Gravitational convergence, shear deformation and rotation of magnetic forcelines

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ABSTRACT
We consider the ‘kinematics’ of space-like congruences and apply them to a family of self-gravitating magnetic forcelines. Our aim is to investigate the convergence and the possible focusing of these lines, as well as their rotation and shear deformation. In so doing, we introduce a covariant 1+2 splitting of the 3-D space, parallel and orthogonal to the direction of the field lines. The convergence, or not, of the latter is monitored by a specific version of the Raychaudhuri equation, obtained after propagating the spatial divergence of the unit magnetic vector along its own direction. The resulting expression shows that, although the convergence of the magnetic forcelines is affected by the gravitational pull of all the other sources, it is unaffected by the field’s own gravity, irrespective of how strong the latter is. This rather counterintuitive result is entirely due to the magnetic tension, namely to the negative pressure the field exerts parallel to its lines of force. In particular, the magnetic tension always cancels out the field’s energy-density input to the Raychaudhuri equation, leaving the latter free of any direct magnetic-energy contribution. Similarly, the rotation and the shear deformation of the aforementioned forcelines are also unaffected by the magnetic input to the total gravitational energy. In a sense, the magnetic lines do not seem to ‘feel’ their own gravitational field no matter how strong the latter may be.

Key words: gravitation – magnetic fields.

1 INTRODUCTION
The Raychaudhuri equation is a fully geometrical expression that has been traditionally used to monitor the convergence (or not) of time-like worldlines in relativistic studies of gravitational collapse (e.g. see Wald 1984; Poisson 2004), or the mean expansion of cosmological space–times (see Raychaudhuri 1955; Dadhich 2005, 2007; Ehlers 2007; Ellis 2007 for representative discussions). Nevertheless, Raychaudhuri’s formula is not a priori restricted to time-like curves and to 4-D space–times. The same is also true for the supplementary equations monitoring the other two ‘optical scalars’ (as they have been historically known – see e.g. Kantowski 1967), namely the shear and the vorticity. Instead, generalized analogues of the Raychaudhuri equations (and of the rest of the supplementary formulae) can be used to study curves of any nature in diverse environments (see e.g. Kar & Sengupta 2007; Abreu & Visser 2011). For example, in astrophysics and cosmology (Dasgupta, Nandan & Kar 2008, 2009; Tsagas & Kadiltzoglou 2013), within quantum and modified-gravity scenarios (Gannouji, Nandan & Dadhich 2011; Harko & Lobo 2012; Das 2014; Mosheni 2015; Pasmatsiou, Tsagas & Barrow 2017), as well as in spaces of arbitrary dimensions (Kuniyal et al. 2015; Pahwa, Nandan & Goswami 2015). The forcelines of a magnetic \((B)\) field are space-like curves, as seen by an observer at rest relative to them. Therefore, confining to the 3-D rest-space of such an observer, one can construct the associated Raychaudhuri equation and the evolution formulae of the remaining optical scalars, to study the convergence (or not) of these lines, their shear deformation and their rotation.

Magnetic fields are rather unique sources and one of their special features is their tension properties. These reflect the elasticity of the magnetic forcelines, which is manifested as negative pressure exerted along the direction of the \(B\)-field (Parker 1979; Mestel 2012). Our aim is to investigate the implications of this unique magnetic property for the ‘kinematic’ behaviour of the field lines themselves. In so doing, we will adopt the so-called \(1+1+2\) covariant approach to general relativity (Greenberg 1970; Tsamparlis & Mason 1983; Mason & Tsamparlis 1985; Zafiris 1997; Clarkson & Barrett 2003; Clarkson 2007). The latter starts by introducing an \(1+3\) splitting of...
the space–time, into time and 3-D space, before proceeding to an additional 1+2 decomposition of the space–like hypersurfaces along a given direction and 2-D surfaces orthogonal to it. This preferred spatial direction also defines a unitary space-like vector parallel to it. The ‘kinematics’ of such a vector field, namely whether its (space-like) tangent curves converge/diverge, rotate or change shape, are determined by a set of ‘propagation’ equations analogous to those of their time-like counterparts (Ellis 1971, 1973). The difference is that, here, the propagation is along a spatial direction, instead of a temporal one. For instance, the convergence/divergence of these space-like curves (along their own direction), is monitored by the associated Raychaudhuri formula. In our study, it is the magneticforcelines that single-out a preferred spatial direction and, in so doing, they also define a unit vector field parallel to them. Then, the associated Raychaudhuri equation and the rest of the propagation formulae determine whether (and under what conditions) these forcelines converge or diverge, whether they rotate relative to each other and whether their shape is deformed.

Perhaps the main difference between the kinematic equations of time-like worldlines and those of a space-like congruence, is in their curvature terms. The former involve the Riemann and the Ricci tensors of the whole space–time, while the latter their 3-D (spatial) counterparts. In empty and static spaces all these tensors vanish identically, but in any other case they differ (sometimes considerably). For our purposes, the key difference appears to come from the pressure contribution. More specifically, although the isotropic pressure of the matter adds to the space–time Ricci tensor, it does not contribute to its spatial analogue. The anisotropic (trace-free) pressure, on the other hand, does. This means that only the magnetic energy density and the anisotropic pressure contribute to the Raychaudhuri equation of the field lines. This brings into play the magnetic tension, which manifests itself as negative pressure in the direction of the B-field. What is important is that the tension contribution to the Raychaudhuri equation always cancels out the input of the magnetic energy density. As a result, the convergence or not of the field lines is not directly affected by their own gravitational energy, no matter how strong the latter may be. The same is also true for the rotation and the shear deformation of these lines. Overall, although the magnetic forcelines respond to the gravitational pull of all the other sources, they do not seem to ‘feel’ their own gravity and this counterintuitive behaviour is exclusively due to their tension properties. This means that a magnetic–line configuration that finds itself at rest in an otherwise empty and static space will remain in equilibrium indefinitely, unless an external agent intervenes.

2 SPACE–TIME DECOMPOSITION

Introducing a time-like 4-velocity field into the 4-D space–time achieves a 1+3 decomposition of the latter into a temporal direction and a 3-D space orthogonal to it. In addition, selecting a space-like direction and then decomposing the spatial sections parallel and orthogonal to it leads to the so-called 1+1+2 space–time splitting (see Greenberg 1970; Tsamparlis & Mason 1983; Mason & Tsamparlis 1985; Zafiris 1997, as well as Clarkson & Barrett 2003; Clarkson 2007).

2.1 1+3 splitting

In a 4-D space–time, with metric $g_{\mu \nu}$, introduce a temporal direction along the time-like 4-velocity $u_\mu$ (normalized so that $u_\mu u^\mu = -1$). Then, the symmetric tensor $h_{\mu \nu} = g_{\mu \nu} + u_\mu u_\nu$ projects into the 3-D (spatial) hypersurfaces orthogonal to $u_\mu$ (i.e. $h_{\mu \nu} u_\mu = 0$ with $h_{\mu \nu} u_\nu = h_u$ and $h_u^a = 3$). The 3-D Levi-Civita tensor is $\varepsilon_{\mu \nu \rho \sigma} = \eta_{\mu \nu \rho \sigma} u^\rho d^\sigma$ (where $\eta_{\mu \nu \rho \sigma}$ is its 4-D counterpart) and satisfies the conditions $\varepsilon_{\mu \nu \rho \sigma} = \varepsilon_{\nu \mu \rho \sigma}$ and $\varepsilon_{\mu \nu \rho \sigma} u^\rho = 3! h_{\mu \nu} u^\rho / 3$. All these allow for an 1+3 splitting of the space–time into time and 3-D space, parallel and orthogonal to $u_\mu$, respectively (Tsagas, Challinor & Maartens 2008). Then, the temporal and spatial derivatives of a general tensor field $T_{\mu \nu \cdots}^{cd\cdots}$ are given by

$$T_{\mu \nu \cdots}^{cd\cdots} = u^\rho \nabla T_{\mu \nu \cdots}^{cd\cdots}$$

(1a)

and

$$D_i T_{\mu \nu \cdots}^{cd\cdots} = h_i^a h_a^b h_b^c \cdots h_f^d \cdots \nabla T_{\mu \nu \cdots}^{cd\cdots}$$

(1b)

respectively. Applying the above to the 3-D projector ($h_{\mu \nu}$) leads to

$$h_{\mu \nu} = 2 u_\mu u_\nu$$

(2)

the second of which shows why $h_{\mu \nu}$ can be used as the metric tensor of the spatial sections (in the absence of rotation – see Tsagas et al. 2008).

Let us now consider a congruence of time-like worldlines tangent to the 4-velocity field $u_\mu$. Using definitions (1a) and (1b), we arrive at the decomposition (Tsagas et al. 2008)

$$\nabla u_\mu = \frac{1}{3} \Theta h_{\mu \nu} + \sigma_{\mu \nu} + \omega_{\mu \nu} - u_\mu u_\nu.$$  

(3)

On the right-hand side we have the irreducible kinematic variables of the congruence’s motion. In particular, $\Theta = \nabla u^\rho u_\rho$ is the volume scalar, $\sigma_{\mu \nu} = D_\mu (u_\nu)$ is the shear tensor, $\omega_{\mu \nu} = D_\mu (u_\nu) u^\rho \nabla u_\rho$ is the vorticity tensor and $u_\mu = u^\rho \nabla u_\rho$ is the 4-acceleration vector (with $\sigma_{\mu \nu} u^\rho = 0 = \omega_{\mu \nu} u^\rho = u_\mu u_\nu$ by construction).\(^1\) Positive values for $\Theta$ mean that the tangent worldlines expand and negative ones imply contraction. The shear describes distortions in the shape of the congruence under constant volume. Non-zero vorticity, on the other hand, indicates that the worldlines are rotating relative to each other. Finally, the 4-acceleration manifests the presence of non-gravitational forces. We also note that, on using the spatial Levi-Civita tensor, we may define the vorticity vector $\omega_\nu = \varepsilon_{\nu \rho \sigma} u^\rho / 2$. The latter determines the rotational axis.

2.2 1+1+2 splitting

Decomposing the 4-D space–time into time and 3-D space may not be enough when the spatial sections are anisotropic. Suppose there is a preferred spatial direction and $n_\nu$ is the unit vector parallel to it. Then, $u_\mu u^\mu = 0$ and $n_\mu n^\mu = 1$ by construction, while the tensor

$$h_{\mu \nu} = n_\mu n_\nu$$

(4)

projects into the 2-D space-like surfaces orthogonal to $n_\mu$. Indeed, following (4), we obtain $h_{\mu \nu} n^\mu = 0 = h_{\mu \nu} n^\nu$, while one can easily verify that $h_{\mu \rho} n^\rho = h_u$ and $h_u^a = 2$. The $n_\nu$-field and the $h_{\mu \nu}$-tensor decompose the 3-D space into a spatial direction parallel to $n_\mu$ and 2-D space-like surfaces (sheets) normal to $n_\nu$ (see Clarkson & Barrett 2003; Clarkson 2007 for details). Therefore, we have achieved an overall 1+1+2 splitting of the space–time into a temporal direction (along $u_\mu$), a spatial direction (parallel to $n_\mu$) and 2-D space-like surfaces orthogonal to both of these vectors. This decomposition is reflected in the following splitting

$$g_{\mu \nu} = h_{\mu \nu} + n_\mu n_\nu - u_\mu u_\nu.$$  

(5)

\(^1\) Round brackets denote symmetrization and square ones antisymmetrization. Angle brackets, on the other hand, indicate the symmetric and traceless part of space-like tensors. For instance, $D_\mu (u_\nu) = D_\nu (u_\mu) - (D_\mu u^\rho / 3) h_{\mu \nu}$. 

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of the space–time metric. Moreover, in direct analogy with definitions (1a) and (1b), the derivatives parallel and orthogonal to the \( n_a \)-field are defined by (Clarkson & Barrett 2003; Clarkson 2007)

\[
T_{ab} = n^c T_{abc}\quad (6a)
\]

and

\[
\tilde{D}_a T_{abc} = \tilde{h}_a \tilde{h}_b \cdots \tilde{h}_p \tilde{h}_d \cdots \tilde{D}_a T_{bdc} \quad (6b)
\]

Applying the operators (6a) and (6b) to the 2-D projector (\( \tilde{h}_{ab} \)), using definition (4) and keeping in mind that \( \tilde{D}_a h_{ab} = 0 \), provides the auxiliary relations

\[
\tilde{h}_{ab} = -2 n'_a n_b \quad \text{and} \quad \tilde{D}_a \tilde{h}_{ab} = 0,
\]

respectively. The latter result implies that the \( \tilde{h}_{ab} \) can act as the metric of the associated 2-surfaces, in the same way \( h_{ab} \) can be seen as the metric of the spatial hypersurfaces. We also note that the vectors \( u_a \) and \( n_a \) are generally orthogonal to the corresponding 3-surfaces and 2-surfaces when they are irrotational. Otherwise their orthogonality is only local.

The 'kinematics' of the \( n_a \)-field are monitored by a set of irreducible variables, obtained in a manner exactly analogous to the one used for the 4-velocity vector (see Section 2.1 before). More specifically, employing definitions (6a) and (6b), gives

\[
D_b n_a = \frac{1}{2} \tilde{\Theta} \tilde{h}_{ab} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} + n'_a n_s,
\]

with \( \tilde{\Theta} = \tilde{D}_a n^a \), \( \tilde{\sigma}_{ab} = \tilde{D}_a n_b - \tilde{D}_b n_a \) and \( n'_a = n^a D_a n_a \). Note that \( \tilde{\sigma}_{ab} n^a = 0 = \tilde{\omega}_{ab} n^b = n'_a n^a \) by construction. The physical/geometrical interpretation of \( \tilde{\Theta}, \tilde{\sigma}_{ab}, \tilde{\omega}_{ab} \) and \( n'_a \) is closely analogous to that of their 3-D counterparts (see Section 2.1 before).

In particular, suppose that the \( n_a \)-field is tangent to a congruence of space-like curves and consider a 2-D cross-section (\( S \)) of this congruence. Then, positive/negative values of the area scalar \( \tilde{\Theta} \) imply that the aforementioned curves converge/diverge. In other words, the congruence expands/contracts and the area of \( S \) increases/decreases accordingly. The symmetric and trace-free 2-tensor \( \tilde{\sigma}_{ab} \) is analogous to the shear tensor defined in the previous section and monitors changes in the shape of \( S \), under constant area. On the other hand, the antisymmetric 2-tensor \( \tilde{\omega}_{ab} \) describes the rotational behaviour of the congruence. Note that the antisymmetry of \( \tilde{\omega}_{ab} \) means that the latter has only one independent component. We may therefore write \( \tilde{\omega}_{ab} = \tilde{\omega}_{gab} \), with \( \tilde{h}_{ab} = \tilde{h}_{gab} n^a n^b \) representing the 2-D Levi-Civita tensor and \( \tilde{\omega} = \tilde{\omega}_{gab} n^a n^b /2 \). Finally, the 2-vector \( n'_a \) vanishes when the curves in question are space-like geodesics.

### 3 KINEMATICS OF SPACE-LIKE CONGRUENCES

As with the time-like worldlines, the kinematics of space-like congruences are determined by a set of propagation formulae, which describe the evolution of the associated area element (\( \tilde{\Theta} \)), surface shear (\( \tilde{\sigma}_{ab} \)) and surface vorticity (\( \tilde{\omega}_{ab} \)), along the direction of the congruence.

#### 3.1 Irreducible kinematic evolution

The kinematic evolution of a time-like congruence follows after applying the 4-D Ricci identity to the corresponding 4-velocity field (see e.g. Section 1.3.1 in Tsagas et al. 2008). In analogy, the kinematics of a space-like vector-field follow from the Ricci identity of the spatial sections. Applied to an arbitrary space-like vector \( u_a \), the latter reads (see e.g. appendix A.3 in Tsagas et al. 2008)

\[
2D_b (\tilde{D}_a v) = -2 \omega_{ab} v^b + \tilde{R}_{abcd} v^d,
\]

with \( \tilde{R}_{abcd} \) being the 3-D Riemann curvature tensor. For zero vorticity the latter satisfies all the symmetries of its space–time counterpart. Otherwise we have \( \tilde{R}_{abcd} = \tilde{R}_{(ab)(cd)} \) (only see Section 1.3.5 in Tsagas et al. 2008 for details). Assuming that \( v = n_a \), where \( n_a \) is a unit space-like vector (i.e. \( n^a n_a = 0 \) and \( n_a n_b = \epsilon_{abc} n^c \)), contracting (9) along \( n_a \) and using decomposition (8), we obtain

\[
(\tilde{D}_b n_a) = \frac{1}{4} \tilde{\Theta}^2 \tilde{h}_{ab} - \tilde{\Theta} (\tilde{\sigma}_{ab} + \tilde{\omega}_{ab}) - \tilde{\sigma}_{ac} \tilde{\sigma}_{cb} - \tilde{\omega}_{ac} \tilde{\omega}_{cb} + 2 \tilde{\sigma}_{(ac) \tilde{\omega}_{bd)} + \tilde{D}_b n^a - 2 n_a \tilde{\sigma}_{bc} n^c + 2 n_a \tilde{\omega}_{bc} n^c + (n_a n_b) - n_a n_b - \tilde{R}_{abcd} n^d n^d
\]

Substituting (8) into the left-hand side of the above and recalling that \( \tilde{h}_{ab} = -2 n_a n_b \) (see equation 7 a in Section 2.2), gives

\[
\frac{1}{2} \tilde{\Theta} \tilde{h}_{ab} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} = -\frac{1}{4} \tilde{\Theta}^2 \tilde{h}_{ab} - \tilde{\Theta} (\tilde{\sigma}_{ab} + \tilde{\omega}_{ab}) - \tilde{\sigma}_{ac} \tilde{\sigma}_{cb} - \tilde{\omega}_{ac} \tilde{\omega}_{cb} + 2 \tilde{\sigma}_{(ac) \tilde{\omega}_{bd)} + \tilde{D}_b n^a - 2 n_a \tilde{\sigma}_{bc} n^c + 2 n_a \tilde{\omega}_{bc} n^c - n_a n_b - \tilde{R}_{abcd} n^d n^d
\]

Finally, projecting orthogonal to \( n_a \) and keeping in mind that \( \tilde{R}_{abcd} = \tilde{R}_{(ab)(cd)} \), we arrive at

\[
\frac{1}{2} \tilde{\Theta} \tilde{h}_{ab} + \tilde{\sigma}_{ab} + \tilde{\omega}_{ab} = -\frac{1}{4} \tilde{\Theta}^2 \tilde{h}_{ab} - \tilde{\Theta} (\tilde{\sigma}_{ab} + \tilde{\omega}_{ab}) - \tilde{\sigma}_{ac} \tilde{\sigma}_{cb} - \tilde{\omega}_{ac} \tilde{\omega}_{cb} + 2 \tilde{\sigma}_{(ac) \tilde{\omega}_{bd)} + \tilde{D}_b n^a - n_a n_b - \tilde{R}_{abcd} n^d n^d + 2 n_a \tilde{\omega}_{bc} n^c
\]

Given that \( \tilde{h}_{ab} \tilde{h}_{bd} \tilde{\sigma}_{ad} = \tilde{h}_{ac} \tilde{h}_{cd} \tilde{\sigma}_{ad} \) and that \( \tilde{h}_{ac} \tilde{h}_{cd} \tilde{\sigma}_{cd} = \tilde{h}_{ac} \tilde{h}_{cd} \tilde{\omega}_{cd} \). This expression monitors the evolution of the space-like congruence tangent to the unitary \( n_a \)-field, along the (spatial) direction of the latter. More specifically, the trace, the projected symmetric trace-free and the projected antisymmetric components of (12) provide the evolution formulae of the area scalar (\( \tilde{\Theta} \)), of the 2-shear tensor (\( \tilde{\sigma}_{ab} \)) and of the 2-vorticity tensor (\( \tilde{\omega}_{ab} \)), respectively.

#### 3.2 Raychaudhuri’s formula for space-like congruences

Taking the trace of (13), while keeping in mind that \( \tilde{h}_{ab} n^a = 0 \) and \( \tilde{\sigma}_{ab} n^a = 0 \), we obtain the following 3-D analogue

\[
\frac{1}{2} \tilde{\Theta} \tilde{h}_{ab} n^a_n = \frac{1}{2} \tilde{\Theta} \tilde{h}_{ab} n^a_n - n_a (\tilde{\sigma}_{bn} - \tilde{\omega}_{bn}) n^c + (n'_a n_b) - n_a n_b
\]
of the Raychaudhuri equation
\[ \Theta' = -\frac{1}{2} \Theta^2 - \mathcal{R}_{ab} n^a n^b - 2 (\hat{\omega}^2 - \hat{\omega}^2) + \tilde{D}_a n^a - n^a n^a + 2 \omega_{ab} n^a n^b, \]
which monitors the evolution of the area scalar \( \Theta \) along the \( n_a \)-direction.\(^5\) Note that \( \mathcal{R}_{ab} = h^c d \mathcal{R}_{cdab} = \mathcal{R}_{abcd} \) defines the 3-D Ricci tensor, which is not necessarily symmetric [see equation (16) below]. Also, \( \hat{\omega}^2 \equiv \sigma_{ab} \omega^{ab} / 2 \) and \( \hat{\omega}^2 \equiv \omega_{\alpha \beta} \omega^{\alpha \beta} / 2 \) by construction. As in the standard Raychaudhuri equation of time-like worldlines, positive terms on the right-hand side of the above force our space-like congruence to diverge, while negative ones lead to its convergence.

When dealing with a congruence of space-like geodesics (i.e. where \( n^a = \mathcal{D}_a n_a = 0 \) by default), expression (14) reduces to
\[ \Theta' = -\frac{1}{2} \Theta^2 - \mathcal{R}_{ab} n^a n^b - 2 (\hat{\omega}^2 - \hat{\omega}^2) + 2 \omega_{ab} n^a n^b. \]

Moreover, when the host space–time is irrotational, the aforementioned extra term (i.e. \( \omega_{ab} = 0 \) and the last term of above vanishes identically. In that case, the antisymmetric component of the 3-Ricci tensor vanishes as well [i.e. \( \mathcal{R}_{ab} = \mathcal{R}_{(ab)} \) – see equation (16) next].

From the purely gravitational point of view, the key variable on the right-hand side of equations (14) and (15) is the 3-Ricci tensor. The latter determines the curvature of the 3-D hypersurfaces orthogonal to \( n_a \) and also carries the effect of the matter fields. Following Tsagas et al. (2008), we note that (unlike its 4-D counterpart) \( \mathcal{R}_{ab} \) is not necessarily symmetric and it is given by
\[ \mathcal{R}_{ab} = \frac{2}{3} \left( \kappa \rho - \frac{1}{3} \Theta^2 + \kappa^2 - \omega^2 \right) h_{ab} + E_{ab} + \frac{1}{2} \kappa \pi_{ab} \]
\[ - \frac{1}{3} \Theta (\sigma_{ab} + \omega_{ab}) + \sigma_{c(a} \omega_{b)c} + \omega_{c(a} \omega_{b)c} \]
\[ - 2 \sigma_{c(a} \omega_{b)c}, \]
(16)
where \( \kappa = 8 \pi G \) is the gravitational constant. Here, \( \Theta, \sigma_{ab} \) and \( \omega_{ab} \) are the irreducible kinematic variables of the \( n_a \)-field (see Section 2.1 earlier), with \( \sigma^2 = \sigma_{ab} \sigma^{ab} / 2 \) and \( \omega^2 = \omega_{ab} \omega^{ab} / 2 \). Also, \( \rho \) and \( \pi_{ab} \) are the energy density and the anisotropic pressure of the total matter, respectively, while \( E_{ab} \) is the electric part of the Weyl tensor (all measured relative to the \( n_a \)-field). The Weyl field monitors the action of gravity at a distance, namely tidal forces and gravitational waves. Finally, we note that \( \pi_{ab} = \pi_{(ab)}, E_{ab} = E_{(ab)} \) and \( \pi_{ab} \pi^{ab} = 0 = E_{dab} t^d \) (see e.g. section 1.3.5 in Tsagas et al. 2008).

3.3 Shear and vorticity evolution

The symmetric trace-free and the antisymmetric parts of (13) govern the evolution of the 2-shear and the 2-vorticity tensors, along the direction of \( n_a \). More specifically, we obtain
\[ h_{(a} h_{b)} \hat{\sigma}_{cd} = -\hat{\Theta} \sigma_{ab} - \sigma_{c(a} \omega_{b)c} - \omega_{c(a} \omega_{b)c} + \tilde{D}_a n^c - n^c n^b + 2 h_{(a} \omega_{b)c} n^c n^b, \]
(17)
and
\[ h_{(a} h_{b)} \hat{\omega}_{cd} = -\hat{\Theta} \omega_{ab} + 2 (\hat{\sigma}_{(a} \omega_{b)c} + \tilde{D}_a n^c - n^c n^b + 2 h_{(a} \omega_{b)c} n^c n^b, \]
(18)
for the 2-shear and 2-vorticity tensors, respectively. When the \( n_a \)-congruence is geodesic and the 4-velocity field is irrotational (i.e. for \( \omega_{ab} = 0 \)), the above two expressions simplify to
\[ \hat{\sigma}_{ab} = -\hat{\Theta} \sigma_{ab} - \sigma_{c(a} \sigma_{b)c} - \sigma_{c(a} \omega_{b)c} - \sigma_{(a} \omega_{b)c} \]
(19)
and
\[ \hat{\omega}_{ab} = -\hat{\Theta} \omega_{ab} + 2 \omega_{(a} \omega_{b)c} - \sigma_{(a} \omega_{b)c} \]
(20)
respectively. Therefore, vorticity sources shear but the opposite is not necessarily true. Also, spatial curvature generally affects the evolution of both \( \sigma_{ab} \) and \( \omega_{ab} \).

As with the Raychaudhuri equation before, the effect of the matter fields is carried by the curvature terms. In a general space–time, the Riemann tensor of the 3-D hypersurfaces is given by the expression (see section 1.3.5 in Tsagas et al. 2008)
\[ \mathcal{R}_{abcd} = -\varepsilon_{a} \varepsilon_{d} E_{b} - \frac{1}{3} \left( \kappa \rho - \frac{1}{3} \Theta^2 \right) (h_{ab} h_{cd} - h_{ac} h_{bd}) \]
\[ + \frac{1}{2} \kappa (h_{ab} \pi_{cd} + \pi_{ab} h_{cd} - h_{ac} \pi_{bd} - h_{ad} \pi_{bc}) \]
\[ - \frac{1}{3} \Theta [h_{ab} (\sigma_{cd} + \omega_{cd}) + (\sigma_{ac} + \omega_{ac}) h_{bd} - h_{ad} (\sigma_{bc} + \omega_{bc}) - (\sigma_{cd} + \omega_{cd}) h_{ab} ] \]
\[ - (\sigma_{ac} + \omega_{ac})(\sigma_{bd} + \omega_{bd}) \]
\[ + (\sigma_{ad} + \omega_{ad})(\sigma_{bc} + \omega_{bc}) \]
(21)
guaranteeing that \( \mathcal{R}_{abcd} = \mathcal{R}_{(ab)(cd)} \) always and that \( \mathcal{R}_{abcd} = \mathcal{R}_{cdab} \) only when \( \omega_{ab} = 0 \). Substituting the above into the right-hand side of (17) and (18) leads to
\[ h_{(a} \hat{h}_{b)} \hat{\sigma}_{cd} = -\hat{\Theta} \sigma_{ab} - \sigma_{c(a} \omega_{b)c} - \omega_{c(a} \omega_{b)c} + \tilde{D}_a n^c - n^c n^b + 2 h_{(a} \omega_{b)c} n^c n^b, \]
\[ + \varepsilon_{a} \varepsilon_{d} E_{b} n_a n_b E_{q_1} + \frac{1}{3} \left( \kappa \rho - \frac{1}{3} \Theta^2 \right) n_{(a} \pi_{b)} \]
\[ + \frac{1}{2} \kappa (\pi_{ab} - 2 n_a \pi_{b}) n_1 \]
\[ + \frac{1}{3} \Theta (\sigma_{ab} - 2 n_a \sigma_{b}) n_1 \]
\[ + \sigma_{a c} \sigma_{b d} n_a n_d + \sigma_{a d} \sigma_{b c} n_a n_d \]
\[ + 2 h_{(a} \omega_{b)c} n^c n^b, \]
(22)
and
\[ h_{(a} \hat{h}_{b)} \hat{\omega}_{cd} = -\hat{\Theta} \omega_{ab} + 2 (\hat{\sigma}_{(a} \omega_{b)c} + \tilde{D}_a n^c - n^c n^b + 2 h_{(a} \omega_{b)c} n^c n^b, \]
(23)
respectively. Note the absence of any geometric or matter terms in the latter expression. This shows that the geometry of the host space–time, namely the gravitational field, does not affect (at least directly) the rotational behaviour of space-like congruences. According to equation (22), on the other hand, this is not the case for shear-like deformations.

Before closing this section we should emphasize that the formulae derived so far are purely geometrical in nature and depend solely on the structure of the 3-D hypersurfaces and on that of their host space–time. Also, no specific assumptions have been made about the material content, the effects of which enter into the equations through the 3-Riemann and the 3-Ricci tensors.
4 THE MAGNETIC-FIELD CASE

Magnetism is an integrable part of the cosmos with a verified presence almost everywhere in the Universe. Also, magnetic fields are rather unique matter sources and what distinguishes them from the rest is their vector nature and tension properties. In what follows, we will use the formalism developed so far to look closer into the implications of these special magnetic features.

4.1 Magnetic pressure and magnetic tension

Consider the 4-D space–time defined in Section 2 earlier. Relative to observers moving with a time-like 4-velocity $u_a$, the electromagnetic tensor ($F_{ab} = F_{(ab)}$) decomposes into its electric and magnetic parts. These are respectively given by (Tsagas et al. 2008)

$$E_a = F_{ab} u^b \quad \text{and} \quad B_a = \frac{1}{2} F_{abc} F^{bc}, \quad (24)$$

with $F_{abc}$ being the 3-D Levi-Civita tensor (see footnote 2 earlier). Then, $E_a u^a = 0 = B_a u^a$, to guarantee that both the electric and the magnetic fields are space-like vectors.

Let us concentrate on the magnetic component of the Maxwell field and switch its electric counterpart off, as it appears in the ideal magnetohydrodynamic (MHD) limit, for example. In such a case, the electromagnetic stress–energy tensor reduces to

$$T_{ab} = \rho_g u_a u_b + p_g h_{ab} + \Pi_{ab}, \quad (25)$$

where $\rho_g = B^2 / 2$ is the energy density, $p_g = B^2 / 6$ is the isotropic pressure and $\Pi_{ab} = (B^2 / 3) h_{ab} - B_a B_b$ is the anisotropic pressure of the $B$-field (with $B^2 = B^a B^a$). The symmetric and trace-free $\Pi_{ab}$-tenensor also carries the tension properties of the magnetic fields. The magnetic tension reflects the elasticity of the magnetic field lines and their tendency to remain ‘straight’. On the other hand, the total pressure exerted by the $B$-field (isotropic plus anisotropic) is encoded in the symmetric Maxwell tensor $M_{ab} = (B^2 / 2) h_{ab} - B_a B_b$ (see e.g. Parker 1979; Mestel 2012).

Suppose now that $\ell_a$ and $k_a$ are unitary space-like vectors orthogonal and parallel to the magnetic field, respectively. Then, $\ell_a u^a = 0 = k_a u^a$, with $\ell^a B_a = 0 = k_a ( B_a B_a )$, where $B_a = \sqrt{\ell^b \ell_b}$.

It is straightforward to show that both $\ell_a$ and $k_a$ are eigenvectors of the Maxwell tensor, though their associated eigenvalues have opposite signs. Indeed, projecting $\mathcal{M}_{ab}$ along $\ell_a$ gives a positive eigenvalue (i.e. $\mathcal{M}_{ab} \ell^b = (1/2) \ell^a$), thus ensuring a (positive) magnetic pressure orthogonal to the field lines. Projecting along $k_a$, on the other hand, leads to a negative eigenvalue ($\mathcal{M}_{ab} k^b = -(1/2) k^a$), which implies that the $B$-field exerts a negative pressure (i.e. a tension) along its own direction. Physically speaking, the magnetic pressure reflects the tendency of the forcelines to push each other apart, while the field’s tension manifests the elasticity of the field lines, namely their tendency to remain “straight” and to react against any agent that distorts them from equilibrium (Parker 1979; Mestel 2012).

4.2 Magnetic-line convergence and focusing

Let us introduce a congruence of magnetic lines tangent to the field vector. Suppose also that $k_a$, with $B_a = B k_a$, is the unitary space-like vector along the direction of the the $B$-field (see Section 4.1 above). Like any other source of energy, the magnetic field contributes to the total gravitational field through its energy density, pressure and tension [see equation (25) in Section 4.1]. The question we would like to address is how gravity affects the convergence/divergence of the magnetic forcelines and, more specifically, whether the $B$-field will collapse under its own gravitational pull or not.

A family of space-like curves will converge and focus when their 2-D cross-sectional area becomes progressively smaller (along their own direction). In the opposite case, the aforementioned congruence will diverge. Assuming that $n_a$ is the unit vector tangent to the aforementioned lines, changes in the size of their cross-section are monitored by the divergence $\Theta = \mathcal{D}_a n^a$, as defined in Section 2.2 earlier. The evolution of $\Theta$ in the direction of the lines, namely along $n_a$, follows from the associated Raychaudhuri formula [see equation (14) in Section 3.2]. When dealing with the forcelines of a magnetic field, that is when $n_a \equiv k_a$, the latter reads

$$\dot{\Theta} = -\frac{1}{2} \Theta^2 - \mathcal{R}_{ab} k^a k^b - 2 \left( \hat{\Sigma}^2 - \hat{\omega}^2 \right) + \mathcal{D}_a k^a - k_a' k^a' + 2 \omega_{ab} k^a k^b, \quad (26)$$

where $\mathcal{R}_{ab}$ is given by (16). Projecting the latter along the direction of the magnetic forcelines, while assuming the presence of other matter sources (with total energy density $\rho$ and anisotropic pressure $\pi_{ab}$), we obtain

$$\mathcal{R}_{ab} k^a k^b = \frac{2}{3} \left[ \kappa (\rho + p_g) - \frac{1}{3} \Theta^2 - \sigma^2 - \omega^2 \right] + E_{ab} k^a k^b$$

+ $1/2 \kappa (\sigma_{ab} + \Pi_{ab}) k^a k^b - \frac{1}{3} \Theta \sigma_{ab} k^a k^b$

+ $\sigma_{ab} \sigma_{cd} k^a k^b + \omega_{(a}(\omega_{b)}/2) k^a k^b$, \quad (27)

given that $\mathcal{R}_{ab} k^a k^b = \kappa k^a k^a$. However, given that $\rho_g = B^2 / 2$ and that $\Pi_{ab} = (1/3) \mathcal{B}_{ab} - B_a B_b$, we find that $(2/3) \kappa \rho_g + (1/2) \kappa \Pi_{ab} k^a k^b = 0$. Then,

$$\mathcal{R}_{ab} k^a k^b = \frac{2}{3} \left[ \kappa \rho - \frac{1}{3} \Theta^2 + \sigma^2 - \omega^2 \right] + E_{ab} k^a k^b + \frac{1}{2} \kappa \pi_{ab} k^a k^b$$

- $\frac{1}{3} \Theta \sigma_{ab} k^a k^b + \sigma_{ab} \sigma_{cd} k^a k^b + \omega_{(a}(\omega_{b)}/2) k^a k^b$. \quad (28)

This ensures that the magnetic energy–density and pressure do not contribute to the right-hand side of equation (26). In other words, although the convergence/divergence of the magnetic forcelines is directly affected by the gravitational pull of the other matter sources, it proceeds unaffected by the $B$-field’s own gravity (i.e. by the magnetic gravitational energy). The reason behind this counterintuitive behaviour is the magnetic tension, which cancels out the field’s energy-density input to the right-hand side of (27), (28) and therefore to equation (26) itself.

The above refer to a general congruence of magnetic forcelines in a general space–time filled with other forms of matter, in addition to the $B$-field. Further physical insight on the role of the magnetic tension can be obtained by considering the idealized case of forcelines that are irrotational and shear-free (space-like) geodesics, resting in an otherwise empty and static space. Then, (26) reduces to

$$\dot{\Theta} + \frac{1}{2} \Theta^2 = -\mathcal{R}_{ab} k^a k^b. \quad (29)$$

6 We use natural and Heaviside–Lorentz units.

7 We remind the reader that the Raychaudhuri formula given in equation (26) monitors the convergence/divergence, of the (space-like) magnetic forcelines along their own (spatial) direction. Therefore, one should not confuse expression (26) with the Raychaudhuri equation monitoring the (time-like) worldlines of charged particles and their temporal evolution in the presence of a magnetic field (see e.g. Raychaudhuri 1975; Kouretsis & Tsagas 2010).
with
\[ R_{ab} = \frac{2}{3} \kappa \rho_b h_{ab} + \frac{1}{2} \kappa \Pi_{ab}. \] (30)

Keeping in mind that \( \rho_b = B^2/2 \) and that \( \Pi_{ab} = (1/3)B^2 h_{ab} - B_i B_i \), the latter of the above gives \( R_{ab} k^a k^b = 0 \), which substituted back into equation (29) leads to
\[ \Theta' = -\frac{1}{2} \Theta^2, \] (31)

ensuring that \( \Theta' = 0 \) at all times when \( \Theta = 0 \) initially. This differential equation integrates immediately giving
\[ \Theta = \Theta(\lambda) = \frac{2 \Theta_0}{1 + \Theta_0 \lambda}, \] (32)

where \( \Theta_0 = \Theta(\lambda = 0) \) and \( \lambda \) may be seen as the proper length measured along the magnetic field lines. Accordingly, we may distinguish between the following three alternatives:

(i) when \( \Theta_0 < 0 \), we find that \( \Theta \to -\infty \) as \( \lambda \to -2/\Theta_0 \)
(ii) when \( \Theta_0 = 0 \), we have \( \Theta = 0 \) at all times
(iii) when \( \Theta_0 > 0 \), we have \( \Theta > 0 \) always

In other words, magnetic forcelines that are initially converging will focus to form caustics within finite proper length. If the lines happen to be stationary, on the other hand, they will remain so and will never converge. Finally, magnetic lines that are initially diverging will continue to do so indefinitely [since \( \Theta \to 0 \) as \( \lambda \to +\infty \) when \( \Theta_0 > 0 \) – see equation (32)]. Note that (unlike typical time-like worldlines) in the last two cases the forcelines remain stationary, or keep diverging, despite the fact that the host 3-space is positively curved.\(^8\) Hence, although the spatial sections have positive mean curvature, the magnetic tension ensures that field lines will not ‘feel’ the pull of their own gravity and therefore their ‘motion’ is fully dictated by their initial condition.

The behaviour of the magnetic forcelines described so far is rather atypical and (to the best of our knowledge) particular to the B-field only. Indeed, consider the (space-like) flow lines of ordinary matter and assume that \( t_i \) is their unit tangent vector. Assuming, for simplicity and demonstration purposes, that these lines are irrational and shear-free geodesics, residing in an otherwise empty and static space–time, the associated Raychaudhuri equation reads
\[ \Theta' + \frac{1}{2} \Theta^2 = -R_{ab} r^a r^b, \] (33)

with
\[ R_{ab} r^a r^b = \frac{2}{3} \kappa \rho + \frac{1}{2} \kappa \pi_{ab} r^a r^b, \] (34)

since \( h_{ab} r^a r^b = t_i t^i = 1 \). In the case of a perfect fluid, with positive energy density \( (\rho > 0) \) and zero viscosity \( (\pi_{ab} = 0) \), we find that \( R_{ab} r^a r^b > 0 \). Therefore, flow lines that are initially static will converge and eventually focus (within finite length) under the pull of their own gravity alone. Also, in contrast to the B-field lines [see alternative no. (iii) above], initially diverging flow lines are not guaranteed to keep diverging. When dealing with an imperfect medium, however, the convergence of the flow lines is not guaranteed, but depends on the sign and the magnitude of the anisotropic-pressure term \( (\pi_{ab} r^a r^b) \) on the right-hand side of equation (34). In particular, for matter with \( \pi_{ab} r^a r^b > -4\rho/3 \) the flow lines will definitely converge, but when \( \pi_{ab} r^a r^b < -4\rho/3 \) the flow lines may instead diverge. It is only for media with a magnetic-like ‘equation of state’ (i.e. for \( \pi_{ab} r^a r^b = -4\rho/3 \)) that the right-hand side of equation (33) vanishes identically.

4.3 Magnetic-line rotation and distortion

Following the evolution formula of the 2-vorticity (see equation (23) in Section 3.3), the geometry and the matter content of the host space–time do not affect the rotation of space-like congruences. Hence, the rotation of the magnetic forcelines is not directly affected by the active gravitational field, including their own. Let us now turn to the 2-shear and apply expression (22) to a set of magnetic forcelines residing in a general space–time. Then, the 2-shear evolution formula reads
\[ h_{ab} f^a f^b \sigma_{cd} = -\Theta(\sigma_{ab} - \tau_{(a|b|c|)}) - \Theta(\sigma_{ab} \tau_{|c|b|} + D_{(a|b|}) - n_{(a|b|}) + \frac{1}{2} \kappa (\rho + \rho_b) - \frac{1}{2} \Theta^2 n_{(a|b|})
\]
\[ - \frac{1}{3} \Theta \left( \tau_{ab} - 2n_{(a|b|}) \right) + \frac{1}{2} \kappa \left( \Pi_{ab} - 2n_{(a|b|}) \right)^c n_{(a|b|}) + \frac{1}{2} \kappa \left( \Pi_{ab} - 2n_{(a|b|}) \right)^c n_{(a|b|}) + \frac{1}{2} \kappa \left( (\rho + \rho_b) - \frac{1}{2} \Theta^2 \right) n_{(a|b|}) + \frac{1}{3} \Theta \left( \tau_{ab} - 2n_{(a|b|}) \right) + \frac{1}{2} \kappa \left( (\rho + \rho_b) - \frac{1}{2} \Theta^2 \right) n_{(a|b|}), \] (35)

where \( \rho, \pi_{ab}, \text{ and } \rho_b \), \( \Pi_{ab} \) are the energy density and the anisotropic pressure of the matter and of the B-field, respectively. Then, given that \( \rho_b = B^2/2 \) and that \( \Pi_{ab} = (1/3)B^2 h_{ab} - B_i B_i \), it is straightforward to show that the above reduces to expression (22) of Section 3.3, with no explicit magnetic terms on the right-hand side \( (\rho \text{ and } \pi_{ab} \text{ refer to the rest of the matter sources}). Again, the absence of any direct magnetic effect is due to the field’s tension, which cancels out the positive contribution from the magnetic energy density and pressure to equation (35).

In summary, the convergence/divergence of the magnetic forcelines, their shear deformation and their rotation proceed unaffected by the B-field’s own gravitational energy. Although the null effect on rotation applies to all space-like congruences, the rest are entirely due to the field’s tension. The latter guarantees that, although the magnetic lines of force respond to the gravitational pull of the other sources, they do not ‘feel’ (at least not directly) their own gravity. This generic magnetic feature implies that (in the absence of other sources) a configuration of field lines that happens to be in equilibrium initially will remain so indefinitely (unless an external agent interferes).

5 DISCUSSION

Magnetic fields are ubiquitous and of rather unique nature, and what distinguishes them from the other known energy sources is their vector status and tension properties. In this work, we have attempted to investigate the implications of the aforementioned features by looking into the ‘kinematics’ of a congruence of magnetic forcelines. We did so by introducing a 1+2 splitting of the 3-D space into a direction parallel to the field lines and 2-D surfaces orthogonal to them. Taking a cross-sectional area of these lines, we defined three variables that monitor the area’s expansion/contraction, rotation and shear deformation. We then derived the equations describing the evolution of these variables along the direction of the magnetic lines of force. Our results showed that, although the magnetic congruence responds to the gravitational pull of the other sources, it is

\(^8\) The mean curvature of the 3-space is decided by the trace of \( R_{ab} \). Recalling that \( \rho_b = B^2/2 \) and that \( \Pi_{ab} = 0 \), we obtain \( R = R_{ab} = B^2 \) to guarantee that the mean 3-curvature is positive (solely due to the magnetic presence).
‘immune’ to its own gravity, no matter how strong the latter may be. More specifically, the kinematics of the magnetic forcelines are unaffected by the field’s own contribution to the total gravitational energy. To the best of our knowledge, no other known matter source shows such a counterintuitive behaviour. The reason behind this unique magnetic conduct is its tension, which always cancels out the input of the field’s energy density and isotropic pressure. In a sense, the magnetic tension ensures that the field lines do not ‘feel’ their own gravitational pull. This also implies that, in a static and otherwise empty space–time, a set of parallel magnetic forcelines will not converge or diverge, it will not rotate and it will not deform. Instead, the aforementioned congruence will remain in equilibrium until an external agent interferes.

These results are reminiscent of the work done several decades ago, in the mid-1960s, by Melvin and Thorne (Melvin 1964, 1965; Thorne 1965a,b). It was shown, in particular, that there exists a stable solution of the Einstein–Maxwell equations that describes a cylindrical configuration of parallel magnetic forcelines in equilibrium, residing in an otherwise empty and static space–time (as in our case – see Sections 4.2 and 4.3 above). This solution is also known as ‘Melvin’s magnetic universe’. It was also argued that ‘a pure magnetic field has a remarkable and previously unsuspected ability to stabilize itself against gravitational collapse’.9 Whether this ability would be enough to avoid the ultimate singularity was left unanswered, but a number of crucial questions regarding the magnetic role during gravitational collapse was raised (Melvin 1964, 1965; Thorne 1965a,b). Our work seems to indicate that the magnetic tension, namely the elasticity of the field lines, may be the physical reason behind such a remarkable ability. This suggestion is also corroborated by other studies showing how the field’s tension gives rise to ever increasing magneto-curvature stresses that resist the gravitational collapse of a magnetized medium (Tsagas 2001, 2005, 2006).

As with the work of Melvin (1964, 1965) and Thorne (1965a,b), however, the complexity of the problem made it impossible to establish whether such stresses would be capable of preventing the singularity from forming. Here, by treating the field lines as a congruence of space–like curves, we have initiated a rather novel approach, which (once again) brought to the fore the role of the field’s tension as a stabilizing agent. Future work will try to exploit the advantages of such a treatment and shed more light on the potential magnetic implications for gravitational collapse.

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9 It was claimed in Melvin (1964, 1965) that pure electric fields show the same behaviour. The symmetry of the source-free Maxwell’s equations guarantees that our results also apply to a pure electric field.