Logarithmic asymptotics for the number of periodic orbits of the Teichmüller flow on Veech’s space of zippered rectangles.

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1 Introduction

The aim of this note is to obtain a logarithmic asymptotics for number of periodic orbits on Veech’s space of zippered rectangles, such that the norm of the corresponding renormalization matrix does not exceed a given value. The space of zippered rectangles is a finite branched cover of a set of full measure in the moduli space of abelian differentials with prescribed singularities. In particular, all periodic orbit in moduli space can be lifted to the space of zippered rectangles. The renormalization matrix corresponds to the action of the corresponding pseudo-Anosov automorphism in relative homology of the underlying surface with respect to the singularities of the abelian differential.

Let $R$ be a Rauzy class of permutations on $m$ symbols, let $\Omega_0(R)$ be Veech’s moduli space of zippered rectangles, and let $P^t$ be the Teichmüller flow on $\Omega_0(R)$. As shown by Veech, to each periodic orbit $\gamma \subset \Omega_0(R)$ there corresponds a renormalization matrix $A(\gamma) \in SL(m, \mathbb{Z})$ (see the next section for precise definitions). The period of the orbit $\gamma$ is the logarithm of the spectral radius of $A(\gamma)$. The aim of this note is to give an asymptotics of the number of orbits such that the norm of the renormalization matrix does not exceed $\exp(T)$. More precisely, for a matrix $A$, write

$$||A|| = \max_j \sum_i |A_{ij}|,$$

and denote by $\text{Per}(R, T)$ the set of periodic orbits $\gamma$ for $P^t$ such that $||A(\gamma)|| \leq \exp(T)$.

**Theorem 1** Let $R$ be a Rauzy class of permutations on $m$ symbols. Then

$$\lim_{T \to \infty} \frac{\log \# \text{Per}(R, T)}{T} = m.$$

**Remark.** By a Theorem of Veech $[11]$, the entropy of the flow $P^t$ is equal to $m$.

The proof proceeds as follows. Veech’s moduli space of zippered rectangles $\Omega_0(R)$ admits a natural Lebesgue measure class, and the Teichmüller flow
$P^t$ preserves an absolutely continuous finite ergodic invariant measure $\mu_\mathcal{R}$ on $\Omega_0(\mathcal{R})$.

The flow $P^t$ has a pair of complementary “stable” and “unstable” foliations. As Maryam Mirzakhani has pointed out to me, the measure $\mu_\mathcal{R}$ has the Margulis property \cite{12} of uniform expansion on unstable leaves. This observation allows to obtain the logarithmic asymptotics of the number of periodic orbits whose period does not exceed $T$ “in the compact part” of the space $\Omega_0(\mathcal{R})$, using mixing of the flow $P^t$, due to Veech \cite{11}, and following the classical argument of Margulis \cite{12}. Note that “in the compact part” of the moduli space the period of the orbit is comparable (up to an additive constant) to the logarithm of the norm of the renormalization matrix.

To estimate the number of periodic orbits “at infinity”, it is convenient to represent the Teichmüller flow as a suspension flow over the natural extension of the Rauzy-Veech-Zorich induction map. Each periodic orbit is then coded by a finite word over a countable alphabet. A compact set in the moduli space can be chosen in such a way that periodic orbits passing through it correspond to words that contain a given subword. But then, to a word coding a periodic orbit, one can assign its concatenation with the word, corresponding to the compact set. The norm of the corresponding matrix grows at most by a multiplicative constant under this procedure. The asymptotics of periodic orbits in the whole space is thus reduced to the asymptotics of orbits passing “through the compact part”.

By a Theorem of Veech \cite{1}, the moduli space $\Omega_0(\mathcal{R})$ of zippered rectangles admits an almost everywhere defined, surjective up to a set of measure zero, finite-to-one projection map on the moduli space of abelian differentials with prescribed singularities (the genus and the orders of the singularities are uniquely determined by the Rauzy class $\mathcal{R}$ \cite{16}). The Teichmüller flow $P^t$ projects to the Teichmüller flow on moduli space of abelian differentials, and periodic orbits are taken to periodic Teichmüller geodesics (perhaps of smaller period). It would be interesting to see if a similar asymptotics could be obtained in moduli space of abelian differentials as well.

2 Rauzy-Veech-Zorich induction.

The Teichmüller flow on Veech’s space of zippered rectangles can be represented as a suspension flow over the natural extension of the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations. Since the Rauzy-Veech-Zorich induction map has a natural countable Markov partition, we obtain a symbolic coding for the Teichmüller flow \cite{1,4,10}. In particular, periodic orbits are represented by periodic symbolic sequences. This representation will be essential for our argument.

In this section, we briefly recall the definitions of the Rauzy-Veech-Zorich induction, of Veech’s space of zippered rectangles, and of the symbolic representation for the Teichmüller flow. For a more detailed presentation, see \cite{1,4,10}.
2.1 Rauzy operations \( a \) and \( b \).

Let \( \pi \) be a permutation on \( m \) symbols. The permutation \( \pi \) will always be assumed irreducible in that \( \pi\{1, \ldots, k\} = \{1, \ldots, k\} \) iff \( k = m \).

Rauzy operations \( a \) and \( b \) are defined by the formulas:

\[
a\pi(j) = \begin{cases} 
\pi j, & \text{if } j \leq \pi^{-1}m; \\
\pi m, & \text{if } j = \pi^{-1}m + 1; \\
\pi(j - 1), & \text{other } j.
\end{cases}
\]

\[
b\pi(j) = \begin{cases} 
\pi j, & \text{if } \pi j \leq \pi m; \\
\pi j + 1, & \text{if } \pi m < \pi j < m; \\
\pi m + 1, & \text{if } \pi j = m.
\end{cases}
\]

These operations preserve irreducibility. The Rauzy class of a permutation \( \pi \) is defined as the set of all permutations that can be obtained from \( \pi \) by repeated application of the operations \( a \) and \( b \).

For \( i, j = 1, \ldots, m \), denote by \( E_{ij} \) an \( m \times m \) matrix of which the \( i, j \)-th element is equal to 1, all others to 0. Let \( E \) be the \( m \times m \)-identity matrix.

Introduce the matrices

\[
A(a, \pi) = \sum_{i=1}^{\pi^{-1}(m)} E_{ii} + E_{\pi^{-1}m+1} \sum_{i=\pi^{-1}m+1}^{m} E_{i,i+1},
\]

\[
A(b, \pi) = E + E_{\pi^{-1}m+1}
\]

For a vector \( \lambda \in \mathbb{R}^m \), \( \lambda = (\lambda_1, \ldots, \lambda_m) \), we write

\[
|\lambda| = \sum_i \lambda_i.
\]

Let \( \Delta_{m-1} \) be the unit simplex in \( \mathbb{R}^m \):

\[
\Delta_{m-1} = \{ \lambda \in \mathbb{R}_+^m : |\lambda| = 1 \}.
\]

The space \( \Delta(\mathcal{R}) \) of interval exchange transformations, corresponding to a Rauzy class \( \mathcal{R} \), is defined by the formula

\[
\Delta(\mathcal{R}) = \Delta_{m-1} \times \mathcal{R}.
\]

Denote

\[
\Delta^+ = \{ \lambda \in \Delta_{m-1} \mid \lambda_{\pi^{-1}m} > \lambda_m \}, \quad \Delta^- = \{ \lambda \in \Delta_{m-1} \mid \lambda_m > \lambda_{\pi^{-1}m} \},
\]

and

\[
\Delta^+ = \bigcup_{\pi' \in \mathcal{R}(\pi)} \Delta^+_{\pi'}, \quad \Delta^- = \bigcup_{\pi' \in \mathcal{R}(\pi)} \Delta^-_{\pi'}.
\]

The Rauzy-Veech induction is a map

\[
\mathcal{T} : \Delta(\mathcal{R}) \to \Delta(\mathcal{R}),
\]
defined by the formula

\[
T(\lambda, \pi) = \begin{cases} \frac{A(\pi,a) - 1}{A(\pi,a) - \lambda}, & \text{if } \lambda \in \Delta^-; \\ \frac{A(\pi,b) - 1}{A(\pi,b) - \lambda}, & \text{if } \lambda \in \Delta^+. \end{cases}
\]

Veech [1] showed that the Rauzy-Veech induction has an absolutely continuous ergodic invariant measure on \( \Delta(\mathbb{R}) \); that measure is, however, infinite.

Following Zorich [4], define the function \( n(\lambda, \pi) \) in the following way.

\[
n(\lambda, \pi) = \begin{cases} \inf \{ k > 0 : T^k(\lambda, \pi) \in \Delta^- \} & \text{if } \lambda \in \Delta^+_\pi \\ \inf \{ k > 0 : T^k(\lambda, \pi) \in \Delta^+ \} & \text{if } \lambda \in \Delta^-_\pi. \end{cases}
\]

The Rauzy-Veech-Zorich induction is defined by the formula

\[
G(\lambda, \pi) = T^{n(\lambda, \pi)}(\lambda, \pi).
\]

**Theorem 2 (Zorich [4])** The map \( G \) has an ergodic invariant probability measure, absolutely continuous with respect to the Lebesgue measure class on \( \Delta(\mathbb{R}) \).

This invariant measure will be denoted by \( \nu \).

### 2.2 Symbolic dynamics for the induction map.

This subsection describes, following [10], the symbolic dynamics for the map \( G \).

Consider the alphabet

\[
A = \{(c, n, \pi) \mid c = a \text{ or } b, n \in \mathbb{N}\}.
\]

For \( w_1 \in A \), \( w_1 = (c_1, n_1, \pi_1) \), we write \( c_1 = c(w_1), \pi_1 = \pi(w_1), n_1 = n(w_1) \). For \( w_1, w_2 \in A \), \( w_1 = (c_1, n_1, \pi_1) \), \( w_2 = (c_2, n_2, \pi_2) \), define the function \( B(w_1, w_2) \) in the following way: \( B(w_1, w_2) = 1 \) if \( c_1^{n_1} \pi_1 = \pi_2 \) and \( c_1 \neq c_2 \) and \( B(w_1, w_2) = 0 \) otherwise.

Introduce the space of words

\[
W_{A,B} = \{ w = w_1 \ldots w_n \mid w_i \in A, B(w_i, w_{i+1}) = 1 \text{ for all } i = 1, \ldots, n \}.
\]

For a word \( w \in W_{A,B} \), we denote by \( |w| \) its length, i.e., the number of symbols in it; given two words \( w(1), w(2) \in W_{A,B} \), we denote by \( w(1)w(2) \) their concatenation. Note that the word \( w(1)w(2) \) need not belong to \( W_{A,B} \), unless a compatibility condition is satisfied by the last symbol of \( w(1) \) and the first symbol of \( w(2) \).

To each word assign the corresponding renormalization matrix as follows. For \( w_1 \in A \), \( w_1 = (c_1, n_1, \pi_1) \), set

\[
A(w_1) = A(c_1, \pi_1)A(c_1, c_1\pi_1) \ldots A(c_1, c_1^{n_1-1}\pi_1),
\]

and for \( w \in W_{A,B} \), \( w = w_1 \ldots w_n \), set

\[
A(w) = A(w_1) \ldots A(w_n).
\]
Words from $W_{A,B}$ act on permutations from $R$: namely, if $w_1 \in A$, $w_1 = (c_1, n_1, \pi_1)$, then we set $w_1\pi_1 = c_1^{n_1}\pi_1$. For permutations $\pi \neq \pi_1$, the symbol $w_1\pi$ is not defined. Furthermore, for $w \in W_{A,B}$, $w = w_1 \ldots w_n$, we write

$$w\pi = w_n(w_{n-1}(\ldots w_1\pi)\ldots),$$

assuming the left-hand side of the expression is defined. Finally, if $\pi' = w\pi$, then we also write $\pi = w^{-1}\pi'$.

Say that $w_1 \in A$ is compatible with $(\lambda, \pi) \in \Delta(R)$ if

1. either $\lambda \in \Delta_+^\pi$, $c_1 = a$, and $a^{n_1}\pi_1 = \pi$
2. or $\lambda \in \Delta_-^\pi$, $c_1 = b$, and $b^{n_1}\pi_1 = \pi$.

Say that a word $w \in W_{A,B}$, $w = w_1 \ldots w_n$ is compatible with $(\lambda, \pi)$ if $w_n$ is compatible with $(\lambda, \pi)$. We shall also sometimes say that $(\lambda, \pi)$ is compatible with $w$ instead of saying that $w$ is compatible with $(\lambda, \pi)$. We can write

$$G^{-n}(\lambda, \pi) = \{t_w(\lambda, \pi) : |w| = n \text{ and } w \text{ is compatible with } (\lambda, \pi)\}.$$

Recall that the set $G^{-n}(\lambda, \pi)$ is infinite.

Now consider the sequence spaces

$$\Omega_{A,B} = \{\omega = \omega_1 \ldots \omega_n \ldots : \omega_n \in A, B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\},$$

and

$$\Omega_{A,B}^Z = \{\omega = \ldots \omega_{-n} \ldots \omega_1 \ldots \omega_n \ldots : \omega_n \in A, B(\omega_n, \omega_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}.$$

Denote by $\sigma$ the right shift on both these spaces. By a Theorem of Veech [5], the dynamical systems $(\Omega_{A,B}, \sigma, \mathbb{P})$ and $(\Delta(R), G, \nu)$ are isomorphic. Indeed, let $w_1 \in A$ and define the set $\Delta(w_1)$ in the following way. If $w_1 = (a, n_1, \pi_1)$, then

$$\Delta(w_1) = \{(\lambda, \pi) \in \Delta^- \mid \exists \lambda' \in \Delta_{a^n_1\pi}^+ \text{ such that } \lambda = \frac{A(w_1)\lambda'}{|A(w_1)\lambda'|}\}.$$

If $w_1 = (b, n_1, \pi_1)$, then

$$\Delta(w_1) = \{(\lambda, \pi) \in \Delta^+ \mid \exists \lambda' \in \Delta_{b^n_1\pi}^- \text{ such that } \lambda = \frac{A(w_1)\lambda'}{|A(w_1)\lambda'|}\}.$$

In other words, for a letter $w_1 = (c_1, n_1, \pi_1)$, the set $\Delta(w_1)$ is the set of all interval exchanges $(\lambda, \pi)$ such that $\pi = \pi_1$ such that the application of the map $G$ to $(\lambda, \pi)$ results in $n_1$ applications of the Rauzy operation $c_1$.

The coding map $\Phi : \Delta(R) \to \Omega_{A,B}$ is given by the formula

$$\Phi(\lambda, \pi) = \omega_1 \ldots \omega_n \ldots \text{ if } G^n(\lambda, \pi) \in \Delta(\omega_n). \quad (1)$$
The $G$-invariant smooth probability measure $\nu$ projects under $\Phi$ to a $\sigma$-
invariant measure on $\Omega_{A,B}$; probability with respect to that measure will be denoted by $P$.

For $w \in W_{A,B}$, $w = w_1 \ldots w_n$, let
\[ C(w) = \{ \omega \in \Omega_{A,B} | \omega_1 = w_1, \ldots, \omega_n = w_n \}. \]

and
\[ \Delta(w) = \Phi^{-1}(C(w)). \]

W. Veech has proved the following

**Proposition 1** Let $q \in W_{A,B}$ be a word such that all entries of the matrix $A(q)$ are positive. Let $\omega \in \Omega_{A,B}$ be a sequence having infinitely many occurrences of the word $q$. Then the set $\Phi^{-1}(\omega)$ consists of one point.

We thus obtain an almost surely bijective symbolic dynamics for the map $G$.

**Lemma 1** Let $q$ be a word such that all entries of the matrix $A(q)$ are positive. Then there exists a constant $\alpha(q)$, depending only on $q$, such that for any word $w \in W_{A,B}$ of the form $w = q\tilde{w}q$, $\tilde{w} \in W_{A,B}$, we have
\[ \alpha(q)^{-1} \leq P(C(w)) \leq \alpha(q). \]

Proof. Recall that the Lebesgue measure of the set $\Delta(w) = \Phi^{-1}(C(w))$ is given by the expression
\[ \frac{1}{(\prod_j \sum_i A(w)_{ij})}. \]

Since $A(w) = A(q)A(\tilde{w})A(q)$, there exists a constant $\beta(q)$, depending only on $q$, such that
\[ \beta(q)^{-1} \leq \frac{\|A(w)\|^m}{(\prod_j \sum_i A(w)_{ij})} \leq \beta(q). \]

Finally, since the density of the invariant measure for the Rauzy-Veech-Zorich map is bounded both from above and from below on $\Delta(q) = \Phi^{-1}(C(q))$ (see [5], [10]), there exists a constant $\alpha(q)$, depending only on $q$, such that
\[ \alpha(q)^{-1} \leq P(C(w)) \|A(w)\|^m \leq \alpha(q), \]
and the Lemma is proved.

### 2.3 Veech’s space of zippered rectangles.

A zippered rectangle associated to the Rauzy class $\mathcal{R}$ is a quadruple $(\lambda, h, a, \pi)$, where $\lambda \in \mathbb{R}^n_+, h \in \mathbb{R}^m_+, a \in \mathbb{R}^m, \pi \in \mathcal{R}$, and the vectors $h$ and $a$ satisfy the following equations and inequalities (one introduces auxiliary components $a_0 = h_0 = a_{m+1} = h_{m+1} = 0$, and sets $\pi(0) = 0, \pi^{-1}(m+1) = m+1$):

\[ h_i - a_i = h_{\pi^{-1}(\pi(i)+1)} - a_{\pi^{-1}(\pi(i)+1)-1}, i = 0, \ldots, m \]  \hspace{1cm} (2)
The area of a zippered rectangle is given by the expression

\[ \lambda_1 h_1 + \cdots + \lambda_m h_m. \]

Following Veech, we denote by \( \Omega(\mathcal{R}) \) the space of all zippered rectangles, corresponding to a given Rauzy class \( \mathcal{R} \) and satisfying the condition

\[ \lambda_1 h_1 + \cdots + \lambda_m h_m = 1. \]

We shall denote by \( x \) an individual zippered rectangle.

Veech [1] defines a flow \( P^t \) and a map \( U \) on the space of zippered rectangles by the formulas:

\[
P^t(\lambda, h, a, \pi) = (e^{t\lambda}, e^{-t}h, e^{-t}a, \pi),
\]

\[
U(\lambda, h, a, \pi) = \begin{cases} 
(A^{-1}(a, \pi)\lambda, A^t(a, \pi)h, a', a\pi), & \text{if } (\lambda, \pi) \in \Delta^- \\
(A^{-1}(b, \pi)\lambda, A^t(b, \pi)h, a'', b\pi), & \text{if } (\lambda, \pi) \in \Delta^+,
\end{cases}
\]

where

\[
a'_i = \begin{cases} 
a_i, & \text{if } j < \pi^{-1}m, \\
h_{\pi^{-1}m} + a_{m-1}, & \text{if } i = \pi^{-1}m, \\
a_{i-1}, & \text{other } i.
\end{cases}
\]

\[
a''_i = \begin{cases} 
a_i, & \text{if } j < m, \\
h_{\pi^{-1}m} + a_{\pi^{-1}m-1}, & \text{if } i = m.
\end{cases}
\]

The map \( U \) is invertible; \( U \) and \( P^t \) commute ([1]). Denote

\[ \tau(\lambda, \pi) = \log(|\lambda| - \min(\lambda_m, \lambda_{\pi^{-1}m})), \]

and for \( x \in \Omega(\mathcal{R}) \), \( x = (\lambda, h, a, \pi) \), write

\[ \tau(x) = \tau(\lambda, \pi). \]

Now, following Veech [1], define

\[ \mathcal{Y}(\mathcal{R}) = \{ x \in \Omega(\mathcal{R}) \mid |\lambda| = 1 \}. \]

and

\[ \Omega_0(\mathcal{R}) = \bigcup_{x \in \mathcal{Y}(\mathcal{R}), 0 \leq t < \tau(x)} P^t x. \]
The space $\Omega_0(\mathbb{R})$ is a fundamental domain for $\mathcal{U}$ and, identifying the points $x$ and $\mathcal{U}x$ in $\Omega_0(\mathbb{R})$, we obtain a natural flow, also denoted by $P^t$, on $\Omega_0(\mathbb{R})$. The space $\Omega_0(\mathbb{R})$ will be referred to as Veech’s moduli space of zippered rectangles, and the flow $P^t$ as the Teichmüller flow on the space of zippered rectangles.

The space $\Omega(\mathbb{R})$ has a natural Lebesgue measure class and so does the transversal $Y(\mathbb{R})$. Veech [1] has proved the following Theorem.

**Theorem 3 (Veech [1])** There exists a measure $\mu_\mathbb{R}$ on $\Omega(\mathbb{R})$, absolutely continuous with respect to Lebesgue, preserved by both the map $\mathcal{U}$ and the flow $P^t$ and such that $\mu_\mathbb{R}(\Omega_0(\mathbb{R})) < \infty$.

The construction of this measure is recalled in the Appendix.

### 2.4 Symbolic representation for the flow $P^t$

Following Zorich [4], define

\[
\Omega^+(\mathbb{R}) = \{x = (\lambda, h, a, \pi) \mid (\lambda, \pi) \in \Delta^+, a_m \geq 0\},
\]

\[
\Omega^-(\mathbb{R}) = \{x = (\lambda, h, a, \pi) \mid (\lambda, \pi) \in \Delta^-, a_m \leq 0\},
\]

\[
Y^+(\mathbb{R}) = Y(\mathbb{R}) \cap \Omega^+(\mathbb{R}), \quad Y^-(\mathbb{R}) = Y(\mathbb{R}) \cap \Omega^-(\mathbb{R}), \quad Y^\pm(\mathbb{R}) = Y^+(\mathbb{R}) \cup Y^-(\mathbb{R}).
\]

Take $x \in Y^\pm(\mathbb{R})$, $x = (\lambda, h, a, \pi)$, and let $F(x)$ be the first return map of the flow $P^t$ on the transversal $Y^\pm(\mathbb{R})$. The map $F$ is a lift of the map $\mathcal{G}$ to the space of zippered rectangles:

\[
F(\lambda, h, a, \pi) = (\lambda', h', a', \pi'), \quad \text{then } (\lambda', \pi') = \mathcal{G}(\lambda', \pi'). \quad (6)
\]

Note that if $x \in Y^+$, then $F(x) \in Y^-$, and if $x \in Y^-$, then $F(x) \in Y^+$.

The map $F$ preserves a natural absolutely continuous invariant measure on $Y^\pm(\mathbb{R})$: indeed, since $Y^\pm(\mathbb{R})$ is a transversal to the flow $P^t$, the measure $\mu_\mathbb{R}$ induces an absolutely continuous measure $\nu$ on $Y^\pm(\mathbb{R})$; since $\mu_\mathbb{R}$ is both $\mathcal{U}$ and $P^t$-invariant, the measure $\nu$ is $F$-invariant. Zorich [4] proved that the measure $\nu$ is finite and ergodic for $F$.

The dynamical system $(Y^\pm, \nu, F)$ is measurably isomorphic to the system $(\Omega^{\mathbb{Z}}_{A,B}, \mathbb{P}, \sigma)$ [1, 10]. The Teichmüller flow $P^t$ on the space $\Omega_0(\mathbb{R})$ is a suspension flow over the map $F$ on $Y^\pm$. Identifying $Y^\pm$ and $\Omega^{\mathbb{Z}}_{A,B}$, we obtain a symbolic dynamics for the Teichmüller flow in the space of zippered rectangles.

### 2.5 Periodic orbits in Veech’s space of zippered rectangles.

A word $w \in W_{A,B}$ will be called admissible if $ww \in W_{A,B}$ and if for some $r \in \mathbb{N}$ all entries of the matrix $A(w)^r$ are positive. Note that under these conditions the sequence

\[
\overline{w}(w) = \ldots w \ldots w \ldots
\]

belongs to $\Omega^{\mathbb{Z}}_{A,B}$ and there is a unique interval exchange transformation, corresponding to it. We denote by $W_{\text{per}}$ the set of all admissible words.

From the definition of the Teichmüller flow on the space of zippered rectangles, we immediately have the following
Proposition 2 1. Let $\gamma$ be a periodic orbit for the flow $P^t$. Then, the intersection $\gamma \cap Y^\pm$ is not empty and every $x \in \gamma \cap Y^\pm$ is a periodic point for the map $F$.

2. Let $x \in Y^\pm$ be a periodic point for the map $F$. Then, there exists an admissible word $w \in W_{A,B}$, such that the symbolic sequence, corresponding to $x$, has the form

$$\ldots w \ldots w \ldots$$

The symbolic sequence $\ldots w \ldots w \ldots$ defines the point $x$ uniquely.

3. If $w \in W_{A,B}$ is admissible, then there exists a unique periodic orbit of the flow $P^t$ on $\Omega_0(\mathcal{R})$, corresponding to $w$.

The periodic orbit, corresponding to an admissible word $w$ will be denoted by $\gamma(w)$; the length of the orbit $\gamma(w)$ will be denoted by $l(w)$. As the Proposition above shows, every periodic orbit $\gamma$ of the flow $P^t$ has the form $\gamma(w)$ for some $w \in W_{\text{per}}$. If $\gamma = \gamma(w)$, then we say that $w$ is a word coding the periodic orbit $\gamma$.

3 Counting admissible words.

Take $w \in W_{\text{per}}$. Consider the one-sided infinite sequence

$$\omega(w) = w \ldots w \ldots \in \Omega_{A,B},$$

and set

$$(\lambda(w), \pi(w)) = \Phi^{-1}(\omega(w)).$$

Periodicity of the sequence $\omega(w)$ implies that $\lambda(w)$ is an eigenvector of $A(w)$. Since all entries of a power of the matrix $A(w)$ are positive, $\lambda(w)$ is an eigenvector with the maximal eigenvalue of the matrix $A(w)$.

We denote by $l(w)$ the logarithm of the maximal eigenvalue of the matrix $A$, and we have then

$$A(w)\lambda(w) = \exp(l(w))\lambda(w).$$

Of course, we have $l(w) \leq \log \|A(w)\|$.

For $T > 0$, set

$$W_{\text{per}}(T) = \{w \in W_{\text{per}} \mid \log \|A(w)\| \leq T\}.$$

Lemma 2 The number of admissible words with norm not exceeding $\exp(T)$ satisfies the following logarithmic asymptotics:

$$\lim_{T \to \infty} \frac{\log \# W_{\text{per}}(T)}{T} = m.$$
Remark. Note that the results of the Lemma and of Theorem\textsuperscript{1} do not depend on the specific matrix norm used.

Lemma\textsuperscript{2} will be proven in the next section. Now we derive Theorem\textsuperscript{1} from Lemma\textsuperscript{2}.

This derivation is not automatic, because, for a given periodic orbit, the coding word \(w\) is not unique: if \(w = w_1 \ldots w_n\) codes a periodic orbit \(\gamma\), then the words \(w^{(2)} = w_2 \ldots w_n w_1\), \(w^{(2)} = w_3 \ldots w_n w_1 w_2\), \ldots, \(w^{(n)} = w_n w_1 \ldots w_{n-1}\) are precisely all words coding \(\gamma\).

In \cite{10} it is proven that the norm of the matrix \(A(w)\) grows exponentially as a function of the number of symbols of the word \(w\). More precisely, Lemma 14 and Corollary 9 in \cite{10} yield the following.

**Lemma 3** There exists a constant \(\alpha_{11}\), depending on the Rauzy class \(\mathcal{R}\) only, such that the following is true. Let \(w \in \mathcal{W}_{A,B}, w = w_1 \ldots w_n\). Then

\[
\|A(w)\| \geq \exp(\alpha_{11} n).
\]

One can therefore estimate from above the number of words coding the same periodic orbit as a power of the length of the orbit.

**Corollary 1** There exist constants \(C_{21}, \alpha_{21}\), depending on the Rauzy class \(\mathcal{R}\) only, such that the following is true. Let \(\gamma\) be a periodic orbit of length \(T\) of the Teichmüller flow on the space of zippered rectangles. Let \(w\) be a coding word for \(\gamma\). Then the number of symbols in \(w\) does not exceed \(C_{21} T^{\alpha_{21}}\).

**Corollary 2** There exists constants \(C_{31}, \alpha_{31}\), depending on the Rauzy class \(\mathcal{R}\) only, such that the following is true. Let \(\gamma\) be a periodic orbit of length \(T\) of the Teichmüller flow on the space of zippered rectangles. Then the number of words, coding \(\gamma\), does not exceed \(C_{31} T^{\alpha_{31}}\).

In view of these Corollaries, Lemma\textsuperscript{2} implies Theorem\textsuperscript{1}. We now proceed to the proof of Lemma\textsuperscript{2}.

### 4 Proof of Lemma \textsuperscript{2}

#### 4.1 “The compact part”: Margulis’s argument.

Take a word \(q \in \mathcal{W}_{\text{per}}, q = q_1 \ldots q_{2l+1}\), such that all entries of the matrix \(A(q_1 \ldots q_l)\) are positive and all the entries of the matrix \(A(q_{l+1} \ldots q_{2l+1})\) are positive. We shall first count the asymptotics of the number of periodic orbits whose coding words contain the subword \(q\).

We begin with two following simple observations.

**Proposition 3** Let \(w \in \mathcal{W}_{\text{per}}\) have the form \(w = qw\) for some \(\hat{w} \in \mathcal{W}_{A,B}\). Then \(w \in \mathcal{W}_{\text{per}}\) if and only if the concatenation \(wq\) also lies in \(\mathcal{W}_{A,B}\).
Proposition 4 There exists a constant $C(q)$, depending only on $q$ and such that the following is true. Let $w \in \mathcal{W}_{per}$ have the form $w = q \tilde{w}$ for some $\tilde{w} \in \mathcal{W}_{A,B}$. Then, for any $(\lambda, \pi) \in \Delta(q)$, we have

$$\frac{1}{C(q)} \leq \frac{|A(w)\lambda|}{\exp(l(w))} \leq C(q).$$

Now we note that for such words the period of the corresponding orbit is comparable (up to an additive constant) to the norm of the renormalization matrix.

Corollary 3 Let $w \in \mathcal{W}_{per}$ have the form $w = q \tilde{w}$ for some $\tilde{w} \in \mathcal{W}_{A,B}$. Then there exists a positive constant $c(q)$, depending on $q$ only and such that

$$\frac{1}{c(q)} \leq \frac{\exp(l(w))}{||A(w)||} \leq c(q).$$

The proof immediately follows from the fact that $(\lambda(w), \pi(w)) \in \Delta(q)$ and therefore a constant $C(q)$, depending only on $q$, may be found in such a way that for all $i, j = 1, \ldots, m$ and for any $w \in \mathcal{W}_{A,B}$ of the form $w = q \tilde{w}$ we have

$$\frac{1}{C(q)} \leq \frac{\lambda_i(w)}{\lambda_j(w)} \leq C(q).$$

Set

$$\mathcal{W}_{per}(q) = \{w \in \mathcal{W}_{A,B} | \exists w(1) \in \mathcal{W}_{A,B} \text{ such that } w = qw(1)\}.$$

For $T > 0$, set

$$\mathcal{W}_{per}(q, T) = \{w \in \mathcal{W}_{per}(q) | ||A(w)|| \leq T\},$$

and, for an interval $[r, s] \subset \mathbb{R}$, set

$$\mathcal{W}_{per}(q, [r, s]) = \{w \in \mathcal{W}_{per}(q) | r \leq ||A(w)|| \leq s\}.$$

We will also need the quantities

$$\mathcal{P}_{per}(q, T) = \{w \in \mathcal{W}_{per}(q) | l(w) \leq T\},$$

and, for an interval $[r, s] \subset \mathbb{R}$, set

$$\mathcal{P}_{per}(q, [r, s]) = \{w \in \mathcal{W}_{per}(q) | r \leq l(w) \leq s\}.$$

We have

$$\#\mathcal{W}_{per}(q, T) \leq \#\mathcal{P}_{per}(q, T),$$

and, by Corollary 3, there exists a constant $\alpha_{35}(q)$ depending on $q$ and such that

$$\#\mathcal{P}_{per}(q, T) \leq \#\mathcal{W}_{per}(q, T + \alpha_{35}).$$
Lemma 4. There exist positive constants $\alpha_{41}, \alpha_{42}, T_0$, depending on $q$ only, such that for all $T > T_0$ we have

$$\alpha_{41}^{-1} \exp(mT) \leq \# \mathcal{P}_{\text{per}}(q, [T - \alpha_{42}, T + \alpha_{42}]) \leq \alpha_{41} \exp(mT).$$

Corollary 4

$$\lim_{T \to \infty} \frac{\log \# W_{\text{per}}(q, T)}{T} = m.$$  

Proof of Lemma 4. As in Margulis’s classical argument [12], the statement of the Lemma follows from the mixing for the Teichmüller flow and the uniform expansion property for conditional measures on stable leaves. The proof below is a symbolic version of that argument, and the almost sure identification of the transversal $Y^\pm(R)$ in $\Omega_0(R)$ and the space $\Omega_{A,B}^Z$ is used. By a slight abuse of notation, it is convenient to consider the space $\Omega_{A,B}^Z$ itself embedded into $\Omega_0(R)$ and to speak of cylinders, etc., in $\Omega_{A,B}^Z$, meaning corresponding subsets in $Y^\pm$.

For a word $w \in W_{A,B}$, $|w| = n$, $w = w_1, \ldots, w_n$, and an integer $a \in \mathbb{Z}$, denote

$$C([a, a + n - 1], w) = \{ \omega \in \Omega_{A,B}^Z | \omega_a = w_1, \ldots, \omega_{a+n-1} = w_n \}.$$  

For an interval $(r, s) \subset \mathbb{R}$, set

$$C([a, a + n - 1], w, (r, s)) = \bigcup_{t \in (r, s)} P^t C([a, a + n - 1], w).$$  

For brevity, set

$$C'(q) = C([-l, l], q),$$  

and, for an interval $(r, s) \subset \mathbb{R}$, write

$$C'(q, (r, s)) = \bigcup_{t \in (r, s)} C'(q).$$  

There exists $\epsilon > 0$, depending only on $q$ and such that

$$P^t C'(q)) \cap C'(q) = \emptyset,$$

for all $t \in [-10\epsilon, 10\epsilon]$. Consider the set

$$C'(q, [-\epsilon, \epsilon]) \subset \Omega_0(R).$$

The measure $\mu_R(C'(q, [-\epsilon, \epsilon]))$ is a constant, depending only on $q$.

By a theorem of Veech [11], the Teichmüller flow is strongly mixing, and the measure

$$\mu_R((C'(q, [-\epsilon, \epsilon]) \cap P^T C'(q, [-\epsilon, \epsilon]))$$

tends to a constant, depending only on $q$, as $T \to \infty$. Therefore, the measure

$$\mu_R((C'(q, [-\epsilon, \epsilon]) \cap \bigcup_{t \in [T-\epsilon, T+\epsilon]} P^t C'(q, [-\epsilon, \epsilon]))$$

for all $T > T_0$.
also tends to a constant, depending only on \( q \).

Since \( \epsilon \) only depends on \( q \), using Proposition \( \text{P} \) and Corollary \( \text{C} \) we conclude that there exist constants \( \alpha_{51}(q) \), \( \alpha_{52}(q) \), depending on \( q \) only and such that

\[
\alpha_{51}(q)^{-1} \leq \bigcup_{w \in W_{\text{per}}(q, [T - \alpha_{52}(q), T + \alpha_{52}(q)]))} \mathbb{P}(wq) \leq \alpha_{51}(q).
\] (7)

Now recall that, by Lemma \( \text{L} \) there exists a constant \( \alpha_{61}(q) \), depending only on \( q \) and such that for any \( w \in W_{\text{per}}(q) \), we have

\[
\alpha_{61}(q)^{-1} \leq \mathbb{P}(wq) ||A(wq)||^m \leq \alpha_{61}(q).
\]

**Remark.** The fact that the measure of a cylinder, corresponding to a periodic orbit, is comparable to a power of the length of the orbit, is the manifestation, in symbolic language, of the uniform expansion property.

Since

\[
||A(w)|| \leq ||A(q)|| \leq ||A(w)|| ||A(q)||^m,
\]

if \( w \in W_{\text{per}}(q) \), then a constant \( \alpha_{71}(q) \), depending only on \( q \), can be chosen in such a way that we have

\[
\alpha_{71}(q)^{-1} \leq \mathbb{P}(wq) ||A(w)||^m \leq \alpha_{71}(q),
\]

whence, by Lemma \( \text{L} \) there exists a constant \( \alpha_{81}(q) \), depending only on \( q \), such that for any \( w \in W_{\text{per}}(q, [T - \epsilon, T + \epsilon]) \), we have

\[
\alpha_{81}(q)^{-1} \leq \mathbb{P}(wq) \exp(mT) \leq \alpha_{81}(q).
\] (8)

From \( \text{P}, \text{C}, \text{L} \), it follows that there exist constants \( \alpha_{41}(q) \), \( \alpha_{42}(q) \), depending only on \( q \) and such that

\[
\alpha_{41}(q)^{-1} \leq \# \{w \in P_{\text{per}}(q, [T - \alpha_{42}(q), T + \alpha_{42}(q)])) \} \exp(mT) \leq \alpha_{41}(q).
\]

The proof of Lemma \( \text{L} \) is complete.

### 4.2 Periodic orbits “at infinity”.

To count periodic orbits “at infinity”, the following trick is used. Take a fixed admissible word \( q \) such that all entries of the matrix \( A(q) \) are positive. Assume that an admissible word \( w \) is such that the word \( qw \) is also admissible. To the word \( w \) assign the word \( qw \).

Since

\[
||A(qw)|| \leq ||A(q)|| ||A(w)||,
\]

the logarithmic asymptotics for \( \#W_{\text{per}}(T) \) is the same as that for \( \#W_{\text{per}}(q, T) \), already known by Lemma \( \text{L} \).

Note, however, that concatenation of two admisible words may not belong to \( W_{A,B} \). One therefore considers words beginning and ending at the same permutation.
For $\pi \in \mathcal{R}$, let $W_{\text{per}}^-(\pi)$ be the set of all words $w \in \mathcal{W}_{A,B}$, $w = w_1 \ldots w_k$, $w_i = (c_i, n_i, \pi_i)$ such that
\[ c_1 = a, c_k = b, \pi_1 = \pi, b^{n_k} \pi_k = \pi, \]
and let $W_{\text{per}}^+(\pi)$ be the set of all words $w \in \mathcal{W}_{A,B}$, $w = w_1 \ldots w_k$, $w_i = (c_i, n_i, \pi_i)$ such that
\[ c_1 = b, c_k = a, \pi_1 = \pi, a^{n_k} \pi_k = \pi. \]

By definition,
\[ W_{\text{per}} = \bigcup_{\pi \in \mathcal{R}} W_{\text{per}}^-(\pi) \cup W_{\text{per}}^+(\pi). \tag{9} \]

Note, however, that, for any $\pi \in \mathcal{R}$,

if $w(1), w(2) \in W_{\text{per}}^-(\pi)$ then $w(1)w(2) \in W_{\text{per}}^-(\pi)$.

and, similarly,

if $w(1), w(2) \in W_{\text{per}}^+(\pi)$ then $w(1)w(2) \in W_{\text{per}}^+(\pi)$.

For $\pi \in \mathcal{R}$, choose an arbitrary $q_{\pi} \in W_{\text{per}}^-(\pi)$, $q_{\pi} = q_1 \ldots q_{2l+1}$, such that all entries of the matrix $A(q_1 \ldots q_l)$, as well as all entries of the matrix $A(q_{l+1} \ldots q_{2l+1})$, are positive.

Denote
\[ W_{\text{per}}(q_{\pi}) = \{ w \in W_{\text{per}}^-(\pi) | \exists \hat{w} \in \mathcal{W}_{A,B} : w = q_{\pi} \hat{w} \}. \]

Note that in this case the word $\hat{w}$ must also belong to $W_{\text{per}}^-(\pi)$. For $T > 0$, set
\[ W_{\text{per}}^-(\pi, T) = \{ w \in W_{\text{per}}^-(\pi) | \| A(w) \| \leq T \}, \]
\[ W_{\text{per}}(q_{\pi}, T) = \{ w \in W_{\text{per}}(q_{\pi}) | \| A(w) \| \leq T \}. \]

Note that if $w \in W_{\text{per}}(q_{\pi}, T)$, then $q_{\pi}w \in W_{\text{per}}(q_{\pi}, T + l(q_{\pi}))$.

For any $T > 0$, we have therefore
\[ \#W_{\text{per}}(q_{\pi}, T + l(q_{\pi})) \geq \#W_{\text{per}}^-(\pi, T) \geq \#W_{\text{per}}(q_{\pi}, T). \tag{10} \]

By Corollary [2] for any $\pi \in \mathcal{R}$ we have
\[ \lim_{T \to \infty} \frac{\log \#W_{\text{per}}(q_{\pi}, T)}{T} = m. \tag{11} \]

Equations (10) and (11) together imply
\[ \lim_{T \to \infty} \frac{\log \#W_{\text{per}}^- (\pi, T)}{T} = m. \]
An identical argument can be conducted for words in $W^+_{\text{per}}(\pi)$ as well, and for the set $W^+_{\text{per}}(\pi,T) = \{ w \in W^+_{\text{per}}(\pi) \mid l(w) \leq T \}$ we also have the asymptotics
\[
\lim_{T \to \infty} \frac{\log \# W^+_{\text{per}}(\pi,T)}{T} = m.
\]
The identity (9) finally yields
\[
\lim_{T \to \infty} \frac{\log \# W_{\text{per}}(T)}{T} = m.
\]
Lemma 2 is proved, and Theorem 1 is proved.

5 Appendix A: Zippered rectangles and abelian differentials.

Veech [1] established the following connection between zippered rectangles and moduli of abelian differentials. A detailed description of this connection is given in [16].

A zippered rectangle naturally defines a Riemann surface endowed with a holomorphic differential. This correspondence preserves area. The orders of the singularities of $\omega$ are uniquely defined by the Rauzy class of the permutation $\pi$ ([1]). For any $R$ we thus have a map
\[
\pi_R : \Omega(R) \to M_\kappa,
\]
where $\kappa$ is uniquely defined by $R$.

Veech [1] proved

**Theorem 4 (Veech)**

1. Upto a set of measure zero, the set $\pi_R(\Omega_0(R))$ is a connected component of $M_\kappa$. Any connected component of any $M_\kappa$ has the form $\pi_R(\Omega_0(R))$ for some $R$.

2. The map $\pi_R$ is finite-to-one and almost everywhere locally bijective.

3. $\pi_R(Ux) = \pi_R(x)$.

4. The flow $P^t$ on $\Omega_0(R)$ projects under $\pi_R$ to the Teichmüller flow $g_t$ on the corresponding connected component of $M_\kappa$.

5. $\pi_R^* \mu_\kappa = \mu_R$.

6. $m = 2g - 1 + \sigma$.

7. For any periodic orbit $\gamma$ of the flow $g_t$ on $M_\kappa$, there exists a periodic orbit $\tilde{\gamma}$ of the flow $P^t$ on $\Omega_0(R)$ such that $\pi_R(\tilde{\gamma}) = \gamma$. 

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6 Appendix B: The uniform expansion property.

This section is devoted to the uniform expansion property for the smooth invariant measure of the Teichmüller flow on Veech’s space of zippered rectangles. I am deeply grateful to Maryam Mirzakhani for explaining this property of the Teichmüller flow to me.

First, Veech’s coordinates on the space of zippered rectangles are modified, following [10]. Take a zippered rectangle \((\lambda, h, a, \pi) \in \Omega(\mathbb{R})\), and introduce the vector \(\delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m\) by the formula
\[
\delta_i = a_{i-1} - a_i, \quad i = 1, \ldots, m
\]
(here we assume, as always, \(a_0 = a_{m+1} = 0\)).

We have then the following lemma from [10].

**Proposition 5** The data \((\lambda, \pi, \delta)\) determine the zippered rectangle \((\lambda, h, a, \pi)\) uniquely.

The parameters \(h\) and \(a\) of the zippered rectangle are expressed through the \(\delta\) by the following formulas from [10]:
\[
h_r = -\sum_{i=1}^{r-1} \delta_i - \sum_{i=1}^{\pi(r)-1} \delta_{\pi^{-1}i}, \quad (12)
\]
\[
a_i = -\delta_1 - \cdots - \delta_{i-1}. \quad (13)
\]

In [10], it is also proven that the relations (2)-(5) defining the zippered rectangle take the following equivalent form in the new coordinates:
\[
\delta_1 + \cdots + \delta_i \leq 0, \quad i = 1, \ldots, m - 1. \quad (14)
\]
\[
\delta_{\pi^{-1}1} + \cdots + \delta_{\pi^{-1}i} \geq 0, \quad i = 1, \ldots, m - 1. \quad (15)
\]
The parameter \(a_m = -(\delta_1 + \cdots + \delta_m)\) can be both positive and negative.

Introduce the following cone in \(\mathbb{R}^m\):
\[
K_{\pi} = \{\delta = (\delta_1, \ldots, \delta_m) : \delta_1 + \cdots + \delta_{i} \leq 0, \delta_{\pi^{-1}1} + \cdots + \delta_{\pi^{-1}i} \geq 0, i = 1, \ldots, m - 1\},
\]

**Proposition 6 ([10])** For \((\lambda, \pi) \in \Delta(\mathbb{R})\) and an arbitrary \(\delta \in K_{\pi}\) there exists a unique zippered rectangle \((\lambda, h, a, \pi)\) corresponding to the parameters \((\lambda, \pi, \delta)\).

In what follows, we shall simply refer to the zippered rectangle \((\lambda, \pi, \delta)\).

Denote by \(\text{Area}(\lambda, \pi, \delta)\) the area of the zippered rectangle \((\lambda, \pi, \delta)\). We have from [10]:
\[
\text{Area}(\lambda, \pi, \delta) = \sum_{i=1}^{m} \delta_i (-\sum_{r=i+1}^{m} \lambda_r + \sum_{r=\pi(i)+1}^{m} \lambda_{\pi^{-1}r}). \quad (16)
\]
Consider the set
\[ \mathcal{V}(\mathcal{R}) = \{ (\lambda, \pi, \delta) : \pi \in \mathcal{R}, \lambda \in \mathbb{R}_+^m, \delta \in K(\pi) \}. \]

In other words, \( \mathcal{V}(\mathcal{R}) \) is the space of all possible zippered rectangles (i.e., not necessarily of those of area 1).

The Teichmüller flow \( P^t \) acts on \( \mathcal{V} \) by the formula
\[ P^t(\lambda, \pi, \delta) = (e^t \lambda, \pi, e^{-t} \delta). \]

The map \( \mathcal{U} \) acts on \( \mathcal{V}(\mathcal{R}) \) by the formula
\[
\mathcal{U}(\lambda, \pi, \delta) = \begin{cases} 
(A(\pi, b)^{-1} \lambda), b\pi, A(\pi, b)^{-1} \delta), & \text{if } \lambda \in \Delta^+_\pi; \\
(A(\pi, a)^{-1} \lambda), a\pi, A(\pi, a)^{-1} \delta), & \text{if } \lambda \in \Delta^-_\pi.
\end{cases}
\]

The volume form \( Vol = d\lambda_1 \cdots d\lambda_m d\pi_1 \cdots d\pi_m \) on \( \mathcal{V}(\mathcal{R}) \) is preserved under the action of the flow \( P^t \) and of the map \( \mathcal{U} \). Now consider the subset
\[ \mathcal{V}^{(1)}(\mathcal{R}) = \{ (\lambda, \pi, \delta) : \text{Area}(\lambda, \pi, \delta) = 1 \}, \]
i.e., the subset of zippered rectangles of area 1; naturally, this set is invariant under the Teichmüller flow. Finally, to represent the moduli space of zippered rectangles in these new coordinates, take the quotient of \( \mathcal{V}^{(1)}(\mathcal{R}) \) by the action of \( \mathcal{U} \) and obtain the space \( \mathcal{V}_0^{(1)}(\mathcal{R}) \), which is isomorphic to \( \Omega_0(\mathcal{R}) \), the linear isomorphism being given by the formulas (12), (13).

Now, given a Borel set \( X \subset \mathcal{V}^{(1)}(\mathcal{R}) \), consider the set \( K(X) \subset \mathcal{V}(\mathcal{R}) \), defined by the formula
\[ K(X) = \{ t(\lambda, \pi, \delta), t \in [0, 1], (\lambda, \pi, \delta) \in X \}. \]
(17)
and introduce a measure \( \tilde{\mu}_R \) on \( \mathcal{V}^{(1)}(\mathcal{R}) \) by the formula
\[ \tilde{\mu}_R(X) = Vol(K(X)). \]

Clearly, the measure \( \tilde{\mu}_R \) belongs to the Lebesgue measure class on \( \mathcal{V}^{(1)} \) and is invariant both under the action of flow \( P^t \) and of the map \( \mathcal{U} \). The corresponding quotient measure \( \mathcal{V}_0^{(1)}(\mathcal{R}) \) will be still denoted by \( \tilde{\mu}_R \). Under the isomorphism \( \mathcal{V}_0^{(1)}(\mathcal{R}) \) and \( \Omega_0(\mathcal{R}) \), the measure \( \tilde{\mu}_R \) on \( \mathcal{V}_0^{(1)}(\mathcal{R}) \) is taken to (a constant multiple of) the measure \( \mu_\mathcal{R} \).

Consider now two foliations on \( \mathcal{V}^{(1)}(\mathcal{R}) \):
\[ \mathcal{F}^- = \{ x \in \mathcal{V}^{(1)}(\mathcal{R}) : x = (\lambda, \pi, \delta) : \lambda = \text{const}, \pi = \text{const} \}, \]
\[ \mathcal{F}^+ = \{ x \in \mathcal{V}^{(1)}(\mathcal{R}) : x = (\lambda, \pi, \delta) : \delta = \text{const}, \pi = \text{const} \}. \]

Note that these foliations are invariant under the action of the map \( \mathcal{U} \).

For a point \( x \in \mathcal{V}^{(1)}(\mathcal{R}) \), we denote by \( \mathcal{F}^+(x) \) the leaf of the foliation \( \mathcal{F}^+ \), passing through \( x \), by \( \mathcal{F}^-(x) \) the leaf of the foliation \( \mathcal{F}^- \), passing through \( x \).

Then, by definition (17) of the measure \( \tilde{\mu}_R \), we have the following properties:
1. each leaf $\eta$ of the foliation $F^+$ carries a globally defined conditional measure $\tilde{\mu}_\eta^+$;

2. each leaf $\xi$ of the foliation $F^-$ carries a globally defined conditional measure $\tilde{\mu}_\xi^-$;

3. $(P^t)_*(\tilde{\mu}_\eta^+) = \exp(+mt)\tilde{\mu}_\eta^+$ for any $t > 0$;

4. $(P^t)_*(\tilde{\mu}_\xi^-) = \exp(-mt)\tilde{\mu}_\xi^-$ for any $t > 0$.

In other words, the measure $\tilde{\mu}_R$ has the Margulis property \[12\] of uniform expansion/contraction with respect to the pair of foliations $F^+, F^-$. In projection to the moduli space of abelian differentials, the foliations $F^+, F^-$ are taken to the stable and the unstable foliation of the Teichmüller flow. The smooth measure on the moduli space thus also satisfies the Margulis property.

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