Research Article

Posterior Propriety of an Objective Prior in a 4-Level Normal Hierarchical Model

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The use of hierarchical Bayesian models in statistical practice is extensive, yet it is dangerous to implement the Gibbs sampler without checking that the posterior is proper. Formal approaches to objective Bayesian analysis, such as the Jeffreys-rule approach or reference prior approach, are only implementable in simple hierarchical settings. In this paper, we consider a 4-level multivariate normal hierarchical model. We demonstrate the posterior using our recommended prior which is proper in the 4-level normal hierarchical models. A primary advantage of the recommended prior over other proposed objective priors is that it can be used at any level of a hierarchical model.

1. Introduction

Bayesian hierarchical models have a wide range of modern applications including engineering [1], astrophysics [2], economics [3], environmental sciences [4], climatology [5], survival analysis [6], and genetics [7]. It is wonderful here to stay, but hyperparameter priors are often chosen in a casual fashion. Statisticians often use improper priors to express ignorance or to provide good frequency properties. However, it is dangerous to implement the Gibbs sampler without checking that the posterior is proper. Formal approaches to objective Bayesian analysis, such as the Jeffreys-rule approach or reference prior approach, are only implementable in simple hierarchical settings. In this paper, we consider a 4-level multivariate normal hierarchical model. We demonstrate the posterior using our recommended prior which is proper in the 4-level normal hierarchical models. A primary advantage of the recommended prior over other proposed objective priors is that it can be used at any level of a hierarchical model.

1. Introduction

Bayesian hierarchical models have a wide range of modern applications including engineering [1], astrophysics [2], economics [3], environmental sciences [4], climatology [5], survival analysis [6], and genetics [7]. It is wonderful here to stay, but hyperparameter priors are often chosen in a casual fashion. Statisticians often use improper priors to express ignorance or to provide good frequency properties. However, it is dangerous to implement the Gibbs sampler without checking that the posterior is proper. As Hobert and Casella [8] pointed, without proper precaution, simple noninformative priors can be misused, sometimes unknowingly, and lead to other difficulties, such as the nonconvergence of the Gibbs sampler. Therefore, it is hazardous to skip the demonstration at the risk of making the inference from an improper posterior distribution. There are many examples of this in the statistical and other literatures. See especially, Wang et al. [9]; Hobert and Casella [8, 10]; Berger and Strawderman [11]; Berger et al. [12]; Speckman et al. [13]; Roy and Dey [14]; Michalak and Morris [15]; Ramos et al. [16]; Tomazella et al. [6]; etc. The importance of the posterior propriety motivates us to explore it.

The normal hierarchical distribution has received enormous attention and is also of substantial importance in contemporary statistical theory and application. Ning [17] proposed a 2-level multivariate normal hierarchical model for the degradation data of multiple units with change point. Wang and Coit [18] handled the reliability prediction problem using a multivariate normal distribution model, considering multiple degradation measures. Heuristically, one could also build the multivariate normal hierarchical model with adding a normal distribution for unknown mean vectors to deal with this reliability prediction problem. The unknown parameters because of either convenience or a lack of prior information are often modeled with improper objective priors. Some references of objective priors can be found in Berger and Strawderman [11]; Berger [19]; Pollo et al. [20]; and Ferreira et al. [21]. However, formal approaches to objective Bayesian analysis, such as the Jeffreys-rule approach or reference prior approach, are only implementable in simple hierarchical settings. It is thus common to
use less formal approaches, such as utilizing formal priors from nonhierarchical models in hierarchical settings. However, this can be fraught with danger. For instance, nonhierarchal Jeffreys-rule priors for variances or covariance matrices result in improper posterior distributions if they are used at higher levels of a hierarchical model (see [12]).

Berger et al. [12] addressed the question of choice of hyperpriors in the following 2-level normal hierarchical model:

$$y \sim N_p(\theta, I_p), \theta \sim N_p(b, Q),$$

where

$$y_{p+1} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \quad \theta_{p+1} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix}, \quad b_{p+1} = \begin{pmatrix} \beta \\ \beta \\ \vdots \\ \beta \end{pmatrix}, \quad Q_{p+1} = I_m \otimes V,$$

for $i = 1, 2, \ldots, m$, in which the $y_i$ are the $k \times 1$ observation vectors and the $\theta_j$ are the $k \times 1$ unknown mean vectors (thus $p = mk$), $\beta$ is a $k \times 1$ unknown “hypermean” vector, and $V$ is an unknown $k \times k$ “hypercovariance matrix.” Some commonly used hyperpriors for the hyperparameter $\beta$ can be considered. For example, the constant prior and conjugate normal prior. The recommended priors for “hypermean” vectors and “hypercovariance matrix.” In Section 3, we demonstrate that the posterior using the recommended prior is proper in the 4-level normal hierarchical model. In Section 4, we consider MCMC computation from the posterior. Section 5 gives the performance of this prior, in comparison with other objective priors that were studied in Berger et al. [12], presenting strong numerical evidence of the superiority of (3). Some concluding remarks are provided in Section 6.

2. A 4-Level Normal Hierarchical Model

Consider the following 4-level normal hierarchical model:

$$\begin{align*}
\text{Level 1:} & \quad y_i = \theta_i + N_k(0, I_k), \\
\text{Level 2:} & \quad \theta_i = Z_i \beta + N_k(0, V), \\
\text{Level 3:} & \quad \beta_j = T_j \eta + N_p(0, W), \\
\text{Level 4:} & \quad \eta_l = \xi + N_q(0, \Sigma),
\end{align*}$$

where the $y_i$ are the $k \times 1$ observation vectors; the $\theta_i$ are the $k \times 1$ unknown mean vectors; the $\beta_j$ are the $p \times 1$ unknown vectors; the $\eta_l$ are the $q \times 1$ unknown vectors; $\xi$ is an unknown $q$-dimensional “hypermean” vectors; the $Z_i$ are the
Consider the one-to-one transformation from \( \xi \rightarrow \mathbf{V} \) and rewrite the prior as

\[
\pi (\xi) \propto \frac{1}{(1 + \|\xi\|)^{(q-1)/2}}, \quad \xi \in \mathbb{R}^q, 
\]

\[
\pi (\mathbf{V}) \propto \frac{1}{|\mathbf{V}|^{1 - 1/(2k)}} \prod_{1 \leq i < j \leq k} (v_i - v_j), \quad V > 0, 
\]

\[
\pi (\mathbf{W}) \propto \frac{1}{|\mathbf{W}|^{1 - 1/(2p)}} \prod_{1 \leq i < j \leq p} (w_i - w_j), \quad W > 0, 
\]

\[
\pi (\Sigma) \propto \frac{1}{|\Sigma|^{1 - 1/(2q)}} \prod_{1 \leq i < j \leq q} (\sigma_i - \sigma_j), \quad \Sigma > 0, 
\]

where \( v_1 > \cdots > v_k > 0 \) are the ordered eigenvalues of \( \mathbf{V} \), \( w_1 > \cdots > w_p > 0 \) are the ordered eigenvalues of \( \mathbf{W} \), and \( \sigma_1 > \cdots > \sigma_q > 0 \) are the ordered eigenvalues of \( \Sigma \).

As discussed by Berger et al. [12], the prior \( \pi (\xi) \) can be represented by the following hierarchical structure:

\[
\xi | \lambda \sim N (0, \lambda I_n), \quad [\lambda] \sim \lambda^{-1/2} e^{-\lambda/2}. 
\]

For both intuitive and technical reasons, it is convenient to write \( \mathbf{V} = \mathbf{O}_1 \mathbf{V}^* \mathbf{O}_1^t, \mathbf{W} = \mathbf{O}_2 \mathbf{W}^* \mathbf{O}_2^t \), and \( \Sigma = \mathbf{O}_3 \mathbf{\Sigma}^* \mathbf{O}_3^t \), where \( \mathbf{V}^* = \text{diag} (v_1, \ldots, v_k), \mathbf{W}^* = \text{diag} (w_1, \ldots, w_p), \mathbf{\Sigma}^* = \text{diag} (\sigma_1, \ldots, \sigma_q) \), and \( \mathbf{O}_i \) being the matrix of eigenvectors corresponding to \( \mathbf{V}^*, \mathbf{W}^*, \) and \( \mathbf{\Sigma}^* \), \( i = 1, 2, 3 \), respectively. Consider the one-to-one transformation from \( \mathbf{V} \) to \( (\mathbf{V}^*, \mathbf{O}_1) \) and rewrite the prior as

\[
\pi (\mathbf{V}) d\mathbf{V} = \pi (\mathbf{V}^*, \mathbf{O}_1) d\mathbf{V}^* d\mathbf{O}_1, 
\]

where \( d\mathbf{V} = \prod_{i \leq j} dv_i dv_j, d\mathbf{V}^* = \prod_{i \leq j} dv_i, \) and \( d\mathbf{O}_1 \) denotes the invariant Haar measure over the space of orthonormal matrices \( \mathcal{O} = \{ \mathbf{O}_1 : \mathbf{O}_1^t \mathbf{O}_1 = \mathbf{I}_k \} \) (see Anderson [33] for definition). From Farrell [34], the functional relationship between \( \pi (\mathbf{V}) \) and \( \pi (\mathbf{V}^*, \mathbf{O}_1) \) is

\[
\pi (\mathbf{V}^*, \mathbf{O}_1) = \pi (\mathbf{O}_1 \mathbf{V}^* \mathbf{O}_1^t) \prod_{i < j} (v_i - v_j). 
\]

Therefore, the prior for \( \mathbf{V} \) becomes the prior of \( (\mathbf{V}^*, \mathbf{O}_1) \):

\[
\pi (\mathbf{V}^*, \mathbf{O}_1) \propto \frac{1}{|\mathbf{V}^*|^{1 - 1/(2k)}}. 
\]

Use of the invariant prior on \( \mathbf{O}_1 \) (essentially a uniform prior over rotations) is natural and noncontroversial. This transformation reveals a significant difficulty of any prior that can be written as a function of \( |\mathbf{V}| \); in the \( (\mathbf{V}^*, \mathbf{O}_1) \) space, such priors contain the term \( \prod_{i < j} (v_i - v_j) \), which gives low mass to close eigenvalues and hence effectively force the eigenvalues apart. It is a criticism of inverse Wishart and Jeffreys priors. This is contrary to the common intuition, in that one often chooses a prior that pushes the eigenvalues closer together.

Similarly, one can obtain the prior of \( (\mathbf{W}^*, \mathbf{O}_2) \) and \( (\mathbf{\Sigma}^*, \mathbf{O}_3) \), given by

\[
\pi (\mathbf{W}^*, \mathbf{O}_2) \propto \frac{1}{|\mathbf{W}^*|^{1 - 1/(2p)}}, 
\]

\[
\pi (\mathbf{\Sigma}^*, \mathbf{O}_3) \propto \frac{1}{|\mathbf{\Sigma}^*|^{1 - 1/(2q)}}. 
\]

3. Posterior Propriety

In this section, we consider the posterior propriety of the recommended prior in the 4-level normal hierarchical model (4).

3.1. The Case \( q \geq 2 \)

**Theorem 1.** Consider the hierarchical Bayes model (4) with \( q \geq 2 \). Assume that \( \mathbf{Z} \) has rank \( ps \) and \( \mathbf{Z}^t \mathbf{Z} \) has rank \( nq \). Then, the posterior distribution is always proper.

**Proof.** For the technical reason, we define

\[
\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \quad \mathbf{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix}, \quad \mathbf{1}_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{Z} = \begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \vdots \\ \mathbf{Z}_m \end{pmatrix}, 
\]

\[
\mathbf{T} = \begin{pmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \vdots \\ \mathbf{T}_s \end{pmatrix}, 
\]

Then, (4) is equivalent to
\[
L(V, W, \Sigma, \lambda) \propto \exp\left( -\frac{1}{2} y' (\Delta + \lambda GG')^{-1} y \right)
\]

\[
\leq \frac{1}{|I_{mk} + I_m \otimes V|^{h_1}} |I_{mk} + Z(I_\theta \otimes W)Z'|^{h_2} |I_{mk} + ZT(I_\theta \otimes \Sigma)T'Z'|^{h_3} |I_{mk} + \lambda GG'|^{h_4}
\]

where \( h_1 = (1/2mk) + \epsilon, h_2 = (1/2p) + \epsilon, h_3 = (1/2) - (1/2mk) - (1/2p) - (1/2q) - 3\epsilon, h_4 = 1 - h_1 - h_2 - h_3 = 1/2 + \epsilon, \) and \( \epsilon \) is a positive constant, assuring \( h_3 \geq 0. \) Note that \( h_1 + h_2 + h_3 + h_4 = 1/2. \) Let \( m(y) \) be the marginal density of \( y. \) Clearly,

\[
m(y) = \int \int \int L(V, W, \Sigma, \lambda) \pi(V) \pi(W) \pi(\Sigma) \pi(\lambda) dV dW d\Sigma d\lambda
\]

\[
\leq \int \int \frac{\pi(V)}{|I_{mk} + I_m \otimes V|^{h_1}} dV \times \int \frac{\pi(W)}{|I_{mk} + Z(I_\theta \otimes W)Z'|^{h_2}} dW \times \int \frac{\pi(\Sigma)}{|I_{mk} + \lambda GG'|^{h_4}} d\Sigma \times \int \frac{\pi(\lambda)}{|I_{mk} + ZT(I_\theta \otimes \Sigma)T'Z'|^{h_3}} d\lambda.
\]

Note that

\[
\int \frac{\pi(V)}{|I_{mk} + I_m \otimes V|^{h_1}} dV = \int \frac{1_{\{\nu > v \rightarrow v_k\}}}{|I_k + V|^{(1/2k) + \epsilon mc}} |V|^{1 - (1/2k)} \prod_{1 \leq i < j \leq k} (v_i - v_j) dV.
\]

From relationship (8) between \( V \) and \( (V^*, O_1), \)

\[
\int \frac{\pi(V)}{|I_{mk} + I_m \otimes V|^{h_1}} dV = \int \frac{1_{\{\nu > v \rightarrow v_k\}}}{|I_k + V^*|^{(1/2k) + \epsilon mc}} |V^*|^{1 - (1/2k)} dV^* dO_1 \leq \prod_{j=1}^{k} \int_0^{\infty} \frac{1}{(1 + v_j)^{(1/2k) + \epsilon mc}} (v_j - v_{j-1})^{-1} dv_j < \infty.
\]

Next, consider the integration over \( W. \) From the matrix identity \(|A + XB X'| = |A||B||B^{-1} + X'A^{-1}X|, \) we obtain

\[
|I_{mk} + Z(I_\theta \otimes W)Z'| = |W| \left( I_\theta \otimes W^{-1} \right) + Z'Z.
\]

Let \( z_0 > 0 \) be the smallest eigenvalue of \( Z'Z, \) then

\[
\left| I_\theta \otimes W^{-1} \right| + Z'Z \geq |W^{-1} + z_0 I_p|^{1/2}.
\]

Thus,

\[
|I_{mk} + Z(I_\theta \otimes W)Z'| \geq |I_p + z_0 W|^{1/2}.
\]

Using the above inequality, it yields

\[
|I_{mk} + \lambda GG'| = |\lambda I_p|^{1/\gamma} + G'G \geq (1 + \nu p \lambda)^{\gamma}.
\]
Therefore,  
\[
\int_0^{\infty} \frac{1}{|I_{mk} + \lambda GG|^{\frac{1}{2}}} \pi(\lambda) d\lambda \leq \int_0^{\infty} \frac{1}{\lambda^{1/2} (1 + u_\lambda)^{(1/2)q} e^{2/\lambda}} d\lambda < \infty.
\]  
(25)

\[
\int \frac{\pi(\Sigma)}{|I_{mk} + ZT(I_0 \otimes \Sigma)T^* Z|^n} d\Sigma \leq \int \frac{1}{I_k + t_0 \Sigma^*} \pi^{n((1/2) - (1/2mk) - (1/2sp) - (1/2q) - 3e)} |\Sigma^*|^{1 - (1/2q)} d\Sigma^*  
\leq \prod_{l=1}^{q} \int_0^{\infty} \frac{1}{(1 + t_0 \sigma_l)^{n((1/2) - (1/2mk) - (1/2sp) - (1/2q) - 3e)} \sigma_l^{1 - (1/2q)}} d\sigma_l,
\]

which is finite if
\[
n(\frac{1}{2} - \frac{1}{2mk} - \frac{1}{2sp} - \frac{1}{2q} - 3e) + 1 - \frac{1}{2q} > 1.
\]  
(28)

Since \( e \) can be chosen arbitrarily small, the integration over \( \Sigma \) is finite if
\[
n(q - 1) > 1 + \frac{mq}{mk} + \frac{mq}{sp} = 1 + \frac{1}{ms} + \frac{1}{s}.
\]  
(29)

Since, \( q, n, m, \) and \( s \geq 2 \), (29) is true. The theorem is proved. \( \square \)

### 3.2. The Case \( q = 1 \)

**Theorem 2.** Consider the hierarchical Bayes model (4) with \( q = 1 \). Assume that \( Z \) has rank \( ps \) and \( ZT \) has rank \( n \). Then, the posterior distribution is proper if
\[
\frac{1}{m} + \frac{1}{s} + \frac{1}{n} + \frac{1}{mn} < 1.
\]  
(30)

**Proof.** In the case \( q = 1 \), note that \( \Sigma \) is just a variance, so that
\[
\Delta = I_{mk} + I_0 \otimes V + Z(I_0 \otimes W)Z' + \Sigma ZTT' Z'.
\]  
(31)

The prior of \( \xi \) becomes
\[
\pi(\xi) \propto 1.
\]  
(32)

Integrating out \( \xi \) in (22) with its constant prior and dropping all exponential terms of the likelihood (as they are less than one), the marginal likelihood for \( V \) and \( W \) satisfies
\[
L(V, W) < \frac{1}{|\Delta|^{1/2} (G' \Delta^{-1} G)^{1/2}}.
\]  
(33)

Last, to consider the integration over \( \Sigma \). Let \( t_0 \) be the smallest eigenvalue of \( T^* ZT \). Then,
\[
|I_{mk} + ZT(I_0 \otimes \Sigma)T^* Z| \geq |I_{q} + t_0 \Sigma^*|^{n}.
\]  
(26)

Thus,
\[
\int |\Delta|^{1/2} (G' \Delta^{-1} G)^{1/2} \pi(V) \pi(W) \pi(\Sigma) d\Sigma dV dW < \infty.
\]  
(34)

Denote \( t_1 \) be the largest eigenvalue of \( T^* ZT \). Let \( \lambda_0(\Delta) \) and \( \lambda_1(\Delta) \) denote the minimum and maximum eigenvalue of \( \Delta \), respectively. Then,
\[
\lambda_0(\Delta) \leq \lambda_1 \left( (1 + v_1) I_{mk} + w_1 ZZ' + \Sigma ZTT' Z' \right)
\leq (1 + v_1 + 1 + w_1 + t_0 \Sigma)
\leq (1 + v_1 + 1 + w_1 + t_0 \Sigma).
\]  
(35)

Noting that \( G \) is an \( mk \times 1 \) vector, it follows that
\[
G' \Delta^{-1} G > \lambda_0(\Delta^{-1}) G' G = \lambda_1^{-1}(\Delta) G' G
\]  
(36)

Clearly,
\[
m(y) \leq C \int \int \left( |I_k + V|^{1/2} |I_p + z_0 W|^{1/2} \right) \frac{1}{|\Delta|^{1/2} (G' \Delta^{-1} G)^{1/2}} dV dW d\Sigma,
\]  
(38)

where \( C = (G' G)^{-(1/2)} \). Note that
\[ \int \frac{(1 + v_1)^{1/2}}{|k + V|^{mb_k} \pi(V) dV} \leq \int \frac{(1 + v_1)^{1/2}}{|k + V|^{mb_k} |V|^{1/2k}} dV' \]

\[ \leq \int_0^\infty \frac{1}{(1 + v_1)^{mb_k - 1/2} v_1^{1/2k}} dV' \times \frac{1}{(1 + v_1)^{mb_k - 1/2k}} dV, \]

(39)

which is finite since \( mb_1 - (1/2) + 1 - (1/2k) = 1 + me > 1 \).

Similarly,

\[ \int \frac{(1 + z_1 w_1)^{1/2}}{|k + W|^{sb_2} \pi(W) dW} \leq \int_0^\infty \frac{(1 + z_1 w_1)^{1/2}}{(1 + w_1)^{sb_1} w_1^{1/2p}} dw_1 \]

\times \prod_{j=2}^{p} \int_0^\infty \frac{1}{(1 + w_j)^{sb_j} w_j^{1/2p}} dw_j, \]

(40)

which is finite since \( sb_2 + 1 - (1/2p) - (1/2) = 1 + se > 1 \).

Next, consider the integration over \( \Sigma \). It follows that

\[ \int_0^\infty \frac{(1 + t_1 \Sigma)^{1/2}}{(1 + t_1 \Sigma)^{n(1/2) - b_1 - b_2}} d\Sigma < \infty, \]

(41)

if

\[ n((1/2) - b_1 - b_2) = n((1/2) - (1/2mk) - (1/2m) - (1/2p) - (1/2s) - 2e) > 1. \]

Since \( e \) can be chosen arbitrarily small, \( p = n \) and \( k = sn \); this is true if

\[ \frac{1}{m} + \frac{1}{s} + \frac{2}{n} + \frac{1}{sn} < 1. \]

(42)

This completes the proof. \( \square \)

4. Computation

In this section, we consider the MCMC computation from the posterior arising from the model (4). The normal hierarchical models are typically handled today by Gibbs sampling. One difficulty of computation is to sample the covariance matrix efficiently. The main new development discussed in Berger et al. [35] is a new and efficient computational algorithm for dealing with priors on covariance matrices as in (3). It overcomes the computational bottleneck mentioned in the introduction.

Fact 1. Here are the full conditional distributions:

(a) For \( i = 1, \ldots, m \), the conditional posterior of \( \theta_i \) given \( \beta \) and \( V \) is the usual conjugate posterior density:

\[ N_{\eta}(\{I_k + V^{-1}\}^{-1}(y_i + V^{-1}Z \beta), (I_k + V^{-1})^{-1}). \]

(43)

(b) Define

\[ M = I \otimes W^{-1} + [(Z'Z)^{-1}Z' (I \otimes V) Z(Z'Z)^{-1}]^{-1}, \]

\[ u = (I \otimes W^{-1})^T \eta + [(Z'Z)^{-1}Z' (I \otimes V) Z(Z'Z)^{-1}]^{-1}(Z'Z)^{-1}Z' \theta. \]

(44)

The conditional posterior of \( \beta \) given \( (\theta, \eta, V, W) \) is

\[ N_{sp}(M^{-1}u, M^{-1}). \]

(c) Defining \( r_1 = (m/2) + 1 - (1/3k) \), the conditional posterior density of \( V \) for given \( (\beta, \theta) \) can be written as

\[ \pi(V \mid \beta, \theta) \propto \frac{1}{|V|^r \prod_{i \in \Omega} (v_i - v_j)} \text{etr}
\[ \left\{ -\frac{1}{2} V^{-1} H_1 \right\}. \]

(45)

Here and in the following etr (A) represents exp(trace(A)) for a squared matrix A, and

\[ H_1 = H_1(\theta, \beta) = \sum_{i=1}^{m} (\theta_i - Z_i \beta) (\theta_i - Z_i \beta)^T. \]

(46)

(d) Define

\[ K = I \otimes \Sigma^{-1} + [(T' T)^{-1} T' (I \otimes W) T(T' T)^{-1}]^{-1}, \]

\[ v = I \otimes \Sigma^{-1} \xi + [(T' T)^{-1} T' (I \otimes W) T(T' T)^{-1}]^{-1} (T' T)^{-1} T' \xi. \]

(47)

The conditional posterior of \( \eta \) given \( (\beta, \xi, W, \Sigma) \) is

\[ N_{pq}(K^{-1}v, K^{-1}). \]

(e) Defining \( r_2 = (s/2) + 1 - (1/2p) \), the conditional posterior density of \( W \) for given \( (\beta, \eta) \) can be written as

\[ \pi(W \mid \beta, \eta) \propto \frac{1}{|W|^r \prod_{i \in \Omega} (w_i - w_j)} \text{etr}
\[ \left\{ -\frac{1}{2} W^{-1} H_2 \right\}, \]

(48)

where \( H_2 = \sum_{i=1}^s (\beta_j - T_i' \eta) (\beta_j - T_i' \eta)^T. \)

(f) To compute with the recommended prior for \( \xi \), use the equivalent representation as follows:

\[ \xi \mid \lambda \sim N_{\eta}(0, \lambda I_s), \]

(49)

\[ [\lambda] \propto \lambda^{-1/2} e^{-1/2 \lambda}. \]

(50)

\[ [\lambda, \xi] \propto \frac{1}{\lambda^{(q+1)/2}} \exp \left\{ -\frac{1 + ||\xi||^2}{2\lambda} \right\}. \]

(51)
(g) Defining \( r_3 = (n/2) + 1 - (1/2q) \), the conditional posterior density of \( \Sigma \) for given \((\beta, \eta)\) can be written as
\[
\pi(\Sigma | \beta, \eta) \propto \frac{1}{|\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2} \Sigma^{-1} \mathbf{H}_3 \right\},
\]
where \( \mathbf{H}_3 = \sum_{i=1}^k (\eta_i - \xi)(\eta_i - \xi)' \).

(i) For sample \( \lambda \) from its full conditional, the Inverse Gamma \(((q - 1)/2, (1 + \|\xi\|^2)/2)\) density
If \( q = 1 \), this step is not needed as the hyperprior, for \( \xi \) is constant.

(ii) Given \((\lambda, \Sigma, \eta, \xi)\), Gibbs sampling of \( \xi \) can be done from its full conditional, which is (when \( q = 1 \), set \( \lambda = \infty \))

Sampling of \( \theta, \beta, \eta, \) and \( \xi \) can simply be carried out with a Gibbs step, as its full conditional will be a normal distribution. To sample the covariance matrix \( \mathbf{V}, \mathbf{W}, \) and \( \Sigma \) from Fact 1, the Metropolis–Hastings Algorithm [36] and Hit-and-Run Method

Chen and Schmeiser [37] could be used, based on proposal distributions that generate full-candidate matrices. But the two well-known methods are both inefficient, especially for high-dimensional data. Fortunately, Berger et al. [35] proposed a new and efficient computational algorithm for sampling the covariance matrix \( \mathbf{V}, \mathbf{W}, \) and \( \Sigma \) from Fact 1, respectively.

Take sampling \( \mathbf{V} \) from (45) as an example. For the conditional density of \( \mathbf{V} \) given in (45), we use the eigenvalue-vector decomposition \( \mathbf{V} = \mathbf{O}_1 \mathbf{V}^* \mathbf{O}_1' \), where \( \mathbf{O}_1 \) is the orthogonal and \( \mathbf{V}^* \) is the diagonal matrix of ordered eigenvalues. For \( \mathbf{H}_1 \) defined in (46), note that \( \text{tr}(\mathbf{V}^{-1} \mathbf{H}_1) = \text{tr}(\mathbf{O}_1 (\mathbf{V}^*)^{-1} \mathbf{O}_1' \mathbf{H}_1) = \text{tr}(\mathbf{H}_1) \), so the conditional density of \( \mathbf{V} \) given in (45) can be transformed to
\[
\pi(\mathbf{V}^*, \mathbf{O}_1 | \beta, \theta, \eta) \propto \frac{1}{|\mathbf{V}|^{1/2}} \exp\left\{ -\frac{1}{2} \mathbf{V}^*^{-1} \mathbf{H}_1 \mathbf{O}_1 \right\} 1_{\{\mathbf{v}_1 > \cdots > \mathbf{v}_k\}}
\]

Gibbs sampling of \( \mathbf{V}^* \): following Lemma 1 in Berger et al. [12], i.e.,
\[
\int g(\mathbf{O}^* \mathbf{V}^* \mathbf{O}_1') d\mathbf{V}^* d\mathbf{O}_1 = \frac{1}{k!} \int g(\mathbf{O}_1 d\mathbf{V}^* \mathbf{O}_1') d\mathbf{V}^* d\mathbf{O}_1,
\]
we can first sample \( \mathbf{V}^* \) given \((\mathbf{O}_1, \mathbf{H}_1)\), from
\[
\pi(\mathbf{V}^* | \mathbf{O}_1, \beta, \theta, \eta) \propto \frac{1}{k!} \exp\left\{ -\frac{1}{2} \mathbf{V}^*^{-1} \mathbf{H}_1 \mathbf{O}_1 \right\}
\]
where \( c_i \) is the \((i,i)\) element of \( \mathbf{O}_1' \mathbf{H}_1 \mathbf{O}_1 / 2 \). Thus, we can directly sample \( \mathbf{v}_i \) independently from inverse gamma \((r_i - 1, c_i)\) distributions, and then simply rearrange the \( \mathbf{v}_i \) so that \( \mathbf{v}_1 \geq \cdots \geq \mathbf{v}_k \).

Gibbs sampling of \( \mathbf{O}_1 \): from (19), the conditional density of \( \mathbf{O}_1 \) given \( \mathbf{V}^* \) and \( \mathbf{H}_1 \) is
\[
\pi(\mathbf{O}_1 | \mathbf{V}^*; \beta, \theta) \propto \exp\left\{ -\frac{1}{2} \mathbf{V}^* \mathbf{H}_1 \mathbf{O}_1 \right\}
\]

Write \( \mathbf{H}_1 = \mathbf{LUL}' \), where \( \mathbf{L} \) is the matrix of normalized eigenvectors \((\mathbf{L}_1' = \mathbf{I}_k)\) and \( \mathbf{U} = \text{diag}(u_1, \ldots, u_k) \) is the diagonal matrix of corresponding eigenvalues with \( u_1 \geq \cdots \geq u_k \). Denote \( \Gamma = \mathbf{L}_1' \mathbf{O}_1 \). Then, the conditional density of \( \Gamma \) given \( \mathbf{V}^* \) and \( \mathbf{H}_1 \) is
\[
\pi(\Gamma | \mathbf{V}^*; \beta, \theta) \propto \exp\left\{ -\frac{1}{2} \mathbf{U} \mathbf{V}^* \mathbf{U}' \right\}
\]

Hoff [31] introduced an MCMC algorithm to sample from the posterior of \( \mathbf{O}_1 \) by updating two randomly selected columns of \( \mathbf{O}_1 \). Alternatively, Berger et al. [35] suggested updating two randomly selected rows of \( \Gamma \) (essentially equivalent to Hoff’s method in full rank cases, but considerably faster in situations of less than full rank). For example, to update the first two rows of \( \Gamma \), we write \( \Gamma = \text{diag}(\mathbf{O}_1, \mathbf{H}_1) \), \( (\Gamma_{12}', \Gamma_{12})' \), where \( \Gamma_{12} \) is the first 2 rows of \( \Gamma \), \( \Gamma_{12} \) is the rest \( k - 2 \) rows of \( \Gamma \), and
\[
\mathbf{O} = \mathbf{D}_c \mathbf{O}_0 = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix}
\]

Here, \( \omega \in (-\pi/2, \pi/2) \) and \( e_i = \pm 1 \) for \( i = 1, 2 \). Write \( \mathbf{U}_1 = \text{diag}(u_1, \ldots, u_k) \) and \( \mathbf{U}_2 = \text{diag}(u_1, \ldots, u_k) \). Then, the conditional posterior of \( \omega, \pi(\omega | \Gamma_{12}, \Gamma_{12}, \mathbf{V}^*; \beta, \theta) \) is
\[
\exp\left\{ -\frac{1}{2} \left( \begin{array}{cc} \mathbf{U}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_2 \end{array} \right) \left( \begin{array}{c} \mathbf{O} \\ \mathbf{0} \end{array} \right) \right\} \left( \begin{array}{c} \Gamma_{12} \\ \Gamma_{12} \end{array} \right) \left( \begin{array}{c} \mathbf{O}' \mathbf{V}^* \mathbf{U}' \\ \mathbf{0} \end{array} \right) \left( \begin{array}{c} \mathbf{0} \\ \mathbf{I}_{k-2} \end{array} \right) \end{bmatrix}
\]

Write
\[
\Gamma_{12} \mathbf{V}^* \mathbf{U}' \left( \begin{array}{c} \cos \phi - \sin \phi \\ \sin \phi \cos \phi \end{array} \right) \left( \begin{array}{c} \mathbf{0} \\ \mathbf{s} \end{array} \right) \left( \begin{array}{c} \cos \phi \sin \phi \\ -\sin \phi \cos \phi \end{array} \right),
\]
where \( \phi \in (-\pi/2, \pi/2) \) and \( s_j \geq 2 \). From (59), the conditional posterior of \( \theta \) is
\[
\pi(\theta | \Gamma_{12}, \mathbf{V}^*; \beta, \theta) \propto \exp\left\{ c_\theta \cos^2 (\omega + \phi) \right\},
\]
where \( c_\theta = -(1/2) (s_1 - s_2) (u_1 - u_2) \leq 0 \). Let \( \alpha = \cos^2 (\omega + \phi) \). Then, the full conditional density of \( \alpha \) is
\[
\pi(\alpha | \Gamma_{12}, \mathbf{V}^*; \beta, \theta) \propto \exp\left\{ \frac{\alpha}{2} \right\},
\]
where \( \alpha \in [0, 1] \).

Sampling \( \alpha \in [0, 1] \) can be done with a rejection sampler with proposal Beta(1/2, 1/2) distribution.
From the studies in Berger et al. [35], the new method substantially outperforms the Metropolis and Hit-and-Run.
Similarly, resulting from the 6 priors in every combination of the dimensions.

From the mean square error (MSE) perspective, we compare Simulation algorithms in moderate dimensions and succeeds for $k$ up to 100, whereas the other methods break down in much lower dimensions.

### 5. Simulation

From the mean square error (MSE) perspective, we compare the performance of 6 objective hyperpriors (considered in Berger et al. [12]), created from the product of three objective hyperpriors for $\xi$:

1. **Constant prior**: $\pi_1(\xi) = 1$
2. **Conjugate prior**: $\pi_2(\xi) \sim N_k(0, I_q)$
3. **Recommended prior**: $\pi_3(\xi) \propto [1 + \|\xi\|^2]^{-(q+1)/2}$ and two objective hyperpriors for $V$
   - **Constant prior**: $\pi(V) = 1, \pi(W) = 1$ and $\pi(\Sigma) = 1$
   - **Recommended reference prior**:
     
     \[
     \pi(V) \propto \frac{1}{V^{1-1/(2k)} \prod_{1 \leq i < j \leq k} (v_i - v_j)} V > 0, \\
     \pi(W) \propto \frac{1}{W^{1-1/(2p)} \prod_{1 \leq i < j \leq p} (w_i - w_j)} W > 0, \tag{63}
     \]
     
     \[
     \pi(\Sigma) \propto \frac{1}{\Sigma^{1-1/(2q)} \prod_{1 \leq i < j \leq q} (\sigma_i - \sigma_j)} \Sigma > 0.
     \]

Except for the constant and recommended reference prior, Berger et al. [12] also studied the nonhierarchical independence Jeffreys prior ($\pi(V) = |V|^{-(k+1)/2}$) and the hierarchical independence Jeffreys prior ($\pi(V) = |I + V|^{-(k+1)/2}$) in the model (1). From Berger et al. [12], the nonhierarchical independence Jeffreys prior for the covariance matrix cannot be used for the hierarchical models since it results in improper posterior. For the hierarchical independence Jeffreys prior, the common sampling methods are very inefficient to sample $V$ from the posterior distribution. Therefore, we do not consider either of the above-mentioned two Jeffreys priors.

Set $q = n = 2, p = s = 4, m = k = 16$. We generate $Z = (z_1, \ldots, z_{mk})$, where $z_i \overset{\text{iid}}{\sim} N_{sp}(0, I_q)$ for $i = 1, \ldots, mk$. Similarly, $T = (t_1, \ldots, t_{sp})$, where $t_j \overset{\text{iid}}{\sim} N_{mq}(0, I_q)$ for $j = 1, \ldots, sp$. We simulate the Bayes risks of the posterior means resulting from the 6 priors in every combination of the following cases:

1. $\bar{x}_1 = 1_q$ or $\bar{x}_2 = 301_q$:

| Prior for $(V, W, \Sigma)$ | Prior for $\xi$ | $(\xi_1, V_1)$ | $(\xi_1, V_2)$ | $(\xi_2, V_1)$ | $(\xi_2, V_2)$ |
|---------------------------|----------------|----------------|----------------|----------------|----------------|
| Constant prior            | $\pi_1(\xi)$  | 234.280        | 301.450        | 244.154        | 312.565        |
|                           | $\pi_2(\xi)$  | 219.234        | 284.716        | 253.791        | 309.367        |
|                           | $\pi_3(\xi)$  | 214.406        | 267.673        | 223.430        | 275.486        |
| Reference prior           | $\pi_1(\xi)$  | 208.342        | 280.193        | 229.257        | 286.841        |
|                           | $\pi_2(\xi)$  | 201.971        | 264.365        | 243.239        | 294.634        |
|                           | $\pi_3(\xi)$  | 197.123        | 255.153        | 208.372        | 265.474        |

Each choice of $(\xi, V, W, \Sigma)$ specifies a “true” hierarchical model for the simulation and we wish to compute the Bayes risk corresponding to the posterior means for $\theta$ that arise from each of the 6 objective priors. To simulate these risks, we generate 2000 random $y_l \sim N_{mk}(I_{mk} \otimes V)$ and generate observations $y_l | \theta_l \sim N_{mk}(I_{mk} \otimes V), l = 1, \ldots, 2000$. To obtain the posterior means, $\delta^*(y_l)$, of $\theta_l$, under the 6 priors, we run 100,000 MCMC cycles after 3,000 burn-in cycles, using the algorithms described above. Finally, we approximate the Bayes risk as the average observed mean squared error (MSE) as follows:

\[
\text{MSE} = \frac{1}{2000} \sum_{l=1}^{2000} (\theta_l - \delta^*(y_l))^T (\theta_l - \delta^*(y_l)). \tag{64}
\]

The results are given in Table 1. The recommended prior (in the last line) dominates all the others in terms of risk. Within each three-row segment, comparison of the first row with the third row is a comparison between using the constant prior for $\xi$ and the recommended prior. The gains with the recommended prior are large. Comparison of each of the first three rows with each of the last three rows is a comparison between using the constant prior for $V$ versus the recommended prior. The gains with the recommended prior are also large. The risk results present the strong numerical evidence of the superiority of the recommended priors (5).

### 6. Conclusion

In this paper, we follow Berger et al.’s [12] work and study a 4-level normal hierarchical model. We demonstrate that the posterior using the recommended prior is still proper in the 4-level normal hierarchical model. Herein, we do not demonstrate Berger et al.’s [32] conjecture, i.e., the posterior using the recommended prior is always proper in any level normal hierarchical models. But our method provides one useful guideline for the completion of the story.

Besides, the normal hierarchical models are typically handled by Gibbs sampling. One difficulty of computation is to sample the covariance matrix efficiently. The common sampling methods for covariance matrices, for example, Metropolis–Hastings algorithm [36] and Hit-and-Run method [25], are inefficient for higher dimensions. To
overcome the computational bottleneck of sampling covariance matrices, we can use a powerful and efficient method proposed by Berger et al. [35] for sampling the conditional density of covariance matrices. Therefore, there is no difficulty in computation for the 4-level normal hierarchical model with the recommended priors (5). In addition, the simulation result presents the strong numerical evidence of the superiority of the recommended priors (5).

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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