4.20 | 6.58 | -0.21 | -0.93 | 0.71 | 0.66 | 0.23 |
| Italian  | Indo-European | Latin | 3.16 | 5.29 | 8.16 | -0.06 | -0.90 | 0.60 | 0.57 | 0.06 |
| Latvian  | Indo-European | Latin | 3.95 | 4.79 | 7.09 | -0.26 | -0.73 | 0.82 | 0.73 | 0.35 |
| Polish   | Indo-European | Latin | 3.82 | 5.27 | 7.87 | -0.17 | -0.94 | 0.72 | 0.64 | 0.18 |
| Portuguese| Indo-European | Latin | 3.14 | 4.53 | 7.49 | -0.19 | -0.91 | 0.69 | 0.68 | 0.21 |
| Romanian | Indo-European | Latin | 3.61 | 5.03 | 7.67 | -0.21 | -0.79 | 0.72 | 0.65 | 0.27 |
| Romandian| Indo-European | Latin | 3.83 | 4.94 | 7.56 | -0.24 | -0.79 | 0.78 | 0.70 | 0.30 |
| Slovenian| Indo-European | Latin | 3.65 | 4.56 | 6.43 | -0.21 | -0.87 | 0.80 | 0.67 | 0.24 |
| Spanish  | Indo-European | Latin | 2.74 | 5.01 | 7.92 | -0.03 | -0.91 | 0.55 | 0.56 | 0.04 |
| Swedish  | Indo-European | Latin | 3.08 | 4.48 | 7.67 | -0.09 | -0.90 | 0.85 | 0.73 | 0.21 |
| Welsh    | Indo-European | Latin | 2.79 | 4.17 | 7.05 | -0.21 | -0.91 | 0.67 | 0.68 | 0.23 |
| Western Frisian | Indo-European | Latin | 3.44 | 4.38 | 7.99 | -0.29 | -0.80 | 0.79 | 0.79 | 0.36 |
| Orya    | Indo-European | Odia | 3.86 | 4.21 | 5.35 | -0.24 | -0.79 | 0.85 | 0.58 | 0.46 |
| Dhivehi  | Indo-European | Thana | 1.97 | 3.32 | 7.61 | -0.16 | -0.84 | 0.59 | 0.76 | 0.19 |
| Georgian | Kartvelian | Georgian | 5.91 | 7.17 | 8.22 | -0.12 | -0.67 | 0.82 | 0.45 | 0.18 |
| Basque   | Language isolate | Latin | 4.44 | 6.41 | 8.89 | -0.16 | -0.91 | 0.69 | 0.56 | 0.18 |
| Mongolian | Mongolic | Mongolian | 4.18 | 5.47 | 7.31 | -0.23 | -0.86 | 0.76 | 0.59 | 0.26 |
| Kinyarwanda| Niger-Congo | Latin | 3.72 | 6.13 | 9.20 | -0.19 | -0.90 | 0.61 | 0.56 | 0.21 |
| Abkhazian | Northwest Caucasian | Cyrillic | 5.59 | 5.94 | 6.42 | -0.32 | -0.55 | 0.94 | 0.58 | 0.59 |
| Hakkha Chin | Sino-Tibetan | Latin | 2.56 | 3.24 | 5.29 | -0.29 | -0.81 | 0.78 | 0.73 | 0.35 |
| Chuvash   | Turkic | Cyrillic | 4.79 | 6.00 | 7.35 | -0.22 | -0.83 | 0.80 | 0.53 | 0.27 |
| Kirghiz   | Turkic | Cyrillic | 4.49 | 6.01 | 7.78 | -0.19 | -0.89 | 0.75 | 0.54 | 0.22 |
| Tajik     | Turkic | Cyrillic | 4.04 | 5.41 | 7.45 | -0.24 | -0.89 | 0.75 | 0.60 | 0.27 |
| Yakut     | Turkic | Cyrillic | 5.02 | 6.32 | 7.99 | -0.26 | -0.74 | 0.79 | 0.56 | 0.36 |
| Turkish   | Turkic | Latin | 4.60 | 6.00 | 8.09 | -0.22 | -0.89 | 0.77 | 0.60 | 0.24 |
| Estonian  | Uralic | Latin | 4.68 | 6.16 | 8.85 | -0.24 | -0.84 | 0.76 | 0.65 | 0.29 |
We use Spearman’s ρ instead of Kendall’s τ in the definition of Ω so as to understand if a different rank-correlation coefficient would lead to a change in the scale of variation of values of Ω, turning them closer to those of Ψ. In Table B4, we show Ωτ, the original definition of Ω, Ωρ, the new definition of Ω that results from replacing τ by ρ, and all the correlations necessary to compute them. The values of Ωρ increase with respect to those of Ωτ, but by no more than 10%, suggesting that the rather low values of Ω compared to those of Ψ are not due to the initial choice of τ to define Ω (Equation 1.4). We suspect that the low values of Ωρ and Ωτ could originate from some similarity between τ and ρ (e.g. both are rank correlation scores and then both may yield low values of Ω) or the fact that the template of the definition of Ωτ, and that of Ωρ, is the same (Ωτ and Ωρ differ only in the choice of the correlation coefficient). However, that issue should be the subject of further research.
Table B4: Kendall τ versus Spearman ρ in the frequency-length correlation and also in Ω in the PUD collection. τ and ρ are the original correlations and τ_{min} and ρ_{min} are the minimum correlations according to the rank ordering minimum baseline. Ω_{τ} is the original value of Ω whereas Ω_{ρ} is the value of Ω that is obtained when Kendall τ is replaced by Spearman ρ in the definition of Ω.

| language | family | script | τ  | ρ  | τ_{min} | ρ_{min} | Ω_{τ} | Ω_{ρ} |
|----------|--------|--------|----|----|---------|---------|-------|-------|
| Arabic   | Afro-Asiatic | Arabic | -0.13 | -0.15 | -0.72 | -0.81 | 0.18 | 0.19 |
| Indonesian | Austronesian | Latin | -0.11 | -0.14 | -0.80 | -0.88 | 0.14 | 0.15 |
| Russian | Indo-European | Cyrillic | -0.19 | -0.23 | -0.64 | -0.73 | 0.30 | 0.32 |
| Hindi | Indo-European | Devanagari | -0.19 | -0.23 | -0.78 | -0.86 | 0.24 | 0.27 |
| Czech | Indo-European | Latin | -0.22 | -0.27 | -0.67 | -0.75 | 0.34 | 0.36 |
| English | Indo-European | Latin | -0.20 | -0.24 | -0.76 | -0.85 | 0.26 | 0.29 |
| French | Indo-European | Latin | -0.17 | -0.21 | -0.76 | -0.84 | 0.22 | 0.24 |
| German | Indo-European | Latin | -0.23 | -0.28 | -0.68 | -0.78 | 0.34 | 0.36 |
| Icelandic | Indo-European | Latin | -0.26 | -0.32 | -0.66 | -0.75 | 0.40 | 0.42 |
| Italian | Indo-European | Latin | -0.18 | -0.22 | -0.76 | -0.84 | 0.23 | 0.26 |
| Polish | Indo-European | Latin | -0.19 | -0.23 | -0.64 | -0.73 | 0.29 | 0.31 |
| Portuguese | Indo-European | Latin | -0.18 | -0.22 | -0.74 | -0.83 | 0.25 | 0.27 |
| Spanish | Indo-European | Latin | -0.16 | -0.20 | -0.74 | -0.83 | 0.22 | 0.24 |
| Swedish | Indo-European | Latin | -0.23 | -0.28 | -0.68 | -0.77 | 0.34 | 0.36 |
| Japanese | Japonic | Japanese | -0.21 | -0.25 | -0.60 | -0.67 | 0.35 | 0.37 |
| Japanese-strokes | Japonic | Japanese | -0.09 | -0.12 | -0.78 | -0.87 | 0.12 | 0.13 |
| Japanese-romaji | Japonic | Latin | -0.15 | -0.19 | -0.80 | -0.87 | 0.19 | 0.21 |
| Korean | Koreanic | Hangul | -0.24 | -0.27 | -0.64 | -0.71 | 0.38 | 0.39 |
| Thai | Kra-Dai | Thai | -0.23 | -0.29 | -0.82 | -0.90 | 0.28 | 0.32 |
| Chinese | Sino-Tibetan | Han (Traditional variant) | -0.24 | -0.27 | -0.55 | -0.59 | 0.44 | 0.45 |
| Chinese-strokes | Sino-Tibetan | Han (Traditional variant) | -0.18 | -0.24 | -0.77 | -0.87 | 0.24 | 0.27 |
| Chinese-pinyin | Sino-Tibetan | Latin | -0.18 | -0.21 | -0.77 | -0.85 | 0.23 | 0.25 |
| Turkish | Turkic | Latin | -0.24 | -0.29 | -0.66 | -0.76 | 0.36 | 0.38 |
| Finnish | Uralic | Latin | -0.23 | -0.28 | -0.60 | -0.69 | 0.39 | 0.41 |

B.3 Stability under the null hypothesis

We estimate E[η], E[Ψ], and E[Ω] under the null hypothesis of a random mapping of word frequencies into lengths by means of a Monte Carlo procedure over a maximum of 10^6 randomizations. The goal is to confirm that Ψ and Ω both yield values tending to zero (Table 3) while checking if η is eventually stable under that null hypothesis.

Table B5: Summary statistics of estimates of E[η], E[Ψ], and E[Ω] under the null hypothesis, for languages in every collection and each definition of length in CV. In PUD, scores in strokes for Chinese and Japanese are excluded for the sake of homogeneity. ‘sd’ stands for standard deviation.

| score | collection | Min. | 1st Qu. | Median | Mean | 3rd Qu. | Max. | sd |
|-------|------------|------|--------|--------|------|--------|------|----|
| E[η]  | PUD-characters | 0.477 | 0.521 | 0.551 | 0.580 | 0.646 | 0.747 | 0.081 |
|       | CV-characters | 0.268 | 0.423 | 0.495 | 0.518 | 0.575 | 0.915 | 0.145 |
|       | CV-duration | 0.213 | 0.426 | 0.504 | 0.518 | 0.603 | 0.882 | 0.139 |
| E[Ψ]  | PUD-characters | -0.000 | -0.000 | 0.000 | -0.000 | 0.000 | 0.000 | 0.000 |
|       | CV-characters | -0.000 | -0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
|       | CV-duration | -0.000 | -0.000 | 0.000 | 0.000 | 0.000 | 0.001 | 0.000 |
| E[Ω]  | PUD-characters | -0.000 | -0.000 | -0.000 | -0.000 | 0.000 | 0.000 | 0.000 |
|       | CV-characters | -0.000 | -0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
|       | CV-duration | -0.000 | -0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

The summary in Table B5 confirms the predictions in section A.6, namely that E[Ψ] and E[Ω] are close to 0, while E[η] takes values spread across its whole domain, ranging from a minimum of 0.21 in CV (duration) to a maximum of 0.91 in CV (characters). However, the values taken by
\( \mathbb{E}[\eta] \) are not arbitrary: the lower bound derived in section A.6 is actually a very accurate estimate of the expectation itself (Figure B1).

**Figure B1:** The Monte Carlo estimate of \( \mathbb{E}[\eta] \) against its theoretical lower bound, namely \( \frac{L_{\text{min}}}{L_r} \). The purple line indicates the identity function, \( \mathbb{E}[\eta] = \frac{L_{\text{min}}}{L_r} \). (a) PUD collection with length measured in characters. (b) CV collection with length measured in characters. (c) CV collection with length measured in duration.

C. Desirable properties of the scores

Here we investigate empirically the desirable properties of scores presented in section 3.5.

C.1 Dependence of the scores on basic parameters

First, we explore the association between the optimality scores \( (\eta, \Psi, \Omega) \) and basic language parameters: observed alphabet size \( A \) (number of distinct characters), observed vocabulary size \( n \) (number of types) and text length \( T \) (number of tokens). The alphabet and vocabulary size observed in the analyzed text samples are lower bounds of the true values in the language due to undersampling. However, the real parameters are likely to be a strictly monotonic function of the observed ones. For this reason, we measure the association with Kendall \( \tau \) correlation, that is known to be invariant under strictly monotonically increasing transformations (Appendix A). Given the recent challenges to the common view about the limits of Pearson \( r \) correlation [35], we also use \( r \) to verify the robustness of the results.

In the parallel corpus PUD, where there is less diversity in terms of basic parameters, no significant association is found for any score, both with Kendall and with Pearson correlation (Figure C1 (a,d)). Concerning the CV collection, \( \eta \) and \( \Omega \) show sensitivity to the alphabet and sample size of a language, especially when length is measured in characters (Figure C1 (b,e)). However, the association of the scores with alphabet size might be a reflection of the strong correlation existing between the amount of tokens and the subset of the real alphabet that can be observed in them. Interestingly, when length is measured in duration, the association with \( \eta \) remains while no significant association is detected between \( \Omega \) and basic parameters, consistently for the two correlation coefficients (Figure C1 (c,f)). Thus, \( \Psi \) is the only score among the analysed ones that does not seem to be related to \( A, n, \) or \( T \), even in a highly heterogeneous collection.
Figure C1: Correlations between optimality scores ($\eta$, $\Psi$, and $\Omega$) and basic language parameters ($T$, $n$, and $A$). Recall that $T$ is the number of tokens, $n$ is the number of types, $A$ is the alphabet size. Correlation tests are two-sided and $p$-values are corrected using Holm-Bonferroni correction. A crossed value indicates a non significant correlation coefficient at a 95% confidence level. For PUD, only immediate word constituents are used to measure word length in Chinese and Japanese. (a) Kendall $\tau$ correlation in PUD collection with word length measured in characters. (b) Kendall $\tau$ correlation in the CV collection with word length measured in characters. (c) Kendall $\tau$ correlation for CV collection with word length measured in duration. (d) Same as (a) using Pearson correlation. (e) Same as (b) using Pearson correlation. (f) Same as (c) using Pearson correlation.

C.2 Convergence speed

We investigate the dependence of $\eta$, $\Psi$, and $\Omega$ on text size $t$ (in number of tokens) within each language. We wish to know whether they converge or not and their convergence speed.

Methods

For a given $t$, we sample $t$ tokens uniformly at random from the whole text. We do not use a prefix of length $t$ of the text as in related research on convergence of entropy estimators [51] because both PUD and CV contain a series of disconnected groups of sentences. In PUD each group of
sentences is taken from a distinct text source and groups consist of a few sentences, often just one sentence. By proceeding in this way, we are easing the convergence of the estimators of the scores that we use, that neglect word order in their definition. We explore $t$ by increasing powers of 2. For each $t$, we perform $10^2$ experiments and compute the average over the available values. Indeed, for small sample sizes, computations could result in divisions by 0, or the impossibility to compute $\tau$ (for $\Omega$) given the low amount of distinct types, thus not yielding a value for that particular experiment.

Results

In Figure C2, we show the results for PUD; in Figure C3 for CV when length is measured in characters; in Figure C4 when length is measured in duration. In all cases $\Omega$ has the widest range of variation, and it keeps rapidly decreasing as sample size grows, making it hard to make inferences about its possible convergence. Both $\eta$ and $\Psi$ evolve within a smaller range, however – while the former mainly shows a slow but decreasing trend – $\Psi$ seems to approach stability in some languages, especially when the sample is large enough. In fact, we can mainly observe this phenomenon in CV, which contains larger samples.

PUD is parallel but texts are rather short. Now we focus on CV because it has the largest samples and then is ideal for investigating convergence. The languages for which convergence is suggested visually are: English, Kinyarwanda, Persian, Tatar and Welsh when length is measured in characters, and French, Kinyarwanda, Persian, Romanian and Tatar when length is measured in duration. The observed trends also suggest that the scores computed in the under-sampled languages are likely to vary largely in a larger sample of the same language, turning comparisons of these scores potentially be misleading in non-parallel sources.
Figure C2: Convergence of the scores. PUD collection. Values of the optimality scores ($\eta$, $\Psi$, and $\Omega$) for increasing number of tokens $t$. For each value of $t$, we show the average value of the score over $10^2$ random experiments (only those for which a value could be computed). The dashed line marks 0.
Figure C3: Convergence of the scores. CV collection with word length measured in characters. Values of the optimality scores ($\eta$, $\Psi$, and $\Omega$) for increasing number of tokens $t$. For each value of $t$, we show the average value of the score over $10^2$ random experiments (only those for which a value could be computed). The dashed line marks 0.
Figure C4: Convergence of the scores. CV collection with word length measured in duration. Values of the optimality scores ($\eta$, $\Psi$, and $\Omega$) for increasing number of tokens $t$. For each value of $t$, we show the average value of the score over $10^2$ random experiments (only those for which a value could be computed). The dashed line marks $0$.

C.3 Can a score be replaced by a simpler one?

To examine the replaceability of a score with a simpler one, we analyze the correlations between scores. In particular, $L$ is considered the simplest score (it does not integrate any baseline), after which comes $\eta$ (as it only integrates the minimum baseline). According to both Kendall and Pearson correlation, the only significant association in PUD is found between $\Omega$ and $\eta$ (Figure C5 (a-d)). This relation is consistently found also for both definitions of length in CV (Figure C5 (b, c, e and f)), suggesting its reality even in non-parallel corpora. In the context of CV, $\Psi$ turns out to be significantly associated to $L$, the simplest score, especially when length is measured in characters.
Despite the significance, CV is not parallel and the associations are not particularly strong (the absolute value of the correlations does not exceed 0.6), suggesting that the additional complexity introduced by $\Psi$ still adds additional information on top of the information provided by $L$.

Figure C5: The correlation between scores. Correlation tests were two-sided the $p$-values were corrected with a Holm-Bonferroni correction. A crossed value signals a non significant correlation coefficient at a 95% confidence level. (a) Kendall $\tau$ correlation in PUD collection with word length measured in characters. (b) Kendall $\tau$ correlation in the CV collection with word length measured in characters. (c) Kendall $\tau$ correlation for CV collection with word length measured in duration. (d) Same as (a) using Pearson correlation. (e) Same as (b) using Pearson correlation. (f) Same as (c) using Pearson correlation.