CONTINUITY OF ATTRACTORS FOR $C^1$ PERTURBATIONS OF A SMOOTH DOMAIN

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ABSTRACT. We consider a family of semilinear parabolic problems with nonlinear boundary conditions
\[
\begin{cases}
  u_t(x,t) = \Delta u(x,t) - au(x,t) + f(u(x,t)), & x \in \Omega_\epsilon \text{ and } t > 0, \\
  \frac{\partial u}{\partial N}(x,t) = g(u(x,t)), & x \in \partial \Omega_\epsilon \text{ and } t > 0,
\end{cases}
\]
where $\Omega_0 \subset \mathbb{R}^n$ is a smooth (at least $C^2$) domain, $\Omega_\epsilon = h_\epsilon(\Omega_0)$ and $h_\epsilon$ is a family of diffeomorphisms converging to the identity in the $C^1$-norm. Assuming suitable regularity and dissipative conditions for the nonlinearities, we show that the problem is well posed for $\epsilon > 0$ sufficiently small in a suitable scale of fractional spaces, the associated semigroup has a global attractor $A_\epsilon$ and the family $\{A_\epsilon\}$ is continuous at $\epsilon = 0$.

1. Introduction

Let $\Omega = \Omega_0 \subset \mathbb{R}^n$ be a $C^2$ domain, $a$ a positive number, $f, g : \mathbb{R} \to \mathbb{R}$ real functions, and consider the family of semilinear parabolic problems with nonlinear Neumann boundary conditions:
\[
\begin{cases}
  u_t(x,t) = \Delta u(x,t) - au(x,t) + f(u(x,t)), & x \in \Omega_\epsilon \text{ and } t > 0, \\
  \frac{\partial u}{\partial N}(x,t) = g(u(x,t)), & x \in \partial \Omega_\epsilon \text{ and } t > 0,
\end{cases} \tag{P_\epsilon}
\]
where $\Omega_\epsilon = \Omega_{h_\epsilon} = h_\epsilon(\Omega_0)$ and $h_\epsilon : \Omega_0 \to \mathbb{R}^n$ is a family of $C^m$, $m \geq 2$ maps satisfying suitable conditions to be specified later.

One of the central questions concerning this problem is the existence and properties of global attractors since, as it is well known, they determine the dynamics of the entire system (see, for example [8] or [19]). The continuity with respect to parameters present in the equation is also of interest, since it can be seen as a desirable property of “robustness” in the model. In many cases, however, the form of the equation is fixed, so the ‘parameter’of interest is the domain where the problem is posed.

The existence of a global compact attractor for the problem $P_\epsilon$ has been proved in [6] and [13], under stronger smoothness hypotheses on the domains and growth and dissipative conditions on the nonlinearities $f$ and $g$.

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The problem of existence and continuity of global attractors for semilinear parabolic problems, with respect to change of domains has also been considered in [3], for the problem with homogeneous boundary conditions

\[
\begin{aligned}
  u_t &= \Delta u + f(x,u) \quad \text{in} \quad \Omega_\epsilon \\
  \frac{\partial u}{\partial N} &= 0 \quad \text{on} \quad \partial \Omega_\epsilon,
\end{aligned}
\]

where \( \Omega_\epsilon, 0 \leq \epsilon \leq \epsilon_0 \) are bounded domains with Lipschitz boundary in \( \mathbb{R}^N, N \geq 2 \). There it is proved that, if the perturbations are such that the convergence of the eigenvalues and eigenfunctions of the linear part of the problem can be shown, than the upper semicontinuity of attractors follow. With the additional assumption that the equilibria are all hyperbolic, the lower semicontinuity is also obtained.

The behavior of the equilibria of \((P_\epsilon)\) was studied in [1] and [2]. In these papers, the authors consider a family of smooth domains \( \Omega_\epsilon \subset \mathbb{R}^N, N \geq 2 \) and \( 0 \leq \epsilon \leq \epsilon_0 \) whose boundary oscillates rapidly when the parameter \( \epsilon \to 0 \) and prove that the equilibria, as well as the spectra of the linearised problem around them, converge to the solution of a “limit problem”.

In [16] the authors prove the continuity of the attractors of \((P_\epsilon)\) with respect to \( C^2 \)-perturbations of a smooth domain of \( \mathbb{R}^n \).

These results do not extend immediately to the case considered here, due to the lack of smoothness of the domains considered and the fact that the perturbations do not converge to the inclusion in the \( C^2 \)-norm.

In this work, we follow the general approach of [16], which consists basically in “pull-backing” the perturbed problems to the fixed domain \( \Omega \) and then considering the family of abstract semilinear problems thus generated. We present a brief overview of this approach in the next section for convenience. Our aim here is then to prove well-posedness, establish the existence of a global attractor \( \mathcal{A}_\epsilon \) for sufficiently small \( \epsilon \geq 0 \) and prove that the family of attractors of is continuous at \( \epsilon = 0 \).

These results were obtained in our previous paper [5] for the family of perturbations of the unit square in \( \mathbb{R}^2 \) given by

\[
h_\epsilon(x_1, x_2) = (x_1, x_2 + x_2 \epsilon \text{sen}(x_1/\epsilon^\alpha))
\]

with \( 0 < \alpha < 1 \) and \( \epsilon > 0 \) sufficiently small, (see figure [1]).

![Figure 1. The perturbed region](image)

\[1\]
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In the present paper, we generalize these results in two directions: we consider the problem in arbitrary spatial dimension and, also, instead of a specific family of perturbations, we consider general families $h_\epsilon : \Omega_0 \to \mathbb{R}^n$ of $C^m$, $m \geq 2$ maps satisfying the following abstract hypotheses:

- $(H_1)$ $\|h_\epsilon - i_{\Omega_0}\|_{C^1(\Omega)} \to 0$ as $\epsilon \to 0$.
- $(H_2)$ The Jacobian determinant $Jh_\epsilon$ of $h_\epsilon$ is differentiable, and $\|\nabla Jh_\epsilon\|_\infty = \sup \{\|\nabla Jh_\epsilon(x)\|, x \in \Omega\} \to 0$ as $\epsilon \to 0$.

We show in section 4 that the family $h_\epsilon$ considered in [5] satisfies the conditions $(H_1)$ and $(H_2)$. Since the domain $\Omega$ is not of class $C^1$, the results obtained here do not immediately apply. However, since the perturbations occur only in a smooth portion of the boundary, they could easily be adapted to this case. We also give there more general examples of families satisfying our properties.

The paper is organized as follows: in section 2 we show how the problem can be reduced to a family of problems in the initial domain and collect some results needed later. In section 3 we give some rather general examples of families satisfying our basic assumptions. In section 4 we show that the perturbed linear operators are sectorial operators in suitable spaces and study properties of the linear semigroup generated by them. In section 5 we show that the problem can be reformulated as an abstract problem in a scale of Banach spaces which are shown to be locally well-posed in section 6 under suitable growth assumptions on $f$ and $g$. In section 7, assuming a dissipative condition for the problem, we use comparison results to prove that the solutions are globally defined and the family of associated semigroups are uniformly bounded. In section 8 we prove the existence of global attractors. In section 9 we show that these attractors behave upper semicontinuously. Finally, in section 10 with some additional properties on the nonlinearities and on the set of equilibria, we show that they are also lower semicontinuous at $\epsilon = 0$.

2. REDUCTION TO A FIXED DOMAIN

One of the difficulties encountered in problems of perturbation of the domain is that the function spaces change with the change of the region. One way to overcome this difficulty is to effect a “change of variables” in order to bring the problem back to a fixed region. This approach was developed by D. Henry in [9] and is the one we adopt here. We describe it briefly here, for convenience of the reader. For a different approach, see [1], [2] and [3].

Given an open bounded $C^m$ region $\Omega \subset \mathbb{R}^n$, $m \geq 1$, denote by $\text{Diff}^m(\Omega)$, $m \geq 0$, the set of $C^m$ embeddings (=diffeomorphisms from $\Omega$ to its image).

We define a topology in $\text{Diff}^m(\Omega)$, by declaring that $\Omega$ is in a $\epsilon$ neighborhood of $\Omega_0$, if $\Omega = h(\Omega_0)$, with $\|h\|_{C^m(\Omega_0)} < \epsilon$. It has been shown in [12] that this topology is metrizable and we denote by $\mathcal{M}_m(\Omega)$ or simply $\mathcal{M}_m$ this (separable) metric space. We say that a function $F$ defined in the space $\mathcal{M}_m$ with values in a Banach space is $C^m$ or analytic if $h \mapsto F(h(\Omega))$ is $C^m$ or analytic as a map of Banach spaces (h near $i_\Omega$ in $C^m(\Omega, \mathbb{R}^n)$). In this sense, we may express problems of perturbation of the boundary of a boundary value problem as problems of differential calculus in Banach spaces.

If $h : \Omega \mapsto \mathbb{R}^n$ is a $C^k$, $k \leq m$ embedding, we may consider the ‘pull-back’ of $h
\[ h^*: C^k(h(\Omega)) \rightarrow C^k(\Omega) \quad (0 \leq k \leq m) \]
defined by \( h^*(\varphi) = \varphi \circ h \), which is an isomorphism with inverse \( h^{-1*} \). Other function spaces can be used instead of \( C^k \), and we will actually be interested mainly in Sobolev spaces and fractional power spaces.

Now, if \( F_{h(\Omega)} : C^m(h(\Omega)) \rightarrow C^0(h(\Omega)) \) is a (generally nonlinear) differential operator in \( \Omega_h = h(\Omega) \) we may consider the operator \( h^*F_{h(\Omega)}h^{-1}^* \), which is a differential operator in the fixed region \( \Omega \).

Let now \( h_\varepsilon : \Omega_0 \rightarrow \mathbb{R}^n \) be a family of maps satisfying the conditions (H1) and (H2) and \( \Omega_\varepsilon = h_\varepsilon(\Omega) \) the corresponding family of “perturbed domains”.

**Lemma 2.1.** If \( \varepsilon > 0 \) is sufficiently small, the map \( h_\varepsilon \) belongs to \( \text{Diff}^m(\Omega) = \text{diffeomorphisms from } \Omega \text{ to its image.} \)

**Proof.** Straightforward.

**Lemma 2.2.** If \( 0 < s \leq m \) and \( \varepsilon > 0 \) is small enough, the map

\[ h_\varepsilon^*: H^s(\Omega_\varepsilon) \rightarrow H^s(\Omega) \quad u \mapsto u \circ h_\varepsilon \]
is an isomorphism, with inverse \( h_\varepsilon^{-1} = (h_\varepsilon^{-1})^* \).

**Proof.** See [5].

Using Lemma 2.1 we may bring the problem \( P_\varepsilon \) back to the fixed region \( \Omega_0 \). For this purpose, observe that \( v(.,t) \) is a solution \( \{P_\varepsilon\} \) in the perturbed region \( \Omega_\varepsilon = h_\varepsilon(\Omega) \), if and only if \( u(.,t) = h_\varepsilon^*v(.,t) \) satisfies

\[
\begin{aligned}
u_t(x,t) &= h_\varepsilon^*\Delta_{\Omega_\varepsilon} h_\varepsilon^{-1} u(x,t) - au(x,t) + f(u(x,t)), \quad x \in \Omega \text{ and } t > 0, \\
h_\varepsilon^* \frac{\partial}{\partial N_{\Omega_\varepsilon}} h_\varepsilon^{-1} u(x,t) &= g(u(x,t)), \quad x \in \partial \Omega \text{ and } t > 0,
\end{aligned}
\]

where \( h_\varepsilon^*\Delta_{\Omega_\varepsilon} h_\varepsilon^{-1} \) and \( h_\varepsilon^* \frac{\partial}{\partial N_{\Omega_\varepsilon}} h_\varepsilon^{-1} \) are defined by

\[ h_\varepsilon^*\Delta_{\Omega_\varepsilon} h_\varepsilon^{-1} u(x) = \Delta_{\Omega_\varepsilon}(u \circ h_\varepsilon^{-1})(h_\varepsilon(x)) \]

and

\[ h_\varepsilon^* \frac{\partial}{\partial N_{\Omega_\varepsilon}} h_\varepsilon^{-1} = \frac{\partial}{\partial N_{\Omega_\varepsilon}}(u \circ h^{-1})(hV(x)) \]

(in appropriate spaces). In particular, if \( \mathcal{A}_\varepsilon \) is the global attractor of \( \{P_\varepsilon\} \) in \( H^s(\Omega_\varepsilon) \), then \( \mathcal{A}_\varepsilon = \{v \circ h \mid v \in \mathcal{A}_\varepsilon\} \) is the global attractor of \( \{2\} \) in \( H^s(\Omega) \) and conversely. In this way we can consider the problem of continuity of the attractors as \( \varepsilon \rightarrow 0 \) in a fixed phase space.

For later use, we now compute an expression for the differential operator \( h_\varepsilon^*\Delta_{\Omega_\varepsilon} h_\varepsilon^{-1} \) in the fixed region \( \Omega \), in terms of \( h_\varepsilon \).
Writing \( h_\epsilon(x) = h_\epsilon(x_1, x_2, \ldots, x_n) = ((h_\epsilon)_1(x), (h_\epsilon)_2(x), \ldots, (h_\epsilon)_n(x)) = (y_1, y_2, \ldots, y_n) = y \), we obtain, for \( i = 1, 2, \ldots, n \)
\[
\left( h^*_\epsilon \frac{\partial}{\partial y_i} h^{-1}_\epsilon(u) \right)(x) = \frac{\partial}{\partial y_i} (u \circ h^{-1}_\epsilon)(h_\epsilon(x))
= \sum_{j=1}^n \left[ \left( \frac{\partial h_\epsilon}{\partial x_j} \right)^{-1} \right]_{j,i}(x) \frac{\partial u}{\partial x_j}(x) 
= \sum_{j=1}^n b'_{ij}(x) \frac{\partial u}{\partial x_j}(x),
\]
where \( b'_{ij}(x) \) is the \( i, j \)-entry of the inverse transpose of the Jacobian matrix of \( h_\epsilon \). From now on, we omit the \( \epsilon \) from the notation for simplicity. Therefore,
\[
h^*_\epsilon \Delta_{\Omega_\epsilon} h^{-1}_\epsilon(u)(x) = \sum_{i=1}^n \left( h^*_\epsilon \frac{\partial^2}{\partial y_i^2} h^{-1}_\epsilon(u) \right)(x)
= \sum_{i=1}^n \left( \sum_{k=1}^n b_{ik} \frac{\partial}{\partial x_k} \left( \sum_{j=1}^n b_{ij} \frac{\partial u}{\partial x_j} \right) \right)(x)
= \sum_{k=1}^n \frac{\partial}{\partial x_k} \left( \sum_{j=1}^n C_{kj} \frac{\partial u}{\partial x_j} \right)(x) - \sum_{j=1}^n \sum_{i=1}^n b_{ik} b_{ij} \frac{\partial}{\partial x_j} \frac{\partial u}{\partial x_j}(x)
= \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \sum_{j=1}^n C_{kj} \frac{\partial u}{\partial x_j} \right)(x) - \sum_{j=1}^n A_j \frac{\partial u}{\partial x_j}(x),
\]
where \( C_{kj} = \sum_{i=1}^n b_{ij} b_{ik} \) and \( A_j = \sum_{i=1}^n b_{ik} b_{ij} \).

We also need to compute the boundary condition \( h^*_\epsilon \frac{\partial}{\partial \nu_{\Omega_\epsilon}} h^{-1}_\epsilon u = 0 \) in the fixed region \( \Omega \) in terms of \( h_\epsilon \). Let \( N_{h_\epsilon(\Omega)} \) denote the outward unit normal to the boundary of \( h_\epsilon(\Omega) := \Omega_\epsilon \). From (3), we obtain
\[
\left( h^*_\epsilon \frac{\partial}{\partial N_{\Omega_\epsilon}} h^{-1}_\epsilon(u) \right)(x) = \sum_{i=1}^n \left( h^*_\epsilon \frac{\partial}{\partial y_i} h^{-1}_\epsilon(u) \right)(N_{\Omega_\epsilon})_i (h_\epsilon(x))
= \sum_{i=1}^n \frac{\partial}{\partial y_i} (u \circ h^{-1}_\epsilon)(h_\epsilon(x)) (N_{\Omega_\epsilon})_i (h_\epsilon(x))
= \sum_{i,j=1}^n b_{ij}(x) \frac{\partial u}{\partial x_j}(x) (N_{\Omega_\epsilon})_i (h_\epsilon(x))
\]
Since
\[
N_{\Omega_\epsilon}(h_\epsilon(x)) = h^*_\epsilon N_{\Omega}(x) = \frac{[h^{-1}_\epsilon]^T N_{\Omega}(x)}{||[h^{-1}_\epsilon]^T N_{\Omega}(x) ||}
\]
(see [9]), we obtain
(N_{\Omega_\epsilon}(h_\epsilon(x)))_i = \frac{1}{||[h_\epsilon^{-1}]_{x}^T N_\Omega(x)||} \sum_{k=1}^{n} b_{ik}(N_\Omega)_k.

Thus, from (5)

\left(h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u\right)(x) = \frac{1}{||[h_\epsilon^{-1}]_{x}^T N_\Omega(x)||} \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} b_{ik} b_{ij}(x) \frac{\partial u}{\partial x_j}(x)\right)(N_\Omega)_k

= \frac{1}{||[h_\epsilon^{-1}]_{x}^T N_\Omega(x)||} \sum_{k=1}^{n} \left(\sum_{j=1}^{n} C_{kj} \frac{\partial u}{\partial x_j}(x)\right)(N_\Omega)_k

(6)

Thus, the boundary condition \left(h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u\right)(x) = 0, becomes

\sum_{j,k=1}^{n} (N_\Omega(x))_k (C_{kj} D_j u) = 0 \text{ on } \partial\Omega.

so the boundary condition is exactly the "oblique normal derivative" with respect to the divergence part of the operator $h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}$.

3. Basic assumptions and examples on domain perturbations

We assume that the unperturbed domain $\Omega_0$ is of class $C^2$, and consider rather general examples of families $h_\epsilon : \Omega_0 \rightarrow \mathbb{R}^n$ of $C^2$ maps satisfying the hypotheses $(H_1)$ and $(H_2)$ stated in the introduction.

Example 3.1. The family $h_\epsilon$ of perturbations of the unit square in $\mathbb{R}^2$ considered in [5] given by

\begin{equation}
    h_\epsilon(x_1,x_2) = (x_1, x_2 + x_2 \epsilon \text{sen}(x_1/\epsilon^\alpha))
\end{equation}

with $0 < \alpha < 1$ and $\epsilon > 0$ sufficiently small, (see figure [7]), satisfy the conditions $(H_1)$ and $(H_2)$. We observe that the unperturbed region is not of class $C^2$ and, therefore, does not strictly satisfy our hypothesis. However, since the perturbation occurs only at a smooth portion of the boundary and the elliptic problem in this case is well posed (see [7]) the problem can actually be included in the framework considered here, with only minor modifications.

In fact, hypothesis $(H_1)$ was shown in [5] (Lemma 2.1). A simple computation gives $\nabla J_{h_\epsilon} = (\epsilon^{(1-\alpha)} \cos(x_1/\epsilon^\alpha), 0)$, from which $(H_2)$ follows easily.

From $(H_1)$, it follows that the boundary Jacobian $\mu_\epsilon = J_{\partial\Omega h_\epsilon|_{\partial\Omega}} \rightarrow 1$ uniformly as $\epsilon \rightarrow 0$.

It can be checked by explicitly computation, as done in [5]:

$$
\mu_\epsilon = \begin{cases}
    \sqrt{1 + \epsilon^{2-2\alpha} \cos^2(x_1/\epsilon^\alpha)} / [1 + \epsilon \sin(x_1/\epsilon^\alpha)] & \text{for } x \in I_1 := \{(x_1, 1) | 0 \leq x_1 \leq 1\}, \\
    1 / [1 + \epsilon \sin(x_1/\epsilon^\alpha)] & \text{for } x \in I_3 := \{(x_1, 0) | 0 \leq x_1 \leq 1\}, \\
    1 & \text{for } x \in I_2 := \{(1, x_2) | 0 \leq x_2 \leq 1\} \text{ and } I_4 := \{(0, x_2) | 0 \leq x_2 \leq 1\}.
\end{cases}
$$
Much more general families satisfying the conditions $(H_1)$ and $(H_2)$ are given in the examples below.

**Example 3.2.**

Let $\Omega \subset \mathbb{R}^n$ be a $C^2$ domain, and $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ a smooth (say $C^1$) vector field defined in an open set containing $\Omega$ and $x(t,x_0)$ the solution of

$$
\begin{cases}
\frac{dx}{dt} = X(x) \\
x(0) = x_0.
\end{cases}
$$

Then, the map

$$x : (t, \xi) \mapsto x(t, \xi) : (-r, r) \times \partial \Omega \to V \subset \mathbb{R}^n$$

is a diffeomorphism for some $r > 0$ and some open neighborhood $V$ of $\partial \Omega$. Let $W$ be a (smaller) open neighborhood of $\partial \Omega$, that is, with $\overline{W} \subset V$ and define $h_\epsilon : W \to \mathbb{R}^n$ by

$$h_\epsilon(x(t, \xi)) = (x(t + \eta(t) \cdot \theta_\epsilon(\xi), \xi)), \quad \text{where } \theta_\epsilon : \partial \Omega \to \mathbb{R}$$

is a $C^1$ function, with $\|\theta_\epsilon\|_{C^1(\partial \Omega)} \to 0$, as $\epsilon \to 0$, $\eta : [-r, r] \to [0, 1]$ is a $C^2$ function, with $\eta(0) = 1$ and $\eta(t) = 0$ if $|t| \geq \frac{\epsilon}{2}$. Observe that $h_\epsilon$ is well defined and $\{h_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$ is a family of $C^1$ maps for $\epsilon_0$ sufficiently small, with $\|h_\epsilon - \text{id} \circ_{B_\epsilon(\partial \Omega)} \|_{C^1(W)} \to 0$ as $\epsilon \to 0$. We may extend $h_\epsilon$ to a diffeomorphism of $\mathbb{R}^n$, satisfying $(H_1)$, which we still write simply as $h_\epsilon$ by defining it as the identity outside $W$.

If $\phi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a local coordinate system for $\partial \Omega$ in a neighborhood of $x_0 \in \partial \Omega$, then the map $\Psi(t, y) = x(t, \phi(y)) : (-r, r) \times U \to \mathbb{R}^n$ is a $C^1$ coordinate system around the point $x_0 \in \mathbb{R}^n$ and $\Psi^{-1}h_\epsilon \Psi(t, y) = (t + \eta(t)\theta_\epsilon(\phi(y)), y)$. By an easy computation, we find that the Jacobian of $\Psi^{-1}h_\epsilon \Psi$ is given by $J(\Psi^{-1}h_\epsilon \Psi(t, y)) = 1 + \eta'(t)\theta_\epsilon(\phi(y))$ and, therefore $Jh_\epsilon(x) = [1 + \eta'(t(x))\theta_\epsilon(\phi(\pi(x)))] : J\Psi(\Psi^{-1}(h_\epsilon(x))) : J\Psi^{-1}(x)$ for $x \in W$. Since $\|h_\epsilon - \text{id} \circ_{B_\epsilon(\partial \Omega)} \|_{C^1} \to 0$, the condition $(H_2)$ follows.

We can also compute $J_{p\Omega}h_\epsilon|_{p\Omega}$, the Jacobian of $h_\epsilon$ restricted to $\partial \Omega$. We drop the subscript $\partial \Omega$ to simplify the notation. Note that the coordinate system $\Psi$ above takes $\{0\} \times U$ into a neighborhood of $x_0 \in \partial \Omega$, and $\Psi^{-1}h_\epsilon|_{p\Omega}(0, y) = (\theta_\epsilon(\phi(y)), y)$.

A straightforward computation then gives $J(\Psi^{-1}h_\epsilon|_{p\Omega}(0, y)) = \sqrt{1 + \|
abla \theta_\epsilon(\phi(y))\|^2}$ and, therefore $Jh_\epsilon|_{p\Omega}(\phi(y)) = \left[1 + \|
abla \theta_\epsilon(\phi(y))\|^2 \right] J\Psi(\Psi^{-1}(h_\epsilon(\phi(y)))) : J\Psi^{-1}(\Psi(0, y))$ for $y \in U$, where $\Psi_0$ and $\Psi_\epsilon$ denote the restriction of $\Psi$ to $\{(0, y) | y \in U\}$ and $\{(\theta_\epsilon(\phi(y)), y) | y \in U\}$, respectively. Since $\|h_\epsilon - \text{id} \circ_{C^1} \|_{C^1}$ and $\|\theta_\epsilon(\xi)\|_{C^1(\partial \Omega)} \to 0$, it follows that $Jh_\epsilon|_{p\Omega}(\phi(y)) \to 1$ as $\epsilon \to 0$, uniformly in $\partial \Omega$. \hfill $\square$

**Example 3.3.**

We can choose the vector field $X$ in the previous example as an extension of $N : \partial \Omega \to \mathbb{R}^n$ the unit outward normal to $\partial \Omega$, $t(x) = \pm \text{dist}(x, \partial \Omega)$, $t$ outside, $-t$ inside, $\phi(x) = \phi(x)$ the point of $\partial \Omega$ nearest to $x$ and $B_\epsilon(\partial \Omega) = \{x \in \mathbb{R}^n | \text{dist}(x, \partial \Omega) < \epsilon\}$.

Then, the map $\rho : (t, \xi) \mapsto \xi + tN(\xi) : (-r, r) \times \partial \Omega \to B_r(\partial \Omega)$ is a diffeomorphism, for some $r > 0$, with inverse $x \mapsto (t(x), \pi(x))$ (see [9]).

Define $h_\epsilon : B_r(\partial \Omega) \to \mathbb{R}^n$ by $h_\epsilon(\rho(t, \xi)) = \xi + tN(\xi) + \eta(t)\theta_\epsilon(\xi)N(\xi) = \rho(t, \xi) + \eta(t)\theta_\epsilon(\xi)N(\xi)$, where $\theta_\epsilon : \partial \Omega \to \mathbb{R}$ is a $C^1$ function, with $\|\theta_\epsilon\|_{C^1(\partial \Omega)} \to 0$ as $\epsilon \to 0$, $\eta : [-r, r] \to [0, 1]$ is a
\(C^2\) function, with \(\eta(0) = 1\) and \(\eta(t) = 0\) if \(|t| \geq \frac{\epsilon}{2}\). Then, \(\{h_\epsilon, \ 0 \leq \epsilon \leq \epsilon_0\}\) is a family of \(C^1\) maps for \(\epsilon_0\) sufficiently small, with \(\|h_\epsilon - i B_\epsilon(\partial \Omega)\|_{C^1} \to 0\) as \(\epsilon \to 0\). We may extend \(h_\epsilon\) to a diffeomorphism of \(\mathbb{R}^n\), satisfying (H1), which we still write simply as \(h_\epsilon\) by defining it as the identity outside \(B_\epsilon(\partial \Omega)\).

If \(\phi : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n\) is a local coordinate system for \(\partial \Omega\) in a neighborhood of \(x_0 \in \partial \Omega\), then the map \(\Psi(t, y) = \phi(y) + t N(\phi(y)) = \rho(t, \phi(y)) : (-r, r) \times U \to \mathbb{R}^n\) is a \(C^1\) coordinate system around the point \(x_0 \in \mathbb{R}^n\) and \(\Psi^{-1}h_\epsilon \Psi(t, y) = (t + \eta(t) \theta_\epsilon(\phi(y)), y)\). The condition (H2) can now be checked as in the previous example.

**Remark 3.4.** We may choose the function \(\theta_\epsilon\) with “oscillatory behavior”, so the example above essentially includes the case considered in [5], since the perturbation there is nonzero only in a smooth portion of the boundary.

4. The Linear Semigroup

In this section we consider the linear semigroups generated by the family of differential operators \(-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{-1} + aI\), appearing in [2].

4.1. Strong form in \(L^p\) spaces. Consider the operator in \(L^p(\Omega), \ p \geq 2\), given by

\[
A_\epsilon := \left( -h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{-1} + aI \right) \tag{8}
\]

with domain

\[
D(A_\epsilon) = \left\{ u \in W^{2,p}(\Omega) \left| h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{-1} u = 0, \ \text{on} \ \partial \Omega \right. \right\}. \tag{9}
\]

(We will denote simply by \(A\) the unperturbed operator \((-\Delta_{\Omega} + aI))\).

**Theorem 4.1.** If \(\epsilon > 0\) is sufficiently small and \(h_\epsilon \in \text{Diff}^1(\Omega)\), then the operator \(A_\epsilon = (-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{-1} + aI)\) defined by (8) and (9) is sectorial.

**Proof.** Consider the operator \(-\Delta_{\Omega_\epsilon}\) defined in \(L^p(h_\epsilon(\Omega))\), with domain

\[
D(-\Delta_{\Omega_\epsilon}) = \left\{ u \in W^{2,p}(\Omega_\epsilon) \left| \frac{\partial}{\partial N_{\Omega_\epsilon}} u = 0, \ \text{on} \ \partial \Omega_\epsilon \right. \right\},
\]

where \(\Omega_\epsilon = h_\epsilon(\Omega)\). It is well known that \(-\Delta_{\Omega_\epsilon}\) is sectorial, with the spectra contained in the interval \([0, \infty) \subset \mathbb{R}\).

If \(\lambda \in \mathbb{C}\) and \(f \in L^2(\Omega)\), we have

\[
(h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{-1} + \lambda I) u(x) = f(x) \tag{10}
\]

\[
\iff (\Delta_{\Omega_\epsilon} + \lambda I) u \circ h_\epsilon^{-1}(h_\epsilon(x)) = f \circ h_\epsilon^{-1}(h_\epsilon(x))
\]

\[
\iff (\Delta_{\Omega_\epsilon} + \lambda I) v(y) = g(y),
\]

Since \(u \mapsto h_\epsilon^* u := u \circ h_\epsilon\) is an isomorphism from \(L^2(\Omega_\epsilon)\) to \(L^2(\Omega)\) with inverse \((h_\epsilon^{-1})^*\), it follows that the first equation is uniquely solvable in \(L^2(\Omega)\) if and only if the last equation is uniquely solvable in \(L^2(\Omega_\epsilon)\).

Suppose \(\lambda\) belongs to \(\rho(-\Delta_{\Omega_\epsilon})\), the resolvent set of \(-\Delta_{\Omega_\epsilon}\). Then, we have.
\[
\|u\|_{L^p(\Omega)}^p = \int_\Omega |u(x)|^p \, dx \\
= \int_\Omega |v \circ h_\epsilon(x)|^p \, dx \\
= \int_{\Omega_\epsilon} |v(y)|^p |Jh_\epsilon^{-1}(y)| \, dy \\
\leq \|Jh_\epsilon^{-1}\|_\infty \|v\|_p^p \\
\leq \|Jh_\epsilon^{-1}\|_\infty \cdot \| (\Delta_{\Omega_\epsilon} + \lambda I)^{-1} \|_{\mathcal{L}(L^p(\Omega_\epsilon))} \cdot \|g\|_{L^p(\Omega_\epsilon)}^p
\]

On the other hand

\[
\|g\|_{L^p(\Omega_\epsilon)}^p = \int_{\Omega_\epsilon} |g(x)|^p \, dy \\
= \int_{\Omega_\epsilon} |f \circ h_\epsilon^{-1}(y)|^p \, dy \\
= \int_\Omega |f(x)|^p |Jh_\epsilon(x)| \, dx \\
\leq \|Jh_\epsilon\|_\infty \|f\|_{L^p(\Omega)}^p
\]

It follows that

\[
\|u\|_{L^p(\Omega)}^p \leq \|Jh_\epsilon\|_\infty \cdot \|Jh_\epsilon^{-1}\|_\infty \cdot \| (\Delta_{\Omega_\epsilon} + \lambda I)^{-1} \|_{\mathcal{L}(L^p(\Omega_\epsilon))} \cdot \|g\|_{L^p(\Omega_\epsilon)}^p
\]

Therefore, \( \lambda \in \rho(-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}) \) and

\[
\| (h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} + \lambda I)^{-1} \|_{\mathcal{L}(L^p(\Omega))} \leq \|Jh_\epsilon\|_\infty \cdot \|Jh_\epsilon^{-1}\|_\infty \cdot \| (\Delta_{\Omega_\epsilon} + \lambda I)^{-1} \|_{\mathcal{L}(L^p(\Omega_\epsilon))}\]  \( (11) \)

Reciprocally, one can prove similarly that \( \lambda \in \rho(-h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1}) \Rightarrow \lambda \in \rho(-\Delta_{\Omega_\epsilon}) \).

Finally, if \( B_\epsilon = -\Delta_{\Omega_\epsilon} + aI \) is sectorial with \( \|(\lambda - B_\epsilon^{-1})\| \leq \frac{M}{|\lambda - a'|} \) for all \( \lambda \) in the sector \( S_{a',\phi_0} = \{ \lambda \mid \phi_0 \leq |\arg(\lambda - a')| \leq \pi, \lambda \neq a' \} \), for some \( a' \in \mathbb{R} \) and \( 0 \leq \phi_0 < \pi/2 \), it follows from (11) that \( A_\epsilon = a - h_\epsilon^* \Delta_{\Omega_\epsilon} h_\epsilon^{*-1} \) satisfies \( \|(\lambda - A)^{-1}\| \leq \frac{M'}{|\lambda - a'|} \) for all \( \lambda \) in the sectoriality of \( A_\epsilon \) follows from the sectoriality of \( B_\epsilon \).

\[
\boxed{}
\]

**Remark 4.2.** From 4.1 and results in [10], it follows that \( A_\epsilon \) generates a linear analytic semigroup in \( L^p(\Omega) \), for each \( \epsilon \leq 0 \).
4.2. Weak form in $L^p$ spaces. One would like to prove that the operators $A_\epsilon$ defined by (8) and (9) become close to the operator $A$ as $\epsilon \to 0$ in a certain sense. This is possible when the perturbation diffeomorphisms $h_\epsilon$ converge to the identity in the $C^2$-norm (see, for example [14] and [16]). To obtain similar results here, we need to consider the problem in weaker topologies, that is, we need to extend those operators. To this end, we now want to consider the operator $A_\epsilon = (-h_\epsilon^* \Delta A h_\epsilon^{-1} + aI)$ as an operator $\widetilde{A}_\epsilon$ in $(W^{1,q}(\Omega))'$ with $D(\widetilde{A}_\epsilon) = W^{1,p}(\Omega)$, where $q$ is the conjugate exponent of $p$, that is $\frac{1}{p} + \frac{1}{q} = 1$.

If $u \in D(A_\epsilon) = \left\{ u \in W^{2,p}(\Omega) \mid h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{-1} u = 0 \right\}$, $\psi \in W^{1,q}(\Omega)$, and $v = u \circ h_\epsilon^{-1}$, we obtain, integrating by parts

$$
\langle A_\epsilon u, \psi \rangle_{-1,1} = -\int_{\Omega} (h_\epsilon^* \Delta A h_\epsilon^{-1} u)(x) \psi(x) \, dx + a \int_{\Omega} u(x) \psi(x) \, dx
$$

$$
= -\int_{\Omega} \Delta_{\Omega_\epsilon} (u \circ h_\epsilon^{-1})(h_\epsilon(x)) \psi(x) \, dx + a \int_{\Omega} u(x) \psi(x) \, dx
$$

$$
= -\int_{\Omega} \Delta_{\Omega_\epsilon} v(y) \psi(h_\epsilon^{-1}(y)) \, dy + a \int_{\Omega} u(h_\epsilon^{-1}(y)) \psi(h_\epsilon^{-1}(y)) \, dy
$$

$$
= -\int_{\Omega} \frac{\partial v}{\partial N_{\Omega_\epsilon}}(y) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, d\sigma(y)
$$

$$
+ a \int_{\Omega} u(h_\epsilon^{-1}(y)) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, dy
$$

$$
= \int_{\Omega} \nabla_{\Omega_\epsilon} v(y) \cdot \nabla_{\Omega_\epsilon} \left[ \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \right] \, dy
$$

$$
+ a \int_{\Omega} u(h_\epsilon^{-1}(y)) \psi(h_\epsilon^{-1}(y)) \frac{1}{|Jh_\epsilon(h_\epsilon^{-1}(y))|} \, dy
$$

$$
= \int_{\Omega} \nabla_{\Omega_\epsilon} v(h_\epsilon(x)) \cdot \nabla_{\Omega_\epsilon} \left[ \psi \circ h_\epsilon^{-1} \frac{1}{|Jh_\epsilon \circ h_\epsilon^{-1}|}(h_\epsilon(x)) \right] \, dx + a \int_{\Omega} u(x) \psi(x) \, dx
$$

$$
= \int_{\Omega} \left( h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} u \right)(x) \cdot \left[ h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} \psi \frac{1}{Jh_\epsilon} \right](x) \, dx + a \int_{\Omega} u(x) \psi(x) \, dx
$$

$$
= \int_{\Omega} \left( h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} u \right)(x) \cdot h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} \psi(x) \, dx + a \int_{\Omega} u(x) \psi(x) \, dx
$$

$$
+ \int_{\Omega} \left( h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} u \right)(x) \cdot \left( h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} Jh_\epsilon \right)(x) \cdot \frac{1}{Jh_\epsilon} \cdot \psi(x) \, dx. \quad (12)
$$

Since (12) is well defined for $u \in W^{1,p}(\Omega)$, we may define an extension $\widetilde{A}_\epsilon$ of $A_\epsilon$, with domain $W^{1,p}(\Omega)$ and values in $(W^{1,q}(\Omega))'$, by

$$
\langle \widetilde{A}_\epsilon u, \psi \rangle_{-1,1} := \int_{\Omega} \left( h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} u \right)(x) \cdot h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{-1} \psi(x) \, dx + a \int_{\Omega} u(x) \psi(x) \, dx
$$
Remark 4.3. If $u$ is regular enough, then $\tilde{A}u = Au$ implies that $u$ must satisfy the boundary condition $h_\epsilon^* \frac{\partial}{\partial N_{\Omega_\epsilon}} h_\epsilon^{*-1} u = 0$, on $\partial \Omega$ but, since this is not well defined in $(W^{1,q}(\Omega))$, the domain of $\tilde{A}$ does not incorporate this boundary condition.

For simplicity, we still denote this extension by $A_\epsilon$, whenever there is no danger of confusion. Also, from now on, we drop the absolute value in $|Jh_\epsilon(x)|$, since the Jacobian of $h_\epsilon$ is positive for sufficiently small $\epsilon$.

We now prove the following basic inequality.

**Theorem 4.4.** $D(A_\epsilon) \supset D(A)$ for any $\epsilon \geq 0$ and there exists a positive function $\tau(\epsilon)$ such that

$$
|| (A_\epsilon - A)u ||_{W^{1,q}(\Omega)} \leq \tau(\epsilon) ||Au||_{W^{1,q}(\Omega)},
$$

for all $u \in D(A)$, with $\lim_{\epsilon \to 0^+} \tau(\epsilon) = 0$.

**Proof.** The assertion about the domain is immediate. The inequality is equivalent to

$$
\langle (A_\epsilon - A)u, \psi \rangle_{-1,1} \leq \tau(\epsilon) ||Au||_{W^{1,q}(\Omega)} ||\psi||_{W^{1,q}(\Omega)},
$$

for all $u \in W^{1,p}(\Omega)$, $\psi \in W^{1,q}(\Omega)$, with $\lim_{\epsilon \to 0^+} \tau(\epsilon) = 0$. We have, for $\epsilon > 0$.

$$
\langle (A_\epsilon - A)u, \psi \rangle_{-1,1} = \int (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x) \cdot [(h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} \psi)(x) - (\nabla_{\Omega_\epsilon} \psi)(x)] \, dx
$$

$$
+ \int (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u - \nabla_{\Omega} u)(x) \cdot (\nabla_{\Omega_\epsilon} \psi)(x) \, dx
$$

$$
+ \int (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u)(x) \cdot (h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} Ju_\epsilon)(x) \cdot \frac{1}{Jh_\epsilon} \cdot \psi(x) \, dx
$$

(14)

Now, writing $|v|_p = (\sum_{i=1}^n |v_i|^p)^{\frac{1}{p}}$, $1 \leq p < \infty$, $|v|_\infty = \sup(|v_i|, i = 1, 2, \cdots, n)$ for the $p$-norm of the vector $v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$, we observe that

$$
|h_\epsilon^* \nabla_{\Omega_\epsilon} h_\epsilon^{*-1} u(x)|_p = \left( \sum_i \left| h_\epsilon^* \frac{\partial}{\partial y_i} h_\epsilon^{*-1} u(x) \right|^p \right)^{\frac{1}{p}} = \left( \sum_i \left( \sum_j b_{i,j}(x) \frac{\partial u}{\partial x_j}(x) \right)^p \right)^{\frac{1}{p}}
$$

$$
\leq \left[ \sum_i \left( \sum_j \left| b_{i,j}(x) \right|^q \right)^{\frac{p}{q}} \left( \sum_j \left( \left| \frac{\partial u}{\partial x_j}(x) \right| \right)^p \right)^{\frac{1}{p}} \right]^{\frac{1}{p}}
$$

$$
\leq \left[ \sum_i \left( \sum_j \left| b_{i,j}(x) \right|^q \right)^{p-1} \right]^{\frac{1}{p}} \left| \nabla u(x) \right|_p
$$
\[| h_\epsilon^* \nabla_{\Omega, \epsilon} h_\epsilon^{-1} u(x) - \nabla u(x)|_p = \left( \sum_i \left| h_\epsilon^* \frac{\partial}{\partial y_i} h_\epsilon^{-1} u(x) - \frac{\partial u}{\partial x_i} (x) \right|^p \right)^{\frac{1}{p}} \]

\[= \left( \sum_i \left( \sum_j \left| (b_{i,j}(x) - \delta_{i,j}) \frac{\partial u}{\partial x_j} (x) \right|^p \right) \right)^{\frac{1}{p}} \]

\[\leq \left[ \sum_i \left( \sum_j |(b_{i,j}(x) - \delta_{i,j})|^q (x) \right) \left( \sum_j \left| \frac{\partial u}{\partial x_j} (x) \right|^p \right) \right]^{\frac{1}{p}} \]

\[\leq \left[ \sum_i \left( \sum_j |(b_{i,j}(x) - \delta_{i,j})|^q (x) \right) \right]^{\frac{1}{p}} \left| \nabla u(x) \right|_p \]

\[\leq \|(b^\epsilon - \delta)\|_\infty \left[ \sum_i n^{p-1} \right]^{\frac{1}{p}} \left| \nabla u(x) \right|_p \]

\[\leq n \|(b^\epsilon - \delta)\|_\infty |\nabla u(x)|_p \]

\[\leq \eta(\epsilon) |\nabla u(x)|_p \]

\[\frac{1}{J_{h_\epsilon}(x)} | h_\epsilon^* \nabla_{\Omega, \epsilon} h_\epsilon^{-1} J_{h_\epsilon}(x) |_\infty = \frac{1}{J_{h_\epsilon}(x)} \sup_i \left\{ \left| h_\epsilon^* \frac{\partial}{\partial y_i} h_\epsilon^{-1} J_{h_\epsilon}(x) \right| \right\} \]

\[= \frac{1}{J_{h_\epsilon}(x)} \sup_i \left\{ \sum_j \left| b_{i,j}(x) \frac{\partial J_{h_\epsilon}(x)}{\partial x_j} \right| \right\} \]

\[= \frac{1}{J_{h_\epsilon}(x)} \| b^\epsilon \|_\infty \sum_j \left| \frac{\partial J_{h_\epsilon}(x)}{\partial x_j} \right| \]

\[\leq \frac{1}{J_{h_\epsilon}(x)} \| b^\epsilon \|_\infty |\nabla J_{h_\epsilon}(x)|_1 \leq n \| b^\epsilon \|_\infty |\nabla J_{h_\epsilon}(x)|_\infty \]

\[\leq \frac{1}{J_{h_\epsilon}(x)} B(\epsilon) |\nabla J_{h_\epsilon}(x)|_\infty \leq \frac{1}{J_{h_\epsilon}(x)} B(\epsilon) |\nabla J_{h_\epsilon}|_\infty \]

\[\leq \mu(\epsilon) \]
\[
\frac{1}{Jh_\epsilon(x)} \left| h_\epsilon^* \nabla \Omega, h_\epsilon^{*-1} Jh_\epsilon(x) \psi(x) \right|_q = \frac{1}{Jh_\epsilon(x)} \left( \sum_i \left| h_\epsilon^* \frac{\partial}{\partial y_i} h_\epsilon^{*-1} Jh_\epsilon(x) \cdot \psi(x) \right|^q \right)^{\frac{1}{q}} \\
\leq \frac{1}{Jh_\epsilon(x)} \left| h_\epsilon^* \nabla h_\epsilon^{*-1} Jh_\epsilon(x) \right|_\infty \left( \sum_i \left| \psi(x) \right|^q \right)^{\frac{1}{q}} \\
\leq n \mu(\epsilon) \psi(x),
\]
where \( \|b\|_\infty = \sup \{ |b_{i,j}|(x), 1 \leq i, j \leq n, x \in \Omega \}, \|b^e - \delta\|_\infty = \sup \{ |b_{i,j}^e - \delta_{i,j}|(x), 1 \leq i, j \leq n, x \in \Omega \}, B(\epsilon) \to n \) and \( \eta(\epsilon), \mu(\epsilon) \to 0 \), as \( \epsilon \to 0 \). by hypotheses \( H_1 \) and \( H_2 \).

In a similar way, we obtain

- \( \left| h_\epsilon^* \nabla \Omega, h_\epsilon^{*-1} \psi(x) \right|_p \leq B(\epsilon) |\nabla \psi(x)|_p \),
- \( \left| h_\epsilon^* \nabla \Omega, h_\epsilon^{*-1} \psi(x) - \nabla \psi(x) \right|_p \leq \eta(\epsilon) |\nabla \psi(x)|_p \),

It follows that

\[
\left| \langle (A_\epsilon - A) u, \psi \rangle \right| \leq B(\epsilon) \left[ \int_\Omega |\nabla u(x)|_p^q \, dx \right]^{\frac{1}{q}} \cdot \eta(\epsilon) \left[ \int_\Omega |\nabla \psi(x)|_q^q \, dx \right]^{\frac{1}{q}} \\
+ \eta(\epsilon) \left[ \int_\Omega |\nabla u(x)|_p^q \, dx \right]^{\frac{1}{2}} \cdot \left[ \int_\Omega |\nabla \psi(x)|_q^q \, dx \right]^{\frac{1}{2}} \\
+ B(\epsilon) n \cdot \mu(\epsilon) \left[ \int_\Omega |\nabla u(x)|_p^q \, dx \right]^{\frac{1}{2}} \cdot \left[ \int_\Omega |\psi(x)|_q^q \, dx \right]^{\frac{1}{2}} \\
\leq (1 + B(\epsilon)) \eta(\epsilon) \cdot \mu(\epsilon) \left\| u \right\|_{W^{1,p}(\Omega)} \cdot \left\| \psi \right\|_{W^{1,q}(\Omega)} \\
\leq K(\epsilon) \left\| u \right\|_{W^{1,p}(\Omega)} \cdot \left\| \psi \right\|_{W^{1,q}(\Omega)},
\]

with \( \lim_{\epsilon \to 0^+} K(\epsilon) = 0 \) (independently of \( u \)). We conclude that

\[
\left\| (A_\epsilon - A) u \right\|_{W^{1,q}(\Omega)} \leq K(\epsilon) \left\| u \right\|_{W^{1,p}(\Omega)} \\
\leq \tau(\epsilon) \left\| Au \right\|_{W^{1,q}(\Omega)} \tag{15}
\]

with \( \lim_{\epsilon \to 0^+} \tau(\epsilon) = 0 \), (and \( \tau(\epsilon) \) does not depend on \( u \)).

4.3. Existence and continuity of the linear semigroup. Using well known facts about the "unperturbed operator" \( A \) and Theorem 4.2, one can now establish existence and continuity of the linear semigroup, based on the following results:
Lemma 4.5. Suppose $A$ is a sectorial operator with $\| (\lambda - A)^{-1} \| \leq \frac{M}{|\lambda - a|}$ for all $\lambda$ in the sector $S_{a, \phi_0} = \{ \lambda \mid \phi_0 \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a \}$, for some $a \in \mathbb{R}$ and $0 \leq \phi_0 < \pi/2$. Suppose also that $B$ is a linear operator with $D(B) \supset D(A)$ and $\| Bx - Ax \| \leq \varepsilon \| Ax \| + K \| x \|$, for any $x \in D(A)$, where $K$ and $\varepsilon$ are positive constants with $\varepsilon \leq \frac{1}{4(1 + LM)}$, $K \leq \frac{\sqrt{5} \sqrt{2L - 1}}{20M L^2 - 1}$, for some $L > 1$.

Then $B$ is also sectorial. More precisely, if $b = \frac{L^2}{L^2 - 1} a - \frac{\sqrt{2L}}{L^2 - 1} |a|$, $\phi = \max \{ \phi_0, \frac{\pi}{4} \}$ and $M' = 2M \sqrt{5}$ then

$$\| (\lambda - B)^{-1} \| \leq \frac{M'}{|\lambda - b|},$$

in the sector $S_{b, \phi} = \{ \lambda \mid \phi \leq |\arg(\lambda - b)| \leq \pi, \lambda \neq b \}$.

Proof. See [16], pg 348. \hfill \Box

Remark 4.6. Observe that $b$ can be made arbitrarily close to $a$ by taking $L$ sufficiently large. In particular, if $a > 0$ then $b > 0$.

Theorem 4.7. Suppose that $A$ is as in Lemma 4.5. $\Lambda$ a topological space and $\{ A_{\gamma} \}_{\gamma \in \Lambda}$ is a family of operators in $X$ with $A_{\gamma_0} = A$ satisfying the following conditions:

1. $D(A_{\gamma}) \supset D(A)$, for all $\gamma \in \Lambda$;
2. $\| A_{\gamma} x - Ax \| \leq \varepsilon(\gamma) \| Ax \| + K(\gamma) \| x \|$ for any $x \in D(A)$, where $K(\gamma)$ and $\varepsilon(\gamma)$ are positive functions with $\lim_{\gamma \to \gamma_0} \varepsilon(\gamma) = 0$ and $\lim_{\gamma \to \gamma_0} K(\gamma) = 0$.

Then, there exists a neighborhood $V$ of $\gamma_0$ such that $A_{\gamma}$ is sectorial if $\gamma \in V$ and the family of (linear) semigroups $e^{-tA_{\gamma}}$ satisfies

$$\| e^{-tA_{\gamma}} - e^{-tA} \| \leq C(\gamma) e^{-bt}$$

$$\| A (e^{-tA_{\gamma}} - e^{-tA}) \| \leq C(\gamma) \frac{1}{t} e^{-bt}$$

$$\| A^\alpha (e^{-tA_{\gamma}} - e^{-tA}) \| \leq C(\gamma) \frac{1}{t^\alpha} e^{-bt}, \quad 0 < \alpha < 1$$ (16)

for $t > 0$, where $b$ is as in Lemma 4.5 and $C(\gamma) \to 0$ as $\gamma \to \gamma_0$.

Proof. See [16], pg 349. \hfill \Box

Theorem 4.8. The operators $A_{\varepsilon}$ given by [13] in the space $X = (W^{1,q})'$, with domain $W^{1,p}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, are sectorial operators with sectors and constant in the sectorial inequality independent of $\varepsilon$, for $\varepsilon_0$ sufficiently small. The family of analytic linear semigroups $e^{-tA_{\varepsilon}}$ generated by $A_{\varepsilon}$ in the “base space” $X$, satisfies (16).

Proof. The first assertion follows from Theorem 4.5 and the second from Theorem 4.7. \hfill \Box
5. The abstract problem in a scale of Banach spaces

Our goal in this section is to pose the problem \( P_A \) in a convenient abstract setting.

We proved in Theorem 4.1 that, if \( \epsilon \) is small, the operator \( A_\epsilon \) in \( L^p(\Omega) \) defined by (8) with domain given in (9) is sectorial and, in Theorem 4.8 that the same is true for its extension \( \tilde{A}_\epsilon \) to \( (W^{1,q})'(\Omega) \).

It is then well-known that the domains \( X^\alpha_\epsilon \) (resp. \( \tilde{X}^\alpha_\epsilon \)) of \( A_\epsilon \) (resp. \( \tilde{A}_\epsilon \)) are Banach spaces, \( X^0_\epsilon = L^p(\Omega) \) (resp. \( \tilde{X}^0_\epsilon = (W^{1,q})'(\Omega) \)), \( X^1_\epsilon = D(A_\epsilon) = W^{2,p}(\Omega) \) (resp. \( \tilde{X}^1_\epsilon = D(\tilde{A}_\epsilon) = W^{1,p}(\Omega) \)), \( X^\alpha_\epsilon \), \( (\tilde{X}^\alpha_\epsilon) \) is compactly embedded in \( X^{\alpha+1}_\epsilon \), \( (\tilde{X}^{\alpha+1}_\epsilon) \) when \( 0 \leq \alpha < \beta < 1 \), and \( X^\alpha_\epsilon = W^{p\alpha} \), when \( 2\alpha \) is an integer number.

Since \( X^{\frac{1}{2}}_\epsilon = \tilde{X}^{\frac{1}{2}}_1 \), it follows easily that \( X^{\frac{1}{2}}_\epsilon = \tilde{X}^{\frac{1}{2}}_\epsilon \), for \( \frac{1}{2} \leq \alpha \leq 1 \) and, by an abuse of notation, we will still write \( X^{\frac{\alpha-1}{2}}_\epsilon \) instead of \( \tilde{X}^{\alpha}_\epsilon \), for \( 0 \leq \alpha \leq \frac{1}{2} \) so we may denote by \( \{X^\alpha_\epsilon, \ -\frac{1}{2} \leq \alpha \leq 1\} = \{X^\alpha_\epsilon, \ 0 \leq \alpha \leq 1\} \cup \{\tilde{X}^\alpha_\epsilon, \ 0 \leq \alpha \leq 1\} \), the whole family of fractional power spaces. We will denote simply by \( X^\alpha \) the fractional power spaces associated to the unperturbed operator \( A \).

For any \( -\frac{1}{2} \leq \beta \leq 0 \), we may now define an operator in these spaces as the restriction of \( \tilde{A}_\epsilon \). We then have the following result

**Theorem 5.1.** For any \( -\frac{1}{2} \leq \beta \leq 0 \) and \( \epsilon \) sufficiently small, the operator \( (A_\epsilon)_\beta \) in \( X^{\beta}_\epsilon \), obtained by restricting \( \tilde{A}_\epsilon \), with domain \( X^{\beta+1}_\epsilon \) is a sectorial operator.

**Proof.** Writing \( \beta = -\frac{1}{2} + \delta \), for some \( 0 \leq \delta \leq \frac{1}{2} \), we have \( (A_\epsilon)_\beta = \tilde{A}^{\epsilon-\delta}_\epsilon \tilde{A}_\epsilon \tilde{A}^{\epsilon+\delta}_\epsilon \). Since \( \tilde{A}^{\epsilon}_\epsilon \) is an isometry from \( X^{\beta}_\epsilon \) to \( X^{\frac{1}{2}}_\epsilon = (W^{1,q}(\Omega))' \), the result follows easily. \( \square \)

We can now pose the problem \( [2] \) as an abstract problem in the scale of Banach spaces \( \{X^\beta_\epsilon, \ -\frac{1}{2} \leq \beta \leq 0\} \).

\[
\begin{align*}
\left\{ \begin{array}{ll}
u = -(A_\epsilon)_\beta u + (H_\epsilon)_\beta u, & t > t_0; \\
u(t_0) = u_0 & \in X^\eta_\epsilon,
\end{array} \right.
\end{align*}
\]

where

\[
(H_\epsilon)_\beta = H(\cdot, \epsilon) := (F(\cdot)_\beta + (G(\cdot)_\beta : X^\eta_\epsilon \to X^\beta_\epsilon, \ \epsilon > 0 \text{ and } 0 \leq \eta \leq \beta + 1, \ (18)
\]

(i) \( (F(\cdot)_\beta = F(\cdot, \epsilon) : X^\eta_\epsilon \to X^\beta_\epsilon \) is given by

\[
\langle F(u, \epsilon), \Phi \rangle_{\beta, -\beta} = \int_\Omega f(u) \Phi dx, \text{ for any } \Phi \in (X^\beta_\epsilon)', \ (19)
\]

(ii) \( (G(\cdot)_\beta = G(\cdot, \epsilon) : X^\eta_\epsilon \to X^\beta_\epsilon \) is given by

\[
\langle G(u, \epsilon), \Phi \rangle_{\beta, -\beta} = \int_{\partial \Omega} g(\gamma(u)) \gamma(\Phi) \left| \frac{J_\partial \Omega h_\epsilon}{\partial h_\epsilon} \right| d\sigma(x), \text{ for any } \Phi \in (X^\beta_\epsilon)', \ (20)
\]

where \( \gamma \) is the trace map and \( J_\partial \Omega h_\epsilon \) is the determinant of the Jacobian matrix of the diffeomorphism \( h_\epsilon : \partial \Omega \to \partial h_\epsilon(\Omega) \).
We will choose $\beta$, small enough in order that $X^\beta_{\epsilon+1}$ does not incorporate the boundary conditions, that is, the closure of the subset defined by smooth functions with Neumann boundary condition is the whole space). It is not difficult to show, integrating by parts, that a regular enough solution of (17), must satisfy (2) (see [6] or [13]).

6. Local well-posedness

In order to prove local well-posedness for the abstract problem, without assuming growth conditions in the nonlinearities, we want to have two somewhat conflicting requirements for our phase space: we need it to be continuously embedded in $L^\infty$ and we also do not want it to incorporate the boundary conditions. To this end, we need to choose $\eta$ and $p$ big enough so that the hypotheses of Theorem 2 hold and, on the other hand, we need $\eta$ small enough so that the normal derivative does not have a well defined trace. To achieve both requirements we will henceforth assume that that

\[ p \text{ and } \eta \text{ are such the inclusion (21) holds, for some } \mu \geq 0 \text{ and } \eta < \frac{1}{2}. \]  

(21)

It is easy to check that (21) holds, for instance, if $p = 2n$, and $\frac{1}{4} < \eta < \frac{1}{2}$. Also, the last inequality is automatically attended if we choose our base space $X^\beta_{\epsilon} = X^{\beta - \frac{1}{2}}_{\epsilon} = (W^{1,q})'$, where $q$ and $p$ are conjugate exponents, since we must have $\beta - \beta < 1$.

**Lemma 6.1.** Suppose that $p$ and $\eta$ are such that (21) holds and $f$ is locally Lipschitz. Then, the operator $(F_\epsilon)_\eta : X^\eta_\epsilon \rightarrow X^{-\frac{1}{2}}_\epsilon$ given by (19) is well defined and Lipschitz in bounded sets.

**Proof.** Suppose $u \in X^\eta_\epsilon$. From (22) and the hypotheses it follows that $u \in L^\infty(\Omega)$ and, therefore, if $L_f$ is the Lipschitz constant of $f$ in the interval $[-\|u\|_\infty, \|u\|_\infty]$, it follows that $|f(u(x) - f(0))| \leq L_f |u(x)|$, for any $x \in \Omega$. If $\Phi \in (X^{-\frac{1}{2}}_\epsilon)' = W^{1,q}$,

\[ |\langle (F_\epsilon)_\eta(u), \Phi \rangle_{\beta,-\beta} | \leq \int_\Omega |f(u)| |\Phi| dx \]

\[ \leq L_f \int_\Omega |u| |\Phi| dx + \int_\Omega |f(0)| |\Phi| dx \]

\[ \leq L_f \|u\|_{L^p(\Omega)} \cdot \|\Phi\|_{L^q(\Omega)} + \|f(0)\|_{L^p(\Omega)} \cdot \|\Phi\|_{L^q(\Omega)} \]

Since $W^{1,q} \subset L^q(\Omega)$ and $X^\eta_\epsilon \subset L^p(\Omega)$ with stronger norms, we have

\[ |\langle (F_\epsilon)_\eta(u), \Phi \rangle_{\beta,-\beta} | \leq L_f \|u\|_{L^p(\Omega)} \|\Phi\|_{W^{1,q}} + \|f(0)\|_{L^p(\Omega)} \|\Phi\|_{W^{1,q}}, \]

so $(F_\epsilon)_\eta$ is well defined and

\[ \|(F_\epsilon)_\eta(u)\|_{W^{1,q}} \leq L_f \|u\|_{L^p(\Omega)} + \|f(0)\|_{L^p(\Omega)} \]

(22)

where $L_f$ is the Lipschitz constant of $f$ in the interval $[-\|u\|_\infty, \|u\|_\infty]$. Alternatively, if $M_f = M_f(u) := \sup\{ |f(x)| : x \in [-\|u\|_\infty, \|u\|_\infty] \}$, it follows that

\[ \|(F_\epsilon)_\eta(u)\|_{(W^{1,q})'} \leq L_f \|u\|_{L^p(\Omega)} + \|f(0)\|_{L^p(\Omega)} \]

(23)
\[ \left| \langle (F_\epsilon)_\eta(u), \Phi \rangle_{\beta,-\beta} \right| \leq \int_\Omega |f(u)| |\Phi| \, dx \]
\[ \leq M_f \int_\Omega |\Phi| \, dx \]
\[ \leq M_f |\Omega|^{\frac{1}{\beta}} |\Phi|_{L^q(\Omega)} \]
\[ \leq M_f |\Omega|^{\frac{1}{\beta}} |\Phi|_{W^{1,q}(\Omega)} \]

Thus
\[ \| (F_\epsilon)_\eta(u) \|_{(W^{1,q})'} \leq M_f |\Omega|^{\frac{1}{\beta}} \]  \hspace{1cm} (24)

Suppose now that if \( u_1, u_2 \) belong to a bounded set \( B \in X_{\eta} \). From (22) and the hypotheses it follows now that \( u_1, u_2 \) belong to a ball of radius \( R = \sup_{u \in B} \|u\|_{\infty} \) in \( L^\infty(\Omega) \) and, therefore, if \( L \) is the Lipschitz constant of \( f \) in the interval \([-R, R]\), we have \(|f(u_1(x)) - f(u_2(x)))| \leq L|u_1(x) - u_2(x)|\), for any \( x \in \Omega \). If \( \Phi \in (X_{\epsilon^{-\frac{1}{2}}})' = W^{1,q} \), we obtain
\[ \left| \langle (F_\epsilon)_\eta(u_1) - (F_\epsilon)_\eta(u_2), \Phi \rangle_{\beta,-\beta} \right| = \left| \int_\Omega [f(u_1) - f(u_2)] \Phi \, dx \right| \]
\[ \leq \int_\Omega L |u_1 - u_2| |\Phi| \, dx \]
\[ \leq L_f \|u_1 - u_2\|_{L^p(\Omega)} \cdot \|\Phi\|_{L^q(\Omega)} \]
\[ \leq L_f \|u_1 - u_2\|_{X_{\eta}^p} \cdot \|\Phi\|_{W^{1,q}(\Omega)} \]

Thus
\[ \| (F_\epsilon)_\eta(u_1) - (F_\epsilon)_\eta(u_2) \|_{(W^{1,q})'} \leq L_f \|u_1 - u_2\|_{L^p(\Omega)} \]  \hspace{1cm} (25)
\[ \leq L_f \|u_1 - u_2\|_{X_{\eta}^p}. \]  \hspace{1cm} (26)

This concludes the proof.

Lemma 6.2. Suppose that \( p \) and \( \eta \) are such that (21) holds and \( g \) is locally Lipschitz. Then, if \( \epsilon_0 \) is sufficiently small, the operator \((G_\epsilon)_\eta : X_{\eta} \to (W^{1,q})'\) given by (20) is well defined, for \( 0 \leq \epsilon < \epsilon_0 \) and bounded in bounded sets.

Proof. Suppose \( u \in X_{\eta} \). From (22) and the hypotheses it follows that \( u \in L^\infty(\Omega) \) and, therefore, if \( L_g \) is the Lipschitz constant of \( g \) in the interval \([-\|u\|_{\infty}, \|u\|_{\infty}]\), it follows that \(|g(\gamma(u)(x)) - g(0)| \leq L_g \gamma(u)(x)\), for any \( x \in \partial \Omega \).

If \( u \in X_{\eta} \) and \( \Phi \in (X_{\epsilon^{-\frac{1}{2}}})' = W^{1,q} \), we have
\[ |\langle G(u, \epsilon), \Phi \rangle_{\beta,-\beta} | \leq \int_{\partial \Omega} |g(\gamma(u))| |\gamma(\Phi)| \left| \frac{J_{\partial \Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x) \]

\[ \leq ||\mu||_\infty \int_{\partial \Omega} L_g |\gamma(u)| |\gamma(\Phi)| + |g(0)| |\gamma(\Phi)| d\sigma(x) \]

\[ \leq ||\mu||_\infty (L_g ||\gamma(u)||_{L^p(\partial \Omega)} \cdot ||\gamma(\Phi)||_{L^q(\partial \Omega)} + ||g(0)||_{L^p(\partial \Omega)} \cdot ||\gamma(\Phi)||_{L^q(\partial \Omega)}) \]

where \( \mu(x, \epsilon) = \left| \frac{J_{\partial \Omega} h_\epsilon}{J h_\epsilon} \right| \), and \( ||\mu||_\infty = \sup \{|\mu(x, \epsilon)| | x \in \partial \Omega, 0 \leq \epsilon \leq \epsilon_0 \} \) is finite by hypothesis (H_1).

By Theorem ??, there

\[ ||\gamma(\Phi)||_{L^q(\partial \Omega)} \leq K_1 ||\Phi||_{W^{1,q}(\Omega)}, \quad ||\gamma(u)||_{L^p(\partial \Omega)} \leq K_2 ||u||_{X_\epsilon^q} \]

for some constants \( K_1, K_2 \). Thus

\[ |\langle G(u, \epsilon), \Phi \rangle_{\beta,-\beta} | \leq ||\mu||_\infty (L_g K_1 ||\gamma(u)||_{L^p(\partial \Omega)} ||\Phi||_{W^{1,q}(\Omega)} + K_1 ||g(0)||_{L^p(\partial \Omega)} \cdot ||\Phi||_{W^{1,q}(\Omega)}) \]

proving that \( (G_\epsilon)_{\beta} \) is well defined and

\[ ||G(u, \epsilon)||_{(W^{1,q}(\Omega))'} \leq ||\mu||_\infty (L_g K_1 ||\gamma(u)||_{L^p(\partial \Omega)} + K_1 ||g(0)||_{L^p(\partial \Omega)}) \]  

(27)

\[ \leq ||\mu||_\infty (L_g K_2 ||u||_{X_\epsilon^q} + K_1 ||g(0)||_{L^p(\partial \Omega)}) \]  

(28)

Alternatively, if \( M_g = M_g(u) := \sup\{|g(x)| | x \in [-||u||_\infty, ||u||_\infty]| \} \), it follows that

\[ |\langle G(u, \epsilon), \Phi \rangle_{\beta,-\beta} | \leq \int_{\partial \Omega} |g(\gamma(u))| |\gamma(\Phi)| \left| \frac{J_{\partial \Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x) \]

\[ \leq ||\mu||_\infty M_g \int_{\partial \Omega} |\gamma(\Phi)| d\sigma(x) \]

\[ \leq ||\mu||_\infty M_g ||\partial \Omega|^{\frac{1}{2}} ||\gamma(\Phi)||_{L^q(\partial \Omega)} \]

\[ \leq ||\mu||_\infty M_g ||\partial \Omega|^{\frac{1}{2}} K_1 ||\Phi||_{W^{1,q}(\Omega)} \]

Thus

\[ ||G(u, \epsilon)||_{(W^{1,q}(\Omega))'} \leq ||\mu||_\infty M_g ||\partial \Omega|^{\frac{1}{2}} K_1 \]  

(29)

Lemma 6.3. Suppose the same hypotheses of Lemma 6.2 hold. Then the operator \( G(u, \epsilon) = G(u) : X_\epsilon^q \times [0, \epsilon_0] \rightarrow (W^{1,q})' \) given by (27) is uniformly continuous in \( \epsilon \), for \( u \) in bounded sets of \( X_\epsilon^q \) and locally Lipschitz continuous in \( u \), uniformly in \( \epsilon \).

Proof. We first show that \( (G_\epsilon)_{\beta} \) is locally Lipschitz continuous in \( u \in X_\epsilon^q \).

Suppose that \( u_1, u_2 \) belong to a bounded set \( B \in X_\epsilon^q \). From (??), the Trace Theorem and the hypotheses, it follows now that \( \gamma(u_1), \gamma(u_2) \) belong to a ball of some radius \( R \) in \( L^\infty(\partial \Omega) \) and, therefore, if \( L_g \) is the Lipschitz constant of \( g \) in the interval \([-R, R]\), we have
\[ |g(\gamma(u_1)(x)) - g(\gamma(u_2)(x))| \leq L_g |\gamma(u_1)(x) - \gamma(u_2)(x)|, \] for any \( x \in \partial \Omega \). If \( \Phi \in (W^{1,q})' \) and \( \epsilon \in [0, \epsilon_0] \), we obtain

\[
\left| \langle G(u_1, \epsilon) - G(u_2, \epsilon), \Phi \rangle_{\beta, -\beta} \right| \leq \int_{\partial \Omega} |g(\gamma(u_1)) - g(\gamma(u_2))| \cdot |\gamma(\Phi)| \left| \frac{J_{\partial \Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x)
\]

\[
\leq \int_{\partial \Omega} L_g |\gamma(u_1) - \gamma(u_2)| \cdot |\gamma(\Phi)| \left| \frac{J_{\partial \Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x)
\]

\[
\leq L_g \|\mu\|_{\infty} \int_{\partial \Omega} |\gamma(u_1) - \gamma(u_2)| \cdot |\gamma(\Phi)| d\sigma(x)
\]

\[
\leq L_g \|\mu\|_{\infty} K_1 K_2 \|u_1 - u_2\|_{X^q} \cdot |\Phi|_{W^{1,q}(\Omega)},
\]

where \( K_1, K_2 \) are the norms of the trace mappings, given by Theorem 2. Therefore,

\[
\|G(u_1, \epsilon) - G(u_2, \epsilon)\|_{(W^{1,q})'} \leq L_g \|\mu\|_{\infty} K_1 \|\gamma(u_1) - \gamma(u_2)\|_{L^p(\partial \Omega)}
\]

\[
\leq L_g \|\mu\|_{\infty} K_1 K_2 \|u_1 - u_2\|_{X^q}
\]

so \( (G_\epsilon)_\beta \) is locally Lipschitz in \( u \).

Now, if \( u \in X^q_0, \Phi \in (W^{1,q})' \) and \( \epsilon_1, \epsilon_2 \in [0, \epsilon_0] \), we have

\[
\left| \langle G(u, \epsilon_1) - G(u, \epsilon_2), \Phi \rangle_{\beta, -\beta} \right| \leq \int_{\partial \Omega} |g(\gamma(u))| \cdot |\gamma(\Phi)| \left( \left| \frac{J_{\partial \Omega} h_{\epsilon_1}}{J h_{\epsilon_1}} \right| - \left| \frac{J_{\partial \Omega} h_{\epsilon_2}}{J h_{\epsilon_2}} \right| \right) d\sigma(x)
\]

\[
\leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} \int_{\partial \Omega} |g(\gamma(u))| \cdot |\gamma(\Phi)| d\sigma(x)
\]

\[
\leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} \int_{\partial \Omega} (L_g |\gamma(u)| + |g(0)|) \cdot |\gamma(\Phi)| d\sigma(x)
\]

\[
\leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} (L_g \|\gamma(u)\|_{L^p(\partial \Omega)} \cdot \|\gamma(\Phi)\|_{L^q(\partial \Omega)} + \|g(0)\|_{L^p(\partial \Omega)} \cdot \|\gamma(\Phi)\|_{L^q(\partial \Omega)})
\]

\[
\leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} (L_g K_1 K_2 \|u\|_{X^q} \cdot |\Phi|_{W^{1,q}(\Omega)} + K_1 \|g(0)\|_{L^p(\partial \Omega)} \cdot \|\Phi\|_{W^{1,q}(\Omega)})
\]

where \( \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} = \sup \left\{ \left| \frac{J_{\partial \Omega} h_{\epsilon_1}}{J h_{\epsilon_1}} \right| - \left| \frac{J_{\partial \Omega} h_{\epsilon_2}}{J h_{\epsilon_2}} \right| : x \in \partial \Omega, \right\} \to 0 \) as \( |\epsilon_1 - \epsilon_2| \to 0 \), by hypothesis (H1) and \( K_1, K_2 \) are trace constants given by Theorem 2.

It follows that

\[
\|G(u, \epsilon_1) - G(u, \epsilon_2)\|_{(W^{1,q}(\Omega))'} \leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} \left( L_g K_1 K_2 \|u\|_{X^q} + K_1 \|g(0)\|_{L^p(\partial \Omega)} \right)
\]

Alternatively, if \( M_g = M_g(u) := \sup \{|g(x)| : x \in [-\|u\|_{\infty}, \|u\|_{\infty}]\} \),

\[
\left| \langle G(u, \epsilon_1) - G(u, \epsilon_2), \Phi \rangle_{\beta, -\beta} \right| \leq \int_{\partial \Omega} |g(\gamma(u))| \cdot |\gamma(\Phi)| \left( \left| \frac{J_{\partial \Omega} h_{\epsilon_1}}{J h_{\epsilon_1}} \right| - \left| \frac{J_{\partial \Omega} h_{\epsilon_2}}{J h_{\epsilon_2}} \right| \right) d\sigma(x)
\]

\[
\leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} \int_{\partial \Omega} |g(\gamma(u))| \cdot |\gamma(\Phi)| d\sigma(x)
\]

\[
\leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} M_g \int_{\partial \Omega} |\gamma(\Phi)| d\sigma(x)
\]
Suppose the hypotheses of Corollary 6.4 hold. Then, for any Theorem 6.5.

Proof.

It follows that

\[ \|G(u, \epsilon_1) - G(u, \epsilon_2)\|_{(W^{1,q}(\Omega))'} \leq \|\mu_\epsilon - \mu_\epsilon\|_\infty M_g|\partial\Omega|^{\frac{1}{p}} K_1 \]  

Corollary 6.4. Suppose the hypotheses of Lemmas 6.1 and 6.2 hold. Then the map \( (H(u, \epsilon))_\eta := (F_\epsilon(u))_\eta + (G(u, \epsilon))_\eta : X^\eta_\epsilon \times [0, \epsilon_0] \rightarrow (W^{1,q})' \) is well defined, bounded in bounded sets uniformly in \( \epsilon \), uniformly continuous in \( \epsilon \) for \( u \) in bounded sets of \( X^\eta_\epsilon \) and locally Lipschitz continuous in \( u \) uniformly in \( \epsilon \).

Proof.

From 22, 23, 27 and 28 we obtain

\[ \| (H_\epsilon)_\eta (u) \|_{(W^{1,q})'} \leq L_f \| u \|_{L^p(\Omega)} + L_g K_1 \| \gamma(u) \|_{L^p(\partial\Omega)} + \| f(0) \|_{L^p(\Omega)} + K_1 \| g(0) \|_{L^p(\partial\Omega)} \]

where \( L_f \) and \( L_g \) are Lipschitz constants of \( f \) and \( g \) in the interval \([-\|u\|_\infty, \|u\|_\infty]\).

Alternatively, if \( M_f = M_f(u) := \sup\{|f(x)| : x \in [-\|u\|_\infty, \|u\|_\infty]\} \), \( M_g = M_g(u) := \sup\{|g(x)| : x \in [-\|u\|_\infty, \|u\|_\infty]\} \), we obtain from (24) and (33)

\[ \| (H_\epsilon)_\eta (u) \|_{(W^{1,q})'} \leq M_f \| \Omega \|^{\frac{1}{p}} + \| \mu \|_\infty M_g |\partial\Omega|^{\frac{1}{p}} K_1 \]  

From 25, 26, 30 and 31,

\[ \| H(u_1, \epsilon) - H(u_2, \epsilon)\|_{(W^{1,q})'} \leq L_g \| \mu \|_\infty K_1 \| \gamma(u_1) - \gamma(u_2)\|_{L^p(\partial\Omega)} + L_f \| u_1 - u_2 \|_{L^p(\Omega)} \]

Alternatively, from (32)

\[ \| H(u, \epsilon_1) - H(u, \epsilon_2)\|_{(W^{1,q}(\Omega))'} \leq \| \mu_\epsilon - \mu_\epsilon\|_\infty (L_g K_1 K_2 \| u \|_{X^\eta_\epsilon} + K_1 \| g(0) \|_{L^p(\partial\Omega)}) \]

Theorem 6.5. Suppose the hypotheses of Corollary 6.4 hold. Then, for any \( (t_0, u_0) \in \mathbb{R} \times X^\eta_\epsilon \), the problem (17) has a unique solution \( u(t, t_0, u_0, \epsilon) \) with initial value \( u(t_0) = u_0 \).
Proof. From Theorem 5.1 it follows that \((A_\epsilon)_{\beta}\) is a sectorial operator in \((W^{1,q})',\) with domain \(X_\epsilon^{1/2} = W^{1,p},\) if \(\epsilon\) is small enough. The result follows then from Corollary 6.4 and results in [10] and [16].

\[
\Box
\]

7. Global existence and boundedness of the semigroup

We will use the notation \(T_\epsilon(t)u_0\) for the (local) solution of the problem (17) given by Theorem 6.5, with initial condition \(u_0\) in some fractional power space of \(A_\epsilon.\) We now want to show that these solutions are globally defined if an additional (dissipative) hypotheses on \(f\) and \(g\) is assumed. Here are these hypotheses:

There exist constants \(c_0\) and \(d_0\) such that

\[
\limsup_{|u| \to \infty} \frac{f(u)}{u} \leq c_0, \quad \limsup_{|u| \to \infty} \frac{g(u)}{u} \leq d_0
\]

and the first eigenvalue \(\mu_1(\epsilon)\) of the problem

\[
\left\{ \begin{array}{l}
- h^*_\epsilon \Delta_{\Omega} h^*_{\epsilon-1} u + (a - c_0) u = \mu u \text{ em } \Omega \\
h^*_\epsilon \frac{\partial u}{\partial N_\Omega} h^*_{\epsilon-1} = d_0 u \text{ em } \partial \Omega
\end{array} \right.
\]

is positive for \(\epsilon\) sufficiently small.

Remark 7.1. Observe that if the hypothesis (43) hold for \(\epsilon = 0\), then this also true for \(\epsilon\) small since the eigenvalues change continuously with \(\epsilon\) by (15).

Remark 7.2. The arguments below are a slight modification of the ones in [13], but we include them here for the sake of completeness. Similar arguments were used in [1] in a somewhat different setting.

In order to use comparison results, we start by defining the concepts of sub- and supersolutions.

Definition 7.3. Suppose \(\Omega\) is a \(C^{1,\alpha}\), domain for some \(\alpha \in (0,1),\) \(L\) is a uniformly elliptic second order differential operator in \(\bar{\Omega},\) \(u_0 \in C^\alpha(\Omega),\) \(T > 0\) and \(\overline{u} : \Omega \subset \mathbb{R}^n\) (\(u\) respectively) a function which is continuous in \([0,T] \times \bar{\Omega},\) continuously differentiable in \(t\) and twice continuously differentiable in \(x\) for \((t,x) \in (0,T] \times \Omega.\) Then \(\overline{u}\) (respectively, \(u\)) is a super-solution (sub-solution) of the problem

\[
\left\{ \begin{array}{l}
u_t = Lu + f(u), \quad \text{in } (0,T] \times \Omega,\\
\frac{\partial u}{\partial N} = g(u), \quad \text{on } \partial \Omega \\
u(0) = u_0.
\end{array} \right.
\]

if it satisfies

\[
\limsup_{|u| \to \infty} \frac{f(u)}{u} \leq c_0, \quad \limsup_{|u| \to \infty} \frac{g(u)}{u} \leq d_0
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
  u_t \geq Lu + f(u), \quad \text{in} \quad (0,T] \times \Omega, \\
  \frac{\partial u}{\partial N} \geq g(u), \quad \text{on} \quad \partial \Omega \\
  u(0) \geq u_0.
\end{array} \right.
\]
( and respectively with the $\geq$ sign replaced by the $\leq$ sign).

A basic result for our arguments is the following

**Theorem 7.4.** (Pao [15])

If \( f \) is locally Lipschitz and \( \bar{u} \) and \( u \) are respectively a super and sub-solution of the problem (43), satisfying

\[
u \leq \bar{u}, \text{ in } \Omega \times (0,T),
\]

then there exists a solution \( u \) of (43) such that

\[
u \leq u \leq \bar{u}, \text{ in } \Omega \times (0,T).
\]

Let now \( \varphi_\epsilon \) be the first positive normalized eigenfunction of (42) and \( m_\epsilon = \min_{x \in \bar{\Omega}} \varphi_\epsilon(x) \). We know that \( m_\epsilon > 0 \). For each \( \theta > 0 \in \mathbb{R} \), define

\[
\Sigma_\theta^\epsilon = \{ u \in X_\epsilon^n : |u(x)| \leq \theta \varphi_\epsilon(x), \text{ for all } x \in \bar{\Omega} \}.
\]

From the dissipative hypothesis (41) on \( f \) and \( g \), we know that there exists \( \xi \in \mathbb{R} \), such that

\[
\frac{f(s)}{s} \leq c_0 \text{ and } \frac{g(s)}{s} \leq d_0,
\]

for all \( s \) with \( |s| \geq \xi \). To simplify the notation, we take the \( \epsilon = 0 \), in the proofs below, since the argument is the same for any \( \epsilon \) such that (42) is true (see Remark 7.1).

**Lemma 7.5.** Suppose, in addition to the hypotheses of Theorem 6.5, that (41) and (42) hold. Then, if \( \theta m_\epsilon \geq \xi \) and \( \epsilon \) is small enough, the set \( \Sigma_\theta^\epsilon \) is a positively invariant set for \( T(t) \).

**Proof.**

Let

\[
\Sigma_\theta^1 = \{ u \in X_\epsilon^n : u(x) \leq \theta \varphi(x), \text{ for all } x \in \bar{\Omega} \}
\]

\[
\Sigma_\theta^2 = \{ u \in X_\epsilon^n : u(x) \geq -\theta \varphi(x), \text{ for all } x \in \bar{\Omega} \}
\]

Since \( \Sigma_\theta = \Sigma_\theta^1 \cap \Sigma_\theta^2 \) it is enough to show that \( \Sigma_\theta^1 \) and \( \Sigma_\theta^2 \) are positively invariant.

Let \( u_0 \in \Sigma_\theta^1 \), and suppose, for contradiction, that there exists \( t_0 \in [0, t_{\text{max}}[ \) and \( x_0 \in \bar{\Omega} \) such that

\[
T(t_0)u_0(x_0) > \theta \varphi(x_0).
\]

Consider \( \bar{v}(t) = e^{-\mu(t-t_0)}\theta \varphi \), where \( \mu \) is the eigenvalue associated with \( \varphi \). We have that
Lemma 7.6. Suppose the hypotheses of Lemma 7.5 hold. If $\epsilon > 0$ and can be chosen independently of $\epsilon$, the solutions with initial condition in $X$ is a sub-solution for the problem (2).

Thus $\bar{v}$ is a super-solution for the problem (2). It follows from Theorem 7.4 that $T(t)u_0 \leq \bar{v}(t)$, in $\bar{\Omega}$ for all $t \in [0, t_0]$.

In particular, $T(t_0)u_0(x_0) \leq \theta \varphi(x_0)$ and we reach a contradiction.

To prove that $\Sigma_\theta$ is positively invariant we proceed in a similar way, using now that $v = -\bar{v}$ is a sub-solution for the problem (2).

Lemma 7.6. Suppose the hypotheses of Lemma 7.5 hold. If $\theta m_\epsilon \geq \xi$, and $\eta \leq \alpha < \frac{1}{2}$, there exists a constant $R = R(\theta, \eta)$, and $T > 0$ independent of $\epsilon$, such that the orbit of any bounded subset $V$ of $X_0^\eta \cap \Sigma_\theta$ under $T_\epsilon(t)$ is in the ball of radius $R$ of $X_0^\alpha$, for $t > T$. In particular, the solutions with initial condition in $X_0^\eta \cap \Sigma_\theta$ are globally defined.

Proof:

Lemma 7.5 implies that $T_\epsilon(t)u_0 \in \Sigma_\theta$, for all $t \in [0, t_{\text{max}}]$ so $\|T_\epsilon(t)u_0\|_\infty \leq \theta \|\varphi\|_\infty$.

Applying the variation of constants formula, we obtain (see [10])

$$\|T(t)u_0\|_{X^\omega} \leq M t^{-\alpha \eta} e^{-\delta t} \|u_0\|_{X^\eta} + M \int_0^t (t - s)^{-\alpha \eta} e^{-\delta(t-s)} \|(H_\epsilon)\eta(T(s)u_0)\|_{X^{-\frac{1}{2}}} ds,$$

where the $M, \delta > 0$ are constants depending only on the decay of the linear semigroup $e^{A_{\omega} t}$, and can be chosen independently of $\epsilon$. By [36]

$$\|(H_\epsilon)\eta(T(s)u_0)\|_{X^{-\frac{1}{2}}} \leq M_f |\Omega|^\frac{1}{p} + \|\mu\|_{\infty} M_g |\partial \Omega|^\frac{1}{p} K_1$$

where now $M_f = M_f(u) := \sup\{|f(x)| \ x \in I\}$, $M_g = M_g(u) := \sup\{|g(x)| \ x \in I\}$ with $[-\theta \|\phi_\epsilon\|_\infty, \phi_\epsilon\|_\infty] \subset I$, for all $\epsilon$ sufficiently small. Thus, writing $K = M_f |\Omega|^\frac{1}{p} + \|\mu\|_{\infty} M_g |\partial \Omega|^\frac{1}{p} K_1$, we obtain

$$\|T_\epsilon(t)u_0\|_{X^\omega} \leq M t^{-\alpha \eta} e^{-\delta t} \|u_0\|_{X^\eta} + K M \int_0^t (t - s)^{-\alpha \eta} e^{-\delta(t-s)} ds,$$

$$\leq M t^{-\alpha \eta} e^{-\delta t} \|u_0\|_{X^\eta} + K M \frac{\Gamma(\frac{1}{2} - \alpha)}{\delta^{\frac{1}{2} - \alpha}}$$

for all $t \in [0, t_{\text{max}}]$. 

Therefore \( \| T_\epsilon(t)u_0 \|_{X^\alpha} \) is bounded by a constant for any \( t > 0 \). Since \( X^\alpha \) is compactly embedded in \( X^\eta \), if \( \alpha > \eta \), it follows that the solution is globally defined. Also, if \( T \) is such that \( t^{-(\alpha-\eta)}e^{-\delta t}\|u_0\|_{X^\eta} \leq K \frac{\Gamma(\frac{1}{2}-\alpha)}{\delta^{\frac{1}{2}-\alpha}} \), then \( \| T_\epsilon(t)u_0 \|_{X^\alpha} \) belongs to the ball of \( X^\alpha \) of radius \( R(\theta) = 2K M \frac{\Gamma(\frac{1}{2}-\alpha)}{\delta^{\frac{1}{2}-\alpha}} \), for \( t \geq T \).

□

8. Existence of Global Attractors

The first step to show the existence of global attractors will be to obtain a “contraction property” of the sets \( \Sigma_\theta \), similar to the property for rectangles, considered by Smoller [18].

**Lemma 8.1.** Suppose that the hypotheses of Lemma 7.5 hold and \( \bar{\theta} \in \mathbb{R} \) satisfy \( \bar{\theta}m_\epsilon > \xi \).

Then, for any \( \theta \) there exists a \( \bar{t} \), which can be chosen independently of \( \epsilon \), such that

\[
T_\epsilon(t)\Sigma_\theta \subset \Sigma_{\bar{\theta}},
\]

for all \( t \geq \bar{t} \).

**Proof:**

Let \( u \in \Sigma_\theta \). We can suppose without loss of generality that \( \theta \geq \bar{\theta} \). Let \( \bar{v} = e^{-t\mu_\epsilon}\theta \varphi \), \( v = -\bar{v} \). As in Lemma 7.5, we can prove that \( \bar{v} \) and \( v \) are super- and sub-solutions respectively. Thus, using Theorem 7.4 and the uniqueness of solution, we have that

\[
v \leq T_\epsilon(t)u \leq \bar{v},
\]

Therefore \( T_\epsilon(t)u \) enters \( \Sigma_{\bar{\theta}} \) after a time depending only on \( \theta \), and on the first eigenvalue \( \mu_\epsilon \) of \( A_\epsilon \) (and not on the particular solution \( u \in \Sigma_\theta \)). Since \( \mu_\epsilon \) is bigger than a constant \( \mu \), for \( \epsilon \) sufficiently small, and \( \Sigma_{\bar{\theta}} \) is positively invariant, the result follows.

□

**Theorem 8.2.** Suppose that the hypotheses of Lemma 7.5 hold. Then the problem (17) has a global attractor \( A_\epsilon \) in \( X_\eta^\epsilon \). Furthermore \( A_\epsilon \subset \Sigma_\bar{\theta}^{\epsilon} \) if \( \theta m_\epsilon \geq \xi \).

**Proof:**

Let \( V \) be a bounded subset of \( X^\eta \), and \( \bar{\theta} \in \mathbb{R} \) be such that \( \bar{\theta}m \geq \xi \). If \( u \) is any element of \( X^\eta \), it follows from the continuity of the embedding \( X^\eta \hookrightarrow C^0(\bar{\Omega}) \) that \( u \in \Sigma_\theta \), for some \( \theta \) and then, applying Lemma 8.1, we conclude that \( T(t)u \in \Sigma_{\bar{\theta}} \), for \( t \) big enough. From Lemma 7.6, it follows that \( V \) enters and remains in a ball of \( X^\alpha \), with \( \alpha > \eta \) of radius \( R(\alpha, \bar{\theta}) \), which does not depend on \( V \). Since this ball is a compact set of \( X^\alpha \), the existence of a global compact attractor \( \mathcal{A} \) follows immediately. Furthermore, since \( \Sigma_{\bar{\theta}} \) is positively invariant by Lemma 7.5, it also follows that \( \mathcal{A} \subset \Sigma_{\bar{\theta}} \), as claimed.

□

**Corollary 8.3.** Suppose that the hypotheses of Lemma 7.5 hold. If \( \epsilon_0 \) is sufficiently small, the attractor \( A_\epsilon \) is uniformly bounded in \( L^\infty \), for \( 0 \leq \epsilon \leq \epsilon_0 \).
CONTINUITY OF ATTRACTIONS FOR $C^1$ PERTURBATIONS

Proof: From (4.4) and results in [11], it follows that the first eigenvalue and eigenfunction of $A_\epsilon$ are continuous in $W^{1,p}$ and, therefore, also in $L^\infty$. Thus the sets $\Sigma_0^\epsilon$ are uniformly bounded in $L^\infty$ and the result follows from Theorem 8.2. \hfill \Box

9. UPPERSEMICOINtinuity Of the family of global attractors

Recall that a family of subsets $A_\lambda$ of a metric space $(X,d)$ is said to be upper-semicontinuous at $\lambda = \lambda_0$ if $\delta(A_\lambda, A_{\lambda_0}) \to 0$ as $\lambda \to \lambda_0$, where $\delta(A,B) = \sup_{x \in A} d(x,B) = \sup_{x \in A} \inf_{y \in B} d(x,y)$ and lower-semicontinuous if $\delta(A_{\lambda_0}, A_\lambda) \to 0$ as $\lambda \to \lambda_0$.

To prove the upper-semicontinuity of the family of attractors $A_\epsilon$, given by Theorem 8.2 in the (fixed) fractional space $X^\eta$, $0 < \eta < \frac{1}{2}$, we will need two main ingredients: the uniform boundedness of the family and the continuity of the nonlinear semigroup $T_t$ with respect to $\epsilon$. This is the content of the next two results. In view of the uniform boundedness of the solutions, proved in Corollary 8.3 we may suppose, without loss of generality, the following hypothesis on the nonlinearites.

- $f$ and $g$ are globally bounded.
- $f$ and $g$ are globally Lipschtiz, with Lipschitz constants $L_f$ and $L_g$ respectively, (45)

Lemma 9.1. Suppose that the hypotheses of Lemma 7.5 and (45) hold. If $\epsilon_0$ is sufficiently small, the family of attractors $A_\epsilon$, given by Theorem 8.2 is uniformly bounded in the (fixed) fractional space $X^\eta$, $0 < \eta < \frac{1}{2}$, for $0 \leq \epsilon \leq \epsilon_0$.

Proof.

Let $b$ be the exponential rate of decay of the linear semigroup generated by $A_\epsilon$, $\epsilon$ for $\epsilon$ small, given by Theorem 4.7. Let $u \in A_\epsilon$. By the variation of constants formula, Lemma 4.2 and Theorem 4.7 we obtain

$$
\|T_\epsilon(t)(u)\|_\eta \leq \|e^{A_\epsilon(t)}u\|_\eta + \int_0^t \|e^{A_\epsilon(t-s)}H_\epsilon(T_\epsilon(s)u)\|_\eta \, ds
$$

$$
\leq \|e^{A(t)}u\|_\eta + \| (e^{A_\epsilon(t)} - e^{A(t)}) \|_\eta + \int_0^t \|e^{A(t-s)}H_\epsilon(T_\epsilon(s)u)\|_\eta \, ds
$$

$$
+ \int_0^t \| (e^{A_\epsilon(t-s)} - e^{A(t-s)}) \|_\eta + \int_0^t \|e^{A_\epsilon(t-s)}H_\epsilon(T_\epsilon(s)u)\|_\eta \, ds
$$

$$
\leq (Ce^{-at} + C(\epsilon)e^{-bt}) \frac{1}{t^{\eta+\frac{1}{2}}} \|u\| + \int_0^t Ce^{-a(t-s)} \frac{1}{(t-s)^\eta} + \frac{1}{2} \|H_\epsilon(T_\epsilon(s)u)\| \, ds
$$

$$
+ \int_0^t Ce^{-b(t-s)} \frac{1}{(t-s)^{\eta+\frac{1}{2}}} \|H_\epsilon(T_\epsilon(s)u)\| \, ds
$$

By (36)

$$
\|(H_\epsilon)_\eta((T(s)u_0))\|_{X^{\frac{1}{2}}} \leq M_1 \|\Omega^{\frac{1}{p}} + \|\mu\|_{\infty} M_2 \|\Omega^{\frac{1}{p}} K_1
$$
Proof.

\( \epsilon \) is continuous at \( \sup \) Suppose that the hypotheses of Lemma 9.1 hold. Then the map

\[
\| T_\epsilon(t)(u) \|_\eta \leq C' e^{-bt} \frac{1}{t^{\eta+\frac{1}{2}}} \| u \|_\infty + C'' \left( \| f \|_\infty \| \Omega \|^{\frac{1}{p}} + \| \mu \|_\infty \| g \|_\infty |\partial \Omega|^{\frac{1}{p}} K_1 \right) \int_0^t e^{-b(t-s)} \frac{1}{(t-s)^\eta + \frac{1}{2}} \, ds,
\]

where the constants \( C' \) and \( C'' \) do not depend on \( \epsilon \).

Since the right hand side is uniformly bounded for \( u \in \mathcal{A}_\epsilon, t > 0 \) and the attractors are invariant, the result follows immediately. \( \Box \)

**Lemma 9.2.** Suppose that the hypotheses of Lemma 9.1 hold. Then the map

\( (u, \epsilon) \in X^\eta \times [0, \epsilon_0] \mapsto T_\epsilon u \in X^\eta \)

is continuous at \( \epsilon = 0 \), uniformly for \( u \) in bounded sets and \( 0 < t \leq T < \infty \).

**Proof.**

Using the variation of constants formula, (36), (40) and (38), we obtain

\[
\| T_\epsilon(t)(u) - T(t)(u) \|_\eta \leq \| e^{A\epsilon(t)} u - e^{A(t)} u \|_\eta \\
+ \int_0^t \| (e^{A\epsilon(t-s)} - e^{A(t-s)}) H_\epsilon(T_\epsilon(s)u) \|_\eta \, ds \\
+ \int_0^t \| e^{A(t-s)} (H_\epsilon(T_\epsilon(s)u) - H(T_\epsilon(s)u)) \|_\eta \, ds \\
+ \int_0^t \| e^{A(t-s)} (H(T_\epsilon(s)u) - H(T(s)u)) \|_\eta \, ds \\
\leq C(\epsilon) e^{-bt} \frac{1}{t^{\eta+\frac{1}{2}}} \| u \| + \int_0^t C(\epsilon) e^{-b(t-s)} \frac{1}{(t-s)^\eta + \frac{1}{2}} \| H_\epsilon(T_\epsilon(s)u) \| \, ds \\
+ \int_0^t C e^{-b(t-s)} \frac{1}{(t-s)^{\eta+\frac{1}{2}}} \| H_\epsilon(T_\epsilon(s)u) - H(T_\epsilon(s)u) \| \, ds \\
+ \int_0^t C e^{-b(t-s)} \frac{1}{(t-s)^{\eta+\frac{1}{2}}} \| H(T_\epsilon(s)u) - H(T(s)u) \| \, ds \\
\leq C(\epsilon) e^{-bt} \frac{1}{t^{\eta+\frac{1}{2}}} \| u \| \\
+ \int_0^t C(\epsilon) e^{-b(t-s)} \frac{1}{(t-s)^\eta + \frac{1}{2}} \left( \| f \|_\infty \| \Omega \|^{\frac{1}{p}} + \| \mu \|_\infty \| g \|_\infty |\partial \Omega|^{\frac{1}{p}} K_1 \right) \, ds \\
+ \int_0^t C e^{-b(t-s)} \frac{1}{(t-s)^{\eta+\frac{1}{2}}} \| (\| \mu \|_\infty - 1) M_g |\partial \Omega|^{\frac{1}{p}} K_1 \| \, ds \\
+ \int_0^t C e^{-b(t-s)} \frac{1}{(t-s)^{\eta+\frac{1}{2}}} \| (L_g \| \mu \|_\infty K_1 K_2 + L_f) \| T_\epsilon(s)u - T(s)u \|_{X^\eta} \| \, ds
\]
Given

\[ A \]

Theorem 9.3. Suppose that the hypotheses of Lemma 9.1 hold. Then the family of attractors

\[ \text{From Lemma 9.1 there exists a bounded set } A \]

is invariant under \( T \) and \( 0 \leq \| \mu \|_{\infty} \mathcal{g} |\partial \Omega|^{\frac{1}{2}} K_1 \) \( d s \)

Thus, by uniqueness of the limit, \( T_0(t)u_0 = u_0 \), for any \( t > 0 \), so \( u_0 \in E_0 \).

**Proof.** From Lemma 9.1 there exists a bounded set \( B \subset X^\eta \) such that \( \bigcup_{0 \leq \varepsilon \leq \varepsilon_0} A_\varepsilon \subset B \).

Given \( \delta > 0 \), there exists \( t_\delta > 0 \) such that \( T(t_\delta)(B) \subset A^\delta_0 \), where \( A^\delta_0 \) is the \( \delta \)-neighborhood of \( A_0 \).

From Lemma 9.2 there exists \( \bar{\varepsilon} > 0 \) such that \( \| T_\varepsilon(t_\delta)u - T(t_\delta)u \|_{X^\eta} \leq \bar{\delta} \), for every \( u \in B \) and \( 0 \leq \varepsilon \leq \bar{\varepsilon} \). It follows that \( T_\varepsilon(t_\delta)B \subset A^\delta_0 \). In particular, \( T_\varepsilon(t_\delta)A_\varepsilon \subset A^\delta_0 \). Since \( A_\varepsilon \) is invariant under \( T_\varepsilon \), we conclude that \( A_\varepsilon \subset A^\delta_0 \), for \( 0 \leq \varepsilon \leq \bar{\varepsilon} \), thus proving the claim.

From the semicontinuity of attractors, we can easily prove the corresponding property for the equilibria.

**Corollary 9.4.** Suppose the hypotheses of Theorem 9.3 hold. Then the family of sets of equilibria \( \{ E_\varepsilon | 0 \leq \varepsilon \leq \varepsilon_0 \} \), of the problem \( 17 \) is upper semicontinuous in \( X^\eta \).

**Proof.** The result is well-known, but we sketch a proof here for completeness. Suppose \( u_n \in A_n \), with \( \lim_{n \to \infty} \varepsilon_n = 0 \). We choose an arbitrary subsequence and still call it \( (u_n) \), for simplicity. It is enough to show that, there exists a subsequence \( (u_{n_k}) \), which converges to a point \( u_0 \in E_0 \). Since \( (u_n) \to A_0 \), there exists \( (v_n) \in A_0 \) with \( \| u_n - v_n \|_{\eta} \to 0 \). Since \( A_0 \) is compact, there exists a subsequence \( (v_{n_k}) \), which converges to a point \( u_0 \in A_0 \), so also \( (u_{n_k}) \to A_0 \). Now, since the flow \( T_\varepsilon(t) \) is continuous in \( \varepsilon \) we have, for any \( t > 0 \)

\[ u_{n_k} \to u_0 \iff T_{\varepsilon_{n_k}}(t)u_{n_k} \to T_0(t)u_0 \iff u_{n_k} \to T_0(t)u_0. \]

Thus, by uniqueness of the limit, \( T_0(t)u_0 = u_0 \), for any \( t > 0 \), so \( u_0 \in E_0 \).

\[ \square \]
For the lower semicontinuity we will need to assume the following additional properties for the nonlinearities.

\[ f \text{ and } g \text{ are in } C^1(\mathbb{R}, \mathbb{R}) \text{ with bounded derivatives} \ . \tag{46} \]

**Lemma 10.1.** Suppose that \( \eta \) and \( p \) are such that (21) holds and \( f \) satisfies (46). Then the operator \( F: X^n \times \mathbb{R} \rightarrow X^{-\frac{1}{2}} \) given by (19) is Gateaux differentiable with respect to \( u \), with Gateaux differential \( \frac{\partial F}{\partial u}(u, \epsilon)w \) given by

\[
\left\langle \frac{\partial F}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{-\frac{1}{2}, \frac{1}{2}} = \int_{\Omega} f'(u)w \Phi \, dx ,
\]

for all \( w \in X^n \) and \( \Phi \in X^{\frac{1}{2}} \).

**Proof.** Observe first that \( F(u, \epsilon) \) is well-defined, since the conditions of Lemma 6.1 are met.

It is clear that \( \frac{\partial F}{\partial u}(u, \epsilon) \) is linear. We now show that it is bounded. In fact we have, for all \( u, w \in X^n \) and \( \Phi \in X^{-\frac{1}{2}} = W^{1,q} \)

\[
\left| \left\langle \frac{\partial F}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{-\frac{1}{2}, \frac{1}{2}} \right| \leq \int_{\Omega} |f'(u)||w||\Phi| \, dx
\leq \| f' \|_{\infty} \int_{\Omega} |w||\Phi| \, dx
\leq \| f' \|_{\infty} \|w\|_{L^p(\Omega)} \|\Phi\|_{L^q(\Omega)} \, dx
\leq \| f' \|_{\infty} \|w\|_{X^n} \|\Phi\|_{X^{\frac{1}{2}}} \, dx ,
\]

where \( \| f' \|_{\infty} = \sup\{ f'(x) | x \in \mathbb{R} \} \). This proves boundedness.

Now, we have, for all \( u, w \in X^n \) and \( \Phi \in X^{\frac{1}{2}} \)

\[
\left| \frac{1}{t} \left\langle F(u + tw, \epsilon) - F(u, \epsilon) - t \frac{\partial F}{\partial u}(u, \epsilon)w, \Phi \right\rangle_{-\frac{1}{2}, \frac{1}{2}} \right|
\leq \frac{1}{|t|} \int_{\Omega} |[f(u + tw) - f(u) - tf'(u)w] \Phi| \, dx
\leq \frac{1}{|t|} \left( \int_{\Omega} |f(u + tw) - f(u) - tf'(u)w|^p \, dx \right)^{\frac{1}{p}} \|\Phi\|_{X^{\frac{1}{2}}}
\leq \left( \int_{\Omega} \left( f'(u + tw) - f'(u) \right) w \, dx \right)^{\frac{1}{p}} \|\Phi\|_{X^{\frac{1}{2}}},
\]

where $0 \leq \tilde{t} \leq t$. Since $f'$ is bounded and continuous, the integrand of $(I)$ is bounded by an integrable function and goes to 0 as $t \rightarrow 0$. Thus, the integral $(I)$ goes to 0 as $t \rightarrow 0$, from Lebesgue’s Dominated Convergence Theorem. It follows that
\[
\lim_{t \to 0} \frac{F(u + tw, \epsilon) - F(u, \epsilon)}{t} = \frac{\partial F}{\partial u}(u, \epsilon)w \quad \text{in} \quad X^{-\frac{1}{2}}, \quad \text{for all} \quad u, w \in X^\eta; \quad \text{so} \quad F \quad \text{is Gateaux differentiable with Gateaux differential given by} \quad (47).
\]

We now want to prove that the Gateaux differential of $F(u, \epsilon)$ is continuous in $u$. Let us denote by $B(X, Y)$ the space of linear bounded operators from $X$ to $Y$. We will need the following result, whose simple proof is omitted.

**Lemma 10.2.** Suppose $X, Y$ are Banach spaces and $T_n : X \to Y$ is a sequence of linear operators converging strongly to the linear operator $T : X \to Y$. Suppose also that $X_1 \subset X$ is a Banach space, the inclusion $i : X_1 \hookrightarrow X$ is compact and let $\overline{T_n} = T_n \circ i$ and $\overline{T} = T \circ i$. Then $\overline{T}_n \to \overline{T}$ uniformly for $x$ in a bounded subset of $X_1$ (that is, in the or norm of $B(X_1, Y)$).

**Lemma 10.3.** Suppose that $\eta$ and $p$ are such that $(21)$ holds and $f$ satisfies $(46)$. Then the Gateaux differential of $F(u, \epsilon)$, with respect to $u$ is continuous in $u$, that is, the map $u \mapsto \frac{\partial F}{\partial u}(u, \epsilon) \in B(X^\eta, X^{-\frac{1}{2}})$ is continuous.

**Proof.** Let $u_n$ be a sequence converging to $u$ in $X^\eta$, and choose $0 < \tilde{\eta} < \eta$, such that the hypotheses still hold. Then, we have for any $\Phi \in X^{\frac{1}{2}}$ and $w \in X^{\tilde{\eta}}$:
\[
\left| \left\langle \left( \frac{\partial F}{\partial u}(u_n, \epsilon) - \frac{\partial F}{\partial u}(u, \epsilon) \right)w, \Phi \right\rangle \right| \leq \int_{\Omega} \left| f'(u) - f'(u_n) \right| w \Phi \quad dx \leq \left( \int_{\Omega} \left| (f'(u) - f'(u_n))w \right|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\Phi|^q dx \right)^{\frac{1}{q}} \leq \left( \int_{\Omega} \left| (f'(u) - f'(u_n))w \right|^p dx \right)^{\frac{1}{p}} \|\Phi\|_{X^{\frac{1}{2}}}.
\]

Now, the integrand in $(I)$ is bounded by the integrable function $\|f'\|_p w^p$ and goes to 0 a.e. as $u_n \to u$ in $X^\eta$. Therefore the sequence of operators $\frac{\partial F}{\partial u}(u_n, \epsilon)$ converges strongly in the space $B(X^{\tilde{\eta}}, X^{-\frac{1}{2}})$ to the operator $\frac{\partial F}{\partial u}(u, \epsilon)$. From Lemma 10.2 the convergence holds in the norm of $B(X^\eta, X^{-\frac{1}{2}})$, since $X^\eta$ is compactly embedded in $X^\eta$. \[\square\]

**Lemma 10.4.** Suppose that $\eta$ and $p$ are such that $(21)$ holds and $g$ satisfies $(46)$. Then the operator $G : X^\eta \times \mathbb{R} \to X^{-\frac{1}{2}}$ given by $(20)$ is Gateaux differentiable with respect to $u$, with Gateaux differential
\[
\left\langle \frac{\partial G}{\partial u}(u, \epsilon) w, \Phi \right\rangle_{-\frac{1}{2}, -\frac{1}{2}} = \int_{\partial \Omega} g'(\gamma(u))\gamma(w) \gamma(\Phi) \left| \frac{J_0 h_\epsilon}{J h_\epsilon} \right| d\sigma(x), \quad (48)
\]
for all \( w \in X^n \) and \( \Phi \in X^{\frac{1}{2}} \).

**Proof.** Observe first that \( G(u, \epsilon) \) is well-defined, since the conditions of Lemma \[6.2\] are met.

It is clear that \( \frac{\partial G}{\partial u} (u, \epsilon) \) is linear. We now show that it is bounded. In fact we have, for all \( u, w \in X^n \) and \( \Phi \in X^{\frac{1}{2}} \)

\[
\left\| \frac{\partial G}{\partial u} (u, \epsilon) w, \Phi \right\|_{-\frac{1}{2}, \frac{1}{2}} = \left| \int_{\partial \Omega} g'(\gamma(u)) \gamma(w) \gamma(\Phi) \frac{J_{\partial \Omega h_\epsilon}}{J h_\epsilon} \, d\sigma(x) \right|
\]

\[
\leq \|\mu\|_{\infty} \|g'\|_{\infty} \int_{\partial \Omega} |\gamma(w)||\gamma(\Phi)| \, d\sigma(x)
\]

\[
\leq \|\mu\|_{\infty} \|g'\|_{\infty} \|\gamma(w)\|_{L^p(\partial \Omega)} \|\gamma(\Phi)\|_{L^p(\partial \Omega)}
\]

\[
\leq K_1 K_2 \|\mu\|_{\infty} \|g'\|_{\infty} \|w\|_{X^n} \|\Phi\|_{X^{\frac{1}{2}}},
\]

where \( \|g'\|_{\infty} = \sup \{g'(x) | x \in \mathbb{R} \} \), \( \|\mu\|_{\infty} = \sup \{\|\mu(x, \epsilon)\| | x \in \partial \Omega \} = \sup \left\{ \frac{J_{\partial \Omega h_\epsilon}(x)}{J h_\epsilon} | x \in \partial \Omega \right\} \) and \( K_1, K_2 \) are embedding constants given by Theorem \[??\]. This proves boundedness.

Now, we have, for all \( u, w \in X^n \) and \( \Phi \in X^{\frac{1}{2}} \)

\[
\left| \frac{1}{t} \int_{\partial \Omega} \|g(\gamma(u + tw)) - g(\gamma(u)) - tg'(\gamma(u))\| \gamma(\Phi) \frac{J_{\partial \Omega h_\epsilon}}{J h_\epsilon} \, d\sigma(x) \right|
\]

\[
\leq \frac{1}{|t|} \int_{\partial \Omega} \left| \int_{\partial \Omega} \left[ g(\gamma(u + tw)) - g(\gamma(u)) - tg'(\gamma(u)) \right] \gamma(w) \, d\sigma(x) \right| \|\Phi\|_{X^{\frac{1}{2}}}
\]

\[
\leq K_1 \|\mu\|_{\infty} \left\{ \int_{\partial \Omega} \left[ g(\gamma(u + tw)) - g(\gamma(u)) - tg'(\gamma(u)) \right] \gamma(w) \, d\sigma(x) \right\}^{\frac{1}{p}} \|\Phi\|_{X^{\frac{1}{2}}}
\]

\[
\leq K_1 \|\mu\|_{\infty} \left\{ \int_{\partial \Omega} \left[ g'(\gamma(u + tw)) - g'(\gamma(u)) \right] \gamma(w) \, d\sigma(x) \right\}^{\frac{1}{2}} \|\Phi\|_{X^{\frac{1}{2}}},
\]

where \( K_1 \) is the embedding constant given by Theorem \[??\] and Lemma \[??\] and \( 0 \leq \bar{t} \leq t \). Since \( g' \) is bounded and continuous, the integrand of \((I)\) is bounded by an integrable function and goes to 0 as \( t \to 0 \). Thus, the integral \((I)\) goes to 0 as \( t \to 0 \), from Lebesgue’s Dominated Convergence Theorem. It follows that \( \lim_{t \to 0} \frac{G(u + tw, \epsilon) - G(u, \epsilon)}{t} = \frac{\partial G}{\partial u} (u, \epsilon) w \) in \( X^{-\frac{1}{2}} \), for all \( u, w \in X^n \); so \( G \) is Gateaux differentiable with Gateaux differential given by \[(48)\].

---

**Lemma 10.5.** Suppose that \( \eta \) and \( p \) are such that \[(21)\] holds and \( g \) satisfies \[(46)\]. Then the Gateaux differential of \( G(u, \epsilon) \), with respect to \( u \) is continuous in \( u \) (that is, the map \( u \mapsto \frac{\partial G}{\partial u} (u, \epsilon) \in B(X^n, X^{-\frac{1}{2}}) \) is continuous) and uniformly continuous in \( \epsilon \) for \( u \) in bounded sets of \( X^n \) and \( 0 \leq \epsilon \leq \epsilon_0 < 1 \).
Proof. Let $0 \leq \epsilon \leq \epsilon_0$, $u_n$ be a sequence converging to $u$ in $X^n$, and choose $0 < \tilde{\eta} < \eta$, still satisfying the hypotheses. Then, we have for any $\Phi \in X^{\frac{1}{2}}$ and $w \in X^{\tilde{\eta}}$:}

\[
\left| \left\langle \left( \frac{\partial G}{\partial u}(u_n, \epsilon) - \frac{\partial G}{\partial u}(u, \epsilon) \right) w, \Phi \right\rangle \right|_{-\frac{1}{2}, \frac{1}{2}} \\
\leq \int_{\partial \Omega} |g'(\gamma(u)) - g'(\gamma(u_n))\gamma(\Phi)| \left| \frac{J_{\Omega} h_\epsilon}{J h_\epsilon} \right| d\sigma(x) \\
\leq \|\mu_\epsilon\|_{\infty} \left\{ \int_{\partial \Omega} |g'(\gamma(u)) - g'(\gamma(u_n))\gamma(\Phi)|^p d\sigma(x) \right\} \left\{ \int_{\partial \Omega} |\gamma(\Phi)|^q d\sigma(x) \right\}^{\frac{1}{q}} \\
\leq K_1\|\mu_\epsilon\|_{\infty} \left\{ \int_{\partial \Omega} |g'(\gamma(u)) - g'(\gamma(u_n))\gamma(\Phi)|^p d\sigma(x) \right\} \|\Phi\|_{X^{\frac{1}{2}}},
\]

where $K_1$ is the constant due to continuity of the trace map from $X^{\frac{1}{2}}$ into $L^2(\partial \Omega)$, as in Lemma 6.2.

Now, the integrand in $(I)$ is bounded by the integrable function $||g'||^2 |\gamma(w)|^2$ and goes to $0$ a.e. as $u_n \to u$ in $X^n$. Therefore the sequence of operators $\frac{\partial G}{\partial u}(u_n, \epsilon)$ converges strongly in the space $B(X^{\tilde{\eta}}, X^{-\frac{1}{2}})$ to the operator $\frac{\partial G}{\partial u}(u, \epsilon)$. From Lemma 10.2 the convergence holds in the norm of $B(X^n, X^{-\frac{1}{2}})$, since $X^n$ is compactly embedded in $X^{\tilde{\eta}}$ (see [10]).

Finally, if $0 \leq \epsilon_1 \leq \epsilon_2 < \epsilon_0$, we have for any $\Phi \in X^{\frac{1}{2}}$ and $w \in X^n$:}

\[
\left| \left\langle \left( \frac{\partial G}{\partial u}(u, \epsilon_1) - \frac{\partial G}{\partial u}(u, \epsilon_2) \right) w, \Phi \right\rangle \right|_{-\frac{1}{2}, \frac{1}{2}} \\
\leq \int_{\partial \Omega} |g'(\gamma(u))\gamma(\Phi)| |\mu_{\epsilon_1} - \mu_{\epsilon_2}| d\sigma(x), \\
\leq \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty} \left\{ \int_{\partial \Omega} |g'(\gamma(u))\gamma(\Phi)|^p d\sigma(x) \right\} \left\{ \int_{\partial \Omega} |\gamma(\Phi)|^q d\sigma(x) \right\}^{\frac{1}{q}} \\
\leq K_1K_2\|g'\|_{\infty} \|w\|_{X^n} \|\Phi\|_{X^{\frac{1}{2}}} \|\mu_{\epsilon_1} - \mu_{\epsilon_2}\|_{\infty},
\]

where $K_2$ is the constant due to continuity of the trace map from $X^n$ into $L^q(\partial \Omega)$, as before. This proves uniform continuity in $\epsilon$. \hfill $\square$

Lemma 10.6. Suppose that $\eta$ and $p$ are such that (21) holds and $f$ and $g$ satisfy (40). Then, the map $(H_\epsilon)_{-\frac{1}{2}} = (F_\epsilon)_{-\frac{1}{2}} + (G_\epsilon)_{-\frac{1}{2}} : X^n \times \mathbb{R} \mapsto X^{-\frac{1}{2}}$ given by (18) is continuously Fréchet differentiable with respect to $u$ and the derivative $\frac{\partial G}{\partial u}$ is uniformly continuous with respect to $\epsilon$, for $u$ in bounded sets of $X^n$ and $0 \leq \epsilon \leq \epsilon_0 < 1$.

Proof. The proof follows from Lemmas 10.3, 10.5 and Proposition 2.8 in [17]. \hfill $\square$
We now prove lower semicontinuity for the equilibria.

**Theorem 10.7.** If \( f \) and \( g \) satisfy the conditions of Theorem 7.5 and also (40), the equilibria of (17) with \( \epsilon = 0 \) are all hyperbolic and \( \frac{1}{4} < \eta < \frac{1}{2} \), then the family of sets of equilibria \( \{ E_\epsilon \mid 0 \leq \epsilon < \epsilon_0 \} \) of (17) is lower semicontinuous in \( X^\eta \) at \( \epsilon = 0 \).

**Proof.** A point \( u \in X^\eta \) is an equilibrium of (17) if and only if it is a root of the map

\[
Z : W^{1,p}(\Omega) \times \mathbb{R} \rightarrow X^{-\frac{1}{2}}
\]

\[
(u, \epsilon) \mapsto (A_\epsilon)_{-\frac{1}{2}}(u) + (H_\epsilon)_{-\frac{1}{2}}(u),
\]

By Lemma 10.6, the map \((H_\epsilon)_{-\frac{1}{2}} : X^\eta \rightarrow X^{-\frac{1}{2}}\) is continuously Fréchet differentiable with respect to \( u \) and by Lemmas 6.3 and 6.1 it is also continuous in \( \epsilon \) if \( \eta = \frac{1}{2} - \delta \), with \( \delta > 0 \) is sufficiently small. Therefore, the same holds if \( \eta = \frac{1}{2} \).

The map \( A_\epsilon = -h_\epsilon^* \Lambda_\Omega h_\epsilon^* + aI \) is a bounded linear operator from \( W^{1,p}(\Omega) \) to \( X^{-\frac{1}{2}} \). It is also continuous in \( \epsilon \) since it is analytic as a function of \( h_\epsilon \in Diff^1(\Omega) \) and \( h_\epsilon \) is continuous in \( \epsilon \).

Thus, the map \( Z \) is continuously differentiable in \( u \) and continuous in \( \epsilon \). The derivative of \( \frac{\partial Z}{\partial u} (e, 0) \) is an isomorphism by hypotheses. Therefore, the Implicit Function Theorem apply, implying that the zeroes of \( Z(\cdot, \epsilon) \) are given by a continuous function \( e(\epsilon) \). This proves the claim. \( \square \)

To prove the lower semi continuity of the attractors, we also need the continuity of local unstable manifolds at equilibria.

**Theorem 10.8.** Suppose that \( \eta \) and \( p \) are such that (21) holds and \( f \) and \( g \) satisfy (40), \( u_0 \) is an equilibrium of (17) with \( \epsilon = 0 \), and for each \( \epsilon > 0 \) sufficiently small, let \( u_\epsilon \) be the unique equilibrium of (17), whose existence is asserted by Corollary 9.4 and Theorem 10.7.

Then, for \( \epsilon \) and \( \delta \) sufficiently small, there exists a local unstable manifold \( W^u_{loc}(u_\epsilon) \) of \( u_\epsilon \), and if we denote \( W^u_\delta(u_\epsilon) = \{ w \in W^u_{loc}(u_\epsilon) \mid \| w - u_\epsilon \|_{X^\eta} < \delta \} \), then

\[
-\frac{1}{2} \left( W^u_\delta(u_\epsilon), W^u_\delta(u_\epsilon) \right) \quad \text{and} \quad -\frac{1}{2} \left( W^u_\delta(u_\epsilon), W^u_\delta(u_\epsilon) \right)
\]

approach zero as \( \epsilon \to 0 \), where \(-\frac{1}{2}(O, Q) = \sup_{o \in O, q \in Q} \| q - o \|_{X^\eta} \) for \( O, Q \subset X^\eta \).

**Proof.** Let \( H_\epsilon(u) = H(u, \epsilon) \) be the map defined by (18) and \( u_\epsilon \) a hyperbolic equilibrium of (17). Since \( H(u, \epsilon) \) is differentiable by Lemma 10.6, it follows that \( H_\epsilon(u_\epsilon + w, \epsilon) = H_\epsilon(u_\epsilon, \epsilon) + H_\epsilon(u_\epsilon, \epsilon)w + r(w, \epsilon) = A_\epsilon u_\epsilon + H_\epsilon(u_\epsilon, \epsilon)w + r(w, \epsilon) \), with \( r(w, \epsilon) = o(\| w \|_{X^\eta}) \), as \( \| w \|_{X^\eta} \to 0 \). The claimed result was proved in [16], assuming the following properties of \( H_\epsilon \):

- a) \( \| r(w, 0) - r(w, \epsilon) \|_{X^{-\frac{1}{2}}} \leq C(\epsilon) \), with \( C(\epsilon) \to 0 \) when \( \epsilon \to 0 \), uniformly for \( w \) in a neighborhood of \( 0 \) in \( X^\eta \).
- b) \( \| r(w_1, \epsilon) - r(w_2, \epsilon) \|_{X^{-\frac{1}{2}}} \leq k(\rho) \| w_1 - w_2 \|_{X^\eta} \), for \( \| w_1 \|_{X^\eta} \leq \rho \), \( \| w_2 \|_{X^\eta} \leq \rho \), with \( k(\rho) \to 0 \) when \( \rho \to 0^+ \) and \( k(\cdot) \) is non-decreasing.

Property a) follows from easily from the fact that both \( H(u, \epsilon) \) and \( H_\epsilon(u, \epsilon) \) are uniformly continuous in \( \epsilon \) for \( u \) in bounded sets of \( X^\eta \), by Lemmas 6.3, 6.1, and 10.6. It remains to prove property b).
If \( w_1, w_2 \in X^n \) and \( \epsilon \in [0, \epsilon_0] \), with \( 0 < \epsilon_0 < 1 \) small enough, we have

\[
\| r(w_1, \epsilon) - r(w_2, \epsilon) \|_{X^{-\frac{1}{2}}} = \| H(u_\epsilon + w_1, \epsilon) - H(u_\epsilon, \epsilon) - H(u_\epsilon, \epsilon)w_1 \\
- H(u_\epsilon + w_2, \epsilon) + H(u_\epsilon, \epsilon) + H(u_\epsilon, \epsilon)w_2 \|_{X^{-\frac{1}{2}}} \\
\leq \| F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F(u_\epsilon, \epsilon)w_1 \\
- F(u_\epsilon + w_2, \epsilon) + F(u_\epsilon, \epsilon) + F(u_\epsilon, \epsilon)w_2 \|_{X^{-\frac{1}{2}}} \\
+ \| G(u_\epsilon + w_1, \epsilon) - G(u_\epsilon, \epsilon) - G(u_\epsilon, \epsilon)w_1 \\
- G(u_\epsilon + w_2, \epsilon) + G(u_\epsilon, \epsilon) + G(u_\epsilon, \epsilon)w_2 \|_{X^{-\frac{1}{2}}}.
\]

We first estimate (49). Since \( f' \) is bounded by (446), we have

\[
\left| \left\langle f'(u_\epsilon + w_1) - f'(u_\epsilon), \Phi \right\rangle \right| \leq \int_\Omega \left( |f'(u_\epsilon + \xi_x) - f'(u_\epsilon)|^p + |f(u_\epsilon + w_2) - f(u_\epsilon)|^p + |f'(u_\epsilon)w_2| \right) \Phi \, dx \\
\leq K_1 \left\{ \int_\Omega |f'(u_\epsilon + \xi_x) - f'(u_\epsilon)|^p (w_1(x) - w_2(x))^p \, dx \right\}^{\frac{1}{p}} \| \Phi \|_{X^\frac{1}{2}} \\
\leq K_1 K_2 \left\{ \int_\Omega |f'(u_\epsilon + \xi_x) - f'(u_\epsilon)|^p \, dx \right\}^{\frac{1}{p}} \| w_1 - w_2 \|_{X^n} \cdot \| \Phi \|_{X^\frac{1}{2}},
\]

where \( K_1 \) is the embedding constant of \( X^{\frac{1}{2}} \) into \( L^p(\Omega) \), \( K_2 \) is the embedding constant of \( X^n \) in \( L^\infty(\Omega) \) and \( w_1(x) \leq \xi_x \leq w_2(x) \) or \( w_2(x) \leq \xi_x \leq w_1(x) \). Therefore, we have

\[
\| F(u_\epsilon + w_1, \epsilon) - F(u_\epsilon, \epsilon) - F(u_\epsilon, \epsilon)w_1 - F(u_\epsilon + w_2, \epsilon) + F(u_\epsilon, \epsilon) + F(u_\epsilon, \epsilon)w_2 \|_{X^{-\frac{1}{2}}} \\
\leq K_1 K_2 \left\{ \int_\Omega |f'(u_\epsilon + \xi_x) - f'(u_\epsilon)|^p \, dx \right\}^{\frac{1}{p}} \| w_1 - w_2 \|_{X^n}.
\]

Now the integrand above is bounded by \( 2^p \| f' \|_{L^p}^p \) and goes a.e. to 0 as \( \rho \to 0 \), since \( \| w_1 \|_{X^n} \leq \rho \), \( \| w_2 \|_{X^n} \leq \rho \) and \( w_1(x) \leq \xi_x \leq w_2(x) \). Thus, the integral goes to 0 by Lebesgue’s bounded convergence Theorem.

We now estimate (50):

\[
\left( G(u_\epsilon + w_1, \epsilon) - G(u_\epsilon, \epsilon) - G(u_\epsilon, \epsilon)w_1 - G(u_\epsilon + w_2, \epsilon) + G(u_\epsilon, \epsilon) + G(u_\epsilon, \epsilon)w_2 \right) \| \Phi \|_{X^{-\frac{1}{2}}} \\
\leq \int_{\partial \Omega} \left| g(\gamma(u_\epsilon + w_1)) - g(\gamma(u_\epsilon)) - g'(\gamma(u_\epsilon))w_1 \\
- g(\gamma(u_\epsilon + w_2)) + g(\gamma(u_\epsilon)) + g'(\gamma(u_\epsilon))w_2 \right| \, d\sigma(x)
\]

\[
- g(\gamma(u_\epsilon + w_2)) + g(\gamma(u_\epsilon)) + g'(\gamma(u_\epsilon))w_2 \gamma(\Phi) \left( \left| \frac{J_{h_{\epsilon^2}}}{J_{h_{\epsilon}}(\Phi)} \right| \right) \, d\sigma(x)
\]
\[
\int_{\partial \Omega} \left| g'(\gamma(u_e + \xi_x)) - g'(\gamma(u_e)) \right| \gamma(w_1(x) - w_2(x)) \gamma(\Phi) \gamma \left( \left| \frac{J_{ \partial \Omega} h_e}{J h_e} \right| \right) \, d\sigma(x)
\]

\[
\leq K_1 \left\{ \int_{\partial \Omega} \left[ (g'(\gamma(u_e + \xi_x)) - g'(\gamma(u_e))) \right]^p \gamma(w_1(x) - w_2(x))^p \, d\sigma(x) \right\}^{\frac{1}{p}} \|\Phi\|_{X_\frac{1}{2}^p},
\]

where \( \mu_e = \left| \frac{J_{ \partial \Omega} h_e}{J h_e} \right| \) is bounded, uniformly in \( \epsilon \) and \( w_1(x) \leq \xi_x \leq w_2(x) \) or \( w_2(x) \leq \xi_x \leq w_3(x) \).

Now the integrand above is bounded by \( 2^p \| g' \|_{X_\frac{1}{2}^p} \) and goes to 0 a.e. as \( \rho \to 0 \), since \( \| w_1 \|_{X^\rho} \leq \rho, \| w_2 \|_{X^\rho} \leq \rho \) and \( w_1(x) \leq \xi_x \leq w_2(x) \). Thus, the integral goes to 0 by Lebesgue’s dominated convergence Theorem.

We are now in a position to prove the main result of this section

**Theorem 10.9.** Assume the hypotheses of Theorem 10.7 hold. Then the family of attractors \( \{A \epsilon \mid 0 \leq \epsilon \leq \epsilon_0 \} \), of the problem \[17\], whose existence is guaranteed by Theorem 8.2 is lower semicontinuous in \( X^\eta \).

**Proof.**

The system generated by \[17\] is gradient for any \( \epsilon \) and its equilibria are all hyperbolic for \( \epsilon \) in a neighborhood of 0. Also, the equilibria are continuous in \( \epsilon \) by Theorem 10.7, the linearisation is continuous in \( \epsilon \) as shown during the proof of Theorem 10.7 and the local unstable manifolds of the equilibria are continuous in \( \epsilon \), by Theorem 10.8. The result follows then from \[16\], Theorem 3.10.

**References**

[1] Arrieta, J.M. and Bruschi, S.M., *Rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a Lipschitz deformation*, Mathematical Models and Methods in Applied Sciences, vol. 17, number 10, (2007).

[2] Arrieta, J.M. and Bruschi, S.M., *Very rapidly varying boundaries in equations with nonlinear boundary conditions. The case of a non uniformly Lipschitz deformation*, Discrete and Continuous Dynamical Systems Serie B, vol. 14, number 2, (2010).

[3] Arrieta, J.M. and Carvalho, A.N., *Spectral convergence and nonlinear dynamics of reaction-diffusion equations under perturbations of the domain*, J. Differential Equations 199 (2004) 143-178.

[4] Arrieta, J.M., Carvalho A.N. and Bernal, A.R., *Attractors for parabolic problems with nonlinear boundary condition. Uniform bounds*, Comm. Partial Differential Equations 25 (1-2), 1-37, (2000).

[5] Barbosa, P.S., Pereira, A.L. and Pereira, M.C., *Continuity of attractors for a family of \( C^1 \) perturbations of the square*, Annali di Matematica Pura ed Applicata, v. 196, p. 1365-1398, (2017).

[6] Carvalho, A.N., Oliva, S.M., Pereira, A.L. and Bernal, A.R., *Parabolic problems with nonlinear boundary conditions*, Journal of Mathematical Analysis and Applications 207, 409-461 (1997).

[7] Grisvard, P., *Elliptic problems in nonsmooth domains*. In: Classics in Applied Mathematics, vol. 69, SIAM, Philadelphia (2011).

[8] Hale, J. K., *Asymptotic behaviour of dissipative systems,*
[9] Henry, D.B., *Perturbation of the boundary in boundary-value problems of partial differential equations*, Cambridge Univ. Press, Cambridge (2005).

[10] Henry, D.B., *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, Berlin (1981).

[11] Kato, T. *Perturbation theory for linear operators*, Springer-Verlag.

[12] Micheletti, A.M., *Pertubazione dello spetro dell operatore de Laplace in relazione ad una variazone del campo*, Ann. Sc. Norm. Super. Pisa 26 (1972) 151-169.

[13] Oliva, S. M. and Pereira, A. L., *Attractors for parabolic problems with nonlinear boundary conditions in fractional power spaces*, Dyn. Contin. Discrete Impuls. Syst. Ser. A. Math. Anal. 9, 551-562, (2002).

[14] Oliveira, L.A.F., Pereira, A. L. and Pereira, M. C., *Continuity of attractors for a reaction-diffusion problem with respect to variations of the domain*, Electron. J. Differential Equations (100), 1-18, (2005).

[15] Pao C. V., *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, (1992).

[16] Pereira, A. L. and Pereira, M. C., *Continuity of attractors for a reaction-diffusion problem with nonlinear boundary conditions with respect to variations of the domain*, Journal of Differential Equations 239, 343-370, (2007).

[17] Rall L.B., *Nonlinear Functional Analysis and Applications*, Academic Press, 1971.

[18] Smoller, J., *Shock waves and reaction-diffusion equations*, Springer-Verlag, 1982.

[19] Teman, R. Smoller, J., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences Series Volume 68, Springer-Verlag New York, 1997, DOI 10.1007/978-1-4612-0645-3.

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