Gravitomagnetism in Teleparallel Gravity

E. P. Spaniol and V. C. de Andrade

Instituto de Física, Universidade de Brasília
C. P. 04385, 70.919-970 Brasília DF, Brazil

(Dated: July 3, 2009)

Abstract

The assumption that matter charges and currents could generate fields, which are called, by analogy with electromagnetism, gravitoelectric and gravitomagnetic fields, dates from the origins of General Relativity (GR). On the other hand, the Teleparallel Equivalent of GR (TEGR), as a gauge theory, seems to be the ideal scenario to define these fields, based on the gauge field strength components. The purpose of the present work is to investigate the nature of the gravitational electric and magnetic fields in the context of the TEGR, where the tetrad formalism behind it seems to be more appropriated to deal with phenomena related to observers. As our main results, we have obtained, for the first time, the exact expressions for the gravito-electromagnetic fields for the Schwarzschild solution that in the linear approximation become the usual expected ones. To improve our understanding about these fields, we have also studied the geometry produced by a spherical rotating shell in slow motion and weak field regime. Again, the expressions obtained are in complete agreement with those of electromagnetism.

PACS numbers: 04.20.Cv, 04.50.Kd
I. INTRODUCTION

Gravitomagnetism has a history that is at least as long as that of GR itself. In fact, it comes from the formal analogy between Newton’s law of gravitation and Coulomb’s law of electromagnetism. Both theories are governed by the same geometric law in a static scheme and the interactions they describe propagate at finite speed. Hence, the concept of a gravitoelectric description of the Newtonian law is quite direct, and from it emerges the intuitive idea that moving masses might generate the gravitational analogue of magnetic fields.

The Maxwell-type gravitational theory was firstly explored by Maxwell himself [1] and subsequently by some authors in the second half of the nineteenth century [2, 3, 4]. Einstein also worked on this parallel concomitantly to the birth of GR [5] and soon after the publication of GR and its prediction of a gravitomagnetic field some astrophysical applications start to be investigated [6, 7].

After that, several authors have been studying the Einstein’s equations when considering a perturbation on flat spacetime resulting in what they call linearized GR. In this context, the remarkable similarity between gravitational field equations and the Maxwell’s becomes evident and we see the theoretical origin of a gravitomagnetic field associated to masses currents [8]. This subject is now well established and discussed in several books of area [9, 10, 11].

The gravitomagnetic ‘dragging of inertial frames’ by rotating matter has played an important part in discussions about the meaning and usefulness of Mach’s principle, in astrophysical models of jets near accreting, rotating black holes and in proposals for testing alternative theories of gravity. All these aspects justify the strong experimental efforts during the past 30 years to measure gravitomagnetism. These attempts have been, however, hampered by a intrinsic difficulty related to the fact that the gravitomagnetic contribution is much smaller than the gravitoelectric one. In fact, the Lense-Thirring precession of planetary orbits is too weak to be measurable at present. On the other hand, since the early 1960’s, the measurement of the precession of a gyroscope has been the goal of the Gravity Probe B experiment [12] and the evidence of the gravitomagnetic field of the Earth (measuring nodal precession) has been offered by Ciufolini et. al [13] by studying the motion of the laser-ranged satellites.
LAGEOS, LAGEOS II and LAGEOS III. Measurements of the gravitomagnetic field around superconductors seem to show its first signs [14].

Within the framework of GR, gravitomagnetism usually allows two different theoretical approaches, which were summarized by Mashhoon in [15]. The first one is the context of linearized GR, mentioned above, where it is obtained essentially the analogous equations to Maxwell’s ones in the linear approximation, that is, by performing an expansion in the metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is the perturbation term. Then, the gravitational potentials $\Phi$ and $\vec{A}$ are identified as $\Phi = \frac{c^2 h_{00}}{4}$ and $A_i = -\frac{c^2 h_{0i}}{2}$, with $h_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h$ and $h = \eta^{\mu\nu} h_{\mu\nu}$, and they in turn define the physical fields. In the second approach, Mashhoon uses Fermi coordinates to write the gravitoelectric and gravitomagnetic fields as components of the curvature tensor. These coordinates, according to Synge, are the correct relativistic generalization of the Newtonian concept of reference frame [16]. Finally, we can mention other formulation of GR known as Quasi-Maxellian, that is based on a dynamics to Weyl tensor [17, 18, 19]. Despite being a fully covariant formalism, it fails precisely because it compares completely different objects [20].

In the present work, a different approach will be adopted to reexamine gravitomagnetism. Due to the fundamental character of the geometric structure underlying gauge theories, the concept of charges and currents and, in particular, the concept of energy and momentum are much more transparent when considered from a gauge point of view [21]. Accordingly, we shall consider gravity to be described as a gauge theory for the translation group [22], which gives rise to the so-called teleparallel equivalent of GR. In this scenario we recover all the aspects predicted by GR and moreover we have all the formal structure of a gauge theory, which is naturally close to electromagnetism due to its abelian character. Therefore, the concepts of gravitoelectric and gravitomagnetic fields emerge, as we will see, in the same way as in the electromagnetic theory, that is, as components of the field strength of the gauge theory. Indeed, the concept of the tetrad fields, that emerges from the gauge theory, and its relation with observers in spacetime [23], results in the ideal scenario to describe gravitoelectric and gravitomagnetic observers-dependent fields.

Finally, we can say that one of our main results, obtained for the first time, are the exact expressions for the gravito-electromagnetic fields in a particular static distribution of matter (Schwarzschild solution). This is a new approach, since the calculations found in literature usually assume some approximation hypothesis.
The paper is divided as follows: in section 2 the gravitational Maxwell equations are introduced in their exact form. The next two sections are devoted to applications of our definitions. Therefore, in section 3 we study the exact and approximated Schwarzschild solution and after that, in section 4, we apply our definition to another case, that is, a spinning massive spherical shell in the linearized approximation. Finally, in section 5 we draw the main conclusions of the paper.

Notation: According to its gauge structure, to each point of spacetime there is attached a Minkowski tangent spacetime (the fiber of the correspondent tangent bundle), on which the translation (gauge) group acts, and whose metric is assumed to be $\eta_{ab} = (+1, -1, -1, -1)$. The spacetime indices will be denoted by the Greek alphabet ($\mu, \nu, \sigma, ... = 0, 1, 2, 3$) and the tangent space indices will be denoted by the first half of the Latin alphabet ($a, b, c.. = 0, 1, 2, 3$). The second half of the Latin alphabet will be used to represent space tensor components, that is, $(i, j, k...) assume the values 1, 2 and 3. Indices in parentheses will also be related to tangent space. We adopt the light velocity as $c = 1$.

A. Teleparallel Gravity: a few concepts

Let us present some of the more important expressions in teleparallel gravity that will be used in the whole paper (for detailed of the teleparallel fundamentals, see for example [22, 24].

The field strength of the theory is defined in the usual form

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu = h^{a\rho} T^{\rho}_{\mu\nu},$$

with $A^a_\mu$ the translational gauge potential and $h^{a\mu} = \partial_\mu x^a + A^a_\mu$ the tetrad field. $T^{\rho}_{\mu\nu}$ is the torsion that represents alone the gravitational field (in opposition to the curvature, that vanishes in the Weitzeböck space) and it also can be identified as the field strength written in the tetrad base.

The dynamics of the gauge fields will be determined by the lagrangian

$$\mathcal{L}_G = \frac{h}{16\pi G} S^{\rho\mu\nu} T^\rho_{\mu\nu},$$

with $h = \det(h^{a\mu})$ and

$$S^{\rho\mu\nu} = -S^{\rho\nu\mu} \equiv \frac{1}{2} [K^{\mu\nu\rho} - g^{\rho\nu} T^{\theta\mu}_{\theta} + g^{\rho\mu} T^{\theta\nu}_{\theta}]$$
which is called superpotential, that will play an important role in theory, as we will see.

The field equations resulting from this lagrangian are

$$\partial_{\sigma}(hS_{a}^{\sigma\rho}) - 4\pi G(hj_{a}^{\rho}) = 0$$  \hspace{1cm} (4)

with

$$j_{a}^{\rho} \equiv \frac{\partial L}{\partial h_{a}^{\rho}} = \frac{h_{a}^{\lambda}}{4\pi G}(F_{c}^{\mu\lambda}S_{c}^{\mu\rho} - \frac{1}{4}\delta^{\lambda}_{\rho}F_{c}^{\mu\nu}S_{c}^{\mu\nu}),$$  \hspace{1cm} (5)

$j_{a}^{\rho}$ stands for the gauge energy-momentum current of the gravitational field.

II. GRAVITATIONAL MAXWELL EQUATIONS

The TEGR, as mentioned in the last section, is an approach to gravitation formulated as an abelian gauge theory, in the same sense as electromagnetism, but associated with a different gauge group, the translational one (in contrast with the U(1) group of electromagnetic theory).

Our idea is to introduce a new version of gravitoelectric and gravitomagnetic fields, by straight analogy with the electric and magnetic fields of electromagnetism. These fields will be proposed based on the gauge theory used, which means that, they emerge in a completely different way that the gravitoelectric/magnetic versions introduced in the GR scenario [15] and therefore the direct comparison can not be performed in a simple manner.

After defining the fields we proceed in the opposite way taken in electromagnetism and, from the covariant teleparallel form of the field equations, we write them in a non covariant form similar to Maxwell’s one that emphasize the phenomenology of what we call the physical fields $\vec{E}_{a}$ and $\vec{B}_{a}$.

The immediate attempt to construct the gravitoelectric and gravitomagnetic fields from this theory would consist in following the usual gauge approach and constructing the fields as components of the field strength of the theory, which means defining $F_{a}^{0i} = E_{a}^{i}$ and $F_{a}^{ij} = \epsilon^{ijk}B_{ak}$. However, we investigate a more carefull assumption, that takes into account the peculiar character of typical gauge theories for gravitation: the possibility of contracting internal (algebra) indices with external (spacetime) ones. Technically, this is ascribed to the presence of a solder form, whose components constitute the tetrad field. This property gives rise to deep changes, contrasting with the usual internal (that is, non-soldered) gauge
theories. In teleparallelism, this effect was well observed, for instance, in the construction of the lagrangian, which acquires some additional terms in comparison with the usual gauge lagrangians (see [21, 22]). The direct consequence of this property, which finds echo in the field equations, is that the quantity playing the role of the field strength, analogous to what happens in Yang–Mills equations, will be, instead, the generalization of the usual field strength, the superpotential $S^a_{\mu\nu}$, given by (3). It is natural, therefore, to consider the gravitoelectric and gravitomagnetic fields as components of this generalized field strength, that is,

$$S^a_{\sigma\rho} = E_a^\sigma,$$  \hspace{1cm} (6)

$$S^a_{ij} = \epsilon^{ijk}B_{ak}.$$

(7)

Although these are the most appropriate definitions following the gauge theories arguments, other approaches on teleparallel gravity are present in literature [25, 26, 27].

A. The first pair of field equations

Let us consider the teleparallel version of the field equations for gravitation outside the sources (in vacuum). They are given by (4):

$$\partial_\sigma (hS_a^{\sigma\rho}) - 4\pi G (hj_a^{\rho}) = 0.$$  

As these are the dynamical equations of the theory, we hope to obtain the analogue of the first pair of Maxwell’s equations of electromagnetism. Hence, for $\rho = 0$, we expect to find the equivalent to Gauss’s law and for $\rho = q$, we expect to obtain the analogue of Ampère’s Law (with Maxwell’s correction).

The field equation for $\rho = 0$, when the model is applied, assumes the general form:

$$\partial_i (hE_a^i) = 4\pi G (hj_a^0).$$  \hspace{1cm} (8)

We thus see that the choice of the superpotential $S_a^{\sigma\rho}$ as the generalized field strength of the theory leads to the analogue of the divergent of $\vec{E}$ in the equation, up to a multiplicative factor given by the determinant of the tetrad field. In writing it in this form, we can directly
interpret $hj_a^\alpha$ as the source of the gravitoelectric field, in accordance with Gauss’s Law. This leads to the idea that 'gravitation generates gravitation'.

Alternatively, we can replace our definitions $[6]$ and $[7]$ in the gauge energy-momentum current $[5]$, explicitly showing all the field contributions to $[8]$. This form makes evident the non-linearity of gravity, that are manifested in the quadratic terms in the gravitoelectric and gravitomagnetic components, as well as in the appearing crossing terms. This complete form are shown in appendix A.

Let us consider now the spatial component of the field equation $\rho = q$. We can write this equation in a compact form:

$$\epsilon^{ijk}\partial_j(hB_{ak}) - \partial_0(hE_a^\rho) = 4\pi G(hj_a^\rho).$$

In the same manner, $hj_a^\rho$ can be interpreted as the source of the equation, in a very similar way to Ampère’s Law (with corrections) of electromagnetism. The equation can also be written in a expanded form, exhibited appendix A.

Hence, the first pair of field equations, in its exact form, is evidently more complex than the corresponding electromagnetic pair. This was in fact expected, due to the non-linearity of gravitation. For some more interesting discussions about the these field equations in the context of gravitational waves, see $[28]$.

**B. The second pair**

Analogously to electromagnetism, the second pair of gravitational Maxwell equations is expected to emerge from the geometric context of the gauge theory. Hence, we shall consider the first Bianchi identity of teleparallel gravity, given by $[29]$:

$$\partial_\rho F^a_{\mu \nu} + \partial_\mu F^a_{\nu \rho} + \partial_\nu F^a_{\rho \mu} = 0.$$  

These equations ought to result in an equivalent of the second pair of electromagnetic Maxwell equations.

We shall start from the relation between the field strength and the superpotential, written in the form

$$F^a_{\gamma \delta} = 2h^b_{\gamma \gamma}g_{\rho \delta}h^a_{\mu \nu}S^b_{\mu \nu} - 2h^b_{\delta \gamma}g_{\rho \gamma}h^a_{\mu \nu}S^b_{\mu \nu} - h^a_{\delta \gamma}g_{\rho \gamma}h^b_{\theta \delta}S^b_{\theta \nu} + h^a_{\gamma \delta}g_{\rho \delta}h^b_{\theta \theta}S^b_{\theta \rho}$$

(11)
and introduce it directly into the teleparallel Bianchi identities \(\text{(10)}\). Substituting the model definition, we get the following equation:

\[
\partial \sigma \left[ \mathcal{O}^{ba \gamma i \delta} E_b^i + \mathcal{P}^{ba \gamma i j \delta} \epsilon^{ijk} B_{bk} \right] \\
+ \partial \gamma \left[ Q^{ba \delta i \sigma} E_b^i + \mathcal{R}^{ba \delta i j \sigma} \epsilon^{ijk} B_{bk} \right] \\
+ \partial \delta \left[ S^{ba \sigma i \gamma} E_b^i + \mathcal{T}^{ba \sigma i j \gamma} \epsilon^{ijk} B_{bk} \right] = 0. \tag{12}
\]

The first coefficients assume the following explicit form

\[
\mathcal{O}^{ba \gamma i \delta} = 2h^b_{\gamma} h^a_{0} g_{i \delta} - 2h^b_{\gamma} h^a_{i} g_{\delta 0} - 2h^b_{\delta} h^a_{0} g_{i \gamma} + 2h^b_{\delta} h^a_{i} g_{0 \gamma} + h^a_{\gamma} h^b_{\delta} g_{i \gamma} - h^a_{\delta} h^b_{\gamma} g_{i 0} + h^a_{\gamma} h^b_{i} g_{0 \delta}, \tag{13}
\]

\[
\mathcal{P}^{ba \gamma i j \delta} = 2h^b_{\gamma} h^a_{i} g_{j \delta} - 2h^b_{\delta} h^a_{i} g_{j \gamma} + h^a_{\delta} h^b_{i} g_{j \gamma} - h^a_{\gamma} h^b_{i} g_{j \delta}. \tag{14}
\]

From \(\mathcal{O}^{ba \gamma i \delta}\) and \(\mathcal{P}^{ba \gamma i j \delta}\) we obtain \(Q^{ba \delta i \sigma}\) and \(\mathcal{R}^{ba \delta i j \sigma}\), by switching the indices \((\gamma \rightarrow \delta\) and \(\delta \rightarrow \sigma)\), and \(S^{ba \sigma i \gamma}\) and \(\mathcal{T}^{ba \sigma i j \gamma}\), by making \((\gamma \rightarrow \sigma\) and \(\delta \rightarrow \gamma)\).

We could, instead, try to make evident the analogy between these equations and Maxwell’s second pair of equations, performing the substitutions \((\sigma \gamma \delta) \rightarrow (012), (013), (023)\), in order to obtain the analogue to Faraday’s Law, or performing the application \((\sigma \gamma \delta) \rightarrow (123)\) to get the gravitomagnetic divergence law. But, the equations, in these exact forms, diverge from the electromagnetic ones drastically.

Equations (A1), (A2) and (12) acquire an aspect totally similar to the Maxwell equations in the weak field limit, that is, when we compare gravitation in a linear form and electromagnetism, a linear theory in essence. It is explicitly shown in the next section.

Finally, it is interesting to note that the exact fields equations in the usual GR gravitomagnetism, for an arbitrary curved spacetime, as can be seen in Mashhoon’s review [15] also exhibit an intricate form, but in that context the gravitoelectric and gravitomagnetic fields are related to the \(R_{0i0j}\) and \(R_{0ijk}\) Riemann tensor components. Only in the limit case (in lowest order in \(|X|/R\), with \(X\) the spacial components of the fermi coordinates and \(R\) the radius of curvature of spacetime), a kind of Maxwell’s equations are recovered. We must emphasize that the conceptual definitions of what would be the gravitoelectric and gravitomagnetic fields in these two approaches are completely different, being the teleparallel one much more consistent with electromagnetic gauge theory definitions.
III. SCHWARZSCHILD SOLUTION

We have made, until now, a theoretical analogy between gravitation and electromagnetism, which was based on the identification of the superpotential components with the gravitoelectric and gravitomagnetic fields. This parallel was drawn in the gauge theories domain. The next step is to investigate whether these fields are associated with physical phenomena similar to those observed and expected in the context of electromagnetism. In short, static electric charges generate electric fields and moving charges are associated with magnetic fields. Similarly, it is desirable to establish an association between the fields suggested in the model and static and moving matter.

A. Geometry

Let us consider the geometry produced by the gravitational field of a spherical symmetric and static distribution of matter, say, a body represented by a mass $m$ and situated at the origin of a coordinate system. This is the Schwarzschild solution of Einstein’s equations, the most popular geometry in gravitation, since it allows us to treat bodies like the sun and other celestial ones with excellent approximation. It is given by the line element:

$$ds^2 = \left(1 - \frac{2mG}{r}\right)dt^2 - \left(1 - \frac{2mG}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$  \hspace{1cm} (15)

The choice of the observer, that corresponds to a specific tetrad field associated with the Schwarzschild metric, is crucial to our conclusions about the fields $\vec{E}^a$ and $\vec{B}^a$ that emerge in theory, as in electromagnetic case. Hence, we adopt a set of tetrads that is adapted to a stationary observer localized at infinity. As such, the tetrad needs to satisfy the time gauge condition and exhibit symmetry in the spatial sector [23].

This set of tetrad fields can be represented by [30].

$$h^a_{\nu} = \begin{pmatrix} \gamma_{00} & 0 & 0 & 0 \\ 0 & \gamma_{11}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \gamma_{11}\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & 0 & \gamma_{11}\cos\theta & -r\sin\theta \end{pmatrix}. \hspace{1cm} (16)$$
In this notation, $\gamma_{00} = \sqrt{g_{00}}$ and $\gamma_{11} = \sqrt{-g_{11}}$. From the usual expression for torsion written in terms of the tetrad

$$T^\sigma_{\mu\nu} = h^a_\sigma \partial_\mu h^a_\nu - h^a_\sigma \partial_\nu h^a_\mu,$$  \hfill (17)

we calculate the components of $T^\sigma_{\mu\nu}$, the non-zero of which are given by:

\begin{align*}
T^{0}_{01} &= -\frac{GM}{r^2 g_{00}^{-1}}, \hfill (18) \\
T^{2}_{12} &= \frac{1}{r}(1 - \gamma_{11}), \hfill (19) \\
T^{3}_{13} &= \frac{1}{r}(1 - \gamma_{11}). \hfill (20)
\end{align*}

Considering the definition (6), we obtain a direct relation between the electric field and torsion:

$$E^i_b = \frac{1}{2} h^0_b T^{ji}. \hfill (21)$$

For $b \neq 0$ it is trivial to verify that the components vanish, that is

$$E^{(i)}_b = 0 \hfill (22)$$

and for $b = 0$ we get

$$E^{(0)}_b = \frac{1}{2} h^{(0)}_b (T^{11} + T^{22} + T^{33}). \hfill (23)$$

Substituting explicitly the tetrad and torsion components, we obtain

\begin{align*}
E^{(0)}_r &= \frac{1}{r}(\gamma_{00} - 1), \hfill (24) \\
E^{(0)}_\theta &= 0, \hfill (25) \\
E^{(0)}_\phi &= 0. \hfill (26)
\end{align*}

This means that only the radial component of the vector $b = 0$ is different from zero, something which is consistent with the spherically symmetric distribution of mass. Considering (7) we get

\begin{align*}
S^{12}_b &= -\frac{1}{2} h^2_b g^{11}[T^0_{10} + T^3_{13}], \hfill (27) \\
S^{13}_b &= -\frac{1}{2} h^3_b g^{11}[T^0_{10} + T^2_{12}], \hfill (28) \\
S^{23}_b &= 0. \hfill (29)
\end{align*}
And finally, we obtain the gravitomagnetic components

\[ B_{(0)\phi} = 0, \]
\[ B_{(1)\phi} = \frac{\cos \theta \cos \phi}{2r^2} (1 - \gamma_{11}^{-1} - \frac{GM}{r}), \]
\[ B_{(2)\phi} = \frac{\cos \theta \sin \phi}{2r^2} (1 - \gamma_{11}^{-1} - \frac{GM}{r}), \]
\[ B_{(3)\phi} = -\frac{\sin \theta}{2r^2} (1 - \gamma_{11}^{-1} - \frac{GM}{r}), \]
\[ B_{(0)\theta} = 0, \]
\[ B_{(1)\theta} = \frac{\sin \phi}{2r^2 \sin \theta} (1 - \gamma_{11}^{-1} - \frac{GM}{r}), \]
\[ B_{(2)\theta} = -\frac{\cos \phi}{2r^2 \sin \theta} (1 - \gamma_{11}^{-1} - \frac{GM}{r}), \]
\[ B_{(3)\theta} = 0. \]

As can be easily seen, the scalar resulting from the contractions of all internal and external indices that can be associated to a 'kind' of modulus of a gravitomagnetic vector does not present angular dependence, which is in agreement to the spherical symmetry of the Schwarzschild solution, that is, \( B^2 \equiv g_{ij} B^i a^j = \frac{1}{\sqrt{g}} (1 - \gamma_{11}^{-1} - \frac{GM}{r})^2 \).

The appearance of non null components of gravitomagnetic components, as shown above, can be associated with the gravitational energy-momentum current \( j_a^\rho \), that represents the non linear effects of gravitation. In this way, the emergence of gravitomagnetism even in static configurations is theoretically consistent and this fact introduces new possibilities of observation in strong field regimes. Finally, we can say that a closer analogy between gravitation and electromagnetism, in the phenomenological point of view, must be investigated when both interactions are treated as linear, that is, when we perform the weak field hypothesis in the gravitational interaction.

**B. A test of consistency: approximate Schwarzschild solution**

Considering now this approximation hypothesis, the spacetime metric will be decomposed into a trivial flat part, \( \eta_{\mu\nu} \), plus a perturbation generated by the presence of matter, \( a_{\mu\nu} \), that is

\[ g_{\mu\nu} = \eta_{\mu\nu} + a_{\mu\nu}. \]
In this specific case, it is convenient, as we will see, to work in Cartesian coordinates, which will simplify substantially our calculations. The correspondent tetrad will be also divided into a trivial (diagonal) part, $H^a_\mu$, plus a contribution due to the presence of matter, $U^a_\mu$, that is

$$h^a_\mu = H^a_\mu + U^a_\mu. \quad (31)$$

The weak field limit means, essentially, that we must discard the terms of second order $(m/r)^2 = O(\epsilon^2)$, that is,

$$\left(\frac{m}{r}\right)^2 << 1 \quad (32)$$

and then we can work with the Taylor expanded version of the studied expressions.

Hence, we get the tetrad fields (16) in cartesian coordinates in first order:

$$h^a_\mu = \begin{pmatrix}
1 - \frac{mG}{\xi} & 0 & 0 & 0 \\
0 & 1 + \frac{mGx^2}{\xi} & \frac{mGxy}{\xi} & \frac{mGxz}{\xi} \\
0 & \frac{mGxy}{\xi} & 1 + \frac{mGy^2}{\xi} & \frac{mGyz}{\xi} \\
0 & \frac{mGxz}{\xi} & \frac{mGyz}{\xi} & 1 + \frac{mGz^2}{\xi}
\end{pmatrix} \quad (33)$$

with $\xi = (x^2 + y^2 + z^2)^{\frac{3}{2}}$. Notice that the tetrad above reduces to the addition of a diagonal part (that corresponds to a Minkowski space only in this coordinate system) and a non trivial part that is due to the presence of matter.

We can therefore calculate the torsion written in terms of the tetrad through (17) and we get the following non null components:

$$T^0_{\ 01} = T^2_{12} = T^3_{13} = -\frac{mGx}{(x^2 + y^2 + z^2)\frac{3}{2}}, \quad (34)$$

$$T^0_{\ 02} = T^1_{21} = T^3_{23} = -\frac{mGy}{(x^2 + y^2 + z^2)\frac{3}{2}}, \quad (35)$$

$$T^0_{\ 03} = T^2_{32} = T^1_{31} = -\frac{mGz}{(x^2 + y^2 + z^2)\frac{3}{2}}. \quad (36)$$

Let us consider again our gravito-eletromagnetic definitions applied to a region in space that satisfies the weak field approximation to the Schwarzschild geometry and also expression (3). For the gravitoelectric components defined in (6), we find the following non-null components:

$$E_{(0)\ x} = \frac{mGx_k}{(x^2 + y^2 + z^2)\frac{3}{2}}, \quad (37)$$
with \( x_k \) the usual spacial cartesian coordinates. This result seems to be really interesting: the components \( a = 0 \) of \( \vec{E}^a \) play a role quite analogous to the coulombian electric field of electromagnetism. The other components \( a = i \) do not contribute to the gravitoelectric field.

Now, looking for the gravitational magnetic components, we see that in this order of approximation, they are not present for a static distribution of matter, that is,

\[
B_{(0)k} = B_{(i)j} = 0
\]  

(38)
supporting the straight analogy between the theories under this condition. We can also observe that in this order of approximation there is no gravitational current contribution, as discussed below. This fact reinforces the argument given earlier that \( j_{a\rho} \) is responsible for the appearance of the gravitomagnetic components in the exact Schwarzschild solution. Furthermore, we can say that these expressions are in total agreement to that obtained in the linearized GR (for example, see [11]), but its origins are conceptually quite different.

C. Field equations in Schwarzschild geometry

We have concluded until now that our model generates components for \( \vec{E}^a \) and \( \vec{B}^a \) that are compatible with the electromagnetic phenomenology in the case of the most simple example of static distribution of matter, that is, the Schwarzschild geometry, considering weak gravitation. The next step is to examine what happens with the equations that describe these fields.

1. The first pair of field equations

The so called dynamical equations of the theory, which correspond to the gravitational analogue of the first pair of Maxwells equations, resulted in the exact form in the expressions (A1) and (A2), that are evidently more complex then the respective equations of electromagnetism. Nevertheless, considering the suggested approximation of weak field limit, we will see, as follows, that the laws become extremely simple, reinforcing the analogy.
Taking the weak field limit, it is trivial to show that all contributions that come from $j_a \rho$ in (A1) and (A2) are of $O(\epsilon^2)$ order, just remaining the derivative term

$$\partial_\sigma (h S_{a}^{\rho \sigma}) = 0$$  \hspace{1cm} \text{(39)}

with $h = 1$. For $\rho = 0$ we get

$$\partial_\sigma (S_{a}^{0 \sigma}) = 0$$  \hspace{1cm} \text{(40)}

which corresponds to

$$\vec{\nabla} \cdot \vec{E}_a = 0$$  \hspace{1cm} \text{(41)}

and for $\rho = i$, we find

$$\partial_0 (S_{a}^{0 i}) + \partial_i (S_{a}^{ij}) = 0,$$  \hspace{1cm} \text{(42)}

that assumes the simple form

$$\vec{\nabla} \times \vec{B}_a = \frac{\partial \vec{E}_a}{\partial t}.$$  \hspace{1cm} \text{(43)}

and which is, in Schwarzschild solution, identically satisfied.

2. The second pair of field equations

The teleparallel Bianchi identities, that give us the second (geometrical) pair of field equations, when calculated in the exact form, have also resulted in an intricate relation, with several coupling terms, unexpected in electromagnetism. But in the same way, the weak field approximation can disappear with these spurious contributions, resulting in the following equations. Let us consider, for example, $\sigma = 0$, $\gamma = 1$ and $\delta = 2$ in the equation \textbf{(12)}. For $a = 0$ we get

$$\partial_x E_{(0)}^y - \partial_y E_{(0)}^x = 0.$$  \hspace{1cm} \text{(44)}

Taking into account the other possibilities, that is, $(\sigma = 0, \gamma = 1, \delta = 3)$ and $(\sigma = 0, \gamma = 2, \delta = 3)$ we obtain an analogous of Faraday’s law with $a = 0$

$$\vec{\nabla} \times \vec{E}_{(0)} = -\frac{\partial \vec{B}_{(0)}}{\partial t}.$$  \hspace{1cm} \text{(45)}
For \( a = 1, 2, 3 \) it is trivially satisfied. On the other hand, equation (12) with \( \sigma = 1, \gamma = 2 \) and \( \delta = 3 \), would correspond to
\[
\vec{\nabla} \cdot \vec{B}_a = 0, 
\]
nevertheless, in this order \( \vec{B}_a = 0 \), the equation becomes identically null.

Finally, we would like to emphasize that these equations, also reproduced in the linearized GR context, were obtained here from the first principles, in a closer and legitimate analogy to electromagnetism.

## IV. THE SPINNING MASSIVE SPHERICAL SHELL

In the last section, we have presented a direct test for our model, in which it was considered a static geometry (Schwarzschild solution) both for exact and approximate solutions.

Our purpose is now to analyze the behavior of the model when applied to another configuration of spacetime, where gravitomagnetic components are expected even in the linearized geometry. Thus, we shall consider the spacetime geometry associated with a spherical mass shell in slow rotation and, in the same fashion performed above, the terms of order superior to \( \epsilon \) shall be discarded, keeping in mind that \( \epsilon = \frac{m}{r} \). The greatest interest in choosing this metric is that it resembles the region outside the Kerr spacetime, with the advantage of having no singularities. It displays therefore regular rotational effects and it can be easily treated mathematically.

### A. The geometry

The metric tensor that represents the spherical mass shell in rotation was firstly introduced by Cohen in [31], and reads, in spherical coordinates \((r, \theta, \phi)\):
\[
ds^2 = -V^2 dt^2 + \psi^4 \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta (d\phi - \Omega dt)^2 \right]. 
\]

It is important to remark that this is a solution of Einsteins equation to first order in \( \Omega \), the angular velocity. In the following terms, \( r_0 \) is the radius of the shell and \( \alpha = \frac{m}{2} \), with \( m \) its mass.
Inside the shell, that is, for the region given by \( r < r_0 \), we have:

\[
V = \frac{r_0 - \alpha}{r_0 + \alpha},
\]

\[
\psi = \psi_0 = 1 + \frac{\alpha}{r_0},
\]

\[
\Omega = \Omega_0.
\]

For \( r > r_0 \):

\[
V = \frac{r - \alpha}{r + \alpha},
\]

\[
\psi = 1 + \frac{\alpha}{r},
\]

\[
\Omega = \left( \frac{r_0 \psi^2}{r \psi^2} \right)^3 \Omega_0.
\]

The constant \( \Omega_0 \) stands for the dragging angular velocity of locally inertial observers inside the shell.

Again, the choice of observers is fundamental to our conclusions, and so we shall adopt a tetrad field adapted to static observers at spacelike infinity. Therefore, the following condition must be satisfied [32]:

\[
h_{(0)}^\mu(t, r, \theta, \phi) = \left( \frac{1}{A}, 0, 0, 0 \right).
\]

A tetrad field that meets this condition is, for example,

\[
h^\alpha_\mu = \begin{pmatrix}
A & 0 & 0 & -C \\
0 & \psi^2 \sin \theta \sin \phi & r \psi^2 \cos \theta \cos \phi & -B \sin \theta \sin \phi \\
0 & \psi^2 \sin \theta \sin \phi & r \psi^2 \cos \theta \cos \phi & B \sin \theta \cos \phi \\
0 & \psi^2 \cos \theta & -r \psi^2 \sin \theta & 0
\end{pmatrix}
\]

with

\[
A = \left( V^2 - r^2 \Omega^2 \psi^4 \sin^2 \theta \right)^{1/2},
\]

\[
C = -\frac{1}{A} \Omega r^2 \psi^4 \sin^2 \theta,
\]

\[
B = \frac{V}{A} r \psi^2.
\]
As the application of the weak field limit results in the simple decomposition of a diagonal tetrad (corresponding to the flat metric) plus a non-trivial term (due to the presence of mass and rotation) only in Cartesian coordinates, we write (55) in this coordinate system as

\[ h^a_\mu = H^a_\mu + U^a_\mu. \]  

(57)

In order to obtain the expression above, it is necessary to make some assumptions. The first one is the same used in the last section, that means, to discard the terms of order \((\frac{\alpha}{r})^2 = \epsilon^2\) We shall also take the limit

\[ \left( \frac{\alpha}{r_0} \right)^2 \ll 1, \]  

(58)

and therefore we can perform a Taylor expansion to obtain expressions that have these terms. Moreover, we shall consider the weak rotation regime, that is

\[ r^2\Omega^2 \ll 1. \]  

(59)

Considering (59), the definitions (56) become

\[ A \approx V, \]
\[ C \approx -\frac{1}{V} \Omega r^2 \psi^4 \sin^2 \theta, \]
\[ B \approx r \psi^2. \]  

(60)

After making these considerations, we can write the tetrad field in Cartesian coordinates and finally get

\[ h^a_\mu = \begin{pmatrix}
1 - \frac{2\alpha}{\chi} & -\frac{r_0^2(\log(6\alpha))}{\chi^4} & -\frac{r_0^2(\log(6\alpha))}{\chi^4} & 0 \\
0 & 1 + \frac{2\alpha}{\chi} & 0 & 0 \\
0 & 0 & 1 + \frac{2\alpha}{\chi} & 0 \\
0 & 0 & 0 & 1 + \frac{2\alpha}{\chi}
\end{pmatrix}, \]  

(61)

with

\[ \chi = \sqrt{x^2 + y^2 + z^2} \]  

(62)

which corresponds to a composition of a flat tetrad and a perturbation, as we wanted.

Thus we can calculate the torsion in first order substituting (57) in (17). These expressions are in appendix B.
B. Gravitomagnetic field of a rotating massive spherical shell

Having all necessary elements, we can evaluate the gravitomagnetic components of a gravitational field produced by a massive spherical shell in slow rotation. According to the definition (7), the $x$ component of the gravitomagnetic field becomes

$$B_{(0)x} = \frac{1}{4} h_{(0)}^0 g^{22} g^{33} (T_{023} + T_{203}) + \frac{1}{2} [h_{(0)}^0 g^{20} g^{33} T_{003} - h_{(0)}^2 g^{00} g^{33} T_{003} - h_{(0)}^2 g^{11} g^{33} T_{113}],$$

(63)

the $y$ component is

$$B_{(0)y} = -\frac{1}{4} h_{(0)}^0 g^{11} g^{33} (T_{013} + T_{103}) - \frac{1}{2} [h_{(0)}^0 g^{10} g^{33} T_{003} + h_{(0)}^1 g^{00} g^{33} T_{003} + h_{(0)}^1 g^{22} g^{33} T_{223}],$$

(64)

and finally, the $z$ component is

$$B_{(0)z} = \frac{1}{2} [h_{(0)}^0 g^{11} g^{00} T_{001} - h_{(0)}^0 g^{11} g^{20} T_{001} + h_{(0)}^0 g^{10} g^{22} T_{002} - h_{(0)}^2 g^{11} g^{33} T_{313} - h_{(0)}^2 g^{22} g^{00} T_{002} + h_{(0)}^1 g^{22} g^{33} T_{323}]$$

$$+ \frac{1}{4} h_{(0)}^0 g^{11} g^{22} [-T_{201} + T_{012} - T_{102}].$$

(65)

All other superpotential combinations are seen to be zero, taking into account our hypothesis.

After substituting the tetrad elements, the metric tensor and torsion components in the equations above, we obtain the following gravitomagnetic $x$, $y$ and $z$ components, respectively,

$$B_{(0)x} = \frac{r_0^2 \Omega_0 (r_0 + 6 \alpha) x z}{2 \chi^5} \left[ \frac{3}{2} - \frac{11 \alpha}{\chi} \right],$$

(66)

$$B_{(0)y} = \frac{r_0^2 \Omega_0 (r_0 + 6 \alpha) y z}{2 \chi^5} \left[ \frac{3}{2} - \frac{11 \alpha}{\chi} \right],$$

(67)

and

$$B_{(0)z} = \frac{r_0^2 \Omega_0 (r_0 + 6 \alpha)}{2 \chi^3} \left[ 1 - \frac{8 \alpha}{\chi} \right] - \frac{r_0^2 \Omega_0 (r_0 + 6 \alpha)}{2 \chi^3} (x^2 + y^2) \left[ \frac{3}{2} - \frac{11 \alpha}{\chi} \right].$$

(68)

It is interesting to perform a comparison between our results and the analogous problem in electromagnetism. Thus, let us consider a spherical shell of radius $R$, bearing a superficial
charge density $\sigma$ and which is placed to move with an angular velocity $\Omega$. After some direct calculations, we obtain the magnetic field of this configuration in Cartesian coordinates to be

$$\vec{B} = \frac{\mu_0 R^4 \Omega \sigma}{3 \chi^5} \left[3(xz\hat{x} + yz\hat{y}) - (x^2 + y^2 - 2z^2)\hat{z}\right]$$

(69)

where $\mu_0$ is the permeability constant.

Comparing the above expressions (66-68) with (69), we note a strong similarity between them. For example, the $x$ and $y$ components of $B^a$ $(a=0)$ are proportional to the product $xz$ and $yz$, respectively, in the same way as the magnetic ones, while the $z$ component has a term proportional to $x^2$ and $y^2$, like in electromagnetism. In fact, we can easily see that the expressions (66-68) correspond in zero order to a magnetic dipole field. Finally, if we stop the rotation, by making $\Omega_0$ equal to zero, we see that the above components will also disappear, that is, if the shell loses its rotation motion, gravitomagnetism disappears. Again we can say that our results are consistent with similar situations studied in literature in the GR version of gravitomagnetism despite using conceptual different definitions of what would be gravitomagnetic field, being the teleparallel ones much more appropriate to draw comparisons with electromagnetism.

V. CONCLUSIONS

Teleparallel gravity, though equivalent to GR, has its fundamentals in an abelian gauge theory, in the same way as electromagnetism. This motivates an attempt to put the dynamical equations in a similar form and to make a closer interpretation of its phenomenology. As the crucial differences between these two fundamental interactions of nature are present, the parallel was carried out as far as possible, as we summarize below.

Those which would be the gravitational Maxwell equations in their exact form proved to be definitely more complex than the electromagnetic ones. This was in fact an expected result, which has its origin in the strong non linearity of gravity. The derivatives of the gravitoelectric and gravitomagnetic fields appearing in the gravitational equations are in general similar to the electromagnetic ones, but we see the emergence of coupling fields terms playing the role of source of gravitation, which characterizes what we have identified as gravitation as source of gravitation. The exact form of the field equations in the standard
GR gravitomagnetism, despite conceptually different, also presents the same complexity.

In an attempt to understand whether the model proposed was associated with physical phenomena analogous to the electromagnetic ones, we first applied our definitions to a well known and simple geometry: the Schwarzschild solution. Initially we found non null gravitomagnetic components in exact calculations, attributed to a second order effect coming from the non-linearity of gravitation. These components vanish when considering linearized gravity, making in fact teleparallelism closer to electromagnetism. Concerning to the gravitoelectric fields, they exhibited the expected form: gravitoelectric components totally analogous to the Coulombian electric field of electromagnetism. In this context, we have found the gravitational field equations presenting strong similarity to the Maxwell’s electromagnetic ones.

A second test was performed with the purpose of making evident the emergence of gravitomagnetism in linear regime, as in electromagnetic case. The chosen geometry was, for a calculational convenience and for easy comparison between theories, the massive spherical shell in slow rotation. The result was surprising when compared with the electromagnetic analogue (the charged spherical shell in rotation), since they are quite similar.

There is another interesting point to be noted: the approach of teleparallelism highlights the role played by the observers, represented here by the tetrad fields. This subtleness is not present in the metric description of gravity. Thus, even for a static distribution of matter, it is in principle expected gravitomagnetic field when considering observers in motion, that is in complete agreement with electromagnetism. We can investigate, for example, which kind of fields emerge in the context of a free falling frame in the Schwarzschild spacetime, that is, a radial accelerated frame (by the gravitational force) going straightly to the singularity. Other interesting possibility is to consider an observer with the same angular velocity of a rotating massive spherical shell. These cases are under investigation.

Finally, we can say that the results obtained in the teleparallel gravitomagnetism, although compatible to those of GR, present much more affinity to electromagnetism from its first principles, based on a gauge similar structure.

New tests for the model are desirable as well as other verifications of theoretical consistency, since it opens real possibilities to improve the comprehension of these two fundamental interactions and has a strong relation with unification efforts.
VI. ACKNOWLEDGMENTS

The authors would like to thank J. W. Maluf for useful discussions and R. F. P. Mendes for the revision of the manuscript. They would like also to thank CNPq for partial financial support.

APPENDIX A: EXPANDED FIRST PAIR OF GRAVITATIONAL FIELD EQUATION

Equivalent to Gauss’s law ($\rho = 0$):

$$\partial_i(hE_a^i) - h[H_{bc}^{\text{aij}}E_b^iE_c^j + g_{cij}h_j^c\epsilon^{rk}(E_c^iB_{ak} - 1/2E_a^iB_{ck})]
+ T_{bc}^{\text{aij}}\epsilon^{jnk}E_c^iB_{bk} + J^{\text{cij}}E_a^iE_c^j + K_{bc}^{\text{arijn}}\epsilon^{ijk}\epsilon^{nrt}B_{ck}B_{bt}] = 0 \quad (A1)$$

with

$$H_{bc}^{\text{aij}} = -g_{00}h_a^0h_b^i h_c^j + h_a^0g_{0j}h_b^i h_c^0 - \frac{1}{2}h_a^0g_{ij}h_b^0 h_c^0 + \frac{1}{2}h_a^0g_{0j}h_c^i h_b^0$$

$$T_{bc}^{\text{aij}} = 2h_a^0g_{0n}h_b^i h_c^j - 2h_a^0g_{nj}h_b^0 h_c^j - h_a^0g_{0n}h_c^i h_b^j + h_a^0g_{nj}h_b^i h_c^j - 2h_a^0h_b^j h_c^j$$

$$J^{\text{cij}} = g_{ij}h_c^0 - 2g_{0j}h_c^i + g_{0j}h_c^i$$

$$K_{bc}^{\text{arijn}} = h_a^0g_{ri}h_j^b h_c^n + h_a^0g_{nj}h_c^j h_b^r.$$

Analogue to Ampère’s law ($\rho = q$):

$$\epsilon^{ijk}\partial_j(hB_{ak}) - \partial_0(hE_a^q) - h[\mathcal{P}_{bc}^{\text{aij}}E_b^iE_c^q + \mathcal{Q}_{bc}^{\text{aij}}\epsilon^{qik}E_b^jB_{ck}]$$

$$+ \mathcal{M}_{bc}^{\text{ijar}}\epsilon^{rjk}\epsilon^{qit}B_{bk}B_{at} + g_{ri}h_j^c\epsilon^{rjk}\epsilon^{qit}B_{bk}B_{ct} - \frac{1}{2}g_{ri}h_j^c\epsilon^{rjk}\epsilon^{qit}B_{bk}B_{at}$$

$$+ \mathcal{U}_{bc}^{\text{ijc}}E_c^q\epsilon^{ijk}B_{bk} + \mathcal{V}_{bc}^{\text{ijc}}E_c^q\epsilon^{ijk}B_{bk} + \mathcal{W}_{bc}^{\text{ijc}}E_a^j\epsilon^{qik}B_{ck}$$

$$- g_{0j}h_c^i\epsilon^{ijk}(E_c^qB_{ak} - E_a^qB_{ck}) + \mathcal{X}_{bc}^{\text{ijk}}E_c^iE_c^q + \frac{1}{2}g_{ri}h_j^c\epsilon^{rjk}\epsilon^{qit}B_{bk}B_{at}$$

$$+ h_{aq}[A_{bc}^{\text{ijk}}E_c^iE_b^j + N_{bc}^{\text{aij}}\epsilon^{jnk}E_c^iB_{bk} + C_{bc}^{\text{aij}}\epsilon^{ijk}E_b^nB_{ck}]$$

$$+ D_{bc}^{\text{aij}}\epsilon^{ijk}\epsilon^{nrt}B_{ck}B_{bt}] = 0 \quad (A2)$$
\[ A_{ij}^{bc} = -g_{00}^{b} h_{i}^{c} h_{j}^{c} + g_{ij}^{b} h_{0}^{c} h_{0}^{c} - \frac{1}{2} g_{ij}^{b} h_{0}^{c} h_{0}^{c} + g_{00}^{b} h_{0}^{c} h_{j}^{c} \]
\[ + \frac{1}{2} g_{00}^{b} h_{i}^{c} h_{j}^{c} - \frac{1}{2} g_{ij}^{b} h_{0}^{c} h_{0}^{c} - \frac{1}{2} g_{00}^{b} h_{0}^{c} h_{j}^{c} , \]
\[ P_{ai}^{bc} = 2 h_{0}^{b} h_{i}^{c} h_{0}^{c} h_{0}^{c} h_{0}^{c} - 2 h_{0}^{b} h_{0}^{c} h_{0}^{c} h_{0}^{c} h_{0}^{c} + h_{i}^{c} h_{0}^{b} h_{a}^{0} - h_{b}^{0} h_{0}^{c} h_{0}^{c} h_{0}^{c} , \]
\[ Q_{aij}^{bc} = -2 h_{i}^{b} h_{0}^{c} h_{a}^{0} + 2 h_{0}^{b} h_{i}^{c} h_{0}^{c} h_{a}^{0} + h_{i}^{c} h_{b}^{0} h_{a}^{0} - h_{b}^{0} h_{i}^{c} h_{a}^{0} , \]
\[ N_{nij}^{bc} = g_{0n}^{b} h_{i}^{c} h_{j}^{c} - g_{ni}^{b} h_{0}^{c} h_{0}^{c} + \frac{1}{2} g_{0j}^{b} h_{0}^{c} h_{n}^{c} - \frac{1}{2} g_{ij}^{b} h_{0}^{c} h_{0}^{c} , \]
\[ C_{nij}^{bc} = g_{ni}^{b} h_{j}^{c} h_{0}^{c} - g_{0n}^{b} h_{j}^{c} h_{0}^{c} + \frac{1}{2} g_{0i}^{b} h_{0}^{c} h_{j}^{c} h_{n}^{c} - \frac{1}{2} g_{ni}^{b} h_{j}^{c} h_{0}^{c} , \]
\[ D_{rjn}^{bc} = h_{r}^{b} h_{j}^{c} h_{0}^{c} h_{0}^{c} + \frac{1}{2} g_{ni}^{b} h_{j}^{c} h_{r}^{c} , \]
\[ M_{ijar}^{bc} = 2 h_{i}^{b} h_{j}^{c} h_{0}^{c} h_{0}^{c} h_{0}^{c} - h_{b}^{0} h_{i}^{c} h_{0}^{c} h_{0}^{c} , \]
\[ U_{ij}^{be} = 2 h_{0}^{b} h_{i}^{c} - h_{i}^{b} h_{0}^{c} , \]
\[ V_{ij}^{b} = g_{0j}^{b} h_{i}^{c} - g_{ij}^{b} h_{0}^{c} , \]
\[ W_{ij}^{b} = 2 g_{ij}^{b} h_{0}^{c} - 2 g_{00}^{b} h_{0}^{c} , \]
\[ X_{i}^{c} = 2 g_{00}^{b} h_{i}^{c} - 2 g_{0i}^{b} h_{0}^{c} , \]
\[ Z_{i}^{b} = 2 g_{0i}^{b} h_{0}^{c} - 2 g_{00}^{b} h_{i}^{c} , \]
APPENDIX B: NON VANISHING TORSION COMPONENTS FOR THE SPINNING MASSIVE SPHERICAL SHELL

\[ T_{001} = T_{221} = T_{331} = -\frac{2\alpha x}{r^3}, \]
\[ T_{002} = T_{112} = T_{332} = -\frac{2\alpha y}{r^3}, \]
\[ T_{003} = T_{113} = T_{223} = -\frac{2\alpha z}{r^3}, \]
\[ T_{101} = -T_{202} = \left(\frac{2\alpha x}{r^3}\right) \left(\frac{r_0^2(r_0 + 6\alpha)\Omega_0 y}{r^3}\right), \]
\[ T_{103} = \left(\frac{2\alpha z}{r^3}\right) \left(\frac{r_0^2(r_0 + 6\alpha)\Omega_0 y}{r^3}\right), \]
\[ T_{203} = -\left(\frac{2\alpha z}{r^3}\right) \left(\frac{r_0^2(r_0 + 6\alpha)\Omega_0 x}{r^3}\right), \]
\[ T_{201} = -\left(\frac{2\alpha x}{r^3}\right) \left(\frac{r_0^2(r_0 + 6\alpha)\Omega_0 x}{r^3}\right), \]
\[ T_{102} = \left(\frac{2\alpha y}{r^3}\right) \left(\frac{r_0^2(r_0 + 6\alpha)\Omega_0 y}{r^3}\right), \]
\[ T_{013} = -\frac{3r_0^2\Omega_0 yz}{r^5} \left(\frac{r_0^2(r_0 + 6\alpha)\Omega_0 y}{r^3}\right), \]
\[ T_{023} = \frac{3r_0^2\Omega_0 xz}{r^5} \left(\frac{r_0^2(r_0 + 6\alpha)\Omega_0 y}{r^3}\right), \]
\[ T_{012} = \left(1 - \frac{2\alpha}{r}\right) \left(\frac{2r_0\Omega_0(r_0 + 6\alpha)}{r^3} - \frac{3r_0^2\Omega_0(r_0 + 6\alpha)(x^2 + y^2)}{r^5}\right). \]

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