Contribution to muon g-2 from the $\pi^0\gamma$ and $\eta\gamma$ intermediate states in the vacuum polarization

N.N. Achasov * and A.V. Kiselev †

Laboratory of Theoretical Physics, Sobolev Institute for Mathematics, Novosibirsk, 630090, Russia

(December 4, 2018)

Abstract

Using new experimental data, we have calculated the contribution to the anomalous magnetic moment of the muon from the $\pi^0\gamma$ and $\eta\gamma$ intermediate states in the vacuum polarization with high precision: $a_\mu(\pi^0\gamma) + a_\mu(\eta\gamma) = (54.7 \pm 1.5) \times 10^{-11}$. We have also found the small contribution from $e^+e^-\pi^0$, $e^+e^-\eta$ and $\mu^+\mu^-\pi^0$ intermediate states equal to $0.5 \times 10^{-11}$.

13.40.Em, 14.60.Ef, 13.65.+i

*achasov@math.nsc.ru
†kiselev@math.nsc.ru
New experimental data \[1\]-\[3\] allows to calculate contribution to the anomalous magnetic moment of the muon \(a_\mu \equiv \frac{g_\mu - 2}{2}\) from the \(\pi^0 \gamma\) and \(\eta \gamma\) intermediate states in the vacuum polarization with high precision. We have also found the contribution from \(e^+e^-\pi^0\), \(e^+e^-\eta\) and \(\mu^+\mu^-\pi^0\) intermediate states.

The contribution to \(a_\mu\) from the arbitrary intermediate state \(X\) (hadrons, hadrons+\(\gamma\), etc.) in the vacuum polarization can be obtained via the dispersion integral

\[
a_\mu = \left(\frac{\alpha m_\mu}{2\pi}\right)^2 \int \frac{ds}{s^2} K(s) R(s). \tag{1}\n\]

\[
R(s) \equiv \frac{\sigma(e^+e^- \rightarrow X)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}, \quad \sigma(e^+e^- \rightarrow \mu^+\mu^-) \equiv \frac{4\pi\alpha^2}{3s}.
\]

\[
K(s > 4m_\mu^2) = \frac{3s}{m_\mu^2} \left\{ x^2(1 - \frac{x^2}{2}) + (1 + x)^2(1 + \frac{1}{x^2}) \left[ \ln(1 + x) - x + \frac{x^2}{2} \right] + \frac{1 + x}{1 - x} x^2 \ln(x) \right\} = \]

\[
= \frac{3}{a^3} \left( 16(a - 2) \ln \frac{a}{4} - 2a(8 - a) - 8(a^2 - 8a + 8) \frac{\text{arctanh}(\sqrt{1 - a})}{\sqrt{1 - a}} \right),
\]

\[
x = \frac{1 - \sqrt{1 - \frac{4m_\mu^2}{s}}}{1 + \sqrt{1 - \frac{4m_\mu^2}{s}}}, \quad a = \frac{4m_\mu^2}{s}.
\]

\[
K(s < 4m_\mu^2) = \frac{3}{a^3} \left( 16(a - 2) \ln \frac{a}{4} - 2a(8 - a) - 8(a^2 - 8a + 8) \frac{\text{arctan}(\sqrt{a - 1})}{\sqrt{a - 1}} \right).
\]

Evaluating integral (1) with the trapezoidal rule for the experimental data from SND \[1,2\], see Fig. 1(a), we found the contribution of \(\pi^0 \gamma\):

\[
a_\mu(\pi^0 \gamma) = (46.2 \pm 6 \pm 1.3) \times 10^{-11}, \quad 600 \text{ MeV} < \sqrt{s} < 1039 \text{ MeV}. \tag{2}\n\]

The first error is statistical, the second is systematic. For the energy region \(\sqrt{s} < 600\) MeV we used theoretical formula for the cross-section:

\[
\sigma(e^+e^- \rightarrow \pi^0 \gamma) = \frac{8\alpha f^2}{3} \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \frac{1}{\left(1 - \frac{s}{m_\pi^2}\right)^2}, \tag{3}\n\]

where \(f^2 = \frac{\pi}{m_{\pi^0}} \Gamma_{\pi^0 \rightarrow \gamma\gamma} \approx 10^{-11}/\text{MeV}^2\) according to \[4\]. Eq. (3) has been written in the approximation

\[
\Gamma_\rho = \Gamma_\omega = 0, \quad m_\rho - m_\omega = 0. \tag{4}\n\]

The \(\gamma^* \rightarrow \pi^0 \gamma\) amplitude is normalized on the \(\pi^0 \rightarrow \gamma\gamma\) one at \(s = 0\). The result is

\[
a_\mu(\pi^0 \gamma) = 1.3 \times 10^{-11}, \quad \sqrt{s} < 600 \text{ MeV}. \tag{5}\n\]

Note that the region \(\sqrt{s} < 2m_\mu\) gives the negligible contribution \(2 \times 10^{-13}\).
We neglect the small errors dealing with the experimental error in the width $\Gamma_{\pi^0 \rightarrow \gamma \gamma}$ (7%) and the approximation (4) (1.5%).

The Eq. (3) agrees with the data in the energy region $\sqrt{s} < 700$ MeV, at higher energies the approximation (4) does not work carefully, see Fig. 1(b).

If we use the point-like model, as in [3], we will get Eq. (3) without factor $\left(1-\frac{s}{m^2}\right)^{-2}$. This formula predicts the contribution from low energies several times less than (5), see also Fig. 1(b).

Treating the data from CMD-2 [3] in the same way, we get contribution of $\eta\gamma$: 

$$a_\mu(\eta\gamma) = (7.1 \pm .2 \pm .3) \times 10^{-11}, \quad 720 \text{ MeV} < \sqrt{s} < 1040 \text{ MeV}. \quad (6)$$

According to the quark model (and the model of vector dominance also), the energy region $\sqrt{s} < 720$ MeV is dominated by the $\rho$-resonance, hence $\sigma(e^+e^- \rightarrow \eta\gamma) \cong \sigma(e^+e^- \rightarrow \rho \rightarrow \eta\gamma)$. So we change Eq. (3) according to this fact, take into account the $\rho$ width and get the small contribution:

$$a_\mu(\eta\gamma) = .1 \times 10^{-11}, \quad \sqrt{s} < 720 \text{ MeV}, \quad (7)$$

Summing (2), (5), (6) and (7), we can write

$$a_\mu(\pi^0\gamma) + a_\mu(\eta\gamma) = (54.7 \pm .6 \pm 1.4) \times 10^{-11}, \quad (8)$$

where statistical and systematic errors are separately added in quadrature. In Table 1 we present our results with statistical and systematic errors added in quadrature. Comparing Eq. (8) with the analogous calculation in [3] (see Table 1), one can see that our result is 27% more and the error is 2.5 times less. The contribution (8) accounts for 1.37 of the projected error of the E821 experiment at Brookhaven National Laboratory $(40 \times 10^{-11})$ or 36% of the reached accuracy $(150 \times 10^{-11})$.

We can also take into account the intermediate state $\pi^0e^+e^-$, using the obvious relation

$$\sigma(e^+e^- \rightarrow \pi^0e^+e^-, s) = \frac{2}{\pi} \int_{2m_e}^{\sqrt{s-m^2}} \frac{dm}{m^2} \Gamma_{\gamma* \rightarrow e^+e^-}(m) \sigma(e^+e^- \rightarrow \pi^0\gamma*, s, m), \quad (9)$$

where $m$ is the invariant mass of the $e^+e^-$ system, $\Gamma_{\gamma* \rightarrow e^+e^-}(m)=(1/2)\alpha_e m(1-\beta_e^2/3)$, $\beta_e=\sqrt{1-4m_e^2/m^2}$, $\sigma(e^+e^- \rightarrow \pi^0\gamma*, s, m) = (p(m)/p(0))^3 \sigma(e^+e^- \rightarrow \pi^0\gamma, s)$, $p(0) = (\sqrt{s}/2)(1-(m_{\pi^0}+m)^2/s)(1-(m_{\pi^0}-m)^2/s)$ is the momentum of $\gamma*$ in s.c.m.

In the same way we can calculate $a_\mu(\mu^+\mu^-\pi^0)$ and $a_\mu(e^+e^-\eta)$. The result is

$$a_\mu(e^+e^-\pi^0) + a_\mu(\mu^+\mu^-\pi^0) + a_\mu(e^+e^-\eta) = (.4 + .026 + .057) \times 10^{-11} = .5 \times 10^{-11}. \quad (10)$$

Note that if $m \gtrsim m_\rho$ we have the effect of the excitation of resonances in the reaction $e^+e^- \rightarrow \pi^0(\rho, \omega) \rightarrow \pi^0e^+e^-$. However this effect increases the final result (10) less than by 10% because of the factor $(p(m)/p(0))^3$, which suppresses the high $m$. So we ignore this correction. We also neglect $a_\mu(\mu^+\mu^-\eta) = 2 \times 10^{-14}$.

As it was noted in [3] and [7], it is necessary to take into account also

$$a_\mu(\text{hadrons} + \gamma, \text{rest}) = a_\mu(\pi^+\pi^-\gamma) + a_\mu(\pi^0\pi^0\gamma) + a_\mu(\text{hadrons} + \gamma, \text{ s} > 1.2 \text{ GeV}^2).$$

3
We take $a_\mu(\pi^+\pi^-\gamma) = (38.6\pm1.0) \times 10^{-11}$ from [4] (see also [3]), $a_\mu(\pi^0\pi^0\gamma) + a_\mu(hadrons + \gamma, s > 1.2 \text{ GeV}^2) = (4 \pm 1) \times 10^{-11}$ from [5]. Adding this to (8), we get

$$a_\mu(hadrons + \gamma, \text{ total}) = (97.3 \pm 2.1) \times 10^{-11}.$$  \hspace{1cm} (11)

The contribution (11) accounts for 2.43 of the projected error of the E821 experiment or 65% of the reached accuracy.

In fact, the errors in (8) and (11) are negligible for any imaginable $(g-2)_\mu$ measurement in near future.
REFERENCES

[1] M.N.Achasov et all. Preprint Budker INP 2001-54, Novosibirsk, 2001 (in Russian).
[2] M.N.Achasov et all. Eur. Phys. J. C12, 25-33 (2000).
[3] R.R.Akhmetshin et al. Phys.Lett. B509 (2001) 217-226, hep-ex/0103043.
[4] Particle Data Group: D. E. Groom et al. Eur. Phys. J. C15, 1 (2000).
[5] J.F. Troconiz, F.J. Yndurain, FTUAM-01-08, hep-ph/0106023.
[6] H. N. Brown et al., Phys. Rev. Letters, 86, 2227 (2001).
[7] A.Hoefer, J. Gluza, F. Jegerlehner, hep-ph/0107154.

Table 1. Contribution to $a_\mu \times 10^{11}$

| State          | Our value | Ref. 5 |
|----------------|-----------|--------|
| $\pi^0\gamma$ | $47.5 \pm 1.4$ | $37 \pm 3$ |
| $\eta\gamma$  | $7.2 \pm 0.4$  | $6.1 \pm 1.4$ |
| $\pi^0\gamma + \eta\gamma$ | $54.7 \pm 1.5$ | $43 \pm 4$ |
| hadrons+$\gamma$, total | $97.3 \pm 2.1$ | $93 \pm 11$ |
 FIG. 1.  a) Plot of the dependence $\sigma(e^+e^- \rightarrow \pi^0\gamma)$, nb upon $\sqrt{s}$, MeV (SND experimental data).  b) Comparison of the theoretical formulas for $\sigma(e^+e^- \rightarrow \pi^0\gamma)$. Eq. (3) is shown with the solid line, point-like model prediction is shown with the dashed line.