ACCURATE SEMICLASSICAL SPECTRAL ASYMPTOTICS FOR A TWO-DIMENSIONAL MAGNETIC SCHRÖDINGER OPERATOR

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Abstract. We revisit the problem of semiclassical spectral asymptotics for a pure magnetic Schrödinger operator on a two-dimensional Riemannian manifold. We suppose that the minimal value $b_0$ of the intensity of the magnetic field is strictly positive, and the corresponding minimum is unique and non-degenerate. The purpose is to get the control on the spectrum in an interval $(hb_0, h(b_0 + \gamma_0)]$ for some $\gamma_0 > 0$ independent of the semiclassical parameter $h$. The previous papers by Helffer-Mohamed and by Helffer-Kordyukov were only treating the ground-state energy or a finite (independent of $h$) number of eigenvalues. Note also that N. Raymond and S. Vu Ngoc have recently developed a different approach of the same problem.

1. Introduction and main results

Let $M$ be a compact connected oriented manifold of dimension $n \geq 2$ (possibly with boundary). Let $g$ be a $C^1$ Riemannian metric and $B \in C(M, \Lambda^2 T^*M)$ a continuous real-valued closed 2-form on $M$. Assume that $B$ is exact and choose a real-valued 1-form $A \in C^1(M, \Lambda^1 T^*M)$ on $M$ such that $dA = B$. Thus, one has a natural mapping

$$u \mapsto ih du + Au$$

from $C^1_c(M)$ to the space $C(M, \Lambda^1 T^*M)$ of continuous, compactly supported one-forms on $M$ and from $C^2_c(M)$ to the space $C(M, \Lambda^1 T^*M)$. The Riemannian metric allows us to define scalar products in these spaces and consider the adjoint operator

$$(ih d + A)^* : C^1(M, \Lambda^1 T^*M) \to C_c(M).$$

A Schrödinger operator with magnetic potential $A$ is defined on $C^2_c(M)$ by the formula

$$H^h = (ih d + A)^*(ih d + A).$$

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Here $h > 0$ is a semiclassical parameter, which is assumed to be small. If $M$ has a non-empty boundary, we will assume that the operator $H^h$ satisfies the Dirichlet boundary conditions.

We are interested in the semiclassical asymptotics of the low-lying eigenvalues of the operator $H^h$. This problem was studied in \[5, 12, 13, 14, 16, 17, 21, 29, 30, 31\] (see also references therein).

We suppose that $M$ is two-dimensional. Then we can write $B = b dx_g$, where $b \in C(M)$ and $dx_g$ is the Riemannian volume form. Let $b_0 = \min_{x \in M} |b(x)|$.

We assume that:

- $b_0 > 0$;
- the set $\{x \in M : |b(x)| = b_0\}$ is a point $x_0$, which is contained in the interior of $M$;
- $b$ is $C^\infty$ in a neighborhood of $x_0$ and there is a constant $C > 0$ such that for all $x$ in some neighborhood of $x_0$ the estimates hold:
  \[ C^{-1}d(x, x_0)^2 \leq |b(x)| - b_0 \leq Cd(x, x_0)^2. \]

We introduce

$$ a = \text{Tr} \left( \frac{1}{2} \text{Hess} b(x_0) \right)^{1/2}, \quad d = \text{det} \left( \frac{1}{2} \text{Hess} b(x_0) \right), $$

and denote by $\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots$ the eigenvalues of the operator $H^h$ in $L^2(M)$.

The following theorem proved in \[17, 14\] gives upper and lower estimates of $\lambda_j(H^h)$ as $h \to 0$. The contribution of \[14\] improves the result of \[17\] which only gives a two-terms asymptotics for the ground state energy in the flat case.

**Theorem 1.1.** Under current assumptions, for any $j \in \mathbb{N}$, there exist $C_j > 0$ and $h_j > 0$ such that, for any $h \in (0, h_j]$,

$$ hb_0 + h^2 \left[ \frac{2d^{1/2}}{b_0} j + \frac{a^2}{2b_0} \right] - C_j h^{19/8} \leq \lambda_j(H^h) \leq hb_0 + h^2 \left[ \frac{2d^{1/2}}{b_0} j + \frac{a^2}{2b_0} \right] + C_j h^{5/2}. $$

The main purpose of this paper is to reinterpret and extend Theorem 1.1 in the following setting. We will consider the magnetic Schrödinger operator $H^h$ in the flat Euclidean space $\mathbb{R}^2$:

\[(1.1)\]

$$ H^h = h^2 D_x^2 + (h D_y + A(x, y))^2. $$

The magnetic field $B$ is given by

$$ B = b dx \wedge dy \text{ with } b(x, y) = \frac{\partial A}{\partial x}(x, y). $$
Let
\[ b_0 = \min_{(x,y) \in \mathbb{R}^2} |b(x,y)|. \]

We assume
\[ |b(x,y)| < b_0 + \eta_0 := \liminf_{|x|+|y| \to \infty} |b(x,y)|, \quad \eta_0 > 0. \]

One can prove (see Theorem 2.1) that, for any \( \eta_1 < \eta_0 \), there exists \( h_1 > 0 \) such that
\[ \sigma(H^h) \cap [0, h(b_0 + \eta_1)) \subset \sigma_d(H^h), \quad \forall h \in (0, h_1]. \]

As above, we assume that:
- \( b_0 > 0; \)
- the set \( \{(x, y) \in \mathbb{R}^2 : |b(x,y)| = b_0\} \) is a single point \((x_0, y_0)\);
- \( b \) is \( C^\infty \) in a neighborhood of \((x_0, y_0)\), and \((x_0, y_0)\) is a non-degenerate minimum:
  \[ \text{Hess} b(x_0, y_0) > 0. \]

We take linear coordinates in \( \mathbb{R}^2 \) such that \((x_0, y_0) = (0, 0)\). We can also assume after possibly a gauge transformation that:
\[ (1.2) \quad A(0,0) = 0 \text{ and } \frac{\partial A}{\partial y}(0,0) = 0. \]

We have a diffeomorphism \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by
\[ \phi(x,y) = (A(x,y), y), \quad (x,y) \in \mathbb{R}^2. \]

We then associate with \( b \) a function \( \hat{b} \in C^\infty(\mathbb{R}^2) \) by
\[ \hat{b} = b \circ \phi^{-1}. \]

Our goal is to prove the following theorem.

**Theorem 1.2.** There exist \( h_0 > 0, \epsilon_0 > 0, \gamma_0 \in (0, \eta_0), h \mapsto \gamma_0(h) \) defined for \((0, h_0]\) such that \( \gamma_0(h) \to \gamma_0 \) as \( h \to 0 \), and a semiclassical symbol \( p_{\text{eff}}(y, \eta, h, z) \), which is defined in a neighborhood \( \Omega \subset \mathbb{R}^2 \) of the set \( \{(y, \eta) \in \mathbb{R}^2 : b(y, \eta) \leq b_0 + \gamma_0\} \) for \( h \in (0, h_0] \) and \( z \in \mathbb{C} \) such that \(|z| < \gamma_0 + \epsilon_0\), of the form
\[ (1.3) \quad p_{\text{eff}}(y, \eta, h, z) \sim \sum_{j \in \mathbb{N}} p_{\text{eff}}^j(y, \eta, z) h^j, \]

with
\[ (1.4) \quad p_{\text{eff}}^0(y, \eta, z) = \hat{b}(y, \eta) - b_0 - z, \]
such that \( \lambda_h \in \sigma(H^h) \cap [0, h(b_0 + \gamma_0(h))) \), if and only if the associated \( h \)-pseudodifferential operator \( p_{\text{eff}}(y, hD_y, h, z(h)) \) has an approximate 0-eigenfunction \( u_h^{qm} \in C^\infty(\mathbb{R}) \), i.e.
\[ (1.5) \quad p_{\text{eff}}(y, hD_y, h, z(h)) u_h^{qm} = \mathcal{O}(h^\infty), \]
with
\[ z(h) = \frac{1}{h} (\lambda h - h b_0) + \mathcal{O}(h^\infty), \]
|z(h)| < \gamma_0(h) for any \( h \in (0, h_0] \), and such that the frequency set of \( u_{h_0}^{qm} \) is non-empty and contained in \( \Omega \).

**Remark 1.3.** Here (1.5) makes sense modulo \( \mathcal{O}(h^\infty) \) by extending first the symbol \( p_{\text{eff}}(y, \eta, h, z) \) outside the neighborhood \( \Omega \) to a semiclassical symbol in \( \mathbb{R}^2 \) and defining then the operators \( p_{\text{eff}}(y, hD_y, h, z) \) by the Weyl calculus. Using the localization of the frequency set of \( u_{h_0}^{qm} \), the left hand side of (1.5) does not depend on the extension up to an error which is \( \mathcal{O}(h^\infty) \).

**Remark 1.4.** For any \( E \in [b_0, b_0 + \gamma_0) \), the spectrum of the operator \( H^h \) (divided by \( h \)) is determined near \( E \) (say in an interval \( (E - Ch^{\frac{1}{2}}, E + Ch^{\frac{1}{2}}) \)) and modulo \( \mathcal{O}(h^{\frac{3}{2}}) \) by the spectrum of \( \hat{b}(y, hD_y) + h b_1(y, hD_y, E) \), where one can use the Bohr-Sommerfeld rule (see [19] or [23] for a mathematical justification) for determining the energy levels.

**Remark 1.5.** Of course \( \gamma_0 \) is such that \( b(x, y) \) is \( C^\infty \) in a neighborhood of \( b^{-1}((0, b_0 + \gamma_0]) \).

Denote by \( \lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots \) the eigenvalues of the operator \( H^h \) in \( [0, h(b_0 + \gamma_0)] \).

**Theorem 1.6.** Under current assumptions, for any \( j \in \mathbb{N} \), there exists a sequence \( (\alpha_{j,\ell})_{\ell \in \mathbb{N}} \) such that
\[ \lambda_j(H^h) \sim h \sum_{\ell=0}^{\infty} \alpha_{j,\ell} h^\ell. \]
In other words, for any \( N \), there exist \( C_{j,N} > 0 \) and \( h_{j,N} > 0 \) such that, for any \( h \in (0, h_{j,N}] \),
\[ |\lambda_j(H^h) - h \sum_{\ell=0}^{N} \alpha_{j,\ell} h^\ell| \leq C_{j,N} h^{N+2}. \]

By the results of [17] (see Theorem 1.1), it follows that
\[ \alpha_{j,0} = b_0, \quad \alpha_{j,1} = \frac{2d^{1/2}}{b_0} j + \frac{a^2}{2b_0}. \]
In [14], it was shown that, in the case of magnetic Schrödinger operator on a two-dimensional Riemannian manifold, each \( \lambda_j \) admits an asymptotic expansion in the form
\[ \lambda_j(H^h) \sim h \sum_{\ell=0}^{\infty} \alpha_{j,\ell/2} h^{\ell/2}. \]

Theorem 1.6 improves this result in the flat case, showing that no odd powers of \( h^{1/2} \) actually occur. It is also proved in [33].
Corollary 1.7. There exists $\gamma_0 \in (0, \eta_0)$, $h_0 > 0$ and $C > 0$ such that
\[ \lambda_{j+1}(H^h) - \lambda_j(H^h) \geq \frac{1}{C} h^2, \quad \forall h \in (0, h_0), \]
for any $j$ such that $\lambda_{j+1}(H^h) < h(b_0 + \gamma_0)$.

The proof of Theorem 1.2 is based on Grushin’s method. As the name “Grushin’s method” indicates, the technique comes back to Grushin [33]. It was popularized by J. Sjöstrand starting from 1974 [36]. The method turned out to be very effective not only in hypoellipticity theory [36], [8], but also in spectral theory [22], [24]. The reader can find a nice presentation of this method in [37].

For the proof, we first make some changes of variables and asymptotic expansions to put the operator $H^h$ in a normal form near $(0, 0)$. Then we construct an appropriate Grushin problem in a neighborhood of $(0, 0)$ and apply Grushin’s method. This approach is local near the minimum point $(0, 0)$.

From a close but different point of view, the problem under consideration was studied by N. Raymond and S. Vu Ngoc [33]. Their proof is reminiscent of Ivrii’s approach (see [26] or in his book [27] in different versions Chapter 18) and uses a Birkhoff normal form. This approach has the advantage to be semi-global and uses more general symplectomorphisms and their quantizations.

Theorem 1.5 in [33] is stronger than our Theorem 1.2 because Theorem 1.2 gives a description of the spectrum of $H^h$ in the interval $[hb_0, h(b_0 + \eta_0)]$ for some $\gamma_0 \in (0, \eta_0)$, whereas, in [33] Theorem 1.5], $\gamma_0 \in (0, \eta_0)$ is arbitrary. On the other hand, the symbol of the effective Hamiltonian in [33] Theorem 1.5] seems to be less explicit than in Theorem 1.2 (see in Section 17). The other point could be that our approach allows us to treat an additional term $h^2V(x, y)$. This will complete the analysis of Helffer-Sjöstrand [22] in the case of the constant magnetic field (strong magnetic case). This kind of approach appears also in [28] (see Remark 3.1 and more specifically (3.5)). The case with an additional term of the form $hV$ could also be interesting. Note that, in the book [27] (see also the announcement in [26]), there is also an interesting normal form corresponding to the case of dimension 3. We hope to come back to this point in a near future.

The paper is organized as follows. In Section 2 we establish some general properties of the magnetic Schrödinger operator in the flat Euclidean space $\mathbb{R}^2$. First, we recall the proof that the spectrum of $H^h$ on the interval $[hb_0, h(b_0 + \eta_0)]$ is discrete. Then we show that, if $A$ is changed at infinity in such a way that a neighborhood of $b^{-1}([b_0, b_0 + \eta_0])$ is unchanged, then this change will only affect the spectrum of $H^h$ on the interval $[hb_0, h(b_0 + \gamma_0)]$ with $\gamma_0 < \eta_0$ by exponentially small corrections. This fact allows us to impose rather strong assumptions on $A$ in our further considerations. More precisely, we will assume that, for any $(k, \ell) \in \mathbb{N}^2$ with $k + \ell > 0$, the derivative $\partial_x^k \partial_y^\ell A$ is uniformly bounded in $\mathbb{R}^2$. In Section 3 using some changes
of variables and asymptotic expansions, we put the operator into a normal form. Section 4 is devoted to some asymptotic properties of eigenfunctions associated with the spectrum of \( H^h \) on the interval \([h b_0, h (b_0 + \eta_0)]\). First, we obtain an information on the frequency set of these eigenfunctions. Then we derive estimates for such eigenfunctions in the Sobolev spaces \( B^k(\mathbb{R}^2) \) defined in (3.15). Section 5 is devoted to construction and investigation of an appropriate Grushin problem. In Section 6, we complete the proof of Theorem 1.2. Finally we discuss in Section 7 possible extensions.

2. Preliminaries on the magnetic Schrödinger operator

In this section, we will discuss some general properties of the magnetic Schrödinger operator \( H^h_A \) in the flat Euclidean space \( \mathbb{R}^2 \) given by

\[
H^h_A = h^2 D_x^2 + (h D_y + A(x, y))^2,
\]

where \( A \in C^1(\mathbb{R}^2) \).^1

Under these assumptions, the operator \( H^h_A \) is essentially self-adjoint in \( L^2(\mathbb{R}^2) \) with initial domain \( C_\infty(\mathbb{R}^2) \) (see, for instance, [5, Theorem 1.2.2]).

Let \( b_0 = \min_{(x, y) \in \mathbb{R}^2} |b(x, y)| \).

We assume

\[
|b(x, y)| < b_0 + \eta_0 := \liminf_{|x| + |y| \to \infty} |b(x, y)|, \quad \eta_0 > 0.
\]

For any self-adjoint operator \( P \) in a Hilbert space \( \mathcal{H} \), we denote by \( \sigma(P) \) the spectrum of \( P \), by \( \sigma_d(P) \) the discrete spectrum of \( P \) and by \( \sigma_{\text{ess}}(P) \) the essential spectrum of \( P \).

**Theorem 2.1.** For any \( \eta_1 < \eta_0 \), we have

\[
\sigma(H^h_A) \cap [0, h (b_0 + \eta_1)) \subset \sigma_d(H^h_A), \quad \forall h > 0.
\]

**Proof.** We recall the estimate

\[
(H^h_A u, u) \geq h \int_{\mathbb{R}^2} |b(x, y)||u(x, y)|^2 \, dx \, dy, \quad \forall u \in H^1_0(\mathbb{R}^2).
\]

The theorem follows immediately from this estimate and Persson’s characterization of the bottom of the essential spectrum of a self-adjoint uniformly elliptic operator \( P \) on \( L^2(\mathbb{R}^n) \) (see [1]),

\[
\text{Inf} \sigma_{\text{ess}}(P) = \lim_{R \to \infty} \inf_{u \in D(P), \|u\| \neq 0, \supp u \subset \{|x| \geq R\}} \frac{(Pu, u)}{\|u\|^2}.
\]

\( \square \)

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^1The optimal condition of regularity of \( A \) is weaker but our assumption permits to avoid writing “almost everywhere” in the condition.
Theorem 2.2. Let \( \eta \in (0, \eta_0) \) with \( \eta_0 \) defined in (2.1). Assume that \( \tilde{A} \in C^1(\mathbb{R}^2) \) is such that \( \tilde{A}(x, y) = A(x, y) \) for any \( (x, y) \in \mathbb{R}^2 \) such that \( b(x, y) < b_0 + \eta \). For any \( \eta_1 \) such that \( 0 < \eta_1 < \eta \), there exist \( h_0 > 0 \), \( \alpha > 0 \), a map \( (0, h_0] \ni h \mapsto \epsilon(h) > 0 \), such that \( \epsilon(h) \to 0 \) as \( h \to 0 \) and, for any \( h \in (0, h_0] \), a one to one map

\[
\Psi^h : \sigma(H^h_A \cap [hb_0, h(b_0 + \eta_1 + \epsilon(h))] \to \sigma(H^h_A \cap [hb_0, h(b_0 + \eta_1 + \epsilon(h))]),
\]

such that

\[
|\Psi^h(\lambda) - \lambda| < e^{-\frac{\alpha}{h^{1/2}}} \quad \forall h \in (0, h_0].
\]

Proof. We choose a function \( \epsilon(h) \) defined for any \( h \in (0, h_0] \) with some \( h_0 > 0 \) so that there exist a function \( a(h) \), \( h \in (0, h_0] \), such that

\[
a(h) = O(h^{N_0})
\]

with some \( N_0 \) and, for any \( h \in (0, h_0] \), we have

\[
\sigma(H^h_A \cap [h(b_0 + \eta_1 + \epsilon(h)), h(b_0 + \eta_1 + \epsilon(h)) + a(h))] = \emptyset,
\]

\[
\sigma(H^h_A \cap [h(b_0 + \eta_1 + \epsilon(h)), h(b_0 + \eta_1 + \epsilon(h)) + a(h))] = \emptyset.
\]

To see the existence of such \( \epsilon(h) \) and \( a(h) \), we use a polynomial upper estimate for the number of eigenvalues of operators \( H^h_A \) and \( H^h_{\tilde{A}} \). Denote by \( N(H^h_A, \lambda) \) the number of eigenvalues of the operator \( H^h_A \) less than or equal to \( \lambda \). By \( [33] \) Lemma 4.6, for any \( C_1 < b_0 + \eta_0 \) there exists \( C > 0 \) such that for all \( h > 0 \), we have

\[
N(H^h_A, C_1 h) \leq C h^{-1}.
\]

Now we just take the interval \( [h(b_0 + \eta_1), h(b_0 + \eta_1 + ch)] \) with some \( c \), divide it into the union of disjoint intervals of the same length \( h^4 \), and, using (2.3), immediately get the existence of an interval of the above form free of eigenvalues for any \( h > 0 \) small enough.

Let \( E \) (resp. \( \tilde{E} \)) be the eigenspace of \( A \) (resp. \( \tilde{A} \)) associated with \( \sigma(H^h_A \cap [hb_0, h(b_0 + \eta_1 + \epsilon(h))] \) (resp. \( \sigma(H^h_A \cap [hb_0, h(b_0 + \eta_1 + \epsilon(h))] \)). By (2.3), we have

\[
dim E = O(h^{-1}), \quad \dim \tilde{E} = O(h^{-1}), \quad h \to 0.
\]

Let \( u_h \) be an eigenfunction of \( H^h_A \) with the corresponding eigenvalue \( \lambda(h) \), satisfying \( \lambda(h) \leq h(b_0 + \eta_1 + \epsilon(h)) \) for \( h \in (0, h_0] \). We observe that

\[
H^h_A u_h = \lambda(h) u_h + r,
\]

where

\[
r = 2(\tilde{A} - A)(hD_y + A)u_h + (hD_y(\tilde{A} - A) + (\tilde{A} - A)^2)u_h.
\]

By assumption, we get

\[
\|r\| \leq C \|u_h\|_{L^2(\mathbb{R}^2 \setminus U_\eta)} + \|u_h\|_{L^2(\mathbb{R}^2 \setminus U_\eta)},
\]

where

\[
U_\eta = \{(x, y) \in \mathbb{R}^2 : b(x, y) < b_0 + \eta\}.
\]
By Agmon estimates (see [16, 17]), there exist $C$ and $\gamma > 0$ such that
\[
\|(hD_y + A)u_h\|_{L^2(\mathbb{R}^2 \setminus U_\eta)} + \|u_h\|_{L^2(\mathbb{R}^2 \setminus U_\eta)} \leq Ce^{-\frac{\gamma}{\sqrt{h^2}}},
\]
that implies the estimate
\[
r = O\left(e^{-\frac{\gamma}{\sqrt{h^2}}}\right).
\]

Now we can apply [20, Proposition 2.5] (see also [10, Proposition 4.1.1]) and obtain the following estimates for the non-symmetric distances between $E$ and $\tilde{E}$:
\[
\tilde{d}(E, \tilde{E}) = \tilde{d}(\tilde{E}, E) = O\left(e^{-\frac{\alpha}{\sqrt{h^2}}}\right),
\]
with some $\alpha > 0$. As soon as we have proven these estimates, the construction of $\Psi_h$ can be done essentially in the same way as a similar construction in the proof of [20, Theorem 2.4] (see also [10, Theorem 4.2.1]).

\[\square\]

Theorem 2.2 allows us to continue our further investigations under very strong conditions on $A$. For instance, we can assume that $b$ is constant outside a compact set in $\mathbb{R}^2$. In the sequel, we will assume that, for any $(k, \ell) \in \mathbb{N}^2$ with $k + \ell > 0$,
\[
\sup_{(x,y) \in \mathbb{R}^2} \left| \frac{\partial^{k+l} A}{\partial x^k \partial y^l}(x,y) \right| < \infty.
\]
In particular, the functions $b$ and $\frac{\partial A}{\partial y}$ belong to the symbol class $S = S(1)$.

3. Towards normal forms

In this section, we will put the operator $H^h$ given by (1.1) in a normal form, using very explicit transformations and asymptotic expansions.

3.1. Some transformations. First, we write the operator $H^h$ in the form
\[
H^h = -X_1^2 - X_2^2,
\]
where
\[
X_1 = h\partial_x, \quad X_2 = h\partial_y + iA(x,y).
\]

3.1.1. Change of variables. Now we make a change of variables
\[
(x_1, y_1) = (A(x,y), y).
\]
In the new coordinates, we have
\[
\mathcal{B} = dx_1 \wedge dy_1.
\]
Define functions $\hat{b}$ and $\hat{A}_y$ on $\mathbb{R}^2$ by
\[
\hat{b}(A(x,y), y) = b(x,y), \quad \hat{A}_y(A(x,y), y) = \frac{\partial A}{\partial y}(x,y), \quad (x,y) \in \mathbb{R}^2.
\]
It is easy to see that $\hat{b}$ and $\hat{A}_y$ belong to the class $S$. By (1.2), it follows that the minimum of $\hat{b}$ is at $(0,0)$ and
\[
\hat{A}_y(0,0) = 0.
\]
For the operator

$$\tilde{H}^h(x_1,y_1,hD_{x_1},hD_{y_1}) = H^h(x,y,hD_x,hD_y),$$

we obtain that

$$\tilde{H}^h(x_1,y_1,hD_{x_1},hD_{y_1}) = -\bar{X}_1^2 - \bar{X}_2^2,$$

where

$$\bar{X}_1(x_1,y_1,hD_{x_1},hD_{y_1}) = \hat{b}(x_1,y_1)h\partial_{x_1},$$

$$\bar{X}_2(x_1,y_1,hD_{x_1},hD_{y_1}) = \hat{A}_g(x_1,y_1)h\partial_{x_1} + h\partial_{y_1} + ix_1.$$  

In the coordinates $ (x_1,y_1) $, the flat Euclidean metric $ g = dx^2 + dy^2 $ is written as

$$g = \hat{b}^{-2}(x_1,y_1)dx_1^2 - 2\hat{A}_g(x_1,y_1)\hat{b}(x_1,y_1)^{-2}dx_1dy_1 + (1 + \hat{b}^{-2}(x_1,y_1)\hat{A}_g(x_1,y_1)^2)dy_1^2.$$ 

The operator $ \tilde{H}^h $ is the magnetic Schrödinger operator associated with this metric and the constant magnetic field. It is self-adjoint with respect to the Riemannian volume form

$$\sqrt{\det g} \, dx_1 \, dy_1 = \hat{b}(x_1,y_1)^{-1} \, dx_1 \, dy_1.$$ 

Now we move the operator $ \tilde{H}^h $ into the Hilbert space $ L^2(\mathbb{R}^2, dx_1 dy_1) $, using the unitary isomorphism

$$u \in L^2(\mathbb{R}^2, \hat{b}(x_1,y_1)^{-1} \, dx_1 \, dy_1) \mapsto \hat{b}(x_1,y_1)^{-1/2} u \in L^2(\mathbb{R}^2, dx_1 dy_1).$$ 

For the corresponding operator $ \hat{H}^h = \hat{b}^{-1/2} \tilde{H}^h \hat{b}^{1/2} $, we obtain that

$$\hat{H}^h(x_1,y_1,hD_{x_1},hD_{y_1}) = -\hat{X}_1^2 - \hat{X}_2^2,$$

where

$$\hat{X}_1(x_1,y_1,hD_{x_1},hD_{y_1}) = \hat{b}^{-1/2} \bar{X}_1(x_1,y_1,hD_{x_1},hD_{y_1}) \hat{b}^{1/2}$$

$$= \hat{b}(x_1,y_1)h\partial_{x_1} + \frac{1}{2} h\partial_{y_1} \hat{b}(x_1,y_1)$$

and

$$\hat{X}_2(x_1,y_1,D_{x_1},D_{y_1}) = \hat{b}^{-1/2} \bar{X}_2(x_1,y_1,D_{x_1},D_{y_1}) \hat{b}^{1/2}$$

$$= \hat{A}_g(x_1,y_1)h\partial_{x_1} + h\partial_{y_1} + ix_1$$

$$+ \frac{1}{2} h[\hat{A}_g \hat{b}^{-1} \partial_1 \hat{b} + \hat{b}^{-1} \partial_2 \hat{b}](x_1,y_1).$$

Here we use notation $ \partial_1 \hat{b}(x_1,y_1) = \frac{\partial \hat{b}}{\partial x_1}(x_1,y_1), \partial_2 \hat{b}(x_1,y_1) = \frac{\partial \hat{b}}{\partial y_1}(x_1,y_1).$
3.1.2. *Metaplectic transformations.* Next, we make some metaplectic transformations.

**Partial $h$-Fourier transform**
Using the partial Fourier transform in $y_1$ ($F : y_1 \rightarrow y_2$), we obtain

$$Q_h(x_1, y_2, hD_{x_1}, hD_{y_2}) = \hat{Q}_h(x_1, -hD_{y_2}, hD_{x_1}, y_2) = -\hat{X}_1^2 - \hat{X}_2^2,$$

where

$$\hat{X}_1(x_1, y_2, hD_{x_1}, hD_{y_2}) = \hat{X}_1(x_1, -hD_{y_2}, hD_{x_1}, y_2)$$

$$= \hat{b}(x_1, -hD_{y_2})h\partial_{x_1} + \frac{1}{2}h\partial_1\hat{b}(x_1, -hD_{y_2})$$

and

$$\hat{X}_2(x_1, y_2, hD_{x_1}, hD_{y_2}) = \hat{X}_1(x_1, -hD_{y_2}, hD_{x_1}, y_2)$$

$$= \hat{A}_y(x_1, -hD_{y_2})h\partial_{x_1} + i(y_2 + x_1)$$

$$+ \frac{1}{2}h[\hat{A}_y\hat{b}^{-1}\partial_1\hat{b} + \hat{b}^{-1}\partial_2\hat{b}](x_1, -hD_{y_2}).$$

A further linear change of variables

$$x = x_1 + y_2, y = -y_2$$

gives for

$$\hat{T}_h(x, y, hD_x, hD_y; h) = Q_h(x + y, -y, hD_x, hD_x - hD_y)$$

the expression

$$\hat{T}_h = -\hat{X}_1^2 - \hat{X}_2^2,$$

where

$$\hat{X}_1(x, y, hD_x, hD_y; h) = \hat{X}_1(x + y, -y, hD_x, hD_x - hD_y)$$

$$= \hat{b}(x + y, hD_y - hD_x)h\partial_x$$

$$+ \frac{1}{2}h\partial_1\hat{b}(x + y, hD_y - hD_x)$$

and

$$\hat{X}_2(x, y, hD_x, hD_y; h) = \hat{X}_1(x + y, -y, hD_x, hD_x - hD_y)$$

$$= \hat{A}_y(x + y, hD_y - hD_x)h\partial_x + ix$$

$$+ \frac{1}{2}h[\hat{A}_y\hat{b}^{-1}\partial_1\hat{b} + \hat{b}^{-1}\partial_2\hat{b}](x + y, hD_y - hD_x).$$

3.1.3. *Scaling.* Finally, we make the dilation $x = h^{\frac{d}{2}}\tilde{x}, y = \tilde{y}$. It should be noted that this transformation is not metaplectic. Therefore, when we apply it we leave the $h$-pseudodifferential calculus and loose the possibility to use all the known results from this theory. Forgetting the tilde, we get, after division by $h$, a more symmetric expression for the operator

$$\tilde{T}_h(x, y, D_x, D_y; h) = h^{-1}\tilde{T}_h(h^{\frac{d}{2}}x, y, h^{-\frac{d}{2}}D_x, D_y; h).$$
We have
\[ \tilde{T}^h = -\tilde{X}_1^2 - \tilde{X}_2^2, \]
where
\[ \tilde{X}_1(x, y, D_x, h D_y; h) = h^{-\frac{1}{2}} \tilde{X}_1(h^{\frac{1}{2}}x, y, h^{-\frac{1}{2}}D_x, D_y; h) \]
\[ = \hat{b}(h^{\frac{1}{2}}x + y, h D_y - h^{\frac{1}{2}}D_x) \partial_x + \frac{1}{2} h^{1/2} \partial_1 \hat{b}(h^{\frac{1}{2}}x + y, h D_y - h^{\frac{1}{2}}D_x) \]
and
\[ \tilde{X}_2(x, y, D_x, h D_y; h) = h^{-\frac{1}{2}} \tilde{X}_2(h^{\frac{1}{2}}x, y, h^{-\frac{1}{2}}D_x, D_y; h) \]
\[ = \hat{A}_y(h^{\frac{1}{2}}x + y, h D_y - h^{\frac{1}{2}}D_x) \partial_x + i x \]
\[ + \frac{1}{2} h^{1/2}[\hat{A}_y \hat{b}^{-1} \partial_1 \hat{b} + \hat{b}^{-1} \partial_2 \hat{b}](h^{\frac{1}{2}}x + y, h D_y - h^{\frac{1}{2}}D_x). \]

The main problem is that the operator \( \tilde{T}_h \) is written as a differential operator in \( x \) and \( D_x \) with pseudodifferential coefficients in \( h^{1/2}x + y, h D_y - h^{1/2}D_x \).

In the next step, we will rewrite it as an \( h \)-pseudodifferential operator in the \( y \) variable with values in the class of differential operators in the \( x \) variable.

### 3.2. Weyl calculus and justification of the expansions.

For any \( h > 0 \), the operators \( h^{1/2}x + y \) and \( h D_y - h^{1/2}D_x \) are commuting self-adjoint unbounded linear operators in \( L^2(\mathbb{R}^2, dx \, dy) \). Spectral theorem allows us to define the operator \( a(h^{1/2}x + y, h D_y - h^{1/2}D_x) \) as a bounded linear operator in \( L^2(\mathbb{R}^2, dx \, dy) \) for any \( a \in S(1) \). In this subsection, we derive an asymptotic expansion for the operator \( a(h^{1/2}x + y, h D_y - h^{1/2}D_x) \) in the form

\[
a(h^{1/2}x + y, h D_y - h^{1/2}D_x) \sim \sum_{j \geq 0} h^{\frac{j}{2}} \sum_{\ell_1, \ldots, \ell_j} b_{j, \ell_1, \ldots, \ell_j}(x, D_x) a_{j, \ell}(y, h D_y)
\]

with some \( b_{j, \ell_1, \ldots, \ell_j} \in S^*(\mathbb{R}^2) \) and \( a_{j, \ell} \in S \). Here \( b(x, D_x) \) denotes the Weyl quantization of the symbol \( b \in S^*(\mathbb{R}^2) \) and \( a(y, h D_y) \) is the semiclassical pseudodifferential operator with Weyl symbol \( a \in S(1) \).

First, we consider the case when \( a \in S(\mathbb{R}^2) \). Then we write

\[
a(h^{1/2}x + y, h D_y - h^{1/2}D_x) = \int \hat{a}(\tau_1, \tau_2) e^{i[\tau_1(h^{1/2}x + y) + \tau_2(h D_y - h^{1/2}D_x)]} d\tau_1 \, d\tau_2,
\]

where \( \hat{a}(\tau_1, \tau_2) \) is the Fourier transform of \( a \).

This formula can be rewritten in the form

\[
a(h^{1/2}x + y, h D_y - h^{1/2}D_x) \]
\[
= \int \hat{a}(\tau_1, \tau_2) e^{i[\tau_1 y + \tau_2 (h D_y)]} e^{ih^{1/2}(\tau_1 x - \tau_2 D_x)} d\tau_1 \, d\tau_2.
\]
Observe that

(3.13) \[ \int \hat{a}(\tau_1, \tau_2) e^{i(\tau_1 y + \tau_2 hD_y)} d\tau_1 d\tau_2 = a(y, hD_y). \]

Indeed, we have

\[ e^{i(\tau_1 y + \tau_2 hD_y)} u(y) = e^{\frac{i}{2} h\tau_1 \tau_2} e^{i\tau_1 y} u(y + h\tau_2). \]

Using this formula, for the operator

\[ \hat{A} = \int \hat{a}(\tau_1, \tau_2) e^{i(\tau_1 y + \tau_2 hD_y)} d\tau_1 d\tau_2, \]

we get

\[ Au(y) = \int \hat{a}(\tau_1, \tau_2) e^{\frac{i}{2} h\tau_1 \tau_2} e^{i\tau_1 y} u(y + h\tau_2) e^{\frac{i}{2} h\tau_1 \tau_2} e^{i\tau_1 y} u(y + h\tau_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2. \]

Now we use the Fourier transform inversion formula in \( \tau_1 \) and \( \xi_1 \):

\[ Au(y) = \frac{1}{2\pi} \int e^{-i\tau_2 \xi_2} a\left(\frac{1}{2} h\tau_2 + y, \xi_2\right) u(y + h\tau_2) e^{\frac{i}{2} h\tau_2 \xi_2} d\tau_1 d\tau_2 d\xi_2. \]

Finally, we make the change of variables \( x = y + h\tau_2 \) and get

\[ Au(y) = \frac{1}{2\pi h} \int e^{\frac{i}{h}(y-x) \xi_2} a\left(\frac{x + y}{2}, \xi_2\right) u(x) dx d\xi_2 = a(y, hD_y), \]

that completes the proof of (3.13).

We can then expand the right hand side of (3.12) in powers of \( h^{1/2} \) and get

(3.14) \[ a(h^{1/2}x + y, hD_y - h^{1/2}D_x) \]

\[ = \sum_{k=0}^{N} \frac{1}{k!} \frac{h^{k/2}}{k!} \int (\tau_1 x - \tau_2 D_x)^k \hat{a}(\tau_1, \tau_2) e^{i[\tau_1 y + \tau_2 hD_y]} d\tau_1 d\tau_2 + R_N(h), \]

where

\[ R_N(h) = \frac{1}{N!} h^{(N+1)/2} \int \hat{a}(\tau_1, \tau_2) e^{i[\tau_1 y + \tau_2 hD_y]} \times \left( \int_0^1 (1-t)^N e^{ith(\tau_1 x - \tau_2 D_x)} (\tau_1 x - \tau_2 D_x)^{N+1} dt \right) d\tau_1 d\tau_2. \]

For \( k \in \mathbb{N} \), consider the Sobolev space \( B^k(\mathbb{R}) \) given by

(3.15) \[ B^k(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) : x^\alpha D_x^\beta u \in L^2(\mathbb{R}) \text{ for } \alpha + \beta \leq k \}. \]
Then for any $h \in (0, 1)$ and for any $s \geq 0$, there exists $C_s > 0$ such that, for any $h \in (0, h_0]$, we have

$$
\|R_N(h) : B^{s+N+1}(\mathbb{R}_x) \times L^2(\mathbb{R}_y) \to B^s(\mathbb{R}_x) \times L^2(\mathbb{R}_y)\| \leq C_s h^{N+1/2}.
$$

Using (3.13), we can compute explicitly the first coefficients in the expansion (3.14). For instance, for $N = 2$, we get

(3.16) 
\[ a(h^{1/2}x + y, hD_y - h^{1/2}D_x) \]
\[ = a(y, hD_y) + ih^{1/2}(x(D_1a)(y, hD_y) - D_x(D_2a)(y, hD_y)) \]
\[ - \frac{1}{2} h(x^2(D_1^2a)(y, hD_y) - (xD_x + D_xx)(D_1D_2a)(y, hD_y) + D_x^2(D_2^2a)(y, hD_y)) \]
\[ + R_2(h). \]

This is what we would have obtained by considering the non commutative Taylor expansion with respect to $h^{1/2}x$ and $h^{1/2}D_x$ at the "point" $(y, hD_y)$. More generally, we get the following result.

**Proposition 3.1.** If $a$ is a semiclassical symbol in $\mathbb{R}^2$, then we have:

(3.17)
\[ a(h^{1/2}x + y, hD_y - h^{1/2}D_x, h) \sim \sum_{j \geq 0} h^{j/2} \left( \sum_{k_1+k_2 \leq j} a_{k_1,k_2,j}(y, hD_y)x^{k_1}D_x^{k_2} \right), \]

where $a_{000}(y, \eta) = a(y, \eta)$, and the remainder $R_N(h)$ defined for any $N \in \mathbb{N}$ by

$$
R_N(h) = a(h^{1/2}x + y, hD_y - h^{1/2}D_x, h)
$$
\[ - \sum_{j=0}^{N} h^{j/2} \left( \sum_{k_1+k_2 \leq j} a_{k_1,k_2,j}(y, hD_y)x^{k_1}D_x^{k_2} \right),
$$
satisfies the following condition: for any $N \in \mathbb{N}$ there exists $h_0 > 0$ such that, for any $h \in (0, h_0]$, and for any $s \geq 0$, we have

$$
\|R_N(h) : B^{s+N+1}(\mathbb{R}_x) \times L^2(\mathbb{R}_y) \to B^s(\mathbb{R}_x) \times L^2(\mathbb{R}_y)\| \leq C_s h^{N+1/2}.
$$

**Remark 3.2.** The (standard) problem is that $a$ is not defined everywhere. But if one has some information on the frequency set of the quasi-mode, one can assume that $a$ is compactly supported (or has an extension to a semiclassical symbol in $S$). The results then will not depend on the choice of the extension.

**Remark 3.3.** On the right hand side of (3.17), the operators will be applied on expression of the form

$$
w(x, y, h) \sim \sum_{\ell} h^{\ell/2} \left( \sum_{k=0}^{k_{\ell}} u_{\ell,k}(y, h)v_{\ell,k}(x) \right),
$$
where the \( u_{\ell,k}(y,h) \) have their \( h \)-microsupport close to \((0,0)\) (our symbols in \((y,\eta)\) are only defined there) and the \( v_{\ell,k} \) are functions in \( \mathcal{S}(\mathbb{R}) \) (actually Gaussians multiplied by polynomials).

**Remark 3.4.** For the treatment of the action of \( \widetilde{T}^h \), we have to compose the expansions obtained in Proposition 3.1 with \( x^k D_x^k \) and sum various terms of this type.

**Remark 3.5.** Together with the maximal estimates on the eigenfunctions (see the estimates (4.4) below), we have the possibility to stop the expansion in degree \( N \), the remainder being controlled. One can then follow formal Grushin’s method using finite expansions in powers of \( h \) and give a non formal meaning to all the constructions modulo an error term of order \( O(h^{\tilde{N}}) \) where \( \tilde{N} \) depends on \( N \) and can be made arbitrarily large by choosing \( N \) large enough.

### 3.3. The explicit expansion

In this subsection, we will use the results of Subsection 3.2 to rewrite the operator \( \widetilde{T}^h \) as an \( h \)-pseudodifferential operator in the \( y \) variable with values in the class of differential operators in the \( x \) variable:

\[
\widetilde{T}^h(x,y, D_x, D_y; h) \sim \sum_{k=0}^{\infty} h^{k/2} S_k(x, y, D_x, hD_y),
\]

(3.18)

For this purpose, we first expand the coefficients in \( h^{1/2}x \) and \( h^{1/2}D_x \) in the formulae (3.10) and (3.11). By (3.16), we obtain that

\[
\frac{1}{i} \tilde{X}_1 = \hat{b}(y, hD_y)D_x
\]

\[
+ i h^{1/2} \left( \frac{1}{2} (xD_x + D_x x)(D_1 \hat{b})(y, hD_y) - D_x^2 (D_2 \hat{b})(y, hD_y) \right)
\]

\[
- h \left( \frac{1}{4} (x^2 D_x + D_x x^2)(D_1^2 \hat{b})(y, hD_y) - \frac{1}{2} (xD_x^2 + D_x^2 x)(D_1 D_2 \hat{b})(y, hD_y) \right)
\]

\[
+ \frac{1}{2} D_x^3 (D_2^2 \hat{b})(y, hD_y) + O(h^{3/2})
\]

and

\[
\frac{1}{i} \tilde{X}_2 = \hat{A}_y(y, hD_y)D_x + x + h^{1/2}([xD_x + D_x x](\partial_1 \hat{A}_y)(y, hD_y)
\]

\[
- D_x^2 (\partial_2 \hat{A}_y)(y, hD_y) + \frac{1}{2} (-D_1(\hat{A}_y \hat{b}^{-1})\hat{b} + \hat{b}^{-1} D_2 \hat{b})(y, hD_y)]
\]

\[
+ h \left[ - \frac{1}{4} (x^2 D_x + D_x x^2)(D_1^2 \hat{A}_y)(y, hD_y) \right]
\]

\[
+ \frac{1}{2} (xD_x^2 + D_x^2 x)(D_1 D_2 \hat{A}_y)(y, hD_y) - \frac{1}{2} D_x^3 (D_2^2 \hat{A}_y)(y, hD_y)
\]

\[
+ \frac{1}{2} i (x(-D_1[D_1(\hat{A}_y \hat{b}^{-1})\hat{b}] + D_1[\hat{b}^{-1} D_2 \hat{b}]))(y, hD_y)
\]

\[
- D_x (-D_2[D_1(\hat{A}_y \hat{b}^{-1})\hat{b}] + D_2[\hat{b}^{-1} D_2 \hat{b}]))(y, hD_y)) + O(h^{3/2}).
\]
Next, we substitute these asymptotic formulas into (3.9) that gives the desired asymptotic expansion (3.18).

We now compute the first two coefficients in this expansion.

For the coefficient $S_0$, we get:

$$
S_0(x, y, D_x, hD_y)
= (\hat{b}^2 + \hat{A}_y^2)(y, hD_y)D_x^2 - \hat{A}_y(y, hD_y)(x D_x + D_x x) + x^2.
$$

The Weyl vector valued $h$-symbol of $S_0$ is given by

$$
\sigma_0(x, D_x, y, \eta) = (\hat{b}^2(y, \eta) + \hat{A}_y^2(y, \eta))D_x^2 - \hat{A}_y(y, \eta)(x D_x + D_x x) + x^2.
$$

For a fixed $(y, \eta)$, this is an harmonic oscillator, whose spectrum is given by

$$
\lambda_k(y, \eta) = (2k + 1)\hat{b}(y, \eta), \quad k \in \mathbb{N}.
$$

This could seem surprising but one way to recognize this simply is to observe that, by a gauge transformation $\exp \left( i \frac{\alpha}{(b^2 + \alpha^2)^{1/2}} x^2 \right)$, with $\alpha = \hat{A}_y(y, \eta)$ and $b = \hat{b}(y, \eta)$, $\sigma_0(x, D_x, y, \eta)$ is unitary equivalent to $(\hat{b}^2 + \alpha^2)D_x^2 + \frac{b^2}{b^2 + \alpha^2}x^2$.

An additional dilation permits us to arrive at $b(D_x^2 + x^2)$. In particular, we get for the $L^2$-normalized ground state of $\sigma_0(x, D_x, y, \eta)$:

$$
h_{y, \eta}(x) = \rho(y, \eta) \exp -\delta(y, \eta)x^2,
$$

with $\rho(y, \eta) > 0$ and $\Re \delta(y, \eta) > 0$, $\rho$ and $\delta$ depending smoothly on $(y, \eta)$.

The coefficient $S_1$ is given by

$$
S_1 = i \left( \frac{1}{2}(x D_x^2 + 2D_x x D_x + D_x^2 x)(\hat{b} D_1 \hat{b})(y, hD_y) - 2D_x^2(\hat{b} D_y \hat{b})(y, hD_y) \right)
+ (\hat{A}_y(y, hD_y)D_x + x) \left[ (x D_x + D_x x)(\partial_1 \hat{A}_y)(y, hD_y) 
- D_x^2(\partial_2 \hat{A}_y)(y, hD_y) + \frac{1}{2}(-D_1(\hat{A}_y \hat{b}^{-1})\hat{b} + \hat{b}^{-1}D_2 \hat{b})(y, hD_y) \right]
+ \left[ (x D_x + D_x x)(\partial_1 \hat{A}_y)(y, hD_y) - D_x^2(\partial_2 \hat{A}_y)(y, hD_y) 
+ \frac{1}{2}(-D_1(\hat{A}_y \hat{b}^{-1})\hat{b} + \hat{b}^{-1}D_2 \hat{b})(y, hD_y) \right] (\hat{A}_y(y, hD_y)D_x + x).
$$

We observe that $S_1$ inverses the parity in the $x$ variable.

Computation of $S_2$ is rather lengthy and we will omit it here. We only observe that $S_2$ respects the parity in the $x$ variable.

These computations could be useful for determining the sub-principal symbol of the effective operator $p_{\text{eff}}(y, hD_y; h, z)$. We will explain this in Section [7]. But for proving the existence of the symbol $p_{\text{eff}}(y, \eta; h, z)$, we need only the structure of the operators $S_j$. 


4. Eigenfunctions estimates

4.1. On the frequency set of eigenfunctions. The frequency set was introduced by V. Guillemin and S. Sternberg [7] but we prefer for our need to refer to the books of D. Robert [31] or M. Zworski [38]. This is the analog of the wave front set of Hörmander in the semi-classical context.

Definition 4.1. Given an open subset $\Omega$ of $\mathbb{R}^m$ and a map $h \in (0, h_0] \mapsto T_h \in \mathcal{D}'(\Omega)$, a point $(x_0, p_0) \in \mathbb{R}^m_x \times \mathbb{R}^m_p$ is not in the frequency set $F[T_h]$ of $T_h$ if there exists $\phi \in C^\infty_c(\mathbb{R}^m)$ such that $\phi(x_0) \neq 0$ and a neighborhood $V_{p_0}$ of $p_0$ such that
\[
\langle \phi(x)e^{-ih^{-1}xp}, T_h \rangle = O(h^\infty), \quad h \to 0,
\]
uniformly with respect to $p \in V_{p_0}$.

There exists also an $h$-pseudodifferential characterization of the frequency set for a family $T_h$ in $L^2$. A point $(x_0, p_0) \in \mathbb{R}^m_x \times \mathbb{R}^m_p$ is not in the frequency set $F[T_h]$ of $T_h$ if there exists an $h$-pseudodifferential operator $\chi(x, hD_x)$ whose symbol is elliptic at $(x_0, p_0)$ such that $\chi(x, hD_x)T_h = O(h^\infty)$ in $L^2$.

The following result is rather standard:

Proposition 4.2. Suppose that $u_h$ is an $L^2$ normalized eigenfunction of $H_h$ corresponding to an eigenvalue $\lambda_h$ such that $\lambda_h \leq Ch$ for $h \in (0, 1)$. Then the frequency set of $u_h$ is non empty and contained in
\[
F[u_h] \subset \{(x, y, \xi, \eta) \in T^*\mathbb{R}^2 : \xi = 0, \eta = -A(x, y), b(x, y) \leq C\}.
\]

Proof. This is just a combination of the elliptic theory for $h$-pseudo-differential operators combined with Agmon estimates (see above around (2.4)). □

Remark 4.3. As a consequence, if we consider a cut-off function $\chi$ equal to 1 on a fixed neighborhood of $b^{-1}((-\infty, C))$ and with support in $b^{-1}((-\infty, C'))$ with some $C' > C$, then $\chi u_h$ has the same frequency set and satisfies:
\[
(H^h - \lambda_h)(\chi u_h) = O(h^\infty).
\]

We can now follow the frequency set by change of coordinates or more generally by the action of $h$-Fourier integral operators (this includes the $h$-Fourier transform) the transformation being given by the associated canonical transformation. Let us consider transformations introduced in Subsection 3.1.

After the change of variables (3.1), we get for the transformed eigenfunction $\tilde{u}_h(x_1, y_1) = u_h(x, y),
\[
F[\tilde{u}_h] \subset \{(x_1, y_1, \xi_1, \eta_1) \in T^*\mathbb{R}^2 : \xi_1 = 0, \eta_1 = -x_1, b(x_1, y_1) \leq C\}.
\]

Now we apply the unitary isomorphism (3.3). For the transformed eigenfunction
\[
u_h(x_1, y_1) = b(x_1, y_1)^{-1/2}u_h(x_1, y_1),
\]
we obtain that
\[
F[v_h] = F[u_h] \subset \{(x_1, y_1, \xi_1, \eta_1) \in T^*\mathbb{R}^2 : \xi_1 = 0, \eta_1 = -x_1, b(x_1, y_1) \leq C\}.
\]
Next we make the partial Fourier transform in $y_1$. So the corresponding eigenfunction $w_h$ is the partial Fourier transform in $y_1$ of $v_h$, and therefore

$$\begin{aligned}
F[w_h] &= \{(x_1, y_2, \xi_1, \eta_2) \in T^*\mathbb{R}^2 : (x_1, -\eta_2, \xi_1, y_2) \in F[v_h]\} \\
&\subset \{(x_1, y_2, \xi_1, \eta_2) \in T^*\mathbb{R}^2 : \xi_1 = 0, y_2 = -x_1, \hat{b}(x_1, -\eta_2) \leq C\}.
\end{aligned}$$

Next we make a change of variables (3.4), which gives for the corresponding eigenfunction

$$\hat{u}_h(x, y, \xi, \eta) = h^{-\frac{1}{2}}\hat{u}_h(h^{-\frac{1}{2}}x, h^{-1}\eta).$$

Note that, in this step, we do not control the frequency set, but, for any natural $k$ and $\ell$, the asymptotic behavior of the norm of $x^kD_x^\ell\hat{u}_h$ as $h \to 0$ is well controlled as will be shown in Subsection 4.2.

4.2. Maximal estimates. In the following, we will use the asymptotic expansions of Subsection 3.2 applied to the eigenfunction $\hat{u}_h$ of the operator $\hat{T}_h$ introduced above. To analyze the action of the remainder in these asymptotic expansions on $\hat{u}_h$, we have to control the $L^2$ norm of $x^\alpha D^\beta_x \hat{u}_h$. Formally the possibility of such a control seems reasonable taking into account the information that $\hat{T}_h^k\hat{u}_h = \lambda_h^k\hat{u}_h = O(h^k)$ in $L^2$. To prove the corresponding estimate rigorously, we observe that before the metaplectic transformations introduced in Subsection 3.1 such an estimate is related to a “regularity” estimate (à la Hörmander) for polynomials of vector fields or more precisely (à la Helffer-Nourrigat) for the iterates of the magnetic Laplacian.

More precisely, in the flat Euclidean space $\mathbb{R}^3$ with coordinates $(x, y, t)$, consider the vector fields

$$Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y} + A(x, y)\frac{\partial}{\partial t}.$$ 

Then we have

$$[Y_1, Y_2](x, y, t) = b(x, y)\frac{\partial}{\partial t} \neq 0.$$ 

Thus, for any $(x, y, t) \in \mathbb{R}^3$, the vector fields $Y_1, Y_2$ satisfy the Hörmander condition $(C.H)_{2,(x,y,t)}$ [18] Chapter I, §1], which means that the vectors $Y_1(x, y, t), Y_2(x, y, t)$ and $[Y_1, Y_2](x, y, t)$ span the tangent space $T_{(x,y,t)}\mathbb{R}^3$.

Consider the operator

$$P = Y_1^2 + Y_2^2.$$ 

By Theorem 1.3 from [18] Chapter IX, we get, for any $u \in C_c^\infty(V \times \mathbb{R})$, where $V$ is a sufficiently small neighborhood of $(0,0)$, that there exists $C$ such that

$$\sum_{\alpha_1 + \alpha_2 \leq 2} \|Y_1^{\alpha_1}Y_2^{\alpha_2}u\|^2 \leq C\|Pu\|^2 + \|u\|^2.$$
Similarly, for any $N$ there exists $C_N$ such that
\[
\sum_{\alpha_1 + \alpha_2 \leq 2N} \|Y_1^{\alpha_1}Y_2^{\alpha_2}u\|^2 \leq C_N(\|P^Nu\|^2 + \|u\|^2).
\]

Taking the partial Fourier transform in the $t$-variable, we get
\[
\sum_{\alpha_1 + \alpha_2 \leq 2N} \left\| \left( \frac{\partial}{\partial x} \right)^{\alpha_1} \left( \frac{\partial}{\partial y} + iA(x,y)\tau \right)^{\alpha_2} u \right\|^2 \\
\leq C_N(\left\| \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} + iA(x,y)\tau \right)^2 u \right\|^2).
\]

Dividing by $\tau^{4N}$ and introducing $h = \tau^{-1}$, we obtain that
\[
\sum_{|\alpha| \leq 2N} h^{4N-2(\alpha_1 + \alpha_2)}\|X_1^{\alpha_1}X_2^{\alpha_2}u\|^2 \leq C_N\left(\left\| (H^h)^Nu \right\|^2 + h^{4N}\|u\|^2 \right).
\]

Remark 4.4. Alternately, we could have used the Boutet de Monvel results on hypoelliptic operators with multiple characteristics (see in [3]) in the symplectic case.

Hence we get

**Proposition 4.5.** If $u_h$ is an $L^2$ normalized eigenfunction of $H^h$ corresponding to an eigenvalue $\lambda_h$ such that $\lambda_h \leq Ch$ for $h \in (0,1)$, then for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ there exists a constant $C_\alpha$ such that, for $h \in (0, 1)$,

\[
\|X_1^{\alpha_1}X_2^{\alpha_2}u_h\|_{L^2(\mathbb{R}^2)} \leq C_\alpha h^{(\alpha_1 + \alpha_2)/2}. \tag{4.2}
\]

The main contribution to the norm in (4.2) of course comes from the set $\{ (x,y) \in \mathbb{R}^2 : b(x,y) \leq C \}$. Outside this set, we have an exponential decay due to Agmon estimates (see [10]).

**Lemma 4.6.** With the same assumptions, if $K$ is a compact such that $K \cap \{ (x,y) \in \mathbb{R}^2 : b(x,y) \leq C \} = \emptyset$, then there exist $\epsilon = \epsilon_K > 0$ and, for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, $C_{K,\alpha}$ and $h_{K,\alpha}$ such that, for $h \in (0, h_{K,\alpha})$,

\[
\|X_1^{\alpha_1}X_2^{\alpha_2}u_h\|_{L^2(K)} \leq C_{K,\alpha} e^{-\frac{\epsilon h^{1/2}}{2}}. \tag{4.3}
\]

We now follow (4.2) in the chain of transformations leading to our normal form (see Subsection 3.1). For the transformed eigenfunction $\hat{u}_h$, we obtain that for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ there exists a constant $C_\alpha$ such that, for $h \in (0, 1)$

\[
\|\hat{X}_1^{\alpha_1}\hat{X}_2^{\alpha_2}\hat{u}_h\|_{L^2(\mathbb{R}^2)} \leq C_\alpha h^{(\alpha_1 + \alpha_2)/2}. \tag{4.3}
\]

By (4.2) and (4.3), it follows that
\[
x = q_{11}\hat{X}_1 + q_{12}\hat{X}_2 + hq_1, \quad hD_x = q_{21}\hat{X}_1 + q_{22}\hat{X}_2 + hq_2,
\]
where \( q_{ij} \) and \( q_i \) are \( h \)-pseudodifferential operators. Using this decomposition, we get from (4.3) that for any \( k \) and \( \ell \) in \( \mathbb{N} \)
\[
x^\ell (hD_x)^m \hat{u}_h = \mathcal{O}(h^{(m+\ell)/2}),
\]
in \( L^2(\mathbb{R}^2) \).
This is what is needed for controlling the expansions which were done after the scaling \( x = h^{1/2} \tilde{x}, y = \tilde{y} \). With this scaling we get that for any \( k \) and \( \ell \) in \( \mathbb{N} \)
\[
(4.4) \quad x^\ell D_x^m \tilde{u}_h = \mathcal{O}(1).
\]

5. The Grushin method

In this section, we construct an appropriate Grushin problem in a neighborhood of the minimum point \((0,0)\) and apply Grushin’s method.

5.1. Classes of pseudo-differential operators. First, let us recall a specific class of pseudo-differential operators which appear to be useful in the analysis of fine spectral properties of globally elliptic operators. We refer to Helffer [9] or Shubin [35] for this specific class which is of course contained in the general class considered in the Weyl calculus by Hörmander [25]. The class \( S^m(\mathbb{R}^2) \) is defined as the set of \( C^\infty \) functions on \( \mathbb{R}^2 \) such that for any natural \( k,\ell \), there exists a constant \( C_{k,\ell} \) such that
\[
|D_x^k D_\xi^\ell a(x,\xi)| \leq C_{k,\ell}(1 + |x| + |\xi|)^{m-k-\ell}.
\]
We associate with a symbol \( a \in S^m(\mathbb{R}^2) \) an operator via the Weyl quantization. We denote by \( \text{Op} S^m \) the corresponding class of operators which are well defined on \( \mathcal{S}(\mathbb{R}) \) and \( \mathcal{S}'(\mathbb{R}) \). When \( m = 0 \), these operators are continuous in \( L^2(\mathbb{R}) \). As usual there is a natural notion of principal symbol and of globally elliptic symbol. For \( m > 0 \), we say that the symbol \( a \in S^m(\mathbb{R}^2) \) is elliptic if there exists a constant \( C > 0 \) such that
\[
a(x,\xi) \geq \frac{1}{C}(1 + |x| + |\xi|)^m - C.
\]
Globally elliptic operators have parametrices, this means that there exists a pseudo-differential operator \( Q = \text{Op}(q) \) in \( \text{Op} S^{-m} \) with principal symbol equal to \( \frac{1}{a} \) for \( |x| + |\xi| \) large enough such that
\[
\text{Op}(q) \circ \text{Op}(a) = I + \mathcal{R},
\]
where \( \mathcal{R} \) is regularizing in the sense that it has a distribution kernel in \( \mathcal{S}(\mathbb{R} \times \mathbb{R}) \) (equivalently that it has a Weyl symbol in \( \mathcal{S}(\mathbb{R}^2) \) or that it can be extended as a map from \( \mathcal{S}' \) into \( \mathcal{S} \)).
In addition, if we know by other means that \( Q \) is invertible then the inverse is itself a pseudo-differential operator in \( \text{Op} S^{-m} \) (this is a special (easier) case of the so-called Beals theorem).

It is also natural to introduce a class of symbols \( S^{0,m}(\mathbb{R}^2 \times \mathbb{R}^2) \) of the form
\[
(x,\xi,y,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto b(x,\xi,y,\eta) \in \mathbb{C},
\]
verifying the following estimates:

\[ |D^{\alpha}_{y,\eta} D^{\beta}_{x,\xi} b(x, \xi, y, \eta)| \leq C_{\alpha, \beta} (1 + |x| + |\xi|)^{m-|\beta|}, \forall (x, \xi) \in \mathbb{R}^2, \forall (y, \eta) \in \mathbb{R}^2. \]

These symbols could also depend on an additional parameter \( h \) and can be possibly expanded in powers of \( h \) (with fixed \( m \)).

With an arbitrary symbol \( b \in S^{0,m}(\mathbb{R}^2 \times \mathbb{R}^2) \), we can associate by the Weyl quantization a global pseudodifferential operator \( b(x, D_x, y, hD_y) \), which is semi-classical in the \( y \) variable. This operator acts on \( \mathcal{S}(\mathbb{R}_x) \hat{\otimes} C^\infty_0(\mathbb{R}_y) \) by the formula

\[
b(x, D_x, y, hD_y)w(x, y, h) := h^{-1} \int b(\frac{x + x'}{2}, \xi, \frac{y + y'}{2}, \eta)w(x', y')e^{i(x-x')\cdot \xi + i(\xi \cdot \eta - x 
\quad \cdot \xi - y \cdot \eta)h} \, dy'd\eta dx'd\xi.
\]

The class of such operators will be denoted by \( \text{Op} S^{0,m} \).

One can consider an operator in the class \( \text{Op} S^{0,m} \) as an \( h \)-pseudodifferential operator on \( C^\infty_0(\mathbb{R}_y) \) with a vector-valued symbol, taking values in the space of global pseudodifferential operators on \( \mathcal{S}(\mathbb{R}_x) \). The Weyl vector valued \( h \)-symbol of the operator \( b(x, D_x, y, hD_y) \) is given by

\[
b(y, \eta)w(x) = b(x, D_x, y, \eta)w(x) := \int b(\frac{x + x'}{2}, \xi, \eta)w(x')e^{i(x-x')\cdot \xi} \, dx'd\xi.
\]

For two operators \( b(x, D_x, y, hD_y) \in \text{Op} S^{0,*} \) and \( c(x, D_x, y, hD_y) \in \text{Op} S^{0,*} \) we will denote by \( b(x, D_x, y, hD_y) \circ c(x, D_x, y, hD_y) \) their composition as operators on \( \mathcal{S}(\mathbb{R}_x) \hat{\otimes} C^\infty_0(\mathbb{R}_y) \) and by \( b(y, \eta)c(y, \eta) \) the (pointwise at \( (y, \eta) \)) composition of their Weyl vector valued \( h \)-symbols \( b(y, \eta) \) and \( c(y, \eta) \) as global pseudodifferential operators on \( \mathcal{S}(\mathbb{R}_x) \).

We introduce the class \( \text{Op} S^{0,m}[h] \), which consists of families \( \{ C(h) : h > 0 \} \) of bounded operators on \( \mathcal{S}(\mathbb{R}_x) \hat{\otimes} L^2(\mathbb{R}_y) \), which can be represented as an asymptotic sum of the following type:

\[
C(h) \sim \sum_{j \geq 0} h^{\frac{s}{2}} c_j(x, D_x, y, hD_y)
\]

where each \( c_j \) belongs to \( S^{0,m_j}(\mathbb{R}^2 \times \mathbb{R}^2) \) with some \( m_j \) and can be represented as a finite sum

\[
c_j(x, D_x, y, hD_y) = \sum b^{(j)}_\ell(y, hD_y) a^{(j)}_\ell(x, D_x)
\]

with some \( b^{(j)}_\ell \in S(1) \) and \( a^{(j)}_\ell \in S^{m_j}(\mathbb{R}) \).

The asymptotic sum means that for any \( N \in \mathbb{N} \) the remainder

\[
R_N(h) = C(h) - \sum_{j=0}^{N} h^{\frac{s}{2}} c^{(j)}_j(x, D_x, y, hD_y)
\]

has the property that there exists \( k(N) \in \mathbb{N} \) and \( h_0 > 0 \) such that, for any \( h \in (0, h_0) \) and for any \( s > 0 \), we have

\[
(5.1) \quad \| R_N(h) : B^{s+k(N)}(\mathbb{R}_x) \hat{\otimes} L^2(\mathbb{R}_y) \to B^s(\mathbb{R}_x) \hat{\otimes} L^2(\mathbb{R}_y) \| \leq C_s h^{\frac{s}{2} - \frac{k(N)}{2}}.
\]
We also consider formal pseudodifferential operators of class $\text{Op} S^{0,*}[\hbar]$, which are formal sums of the following type:

$$C(\hbar) = \sum_{j \geq 0} \hbar^j c_j(x, D_x, y, hD_y),$$

where each $c_j$ belongs to $S^{0,m_j}(\mathbb{R}^2 \times \mathbb{R}^2)$ with some $m_j$.

By Proposition 3.1, it follows that, for a semiclassical symbol $a$ on $\mathbb{R}^2$, the operator $a(h^{1/2}x + y, hD_y - h^{1/2}D_x, h)$ belongs to $\text{Op} S^{0,*}[\hbar]$. By (3.9), this implies that the operator $\tilde{T}^h$ belongs to $\text{Op} S^{0,2}[\hbar]$.

**Definition 5.1.** Let $\Omega$ be an open subset of $\mathbb{R}^2$. For an operator $C \in \text{Op} S^{0,*}[\hbar]$, we say that $C = O_\Omega(\hbar^{k/2})$ with some $k \in \mathbb{N}$ if, for $j = 0, \ldots, k-1$,

$$c_j(x, D_x, y, \eta) = 0, \quad \forall (y, \eta) \in \Omega.$$

Using the fact that the composition of the semiclassical symbols is a local operation, we easily get that, if $A \in \text{Op} S^{0,*}[\hbar]$ and $B = O_\Omega(\hbar^{k/2})$, then $A \circ B = O_\Omega(\hbar^{k/2})$ and $B \circ A = O_\Omega(\hbar^{k/2})$.

**5.2. Initialization.** We will use the variables $(x, y)$ introduced in Section 3.3. Our Grushin problem takes the form

$$(5.2) \quad P_h(z) = \begin{pmatrix} \tilde{T}^h - b_0 - z & R_- \\ R_+ & 0 \end{pmatrix},$$

where $\tilde{T}^h$ was introduced in (3.9)-(3.18), the operator $R_- : S(\mathbb{R}) \to S(\mathbb{R}^2)$ is given by

$$(5.3) \quad R_- f(x, y) = h_0(x)f(y),$$

with $h_0(x) = \pi^{-1/2} b_0^{-1/2} e^{-b_0 x^2/2}$ being the normalized first eigenfunction of the harmonic oscillator

$$T = b_0^2 D_x^2 + x^2,$$

and the operator $R_+ : S(\mathbb{R}^2) \to S(\mathbb{R})$ is given by

$$(5.4) \quad R_+ \phi(y) = \int h_0(x) \phi(x, y) dx.$$

Note that $R_-$ and $R_+$ have a very simple structure, which simplifies the analysis. But the counterpart is that, since our considerations are perturbative near the bottom, we are obliged to choose $\gamma_0$ to be small enough.

**5.3. Towards an inverse.** First, we will work at the level of symbols in $(y, \eta)$. We subtract $b_0$ from $\sigma_0$ and introduce

$$\Theta_0(x, D_x, y, \eta) := \sigma_0(x, D_x, y, \eta) - b_0.$$

We now look at the Grushin problem

$$(5.5) \quad Q(x, D_x, y, \eta, z) := \begin{pmatrix} \Theta_0(x, D_x, y, \eta) - z & R_- \\ R_+ & 0 \end{pmatrix}.$$

We first look at the invertibility for \((y, \eta) = (0, 0)\).

Put
\[
T = \sigma_0(0, 0) = b_0^2 D_x^2 + x^2,
\]
which is the value of the Weyl vector valued \(h\)-symbol \(\sigma_0(y, \eta)\) of \(S_0\) at \((0, 0)\) (see (3.19) and (3.20)).

Consider the operator
\[
P_0 := Q(x, D_x, 0, 0, 0) = (T - b_0) R_- + R_+ 0,
\]
on \(L^2(\mathbb{R}) \times \mathbb{C}\). Its left inverse as an operator on \(L^2(\mathbb{R}) \times \mathbb{C}\) has the form
\[
(5.6) \quad \mathcal{E}^0 = \begin{pmatrix} U_0 & R_- \\ R_+ & 0 \end{pmatrix},
\]
where
\[
(5.7) \quad U_0 \text{ is the regularized inverse of the harmonic oscillator } T - b_0.
\]

**Lemma 5.2.** The operator \(U_0\) introduced in (5.7) is a pseudodifferential operator with symbol in \(S^{-2}(\mathbb{R}^2)\).

**Proof.** The projector \(\Pi_0\) on the first eigenspace is a pseudodifferential operator of order 0 with symbol in \(S(\mathbb{R}^2)\). Let us look at \(T - b_0 + \Pi_0\). This is a globally elliptic pseudodifferential operator of order 2, which is invertible. Hence, its inverse \(U_1\) is a pseudodifferential operator in \(\text{Op } S^{-2}\). It is then enough to observe that \(U_0 = (I - \Pi_0)U_1\) which is also a pseudodifferential operator with the same principal symbol. \(\square\)

Observe the identities:
\[
U_0 \Theta_0(0, 0) + R_- R_+ = I, \quad U_0 R_- = 0, \quad R_+ R_- = 1, \quad R_+ \Theta_0(0, 0) = 0.
\]

**5.4. Grushin’s problem: step 2.** Now starting from our inverse of \(Q(x, D_x, y, \eta, z)\) for \((y, \eta) = (0, 0)\) and \(z = 0\) constructed explicitly in (5.6), we will construct the inverse for \((y, \eta, z)\) in a neighborhood of \((0, 0, 0)\).

We can first consider
\[
(5.8) \quad \mathcal{E}^0 Q(x, D_x, y, \eta, z) = I + \begin{pmatrix} U_0(\Theta_0(y, \eta) - \Theta_0(0, 0)) - zU_0 & 0 \\ R_+(\Theta_0(y, \eta) - \Theta_0(0, 0) - z) & 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} I + U_0(\Theta_0(y, \eta) - \Theta_0(0, 0)) - zU_0 & 0 \\ R_+(\Theta_0(y, \eta) - \Theta_0(0, 0) - z) & 1 \end{pmatrix}.
\]
(Note that \(R_+ \Theta_0(0, 0) = 0\) but we prefer to keep the expression \(\Theta_0(y, \eta) - \Theta_0(0, 0)\) in the formula).

The operator on the right hand side is invertible as an operator in \(\mathcal{L}(L^2(\mathbb{R}) \times \mathbb{C})\).

It has the form
\[
\begin{pmatrix} A & 0 \\ b & 1 \end{pmatrix},
\]
where
\[
A = 1 + \frac{b_0^2}{b_0^2 - 1}.
\]
where $A$ is a global pseudodifferential operator of degree 0 on $\mathcal{S}(\mathbb{R})$ and $b : L^2(\mathbb{R}) \to \mathbb{C}$ is given by $u \mapsto \langle u, b \rangle$.

It is invertible if $A$ is invertible and then the left inverse reads:

$$
\begin{pmatrix}
A^{-1} & 0 \\
-bA^{-1} & 1
\end{pmatrix}.
$$

We can be more explicit by using the pseudodifferential calculus. For fixed $(y, \eta, z)$,

$$
A(y, \eta, z) = I + U_0(\Theta_0(y, \eta) - \Theta_0(0, 0)) - zU_0
$$

is a pseudodifferential operator of order 0 with $C^\infty$ coefficients with respect to $y, \eta, z$.

It can be shown (see [3, Section 4], which treats a much more complicated case) that there exists an open neighborhood $\Omega_1 \subset \mathbb{R}^2$ of $(0, 0)$ and $\alpha_0 > 0$ such that, for any $(y, \eta) \in \Omega_1$ and $z \in \mathbb{C}$ such that $|z| < \alpha_0$ the operator $A(y, \eta, z)$ is invertible as an operator in $L^2(\mathbb{R})$ and its inverse

$$
(5.9) \quad W_0(y, \eta, z) = (I + U_0(\Theta_0(y, \eta) - \Theta_0(0, 0)) - zU_0)^{-1},
$$

is also a pseudodifferential operator of order 0 whose symbol depends smoothly on $(y, \eta, z)$. We note that

$$
W_0(0, 0, z) = (I - zU_0)^{-1},
$$

and that the left inverse of the system in the right hand side of (5.8) takes the form

$$
\begin{pmatrix}
W_0(y, \eta, z) & 0 \\
-R_+(\Theta_0(y, \eta) - \Theta_0(0, 0) - z) & 1
\end{pmatrix} \circ E_0 =
\begin{pmatrix}
\epsilon_0^+ & \epsilon_0^- \\
\epsilon_+ & \epsilon_-
\end{pmatrix}.
$$

We observe that the first term

$$
\epsilon_0(y, \eta, z) = W_0(y, \eta, z) U_0
$$

is a global pseudo-differential operator of order $-2$, whose symbol depends smoothly on $(y, \eta, z)$.

The second term

$$
\epsilon_-(y, \eta, z) = W_0(y, \eta, z) R_-
$$
is an Hermite operator from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R}^2)$ of the form $\Psi_-R_-$ with some $\Psi_- \in \mathcal{S}^{0,*}$.

The third term

$$\epsilon_1^0(y, \eta, z) = R_+(1 - (\Theta_0(y, \eta) - \Theta_0(0,0) - z)W_0(y, \eta, z)U_0)$$

is an operator from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{R})$ of the form $R_+\Psi_+$ with some $\Psi_+ \in \mathcal{S}^{0,*}$.

We note that for $u \in L^2(\mathbb{R})$ we have:

$$R_+\Psi_+ u = \langle u, \Psi_+^* h_0(x) \rangle_{L^2(\mathbb{R})}.$$

Finally, the fourth term of the matrix is a (scalar) $C^\infty$ function of $(y, \eta, z)$:

$$(5.12) \quad \epsilon_0^0(y, \eta, z) = R_-(\Theta_0(y, \eta) - \Theta_0(0,0) - z)W_0(y, \eta, z)R_-$$

which for $(y, \eta) = (0,0)$ is equal to

$$\epsilon_0^0(0,0, z) = -zR_+W_0(0,0, z)R_- = -zR_+(I - zU_0)^{-1}R_- = -z.$$

We can write it in the form:

$$(5.13) \quad \epsilon_0^0(y, \eta, z) = \langle (\Theta_0(y, \eta) - \Theta_0(0,0) - z)W_0(y, \eta, z)h_0, h_0 \rangle_{L^2(\mathbb{R})}.$$

**Remark 5.3.** Using the fact that $(0,0)$ is a critical point of $\hat{b}$, one can show, by direct computations, that

$$\frac{\partial \epsilon_0^0}{\partial y}(0,0, z) = \frac{\partial \epsilon_0^0}{\partial \eta}(0,0, z) = 0$$

and

$$\frac{\partial^2 \epsilon_0^0}{\partial y^2}(0,0,0) = \frac{\partial^2 \hat{b}}{\partial y^2}(0,0), \quad \frac{\partial^2 \epsilon_0^0}{\partial y \partial \eta}(0,0,0) = \frac{\partial^2 \hat{b}}{\partial y \partial \eta}(0,0),$$

$$\frac{\partial^2 \epsilon_0^0}{\partial \eta^2}(0,0,0) = \frac{\partial^2 \hat{b}}{\partial \eta^2}(0,0).$$

**Remark 5.4.** Note that in this subsection, we have to choose $\gamma_0$ small enough in order to stay in a sufficiently small neighborhood of $(0,0)$.

### 5.5. From one Grushin problem to another

In [8], there is a computation of the symbol of $\epsilon_\pm^0$. In particular it is proven that one can compute $\epsilon_\pm^0$ for a suitable Grushin problem. The point is that the Grushin problem considered in [8] is not the same as above (see (5.5)). In [8], the Grushin problem depends on $(y, \eta)$ in the sense that one uses the eigenfunction of the harmonic oscillator $L = \Theta_0(y, \eta) - z$ corresponding to the eigenvalue $\hat{b}(y, \eta) - b_0 - z$. Hence it is necessary to control the link between two different Grushin problems. We will use the index 0 for the Grushin problem defined in (5.5) and the index 1 for another Grushin problem.

Thus, consider two Grushin problems

$$Q^0(y, \eta, z) = \begin{pmatrix} L & R_0^0 \\ R_0^0 & 0 \end{pmatrix}, \quad Q^1(y, \eta, z) = \begin{pmatrix} L & R_1^1 \\ R_1^1 & 0 \end{pmatrix}.$$
Let $E^0(y, \eta, z)$ and $E^1(y, \eta, z)$ be the inverses of $Q^0(y, \eta, z)$ and $Q^1(y, \eta, z)$ respectively:

$$E^0 = \begin{pmatrix} e^0_0 & e^0_1 \\ e^0_1 & e^0_\pm \end{pmatrix}, \quad E^1 = \begin{pmatrix} e^1_0 & e^1_1 \\ e^1_1 & e^1_\pm \end{pmatrix}. $$

A relation between $e^1_\pm(y, \eta, z)$ and $e^0_\pm(y, \eta, z)$ is given by the following lemma.

**Lemma 5.5.** If $R^1_+(0,0) = R^0_+(0,0)$ and $R^1_-(0,0) = R^0_-(0,0)$, there exists $q(y, \eta, z)$ elliptic for $(y, \eta, z)$ close to $(0,0,0)$ such that

$$e^0_\pm(y, \eta, z) = q(y, \eta, z) e^1_\pm(y, \eta, z),$$

and

$$q(0,0,z) = 1. $$

**Proof.** To simplify notation, we omit the reference to $(y, \eta, z)$. Computing $E^0 Q^1 E^1$ in two different ways, we get the identity

$$E^0 = E^1 + E^0 (Q^1 - Q^0) E^1. $$

For the lower right entry in this matrix identity, we get

$$e^0_\pm = e^1_\pm + e^0_\pm (R^1_+ - R^0_+) e^1_\pm + e^0_\pm (R^0_- - R^0_+) e^1_\pm. $$

Since $R^1_-(0,0) = R^0_-(0,0)$, the operator $1 + e^0_\pm (R^1_- - R^0_-)$ is invertible for $(y, \eta)$ close to $(0,0)$, and we obtain that

$$(1 + e^0_\pm (R^1_- - R^0_-))^{-1} e^0_\pm (1 - (R^1_- - R^0_-) e^1_\pm) = e^1_\pm,$$

that immediately completes the proof. \qed

The consequence of this lemma is that if we find easier to compute the symbol of $e^1_\pm$ we will get the symbol of $e^0_\pm$ up to the multiplication by an elliptic symbol.

We took as $R^1_+(y, \eta)$ the operator from $C$ into $L^2(\mathbb{R}_x)$:

$$C \ni \lambda \mapsto R^1_+(y, \eta) \lambda = \lambda h_{y,\eta} (\cdot),$$

$h_{y,\eta}$ being the normalized first eigenfunction of $\sigma_0(x, D_x, y, \eta)$ associated with the eigenvalue $b(y, \eta)$ (see [3.22]). We took as $R^1_+(y, \eta)$ the adjoint of $R^1_- (y, \eta)$.

For the inverse of $Q^1$, a direct computation gives

$$E^1(y, \eta, z) = \begin{pmatrix} e^1_1(y, \eta, z) & R^1_+(y, \eta) \\ R^1_-(y, \eta) & b(y, \eta) - b_0 - z \end{pmatrix},$$

where $e^1_1(y, \eta, z)$ is the inverse of $L = \Theta_0(y, \eta) - z$ when restricted to the orthogonal of $h_{y,\eta}$ and $0$ on $h_{y,\eta}$.

**Remark 5.6.** Note here that a natural condition on $z$ is that

$$z < 2b(y, \eta),$$

in order to avoid the second eigenvalue $2b(y, \eta)$ of $\sigma_0(x, D_x, y, \eta) - b(y, \eta)$.

Hence we finally get:
Proposition 5.7. There exists $q_0(y, \eta, z)$ elliptic for $(y, \eta, z)$ close to $(0, 0, 0)$ such that

\begin{equation}
\epsilon_{+}(y, \eta, z) = q_0(y, \eta, z)(\hat{b}(y, \eta) - b_0 - z),
\end{equation}

and

\[ q_0(0, 0, 0) = 1. \]

In particular, we recover in another way the statements of Remark 5.3.

5.6. Grushin’s problem: final step. In this subsection, we will complete our study of Grushin’s problem $P_h(z)$ given by (5.2), considering it at the level of operators.

First, we introduce an appropriate algebra of operators. Consider the space $\mathfrak{A}$ of operators on $(S(\mathbb{R}_x) \hat{\otimes} C_0^\infty(\mathbb{R}_y)) \times C_0^\infty(\mathbb{R}_y)$ of the form

\[ A(h) = \left( \begin{array}{cc}
\alpha_0(x, D_x, y, hD_y, h) & a_-(x, D_x, y, hD_y, h)R_-
\\
R_+(a_+ + a_0 \circ I)(x, D_x, y, hD_y, h) & a_+(y, hD_y, h)
\end{array} \right), \]

where $a_0, a_-, a_+ \in \text{Op} S^0, \alpha_0 \in \text{Op} S^\infty[h]$, and $a_\pm \in \text{Op} S^* h$.

Here $R_+(a_+ + a_0 \circ I)(x, D_x, y, hD_y, h)$ can be defined in the following way:

\[ S(\mathbb{R}_x) \hat{\otimes} C_0^\infty(\mathbb{R}_y) \ni \exists u \mapsto R_+(a_+ + a_0 \circ I)(x, D_x, y, hD_y, h)u \in C^\infty(\mathbb{R}_y), \]

with the property that, $\forall v \in C^\infty(\mathbb{R}_y)$,

\[ \langle R_+ a_+(x, D_x, y, hD_y, h)u, v \rangle = \langle a_+(x, D_x, y, hD_y, h)u, h_0 \otimes v \rangle_{L^2(\mathbb{R}_y^2)}. \]

Observe that, for any $a \in S^*$, we have

\[ (I \otimes a(y, hD_y, h))R_- = R_- a(y, hD_y, h), \]

and

\[ R_+(I \otimes a(y, hD_y, h)) = a(y, hD_y, h)R_. \]

We also have

\[ R_+ \theta(x, D_x, y, hD_y, h)R_- = \phi(y, hD_y, h). \]

One can see that $\mathfrak{A}$ is an algebra. For $A, B \in \mathfrak{A}$ we have

\[ A \circ B = \left( \begin{array}{cc}
a_0 \circ b_0 + a_- \circ R_- R_+ \circ b_+ & (a_0 \circ b_+ + a_- \circ (I \otimes b_1) \circ R_-)
\\
R_+(a_+ \circ b_0 + (I \otimes a_0) \circ b_+) & a_+ \circ b_+ + R_+ a_+ \circ b_- R_-
\end{array} \right). \]

For any $A \in \mathfrak{A}$ of the form

\[ A(h) = \left( \begin{array}{cc}
\alpha_0(x, D_x, y, hD_y, h) & a_-(x, D_x, y, hD_y, h)R_-
\\
R_+(a_+ \circ I)(x, D_x, y, hD_y, h) & a_+(y, hD_y, h)
\end{array} \right), \]

we define the Weyl vector valued $h$-symbol of $A$ as a function on $\mathbb{R}^2$ whose value at $(\eta, \eta) \in \mathbb{R}^2$ is an operator in $L^2(\mathbb{R}) \times \mathbb{C}$ given by

\[ A(y, \eta, h) = \left( \begin{array}{cc}
a_0(x, D_x, y, \eta, h) & a_-(x, D_x, y, \eta, h)R_-
\\
R_+(a_+ \circ I)(x, D_x, y, \eta, h) & a_+(y, \eta, h)
\end{array} \right). \]

We observe that Grushin’s problems $P_h(z)$ and $Q(z)$ belong to $\mathfrak{A}$. 

We take as the first approximate inverse for $\mathcal{P}_h(z)$ the operator $\mathcal{E}_0^\chi(z) \in \mathfrak{A}$, whose Weyl vector valued $h$-symbol is

$$\mathcal{E}_0^\chi(x, D_x, y, \eta, z) := \chi(y, \eta) \mathcal{E}_0(x, D_x, y, \eta, z),$$

where $\mathcal{E}_0(x, D_x, y, \eta, z)$ is constructed in Section 5.4 and $\chi \in C^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} \chi \subset \Omega_1$ and $\chi \equiv 1$ on $\Omega$ where $\Omega$ is an open neighborhood of $(0, 0)$ in $\mathbb{R}^2$ such that $\Omega \subset \Omega_1$. Then we get

$$\mathcal{E}_0^\chi(z) \circ \mathcal{P}_h(z) = I + \Sigma(z),$$

where $\Sigma(z) \in \mathfrak{A}$, $\Sigma(z) = O_{\Omega}(h^{1/2})$. One can easily see that

$$\Sigma(z) = \sum_{j=1}^{+\infty} h^{j/2} \Sigma_j(z) + O_{\Omega}(h^{\infty}),$$

where

$$\Sigma_j = \mathcal{E}_0^\chi(z) \circ \begin{pmatrix} S_j & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \chi \mathcal{E}_0^0 \circ S_j & 0 \\ \chi \mathcal{E}_0^0 \circ S_j & 0 \end{pmatrix}.$$

Next, we construct the inverse of $I + \Sigma(z)$ in $\mathfrak{A}$. Formally, using the Neumann series, we first get that

$$(I + \Sigma(z))^{-1} \sim I + \mathcal{T}(z),$$

where

$$\mathcal{T}(z) \sim \sum_{j=1}^{+\infty} (-1)^j \Sigma(z)^j$$

with $\Sigma(z)^j = O_{\Omega}(h^{j/2})$, and then the inverse $\mathcal{E}_h(z)$ of $\mathcal{P}_h(z)$ is obtained as

$$\mathcal{E}_h(z) = (I + \mathcal{T}(z)) \circ \mathcal{E}_0^\chi(z).$$

The problem is that the order of $\Sigma(z)^j$ as a global differential operator in $x$ goes to $+\infty$ as $j \to \infty$, so the sum $\mathcal{T}(z)$, if existing, should be of infinite order in $(x, D_x)$. Therefore, we cut the formal expansion for $\mathcal{T}(z)$, choosing some natural $N$ and putting

$$\mathcal{T}_N(z) := \sum_{j=1}^{N-1} (-1)^j \Sigma(z)^j \in \mathfrak{A}.$$ 

We obtain that $\mathcal{T}_N(z) = O_{\Omega}(h^{1/2})$ and $I + \mathcal{T}_N(z)$ is the inverse of $I + \Sigma(z)$ in $\mathfrak{A}$ modulo $O_{\Omega}(h^{N/2})$:

$$(I + \Sigma(z)) \circ (I + \mathcal{T}_N(z)) = (I + \mathcal{T}_N(z)) \circ (I + \Sigma(z)) = I + \Sigma(z)^N = I + O_{\Omega}(h^{N/2}).$$

Finally, we put

$$\mathcal{E}_h^N(z) := (I + \mathcal{T}_N(z)) \circ \mathcal{E}_0^\chi(z) \in \mathfrak{A}.$$ 

So we obtain that

$$\mathcal{E}_h^N(z) \circ \mathcal{P}_h(z) = (I + \mathcal{T}_N(z)) \circ (I + \Sigma(z)) = I + O_{\Omega}(h^{N/2}).$$
Thus, we have found a left inverse $E^N_h(z)$ for the operator $P_h(z)$ in the algebra $\mathfrak{A}$ modulo $O_\Omega(h^{N/2})$.

Similarly, we can construct a right inverse for the operator $P_h(z)$ in the algebra $\mathfrak{A}$ modulo $O_\Omega(h^{N/2})$, which implies that $E^N_h(z)$ is a two-sided inverse for the operator $P_h(z)$ in the algebra $\mathfrak{A}$ modulo $O_\Omega(h^{N/2})$:

$$P_h(z) \circ E^N_h(z) = I + O_\Omega(h^{N/2}).$$

Denote

$$E^N_h(z) = \begin{pmatrix} e^N(z) & e^N_+(z) \\ e^N_-(z) & e^N_+(z) \end{pmatrix}.$$  

If we write

$$E^N_h(z) \sim E^N_0(z) + h \frac{1}{2} E^N_1(z) + h E^N_2(z) + \ldots,$$

then we have

$$E^N_j(y, \eta, z) = E_j(y, \eta, z), \quad (y, \eta) \in \Omega, \quad j = 1, 2, \ldots, N - 1,$$

where $E_j(z)$ are the coefficients of the formal expansion:

$$E_h(z) \sim E_0(z) + h \frac{1}{2} E_1(z) + h E_2(z) + \ldots.$$

Writing the formal expansion $E_h(z)$ in a block form:

$$E_h(z) = \begin{pmatrix} e(z) & e_-(z) \\ e_+(z) & e_+(z) \end{pmatrix},$$

we observe that $e_\pm(z)$ can be considered not only as a formal series, but also as an $h$-pseudodifferential operator on $L^2(\mathbb{R})$:

$$e_\pm(z) \sim \sum_{j=0}^{\infty} h^j e_{j,\pm}(y, hD_y, z).$$

We have

$$e_\pm(y, hD_y, z) = e^N_\pm(y, hD_y, z) + O_\Omega(h^{N/2}),$$

or equivalently

$$e_{j,\pm}(y, \eta, z) = e^N_{j,\pm}(y, \eta, z), \quad (y, \eta) \in \Omega, \quad j = 1, 2, \ldots, N - 1.$$

Let us show how to compute the first two coefficients of (5.25).

The coefficient of $h^{1/2}$ is given by

$$E_1 = -\Sigma_1 \circ e^0_\chi.$$

Using (5.20), we get

$$E_1 = -\begin{pmatrix} \chi e^0_0 \circ S_1 & 0 \\ 0 & \chi e^0_0 \circ S_1 \end{pmatrix} \circ \begin{pmatrix} \chi e^0_0 & \chi e^0_0 \\ \chi e^0_0 & \chi e^0_0 \end{pmatrix}$$

$$= -\begin{pmatrix} \chi e^0_0 \circ S_1 \circ \chi e^0_0 & \chi e^0_0 \circ S_1 \circ \chi e^0_0 \\ \chi e^0_0 \circ S_1 \circ \chi e^0_0 & \chi e^0_0 \circ S_1 \circ \chi e^0_0 \end{pmatrix}.$$
So the correction $\epsilon_{1\pm}(y, hD_y, z)$ is a $h$-pseudodifferential operator given by:
\[
\epsilon_{1\pm} = -R_+ \chi(1 - (\Theta_0 - \Theta_0(0,0) - z)W_0U_0) \circ S_1 \circ \chi W_0 R_-. 
\]
Since the operators $\Theta_0$, $W_0$ and $U_0$ respect the parity and $S_1$ changes the parity in $x$, we obtain\footnote{This type of argument appears in Sjöstrand \cite{Sjostrand} who refers to Grushin \cite{Grushin}, and then in the paper of B. Helffer \cite{Helffer} devoted to the hypoellipticity with loss of $3/2$ derivatives.} that
\[
\epsilon_{1\pm}(y, hD_y, z) = 0.
\]

**The coefficient of $h$** is given by
\[
\mathcal{E}_2 = (\Sigma_1^2 - \Sigma_2) \circ \mathcal{E}_0. 
\]
Using (5.20), we get
\[
\mathcal{E}_2 = (\chi \epsilon_0^0 \circ (S_1 \circ \chi \epsilon_0^0 \circ S_1 - S_2) - (S_1 \circ \chi \epsilon_0^0 \circ S_1 - S_2)) \circ (\chi \epsilon_0^0 \chi \epsilon_0^0). 
\]

The correction $\epsilon_{2\pm}(y, hD_y, z)$ is given by
\[
\epsilon_{2\pm} = \chi \epsilon_+^0 \circ (S_1 \circ \chi \epsilon_0^0 \circ S_1 - S_2) \circ \chi \epsilon_-. 
\]

It looks rather difficult to compute this coefficient explicitly. But of course, this is just a rather routine long computation. If we are interested in the low lying eigenvalues, the approach used in our previous work \cite{PreviousWork} is better. Hence we will not pursue in this direction.

### 6. Grushin’s problem and quasimodes

By the Grushin method, we formally arrive at a statement of the type $z \in \sigma(\tilde{T}_h)$ is equivalent to $0 \in \sigma(\epsilon_{\pm}(y, hD_y, z; h))$. This kind of problem is treated in \cite{GeneralGrushin}, where the notion of $\mu$-spectrum is introduced (see Definition 3.2).

Let us choose $\gamma_0 > 0$ such that $\gamma_0 < \alpha_0$ and the set \{(y, \eta) \in \mathbb{R}^2 : \hat{b}(y, \eta) < b_0 + \gamma_0\} is connected and contained in $\Omega$.

By Proposition 5.7, we have
\[
\epsilon_{0\pm}(y, \eta, z) = q_0(y, \eta, z)(\hat{b}(y, \eta) - b_0 - z),
\]
where $q_0(y, \eta, z)$ is elliptic for $(y, \eta) \in \Omega$ and $|z| < \alpha_0$. Let us extend $q_0$ to an elliptic semiclassical symbol from $S(1)$. The operator $Q_0(z) = q_0(y, hD_y, z)$ is invertible as an operator in $L^2(\mathbb{R})$ and the inverse $Q_0(z)^{-1}$ is an elliptic $h$-pseudodifferential operator. Consider an $h$-pseudodifferential operator $p_{\text{eff}}(z) = p_{\text{eff}}(y, hD_y, h, z)$ given by
\[
p_{\text{eff}}(z) = Q_0(z)^{-1} \circ \epsilon_{\pm}(z).
\]
Then we have
\[
p_{\text{eff}}(y, \eta, h, z) \sim \sum_{j \in \mathbb{N}} p_{\text{eff}}^j(y, \eta, z) h^j,
\]
with
\[ p_{\text{eff}}^0(y, \eta, z) = \tilde{b}(y, \eta) - b_0 - z \]
for \((y, \eta) \in \Omega\) and \(|z| < \alpha_0\).

6.1. The direct statement. Here we follow Helffer-Sjöstrand (23, 24) for the 1D-problem and Fournais-Helffer [4].

Choose \(\gamma_0(h) \in (b_0, \alpha_0)\) defined for \(h \in (0, h_0]\) such that \(\gamma_0(h) \to \gamma_0\) as \(h \to 0\) and there exists \(a(h) > 0, h \in (0, h_0]\) such that \(a(h) = O(h^{N_0})\) and

\[ \sigma(H^h) \cap (h(b_0 + \gamma_0(h)), h(b_0 + \gamma_0(h) + a(h))) = \emptyset. \]

Suppose that we have found \(z = z(h)\), satisfying \(|z(h)| < \gamma_0(h)\) for \(h \in (0, h_0]\) and the corresponding approximate 0-eigenfunction \(u_h^{qm} \in C^\infty(\mathbb{R})\) of the operator \(p_{\text{eff}}(y, hD_y, h, z(h))\), i.e.

\[ p_{\text{eff}}(z(h))u_h^{qm} = O(h^\infty), \]

such that \(\|u_h^{qm}\| = 1 + O(h^\infty)\) and the frequency set of \(u_h^{qm}\) is non-empty and contained in \(\Omega\). Then \(u_h^{qm}\) is the approximate 0-eigenfunction of the operator \(\epsilon_\pm(z)\): \(\epsilon_\pm(z)u_h^{qm} = Q_0(z)(p_{\text{eff}}(z)u_h^{qm}) = O(h^\infty)\).

Define the function \(\psi_h \in S^2(\mathbb{R}^2)\) by

\[ \psi_h = \epsilon^N_-(z)u_h^{qm}, \quad h \in (0, h_0]. \]

Using the fact that \(E^N_h(z)\) is the right inverse for \(P_h(z)\) in \(\mathfrak{A}\) modulo \(O(h^{N/2})\), by (5.28), we obtain that

\[ (\widetilde{T}_h - b_0 - z(h))\epsilon^N_-(z) + R_- \epsilon^N_\pm(z) = K_- \in O(h^{N/2}). \]

We get

\[ (\widetilde{T}_h - b_0 - z(h))\psi_h + R_- \epsilon^N_\pm(z)u_h^{qm} = K_- u_h^{qm}. \]

Since the frequency set of \(u_h^{qm}\) is contained in \(\Omega\) and \(K_- \in O(h^{N/2})\), we have

\[ K_- u_h^{qm} = O(h^{N/2}). \]

By (5.28) and the fact that the frequency set of \(u_h^{qm}\) is contained in \(\Omega\), we also have

\[ \epsilon^N_\pm(z)u_h^{qm} = \epsilon_\pm(z)u_h^{qm} + O(h^{N/2}) = O(h^{N/2}). \]

So we arrive at

\[ (\widetilde{T}_h - h^{-1}\mu_h)\psi_h = O(h^{N/2}), \]

where \(\mu_h = h(b_0 + z(h)) \in [hb_0, h(b_0 + \gamma_0(h))].\)

To control the norm of \(\psi_h\), using (5.21) and the fact that the frequency set of \(u_h^{qm}\) is contained in \(\Omega\), we observe that

\[ \|\psi_h\| = \|\epsilon_\pm^0(z)u_h^{qm}\|(1 + O(h^{1/2})). \]

Then we see that

\[ \epsilon_-^0(z) = W_0 R_- = (I + U_0 \circ (\Theta_0(y, hD_y) - \Theta_0(0, 0)) - zU_0)^{-1} R_- \]
Therefore
\[ \|\psi_h\| = \|e^0(z)u_h^q\| \geq C\|R_-u_h^q\| = C\|u_h^q\| = C(1 + O(h^\infty)). \]
Put
\[ \tilde{v}_h = \frac{\psi_h}{\|\psi_h\|}. \]
Then we have \( \|\tilde{v}_h\| = 1 \) and
\[ (\tilde{T}_h - h^{-1}\mu_h)\tilde{v}_h = O(h^{N/2}). \]
Coming back to the initial variables (see Subsection 4.1), we obtain a function \( v_h \) such that \( \|v_h\| = 1 \) and
\[ (H^h - \mu_h)v_h = O(h^{N/2+1}). \]
Since \( N \) is arbitrary, by Spectral Theorem, for any \( h \in (0, h_0) \), there exists \( \lambda_h \in \sigma(H^h) \cap [hb_0, h(b_0 + \gamma_0(h))] \) such that
\[ \lambda_h - \mu_h = \lambda_h - h(b_0 + z(h)) = O(h^\infty). \]
By (6.1), it follows that \( \lambda_h \in [hb_0, h(b_0 + \gamma_0(h)] \).

6.2. The converse statement. This time we start from an \( L^2 \) eigenfunction \( u_h \) of \( H^h \) associated with \( \lambda_h \in [hb_0, h(b_0 + \gamma_0)] \) for any \( h \in (0, h_0) \) with \( \gamma_0 > 0 \) as above. The aim is to construct an approximate 0-eigenfunction for the operator \( p_{\text{eff}}(z) \) with \( z(h) = \frac{1}{h}(\lambda_h - hb_0) \).

Performing metaplectic transformations as in Subsection 3.1, we arrive at an \( L^2 \) eigenfunction \( \tilde{u}_h \) of \( \tilde{T}_h \) associated with \( h^{-1}\lambda_h \). Now we fix some natural \( N \) and use the fact that the operator \( E^N_h(z) \in \mathfrak{A} \) is the left inverse for \( P_h(z) \) in \( \mathfrak{A} \) modulo \( O(h^{N/2}) \). In particular, (5.22) reads:
\[ e^N_h(z)(\tilde{T}_h - h^{-1}\lambda_h) + e^N_h(z)R_+ = K_+, \]
where \( K_+ = R_+a_+(x, D_x, y, hD_y, h) \) with \( a_+ \in \text{Op} \, S^{0,\ast}(h), a_+ = O_{\Omega}(h^{N/2}). \)

By definition, we can write
\[ a_+(x, D_x, y, hD_y, h) = A_{N-1}(h) + R_{N-1}(h), \]
where
\[ A_{N-1}(h) = \sum_{j=0}^{N-1} h^{2j}a_j(x, D_x, y, hD_y), \]
each \( a_j \) belongs to \( S^{0,m+j}(\mathbb{R}^2 \times \mathbb{R}^2) \) and can be represented as a finite sum
\[ a_j(x, D_x, y, hD_y) = \sum b_j^{(j)}(x, D_x)c_j^{(j)}(y, hD_y) \]
with some \( b_j^{(j)} \in S^{m+j}(\mathbb{R}) \) and \( c_j^{(j)} \in S(1) \), and \( R_{N-1}(h) \) satisfies (5.1). Moreover, \( c_j^{(j)}(y, \eta) = 0 \) for any \( (y, \eta) \in \Omega \) and \( j = 0, 1, \ldots, N - 1 \).

We know that \( \tilde{u}_h = S^h\tilde{u}_h \), where \( S^h \) is a unitary operator in \( L^2(\mathbb{R}^2) \) given by \( S^hf(x, y) = h^{1/4}f(h^{1/2}x, y) \). It is easy to see that
\[ c_j^{(j)}(y, hD_y) \circ S^h = S^h \circ c_j^{(j)}(y, hD_y). \]
Therefore, we have
\[ R_+A_{N-1}(h)\tilde{u}_h = \sum R_+ b^{(j)}(x, D_x)S^h c_k^{(j)}(y, hD_y)\tilde{u}_h. \]
Since the frequency set of \( \hat{u}_h \) is contained in \( \Omega \), we have
\[ c_k^{(j)}(y, hD_y)\hat{u}_h = O(h^{N/2}). \]
Since the operator \( R_+ b^{(j)}(x, D_x)S^h \) is uniformly bounded in \( h \) as \( \Omega \) operator from \( L^2(\mathbb{R}^2) \) to \( L^2(\mathbb{R}) \), we obtain that
\[ R_+A_{N-1}(h)\tilde{u}_h = O(h^{N/2}). \]
Using (5.1) and (4.4), we conclude that
\[ R_{N-1}(h)\tilde{u}_h = O(h^{N/2}). \]
By (6.3) and (6.4), we obtain that
\[ K_+\tilde{u}_h = O(h^{N/2}). \]
By (6.2) and (6.5), we have
\[ \epsilon_+^N(z)R_+\tilde{u}_h = O(h^{N/2}). \]
It remains to show that \( R_+\tilde{u}_h \) is not too small in order to get effectively a quasi-mode. For this, we need first the following proposition:

**Proposition 6.1.** Let \( v_h, h \in (0, h_0] \), be a family of functions in \( L^2(\mathbb{R}^2) \) such that \( \|v_h\| = 1 + O(h) \). Let
\[ \Phi_h(y) := h^{-1/4}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2h\eta}} v_h(x, y)dx. \]
Then the frequency set of \( \Phi_h \) is contained in the set of \( (y, \eta) \) such that \( (0, 0, y, \eta) \) belongs to the frequency set of \( v_h \).

**Proof.** Fix \( (y_0, \eta_0) \in \mathbb{R}^2 \). Suppose that \( \chi \in C_c^\infty(\mathbb{R}^2) \) be such that \( \chi \equiv 1 \) in a neighborhood of \( (0, y_0) \). Then for any \( y \) in some neighborhood of \( y_0 \) we have as \( h \to 0 \)
\[ \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2h\eta}} v_h(x, y)dx = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2h\eta}} \chi(x, y)v_h(x, y)dx + O(h^{\infty}). \]
Therefore, without loss of generality, we can assume that \( v_h \) is supported in a (arbitrary small) neighborhood of \( (0, y_0) \). Then \( \Phi_h \) is supported in a neighborhood of \( y_0 \) and we have the formula
\[ \langle e^{-ih^{-1}y\eta}, \Phi_h(y) \rangle_y = \int_{-\infty}^{+\infty} e^{-\frac{b_n^2}{2h}} \langle e^{-ih^{-1}(x\xi+y\eta)}, v_h(x, y) \rangle_{x,y}d\xi. \]
Thus if \( (0, 0, y_0, \eta_0) \) is not in the frequency set of \( v_h \), then, by definition, there exist \( \varepsilon_0 > 0 \) and a neighborhood \( V_{\eta_0} \) of \( \eta_0 \) in \( \mathbb{R} \) such that
\[ \langle e^{-ih^{-1}(x\xi+y\eta)}, v_h(x, y) \rangle_{x,y} = O(h^{\infty}), \quad h \to 0, \]
uniformly on $\xi, |\xi| < \varepsilon_0$ and $\eta \in V_{\eta_0}$, that immediately implies (observing that for $|\xi| \geq \varepsilon_0$ the contribution is exponentially small) that
\[
\langle e^{-ih^{-1}y}, \Phi_h(y) \rangle_y = \mathcal{O}(h^\infty), \quad h \to 0,
\]
uniformly on $\eta \in V_{\eta_0}$, and, therefore, $(y_0, \eta_0)$ is not in the frequency set of $\Phi_h$. \hfill \Box

Let us rewrite the formula
\[
R_+ \tilde{u}_h(y) = \pi^{-1/4} b_0^{-1/2} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2b_0}} \tilde{u}_h(x, y) dx
\]
in the form
\[
R_+ \tilde{u}_h(y) = \pi^{-1/4} b_0^{-1/2} h^{-1/4} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2b_0}} \tilde{u}_h(x, y) dx,
\]
which corresponds simply to the change of variable $x = h^{-1/2} \tilde{x}$ in the integral. Here we note that
\[
\tilde{u}_h(x, y) = h^{1/4} \hat{\tilde{u}}_h(h^{1/2} x, y).
\]

Now we apply Proposition 6.1 with $v_h = \hat{\tilde{u}}_h$. By (4.1), we know that the frequency set of $v_h$ is contained in $\{(x, y, \xi, \eta) \in T^*\mathbb{R}^2 | (x, \xi) = (0, 0), \hat{b}(y, \eta) < b_0 + \gamma_0\}$. Proposition 6.1 implies that the frequency set of $R_+ \tilde{u}_h$ is contained in $\{(y, \eta) \in T^*\mathbb{R} | \hat{b}(y, \eta) < b_0 + \gamma_0\}$. Using this fact, (5.28) and (6.6), we obtain that
\[
\varepsilon_{\pm}(z) R_+ \tilde{u}_h = \varepsilon_{\pm}^N(z) R_+ \tilde{u}_h + \mathcal{O}(h^{N/2}) = \mathcal{O}(h^{N/2}).
\]

To control the norm of $R_+ \tilde{u}_h$, we use (5.19) and write
\[
\varepsilon^0(z)(I_h - h^{-1} \lambda_h) + \varepsilon^0(z)R_+ = I + \Sigma^0,
\]
where $\Sigma^0 \in \text{Op} S^{0,*}[h]$, $\Sigma^0 = \mathcal{O}(h^{1/2})$. Applying this identity to $\tilde{u}_h$, we obtain that
\[
\varepsilon^0(z) R_+ \tilde{u}_h = \tilde{u}_h + \Sigma^0 \tilde{u}_h.
\]
As above (see the proof of (5.5)), one can show that
\[
\Sigma^0 \tilde{u}_h = \mathcal{O}(h^{1/2}).
\]
Therefore, we have
\[
\| \varepsilon^0(z) R_+ \tilde{u}_h \| = 1 + \mathcal{O}(h^{1/2}).
\]
Since $\varepsilon^0(z)$ is bounded as an operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}^2)$ uniformly on $h \in (0, 1]$, this implies that there exist $h_0 > 0$ and $C > 0$ such that, for any $h \in (0, h_0]$,
\[
\| R_+ \tilde{u}_h \| > C.
\]
Thus, the function
\[
u^m_h = \frac{R_+ \tilde{u}_h}{\| R_+ \tilde{u}_h \|}
\]
is the approximate $L^2$ normalized eigenfunction for $\epsilon_{\pm}(z)$:

$$
\epsilon_{\pm}(z) u_h^{q_m} = O(h^{N/2}), \quad \|u_h^{q_m}\| = 1.
$$

Since $N$ is arbitrary, we obtain that

$$
\epsilon_{\pm}(z) u_h^{q_m} = O(h^{\infty}),
$$

and

$$
p_{\text{eff}}(z) u_h^{q_m} = Q_0(z)^{-1}(\epsilon_{\pm}(z) u_h^{q_m}) = O(h^{\infty}).
$$

7. Concluding remarks

In the treatment of the Grushin problem in Section 5, one can distinguish two parts. The first part is a purely formal computation of vector-valued $h$-symbols in $(y, \eta)$ at the level of complete formal expansions in powers of $h$ where we use the formal Weyl composition law $\#_h$. The second part is a way of associating with these symbols well defined operators on a Hilbert space. This forces us to introduce a cut-off function $\chi$ and to consider finite sums (choice of $N$) instead of formal infinite sums. It is only in the last part that the choice of the Grushin problem with $R_-$ and $R_+$ is useful.

If we are interested in the explicit computation of symbols, it is better, in the spirit of what we have done in Subsection 5.5, to consider instead the Grushin problem associated with the pair $(R^1_-, R^1_+)$ introduced in (5.16). Now if we make the construction done in Subsection 5.6 with this pair (taking $\chi = 1$, $N = +\infty$ and using the formal Weyl composition law $\#_h$ instead of the composition of operators $\circ$), we will arrive at an explicit formal inverse $\mathcal{E}^{(1, \infty)}(y, \eta; z)$ in the same form as in (5.26), defined for any $(y, \eta)$ in $\Omega_{\eta_0} := \{(y, \eta), \hat{b}(y, \eta) < \eta_0\}$,

$$
\mathcal{E}^{(1, \infty)}_h(y, \eta; z) = \begin{pmatrix}
\epsilon^{(1, \infty)}_+(y, \eta; z) & \epsilon^{(1, \infty)}_-(y, \eta; z) \\
\epsilon^{(1, \infty)}_+(y, \eta; z) & \epsilon^{(1, \infty)}_-(y, \eta; z)
\end{pmatrix},
$$

where $z$ should satisfy (see (5.17)):

$$
z \in I := [0, \inf(\eta_0, 2b_0)],
$$

and assuming that $\hat{b}$ is $C^\infty$ in $\Omega_{\eta_0}$.

To relate $\epsilon^{(1, \infty)}_{\pm}(y, \eta; z)$ with $\epsilon_{\pm}(y, \eta; z)$ defined for $(y, \eta) \in \Omega$ in (5.26), we can proceed in the same spirit as in Subsection 5.5, but this time working with the formal vector valued $h$-symbols defined for $(y, \eta) \in \Omega$ and the formal Weyl composition law $\#_h$. We get, for $(y, \eta) \in \Omega$ (cf. (5.15)):

$$
(1 + \epsilon_+ \#_h (R^1_- - R_-))^{-1} \#_h \epsilon_{\pm} (1 - (R^1_+ - R_+) \#_h \epsilon^{(1, \infty)}_{\pm}) = \epsilon^{(1, \infty)}_{\pm}.
$$

Now, the invertibility of $1 + \epsilon_+ \#_h (R^1_- - R_-)$ is just a question of looking at the principal symbol and this is exactly what was done in Subsection 5.5.

Note that $\epsilon^{(1, \infty)}_{\pm}(y, \eta, z)$ is defined for any $(y, \eta, z) \in \Omega_{\eta_0} \times I$, not necessarily close to $(0, 0, 0)$. This symbol can be theoretically computed following the composition law of the symbols. Practically this remains to be difficult.
and the computation of the subprincipal symbol of $\epsilon_\pm^{(1)}(1,\infty)$ should involve the computation of $S_2$ at the end of Subsection 3.3.

We think that $\epsilon_\pm^{(1)}(1,\infty)$ is the right effective Hamiltonian which could permit us to analyze the spectrum in $[h\mu_0, h\inf(\mu_0 + \eta_0, 3\mu_0)]$ like in [33], but we are at the moment obliged to go back to the effective Hamiltonian $\epsilon_\pm$ for technical reasons and hence have limited our statements to the bottom of the spectrum.

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