Quantum mechanics of higher derivative systems
and total derivative terms

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Abstract

A general theory is presented of quantum mechanics of singular, non-autonomous, higher derivative systems. Within that general theory, $n$-th order and $m$-th order Lagrangians are shown to be quantum mechanically equivalent if their difference is a total derivative.

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I. INTRODUCTION

Higher derivative theories occur in various aspects of modern physics—gravity, strings, particle phenomenology, and so on. It is of importance to clarify general properties of such theories.

The purpose of this paper is to prove a very simple, but important theorem for the quantum mechanics of higher derivative theories: *Total derivative terms in a Lagrangian never affect the quantum mechanics*. The theorem is proven within the most general, i.e. singular and non-autonomous (explicitly time-dependent), situation. Needless to say, the classical version of the theorem is well known and is trivial. And the classical result often plays an important role in various theories. Surprisingly enough, the quantum version, despite its importance, has not been proven up to now. It is in fact non-trivial and requires a proof.

Our plan is as follows:

In Sec.II, the canonical theory, now known as the Ostrogradski formalism [1], is reviewed and the canonical quantization à la Dirac is performed. Sec.III is devoted to the proof of the theorem.

II. GENERAL THEORY OF HIGHER DERIVATIVE SYSTEMS

A. The Ostrogradski formalism

Let us consider a Lagrangian which depends on the coordinates $q^i$ and their time derivatives up to the $N$-th order,

$$L(q^{(0)}, q^{(1)}, \ldots, q^{(N)}, t),$$  \hspace{1cm} (2.1)

where $q^{(I)}$'s are abbreviations for $q^{(I)i} := d^I q^i / dt^I$, $I = 0, 1, \ldots, N$. The Euler-Lagrange equations for (2.1) are given by

$$\sum_{I=0}^{N} \left( -\frac{d}{dt} \right)^I \frac{\partial L}{\partial q^{(I)i}} = 0.$$ \hspace{1cm} (2.2)
We introduce canonical variables,

\[ q^{I_i} := q^{(I_i)}, \]
\[ p_{I_i} := \sum_{K=I+1}^{N} \left(-\frac{d}{dt}\right)^{K-I-1} \frac{\partial L}{\partial q^{(K)_{I_i}}}, \]

(2.3a) (2.3b)

to parametrize the \(2N\) dimensional (for each \(i\)) phase space of (2.2). Throughout the present paper, we take the convention that \(I, J,\) and \(K\) run from 0 to \(N-1\), unless otherwise stated.

There are relations as follows,

\[ \frac{\partial L}{\partial q^{(0)_{I_i}}} = \frac{dp_{I_i}}{dt}, \]
\[ p_{A_i} = \frac{\partial L}{\partial q^{(A+1)_{I_i}}} - \frac{d}{dt}p_{A+1,i}, \]

(2.4a) (2.4b)

with \(A = 0, 1, \ldots, N-2\). Eqs.(2.4a) are the Euler-Lagrange equations (2.2).

The Lagrangian (2.1) with \(\det(\partial^2 L/\partial q^{(N)_{I_i}} \partial q^{(N)_{I_j}}) = 0\) is singular, which means there exist primary constraints,

\[ \phi_m(q, p, t) = 0, \]

(2.5)

where \(q\) and \(p\) are abbreviations for \(q^{I_i}\)’s and \(p_{I_i}\)’s. The number of primary constraints is \textit{larger than} or equal to the nullity of the Hessian matrix \(\|\partial^2 L/\partial q^{(N)_{I_i}} \partial q^{(N)_{I_j}}\|\); the larger case may occur when \(N \geq 2\).

The canonical Hamiltonian is defined by

\[ H(q, p, t) := \sum_{A=0}^{N-2} p_{A_i}q^{A+1,i} + p_{N-1,i}q^{N-1,i} - L(q, q^{N-1}, t), \]

(2.6)

which is conserved if the system is autonomous:

\[ \frac{dH}{dt} + \frac{\partial L}{\partial t} = 0. \]

(2.7)

Note that the canonical Hamiltonian (2.6) have some ambiguity as a function of \(q, p,\) and \(t\) because of the relations \(q^{I+1,i} = \dot{q}^{I_i}\). We fix the ambiguity by distinguishing \(q^{I+1,i}\)’s and \(\dot{q}^{I_i}\)’s as appeared in Eq.(2.6). With this distinction, Hamilton’s canonical equations of motion become equivalent to the original Euler-Lagrange equations (2.2). If we chose another
form, we would obtain a Hamiltonian formalism not equivalent to the original Lagrangian formalism.

Once we reach here, higher derivative theories do not differ much from usual theories with first order derivatives. The well-known Dirac procedure \[1,2\] for singular Lagrangians is applied to higher derivative systems without any modifications:

All the constraints (the primary and the secondary ones) are classified into first-class, $\gamma_\alpha \approx 0$, and second class, $\chi_\alpha \approx 0$. The Poisson bracket is defined by

\[
\{F,G\} := \frac{\partial F}{\partial q^I_i} \frac{\partial G}{\partial p_{Ii}} - \frac{\partial F}{\partial p_{Ii}} \frac{\partial G}{\partial q^I_i},
\] (2.8)

with which the equations of motion for an arbitrary quantity $F(q,p,t)$ is

\[
\dot{F} \approx \{F, H_T\} + \frac{\partial F}{\partial t}.
\] (2.9)

The total Hamiltonian $H_T$ is defined as

\[
H_T := H + u^{a_1} \chi_{a_1} + \lambda^{a_1} \gamma_{a_1},
\] (2.10)

where $H$ is the canonical Hamiltonian (2.6), $u^{a_1}$’s are functions of $q$, $p$, and $t$ determined by consistency, and $\lambda^{a_1}$’s are the arbitrary Lagrange multipliers. The indeces $a_1$ and $a_1$ run only on the primary constraints. The second-class constraints $\chi_\alpha \approx 0$ become strong equations $\chi_\alpha = 0$ in terms of the Dirac bracket,

\[
\{F,G\}^* := \{F,G\} - \{F,\chi_\alpha\} C^{-1\alpha\beta} \{\chi_\beta, G\},
\] (2.11)

with $C_{\alpha\beta} := \{\chi_\alpha, \chi_\beta\}$.

B. Quantum mechanics

The quantization is formally performed by replacing the Dirac bracket $\{ , \}^*$ by the quantum commutator $(i\hbar)^{-1}[ , ]$. Unfortunately, it is incredibly difficult to find out the operator representation of the Dirac bracket. In order to proceed further, it is desirable
to circumvent the Dirac bracket. Here let us remember a general result [2] that second-
class constraints can always be turned into first-class constraints, if necessary, by adding
extravariables. Without losing generality, therefore, we assume there is no second-class
constraints. Then we have only to consider the Poisson bracket. Quantization is performed
by replacing the Poisson bracket by the commutator, and the first-class constraints by the
subsidiary conditions on the wave function.

Let us take the Schrödinger picture with coordinate representation. The commutator
algebra is represented by $\hat{q}^{Ii} = q^{Ii}$, and $\hat{p}_{Ii} = -i\hbar \partial / \partial q^{Ii}$. One obtains the Schrödinger
equation
\[i\hbar \frac{\partial \psi(q, t)}{\partial t} = \hat{H}(q, -i\hbar \frac{\partial}{\partial q}, t) \psi(q, t),\]  
and the subsidiary conditions
\[\dot{\gamma}_a(q, -i\hbar \frac{\partial}{\partial q}, t) = 0,\]  
in which $q$ and $\partial / \partial q$ are abbreviations for $q^{Ii}$’s and $\partial / \partial q^{Ii}$’s.

Finally I finish this section by emphasizing peculiarity of the quantum mechanics of
higher derivative systems: Since the coordinates and velocities are both treated as coor-
dinates in the Ostrogradski formalism, the uncertainty principle does not work between
them!

### III. THE REDUCTION THEOREM

#### A. Problem setting

Consider an $(N - 1)$-th order Lagrangian $L^\sharp$ that may be singular. We define an $N$-th
order Lagrangian $L$ as follows:

\[L(q^{(0)}, q^{(1)}, \ldots, q^{(N)}, t) := L^\sharp(q^{(0)}, q^{(1)}, \ldots, q^{(N-1)}, t) + \frac{d}{dt} W(q^{(0)}, q^{(1)}, \ldots, q^{(N-1)}, t),\]  
where $W$ is an arbitrary function of $q^{(Ii)}$’s and $t$. Needless to say, $L$ and $L^\sharp$ are classically
equivalent because the total derivative term is turned, in the action integral, into the surface
term, which does not affect the classical equations of motion. Nevertheless their quantum equivalence is non-trivial. Note that $L$ and $L^♯$ belong to different order Lagrangians. As is stated in Sec.[2], different order Lagrangians lead to different conjugate pairs, which means the different uncertainty principle. The canonical variables \((2.3)\) for the Lagrangian \((3.1)\) is given as

\begin{align}
q^I_i &= q^{(I)i}, \tag{3.2a} \\
p_{N-1,i} &= \frac{\partial W}{\partial q^{(N-1)i}}, \tag{3.2b} \\
p_{A_i} &= p_{A_i}^♯ + \frac{\partial W}{\partial q^{(A)i}}. \tag{3.2c}
\end{align}

We take the convention that $I, J, K$ run from 0 to $N-1$, and $A, B, C$ run from 0 to $N-2$, unless otherwise stated. Here $p_{A_i}^♯$'s are the canonical momentum in the $L^♯$-theory,

\begin{align}
p_{N-2,i}^♯ &:= \frac{\partial L^♯}{\partial q^{(N-1)i}}, \tag{3.3a} \\
p_{A_i}^♯ &:= \frac{\partial L^♯}{\partial q^{(A+1)i}} - \frac{d}{dt}p_{A_i+1}^♯, \tag{3.3b}
\end{align}

with $A = 0, 1, \ldots, N-3$. The canonical Hamiltonian \((2.6)\) becomes

\begin{equation}
H(q, p, t) = p_{A_i}q^{A+1,i} + p_{N-1,i}q^{N-1,i} - L(q, \dot{q}^{N-1}, t) \\
= H^♯(q, p^♯, t) - \frac{\partial W(q, t)}{\partial t}, \tag{3.4}
\end{equation}

with

\begin{equation}
H^♯(q, p^♯, t) := p_{A_i}q^{A+1,i} - L^♯(q, t). \tag{3.5}
\end{equation}

We should distinguish $H^♯(q, p^♯, t)$ defined here from the canonical Hamiltonian in the $L^♯$-theory,

\begin{equation}
H^♯(q^♯, p^♯, t) := \sum_{A=0}^{N-3} p_{A_i}q^{A+1,i} + p_{N-2,i}q^{N-2,i} - L^♯(q^♯, \dot{q}^{N-2}, t), \tag{3.6}
\end{equation}

where $q^♯$ is an abbreviation for $q^{A_i}$'s, while $q$ without $♯$ is the abbreviation for $q^{I_i}$'s.

These two $H^♯$'s, \((3.3)\) and \((3.6)\), are the same in their value but different as functions of canonical variables. This implies the following: While Eq.\((3.4)\) with Eq.\((3.5)\) defines a
Hamiltonian system equivalent to the original Lagrangian system (3.1), the use of Eq.(3.6) instead of Eq.(3.5) results in another Hamiltonian system not equivalent to the original Lagrangian system. In other words, Eq.(3.6) have forgotten the relations

\[ \dot{q}^{N-2,i} = q^{N-1,i}. \]  

(3.7)

Eq.(3.5), on the other hand, remember them as canonical equations, \( \dot{q}^{N-2,i} = \frac{\partial H(q,p,t)}{\partial q^{N-1,i}}. \) Thus Eq.(3.6) defines a larger theory, in which the original one is contained as a special case. If we impose, by hand, the relation (3.7) on the larger theory, then it reduces to the original one.

Since the larger theory is quite useful for our purposes, we use, in the following, (3.6) instead of (3.5). As will be stated in Sec.III C, the larger theory is a kind of gauge theory and we can impose Eq.(3.7) as a gauge fixing condition to it.

**B. Constraint analysis**

Equations (3.2b) are primary constraints:

\[ \gamma_i(q,p,t) := p_{N-1,i} - \frac{\partial W}{\partial q^{N-1,i}} \approx 0. \]  

(3.8)

When \( L^i \) is singular, i.e. \( \det(\partial^2 L^i / \partial q^{(N-1)i} q^{(N-1)j}) = 0 \), there are other primary constraints in addition to \( \gamma_i \)’s:

\[ \gamma^\sharp_a(q^\sharp, p^\sharp, t) \approx 0, \]  

(3.9)

which stem from (3.2c) together with (3.3). It is not a hard task to show

\[ \{ \gamma_i, \gamma_j \} = -\left\{ p_{N-1,i}, \frac{\partial W}{\partial q^{N-1,j}} \right\} - \left\{ \frac{\partial W}{\partial q^{N-1,i}}, p_{N-1,j} \right\} \]

\[ = \frac{\partial^2 W}{\partial q^{N-1,i} \partial q^{N-1,j}} - \frac{\partial^2 W}{\partial q^{N-1,j} \partial q^{N-1,i}} \]

\[ = 0, \]

\[ \{ \gamma_i, \gamma^\sharp_a \} = \left\{ p_{N-1,i}, q^{N-1,k} \right\} \frac{\partial \gamma^\sharp_a}{\partial p_{Aj}} \frac{\partial p_{Aj}}{\partial q^{N-1,k}} - \left\{ \frac{\partial W}{\partial q^{N-1,i}}, p_{Bk} \right\} \frac{\partial \gamma^\sharp_a}{\partial p_{Aj}} \frac{\partial p_{Aj}}{\partial p_{Bk}} \]  

(3.10a)
\[
\frac{\partial \gamma^a_i}{\partial p^A_{Aj}} \frac{\partial^2W}{\partial q^{N-1,i} \partial q^{Aj}} - \frac{\partial^2W}{\partial q^{A^i} \partial q^{N-1,i}} \frac{\partial \gamma^a_i}{\partial p^A_{Aj}} = 0.
\]

(3.10b)

The total Hamiltonian is given by

\[
H_T = H + \lambda^a \gamma^a_i + \lambda^i \gamma_i,
\]

(3.11)

where \(\lambda^a\)'s and \(\lambda^i\)'s are the Lagrange multipliers. Straightforward calculation shows that \(\gamma_i\)'s do not produce secondary constraints:

\[
\dot{\gamma}_i \approx \{\gamma_i, H_T\} + \frac{\partial \gamma_i}{\partial t} \approx \{\gamma_i, H^z - \frac{\partial W}{\partial t}\} + \frac{\partial \gamma_i}{\partial t} = \frac{\partial H^z}{\partial p^A_{Aj}} \frac{\partial^2W}{\partial q^{N-1,i} \partial q^{Aj}} + \frac{\partial^2W}{\partial q^{A^i} \partial q^{N-1,i}} \frac{\partial H^z}{\partial t} \frac{\partial H^z}{\partial q^{N-1,i}} - \frac{\partial^2W}{\partial t \partial q^{N-1,i}} = 0.
\]

(3.12)

Therefore one concludes that \(\gamma_i\)'s are first-class. This conclusion were not obtained if we would adopt Eq.(3.5) in Eq.(3.4). As for \(\gamma^a\)'s, one obtains

\[
\dot{\gamma}^a_i \approx \{\gamma^a_i, H_T\} + \frac{\partial \gamma^a_i}{\partial t} \approx \{\gamma^a_i, H^z\} + \frac{\partial \gamma^a_i}{\partial p^A_{Ai}} \frac{\partial^2W}{\partial q^{N-1,i} \partial q^{A_i}} + \lambda^i \{\gamma^a_i, \gamma^a_i\} - \frac{\partial^2W}{\partial p^A_{Ai}} \frac{\partial \gamma^a_i}{\partial t} \frac{\partial q^{N-1,i}}{\partial q^{A_i}} + \left(\frac{\partial \gamma^a_i}{\partial t}\right)^z,
\]

(3.13)

where

\[
H^z_T := H^z + \lambda^a \gamma^a_i
\]

(3.14)

is the total Hamiltonian in the \(L^z\)-theory. The symbol \((\partial/\partial t)^z\) represents partial derivative by \(t\) with \(q^z\) and \(p^z\) fixed. One can prove

\[
\{q^{A_i}, p^z_{Bj}\} = \delta^A_B \delta^i_j,
\]

(3.15a)

\[
\{q^{A_i}, q^z_{Bj}\} = \{p^z_{A_i}, p^z_{Bj}\} = 0,
\]

(3.15b)
which mean
\[
\{ F^\sharp, G^\sharp \} = \{ F^\sharp, G^\sharp \}_z, \tag{3.16}
\]
where \( F^\sharp \) and \( G^\sharp \) are arbitrary functions of \( q_\sharp, p_\sharp, \) and \( t. \) The symbol \( \{ , \}_z \) is the Poisson bracket in the \( L^\sharp \)-theory. Using (3.16), one can rewrite (3.13) as
\[
\dot{\gamma}_a^\sharp \approx \{ \gamma_a^\sharp, H^\sharp \}_z + \left( \frac{\partial \gamma_a^\sharp}{\partial t} \right)_z. \tag{3.17}
\]
Note that the only property we assumed in deriving the above equations is that \( \gamma_a^\sharp \)'s are functions of \( q_\sharp, p_\sharp, \) and \( t. \) Thus Eqs.(3.17) remain valid even if secondary constraints are substituted for \( \gamma_a^\sharp \)'s. Thus we have proven that all the secondary constraints emerge from \( \gamma_a^\sharp \)'s are just the same as the ones derived in the \( L^\sharp \)-theory.

As is stated in Sec.II B, in the following, we do assume that \( \gamma_a^\sharp \)'s and the secondary constraints derived from them are all first-class. Let us write them again as \( \gamma_a^\sharp \approx 0. \) Then all the constraints in our Lagrangian (3.1) are exhausted by (3.8) and (3.9). The final form of the total Hamiltonian is (3.11), in which the summation on \( a \) should be taken only on the primary constraints. (The summation on all the first-class constraints defines the extended Hamiltonian formalism \[2\]; our discussion in what follows remains valid even if we take the extended formalism.)

C. Gauge transformations

In this subsection we investigate the gauge transformation derived from \( \gamma_i \)'s. For an arbitrary quantity \( F(q, p, t), \) the gauge transformation is defined as
\[
\delta F := \varepsilon^i \{ F, \gamma_i \}, \tag{3.18}
\]
where \( \varepsilon^i \)'s are arbitrary functions of \( t, \) but are independent of the canonical variables. As is well known, physical quantities must be gauge invariant.

One can show that quantities related to the \( L^\sharp \)-theory, for example \( q^{Ai}, p^j_{Ai}, \) and \( H^\sharp, \) are all gauge invariant. Whereas quantities proper to the \( L \)-theory are, in general, non-invariant. For example, one obtains
\[ \delta p_i = \varepsilon^j \frac{\partial^2 W}{\partial q^{N-1,j} \partial q^{N-1,j}}, \]  
\[ \delta q^{N-1,i} = \varepsilon^i, \]  
\[ \delta W = \varepsilon^i \frac{\partial W}{\partial q^{N-1,i}}, \]  
\[ \delta H_T = \delta H = -\varepsilon^i \frac{\partial^2 W}{\partial t \partial q^{N-1,i}}, \]

which shows that \( p_i \)'s, \( q^{N-1,i} \)'s, \( W \), \( H_T \), and \( H \) are all non-invariant and unphysical.

Needless to say, true physical quantities must be gauge invariant under the gauge transformations derived from \( \gamma^i_a \)'s as well. Further investigation of them requires the specification of \( L^i \)'s concrete form. So we do not pursue it any more.

Notice: Remember that our theory is equivalent to the original Lagrangian system (3.4) when we impose Eqs.(3.7) in addition. Since Eqs.(3.7) are not gauge invariant, they work as a gauge fixing condition. The gauge transformations derived from \( \gamma_i \)'s are the symmetry of the larger theory but not the symmetry of the original Lagrangian (3.1). Nevertheless they are important; gauge invariance of our theory makes it clear that the conditions (3.7) are in fact unessential and do not affect the physics.

### D. Proof of the theorem

We now turn our attention to the quantum mechanics. The Schrödinger equation is given by (2.12) with the Hamiltonian operator,

\[ \hat{H} \left( q, -\frac{\partial}{\partial q}, t \right) = \hat{H}^z \left( q^z, -i\hbar \frac{\partial}{\partial q^z}, t \right) - \frac{\partial W}{\partial t}, \]  

derived from (3.4) with (3.6) and (3.2c). The subsidiary conditions (2.13) are given by

\[ \hat{\gamma}_i \psi(q, t) = \left( -i\hbar \frac{\partial}{\partial q^{N-1,i}} - \frac{\partial W}{\partial q^{N-1,i}} \right) \psi(q, t) = 0, \]  
\[ \hat{\gamma}_a \left( q^z, -i\hbar \frac{\partial}{\partial q^z}, t \right) \psi(q, t) = 0. \]

Eqs.(3.21) are solved as follows:
\( \psi(q, t) = \psi^\sharp(q^\sharp, t) \exp \frac{iW(q, t)}{\hbar}, \quad (3.23) \)

where \( \psi^\sharp \) is the arbitrary function of \( q^{Ai} \)'s and \( t \). Eq. (3.23) gives the general form of the physical state. Note that \( \psi^\sharp(q^\sharp, t) \) is gauge invariant, while \( \psi(q, t) \) is not.

It is not hard to verify the identities

\[
\left( -i\hbar \frac{\partial}{\partial q^{Ai}} - \frac{\partial W}{\partial q^{Ai}} \right)^n \psi = \left[ \left( -i\hbar \frac{\partial}{\partial q^{Ai}} \right)^n \psi^\sharp \right] \exp \frac{iW}{\hbar}, \quad (3.24)
\]

on the physical state (3.23). Here \( n \) is a non-negative integer. These identities imply the following identity:

\[
\hat{F} \left( q^\sharp, -i\hbar \frac{\partial}{\partial q^\sharp}, t \right) \psi^\sharp = \left[ \hat{F} \left( q^\sharp, -i\hbar \frac{\partial}{\partial q^\sharp}, t \right) \psi^\sharp \right] \exp \frac{iW}{\hbar}. \quad (3.25)
\]

Here \( F(q^\sharp, p^\sharp, t) \) is an arbitrary quantity which is a polynomial with respect to \( p^{\sharp Ai} \)'s. Assuming that \( H^\sharp(q^\sharp, p^\sharp, t) \) and \( \gamma^\sharp_\alpha(q^\sharp, p^\sharp, t) \)'s are polynomials with respect to \( p^{\sharp Ai} \)'s, we apply (3.25) for them.

Inserting (3.23) into the Schrödinger equation (2.12) with (3.20), and using (3.25), we finally obtain

\[
i\hbar \frac{\partial \psi^\sharp(q^\sharp, t)}{\partial t} = \hat{H} \left( q^\sharp, -i\hbar \frac{\partial}{\partial q^\sharp}, t \right) \psi^\sharp(q^\sharp, t). \quad (3.26)
\]

This is nothing but the Schrödinger equation in the \( L^\sharp \)-theory. As for the condition (3.22), it simply becomes

\[
\gamma^\sharp_\alpha \left( q^\sharp, -i\hbar \frac{\partial}{\partial q^\sharp}, t \right) \psi^\sharp(q^\sharp, t) = 0. \quad (3.27)
\]

Eq. (3.26) together with Eqs. (3.27) constitutes the full quantum mechanics for the \( L^\sharp \)-theory. Here \( \psi^\sharp(q^\sharp, t) \) is identified as the wave function of the \( L^\sharp \)-theory.

We have proven the following proposition:

**Proposition 1** \( n \)-th order and \( (n - 1) \)-th order Lagrangians, which may be singular and non-autonomous, lead to the same quantum mechanics if their difference is a total time derivative.
Proposition 1 implies the following theorem:

**Theorem 2** $n$-th order and $m$-th order Lagrangians, which may be singular and non-autonomous, lead to the same quantum mechanics if their difference is a total time derivative.

**Proof:** Let us assume $n \geq m$, and put

$$L_n = L^\sharp_m + \frac{d}{dt} W_{n-1}. \quad (3.28)$$

Here, the subscript of $L$, $L^\sharp$, and $W$ denote the order of highest derivatives contained. We use the same notation for $L$ and $W$, which appear in what follows, as well.

(i) $n = m$ case

Let us introduce an arbitrary function $W_n$, and define a new Lagrangian $L_{n+1}$ as follows:

$$L_{n+1} := L_n + \frac{d}{dt} W_n. \quad (3.29)$$

Then (3.28) is rewritten as

$$L_{n+1} = L^\sharp_n + \frac{d}{dt} (W_{n-1} + W_n). \quad (3.30)$$

These equations together with Proposition 1 say

$$L_n \sim L_{n+1} \sim L^\sharp_n, \quad (3.31)$$

where $\sim$ means quantum mechanically equivalent.

(ii) $n = m + 1$ case

This case is just the same as Proposition 1.

(iii) $n \geq m + 2$ case

Let us introduce $n - m - 1$ arbitrary functions $W_i$, $i = m, m + 1, \ldots, n - 2$, and define the same number of new Lagrangians $L_i$, $i = m + 1, m + 2, \ldots, n - 1$, as follows:
\[ \mathcal{L}_{m+1} := L_m^z + \frac{d}{dt}W_m, \]
\[ \mathcal{L}_{m+2} := \mathcal{L}_{m+1} + \frac{d}{dt}(W_{m+1} - W_m), \] (3.32)

\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \]
\[ \mathcal{L}_{n-1} := \mathcal{L}_{n-2} + \frac{d}{dt}(W_{n-2} - W_{n-3}). \]

Consistency between these equations and (3.28) requires
\[ L_n = \mathcal{L}_{n-1} + \frac{d}{dt}(W_{n-1} - W_{n-2}). \] (3.33)

Therefore we have proven
\[ L_m^z \sim \mathcal{L}_{m+1} \sim \mathcal{L}_{m+2} \sim \cdots \sim \mathcal{L}_{n-2} \sim \mathcal{L}_{n-1} \sim L_n. \] (3.34)

This completes the proof. \[ \square \]

**Corollary 3 (Grosse-Knetter)** An \( m \)-th order Lagrangian, and the same Lagrangian formally treated as if it were an \( n(\geq m) \)-th order Lagrangian lead to the same quantum mechanics.

**Proof:** This is the special case of Theorem 2 with the vanishing total derivative term. \[ \square \]

Corollary 3 has been proven, using the path integral, by Grosse-Knetter for the special case of autonomous Lagrangians. Our Corollary 3 is the generalization of his result to non-autonomous Lagrangians.

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