Renormalized solutions for the fractional $p(x)$-Laplacian equation with $L^1$ data

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Abstract

In this paper, we prove the existence and uniqueness of nonnegative renormalized solutions for the fractional $p(x)$-Laplacian problem with $L^1$ data. Our results are new even in the constant exponent fractional $p$-Laplacian equation case.

Keywords: Variable exponent Sobolev fractional space; fractional $p(x)$-Laplacian; renormalized solutions.

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1 Introduction and main result

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. In this paper, we consider the following nonlocal fractional $p(x)$-Laplacian equation:

$$
\begin{align*}
    Lu(x) &= f(x), & \text{in } \Omega, \\
    u &= 0, & \text{on } \partial\Omega.
\end{align*}
$$

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Here we assume that
\[ 0 \leq f \in L^1(\Omega). \quad (1.2) \]

The operator \( L \) is given by
\[ L u(x) := (-\Delta)^s_{p(x)} u(x) = \text{P.V.} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy, \quad x \in \Omega, \]
where P.V. is a commonly used abbreviation in the principal value sense, \( 0 < s < 1 \),
\( p : \overline{\Omega} \times \overline{\Omega} \to (1,\infty) \) is a continuous functions with \( sp(x,y) < N \) for any \( (x,y) \in \overline{\Omega} \).
This operator was first introduced by Kaufmann, Rossi and Vidal in [14], in which they established a compact embedding theorem and proved the existence and uniqueness of weak solutions for the fractional \( p(x) \)-Laplacian problem
\[
\begin{cases}
L u(x) + |u(x)|^{q(x)-2}u(x) = f(x), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\quad (1.3)
\]
provided \( f \in L^{a(x)}(\Omega) \) for some \( a(x) > 1 \).

In the constant exponent case, the operator \( L \) is known as the regional fractional \( p \)-Laplacian, see [9]. The regional fractional Laplacian arises, for instance, from the Feller generator of the reflected symmetric stable process ([6, 7, 12, 13]). On the other hand, this operator is also a fractional version of the \( p(x) \)-Laplacian, given by \( \text{div}(|\nabla u|^{p(x)-2}\nabla u) \), which is associated with the variable exponent Sobolev space.

Regarding the non-local \( p \)-Laplacian operator \((-\Delta)^s_p\), the linear elliptic case \( p = 2 \) has been studied in [1, 15, 17]. In particular, the existence and uniqueness of renormalized solutions for the problems of the kind
\[ \beta(u) + (-\Delta)^s u \ni f \quad \text{in } \mathbb{R}^N \]
was proved by Alibaud, Andreianov and Bendahmane in [3], where \( f \in L^1(\mathbb{R}^N) \) and \( \beta \) is a maximal monotone graph in \( \mathbb{R} \). Using a duality argument, in the sense of Stampacchia, Kenneth, Petitta and Ulusoy in [15] proved the existence and uniqueness of solutions to non-local problems like \((-\Delta)^s u = \mu \) in \( \mathbb{R}^N \) with \( \mu \) being a bounded Radon measure whose support is compactly contained in \( \mathbb{R}^N \). In [16], Kuusi, Mingione and Sire discussed the
elliptic non-local case $p \neq 2$ with measure data and developed an existence of SOLA, regularity and Wolf potential theory. In addition, Abdellaoui et al in [1] investigated the fractional elliptic $p$-Laplacian equations with weight and general datum and showed that there exists a unique nonnegative entropy solution.

In this paper, we focus our attention on the existence and uniqueness of renormalized solutions for the fractional $p(x)$-Laplacian problem (1.1). It is well-known that the notion of renormalized solutions was first introduced by DiPerna and Lions [11] in their study of the Boltzmann equation. Our results can be seen as a continuation of the paper [14] and are new even in the constant exponent fractional $p$-Laplacian equation case. We construct an approximate solution sequence and establish some a priori estimates in order to draw a subsequence to obtain a limit function. Then based on the strong convergence of the truncations of approximate solutions and the decomposition for the region of integration according to the different contributions, we prove that this function is a renormalized solution. Moreover, the uniqueness of renormalized solutions follows by choosing suitable test functions.

We denote $u \in T_{0}^{s,p(x,y)}(\Omega)$ if $u : \Omega \rightarrow \mathbb{R}$ is measurable and $T_{k}(u) \in W_{0}^{s,p(x,y)}(\Omega)$ for any $k > 0$ (see Section 2), where the truncation function $T_{k}$ is defined by

$$T_{k}(t) = \max\{-k, \min\{k, t\}\},$$

for any $t \in \mathbb{R}$.

Next we give the definition of renormalized solutions to problem (1.1) which is influenced by [3] and [18].

**Definition 1.1.** We say that $u \in T_{0}^{s,p(x,y)}(\Omega)$ is a renormalized solution to (1.1) if the following conditions are satisfied:

(i) $$\lim_{h \to \infty} \int_{\{(x,y) \in \Omega \times \Omega : (u(x), u(y)) \in R_{h}\}} \frac{|u(x) - u(y)|^{p(x,y)-1}}{|x - y|^{N+sp(x,y)}} dxdy = 0,$$

where

$$R_{h} = \{(v, w) \in \mathbb{R}^{2} : \max\{|v|, |w|\} \geq h + 1 \text{ and } (\min\{|v|, |w|\} \leq h \text{ or } vw < 0)\}.$$
For any $\varphi \in C_0^\infty(\Omega)$ and $S \in W^{1,\infty}(\mathbb{R})$ with compact support,
\[
\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y) - 2}(u(x) - u(y))[(S(u)\varphi)(x) - (S(u)\varphi)(y)]}{|x - y|^{N + sp(x,y)}} \, dx \, dy = \int_{\Omega} fS(u)\varphi \, dx \tag{1.4}
\]
holds.

The main result of this work is the following theorem:

**Theorem 1.1.** Under the integrability condition (1.2), there exists a unique nonnegative renormalized solution to problem (1.1).

The rest of this paper is organized as follows. In Section 2, we collect some basic properties for variable exponent Sobolev fractional spaces which will be used later. We will prove the main result in Section 3.

## 2 Preliminaries

For the convenience of the readers, we recall some definitions and basic properties of variable exponent Sobolev fractional spaces. For a deeper treatment on these spaces, we refer to [8] and [14]. For a smooth bounded domain $\Omega \subset \mathbb{R}^N$, let
\[
p : \overline{\Omega} \times \overline{\Omega} \to (1, \infty)
\]
and
\[
q : \overline{\Omega} \to (1, \infty)
\]
be two continuous functions. We assume that $p$ is symmetric, i.e. $p(x,y) = p(y,x)$ and
\[
1 < p_- = \inf_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) \leq p_+ = \sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) < \infty,
\]
and
\[
1 < q_- = \inf_{x \in \overline{\Omega}} q(x) \leq q_+ = \sup_{x \in \overline{\Omega}} q(x) < \infty.
\]
For $0 < s < 1$, the variable exponent Sobolev fractional space $W^{s,q(x),p(x,y)}(\Omega)$ is the class of all functions $u \in L^{q(x)}(\Omega)$ such that

$$
\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy < \infty,
$$

for some $t > 0$, where $L^{q(x)}(\Omega)$ is the variable exponent Lebesgue space.

Define

$$
[u]_{s,p(x,y)}(\Omega) = \inf \left\{ t > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \leq t \right\}.
$$

It is the variable exponent seminorm. For simplicity, we omit the set $\Omega$ from the notation.

We could get the following properties:

**Lemma 2.1.** (1) If $1 \leq [u]_{s,p(x,y)} < \infty$, then

$$
([u]_{s,p(x,y)})^{p^{-}} \leq \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \leq ([u]_{s,p(x,y)})^{p^{+}};
$$

(2) If $[u]_{s,p(x,y)} \leq 1$, then

$$
([u]_{s,p(x,y)})^{p^{+}} \leq \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \leq ([u]_{s,p(x,y)})^{p^{-}}.
$$

**Remark 2.1.** Similarly to the discussion of the norm in variable exponent space, we could get the above results. Here we omit the proof of Lemma 2.1.

The space $W^{s,q(x),p(x,y)}(\Omega)$ is a Banach space with the norm

$$
\|u\|_{W^{s,q(x),p(x,y)}(\Omega)} = \|u\|_{L^{q(x)}(\Omega)} + [u]_{s,p(x,y)}.
$$

By $W_{0}^{s,q(x),p(x,y)}(\Omega)$ we denote the subspace of $W^{s,q(x),p(x,y)}(\Omega)$ which is the closure of compactly supported functions in $\Omega$ with respect to the norm $\| \cdot \|_{W^{s,q(x),p(x,y)}(\Omega)}$. Especially, if $q(x) = p(x,x)$, we denote $W^{s,q(x),p(x,y)}(\Omega)$ and $W_{0}^{s,q(x),p(x,y)}(\Omega)$ by $W^{s,p(x,y)}(\Omega)$ and $W_{0}^{s,p(x,y)}(\Omega)$ (see [8]), respectively.

For any $u \in W^{s,q(x),p(x,y)}(\Omega)$, define

$$
\rho(u) = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy + \int_{\Omega} |u|^{q(x)} \, dx \quad (2.1)
$$

and

$$
\|u\|_{\rho} = \inf \left\{ \lambda > 0 : \rho \left( \frac{u}{\lambda} \right) \leq 1 \right\} \quad (2.2)
$$

It is easy to see that $\| \cdot \|_{\rho}$ is a norm which is equivalent to the norm $\| \cdot \|_{W^{s,q(x),p(x,y)}(\Omega)}$. 


Lemma 2.2. \((W^{s,q(x),p(x,y)}(\Omega), \| \cdot \|_\rho)\) is uniformly convex and \(W^{s,q(x),p(x,y)}(\Omega)\) is a reflexive Banach space.

Proof. The result is essentially known. Here is a short proof of it. As \(p_+ < \infty\), from the definition of \(\rho\) we know that \(\rho\) satisfies \(\Delta_2\)-condition, i.e. there exists \(K \geq 2\) such that

\[
\rho(2u) \leq K \rho(u)
\]

for all \(u \in W^{s,q(x),p(x,y)}(\Omega)\).

Since \(p_- > 1\), similar to the proof of Theorem 3.4.9 in [10], we could verify that \(\rho\) is a uniformly convex semimodular, i.e. for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that

\[
\rho\left(\frac{u-v}{2}\right) \leq \varepsilon \frac{\rho(u) + \rho(v)}{2}
\]

or

\[
\rho\left(\frac{u+v}{2}\right) \leq (1-\delta) \frac{\rho(u) + \rho(v)}{2}
\]

for all \(u, v \in W^{s,q(x),p(x,y)}(\Omega)\).

Theorem 2.4.14 in [10] further implies that the norm \(\| \cdot \|_\rho\) is uniformly convex and \((W^{s,q(x),p(x,y)}(\Omega), \| \cdot \|_\rho)\) is uniformly convex. Hence, \(W^{s,q(x),p(x,y)}(\Omega)\) is a reflexive Banach space by virtue of Theorem 1.20 in [2].

In the following, we give a compact embedding theorem into the variable exponent Lebesgue spaces.

Lemma 2.3. Let \(\Omega \subset \mathbb{R}^N\) be a smooth bounded domain and \(s \in (0,1)\). Let \(q(x), p(x,y)\) be continuous variable exponents with \(sp(x,y) < N\) for \((x,y) \in \overline{\Omega} \times \Omega\) and \(q(x) \geq p(x,y)\) for \(x \in \overline{\Omega}\). Assume that \(r : \overline{\Omega} \to (1, \infty)\) is a continuous function such that

\[
p^*(x) := \frac{Np(x,x)}{N - sp(x,x)} > r(x) \geq r_- > 1,
\]

for \(x \in \overline{\Omega}\). Then, there exists a constant \(C = C(N,s,p,q,r,\Omega)\) such that for every \(u \in W^{s,q(x),p(x,y)}(\Omega)\), it holds that

\[
\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_{W^{s,q(x),p(x,y)}(\Omega)}.
\]
That is, the space $W^{s,q}(x,p(x,y)}(\Omega)$ is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

In addition, if $u \in W^{s,q}(x,p(x,y)}(\Omega)$, it holds that

$$
\|u\|_{L^{r(x)}(\Omega)} \leq C[u]_{s,p(x,y)}.
$$

**Remark 2.2.** (1) We would like to mention that the compact embedding theorem has been proved in [14] under the assumption $q(x) > p(x,x)$. Here we give a slightly different version of compact embedding theorem assuming that $q(x) \geq p(x,x)$ which can be obtained by following the same discussions in [14].

(2) Since $\frac{Np(x,x)}{N-sp(x,x)} > \bar{p}(x) > p_1 > 1$, Lemma 2.3 implies that $[u]_{s,p(x,y)}$ is a norm on $W^{s,p(x,y)}(\Omega)$, which is equivalent to the norm $\| \cdot \|_{W^{s,p(x,y)}(\Omega)}$.

3 Proof of the main result

In order to discuss Eq. (1.1), we restrict ourselves to $sp_+ > 1$ to have a well defined trace on $\partial \Omega$. In fact, there exist $\bar{s} \in (0,s)$ and $r \in (0,p_-)$ such that $\bar{s}r \in (0,N)$. Therefore, $W^{s,p(x,y)}(\Omega)$ is continuously embedded in $L^q(\partial \Omega)$ for all $q \in [1,\frac{(N-1)r}{N-\bar{s}r}]$ (see [8]). That is, for any $u \in W^{s,p(x,y)}(\Omega)$, $u|_{\partial \Omega}$ is well defined.

Now we are ready to prove the main results. Some of the reasoning is based on the ideas developed in [1, 18].

We first introduce the approximate problems. Define $T_n(f) = f_n$, we know that $0 \leq f_n \leq f$ such that

$$
f_n \to f \quad \text{strongly in } L^1(\Omega).
$$

Consider the following approximate problem of (1.1)

$$
\begin{cases}
\mathcal{L}u = f_n(x), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
$$

(3.1)
Lemma 3.1. For any $n \in \mathbb{N}$, there exists a unique weak solution $u_n \in W_0^{s,p(x,y)}(\Omega)$ to
\begin{equation}
\int_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)|^{p(x,y) - 2}(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N + sp(x,y)}} \, dx \, dy = \int_{\Omega} f_n v \, dx.
\end{equation}

Besides, $\{u_n\}_n$ is an increasing nonnegative sequence.

Proof. For any $u \in W_0^{s,p(x,y)}(\Omega)$, define
\begin{align*}
F(u) &= \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy - \int_{\Omega} f_n u \, dx.
\end{align*}

Then, for any $u \in W_0^{s,p(x,y)}(\Omega)$ with $[u]_{s,p(x,y)} \geq 1$, by using Lemmas 2.1 and 2.3 we derive that
\begin{align*}
F(u) &\geq \frac{1}{p^+} \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy - \|f_n\|_{L^{(p(x))'}(\Omega)} \|u\|_{L^{p(x)}(\Omega)} \\
&\geq \frac{1}{p^+}([u]_{s,p(x,y)})^{p-} - C[u]_{s,p(x,y)},
\end{align*}

which implies that $F$ is coercive on $W_0^{s,p(x,y)}(\Omega)$. Then, there is a unique minimizer $u_n$ of $F$. Similar to the proof of Theorem 1.4 in [14], we could also verify that $u_n$ is a weak solution to problem (1.1).

Thanks to $f \geq 0$ and $f_n = T_n(f)$, we get that $\{u_n\}_n$ is an increasing nonnegative sequence. 

Let $u : \Omega \rightarrow \mathbb{R}$ be a measure function. In the following, for simplicity, we denote
\begin{align*}
U_n(x,y) &= |u_n(x) - u_n(y)|^{p(x,y) - 2}(u_n(x) - u_n(y)), \\
\{u > t\} &= \{x \in \Omega : u(x) > t\}, \quad \{u \leq t\} = \{x \in \Omega : u(x) \leq t\},
\end{align*}

and denote $|E|$ by the Lebesgue measure of a measurable set $E$ and
\begin{equation}
d\nu = \frac{dxdy}{|x - y|^{N + sp(x,y)}}.
\end{equation}

Lemma 3.2. There exists $u \in T_0^{s,p(x,y)}(\Omega)$ such that $u_n \rightarrow u$ in measure and $u_n \rightarrow u$ a.e. in $\Omega$, as $n \rightarrow \infty$. 

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Proof. Taking $T_k(u_n)$ as a test function in (3.1), we get
\[
\int_{\Omega \times \Omega} U_n(x,y)[T_k(u_n)(x) - T_k(u_n)(y)] \, d\nu = \int_{\Omega} f_n T_k(u_n) \, dx \leq k \int_{\Omega} f_n \, dx \leq Ck, \tag{3.2}
\]
where $C$ is independent of $k$ and $n$.

Note that
\[
|T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)} \leq U_n(x,y)[T_k(u_n)(x) - T_k(u_n)(y)],
\]
from (3.2) we obtain
\[
\int_{\Omega \times \Omega} |T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)} \, d\nu \leq Ck. \tag{3.3}
\]
Then, $\{T_k(u_n)\}_n$ is bounded in $W_0^{s,p(x,y)}(\Omega)$. In the following, we assume that
\[
T_k(u_n) \rightharpoonup v \text{ weakly in } W_0^{s,p(x,y)}(\Omega),
\]
as $n \to \infty$. As $W_0^{s,p(x,y)}(\Omega) \hookrightarrow L^{\bar{p}(x)}(\Omega)$ is compact due to Lemma 2.3, $T_k(u_n) \to v$ strongly in $L^{\bar{p}(x)}(\Omega)$. Passing to a subsequence, still denoted by $\{u_n\}_n$, we assume that
\[
T_k(u_n) \to v \text{ a.e. in } \Omega.
\]
From (3.3), for any $k \geq 1$, we have
\[
\int_{\Omega \times \Omega} \left| \frac{T_k(u_n)(x) - T_k(u_n)(y)}{k^{\frac{1}{\bar{p}(x)}}} \right|^{p(x,y)} \, d\nu \leq C.
\]
As $[u]_{s,p(x,y)}$ is a norm on $W_0^{s,p(x,y)}$, it follows from Lemmas 2.1 and 2.3 that
\[
\|k^{-\frac{1}{r}} T_k(u_n)\|_{L^{\bar{p}(x)}(\Omega)} \leq C,
\]
where $C$ is independent of $k$ and $n$. Then,
\[
\left| \{u_n \geq k\} \right| = \left| \{T_k(u_n) = k\} \right| \\
\leq \int_{\Omega} \left| \frac{T_k(u_n)}{k} \right|^{ar{p}(x)} \, dx \\
\leq k^{1-p} \int_{\Omega} \left| k^{-\frac{1}{r}} T_k(u_n) \right|^{ar{p}(x)} \, dx \leq Ck^{1-p},
\]

which implies

\[
\lim_{k \to \infty} \limsup_{n \to \infty} |\{u_n \geq k\}| = 0. \tag{3.4}
\]

For any \(t > 0\), we get

\[
|\{|u_n - u_m| > t\}| \leq |\{u_n > k\}| + |\{u_m > k\}| + |\{|T_k(u_n) - T_k(u_m)| > t\}|. \tag{3.5}
\]

Note that

\[
|\{|T_k(u_n) - T_k(u_m)| > t\}| \\
\leq \int_{\{|T_k(u_n) - T_k(u_m)| > t\}} \frac{|T_k(u_n) - T_k(u_m)| \bar{\beta}(x)}{t} \, dx \\
\leq (t^{-p} + t^{-p^*}) \int_{\{|T_k(u_n) - T_k(u_m)| > t\}} |T_k(u_n) - T_k(u_m)| \bar{\beta}(x) \, dx,
\]

we have

\[
\lim_{m,n \to \infty} \left|\{|T_k(u_n) - T_k(u_m)| > t\}\right| = 0. \tag{3.6}
\]

From (3.4)–(3.6), then

\[
\lim_{m,n \to \infty} \left|\{|u_n - u_m| > t\}\right| = 0,
\]

which implies that \(u_n \to u\) in measure and \(u_n \to u\) a.e. in \(\Omega\), as \(n \to \infty\).

Then \(v = T_k(u) \in W_0^{s,p(x,y)}(\Omega)\) and \(T_k(u_n) \to T_k(u)\) strongly in \(L^{\bar{\beta}(x)}(\Omega)\). As \(\{u_n\}_n\) is increasing, we have \(u_n(x) \leq u(x)\) a.e. in \(\Omega\). \(\square\)

**Lemma 3.3.** For any \(k > 0\), \(T_k(u_n) \to T_k(u)\) strongly in \(W_0^{s,p(x,y)}(\Omega)\) as \(n \to \infty\).

**Proof.** Taking \(T_k(u_n) - T_k(u)\) as a test function in (3.1) to yield that

\[
\langle Lu_n, T_k(u_n) - T_k(u) \rangle = \int_{\Omega} f_n(T_k(u_n) - T_k(u)) \, dx. \tag{3.7}
\]

Denote

\[
I_{1,n} = \int_{\Omega \times \Omega} U_n(x, y)[T_k(u_n)(x) - T_k(u_n)(y)] \, dv
\]

and

\[
I_{2,n} = \int_{\Omega \times \Omega} U_n(x, y)[T_k(u)(x) - T_k(u)(y)] \, dv.
\]
From (3.7), we have

\[ I_{1,n} = I_{2,n} + \int_{\Omega} f_n(T_k(u_n) - T_k(u)) \, dx. \] (3.8)

Denote

\[ T_{n,k}(x, y) = |T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)} \left( T_k(u_n)(x) - T_k(u_n)(y) \right). \]

Then

\[ I_{1,n} = \int_{\Omega \times \Omega} |T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)} \, d\nu \]

\[ + \int_{\Omega \times \Omega} \left( U_n(x, y) - T_{n,k}(x, y) \right) [T_k(u_n)(x) - T_k(u_n)(y)] \, d\nu \]

and

\[ I_{2,n} = \int_{\Omega \times \Omega} T_{n,k}(x, y) [T_k(u)(x) - T_k(u)(y)] \, d\nu \]

\[ + \int_{\Omega \times \Omega} \left( U_n(x, y) - T_{n,k}(x, y) \right) [T_k(u)(x) - T_k(u)(y)] \, d\nu \]

\[ \leq \int_{\Omega \times \Omega} \frac{1}{p(x,y)} |T_k(u)(x) - T_k(u)(y)|^{p(x,y)} \, d\nu \]

\[ + \int_{\Omega \times \Omega} \frac{p(x,y) - 1}{p(x,y)} |T_{n,k}(x, y)|^{\frac{p(x,y)}{p(x,y)} - 1} \, d\nu \]

\[ + \int_{\Omega \times \Omega} \left( U_n(x, y) - T_{n,k}(x, y) \right) [T_k(u)(x) - T_k(u)(y)] \, d\nu. \]

From (3.8), we have

\[ \int_{\Omega \times \Omega} \frac{|T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)}}{p(x,y)} \, d\nu \]

\[ + \int_{\Omega \times \Omega} \left( U_n(x, y) - T_{n,k}(x, y) \right) [T_k(u_n)(x) - T_k(u_n)(y) + T_k(u)(y)] \, d\nu \]

\[ \leq \int_{\Omega \times \Omega} \frac{|T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x,y)} \, d\nu \]

\[ + \int_{\Omega} f_n(T_k(u_n) - T_k(u)) \, dx. \]

In the following, we will verify that the second term on the left-hand side of the above inequality is nonnegative. We divide \( \Omega \times \Omega \) into the following four parts:

\[ A_1 = \{ (x, y) \in \Omega \times \Omega : u_n(x) \leq k, u_n(y) \leq k \}, \]

\[ A_2 = \{ (x, y) \in \Omega \times \Omega : u_n(x) \geq k, u_n(y) \geq k \}, \]
\[ A_3 = \{(x, y) \in \Omega \times \Omega : u_n(x) \leq k, u_n(y) \geq k\}, \]
\[ A_4 = \{(x, y) \in \Omega \times \Omega : u_n(x) \geq k, u_n(y) \leq k\}. \]

Similar to the proof of Lemma 3.6 in [1], we could verify that
\[(U_n(x, y) - T_{n,k}(x, y))[T_k(u_n)(x) - T_k(u_n)(y) + T_k(u)(y)] \geq 0\]
a.e. in \(A_1 \cup A_2 \cup A_3 \cup A_4\). Then
\[
\int_{\Omega \times \Omega} (U_n(x, y) - T_{n,k}(x, y))[T_k(u_n)(x) - T_k(u)(x) - T_k(u_n)(y) + T_k(u)(y)] \, d\nu \geq 0,
\]
which implies
\[
\int_{\Omega \times \Omega} \frac{|T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)}}{p(x,y)} \, d\nu \leq \int_{\Omega \times \Omega} \frac{|T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x,y)} \, d\nu + \int_{\Omega} f_n(T_k(u_n) - T_k(u)) \, dx.
\]
(3.9)

As \(T_k(u_n) \to T_k(u)\) strongly in \(L^{p(x)}(\Omega)\), we derive that as \(n \to \infty\),
\[
\int_{\Omega} f_n(T_k(u_n) - T_k(u)) \, dx \to 0.
\]

It follows from Fatou lemma and (3.9) that
\[
\int_{\Omega \times \Omega} \frac{|T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x,y)} \, d\nu = \int_{\Omega \times \Omega} \liminf_{n \to \infty} \frac{|T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)}}{p(x,y)} \, d\nu 
\]
\[
\leq \liminf_{n \to \infty} \int_{\Omega \times \Omega} \frac{|T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)}}{p(x,y)} \, d\nu 
\]
\[
\leq \limsup_{n \to \infty} \int_{\Omega \times \Omega} \frac{|T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)}}{p(x,y)} \, d\nu 
\]
\[
\leq \int_{\Omega \times \Omega} \frac{|T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x,y)} \, d\nu,
\]
which yields
\[
\lim_{n \to \infty} \int_{\Omega \times \Omega} \frac{|T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x,y)} \, d\nu = \int_{\Omega \times \Omega} \frac{|T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x,y)} \, d\nu.
\]
Note that
\[
\left| [T_k(u_n)(x) - T_k(u_n)(y)] - [T_k(u)(x) - T_k(u)(y)] \right|^{p(x,y)} \\
\leq 2^{p+} \left( [T_k(u_n)(x) - T_k(u_n)(y)]^{p(x,y)} + [T_k(u)(x) - T_k(u)(y)]^{p(x,y)} \right),
\]
then by Fatou lemma, we have
\[
\int_{\Omega \times \Omega} \frac{2^{p+} |T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x, y)} \ dv \\
= \int_{\Omega \times \Omega} \frac{1}{p(x, y)} \ lim \inf_{n \to \infty} \left( 2^{p+} |T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)} + 2^{p+} |T_k(u)(x) - T_k(u)(y)|^{p(x,y)} \right. \\
- \left. |T_k(u_n)(x) - T_k(u_n)(y)| - |T_k(u)(x) - T_k(u)(y)| \right)^{p(x,y)} \ dv \\
\leq lim \inf_{n \to \infty} \int_{\Omega \times \Omega} \frac{1}{p(x, y)} \left( 2^{p+} |T_k(u_n)(x) - T_k(u_n)(y)|^{p(x,y)} + 2^{p+} |T_k(u)(x) - T_k(u)(y)|^{p(x,y)} \right. \\
- \left. |T_k(u_n)(x) - T_k(u_n)(y)| - |T_k(u)(x) - T_k(u)(y)| \right)^{p(x,y)} \ dv \\
\leq \int_{\Omega \times \Omega} \frac{2^{p+} |T_k(u)(x) - T_k(u)(y)|^{p(x,y)}}{p(x, y)} \ dv \\
- \lim sup_{n \to \infty} \int_{\Omega \times \Omega} \frac{|T_k(u_n)(x) - T_k(u_n)(y)| - |T_k(u)(x) - T_k(u)(y)|}{p(x, y)} \ dv.
\]
Thus,
\[
\int_{\Omega \times \Omega} \frac{|T_k(u_n)(x) - T_k(u_n)(y)| - |T_k(u)(x) - T_k(u)(y)|}{p(x, y)} \ dv \to 0.
\]
As \([u]_{s,p(x,y)}\) is a norm on \(W_0^{s,p(x,y)}\), it follows from Lemma 2.1 that \(T_k(u_n) \to T_k(u)\) strongly in \(W_0^{s,p(x,y)}(\Omega)\).

\[\square\]

**Theorem 3.1.** The function \(u\) obtained in Lemma 3.2 is a unique renormalized solution to problem (1.1).

**Proof.** (i) Existence of renormalized solutions. We will divide the proof into the following two steps.

**Step 1.** We will verify that
\[
\lim_{h \to \infty} \int_{\{u(x), u(y)\} \in R_h} |u(x) - u(y)|^{p(x,y)-1} \ dv = 0.
\]
For any \( h > 0 \), denote \( G_h(t) = t - T_h(t) \). Taking \( T_1(G_h(u_n)) \) as a test function in (3.1), we have

\[
\int_{\Omega \times \Omega} U_n(x, y)[T_1(G_h(u_n))](x) - T_1(G_h(u_n))(y) \, d\nu = \int_{\Omega} f_n T_1(G_h(u_n)) \, dx \\
\leq \int_{\{u_n > h\}} f_n \, dx \leq \int_{\{u_n > h\}} f \, dx.
\]

If \((u_n(x), u_n(y)) \in R_h\),

\[
|u_n(x) - u_n(y)|^{p(x,y)-1} \leq U_n(x, y)[T_1(G_n(u_n))](x) - T_1(G_n(u_n))(y).
\]

Then

\[
\int_{\{(u_n(x), u_n(y)) \in R_h\}} |u_n(x) - u_n(y)|^{p(x,y)-1} \, d\nu \leq \int_{\{u_n > h\}} f \, dx. \quad (3.10)
\]

By Fatou lemma,

\[
\int_{\Omega \times \Omega} |u(x) - u(y)|^{p(x,y)-1} \chi_{\{(u(x), u(y)) \in R_h\}} \, d\nu = \int_{\Omega \times \Omega} \liminf_{n \to \infty} |u_n(x) - u_n(y)|^{p(x,y)-1} \chi_{\{(u_n(x), u_n(y)) \in R_h\}} \, d\nu \\
\leq \liminf_{n \to \infty} \int_{\Omega \times \Omega} |u_n(x) - u_n(y)|^{p(x,y)-1} \chi_{\{(u_n(x), u_n(y)) \in R_h\}} \, d\nu \\
\leq \liminf_{n \to \infty} \int_{\{u_n > h\}} f \, dx.
\]

As \( f \in L^1(\Omega) \), by (3.4) we obtain that

\[
\lim_{h \to \infty} \int_{\Omega \times \Omega} |u(x) - u(y)|^{p(x,y)-1} \chi_{\{(u(x), u(y)) \in R_h\}} \, d\nu \\
\leq \lim_{h \to \infty} \liminf_{n \to \infty} \int_{\{u_n > h\}} f \, dx = 0.
\]

**Step 2.** For any \( \varphi \in C_0^\infty(\Omega) \) and \( S \in W^{1,\infty}(\mathbb{R}) \) with compact support, we will verify that \( u \) satisfies (1.4).

Taking \( S(u_n)\varphi \) as a test function in (3.1), we have

\[
\int_{\Omega \times \Omega} U_n(x, y)[S(u_n)\varphi](x) - (S(u_n)\varphi)(y) \, d\nu = \int_{\Omega} f_n S(u_n)\varphi \, dx. \quad (3.11)
\]
Note that
\[
\begin{align*}
\int_{\Omega \times \Omega} U_n(x,y) [(S(u_n)\varphi)(x) - (S(u_n)\varphi)(y)] d\nu \\
= \int_{\Omega \times \Omega} U_n(x,y) [S(u_n)(x) - S(u_n)(y)] \cdot \frac{\varphi(x) + \varphi(y)}{2} d\nu \\
+ \int_{\Omega \times \Omega} U_n(x,y) (\varphi(x) - \varphi(y)) \cdot \frac{S(u_n)(x) + S(u_n)(y)}{2} d\nu \\
:= I_1 + I_2.
\end{align*}
\]

In the following, we assume that supp $S \subset [-M,M]$, where $M > 0$ and define the following subdomains of $\Omega \times \Omega$:

\[
\begin{align*}
B_{1,n} &= \{(x,y) \in \Omega \times \Omega : u_n(x) \geq M, u_n(y) \geq M\}, \\
B_{2,n} &= \{(x,y) \in \Omega \times \Omega : u_n(x) \leq M, u_n(y) \leq M\}, \\
B_{3,n} &= \{(x,y) \in \Omega \times \Omega : M \leq u_n(x) \leq M + 1, u_n(y) \leq M\}, \\
B_{4,n} &= \{(x,y) \in \Omega \times \Omega : u_n(x) \geq M + 1, u_n(y) \leq M\}, \\
B_{5,n} &= \{(x,y) \in \Omega \times \Omega : u_n(x) \leq M, M \leq u_n(y) \leq M + 1\}, \\
B_{6,n} &= \{(x,y) \in \Omega \times \Omega : u_n(x) \leq M, u_n(y) \geq M + 1\}.
\end{align*}
\]

First, we will estimate $I_1$ and denote
\[
\begin{align*}
G_n(x,y) &= \frac{U_n(x,y) [S(u_n)(x) - S(u_n)(y)] \varphi(x) + \varphi(y)}{|x-y|^{N+sp(x,y)}} \cdot \frac{2}{\varphi(x) + \varphi(y)}, \\
G(x,y) &= \frac{U(x,y) [S(u)(x) - S(u)(y)] \varphi(x) + \varphi(y)}{|x-y|^{N+sp(x,y)}} \cdot \frac{2}{\varphi(x) + \varphi(y)}.
\end{align*}
\]

where
\[U(x,y) = |u(x) - u(y)|^{p(x,y) - 2}(u(x) - u(y)).\]

(1) In $B_{1,n}$, $S(u_n)(x) = S(u_n)(y) = 0$. Then, $G_n(x,y) = 0$.

(2) In $B_{2,n}$, $T_M(u_n)(x) = u_n(x)$ and $T_M(u_n)(y) = u_n(y)$. Then
\[
\begin{align*}
U_n(x,y) [S(u_n)(x) - S(u_n)(y)] \\
= |T_M(u_n)(x) - T_M(u_n)(y)|^{p(x,y) - 2}[T_M(u_n)(x) - T_M(u_n)(y)] \\
\cdot [S(T_M(u_n))(x) - S(T_M(u_n))(y)].
\end{align*}
\]
Note that

\[ S(T_M(u_n))(x) - S(T_M(u_n))(y) = S'(\xi)[T_M(u_n)(x) - T_M(u_n)(y)], \]

where \( \xi \) is between \( T_M(u_n)(x) \) and \( T_M(u_n)(y) \). We could verify that

\[
\left\{ \frac{|S(T_M(u_n))(x) - S(T_M(u_n))(y)|}{|x - y|^{N + sp(x,y)}} \cdot \frac{\varphi(x) + \varphi(y)}{2} \cdot \chi_{B_{2,n}} \right\} _{n}
\]

is bounded in \( L^{p(x,y)}(\Omega \times \Omega) \). Besides, it follows from Lemma 3.3 that

\[
\frac{|T_M(u_n)(x) - T_M(u_n)(y)|^{p(x,y)} - 2|T_M(u_n)(x) - T_M(u_n)(y)|}{|x - y|^{(N + sp(x,y))\frac{p(x,y) - 1}{p(x,y)}}}
\]

\[
\to \frac{|T_M(u)(x) - T_M(u)(y)|^{p(x,y)} - 2|T_M(u)(x) - T_M(u)(y)|}{|x - y|^{(N + sp(x,y))\frac{p(x,y) - 1}{p(x,y)}}}
\]

strongly in \( L^{p(x,y)}(\Omega \times \Omega) \).

Then, we obtain

\[
\int_{B_{2,n}} G_n(x,y) \, dx \, dy
\]

\[
= \int_{\Omega \times \Omega} |T_M(u_n)(x) - T_M(u_n)(y)|^{p(x,y)} - 2|T_M(u_n)(x) - T_M(u_n)(y)|
\]

\[
\cdot \frac{\varphi(x) + \varphi(y)}{2} \chi_{B_{2,n}} \, d\nu
\]

\[
\to \int_{\Omega \times \Omega} |T_M(u)(x) - T_M(u)(y)|^{p(x,y)} - 2|T_M(u)(x) - T_M(u)(y)|
\]

\[
\cdot \frac{\varphi(x) + \varphi(y)}{2} \chi_{\{u(x) \leq M, u(y) \leq M\}} \, d\nu.
\]

(3) In \( B_{3,n} \), similar to the discussion of (2), we verify that

\[
\lim_{n \to \infty} \int_{B_{3,n}} G_n(x,y) \, dx \, dy = \int_{\{u \leq u(x) \leq M + 1, u \leq u(y) \leq M\}} G(x,y) \, dx \, dy.
\]

(4) In \( B_{4,n} \), we have

\[
\max\{u_n(x), u_n(y)\} \geq M + 1 \text{ and } \min\{u_n(x), u_n(y)\} \leq M,
\]

which implies \( (u_n(x), u_n(y)) \in R_M \). By \( 3.10 \), we conclude that

\[
\lim_{M \to \infty} \lim_{n \to \infty} \int_{B_{4,n}} |u_n(x) - u_n(y)|^{p(x,y) - 1} \, d\nu = 0.
\]

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Thus
\[ \lim_{M \to \infty} \lim_{n \to \infty} \int_{B_{4,n}} G_n(x, y) \, dx \, dy = 0. \]

Since
\[ \int_{B_{3,n}} G_n(x, y) \, dx \, dy = \int_{B_{5,n}} G_n(x, y) \, dx \, dy \]
and
\[ \int_{B_{4,n}} G_n(x, y) \, dx \, dy = \int_{B_{6,n}} G_n(x, y) \, dx \, dy, \]
we have
\[ I_1 = \left( \int_{B_{1,n}} + \int_{B_{2,n}} + 2 \int_{B_{3,n}} + 2 \int_{B_{4,n}} \right) G_n(x, y) \, dx \, dy. \]

It follows from (1)–(4) that
\[
\lim_{n \to \infty} I_1 = \lim_{M \to \infty} \lim_{n \to \infty} I_1 = \\
= \lim_{M \to \infty} \int_{\{u(x) \leq M, u(y) \leq M\}} G(x, y) \, dx \, dy \\
+ 2 \lim_{M \to \infty} \int_{\{M \leq u(x) \leq M+1, u(y) \leq M\}} G(x, y) \, dx \, dy \\
+ \lim_{M \to \infty} 2 \lim_{n \to \infty} \int_{B_{4,n}} G_n(x, y) \, dx \, dy \\
= \int_{\Omega \times \Omega} G(x, y) \, dx \, dy.
\]

Similarly, we could verify that
\[
I_2 \to \int_{\Omega \times \Omega} U(x, y)(\varphi(x) - \varphi(y)) \cdot \frac{S(u(x) + S(u)(y))}{2} \, d\nu.
\]

Besides,
\[ \int_{\Omega} f_n S(u_n) \varphi \, dx \to \int_{\Omega} f S(u) \varphi \, dx. \]

Thus by (3.11), we find
\[
\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} [(S(u)\varphi)(x) - (S(u)\varphi)(y)] \, dx \, dy = \int_{\Omega} f S(u) \varphi \, dx.
\]

Combining with Step 1 and Step 2, we verify that \( u \) is a renormalized solution to (1.1).

(ii) Uniqueness of renormalized solutions.
Now we prove the uniqueness of renormalized solutions to problem (1.1) by choosing an appropriate test function motivated by [4, 5, 18]. Let \( u \) and \( v \) be two renormalized solutions to problem (1.1). Fix a positive number \( k \). For \( \sigma > 0 \), let \( S_\sigma \) be the function defined by

\[
S_\sigma(r) = \begin{cases} 
  r & \text{if } |r| < \sigma, \\
  (\sigma + \frac{1}{2}) \mp \frac{1}{2} (r \mp (\sigma + 1))^2 & \text{if } \sigma \leq \pm r \leq \sigma + 1, \\
  \pm (\sigma + \frac{1}{2}) & \text{if } \pm r > \sigma + 1.
\end{cases}
\]  

(3.12)

It is obvious that

\[
S'_\sigma(r) = \begin{cases} 
  1 & \text{if } |r| < \sigma, \\
  \sigma + 1 - |r| & \text{if } \sigma \leq |r| \leq \sigma + 1, \\
  0 & \text{if } |r| > \sigma + 1.
\end{cases}
\]

It is easy to check \( S_\sigma \in W^{1,\infty}(\mathbb{R}) \) with \( \text{supp} S'_\sigma \subset [-\sigma - 1, \sigma + 1] \). Therefore, we may take \( S = S_\sigma \) in (1.4) to have

\[
\int_{\Omega \times \Omega} U(x, y)(\varphi(x) - \varphi(y)) \cdot \frac{S'_\sigma(u)(x) + S'_\sigma(v)(y)}{2} \, d\nu \\
+ \int_{\Omega \times \Omega} U(x, y)(S'_\sigma(u)(x) - S'_\sigma(v)(y)) \cdot \frac{\varphi(x) + \varphi(y)}{2} \, d\nu \\
= \int_{\Omega} f S'_\sigma(u) \varphi \, dx
\]

and

\[
\int_{\Omega \times \Omega} V(x, y)(\varphi(x) - \varphi(y)) \cdot \frac{S'_\sigma(v)(x) + S'_\sigma(u)(y)}{2} \, d\nu \\
+ \int_{\Omega \times \Omega} V(x, y)(S'_\sigma(v)(x) - S'_\sigma(u)(y)) \cdot \frac{\varphi(x) + \varphi(y)}{2} \, d\nu \\
= \int_{\Omega} f S'_\sigma(v) \varphi \, dx,
\]

where

\[ V(x, y) = |v(x) - v(y)|^{\rho(x,y) - 2}(v(x) - v(y)). \]

For every fixed \( k > 0 \), we plug \( \varphi = T_k(S_\sigma(u) - S_\sigma(v)) \) as a test function in the above equalities and subtract them to obtain that

\[ J_1 + J_2 = J_3, \]

(3.13)
where

\[
J_1 = \int_{\Omega \times \Omega} \left[ S'_\sigma(u)(x) + S'_\sigma(u)(y) U(x, y) - \frac{S'_\sigma(v)(x) + S'_\sigma(v)(y)}{2} V(x, y) \right]
\cdot \left[ T_k(S_\sigma(u) - S_\sigma(v))(x) - T_k(S_\sigma(u) - S_\sigma(v))(y) \right] d\nu,
\]

\[
J_2 = \int_{\Omega \times \Omega} \left[ U(x, y)(S'_\sigma(u)(x) - S'_\sigma(u)(y)) - V(x, y)(S'_\sigma(v)(x) - S'_\sigma(v)(y)) \right]
\cdot \frac{T_k(S_\sigma(u) - S_\sigma(v))(x) + T_k(S_\sigma(u) - S_\sigma(v))(y)}{2} d\nu,
\]

\[
J_3 = \int_{\Omega} f(S'_\sigma(u) - S'_\sigma(v)) T_k(S_\sigma(u) - S_\sigma(v)) dx.
\]

We estimate \( J_1, J_2 \) and \( J_3 \) one by one. Writing

\[
J_1 = \int_{\Omega \times \Omega} (U(x, y) - V(x, y)) \cdot \left[ T_k(S_\sigma(u) - S_\sigma(v))(x) - T_k(S_\sigma(u) - S_\sigma(v))(y) \right] d\nu
\]

\[
+ \int_{\Omega \times \Omega} \left( 1 - \frac{S'_\sigma(u)(x) + S'_\sigma(u)(y)}{2} \right) U(x, y)
\cdot \left[ T_k(S_\sigma(u) - S_\sigma(v))(x) - T_k(S_\sigma(u) - S_\sigma(v))(y) \right] d\nu
\]

\[
+ \int_{\Omega \times \Omega} \left( \frac{S'_\sigma(v)(x) + S'_\sigma(v)(y)}{2} - 1 \right) V(x, y)
\cdot \left[ T_k(S_\sigma(u) - S_\sigma(v))(x) - T_k(S_\sigma(u) - S_\sigma(v))(y) \right] d\nu
\]

\[
:= J_1^1 + J_1^2 + J_1^3,
\]

and setting \( \sigma \geq k \), we have

\[
J_1^1 \geq \int_{\{|u-v|\leq k\} \cap \{|u|,|v|\leq k\}} (U(x, y) - V(x, y))
\cdot \left[ (u(x) - v(x)) - (u(y) - v(y)) \right] d\nu.
\]  

(3.14)

By the Lebesgue dominated convergence theorem, we conclude that

\[
J_1^2, J_1^3 \to 0, \quad \text{as } \sigma \to +\infty.
\]
Furthermore, we have
\[ |J_2| \leq C \left( \int_{\{(u(x),u(y))\in R_\sigma\}} |u(x) - u(y)|^{p(x,y)-1} \, dv + \int_{\{(v(x),v(y))\in R_\sigma\}} |v(x) - v(y)|^{p(x,y)-1} \, dv \right). \]

From the above estimates and (i) in Definition 1.1, we obtain
\[ \lim_{\sigma \to +\infty} (|J_1^2| + |J_1^3| + |J_2|) = 0. \]

Observing
\[ f \left( S_\sigma'(u) - S_\sigma'(v) \right) \to 0 \quad \text{strongly in} \ L^1(\Omega) \]
as \( \sigma \to +\infty \) and using the Lebesgue dominated convergence theorem, we deduce that
\[ \lim_{\sigma \to +\infty} |J_3| = 0. \]

Therefore, sending \( \sigma \to +\infty \) in (3.13) and recalling (3.14), we have
\[ \int_{\{|u| \leq \frac{k}{2}, |v| \leq \frac{k}{2}\}} (U(x,y) - V(x,y)) \cdot [(u(x) - v(x)) - (u(y) - v(y))] \, dv = 0, \]
which implies \( u = v \) a.e. on the set \( \{|u| \leq \frac{k}{2}, |v| \leq \frac{k}{2}\} \). Since \( k \) is arbitrary, we conclude that \( u = v \) a.e. in \( \Omega_T \). This finishes the proof of Theorem 1.1. \( \square \)

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References

[1] B. Abdellaoui, A. Attar and R. Bentifour, On the fractional \( p \)-Laplacian equations with weight and general datum, Adv. Nonlinear Anal. (2016), https://doi.org/10.1515/anona-2016-0072.
[2] R. A. Adams, Sobolev spaces, Academic Press, New York, 1975.

[3] N. Alibaud, B. Andreianov and M. Bendahmane, Renormalized solutions of the fractional Laplace equation, C. R. Acad. Sci. Paris, Ser. I 348 (2010) 759–762.

[4] M. Bendahmane, P. Wittbold and A. Zimmermann, Renormalized solutions for a nonlinear parabolic equation with variable exponents and $L^1$ data, J. Differential Equations 249 (6) (2010) 1483–1515.

[5] D. Blanchard, F. Murat and H. Redwane, Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems, J. Differential Equations 177 (2) (2001) 331–374.

[6] K. Bogdan, K. Burdzy and Z. Chen, Censored stable processes, Probab. Theory Related Fields 127 (2003) 89–152.

[7] Z-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like process on $d$-Sets, Stochastic Process Appl. 108 (2003) 27–62.

[8] L. M. Del Pezzo and J. D. Rossi, Traces for fractional Sobolev spaces with variable exponents, arXiv:1704.02599.

[9] L. M. Del Pezzo and A. M. Salort, The first non-zero Neumann $p$-fractional eigenvalue, Nonlinear Anal. 118 (2015) 130–143.

[10] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, Vol. 2017, Springer-Verlag, Heidelberg, 2011.

[11] R. J. DiPerna and P. L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability, Ann. Math. 130 (1989) 321–366.

[12] Q. Guan, Integration by parts formula for regional fractional Laplacian, Comm. Math. Phys. 266 (2006) 289–329.
[13] Q. Guan and Z. Ma, Reflected symmetric $\alpha$-stable processes and regional fractional Laplacian, Probab. Theory Related Fields 134 (2006) 649–694.

[14] U. Kaufmann, J. D. Rossi and R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, http://mate.dm.uba.ar/~jrossi/krvP.pdf.

[15] K. H. Kenneth, F. Petitta and S. Ulusoy, A duality approach to the fractional Laplacian with measure data, Publ. Mat. 55 (1) (2011) 151–161.

[16] T. Kuusi, G. Mingione and Y. Sire, Nonlocal equations with measure data, Comm. Math. Phys. 337 (2015) 1317–1368.

[17] T. Leonori, I. Peral, A. Primo and F. Soria, Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations, Discrete Contin. Dyn. Syst. 35 (12) (2015) 6031–6068.

[18] C. Zhang and S. Zhou, Entropy and renormalized solutions for the $p(x)$-Laplacian equation with measure data, Bull. Aust. Math. Soc. 82 (3) (2010) 459–479.