Hysteresis and multistability in networks of bistable stochastic elements with global interaction

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Abstract. We demonstrate the existence of the first-order phase transitions and hysteresis in a network of bistable stochastic elements with global interaction subject to additive white noise. Using the Fokker–Planck equation approach, we present a method which allows one to use a continuation technique (AUTO) to follow the stationary one-particle distribution density in the space of system parameters. In addition, the Gaussian approximation is employed to compute the loci of the bifurcation points.

Networks of interacting bistable stochastic elements are of great importance for studying critical phenomena [1, 2], phase transitions in spin systems [3], neural networks [4, 5] and decision-making processes in the groups of individuals [6].

The state of the $i$th element is characterized by a scalar state variable $x_i(t)$ which changes continuously in time and has two stable states given by two minima of an external bistable potential $U(x)$. In the simplest case, the coupling between the elements is realized through the mean field, i.e. through the ensemble average of the state variable $x$, $\langle x \rangle = (1/N) \sum_i x_i$, where $N$ denotes the total number of elements in the network [4, 5, 7, 8]. The global stationary state of such a network can be characterized by the one-particle distribution density $P_i(x)$ which is an even function in the symmetric phase and a non-symmetric function with one well-pronounced maximum in the non-symmetric phase. If $\langle x \rangle$ changes continuously (discontinuously) at the point of transition, the transition is known to be of the second (first) order. Phase transitions of the first and second orders in the presence of noise can be also found in networks of interacting phase oscillators. A prominent example of the latter is the noisy Kuramoto model (for a review, see e.g. [9]). It should be noticed that the state variables $x_i$, in the Kuramoto model is confined to the interval $[0; \pi]$, whereas the state variable $x$ of a bistable element can take arbitrary (positive or negative) values.
The case of mean-field coupling was studied in detail in [8, 10]. It was demonstrated that the order of the phase transition depends crucially on the stochastic force. According to [8], the first-order phase transition was observed for the multiplicative noise term only. Additive noise and mean-field coupling do not induce multistability.

Here, we show that additive white noise can lead to multistability and hysteresis in a network, where interaction between the elements is global and depends on the difference between their state variables. Such interaction is often called global pair interaction. We consider a network of \(N\) bistable elements interacting globally through a pair potential \(W(x_i - x_k)\). The force acting between the elements \(i\) and \(k\) is given by \(-\partial W(x_i - x_k)/\partial x_i\). Time evolution of the state variable \(x_i\) obeys the Langevin equation in the overdamped limit

\[
\dot{x}_i = -\frac{dU(x_i)}{dx} - \frac{1}{N} \sum_{k=1}^{N} \frac{\partial W(x_i - x_k)}{\partial x_i} + \sqrt{2D\xi_i(t)},
\]

(1)

where \(\xi_i(t)\) is a Gaussian white noise with the correlation function given by \(\langle \xi_i(t)\xi_k(t') \rangle = \delta_{ik}\delta(t-t')\). \(D\) is the strength of the noise term and \(U(x) = x^4/4 - x^2/2\) is a bistable potential. It should be stressed that the above model can also be used to describe systems of interacting particles in the overdamped limit, such as polymer and colloidal solutions [11].

The Fokker–Planck equation for the one-particle distribution function \(P_1(x, t)\)

\[
\frac{\partial P_1(x, t)}{\partial t} = \frac{\partial}{\partial x} \left[ P_1(x, t) \frac{dU(x)}{dx} + D \frac{\partial P_1(x, t)}{\partial x} + \int_{-\infty}^{\infty} P_2(x, x', t) \frac{\partial W(x - x')}{\partial x} dx' \right]
\]

(2)

is coupled to the two-particle distribution density \(P_2(x, x', t)\). When the number of particles \(N\) is large, the average distance between the particles is much smaller than the characteristic length of the interparticle interaction and also smaller than the distance between the two wells of the potential energy \(U(x)\). As it was demonstrated in [12, 13], in this limit \((N \to \infty)\) one can use the molecular chaos approximation and set \(P_2(x, x', t) = P_1(x, t)P_1(x', t)\) which leads to a closed equation for \(P_1(x, t)\). In the density functional theory, one refers to it as a random phase approximation [14, 15]. Equation (2) must be supplemented by natural boundary conditions for \(P_1(x, t)\), i.e. \(\lim_{x \to \pm\infty} P_1(x, t) = 0\).

The presence of the non-local integral term in equation (2) makes it difficult to treat. Various approximations have been suggested to deal with the non-locality. In the approximation of strictly local coupling [16, 17] when any single particle experiences influence only from its nearest neighbours but not from the rest of the network, the force \(-\partial W(x-x')/\partial x\) which acts between two elements in the states \(x\) and \(x'\), becomes strictly zero if the difference \(|x-x'|\) exceeds a certain fixed value. In that case one can approximately reduce the integral in equation (2) to \(gP_1(x, t)\partial P_1(x, t)/\partial x\), with the coefficient \(g\) given by \(g = \int W(x-x') dx'\). The Fokker–Planck equation in the local approximation then reads

\[
\frac{\partial P_1(x, t)}{\partial t} = \frac{\partial}{\partial x} \left[ P_1(x, t) \frac{dU(x)}{dx} + D \frac{\partial P_1(x, t)}{\partial x} + gP_1(x, t) \frac{\partial P_1(x, t)}{\partial x} \right].
\]

(3)

This approximation has been used to study transport properties of rocking ratchets [16, 17].

It is worthwhile noticing that in the local approximation [16, 17], the stationary solution \(P_s^{\text{loc}}(x)\) of the Fokker–Planck equation, equation (3), with a symmetric potential \(U(x)\) is always an even function of \(x\) and, therefore, there can be no phase transitions in it. To show this,
we solve the stationary Fokker–Planck equation, equation (3), using zero probability current condition $J = 0$.

$$J = P_s^{\text{loc}} \frac{dU(x)}{dx} + D \frac{\partial P_s^{\text{loc}}}{\partial x} + g P_s^{\text{loc}} \frac{\partial P_s^{\text{loc}}}{\partial x} = 0. \quad (4)$$

From equation (4) $P_s^{\text{loc}}(x)$ is given by a solution of the transcendental equation

$$P_s^{\text{loc}}(x) \exp \left( \frac{g P_s^{\text{loc}}(x)}{D} \right) = C_0 \exp \left( -\frac{U(x)}{D} \right), \quad (5)$$

where $C_0$ is a normalization constant which ensures that $\int_{-\infty}^{\infty} P_s^{\text{loc}}(x) \, dx = 1$. Assume now that $U(x)$ is a symmetric function and $g$ is negative which corresponds to an attractive potential $W$. For any sufficiently large $x$, there are two solutions of equation (5). However, only one of them satisfies the natural boundary condition. This shows that the stationary density $P_s^{\text{loc}}(x)$ is also a symmetric function of $x$. Therefore, the phase transitions, i.e. the transitions from a symmetric $P_s^{\text{loc}}(x)$ to a non-symmetric $P_s^{\text{loc}}(x)$ are not captured by the local approximation used in [16, 17].

Here we show that global pair interaction does allow for the phase transitions in the network. Moreover, for a certain range of parameters, the phase transition can be of the first order, resulting in the coexistence of stable symmetric and stable non-symmetric stationary distributions $P_s(x)$. This gives rise to a hysteresis, as discussed in detail below.

Further we assume short-range interaction potential decreasing exponentially with its argument $W(x - x') = -\alpha \exp(-\lambda |x - x'|)$, where $\alpha$ is the strength of the interaction and $(1/\lambda)$ represents a characteristic ‘length’ of interaction. The sign in front of the interaction strength $\alpha$ was chosen in such a way that positive values of $\alpha$ correspond to an attractive potential.

Note that the case of the repulsive potential $W$ (negative $\alpha$) is rather trivial, as it does not lead to phase transitions. For negative $\alpha$ the stationary one-particle distribution density is broadened due to repulsion, remaining however, symmetric (not shown).

The stationary distribution density $P_s$ yields zero probability current $\partial J/\partial x = 0$, i.e.

$$0 = \frac{\partial}{\partial x} \left[ P_s(x) \frac{dU(x)}{dx} + \frac{\partial P_s(x)}{\partial x} + \alpha P_s(x) \int P_s(x') \frac{\partial W(x - x')}{\partial x'} \, dx' \right]. \quad (6)$$

Our aim is to study the bifurcations of the stationary distribution density $P_s$ in the parameter plane $(\alpha, \lambda)$. A direct way to find $P_s(x)$ at fixed values of $\lambda$ and $\alpha$ is to numerically solve the time-dependent equation (2) in the molecular chaos approximation. However, this method is very time-consuming as one has to wait until the initial distribution has relaxed to the stationary one. The relaxation time can be extremely large close to the bifurcation point rendering the method inappropriate. Moreover, it does not allow one to find unstable stationary solutions which play an important role in the observed phenomena.

Instead, we solve equation (6) using the continuation technique. Namely, any solution of equation (6) which is known numerically or analytically at a certain set of parameters can be followed (‘continued’) in the space of system parameters using a collocation method and subsequent evaluation of the solution using Newton’s iteration scheme. On each continuation step adaptive mesh is used to discretize the solution. This technique is implemented within the freely available software package AUTO [18], which we used here.

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In order to be able to use AUTO [18], we need to approximate the integral in equation (6) by a certain function of $x$. This can be achieved by observing that the above integral is a convolution of two functions $P_s$ and $\partial W/\partial x$. Both these functions are expanded in a Fourier series on the interval $[-L; L]$ using the first $M$ Fourier modes. With account of the fact that $\partial W/\partial x$ is an odd function, we obtain

$$P_s(x) \approx \sum_{n=0}^{M} \left[ A_n \cos (\omega_n x) + B_n \sin (\omega_n x) \right],$$

$$\frac{\partial W}{\partial x} \approx \sum_{k=1}^{M} \tilde{W}_n \sin(\omega_n x),$$

where $\omega_n = 2\pi n / L$. The length of the interval $L$ must be chosen sufficiently large to satisfy the natural boundary conditions, i.e. $P_s(\pm L) \approx 0$. Then we can use the convolution theorem for the Fourier transforms to reduce the integral in equation (6) to

$$I(x) = \int_{-L}^{L} P_s(x') \frac{\partial W(x - x')}{\partial x} \, dx'$$

$$= L \sum_{n=1}^{M} \tilde{W}_n \left[ A_n \sin (\omega_n x) - B_n \cos (\omega_n x) \right].$$

Equation (6) can now be treated as a boundary value problem with a non-homogeneous term $I(x)$ given by equation (8) and $(2M+1)$ integral conditions for the coefficients $A_0$, $A_n$, $B_n$, $n = 1, \ldots, M$

$$A_0 = \int_{-L}^{L} P_s(x) \, dx,$$

$$A_n = \frac{1}{L} \int_{-L}^{L} P_s(x) \cos (\omega_n x) \, dx,$$

$$B_n = \frac{1}{L} \int_{-L}^{L} P_s(x) \sin (\omega_n x) \, dx, \quad n = 1, \ldots, M.$$

To obtain the starting point for the continuation, we numerically solve the time-dependent equation (2) in the molecular chaos approximation with random initial conditions until the solution has approached a stationary distribution. To numerically solve equation (2), we use semi-implicit pseudo-spectral integration scheme as explained in detail in [19]. We take the first $M = 32$ modes into account and check the validity of the truncation by repeating the same calculations for the doubled number of modes $M = 64$.

Figure 1(a) shows the stationary average $\langle x \rangle = \int_{-\infty}^{\infty} x P_s(x) \, dx$ as a function of the coupling strength $\alpha$ for $D = 0.5$ and different $\lambda$, as indicated by a number near each line. The non-symmetric distribution corresponds to a non-vanishing $\langle x \rangle \neq 0$, as shown by dashed and dotted lines in figure 1(b). At fixed $\lambda$, the transition from the symmetric to the non-symmetric phase is achieved by changing the interaction strength $\alpha$. As we see, for $\lambda < 5.0$, the phase transition is of the second order, implying that the bifurcation is supercritical, i.e. that at the point of transition the order parameter $\langle x \rangle$ changes continuously. However, for larger values of $\lambda$, the bifurcation is subcritical, which results in the coexistence of stable symmetric phase $\langle x \rangle = 0$ and stable non-symmetric phase $\langle x \rangle \neq 0$ (see figure 1(b)). The unstable stationary solutions belong to the subcritical branch, i.e. to the branch between the PF point and the SN point.
Interestingly the critical coupling strength $\alpha_{PF}$ at the point of the (sub- or supercritical) PF bifurcation depends non-monotonically on $\lambda$. If $\lambda$ is large ($\lambda > 1$) and decreases, the coupling strength $\alpha_{PF}$ decreases until $\lambda$ becomes small ($\lambda < 1$). After this point the further decrease of $\lambda$ results in the increase of $\alpha_{PF}$, as shown by dashed lines in figure 1(a).

To understand hysteresis, we assume that $\lambda$ is kept fixed at $\lambda = 5.5$ and $\alpha$ is being gradually increased from $\alpha = 3.4$ to $\alpha = 3.8$. When passing the bifurcation point at around $\alpha_c = 3.65$ (this point is denoted by $A$ in figure 1(c)), the symmetric distribution becomes unstable. At this point the system undergoes a spontaneous transition to a non-symmetric phase, where the state variable $x$ of most of the elements is in one of the two stable states. Importantly, $\langle x \rangle$ directly at the transition point changes abruptly as the interaction strength $\alpha$ increases. Conversely, if $\alpha$ is gradually decreased in the same limits, the system remains in the non-symmetric phase until $\alpha$ reaches the point of the SN bifurcation, denoted by $B$ in figure 1(c). At this point $\langle x \rangle$ instantly drops to zero and the distribution function becomes symmetric. Such hysteretic behaviour is confirmed by numerical simulation of the Langevin equations, equation (1), with $N = 1000$ elements, as shown by symbols in figure 1(c). Notice that hysteretic behaviour can also be achieved by cooling or heating the system, i.e. by gradually changing the noise intensity $D$. 

**Figure 1.** (a) Stationary average $\langle x \rangle$ versus coupling strength $\alpha$ for different values of $\lambda$ and fixed noise intensity $D = 0.5$. Corresponding values of $\lambda$ are given by numbers near each curve. Boxes indicate the points where the pitchfork (PF) bifurcation occurs, circles correspond to the saddle-node (SN) bifurcation. (b) Stationary distribution densities $P_s(x)$ at $\alpha = 3.41$, $D = 0.5$ and $\lambda = 5.0$: solid line shows symmetric distribution $\langle x \rangle = 0$, dashed and dotted lines correspond to the non-symmetric distributions with $\langle x \rangle > 0$ and $\langle x \rangle < 0$, respectively. (c) First-order phase transition at $\alpha = 3.41$, $D = 0.5$ and $\lambda = 5.5$. Symbols represent the results of numerical simulation of the Langevin equations, equation (1). Filled triangles indicate unstable branches of stationary solutions.

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with time. This was demonstrated in \cite{20} for a network of bistable stochastic elements with mean-field coupling. Hysteresis occurring when the parameter is being slowly changed across the bifurcation point is known in bifurcation theory and is associated with dynamical bifurcations.

A slightly larger hysteresis region obtained from the Langevin equations, equation (1) as compared to equation (6) can be explained as follows. The symmetric state becomes only weakly unstable beyond the PF bifurcation point, and, if one sets initial conditions close to this state, the characteristic growth time of the instability appears to be larger than the observation time accessible in the simulations.

To study the phase transitions analytically, we employ the Gaussian approximation and assume that the time-dependent distribution density is given by

\[ P(x, t) = \frac{1}{\sqrt{2\pi\sigma(t)}} \exp\left[-\frac{(x - m(t))^2}{2\sigma(t)}\right], \]

where \( m \) and \( \sigma \) are the average and the variance of the state variable \( x \), respectively. Using this ansatz and the time-dependent Fokker–Planck equation equation (2), we derive the time-evolution equations for \( m \) and \( \sigma \):

\[
\dot{m} = m(1 - m^2 - 3\sigma) + \int P_s(x)P_s(x') \frac{\partial W(x - x')}{\partial x} \, dx \, dx',
\]

\[
\dot{\sigma} = 2\sigma(1 - 3\sigma) - 6m^2\sigma + 2D - 2\int xP_s(x)P_s(x') \frac{\partial W(x - x')}{\partial x} \, dx \, dx'.
\]

(10)

With \( W(x - x') = -\alpha \exp(-\lambda|x - x'|) \) and \( P(x, t) \) Gaussian, the integrals in equations (10) can be solved analytically to yield

\[
\dot{m} = m(1 - m^2 - 3\sigma),
\]

\[
\dot{\sigma} = 2\sigma(1 - 3\sigma) - 6m^2\sigma + 2D - 2\lambda\alpha \left[ \sqrt{\frac{\sigma}{\pi}} + \lambda\sigma \exp(\lambda^2\sigma) \left( \text{erf}(\lambda\sqrt{\sigma}) - 1 \right) \right].
\]

(11)

It should be emphasized that the bifurcation diagram obtained in the Gaussian approximation can be compared only on the qualitative level with the exact results. However, this approximation proved to be a useful and simple tool for studying phase transitions in networks of interacting elements, as has been demonstrated by many authors \cite{4, 5, 21–24}.

From figure 1(b), it is clear that the shape of the distribution density \( P_s(x) \) in the non-symmetric phase is closer to Gaussian than it is in the symmetric phase. Therefore, we expect that the Gaussian approximation predicts more accurately the point of the SN bifurcation than that of the PF bifurcation as the former occurs at finite \( m \neq 0 \).

In figure 2(a), the locus of the SN as well as the PF bifurcations is shown on the parameter plane \((\alpha, \lambda)\) at fixed noise intensity \( D = 0.5 \). Figure 2(b) demonstrates how the position of the SN line changes with the noise intensity \( D \), which is given by a number near each line. With increasing \( D \), the critical coupling strength at the point of the SN bifurcation \( \alpha_{\text{SN}} \) grows, showing that it becomes harder for the system to undergo the phase transition.

We would like to emphasize that the case of an exponential interaction potential considered here represents an intermediate situation between two limiting cases. The case of the mean-field coupling is formally obtained by setting the interaction length \( 1/\lambda \) to infinity \( (\lambda = 0) \), whereas the local approximation \cite{16, 17} is recovered for large values of \( \lambda \).
Figure 2. (a) SN and PF bifurcations of the steady state in the Gaussian approximation equations (11) at fixed noise intensity \( D = 0.5 \). (b) Several SN lines at different values of \( D \), as indicated by a number near each line.

From figure 2(a) we can see that as \( \lambda \) decreases (limit of the mean-field coupling), the two bifurcation lines (SN) and (PF) collide which implies that multistability disappears and the bifurcation becomes supercritical. This is in agreement with the previous findings for the mean-field coupling [4, 5, 8].

The opposite case of large \( \lambda \) is more intriguing. When \( D \) is fixed and \( \lambda \) is being increased, the critical coupling strength of the SN bifurcation \( \alpha_{SN} \) remains finite. This shows that for a very short-range interaction (large but finite \( \lambda \)), the phase transition is still possible, and it is of the first order. In other words, if a sufficiently large (macroscopic) number of elements in the network will be forced into one of the two stable states, the rest of the elements will follow this trend, resulting in a phase transition to a new stable non-symmetric phase.

However, if interaction is localized to the extent that it completely vanishes at finite distances \( (x - x') \) and is thus not described by any finite \( \lambda \) however large, as in [16, 17], no phase transition occurs at all. The transition between intermediate global and strictly local coupling can be mathematically associated with the hysteresis in figure 1(a) becoming more pronounced until the inner and outer SN lines merge and disappear.

To conclude, we have shown that a non-local pair interaction between bistable elements driven by Gaussian white noise enables both first- and the second-order phase transitions in the network. At the point of the first-order transition, the symmetric one-particle distribution density becomes unstable leading to a spontaneous synchronous transition of the majority of the elements to a state with nonzero average \( \langle x \rangle \). Multistability is observed for small but finite values of the interaction length \( 1/\lambda \). For the values of the interaction strength \( \alpha \) between \( \alpha_{SN} \) and \( \alpha_{PF} \), the stable symmetric phase coexists with a stable non-symmetric phase, which gives rise to a hysteresis.

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References

[1] Hohenberg P C and Halperin B I 1977 Theory of dynamic critical phenomena Rev. Mod. Phys. 49 435
[2] Dawson D 1983 Critical dynamics and fluctuations for a mean-field model of cooperative behavior J. Stat. Phys. 29 31
[3] Jung P, Behn U, Pantazelou E and Moss F 1992 Collective response in globally coupled bistable systems Phys. Rev. A 46 R1709
[4] Huber D and Tsimring L S 2003 Dynamics of an ensemble of noisy bistable elements with global time delayed coupling Phys. Rev. Lett. 91 260601
[5] Huber D and Tsimring L S 2005 Cooperative dynamics in a network of stochastic elements with delayed feedback Phys. Rev. E 71 036150
[6] Zanette D H 1997 Dynamics of globally coupled bistable elements Phys. Rev. E 55 5315
[7] Garcia-Ojalvo J and Sancho J M 1999 Noise in Spatially Extended Systems (New York: Springer)
[8] Buceta J and Lindenberg K 2004 Comprehensive study of phase transitions in relaxational systems with field-dependent coefficients Phys. Rev. E 69 011102
[9] Acebron J A, Bonilla L L, Vicente C J P, Ritort F and Spigler R 2005 The Kuramoto model: a simple paradigm for synchronization phenomena Rev. Mod. Phys. 77 137
[10] Desai R C and Zwanzig R 1978 Statistical mechanics of a nonlinear stochastic model J. Stat. Phys. 19 1
[11] Likos C N 2001 Effective interactions in soft condensed matter physics Phys. Rep. 348 267
[12] Savel’ev S, Marchesoni F and Nori F 2004 Stochastic transport of interacting particles in periodically driven ratchets Phys. Rev. E 70 061107
[13] Savel’ev S and Nori F 2005 Controlling the motion of interacting particles: homogeneous systems and binary mixtures Chaos 15 026112
[14] Hansen J P and McDonald I R 2006 Theory of Simple Liquids (London: Academic)
[15] Archer A J and Evans R 2004 Dynamical density functional theory and its application to spinodal decomposition J. Chem. Phys. 121 4246–54
[16] Savel’ev S, Marchesoni F and Nori F 2004 Controlling transport in mixtures of interacting particles using Brownian motors Phys. Rev. Lett. 91 010601
[17] Savel’ev S, Marchesoni F and Nori F 2005 Interacting particles on a rocked ratchet: rectification by condensation Phys. Rev. E 71 011107
[18] Doedel E J, Paffenroth R C, Champneys A R, Fairgrieve T F, Kuznetsov Yu A, Sandstede B and Wang X 2001 Auto 2000: continuation and bifurcation software for ordinary differential equations (with homcont) Technical Report, Caltech Available online at http://cmvl.cs.concordia.ca/auto/
[19] Bestehorn M, Pototsky A and Thiele U 2003 3D large scale Marangoni convection in liquid films Eur. Phys. J. B 33 457–67
[20] Casado J M and Morillo M 1990 Phase transitions in a nonlinear stochastic model: a numerical simulation study Phys. Rev. A 42 1875
[21] Zaks M A, Sailer X, Schimansky-Geier L and Neiman A B 2005 Noise induced complexity: from subthreshold oscillations to spiking in coupled excitable systems Chaos 15 026117
[22] Pikovsky A, Zaikin A and de la Casa M A 2002 System size resonance in coupled noisy systems and in the Ising model Phys. Rev. Lett. 88 050601
[23] Drozdov A N and Morillo M 1996 Solution of nonlinear Fokker–Planck equations Phys. Rev. E 54 931
[24] Drozdov A N and Morillo M 1996 Validity of basic concepts in nonlinear cooperative Fokker–Planck models Phys. Rev. E 54 3304