Does curvature-dilaton coupling with Kalb Ramond field lead to an accelerating Universe?

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In this work we show that the Universe evolving in a spacetime with torsion (originated from a second rank antisymmetric Kalb-Ramond field) and dilaton is free from any big bang singularity and can have acceleration during the evolution. Both the matter and radiation dominated era have been considered and the role of the dilaton to explain the decelerating phase in the earlier epoch has also been discussed.

I. INTRODUCTION

Eversince Einstein-Cartan (EC) theory was proposed, spacetime torsion has provided a substantial amount of interest both in gravity and cosmology. Such a theory, however, is faced with serious difficulties in absence of any confirmed experimental signature in favour of torsion. There was a revival in the interest for such a theory after the development of String theory where the massless second-rank antisymmetric Kalb-Ramond (KR) field \( B_{\mu \nu} \) [1] arising in the heterotic string spectrum has a natural explanation in the form of spacetime torsion. It has recently been shown [2] that in the string theoretic scenario, spacetime torsion can in fact be identified with KR field strength augmented with the Chern-Simons (CS) extension. Such an extension, which originates on account of cancellation of gauge anomaly in the context of string theory, plays the crucial role in restoring electromagnetic \( U(1) \) gauge invariance which is normally lost in the standard EC framework [2]. This enables one to study the effect of torsion on the propagation of electromagnetic waves preserving the gauge symmetry. Investigations have also been carried out in the context of theories with large extra dimensions, namely those of Arkani Hamed–Dimopoulos–Dvali (ADD) [3] and Randall–Sundrum (RS) [4]. An interesting possibility in such models is that of torsion existing in the bulk together with gravity, while all the standard model fields are confined to a 3-brane. In order to study the effect of extra dimensions on the spacetime torsion in the context of RS scenario it has been shown by Mukhopadhyaya et.al. [5] that in a RS type of model where the torsion has the same status as gravity in the bulk, the effects of massless torsion becomes heavily suppressed on the standard model brane, thus producing the illusion of a torsionless Universe. This explains why the detection of torsion has been eluding us for so long.

In recent years, it has already been shown that the presence of KR field in the background spacetime may lead to various interesting astrophysical/cosmological phenomena like cosmic optical activity, neutrino helicity flip, parity violation etc [6]. This motivates us to address some of the important problems associated with the standard Friedmann-Robertson-Walker (FRW) model in the light of KR cosmology. The most longstanding problem is the existence of so-called ‘big bang singularity’ and the most recent one concerns the observational evidences from type Ia Supernova data [7,8] showing acceleration at a late time (may even be at the present epoch considering the data from the WMAP [9]) in course of evolution of the Universe. A considerable amount of work has been carried out in this regard in the past in the framework of FRW cosmology or in alternative cosmological

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models [10]. However, the models obeying the ‘cosmological principle’, i.e., the large scale homogeneity and isotropy of the Universe, are of more significance for obvious observational reasons. Although some bouncing solutions were found in presence of torsion in the past [11], the present work reexamines these scenarios in a homogeneous and isotropic cosmological model originated from a background spacetime with torsion along with dilaton.

II. GENERAL FORMALISM

Following the formalism in [2], the action for gauge invariant Einstein-Cartan-Kalb-Ramond coupling in presence of external matter fields is given by

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R(g,T)}{2\kappa} - c_1 H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \frac{c_2}{\sqrt{\kappa}} T_{\mu\nu\lambda} H^{\mu\nu\lambda} + L_m \right] \]

(1)

where \( \kappa = 8\pi G/c^4 \) is the gravitational coupling constant and \( T_{\mu\nu\lambda} \) is the torsion tensor which is the antisymmetrization of the affine connection in EC spacetime and is chosen to be antisymmetric in all its indices. \( H_{\mu\nu\lambda} \) is the strength of the KR field \( B_{\mu\nu} \) plus the Chern-Simons (CS) term \( \Omega_{\kappa\lambda} \) which keeps the corresponding quantum theory anomaly-free: \( H = dB + \Omega_{\kappa\lambda} \). \( \Omega_{\kappa\lambda} \), however, contains a suppression factor of the order of Planck mass relative to \( dB \) and therefore can safely be dropped from \( H \) in the present analysis. In the above action \( L_m \) is the external matter Lagrangian density and \( c_1 \) and \( c_2 \) are respectively the coupling constants for self-coupling of the three-form \( H \) and the \( H - T \) coupling. The Ricci scalar \( R(g,T) \) in the torsioned spacetime is related to the pure Einsteinian scalar curvature \( R(g) \)

\[ R(g,T) = R(g) - T_{\mu\nu\lambda} T^{\mu\nu\lambda} \]

(2)

The torsion tensor \( T_{\mu\nu\lambda} \) being an auxiliary field in the action (1), satisfies the constraint relation

\[ T_{\mu\nu\lambda} = \frac{2c_1}{c_2} \frac{1}{\sqrt{\kappa}} H_{\mu\nu\lambda} \]

(3)

which implies that the augmented KR field strength 3-tensor acts as the source of torsion [12]. The resulting action, which has direct correspondence with heterotic string theory, is then given by

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R(g)}{2\kappa} + \frac{c_T}{H_{\mu\nu\lambda}H^{\mu\nu\lambda}} + L_m \right] \]

(4)

where \( c_T = c_1(1 - 2c_1/c_2^2) \).

Now, in the string scenario the low-energy effective action compactified to four spacetime dimensions contains alongwith the KR field, the massless scalar dilaton field \( \phi \). The low energy heterotic string action in the String frame is then expressed as

\[ S = \int d^4x \sqrt{-\tilde{g}} \left[ e^{-\phi} \left( \frac{R(\tilde{g}) + [D\phi]^2}{2\kappa} + c_T \tilde{H}^2 \right) + L_m \right] \]

(5)

where \( \tilde{H} = H_{\mu\nu\lambda} H^{\mu\nu\lambda} \) and \( [D\phi]^2 = D_{\mu}\phi D^{\mu}\phi \); \( D_{\mu} \) being the covariant derivative defined in terms of the usual Christoffel connection. \( L_m \) is the Lagrangian density for external matter fields, for example, the standard cosmological matter (perfect fluid).

Now, as a general convention, specifically in observational studies, it is useful to work in the Einstein frame which can be obtained from the String frame by making a conformal transformation \( g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{-\phi}g_{\mu\nu} \). The resulting 4-dim effective string action is given by

\[ \tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R}(\tilde{g}) - \frac{\frac{1}{2}[D\phi]^2}{2\kappa} + c_T e^{-2\phi} \tilde{H}^2 + \tilde{L}_m \right] \]

(6)

where \( \tilde{H}_{\mu\nu\lambda} = H_{\mu\nu\lambda} \) and \( \tilde{L}_m = e^{2\phi}L_m \), the untilded quantities refer to the String frame. Dropping the tildes for simplicity one can express the corresponding field equations as

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = \kappa \left\{ T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(m)} + e^{-2\phi} T_{\mu\nu}^{(H)} \right\} \]

(7)

\[ D_{\mu} \left( e^{-2\phi} H_{\mu\nu\lambda} \right) = 0 \]

(8)

\[ D_{\mu}\phi = 4\kappa \left( c_T e^{-2\phi} \tilde{H}^2 - T_{\mu\nu}^{(m)} \right) \]

(9)
where $\phi_\mu \equiv \partial_\mu \phi$. $T_{\mu\nu}^{(m)}$ is the energy-momentum tensor corresponding to the background matter distribution which for our cosmological model is taken to be a perfect fluid, and $T_{\mu\nu}^{(\phi)}$ and $T_{\mu\nu}^{(H)}$ are the analogous contributions due to the dilaton and the KR field:

$$T_{\mu\nu}^{(\phi)} = \frac{1}{2\kappa} (\phi_\mu \phi_\nu - \frac{1}{2} g_{\mu\nu} \phi_\alpha \phi^\alpha)$$  \hspace{1cm} (10)

$$T_{\mu\nu}^{(m)} = [(\rho c^2 + p) u_\mu u_\nu - p g_{\mu\nu}]$$  \hspace{1cm} (11)

$$T_{\mu\nu}^{(H)} = -6 c_T (3H_{\alpha\beta\mu} H^{\alpha\beta}_\nu - \frac{1}{2} g_{\mu\nu} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma})$$  \hspace{1cm} (12)

$p$ and $\rho$ being the pressure and mass-density respectively, and $u_\mu = (1, 0, 0, 0)$ is the hypersurface-orthogonal four-velocity vector.

Now, with the CS term neglected, the three form $H$ being equal to the exterior derivative of the two form $B$, i.e., $H = dB$, one can verify the Bianchi identity in 4-dim

$$dH = \epsilon^{\mu\nu\lambda\beta} \partial_\beta [\partial_\mu B_\nu \lambda] = 0$$  \hspace{1cm} (13)

which along with Eq.(8) leads to the well-known duality relationship in string theory between the three form $H$ and the pseudoscalar (axion) $\xi$:

$$H^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\beta} e^{2\phi} \xi_\beta.$$  \hspace{1cm} (14)

where $\xi_\mu \equiv \partial_\mu \xi$ and $\xi$ satisfies the wave equation

$$D_\mu (e^{2\phi} \xi^\mu) = 0.$$  \hspace{1cm} (15)

In our subsequent analysis we shall be using the axion $\xi$ frequently rather than the KR field strength $H$ using the above duality.

In order to have a large-scale homogeneous and isotropic cosmological model, we consider the standard Robertson-Walker metric structure:

$$ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]$$  \hspace{1cm} (16)

$a(t)$ being the scale-factor and $k$ is the curvature. It can now easily be shown that for this type of metric the consistency of the field equations immediately demands that both the axion $\xi$ and the dilaton $\phi$ depend on time only. Eq.(15) then yields

$$\dot{\xi}_0 \equiv \frac{d}{c} \frac{d}{dt} \xi = \frac{\alpha}{a^3(t)} e^{-2\phi}$$  \hspace{1cm} (17)

$\alpha$ being a parameter determining the strength of the axion. The effective Kalb-Ramond energy density is then given by

$$\rho_{KR} = \frac{1}{c^2} \pi_0^{(KR)} = -\frac{6 c_T \alpha^2}{c^2 a^6(t)}.$$  \hspace{1cm} (18)

The sign of $c_T$ determines whether the KR energy density is positive or negative. The field equations now reduce to

$$\frac{\ddot{a}}{a^2} = \frac{k c^2}{a^2} = \frac{\dot{\phi}^2}{12} + \frac{8\pi G}{3c^2} \left( \rho c^2 - \frac{6 c_T \alpha^2 e^{-2\phi}}{a^6} \right)$$  \hspace{1cm} (19)

$$\frac{\ddot{a}}{a} = -\frac{\dot{\phi}^2}{6} - \frac{8\pi G}{3c^2} \left( \frac{\rho c^2 + 3p}{2} - \frac{6 c_T \alpha^2 e^{-2\phi}}{a^6} \right)$$  \hspace{1cm} (20)

$$\frac{d}{dt} (a^3 \dot{\phi}) = -\frac{32\pi G}{c^2} \left( (\rho c^2 - 3p) a^3 + \frac{6 c_T \alpha^2 e^{-2\phi}}{a^3} \right)$$  \hspace{1cm} (21)

where overhead dot stands for derivative with respect to time. For a general equation of state $p = \omega \rho c^2$ we rewrite the field equations as
\[
\frac{b^2}{b^2} + \frac{\chi}{b^2} = \frac{\dot{\phi}^2}{12} + \frac{\gamma}{3} \left( \zeta - \frac{\sigma}{b^6} e^{-2\phi} \right)
\]

\[
\frac{\dot{b}}{b} = -\frac{\dot{\phi}^2}{6} - \frac{\gamma}{3} \left( \frac{1 + 3\omega}{2} \zeta - \frac{2\sigma}{b^6} e^{-2\phi} \right)
\]

\[
\frac{d}{dt}(b^3 \dot{\phi}) = -4\gamma \left\{ (1 - 3\omega) b^3 + \frac{\sigma}{b^6} e^{-2\phi} \right\}
\]

where, for convenience, we have used dimensionless quantities:

\[
b = \frac{a}{a_0}, \quad \zeta = \frac{\rho}{\rho_0}, \quad \chi = \frac{k e^2}{a_0^2}, \quad \gamma = 8\pi G \rho_0, \quad \sigma = \frac{6 c_T \alpha^2 a_0^{-6}}{\rho_0 c^2}.
\]

The subscript 0 generally stands for the values of the quantities at the present epoch \(t_0\). Recalling Eq.(18), one finds the quantity \(\sigma\) defined above gives a relative measure of the present values of KR field energy density and the energy density of the Universe. From the above field equations it is easy to show that \(\zeta\) satisfies the energy-momentum conservation relation

\[
\frac{d}{db}(\zeta b^3) = -3\omega \zeta b^3 \left\{ \frac{3\omega}{b} - 2(1 - 3\omega) \frac{d\phi}{db} \right\}
\]

with solution

\[
\zeta = \frac{\phi^{2(1-3\omega)}}{b^{6(1+\omega)}}.
\]

Now, in the development of a cosmological model the quantity \(\sigma\) may be positive or negative depending on the sign of the torsion-KR coupling parameter \(c_T\). This implies that the effective KR energy density \(\rho_{KR}\) can be negative as well as positive. In the case of low energy string field effective action the term quadratic in the three-form \(\mathbf{H}\) appears with a negative \((=-1/12)\) sign which ensures positive energy density for the KR field. Therefore, if one persists in considering the spacetime torsion to be identified with the KR field strength then \(\sigma\) can only take negative values. On the contrary, a repulsive (anti-gravitating) character of torsion is well known in the context of cosmologies based on Einstein-Cartan theory. Bouncing cosmological solutions have already been shown to exist in Bianchi type torsioned spacetimes [11]. Thus one is led to the fact that even when torsion is generated by the hitherto attractive KR field in string background geometries of Riemannian type, the repulsive character of torsion may still be restored via the nature of its coupling with the KR field within the minimal coupling prescription when \(\sigma > 0\) (or, in other words when the effective KR energy density is negative). This may well be treated as a departure from the usual string theoretic picture and can be looked upon as an alternative theory of gravity. However, regardless of the sign of \(\sigma\) we show here the existence of a theory of gravity with torsion where the background metric continues to be homogeneous and isotropic and is described by a structure similar to Robertson-Walker spacetime. We seek specific solutions of the above field equations in a scenario where \(\sigma\) can be positive as well as negative, separately for the present matter-dominated (dust) Universe and for the early radiation-dominated Universe.

### III. MATTER-DOMINATED UNIVERSE

Presently, in the Universe \(p \ll \rho c^2\), i.e., effectively \(\omega = 0\), and hence \(\zeta = e^{2\phi}/b^3\). The dilatonic evolution rate \(\dot{\phi}\) as well as the Hubble parameter \(\mathcal{H}\) and the deceleration parameter \(q\) can be obtained from the field equations (25) - (27):

\[
\dot{\phi}^2 = \frac{4\gamma}{b^6} \left\{ \sigma e^{-2\phi} - b^3 e^{2\phi} + 3f(b) + \Lambda_m \right\}; \quad f(b) = \int e^{2\phi} b^2 db
\]

\[
\mathcal{H} \equiv \frac{\dot{b}}{b} = \left[ -\frac{\chi}{b^2} + \frac{\gamma}{3b^6} \left( 3f(b) + \Lambda_m \right) \right]^{1/2}
\]

\[
q \equiv -\frac{1}{\mathcal{H}^2} \frac{\dot{b}}{b} = \frac{\gamma}{3\mathcal{H}^2 b^6} \left\{ 6f(b) + 2\Lambda_m - \frac{3}{2} b^3 e^{2\phi} \right\}.
\]

Imposing the limiting conditions: \(\phi = \phi_0\), \(\dot{\phi}^2 = \lambda_0\), \(\mathcal{H} = \mathcal{H}_0\) at the present epoch \(t = t_0\) \((b = 1)\) whence \(f(b) = f(1)\), the constant \(\gamma\) as well as the integration constant \(\Lambda_m\) turn out to be
\[
\gamma = \frac{12(H_0^2 + \chi) - \lambda_0}{4(\epsilon^{2\phi_0} - e^{-2\phi_0})}, \quad \sigma \neq e^{4\phi_0} \\
\Lambda_m = \frac{3}{\gamma} \{H_0^2 + \chi - \gamma f(1)\} 
\]

The Hubble parameter and the deceleration parameter reduce to
\[
\mathcal{H} = \frac{1}{b^3} \left[ H_0^2 + \gamma \{f(b) - f(1)\} + \chi (1 - b^4) \right]^{1/2} \quad (32)
\]
\[
q = 2 - \frac{\gamma e^{2\phi} - 4\chi b}{2H^2b^3} \quad (33)
\]

As can be seen from the above equations that an accelerating Universe \((\ddot{b} > 0, \ q < 0)\) is only possible when \(\gamma > 4(H^2b^3 + \chi b)e^{-2\phi}\). In late stages of the evolution of the Universe this implies that for \(\sigma < 0\) i.e., \(\rho_{KR} > 0\), acceleration is never possible. This can be clearly be observed from Eq.(23) which for the matter-dominated model takes the form
\[
\frac{\ddot{b}}{b} = -\frac{\dot{b}^2}{6} - \frac{\gamma}{3} \left( \frac{e^{2\phi}}{2b^3} - \frac{2\sigma e^{-2\phi}}{b^6} \right). \quad (34)
\]

As \(\dot{\phi}^2\) is known to be positive in the string theoretic scenario, an accelerating Universe at any of stage of its evolution is not possible in pure stringy cosmological models where \(\sigma < 0\) as pointed out earlier. However, if the nature of the KR-torsion coupling be such that \(\sigma > 0\) then torsion would exhibit an anti-gravitating character and in such case an accelerating cosmological model is indeed possible whenever the KR field is sufficiently large to overcome the effect of the dilaton (considered to be of positive energy density) and ordinary gravitating matter. We now make a more critical examination of the scenario resorting to a rather typical dilatonic evolution.

**Frozen dilaton:**

As we are primarily interested in the effect of KR field on the standard Robertson-Walker cosmology especially in late times, we consider dilatonic evolution rate gradually gets damped from its value in the early stages and finally the dilaton ‘freezes out’ to a constant vacuum expectation value (vev) = \(\phi_0\) at an epoch \(t_f < t_0\). In such a scenario one has the limiting conditions:

(i) \(\phi = \phi_0\) for the entire period \(t_f \leq t \leq t_0\) i.e., for \(b_f \leq b \leq 1\) where \(b_f = b \mid t = t_f\);

(ii) \(f(b) = e^{2\phi_0}b^3/3\) and \(\dot{\phi} = 0\) for \(t_f < t \leq t_0\) \((b_f < b \leq 1\).

These imply \(f(1) = e^{2\phi_0}/3\) and \(\lambda_0 \equiv \dot{\phi}^2 \mid t = t_0 = 0\). The Hubble parameter and the deceleration parameter take the forms:
\[
\mathcal{H} = \left[ -\frac{\chi}{b^2} + \frac{\gamma}{3b^6} \left( 3f(b) - \sigma e^{-2\phi_0} \right) \right]^{1/2}; \quad \gamma \equiv 8\pi G \rho_0 = \frac{3(H_0^2 + \chi)}{e^{4\phi_0} - \sigma}; \quad (\sigma \neq e^{4\phi_0}) \quad (35)
\]
\[
q = 2 - \frac{3}{2} \left[ b^3 e^{2(\phi + \phi_0)}(H_0^2 + \chi) - \frac{4}{3} \chi b^4 (e^{4\phi_0} - \sigma) \right] \left( \frac{3f(b)e^{2\phi_0} - \sigma}{(H_0^2 + \chi)} - \chi b^4 (e^{4\phi_0} - \sigma) \right) \quad (36)
\]

In addition one obtains from Eq.(28) the expression for the dilatonic evolution rate \((d\phi/db)\):
\[
\left( \frac{d\phi}{db} \right)^2 = \frac{12H_0^2}{b^3} \left\{ 1 + \frac{3f(b) - b^3e^{2\phi_0}}{e^{2\phi_0} - \sigma e^{-2\phi_0}} \right\}; \quad f(b) = \int_{t_f}^{b} e^{2\phi(b)} b^2 db; \quad t \leq t_f \ (b \leq b_f)
\]
\[
\left( \frac{d\phi}{db} \right)^2 = 0; \quad f(b) = \frac{e^{2\phi_0}}{3} b^3; \quad t_f < t \leq t_0 \ (b_f < b \leq 1). \quad (37)
\]

As a simplification we consider only the spatially flat matter-dominated model setting the curvature \(\chi = k c^2/a_0^2\) equal to zero. This is fairly justified considering the fact that in the standard Friedman-Robertson-Walker (FRW) framework the matter-dominated Universe is known to have negligible curvature, one expects that the effect of curvature on our results for a spatially flat model involving the dilaton and the axion may not be of much significance.

**I. \(b > b_f\) era:**
The expressions for $\mathcal{H}$ and $q$ can be given for $\chi = 0$ and $b > b_f$ as

$$\mathcal{H} = \frac{\dot{b}}{b} = \frac{1}{b^3} \left[ \frac{\gamma}{3} (b^3 e^{2\phi_0} - \sigma e^{-2\phi_0}) \right]; \quad \gamma = 8\pi G \rho_0 = \frac{3\mathcal{H}_0^2 e^{2\phi_0}}{e^{4\phi_0} - \sigma} \quad (38)$$

$$q = 2 - \frac{3}{2} \frac{b^3 e^{4\phi_0}}{(b^3 e^{4\phi_0} - \sigma)} \quad (39)$$

For $\sigma > 0$, the constant $\gamma = 8\pi G \rho_0$ being positive definite, one obtains an upper bound on $\sigma$: $\sigma < e^{4\phi_0}$. This maximum limit on $\sigma$ is further reduced on taking into account the fact that $\mathcal{H}^2$ is positive always for all $b > b_f$, which implies $\sigma < b_f^3 e^{4\phi_0}$. Moreover, to have an accelerating Universe ($q < 0$) at any value $b_{accl}$ ($> b_f$) the value of $\sigma$ should be limited by $\sigma > (1/4)b_{accl}^3 e^{4\phi_0}$. Consistency of these two bounds on $\sigma$ then demands $b_{accl}^3 > 4b_f^3$. The ultimate bounds posed on $\sigma$ and $b_f$ in the context of present-day accelerating Universe ($q_0 < 0$) therefore turn out to be

$$\frac{1}{4} e^{4\phi_0} < \sigma < b_f^3 e^{4\phi_0} \quad; \quad b_f > 4^{-1/3}. \quad (40)$$

For $\sigma < 0$, $\gamma$ and $\mathcal{H}^2$ being positive by construction, no limit is imposed on $\sigma$. Moreover, from Eq.(39) $q$ is always positive and greater than 0.5 (i.e., it's value in spatially flat FRW model) which in turn signifies absence of any accelerating phase of the Universe all the way through the post-frozen dilaton era $b > b_f$.

Regardless of the sign of $\sigma$, the functional form of the scale factor $b(t)$ can be obtained straightaway on solving Eq.(38). Given the boundary conditions $b = b_f$, $\phi = \phi_0$ at $t = t_f$ and $b = 1$, $\phi = \phi_0$, $\mathcal{H} = \mathcal{H}_0$ at $t = t_0$ the solution is expressed as

$$b(t) = \left[ b_f^3 + 3\mathcal{H}_0 \left( \frac{b_f^3 e^{4\phi_0} - \sigma}{e^{4\phi_0} - \sigma} \right)^{1/2} (t - t_f) + \frac{9\mathcal{H}_0^2}{4} \left( e^{4\phi_0} - \sigma \right)^{1/3} (t - t_f) \right]^{1/3}. \quad (41)$$

In the limit $\sigma \to 0$, $b \to \{b_f^{3/2} + (3\mathcal{H}_0/2)(t - t_f)\}^{2/3}$ which shows a similar behaviour as the spatially flat matter-dominated FRW Universe ($b_{FRW} \sim t^{2/3}$) especially when the dilaton freezes at a very early epoch $t_f \ll t_0$ ($b_f \ll 1$).

The post-frozen dilaton epoch time lapse is given by

$$T_f \equiv (t_0 - t_f) = \frac{2}{3\mathcal{H}_0} (1 - \sigma e^{-4\phi_0}) \left\{ 1 - \left( \frac{b_f^3 e^{4\phi_0} - \sigma}{e^{4\phi_0} - \sigma} \right)^{1/2} \right\} \quad (42)$$

which has the limiting form $T_f = 2(1 - b_f^{3/2})/3\mathcal{H}_0$ as $\sigma \to 0$, and is almost the same as the age of the matter-dominated Universe in the FRW framework if the dilaton freezes in a very early epoch ($b_f \ll 1$).

The variations of the scale factor $b$ and the deceleration parameter $q$ with time in the post-frozen dilaton era has been depicted in Fig.1 for three characteristic values of $\sigma$ ($-0.35, 0, +0.35$). The time scale is as usual taken to be the inverse Hubble constant $\mathcal{H}_0$. Although the value of $b_f$ may depend on $\sigma$ as well the nature of the evolution of the Universe in the pre-frozen dilaton era, we have for simplicity chosen a typical parametric value ($b_f = 0.75$) for all the three values of $\sigma$. The nature of the curves are however not altered by this simplification.
FIG. 1. $b - t$ and $q - t$ plots for spatially flat matter-dominated Universe in the post-frozen dilaton era $b > b_f$ for three parametric values of $\sigma$, +0.35, 0, -0.35. The time-scale is, as usual, the inverse Hubble constant and $b_f$ is characteristically chosen to be equal to 0.75. The vertical grids denote the present time-slice $t_0 = \text{const.}$ for various $\sigma$ while the present scale factor is given by $b = 1$.

It should also be noted here that considering the late time accelerating expansion of the Universe as confirmed by the Supernova Ia data [7,8] and the recent WMAP results we focus primarily on a model with $\sigma > 0$ (i.e., negative effective energy density of the KR field) where one can find such acceleration as shown above.

II. $b \leq b_f$ era:

The expressions for $\mathcal{H}$ and $q$ for $\chi = 0$ and $b \leq b_f$ are given by

$$\mathcal{H} \equiv \frac{\dot{b}}{b} = \frac{1}{b^3} \sqrt{\frac{\gamma}{3} \left\{ 3f(b) - \sigma e^{-2\phi_0} \right\}}; \quad \gamma \equiv 8\pi G \rho_0 = \frac{3H_0^2 e^{2\phi_0}}{e^{4\phi_0} - \sigma} \quad (43)$$

$$q = 2 - \frac{3}{2} \left\{ \frac{b^2 e^{2(b+\phi_0)}}{3 e^{2\phi_0} f(b) - \sigma} \right\} \quad (44)$$

Since $\gamma > 0$ we again find that if $\sigma$ is positive then it should have a value less than $e^{4\phi_0}$. The fact that $\mathcal{H}^2$ is positive for all values of $b$ then puts a further stringent upper bound on $\sigma$: $\sigma < 3 e^{2\phi_0} f_{\text{min}}$, where $f_{\text{min}}$ is the minimum value of the integral $f(b) = \int e^{2b} b^2 db$, i.e., the value of $f$ when $b = b_m$ - the root of the equation $e^{2\phi(b_m)} b_m^2 = 0$.

The expression for evolution rate of the dilaton [Eq.(37)] in the present circumstances reduces to

$$\left( \frac{d\phi}{db} \right)^2 = \frac{12}{b^2} \left( 1 - \frac{b^2 e^{2\phi} - \sigma e^{-2\phi}}{3f(b) - \sigma e^{-2\phi}} \right). \quad (45)$$

Using this and the definition of $f(b)$ one finally obtains

$$\mathcal{H}^2 = \frac{\gamma e^{2\phi}}{3b^2} \left( 1 - \frac{\sigma e^{-4\phi}}{b^2} \right); \quad \gamma = \frac{3H_0^2 e^{2\phi_0}}{e^{4\phi_0} - \sigma} \quad (46)$$

$$q = 2 - \frac{3}{2} \left( \frac{1 - \frac{b^2 e^{2\phi}}{12}}{1 - \frac{b^2 e^{2\phi}}{b^2}} \right) \quad (47)$$

where $\phi' = d\phi/db$.

As for the reasons mentioned earlier in the context of an accelerating Universe, we now concentrate on the case where $\sigma > 0$ for which one can anticipate such acceleration. Moreover, some renewed study of the Supernova Ia data provides evidence of not only a late-time accelerating Universe but a decelerating Universe in the remote past as well [13]. This is particularly relevant in order to explain the structure formation in the early Universe.

To understand such a situation in the present model with $\sigma > 0$ we refer to the above expressions for the Hubble parameter and the deceleration parameter. The dilaton $\phi$ is supposed to fall with time (i.e., with $b$ since for an expanding Universe $b$ always increases with time), from a high value in the early Universe until it gets frozen to $\phi_0$ at $b = b_f$ while $\sigma$ is universally a constant. The $\phi'^2$ term in Eq.(47) shows that the dilaton only has a positive contribution to the deceleration parameter $q$ contrary to that of $\sigma$ which gives a negative contribution by reducing the denominator of the second term on the right of Eq.(47). However, in the early Universe, $\phi$ being larger the quantity involving $\sigma$ in Eq.(47) is more suppressed by the factor $e^{4\phi}$. Therefore, as time progresses the gradual reduction of the dilatonic effect enables the KR field axion to dominate and this may produce an overall negative contribution to $q$ at late stages. It should also be noted here that one can explicitly check that the dilatonic evolution equation (45) in the pre-freezeout era is satisfied identically for any functional form of $\phi$ only under proper consideration of the limiting constraints. However, to make at least a qualitative assessment of the whole scenario so that the functional form of $q$ for $b \leq b_f$ exactly matches with that given in Eq.(39) for $b > b_f$ at the freeze-out epoch $t = t_f$ ($b = b_f$), we consider a typical power law type dilatonic fall-off

$$\phi = \phi_0 + \left( \frac{b_f}{b} - 1 \right)^n \quad (48)$$

where $n$ is a positive index. The expressions for $\mathcal{H}$ and $q$ turn out to be
The nature of $H$ vs $b$ and $q$ vs $b$ curve is shown in Fig. 2 with parametrically chosen values of the constants: $b_f = 0.75$, $\phi_0 = 0.01$, $\sigma = 0.35$, $n = 1.25$.

IV. RADIATION-DOMINATED UNIVERSE

Extrapolation of the matter-dominated model to a very early epoch $t \ll t_0$ to study the effects of the dilaton and the axion may not be very reliable because of the radiation dominance at the early ages. In a radiation-dominated model we have $p = \frac{1}{3} \rho c^2$, i.e., $\omega = 1/3$, and Eq.(27) yields $\zeta = 1/b^4$. We obtain from the field equations (22) - (24) the expressions for the dilatonic evolution rate $\dot{\phi} \equiv d\phi/dt$ as well the expressions for the Hubble parameter $H$ and the deceleration parameter $q$:

$$\dot{\phi}^2 = \frac{\gamma(4\sigma e^{-2\phi} + \Lambda_r)}{b^6}$$  \hspace{1cm} (51)

$$H \equiv \frac{\dot{b}}{b} = \left[ -\frac{\chi}{b^2} + \frac{\gamma}{12b^6}(4b^2 + \Lambda_r) \right]^{1/2}$$  \hspace{1cm} (52)

$$q \equiv -\frac{1}{H^2} \frac{\ddot{b}}{b} = \frac{\gamma}{6b^6H^2} (2b^2 + \Lambda_r)$$  \hspace{1cm} (53)

where the parameter $\sigma$ is now the ratio of the KR field energy density and the present radiation density $\rho_0^{(r)}$ of the Universe and $\gamma = 8\pi G\rho_0^{(r)}$. However, since the Universe is presently not radiation-dominated one cannot determine the integration constant $\Lambda_r$ in terms of the present values of the physical parameters like the Hubble constant, etc. We, instead, take the initial values $b = b_i$, $\phi = \phi_i$, $\dot{\phi}^2 = \lambda_i$ at the origin of time $t = t_i$. In terms of these initial values $\Lambda_r$ is given by
\[ \Lambda_r = \frac{\lambda_i b_i^6}{\gamma} - 4\sigma e^{-2\phi_i} \]  

Once again we find from Eqs.(53) and (54) that an accelerated expansion of the Universe at any phase during its evolution is not possible when \( \sigma < 0 \), i.e., the KR field energy is positive — the case in usual string background geometries without torsion. However, our point of interest here is in a KR field induced torsioned background where a positive value of \( \sigma \) is possible. Only in such case one can have \( \Lambda_r \) negative whenever the condition \( \sigma > (4\gamma)^{-1}\lambda_i b_i^6 e^{2\phi_i} \) is satisfied and the Universe can have an accelerating phase that sets in at some value of \( t = t_{accl} \) whence, \( b = b_{accl} \) and \( b_{accl}^2 < 2|\Lambda_r| \), i.e.,

\[ \sigma > \frac{e^{2\phi_i}}{4} \left( \frac{\lambda_i b_i^6}{\gamma} + b_{accl}^2 \right). \]  

However, for an expanding Universe \( b \) being an increasing function of time, it is expected that the above inequality may break and such an accelerating phase in the early radiation-dominated era may not last long when \( b \) gets much larger than \( b_{accl} \).

For zero curvature \( \chi \) (spatially flat Universe) the field equation (52) can be solved exactly. We write down the solution taking \( \Lambda_r \) to be negative (which is only possible for \( \sigma > 0 \)) in a parametric form as

\[
  b(\eta) = \sqrt{\frac{\gamma}{3}} \left( \eta^2 + x \right)^{1/2}
\]

\[
  t(\eta) = \frac{\gamma}{12} \left\{ \eta \sqrt{\eta^2 + x} + \ln \left( \frac{\eta + \sqrt{\eta^2 + x}}{\sqrt{x}} \right) \right\}
\]

where \( x = -3(4\gamma)^{-1} \Lambda_r \). One can explicitly check that the above solution assumes the FRW form \( b \sim t^{1/2} \) if we ignore the effects of the axion and the dilaton, i.e., \( \sigma \rightarrow 0 \), \( \lambda_i = \phi_i^2 |_{\eta} \rightarrow 0 \) which implies that \( \Lambda_r \) vanishes identically. In fact, the FRW solution is always recovered in the limit \( \Lambda_r \rightarrow 0 \), which is possible even for non-vanishing contributions from the dilaton and the axion whenever \( \lambda_i b_i^6 e^{2\phi_i} \rightarrow 4\gamma \sigma \). For non-vanishing \( \Lambda_r \) (taken to be negative which implies \( \sigma > 0 \)), the above solution shows that \( b \) remains non-vanishing for all values of \( t \). On back-tracing this spatially-flat radiation-dominated model to the origin of time \( t = t_i \), \( \eta = \eta_i \), where both \( t_i \) and \( \eta_i \) can be set to be equal to zero in view of the limiting FRW case, one finds that the scale factor \( b \) has a non-zero value \( (1/2) \sqrt{|\Lambda_r|} \) and hence the energy density \( \zeta = 1/b^4 \) is finite at that epoch. This implies the removal of the so-called ’big bang singularity’ (that features in FRW cosmology) in a torsioned spacetime.

The deceleration parameter \( q \) have been calculated using the above solution and the nature of \( b - t \) and \( q - t \) plots are depicted in Fig.3 for characteristic values \( \sigma = 0.125 \), \( \phi_i = 1 \), \( \lambda_i = 1 \) of the physical parameters.

![FIG. 3. b – t and q – t plots for spatially-flat radiation-dominated Universe. The broken line is for FRW model (\( \sigma = 0 \)) while the solid line is for a model involving the axion and the dilaton (\( \sigma = 0.125 \), \( \phi_i = 1 \), \( \lambda_i = 1 \), parametrically chosen).](image)

For a spatially non-flat radiation-dominated Universe in presence of the dilaton and the KR field, parametric solutions of Eq.(52) can be given respectively for closed (\( \chi > 0 \)) and open (\( \chi < 0 \)) models as

\[
  b(\eta) = \frac{\gamma}{6\chi} \left[ 1 - \sqrt{1 - \frac{3\chi \Lambda_r e^{2\phi_i}}{2\gamma} \cos(2\eta \sqrt{\chi})} \right]^{1/2} \quad (\chi > 0)
\]
\[
b(\eta) = \frac{\gamma}{6|\chi|} \left[ \sqrt{1 + \frac{3|\chi| \Lambda e^{2\phi}}{2\gamma} \cosh(2\eta \sqrt{|\chi|})} - 1 \right]^{1/2} \quad (\chi < 0) \tag{59}
\]

where \( \eta \) is related to the cosmic time \( t \) as \( dt(\eta) = b(\eta)d\eta \).

We observe that there is, in fact, no value of \( \eta \) for which \( b \) vanishes. In other words, even when one finds a value of \( \eta \) for which \( t = 0 \), then for that value of \( \eta \) the scale factor \( b \) remains non-zero. Thus even at the epoch \( t = 0, b \neq 0 \) and \( \zeta \) finite — there is no big bang singularity.

V. CONCLUSION

In conclusion, we have found that in a spacetime with torsion the axion (dual to the KR field strength) provides a natural solution to the problem of big bang singularity present in the FRW model. Such an axion appears naturally in heterotic string theory. Different kinds of KR-torsion couplings have been considered and we have shown that for a certain kind of coupling the Universe passes through an accelerating phase which at least qualitatively explains the recent experimental findings through the Supernova Ia. The possibility of acceleration at different epochs as well as the size of the Universe at the initial time in such scenario has been estimated. We have further shown that the other scalar massless mode of string theory, namely the dilaton, also plays a crucial role to control the expansion rate at various epochs. Our work differ from the previous works \cite{10} significantly. Here we explicitly solve for the axion by using its equation of motion in different sectors of it’s coupling with torsion. Although both the phenomena of removal of big bang singularity and late-time accelerating Universe are found to depend on the dilaton and the KR field strength, in this work we have focused primarily to extract the effect of the antisymmetric KR field on these two phenomena. This work thus proposes a possible resolution of two of the most important problems prevailing in the present cosmological scenario.

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