Torques without Rotation: the Right-Angle Lever

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An extended body subject to external forces which exert zero net force and zero total torque in the rest frame, may experience a nonzero torque in another inertial frame, and nonetheless does not rotate. Long known as the Trouton-Noble or right-angle lever paradox, there has been extensive discussion and indeed controversy, but a clear understanding comes from a suitable treatment of angular momentum and simultaneity.

I. INTRODUCTION

Special relativity presents many situations with consequences that seem paradoxical to someone accustomed to classical, Galilean kinematics. Most often the relativity of simultaneity is the source of the problem. Elementary courses discuss as examples the twin paradox, the pole-vaulter entering the barn, and the sled falling through a smaller hole in the ice. But the curious behavior of torques on a rigid body, though discovered and explained in the first decade of relativity, is not generally discussed in undergraduate courses.

The paradox arose from an attempt before Einstein’s paper to find the motion of the Earth through the aether by the torque aligning a charged parallel plate capacitor perpendicular to the motion. Of course no effect was found. An equivalent purely mechanical situation has a rod under compression or tension.

In exploring the angular momentum of a rigid body in elementary courses, we claim that internal forces do not contribute to the total torque and do not affect the angular momentum. This is because we assume action and reaction are equal and opposite, and also act along a common line, so as to have the same moment arm. This is called the strong form of Newton’s third law.

When two equal and opposite external forces act on different points of a rigid body, such as two forces trying to stretch a thin rod, whether they act along the same line or not depends on reference frame. If, in the rest frame of the rod, they act along the direction of the rod, they will produce no torque, and the momentum and angular momentum of the rod will be conserved, in that frame. But transforming to a frame with respect to which the rod is moving at an angle α relative to its length, the forces change direction and are generally not collinear, and there is a torque.

II. THE RIGHT-ANGLE LEVER

An equivalent situation which is simpler to analyze, known as the “right-angle lever paradox” was first discussed by Lewis and Tolman in 1909. Consider a rigid angle lever as shown, with

- a force \( F_x \) acting at the upper end A, at height \( y \) above the origin at point O,
- a force \( F_y \) acting at the right end B, a distance \( x \) from the origin, and
- a counterbalancing pair of forces \( R_x \) and \( R_y \).

\[ F_x \]
\[ F_y \]

In the rest frame of the angle lever, let us take \( x = y = L, F_x = F_y = R_x = R_y \) in magnitudes (which we will call \( F \)), so that the is no net force or net torque on the lever, and therefore the lever remains at rest with zero momentum and zero angular momentum.

Now consider this from an observer \( O' \) for whom the lever is moving right with velocity \( v \). Distances perpendicular to the motion are unchanged, so \( y' = L \), but distances parallel are contracted, so \( x' = L/v \).

How do forces transform? The Minkowski 4-force \( f^\mu \) is a 4-vector, but this is not the usual force. For a particle, the force \( \vec{F} \) as measured in some frame is \( d\vec{P}/dt \) where \( \vec{P} \) and \( t \) are the particle’s momentum and the frame’s time, respectively, but the 4-force is \( f^\mu = dP^\mu/dr \), where \( dr = dt/\gamma \) is the proper time for the particle on which the force acts. If we are considering a force acting on one point of a rigid body, we mean the proper time as measured by that point of the body. So for the spatial components, \( \vec{f} = \gamma \vec{F} \).

Now for our lever in its rest frame, \( \gamma = 1, f_x^A = F = f_y^A = -r^x = -r^y \), and all other components of the 4-forces are zero, including the zeroth components, which are zero because the forces are doing no work. The Lorentz transformation to the \( O' \) frame is

\[ f^0 = \beta \gamma f^1, \quad f^1 = f^1, \quad f^2 = f^2, \quad f^3 = f^3 = 0, \]

so the 4-force at A is \( f^\mu_A = (\beta \gamma F, \gamma F, 0, 0) \), and \( r'_x = -f'_x \), while the 4-force at the right is \( f^\mu_B = (0, 0, F, 0) \). As the
3-force is \(1/\gamma \times \) the 4-force spatial components, we have \(F'_{Ax} = F = -R_x', \quad F'_{By} = F'_{\gamma y} = -R_y'\), and the total clockwise torque is \(y'F'_{Ax} - x'F'_{By} = LF(1-\gamma^{-2}) = \beta^2 LF \neq 0\).

This seems strange for two reasons. The torque is supposed to give the rate of change of the angular momentum. Even in the \(O'\) frame, each atom of the moving wedge is moving with constant velocity in the \(x\) direction only, so there is no rotation. Shouldn’t that mean the the angular momentum is constant? Secondly, the angular momentum is part of a Minkowski tensor \(\mathbb{L}^{\mu\nu}\), which we might expect to transform linearly, so as the spatial part is zero in the rest frame, and the \(L^{0i}\) part is at least constant in that frame, it again seems the \(O'\) ought to see constant angular momentum components.

One way some have chosen out of this conundrum is to argue that the rate of change of angular momentum is given by the total torque, which needs to include the torque due to internal forces. If we assume atoms in different parts of the body apply equal and opposite forces on each other, collinear in their mutual rest frame, this gives no additional torque in that frame, but this pair of forces does produce a net torque in frames with respect to which they are moving. This is the approach taken by Nickerson and McAdory, who take the angular momentum to mean the sum over atoms of their individual \(L_i = \vec{r}_i \times \vec{p}_i\), where \(\vec{p}_i = E_i \vec{u}_i / c^2\), and \(E_i\) is the total energy of the \(i\)th particle. Obviously this angular momentum is constant, and the total torque, external plus internal, must sum to zero. But this approach is unsatisfactory for several reasons. First, a torque which requires knowledge of all the internal forces in a system of particles is not very useful. Second, the particles in an interacting system do not individually have a well-defined energy, and taking the system to include only the energy and momentum of each particle as if there were no interactions between them is inappropriate. Finally, the treatment of a relativistic system as composed of particles with action at a distance forces is dubious.

### III. PROPER TREATMENT OF ANGULAR MOMENTUM

Our right-angle lever is stressed, and there is energy in that stress which is not included in the rest energy of each of the atoms. And while the momentum being injected into the system by the external forces totals to zero, it is clearly being deposited in different locations from where it is being extracted, so there is a flow of momentum, even in the rest frame. A proper treatment needs to consider things locally.

We can resolve part of our conundrum when we recognize the angular momentum is not due only to the motion of the atoms of the lever. In relativity global properties must be considered integrals over local densities, and in particular the total momentum is the integral over the lever of the momentum density. The momentum density is a piece of the four-dimensional stress-energy tensor \(T^{\mu\nu}\), which has components

\[
T^{00} = \text{energy density} \\
T^{y0} = c \times \text{density of } (\vec{P})_j, j = 1, 2, 3 \\
T^{0j} = \frac{1}{c} \times \text{flux of energy in the } j \text{ direction} \\
T^{ij} = \text{flux of } (\vec{P})_i, \text{ in the } j \text{ direction}.
\]

The 4-momentum \(P^\mu = (E/c, \vec{P}) = \int F^{\mu\alpha}(\vec{x}, t) d^3x\) of an isolated system is conserved because the stress-energy tensor is then a conserved current, \(\sum_\nu \partial_\nu T^{\mu\nu} = 0\).

The angular momentum is given by integrating the moments of the momentum density,

\[
L^{\mu\nu}(t) = \frac{1}{c} \int d^3x \, \mathcal{M}^{\mu\nu}(\vec{x}, t)
\]

where \(\mathcal{M}^{\mu\nu}(x) = x^\nu T^{\mu\rho} - x^\mu T^{\nu\rho}\).

For an isolated system, conservation of angular momentum requires that the angular momentum current is conserved,

\[
\partial_\nu \mathcal{M}^{\mu\nu} = 0 = \frac{1}{c} \left( \delta_\nu^\mu T^{\rho\rho} + x^\nu \partial_\nu T^{\mu\rho} - \delta_\nu^\mu T^{\nu\rho} - x^\mu \partial_\nu T^{\nu\rho} \right) = \frac{1}{c} \left( T^{\mu\nu} - T^{\nu\mu} \right),
\]

where we have used that the energy-momentum current is itself conserved, \(\partial_\nu T^{\mu\nu} = 0\). Thus angular momentum conservation requires that the stress-energy tensor is symmetric, which we assume.

For observer \(O'\) there is an energy flow, because the \(F_A\) force at the top is doing work, injecting energy at a rate \(F'_{Av} = Fv\) at the top, and the force \(R_x\) does negative work at the same rate, extracting energy at the bottom. Thus while the two do not change the total energy, they do produce a flow of energy. Across any \(y = \text{constant surface between them there is a flow of energy } -Fv = cT^{0y}\). There is no such net flow of energy in the \(x\) direction due to the external forces. Using the symmetry of the stress-energy tensor, we see there is a momentum density \(T^{y0}\) and

\[
L_z = L^{xy} = \frac{1}{c} \int d^3x \left( xT^{y0} - yT^{x0} \right)
\]

gets a contribution \(\Delta L^{xy} = -XFvL/c^2\), where \(X\) is the mean \(x\) coordinate of the energy-flow down the arm. As this position is growing at a rate \(v\), we have

\[
\frac{dL^{xy}}{dt} = -\frac{1}{c^2} Fv^2 L = -\beta^2 LF,
\]

consistent with the torque calculated from the forces in the \(O'\) frame.

What about spatial momentum? \(F_x\) is injecting momentum in the \(x\) direction in either reference frame, and it is being extracted by \(R_x\), while \(F_y\) and \(R_y\) do the same in the \(y\) direction. Thus there is a flow of momentum, but this does not contribute to \(T^{y0}\).
So we have preserved that torque is the rate of change of angular momentum. The necessity of considering the angular momentum due to the energy flow, and thus resolving the paradox of nonzero torque with no rotation, was first noticed by Laue.

But what about tensor nature of $L^\mu\nu$? Our lever at rest had no $L^2$ for spatial components, but might it have had nonzero

$$L^{0j} = \frac{1}{c} \int d^3 x \left( x^i T^{0j}(\vec{r}) - x^0 T^{j0}(\vec{r}) \right).$$

In the rest frame the first term is just $(Mc\vec{R})^j$, where $M$ is the total mass (energy/c$^2$) and $\vec{R}$ the center of mass. The second term vanishes as $T^{j0}$ is the momentum density, which is zero. To $O'$, however, the first term is $\vec{R}'(t')Mc\gamma$, while the second term is $-t'cP^j = -t'Mc\gamma v$, so they add to a constant $Mc\gamma \vec{R}'(0)$. Thus in either frame $T^{j0}$ is constant, while $T^{ij}$ is constant in the rest frame but not the moving frame. Thus the angular momenta we have calculated are inconsistent with covariance.

IV. COVARIANCE OF GLOBAL PROPERTIES

Now a well-defined physical property described by a covariant tensor should not behave in this fashion. The problem is that we do not have one well-defined property here. If we were talking about the momentum and angular momentum of a point particle at a given space-time event, the two observers would measure values consistent with covariance, even if there were nonzero forces and torques acting on the particle, because we would be comparing the values at the same event. But for an extended object, when we describe a global property such as the total momentum or angular momentum, we are not talking about a single event, but rather a sum or integral over different spatial points at some set of times. Usually each observer uses the same time, according to his clocks, for all the points, and what is a single time for one observer is not for another. The momentum and angular momentum are integrals of a zero component of currents $J^\mu$, and if $O$ examines the transformation back to his reference frame of what $O'$ did to calculate his values, he would find $O'$ had integrated $J^\mu dS_\nu$ over a hypersurface $S$ of constant $t'$ and not one of constant $t$. If no external forces were acting, the current would be conserved, $\partial_\mu J^\mu = 0$, so Gauss’ law would say the the difference of these integrals is given by a contribution from a surface connecting the two hyperplanes, which could be taken outside the body and thus over a region with $J^\mu = 0$. So both observers would agree. But as there are external forces, this does not happen, $\partial_\mu J^\mu \neq 0$ in the region between the two hypersurfaces, so a contribution from the force at one end is included in the $O'$ calculation, while the corresponding force (synchronous according to $O$) is not.

This issue is raised in Gamba and Cavalleri and Salgarelli, who call our our approach definition (b) and synchronous respectively, and contrast it to definition (a) or asynchronous definitions of the total, which would have us integrate only over the hypersurface of times synchronous in the rest frame of the lever. Only using that second definition of global quantities would one expect to get covariance and constant angular momentum. Cavalleri et al. argue that failure to make this distinction led Nickerson and McAdory astray.

I would argue for the legitimacy of the synchronous definitions, as long as one keeps in mind that using them, a time-dependent global property of an extended system is not a covariant object, even if the density of that property is covariant.

In a recent paper Masud Mansuripur has claimed that the Lorentz force law is incompatible with special relativity. He gives a simple example of a charge and magnetic dipole at rest in one frame, with the dipole moment perpendicular to the line separating them. In the rest frame there is no torque, but in a frame moving along their line of separation the Lorentz force gives a net torque. It is claimed that this proves the inadequacy of the Lorentz law, but our example shows this is not sufficient to conclude that.

V. AN ALTERNATE EXAMPLE CLARIFIES THINGS

To avoid the problem of computing the total angular momentum when external forces are acting, consider the alternate scenario where the forces act as described but only as an impulse at time $t = 0$, simultaneously in the $O$ frame. That is, the force at A is $FT\delta(t)\hat{e}_x$ acting at $\vec{x} = (0, L, 0)$. We take the origin of $O$ frame at O, and assume that of $O'$ coincides with O at $t = t' = 0$. In the $O$ frame, the forces deposit 4-momentum $(P^\mu = (E/c, \vec{P}))$ in the amounts of $\Delta P^\mu = (0, FT, 0, 0)$ at A: $x^\mu = (0, 0, L, 0)$; $\Delta P^\mu = (0, -FT, -FT, 0)$ at O: $x^\mu = (0, 0, 0, 0)$; and $\Delta P^\mu = (0, 0, FT, 0)$ at B: $x^\mu = (0, L, 0, 0)$.

In the $O$ frame these are all simultaneous and there is no change in total momentum or angular momentum. In the $O'$ frame, however, the momentum deposits and their space-time locations are Lorentz transformed. At A: $\Delta P^\mu = (\beta\gamma FT, \gamma FT, 0, 0)$ at $x'^\mu = (0, 0, L, 0)$; at O: $\Delta P^\mu = (-\beta\gamma FT, -\gamma FT, 0, 0)$ at $x'^\mu = (0, 0, 0, 0)$; and at B: $\Delta P^\mu = (0, 0, FT, 0)$ at $x'^\mu = (0, \gamma L, 0, 0)$.

Before we discuss the angular momentum, let us comment on the position of what we usually call the center of mass, but which is really the center of energy,

$$\vec{R}' = \int d^3 r' \hat{r}' T^{00}(r') \left/ \int d^3 r' T^{00}(r') \right.,$$

as $cT^{00}$ is the energy density (for $O'$). As the impulse at B occurs $\beta\gamma L/c$ after the other impulses, during that time interval, the wedge had momentum $P_y = -FT$. 

which meant during that interval the numerator in the center of mass calculation was decreasing at a rate $cF T$ so it decreased by $\beta\gamma LF T$. However the force at A injected energy $c\beta\gamma F T$ at $y' = L$, which increased the numerator by $\beta\gamma LF T$ at $t' = 0$, so there is no net change at the end, and the $y$ position of the center of mass is unchanged.

We have already explained why the angular momentum can change even without any rotation. For $O'$, the torque impulse at $t' = 0$ from A is $\Delta L_z = -L(\gamma FT)$ at time $t' = 0$, and $+(\gamma L)(FT)$ at $t' = \beta\gamma L/c$ from B, so overall there is no change. There are, however instantaneous changes at the ends, so during the interval $t' \in [0, \beta\gamma L/c]$, there is a changed angular momentum.

Initially, this is due to the instantaneous increase in $y_{cm}$, but during the interval this height is diminishing, and at the same time there is a flow of negative $P_y$ injected at O towards B, where it will be cancelled by the impulse at B. This flow diminishes $L_z$ during the interval, so as to keep $L_z$ constant, until the torque impulse from B instantaneously restores $L_z$ to its original value.

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