ON UPPER BOUNDS OF MANIN TYPE

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Abstract. We introduce a certain birational invariant of a polarized algebraic variety and use that to obtain upper bounds for the counting functions of rational points on algebraic varieties. Using our theorem, we obtain new upper bounds of Manin type for 28 deformation types of smooth Fano 3-folds of Picard rank \( \geq 2 \) following Mori-Mukai’s classification. We also find new upper bounds for polarized K3 surfaces \( S \) of Picard rank 1 using Bayer-Macrì’s result on the nef cone of the Hilbert scheme of two points on \( S \).

1. Introduction

A driving question in diophantine geometry is to prove asymptotic formulae for the counting function of rational points on a projective variety. Manin’s conjecture, originally formulated in [BM90], predicts a precise asymptotic formula when the underlying variety is smooth Fano, or more generally smooth and rationally connected. This asymptotic formula has a description in terms of the geometric invariants of the underlying variety. A refinement has been formulated by Peyre [Pey95] and Batyrev-Tschinkel [BT98], and the most recent formulation predicting the exceptional set has been proposed by the author with Lehmann and Sengupta in [LST18].

In this paper we discuss the following weaker version of the conjecture which is called weak Manin’s conjecture: let \( X \) be a geometrically uniruled smooth projective variety defined over a number field \( k \) and let \( L \) be a big and nef divisor on \( X \). One can associate a height function

\[ H_L : X(k) \to \mathbb{R}_{>0}, \]

to \( (X, L) \), and we consider the following counting function:

\[ N(U, L, T) = \# \{ P \in U(k) | H_L(P) \leq T \} \]

for an appropriate Zariski open subset \( U \subset X \). Weak Manin’s conjecture predicts that this function is governed by the following geometric invariant of \( (X, L) \):

\[ a(X, L) = \inf \{ t \in \mathbb{R} | K_X + tL \in \mathbb{N}^1(X) \}, \]

where \( \mathbb{N}^1(X) \) is the cone of pseudo-effective divisors on \( X \). Here is the statement of weak Manin’s conjecture:

**Conjecture 1.1** (Weak Manin’s conjecture/Linear growth conjecture). Let \( X \) be a geometrically uniruled smooth projective variety defined over a number field \( k \) and let \( L \) be a big and nef divisor on \( X \). Then there exists a non-empty Zariski open subset \( U \subset X \) such that for any \( \epsilon > 0 \)

\[ N(U, L, T) = O_{\epsilon}(T^{a(X,L)+\epsilon}). \]

The starting point of the current research is [McK11]. In this paper, McKinnon shows that Vojta’s conjecture implies weak Manin’s conjecture for K3 surfaces and more generally
varieties with Kodaira dimension 0 assuming the Non-Vanishing conjecture in the minimal model program. (For such varieties, the $a$-invariant is 0.) While McKinnon’s result is conditional on Vojta’s conjecture, our results are unconditional: they do not rely on Vojta’s conjecture. In our approach, instead of appealing to Vojta’s conjecture, we use the positivity of divisors by introducing the following invariant measuring the local positivity of big divisors:

**Definition 1.2.** Let $X$ be a normal projective variety defined over an algebraically closed field of characteristic 0 and $H$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We consider $W = X \times X$ and denote each projection by $\pi_i : W \to X_i$. Let $\alpha : W' \to W$ be the blow up of the diagonal and denote its exceptional divisor by $E$, i.e., the pullback of the diagonal. For any $\mathbb{Q}$-Cartier divisor $L$ on $X$ we denote $\alpha^* \pi_1^* L + \alpha^* \pi_2^* L$ by $L[2]$. We define the following invariant

$$
\delta(X, H) = \inf \left\{ s \in \mathbb{R} \left| \begin{array}{c}
\text{for any component } V \subset \text{SB}(sH[2] - E) \text{ not contained in } E, \\
\text{one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i
\end{array} \right. \right\},
$$

where $\text{SB}(sH[2] - E)$ is the stable base locus of a $\mathbb{R}$-divisor $sH[2] - E$. We call this invariant the $\delta$-invariant.

McKinnon proves a Repulsion principle for rational points assuming Vojta’s conjecture. We instead prove a different Repulsion principle using the $\delta$-invariant. Here is our theorem:

**Theorem 1.3** (Repulsion principle). Let $X$ be a normal projective variety defined over a number field $k$. We fix a place $v$ of $k$. Let $A$ be a big $\mathbb{Q}$-Cartier divisor on $X$. Then for any $\epsilon > 0$ there exists a constant $C = C_\epsilon > 0$ and a non-empty Zariski open subset $U = U(\epsilon) \subset X$ such that we have

$$
\text{dist}_v(P, Q) > C(H_A(P)H_A(Q))^{-(\delta(X, A) + \epsilon)},
$$

for any $P, Q \in U(k)$ with $P \neq Q$, where $\text{dist}_v(P, Q)$ is the $v$-adic distant function on $X$.

Inspired by [McK11], we obtain the following general result on the counting functions of rational points on algebraic varieties:

**Theorem 1.4.** Let $X$ be a normal projective variety of dimension $n$ defined over a number field $k$ and $L$ be a big $\mathbb{Q}$-Cartier divisor on $X$. Then for any $\epsilon > 0$ there exists a non-empty Zariski open subset $U = U(\epsilon) \subset X$ such that we have

$$
N(U, L, T) = O_\epsilon(T^{2n\delta(X, L) + \epsilon}).
$$

Previous results applying to general projective varieties are the results related to dimension growth conjecture obtained by Browning, Heath-Brown, and Salberger ([BHBS06], and Salberger ([Sal07]). For many examples of Fano varieties $X$ we have $\delta(X, -K_X) = \frac{1}{2}$, thus this theorem recovers weaker statements of [BHBS06] and [Sal07]. However [BHBS06] and [Sal07] are better in the sense that they obtain a bound for $N(X, L, T)$ and their constants only depend on the dimension of $X$, $\epsilon$, and the dimension of the ambient projective space where $X$ is embedded into. On the other hand, our method also has the advantage in the sense that our theorem applies to arbitrary big divisor and in the case that $\delta(X, L)$ is the minimum, then one does not need to introduce $\epsilon > 0$ in the above theorem. For Fano conic bundles, we can obtain better bounds using conic bundle structures.

**Theorem 1.5.** Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. We assume that $X$ and $S$ are smooth Fano. Let $L = -K_X$. Let $W = X \times X$ and $W'$ be the blow up of $W$ along the diagonal with the exceptional divisor $E$. We denote each
projection $W' \to X_i$ by $\pi_i$. Let $\alpha, \beta$ be positive real numbers such that $2\alpha - 2\beta = 1$. We further make the following assumptions:

1. Weak Manin’s conjecture for $(S, -K_S)$ holds,
2. for any component $V$ of the stable locus of the following divisor
   $$-\alpha K_{X/S}[2] - \beta f^* K_S[2] - E,$$
   such that $V$ is not contained in $E$, one of projections $\pi_i|_V$ is not dominant.

Then there exists a non-empty Zariski open subset $U \subset X$ such that for any $\epsilon > 0$, there exists $C = C_\epsilon > 0$ such that
$$N(U, L, T) < CT^{2\alpha + \epsilon}.$$  

Using this theorem, new upper bounds for 28 deformation types of Fano 3-folds are obtained. Here are some examples of Fano 3-folds which our theorem applies to:

**Example 1.6.** Let $X$ be the blow-up of a quadric threefold $Q$ defined over a number field $k$ with center a line defined over the same ground field. Let $H$ be the pullback of hyperplane class from $Q$ and we denote the exceptional divisor by $D$. Then the linear system $|H - D|$ defines a $\mathbb{P}^1$-fibration over $\mathbb{P}^2$. One can prove that $\alpha = 5/6$ satisfies the assumptions of Theorem 1.5, thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have
$$N(U, -K_X, T) = O_\epsilon(T^{5/3+\epsilon}).$$

**Example 1.7.** Let $V_7$ be the blow-up of $\mathbb{P}^3$ at a point $P$. This is isomorphic to $\mathbb{P}(O \oplus O(1))$ over $\mathbb{P}^2$. Let $X$ be the blow-up of $V_7$ with center the strict transform of a conic passing through $P$. Then $X$ is a Fano conic bundle with singular fibers. One can prove that $\alpha = 5/6$ satisfies the assumptions of Theorem 1.5, thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have
$$N(U, -K_X, T) = O_\epsilon(T^{5/3+\epsilon}).$$

**Example 1.8.** Let $X$ be the blow-up of $\mathbb{P}^3$ with center a disjoint union of a line and a twisted cubic. Then $X$ is a Fano conic bundle with singular fibers. One can prove that $\alpha = 4/3$ satisfies the assumptions of Theorem 1.5, thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have
$$N(U, -K_X, T) = O_\epsilon(T^{8/3+\epsilon}).$$

**Example 1.9.** Let $X$ be the blow-up of $\mathbb{P}^3$ with center a disjoint union of three lines. Then $X$ is a Fano conic bundle with singular fibers. One can prove that $\alpha = 1$ satisfies the assumptions of Theorem 1.5, thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have
$$N(U, -K_X, T) = O_\epsilon(T^{2+\epsilon}).$$

Note that for del Pezzo surfaces, there are many better results on bounds of the counting functions, see, e.g., [HB97], [Bro01], [BSJ14], [FLS18], and [BS18]. Next we establish weak Manin’s conjecture for non-anticanonical height functions in some cases:

**Theorem 1.10.** Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. We assume that $X$ and $S$ are Fano. Let $L = -K_X - tf^* K_S$. We make the following assumptions:
(1) Weak Manin’s conjecture for \((S, -K_S)\) holds,
(2) \( t \geq \frac{1}{2\delta(X, -K_X)} \).

Then for any \( \epsilon > 0 \) there exists a non-empty Zariski open subset \( U = U(\epsilon) \subset X \) and \( C = C_\epsilon > 0 \) such that

\[ N(U, L, T) < CT^{2\delta(X, -K_X)+\epsilon}. \]

In particular when \( \delta(X, -K_X) = 1/2 \), Conjecture \( \text{I.1} \) holds for \((X, L)\) except independence of \( U \) on \( \epsilon \).

For such a height function, [FL17] establishes Manin’s conjecture when the base is the projective space using conic bundle structures. Finally we apply our results to K3 surfaces. Here is our theorem:

**Theorem 1.11.** Let \( S \) be a K3 surface defined over a number field \( k \) with a polarization \( H \) of degree \( 2d \) such that \( \text{Pic}(S) = \mathbb{Z}H \). Then for any \( \epsilon > 0 \), we have

\[ N(S, H, T) = O(\epsilon(T^4(\sqrt{\frac{4}{d^2}} + \frac{5d}{4} + \epsilon)). \]

Our proof is relied on the work of [BM14] on the nef cone of the Hilbert scheme of 2 points \( \text{Hilb}^2(S) \). Indeed, \( \delta(S, H) \) is bounded by the \( s \)-invariant of \( H \), and the computation of the \( s \)-invariant can be done using the description of the nef cone of \( \text{Hilb}^2(S) \). Similar bounds are obtained for hypersurfaces in \( \mathbb{P}^n \) by Heath-Brown [H1302].

Here is the road map of this paper: in Section 2 we recall basic properties and results of the \( a \)-invariants. In Section 3 we discuss some basic properties of the \( \delta \)-invariants and compute them for some examples, e.g., del Pezzo surfaces. In Section 4 we prove the Repulsion principle for projective varieties (Theorem 1.3). In Section 5 we establish Theorem 1.4. In Section 6 we prove Theorem 1.5 and Theorem 1.10. In Section 7 we study 3-dimensional Fano conic bundles using Theorem 1.5. In Section 8 we study K3 surfaces and prove Theorem 1.11.

**Acknowledgement** The author would like to thank Brian Lehmann for helpful conversations, and careful reading and comments on an early draft of this paper. In particular the author thanks Brian for his suggestions regarding a relation of the \( \delta \)-invariant to the Seshadri constant and the \( s \)-invariant, and applications of his works to K3 surfaces. The author also would like to thank Takeshi Abe for answering his question regarding the second chern form. The author would like to thank Shigefumi Mori and Shigeru Mukai for teaching him about the work of Matsuki on the cone of curves on Fano 3-folds. ([Mat95]) The author thanks Tim Browning and Yuri Tschinkel for comments on this paper. Sho Tanimoto is partially supported by MEXT Japan, Leading Initiative for Excellent Young Researchers (LEADER)

2. **The Fujita invariant in Manin’s conjecture**

Here we assume that our ground field \( k \) is a field of characteristic zero, but not necessarily algebraically closed. In this paper, a variety defined over \( k \) means a geometrically integral separated scheme of finite type over \( k \). Recently the geometric study of Fujita invariants has been conducted in a series of papers [HTT15], [LTT18], [HJ17], [LT17b], [LT17a], [Sen17], [LST18], [LT18a], [LT18b]. We recall its definition here.
Definition 2.1. Let $X$ be a smooth projective variety defined over $k$. Let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. We define the Fujita invariant (or $a$-invariant) by

$$a(X, L) = \inf\{ t \in \mathbb{R} \mid K_x + tL \in \Eff(X) \},$$

where $\Eff(X)$ is the cone of pseudo-effective divisors on $X$. By $\text{BDPP13}$, $a(X, L) > 0$ if and only if $X$ is geometrically uniruled. When $L$ is not big, we simply set $a(X, L) = +\infty$. When $X$ is singular, we take a resolution $\beta : X' \to X$ and we define the Fujita invariant by

$$a(X, L) := a(X', \beta^*L).$$

This is well-defined because the Fujita invariant is a birational invariant ($\text{HTT15}$ Proposition 2.7)).

This invariant plays a central role in Manin’s conjecture. For example, one can predict the exceptional set of Manin’s conjecture by studying this invariant and the following result is a consequence of Birkar’s celebrated papers $\text{Bir16a}$ and $\text{Bir16b}$:

Theorem 2.2 ($\text{LT18}$, $\text{HTT15}$, $\text{HJ17}$, $\text{LT17a}$). Assume that our ground field is algebraically closed. Let $X$ be a smooth projective uniruled variety and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Let $V$ be the union of subvarieties $Y$ with $a(Y, L) > a(X, L)$. Then $V$ is a proper closed subset of $X$.

For computations of this exceptional set $V$ for some examples, see $\text{LT18}$ and $\text{LTT18}$.

3. The invariant $\delta(X, H)$

Here we assume that our ground field $k$ is an algebraically closed field of characteristic 0. Let $X$ be a normal projective variety and $H$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We consider $W = X \times X$ and denote each projection by $\pi_i : W \to X_i$. Let $\alpha : W' \to W$ be the blow up of the diagonal and we denote its exceptional divisor by $E$. For any $\mathbb{Q}$-Cartier divisor $L$ on $X$ we denote $\alpha^*\pi_1^*L + \alpha^*\pi_2^*L$ by $L[2]$.

Definition 3.1. Let $X, W', E$ as above. We define the following invariant

$$\delta(X, H) = \inf \left\{ s \in \mathbb{R} \mid \begin{array}{l}
\text{for any component } V \subset \text{SB}(sH[2] - E) \text{ not contained in } E, \\
\text{one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i.
\end{array} \right\},$$

where $\text{SB}(sH[2] - E)$ is the stable base locus of a $\mathbb{R}$-divisor $sH[2] - E$.

It is clear from the definition that when the rational map associated to $|H|$ is birational, we have $\delta(X, H) \leq 1$. Also it follows from the definition that $\delta(X, H)H[2] - E$ is pseudoeffective.

Example 3.2. Let $X = \mathbb{P}^n$ and $H$ be the hyperplane class. Then $\delta(X, H) = 1$. Indeed, it is clear that $\delta(X, H) \leq 1$. On the other hand, let $F_i$ be a general fiber of the projection $\pi_i \circ \alpha : W' \to X_i$ at $x \in X_i$ and $\ell$ be the strict transform of a line passing through $x$. Then we have $(H[2] - E)\cdot \ell = 0$. Since such $\ell$ deforms to cover $W'$, this means that $H[2] - E$ is not big. Thus our assertion follows.

Example 3.3. Let $X \subset \mathbb{P}^n$ be a normal projective variety and $H$ be the hyperplane class. Suppose that $X$ is covered by lines. Then the same proof of the above example shows that $\delta(X, H) = 1$. 

5
Next we show that the invariant $\delta(X, H)$ is a birational invariant.

**Lemma 3.4.** Let $X$ be a normal projective variety and $H$ be a big $\mathbb{Q}$-Cartier divisor on $X$. Let $\beta : X' \to X$ be a birational morphism between normal projective varieties. Then we have $\delta(X', \beta^* H) = \delta(X, H)$.

**Proof.** Let $W_X$ be the blow-up of $X \times X$ along the diagonal and $W_{X'}$ be the blow-up of $X' \times X'$ along the diagonal. We denote their exceptional divisors by $E_X$ and $E_{X'}$ respectively. Then we have a birational map

$$\phi : W_{X'} \to W_X$$

which is a birational contraction and the indeterminacy of this map is not dominant to both $X'$. Also for a component $V$ of the non-isomorphic loci of this map such that $V$ is not contained in $E_{X'}$, one of projections is not dominant.

Fix $\epsilon > 0$. Suppose that the stable locus of $(\delta(X, H) + \epsilon)\beta^* H[2] - E_{X'}$ contains a subvariety $Y \subset W_{X'}$ such that $Y \not\subset E_{X'}$ and $Y$ maps dominantly to both $X'$. By the definition, $(\delta(X, H) + \epsilon)H[2] - E_X$ does not contain $\phi(Y)$ in the stable locus so that there exists $0 \leq D \sim_{\mathbb{R}} (\delta(X, H) + \epsilon)H[2] - E_X$ such that $\phi(Y) \not\subset \text{Supp}(D)$. Then we have $\phi_* D \sim_{\mathbb{R}} (\delta(X, H) + \epsilon)\beta^* H[2] - E_{X'}$ because $\phi_* E_X = E_{X'}$. Furthermore we have $Y \not\subset \text{Supp}(\phi_* D)$. This contradicts with our assumption. Thus we conclude

$$\delta(X', \beta^* H) \leq \delta(X, H).$$

Suppose that the stable locus of $(\delta(X', H) + \epsilon)H[2] - E_X$ contains a subvariety $Y \subset W_X$ such that $Y \not\subset E_X$ and $Y$ maps dominantly to both $X'$. We take the strict transform $Y' \subset W_{X'}$ of $Y$. By the definition, $(\delta(X', H) + \epsilon)\beta^* H[2] - E_{X'}$ does not contain $Y'$ in the stable locus so that there exists $0 \leq D \sim_{\mathbb{R}} (\delta(X', H) + \epsilon)\beta^* H[2] - E_{X'}$ such that $Y' \not\subset \text{Supp}(D)$. Then we have $\phi_* D \sim_{\mathbb{R}} (\delta(X', H) + \epsilon)\beta^* H[2] - E_X$. Furthermore we have $Y \not\subset \text{Supp}(\phi_* D)$. This contradicts with our assumption. Thus we conclude

$$\delta(X', \beta^* H) \geq \delta(X, H).$$

Thus our assertion follows. \qed

Here is a relation between $\delta(X, H)$ and $a(X, H)$.

**Proposition 3.5.** Let $X$ be a smooth weak Fano variety and $H$ be a big and nef divisor on $X$. Then we have

$$\delta(X, H) \leq a(X, H)\delta(X, -K_X).$$

**Proof.** We write $a(X, H)H + K_X \sim_{\mathbb{Q}} D \geq 0$. Fix $\epsilon > 0$. Then we have

$$a(X, H)(\delta(X, -K_X) + \epsilon)H[2] - E \sim_{\mathbb{Q}} -(\delta(X, -K_X) + \epsilon)K_X[2] + (\delta(X, -K_X) + \epsilon)D[2] - E.$$

Thus we see that the stable locus of $|a(X, H)(\delta(X, -K_X) + \epsilon)H[2] - E|$ does not contain any dominant component possibly other than subvarieties in $E$. Thus our assertion follows. \qed

Next we consider the $s$-invariants and its relation to the $\delta$-invariants:

**Definition 3.6.** Let $X$ be a smooth projective variety and $H$ be an ample divisor on $X$. Let $W$ be the blow-up of $X \times X$ along the diagonal and we denote its exceptional divisor by $E$. The $s$-invariant of $H$ is defined by

$$s(X, H) = \inf\{s \in \mathbb{R} | sH[2] - E \text{ is nef}\}.$$
This is a positive real number in general. See \cite{Laz04} Section 5.4] for many properties of this invariant.

**Proposition 3.7.** Let $X$ be a smooth projective variety and $H$ be an ample divisor on $X$. Then we have

$$\delta(X, H) \leq s(X, H).$$

**Proof.** For sufficiently small $\epsilon > 0$, $(s(X, H) + \epsilon)H[2] - E$ is ample so its stable base locus is empty. Thus our assertion follows. \hfill \Box

### 3.1. Del Pezzo surfaces.

Next we discuss del Pezzo surfaces. Let $S$ be a smooth del Pezzo surface. We consider $W = S \times S$ and we denote each projection by $\pi_i : W \to S_i$. Let $\alpha : W' \to W$ be the blow up of the diagonal and we denote its exceptional divisor by $E$.

First we record a lower bound for the $\delta$-invariant:

**Lemma 3.8.** Let $S$ be a smooth del Pezzo surface. Then we have

$$\delta(S, -K_S) \geq \frac{1}{\epsilon(-K_S, x)}$$

for any general point $x \in X$ where $\epsilon(-K_S, x)$ is the Seshadri constant of $-K_S$ at $x$.

**Proof.** The Seshadri constant for the anticanonical divisor on a smooth del Pezzo surface is computed in \cite{Bro06}. According to this paper, $\epsilon(-K_S, x)$ is constant for a general point $x \in S$ and for such a $x$ we have

$$\epsilon(-K_S, x) = \min_{x \in CCS} \frac{-K_S.C}{\text{mult}_x(C)}$$

Moreover, curves achieving the minimum are completely described and they are members of one family from the Hilbert scheme.

Let $x \in S_1$ be a general point and let $C_x$ be the strict transform of a curve in $\{x\} \times S_2$ achieving the minimum $\epsilon(-K_S, x)$. Then we have

$$(-K_S[2] - \epsilon(-K_S, x)E).C_x = -K_X.C_x - \epsilon(-K_S, x)\text{mult}_x(C_x) = 0.$$ 

Thus $C_x$ is contained in the stable locus of $-K_S[2] - (\epsilon(-K_S, x) + \eta)E$ for any $\eta > 0$. As $x$ varies, $C_x$ deforms to cover both $S_i$, proving the claim. \hfill \Box

Now we compute $\delta(S, -K_S)$ for a del Pezzo surface $S$:

**Proposition 3.9.** Let $S$ be a del Pezzo surface of degree $d$ where $4 \leq d \leq 8$. Then we have

$$\delta(S, -K_S) = \frac{1}{2}.$$ 

**Proof.** We only discuss the cases of degree 4 del Pezzo surfaces. Other cases are easier.

Suppose that $S$ is a del Pezzo surface of degree 4. Let $F_1$ be a conic on $S$ and $F_2$ be another conic on $S$ such that $-K_S \sim F_1 + F_2$. Then $F_i[2] - E$ is linearly equivalent to a unique effective divisor. We denote it by $\Delta_{F_i}$. Let $D_1$ be a third conic such that $D_1.F_1 = D_1.F_2 = 1$. Let $D_2$ be the class of conics such that $-K_S \sim D_1 + D_2$. Then we have $D_2.F_1 = D_2.F_2 = 1$. We also consider $\Delta_{D_i}$. Then obviously we have

$$\text{SB}(-K_X[2] - 2E) \subset (\Delta_{F_1} \cup \Delta_{F_2}) \cap (\Delta_{D_1} \cup \Delta_{D_2})$$

but the only possible dominant component of $\Delta_{F_i} \cap \Delta_{D_i}$ is contained in $E$. Thus we conclude that $\delta(X, H) \leq 1/2$. The opposite inequality follows from Lemma 3.8 and \cite{Bro06}. \hfill \Box


Remark 3.10. The above proof actually shows that for any component $V \subset SB(-K_X[2] - 2E)$ not contained in $E$, the projection $\pi_i \circ \alpha|_V$ is not dominant. In particular we have
\[
\delta(S, -K_X) = \inf \left\{ s \in \mathbb{R} \mid \text{for any component } V \subset SB(-sK_X[2] - E) \text{ not contained in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } S_i. \right\}
\]
is the minimum.

Proposition 3.11. Let $S$ be a del Pezzo surface of degree 3. Then we have $\delta(S, -K_S) = \frac{2}{3}$.

Proof. Let $F_1$ be a conic. Then one can write $-K_S - \frac{1}{2}F_1 \sim \frac{1}{2}D$ where $D$ is the pullback of the anticanonical class from a degree 4 del Pezzo surface. Now we have
\[
-2K_S[2] - 3E = D[2] - 2E + F_1[2] - E.
\]
Then it follows from the proof of Proposition 3.9 that the stable locus of $-2K_S[2] - 3E$ minus $E$ is not dominant. Thus the stable locus of $-2K_S[2] - 3E$ is contained in $\Delta_{F_1}$. By considering another conic as before, we conclude that $\delta(S, -K_S) \leq 2/3$. The opposite inequality follows from Lemma 3.8 and [Bro06].

Remark 3.12. We again show that $\delta(S, -K_S)$ is not just the infimum, but actually the minimum.

Proposition 3.13. Let $S$ be a del Pezzo surface of degree 2. Then we have $\delta(S, -K_S) = 1$.

Proof. Note that $-3K_S$ can be expressed as
\[
-3K_S \sim -f_1^*K_{S_1} - f_2^*K_{S_2},
\]
where $f_i : S \to S_i$ is the blow down to a cubic surface. Thus arguing as Proposition 3.11, we prove that the stable locus of $-K_S[2] - E$ does not contain any dominant component other than $E$. This shows that $\delta(S, -K_S) \leq 1$. On the other hand, let $\phi : S \to \mathbb{P}^2$ be the anticanonical double cover. We denote the involution associated to $\phi$ by $\iota$ and we consider the image $S^\iota$ of the following map
\[
S \to S \times S, P \mapsto (P, \iota(P)).
\]
Then one can show that for any curve $C$ in $S^\iota$ and any $\epsilon > 0$ we have
\[
(-K_S[2] - (1 + \epsilon)E).C < 0
\]
Thus $C$ is contained in the stable locus of $-K_S[2] - (1 + \epsilon)E$, proving the claim.

Remark 3.14. We again show that $\delta(S, -K_S)$ is not just the infimum, but actually the minimum.

Proposition 3.15. Let $S$ be a del Pezzo surface of degree 1. Then we have $\delta(S, -K_S) = 2$.

Proof. First of all, note that $-4K_S$ can be expressed as
\[
-4K_S \sim -f_1^*K_{S_1} - f_2^*K_{S_2},
\]
where $f_i : S \to S_i$ is the blow down to a degree 2 del Pezzo surface. Thus by Proposition 3.13, one may conclude that $\delta(S, -K_S) \leq 2$. The opposite inequality follows from Lemma 3.8 and [Bro06].
Remark 3.16. We again show that $\delta(S, -K_S)$ is not just the infimum, but actually the minimum.

4. Repulsion principle for projective varieties

Assuming Vojta’s conjecture and the Non-Vanishing conjecture in the minimal model program, McKinnon shows a Repulsion principle for varieties of non-negative Kodaira dimension in [McK11]. In this paper we develop a weaker Repulsion principle for projective varieties in general. We introduce some notations. We refer readers to [Sil87] for the definitions and their basic properties.

Let $k$ be a number field and $M_k$ denote the set of places of $k$. For each place $v \in M_k$, $k_v$ denotes its completion with respect to $v$. Suppose that we have a projective variety $X$ defined over $k$ and a big $\mathbb{Q}$-divisor $L$ on $X$. Let $D$ be a closed subscheme on $X$. We denote the logarithmic height functions for $L$ by $h_L$ and their multiplicative height functions by $H_L$.

Let $h_{D,v}$ be a local height function for $D$ with respect to $v$. Note that in this paper, we use unnormalized heights, i.e., we do not normalize heights by the degree of $k$.

Let $\Delta$ be the diagonal of $X \times X$. We define the $v$-adic distant function by $h_{\Delta,v}(P, Q) = -\log \text{dist}_v(P, Q)$. See [Sil87] for basic properties of this function.

Let $X$ be a normal projective variety defined over a number field $k$ and $L$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We set $\delta(X, L) = \delta(X, L)$ where $X$ and $L$ are the base change of $X$, $L$ to an algebraic closure. Here is our main theorem:

**Theorem 4.1** (the Repulsion principle). Let $X$ be a normal projective variety defined over a number field $k$. Let $v$ be a place of $k$. Let $A$ be a big Cartier divisor on $X$. We let $W = X \times X$ with projections $\pi_i : W \to X_i$ and we let $L = \pi_1^* A + \pi_2^* A$. We denote the blow up of the diagonal by $\alpha : W' \to W$ and its exceptional divisor by $E$.

Then for any $\epsilon > 0$ there exists a constant $C = C_\epsilon > 0$ and a non-empty Zariski open subset $U = U(\epsilon) \subset X$ such that we have

$$\text{dist}_v(P, Q) > C(H_A(P) H_A(Q))^{-(\delta(X, A) + \epsilon)},$$

for any $P, Q \in U(k)$ with $P \neq Q$.

**Proof.** Fix $\epsilon > 0$. Let $B \subset W$ be the stable locus $\text{SB}((\delta(X, A) + \epsilon)\alpha^* L - E)$ where $W'$ is the base change to an algebraic closure. One can express $B$ as the intersection of supports of finitely many effective $\mathbb{R}$-divisors which are $\mathbb{R}$-linearly equivalent to $(\delta(X, A) + \epsilon)\alpha^* L - E$. After taking some finite extension $k'$ of $k$, we may assume that these divisors are defined over $k'$ so does $B$. We denote the union of the Galois orbits of $B$ by $B'$. Then it is a property of height functions that for any $(P, Q) \in W'(k) \setminus B'(k)$, we have

$$0 \leq h_{((\delta(X, A) + \epsilon)\alpha^* L - E)}(P, Q) + O(1).$$

From this, we may conclude that

$$h_{E,v}(P, Q) \leq h_E(P, Q) + O(1) \leq h_{((\delta(X, A) + \epsilon)\alpha^* L)}(P, Q) + O(1).$$
Let $V \subset B'$ be a component not contained in $E$. Then one of projections $\pi_i \circ \alpha|_V$ is not dominant, and we denote its image by $F_V$. Now we define $U$ by $X \setminus \cup_V F_V$. Our assertion follows for this $U$. \hfill \Box

**Remark 4.2.** Note that $\delta(X, L)$ is defined as

$$
\delta(X, L) = \inf \left\{ s \in \mathbb{R} \mid \text{for any component } V \subset \text{SB}(sa^*L - E) \text{ not contained in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i. \right\}
$$

If this is the minimum, then in the above proof, one does not need to introduce $\epsilon > 0$.

**Remark 4.3.** When $L$ is ample, we may replace $\delta(X, L)$ by $s(X, L)$ in the above theorem. In this situation, one can take our exceptional set to be empty.

### 5. Counting problems: general cases

In this section, we discuss some applications of Theorem 4.1 to the counting problems of rational points on algebraic varieties.

#### 5.1. Local Tamagawa measures.

Here we record some auxiliary results for local Tamagawa measures. Let $X$ be a smooth projective variety defined over a number field $k$. Let $v$ be a place of $k$. We fix a $v$-adic metrization on $\mathcal{O}(K_X)$ and it induces the Tamagawa measure $\tau_{X, v}$ on $X(k_v)$. We refer readers to [CLT10] for its definition.

**Lemma 5.1.** Let $n = \dim X$. There exists $C > 0$ such that for sufficiently small $T$ and $P \in X(k_v)$, we have

$$
CT^n < \tau_{X, v}(\{Q \in X(k_v) \mid \text{dist}_v(P, Q) < T\}).
$$

**Proof.** Let $Y = X \times X$. We take a finite open cover $\{U_i\}$ of $Y$ such that on $U_i$, $\Delta$ is the scheme-theoretic intersection of $D_{i, 1}, \ldots, D_{i, n}$ where $D_i = \sum D_{i, j}$ is a strict normal crossings divisor on $U_i$. On $U_i(k_v)$, there exists $C > 0$ such that

$$
\text{dist}_v(P, Q) < C \max_j \{H_{D_{i, j}}^{-1}(P, Q)\}
$$

for all $(P, Q) \in U_i(k_v)$. For each $P \in X_i(k_v)$, there exists a $v$-adic open neighborhood $V_P \subset X(k_v)$ such that $\overline{V}_P \times \overline{V}_P \subset U_i(k_v)$ for some $i$ and $D_{i, j}$ induces local coordinates $x_{i, j}$ on $V_P$. Since $X(k_v)$ is compact, finitely many $V_P$ covers $X(k_v)$. We denote them by $V_i$. For each $l$ let

$$
\omega_l = dx_{1, l} \wedge \cdots \wedge dx_{l, l}.
$$

Then on $V_i$ we have a uniform upper bound $C' > 0$ such that

$$
\|\omega_l\|_v < C'.
$$

Also let $d_i(P) = \min\{\text{dist}_v(P, Q) \mid Q \in V_i^0\}$ and we define $d(P) = \max_l \{d_i(P)\}$. Then $d(P) > 0$ for any $P \in X(k_v)$ so there is the minimum $d_m = \min\{d(P)\} > 0$. Now by the definition of the Tamagawa measure, for $0 < T < d_m$, we have

$$
\tau_{X, v}(\{Q \in X(k_v) \mid \text{dist}_v(P, Q) < T\}) > \int_{\{\max_i \{\text{dist}_i(P, Q)\} < C^{-1}T\}} \|\omega_l\|_v^{-1} |\omega_l|
$$

Thus our assertion follows. \hfill \Box
Let $X$ be a smooth variety defined over $k$. We fix a $v$-adic metrization on $\mathcal{O}(K_X)$. We can define the Tamagawa measure $\tau_{X,v}$. Then the local Tamagawa number is defined by

$$\tau_v(X) = \tau_v(X(k_v)).$$

See [CLT10] for more details.

### 5.2. General estimates.

Let $X$ be a projective variety defined over a number field $k$ and $L$ be a big $\mathbb{Q}$-divisor on $X$. We fix an adelic metrization on $\mathcal{O}(L)$ and consider the induced height:

$$H_L : X(F) \to \mathbb{R}_{>0}.$$ 

For each Zariski open subset $U \subset X$ we define the counting function:

$$N(U, L, T) = \# \{ P \in U(k) \mid H_L(P) \leq T \}.$$

Here is a general result using Repulsion principle:

**Theorem 5.2.** Let $X$ be a normal projective variety of dimension $n$ defined over a number field $k$ and $L$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We fix an adelic metrization on $\mathcal{O}(L)$. Then for any $\epsilon > 0$ there exists a non-empty Zariski open subset $U = U(\epsilon) \subset X$ such that we have

$$N(U, L, T) = \mathcal{O}(T^{2n\delta(X,L)+\epsilon}).$$

**Proof.** We may assume that $X$ is smooth after applying a resolution. By Lemma 3.4, $\delta(X, L)$ is invariant under a resolution. Let $v$ be an archimedean place of $k$. By Theorem 4.1, for any $\epsilon > 0$ there exists a non-empty Zariski open subset $U(\epsilon) \subset X$ such that there exists $C = C_\epsilon > 0$ such that

$$\text{dist}_v(P, Q) > C(H_L(P)H_L(Q))^{-\delta(X,L)+\epsilon}.$$

We define

$$A_T = \{ P \in U(k) \mid H_L(P) \leq T \}.$$

For $P \in A_T$, we define the $v$-adic ball by

$$B_T(P) = \{ R \in U(k_v) \mid \text{dist}_v(P, R) < \frac{1}{2} CT^{-2(\delta(X,L)+\epsilon)} \}.$$

Then $\bigcup_{P \in A_T} B_T(P)$ is disjoint because of the triangle inequality. Hence we have

$$\tau_v(X) > \sum_{P \in A_T} \tau_{X,v}(B_T(P)) \gg N(U, L, T)T^{-(2n(\delta(X,L)+\epsilon)}$$

by Lemma 5.1. Thus our assertion follows.

**Remark 5.3.** In the case that $\delta(X, L)$ is the minimum, then one does not need to introduce $\epsilon$ in the above theorem.

**Remark 5.4.** In the above theorem, assuming $L$ is ample we may replace $\delta(X, L)$ by $s(X, L)$. In this case, one can take $U = X$.

In view of Manin’s conjecture, we expect the following is true:

**Conjecture 5.5.** Let $X$ be a geometrically rationally connected smooth projective variety of dimension $n$ and $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Then we have

$$a(X, L) \leq 2n\delta(X, L).$$
6. Manin type upper bounds for Fano conic bundles

In this section, we study the counting problems of rational points on conic bundles.

Definition 6.1. Let \( f : X \to S \) be a flat projective morphism between smooth projective varieties. The fibration \( f \) is a conic bundle if every fiber is isomorphic to a conic in \( \mathbb{P}^2 \).

Lemma 6.2. Let \( f : X \to S \) be a conic bundle and \( H \) be a big \( \mathbb{Q} \)-divisor on \( X \). Suppose that for a fiber \( C_s \) of \( f \), we have \( H.C_s = 2 \), i.e., \( C_s \) is a \( H \)-conic. Then we have

\[
\delta(X,H) \geq \frac{1}{2}.
\]

Proof. Let \( W = X \times X \) and \( \alpha : W' \to W \) be the blow up of the diagonal. We denote its exceptional divisor by \( E \). Let \( C_x \) be a conic in the fiber at \( x \in X_1 \) passing through \( x \). Then we have

\[
(H[2] - 2E).C_x = 0.
\]

As \( x \) varies over \( X_i \) \( C_x \) forms a subvariety \( D \) in \( W' \) which is dominant to both \( X_i \) and \( X_2 \). It follows that for any \( \epsilon > 0 \), \( (H[2] - (2 + \epsilon)E) \) contains \( D \) in its stable locus. Thus our assertion follows.

Proposition 3.11 shows that in general, \( \delta(X,H) \) may not be \( 1/2 \).

6.1. Local Tamagawa measures of conics in families. Here we study the behavior of local Tamagawa measures of conics in a family. Let \( f : X \to S \) be a conic bundle defined over a number field \( k \). Let \( S^0 \) be the complement of the discriminant locus \( \Delta_f \) of \( f \).

Let \( u \) be a place of \( k \). We fix a \( v \)-adic metrization on \( \mathcal{O}(K_X) \) and \( \mathcal{O}(K_S) \). This induces a \( v \)-adic metrization on \( \mathcal{O}(K_{X/S}) \). For each local 1-form \( dt \in \Omega^1_{X/S} \), one can define the local Tamagawa measure \( \tau_{C_s,v} \) on a conic \( C_s \) for any \( s \in S^0(k_v) \) by

\[
\tau_{C_s,v}(U) = \int_U \frac{|dt|}{||dt||_v}
\]

which is independent of a choice of \( dt \).

Lemma 6.3. Suppose that \( f : X \to S \) admits a rational section. Then there exists \( C > 0 \) such that for sufficiently small \( T \), any \( s \in S^0(k_v) \), \( P \in C_s(k_v) \), we have

\[
C \text{dist}_v(\Delta_f, f(P))T < \tau_{C_s,v}(\{Q \in C_s(k_v) \mid \text{dist}(P, Q) < T\})
\]

Proof. We fix a rational section \( S_0 \) and an ample divisor \( A \) on \( S \). Let \( S_m = S_0 + mf^*A \). Now \( f_*(\mathcal{O}(S_0)) \otimes \mathcal{O}(mA) \) is globally generated for \( m \gg 0 \). Using this for each point \( p \in S \), one may find a rational section \( S_p \sim S_m \) such that \( S_p \) is a local section in a neighborhood of the point \( p \in S \). It is a well-known fact that one can embed \( f : X \to S \) into a projective bundle \( \mathbb{P}(f_*(\mathcal{O}(-K_X))) \). Take a finite open affine covering \( \{U_i\} \) of \( S \) so that over \( U_i \), \( \mathbb{P}(f_*(\mathcal{O}(-K_X/S)))|_{U_i} \) is trivialized, i.e., isomorphic to \( U_i \times \mathbb{P}^2 \). Taking a finer finite open covering, we may assume that \( f \) admits a local section \( S_i \) over \( U_i \). By taking a finer finite open covering and applying a change of coordinates, one can assume that the local section \( S_i \) corresponds to \( (1 : 0 : 0) \) in \( \mathbb{P}^2 \). Moreover we may assume that the tangent line of \( C_s \) at \( (1 : 0 : 0) \) is given by \( x_1 = 0 \). Let \( A_j (j = 0, 1, 2) \) be the standard affine charts of \( \mathbb{P}^2 \) and we define \( V_{i,j} = f^{-1}(U_i) \cap (U_i \times A_j) \) which is affine.
Now we take a finite \( v \)-adic open covering \( B_l \) of \( X(k_v) \) such that \( \overline{B_l} \) is contained in some \( V_{i,j}(k_v) \). Then on \( B_l \) there exists a positive constant \( C_1 \) such that for any \( (s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v) \) such that
\[
\text{dist}_v(P, Q) \leq C_1 \max\{|x_j(P) - x_j(Q)|_v, |y_j(P) - y_j(Q)|_v\}, \tag{6.1}
\]
where \( x_j, y_j \) is the coordinates of \( A_j \).

Now we are going to parametrize conics in the family. By our construction, \( f^{-1}(U_i) \subset U_i \times \mathbb{P}^2 \) is defined by the following equation:
\[
d(s)y^2 + f(s)z^2 + 2xy + 2e(s)yz = 0,
\]
where \( d, f, e \) are functions on \( U_i \). Note that the discriminant locus \( \Delta_f \) is defined by \( f = 0 \) and it is a smooth divisor by our assumption. After further simplifications, we may assume that the equation is given by
\[
f(s)z^2 + 2xy = 0.
\]
Lines \( uy - vz = 0 \) passing through \( (1 : 0 : 0) \) are parametrized by \( (u : v) \in \mathbb{P}^1 \). Then the rational parameterization of conics is given by
\[
(fu^2 : -2u^2 : -2uv).
\]
In particular, any smooth conic \( C_s \) over \( s \in U_i(k_v) \) is covered by \( V_{i,0} \) and \( V_{i,1} \). Also note that while this rational parametrization is not valid along singular fibers, a rational map mapping \( (s, P) \in f^{-1}(U_i) \) to \( (u(P) : v(P)) \in \mathbb{P}^1 \) is a well-defined morphism.

Suppose that \( B_l \) is contained in \( V_{i,0} \). The inequality (6.1) shows that there exists \( C_2 > 0 \) such that any \( (s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v) \)
\[
\text{dist}_v(P, Q) < C_2 \text{dist}_v(\Delta_f, f(P))^{-1}|t(P) - t(Q)|_v
\]
where \( t = v/u \) and \( \text{dist}_v(\Delta_f, f(P)) \) is the distant function of \( \Delta_f \).

Suppose that \( B_l \) is contained in \( V_{i,1} \). The inequality (6.1) shows that there exists \( C_3 > 0 \) such that any \( (s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v) \)
\[
\text{dist}_v(P, Q) < C_3|t(P) - t(Q)|_v
\]
where \( t = u/v \).

Suppose that \( B_l \) is contained in \( V_{i,2} \). The inequality (6.1) shows that there exists \( C_4 > 0 \) such that any \( (s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v) \)
\[
\text{dist}_v(P, Q) < C_4|t(P) - t(Q)|_v
\]
where \( t = v/u \). Now by arguing as in Lemma 5.1 our assertion follows.

\[ \square \]

**Lemma 6.4.** Let \( f : X \to S \) be a conic bundle defined over a number field \( k \) with a rational section. Let \( v \) be an archimedean place of \( k \). We fix a \( v \)-adic metrization on \( \mathcal{O}(K_X) \) and \( \mathcal{O}(K_S) \). Then for any sufficiently small \( \epsilon > 0 \) there exists a constant \( C_\epsilon > 0 \) such that for any \( s \in S^0(k_v) \) we have
\[
\tau_v(C_s) < C\text{dist}_v(\Delta_f, s)^{1-\epsilon}
\]

**Proof.** This follows from the descriptions in the proof of Lemma 6.3 and an explicit computations of local Tamagawa numbers using the naive metrization. \[ \square \]
6.2. Fano conic bundles: the anticanonical height. In this section, we discuss upper bounds of Manin type for the anticanonical height of Fano conic bundles. Here is a theorem:

**Theorem 6.5.** Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. We assume that $X$ and $S$ are Fano. Let $L = -K_X$. Let $W = X \times X$ and $W'$ be the blow up of $W$ along the diagonal with the exceptional divisor $E$. We denote each projection $W' \to X_i$ by $\pi_i$. Let $\alpha, \beta$ be positive real numbers such that $2\alpha - 2\beta = 1$. We further make the following assumptions:

1. Weak Manin’s conjecture for $(S, -K_S)$ holds,
2. for any component $V$ of the stable locus of $\pi : \pi^{-1}(S) \to S$ such that $V$ is not contained in $E$, one of projections $\pi_i|_V$ is not dominant.

Then there exists a non-empty Zariski open subset $U \subseteq X$ such that for any $\epsilon > 0$ there exists $C = C_\epsilon > 0$ such that

$$N(U, L, T) < CT^{2\alpha + \epsilon}.$$

**Proof.** First of all note that the assumption (2) implies that $-2\alpha K_X + f^*K_S$ is big. Let $v$ be an archimedean place and fix $v$-adic metrizations on $O(K_X)$ and $O(K_S)$. Let $m$ be a positive integer such that $mL - f^*\Delta_f$ is ample. Fix $\epsilon > 0$. Arguing as Theorem 4.1 the assumption (3) implies that there exists $U \subseteq X$ and $C$ such that for any $P, Q \in U(k)$ with $P \neq Q$ and $f(P) = f(Q) = s$ we have

$$\text{dist}_v(P, Q) > C(H_{-K_S}(s))^{-2\beta}(H_{-K_{X/S}}(P)H_{-K_{X/S}}(Q))^{-\alpha}. \quad (6.2)$$

We define

$$A_T = \{ P \in U(k) \mid H_L(P) \leq T \}.$$ 

For $P \in A_T$, we define the $v$-adic ball by

$$B_T(P) = \{ R \in C_s \cap U(k_v) \mid f(P) = s, \text{dist}_v(P, R) < \frac{1}{2} CT^{-2\alpha} H_{-K_S}(s) \}$$

Then $\cup B_T(P)$ is disjoint because of (6.2) and the triangle inequality. Note that after removing some closed subset, $T^{-2\alpha} H_{-K_S}(s)$ goes to 0 as $T \to \infty$ because of our assumption (2). Thus we have

$$\text{dist}_v(\Delta_f, s)^{1-\epsilon} \gg \epsilon \tau_v(C_s) > \sum_{P \in A_T \cap C_s(k)} \tau_v(B_T(P))$$

$$\gg \epsilon N(U \cap C_s, L, T) \text{dist}_v(\Delta_f, s) T^{-2\alpha} H_{-K_S}(s)$$

by Lemma 6.3 and Lemma 6.4. Hence we conclude that

$$N(U \cap C_s, L, T) \ll \epsilon T^{2\alpha} H_{-K_S}(s)^{-1} H_{\Delta_f, v}(s)^{\epsilon} \ll T^{2\alpha + \epsilon} H_{-K_S}(s)^{-1},$$

where $P \in A_T \cap C_s$. Since $-kK_X \geq -f^*K_S$ for some $k$, our assertion follows from the fact that $S$ satisfies Weak Manin’s conjecture and Tauberian theorem. 

\[ \square \]
6.3. Fano conic bundles: the non-anticanonical heights. In this section, we discuss weak Manin’s conjecture for non-anticanonical height functions in some cases:

**Theorem 6.6.** Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. We assume that $X$ and $S$ are Fano. Let $L = -K_X - tf^*K_S$. We make the following assumptions:

1. Weak Manin’s conjecture for $(S, -K_S)$ holds,
2. $t \geq \frac{1}{2\delta(X, -K_X)}$.

Then for any $\epsilon > 0$ there exists a non-empty Zariski open subset $U = U(\epsilon) \subset X$ and $C = C_\epsilon > 0$ such that

$$N(U, L, T) < CT^{2\delta(X, -K_X)+\epsilon}.$$  

In particular when $\delta(X, -K_X) = 1/2$, Conjecture L.1 holds for $(X, L)$ except independence of $U$ on $\epsilon$.

**Proof.** Let $v$ be an archimedean place and fix $v$-adic metrizations on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. Let $m$ be a positive integer such that $mL - f^*\Delta_f$ is ample. Fix $\epsilon > 0$. Theorem 4.1 implies that there exists $U \subset X$ and $C$ such that for any $P, Q \in U(k)$ with $P \neq Q$ and $f(P) = f(Q) = s$ we have

$$\text{dist}_v(P, Q) > C(H_{-K_X}(P)H_{-K_X}(Q))^{-(\delta(X, -K_X)+\epsilon)}.$$  

(6.3)

We define

$$A_T = \{P \in U(k) \mid H_L(P) \leq T\}.$$  

For $P \in A_T$, we define the $v$-adic ball by

$$B_T(P) = \{R \in C_s \cap U(k_v) \mid f(P) = s, \text{dist}_v(P, R) < \frac{1}{2}CT^{2\delta(X, -K_X)+\epsilon}(H_{-K_S}(s))^{2t(\delta(X, -K_X)+\epsilon)}\}.$$  

Then $\cup B_T(P)$ is disjoint because of (6.3) and the triangle inequality. Note that

$$T^{2(\delta(X, -K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X, -K_X)+\epsilon)}$$

goes to 0 as $T \to \infty$. Thus we have

$$\text{dist}_v(\Delta_f, s)^{1-\epsilon} \gg \epsilon \tau_v(C_s) > \sum_{P \in A_T \cap C_s(k)} \tau_{C_s, v}(B_T(P))$$

$$\gg \epsilon N(U \cap C_s, L, T)\text{dist}_v(\Delta_f, s)T^{-2(\delta(X, -K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X, -K_X)+\epsilon)}$$

by Lemma 6.3 and Lemma 0.4. Hence we conclude that

$$N(U \cap C_s, L, T) \ll \epsilon T^{2(\delta(X, -K_X)+\epsilon)+me}H_{-K_S}(s)^{-2t(\delta(X, -K_X)+\epsilon)},$$

where $P \in A_T \cap C_s$. Since $L \geq -tf^*K_S$, our assertion follows from the fact that $S$ satisfies Weak Manin’s conjecture, Tauberian theorem, and Lemma 6.2.

\[ \square \]

**Remark 6.7.** If $\delta(X, -K_X)$ is the minimum, then one can take $U$ to be independent of $\epsilon$. 

---

\[ \square \]
7. 3-dimensional Fano conic bundles

In this section we list smooth 3-dimensional Fano conic bundles and the smallest $2\alpha$ satisfying the conditions of Theorem 6.5. Fano 3-folds with Picard rank $\geq 2$ are classified by Mori-Mukai in [MM82], [MM83], and [MM03]. We follow their classification. We assume that our ground field is an algebraically closed field of characteristic 0. In our computations of $\delta(X, -K_X)$ and the minimum $2\alpha$ satisfying the conditions of Theorem 6.5, it is important to know a description of the nef cone of divisors of $X$. Such a description has been obtained in [Mat95]. We freely use the results in this lecture note.

7.1. Fano threefolds with Picard rank 2. According to [MM82], there are 36 deformation types of smooth Fano 3-folds with Picard rank 2. Among them there are 16 deformation types of smooth Fano 3-folds which come with conic bundle structures. Since Fano 3-folds have Picard rank 2, these conic bundle structures are extremal contractions. Thus in these cases, a conic bundle structure comes with a rational section if and only if there is no singular fiber. Thus there are 7 deformation types of smooth Fano 3-folds which come with a conic bundle structure with a rational section. Here is the list of these Fano 3-folds from Table 2 of [MM82]:

| no | $(-K_X)^3$ | $X$ | $\delta(X, -K_X)$ | $2\alpha$ |
|----|-------------|-----|------------------|----------|
| 24 | 30          | a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (1, 2) | $\leq 1$ | $\leq 5$ |
| 27 | 38          | the blow-up of $\mathbb{P}^3$ with center a twisted cubic | $1/2$ | 2 |
| 31 | 46          | the blow-up of $Q \subset \mathbb{P}^4$ with center a line on it | $1/2$ | $5/3$ |
| 32 | 48          | a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (1, 1) | $1/2$ | $5/2$ |
| 34 | 54          | $\mathbb{P}^1 \times \mathbb{P}^2$ | $1/2$ | $5/3$ |
| 35 | 56          | $V_7 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbb{P}^2$ | $1/2$ | $5/4$ |
| 36 | 62          | $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbb{P}^2$ | $1/2$ | $5/3$ |

Here $Q \subset \mathbb{P}^4$ is a smooth quadric 3-fold. Note that no 34, 35, and 36 are toric thus Manin’s conjecture is known for these cases by [BT96] and [BT98a]. [BBS18] proves Manin’s conjecture for an example of Fano 3-folds of no 24.

Let us illustrate the computation of $\delta(X, -K_X)$ and $2\alpha$ in some cases:

**Example 7.1** (no 32). Let $W$ be a smooth divisor of $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (1, 1). We denote each projection by $\pi_i : W \to \mathbb{P}^2$ and let $H_i$ be the pullback of the hyperplane class via $\pi_i$. Then we have

$$-K_X = 2H_1 + 2H_2.$$ 

Since $H_1 + H_2$ is very ample, it follows that $\delta(X, -K_X) = 1/2$ by Lemma 6.2.

Next we consider

$$\alpha(2H_1 + 2H_2 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = \frac{4\alpha - 3}{2} H_1 + 2\alpha H_2.$$ 

Then when $\frac{4\alpha - 3}{2} \geq 1$, i.e., $\alpha \geq 5/4$, the above divisor satisfies the assumptions of Theorem 6.5. On the other hand, for each $x \in X$, let $C_x$ be a fiber of $\pi_2$ meeting with $x$. Then we have $C_x.H_1 = 1$ and $C_x.H_2 = 0$. Thus we have

$$\left(\frac{4\alpha - 3}{2} H_1 + 2\alpha H_2\right) \cdot C_x = \frac{4\alpha - 3}{2}.$$
Thus by arguing as Lemma 6.2, we conclude that $\alpha = 5/4$ is the minimum value satisfying the assumptions of Theorem 6.5.

**Example 7.2** (no 31). Let $X$ be the blow-up of $Q$ along a line. Then $X$ has Picard rank 2 so it comes with two extremal contractions, one is a $\mathbb{P}^1$-bundle $\pi_1 : X \to \mathbb{P}^2$, and another is a divisorial contraction $\pi_2 : X \to Q$. Let $H_i$ be the pullback of the hyperplane class via $\pi_i$. Then it follows from [MM83, Theorem 5.1] that

$$-K_X = H_1 + 2H_2.$$ 

Since $\delta(X, H_2) = \delta(Q, H) = 1$ and $\pi_2$ is birational, it follows that $\delta(X, -K_X) \leq 1/2$. Thus by Lemma 6.2, $\delta(X, -K_X) = 1/2$ is proved.

Next we have

$$\alpha(H_1 + 2H_2 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = 2\alpha H_2 + \frac{2\alpha - 3}{2} H_1.$$ 

Let $D$ be the exceptional divisor of $\pi_2$. Then we have $H_1 = H_2 - D$. Thus the above divisor becomes

$$\frac{6\alpha - 3}{2} H_2 - \frac{2\alpha - 3}{2} D.$$ 

Thus when $\frac{6\alpha - 3}{2} \geq 1$ and $2\alpha - 3 \leq 0$, i.e., $5/6 \leq \alpha \leq 3/2$ the assumption of Theorem 6.5 holds. On the other hand let $\ell \subset X$ be the strict transform of a line on $Q$ not meeting with center of $\pi_2$. Then we have

$$\left(\frac{6\alpha - 3}{2} H_2 - \frac{2\alpha - 3}{2} D\right) \cdot \ell = \frac{6\alpha - 3}{2}.$$ 

Thus since such $\ell$ deforms to cover $X$, by arguing as Lemma 6.2 we conclude that $\alpha = 5/6$ is the minimum value satisfying the assumptions of Theorem 6.5.

**7.2. Fano threefolds with Picard rank 3.** According to [MM82], there are 31 deformation types of smooth Fano 3-folds with Picard rank 3. It follows from [MM83, p. 125, (9.1)] that all such Fano 3-folds come with a conic bundle structure except the blow-up of $\mathbb{P}^3$ along a disjoint union of a line and a conic. Again a conic bundle structure with singular fibers which is extremal never comes with a rational section. Note that if $X$ is a Fano conic bundle which does not admit a divisorial contraction to a Fano conic bundle of Picard rank 2 with a rational section, then its extremal conic bundle structure admits singular fibers. For such a 3-fold, one can conclude that it does not admit a rational section. This implies that there are 25 deformation types of 3 dimensional Fano conic bundles with a rational section. Here is the list of these Fano 3-folds from Table 3 of [MM82].
| no | $X$ | $\delta$ | $2\alpha$ |
|----|------|--------|--------|
| 3  | a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 2)$ | $\leq 1$ | $\leq 5$ |
| 5  | the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a curve $C$ of bidegree $(5, 2)$ such that the projection $C \to \mathbb{P}^2$ is an embedding | $\leq 1$ | $\leq 5$ |
| 7  | the blow up of $W$ (no 32) with center an intersection of two members of $| - \frac{1}{2}K_W |$ | $\leq 2/3$ | $\leq 3$ |
| 8  | a member of the linear system $| p_1^*g^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(2) |$ on $\mathbb{F}_1 \times \mathbb{F}^2$ where $p_i$ is the projection to each factor and $g : \mathbb{F}_1 \to \mathbb{P}^2$ is the blowing up | $\leq 1$ | $\leq 5$ |
| 9  | the blowing up of the cone $\overline{H}_4 \subset \mathbb{P}^6$ over the veronese surface $R_4 \subset \mathbb{P}^3$ with center a disjoint union of the vertex and quartic in $R_4 = \mathbb{P}^2$ | $1/2$ | $\leq 7/5$ |
| 11 | the blow up of $\mathbb{V}_7$ (no 35) with center an intersection of two members of $| - \frac{1}{2}K_{\mathbb{V}_7} |$ | $1/2$ | $5/2$ |
| 12 | the blow up of $\mathbb{P}^3$ with center a disjoint union of a line and a twisted cubic | $1/2$ | $8/3$ |
| 13 | the blow up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ with center a curve $C$ of bidegree $(2, 2)$ on it such that each projection from $C$ to $\mathbb{P}^2$ is an embedding | $1/2$ | $\leq 5/2$ |
| 14 | the blow up of $\mathbb{P}^3$ with center a disjoint union of a point and a plane cubic | $\leq 1$ | $\leq 9/5$ |
| 15 | the blow up of $Q \subset \mathbb{P}^4$ with center a disjoint union of a line and a conic | $1/2$ | $5/2$ |
| 16 | the blow up of $\mathbb{V}_7$ with center the strict transform of a twisted cubic passing through the center of the blow up $\mathbb{V}_7 \to \mathbb{P}^3$ | $1/2$ | $5/2$ |
| 17 | a smooth divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 1)$ | $1/2$ | $5/2$ |
| 19 | the blow up of $Q \subset \mathbb{P}^4$ with center two points which are not colinear | $1/2$ | $5/3$ |
| 20 | the blow up of $Q \subset \mathbb{P}^4$ with center a disjoint union of two lines | $1/2$ | $5/2$ |
| 21 | the blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a curve of bidegree $(2, 1)$ | $1/2$ | $5/2$ |
| 22 | the blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a conic in $\{t\} \times \mathbb{P}^2$ | $1/2$ | $5/3$ |
| 23 | the blow up of $\mathbb{V}_7$ with center the strict transform of a conic passing through the center of the blow up $\mathbb{V}_7 \to \mathbb{P}^3$ | $1/2$ | $5/3$ |
| 24 | the fiber product $W \times_{\mathbb{P}^2} \mathbb{F}_1$, where $W$ is no 32 $W \to \mathbb{P}^2$ is the $\mathbb{P}^1$-bundle and $\mathbb{F}_1 \to \mathbb{P}^2$ is the blowing up | $1/2$ | $5/2$ |
| 25 | $\mathbb{P}(\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$ | $1/2$ | $3/2$ |
| 26 | the blow up of $\mathbb{P}^3$ with center a disjoint union of a point and a line | $1/2$ | $5/3$ |
| 27 | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | $1/2$ | $2$ |
| 28 | $\mathbb{P}^1 \times \mathbb{P}_1$ | $1/2$ | $2$ |
| 29 | the blow up of $\mathbb{V}_7$ with center a line on the exceptional set $D = \mathbb{P}^2$ of the blow up $\mathbb{V}_7 \to \mathbb{P}^3$ | $1/2$ | $\leq 7/5$ |
| 30 | the blow up of $\mathbb{V}_7$ with center the strict transform of a line passing through the center of the blow up $\mathbb{V}_7 \to \mathbb{P}^3$ | $1/2$ | $4/3$ |
| 31 | $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1, 1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$ | $1/2$ | $\leq 4/3$ |

Note that no 24-31 are toric, thus Manin’s conjecture is known for these cases by [BT96] and [BT98a]. Let us demonstrate the computation of $\delta(X, -K_X)$ and $\alpha$ in some cases:

**Example 7.3** (no 23). Let $X$ be a Fano 3-fold of no 23. Then $X$ admits a divisorial contraction $\beta : X \to \mathbb{V}_7$ with the exceptional divisor $D_1$. The Fano 3-fold $\mathbb{V}_7$ admits two extremal contractions: one is a $\mathbb{P}^1$-bundle $\pi_1 : \mathbb{V}_7 \to \mathbb{P}^2$ and another is the blow down...
$V_7 \to \mathbb{P}^3$. We denote the pullback of the hyperplane class via $\pi_i$ by $H_i$. One can conclude that the only conic bundle structure on $X$ is $\pi_1 \circ \beta$. It follows from [MM83 Theorem 5.1] that

$$-K_{V_7} = 2H_1 + 2H_2.$$  

Thus we have

$$-K_X = 2\beta^*H_1 + 2\beta^*H_2 - D_1.$$  

Since $\beta^*H_1 - D_1$ is effective and the morphism associated to $|H_2|$ is birational, one can conclude that $\delta(X, -K_X) = 1/2$ by Lemma 6.2.

Let $D_2$ be the strict transform of the exceptional divisor of $\pi_2$. Then we have

$$\alpha(2\beta^*H_1 + 2\beta^*H_2 - D_1 - 3\beta^*H_1) + \frac{2\alpha - 1}{2} \cdot 3\beta^*H_1 = 2\alpha\beta^*H_2 - \alpha D_1 + \frac{4\alpha - 3}{2} \beta^*H_1$$  

Since we have $\beta^*H_2 = \beta^*H_1 + D_2$, the above divisor becomes

$$\beta^*H_2 + (2\alpha - 1)D_2 + \frac{8\alpha - 5}{2} \beta^*H_1 - \alpha D_1$$  

Since $\beta^*H_1 - D_1 \geq 0$, in the case of $\frac{8\alpha - 5}{2} \geq \alpha$, i.e., $\alpha \geq 5/6$, the above divisor satisfies the assumption of Theorem 6.5. On the other hand let $\ell$ be the strict transform of a line meeting with the center of $D_1$. When $\alpha = 5/6$, we have

$$\left(\beta^*H_2 + (2\alpha - 1)D_2 + \frac{8\alpha - 5}{2} \beta^*H_1 - \alpha D_1\right) \cdot \ell = 1$$  

Thus since such $\ell$ deforms to cover $X$, by arguing as Lemma 6.2, we conclude that $\alpha = 5/6$ is the minimum value satisfying the assumption of Theorem 6.5.

**Example 7.4** (no 12). Let $X$ be a Fano 3-fold of no 12. Then it admits a conic bundle structure $\pi_1 : X \to \mathbb{P}^2$, a birational morphism $\pi_2 : X \to \mathbb{P}^3$, and a del Pezzo fibration $\pi_3 : X \to \mathbb{P}^1$. Let $H_i$ be the pullback of the hyperplane class via $\pi_i$. Then we have

$$-K_X = H_1 + H_2 + H_3$$  

Since $|H_2|$ defines a birational morphism to $\mathbb{P}^3$ and $|H_1 + H_3|$ defines a birational morphism to $\mathbb{P}^2 \times \mathbb{P}^1$ we can conclude that $\delta(X, -K_X) \leq 1/2$. Thus by Lemma 6.2 we have $\delta(X, -K_X) = 1/2$. Next we consider the following divisor

$$\alpha(H_1 + H_2 + H_3 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = (3\alpha - 3)H_2 + \alpha H_3 - \frac{2\alpha - 3}{2} D_1$$  

where $D_1$ is the exceptional divisor of $\pi_2$ whose center is a twisted cubic. When $3\alpha - 3 \geq 1$ and $2\alpha - 3 \leq 0$, i.e., $4/3 \leq \alpha \leq 3/2$ the assumptions of Theorem 6.5 holds. On the other hand let $\ell$ be the strict transform of a general line meeting with the line which is the center of $\pi_2$. Then we have

$$\left((3\alpha - 3)H_2 + \alpha H_3 - \frac{2\alpha - 3}{2} D_1\right) \cdot \ell = 3\alpha - 3.$$  

Thus since such $\ell$ deforms to cover $X$, by arguing as Lemma 6.2, we conclude that $\alpha = 4/3$ is the minimum value satisfying the assumption of Theorem 6.5.
7.3. Fano threefolds with Picard rank 4 or 5. For Fano 3-folds in this range, all of them admit conic bundle structures with a rational section. Here is the list of Fano 3-folds of Picard rank 4 from [MM82, Table 4] as well as [MM03]:

| no | \(X\) | \(\delta\) | \(2\alpha\) |
|----|--------|--------|--------|
| 1  | a smooth divisor on \((\mathbb{P}^1)^4\) of multidegree \((1,1,1,1)\) | \(1/2\) | 3 |
| 2  | the blow-up of the blow-up of the cone over a quadric surface \(S \subset \mathbb{P}^2\) with center a disjoint union of the vertex and an elliptic curve on \(S\) | \(\leq 1\) | \(\leq 2\) |
| 3  | the blow-up of \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) with center a curve of tridegree \((1,1,2)\) | \(1/2\) | \(\leq 2\) |
| 4  | the blow-up of \(Y\) (no 19 Table 3) with center the strict transform of a conic passing through \(p\) and \(q\) | \(\leq 1\) | \(\leq 2\) |
| 5  | the blow-up of \(\mathbb{P}^1 \times \mathbb{P}^2\) with center two disjoint curves of bidegree \((2,1)\) and \((1,0)\) | \(\leq 1\) | 2 |
| 6  | the blow-up of \(\mathbb{P}^3\) with center three disjoint lines | \(1/2\) | 2 |
| 7  | the blow-up of \(W \subset \mathbb{P}^2 \times \mathbb{P}^2\) with center two disjoint curves of bidegree \((0,1)\) and \((1,0)\) | \(1/2\) | 5/2 |
| 8  | the blow-up of \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) with center a curve of tridegree \((0,1,1)\) | \(1/2\) | 2 |
| 9  | the blow-up of \(Y\) (no 25 Table 3) with center an exceptional line of the blowing up \(Y \to \mathbb{P}^3\) | \(1/2\) | 2 |
| 10 | \(\mathbb{P}^1 \times S_t\) | \(1/2\) | 2 |
| 11 | the blow-up of \(\mathbb{P}^1 \times \mathbb{F}^1\) with center \(t \times e\) where \(t \in \mathbb{P}^1\) and \(e\) is an exceptional curve on \(\mathbb{F}^1\) | \(1/2\) | 2 |
| 12 | the blow-up of \(Y\) (no 33 Table 2) with center two exceptional lines of the blowing up \(Y \to \mathbb{P}^3\) | \(1/2\) | 2 |
| 13 | the blow-up of \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\) with center a curve of tridegree \((1,1,3)\) | \(\leq 1\) | \(\leq 3\) |

Here \(S_t\) is a smooth del Pezzo surface of degree 7. Note that no 9-12 are toric, so Manin’s conjecture is known for these cases by [BT96] and [BT98a]. Again let us illustrate the computation of \(\delta(X, -K_X)\) and \(\alpha\) in some cases:

**Example 7.5** (no 6). Let \(X\) be a Fano 3-fold of no 6. Then \(X\) admits three del Pezzo fibrations \(\pi_i : X \to \mathbb{P}^1\). It also admits a birational morphism \(\pi : X \to \mathbb{P}^3\). Let \(H_i\) be the pullback of the hyperplane class via \(\pi_i\). Let \(H\) be the pullback of the hyperplane class via \(\pi\). Then we have

\[-K_X = H + H_1 + H_2 + H_3.\]

Since both \(|H|, |H_1 + H_2 + H_3|\) are birational, we conclude that \(\delta(X, -K_X) = 1/2\).

Next any conic bundle structure on \(X\) is given by \(|H_i + H_j|\) where \(i \neq j\). So we look at the conic bundle structure defined by \(|H_1 + H_2|\). We consider

\[\alpha(H + H_3 - H_1 - H_2) + (2\alpha - 1)(H_1 + H_2) = \alpha H + \alpha H_3 + (\alpha - 1)H_1 + (\alpha - 1)H_2.\]

From this description we may conclude that \(\alpha = 1\) satisfies the assumptions of Theorem 6.5. On the other hand, by looking at the strict transform \(C\) of a line such that \(H_3.C = 0\), we may conclude that \(\alpha = 1\) is the minimum value satisfying the assumptions of Theorem 6.5.

**Example 7.6** (no 4). Let \(V_7\) be the blow-up of \(\mathbb{P}^3\) at a point \(p\). It admits two extremal rays and we denote each contraction morphism by \(\pi_1 : V_7 \to \mathbb{P}^2\) and \(\pi_2 : X \to \mathbb{P}^3\). Let \(H_i\) be the pullback of the hyperplane via \(\pi_i\). Let \(X'\) be the blow-up of a fiber of \(\pi_1\) which is the strict
transform of a line \( \ell \) passing through \( p \). It admits a conic bundle structure over \( \mathbb{P}_1 \). Let \( X \) be the the blow up of \( X' \) along the strict transform of a conic \( C_0 \) not meeting with \( \ell \). Then \( X \) is a smooth Fano 3-fold of no 4. We denote the strict transform of the exceptional divisor of \( X' \to V_7 \) by \( D_1 \) and the exceptional divisor of \( X \to X' \) by \( D_2 \). Then we have

\[
-K_X = 2H_1 + 2H_2 - D_1 - D_2.
\]

Since \( 2H_1 + H_2 - D_1 - D_2 \) is linearly equivalent to an effective divisor, it follows that \( \delta(X, -K_X) \leq 1 \).

Next we consider

\[
\alpha(-K_X - (3H_1 - D_1)) + \frac{2\alpha - 1}{2} (3H_1 - D_1) = 2\alpha H_2 - \alpha D_2 + \frac{4\alpha - 3}{2} H_1 - \frac{2\alpha - 1}{2} D_1.
\]

Since \( H_2 - D_2 \) and \( H_1 - D_1 \) are effective, it follows that \( \alpha = 1 \) satisfies the assumptions of Theorem 6.5.

Finally we discuss the case of Picard rank 5. Here is the list of Fano 3-folds of Picard rank 5 from [MM82, Table 5]

| no | \( X \) | \( \delta \) | \( 2\alpha \) |
|----|---------|-------------|-------------|
| 1  | the blow up of \( Y \) (no 29 Table 2) with center three exceptional lines of the blowing up \( Y \to Q \) | 1/2 | \( \leq 2 \) |
| 2  | the blow up of \( Y \) (no 25 Table 3) with center two exceptional lines \( \ell \) and \( \ell' \) of the blowing up \( \phi : Y \to \mathbb{P}^3 \) such that \( \ell \) and \( \ell' \) lie on the same irreducible component of the exceptional set for \( \phi \) | 1/2 | \( \leq 2 \) |
| 3  | \( \mathbb{P}^1 \times S_6 \) | 1/2 | 2 |

Here \( S_6 \) is a smooth del Pezzo surface of degree 6. Note that no 2, 3 are toric, so Manin’s conjecture is known by [BT96] and [BT98a].

7.4. **Fano threefolds with Picard rank \( \geq 6 \).** We start this section by the following theorem of Mori-Mukai:

**Theorem 7.7.** [MM83, Theorem 1.2] Let \( X \) be a smooth Fano 3-fold and we denote its Picard rank by \( \rho(X) \). Suppose that \( \rho(X) \geq 6 \). Then \( X \) is isomorphic to \( \mathbb{P}^1 \times S_{11-\rho(X)} \) where \( S_d \) is a smooth del Pezzo surface of degree \( d \).

Thus in this section, we study the product of a smooth del Pezzo surface \( S_d \) with \( \mathbb{P}^1 \). Note that weak Manin’s conjecture for the product follows as soon as weak Manin’s conjecture is known for \( S_d \) by [FMT89], so we omit the discussion of \( \alpha \) in this section.

**Proposition 7.8.** Let \( X = \mathbb{P}^1 \times S \) where \( S \) is a smooth del Pezzo surface of degree \( d \) with \( 1 \leq d \leq 8 \). Then we have \( \delta(X, -K_X) = \delta(S, -K_S) \).

**Proof.** Let \( W_X = X \times X \) and \( \alpha : W_X' \to W_X \) be the blow up of the diagonal. We use the same notation for \( S \) as well. Let \( H_1 \) be the pullback of the ample generator via \( p_1 : X \to \mathbb{P}^1 \) and let \( H_2 \) be the pullback of the anticanonical divisor via \( p_2 : X \to S \). Then the anticanononical divisor of \( X \) is

\[
-K_X = 2H_1 + H_2.
\]
Fix $\epsilon > 0$ and we consider
\[-(\delta(S, -K_S) + \epsilon)K_X[2] - E = 2(\delta(S, -K_S) + \epsilon)H_1[2] + \epsilon H_2[2] + \delta(S, -K_S)H_2[2] - E.
\]
Since $2(\delta(S, -K_S) + \epsilon)H_1[2] + \epsilon H_2[2]$ is semi-ample, the stable locus of $-(\delta(S, -K_S) + \epsilon)K_X[2] - E$ is contained in the stable locus of $\delta(S, -K_S)H_2[2] - E$. The possible dominant components of the stable locus of $\delta(S, -K_S)H_2[2] - E$ are $E$ and the strict transform of
\[\mathbb{P}^1 \times \mathbb{P}^1 \times \Delta_S\]
where $\Delta_S$ is the diagonal of $W_S$. Next we consider
\[-(\delta(S, -K_S) + \epsilon)K_X[2] - E = 2\epsilon H_1[2] + (\delta(S, -K_S) + \epsilon)H_2[2] + 2\delta(S, -K_S)H_1[2] - E.
\]
Since $2\epsilon H_1[2] + (\delta(S, -K_S) + \epsilon)H_2[2]$ is again semi-ample, the stable locus of $-(\delta(S, -K_S) + \epsilon)K_X[2] - E$ is contained in the stable locus of $2\delta(S, -K_S)H_1[2] - E$. Since $\delta(S, -K_S) \geq 1/2$, it follows that the stable locus of $2\delta(S, -K_S)H_1[2] - E$ is contained in the strict transform $Z$ of
\[\Delta_{\mathbb{P}^1} \times W_S \subset W_X\]
where $\Delta_{\mathbb{P}^1}$ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. The variety $Z$ is isomorphic to $\mathbb{P}^1 \times W'_S$. From this one may conclude that $\delta(X, -K_X) \leq \delta(S, -K_S)$ by taking the intersection of two loci.

On the other hand the discussion in Section 5.1 shows that in each case there are curves $C$ on $W'_S$ such that (i) $C$ deforms to dominate both $S$ and (ii) $-(\delta(S, -K_S)K_S[2] - E)C = 0$. Thus we conclude that $\delta(X, -K_X) \geq \delta(S, -K_S)$. Thus our assertion follows.

\[\square\]

8. K3 surfaces of Picard rank 1

Let $S$ be a K3 surface with a polarization $H$ of degree $2d$. In this section, we obtain an upper bound for $s(X, H)$ using [BM14]. Let $W$ be the blow-up of $S \times S$ along the diagonal and we denote the exceptional divisor by $E$. We also consider the Hilbert Scheme of two points on $S$, i.e., $\text{Hilb}^{[2]}(S)$. The variety $\text{Hilb}^{[2]}(S)$ comes with the divisor $H(2)$ induced by $H$ and a divisor class $B$ such that $2B$ is the class of the exceptional divisor of the Hilbert-Chow morphism. The variety $W$ admits a degree 2 finite morphism $f : W \rightarrow \text{Hilb}^{[2]}(S)$ and we have
\[f^*H(2) = H[2], \quad f^*B = E\]
It is easy to see that
\[sH[2] - E \text{ is nef} \iff sH(2) - B \text{ is nef},\]
and the nef cone of $\text{Hilb}^{[2]}(S)$ is studied in [HT01], [HT09], and [BM14]. We use results from [BM14] for the nef cone of $\text{Hilb}^{[2]}(S)$. Here is the theorem:

**Theorem 8.1.** [BM14] Let $S$ be a K3 surface with a polarization $H$ of degree $2d$ such that $\text{Pic}(S) = \mathbb{Z}H$. Then we have
\[s(S, H) \leq \sqrt{\frac{4}{d} + \frac{5}{d^2}}.\]

**Proof.** We recall a result on the nef boundary of $sH(2) - B$ based on properties of certain Pell’s equation. First we consider
\[X^2 - 4dY^2 = 5.\]
Suppose that there is a non-trivial solution \((x_1, y_1)\) with \(x_1 > 0\) minimal and \(y_1 > 0\) even. Then it follows from \([BM14]\) Lemma 13.3 that

\[
s(S, H) = \frac{x_1}{dy_1} \leq \sqrt{\frac{4}{d} + \frac{5}{d^2}}.
\]

Next suppose that there is no non-trivial solution to the above Pell’s equation. Then for \(\text{Hilb}^{[2]}(S)\), the nef cone and the movable cone coincides by \([BM14]\) Lemma 13.3. Suppose that \(d\) is a square. Then it follows from \([BM14]\) Proposition 13.1 that

\[
s(S, H) = \frac{1}{\sqrt{d}}.
\]

Next suppose that \(d\) is not a square. We consider the following Pell’s equation:

\[
X^2 - dy^2 = 1.
\]

This has a solution. Let \(x_1, y_1 > 0\) be the solution with \(x_1\) minimal. Then by \([BM14]\) Proposition 13.1, we have

\[
s(S, H) = \frac{x_1}{dy_1} \leq \sqrt{\frac{1}{d} + \frac{1}{d^2}}.
\]

Thus our assertion follows.

Now Theorem 1.11 follows from the above theorem and Remark 5.3.

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