The Isodiametric Problem with Lattice-Point Constraints*

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Abstract

In this paper, the isodiametric problem for centrally symmetric convex bodies in the Euclidean $d$-space $\mathbb{R}^d$ containing no interior non-zero point of a lattice $L$ is studied. It is shown that the intersection of a suitable ball with the Dirichlet-Voronoi cell of $2L$ is extremal, i.e., it has minimum diameter among all bodies with the same volume. It is conjectured that these sets are the only extremal bodies, which is proved for all three dimensional and several prominent lattices.

2000 Mathematical Subject Classification: Primary 52A20, 52C07; Secondary 52A40

Keywords: Isodiametric problem, lattices, Dirichlet-Voronoi cells, parallelohedra

1 Introduction

Let $\mathbb{R}^d$ be the $d$-dimensional Euclidean space endowed with standard norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$. We denote the set of full rank lattices by $\mathcal{L}^d$, where a (full rank) lattice $L$ of $\mathbb{R}^d$ is a set of the form $L = AZ^d$ with $A \in \text{GL}_d(\mathbb{R})$. The columns of $A$ are called a basis of $L$. The determinant $\det L = |\det A|$ of the lattice is independent of the chosen basis.

*The first author was supported in part by Dirección General de Investigación (MEC) MTM2004-04934-C04-02 and by Fundación Séneca (C.A.R.M.) 00625/PI/04. The second and the third author were supported by the Deutsche Forschungsgemeinschaft (DFG) under grant SCHU 1503/4-2. During the work on this paper the third author was partially supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany), and he was partially supported by the Netherlands Organization for Scientific Research under grant NWO 639.032.203.
For a lattice $L$, let $\mathcal{K}_L$ be the family of all centrally symmetric convex bodies, that is, compact convex sets $K$ with $K = -K$, which do not contain a non-zero lattice point in their interior. We denote the volume ($d$-dimensional Lebesgue measure) of a convex body $K \in \mathcal{K}_L$ by $\text{vol} K$ and its diameter by $\text{diam} \ K = 2 \max_{x \in K} \|x\|$. Notice that for centrally symmetric sets the diameter is twice the circumradius.

*Minkowski’s first fundamental theorem* (see e.g. [5], § 5, Th. 2] and [9]) gives an upper bound for the volume of a convex body $K \in \mathcal{K}_L$:

**Theorem 1.1** (Minkowski, 1891). If $L \in \mathcal{L}^d$ and $K \in \mathcal{K}_L$ then

$$\text{vol} K \leq 2^d \det L.$$  

Here we consider the following problem:

**Problem 1.2** (The isodiametric problem with lattice point constraints). Given a lattice $L \in \mathcal{L}^d$ and a real number $V \in (0, 2^d \det L]$, determine the minimum diameter

$$\text{diam}_L(V) = \min \{ \text{diam} K : K \in \mathcal{K}_L, \text{vol} K = V \}$$

and the bodies $K \in \mathcal{K}_L$ for which this minimum is attained.

Since the problem is trivial for $d = 1$, we consider it only for $d \geq 2$. In Theorem 2.1 we solve it partially by giving a description of convex bodies attaining the minimum. We conjecture (Conjecture 2.2) that these convex bodies are the only ones attaining the minimum and we show that the conjecture is valid in many cases, e.g., for a wide class of lattices (Corollary 1.4 and Proposition 1.6) and for many values of $V$ (Corollary 1.3). In particular we give a complete answer for all lattices of dimension $d \leq 3$, for the integral lattice $\mathbb{Z}^d$, the Leech lattice $\Lambda_{24}$, all root lattices $A_d$ with $d \geq 2$, $D_d$ with $d \geq 3$, $E_d$ with $d = 6, 7, 8$ and all their reciprocals (Theorem 2.3). The 2-dimensional case was already completely solved in [6] by different methods.

## 2 Main Results

Before stating our main results we need some more notation. For a lattice $L \in \mathcal{L}^d$ we consider its *Dirichlet-Voronoi cell*

$$\text{DV}(L) = \{ x \in \mathbb{R}^d : \|x\| \leq \|x - y\| \text{ for all } y \in L \}.$$
Its volume is equal to \( \det L \), its circumradius is equal to the \textit{inhomogeneous minimum} (also called covering radius) of \( L \),

\[
\mu(L) = \min \max_{y \in L} \|x - y\| = \min \{ \mu \in \mathbb{R} : L + B_d(\mu) = \mathbb{R}^d \},
\]

and its inradius (also called packing radius) is half of the \textit{homogeneous minimum} of \( L \),

\[
\lambda(L) = \min_{y \in L \setminus \{0\}} \|y\| = \min \{ \lambda \in \mathbb{R} : B_d(\lambda) \cap L \neq \{0\} \}.
\]

Here \( B_d(r) = \{ x \in \mathbb{R}^d : \|x\| \leq r \} \) denotes the \( d \)-dimensional ball centered at the origin with radius \( r \). For sets \( A, B \subset \mathbb{R}^d \) we write \( A + B = \{ a + b : a \in A, b \in B \} \) to denote the Minkowski addition (vector sum). We write \( y + K \) instead of \( \{y\} + K \). For a set \( A \subset \mathbb{R}^d \) and a real number \( \alpha \) we define \( \alpha A = \{ \alpha a : a \in A \} \). Then, if \( K \) is a \( d \)-dimensional convex body we have \( \text{vol}(\alpha K) = \alpha^d \text{vol} K \) and \( \text{diam}(\alpha K) = \alpha \text{diam} K \).

Clearly, \( \text{DV}(L) \) is a \textit{parallelohedron}, i.e., a convex polytope which tiles \( \mathbb{R}^d \) by (lattice) translations. Furthermore, every \((d-1)\)-dimensional face (facet) of \( \text{DV}(2L) \) contains exactly one lattice point in its relative interior, and it is centrally symmetric with respect to this point (see e.g. [5, §12]).

For \( V \in (0, 2^d \det L] \) we define the convex body \( K_L(V) \) by choosing the (unique) positive real number \( r_L(V) \) such that

\[
K_L(V) = B_d(r_L(V)) \cap \text{DV}(2L)
\]

has volume \( V \) (see Figure 4 for an example of \( K_L(V) \) in \( \mathbb{R}^3 \)).

\textbf{Theorem 2.1.} Let \( L \in \mathcal{L}^d \) be a lattice and let \( V \in (0, 2^d \det L] \). Then for all \( K \in \mathcal{K}_L \) with \( \text{vol} K = V \) the inequality \( \text{diam} K_L(V) \leq \text{diam} K \) holds.

We present a proof in Section 3. We say that a convex body \( K \in \mathcal{K}_L \) is an \textit{extremal body} if it is a solution of Problem 1.2 hence if \( \text{diam} K = \text{diam} K_L(V) \).

\textbf{Conjecture 2.2.} For \( L \in \mathcal{L}^d \) and \( V \in (0, 2^d \det L] \), \( K_L(V) \) is the unique extremal body.

Notice that Conjecture 2.2 is trivially true if \( r_L(V) \leq \lambda(L) \) because of the classical isodiametric inequality without lattice-point constraints (see e.g. [1, p. 83]), which says that for a fixed volume the ball is the only set with minimum diameter. In Section 4 we verify Conjecture 2.2 for many particular cases:
Theorem 2.3. \( K_L(V) \) is the only extremal body for any \( V \in (0, 2^d \det L] \) and for the following lattices (for explicit descriptions we refer to [2]): all lattices in dimension 2 and 3, the integral lattice \( \mathbb{Z}^d \), the Leech lattice \( \Lambda_{24} \), all root lattices \( A_d \) with \( d \geq 2 \), \( D_d \) with \( d \geq 3 \), \( E_d \) with \( d = 6, 7, 8 \) and all their reciprocals.

3 Proof of Theorem 2.1

Our proof of Theorem 2.1 relies on an equivalent point of view. Instead of minimizing the diameter among all convex bodies in \( K_L \) with fixed volume, we maximize the volume among all convex bodies in \( K_L \) with fixed diameter. It turns out that extremal bodies also maximize volume among all convex bodies in \( K_L \) with fixed diameter.

For any \( V \in (0, 2^d \det L] \) the minimum diameter of a body \( K \in K_L \) with volume \( V \) is at most the diameter of \( D_L(2^L) \), which is equal to \( 4\mu(L) \). Theorem 2.1 is equivalent to the following statement.

Theorem 3.1. Let \( L \in \mathcal{L}^d \) be a lattice and let \( D \) be at most \( 4\mu(L) \). Define \( V \) by \( 2r_L(V) = D \). Then for every \( K \in K_L \) with \( \text{diam} K = D \) the inequality \( \text{vol} K \leq \text{vol} K_L(V) \) holds.

Here we show that Theorem 3.1 implies Theorem 2.1. Using an analogous argument the other direction can be proved. For \( V \in (0, 2^d \det L] \) let \( K \in K_L \) be a convex body with \( \text{vol} K = V \). Suppose that \( \text{diam} K_L(V) > \text{diam} K \). Then Theorem 3.1 yields a contradiction:

\[
V = \text{vol} K_L(V) \geq \text{vol} \left( \frac{\text{diam} K_L(V)}{\text{diam} K} K \right) > \text{vol} K = V.
\]

Hence, \( \text{diam} K_L(V) \leq \text{diam} K \).

For the proof of Theorem 3.1 we recall some standard notions from the Geometry of Numbers. A lattice \( L \in \mathcal{L}^d \) is called \emph{admissible} for a subset \( K \subset \mathbb{R}^d \) if \( K \) has no lattice point except the origin in its interior. In particular the lattice \( L \) is admissible for all convex bodies in \( K_L \). On the other hand, a lattice \( L \in \mathcal{L}^d \) is called \emph{packing lattice} for \( K \) if \( \text{int}(K + x) \cap \text{int}(K + y) = \emptyset \) for all distinct \( x, y \in L \). Then for a centrally symmetric convex body \( K \), a lattice \( L \) is a packing lattice for \( K \) if and only if it is admissible for \( 2K \) (see e.g. [5], §20, Th. 1).

\textbf{Proof of Theorem 3.1.} As mentioned above, the case \( r_L(V) \leq \lambda(L) \) is covered by the classical isodiametric inequality without lattice-point constraints. Thus we suppose \( r_L(V) > \lambda(L) \).
Let $K \in \mathcal{K}_L$ be a convex body with diam $K = 2r_L(V)$. By definition of $\mathcal{K}_L$ the lattice $L$ is admissible for $K$. Therefore $2L$ is a packing lattice for $K$. Since $DV(2L)$ is a parallelohedron,

$$\text{vol } K = \text{vol } ((2L + K) \cap DV(2L))$$

(roughly speaking, the Dirichlet-Voronoi cell $DV(2L)$ contains the full set $K$ “in pieces”). On the other hand, since diam $K = 2r_L(V)$ we have $2y + K \subseteq 2y + B_d(r_L(V))$ for all $y \in L$. Then,

$$(2L + K) \cap DV(2L) \subseteq [2L + B_d(r_L(V))] \cap DV(2L).$$ (2)

The right hand side of (2) is equal to $B_d(r_L(V)) \cap DV(2L) = K_L(V)$, because if $x \in [2y + B_d(r_L(V))] \cap DV(2L)$ for $y \in L$, then by the definition of the Dirichlet-Voronoi cell we have $\Vert x \Vert \leq \Vert x - 2y \Vert \leq r_L(V)$. Therefore, $\text{vol } K_L(V) \geq \text{vol } ((2L + K) \cap DV(2L)) = \text{vol } K$, as required.

## 4 Equality Cases

In this section we further investigate equality cases. We give geometric conditions on the vertices (Corollary 4.4) and on the facets (Proposition 4.6) of the Dirichlet-Voronoi cell $DV(2L)$ which assure the validity of Conjecture 2.2 for the corresponding lattice $L$. As a consequence of these results we obtain a proof of Theorem 2.3.

Let $L \in \mathcal{L}^d$ be a lattice and let $V \in (0, 2^d \det L]$. Then $r_L(V) \leq 2\mu(L)$. We define

$$\mathcal{R} = \{ x \in \mathbb{R}^d : \Vert x - y \Vert > r_L(V), \text{ for all } y \in 2L \}. \tag{3}$$

**Lemma 4.1.** Let $K \in \mathcal{K}_L$ be an extremal body. Then

i) $\mathbb{R}^d = (2L + K_L(V)) \cup \mathcal{R},$

ii) the intersection $(2L + K) \cap \mathcal{R}$ is empty,

iii) $\mathbb{R}^d = (2L + K) \cup \mathcal{R}$.

**Proof.** For every $x \in \mathbb{R}^d$ there exists $y \in L$ such that $x \in 2y + DV(2L)$. From the definition of $DV(2L)$ we know that $2y$ is a nearest point of the lattice $2L$ to $x$. If $x \notin 2y + K_L(V)$, then $x \notin 2y + B_d(r_L(V))$. Consequently, since $2y$ is a nearest point to $x$, we have $x \notin 2L + B_d(r_L(V))$. Hence, $x \in \mathcal{R}$. This shows i).
ii) is obvious since diam $K = 2r_L(V)$.

For iii) suppose that $(2L + K) \cup R$ is strictly contained in $\mathbb{R}^d$. Then there exists a set $B$ with positive volume such that $2L + B$ does not intersect $(2L + K) \cup R$. The latter is invariant with respect to translations of $2L$ which contradicts the extremality of $K$.

Let $K \in K_L$ be an extremal body with respect to the lattice $L$ and a value $V \in (0, 2^d \det L)$. Since $2L$ is admissible for $K$, there exists a closed halfspace $G_y^-$ for every $y \in L \setminus \{0\}$ with bounding hyperplane $G_y$ through $y$ so that $K \subseteq G_y^-$. Since $K$ is contained in $B_d(r_L(V))$, we have the representation

$$K = \bigcap_{y \in L \setminus \{0\}, \|y\| < r_L(V)} G_y^- \cap B_d(r_L(V)). \quad (4)$$

Notice that we do not have to care about separation of lattice points outside or on the boundary of $B_d(r_L(V))$.

We now consider those $y \in L \setminus \{0\}$ with $\|y\| < r_L(V)$ which are facet-centers of the Dirichlet-Voronoi cell $DV(2L)$. Notice that all the facets of $DV(2L)$ have a center since they are centrally symmetric. In some cases we can prove that facet defining hyperplanes of $DV(2L)$ coincide with those of $K$. In the following $C_L \subset L$ denotes the set of lattice points being centers of facets of $DV(2L)$. Let the facets of $DV(2L)$ be defined by hyperplanes $H_y$ passing through $y \in C_L$ and let $H_y^-$ denote the corresponding closed halfspaces bounded by $H_y$ and containing $DV(2L)$, so that

$$DV(2L) = \bigcap_{y \in C_L} H_y^- \quad (5)$$

is a non-redundant description of $DV(2L)$.

The following proposition allows to prove Conjecture 2.2 for many values of $V$ and many lattices.

**Proposition 4.2.** Let $L \in \mathcal{L}^d$ be a lattice and let $D$ be at most $4\mu(L)$. Define $V$ by $2r_L(V) = D$. Let $K \in K_L$, given as in (4), be extremal with respect to $L$ and $V$, and let $y \in C_L$ with $\|y\| < r_L(V)$. If the facet $F_y = H_y \cap DV(2L)$ of $DV(2L)$ intersects $\mathbb{R}^d \setminus B_d(r_L(V))$ then $G_y = H_y$.

**Proof.** Suppose $G_y \neq H_y$ for some $y \in C_L$ satisfying the assumptions of the proposition. We construct a ball $B$ having the following properties:

i) $B \subset K_L(V)$,
ii) $B \cap K = \emptyset,$

iii) $B \cap [2L \setminus \{0\} + B_d(r_L(V))] = \emptyset.$

Then the lattice $2L$ is a packing lattice for $K \cup B$. Using the argument which was applied in the proof of Theorem 3.1 we obtain the strict inequality $\text{vol } K < \text{vol } (K \cup B) < \text{vol } K_L(V)$, which contradicts the assumption that $K$ is extremal.

It remains to construct $B$. Since the facet $F_y$ of $DV(2L)$ intersects $\mathbb{R}^d \setminus B_d(r_L(V))$ by our hypothesis, there exists a vertex $x$ of $F_y$ with $\|x\| > r_L(V)$ (see Figure 1).

![Figure 1: The ball $B$ lies inside the shaded region.](image)

Since $F_y$ is centrally symmetric with respect to $y$, either $x$ or $2y - x$ lies in the open halfspace $G_y^+ = \mathbb{R}^d \setminus G_y^-$. Without loss of generality we assume $x \in G_y^+$.

On the line segment connecting $x$ and $y$ there is a point $z$ with $\|z\| = r_L(V)$ (see Figure 1). We have $\|z - 2y\| = r_L(V)$ and, since $z$ lies in the relative interior of a facet of a Dirichlet-Voronoi cell, for any other $y' \in L \setminus \{0, y\}$ it holds $\|z - 2y'\| > r_L(V)$. Let $c = z - \varepsilon y$ where $\varepsilon > 0$ is chosen so that $c$ lies in the interior of $DV(2L) \cap G_y^+ \cap B_d(r_L(V))$. Thus there exists $\delta > 0$ sufficiently small such that the ball $B$ centered in $c$ with radius $\delta$ satisfies properties i)–iii).

The following result is an immediate consequence of Proposition 4.2.

**Corollary 4.3.** Let $L \in \mathcal{L}^d$ be a lattice and let $D$ be at most $4\mu(L)$. Define $V$ by $2r_L(V) = D$. If every facet of $DV(2L)$ contains a vertex $x$ with $\|x\| > r_L(V)$, then $K_L(V)$ is the unique extremal body with respect to $L$ and $V$.

So Corollary 4.3 proves Conjecture 2.2 for many values of $V$. We can apply Corollary 4.3 also to prove Conjecture 2.2 for certain classes of lattices and any value of $V \in (0, 2^d \det L]$.

**Corollary 4.4.** Let $L \in \mathcal{L}^d$ be a lattice such that every facet of $DV(2L)$ contains a vertex $x$ with $\|x\| = 2\mu(L)$. Then $K_L(V)$ is the unique extremal body with respect to $L$ and any $V \in (0, 2^d \det L]$. 

7
Proof. The case $D < 4\mu(L)$ is covered by Corollary 4.3. So we assume that $D = 4\mu(L)$. If $K$ is an extremal body for the diameter $D = 4\mu(L)$ then $\mathbb{R}^d = 2L + K$. Hence for arbitrarily small $\varepsilon > 0$ the convex body $K_\varepsilon = K \cap B_d(D - \varepsilon)$ is extremal for diameter $D - \varepsilon$, since $\mathbb{R}^d = (2L + K_\varepsilon) \cup K$. Thus applying Proposition 4.2 to every facet of $DV(2L) \cap B_d(D - \varepsilon)$ we get that $K_\varepsilon = DV(2L) \cap B_d(D - \varepsilon)$. Hence $K = DV(2L)$.

Notice that even in dimension 3 there are lattices to which Corollary 4.4 cannot be applied. We give an explicit example using the notation of Selling parameters from [3]:

Remark 4.5. Let $L \in \mathcal{L}^3$ be the lattice defined by the Selling parameters $p_{01} = 2$ and $p_{ij} = 1$ for $i, j = 0, \ldots, 3$ with $i \neq j$ and the pair $(i, j) \neq (0, 1)$. A Gram matrix of a basis $A$ of $L$ is for example

$$A^\top A = \begin{pmatrix} 4 & -2 & -1 \\ -2 & 4 & -1 \\ -1 & -1 & 3 \end{pmatrix}. $$

The Dirichlet-Voronoi cell of $L$ is a truncated octahedron (or permutohedron, see Figure 2). All the vertices of its 4-gonal facet given by 1023, 0123, 0132, 1032 have norm $\sqrt{35/24}$ whereas the vertex denoted by 0213 has norm $\sqrt{3/2} = \mu(L) > \sqrt{35/24}$.

![Figure 2: The truncated octahedron.](image)

In order to overcome this problem in $\mathbb{R}^3$ and to solve more equality cases, we prove the following proposition. It uses the fact that the Dirichlet-Voronoi cell is a parallelotope and hence every projection along a $(d-2)$-face (ridge) is a centrally symmetric hexagon or, as a limiting case, a parallelogram. This was independently proved by McMullen [7, 8] and Venkov [10]. The facets of the parallelotope adjacent to the 4 or 6 translates of a ridge are said to form a 4-belt or 6-belt respectively.
Proposition 4.6. Let $L \in \mathcal{L}^d$ be a lattice and let $D$ be at most $4\mu(L)$. Define $V$ by $2r_L(V) = D$. Let $K \in \mathcal{K}_L$, given as in (4), be extremal with respect to $L$ and $V$. Let $DV(2L)$ be given as in (5) and let $y \in C_L$ be the center of a facet belonging to a 6-belt of $DV(2L)$. If $G_z = H_z$ for all $z \in C_L \setminus \{y\}$, then also $G_y = H_y$ and hence $K = K_L(V)$.

Proof. We suppose that $G_y \neq H_y$, respectively $K \neq K_L(V)$. Consider the sets $A = G_y^+ \cap K_L(V)$ and $B = G_y^− \cap (2y + K_L(V))$, see Figure 3. Since $K_L(V)$ is centrally symmetric, the isometry $x \mapsto 2y - x$ maps the closure $\overline{A}$ to $\overline{B}$ and hence $\text{vol}(A) = \text{vol}(B)$. Clearly, $\text{vol}(A \cap K) = 0$.

By assumption, the facet $F_y = H_y \cap DV(2L)$ of $DV(2L)$ with center $y$ contains two ridges $F$ and $2y - F$ which are also ridges of other facets in a 6-belt. Since $F_y$ is centrally symmetric with respect to $y$ we find a relative interior point $x$ of either $F$ or $2y - F$ which is contained in $G_y^− \setminus G_y$. Otherwise $G_y$ would be equal to the affine hull of $\{y\} \cup F$ and hence equal to $H_y$.

Without loss of generality we assume $x \in F$ and $F = H_y^+ \cap F_y^−$. We can suppose that the points of $F$ all lie in $B_d(r_L(V))$ and $2y + B_d(r_L(V))$, because otherwise we could apply Corollary 4.3 to show $G_y = H_y$. Therefore there exists $\varepsilon > 0$ such that $x + \varepsilon y$ lies in the interior of $G_y^− \cap H_y^+$ and $2y + K_L(V)$. So there exists a ball centered in $x + \varepsilon y$ and contained in $B \setminus K$ with positive volume. It shows that $\text{vol}(B \cap K) < \text{vol}(B)$. Hence $K_L(V)$ has larger volume than $K$ which contradicts the extremality of $K$. \hfill \square

Corollary 4.4 and Proposition 4.6 can be applied to many lattices so that the isodiametric problem is solved for them completely, that is, for every volume respectively every diameter. In the proof of Theorem 2.3 below we work out this argument for several prominent lattices, where we did not try to be exhaustive.
**Proof of Theorem 2.3.** If the automorphism group of a lattice $L$ acts transitively on the facet centers of $DV(L)$, then every facet of $DV(2L)$ contains a vertex $x$ with $\|x\| = 2\mu(L)$. Lattices with this transitivity property are for example the integral lattice $\mathbb{Z}^d$ and the root lattices $A_d$, $D_d$ and $E_d$ (see [2, Chap. 4, Chap. 22, Cor. to Th. 5]).

With a similar argument it can be shown that Corollary 4.4 applies to the Leech lattice $\Lambda_{24}$ and the lattices $D^*_d$. The facets of $DV(\Lambda_{24})$ are given by the lattice vectors of squared length 4 and 6. The automorphism group of $\Lambda_{24}$ acts transitively on each of these two sets. A vertex $x$ of “type $A_{24}$” (see [2, Chap. 23]) satisfies $\|x\| = \mu(\Lambda_{24})$. This vertex is incident to 275 facets which correspond to vectors of length 4 and to 25 facets which correspond to vectors of length 6. Hence in every facet a vertex of type $A_{24}$ can be found. The case $D^*_d$ is analogous.

Other lattices where Corollary 4.4 can be applied are all the 2-dimensional lattices and the reciprocals $A^*_2$, $E^*_6$, $E^*_7$ of the root lattices, since they are examples of lattices $L$ for which every vertex $x$ of $DV(2L)$ satisfies $\|x\| = 2\mu(L)$ (see [2, Chap. 4, Chap. 22, Th. 7]).

Finally in order to get the solution for $d = 3$, we can use the knowledge of all combinatorial types of Dirichlet-Voronoi cells in dimension 3. It is well known (Fedorov, [4]) that there are five combinatorial types of Dirichlet-Voronoi cells: cube, hexarhombic dodecahedron, rhombic dodecahedron, hexagonal prism and the truncated octahedron. The first four are degenerations of the last one.

The vertices of $DV(2L)$ can be partitioned into equivalence classes by identifying those ones which are lattice translates or point reflections of each other. Vertices of the same class have all the same distance to the origin. For a vertex $x$ of a facet with center $y$, the opposite vertex $2y - x$ of that facet belongs to the same class. Using this symmetry it is easily checked for all but the truncated octahedron, that every facet of $DV(2L)$ contains a vertex $x$ with $\|x\| = 2\mu(L)$. Then Corollary 4.4 can be applied to derive that $K_L(V)$ is the unique extremal set.

In the remaining type, i.e., the one corresponding to the truncated octahedron (see Figure 2), we can apply Proposition 4.6. We find that $DV(2L)$ has three equivalence classes of eight vertices each. Each hexagonal facet of the truncated octahedron contains two opposite vertices of every class and each vertex class has vertices in four quadrilateral facets (see points on vertices in Figure 2). Assuming that only one class attains the radius $2\mu(L)$, we know by Corollary 4.3 that the condition of Proposition 4.6 holds, and therefore we obtain $K_L(V)$ is the unique extremal set. $\square$
5 Some Consequences and Remarks

For a lattice $L \in \mathcal{L}^d$, the explicit isodiametric inequality for convex bodies $K \in \mathcal{K}_L$ can be stated by computing the volume $V = \text{vol} K_L(V)$ in terms of the diameter $\text{diam} K_L(V)$. Notice that the function defined as $f(r) = \text{vol}(B_d(r) \cap \text{DV}(2L))$ is clearly an increasing function of $r$. Then it follows that for any convex body $K \in \mathcal{K}_L$

$$\text{vol} K \leq f\left(\frac{\text{diam} K}{2}\right),$$

(6)

and equality holds when (and, in many cases, only when) $K = K_L(V)$.

For instance, in the case of the 3-dimensional Euclidean space and the integral lattice $\mathbb{Z}^3$, the isodiametric inequality is expressed in the following way (we write $D := \text{diam} K$ for the sake of brevity):

If $0 < D \leq 2$

$$\text{vol} K \leq \frac{\pi}{6} D^3$$

If $2 \leq D \leq 2\sqrt{2}$

$$\text{vol} K \leq 2\pi \left(-\frac{D^3}{6} + \frac{3D^2}{4} - 1\right)$$

If $2\sqrt{2} \leq D \leq 2\sqrt{3}$, \text{vol} K \leq 4\sqrt{D^2 - 8 + (3D^2 - 4) \arctan \frac{12 - D^2}{4\sqrt{D^2 - 8} - 2\frac{3}{D^3} \arctan \frac{D(12 - D^2)}{(D^2 + 4)\sqrt{D^2 - 8}}}

The extremal sets for these inequalities, i.e., the sets with maximum volume for different values of the diameter, are shown in Figure 4.

![Figure 4: Extremal sets in \( \mathbb{R}^3 \) for the integral lattice: when \( \text{diam} K \leq 2\sqrt{2} \) (left) and \( \text{diam} K \geq 2\sqrt{2} \) (right).](image)

Notice that, since the diameter is twice the circumradius for centrally symmetric $K$, all of the inequalities above also relate volume and circumradius of $K$.
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