ON THE HÖPF BOUNDARY LEMMA FOR QUASILINEAR PROBLEMS
INVOLVING SINGULAR NONLINEARITIES AND APPLICATIONS

FRANCESCO ESPOSITO* AND BERARDINO SCIUNZI*

ABSTRACT. In this paper we consider positive solutions to quasilinear elliptic problem with singular nonlinearities. We provide a Höpf type boundary lemma via a suitable scaling argument that allows to deal with the lack of regularity of the solutions up to the boundary.

1. Introduction

We deal with positive weak solutions to the singular quasilinear elliptic problem:

\begin{equation}
\begin{aligned}
-\Delta_p u &= \frac{1}{u^\gamma} + f(u) \quad \text{in } \Omega \\
u &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\end{equation}

where \( p > 1, \gamma > 1, \Omega \) is a \( C^{2,\alpha} \) bounded domain of \( \mathbb{R}^N \) with \( N \geq 1 \) and \( f : \Omega \to \mathbb{R} \) locally Lipschitz continuous. A key point to have in mind in the study of semilinear or quasilinear problems involving singular nonlinearities is the fact that the source term loses regularity at zero, namely the problem is singular near the boundary. As a first consequence, solutions are not smooth up to the boundary (see [21]) and the gradient generally blows up near the boundary in such a way that \( u \notin W^{1,p}_0(\Omega) \). Therefore, here and in all the paper, we mean that \( u \in C^{1,\alpha}(\Omega) \) is a solution to \((\mathcal{P})\) in the weak distributional meaning according to Definition 2.1. Existence and uniqueness results regarding problem \((\mathcal{P})\) can be found e.g. in [3, 5, 6, 8, 9, 10, 19, 24, 25].

In this setting we prove here a Höpf type boundary lemma regarding the sign of the derivatives of the solution near the boundary and in the interior of the domain. To state our result we need some notation thus we shall denote with \( I_\delta(\partial\Omega) \) a neighborhood of \( \partial\Omega \) with the unique nearest point property (see e.g. [16]). For \( x \in I_\delta(\partial\Omega) \) we denote by \( \hat{x} \in \partial\Omega \) the point such that \( |x - \hat{x}| = \text{dist}(x, \partial\Omega) \) and we set:

\begin{equation}
\eta(x) := \frac{x - \hat{x}}{|x - \hat{x}|}.
\end{equation}

With this notation we have the following:

**Theorem 1.1.** Let \( u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega}) \) be a positive solution to \((\mathcal{P})\). Then, for any \( \beta > 0 \), there exists a neighborhood \( I_\delta(\partial\Omega) \) of \( \partial\Omega \), such that

\begin{equation}
\partial_{\nu(x)} u > 0 \quad \forall x \in I_\delta(\partial\Omega)
\end{equation}

whenever \( \nu(x) \in \mathbb{R}^N \) with \( \|\nu(x)\| = 1 \) and \( (\nu(x), \eta(x)) \geq \beta \).

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The technique of E. H"{o}pf [18] (see also [20]) has been already developed, and improved, in the quasilinear setting. We refer the readers to [26] (see also [29]). Nevertheless the proof of Theorem 1.1, namely the proof of the lemma in the case when it appears the singular term \( u^{-\gamma} \), cannot be carried out in the standard way mainly because the solutions are not of class \( C^1 \) up to the boundary. More precisely the proofs in [18, 20, 26, 29] has the common feature of basing on the comparison of the solution with subsolutions that have a known behaviour on the boundary. This approach, with some difficulty to take into account, can be exploited also in the singular case since \( u^{-\gamma} \) has the right monotonicity behaviour. This actually leads to control the behaviour of the solution near the boundary with a comparison based on the distance function. This is in fact also behind our Theorem 3.2. Although some of the underlying ideas in our approach have a common flavour with the ones exploited in [7], the proofs that we exploit are new and adapted to the degenerate nonlinear nature of the \( p \)-laplacian.

We are mainly concerned with the study of the sign of the derivatives near the boundary. Such a control is generally deduced a posteriori, by contradiction, assuming that the solution is \( C^1 \) up to the boundary. In our setting this is not a natural assumption and we develop a different technique that in any case exploits very basic arguments of common use. In fact we carry out a scaling argument near the boundary that leads to a limiting problem in the half space.

\[
\begin{aligned}
-\Delta_p u &= \frac{1}{u^\gamma} \quad \text{in } \mathbb{R}^N_+ \\
u &> 0 \quad \text{in } \mathbb{R}^N_+ \\
u &= 0 \quad \text{on } \partial \mathbb{R}^N_+ 
\end{aligned}
\]

where \( p > 1, \gamma > 1, N \geq 1, \mathbb{R}^N_+ := \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0 \} \) and \( u \in C^{1,\alpha}(\mathbb{R}^N_+) \cap C(\mathbb{R}^N_+) \).

Our scaling argument leads in fact to the study of a limiting profile which is a solution to \( (P_1) \) and obeys to suitable decay assumptions. It is therefore crucial for our technique, and may also have an independent interest, the following classification result:

**Theorem 1.2.** Let \( \gamma > 1 \) and let \( u \in C^{1,\alpha}(\mathbb{R}^N_+) \cap C(\mathbb{R}^N_+) \) be a solution to problem \( (P_1) \) such that

\[
 c x^\beta_N \leq u(x) \leq \overline{c} x^\beta_N \quad \text{with } \beta := \frac{p}{\gamma + p - 1}
\]

and \( c, \overline{c} \in \mathbb{R}^+ \). Then

\[
u(x) = u(x_N) = M x^\beta_N \quad \text{with } M := \left[ \frac{(\gamma + p - 1)^p}{p^{p-1} (p-1)(\gamma - 1)} \right]^{1/p-1}
\]

The H"{o}pf boundary lemma is a fundamental tool in many applications. We exploit it here to develop the moving plane method (see [1, 3, 17, 27]) for problem \( (P) \) obtaining the following:

**Theorem 1.3.** Let \( \Omega \) be a bounded smooth domain of \( \mathbb{R}^N \) which is strictly convex in the \( x_1 \)-direction and symmetric with respect to the hyperplane \( \{ x_1 = 0 \} \). Let \( u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega}) \) be a positive solution of problem \( (P) \) with \( f(s) > 0 \) for \( s > 0 \) \( (f(0) \geq 0) \). Then it follows that \( u \) is symmetric with respect to the hyperplane \( \{ x_1 = 0 \} \) and increasing in the \( x_1 \)-direction.
in $\Omega \cap \{x_1 < 0\}$.

In particular if the domain is a ball, then the solution is radial and radially decreasing.

The proof is based on the fact that the H"{o}pf type boundary lemma (Theorem 1.1) allows to exploit and improve the results in [12, 13]. The key point is the fact that the monotonicity of the solution near the boundary is provided by the H"{o}pf lemma and the moving plane procedure can be exploited working in the interior of the domain where the nonlinearity is no more singular. In the semilinear case, a similar result is proved in [8, 9].

For the reader’s convenience we sketch the proofs here below.

- In Section 2 we prove 1D-symmetry result in half spaces for problem (P), see Theorem 2.2. Mainly we develop a comparison principle to compare the solution $u$ and its translation $u_\tau := u(x - \tau e_1)$. Even if the source term is decreasing, a quite technical approach is needed because the operator is nonlinear and we are reduced to work in unbounded domains. The 1D-symmetry result obtained leads us to the study of a one dimensional problem in $\mathbb{R}^+$. We carry out this analysis proving a uniqueness result (see Proposition 2.4) that provides, as a corollary, the proof of Theorem 1.2.

- Section 3 is the core of the paper. We prove here Theorem 1.1 developing the scaling argument that leads to the problem in the half space. To this aim we strongly exploit the asymptotic estimates deduced in Theorem 3.2. The proof follows by contradiction thanks to the classification result Theorem 1.2.

- Finally, in Section 3, we apply our H"{o}pf type boundary lemma to prove the symmetry and monotonicity result stated in Theorem 1.3. The proof is based on the joint use of the moving plane method and the monotonicity information near the boundary provided by Theorem 1.1 that allow to avoid the region where the problem is singular.

- In the Appendix we prove Lemma 2.3 that is a very useful tool in the ODE analysis. Moreover we run through again, in the quasilinear setting, the technique of [21] to provide asymptotic estimates for the solutions near the boundary in terms of the distance function, see Theorem 3.2. Finally we prove Lemma 3.1 that is a weak comparison principle in bounded domain that we used in the proof of Theorem 3.2.

2. 1D SYMMETRY IN THE HALF SPACE, ODE ANALYSIS AND PROOF OF THEOREM 1.2

Solutions to $p$-Laplace equations are generally of class $C^{1,\alpha}$, see [14, 28]. Therefore a solution to (2) has to be understood in the weak distributional meaning taking into account the singular nonlinearity. We state the following:

**Definition 2.1.** We say that $u \in W_{loc}^{1,p}(\Omega) \cap C(\overline{\Omega})$ is a (positive) weak solution to problem (2) if

$$
\int_{\Omega} |\nabla u|^{p-2}(\nabla u, \nabla \varphi) \, dx = \int_{\Omega} \frac{\varphi}{u^\gamma} \, dx + \int_{\Omega} f(u)\varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega).
$$

We say that $u \in W_{loc}^{1,p}(\Omega) \cap C(\overline{\Omega})$ is a weak subsolution of problem (2) if

$$
\int_{\Omega} |\nabla u|^{p-2}(\nabla u, \nabla \varphi) \, dx \leq \int_{\Omega} \frac{\varphi}{u^\gamma} \, dx + \int_{\Omega} f(u)\varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega), \varphi \geq 0.
$$
Similarly, we say that \( u \in W^{1,p}_\text{loc}(\Omega) \cap C(\overline{\Omega}) \) is a weak supersolution of problem \((P)\) if
\[
\int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla \varphi) \, dx \geq \int_\Omega f(u) \varphi \, dx + \int_\Omega f(u) \varphi \, dx \quad \forall \varphi \in C^\infty_c(\Omega), \varphi \geq 0.
\]

In the following we further use the following inequalities: \( \forall \eta, \eta' \in \mathbb{R}^N \) with \( |\eta| + |\eta'| > 0 \)
there exists positive constants \( C_1, C_2, C_3, C_4 \) depending on \( p \) such that
\[
|\eta|^{p-2} \eta - |\eta'|^{p-2} \eta'| \geq C_1 (|\eta| + |\eta'|)^{p-2} |\eta - \eta'|^2,
\]
\[
\|\eta|^{p-2} \eta - |\eta'|^{p-2} \eta'| \leq C_2 (|\eta| + |\eta'|)^{p-2} |\eta - \eta'|,
\]
\[
|\eta|^{p-2} \eta - |\eta'|^{p-2} \eta'| \geq C_3 |\eta - \eta'|^p \quad \text{if} \quad p \geq 2,
\]
\[
\|\eta|^{p-2} \eta - |\eta'|^{p-2} \eta'| \leq C_4 |\eta - \eta'|^{p-1} \quad \text{if} \quad 1 < p \leq 2.
\]

**Theorem 2.2.** Let \( \gamma > 1 \) and let \( u \in C^{1,\alpha}(\mathbb{R}_+^N) \cap C(\overline{\mathbb{R}_+^N}) \) be a solution to problem \((P)\)
such that
\[
x_\nu^\beta \leq u(x) \leq \overline{x}_N^\beta \quad \forall x \in \mathbb{R}_+^N
\]
with \( \beta := \frac{p}{\gamma + p - 1} \). Then
\[
u_x, \equiv 0
\]
for every \( i = 1, \ldots, N - 1 \). Namely, \( u(x) = u(x_N) \).

**Proof of Theorem 2.2.** We start with a gradient estimate showing that
\[
|\nabla u(x)| \leq \frac{C_b}{x_N^{\gamma + p - 1}}.
\]

To prove this fact we use the notation \( x = (x_1, \ldots, x_N) = (x', x_N) \in \mathbb{R}^N \) and, with no loss of generality we consider a point \( P_c := (0', x_N^c) \). Setting
\[
w(x) := \frac{u(x_N^c \cdot x)}{(x_N^c)^\beta}
\]
it follows that
\[
- \Delta w = \frac{1}{w^\gamma} \quad \text{in} \ \mathbb{R}_+^N.
\]

We restrict our attention to the problem
\[
\begin{cases}
- \Delta w = \frac{1}{w^\gamma} & \text{in} \ B_{1/2}(0', 1) \\
w > 0 & \text{in} \ B_{1/2}(0', 1)
\end{cases}
\]
so that, by \((2.9)\), it follows that \( w \) is bounded and \( \frac{1}{w^\gamma} \in L^\infty(B_{1/2}(0', 1)) \). Therefore, by standard \( C^{1,\alpha} \) estimates \(1428\), we deduce that
\[
\|w\|_{C^{1}(B_{1/4}(0', 1))} \leq C_b.
\]

Scaling back we get \((2.11)\).
Arguing by contradiction, without loss of generality, we assume that there exists $P_0 \in \mathbb{R}^N$ such that $u_{x_1}(P_0) > 0$. Hence there exists $\delta > 0$ sufficiently small such that $u_{x_1}(x) > 0$ for all $x \in B_\delta(P_0)$. Now we define
\begin{equation}
(2.14) \quad u_\tau(x) := u(x - \tau e_1)
\end{equation}
where $0 < \tau < \delta$. Hence by the Mean Value Theorem it follows that
\begin{equation}
(2.15) \quad u(P_0) - u_\tau(P_0) = u_{x_1}(\xi)\tau > \hat{C}\tau > 0
\end{equation}
where $\xi \in \{tP_0 + (1-t)(P_0 - \tau e_1), t \in [0,1]\}$. Moreover, there exists $k > 0$ sufficiently large such that, by the Mean Value Theorem and $(2.11)$, we have
\begin{equation}
(2.16) \quad |u - u_\tau| \leq \frac{\hat{C}\tau}{x_N^{p-1}} \quad \text{in } \mathbb{R}_+^N \cap \{x_N \geq k\}.
\end{equation}
Now we set
\begin{equation}
(2.17) \quad S := \sup_{x \in \mathbb{R}_+^N} (u - u_\tau) > 0.
\end{equation}
We also note that $S < +\infty$ by $(2.9)$ and $(2.16)$. Let us consider
\begin{equation}
(2.18) \quad w_{\tau,\varepsilon}(x) := [u - u_\tau - (S - \varepsilon)]^+
\end{equation}
for every $\varepsilon > 0$ small enough. We notice that, by $(2.9)$ and $(2.16)$,
\begin{equation}
(2.19) \quad \text{supp}(w_{\tau,\varepsilon}) \subset \subset \{\hat{k} \leq x_N \leq \hat{K}\}
\end{equation}
for some $\hat{k}, \hat{K} > 0$. We consider a standard cutoff function $\varphi_R := \varphi_R(x')$ such that $\varphi_R = 1$ in $B'_R(0)$, $\varphi_R = 0$ in $(B'_{2R}(0))^c$ and $|\nabla \varphi_R| \leq \frac{2}{R}$ in $B'_{2R}(0) \setminus B'_R(0)$, where $B'_R(0)$ denotes the $(N - 1)$-dimensional ball of center 0 and radius $R$.

We distinguish two cases:

**Case 1:** $1 < p < 2$. We set
\begin{equation}
(2.20) \quad \psi := w_{\tau,\varepsilon}^\alpha \varphi_R^2
\end{equation}
where $\alpha > 0$, $w_{\tau,\varepsilon}$ is defined in $(2.18)$ and $\varphi_R$ is the cutoff function defined here above. First of all we notice that $\psi$ belongs to $W_0^{1,p}(\mathbb{R}_+^N)$. By density argument we can take $\psi$ as test function in the weak formulation of problem $(P_1)$, see Definition 2.1 so that, subtracting the equation for $u$ and $u_\tau$, we obtain
\begin{align}
\alpha \int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla w_{\tau,\varepsilon}) w_{\tau,\varepsilon}^{\alpha-1} \varphi_R^2 \, dx
\end{align}
\begin{equation}
(2.21)
= -2 \int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} (|\nabla u|^{p-2}\nabla u - |\nabla u_\tau|^{p-2}\nabla u_\tau, \nabla \varphi_R) w_{\tau,\varepsilon}^\alpha \varphi_R \, dx
\end{equation}
\begin{align}
&+ \int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} \left( \frac{1}{u^\gamma} - \frac{1}{u_\tau^\gamma} \right) w_{\tau,\varepsilon}^\alpha \varphi_R^2 \, dx.
\end{align}
From (2.21), using (2.8) and the Mean Value Theorem, we obtain

\[
(2.22) \quad \alpha C_1 \int_{R^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \varphi_R^2 \, dx
\]

\[
\leq \alpha \int_{R^N_+ \cap \text{supp}(\psi)} (|\nabla u|^{p-2} \nabla u - |\nabla u_\tau|^{p-2} \nabla u_\tau, \nabla w_{\tau,\varepsilon}) w_{\tau,\varepsilon}^{\alpha-1} \varphi_R^2 \, dx
\]

\[
= -2 \int_{R^N_+ \cap \text{supp}(\psi)} (|\nabla u|^{p-2} \nabla u - |\nabla u_\tau|^{p-2} \nabla u_\tau, \nabla \varphi_R) w_{\tau,\varepsilon}^\alpha \varphi_R \, dx
\]

\[+ \int_{R^N_+ \cap \text{supp}(\psi)} \left( \frac{1}{u^\gamma} - \frac{1}{u_\tau^\gamma} \right) w_{\tau,\varepsilon}^\alpha \varphi_R^2 \, dx
\]

\[
\leq 2C_4 \int_{R^N_+ \cap \text{supp}(\psi)} |\nabla (u - u_\tau)|^{p-1} |\nabla \varphi_R| |w_{\tau,\varepsilon}^\alpha \varphi_R| \, dx - \gamma \int_{R^N_+ \cap \text{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w_{\tau,\varepsilon}^\alpha \varphi_R^2 \, dx
\]

where $\xi$ belongs to $\overline{(u, u_\tau)}$. Hence, recalling also (2.11), we deduce that

\[
(2.23) \quad \alpha C_1 \int_{R^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \varphi_R^2 \, dx
\]

\[
\leq 2C_4 \int_{R^N_+ \cap \text{supp}(\psi)} |\nabla (u - u_\tau)|^{p-1} |\nabla \varphi_R| |w_{\tau,\varepsilon}^\alpha \varphi_R| \, dx - \gamma \int_{R^N_+ \cap \text{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} w_{\tau,\varepsilon}^{\alpha+1} \varphi_R^2 \, dx
\]

\[
\leq \tilde{C} \int_{R^N_+ \cap \text{supp}(\psi)} |\nabla \varphi_R| w_{\tau,\varepsilon}^\alpha \, dx
\]

where $\tilde{C} := 2C_4 \| \nabla (u - u_\tau) \varphi_R \|_{L^{p-1}(R^N_+ \cap \text{supp}(\psi))}$. Exploiting the weighted Young inequality with exponents \(\left( \frac{\alpha + 1}{\alpha}, \, \alpha + 1 \right)\) we obtain

\[
(2.24) \quad \alpha C_1 \int_{R^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \varphi_R^2 \, dx
\]

\[
\leq \tilde{C} \int_{R^N_+ \cap \text{supp}(\psi)} |\nabla \varphi_R| w_{\tau,\varepsilon}^\alpha \, dx
\]

\[
\leq \tilde{C} \int_{R^N_+ \cap \text{supp}(\psi)} |\nabla \varphi_R|^{\alpha+1} \, dx + \frac{\tilde{C} \alpha}{\alpha + 1} \int_{R^N_+ \cap \text{supp}(\psi)} w_{\tau,\varepsilon}^{\alpha+1} \, dx
\]

\[
\leq \frac{\tilde{C}}{R^{\alpha-(N-2)}} + \frac{\tilde{C} \alpha}{\alpha + 1} \int_{R^N_+ \cap \text{supp}(\psi)} \left[ w_{\tau,\varepsilon}^{\alpha+1} \right] \, dx.
\]
From (2.24) and exploiting the Poincaré inequality in the $x_N$-direction it follows that

\[
\alpha C_1 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \varphi_R^2 \, dx \\
\leq \frac{\hat{C}}{R^{\alpha-(N-2)}} + \frac{\hat{C} \alpha}{\alpha + 1} \sigma \frac{\alpha}{\alpha + 1} \int_{B_{\delta R}(0)} \left( \int_{\{y \leq k\}} \left[ w_{\tau,\varepsilon}^\alpha \right]^2 \, dy \right) \, dx' \\
\leq \frac{\hat{C}}{R^{\alpha-(N-2)}} + \frac{\hat{C} \alpha}{\alpha + 1} \sigma \frac{\alpha}{\alpha + 1} C_p^2(k) \left( \frac{\alpha + 1}{2} \right)^2 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \, dx \\
\leq \frac{\hat{C} \alpha}{\alpha + 1} \sigma \frac{\alpha}{\alpha + 1} C_p^2(k) \left( \frac{\alpha + 1}{2} \right)^2 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{2(p-2)} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \, dx \\
+ \frac{\hat{C}}{R^{\alpha-(N-2)}}
\]

where $C_p$ is the Poincaré constant. Let us point out that, by (2.20), (2.19) and standard regularity theory [13] [28], it follows that

\[
|\nabla u| + |\nabla u_\tau| \leq C \quad \text{in supp}(w_{\tau,\varepsilon}) \subset \subset \{ \hat{k} \leq x_N \leq \hat{K} \}.
\]

Hence we have

\[
\int_{C(R)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \, dx \\
\leq \vartheta \int_{C(2R)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \, dx + \frac{\hat{C}}{R^{(N+2)}}
\]

where $C(R) := (\mathbb{R}^N_+ \cap (B_R(0) \times \mathbb{R}))$ and $\vartheta := \frac{\hat{C} \alpha}{\alpha + 1} \sigma \frac{\alpha}{\alpha + 1} C_p^2(k) \left( \frac{\alpha + 1}{2} \right)^2 ||(|\nabla u| + |\nabla u_\tau|)^{2-p}||_\infty$.

We set

\[
g(R) := \frac{\hat{C}}{R^{(N+2)}}
\]

and

\[
L(R) := \int_{C(R)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\varepsilon}|^2 w_{\tau,\varepsilon}^{\alpha-1} \, dx
\]

so that

\[
L(R) \leq \vartheta L(2R) + g(R).
\]

Now we fix $\alpha$ sufficiently large so that $g(R) \to 0$ as $n \to +\infty$ and, consequently, we take $\sigma$ small enough so that $\vartheta < 2^{(\alpha-\gamma)\beta+1}$. This allows to exploit Lemma 2.1 of [15] it follows that

\[
L(R) = 0
\]

for any $R > 0$. This proves that actually $w_{\tau,\varepsilon}$ is constant and therefore $w_{\tau,\varepsilon} = 0$ since it vanishes near the boundary. This is a contradiction with (2.15) thus proving the result in the case $1 < p < 2$.

**Case 2:** $p \geq 2$. We set

\[
\psi := w_{\tau,\varepsilon} \varphi_R^2
\]
with \( w_{\tau,\varepsilon} \) and \( \varphi_R \) defined as in the previous case \( 1 < p < 2 \). Arguing exactly as in the case \( 1 < p < 2 \) we arrive to

\[
\int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u|^p - |\nabla u_\tau|^p - \nabla w_{\tau,\varepsilon}) \varphi_R^2 \, dx
\]

\[\tag{2.29}\]

\[
= -2 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u|^p - |\nabla u_\tau|^p - \nabla \varphi_R \nabla \varphi) w_{\tau,\varepsilon} \varphi_R^2 \, dx
\]

\[+ \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} \left( \frac{1}{u^\gamma} - \frac{1}{u_\tau^\gamma} \right) w_{\tau,\varepsilon} \varphi_R^2 \, dx
\]

From (2.29), using (2.28) and the Mean Value Theorem, we deduce that

\[
C_1 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^p - |\nabla w_{\tau,\varepsilon}|^2 \varphi_R^2 \, dx
\]

\[
\leq \left| \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u|^p - |\nabla u_\tau|^p - \nabla \varphi_R \nabla \varphi) w_{\tau,\varepsilon} \varphi_R^2 \, dx
\]

\[+ \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} \left( \frac{1}{u^\gamma} - \frac{1}{u_\tau^\gamma} \right) w_{\tau,\varepsilon} \varphi_R^2 \, dx
\]

\[\leq 2C_2 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^p - |\nabla w_{\tau,\varepsilon}| \varphi_R |\nabla \varphi_R| \varphi_R \, dx
\]

\[- \gamma \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w_{\tau,\varepsilon} \varphi_R \varphi_R \, dx
\]

where \( \xi \) belongs to \( (u, u_\tau) \). Exploiting the Young inequality to the right hand side we have

\[
C_1 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^p - |\nabla w_{\tau,\varepsilon}|^2 \varphi_R^2 \, dx
\]

\[
\leq 2C_2 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^p - |\nabla w_{\tau,\varepsilon}| |\nabla \varphi_R| \varphi_R \, dx
\]

\[- \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w_{\tau,\varepsilon} \varphi_R \varphi_R \, dx
\]

\[\leq \sigma C_2 \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^p - |\nabla w_{\tau,\varepsilon}|^2 \, dx
\]

\[+ \frac{C_2}{\sigma} \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^p - |\nabla \varphi_R|^2 \varphi_R^2 \varphi_R \, dx
\]

\[- \gamma \int_{\mathbb{R}^N_+ \cap \text{supp}(\psi)} \frac{1}{\xi^{\gamma+1}} (u - u_\tau) w_{\tau,\varepsilon} \varphi_R \varphi_R \, dx
\]
As above we shall exploit the fact that $|\nabla u|$ and $|\nabla u_\tau|$ are uniformly bounded in $\mathbb{R}_+^N \cap \text{supp}(\psi)$, see (2.26). Therefore we get

$$C_1 \int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\epsilon}|^2 \varphi_R^2 \, dx \leq \sigma C_2 \int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\epsilon}|^2 \, dx$$

$$+ \left( \frac{\bar{C}}{\sigma R^2} - \frac{\dot{C}}{\sigma R^2} \right) \int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} w_{\tau,\epsilon}^2 \varphi_R^2 \, dx \quad (2.32)$$

where $\bar{C}$ and $\dot{C}$ are positive constants. By taking $R_0 > 0$ sufficiently large it follows that $\frac{\bar{C}}{\sigma R^2} - \frac{\dot{C}}{\sigma R^2} < 0$ for every $R \geq R_0$. Hence we have

$$\int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\epsilon}|^2 \varphi_R^2 \, dx \leq \sigma C_2 \int_{\mathbb{R}_+^N \cap \text{supp}(\psi)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\epsilon}|^2 \, dx \quad (2.33)$$

As above, for $\mathcal{C}(R) := (\mathbb{R}_+^N \cap (B_R'(0) \times \mathbb{R}))$, we set

$$\mathcal{L}(R) := \int_{\mathcal{C}(R)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\epsilon}|^2 \, dx$$

so that

$$\mathcal{L}(R) \leq \vartheta \mathcal{L}(2R) .$$

where $\vartheta := \frac{2C_2}{C_1} > 0$ is sufficiently small when $\sigma > 0$ is sufficiently small. Applying again Lemma 2.1 of [15] it follows that

$$\int_{\mathcal{C}(R)} (|\nabla u| + |\nabla u_\tau|)^{p-2} |\nabla w_{\tau,\epsilon}|^2 \, dx = 0$$

for any $R \geq R_0$. This provides a contradiction exactly as in the case $1 < p < 2$ so that the thesis follows also in the case $p \geq 2$. \qed

The 1D-symmetry result in Theorem 2.2 leads to the study of the one dimensional problem:

$$\begin{cases}
 - (|u'|^{p-2}u')' = \frac{1}{u^\gamma} & t \in \mathbb{R}_+^+ \\
 u > 0 & t \in \mathbb{R}_+^+
\end{cases}$$

$$u(0) = 0 \quad (2.34)$$

where $\gamma > 1$ and $u \in C^{1,\alpha}(\mathbb{R}_+^+) \cap C(\mathbb{R}_+^+ \cup \{0\})$. As a consequence of the common feeling we expect uniqueness for such a problem, since the source term is decreasing. By the way the proof is not straightforward since the source term is decreasing but singular at zero. Now we present the following lemma, postponing the proof in the appendix.
Lemma 2.3. Let \( u \in C^{1,\alpha}(\mathbb{R}^+) \cap C(\mathbb{R}^+ \cup \{0\}) \) be a solution to (2.31). Assume that there exists a positive constant \( C_u \) such that

\[
\frac{t^\beta}{C_u} \leq u(t) \leq C_u t^\beta
\]

for \( t \) sufficiently large and \( \beta := \frac{p}{\gamma + p - 1} \). Then there exists a positive constant \( C'_u \) such that

\[
\frac{t^{\beta - 1}}{C'_u} \leq u'(t) \leq C'_u t^{\beta - 1}
\]

for \( t \) large enough.

Now we are ready to prove our uniqueness result:

Proposition 2.4. Problem (2.31) admits a unique solution \( u \in C^{1,\alpha}(\mathbb{R}^+) \cap C(\mathbb{R}^+ \cup \{0\}) \) satisfying (2.35) given by

\[
u(t) = Mt^\beta
\]

where \( M := \left[ \frac{(\gamma + p - 1)p}{p^p(p - 1)(\gamma - 1)} \right]^{1/p - 1} \) and \( \beta := \frac{p}{\gamma + p - 1} \).

Proof. Arguing by contradiction we assume that there exist two positive solutions \( u, v \in C^{1,\alpha}(\mathbb{R}^+) \cap C(\mathbb{R}^+ \cup \{0\}) \) to problem (2.31) such that \( u \neq v \). Let us consider the cutoff function \( \varphi_R \in C_c^\infty(\mathbb{R}) \), \( R > 0 \), such that \( \varphi_R(t) = 1 \) if \( t \in [-R,R] \), \( \varphi_R(t) = 0 \) if \( t \in (-\infty,-2R) \cup (2R,\infty) \) and \(|\varphi'(t)| < \frac{2}{R}\) for every \( t \in (-2R,-R) \cup (R,2R) \). For \( \varepsilon > 0 \) (small) we set

\[
w_\varepsilon = (u - v - \varepsilon)^+ \quad \text{and} \quad \psi = [(u - v - \varepsilon)^+]^\alpha \varphi_R^2
\]

with \( \alpha > 0 \) (large). Passing through the weak formulation of problem (2.31) for \( u \) and \( v \), subtracting and using standard elliptic estimates estimates and (2.8) we obtain

\[
\alpha C_1 \int_0^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \varphi_R^2 dt \leq \alpha \int_0^{2R} (|u'|^{p-2}u' - |v'|^{p-2}v', w_\varepsilon)w_\varepsilon^{\alpha-1} \varphi_R^2 dt
\]

\[
= -2 \int_R^{2R} (|u'|^{p-2}u' - |v'|^{p-2}v', \varphi_R)w_\varepsilon^{\alpha} \varphi_R dt + \int_0^R \left( \frac{1}{u'} - \frac{1}{v'} \right) w_\varepsilon^{\alpha} \varphi_R^2 dt
\]

\[
\leq 2C_2 \int_R^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 |w_\varepsilon|^{\alpha-1} \varphi_R \varphi_R dt - \gamma \int_R^{2R} \frac{1}{\xi^{\gamma+1}} (u - v) w_\varepsilon^{\alpha} \varphi_R^2 dt
\]

with \( \xi \in (u,v) \). Exploiting the weighted Young inequality to the right hand side we have

\[
\alpha C_1 \int_0^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \varphi_R^2 dt
\]

\[
\leq LC_2 \int_R^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} dt
\]

\[
+ \frac{C_2}{LR^2} \int_R^{2R} (|u'| + |v'|)^{p-2} w_\varepsilon^{\alpha+1} \varphi_R^2 dt
\]

\[
- \gamma \int_R^{2R} \frac{1}{\xi^{\gamma+1}} w_\varepsilon^{\alpha+1} \varphi_R^2 dt
\]
By Lemma 2.3 it follows that

\[ \alpha C_1 \int_0^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \varphi_R^2 \, dt \]
\[ \leq L C_2 \int_R^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \, dt \]
\[ + \frac{\hat{C}}{L} \int_R^{2R} t^{(\beta-1)(p-2)-2} w_\varepsilon^{\alpha+1} \varphi_R^2 \, dt \]
\[ - \hat{C} \int_R^{2R} t^{-\beta(\gamma+1)} w_\varepsilon^{\alpha+1} \varphi_R^2 \, dt \]
\[ \leq L C_2 \int_R^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \, dt \]
\[ + \left( \frac{\hat{C}}{L} - \hat{C} \right) \frac{1}{R^{\beta(\gamma+1)}} \int_R^{2R} w_\varepsilon^{\alpha+1} \varphi_R^2 \, dt \]  

(2.40)

where we also used the fact that \( t/2 \leq R \leq t \) when \( t \in [R, 2R] \). Now we fix \( L \) sufficiently large such that \( \left( \frac{\hat{C}}{L} - \hat{C} \right) \leq 0 \) so that

\[ \int_0^R (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \, dt \leq L C_2 \int_0^{2R} (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \, dt. \]

Hence we define

\[ \mathcal{L}(R) := \int_0^R (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \, dt. \]

By Lemma 2.3 we deduce that \( \mathcal{L}(\cdot) \) has polynomial growth, namely

\[ \mathcal{L}(R) \leq C R^{(\beta-1)(p-2) \gamma^2 - \beta \alpha - \gamma + 1} \int_0^R (|u'| + |v'|)^{p-2} |w_\varepsilon'|^2 w_\varepsilon^{\alpha-1} \, dt = C R^{R \sigma}. \]

with \( \sigma := (\alpha - \gamma) \beta + 1. \) We take \( \alpha > 0 \) sufficiently large so that \( \sigma > 0 \) and \( \frac{dC_1}{d\alpha} < 2^{-\sigma} \) so that Lemma 2.1 of [15] apply and shows that \( \mathcal{L}(R) = 0. \)

From this it follows that \( u \leq v + \varepsilon \) for every \( \varepsilon > 0, \) hence \( u \leq v. \) Arguing in the same way it follows that \( u \geq v \) and this proves the uniqueness result. To conclude the proof it is now sufficient to check that the function defined in (2.37) solve the problem. \( \square \)

**Proof of Theorem 1.2.** Once that Theorem 2.2 is in force, the proof of Theorem 1.2 is a consequence of Proposition 2.4. \( \square \)

3. Asymptotic analysis near the boundary and proof of Theorem 1.1

We start this section considering the auxiliary problem:

(3.42)
\[ \begin{cases} 
-\Delta_p u = \frac{p(x)}{u^\gamma} & \text{in } \mathcal{D} \\
\quad u > 0 & \text{in } \mathcal{D} 
\end{cases} \]

where \( \mathcal{D} \) is a bounded smooth domain of \( \mathbb{R}^N, \) where \( p \in L^\infty(\mathcal{D}) \) and \( p(x) \geq c > 0 \) a.e. in \( \mathcal{D}, \) \( \gamma > 1 \) and \( u \in W^{1,p}_{loc}(\mathcal{D}) \cap C^0(\bar{\mathcal{D}}). \) For this kind of problems, generally, the weak comparison
principle holds true. This is mainly due to the monotonicity properties of the source term. In spite of this remark, the proof is not straightforward when considering sub/super solutions that are not smooth up the boundary. Therefore we provide here below a self contained proof of a comparison principle that we shall exploit later on.

**Lemma 3.1.** Let \( u \in W^{1,p}_{loc}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}}) \) be a subsolution of problem \((3.42)\) in the sense of \((2.6)\) and let \( v \in W^{1,p}_{loc}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}}) \) be a supersolution of problem \((3.42)\) in the sense of \((2.7)\). Then, if \( u \leq v \) on \( \partial \mathcal{D} \) it follows that \( u \leq v \) in \( \mathcal{D} \).

The proof of this lemma is contained in the appendix. We exploit now Lemma 3.1 to study the boundary behaviour of the solutions to \((2.7)\). The proof is actually the one in [21]. Since we could not find an appropriate reference for the estimates that we need, we repeat the argument. We denote with \( \phi_1 \) the first (positive) eigenfunction of the \( p \)-laplacian in \( \Omega \). Namely

\[
(3.43) \quad \begin{cases} -\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial \Omega. \end{cases}
\]

Having in mind Lemma 3.1 we can prove a similar result to the one in [21], but in the quasilinear setting.

**Theorem 3.2.** Let \( u \in C^{1,\alpha}_{loc}(\Omega) \cap C(\overline{\Omega}) \) be a positive solution to \((2.7)\). Then there exist two positive constants \( m_1, m_2 \) and there exists \( \delta > 0 \) sufficiently small such that

\[
(3.44) \quad m_1 \phi_1(x)^{\frac{p}{p-1}} \leq u(x) \leq m_2 \phi_1(x)^{\frac{p}{p-1}} \quad \forall \ x \in I_\delta(\partial \Omega).
\]

We postpone the proof of Theorem 3.2 in the appendix. We are now ready to prove Theorem 1.1 exploiting the previous preliminary results.

**Proof of Theorem 1.1.** Since the domain is of class \( C^{2,\alpha} \) we may and do reduce to work in a neighborhood of the boundary \( I_\delta(\partial \Omega) \) where the unique nearest point property holds (see e.g. [16]). Arguing by contradiction, let us assume that there exists a sequence of points \( \{x_n\} \) in \( I_\delta(\partial \Omega) \), such that \( x_n \to x_0 \in \partial \Omega \), as \( n \to +\infty \), and

\[
(3.45) \quad \partial_\nu(x_n)u(x_n) \leq 0, \quad \text{with } (\nu(x_n), \eta(x_n)) \geq \beta > 0.
\]

Without loss of generality, we can assume that \( x_0 = 0 \in \partial \Omega \) and \( \eta(x_n) = e_N \). This follows by the fact that the \( p \)-Laplace operator is invariant under isometries. More precisely, for each \( n \in \mathbb{N} \), we can consider an isometry \( T_n : \mathbb{R}^N \to \mathbb{R}^N \) with the above mentioned properties just composing a translation and a rotation of the axes. This procedure generates a new sequence of points \( \{y_n\} \), where \( y_n := T_n x_n \), such that every \( y_n \in \text{span}\{e_N\} \) and \( y_n \to 0 \) as \( n \to +\infty \). Setting \( u_n(y) := u(T_n^{-1}(y)) \), it follows that

\[
(3.46) \quad -\Delta_p u_n = \frac{1}{u_n} + f(u_n) \quad \text{in } \Omega_n = T_n(\Omega).
\]

Now we set

\[
(3.47) \quad w_n(y) := \frac{u_n(\delta_n y)}{M_n}
\]

where \( \delta_n := \text{dist}(x_n, \partial \Omega) = \text{dist}(T_n x_n, 0) \) and \( M_n := u_n(\delta_n e_N) = u(x_n) \). It follows that \( \delta_n \to 0 \) as \( n \to +\infty \) and

\[
\text{as } n \to +\infty.
\]
- $w_n$ is defined in $\Omega^*_n := \Omega_n / \delta_n$.
- $w_n(c_N) = 1$.
- $M_n \to 0$, as $n \to +\infty$.

It is easy to see that $w_n$ weakly satisfies

\begin{equation}
- \Delta_p w_n = \frac{\delta_n^p}{M_n^{\gamma + p - 1}} \left( \frac{1}{w_n(y)} + M_n \gamma f(u_n(\delta_n y)) \right) \quad \text{in } \Omega^*_n.
\end{equation}

The key idea of the proof is to argue by contradiction exploiting a limiting profile, that we shall denote by $u_\infty$, which is a solution to a limiting problem in a half space. The contradiction will then follows applying the classification result in Theorem 1.2. Here below we develop this argument and we suggest to the reader to keep in mind that $f$ is bounded, the term $M_n^\gamma f(u_n(\delta_n y))$ will vanish since $M_n$ goes to zero and $\frac{\delta_n^p}{M_n^{\gamma + p - 1}}$ is bounded as a consequence of Theorem 3.2. Therefore the expected limiting equation is:

\begin{equation}
- \Delta_p w_\infty = \frac{\tilde{C}}{w_\infty} \quad \text{in } \mathbb{R}_+^N.
\end{equation}

Let us provide the details needed to pass to the limit. We claim that:

- $w_n \overset{C^{1,\alpha}}{\to} w_\infty$, as $n \to +\infty$, in any compact set $K$ of $\mathbb{R}_+^N$.
- $w_\infty \in C^{1,\alpha}(\mathbb{R}_+^N) \cap \overline{C(\mathbb{R}_+^N)}$.
- $w_\infty = 0$ on $\partial \mathbb{R}_+^N$.

To prove this let us consider a compact set $K \subset \mathbb{R}_+^N$. For $n \in \mathbb{N}$ large we can assume that $K \subset I_{\delta_n}(\partial \Omega^*_n)$ so that Theorem 3.2 can be exploited.

Claim 1. We claim that $w_n(y) > 0$ for all $y \in K$ and for $n \in \mathbb{N}$ large.

Let $y \in K$. Hence, by Theorem 3.2

$$w_n(y) := \frac{u_n(\delta_n y)}{M_n} \geq L \frac{\text{dist}(\delta_n y, \partial \Omega_n)}{M_n^{\gamma + p - 1}}.$$ 

In particular, by the fact that $\text{dist}(\delta_n y, \partial \Omega_n) \geq C \delta_n$, it follows that

\begin{equation}
w_n(y) \geq L \frac{(C \delta_n)^{\gamma + p - 1}}{M_n} \geq C(K, \gamma, m_1) > 0.
\end{equation}

Claim 2. We claim that $w_n \overset{C^{1,\alpha}}{\to} w_\infty$, as $n \to +\infty$, in any compact set $K$ of $\mathbb{R}_+^N$. 

Since \( \text{dist}(y, \partial \Omega^*_n) \leq C \) for every \( y \in K \), by Theorem 3.2 it follows that

\[
    w_n(y) = \frac{u_n(\delta_n y)}{M_n} \leq L m_2 \frac{[\text{dist}(\delta_n y, \partial \Omega_n)]^{\frac{p}{p-1}}}{M_n}
    = L m_2 \frac{\delta_n^{\frac{p}{p-1}} [\text{dist}(y, \partial \Omega^*_n)]^{\frac{p}{p-1}}}{M_n}
    \leq L m_2 C^{\frac{p}{p-1}} \delta_n^{\frac{p}{p-1}}
    \leq L C^{\frac{p}{p-1}} C(K, m_2),
\]

(3.51)

Hence

\[
    \|w_n\|_{L^\infty(K)} \leq C(K)
\]

for any compact set \( K \) of \( \mathbb{R}^N \). By standard regularity theory (see e.g. [20]) it follows that \( w_n \) is uniformly bounded in \( C^{1,\alpha}(K') \) for any compact set \( K' \subset K \). Therefore, by Ascoli’s Theorem, we can pass to the limit in any compact set and with \( C^{1,\alpha}_{\gamma} \) convergence.

Exploiting a standard diagonal process we can therefore define the limiting function \( w_\infty \) that turns out to be a solution to (3.49) in the half space. The fact that \( \Omega^*_n \delta_n \) leads to the limiting domain \( \mathbb{R}^N_+ \) as \( n \to +\infty \) follows by standard arguments that we omit.

It remains to verify the Dirichlet datum for the limiting profile \( w_\infty \). More precisely we have to show that \( w_\infty = 0 \) on \( \partial \mathbb{R}^N_+ \). By Theorem 3.2 it follows that

\[
    w_\infty(y) = \frac{u_\infty(\delta_\infty y)}{M_\infty} \leq L m_2 \frac{[\text{dist}(\delta_\infty y, \partial \Omega^*_\infty)]^{\frac{p}{p-1}}}{M_\infty}
    = L m_2 \frac{\delta_\infty^{\frac{p}{p-1}} [\text{dist}(y, \partial \Omega^*_\infty)]^{\frac{p}{p-1}}}{M_\infty}
    \leq C(K, L, m_2, m_1) [\text{dist}(y, \partial \mathbb{R}^N_+)]^{\frac{p}{p-1}}
\]

in \( \Omega^*_n \).

Since \( \Omega^*_n \to \mathbb{R}^N_+ \), as \( n \) goes to +\( \infty \), by (3.52) and (3.50), passing to the limit we have that

\[
    0 \leq w_\infty(y) \leq C(K, L, m_2, m_1) [\text{dist}(y, \partial \mathbb{R}^N_+)]^{\frac{p}{p-1}}.
\]

In a similar fashion, and exploiting again Theorem 3.2, we also deduce that

\[
    w_\infty(y) \geq C(K, L, m_2, m_1) [\text{dist}(y, \partial \mathbb{R}^N_+)]^{\frac{p}{p-1}}.
\]

(3.54)

By (3.53) it follows that \( w_\infty(y) = 0 \) as claimed. Nevertheless, collecting (3.53) and (3.54), we deduce that \( w_\infty \) has the right asymptotic behaviour needed to apply Theorem 1.2 see (1.3). This shows that that \( w_\infty \) is the unique solution to (3.49) given by

\[
    w_\infty(x) = w_\infty(x_N) = \left[ \frac{\hat{C}}{[p^{p-1} - (p-1)(\gamma - 1)]} \right]^{\frac{1}{\gamma p - 1}} (x_N)^{\frac{p}{p-1}}.
\]

(3.55)

On the other hand, passing to the limit in (3.35), it would follows that

\[
    \partial_\nu w_\infty(e_N) \leq 0
\]
for some \( \bar{\nu} \in \mathbb{R}^N \) with \( (\bar{\nu}, e_N) > 0 \). Clearly this is a contradiction with (3.55) thus proving the result.

Now using the Theorem 1.1 we want to prove the symmetry result.

4. Symmetry: Proof of Theorem 1.3

In this section we prove our symmetry (and monotonicity) result. Actually we provide the details needed for the application of the moving plane method. For the semilinear case see [7, 8, 9], in the quasilinear setting we use the technique developed in [13].

We start with some notation: for a real number \( \lambda \) we set

\[
\Omega_{\lambda} = \{ x \in \Omega : x_1 < \lambda \}
\]

\[
x_{\lambda} = R_{\lambda}(x) = (2\lambda - x_1, x_2, \ldots, x_n)
\]

which is the reflection through the hyperplane \( T_{\lambda} := \{ x \in \mathbb{R}^n : x_1 = \lambda \} \). Also let

\[
a = \inf_{x \in \Omega} x_1.
\]

Finally we set

\[
u_{\lambda}(x) = u(x_{\lambda}).
\]

Finally we define

\[
\Lambda_0 = \{ a < \lambda < 0 : u \leq u_t \text{ in } \Omega \text{ for all } t \in (a, \lambda) \}.
\]

In the following the critical set of \( u \)

\[
Z_{u} := \{ \nabla u = 0 \}
\]

will play a crucial role. Let us first note that, as a consequence of Theorem 1.1 we know that

\[
Z_{u} \subset \subset \Omega.
\]

This fact allows to exploit the results of [13] since the solution is positive in the interior of the domain (and the nonlinearity is no more singular there). Therefore we conclude that

\[
|Z_{u}| = 0 \quad \text{and} \quad \Omega \setminus Z_{u} \text{ is connected}.
\]

Proof of Theorem 1.3. The proof follows via the moving plane technique. We start showing that:

\[
\Lambda_0 \neq \emptyset.
\]

To prove this, let us consider \( \lambda > a \) with \( \lambda - a \) small. By Theorem 1.1 it follows that

\[
\frac{\partial u}{\partial x_1} > 0 \quad \text{in} \quad \Omega_{\lambda} \cup R_{\lambda}(\Omega_{\lambda}),
\]

and this immediately proves that \( u < u_{\lambda} \) in \( \Omega_{\lambda} \).

Now we define

\[
\lambda_0 := \sup \Lambda_0.
\]

We shall show that \( u \leq u_{\lambda} \) in \( \Omega_{\lambda} \) for every \( \lambda \in (a, 0] \), namely that:

\[
\lambda_0 = 0.
\]
To prove this, we assume that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \leq u_{\lambda_0 + \nu}$ in $\Omega_{\lambda_0 + \nu}$ for any $0 < \nu < \tilde{\nu}$ for some $\tilde{\nu} > 0$ (small). By continuity we know that $u \leq u_{\lambda_0}$ in $\Omega_{\lambda_0}$. The strong comparison principle (see e.g. [26, 29]) holds true in $\Omega_{\lambda_0} \setminus Z_u$, providing that

$$u < u_{\lambda_0} \quad \text{in} \quad \Omega_{\lambda_0} \setminus Z_u.$$  

Note in fact that, in each connected component $C$ of $\Omega_{\lambda_0} \setminus Z_u$, the strong comparison principle implies that $u < u_{\lambda_0}$ in $C$ unless $u \equiv u_{\lambda_0}$ in $C$. Actually the latter case is not possible. In fact, if $\partial C \cap \partial \Omega \neq \emptyset$ this is not possible in view of the zero Dirichlet boundary datum since $u$ is positive in the interior of the domain. If else $\partial C \cap \partial \Omega = \emptyset$ then we should have a local symmetry region causing $\Omega \setminus Z_u$ to be not connected, against what we already remarked above.

Therefore, given a compact set $K \subset \Omega_{\lambda_0} \setminus Z_u$, by uniform continuity we can ensure that $u < u_{\lambda_0 + \nu}$ in $K$ for any $0 < \nu < \tilde{\nu}$ for some small $\tilde{\nu} > 0$. Moreover, by Theorem [11] and taking into account the zero Dirichlet boundary datum, it is easy to show that, for some $\delta > 0$, we have that

$$u < u_{\lambda_0 + \nu} \quad \text{in} \quad I_\delta(\partial \Omega) \cap \Omega_{\lambda_0 + \nu}$$

for any $0 < \nu < \tilde{\nu}$. This is quite standard once that Theorem [11] is in force. The hardest part is the study in the region near $\partial \Omega \cap T_{\lambda_0 + \nu}$. Here we exploit the monotonicity properties of the solutions proved in Theorem [11] that works once we note that $(\epsilon_1, \eta(x)) > 0$ in a neighborhood of $\partial \Omega \cap T_{\lambda_0 + \nu}$ since the domain is smooth and strictly convex.

Now we define

$$w_{\lambda_0 + \nu} := (u - u_{\lambda_0 + \nu})^+$$

for any $0 < \nu < \tilde{\nu}$. We already showed in [12] that $\text{supp}(w_{\lambda_0 + \nu}) \subset \subset \Omega_{\lambda_0 + \nu}$. Moreover $w_{\lambda_0 + \nu} = 0$ in $K$ by construction.

For any $\tau > 0$ fixed, we can choose $\tilde{\nu}$ small and $K$ large so that

$$|\Omega_{\lambda_0 + \nu} \setminus K| < \tau.$$  

Here we are also exploiting the fact that the critical set $Z_u$ has zero Lebesgue measure (see [13]).

In particular we take $\tau$ sufficiently small so that the weak comparison principle in small domains (see [13]) works, showing that

$$w_{\lambda_0 + \nu} = 0 \quad \text{in} \quad \Omega_{\lambda_0 + \nu}$$

for any $0 < \nu < \tilde{\nu}$ for some small $\tilde{\nu} > 0$. But this is in contradiction with the definition of $\lambda_0$. Hence $\lambda_0 = 0$.

The desired symmetry (and monotonicity) result follows now performing the procedure in the same way but in the opposite direction.  

$\square$

### Appendix

We start this appendix, by proving Lemma 2.3 that is an essential tool in the ODE analysis.
Proof of Lemma 2.3. We first claim that \( u'(t) \geq 0 \) for every \( t > 0 \). To prove this fact we argue by contradiction and assume that there exist \( t_0 \geq 0 \) such that \( u'(t_0) < 0 \). Setting

\[
    w(t) := |u'(t)|^{p-2} u'(t)
\]

it follows by the equation in (2.34) that \( w \) is a strictly decreasing function. Therefore \( u'(t) \leq -C := u'(t_0) < 0 \) for every \( t \geq t_0 \) and

\[
    u(t) = u(t_0) + \int_{t_0}^{t} u'(s) \, ds \leq u(t_0) - \int_{t_0}^{t} C \, ds = -Ct + Ct_0 + u(t_0).
\]

This would force \( u \) to be negative for \( t \) large in contradiction with the fact that \( u \) is positive by assumption. Therefore we deduce that \( u'(t), w(t) \geq 0 \) for \( t \) sufficiently large.

Recalling that \( w \) is a strictly decreasing function, we deduce that actually \( u'(t), w(t) > 0 \). Furthermore \( w(t) \to M \geq 0 \) as \( t \) goes to \(+\infty\). It is easy to show that \( M = 0 \). If \( M > 0 \) in fact, arguing as in (4.61), we would have

\[
    u(t) \geq Mt + c
\]

for \( t \) sufficiently large. This gives a contradiction with our initial assumption (2.35), hence \( M = 0 \).

Let us now set

\[
    h(t) := \frac{t^{-\beta \gamma + 1}}{\beta \gamma - 1}.
\]

By Cauchy’s Theorem we have that for \( t \) large enough and \( k > t \) fixed there exists \( \xi_t \in (t, t + k) \) such that

\[
    \frac{w(t) - w(t + k)}{h(t) - h(t + k)} = \frac{w'(|\xi_t|)}{h'(|\xi_t|)}.
\]

Letting \( k \to +\infty \) in (4.62) we obtain

\[
    \frac{w(t)}{h(t)} = \frac{([u']^{p-1})'(|\xi_t|)}{(\xi_t)^{-\beta \gamma}} = \frac{t^{\beta \gamma}}{u'}. \tag{4.63}
\]

for \( t \) large enough. By (2.35) and (4.63) we deduce that \( \frac{w(t)}{h(t)} \) is bounded at infinity, thus proving (2.36). \( \square \)

Now we are ready to prove a useful weak comparison principle in bounded domains (Lemma 3.1).

Proof of Lemma 3.1. Let us set:

\[
    w_{\varepsilon} := (u - v - \varepsilon)^+
\]

where \( \varepsilon > 0 \). We notice that \( w_{\varepsilon} \) is suitable as test function since \( \text{supp}(w_{\varepsilon}) \subset \subset D \) and \( u, v \in W^{1,p}_{\text{loc}}(D) \). Hence \( w_{\varepsilon} \in W^{1,p}_{0}(D) \) and, by density arguments, we can plug \( w_{\varepsilon} \) as test function in (2.6) and (2.7) and by subtracting we obtain

\[
    \int_{D \cap \text{supp}(w_{\varepsilon})} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla w_{\varepsilon}) \, dx \leq \int_{D \cap \text{supp}(w_{\varepsilon})} p(x) \left( \frac{1}{u^+} - \frac{1}{v^+} \right) w_{\varepsilon} \, dx. \tag{4.65}
\]
Taking into account the fact that \( u - v \geq u - v - \varepsilon \), the fact that \( p(\cdot) \) is positive and \( u^{-\gamma} \) is decreasing, it follows that

\[
\int_{D \cap \text{supp}(w_\varepsilon)} (|\nabla u| + |\nabla v|)^{p-2} |\nabla w_\varepsilon|^2 \, dx \leq 0.
\]

By Fatou Lemma, as \( \varepsilon \) tends to zero, we deduce that

\[
\int_D (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u - v)^+|^2 \, dx \leq 0
\]

showing that \((u - v)^+\) is constant, and therefore zero by the boundary data. Thus we deduce that \( u \leq v \) in \( D \) proving the thesis. \( \square \)

Now we are ready to prove the analogous of the result contained in [21], but in the quasilinear setting.

**Proof of Theorem 3.2.** We rewrite the equation in \((P)\) as

\[
-\Delta_p u = \frac{1}{u^\gamma} + f(u) = \frac{p(x)}{u^\gamma} \quad \text{in } \Omega
\]

where \( p(x) := 1 + u(x)^\gamma f(u(x)) \). In the following we assume that \( \delta \) is small enough so that \( p(x) > 0 \quad \forall \, x \in I_\delta(\partial \Omega) \).

Arguing as in [21], we exploit the principal eigenfunction \( \phi_1 \) of problem (3.43) and the fact that \( \phi_1 \in C^{1,\alpha}(\Omega) \) (see e.g. [2, 22, 23]) and

\[
\nabla \phi_1(x) \neq 0 \quad \forall \, x \in \partial \Omega.
\]

For \( t := \frac{p}{\gamma + p - 1} \) we set \( \Psi := s \phi_1^t, \, s > 0 \). It is easy to see that

\[
-\Delta_p \Psi = \frac{g(x,s)}{\Psi^{\gamma}} \quad \text{in } I_\delta(\partial \Omega)
\]

where

\[
g(x,s) := s^{\gamma + p - 1} t^{p-1} \left[ \frac{(\gamma - 1)(p - 1)}{\gamma + p - 1} |\nabla v_1(x)|^p + \lambda_1 v_1(x)^p \right].
\]

Since \( 0 < t < 1 \), we can choose two positive constants \( s_1 \) and \( s_2 \) such that \( 0 < s_1 < s_2 \) and

\[
g(x,s_1) < p(x) < g(x,s_2) \quad \forall \, x \in I_\delta(\partial \Omega).
\]

Hence, setting \( u_1 := s_1 \phi_1^t \) and \( u_2 := s_2 \phi_1^t \), we have that

\[
-\Delta_p u_1 < \frac{p(x)}{u_1^\gamma} \quad \text{in } I_\delta(\partial \Omega)
\]

and

\[
-\Delta_p u_2 > \frac{p(x)}{u_2^\gamma} \quad \text{in } I_\delta(\partial \Omega)
\]

In order to control the datum on the boundary of \( I_\delta(\partial \Omega) \) (in the interior of the domain), we need to switch from \( u \) to \( u_\beta := \beta u \) observing that

\[
-\Delta_p u_\beta = \beta^{\gamma + p - 1} \frac{p(x)}{u_\beta^\gamma}.
\]
For $\beta_1 > 0$ large it follows that $u_{\beta_1}$ and $u_1$ satisfy the following problem:

\[
\begin{cases}
-\Delta_p u_{\beta_1} \geq \frac{p(x)}{u_{\beta_1}^\gamma} & \text{in } I_\delta(\partial\Omega) \\
-\Delta_p u_1 < \frac{p(x)}{u_1^\gamma} & \text{in } I_\delta(\partial\Omega) \\
u_{\beta_1} \geq u_1 & \text{on } \partial I_\delta(\partial\Omega).
\end{cases}
\]

(4.72)

By Lemma 3.1 it follows now that

\[
u_{\beta_1} = \beta_1 u \geq u_1 \text{ in } I_\delta(\partial\Omega).
\]

(4.73)

Similarly, for $\beta_2 > 0$ small, it follows that $u_{\beta_2}$ and $u_2$ satisfy the problem:

\[
\begin{cases}
-\Delta_p u_{\beta_2} \leq \frac{p(x)}{u_{\beta_2}^\gamma} & \text{in } I_\delta(\partial\Omega) \\
-\Delta_p u_2 > \frac{p(x)}{u_2^\gamma} & \text{in } I_\delta(\partial\Omega) \\
u_{\beta_2} \leq u_2 & \text{on } \partial I_\delta(\partial\Omega).
\end{cases}
\]

(4.74)

By Lemma 3.1 it follows that

\[
u_{\beta_2} = \beta_2 u \leq u_2 \text{ in } I_\delta(\partial\Omega).
\]

(4.75)

Hence the thesis is proved with $m_1 := \frac{s_1}{\beta_1}$ and $m_2 := \frac{s_2}{\beta_2}$. □

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* Dipartimento di Matematica e Informatica, UNICAL, Ponte Pietro Bucci 31B, 87036 Arcavacata di Rende, Cosenza, Italy.

E-mail address: esposito@mat.unical.it

E-mail address: sciunzi@mat.unical.it