A CHARACTERIZATION OF THE DIFFERENTIAL IN SEMI-INFINITE COHOMOLOGY

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INTRODUCTION

Semi-infinite cohomology is by now firmly established as a means of describing certain aspects of string theory in modern mathematical physics [5]. For mathematicians, the first rigorous formulation was given by Feigin [1], and further clarification and explicit calculations came with [3]. Several articles ([8],...) have appeared on the subject (also studied as BRST cohomology), and yet so far, existence arguments for the differential have been given solely via formulas, or “by analogy”. Moreover, one has to show that the square of the differential is zero each time, the length of the proof depending on the complexity of the author’s favorite module and on the square of the width of the formula. Hence, complete proofs practically never get published and semi-infinite cohomology remains exotic
and inaccessible to most mathematicians.

The main goal of this article is to characterize a universal differential, \( d \), which will encode the properties of a given \( \mathbb{Z} \)-graded Lie algebra

\[
\mathcal{G} = \oplus_{n \in \mathbb{Z}} \mathcal{G}_n
\]

with \( \dim \mathcal{G}_n < \infty \) for all \( n \); satisfy the minimal properties that any self-respecting differential should possess; and induce the cohomology operator for a very large class of (projective) representations of \( \mathcal{G} \). The operators of semi-infinite cohomology theory will be realized as inner derivations of an associative algebra \( Y^\infty \mathcal{G} \), which is (almost) the completed universal enveloping algebra of the Lie superalgebra

\[
\tilde{s}(\mathcal{G}) = \pi(\mathcal{G}) \oplus \iota(\mathcal{G}) \oplus \epsilon(\mathcal{G}') \oplus K k
\]

where \( \tilde{G} \) is a certain central extension of \( \mathcal{G} \), \( \mathcal{G}' \) is the restricted dual of \( \mathcal{G} \), and \( \iota(\mathcal{G}) \oplus \epsilon(\mathcal{G}') \oplus Kk \) \((K \text{ fixed field of characteristic zero, } k \text{ central})\) is the super Heisenberg Lie algebra based on the underlying vector space of \( \mathcal{G} \). The two gradings \( * \)-deg= superdegree and \( \text{deg} \)= secondary degree on \( \tilde{s}(\mathcal{G}) \), coming from \( * \)-deg \( \iota(x) = -1 \) and \( * \)-deg \( \epsilon(x') = 1 \) and the grading on \( \mathcal{G} \) respectively, will induce gradings on \( Y^\infty \mathcal{G} \) and its derivations by the Poincaré-Birkhoff-Witt Theorem (PBW). One group of such derivations will be \( \{\theta(x)\}_{x \in \mathcal{G}} \), corresponding to elements \( \{\theta(x)\}_{x \in \mathcal{G}} \) of \( Y^\infty \mathcal{G} \), which constitute a natural representation of \( \mathcal{G} \) on \( Y^\infty \mathcal{G} \). All these inner derivations will now act on \( Y^\infty \mathcal{G} \) modules of the form

\[
M \otimes \wedge^\infty/2+* \mathcal{G}',
\]

\( M \) a smooth \( \tilde{G} \) module, \( \wedge^\infty/2+* \mathcal{G}' \) the module of semi-infinite forms on \( \mathcal{G} \). Modulo the precise definitions in the following sections and the choice of a special homogeneous basis \( \{e_i\}_{i \in \mathbb{Z}} \) for \( \mathcal{G} \), the main result can now be stated as

**THEOREM.** (a) There exists a unique superderivation \( d \) of \( Y^\infty \mathcal{G} \) with \( * \)-deg 1, \( \text{deg} \) 0, satisfying

\[
di(x) = \theta(x) \quad \forall x \in \mathcal{G}.
\]

(b) Moreover,

\[
d\theta(x) = 0 \quad \forall x \in \mathcal{G}
\]
and
\[ d^2 = 0. \]

(c) There is a unique element \( d \) of the same degrees in \( Y^\infty \mathcal{G} \), namely
\[
d = \sum_i \pi(e_i)e(e_i') + \sum_{i<j} :u([e_i,e_j])e(e_j')e(e_i'):,
\]
for which
\[ du = [d,u], \quad u \in Y^\infty \mathcal{G}, \]
and \( d^2 = 0. \)

(d) Hence, if \( M \) is any smooth \( \tilde{\mathcal{G}} \) module on which the central element acts by 1, then \( d \) induces a well-defined differential on the \( Y^\infty \mathcal{G} \) module
\[ M \otimes \wedge^{\infty} (\mathcal{G}', \mathcal{G}). \]
The cohomology \( H^\bullet(M \otimes \wedge^{\infty} (\mathcal{G}', \mathcal{G}), d) \) is by definition the semi-infinite cohomology of \( \mathcal{G} \) with coefficients in \( M \).

We will show the existence, uniqueness, and nilpotency of a canonical differential \( d \) once and for all, starting with the only piece of data we need, namely \( \mathcal{G} \), and using only the language of derivations. In the end, we will obtain the familiar but generic and - algebraically speaking- well motivated formula that establishes \( d \) as an inner derivation.

Note that by the Theorem (and by the definition of completion) \( d \) has to yield the classical differential in case of an ungraded finite dimensional Lie algebra \( \mathcal{G} \). Hopefully some physics terminology will be demystified in the process.

PRELIMINARIES

Unlike the classical theory, derivations of associative and Lie algebras have never been utilized systematically in semi-infinite cohomology. If \( \mathcal{A}, \mathcal{B} \) are (super) \( \mathbf{Z} \)-graded associative
or Lie algebras, with a degree-preserving algebra map \( \phi : \mathcal{A} \rightarrow \mathcal{B} \), a \textit{superderivation} \( D : \mathcal{A} \rightarrow \mathcal{B} \) of \( \ast\)-deg \( N \) with respect to \( \phi \) is a linear map of \( \ast\)-deg \( N \) satisfying

\[
D(u \cdot v) = (Du) \cdot (\phi v) + (-1)^{N(\ast \text{deg } u)} (\phi u) \cdot (Dv)
\]

for \( u, v \in \mathcal{A} \), \( u \) homogeneous. In case of secondary \( \mathbb{Z} \)-gradings also preserved by \( \phi \), we will still talk about a superderivation of \( \ast\)-deg \( M \) and (secondary) deg \( N \) even though this new grading may not be compatible with the “super” structure. Note that any associative algebra derivation is a Lie algebra derivation in a natural way. The converse is not true.

A reliable way of constructing a derivation is extending another derivation, or just a linear map, to an associative algebra generated in some way by the original space. For example, any linear map \( f : E \rightarrow \mathcal{A} \) into an associative algebra extends to an algebra derivation \( f : T\mathcal{E} \rightarrow \mathcal{A} \) from the tensor algebra, if we first define \( p \)-linear maps

\[
E \times \cdots \times E \rightarrow \mathcal{A}
\]

with

\[
(u_1, \ldots, u_p) \mapsto \sum_{k=1}^{p} \phi(u_1) \cdots f(u_k) \cdots \phi(u_p).
\]

Here \( \phi : E \rightarrow \mathcal{A} \) is a given linear map and \( \phi : T\mathcal{E} \rightarrow \mathcal{A} \) the corresponding algebra map. Using this prototype as an intermediate step, a Lie algebra derivation \( f : \mathfrak{l} \rightarrow \mathcal{A} \) into an associative algebra can be extended to an associative algebra derivation \( f : \mathcal{U}\mathfrak{l} \rightarrow \mathcal{A} \) from the universal enveloping algebra (again \( \phi : \mathfrak{l} \rightarrow \mathcal{A} \) given Lie algebra map, with extension \( \phi : \mathcal{U}\mathfrak{l} \rightarrow \mathcal{A} \) by the universal property of \( \mathcal{U}\mathfrak{l} \)) : The derivation \( f : T\mathfrak{l} \rightarrow \mathcal{A} \) factors through \( T\mathfrak{l} \rightarrow \mathcal{U}\mathfrak{l} = T\mathfrak{l}/\text{ideal} \) since elements of the form

\[
u \otimes (x \otimes y - y \otimes x - [x, y]) \otimes v = u \otimes ([x, y]_{T\mathfrak{l}} - [x, y]_{\mathfrak{l}}) \otimes v
\]

with \( x, y \in \mathfrak{l}, u, v \in T\mathfrak{l} \) are annihilated. The result also holds for super Lie algebras, so we immediately get extensions of linear maps \( E \rightarrow \mathcal{A} \) to associative algebra derivations \( SE \rightarrow \mathcal{A} \) and \( \wedge E \rightarrow \mathcal{A} \) (here \( SE, \wedge E \) are the symmetric and exterior algebras) as special cases. Once we define completions of certain enveloping algebras, our Lie derivations will extend uniquely to the completion and we will not distinguish between the generating map and the full one by virtue of this discussion. Derivations form a super Lie algebra.
themselves under the obvious superbracket. The map $\phi$ will be an inclusion most of the time, and we will sometimes omit the prefix $\text{super}$.

Let us now define the completion of a tame $\mathbb{Z}$-graded (super) Lie algebra $\mathfrak{L}$ with a secondary $\mathbb{Z}$-grading

$$\mathfrak{L} = \oplus_n \mathfrak{L}_n.$$ 

“Tame” means each $\mathfrak{L}_n$ is finite dimensional (the terminology is Zuckerman’s). Let

$$\mathfrak{L}_{\leq 0} = \oplus_{n \leq 0} \mathfrak{L}_n$$

and

$$\mathfrak{L}_+ = \oplus_{n > 0} \mathfrak{L}_n.$$

By PBW their universal enveloping algebras are related by

$$\mathcal{U}\mathfrak{L} \cong \mathcal{U}\mathfrak{L}_{\leq 0} \otimes_{\mathbb{K}} \mathcal{U}\mathfrak{L}_+$$

as vector spaces. The grading in $\mathcal{U}\mathfrak{L}_+$ induces a subsidiary grading which we denote by sdeg, as opposed to deg, which comes from the total (secondary) grading on $\mathfrak{L}$. A typical homogeneous element of $\mathcal{U}\mathfrak{L}$ is of the form

$$\sum_{k=1}^{r} u_k v_k$$

with $u_k \in \mathcal{U}\mathfrak{L}_{\leq 0}, v_k \in \mathcal{U}\mathfrak{L}_+$, and $\deg(u_k v_k) = \deg(u_l v_l) \forall k, l$. If infinite expressions

$$\sum_{k=1}^{\infty} u_k v_k$$

with $\deg v_k = \text{sdeg}(u_k v_k) \to \infty$ as $k \to \infty$ are also allowed, one gets a canonical completion $\mathcal{U}^{\infty}\mathfrak{L}$ of $\mathcal{U}\mathfrak{L}$ with respect to $\deg [10]$. Completions of universal enveloping algebras have appeared in the work of Kac [6] and Zhelobenko [9], for example, but the construction is somewhat different and the grading used is the so-called primary grading for Kac-Moody algebras. Probably the object closest to this one in literature is the “universal enveloping algebra of a vertex operator algebra” in the Frenkel-Zhu paper [4], where infinite sums are built out of modes of vertex operators. In general, it can be shown that products exist and are exactly what we think they ought to be [10]. $\mathcal{U}^{\infty}\mathfrak{L}$ is a topological associative algebra
with $U\ell$ as a dense subalgebra, and both the multiplication and the induced Lie bracket are continuous. Note that

$$U^\infty \ell = \oplus_n (U\ell_n)^\infty$$

by definition, that is, each homogeneous piece is completed separately.

$U^\infty \ell$ provides a context for the puzzling infinite sums of mathematical physics in terms of elements of $\ell$. The normal ordering $:\cdot\cdot$ that appears in most sums merely (another algebraist’s belittlement) ensures the validity of such expressions as elements of the completed algebra. What we really want to do with them is to take their supercommutators, and $:\cdot\cdot$ is superfluous once inside the bracket. Since $U^\infty \ell$ acts on every smooth $\ell$ module (a $\mathbb{Z}$-graded $\ell$ module where each $U\ell_+$ orbit is finite dimensional) by its very definition, Lie algebra maps

$$\mathcal{G} \to U^\infty \mathcal{H},$$

which abound in physics ($\mathcal{G} =$ constraint algebra, $\mathcal{H} =$ current algebra), are used to generate an action of $\mathcal{G}$ on smooth $\mathcal{H}$ modules. Such a map always extends to

$$U^\infty \mathcal{G} \to U^\infty \mathcal{H}$$

by abstract nonsense. More about this later.

At this point we reluctantly fix a homogeneous basis $\{e_i\}_{i \in \mathbb{Z}}$ for $\mathcal{G}$ such that whenever $e_i \in \mathcal{G}_n$, either $e_{i+1} \in \mathcal{G}_n$ or $e_{i+1} \in \mathcal{G}_{n+1}$, with the ultimate purpose of generating formulas. What precedes that will be basis independent. Following the notation of [1], [3], and [10], let

$$\mathcal{G}' = \oplus_n \mathcal{G}'_n$$

be the restricted dual of $\mathcal{G}$, with

$$\mathcal{G}'_n = \text{Hom}_K(\mathcal{G}_{-n}, K).$$

Then $\{e_i\}$ is the dual basis in $\mathcal{G}'$. Let

$$s = s(\mathcal{G}) = \mathcal{G} \oplus \mathcal{G}' \oplus Kk$$

be the super Heisenberg Lie algebra with center $Kk$ in which elements of $\mathcal{G}$ and $\mathcal{G}'$ will be distinguished by the additional symbols $\iota$ and $\epsilon$, emphasizing that $\mathcal{G}$ and $\mathcal{G}'$ are now the
adjoint and coadjoint representations. The ∗-degrees for elements of $\mathcal{G}$, $\mathcal{G}^{-}$, and $Kk$ are $-1$, $1$, and $0$ respectively, and both commutators and anticommutators will be denoted by the superbracket $[\ ,
abla]$. Thus

$$[\iota(x), \iota(y)] = [\epsilon(x\prime), \epsilon(y\prime)] = 0$$

and

$$[\epsilon(x\prime), \iota(y)] = < x\prime, y > k$$

for all $x, y \in \mathcal{G}$, $x\prime, x\prime \in \mathcal{G}^{-}$. A secondary $\mathbb{Z}$-grading is given by

$$\deg \iota(x) = n, \quad x \in \mathcal{G}_n, \quad \deg(x\prime) = -n, \quad x\prime \in \mathcal{G}^{-}_n \quad \deg k = 0.$$ We will use the same letter to denote dual elements, e.g. $\iota(x)$ and $\epsilon(x\prime)$.

In the abstract bigraded algebra $s$, the two compatible gradings have different uses: The super identities

$$[u, v] + (-1)^{\ast\deg u}(\ast\deg v)[v, u] = 0$$

and

$$(-1)^{\ast\deg u}(\ast\deg w)[u, [v, w]]$$

$$+(-1)^{\ast\deg v}(\ast\deg u)[v, [w, u]]$$

$$+(-1)^{\ast\deg w}(\ast\deg v)[w, [u, v]] = 0$$

hold for ∗-deg, and we complete $Us$ with respect to deg. Let

$$C^\infty \mathcal{G} = Us/(k = 1),$$

the completed Clifford algebra on $\mathcal{G}$. Then by PBW

$$C^\infty \mathcal{G} \cong (\wedge \mathcal{G} \otimes \mathcal{G}^{-})^\infty$$

as vector spaces. (Another approach would be to define $C^\infty \mathcal{G}$ to be $Us[1/k]$ and modify all formulas by a factor of $1/k$.)

$\wedge^\infty /2^{+\ast} \mathcal{G}^{-}$ is the $\mathcal{C} \mathcal{G} = Us(\mathcal{G})/(k = 1)$ module induced from the one dimensional trivial representation of the subalgebra

$$\wedge(\mathcal{G}_{\geq 0} \oplus \mathcal{G}_{\leq})$$
whose elements are called annihilation operators. This module was first realized as the span of the semi-infinite forms

\[ e'_{i_1} \wedge e'_{i_2} \wedge \cdots \wedge e'_{i_k} \cdots \]

where \( i_1 > i_2 > \cdots \) and \( i_{k+1} = i_k - 1 \) from some point on. The completed algebra also acts on \( \wedge^{\infty/2+\ast}G' \).

The following results, together with Lemma 3, an extension of the second one, essentially form the proof of the Theorem. As an unexpected bonus, they help one guess and verify formulas for derivations, as well as take commutators of normal ordered expressions easily.

**Lemma 1.** The center of \( C^\infty G \) is \( K \).

**Proof.** Any PBW monomial in \( C^\infty G \) is uniquely determined by its commutators with all the \( \iota(x) \) and \( \epsilon(y') \): If there exists an \( \epsilon(e'_i) \) in the monomial, \([\iota(e_i), \_] \) replaces it with \( \pm 1 \). Otherwise it kills the monomial. Similarly \([\epsilon(e'_i), \_] \) replaces an existing \( \iota(e_i) \) by \( \pm 1 \) or else kills the monomial. A central element commutes with all \( \iota(x) \) and \( \epsilon(y') \), hence cannot contain a nonconstant monomial as a summand. So \( \text{Center } C^\infty G = K \), of \( \ast \)-deg and deg 0.

**Lemma 2.** Let \( D : s \to C^\infty G \) (equivalently, \( C^\infty G \to C^\infty G \)) be a superderivation of \( \ast \)-degree \( N \geq 0 \). Then \( D \) is determined by its values on \( \iota(G) \oplus Kk \subset s(G) \).

**Proof.** A linear map \( D : s \to C^\infty G \) of \( \ast \)-degree \( N \) is a derivation iff

\[ D[u, v]_s = [Du, v]_{C^\infty G} + (-1)^{N(\ast \text{deg } u)}[u, Dv]_{C^\infty G} \]

for all \( u, v \in s \), \( u \) homogeneous. If \( Dk = 0 \), this is reduced to

\[ [Du, v] = (-1)^{N(\ast \text{deg } u)+1}[u, Dv]. \]

Let \( D_1, D_2 \) be derivations of \( \ast \)-deg \( N \) such that \( D_1 = D_2 \) on \( G \oplus K \). Then for \( D = D_1 - D_2 \), we have

\[ [De(x'), \iota(y)] = (-1)^{-N+1}[e(x'), Du(y)] = 0 \quad \forall x' \in G', y \in G. \]
This shows that $D\epsilon(x')$, which is of $\ast$-deg $N+1 \geq 0$, does not have any $\epsilon$’s. Constants are also excluded by degree considerations so $D\epsilon(x') = 0$. Therefore $D_1 = D_2$ on $s$.

Remarks. (1) A similar result holds for $\ast$-degree $N \leq 0$ and $\mathcal{G}' \oplus Kk$.

(2) Since $C^\infty \mathcal{G}$ is generated by $s$, the extension of any derivation of $s$ into $C^\infty \mathcal{G}$ carries the center to the center. Hence any derivation of nonzero $\ast$-degree is trivial on $Kk$.

The new techniques can best be demonstrated by constructing an action of $\mathcal{G}$ (rather, of a central extension) on $C^\infty \mathcal{G}$ by inner derivations. The action

$$\iota(y) \mapsto \iota(ad(x) \cdot y), \quad \epsilon(y') \mapsto \epsilon(ad'(x) \cdot y'), \quad 1 \mapsto 0$$

of $x \in \mathcal{G}$ in $s$ extends to a derivation of $C^\infty \mathcal{G}$ by previous arguments. If we can produce an element, say $\underline{\rho}(x)$, such that $\rho(x) = [\underline{\rho}(x), \ ]$ has the correct degrees and agrees with the action of $x$ on $\iota(\mathcal{G}) \oplus Kk$, then the extension is exactly $\rho(x)$ by Lemma 2. (We will consistently underline elements corresponding to inner derivations.) Our candidate is

$$\underline{\rho}(x) = \sum_{i \in \mathbb{Z}} :\iota(ad(x) \cdot e_i)\epsilon(e'_i) : \in C^\infty \mathcal{G}$$

(the formula looks slightly different than the traditional one, say in [3], but this order is more natural and leads directly to the correct formula for $d$). The normal ordering $\ : \ : \ : \ : \ :, \ $ means a factor with positive degree should appear on the right, and if we have to change the given order, we had better multiply that term by $-1$. If we take the commutator of $\underline{\rho}(x)$ with an element of $s$, we will be allowed to do it term by term by continuity, and the central term arising from the possible change of order will vanish. The sum over $\mathbb{Z}$ is shorthand for two different sums over nonnegative integers. Also recall that

$$\deg \epsilon(e'_i) = \deg e'_i = -\deg e_i$$

and

$$\deg \iota(ad(x) \cdot e_i) = \deg[x, e_i] = \deg x + \deg e_i,$$

so that $\underline{\rho}(x)$ is a valid expression. From now on we will write such sums with a clear conscience and not offer explanations. We immediately check that

$$[\rho(x), \iota(y)] = \iota(ad(x) \cdot y) \quad x, y \in \mathcal{G},$$

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hence \( \rho(x) \) and the action of \( x \) are the same derivation by Lemma 2. Here is part of the computation:

\[
\begin{align*}
\{} : \iota(ad(x) \cdot e_i)\epsilon(e'_i) : , \iota(e_j) \} &= \{ \iota(ad(x) \cdot e_i)\epsilon(e'_i), \iota(e_j) \} \\
&= \iota(ad(x) \cdot e_i)\epsilon(e'_i), \iota(e_j) \\
&= \iota(ad(x) \cdot e_i)\delta_{ij}.
\end{align*}
\]

Of course, addition of central terms to \( \underline{\rho}(x) \) will not change \( \rho(x) \), but we always impose the condition that \( \deg \underline{\rho}(x) \) be the same as that of \( x \). By Lemma 1, \( \rho(x) \) is uniquely represented by \( \underline{\rho}(x) \) as long as \( \deg x \neq 0 \).

The span \( \{ \rho(x) \} \subset \text{Der}C^\infty G \) closes as a Lie algebra, but \( \{ \underline{\rho}(x) \} \subset C^\infty G \) does not in general. We have

\[
[[\rho(x), \rho(y)], \iota(z)] = [\rho([x, y]), \iota(z)] = \iota([[x, y], z])
\]

for all \( x, y, z \in G \). For reasons to become clear later we would like to have

\[
\gamma(x, y) = \text{def} \ [\underline{\rho}(x), \underline{\rho}(y)] - \underline{\rho}([x, y]) = 0
\]

for \( x, y \in G \). It is easy to see that \( \gamma \) is a cocycle in the classical sense, i.e.

\[
\gamma(x, [y, z]) + \gamma(y, [z, x]) + \gamma(z, [x, y]) = 0.
\]

The calculation makes use of the Jacobi identity as well as the \(*\)-deg of \( \underline{\rho}(x) \) (zero). We can manage to have \( \gamma \equiv 0 \) when \( H^2G = 0 \) (e.g. \( G = \text{Virasoro} \), or \( G = \ell \otimes K[t, t^{-1}] \oplus Kc \), affine Kac-Moody algebra) if we redefine \( \underline{\rho} \) by

\[
\underline{\rho}(x) = \sum_i : \iota(ad(x) \cdot e_i)\epsilon(e'_i) : + \beta(x),
\]

where \( \beta \) is a special element of \( G'_0 \) ([3], [7] Chapter 7). On the other hand, one can choose to leave \( \underline{\rho}(x) \) as it is, for example when \( G = \text{Witt} = \text{Der} K[t, t^{-1}] \) or \( G = \ell \otimes K[t, t^{-1}] \) might be more appropriate than their famous central extensions. We will adopt the second method and make up for the inconvenience later.

On a final note, one can always pretend that the formula for \( \underline{\rho}(x) \) is inspired by Lemma 2, where \( \epsilon(e'_i) \) deletes \( \iota(e_i) \), to be replaced by \( \iota([x, e_i]) \).
FIRST STEP

We will start by characterizing the part of $d$ that acts on $C^\infty G$, namely $d_0$. It will be the unique derivation of $C^\infty G$ of $*-\deg 1$, $\deg 0$ mapping $\iota(x)$ to $\rho(x)$. It suffices to define $d_0$ on $s(G)$, i.e. we have to figure out the image of $\epsilon(x')$. This differential only has to satisfy

$$[d_0\iota(x), \iota(y)] = [\iota(x), d_0\iota(y)] \quad (1)$$
$$[d_0\epsilon(x'), \epsilon(y')] = [\epsilon(x'), d_0\epsilon(y')] \quad (2)$$
$$[d_0\epsilon(x'), \iota(y)] = [\epsilon(x'), d_0\iota(y)] \quad (3)$$

for all $x, y \in G, x', y' \in G'$. Condition (1) is satisfied since $d_0\iota(x) = \rho(x)$. Next, we assume

$$[d_0\epsilon(x'), \epsilon(y')] = 0 \quad \forall x', y' \in G' \quad (4)$$

and hope that this works. After all, $d_0\epsilon(x')$ is of $*-\deg 2$ and every term has two more $\epsilon$'s than $\iota$'s. (4) merely says that there are exactly two $\epsilon$'s and no $\iota$'s. So

$$d_0\epsilon(x') = \sum_{\deg e_i + \deg e_j = \deg x} \lambda_{ij} : \epsilon(e'_i)\epsilon(e'_j) : , \quad \lambda_{ij} \in K,$$

which can be completely determined if all $[d_0\epsilon(x'), \iota(y)]$ are known. But then we simply define

$$[d_0\epsilon(x'), \iota(y)] = [\epsilon(x'), d_0\iota(y)]$$
$$= [\epsilon(x'), \rho(y)]$$
$$= -\epsilon(ad'(y) \cdot x')$$

via condition (3). Incidentally, this gives us the formula

$$d_0\epsilon(x') = -1/2 \sum_i : \epsilon(ad'(e_i) \cdot x')\epsilon(e'_i) : .$$

The factor $1/2$ is there because there are two $\epsilon$'s for an $\iota$ to cross. We have

$$-1/2 \sum_i : \epsilon(ad'(e_i) \cdot x')\epsilon(e'_i) :, \iota(e_j)] = -\epsilon(ad'(e_j) \cdot x')$$

as the result of some mild linear algebra, which verifies our guess.
Now $d^2_0 = 1/2 [d_0, d_0]$ is a superderivation of $C^\infty \mathcal{G}$ of $*\text{-deg } 2$, deg 0, and is determined by its values on $\mathcal{G}$. Then

$$d^2_0 = 0 \iff d^2_0 \iota(x) = 0 \quad \forall x \in \mathcal{G} \iff d_0 \rho(x) = 0 \quad \forall x \in \mathcal{G},$$

but $d_0 \rho(x)$ is in turn an element of $*\text{-deg } 1$ and again Lemma 2 applies:

$$[d_0 \rho(x), \iota(y)] = d_0 [\rho(x), \iota(y)] - [\rho(x), d_0 \iota(y)]$$

$$= d_0 \iota([x, y]) - [\rho(x), \rho(y)]$$

$$= \rho([x, y]) - [\rho(x), \rho(y)]$$

$$= - \gamma(x, y).$$

So $d^2_0 = 0$ is equivalent to the closure of $\{\rho(x)\}_{x \in \mathcal{G}}$, which was remarked upon earlier. This approach makes it unnecessary to square giant sums.

A formula for $d_0$? Why not. We find an inner derivation of $C^\infty \mathcal{G}$ of $*\text{-deg } 1$ and deg 0, which is the same as $d_0$ on $\mathcal{G}$. Let

$$d_0 = 1/2 \sum_i : \rho(e_i) e(e'_i) :$$

$$= 1/2 \sum_{i \neq j} : \iota([e_i, e_j]) e(e'_j) e(e'_i) :$$

$$= \sum_{i < j} : \iota([e_i, e_j]) e(e'_j) e(e'_i) :$$

(add $e(\beta)$ if $\rho(x)$ is modified by $\beta(x)$ for deg $x = 0$).

Then

$$[d_0, \iota(e_r)] = 1/2 \sum_{i \neq j} : \iota([e_i, e_j]) e(e'_j) e(e'_i), \iota(e_r) :$$

$$= 1/2 \sum_{i \neq j} : \iota([e_i, e_j]) \{\delta_{ir} e(e'_j) - \delta_{jr} e(e'_i)\} :$$

$$= 1/2 \{\sum_j : \iota([e_r, e_j]) e(e'_j) : - \sum_i : \iota([e_i, e_r]) e(e'_i) :\}$$

$$= 1/2 \{\rho(e_r) + \rho(e_r)\} = \rho(e_r).$$

**FINAL TOUCHES**
The cocycle $-\gamma(x, y)$ determines a central extension of $\mathcal{G}$, say $\tilde{\mathcal{G}}$. We extend the chosen basis of $\mathcal{G}$ to $\{e_i\} \cup \{c\}$, but now

$$[e_i, e_j]_{\tilde{\mathcal{G}}} = [e_i, e_j]_{\mathcal{G}} - \gamma(e_i, e_j)c.$$ 

Form the Lie superalgebra $\tilde{s}$ as the direct sum

$$\tilde{s} = \tilde{s}(\mathcal{G}) = \tilde{\mathcal{G}} \oplus s(\mathcal{G})$$

where $\tilde{\mathcal{G}}$ is of $*$-deg 0 and the rest is as before. Elements of $\tilde{\mathcal{G}}$ will be denoted by $\pi(x)$ and will have secondary grading induced from $\mathcal{G}$, with $\deg \pi(c) = 0$. $\tilde{\mathcal{G}}$ will be “itself” in $\tilde{s}$ unlike $\mathcal{G} \subset s(\mathcal{G})$, that is,

$$[\pi(x), \pi(y)]_{\tilde{s}} = \pi([x, y]_{\mathcal{G}}).$$

Define

$$Y^\infty \mathcal{G} = \mathcal{U}^\infty \tilde{s}(\mathcal{G})/(k = 1, \pi(c) = 1)$$

which is isomorphic to

$$(\mathcal{U}\tilde{\mathcal{G}} \otimes \mathcal{G} \otimes \mathcal{G}')^\infty$$

by PBW. We will consider derivations $D : \tilde{s} \to Y^\infty \mathcal{G}$ and imitate the former construction.

**Lemma 3.** Let $D : \tilde{s} \to Y^\infty \mathcal{G}$ be a superderivation of $*$-deg $N \geq 1$ (in particular $k \mapsto 0$). Then $D$ is determined by its values on $\mathcal{G}$ only.

**Proof.** Let $D_1 = D_2$ on $\mathcal{G}$, and $D = D_1 - D_2$. We want to show

$$D\epsilon(x') = 0, \quad D\pi(x) = 0 \quad \forall x \in \mathcal{G}, x' \in \mathcal{G}'.$$ 

Again,

$$[D\epsilon(x'), \iota(y)] = \pm[\epsilon(x'), D\iota(y)] = 0.$$ 

But $D\epsilon(x')$ has at least two $\epsilon$’s in each term, so $D\epsilon(x') = 0$. Similarly, from $[\pi(x), \iota(y)] = 0$, we get

$$[D\pi(x), \iota(y)] = -[\pi(x), D\iota(y)] = 0,$$

and $D\pi(x)$, with $*$-deg at least 1, has no $\epsilon$’s. Then $D\pi(x) = 0$ and $D_1 = D_2$. 

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Define
\[\theta(x) = \pi(x) + \rho(x) \quad \forall x \in \mathcal{G},\]
where \(\rho(x)\) has no \(\beta(x)\) term. This time
\[
[\theta(x), \theta(y)] = [\pi(x), \pi(y)] + [\rho(x), \rho(y)]
\]
\[= \pi([x, y]_{\mathcal{G}}) + \rho([x, y]) + \gamma(x, y) \]
\[= \pi([x, y]_{\mathcal{G}}) - \gamma(x, y)\pi(c) + \rho([x, y]) + \gamma(x, y) \]
\[= \theta([x, y]) + \gamma(x, y)(1 - \pi(c)) \]
\[= \theta([x, y]),\]
which is the phenomenon known as cancellation of anomalies. The corresponding inner derivations induce a genuine action of the graded Lie algebra \(\mathcal{G}\) on modules \(M \otimes \wedge^\infty/2^+ \mathcal{G}'\), and this will give us a square-zero differential.

PROOF OF THE THEOREM

As before, we define \(d\) on \(\epsilon(x')\) and \(\pi(x)\) only, making sure it is a derivation. Again assume \([d\epsilon(x'), \epsilon(y')] = 0\), and define \(d\epsilon(x')\) via
\[
[d\epsilon(x'), \iota(y)] = [\epsilon(x'), d\iota(y)] = -\epsilon(ad'(y) \cdot x'),
\]
and \(d\pi(x)\) via
\[
[d\pi(x), \iota(y)] = -[\pi(x), d\iota(y)] = -[\pi(x), \theta(y)]
\]
\[= -[\pi(x), \pi(y)] = -\pi([x, y]_{\mathcal{G}}),\]
thanks to Lemma 3. Then \(d\epsilon(x')\) is the same as \(d_0 \epsilon(x')\) and
\[d\pi(x) = -\sum_i \pi([x, e_i]_{\mathcal{G}})\epsilon(e'_i).\]
It remains to check the relations
\[
[d\pi(x), \epsilon(y')] + [\pi(x), d\epsilon(y')] = 0 \quad (7)
\]
and
\[ [d\pi(x), \pi(y)] + [\pi(x), d\pi(y)] = d[\pi(x), \pi(y)] = d\pi([x, y]_{\tilde{G}}) \quad (8). \]

Eq. (7) is just \(0 + 0 = 0\). As for (8),
\[
- \sum_i \pi([[x, e_i]_{\tilde{G}}, y]_{\tilde{G}})\epsilon(e'_i) - \sum_i \pi([x, [y, e_i]_{\tilde{G}}]_{\tilde{G}})\epsilon(e'_i)
= - \sum_i \pi([[x, y]_{\tilde{G}}, e_i]_{\tilde{G}})\epsilon(e'_i)
\]
by the Jacobi identity.

Once more,
\[ d^2 = 0 \quad \text{and} \quad d\theta(x) = 0 \]
follows from:
\[
[d\theta(x), \iota(y)] = d[\theta(x), \iota(y)] - [\theta(x), d\iota(y)]
= d[\rho(x), \iota(y)] - [\theta(x), \theta(y)]
= d\iota([x, y]) - [\theta(x), \theta(y)]
= \theta([x, y]) - [\theta(x), \theta(y)]
= 0.
\]

Finally,
\[
[d, \iota(e_r)] = \pi(e_r) + [d_0, \iota(e_r)]
= \pi(e_r) + \rho(e_r)
= \theta(e_r),
\]
which shows \(d\) is an inner derivation on \(Y^\infty \tilde{G}\); moreover \(d\) is unique subject to the degree conditions. Since \([D_1, D_2] = [D_1, D_2]\) in general, we have \(d^2 = 0\).

AN EXAMPLE: THE SEMI-INFINITE WEIL COMPLEX

The semi-infinite Weil complex \(W^\infty/2 \tilde{G}\) associated to a \(\mathbb{Z}\)-graded tame Lie algebra \(\tilde{G}\) is the tensor product of the semi-infinite symmetric and exterior modules \(S^\infty/2 \tilde{G}'\) and
\[^\wedge_{0}^{\infty}/2\mathcal{G}'\]

\[\text{together with the semi-infinite differential induced by } d.\]

The construction of these modules is as follows: Starting with the Heisenberg and super Heisenberg algebras \(h(\mathcal{G}) = \mathcal{I}(\mathcal{G}) \oplus \mathcal{E}(\mathcal{G}') \oplus Kk_1\) and \(s(\mathcal{G}) = \iota(\mathcal{G}) \oplus \epsilon(\mathcal{G}') \oplus Kk_2\) with the secondary \(\mathbb{Z}\)-grading, we choose two subalgebras \(S(\mathcal{G}_M \oplus \mathcal{G}'_{M})\) and \(\wedge(\mathcal{G}_N \oplus \mathcal{G}'_{N})\) as the algebras of annihilation operators with trivial representations \(K\nu_S, K\nu_\wedge\). Then \(S^{\infty}/2\mathcal{G}'\) and \(\wedge^{\infty}/2\mathcal{G}'\) are defined to be the induced representations for \(h(\mathcal{G})\) and \(s(\mathcal{G})\) respectively, and are isomorphic (by PBW) to the algebras of creation operators \(S(\mathcal{G}_M \oplus \mathcal{G}'_{M})\) and \(\wedge(\mathcal{G}_N \oplus \mathcal{G}'_{N})\), which commute with each other and act freely on the vacuum \(\nu = \nu_S \otimes \nu_\wedge\). \((M = N = 0\) is usually referred to as the standard choice of vacuum. This convention will be in effect for the rest of this example.\)

An algebra more specific than \(Y^\infty \mathcal{G}\), namely

\[A^\infty \mathcal{G} = \mathcal{U}^\infty (h(\mathcal{G}) \oplus s(\mathcal{G}))/ (k_1 = k_2 = 1),\]

will be home to our operators. It is well-known that the cocycles corresponding to the two realizations

\[\pi(x) = \sum_i : \mathcal{I}([x, e_i]) \mathcal{E}(e'_i) :\]

and

\[\rho(x) = \sum_i : \iota([x, e_i]) \epsilon(e'_i) :\]

of \(\mathcal{G}\) cancel [2], hence \(\tilde{\theta}(x) = \pi(x) + \rho(x)\) automatically induces a representation of \(\mathcal{G}\) in the Weil complex. There is a degree derivation on \(A^\infty \mathcal{G}\) represented by

\[\operatorname{deg} = \sum_i i : \mathcal{I}(e_i) \mathcal{E}(e'_i) : + \sum_i i : \iota(e_i) \epsilon(e'_i) :\]

which multiplies monomials by their (secondary) degrees. By Lemma 2 and its symmetric version, it suffices to check the formula only on \(\mathcal{I}(\mathcal{G}) \oplus \iota(\mathcal{G})\) ! (Normal ordering for the symmetric case does not require a factor of \(-1\).) This operator commutes with the differential and hence the cohomology \(H^*(W^{\infty}/2 \mathcal{G}, d)\) of the complex is the direct sum of cohomologies of the eigenspaces of \(\text{deg}\) for nonpositive integer eigenvalues. The calculation of \(H^*(W^{\infty}/2 \mathcal{G}, d)\) for arbitrary \(\mathcal{G}\) is in general an extremely difficult problem, but using the techniques developed in this paper it is not too hard to prove the following results:
PROPOSITION. For any $\mathbb{Z}$-graded tame Lie algebra $\mathcal{G}$ we have
\[(W^\infty/2\mathcal{G})_{\deg=0}, d) \cong (W\mathcal{G}_0, d)\]
and
\[H^*(W^\infty/2\mathcal{G})_{\deg=0}, d) \cong H^*(W\mathcal{G}_0, d)\].

COROLLARY. Let $\mathcal{G}$ be the Witt algebra $\oplus_n KL_n$ where $[L_m, L_n] = (m - n)L_{m+n}$. Then
\[H^*(W^\infty/2\mathcal{G}, d) \cong W(KL_0)\].

Remark. We know that $H^*(W\ell, d)$ is exactly
\[(S\ell')^\ell \otimes (\wedge\ell')^\ell\]
for a finite dimensional reductive Lie algebra $\ell$, so the Proposition gives us a start for cohomological computations in a lot of interesting cases. The Corollary follows immediately from the Cartan identity
\[d\iota(L_0) + \iota(L_0)d = \theta(L_0)\]
which reduces the cohomology to the degree zero part since $-\theta(L_0)$ is the degree operator. If we change the vacuum, the cohomology changes dramatically as shown in [2]. In case of Lie algebras that do not naturally possess an element that acts by degree, the remaining part of the cohomology may have a very rich structure, even for the standard vacuum.

Results concerning the $d$-cohomology of the Weil complex of the loop algebra of $sl(2, K)$ and more will be presented in the author’s Ph.D. Dissertation (Yale 1993).

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