0 Introduction

Consider two balls of different sizes, rolling on each other, without slipping or spinning. The configuration space for this system is a 5-dimensional manifold $Q \cong \text{SO}_3 \times S^2$ on which the no-slip/no-spin condition defines a rank 2 distribution $D \subset TQ$, the “rolling-distribution”.

Now $D$ is a non-integrable distribution (unless the balls are of equal size) which has an “obvious” 6-dimensional transitive symmetry group $\text{SO}_3 \times \text{SO}_3$ arising from the isometry groups of each ball, but for balls whose radii are in the ratio 3:1, and only for this ratio, something strange happens: the local symmetry group of the distribution increases from $\text{SO}_3 \times \text{SO}_3$ to $G_2$, a 14-dimensional Lie group.

More precisely, let $\mathfrak{g}_2$ be the real “split form” of the complex 14-dimensional exceptional simple Lie algebra $\mathfrak{g}_{2\mathbb{C}}$. There are precisely two connected Lie groups whose Lie algebras are $\mathfrak{g}_2$. (See Appendix A) We choose the one corresponding
to the adjoint representation, and call it \( G_2 \). (The other one is \( \tilde{G}_2 \), the universal cover of the one we chose.) \( G_2 \) is a subgroup of \( \text{SO}(3, 4) \) and its maximal compact subgroup \( K \subset G_2 \) is isomorphic to \( \text{SU}_2 \times \text{SU}_2 / \{ \pm(1, 1) \} \) which double-covers \( \text{SO}_3 \times \text{SO}_3 \). Let \( \tilde{Q} = S^3 \times S^2 \) be the universal cover of \( Q \) equipped with the distribution \( \tilde{D} \) induced by the double covering \( \tilde{Q} \to Q \). Let \( \text{Aut}(\tilde{Q}, \tilde{D}) \) be the group of diffeomorphisms of \( \tilde{Q} \) leaving \( \tilde{D} \) invariant. Then we have

**Theorem 1.** The connected component of the identity in \( \text{Aut}(\tilde{Q}, \tilde{D}) \) for radius ratio 3:1 or 1:3 is isomorphic to \( G_2 \). The \( G_2 \) action on \( \tilde{Q} \) does not descend to \( Q \), but its restriction to the maximal compact \( K \subset G_2 \) does, covering the \( \text{SO}_3 \times \text{SO}_3 \) action on \( Q \). For any other radius ratio (other then 1:1) \( \text{Aut}(\tilde{Q}, \tilde{D}) \) is isomorphic to \( K \).

This theorem was communicated to us by Robert Bryant, for whom it is but a variation on a theme of E. Cartan’s work on the method of equivalence, contained in his notoriously difficult “Five Variables Paper” [5] from 1910. Bryant wrote to us recently:

“Cartan himself gave a geometric description of the flat \( G_2 \)-structure as the differential system that describes space curves of constant torsion 2 or 1/2 in the standard unit 3-sphere. (See the concluding remarks of Section 53 in Paragraph XI in the Five Variables Paper.) One can easily transform the rolling balls problem (for arbitrary ratios of radii) into the problem of curves in the 3-sphere of constant torsion and, in this guise, one can recover the 3:1 or 1:3 ratio as Cartan’s torsion 2 or 1/2 with a minimum of fuss. Thus, one could say that Cartan’s calculation essentially covers the rolling ball case.”
Our main purpose in this note is to try to explain this beautiful and mysterious theorem in a direct manner which does not appeal to Cartan’s method of equivalence. We consider it an expansion of Section 4 in Bryant’s lecture notes [4]. Our contribution consists basically of a description of two constructions of \((\tilde{Q}, \tilde{D})\) with a built-in \(G_2\)-invariance. Using these constructions we show here that, for radius ratio 3:1 or 1:3, \(G_2\) is contained in \(\text{Aut}(\tilde{Q}, \tilde{D})\). But we do not know how to show, without the more sophisticated Cartan’s methods (or its variants such as those of Tanaka) that \(G_2\) is the full identity component of \(\text{Aut}(\tilde{Q}, \tilde{D})\), nor that for radius ratio different from 3:1, 1:3 or 1:1, \(\text{Aut}(\tilde{Q}, \tilde{D})\) is not larger than \(K\).

A secondary purpose of this article is to correct an error appearing in the book [12] by one of us. We had mistakenly said there that the symmetry group for the rolling distribution for a ball on a plane (ratio 1 : \(\infty\)) was \(G_2\).

A tertiary purpose is to obtain a bit of a feel for the simplest exceptional Lie algebra \(g_2\) and its Lie groups, and to provide a refresher course on roots and weights.

**Structure of Paper.** In the next section (section 1) we describe the background and wider context of the problem, with references to the literature. In section 2 we give a detailed description of the distributions associated with the rolling of balls, noting their \(\text{SO}_3 \times \text{SO}_3\)-symmetries. In section 3 we describe the homogeneous distributions of a Lie group \(G\) in terms of data \((G, H, W)\), where \(H \subset G\) is a closed subgroup and \(W \subset g/h\) is an \(H\)-invariant subspace. We then identify this data for the rolling distribution with respect the group \(G = \text{SO}_3 \times \text{SO}_3\). In section 4 we use the root diagram of \(G_2\) to give our first construction of a \(G_2\)-invariant distribution data \((G_2, P, W)\). The identification of the resulting \(G_2\)-homogenous distribution on \(G_2/P\) with \((\tilde{Q}, \tilde{D})\) amounts to the embedding of \(\text{so}_3 \times \text{so}_3\) in \(g_2\) and is the subject of section 5 (and Appendix B) which forms the heart of this article. In section 6 we give the second \(G_2\)-invariant construction of \((\tilde{Q}, \tilde{D})\) an explicit construction applying projective geometry to the space of purely imaginary split octonions \(V\), the lowest dimensional non-trivial representation space for \(G_2\). Appendix C is historical. Following suggestions by Bryant we looked into Cartan’s thesis and found that much of content of section 6, and hence of the rolling distribution already appears there.

**Open problem.** Find a geometric or dynamical interpretation for the “3” of the 3 : 1 ratio.

For work in this direction see Agrachev [1] and also Kaplan and Levstein [11].
Acknowledgements. Robert Bryant has been crucial, at various key steps along the way, in steering us in the right direction. Marti Weissmann supplied us with key information regarding $G_2$, and the crucial Vogan reference.

1. History and Background

On distributions. By a distribution we mean here a linear subbundle of the tangent bundle of a manifold. The distributions first encountered are usually the integrable and the contact distributions and have infinite dimensional symmetry groups. In dimension 5 we first encounter distributions whose symmetry groups are finite-dimensional. Indeed, the generic distribution of rank 2 or 3 in 5 dimension has no local symmetries. Cartan [5] investigated rank 2 and 3 distributions in 5 dimensions in detail. The growth vector of a generic rank 2 distribution on a 5-dimensional manifold, at a generic point of that manifold, is $(2, 3, 5)$. This is the growth vector of a distribution at a point means that if $X, Y$ are any local vector fields spanning the distribution in a neighborhood of the point, then $[X, Y] = Z$ is pointwise linearly independent of $X, Y$ (in a neighborhood of the point) and $X, Y, Z, [X, Z], [Y, Z]$ span the tangent bundle in a neighborhood of the point. Cartan worked out the complete local invariants – analogues of the Riemann curvature tensor – for these (2,3,5) distributions. For the distribution’s symmetry group to act transitively all of Cartan’s invariants must be constant. To get the maximal dimensional symmetry group all Cartan’s invariants must vanish, in which case we call the distribution “flat”. Any such distribution is locally diffeomorphic to that of the “Carnot group” distribution associated to the unique graded nilpotent Lie group $n = n_{2,3,5}$ of this same growth, and its local symmetry algebra is $g_2$. (By the “local symmetry algebra” of a distribution we mean the algebra of vector fields $X$ satisfying $[X, \Gamma(D)] \subset \Gamma(D)$ where $\Gamma(D)$ is the sheaf of local sections of vector fields tangent to the distribution.)

As mentioned in the above quote from Bryant, Cartan [5] presented several geometric realizations of the flat case. Bryant and Hsu [3] (see section 3.4) pointed out the rolling incarnation of $G_2$. A (2,3,5) distribution will arise whenever one rolls one Riemannian surface on another provided their Gaussian curvatures are not equal. The Cartan invariants vanish if and only if the ratio of their curvatures are $1:9$. Hence the $1:3$ radii for spheres. We could also achieve the maximal local symmetry algebra $g_2$ by rolling two hyperbolic planes along each other, provided their “radii” are in the ratio $i:3i$. More history, and more instances of the flat $G_2$ system are explained in Bryant [4].

Non-integrable rank 2 distributions in dimension $n (n > 3)$ admit special families of integral curves known as “singular” or “abnormal”) [11]. These are curves which admit no local variations through integral curves and having endpoints fixed. In the case of (2,3,5) distributions there is precisely 1 singular curve (up to reparametrization) through every point in every direction tangent to $D$. In the case of rolling one Riemannian surface along another, these singular curves correspond to rolling along geodesics. Using the symplectic geometry associated to variations of singular curves Zelenko and Agrachev have been able to rederive Cartan’s (2,3,5) invariants. See [1] and references therein.

Tanaka and his school have established a wonderful generalization of the passage from the flat nilpotent model $n_{2,3,5}$ to $g_2$. Associated to each point $p$ of a manifold endowed with a non-integrable distribution there is a graded nilpotent Lie algebra $m = m(p)$ called the ‘nilpotentization’ of the distribution, or sometimes
the “symbol algebra”. The dimension of \( m \) is that of the underlying manifold. Call
the distribution “of type \( m \)” if the different algebras \( m(p) \) are all isomorphic to the
same \( m \), i.e. the isomorphism type does not change from point to point. (Every
\( (2,3,5) \) distribution is of type \( n \).) Associated to each graded nilpotent \( m \) there is
graded Lie algebra \( g \supset m \), possibly infinite dimensional, called the ‘prolongation’ of
\( m \) and built from \( m \) in a purely algebraic manner. This \( g \) represents, roughly speak-
ing, the maximal possible symmetry of a distribution of type \( m \): every symmetry
algebra for a type \( m \)-distribution, after applying a grading process to it, must be
a subalgebra of \( g \). The prolongation of the \( (2,3,5) \) algebra is \( g_2 \), and this fact can
be viewed as the algebraic restatement of Cartan’s work on the flat model. This
Tanaka prolongation method thus yields a proof that \( \text{Aut}(\tilde{Q},\tilde{D}) \subset G_2 \) in theorem
1, alternative to Cartan’s proof. Yamaguchi [17] has classified all \( m \)’s whose \( g \)’s are
simple. To each of these pairs \( (m,g) \) is associated an intricate differential geometry
and most of these have not been explored in any detail.

On \( G_2 \). The Lie algebra \( g_2 \) is the smallest of the exceptional simple Lie algebras.
In 1894 Killing uncovered the existence of the root lattice for \( g_2 \)’s, but without
establishing the existence of the corresponding Lie algebra. Cartan, in his thesis,
established the existence of \( g_2 \) in one page of his thesis [6]. He did so by constructing
the 7-dimensional representation of \( g_2 \), in a way which is closely related to our
second “projective split octonion” model for \( \tilde{Q} \), the universal cover of the rolling
space. We have devoted appendix C to this page of his thesis and its connection
with this second model. In 1914 Cartan [7] showed that \( G_2 \) can be realized as the
automorphism group of the octonions. For our split \( G_2 \) he used ‘split octonions’.
The compact form of \( G_2 \) appears in the Berger list of potential holonomy groups of
Riemannian metrics. Recently, the compact \( G_2 \) has been featured in string theories,
but perhaps that fad has passed already.

2. Distribution for rolling balls

2.1. The distribution.
Take the first ball to be stationary, of radius \( R \), with its center at the origin.
Roll a second ball of radius \( r \) on the first ball. The position of the second ball is
given by an isometry (rigid motion) \( \varphi_{(g,x)} : \mathbb{R}^3 \to \mathbb{R}^3 \), mapping a point \( P \) to
\[
\mathbf{p} = \varphi_{(g,x)}(\mathbf{P}) = g\mathbf{P} + (R+r)x,
\]
where \( (g,x) \in \text{SO}_3 \times S^2 \). Here, \( Rx \) is the point of contact of the two balls, \( (R+r)x \)
is the center of the second ball and \( g \in \text{SO}_3 \) describes the rotation of the second
ball relative to its initial position. See figure 1 in the introduction. Thus the
configuration space \( Q \) for our rolling problem has been identified with the manifold
\( \text{SO}_3 \times S^2 \). (For a visceral account of rolling a sphere on a plane, accessible to upper
division undergraduates, we recommend [9].)

Let \( (g_t,x_t) \in Q \) be a differentiable rolling motion. Let \( \omega_t \in \mathbb{R}^3 \cong \mathfrak{so}_3 \) be the
angular velocity of the rolling ball relative to its center, measured with respect to
inertial axes. In other words, if \( P \) is a material point fixed on the second ball,
\( \dot{P} = 0 \), and if we write \( \mathbf{p}_t = g_t\mathbf{P} \), then \( \dot{\mathbf{p}} = \dot{g}g^{-1}\mathbf{p} = \omega \times \mathbf{p} \). Then we have
**Proposition 1.** Let $Q = \SO_3 \times S^2$ be the configuration space of two rolling balls of radii $R$ and $r$. Let $\rho = R/r$. Then a curve $(g_t, x_t) \in Q$ describes a rolling motion without slipping and spinning iff

1. $(\rho + 1)\dot{x} = \omega \times x$ (no-slip condition),
2. $\langle \omega, x \rangle = 0$ (no-spin condition, i.e. $\omega$ need to be tangent to the stationary ball at $Rx$).

**Proof.** (1) The contact point between the two balls is $p = Rx$ on the first ball, $P = -g^{-1}r x$ with respect to the second ball. For non-slip, their velocities must match: $\dot{p} = g \dot{P}$. Now $\dot{p} = R \dot{x}$ and

$$\dot{P} = \left[ -\frac{d}{dt} g^{-1} r x - g^{-1} r \dot{x} = g^{-1} g' g^{-1} r x - g^{-1} r \dot{x} = g^{-1} r (\omega \times x - \dot{x}), \right]$$

hence the non-slip condition $\dot{p} = g \dot{P}$ is equivalent to $R \dot{x} = r (\omega \times x - \dot{x})$, from which (1) follows.

(2) Let $P$ be a material point fixed on the second ball ($\dot{P} = 0$). From the inertial point of view, which is to say, from the point of view of the first ball with origin at its center, the position of this material point is $p = g P + (R + r) x$, and so its velocity

$$\dot{p} = \dot{g} P + (R + r) \dot{x} = \dot{g} g^{-1} [p - (R + r) x] + (R + r) \dot{x} = \omega \times [p - (R + r) x] + (R + r) \dot{x}.$$ 

Using the no-slip equation, $(R + r) \dot{x} = r \omega \times x$, we get

$$\dot{p} = \omega \times [p - (R + r) x] + r \omega \times x = \omega \times (p - R \dot{x}).$$ 

The equation $\dot{p} = \omega \times (p - R \dot{x})$ asserts that the instantaneous motion of the second ball is a rotation whose axis of rotation (a line) passes through the point of contact $R \dot{x}$, in the direction of $\omega$ and with angular velocity of magnitude $||\omega||$. The no-spin condition is that the second ball does not spin about the point of contact of the two balls, which is to say that $\omega$ should have no component orthogonal to the common tangent plane of the two balls, i.e. $\langle \omega, x \rangle = 0$. □

The two conditions in the last Proposition define together a rank 2 distribution on $Q$. This is the rolling distribution.

2.2. The "obvious" symmetry. The group $\SO_3 \times \SO_3$ acts on $Q$ by $\varphi_{(g,x)} \mapsto g' \circ \varphi_{(g,x)} \circ g''^{-1}$, where $g', g'' \in \SO_3$. In terms of $(g, x)$ this action is

$$(g, x) \mapsto (g' gg''^{-1}, g'x), \quad g', g'' \in \SO_3.$$ 

This action is transitive and preserves the rolling distribution $D$ for any value of $\rho = R/r$. The proofs of these assertions are easy and left as exercises.

3. Group theoretic description of the rolling distribution

In the previous section we wrote down a distribution $D$ on $Q = \SO_3 \times S^2$, depending on the real parameter $\rho$. We showed that $Q$ admits an $\SO_3 \times \SO_3$-transitive action which preserves the distribution. Our aim in this paper is to show that for two specific values of the parameter, $\rho = 3$ and $\rho = 1/3$, the distribution admits a larger local group of symmetries, namely the group $G_2$. We do so by defining a $G_2$-homogeneous space $\bar{Q} = G_2 / P$, together with a $G_2$-invariant rank 2 distribution $\bar{D}$ on it. We then define a 2:1 covering map $\bar{Q} \to Q$ which maps $\bar{D}$
to $D$. Furthermore, the group $G_2$ contains a maximal compact subgroup $K \subset G_2$ which is a double cover of $SO_3 \times SO_3$, such that the map $\tilde{Q} \to Q$ is $K$-equivariant (with respect to the covering homomorphism $K \to SO_3 \times SO_3$). The constructions are most easily done on the group level or on the Lie algebra level. We describe in what follows the general set up required for “working on the group level” and then calculate the group theoretic data corresponding to the rolling distribution.

Let $G$ be a Lie group. A “$G$-homogeneous distribution” is a pair $(Q, D)$ where $Q$ is a manifold on which $G$ acts transitively and $D \subset TQ$ is a $G$-invariant distribution. Fixing a base point $q_0 \in Q$ with isotropy $H \subset G$ we obtain a $G$-equivariant identification $Q \cong G/H$ and an $H$-equivariant identification $T_{q_0}Q \cong g/h$ where $h \subset g$ denote the Lie algebras corresponding to $H \subset G$. Then $D_{q_0} \subset T_{q_0}Q$ corresponds to an $H$-invariant subspace $W \subset g/h$. In this way every $G$-homogeneous distribution $(Q, D)$ corresponds to data $(G, H, W)$, where $H \subset G$ is a closed subgroup with Lie algebra $h \subset g$ and $W \subset g/h$ is an $H$-invariant subspace. The adjoint action of $G$ defines an equivalence relation on the set of pairs $(H, W)$ so that different choices of base points on $Q$ correspond to equivalent pairs $(H, W) \sim (H', W')$. Conversely, given the data $(G, H, W)$, we can construct a $G$-homogeneous distribution $(Q, D)$ by letting $G$ act by left translations on the right $H$-coset space $Q := G/H$, and define a $G$-equivariant distribution $D \subset TQ$ using the $G$-action to push $D_{[e]} := W \subset g/h \cong T_{[e]}(G/H)$ around to all other points of $Q$.

On the level of Lie algebras, the data $(g, h, W)$ determines $(Q, D)$ up to a cover. If, as in our case of $g = so_3 \oplus so_3$, the simply connected Lie group $G$ realizing $g$ is compact, then there are only finitely many homogeneous distributions $(G, H, W)$ which realize the given Lie algebraic data $(g, h, W)$.

We now determine the data $(G, H, W)$ corresponding to the rolling distribution $(Q, D)$ of section 2.1. Here $G = SO_3 \times SO_3$, $Q = SO_3 \times S^2$, dim $H = 1$, dim $W = 2$. Identify the Lie algebra of $SO_3 \times SO_3$ with $\mathbb{R}^3 \times \mathbb{R}^3$, the set of pairs of angular velocities $(\omega', \omega'')$, with Lie bracket given by the cross product:

$$[(\omega', \omega''), (\eta', \eta'')] = (\omega' \times \eta', \omega'' \times \eta'').$$

The first factor $\omega'$ corresponds to the first (stationary) sphere, of radius $R$, while the second $\omega''$ factor corresponds to second (rolling) sphere of radius $r$.

Fix a base point, say $(1, e_3) \in SO_3 \times S^2 = Q$. The isotropy at this base point is the circle subgroup $H$ consisting of elements of the form $(h, h)$, where $h$ is a rotation around the $e_3$ axis, so $h = \mathbb{R}(e_3, e_3) \subset \mathbb{R}^3 \times \mathbb{R}^3$. Using the Killing metric on $g = so_3 \times so_3 = \mathbb{R}^3 \times \mathbb{R}^3$ we can identify $g/h \cong h^\perp$, so that the plane of the distribution at the base point is given by some 2-plane in $h^\perp$. Let us determine explicitly this 2-plane.

**Proposition 2.** The rolling distribution on $SO_3 \times S^2$ corresponding to rolling a ball of radius $r$ along one of radius $R$ is given by the 2-plane in $\mathbb{R}^3 \times \mathbb{R}^3$ (the Lie algebra of $SO_3 \times SO_3$) defined by the equations

$$\langle \omega', e_3 \rangle = \langle \omega'', e_3 \rangle = 0, \quad \rho \omega' + \omega'' = 0,$$

where $\rho = R/r$.

**Proof.** Since $h \subset \mathbb{R}^3 \times \mathbb{R}^3$ is generated by the vector $(\omega', \omega'') = (e_3, e_3)$ and the Killing metric corresponds to some multiple of the standard metric on $\mathbb{R}^3 \times \mathbb{R}^3$, $h^\perp \subset \mathbb{R}^3 \times \mathbb{R}^3$ is given by the equation $\langle \omega' + \omega'', e_3 \rangle = 0$. 


From the formula for the $SO_3 \times SO_3$-action in §2.2 we get the infinitesimal action at the base point
\[ \omega = \omega' - \omega'', \quad \dot{x} = \omega' \times e_3. \]
Substituting these into the rolling conditions at the base point (see §2.1),
\[ \langle \omega, e_3 \rangle = 0, \quad (R + r)\dot{x} = r\omega \times e_3, \]
we obtain
\[ \langle \omega' - \omega'', e_3 \rangle = 0, \quad [R\omega' + r\omega''] \times e_3 = 0. \]
Adding the condition of orthogonality to $h$,
\[ \langle \omega' + \omega'', e_3 \rangle = 0, \]
we obtain the above equations. □

3.1. Shrinking the group. The following observation will be key to proving that part of theorem 1 which we are going to prove, namely that $G_2 \subset Aut(\tilde{Q}, \tilde{D})$ for $\rho = 3$ or $1/3$. Suppose that $(Q, D)$ is a $G$-homogeneous distribution with $G$-data $(H, W)$. Let $G_1 \subset G$ be a subgroup for which the restriction of the $G$-action on $Q$ to $G_1$ is still transitive. Then $(Q, D)$ is also $G_1$-homogeneous distribution and its $G_1$-data is $(H_1, W_1)$ where $H_1 = H \cap G_1$ and $W_1 \subset g_1/\mathfrak{h}_1$ corresponds to $W$ under the linear isomorphism $g_1/\mathfrak{h}_1 \rightarrow g/\mathfrak{h}$ induced by the diffeomorphism $Q = G/H = G_1/\mathfrak{h}_1$. Since $(Q, D)$ has not been changed, it follows that the $G$-data $(H, W)$ and the $G_1$-data $(H_1, W_1)$ yield diffeomorphic manifolds with distributions. At the Lie algebra level, this discussion asserts that $(g_1, \mathfrak{h}_1, W_1)$ and $(g, \mathfrak{h}, W)$ define manifolds-with-distributions which are diffeomorphic up to a cover. To prove that $G_2 \subset Aut(\tilde{Q}, \tilde{D})$ we will be applying this observation to the case $g_1 = \mathfrak{so}_3 \oplus \mathfrak{so}_3 \subset g_2$.

4. A $G_2$-homogeneous distribution

We now describe the other main actor in this paper, a distribution with Lie algebraic data $(g_2, p, W)$. Please see the root diagram of $g_2$ in figure 2. This diagram will be explained immediately below. The decorations on the diagram are used to indicate the Lie algebraic data and will be explained a bit later.

![Figure 2: The root diagram of $g_2$.](image)

A reminder of the meaning of the root diagram. The plane in which the diagram is drawn is the dual of a Cartan subalgebra $t \subset g_2$. A Cartan subalgebra of a semi-simple Lie algebra $g$ is a maximal abelian subalgebra $t \subset g$ of semi-simple elements, i.e. each $ad(T) \in \text{End}(g)$, $T \in t$, is diagonalizable. In the case of $g = g_2$, $t$ is 2-dimensional, hence the subscript 2 in $G_2$, the rank of the group. The root
The diagram of $g$ encodes the adjoint action of $t$ on $g$, from which one can recover the whole structure of $g$.

The commutativity of the Cartan subalgebra $t$ implies that the diagonalizable endomorphisms $\text{ad}(T) \in \text{End}(g)$, $T \in t$, are *simultaneously* diagonalizable, resulting in a $t$-invariant decomposition

$$g = t \oplus \sum_\alpha g_\alpha,$$

where each $g_\alpha \subset g$ is a 1-dimensional subspace of $t$-common eigenvectors called a *root space*. The corresponding eigenvalue depends linearly on the acting element of $t$, so is given by a linear functional $\alpha \in t^*$, called *root*. Thus

$$[T, X] = \alpha(T)X, \quad T \in t, \quad X \in g_\alpha.$$

When we draw the root diagram in $t^*$ we use the Killing metric in $g$ to determine the size of the roots and especially the angles between them. The Killing metric in $g$ is the inner product $\langle X, Y \rangle = -\text{tr}(\text{ad}(X)\text{ad}(Y))$. It is non-degenerate (this is equivalent to semi-simplicity) and its restriction to $t$ is positive definite.

**Example of $g = \mathfrak{sl}_3(\mathbb{R})$.** The more familiar example of $\mathfrak{sl}_3(\mathbb{R})$ is useful to keep in mind before proceeding with $g_2$. The Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ is the vector space of of 3 by 3 traceless real matrices with Lie bracket the usual matrix Lie bracket. It is Lie algebra of the Lie group $\text{SL}_3(\mathbb{R})$ of 3 by 3 real matrices with determinant 1. Like $g_2$, the Lie algebra $\mathfrak{sl}_3(\mathbb{R})$ is a non-compact split form of its complexification ($\mathfrak{sl}_3(\mathbb{C})$) and has rank 2. We take as a Cartan subalgebra the subspace $t \subset \mathfrak{sl}_3(\mathbb{R})$ of traceless diagonal matrices,

$$t := \{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mid t_1 + t_2 + t_3 = 0, t_i \in \mathbb{R} \}. $$

$\mathfrak{sl}_3(\mathbb{R})$ has 6 roots:

$$\alpha_{ij} := t_i - t_j \in t^*, \quad i \neq j, \quad i, j \in \{1, 2, 3\},$$

with corresponding root spaces

$$g_{\alpha_{ij}} = \mathbb{R}E_{ij},$$

where $E_{ij}$ is the matrix whose $ij$ entry is 1 and all of whose other entries are 0. The corresponding root space decomposition

$$\mathfrak{sl}_3 = t \oplus \sum_{i \neq j} g_{\alpha_{ij}},$$

is just the decomposition of a matrix as a diagonal matrix plus its off diagonal terms. The metric induced on $t$ by the Killing metric is some multiple of the standard euclidean metric, so that $\langle T, T' \rangle = c \sum_i t_i t'_i$ for some $c > 0$. 


Reading the root diagram. One can read much of the structure of $\mathfrak{g}$ from its root diagram in a formula-free manner. Here is the key observation. Let $\alpha, \beta$ be two roots with (non-zero) root vectors $E_\alpha \in \mathfrak{g}_\alpha$, $E_\beta \in \mathfrak{g}_\beta$. That is,

$$[T, E_\alpha] = \alpha(T) E_\alpha, \quad T \in \mathfrak{t},$$

and similarly for $\beta$. It then follows immediately from the Jacobi identity that

$$[T, [E_\alpha, E_\beta]] = (\alpha + \beta)(T) [E_\alpha, E_\beta].$$

This means that

1. if $\alpha + \beta \neq 0$ and is not a root then $[E_\alpha, E_\beta] = 0$;
2. if $\alpha + \beta \neq 0$ and is a root then $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha + \beta}$;
3. if $\alpha + \beta = 0$, i.e. $\beta = -\alpha$, then $[E_\alpha, E_\beta] \in \mathfrak{t}$.

This set of 3 conclusions permit us to see at a glance from the diagram a fair amount of the structure of $\mathfrak{g}$. In the last two cases one can further show that $[E_\alpha, E_\beta]$ is non-zero and determine, with some calculations, the actual bracket, as will be illustrated in Appendix B.

Example: reading the root diagram of $\mathfrak{sl}_3$. Let us consider the subspace $\mathfrak{p} \subset \mathfrak{sl}_3$ spanned by $\mathfrak{t}$ and the root spaces corresponding to the roots marked with dark dots in figure 3.

The diagram shows that $\mathfrak{p}$ is a 5-dimensional subalgebra, i.e. it is closed under the Lie bracket (there are 4 dark dots, but remember that the thick dot at the origin stands for the 2-dimensional Cartan subalgebra). Indeed, $\mathfrak{p}$ is the subalgebra of upper triangular matrices (including diagonal ones), with corresponding subgroup $P \subset \text{SL}_3$, the subgroup of upper triangular matrices with determinant=1. The quotient space $\text{SL}_3(\mathbb{R})/P$ can be identified with the space $F$ of full flags in $\mathbb{R}^3$. A full flag is a pairs $(l, \pi)$, where $l$ is a line and $\pi$ is a plane, and $l \subset \pi \subset \mathbb{R}^3$. The “standard flag” consisting of the $x$ axis sitting inside the $xy$ plane has isotropy group $P$. The tangent space to $F$ at this base point is naturally identified with $\mathfrak{sl}_3/\mathfrak{p}$, represented in the root diagram by the remaining three light dots. Two of the light dots are marked +. The diagram shows that the root spaces corresponding to these roots span a $\mathfrak{p}$-invariant 2-dimensional subspace of $\mathfrak{sl}_3/\mathfrak{p}$ which Lie generates the root space associated with the third light dot. This means that we have on $F$ an $\text{SL}_3(\mathbb{R})$-invariant rank 2 contact distribution, i.e. a non-integrable distribution that Lie generates the tangent bundle.
This distribution can be geometrically interpreted as the “tautological” contact distribution on $F$. This distribution is spanned by two vector fields, corresponding to the two $+$s in figure 3. One vector field generates the flow in which the line $l$ spins within the plane $\pi$, while the plane remains fixed. The other vector field generates the flow in which the plane $\pi$ rotates about the line $l$, while the line remains fixed.

**Reading the $g_2$ diagram.** Now let us draw conclusions in a similar fashion from the $g_2$ diagram. There are twelve roots in the diagram (figure 2) and so 12 root spaces. The rank of $g_2$ is 2 and so the dimension of $g_2$ is $14 = 2 + 12$. Consider the 9-dimensional subspace $p \subset g_2$ spanned by $t$ and the root spaces associated with the roots marked by the dark dots in the diagram of figure 2. Then the diagram shows that

- $p$ is closed under the Lie bracket, i.e. is a subalgebra (a so-called parabolic subalgebra, a subalgebra containing a Cartan subalgebra).
- Let $P \subset G_2$ be the corresponding subgroup. It follows that $G_2$ has a 5-dim homogeneous space $G_2/P$, whose tangent space $g_2/p$ at a point is represented by the remaining 5 light dots.
- Two of the light dots are marked with $+$. The diagram shows that their root spaces generate a 2-dim $p$-invariant subspace $W_1 \subset g_2/p$, hence a $G_2$-invariant rank 2 distribution on $G_2/P$.
- This distribution is not integrable, in fact, it is a distribution of type $(2, 3, 5)$, since the diagram shows that bracketing once gives the light dot marked with $\sigma_3$ and bracketing again gives the remaining two light dots.

To summarize, we have assembled the ingredients for the data $(G_2, P, W_1)$. To prove the theorem is to provide the geometric interpretation of this distribution as $(\tilde{Q}, \tilde{D})$ of theorem 1. The first step in doing so is to embed $so_3 \oplus so_3$ in $g_2$.

5. **The maximal compact subgroup of $G_2$**

5.1. **Algebraic strategy of the proof.** In the previous sections we assembled the Lie algebraic data, $(so_3 \oplus so_3, h, D)$ and $(g_2, p, W)$ with corresponding group data $(K, H, D)$ and $(G_2, P, W)$. The key to theorem 1 is to embed $so_3 \oplus so_3$ in $g_2$. This embedding is constructed in the next section. Once established, we obtain a diffeomorphism between corresponding distributions by following the observation made in section 3.1.

We recap that observation, adding a bit of topology. Suppose that $\mathfrak{h} \subset g$ and $p \subset g$ are Lie subalgebras of the Lie algebra $g$. Suppose that the natural map $\mathfrak{h}/\mathfrak{h} \cap p \rightarrow g/p$ is a linear isomorphism. Let $W \subset g/p$ be an $ad$- $p$-invariant subspace and $W_1 \subset \mathfrak{h}/\mathfrak{h} \cap p$ the corresponding subspace. Then we will say that the Lie algebraic distributional data $(g, p, W)$ and $(\mathfrak{h}, \mathfrak{h} \cap p, W_1)$ are isomorphic. If the corresponding connected Lie groups are $K \subset G$ and if $K$ is compact, then we can conclude that the data $(G; P, W)$ and $(K; K \cap P, W_1)$ define isomorphic manifolds with distributions. For when $K$ is compact we have that $K/K \cap P$ is a compact and open submanifold of $G/P$ and hence is diffeomorphic to $G/P$. And under this diffeomorphism the distribution corresponding to $W$ is the same as the one represented by $W_1$.

The compactness assumption on $K$ is necessary to conclude that $G/P = K/(K \cap P)$. Think of the case $K = \mathbb{C}^* \subset G = SL(2, \mathbb{C})$ where $G$ acts on the sphere $Q = \mathbb{C} \cup \{\infty\}$ by Mobius transformations and where $\mathbb{C}^*$ corresponds to the complex...
scalings \( z \mapsto \lambda z, \lambda \neq 0 \). The fixed points of the \( \mathbb{C}^* \)-action are 0, \( \infty \). The \( \mathbb{C}^* \) orbit through any point \( z_0 \neq 0, \infty \) is open, being the whole sphere minus the two fixed points. Thus \( G/P \neq K/(K \cap P) \) where \( P \) is the isotropy group of \( z_0 \). But we still have \( \frak{g}/(\frak{g} \cap p) = g/p \) since the orbit of \( z_0 \) is open.

Lie algebraic data defines the corresponding Lie group data only up to a covering. We can insure that there are only finitely many such coverings by knowing that \( \frak{h} \), like \( \frak{so}_3 \oplus \frak{so}_3 \), is the compact real form of its corresponding complex Lie algebra. For in this case there are only a finite number of connected Lie groups \( K \) with Lie algebra \( \frak{h} \), all of these being compact and covered by the simply connected \( K \). To say this in another way, suppose we are given Lie algebra data \((\frak{g}, p, W)\) and \((\frak{h}, \frak{r} \cap p, W_1)\) as above, and suppose that \( \frak{h} \) is a compact real form. Let \( (G, P, W) \) and \((K, H, W_1)\) denote any Lie-group data realizing these respective Lie algebraic data where we are no longer assuming that \( K \subset G \). Then the two manifolds-with-distribution which they stand for are isomorphic up to a finite cover. By this we mean, there is a third manifold-with-distribution \((X, E)\) and covering maps \( \pi_G : X \to G/P, \pi_K : X \to K/H \) such that \( \pi_G^* W = \pi_K^* W_1 = E \). Indeed, we can take \( X \) to be \( G/\tilde{P} \) where \( G \) is the unique simply connected Lie group with algebra \( \frak{g} \).

To establish theorem 1 we will apply these considerations to the case \( G = G_2 \) and \( K \subset G_2 \) its maximal compact subgroup. We will show that \((\frak{h}, \frak{r} \cap p)\) is isomorphic to \((\frak{so}_3 \times \frak{so}_3, \frak{h} = \mathbb{R}(\frak{e}_3, \frak{e}_3))\). And we show that under this isomorphism \( W_1 \subset \frak{r}/\frak{r} \cap p \) corresponds to the rolling distribution when the ratios of the rolling spheres are 1 : 3.

5.2. Finding Maximal compacts. How can we “see” a maximal compact subgroup of \( G_2 \) tangled within its root diagram? Let us look back again at the example of \( \text{SL}_3(\mathbb{R}) \). Here the maximal compact subgroup is \( \text{SO}_3 \), with Lie algebra \( \frak{so}_3 \), the set of 3 by 3 antisymmetric matrices. These are spanned by the vectors \( E_{ij} - E_{ji}, \) \( i > j \). So we see that corresponding to each pair of “antipodal” roots \( \pm \alpha_{ij} \) we have one generator of \( \frak{h} \), lying in the sum of the two corresponding root spaces.

More generally, for the “split” real form of any semi-simple Lie algebra (such as our \( \frak{g}_2 \)), the situation is similar: we get the Lie algebra \( \frak{h} \) of a maximal compact subgroup \( K \subset G \) by taking the sum of 1-dimensional subspaces, one subspace for each pair of antipodal roots \( \pm \alpha \). In fact, there is a certain particularity “nice” choice of root vectors \( E_\alpha \in \frak{g}_\alpha \) (sometimes called a “Weyl basis”), so that the sought-for line is given by \( \mathbb{R}(E_\alpha - E_{-\alpha}) \), as in the \( \frak{sl}_3 \) case.

In the case of \( \frak{g}_2 \) we thus have that

- \( \frak{h} \) is the sum of six 1-dimensional subspaces \( \frak{a}_i, \frak{l}_i, \) \( i = 1, 2, 3 \), where \( \frak{a}_i \) lies in the sum of the root spaces corresponding to \( \pm \sigma_1 \), and \( \frak{l}_i \) lies in the sum of the root spaces corresponding to \( \pm \lambda_i \).
- The isotropy of the \( K \)-action, \( H = K \cap P \subset K \), is given in the diagram by the vertical segment, \( \frak{h} = \frak{l}_3 \).
- The distribution plane \( W \subset \frak{r}/\frak{h} \) is generated by \( \frak{a}_1, \frak{a}_2 \) (mod \( \frak{h} \)).

We have thus assembled the required ingredients for a “distribution data” \((\frak{h}, \frak{h}, W)\).

5.3. \( \frak{r} \simeq \frak{so}_3 \oplus \frak{so}_3 \). Our task here is to define an isomorphism \( \frak{r} \simeq \frak{so}_3 \oplus \frak{so}_3 \) that maps \((\frak{r}, \frak{h}, W)\) to the data of §4 with \( \rho = 3 \) or 1/3. This entails the decomposition of \( \frak{r} \) into the direct sum of two ideals, each isomorphic to \( \frak{so}_3 \). It would have been quite
nice and simple if the sought-for decomposition of \( \mathfrak{g} \) had been the decomposition into “long” (\( l_i \)) and “short” (\( s_i \)). But this is not the case. For the diagram shows that although the \( l_i \) generate an \( \mathfrak{so}_3 \) subalgebra of \( \mathfrak{g} \), this subalgebra is not an ideal, so is not one of the summands in the decomposition. And the \( s_i \) do not generate even a subalgebra. So we have to work harder, i.e. write down the precise commutation relations.

**Proposition 3.** There is a basis \( \{ S_i, L_i | i = 1, 2, 3 \} \) of \( \mathfrak{g} \), with \( S_i \in \mathfrak{s}_i \) and \( L_i \in \mathfrak{l}_i \), such that

\[
[L_i, L_j] = \epsilon_{ijk} L_k, \quad [L_i, S_j] = \epsilon_{ijk} S_k, \quad [S_i, S_j] = \epsilon_{ijk} (\frac{3}{4} L_k - S_k),
\]

where \( \epsilon_{ijk} \) is the “totally antisymmetric tensor on 3 indices” (\( \epsilon_{ijk} = 1 \) if \( ijk \) is a cyclic permutation of \( 123 \), \( -1 \) if anticyclic permutation, and \( 0 \) otherwise).

The proof of this proposition is relegated to Appendix B. It consists of simple but tedious calculations which we could not “see” in the diagram. We tried. We were reduced to picking up as nice as possible basis for \( \mathfrak{g}_2 \) and calculating the corresponding structure constants with the help of Serre [13].

Now set

\[
e_i' := \frac{3L_i + 2S_i}{4}, \quad e_i'' := \frac{L_i - 2S_i}{4}, \quad i = 1, 2, 3.
\]

These 6 vectors form a new basis for \( \mathfrak{g} \) and satisfy the standard \( \mathfrak{so}_3 \oplus \mathfrak{so}_3 \) commutation relations

(1) \[ [e_i', e_j'] = \epsilon_{ijk} e_k, \quad [e_i'', e_j''] = \epsilon_{ijk} e_k, \quad [e_i', e_j''] = 0, \]

thus establishing the desired Lie algebra isomorphism \( \mathfrak{g} \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3 \).

**Corollary 1.** The map \( \mathfrak{g} \to \mathfrak{so}_3 \oplus \mathfrak{so}_3 \) defined by \( e_i' \mapsto (e_i, 0), e_i'' \mapsto (0, \epsilon_i) \), \( i = 1, 2, 3 \), is a Lie algebra isomorphism. It maps \( \mathfrak{h} = \mathbb{R} L_3 \) to \( \mathbb{R} (e_3, \mathfrak{e}_3) \) and the 2-plane in \( \mathfrak{g} \) generated by \( S_1, S_2 \) to the 2-plane in \( \mathfrak{so}_3 \times \mathfrak{so}_3 \) defined in the Proposition of §4 for \( \rho = 3 \). Interchanging the summands in \( \mathfrak{so}_3 \oplus \mathfrak{so}_3 \), i.e. mapping \( e_i' \mapsto (0, e_i) \), \( e_i'' \mapsto (e_i, 0) \), corresponds to \( \rho = 1/3 \).

The first assertion is eq [11]. The second assertion is easily verified using the last Proposition. We have thus defined a \( G_2 \)-action on some finite cover of the rolling configuration, one which preserves the pulled-back distribution when the radii of the two balls are in the ratio \( 3 : 1 \) or \( 1 : 3 \). QED

**How we came up with the formulae for** \( e_i', e_i'' \). The first thing to observe is that since \( L_3 \) generates the isotropy \( H = P \cap K \) we should have \( L_3 = e_3' + e_3'' \). Since everything is symmetric in \( 1, 2, 3 \) we conclude that \( L_i = e'_i + e''_i, i = 1, 2, 3 \). Next since \( S_3 \) commutes with \( L_3 \) we should have \( S_3 = a e'_i + b e''_i \) for some constants \( a, b \), and again by symmetry \( S_i = a e'_i + b e''_i, i = 1, 2, 3 \). Now by using the sought-after commutations relations for the \( e_i', e_i'' \) and the known commutations for \( L_i, S_i \) we get that \( a, b \) are roots of the equation \( x^2 + x - 3/4 = 0 \), i.e. \( a = 1/2, b = -3/2 \). Hence,

\[
L_i = e'_i + e''_i, \quad S_i = (e'_i - 3 e''_i)/2, \quad i = 1, 2, 3.
\]

Inverting these equations we obtain the above equations for \( e_i', e_i'' \).
6. Split Octonions and the projective quadric realization of $\hat{Q}$

To show that $G_2 \subset Aut(\hat{Q}, \hat{D})$ (theorem 1), it remains to identify $G_2/P$ with the $\hat{Q} = S^3 \times S^3$ of the theorem and to show that the covering map $S^3 \times S^3 \to \hat{Q}$ corresponds to the identification $G_2/P = K/H$ composed with the projection $K/H \to (\pm 1, \pm 1) \backslash K/H$. In order to do these things will use the fact, discovered by Cartan [7] in 1914, that $G_2$ is the group of automorphisms of the “split octonions” $\hat{O}$. We will follow the treatment in the book [10], in the section “The Cayley-Dickson process” (p.104). There further consequences and motivation can be found.

The split octonions $\hat{O}$ are a real eight-dimensional algebra with unit and which is neither associative nor commutative. We identify $\hat{O}$ with $\mathbb{H}^2$, the 2-dimensional quaternionic vector space. Its multiplication law is

$$ (a, b)(c, d) = (ac + \bar{a}b, da + bc), \quad a, b, c, d \in \mathbb{H} $$

The unit $1 \in \hat{O}$ is $(1, 0) \in \mathbb{H}^2$

The automorphism group of a real algebra is $A$ is defined to be the space of nonzero real invertible linear maps $g : A \to A$ satisfying $g(xy) = g(x)g(y)$ for all $x, y \in A$. $G_2$ is the automorphism group of $\hat{O}$.

The unit 1 is automatically invariant under any automorphism of $\hat{O}$, so that $\mathbb{R} = \mathbb{R}1 \subset \hat{O}$ is a $G_2$-invariant subspace. This subspace has a $G_2$-invariant complement:

$$ \hat{O} = \mathbb{R}1 \oplus V = Re(\hat{O}) \oplus Im(\hat{O}) $$

In quaternionic terms:

$$ V = Im\hat{O} = Im\mathbb{H} \oplus \mathbb{H} \subset \mathbb{H}^2. $$

To see the invariant nature of $V$, we use the split-octonion conjugation $x \mapsto \bar{x}$ defined by $x = (a, b) \in \hat{O} \mapsto \bar{x} = (\bar{a}, -b)$ for $x \in \hat{O}$. Then $x = Re(x) + Im(x)$, $Re(x) = (x + \bar{x})/2 \in \mathbb{R}1$, and $Im(x) = (x - \bar{x})/2$. Also $x\bar{x} = -\langle x, x \rangle 1$ where $\langle x, y \rangle = Re(x\bar{y})$ is an inner product of signature 4,4 on $\hat{O}$ which is invariant under the action of $G_2$. $V$ is the orthogonal complement of $1 \in \hat{O}$ relative to this inner product. Alternatively, it can be shown that $x \in V$ if and only if $x^2 = \langle x, x \rangle 1$ (see [10], lemma 6.67), proving the $G_2$-invariance of $V$. $V$ forms a 7-dimensional inner product space of signature $(3, 4)$ relative to the restriction of $\langle \cdot, \cdot \rangle$. The $G_2$ action on $V$ leaves this inner product invariant, so that $G_2$ is realized as a subgroup of $SO(3, 4)$ through its representation on $V$.

The maximal compact of $G_2$ is $K = SO(4) = (SU(2) \times SU(2))/ \pm (1, 1)$. See Appendix B and [10]. Upon restricting from $G_2$ to $K$, the representation $V$ decomposes into irreducibles according to [3]. In other words, thinking of $SU(2)$ as unit quaternions, $(q_1, q_2) \in SU(2) \times SU(2) = \bar{K}$ (the universal cover of $K$) and $(a, b) \in Im(\mathbb{H}) \oplus \mathbb{H} = V$ we have $(q_1, q_2) \cdot (a, b) = (q_1aq_1, q_1bq_2)$.

In quaternionic terms [3] the quadratic form associated to our $(3, 4)$ inner product on $V$ is

$$ \langle (v, q), (v, q) \rangle = -|v|^2 + |q|^2. $$

Since $K$ acts transitively on the product of spheres $S^2 \times S^3 \subset Im(\mathbb{H}) \oplus \mathbb{H} = \hat{O}$ we have that $G_2$ acts transitively on the null cone $\{ x = (v, h) : (x, x) = 0, x \neq 0 \}$.

(To see that we can change the ‘length’ of an $x$ in the null cone using $G_2$, use the fact that each such null vector is a nonzero weight vector relative to some choice of maximal Cartan $T \subset G_2$. This maximal Cartan then acts on $x$ by scaling. See the
description following eq. (6) below.) Thus $G_2$ acts transitively on the space of null rays

$$C = \{ \mathbb{R}^+ x \subset V | \langle x, x \rangle = 0, x \neq 0 \} \subset P^+(V) := \text{rays in } V$$

This $C$ is a nondegenerate 5-dimensional quadric sitting in the 6-dimensional real ray space $P^+(V)$ (diffeomorphic to $S^6$). We can describe points of the ray space $P^+(V)$ using homogeneous coordinates $[x] = [v, h] = [\lambda v, \lambda h], \lambda \in \mathbb{R}^+$ with $(v, h) \in Im \mathbb{H} \oplus \mathbb{H} = V$. $C$ is defined by the homogeneous equation $\|v\|^2 = \|h\|^2$.

Using the $\mathbb{R}^+$ action, we normalize $\|v\| = 1$, proving that $C$ is diffeomorphic to the product of spheres $S^2 \times S^3 \cong S^3 \times S^2 = \tilde{Q}$ which appears in theorem 1.

Given a point $\mathbb{R}^+ x = [x] \in C$, set

$$x^\perp = \{ y \in V | \langle x, y \rangle = 0 \}, \quad x^0 = \{ y \in V | xy = 0 \}.$$

Then

**Proposition 4.**

$$\mathbb{R} x \subset x^0 \subset (x^0)^\perp \subset x^\perp \subset V,$$

and the dimensions are 1, 3, 4, 6, 7.

**Proof.** Use the definitions of the split octonion product (eq (2)) and inner product. \qed

When we projectivize, $x^\perp$ maps to the tangent plane $T[x]C$ to $C$ at $[x]$, and $x^0$ maps to a 2-dimensional subspace $D[x] \subset T[x]C$. Letting $[x]$ vary over $C$ we have defined a rank 2 distribution $D \subset TC$. This construction of $(C, D)$ depends only on the algebraic structure of $\tilde{O}$, so that $G_2 = \text{Aut}(\tilde{O})$ acts on $C$ preserving $D$.

**Proposition 5.** The (ray) projective quadric $C$ is a homogeneous space for $G_2$.

$C$ is diffeomorphic to $\tilde{Q} = S^3 \times S^2$ of theorem 1, and is naturally endowed with a $G_2$-invariant distribution $D$ of rank 2. Viewed as a $G_2$-homogeneous space, the data for $(C, D)$ coincides with the data $(G_2, P, W)$ of section 3.3. Viewed as a $K$-homogeneous space, its Lie algebraic data coincides with that of the rolling distribution $(\mathbb{R}, h, D)$ for the ratio $1 : 3$. The distribution on $C$ pushes down to the rolling distribution for ratios $3 : 1$ under the two-to-one cover $C = S^3 \rightarrow S^2 \rightarrow Q = SO_3 \times S^2$

This proposition immediately implies that part of the theorem we are going to prove: that $G_2 \subset \text{Aut}(\tilde{Q}, \tilde{D})$.

**Steps of the proof.** In the paragraph preceding the proposition we proved that $C$ is a homogeneous space for $G_2$, that $D$ is invariant under this $G_2$ action, and that $C$ is diffeomorphic to $\tilde{Q}$. Next, we will prove that the coincidence of the $g_2$-data for $(C, D)$ and the data $(p, W)$ of the previous section. For this we will use the weights for the $G_2$-representation space $V = Im(\tilde{O})$.

**Weights for the 7-dimensional representation.**

Here is the weight diagram for this representation.
The weights of the representation $V$ form a subset of the roots of $\mathfrak{g}_2$. In figure 4 we redrew the root diagram of $\mathfrak{g}_2$, marking those roots which are weights for $V$ with bullseye’s. They are the six short roots and one zero root. The corresponding weight spaces $V_w$ are all one-dimensional. The black dot is a selected weight vector and corresponds to a ‘choice of base point’ for $C$. The meaning of the X’s will be given below.

**A reminder of the meaning of the weight diagram.**

We recall the general case of a representation $V$ of a semi-simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{t}$. A weight for the representation $V$ of $\mathfrak{g}$ is an element $w \in \mathfrak{t}^*$ such that there is a nonzero vector $v \in V$ with the property that $\zeta \cdot v = w(\zeta)v$ for all $\zeta \in \mathfrak{t}$. The space of $v$’s for a given weight $w$ is called the weight space for $w$ and is denoted $V_w$. If, for given $w \in \mathfrak{t}^*$ there is no such nonzero $v$ then we set $V_w = 0$. For a finite-dimensional representation the set of weights is finite. We have

$$V = \bigoplus_{w \in \mathfrak{t}^*} V_w.$$ 

The roots of $\mathfrak{g}$ are the non-zero weights of the adjoint representation.

If, as in our situation, the roots for $\mathfrak{t}$ are real, then its ‘torus’ $T = \exp(\mathfrak{t})$ is noncompact and acts on the weight spaces by scaling, as follows. If $\lambda = \exp(\xi) \in T$, with $\xi \in \mathfrak{t}$, then $\lambda w = \exp(w(\xi))v$ for $w \in V_w$.

From $\zeta \cdot \xi \cdot v = \xi \cdot (\zeta \cdot v + [\zeta, \xi] \cdot v)$ it follows that if $v$ is in the weight space for $w$ and $\xi \in \mathfrak{g}_\alpha$ is in the root space for $\alpha$ then $\xi v$ is in the weight space for $w + \alpha$ (which, as above, could be zero). In other words: $\mathfrak{g}_\alpha V_w \subset V_{w+\alpha}$. This inclusion is half of the rule:

$$w \text{ a weight, } \alpha \text{ a root } \Rightarrow \mathfrak{g}_\alpha V_w = V_{w+\alpha}$$

which is true for $V$. It follows in particular that if $v \in V_w$ and $\xi \in \mathfrak{g}_\alpha$ and if $w + \alpha$ is not a weight for the representation, then $\xi(v) = 0$. We will use this fact momentarily.

We now construct the weight spaces and the action of the torus for our $G_2$-representation $V = \text{Im}(\mathfrak{O})$. Let $n$ be an imaginary quaternion. Then $(n, n)$ and $(n, -n)$ are both null vectors in $V$. Take as basis for $V$:

$$e_1 = \frac{1}{2}(i, i), e_2 = \frac{1}{2}(j, j), e_3 = \frac{1}{2}(k, k); f_1 = \frac{1}{2}(i, -i), f_2 = \frac{1}{2}(j, -j), f_3 = \frac{1}{2}(k, -k)$$

and

$$U = (0, 1).$$
Then we have the multiplication table:

\[
e_i^2 = f_i^2 = 0
\]

\[
e_i f_j = f_j e_i = 0, \text{ if } i \neq j
\]

\[
e_i e_j = f_k; i, j, k \text{ a cyclic permutation of 1, 2, 3}
\]

\[
f_i f_j = e_k; i, j, k \text{ a cyclic permutation of 1, 2, 3}
\]

\[
e_i f_i = -\frac{1}{2} + \frac{1}{2}U
\]

\[
f_i e_i = -\frac{1}{2} - \frac{1}{2}U
\]

\[
e_i U = e_i
\]

\[
f_i U = -f_i
\]

To complete the multiplication table, use that the conjugate of \(xy\) is \(\bar{x}\bar{y}\), so that if \(x, y \in V = Im(\tilde{O})\) we have \(yx = \bar{z}\) where \(z = xy\). Thus, for example since \(f_k = -f_k\) we see that \(e_j e_i = -f_k\), for \(i, j, k \text{ a cyclic permutation of 1, 2, 3}\). Now let \(\lambda_1, \lambda_2, \lambda_3\) be nonzero reals with \(\lambda_1 \lambda_2 \lambda_3 = 1\). Let \(\alpha_i, \beta_i, \gamma_i\) and \(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i\) be real exponents for \(i = 1, 2, 3\) satisfying \(\alpha_i + \beta_i + \gamma_i = 0\) Then the scaling transformation

\[
e_i \mapsto \lambda_1^{\alpha_i} \lambda_2^{\beta_i} \lambda_3^{\gamma_i} e_i
\]

\[
f_i \mapsto \lambda_1^{\tilde{\alpha}_i} \lambda_2^{\tilde{\beta}_i} \lambda_3^{\tilde{\gamma}_i} f_i
\]

together with \(z \mapsto \bar{z}\) preserves the multiplication table, and hence defines an element of \(G_2\), provided

\[
\tilde{\alpha}_i = -\alpha_i, \tilde{\beta}_i = -\beta_i, \tilde{\gamma}_i = -\gamma_i
\]

and provided that \((\alpha_i, \beta_i, \gamma_i)\) are multiples of the values from the following weight table

|   | \(\alpha_i\) | \(\beta_i\) | \(\gamma_i\) |
|---|-------------|-------------|-------------|
| 1 | 2           | -1          | -1          |
| 2 | -1          | 2           | -1          |
| 3 | -1          | -1          | 2           |

These scaling transformations generate the Cartan \(T\) of \(G_2\), and the table gives the corresponding weights. Thus for example \(e_1\) is a weight vector with corresponding weight, relative to the basis for \(t\), being \((2, -1, -1)\). Here we view \(t\) as being the collection of vectors \((a, b, c)\) with \(a + b + c = 0\). Looking at the inner products of these vectors we see that they are arranged on the weight diagram according to:

![Figure 5: The weight space basis.](image-url)
We are now in a position to compute the $g_2$-data associated to $(C, D)$ from the proposition.

**Weight vectors are null vectors:** Because the inner product is $G_2$-invariant, the $g_2$ action on $V$ satisfies $\langle \xi, x \rangle = 0$ for any $\xi \in g_2$, $x \in V$. Take $x$ a weight vector with nonzero weight $w$, and take $\xi \in \mathfrak{t}$ with $w(\xi) \neq 0$. From $\langle \xi, x \rangle = w(\xi)\langle x, x \rangle$ we have that $x$ is a null-vector.

**Computing the isotropy data.** Set $c_0 = [e_1]$, the ray through $e_1$. We must show that the isotropy group of $c_0$ is $P$.

We begin by showing that the isotropy algebra $g_{c_0}$ of $c_0 = [e_1]$ is $\mathfrak{p}$. We have that $g_{c_0} = \{\xi \in g_2 : \xi e_1 = \lambda e_1 \text{ for some real number } \lambda\}$. The black dot in figure 4 indicates the weight space spanned by $e_1$, with corresponding weight by $w_1$. (This weight is the root marked $-\sigma_3$ in Figure 3.) According to the addition rule, (4) if $\alpha \in \mathfrak{t}^*$ is a root and $w_1 + \alpha$ is not a weight for $V$, then $\xi_\alpha x_0 = 0$. Those roots for $w_0 + \alpha$ is not a weight are marked by X’s in figure 4. The sum of the corresponding $g_\alpha \subset g_2$ is a vector space of elements $\xi$ satisfying $\xi(e_1) = 0$. Now the isotropy algebra $g_{c_0}$ of the ray through $e_1$ consists of all those $\xi$ such that $\xi e_1 = \lambda e_1$ for some real scalar $\lambda$. The elements $H \in \mathfrak{t}$ act on $e_1$ by scalar multiplication by $\lambda = w_1(H)$. Referring to the diagram then, we see that $\mathfrak{p} \subset g_{c_0}$. But there is no subalgebra of $g_2$ lying between $\mathfrak{p}$ and all of $g_2$. It follows that the isotropy algebra for the ray is $\mathfrak{p}$.

It follows from this Lie algebra computation that the isotropy subgroup $G_{c_0}$ contains $P$ and has Lie algebra equalling the Lie algebra $\mathfrak{p}$ of $P$. Now $P$ was defined to be the connected Lie subgroup of $G_2$ whose Lie algebra is $\mathfrak{p}$, thus to show $G_{c_0} = P$ it suffices to show that $G_{c_0}$ is connected. We demonstrate connectivity by applying the homotopy exact sequence to the fiber bundle $G_{c_0} \rightarrow G_2 \rightarrow C = G_2/G[x]$. This exact sequence is $\ldots \rightarrow \pi_1(C) \rightarrow \pi_0(G_{c_0}) = \pi_0(G_2) \rightarrow \pi_0(C)$. Since $C$ is simply connected and connected we get that $\pi_0(G_{c_0}) = \pi_0(G_2)$ and since $\pi_0(G_2) = 0$ we have our connectivity: $\pi_0(G_{c_0}) = 0$.

We have established the isotropy $(P \text{ part})$ of the data for $(C, D)$.

**Computing the distribution data.** The distribution plane $D(c_0)$ at $c_0$ corresponds to $e_1^0$ - the subspace $S \subset V$ consisting of those vectors $y \in V$ for which $e_1 y = 0$. From the multiplication table following the description of our basis we see that $S = \text{span}\{e_1, f_2, f_3\}$. From Figure 4, we see that weights corresponding to $f_2, f_3$, say $w_2, w_3$, are given by $w_2 = w_1 + \sigma_1$, $w_3 = w_1 + (\sigma_2)$. Compare Figure 3. Let $x_1, y_1 \in g_2$ be the corresponding nonzero root vectors for $\pm \sigma_1, -\sigma_2$. (We follow the $x, y$ notation from Figure 5, Appendix 2.) It follows from rule (4) that $f_2$ is a multiple of $x_1(c_0)$ and $f_2$ is a multiple of $y_2(c_0)$. In other words, $S = W(e_1) \text{ mod } \mathfrak{p}(e_1)$ where $W$ is the space spanned by the roots indicated by the pluses in Figure 2. We have proved that the Lie algebraic data for $(C, D)$ is $(g_2, \mathfrak{p}, W)$.

**The covering map.** On the Lie algebra level we have shown that the data for $(C, D)$ is $(g_2, \mathfrak{p}, W)$. As computed in section 3.3, Cor. 1, upon restricting the action of $G_2$ to $K$ this Lie algebraic data $(g_2, \mathfrak{p}, W)$ corresponds to the data $(\mathfrak{so}_3 \oplus \mathfrak{so}_3, \mathfrak{h}, D(3; 1))$. Thus, up to a finite cover, $(C, D)$ is the rolling distribution. Now $C$ is simply connected, and $2 : 1$ covers $Q$. This covering map $C = S^3 \times S^2 \rightarrow Q = SO_3 \times S^2$ is realized by forming the quotient of $C$ by the $Z_2$ subgroup generated by image of $\sigma = (\pm 1, 1) \in K = SU_1 \times_{\pm (1,1)} SU_2$. Being an element of the symmetry group $\sigma$ preserves the distribution $D$ on $C$, and so $D$ does push down to the rolling
space $Q$. The $\mathfrak{r}$ data of the pushed-down distribution remains $(\mathfrak{so}_3 \oplus \mathfrak{so}_3, \mathfrak{h}, D(3; 1))$. Thus the pushed-down distribution is isomorphic to the rolling distribution on $Q$.

QED

7. Summary. Lack of action on the rolling space. The theorem is done.

We have proved that $G_2 \subset Aut(\hat{Q}, \hat{D})$ and that is all that we are going to prove of theorem 1, with the exception of the fact that the $G_2$-action does not descend to $Q$. (Recall from the introduction we are not going to prove that $G_2 = Aut(\hat{Q}, \hat{D})$.) To prove that the $G_2$ action does not descend to $Q$, we realize as above that $Q = \mathbb{Z}_2 \backslash C$ where the $\mathbb{Z}_2 \subset K \subset G_2$ is generated by $\sigma = (\pm 1, 1)$. Now we use the following fact about group actions. Suppose that a group $G$ (here $G_2$) acts effectively on a set $C$ and that $\Gamma \subset G$. (“Effectively” means that the only group element acting as the identity on $C$ is the identity.) Then the action of an element $g \in G$ descends to the quotient space $\Gamma \backslash C$ if and only if $g\Gamma g^{-1} = \Gamma$. In particular, if $\Gamma$ is not normal in $G$ then the action of all of $G$ does not descend to the quotient $\Gamma \backslash C$. Returning to our situation, we see that if the $G_2$ action were to descend then this $\mathbb{Z}_2$ generated by $\sigma$ would have to be normal. But a discrete normal subgroup of a connected Lie group is central, and $G_2$ has no center. See Appendix A, or [16]. So our $\mathbb{Z}_2$ is not normal, and the $G_2$ action does not descend.

Remark. Had we used lines instead of rays when constructing $C = \hat{Q}$, we would have arrived at a quadric $Q_f$ in the standard real projective space $P(V)$ which is double covered by $C = \hat{Q}$. (The subscript ‘f’ is for ‘false.’) $Q_f \subset P(V)$ is diffeomorphic to $S^3 \times_{\mathbb{Z}_2} S^2 = \pm I \backslash C$ where the notation $\times_{\mathbb{Z}_2}$ indicates that we divide out by the action of the involution $(v, h) \mapsto (-v, -h)$. (This involution does not lie in $G_2$.) $C = \hat{Q}$ double-covers both $Q_f$ and $Q$, and the distribution $\hat{D}$ pushes down to both covered spaces. But $Q_f$ is topologically distinct from $Q$. Both $Q$ and $Q_f$ are $SO_3$-bundles over $S^2$. $Q$ is the trivial $SO(3)$-bundle. $Q_f$ is the other one. (Since $\pi_1(SO_3) = \mathbb{Z}_2$ there are precisely two topologically distinct $SO_3$ bundles over $S^2$.) Because $-I \in GL(V)$ commutes with the $G_2$ action on $V$ the $G_2$-action on $\hat{Q}$ does descend to $Q_f$. We find it curious that the action of $G_2$ on $\hat{Q}$ does descend to this ‘false’ rolling configuration space $Q_f$, but not to the real one $Q$.

Appendix A. Covers. Two $G_2$’s.

To understand our results, it helps to understand that up to isomorphism, there are precisely two connected $G_2$’s: the adjoint one which is the one we have been using, and the simply connected one, which is the universal cover of the adjoint one.

For a general semi-simple Lie algebra $\mathfrak{g}$ we can always form the simply connected Lie group $\tilde{G}$ having $\mathfrak{g}$ as its Lie algebra. If $Z$ is the center of $\tilde{G}$, then $Ad(\tilde{G}) = \tilde{G}/Z$ where $Ad(\tilde{G})$ is the image of $\tilde{G}$ under the adjoint map from $\tilde{G}$ to $\text{Hom}(\mathfrak{g})$. If $Z \neq I$ then $\tilde{G} \neq Ad(\tilde{G})$. There are as many distinct connected Lie groups with algebra $\mathfrak{g}$ as there are distinct subgroups of $Z$, these being the connected topological groups covered by $\tilde{G}$ and covering $Ad(\tilde{G})$. So, when $Z = \mathbb{Z}_2$ there are precisely two such Lie groups, $\tilde{G}$, the simply connected one, and $G = Ad(\tilde{G})$, the adjoint one.

We find on p. 3 of Vogan [16] that the center of the simply connected $G_2$ is indeed $\mathbb{Z}_2$, and hence we have precisely two $G_2$’s. It will be useful to explain a few details of this computation of $Z(G_2)$. The universal cover of any $G$ contracts
onto its maximal compact. Thus, if the maximal compact of $Ad(G)$ has finite fundamental group, then the universal cover $\tilde{G} \to Ad(G) = \tilde{G}/Z$ is a finite cover, and so the center $Z$ must be finite. (At the other extreme, the maximal compact of $SL(2, \mathbb{R})$ is a circle group, corresponding to the fact that its universal cover has infinite center $\mathbb{R}$.)

We saw above that the Lie algebra of the maximal compact of any $G_2$ realizing $g_2$ is $\mathfrak{r} = \mathfrak{so}_3 \times \mathfrak{so}_3$. The connected Lie groups $K$ having $\mathfrak{r}$ as Lie algebra have fundamental groups consisting of either 1, 2 or 4 elements. It follows that the center $Z(G_2)$ of any $G_2$ is finite, and hence compact. Being compact and central, this center lies in every maximal compact: $Z(G_2) \subset K \subset G_2$. If we take the simply connected $G_2$, call it $\tilde{G}_2$, then its maximal compact is $\tilde{K} = SU_2 \times SU_2$. The center of $\tilde{K}$ is the group of the four elements $(\pm 1, \pm 1)$. The center of $\tilde{K}$ need not be the center of $G_2$ but it must contain it: $Z(\tilde{G}_2) \subset Z(\tilde{K})$. To see what the actual center of $\tilde{G}_2$ is, it suffices to see how $Z(\tilde{K}) \subset \tilde{K}$ acts on the Lie algebra $g_2$ under the adjoint action. This can be done using roots. The center of $\tilde{G}_2$ is that part of $Z(\tilde{K})$ which acts trivially on $g_2$. A computation using roots and the restriction of the adjoint representation to $\tilde{K}$ shows that this part is $(1, 1)$ and $-(1, 1)$.

Appendix B. The isomorphism of $\mathfrak{r}$ and $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ from Proposition 3.

We complete the proposition 3 from section 5, in which the explicit identification of $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ as the Lie algebra $\mathfrak{r}$ of the maximal compact in $g_2$. We follow Serre [13], page VI-11: $g_2$ is Lie-generated by the elements $x, y, h, X, Y, H$, subject to the following relations, which one can read off the root diagram.

\[
\begin{align*}
[x, y] &= h, & [h, x] &= 2x, & [h, y] &= -2y, \\
[X, Y] &= H, & [H, X] &= 2X, & [H, Y] &= -2Y; \\
[h, X] &= -3X, & [h, Y] &= 3Y; & [H, x] &= -x, & [H, y] &= y; \\
[x, Y] &= [X, y] = [h, H] = 0; & \quad & \quad & \\
[ad(x)]^4 X &= 0; & [ad(Y)]^2 x &= 0; \\
[ad(y)]^3 Y &= 0; & [ad(Y)]^2 x &= 0.
\end{align*}
\]

Taking Lie brackets of the vectors $x, y, h, X, Y, H$ we generate a complete set \{\(x_1, x_2, y_1, y_2, y_3, x_3\)\} of root vectors for $g_2$, which, together with the basis $h, H$ for the Cartan subalgebra form a basis for $g_2$ as follows:

\[
\begin{align*}
x_3 &= x, & X_1 &= X, & x_2 &= [x, X_1], & x_1 &= [x, x_2], & X_2 &= [x, x_1], & X_3 &= [X_1, X_2]; \\
y_3 &= y, & Y_1 &= Y, & y_2 &= [y, Y_1], & y_1 &= [y, y_2], & Y_2 &= [y, y_1], & Y_3 &= [Y_1, Y_2].
\end{align*}
\]

We label each root in the diagram with the corresponding root vector.
We end up with a “nice” basis wrt which the structure constants are particularly pleasant; they are integers and have symmetry properties which facilitate greatly the work involved in their determination; you can also apply some elementary \( \mathfrak{sl}_2 \) representation theory that further facilitate the calculation; it helps to work with the root diagram nearby.

**Symmetry properties of the structure constants.** Suppose \( \alpha, \beta \) are two roots such that \( \alpha + \beta \) is also a root. Let \( E_\alpha, E_\beta \) be the corresponding root vectors, as chosen above. Then \( [E_\alpha, E_\beta] = c_{\alpha,\beta}E_{\alpha+\beta} \), for some non-zero constant \( c_{\alpha,\beta} \in \mathbb{Z} \).

The nice feature of our base is that the structure constants satisfy

\[
c_{-\alpha,-\beta} = -c_{\alpha,\beta}.
\]

This cuts in half the amount of work involved, since you need only consider say \( \alpha > 0 \) (the positive roots are the six dots in the last root diagram marked with \( x \)'s and \( X \)'s). Combining this with the obvious \( c_{\alpha,\beta} = -c_{\beta,\alpha} \) (antisymmetry of Lie bracket) you obtain

\[
c_{\alpha,-\beta} = c_{\beta,-\alpha}.
\]

This cuts in half again the amount of work.

**Proposition 6.** The structure constants of \( \mathfrak{g}_2 \), with respect to the basis of root vectors \( \{x_i, X_i, y_i, Y_i | i = 1, 2, 3\} \) and the Cartan algebra elements \( \{h, H\} \) are given as follows. The basis elements are grouped in three sets: positive (three \( x \)'s and three \( X \)'s), negative (three \( y \)'s and three \( Y \)'s), and Cartan subalgebra elements (\( h \) and \( H \)).

- **[Positive, positive]:** other then the ones given above, and those which are zero for obvious reasons from the root diagram (sum of roots which is not a root):
  \[
  [x_1, x_2] = X_3.
  \]

- **[Positive, negative]:**

| \( c_{\alpha,\beta} \) | \( y_1 \) | \( y_2 \) | \( y_3 \) | \( Y_1 \) | \( Y_2 \) | \( Y_3 \) |
|-----------------|------|------|------|------|------|------|
| \( x_1 \)      | 1    | 4    | -4   | 0    | 12   | -12  |
| \( x_2 \)      | 4    | 1    | -3   | 1    | 0    | 3    |
| \( x_3 \)      | -4   | -3   | 1    | 0    | -3   | 0    |
| \( X_1 \)      | 0    | 1    | 0    | 1    | 0    | -1   |
| \( X_2 \)      | 12   | 0    | -3   | 0    | 1    | 36   |
| \( X_3 \)      | -12  | 3    | 0    | -1   | 36   | 1    |

The 1’s on the diagonal stand for the relations \( [x_i, y_i] = h_i, \) \( [X_i, Y_i] = H_i \), where, in terms of our basis \( \{h, H\} \) for the Cartan subalgebra,

\[
  h_1 = 8h + 12H, \quad h_2 = h + 3H, \quad h_3 = h,
\]

\[
  H_1 = H, \quad H_2 = 36(h + H), \quad H_3 = 36(h + 2H).
\]
• [Cartan, anything]: this is coded directly by the root diagram:
  - \( \text{ad}(x) \) has eigenvalues and eigenvectors

| eigenvalue | 3 | 2 | 1 | 0 | -1 | -2 | -3 |
|------------|---|---|---|---|----|----|----|
| eigenvectors | \( X_2, Y_1 \) | \( x_3 \) | \( x_1, y_2 \) | \( X_3, Y_3, h, H \) | \( x_2, y_1 \) | \( y_3 \) | \( X_1, Y_2 \) |

- \( \text{ad}(X) \) has eigenvalues and eigenvectors

| eigenvalue | 2 | 1 | 0 | -1 | -2 |
|------------|---|---|---|----|----|
| eigenvectors | \( X_1 \) | \( X_3, x_2, y_3, Y_2 \) | \( x_1, y_1, h, H \) | \( X_2, x_3, y_2, Y_3 \) | \( Y_1 \) |

**Proof.** This is elementary, using only the Jacobi identity, but takes time. We will give as a typical example the calculation of \([x_1, x_2]\):

\[
[x_1, x_2] = [x_1, [x, X]] = [x, [x_1, X]] + [X, [x, x_1]] = [X, [x, x_1]] = [X, X] = X_3
\]

(by defnition of \(x_2\))

(by Jacob identity)

since \([x_1, X] = 0\)

(by defnitions of \(X_2, X_3\)).

The rest of the relations are derived in a similar fashion. \(\square\)

Now we are ready to define the generators of the Lie algebra of a maximal compact subgroup \(K \subset G_2\). Let

\[
L_1 = X_1 - Y_1, \quad L_2 = \frac{X_2 - Y_2}{6}, \quad L_3 = \frac{X_3 - Y_3}{6},
\]

\[
S_1 = \frac{x_1 - y_1}{4}, \quad S_2 = \frac{x_2 - y_2}{2}, \quad S_3 = \frac{x_3 - y_3}{2}.
\]

Using the commutation relations of the last Proposition one checks easily that

\[
[L_i, L_j] = \epsilon_{ijk} L_k, \quad [L_i, S_j] = \epsilon_{ijk} S_k, \quad [S_i, S_j] = \epsilon_{ijk} \left(\frac{3}{4} L_k - S_k\right).
\]

Note: the strange-looking coefficients 2, 4, 6 in the definition of the \(L_1, S_i\) are chosen precisely so that we get these pleasing commutation relations.

**Appendix C. The Rolling Distribution in Cartan’s Thesis**

C.1. Cartan’s constructions and claims. In E. Cartan’s thesis [6], p.146, we find the following constructions: consider \(V = \mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}\) with coordinates \((x, y, z)\), where \(x, y \in \mathbb{R}^3, z \in \mathbb{R}\), and the following 15 linear vector fields (hence linear operators) on \(V\):

- \(X_i = -x_i \partial_{x_i} + y_i \partial_{y_i} + \frac{1}{4} \sum_{j=1}^3 (x_j \partial_{x_j} - y_j \partial_{y_j}), \ i = 1, 2, 3\).
- \(X_0 = 2z \partial_z - y_2 \partial_{y_2} + x_2 \partial_{x_2}, (ijk) \in A_3 = \{(123), (231), (312)\}\).
- \(X_0 = -2z \partial_z - y_2 \partial_{y_2} + x_2 \partial_{x_2} - y_2 \partial_{y_2}, (ijk) \in A_3,\)
- \(X_{ij} = -x_j \partial_{x_i} + y_i \partial_{y_j}, i \neq j, i, j = 1, 2, 3\).

Cartan makes the following claims without proof:

1. The linear span of these 15 operators is a 14 dimensional Lie subalgebra \(\mathfrak{g} \subset \text{End}(V)\) isomorphic to \(\mathfrak{g}_2\).
2. \(\mathfrak{g}\) preserves the quadratic form on \(V\) given by

\[
J = z^2 + x \cdot y.
\]

3. The linear group \(G \subset \text{GL}(V)\) generated by \(\mathfrak{g}\) acts transitively on the projectivized null cone of \(J\).
(4) $G$ preserves the system of 6 Pfaffian equations on $V$, given by the 6 components of

$$
\begin{align*}
\alpha &= zdx - xdz + y \times dy = 0, \\
\beta &= zdy - ydz + x \times dx = 0,
\end{align*}
$$

which have as a consequence

$$
\begin{align*}
\gamma_1 &= zdz + x \cdot dy = 0, \\
\gamma_2 &= zdz + y \cdot dx = 0.
\end{align*}
$$

(5) $G$ preserves a 5 parameter family of 3 dimensional linear subspaces of $V$, contained in the null cone of $J$,

$$
\begin{align*}
\{ x - za + b \times y = 0, \\
y - zb + a \times x = 0 \}
\end{align*}
$$

where $a \cdot b + 1 = 0$.

Our goal in this appendix is to sketch proofs of these claims, provide a minor correction in one place, relate Cartan’s construction to the octonions, and show how they contain, in essence, the construction of the rolling distribution $\tilde{Q}$ via projective geometry, as in the proposition 5 from section 5.

C.2. Relation with Octonions.

Recall the basis $e_i, f_i, U$ of section 4 for $V$ (imaginary split octonions) with its consequent multiplication table. Make the change of basis $e_i \mapsto - e_i$, keeping $f_i, U$ as they were, thus changing the signs of some entries of the multiplication table. Use this new basis $E_i = - e_i, f_i, U$ to identify $V$ with $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ by setting $(x, y, z) = \Sigma x_i E_i + \Sigma y_i f_i + z U \in V$. Referring to the multiplication table we compute

$$
(x, y, z)(x', y', z') = (-y \times y' - z x' + z' x, x \times x' + z y' - z' y, \frac{1}{2} (x \cdot y' - x' \cdot y)) + 1\{zz' + \frac{1}{2} (x \cdot y' - x' \cdot y)\}.
$$

The last term is in the real part of the split octonions, and not in $V$. It follows from this formula that $(x, y, z)^2 = J$, of Cartan’s claim 2 in the preceding paragraph. Multiplying out $(x, y, z)(dx, dy, dz)$ we find that

$$
(x, y, z)(dx, dy, dz) = (\alpha, \beta, \frac{1}{2}(\gamma_1 - \gamma_2)) + 1(\frac{1}{2}(\gamma_1 + \gamma_2)),
$$

where $\alpha, \beta, \gamma_1, \gamma_2$ are as in Cartan’s claim 4 of the previous paragraph. It follows that the any element of $G_2 = Aut(\tilde{O})$ preserves $J$ and preserves the Pfaffian system of Cartan’s claim 4. The distribution $D$ defined by this system is, upon restriction to the null cone $\{J = 0\} \setminus \{0\}$, precisely the distribution $D$ which we defined in the final section of our paper: $D(x, y, z) := \{ (a, b, c) : (x, y, z)(a, b, c) = 0 \}$. It follows that Cartan’s construction, pushed down to the space of rays using the $\mathbb{R}^+$-action, yields precisely our $\tilde{Q}$.

C.3. Commentary and proofs of Cartan’s claims.
Definition of $g_2$. The Cartan subalgebra. The first 3 operators of claim 1 are linearly dependent since $\sum X_{ii} = 0$. This is the only linear relation (proof below) and explains why $g$ is 14 dimensional and not 15 dimensional. The flows of $3X_{ii}$ generate the scalings $x_i \mapsto \lambda_1^{\alpha_i} \lambda_2^{\beta_i} \lambda_3^{\gamma_i} x_i$ $y_i \mapsto \lambda_1^{-\alpha_i} \lambda_2^{-\beta_i} \lambda_3^{-\gamma_i} y_i$ $z \mapsto z$ as described in section 5. Hence these operators should span the Cartan of $g = g_2$.

**Proposition 7.** $g$ is a 14 dimensional Lie subalgebra of $\text{End}(V)$, isomorphic to $g_2$, with Cartan subalgebra as just described.

**Proof.** It is convenient to put $g$ in block matrix form. For each $u \in \mathbb{R}^3$ let $R_u \in \text{End}(\mathbb{R}^3)$ be given by $v \mapsto u \times v$; i.e.

$$R_u = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$ 

Define the linear map $\rho : \mathfrak{sl}_3(\mathbb{R}) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \text{End}(V)$ by

$$\rho(A, b, c) = \begin{pmatrix} A & R_c & 2b \\ -R_b & -A^t & -2c \\ c^t & -b^t & 0 \end{pmatrix}.$$ 

Now $\rho$ is clearly injective, hence its image is a 14 dimensional linear subspace of $\text{End}(V)$. Denote the components of $A, b, c$ by $a_{ij}, b_i, c_i$ (resp.), then it is easy to check that

$$\rho(A, b, c) = -\sum_{i,j} a_{ij} X_{ij} + \sum_i b_i X_{i0} + \sum_i c_i X_{0i}.$$ 

This shows that $g$ is the image of $\rho$ and hence a 14 dimensional subspace of $\text{End}(V)$.

To show that $g$ is a lie algebra one calculates that

$$[\rho(A, b, c), \rho(A', b', c')] = \rho(A'', b'', c''),$$

where

$$A'' = [A, A'] + 3(bc'^t - b'c^t) - [b \cdot c' - b' \cdot c]I,$$

$$b'' = Ab' - A'b - 2c \times c',$$

$$c'' = -A'c' + A''c + 2b \times b'.$$

These formulae show that $\{\rho(A, 0, 0) | A \in \mathfrak{sl}_3(\mathbb{R})\}$ forms a lie subalgebra of $g$ isomorphic to $\mathfrak{sl}_3(\mathbb{R})$. This subalgebra corresponds to the sum of the long root spaces in the root diagram, and the Cartan subalgebra (the sum of the $X_{ij}$) as identified earlier. The formulae also show that the images of the $\rho(0, b, 0)$ and $\rho(0, 0, c)$ are stable under the adjoint action of the Cartan, hence they must correspond to the remaining short roots.

A tedious computation now yields the root diagram and the structure constants of $g_2$. 
C.3.2. Invariance of $J$. Let $G_2 \subset \text{GL}_7(\mathbb{R})$ be the subgroup generated by $g$.

**Proposition 8.** $J$ is $G_2$-invariant.

**Proof.** This is equivalent to showing that every $X \in g$ is $J$-antisymmetric, i.e. that $X$ anti-commutes with
\[
\begin{pmatrix}
0 & I/2 & 0 \\
I/2 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

One now checks easily that the set of $J$-antisymmetric matrices consists of the matrices of the form
\[
\begin{pmatrix}
A & R_c & 2b \\
-R_b & -A^t & -2\tilde{c} \\
\tilde{c}^t & -\tilde{b}^t & 0
\end{pmatrix},
\]
where $A \in \text{End}(\mathbb{R}^3)$ and $b, \tilde{b}, c, \tilde{c} \in \mathbb{R}^3$. Looking at the formula for $\rho(A, b, c)$ we see that $g$ is the subset of the $J$-antisymmetric matrices satisfying $\text{tr}A = 0, b = \tilde{b}, c = \tilde{c}$ (a codimension 7 condition). $\square$

C.3.3. Invariance of the Pfaffian system.

**Generalities.** A “Pfaffian system” on a manifold $M$ is given locally by the common kernels of a finite set of 1-forms, $\alpha_1 = \ldots = \alpha_m = 0$.

Two sets of 1-forms $\{\alpha_1, \ldots, \alpha_m\}, \{\beta_1, \ldots, \beta_n\}$, give equivalent systems if one can express each element of one set as a linear combination (with coefficients in $C^\infty(M)$) of the elements of the other set. We write this as
$\alpha_i \equiv 0 \mod \beta_1, \ldots, \beta_n, \quad i = 1, \ldots, m,$
and similarly for the $\beta$’s.

Consequently, if we want to prove that a system is preserved by some diffeomorphism $f : M \to M$ we must show that
$\alpha_i \equiv 0 \mod \beta_1, \ldots, \beta_n, \quad i = 1, \ldots, m,$
and if we want to show that the flow of some vector field $X$ on $M$ preserves the system we must show that
$\mathcal{L}_X \alpha_i \equiv 0 \mod \alpha_1, \ldots, \alpha_m, \quad i = 1, \ldots, m.$

Given such a system we can consider the common kernels $D_x \subset T_x M$ of the 1-forms at each point $x \in M$. This is well defined independently of the 1-forms chosen to represent the system. If $\dim D_x$ (the rank of the system) is constant we obtain a distribution $D \subset TM$ (a subbundle of the tangent bundle). But the rank may vary. For example, the system on $\mathbb{R}$ given by $xdx = 0$ has rank 1 at $x = 0$ and rank 0 for $x \neq 0$. However, if $G$ acts on $M$ preserving a Pfaffian system, then the rank must clearly be constant along the $G$-orbits.

**Cartan’s Pfaffian system. Rank jumps. A correction.** Due to jumping of rank, as discussed in the last remark, the Pfaffian system which Cartan defined by the vanishing of the 6 components of $\alpha, \beta$ cannot be $G_2$ invariant, even when restricted to $\tilde{C}$, the $J$ null cone. For at $(e_1, 0, 0)$ the system reduces to $dx_2 =
\[ dx_3 = dz = 0 \] and so has rank 4. On the other hand, at the point \((e_1, e_2, 0)\) the system is equivalent to \(dy_1 = dx_2 = dz - dy_3 = dz + dx_3 = 0\), and so has rank 3. And both points lie in \(\bar{C} \setminus \{0\}\), which is a single \(G_2\)-orbit, contradicting \(G_2\) invariance. A related problem with Cartan’s claim 4 of subsection \(\text{C.1}\) is his claim that \(\gamma_1 = \gamma_2 = 0\) is a consequence of \(\alpha = \beta\). But this is true only on the \(z \neq 0\) part of \(C\).

Both errors are fixed by imposing the extra equation \(\gamma := \gamma_1 - \gamma_2 = 0\). Then, as in section \(\text{C.2}\), we do obtain a \(G_2\)-invariant system on \(V\). Furthermore, as proved immediately below, the two equations \(\gamma_1 = \gamma_2 = 0\) are indeed a consequence of \(\alpha = \beta = 0, \gamma = 0\) on \(\bar{C}\), and are a consequence \(\alpha = \beta = 0\) on the subset \(z \neq 0\) of \(\bar{C}\). So Cartan’s claim is correct on the open dense set \(z \neq 0\) of the null cone \(\bar{C} \subset V\).

(See also page 11 of Bryant’s paper on Geometric Duality \([\text{B}]\), where he adds the equation \(\gamma = 0\) to \(\alpha = \beta = 0\).)

**Proposition 9.** The Pfaffian system on \(V\) given by \(\alpha = \beta = 0, \gamma = 0\) is \(G_2\)-invariant. On \(\bar{C}\) the system is equivalent to \(\alpha = \beta = 0, \gamma_1 = \gamma_2 = 0\). On the subset \(z \neq 0\) of \(\bar{C}\) it is equivalent to \(\alpha = \beta = 0\).

**Proof.** We prove the claims of the last two sentences first. Note that \(\gamma_1 + \gamma_2 = dJ\).

It follows that on \(\bar{C}\), where \(J = 0\), we have that \(\gamma_1 = \gamma_2 = 0\) is a consequence of \(\gamma := \gamma_1 - \gamma_2 = 0\). Thus, restricted to \(\bar{C}\), the system \(\alpha = \beta = 0, \gamma = 0\) is equivalent to \(\alpha = \beta = 0, \gamma_1 = \gamma_2 = 0\). Next, note that \(x \cdot \beta - y \cdot \alpha = z \gamma\). It follows that on \(z \neq 0\) the equation \(\gamma = 0\) is a consequence of \(\alpha = \beta = 0\).

It remains to establish invariance. We need to show that

\[ L_X \alpha_i \equiv L_X \beta_j \equiv L_X \gamma \equiv 0 \mod \alpha_i, \beta_j, \gamma, \]

for all \(X = \rho(A, b, c) \in \mathfrak{g}\). Divide into 3 cases, corresponding to \((A, 0, 0), (0, a, 0)\) and \((0, 0, b)\) in our coordinatization of \(\mathfrak{g}\).

- **case 1:** \(X = \rho(A, 0, 0), A \in \text{sl}_3(\mathbb{R})\).

**Lemma 1.** If \(A \in \text{End}(\mathbb{R}^3)\) and \(u, v \in \mathbb{R}^3\), then

\[ A(u \times v) + A'u \times v + u \times A'v = \text{tr}(A(u \times v)). \]

**Proof.** Sketch: divide in 2 cases. If \(A' = -A\) then \(\text{tr}A = 0\) and the identity is a consequence of the fact the \(SO_3\) preserves de cross product and that \(so_3\) are the antisymmetric matrices. If \(A' = A\) then can assume w.l.o.g. that \(A\) is diagonal and do an explicit easy calculation.

Now since

\[ X(x, y, z) = (Ax, -A'y, 0), \quad \alpha = zd\bar{x} - x \cdot dy, \]

we get, using the lemma and \(\text{tr}A = 0\), that

\[ L_X \alpha = zd\bar{x} - x \cdot dy = A(\alpha) \equiv 0 \mod \alpha. \]

Similarly, \(L_X \beta = -A' \beta \equiv 0 \mod \beta\).

Finally, \(L_X \gamma = (Ax) \cdot dy - x \cdot (A' dy) = 0\).

- **case 2:** \(X = \rho(0, b, 0), b \in \mathbb{R}^3\).

Here

\[ X(x, y, z) = (2bz, -b \times x, -b \cdot y), \]
and one calculates that
\[ \mathcal{L}_X \alpha = b \gamma, \quad \mathcal{L}_X \beta = b \times \beta, \quad \mathcal{L}_X \gamma = b \cdot \beta. \]

- case 3: \( X = \rho(0,0,c), c \in \mathbb{R}^3 \). The proof for this case is very similar to the previous case. Just interchange \( x \) and \( y \), and \( b \) and \( c \).

This completes the proof of invariance, and hence the proof of the proposition.

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