Numerical Integration Of The Integrals Based On Haar Wavelets

Anvarjon A. Ahmedov\textsuperscript{1,2}, Mohammad Hasan bin Abd Sathar\textsuperscript{2}

\textsuperscript{1}Department of Process and Food Engineering, Faculty of Engineering, University Putra Malaysia  
\textsuperscript{2}Institute of Mathematical Research, Universiti Putra Malaysia  
E-mail: anvar@eng.upm.edu.my, hanaz_3704@yahoo.com

Abstract. In this work, we present a computational method for solving double and triple integrals with variable limits of integrations which is based on Haar wavelets. This approach is the generalization and improvement of the methods [3]. The advantage of this new methods is its more efficient and simple applicability than the previous methods. Error analysis for the case of two dimension are considered. Finally, we also give some numerical examples to compared with existing methods.

1. Introduction

Integrals and derivatives are basic tools of calculus, with numerous applications in science and engineering. Many problems in mathematics, physics, and engineering involve integration where an explicit formula for the integral is desired. The integrals encountered in a basic calculus course are deliberately chosen for simplicity; those found in real applications are not always so accommodating. Some integrals cannot be found exactly, some require special functions which themselves are a challenge to compute, and others are so complex that finding the exact answer is too slow. This motivates the study and application of numerical methods for approximating integrals, which today use floating-point arithmetic on digital electronic computers. Therefore numerical integration is one of the convenient ways to find approximate values of the functions, see for example [1,2]. In recent years, wavelets become one of the most promising, powerful and reliable tools in this area. There are many kinds of wavelets with different properties, some examples are the Haar wavelets, Daubechies’ orthonormal of compact support and Meyer’s wavelets [3,4,5]. Haar wavelets which are mathematically the most simple orthonormal wavelets with compact support and they have been used in different numerical approximation problems, for example the Haar wavelets methods for solving linear and nonlinear integral equations was proposed in [6,7,8].In this paper we have modified the method suggested in [3] and the advantage of this methods is its more efficient and simple applicability. The organization of this paper is as follows. In Section 2, numerical integration using Haar wavelets is described and in Section 3 new techniques for solving the double and triple integral equations. Error analysis for Haar Wavelets is given in Section 4 and numerical results are reported in Section 5. Some conclusions are drawn in Section 6.
2. Numerical integration based on Haar wavelets

A wavelet is a wave-like oscillation with an amplitude that starts out at zero, increases, and then decreases back to zero. Generally, wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Wavelets can be combined, using a "reverse, shift, multiply and sum" technique called convolution, with portions of an unknown signal to extract information from the unknown signal. As a mathematical tool, wavelets can be used to extract information from many different kinds of data, including - but certainly not limited to - audio signals and images. Sets of wavelets are generally needed to analyze data fully. A set of "complementary" wavelets will deconstruct data without gaps or overlap so that the deconstruction process is mathematically reversible. Thus, sets of complementary wavelets are useful in wavelet based compression/decompression algorithms where it is desirable to recover the original information with minimal loss. In formal terms, this representation is a wavelet series representation of a square-integrable function with respect to either a complete, orthonormal set of basis functions, or an overcomplete set or frame of a vector space, for the Hilbert space of square integrable functions.

2.1. Haar wavelets

The mother wavelet for the Haar wavelets family is defined on the interval \([a,b)\), as:

\[
H \left( 2^j \frac{x - a}{b - a} - k \right) = \begin{cases} 
1 & \text{for } x \in \left[ a + k \frac{(b-a)}{2^j}, a + (k + \frac{1}{2}) \frac{(b-a)}{2^j} \right) \\
-1 & \text{for } x \in \left[ a + (k + \frac{1}{2}) \frac{(b-a)}{2^j}, a + (k + 1) \frac{(b-a)}{2^j} \right) \\
0 & \text{elsewhere},
\end{cases}
\]

where \(n = 1, 2, 3, \ldots\) and write \(n = 2^j + k\) with \(j = 0, 1, 2\) and \(k = 0, 1, \ldots, 2^j - 1\) and defined

\[
h_n(x) = H \left( 2^j \frac{x - a}{b - a} - k \right).
\]

All the functions in the Haar wavelet family are defined on subintervals of \([a, b)\) and are generated from by the operations of dilation and translation. The integer \(j\) indicates the level of the wavelet and \(k\) is the translation parameter. The Haar wavelet functions form the orthogonal functions to each on \([a, b)\)

\[
\int_a^b h_n(x) h_\ell(x) \, dx = \begin{cases} 
(b - a)2^{-j} & n = \ell = 2^j + k \\
0 & n \neq \ell.
\end{cases}
\]

Also we defined the scalar wavelet \(h_0(x) = 1\) for all \(x\). It can be shown that the sequence is a complete orthogonal system in \(L_2(a, b)\) and for \(f \in C[a, b]\), the series \(\sum_n \langle f, h_n \rangle h_n\) converges to \(f\) [10], where \(\langle f, h_n \rangle = \int_a^b f(x) h_n(x) \, dx\).

2.2. Numerical integration based on Haar wavelets

Here, we consider numerical integration for single, double and triple integrals using Haar wavelets.

2.2.1. Numerical technique for single integrals.

We consider the integral

\[
\int_a^b f(x) \, dx
\]
over the interval \([a, b)\). Thus any function \(f(x)\) which is square integrable in the interval \([a, b)\) may be expanded as:

\[
f(x) = \sum_{n=0}^{\infty} a_n h_n(x).
\]

In practice, only the first \(2M - 1\) term of the sum is considered, where \(M = 2^{J_1}\) and \(J_1\) is the maximum level of Haar wavelets that is,

\[
f(x) \approx \sum_{n=0}^{2M-1} a_n h_n(x).
\] (1)

**Lemma 1.** The approximate value of the integral is

\[
f(x) \approx a_0 (b - a).
\]

**Proof.** Since

\[
\int_{a}^{b} h_n(x) \, dx = 0, \quad n = 1, 2, 3, \ldots,
\]

and

\[
\int_{a}^{b} h_0(x) \, dx = (b - a),
\]

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{n=0}^{2M-1} a_n \int_{a}^{b} h_n(x) \, dx \approx a_0 (b - a).
\] (2)

It is clear from equation (2) that, Haar approximation involves only one coefficient in the evaluation of the definite integral. To calculate the Haar coefficient \(a_0\) we consider the nodal points

\[
x_k = a + (b - a) \frac{k + 0.5}{2M}, \quad k = 0, 1, \ldots, 2M - 1.
\]

The discretized form of (1) can be written as

\[
f(x_k) \approx \sum_{n=0}^{2M-1} a_n h_n(x_k), \quad k = 0, 1, \ldots, 2M - 1.
\] (3)

The second advantage of Haar wavelet approximation is that we do not need to solve the above system which is computationally expensive for large values of \(M\). The next lemma gives us an easy formula to calculate the value of the Haar coefficient \(a_0\).

**Lemma 2.** The solution in of the system (3) for \(a_0\) is given by

\[
a_0 = \frac{1}{2M} \sum_{k=0}^{2M-1} f(x_k).
\]

For the proof see [9].

We apply Lemma 2 to the integral (2) to obtain the following formula for numerical integral equation

\[
\int_{a}^{b} f(x) \, dx \approx \frac{(b - a)}{2M} \sum_{k=0}^{2M-1} f(x_k) = \frac{(b - a)}{2M} \sum_{k=0}^{2M-1} f \left( a + (b - a) \frac{k + 0.5}{2M} \right).
\]

3
2.2.2. **Numerical technique for double integrals and triple integrals**

The formula derived for numerical integration of double integrals with constant limit in [9] given by

\[
\int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy \approx a_{00}(b-a)(d-c) = \frac{(b-a)(d-c)}{4M^2} \sum_{l=0}^{2M-1} \sum_{k=0}^{2M-1} f(x_k, y_l),
\]

where

\[
f(x, y) \approx \sum_{j=0}^{2M-1} \sum_{i=0}^{2M-1} a_{ij} h_i(x) h_j(y),
\]

and

\[
a_{00} = \frac{1}{4M^2} \sum_{l=0}^{2M-1} \sum_{k=0}^{2M-1} f(x_k, y_l).
\]

This formula can be extended to triple integral, and given by

\[
\int_{e}^{f} \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy \approx \frac{(b-a)(d-c)(f-e)}{8M^3} \sum_{l=0}^{2M-1} \sum_{k=0}^{2M-1} \sum_{j=0}^{2M-1} f(x_j, y_k, z_l).
\]

3. **Present method (PM) for double and triple integral with variable limits**

3.1. **Numerical technique for double integrals with variable limits.**

Consider double integral with variable limits of the type

\[
\int_{a}^{b} \int_{f_{1}(y)}^{f_{2}(y)} F(x, y) \, dx \, dy, \quad f_{1}(y) \leq x \leq f_{2}(y), \quad a \leq y \leq b.
\]

After substitution \( x = X \) \((f_{2}(y) - f_{1}(y)) + f_{1}(y)\) in integral equation, we obtain

\[
\int_{a}^{b} \int_{0}^{1} F(X, f_{2}(y) - f_{1}(y)) + f_{1}(y), y) (f_{2}(y) - f_{1}(y)) \, dX \, dy, \quad 0 \leq X \leq 1, \quad a \leq y \leq b.
\]

We apply formula (4) for \( a_{00} \) to approximate the integral equation

\[
\int_{a}^{b} \int_{0}^{1} F(X, f_{2}(y) - f_{1}(y)) + f_{1}(y), y) (f_{2}(y) - f_{1}(y)) \, dX \, dy \approx a_{00}(b-a)
\]

\[
= \frac{(b-a)}{4M} \sum_{l=0}^{2M-1} \sum_{k=0}^{2M-1} F(X_k, f_{2}(y_l) - f_{1}(y_l)) + f_{1}(y_l), y_l) (f_{2}(y_l) - f_{1}(y_l))
\]

where

\[
X_k = \frac{k + 0.5}{2M}, \quad k = 0, 1, ..., 2M - 1,
\]

\[
y_l = a + (b-a) \frac{l + 0.5}{2N}, \quad l = 0, 1, ..., 2N - 1.
\]
Consider triple integral with variable limits of the type

\[ \int_a^b \int_{f_1(z)}^{f_2(z)} \int_{g_1(y,z)}^{g_2(y,z)} F(x, y, z) \, dx \, dy \, dz. \]

The formula for triple integrals with variable limits can be derived in a similar way and is given by substitution. By Parceval’s formula, we have

\[ \left( \int_a^b f(x) \, dx \right)^2 = \sum_{k=0}^{2N-1} \sum_{j=0}^{2M-1} F\left( X_j \right) \left( g_k f_2(z_i) - f_1(z_i) \right) \left( g_k f_2(z_i) - f_1(z_i) \right) \]

Therefore, the formula for approximating triple integrals is given by

\[ \int_a^b \int_0^1 \int_0^1 F(X(x, y, z)) + g_1, Y(f_2 - f_1, + f_1, z) + f_2 - f_1, (g_2 - g_1) dX \, dy \, dz. \]

4. Error Analysis

In this section we assume that \( f(x, y) \in C^2([a, b] \times [a, b]) \) and

\[ \exists K > 0; \left| \frac{\partial^2 f(x, y)}{\partial x \partial y} \right| \leq K, \forall x, y \in [a, b] \times [a, b]. \]

We may proceed as follows, suppose \( f_{k\ell}(x, y) \) is the following approximation of \( f(x, y) \)

\[ f_{k\ell}(x, y) = \sum_{n=0}^{k-1} \sum_{m=0}^{\ell-1} a_{nm}h_n(x)h_m(y), \]

where \( k = 2^{\alpha+1}, \alpha = 0, 1, 2, \ldots \) and \( \ell = 2^{3+1}, \beta = 0, 1, 2, \ldots \) then

\[ f(x, y) - f_{k\ell}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{nm}h_n(x)h_m(y) + \sum_{n=0}^{\ell-1} \sum_{m=0}^{\ell-1} a_{nm}h_n(x)h_m(y) + \sum_{n=0}^{k-1} \sum_{m=0}^{\ell} a_{nm}h_n(x)h_m(y). \]

By Parceval’s formula we have

\[ ||f(x, y) - f_{k\ell}(x, y)||^2 = \int_a^b \int_0^1 (f(x, y) - f_{k\ell}(x, y))^2 \, dx \, dy \]

\[ = \sum_{p=k}^{\infty} \sum_{s=k}^{\infty} \sum_{n=k}^{\infty} a_{nm}a_{ps} f_a h_n(x)h_p(x) \, dx \, \int_a^b h_m(y)h_s(y) \, dy \]

\[ \quad + \sum_{p=k}^{\ell-1} \sum_{s=p}^{\ell-1} \sum_{n=k}^{\ell-1} a_{nm}a_{ps} f_a h_n(x)h_p(x) \, dx \, \int_a^b h_m(y)h_s(y) \, dy \]

\[ \quad + \sum_{p=0}^{k-1} \sum_{s=0}^{k-1} \sum_{n=0}^{\infty} a_{nm}a_{ps} f_a h_n(x)h_p(x) \, dx \, \int_a^b h_m(y)h_s(y) \, dy \]

\[ = \sum_{n=k}^{\infty} \sum_{m=\ell}^{\infty} a_{nm}^2 + \sum_{n=k}^{\ell-1} \sum_{m=\ell}^{\ell-1} a_{nm}^2 + \sum_{n=0}^{k-1} \sum_{m=\ell}^{\infty} a_{nm}^2, \]
Hence, using the mean value theorem we have:
\[ a = \frac{1}{2} b - a \]

Consider
\[ a = 2 \]
\[ a = 2 - 1 = 1 \]

Using the mean value theorem we have:
\[ a + \frac{b}{2} \leq y_1 \leq a + \left( k + \frac{1}{2} \right) \frac{b - a}{2^j}, a + \left( k + \frac{1}{2} \right) \frac{b - a}{2^j} \leq y_2 \leq a + (k + 1) \frac{b - a}{2^j} \]

Hence,
\[ a_{nm} = (b - a) \int_a^b (f(x,y)) \frac{1}{2} \beta^2 - 1 - f(x,y) \right) \frac{2^{i-1} - 1}{b-a} - k \right) dx \]

Using the mean value theorem
\[ a_{nm} = 2^{-i-1} (b-a) \int_a^b \frac{\partial f(x,y)}{\partial y} (y_1 - y_2) \right) \frac{2^{i-1} - 1}{b-a} - k \right) dy, y_1 \leq y_2 \leq y_2 \]

Again we use the mean value theorem as the above step
\[ a_{nm} = 2^{-i-1} (b-a)^2 (y_1 - y_2) \left( 2^{i-1} \left( \frac{\partial f(x,y)}{\partial y} - \frac{\partial f(x,y)}{\partial y} \right) \right) \]

Therefore
\[ \| f(x,y) - f_k((x,y)) \|^2 = \sum_{n=k}^{\infty} \sum_{m=\ell}^{\infty} a_{nm}^2 + \sum_{n=0}^{\ell-1} \sum_{m=0}^{\infty} a_{nm}^2 + \sum_{n=0}^{\ell-1} \sum_{m=\ell}^{\infty} a_{nm}^2 \]

\[ \sum_{n=k}^{\infty} \sum_{m=\ell}^{\infty} a_{nm}^2 = \sum_{n=2^{\beta+1}}^{\infty} \sum_{m=2^{\beta+1}}^{\infty} 2^{4i-4j-4} (b-a)^8 K^2 \]

\[ \sum_{n=2^{\alpha+1}}^{\infty} \sum_{m=2^{\alpha+1}}^{\infty} 2^{4i-4j-4} (b-a)^8 K^2 \]

\[ \sum_{n=2^{\alpha+1}}^{\infty} \sum_{m=2^{\alpha+1}}^{\infty} 2^{3i-4j-4} \]

\[ (b-a)^8 K^2 \sum_{j=\alpha+1}^{\infty} \sum_{n=2^j}^{2^{i+1}} 2^{-4j} \]

\[ \frac{(b-a)^8}{14K} K^2 \sum_{j=\alpha+1}^{\infty} 2^{-3j} \]

\[ \frac{4(b-a)^8}{14K} K^2 \]
\[ \sum_{n=k}^{\infty} \sum_{m=0}^{\ell-1} a_{nm}^2 = \sum_{n=2^{\varphi+1}}^{\infty} \sum_{m=0}^{(\ell-1)} 2^{-4i-4j-4}(b-a)^8 K^2 \]
\[ = \sum_{n=2^{\varphi+1}}^{\infty} \left( (b-a)^8 K^2 \beta \sum_{i=0}^{2^{\varphi+1}-1} \sum_{m=2^i}^{2^i-1} 2^{-4i-4j-4} \right) \]
\[ = \sum_{n=2^{\varphi+1}}^{\infty} \left( (b-a)^8 K^2 2^{-4j-4} \sum_{i=0}^{\beta} 2^{-4i} (2^i + 1) \right) \]
\[ \leq \sum_{n=2^{\varphi+1}}^{\infty} \left( (b-a)^8 K^2 2^{-4j-4} \sum_{i=0}^{\infty} 2^{-4i} (2^i + 1) \right) \]
\[ = \sum_{n=2^{\varphi+1}}^{\infty} \left( \frac{29}{270} (b-a)^8 K^2 2^{-4j} \right) \]
\[ = \frac{29(b-a)^8}{270} K^2 \sum_{j=\alpha+1}^{\infty} \left( \sum_{n=2^j}^{2^{j+1}-1} 2^{-4j} \right) \]
\[ = \frac{(b-a)^8}{14k^3} K^2 \sum_{j=\alpha+1}^{\infty} 2^{-3j} \]
\[ = \frac{116(b-a)^8}{735k^3} K^2 \]
\[ \sum_{n=0}^{k-1} \sum_{m=\ell}^{\infty} a_{nm}^2 = \frac{116(b-a)^8}{735\ell^3} K^2 \]
\[ \| f(x, y) - f_{kl}(x, y) \| \leq 4(b-a)^8 K^2 \left( \frac{1}{49k^3\ell^3} + \frac{29}{735k^3} + \frac{29}{735\ell^3} \right) \]

Therefore it is obvious that the error bound will approaches 0 as \( k \to \infty \) and \( \ell \to \infty \).

5. Numerical examples
Example 1
\[ \int_0^1 \int_0^y e^{x+y-1} \, dx \, dy = \int_0^1 \int_0^1 ye^{yX+y-1} \, dX \, dy \]

| Haar | Absolute errors (PM) | Absolute errors (NM) | NM-PM |
|------|----------------------|----------------------|-------|
| M = 4, N = 5 | 0.0016425210 | 0.0016425205 | -5.0000000017 \times 10^{-10} |
| M = 5, N = 10 | 0.0004951200 | 0.0004951195 | -5.0000000006 \times 10^{-10} |
| M = 10, N = 15 | 0.0001980110 | 0.0001980118 | 8.0000000000 \times 10^{-10} |
| M = 16, N = 16 | 0.0001185685 | 0.0001185682 | -3.0000000000 \times 10^{-10} |

The errors are shown in Table 1. From Table 1 it is clear that the new method (NM) performs very well then the previous method (PM) [3].
Example 2

\[ \int_0^\pi \int_0^\frac{\sin y}{\sqrt{1-x^2}} \, dx \, dy = \int_0^1 \int_0^1 \frac{\sin(y)}{\sqrt{1-(\sin y)^2}} \, dX \, dy \]

**Table 2.** Absolute Errors of Example 2.

| Haar  | Absolute errors (PM) | Absolute errors (NM) | NM-PM                  |
|-------|-----------------------|-----------------------|------------------------|
| M = 3, N = 3 | 0.0001711541          | 0.0001711538          | −2.9999999999 × 10^{−10} |
| M = 4, N = 3 | 0.0000978111          | 0.0000966500          | −0.000001161099         |
| M = 5, N = 4 | 0.0000630685          | 0.0000627182          | −3.5029999999 × 10^{−7} |
| M = 6, N = 5 | 0.0000439774          | 0.0000438437          | −1.3369999999 × 10^{−7} |

Example 3

\[ \int \int \frac{(x + y)}{\sqrt{7y-7}} \, dx \, dy \approx 3.549613026789713 \]

where \( R \) is a quadrilateral region connecting the points (-1,2),(2,1),(3,3) and (1,4). Here divide the region \( R \) into three subregions so that the given integrals can be written as

\[
\int \int_R \frac{(x + y)}{\sqrt{7y-7}} \, dx \, dy = \int_1^2 \int_{5-3y}^{\frac{y+3}{y-3}} \frac{(x + y)}{\sqrt{7y-7}} \, dx \, dy + \int_2^3 \int_{\frac{y+3}{y-3}}^{\frac{9+y}{y-3}} \frac{(x + y)}{\sqrt{7y-7}} \, dx \, dy + \int_3^4 \int_{\frac{y+3}{y-3}}^{\frac{9+y}{y-3}} \frac{(x + y)}{\sqrt{7y-7}} \, dx \, dy.
\]

**Table 3.** Absolute Errors of Example 3.

| Haar  | Absolute errors (PM) | Absolute errors (NM) | NM-PM                  |
|-------|-----------------------|-----------------------|------------------------|
| M = 2, N = 2 | 1.1496990287971 × 10^{−2} | 1.1496892897971 × 10^{−2} | −1.0000000827 × 10^{−9} |
| M = 4, N = 4 | 2.987381789712 × 10^{−3} | 2.98735289712 × 10^{−3} | 3.4999998455 × 10^{−9} |
| M = 8, N = 8 | 7.550162897129 × 10^{−4} | 7.550119897126 × 10^{−4} | −4.3000003557 × 10^{−9} |
| M = 16, N = 16 | 1.892872897131 × 10^{−4} | 1.892849897129 × 10^{−4} | −2.300001903 × 10^{−9} |
| M = 32, N = 32 | 4.734928971261 × 10^{−5} | 4.735778971287 × 10^{−5} | 8.500002592 × 10^{−9} |
| M = 64, N = 64 | 1.183178971286 × 10^{−5} | 1.184078971272 × 10^{−5} | 8.999998564 × 10^{−9} |
| M = 128, N = 128 | 2.950289712533 × 10^{−6} | 2.928289712933 × 10^{−6} | −2.1999999599 × 10^{−8} |
Example 4

\[ \int_0^\pi \int_0^z \int_0^{zy} \frac{1}{y} \sin \left( \frac{x}{y} \right) \, dx \, dy \, dz \ngtr \int_0^\pi \int_0^1 \int_0^1 z^2 \sin \left( \frac{X}{Y} \right) \, dX \, dY \, dz \]

| Haar          | Absolute errors (PM) | Absolute errors (NM) | NM-PM               |
|---------------|----------------------|----------------------|---------------------|
| M = 4, N = 4  | 0.033053436          | 0.033053436          | 0                   |
| M = 8, N = 8  | 0.030061100          | 0.030061104          | 4.00000000000000000136 × 10^{-9} |
| M = 16, N = 16 | 0.000773945         | 0.000773939          | -5.99999999998910^{-9} |
| M = 32, N = 32 | 0.011996531         | 0.011996550          | 1.9000000004 × 10^{-8} |
| M = 64, N = 64 | 0.001599302         | 0.001599289          | -1.29999999998 × 10^{-8} |
| M = 128, N = 128 | 0.001430520       | 0.001430510          | -9.99999999994 × 10^{-9} |

6. Conclusion

In this paper a new method based on Haar wavelets is applied for numerical integration of double and triple integrals with variable limits. This approach is the generalization and improvement of the methods [3]. From Table 1-4, it can be noticed that the approximate value is better compared to method [3]. Example 1-3 are taken exactly same from [3]. Finally error analysis for the case of two dimensional shows that the approximation become more accurate when \( k \) and \( \ell \) are increased. Therefore for better results, using a large value of \( k \) and \( \ell \) are recommended.

Acknowledgments

This work has been supported by UPM under the Research University Grant (RUGS), project number 05-01-12-1630RU.

References

[1] Burden R L and Faires J D 1993 Numerical Analysis (USA: PWS-KENT).
[2] Ferziger J H 1998 Numerical Methods for Engineering Application (New York: John Wiley and Sons Inc).
[3] Imran Aziz, Siraj-ul-Islam and Wajid Khan 2011 comput.Math.Appl. 61 2770.
[4] Babolian E and Shahsavaran A 2009 J. Comput. Appl.Math. 225 87.
[5] Daubechies I 1992 Ten Lectures on Wavelets (Philadelphia:SIAM).
[6] Maleknejad K and Mirzaee F 2003 Int. J. Comput. Math. 8 (11) 1397.
[7] Lepik U and Tamme E 2004 Proc (Turkey:Antalaya) p494.
[8] Lepik U and Tamme E 2007 Proc. Estonian Acad.Sci.Phys.Math. 56 17.
[9] Imran Aziz, Siraj-ul-Islam and Wajid Khan 2010 comput.Math.Appl. 59 2026.
[10] Wojtaszczyk P 1997, A Mathematical Introduction to Wavelets (Cambridge: University Press).