Thirring’s low-energy theorem and its generalizations in the electroweak Standard Model

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Abstract:
The radiative corrections to Compton scattering vanish in the low-energy limit in all orders of perturbation theory. This theorem, which is well-known for Abelian gauge theories, is proved in the electroweak Standard Model. Moreover, analogous theorems are derived for photon scattering off other charged particles, in particular W bosons. Since the theorems follow from gauge invariance and on-shell renormalization, their derivation is most conveniently performed in the framework of the background-field method.
1 Introduction

In all models for particle interactions that contain QED, the electric charge \( -e \) of the electron is usually fixed in the Thomson limit, i.e., by low-energy Compton scattering, \( e^- \gamma \rightarrow e^- \gamma \). This definition ensures that \( e \) indeed represents the electric charge that describes the interaction between electrons and the electromagnetic field in classical electrodynamics. However, in the commonly adopted on-shell renormalization scheme, \( e \) is formally defined as the coupling of a photon with momentum zero to an on-shell electron. In QED the equivalence of the two definitions of \( e \) was first proved by Thirring [1], who showed that all radiative corrections to Compton scattering vanish in the low-energy limit within the on-shell scheme. Subsequently, this theorem was rederived and modified for more general U(1) gauge theories by several authors [2]. The validity of the theorem is commonly assumed also for theories like the electroweak Standard Model (SM), where QED is embedded in a non-Abelian gauge symmetry, although—to the best of our knowledge—no field-theoretical proof has been given so far. The purpose of this paper is to close this gap for the SM and to generalize the theorem to elastic low-energy scattering between photons and charged particles other than electrons, for instance to \( W^+ \gamma \rightarrow W^+ \gamma \).

Thirring’s theorem is a consequence of on-shell renormalization and gauge invariance, which implies relations between Green functions. In QED, these relations are the well-known Ward identities. In non-Abelian gauge theories the BRS symmetry, which replaces the original gauge symmetry after the usual Faddeev–Popov quantization, leads to much more complicated relations. These are known as Slavnov–Taylor identities, which in general involve explicit contributions of Faddeev–Popov ghost fields. In particular, the BRS symmetry of the SM does not lead to those Ward identities that imply Thirring’s theorem in QED. This is due to the fact that the electromagnetic current is not strictly conserved in the non-Abelian case. More precisely, the divergence of this current acts as the null operator only on physical states, because it is the BRS transform of an appropriate operator. Therefore, the derivations of the low-energy theorems presented in Ref. [2], which rely on strict electromagnetic current conservation, are not valid in theories such as the SM. In such cases a proof has to be based on the underlying BRS symmetry in the conventional Faddeev–Popov approach. However, if non-Abelian gauge theories are quantized in the framework of the background-field method (BFM) [3, 4, 5, 6, 7, 8, 9], the vertex and Green functions obey simple QED-like Ward identities. Since this feature greatly simplifies the investigation of the consequences of gauge invariance, we adopt this formalism in the following.

The outline of the article is as follows: After some preliminary remarks about our calculational framework in Section 2, we inspect the low-energy limit of Compton scattering within the SM in Section 3. In Section 4 we show how the treatment of the electron carries over to other charged particles, where particular attention is paid to the W boson. A summary is presented in Section 5.

2 Preliminary remarks

As the BFM is not yet as well-known as the conventional quantization procedure, we first make some brief remarks on the basis and the starting point of our analysis.
The BFM represents an alternative formalism for the quantization of non-Abelian gauge theories, which was invented in order to retain the gauge invariance of the effective action. The method was developed in different ways in Refs. [3, 4, 5] for pure, massless Yang–Mills theories and more recently applied to the SM in Refs. [6, 7, 8, 9], where Abbott’s formulation [4] was used as guideline.

The first step in the BFM approach is the construction of an effective action that is invariant under gauge transformations. To this end, each field in the classical Lagrangian is split into a background part and a quantum part, where only the latter is quantized, and the former appears as auxiliary field. The Faddeev–Popov quantization of the quantum fields requires the fixing of their gauge. The corresponding gauge-fixing conditions are chosen such that the complete path integral remains invariant under gauge transformations of the background fields. The background fields are identified with the independent variables of the BFM effective action, which is thus gauge-invariant. Diagrammatically this means that quantum fields appear on internal lines of one-particle-irreducible loop diagrams, whereas background fields represent the external legs. The gauge invariance of the BFM effective action implies that the vertex functions, which follow from this effective action by taking functional derivatives, obey the simple Ward identities that are related to the classical Lagrangian. In Refs. [6, 7] it was shown that the BFM Ward identities are compatible with the on-shell renormalization of the SM when the field renormalization is chosen appropriately. In the following all vertex functions \(\Gamma\) and self-energies \(\Sigma\) are assumed to be renormalized according to the scheme of Ref. [7].

One-particle-reducible connected Green functions and \(S\)-matrix elements are constructed by forming trees with the (one-particle-irreducible) vertex functions joined by propagators for the background fields [5]. The background-field propagators are defined by introducing a gauge-fixing term for the background gauge fields, which is, however, not related to the one that fixes the gauge of the quantum fields. The associated additional gauge parameters \(\xi\) enter only via tree-level quantities, but not via the higher-order contributions to the vertex functions. Therefore, the simple Ward identities for the vertex functions \(\Gamma\) translate into simple relations between connected Green functions \(G\). For a 't Hooft gauge-fixing term for the background fields in the SM, these identities were explicitly derived in Ref. [9].

Our notations and conventions for vertex and Green functions exactly follow the ones of Refs. [6, 7, 9], i.e. for instance all labelled fields and momenta are incoming. Background fields are marked by carets, except for fermion fields, where background and quantum fields need not be distinguished.

### 3 Low-energy Compton scattering—Thomson limit

In order to prove the low-energy theorem for Compton scattering, \(e^-\gamma \rightarrow e^-\gamma\), we start by deriving the Ward identity for the corresponding connected Green function \(G_{\mu\nu}^{A\bar{A}ee}\), where both photon legs are contracted with their momenta. This identity is easily obtained from the first relation of Eq. (11) in Ref. [7], which is the identity for the generating
functional expressing electromagnetic gauge invariance. Taking derivatives with respect to the sources of the fields $\bar{e}$, $e$ and $\hat{A}$, $\hat{\bar{e}}$, $\hat{e}$, respectively, in momentum space one gets

$$\frac{i k^2}{\xi_A} k^\alpha G_{\alpha e e}^\hat{A}(k, \bar{p}, p) = e G_{\bar{e} e e}(\bar{p}, -\bar{p}) - e G_{\bar{e} e e}(-p, p),$$  \hspace{1cm} (1)

$$\frac{i k_1^2 k_2^2}{\xi_A} k_1^\mu G_{\mu \nu}^{\hat{A} \bar{e} e}((k_1, k_2, \bar{p}, p)) = e G_{\nu e e}(k_2, \bar{p}, -\bar{p} - k_2) - e G_{\nu e e}(k_2, -p - k_2, p).$$  \hspace{1cm} (2)

Upon contracting (2) with $k_2^\nu$ and using (1), one obtains the desired identity,

$$\frac{i k_1^2 k_2^2}{\xi_A} k_1^\mu G_{\mu \nu}^{\hat{A} \bar{e} e}((k_1, k_2, \bar{p}, p)) = e^2 G_{\bar{e} e e}(\bar{p}, -\bar{p}) - e^2 G_{\bar{e} e e}(\bar{p} + k_1, p + k_2) - e^2 G_{\bar{e} e e}(\bar{p} + k_2, p + k_1) + e^2 G_{\bar{e} e e}(-p, p).$$  \hspace{1cm} (3)

The next step consists in amputating the external photon propagators in $G_{\mu \nu}^{\hat{A} \bar{e} e}$. Denoting amputated external fields by a lowered field index, we have

$$G_{\alpha}^{\hat{A} \bar{e} e}(k, \ldots) = \sum_X G_{\alpha}^{\hat{A} X}(k, -k) G_{X}(k, \ldots),$$  \hspace{1cm} (4)

where $X$ stands for all (background) fields that can mix with the (background) photon field $\hat{A}$. In the SM the sum over $X$ extends over $\hat{A}$, $\hat{Z}$, $\hat{H}$, and $\hat{\chi}$. Substituting $G_{\mu \nu}^{\hat{A} \bar{e} e}$ in the l.h.s. of (3) according to (4), and using the Ward identities for the two-point functions $G_{\alpha}^{\hat{A} X}$ [9],

$$k^\alpha G_{\alpha \beta}^{\hat{A} \bar{e} e}(k, -k) = -\frac{i \xi}{k^2} k_\beta,$$

$$k^\alpha G_{\alpha}^{\hat{A} X}(k, -k) = 0 \hspace{1cm} \text{for} \hspace{1cm} X = \hat{Z}, \hat{H}, \hat{\chi},$$  \hspace{1cm} (5)

we find

$$k_{1, \mu} k_{2, \nu} G_{\mu \nu}^{\bar{e} e e}(k_1, k_2, \bar{p}, p) = \frac{i k_1^2}{\xi_A} k_2^2 k_1^\mu k_2^\nu G_{\mu \nu}^{\hat{A} \bar{e} e}(k_1, k_2, \bar{p}, p),$$  \hspace{1cm} (6)

which translates (3) into a Ward identity for $G_{\mu \nu}^{\bar{e} e e}$. We note in passing that the counterparts of the BFM Ward identities (3), (5), (6) in the conventional field-theoretical approach are much more complicated and involve explicit contributions of Faddeev–Popov ghosts. In order to get a low-energy limit of $G_{\mu \nu}^{\bar{e} e e}$, we keep the momentum $p$ fixed, take derivatives from (6) with respect to $k_{1, \mu}$ and $k_{2, \nu}$, and take the limit $k_1, k_2 \rightarrow 0$. Note that the momentum $\bar{p}$ is not independent, but $\bar{p} = -p - k_1 - k_2$. One obtains

$$G_{\mu \nu}^{\bar{e} e e}(0, 0, -p, p) = e^2 \frac{\partial^2}{\partial p_\mu \partial p_\nu} G_{\bar{e} e e}(-p, p).$$  \hspace{1cm} (7)

For the fully amputated Green function, which is needed for the $S$-matrix element, also the electron legs must be amputated by multiplication with the inverse propagators. Using the relation between electron propagator $G_{\bar{e} e e}$ and two-point vertex function $\Gamma_{\bar{e} e e}$,

$$G_{\bar{e} e e}(-p, p) \Gamma_{\bar{e} e e}(-p, p) = -1,$$  \hspace{1cm} (8)
we have
\[ G_{\Lambda\lambda\lambda\lambda}^{\nu}(0,0,-p,p) = e^2 \Gamma^{\lambda\lambda\lambda\lambda}(-p,p) \left[ \frac{\partial^2}{\partial p_\mu \partial p_\nu} G^{\lambda\lambda\lambda\lambda}(-p,p) \right] \Gamma^{\lambda\lambda\lambda\lambda}(-p,p), \] (9)
according to (3). The r.h.s. of this equation can explicitly be expressed in terms of the electron self-energy \( \Sigma^{\lambda\lambda\lambda\lambda} \), which is related to \( \Gamma^{\lambda\lambda\lambda\lambda} \) by
\[ \Gamma^{\lambda\lambda\lambda\lambda}(-p,p) = i \left[ p - m_e + \Sigma^{\lambda\lambda\lambda\lambda}(p) \right] = i \left[ p + \sum_\sigma \Sigma^{\lambda\lambda\lambda\lambda}(p^2) - m_e + m_e \Sigma^{\lambda\lambda\lambda\lambda}(p^2) \right]. \] (10)
Here the sum over \( \sigma = +/\) runs over the right- and left-handed contributions \( \Sigma^{\lambda\lambda\lambda\lambda}_R \) and \( \Sigma^{\lambda\lambda\lambda\lambda}_L \) to the \( \lambda \) part of \( \Sigma^{\lambda\lambda\lambda\lambda} \), respectively, and \( \omega_\pm = \frac{1}{2}(1 \pm \gamma_5) \) are the chirality projectors. The actual evaluation of (10) is facilitated by shifting the derivatives from \( G^{\lambda\lambda\lambda\lambda} \) to \( \Gamma^{\lambda\lambda\lambda\lambda} \) with the identity
\[ \Gamma^{\lambda\lambda\lambda\lambda} \left[ \frac{\partial^2}{\partial p_\mu \partial p_\nu} \right] \Gamma^{\lambda\lambda\lambda\lambda} = \left[ \frac{\partial \Gamma^{\lambda\lambda\lambda\lambda}}{\partial p_\mu} \right] G^{\lambda\lambda\lambda\lambda} \left[ \frac{\partial \Gamma^{\lambda\lambda\lambda\lambda}}{\partial p_\nu} \right] + \left[ \frac{\partial \Gamma^{\lambda\lambda\lambda\lambda}}{\partial p_\nu} \right] G^{\lambda\lambda\lambda\lambda} \left[ \frac{\partial \Gamma^{\lambda\lambda\lambda\lambda}}{\partial p_\mu} \right] + \frac{\partial^2 \Gamma^{\lambda\lambda\lambda\lambda}}{\partial p_\mu \partial p_\nu}, \] (11)
which can be easily derived by taking derivatives from (3). The final result for the amputated Green function \( G_{\Lambda\lambda\lambda\lambda}^{\mu\nu}(0,0,-p,p) \) with photons of momentum zero reads
\[ G_{\Lambda\lambda\lambda\lambda}^{\mu\nu}(0,0,-p,p) = 2e^2 g^{\mu\nu} \left[ \frac{\Gamma^{\lambda\lambda\lambda\lambda}(0,0)}{p^2 - m_e^2 f(p^2) + i\sum_\sigma \Sigma^{\lambda\lambda\lambda\lambda}_\sigma(p^2) + im_e \Sigma^{\lambda\lambda\lambda\lambda}_S(p^2)} \right] \]
+ \( (\text{terms involving } p^\mu \text{ or } p^\nu) \), \] (12)
where
\[ f(p^2) = \frac{[1 - \Sigma^{\lambda\lambda\lambda\lambda}_S(p^2)]^2}{[1 + \Sigma^{\lambda\lambda\lambda\lambda}_R(p^2)][1 + \Sigma^{\lambda\lambda\lambda\lambda}_L(p^2)]}, \] (13)
and \( \Sigma'(p^2) \equiv \partial \Sigma(p^2)/\partial p^2 \). Here we have ignored terms with explicit factors \( p^\mu \text{ or } p^\nu \), because they turn out to be irrelevant for the \( S \)-matrix element, which is to be constructed in the last step.

The particles of the Compton process are labelled according to
\[ e^-(p,\kappa) \rightarrow e^-(p',\kappa') \rightarrow e^-(p,\kappa') \rightarrow e^-(p',\kappa'), \] (14)
where the parentheses contain the momenta \( p, k, k', k' \) and the polarizations \( \kappa, \lambda, \kappa', \lambda' \) of the respective particles. In the low-energy limit we have \( p' = p \) and \( k = k' = 0 \). We denote the electron spinors by \( u(p,\kappa) \) and \( \bar{u}'(p,\kappa') \), and the polarization vectors of the photons by \( \varepsilon(\lambda) \) and \( \varepsilon' \alpha(\lambda) \). The \( S \)-matrix element is directly obtained from (12) by going on shell with the electrons and multiplying with the corresponding wave functions. At this point, the on-shell renormalization \( \left[ 7 \right] \) (see also Ref. \( \left[ 14 \right] \)) comes into play. Within the on-shell scheme, \( m_e^2 \) is identified with the pole position in the electron propagator, and the wave-function renormalization of the right- and left-handed field components fixes the residue of the pole to 1. In this context, it should be noted that the electron self-energies (and their derivatives) at \( p^2 = m_e^2 \) are real quantities, since the electron is a stable particle. This implies that the pole of the propagator lies on the real axis. In terms of the renormalized self-energies \( \Sigma^{\lambda\lambda\lambda\lambda}_{R,L,S} \), the three on-shell conditions imply
\[ \Sigma^{\lambda\lambda\lambda\lambda}_S(m_e^2) = -\Sigma^{\lambda\lambda\lambda\lambda}_R(m_e^2) = -\Sigma^{\lambda\lambda\lambda\lambda}_L(m_e^2) = m_e^2 \left[ \Sigma^{\lambda\lambda\lambda\lambda}_R(m_e^2) + \Sigma^{\lambda\lambda\lambda\lambda}_L(m_e^2) + 2\Sigma^{\lambda\lambda\lambda\lambda}_S(m_e^2) \right]. \] (15)
We recall that the photon wave function is already properly renormalized, since the on-shell charge renormalization [6, 7, 8, 9] fixes the residue of the photon propagator to 1. Making use of the relations (15) and the Dirac equations for the electron spinors, all higher-order corrections in $G_{AA\bar{e}e}^{\mu\nu}$ of (12) drop out in the on-shell limit. The terms in (12) which contain explicit factors $p^\mu$ or $p^\nu$ vanish after contraction with the polarization vectors $\varepsilon$ and $\varepsilon'^*\bar{\epsilon}_e^{\nu}$, because we can choose their gauge such that $\varepsilon \cdot p = \varepsilon'^* \bar{\epsilon}_e^{\nu} \cdot p = 0$. The final result is

$$\langle e^-(p, \kappa'), \gamma(0, \lambda') | S | e^-(p, \kappa), \gamma(0, \lambda) \rangle = \varepsilon_{\nu}'(\lambda') \bar{u}(p, \kappa')G_{AA\bar{e}e}^{\mu\nu}(0, 0, -p, p)u(p, \kappa) \varepsilon_\mu(\lambda)$$

$$= \frac{ie^2}{m_e} \bar{u}(p, \kappa')u(p, \kappa) \varepsilon(\lambda) \cdot \varepsilon'^*\bar{\epsilon}_e^{\nu}(\lambda'), \quad (16)$$

which is the usual Thomson scattering amplitude. Averaging over the electron polarization, which is conserved, this yields the well-known Thomson cross-section

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{Thomson}} = \frac{\alpha^2}{m_e^2} |\varepsilon(\lambda) \cdot \varepsilon'^*\bar{\epsilon}_e^{\nu}(\lambda')|^2, \quad (17)$$

where $\alpha = e^2/4\pi$ denotes the fine-structure constant.

Before turning to the generalization of the above reasoning, it is mandatory to consider the question of photonic bremsstrahlung and the IR problem. We have implicitly assumed that IR divergences are regulated in a gauge-invariant manner, i.e. that the IR regulation preserves the Ward identities. Usually an infinitesimally small photon mass $m_\gamma$ is introduced in actual calculations, which preserves the Ward identities up to terms of $O(m_\gamma)$. The above derivation has shown that all virtual corrections vanish in the low-energy limit before the IR regulation is released. Indeed this is not the usual order in which kinematical limits of physical observables are taken. Taking these two limits in the same order for the bremsstrahlung, one observes that photon emission is impossible in the low-energy limit, because there is no phase space. In other words, the bremsstrahlung correction to Thomson scattering vanishes. This conclusion is based on the assumption that the exchange of the order of the two limits is allowed. This step can be justified by inspecting the form of the IR divergences at finite energies, which is explicitly known from QED [11], where the situation is the same for the Compton process. Indeed it turns out that the logarithmically IR-divergent terms can be separated order by order in the low-energy expansion. For a more detailed discussion of this subject, see Ref. [12].

The one-loop corrections and the corresponding photonic bremsstrahlung to Compton scattering are known both for QED [13] and for the electroweak SM [14]. In agreement with the considered theorem these corrections vanish in the low-energy limit, which can be explicitly checked by inspecting the analytical results given in the literature.

4 Generalization to other charged particles

The generalization of the above derivation to the scattering between a photon and any fermion $f$ with charge $Q_fe$ is quite obvious. One merely has to replace the electron field $e$ by $f$ and the charge $e$ by $-Q_fe$ in each step. While the above treatment remains exactly valid up to (12) for any fermion $f$, differences occur when constructing the $S$-matrix
element from the amputated Green function if \( f \) is unstable. In this case the \( f \) propagator possesses a complex pole, the position of which is not identical with the renormalized mass parameter \( m_f^2 \) in the on-shell renormalization scheme \([2, 10]\). It is well-known that this difference starts to play a role at the two-loop level so that mutatis mutandis \([16]\) is valid for all fermions \( f \) at least in the one-loop order. In this approximation also the above remarks on the photonic bremsstrahlung corrections and the IR problem apply. The generalization of the theorem to higher orders requires a proper definition of on-shell renormalization and of the \( S \)-matrix for unstable particles \([13]\), which is a highly non-trivial subject and clearly out of the scope of this article.

Next we turn to the process \( W^+\gamma \rightarrow W^+\gamma \) in the SM, where we meet analogous problems owing to the instability of the \( W \) bosons. Although the derivation of the low-energy theorem proceeds along the same lines as in the electron case, we again consider the single steps of the calculation in order to see where the restriction to the one-loop level becomes relevant. Moreover, the example of the \( W \) boson nicely illustrates the elegance of the BFM.

In a general ’t Hooft gauge for the background \( W \)-boson field, the relevant Ward identities are more involved owing to the mixing between the background gauge fields \( \hat{W}^\pm \) and the background Goldstone fields \( \hat{\phi}^\pm \). Therefore, we choose the unitary gauge for the \( \hat{W}^\pm \) fields, which is formally reached by sending the background gauge parameter \( \xi_W \) to infinity. In this limit the \( \hat{\phi}^\pm \) fields completely drop out. We recall that imposing the unitary gauge for a background gauge field in the BFM does not lead to any difficulties with renormalizability, as only tree-level quantities are concerned. This is in contrast to the conventional quantization procedure, where in general Green functions are non-renormalizable in the unitary gauge.

The desired Ward identity for the Green function \( G_{\mu\nu\rho\sigma}^{\hat{A}\hat{A}\hat{W}^+\hat{W}^-} \) is again obtained from the first relation of Eq. (11) in Ref. \([3]\). Note that in this equation the unitary gauge for the \( \hat{W}^\pm \) fields is reached by disregarding all terms involving \( \hat{\phi}^\pm \) and taking \( \xi_W \rightarrow \infty \). The final Ward identity possesses the same structure as (3) for the electron case:

\[
\frac{ik_1^2}{\xi_A} \frac{ik_2^2}{\xi_A} k_1^\mu k_2^\nu G_{\mu\nu\rho\sigma}^{\hat{A}\hat{A}\hat{W}^+\hat{W}^-} (k_1, k_2, k_+, k_-) = e^2 G_{\rho\sigma}^{\hat{W}^+\hat{W}^-} (k_+, -k_+) - e^2 G_{\rho\sigma}^{\hat{W}^+\hat{W}^-} (k_+ + k_1, k_- + k_2) - e^2 G_{\rho\sigma}^{\hat{W}^+\hat{W}^-} (k_+ + k_2, k_- + k_1) + e^2 G_{\rho\sigma}^{\hat{W}^+\hat{W}^-} (-k_-, k_-). \tag{18}
\]

The amputation of the external photon propagators according to (4) and (5) leads to

\[
\frac{ik_1^2}{\xi_A} \frac{ik_2^2}{\xi_A} k_1^\mu k_2^\nu G_{\mu\nu\rho\sigma}^{\hat{A}\hat{A}\hat{W}^+\hat{W}^-} (k_1, k_2, k_+, k_-) = k_{1,\mu} k_{2,\nu} G_{\mu\nu\rho\sigma}^{\hat{A}\hat{A}\hat{W}^+\hat{W}^-} (k_1, k_2, k_+, k_-). \tag{19}
\]

The low-energy limit of \( G_{\mu\nu\rho\sigma}^{\hat{A}\hat{A}\hat{W}^+\hat{W}^-} \) is obtained by taking derivatives from (18) and (19) with respect to \( k_{1,\mu} \) and \( k_{2,\nu} \),

\[
G_{\mu\nu\rho\sigma}^{\hat{A}\hat{A}\hat{W}^+\hat{W}^-} (0, 0, k_+, -k_+) = e^2 \frac{\partial^2}{\partial k_{+,\mu} \partial k_{+,\nu}} G_{\rho\sigma}^{\hat{W}^+\hat{W}^-} (k_+, -k_+). \tag{20}
\]
In the unitary gauge for the background W-boson fields, the amputation of each external \( W^\pm \) leg simply amounts to the multiplication of \( G_{A\tilde{A}W^+\tilde{W}^-}^{\mu\nu} \) with the inverse of the corresponding \( \tilde{W}^\pm \) propagator \( G_{\alpha\beta}^{\tilde{W}^+\tilde{W}^-} \), which is the negative of the \( \tilde{W}^\pm \) two-point function \( \Gamma_{\alpha\beta}^{\tilde{W}^+\tilde{W}^-} \),

\[
[G_{\alpha\beta}^{\tilde{W}^+\tilde{W}^-}(k_+, -k_+)]^{-1} = -\Gamma_{\alpha\beta}^{\tilde{W}^+\tilde{W}^-}(k_+, -k_+). \tag{21}
\]

Hence we have

\[
G_{A\tilde{A}W^+\tilde{W}^-}^{\mu\nu\rho\sigma}(0, 0, k_+, -k_+) = \frac{\partial^2}{\partial k_+^\mu \partial k_-^\rho} G_{\alpha\beta}^{\tilde{W}^+\tilde{W}^-}(k_+, -k_+)[\Gamma^{\tilde{W}^+\tilde{W}^-\beta\sigma}(k_+, -k_+). \tag{22}
\]

The higher-order corrections to \( \Gamma_{\alpha\beta}^{\tilde{W}^+\tilde{W}^-} \) are represented by the transverse and longitudinal self-energies \( \Sigma_T^{\tilde{W}^+\tilde{W}^-} \) and \( \Sigma_L^{\tilde{W}^+\tilde{W}^-} \), respectively,

\[
\Gamma_{\alpha\beta}^{\tilde{W}^+\tilde{W}^-}(k_+, -k_+) = -i\left(g_{\alpha\beta} - \frac{k_{\alpha\beta}}{k_+^2}\right) \left[k_+^2 - M_W^2 + \Sigma_T^{\tilde{W}^+\tilde{W}^-}(k_+)\right] + i\frac{k_{\alpha\beta}}{k_+^2} \left[M_W^2 - \Sigma_L^{\tilde{W}^+\tilde{W}^-}(k_+)\right]. \tag{23}
\]

In terms of these self-energies, (22) reads

\[
G_{A\tilde{A}W^+\tilde{W}^-}^{\mu\nu\rho\sigma}(0, 0, k_+, -k_+) = -2ie^2 g_{\mu\nu} g_{\rho\sigma} \left[1 + \Sigma_T^{\tilde{W}^+\tilde{W}^-}(k_+)\right]
- \frac{i e^2}{k_+^2} (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\rho\nu}) \left[k_+^2 - M_W^2 + \Sigma_T^{\tilde{W}^+\tilde{W}^-}(k_+)\right] \frac{k_+^2 + \Sigma_T^{\tilde{W}^+\tilde{W}^-}(k_+)^2 - \Sigma_T^{\tilde{W}^+\tilde{W}^-}(k_+)^2}{M_W^2 - \Sigma_L^{\tilde{W}^+\tilde{W}^-}(k_+)^2} + \left(\text{terms involving } k^\mu, k_+^\nu, k_+^\rho, \text{ or } k_+^\sigma\right). \tag{24}
\]

Up to this point no restriction to the one-loop level is necessary.

The \( S \)-matrix element of the process

\[
W^+(k_+, \lambda_+) + \gamma(k_+; \lambda) \rightarrow W^+(k'_+, \lambda'_+) + \gamma(k'_+; \lambda') \tag{25}
\]

for low-energetic photons, i.e. \( k = k' = 0 \) and \( k'_+ = k_+ \), is obtained from \( G_{A\tilde{A}W^+\tilde{W}^-}^{\mu\nu\rho\sigma} \) in (24) by taking \( k_+^2 \rightarrow M_W^2 \), contraction with the respective polarization vectors \( \varepsilon_{+\rho}(\lambda_+), \varepsilon_{+\sigma}(\lambda'_+), \varepsilon_{+\sigma}(\lambda), \) and multiplication with the wave-function correction factor \( \sqrt{R_{\tilde{W}}} \) for each external \( \tilde{W}^\pm \) field [1],

\[
\langle W^+(k_+, \lambda'_+) \big| S \big| W^+(k_+, \lambda_+) \rangle \gamma(0, \lambda) \rangle = R_{\tilde{W}} \varepsilon_{+\sigma}(\lambda'_+) \varepsilon_{+\rho}(\lambda) \frac{\Gamma_{\mu\nu\rho\sigma}^{A\tilde{A}W^+\tilde{W}^-}(0, 0, k_+, -k_+) \varepsilon_{+\rho}(\lambda_+) \varepsilon_{+\mu}(\lambda)}{\varepsilon_{+\rho}(\lambda'_+) \varepsilon_{+\sigma}(\lambda) \varepsilon_{+\sigma}(\lambda'_+)} \varepsilon_{+\rho}(\lambda_+) \varepsilon_{+\sigma}(\lambda) \varepsilon_{+\mu}(\lambda). \tag{26}
\]

The parts of \( G_{A\tilde{A}W^+\tilde{W}^-}^{\mu\nu\rho\sigma} \) containing explicit \( k'_+^\mu \) factors etc., which are already suppressed in (24), do not contribute to the \( S \)-matrix element because of the transversality of the W-boson polarization vectors, \( \varepsilon_+ \cdot k_+ = \varepsilon_+^\nu \cdot k_+ = 0 \), and the gauge choice for the photon
polarization vectors, $\varepsilon \cdot k_+ = \varepsilon' \cdot k_+ = 0$. For the other terms in (24) we have to inspect the explicit form of the renormalization conditions. In the BFM renormalization scheme of Ref. [9], the W-boson mass is determined by the usual on-shell condition [10]

$$\text{Re} \{\Sigma_T^{W+}(-M_W^2)\} = 0$$

(27)

for the transverse part of the renormalized self-energy. The wave-function correction factor $R_W$ [9] is given by

$$R_W = \left[1 + \text{Re} \{\Sigma_T^{W+}(-M_W^2)\}\right]^{-1}.$$  

(28)

Making use of these relations when going on shell in (24), one finds

$$\langle W^+(k_+, \lambda'_+), \gamma(0, \lambda') | S | W^+(k_+, \lambda_+), \gamma(0, \lambda) \rangle$$

$$= -2ie^2\varepsilon'^\mu_+\varepsilon^{\sigma_+}(\lambda'_+) \varepsilon^{\rho_+}(\lambda) \varepsilon_+ \varepsilon_\rho(\lambda) \left[g^{\mu\sigma}g^{\rho\sigma} + i \text{Re} \{\mathcal{O}(e^2)\}\right],$$

(29)

which implies that the corrections of relative order $\mathcal{O}(e^2)$ only yield an unobservable phase factor, because they do not interfere with the Born amplitude. Therefore, the squared $S$-matrix element reads

$$\left|\langle W^+(k_+, \lambda'_+), \gamma(0, \lambda') | S | W^+(k_+, \lambda_+), \gamma(0, \lambda) \rangle\right|^2$$

$$= 4e^4 |\varepsilon_+ (\lambda_+) \cdot \varepsilon'_{+}(\lambda'_+)|^2 |\varepsilon(\lambda) \cdot \varepsilon'(\lambda')|^2 + \mathcal{O}(e^8),$$

(30)

leading to the Thomson cross-section [17] after averaging over the (conserved) W-boson polarization. The bremsstrahlung corrections of relative order $\mathcal{O}(e^2)$ also vanish, as in the cases considered above. This is due to the fact that the IR behaviour of these corrections is exactly the same as for fermions (see for instance the last paper of Ref. [10]).

Finally, we point out that it is straightforward to carry over the above reasoning, which was explained in detail for the reactions $e^- \gamma \rightarrow e^- \gamma$ and $W^+ \gamma \rightarrow W^+ \gamma$, to other charged particles in various extensions of the SM. For instance, adding a scalar SU(2)×U(1) multiplet containing the charged boson $S^\pm$ with electric charge $\pm Q_S$, one obtains

$$\left|\langle S^\pm(k), \gamma(0, \lambda') | S | S^\pm(k), \gamma(0, \lambda) \rangle\right|^2 = 4Q^4_S e^4 |\varepsilon(\lambda) \cdot \varepsilon'(\lambda')|^2 + \mathcal{O}(e^8).$$

(31)

for $S^+ \gamma \rightarrow S^+ \gamma$ at low energies.

5 Summary

We have proved that the electroweak radiative corrections to Compton scattering vanish in the low-energy limit in all orders of perturbation theory. This generalizes Thirring’s theorem for QED to the electroweak Standard Model. The theorem guarantees that the on-shell definition of the electric charge of the electron corresponds to the electromagnetic coupling measured via Thomson scattering. An analogous theorem for photon scattering off W bosons has also been derived. However, the instability of the W bosons restricted the simple formulation of the theorem to the one-loop level.

Since these low-energy theorems are a consequence of gauge invariance and on-shell renormalization, their generalization from Abelian gauge theories such as QED to non-Abelian ones is non-trivial. We have formulated our derivation in the framework of the background-field method, rendering the formal proof very simple and transparent, because vertex and Green functions obey, in this formalism, QED-like Ward identities.
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