THE PHYSICAL APPROACH ON THE SURFACES OF ROTATION IN $E^4_2$

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Abstract. In this paper, some physical expressions as the specific energy and the specific angular momentum on these surfaces of rotation are investigated using conditions being geodesic on rotational surfaces with the help of Clairaut’s theorem.

1. Introduction

The geodesics have been studied Riemannian geometry widely, metric geometry and general relativity. Furthermore, the equation of motion containing the energy and angular momentum is a natural topic using many applications. The mass $m$ of the particle whose motion traces out a geodesic path is unconnected in this problem, these physical features as energy and momentum that they include the mass as well proportioned factor will instead by changed by the specific features supplied by dividing out the mass. Therefore, since the kinetic energy is $E = \frac{1}{2m^2}$, the specific kinetic energy because of its motion in space physics, which the motion is very important in terms of its specific energy and angular momentum in [15, 16]. If a force is accountable for this acceleration, that is to say the normal force that it supplies the particle on the surface, it is perpendicular to the velocity of the particle. Therefore, the specific energy and the speed $V$ must be constant along a geodesic. Because the existence of this constant is a result of the one parameter rotational group of symmetries of the surface, as a constant of the movement introduces a new thing since the surface is invariant under any one parameter group of symmetries, [11]. In [1], the brief description of rotational surfaces is defined in Galilean 4-space by the authors. In [2], time-like geodesics are expressed using Clairaut’s theorem on the hyperbolic and elliptic rotational surfaces in $E^4_2$ by the authors. In [3], the magnetic rotated surfaces are defined in null cone $Q^2 \subset E^4_1$ by the authors. In [4], by using the conditions of being geodesic on the tubular surface obtained with the help of Clairaut’s theorem, the specific energy and the angular momentum are expressed by the authors. In [5], a brief description of the hyperbolic and elliptic rotational surfaces is defined using a curve and matrices in semi-Euclidean 4-space and different types of rotational matrices which are the subgroups of M by rotating the axis in $E^4_1$ are given by the authors. In [8], A new type of surfaces in Euclidean and Minkowski 4-space is constructed by performing two simultaneous rotations on a planar curve by the authors. Also, classification theorems of flat double rotational surfaces are proved by the authors.

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In [9], the authors discuss some issues of displaying 2D surfaces in 4-space, including the behaviour of surface normals under projection.

2. Preliminaries

Let $E_4^4$ denote the 4–dimensional pseudo-Euclidean space with signature $(2, 4)$, that is, the real vector space $\mathbb{R}^4$ endowed with the metric $(,)_{E_4^4}$ which is defined by

\[(,)_{E_4^4} = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,\]

or

\[g = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},\]

where $(x_1, x_2, x_3, x_4)$ is a standard rectangular coordinate system in $E_4^4$.

Recall that an arbitrary vector $v \in E_4^4 \backslash \{0\}$ can have one of three characters: it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null if $g(v, v) = 0$ and $v \neq 0$.

The norm of a vector $v$ is given by $\|v\| = \sqrt{g(v, v)}$ and two vectors $v$ and $w$ are said to be orthogonal if $g(v, w) = 0$. An arbitrary curve $x(s)$ in $E_4^4$ can locally be space-like, time-like or null.

A space-like or time-like curve $x(s)$ has unit speed, if $g(x', x') = \pm 1$.

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)$ be any three vectors in $E_4^4$. The pseudo-Euclidean cross product is given as

\[x \wedge y \wedge z = \begin{pmatrix}
-i_1 & -i_2 & i_3 & i_4 \\
x_1 & x_2 & x_3 & x_4 \\
y_1 & y_2 & y_3 & y_4 \\
z_1 & z_2 & z_3 & z_4
\end{pmatrix},\]

where $i_1 = (1, 0, 0, 0), i_2 = (0, 1, 0, 0), i_3 = (0, 0, 1, 0), i_4 = (0, 0, 0, 1)$.

The pseudo-Riemannian sphere $S^3_2(m, r)$ centred at $m \in E_4^4$ with radius $r > 0$ of $E_4^4$ is defined by

\[S^3_2(m, r) = \{x \in E_4^4 : (x - m, x - m) = r^2\}.\]

The pseudo-hyperbolic space $H^3_1(m, r)$ centred at $m \in E_4^4$ with radius $r > 0$ of $E_4^4$ is defined by

\[H^3_1(m, r) = \{x \in E_4^4 : (x - m, x - m) = -r^2\}.\]

The pseudo-Riemannian sphere $S^3_2(m, r)$ is diffeomorphic to $\mathbb{R}^2 \times S$ and the pseudo-hyperbolic space $H^3_1(m, r)$ is diffeomorphic to $S^1 \times \mathbb{R}^2$. The hyperbolic space $H^3(m, r)$ is given by

\[H^3(m, r) = \{x \in E_4^4 : (x - m, x - m) = -r^2, x_1 > 0\},\]

[7] [12], [13].

Definition 1. [10]. A one-parameter group of diffeomorphisms of a manifold $M$ is a regular map $\psi : M \times \mathbb{R} \to M$, such that $\psi_t(x) = \psi(x, t)$, where

1. $\psi_t : M \to M$ is a diffeomorphism
2. $\psi_0 = id$
3. $\psi_{s+t} = \psi_s \circ \psi_t$. 


This group is attached to a vector field $W$ given by $\frac{d}{dt}\psi_t(x) = W(x)$, and the group of diffeomorphism is said to be as the flow of $W$.

**Definition 2.** If a one-parameter group of isometries is generated by a vector field $W$, then this vector field is called as a Killing vector field, [10].

**Definition 3.** Let $W$ be a vector field on a smooth manifold $M$ and $\psi_t$ be the local flow generated by $W$. For each $t \in \mathbb{R}$, the map $\psi_t$ is diffeomorphism of $M$ and given a function $f$ on $M$, one considers the Pull-back $\psi_t f$, the Lie derivative of the function $f$ as defined as to $W$ by

\begin{equation}
L_W f = \lim_{t \to 0} \left( \psi_t f - f \right) = \frac{d}{dt} \bigg|_{t=0} \psi_t f.
\end{equation}

Let $g_{\xi \theta}$ be any pseudo-Riemannian metric, then the derivative is given as

$$L_W g_{\xi \theta} = g_{\xi \theta} W z + g_{\xi \theta} W^z_x + g_{\xi \theta} W^z_\xi.$$

In Cartesian coordinates in Euclidean spaces where $g_{\xi \theta, z} = 0$, and the Lie derivative is given by

$$L_W g_{\xi \theta} = g_{\xi \theta} W^z_x + g_{\xi \theta} W^z_\xi,$$

the vector $W$ generates a Killing field if and if only

$$L_W g = 0.$$
$H = \left( f_1 f_4 \left( \frac{\ddot{x} \alpha + x \dot{x} \ddot{\alpha}}{2 f_1^2 \dot{x}^2 - f_1^2 \dot{\alpha}^2} \right) + f_1 f_4 - f_1^2 f_4 \alpha^2 \right) e_3 + \left( f_1^4 f_4^2 - f_1^2 f_4 \right) e_4 \over 2 \sqrt{-f_1^2 + f_4^2} \right)

where

$e_3 = \left( f_1 \cos \alpha x, f_1 x \cos \alpha, f_1 x \sin \alpha \right), e_4 = \left( f_1 \cos \alpha x, f_1 x \sin \alpha, f_1 x \cos \alpha \right).

For the rotations $\Omega_2 = 2 \partial \varphi + \rho \partial \theta$, the hyperbolic surface of rotation is given as

$S_{23}(y, z, s) = \left( f_1 \cosh y + f_4 \sinh y, f_2 \cosh z + f_3 \sinh z, f_1 \sinh y + f_4 \cosh y \right)

and for the planar curve $\gamma(s) = (f_1(s), f_2(s), 0, 0)$ the Gaussian curvature $K$ and the mean curvature vector $H$ of the rotational surface $S_{23}(y(t), z(t), s) = (f_1 \cosh y, f_2 \cosh z, f_2 \sinh z, f_1 \sinh y)$ are given as

$K = - \left( \frac{f_1 f_2 + f_1 f_4 (y \dot{y})^2}{f_1 f_2^2 + f_1 f_4^2} \right); H = \left( \frac{f_1 f_2 (y \dot{y} + y \ddot{y})}{f_1 f_2^2 + f_1 f_4^2} e_3 \right)

where

$e_3 = \left( f_1 f_2 y, f_1 y \dot{z}, f_1 y \ddot{z} \right), e_4 = \left( f_1 f_2 y, f_1 y \dot{z}, f_1 y \ddot{z} \right).

For the rotations $\Omega_3 = \xi \partial \varphi - \rho \partial \theta$ and $\Omega_4 = \vartheta \partial \varphi - \eta \partial \theta$, the elliptic surface of rotation is given as

$S_{56}(\beta, \theta, s) = \left( f_1 \cos \beta + f_2 \sin \beta, f_3 \cos \theta + f_4 \sin \theta, f_1 \sin \beta + f_3 \sin \theta \right)

and for the planar curve $\gamma(s) = (0, f_2(s), 0, f_4(s))$ the Gaussian curvature $K$ and the mean curvature vector $H$ of the rotational surface $S_{56}(\beta(t), \theta(t), s) = (f_2 \sin \beta, f_2 \cos \beta, f_4 \sin \theta, f_4 \cos \theta)$ are given as

$K = - \left( \frac{f_2 f_4 (\ddot{y} \dot{\beta} - \ddot{\beta} \dot{\beta})}{f_2^2 \dot{\beta}^2 + f_4^2 \dot{\beta}^2} \right); H = \left( \frac{f_2 f_4 \dot{\beta}^2 - f_2^2 f_4 \dot{\beta}^2 + f_4^2 f_4^2 - f_2^2 f_4^2}{2 \sqrt{f_2^2 + f_4^2} \dot{\beta}^2} e_3 \right)

where

$e_3 = \left( \frac{-f_4^2 \beta \dot{y} \dot{\beta} - f_4^2 \dot{\beta} \dot{y} \dot{\beta}}{\sqrt{-f_2^2 + f_4^2} \dot{\beta}^2} \right), e_4 = \left( \frac{f_4^2 \beta \dot{y} \dot{\beta} - f_4^2 \dot{\beta} \dot{y} \dot{\beta}}{\sqrt{-f_2^2 + f_4^2} \dot{\beta}^2} \right).

Theorem 4. Let $\gamma(t) = (f_1(t), 0, 0, f_4(t))$ or $\gamma(t) = (0, f_2(t), f_3(t), 0))$, $f_1 \in C^\infty$ be a time-like geodesic curve on the hyperbolic surface of rotation $\mathbb{Y}^1$ in the $E^4_2$, let $f_1$ and $f_4$ be the distance functions from the axis of rotation to a point on the surface. Therefore, $2 f_1 \cos \varphi_1$ and $-2 f_4 \cosh \theta_1 \sin \varphi_1$ are constant along the curve $\gamma$ where $\varphi_1$ and $\theta_1$ are the angles between the meridians of the surface and the time-like geodesic $\gamma$. Conversely, if $2 f_1 \cos \varphi_1$ and $-2 f_4 \cosh \theta_1 \sin \varphi_1$ are constant along $\gamma$, if no part of some parallels of the surface of rotation, then $\gamma$ is time-like geodesic $[2]$. 
The general equation of geodesics on the rotational surface $\Upsilon^1 \subset E_2^4$, and for the parameters $\dot{x} = \frac{1}{f_1} \cos \varphi_1$ and $\dot{\alpha} = \frac{1}{f_4} \cosh \theta_1 \sin \varphi_1$, are given by
$$\frac{dt}{dx} = f_1 \sqrt{1 - \cosh^2 \theta_1 \tan^2 \varphi_1 - L \sec^2 \varphi_1}$$
or
$$\frac{dt}{d\alpha} = f_2 \sqrt{\cot^2 \varphi_1 \tan h^2 \theta_1 - L \sech^2 \varphi_1 \cosec^2 \varphi_1}.$$

**Theorem 6.** Let $\gamma(t) = (f_3(t), f_4(t), 0, 0)$ (or $\gamma(t) = (0, 0, f_3(t), f_4(t))$, $f_i \in C^\infty$ be a time-like geodesic curve on the hyperbolic surface of rotation $\Upsilon^2$ in the $E_2^4$, and let $f_1$ and $f_2$ be the distance functions from the axis of rotation to a point on the surface. Then, $2f_1 \cos \theta_2 \sinh \varphi_2$ and $2f_2 \sin \theta_2 \sinh \varphi_2$ are constant along the curve $\gamma$ where $\varphi_2$ and $\theta_2$ are the angles between the meridians of the surface and the time-like geodesic curve $\gamma$. Conversely, if $2f_1 \cos \theta_2 \sinh \varphi_2$ and $2f_2 \sin \theta_2 \sinh \varphi_2$ are constant along the curve $\gamma$, if no part of some parallels of the surface of rotation, then $\gamma$ is time-like geodesic [2].

**Theorem 7.** [2] The general equation of geodesics on the rotational surface $\Upsilon^2 \subset E_2^4$, and for the parameters $\dot{y} = \frac{\cos \theta_3 \sinh \varphi_2}{f_1}$ and $\dot{z} = \frac{\sinh \varphi_3 \sin \theta_2}{f_2}$, are given by
$$\frac{dt}{dx} = \frac{f_1}{\cos \theta_2 \sinh \varphi_2} \sqrt{\sinh^2 \varphi_2 - L}; \quad \frac{dt}{dz} = \frac{f_2}{\sinh \varphi_2 \sin \theta_2} \sqrt{\sinh^2 \varphi_2 - L}.$$

**Theorem 8.** Let $\gamma(t) = (0, f_3(t), 0, f_4(t))$ (or $\gamma(t) = (f_3(t), 0, f_3(t), 0)$, $f_i \in C^\infty$ be a time-like geodesic curve on the elliptic surface of rotation $\Upsilon^3 \subset E_2^4$, and let $f_2$ and $f_4$ be the distance functions from the axis of rotation to a point on the surface. Then, $2f_2 \sin \varphi_3 \cosh \theta_3$ and $2f_4 \sin \theta_3 \sin \varphi_3$ are constant along the curve $\gamma$ where $\varphi_3$ and $\theta_3$ are the angles between the meridians of the surface and the time-like geodesic curve $\gamma$. Conversely, if $2f_2 \sin \varphi_3 \cosh \theta_3$ and $2f_4 \sin \theta_3 \sin \varphi_3$ are constant along the curve $\gamma$, if no part of some parallels of the surface of rotation, then $\gamma$ is time-like geodesic curve [2].

**Theorem 9.** [2] The general equation of geodesics on the rotational surface $\Upsilon^3 \subset E_2^4$, and for the parameters $\dot{\beta} = \frac{\sin \varphi_3 \cosh \theta_3}{f_2}$ and $\dot{\nu} = \frac{\sin \theta_3 \sin \varphi_3}{f_4}$, are given by
$$\frac{dt}{d\beta} = i \frac{f_2}{\sin \varphi_3 \cosh \theta_3 \varphi_3} \sin \theta_3 \sin \varphi_3; \quad \frac{dt}{d\nu} = i \frac{f_4}{\sinh \theta_3 \sin \varphi_3} \sqrt{\sin^2 \varphi_3 + L}.$$

3. Physical approach on the surfaces of rotation in $E_2^4$

In this section, one investigates the point of view of a physicist, imagining tracing out a geodesic by identifying the affine parameter $s$ with the time, so that the picture is now of a point particle moving on the surface, tracing out a path called the orbit of the particle.

By using the variational approach, which produces the geodesics by extremizing an action functional on the space of all curves connecting any two fixed points on the surfaces of rotation (the hyperbolic surfaces of rotation $\Upsilon^1(x, \alpha, t)$, $\Upsilon^2(y, z, t)$ and the elliptic surface of rotation $\Upsilon^3(\beta, \theta, t)$). Hence, one can go a step further than all the Riemannian geometry discussions about covariant differentiation and parallel transport.

1) For the hyperbolic surface of rotation $\Upsilon^1$; one will try to obtain specific energy equations on this surface. Then, let $\Upsilon^1(x(s), \alpha(s), t(s))$ be a parametrized curve
on the surface. One can either extremes the arc length of the curve, which is
the integral of the length of the tangent of the curve, which is the integral of the
length of the tangent vector in any parameterization of the curve, namely the speed
function is given as
$$I_1 = \int ds = \int \frac{ds}{d\pi} d\pi = \int \sqrt{\left(\frac{dx}{d\pi}\right)^2 + \left(\frac{d\alpha}{d\pi}\right)^2 + \left(\frac{dt}{d\pi}\right)^2} d\pi,$$
which is clearly independent of a change of parameterization or the integral of half
the length squared of the tangent vector
$$I_2 = \frac{1}{2} \int \left(\frac{ds}{d\pi}\right)^2 d\pi = \frac{1}{2} \int \left(\left(f_1 \frac{dx}{d\pi}\right)^2 - \left(f_2 \frac{d\alpha}{d\pi}\right)^2 - \left(\frac{dt}{d\pi}\right)^2\right) d\pi,$$
which is equivalent to the previous case only for affinely parametrized curves for
which the speed $\frac{ds}{d\pi}$ is constant. The integrate is called a Lagrangian function, and
it is a function of the curve and its tangent vector. The second Lagrangian function
is just the energy function
$$L_2 = \left(x, \alpha, t, \frac{dx}{d\pi}, \frac{d\alpha}{d\pi}, \frac{dt}{d\pi}\right) = \frac{1}{2} \left(f_1 \dot{x}^2 - \frac{1}{2} \left(f_2 \dot{\alpha}^2 - \frac{1}{2} \left(\dot{t}\right)^2\right) = E^i,$$
while the first Lagrangian $L_1 = \frac{ds}{d\pi}$ is speed function given as
$$L_1 = \left(x, \alpha, t, \dot{x}, \dot{\alpha}, \dot{t}\right) = \sqrt{\left(f_1 \dot{x}\right)^2 - \left(f_2 \dot{\alpha}\right)^2 - \left(\dot{t}\right)^2}.$$
Both are independent of the azimuthal angle because of the rotational invariance
of the problem. Then, the Lagrange equation of motion of a particle, analogues to
equation defined in terms of the Lagrangian $L^i$ with the non-scalar time variable
$\pi$ as the parameter, are given by
$$\frac{d}{d\pi} \left(\frac{\partial L^i}{\partial \dot{a}^j}\right) = \frac{\partial L^i}{\partial a^j}, i, j = 1, 2, 3$$
[16], the second Lagrangian this produces the equations used in the main body of
the article, with the angular equation directly giving the constancy of the angular
momentum $l_i = \frac{\partial L^i}{\partial \alpha^j}$. Hence, the constancy of the momentum conjugate to $a$ is
written as
$$p^a = \frac{\partial L^i}{\partial \dot{a}^j},$$
[16], and let us now calculate the total time derivative of the Lagrangian $L^i$ as
follows
$$\frac{\partial L^i}{\partial \pi} = \frac{\partial L^i}{\partial a^j} \frac{\partial a^j}{\partial \pi} + \frac{\partial L^i}{\partial v^j} \frac{\partial v^j}{\partial \pi} = v^j,$$
by using the equations of motion and the definition of the three dimensional velocity
can be written as
$$\frac{\partial}{\partial \pi} \left(\frac{\partial L^i}{\partial v^j} v^j - L^i\right) = 0; \frac{\partial L^i}{\partial v^j} v^j - L^i = constant.$$
For the curve $Y^1(x(s), \alpha(s), t(s))$, to calculate the derivative of this tangent
vector along the curve on the surface, we need the product and chain rules. Thus,
the tangent vector of this curve can be evaluated using the chain rule and using theorem 4, one gets

\[
\frac{dY^1(x(s), \alpha(s), t(s))}{ds} = \frac{dx(s)}{ds} Y^1_x + \frac{d\alpha(s)}{ds} Y^1_\alpha + \frac{dt(s)}{ds} Y^1_t,
\]

(3.1)

\[
\gamma = N_x \cos \varphi_1 + N_z^2 \sin \varphi_1 = \dot{x} T^1_x + \dot{\alpha} T^1_\alpha + i T^1_t
\]

\[
\gamma = f_1 N_x \dot{x} + (f_2 N_\alpha \dot{\alpha} + t N_t) = N_x \cos \varphi_1 + N_z^2 \sin \varphi_1;
\]

(3.2)

\[
\gamma = \cos \varphi_1 N_x + \sin \theta_1 \sin \varphi_1 N_\alpha + \sinh \theta_1 \sin \varphi_1 N_t.
\]

(3.3)

The tangent vector of the geodesic is called the velocity and it is given as

\[
\vec{V}_1 = \frac{dY^1}{ds} = V^1_x T^1_x + V^1_\alpha T^1_\alpha + V^1_t T^1_t
\]

and we might as well write two component vectors notation for components with respect to the basis vectors \( T^1_x, T^1_\alpha, T^1_t \) as

\[
V^1_j = \frac{dz^j}{ds}; \langle V^1_x, V^1_\alpha \rangle = \left\langle \frac{dx}{ds}, \frac{d\alpha}{ds} \right\rangle
\]

and its magnitude \( V_1 = \sqrt{\langle V^1_x, V^1_\alpha \rangle} = \sqrt{g_{ij} \frac{dz^i}{ds} \frac{dz^j}{ds}} \) is the speed, which is just the time rate of change of the arc length along the curve \( \gamma \).

Think that \( V^1_x = V_1 \cos \varphi_1 \) and \( V^1_\alpha = V_1 \sin \theta_1 \sin \varphi_1 \) are just the radial vertical velocity while \( V^1_t \) is the horizontal angular velocity and \( V^1_\varphi = V_1 = V_1 \sin \theta_1 \sin \varphi_1 \) is the horizontal component of the velocity vector. The velocity can be represented in terms of polar coordinates in the tangent plane to make explicit its magnitude and slope angle with respect to the radial direction on the surface.

One represents the orthonormal components in terms of the usual polar coordinate variables in this velocity plane in which \( V^1_x \) is along the first axis, \( V^1_\alpha \) is along the second axis and \( V^1_t \) is along the third axis.

The speed plays the role of the radial variable in this velocity plane, while the angles \( \theta_1 \) and \( \varphi_1 \) give the direction of the velocity according to the direction \( Y^1_x \) in the counter clockwise sense in this plane. Also, one can say that the speed is constant along the geodesic for affinely parametrized geodesics.

It is to understand the system of two second order geodesic equations that one can use a standard physics technique of partially integrating them and so lessen them to two first order equations by using two constants of the movement. Therefore, it will be manipulated specific equations to accomplish. From the mass \( m \) of the point particle whose movement traces out a geodesic path is insufficient in this problem. Thus, the specific kinetic energy can be written given as follows

\[
E_{\text{specific energy}} = \frac{1}{2} V_1^2 = \frac{1}{2} (V_1^2 \cos^2 \varphi_1 + V_1^2 \cosh^2 \theta_1 \sin^2 \varphi_1 - V_1^2 \sinh^2 \theta_1 \sin^2 \varphi_1)
\]

(3.4)

\[
= \frac{1}{2} \left( f_1 \frac{dx}{ds} \right)^2 - \frac{1}{2} \left( f_1 \frac{d\alpha}{ds} \right)^2 - \frac{1}{2} \left( \frac{dt}{ds} \right)^2,
\]

by using the right side of the previous equations we can say that both the specific energy and speed are constant along a geodesic.
In the physics approach the specific energy of the particle is constant because of the point of view of its motion in space, it is only accelerated perpendicular to the surface. Therefore, its energy and hence specific kinetic energy \( E^1 \) must be constant and the speed \( V_1 = \sqrt{2E^1} \) must be constant along a geodesic according to this cause.

From Theorem 4 and Theorem 5, for \( x = \int \frac{1}{R} \cos \varphi_1 ds \) and \( \alpha = \int \frac{1}{R} \cosh \theta_1 \sin \varphi_1 ds \) one can define exactly as in the case of circular motion around an axis with radius \( R = f_1 \) and \( R_2 = f_4 \) or \( R_1 = f_1 \hat{e}_1 \) and \( R_2 = f_4 \hat{e}_2 \).

That is, to know the velocity \( V_1^{\alpha*} = V_1 \cos \varphi_1 = f_1 \frac{dx}{ds} \) and the velocity \( V_1^{\alpha*} = -V_1 \cosh \theta_1 \sin \varphi_1 = f_4 \frac{dt}{ds} \) the velocity \( V_1^{\alpha*} = V_1 \sinh \theta_1 \sin \varphi_1 = \frac{dz}{ds} \) in the angular direction multiplied by the radius \( f_2 \) and \( f_4 \). Physically, since the second geodesic equation, one writes the following equations:

\[
\frac{l_{\text{specific angular momentum}}}{\partial t} = V_1 f_1 = -2t = -2 \sinh \theta_1 \sin \varphi_1 V_1 \Rightarrow \frac{l_1}{2} = t.
\]

The specific angular momentum about the axis of symmetry is constant along a geodesic. This expression can be used to rewrite the variable angular velocity \( dx/ds \) and \( d\alpha/ds \) in the specific energy formula according to the constant specific angular momentum to obtain the constant specific kinetic energy that is given according to the radial motion and another constant of the motion is given as

\[
E_{\text{specific energy}} = \frac{V_1^2}{2} (\cos^2 \varphi_1 - \cosh^2 \theta_1 \sin^2 \varphi_1) - \frac{l_1^2}{8},
\]

2) For the hyperbolic surface of rotation \( \Upsilon^2(y(s), z(s), t(s)) \); similarly, one can write the speed function

\[
I_1^2 = \int ds = \int \frac{ds}{d\pi} d\pi = \int \sqrt{\left( \frac{dy}{d\pi} \right)^2 + \left( \frac{dz}{d\pi} \right)^2 + \left( \frac{dt}{d\pi} \right)^2} d\pi,
\]

which is clearly independent of a change of parameterization or the integral of half the length squared of the tangent vector

\[
I_2^2 = \frac{1}{2} \int \left( \frac{dx}{d\pi} \right)^2 d\pi = \frac{1}{2} \int \left( f_1 \frac{dy}{d\pi} \right)^2 + \left( f_2 \frac{dz}{d\pi} \right)^2 - \left( \frac{dt}{d\pi} \right)^2 d\pi,
\]

which is equivalent to the previous case only for affinely parametrized curves for the speed \( \frac{dx}{d\pi} \) being constant and is given as

\[
I_2^2 = (y, z, t, \dot{y}, \dot{z}, \dot{t}) = \sqrt{(f_1 \dot{y})^2 + (f_2 \dot{z})^2 - (\dot{t})^2}
\]

and the integrate is a Lagrangian function that is a function of the curve and its tangent vector. The second Lagrangian function is the energy function given as

\[
L_2 = \left( y, z, t, \frac{dy}{d\pi}, \frac{dz}{d\pi}, \frac{dt}{d\pi} \right) = \frac{1}{2} (f_1 \dot{y})^2 + \frac{1}{2} (f_2 \dot{z})^2 - \frac{1}{2} (\dot{t})^2 = E^2.
\]

Also, in order to calculate the derivative of this tangent vector along the curve \( \Upsilon^2(y(s), z(s), t(s)) \). Thus, the tangent vector of this curve can be evaluated using the chain rule

\[
\dot{\gamma} = \cosh \varphi_2 N_t + \cos \theta_2 \sinh \varphi_2 N_y + \sinh \varphi_2 \sin \theta_2 N_z
\]
and its magnitude $V_2$ is the speed, which is just the time rate of change of the arc length along the curve $\gamma$. Hence, by using theorem 6 and theorem 7, $V_2^\gamma = f_1V_2^\gamma$ = $V_2\cos\theta_2\sinh\varphi_2$ and $V_2^\zeta = f_2V_2^\zeta = V_2\sinh\varphi_2\sin\theta_2$ are just the radial vertical velocity while $V_2^t$ is the horizontal angular velocity and $V_2^\tau = V_2^t = V_2\cosh\varphi_2$ is the horizontal component of the velocity vector. Similarly, $V_3^\gamma$ is along the first axis, $V_3^\zeta$ is along the second axis and $V_3^\tau$ is along the third axis. Therefore, the specific kinetic energy can be given as

$$E_{\text{specific energy}}^2 = \frac{1}{2}V_2^2 = \frac{1}{2}\left(-V_2^2\cos^2\theta_2\sinh^2\varphi_2 - V_2^2\sinh^2\varphi_2\sin^2\theta_2 + V_2^2\cosh^2\varphi_2\right)$$

(3.7)

by using the right side of the previous equations, the specific energy and speed are constant along a geodesic. That is, its energy and hence specific kinetic energy $E^2$ are constant and the speed $V_2 = \sqrt{2E^2}$ is constant along a geodesic, the velocities $V_2^\gamma = V_2\cos\theta_2\sinh\varphi_2 = f_1\frac{dy}{ds}$, $V_2^\zeta = V_2\sinh\varphi_2\sin\theta_2 = f_2\frac{dz}{ds}$ and $V_2^\tau = V_2\cosh\varphi_2 = \frac{dt}{ds}$ are in the angular direction multiplied by the radius $f_2$ and $f_1$ and from the second geodesic equation, one writes

$$l_{\text{specific angular momentum}} = \frac{\partial L_2}{\partial t} = -2t = -2\cosh\varphi_2V_1 = -2\cosh\varphi_2\sqrt{2E^2} \Rightarrow \frac{-l_2}{2} = t.$$

Since the specific angular momentum is constant along a geodesic, one can write the angular velocities $\frac{dy}{ds}$ and $\frac{dz}{ds}$ in the specific energy formula according to the constant specific angular momentum and according to the radial motion and another constant of the motion is obtained as follows

$$E_{\text{specific energy}} = \frac{V_2^2}{2}\left(\sinh^2\varphi_2 - \frac{l_2^2}{8}\right).$$

(3.8)

3) For the elliptic surface of rotation $T^3$; if one wants to obtain specific energy equations on this surface, one has to think the integral of the length of the tangent vector in any parametrization of the curve $T^3(\beta(s), \theta(s), t(s))$, then the speed function is given as follows

$$I_1^3 = \int ds = \int \frac{ds}{d\pi}d\pi = \int \sqrt{\left(\frac{d\beta}{d\pi}\right)^2 + \left(\frac{d\theta}{d\pi}\right)^2 + \left(\frac{dt}{d\pi}\right)^2}d\pi$$

and since this integrate is independent of a change of parametrization or the integral of half the length squared of the tangent vector, one gets

$$I_2^3 = \frac{1}{2} \int \left(\frac{ds}{d\pi}\right)^2d\pi = \frac{1}{2} \int \left(-\left(f_2\frac{d\beta}{d\pi}\right)^2 + \left(f_4\frac{d\theta}{d\pi}\right)^2 - \left(\frac{dt}{d\pi}\right)^2\right)d\pi,$$

and it is a function of its tangent vector of the curve. The second Lagrangian function is the energy function and is written as

$$L_2^3 = \left(\beta, \theta, t, \frac{d\beta}{d\pi}, \frac{d\theta}{d\pi}, \frac{dt}{d\pi}\right) = -\frac{1}{2}\left(f_2\beta\right)^2 + \frac{1}{2}\left(f_4\theta\right)^2 - \frac{1}{2}t^2 = E^3,$$
since the first Lagrangian \( L_1^3 \) is speed function one can write as

\[
L_1^3 = (\beta, v, t, \beta, \dot{v}, \dot{t}) = \sqrt{-\left(f_2 \dot{\beta}\right)^2 + \left(f_4 \dot{\theta}\right)^2 - t^2},
\]

and with the second Lagrangian, the angular equation is directly given the constancy of the angular momentum \( l_3 \). Also, in order to calculate the derivative of this tangent vector along the curve \( T^3(\beta(s), \theta(s), t(s)) \), by using the product and chain rules the tangent vector of this curve can be written as follows

\[
(3.9) \quad \dot{\gamma} = \cos \varphi_3 N_t + \sin \varphi_3 \cosh \theta_3 N_\beta + \sinh \theta_3 \sin \varphi_3 N_\theta.
\]

Also, the tangent vector(velocity) of the geodesic on \( T^3 \) is written as

\[
\frac{\vec{V}_3}{ds} = V_3^\beta Y_3^\beta + V_3^\theta Y_3^\theta + V_3^t Y_3^t
\]

and its magnitude \( V_3 \) is the speed. Also, by using theorem 8 and theorem 9, \( V_3^\beta = f_2 V_3^\beta = V_3 \sin \varphi_3 \cosh \theta_3 \) and \( V_3^\theta = f_4 V_3^\theta = V_3 \sinh \theta_3 \sin \varphi_3 \) are the radial velocity while \( V_3^t \) is the horizontal angular velocity and \( V_3^t = V_3^t \) is the horizontal component of the velocity vector. Here, \( V_3^\beta \) is along the first axis, \( V_3^\theta \) is along the second axis and \( V_3^t \) is along the third axis.

Similarly, the angles \( \theta_3 \) and \( \varphi_3 \) give the direction of the velocity according to the direction \( Y_3^\beta \). Also, the speed is constant along the geodesic. Therefore, the specific kinetic energy can be written as follows

\[
E_{\text{specific energy}}^3 = \frac{1}{2} V_3^2 = \frac{1}{2} \left( V_3^2 \sin^2 \varphi_3 \cosh^2 \theta_3 - V_3^2 \sin^2 \theta_3 \sin^2 \varphi_3 + V_3^2 \cos^2 \varphi_3 \right)
\]

\[
= -\frac{1}{2} \left( f_2 \frac{d\beta}{ds} \right)^2 + \frac{1}{2} \left( f_4 \frac{d\theta}{ds} \right)^2 - \frac{1}{2} \left( \frac{dt}{ds} \right)^2,
\]

the specific energy and speed also are constant along geodesic. Physically, the specific energy of the particle is constant because of the point of view of its motion in space. Since its specific kinetic energy \( E^3 \) is constant and the speed \( V_3 = \sqrt{2E^3} \) is constant along a geodesic. Hence, \( V_3^\beta = V_3 \sin \varphi_3 \cosh \theta_3 = f_2 \frac{d\beta}{ds}, V_3^\theta = -V_3 \sinh \theta_3 \sin \varphi_3 = f_4 \frac{d\theta}{ds} \) are velocities in the angular direction multiplied by the radius \( f_2 \) and \( f_4 \). Physically, by thinking the second geodesic equation given as

\[
l_3 = 2 \frac{\partial L_3}{\partial t} = -2t = -2 \cos \varphi_3 V_3 = -2 \cos \varphi_3 \sqrt{2E^3} \Rightarrow \frac{l_3}{2} = l,
\]

and by using the variable angular velocities \( d\beta/ds, d\theta/ds \) in the specific energy formula according to the radial motion and another constant of the motion are written as

\[
(3.11) \quad \frac{E_{\text{specific energy}}}{2} = \frac{V_3^2 \sin^2 \varphi_3}{2} - \frac{l_3^2}{8}.
\]

4. Conclusion

In taking into consideration the mathematical problem of the geodesics on a surface, there is an enormous advantage in conceptual comprehending that results from taking the point of view of a physicist by explaining parametrized geodesics
as the paths traced out in time by the motion of a point on the surface, this combination of the constants of the motion is of course also constant along a geodesic. The existence of this constant is a conclusion of the one-parameter rotational group of symmetries of the rotational surfaces, like this constant of the movement introduces a new thing when the surface is invariant under any one-parameter group of symmetries, which is seen in the variation approximate to the geodesic equations easily. Mathematically, this quantity is a constant obtained by Clairaut for geodesic movement on surface defined in a coordinate system adapted to this one-parameter group of symmetries, [13]. Thinking energy levels in impact potential for the decreased movement then supply to be an extremely useful tool in studying the treatment and features of the geodesics.

In this paper, the results show that the specific energy and the specific angular momentum on the surfaces of rotation can be expressed in $E^4_2$ using some certain results describing the geodesics on the rotational surface are given. The physical meaning of the specific energy and the angular momentum is of course related to the physical meaning itself. As a first instance, using the conditions of being geodesic, in which the curves can be chosen to be time-like curves, which allows us to constitute the specific energy and specific angular momentum.

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CONFLICT OF INTEREST

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