Space-time symmetry restoration in cosmological models with Kalb–Ramond and scalar fields

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Abstract

We study symmetry of space-time in presence of a minimally coupled scalar field interacting with a Kalb–Ramond tensor fields in a homogeneous but initially anisotropic universe. The analysis is performed for the two relevant cases of a pure cosmological constant and a minimal quadratic, renormalizable, interaction term. In both cases, due to expansion, a complete spatial symmetry restoration is dynamically obtained.

The most recent measurements of the anisotropy of cosmic background radiation (CBR) [1] strongly support the inflationary scenario for the dynamics of the early universe, where a scalar field, dominating the energy density of the primordial plasma, drives the fast expansion of the universe scale factor. If inflation occurs at very high energy, it is likely that this phenomenon experiences the non trivial structure of space time, if any, as the one predicted by noncommutative geometry. A simple way to implement effectively such a noncommutative structure has been proposed in Ref.s [2, 3], where an antisymmetric tensor (Kalb–Ramond) field takes into account the non-vanishing commutator of space-time coordinates.

In the present letter, we analyze the behaviour of a homogeneous but initially anisotropic universe, filled with a Kalb–Ramond tensor $\theta_{\mu\nu}$ which interacts with a scalar field, minimally coupled with gravity. In particular we look for space-time symmetry restoration, as an effect of the matter fields evolution, for two relevant cases, namely a pure cosmological constant and a quadratic renormalizable interaction term. For such an involved physical system, the knowledge of the constants of motion is particularly useful in order to solve the dynamics and in this concern, the procedure described in Ref.s [4, 5, 6, 7] to find the Noether symmetries can be applied.

In order to describe a field dynamics in a homogeneous but spatially anisotropic space-time, the appropriate metric is a Bianchi I. In this case the line element can be written as

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2(t)(dx^i)^2,$$

where $a_i$ is the scale factor for the $i$-th spatial direction. The corresponding nonvanishing connection coefficients are (no summation over $i$ is here meant)

$$\Gamma^0_{ij} = \delta_{ij} \dot{a}_i \dot{a}_i, \quad \Gamma^i_{0i} = \frac{\dot{a}_i}{a_i},$$

and for the Ricci tensor one has

$$R^0_0 = -\sum_{i=1}^{3} \frac{\ddot{a}_i}{a_i}, \quad R^i_j = -\delta^i_j \left( \frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_i}{a_i} \sum_{k \neq i} \frac{\dot{a}_k}{a_k} \right).$$
We consider a model with an antisymmetric tensor, \( \theta_{\mu\nu} \), responsible for the noncommutativity of space-time, and a minimally coupled scalar field, \( \varphi \), which drives the inflation [3]. The corresponding action reads

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{R}{16\pi G} + \frac{1}{12} H_{\mu\nu\sigma} H^{\mu\nu\sigma} + \frac{1}{2} \dot\varphi_{\mu}\dot\varphi^{\mu} - V(\varphi) - W(\varphi, \theta_{\mu\nu}\theta^{\mu\nu}) \right],
\]

where

\[
H_{\mu\nu\sigma} \equiv \nabla_\mu \theta_{\nu\sigma} + \nabla_\nu \theta_{\mu\sigma} + \nabla_\sigma \theta_{\mu\nu},
\]

is the field strength associated to the antisymmetric tensor \( \theta_{\mu\nu} = -\theta_{\nu\mu} \).

By using the space-time homogeneity we can safely assume that all fields depend on time only. Moreover, one can show that \( \theta_{0i} = \theta_{0i} = 0 \) and that it is possible to take only one component of \( \theta_{ij} \) to be non vanishing, like for example \( \theta_{12} = \theta \) [3]. Indeed this condition guarantees that the energy momentum tensor is diagonal. Therefore it should be imposed as a constraint since we assume a Bianchi I metric. In this case the Lagrangian density results

\[
\mathcal{L} = -M^2 a_1 a_2 a_{\varphi L} + V(\varphi) + \frac{1}{2} \dot\varphi_{\mu}\dot\varphi^{\mu} - W(\varphi, a_1 a_2 a_{\varphi} + \frac{1}{2} \dot\varphi_{\mu}\dot\varphi^{\mu})\]

where according to our notations \( a_1 = a_3 \) and \( M^2 = (8\pi G)^{-1} \).

As in [4, 5, 6, 7], we investigate the Noether symmetries of \( \mathcal{L} \) which can help in solving the dynamical system. We restrict our analysis to those transformations on the tangent space which preserve the second order character of field dynamics. To this aim, let us consider a generic vector field \( \mathbf{X} \) on the tangent space \( TQ \)

\[
\mathbf{X} = a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2} + \beta \frac{\partial}{\partial \varphi} + \gamma \frac{\partial}{\partial \theta} + \delta \frac{\partial}{\partial \dot\varphi} + \epsilon \frac{\partial}{\partial \dot\theta}
\]

and such that the corresponding Lie derivative vanishes

\[
L_{\mathbf{X}} \mathcal{L} = 0.
\]

It is easy to see that condition (8) leads to the following set of differential equations

\[
\left( \frac{a_1}{a_1} + \frac{a_2}{a_2} + \frac{\beta}{a_L} \right) (V + W) + \gamma (V_{\varphi} + W_{\varphi}) + \delta W_{\theta} + \alpha_1 W_{a_1} + \alpha_2 W_{a_2} = 0,
\]

\[
a_1 a_2 a_L + a_2 a_1 a_L + 2a_1 a_2 a_L \frac{\partial \gamma}{\partial \varphi} + \beta a_1 a_2 = 0,
\]

\[
-\alpha_1 a_2 a_L - \alpha_2 a_1 a_L + \beta a_1 a_2 + 2 a_L a_1 a_2 \frac{\partial \delta}{\partial \theta} = 0,
\]

\[
a_3 \frac{\partial a_2}{\partial a_1} + a_2 \frac{\partial \beta}{\partial a_1} = 0,
\]

\[
a_3 \frac{\partial a_1}{\partial a_2} + a_1 \frac{\partial \beta}{\partial a_2} = 0,
\]

\[
a_2 \frac{\partial a_1}{\partial a_3} + a_1 \frac{\partial a_2}{\partial a_3} = 0,
\]

\[
a_L \frac{\partial a_2}{\partial \varphi} + a_2 \frac{\partial \beta}{\partial \varphi} - \frac{a_1 a_2 a_L}{M^2} \frac{\partial \gamma}{\partial a_1} = 0,
\]

\[
a_L \frac{\partial a_1}{\partial \varphi} + a_1 \frac{\partial \beta}{\partial \varphi} - \frac{a_1 a_2 a_L}{M^2} \frac{\partial \gamma}{\partial a_2} = 0,
\]

The final step is to solve the system.
\[ a_2 \frac{\partial \alpha_2}{\partial \varphi} + a_1 \frac{\partial \alpha_1}{\partial \varphi} - \frac{a_1 a_2 a_L}{M^2} \frac{\partial \gamma}{\partial a_L} = 0 , \] (17)
\[ a_L \frac{\partial \alpha_2}{\partial \theta} + a_2 \frac{\partial \beta}{\partial \theta} = \frac{a_L}{a_1 a_2 M^2} \frac{\partial \delta}{\partial a_1} = 0 , \] (18)
\[ a_L \frac{\partial \alpha_1}{\partial \theta} + a_1 \frac{\partial \beta}{\partial \theta} = \frac{a_L}{a_1 a_2 M^2} \frac{\partial \delta}{\partial a_2} = 0 , \] (19)
\[ a_2 \frac{\partial \alpha_2}{\partial \theta} + a_1 \frac{\partial \alpha_1}{\partial \theta} - \frac{a_L}{a_1 a_2 M^2} \frac{\partial \delta}{\partial a_L} = 0 , \] (20)
\[ \frac{1}{a_1 a_2} \frac{\partial \delta}{\partial \varphi} + a_1 a_2 \frac{\partial \gamma}{\partial \theta} = 0 , \] (21)
\[ \alpha_1 + a_2 \frac{\partial \alpha_1}{\partial a_2} + a_L \frac{\partial \alpha_1}{\partial a_L} + a_1 \frac{\partial \alpha_2}{\partial a_2} + a_1 \frac{\partial \beta}{\partial a_3} = 0 , \] (22)
\[ \alpha_2 + a_2 \frac{\partial \alpha_2}{\partial a_1} + a_1 \frac{\partial \alpha_2}{\partial a_1} + a_L \frac{\partial \alpha_2}{\partial a_L} + a_2 \frac{\partial \beta}{\partial a_L} = 0 , \] (23)
\[ \beta + a_1 \alpha_2 + a_1 \alpha_2 + a_2 \frac{\partial \beta}{\partial a_2} = 0 . \] (24)

It is possible to show that the general solution of Eqs. (9)-(24) is given by
\[ \alpha_1 = A a_1 , \] (25)
\[ \alpha_2 = -A a_2 , \] (26)
\[ \beta = 0 , \] (27)
\[ \gamma = B , \] (28)
\[ \delta = C , \] (29)

where \( A, B, C \) are arbitrary constants. Thus the general vector fields \( X \), which satisfies \( L_X L = 0 \) is given by
\[ X = A X_\perp + B X_\varphi + C X_\theta , \] (30)

where the independent and commuting vectors are
\[ X_\perp = a_1 \frac{\partial}{\partial a_1} - a_2 \frac{\partial}{\partial a_2} + a_1 \frac{\partial}{\partial a_1} - \dot{a}_1 \frac{\partial}{\partial \dot{a}_2} , \quad X_\varphi = \frac{\partial}{\partial \varphi} , \quad X_\theta = \frac{\partial}{\partial \theta} . \] (31)

with the constraint
\[ A (a_1 \Omega_{a_1} - a_2 \Omega_{a_2}) + B \Omega_{a_1} + C \Omega_{a_2} = 0 , \] (32)

where \( \Omega = V + W \). Note that from covariance \( \Omega = \Omega (\varphi, \theta_{12} \theta_{12}) = \Omega (\varphi, \theta^2 / a^2_{12}) \) and thus the compatibility condition (32) becomes
\[ B \Omega_{a_1} + C \Omega_{a_2} = 0 , \] (33)

which is independent of \( A \). The constant of motion associated to \( X_\perp, X_\varphi, X_\theta \) are the following:
\[ K_\perp = -M^2 a_3 (a_1 \dot{a}_2 - a_2 \dot{a}_1) , \] (34)
\[ K_\theta = \frac{a_L}{a_1 a_2} \dot{\theta} , \] (35)
\[ K_\varphi = a_1 a_2 a_L \dot{\varphi} . \] (36)

in particular condition (33) implies that \( K_\perp \) is always preserved for any \( A \neq 0 \). Moreover, in order to satisfy Eq. (33) we have the following possibilities only:
a) $\Omega = \text{const}$. In this case we have three free parameters, namely $A$, $B$ and $C$ and thus three constants of motion $K_\perp$, $K_\varphi$ and $K_\theta$;

b) $B = 0$, $\Omega = \Omega(\varphi)$, and two generic values for $A$ and $C$. In this case one has the two constants of motion $K_\perp$ and $K_\varphi$.

c) $C = 0$ and $\Omega(\theta)$, and two generic values for $A$ and $B$. Hence two constants of motion $K_\perp$ and $K_\varphi$.

The presence of constants of motion allows to reduce the dimension of the configuration space and thus simplifies the dynamics of the system.

Let us consider as a relevant but simple model the case of a cosmological constant $\Omega = V + W = \text{const}$. In this case, the dynamical system can be reduced to a two dimensional one, and solved by using the equation of motion for the not cyclic variables together with the constraint energy equation which fixes the initial conditions. In particular one can choose a local set of coordinate transformations such that

$$q_1 \equiv \log(a_1), \quad q_2 \equiv \log(a_1 a_2), \quad a \equiv (a_1 a_2 a_L)^{1/3}. \quad (37)$$

In terms of these new variables the Lagrangian of Eq.(6) becomes

$$\mathcal{L} = a^3 \left\{ M^2 \left[ q_1'^2 + q_2'^2 - q_1 q_2 - 3 \frac{\dot{a}}{a} q_2 \right] + \frac{\dot{\varphi}^2}{2} + e^{-2q_2} \frac{\dot{\theta}^2}{2} - \Omega \right\}, \quad (38)$$

which is independent of $q_1$, $\varphi$ and $\theta$. As stated before, we have three dimensionless constants of motion

$$\hat{K}_\perp = \frac{K_\perp}{M^3} \equiv \frac{1}{M^3} \frac{\partial \mathcal{L}}{\partial q_1} = \frac{1}{M} a^3 \left[ 2q_1 - q_2 \right], \quad (39)$$

$$\hat{K}_\varphi = \frac{K_\varphi}{M^2} \equiv \frac{1}{M^2} \frac{\partial \mathcal{L}}{\partial \varphi} = \frac{1}{M^2} a^3 \varphi, \quad (40)$$

$$\hat{K}_\theta = \frac{K_\theta}{M^2} \equiv \frac{1}{M^2} \frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{M^2} a^3 e^{-2q_2} \dot{\theta}. \quad (41)$$

Hence we have

$$\dot{q}_1 = \frac{1}{2} \left[ \dot{q}_2 + M \frac{\hat{K}_\perp}{a^3} \right], \quad (42)$$

$$\varphi = M^2 \frac{\hat{K}_\varphi}{a^3}, \quad (43)$$

$$\dot{\theta} = M^2 \frac{\hat{K}_\theta}{a^3} e^{2q_2}. \quad (44)$$

Therefore, we have reduced the problem to a 2-dimensional one, since once $a(t), q_2(t)$ are found it is possible to get $q_1(t), \varphi(t), \theta(t)$ by direct integration of the previous equations. If we introduce the dimensionless time $\tau = M t$ and potential $\omega = \Omega/M^4$, the remaining set of equations for $a(\tau)$ and $q_2(\tau)$ to be solved now reads

$$q_2'' = -\frac{3}{4} q_2'^2 + \omega + \frac{1}{2 a^6} \left[ \frac{1}{2} \hat{K}_\perp^2 + \hat{K}_\varphi^2 + \hat{K}_\theta^2 e^{2q_2} \right], \quad (45)$$

$$\frac{a''}{a} = -2 \left( \frac{a'}{a} \right)^2 + 3 \left( \frac{a'}{a} \right) q_2' - \frac{3}{8} q_2'^2 + \frac{\omega}{2} + \frac{1}{4 a^6} \left[ \frac{1}{2} \hat{K}_\perp^2 + \hat{K}_\varphi^2 - \frac{1}{3} \hat{K}_\theta^2 e^{2q_2} \right], \quad (46)$$

where the prime-index stands for $d/d\tau$.

A comment is in turn. The quantity $\hat{K}_\perp = a^3 |H_1 - H_2|/M$ where $H_1 \equiv \dot{a}_1/a_1$ and $H_2 \equiv \dot{a}_2/a_2$
Figure 1: The Hubble parameters along the direction 1 and 2, namely $H_1$ (solid line), $H_2$ (dashed line) are here plotted versus $\tau$. It is also reported as a long-dashed line the behaviour of $H_L$. The evolution corresponds to the initial conditions $a(0) = 1, a'(0) = 1, q_2(0) = 1, q'_2(0) = 1$ is a measure of the initial spatial anisotropy in the 1-2 plane. For the solutions with increasing $a = a(t)$, the constancy of $K_\perp$ implies that $|H_1 - H_2| \to 0$. Thus we have an expanding universe that becomes asymptotically isotropic along 1, 2-directions and in the meanwhile, due to the constancy of $K_\theta$, the $\theta$-field, contributing to anisotropy goes to zero. The isotropization is complete since it involves $a_L$ as well. This can be easily understood by observing that for increasing $a = a(t)$ the contribution of the square brackets to the r.h.s. of Eq.s (45) and (46) asymptotically vanishes. In this condition $q'_2 \to 2(a'/a)$ and thus $H_1 = H_2 = H_L$ which means complete spatial isotropization.

Figure 1 shows the behaviour of the three directional Hubble parameters $H_1(\tau)$, $H_2(\tau)$, and $H_L(\tau)$ versus $\tau$, respectively. The evolution corresponds to the initial conditions $a(0) = 1, a'(0) = 1, q_2(0) = 1, q'_2(0) = 1$ but the main features shown for these particular choice of initial conditions are quite general. In particular, as can be seen by looking at the behaviour of $H_1$, $H_2$ and $H_L$, the presence of the cosmological constant $\omega$ asymptotically yields to a complete spatial isotropization and thus to a space-time symmetry restoration.

As a second relevant case we can assume for the potential $\Omega$ in the lagrangian density (6) the following form

$$\Omega(\varphi, \theta^{\mu\nu}) = \xi \varphi^2 \theta^{\mu\nu} = \xi \varphi^2 \frac{\theta^2}{a_1^2 a_2^2}.$$  \hfill (47)

In this case $K_\perp$ is the only constant of motion.

As in the previous case, the Lagrangian of Eq.(6) becomes

$$\mathcal{L} = a^3 \left\{ M^2 \left[ q_2^2 + q_1^2 - \dot{q}_2 \dot{q}_1 - 3\frac{\dot{a}}{a} \dot{q}_2 \right] + \frac{\dot{\varphi}^2}{2} + e^{-2q_2} \frac{\dot{\theta}^2}{2} - \xi e^{-2q_2} \varphi^2 \theta^2 \right\}.$$  \hfill (48)

Therefore, we have reduced the problem to a 4-dimensional one, since once $a(t)$, $q_2(t)$, $\varphi(t)$, $\theta(t)$ are found it is possible to get $q_1(t)$ by direct integration of eq. (42). As in the previous case, we
introduce the dimensionless time $\tau \equiv M t$ and $\tilde{\varphi}(\tau) = M \varphi, \tilde{\theta}(\tau) = \theta$, the remaining set of equations for $a(\tau), q_2(\tau), \tilde{\varphi}(\tau), \tilde{\theta}(\tau)$ to be solved now reads:

\begin{align}
\varphi'' &= -2\xi e^{-2q_2} \tilde{\varphi} \tilde{\theta}^2 - 3\varphi' \frac{a'}{a}, \\
\tilde{\theta}'' &= -2\xi \tilde{\varphi}^2 \tilde{\theta} - 3\tilde{\theta}' \frac{a'}{a} + 2q_2' \tilde{\theta}', \\
q_2'' &= -\frac{3}{4} q_2^2 - \frac{1}{2} \tilde{\theta}'^2 + \left[ \frac{1}{4 a^6} \tilde{K}_\perp^2 - \frac{1}{2} \tilde{\theta}'^2 e^{-2q_2} + \xi e^{-2q_2} \tilde{\varphi}^2 \tilde{\theta}^2 \right], \\
\frac{a''}{a} &= -2 \left( \frac{a'}{a} \right)^2 + \frac{3}{2} \left( \frac{a'}{a} \right) q_2' - \frac{3}{8} q_2'^2 - \frac{1}{4} \tilde{\theta}'^2 \\
&\quad + \left[ \frac{1}{8 a^6} \tilde{K}_\perp^2 + \frac{1}{12} \tilde{\theta}'^2 e^{-2q_2} - \frac{1}{6} \xi e^{-2q_2} \tilde{\varphi}^2 \tilde{\theta}^2 \right].
\end{align}

Figures 2 and 3 show the behaviour of $a_1(\tau), a_2(\tau), a_L(\tau)$ and $H_1(\tau), H_2(\tau), H_L(\tau)$ versus $\tau$. The evolution corresponds to $\xi = 0.5$ and to the initial conditions $a(0) = 1, a'(0) = 1, q_2(0) = 1, q_2'(0) = 1$ but the main features shown are quite general. Even in this case, the contribution of the square brackets to the r.h.s. of Eq.s (51) and (52) asymptotically vanishes and thus $q_2' \to 2(a'/a)$ and thus $H_1 = H_2 = H_L$, which again yields to complete spatial isotropization.

To conclude we have studied the space-time symmetry restoration in a cosmological model where a minimally coupled scalar field is interacting with a K-R field. The analysis has been performed in a Bianchi I universe and restricted to the simplest cases of a cosmological constant (not interacting fields) and in presence of a pure quadratic and renormalizable interaction term. In both cases, even if the two systems are characterized by a different number of constants of motion and thus by a different level of dynamical symmetry, an asymptotically complete spatial symmetry
Figure 3: The Hubble parameters along the direction 1 and 2, namely $H_1$ (solid line), $H_2$ (dashed line) are here plotted versus $\tau$. It is also reported as a long-dashed line the behaviour of $H_L$. The evolution corresponds to the initial conditions $a(0) = 1, a'(0) = 1, q_2(0) = 1, q'_2(0) = 1$.

restoration is obtained. This is due to the universe expansion which dilutes the contribution of the interaction terms responsible for a possible spatial anisotropy.

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