Inversion Formulas for the Spherical Radon–Dunkl Transform

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Abstract. The spherical Radon–Dunkl transform $R_\kappa$, associated to weight functions invariant under a finite reflection group, is introduced, and some elementary properties are obtained in terms of $h$-harmonics. Several inversion formulas of $R_\kappa$ are given with the aid of spherical Riesz–Dunkl potentials, the Dunkl operators, and some appropriate wavelet transforms.

Key words: spherical Radon–Dunkl transform; $h$-harmonics; inversion formula; wavelet

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1 Introduction

Let $\langle x, y \rangle$ denote the usual Euclidean inner product of $x, y \in \mathbb{R}^{d+1}$, and $S^d = \{ x : \|x\| = 1 \}$ the unit sphere in $\mathbb{R}^{d+1}$. We use $d\omega_k$ to denote the surface (Lebesgue) measure on a $k$-dimensional sphere. The spherical Radon transform $R$ is one of the tools in integral geometry, which is defined, for $f \in C(S^d)$, by

$$Rf(x) = \Lambda_{d-1} \int_{\langle x, y \rangle = 0} f(y) d\omega_{d-1}(y), \quad x \in S^d,$$

where $\Lambda_{d-1}$ is the surface area of $S^{d-1}$. There are a number of papers devoting to the study of the spherical Radon transform $R$ by different methods (see [1, 13, 14, 15, 16, 17, 21, 22, 23, 24, 25, 26, 27, 28]), and to its applications to various problems (see [10, 11]). Some deep results about $R$ were obtained with the aid of spherical harmonics (see [21, 22, 23, 24, 25, 26, 27, 28]), and furthermore, $R$ is a special case of the spherical means

$$M_\tau f(x) = \frac{\Lambda_{d-1}}{(1 - \tau^2)^{(d-1)/2}} \int_{\langle x, y \rangle = \tau} f(y) d\omega_{d-1}(y), \quad x \in S^d,$$

by taking $\tau = 0$, and also of the spherical Riesz potentials

$$I^\alpha f(x) = \frac{\Gamma((1 - \alpha)/2)}{2\pi d/2 \Gamma(\alpha/2)} \int_{S^d} |\langle x, y \rangle|^{\alpha-1} f(y) d\omega_d(y), \quad x \in S^d,$$

by taking the limit $\lim_{\alpha \to 0^+} I^\alpha f = Rf$ in some sense (see [22]). The former is a tool in approximation on the sphere $S^d$, and the later is one of the research objectives in harmonic analysis on $S^d$.

The purpose of the present paper is to study an analogous model of the spherical Radon transform $R$ in Dunkl's theory. This is based on the definition of the generalized spherical
means $M_\kappa^\ast f(x)$ due to [33] (instead of $M_\tau^\ast$, the notation $T_\theta^\ast$ with $\tau = \cos \theta$ was used there), in terms of the equation

$$
\int_{-1}^{1} M_\kappa^\ast f(x) g(\tau) w_{\kappa}(\tau) d\tau = c_\kappa \int_{\mathbb{S}^d} f(y) V_\kappa[g((x, \cdot))](y) h_\kappa^2(y) d\omega_d(y)
$$

for any $g \in L^1([-1, 1]; w_{\kappa})$, where $w_{\kappa}(t) = \tilde{c}_{\kappa-1/2}(1-t^2)^{1/2}$, $V_\kappa$ is the intertwining operator associated to a given finite reflection group, and $h_\kappa^2$ is the related weight function (for details concerning them and other notations in the equation, see the next section). We define $R_\kappa$ by $R_\kappa f = M_\kappa^0 f$, and call $R_\kappa$ the spherical Radon–Dunkl transform. Although $M_\kappa^\ast$ is defined implicitly, it is a proper extension of $M_\tau$ and $M_\kappa^0 = M_\tau$, and moreover, from [2] and [33, 34, 35, 36], $M_\kappa^\ast$ shares many properties with $M_\tau$ and plays the same roles in weighted approximation and related harmonic analysis on the sphere $\mathbb{S}^d$. One could expect that the spherical Radon–Dunkl Transform $R_\kappa$ would have similar features to $R$ and be a suitable tool in reconstruction of functions in weighted spaces. This is the motivation of the paper. Despite less closed representation, a further work worth doing is to find applications of $R_\kappa$ in geometry or other fields.

The paper is organized as follows. In Section 2 some necessary facts in Dunkl’s theory are reviewed, and in Section 3 the spherical Radon–Dunkl transform $R_\kappa$ is defined and some of elementary properties are obtained in terms of $h$-harmonics. Sections 4 and 5 are devoted to inversion formulas of $R_\kappa$, which are given by means of spherical Riesz–Dunkl potentials $I_\kappa^\ast$, the Dunkl operators, and some appropriate wavelet transforms. These conclusions generalize part of those in [21, 22, 23].

## 2 Some facts in Dunkl’s theory

Let $G$ be a finite reflection group on $\mathbb{R}^{d+1}$ with a fixed positive root system $R_+$, normalized so that $\langle v, v \rangle = 2$ for all $v \in R_+$. It is known that $G$ is a subgroup of $O(d+1)$ generated by $\{\sigma_v : v \in R_+\}$, where $\sigma_v$ denotes the reflection with respect to the hyperplane perpendicular to $v$, i.e. $x \sigma_v = x - 2\langle x, v \rangle / \langle v, v \rangle v$ for $x \in \mathbb{R}^{d+1}$. Let $\kappa$ be a multiplicity function $\kappa : R_+ \rightarrow [0, +\infty)$ defined on $R_+$, with invariance under the action of $G$. Thus $\{\kappa_v : v \in R_+\}$ has different values only as many as the number of $G$-orbits in $R_+$.

The Dunkl operators are a family of first-order differential-reflection operators $D_j$, $1 \leq j \leq d+1$, defined by (see [5])

$$
D_j f(x) := \partial_j f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x \sigma_v)}{\langle x, v \rangle} \langle v, e_j \rangle,
$$

for $f \in C^1(\mathbb{R}^{d+1})$, where $\{e_i : 1 \leq i \leq d+1\}$ is the usual standard basis of $\mathbb{R}^{d+1}$. As substitutes of partial differentiations $\partial_j$, these operators are mutually commutative. The associated Laplacian, called $h$-Laplacian, is defined by $\Delta_h = D_1^2 + \cdots + D_{d+1}^2$, which plays roles similar to that of the usual Laplacian $\Delta = \Delta_0$ (see [4]). In terms of the polarspherical coordinates $x = rx'$, $r = \|x\|$, the operator $\Delta_h$ can be expressed as (see [33])

$$
\Delta_h = \frac{\partial^2}{\partial r^2} + \frac{2\lambda_\kappa + 1}{r} \partial_r + \frac{1}{r^2} \Delta_{h,0},
$$

where $\Delta_{h,0}$ is the associated Laplace-Beltrami operator on $\mathbb{S}^d$, and $\lambda_\kappa = \kappa + (d - 1)/2$ with $\gamma_\kappa = \sum_{v \in R_+} \kappa_v$.

If $\gamma_\kappa = 0$, i.e. $\kappa_v \equiv 0$, then $D_j = \partial_j$, $1 \leq j \leq d + 1$. In the following, we assume that $\gamma_\kappa > 0$, and so $\lambda_\kappa > 0$.

For each multiplicity function $\kappa$, there is a linear operator $V_\kappa$ intertwining the partial differentiations and the Dunkl operators (see [6]). Precisely, if $P_n = P_n^{d+1}$ denotes the set of homogeneous polynomials of degree $n$ in $d+1$ variables, then the intertwining operator $V_\kappa$ is determined
uniquely by $V_\kappa P_n \subseteq P_n$, $V_\kappa 1 = 1$ and $D_j V_\kappa = V_\kappa \partial_j$, $1 \leq i \leq d + 1$. $V_\kappa$ commutes with the group action and is a linear isomorphism on each $P_n$ ($n = 0, 1, \ldots$). Moreover it is a positive operator (see [18]) and can be extended to the space of smooth functions and even to the space of distributions (see [29, 30]).

The intertwining operator $V_\kappa$ allows to introduce some useful tools in Dunkl’s theory. For example, the Dunkl transform $F_\kappa$ is defined in [7] associated with the measure $h_\kappa^2 dx$ on $\mathbb{R}^{d+1}$, where

$$F_\kappa = \prod_{v \in R_+} \| \langle x, v \rangle \|^\kappa_v.$$  

$F_\kappa$ is a generalization of the Fourier transform $F = F_0$ and enjoys properties similar to those of $F$ (see [3, 7, 19]).

For $1 \leq p < \infty$, denote by $\|f\|_{\kappa, p}$ the norm of $f \in L^p(\mathbb{S}^d; h_\kappa^2)$, with $c_\kappa^{-1} = \int_{\mathbb{S}^d} h_\kappa^2 d\omega_d$, and by $\|\phi\|_{\lambda, p} = \left\{ \int_{\mathbb{S}^d} |\phi|^p w_{\lambda} dt \right\}^{1/p}$ the norm of $\phi \in L^p([-1, 1], w_{\lambda})$, where $w_{\lambda}(t) = c_{\lambda+1/2}(1-t^2)^{\lambda-1/2}$, $c_{\lambda} = \pi^{-1/2} \Gamma(\lambda + 1/2)/\Gamma(\lambda)$. When $p = \infty$, $\|f\|_\infty = \|f\|_{\kappa, \infty}$ and $\|\phi\|_\infty = \|\phi\|_{\kappa, \infty}$ are defined as usual.

The functions in $\mathcal{H}_n^{h, d+1} = P_n^{d+1} \cap \ker \Delta_h$ are called $h$-harmonic polynomials of degree $n$, and the spherical $h$-harmonics of degree $n$ are their restrictions on $\mathbb{S}^d$. The orthogonality theorem in [4] asserts that if $P \in P_n^{d+1}$, then $\int_{\mathbb{S}^d} PQ h_\kappa^2 d\omega_d = 0$ for all $Q \in \bigcup_{k=0}^{n-1} P_k^{d+1}$, if and only if $P$ is $h$-harmonic, i.e. $\Delta_h P = 0$. Moreover $L^2(\mathbb{S}^d; h_\kappa^2) = \bigoplus_{n=0}^\infty \mathcal{H}_n^{h, d+1}$.

If $Y_n(h_\kappa^2; f; x)$ is the projection of $f \in L^1(\mathbb{S}^d; h_\kappa^2)$ to $\mathcal{H}_n^{h, d+1}$, then the $h$-harmonic expansion of $f$ is given by

$$f(x) \sim \sum_{n=0}^\infty Y_n(h_\kappa^2; f; x), \quad x \in \mathbb{S}^d. \quad (1)$$

The projection $Y_n(h_\kappa^2; f; x)$ takes the form

$$Y_n(h_\kappa^2; f; x) = c_\kappa \int_{\mathbb{S}^d} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) d\omega_d(y), \quad (2)$$

where $P_n(h_\kappa^2; x, y)$ is the reproducing kernel of the space $\mathcal{H}_n^{h, d+1}$. A compact formula of $P_n(h_\kappa^2; x, y)$ is (see [32])

$$P_n(h_\kappa^2; x, y) = \frac{n + \lambda_k}{\lambda_k} V_\kappa \left[ C_n^{\lambda_k} (\langle x, \cdot \rangle) \right](y), \quad (3)$$

with $C_n^{\lambda_k}$, the Gegenbauer polynomial of degree $n$ with parameter $\lambda_k$. It is noted that (see [33])

$$\Delta_{h,0} Y_n = -n(n+2\lambda_k) Y_n, \quad Y_n \in \mathcal{H}_n^{h, d+1}. \quad (4)$$

When $\kappa_v = 0$ for all $v \in R_+$, we have $V_0 = id$, and hence, $P_n(h_\kappa^2; x, y)$ reduces to the usual zonal polynomial for the ordinary spherical harmonics $P_n(x, y) = \frac{n+(d-1)/2}{(d-1)/2} C_n^{(d-1)/2}(\langle x, y \rangle)$.

A useful integration formula for the intertwining operator $V_\kappa$ is

$$\int_{\mathbb{S}^d} V_\kappa f(x) h_\kappa^2(x) d\omega_d(x) = \frac{c_\kappa^{-1} \Gamma(\lambda_k + 1)}{\pi^{(d+1)/2} \Gamma(\gamma_k)} \int_{\mathbb{S}^{d+1}} f(x) (1-|x|^2)^{\gamma_k-1} dx. \quad (5)$$

The formula is proved in [31] when $f$ is a polynomial. Applying density of polynomials and positivity of $V_\kappa$, this allows us to extend the intertwining operator $V_\kappa$ acting on those functions $f$
on the sphere \( S^d \) which are restrictions of functions in \( L^1(\mathbb{B}^{d+1}; (1 - |x|^2)\gamma_n^{-1}) \), and moreover, the formula (5) is true for these functions too and \( V_\kappa f \in L^1(S^d; h_\kappa^2) \). In particular, if \( \phi \in L^1([-1, 1]; w_{\lambda_n}) \), then for each \( y \in S^d \),

\[
\int_{S^d} \phi(x,y)(1 - |x|^2)\gamma_n^{-1} dx = \frac{\pi^{(d+1)/2}\Gamma(\gamma_n)}{\Gamma(\lambda_n + 1)} \int_{-1}^1 \phi(t)w_{\lambda_n}(t)dt, \tag{6}
\]

i.e. \( f(x) = \phi((x,y)) \in L^1(\mathbb{B}^{d+1}; (1 - |x|^2)\gamma_n^{-1}) \), and hence \( V_\kappa[\phi(\cdot, y)] \) is well defined and in \( L^1(S^d, h_\kappa^2) \). In addition, we have the following symmetric relation

\[
V_\kappa[\phi(\cdot, y)](x) = V_\kappa[\phi((x, \cdot))](y), \quad \text{for a.e. } (x, y) \in S^d \times S^d. \tag{7}
\]

The validity of (7) for polynomials and for all \((x, y) \in S^d \times S^d\) follows from Gegenbauer expansions and the symmetry of the reproducing kernels \( P_\kappa(h_\kappa^2; x, y) \). If \( \phi \in L^1([-1, 1]; w_{\lambda_n}) \) and \( \phi_1 \) is a univariate polynomial, then applying (5) and (6),

\[
\int_{S^d} \int_{S^d} |V_\kappa[\phi(\cdot, y)](x) - V_\kappa[\phi((x, \cdot))](y)|h_\kappa^2(x)h_\kappa^2(y)d\omega_d(x)d\omega_d(y)
\]

\[
= \int_{S^d} \int_{S^d} |V_\kappa[(\phi - \phi_1)(\cdot, y)](x) - V_\kappa[(\phi - \phi_1)((x, \cdot))](y)|h_\kappa^2(x)h_\kappa^2(y)d\omega_d(x)d\omega_d(y)
\]

\[
\leq 2c_\kappa^{-2}\int_{-1}^1 |\phi(t) - \phi_1(t)|w_{\lambda_n}(t)dt,
\]

which implies (7) by the density of polynomials in \( L^1([-1, 1]; w_{\lambda_n}) \). Following the above remarks, the Funk–Hecke formula for \( h\)-harmonics proved in [32] (for continuous functions there only) holds also for \( \phi \in L^1([-1, 1]; w_{\lambda_n}) \), that is

\[
c_\kappa\int_{S^d} V_\kappa[\phi(\cdot, y)](x)H_n(x)h_\kappa^2(x)d\omega_d(x) = L_n(\phi)H_n(y) \tag{8}
\]

for each \( H_n \in \mathcal{H}_n^{h,d+1} \) and \( y \in S^d \), where

\[
L_n(\phi) = \int_{-1}^1 \phi(t)\frac{C_n^\lambda(t)}{C_n^\lambda(1)}w_{\lambda_n}(t)dt. \tag{9}
\]

The convolution \( f * \kappa \phi \) of two functions \( f \in L^1(S^d; h_\kappa^2) \) and \( \phi \in L^1([-1, 1]; w_{\lambda_n}) \) is defined in [33], by

\[
f * \kappa \phi(x) = c_\kappa \int_{S^d} f(y)V_\kappa[\phi((x, \cdot))](y)h_\kappa^2(y)d\omega_d(y). \tag{10}
\]

The Young inequality concerning such convolution is proved in [34], that is, for \( p, q, r \geq 1 \) with \( r^{-1} = p^{-1} + q^{-1} - 1 \),

\[
\|f * \kappa \phi\|_{\kappa,r} \leq \|f\|_{\kappa,p}\|\phi\|_{\kappa,q}. \tag{11}
\]

A typical example of Dunkl’s theory is the case when \( G = \mathbb{Z}_2^{d+1} \), for which, the function \( h_\kappa(x) \) has the form \( h_\kappa(x) = |x_1|^{\kappa_1} \cdots |x_{d+1}|^{\kappa_{d+1}} \) and the intertwining operator \( V_\kappa \) is given by

\[
V_\kappa f(x) = \tilde{c}_\kappa \int_{[-1, 1]^{d+1}} f(x_1t_1, \ldots, x_{d+1}t_{d+1})\prod_{i=1}^{d+1}(1 + t_i)(1 - t_i)^{\kappa_i-1}dt_1 \cdots dt_{d+1}, \tag{12}
\]

where \( \tilde{c}_\kappa = \tilde{c}_{\kappa_1} \cdots \tilde{c}_{\kappa_{d+1}} \).
3 The spherical Radon–Dunkl transform

For $f \in L^1(S^d; h_κ^2)$, its generalized spherical mean $M^κ_f(x)$ due to \cite{33} is defined by the equation

$$\int_{-1}^1 M^κ_τ f(x)\phi(τ)w_{λ,κ}(τ)dτ = f *_κ φ(x) \quad (13)$$

for any $φ$ in $L^1([-1,1]; w_{λ,κ})$. Since, for $φ \in L^∞(S^d)$,

$$\left| \int_{-1}^1 M^κ_τ f(x)φ(τ)w_{λ,κ}(τ)dτ \right| \leq |f|_κ,1|φ|_∞,$$

it follows that, for each $x \in S^d$, the function $φ_x(τ) = M^κ_τ f(x) \in L^1([-1,1]; w_{λ,κ})$. This shows that for almost all $τ \in [-1,1]$, $M^κ_τ f$ is well defined. To give further illustration of $M^κ_τ$, we introduce the space $W_m^p(S^d; h_κ^2)(\subseteq L^p(S^d; h_κ^2))$ of functions for $m \geq 0$, such that for $f \in W_m^p(S^d; h_κ^2)$, there exist some $g \in L^p(S^d; h_κ^2)$ satisfying $Y_0(h_κ^2; f) = Y_0(h_κ^2; g)$ and $|n(n + 2λκ)|^{m/2}Y_n(h_κ^2; f) = Y_n(h_κ^2; g)$ for all $n = 1, 2, \ldots$. In view of \cite{14}, we formally write $g = (-Δ_h,φ)^{m/2} f$. It is noted that for even $m$, $C^m(S^d) \subseteq W_m^p(S^d; h_κ^2) (1 \leq p \leq ∞)$. The following properties of $M^κ_τ$ are proved in \cite{33,34}.

**Proposition 1.**

(i) If $f_0(x) \equiv 1$, then $M^κ_τ f_0(x) \equiv 1$.

(ii) For each $τ \in [-1,1]$, there is an extension of $M^κ_τ$ to $L^p(S^d; h_κ^2)$ ($1 \leq p < ∞$), or $C(S^d)$ ($p = ∞$), such that

$$|M^κ_τ f|_κ,p \leq |f|_κ,p, \quad τ \in [-1,1].$$

(iii) For $f \in L^1(S^d; h_κ^2)$,

$$Y_n(h_κ^2; M^κ_τ f; x) = \frac{C^κ_n(τ)}{C^κ_n(1)} Y_n(h_κ^2; f; x),$$

and in particular, $Δ_{h,0}(M^κ_τ f) = M^κ_τ(Δ_{h,0} f)$ if $Δ_{h,0} f \in L^1(S^d; h_κ^2)$.

**Proof.** Here we give an independent, but simpler proof for part (ii) as follows. For all $φ \in L^1([-1,1]; w_{λ,κ})$ and $g \in L^p(S^d; h_κ^2)$, it follows from \cite{13} that

$$\int_{-1}^1 φ(τ)ξ(τ)w_{λ,κ}(τ)dτ = c_κ \int_{S^d} (f *_κ φ)(x)g(x)h_κ^2(x)dω_d(x),$$

where $ξ(τ) = c_κ \int_{S^d} M^κ_τ f(x)φ(x)h_κ^2(x)dω_d(x)$. By using \cite{7} and \cite{10}, the right-hand side above becomes $c_κ \int_{S^d} f(y)(g *_κ φ)(y)h_κ^2(φ)dω_d(y)$, and then, by applying Hölder’s inequality and the Young inequality \cite{11}, its absolute value is dominated by $|f|_κ,p|g|_κ,p,|φ|_∞$. This gives that sup$τ\in[-1,1] |ξ(τ)| \leq |f|_κ,p|g|_κ,p,|φ|_∞$, which means that, for almost all $τ \in [-1,1]$, $|M^κ_τ f|_κ,p \leq |f|_κ,p$. If $f$ is a polynomial, part (iii) implies that $M^κ_τ f(x)$ is a continuous function of $(τ,x) \in [-1,1] \times S^d$, so that $|M^κ_τ f|_κ,p \leq |f|_κ,p$ is true for all $τ \in [-1,1]$ in this case. Finally, from density of the set of polynomials, for each $τ \in [-1,1]$, $M^κ_τ$ can be extended to all functions in $L^p(S^d; h_κ^2)$ ($1 \leq p < ∞$), or $C(S^d)$. Following this, part (iii) also holds for $f \in L^1(S^d; h_κ^2)$ and each $τ \in [-1,1]$, and moreover,

$$M^κ_τ f \sim \sum_{n=0}^{∞} \frac{C^κ_n(τ)}{C^κ_n(1)} Y_n(h_κ^2; f; x). \quad (14)$$

We note that when $f$ is even in $S^d$, $M_τ f$ is even for $τ \in (-1,1)$.
The following proposition gives a pointwise description of $M^\kappa_n$ for a larger class of functions.

**Proposition 2.** For $f \in W^2_m(S^d; h^2_\kappa)$ with $m > \lambda_\kappa + 1$, $M^\kappa_n f(x)$ is a continuous function of $(\tau, x) \in [-1, 1] \times S^d$ and

$$M^\kappa_n f(x) = \sum_{n=0}^{\infty} C_n^\kappa(x) Y_n(h^2_\kappa; f; x),$$

the series on the right-hand side being absolutely and uniformly convergent.

**Proof.** It is noted that for $t \in [-1, 1]$, $|C_n^\kappa(t)| \leq C_n^\kappa(1) = (2\lambda_\kappa)_n/n! \simeq n^{2\lambda_\kappa-1}$ [3, p. 19]. From (2), for $g \in L^2(S^d; h^2_\kappa)$ and all $x \in S^d$ we have $|Y_n(h^2_\kappa; g; x)| \leq \|g\|_{\kappa,2} \|P_n(h_\kappa; x, \cdot)\|_{\kappa,2}$. In view of orthogonality of $h$-harmonics and from (3),

$$\|P_n(h_\kappa; x, \cdot)\|^2_{\kappa,2} = \|P_n(h_\kappa; x)\|^2 = \lambda_\kappa^{-1}(n + \lambda_\kappa)C_n^{\lambda_\kappa}(1) \simeq \lambda_\kappa^{-1} n^{2\lambda_\kappa}.$$

Therefore $|Y_n(h^2_\kappa; g; x)| \leq c\|g\|_{\kappa,2} n^{\lambda_\kappa}$. If $f \in W^2_m(S^d; h^2_\kappa)$, then for $n \geq 1$, $Y_n(h^2_\kappa; f; x) = [n(n + 2\lambda_\kappa)]^{-m/2} Y_n(h^2_\kappa; (-\Delta_{h_\kappa})^{m/2} f; x)$, so that $|Y_n(h^2_\kappa; f; x)| \leq c\|(-\Delta_{h_\kappa})^{m/2} f\|_{\kappa,2} n^{\lambda_\kappa-m}$. Hence, when $m > \lambda_\kappa + 1$, the series in (14) converges absolutely and uniformly for $(\tau, x) \in [-1, 1] \times S^d$. In view of the uniqueness of $h$-harmonic expansion following from its Cesàro summability (see [3]), the conclusions in the proposition are proved. 

Now we define the transform $R^\kappa$ by

$$R^\kappa f = M^\kappa_0 f,$$

and call $R^\kappa$ the spherical Radon–Dunkl transform. By Proposition 1 (ii), $R^\kappa f$ is well defined for $f \in L^1(S^d; h^2_\kappa)$, and moreover, from Propositions 1 and 2 we have the following corollary.

**Corollary 1.**

(i) For $f \in L^p(S^d; h^2_\kappa)$ $(1 \leq p < \infty)$, or $f \in C(S^d)$ $(p = \infty)$, we have $\|R^\kappa f\|_{\kappa,p} \leq \|f\|_{\kappa,p}$, and

$$R^\kappa f \sim \sum_{n=0}^{\infty} b_n Y_n(h^2_\kappa; f; x),$$

where

$$b_n = \begin{cases} (-1)^{\frac{n}{2}} \frac{\Gamma(\lambda_\kappa+1/2)}{\Gamma(1/2)} \frac{\Gamma((n+1)/2)}{\Gamma(\lambda_\kappa+(n+1)/2)}, & \text{for } n \text{ even;} \\ 0, & \text{for } n \text{ odd.} \end{cases}$$

(ii) For $f \in W^2_m(S^d; h^2_\kappa)$ with $m > \lambda_\kappa + 1$, $R^\kappa f(x)$ is a continuous function on $S^d$ and

$$R^\kappa f(x) = \sum_{n=0}^{\infty} b_n Y_n(h^2_\kappa; f; x),$$

where the series on the right-hand side is absolutely and uniformly convergent.

The numbers $b_n = C_n^\kappa(0)/C_n^\kappa(1)$ are computed by using 10-9(3) and 10-9(19) in [3].

The following is a nontrivial example of $R^\kappa$. We consider the group $G = Z^d_{2^d+1}$, with $\kappa = (\kappa_1, 0, \ldots, 0)$ and $\kappa_1 > 0$. In this case, $h_\kappa(x) = |x_1|^\kappa_1$ and the intertwining operator $V_\kappa$ in [12] reduces to

$$V_\kappa f(x) = \tilde{c}_\kappa \int_{-1}^{1} f(x_t \tilde{x})(1 + t)(1 - t^2)^{\kappa_1-1} dt,$$

where $x = (x_1, \tilde{x})$ with $\tilde{x} = (x_2, \ldots, x_{d+1}) \in \mathbb{R}^d$. 

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We shall show that, for \( f \in C(S^d) \) and for \( x_1 \neq 0 \),
\[
M^\tau f(x) = \frac{c_{c_0}c_{\lambda_1}}{|x_1|^{2\lambda_1}} \int_{\Omega_0} f(y)|\langle x, y \rangle - \tau|^{|\lambda_1| - 1}|\langle x\sigma, y \rangle - \tau|^{|\lambda_1|}d\omega_d(y),
\]
(17)
where \( \Omega_\tau = \{ y \in S^d : y = (y_1, \bar{y}) \text{ with } |\langle \bar{x}, \bar{y} \rangle - \tau| < |x_1 y_1| \} \), and \( \sigma \) is the reflection such that \( x\sigma = (-x_1, x_2, \ldots, x_{d+1}) \). Indeed, from the above formula for \( V_\kappa \),
\[
V_\kappa[\phi((x, \cdot))](y) = c_{\lambda_1} \int_{-1}^1 \phi(x_1 y_1 t + (\bar{x}, \bar{y}))(1 + t)(1 - t^2)^{|\lambda_1| - 1}dt.
\]
When \( y_1 \neq 0 \), taking the substitution of variables \( t = (\tau - \langle \bar{x}, \bar{y} \rangle)/(x_1 y_1) \), we get
\[
V_\kappa[\phi((x, \cdot))](y) = \frac{c_{\lambda_1}}{|x_1 y_1|^{2\lambda_1}} \int_{(\bar{x}, \bar{y}) - |x_1 y_1|} (\bar{x}, \bar{y})\phi(\tau)|\langle x, y \rangle - \tau|^{|\lambda_1| - 1}|\langle x\sigma, y \rangle - \tau|^{|\lambda_1|}d\tau.
\]
Substituting this into the definition (10) of \( f *_\kappa \phi \), we have
\[
f *_\kappa \phi(x) = \int_{-1}^1 A_\tau f(x)\phi(\tau)w_{\lambda_1}(\tau)d\tau
\]
for all \( \phi \in L^1([-1, 1]; w_{\lambda_1}) \), where \( A_\tau f(x) \) denotes the expression on the right-hand side of (17). Then from (13), \( A_\tau f(x) = M^\tau f(x) \), so that (17) is proved.

Taking \( \tau = 0 \) in (17), we get that, for \( f \in C(S^d) \) and for \( x_1 \neq 0 \),
\[
R_\kappa f(x) = \frac{c_{c_0}c_{\lambda_1}}{|x_1|^{2\lambda_1}} \int_{\Omega_0} f(y)|\langle x, y \rangle|^{|\lambda_1| - 1}|\langle x\sigma, y \rangle|^{|\lambda_1|}d\omega_d(y).
\]

4 Inversion formulas for \( R_\kappa \)
by means of spherical Riesz–Dunkl potentials

For \( f \in L^1(S^d; h_\kappa^2) \) and \( \Re \alpha > 0, \alpha \neq 1, 3, 5, \ldots \), we define its spherical Riesz–Dunkl potential \( I^\alpha_\kappa f \) by
\[
I^\alpha_\kappa f(x) = C_{\kappa, \alpha} \int_{S^d} f(y)V_\kappa(|\langle x, \cdot \rangle|^{|\alpha| - 1})(y)h_\kappa^2(y)d\omega_d(y),
\]
(18)
where \( C_{\kappa, \alpha} = \frac{\sqrt{\pi^\alpha(1 - \alpha)/2}}{\Gamma(n_\kappa + 1)\Gamma(\alpha/2)}c_\kappa \).

**Proposition 3.** For \( \Re \alpha > 0, \alpha \neq 1, 3, 5, \ldots \), \( I^\alpha_\kappa f \) is well defined for each \( f \in L^1(S^d; h_\kappa^2) \), and moreover, we have the following statements:

(i) for \( 1 \leq p \leq \infty \), there exists a constant \( c > 0 \), such that for all \( f \in L^p(S^d; h_\kappa^2) \), \( \|I^\alpha_\kappa f\|_{\kappa, p} \leq c\|f\|_{\kappa, p} \);

(ii) if the \( h \)-harmonic expansion of a function \( f \in L^1(S^d; h_\kappa^2) \) is given by (11), then \( I^\alpha_\kappa f \) has the following expansion
\[
I^\alpha_\kappa f(x) \sim \sum_{n=0}^{\infty} b_{n, \alpha} Y_n(h_\kappa^2; f; x), \quad x \in S^d,
\]
(19)
where
\[
b_{n, \alpha} = \begin{cases} (-1)^{\frac{n}{2}} \frac{\Gamma((n + 1 - \alpha)/2)}{\Gamma(\lambda_n + (n + 1 + \alpha)/2)} & \text{for } n \text{ even;} \\ 0, & \text{for } n \text{ odd.} \end{cases}
\]
(20)
The conclusions in the proposition are contained in Proposition 2.9 of [34]. Here we give a short presentation. Since $I^α_{m} f = cf * φ$ with $φ(t) = |t|^{α-1} \in L^1([-1,1]; w_{λ,κ}(t))$, part (i) follows from the Young inequality (11) immediately. For part (ii), since $V_n(|\cdot|^{α-1}) \in L^1(S^d; h^2)$ from (13) and (6), we have, using (2), (8) and (9),

$$Y_n(h^2_{κ}; V_n(|\cdot|^{α-1}); x) = L_n P_n(h^2_{κ}; x, y),$$

where

$$L_n = \int_{-1}^{1} |t|^{α-1} \frac{C_{λ,κ}(t)}{C_{λ,κ}(1)} w_{λ,κ}(t) dt.$$

From (2), (7) and (13), one can get

$$Y_n(h^2_{κ}; I^α_{m} f; x) = C_{κ,α} \int_{S^d} f(z) Y_n(h^2_{κ}; V_n(|\cdot|^{α-1}); x) h^2_{κ}(z) dω_d(z),$$

and then applying (21), $Y_n(h^2_{κ}; I^α_{m} f; x) = b_{n,α} Y_n(h^2_{κ}; f; x)$ with $b_{n,α} = c_{κ}^{-1} C_{κ,α} L_n$. It is clear that $b_{n,α} = 0$ for odd $n$. When $n$ is even, we use 7.311(2) in [12], part (v) in [8, p. 19], and some properties of the gamma function, to get the stated value of $b_{n,α}$.

It is easy to see that (19) and (20) allow us to extend the family $\{I^α_{m} : βκ > 0, α \neq 1, 3, 5, \ldots\}$ to a larger one, which leads to the following definition. We put $Π = \{α \in C : βκ \neq 1, 3, 5, \ldots\}$.

**Definition 1.** Let $α \in Π$. For $f \in L^1(S^d; h^2_{κ})$, we define $I^α_{m} f$ by the following $h$-harmonic expansion

$$I^α_{m} f \sim \sum_{n=0}^{∞} b_{n,α} Y_n(h^2_{κ}; f; x), \quad x \in S^d,$$

where $b_{n,α}$ is given by (20).

It is clear that $I^α_{m} f$ is well defined for $f \in C^∞(S^d)$. In general, $I^α_{m} f$ may be a distribution on $S^d$. Since $|b_{n,α}| \leq cn^{-λ,κ-βκ}$, then $I^α_{m} f \in L^2(S^d; h^2_{κ})$ when $f \in L^2(S^d; h^2_{κ})$ and $βκ \geq -λ,κ$. For $βκ < -λ,κ$ and $m \geq -λ,κ - βκ$, since for $n \geq 1$, $Y_n(h^2_{κ}; f; x) = [n(n+2λ,κ)]^{-m/2} Y_n(h^2_{κ}; (-Δ_{h,κ})^{m/2}; f; x)$, we also have $I^α_{m} f \in L^2(S^d; h^2_{κ})$ when $f \in W_{m,κ}^2(S^d; h^2_{κ})$. We denote by $W_{m,κ}^2(S^d; h^2_{κ})$ the subspace of even functions of $W^{2}_{m,κ}(S^d; h^2_{κ})$.

**Theorem 1.** If $α, -2λ,κ - α \in Π$ and $m \geq \max\{0, -λ,κ - βκ\}$, then $I^α_{m} f$ is an isomorphism between $W^{2}_{m,κ}(S^d; h^2_{κ})$ and $W^{2}_{m+λ,κ+βκ,κ}(S^d; h^2_{κ})$, and

$$(I^α_{m})^{-1} = I^{-2λ,κ-α}_{m}.$$

In fact, proceeding the above process, it is not difficult to show that for $f \in W^{2}_{m,κ}(S^d; h^2_{κ})$ with $m \geq \max\{0, -λ,κ - βκ\}$, we have $I^{-2λ,κ-α}_{m} f = f$. For $f \in W^{2}_{m',κ}(S^d; h^2_{κ})$ with $m' = m+λ,κ+βκ$, since $m' \geq \max\{0, -λ,κ - βκ'\}$ with $α' = -2λ,κ - α$, we again have $I^{-2λ,κ-α'}_{m'} f = f$, i.e. $I^α_{m} I^{-2λ,κ-α}_{m} f = f$. Combining the two cases proves the theorem.

To go further, for $r \in Z_+$ (nonnegative integers) we define

$$P_{r,α}(Δ_{h,κ}) = \begin{cases} 
\text{the identity operator}, & r = 0, \\
4-r \prod_{j=1}^{r} (-Δ_{h,κ} + a_j), & r \geq 1,
\end{cases}$$

where $a_j = (2λ,κ - 2r + 2j + α - 1)(2r - 2j + 1 - α)$. 
Lemma 1. If \( \alpha \in \Pi \), and \( r \in \mathbb{Z}_+ \) such that \( 2r - 2\lambda_\kappa - \alpha \in \Pi \), then for even \( n \) and \( Y_n \in \mathcal{H}_{n}^{h,d+1} \),

\[
Pr_{r,\alpha}(\Delta_{h,0})f_{\kappa}^{2r-2\lambda_\kappa-\alpha}f_{\kappa}^\alpha Y_n = Y_n.
\]

Proof. From (19) and (20), we have

\[
I_{\kappa}^{2r-2\lambda_\kappa-\alpha}f_{\kappa}^\alpha Y_n = \frac{\Gamma((n + 2\lambda_\kappa - 2r + \alpha + 1)/2)\Gamma((n + 1 - \alpha)/2)}{\Gamma((n + 2r - \alpha + 1)/2)\Gamma((n + 2\lambda_\kappa + \alpha + 1)/2)} Y_n.
\]

Furthermore, from (14),

\[
P_{r,\alpha}(\Delta_{h,0})Y_n = \prod_{j=1}^{r} \left( \frac{n + 2\lambda_\kappa + \alpha - 1}{2} - r + j \right) \left( \frac{n + 1 - \alpha}{2} + r - j \right) Y_n.
\]

Using the properties of \( \Gamma \)-functions, the result is obtained.

The following theorem is a direct consequence of the above lemma.

Theorem 2. If \( \alpha \in \Pi \), and \( r \in \mathbb{Z}_+ \) such that \( 2r - 2\lambda_\kappa - \alpha \in \Pi \) and \( r \geq \lambda_\kappa + \Re \alpha/2 \), then for even \( f \in C^\infty(S^d) \) and \( g = I_{\kappa}^\alpha f \), we have the inversion formula

\[
f = P_{r,\alpha}(\Delta_{h,0})I_{\kappa}^{2r-2\lambda_\kappa-\alpha}g.
\]

Now we turn to the inversion problem of the spherical Radon–Dunkl transform \( R_\kappa \). From (15), (16), (20) and (22), we see that

\[
R_\kappa f = \pi^{-1/2}\Gamma(\lambda_\kappa + 1/2)I_{\kappa}^0 f.
\]

This consistency can be also seen from the following equalities

\[
\frac{\Gamma(\lambda_\kappa + 1)\Gamma(\alpha/2)}{\sqrt{\pi\Gamma(1 - \alpha)/2}} I_{\kappa}^\alpha f(x) = f \ast_\kappa \phi = \int_{-1}^{1} M_{\kappa} f(x)\phi(\tau)w_{\lambda_\kappa}(\tau)d\tau
\]

in view of (13) and (18), where \( \phi(t) = |t|^{\alpha-1} \in L^1([-1,1];w_{\lambda_\kappa}) \). Assume that \( f \in W_m^2(S^d; h_\kappa^2) \) with \( m > \lambda_\kappa + 1 \). By Proposition 2, for each \( x \in S^d \), \( M_{\kappa} f(x) \) is a continuous function of \( \tau \in [-1,1] \). Dividing each part of (24) by \( \Gamma(\alpha/2) \) and taking limit for \( \alpha \to 0^+ \), we regain the relation (23).

From Theorems 1 and 2, we obtain the inversion formulas for the spherical Radon–Dunkl transform \( R_\kappa \).

Theorem 3. \( R_\kappa \) is an isomorphism between \( W_m^2(S^d; h_\kappa^2) \) and \( W_{m+\lambda_\kappa}^2(S^d; h_\kappa^2) \) with \( m \geq 0 \), and

\[
R_\kappa^{-1} = \frac{\sqrt{\pi}}{\Gamma(\lambda_\kappa + 1/2)} I_{\kappa}^{-2\lambda_\kappa}.
\]

Theorem 4. If \( r \in \mathbb{Z}_+ \) such that \( 2r - 2\lambda_\kappa \in \Pi \) and \( r \geq \lambda_\kappa \), then for even \( f \in C^\infty(S^d) \) and \( g = R_\kappa f \), we have the inversion formula

\[
f = \frac{\sqrt{\pi}}{\Gamma(\lambda_\kappa + 1/2)} P_{r,0}(\Delta_{h,0})I_{\kappa}^{2r-2\lambda_\kappa}g.
\]

For a special case, we have some simple inversion formulas for \( R_\kappa \), which are interesting generalizations of those about the usual spherical Radon transform (see [15, 16, 22]).
Corollary 2. If $\lambda_\kappa$ is a positive integer, then an even $f \in C^\infty(\mathbb{S}^d)$ can be recovered by

(i) $f = c' P_{r,0}(\Delta_{h,0}) R_\kappa R_\kappa f$,

with $r = \lambda_\kappa$, $c' = \pi / \Gamma(\lambda_\kappa + 1/2)^2$, and

$$P_{r,0}(\Delta_{h,0}) = 4^{-r} \prod_{j=1}^{r} [-\Delta_{h,0} + (2j - 1)(2r - 2j + 1)]$$

and

(ii) $f = c'' P_{r,0}(\Delta_{h,0}) \left[ \int_{\mathbb{S}^d} R_\kappa f(y)V_\kappa (\|\langle x, \cdot \rangle\|)(y)h_\kappa^2(y) dw_d(y) \right]$,

with $r = \lambda_\kappa + 1$, $c'' = -2\pi^{3/2} c_\kappa / [\Gamma(\lambda_\kappa + 1)\Gamma(\lambda_\kappa + 1/2)^2]$, and

$$P_{r,0}(\Delta_{h,0}) = 4^{-r} \prod_{j=1}^{r} [-\Delta_{h,0} + (2j - 3)(2r - 2j + 1)]$$

5 Inversion formulas for $R_\kappa$ by means of associated wavelets

In this section, we shall use, for a suitably chosen $\psi$ defined on $[0, \infty)$, the wavelet-like transform

$$W_\kappa f(t, x) = f \ast_\kappa \psi_t(x), \quad \psi_t(\tau) = t^{-1} \psi(\tau/t),$$

for $(t, x) \in (0, \infty) \times \mathbb{S}^d$, to present the inverse of the spherical Radon–Dunkl transform $R_\kappa$ and itself. Although $R_\kappa$ is defined implicitly and the intertwining operator $V_\kappa$ is involved in the definition of $W_\kappa$, the approaches in studying the usual spherical Radon transform (see [21], for example) could be transplanted to $R_\kappa$.

The first lemma below reveals a relation of the spherical Radon–Dunkl transform $R_\kappa$ with the one-dimensional fractional integral, and the second gives a representation of the successive action of $R_\kappa$ and $W_\kappa$ to a function. We shall use a modified notation of the fractional integral as

$$B_\delta \phi(u) = \frac{2}{\Gamma(\delta)} \int_0^\infty \phi(v)(u - v^2)^{\delta - 1} dv, \quad u > 0,$$

for $\delta > 0$, which will simplify some expressions.

Lemma 2. For even function $f \in L^1(\mathbb{S}^d; h_\kappa^2)$ and $0 < s < 1$, we have

$$M_s^\kappa(R_\kappa f) = \frac{\lambda_\kappa!}{w_{\lambda_\kappa}(s)} B_{\lambda_\kappa} (M_{\tau}^s f)(1 - s^2),$$

where the action of $B_{\lambda_\kappa}$ to $M_{\tau}^s f$ is associated with $\tau$-variable.

Proof. From the product formula of the Gegenbauer polynomial $C_{2n}^{\lambda_\kappa}$ (see [5] p. 203], we have

$$\frac{C_{2n}^{\lambda_\kappa}(s)}{C_{2n}^{\lambda_\kappa}(1)} \frac{C_{2n}^{\lambda_\kappa}(0)}{C_{2n}^{\lambda_\kappa}(1)} = 2 \int_0^1 \frac{C_{2n}^{\lambda_\kappa}(u\sqrt{1 - s^2})}{C_{2n}^{\lambda_\kappa}(1)} w_{\lambda_\kappa - 1/2}(u) du.$$  

By Proposition 3(iii), the three quotients above are the coefficients of a member $Y_{2n}$ in $H_{2n}^{h, d+1}$ under action of $M_s^\kappa$, $R_\kappa(= M_0^\kappa)$, and $M_{\sqrt{1 - s^2}}$, respectively. Therefore,

$$M_s^\kappa(R_\kappa Y_{2n}) = 2 \int_0^1 (M_{\sqrt{1 - s^2}} Y_{2n}) w_{\lambda_\kappa - 1/2}(u) du.$$  

Making substitution of variables $u = v/\sqrt{1 - s^2}$, (27) is proved for $Y_{2n}$. By Proposition 3(ii), both sides of (27) are bounded operators in $L^1(\mathbb{S}^d; h_\kappa^2)$, and hence, the validity of (27) for general even $f \in L^1(\mathbb{S}^d; h_\kappa^2)$ follows from density of the set of $h$-harmonics.  □
Lemma 3. For even \( f \in L^1(\mathbb{S}^d; h^2_\kappa) \) and \( \psi \in L^1([0, \infty); dx) \), we have

\[
W_\kappa(R_\kappa f)(t, x) = \frac{2\lambda_\kappa}{\pi} \int_0^1 M_\kappa^s f(x)(B_{\lambda_\kappa} \psi_t)(1 - s^2)ds,
\]

provided the integral on the right-hand side exists with \(|f|\) and \(|\psi|\) instead of \( f, \psi \).

**Proof.** From [13, 25] and [27], we have

\[
W_\kappa(R_\kappa f)(t, x) = \int_{-1}^1 M_\kappa^s(R_\kappa f)(x) \cdot \psi_t(s)w_{\lambda_\kappa}(s)ds = \frac{2\lambda_\kappa}{\pi} \int_0^1 B_{\lambda_\kappa}(M_\kappa^s f)(1 - s^2) \cdot \psi_t(s)ds,
\]

and then, substituting the formula for \( B_{\lambda_\kappa}(M_\kappa^s f) \) from [26], and making changes of variables, we prove the equality in [28].

**Theorem 5.** Let

\[
\int_0^\infty s^j\psi(s)ds = 0 \quad \text{for all} \quad j = 0, 2, 4, \ldots, 2[\lambda_\kappa], \quad (29)
\]

\[
\int_1^\infty s^\beta |\psi(s)|ds < \infty \quad \text{for some} \quad \beta > 2\lambda_\kappa. \quad (30)
\]

Then for even \( f \in L^p(\mathbb{S}^d; h^2_\kappa) \) \((1 \leq p < \infty)\), or \( C(\mathbb{S}^d) \) \((p = \infty)\), we have

\[
\lim_{\epsilon \to 0+} \| T_\epsilon f - f \|_{\kappa, p} = 0, \quad (31)
\]

where

\[
T_\epsilon f(x) = \tilde{C}_\psi^{-1} \int_\epsilon^\infty \frac{W_\kappa g(t, x)}{t^{2\lambda_\kappa+1}} dt, \quad \epsilon > 0, \quad (32)
\]

with \( g = R_\kappa f \) and

\[
\tilde{C}_\psi = \left\{
\begin{array}{ll}
-\frac{2\Gamma(1 - \lambda_\kappa)}{\pi} \int_0^\infty s^{2\lambda_\kappa} \psi(s)ds, & \text{if} \quad \lambda_\kappa \in \mathbb{N}, \\
\frac{4(-1)^{\lambda_\kappa+1}}{\pi \Gamma(\lambda_\kappa)} \int_0^\infty s^{2\lambda_\kappa} \psi(s) \log s ds, & \text{if} \quad \lambda_\kappa \in \mathbb{N}.
\end{array}
\right.
\]

(33)

In addition, \( \lim_{\epsilon \to 0+} T_\epsilon f(x) = f(x) \) for almost all \( x \in \mathbb{S}^d \).

**Proof.** Under the assumptions, by [20, Lemma 4.12], we have \( \int_0^\infty |B_{\lambda_\kappa} \psi(s)|ds < \infty \). To prove (31) in general, we only need to show that it is valid for \( Y_{2n} \in \mathcal{H}_n^{h,D+1} \), and

\[
\| T_\epsilon f \|_{\kappa, p} \leq c\| f \|_{\kappa, p}, \quad \epsilon > 0, \quad (34)
\]

where the constant \( c \) is independent of \( \epsilon \). The key step is to rewrite \( T_\epsilon \) into a convolution operator with an approximate identity, that is,

\[
T_\epsilon f(x) = \frac{2\lambda_\kappa(\lambda_\kappa + 1)}{\pi(2\lambda + 1)} \tilde{C}_\psi f *_\kappa K_\epsilon, \quad (35)
\]

where

\[
K_\epsilon(\tau) = [w_{\lambda_\kappa+1}(\tau)]^{-1}(B_{\lambda_\kappa+1} \psi)(\epsilon^{-2}(1 - \tau^2)) \quad . \quad (36)
\]
Indeed, applying Lemma 3 to (32) gives that
\[ \tilde{C}_Ψ T_ε f(x) = \frac{2λ_κ}{π} \int_0^1 M_κ^f(x) K_ε(s) ds, \]
where \( \tilde{K}_ε(s) = \int_ε^∞ \frac{(B_{λ_κ+1})^2_ε(1-s^2)}{ε} dt. \) Inserting the formula of \( B_{λ_κ}Ψ_t \) from (26), and then, making changes of variables by \( t = ξ^{-1/2} \) and \( v = ηξ^{-1/2} \), we get
\[ \tilde{K}_ε(s) = \frac{1}{Γ(λ_κ)} \int_0^{ε^{-2}} \int_0^{\sqrt{ξ(1-s^2)}} ψ(η)[ξ(1-s^2) - η^2]^{λ_κ-1} dηdξ. \]
Changing order of the integrals, it follows that \( \tilde{K}_ε(s) = \frac{λ_κ+1}{2λ_κ+1} K_ε(s) w_{λ_κ}(s) \). Substituting this into (37) and using (13) yields (35).

By Lemma 2.4 in (21), we have \( \| f - \tilde{f} \|_{L^∞(\mathbb{R}^d)} \leq 2 \| f - \tilde{f} \|_{H^λ(\mathbb{R}^d)} \), which implies the following estimate for \( u \).

In order to prove (34), by (11), it suffices to show
\[ \| f - \tilde{f} \|_{L^∞(\mathbb{R}^d)} \leq c \| f - \tilde{f} \|_{H^λ(\mathbb{R}^d)}, \]
which approaches to \( \pi(λ_κ+1/2) \tilde{C}_Ψ \) as \( ε \to +0 \). For \( Y_2n ∈ H_2^h + \), from (8) we have \( Y_2n \ast_κ K_ε = L_2n(K_ε)Y_2n, \) and by (35), \( \lim_{ε \to +0} Y_n = Y_n \) uniformly on \( S^d \).

To prove (51), by (11), it suffices to show \( \| K_ε \|_{L^∞(\mathbb{R}^d)} \leq c \) uniformly for \( ε > 0 \) (essentially for \( 0 < ε ≤ 1 \)). In fact, similarly to (35), we have \( \| K_ε \|_{L^∞(\mathbb{R}^d)} = L_0(\| K_ε \|), \) approaching to
\[ \frac{λ_κ + 1/2}{λ_κ + 1} \int_0^∞ u^{-1} |(B_{λ_κ+1}Ψ)(u)| du < +∞, \]
as \( ε \to +0 \). Thus (34), and so (31), are proved.

In order to prove \( T_ε f \) to be convergent almost everywhere, we need the associated maximal function \( T_ε f(x) = \sup_{0 < t ≤ 1} |T_ε f(x)|. \) We shall show that \( T_ε f \) is dominated by the maximal function introduced in (35)
\[ M_κ f(x) = \sup_{0 < θ ≤ 1} \int_0^θ (M_{cos θ}^f)(x) (sin θ)^{2λ_κ} dθ, \]
for \( f ∈ L^1(S^d, h^2_κ) \), that is
\[ T_ε f(x) ≤ cM_κ f(x), \quad x ∈ S^d. \]
(39)

The pointwise estimates of \( B_{λ_κ+1}Ψ(u) \) can be written as \( B_{λ_κ+1}Ψ(u) = O(u^{λ_κ}(u + 1)^{-λ_κ-ρ}) \), which implies the following estimate for \( K_ε(\cos θ) \)
\[ K_ε(\cos θ) = O(m_ε(θ)), \quad m_ε(θ) = \frac{ε^{2ρ}(sin θ)^{-1}}{ε + sin θ}^{2λ_κ+2ρ}, \]
with \( ρ > 0 \). The function \( m_ε(θ) \) does not suit the process of integration by part in the proof of Theorem 2.6 in (35), since \( m(0) = 0 \). Here we give a proof for the case.
From (13) and (35),
\[ |T_\epsilon f(x)| \leq c \int_0^{\pi/2} (M^\kappa_{\cos \theta} |f|)(x)m_\epsilon(\theta)(\sin \theta)^{2\lambda_\kappa} d\theta, \]  
(40)

where the evenness of \( M^\kappa_{\epsilon} f \) is used. Splitting the interval \([0, \pi/2]\) into \( \bigcup_j [2^j \epsilon, 2^{j+1} \epsilon] \), we evaluate each integral \( U_j = \int_{2^j \epsilon}^{2^{j+1} \epsilon} \) separately. For \( j \leq 0 \), since \( \epsilon + \sin \theta \geq \epsilon \), we have
\[ U_j \leq \frac{c2^{-j}}{c^{2\lambda_\kappa+1}} \int_0^{2^{j+1} \epsilon} (M^\kappa_{\cos \theta} |f|)(x)(\sin \theta)^{2\lambda_\kappa} d\theta \leq c2^{2\lambda_\kappa j} M_\kappa f(x); \]
and for \( j > 0 \), since \( \epsilon + \sin \theta \approx \theta \),
\[ U_j \leq \frac{ce^{2p}}{(2^j \epsilon)^{2\lambda_\kappa+2p+1}} \int_0^{2^{j+1} \epsilon} (M^\kappa_{\cos \theta} |f|)(x)(\sin \theta)^{2\lambda_\kappa} d\theta \leq c2^{-2^p j} M_\kappa f(x). \]

Collecting these estimates into (40) yields (39).

By Theorem 2.1 in [2], \( T_\epsilon \) is of weak (1,1), and strong \((p,p)\) boundedness. Combining with the uniformly convergence of \( T_\epsilon \) for \( h\)-harmonics, for general \( f \in L^1(S^d; h^2_\kappa) \), \( T_\epsilon f \) converges to \( f \) almost everywhere. The proof of Theorem 5 is completed. \( \blacksquare \)

In the following, we state two theorems, without proof, which are analogs of Theorems 1.2 and 1.4 in [24]. One is about the reproducing property of the spherical Radon–Dunkl transform \( R_\kappa \), and the other illustrates the range \( R_\kappa(L^1(S^d; h^2_\kappa)) \).

**Theorem 6.** Let
\[ \int_0^\infty \psi(s)ds = 0, \quad \int_0^\infty |\psi(s)| \log s ds < \infty. \]

Then for \( f \in L^p(S^d; h^2_\kappa) \) \((1 \leq p < \infty)\), or \( C(S^d) \) \((p = \infty)\), we have
\[ \lim_{\epsilon \to 0^+} \|\tilde{T}_\epsilon f - R_\kappa f\|_{\kappa,p} = 0, \]
where \( \tilde{T}_\epsilon f(x) = \tilde{C}_\phi^{-1} \int_\epsilon^\infty t^{-1}(W_\kappa f)(t,x)dt \) \((\epsilon > 0)\), with \( \tilde{C}_\phi = 2c_{\lambda_\kappa} \int_0^\infty \psi(s) \log \frac{1}{s} ds. \)

**Theorem 7.** Let \( \psi \) satisfy conditions (29) and (31), \( g \in L^p(S^d; h^2_\kappa) \) \((1 \leq p < \infty)\), or \( C(S^d) \) \((p = \infty)\), and \( \tilde{C}_\phi \neq 0 \) be the constant in (33). Then the following statements are equivalent:

(i) \( g \in R_\kappa(L^p(S^d; h^2_\kappa)) \);

(ii) the integrals \( S_\epsilon g = \int_\epsilon^\infty t^{-2\lambda_\kappa-1}(W_\kappa g)(t,x)dt \) converge in the \( L^p(S^d; h^2_\kappa) \)-norm.

If \( 1 < p < \infty \), then (i) and (ii) are equivalent to
\[ \sup_{\epsilon>0} \|S_\epsilon g\|_{\kappa,p} < \infty. \]

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