Translation lengths in $\text{Out}(F_n)$

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Abstract

We prove that all elements of infinite order in $\text{Out}(F_n)$ have positive translation lengths; moreover, they are bounded away from zero. Consequences include a new proof that solvable subgroups of $\text{Out}(F_n)$ are finitely generated and virtually abelian and the new result that such subgroups are quasi-convex.

1 Introduction

In this paper we will study the translation lengths of outer automorphisms of a free group. Following [GS91] we define the translation length $\tau_{X,G}(g)$ of $g \in \Gamma$ to be

$$\lim_{n \to \infty} \frac{\|g^n\|}{n}$$

where $\Gamma$ is a group with finite generating set $X$, and $\|g\|$ denotes the length of $g$ in the word metric on $\Gamma$ associated to $X$.

Farb, Lubotzky and Minsky proved that Dehn twists (more generally, all elements of infinite order) in $\text{Mod}(\Sigma_g)$ have positive translation length ([FLM]). We prove

**Theorem 1.1.** Every infinite order element $O \in \text{Out}(F_n)$ has positive translation length. Furthermore, there exists a positive constant $\varepsilon_n$ such that $\tau(O) \geq \varepsilon_n$.

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Once more we can see the strong analogy between mapping class group of a surface, $\text{Mod}(\Sigma_g)$, and outer automorphism group of a free group, $\text{Out}(F_n)$.

To prove their theorem, Farb, Lubotzky and Minsky found a way to measure how much a Dehn twist is ‘twisted’ by looking at simple closed curves and their intersection number. Such an approach cannot work in the case of $\text{Out}(F_n)$ as we do not have an analogue of the intersection number.

As a consequence of our main result we have

**Corollary 1.2.** Every solvable subgroup of $\text{Out}(F_n)$ is finitely generated and virtually abelian.

Corollary 1.2 was proved in [BFH99a], but Theorem 1.1 offers an alternative proof.

**Corollary 1.3.** Every abelian subgroup $A$ of $\text{Out}(F_n)$ is quasi-convex.

The proofs use techniques of [GS91] and follow the same lines as the corresponding proofs in [Bes99].

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## 2 Translation lengths

From the definition of translation length we can see that it depends on the choice of generating set for a group $\Gamma$. We will omit the reference to the generating set, since it will be clear which one we are using.

We list some properties of translation lengths which can be found in [GS91].

**Proposition 2.1.** Let $X$ be a generating set for a group $\Gamma$.

1. $0 \leq \tau(g) \leq \|g\|
2. For all $x, g \in G$, $\tau(xgx^{-1}) = \tau(g)$.
3. $\tau(g^n) = n \cdot \tau(g)$ $\forall n \in \mathbb{N}$.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of generators of a free group $F_n$. Let $Y$ be the set of generators for $\text{Aut}(F_n)$ consisting of:
1. permutations \((x_i \mapsto x_j, x_j \mapsto x_i, x_k \mapsto x_k \text{ for all } k \neq i, j)\),

2. inversions \((x_i \mapsto x_i^{-1}, x_j \mapsto x_j \text{ for all } j \neq i)\),

3. Dehn twists \((x_i \mapsto x_i x_j, x_k \mapsto x_k \text{ for all } k \neq i)\).

Let \(\tilde{Y}\) denote the generating set for \(\text{Out}(F_n)\) consisting of equivalence classes of elements of \(Y\).

Our goal is to prove that every element of infinite order in \(\text{Out}(F_n)\) has positive translation length. Since \(\text{Aut}(F_n)\) embeds into \(\text{Out}(F_{n+1})\), it will follow that every infinite order element of \(\text{Aut}(F_n)\) has positive translation length.

We will need the following definition for our proof:

**Definition 2.2.** Define a map \(\alpha : F_n \to \mathbb{N}\) by

\[
\alpha(w) = \max\{ |p| : \tilde{w}^p \text{ is a subword of } w \},
\]

where elements of \(F_n\) are regarded as reduced words in the generators and their inverses. We also define

\[
\tilde{\alpha}([w]) = \max\{ \alpha(u) : u \text{ is a cyclically reduced conjugate of } w \}
\]

for the conjugacy class, \([w]\), of \(w\).

**Lemma 2.3.** There exists a constant \(C > 0\) such that for any \(\tilde{g} \in \tilde{Y}\) and any cyclically reduced word \(w \in F_n\) we have

\[
\tilde{\alpha}(\tilde{g}([w])) \leq \tilde{\alpha}([w]) + C.
\]

**Proof.** Note that inversions and permutations do not affect \(\alpha(w)\), so we need only consider the case where \(g\) is a Dehn twist.

Let \(w \in F_n\) be a cyclically reduced element of length \(n\). Let \(w = A \tilde{w}^p B\), where \(\alpha(w) = |p|\). Consider

\[
g(w) = [[g(A)]] [[g(\tilde{w})^p]] [[g(B)]]
\]

where \([[g(w)^p]]\) denotes the reduced word obtained from \(g(w)^p\). By the *Bounded Cancellation Lemma* ([Coo87]) there is a constant \(C(g)\) such that at most \(C(g)\) cancellations occur after concatenation of the words \([[g(A)]]\) and \([[g(\tilde{w})^p]]\). Hence \(p\) can decrease by at most \(2C(g)\) (cancellations may occur at
the beginning and at the end of \([g(\tilde{w})]\). Let \(C_g = 2 \max\{C(g), C(g^{-1})\}\). We now have
\[
\begin{align*}
\alpha([g(w)]) & \geq \alpha(w) - C_g \\
\alpha(w) & = \alpha(g^{-1}(g(w))) \geq \alpha([g(w)]) - C_g \\
\alpha([g(w)]) & \leq \alpha(w) + C_g.
\end{align*}
\]
If we take \(\tilde{C} = \max\{C_g : g \text{ a Dehn twist in } Y\}\), our claim is proved for elements of \(Y\). Using a similar argument, we see that there is a constant \(C\) such that
\[
\tilde{\alpha}(\tilde{g}([w])) \leq \tilde{\alpha}([w]) + C.
\]

Example 2.4. We illustrate the idea of the proof of Theorem 1.1 with an example of a Dehn twist. Let \(g\) be a Dehn twist which sends \(x_2\) to \(x_2x_1\) and fixes all other generators of \(F_n\).
\[
\alpha(g^n(x_2)) = \alpha(x_2x_1^n) = n.
\]
If \(g^n = g_1 \cdots g_m\), then \(\|g^n\| = m\). By Lemma 2.3, we have that
\[
\begin{align*}
n & = \alpha(g^n(x_2)) \leq \alpha(x_2) + mC = mC + 1, \\
\tau(g) & = \lim_{n \to \infty} \frac{\|g^n\|}{n} \geq \lim_{n \to \infty} \frac{n - 1}{nC} = \frac{1}{C} > 0.
\end{align*}
\]
So \(g\) has positive translation length.

We give a short list of definitions which will be used throughout the rest of the paper, but we suggest that the reader look at [BFH99].

Every element \(O \in Out(F_n)\) can be represented by a homotopy equivalence \(f: G \to G\) of a graph \(G\) whose fundamental group is identified with \(F_n\). A map \(\sigma: J \to G\) (\(J\) is an interval) is called a path if it is either locally injective or a constant map (we also require that the endpoints of \(\sigma\) are at vertices). Every map \(\sigma: J \to G\) is homotopic (relative endpoints) to a path \([[\sigma]]\).

If \(\sigma = \sigma_1 \cdots \sigma_l\) is a decomposition of a path or a circuit \(\sigma\) into nontrivial subpaths we say that it is a \(k\)-splitting if
\[
f^k(\sigma) = [[f^k(\sigma_1)]][[f^k(\sigma_2)]] \cdots [[f^k(\sigma_l)]].
\]
is a decomposition into subpaths and is a *splitting* if it is a $k$-splitting for all $k > 0$.

We say that a nontrivial path $\sigma \in G$ is a *Nielsen path* for $f : G \to G$ if $[[f(\sigma)]] = \sigma$. The Nielsen path $\sigma$ is *indivisible* if it cannot be written as a concatenation of nontrivial Nielsen paths.

Let $G = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_K = G$ be a filtration of $G$ by $f$-invariant subgraphs, and let $H_i = G_i \setminus G_{i-1}$. Suppose $H_i$ is a single edge $E_i$ and $f(E_i) = E_i \cdot u_i$ for some closed indivisible Nielsen path $u_i \subseteq G_{i-1}$ and some $l > 0$. The *exceptional paths* are paths of the form $E_i \cdot v^k E_j$ or $E_i \cdot \bar{v}^k E_j$, where $k \geq 0$, $j \leq i$ and $f(E_j) = E_j \cdot v^m$, for $m > 0$.

We remind the reader that every element of $Out(F_n)$ of infinite order has either exponential or polynomial growth ([BH92]). A polynomially growing outer automorphism $O \in Out(F_n)$ is unipotent if its action in $H_1(F_n; \mathbb{Z})$ is unipotent (UPG automorphism).

The following Theorem can be found in [BFH00] (page 564).

**Theorem 2.5.** Suppose that $O \in Out(F_n)$ is a UPG automorphism. Then there is a topological representative $f : G \to G$ of $O$ with the following properties:

1. Each $G_i$ is the union of $G_{i-1}$ and a single edge $E_i$ satisfying $f(E_i) = E_i \cdot u_i$ for some closed path $u_i$ that crosses only edges in $G_{i-1}$ (· indicates that the decomposition in question is a splitting).

2. If $\sigma$ is any path with endpoints at vertices, then there exists $M = M(\sigma)$ so that for each $m \geq M$, $[[f^m(\sigma)]]$ splits into subpaths that are either single edges or exceptional subpaths.

**Lemma 2.6.** Let $O \in Out(F_n)$ be a UPG automorphism of infinite order and let $f : G \to G$ be its topological representative as in Theorem 2.5. For every path $\gamma$ in $G$ for which $[[f(\gamma)]] \neq \gamma$ there exists $a \in \mathbb{R}$ such that

$$\alpha([[f^k(\gamma)]])) \geq k + a.$$  

**Proof.** We prove our claim by induction on the (minimal) index, $m$, of the filtration element that contains a path $\gamma$.

If $\gamma \subset G_1$ there is nothing to be proved since $G_1$ contains only one edge $E_1$ which is fixed by $f$. 

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Suppose the claim is true for the subpaths contained in $G_{m-1}$ that satisfy the hypothesis, and let $\gamma$ be a path in $G_m$ for which $[[f(\gamma)]] \neq \gamma$. By Theorem 2.5 for every $m \geq M(\gamma)$, $[[f^m(\gamma)]]$ splits into subpaths that are either single edges or exceptional paths. Denote $[[f^M(\gamma)]]$ by $\tilde{\gamma}$, so that $\tilde{\gamma} = \gamma_1 \cdots \gamma_p$, where $\gamma_i$ is either a single edge or an exceptional path.

Assume there is an exceptional path $\gamma_t$ which is not fixed by $f$. Without loss of generality we may assume that $\gamma_t = E_i u^l E_j$, where $f(E_i) = E_i u^l$ ($l > 0$), $f(E_j) = E_j u^s$ ($s > 0$) and $j \leq i$. Now we have that $[[f^k(\gamma_t)]] = E_i u^{k(l-s)+r} E_j$, and

$$\alpha([[f^k(\gamma_t)]])) \geq k(l-s) + r, \quad \text{if } l-s > 0,$$

$$\alpha([[f^k(\gamma_t)]])) \geq k(s-l) - r, \quad \text{if } l-s < 0.$$ 

Since $\gamma_t$ is not fixed, $l$ and $s$ cannot be equal. Therefore

$$\alpha([[f^k(\gamma_t)]])) \geq k \pm r,$$

$$\alpha([[f^k(\gamma_t)]])) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq k - M(\gamma) \pm r.$$ 

If all exceptional paths in $\tilde{\gamma}$ are fixed, there exists an edge $\gamma_t = E_i$ which is not fixed by $f$. We know that $f(E_i) = E_i u_i$, where $u_i$ is a closed path contained in $G_{m-1}$.

If $[[f(u_i)]] = u_i$, our claim is proven since $[[f^k(E_i)]] = E_i u_i^k$ and so

$$\alpha([[f^k(\tilde{\gamma})]]) \geq k,$$

$$\alpha([[f^k(\gamma)]])) \geq k - M(\gamma).$$

If $[[f(u_i)]] \neq u_i$, there exists $a \in \mathbb{R}$ such that $\alpha([[f^k(u_i)]])) \geq k + a$. We now have

$$\alpha([[f^k(\gamma)]])) = \alpha([[f^{k-M(\gamma)}(\tilde{\gamma})]]) \geq \alpha([[f^{k-M(\gamma)}(u_i)]])) \geq k - M(\gamma) + a.$$

$\Box$
Lemma 2.7. Let $\mathcal{O}$ be a UPG automorphism of $F_n$ of infinite order. There exist a closed path $\sigma$ in $G$, and $b \in \mathbb{R}$ such that

$$\bar{\alpha}(\mathcal{O}^k(\sigma)) \geq k + b.$$ 

Proof. Let $f : G \to G$ be as in Theorem 2.5. Since $\mathcal{O} \neq id$ there is a closed path $\sigma$ which is not fixed by $f$. We know that for every $m \geq M(\sigma)$, $[[f^m(\sigma)]] = \sigma_1 \cdot \ldots \cdot \sigma_p$ splits into subpaths that are either single edges or exceptional paths. Denote $[[f^M(\sigma)(\sigma)]]$ by $\tilde{\sigma}$, so that $\tilde{\sigma} = \sigma_1 \cdot \ldots \cdot \sigma_p$.

If there is an exceptional path $\sigma_t$ in this splitting which is not fixed by $f$, we get

$$\bar{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k \pm r$$

as in Lemma 2.6.

If all exceptional paths in $\tilde{\sigma}$ are fixed, there exists an edge $\sigma_t = E_i$ such that $f(E_i) = E_i \cdot u_i$, where $u_i$ is a closed path contained in $G_{i-1}$. By Lemma 2.6 there exists $a \in \mathbb{R}$ such that

$$\alpha(f^k(E_i)) \geq k + a.$$ 

Hence, in all the above cases, there is $b \in \mathbb{R}$ such that

$$\bar{\alpha}(\mathcal{O}^k(\tilde{\sigma})) \geq k + b.$$ 

\[\blacksquare\]

3 Proof of Theorem 1.1

We consider the cases of exponentially and polynomially growing outer automorphisms separately.

Case 1. Let $\mathcal{O}$ be an exponentially growing outer automorphism of $F_n$. There exist $\lambda > 1$ and a cyclically reduced word $w$ such that $\ell(\mathcal{O}^k([w])) \geq \lambda^k \ell([w])$, for all $k \geq 1$, where $\ell$ denotes the cyclic word length (see [BH92]). Suppose that $\mathcal{O}^k$ can be written as $\tilde{g}_1 \ldots \tilde{g}_m$, for some $\tilde{g}_i \in \tilde{Y}$. It is straightforward to show that for all $\tilde{g} \in \tilde{Y}$ and any cyclically reduced word $w$ we have

$$\ell(\tilde{g}([w])) \leq 2 \ell([w])$$
Using this inequality we obtain:
\[ \lambda^k \ell([w]) \leq \ell(O^k([w])) \leq 2^m \ell([w]) \]

Hence
\[ m \geq \frac{\log \lambda^k}{\log 2} \]

which implies
\[ \tau(O) \geq \frac{\log \lambda}{\log 2} > 0 \]

There is a constant \( c_1 > 1 \) such that \( \lambda \geq c_1 \) \cite{BH92}. Therefore \( \tau(O) \) is bounded away from zero.

**Case 2.** Let \( O \) be a \( UPG \) automorphism. Again assume that \( O^k \) can be written as \( \tilde{g}_1 \ldots \tilde{g}_m \), for some \( \tilde{g}_i \in \tilde{Y} \). By Lemma 2.7 there is a closed path \( \sigma \) in \( G \) such that
\[ \tilde{\alpha}(O^k(\sigma)) \geq k + b \]

Let \( u_j = \tilde{g}_j \ldots \tilde{g}_m \). Applying Lemma 2.3 we get
\[ \tilde{\alpha}(u_i(\sigma)) \leq \tilde{\alpha}(u_{i+1}(\sigma)) + C \]

which yields
\[ k + b \leq \tilde{\alpha}(O^k(\sigma)) \leq mC + \tilde{\alpha}(\sigma) \]
\[ \frac{k + b - \tilde{\alpha}(\sigma)}{C} \leq m. \]

We have
\[ \tau(O) \geq \lim_{k \to \infty} \frac{k + b - \tilde{\alpha}(\sigma)}{kC} = \frac{1}{C}. \]

Finally, if \( O \) is any polynomially growing outer automorphism, then there exists \( s \geq 1 \) (bounded above by some \( c_2 \)), such that \( O^s \) is a \( UPG \) automorphism. Then
\[ \tau(O) = \frac{1}{s} \tau(O^s) \geq \frac{1}{Cs} > 0. \]

Since \( s \) is bounded by \( c_2 \), we get \( \tau(O) \geq \frac{1}{Cc_2} > 0. \)

This completes the proof.
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