Loop Variables and Gauge Invariance in (Open) Bosonic String Theory.

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Abstract

We give a simplified and more complete description of the loop variable approach for writing down gauge invariant equations of motion for the fields of the open string. A simple proof of gauge invariance to all orders is given. In terms of loop variables, the interacting equations look exactly like the free equations, but with a loop variable depending on an extra parameter, thus making it a band of finite width. The arguments for gauge invariance work exactly as in the free case. We show that these equations are Wilsonian RG equations with a finite world-sheet cutoff and that in the infrared limit, equivalence with the Callan-Symanzik $\beta$-functions should ensure that they reproduce the on-shell scattering amplitudes in string theory. It is applied to the tachyon-photon system and the general arguments for gauge invariance can be easily checked to the order calculated. One can see that when there is a finite world sheet cutoff in place, even the $U(1)$ invariance of the equations for the photon, involves massive mode contributions. A field redefinition involving the tachyon is required to get the gauge transformations of the photon into standard form.
1 Introduction

The renormalization group equations (beta functions) for the 2-dimensional action of a string in a non-trivial background is expected to give the equations of motion for the modes of the string \([1]-[11]\). This is expected to be true both for the closed string modes as well as open string modes. For massless modes, which were the first to be studied, this is relatively easy. In certain limits it can be done to all orders \([19, 20]\). For the tachyon also it has been done in some detail \([1, 25]\) and in some limits can be done to all orders \([26, 27, 28, 31]\). It is not too difficult because there are no issues of gauge invariance. The question then arises: How does one do this for the (interacting) massive modes? This question has been addressed in many places, for instance in \([7, 9, 10, 26, 27]\). For the open string, we argue that the loop variable approach gives an answer to this question.

At the free level, equations were written down in \([12]\). A prescription for the interacting case was given in \([13, 14]\) and many details were worked out in \([15, 16, 18]\). In this paper we give a simplified and more complete treatment of the problem. A field redefinition at the loop variable level turns out to simplify all the arguments in the earlier papers and gauge invariance is much more transparent. It is easy to show that the final system of equations has the property of being gauge invariant off shell. The relation between these equations and the equations that produce the correct scattering amplitudes for the on-shell physical states, is the same as that between the Wilson renormalization group equations with finite cutoff and the Callan-Symanzik beta function. Thus one can expect that when one solves for the irrelevant operators one will reproduce the on-shell scattering amplitudes. As an illustration we also check the gauge invariance by explicit calculation in the case of the tachyon-photon system. This method would thus seem (at tree level) to be an alternative to BRST string field theory \([22, 23, 24]\).

We have not investigated what happens at the one loop level. For closed strings the free equations seem to be obtainable in this approach \([17]\). The interactions have not been investigated.

2 Loop Variable
2.1 Free Theory

We write the string field as a generalized Fourier transform.

\[ \Phi[X(z+s)] = \int Dk(s) e^{ia \int c k(s) \partial_z X(z+as)ds + ik_0 X(z)} \Psi[k(s)] \] (2.1.1)

The object \( e^{ia \int c k(s) \partial_z X(z+as)ds + ik_0 X(z)} \) is what is referred to as the loop variable and can also be thought of as a collection of all the vertex operators of the bosonic string. \( \Psi[k(s)] \) is a wave functional that describes a particular state of the string. \( k^\mu(s) \) is a generalized momentum and can be expanded as (suppressing the Lorentz index):

\[ k(s) = k_0 + \frac{k_1}{s} + \frac{k_2}{s^2} + \ldots \]
\[ = \sum_{n\geq 0} k_n s^{-n} \] (2.1.2)

Similarly one can Taylor expand \( \partial_z X(z+s) \) as

\[ \partial X(z+s) = \sum_{n>0} s^{n-1} \frac{\partial^n X(z)}{(n-1)!} \]
\[ \equiv \sum_{n>0} s^{n-1} \tilde{Y}_n(z) \] (2.1.3)

Thus (\( a = 1 \)),

\[ e^{ia \int c k(s) \partial_z X(z+as)ds + ik_0 X(z)} = e^{i \sum_{n\geq 0} k_n \tilde{Y}_n} \] (2.1.4)

We use the notation \( \tilde{Y}_0 = X \). For the bosonic string, one expects \( \mu \) to run from 0 to 25. However we shall let it run from 0 to 26. We will use the 27th coordinate as the equivalent of the bosonized ghost coordinate, necessary for representing all the auxiliary fields in the covariant representation of the string fields \[23\]. We are not going to identify it with the ghost coordinate itself because there is no need to do so and also because we do not wish to be forced into any particular representation. The fields are all taken to be massless in 27 dimensions. Dimensional reduction to 26 dimensions is then required. \( k_0^{26} \) will be set equal to the mass of the field, in the free equation. In the interacting equation the prescription will be given below. This reduction is quite different from Kaluza-Klein reduction.
The \( k_n \) define space time fields:

\[
\langle k^\mu_n \rangle \equiv \int \prod_{n>0} dk_n \Psi [k_0, k_1, k_2, ..., k_m, ...] \frac{k^\mu_n}{S^\mu_n(0)} = S^\mu_n(0)
\]

etc. In order to make the theory gauge invariant we introduce the einbein \( \alpha(s) \) in the loop variable:

\[
e^i \int c \alpha(s) \partial_z X(z+s) ds + ik_0 X(z)
\]

with the mode expansion:

\[
\alpha(s) = \sum_{n \geq 0} \alpha_n s^{-n} \equiv e^\sum_{n \geq 0} x_n s^{-n}
\]

We set \( \alpha_0 = 1 \). For reasons explained in [12, 21] one has to integrate over all \( \alpha(s) \). We assume that \([D\alpha(s)] = [\prod_n dx_n]\).

The \( \alpha_n \) obey

\[
\frac{\partial \alpha_n}{\partial x_m} = \alpha_{n-m}
\]

Defining

\[
Y = X + \sum_{n \geq 0} \alpha_n \frac{\partial^n X}{(n-1)!} \equiv \sum_{n \geq 0} \alpha_n \tilde{Y}_n
\]

and \( \tilde{Y}_n = \frac{\partial Y}{\partial x_n} \), we see that

\[
e^i \int c \alpha(s) \partial_z X(z+s) ds + ik_0 X(z) = e^i \sum_{n \geq 0} k_n \tilde{Y}_n(z)
\]

2.2 Interacting Theory

The theory is made interacting by the simple modification of making everything a function of an additional parameter \( t \):

\[
k(s) \rightarrow k(s, t)
\]

(Thus \( k_n \rightarrow k_n(t) \))

\[
X(z) \rightarrow X(z(t))
\]

The parameter \( t \) is only a label for the vertex operator. There is no functional dependence on \( t \). It only enters when we take expectation values \( \langle ... \rangle \)
(see (2.2.14) below). We do not want to do this for the \( \alpha(s) \) because the theory does not possess such a large gauge invariance. In order to make the choice unambiguous we will translate all \( X \)’s to \( z = 0 \) and introduce the einbein there.

Thus we first write

\[
\sum_{n \geq 0} \kappa_n(t) \tilde{Y}_n(z(t)) = \sum_{n \geq 0} \tilde{k}_n(t, -z(t)) \tilde{Y}_n(0) \tag{2.2.8}
\]

This defines \( \tilde{k}_n(-z(t)) \) to be

\[
\tilde{k}_q(-z) = \sum_{n=0}^{n=q} k_q D_n^q z^{q-n}
\]

where

\[
D_n^q = \begin{cases} 
q^{-1} C_{n-1}, & n, q \geq 1 \\
1, & n = 0 \\
q, & n = q = 0
\end{cases}
\tag{2.2.9}
\]

Now we can write the gauge invariant loop variable \( \kappa_n \) as

\[
\sum_{n \geq 0} i \tilde{k}_n(t, -z) Y_n(0)
\tag{2.2.11}
\]

One can also rewrite this as a loop variable analogous to (2.1.6). Define first, \( k(s-z) = \sum_{n \geq 0} k_n(-z) s^{-n} \). Consider

\[
\sum_{n>0} k_n(-z) \tilde{Y}_n(0) + k_0 X(z) = \sum_{n>0} (k_n(-z) + k_0 \tilde{z}_n/n) \tilde{Y}_n(0) + k_0 \tilde{Y}(0)
\]

(The variable in brackets is in fact \( \tilde{k}_n(-z) \) defined earlier in (2.2.9)

\[
= \sum_{n>0} k_n(-z) \tilde{Y}_n(0) + k_0 X(0)
\]

\[
= \int ds \sum_{n>0} \tilde{k}_n(-z) s^{-n} \partial X(s) + k_0 X(0)
\]

\[\text{These variables} \ k_n(-z) \ \text{are related to corresponding variables used in [16, 18], but the relation involves} \ \alpha_n. \ \text{Thus when we treat} \ k_n(-z) \ \text{as independent variables, this implies a change of variables. In terms of space-time fields this is a fairly complicated field transformation.} \]
\[ \int ds \bar{k}(s, -z) \partial X(s) + k_0 X(0) \]  

(2.2.12)

This equation defines \( \bar{k}(s, -z) \). We can now introduce an einbein \( \alpha(s) \) to get

\[ \int ds \bar{k}(s, -z) \alpha(s) \partial X(s) + k_0 X(0) \]  

(2.2.13)

This should be compared with (2.1.6) of the free theory.

The definition of space-time fields is analogous to (2.1.5) (we will write \( z_i \) for \( z(t_i) \)),

\[ \langle k_{\mu}^n(t, -z) \rangle \equiv \int [ \prod_{n>0} dk_n(t) ] \Psi[k_n(t)] k_{\mu}^n(t) = S_{\mu n}(k_0) \]

\[ \langle k_n(t_1, -z_1) k_{\mu}^n(t_2, -z_2) \rangle = \sum_{\nu} S_{\mu\nu,n,m}(k_0) \delta(t_1 - t_2) + S_{\mu n}(k_0(t_1)) S_{\nu m}(k_0(t_2)) \]  

(2.2.14)

Thus when \( t_1 = t_2 \) it describes a higher excitation of one string, and when \( t_1 \neq t_2 \) it describes two string modes interacting.

One can also, if one wants, simplify the notation somewhat by setting \( z(t) = t \). In the open string \( z \) is real, so this is allowed. This was done in [18].

3 Gauge Transformation

3.1 Free Theory

The gauge transformation in the free case is given by [12],

\[ k(s) \rightarrow k(s) \lambda(s) \]  

(3.1.15)

Here \( \lambda(s) \) is a gauge transformation parameter with an expansion

\[ \lambda(s) = \sum_{n \geq 0} \lambda_n s^{-n} \]  

(3.1.16)

We set \( \lambda_0 = 1 \).

In terms of modes we get:

\[ k_n \rightarrow k_n + \sum_{p=1}^{n} \lambda_p k_{n-p} \]  

(3.1.17)

In order to translate (3.1.17) into space-time fields we will assume that \( \Psi[k(s), \lambda(s)] \) is also a functional of \( \lambda(s) \).
Thus taking $\langle \ldots \rangle$ on both sides of the equation one gets:

$$S_\mu^\mu_n(k_0) \rightarrow k_0^\mu \Lambda_n(k_0) + \sum_{p=1}^{n} \Lambda_\mu^{p,n} - p(k_0)$$  \hspace{1cm} (3.1.18)

where we have set

$$\langle \lambda_n k_\mu^\mu_m \rangle = \Lambda_\mu^{n,m}(k_0)$$

Note that the photon is $S_1^\mu$ in the above notation and has the usual Abelian gauge transformation. We will denote it by $A^\mu$ hereafter.

### 3.2 Interacting Theory

In the interacting case a simple generalization of (3.1.15) gives the following

$$\bar{k}(s, t, -z(t)) \rightarrow \int dt' \lambda(s, t') \bar{k}(s, t, -z(t))$$ \hspace{1cm} (3.2.19)

This is very similar to what was suggested in [13]. However, there the $k$'s were not $z$-dependent, and consequently only a subset of the interactions were obtained.

In terms of modes:

$$\bar{k}_n(t, -z(t)) \rightarrow \bar{k}_n(t, -z(t)) + \int dt' \sum_{p=1}^{n} \bar{k}_{n-p}(t, -z(t)) \lambda_p(t')$$ \hspace{1cm} (3.2.20)

To translate this to space-time fields one simply takes expectation values on both sides. Since the LHS involves, in general, many space-time fields, one has to recursively calculate the gauge transformations of higher level fields after fixing the gauge transformations of all the lower ones.

The $z_i$'s are variables of integration, and in any term in the equation of motion they are integrated over a fixed range. So these integrals are understood on both sides of any equation.

### 4 Equations of Motion

#### 4.1 Free Theory

One first defines the analogue of the Liouville mode. The Polyakov functional integral defines the two dimensional conformal field theory. The two point function is

$$< X(z) X(w) > \approx \frac{1}{2} ln \left( (z - w)^2 + \epsilon^2 \right)$$ \hspace{1cm} (4.1.21)
where $\epsilon$ is a world sheet cutoff. On a world sheet where the Liouville mode is $\sigma$ one can let $\epsilon \to \epsilon e^\sigma$. Thus, when $z = w$,

$$< X(z)X(z) > \approx \ln \epsilon + \sigma(z) \quad (4.1.22)$$

By analogy with (4.1.22) we define

$$\Sigma(z) = < Y(z)Y(z) > \quad (4.1.23)$$

neglecting the $\ln \epsilon$ piece. This piece will be retained in the interacting case. (An alternative way of defining $\Sigma$ is given in [15].) $\Sigma$ is a function of $\sigma$ and also $\alpha_n$. When $\alpha(s) = 1$, $\Sigma$ reduces to $\sigma$.

In the RG approach, the equations of motion are obtained by requiring the vanishing of all anomalous $\sigma$ dependences. In the present case we require the vanishing of $\Sigma$ dependence. Thus consider

$$e^A = e^{i \sum_{n \geq 0} k_n Y_n + k_0 \sum_{n > 0} k_n Y_n + \sum_{n, m > 0} k_n k_m Y_n Y_m} \quad (4.1.24)$$

The contractions are the result of normal ordering. Use

$$< Y_n(z)Y(z) > = \frac{1}{2} \frac{\partial \Sigma(z)}{\partial x_n}$$

$$< Y_n(z)Y_m(z) > = \frac{1}{2} \left( \frac{\partial^2 \Sigma(z)}{\partial x_n \partial x_m} - \frac{\partial \Sigma(z)}{\partial x_n} \frac{\partial \Sigma(z)}{\partial x_m} \right) \quad (4.1.25)$$

To get

$$e^A = e^{i \sum_{n \geq 0} k_n Y_n(z) + k_0 \sum_{n > 0} k_n Y_n(z) + \sum_{n, m > 0} k_n k_m \frac{1}{2} \frac{\partial \Sigma(z)}{\partial x_n} + \sum_{n, m > 0} k_n k_m \frac{1}{2} \left( \frac{\partial^2 \Sigma(z)}{\partial x_n \partial x_m} - \frac{\partial \Sigma(z)}{\partial x_n} \frac{\partial \Sigma(z)}{\partial x_m} \right)} \quad (4.1.26)$$

The equations of motion are simply obtained by [12]

$$\left( \frac{\delta}{\delta \Sigma} e^A \right) |_{\Sigma=0} = 0 \quad (4.1.27)$$

where integration by parts on all the $x_n$ are allowed.

### 4.2 Interacting Theory

In the interacting case we define

$$< Y(0)Y(0) > = G_\epsilon(0) + \Sigma(0) \quad (4.2.28)$$
Here $G_\epsilon$ is the coincident two point function of $Y$ on the flat world sheet. It is a function of $\epsilon$ and also $\alpha_n$. It reduces to $\ln \epsilon$ when $\alpha(s) = 1$. It is crucial in everything we do, that $\epsilon$ is finite and non-zero. Off-shell description of string theory requires this. Otherwise we get singularities. We can only take $\epsilon$ to zero in on-shell amplitude calculations. This equation is a simple generalization of (4.1.22). We then replace $\Sigma$ in (4.1.26) by $G + \Sigma$. Also replace $k_n$ by $\delta_n(t, -z(t))$. With these replacements eqn (4.1.27) gives the equations of motion. Note that in the free case we did not include $G(0)$. This would introduce terms in the equation with different powers of $\epsilon$. However in the free theory different powers of $\epsilon$ are not mixed by gauge transformations, so they can be safely set to zero. However in the interacting case they will be retained.

5 Gauge Invariance

5.1 Free Theory

A simple way to understand gauge invariance is to note that the gauge transformation (3.1.15) can be compensated by an inverse scaling of $\alpha(s)$. But since we are integrating over all $\alpha(s)$ (equivalently all $x_n$), this does not affect the functional integral. This assumes that the measure is invariant. This is true because if we write $\lambda(s) = e^{\sum_n y_n s^{-n}}$ the gauge transformation simply translates all the $x_n$ by an amount $y_n$ which leaves the measure invariant. This is equivalent to saying that $A$ changes by a total derivative of the form $\frac{\partial}{\partial x_p} C$ under gauge transformation by $\lambda_p$.

Thus if $\delta A = \partial C = \partial (f(\Sigma)B) = (\partial f)B + f\partial B$, and we vary w.r.t. $\Sigma$, we get, on integrating by parts, $-f' \partial B + f' \partial B = 0$.

However there is a subtlety. $\Sigma$ satisfies some constraints (arising from its definition) and is not a completely unconstrained. In fact these constraints have to be used to prove that $A$ changes by a total derivative.

If one studies the exact expression for $A$ it is easy to see [17] that if we do not assume any special properties for $\Sigma$, there are some terms proportional to $\lambda_p k_n, k_m$ (with both $n, m > 0$) that have to be set to zero if $A$ is to change by a total derivative. Thus we will impose these constraints on the gauge parameters. These are the familiar “tracelessness” constraints for higher spin gauge fields. Thus we conclude that if we impose these constraints the

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2This is just one convenient choice. Any other ultraviolet cutoff propagator would do just as well.
5.2 Interacting Theory

The structure of $A$ in the interacting case being exactly the same up to the replacements given above (i.e. $k_n \rightarrow \bar{k}_n(t, -z(t))$, $\Sigma(0) \rightarrow G_\varepsilon(0) + \Sigma(0)$) and the form of the gauge transformations also being the same, the arguments for gauge invariance given above for the free case, go through here also. The only change is that the constraints have the form $\lambda \mu \nu(t) \bar{k}_n(t_1, -z_1) \bar{k}_m(t_2, -z_2) = 0$. Integrations over all variables, $t_i, z_i$, are understood ($t_i$ can be integrated from 0-1, $z_i$ from 0 to $-a$).

6 Dimensional Reduction

6.1 Free Theory

This was described in [12]. Let us denote the 27th dimension by the index $'V'$. We simply set $k_0^V = m$, the mass of the state. The kinetic term $k_\mu^0 k_0^\mu \rightarrow k_\mu^0 k_0^\mu + k_0^V k_0^V$. Here $\mu$ runs from 0 to 26 on the LHS and 0 to 25 on the RHS.

The gauge transformation law for $k_n^V$ under $\lambda_n$ remains

$$k_n^V \rightarrow k_n^V + k_0^V \lambda_n = k_n^V + \sqrt{n - 1} \lambda_n.$$  \hspace{1cm} (6.1.29)

At the free level all the fields belong to the same level i.e. they have the same value of $k_0^V$. There is no inconsistency in setting $k_0^V$ to a particular value since gauge invariance is maintained.

Also it was shown in [24] that in order to get the right number of auxiliary fields the first oscillator of the bosonized ghost has to be set to zero. This counting was implemented in [12, 13] by imposing constraints that related terms involving $k_1^V$ to terms that didn’t involve it. Thus for instance.

$$k_1^V k_1^\mu = k_2^\mu k_0^V$$

$$k_1^V k_1^V = k_2^V k_0^V$$ \hspace{1cm} (6.1.30)

The basic idea is to find combinations such that the gauge transformations match. Of course one also has to find suitable identifications for gauge parameters of the form $k_1^V \lambda_n$ etc. For instance, it is easy to see [12, 13] that

$$k_1^V \lambda_1 = \lambda_2 k_0^V$$ \hspace{1cm} (6.1.31)

is required for the consistency of the identification (6.1.30). These combinations can then be eliminated from the equations of motion.
6.2 Interacting Theory

In order to make contact with string theory it is clear that reproducing the Veneziano amplitude, which involves integrals of the form $(z - w)^{p-q}$, where $p, q$ are 26 dimensional and \textit{not} 27 dimensional, requires that

\[ < Y^V(z)Y^V(w) > = 0 \quad z \neq w \]

We also need

\[ < Y^V(z)Y^V(z) > = \Sigma(z) \quad (6.2.32) \]

In order to implement both of the above we will simply assume that $Y^V(z) = Y^V(0)$ and is not a function of $z$. Thus $\tilde{k}^V_n(t, -z(t)) = \tilde{k}^V_n(t, 0)$. Since we do not have Lorentz invariance in the 27th dimension we are free to do this while retaining the earlier expression for $\mu : 0 - 25$. Note that on-shell scattering of physical states is not affected by anything we do to $Y^V$.

\[ k^V_0 \quad (= \sum_{i=1}^{N} k^V_0(t_i)) \quad \text{the total momentum in the 27th dimension, in any given term involving } \prod_{i=1}^{N} k_n(t_i) \text{, will be set equal to } \sqrt{\sum_{i=1}^{N} n_i - 1}. \]

This counts the number of powers of the cutoff and is equal to the dimension of the term in the sense of 2-d conformal field theory. This guarantees that the coefficient $k_0.k_0(= k_0^\mu.k_0^\mu + k_0^V.k_0^V)$, of $\Sigma$ is nothing but the RG-scaling operator $\epsilon \frac{d}{d\epsilon}$. We see this as follows:

In the coefficient of $k_0.k_0\Sigma$, powers of $\epsilon$ can come from the following sources:

1) terms of the form

\[ e^{p.qG(z-w)} = e^{p.qln(\epsilon^2 + (z-w)^2)} \]

$p, q$ being 26-dimensional momenta. If we expand in powers of $\epsilon$ we get $(\epsilon)^{p.q}$ as well as all powers of $z, w$.

2) terms of the form

\[ k^\mu_n.k^\mu_m(\frac{\partial^2}{\partial x_n \partial x_m} - \frac{\partial}{\partial x_{n+m}})G_\epsilon(z - w) \]

$\mu$ goes from $0 - 25$.

1) and 2) are responsible for the $z$-dependence of the $k^\mu_n(-z) \quad (\mu : 0 - 25)$.

3) The uncontracted $Y_n$ has scaling dimension $n$. This can be made explicit by measuring all $z$’s in units of $\epsilon$. Thus we can write $Y_n(z) = \epsilon^{-n}Y_n(\frac{z}{\epsilon})$. The overall power of $\epsilon$ can then be assigned to $k_n$ that multiplies
Thus \( k_n(-z) \) collects all terms with a given scaling dimension \( n \), in the RG equation.

The expression \( k^\mu_0 k^\nu_0 + k_0^V k_0^V \) counts all the powers of \( \epsilon \) described in 1), 2) and 3) above.

We also need some convention for assigning values to the individual momenta in the \( V \) direction. We will simply assume that every field in an interaction term in the equation of motion has equal amounts of it.

The argument that these equations are physically equivalent to those obtained from a scattering amplitude calculation can now be made in three steps.

First, only the term in the equation of motion coming from \( k_0^\mu k_0^\nu \Sigma \) multiplying anything else, is relevant to the scattering of physical states. All the other terms obtained from derivatives of \( \Sigma \) are necessary only for gauge invariance. So we can set them to zero for the purposes of this argument, and recover them at the end in a unique way because the formalism is gauge invariant.

Second, set \( \alpha(s) = 1 \). Put the \( z \)'s back into the \( \tilde{Y} \)'s (thus undoing the Taylor expansion) put everything on-shell and take the limit \( \epsilon \to 0 \). We get an RG Callan-Symanzik \( \beta \)-function equation for the coefficient of a marginal vertex operator. This, by the usual arguments (see for eg [23, 24]), are equivalent to the scattering amplitudes of string states to all orders. See also [16] for some explicit calculations.

Third, now do the Taylor expansion with finite \( \epsilon \) as described in this paper. (Removing the \( z \)-dependence of \( Y_n \) and putting it into \( k_n \) amounts to a Taylor expansion.) We have shown that we get again the RG equations, but now in their Wilsonian form involving not only marginal but all irrelevant operators. The “Magic of the Renormalization Group” in field theory ensures that when we solve for the irrelevant couplings and get an equation for the marginal ones, in the limit of going to the infrared limit, we are guaranteed to get the \( \beta \)-function calculated directly using Feynman diagrams.

Thus the equations obtained here are physically equivalent to those obtained from string amplitudes.
7 Tachyon-Photon System

The tachyon can be included by the simple device of adding to the loop variable a term \( \int dt J(t) \), with the rules

\[
\langle J(t) \rangle = \phi(k_0) \\
\langle J(t_1)J(t_2) \rangle = \phi(k_0(t_1))\phi(k_0(t_2))
\]

and so on.

The equations are obtained by the following steps:

**Step 1:**

We first write down terms coming from evaluation (4.1.27) that are proportional to \( Y^2_{\mu} \). We keep terms up to level three, i.e. involving \( k_1k_1k_1, k_1k_2, \) and \( k_3 \). We find the following terms: (There is an overall factor of \( (\epsilon^2)^2 \) multiplying every term, where \( k_0 \) is the total 26-dimensional momentum in any given term):

**Level 1:**

\[
(k_0(t_1).k_0(t_2)i\tilde{k}^\mu_1(t_3,-z_3) - \tilde{k}_1(t_1,-z_1).k_0(t_2)i\tilde{k}_0^\mu(t_3))(1 + J(t_4))
\]

**Level 3:**

\[
a) \left( -\frac{4}{\epsilon^2} \tilde{k}_1(t_1,-z_1).k_0(t_2)\tilde{k}_2(t_3,-z_3).k_0(t_4)i\tilde{k}_0^\mu(t_5)(1 + J(t_6)) \right) \\
b) \left( \frac{2}{\epsilon^2} \tilde{k}_1(t_1,-z_1).k_0(t_2)\tilde{k}_1(t_3,-z_3).k_1(t_4,-z_4)i\tilde{k}_0^\mu(t_5)(1 + J(t_6)) \right) \\
c) \left( 4\frac{1}{\epsilon^2} \tilde{k}_2(t_1,-z_1).\tilde{k}_1(t_2,-z_2)k_0(t_3).k_0(t_4)i\tilde{k}_0^\mu(t_5)(1 + J(t_6)) \right)
\]

Gauge invariance is easy to check. The gauge parameters obey the constraint \( \lambda_1(t)\tilde{k}_1(t_1,-z_1)\tilde{k}_1(t_2,-z_2)(1 + J(t_3)) = 0 \) and this has to be used while checking gauge invariance.

**Step 2:**

We dimensionally reduce by setting (when evaluating \( \langle .. \rangle \) to convert to space-time fields) \( k_0^V = 0 \) in the level-1 terms, and also \( k_1^V = 0 \). This is consistent since it’s gauge transformation involves \( k_0^V \). In the level-3 terms
we set \( k^V_0 = \sqrt{2} \) for the total momentum. The individual momenta are equal fractions of this.

Before converting to space-time fields we rewrite terms involving \( k^V_1 \) in terms of other variables. The following identifications preserve gauge invariance [13]: (The \( z \)-dependences have been suppressed. Note that because of (6.2.32) \( k^V_1 \) has no \( z \)-dependence.)

\[
2\bar{k}^V_1 k^\mu_1 k^\nu_1 = (\bar{k}^\mu_1 k^\nu_2 + \bar{k}^\nu_1 k^\mu_2) k^V_0
\]

\[
2\lambda_1 k^V_1 k^\mu = (\lambda_2 k^\mu_1 + \lambda_1 k^\mu_2) k^V_0
\]  

(7.0.35)

\[
\bar{k}^V_1 \bar{k}^\mu_1 = 2\bar{k}^\rho_0 k^\mu - \bar{k}^V_1 \bar{k}^\rho_1
\]

\[
\bar{k}^V_1 \bar{k}^V_1 = k^\mu_0 k^\rho_1 = k^\rho_1 (k^V_0)^2
\]

\[
\lambda_1 \bar{k}^V_1 \bar{k}^V_1 = \lambda_1 \bar{k}^V_0 \bar{k}^V_0 = \lambda_2 \bar{k}^V_1 \bar{k}^V_0 = \lambda_3 (k^V_0)^2
\]  

(7.0.36)

**Step 3:**

We convert to space time fields by taking expectation values \( \langle \ldots \rangle \).

**Step 4:**

Gauge transformation of space time fields is determined at each level recursively, using the transformation of lower levels as inputs as explained in Section 3. For instance, the combination \( \bar{k}^\mu_1 k^\mu_1 (1 + J(t_4)) \) is used to determine the gauge transformation law of \( S^\mu_{111} \). As inputs we use the previously determined laws of \( S^\mu_{11} \) and \( S^\mu_{1} (\equiv A^\mu) \) (photon). Since it is the same combination that occurs in the equations of motion (with contracted indices or multiplied by momenta) gauge invariance of the equations at the loop variable level guarantees invariance at the level of space-time fields also. If one looks at the space-time field equations and their unwieldy gauge transformation laws, this invariance is far from obvious, though of course it does hold.

As an illustration we write out some of the terms coming from Level-1 and Level-3 (7.0.34).

**Level-1:**

\[
(e^2)k^3_0 [k^2 A^\mu(k) - A(k).k^\mu] + (p+r)^2 iA^\mu(p)\phi(r) - A(p).(p+r)i(p+r)^\mu\phi(r)
\]  

(7.0.37)
The gauge transformation of the photon is
\[ \delta A^\mu(k) = k^\mu \Lambda_1(k) + r^\mu \Lambda_1(p) \phi(r) \quad (7.0.38) \]
Integration over momenta and momentum conserving \( \delta \)-functions are understood in all expressions. Thus \( p + r = k \) in the above expressions. Also, in the above expression \( k^0_V = 0 = A^V \). Thus the indices run over 26 dimensions only.

By inserting an arbitrary number of tachyons one sees that the gauge transformation of the photon can be expressed as follows:
\[ \delta A^\mu(X)e^{\phi(X)} = \partial^\mu(\Lambda(X)e^{\phi(X)}) \quad (7.0.39) \]
Thus the photon field with the canonical transformation law is \( Ae^{\phi} \). If the normalization of the photon \( A \) is fixed due to interactions with the closed string sector, then at the tachyon minimum, \( \phi = -\infty \), the canonical photon field becomes zero. This supports the arguments for the absence of open string excitations in the closed string vacuum [29, 30].

**Level-3** :
On making the substitutions given in (7.0.36) we find

a) 
\[ (\epsilon^2)^k^2(-\frac{4}{\epsilon^2})[(k_1.k_0 + 2k_1.k_0(k_0^V)^2 + k_3^V(k_0^V)^3)(1 + J)]i\kappa^\mu_0 \quad (7.0.40) \]
b) 
\[ (\epsilon^2)^k_0^2(\frac{2}{\epsilon^2})[(k_1.k_0 + 2(k_0^V)^2k_1.k_2 + (k_0^V)^3k_3^V)(1 + J)]i\kappa^\mu_0 \quad (7.0.41) \]
c) 
\[ (\epsilon^2)^k_0^2(\frac{4}{\epsilon^2})[(k_2.k_1(k_0^V)^2 + (k_0^V)^2) + k_1^V k_3^V(k_0^V)(k_0^V)^2)](1 + J)]i\kappa^\mu_0 \quad (7.0.42) \]
The next step is to convert to space-time fields by taking expectation values using (2.2.14).
We will illustrate this on some of the terms in Level-3 a). The others can be done similarly.
The leading \( z \)-independent term of Level-3 a) gives (converting to position space):
\[ (\epsilon^2)^k_0^2(-\frac{4}{\epsilon^2})\{(1+\phi)(S^\sigma_{2,1} + S^\sigma_{2,2}A^\sigma)\} + 4i\partial^\mu[i\partial^\rho[S^\rho_{3,1}(1+\phi)] + 2\sqrt{2}\partial^\mu(S_3(1+\phi)) \quad (7.0.43) \]
We have set \((k_0^2)^2 = 2\).

As another example let us look at the \(O(z^2)\) piece in \(k_1 k_0 \bar{k}_1 \bar{k}_1\). Writing out the \(z\)-dependence gives:

\[
(\epsilon^2) k_0^2 (-\frac{4}{\epsilon^2})(k_1(t_1)+z_1 k_0(t_1)).k_0(t_2)(k_2(t_3)+z_3 k_1(t_3))k_0(t_4)i k_0^\mu(t_5)(1+J(t_6))
\]

(7.0.44)

Note that \(z_i\) are all variables of integration and are all being integrated over the same range. Thus the following are true: \(\int dz_i(z_i)^n\), \(\int dz_idz_j(z_i^2 - z_j z_i)\). Using these and (2.2.14), we find for the \(O(z^2)\) piece:

\[
(\epsilon^2) k_0^2 \left\{ -6\frac{z^2}{\epsilon^2} \partial_\mu \partial^\sigma [A^\mu (1+\phi)] + 2 \frac{(z - z')^2}{\epsilon^2} \partial_\mu \partial^\sigma [A^\nu \partial^\sigma \phi] \right\}
\]

The rest of the terms can similarly be evaluated using the same techniques. We do not give the expressions since they are quite long and not particularly illuminating.

Note that the \(O(z^2)\) term is one of the higher derivative terms in the tachyon-photon interaction. The Koba-Nielsen variables, \(z\), are understood to be integrated over some well defined range, say, \(0 - a\). Thus the final answers will have in them a dimensionless number \(\frac{a}{\epsilon}\) (after a suitable rescaling of the \(k_n\)). This number is a free parameter and is analogous to the level expansion parameter \(\frac{4}{3\sqrt{3}}\) in BRST string field theory. It is a measure of how irrelevant an irrelevant operator really is. One can also set \(a = 1\) and then effectively \(\epsilon\) becomes that parameter.

What is also noteworthy is that the massive modes \(S_{1,1,1}, S_{2,1}\) contribute in a non trivial way to the Abelian gauge invariance (whose parameter is \(\Lambda_1\)). This is clear from the gauge transformation laws given below. As was pointed out in [53] this is due to the finite cutoff on the world sheet.

The gauge transformation law for the fields are given below:

\[
\delta S_{1,1,1}^{\mu\nu} = \Lambda_{1,1}^{\mu}(k)k^\nu + \Lambda_1(1+p)A^{\nu}(q)q^\mu + \nu^{(\mu} \Lambda_{1,1}^{\nu)}(p)\phi(r)
\]

\[
\delta S_{2}^{\mu} = \Lambda_{1,1}^{\mu} + \Lambda_1(1+p)A^{\mu}(q)
\]

\[
\delta S_{2,1}^{\mu\nu}(k) = \Lambda_{1,1}^{\mu}(k) + \Lambda_{1,2}^{\mu}(k)k^\nu + \Lambda_{2,1}^{\mu}(k)k^\mu + \Lambda_1(1+p)S_{1,1,1}^{\mu\nu}(q) + \Lambda_1(1+p)q^\nu S_{1,1,1}^{\mu\nu}(q) + \Lambda_{1,1}^{\nu}(p)A^{\mu}(q) + \nu^{\mu} \Lambda_2(1)(p)A^{\nu}(q)
\]

\[
+ \Lambda_{1,2}^{\mu}(p)r^\nu \phi(r) - \frac{(z_1 - z_3)^2}{2} \nu^{\mu} \Lambda_1(1)(p)\phi(r)
\]

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\begin{align*}
\delta S_{1,1,1}^{\mu\nu\rho} &= k(\mu)\Lambda_{1,1,1}^{\mu} + \Lambda_1(p)q(\mu)S_{1,1,1}^{\mu}(q) + [[\Lambda_{1,1}^{\mu}(p)A^\mu(q) + (\rho \leftrightarrow \nu)]q^\mu + \text{two permutations }] +
z[\Lambda_1(p)q^\mu(p + q)^\nu A^\rho + (\rho \leftrightarrow \nu) + \text{two permutations }] + r(\mu)\Lambda_{1,1}^{\mu}p^\rho \\
-\mu A_1(p)\phi(r)|((r^\mu(p + q)^\rho + r^\rho(p + q)^\mu + r^\mu r^\rho) + \text{two permutations }] \\
& - (z_4 - z)^2\Lambda_1(p)\phi(r)r(\mu)\rho^\rho
\delta S_{2} &= 2z_2 \\
\delta S_{2,1}^{\mu} &= \frac{3}{\sqrt{2}}\Lambda_{2,1}^{\mu}(k) + \frac{3}{\sqrt{2}}2\Lambda_2(p)A^\mu(q) - S_2(p)q^\mu A_1(q) +
\frac{1}{\sqrt{2}}\Lambda_{1,2}^{\mu} + \frac{1}{\sqrt{2}}\Lambda_1(p)S_{2}^{\mu}(q) + \sqrt{2}\Lambda_3(k)k^\mu +
(\frac{3}{\sqrt{2}} - 2)z\Lambda_2(k)k^\mu + \frac{1}{\sqrt{2}}z\Lambda_{1,1}^{\mu} + \frac{1}{\sqrt{2}}z\Lambda_1(p)A^\mu(q)
\delta S_{3} &= 3\sqrt{2}\Lambda_3.
\end{align*}

The gauge invariance of the equations of motion are much easier to verify before dimensional reduction. We have done this explicitly for some of the gauge invariances: \(\Lambda_{1,1,1}\) at \(O(z^0)\), \(\Lambda_{1,1}\) at \(O(z)\) and \(\Lambda_1\phi\) at \(O(z^2)\). As explained earlier this follows necessarily from the gauge invariance at the loop variable level.

8 Summary and Conclusions

In this paper we have described a solution to the problem of obtaining gauge invariant equations of motion for the modes of the open bosonic string using the RG equations of the world sheet conformal field theory. This approach involves defining variables on a curve ("loop variable"). In this approach, there are several intriguing features. First, the theory is formally written as a massless theory in 27 dimensions and masses are obtained by a dimensional reduction prescription (that is quite a different one from the usual Kaluza-Klein reduction). Second, the structure of the interacting theory, both the form of the equations and the gauge transformation law, is similar
to that of the free theory. The loop is just thickened to a band and the loop variables acquire a dependence on the positions of the vertex operators. Third, the gauge transformation law, in terms of loop variables has a simple interpretation of space-time scale transformations. This supports the speculation \cite{12} that the space-time Renormalization Group on a lattice with finite spacing, is part of the invariance group of string theory. Finally, space-time gauge invariance of the equations obtained this way does not seem tied down to any special world sheet properties, unlike in BRST string field theory where it follows from BRST invariance. To that extent it need not describe a string theory. Only the special choice of Green’s function enforces the string theory connection.

There are many open questions. We have not investigated the issue of whether there is a simple generalization that works for loops. The precise relation to BRST string field theory is not clear. The theory is so much simpler in terms of loop variables that it would be interesting to work out solution generating techniques in terms of these variables rather than in terms of space-time fields. Finally, it would be interesting to find a physical explanation of the “intriguing” features mentioned above.
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