Universal upper bound for the Holevo information induced by a quantum operation

by

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Abstract

Let \(H_A \otimes H_B\) be a bipartite system and \(\rho_{AB}\) a quantum state on \(H_A \otimes H_B\). \(\rho_A = \text{Tr}_B(\rho_{AB})\), \(\rho_B = \text{Tr}_A(\rho_{AB})\). Then each quantum operation \(\Phi_B\) on quantum system \(H_B\) can induce a quantum ensemble \(\{ (p_\mu, \rho_{A,\mu}) \}\) on quantum system \(H_A\). In this paper, we show that the Holevo quantity \(\chi(\{ (p_\mu, \rho_{A,\mu}) \})\) of the quantum ensemble \(\{ (p_\mu, \rho_{A,\mu}) \}\) can be upper bounded by \(S(\rho_B)\). By using the result, we answer partly a conjecture of Fannes, de Melo, Roga and Życzkowski.

Keywords: Quantum state, Quantum operation, von Neumann entropy, Holevo quantity.

1 Introduction and preliminaries

Let \(\mathcal{H}\) be a finite dimensional complex Hilbert space. A quantum state \(\rho\) on \(\mathcal{H}\) is a positive semi-definite operator of trace one, in particular, for each unit vector \(|\psi\rangle \in \mathcal{H}\), the
operator $\rho = |\psi\rangle\langle\psi|$ is said to be a pure state. The set of all quantum states on $\mathcal{H}$ is denoted by $D(\mathcal{H})$. For each quantum state $\rho \in D(\mathcal{H})$, its von Neumann entropy is defined by $S(\rho) = - \text{Tr}(\rho \log_2 \rho)$. A quantum operation $\Phi$ on $\mathcal{H}$ is a trace-preserving completely positive linear mapping defined over the set $D(\mathcal{H})$. It follows from ([?], Prop. 5.2 and Cor. 5.5) that there exist linear operators $\{M_\mu\}_{\mu=1}^K$ on $\mathcal{H}$ such that $\sum_{\mu} M_\mu^\dagger M_\mu = \mathbb{1}$ and $\Phi = \sum_\mu \text{Ad}_{M_\mu}$, that is, for each quantum state $\rho$, we have the Kraus representation

$$\Phi(\rho) = \sum_{\mu=1}^K M_\mu \rho M_\mu^\dagger.$$ 

Let $E = \{(p_\mu, \rho_\mu)\}$ be a quantum ensemble on $\mathcal{H}$, that is, each $\rho_\mu \in D(\mathcal{H})$, $p_\mu > 0$, and $\sum_\mu p_\mu = 1$. The Holevo quantity of the quantum ensemble $\{(p_\mu, \rho_\mu)\}$ is defined by the following expression:

$$\chi\{(p_\mu, \rho_\mu)\} = S(\sum_\mu p_\mu \rho_\mu) - \sum_\mu p_\mu S(\rho_\mu). \quad (1.1)$$

Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite system and $\rho_{AB}$ a quantum state on $\mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_A = \text{Tr}_B(\rho_{AB})$, $\rho_B = \text{Tr}_A(\rho_{AB})$, $\Phi_B = \sum_\mu \text{Ad}_{M_{B,\mu}}$ be a quantum operation on quantum system $\mathcal{H}_B$. Then $\Phi = \sum_\mu \text{Ad}_{1_A \otimes M_{B,\mu}}$ is a quantum operation on the bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$.

Let

$$p_\mu = \text{Tr}\left( (\mathbb{1}_A \otimes M_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes M_{B,\mu}^\dagger) \right).$$

Then $p_\mu \geq 0$ and $\sum_\mu p_\mu = 1$. Without loss of generality, we assume that $p_\mu > 0$. Let

$$\rho_{A,\mu} = p_\mu^{-1} \text{Tr}_B\left( (\mathbb{1}_A \otimes M_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes M_{B,\mu}^\dagger) \right) = p_\mu^{-1} \text{Tr}_B\left( (\mathbb{1}_A \otimes \sqrt{M_{B,\mu}^\dagger M_{B,\mu}}) \rho_{AB} (\mathbb{1}_A \otimes \sqrt{M_{B,\mu}^\dagger M_{B,\mu}}) \right).$$

Then $\rho_{A,\mu}$ is a quantum state on $\mathcal{H}_A$. Thus, quantum operation $\Phi_B$ induced a quantum ensemble $\{(p_\mu, \rho_{A,\mu})\}$ on quantum system $\mathcal{H}_A$.

In this paper, the following result is obtained:

**Theorem 1.1.** $\chi\{(p_\mu, \rho_{A,\mu})\} \leq \min\{S(\rho_A), S(\rho_B)\}$.

By using this result, we answer partly a conjecture of Fannes, de Melo, Roga and Życzkowski.
2 The proof of Theorem ??

Clearly, \( \chi\{ (p_\mu, \rho_{A,\mu}) \} \leq S(\rho_A) \) is trivial by the definition of the Holevo information. It remains to prove \( \chi\{ (p_\mu, \rho_{A,\mu}) \} \leq S(\rho_B) \). The nontrivial part of the proof is divided into three parts as follows:

(i) If \( \{|\psi_{B,\mu}\rangle\}_{\mu=1}^K \) is a standard orthonormal basis of \( \mathcal{H}_B \) and \( M_\mu = |\psi_{B,\mu}\rangle\langle \psi_{B,\mu}| \), then it follows, from Theorem 3.1 in [? and its proof, that \( \chi\{ (p_\mu, \rho_{A,\mu}) \} \leq S(\rho_B) \).

(ii) If \( M_{B,\mu} = P_{B,\mu} \), where \( P_{B,\mu} \) is a projector on \( \mathcal{H}_B \). Note that \( \sum_\mu P_{B,\mu} = \mathbb{1}_B \), so there is a standard orthonormal basis \( \{ |u_{\mu,i}\rangle \} \) of \( \mathcal{H}_B \) such that

\[
P_{B,\mu} = \sum_i |u_{\mu,i}\rangle\langle u_{\mu,i}|
\]

for each \( \mu \).

Denote \( p_{\mu,i} = \langle u_{\mu,i} | \rho_B | u_{\mu,i} \rangle \), without loss of generality, we assume that \( p_{\mu,i} > 0 \), and denote

\[
\rho_{A,\mu,i} = p_{\mu,i}^{-1} \langle u_{\mu,i} | \rho_{AB} | u_{\mu,i} \rangle.
\]

Thus

\[
p_{\mu} = \text{Tr} \left( (\mathbb{1}_A \otimes P_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes P_{B,\mu}) \right) = \text{Tr} \left( (\mathbb{1}_A \otimes P_{B,\mu}) \rho_{AB} \right) = \sum_i \text{Tr} \left( \langle u_{\mu,i} | \rho_{AB} | u_{\mu,i} \rangle \right) = \sum_i p_{\mu,i}
\]

\[ (1.21) \]

and

\[
(\mathbb{1}_A \otimes P_{B,\mu}) \rho_{AB} (\mathbb{1}_A \otimes P_{B,\mu}) = \sum_{i,i'} \langle u_{\mu,i} | \rho_{AB} | u_{\mu,i'} \rangle \otimes |u_{\mu,i} \rangle\langle u_{\mu,i'}|.
\]

\[ (2.2) \]

It follows from Eq. (1.21) and Eq. (2.2) that

\[
p_{\mu} \rho_{A,\mu} = \sum_i p_{\mu,i} \rho_{A,\mu,i}.
\]

Therefore, by the concavity of von Neumann entropy, we have

\[
p_{\mu} \mathcal{S}(\rho_{A,\mu}) \geq \sum_i p_{\mu,i} \mathcal{S}(\rho_{A,\mu,i}).
\]

So,

\[
\sum_{\mu} p_{\mu} \mathcal{S}(\rho_{A,\mu}) \geq \sum_{\mu} \sum_i p_{\mu,i} \mathcal{S}(\rho_{A,\mu,i}).
\]

Thus, the desired inequality is obtained.
(iii) Now we prove the theorem generally. By the Naimark theorem [?], there exists a quantum system $\mathcal{H}_C$, a unit vector $|0_C\rangle \in \mathcal{H}_C$ and a projector $\{P_{BC,\mu}\}$ on the bipartite system $\mathcal{H}_B \otimes \mathcal{H}_C$ such that $\langle 0_C | P_{BC,\mu} | 0_C \rangle = M_{B,\mu}^* M_{B,\mu}$. Thus,

$$p_\mu \rho_{A,\mu} = \text{Tr}_B \left( \left( \mathbb{1}_A \otimes \sqrt{M_{B,\mu}^* M_{B,\mu}} \right) \rho_{AB} \left( \mathbb{1}_A \otimes \sqrt{M_{B,\mu}^* M_{B,\mu}} \right) \right) = \text{Tr}_{BC} \left( \left( \mathbb{1}_A \otimes P_{BC,\mu} \right) (\rho_{AB} \otimes |0\rangle \langle 0|_C) (\mathbb{1}_A \otimes P_{BC,\mu}) \right).$$

So, the quantum ensemble $\{ (p_\mu, \rho_{A,\mu}) \}$ which is induced by the quantum operation $\Phi_B$ can be considered as one which is induced by the quantum operation $\Phi_{BC} = \sum_\mu \text{Ad}_{P_{BC,\mu}}$ over $\mathcal{H}_B \otimes \mathcal{H}_C$. Thus, it follows from (ii) that

$$\chi \{ (p_\mu, \rho_{A,\mu}) \} \leq S(\rho_B \otimes |0\rangle \langle 0|_C) = S(\rho_B).$$

### 3 The conjecture of Fannes, de Melo, Roga and Życzkowski

Let $\mathcal{E}_N = \{(p_i, \rho_i)\}_{i=1}^N$ be a quantum ensemble on a finite dimensional quantum system $\mathcal{H}$, $F_{ij} = F(\rho_i, \rho_j) = (\text{Tr}(|\sqrt{\rho} \sqrt{\sigma}|))^2$ be the fidelity between $\rho_i$ and $\rho_j$. The matrix

$$C_{\sqrt{F}}(\mathcal{E}_N) = \left[ \sqrt{p_ip_jF_{ij}} \right]_{ij}$$

is said to be a correlation matrix of the quantum ensemble $\mathcal{E}_N = \{(p_i, \rho_i)\}_{i=1}^N$.

For $N = 2$ or $3$, the correlation matrix $C_{\sqrt{F}}(\mathcal{E}_N) = \left[ \sqrt{p_ip_jF_{ij}} \right]_{ij}$ is a legitimate state. However, if $N \geq 4$, then $C_{\sqrt{F}}(\mathcal{E}_N) = \left[ \sqrt{p_ip_jF_{ij}} \right]_{ij}$ fails to be a positive semi-definite matrix in general [?]. For $N = 2$, the correlation matrix

$$C_{\sqrt{F}}(\mathcal{E}_2) = \begin{bmatrix} p_1 & \sqrt{p_1p_2F(\rho_1, \rho_2)} \\ \sqrt{p_1p_2F(\rho_1, \rho_2)} & p_2 \end{bmatrix}$$

was shown to satisfy the following inequality [?]:

$$\chi(\mathcal{E}_2) \leq S(C_{\sqrt{F}}(\mathcal{E}_2)).$$

Moreover, the upper bound $S(C_{\sqrt{F}}(\mathcal{E}_2))$ is the tighter one in the above inequality.

Fannes, de Melo, Roga and Życzkowski conjectured that for $N = 3$, $\chi(\mathcal{E}_3) \leq S(C_{\sqrt{F}}(\mathcal{E}_3))$ is also true [?].

In what follows, we apply Theorem ?? to answer partly the conjecture.
Lemma 3.1 ([?]). Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be three finite dimensional complex Hilbert spaces. Then the block operator
\[
\begin{bmatrix}
A & D & E \\
D^\dagger & B & F \\
E^\dagger & F^\dagger & C
\end{bmatrix}
\]
defined on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ is positive semi-definite if and only if the following statements are valid:

(i) $A \succeq 0, B \succeq 0, C \succeq 0$;

(ii) there exist three contractive operators $R_1, R_2$ and $R_3$ such that $D = \sqrt{A}R_1\sqrt{B}, F = \sqrt{B}R_2\sqrt{C}$, and
\[
E = \sqrt{A}R_1\text{supp}(B)R_2\sqrt{C} + \sqrt{A - \sqrt{A}R_1\text{supp}(B)R_1^\dagger}\sqrt{A}R_3\sqrt{C - \sqrt{A}R_1\text{supp}(B)R_1^\dagger}\sqrt{C},
\]
where $\text{supp}(B)$ stands for the support projection of $B$.

Lemma 3.2. Let $U$, $V$ and $W$ be three unitary operators on finite dimensional complex Hilbert space $\mathcal{H}$ and $\mathbf{1}$ be the identity operator on $\mathcal{H}$. Then the operator
\[
\begin{bmatrix}
\mathbf{1} & U & V \\
U^\dagger & \mathbf{1} & W \\
V^\dagger & W^\dagger & \mathbf{1}
\end{bmatrix}
\]
is positive semi-definite if and only if $V = UW$.

Proof. Taking $D = U, E = V, F = W$ and $A = B = C = \mathbf{1}$ in Lemma ??, we have that $R_1 = U, R_2 = W, \text{supp}(B) = \mathbf{1}$ and $R_3$ is a contractive operator. Moreover, $V = UW$. That is
\[
\begin{bmatrix}
\mathbf{1} & U & V \\
U^\dagger & \mathbf{1} & W \\
V^\dagger & W^\dagger & \mathbf{1}
\end{bmatrix} \succeq 0 \iff V = UW.
\]

Remark 3.3. The alternative proof of Lemma ?? may be given by Theorem 3.1 in [?].
**Theorem 3.4.** Let $\mathcal{E}_3 = \{(p_1, \rho_1), (p_2, \rho_2), (p_3, \rho_3)\}$ be a quantum ensemble on the finite-dimensional quantum system $\mathcal{H}$. It follows from the polar decomposition theorem that there exist three unitary operators $V, U$ and $W$ on $\mathcal{H}$ such that

$$
|\sqrt{\rho_2}\sqrt{\rho_1}| = U\sqrt{\rho_2}\sqrt{\rho_1}, \\
|\sqrt{\rho_3}\sqrt{\rho_1}| = V\sqrt{\rho_3}\sqrt{\rho_1}, \\
|\sqrt{\rho_3}\sqrt{\rho_2}| = W\sqrt{\rho_3}\sqrt{\rho_2}.
$$

If $V=UW$, then

$$
\chi(\mathcal{E}_3) \leq S(C_{\sqrt{\mathcal{F}}}(\mathcal{E}_3)).
$$

**Proof.** By the conditions, it follows that

$$
\text{Tr} (\sqrt{\rho_1}U\sqrt{\rho_2}) = \sqrt{F_{12}}, \\
\text{Tr} (\sqrt{\rho_1}V\sqrt{\rho_3}) = \sqrt{F_{13}}, \\
\text{Tr} (\sqrt{\rho_2}W\sqrt{\rho_3}) = \sqrt{F_{23}}.
$$

Let $\mathcal{H}_A = \mathcal{H}, \mathcal{H}_B = \mathbb{C}^3$, and

$$
\rho_{AB} = 
\begin{bmatrix}
 p_1\rho_1 & \sqrt{p_1p_2}\sqrt{\rho_1}

 p_2\rho_2 & \sqrt{p_2p_3}\sqrt{\rho_2}

 p_3\rho_3 & \sqrt{p_3}\sqrt{\rho_3}
\end{bmatrix}.
$$

Now, we only need to show that $\rho_{AB}$ is a positive semi-definite operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. Note that

$$
\rho_{AB} = 
\begin{bmatrix}
 \sqrt{p_1p_1} & 0 & 0 \\
 0 & \sqrt{p_2p_2} & 0 \\
 0 & 0 & \sqrt{p_3p_3}
\end{bmatrix} 
\begin{bmatrix}
 1 & U & V \\
 U^* & 1 & W \\
 V^* & W^* & 1
\end{bmatrix} 
\begin{bmatrix}
 \sqrt{p_1p_1} & 0 & 0 \\
 0 & \sqrt{p_2p_2} & 0 \\
 0 & 0 & \sqrt{p_3p_3}
\end{bmatrix},
$$

and it follows from Lemma ?? that $\rho_{AB} \succeq 0$ is equivalent to

$$
\begin{bmatrix}
 1 & U & V \\
 U^* & 1 & W \\
 V^* & W^* & 1
\end{bmatrix} \succeq 0 \iff V = UW.
$$

Moreover, it is easy to show that

$$
\rho_A = \text{Tr}_B (\rho_{AB}) = \sum_{i=1}^3 p_i \rho_i, \quad \rho_B = C_{\sqrt{\mathcal{F}}}(\mathcal{E}_3).
$$
Since \( \dim(\mathcal{H}_B) = 3 \), take a standard orthogonal basis \( \{ |\mu_B\rangle \} \) of \( \mathcal{H}_B \) such that \( p_\mu \rho_{A,\mu} = \langle \mu_B | \rho_{AB} | \mu_B \rangle \). By Theorem ??, we have

\[ \chi(\mathcal{E}_3) = \chi\{ (p_\mu \rho_{A,\mu}) \} \leq S(\rho_B) = S(C_{\sqrt{t}}(\mathcal{E}_3)). \]

This completes the proof. \( \square \)

**Remark 3.5.** In fact, Lemma ?? can be easily generalized to the case where 3-by-3 block matrix is replaced by \( K \)-by-\( K \) (\( K \geq 3 \)) block matrix of unitary entries. The generalization is described as follows:

Assume that the following \( K \times K \) block matrix of unitary entries is positive semi-definite:

\[
\begin{bmatrix}
U_{11} & U_{12} & \cdots & U_{1K} \\
U_{21} & U_{22} & \cdots & U_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
U_{K1} & U_{K2} & \cdots & U_{KK}
\end{bmatrix} \equiv P \geq 0.
\]

Then these unitary operators satisfy the conditions:

- \( U_{ii} = 1 \) for each index \( i \); \( U_{ji} = U_{ij}^\dagger \) for all indices \( i, j \).

Thus

\[
P = \begin{bmatrix}
1 & U_{12} & \cdots & U_{1K} \\
U_{12}^\dagger & 1 & \cdots & U_{2K} \\
\vdots & \vdots & \ddots & \vdots \\
U_{1K}^\dagger & U_{2K}^\dagger & \cdots & 1
\end{bmatrix}.
\]

Furthermore, we have that \( P \) is of the following forms:

(a)

\[
P = \begin{bmatrix}
1 & U_1 & U_1 U_2 & U_1 U_2 U_3 & \cdots & \cdots & U_1 U_2 \cdots U_{K-1} \\
U_1^\dagger & 1 & U_2 & U_2 U_3 & U_2 U_3 U_4 & \cdots & \cdots \\
U_1^\dagger U_1^\dagger & U_1^\dagger & 1 & U_3 & U_3 U_4 & \cdots & \cdots \\
U_1^\dagger U_1^\dagger U_1^\dagger & U_1^\dagger U_1^\dagger & U_1^\dagger & 1 & \cdots & \cdots & \cdots \cdots U_{K-3} U_{K-2} U_{K-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \ddots \cdots U_{K-2} U_{K-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \ddots \ddots \cdots U_{K-1} \\
U_{K-1}^\dagger U_{K-2}^\dagger U_{K-1}^\dagger & \cdots & \cdots & \cdots & \cdots & U_{K-1}^\dagger U_{K-2}^\dagger U_{K-1}^\dagger & U_{K-1}^\dagger U_{K-2}^\dagger U_{K-1}^\dagger & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix}
\]

for a collection of unitary operators \( \{ U_i : i = 1, \ldots, K-1 \} \) on \( \mathcal{H} \),
or

\[
P = \begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_K
\end{bmatrix}
\begin{bmatrix}
V_1^\dagger & V_2^\dagger & \cdots & V_K^\dagger
\end{bmatrix}
\]

for a collection of unitary operators \( \{V_i : i = 1, \ldots, K\} \) on \( \mathcal{H} \).

The outline of the proof is the following. The fact that \( P \) is of the form (a) can be easily derived by applying repeatedly the Theorem 3.1 in [?] to a block matrix. Indeed, we first apply it to the new block matrix:

\[
\begin{bmatrix}
1 & X \\
X^\dagger & A
\end{bmatrix} \succeq 0,
\]

where \( X = [U_{12}, \ldots, U_{1K}] \) and

\[
A = \begin{bmatrix}
1 & U_{23} & \cdots & U_{2K} \\
U_{23}^\dagger & 1 & \cdots & U_{3K} \\
\vdots & \vdots & \ddots & \vdots \\
U_{2K}^\dagger & U_{3K}^\dagger & \cdots & 1
\end{bmatrix}.
\]

Then apply it again to a similar block structure for \( A \), and so on. Finally we obtain the form (a) of \( P \). The forms (a) and (b) are equivalent via the following identification:

\[
U_1 = V_1V_2^\dagger, U_2 = V_2V_3^\dagger, \ldots, U_{K-1} = V_{K-1}V_K^\dagger.
\]

4 Concluding remarks

In this paper, we obtained an universal upper bound for the Holevo quantity which is induced by a quantum operation and proved that for a given quantum ensemble which consists of \( N \) quantum states on the same space, a so-called correlation matrix \( C_{\sqrt{F}(E_N)} \) can be constructed. Its von Neumann entropy is shown to be a upper bound of the Holevo quantity for \( N = 3 \) under some constraints. We also generalized Lemma ?? and obtained an interesting characterization of positivity of special operator matrix, which may shed new light on solving other related problems in quantum information theory.
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