A fundamental solution to the time-periodic Stokes equations

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The concept of a fundamental solution to the time-periodic Stokes equations in dimension $n \geq 2$ is introduced. A fundamental solution is then identified and analyzed. Integrability and pointwise estimates are established.

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1 Introduction

Classically, fundamental solutions are defined for systems of linear partial differential equations in $\mathbb{R}^n$. Specifically, a fundamental solution to the Stokes system ($n \geq 2$)

$$
\begin{align*}
-\Delta v + \nabla p &= f & \text{in } \mathbb{R}^n, \\
\text{div } v &= 0 & \text{in } \mathbb{R}^n,
\end{align*}
$$

(1.1)

with unknowns $v : \mathbb{R}^n \to \mathbb{R}^n$, $p : \mathbb{R}^n \to \mathbb{R}$ and data $f : \mathbb{R}^n \to \mathbb{R}^n$, is a tensor-field

$$
\Gamma_{\text{Stokes}} := \begin{pmatrix}
\Gamma_{11}^s & \cdots & \Gamma_{1n}^s \\
\vdots & \ddots & \vdots \\
\Gamma_{n1}^s & \cdots & \Gamma_{nn}^s \\
\gamma_1^s & \cdots & \gamma_n^s
\end{pmatrix} \in \mathcal{S}'(\mathbb{R}^n)^{(n+1)\times n}
$$

that satisfies\(^1\)

$$
\begin{align*}
-\Delta \Gamma_{ij}^s + \partial_i \gamma_j^s &= \delta_{ij} \delta_{\mathbb{R}^n}, \\
\partial_j \Gamma_{ij}^s &= 0,
\end{align*}
$$

(1.2)

\(^1\)We make use of the Einstein summation convention and implicitly sum over all repeated indices.
where $\delta_{ij}$ and $\delta_{Gn}$ denotes the Kronecker delta and delta distribution, respectively. For arbitrary $f \in \mathcal{S}(\mathbb{R}^n)^n$, a solution $(v, p)$ to (1.1) is then given by the componentwise convolution

$$
\begin{pmatrix}
v \\
p
\end{pmatrix} := \Gamma_{\text{Stokes}} \ast f,
$$

which at the outset is well-defined in the sense of distributions. In the specific case of the Stokes fundamental solution $\Gamma_{\text{Stokes}}$ above, $L^p$-integrability and pointwise decay estimates for $(v, p)$ can be established from (1.3). We refer to the standard literature such as [3] and [7] for these well-known results.

The aim of this paper is to identify a fundamental solution to the time-periodic Stokes system

$$
\begin{cases}
\partial_t u - \Delta u + \nabla p = f & \text{in } \mathbb{R}^n \times \mathbb{R}, \\
\text{div } u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}, \\
u(x, t) = u(x, t + T)
\end{cases}
$$

with unknowns $u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $p : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ corresponding to time-periodic data $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ with the same period, that is, $f(x, t) = f(x, t + T)$. Here $T \in \mathbb{R}$ denotes the (fixed) time-period. Moreover, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ denotes the spatial and time variable, respectively. The main objective is to establish a framework which enables us to define and identify a fundamental solution $\Gamma_{\text{TPStokes}}$ to (1.4) with the property that a solution $(u, p)$ is given by a convolution

$$
\begin{pmatrix}
v \\
p
\end{pmatrix} := \Gamma_{\text{TPStokes}} \ast f.
$$

Having obtained this goal, we shall then examine to which extent regularity such as $L^p$-integrability and pointwise estimates of the solution can be derived from (1.5).

Since time-periodic data $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, $(x, t) \rightarrow f(x, t)$ are non-decaying in $t$, a framework based on classical convolution in $\mathbb{R}^n \times \mathbb{R}$ cannot be applied. Instead, we reformulate (1.4) as a system of partial differential equations on the locally compact abelian group $G := \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z}$. More specifically, we exploit that $T$-time-periodic functions can naturally be identified with mappings on the torus group $T := \mathbb{R}/T\mathbb{Z}$ in the time variable $t$. In the setting of the Schwartz-Bruhat space $\mathcal{S}(G)$ and corresponding space of tempered distributions $\mathcal{S}'(G)$, we can then define a fundamental solution $\Gamma_{\text{TPStokes}}$ to (1.4) as a tensor-field

$$
\Gamma_{\text{TPStokes}} := \begin{pmatrix}
\Gamma_{\text{TPS}11} & \cdots & \Gamma_{\text{TPS}1n} \\
\vdots & \ddots & \vdots \\
\Gamma_{\text{TPS}n1} & \cdots & \Gamma_{\text{TPS}nn} \\
\gamma_{11} & \cdots & \gamma_{nn}
\end{pmatrix} \in \mathcal{S}'((n+1) \times n)
$$

that satisfies

$$
\begin{cases}
\partial_t \Gamma_{ij}^{\text{TPS}} - \Delta \Gamma_{ij}^{\text{TPS}} + \partial_i \gamma_{ij}^{\text{TPS}} = \delta_{ij} \delta_G, \\
\partial_i \Gamma_{ij}^{\text{TPS}} = 0
\end{cases}
$$

(1.7)
Theorem 1.1. Let the time-periodic Stokes equations (1.4) in the sense of \( \mathcal{S}'(G) \)-distributions. A solution to the time-periodic Stokes system (1.4) is then given by (1.5), provided the convolution is taken over the group \( G \).

The aim in the following is to identify a tensor-field \( \Gamma_{TPStokes} \in \mathcal{S}'(G)^{(n+1)\times n} \) satisfying (1.7). We shall describe \( \Gamma_{TPStokes} \) as a sum of the steady-state Stokes fundamental solution \( \Gamma_{Stokes} \) and a remainder part satisfying remarkably good integrability and pointwise decay estimates. It is well-known that the components of the velocity part \( \Gamma^\varepsilon \in \mathcal{S}'(\mathbb{R}^n)^n \) and pressure part \( \gamma^\varepsilon \in \mathcal{S}'(\mathbb{R}^n)^n \) of \( \Gamma_{Stokes} \) are functions

\[
\Gamma^\varepsilon_{ij}(x) := \begin{cases} 
\frac{1}{2\omega_n} \left( \delta_{ij} \log \left( |x|^{-1} \right) + \frac{x_i x_j}{|x|^2} \right) & \text{if } n = 2, \\
\frac{1}{2\omega_n} \left( \delta_{ij} \frac{1}{n-2} |x|^{2-n} + \frac{x_i x_j}{|x|^n} \right) & \text{if } n \geq 3,
\end{cases}
\]

\[
\gamma^\varepsilon_i(x) := \frac{1}{\omega_n} \frac{x_i}{|x|^n},
\]

respectively; see for example [3, IV.2]. Here, \( \omega_n \) denotes the surface area of the \((n-1)\)-dimensional unit sphere in \( \mathbb{R}^n \). Our main theorem reads:

**Theorem 1.1.** Let \( n \geq 2 \). There is a fundamental solution \( \Gamma_{TPStokes} \in \mathcal{S}'(G)^{(n+1)\times n} \) to the time-periodic Stokes equations (1.4) on the form (1.6) satisfying (1.7) and

\[
\begin{align*}
\Gamma^{TPS} &= \Gamma^s \otimes 1_T + \Gamma^\perp, \\
\gamma^{TPS} &= \gamma^s \otimes \delta_T
\end{align*}
\]

with \( \Gamma^\perp \in \mathcal{S}'(G)^{n\times n} \) satisfying

\[
\forall q \in \left(1, \frac{n}{n-1}\right) : \quad \Gamma^\perp \in L^q(G)^{n\times n},
\]

\[
\forall r \in [1, \infty) \forall \varepsilon > 0 \exists C > 0 \forall |x| \geq \varepsilon : \quad \|\Gamma^\perp(x, \cdot)\|_{L^r(T)} \leq \frac{C}{|x|^n},
\]

\[
\forall q \in (1, \infty) \exists C > 0 \forall f \in \mathcal{S}(G) : \quad \|\Gamma^\perp \ast f\|_{W^{2,1,q}(G)} \leq C \|f\|_{L^q(G)},
\]

where \( T \) denotes the torus group \( T := \mathbb{R}/\mathbb{Z} \), \( 1_T \in \mathcal{S}'(T) \) the constant 1, \( \delta_T \in \mathcal{S}'(T) \) the Dirac delta distribution on \( T \), \( \ast \) the convolution on \( G \), and \( W^{2,1,q}(G) \) the Sobolev space of order 2 in \( x \) and order 1 in \( t \).

**Remark 1.2.** We shall briefly demonstrate how the fundamental solution (1.8)–(1.9) can be applied in a more classical setting of the time-periodic Stokes equations to obtain a representation formula, integrability properties and decay estimates of a solution. The time-periodic Stokes equations are typically studied in a function analytical framework based on the function space

\[
C^\infty_{0,per}(\mathbb{R}^n \times \mathbb{R}) := \{ f \in C^\infty(\mathbb{R}^n \times \mathbb{R}) \mid f(x, t + T) = f(x, t) \quad \forall f \in C^\infty_0(\mathbb{R}^n \times [0, T]) \},
\]

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upon which \( \|f\|_q := \|f\|_{L^q(\mathbb{R}^n \times [0,T])} \) is a norm. Time-periodic Lebesgue and Sobolev spaces are defined as

\[
L^q_{\text{per}}(\mathbb{R}^n \times \mathbb{R}) := C^\infty_{0,\text{per}}(\mathbb{R}^n \times \mathbb{R})^q, \quad W^{2,1,q}_{\text{per}}(\mathbb{R}^n \times \mathbb{R}) := C^\infty_{0,\text{per}}(\mathbb{R}^n \times \mathbb{R})^{2,1,q}, \quad \|f\|_{2,1,q} := \left( \sum_{|\alpha| \leq 2} \|\partial^\alpha_x f\|_q^2 + \sum_{|\beta| \leq 1} \|\partial^\beta_t f\|_q^2 \right)^{\frac{1}{q}}.
\]

It is easy to see that \( L^q_{\text{per}}(\mathbb{R}^n \times \mathbb{R}) \) and \( W^{2,1,q}_{\text{per}}(\mathbb{R}^n \times \mathbb{R}) \) are isometrically isomorphic to \( L^q(G) \) and \( W^{2,1,q}(G) \), respectively. Regarding \( \Gamma^\perp \) as a tensor-field in \( L^q_{\text{per}}(\mathbb{R}^n \times \mathbb{R}) \), we obtain by Theorem 1.1 for any sufficiently smooth vector-field \( f \), say \( f \in C^\infty_{0,\text{per}}(\mathbb{R}^n \times \mathbb{R})^n \), a solution \( (u,p) \) to (1.4) given by 

\[
\begin{align*}
    u_1 &:= \left[ \Gamma^\perp \ast_{\mathbb{R}^n} \left( \frac{1}{T} \int_0^T f(\cdot,s) \, ds \right) \right](x,t), \\
    u_2 &:= \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T \Gamma^\perp(x-y,t-s) f(y,s) \, ds \, dy
\end{align*}
\]

and \( p(x,t) := [\gamma^\perp \ast_{\mathbb{R}^n} f(\cdot,t)](x) \). Properties of \( u_1 \) and \( p \) can be derived directly from the Stokes fundamental solution \( (\Gamma^\perp, \gamma^\perp) \), which, given the simple structure of \( (\Gamma^\perp, \gamma^\perp) \), is elementary and can be found in standard literature such as [3] and [7]. To fully understand the structure of a time-periodic solution, it therefore remains to investigate \( u_2 \). For this purpose, (1.10)–(1.12) of Theorem 1.1 is useful. For example, (1.12) yields integrability \( u_2 \in W^{2,1,q}_{\text{per}}(\mathbb{R}^n \times \mathbb{R}) \), and from (1.11) the pointwise decay estimate \( |u_2(x,t)| \leq C|x|^{-n} \) can be derived for large values of \( x \).

**Remark 1.3.** Theorem 1.1 implies that \( \Gamma^\perp \) decays faster than \( \Gamma^\perp \) as \( |x| \to \infty \); both in terms of summability (1.10) and pointwise (1.11). This information provides us with a valuable insight into the asymptotic structure as \( |x| \to \infty \) of a time-periodic solution to the Stokes equations. More precisely, from the representation formula \( u = u_1 + u_2 \) with \( u_1 \) and \( u_2 \) given by (1.13), and the fact that \( \Gamma^\perp \) decays faster than \( \Gamma^\perp \) as \( |x| \to \infty \), it follows that the leading term in an asymptotic expansion of \( u \) coincides with the leading term in the expansion of \( u_1 \). Since \( u_1 \) is a solution to a steady-state Stokes problem, it is well-known how to identify its leading term. In conclusion, Theorem 1.1 tells us that time-periodic solutions to the Stokes equations essentially have the same well-known asymptotic structure as \( |x| \to \infty \) as steady-state solutions—a nontrivial fact, which is not clear at the outset.

The Stokes system is a linearization of the nonlinear Navier-Stokes system. A fundamental solution to the time-periodic Stokes equations can therefore be used to develop a linear theory for the time-periodic Navier-Stokes problem. The study of the time-periodic Navier-Stokes equations was initiated by Serrin [6], Prodi [5], and Yudovich [8]. Since then, a number of papers have appeared based on the techniques proposed...
by these authors. The methods all have in common that the time-periodic problem is investigated in a setting of the corresponding initial-value problem, and time-periodicity of a solution only established a posterior. With an appropriate time-periodic linear theory, a more direct approach to the time-periodic Navier-Stokes problem can be developed, which may reveal more information on the solutions. The asymptotic structure mentioned in Remark 1.3 is but one example.

2 Preliminaries

Points in $\mathbb{R}^n \times \mathbb{R}$ are denoted by $(x, t)$ with $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We refer to $x$ as the spatial and to $t$ as the time variable.

We denote by $B_R := B_R(0)$ balls in $\mathbb{R}^n$ centered at 0. Moreover, we let $B_{R,r} := B_R \setminus \overline{B}_r$ and $B^R := \mathbb{R}^n \setminus \overline{B}_r$

For a sufficiently regular function $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, we put $\partial_t u := \partial_x u$. The differential operators $\Delta$, $\nabla$ and $\text{div}$ act only in the spatial variables. For example, $\text{div} u := \sum_{j=1}^n \partial_j u_j$ denotes the divergence of $u$ with respect to the $x$ variables.

We let $G$ denote the group $G := \mathbb{R}^n \times \mathbb{T}$, with $\mathbb{T}$ denoting the torus group $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. $G$ is equipped with the quotient topology and differentiable structure inherited from $\mathbb{R}^n \times \mathbb{R}$ via the quotient mapping $\pi : \mathbb{R}^n \times \mathbb{R} \to G$, $\pi(x, t) := (x, [t])$. Clearly, $G$ is a locally compact abelian group with Haar measure given by the product of the Lebesgue measure $dx$ on $\mathbb{R}^n$ and the (normalized) Haar measure $dt$ on $\mathbb{T}$. We implicitly identify $\mathbb{T}$ with the interval $[0, \mathcal{T})$, whence the (normalized) Haar measure on $\mathbb{T}$ is determined by

$$\forall f \in C(\mathbb{T}) : \int_{\mathbb{T}} f \, dt := \frac{1}{\mathcal{T}} \int_0^\mathcal{T} f(t) \, dt.$$  

We identify the dual group $\hat{G}$ with $\mathbb{R}^n \times \mathbb{Z}$ and denote points in $\hat{G}$ by $(\xi, k)$.

We denote by $\mathcal{S}(G)$ the Schwartz-Bruhat space of generalized Schwartz functions; see [2]. By $\mathcal{S}'(G)$ we denote the corresponding space of tempered distributions. The Fourier transform on $G$ and its inverse takes the form

$$\mathcal{F}_G: \mathcal{S}(G) \to \mathcal{S}(\hat{G}), \quad \mathcal{F}_G[u](\xi, k) := \int_{\mathbb{R}^n} \int_{\mathbb{T}} u(x, t) \, e^{-ix\xi - ik\frac{2\pi}{\mathcal{T}} t} \, dt \, dx,$$

$$\mathcal{F}_G^{-1}: \mathcal{S}(\hat{G}) \to \mathcal{S}(G), \quad \mathcal{F}_G^{-1}[w](x, t) := \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} w(\xi, k) \, e^{ix\xi + ik\frac{2\pi}{\mathcal{T}} t} \, d\xi,$$

respectively, provided the Lebesgue measure $d\xi$ is normalized appropriately. By duality, $\mathcal{F}_G$ extends to a homeomorphism $\mathcal{F}_G: \mathcal{S}'(G) \to \mathcal{S}'(\hat{G})$. Observe that $\mathcal{F}_G = \mathcal{F}_{\mathbb{R}^n} \circ \mathcal{F}_T$.

We denote by $\delta_{\mathbb{R}^n}$, $\delta_{\mathbb{T}}$, $\delta_{\mathbb{Z}}$ the Dirac delta distribution on $\mathbb{R}^n$, $\mathbb{T}$ and $\mathbb{Z}$, respectively. Observe that $\delta_{\mathbb{Z}}$ is a function with $\delta_{\mathbb{Z}}(k) = 1$ if $k = 0$ and $\delta_{\mathbb{Z}}(k) = 0$ otherwise. Also note that $\mathcal{F}_{\mathbb{T}}[1_{\mathbb{T}}] = \delta_{\mathbb{Z}}$.

Given a tensor $\Gamma \in \mathcal{S}'(G)^{n \times m}$, we define the convolution of $\Gamma$ with vector field $f \in \mathcal{S}'(G)^{m}$ as the vector field $\Gamma \ast f \in \mathcal{S}'(G)^{n}$ with $[\Gamma \ast f]_i := \Gamma_{ij} \ast f_j$. 

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The $L^q(G)$-spaces with norm $\|\cdot\|_q$ are defined in the usual way via the Haar measure $dx dt$ on $G$. We further introduce the Sobolev space

$$W^{2,1,q}(G) := C_0^\infty(G) \cap L^2(G) \cap L^{1,q}(G),$$

where $C_0^\infty(G)$ denotes the space of smooth functions of compact support on $G$.

We emphasize at this point that a framework based on $G$ is a natural setting for the time-periodic Stokes equations. It is easy to see that lifting by the restriction $\pi|_{\mathbb{R}^n \times [0,T)}$ of the quotient mapping provides us with an equivalence between the time-periodic Stokes problem (1.4) and the system

$$\begin{cases}
\partial_t u - \Delta u + \nabla p = f & \text{in } G, \\
\text{div } u = 0 & \text{in } G.
\end{cases}$$

An immediate advantage obtained by writing the time-periodic Stokes problem as system of equations on $G$ is the ability to then apply the Fourier transform $\mathcal{F}_G$ and re-write the problem in terms of Fourier symbols. We shall take advantage of this possibility in the proof of the main theorem below.

We use the symbol $C$ for all constants. In particular, $C$ may represent different constants in the scope of a proof.

3 Proof of main theorem

Proof of Theorem 1.1. Put

$$\Gamma^\perp := \mathcal{F}_G^{-1} \left[ \frac{1 - \delta_Z(k)}{\xi^2 + i \frac{2\pi}{T} k} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right],$$

where $I \in \mathbb{R}^{n \times n}$ denotes the identity matrix. Since

$$M : \hat{G} \to \mathbb{C}, \quad M(\xi,k) := \frac{1 - \delta_Z(k)}{|\xi|^2 + i \frac{2\pi}{T} k}$$

is bounded, that is, $M \in L^\infty(\hat{G})$, the inverse Fourier transform in (3.1) is well-defined as a distribution in $\mathcal{S}'(G)^{n \times n}$. Now define $\Gamma_{\text{TPS}}$ and $\gamma_{\text{TPS}}$ as in (1.8) and (1.9). It is then easy to verify that $(\Gamma_{\text{TPS}}, \gamma_{\text{TPS}})$ is a solution to (1.7).

It remains to show (1.10)–(1.12). For this purpose, we introduce for $k \in \mathbb{Z} \setminus \{0\}$ the function

$$\Gamma_{\text{SSR}}^k : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}, \quad \Gamma_{\text{SSR}}^k(x) := \frac{i}{4} \left( \sqrt{-i \frac{2\pi}{T} k} \right)^{\frac{n-2}{2}} H_{\frac{n}{2}-1}^{(1)} \left( \sqrt{-i \frac{2\pi}{T} k} \cdot |x| \right),$$

(3.3)
where $H^{(1)}_\alpha$ denotes the Hankel function of the first kind, and $\sqrt{z}$ the square root of $z$ with positive imaginary part. As one readily verifies, $\Gamma^k_{\text{SSR}}$ is a fundamental solution to the Helmholtz equation

$$
\left(-\Delta + i\frac{2\pi}{T}k\right)\Gamma^k_{\text{SSR}} = \delta_{\mathbb{R}^n} \quad \text{in } \mathbb{R}^n.
$$

(3.4)

Clearly, $\Gamma^k_{\text{SSR}} \in \mathcal{S}'(\mathbb{R}^n)$. Moreover, its Fourier transform is given by the function

$$
\mathcal{F}_{\mathbb{R}^n}[\Gamma^k_{\text{SSR}}](\xi) = \frac{1}{|\xi|^2 + i\frac{2\pi}{T}k}.
$$

(3.5)

From the estimates in Lemma 3.1 below, we see that

$$
\int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma^k_{\text{SSR}}|^2 \right)^{\frac{q}{2}} \, dx < \infty
$$

for $q \in (1, 1)$. By Hölder’s inequality and Parseval’s theorem, we thus deduce

$$
\int_{\mathbb{R}^n} \left( \int_{\mathbb{T}} |\mathcal{F}^{-1}_T[(1 - \delta_Z(k)) \cdot \Gamma^k_{\text{SSR}}(x)](t)|^q \, dt \right) \, dx \\
\leq C \int_{\mathbb{R}^n} \left( \int_{\mathbb{T}} |\mathcal{F}^{-1}_T[(1 - \delta_Z(k)) \cdot \Gamma^k_{\text{SSR}}]|^2 \, dt \right)^{\frac{q}{2}} \, dx
$$

$$
\leq C \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma^k_{\text{SSR}}|^2 \right)^{\frac{q}{2}} \, dx < \infty.
$$

It is well-known that the Riesz transform $\mathcal{R}_k(f) := \mathcal{F}^{-1}_{\mathbb{R}^n}\left[\frac{\xi_k}{|\xi|} \cdot \mathcal{F}_{\mathbb{R}^n}[f]\right]$ is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1, \infty)$. Consequently, we obtain $\mathcal{R}_i \circ \mathcal{R}_j(\mathcal{F}^{-1}_T[(1 - \delta_Z(k)) \cdot \Gamma^k_{\text{SSR}}]) \in L^q(\mathbb{R}^n)$ for $q \in (1, \frac{n}{n-1})$. Recalling (3.5), we compute

$$
[\delta_{ij} \mathcal{R}_h \circ \mathcal{R}_h - \mathcal{R}_i \circ \mathcal{R}_j](\mathcal{F}^{-1}_T[(1 - \delta_Z(k)) \cdot \Gamma^k_{\text{SSR}}]) = \Gamma^L_{ij}
$$

and conclude (1.10).

In order to show (1.11), we further introduce

$$
\Gamma_L : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{C}, \quad \Gamma_L := \begin{cases} 
-\frac{1}{2\pi} \log |x| & \text{if } n = 2, \\
\frac{1}{(n-2)\omega_n} |x|^{2-n} & \text{if } n > 2,
\end{cases}
$$

which is the fundamental solution to the Laplace equation $\Delta \Gamma_L = \delta_{\mathbb{R}^n}$. As one may verify directly from the pointwise definitions of $\Gamma^k_{\text{SSR}}$ and $\Gamma_L$, the convolution integral

$$
\int_{\mathbb{R}^n} \Gamma_L(x - y) \Gamma^k_{\text{SSR}}(y) \, dy =: \Gamma_L \ast \Gamma^k_{\text{SSR}}(x)
$$

(3.6)
exists for all \( x \in \mathbb{R}^n \setminus \{0\} \). In fact, the function given by \( \Gamma_L \ast \Gamma^k_{SSR} \) belongs to \( L^1_{loc}(\mathbb{R}^n) \) and defines a tempered distribution in \( \mathcal{S}'(\mathbb{R}^n) \). One may further verify that also the second order derivatives of \( \Gamma_L \ast \Gamma^k_{SSR} \) are given by convolution integrals

\[
\partial_i \partial_j [\Gamma_L \ast \Gamma^k_{SSR}](x) = \int_{\mathbb{R}^n} \partial_i \Gamma_L(x - y) \partial_j \Gamma^k_{SSR}(y) \, dy,
\]

from which it follows that their Fourier transform are functions

\[
\mathcal{F}_{\mathbb{R}^n} \left[ \partial_i \partial_j [\Gamma_L \ast \Gamma^k_{SSR}] \right](\xi) = \frac{\xi_i \xi_j}{|\xi|^2} \frac{1}{|\xi|^2 + i \frac{2\pi}{T} |k|}.
\]

We infer from the expression above that

\[
\Gamma^\perp_{ij} = \mathcal{F}^{-1}_T \left[ (1 - \delta_Z(k)) \cdot [\delta_{ij} \partial_h \partial_h - \partial_i \partial_j] [\Gamma_L \ast \Gamma^k_{SSR}] \right].
\]

Employing Hausdorff-Young’s inequality in combination with the pointwise estimate from Lemma 3.2 below, we obtain for \( r \in [2, \infty) \)

\[
\| \Gamma^\perp(x, \cdot) \|_{L^r(\mathbb{T})} \leq \left( \sum_{k \in \mathbb{Z}} |(1 - \delta_Z(k)) \cdot [\delta_{ij} \partial_h \partial_h - \partial_i \partial_j] [\Gamma_L \ast \Gamma^k_{SSR}]|^r \right)^{\frac{1}{r}} \leq C |x|^{-n} \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-r} \right)^{\frac{1}{r}} \leq C |x|^{-n},
\]

which concludes (1.11).

The convolution \( \Gamma^\perp \ast f \) can be expressed in terms of a Fourier multiplier

\[
\Gamma^\perp \ast f = \mathcal{F}^{-1}_G \left[ M(\xi, k) \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \mathcal{F}_G[f] \right],
\]

with \( M \) given by (3.2). As already mentioned, \( M \in L^\infty(\hat{G}) \). As one may verify, also second order spatial derivatives \( \partial_i \partial_j M \in L^\infty(\hat{G}) \) and the time derivative \( \partial_t M \in L^\infty(\hat{G}) \) are bounded. Based on this information, (1.12) can be established. For the details of the argument, we refer the reader to [4, Proof of Theorem 4.8].

**Lemma 3.1.** The function \( \Gamma^k_{SSR} \) defined in (3.3) satisfies

\[
\left( \sum_{k \in \mathbb{Z} \setminus \{0\}} |\Gamma^k_{SSR}(x)|^2 \right)^{\frac{1}{2}} \leq C |x|^{1-n} e^{-\frac{1}{2} \sqrt{T} |x|}. \tag{3.8}
\]

**Proof.** The estimates are based on the asymptotic properties of Hankel functions summarized in Lemma 3.3 below. We start with the case \( n > 2 \). Employing (3.16) with \( \varepsilon = 1 \), we deduce

\[
\forall k \in \mathbb{Z} \forall |x| \geq \sqrt{\frac{T}{2\pi}}: \quad \left| H^{(1)}_{\frac{1}{2} - 1} \left( \sqrt{-i \frac{2\pi}{T} k} \cdot |x| \right) \right| \leq C |k|^{-\frac{1}{2}} |x|^{-\frac{1}{2}} e^{-\frac{1}{2} \sqrt{T} |k|^{\frac{1}{2}} |x|}. \tag{3.9}
\]
Employing (3.17) with $R = 1$, we obtain:

$$\forall k \in \mathbb{Z} \ \forall |x| \leq \sqrt{\frac{T}{2\pi}} |k|^{-\frac{1}{2}} : \ \left| H_{k/2-1}^{(1)} \left( \sqrt{-i\frac{2\pi}{T} k \cdot |x|} \right) \right| \leq C |k|^{-\frac{n+2}{2}} |x|^{-\frac{n+2}{2}}. \quad (3.10)$$

It follows that

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} |I_{SSR}^k(x)|^2 \leq C \left( \sum_{|k| \leq \frac{T}{2\pi} |x|^{-2}} |k|^{\frac{n+2}{2}} |x|^{2-n} |k|^{\frac{n+2}{2}} |x|^{2-n} \right. \left. + \sum_{|k| > \frac{T}{2\pi} |x|^{-2}} |k|^{\frac{n+2}{2}} |x|^{2-n} |k|^{-\frac{1}{2}} |x|^{-1} e^{-2\sqrt{T} |k|^\frac{1}{2} |x|} \right) \leq C \left( |x|^{-2} \cdot |x|^{2(2-n)} \cdot \chi_{[0, \sqrt{T}]}(|x|) \right. \left. + \sum_{|k| \geq 1} |k|^{\frac{n+2}{2}} |x|^{1-n} e^{-2\sqrt{T} |k|^\frac{1}{2} |x|} \right). \quad (3.11)$$

For $|q| < 1$ we observe that

$$\sum_{k \geq 1} |k|^{\frac{n-3}{2}} q^{k^2} \leq \sum_{j=1}^\infty \sum_{k=j^2}^{(j+1)^2-1} k^{\frac{n-3}{2}} q^{k^2} \leq \sum_{j=1}^\infty (j+1)^{n-3} q^j = \sum_{j=1}^\infty j (j+1)^{n-3} q^j \leq q \sum_{j=1}^\infty j (j+1) (j+2) \ldots (j+n-3) q^{j-1} = q^{-n-1} \sum_{j=1}^\infty q^{j+n-3} = q^{-n-2} \sum_{j=1}^\infty q^j = q^{-n-2} [(1-q)^{-1}] = (n-2)! \cdot q^{-1} (1-q)^{1-n},$$

from which we deduce

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} |I_{SSR}^k(x)|^2 \leq C \left( |x|^{2(1-n)} \cdot \chi_{[0, \sqrt{T}]}(|x|) \right. \left. + |x|^{1-n} e^{-2\sqrt{T} |x|} (1 - e^{-2\sqrt{T} |x|})^{1-n} \right) \leq C |x|^{2(1-n)} e^{-\sqrt{T} |x|}. \quad 9$$
and consequently (3.8) in the case \( n > 2 \). In the case \( n = 2 \), we employ (3.18) to deduce
\[
\forall k \in \mathbb{Z} \forall |x| \leq \sqrt{\frac{T}{2\pi}}|k|^{-\frac{1}{2}} : \quad |H^{(1)}_0 \left( \sqrt{-i \frac{2\pi}{T}} k \cdot |x| \right)| \leq C \log \left( \sqrt{\frac{2\pi}{T}} |k|^\frac{1}{2} |x| \right). \tag{3.12}
\]
It follows in the case \( n = 2 \) that
\[
\sum_{|k| \leq \frac{T}{2\pi} |x|^{-2}} |\Gamma_{\text{SSR}}^k(x)|^2 \leq C \sum_{|k| \leq \frac{T}{2\pi} |x|^{-2}} \left| \log \left( \sqrt{\frac{2\pi}{T}} |k|^\frac{1}{2} |x| \right) \right|^2 \\
\leq C \int_0^1 \left| \log \left( \sqrt{\frac{2\pi}{T}} t^\frac{1}{2} |x| \right) \right|^2 dt \cdot \chi_{[0, \sqrt{\frac{T}{2\pi}}]}(|x|) \tag{3.13}
\]
\[
\leq C|x|^{-2} \int_0^1 \left| \log(s) \right|^2 s \, ds \cdot \chi_{[0, \sqrt{\frac{T}{2\pi}}]}(|x|) \\
\leq C|x|^{-2} \cdot \chi_{[0, \sqrt{\frac{T}{2\pi}}]}(|x|).
\]
Estimate (3.9) is still valid in the case \( n = 2 \). We can thus proceed as in (3.11) and obtain (3.8) also in the case \( n = 2 \). \( \square \)

**Lemma 3.2.** The convolution \( \Gamma_L * \Gamma_{\text{SSR}}^k \) defined in (3.6) satisfies
\[
\forall \varepsilon > 0 \exists C > 0 \forall |x| \geq \varepsilon : \quad |\partial_i \partial_j [\Gamma_L * \Gamma_{\text{SSR}}^k](x)| \leq C |k|^{-1} |x|^{-n}. \tag{3.14}
\]

**Proof.** Fix \( \varepsilon > 0 \) and consider some \( x \in \mathbb{R}^n \) with \( |x| \geq \varepsilon \). Put \( R := \frac{|x|}{2} \). Let \( \chi \in C_0^\infty(\mathbb{R}; \mathbb{R}) \) be a “cut-off” function with
\[
\chi(r) = \begin{cases} 
0 & \text{when } 0 \leq |r| \leq \frac{1}{2}, \\
1 & \text{when } 1 \leq |r| \leq 3, \\
0 & \text{when } 4 \leq |r|. 
\end{cases}
\]
Define \( \chi_R : \mathbb{R}^n \to \mathbb{R} \) by \( \chi_R(y) := \chi(R^{-1}|y|) \). We use \( \chi_R \) to decompose the integral in (3.7) as
\[
\partial_i \partial_j [\Gamma_L * \Gamma_{\text{SSR}}^k](x) = \int_{B_{4R,R/2}} \partial_i \Gamma_L(x-y) \partial_j \Gamma_{\text{SSR}}^k(y) \chi_R(y) \, dy \\
+ \int_{B_R} \partial_i \Gamma_L(x-y) \partial_j \Gamma_{\text{SSR}}^k(y) \left( 1 - \chi_R(y) \right) \, dy \\
+ \int_{B_{4R}} \partial_i \Gamma_L(x-y) \partial_j \Gamma_{\text{SSR}}^k(y) \left( 1 - \chi_R(y) \right) \, dy \\
=: I_1(x) + I_2(x) + I_3(x).
\]

\footnote{I would like to thank Prof. Toshiaki Hishida for suggesting this estimate to me and thereby improving my original proof.}
Recalling the definition (3.3) of $I_{SSR}^k$ as well as the property (3.15) and the estimate (3.16) of the Hankel function, we can estimate for $|y| \geq R/2$:

$$\left| \partial_j I_{SSR}^k(y) \right| \leq C |k|^{-\frac{n+2}{2}} \left( \left| \partial_j \left[ |y|^{\frac{2-n}{2}} H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{T} k \cdot |y|} \right) \right] \right|$$

$$+ \left| |y|^{\frac{2-n}{2}} \partial_j \left[ H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{T} k \cdot |y|} \right) \right] \right|$$

$$\leq C \left( |k|^{\frac{n-2}{2}} |y|^{-\frac{n-2}{2}} + |k|^{\frac{n-2}{2}} |y|^{-\frac{n}{2} + \frac{1}{2}} \right) e^{-\sqrt{T/\pi} |k|^2 |y|}$$

$$\leq C |k|^{-1} |y|^{-(n+1)}.$$

Consequently, we obtain:

$$|I_1(x)| \leq C \int_{B_{4R, R/2}} |x - y|^{-n} |k|^{-1} |y|^{-(n+1)} \, dy \leq C |k|^{-1} R^{-n}.$$

To estimate $I_2$, we integrate partially and employ polar coordinates to deduce

$$|I_2(x)| \leq C \int_{B_R} |\partial_j \partial_t \Gamma^t(x - y) | I_{SSR}^k(y) + |\partial_t \Gamma^t(x - y) | I_{SSR}^k(y) | R^{-1} \, dy$$

$$\leq C \int_{B_R} R^{-n} |I_{SSR}^k(y)| \, dy$$

$$\leq C \int_{B_R} R^{-n} |k|^\frac{n-2}{2} |y|^{\frac{2-n}{2}} \left| H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{T} k \cdot |y|} \right) \right| \, dy$$

$$\leq C \int_0^R R^{-n} |k|^\frac{n-2}{2} r^{\frac{n}{2}} \left| H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{T} k \cdot r} \right) \right| \, dr$$

$$\leq C \int_0^\infty R^{-n} |k|^{-1} s^{\frac{n}{2}} \left| H_{\frac{n-2}{2}}^{(1)} \left( \sqrt{-i \frac{2\pi}{T} s/s |k|} \right) \right| \, ds.$$

Employing in the case $n > 2$ estimate (3.17) in combination with (3.16), we obtain

$$|I_2(x)| \leq C R^{-n} |k|^{-1} \left( \int_0^1 s^{\frac{n-2}{2}} s^{\frac{2-n}{2}} ds + \int_1^\infty s^{\frac{n}{2}} s^{-\frac{1}{2}} e^{-\sqrt{T/\pi} s} ds \right) \leq C R^{-n} |k|^{-1}.$$

When $n = 2$, we use estimate (3.18) in combination with (3.16) and obtain also in this case

$$|I_2(x)| \leq C R^{-n} |k|^{-1} \left( \int_0^1 s \cdot \log \left( \sqrt{T/\pi} s \right) ds + \int_1^\infty s^{\frac{1}{2}} e^{-\sqrt{T/\pi} s} ds \right) \leq C R^{-n} |k|^{-1}.$$
In order to estimate $I_3$, we again integrate partially and utilize (3.16):

\[
|I_3(x)| \leq C \int_{B^{3R}} \left| \partial_j \partial_i \Gamma_L(x-y) \right| |F^{k}_{\text{SSR}}(y)| + \left| \partial_i \Gamma_L(x-y) \right| |F^{k}_{\text{SSR}}(y)| R^{-1} dy
\]

\[
\leq C \int_{B^{3R}} R^{-n} |F^{k}_{\text{SSR}}(y)| dy
\]

\[
\leq C \int_{B^{3R}} R^{-n} |k|^{\frac{n-2}{4}} |y|^{\frac{2-n}{2}} \left| H_{\frac{1}{2}n}^{(1)} \left( \sqrt{-i T k} \cdot |y| \right) \right| \left| \partial_j \partial_i \Gamma_L(x-y) \right| \left| \partial_j \partial_i \Gamma_L(x-y) \right| R^{-1} dy
\]

\[
\leq C \int_{B^{3R}} R^{-n} |k|^{\frac{n-2}{4}} |y|^{\frac{1-n}{2}} e^{\sqrt{T} |k|^2 |y|} dy
\]

\[
\leq C \int_{B^{3R}} R^{-n} |k|^{\frac{n-2}{4}} |y|^{\frac{1-n}{2}} (\frac{1}{2} |y|)^{-\frac{n+1}{4}} dy \leq C R^{-n} |k|^{-\frac{1}{2}} \leq C R^{-n} |k|^{-1}.
\]

Since $|x| = 2R$, we conclude (3.14) by collecting the estimates for $I_1$, $I_2$ and $I_3$.

\[\square\]

**Lemma 3.3.** Hankel functions are analytic in $\mathbb{C} \setminus \{0\}$ with

\[
\forall \nu \in \mathbb{C} \forall z \in \mathbb{C} \setminus \{0\}: \quad \frac{d}{dz} H^{(1)}_{\nu}(z) = H^{(1)}_{\nu-1}(z) - \frac{\nu}{z} H^{(1)}_{\nu}(z). \quad (3.15)
\]

The Hankel functions satisfy the following estimates:

\[
\forall \nu \in \mathbb{C} \forall z \geq \varepsilon : \quad |H^{(1)}_{\nu}(z)| \leq C |z|^{-\frac{1}{2}} e^{-\text{Im} z}, \quad (3.16)
\]

\[
\forall \nu \in \mathbb{R}^+ \forall z \leq R : \quad |H^{(1)}_{\nu}(z)| \leq C |z|^{-\nu}, \quad (3.17)
\]

\[
\forall z \leq R : \quad |H^{(1)}_{0}(z)| \leq C \log(|z|). \quad (3.18)
\]

**Proof.** The recurrence relation (3.15) is a well-known property of various Bessel functions; see for example [1, 9.1.27]. We refer to [1, 9.2.3] for the asymptotic behaviour (3.16) of $H^{(1)}_{\nu}(z)$ as $z \to \infty$. See [1, 9.1.9 and 9.1.8] for the asymptotic behaviour (3.17) and (3.18) of $H^{(1)}_{\nu}(z)$ as $z \to 0$.

\[\square\]

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