In this work, we discuss the numerical method for the solution of the Black-Scholes model. First of all, the asymptotic convergence for the solution of Black-Scholes model is proved. Second, we develop a linear, unconditionally stable, and second-order time-accurate numerical scheme for this model. By using the finite difference method and Legendre-Galerkin spectral method, we construct a time and space discrete scheme. Finally, we prove that the scheme has second-order accuracy and spectral accuracy in time and space, respectively. Several numerical experiments further verify the convergence rate and effectiveness of the developed scheme.

1. Introduction

The Black-Scholes (B–S) model is a classical pricing model of European options, which was developed by the economists Black and Scholes [1]. Because of their special contribution to the model, Merton and Scholes were awarded the Nobel Prize in 1997. This model can effectively price financial derivatives such as stocks, currencies, and bonds. Merton [2, 3] constructed the theoretical framework of option pricing and gave corresponding examples to verify the theory. Nowadays, B–S model is playing an increasingly important role in derivative financial instruments; it has been widely used in actual trading, which has inspired many options, trading strategies, and hedging strategies [4, 5, 5–7].

Based on the B–S theoretical framework, Duffie [8] proposed a formula of the market value of securities, in which the arbitrage-free value of derivative securities is obtained by solving partial differential equations. Dennis and Antonio [9] presented a new method to approximately assess price under the background of continuous time model. They discussed this method under various conditions, including option pricing with random fluctuation, Greek calculation, and term structure of interest rate. He [10] discussed the convergence of contingent claim price model from discrete time to continuous time. These results show that the contingent claim price and dynamic replication portfolio strategy converge to the continuous time limit.

Kim et al. [11] considered the pricing of a European option on a new multiscale mixed structure of underlying assess price fluctuation, and the result indicates that the option price has an ideal modification to Black-Scholes formula. In addition, this kind of correction can bring significant improvement in fitting the surface of implied volatility through calibration exercises. Lai [12] studied the influence of time discretization on European option pricing. The correction and discrete-time rebalancing strategies caused by discrete transactions are reconsidered, and the higher-order terms are expanded by Taylor series, and the corresponding correction source terms are derived. Yousuf et al. [13] proposed a second-order exponential time differencing scheme to solve nonlinear Black-Scholes model with transaction cost.
In this paper, we will study the asymptotic property of the solution of B–S model. In addition, we develop a time and space discrete scheme, where the time direction has second-order accuracy and the space direction has spectral accuracy. The stability and convergence of fully discrete scheme are also proved. Finally, numerical examples were shown to verify the accuracy of theoretical analysis.

This paper is organized as follows. In Section 2, the B–S model will be introduced. In Section 3, we will present second-order numerical method. The error estimates will be presented in Section 4. In Section 5, several numerical examples will be used to demonstrate the effectiveness of the fully discrete method. We will show some conclusions in the last section.

## 2. Black-Scholes Model

We consider the B–S model as follows:

\[
\frac{\partial V(S,t)}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + r S \frac{\partial V(S,t)}{\partial S} - r V(S,t) = 0, \quad (S,t) \in (0, +\infty) \times (0,T),
\]

\[
V(0,t) = p(t), V(+\infty,t) = q(t),
\]

\[
V(S,T) = w(S),
\]

where \( V \) is the price of the option, \( S \) is the price of the underlying asset, \( r \) is the interest rate, \( t \) is the time, and \( \sigma \) is the volatility of the stock price.

**Remark 1.** The B–S model is actually a stochastic partial differential equation; this model has a closed solution (we should know that most partial differential equations do not have it). That is to say, the option price can be accurately expressed by a function \( V = f(S, t) \), and the option price can be directly calculated by substituting the value of the independent variables \( S, t \). In fact, it is difficult to get the desired theoretical and numerical results by directly analyzing the above equations. Therefore, we need to make some equivalent transformations to transform the equation into a more general form.

We set \( \tau = T - t \); and \( V(S, \tau) = u(e^{\tau}, T - \tau) \). Thus, we can convert model (1) into the following form:

\[
\partial_t u(x, \tau) - \alpha \partial^2_x u(x, \tau) - \beta \partial_x u(x, \tau) + ru(x, \tau) = 0, \quad (x, \tau) \in (0, +\infty) \times (0,T),
\]

where \( \alpha = 1/2\sigma^2, \beta = r - \alpha \), and with the following boundary (barrier) and initial conditions,

\[
u(-\infty, \tau) = p(\tau), u(+\infty, \tau) = q(\tau),
\]

\[
u(x, 0) = u_0(x), a < x < b.
\]

In order to solve the above model better, we consider constructing a numerical method on the finite interval \((a, b)\), and then the considered model becomes

\[
\partial_t u(x, \tau) - \alpha \partial^2_x u(x, \tau) - \beta \partial_x u(x, \tau) + ru(x, \tau) = 0, \quad (x, \tau) \in (a, b) \times (0,T),
\]

\[
u(a, \tau) = u(b, \tau) = 0, \partial_x u(a, \tau) = \partial_x u(b, \tau) = 0,
\]

\[
u(x, 0) = u_0(x), a < x < b.
\]

For the solution of models (6)–(8), we have the following asymptotic estimation results.

**Lemma 1.** Suppose \( u \) is the solution of (6)–(8), then we find

\[
||u||^2 \leq e^{-2r\tau} ||u_0||^2.
\]
$$e^{2r_t} \frac{d}{dt} \|u\|^2 + 2re^{2r_t} \|u\|^2 \leq 0.$$ (11)

Let \( H(\tau) = e^{2r\tau} \|u\|^2 \), then we find
$$\frac{d}{d\tau} H(\tau) \leq 0. \quad \text{(12)}$$

Integrating with respect to time from 0 to \( \tau \), we have
$$H(\tau) \leq H(0). \quad \text{(13)}$$

Multiplying both sides by \( e^{-2r\tau} \), we find
$$\|u\|^2 \leq e^{-2r\tau} \|u\|^2. \quad \text{(14)}$$

This concludes the proof. \( \square \)

3. Second-Order Numerical Method

Let \( K \) be a positive integer, \( \Delta t = T/K \) be the time step, and \( \tau_n = n\Delta t, n = 0, 1, \ldots, K \) be mesh point. Using Crank-Nicolson formula to time discretization, we can obtain the following time-discrete scheme:
$$u^{n+1} - u^n = a\Delta t u^{n+1/2} - \beta \alpha u^{n+1/2} + r u^{n+1/2} = 0. \quad \text{(15)}$$

Here, \( u^{n+1/2} = u^{n+1} + u^n/2, n \geq 0 \).

We can prove the following unconditional energy stability theorem for scheme (15).

Theorem 1. The time discretization scheme (15) is unconditionally stable. It satisfies the following properties:
$$\|u^n\| \leq \|u^0\|, n = 1, 2, \ldots, K. \quad \text{(16)}$$

Proof. Taking the \( L^2 \) inner product of the (15) with \( \Delta t (u^{n+1} + u^n) \), we arrive at
$$e_1^n = \pi_N u(t^n) - u^n, \quad e_2^n = u(t^n) - \pi_N u(t^n), \quad e_3^n = e_1^n + e_2^n = u(t^n) - u_N^n. \quad \text{(23)}$$

Then, we can develop the following full-discrete scheme:
$$\left( u_N^{n+1} - u_N^n, \phi_N \right) - a(\alpha u_N^{n+1/2}, \phi_N) - \beta(\alpha u_N^{n+1/2}, \phi_N) + r(u_N^{n+1/2}, \phi_N) = 0, \phi_N \in S_N. \quad \text{(24)}$$

We now state the stability results for full-discrete scheme (23).

Theorem 2. Let \( \{u_N^n\}_{n=1}^{N-1} \) be solution of (23), then we derive that
$$\|u_N^{n+1}\| \leq \|u_N^n\|. \quad \text{(25)}$$

4. Error Estimate

In this part, we consider Legendre-Galerkin spectral method for the time-discrete scheme (12). We will present some error estimates for full-discretization schemes in \( L^2 \) norm. First, we denote \( S_N \) is the Legendre polynomial space, and denote \( \pi_N: L^2(\Omega) \rightarrow S_N \) is the \( L^2 \)-projection operator which satisfies as follows:
$$\langle \pi_N \phi - \phi, \psi \rangle = 0, \forall \psi \in S_N. \quad \text{(19)}$$

We have the following estimate [14]:
$$\|\phi - \pi_N \phi\| \leq C N^{l-m} \|\phi\|_m, \forall \phi \in H^m(\Omega), m > l \geq 0. \quad \text{(20)}$$

Next, we begin to analyze the error estimates of the full-discrete scheme (23). We denote the truncation error as follows:
$$R^{n+1/2} = \frac{u(t^n) - u(t^{n+1}), \Delta t}{\Delta t} - \beta(\alpha u_N^{n+1/2}, \phi_N). \quad \text{(21)}$$

We know that \( R^{n+1/2} \) satisfies
$$\|R^{n+1/2}\| \leq C\Delta t^2. \quad \text{(22)}$$

We denote error functions:

Then, we have
$$\|u_N^{n+1}\|^2 - \|u_N^n\|^2 + 2\Delta t a \|\partial_x u_N^{n+1/2}\|^2 + 2\Delta t r \|u_N^{n+1/2}\|^2 = 0. \quad \text{(17)}$$

Giving up the last two terms of the left hand side, we have
$$\|u_N^{n+1}\|^2 - \|u_N^n\|^2 \leq 0. \quad \text{(18)}$$

This yields (16). \( \square \)
Finally, we obtain the desired result (24).

**Theorem 3.** For the constructed numerical scheme (24), we have the following error estimate:

\[
\| \mathbf{u}(\tau^k) - \mathbf{u}_N^k \| \leq C(\Delta \tau^2 + N^{1-m}), k = 0, 1, \ldots, K = T/\Delta \tau.
\]  

**Proof.** For \( n = 0 \), equation (23) can be written as

\[
\frac{1}{\Delta \tau} (\mathbf{u}_N^0 - \mathbf{u}_N^0, \phi_N) - \alpha (\partial_x^2 \mathbf{u}_N^1/2, \phi_N) - \beta (\partial_x \mathbf{u}_N^1/2, \phi_N) + r(\mathbf{u}_N^1/2, \phi_N) = 0.
\]  

Subtracting (28) from (7) at \( \tau_1 \), we note that

\[
\begin{align*}
\partial_x \mathbf{u}(\cdot, \tau^{1/2}) - \mathbf{u}_N(\cdot, \tau^{1/2}) &= \partial_x \mathbf{u}(\cdot, \tau^{1/2}) - \mathbf{u}(\cdot, \tau_1) + \frac{\Delta \tau}{\Delta \tau} \mathbf{u}(\cdot, \tau^{1/2}) - \mathbf{u}(\cdot, \tau^0) \\
+ \frac{\Delta \tau}{\Delta \tau} (\mathbf{u}(\cdot, \tau^1) - \mathbf{u}(\cdot, \tau^0) - \pi_N \mathbf{u}(\cdot, \tau^1) - \pi_N \mathbf{u}(\cdot, \tau^0)) \\
+ \pi_N \mathbf{u}(\cdot, \tau^1) - \pi_N \mathbf{u}(\cdot, \tau^0) - \frac{\mathbf{u}_N^0 - \mathbf{u}_N^1}{\Delta \tau} = \tau^{1/2} \\
+ \frac{1}{\Delta \tau} (I - \pi_N)(\mathbf{u}(\cdot, \tau^0) - \mathbf{u}(\cdot, \tau^0)) + \frac{1}{\Delta \tau} (e_1^0 - e_1^0),
\end{align*}
\]

\[
\begin{align*}
\partial_x^2 \mathbf{u}(\cdot, \tau^{1/2}) - \partial_x^2 \mathbf{u}_N^{1/2} &= \partial_x^2 \mathbf{u}(\cdot, \tau^{1/2}) - \pi_N \partial_x^2 \mathbf{u}(\cdot, \tau^{1/2}) + \pi_N \partial_x^2 \mathbf{u}(\cdot, \tau^{1/2}) - \partial_x^2 \mathbf{u}_N^{1/2} \\
= \partial_x^2 [(I - \pi_N) \mathbf{u}(\cdot, \tau^{1/2})] + \partial_x^2 e_1^{1/2},
\end{align*}
\]

\[
\begin{align*}
\partial_x \mathbf{u}(\cdot, \tau^{1/2}) - \partial_x \mathbf{u}_N^{1/2} &= \partial_x \mathbf{u}(\cdot, \tau^{1/2}) - \pi_N \partial_x \mathbf{u}(\cdot, \tau^{1/2}) + \pi_N \partial_x \mathbf{u}(\cdot, \tau^{1/2}) - \partial_x \mathbf{u}_N^{1/2} \\
= \partial_x [(I - \pi_N) \mathbf{u}(\cdot, \tau^{1/2})] + \partial_x e_1^{1/2},
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}(\cdot, \tau^{1/2}) - \mathbf{u}_N^{1/2} &= \mathbf{u}(\cdot, \tau^{1/2}) - \pi_N \mathbf{u}(\cdot, \tau^{1/2}) + \pi_N \mathbf{u}(\cdot, \tau^{1/2}) - \mathbf{u}_N^{1/2} \\
= (I - \pi_N) \mathbf{u}(\cdot, \tau^{1/2}) + e_1^{1/2}.
\end{align*}
\]

Then, we have

\[
\begin{align*}
\left( e_1^1 - e_1^0, \phi_N \right) + \Delta \tau \alpha (\partial_x e_1^{1/2}, \partial_x \phi_N) - \Delta \tau \beta (\partial_x e_1^{1/2}, \phi_N) + \Delta \tau r(e_1^{1/2}, \phi_N) \\
= -\Delta \tau R_{1/2} \mathbf{u}(\cdot, \tau^0) + ((\pi_N - I)(\mathbf{u}(\cdot, \tau^1) - \mathbf{u}(\cdot, \tau^0), \phi_N) + \Delta \tau r((\pi_N - I)\partial_x \mathbf{u}(\cdot, \tau^{1/2}), \partial_x \phi_N) \\
- \Delta \tau \beta ((I - \pi_N) \mathbf{u}(\cdot, \tau^{1/2}), \partial_x \phi_N) + \Delta \tau r((\pi_N - I) \mathbf{u}(\cdot, \tau^{1/2}), \phi_N), \phi_N \in S_N.
\end{align*}
\]

Setting \( \phi_N = 2e_1^1 \), we have...
Note that

\[ \sum_{n} \leq e \]

By using discrete Gronwall lemma, we get

\[ \Delta r \left( \frac{1}{2} \| R^{1/2} \|^2 + r \| e_1 \|^2 \right) + \Delta r \alpha \left( \| (\pi_N - I) \partial_x u(\cdot \tau) \|^2 + \| \partial_x e_1 \|^2 \right) \]

\[ + \Delta r \beta \left( \frac{\beta}{\alpha} \right) \left( \| I - \pi_N \| u(\cdot \tau) \|^2 + \| \partial_x e_1 \|^2 \right). \]

Dropping some positive terms, we find

\[ \| e_1 \|^2 \leq C_1 \Delta r^5 + C_2 N^{2-2m}. \quad (33) \]

Subtracting (23) from a reformulation of (7) at \( \tau^{n+1/2} \), we find

\[ \left( e_1^{n+1} - e_1^n, \phi_N \right) + \Delta r \alpha \left( \partial_x e_1^{n+1/2}, \partial_x \phi_N \right) - \Delta r \beta \left( \partial_x e_1^{n+1/2}, \phi_N \right) + \Delta r \tau \left( e_1^{n+1/2}, \phi_N \right) \]

\[ = -\Delta r \left( R^{n+1/2}, e_1^{n+1/2} \right) + \Delta r \alpha \left( (\pi_N - I) \partial_x u(\cdot \tau^{n+1/2}), \partial_x e_1^{n+1/2} \right) \]

\[ - 2 \Delta r \tau \beta \left( (I - \pi_N) u(\cdot \tau^{n+1/2}), \partial_x e_1^{n+1/2} \right) \]

\[ \leq \Delta r \left( \left\| R^{n+1/2} \right\|^2 + \| e_1^{n+1/2} \|^2 \right) + \Delta r \alpha \left( \left\| (\pi_N - I) \partial_x u(\cdot \tau^{n+1/2}) \right\|^2 + \| \partial_x e_1^{n+1/2} \|^2 \right) \]

\[ + \Delta r \beta \left( \frac{\beta}{\alpha} \right) \left( \| I - \pi_N \| u(\cdot \tau^{n+1/2}) \|^2 + \| \partial_x e_1^{n+1/2} \|^2 \right). \quad (34) \]

Letting \( \phi_N = 2e_1^{n+1/2} \), we have

\[ \| e_1^{n+1} \|^2 - \| e_1^n \|^2 \leq C_1 \Delta r^5 + C_2 \Delta r N^{2-2m} + C_3 \Delta \| e_1^{n+1} \|^2. \quad (36) \]

Summing up for \( n = 1, \ldots, k \), we find

\[ \| e_1^k \|^2 \leq \| e_1^0 \|^2 + C \left( \Delta t^4 + N^{2-2m} \right) + C_3 \Delta t \sum_{n=1}^{k} \| e_1^n \|^2. \quad (37) \]

By using discrete Gronwall lemma, we can get

\[ \| e_1^{k+1} \|^2 \leq C \left( \Delta t^4 + N^{2-2m} \right). \quad (38) \]

Thus, we have

\[ \| e_1^{n+1} \|^2 - \| e_1^n \|^2 \leq C_1 \Delta r^5 + C_2 \Delta r N^{2-2m} + C_3 \Delta r \| e_1^{n+1} \|^2. \]

\[ \sum_{n=1}^{k} \Delta t \leq C \Delta r^4 + N^{2-2m}. \quad (39) \]

5. Numerical Examples

5.1. Verification of the Convergence Order. In this part, we will test the accuracy and validity of the full-discrete scheme (23). Actually, in the B–S model, researches usually choose \( \alpha = 0.5\sigma^2; \beta = r - \alpha \). Set \( T = 1, N = 128 \) and \( \sigma_0 = \tan(x) + x, \sigma = 0.1, r = 0.2 \). In Table 1, we list \( L^2 \) error and convergence order for different time step size. We can see that the time direction is obvious of second-order accuracy, while the space direction has a good convergence property.

5.2. Effect of Various Parameters. Next, we will test the asymptotic decay property of the solution. We fix \( N = 50, \Delta t = 0.01, T = 1, \Lambda = (0, 6), \) and \( r = 0.1 \) and leave the initial value unchanged, and at the same time, let \( \beta \) constantly change. In Figure 1–4, we list the change process of numerical solution with \( \beta = 0.2, 0.6, 1.0, 5.0 \), and we find...
Table 1: The $L^2$, numerical errors, and time convergence order for various temporal steps.

| $\Delta \tau$ | $L^2$-error | Order |
|---------------|--------------|-------|
| 0.02          | 0.0365       | 2.0045|
| 0.01          | 0.0091       | 2.0011|
| 0.005         | 0.0023       | 2.0003|
| 0.001         | 9.005e-05    | 2.0000|
| 0.0005        | 2.2580e-05   | 2.0000|

Figure 1: Change process of numerical solution with $\beta = 0.2$.

Figure 2: Change process of numerical solution with $\beta = 0.6$. 
Figure 3: Change process of numerical solution with $\beta = 1.0$.

Figure 4: Change process of numerical solution with $\beta = 5.0$. 
that when $\beta$ becomes larger, the solution gradually tends to 0.

Fix $u_0 = |\tan(x) + x|$. In the following numerical experiments, we also test the influence of $\sigma$ and $r$ on the numerical solution. In Figure 5–6, the variation process of numerical solution with $\sigma$ and $r$ is given. We find that the numerical solution experiences four peaks and valleys; at the same time, with the increase of $R$, the fluctuation of numerical solution gradually increases.

6. Concluding Remarks

We construct an effective fully discrete scheme for the B–S model based on Legendre-Galerkin scheme for spatial
discretization. The scheme is also linear, unconditionally stable, and has second-order accuracy in time, where the Crank-Nicolson method is used for time discretization. Through the implementations of several numerical examples, we demonstrate the accuracy and effectiveness of the developed scheme, numerically.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

This paper was supported by CUFU postgraduate students support program for the integration of research and teaching, the Guizhou Key Laboratory of Big Data Statistical Analysis (Nos. [2019]5103, BDSA20200114), the China Scholarship Council (No. 202008520027), and the Academic Research Projects of Beijing Union University (No. SK80202105).

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