Occupation times of discrete-time fractional Brownian motion

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Abstract

We prove a conditional local limit theorem for discrete-time fractional Brownian motions (dBm) with Hurst parameter $\frac{3}{4} < H < 1$. Using results from infinite ergodic theory it is then shown that the properly scaled occupation time of dBm converges to a Mittag-Leffler distribution.

1 Introduction and Main Results

In 1957, Darling and Kac [8] established limit theorems for the occupation times of Markov processes with stationary transition probabilities, proving that under a “Darling-Kac condition”, the limit distributions are necessarily Mittag-Leffler distributions with appropriate indices. Earlier and weaker results were obtained by Dobrushin [10] and Chung and Kac [5]. The theory is applicable to Markov chains and in particular, to random walks, the sum of independent, identically distributed random variables $\{X_n\}$ with common distribution function $F$. $X_n$ may take a lattice or a non-lattice distribution. When $F$ has mean 0 and belongs to the domain of attraction of some stable law with index $d$, it is known that $S_n$, the partial sum of $\{X_n\}$ obeys a local limit theorem [12], which plays an important role in proving the limiting distribution of the occupation time. This is because the local limit theorem implies the “Darling-Kac condition” of i.i.d. random variables $\{X_n\}$: 

$$\sum_{k=0}^{n} P(S_k \in A) \sim |A| n^{1-\frac{1}{d}} L(n),$$

where $L(n)$ is some slowly varying function\(^1\). By Darling and

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\(^1\)In the whole article, $A_n \sim B_n$ means $\frac{A_n}{B_n} \to 1$ as $n \to \infty$. 

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Kac [8], the “Darling-Kac condition” implies that the normalized occupation time of $S_n$ converges to a Mittag-Leffler distribution.

The Darling-Kac theorem can also be extended to the sum of weakly dependent random variables, one of such dependence is the Renyi-mixing sequence [7]. Only little seems to be known on this topic, a recent study of the two authors [9,17] sheds some light onto this question by connecting occupation times of ergodic sums in Gibbs-Markov dynamical systems to group extensions over these systems (see [3]). The present paper is about the limiting distribution of occupation times of the discrete-time fractional Brownian motion $(B^H(n))_{n \in \mathbb{Z}}$ which is not weakly dependent but has long range dependence. It will be shown that $B^H(n)$ satisfies a conditional local limit theorem. Next, the occupation times will be represented as partial sums of iteratives of a transformation $\tilde{T}$ on the infinite measure space $\Omega \times \mathbb{R}$, where $(\Omega, \mathcal{B}_0, P)$ is the probability space carrying the Gaussian process. The conditional local limit theorem ensures that $\tilde{T}$ is a pointwise dual ergodic transformation with respect to the canonical product measure on the product space $\Omega \times \mathbb{R}$. Finally, the limiting distribution of the occupation time can be shown to be Mittag-Leffler distribution by [1].

To be more precise, let $\{B^H(t)\}$ be a fractional Brownian motion with Hurst index $\frac{1}{2} < H < 1$, and define $\{X_n\}$ to be the increment of the fBm: $X_n = B^H(n) - B^H(n - 1)$ for $n \in \mathbb{Z}^+$. $\{X_n\}$ is also called discrete-time fractional Gaussian noise (DFGN). Then for any $n$, $(X_1, X_2, \ldots, X_n)$ has a multivariate normal distribution with mean 0 and the covariance function $b(i - j) = E[X_i X_j] = E[X_j X_i]$ satisfying

$$b(t) = \frac{1}{2}((t + 1)^{2H} - 2t^{2H} + (t - 1)^{2H}),$$

and $b(0) = E(X_i)^2 = 1$. This is because $E[B^H(s)B^H(t)] = \frac{1}{2}[|s|^{2H} + |t|^{2H} - |t - s|^{2H}]$. So $\{X_n\}$ are Gaussian random variables with a long-range dependence structure.

It is proved in [16] that for $\overline{d}_N^2 \sim N^{2H}$, $Z_N(t) := \frac{1}{d_N} \sum_{i=1}^{[Nt]} X_i = \frac{1}{d_N} B^H([Nt])$ converges weakly to $B^H(t)$ in the Skorohod space $\mathcal{D}([0,1])$ as $N \to \infty$. Denote by $S_n$ the partial sum of $\{X_n\}$:

$$S_n := \sum_{i=1}^{n} X_i,$$

which actually equals $B^H(n)$.

Let $V$ be a non-negative function over $\mathbb{R}$, the state space of the DFGN $\{X_n\}$. In this article, we study the limiting distribution of the random variable $\sum_{i=1}^{n} V(S_i)$ as $n \to \infty$. If $V(x)$ is the characteristic function of some Borel set $B \subset \mathbb{R}$, then $\sum_{i=1}^{n} V(S_i)$ becomes the occupation time of $S_n$ of the set $B$: i.e. $\sum_{i=1}^{n} V(S_i) = \#\{i \leq n : S_i \in B\}$. What plays an important role in finding the limiting distribution of the occupation time is the “Darling-Kac” condition: $\{S_n\}$ has a conditional local limit theorem when $\frac{4}{3} < H < 1$. We state it below as our first main result.
Theorem 1.1 (Conditional Local Limit Theorem) Suppose \( \{X_n\} \) is a sequence of stationary Gaussian random variables with mean 0 and covariance function \( b(i - j) = E(X_iX_j) \) satisfying \( b(t) = \frac{1}{2}[(t + 1)^{2H} - 2t^{2H} + (t - 1)^{2H}] \), where \( 3/4 < H < 1 \). Then there exists a normalization sequence \( \{d_n\} \), satisfying \( d_n^2 = n^{2H}L(n) \) as \( n \to \infty \), where \( L(n) \) is slowly varying and converging to a constant, such that for any interval \( (a, b) \subset \mathbb{R} \) and any sequence \( \{q_n\} \) and \( \kappa \in \mathbb{R} \), such that \( \frac{q_n}{d_n} \to \kappa \) as \( n \to \infty \), the conditional probability satisfies

\[
\lim_{n \to \infty} d_n P \left( S_n \in (q_n + a, q_n + b) | (X_{n+1}, X_{n+2}, \ldots) \right) = (b - a)g(\kappa), \text{ a.s.},
\]

where \( g \) is the density function of the standard normal distribution.

In case that \( q_n = 0 \), the convergence is uniform for almost all \( \omega \) and intervals \( (a, b) \).

Then the second main result of the paper follows: the normalized occupation time of \( S_n \) converges to Mittag-Leffler distribution.

Theorem 1.2 (Limiting distribution of the occupation time of \( S_n \)) Let \( (X_n)_{n \in \mathbb{Z}} \) be as in Theorem 1.1. Denote the occupation time of \( S_k \) in the interval \( (a, b) \) at time \( n \) by \( \ell_n([a, b]) = \sum_{i=1}^n 1_{(a, b)}(S_i) \). Then there exists a sequence of numbers \( a_n = O(n^{-H}) \) such that

\[
\frac{1}{2 \epsilon} \int_{-\epsilon}^{\epsilon} \int_{\Omega} \left( \ell_n([a - x, b - x]) \Phi(\omega) dP(\omega) \right) dx \to E[v((b - a)Y_\alpha)], \quad (1.1)
\]

for any \( \epsilon > 0 \), any bounded and continuous function \( V : \mathbb{R} \to \mathbb{R} \), any probability density function \( \Phi \in L^2(P) \), where \( Y_\alpha \) is a random variable having the Mittag-Leffler distribution with index \( \alpha = 1 - H \).

Remarks (1) Taking \( \Phi = 1 \) one could try to evaluate the left hand side when \( \epsilon \to 0 \). This could show that the occupation times have a weak limit which is Mittag-Leffler. We do not know this, but the result shows that convergence in the weak* sense in \( L^\infty(dx) \).

(2) We do not know the precise connection of this result to the local time of fractional Brownian motion. In [13] it is remarked that the law of the local time of a fractional Brownian motion is not a Mittag-Leffler distribution unless it is Brownian motion, although Kono’s result in [14] suggested that it may be true. Theorem 1.2 may give a hint to explain this phenomenon. Kasahara and Matsumoto have found that the limiting distribution of the occupation time of \( B^H \) is similar but not equal to a Mittag-Leffler distribution. In the proof that the limiting distribution is not Mittag-Leffler, it is assumed that \( d \geq 2, 0 < Hd < 1 \). However, their proof is still available when \( H \neq \frac{1}{2} \) and \( d = 1 \).
This paper is structured as follows. Section 2 is devoted to proving the conditional local limit theorem of $S_n$. In Section 3, the occupation time of $S_n$ is represented as an ergodic partial sum by introducing a skew product transformation $\hat{T}$ on $\Omega \times \mathbb{R}$. The skew product has an ergodic decomposition, and each component is pointwise dual ergodic. It follows that the normalized occupation time of $S_n$ converges to Mittag-Leffler distribution as described in Theorem 1.2.

2 Conditional Local Limit Theorem

2.1 Proof of the conditional local limit theorem

In this part, we state two claims which are the key points in the proof of Theorem 1.1 and provide the proof modulo these conditions.

Proof of Theorem 1.1. For fixed positive integers $k$ and $n$, the conditional probability $P(S_n \in (q_n + a, q_n + b)|(X_{n+1}, X_{n+2}, ..., X_{n+k}))$ is given by a normal distribution. Indeed, let the $(n+k)$-dimensional random variable $X = (X_1, \cdots, X_{n+k})^T$ be partitioned as $\begin{bmatrix} X[1] \\ X[2] \end{bmatrix}$ with sizes $n$ and $k$ respectively. The covariance matrix of $X$ is denoted by

$$
\Sigma = \begin{bmatrix} \Sigma_{11}(n, n) & \Sigma_{12}(n, k) \\ \Sigma_{21}(k, n) & \Sigma_{22}(k, k) \end{bmatrix},
$$

where $\Sigma_{11}$ and $\Sigma_{22}$ are symmetric Toeplitz matrixes, $\Sigma_{11}(n, n) = [b(|i - j|)]_{0 \leq i, j < n}$, $\Sigma_{22}(k, k) = [b(|i - j|)]_{0 \leq i, j < k}$, and where $\Sigma_{12}(n, k) = \Sigma_{21}(k, n) = [b(n - i + j)]_{0 \leq i < n; 0 \leq j < k}$.

Let $D$ be the $(k+1) \times (n+k)$ matrix, defined by $D = \begin{bmatrix} e(n) & 0, \cdots, 0 \\ 0 & I_k \end{bmatrix}$, where $e(n) = (1,1,\ldots,1)$

and let $I_k$ be the identity matrix of dimension $k$. Then $DX \sim N(0, D\Sigma D^T)$, i.e.

$$
\begin{bmatrix} S_n \\ X[2] \end{bmatrix} \sim N \left( 0, \begin{bmatrix} e(n)\Sigma_{11}e(n)^T & e(n)\Sigma_{12} \\ e(n)\Sigma_{21}e(n)^T & \Sigma_{22} \end{bmatrix} \right).
$$

By the conditional normal formula (see for example [4], Section 5.5), when $\Sigma_{22}$ is of full rank,

$$
(S_n|X[2]) \sim N(\mu(n, k), \sigma^2(n, k)),
$$

where

$$
\mu(n, k) = e(n)\Sigma_{12}(n, k)\Sigma_{22}^{-1}(k, k)X[2]
$$
and
\[ \sigma^2(n, k) = e(n)\Sigma_{11}(n, n)e(n)^T - e(n)\Sigma_{12}(n, k)\Sigma_{22}^{-1}(k, k)\Sigma_{21}(k, n)e(n)^T. \]

That is, \( P(S_n \in A|X_{[2]}) = \int_A f(y_1|X_{[2]})dy_1, \) with \( f(y_1|X_{[2]}) = \frac{1}{\sqrt{2\pi}\sigma(n, k)} \exp\left(\frac{-(y_1-\mu(n, k))^2}{2\sigma^2(n, k)}\right). \)

Let \( B = (B(1), B(2), \cdots, B(k))^T := \Sigma_{21}(k, n)e(n)^T, \) so \( B(s) = \sum_{i=s}^{n+s-1} b(i). \) Then the mean and the variance become
\[ \mu(n, k) = B^T\Sigma_{22}^{-1}(k, k)X_{[2]} \]
and
\[ \sigma^2(n, k) = e(n)\Sigma_{11}e(n)^T - B^T\Sigma_{22}^{-1}B. \]

It follows that
\[
\sigma(n, k)P\left(S_n \in (q_n + a, q_n + b)((X_{n+1}, X_{n+2}, \ldots, X_{n+k}) \right) \\
= \frac{1}{\sqrt{2\pi}} \int_{a+q_n}^{b+q_n} \exp\left(-\frac{(x - B^T\Sigma_{22}^{-1}(k, k)X_{[2]})^2}{2\sigma^2(n, k)}\right) dx \\
= \sigma(n, k) \frac{1}{\sqrt{2\pi}} \int_{(a+q_n)/\sigma(n, k)}^{(b+q_n)/\sigma(n, k)} \exp\left(-\frac{1}{2}(y - \frac{1}{\sigma(n, k)}B^T\Sigma_{22}^{-1}(k, k)X_{[2]})^2\right) dy \\
= \sigma(n, k) \frac{1}{\sqrt{2\pi}} \frac{(b - a)/\sigma(n, k)}{\exp\left(-\frac{1}{2}\frac{\xi + q_n}{\sigma(n, k)} - \frac{1}{\sigma(n, k)}B^T\Sigma_{22}^{-1}(k, k)X_{[2]}^2\right)},
\]
where the mean value theorem is used in the last step and \( \xi \in [a, b]. \)

Now we make two claims which will be proved in Sections 2.2 and 2.3 below.

**Claim 1:** For fixed \( n, d_n^2 = \lim_{k \to \infty} \sigma^2(n, k) \) exists and \( d_n^2 = n^{2H}L(n) \) as \( n \to \infty, \) where \( L(n) \) is slow varying and converges to a constant.

**Claim 2:** For fixed \( n, \lim_{k \to \infty} \frac{1}{\sigma(n, k)}B^T\Sigma_{22}^{-1}(k, k)X_{[2]} = 0 \) almost surely.

As a consequence, \( \lim_{k \to \infty} \frac{\xi + q_n}{\sigma(n, k)} = \frac{\xi}{d_n} + \frac{q_n}{d_n} =: \kappa(n). \) Since \( \xi \in [a, b] \) and \( \frac{q_n}{d_n} \to \kappa \) as \( n \to \infty, \)
\[ \lim_{n \to \infty} \kappa(n) = \kappa. \]

Hence
\[
\lim_{k \to \infty} \sigma(n, k)P\left(S_n \in (q_n + a, q_n + b)((X_{n+1}, X_{n+2}, \ldots, X_{n+k}) \right) \\
= \frac{1}{\sqrt{2\pi}}(b - a) \exp\left(-\frac{\kappa(n)^2}{2}\right) = g(\kappa(n))(b - a),
\]

where \( g \) is the density function of the standard normal random variable.

On the other hand, by Doob’s martingale convergence theorems, almost surely,

\[
\lim_{k \to \infty} \sigma(n, k) P\left(S_n \in (q_n + a, q_n + b) | (X_{n+1}, X_{n+2}, \ldots, X_{n+k})\right) = d_n P\left(S_n \in (q_n + a, q_n + b) | (X_{n+1}, X_{n+2}, \ldots)\right).
\]

Hence almost surely,

\[
d_n P\left(S_n \in (q_n + a, q_n + b) | (X_{n+1}, X_{n+2}, \ldots)\right) = (b - a) g(\kappa(n))
\]

It follows that

\[
\lim_{n \to \infty} d_n P\left(S_n \in (q_n + a, q_n + b) | (X_{n+1}, X_{n+2}, \ldots)\right) = g(\kappa)(b - a).
\]

When \( q_n = 0 \), almost surely,

\[
d_n P\left(S_n \in (a, b) | (X_{n+1}, X_{n+2}, \ldots)\right) = (b - a) g(0).
\]

So

\[
\lim_{n \to \infty} d_n P\left(S_n \in (a, b) | (X_{n+1}, X_{n+2}, \ldots)\right) = g(0)(b - a),
\]

uniformly for almost all \( \omega \in \Omega \) and \((a, b)\).

In the following sections, we give the proof of the two claims.

\subsection{2.2 Estimating the variance}

In this part, we prove Claim 1: \( \lim_{k \to \infty} \sigma^2(n, k) = d_n^2 = n^{2H} L(n) \). Since by definition of the \( b(i) \) we have that the first term of \( \sigma^2(n, k), e(n) \Sigma_{11}(n, n) e(n)^T = n^{2H} L(n) \), it is sufficient to prove that the second term of \( \sigma^2(n, k) \) converges to 0 as \( k \to \infty \), i.e.

\[
\lim_{k \to \infty} B^T \Sigma_{22}^{-1}(k, k) B = 0.
\]

We shall write \( \Sigma_{22} \) for \( \Sigma_{22}(k, k) \) in this subsection to simplify notation. First we give an estimate of the element \( B(s) = \sum_{i=s}^{n+s-1} b(i) \) of the vector \( B \).

\textbf{Lemma 2.1} It holds that

\[
B(s) = \binom{2H}{2} n(s + n)^{2H-2} \left( 1 + O\left(\frac{1}{s}\right)\right), \quad \text{as} \; s \to \infty
\]

and therefore \( \sum_{i=1}^{k} B^2(i) = O(n^2 (k + n)^{4H-3}) \) as \( k \to \infty \).
Proof By Taylor expansion, \((1+x)^a = \sum_{i=0}^{\infty} \binom{a}{i} x^i\) when \(|x| < 1\), where \(\binom{a}{i} = \frac{a(a-1)\cdots(a-i+1)}{i!}\).

So by definition of \(B(s)\) and \(b(t)\),

\[
2(sn)^{-2H} B(s) = (sn)^{-2H} \left[ (s+n)^{2H} - (s+n-1)^{2H} - s^{2H} + (s-1)^{2H} \right] = \left( 1 - \frac{sn-s-n}{sn} \right)^{2H} - \left( 1 - \frac{sn-s-n+1}{sn} \right)^{2H} - \left( 1 - \frac{sn-s}{sn} \right)^{2H} + \left( 1 - \frac{sn-s+1}{sn} \right)^{2H}
\]

\[
= \sum_{i=2}^{\infty} \binom{2H}{i} (-1)^i f(s, n, i)(sn)^{-i}
\]

where \(f(s, n, i) = (sn-s-n)^i - (sn-s-n+1)^i - (sn-s)^i + (sn-s+1)^i\). Using the binomial formula, we can rewrite \(f(s, n, i) = \sum_{j=1}^{i} \binom{i}{j} (sn-s)^{i-j} - (sn-s-n)^{i-j}\). Since

\[
(sn-s)^{i-j} - (sn-s-n)^{i-j} = \sum_{j=1}^{i-j} \binom{i-j}{l} (sn-s-n)^{i-j-l} n^l
\]

\[
= n(i-j)(sn-s-n)^{i-j-1} \left( 1 + O\left(\frac{n}{sn-s-n}\right) \right), \quad \text{as } \frac{n}{sn-s-n} \to 0,
\]

a straight forward calculation furthermore shows that

\[
f(s, n, i) = i(i-1)n(sn-s-n)^{i-2} \left( 1 + O\left(\frac{n}{ns-s-n}\right) \right)
\]

as \(\frac{n}{sn-s-n} \to 0\) and \(\frac{1}{sn-s-n} \to 0\).

Inserting into equation (2.1) we arrive at

\[
2(sn)^{-2H} B(s) = \sum_{i=2}^{\infty} \binom{2H}{i} (-1)^i i(i-1)n(sn-s-n)^{i-2} (sn)^{-i} (1 + O\left(\frac{n}{ns-s-n}\right))
\]

\[
= (2H)(2H-1) \frac{1}{s^2 n\left(\frac{1}{s} + \frac{1}{n}\right)^{2H-2}} \left( 1 + O\left(\frac{n}{ns-s-n}\right) \right).
\]

This shows that \(B(s) = \binom{2H}{\frac{1}{2}} n(s+n)^{2H-2} (1 + O\left(\frac{1}{s}\right))\) as \(s \to \infty\). \(\square\)

The main idea of estimating \(B^T \Sigma_{22}^{-1} B\) is to write

\[
B^T \Sigma_{22}^{-1} B = c(k) B^T A^{-1} B = c(k) B^T \sum_{l=0}^{\infty} (I - A)^l B = c(k) \sum_{l=0}^{\infty} B^T (I - A)^l B,
\]

where \(c(k)\) is an appropriate constant chosen below satisfying \(||I - A||_2 < 1\) for \(A = c(k)\Sigma_{22}\).
We consider the minimal and maximal eigenvalues of $\Sigma_{22}$, denoted by $\lambda_{\min}(\Sigma_{22})$ and $\lambda_{\max}(\Sigma_{22})$, before determining $c(k)$, which are closely related to the norm of $\Sigma_{22}^{-1}$.

**Lemma 2.2** Suppose $H > \frac{1}{2}$, then $\lambda_{\min}(\Sigma_{22}) \to c = \text{essinf} f > 0$ as $k \to \infty$.

**Proof** One can define \cite{15} the power spectrum (see \cite{6}, chapter 14) with a singularity at $\lambda = 0$ by

$$f(\lambda) := \sum_{k=-\infty}^{\infty} b(k) e^{-i2\pi \lambda k}, \quad -\frac{1}{2} < \lambda < \frac{1}{2},$$

(also known as the spectral density function or spectrum of the stationary process $\{X_n\}$) where $b(k) = E(X_n X_{n+k})$ is the covariance function as before. $f(\lambda)$ has an inverse transformation:

$$b(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(\lambda) e^{2\pi i k \lambda} d\lambda.$$

For the fractional Gaussian noise $\{X_k\}$, $f(\lambda)$ has the form (cf. \[15\], Section 2.3):

$$f(\lambda) = C|1 - e^{i2\pi \lambda}|^2 \sum_{m=-\infty}^{\infty} \frac{1}{|\lambda + m|^{1+2H}} = C(1 - \cos(2\pi \lambda)) \sum_{m=-\infty}^{\infty} \frac{1}{|\lambda + m|^{1+2H}}.$$

From \[11\] page 64/65, $\lim_{k \to \infty} \lambda_{\min}(\Sigma_{22}) = \text{essinf} f$ and $\lim_{k \to \infty} \lambda_{\max}(\Sigma_{22}) = \text{esssup} f$ (including the cases $\pm \infty$). Since $\text{essinf}_\lambda f(\lambda) > 0$, the lemma is proved. \hfill \blacksquare

**Lemma 2.3** Let $c(k) = \max_{m} \frac{1}{\lambda_{k}^{m}}$, where $m$ is a constant independent of $k$. If $m$ is large enough then $||I - A||_2 = ||I - c(k)\Sigma_{22}||_2 < 1$.

**Proof** By Lemma 2.1, on the one hand, $||\Sigma_{22}||_2 \leq \sqrt{||\Sigma_{22}||_1 ||\Sigma_{22}||_\infty} = O(k^{2H-1})$, and on the other hand, $\frac{B^T \Sigma_{22} B}{B^T B} \leq \lambda_{\max}(\Sigma_{22}) = ||\Sigma_{22}||_2$ and $\frac{B^T \Sigma_{22} B}{B^T B} = O(k^{2H-1})$. Hence $||\Sigma_{22}||_2 = O(k^{2H-1})$.

The eigenvalues of $I - A := I - c(k)\Sigma_{22}$ are $\{1 - c(k)\lambda_i(\Sigma_{22})\}$, $\lambda_i(\Sigma_{22})$ denoting the eigenvalues of $\Sigma_{22}$. Therefore, choosing $m$ large enough, $|1 - c(k)\lambda_{\max}(\Sigma_{22})|$ and $|1 - c(k)\lambda_{\min}(\Sigma_{22})|$ can be both made less than 1, independently of $k$. Hence $||I - c(k)\Sigma_{22}||_2 < 1$. \hfill \blacksquare

With the preparation above, we can return to the estimate of $B^T \Sigma_{22}^{-1} B = c(k) \sum_{l=0}^{\infty} B^T (I - c(k)\Sigma_{22})^l B$.

**Lemma 2.4** If $\frac{3}{4} < H < 1$, then

$$\lim_{k \to \infty} B^T \Sigma_{22}^{-1} B = 0.$$
It follows from the lemma that \( d_n^2 := \lim_{k \to \infty} \sigma^2(n, k) = O(n^{2H}). \)

**Proof** We first derive an recursive equation, which will be used frequently.

For any \( k \)-dimensional column vector \( V = (V(1), V(2), \ldots, V(k))^T \), define \( V^{(l)} := (I - c(k)\Sigma_{22})^l V = (I - c(k)\Sigma_{22})V^{(l-1)}, \) \( V^{(0)} = V. \) Recall that \( B^T = (B(1), B(2), \ldots, B(k)) \), then

\[
B^T V^{(l)} = B^T (I - c(k)\Sigma_{22}) V^{(l-1)} = \sum_{s=1}^k V^{(l-1)}(s)B(s)
\]

Hence we get a recursive equation for any vector \( V: \)

\[
\sum_{s=1}^k B(s)V^{(l)}(s) = \sum_{s=1}^k B(s)V^{(l-1)}(s) \left( 1 - q(s) \right), \quad l \geq 1
\]

where

\[
q(s) = c(k) \sum_{i=1}^k \frac{B(i)}{B(s)} b(i - s) = c(k) \left( 1 + \sum_{1 \leq i \leq k, i \neq s} \frac{B(i)}{B(s)} b(i - s) \right)
\]

By Lemma 2.1 and \( b(t) = \left( \frac{2H}{2} \right)|t|^{2H-2}(1 + \left( \frac{2H}{2} \right) \frac{1}{t^2} + \cdots) \) for \( t \geq 1, \)

\[
q(s) \geq c(k) \left( \int_2^{s-2} \frac{B(i)}{B(s)} b(s - i) \, di + \int_{s+2}^{k+1} \frac{B(i)}{B(s)} b(s - i) \, di \right)
\]

\[
\geq c(k) \left( C \int_2^{s-2} \frac{(i + n)^{2H-2}}{(s + n)^{2H-2}} (s - i)^{2H-2} \, di + \int_{s+2}^{k+1} \frac{(i + n)^{2H-2}}{(s + n)^{2H-2}} (i - s)^{2H-2} \, di \right)
\]

\[
= \frac{C}{m} \frac{(n + s)^{2-2H}}{k^{2-2H}} \left( \int_{\frac{s+2}{k}}^{\frac{s+2}{k}} (x + \frac{n}{k})^{2H-2} \, dx + \int_{\frac{s+2}{k}}^{\frac{s+2}{k}} (x + \frac{n}{k})^{2H-2} \, dx \right)
\]

\[
= \frac{1}{m} \frac{(n + s)^{2-2H}}{k^{2-2H}} I(k, s, n).
\]

The integral \( I(k, s, n) \) is bounded below by some constant \( K \) independent of \( k, s \) and \( n. \)

For all \( s \) satisfying

\[
(s + n)^{2-2H} \geq n^{2-2H} k^{(2-2H)}
\]

where \( \gamma \in [0, 1] \) is determined below, that is

\[
s \geq s^* := nk^\gamma - n,
\]

\[
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\]
we get
\[ q(s) \geq q := \frac{n^{2-2H}}{k^{(2-2H)(1-\gamma)}} \frac{K}{n^H} \] when \( s \geq s^* \).

In the recursive equation (2.2), put \( V(s) = B(s), s = 1, 2, \cdots, k \), so \( V^{(l)} = (I - c(k)\Sigma_{22})^lB = (I - c(k)\Sigma_{22})V^{(l-1)} \). Then one has
\[
\sum_{s=1}^{k} B(s)V^{(l)}(s) = \sum_{s=1}^{k} (1 - q(s))B(s)V^{(l-1)}(s)
\]
\[
\leq \sum_{s=1}^{k} (1 - q)B(s)V^{(l-1)}(s) + \sum_{s<s^*} (q - q(s))B(s)V^{(l-1)}(s)
\]
\[
\leq (1 - q) \sum_{s=1}^{k} B(s)V^{(l-1)}(s) + (q - \min_{s<s^*} q(s)) \sum_{s<s^*} B(s)V^{(l-1)}(s).
\]

The idea is to incorporate the second term (when \( s < s^* \)) into the first one. For any \( \epsilon > 0 \), define
\[
L^* = \inf \{ l : \sum_{s=1}^{s^*-1} B(s)V^{(l)}(s) \geq \epsilon \sum_{s=1}^{k} B(s)V^{(l)}(s) \}.
\]

If \( L^* = \infty \), then \( \sum_{s=1}^{s^*-1} B(s)V^{(l)}(s) < \epsilon \sum_{s=1}^{k} B(s)V^{(l)}(s) \) for all \( l \). If \( \epsilon \) is small enough, it follows that for all \( l \geq 1 \),
\[
\sum_{s=1}^{k} B(s)V^{(l)}(s) \leq (1 - q) \sum_{s=1}^{k} B(s)V^{(l-1)}(s)
\]
\[
+ \epsilon(q - \min_{s} q(s)) \sum_{s=1}^{k} B(s)V^{(l-1)}(s)
\]
\[
= (1 - q^*) \sum_{s=1}^{k} B(s)V^{(l-1)}(s)
\]
\[
\leq (1 - q^*)^2 \sum_{s=1}^{k} B(s)V^{(0)}(s)
\]
where \( q^* = q - \epsilon(q - \min_{s} q(s)) \). By Lemma 2.1, for some constants \( C_1, C_2 > 0 \)
\[
c(k) \sum_{l=1}^{\infty} \sum_{s=1}^{k} B(s)V^{(l)}(s) \leq c(k) \sum_{l=1}^{\infty} (1 - q^*)^l \sum_{s=1}^{k} B(s)^2
\]
\[
\leq c(k) \frac{C_1}{q^*} n^2 (k + n)^{4H-3}.
\]
\[
\leq c(k) \frac{C_1}{(1 - \epsilon)^q} \left( k + n \right)^{4H - 3} = C_2 n^{2H} k^{(2 - 2H)(-\gamma)}.
\]

When \( l = 0 \), for some constant \( C_3 > 0 \)
\[
c(k) \sum_{s=1}^{k} B(s) V^{(0)}(s) = c(k) \sum_{s=1}^{k} B^2(s) \leq c(k) m C_3 \left( n^2 (k + n)^{4H - 3} \right) \leq C_3 (k^{-2(1-H)} n^2).
\]

Hence,
\[
B^T \Sigma^{-1}_{22} B = c(k) \sum_{l=0}^{\infty} B^T V^{(l)} = O(k^{-2\gamma(1-H)}), \text{ as } k \to \infty.
\]

Therefore, if \( L^* = \infty \), then \( \lim_{k \to \infty} B^T \Sigma^{-1}_{22} B = 0 \).

If \( L^* < \infty \), then for \( l < L^* \),
\[
\sum_{s=1}^{s^* - 1} B(s) V^{(l)}(s) < \epsilon \sum_{s=1}^{k} B(s) V^{(l)}(s),
\]
and
\[
\sum_{s=1}^{s^* - 1} B(s) V^{(L^*)}(s) \geq \epsilon \sum_{s=1}^{k} B(s) V^{(L^*)}(s). \tag{2.3}
\]

In this case, we split \( \sum_{l=1}^{\infty} B^T V^{(l)} \) into two parts:
\[
\sum_{l=1}^{\infty} \sum_{s=1}^{k} B(s) V^{(l)}(s) = \sum_{l=1}^{L^* - 1} \sum_{s=1}^{k} B(s) V^{(l)}(s) + \sum_{l=L^*}^{\infty} \sum_{s=1}^{k} B(s) V^{(l)}(s).
\]

The first term can be handled with in the same way as the case when \( L^* = \infty \):
\[
c(k) \sum_{l=1}^{L^* - 1} \sum_{s=1}^{k} B(s) V^{(l)}(s) \leq c(k) \sum_{l=1}^{L^* - 1} (1 - q^*)^l \sum_{s=1}^{k} B^2(s) \leq c(k) \frac{1}{q^*} \sum_{s=1}^{k} B^2(s) \leq C_2 n^{2H} k^{-2\gamma(1-H)}.
\]

For the other term, in the iteration equation (2.2) take \( V \) to be \( V_{new} := (I - c(k) \Sigma_{22})^{L^*} B = V^{(L^*)} \), then by changing variable \( d = l - L^* \), one has
\[
\sum_{l=L^*}^{\infty} B^T V^{(l)} = \sum_{l=L^*}^{\infty} B^T (I - c(k) \Sigma_{22})^{l-L^*} V_{new} = \sum_{l=L^*}^{\infty} B^T V_{new}^{(l-L^*)}
\]

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\[
\sum_{d=0}^{\infty} \sum_{s=1}^{k} B(s)V_{new}^{(d)}(s) \\
= \sum_{s=1}^{k} B(s)V_{new}^{(0)}(s) + \sum_{d=1}^{\infty} \sum_{s=1}^{k} (1 - q(s))B(s)V_{new}^{(d-1)}(s) \\
\leq \sum_{d=0}^{\infty} (1 - \min q(s))d \sum_{s=1}^{k} B(s)V^{(L^*)}(s) \\
\leq \sum_{d=0}^{\infty} (1 - \min q(s))d \cdot \frac{1}{\ell} \sum_{s=1}^{s_*-1} B(s)V^{(L^*)}(s) \text{ by (2.3)} \\
\leq \frac{1}{\min q(s)\epsilon} \sum_{s=1}^{s_*-1} B(s)^2 \\
\leq \frac{C_4}{\min q(s)} n^{4H-1} k^{\gamma(4H-3)},
\]

where \(C_4 > 0\) and Lemma 2.1 is used.

Since \(q(s) \geq \frac{1}{m}(\frac{n+1}{k})^{2-2H} K\), \(\min q(s) \geq \frac{K n^{2H-2}}{m^{2-H}}\), one has

\[
c(k) \sum_{l=L^*}^{\infty} B^T V^{(l)} \leq c(k) \frac{C_4}{\min q(s)\epsilon} n^{4H-1} k^{\gamma(4H-3)} = O(k^{-(1-\gamma)(4H-3)}).
\]

Combining (2.4) and (2.5), one has \(B^T \Sigma_{22}^{-1} B \leq C(k^{-(1-\gamma)(4H-3)} + k^{-2\gamma(1-H)})\). Since \(H \in \left(\frac{3}{4}, 1\right)\), \(\lim_{k \to \infty} B^T \Sigma_{22}^{-1} B = 0\) follows.

\[\blacksquare\]

### 2.3 Estimate of the mean

We continue writing \(\Sigma_{22}\) for \(\Sigma_{22}(k,k)\).

**Lemma 2.5** Suppose \(3/4 < H < 1\), then almost surely,

\[
\lim_{k \to \infty} B^T \Sigma_{22}^{-1} X_{[2]} = 0,
\]

where \(X_{[2]} = (X_{n+1}, X_{n+2}, \ldots, X_{n+k})^T\) and \(B = (B(1), \ldots, B(k))^T\) depend on \(k\) as before and \(n\) is fixed.

**Proof** The random variable \(B^T \Sigma_{22}^{-1} X_{[2]}\) has a normal distribution. Its mean and variance are 0 and \((B^T \Sigma_{22}^{-1}) \Sigma_{22} (\Sigma_{22}^{-1} B) = B^T \Sigma_{22}^{-1} B\), respectively. Hence for any \(\epsilon > 0\), and supressing the running
index $k$,
\[
\sum_{k=1}^{\infty} P(|B^T\Sigma_{22}^{-1}X_{[2]}| > \epsilon) = \frac{2}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \int_{\epsilon/(B^T\Sigma_{22}^{-1}B)^{1/2}}^{\infty} e^{-\frac{x^2}{2}} \, dx \\
\leq \frac{2}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \exp \left\{ -\frac{(\epsilon/(B^T\Sigma_{22}^{-1}B)^{1/2})^2}{2} \right\} \\
= \frac{2}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \exp \left( -\frac{\epsilon^2}{2} (B^T\Sigma_{22}^{-1}B)^{-1} \right).
\]
Recall that by the proof of Lemma 2.4 $B^T\Sigma_{22}^{-1}B = O(k^{-\gamma})$ for some $\gamma > 0$, so that by the Borel-Cantelli Lemma it follows that $B^T\Sigma_{22}^{-1}X_{[2]} \to 0$ as $k \to \infty$ almost surely.

\section{Limit Theorem of the Occupation Times of $\{S_n\}$}

Recall that the occupation time of $S_n = \sum_{i=1}^{n} X_i$ is defined as $\lambda(n, A) = \sum_{i=1}^{n} 1_A(S_i)$. The main result is that for some $a_n$, the normalized occupation time $\frac{1}{n} \lambda(n, A)$ converges to Mittag-Leffler distribution in the sense of Theorem 1.2. In this section, we give the proof of Theorem 1.2 via Theorem 1.1.

\subsection{Representation of the occupation time}

In this section, we consider the stationary Gaussian random variables $\{X_n\}$ as above to be represented as the coordinate process of a shift dynamical systems.

Without loss of generality, suppose the random variables $\{X_n\}$ are defined on the probability space $(\Omega, \mathcal{B}_0, P)$ with $\Omega = \mathbb{R}^\mathbb{N}$ and $\mathcal{B}_0$ is the $\sigma$-algebra generated by the cylinder sets of $\mathbb{R}^\mathbb{N}$. Let $T$ denote the shift operator: $T : \Omega \to \Omega$, $(T\omega)_i = \omega_{i+1}$ where $\omega = (\omega_1, \omega_2, \ldots) \in \mathbb{R}^\mathbb{N}$. Define $\phi : \Omega \to \mathbb{R}$ as $\phi(\omega) := \omega_1$, $\int\phi \, dP < \infty$ and $\int\phi \, dP = 0$. The probability $P$ is the distribution of the stochastic process $\{X_n\}$, which then is represented as $X_n(\omega) := \phi \circ T^{(n-1)}(\omega) = \omega_n$, $n \in \mathbb{N}$, and has the joint Gaussian probability distribution with zero mean: for any family of Borel sets $C_1, C_2, \ldots C_r \subset \mathbb{R}$, $P(\{\omega \in \Omega : X_{n_1}(\omega) \in C_1, X_{n_2}(\omega) \in C_2, \ldots X_{n_r}(\omega) \in C_r\}) = \int_{C_1 \times C_2 \times \ldots C_r} p(t_1, t_2, \ldots, t_r) \, dt_1 \, dt_2 \ldots \, dt_r$. Here $p$ is the normal probability density function: $p(t_1, t_2, \ldots, t_r) = C \exp\left(-\frac{1}{2}Dt, t)\right)$, where $D$ is the matrix inverse to the covariance matrix $B = (b(n_i - n_j))$ and $b(n_i - n_j) = E[X_{n_i}X_{n_j}]$.

We represent the occupation time of $\{S_n\}$ by introducing the skew product: Let $(X, \mathcal{B}, \mu) = (\Omega \times \mathbb{R}, \mathcal{B}_0 \times \sigma(\mathbb{R}), P \times m_{\mathbb{R}})$, where $\sigma(\mathbb{R})$ is the Borel $\sigma$-algebra and $m_{\mathbb{R}}$ denotes the Lebesgue measure on $\mathbb{R}$. 

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Define $\tilde{T} : \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}$ by $\tilde{T}(\omega, r) := (T(\omega), r + \phi(\omega))$. By induction, $\tilde{T}^n(\omega, r) = (T^n(\omega), r + S_n(\omega))$, where $S_n$ is the partial sum of $\{X_n\}$.

Define $S_n(\tilde{T}(f)) := \sum_{k=0}^{n-1} f \circ \tilde{T}^k$ for any $f : \Omega \to \mathbb{R}$. Specifically, $S_n^T(\phi) = S_n$. Let $A = \Omega \times \mathbb{1}_D$, then the occupation time of $\{S_n\}$ for set $D \subset \mathbb{R}$ has the following representation:

$$\lambda(n, D) = \sum_{i=1}^{n} \mathbb{1}_{\{S_i \in D\}} = \sum_{i=1}^{n} \mathbb{1}_{\{A\}}(\tilde{T}^i(\omega, 0)) = S_n^T(\mathbb{1}_{\{A\}})(\omega, 0).$$

### 3.2 Proof of the main theorem

**Theorem 3.1 (Conservative and Ergodic Decomposition)** For the dynamic system $(X, \mathcal{B}, \mu, \tilde{T})$ defined above, one has

1. $\tilde{T}$ is a conservative measure preserving transformation of $(X, \mathcal{B}, \mu)$.

2. There exists a probability space $(Y, \mathcal{C}, \lambda)$ and a collection of measures $\{\mu_y : y \in Y\}$ on $(X, \mathcal{B})$ such that

   (a) for $\lambda$–almost all $y \in Y$, $\tilde{T}$ is a conservative, ergodic measure-preserving transformation of $(X, \mathcal{B}, \mu_y)$;

   (b) for $A \in \mathcal{B}$, the map $y \to \mu_y(A)$ is measurable and

   $$\mu(A) = \int_Y \mu_y(A) d\lambda(y).$$

3. And $\lambda$–almost surely, $(X, \mathcal{B}, \mu_y, \tilde{T})$ is pointwise dual ergodic.

**Proof** 1. By Corollary 8.1.5 in [2], in order to prove that $\tilde{T}$ is conservative, it is sufficient to show that

1. $\phi : \Omega \to \mathbb{R}$ is integrable, and $\int_\Omega \phi dP = 0$.
2. $T$ is ergodic and probability preserving on $(\Omega, \mathcal{B}_0, P)$.

1 holds by our assumption on $\{X_n\}$. For 2, by [6] (page 369), $\lim_{n \to \infty} b(n) = 0$ is a necessary and sufficient condition that $T$ is mixing: $|P(A \cap T^{-n}B) - P(A)P(B)| \to 0$ as $n \to \infty$ for any $A, B \in \mathcal{B}_0$. It implies that $T$ is ergodic. $T$ is also a probability preserving transformation, since $\{X_n\}$ is a stationary process. Hence $\tilde{T}$ is conservative.

$\tilde{T}$ is measure preserving since $T$ is measure preserving.
2. The proof of the ergodicity decomposition is an adaptation of the corresponding argument of Section 2.2.9 of [2] (page 63).

3. Next we prove that \((X, \mathcal{B}, \mu_y, \tilde{T})\) is pointwise dual ergodic.

We claim that
\[
\sum_{n=0}^{\infty} P_{\tilde{T}^n} 1_{\Omega} \otimes 1_{(a,b)} = \infty \quad \mu - a.s.,
\]
where \(P_{\tilde{T}^n} : L^1(\mu) \to L^1(\mu)\) is the Frobenius-Perron operator of \(\tilde{T}^n\):
\[
\int_X (P_{\tilde{T}^n} f) \cdot g \, d\mu = \int_X f \cdot (g \circ \tilde{T}^n) \, d\mu
\]
for any \(g \in L^\infty(\mu)\). Likewise we denote the Frobenius-Perron operator for \(T^n\) by \(P_{T^n}\). Now,
\[
P_{\tilde{T}^n} \left(1_{\Omega} \otimes 1_{(a,b)}\right)(\omega, z) = P_{T^n} \left(1_{(z-b, z-a)} \left(S_n(\cdot)\right)\right)(ω), \ \mu - a.s. (ω, z) \in X,
\]
and
\[
P_{T^n} \left(1_{(z-b, z-a)} \left(S_n(\cdot)\right)\right)(ω) = P(S_n \in (z-b, z-a) | T^n(\cdot) = ω).
\]

By Theorem 1.1,
\[
\sum_{n=0}^{N} P_{\tilde{T}^n} \left(1_{\Omega} \otimes 1_{(a,b)}\right)(ω, z) = \sum_{n=0}^{N} P(S_n \in (z-b, z-a) | T^n(\cdot) = ω) \sim \sum_{n=0}^{N} (b-a) \frac{g(0)}{d_n} \mu - a.s. (ω, z) \in X.
\]

Let \(a_N = \sum_{n=0}^{N} \frac{g(0)}{d_n}\), then \(a_N \to \infty\) as \(N \to \infty\), and
\[
\sum_{n=0}^{N} P_{\tilde{T}^n} \left(1_{\Omega} \otimes 1_{(a,b)}\right) \sim a_N (b-a), \ \mu - a.s.
\]
It follows that \(\lambda\)-a.s. \(y\),
\[
\sum_{n=0}^{N} P_{T^n} \left(1_{\Omega} \otimes 1_{(a,b)}\right) \sim a_N (b-a) \mu_y - a.s.
\]

Since for \(\lambda\)-a.e. \(y\), \(\tilde{T}\) is conservative on \((X, \mathcal{B}, \mu_y)\), by Hurewicz’s ergodic theorem (see for example, [2]), one has for all \(f \in L^1(\mu_y)\), \(\mu_y\)-almost surely,
\[
\frac{1}{a_N} \sum_{n=0}^{N} P_{\tilde{T}^n} f \sim \frac{(b-a) \sum_{n=0}^{N} P_{\tilde{T}^n} f}{\sum_{n=0}^{N} P_{\tilde{T}^n} \left(1_{\Omega} \otimes 1_{(a,b)}\right)}
\]

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\[
\int_{\Omega \times \mathbb{R}} (b-a) f d\mu_y^{\omega, x} \\
= \int_{\Omega \times \mathbb{R}} \mathbb{1}_{\omega} \otimes \mathbb{1}_{(a,b)}(y) d\mu_y^{\omega, x} \\
= \frac{(b-a)}{\mu_y^{\omega}((\omega \times (a,b)))} \int_{\Omega \times \mathbb{R}} f d\mu_y^{\omega, x}.
\]

Since we may change the interval \((a, b)\) to be some other interval \((c, d)\), one finds that \(\frac{(b-a)}{\mu_y^{\omega}}\) does not depend on \((a, b)\), hence \(\frac{(b-a)}{\mu_y^{\omega}((\omega \times (a,b)))}\) is a constant, denoted by \(C(y)\).

Thus, \((X, \mathcal{B}, \mu_y, \tilde{T})\) is pointwise dual ergodic with return sequence \(a_nC(y)\).

We end the paper with the proof of Theorem 1.2.

**Proof** Since \(\tilde{T}\) is pointwise dual ergodic with respect to measure \(\mu_y\), suppose \(a_n\) is regularly varying with index \(\alpha = 1 - H\) and has the same order as \(\sum_{i=0}^{n} \frac{\alpha(i)}{a_i}\), then by Corollary 3.7.3 in [2], \(\frac{S_n^{\tilde{T}}}{C(y)a_n}\) converges strongly in distribution, i.e.,

\[
\int_X v \left( \frac{S_n^{\tilde{T}}(f)(\omega, x)}{C(y)a_n} \right) h_y(\omega, x) d\mu_y(\omega, x) \to E[v(\mu_y(f)Y_\alpha)],
\]

or equivalently,

\[
\int_X v \left( \frac{S_n^{\tilde{T}}(f)(\omega, x)}{a_n} \right) h_y(\omega, x) d\mu_y(\omega, x) \to E[v(C(y)\mu_y(f)Y_\alpha)],
\]

for any bounded and continuous function \(v : \mathbb{R} \to \mathbb{R}\) and for any \(h_y \in L^1(\mu_y)\) and \(\int_X h_y d\mu_y = 1\), where \(S_n^{\tilde{T}}(f) = \sum_{i=1}^{n} f \circ \tilde{T}^{-i-1}\), and \(Y_\alpha\) has the normalized Mittag-Leffler distribution of order \(\alpha = 1 - H\).

Let \(f = 1_{\Omega} \times 1_{(a,b)}\), then \(S_n^{\tilde{T}}(f)(\omega, x) = \sum_{i=1}^{n} 1_{(a,b)}(x + S_i(\omega))\), which is the occupation time of \(S_n\) at time \(n\) on interval \((a-x, b-x)\). Since \(C(y) = \frac{b-a}{\mu_y^{\omega}(x (\omega \times (a,b)))}\), \(C(y)\mu_y(1_{\Omega} \otimes 1_{(a,b)}) = \int_R 1_{(a,b)} dm\), then the right hand side of (3.1) is simplified to be \(E\left[v \left( (b-a)Y_\alpha \right) \right]\).

If \(\Phi_1(\omega, x)\) is any probability density function on \((X, \mathcal{B}, \mu)\), for each \(y\), define

\[
\phi_y(\omega, x) = \begin{cases} 
\frac{1}{\int_X \Phi_1(\omega, x) d\mu_y^{\omega, x}} \Phi_1(\omega, x), & \int_X \Phi_1(\omega, x) d\mu_y^{\omega, x} \neq 0; \\
0, & \int_X \Phi_1(\omega, x) d\mu_y^{\omega, x} = 0.
\end{cases}
\]

\(\phi_y(\omega, x)\) is a density function on \((X, \mathcal{B}, \mu_y)\) for \(y \in U\) where \(U = \{y \in Y : \int_X \Phi_1(\omega, x) d\mu_y^{\omega, x} \neq 0\}\). By (3.1) and Theorem 3.1, one has

\[
\int_X v \left( \frac{\sum_{i=1}^{n} 1_{(a,b)}(x + S_i(\omega))}{a_n} \right) \Phi_1(\omega, x) d\mu_y^{\omega, x}
\]
\[
\int_U \int_X v \left( \frac{\sum_{i=1}^n \mathbb{1}_{(a,b)}(x + S_i(\omega))}{a_n} \right) \Phi_1(\omega, x) d\mu_y(\omega, x) d\lambda(y) \\
= \int_U \left( \int_X \Phi_1(\omega, x) d\mu_y(\omega, x) \right) \int_X v \left( \frac{\sum_{i=1}^n \mathbb{1}_{(a,b)}(x + S_i(\omega))}{a_n} \right) \phi_y(\omega, x) d\mu_y(\omega, x) d\lambda(y) \\
= \int_Y \left( \int_X \Phi_1(\omega, x) d\mu_y(\omega, x) \right) \int_X v \left( \frac{\sum_{i=1}^n \mathbb{1}_{(a,b)}(x + S_i(\omega))}{a_n} \right) \phi_y(\omega, x) d\mu_y(\omega, x) d\lambda(y) \\
\rightarrow \int_Y \mu_y(\Phi_1) E[v((b - a)Y_\alpha)] d\lambda(y) \text{ by the dominated convergence theorem} \\
= E[v((b - a)Y_\alpha)].
\]

Let \( \Phi_1(x, \omega) = \frac{1}{2\epsilon} \mathbb{1}_{(-\epsilon, \epsilon)}(x) \otimes \Phi(\omega) \) where \( \epsilon > 0 \) and \( \Phi(\omega) \) is a probability density function on \((\Omega, \mathcal{B}_0, P)\). Then one obtains as \( n \to \infty \),

\[
\frac{1}{2\epsilon} \int_{-\epsilon}^\epsilon \left[ \int_{\Omega} v \left( \frac{\sum_{i=1}^n \mathbb{1}_{(a-x,b-x)}(S_i(\omega))}{a_n} \right) \Phi(\omega) dP(\omega) \right] \ dx \rightarrow E[v((b - a)Y_\alpha)].
\]

\[\blacksquare\]

References

[1] Aaronson, J.: The asymptotic distributional behaviour of transformations preserving infinite measures. J. Analyse Math. 39 (1981): 203-234.

[2] Aaronson, J.: An introduction to infinite ergodic theory. Mathematical Surveys and Monographs 50, Amer. Math. Soc., Providence, RI, 1997.

[3] Aaronson, J. and Denker, M.: Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. Stochastics & Dynamics 1 (2001): 193-237.

[4] Bilodeau, M. and Breuer, D.: Theory of multivariate statistics. Springer Texts in Statistics, Springer-Verlag, New York 2009.

[5] Chung, K. L.; Kac, M.: Remarks on fluctuations of sums of independent random variables. Mem. Amer. Math. Soc. 6 (1951), 11 pp. Corrections: Proc. Amer. Math. Soc. 4, (1953): 560–563.

[6] Cornfeld I.P., Fomin, S.V. and Sinai, Y.G.: Ergodic theory. Grundlehren der mathematischen Wissenschaften 245, Springer-Verlag, New York, 1982.

[7] Csörgő, S.: Renyi-mixing and occupation times. In: Asymptotic Methods in Probability and Statistics (Ottawa 1997), 3-12, North-Holland, Amsterdam 1998.
[8] Darling, D.A. and Kac, M.: *On occupation times for Markoff processes*. Trans. Amer. Math. Soc. 84 (1957): 444-458.

[9] Denker, M. and Zheng, X.: *On local times for stationary mixing processes*. Preprint 2017.

[10] Dobrushin, R.L.: *Two limit theorems for the simplest random walk on a line*. (Russian) Uspehi Mat. Nauk (N.S.) 103(65) (1955): 139–146.

[11] Grenander, U. and Szegö, G.: *Toeplitz forms and their applications*. University of California Press 321, 1958.

[12] Ibragimov, I.A. and Linnik, Yu. V.: Independent and stationary sequences of random variables. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov. Translation from the Russian edited by J. F. C. Kingman. Wolters-Noordhoff Publishing, Groningen, 1971.

[13] Kasahara, Y. and Matsumoto Y.: *On Kallianpur-Robbins law for fractional Brownian motion*. J. of Math., Kyoto University 36 (1996): 815-824.

[14] Kono, N.: *Kallianpur-Robbins law for fractal Brownian motion*. In: Probability theory and mathematical statistics. Proceedings of the 7th Japan-Russia Symposium held in Tokyo, July 26–30, 1995. Eds: S. Watanabe, M. Fukushima, Yu. V. Prohorov and A. N. Shiryaev. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[15] Qian, H.: *Fractional Brownian motion and fractional Gaussian noise*. In: Processes with Long-Range Correlations. Lecture Notes in Physics 621: 22-33. Eds. G. Rangarajan, M.Z. Ding. Springer, Berlin Heidelberg 2003.

[16] Taqqu, M.S.: *Weak convergence to fractional Brownian motion and to the Rosenblatt process*. Probability Theory and Related Fields 31(4) (1975): 287-302.

[17] Zheng, X.: Studies on the local times of discrete-time stochastic processes. PhD thesis, The Pennsylvania State University 2017.