Nonsupersymmetric multibrane solutions

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Abstract

Gravity coupled to an arbitrary number of antisymmetric tensors and scalar fields in arbitrary spacetime dimensions is studied in a context of general, static, spherically symmetric solutions with many orthogonally intersecting branes. Neither supersymmetry nor harmonic gauge is assumed. It is shown that the system reduces to a Toda-like system after an adequate redefinition of transverse radial coordinate $r$. Duality $r \to 1/r$ in the set of solutions is observed.

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1 Introduction

Recently branes are subject of strong interest mainly because of their crucial role in superstring theories where they can carry Ramond-Ramond charges. However it is possible to view them from a classical point of view as solutions of field equations in supergravity or more generally in systems with gravity coupled to antisymmetric tensors and other fields. Here we consider static, spherically symmetric solutions (see \[1\] [2] [3] and references therein). Most of solutions of this type were previously found in so-called harmonic gauge (linearity condition) i.e. assuming that supersymmetry is not entirely broken \[4\] [5]. There were also attempts to relax this condition \[6\] [9]. In a case of a system with single scalar and single antisymmetric tensor a complete solution was presented in \[10\]. It was also observed that studying multibranes systems is equivalent to solving a Toda-like system what turned out to be a very helpful tool in derivation of many classes of exact solutions \([6, 7, 8]\) and references in \[3\]). Such equivalence however was proved with a postulate of Poincar´e invariance in all directions parallel to at least one brane \([6]\) or in harmonic gauge \([7, 8]\).

In this paper we continue exploring this idea. We consider \(D\)-dimensional gravity coupled to several antisymmetric tensors and scalars, assuming that each antisymmetric tensor supports only one brane. We do not postulate Poincaré invariance in the whole subspace parallel to at least one brane, but divide it into several smaller subspaces each uniquely described as parallel to some of the branes and transversal to the others. The geometry of each of the subspaces is governed by an independent factor in the metric tensor. In the directions transversal to all branes \(SO(D - d)\) symmetry is assumed. We show that the classical equations of motion are equivalent to a Toda-like system after an adequate redefinition of radial coordinate: \(r \rightarrow \vartheta(r)\), where \(\vartheta(r)\) is in general not harmonic. A duality in the set of solutions is observed. Each solution described by given values of parameters has a partner which is numerically equal to it if a sign of \(\vartheta\) is reversed (or equivalently if \(r \rightarrow 1/r\)). In particular, the solution dual to the supersymmetric (harmonic) is not supersymmetric (not harmonic).

2 The model

Consider a \(D\) dimensional theory having in the classical limit the following action:

\[
\int_{\mathcal{M}} d^D X \sqrt{|\det g|} \left( R - \frac{1}{2} \sum_{\alpha=1}^{N_\phi} \partial_M \phi_\alpha \partial^M \phi_\alpha - \sum_{i=1}^{N_A} \frac{\exp(\sum_{\alpha=1}^{N_\phi} a_{i\alpha} \phi_\alpha)}{2n_i!} F_{M_1 \ldots M_n}^i F^{iM_1 \ldots M_n} \right),
\]

were \(F^i\) are antisymmetric \(n_i\)-forms, \(\phi^\alpha\) - scalar fields, \(a_{i\alpha}\) - constants, \(\mathcal{M}\) - a manifold of dimension \(D\) and \((X^M)\) coordinates on it. For example the bosonic sector of most of supergravity theories is well described by the above action if additional assumptions leading to cancellation of Chern-Simons term are made. The equations of motions derived from \([1]\) are:

\[
R_{MN} = \frac{1}{2} \sum_{\alpha} \partial_M \phi_\alpha \partial_N \phi_\alpha + \sum_{i} \frac{\epsilon^{\sum_{\alpha} a_{i\alpha} \phi_\alpha}}{2(n_i - 1)!} S^i_{MN},
\]

\[
0 = \nabla_M \left( \epsilon^{\sum_{\alpha} a_{i\alpha} \phi_\alpha} F^i_{MR_1 \ldots R_{n_i} - 1} \right),
\]

\[
\nabla_M \nabla^M \phi_\alpha = \sum_{i} \frac{a_{i\alpha}}{2n_i} \epsilon^{\sum_{\beta} a_{i\beta} \phi_\beta} F^i_{R_1 \ldots R_{n_i}} F^{iR_1 \ldots R_{n_i}},
\]

with:

\[
S^i_{MN} = F^i_{MR_1 \ldots R_{n_i} - 1} F^i_{N R_1 \ldots R_{n_i} - 1} - \frac{n_i - 1}{n_i(D - 2)} F^i_{R_1 \ldots R_{n_i}} F^i_{R_1 \ldots R_{n_i} g_{MN}},
\]

where \(\sum_i\) and \(\sum_{\alpha}\) are sums over all possible values of \(i = 1, \ldots, N_A\) and \(\alpha = 1, \ldots, N_\phi\).

We search for a solution which allows \(N_A\) orthogonally intersecting elementary (electric) or solitonic (magnetic) branes, where each \(V_i\) - a worldvolume of the \(i\)-th brane is supported by a potential of the adequate
\( F^i \) or \(* F^i \). We define indices \( I, J, \ldots \) running through the set of all non-empty subsets of \( \{ 1, \ldots, N_A \} \), subspaces of \( \mathcal{M} \):

\[
\begin{align*}
V_I &= \{ 0 \} \cup \left( \bigcap_{I \in J} V_J \right) \setminus \left( \sum_{j \notin I} V_j \right), \quad V = \bigoplus V_I \\
V_0 &= \mathcal{M} / V,
\end{align*}
\]

(in other words, \( V_I \) with \( I = \{ i_1, \ldots, i_k \} \) is a subspace span by vectors simultaneously parallel to all \( V_{i_1}, \ldots, V_{i_k} \) and transversal to all \( V_{i_{k+1}}, \ldots, V_{i_{N_A}} \)) and numbers of subspace’s dimensions:

\[
d_I = \dim V_I, \quad D_i = \dim V_I = \sum_{I : i \in I} d_I, \quad \hat{D}_i = \dim \hat{V}_I = \sum_{I : i \notin I} d_I, \quad d = \dim V = \sum_I d_I.
\]

For particular brane configurations some of \( V_I \) can be zero-dimensional. (For example, if \( V_i \subset V_j \) for any \( i, j \), then \( V_{\{i\}} = \{0\} \). We introduce also a mapping \( \tilde{\cdot} \) defined by \( \tilde{n} = D - n - 2 \).

Because all the branes are assumed to propagate in time and the solution is trivial in case \( V_0 = \{0\} \) (all fields constant) therefore:

\[
1 \leq d_{\{1, \ldots, N_A\}} \leq D - 1, \quad -1 \leq \hat{d} \leq D - 3, \quad 0 \leq d_I \leq D - 2 \quad \text{for} \quad I \neq \{1, \ldots, N_A\}.
\]

Additionally it is postulated that:

\[
\hat{d} \neq 0.
\]

(This assumption can be omitted if an appropriate limiting procedure \( \hat{d} \to 0 \) is applied.)

We call by \( (x^\mu) \) the coordinates on \( V_I \), by \( (x^{\hat{\mu}}) \) – the coordinates on \( \hat{V}_I \), by \( (y^m) \) – on \( V_0 \) and we use sum convention for all indices enumerating coordinates but not for indices like \( i \) or \( I \). We introduce also symbols \( r = \sqrt{y^m y^n \delta_{mn}} \) and \( f' = df/dr \) for any function \( f \).

We assume that all the fields depend nontrivially only on the radial coordinate (in the transverse space) \( r \). For scalar fields it means:

\[
\phi_\alpha(X) = \phi_\alpha(r).
\]

Metric tensor is assumed to be divided into \( N_g \) segments related to \( V_0 \) and those \( V_I \) which are at least one-dimensional (\( 2 \leq N_g \leq 2^{N_A} \)):

\[
ds^2(X) = \sum_I e^{2A_I(r)} dx^{\mu} dx^{\nu} \eta^{\mu \nu} + e^{2B_I(r)} \left( (dr)^2 + r^2 d\Omega^2 \right),
\]

where \( d\Omega \) is the space interval of the \( \hat{d} + 1 \) dimensional unit sphere and \( \eta^{\mu \nu} = \delta^{\mu \nu} \) if \( I \neq \{1, \ldots, N_A\} \). In the formula \( (11) \) and any formulae below by \( \sum_{I : r(I) = r} \) we denote a sum over those \( I \) for which \( V_I \) are at least one-dimensional and which satisfy the restriction \( r(I) \).

For the antisymmetric tensor fields \( F^i \), two cases should be distinguished. If a brane is elementary, the only nonzero components of \( F^i \) have a form:

\[
F^i_{\mu_1^{D_i} \ldots \mu_{D_i}^{D_i}}(X) = \epsilon_{\mu_1^{D_i} \ldots \mu_{D_i}^{D_i}} \partial_m \exp(C_i(r))
\]

when for a solitonic brane only:

\[
F^i_{\mu_1^{D_i} \ldots \mu_{D_i}^{D_i} m_1 \ldots m_{\hat{d}+1}}(X) = \epsilon_{\mu_1^{D_i} \ldots \mu_{D_i}^{D_i} m_1 \ldots m_{\hat{d}+1}} \frac{\lambda_i y_n}{y^{\hat{d}+2}}
\]

do not vanish, where \( \lambda_i \) is an arbitrary nonzero real constant.

Now it is possible to rewrite the equations of motion \( (12, 13) \) in terms of the scalar functions \( A_I, B, \phi_\alpha \) and (only for the elementary branes) \( C_i \) introduced by \( (11, 12) \).
\[ A''_l + A'_l \left( \sum_j d_j A'_j + \tilde{d}B' + \frac{\tilde{d} + 1}{r} \right) = \frac{\sum_{i \in I} \tilde{D}_i (S'_i)^2 - \sum_{i \notin I} D_i (S'_i)^2}{2(D - 2)}, \] (14)

\[ B'' + \tilde{d}(B')^2 + \frac{2\tilde{d} + 1}{r} B' + (B' + \frac{1}{r} \sum I d_I A'_I = \frac{-\sum D_i (S'_i)^2}{2(D - 2)}, \] (15)

\[ \phi''_\alpha + \phi'_\alpha \left( \sum I d_I A'_I + \tilde{d}B' + \frac{\tilde{d} + 1}{r} \right) = \frac{1}{2} \sum_i s_i \epsilon_i (S'_i)^2, \] (16)

\[ \tilde{d}B'' - \tilde{d}(B')^2 - \frac{\tilde{d}}{r} B' + \sum I d_I \left( A''_I - \frac{1}{r} A'_I - 2A'_IB' + (A_I)^2 \right) + \frac{1}{2} \sum_\alpha (\phi'_\alpha)^2 = \frac{1}{2} \sum (S'_i)^2, \] (17)

\[ C''_i + C'_i \left( C'_i - \sum_{I : i \in I} d_I A'_I + \sum_{I : i \notin I} d_I A'_I + \tilde{d}B' + \sum_\alpha \epsilon_i \phi'_\alpha + \frac{\tilde{d} + 1}{r} \right) = 0, \] (18)

where:

\[ s_i = \begin{cases} +1 & \text{(elem.)}, \\ -1 & \text{(solit.)}, \end{cases} \] (19)

\[ S'_i = \begin{cases} \exp(\frac{1}{2} \sum \epsilon_i \phi'_\alpha - \sum_{I : i \in I} d_I A'_I (\epsilon C'_i)' & \text{(elem.)}, \\ \exp(\frac{1}{2} \sum \epsilon_i \phi'_\alpha - \sum_{I : i \notin I} d_I A'_I - \tilde{d}B) \frac{\Delta \phi'_\alpha}{\epsilon\phi'_\alpha} & \text{(solit.)}. \end{cases} \] (20)

### 3 Harmonicity, supersymmetry and \( \hat{\vartheta} \) coordinate

Assuming that \( \tilde{d} \neq 0 \), define:

\[ \chi = \sum_I d_I A_I + \tilde{d}B, \] (21)

Relation \( \chi' = 0 \) is usually called linearity condition or harmonic gauge because it leads to a solution expressed in terms of harmonic functions on \( V_\vartheta \). In case of supergravity theories it is necessary but not sufficient for preserving supersymmetry. Here we do not make any a priori assumption on \( \chi \), so the results presented below remain valid in more general classes of non-supersymmetric solutions and solutions not governed by harmonic functions.

Summing (14)-(15) one can see that \( \chi \) has to satisfy the following equation:

\[ \chi'' + (\chi')^2 + \frac{2\tilde{d} + 1}{r} \chi' = 0, \] (22)

which can be solved as:

\[ \chi(r) = \ln \left| \frac{c_\chi - 1/r^{2d}}{c_\chi - c_0} \right| + \epsilon_\chi, \] (23)

where \( \epsilon_\chi \) is an arbitrary real constant and \( c_\chi \) takes real as well as infinite values. Since \( \lim_{c_\chi \to +\infty} \chi = \lim_{c_\chi \to -\infty} \chi \), points \( c_\chi = +\infty \) and \( c_\chi = -\infty \) in the parameter space can be identified. In the discussion below we choose \( c_0 = 1 \), but it can be generalised to arbitrary \( c_0 \).

Let us introduce new parameters: \( R \in [0, +\infty] \) and \( s_\chi \in \{-1, +1\} \), such that:

\[ s_\chi R^{2d} = 1/c_\chi \] (24)
(if \( R = 0 \) or \( R = \infty \) both possible signs of \( s_\chi \) describe the same point in the parameter space) and define a function:

\[
\vartheta(r) = \begin{cases} 
\frac{1}{2d} \left( \frac{R^d}{r^d} + R^d \right) \left( \arctan((\frac{R}{r})^d) - \arctan(R^d) \right), & s_\chi = -1, \\
\frac{1}{2d} \left( \frac{1}{r^d} - R^d \right) \left( \arctanh((\frac{r}{R})^d) - \arctanh(R^d) \right), & s_\chi = +1,
\end{cases} \tag{25}
\]

where:

\[
\text{Arth}(x) = \begin{cases} 
\text{arctanh}(x), & \text{for } |x| < 1, \\
-\text{arcoth}(x), & \text{for } |x| > 1.
\end{cases}
\tag{26}
\]

Formula (25) exhibits the following duality:

\[
\vartheta(r; R, s_\chi) = -\vartheta(1/r; 1/R, s_\chi), \tag{27}
\]

what gives a relation between \( \vartheta \) defined for different parameter values. \( R = 0 \) is equivalent to harmonic gauge and then and only then \( \vartheta = \frac{1}{2d} \left( \frac{1}{r^d} - 1 \right) \) what is harmonic function of \( r \). So, the parameter \( R \) (or \( c_\chi \)) can be treated as a measure how distant is a given case from the harmonic one. \( R = \infty \), is a partner of \( R = 0 \) under (27) and then \( \vartheta = -\frac{1}{2d} \left( r^d - 1 \right) \).

Function \( \vartheta \) can be understood as a space coordinate instead of \( r \) and the coordinate change is singular only at \( r = R, s_\chi = +1 \). The space-time interval expressed in terms of \( \vartheta \) is:

\[
d\sigma^2(\vartheta) = \sum_I e^{2A_I(\vartheta)} dx^\mu_i dx^\nu_i + e^{2B_\vartheta(\vartheta)} \left(d\vartheta^2 + \rho(\vartheta)^2 d\Omega^2\right), \tag{28}
\]

where

\[
\exp(B_\vartheta(\vartheta)) = \left( \frac{1}{2} \exp(-\sum_I d_I A_I(\vartheta) + \epsilon_\chi) \right)^{1/d} \rho(\vartheta)^{-(1+1/d)},
\]

\[
\rho(\vartheta) = \begin{cases} 
\frac{1}{2}((1/R)^d + R^d) \sinh(\frac{4\vartheta}{(1/R)^d + R^d}) + 2 \arctan R^d, & s_\chi = -1, \\
\frac{1}{2}((1/R)^d - R^d) \sinh(\frac{4\vartheta}{(1/R)^d - R^d}) + 2 \arctanh R^d, & s_\chi = +1
\end{cases}, \tag{30}
\]

and for \( \rho \) one has:

\[
\rho(\vartheta; R, s_\chi) = \rho(-\vartheta; 1/R, s_\chi). \tag{31}
\]

Note, that (28) is well defined even for such values of \( \vartheta \) which cannot be related to any \( r \) by (26). In other words, \( \vartheta \) covers wider area of space-time than \( r \). We discuss some aspects of the fact in the last section of the paper.

The equations (15-18) can be translated to:

\[
\ddot{A}_I = \frac{\sum_{i \in I} \dot{D}_i \dot{S}_i - \sum_{i \in l} D_i \dot{S}_i}{2(D - 2)}, \tag{32}
\]

\[
\ddot{\phi}_\alpha = -\frac{1}{2} \sum_i \dot{a}_i a_{i\alpha}(\dot{S}_i)^2, \tag{33}
\]

\[
0 = \ddot{S}_i + \left( \sum_\alpha a_{i\alpha} \dot{\phi}_\alpha - \sum_{I; i \in I} d_I \dot{A}_I \right) \dot{S}_i, \text{ (only elem.)} \tag{34}
\]

\[
\frac{1}{d} \left( \sum_I d_I \dot{A}_I \right)^2 + \sum_\alpha \left( \dot{A}_I(\dot{A}_I) + \frac{1}{2} \sum_\alpha (\dot{\phi}_\alpha)^2 \right) + \Lambda_\chi = \frac{1}{d} \sum_i (\dot{S}_i)^2, \tag{35}
\]

\[
\rho(\vartheta; R, s_\chi) = \rho(-\vartheta; 1/R, s_\chi). \tag{31}
\]
where:
\[
\dot{S}_i = \begin{cases} 
\exp\left(\frac{1}{2} \sum_{\alpha} a_{i\alpha} \phi_\alpha - \sum_{I: i \in I} d_I A_I\right) (e^{C_i}) \gamma_i & \text{(elem.)}, \\
-2\lambda_i e^{-\epsilon x} \exp\left(\frac{1}{2} \sum_{\alpha} a_{i\alpha} \phi_\alpha + \sum_{I: i \in I} d_I A_I\right) & \text{(solit.)},
\end{cases}
\] (36)
\[
\Lambda_\chi = -\frac{16 \tilde{d} (\tilde{d} + 1) c_\chi (c_\chi - 1)^2}{(c_\chi - 1)^2} \] (37)

and the "dots" describe derivatives with respect to \(\vartheta\).

The system (32–37) together with (29–30) carries complete information originally contained in (14–18, 20). Since (14–18, 20) drastically simplifies when harmonic gauge is imposed, it is interesting what happens to (29–30, 32–37) in analogous situation. If one treats \(\vartheta\) as a fundamental coordinate then all dependence of the solution on parameter \(R\) (so also all differences between the harmonic and general cases) enters only in \(\rho\) function (30) which influences a form of the metric on \(V_\emptyset\) and in \(\Lambda_\chi\) constant (37) appearing in (35). However (35) is not a dynamic equation but rather a constraint decreasing by one a number of integration constants. What is more, if \(R \neq 1\) two different \(R\) gives the same value of \(\Lambda_\chi\). In particular, for both \(R = 0\) and \(R = \infty\) one has \(\Lambda_\chi = 0\) and \(\rho(\vartheta) = |\tilde{d} \vartheta|\).

4 Toda-like system

Define:
\[
|\omega_i| = \exp\left(\frac{1}{2} \sum_{\alpha} \varsigma_i a_{i\alpha} \phi_\alpha - \sum_{I: i \in I} d_I A_I\right), \] (38)

With these functions one can find that (34) leads to:
\[
\dot{S}_i = p_i / |\omega_i|, \] (39)

where \(p_i\) are nonzero real integration constants. Simultaneously (30) for solitonic branes gives \(\dot{S}_i = -2\lambda_i e^{-\epsilon x} / |\omega_i|\), so, after identification \(p_i = -2\lambda_i e^{-\epsilon x} / |\omega_i|\), relation (39) is valid for elementary as well as for solitonic branes.

It can be checked from (32–33) that \(\omega_i\) have to satisfy the following system of equations:
\[
\frac{d^2}{d\vartheta^2} (\ln|\omega_i|) = -\sum_j \Delta_{ij} \frac{p_j^2}{4 |\omega_j|^2}, \] (40)

where elements of \(\Delta\) matrix are:
\[
\Delta_{ij} = \frac{2}{D-2} \left( \sum_{I: i,j \in I} d_I \sum_{J: i,j \notin J} d_J - \sum_{I: i \notin I, j \in J} d_I \sum_{J: j \notin J} d_J \right) + \sum_{\alpha} \varsigma_i a_{i\alpha} \varsigma_j a_{j\alpha}, \] (41)

indices \(\tilde{I}, \tilde{J}\) run through all values allowed for \(I, J\) and additionally \(\emptyset\), and by \(d_\emptyset\) is understood \(\tilde{d}\) (but not \(\dim V_\emptyset\)). The diagonal elements of \(\Delta\) are:
\[
\Delta_{ii} = \frac{2D_i \tilde{D}_i}{D-2} + \sum_{\alpha} a_{i\alpha}^2, \] (42)

and the non-diagonal ones are bounded by:
\[
\Delta_{ij} \leq \frac{1}{2} (\Delta_{ii} + \Delta_{jj}), \] (43)
The equations (40) are equivalent to a Toda-like system. After substituting \( x_i = 2 \ln(\frac{2\sqrt{\omega_i}}{2\omega_i}) \) they transform to:

\[
\ddot{x}_i = -\sum_j K_{ij} \exp(x_j).
\]  

(44)

where \( K_{ij} = -\frac{2\Delta_{ij}}{\Delta} \). The original Toda (molecule) system is defined by (44) with \( K \) being a Cartan matrix [12]. But in our case \( K \) is not in general a Cartan matrix.

If \( \det(\Delta) \neq 0 \) then it is possible to express all functions \( A_I, B_\vartheta \) \([28]\), \( \phi_\alpha \) \([11]\), \( C_i \) \([12]\) in terms of \( \omega_i \):

\[
\exp(A_I(\vartheta)) = E_I \left( \prod_i |\omega_i(\vartheta)|^{\gamma_i^I} \right) \exp(c_I \vartheta),
\]

(45)

\[
\exp(B_\vartheta(\vartheta)) = E_B \left( \prod_i |\omega_i(\vartheta)|^{\gamma_i^B} \right) \exp(c_B \vartheta)(\vartheta)^{-1+1/d},
\]

(46)

\[
\exp(\phi_\alpha(\vartheta)) = E_\alpha \left( \prod_i |\omega_i(\vartheta)|^{\gamma_i^\alpha} \right) \exp(c_\alpha \vartheta),
\]

(47)

\[
\frac{d}{d\vartheta} \exp(C_i(\vartheta)) = p_i |\omega_i(\vartheta)|^{-2},
\]

(48)

where \( \gamma_i^I, \gamma_i^B \) and \( \gamma_i^\alpha \) have to satisfy:

\[
\frac{D-2}{2} \sum_i \Delta_{ij} \gamma_i^j = \left\{ \begin{array}{ll}
-\tilde{D}_j & \text{if } j \in I, \\
\tilde{D}_j & \text{if } j \notin I,
\end{array} \right.
\]

\[
\sum_i \Delta_{ij} \gamma_i^j = 2a_{j\alpha},
\]

\[
\gamma_i^B = -\frac{1}{d} \sum_I d_I \gamma_i^I
\]

(49)

and values of real constants \( c_I, c_B, c_\alpha \) and positive constants \( E_I, E_B, E_\alpha \) are restricted by:

\[
0 = \frac{1}{2} \sum_I \gamma_i^{a_\alpha c_\alpha} - \sum_{I;i \notin I} d_I c_I, \quad \Pi_{I;i \in I} E_I^{d_I} = \prod_\alpha E_\alpha^{\gamma_i^{a_i c_i}}.
\]

(50)

So, the problem of finding brane solution in gravity coupled to an arbitrary number of antisymmetric tensors and scalar fields without assumption of harmonic gauge can be reduced to solving a Toda-like system \([40]\) with a condition derived from \([35]\):

\[
\sum_I N_I \frac{d}{d\vartheta} (\ln |\omega_i|) + \frac{1}{2} (\Lambda_\chi + \Lambda_\epsilon) = \sum_i \frac{\rho_i^2}{4\omega_i^2},
\]

(51)

where \( \Lambda_\epsilon \) and \( N_I(\vartheta) \) are defined by:

\[
\Lambda_\epsilon = \sum_I d_I c_I^2 + \tilde{d}_B c_B^2 + \frac{1}{2} \sum_\alpha c_\alpha^2, \quad \sum_j \Delta_{ij} N_j = \frac{d}{d\vartheta} (\ln |\omega_i|).
\]

(52)

5 An example of a solution

Consider an example when matrix \( \Delta \) is diagonal and nonsingular. Then \([40]\) gives:

\[
|\omega_i(\vartheta)| = \left\{ \begin{array}{ll}
\frac{\rho_i \sqrt{-\kappa_i}}{2\sqrt{-\kappa_i}} \sin(\sqrt{-\kappa_i}(\vartheta - \theta_i)) & \text{for } \kappa > 0, \\
\frac{\rho_i \sqrt{\kappa_i}}{2\sqrt{\kappa_i}} (\vartheta - \theta_i) & \text{for } \kappa = 0, \\
\frac{\rho_i \sqrt{-\kappa_i}}{2\sqrt{-\kappa_i}} \sinh(\sqrt{-\kappa_i}(\vartheta - \theta_i)) & \text{for } \kappa < 0,
\end{array} \right.
\]

(53)
where real phases \( \theta_i \) are independent but \( \kappa_i \) gives a restriction on \( \kappa_i \):

\[
\sum_i \frac{\kappa_i}{\Delta_i} = \frac{1}{2} (\Lambda_\chi + \Lambda_e) .
\]  

(54)

Substituting (53) into (45–47) and solving (48):

\[
\exp(C_\ell(\vartheta)) = \begin{cases} 
E_1 & \text{for } \kappa_1 > 0 , \\
E_2 & \text{for } \kappa_1 = 0 , \\
E_3 & \text{for } \kappa_1 < 0 ,
\end{cases}
\]  

(55)

where \( E_1 \) are integration constants. A hint that the solution can be extended beyond point \( r = 0 \) gives a restriction on \( \kappa_i \).

In this paper we have shown that the system of equations of motions for gravity coupled to antisymmetric tensors and scalar fields reduces to Toda-like equations. The assumption of harmonic gauge usually made in connection with supersymmetry is not needed if an adequate redefinition of radial coordinate \( r \) is done. A duality in the set of solutions \( \vartheta \rightarrow -\vartheta \) (or equivalently \( r \rightarrow 1/r \)) was noticed.

The set of assumptions made in this paper follows the analogous set usually made in supergravity: there are several scalar and antisymmetric fields coupled to gravity and the solutions are not necessarily supersymmetric. Under these assumptions the extremely complicated system of equations reduces to a well-known Toda-like system.

6 Conclusions

In this paper we have shown that the system of equations of motions for gravity coupled to antisymmetric tensors and scalar fields reduces to Toda-like equations. The assumption of harmonic gauge usually made in connection with supersymmetry is not needed if an adequate redefinition of radial coordinate \( r \) is done. A duality in the set of solutions \( \vartheta \rightarrow -\vartheta \) (or equivalently \( r \rightarrow 1/r \)) was noticed.

The set of assumptions made in this paper follows the analogous set usually made in supergravity: there are several scalar and antisymmetric fields coupled to gravity and the solutions are not necessarily supersymmetric. Under these assumptions the extremely complicated system of equations reduces to a well-known Toda-like system.
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