Modeling Supply Chains and Business Cycles as Unstable Transport Phenomena

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Physical concepts developed to describe instabilities in traffic flows can be generalized in a way that allows one to understand the well-known instability of supply chains (the so-called “bullwhip effect”). That is, small variations in the consumption rate can cause large variations in the production rate of companies generating the requested product. Interestingly, the resulting oscillations have characteristic frequencies which are considerably lower than the variations in the consumption rate. This suggests that instabilities of supply chains may be the reason for the existence of business cycles. At the same time, we establish some link to queuing theory and between micro- and macroeconomics.

Concepts from statistical physics and non-linear dynamics have been very successful in discovering and explaining dynamical phenomena in traffic flows [1]. Many of these phenomena are based on mechanisms such as delayed adaptation to changing conditions and the competition for limited resources, which are relevant for other systems as well. This includes pattern formation such as segregation in driven granular media [2] and lane formation in colloid physics [3] or biological physics (pedestrians, ants) [4]. Another example are clogging phenomena at bottlenecks in freeway traffic [5], panicking pedestrian crowds [6], or granular media [7]. In the following study, we will focus on the phenomenon of stop-and-go traffic [8] and its analogies.

Recently, economists and traffic scientists have wondered, whether traffic dynamics has also implications for the stability and management of supply chains [9,10] or for the dynamics of business cycles [12]. To explain business cycles, many theoretical concepts have been suggested over the decades, such as the Schumpeter clock [11]. These are usually based on macroeconomic variables such as investment, income, consumption, public expenditure, or the employment rate, and their interactions. In contrast, Witt et al. [12] have recently suggested to interpret business cycles as self-organization phenomenon due to a linear instability of production dynamics related to stop-and-go waves in traffic or driven many-particle systems. In order to illustrate their idea, they have transferred a continuous macroscopic traffic model and re-interpreted the single terms and variables.

The author believes that this is a very promising approach to understand business cycles, but instead of simply transferring macroscopic traffic models, suggests to derive equations for business cycles from first principles, which means to derive the dynamics on the macroscopic level from microscopic interactions. This would also make some contribution to the goal of understanding macroeconomics based on microeconomics or, even more, based on the “elementary interactions” of individuals.

In order to make some progress in this direction, we will generalize some ideas suggested by Daganzo to describe the dynamics of supply chains [9]. Like the work by Armbruster et al. [10], his approach is related to traffic models as well, but he focusses on models in discrete space in order to reflect the discreteness of successive production steps. In order to be able to reflect the economics of a country, we will have to generalize these ideas to complex production and supply networks. This will be done in Sec. I. Section II will, then, focus on business cycles in an sectorally structured economy. This manuscript can, of course, only be a first step into the direction of simulating complex production processes or the whole economy of a country. Some further research directions are indicated in the outlook of Sec. III.

I. MODELLING SUPPLY NETWORKS

The production units: We will investigate a system with \( u \) production units (e.g. machines or factories) \( a, b, c \in \{1,2,\ldots, u\} \) producing \( p \) products \( i, j, k \in \{1,2,\ldots, p\} \). The respective production process is characterized by parameters \( n^i_b \) and \( f^i_b \); In each production step, production unit \( b \) requires \( n^i_b \) products (educts) \( j \in \{1,\ldots, p\} \) and produces \( f^i_b \) products \( i \in \{1,\ldots, p\} \). The number of production steps of production unit \( b \) per unit time shall be \( P_b(t) \). It is the product of the processing (departure) rate \( \mu_b \) and the probability \( p_b \) of the production unit being occupied:

\[
P_b(t) = \mu_b(t)p_b(\rho_b, c_b, s_b) = \rho_b = \lambda_b/\mu_b.
\]

Herein,

\[
\rho_b = \lambda_b/\mu_b
\]

denotes the channel utilization, \( \lambda_b \) the feeding rate, \( c_b \) the number of parallel channels of the production unit, and \( s_b \) the storage capacity. Changing the number \( c_b \) of channels or the storage capacity \( s_b \) is costly. For the time being, we will therefore assume \( c_b \) to be constant. The expression for \( p_b(\rho_b, c_b, s_b) \) is well-known for the stationary state of many queuing systems [13]. For example, for a \( M/M/1 : (\infty/FIFO) \) process (i.e. one channel with first-in-first-out serving, unlimited storage capacity, Poisson-distributed arrival times and exponentially distributed service intervals), one finds

\[
p_b(\rho_b, 1, \infty) = \min(\rho_b, 1),
\]

so that, in this particular case, we have the relation
\(P_b(t) = \mu_b(t) \min[\rho_b(t), 1] = \min[\lambda_b(t), \mu_b(t)].\) \hspace{1cm} (4)

**Feeding rates:** The feeding rate \(\lambda_b\) is not just the product of the concentrations of educts required for a product, as that would be the case for chemical reactions. Instead, the feeding rate is determined by the minimum arrival rate \(A_{ab}^i\) of required educts \(j\), divided by the number \(n_{ij}\) of educts required for one production step. (The difference is comparable to the Probabilistic AND and the Fuzzy AND in Fuzzy Logic.) As the arrival rates \(A_{ab}^i\) are given by the input buffer flows \(v_b^i I_b^i\) (where \(v_b^i\) is the rate of getting educt \(j\) from the input buffer into the production unit \(b\) and \(I_b^i\) is the number of educts stored in the input buffer), the resulting feeding rate is

\[
\lambda_b(t) = \min_i \left( \frac{v_b^i(t) I_b^i(t)}{n_b^i} \right). \hspace{1cm} (5)
\]

Note that, due to stochastic variations (which can cause queuing effects), effective production is related to \(\lambda_b \leq r\mu_b\) with \(r > 0\) \([13]\). Therefore, it would be reasonable to use transport rates \(v_b^i\) according to \(v_b^i = r\mu_b n_b^i / I_b^i\). However, there are capacity constraints \(V_b^j\) in the transport of product \(j\) in production unit \(b\), which may be reflected by a function \(U(x, V)\) such as

\[
U(x, V) = \frac{x}{1 + x/V} \approx \begin{cases} x & \text{for } x \ll V \\ V & \text{for } x \gg V. \end{cases} \hspace{1cm} (6)
\]

Therefore, we will make the specification

\[
\nu_b^i(t) = U \left( \frac{r\mu_b(t)n_b^i}{I_b^i(t)}, V_b^j(t) \right). \hspace{1cm} (7)
\]

**Input and output buffers:** Let us assume, each production unit \(b\) has input buffers for required educts \(i\) and output buffers (a warehouse) for the products. We will assume the input buffers are filled with \(I_b^i(t)\) educts \(i \in \{1, \ldots, p\}\) and the output buffers with \(B_b^j(t)\) products waiting to be delivered. If \(D_{ab}^i(t)\) denotes the delivery flow of products \(i\) from production unit \(a\) to \(b\), the change of an input buffer stock with time is given by the conservation equation

\[
\frac{dI_b^i}{dt} = \sum_a D_{ab}^i(t) - n_b^i P_b(t), \hspace{1cm} (8)
\]

as \(\sum_a D_{ab}^i(t)\) is the number of products \(i\) delivered from various sources (production units) \(a\) and \(n_b^i P_b(t)\) is the number of educts \(i\) used up for production per unit time. Analogously, the dynamics of an output buffer stock is determined by the equation

\[
\frac{dB_b^j}{dt} = f_b^j P_b(t) - \sum_c D_{bc}^j(t), \hspace{1cm} (9)
\]

as \(f_b^j P_b\) is the number of newly generated products \(j\), and \(\sum_c D_{bc}^j(t)\) are deliveries to other production units \(c\). The delivery flow \(D_{ab}^i\) is given by the delivery rate \(\nu_{ab}^i\) times the available products \(B_b^j\) in the buffer:

\[
D_{ab}^i(t) = \nu_{ab}^i(t) B_b^j(t). \hspace{1cm} (10)
\]

The delivery flows are adapted to the order flows \(O_{ab}^i\). Ideally, one would have the relation \(D_{ab}^i = O_{ab}^i\), but due to capacity constraints \(V_{ab}^i\), we will again assume

\[
\nu_{ab}^i(t) = U \left( \frac{O_{ab}^i(t)}{B_b^j(t)}, V_{ab}^i(t) \right), \hspace{1cm} (11)
\]

which implies \(D_{ab}^i \leq O_{ab}^i\).

**Adaptation of capacities:** One important aspect of production is the adaptation of production capacities. This requires usually a lot of time compared to the other time scales of production. The adaptation time \(T_b\) of the processing rate \(\mu_b\) is much greater than the other adaptation times, so that we will assume the relation

\[
\frac{d\mu_b}{dt} = \frac{1}{T_b} \left[ \left( \sum_j f_b^j \right) B_b^j(t) \right] - \mu_b. \hspace{1cm} (12)
\]

Herein, \(W(y, \ldots)\) is some control function reflecting the strategy by the production management to adapt the processing rate \(\mu_b\), e.g. as a function of the output buffer stocks \(B_b^j\) or other variables. Normally, it will increase with decreasing output buffer stocks \(B_b^j\), but the function \(W(y, \ldots)\) saturates due to financial, spatial, or technological limitations and inefficiencies in the processing of high order flows. In this study, we will assume a function of the form

\[
W(1/z, \ldots) = A \frac{1 + Bz}{1 + Cz + Dz^2} \hspace{1cm} (13)
\]

with \(z = 1/y\) and suitable parameters \(A, B, C,\) and \(D\).

Of course, the transportation capacities \(V_b^j\) and \(V_{ab}^i\) are adapted as well. One may set up separate equations for this, but for simplicity, we will here assume the proportionality relations

\[
V_b^j(t) = \hat{V}_b^j \mu_b(t) \hspace{1cm} \text{and} \hspace{1cm} V_{ab}^i(t) = \hat{V}_{ab}^i \mu_b(t). \hspace{1cm} (14)
\]

**Order flows, delivery networks, and price dynamics:** The flow of orders is basically given by the flow \(P_b n_b^i\) of educts required for the production by unit \(b\). If the proportions \(q_{ab}^i\) with \(\sum_a q_{ab}^i = 1\) reflect orders from different producers, we have

\[
O_{ab}^i = q_{ab}^i P_b n_b^i. \hspace{1cm} (15)
\]

The fractions \(q_{ab}^i\) characterize the delivery or supply network. One can imagine various scenarios. For example, the fractions could be modelled by an evolutionary selection equation with a selection rate \(\nu_q\) \([14]\):
\[
\frac{dq_{ab}^i}{dt} = \nu_q \left( F_{ab}^i(t) - \sum_{a'} F_{a'\backslash b}^i(t) q_{a'\backslash b}^i(t) \right) q_{ab}^i(t). \tag{16}
\]

It makes sense to relate the fitness \( F_{ab}^i \) to the inverse price \( p_{ab}^i \) of product \( i \), which producer \( a \) takes from production unit \( b \):

\[
F_{ab}^i(t) = 1/p_{ab}^i(t). \tag{17}
\]

There are many pricing strategies, but the common element seems to be that the price changes according to the law of supply and demand. Therefore, one conceivable specification would be

\[
p_{ab}^i(t) = p_0^i \frac{O_{ab}^i(t)}{D_{ab}^i(t)}, \tag{18}
\]

where \( p_0^i \) is the price when supply \( D_{ab}^i \) and demand \( O_{ab}^i \) agree.

Other specifications are, of course, possible as well, as the above formulas partly depend on the strategies of the human decision makers involved.

**II. MARKETS AND BUSINESS CYCLES**

We will now introduce a simplified and aggregated description of an economy. For this, we will summarize a whole economic sector by one production unit \( b \) and replace input and output buffers by markets for the products \( i \) of the \( u \) different economic sectors. To describe this, we define the macroscopic stock level of products \( i \) in the market by

\[
N_i(t) = \sum_b I_b^i(t) + \sum_b B_b^i(t). \tag{19}
\]

According to the balance equations (8) and (9), we find

\[
\frac{dN_i}{dt} = \sum_b (f_b^i - n_b^i) p_b(t). \tag{20}
\]

Furthermore, in the arguments of the remaining mathematical relations, we have to replace input buffer stocks \( I_b^i \) and output buffer stocks \( B_b^i \) by the respective number \( N_i \) of products in the market, as the market replaces the buffers. In this way, we obtain the feeding rates

\[
\lambda_b(t) = \min_j \left( \frac{v_{b,j}^i(t) N_j(t)}{n_b^i} \right), \tag{21}
\]

which enter \( p_b \) according to equation (1), the transport rates

\[
v_{b,j}^i(t) = U \left( \frac{r \mu_b(t) n_b^i}{N_j(t)} , V_{b,j}^i \right), \tag{22}
\]

and the processing rates

\[
\frac{d\mu_b}{dt} = \frac{1}{T_b} \left[ W \left( \sum_j f_b^j N_j(t) \ldots \right) - \mu_b \right]. \tag{23}
\]

For \( P_b \approx \lambda_b \), one would have to solve this equation together with

\[
\frac{dN_i}{dt} = \sum_b (f_b^i - n_b^i) \mu_b(t) \min_j \left[ U \left( \frac{r n_b^i}{N_j(t)} , V_{b,j}^i \right) \frac{N_j(t)}{n_b^i} \right]. \tag{24}
\]

For a linear supply chain with \( n_b^i = \delta_{b,i+1} \) and \( f_b^i = \delta_{b,i} \) (where \( \delta_{k,l} = 1 \), if \( k = l \), otherwise 0), this leads to

\[
\frac{dN_i}{dt} = V_i(t) N_{i-1}(t) - V_{i+1}(t) N_i(t) \tag{25}
\]

with

\[
V_i(t) = \mu_i(t) U \left( \frac{r}{N_{i-1}(t)} , V_{i-1}^i \right). \tag{26}
\]

Interestingly, Eqs. (23) and (25) basically agree with the traffic flow model by Hilliges and Weidlich [15], where (25) is analogous to the equation for the vehicle density and (23) corresponds to the velocity equation. These equations are known to behave linearly unstable with respect to perturbations of \( N_i(t) \), if \( T_b \) exceeds a certain threshold which depends on the maximum slope \( \frac{dW}{dN} \) [15]. This is the reason why one can frequently observe an instability of supply chains, called the "bullwhip effect" (e.g. in the "beer distribution game" [16]). A thorough analysis shows that the Hilliges-Weidlich model would, in fact, be more suitable for the description of supply chains than of traffic: First, their velocity equation does not contain a convection term, as it should be the case for traffic. Second, their equations normally do not show metastability [8], i.e. a region in which traffic flow breaks down, when a perturbation exceeds a critical amplitude, but where smaller perturbations fade away. For this reason, a supply chain of the above kind is expected to behave either stable or linearly unstable, no matter how large the perturbation amplitude is. The most interesting point, however, is the reaction of the system to a periodic perturbation in the consumption rate \( V_{p+1}(t) \), as the resulting variations in the stock levels are synchronized and much slower (see Fig. 1). For this reason, they may explain business cycles as self-organized phenomenon.

**III. SUMMARY AND OUTLOOK**

In this contribution, a theory of supply networks has been sketched, which may open a new area of econophysics [17]. This theory is developed to help understand the dynamical phenomena, breakdowns, instabilities and inefficiencies of production processes and supply networks. Future work will have to address questions
such as the relevance of the network structure for the resulting dynamics, possible control strategies, the role of the market and pricing mechanism, etc. This specifically concerns the choice of the factor $r$, the specification of the functions $U$, $W$, and the equations suggested in the paragraph on delivery networks and price dynamics. Here, we have focussed on an application to markets, i.e. macroeconomics. For this, we have replaced input and output buffers by markets for different products [see Eq. (19)]. One may view this as some kind of micro-macro link, as we have started with equations for single firms (a microeconomic level), e.g. Eqs. (8), (9), and ended up with equations (20) for market sectors. Assuming a sectoral structure of economics, one can relate the resulting equations with the Hilliges-Weidlich model, which has originally been developed for traffic flow. These equations can describe the “bullwhip effect” due to their linear instability in a certain regime of operation. The underlying mechanism is the slow adaptation of the processing rate to changes in the order flows or stock levels in the market. Interestingly, the resulting oscillations in the stock levels of the different products $i$ have a characteristic frequency, which can be much lower than the underlying fast variations in the consumption rate. These oscillations synchronize among different economic sectors and may explain business cycles as a self-organized phenomenon. As the variables in the model are operational and measurable, the model can be tested and calibrated with empirical data. Apart from some equations which were not further applied in this study, most of the proposed model equations were conservation equations, equations given by the product flows, or relations derived with stochastic concepts used in queuing theory. They reflect the transport and interaction of products, so that the physics of driven many-particle systems and of complex systems can make some significant contributions to the new multidisciplinary field of self-organization phenomena in production and supply networks.

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![FIG. 1. Variations of the stock levels $N_i(t)$ of $u = p = 10$ economic sectors $i$, triggered by a small perturbation $\delta V_{11}(t) = 0.1 \sin(0.1t)$ of the constant consumption rate $V_{01}^i = 0.1 W(1/20)$. The model parameters are $A = 10$, $B = 0.01$, $C = 0.01$, $D = 0.02$, $\hat{V}_i^{-1} = 0.1$, and $T_i = 90$. The initial and boundary conditions are $N_i(0) = 20 = N_0(t)$, $V_i(0) = \hat{V}_i^{-1} W(1/N_i(0))$, and $V_{11}(t) = V_{11}^0 + \delta V_{11}(t)$.](image-url)