Effect of retardation in the atom–field interaction on entanglement in a double Jaynes–Cummings system

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Abstract

The effect of retardation in the atom–field interaction on the dynamics of entanglement in a double Jaynes–Cummings system is investigated. We consider large cavities in which a finite time necessary for light to travel between the atoms and the cavity mirrors may result in retardation effects. Our results demonstrate the qualitatively new behaviour observable in the time evolution of entanglement when the retardation effects are included. Solutions for single and double excitations in the system are presented. We follow the temporal evolution of an initial entanglement and find that the evolution is affected drastically by the retardation effects. In particular, the harmonic oscillations of the atomic populations and the concurrence, characteristic of single-mode Jaynes–Cummings systems, are suppressed when the retardation effects are included. The process of revival of the entanglement degrades with an increasing number of the cavity modes to which the atoms are coupled. It is also found that the effect of the retardation on the doubly excited states is more drastic than that on the single-excitation states and that at relatively short times, the retardation leads to a complete distortion of entanglement carried by a doubly excited state.

(Some figures may appear in colour only in the online journal)

1. Introduction

Partly because of the successful experimental realization of quantum gates, and the opportunity it provides for applications in quantum information and quantum computation, the problem of the entanglement evolution and transfer in a network of quantum systems has recently received a great deal of attention [1, 2]. The theoretical description of such systems is generally complicated, in particular when one includes decoherence processes resulting from the coupling of a system to an external environment. The complete treatments of these situations are available. However, the physics of the coherent processes involved in the transfer of entanglement is often lost in details or masked by the incoherent processes [3–9].

Although less important from the practical situation, studying the evolution and transfer of entanglement in simpler systems such as Jaynes–Cummings (JC) models [10, 11] involving single atoms interacting with a single-mode cavity field is significant in terms of giving a greater insight into the fundamentals of the transfer process. In particular, the relation between the form of entangled states and the efficiency of the transfer can be studied. A large number of studies of such simple systems have been carried out for independent [12–20] and also coupled JC models [21–32]. In these models, simple analytical formulas can be derived for the state vector or density operator of the system and the relations between the coherence properties and the efficiency of transferring a given state can be studied.

It has been demonstrated that the evolution of singly and doubly excited entangled states in independent or coupled JC systems can be significantly different and that the singly excited states can be continuously and completely transferred between the atoms and the cavity modes, whereas the doubly excited states may suffer an abrupt loss during the evolution, the phenomenon known as sudden death of entanglement [33–35]. Although the single-mode
approximation of the interaction of an atom with a cavity field is often entirely adequate to describe a practical situation, there are circumstances when the interaction of the atom with a large number of modes cannot be neglected, for example when sizes of cavities are much larger than the wavelength of the atomic transition. In this case, a finite number of the cavity modes could fit into the atomic linewidth leading to an effective multi-mode interaction of the atom with the cavity field. The atom can then be re-excited at finite time and at multiple finite times required for a photon emitted by the atom to cycle through the cavity and be reabsorbed. The finite interaction time between the atom and the cavity field, arising from a finite time required for the photon to travel between the atom and the cavity mirrors, could lead to retardation in the excitation of the atom [36–40].

In this paper, we show how various results relating to the evolution of entanglement in a system composed of two isolated JC models can be affected by the retardation. We calculate concurrence, the measure of entanglement between two qubits, and investigate the effect of retardation on the evolution of single- and double-excitation entangled states. We find that the evolution differs qualitatively from that predicted before in the absence of the retardation. In particular, the evolution is essentially nonoscillatory. A simple physical interpretation is given based on the energy–time uncertainty relation.

The paper is organized as follows. We begin in section 2 by introducing the model and explaining in details the meaning of retardation in the atom–field interaction. In section 3, we derive equations of motion for the probability amplitudes of single- and double-excitation states. Following the description of single- and double-excitation systems, in section 4 we derive expressions for the concurrence between the atoms. We pay particular attention to the role of retardation in modifying the evolution of an initial entanglement. We show that the evolution of entangled states is affected drastically by the retardation effects that the harmonic oscillations of the concurrences characteristic of the single-mode interaction are suppressed in the presence of the retardation effects. At the same time, we show that the evolution of the concurrence from an initial state in which atoms are entangled follows the evolution of the atomic population. The numerical results for various special cases of the time evolution of the concurrence are illustrated. Finally, in section 5 we summarize our results.

2. The system and meaning of retardation

We consider a system composed of two independent JC cavities [10, 11]. Each JC system is composed of a one-dimensional cavity containing an atom located at a position \( \vec{x} \) inside the cavity modes. The atoms are modelled as two-level systems with the excited \( |e_i⟩ \) and ground \( |g_i⟩ \) states separated by the transition frequency \( \omega_{0i} \), as illustrated in figure 1. The cavities are considered as composed of a multi-mode field with the frequency difference between adjacent modes (free spectral range), such that multiple modes are supported within the atomic resonance line width. The need to consider a multi-mode rather than single-mode cavity results from the time–energy uncertainty relation that a finite interaction time between the atom and the cavity field leads to an unavoidable spread of the achievable energy distribution.

The interaction Hamiltonian between the atom and the multi-mode field, in the rotating-wave approximation, is of the form

\[
H_{df} = i\hbar \sum_{n} \sum_{j=1}^{2} (g_{jn}(\vec{x}) a_n \hat{S}_j^+ - H.c.),
\]

where \( g_{jn}(\vec{x}) \) is the position-dependent Rabi frequency which determines the strength of the coupling of the \( j \)-th atom to the mode \( n \) of the cavity field. The Rabi frequency is proportional to the density of the cavity modes to which the atom is coupled and can be written in a form

\[
g_{jn}(\vec{x}) = g(\omega_n)(\vec{d}_j \cdot \hat{e}_n)e^{i\vec{x} \cdot \vec{d}_j},
\]

where \( g(\omega_n) \) characterize the density of the cavity modes of frequencies \( \omega_n \) and \( \hat{e}_n \) is the unit polarization vector of the mode \( n \).

The function can be identified as the Airy function of the cavity so that the form of \( g(\omega_n) \) depends on the type of the cavity. For a high-finesse optical cavity, the function is of appreciable magnitude only at a single frequency \( \omega_0 \) called the cavity resonance. The function can be approximated by a delta function \( \delta(\omega_0) \) if the cavity is composed of perfectly reflecting mirrors, or by a Lorentzian of bandwidth \( \kappa \), if the cavity losses through the mirrors with a rate \( \kappa \) are included. For larger cavities (microwave cavities) with plane perfectly reflecting mirrors, or by a Lorentzian of bandwidth \( \kappa \), if the cavity losses through the mirrors with a rate \( \kappa \) are included. For larger cavities (microwave cavities) with plane perfectly reflecting mirrors, or by a Lorentzian of bandwidth \( \kappa \), if the cavity losses through the mirrors with a rate \( \kappa \) are included. For larger cavities (microwave cavities) with plane perfectly reflecting mirrors, or by a Lorentzian of bandwidth \( \kappa \), if the cavity losses through the mirrors with a rate \( \kappa \) are included. For larger cavities (microwave cavities) with plane perfectly reflecting mirrors, or by a Lorentzian of bandwidth \( \kappa \), if the cavity losses through the mirrors with a rate \( \kappa \) are included.

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can be re-excited at finite time $L/c$ and also at multiples of $L/c$. In other words, multiple finite times are required for a photon emitted by the atom to cycle through the cavity and be reabsorbed by the atom. Thus, the retardation may result in the excitation arising from a finite time required for the photon to travel between the atom and the cavity mirrors.

3. Evolution of entangled states

The main motivation of considering the multimode interaction between the atom and the cavity field is to study the drastic departure of the dynamics of entanglement from the usual single-mode behaviour [12–17]. We consider situations only when single and double excitations are present in the system. In the case of single excitation, we assume that either the atoms or the cavity modes are initially prepared in a single-excitation entangled state and discuss in details the dynamics of entanglement for two initial states

$$|\psi(0)⟩ = (\cos \theta |e⟩|g⟩ + \sin \theta |g⟩|e⟩) \otimes |0⟩_μ|0⟩_ν,$$  
(4)

which is a tensor product of a single-excitation entangled state of the atoms and the vacuum states of both cavities, and

$$|\varphi(0)⟩ = (\cos \theta |e⟩|g⟩ + \sin \theta |g⟩|e⟩) \otimes |g⟩|g⟩,$$  
(5)

which is a tensor product of a single-excitation entangled state of the cavities and the atomic ground states. Here, $θ$ determines the degree of entanglement in the initial state, and we use the index $μ$ to label modes of the cavity $a$, and $ν$ those of the cavity $b$.

In the case of double excitation, we choose an initial state of the form

$$|\phi(0)⟩ = (\cos \theta |e⟩|g⟩ + \sin \theta |g⟩|e⟩) \otimes |0⟩_μ|0⟩_ν,$$  
(6)

which is a tensor product of a double-excitation entangled state of the atoms and the vacuum states of both cavities.

3.1. Evolution of single-excitation states

First, suppose that only a single excitation is present in the system. In this case, the Schrödinger equation

$$ih \frac{d}{dt} |ψ(t)⟩ = H_{af} |ψ(t)⟩,$$  
(7)

for the evolution of the state vector

$$|ψ(t)⟩ = C_1(t) |e⟩|g⟩|0⟩_μ|0⟩_ν + C_2(t) |g⟩|e⟩|0⟩_μ|0⟩_ν + \sum_μ C_{aμ}(t) |1⟩_μ|0⟩_ν$$
$$+ \sum_ν C_{bν}(t) |0⟩_μ|1⟩_ν \otimes |g⟩|g⟩,$$  
(8)

leads to a simple set of coupled differential equations for the probability amplitudes

$$\dot{C}_1(t) = \sum_μ g_{1μ} C_{aμ}(t),$$  
(9)

$$\dot{C}_{aμ}(t) = -i\Delta_μ C_{aμ}(t) - g_{1μ}^* C_1(t),$$  
(10)

and

$$\dot{C}_2(t) = \sum_ν g_{2ν} C_{bν}(t),$$  
(11)

$$\dot{C}_{bν}(t) = -i\Delta_ν C_{bν}(t) - g_{2ν}^* C_2(t),$$  
(12)

where sums are taken over the discrete modes of the cavities, $Δ_μ = ω_μ - ω_a$ and $Δ_ν = ω_ν - ω_b$ are detunings of the mode frequencies $ω_μ$ and $ω_ν$ of the cavity $a$ and $b$ from the atomic transition frequency $ω_a$, respectively. For simplicity, we assume that $ω_a$ coincides with the central mode frequency of each cavity. Also, we note that the state (8) involves superpositions of the cavity modes oscillating with amplitudes $C_{aμ}(t)$ and $C_{bν}(t)$.

The formal integration of (10) and (12) yields

$$C_{aμ}(t) = C_{aμ}(0) e^{-i\Delta_μ t} - \int_0^t dt′ g_{1μ}^* C_1(t′) e^{-i\Delta_μ(t′-t)},$$  
(13)

$$C_{bν}(t) = C_{bν}(0) e^{-i\Delta_ν t} - \int_0^t dt′ g_{2ν}^* C_2(t′) e^{-i\Delta_ν(t′-t)}.$$  
(14)

On substituting for $C_{aμ}(t)$ from (13) in (9), and for $C_{bν}(t)$ from (14) in (11), we arrive at the following two integro-differential equations:

$$\dot{C}_1(t) = \sum_μ g_{1μ} C_{aμ}(0) e^{-i\Delta_μ t}$$
$$- \int_0^t dt′ \sum_μ |g_{1μ}|^2 C_1(t′ - t) e^{-i\Delta_μ t′},$$  
(15)

$$\dot{C}_2(t) = \sum_ν g_{2ν} C_{bν}(0) e^{-i\Delta_ν t}$$
$$- \int_0^t dt′ \sum_ν |g_{2ν}|^2 C_2(t′ - t) e^{-i\Delta_ν t′}.$$  
(16)

The effect of retardation in the atom–field interaction is clearly visible in terms of the detunings $Δ_μ$ and $Δ_ν$, which in general are all different, because the frequencies $ω_μ$ and $ω_ν$ are different. Since a large cavity of the length $L$ contains a finite number of discrete modes whose frequencies differ by $2πc/L$, the detunings also differ by integer multiples of $2πc/L$. Thus, the dynamics of the system response will exhibit sharp peaks at discrete times due to the constructive interference of all modes.

3.2. Evolution of double-excitation states

We now turn to the problem of determining how the retardation in the atom–field interaction can affect the evolution of double-excitation states. For this purpose, we consider the situation when the excitations are evenly redistributed through the system, i.e., each of the JC cavities contains a single excitation. In this case, the state vector of the system can be written as

$$|φ(t)⟩ = D_{11}(t)|e⟩|e⟩|0⟩_μ|0⟩_ν + \sum_ν D_{12}(t)|e⟩|g⟩|0⟩_μ|1⟩_ν$$
$$+ \sum_μ D_{31}(t)|g⟩|e⟩|1⟩_μ|0⟩_ν + D_{10}(t)|0⟩_μ|0⟩_ν$$
$$+ \sum_μ \sum_ν D_{4μν}(t)|1⟩_μ|1⟩_ν \otimes |g⟩|g⟩.$$  
(17)
where \([m]_a[n]_b\) denotes the state of the cavity modes with \(m\) photons in the mode \(\mu\) of the cavity \(a\), and \(n\) photons in the mode \(\nu\) of the cavity \(b\). We have also included the auxiliary state \((\tilde{g}_1 g_2 | 0\rangle_a | 0\rangle_b)\) with zero excitation that as one can see from (17) creates superpositions of the cavity modes crucial for the presence of an entanglement in the system.

On making use of the Schrödinger equation, we arrive at the following set of coupled equations of motion for the probability amplitudes

\[
\begin{align*}
\hat{D}_{00}(t) &= 0, \\
\hat{D}_{11}(t) &= -\sum_\nu g_{1\nu}^* D_{1\nu}(t) - \sum_\mu g_{1\mu} D_{1\mu}(t), \\
\hat{D}_{22}(t) &= -i\Delta c D_{22}(t) + g_{21} D_{11}(t) - \sum_\mu g_{1\mu}^* D_{4\mu}(t), \\
\hat{D}_{33}(t) &= -i\Delta c D_{33}(t) + g_{12} D_{21}(t) - \sum_\mu g_{1\mu}^* D_{3\mu}(t), \\
\hat{D}_{44}(t) &= -i(\Delta_\mu + \Delta_\nu) D_{44}(t) + g_{1\mu} D_{21}(t) + g_{2\nu} D_{33}(t).
\end{align*}
\]

From which it is straightforward to evaluate the concurrence

\[
C_{AB}(t) = 2 \max\{0, \hat{C}_{AB}(t)\},
\]

where

\[
\hat{C}_{AB}(t) = |D_{11}(t)||D_{00}(t)| - \sqrt{\left(\sum_\nu |D_{2\nu}(t)|^2\right)\left(\sum_\mu |D_{3\mu}(t)|^2\right)}.
\]

The entangled properties of the system are clearly exhibited by the off-diagonal elements of the density matrix (22). It is apparent from (24) that the presence of the auxiliary state with zero excitation, \(D_{00}(t) \neq 0\), is necessary to produce entanglement between the atoms.

4. Evolution of entanglement

Since we are interested in the dynamics of entanglement between the atoms and also between the cavity modes that behave like two-qubit systems, we shall consider the concurrence as a measure of entanglement. The concurrence is defined as [42]

\[
\mathcal{C} = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\},
\]

where \(\lambda_i\) are the eigenvalues (in descending order) of the Hermitian matrix \(R = \rho \hat{\rho}\), in which \(\rho\) is the density matrix of the system, \(\hat{\rho}\) = \(\sigma_3 \otimes \sigma_3 \rho^* \sigma_3 \otimes \sigma_3\) and \(\sigma_3\) is a Pauli matrix. The concurrence varies between 0 when qubits are separable and 1 when they are maximally entangled.

In the case of a single excitation present in the system, the reduced density operator of the atoms \(\rho_{AB} = \text{Tr}_C(\rho)\) can be written in the basis of the product states, \([1] = |e_1 e_2\rangle, [2] = |g_1 g_2\rangle, [3] = |g_1 e_2\rangle, [4] = |g_1 g_2\rangle\), in which it has a form

\[
\rho_{AB} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & C_1(t) & 0 & 0 \\
0 & 0 & C_2(t) & 0 \\
0 & 0 & 0 & \sum_\mu |C_{\mu}(t)|^2 + \sum_\nu |C_{\nu}(t)|^2
\end{pmatrix},
\]

from which we can easily find the concurrence

\[
C_{AB}(t) = 2 |C_1(t)|^2 |C_2(t)|^2 = 2 |D_{21}(t)|^2.
\]

Thus, the necessary and sufficient condition for an entanglement between the atoms is the presence of the one-photon coherence \(|D_{21}(t)|^2\).

In the case of a double excitation, again by tracing over the cavity modes, the reduced density operator of the atoms written in the basis of the product states takes a matrix form

\[
\rho_{AB} = \frac{1}{2} \begin{pmatrix}
|D_{11}(t)|^2 & 0 & 0 & 0 \\
0 & \sum_\nu |D_{0\nu}(t)|^2 & 0 & 0 \\
0 & 0 & \sum_\mu |D_{1\mu}(t)|^2 & 0 \\
0 & 0 & 0 & |D_{33}(t)|^2 + \sum_\mu |D_{3\mu}(t)|^2
\end{pmatrix}.
\]

4.1. The case of single-mode cavities

Let us first present the results for the evolution of entanglement in the simplified double JC system composed of two single-mode \((\mu = \nu = 1)\) cavities [12–17]. Figure 2 shows the time evolution of the concurrences for several different values of the parameter \(\theta\). It is seen that an initial entanglement periodically oscillates in time and vanishes at certain discrete times \(t_n = n\pi/2\), \((n = 1, 3, 5, \ldots)\) when it is completely transferred to the cavity modes. This behaviour is seen for both maximally \((\theta = \pi/4)\) and non-maximally entangled states \((\theta \neq \pi/4)\), but the effect of going off maximally entangled initial states is clearly to decrease the amount of entanglement [12–14, 17, 39].

Similar behaviour of the periodic exchange of entanglement is observed for double-excitation states with a difference that in this case the phenomenon of the sudden death of entanglement is observed. Therefore, we may conclude that in the case of single-mode cavities the periodic oscillatory behaviour of the concurrence can be observed as resulting from an instantaneous exchange of an excitation between the atoms and the cavity modes. This is true, however, only for sufficiently small cavities in which a light signal can pass between the atom and the cavity boundaries in a time short compared with the time \(\Delta t\) required for appreciable (secular) changes in the atomic states. In other words, the length \(L\) of the cavity must be much smaller than \(c\Delta t\).

4.2. The case of multi-mode cavities

We now examine the effect of retardation on the dynamics of entanglement for different numbers of modes and for two different initial states. Figure 3 shows the time evolution of the concurrence for the initial state (4) in which the atoms are initially entangled, whereas in figure 4 the initial state is (5) in which the cavity modes are initially entangled. The concurrence is determined from (21), where the time evolution of the probability amplitudes is found by solving numerically the sets of coupled differential equations (15) and (16).

The evolution of the concurrence in the presence of the retardation is seen to be qualitatively different from the previous case of the instantaneous exchange of an excitation between the atoms and the cavity modes, shown in figure 2. Firstly, the evolution is not oscillatory with time, and secondly...
Figure 2. The concurrences $C_{AB}(t)$ plotted as a function of dimensionless time $\Omega_0 t$ for the initial single-excitation state (4) (left frame) and the double-excitation state (6) (right frame) with $\theta = \pi/4$ (solid blue line), $\theta = \pi/6$ (dashed red line) and $\theta = \pi/12$ (solid black line).

Figure 3. The time evolution of $C_{AB}(t)$ for the initial state (4) with $\theta = \pi/4$ (solid blue line) and $\theta = \pi/12$ (solid red line) and the number of the cavity modes $n = 19$ (left frame) corresponding to $L = 6.7 \times 10^2 \lambda_a$ and $n = 99$ (right frame) corresponding to $L = 3.48 \times 10^3 \lambda_a$. The other parameters are $\omega_a = 4.84 \times 10^3 \Omega_0$, where $\lambda_a$ is the wavelength of the atomic transition and $\Omega_0$ is the vacuum Rabi frequency of the central cavity mode.

Figure 4. The time evolution of the concurrence $C_{AB}(t)$ for $\omega_a = 1.11 \times 10^3 \Omega_0$ and a different number of modes: $n = 19$ (solid blue line) corresponding to $L = 6.7 \times 10^2 \lambda_a$ and $n = 99$ (solid red line) corresponding to $L = 3.48 \times 10^3 \lambda_a$. The system was initially in the state (5) with $\theta = \pi/4$.

The initial entanglement is not preserved in time, it decreases as time progresses. At the initial time, the concurrence decays almost exponentially to zero and remains zero until a finite time at which a nonzero entanglement suddenly revives. The process repeats with increasing distortions of the concurrence. The effect of an increasing number of modes to which the atom couples is to decrease the periodicity of the entanglement revival. In other words, the process of the revival of the entanglement degrades with an increasing number of modes to which the atoms are coupled. Equivalently, we may say that the memory effects of the atom–field interaction degrade with an increasing number of field modes.

Suppose now that the system starts at $t = 0$ from the state (5) in which the cavity modes are entangled and the atoms are in their ground states. In this case, the transfer of the initial entanglement may depend strongly on the number of cavity modes to which the atoms are coupled. Figure 4 shows the time evolution of the concurrence between the atoms for two different numbers of cavity modes. The effect of increasing the number of modes is seen to decrease the efficiency of transferring entanglement from the field to the atoms, namely the entanglement between the atoms decreases in magnitude and occurs in more restricted intervals of time. This is readily understood if it is recalled that increasing the number of modes leads to a smaller efficiency of transferring an initial entanglement existing between two modes of the cavities to the atoms. One may notice that the concurrence exhibits collapses and revivals on time scales much larger than the Rabi period, the phenomenon characteristics of the JC system [43].
Figure 5. The time evolution of the probabilities $|C_1(t)|^2$ (solid red line) and $|C_2(t)|^2$ (dashed blue line) for $n = 19$ (left frame) and $n = 99$ (right frame). The system was initially in the state (4) with $\theta = \pi/4$.

Figure 6. The time evolution of the concurrence $C_{AB}(t)$ for $n = 49$ corresponding to $L = 1.72 \times 10^3 \lambda_a$ and $\omega_a = 1.11 \times 10^2 \Omega_0$. The system was initially in the state (6) with $\theta = \pi/4$ (solid black line) and $\theta = \pi/6$ (dashed red line).

It is interesting to note that the evolution of the concurrence from an initial state in which atoms are entangled follows the evolution of the atomic population. To see it, we plot in figure 5 the time evolution of the probabilities $|C_1(t)|^2$ and $|C_2(t)|^2$ for the same parameters as in figure 3. Clearly, the initially excited atoms decay almost exponentially in time [39]. However, at the particular times that correspond to $nL/c$, where $n$ is an integer, a sudden change (jump) in the probabilities occurs. These are just the times when the radiation field emitted into the cavity modes returns to the atoms. Correspondingly, the entanglement revives and remains nonzero for a finite time the excitation resides in the atoms.

The sudden jumps continue in time. However, the periodic maxima of the populations are reduced in magnitude and become more broadened as $t$ increases. This effect can be explained using the energy–time uncertainty arguments and is readily understood if it is recalled that the excitation wave packet spreads during the evolution, that the excitation becomes less localized as time progresses. This delocalization of the excitation, it turns out, is sufficient to wipe out the entanglement.

Figure 6 shows the time evolution of the concurrence for two initial conditions of the maximally and non-maximally entangled doubly excited state (6). At very early times, the concurrence decays almost exponentially to zero, remains zero for a finite period of time and then revives. The concurrence remains nonzero for a finite time, and the process repeats with an increasing distortion. The distortion is significantly much stronger than that observed previously for the evolution of the single-excitation states. The reason is that the evolution of the excitations occurs in two JC cavities with no fixed relation between them. At short times the excitations in the two JC cavities are unresolved resulting in the revival of the entanglement, but after a long time, the excitations can be resolved which gives no inkling of correlations between the atoms.

5. Summary
We have studied the effect of retardation on the evolution of entanglement in a system composed of two independent Jaynes–Cummings (JC) cavities. The retardation effects could be important in large cavities where the mode spacing is small enough such that an atom located inside the cavity can concurrently couple to many modes of the cavity field. We have used the concurrence to quantify entanglement between the atoms and the cavity modes and have shown that the multi-mode structure of the field inside the cavity affects considerably the harmonic evolution of the atomic population and entanglement characteristic of single-mode JC systems. We have considered both single- and double-excitation cases and have shown that the retardation leads to sudden death of the initial entanglement and abrupt revival at times corresponding to intervals required for a round trip of the excitation in the cavity. This demonstrates that in the presence of the retardation, the complete transfer of entanglement during the evolution of the independent JC systems never occurs, which is the essential feature of single-mode JC systems. Finally, we have shown that single-excitation entangled states are more robust against the retardation than doubly excited states.

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