A Sharp Restricted Isometry Constant Bound of Orthogonal Matching Pursuit

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Abstract

We shall show that if the restricted isometry constant (RIC) $\delta_{s+1}(A)$ of the measurement matrix $A$ satisfies

$$\delta_{s+1}(A) < \frac{1}{\sqrt{s+1}},$$

then the greedy algorithm Orthogonal Matching Pursuit (OMP) will succeed. That is, OMP can recover every $s$-sparse signal $x$ in $s$ iterations from $b = Ax$. Moreover, we shall show the upper bound of RIC is sharp in the following sense. For any given $s \in \mathbb{N}$, we shall construct a matrix $A$ with the RIC

$$\delta_{s+1}(A) = \frac{1}{\sqrt{s+1}}$$

such that OMP may not recover some $s$-sparse signal $x$ in $s$ iterations.

Index Terms

Compressed sensing, restricted isometry property, orthogonal matching pursuit, sparse signal reconstruction.

I. INTRODUCTION

We shall give a sharp RIC bound of OMP in this paper. First let us review some basic concepts. Suppose we have a signal $x$ in $\mathbb{R}^N$ where $N$ is very large. For instance, $x$ represents a digit photo of $1024 \times 1024$ pixels and $N = 1024^2 \approx 10^6$. Denote $\|x\|_0$ to be the number of nonzero entries of $x$. We say $x$ is $s$-sparse if $\|x\|_0 \leq s$; and $x$ is sparse if $s \ll N$. In real world, there

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are many signals are sparse or can be well approximated by sparse signals, either under the canonical basis or other special basis/frames. For simplicity, we assume that $x$ is sparse under the canonical basis in $\mathbb{R}^N$.

Suppose we have some linear measurements of a signal $x$. That is, we have $b = Ax$ where $A \in \mathbb{R}^{m \times N}$ and $b \in \mathbb{R}^m$ are given. The key idea of compressed sensing [3], [8] is that it is highly possible to retrieve the sparse signal $x$ from those linear measurements, while the number of measurements is far less than the dimension of the signal, i.e., $m \ll N$.

To retrieve such a sparse signal $x$, a natural method is to solve the following $l_0$ problem

$$\min_x \|x\|_0 \quad \text{subject to} \quad Ax = b \quad (1)$$

where $A$ and $b$ are known. To ensure the $s$–sparse solution is unique, we would like to use the restricted isometry property (RIP) which was introduced by Candès and Tao in [4]. A matrix $A$ satisfies the RIP of order $s$ with the restricted isometry constant (RIC) $\delta_s = \delta_s(A)$ if $\delta_s$ is the smallest constant such that

$$(1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2 \quad (2)$$

holds for all $s$-sparse signal $x$.

If $\delta_{2s}(A) < 1$, the $l_0$ problem has a unique $s$-sparse solution [4]. The $l_0$ problem is equivalent to the $l_1$ minimization problem when $\delta_{2s}(A) < \sqrt{2}/2$, please see [2], [13], [1] and the references therein.

OMP is an efficient greedy algorithm to solve the $l_0$ problem [7], [16], [17]. To introduce this algorithm, we denote $A_\Omega$ by the matrix $A$ with indices of its columns in $\Omega$ and we denote $x_\Omega$ as the similar restriction of some vector $x$. The following iterative algorithm shows the framework of OMP.

**Input:** $A$, $b$

**Set:** $\Omega_0 = \emptyset$, $r_0 = b$, $k = 1$

**while not converge**

- $\Omega_k = \Omega_{k-1} \cup \arg \max_i |\langle r_{k-1}, Ae_i \rangle|$
- $x_k = \arg \min_z \|A_{\Omega_k}z - b\|_2$
- $r_k = b - A_{\Omega_k}x_k$
- $k = k + 1$
end while
\[ \hat{x}_{\Omega_k} = x_k, \hat{x}_{\Omega_k^c} = 0 \]

Return \[ \hat{x} \]

Now let us consider the conditions in terms of RIC for OMP to exactly recover any \( s \)-sparse signal in \( s \) iterations. Davenport and Wakin [6] have proven that \( \delta_{s+1}(A) < 1/(3\sqrt{s}) \) is sufficient. Later, Liu and Temlyakov [12] have improved the condition to \( \delta_{s+1}(A) < 1/((\sqrt{2} + 1)\sqrt{s}) \). Furthermore, Mo and Shen [14] have pushed the sufficient condition to \( \delta_{s+1}(A) < 1/(\sqrt{s} + 1) \) and have shown by detailed examples that a necessary condition is \( \delta_{s+1}(A) < 1/\sqrt{s} \). Later, Yang and Hoog [10] improved the sufficient condition to \( \delta_s(A) + \sqrt{s} \delta_{s+1}(A) < 1 \).

There is a small gap open between the sufficient condition and the necessary condition of the best results. And we would like to fill out the gap in this paper.

The content of this paper is consisted by two parts.

- We shall prove that the condition
  \[ \delta_{s+1}(A) < \frac{1}{\sqrt{s+1}} \]
  is sufficient for OMP to exactly recover any \( s \)-sparse \( x \) in \( s \) iterations.

- For any positive integer \( s \), we shall construct a matrix with \( \delta_{s+1} = 1/\sqrt{s+1} \) such that OMP may fail for at least one \( s \)-sparse signal in \( s \) iterations.

II. Preliminaries

Before going further, let us introduce some notations. For \( k = 1, 2, \ldots, N \), define \( e_k \) to be the \( k \)-th element in the canonical basis of \( \mathbb{R}^N \). That is, \( e_k \) is the \( 1 \)-sparse vector in \( \mathbb{R}^N \) such that only the \( k \)-th entry of \( e_k \) is 1. For a given matrix \( A \) and a given signal \( x \), define

\[ S_k := \langle Ax, Ae_k \rangle, \quad k = 1, \ldots, N. \]

Denote \( S_0 := \max_{k \in \{1, \ldots, s\}} |S_k| \). The following two lemmas are useful in our analysis.

**Lemma II.1.** For any \( s > 0 \), define \( t := \pm(\sqrt{s+1} - 1)/\sqrt{s} \). Then we have \( t^2 < 1 \) and

\[
\|A(x + te_k)\|_2^2 - \|A(t^2x - te_k)\|_2^2 = (1 - t^4) (\langle Ax, Ax \rangle + \sqrt{s} \langle Ax, Ae_k \rangle) \quad \forall k = 1, \ldots, N.
\]
Proof: By the definition of $t$, $t^2 = (\sqrt{s+1} - 1)^2/s = (\sqrt{s+1} - 1)/(\sqrt{s+1} + 1) < 1$. Moreover, by direct expanding and simplification, we have

$$\|A(x + te_k)\|_2^2 - \|A(t^2 x - te_k)\|_2^2 = (1 - t^4) \left( \langle Ax, Ax \rangle + \frac{2t}{1 - t^2} \langle Ax, Ae_k \rangle \right).$$

Now we only need to verify that $2t/(1 - t^2) = \pm \sqrt{s}$. This is indeed true since

$$\frac{2t}{1 - t^2} = \frac{\pm \sqrt{s+1} - 1}{\sqrt{s}} / \left(1 - \frac{\sqrt{s+1} - 1}{\sqrt{s+1} + 1}\right) = \pm \sqrt{s}.$$

Lemma II.2. If the restricted isometry constant $\delta_{s+1}(A)$ satisfies

$$\delta_{s+1}(A) < \frac{1}{\sqrt{s+1}}$$

and the support of the signal $x$ is a non-empty subset of $\{1, 2, \ldots, s\}$, then $S_0 > |S_k|$ for $k > s$.

Proof: Notice this lemma keeps unchanged if we replace $x$ by $cx$ with some non-zero scalar $c$. Therefore, we can assume that $\|x\|_2 = 1$. For the given $s$-sparse $x$, we obtain

$$\langle Ax, Ax \rangle = \left\langle A \sum_{k=1}^{s} x_k e_k, Ax \right\rangle$$

$$= \sum_{k=1}^{s} x_k \langle Ae_k, Ax \rangle$$

$$= \sum_{k=1}^{s} x_k S_k$$

$$\leq S_0 \|x\|_1$$

$$\leq S_0 \sqrt{s} \|x\|_2$$

$$= \sqrt{s} S_0.$$

Define $t := -(\sqrt{s+1} - 1)/\sqrt{s}$. Then by the above inequality and Lemma II.1, we have

$$(1 - t^4) \sqrt{s}(S_0 - S_k) \geq (1 - t^4)(\langle Ax, Ax \rangle - \sqrt{s} \langle Ax, Ae_k \rangle)$$

$$= \|A(x + te_k)\|_2^2 - \|A(t^2 x - te_k)\|_2^2.$$
Moreover, for any \( k > s \), by the definition of \( \delta_{s+1}(A) \), we obtain
\[
\|A(x + te_k)\|_2^2 - \|A(t^2 x - te_k)\|_2^2 \\
\geq (1 - \delta_{s+1}(A)) (\|x\|_2^2 + t^2) - t^2 (1 + \delta_{s+1}(A))(t^2 \|x\|_2^2 + 1) \\
= (1 + t^2) (1 - t^2 - (1 + t^2) \delta_{s+1}(A)) \\
= (1 + t^2)^2 ((1 - t^2)/(1 + t^2) - \delta_{s+1}(A)).
\] (6)

By the definition of \( t \), we have
\[
(1 - t^2)/(1 + t^2) = \left(1 - \frac{\sqrt{s + 1} - 1}{\sqrt{s + 1} + 1}\right) / \left(1 + \frac{\sqrt{s + 1} - 1}{\sqrt{s + 1} + 1}\right) = 1/\sqrt{s + 1}.
\] (7)

Now combining (5), (6), (7) and condition (4), we obtain
\[
(1 - t^4)\sqrt{s}(S_0 - S_k) \\
\geq \|A(x + te_k)\|_2^2 - \|A(t^2 x - te_k)\|_2^2 \\
\geq (1 + t^2)^2 \left((1 - t^2)/(1 + t^2) - \delta_{s+1}(A)\right) \\
= (1 + t^2)^2 \left(1/\sqrt{s + 1} - \delta_{s+1}(A)\right) \\
> 0.
\]

Therefore, we have \( S_0 > S_k \) for all \( k > s \). Similarly we can prove that \( S_0 > -S_k \) for all \( k > s \). Hence, \( S_0 > |S_k| \) for all \( k > s \). \[\blacksquare\]

### III. Main Results

Now we are ready to show the main results of this paper.

**Theorem III.1.** Suppose that \( A \) satisfies
\[
\delta_{s+1}(A) < \frac{1}{\sqrt{s + 1}},
\] (8)
then for any \( s \)-sparse signal \( x \), OMP will recover \( x \) from \( b = Ax \) in \( s \) iterations.

**Proof:** Without loss of generality, we assume that the support of \( x \) is a subset of \( \{1, 2, \ldots, s\} \). Thus, the sufficient condition for OMP choosing an index from \( \{1, \ldots, s\} \) in the first iteration is
\[
S_0 > |S_k| \text{ for all } k > s.
\]
By Lemma II.2, $\delta_{s+1} < \frac{1}{\sqrt{s+1}}$ guarantees the success of the first iteration of OMP. OMP makes an orthogonal projection in each iteration. Hence, it can be proved by induction that OMP always selects an index from the support of $x$ in $s$-iterations.

**Theorem III.2.** For any given positive integer $s$, there exist a $s$-sparse signal $x$ and a matrix $A$ with the restricted isometry constant

$$\delta_{s+1}(A) = \frac{1}{\sqrt{s+1}}$$

such that OMP may fail in $s$ iterations.

**Proof:** For any given positive integer $s$, let

$$A = \begin{pmatrix}
1 & \sqrt{s(s+1)} & \cdots & \sqrt{s(s+1)} \\
\sqrt{s+1} I_s & \sqrt{s+1} I_s & \cdots & \sqrt{s+1} I_s \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1
\end{pmatrix}_{(s+1) \times (s+1)}.$$

By simple calculation, we get

$$A^T A = \begin{pmatrix}
\frac{s}{s+1} I_s & \frac{1}{s+1} \\
\frac{1}{s+1} I_s & \frac{1}{s+1} \\
\frac{1}{s+1} & \frac{1}{s+1} & \cdots & 1 + \frac{1}{s+1}
\end{pmatrix}_{(s+1) \times (s+1)},$$

where $A^T$ denotes the transpose of $A$. By direct calculation, we can verify that $A^T A x = s x / (s + 1)$ for all $x \in \mathbb{R}^{s+1}$ satisfying $x_{s+1} = 0$ and $\sum_{k=1}^s x_k = 0$. Therefore, $s / (s + 1)$ is an eigenvalue of $A^T A$ with multiplicity of $s - 1$. Moreover, by direct calculation, we can see that $A^T A x = (1 \pm 1 / \sqrt{s+1}) x$ with $x = (1, 1, \ldots, 1, 1 \pm 1 / \sqrt{s+1})^T$. Therefore, $A^T A$ has another two eigenvalues $1 \pm 1 / \sqrt{s+1}$. Thus, the restricted isometry constant $\delta_{s+1}(A)$ is equal to $\frac{1}{\sqrt{s+1}}$. Now let

$$x = (1, 1, \ldots, 1, 0)^T \in \mathbb{R}^{s+1}.$$  

We have

$$S_k = \langle Ae_k, Ax \rangle = \langle A^T Ae_k, x \rangle = s / (s + 1) \text{ for all } k \in \{1, \ldots, s + 1\}.$$
This implies OMP may fail in the first iteration. Since OMP chooses one index in each iteration, we conclude that OMP may fail in \( s \) iterations for the given matrix \( A \) and the \( s \)-sparse signal \( x \).

**Remark III.3.** For the case of measurements with noise, please see [9], [11], and the references therein.

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