Quantum wormhole as a Ricci flow

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The idea is considered that a quantum wormhole in a spacetime foam can be described as a Ricci flow. In this interpretation the Ricci flow is a statistical system and every metric in the Ricci flow is a microscopical state. The probability density of the microscopical state is connected with a Perelman’s functional of a rescaled Ricci flow.

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I. INTRODUCTION

Ricci flows are the tool for the investigation of the topology of manifolds. Generally Ricci flow creates on a manifold a singularity (or singularities) for a finite parameter \( \lambda_0 \). Fig. 1 gives a schematic picture of the partially singular metric \( g(\lambda_0) \) on the manifold \( \mathcal{M} \). The metric \( g(\lambda_0) \) is smooth on a maximal domain \( \Omega \subset \mathcal{M} \), where the curvature is locally bounded but is singular, i.e. ill-defined, on the complement where the curvature blows-up as \( \lambda \to \lambda_0 \).

If \( \Omega \neq \emptyset \), then the main point is that small neighborhoods of the boundary \( \partial \Omega \) consist of horns. A horn is a metric on \( S^2 \times [0, \delta] \) where the \( S^2 \) factor is approximately round of radius \( \rho(r) \), with \( \rho(r) \) small and \( \rho(r)/r \to 0 \) as \( r \to 0 \). Fig. 1 represents a partially singular metric on the smooth manifold \( S^2 \times I \), consisting of a pair horns joined by a degenerate metric.

This is exactly the same what physicists are talking about a quantum wormhole in a spacetime foam. It allows us to think that Ricci flows are the mathematical tool for the description of the wormhole in the spacetime foam. Here we offer the idea that for every \( \lambda \) the 3D space-like metric \( g(\lambda) \) is realised with some probability \( \rho(\lambda) \) where the parameter \( \lambda \) describes the evolution of the metric \( g \) under the Ricci flow. Usually such probability is connected with path integral. We offer the idea that this probability is connected with a Perelman’s functional \( W \) on a rescaled Ricci flow as \( \rho \propto \frac{dW}{d\lambda} \) in the consequence of the property \( \frac{dW}{d\lambda} \geq 0 \). Then the Ricci flow is a statistical system where every metric \( g(\lambda) \) is a microscopical state.

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The notion of a spacetime foam was introduced by Wheeler [1] for the description of the possible complex structure of spacetime on the Planck scale ($l_{Pl} \approx 10^{-33} cm$). The exact mathematical description of this phenomenon is very difficult and even though there is a doubt: does the Feynman path integral in the gravity contain a topology change of the spacetime? This question spring up as (according to the Morse theory) the singular points must arise by topology changes. In such points the time arrow is undefined that leads in difficulties at the definition of the Lorentzian metric, curvature tensor and so on.

The Ricci flows were introduced by Hamilton [2] over 25 years ago. It plays an important role in the proof of the Poincare conjecture [3]. In Ref. [4] the evolution of wormhole geometries under Ricci flow is studied. Depending on value of initial data parameters, wormhole throats either pinch off or evolve to a monotonically growing state. The connection between Ricci flow and quantum mecanins is considered in Ref’s [5] [6]. The physical application of Ricci flow can be found in Ref’s 7 - 8 as well. Very simple introduction to Ricci flows for non-specialists can be found in Ref. [12].

Quantum gravity remains a theorists playground, an arena for theoretical experiments which may or may not stand the test of time. The two leading candidates for a quantum theory of gravity today are string theory and loop quantum gravity. One can find a thorough introduction to string theory in textbook [10], and a review of loop quantum gravity in Ref. [11]. Despite some 70 years of active research, no one has yet formulated a consistent and complete quantum theory of gravity. The failure to quantize gravity rests in part on technical difficulties. General relativity is a complicated and highly nonlinear theory. But the real problems are almost certainly deeper: quantum gravity requires a quantization of spacetime itself, and at a fundamental level we do not know what that means.

Our point of view is that in quantum gravity it is necessary to separate two problems: the first one is the quantization of very strongly self-interacting gravitational field (metric), the second one is the problem of topology change in quantum gravity. We think that it is two different problems. For the first problem we need a non-perturbative quantization technique (in some more weakly sense this problem one can find in quantum chromodynamics – so called confinement problem). In this paper we propose an approach for the second problem: we conjecture that a quantum wormhole in a spacetime foam may be described on the Ricci flow language. The mathematicians use this language for the description of topology change. Our approach is that the quantum wormhole can be described as a statistical system and corresponding probability is connected with Perelman functional.

II. RICCI FLOWS

In this section we follow to Ref. [12]. Ricci flow is a means of processing the metric $g_{ab}$ by allowing it to evolve under

$$\frac{\partial g_{ab}(x^c, \lambda)}{\partial \lambda} = -2R_{ab}(x^c, \lambda)$$

(1)

where $R_{ab}$ is the Ricci curvature; $\lambda$ is a parameter; $a, b = 1, 2, 3$; $x^c$ is the coordinate on a manifold $\mathcal{M}$. The Ricci flow describes the evolution of the metric $g_{ab}$ in during of the parameter $\lambda$. In Ref. [12] such reply for the question “Ricci flow: what is it, and from where did it come” is given: “... the flow can be used to deform $g_{ab}$ into a metric distinguished by its curvature. For example, if $\mathcal{M}$ is two-dimensional, the Ricci flow deforms a metric conformally to one of constant curvature, and thus gives a proof of the two-dimensional uniformisation theorem. More generally, the topology of $\mathcal{M}$ may preclude the existence of such distinguished metrics, and the Ricci flow can then be expected to develop a singularity in finite time [13]. Nevertheless, the behavior of the flow may still serve to tell us much about the topology of the underlying manifold. The strategy of the investigation is to stop a flow, and then carefully perform “surgery” on the evolved manifold, exciting any singular regions before continuing the flow.

Provided we understand the structure of finite time singularities sufficiently well, we may hope to keep track of the change in topology of the manifold under surgery, and reconstruct the topology of the original manifold from a limiting flow, together with the singular regions removed.”

Let us introduce the functional

$$W(g_{ab}, f, \tau) := \int \left[ \tau \left( R + |f|^2 \right) + f - n \right] u \ dV$$

(2)

where $f : \mathcal{M} \rightarrow \mathbb{R}$ is a smooth function; $R$ is the Ricci scalar; $\tau > 0$ is a scale parameter; $n = \dim \mathcal{M}$, and $u$ is defined by

$$u := (4\pi\tau)^{-n/2} e^{-f}.$$  

(3)
Theorem II.1 If \( \mathcal{M} \) is closed, and \( g_{ab}, f \) and \( \tau \) evolve according to

\[
\frac{\partial g_{ab}}{\partial \lambda} = -2R_{ab},
\]
\[
\frac{d\tau}{d\lambda} = -1,
\]
\[
\frac{\partial f}{\partial \lambda} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}
\]

then the functional \( W \) increases according to

\[
\frac{d}{d\lambda} W(g_{ab}, f, \tau) = 2\tau \int \left| R_{ab} + \frac{\partial^2 f}{\partial x^a \partial x^b} - \frac{g_{ab}}{2\tau} \right|^2 udV \geq 0
\]

where \( R_{ab} \) is the Ricci tensor.

Under evolution (4)-(6)

\[
\frac{d}{d\lambda} \int udV = 0.
\]

It means that

\[
\int udV = \text{const}
\]

and \( u(\lambda) \) represents the probability density of a particle evolving under Brownian motion, backwards in time. It allows us to define the classical, or “Boltzman - Shannon” entropy

\[
S = -\int_{\mathcal{M}} u \ln u \, dV
\]

or a renormalized version of the classical entropy

\[
\tilde{S} = S - \frac{n}{2} \{ 1 + \ln [4\pi (\lambda_0 - \lambda)] \}.
\]

The \( W \)-functional applied to this backwards Brownian diffusion on a Ricci flow also arises via the renormalized classical entropy \( \tilde{S} \)

\[
W(\lambda) = -\frac{d}{d\lambda} \left( \tau \tilde{S} \right).
\]

We would like to offer the following physical interpretation of the Ricci flow:

- the Ricci flow is a statistical system and describes the topology change in quantum gravity;
- for every \( \lambda \), \( g_{ab}(\lambda) \) is a microscopical state in the statistical system;
- \( \frac{d}{d\lambda} W(g_{ab}, f, \tau) \) is a probability density for the microscopical state \( g_{ab}(\lambda) \).

The problem here is that at \( \lambda \to \lambda_0 \) (where \( \lambda_0 \) is some parameter depending on \( g_{ab}(\lambda = 0) \)) the metric blow up and a singularity appears. The possible solution of this problem is rescaling or renormalizing the Ricci flow.

The curvature of a Ricci flow blow up in magnitude at a singularity and it is necessary to work towards a theory of “blowing-up” that we can rescale a flow more and more as we get closer and closer to a singularity. In Ref. [9] we read: “The usual method to understand the structure of singularities is to rescale or renormalize the solution on a sequence converging to the singularity to make the solution bounded and try to pass to a limit of the renormalization. Such a limit solution serves as a model for the singularity, and one hopes... that the singularity models have special features making them much simpler than an arbitrary solution of the equation.”

In Appendix A the mathematical definitions and theorems are given which are necessary for mathematical understanding of Ricci flow rescaling. For the physical understanding of the idea presented here the Theorem A.3 is necessary only. The theorem states that there exists a regular “singularity” model \((\mathcal{N}, \tilde{g}_{ab}(t), p)\) in the notations of Appendix A. Such singularity models do in fact have important features making them much simpler than general solutions of the Ricci flow. If \( \mathcal{N} \) is noncompact, then \( \mathcal{N} \) is diffeomorphic to \( \mathbb{R}^3 \), \( S^2 \times \mathbb{R} \) or a quotient of these spaces. If \( \mathcal{N} \) is compact, then \( \mathcal{N} \) is diffeomorphic to \( S^1/\Gamma \), \( S^2 \times S^1 \) or \( S^2 \times_{\mathbb{Z}_2} S^1 \).

Thus we modify the physical interpretation of the Ricci flow as follows:
• the Ricci flow is a statistical system and describes the topology change in quantum gravity;
• for every $\lambda$, $g_{ab}(\lambda)$ is a microscopical state in the statistical system;
• a probability density for the microscopical state $g_{ab}(\lambda)$ is defined as $\frac{d\lambda}{\lambda} \tilde{W}(\hat{g}_{ab}, f, \tau)$ where $\tilde{W}(\hat{g}_{ab}, f, \tau)$ is the Perelman’s functional for a rescaled Ricci flow $(N, g_{ab}(t), p)$.

The last item means that the Perelman’s functional is calculated on above mentioned singularity model.

III. QUANTUM WORMHOLE IN A SPACETIME FOAM

In this section we would like to apply the Ricci flow for the description of appearing/disappearing quantum wormhole in 4D Einstein gravity with characteristic sizes in Planck region; on the second step we should obtain a Ricci flow with initial conditions as above mentioned wormhole.

A. Wormhole supported by two interacting scalar fields

In Ref. [13] it is found a wormhole solution supporting with two phantom scalar fields. The Lagrangian is

$$L = -\frac{\mathcal{R}}{16\pi G} + \epsilon \left[ \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi + \frac{1}{2} \partial_{\mu}\chi \partial^{\mu}\chi - V(\phi, \chi) \right], \quad (13)$$

where $\mathcal{R}$ is the 4D scalar curvature, $G$ is the Newtonian gravity constant and the constant $\epsilon = -1$ means that we consider phantom scalar fields $\phi, \chi$. The potential $V(\phi, \chi)$ is

$$V(\phi, \chi) = \frac{\lambda_1}{4}(\phi^2 - m_1^2)^2 + \frac{\lambda_2}{4}(\chi^2 - m_2^2)^2 + \phi^2 \chi^2 - V_0 \quad (14)$$

where $\phi, \chi$ are two scalar fields with the masses $m_1$ and $m_2$, $\lambda_1, \lambda_2$ are the self-coupling constants and $V_0$ - some constant. The field equations are

$$\mathcal{R}^k_i - \frac{1}{2} \delta^k_i \mathcal{R} = 8\pi G T^k_i, \quad (15)$$

$$\frac{1}{\sqrt{-g}} \partial_{\nu} \left[ \sqrt{-g} g_{\nu}^{\sigma} \partial_{\sigma}(\phi, \chi) \right] = -\frac{\partial V}{\partial(\phi, \chi)}, \quad (16)$$

where $\mathcal{R}^k_i$ is the 4D Ricci tensor; $T^k_i$ is the energy-momentum tensor for scalar fields $\phi, \chi; g_{\mu\nu}$ is 4D spacetime metric [10]. The wormhole metric is

$$ds^2 = B(r)dt^2 - dr^2 - A(r)(d\theta^2 + \sin^2\theta d\phi^2) \quad (17)$$

where $A(r), B(r)$ are the even functions depending only on the coordinate $r$ which covers the entire range $-\infty < r < +\infty$. Using this metric, one can obtain from Eq’s [15] the following equations

$$\frac{A''}{A} - \frac{1}{2} \left( \frac{A'}{A} \right)^2 - \frac{1}{2} \frac{A' B'}{B} = \varphi'^2 + \chi'^2, \quad (18)$$

$$\frac{A''}{A} + \frac{1}{2} \frac{A' B'}{B} - \frac{1}{2} \left( \frac{A'}{A} \right)^2 - \frac{1}{2} \left( \frac{B'}{B} \right)^2 + \frac{B''}{B} = 2 \left[ \frac{1}{2} (\varphi'^2 + \chi'^2) + V \right], \quad (19)$$

$$\frac{1}{4} \left( \frac{A'}{A} \right)^2 - \frac{1}{A} + \frac{1}{2} \frac{A' B'}{B} = \frac{1}{2} (\varphi'^2 + \chi'^2) + V, \quad (20)$$

where a prime denotes differentiation with respect to $r$. The corresponding field equations from [10] are

$$\varphi'' + \left( \frac{A'}{A} + \frac{1}{2} \frac{B'}{B} \right) \varphi' = \varphi \left[ 2\chi'^2 + \lambda_1 (\varphi^2 - m_1^2) \right], \quad (21)$$

$$\chi'' + \left( \frac{A'}{A} + \frac{1}{2} \frac{B'}{B} \right) \chi' = \chi \left[ 2\varphi'^2 + \lambda_2 (\chi^2 - m_2^2) \right]. \quad (22)$$
In the equations (18)-(22) the following rescaling are used: \( r \rightarrow \sqrt{8\pi G} r, \varphi \rightarrow \varphi/\sqrt{8\pi G}, \chi \rightarrow \chi/\sqrt{8\pi G}, m_{1,2} \rightarrow m_{1,2}/\sqrt{8\pi G} \).

The boundary conditions are choosing with account of \( Z_2 \) symmetry in the following form:

\[
\begin{align*}
\varphi(0) &= \sqrt{3}, & \varphi'(0) &= 0, \\
\chi(0) &= \sqrt{0.6}, & \chi'(0) &= 0, \\
A(0) &= -\frac{1}{V(\phi(0), \chi(0))}, & A'(0) &= 0, \\
B(0) &= 1.0, & B'(0) &= 0,
\end{align*}
\]  

(23)

where the condition for \( A(0) \) is choosing to satisfy the constraint \( (20) \) at \( r = 0 \), \( V(\phi(0), \chi(0)) \) is the value of the potential at \( r = 0 \) and the self-coupling constants \( \lambda_1 = 0.1 \) and \( \lambda_2 = 1 \).

The results are presented in Fig’s. 2-4. These results are obtained for the masses \( m_1 \approx 2.661776085 \) and \( m_2 \approx 2.928340304 \).

The asymptotical behavior of the wormhole solution is

\[
\begin{align*}
A &\approx r^2 + r_0^2, \\
B &\approx B_{\infty} \left(1 - \frac{r_0}{r}\right)
\end{align*}
\]  

(24) (25)

where \( r_0 \) and \( B_{\infty} \) are some constants.

![Graph](image)

FIG. 2: The scalar fields \( \varphi, \chi \) in the wormhole model for the boundary conditions given in (23).

**B. Ricci flow started from the wormhole**

We consider 3D part of the 4D wormhole metric \( (17) \)

\[
dt^2 = e^{2u(r,\lambda)} dr^2 + e^{2v(r,\lambda)} (d\theta^2 + \sin^2 \theta d\phi^2).
\]  

(26)

According to above presented wormhole solution

\[
u(r,0) = 0; \quad e^{2v(r,0)} = A(r).
\]  

(27)

The corresponding Ricci flow is

\[
\begin{align*}
\frac{\partial u}{\partial \lambda} &= 2e^{-2u} \left(v'' - u'v' + v'^2\right), \\
\frac{\partial v}{\partial \lambda} &= e^{-2u} \left(v'' - u'v' + 2v'^2\right) - e^{-2v}.
\end{align*}
\]  

(28) (29)
1. **Ricci soliton**

A Ricci soliton is defined as

\[
\frac{\partial u}{\partial \lambda} = \frac{\partial v}{\partial \lambda} = 0. \tag{30}
\]

The solution of Eq’s (30) is

\[
(e^v)' = \pm e^u. \tag{31}
\]

In this case one can introduce new coordinate \( x \) in following way

\[
\pm e^u dr = (e^v)' dr = dx. \tag{32}
\]

After this the metric (26) is the metric of 3D Euclidean space

\[
dl^2 = dx^2 + x^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{33}
\]

2. **Numerical solution**

In this section we would like to present the numerical solution of Eq’s (28) (29). The boundary conditions are

\[
u(r, 0) = 0; \quad \frac{\partial u(0, \lambda)}{\partial r} = 0; \quad u(\infty, \lambda) = 0;
\]

\[
v(r, 0) = \frac{1}{2} \ln A(r); \quad \frac{\partial v(0, \lambda)}{\partial r} = 0; \quad v(\infty, \lambda) = \ln r. \tag{35}
\]

The numerical solution for the Ricci flow is presented in Fig’s 5 and 6. One can say that this result is in agreement with the investigation presented in Ref. [4]: our initial data parameters leads to pinching off of a wormhole mouth. We would like to note that in Ref. [14] the topology change is considered as well but not using the Ricci flow.

### IV. DISCUSSION AND CONCLUSIONS

In this paper we have offered the idea about physical interpretation of the Ricci flows. The Ricci flow has the statistical interpretation as a quantum wormhole in a spacetime foam. For every \( \lambda \) the metric \( g(\lambda) \) is a microscopical...
state realized with some probability density $\rho(\lambda)$ connected with a Perelman’s functional $\hat{W}(\lambda)$ of a renormalized Ricci flow. This interpretation is based on the fact that the functional $\hat{W}(\lambda)$ is non-decreasing one. Such property allows us to suppose that $\hat{W}(\lambda)$ is the probability for the metric $g(\lambda)$ to be in the region $g(\lambda) \in [g(0), g(\lambda)]$. Accordingly $\frac{d\hat{W}(\lambda)}{d\lambda}$ is proportional to corresponding probability density.

We would like to list the problems for the future investigation in this direction:

1. The Ricci flow considered in section III is not covariant from 4D point of view. Consequently it is necessary to investigate the question about 4D covariance of the Ricci flow.

2. It is necessary to calculate a Perelman’s functional for a rescaled Ricci flow.

3. The Ricci flows considered in Ref. [12] are defined on compact manifolds but the quantum wormholes are non-compact manifolds. Consequently it is necessary to have in non-compact case the theorem similar to compactness theorem A.3 of Ricci flows.

4. In the statistical mechanics the probability density of a microscopical state is connected with Hamiltonian of some physical system. The question is: there exists some physical system whose statistical properies leads to the Perelman’s functional?

For the item (1) we may note that probably the rescaled Ricci flow does not depend on 3+1 decomposition which we have used in section III. It is possible that in this case we will have 4D invariance.

In conclusion we would like to emphasize that we consider only the problem connected with the topology change in quantum gravity. In our opinion this problem is not connected with the problem of non-perturbative field quantization of a gravitational field (metric).

V. ACKNOWLEDGMENTS

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APPENDIX A: RESCALING OF A RICCI FLOW

In this section we follow to Ref. [12].

Definition A.1 A sequence $(\mathcal{M}_i, (g_{ab})_i, p_i)$ of smooth, complete, pointed Riemannian manifolds (that is, Riemannian manifolds $(\mathcal{M}_i, (g_{ab})_i)$ and points $p_i \in \mathcal{M}$) is said to converge (smoothly) to the smooth, complete, pointed manifold $(\mathcal{M}, g_{ab}, p)$ as $i \to \infty$ if there exist
1. a sequence of compact sets \( \Omega_i \in \mathcal{M} \), exhausting \( \mathcal{M} \) (that is, so that any compact set \( K \subset \mathcal{M} \) satisfies \( K \subset \Omega_i \) for sufficiently large \( i \)) with \( p \in \text{int} (\Omega_i) \) for each \( i \);

2. a sequence of smooth maps \( \phi_i : \Omega_i \to \mathcal{M}_i \) which are diffeomorphic onto their image and satisfy \( \phi_i(p) = p_i \) for all \( i \);

such that,

\[
\phi_i^* (g_{ab})_i \to g_{ab} \tag{A1}
\]

smoothly as \( i \to \infty \) in the sense that for all compact sets \( K \subset \mathcal{M} \), the tensor \( \phi_i^* (g_{ab})_i - g_{ab} \) and its covariant derivatives of all orders (with respect to any fixed background connection) each converge uniformly to zero on \( K \).

Two consequences of the convergence \( (\mathcal{M}_i, (g_{ab})_i , p_i) \to (\mathcal{M}, g_{ab}, p) \) are that

1. for all \( s > 0 \) and \( k \in \{0\} \cup \mathbb{N} \),

\[
\sup_{i \in \mathbb{N}} \sup_{B_{p_i}(p_i,s)} |\nabla^a R^b_{cde}(g_i)| < \infty; \tag{A2}
\]

2. \[ \inf_i \text{inj} (\mathcal{M}_i, (g_{ab})_i , p_i) > 0, \tag{A3} \]

where \( \text{inj} (\mathcal{M}_i, (g_{ab})_i , p_i) \) denotes the injectivity radius of \( (\mathcal{M}_i, (g_{ab})_i) \) at \( p_i \).

**Theorem A.1** (Compactness - manifolds). Suppose that \( (\mathcal{M}_i, (g_{ab})_i , p_i) \) is a sequence of complete, smooth, pointed Riemannian manifolds (all of dimension \( n \)) satisfying (A2) and (A3). Then exists a complete, smooth, pointed Riemannian manifold \( (\mathcal{M}, g_{ab}, p) \) (of dimension \( n \)) such that after passing to some subsequence in \( i \),

\[
(\mathcal{M}_i, (g_{ab})_i , p_i) \to (\mathcal{M}, g_{ab}, p) \tag{A4}
\]

One can derive, from the compactness theorem for manifolds (theorem 1) a compactness theorem for Ricci flows.

**Theorem A.2** Let \( (\mathcal{M}_i, (g_{ab})_i (t)) \) be a sequence of smooth families of complete Riemannian manifolds for \( t \in (a, b) \) where \( -\infty \leq a < 0 < b \leq \infty \). Let \( p_i \in \mathcal{M}_i \) for each \( i \). Let \( (\mathcal{M}, g_{ab}(t)) \) be a smooth family of complete Riemannian manifolds for \( t \in (a, b) \) and let \( p \in \mathcal{M} \). We say that

\[
(\mathcal{M}_i, (g_{ab})_i (t), p_i) \to (\mathcal{M}, g_{ab}(t), p) \tag{A5}
\]

as \( i \to \infty \) if there exist

1. a sequence of compact \( \Omega_i \subset \mathcal{M} \) exhausting \( \mathcal{M} \) and satisfying \( p \in \text{int} (\Omega_i) \) for each \( i \);

2. a sequence of smooth maps \( \phi_i : \Omega_i \to \mathcal{M}_i \), diffeomorphic onto their image, and with \( \phi(p_i) = p_i \);

such that

\[
\phi_i^* (g_{ab})_i (t) \to g_{ab}(t) \tag{A6}
\]

as \( i \to \infty \) in the sense that \( \phi_i^* (g_{ab})_i (t) - g_{ab}(t) \) and its derivatives of every order (with respect to time as well as covariant space derivatives with respect to any fixed background connection) converge uniformly to zero on every compact subset of \( \mathcal{M} \times (a, b) \).

One can prove the following result

**Theorem A.3** (Compactness of Ricci flows.) Let \( \mathcal{M}_i \) be a sequence of manifolds of dimension \( n \), and let \( p_i \in \mathcal{M}_i \) for each \( i \). Suppose that \( (g_{ab})_i \) is a sequence of complete Ricci flows on \( \mathcal{M}_i \) for \( t \in (a, b) \), where \( -\infty \leq a < 0 < b \leq \infty \). Suppose that

1. \[
\sup_i \sup_{x \in \mathcal{M}_i, t \in (a, b)} |R^a_{bcd} (g_i(t))| (x) < \infty; \tag{A7}
\]
Define rescaled (and translated) flows \( (g_{ab}(i)) \) such that, after passing to a subsequence in \( i \),

\[
(\mathcal{M}_i, (g_{ab}(i)) (0), p_i) \rightarrow (\mathcal{M}, g_{ab}(t), p)
\]

as \( i \rightarrow \infty \). Then there exist a manifold \( \mathcal{M} \) of dimension \( n \), a complete Ricci flow \( g_{ab}(\lambda) \) on \( \mathcal{M} \) for \( t \in (a, b) \), and a point \( p \in \mathcal{M} \) such that, after passing to a subsequence in \( i \),

\[
(\mathcal{M}_i, (g_{ab}(i)) (t), p_i) \rightarrow (\mathcal{M}, g_{ab}(t), p)
\]

as \( i \rightarrow \infty \).

The application of the compactness of Ricci flows in Theorem A.3 is to analyze rescaling of Ricci flows near their singularities. Let \( (\mathcal{M}, g_{ab}(t)) \) be a Ricci flow with \( \mathcal{M} \) closed, on the maximal interval \( [0, \lambda_0) \). In the consequence of a singularity

\[
\sup_{\mathcal{M}} |R_{bcd}^a| (\cdot, \lambda) \rightarrow \infty
\]

as \( \lambda \rightarrow \lambda_0 \). Let us choose points \( p_i \in \mathcal{M} \) and \( \lambda_i \rightarrow \lambda_0 \) such that

\[
|R_{bcd}^a| (p_i, \lambda_i) = \sup_{x \in \mathcal{M}, \lambda \in [0, \lambda_i]} |R_{bcd}^a| (x, \lambda).
\]

Define rescaled (and translated) flows \( (g_{ab}(i)) \) by

\[
(g_{ab}(i))(\lambda) = |R_{bcd}^a| (p_i, \lambda_i) g_{ab} \left[ \lambda_i + \frac{\lambda}{|R_{bcd}^a| (p_i, \lambda_i)} \right]
\]

One can show that \( (\mathcal{M}, (g_{ab}(i)) \) is a Ricci flow on the interval \( [\lambda, |R_{bcd}^a| (p_i, \lambda_i), (\lambda_0 - \lambda_i) |R_{bcd}^a| (p_i, \lambda_i)] \).

One can show that for all \( a < 0 \) and some \( b > 0 \), \( (g_{ab}(i)) \) is defined for \( \lambda \in (a, b) \) and

\[
\sup_{i} \sup_{\mathcal{M} \times (a, b)} |R_{bcd}^a ((g_{ab}(i))] < \infty.
\]

By Theorem A.3 one can pass to a subsequence in \( i \), and get convergence \( (\mathcal{M}, (g_{ab}(i))(\lambda), p_i) \rightarrow (\mathcal{N}, \hat{g}_{ab}(\lambda), p_\infty) \) to a “singularity model” Ricci flow \( (\mathcal{N}, \hat{g}_{ab}(\lambda)) \), provided that we can establish the injectivity radius estimate

\[
\inf_{i} \inf_{\mathcal{M}} (g_{ab}(i)(0), p_i) > 0.
\]

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