A Riemann mapping type Theorem in higher dimensions
Part I : the conformally flat case with umbilic boundary

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Abstract

In this paper we prove that every Riemannian metric on a locally conformally flat manifold with
umbilic boundary can be conformally deformed to a scalar flat metric having constant mean curvature.
This result can be seen as a generalization to higher dimensions of the well known Riemann mapping
Theorem in the plane.

Key Words: critical trace Sobolev exponent, curvature, conformal invariance, lack of compactness,
critical point at infinity

1 Introduction

In [14], José F. Escobar raised the following question: Given a compact Riemannian manifold with
boundary, when it is conformally equivalent to one that has zero scalar curvature and whose boundary
has a constant mean curvature ? This problem can be seen as a “generalization” to higher dimensions
of the well known Riemannian mapping Theorem. The later states that an open, simply connected
proper subset of the plane is conformally diffeomorphic to the disk. In higher dimensions few regions
are conformally diffeomorphic to the ball. However one can still ask whether a domain is conformal to a
manifold that resembles the ball into ways : namely, it has zero scalar curvature and its boundary has
constant mean curvature. In the above the term “generalization” has to be understood in that sens. The
above problem is equivalent to finding a smooth positive solution to the following nonlinear boundary
value problem on a Riemannian manifold with boundary $(M^n, g)$, $n \geq 3$:

\begin{equation}
\begin{aligned}
-\Delta_g u + \frac{(n-2)}{4(n-1)} R_g u &= 0, \quad u > 0 \quad \text{in } \bar{M}; \\
\partial_{\nu} u + \frac{n-2}{2} h_g u &= Q(M, \partial M) u^{\frac{n}{n-2}}, \quad \text{on } \partial M.
\end{aligned}
\end{equation}

where $R$ is the scalar curvature of $M$, $h$ is the mean curvature of $\partial M$, $\nu$ is the outer normal vector with
respect to $g$ and $Q(M, \partial M)$ is a constant whose sign is uniquely determined by the conformal structure.
Indeed if $\overline{g} = u^{\frac{2}{n-2}} g$, then the metric $\overline{g}$ has zero scalar curvature and the boundary has constant mean
curvature with respect to $\overline{g}$.

Solutions of equation (P) correspond , up to a multiple constant, to critical points of the following
functional $J$ defined on $H^1(M) \setminus \{0\}$

\begin{equation}
J(u) = \left( \int_M \left( |\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dV_g + \frac{n-2}{2} \int_{\partial M} h_g u^2 d\sigma_g \right)^{\frac{n-1}{n}}
\end{equation}

where $dV_g$ and $d\sigma_g$ denote the Riemannian measure on $M$ and $\partial M$ induced by the metric $g$.

The regularity of the $H^1$ solutions of (P) was established by P. Cherrier [11]; and related problems
regarding conformal deformations of metrics on manifold with boundary were studied in [1], [9], [12],
[7] , [13], [19] , [20], [22] , [23] and the references therein.
The exponent $\frac{2(n-1)}{n-2}$ is critical for the Sobolev trace embedding $H^1(M) \to L^q(\partial M)$. This embedding being not compact, the functional $J$ does not satisfy the Palais Smale condition. For this reason standard variational methods cannot be applied to find critical points of $J$.

Following the original arguments introduced by T. Aubin [2], [3] and R. Schoen [27] to prove Yamabe conjecture on closed manifolds, Escobar proved the existence of a smooth positive solution $u$ of (P) on $(M^n, g), n \geq 3$ for many cases. To state his results we need some preliminaries:

**Definition 1.1** A point $q \in \partial M$ is called an umbilic point if $U = 0$ at $q$. $\partial M$ is called umbilic if every point of $\partial M$ is umbilic.

Regarding the above problem Escobar proved the following Theorem [14, 16]:

**Theorem 1.1** Let $(M^n, g)$ be a compact Riemannian manifold with boundary, $n \geq 3$. Assume that $M^n$ satisfies one of the following conditions:

(i) $n \geq 6$ and $M$ has a nonumbilic point on $\partial M$

(ii) $n \geq 6$ and $M$ is conformally locally flat with umbilic boundary

(iii) $n = 4, 5$ and $\partial M$ is umbilic

(iv) $n = 3$

then there exists a smooth metric $u^{-\frac{4}{n-2}}g, u > 0$ on $M$ of zero scalar curvature and constant mean curvature on $\partial M$.

In his proof Escobar uses strongly an extension of the positive mass Theorem of R. Schoen and S.T. Yau [29], [28] to some type of manifolds with boundary. Such an extension was proved by Escobar in [15]. Besides the proof of T.Aubin and R.Schoen of the Yamabe conjecture, another proof by A. Bahri [5] and A.Bahri and H. Brezis [6] of the same conjecture is available by techniques related to the Theory of critical point at Infinity of A. Bahri [4].

We plan to give a complete positive answer to the above problem based on the topological argument of Bahri-Coron [7], as Bahri and Brezis did for the Yamabe conjecture. In this first part we study the case where the manifold is locally conformally flat with umbilic boundary. Namely we prove the following Theorem

**Theorem 1.2** Suppose that $(M^n, g), n \geq 3$ is a compact locally conformally flat manifold with umbilic boundary, then equation (P) has a solution.

Let us observe that while the solution obtained by Escobar is a minimum of $J$, our solution is in general, a critical point of $J$ of higher Morse index, more precisely we have the following characterization of solutions obtained by Bahri-Coron existence scheme (see [10]):

**Theorem 1.3** The solution $u$ obtained in Theorem 1.2 satisfies, for some nonnegative integer $p_0$:

(i) $p_0^{-\frac{4}{n-2}}S \leq J(u) \leq (p_0 + 1)^{-\frac{4}{n-2}}S$

(ii) $\text{ind}(J, u) \leq (p_0 + 1)(n - 1) + p_0, \text{ind}(J, u) + \text{dimker}d^2 J(u) \geq p_0(n - 1) + p_0$

(iii) $u$ induces some difference of topology at the level $p_0^{-\frac{4}{n-2}}S$.

where $\text{ind}(J, u)$ is the Morse index of $J$ at $u$, and $S = 2^{1-n}\omega_{n-1}$, where $\omega_{n-1}$ is the volume of the $n-1$ dimensional unit sphere. Moreover, if (P) has only nondegenerate solutions, $\text{ind}(J, u) = (n - 1)p_0 + p_0$. 

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The remainder of the paper is organized as follows: in section 2 we construct some “almost solutions” which are solutions of the “problem at Infinity”. In section 3 we collect some standard results regarding the description of the lack of compactness and some local deformation Lemma. In section 4 we perform an expansion of $J$ at 'Infinity' and we give the proof of Theorem 1.2 in section 5. Lastly we devote the appendix to establish some technical Lemma and to recall some well known results.

Acknowledgements

The author is indebted to Pr. Abbas Bahri for teaching him his Theory of critical point at Infinity and he is grateful to Pr. Antonio Ambrosetti for his interest in his work and his constant support.

2 Construction of “almost solutions”

In this paper we assume that $(M^n, g)$ is a compact Riemannian manifold with boundary and dimension $n \geq 3$. Let $R_{pq}$ and $R = g^{pq}R_{pq}$ be the Ricci curvature and the scalar curvature, respectively; let $h_{ij}$ and $h = \frac{1}{n}g^{ij}h_{ij}$ be the second fundamental form of the boundary of $M$, $\partial M$ and the mean curvature, respectively. Let $\tilde{g} = u^{\frac{4}{n-2}}g$ be a metric conformally related to $g$. We denote by a tilde all quantities computed with respect to the metric $\tilde{g}$. The transformation law for the scalar curvature is

$$\tilde{R} = 4(n-1) \frac{Lu}{u^{\frac{n+2}{2}}}$$

where $L$ is the conformal Laplacian $L = \Delta - \frac{n-2}{4(n-1)} R$ on $M$; while the transformation law for the mean curvature is

$$\tilde{h} = 2 \frac{Bu}{n-2} u^{\frac{n-2}{2}}$$

where $B$ is the boundary operator $B = \frac{\partial}{\partial w} + \frac{n-2}{2} h$ on $\partial M$.

Consider now the following eigenvalue problem on $(M, g)$ :

$$\begin{align*}
Lg u &= \lambda u & \text{on } & \tilde{M} \\
B_g u &= 0 & \text{on } & \partial M
\end{align*}$$

(E)

Let $\lambda_1$ the first eigenvalue of (E).

Definition 2.1 We say that a manifold $M$ is of positive(negative, zero) type if $\lambda_1 > 0 (< 0, = 0)$.

As it is well known the existence problem is easy when the manifold is of negative or zero type, so we treat only in this paper the case of manifold of positive type. Now we construct some almost solutions of (P), which will play a central role in the description of the lack of compactness.

Let $f_1$ denote a positive eigenfunction corresponding to the first eigenvalue of (E), and consider $g_1 = (f_1)\frac{4}{n+2} g$ then, according to (2) and (3) we have: $R_{g_1} > 0$ and $h_{g_1} = 0$ on $\partial M$. We can work with $g_1$ instead of $g$, but for simplicity we still denote it by $g$. Let $a \in \partial M$; since $M$ is a compact locally conformally flat manifold one can find a neighborhood of $a$, $\mathcal{U}(a) \supset B^M_\rho(a)$, $\rho > 0$ uniform and a conformal diffeomorphism $\varphi$ which maps $B^M_\rho(a)$ into $\mathbb{R}^n$ with $\varphi(0) = a$. Therefore, denoting $g_0$ the flat metric on $\mathbb{R}^n$, there exist a positive function $u_a$ such that $\varphi^*(g_0) = u_a^{\frac{4}{n-2}} g$. Since the boundary is umbilic, $\varphi(\partial M \cap B^M_\rho(a))$ has to be a piece of sphere or a piece of a hyperplane (See [30]) and since spheres and hyperplanes are locally conformal to each other, we can assume without loss of generality that $\partial B^\rho_2(0) \cap \partial \mathbb{R}^n_+ \subset \varphi(\partial M \cap B^M_\rho(a))$ and $\varphi(M \cap B^M_\rho(a)) \subset \mathbb{R}^n_+$. Since $\partial B^\rho_2(0) \cap \partial \mathbb{R}^n_+$ has zero mean
curvature in $\overline{B^2}$, we deduce from (3) that $\frac{\partial u}{\partial \nu} = 0$ on $\partial M \cap B^M_\rho(a)$. We extend $u_a$ to be a smooth positive function on $M$ such that $\frac{\partial u}{\partial \nu} = 0$ on $\partial M$ and $u_a = 0$ on $M \setminus B^M_\rho(a)$. Consider now the conformal metric $\overline{\gamma}_0 = u^{\frac{4}{n-2}} \gamma$, then $\overline{\gamma}_0$ has the property that $\overline{\gamma}_0 = 0$ and it is Euclidean in $B^M_\rho(a)$. Moreover this metric can be chosen to depend smoothly on $a$ (see [5]).

For $a \in \partial M$, define the function:

$$\delta_a,\lambda(y) = \frac{\lambda^{\frac{n-2}{2}}}{(1 + \lambda x^n)^{\frac{n-2}{2}} + \lambda^2 |x'|^2} \frac{\hat{g}(y)}{\lambda^{\frac{n-2}{2}}}$$

where $(x', x^n) = \varphi(y)$, and $\hat{g}$ is chosen such that $\delta_a,\lambda$ satisfies the following equation

$$\begin{cases}
-\Delta_{\overline{\gamma}_0} u = 0, & \text{in } B^M_\rho \cap \hat{M}; \\
\partial_\nu u = \delta_a,\lambda, & \text{on } B^M_\rho \cap \partial M
\end{cases}$$

Set $\hat{\delta}_a,\lambda = \omega_a u_a \delta_a,\lambda$ where $\omega_a$ is a cutoff function $\omega_a = 1$ on $B^M_\rho(a)$ and $\omega_a = 0$ on $M \setminus B^M_\rho$. We define now a family of almost solutions $\varphi_{a,\lambda}$ to be the unique solution of

$$\begin{cases}
-L_g u = 0, & \text{in } \hat{M}; \\
B_g u = \hat{\delta}_a,\lambda, & \text{on } \partial M
\end{cases}$$

Let us recall that the operators $L_g$ and $B_g$ are conformally invariant under the conformal change of metrics, namely we have:

**Lemma 2.1** (3)

Let $\psi \in C^2(B^\rho_\rho(a))$, we have

$$L_g(u_a \psi) = u_a^{\frac{n+2}{2}} L_{\overline{\gamma}_0}(\psi)$$

and

$$B_g(u_a \psi) = u_a^{\frac{n}{2}} B_{\overline{\gamma}_0}(\psi)$$

In the remainder of this section we establish some properties of our almost solutions $\varphi_{a,\lambda}$.

**Lemma 2.2** There are two positive constants $C$ and $B$, such that for all $a \in \partial M$ and $\lambda \geq B$, we have

$$\left| \varphi_{a,\lambda} - \hat{\delta}_{a,\lambda} \right|_{\infty} \leq \frac{C}{\lambda^{\frac{n+2}{2}}}$$

**Proof.**

Let $H_{a,\lambda} = \lambda^{\frac{n-2}{2}} (\varphi_{a,\lambda} - \hat{\delta}_{a,\lambda})$, we have

$$L_g H_{a,\lambda} = \lambda^{\frac{n-2}{2}} L_g (\omega_a u_a \delta_{a,\lambda}) = \lambda^{\frac{n-2}{2}} \frac{n+2}{u_a^{\frac{n+2}{2}}} L_g (\omega_a \delta_{a,\lambda})$$

Since on $B_\rho$, $\omega_a = 0$, we deduce that on $B_\rho$ we have $L_g H_{a,\lambda} = 0$, whereas on $M \setminus B_\rho$ there holds $L_g H_{a,\lambda} \leq C$.

From another part

$$B_g H_{a,\lambda} = \lambda^{\frac{n-2}{2}} \left[ B_g \varphi_{a,\lambda} - B_g (\omega_a u_a \delta_{a,\lambda}) \right] = \lambda^{\frac{n-2}{2}} \left[ \hat{\delta}_{a,\lambda} - u_a^{\frac{n+2}{2}} B_g (\omega_a \delta_{a,\lambda}) \right]$$

on $B_\rho(a) \cap \partial M$, $\omega_a = 1$, therefore $B_g H_{a,\lambda} = 0$, while on $M \setminus B_\rho$ there holds $B_g H_{a,\lambda} \leq C$. Thus our Lemma follows from Lemma 3 quoted in the appendix.
Lemma 2.3 There are two positive constants $C$ and $B$, such that for all $a \in \partial M$ and $\lambda \geq B$, we have

$$\varphi_{a,\lambda} \geq \frac{C}{\lambda^{n-2}}$$

Proof.

Using Lemma 2.2, we know that if $\rho_1 < \rho$ is chosen small enough, independent of $\lambda$, the following inequality holds on $B(a, \rho_1)$

$$\varphi_{a,\lambda} \geq \hat{\delta}_{a,\lambda} - \frac{C}{\lambda^{n-2}} \geq \frac{C}{\lambda^{n-2}}$$

Let $\Sigma_1 = \partial B(a, \rho) \cap \hat{M}$ and $\Sigma_2 = \partial M \setminus \Sigma_1$. Then we have

$$\begin{cases}
L_g(\varphi_{a,\lambda} - \frac{C}{\lambda^{n-2}}) \leq 0 & \text{in } \hat{M} \\
\varphi_{a,\lambda} - \frac{C}{\lambda^{n-2}} \geq 0, & \text{on } \Sigma_1 \\
\frac{\partial}{\partial \nu}(\varphi_{a,\lambda} - \frac{C}{\lambda^{n-2}}) \geq 0 & \text{on } \Sigma_2
\end{cases} \quad (4)$$

Then by the Hopf maximum principle, we deduce from (4) that

$$\varphi_{a,\lambda} \geq \frac{C}{\lambda^{n-2}} \text{ for } x \in M$$

Lemma 2.4 Let $\theta > 0$ be given. There are positive constants $C$ and $B$, such that the following estimates hold, provided $\lambda \geq B$

(i) \[ \left| \int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g \right| - \frac{2(n-1)}{\lambda^{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{(1 + |x|^2)^{n-1}} \leq \frac{C}{\lambda^{n-2}} \text{ for } a \in \partial M \]

(ii) \[ \left| \int_{\partial M} \varphi_{a,\lambda}^{2(n-1)} d\sigma_g \right| - \frac{2(n-1)}{\lambda^{n-2}} \int_{\mathbb{R}^{n-1}} \frac{dx}{(1 + |x|^2)^{n-1}} \leq \frac{C}{\lambda^{n-2}} \text{ for } a \in \partial M \]

(iii) \[ \int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} d\sigma_g \geq \frac{C}{\lambda^{n-2}} \text{ for } a_1, a_2 \in \partial M \]

(vi) \[ \int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g \leq (1 + \theta) \int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} d\sigma_g \]

Proof.

Proof of (i)

From the definition of $\varphi_{a,\lambda}$, we derive:

$$\int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g = \int_{\partial M} (\omega_a \delta_{a,\lambda} u_a) \frac{n}{\lambda^{n-2}} \varphi_{a,\lambda} d\sigma_g$$
Using Lemma 2.2 we deduce:

\[
\int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g = \int_{\partial M \cap B_\rho} (\omega_a \delta_{a,\lambda} u_a)^{\frac{2(\alpha-1)}{\alpha-2}} d\sigma_g + O(\frac{1}{\lambda^{n-2}}) \int_{\partial M \cap B_\rho} (\omega_a \delta_{a,\lambda} u_a)^{\frac{2}{\alpha-2}} d\sigma_g \\
= \int_{\partial M \cap B_\rho} (\delta_{a,\lambda})^{\frac{2(\alpha-1)}{\alpha-2}} dv + O(\frac{1}{\lambda^{n-2}}) \int_{\partial M \cap B_\rho} (\omega_a \delta_{a,\lambda} u_a)^{\frac{2}{\alpha-2}} dv_{g_\rho} + O(\lambda^{n-1}) \\
(5) \\
= \frac{2(\alpha-1)}{\alpha-2} \int_{\mathbb{R}^{n-1}} \frac{dx}{(1 + |x|^2)^{n-1}} + O(\frac{1}{\lambda^{n-2}})
\]

The proof of (ii) is essentially reduced, up to minor differences to the same computations involved in the proof of (ii).

Proof of (iii)

From Lemma 2.3 we deduce

\[
\int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} d\sigma_g \geq \frac{1}{C \lambda^{n-2}} \int_{\partial M} \varphi_{a_1,\lambda} d\sigma_g
\]

Then from Lemma 2.2 and Lemma 2.3 we derive

\[
\int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} d\sigma_g \geq \frac{1}{C \lambda^{n-2}} \int_{B_{\rho(a)} \cap \partial M} \varphi_{a_1,\lambda} d\sigma_g \geq \frac{1}{C \lambda^{n-2}} \int_{B_{\rho(a)} \cap \partial M} \varphi_{a_1,\lambda} d\sigma_g = O(\frac{1}{\lambda^{n-2}})
\]

Proof of (vi)

\[
\int_{\partial M} B_g \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} dv_g = \int_{\partial M} \delta_{a_1,\lambda} \varphi_{a_2,\lambda} dv_g \\
= \int_{\partial M \cap B_\rho} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} + O(\frac{1}{\lambda^{n-2}}) \int_{\partial M \cap B_\rho} \varphi_{a_2,\lambda} \\
= \int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} d\sigma_g + O(\frac{1}{\lambda^{n-2}}) \int_{\partial M} \varphi_{a_2,\lambda} d\sigma_g + O(\frac{1}{\lambda^{n-1}}) \\
= \int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} dv_g \varphi_{a_2,\lambda} + O(\frac{1}{\lambda^{n-2}}) \int_{\partial M} \varphi_{a_2,\lambda} d\sigma_{g_0} \\
= \int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} dv_g + O(\frac{1}{\lambda^{n-1}}) \int_{\partial M} \varphi_{a_2,\lambda} d\sigma_{g_0} + O(\frac{1}{\lambda^{n-1}})
\]

Now from Lemma 6.1 in the Appendix we deduce:

\[
O(\frac{1}{\lambda^{n-2}}) \int_{\partial M \cap B_{\rho(a)}} \delta_{a_1,\lambda} \varphi_{a_2,\lambda} d\sigma_{g_0} = o \left( \int_{\partial M} \delta_{a_1,\lambda} \varphi_{a_2,\lambda} d\sigma_{g_0} \right)
\]

Therefore using (iii) we have

\[
\int_{\partial M} B_g \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} dv_g = \int_{\partial M} \varphi_{a_1,\lambda} \varphi_{a_2,\lambda} dv_g (1 + o(1))
\]

The proof of (vi) and the proof of Lemma 2.4 are thereby completed. ■

### 3 Some standard facts

We recall that solutions of Problem (P) arises, up to a constant, as critical points of the functional $J$ is defined by

\[
\int_{\partial M} B_g \varphi_{a,\lambda} \varphi_{a,\lambda} d\sigma_g
\]
where \( u \) belongs to \( \Sigma^+ \) defined as follows:

\[
\Sigma^+ = \{ u \in H^1(M), u \geq 0, \|u\| = 1 \}
\]

Let us observe that \( \Sigma^+ \) is invariant by the flow of \(-\partial J\).

The functional \( J \) is known to not satisfy Palais Smale condition (PS for short), which leads to the failure of classical existence mechanism. In order to describe this failure we need some notation.

For \( \varepsilon > 0 \) and \( p \geq 1 \), let

\[
V(p, \varepsilon) = \left\{ u \in \Sigma^+ \text{ such that } \exists (a_1, \ldots, a_p) \in (\partial M)^p \text{ and } \exists (\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}^*_+)^p \text{ such that } \right.
\]

\[
\left. \left\| u - \frac{\sum_{i=1}^p \varphi_{a_i, \lambda_i}}{\sum_{i=1}^p \varphi_{a_i, \lambda_i}} \right\| < \varepsilon, \text{ with } \lambda_i \geq \frac{1}{\varepsilon} \text{ and } \varepsilon_{ij} < \varepsilon \right\}
\]

where \( \varepsilon_{ij} = \frac{1}{\frac{n}{2} + |a_i - a_j| d(a_i, a_j)^2} \) and \( d \) denotes the geodesic distance.

If \( u \) is a function in \( V(p, \varepsilon) \), one can find an optimal representation, arguing as in Proposition 7 of [6], namely we have:

**Lemma 3.1** For every \( p \geq 1 \), there exists \( \varepsilon > 0 \) such that \( \forall u \in V(p, \varepsilon) \) the minimization problem

\[
\inf_{\alpha_i, b_i, \mu_i} \left\| u - \sum_{i=1}^p \alpha_i \varphi_{b_i, \mu_i} \right\|
\]

has a unique solution, up to permutation on the set of indices \( \{1, \ldots, p\} \)

At this point we introduce the following notations:

Let \( S \) be defined as \( S = \left( \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |x|^2)^p} \right)^{\frac{1}{2}} \), \( b_p = \frac{1}{n-1} S \) and

\[
W_p = \{ u \in \Sigma^+, \text{such that } J(u) < b_{p+1} \}
\]

We are ready now to state the characterization of the Palais Smale sequences failing the P.S condition.

**Proposition 3.1** Under the assumption that \((P)\) has no solution, let \( u_k \in \Sigma^+ \) be a sequence satisfying \( J(u_k) \to c \), a positive number and \( \partial J(u_k) \to 0 \). There exist an integer \( p \geq 1 \) and a sequence \( (\varepsilon_k)_k \) such that \( u_k \in V(p, \varepsilon) \). Conversely, let \( p \in \mathbb{N}^+ \), let \( \varepsilon_k \) be a positive sequence with \( \lim_{k \to +\infty} \varepsilon_k = 0 \) and let \( u_k \in V(p, \varepsilon) \) then \( \partial J(u_k) \to 0 \) and \( J(u_k) \to b_p \).

**Proof.** The proof of this Proposition is by now standard, taking into account the uniqueness result of Li-Zhu [24], see also [13] and using the Liouville Theorem to rule out the possibility of interior blow up.

We have also the following local deformation Lemma, similar to Lemma 17 in [6]:

**Lemma 3.2** Under the assumption that \((P)\) has no solution, for \( \varepsilon > 0 \), the pair \((W_p, W_{p-1})\) retracts by deformation onto the pair \((W_{p-1} \cup A_p, W_{p-1})\) where \( A_p \subset V(p, \varepsilon) \).
4 Expansion of the functional near its potential critical point at infinity

This section is devoted to an asymptotic expansion of $J$ in the neighborhood of its potential critical points at infinity, that is in some $V(p, \varepsilon)$. This expansion displays the fact that when the number of the bubbles are large enough, their interaction increases to force their energy to be under their critical level. Such a fact is a key point in the topological argument.

Lemma 4.1 There holds:

(i) For every $p \in \mathbb{N}^*$ and every $\varepsilon_1 > 0$, there exists $\lambda_p = \lambda(p, \varepsilon_1)$ such that for any $(\alpha_1, \ldots, \alpha_p)$ satisfying $\alpha_i \geq 0$, $\sum_{i=1}^{p} \alpha_i = 1$, for any $(x_1, \ldots, x_p) \in (\partial M)^p$ for any $\lambda \geq \lambda_p$, we have:

$$\text{If } \sum_{i \neq j} \int_{\partial M} \varphi_{a_i, \lambda} \varphi_{a_j, \lambda} \geq \varepsilon_1 \text{ then } J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda}\right) \leq p^{\frac{1}{p+2}} S.$$

(ii) There exist $0 < \theta_0 < 1$, $C > 0$, and $\bar{\varepsilon}_1 \geq \varepsilon_1$, such that for any $p \in \mathbb{N}^*$, for any $(\alpha_1, \ldots, \alpha_p)$ satisfying $\alpha_i \geq 0$, $\sum_{i=1}^{p} \alpha_i = 1$, $\frac{\alpha_i}{\alpha_j} \geq \theta_0$, for any $(\alpha_1, \ldots, \alpha_p) \in (\partial M)^p$, for any $\lambda \geq 1$, the following inequality holds:

$$J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda}\right) \leq p^{\frac{1}{p+2}} S\left(1 + O\left(\frac{1}{\lambda^{n-2}}\right) + \frac{(p+1)C}{\lambda^{n-2}}\right)$$

If

$$\sum_{i \neq j} \int_{\partial M} \varphi_{a_i, \lambda} \varphi_{a_j, \lambda} \leq \bar{\varepsilon}_1$$

(iii) If we drop in (ii) the condition $\frac{\alpha_i}{\alpha_j} \geq \theta_0$, then the following weaker inequality still holds:

$$J\left(\sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda}\right) \leq p^{\frac{1}{p+2}} S\left(1 + \frac{1}{C\lambda^{n-2}} + \frac{1}{C} \sum_{i \neq j} \int_{\partial M} \varphi_{a_i, \lambda} \varphi_{a_j, \lambda}\right)$$

Proof.

Let

$$\begin{cases} b_i = \frac{\alpha_i \delta_{a_i, \lambda}}{\sum_{j=1}^{p} \alpha_j \delta_{a_j, \lambda}} & \text{if } \delta_{a_i, \lambda} \neq 0 \\ b_i = 1 & \text{if } \delta_{a_i, \lambda} = 0 \end{cases}$$

then Lemma B2 of [7], which extends to our functional, implies that

$$J\left(\sum_{i=1}^{p} \varphi_{a_i, \lambda}\right) \leq \left(\int_{\partial M} \left(\sum_{i=1}^{p} \delta_{a_i, \lambda} \varphi_{a_i, \lambda}\right)^{\frac{2(n-1)}{n}}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{p} b_i \int_{\partial M} \delta_{a_i, \lambda}^{\frac{2(n-1)}{n}}\right)^{\frac{1}{2}}$$

Using Lemma 2.2 and $\int_{\partial M} \delta_i^{\frac{n-2}{2}} \, dv_g = O\left(\frac{1}{\lambda^{n-2}}\right)$, we derive easily

$$J\left(\sum_{i=1}^{p} \varphi_{a_i, \lambda}\right) \leq (1 + O\left(\frac{1}{\lambda^{n-2}}\right)) \left(\sum_{i=1}^{p} b_i \int_{\partial M} \delta_{a_i, \lambda}^{\frac{2(n-1)}{n}}\right)^{\frac{1}{2}}$$
Thus we obtain

\[ J(\sum_{i=1}^{p} \varphi_{a_{i},\lambda}) \leq (1 + O(\frac{1}{\lambda^{n-2}})) \left( (p - 1)S^{n-2} + \int_{\partial M} \frac{\alpha_{1}\hat{\delta}_{a_{1},\lambda}}{\alpha_{1}\delta_{a_{1},\lambda} + \alpha_{2}\delta_{a_{2},\lambda}} \right) \]

We may assume without loss of generality that

(i) \( \frac{\alpha_{1}}{\alpha_{2}} \leq 1 \)

(ii) \[
\int_{\partial M} (\varphi_{\hat{a}_{1},\lambda}^{\frac{n}{\lambda - 2}} \varphi_{a_{2},\lambda} + \varphi_{a_{1},\lambda}^{\frac{n}{\lambda - 2}} \varphi_{\hat{a}_{2},\lambda}) = \sup_{i \neq j} \int_{\partial M} \frac{\varphi_{\hat{a}_{1},\lambda}^{\frac{n}{\lambda - 2}} \varphi_{a_{2},\lambda} + \varphi_{a_{1},\lambda}^{\frac{n}{\lambda - 2}} \varphi_{\hat{a}_{2},\lambda}}{\varphi_{\hat{a}_{2},\lambda}^{\frac{n}{\lambda - 2}} \varphi_{a_{2},\lambda} + \varphi_{a_{1},\lambda}^{\frac{n}{\lambda - 2}} \varphi_{\hat{a}_{1},\lambda}}
\]

We then have using that \( \int_{\partial M \setminus B_{\hat{a}}(a_{1})} \frac{2(n-1)}{\hat{a}_{1},\lambda} \varphi_{a_{2},\lambda} d\sigma = O(\frac{1}{\hat{a}_{1},\lambda}) \)

\[ J(\sum_{i=1}^{p} \varphi_{a_{i},\lambda}) \leq (1 + O(\frac{1}{\lambda^{n-2}})) \left( (p - 1)S^{n-2} + \int_{\partial M} \frac{\alpha_{1}\hat{\delta}_{a_{1},\lambda}}{\alpha_{1}\delta_{a_{1},\lambda} + \alpha_{2}\delta_{a_{2},\lambda}} \right) \]

The continuity of \( u_{y} \) with respect to \( y \) implies the existence of \( \eta > 0 \) such that

\[ \frac{1}{2} \leq \frac{u_{a_{1}}}{u_{a_{2}}} \leq 2 \quad \text{if} \quad x_{2} \in B_{\eta}(a_{1}) \]

Thus if \( d(a_{2},a_{2}) \leq \eta \), we have

\[ J(\sum_{i=1}^{p} \varphi_{a_{i},\lambda}) \leq (1 + O(\frac{1}{\lambda^{n-2}})) \left( (p - 1)S^{n-2} + \int_{\partial M} \frac{\alpha_{1}\hat{\delta}_{a_{1},\lambda}}{\alpha_{1}\delta_{a_{1},\lambda} + \alpha_{2}\delta_{a_{2},\lambda}} \right) \]

\[ \leq (1 + O(\frac{1}{\lambda^{n-2}})) \left( (p - 1)S^{n-2} - \int_{\partial M \setminus B_{\hat{a}}(a_{1})} \frac{\alpha_{1}\hat{\delta}_{a_{1},\lambda}}{\alpha_{1}\delta_{a_{1},\lambda} + \alpha_{2}\delta_{a_{2},\lambda}} \right) \]

At this point we state the following Claim which proof is postponed until the end of this section.

**Claim** : There exists \( \varepsilon_{0} \) small enough, and a positive constant \( C \) such that

\[
\int_{\partial M} \frac{\delta_{a_{2},\lambda}}{\delta_{a_{2},\lambda} + \delta_{a_{1},\lambda}} \frac{2(n-1)}{\lambda - 2} d\sigma_{\varepsilon_{0}} \geq C\varepsilon_{0} \int_{\partial M} \frac{n_{\lambda - 2}}{\varphi_{\lambda - 2}} \varphi_{a_{2},\lambda} + \varphi_{a_{1},\lambda} \varphi_{a_{2},\lambda} d\sigma_{\varepsilon_{0}}
\]

From another part, by assumption, we know that

\[
\sum_{i \neq j} \int_{\partial M} (\frac{n}{\lambda - 2} \varphi_{\lambda - 2} \varphi_{a_{j}} d\sigma_{\varepsilon_{0}}) \leq p^{2} \int_{\partial M} (\frac{n}{\lambda - 2} \varphi_{\lambda - 2} \varphi_{a_{2},\lambda} + \varphi_{a_{1},\lambda} \varphi_{a_{2},\lambda}) d\sigma_{\varepsilon_{0}}
\]

Thus we derive from the above claim and (6), that:

\[
(7) \quad J(\sum_{i=1}^{p} \alpha_{i} \varphi_{a_{i},\lambda}) \leq (1 + O(\frac{1}{\lambda^{n-2}})) \left( pS^{n-2} - \frac{C\varepsilon_{0}}{p^{2}} \sum_{i \neq j} \int_{\partial M} \frac{n}{\lambda - 2} \varphi_{\lambda - 2} \varphi_{a_{j}} d\sigma_{\varepsilon_{0}} \right)
\]

\[ \leq p^{\frac{n-1}{n}} S(1 + O(\frac{1}{\lambda^{n-2}})) - \frac{C_{1}\varepsilon_{0}\varepsilon_{1}}{p^{3}} \]

(8)
clearly implies (i), which is therefore proven if \(d(a_1, a_2) \leq \eta\). Now we rule out the case where \(d(a_1, a_2) \geq \eta\) as follows:

If \(d(a_1, a_2) \geq \eta\) then

\[
\int_{\partial M} \left( \varphi_{a_1, \lambda}^{n-2} \varphi_{a_2, \lambda} + \varphi_{a_2, \lambda}^{n-2} \varphi_{a_1, \lambda} \right) = O(\frac{1}{\lambda^{n-2}})
\]

Therefore

\[
\sum_{i \neq j} \varphi_{a_i, \lambda}^{n-2} \varphi_{a_j, \lambda} \leq \eta^2 \cdot O(\frac{1}{\lambda^{n-2}}) = O(\frac{1}{\lambda^{n-2}})
\]

Taking \(\lambda\) very large we have \(\sum_{i \neq j} \varphi_{a_i, \lambda}^{n-2} \varphi_{a_j, \lambda} < \varepsilon_1\) therefore the proof of (i) is reduced to the case \(d(a_1, a_2) \leq \eta\). The proof of (i) is thereby established.

The proofs of (ii) and (iii) will be completed together since they rest on the same expansion of the functional \(J\).

Using Lemma 2.2 we derive the following inequality

\[
J\left( \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda} \right) \leq \frac{\left( \sum_{i=1}^{p} \alpha_i^2 \int_{\partial M} \delta_{\varphi_{a_i, \lambda}}^{n-2} \varphi_{a_i, \lambda} + \sum_{i \neq j} \alpha_i \alpha_j \int_{\partial M} \delta_{\varphi_{a_i, \lambda}}^{n-2} \varphi_{a_j, \lambda} \right)^{\frac{n-1}{n-2}}}{\sum_{i=1}^{p} \alpha_i \int_{\partial M} \varphi_{a_i, \lambda}^{n-2} + \sum_{i \neq j} \frac{\gamma(n-1)}{n-2} \alpha_i \alpha_j \int_{\partial M} \varphi_{a_i, \lambda}^{n-2} \varphi_{a_j, \lambda} - \sum_{i=1}^{p} \alpha_i \frac{2^{(n-1)}}{\lambda^{n-2}}}
\]

Then (3) implies using Lemma 2.2 and Lemma 2.4

\[
J\left( \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda} \right) \leq \frac{\left( \sum_{i=1}^{p} \alpha_i^2 S^{n-2} + (1 + \theta) \sum_{i \neq j} \alpha_i \alpha_j \int_{\partial M} \varphi_{a_i, \lambda}^{n-2} \varphi_{a_j, \lambda} + \sum_{i=1}^{p} \alpha_i^2 \frac{\gamma(n-1)}{n-2} \right)^{\frac{n-1}{n-2}}}{\sum_{i=1}^{p} \alpha_i S^{n-2} + \sum_{i \neq j} \frac{\gamma(n-1)}{n-2} \alpha_i \alpha_j \int_{\partial M} \varphi_{a_i, \lambda}^{n-2} \varphi_{a_j, \lambda} - \sum_{i=1}^{p} \alpha_i \frac{2^{(n-1)}}{\lambda^{n-2}}}
\]

where \(C\) and \(C'\) are positive constants independent of \((\alpha_1, \cdots, \alpha_p), \lambda\) and \(p\).

Let us assume that

\[
\sum_{i \neq j} \int_{\partial M} \varphi_{a_i, \lambda}^{n-2} \varphi_{a_j, \lambda} < \varepsilon_1.
\]

If \(\varepsilon_1\) is chosen small enough, then for \(\lambda\) large enough , we have

\[
\int_{\partial M} \left( \varphi_{a_1, \lambda}^{n-2} \varphi_{a_2, \lambda} + \varphi_{a_2, \lambda}^{n-2} \varphi_{a_1, \lambda} \right) = O(\frac{1}{\lambda^{n-2}})
\]

Making an expansion we have

\[
J\left( \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda} \right) \leq \frac{\left( \sum_{i=1}^{p} \alpha_i^2 S^{n-2} + (1 + \theta) \sum_{i \neq j} \alpha_i \alpha_j \int_{\partial M} \varphi_{a_i, \lambda}^{n-2} \varphi_{a_j, \lambda} + \sum_{i=1}^{p} \alpha_i^2 \frac{\gamma(n-1)}{n-2} \right)^{\frac{n-1}{n-2}}}{\sum_{i=1}^{p} \alpha_i \frac{2^{(n-1)}}{\lambda^{n-2}}}
\]

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Finally we obtain:

\[
J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda}) \leq \frac{(\sum_{i=1}^{p} \alpha_i^2)^{1/2}}{\sum_{i=1}^{p} n \alpha_i} \left(1 + \frac{n-1}{n-2} \sum_{i \neq j} \frac{\alpha_i \alpha_j}{\alpha_i^{n-1}S^{n-2}} \int_{\partial M} \varphi_{a_i, \lambda} \varphi_{a_j, \lambda} \right)
- \frac{n-1}{n-2} \sum_{i \neq j} \frac{\alpha_i \alpha_j}{\alpha_i^{n-1}S^{n-2}} \int_{\partial M} \varphi_{a_i, \lambda} \varphi_{a_j, \lambda} + \frac{C'}{\lambda^{n-2}}
\]

Let us observe that (iii) follows from (14), so it remains only to prove (ii).

Proof of (ii)

Let us now assume that there exists \( \theta_0 \), \( 0 < \theta_0 < 1 \) and \( \frac{\alpha_i}{\alpha_j} \geq \theta_0 \) for any \((i, j)\), then

\[
\frac{\alpha_i \alpha_j}{\sum_{r=1}^{p} \alpha_r^2} \leq \frac{1}{p\theta_0^2} \quad \text{and} \quad \frac{\alpha_i^{n-1} \alpha_j}{\sum_{r=1}^{p} n \alpha_r^{n-1} S^{n-2}} \geq \frac{2(n-1)}{p} \theta_0^{n-2}
\]

Now we choose \( 0 < \theta_0 < 1 \) and \( \theta > 0 \) such that \( (1 + \theta) \frac{1}{\theta_0} - \gamma \theta_0^{2(n-1)} < 0 \).

Let \( \delta = S^{n-2} \left( \frac{\gamma (n-1)}{n-2} \theta_0^{2(n-1)} - \frac{n-1}{n-2} \frac{1+\theta}{\theta_0} \right) \)

We then derive

\[
\sum_{i=1}^{p} \frac{\alpha_i \varphi_{a_i, \lambda}}{\sum_{r=1}^{p} \alpha_r^{n-1} S^{n-2}} S \left(1 - \frac{\delta}{p} \sum_{i \neq j} \frac{\alpha_i \alpha_j}{\alpha_i^{n-1}S^{n-2}} \int_{\partial M} \varphi_{a_i, \lambda} \varphi_{a_j, \lambda} + \frac{C'}{\lambda^{n-2}} \right)
\]

Then using (iii) of Lemma 2.4, we derive (ii) from (14). The proof of Lemma 4.1 is thereby complete.

Proof of the claim

Let \( \varepsilon_0 > 0 \) be given and let

\[
E_0 = \{ x \in \partial M \text{ such that } \delta_{a_1, \lambda} (x) \geq \varepsilon_0 (2\delta_{a_1, \lambda} + \delta_{a_2, \lambda}) (x) \}
\]

We have:

\[
\int_{\partial M} \frac{\delta_{a_2, \lambda}}{2\delta_{a_1, \lambda} + \delta_{a_2, \lambda}} \geq \varepsilon_0 \int_{E_0} \frac{\delta^{\frac{n-1}{2}}_{a_1, \lambda} \delta^{\frac{n-1}{2}}_{a_2, \lambda}}{\beta_0}
\]

Let us observe that

\[
\int_{\partial M} \delta^{\frac{n-1}{2}}_{a_1, \lambda} \delta^{\frac{n-1}{2}}_{a_2, \lambda} = \int_{\partial M} \delta^{\frac{n-1}{2}}_{a_2, \lambda} \delta^{\frac{n-1}{2}}_{a_1, \lambda}
\]

and

\[
\int_{\partial M} \delta^{\frac{n-1}{2}}_{a_1, \lambda} \delta^{\frac{n-1}{2}}_{a_2, \lambda} \leq \frac{1}{2} \left( \int_{\partial M} \delta^{\frac{n-1}{2}}_{a_1, \lambda} \delta^{\frac{n-1}{2}}_{a_2, \lambda} + \delta_{a_1, \lambda} \delta^{\frac{n-1}{2}}_{a_2, \lambda} \right)
\]

Then (15) becomes
\[ (16) \quad \int_{\partial M} \frac{\delta_{a_2,\lambda}}{2\delta_{a_1,\lambda} + \delta_{a_2,\lambda}} \geq \frac{\varepsilon_0}{2} \left( 1 - \left( \frac{\varepsilon_0}{1 - \varepsilon_0} \right)^{\frac{1}{n-2}} \right) \int_{\partial M} \delta_{a_1,\lambda}^{\frac{n}{n-2}} \delta_{a_2,\lambda} + \delta_{a_1,\lambda} \delta_{a_2,\lambda} \]

Thus for \( \varepsilon_0 \) small enough, we have:

\[ \int_{\partial M} \frac{\delta_{a_2,\lambda}}{2\delta_{a_1,\lambda} + \delta_{a_2,\lambda}} \geq C\varepsilon_0 \int_{\partial M} \delta_{a_1,\lambda}^{\frac{n}{n-2}} \delta_{a_2,\lambda} + \delta_{a_1,\lambda} \delta_{a_2,\lambda}. \]

\[ \geq C'\varepsilon_0 \int_{\partial M \cap B_p(a_1)} \delta_{a_1,\lambda}^{\frac{n}{n-2}} \delta_{a_2,\lambda} + \delta_{a_1,\lambda} \delta_{a_2,\lambda} \]

\[ \geq C''\varepsilon_0 \int_{\partial M \cap B_p(a_1)} \varphi_{a_1,\lambda}^{\frac{n}{n-2}} \varphi_{a_2,\lambda} + \varphi_{a_1,\lambda} \varphi_{a_2,\lambda}. \]

\[ (17) \]

Hence our claim is proved.

Lemma 4.1 implies the following proposition:

**Proposition 4.1** There exists an integer \( p_0 \) and a positive real number \( \lambda_0 > 0 \) such that for any \((\alpha_1, \cdots, \alpha_p)\) satisfying \( \alpha_i \geq 0 \), \( \sum_{i=1}^{p} \alpha_i = 1 \), for any \((a_1, \cdots, a_p) \in \partial M\), for any \( \lambda \geq \lambda_0 \), we have

\[ J\left( \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda} \right) \leq \frac{1}{p} S \]

**Proof.** The proof of Proposition 4.1 follows from (i), (ii) and (iii) of Lemma 4.1. We first choose \( 0 < \varepsilon_1 < \frac{1}{\lambda_1} \) and \( \lambda_0 \) so that:

\[ \left( \sum_{i=1}^{p} \alpha_i^2 \right)^{\frac{n-1}{n-2}} \left( 1 + O\left( \frac{1}{\lambda^{n-2}} \right) + \frac{\varepsilon_1}{C} \right) < p^{\frac{1}{n-2}} \]

Considering \((\alpha_1, \cdots, \alpha_p), (a_1, \cdots, a_p)\) and \( \lambda \geq \lambda_0 \), we study various cases:

**1st case:** There exists \((i_0, j_0)\) such that \( \frac{\alpha_{i_0}}{\alpha_{j_0}} \leq \theta_0 \), then taking, \( \lambda \geq \lambda_{p_0} = \sup(\lambda(p_0, \varepsilon), \lambda_0) \) where \( \lambda(p_0, \varepsilon) \) is given by (i) of Lemma 4.1, we derive:

\[ J\left( \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda} \right) \leq \frac{1}{p} S \quad \text{if} \quad \sum_{i \neq j} \int_{\partial M} \varphi_{a_i,\lambda} \varphi_{a_j,\lambda} d\sigma_g \geq \varepsilon_1 \]

If on the contrary \( \sum_{i \neq j} \int_{\partial M} \varphi_{a_i,\lambda} \varphi_{a_j,\lambda} d\sigma_g \leq \varepsilon_1 \) we apply (iii). Since we have choose \( \varepsilon_1 \) such that:

\[ \left( \sum_{i=1}^{p} \alpha_i^2 \right)^{\frac{n-1}{n-2}} \left( 1 + O\left( \frac{1}{\lambda^{n-2}} \right) + \frac{\varepsilon_1}{C} \right) < p^{\frac{1}{n-2}} \]

we derive that \( J\left( \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\lambda} \right) \leq \frac{1}{p} S \) and the proof of Proposition 4.1 is established in this case.
2nd case: Let us assume that \( \frac{\alpha_i}{\alpha_j} > \theta_0 \) for any \((i,j)\), then either (i) or (ii) of Lemma 4.1 holds. If (i) holds then Proposition 4.1 holds, so let assume that (ii) holds and then choose \( p_0 \) such that:

\[
(p_0 + 1)c^2 > 1
\]

Then

\[
J\left( \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda} \right) \leq \frac{p_0}{p_0 + 1} S
\]

The proof of Proposition 4.1 is thereby complete. ■

5 Proof of Theorem 1.2

For the proof of the Theorem 1.2, we introduce the following notations:

For any \( p \geq 1 \) and \( \lambda > 0 \), let

\[
B_p = B_p(\partial M) = \left\{ \sum_{i=1}^{p} \alpha_i \delta_{a_i}, \alpha_i \geq 0, \sum_{i=1}^{p} \alpha_i = 1, a_i \in \partial M \right\}
\]

and \( B_0 = B_0(\partial M) = \emptyset \)

Set also \( f_p(\lambda) \) to denote the map from \( B_p(\partial M) \) to \( \Sigma^+ \) defined by

\[
f_p(\lambda)\left( \sum_{i=1}^{p} \alpha_i \delta_{a_i} \right) = \frac{\sum_{i=1}^{p} \varphi_{a_i, \lambda}}{\left\| \sum_{i=1}^{p} \varphi_{a_i, \lambda} \right\|}
\]

Clearly we have \( B_{p-1} \subset B_p \) and \( W_{p-1} \subset W_p \).

Moreover \( f_p(\lambda) \) enjoys the following properties:

**Proposition 5.1** The function \( f_p(\lambda) \) has the following properties:

(i) For any integer \( p \geq 1 \), there exists a real number \( \lambda_p > 0 \) such that

\[
f_p(\lambda) : B_p(\partial M) \rightarrow W_p \text{ for any } \lambda \geq \lambda_p
\]

(ii) There exists an integer \( p_0 > 1 \), such that for any integer \( p \geq p_0 \), and for any \( \lambda \geq \lambda_{p_0} \), the map of pairs \( f_p(\lambda) : (B_p, B_{p-1}) \rightarrow (W_p, W_{p-1}) \) satisfies \( (f_p)_*(\lambda) \equiv 0 \) where

\[
(f_p(\lambda))_* : H_*(B_p, B_{p-1}) \rightarrow H_*(W_p, W_{p-1})
\]

and \( H_* \) is the \( * \)th homology group with \( \mathbb{Z}_2 \) coefficients.

**Proof.**

(i) is a direct consequence of the inequalities (i), (ii) and (iii) of Lemma 4.1, indeed:

\[
J(f_p(\lambda)\left( \sum_{i=1}^{p} \alpha_i \delta_{a_i} \right)) = J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i, \lambda_i})
\]

(ii) follows from Proposition 4.1. ■

For the sequel we need the following notations: Let \( \Delta_{p-1} = \{(\alpha_1, \cdots, \alpha_p), \alpha_i \geq 0, \sum_{i=1}^{p} \alpha_i = 1\} \) and \( F_p = \{(a_1, \cdots, a_p) \in (\partial M)^p \text{ such that } 3 i \neq j \text{ with } a_i = a_j\} \). Let \( \sigma_p \) be the symmetric group of order \( p \), which acts on \( F_p \), and let \( T_p \) be a \( \sigma_p \)-equivariant tubular neighborhood of \( F_p \), in \( (\partial M)^p \) (The existence of a such neighborhood is derived in the book of G. Bredon [8]).
From another part, considering the topological pair \((B_p, B_{p-1})\) we observe that \((B_p \setminus B_{p-1})\) can be described as \(((\partial M)^p)^* \times_{\sigma_p} (\Delta_p \setminus \partial \Delta_{p-1})\) where \((\partial M)^p)^* = \{(a_1, \cdots, a_p) \in (\partial M)^p\} such that \(a_i \neq a_j, \forall i \neq j\). We notice that \(((\partial M)^p)^* \times_{\sigma_p} (\Delta_p \setminus \partial \Delta_{p-1})\) is a noncompact manifold of dimension \((n-1)p + p - 1\). Let for \(0 < \theta < 1, M_p = V_p \times_{\sigma_p} \Delta_p^\theta, V_p = M_p \setminus T_p\), and \(\Delta_p^\theta = \{(a_1, \cdots, a_p) \in \Delta_p \setminus \partial \Delta_{p-1}|\sum \theta_j \in [1-\theta, 1+\theta], \forall i, \forall j\}\). \(M_p\) is a manifold which can be seen as a subset of \(B_p\), and the topological pair \((B_p, M^c)\) retracts by deformation onto \((B_p, B_{p-1})\), we thus have

\[
H_*(B_p, B_{p-1}) = H_*(B_p, M^c_p)
\]

Thus by excision we have

\[
H_*(B_p, B_{p-1}) = H_*(M, \partial M_p)
\]

Since any manifold is orientable modulo its boundary with \(\mathbb{Z}_2\) coefficients, we have a nonzero orientation class in \(H_{n-1}(\partial M_p)\) which we denote by \(\omega_p\).

In contrast with Proposition 5.1, we have the following Proposition:

**Proposition 5.2** Under the assumption that \((P)\) has no solution, we have,

\[
\text{for every } p \in \mathbb{N}^+ \quad (f_p(\lambda))_*(\omega_p) \neq 0.
\]

**Proof.**
An abstract topological argument displayed in [6], pp 260-265, see also [2], which extends virtually to our framework shows that:

\[
\text{If } (f_1(\lambda))_* \neq 0 \text{ then } (f_p(\lambda))_*, \forall p \geq 2.
\]

Since \(J_{S+\epsilon}\) for \(\epsilon > 0\) small enough satisfies \(J_{S+\epsilon} \subset V(1, \delta)\), where \(\delta \to 0\) if \(\epsilon \to 0\), one can define using Lemma 3.3 a continuous map \(s : J_{S+\epsilon} \to \partial M\) which associates to \(u = (\pi, \lambda, \lambda) + v \in J_{S+\epsilon}\) to \(\sum \pi_i = a \in \partial M\).

Here \((\pi, \lambda, \lambda)\) are the unique solution of the minimization:

\[
\min \{|u - \alpha \varphi_{a, \lambda}|, \alpha \geq 0, \lambda > 0, a \in \partial M\}
\]

So if \(r : W_1 \to J_{S+\epsilon}\) denotes the retraction by deformation of \(W_1\) onto \(J_{S+\epsilon}\), the existence of such retraction by deformation follows from the assumption that \((P)\) has no solution from one part and from Proposition 5.1 from another part. Let us observe that \(s \circ r \circ f_1(\lambda) = id_{\partial M}\) hence \((f_1(\lambda))_*(\omega_1) \neq 0\), where \(\omega_1\) is the orientation class of \(\partial M\). Therefore the proof of Proposition 5.2 is reduced to the abstract topological argument of Bahri-Coron [4].

**Proof of Theorem 1.2 completed.**
Proposition 5.3 is in contradiction with Proposition 5.1. Therefore \((P)\) has a solution and Theorem 1.2 is thereby established.

6 Appendix

**Lemma 6.1** There holds

\[
\frac{1}{\lambda^2} \int_{\partial M \cap B_{\rho}(a_1)} \delta^2_{a_1, \lambda} \delta_{a_2, \lambda} dv_{g_0} = o(\int_{\partial M} \delta^2_{a_1, \lambda} \delta_{a_2, \lambda} dv_{g_0})
\]

**Proof.** For \(\epsilon > 0\) a fixed number, let

\[
A_\epsilon = \{x \in B_p(a_1) \cap \partial M; \delta_{a_1, \lambda} \geq \frac{1}{\epsilon \lambda^2}\}
\]

Then
Since $\frac{n}{2} + \frac{(n-2)^2}{2n} > n-2$ and since
\[ \int_{\partial M} \delta_{a_1, \lambda} \delta_{a_2, \lambda} dv_{g_0} \geq \frac{C}{\lambda^{n-2}} \]
then our Lemma follows. 

**Lemma 6.2 [5]** Let $q > 2$ be given. There exists $\gamma > 1$ such that for any $(a_1, \cdots, a_p), a_i > 0$, we have
\[ \left( \sum_{i=1}^{p} \alpha_i \right)^q \geq \sum_{i=1}^{p} \alpha_i^p + \frac{\gamma q}{2} \sum_{i \neq j} a_i^{q-1} a_j \]

**Lemma 6.3 [16] [Maximum Principle]**

Under the assumption that $R_g \geq 0$, $h_g \geq 0$ and $R_g > 0$, or $h_g > 0$, let $u \in C^2(\tilde{M}) \cap C^1(\partial M)$ satisfying
\[ L_g u \geq 0 \quad \text{on} \quad \tilde{M} \quad \text{and} \quad B_g u \leq 0 \quad \text{on} \quad \partial M \]
Then $u \leq 0$ on $M$.

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