Infraparticle states in the massless Nelson model – revisited

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Abstract

We provide a new construction of infraparticle states in the massless Nelson model. The approximating sequence of our infraparticle state does not involve any infrared cut-offs. Its derivative w.r.t. the time parameter \( t \) is given by a simple explicit formula. The convergence of this sequence as \( t \to \infty \) to a non-zero limit is then obtained by the Cook method combined with stationary phase estimates. To apply the latter technique we exploit recent results on regularity of ground states in the massless Nelson model, which hold in the low coupling regime.

Keywords: Nelson model, scattering theory, infrared problems.

1 Introduction

The massless Nelson model is a time-honoured theoretical laboratory for the infrared aspects of QED. One of its variants, which we consider in this work, describes one non-relativistic massive particle (‘the electron’), interacting with massless scalar bosons (‘the photons’). The coupling between the electrons and photons is chosen in such a way that the model exhibits the infraparticle problem, i.e., it does not contain physical states describing
the electron in empty space. In other words, the electron is always accompanied by (soft) photons and it is a challenge to mathematically describe the resulting composite object, usually called an infraparticle. Two milestones in rigorous understanding of this problem are works of J. Fröhlich [Fr73, Fr74] and A. Pizzo [Pi03, Pi05]. The latter two papers actually give a complete discussion of the infraparticle in the Nelson model and of its collisions with (hard) photons. Also collisions of an infraparticle with a Wigner-type particle (‘an atom’) in a Nelson model with two massive particles are under control [DP19]. However, scattering of several infraparticles appears steeply difficult in the conventional approach from [Pi05], as discussed in detail in [DP19, Introduction]. One reason is that the approximating sequence of the infraparticle state from [Pi05] and the proof of its convergence are technically quite intricate, which may be due to limited spectral information on the model available back then. Given intervening advances in the spectral theory [AH12, DP18, DP18.1], we revisit the subject and propose a simpler approximating sequence of the infraparticle in the Nelson model. Its convergence to a non-trivial limit is relatively straightforward, given the currently available spectral ingredients.

To explain our construction, let us recall the definition of the Nelson model. The Hilbert space of the model is $\mathcal{H} = L^2(\mathbb{R}^3, \mathcal{F})$, where $\mathcal{F}$ is the symmetric Fock space over $L^2(\mathbb{R}^3_+)$. Thus we will treat $\psi \in \mathcal{H}$ as $\mathcal{F}$-valued square-integrable functions $\{\psi(x)\}_{x \in \mathbb{R}^3}$, whose scalar product has the form

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{H}} = \int d^3x \langle \psi_1(x), \psi_2(x) \rangle_F. \quad (1.1)$$

The creation and annihilation operators on $\mathcal{F}$ are denoted by $a^{(\psi)}(f)$, $f \in L^2(\mathbb{R}^3_+)$, and their sharp variants by $k \mapsto a^{(\psi)}(k)$. The Hamiltonian of the Nelson model has the form

$$H = \frac{(-i\nabla_x)^2}{2} + H_t + a(v_x) + a^*(v_x). \quad (1.2)$$

Here $x$ and $-i\nabla_x$ are the position and momentum operators on $L^2(\mathbb{R}^3)$, $(H_t, P_t) := (d\Gamma(|k|), d\Gamma(k))$ are the energy-momentum operators of non-interacting photons and $v_x(k) = v(k)e^{-ikx}$, where $v(k) := \frac{\lambda v(k)}{\sqrt{2\lambda}}$ and $|k| \in (0, \lambda_0]$ is the coupling constant, whose maximal value $\lambda_0$ will be sufficiently small but non-zero. Here $\chi \in C_0^\infty(\mathbb{R}^3)$ is a smooth approximate characteristic function of the ball of radius $\kappa = 1$. We choose this function rotation invariant, supported in the ball of radius $\kappa$ and equal to one on a ball of a slightly smaller radius $(1 - \varepsilon_0)\kappa$. By the Kato-Rellich theorem, $H$ is a self-adjoint operator on $D(\frac{1}{2}(-i\nabla_x)^2 + H_t)$. Recalling that the model is translation invariant, we denote by $\{H_p\}_{p \in \mathbb{R}^3}$ the usual fiber Hamiltonians acting on the fiber Fock space $\mathcal{F}_p$, satisfying

$$H = \Pi' \left( \int d^3p H_p \right) \Pi, \quad \Pi = Fe^{iP_t \cdot x}, \quad (1.3)$$

where $F$ is the Fourier transform in the $x$ variable. In our construction of infraparticle scattering states we will identify the fiber Fock space $\mathcal{F}_p$ with the physical Fock space $\mathcal{F}$ which is the reason for the appearance of the unitary $\Pi$ explicitly in formula (1.6) below. After this identification, the fiber Hamiltonians are the following self-adjoint operators on $D(P_t^2 + H_t) \subset \mathcal{F}$

$$H_p := \frac{1}{2}(p - P_t)^2 + H_t + a^*(v) + a(v), \quad p \in \mathbb{R}^3. \quad (1.4)$$

We denote the infimum of the spectrum of $H_p$ by $E_p$. One manifestation of the infraparticle problem is that $E_p$ is not an eigenvalue. This has been established in considerable generality [Da18, Pi03, Fr74], and holds, in

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1 Although $\kappa = 1$, it is convenient to keep it in the notation.
particular, for $p \in S := \{ p' \in \mathbb{R}^3 \mid |p'| < 1/3 \}$ and $\lambda_0$ sufficiently small. In this range of parameters we know from [AH12] that $p \mapsto E_p$ is real analytic. It is also well known that the modified Hamiltonian $H^w_p$, obtained from $H_p$ by the Bogolubov transformation

$$a^\dagger(k) \mapsto a^\dagger(k) - f_p(k), \quad f_p(k) := \lambda \frac{\chi_p(k)}{\sqrt{2k |k| (1 - e_k \cdot \nabla E_p)}}, \quad e_k := k/|k|,$$

is self-adjoint on $D(P_t^2 + H_f)$ and $E_p$ is its eigenvalue at the bottom of the spectrum corresponding to an eigenvector $\phi_p$. (Its phase is chosen in the following in accordance with [DP18, Definition 5.2]).

After these preparations we are ready to define the approximating sequences of the infraparticle states. For any $h \in C_0^\infty(\mathbb{R}^3)$ supported in $S$ and any time parameter $t \in \mathbb{R}$ we set

$$\psi_t(x) := e^{iHt} e^{-i P_t x} \frac{1}{(2\pi)^{3/2}} \int d^3 p \ e^{i(p \cdot x - E_p t)} e^{i\gamma(p,x,t)} h(p) W(f_p(e^{-i\lambda t} + ik \cdot x - 1)) \phi_p,$$

$$\gamma(p, x, t) := \int d^3 k f_p(k)^2 \sin(|k| t - k \cdot x).$$

Clearly, the Weyl operator $W(g) := e^{i\gamma(g) - \alpha(g)}$ is well defined for $g(k) := f_p(k)(e^{-i\lambda t} + ik \cdot x - 1)$, for any $(t, x) \in \mathbb{R}^4$. The integral in (1.6) is well-defined in $F$, since $S \ni p \mapsto \phi_p$ is Hölder continuous in norm by [Pi03] (which can also be seen by [DP18, formulas (1.8), (A.4) and Corollary 5.6] combined with Lemma C.3 below). This integral is an element of $L^2(\mathbb{R}^3; F)$ by Lemma 3.5. Our main result is the following:

**Theorem 1.1.** There is such $\lambda_0 > 0$ that the following holds: For any $t \in \mathbb{R}$ the vector $\psi_t$ given by (1.6) belongs to $L^2(\mathbb{R}^3; F)$. The derivative $\partial_t \psi_t$ exists in norm in $L^2(\mathbb{R}^3; F)$ and we have

$$\partial_t \psi_t = e^{iHt} e^{-i P_t x} \frac{1}{(2\pi)^{3/2}} \int d^3 p \ e^{i(p \cdot x - E_p t)} e^{i\gamma(p,x,t)} i\gamma_{int}(p, x, t) h(p) W(f_p(e^{-i\lambda t} + ik \cdot x - 1)) \phi_p,$$

where $\gamma_{int}(p, x, t) := 2 \int d^3 k f_p(k)^2 (|k| - k \cdot \nabla E_p) \cos(|k| t - k \cdot x)$ is rapidly decreasing in the region $|x|/t < 1$ (cf. Lemma 3.7). Furthermore,

$$\int_0^\infty dt \|\partial_t \psi_t\|_H < \infty,$$

hence $\psi^+ := \lim_{t \to \infty} \psi_t$ exists in the norm of $L^2(\mathbb{R}^3; F)$. For $h \neq 0$ and $|\lambda| \in (0, \lambda_0]$ sufficiently small, $\psi^+ \neq 0$. Analogous statements hold for incoming scattering states.

The most remarkable part of the theorem is the explicit formula for $\partial_t \psi_t$ given in (1.8). It can be anticipated by formal computations on $F$ noting the key relation

$$T(p, x, t)^* \ (-i \nabla_x - P_t) \ T(p, x, t) = -i \nabla_x - P_t^w \quad \text{for} \quad T(p, x, t) := W(f_p(e^{-i\lambda t} + ik \cdot x - 1)) e^{i\gamma(p,x,t)},$$

where $P_t^w$ is obtained from $P_t$ via the Bogolubov transformation (1.5). Relation (1.10) allows to reconstruct $H^w_p$ in front of $\phi_p$ and make use of $H^w_p \phi_p = E_p \phi_p$. It dictates our choice of the phase $\gamma$ and it is noteworthy that the resulting $\gamma_{int}$ enjoys a rapid decay in $t$ in the physical region of velocities of the electron. This coincidence suggests that our approximating vector (1.6) captures optimally the asymptotic dynamics of the Nelson model in the infrared regime. The decay of $\gamma_{int}$ is the driving force of our convergence argument based on the Cook method. It also allows for a simple proof of non-triviality of the limit for small $|\lambda|$.
Given formula (1.8) and the above remarks, it may seem very easy to prove the theorem. But it should be kept in mind, that estimate (1.9) must hold in the norm of $L^2(\mathbb{R}^3, \mathcal{F})$, which involves the integral over whole space, cf. formula (1.1) above. To control this integral we use the stationary phase method, which generates derivatives w.r.t. $p$ up to the second order (cf. Lemma 3.1 below). Since differentiability of $p \mapsto \phi_p$ is not settled, we have to approximate $\phi_p$ with $\phi_{p,\sigma}$, which come from the Nelson model with an infrared cut-off $\sigma > 0$ in the interaction. The function $p \mapsto \phi_{p,\sigma}$ is differentiable and its derivatives up to the second order have only a mild infrared divergence of the form
\[ \| \partial_p^\alpha \phi_{p,\sigma} \|_\mathcal{F} \leq c\sigma^{-\delta_0}, \quad |\alpha| = 0, 1, 2, \] (1.11)
where $\delta_0 > 0$ tends to zero with $\lambda_0 \to 0$. This estimate, and similar bounds for the wave functions of $\phi_{p,\sigma}$, rely on technical advances from [DP18, DP18.1]. Thus, at our present level of understanding, we can eliminate the infrared cut-off from the formulation of Theorem 1.1, but not from its proof.

This paper is organized as follows: In Section 2 we provide some technical information, in particular about the model with infrared cut-off. Section 3 is devoted to the proof of Theorem 1.1. In Conclusions we provide a brief comparison of our infraparticle states with the Faddeev-Kulish approach. More technical parts of the discussion are postponed to Appendices.

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2 Preliminaries

Recall that $\{H_p\}_{p \in \mathbb{R}^3}$ are the fiber Hamiltonians (1.4) and let $\{H_{p,\sigma}\}_{p \in \mathbb{R}^3}$ be their counterparts at an infrared cut-off $0 < \sigma \leq \kappa$. This means that the form factor $v$, appearing in (1.2), is replaced with $v^\sigma$ given by
\[ v^\sigma(k) := \lambda \frac{\chi_{[\sigma,\infty)}(k)}{\sqrt{2|k|}}. \] (2.1)
Here $\chi_{[\sigma,\infty)}(k) = 1_{B'_\sigma}(k) \chi_\delta(k)$, $B'_\sigma$ is the complement of the ball of radius $\sigma$ and $1_\Delta$ is the characteristic function of a set $\Delta$. We remark that $\{H_{p,\sigma}\}_{p \in \mathbb{R}^3}$ act on a dense domain in $\mathcal{F}$, that is no infrared cut-off is introduced on the Fock space. We will work in the range of parameters for which the technical results of [DP18, DP18.1, DP19] hold. That is,
\[ \lambda \in \mathcal{B}_0, \quad \sigma \in (0, \kappa_{\lambda_0}], \quad p \in S := \{ p' \in \mathbb{R}^3 \mid |p'| < 1/3 \}, \] (2.2)
where $\lambda_0$ is sufficiently small and $0 < \kappa_{\lambda_0} \leq \kappa$. As the fiber Hamiltonians $H_p, H_{p,\sigma}$ are bounded from below, we can define
\[ E_p := \inf \sigma(H_p), \quad E_{p,\sigma} := \inf \sigma(H_{p,\sigma}), \] (2.3)
where $\sigma$ denotes the spectrum. (Occasionally we will write $E_p^{(i)}, E_{p,\sigma}^{(i)}$ etc. if the dependence on $\lambda$ will play a role). $E_p$ enters our definition of the infraparticle state (1.8) and our analysis relies on the following result:

Lemma 2.1. [AH12] The function $S \times \mathbb{B}_0 \ni (p, \lambda) \mapsto E_p^{(i)}$ is real-analytic and non-constant. It satisfies $|\nabla E_p^{(i)}| \leq 1/2$ and its Hessian matrix in the $p$-variable is strictly positive in $S$ uniformly in $\lambda$. 

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We recall that the modified Hamiltonians $H_p^w$ are obtained from $H_p$ by the Bogolubov transformation (1.5) and their ground states are denoted $\phi_p$. Similarly, the modified Hamiltonians $H_{p,\sigma}$ are obtained from $H_{p,\sigma}$ by the transformation

$$a^{(\sigma)}(k) \mapsto a^{(\sigma)}(k) - f_{p,\sigma}(k), \quad f_{p,\sigma}(k) := \lambda^{1/2} \left(\frac{k}{|k|} \cdot \nabla E_{p,\sigma}\right)$$

and their ground states are denoted $\phi_{p,\sigma}$. Both $\phi_p$ and $\phi_{p,\sigma}$ are in the domain of any power of $H_\ell$ (cf. Lemma C.3) and in addition $\phi_{p,\sigma}$ are in the domain of any power of the number operator $N := d\Gamma(1)$. For a choice of the phases of $\phi_p, \phi_{p,\sigma}$ as in [DP18, Definition 5.2] the following estimate holds

$$\| (H_\ell)\ell (\phi_p - \phi_{p,\sigma}) \|_F \leq c\sigma^{1/5}, \quad p \in S, \quad \ell \in \mathbb{N}_0,$$

provided that $\lambda_0 > 0$ is readjusted for each $\ell$. It is well known for $\ell = 0$ [Pi03], [DP19, Corollary 5.6 (a)] and for $\ell \in \mathbb{N}$ it is shown in Appendix C. We will also need the following lemma:

**Lemma 2.2.** Let $\ell_1, \ell_2 \in \mathbb{N}_0$. Then, for $\sigma \in (0, \kappa_{\lambda_0}]$,

$$\| H_\ell N^{\ell_2} \partial_\sigma^p \phi_{p,\sigma} \|_F \leq \frac{c}{\sigma^{\ell_0}} \quad \text{for} \quad |\sigma| = 0, 1, 2.$$

The function $\lambda_0 \mapsto \delta_{\lambda_0}$ is positive and satisfies $\lim_{\lambda_0 \to 0} \delta_{\lambda_0} = 0$. This function and the constant $c$ are independent of $p, \sigma$ within the assumed restrictions, but may depend on $\ell_1, \ell_2$.

In Appendix B we show how to extract the proof of Lemma 2.2 from [DP18, DP18.1]. We remark that Lemma 2.1, bound (2.5), and Lemma 2.2 are the technical basis for our discussion in the next section.

**Notation.** As we will discuss only outgoing scattering states, we set $t \geq 1$. We denote by $c$ numerical constants which may change from line to line. These constants are independent of $\sigma, p, \lambda, t, x$ within the assumed restrictions, but may depend on $\lambda_0, \varepsilon_0$. Similarly, functions denoted $\lambda_0 \mapsto \delta_{\lambda_0}$ are positive and satisfy $\lim_{\lambda_0 \to 0} \delta_{\lambda_0} = 0$. They are independent of $\sigma, p$ within the assumed restrictions but may depend on $\varepsilon_0$. These functions may change from line to line.

### 3 Inf suparficulate states

The goal of this section is to provide a proof of Theorem 1.1. Our main tool will be the stationary phase method. The estimates suitable for our purposes are stated in the following lemma, which is proven in Appendix D.

**Lemma 3.1.** Let $p \mapsto g(p) \in \mathcal{F}$ be weakly infinitely differentiable on some dense domain and compactly supported in $S$. Let $c_0$ be s.t. $|\nabla E_p| < c_0 < 1$ for $p \in \text{supp } g$. Then, for any $0 \leq \varepsilon \leq 1/2$,

$$\left( \int_{|x| \leq 1} d^3x \| \int d^3p e^{i(p \cdot x - E_p t)} g(p) \|_F^2 \right)^{1/2} \leq c \sum_{|\lambda| \leq 2} \sup_{p,|\lambda| \leq c_0} \| \partial_\sigma^p g(p) \|_F,$$

$$\left( \int_{|x| \geq c_0} d^3x \| \int d^3p e^{i(p \cdot x - E_p t)} g(p) \|_F^2 \right)^{1/2} \leq ct^{-1/2+\varepsilon} \sum_{|\lambda| \leq 2} \sup_{p,|\lambda| \leq c_0} \left( \frac{1}{(1 + |t| + |x|)^\varepsilon} \| \partial_\sigma^p g(p) \|_F \right).$$

The function $g$ above may depend on $(x, t).$
Lemma 3.1 immediately gives the following estimate
\[
\left( \int d^3x \left\| \int d^3p \, e^{i \langle p, x - E_p(t) \rangle} g(p) \right\|_F^2 \right)^{1/2} \leq c t^{1/2} \sum_{|p| \leq 2} \sup_{p, x} \left( \frac{1}{(1 + |x|)^{1/2}} |\partial_p g(p)|_F \right). \tag{3.3}
\]
which will be useful for analyzing vectors (1.6) at finite \( t \). We note that we cannot apply (3.3) or Lemma 3.1 directly to the infraparticle vector (1.6), since differentiability of \( p \mapsto \phi_p \) is out of control. In the course of our discussion we will approximate \( \phi_p \) with \( \phi_{p, \sigma} \) in a suitable manner.

As a first step of our analysis, we compute and estimate derivatives of \( e^{i \gamma(p, x, t)} \) w.r.t. \( p, x, t \). The following is a result of a straightforward computation:
\[
\partial_t e^{i \gamma(p, x, t)} = e^{i \gamma(p, x, t)} i \int d^3k \, f_p(k)^2 |k| \cos(|k| t - k \cdot x),
\]
(3.4)
\[
\partial_x e^{i \gamma(p, x, t)} = -e^{i \gamma(p, x, t)} i \int d^3k \, f_p(k)^2 k_i \cos(|k| t - k \cdot x),
\]
(3.5)
\[
\partial_{x_j} \partial_x e^{i \gamma(p, x, t)} = -e^{i \gamma(p, x, t)} \int d^3k \, f_p(k)^2 k_j \cos(|k| t - k \cdot x) \int d^3k f_p(k)^2 k_i \cos(|k| t - k \cdot x)
\]
(3.6)
\[
- e^{i \gamma(p, x, t)} i \int d^3k f_p(k)^2 k_j \sin(|k| t - k \cdot x).
\]
(3.7)

Now we estimate the above expressions together with their derivatives w.r.t. \( p \).

Lemma 3.2. The following bounds hold
\[
|\partial_p^\alpha \partial_t^\ell e^{i \gamma(p, x, t)}| \leq c (1 + \log(1 + |t| + |x|))^2,
\]
(3.8)
\[
|\partial_p^\alpha \partial_x^\ell e^{i \gamma(p, x, t)}| \leq c (1 + \log(1 + |t| + |x|))^2,
\]
(3.9)
for \( |\alpha|, |\beta| \leq 2, \ell \leq 1 \).

Proof. We see from (3.4)-(3.7) that the derivatives w.r.t. \( x, t \) produce expressions which are uniformly bounded in \( x, t \) due to the additional factors \( k_i, |k| \), which regularize the singularity of \( f_p^2 \) at \( |k| = 0 \). Hence, it suffices to study the expression
\[
\partial_p \partial_{p, i} e^{i \gamma(p, x, t)} = \partial_p (e^{i \gamma(p, x, t)} i \partial_{p, i} \gamma(p, x, t))
\]
(3.10)
\[
= e^{i \gamma(p, x, t)} (i \partial_{p, i} \gamma(p, x, t))(i \partial_{p, i} \gamma(p, x, t)) + e^{i \gamma(p, x, t)} i \partial_{p, i} \partial_{p, j} \gamma(p, x, t).
\]
Making use of (E.4), we obtain
\[
|\partial_{p, i} \partial_{p, j} e^{i \gamma(p, x, t)}| \leq c (1 + \log(1 + |t| + |x|))^2,
\]
(3.11)
where the dependence of \( c \) on parameters is as discussed in Section 2. This concludes the proof. \( \Box \)

As a next step of our discussion we compute derivatives of the following auxiliary vector
\[
\hat{g}_{(t, x)}(p) = W(f_p m(t, x)) \phi_p, \quad m(t, x) := u(t, x) - 1, \quad u(t, x) := e^{-i |k| x + ik x}
\]
(3.12)
w.r.t. \( (t, x) \) up to the second order. We will abbreviate \( m := m(t, x), u := u(t, x) \).
Lemma 3.3. The function \((t, x) \mapsto \hat{g}_{(t, x)}(p)\) is infinitely often partially differentiable in the norm of \(F\) and the following formulas hold

\[
\begin{align*}
\partial_t \hat{g}_{(t,x)}(p) &= W(f_{\rho}m)i((\Phi(f_{\rho}\partial_tm) + \text{Im}(f_{\rho}m, f_{\rho}\partial_tm))\phi_{\rho}, \\
\partial_t^2 \hat{g}_{(t,x)}(p) &= -W(f_{\rho}m)(\Phi(f_{\rho}\partial_tm) + \text{Im}(f_{\rho}m, f_{\rho}\partial_tm))^2\phi_{\rho} + W(f_{\rho}m)i((\Phi(f_{\rho}\partial_tm) + \text{Im}(f_{\rho}m, f_{\rho}\partial_tm))\phi_{\rho}, \\
\partial_x \hat{g}_{(t,x)}(p) &= W(f_{\rho}m)i((\Phi(f_{\rho}\partial_xm) + \text{Im}(f_{\rho}m, f_{\rho}\partial_xm))\phi_{\rho}, \\
\partial_x \partial_x \hat{g}_{(t,x)}(p) &= W(f_{\rho}m)i((\Phi(f_{\rho}\partial_xm, f_{\rho}\partial_xm) + \text{Im}(f_{\rho}m, f_{\rho}\partial_xm, f_{\rho}\partial_xm))\phi_{\rho} + W(f_{\rho}m)i((\Phi(f_{\rho}\partial_xm, f_{\rho}\partial_xm, f_{\rho}\partial_xm))\phi_{\rho},
\end{align*}
\]

where \(\Phi(F) := a(-iF) + a(-iF), F \in L^2(\mathbb{R}^3_\lambda)\).

Proof. We note that, by Lemma C.3, \(\phi_{\rho}\) belongs to \(D(H^2_\ell)\) for any \(\ell \in \mathbb{N}\). We observe that for any fixed \((t, x)\) the function \(f_{\rho}m(t, x) \in L^2_0(\mathbb{R}^3_\lambda)\) and it is infinitely differentiable in \((t, x)\) in the norm of \(L^2_0(\mathbb{R}^3_\lambda)\) (see Appendix A). For the first derivative w.r.t. \(x_i\) this follows from

\[
m(t, x + (\Delta x)e_i) = m(t, x) + (\Delta x)(\partial_{x_i}m)(t, x) + (\Delta x)^2\int_0^1 ds (1 - s) (\partial^2_{x_i}m)(t, x + s(\Delta x)e_i),
\]

and the fact that \(|k|^{-1}\partial^\ell_{x_i}m(t, x)\) is bounded in \(k\) for any \(\ell \in \mathbb{N}_0\). For higher derivatives we simply replace \(m\) with \(\partial^\ell_{x_i}m\) in (3.17). The arguments regarding the derivatives w.r.t. \(t\) are analogous. Thus we can compute the derivatives using Lemma A.2, which gives the formulas from the statement of the lemma. \(\square\)

Now we analyze the regularized variants of the vectors from (3.12)

\[
\hat{g}^\sigma_{(t,x)}(p) := W(f_{\rho}m(t, x))\phi_{\rho, \sigma}.
\]

We note the following fact:

Lemma 3.4. There hold the bounds

\[
\begin{align*}
\|\partial^\ell_{\rho}\partial^\ell_t \hat{g}^\sigma_{(t,x)}(p)\|_F &\leq c \frac{(1 + \log(1 + |x| + |t|))^3}{\sigma^3}, \\
\|\partial^\ell_{\rho}\partial^\ell_{x_i} \hat{g}^\sigma_{(t,x)}(p)\|_F &\leq c \frac{(1 + \log(1 + |x| + |t|))^3}{\sigma^3},
\end{align*}
\]

for \(\ell, |\alpha|, |\beta| \leq 2\) and \(\sigma \in (0, \kappa_0]\). The \(x\) and \(t\) derivatives exist in the norm of \(F\). The derivatives w.r.t. \(p\) exist in the weak sense on the domain of finite particle vectors with compactly supported wave functions (cf. [RS2, p. 208]). The bound (3.20) still holds if \(\partial^\ell_{x_i}\sigma\) is replaced with \(H_i, P_{i,x}, P^2_{i,x}\) or \(\partial^\ell_{x_i}p\).

Proof. We consider only (3.20) for \(|\alpha| = 2, |\beta| = 2\) as the remaining cases are analogous and simpler. To handle the resulting expressions, it is convenient to define, for \(s \mapsto F_s\) as in Lemma A.2,

\[
\overline{\Phi}_s(F) := \Phi(\partial_j F_s) + \text{Im}(F, \partial_j F_s).
\]

Using this notation and recalling (3.16), we can write

\[
\partial_{x_j} \partial_j \hat{g}^\sigma_{(t,x)}(p) = W(f_{\rho}m)\left(i\overline{\Phi}_{x_j}(f_{\rho}m)i\overline{\Phi}_{x_j}(f_{\rho}m) + i\partial_{x_j} \overline{\Phi}_{x_j}(f_{\rho}m)\right)\phi_{\rho, \sigma} = W(f_{\rho}m)P_{x_j, x_j}(f_{\rho}m)\phi_{\rho, \sigma},
\]

(3.22)
where in the last step we denoted the expression in curly bracket by $P_{v,x}(f_p m)$ to further abbreviate the notation. Now we compute the first derivative w.r.t. momentum. We recall that these derivatives must only exist weakly on the domain of finite particle vectors, i.e., after taking a scalar product with such vectors. This will control the unbounded operators acting on $\phi_{p,\sigma}$ below and, in particular, allow us to differentiate $p \mapsto \phi_{p,\sigma}$ in (3.25) below. In this sense, we compute:

$$\partial_p \partial_{x_j} \partial_{x_j} \hat{g}^{\sigma(p)} = W(f_p m) i \tilde{\Phi}_{P_{p_j}}(f_p m) P_{x_i x_j}(f_p m) \phi_{p,\sigma} + W(f_p m) \partial_{P_{p_j}}(P_{x_i x_j}(f_p m)) \phi_{p,\sigma} + W(f_p m) P_{x_i x_j}(f_p m) \partial_p \phi_{p,\sigma}.$$  

(3.23)

Now we compute the respective contributions to $\partial_p \partial_{p_j} \partial_{x_j} \hat{g}^{\sigma(p)}$: (3.23) gives

$$\partial_p (W(f_p m) i \tilde{\Phi}_{P_{p_j}}(f_p m) P_{x_i x_j}(f_p m) \phi_{p,\sigma}) = W(f_p m) i \tilde{\Phi}_{P_{p_j}}(f_p m) \partial_{P_{p_j}}(P_{x_i x_j}(f_p m)) \phi_{p,\sigma} + W(f_p m) \partial_{P_{p_j}}(P_{x_i x_j}(f_p m)) \phi_{p,\sigma} + W(f_p m) P_{x_i x_j}(f_p m) \partial_p \phi_{p,\sigma}.$$  

(3.26)

(3.24)

From (3.24) we obtain

$$\partial_p (W(f_p m) \partial_{P_{p_j}}(P_{x_i x_j}(f_p m) \phi_{p,\sigma})) = W(f_p m) i \tilde{\Phi}_{P_{p_j}}(f_p m) \partial_{P_{p_j}}(P_{x_i x_j}(f_p m)) \phi_{p,\sigma} + W(f_p m) \partial_{P_{p_j}}(P_{x_i x_j}(f_p m)) \phi_{p,\sigma} + W(f_p m) P_{x_i x_j}(f_p m) \partial_p \phi_{p,\sigma}.$$  

(3.27)

(3.28)

From (3.25) we get

$$\partial_p (W(f_p m) P_{x_i x_j}(f_p m) \partial_p \phi_{p,\sigma}) = W(f_p m) i \tilde{\Phi}_{P_{p_j}}(f_p m) P_{x_i x_j}(f_p m) \partial_p \phi_{p,\sigma} + W(f_p m) \partial_{P_{p_j}}(P_{x_i x_j}(f_p m)) \partial_p \phi_{p,\sigma} + W(f_p m) P_{x_i x_j}(f_p m) \partial_p \partial_p \phi_{p,\sigma}.$$  

(3.29)

(3.30)

To estimate these expressions, we recall from Lemma 2.2 that $\partial_p \phi_{p,\sigma}$ are in the domain of any power of $N$ and $\|N^l \partial^p \phi_{p,\sigma}\|_F \leq c_l \sigma^{-\delta_0}$. Thus making use of the number bounds (A.2), we have

$$\|\partial_p \partial_{p_j} \partial_{x_j} \hat{g}^{\sigma(p)}\|_F \leq P(\|f_p m\|_2, \|f_p \partial_{x_j} m\|_2, \|f_p \partial_{x_j} \partial_{x_j} m\|_2) \sigma^{-\delta_0}.$$  

(3.31)

Here $P$ is a certain polynomial in the specified norms, which also includes $\|\partial_p f m\|_2$. We recall, however, that $f_p(k) := \lambda \frac{1}{\sqrt{2 \pi}} e^{\frac{-i}{\sigma^2} k \cdot e_k} |k|^{-1} e^{\frac{E_p}{2}}$, thus derivatives of $f_p$ w.r.t. $p$ only change the behaviour of this function in the angular variable $e_k$ but not in the $|k|$-variable. As our estimates are insensitive to the angular behaviour, we omitted these derivatives in the notation in (3.31). Making use of Lemma E.3, we have

$$\|f_p m\|_2 \leq c |\lambda| (1 + \log(1 + |x| + |t|))^{1/2}, \quad \|f_p \partial_{x_j} m\|_2 \leq c |\lambda|, \quad \|f_p \partial_{x_j} \partial_{x_j} m\|_2 \leq c |\lambda|.$$  

(3.32)

By inspection, we see that $P$ is at most of the sixth order in $\|f_p m\|_2$ (cf. (3.26)), which concludes the proof of estimates (3.19), (3.20).

As for the last statement of the lemma, the case of $H_t, P_{t,i}, P_{t,i}^2$ is covered by the fact that the derivatives w.r.t. $p$ should exist only weakly on vectors which belong to domains of these operators. After computing these derivatives one pulls $H_t, P_{t,i}, P_{t,i}^2$ to the right through the Weyl operator according to

$$H_t W(f_p m) = W(f_p m)(H_t + a^*(|k| f_p m) + a(|k| f_p m) + |||k|^{1/2} f_p m|^2).$$  

(3.33)
and applies Lemmas 2.2 and A.1. The case of \( P_{t,i} \partial_x \) requires more consideration as the derivative w.r.t. \( x_i \) should exist in the norm of \( \mathcal{F} \). To check that \( P_{t,i}W(f_pm)\phi_{p,\sigma} \) is partially differentiable w.r.t. \( x \) in the norm of \( \mathcal{F} \), we write, analogously to (3.33),

\[
P_{t,i}W(f_pm)\phi_{p,\sigma} = W(f_pm)(P_{t,i} + a^*(k_if_pm) + a(k_if_pm) + (f_p,jf_pm))\phi_{p,\sigma}
\]

and refer to Lemma A.2. By a computation we obtain

\[
\partial_x P_{t,i}W(f_pm)\phi_{p,\sigma} = P_{t,i}\partial_x(W(f_pm))\phi_{p,\sigma},
\]

where \( \partial_x(W(f_pm)) \) is the explicit formula from Lemma A.2, and then proceed as in the discussion of \( H_t, P_{t,i}, P_{t,i}^2 \) above. \( \square \)

Now we are ready to analyze the infraparticle vector (1.6).

**Lemma 3.5.** There is such \( \lambda_0 > 0 \) that for any \( |\lambda| \in (0, \lambda_0] \) and \( t \in \mathbb{R} \), the integral

\[
\Psi_t(x) := \int d^3p e^{i(p \cdot x-E_p t)} e^{-i\gamma(p,x,t)} h(p)W(f_pm(t,x))\phi_p
\]

has the following properties:

(a) \( \Psi_t \in L^2(\mathbb{R}^3; \mathcal{F}) \).

(b) \( \Psi_t \) is differentiable in \( t \) in the norm of \( L^2(\mathbb{R}^3; \mathcal{F}) \) and

\[
\partial_t \Psi_t(x) = \int d^3p e^{i(p \cdot x-E_p t)} (-iE_p + i\partial_t\gamma(p,x,t) + i\text{Im}(f_pm, f_p\partial_t m))e^{-i\gamma(p,x,t)} h(p)W(f_pm)\phi_p
\]

\[
+ \int d^3p e^{i(p \cdot x-E_p t)} e^{-i\gamma(p,x,t)} h(p)W(f_pm)(a^*(f_p\partial_t m) - a(f_p\partial_t m))\phi_p.
\]

**Proof.** As for (a), to prove that \( x \mapsto \Psi_t(x) \) is square integrable, we intend to apply Lemma 3.1. However, we lack information about the differentiability of \( p \mapsto \phi_p \). To circumvent this problem, we introduce an \( x \)-dependent cut-off \( \sigma_x := \kappa_{\lambda_0}/(1 + |x|)^M \), where \( M \) is sufficiently large but fixed. We insert into (3.36)

\[
\phi_p = (\phi_p - \phi_{p,\sigma_x}) + \phi_{p,\sigma_x}
\]

and obtain

\[
\Psi_t(x) = \int d^3p e^{i(p \cdot x-E_p t)} e^{-i\gamma(p,x,t)} h(p)W(f_pm)(\phi_p - \phi_{p,\sigma_x}) + \Psi^{p_{\sigma_x}}_t(x).
\]

Here \( \Psi^{p_{\sigma_x}}_t(x) \) is given by (3.36) with \( \phi_p \) replaced with \( \phi_{p,\sigma_x} \). Concerning the first term on the r.h.s. of (3.39), we have by (2.5)

\[
\| \int d^3p e^{i(p \cdot x-E_p t)} e^{-i\gamma(p,x,t)} h(p)W(f_pm)(\phi_p - \phi_{p,\sigma_x}) \|_{\mathcal{F}} \leq \frac{c(\kappa_{\lambda_0})^{1/5}}{(1 + |x|)^{M/5}}.
\]

Thus this term is manifestly in \( L^2(\mathbb{R}^3; \mathcal{F}) \) for \( 2M/5 > 3 \). As for the last term on the r.h.s. of (3.39), estimate (3.3) gives

\[
\|\Psi^{p_{\sigma_x}}_t\|_{\mathcal{H}} \leq ct^{1/2} \sup_{|\lambda| \leq 2} \frac{1}{(1 + |x|)^{1/2}} \| \partial_p e^{i\gamma(p,x,t)} \hat{g}_{\sigma_x}(p) \|_{\mathcal{F}}.
\]
The expression on the r.h.s. above is finite for any fixed $t$ by Lemmas 3.2, 3.4, provided $\delta_{\lambda_0}$ of Lemma 3.4 satisfies $M\delta_{\lambda_0} < 1/2$. This concludes the proof of (a).

Part (b) is a straightforward computation, provided we can show differentiability in the norm of $L^2(\mathbb{R}^3; \mathcal{F})$. To this end, we use the Taylor theorem (cf. formula (3.17))

$$\int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) \left( \frac{W(f_p m(t + \Delta t, x)) - W(f_p m(t, x))}{\Delta t} - \partial_t W(f_p m(t, x)) \right) \phi_p = \Delta t \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) \int_0^1 ds \, (1 - s) \left\{ \partial^2_t W(f_p m(\tau, x)) \right\} \phi_p. \quad (3.42)$$

Since $\phi_p$ is in the domain of any power of $H_t$ (cf. Lemma C.3), we can compute $\partial^2_t W(f_p m(\tau, x))$ using Lemma A.2. Next, exploiting the energy bounds (A.3) to control the creation and annihilation operators acting on $\phi_p$, we apply the shift (3.38) and estimate (2.5). Then, proceeding analogously as in part (a), we show that (3.42) tends to zero with $\Delta t \to 0$ in the norm of $L^2(\mathbb{R}^3; \mathcal{F})$. Differentiability in the norm of $L^2(\mathbb{R}^3; \mathcal{F})$ of other ingredients of (3.36) can be shown by analogous and simpler arguments. Now formula (3.37) follows by an application of Lemma A.2. □

**Lemma 3.6.** Vectors $\Psi_t \in L^2(\mathbb{R}^3; \mathcal{F})$, $t \in \mathbb{R}$, defined in (3.36) have the following properties:

(a) $\Psi_t$ is in the domain of $P_{t,i}$, $P^2_{t,i}$, $H_t$ and the following formula holds

$$\langle H_t \Psi_t \rangle(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) W(f_p m)(H_t^w + a^*(|k|f_p u) + a(|k|f_p u) \langle f_p, |k|f_p u \rangle - 2\text{Re}(f_p, |k|f_p u)) \phi_p. \quad (3.43)$$

(b) $\Psi_t$ is in the domain of $-i\partial_{x_i}$, $-i\partial_{\xi_i}$, $-i\partial_{x_i} P_{t,i}$ and the following formula holds

$$(-i\partial_{x_i} - P_{t,i})^2 \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) W(f_p m) \rho^w_{i,i} \phi_p. \quad (3.44)$$

(c) $\Psi_t$ is in the domain of $(a^*(v) + a(v))$ and the following formula holds

$$(a^*(v) + a(v)) \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) W(f_p m)((a^*(v) + a(v))^w + 2\text{Re}(f_p u, v)) \phi_p. \quad (3.45)$$

**Proof.** We start with some computations on $\mathcal{F}$ which are justified by Lemma A.2. Since $W(f_p m)\phi_p$ is in the domain of $H_t$, we can write for any fixed $t$

$$\begin{align*}
H_t \Psi_t(x) &= \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) H_t W(f_p m) \phi_p \\
&= \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) W(f_p m) (H_t + a^*(|k|f_p m) + a(|k|f_p m) + ||k||^2 f_p m ||^2 \phi_p \\
&= \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i(y(p,x,t))} h(p) W(f_p m) (H_t^w + a^*(|k|f_p m) + a(|k|f_p m) + \langle f_p, |k|f_p m \rangle - 2\text{Re}(f_p, |k|f_p m)) \phi_p, \quad (3.46)
\end{align*}$$

where we made use of $H_t^w = H_t - a^*(|k|f_p m) - a(|k|f_p m) + ||k||^2 f_p m ||^2$ and

$$-||k||^2 f_p m ||^2 + ||k||^2 f_p m ||^2 = \langle f_p, |k|f_p m \rangle - 2\text{Re}(f_p, |k|f_p m). \quad (3.47)$$
Analogously, we obtain for $\ell \in \{1, 2\}$,

$$
(P_{t,i})^{\ell} \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p) W(f_p m) \times 
\times \left( P_{t,i}^{\nu} + a^*(k_i f_p u) + a(k_i f_p u) + \langle f_p, k_i f_p \rangle - 2\text{Re}(f_p, k_i f_p u) \right)^{\ell} \phi_p. 
$$

(3.48)

Furthermore, by similar considerations as in (3.42), we can exchange $-i\partial_{x_i}$ with the $p$-integral and obtain the following

$$
-i\partial_{x_i} \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p)(p_i + \partial_{x_i} \gamma(p, x, t) + \text{Im}(f_p m, f_p \partial_{x_i} m)) W(f_p m) \phi_p 
+ \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p)(a^*(k_i f_p u) + a(k_i f_p u)) \phi_p. 
$$

(3.49)

Combining the above computations we also obtain

$$
-i\partial_{x_i} P_{t,i} \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p)(p_i + \partial_{x_i} \gamma(p, x, t) + \text{Im}(f_p m, f_p \partial_{x_i} m)) \times 
\times W(f_p m)(P_{t,i} + a^*(k_i f_p m) + a(k_i f_p m)) \phi_p 
+ \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p) W(f_p m)(P_{t,i} + a^*(k_i f_p m) + a(k_i f_p m)) \times 
\times (a^*(k_i f_p u) + a(k_i f_p u)) \phi_p = P_{t,i}(-i\partial_{x_i} \Psi_t(x)). 
$$

(3.50)

Thus we get from (3.48) and (3.49)

$$
(-i\partial_{x_i} - P_{t,i}) \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p) W(f_p m) \times 
\times (- P_{t,i}^{\nu} + p_i + \partial_{x_i} \gamma(p, x, t) + \text{Im}(f_p m, f_p \partial_{x_i} m) - \langle f_p, k_i f_p \rangle + 2\text{Re}(f_p, k_i f_p u) \phi_p 
= \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p) W(f_p m)(p_i - P_{t,i}^{\nu}) \phi_p, 
$$

(3.51)

where we used that

$$
\text{Im}(f_p m, f_p \partial_{x_i} m) - \langle f_p, k_i f_p \rangle + 2\text{Re}(f_p, k_i f_p u) = \text{Re}(f_p, k_i f_p u) = -\partial_{x_i} \gamma(p, x, t). 
$$

(3.52)

By iteration of (3.51)

$$
(-i\partial_{x_i} - P_{t,i})^\ell \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p) W(f_p m)(p_i - P_{t,i}^{\nu})^\ell \phi_p. 
$$

(3.53)

Finally, we obtain

$$
(a^*(v) + a(v)) \Psi_t(x) = \int d^3 p \, e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x)} h(p) W(f_p m)((a^*(v) + a(v))^w + 2\text{Re}(f_p u, v)) \phi_p. 
$$

(3.54)

One can see, by analogous arguments as in the proof of Lemma 3.5 (a), that all vectors above are in $L^2(\mathbb{R}^3_x, \mathcal{F})$: First, we apply the shift (3.39) and estimate the term involving $\phi_p - \phi_{p,ir_x}$ with the help of the bound (2.5). The presence of $H^I_t$ in (2.5) allows us to control both $P_{t,i}$ and the creation and annihilation operators acting on $\phi_p - \phi_{p,ir_x}$.
as for example in the case of (3.50). To the latter operators we apply the energy bounds (A.3) and note that all the resulting \( \| \cdot \|_{\omega} \)-norms are finite. Next, we study the term proportional to \( \phi_{p,\sigma} \), using Lemma 3.1. Staying with the case of (3.50), we can rewrite the relevant vector as \( P_{t_x}(-i\partial_x \Psi_{t_x}^r) \) and estimate the r.h.s. of (3.3) using Lemmas 3.2, 3.4. In particular, the last part of Lemma 3.4 plays a role here. From (3.53), (3.50) we also obtain that \( \{(i\partial_x)^2\Psi_t(x)\}_{x \in \mathbb{R}^3} \) is in \( L^2(\mathbb{R}^3_x; \mathcal{F}) \). This concludes the proof. \( \square \)

**Proof of Theorem 1.1.** We recall that \( \psi_t(x) := \frac{1}{(2\pi)^{3/2}} e^{iH_t} e^{-iP_{t_x}x} \phi_{t_x} \). By Lemma 3.5, \( t \mapsto \Phi_t \) is differentiable in the norm in \( L^2(\mathbb{R}^3_x; \mathcal{F}) \). Next, by applying the Stone theorem to \( e^{iH_t} \), we obtain the differentiability of \( t \mapsto \psi_t \) in the norm of \( L^2(\mathbb{R}^3_x; \mathcal{F}) \), provided that the vector \( \{e^{-iP_{t_x}x} \Phi_t(x)\}_{x \in \mathbb{R}^3} \in L^2(\mathbb{R}^3_x; \mathcal{F}) \) is in the domain of \( H \). This is easily checked using Lemma 3.6. In particular, to verify that this vector is in the domain of \( (i\partial_x)^2 \), we apply the Stone theorem to \( x \mapsto e^{-iP_{t_x}x} \) and use that \( \Phi_t \) is in the domain of \( P_x^2 \). Now we compute

\[
\partial_t \psi_t(x) = \frac{1}{(2\pi)^{3/2}} e^{iH_t} iH e^{-iP_{t_x}x} \Phi_t(x) + \frac{1}{(2\pi)^{3/2}} e^{iH_t} e^{-iP_{t_x}x} \sigma \Phi_t(x) \]

\[
= \frac{1}{(2\pi)^{3/2}} e^{iH_t} e^{-iP_{t_x}x} \left( \frac{1}{2}(i\partial_x - P_{t_x})^2 \Phi_t(x) + H_t \Phi_t(x) + (a^*(v) + a(v)) \Phi_t(x) - i\sigma \Phi_t(x) \right) \]

\[
= \frac{1}{(2\pi)^{3/2}} e^{iH_t} e^{-iP_{t_x}x} \int d^3p e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x, t)} \gamma_{\text{int}}(p, x, t) h(p) W(f_p m) \phi_p, \tag{3.55}
\]

where in the last step we made use of the formulas in Lemmas 3.5, 3.6, the fact that \( H^w \phi_p = E_p \phi_p \), and of the relations

\[
\langle f_p, |k| f_p \rangle - 2\text{Re} \langle f_p, |k| f_p u \rangle + \partial_t \gamma(p, x, t) + \text{Im} \langle f_p m, f_p \sigma \rangle = 0, \quad 2\text{Re} \langle f_p u(t, x), v \rangle = \gamma_{\text{int}}(p, x, t), \tag{3.56}
\]

where \( v \) appeared in (1.2). To show (1.9), we proceed similarly as in the proof of Lemma 3.5: We choose a \((t, x)\)-dependent cut-off as follows: \( \sigma_{(t, x)} = \kappa_{\delta_2}/(1 + |t| + |x|) \) where \( M \in \mathbb{N} \) is fixed. We make a shift \( \phi_p = (\phi_p - \phi_{p,\sigma_{(t, x)}}) + \phi_{p,\sigma_{(t, x)}} \) and insert it into the formula for the norm of \( \partial_t \psi_t \):

\[
\| \partial_t \psi_t \|_H \leq \frac{1}{(2\pi)^{3/2}} \left\{ \int d^3p e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x, t)} \gamma_{\text{int}}(p, x, t) h(p) W(f_p(e^{-i|k|t} + ik \cdot x - 1))(\phi_p - \phi_{p,\sigma_{(t, x)}}) \right\}_{x \in \mathbb{R}^3} \|_H \]

\[
+ \frac{1}{(2\pi)^{3/2}} \left\{ \int d^3p e^{i(p \cdot x - E_p t)} e^{i\gamma(p, x, t)} \gamma_{\text{int}}(p, x, t) h(p) W(f_p(e^{-i|k|t} + ik \cdot x - 1))\phi_{p,\sigma_{(t, x)}} \right\}_{x \in \mathbb{R}^3} \|_H. \tag{3.57}
\]

We note that by (2.5) the term involving \( (\phi_p - \phi_{p,\sigma_{(t, x)}}) \) is integrable in \( t \) in the norm of \( L^2(\mathbb{R}^3_x; \mathcal{F}) \) for \( M \) sufficiently large. Our strategy to estimate the second term on the r.h.s. of (3.57) is to combine Lemma 3.1, Lemma 3.7 and Lemma 2.2. In our case \( g \) of Lemma 3.1 has the form

\[
g_{(t, x)}(p) := e^{i\gamma(p, x, t)} \gamma_{\text{int}}(p, x, t) h(p) W(f_p(e^{-i|k|t} + ik \cdot x - 1))\phi_{p,\sigma_{(t, x)}}. \tag{3.58}
\]

We rewrite this expression as follows:

\[
g_{(t, x)}(p) = e^{i\gamma(p, x, t)} \gamma_{\text{int}}(p, x, t) h(p) \delta_{\phi_{(t, x)}}(p), \quad \delta_{\phi_{(t, x)}}(p) := W(f_p(e^{-i|k|t} + ik \cdot x - 1))\phi_{p,\sigma}. \tag{3.59}
\]

First, we note that by Lemma 3.7 below, for \( c_0 \) as in Lemma 3.1,

\[
|\partial_p^2 \gamma_{\text{int}}(p, x, t)| \leq |p|^2 \frac{c_0}{t^M} \text{ for } |x|/t \leq c_0 < 1, \tag{3.60}
\]

\[
|\partial_p^2 \gamma_{\text{int}}(p, x, t)| \leq |p|^2 \frac{c_0}{t} \log t \text{ for } |x|/t \geq c_0, \tag{3.61}
\]
and $|\alpha| = 0, 1, 2$. Furthermore, we have by Lemma 3.2

$$|\partial_\rho e^{i\rho(p, x, t)}| \leq c(1 + \log(1 + |t| + |x|))^2.$$  \hspace{1cm} (3.62)

Given (3.60)–(3.62) and Lemma 3.1 and Lemma 3.4, for any $0 < \varepsilon < 1/2$ we can choose $\lambda_0$ so small, that

$$\|\partial_t \psi_t\|_{H^1} \leq |\lambda|^2 \frac{c}{T^{3/2 - \varepsilon}}$$  \hspace{1cm} (3.63)

which concludes the proof of (1.9). Hence, by the Cook method, we obtain the existence of the limit $\psi^+$. To see that $\psi^+ \neq 0$ under the specified conditions, we write

$$\|\psi^{+, (l)}\|_{H^1} \geq \|\psi^{+, (l)}_{t = 0}\|_{H^1} - \int_0^\infty dt \|\partial_t \psi^{+, (l)}_t\|_{H^1},$$  \hspace{1cm} (3.64)

where we included the dependence on $\lambda$ explicitly in the notation. We recall that all constants in our discussion are uniformly bounded in $|\lambda| \in (0, \lambda_0]$. Thus by estimate (3.63), the second term on the r.h.s. of (3.64) tends to zero as $\lambda \to 0$. So it suffices to show that $\|\psi^{+, (l)}_{t = 0}\|_{H^1}$ is bounded from below uniformly in $\lambda$ from some neighbourhood of zero. We collect the relevant ingredients: First, we recall that by [DP18, formula (5.2)]

$$\|\phi_p^{+, (l)} - \Omega\|_F \leq c|\lambda|^{1/4}.$$  \hspace{1cm} (3.65)

Furthermore, we obtain from (3.32), (E.4)

$$\|f_p^{+, (l)}(e^{-i(p + ik \cdot x)} - 1)\|_2 \leq c|\lambda|(1 + \log(1 + |t| + |x|))^{1/2}, \quad |\gamma(p, x, t)| \leq c|\lambda|^2(1 + \log(1 + |t| + |x|)).$$  \hspace{1cm} (3.66)

Considering the above, we have

$$\psi^{+, (l)}_{t = 0}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 p e^{ip \cdot x} e^{iy^{+, (l)}(p, x, 0)} h(p) W(f_p^{+, (l)}(e^{ik \cdot x} - 1)) \phi_p^{+, (l)}$$  \hspace{1cm} (3.67)

$$\quad = \frac{1}{(2\pi)^{3/2}} \int d^3 p e^{ip \cdot x} e^{iy^{+, (l)}(p, x, 0)} h(p) W(f_p^{+, (l)}(e^{ik \cdot x} - 1))(\phi_p^{+, (l)} - \Omega)$$  \hspace{1cm} (3.68)

$$\quad + \frac{1}{(2\pi)^{3/2}} \int d^3 p e^{ip \cdot x} e^{iy^{+, (l)}(p, x, 0)} h(p)(W(f_p^{+, (l)}(e^{ik \cdot x} - 1)) - 1)\Omega$$  \hspace{1cm} (3.69)

$$\quad + \frac{1}{(2\pi)^{3/2}} \int d^3 p e^{ip \cdot x} (e^{iy^{+, (l)}(p, x, 0)} - 1)h(p)\Omega$$  \hspace{1cm} (3.70)

$$\quad + \frac{1}{(2\pi)^{3/2}} \int d^3 p e^{ip \cdot x} h(p)\Omega.$$  \hspace{1cm} (3.71)

Thus it is manifest from estimates (3.66), (3.65), combined with an argument as in (A.9), that

$$\psi^{+, (l)}_{t = 0}(x) = (F^{-1} h)(x)\Omega + O(|\lambda|^{1/4}(1 + \log(1 + |x|))),$$  \hspace{1cm} (3.72)

where $O(|\lambda|^{1/4}(1 + \log(1 + |x|)))\|_F \leq c|\lambda|^{1/4}(1 + \log(1 + |x|))$. Clearly, we can write for any compact subset $\Delta \subset \mathbb{R}^3$

$$\|\psi^{+, (l)}_{t = 0}\|_{H^1} \geq \left( \int_\Delta d^3 x \|\psi^{+, (l)}_{t = 0}(x)\|_F^2 \right)^{1/2}$$  \hspace{1cm} (3.73)

$$\geq \left( \int_\Delta d^3 x (F^{-1} h)(x)^2 \right)^{1/2} - c|\lambda|^{1/4}\left( \int_\Delta d^3 x (1 + \log(1 + |x|))^2 \right)^{1/2}.$$  \hspace{1cm} (3.73)

For any $\Delta$ intersecting with the support of $F^{-1} h$ the first term in the second line of (3.73) is positive and independent of $\lambda$. As the second term tends to zero as $\lambda \to 0$, this concludes the proof. $\Box$
Lemma 3.7. Consider the expression
\[ \gamma_{\text{int}}(p,x,t) := 2 \int d^3 k \, f_p(k)^2 (|k| - k \cdot \nabla E_p) \cos(|k|t - k \cdot x). \] (3.74)

The following bounds hold:

(a) Fix some \(0 < c_0 < 1\). For any \(M \in \mathbb{N}\) there exists a constant \(c_M\), uniform in \(p \in S\), s.t.
\[ \sup_{|k|/t \leq c_0} |\gamma_{\text{int}}(p,x,t)| \leq |t|^2 \frac{c_M}{t^M}. \] (3.75)

The same is true if the supremum is taken over \(|k|/t \geq c_1 > 1\).

(b) For all \(p \in S\) and \((t,x) \in \mathbb{R}^4\)
\[ |\gamma_{\text{int}}(p,x,t)| \leq |t|^2 \frac{c}{t} \log t. \] (3.76)

Analogous estimates hold if we replace \(f_p(k)^2 (|k| - k \cdot \nabla E_p)\) in (3.74) by its arbitrary derivatives w.r.t. \(p\).

Proof. Proceeding to spherical coordinates, we have
\[ \gamma_{\text{int}}(p,x,t) = \int d\Omega(e_k) \int_0^\infty d|k| \, f(|k|, e_k, p) \cos(|k|t - e_k \cdot v)) \cdot f(|k|, e_k, p) := |t|^2 X_k(k)^2 \frac{1}{2 (1 - e_k \cdot \nabla E_p)}, \] (3.77)
where we set \(v := x/t\). We suppose that \(|v| - 1| \geq \varepsilon > 0\) and consider part (a) of the lemma. By integrating by parts w.r.t. \(|k|\) and exploiting that sine vanishes at zero, we obtain
\[ \gamma_{\text{int}}(p,x,t) = - \int d\Omega(e_k) \int_0^\infty d|k| \, \partial_{|k|} f(|k|, e_k, p) \frac{1}{t(1 - e_k \cdot v)} \sin(|k|t(1 - e_k \cdot v)). \] (3.78)

Concerning (a), we can continue integrating by parts, exploiting that \(\partial_{|k|} f\) vanishes in a fixed neighbourhood of zero. The fact that \(1 - e_k \cdot v\) is never zero in this case gives the claim.

Proceeding to (b), we suppose that \(|v| - 1| \leq \varepsilon\) for some \(0 < \varepsilon < 1\), in particular \(|v|\) is isolated from zero by some interval independent of \(p, x, t\). We choose the third axis in the direction of \(v\) and write
\[
\begin{align*}
\gamma_{\text{int}}(p,x,t) &= \int_{|k| \geq 1/t} d|k| \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta) f(|k|, e(\cos(\theta), \varphi), p) \cos(|k|t(1 - |v| \cos(\theta))) + O(t^{-1}) \\
&= - \int_{|k| \geq 1/t} d|k| \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta) f(|k|, e(\cos(\theta), \varphi), p) \frac{1}{t|k||v|} \sin(|k|t(1 - |v| \cos(\theta))) + O(t^{-1}) \\
&= - \int_{|k| \geq 1/t} d|k| \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta) f(|k|, e(\cos(\theta), \varphi), p) \frac{1}{t|k||v|} \sin(|k|t(1 - |v| \cos(\theta))) |_{\cos \theta = 1}^{\cos \theta = 1} + O(t^{-1}) \\
&+ \int_{|k| \geq 1/t} d|k| \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos(\theta) \left( \frac{d}{d \cos(\theta)} f(|k|, e(\cos(\theta), \varphi), p) \frac{1}{t|k||v|} \sin(|k|t(1 - |v| \cos(\theta))) \right). \tag{3.79}
\end{align*}
\]

By estimating \(|\sin(|k|t(1 - |v| \cos(\theta)))| \leq 1\) everywhere above and using that the integration in \(|k|\) is over a compact set, the claim follows from
\[ \frac{1}{t} \int_{|k| \geq 1/t} d|k| \frac{1}{|k|} \leq c' \frac{|\log(t)|}{t}. \] (3.80)

This concludes the proof. \(\square\)
4 Conclusions

In this paper we proposed a new construction of infraparticle states in the massless Nelson model. The approximating sequence does not involve infrared cut-offs and the proof of convergence is relatively simple: Taking the spectral results from [AH12, DP18, DP18.1] for granted, it amounts to the Cook method combined with the stationary phase method, like for basic Schrödinger operators. It is legitimate to ask how the new infraparticle state compares with the established knowledge on the infrared problem in the Nelson model. To partially answer this question we provide some heuristic remarks on the relation of our states to the Faddeev-Kulish approach. First, we note that the asymptotically dominant part of the wave packet (1.6) should propagate along the ballistic trajectory \( x = \nabla E_p t \), thus \( \psi_t \) should have the same limit as

\[
\psi_t^D(x) := e^{iH_1 t} \frac{1}{(2\pi)^{3/2}} \int d^3 p \, h(p) \, e^{-i(E_p + H_1)t} e^{i\gamma(p, \nabla E_p t)} W(f_p(1 - e^{i[\mathcal{L} - ik \cdot \nabla E_p]t})) e^{iH_1 t} \frac{1}{(2\pi)^{3/2}} e^{i(p - P) \cdot x} \phi_p. \tag{4.1}
\]

To proceed, let us second quantize also the electrons, denote their creation and annihilation operators by \( b^\dagger \) and the common vacuum of the electrons and photons by \( \Omega \). Expressing \( \phi_p \in \mathcal{F} \) by its \( n \)-particle wave functions \( f_{w,p} \), we define its renormalized creation operator in a standard manner [Al73]:

\[
\hat{b}_{w}^\dagger(p) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int d^3 k \, f_{w,p}^n(k_1, \ldots, k_n) a^\dagger(k_1) \ldots a^\dagger(k_n) \, b^\dagger(p - (k_1 + \cdots + k_n)), \tag{4.2}
\]

so that \( \frac{1}{(2\pi)^{3/2}} e^{i(p - P) \cdot x} \phi_p \) can be identified with \( \hat{b}_{w}^\dagger(p) \Omega \). Now recalling that \( f_p(k) = \gamma(k) \frac{1}{|k| - k \cdot \nabla E_p} \), we can write

\[
W(f_p(1 - e^{i[\mathcal{L} - ik \cdot \nabla E_p]t})) = \exp \left(-i \int_0^t d\tau \, e^{iH_1\tau}\{a^\dagger(ve^{-ik \cdot \nabla E_p}\tau) + a(ve^{-ik \cdot \nabla E_p}\tau)\}e^{-iH_1\tau}\right) = e^{iC_p t} e^{i\gamma(p, \nabla E_p, t) + t} \text{Exp} \left(-i \int_0^t d\tau \, e^{iH_1\tau}\{a^\dagger(ve^{-ik \cdot \nabla E_p}\tau) + a(ve^{-ik \cdot \nabla E_p}\tau)\}e^{-iH_1\tau}\right). \tag{4.3}
\]

where \( C_p := \int d^3 k \, \frac{|\gamma(k)|^2}{|k| - k \cdot \nabla E_p} \) is finite and the time-ordered exponential \( U_D(t) := \text{Exp}(\ldots) \) is the Dollard modifier of the Nelson model, cf [Dy17, formula (3.6)]. Thus (4.1) can be rewritten as

\[
\psi_t^D = e^{iH_1 t} \int d^3 p \, h(p) \, e^{-i(E_p + H_1 - C_p)t} U_D(t) e^{iH_1 t} \hat{b}_{w}^\dagger(p) \Omega. \tag{4.4}
\]

We recall from [Dy17], that a direct application of the Faddeev-Kulish prescription to the Nelson model leads to a formula which differs from (4.4) only by a substitution \( \hat{b}_{w}^\dagger(p) \rightarrow b^\dagger(p) \). We believe that this discrepancy can be attributed to the quantum mechanical origin of the Dollard formalism which makes it difficult to reconcile with the electron mass renormalization present in the model. We think that formula (4.4) is a correct implementation of the Faddeev-Kulish formalism in the Nelson model and hope that the findings of the present paper will lead to a rigorous proof of convergence of \( \psi_t^D \) as \( t \to \infty \).

A Energy bounds and derivatives of the Weyl operators

We introduce the following subspace of \( L^2(\mathbb{R}^3_+) \):

\[
L^2_{\omega}(\mathbb{R}^3_+) := \{ f \in L^2(\mathbb{R}^3_+) \mid ||f||_\omega := ||(1 + |\mathcal{L}|^{-1/2})f||_2 < \infty \}. \tag{A.1}
\]

We recall that \( N := d\Gamma(1), H_1 := d\Gamma(|\mathcal{L}|) \) and state the standard energy and number bounds [BR2]:
Lemma A.1. Let \( f_1, \ldots, f_n \in L^2(\mathbb{R}^3_k) \). Then
\[
\|a(\omega) f_1 \cdots a(\omega) f_n (1 + N)^{-n/2}\|_F \leq c_n \|f_1\|_2 \cdots \|f_n\|_2.
\] (A.2)

Let \( f_1, \ldots, f_n \in L^2_\omega(\mathbb{R}^3_k) \). Then
\[
\|a(\omega) f_1 \cdots a(\omega) f_n (1 + H_1)^{-n/2}\|_F \leq c_n \|f_1\|_\omega \cdots \|f_n\|_\omega.
\] (A.3)

Formula (A.4) below is also well-known, but we provide a proof for the reader’s convenience.

Lemma A.2. Let \( \mathbb{R} \ni s \mapsto F_s \in L^2_\omega(\mathbb{R}^3_k) \) be differentiable in the norm \( \| \cdot \|_\omega \). Then \( s \mapsto W(F_s) \psi, \psi \in D(H_1^{1/2}) \), is differentiable in the norm of \( \mathcal{F} \) and
\[
\partial_s W(F_s) \psi = W(F_s) (a(\partial_s F_s) - a(\partial_s F_s) + i \text{Im}(F_s, \partial_s F_s)) \psi.
\] (A.4)

Also, \( s \mapsto a(\omega)(F_s) \psi \) is differentiable w.r.t. \( s \) in the norm of \( \mathcal{F} \) and \( \partial_s a(\omega)(F_s) \psi = a(\omega)(\partial_s F_s) \psi \). If \( \psi \in D(N^{1/2}) \) then analogous statements hold for \( s \mapsto F_s \) differentiable in the norm \( \| \cdot \|_2 \).

**Proof.** Using the Weyl relations \( W(F) W(G) = e^{-i \text{Im}(F,G)} W(F + G) \), \( F, G \in L^2(\mathbb{R}^3_k) \),
\[
\frac{1}{\Delta s} (W(F_{s + \Delta s}) - W(F_s)) = W(F_s) \frac{1}{\Delta s} (W(-F_s) W(F_{s + \Delta s}) - 1)
\]
\[
= W(F_s) \frac{1}{\Delta s} (e^{i \text{Im}(F_s, F_{s + \Delta s} - F_s)} W(F_{s + \Delta s} - F_s) - 1)
\]
\[
= W(F_s) \frac{1}{\Delta s} (e^{i \text{Im}(F_s, F_{s + \Delta s} - F_s)} - 1) W(F_{s + \Delta s} - F_s) + W(F_s) \frac{1}{\Delta s} (W(F_{s + \Delta s} - F_s) - 1).
\] (A.5)

Considering (A.5), we obtain immediately
\[
\lim_{\Delta s \to 0} \frac{1}{\Delta s} (e^{i \text{Im}(F_s, F_{s + \Delta s} - F_s)} - 1) = i \text{Im}(F_s, \partial_s F_s).
\] (A.7)

Furthermore, it is easy to see that in the norm of \( \mathcal{F} \)
\[
\lim_{\Delta s \to 0} W(F_{s + \Delta s} - F_s) \psi = \psi.
\] (A.8)

In fact, denoting \( \Phi(F) := a(\omega)(-iF) + a(-iF) \), we can write
\[
W(F_{s + \Delta s} - F_s) \psi = \psi + \left( \frac{e^{\Phi(F_{s + \Delta s} - F_s)} - 1}{\Phi(F_{s + \Delta s} - F_s)} \right) \Phi(F_{s + \Delta s} - F_s) \psi.
\] (A.9)

By the spectral theorem, the norm of the expression in curly bracket above is bounded uniformly in \( \Delta s \). On the other hand, by the assumed form of differentiability
\[
\Phi(F_{s + \Delta s} - F_s) \psi = \Delta s \Phi(\partial_s F_s) \psi + \Phi(o(\Delta s)) \psi,
\] (A.10)

where \( \partial_s F_s \in L^2_\omega(\mathbb{R}^3_k) \) and the rest term satisfies
\[
\lim_{\Delta s \to 0} \frac{\|o(\Delta s)\|_\omega}{\Delta s} = 0.
\] (A.11)
Thus, by the energy bounds of Lemma A.1 we obtain that (A.10) tends to zero in the norm of \( \mathcal{F} \) as \( \Delta s \to 0 \) which gives (A.8).

Concerning (A.6), we write again \( F_{s+\Delta s} - F_s = \Delta s \partial_s F_s + o(\Delta s) \), which gives
\[
\frac{1}{\Delta s} (W(F_{s+\Delta s} - F_s) - 1) \psi = \frac{1}{\Delta s} (e^{-i\text{Im}(\Delta s \partial_s F_s, o(\Delta s))} W(\Delta s \partial_s F_s) W(o(\Delta s)) - 1) \psi.
\]

(A.12)

To take the limit \( \Delta s \to 0 \) above, we note
\[
\lim_{\Delta s \to 0} \frac{1}{\Delta s} (e^{-i\text{Im}(\Delta s \partial_s F_s, o(\Delta s))} - 1) = 0, \quad \lim_{\Delta s \to 0} \frac{1}{\Delta s} (W(o(\Delta s)) - 1) \psi = 0,
\]

(A.13)

where the latter limit is computed as in (A.10) using (A.11). Also, we exploit that by the Stone theorem
\[
\lim_{\Delta s \to 0} \frac{1}{\Delta s} (W(\Delta s \partial_s F_s) - 1) \psi = i\Phi(\partial_s F_s) \psi.
\]

(A.14)

Finally, substituting (A.14), (A.7) to (A.5), (A.6), we obtain (A.4). The last statement of the lemma is proven by analogous arguments. \( \square \)

## B Proof of Lemma 2.2

We write \( \phi_{p,\sigma} = \{f_{w,p,\sigma}^n\}_{n \in \mathbb{N}_0} \) in terms of the Fock space wave functions. Given Lemma B.1 and formula (B.3) below, we can write
\[
\|H_1^f N_{\ell,2} \partial_\sigma^p \phi_{p,\sigma}\|_\mathcal{F} \leq \left( \sum_{n=0}^{\infty} n^{2(\ell_1+\ell_2)} \|\partial_\sigma^p f_{w,p,\sigma}^n\|_2^2 \right)^{1/2} \leq \frac{c}{\sigma^{\delta_{\lambda_0}}} \left( \sum_{n=0}^{\infty} \frac{1}{n!} (c_{\ell_1,\ell_2} \kappa)^n \log \sigma^n \right)^{1/2} \leq \frac{c}{\sigma^{\delta_{\lambda_0}}},
\]

(B.1)

for some constants \( c_{\ell_1,\ell_2} \) and \( \delta_{\lambda_0} > 0 \) which tends to zero as \( \lambda_0 \to 0 \). To handle the powers of \( H_1 \) we used that the UV cut-off \( \kappa = 1 \) and consequently the wave functions \( f_{w,p,\sigma}^n \) are supported in unit balls in each variable \( k_1, \ldots, k_n \) separately. This gives Lemma 2.2.

In preparation for the proof of Lemma B.1, we state a general relation for wave functions of a Fock space vector:
\[
f_{w,p,\sigma}^n(k_1, \ldots, k_n) = \frac{1}{\sqrt{n!}} \langle \Omega, a(k_1) \ldots a(k_n) \phi_{p,\sigma} \rangle.
\]

(B.2)

This formula is meaningful by considerations in [DP18, Appendix D]. Let us now introduce the following auxiliary functions:
\[
g_0^\sigma := c \quad \text{and} \quad g_\sigma^n(k_1, \ldots, k_n) := \prod_{i=1}^n \frac{c_i \kappa_1^{(\sigma,\kappa_1)(k_i)}}{|k_i|^{3/2}}, \quad n \geq 1,
\]

(B.3)

where \( \kappa_1 := (1 - \varepsilon_0)^{-1} \kappa \) is slightly larger than \( \kappa \) and \( 0 < \varepsilon_0 < 1 \) was introduced below (1.2).

**Lemma B.1.** The following estimates hold
\[
|\partial_\sigma^p f_{w,p,\sigma}^n(k_1, \ldots, k_n)| \leq \frac{1}{\sqrt{n!}} \left( \frac{1}{\sigma^{\delta_{\lambda_0}}} \right)^{|\sigma|} g_\sigma^n(k_1, \ldots, k_n) \quad \text{for} \quad |\sigma| \leq 2.
\]

(B.4)
Proof. In [DP18.1, formula (4.42)] the following functions are introduced\(^{2}\)

\[
\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n) := W^*(f_{p,\sigma})a(k_1) \ldots a(k_n)\phi_{p,\sigma},
\]

where \(W(f_{p,\sigma}) := e^{f_{p,\sigma} - a f_{p,\sigma}}\) and the r.h.s. above is well-defined by considerations from [DP18, Appendix D]. In Proposition 4.7 of [DP18.1] and in the subsequent discussion in this reference the following bounds are shown

\[
||\partial_\alpha \hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)||_F \leq \left( \frac{1}{\alpha!} \right) \delta_{\alpha}(k_1, \ldots, k_n) \quad \text{for} \quad ||\alpha|| \leq 2.
\]

In view of (B.3) the r.h.s. depends on numerical constants, whose dependence on various parameters is specified at the end of Section 2. We note that for \(||\alpha|| = 0\) (B.4) follows immediately from (B.6) and (B.2).

As for the case \(||\alpha|| = 1\), we can write

\[
\partial_{p_i}(a(k_1) \ldots a(k_n)\phi_{p,\sigma}) = \partial_{p_i}(W(f_{p,\sigma})\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)).
\]

(B.7)

The term in which the derivative acts on \(\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)\) is immediately estimated using (B.6) for \(||\alpha|| = 1\). As for the remaining term, we estimate

\[
||\partial_{p_i}(W(f_{p,\sigma})\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)||_F
\leq 2||a(\partial_{p_i}f_{p,\sigma})W(f_{p,\sigma})\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)||_F + ||\partial_{p_i}f_{p,\sigma}||_2 ||\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)||_F
\leq 2 \int d^3k_0 ||(\partial_{p_i}f_{p,\sigma})(k_0)|| \hat{f}_n^{p,\sigma+1}(k_0, k_1, \ldots, k_n)||_F + c|\log(\sigma)|^{1/2} \hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)||_F.
\]

(B.8)

For differentiability of the Weyl operator we refer to Lemma A.2 and the fact that \(\hat{f}_n^{p,\sigma}\) is in the domain of \(H_t^{1/2}\) (cf. [DP18, formula (D.8)]). The bound on \(||\partial_{p_i}f_{p,\sigma}||_2\) follows from Lemma E.4. This, together with (B.6), gives (B.4) for \(||\alpha|| = 1\).

Now we consider the case \(||\alpha|| = 2\). Again, we can write

\[
\partial_{p_i}\partial_{p_j}(a(k_1) \ldots a(k_n)\phi_{p,\sigma}) = \partial_{p_i}\partial_{p_j}(W(f_{p,\sigma})\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)).
\]

(B.10)

The term in which both derivatives act on \(\hat{f}_n^{p,\sigma}\) is immediately estimated using (B.6). Let us consider the term in which one derivative acts on \(W(f_{p,\sigma})\) and another on \(\hat{f}_n^{p,\sigma}\). Similarly as in (B.8), we have

\[
||\partial_{p_i}(W(f_{p,\sigma})\hat{f}_n^{p,\sigma} (k_1, \ldots, k_n)||_F \leq 2||a(\partial_{p_i}f_{p,\sigma})\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)||_F + ||\partial_{p_j}f_{p,\sigma}||_2 ||\hat{f}_n^{p,\sigma}(k_1, \ldots, k_n)||_F.
\]

(B.11)

The last term on the r.h.s. of (B.11) clearly satisfies the required bound by (B.6) and Lemma E.4. As for the first term above, we note that

\[
a(\partial_{p_i}f_{p,\sigma})\partial_{p_j}(\hat{f}_n^{p,\sigma})(k_1, \ldots, k_n) = -\left( \int d^3k_0 (\partial_{p_i}f_{p,\sigma})(k_0) f_{p,\sigma}(k_0) \right) \partial_{p_j}(W(f_{p,\sigma})^*a(k_1) \ldots a(k_n)\phi_{p,\sigma})
+ \int d^3k_0 (\partial_{p_i}f_{p,\sigma})(k_0) \partial_{p_j}(W(f_{p,\sigma})^*a(k_0)a(k_1) \ldots a(k_n)\phi_{p,\sigma}).
\]

(B.12)

\(^{2}\)For consistency with the notation from [DP18, DP18.1, DP19], we use similar symbols for several different functions: \(f_{p,\sigma}\) are defined in (2.4), \(\hat{f}_n^{p,\sigma}\) in (B.5) and \(f_n^{p,\sigma}\) are the wave functions of \(\phi_{p,\sigma}\).
where we first computed the derivative of \( \hat{f}^n \) and then used \( a(g)W(f_{p,\sigma})^* = W(f_{p,\sigma})^*(a(g) - \langle g, f_{p,\sigma} \rangle) \) for \( g = \partial_{p,j}f_{p,\sigma} \). The last expression is immediately estimated using (B.6) for \( |\alpha| = 1 \). We still have to estimate a contribution to (B.10), where both derivatives act on \( W(f_{p,\sigma}) \):

\[
\|\partial_{p,j}\partial_{p,j}(W(f_{p,\sigma}))\hat{f}^n_{p,\sigma}(k_1, \ldots, k_n)\|_F \\
\leq \|\langle a^*(\partial_{p,j}f_{p,\sigma}) - a(\partial_{p,j}f_{p,\sigma}) \rangle W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1, \ldots, k_n)\|_F + \|\langle a^*(\partial_{p,j}f_{p,\sigma}) - a(\partial_{p,j}f_{p,\sigma}) \rangle W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1, \ldots, k_n)\|_F.
\]

(B.13)

This expression can be estimated by a linear combination of terms of the form:

\[
\|\partial_{p,j}^{\alpha_1}f_{p,\sigma}\|_2 \|\partial_{p,j}^{\alpha_2}f_{p,\sigma}\|_2 \|\hat{f}^n_{p,\sigma}(k_1, \ldots, k_n)\|_F,
\]

(B.15)

\[
\|\partial_{p,j}^{\alpha_1}f_{p,\sigma}\|_2 \|a(\partial_{p,j}^{\alpha_2}f_{p,\sigma})W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1, \ldots, k_n)\|_F,
\]

(B.16)

\[
\|a(\partial_{p,j}^{\alpha_1}f_{p,\sigma})a(\partial_{p,j}^{\alpha_2}f_{p,\sigma})W(f_{p,\sigma})\hat{f}^n_{p,\sigma}(k_1, \ldots, k_n)\|_F,
\]

(B.17)

where \( |\alpha_1|, |\alpha_2| \leq 2 \). Expression (B.15) is estimated using (B.6) for \( |\alpha| = 0 \) and Lemma E.4. Expression (B.16) is estimated as in (B.9). As for (B.17), it can be bounded by

\[
(B.17) \leq \int d^3k'd^3k'' |\partial_{p,j}^{\alpha_1}f_{p,\sigma}(k')||\partial_{p,j}^{\alpha_2}f_{p,\sigma}(k'')|\|\hat{f}^{n+2}_{p,\sigma}(k', k'', k_1, \ldots, k_n)\|_F,
\]

(B.18)

which is estimated with the help of (B.6) for \( |\alpha| = 0 \) and Lemma E.4. □

\section*{C Proof of estimate (2.5)}

\textbf{Lemma C.1.} For any \( \ell \in \mathbb{N}_0 \) the maximal coupling constant \( \lambda_0 > 0 \) can be chosen sufficiently small, so that there exists a constant \( c \) such that

\[
\|H^\ell_\tau(\phi_p - \phi_{p,\sigma})\|_F \leq c\sigma^{1/5}.
\]

(C.1)

\textbf{Proof.} First, we will prove

\[
\|H^w_\tau(\phi_p - \phi_{p,\sigma})\|_F \leq c\sigma^{1/5}.
\]

(C.2)

To this end, we recall from [Pi03], [DP18, Lemma 3.6] the form of the modified Hamiltonian on \( D(P^2_\tau + H_\tau) \)

\[
H^w_{p,\sigma} = \frac{1}{2}T^2_{p,\sigma} + \int d^3k \alpha_{p,\sigma}(e_k)|k|a^*(k)a(k) + c^\sigma_p,
\]

(C.3)

where

\[
\Gamma_{p,\sigma} := \nabla E_{p,\sigma} - (P - P^w_{p,\sigma}), \quad P^w_{p,\sigma} := W(f_{p,\sigma})P\tau W(f_{p,\sigma})^*, \quad \alpha_{p,\sigma}(e_k) := (1 - e_k \cdot \nabla E_{p,\sigma}),
\]

(C.4)

\[
c^\sigma_p := \frac{1}{2}p^2 \frac{1}{2}(p - \nabla E_{p,\sigma})^2 - \lambda^2 \int d^3k \frac{\chi_{|\cdot|}(k)}{2k^2\alpha_{p,\sigma}(e_k)}.
\]

(C.5)

The corresponding quantities at \( \sigma = 0 \) are denoted by dropping \( \sigma \) in the notation. We will use the standard bounds from [Pi03]

\[
|E_p - E_{p,\sigma}| \leq c\sigma, \quad |\nabla E_p - \nabla E_{p,\sigma}| \leq c\sigma^{1/4}, \quad \|\phi_p - \phi_{p,\sigma}\|_F \leq c\sigma^{1/2},
\]

(C.6)
see also [DP18, Theorem 2.1 (b), Corollary 5.6], [DP19, Proposition A.2].

Now we proceed by induction: for \( \ell = 0 \) the estimate holds by the third bound in (C.6). For the inductive step we compute

\[
\|(H_p^w)^\ell (\phi_p - \phi_{p,\sigma})\|_F = \|(H_p^w)^{\ell-1} (E_p \phi_p - H_p^w \phi_{p,\sigma})\|_F \\
\leq \|(H_p^w)^{\ell-1} (E_p \phi_p - E_p \phi_{p,\sigma})\|_F + \|(H_p^w)^{\ell-1} (H_p^w - H_p^{w,\alpha}) \phi_{p,\sigma}\|_F. \tag{C.7}
\]

The first term on the r.h.s. of (C.7) is \( O(\sigma^{1/5}) \) by the induction hypothesis and the first estimate in (C.6). Concerning the last term on the r.h.s. of (C.7), we note that there are three contributions to \( H_p^w - H_p^{w,\alpha} \) coming from the three terms in the Hamiltonian (C.3). They have the following properties: First, by (C.6), \(|c_p - c_p^\alpha| \leq c\sigma^{1/4}\). Thus, by Lemma C.2 below,

\[
\|(H_p^w)^{\ell-1} (c_p - c_p^\alpha) \phi_{p,\sigma}\|_F \leq c\sigma^{1/4 - \delta_{\lambda_0}}. \tag{C.8}
\]

Clearly, for \( \lambda_0 > 0 \) sufficiently small, the last expression is \( O(\sigma^{1/5}) \). The second contribution is \( d\Gamma((\alpha_{p,\sigma}(e_k) - \alpha_p(e_k))|k|) \), and \(|\alpha_{p,\sigma}(e_k) - \alpha_p(e_k)| \leq c\sigma^{1/4}\) by (C.6). Thus Lemma C.2 gives

\[
\|(H_p^w)^{\ell-1} d\Gamma((\alpha_{p,\sigma}(e_k) - \alpha_p(e_k))|k|) \phi_{p,\sigma}\|_F \leq c\sigma^{1/4 - \delta_{\lambda_0}}. \tag{C.9}
\]

Concerning the third contribution, \((\Gamma_p^2 - \Gamma_{p,\sigma}^2)\), we note that, on \( D(P_{\ell}^w + H_\ell)\),

\[
(P_{\ell,\sigma}^w)_i = W(f_{p,\sigma}) P_{\ell,\sigma} W(f_{p,\sigma})^* = (P_{\ell})_i - a^*(k_i f_{p,\sigma}) - a(k_i f_{p,\sigma}) + (f_{p,\sigma}, k_i f_{p,\sigma}) \\
= (P_{\ell}^w)_i - a^*(k_i (f_{p,\sigma} - f_p)) - a(k_i (f_{p,\sigma} - f_p)) + (f_{p,\sigma}, k_i f_{p,\sigma}) - (f_p, k_i f_{p}). \tag{C.10}
\]

Consequently, \( \Gamma_{p,\sigma} = \Gamma_p + \Delta \Gamma_{p,\sigma}\), where

\[
(\Delta \Gamma_{p,\sigma})_i = -a^*(k_i (f_{p,\sigma} - f_p)) - a(k_i (f_{p,\sigma} - f_p)) + (f_{p,\sigma}, k_i f_{p,\sigma}) - (f_p, k_i f_{p}) + (\nabla E_{p,\sigma} - \nabla E_p)_i. \tag{C.11}
\]

Considering that \((\Gamma_p^2 - \Gamma_{p,\sigma}^2) = -\Gamma_p \cdot \Delta \Gamma_{p,\sigma} - \Delta \Gamma_{p,\sigma} \cdot \Gamma_p - (\Delta \Gamma_{p,\sigma})^2\), Lemmas C.2 and E.5 give

\[
\|(H_p^w)^{\ell-1} (\Gamma_p^2 - \Gamma_{p,\sigma}^2) \phi_{p,\sigma}\|_F \leq c\sigma^{1/4 - \delta_{\lambda_0}}. \tag{C.12}
\]

This concludes the proof of (C.2). Now (C.1) follows from Lemma C.3. \( \square \)

**Lemma C.2.** Let \( h_1, \ldots, h_\ell \) be real-valued measurable functions (in momentum space) which are bounded on compact sets. Then

\[
\|(d\Gamma(h_1) \ldots d\Gamma(h_\ell)) \phi_{p,\sigma}\|_F \leq \frac{c_{\ell,\delta_{\lambda_0}}}{\sigma^{\delta_{\lambda_0}}} (\sup_{|k_1| \leq \kappa_*} |h_1(k_1)| \ldots \sup_{|k_\ell| \leq \kappa_*} |h_\ell(k_\ell)|), \tag{C.13}
\]

where \( c_{\ell,\delta_{\lambda_0}} \) may depend on \( \ell \). Furthermore, if \( f_1, \ldots, f_\ell \in L^2(\mathbb{R}^3_\kappa) \) are supported in a ball of radius \( \kappa_* \), then we get

\[
\|(d\Gamma(h_1) \ldots d\Gamma(h_\ell)) a^{(\gamma)}(f_1) \ldots a^{(\gamma)}(f_\ell) \phi_{p,\sigma}\|_F \leq \frac{c_{\ell,\tilde{\ell},\delta_{\lambda_0}}}{\sigma^{\delta_{\lambda_0}}} (\sup_{|k_1| \leq \kappa_*} |h_1(k_1)| \ldots \sup_{|k_\ell| \leq \kappa_*} |h_\ell(k_\ell)|) (\|f_1\|_2 \ldots \|f_\ell\|_2), \tag{C.14}
\]

where \( c_{\ell,\tilde{\ell},\delta_{\lambda_0}} \) may depend on \( \ell, \tilde{\ell} \). The estimate also holds for an arbitrary permutation of the \((\ell + \tilde{\ell})\)-element set of operators on the l.h.s. of (C.14).
Proof. We consider (C.13) for $\ell = 1$. Let $f_{w,p,\sigma}^n$ be the $n$-photon wave functions of $\phi_{p,\sigma}$. Then, by [DP19, Proposition A.4], we have

$$|f_{w,p,\sigma}^n(k_1, \ldots, k_n)| \leq \frac{1}{\sqrt{n!}} g^n_{\sigma}(k_1, \ldots, k_n),$$

(C.15)

where $g^n_{\sigma}$ are defined as in (B.3). Thus

$$\|d\Gamma(h_1)\phi_{p,\sigma}\|^2_\mathcal{F} \leq \sum_{n=1}^\infty \frac{1}{n!} \int d^3 n k (h_1(k_1) + \cdots + h_1(k_n))^2 |g^n_{\sigma}(k_1, \ldots, k_n)|^2$$

$$\leq \left( \sup_{|k| \leq \kappa_{\sigma}} |h(k)| \right)^2 \sum_{n=1}^\infty \frac{n^2}{n!} \int d^3 n k |g^n_{\sigma}(k_1, \ldots, k_n)| \leq \frac{c}{\sigma^{\delta_{h_0}} \sup_{|k| \leq \kappa_{\sigma}}} |(C.16)$$

where we estimated as in (B.1). Generalization to arbitrary $\ell$ is straightforward.

As for (C.14), we first commute all the operators $a^{(\ast)}(f_j^\ell)$ to the left and thus get a linear combination of terms of the form

$$\|a^{(\ast)}(f'_1) \cdots a^{(\ast)}(f'_n)(1 + N)^{-\ell}(1 + N)^\ell (d\Gamma(h_{i_1}) \cdots d\Gamma(h_{i_n})) \phi_{p,\sigma}\|_\mathcal{F},$$

(C.17)

where $f'_j(k) = h_{i_j}(k) \cdots h_{i_j}(k) f_i(k)$ and the functions $h_j$ included in $f'_j$ do not appear in the product of $d\Gamma(h_{i_j})$ in (C.15). Now using the number bounds (A.2) on creation and annihilation operators, assumption on the supports of $f_i$ and (C.13), we obtain the claim. □

Lemma C.3. For any $\ell \in \mathbb{N}$, the operators $H_p^\ell(i + H_p^w)^{-\ell}$ are bounded.

Proof. Let $\psi \in \mathcal{F}$, $\|\psi\|_\mathcal{F} = 1$, be in the domain of $H_p^\ell$. Then we can write

$$\|(1 + H_p^w)^{-\ell}H_p^\ell \psi\| \leq \|((1 + H_p^w)^{-\ell} - (1 + H_p^w)^{-\ell})H_p^\ell \psi\|_\mathcal{F}$$

$$+\|((1 + H_p^w)^{-\ell}W(f_{p,\sigma}) H_p^\ell W(f_{p,\sigma})^\ast)f\|_\mathcal{F}.$$

(C.18)

Exploiting the concrete expression for $W(f_{p,\sigma}) H_p^\ell W(f_{p,\sigma})$ and standard energy bounds for the Hamiltonian $H_{p,\sigma}$, i.e. the boundedness of $(1 + H_{p,\sigma})^{-\ell}H_p^\ell$ (cf. [FGS01, Appendix D]), we obtain that the last term is uniformly bounded in $\sigma$. Now since $\lim_{\sigma \to 0} H_p^w = H_p^w$ in the norm-resolvent sense, we complete the proof by first taking $\sigma \to 0$ on the r.h.s. and then taking supremum over $\psi$. (The statement about the norm-resolvent convergence is verified using the resolvent identity and explicit formulas for $H_p^w - H_p^w$, appearing in the proof of Lemma C.1). □

D Proof of Lemma 3.1

Let $V := \{ x \in V_{\delta} \}$ and $V_{\delta}$ be its slightly larger neighbourhood. Since $|\nabla E_p| < c_0 < 1$ for $p \in S$, we can ensure that $V_{\delta}$ is in the interior of the ball of radius $c_0$ centered at zero. Arguing by the non-stationary phase method (cf. Theorem XI.14 from [RS3] and its Corollary) we obtain for $x/t \notin V_{\delta}$ and any $\psi$ of norm one from the dense domain in the statement of the theorem:

$$|\chi_{(x/t)}(x) \int d^3 p \ e^{i(p \cdot x - E_p t)} \langle \psi, g(t,x,p) \rangle_\mathcal{F}| \leq c(1 + |x| + |t|)^{-2} \chi_{(x/t) \notin V_{\delta}}(x) \sup_{|x| \leq 2} \langle \psi, \partial_p^2 g(t,x,p) \rangle_\mathcal{F},$$

(D.1)
where we write the \((t, x)\)-dependence of \(g\) explicitly. Hence, considering that \(|x|/t \geq c_0\) implies \((x/t) \notin V_\delta\), we obtain for any \(0 < \varepsilon < 1\),

\[
\int_{|x| \geq ct} d^3x \sup_{||\phi|| \leq 1} \left| \langle \psi, \int d^3p e^{i(p \cdot x - E_p t)} g_{(t, x)}(p) \rangle \right|^2 \leq \int_{|x| \geq ct} d^3x c(1 + |x|-4+2\varepsilon) \left( \sum_{|\alpha| \leq 2} \sup_{p, |x'| \leq ct} \frac{1}{1 + |x'| + |t|} \varepsilon \left| \partial^\alpha p g_{(t, x')}(p) \right|_F^2 \right),
\]

which gives (3.2).

Next, by the stationary phase method (cf. Theorem XI.15 from [RS3] and its Corollary) we have for all \(x \in \mathbb{R}^3\)

\[
\left| \int d^3p e^{i(p \cdot x - E_p t)} \langle \psi, g_{(t, x)}(p) \rangle \right|_F^2 \leq c_{t^{-2/3}} \sum_{|\alpha| \leq 2} \sup_{p} \left| \langle \psi, \partial^\alpha p g_{(t, x)}(p) \rangle \right|_F,
\]

where we used that the Hessian matrix of \(p \mapsto E_p\) is invertible in \(S\) (cf. Lemma 2.1). Thus we get

\[
\int_{|x| \leq ct} d^3x \sup_{||\phi|| \leq 1} \left| \langle \psi, \int d^3p e^{i(p \cdot x - E_p t)} g_{(t, x)}(p) \rangle \right|^2 \leq \int_{|x| \leq ct} d^3x c(t^{-3})(\sum_{|\alpha| \leq 2} \sup_{p, |x'| \leq ct} ||\partial^\alpha p g_{(t, x')}(p)||_F^2),
\]

which gives (3.1).

To prove that the constants \(c\) are uniform in \(|\lambda| \in (0, \lambda_0]\), we use the last statement in [RS3, Theorem XI.15] and argue as in the corollary of this theorem. This argument requires, that \(\lambda \mapsto \{E_p^{(1)}\}_{p \in S}\) is continuous in the topology of \(C^\ell(S)\) (cf. [RS3, p.37]). This follows from Lemma 2.1. □

### E Properties of functions \(f_p, f_{p, \sigma}\)

We recall from (1.5), (2.4) the definitions:

\[
f_p(k) := \lambda \frac{c_{p, \sigma}(k)}{\sqrt{2|k|}} \frac{1}{|1 - e_k \cdot \nabla E_p|}, \quad f_{p, \sigma}(k) := \lambda \frac{c_{p, \sigma}(k)}{\sqrt{2|k|}} \frac{1}{|1 - e_k \cdot \nabla E_{p, \sigma}|}.
\]

We start with the following preparatory lemma.

**Lemma E.1.** For \(n = 1, 2, \ldots\) there holds the bound

\[
\int \frac{c_{p, \sigma}(k)^2}{|k|^3} |e^{-ikl+ikx} - 1|^n d^3k \leq c_n (1 + \log(1 + |t| + |x|)).
\]

**Proof.** We estimate

\[
\int \frac{c_{p, \sigma}(k)^2}{|k|^3} |e^{-ikl+ikx} - 1|^n d^3k \leq \sum_{\varepsilon = \pm} \int_{|e^{-k \cdot e_k + x}| \geq 0} d\Omega(e_k) \int_0^\infty \frac{d|k|}{|k|} |e^{-ikl+ikx} - 1|^n \leq \sum_{\varepsilon = \pm} \int_{|e^{-k \cdot e_k + x}| \geq 0} d\Omega(e_k) \int_0^{\infty(1 + |t| + |x|)} \frac{d|k|}{|k|} |e^{-ikl} - 1|^n \leq c \int_0^\infty \frac{d|k|}{|k|} |e^{-ikl} - 1|^n + c2^n \int_0^{\infty(1 + |t| + |x|)} \frac{d|k|}{|k|} \leq c_n (1 + \log(1 + |t| + |x|)).
\]

This completes the proof. □
**Lemma E.2.** There holds the following bound for $|\alpha| = 0, 1, 2$

$$|\partial^{\alpha}_p \gamma(p, x, t)| \leq c|\lambda|^2 (1 + \log(1 + |t| + |x|)). \tag{E.4}$$

**Proof.** We have

$$\partial_p f_p(k) = \frac{\lambda}{2} \frac{X(k)}{2|k|^{3/2}} \frac{1}{(1 - e_k \cdot \nabla E_p)^2} \partial_p(e_k \cdot \nabla E_p), \tag{E.5}$$

$$\partial_{p_j} \partial_p f_p(k) = \frac{\lambda}{2} \frac{X(k)}{2|k|^{3/2}} \left\{ \frac{1}{(1 - e_k \cdot \nabla E_p)^2} \partial_p(e_k \cdot \nabla E_p) \partial_{p_j}(e_k \cdot \nabla E_p) \right. \right.$$

$$\left. \left. + \frac{1}{(1 - e_k \cdot \nabla E_p)^2} \partial_{p_j} \partial_p(e_k \cdot \nabla E_p) \right\}. \tag{E.6}$$

Thus by Lemma 2.1, we have

$$|f_p(k)|, |\partial_p f_p(k)|, |\partial_{p_j} \partial_p f_p(k)| \leq c \lambda \frac{X(k)}{2|k|^{3/2}}. \tag{E.7}$$

Now we write

$$\partial^{\alpha}_p \gamma(p, x, t) = \int d^3k \partial^{\alpha}_p(f_p(k)^2) \sin(|k|t - k \cdot x). \tag{E.8}$$

Clearly, estimates (E.7) and Lemma E.1 give (E.4). (Here we made use of $|\sin y| = |\text{Im} e^{iy}| = |\text{Im}(e^{iy} - 1)| \leq |e^{iy} - 1|$).

□

**Lemma E.3.** Let $m(t, x) := (e^{-i|k|t} + ik - 1).$ Then, for $|\alpha| \leq 2, |\beta| \leq 2$

$$|\partial^{\alpha}_p f_p m(t, x), \partial^{\alpha}_p f_p m(t, x)| \leq c|\lambda|^2 (1 + \log(1 + |t| + |x|)). \tag{E.9}$$

**Proof.** Follows immediately from (E.7) and Lemma E.1. Indeed, we have

$$\int d^3k |(\partial^{\alpha}_p f_p(k)\partial^{\alpha}_p f_p(k)| e^{i|k|t} + ik - 1|^2 \leq c|\lambda|^2 \int d^3k \frac{X(k)^2}{2|k|^3} |e^{i|k|t} + ik - 1|^2 \leq c|\lambda|^2 (1 + \log(1 + |t| + |x|)). \tag{E.10}$$

which concludes the proof. □

**Lemma E.4.** The following bounds hold

$$|\partial^{\alpha}_p f_{p, \sigma}(k)| \leq c_0 \frac{X(k)}{\sqrt{2}|k|^{3/2}} \text{ for } |\alpha| = 0, 1 \text{ and } |\partial^{\alpha}_p f_{p, \sigma}(k)| \leq \frac{c}{\sigma \delta_0} \frac{X(k)}{\sqrt{2}|k|^{3/2}} \text{ for } |\alpha| = 2. \tag{E.11}$$

**Proof.** The estimates follow from definition (2.4) via computations analogous to (E.5)–(E.6). For the relevant estimates on derivatives of $S \ni p \mapsto E_{p, \sigma}$ up to the third order, see [DP18, Theorem 2.1]. □

**Lemma E.5.** The following bounds hold

$$|k_i(f_{p, \sigma} - f_p)|_2 \leq c\sigma^{1/4}, \quad |(f_{p, \sigma}, k_i f_{p, \sigma}) - (f_p, k_i f_p)| \leq c\sigma^{1/4}. \tag{E.12}$$
Proof. Definitions (E.1) give

\[
 f_p(k) - f_{p,\sigma}(k) = \lambda \chi_{[0,\sigma]}(k) \frac{1}{\sqrt{2|k|^3}} (1 - e_k \cdot \nabla E_p) + \lambda \chi_{[\sigma,\infty]}(k) \left( \frac{1}{(1 - e_k \cdot \nabla E_p)} - \frac{1}{(1 - e_k \cdot \nabla E_{p,\sigma})} \right),
\]  

(E.13)

where \( \chi_{[0,\sigma]} \) is the characteristic function of a ball of radius \( \sigma \) centered at zero. Now considering that \( |\nabla E_p - \nabla E_{p,\sigma}| \leq c \sigma^{1/4} \) (cf. (C.6) above), we can write for any \( \beta > 0 \)

\[
 \|k^{\beta} (f_{p,\sigma} - f_p)\|_2 \leq c(\sigma^\beta + \sigma^{1/4}).
\]  

(E.14)

Setting \( \beta = 1 \) we obtain the first bound in (E.12). As for the second estimate, we note that

\[
 |\langle f_{p,\sigma}, k_i f_{p,\sigma} \rangle - \langle f_p, k_i f_p \rangle| \leq \int d^3 k |k| (f_{p,\sigma}(k) - f_p(k))(f_{p,\sigma}(k) + f_p(k))
\]

\[
 \leq \|k|^{1/2} (f_{p,\sigma} - f_p)\|_2 (\|k|^{1/2} f_{p,\sigma}\|_2 + \|k|^{1/2} f_p\|_2).
\]  

(E.15)

Applying (E.14) with \( \beta = 1/2 \) and considering that \( \|k|^{1/2} f_{p,\sigma}\|_2, \|k|^{1/2} f_p\|_2 \leq c \) we conclude the proof. □

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