Non abelian cohomology: the point of view of gerbed tower

Aristide Tsemo

November 11, 2018

Abstract

We define in this paper the notion of gerbed tower. This enables us to interpret geometrically cohomology classes without using the notion of $n$-category. We use this theory to study sequences of affine maps between affine manifolds, and the cohomology of manifolds.

keywords gerbes, non abelian cohomology.

Classification A.M.S. 18D05, 57R20.

1 Introduction

In mathematics, a theory is defined by axioms which describe relations between elements of a set. The purpose of mathematicians is to classify these elements by defining structures modelled on reference objects. In geometry, a structure modelled on the space $L$, is defined on a topological set $N$, by a Cech 0-chain whose boundary reflects properties of $L$. For example, a $n$-differentiable manifold is defined by an atlas $(U_i)_{i \in I}$, and charts $\phi_i : U_i \to \mathbb{R}^n$, such that $\phi_j \circ \phi_i^{-1}$ is a differentiable map, here the model $L$ is $\mathbb{R}^n$, and the property reflected is the differentiability. The manifold is obtained by gluing the sets $\phi_i(U_i)$ using the 1-cocycle $h_{ij} = \phi_j \circ \phi_i^{-1}$. Alternatively, a structure is defined by gluing a family of sets $N_i$ using a cocycle $h_{ij}$. Often the sets $N_i$ are related to the model in the sense that each of them is endowed with a $L$-structure. The natural problem to determine if a given topological space $N$ can be endowed with a structure modelled on $L$, leads to the notion of sheaf of categories. When the structure exists locally, that is when there exists an open cover $(N_i)_{i \in I}$ of $N$, such that each $N_i$ is endowed with a $L$-structure, the existence of the $L$-structure on $N$ is equivalent to determine whether the cohomology class of a 2-Cech cocycle is trivial. This has motivated the definition of a 2-structure called gerbe, which is classified in geometry by a 2-Cech cocycle. The natural problem which occurs is to provide geometric conditions which insure a 2-Cech chain to define a 2-type structure, ..., a $n$-Cech chain to define a $n$-type structure. On this purpose, one needs to give a geometric interpretation of Cech cohomology classes. Unfortunately, the notion of $n$-category needed to define $n$-structures is not well-understood. The main goal of this paper is to interpret Cech classes geometrically, by defining the notion of commutative $n$-gerbed tower. These are sequences of 2-categories $F_n \to F_{n-1}...F_2 \to F_1$, where $F_1$ is a gerbe.
defined on a topos $N$. A commutative $n$-gerbed tower satisfy conditions which allow to attach to it a family of $p$-Cech cohomology classes $([f_2], \ldots, [f_{n+1}])$, where $[f_p] \in H^p(N, L_{p-1})$, and $L_p$ are commutative sheaves defined on $N$. This notion represents geometrically the connecting morphism in cohomology. More precisely we have:

**Theorem 4.2.6.**

Let $u_n = F_n \to F_{n-1} \ldots F_2 \to F_1$ be a commutative $n$-gerbed tower defined on a topos to which is associated the family of cohomology classes $([f_2], \ldots, [f_{n+1}])$. Suppose that there exists an exact sequence of sheaves $0 \to L_{n+1} \to L'_{n+1} \to L_n \to 0$, then the family of cohomology classes $([f_2], \ldots, [f_{n+2}])$ where $[f_{n+2}]$ is the image of $[f_{n+1}]$ by the connecting morphism $H^{n+1}(N, L_n) \to H^{n+2}(N, L_{n+1})$, is associated to a $n+1$-gerbed tower.

An example of a $n$-gerbed tower appears in the theory of affine manifolds. An affine manifold $(N, \nabla_N)$ is a differentiable manifold $N$, endowed with a connection $\nabla_N$, whose curvature and torsion forms vanish identically. We say that the $n$-dimensional affine manifold $(N, \nabla_N)$, is complete, if and only if it is the quotient of the affine space $\mathbb{R}^n$, by a subgroup $\Gamma_N$ of $\text{Aff}(\mathbb{R}^n)$ which acts properly and freely on $\mathbb{R}^n$. L. Auslander has conjectured that the fundamental group of a compact and complete affine manifold is polycyclic. In [26] we have conjectured that a finite Galois cover of a compact and complete affine manifold is the domain of a non trivial affine map. This leads to the following problem: classify sequences $(N_n, \nabla_{N_n}) \to \ldots \to (N_1, \nabla_{N_1})$ where each map $f_i : (N_{i+1}, \nabla_{N_{i+1}}) \to (N_i, \nabla_{N_i})$ is an affine fibration whose domain is compact and complete: This means that $f_i$ is a surjective map and each affine manifold $(N_i, \nabla_{N_i})$ is a compact and complete affine manifold. The classification of affine fibrations has been done using gerbe theory (see [28]). It is normal to think that composition sequences of affine manifolds are related to $n$-gerbes. We define a $n$-gerbed tower which appears naturally in this context.

Characteristic classes are used in mathematics to study many objects, for example, Witten has used characteristic classes to study the Jones polynomial. This shows the necessity to give a geometric interpretation of characteristic classes. On this purpose, we have to interpret geometrically the integral cohomology of a differentiable manifold $N$. The theory of Kostant-Weil gives an interpretation of the group $H^2(N, \mathbb{Z})$: It is the set of equivalence classes of complex line bundles over $N$. In [5], is given an interpretation of $H^3(N, \mathbb{Z})$ in terms of equivalence classes of Dixmier-Douady groupoids. Our theory enables us to interpret a subgroup of $H^{n+2}(N, \mathbb{Z})$ as the set of equivalence classes of a family of $n$-gerbed towers.

This is the plan of our paper:
1. Introduction.
2. The notion of gerbe.
3. Notations
4. The notion of gerbed towers.
5. Spectral sequences and gerbed towers.
6. Applications of gerbed towers to affine geometry.
7. Interpretation of the integral cohomology of a manifold.
8. A definition of a notion of sheaf of $n$-categories.
2 The notion of gerbe.

In this part we present the notion of gerbe studied by Giraud [11].

Definitions 2.0.1.
Let $E$ be a category, a **sieve** is a subclass $R$ of the class of objects $Ob(E)$ of $E$ such that if $f : X \to Y$ is a map of $E$, such that $Y \in R$, then $X \in R$.

Let $f : E' \to E$ be a functor, and $R$ a sieve of $E$, we denote by $R^f$, the sieve defined by $R^f = \{ X \in Ob(E') : f(X) \in R \}$.

For each object $T$ of $E$, we denote by $E_T$, the category whose objects are arrows $u : U \to T$, a morphism of $E_T$ between $u_1 : U_1 \to T$, and $u_2 : U_2 \to T$, is a map $h : U_1 \to U_2$ such that $u_2 \circ h = u_1$.

Definition 2.0.2.
A **topology** on $E$ is defined as follows: to each object $T$ of $E$, we associate a non empty set $J(T)$ of sieves of the category $E_T$ of $E$, above $T$ such that:

(i) For each map $f : T_1 \to T_2$, and for each element $R$ of $J(T_2)$, $R^f \in J(T_1)$. (The morphism $f$ induces a functor between $E_{T_1}$ and $E_{T_2}$ abusively denoted $f$).

(ii) The sieve $R$ of $E_T$ is an element of $J(T)$, if for every map $f : T' \to T$ of $E$, $R^f \in J(T')$.

A category endowed with a topology is called a site.

Definitions 2.0.3.
A **sheaf of sets** $L$ defined on the category $E$ endowed with the topology $J$, is a contravariant functor $L : E \to Set$, where $Set$ is the category of sets, such that for each object $U$ of $E$, and each element $R$ of $J(U)$, the natural map:

$$L(U) \to \lim (L \mid R)$$

is bijective, where $(L \mid R)$ is the correspondence defined on $R$ by $(L \mid R)(f) = L(T)$ for each map $f : T \to U$ in $R$.

Let $h : F \to E$ be a functor, for each object $U$ of $E$, we denote by $F_U$ the subcategory of $F$ defined as follows: an object $T$ of $F_U$ is an object of $F$ such that $h(T) = U$. A map $f : T \to T'$ between a pair of objects $T$ and $T'$ of $F_U$, is a map of $F$ such that $h(f)$ is the identity of $U$. The category $F_U$ is called the **fiber** of $U$. For each objects $X$, and $Y$ of $F_U$, we will denote by $Hom_U(X,Y)$ the set of morphisms of $F_U$ between $X$ and $Y$.

Definitions 2.0.4.
Let $h : F \to E$ be a functor, $m : x \to y$ a map of $F$, and $f = h(m) : T \to U$ its projection by $h$. We will say that $m$ is **cartesian**, or that $m$ is the **inverse image** of $f$ by $h$, or $x$ is an inverse image of $y$ by $h$, if for each element $z$ of $F_T$, the map

$$Hom_T(z,x) \to Hom_f(z,y)$$

$$n \to mn$$

is bijective, where $Hom_f(z,y)$ is the set of maps $g : z \to y$ such that $h(g) = f$. 

Non abelian cohomology: the point of view of gerbed towers
A functor $h : F \to E$ is a **fibred category** if and only if each map $f : T \to U$, has an inverse image, and the composition of two cartesian maps is a cartesian map.

We will say that the category is fibered in groupoids, if for each diagram

$$
\begin{array}{ccc}
  x & \xrightarrow{f} & z \\
  \downarrow & & \Downarrow{g} \\
  y & \leftarrow & \phi \\
\end{array}
$$

of $F$ above the diagram of $E$,

$$
U \xrightarrow{\phi} W \leftarrow \psi
$$

and for each map $m : U \to V$ such that $\psi m = \phi$, there exists a unique map $p : x \to y$, such that $gp = f$, and $h(p) = m$.

This implies that the inverse image is unique up to isomorphism.

Consider a map $\phi : U \to V$ of $E$, we can define a functor $\phi^* : F_V \to F_U$, such that for each object $y$ of $F_V$, $\phi^*(y)$ is defined as follows: we consider a cartesian map $f : x \to y$ above $\phi$ and set $\phi^*(y) = x$. Remark that although the definition of $\phi^*(y)$ depends of the chosen inverse image $f$, the functors $(\phi \psi)^*$ and $\psi^* \phi^*$ are isomorphic $\bullet$

**Definitions 2.0.5.**

A **section** of a fibered category $h : F \to E$, is a correspondence defined on the class of arrows of $E$ as follows: to each map $f : U \to T$, we define a cartesian map: $l^f : x_U \to y_T$ of $F$, whose image by $h$ is $f$ such that: $l^f f = l^f \circ l^f$.

Consider the diagram

$$
\begin{array}{ccc}
  F & G \\
  \downarrow{f} & \downarrow{g} \\
  E_1 & \xrightarrow{u} & E_2 \\
\end{array}
$$

where the functors $f$ and $g$ are fibered categories. We denote by $\text{Hom}_u(F,G)$ the subcategory of $\text{Hom}(F,G)$ whose objects are functors $v : F \to G$ which verify $gv = uf$. The maps of this category are morphisms $m : v \to v'$ such that $gm$ is the identity morphism of the functor $uf$.

We denote by $\text{Cart}_u(F,G)$, the subcategory of $\text{Hom}_u(F,G)$ whose objects are cartesian functors: These are functors which transform cartesian maps to cartesian maps $\bullet$

Let $E$ be a category endowed with a topology $J$, and $F \to E$ a fibered category, for each object $U$ of $E$, and each element $R$ of $J(U)$, we consider the canonical functors $E_U \to E$, and $R \to E$. We can define the set of cartesian functors $\text{Cart}_{Id_E}(E_U, F)$ and $\text{Cart}_{Id_E}(R, F)$. There exists a canonical restriction functor $\text{Cart}_{Id_E}(E_U, F) \to \text{Cart}_{Id_E}(R, F)$.

**Definition 2.0.6.**

Let $E$ be a category endowed with a topology, a **sheaf of categories** on $E$, is a fibered category $F \to E$, such that for each sieve $R$, the cartesian functor $\text{Cart}_{Id_E}(E_U, F) \to \text{Cart}_{Id_E}(R, F)$ defined at the paragraph above is an equivalence of categories $\bullet$

**Proposition-Definition 2.0.7.**

Suppose that $E$ is a topos whose topology is generated by a contractible covering family $(U_i \to U)_{i \in I}$, and $h : F \to E$ a fibered category in groupoids. For each map $f : U \to V$ of $E$, we consider the functor $r_{U,V}(f) : F_V \to F_U$ defined as follows: For each object $y$ of $F_V$, $r_{U,V}(f)(y)$ is an object $x$ of $F_U$ such that there exists a cartesian map $n : x \to y$ such that $h(n) = f$. Consider the
maps $v_1 : U_1 \to U_2$, and $v_2 : U_2 \to U_3$ of $E$, the functors $r_{U_1, U_2}(v_1) \circ r_{U_2, U_3}(v_2)$ and $r_{U_1, U_3}(v_2 v_1)$ are isomorphic (see [11]). The functor $h : F \to E$ is a sheaf of categories if and only if the correspondence $U \to F_U = F(U)$ satisfies the following properties:

(i) Gluing condition for arrows.

Let $U$ be an object of $E$, and $x, y$ objects of $F(U)$. The functor from $E_U$, endowed with the restriction of the topology $J$, to the category of sets which associates to an object $f : V \to U$ the set $\text{Hom}_V(r_{V, U}(f)(x), r_{V, U}(f)(y))$ is a sheaf of sets.

(ii) Gluing condition for objects.

Consider a covering family $(U_i \to U)_{i \in I}$ of an object $U$ of $E$, and for each $U_i$, an object $x_i$ of $F(U_i)$. Let $t_{ij} : x_j^i \to x_i^j$, a map between the respective restrictions of $x_j$ and $x_i$ to $U_i \times_U U_j$ such that on $U_{i_1} \times_{U} U_{i_2} \times_{U} U_{i_3}$, the restrictions of the arrows $t_{i_1 i_3}$ and $t_{i_1 i_2} t_{i_2 i_3}$ are equal. There exists an object $x$ of $F(U)$ whose restriction to $F(U_i)$ is $x_i$.

If moreover the following properties are verified:

(iii) There exists a covering family $(U_i \to U)_{i \in I}$ of $E$ such that $F(U_i)$ is not empty.

(iv) For each pair of objects $x, y$ of $F(U_i)$, $\text{Hom}_U(x, y)$ is not empty (local connectivity).

(v) The elements of $\text{Hom}_U(x, y)$ are invertible. The fibered category is called a gerbe.

(vi) We say that the gerbe is bounded by the sheaf $L_F$ defined on $E$, or that $L_F$ is the band of the gerbe, if and only if there exists a sheaf of groups $L_F$ defined on $E$ such that for each object $x$ of $F(U)$ we have an isomorphism:

$$L_F(U) \to \text{Hom}_U(x, x)$$

which commutes with restrictions, and with morphisms between objects.

### 2.1 Classifying cocycle and classification of gerbes.

In this paragraph, we recall the definition of the classifying cocycle of a gerbe defined on the topos $E$ and bounded by the sheaf $L$.

**Definitions 2.1.1.**

- A gerbe $F \to E$ is trivial if it has a section. This means that $F_E$ is not empty.
- Two gerbes $F \to E$, and $F' \to E$ whose band is $L$, are equivalent if and only if there exists an isomorphism between the underlying fibered categories which commutes with the action of $L$. We denote by $H^2(E, L)$ the set of equivalence classes of gerbes defined on $E$ bounded by $L$.

Suppose that the topology of $E$ is defined by the covering family $(U_i \to U)_{i \in I}$, the class of objects of $F_{U_i}$ is not empty, and each objects $x$ and $y$ of $F_{U_i}$ are isomorphic. Let $(x_i)_{i \in I}$ be a family of objects of $F$, such that $x_i$ is an object of $F_{U_i}$. There exists a map $u_{ij} : x_j^i \to x_i^j$ between the respective restrictions of $x_j$ and $x_i$ to $F_{U_i}$. We denote by $u_{ij}^k$ the map between the respective restrictions of $x_{i_2}$ and $x_{i_1}$ to $F_{U_{i_2}}$.

**Theorem [11] 2.1.2.**

The family of maps $c_{ij} = u_{ij}^{i_3} u_{i_1 i_2}^{i_3} u_{i_1 i_3}^{i_2}$ is the classifying 2-Cech cocycle of the gerbe. If the band $L$ is commutative, then the set of equivalence classes of gerbes over $E$ whose band is $L$, is one to one with the Cech cohomology group $H^2(E, L)$. 

3 Notations.

Let $U_{i_1}, \ldots, U_{i_p}$ be objects of a topos $E$, and $C$ a presheaf of categories defined on $E$. We will denote by $U_{i_1} \ldots \cdot U_{i_p}$ the fiber product of $U_{i_1}, \ldots, U_{i_p}$ over the final object. If $e_{i_1}$ is an object of $C(U_{i_1})$, $e_{i_1} \cdot \ldots \cdot e_{i_p}$ will be the restriction of $e_{i_1}$ to $U_{i_1} \ldots \cdot U_{i_p}$. For a map $h : e \to e'$ between two objects of $C(U_{i_1} \ldots \cdot U_{i_p})$, we denote by $h_{i_1}^{1 \ldots \cdot i_p}$ the restriction of $h$ to a morphism between $e_{i_1}^{1 \ldots \cdot i_p} \to e'_{i_1}^{1 \ldots \cdot i_p}$.

4 Gerbed tower.

The purpose of this part is to generalize the notion of gerbe to the notion of gerbed tower. This notion will allow us to define, and to represent geometrically higher non abelian cohomological classes. In the sequel we assume known the notion of 2-category or bicategory defined by Benabou [1]. Recall that a 2-category $C$ is defined by a class of objects $\text{Obj}(C)$, and for each objects $x$ and $y$ of $C$, a category $\text{Hom}_C(x, y)$ called the category of morphisms. The objects of $\text{Hom}_C(x, y)$ are called 1-arrows, and the arrows are called 2-arrows. There exists a composition functor:

$$c(u_1, u_2, u_3) : \text{Hom}(u_2, u_3) \times \text{Hom}(u_1, u_2) \to \text{Hom}(u_1, u_3)$$

For each quadruple $(u_1, u_2, u_3, u_4)$ in $C$, there exists an isomorphism $c(u_1, u_2, u_3, u_4)$ between the functors

$$(\text{Hom}(u_3, u_4) \times \text{Hom}(u_2, u_3)) \times \text{Hom}(u_1, u_2) \to \text{Hom}(u_1, u_4)$$

and

$$\text{Hom}(u_3, u_4) \times (\text{Hom}(u_2, u_3) \times \text{Hom}(u_1, u_2)) \to \text{Hom}(u_1, u_4)$$

which satisfies more compatibility axioms which can be found in Benabou. We will suppose that $c(u_1, u_2, u_3, u_4)$ is the identity on objects. This implies that we can define the category $C_1$ whose objects are the objects of $C$, and such that for each pair of objects $x, y$ of $C_1$, $\text{Hom}_{C_1}(x, y)$ is the set of objects of $\text{Hom}_C(x, y)$. Let $h : F \to E$ be a gerbe. We can define the 2-category $C(E, F)$ whose objects are objects of $F$. Let $x$ and $y$ be a pair of objects of $C(E, F)$, an object of $\text{Hom}_{C(E, F)}(x, y)$ is an arrow between $h(x)$ and $h(y)$. A 2-arrow between the objects $x$ and $y$ is a cartesian map between $x$ and $y$.

**Definition 4.0.1.**

A bicategory $C$, endowed with a topology $J$, is a bicategory whose objects are toposes, and for each pair of objects $x$ and $y$ of $C$. The set of 2-arrows between $x$ and $y$ is contained in the space of continuous maps between $x$ and $y$ ●

**Definition 4.0.2.**

A $n$-gerbed tower is defined by:

1. A family $F_n, F_{n-1}, \ldots, F_2, F_1$ of 2-categories respectively endowed with topologies $J_n, \ldots, J_1$, and a family of 2-functors $p_l : F_l \to F_{l-1}$, $l \in \{2, \ldots, n\}$ which satisfy the following conditions:
2. $F_1$ is a gerbe $p_1 : F \to E$, since we can assume that $E$ is a 2-category such that for each objects $U$ and $V$ of $E$, the set of arrows between a pair of elements $f$ and $f'$ of $\text{Hom}_E(U, V)$ is a singleton, we will often consider the sequence of 2-categories $F_n \to .. F_1 \to E$. We suppose that the 2-arrows of $F_p, p \in \{1, ..., n\}$ are invertible.

3. Let $U$ be an object of $F_p$, and $l \geq p$ a pair of integers inferior to $n$. We denote by $F_{lpU}$, the 2-category whose class of objects is contained in the class of objects of $F_1$, and such that $V$ is an object of $F_{lpU}$ if and only if $p_{p+1}..p_l(V) = U$. The category of morphisms $\text{Hom}_{F_{lpU}}(X, Y)$ between a pair of objects $X$ and $Y$ is the subcategory of $\text{Hom}_{F_1}(X, Y)$ such that the projections of 1-arrows of $\text{Hom}_{F_{lpU}}(X, Y)$ by $p_{p+1}..p_l$ is $\text{Id}_U$ of $U$, and the projections of 2-arrows of $\text{Hom}_{F_{lpU}}(X, Y)$ by the same functor is the identity of $\text{Id}_U$. We denote $F_{0U}$ by $F_{1U}$. We suppose that for each arrow $f : U \to V$ of $E$, there exists a restriction functor $r_{U,V}^l(f) : F_{1V} \to F_{1U}$ such that for every map $g : V \to V'$, $r_{U,V}^l(f) \circ r_{V,V'}(g) = r_{U,V'}^l(gf)$.

4. There exists a family of sheaves $L_1,...,L_n$ defined on $E$. The sheaf $L_{l+1}$ induces a sheaf $L_{l+1U_1}$ on the object $U_1$ of $F_1$ (recall that $U_1$ is a topos) defined by its global sections $L_{l+1U_1} = L_{l+1}(p_{l+1}..p_l(U))$. For each object $U_{l-1}$ of $F_{l-1}$, we suppose that the fiber of $F_{ul-1U_{l-1}}$ is a gerbe defined on the topos $U_{l-1}$ bounded by $L_{lU_{l-1}}$.

5. Let $U_{1l}$ and $U_{2l}$ be a pair of objects of $F_1$, and $u_{1l}^2 : h_{1l}^1 \to h_{1l}^2$. a 2-arrow, between $U_{1l}$ and $U_{2l}$, that is an arrow of the category $\text{Hom}_{F_1}(U_{1l}, U_{2l})$ between the objects $h_{1l}^1$ and $h_{1l}^2$. Recall that $u_{1l}^2$ is a continuous functor between the topos $U_{1l}$ and $U_{2l}$. We suppose that for every object $U_{2l}$ of $F_{2l+U_{2l}}$ there exists an object $U_{1l}'$ of $F_{1l+U_{2l}}$, and a 2-arrow $u_{1l}'^2 : U_{1l}' \to U_{2l}'$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
U_{1l}' & \xrightarrow{u_{1l}'^2} & U_{2l}' \\
\downarrow p_{l} & & \downarrow p_{l} \\
U_{1l} & \xrightarrow{u_{1l}^2} & U_{2l}
\end{array}
$$

This implies that for every 2-arrow $v_{1l}^2 : h_{1l}^2 \to h_{1l}^3$ between the pair of objects $U_{2l}$ and $U_{3l}$, for every object $U_{3l}'$ of $F_{1l+U_{3l}}$, and every 2-arrow $v_{1l}'^2$ of $F_{1l+1}$ defined by the diagram above, there exists an automorphism over the identity $c(u_{1l}^2, v_{1l}^2)$ of $U_{3l}'$ such that

$$
v_{1l}'^2 \circ u_{1l}'^2 = c(u_{1l}^2, v_{1l}^2)(u_{1l}^2 \circ v_{1l}^2)^*,
$$

since we have supposed that the 2-arrows are invertible morphisms of topoi.

**Definition 4.0.3.**

An ∞-gerbed tower, is a sequence of functors between 2-categories $.. F_n \to F_{n-1} \to .. F_1 \to E$ such that for each integer $n$, $F_n \to F_{n-1} \to .. F_1 \to E$ is a gerbed tower.

**Definitions 4.0.4.**

- A morphism $F$ between the gerbed towers $f = F_n \to F_{n-1} \to .. F_1 \to E$ and $f' = F'_n \to F'_n \to F'_1 \to E$, is defined by a family of 2-functors $f_l : F_l \to F'_l$ such that for each $l$, the following diagram is commutative:

$$
\begin{array}{ccc}
F_l & \xrightarrow{f_l} & F'_l \\
\downarrow p_l & & \downarrow p'_l \\
F_{l-1} & \xrightarrow{f_{l-1}} & F'_{l-1}
\end{array}
$$
and for each object $U_i$ of $F$, the induced morphism: $F_{i+1|U_i} \to F'_{i+1|f_i(U_i)}$ is a morphism of gerbes.

- The morphism defined by the family of 2-functors $(f_n, ..., f_1)$ is an isomorphism, if and only if there exists a morphism between the gerbed tower $F'_n \to .. \to F'_1 \to E$ and $F_n \to .. \to F_1 \to E$ defined by the family of 2-functors $f'_n, ..., f'_1$ such that for each $l$, $f'_l \circ f_l = Id_{F'_l}$, and $f'_l \circ f'_l = Id_{F'_l}$.

We say that the gerbed towers $f'$ and $f$ are weakly equivalent if and only if $f'_l \circ f_l$ is isomorphic to $Id_{F'_l}$ and $f_l \circ f'_l$ is isomorphic to $Id_{F'_l}$. We denote by $H^{n+1}(E, L_n)$ the set of weakly equivalence classes of gerbed towers bounded by $(L_1, ..., L_n)$. Here $L_n$ is a fixed sheaf defined on $E$.

4.1 Non commutative cohomology of groups.

Let $H$ be a group, $V$ a vector space, $Gl(V)$ the group of linear automorphisms of $V$, and $\rho : H \to Gl(V)$ a representation. To define the cohomology groups $H^n(H, V, \rho)$ of the representation $\rho$, one can consider $EH$ the 1-Eilenberg-Maclane space defined by $H$, the representation $\rho$ defines on $EH$ a flat $V$-bundle whose holonomy is $\rho$. The cohomology groups $H^n(H, V, \rho)$, are the $n$-cohomology groups, of the sheaf of locally constant sections of this flat bundle. This motivates the following definition:

Definition 4.1.1.

Consider the groups $H$ and $G$, $Aut(G)$ the group of automorphisms of $G$, and $\rho : H \to Aut(G)$ a representation. The representation $\rho$ defines on $EH$ a flat $G$-bundle $p_G$. We denote by $L_{p_G}$ the sheaf of locally constant sections of this bundle. We define $H^{n+1}(H, G, \rho)$ to be set of weakly equivalence classes of gerbed towers of rank $n$ $F_n \to .. \to F_1 \to EH$ bounded by a sequence $(L_1, ..., L_{n-1}, L_{p_G})$.

4.2 The classifying cocycle of a gerbed tower.

Let $f = F_n \to F_{n-1} \to ... F_1 \to E$ be a gerbed tower. We will define in this part the classifying cocycle of $f$. We suppose that the sheaves $L_1, ..., L_n$ are commutative, and there exists commutative sheaves $L'_1, ..., L'_n$ defined on $E$ such that for every objects $U_i$ of $F_1, U_{i+1}$ of $F_{i+1|U_i}$, and $h$ an object of $Hom_{F_{i+1}}(U_{i+1}, U_{i+1})$, $Aut(F_{i+1|h_{i+1}}) = L'_{i+1}(p_{i+1}(U_i))$ where $Aut(F_{i+1|p_{i+1}})$ is the group of automorphisms of the object $h_{i+1}$ whose image by $p_{i+1}$ are elements of $L_i(p_{i+1}(U_i))$.

Suppose that the topology of $E$ is defined by the covering family $(U_i \to U)_{i \in I}$ such that for each $i$, $F_{i|U_i}$ is not empty, and its objects are isomorphic. Let $u_i$ be an object of $F_{i|U_i}$, and $v_{i_1|i_2}$ an arrow between $u_{i_1}$ and $u_{i_2}$. The family of arrows $c_{i_1|i_2} = v_{i_2|i_1} v_{i_1|i_2}^{-1} v_{i_2|i_1}$ is the classifying cocycle of the gerbe $F_1 \to E$.

If we identify $F_1 \to E$ with a 2-category, then $c_{i_1|i_2}$ is a 2-arrow. Let $u_{i_1|i_2}$ be an object of the fiber $F_{21|u_{i_1|i_2}}$. The property 5 of the definition of gerbed towers implies the existence of a 2-arrow $c_{i_1|i_2|i_3}$ of $Hom_{F_{21}}(u_{i_1|i_2}, u_{i_1|i_2})$ over $c_{i_1|i_2}$. We can define the automorphism $c_{i_1|i_2|i_3}$ of the object $u_{i_1|i_2}$ by:

$$c_{i_1|i_2|i_3} = c_{i_1|i_2|i_3}^* - c_{i_1|i_2|i_3}^* + c_{i_1|i_2|i_3} - c_{i_1|i_2|i_3}^*$$
each member of the right part of the previous equality can be supposed to be a morphism of the same object of $\text{Hom}_{F_2}(u_{i_1i_2i_3i_4}, u_{i_1i_2i_3i_4})$. The property 5 of the definition of gerbed towers implies that $c_{i_1i_2i_3i_4}$ is an element of $L_2(U_{i_1} \times E U_{i_2} \times E U_{i_3} \times E U_{i_4})$.

**Proposition 4.2.1.**
The family $c_{i_1i_2i_3i_4}$ that we have just defined is a 2-Cech cocycle.

**Proof.**
The Cech boundary of $c_{i_1i_2i_3i_4}$ is:

$$\partial(c_{i_1...i_4}) = \sum_{p=1}^{p=5} (-1)^p c_{i_1...i_p...i_5}$$

$$= \sum_{p=1}^{p=5} (-1)^p (\sum_{c=1}^{c=p-1} (-1)^c c^*_{i_1...i_c...i_p...} + \sum_{c=p}^{c=5} (-1)^{c+1} c^*_{i_1...i_p...i_c...i_5}) = 0.$$  

The last sum is zero because the sheaf $L'_1$ is commutative

Suppose defined the classifying cocycles $c_{i_1i_2i_3, ... , i_1i_2i_3}$, $l \geq 2$ of the gerbed tower $F_l \rightarrow ... \rightarrow F_1 \rightarrow F$. The arrow $c_{i_1...i_{l+2}}$ is a 2-arrow of $u_{i_1...i_{l+2}}$. Let $u_{i_1...i_{l+2}}$ be an object of $F_{l+1} u_{i_1...i_{l+1}i_{l+2}}$. The property 5 implies the existence of an automorphism $c_{i_1...i_{l+2}}^*$ of a 1-arrow of the object $u_{i_1...i_{l+2}}$ over $c_{i_1...i_{l+2}}$. We can define:

$$c_{i_1...i_{l+3}} = \sum_{p=1}^{p=l+3} (-1)^p c_{i_1...i_p...i_{l+3}}^*$$

We can apply the property 5 to identify $c_{i_1...i_{l+3}}$ to a 2-arrow of $u_{i_1...i_{l+2}}i_{l+3}$.

**Proposition 4.2.2.**
The family of arrows $c_{i_1...i_{l+3}}$ that we have just defined is a $l+1$-Cech cocycle.

**Proof.**
The Cech boundary of $c_{i_1...i_{l+3}}$ is:

$$\partial(c_{i_1...i_{l+3}}) = \sum_{p=1}^{p=l+4} (-1)^p c_{i_1...i_p...i_{l+4}}$$

$$= \sum_{p=1}^{p=l+4} (-1)^p (\sum_{c=1}^{c=p-1} (-1)^c c^*_{i_1...i_c...i_p...} + \sum_{c=p}^{c=l+4} (-1)^{c+1} c^*_{i_1...i_c...i_{l+4}}) = 0.$$  

The last sum is zero because the sheaf $L'_{l+1}$ is commutative

**Proposition 4.2.3.**
The cohomology class $[c_{l+2}]$, is the image of $[c_{l+1}]$ by the connecting morphism $H^{l+1}(E, L_l) \rightarrow H^{l+2}(E, L_{l+1})$ of the exact sequence $0 \rightarrow L_{l+1} \rightarrow L'_{l+1} \rightarrow L_l \rightarrow 0$. In particular this shows that the cohomology classes of the cocycles $c_l$, $2 \leq l \leq n+1$ attached to the gerbed tower $F_n \rightarrow F_{n-1} \rightarrow ... \rightarrow F_1 \rightarrow E$ are independent of the choices made to construct them.

**Proof.**
To construct the cocycle $c_{l+2}$ we pick elements $c_{i_1...i_{l+2}}$ which represents 2-arrows of an object $u_{i_1...i_{l+2}}$ that we lift to 2-arrows $c_{i_1...i_{l+2}}$ of $F_{l-1}u_{i_1...i_{l+2}}$. The representant $c_{i_1...i_{l+2}}$ of $[c_{l+2}]$ are the Cech boundary of the family $c_{i_1...i_{l+2}}$ acting on $u_{i_1...i_{l+2}}$. This is by definition the construction of the connecting morphism $H^{l+1}(E, L_l) \rightarrow H^{l+2}(E, L_{l+1})$ of the exact sequence $0 \rightarrow L_{l+1} \rightarrow L'_l \rightarrow L_l \rightarrow 0 \bullet$

**Definition 4.2.4.**

A gerbed tower $f_n = F_n \rightarrow F_{n-1}...F_1 \rightarrow E$ is trivial if and only if there exists a gerbed tower $f_{n-1} = F'_{n-1} \rightarrow F'_{n-2} \rightarrow ...F'_1 \rightarrow E$ and an element $p \in \{1,..,n-2\}$ such that $F'_p$ is $F_l$ if $l \leq p$, $F'_{p+1}$ is a sub 2-category of $F_{p+1}$, for each 2-arrow $h$ of $F_{p-1}$, the image of the arrow $h^*$ of $F'_p$ by $p_{p+1}$ is an arrow $u^*$, of $F'_p$, defined by the axiom 5, defined by a 2-arrow of $u$ of $F_p$. For $l > p$, the 2-category $F'_l$ a subcategory of $F_{l+1}$. The gerbed tower $f_{n-1}$ is called a trivialization of $f_n$.

**Proposition 4.2.5.**

The class $[c_{n+1}]$ of a trivial gerbed tower $F_n \rightarrow .. \rightarrow F_1 \rightarrow E$ is zero.

**Proof.**

Let $f_n = F_n \rightarrow F_{n-1}...F_1 \rightarrow E$ be a trivial gerbed tower, and $F'_{n-1} \rightarrow F'_{n-2}... \rightarrow F'_1 \rightarrow E$ a trivialization of $f_n$. Suppose that the integer $p$ of the definition above is $n-2$, this means that if $l \leq n-2$, and $F'_l$ is $F_l$. We denote by $(L_1,...,L_n)$ the band of the gerbed tower $F_n \rightarrow ...F_1 \rightarrow E$, and by $L'_n$ the sheaf such that the group $Aut(F'_n-1_n-2h_{n-1})$ of automorphisms of a 2-arrow $h_{n-1}$ of the object $U_{n-1}$ of $F'_n-1_n$ which project by $p_1...p'_{n-1}$ to elements of $L_{n-2}(p_1...p'_{n-1}(U_{n-1}))$ is $L'_n(p_1...p'_{n-1}(U_{n-1}))$. We have the commutative:

\[
\begin{array}{ccc}
0 & \rightarrow & L_n \\
\downarrow & & \downarrow \\
0 & \rightarrow & L'_n \\
\end{array}
\]

The map of $L'_n \rightarrow L''_n$ is defined by the restriction of the morphisms $u^*$, where $u \in L_{n-1}$, and the map $L_n \rightarrow L_{n-2}$ is zero. This exact sequence gives rise to the commutative diagram:

\[
\begin{array}{ccc}
H^n(E, L_{n-1}) & \rightarrow & H^n(E, L_{n-2}) \\
\downarrow & & \downarrow \\
H^{n+1}(E, L_n) & \rightarrow & H^{n+1}(E, L_n)
\end{array}
\]

Since the map $H^n(E, L_{n-1}) \rightarrow H^n(E, L_{n-2})$ is zero, and the map $H^{n+1}(E, L_n) \rightarrow H^{n+1}(E, L_n)$ is the identity, we deduce that the class $[c_{n+1}]$ of the classifying cocycle of the gerbed tower $F_n \rightarrow F_{n-1}... \rightarrow F_1 \rightarrow E$ is zero.

If $p$ is not $n-1$, the last argument show that $[c_{p+1}]$ is zero. This implies that $[c_l] = 0$ for $l \geq p + 1 \bullet$

**Theorem 4.2.6.**

Let $F_n \rightarrow F_{n-1}...F_1 \rightarrow E$ be a gerbed tower bounded by the family of sheaves $(L_1,...,L_n)$ whose classifying cocycles are $(c_2,...,c_{n+1})$. Consider an exact sequence of sheaves $0 \rightarrow L_{n+1} \rightarrow L'_{n+1} \rightarrow L_n \rightarrow 0$. Then there exists a gerbed tower $F_{n+1} \rightarrow F_n... \rightarrow F_1 \rightarrow E$ whose classifying cocycles are $c_2,...,c_{n+1},c_{n+2}$, where $c_{n+2}$ is a $n+1$-cocycle whose cohomology class is the image of the class of $c_{n+1}$ by the connecting map $H^{n+1}(E, L_n) \rightarrow H^{n+2}(E, L_{n+1})$ defined by the previous exact sequence.
Proof.

Let $U_n$ be an object of $F_n$, and $U_{n-1}$ its image by the projection map $p_n : F_n \to F_{n-1}$. The topos $U_n$ is a $L_n$-torsor defined over an object of the topos $U_{n-1}$ since $F_{nn-1}U_{n-1}$ is a $L_n$-gerbe defined on $U_{n-1}$. An object $V_n$ of $F_{nn-1}U_{n-1}$ is a $L_n$-torsor defined over an object $U'_{n-1}$ of the topos $U_{n-1}$. This torsor is defined by a trivialization $(V_i)_{i \in I}$, and coordinate changes $u_{ij} : V_i \times_{U_{n-1}} V_j \to L_n$. This coordinate changes define a principal $L_n$-torsor over $U'_{n-1}$ by gluing $V_i \times L_n$. Without restricting the generality, we can suppose that the objects of $F_{nn-1}U_{n-1}$ are $L_n$-principal torsors. We define the fiber $F_{n+1}U_n$ to be the gerbe bounded by $L_{n+1}$ defined on $U_n$ which represents the obstruction to lifts the $L_n$-gerbe defined on $U'_{n-1}$ to a $L'_{n+1}$-tusbor whose quotient by $L_{n+1}$ is the previous $L_n$-tusbor $pU_n : U_n \to U'_n$.

The objects of the category of morphisms $\text{Hom}_{F_{n+1}}(U_{n+1}, U'_{n+1})$ between the objects $U_{n+1}$ of $F_{n+1}U_n$ and $U'_{n+1}$ of $F_{n+1}U'_n$ are the 2-arrows between $U_n$ and $U'_n$. The 2-arrows are the morphisms of torsors $u^2_{n+1}$ such that there exists a 2-arrow $u^1_n : U_n \to U'_n$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
U_{n+1} & \xrightarrow{u^2_n} & U'_{n+1} \\
\downarrow p_{n+1} & & \downarrow p_{n+1} \\
U_n & \xrightarrow{u^1_n} & U'_n
\end{array}
\]

We show now that $F_{n+1} \to F_{n+1} \to \cdots \to F_1 \to E$ is a gerbed tower.

Let $U_n$ be an object of $F_n$, and $U$ the object $p_1 \cdots p_1(U_n)$ of $E$. For each map $f : V \to U$, we can define the restriction $r^n_{V,U}(f) : F_{n+1} \to F_{nV}$. The restriction $r^{n+1}_{V,U}$ is defined on $F_{n+1}U_n$ by the pull-back of $U_{n+1}$ by the arrow $r^n_{V,U}(f)$.

The definition of $F_{n+1}$ implies that for every object $U_n$ of $F_n$, the gerbe $F_{n+1}U_n$ is bounded by $L_{n+1}$.

For every 2-arrow $u^2_n : h^1_n \to h^2_n$, (recall that $u^2_n$ is a morphism between the topoi $U_{1n}$ and $U_{2n}$) of $F_n$, the functor $u^2_n$ is defined as follows: Without restricting the generality, we can suppose that $U_{1n}$ is the trivial tosr to $V_1 \times L_n$ and $U_{2n}$ the trivial tosr $V_2 \times L_n$; $u^2_n$ is then a morphism of $L_n$-torsors, $u^2_n$ is a morphism such that the following diagram is commutative:

\[
\begin{array}{ccc}
V_1 \times L'_{n+1} & \xrightarrow{u^2_n^*} & V_2 \times L'_{n+1} \\
\downarrow & & \downarrow \\
V_1 \times L_n & \xrightarrow{u^2_n} & V_2 \times L_n
\end{array}
\]

This shows that $F_{n+1} \to \cdots \to F_1 \to E$ is a gerbed tower.

The classifying cocycle of this gerbed tower is constructed by considering the automorphism $c_{i_1 \cdots i_{n+2}}$ of the object $u_{i_1 \cdots i_{n+2}}$ of $F_{n+1}$, that we suppose to be isomorphic to a trivial tosr $V_{i_1 \cdots i_{n+2}} \times L_n$, the morphism $c_{i_1 \cdots i_{n+2}}$ can be lifted to an element $c_{i_1 \cdots i_{n+2}}$ of $V_{i_1 \cdots i_{n+2}} \times L'_{n+1}$. The Cech boundary of $c_{i_1 \cdots i_{n+2}}$ is the classifying cocycle of the gerbed tower. The cohomology class of this cocycle is the image of the cohomology class of $c_{n+1}$ by the connecting morphism $H^{n+1}(E, L_n) \to H^{n+2}(E, L_{n+1})$. 

5 Spectral sequences and gerbed towers.

The goal of this part is to apply spectral sequences to study commutative gerbed towers.

Let $E(L_1, \ldots, L_n, \ldots)$ be an $\infty$-gerbed tower, where $(L_n)_{n \in \mathbb{N}}$ is a family of commutative sheaves defined on $E$. We suppose that the topology of $E$ is defined by the covering family $(X_i \to X)_{i \in I}$. We define $L = \oplus_{i \geq 1} L_i$, and denote by $(C^*(X, L), d)$ the complex of Cech $L$-chains defined on $E$. We can endow this chain complex with the following filtration:

$$V_p = C(X, \oplus L_{i \geq p}),$$

and with the graduation

$$V^p = C^p(X, L)$$

We will calculate the terms associated to the spectral sequence associated to this graduation.

Denote by $Z^p_r = \{ x \in V_p : d(x) \in V_{p+r} \}$, $B^p_r = d(V_{p-r}) \cap V_p$, and $E^p_r = \frac{Z^p_r}{Z^p_{r-1} + B^p_{r-1}}$. We suppose in the sequel that $r \geq 1$.

**Determination of $Z^p_r$.**

Let $x$ be an element of $V_p$, $d(x)$ is an element of $Z^p_r$ if and only if $d(x)$ is an element of $V_{p+r}$. We can write $x = x_{i_p} + \cdots + x_{i_n}$, where $x_{i_j}$ is the homogeneous component of $x$ which takes value in $L_i$, $d(x_l)$ is an element of $V_{p+r}$ if and only if $d(x_l) = 0$ if $l \leq p + r$. We deduce that $x$ is an element of $Z^p_r$ if and only if its components $x_{i_j}$, such that $j < p + r$ are cocycles.

**Determination of $B^p_r$.**

Let $x$ be an element of $V_{p-r}$, and $x_l$ its component which takes values in $L_l$. The image by $d$ of $x_l$ is an element of $B^p_{r}$ if and only if $d(x_l)$ is an element of $V_p$. This equivalent to saying that $d(x_l)$ is zero, or $l \geq p$. This implies that $B^p_r = d(V_p)$.

**Determination of $E^p_r$.**

We have $Z^p_r = Z^{p+1}_{r-1} \oplus Z(V_p) \cap C(X, L_p)$. We deduce that $E^p_r = H(X, L_p)$.

Now we set $Z^{pq}_r = Z^p_r \cap V^{p+q}$, $B^{pq}_r = B^p_r \cap V^{p+q}$ and $E^{pq}_r = \frac{Z^{pq}_r}{B^{pq}_{r-1} + Z^{pq}_{r-1}}$.

**Determination of $Z^{pq}_r$.**

Let $x$ be an element of $Z^p_r$, one of its homogeneous components $x_l$ which takes values in $L_l$, is an element of $Z^{pq}_r$, if $x_l$ is a $V_p$ $p + q$-chain, and $d(x_l)$ is an element of $V_{p+r}$. We have seen that $d(x_l)$ is an element of $Z^p_r$ if and only if $x_l$ is a cocycle or $r \geq p + r$. We deduce from this fact that $Z^{pq}_r = C^{p+q}(V_{p+r}) \oplus Z^{p+q}(X, L_1 \oplus \cdots \oplus L_{p+r-1})$.

**Determination of $B^{pq}_r$.**

Let $x$ be an element of $B^{pq}_r$, one of its homogeneous components, $x_l$ which takes value in $L_l$ is an element of $B^{pq}_r$ if and only if it is a $p + q$-chain, and there exists an element $y$ in $V_{p-r}$ such that $d(y) = x_l$. We deduce that $B^{pq}_r = d(C^{p+q-1}(X, V_p))$.

**Determination of $E^{pq}_r$.**
The vector space $Z^p_q$ is the summand of $Z^{p+1,q-1}_r$ and $Z^{p+q}(X,L_p)$. We deduce that $E_r^{pq} = H^{p+q}(X,L_p)$.

Now, we will denote by $Z^p_r$, the set of cocycles contained in $V_p$, by $B^p_r$ the set of boundaries contained in $V_p$, and by $E^p_r = \frac{Z^p_r}{Z^p_r + B^p_r}$. We remark that $E^p_r = H(X,L_p)$.

The following proposition can be deduced from [19] p. 84 Theorem 4.6.1.

**Proposition 4.0.1.**

Suppose that there exists an integer $n \geq r$ such that $H^{p+q}(X,L_p) = E_p^{pq} = 0$ for $p \neq 0$, $n$ and an integer $s$ such that $L_n = 0$ if $n > s$, then we have the following exact sequence

$$...	o H^1(X,L_n) \to H^1(X,L) \to H^1(X,L_1) \to H^1(X,L_n) \to H^1(X,L) \to ...$$

6 Application of gerbed towers to affine manifolds.

An affine manifold $(N,\nabla_N)$, is a differentiable manifold $N$, endowed with a connection $\nabla_N$ whose curvature and torsion forms vanish identically. The connection $\nabla_N$ defines on $N$ an atlas whose coordinate changes are affine transformations. Auslander has conjectured that the fundamental group of a compact and complete affine manifold is polycyclic. Let $(N,\nabla_N)$ and $(N',\nabla_{N'})$ be two affine manifolds of respective dimension $n$ and $n'$ whose affine structures are defined by the respective atlases $(U_i,\varphi_i)$, and $(U'_j,\varphi'_j)$. An affine map $f : (N,\nabla_N) \to (N',\nabla_{N'})$ is a differentiable map $f : N \to N'$ such that $\varphi'_j \circ f \circ \varphi_i^{-1}$ is a restriction of an affine map from $\mathbb{R}^n$ to $\mathbb{R}^{n'}$. Suppose that $(N,\nabla_N)$ and $(N',\nabla_{N'})$ are complete and compact. It is shown in Tsemo [26] that in this case, there exists a compact and complete affine manifold $(N_1,\nabla_{N_1})$ of dimension $n'$, and an affine submersion $f_1 : (N,\nabla_N) \to (N_1,\nabla_{N_1})$. Ehresman has shown that submersions between compact manifolds are locally trivial differentiable fibrations. The typical fiber $F$ of the fibration $N \to N_1$ inherits from $N$ complete affine structures. The homotopy exact sequence of this fibration gives rise to the sequence:

$$1 \to \pi_1(F) \to \pi_1(N) \to \pi_1(N_1) \to 1$$

If the fundamental groups of $\pi_1(F)$ and $\pi_1(N_1)$ are polycyclic, then $\pi_1(N)$ is also polycyclic. This has motivated the following conjecture:

**Conjecture 5.0.1.**

Let $(N,\nabla_N)$ be a $n$-dimensional compact and complete affine manifold, then there exists a complete affine structure $(N',\nabla_{N'})$ defined on a finite Galois cover $N'$ of $N$, and a non trivial affine map $f : (N',\nabla_{N'}) \to (N_1,\nabla_{N_1})$. Non trivial means that the dimension of the fibers of $f$ are different from zero, and $n$.

This conjecture implies the Auslander conjecture, and leads to the problem of classifying sequences of affine submersions $(N_n,\nabla_{N_n}) \to \ldots \to (N_1,\nabla_{N_1})$. This if the last conjecture is true, will allow to know the topology of all the compact and complete affine manifolds. The theory of gerbed towers has been first constructed to study this classification problem.
Definition 5.0.2.

An affinely locally trivial affine fibration, whose typical fiber is the affine manifold \((F, \nabla_F)\), is an affine map \(f : (N_1, \nabla_{N_1}) \to (N, \nabla_N)\) which is the total space of a bundle whose fibers inherit from \((N_1, \nabla_{N_1})\), affine structures whose holonomies is the holonomy of the affine structure \((F, \nabla_F)\).

We will restrict to the study of sequences \((N_n, \nabla_{N_n}) \to \cdots (N_1, \nabla_{N_1}) \to (N, \nabla_N)\) where each map \(f_p : (N_p, \nabla_{N_p}) \to (N_{p-1}, \nabla_{N_{p-1}})\) is an affinely locally trivial affine fibration.

Let \(f : (N_1, \nabla_{N_1}) \to (N, \nabla_N)\) be an affinely locally trivial affine fibration whose typical fiber is the affine manifold \((F, \nabla_F)\). We suppose that the affine structure of \((N_1, \nabla_{N_1})\) is complete. This implies that the affine structure of \((N, \nabla_N)\) is complete see Tsemo [27]. We can identify \(\pi_1(N_1)\) with its image by the holonomy morphism of \((N_1, \nabla_{N_1})\). Suppose that the dimension of \(N_1\) and \(N\) are respectively \(n_1\) and \(n\). Let \(h : \pi_1(N_1) \to Aff(R^{n_1})\) be the holonomy representation of \((N_1, \nabla_{N_1})\). We can write see Tsemo [27] \(R^{n_1} = R^n \times R^p\), and for each element \(\gamma\) of \(\pi_1(N_1)\), \(\gamma(x, y) = (L_1, y + l_1(x), L_2, y + l_2(x))\). Where \(L_1, y\) and \(L_2, y\) are respective automorphisms of \(R^n\) and \(R^p\), \(L_i : R^n \to R^p\) is a linear map, and \(l_1, y\) and \(l_2, y\) are respective elements of \(R^n\) and \(R^p\). An element \(\gamma\) of \(\pi_1(F)\) is an element \(\gamma\) of \(\pi_1(N_1)\) such that \(L_1, y\) is the identity of \(R^n\), and \(l_1, y\) is zero, and \(L_3, y\) is zero. We can identify \(\pi_1(N)\) to the set of affine transformations \((L_1, y, l_1, y)\).

Let \(T_F\) be the translation group of \((F, \nabla_F)\), that is the group of affine automorphisms of \((F, \nabla_F)\) whose elements lift to translations of \(R^p\). Since the group \(\pi_1(F)\) is a normal subgroup of \(\pi_1(N_1)\), the holonomy of \((N_1, \nabla_{N_1})\) induces a representation \(\pi_1(N) \to Aff(F, \nabla_F)/T_F\), which defines a flat \(Aff(F, \nabla_F)/T_F\)-bundle \(p_F\) over \((N, \nabla_N)\). The composition of the holonomy of \((N_1, \nabla_{N_1})\), and the conjugation of \(Aff(F, \nabla_F)\) defined a flat bundle \(T_F\)-bundle \(p'_F\) over \((N, \nabla_N)\) see Tsemo [27]. An isomorphism \(h : e \to e'\) between a pair of locally trivial \((F, \nabla_F)\)-affine bundles \(e\) and \(e'\) defined over \((N, \nabla_N)\), is an affine map \(h : e \to e'\) which is an isomorphism of bundles which gives rise to the identity of \(p_F\).

Given affine manifolds \((F, \nabla_F)\) and \((N, \nabla_N)\) and a flat \(Aff(F, \nabla_F)/T_F\)-bundle \(p_F\), we can define the first extension problem as follow: study the existence and classify affinely locally trivial affine fibrations \(f : (N_1, \nabla_{N_1}) \to (N, \nabla_N)\) which give rise to the flat bundle \(p_F\).

Proposition 5.0.3.

Let \((F, \nabla_F)\) and \((N, \nabla_N)\) be compact and complete affine manifolds, and \(p_F\) a flat \(Aff(F, \nabla_F)/T_F\)-bundle defined on \(N\). The bundle \(p_F\) induces a flat \(T_F\)-bundle \(p'_F\). For each open subset \(U\) of \(N\), define the category \(C_F(U)\) to be the category whose objects are affinely locally trivial \((F, \nabla_F)\)-affine bundles which induce the restriction of \(p_F\) to \(U\). A map \(h : e_U \to e'_U\) between two objects of \(C_F(U)\) is an isomorphism of affine bundles which gives rise to the identity of the restriction of \(p_F\) to \(U\). The correspondence defined on the category of open subsets of \(N\), by \(U \to C_F(U)\), is a gerbe whose classifying cocycle represents the obstruction to the existence of an affinely locally trivial affine bundle, which gives rise to \(p_F\). The gerbe \(C_F(U)\) is a gerbe bounded by the sheaf of affine sections of \(p'_F\), that we denote \(L_F\).

Proof.

Gluing property for objects.

Let \((U_i)_{i \in I}\) be an open covering of an open subset \(U\) of \(N\), \(e_i\) an object of \(C_F(U_i)\), and \(u_{ij} : e_j \to e_i\) an isomorphism such that \(u_{12}u_{23}u_{31} = u_{13}u_{12}\). The definition of a bundle implies
the existence of a bundle \( e \) defined over \( U \) whose restriction to \( U_i \) is \( e_i \). Since the coordinate changes \( u_{ij} \) are affine isomorphisms between affinely locally trivial \( (F, \nabla_F) \)-affine bundles, this implies that \( e \) is an affinely locally trivial \( (F, \nabla_F) \)-affine bundle.

Gluing conditions for arrows.

Let \( e \) and \( e' \) be two objects of \( C_F(U) \), the correspondence defined on the open subsets of \( U \) by \( V \to Hom(e|_V, e'|_V) \), where \( e|_V \) and \( e'|_V \) are the respective restrictions of \( e \) and \( e' \) to \( V \) is a sheaf of sets, since it is a sheaf of morphisms between two bundles.

This shows that \( C_F \) is a sheaf of categories. It remains to show that \( C \) is a gerbe.

Let \((U_i)_{i \in I}\) be an open covering of \( N \) by contractible open subsets which are domain of affine charts. Then \( U_i \times (F, \nabla_F) \) is an object of \( C_F(U_i) \).

Let \( U \) be an open subset of \( N \), and \( e \) and \( e' \) a pair of objects of \( C_F(U) \). The respective restrictions \( e|_{U \cap U_i} \) and \( e'|_{U \cap U_i} \) of \( e \) and \( e' \) to \( U_i \cap U \) are isomorphic to \( U_i \cap U \times (F, \nabla_F) \).

An isomorphism \( h \), of an object \( e \) of \( C_F(U) \), is an isomorphism of affine bundle which gives rise to the identity of the restriction of \( p_F \) to \( U \). The restriction of \( h \) to \( e|_{U \cap U_i} \) is an isomorphism \( h_i \) of the trivial bundle \( U \cap U_j \times (F, \nabla_F) \). The fact that \( h \) gives rise to the identity of \( p_F \), is equivalent to the fact that its restriction to a fiber yields to the identity of \( Aff(F, \nabla_F)/T_F \). This implies that \( h_i \) is a \( T_F \) valued affine map, and \( h \) is a section of \( p'_F \).

**Proposition 5.0.4.**

*Suppose that the gerbe \( C_F \) is trivial, then the objects of \( C_F(N) \) are diffeomorphic manifolds.*

**Proof.**

Suppose that the gerbe \( C_F \) is trivial, then the holonomy of a global object \((N_1, \nabla_{N_1})\) is defined by a representation \( h_\gamma(x, y) = (L_{1\gamma}(x)+l_{1\gamma}, L_{2\gamma}(y)+L_{3\gamma}(x)+l_{2\gamma}) \) which defines an \((F, \nabla_F)\)-bundle.

The objects of \( C_F(N) \) are classified by \( H^1(N, p'_F) \), the 1-cohomology group of the sheaf of affine sections of \( p'_F \). An element of \( H^1(N, p'_F) \) is defined by an affine \( C_3 : \mathbb{R}^3 \to \mathbb{R}^3 \) which is a 1-cocycle for the action of \( \pi_1(N) \) on \( Aff(\mathbb{R}^n, \mathbb{R}^n) \) defined by \( \gamma(C_3) = L_{2\gamma} \circ C_3 \circ (L_{1\gamma}, l_{1\gamma})^{-1} \). The bundle defined by the representation \( h'_t(x, y) = (L_{1\gamma}(x)+l_{1\gamma}, L_{2\gamma}(y)+L_{3\gamma}(x)+tC_3((L_{1\gamma}, l_{1\gamma})(x)) + l_{2\gamma}) \), \( 0 \leq t \leq 1 \) defines an homotopy between the bundle \( h'_1 \) defined by \( h^1 \) and the one \( h'_0 \) defined by \( h^0 \). We deduce that \( h'_0 \) and \( h'_1 \) are isomorphic differentiable bundles. This implies that \( N_1 \) is diffeomorphic to the \( F \)-bundle defined by \( h^0 \).

Let \((N, \nabla_F) = (F_0, \nabla_{F_0}), (F_1, \nabla_{F_1}), ..., (F_n, \nabla_{F_n}) \) be affine manifolds. We are going to define a gerbed tower which will allow us to study the classification of sequences \((N_n, \nabla_{N_n}) \to (N_{n-1}, \nabla_{N_{n-1}}) \to \cdots \to (N, \nabla_N) \), where \( (h_t : (N_t, \nabla_{N_t}) \to (N_{t-1}, \nabla_{N_{t-1}}) \) is an affinely locally trivial affine bundle).

Denote by \( T_{F_1} \) the group of translations of \((F_1, \nabla_{F_1}) \). We suppose defined a flat \( Aff(F_1, \nabla_{F_1})/T_{F_1} \)-bundle \( p_{F_1} \) over \((F_1-1, \nabla_{F_1-1}) \). The bundle \( p_{F_1} \) induces a gerbe \( C_1 \) defined over \( F_{1-1} \) (see proposition 4.3).

**Definition 5.0.5.**

We define \( L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \) to be the sequence of 2-categories such that \( L_1 \to L_0 \) is \( C_1 \), supposed defined \( L_p \), an object of \( L_p \) is an affinely locally trivial \((F_p, \nabla_{F_p})\)-bundle. A 1-arrow between a pair of objects \( e_p \) and \( e'_p \) of \( L_p \) is an affine map between their base space. A 2-arrow is an
isomorphism of affinely locally trivial bundle which cover a 1-arrow. We denote by \( p_l : L_l \to L_{l-1} \)
the projection.

An object of \( L_{p+1} \) is an affinely locally trivial \((F_{p+1}, \nabla_{F_{p+1}})\)-bundle \( e_{p+1} \) defined over an open
subset \( U_p \) of an object \( e_p \) of \( L_p \) such that the restriction to each fiber of \( U_p \to p_p(e_p) \) of the
\( Aff(F_{p+1}, \nabla_{F_{p+1}})/TF_{p+1} \)-bundle induced, is the restriction of the \( Aff(F_{p+1}, \nabla_{F_{p+1}})/TF_{p+1} \)-bundle
induced is \( pF_{p+1} \). A 1-arrow \( h^1_{p+1} : e_{p+1} \to e'_{p+1} \) between a pair of objects \( e_{p+1} \) and \( e'_{p+1} \) of \( L_{p+1} \) is
a affine map between their respective base spaces \( e_p \) and \( e_p' \) induced by a 2-arrow of \( L_p \). A 2-arrow
between \( e_{p+1} \) and \( e'_{p+1} \) is an isomorphism of affinely locally trivial \((F_{p+1}, \nabla_{F_{p+1}})\)-bundles which
cover a 1-arrow. We suppose that the 2-arrows depend only of the open subset \( p_1..p_{p+1}(e_{p+1}) \) and
\( p_{1..p_{p+1}}(e'_{p+1}) \). Let \( h^2_p \) be a 2-arrow of \( L_p \), we define \( h^2_p^* \) to be a 2-arrow of \( L_{p+1} \) which cover \( h^2_p \) •

Proposition 5.0.6.

The sequence \( L_n \to L_{n-1}... \to L_1 \to L_0 \) that we have just defined is a gerbed tower.

Proof.

The fibered category \( L_1 \to L_0 \) is a gerbe as shows proposition 4.4. Let \( V \) be an open subset
of \( N \), and \( U \) an open subset of \( V \), the restriction functor \( r_{U,V} : L_{pU} \to L_{pV} \) is defined by the
restriction of bundles.

Let \( U \) be an open subset of \( N \), and \( e_l \) be an object of \( L_{lU} \), the band of \( L_{l+1}e_l \) is the sheaf
of sections of the \( T_{F_{l+1}} \)-bundle \( p'_e_l \) defined on \( e_l \) induced by \( p_l \), this sheaf does not depend of the
objects chosen in the fibre \( L_{l+1}e_l \), since we have supposed that the 2-arrows depend only of \( N \).

Consider 2-morphisms \( u_p : e_p \to e'_p \), and \( u'_p : e'_p \to e''_p \) of \( L_p \). We have defined in the paragraph
above the proposition a morphism \( u_p^* \). These morphisms satisfy \( u'_p u_p^* = c(u_p,u'_p)(u'_p u_p)^* \), where
c\( (u_p,u'_p) \) is an automorphism of an object of the gerbe \( L_{p+1}e_p \) induced by the band •

7 Interpretation of the integral cohomology of a manifold.

Characteristic classes have been used by many mathematicians to study geometric objects. On
this purpose, we have to give a geometric interpretation of the group \( H^*(N, \mathbb{Z}) \). This is what we propose to do in this part.

It is a well-known fact that the group \( H^2(N, \mathbb{Z}) \) is the set of equivalence classes of complex line bundles over \( N \). Brylinski has defined an equivalence between the space of equivalence classes of complex line gerbes and \( H^3(N, \mathbb{Z}) \).

Consider \( \mathcal{L}_N^* \) the sheaf of differentiable \( \mathcal{L} - \{0\} = \mathcal{L}^* \)-functions defined on \( N \). We say that a
class \( [c_n] \) of \( H^n(N, \mathcal{L}_N) \) is geometric, if and only if there exists an \((n-1)\)-gerbed tower which
classifying cocycle is \( c_n \). A sufficient condition for a class \( c_n \) to be geometric is the following: there
exists a classifying cocycle \( c_{n-1} \) of a commutative \((n-2)\)-gerbed tower \( E(L_1,...,L_{n-2}) \) which is an
element of \( H^{n-1}(N,L_{n-2}) \), an exact sequence of sheaves \( 0 \to \mathcal{L}_N^* \to L \to L_{n-2} \to 0 \) such that
\([c_n] \) is the image of \( [c_{n-1}] \) by the boundary map \( \delta : H^{n-1}(N,L_{n-2}) \to H^n(N, \mathcal{L}_N^*) \).

We have the exact sequence

\[
0 \to \mathbb{Z} \to \mathcal{L}_N^* \xrightarrow{\exp} \mathcal{L}_N^* \to 0.
\]
where $i$ is the canonical injection, and $exp$ the exponential map. It results from this sequence an isomorphism between $H^n(N, C^*_N)$ and $H^{n+1}(N, \mathbb{Z})$. An element of $H^{n+1}(N, \mathbb{Z})$ will be said geometric if and only if it is the image of an element of $H^n(N, C^*_N)$ which is the classifying cocycle of a $(n - 1)$-gerbed tower by the preceding isomorphism.

We have the following result:

**Theorem 6.0.1.**

Let $N$ be a differentiable manifold, then each geometric class of $H^{n+2}(N, \mathbb{Z})$ is the classifying cocycle of a $n$-gerbed tower defined on $N$.

### 8 n-categories, and sheaves of $n$–categories.

In this part, we will define a notion of sheaf of $n$-categories over a topos $N$.

**Definition 7.0.1.**

A 0-pseudo-category is a set, a 1-pseudo-category $C_1$, is a category.

Suppose defined the notion of $n$-pseudo-category.

An $(n+1)$-pseudo-category $C_{n+1}$, is defined by, a class of objects $\text{Ob}(C_{n+1})$, for each objects $x$, and $y$, the $n$-pseudo-category of morphisms $\text{Hom}(x, y)$. For each objects $u_1, u_2$ and $u_3$ of $C_{n+1}$, there exists a composition $n$-functor:

$$\text{Hom}(u_2, u_3) \times \text{Hom}(u_1, u_2) \rightarrow \text{Hom}(u_1, u_3)$$

We suppose the existence of an object $1_x$ of $\text{Hom}(x, x)$, such that for each arrow $h : x \rightarrow y, h \circ 1_x$ is isomorphic to $h$, and for each arrow $h' : y' \rightarrow x, 1_x h'$ is isomorphic to $h'$.

An isomorphism between the objects $x$ and $y$ of an $n+1$–pseudo-category is a map $f : x \rightarrow y$, such that there exists $h : y \rightarrow x$ such that $hf$ is isomorphic to $1_x$, and $fh$ to $1_y$.

A functor between two $n$-pseudo-categories $C_n$ and $C'_n$, is defined as follows:

(i) A map $F : \text{Ob}(C_n) \rightarrow \text{Ob}(C'_n)$, and for each arrow $f : x \rightarrow y$, a morphism $F(f) : F(x) \rightarrow F(y)$ such that $F(f \circ f')$ is isomorphic to $F(f) \circ F(f')$.

(ii) A natural transformation between two functors $F$ and $F'$, is defined by a family of maps $u_x : F(x) \rightarrow F'(x)$ such that for each map $f : x \rightarrow y$, $u_y F(f)$ is isomorphic to $F'(f) u_x$ At this stage, we do not precise the gluing datas.

**Definition 7.0.2.**

A 0-sheaf of sets defined on $N$, will be a sheaf of sets. Suppose defined the notion of sheaves of $n–1$-pseudo-categories. A sheaf of $n$-pseudo-categories, will be defined by the following data: for each pair of objects $U$ and $V$ of $N$, and a map $h : U \rightarrow V$, a restriction functor $r^{C_n}_{U, V}(h) : C_n(V) \rightarrow C_n(U)$ such that for each triple of objects $U_1, U_2$ and $U_3$, there exists an isomorphism $\epsilon(U_1, U_2, U_3)$ between $r^{C_n}_{U_1, U_2}(h) r^{C_n}_{U_2, U_3}(f)$ and $r^{C_n}_{U_1, U_3}(fh)$.

Gluing condition for objects:

Let $(U_i)_{i \in I}$ be an open cover of the object $U$ of $N$, $x_i$ an object of $C_n(U_i)$. If there exists, a family of maps $u_{ij} : r^{C_n}_{U_i \times_n U_j, U_i}(x_i) \rightarrow r^{C_n}_{U_i \times_n U_j, U_i}(x_j), a \text{ sheaf of } n–1\text{-pseudo-categories}$
$C_{n-1}$ defined on $U$, such that the restrictions map $r^{C_{n-1}_{U_i \times \cup U_j, U_i}}$ are $u_{ji}$, then there exists an object $x$ of $C_{n}(U)$ such that the restriction of $x$ to $U_i$ is $x_i$.

Gluing conditions for arrows:

For each $x$, and $y$ in $C_{n}(U)$, the map defined on sub-objects of $U$ by $V \rightarrow Hom(x|_V, y|_V)$ is a sheaf of $(n-1)$-categories.

We denote $IN_n$ the pseudo-category whose objects are the elements of the set $\{1, ..., n\}$, $Hom(j_1, j_2)$ has one element if $j_1$ inferior to $j_2$, if not it is empty. We endow it with the topology such that the covering family of $l$ are the integers inferior to $l$.

**Definition 7.0.3.**

An $n$-category, is a $n$-pseudo-category, such that for any objects $x_0, ..., x_n, ..$ of $C$, for each family of maps $u_{ij}: x_j \rightarrow x_i$ such that $u_{i_1 i_2} u_{i_2 i_3} = u_{i_1 i_3}$, the map defined on $IN, i \rightarrow Hom(x_i, x_0)$, is a sheaf of $n-1$-pseudo-categories whose restrictions functors $u_{ij}^*: Hom(x_j, x_0) \rightarrow Hom(x_i, x_0)$ are defined by: $h \rightarrow hu_{ji}$.

**References**

[1] Benabou, Some remarks on 2-categorical algebra I, Bul. Soc. Math. Belg., 41 (1989) pp. 127-194.

[2] A. Borel, Linear algebraic groups, W.A. Benjamin, Inc, New York-Amsterdam., (1969).

[3] L. Breen, On the classification of 2-gerbes and 2-stacks, Asterisque., 225 (1994).

[4] G. E. Bredon, Sheaf theory, McGraw-HillBook Co., (1967).

[5] J. L. Brylinski, J.L Loops spaces, Characteristic Classes and Geometric Quantization, Progr. Math. 107, Birkhauser., (1993).

[6] J. L. Brylinski, and D. A. Mc Laughlin, The geometry of degree four characteristic classes and of line bundles on loop spaces I, Duke Math. Journal., 75 (1994), pp. 603-637.

[7] Y. Carriere, Autour de la conjecture de L. Markus sur les variétés affines, Invent. Math., 95 (1989), pp. 615-628.

[8] P. Deligne, Theorie de Hodge III, Inst. Hautes Etudes Sci. Publ. Math., 44 (1974), pp. 5-77.

[9] J. Duskin, An outline of a theory of higher dimensional descent, Bull. Soc. Math. Bel. Série A., 41 (1989), pp. 249-277.

[10] D. Fried, Closed similarity affine manifolds, Comment. Math. Helv., 55 (1980), pp. 576-582.

[11] J. Giraud, Cohomologie non abélienne

[12] D. Fried and W. Goldman, Three-dimensional affine crystallographic groups, Advances in Math., 47 (1983), pp. 1-49.

[13] D. Fried, W. Goldman, and M. Hirsch, Affine manifolds with nilpotent holonomy, Comment. Math. Helv., 56 (1981), pp. 487-523.
Non abelian cohomology: the point of view of gerbed towers

[14] W. Goldman, Two examples of affine manifolds, Pacific J. Math., 94 (1981), pp. 327-330.

[15] W. Goldman, The symplectic nature of fundamental groups of surfaces, Advances in in Math., 54 (1984), pp. 200-225.

[16] W. Goldman, Geometric structure on manifolds and varieties of representations, 169-198, Contemp. Math., 74.

[17] W. Goldman and M. Hirsch, The radiance obstruction and parallel forms on affine manifolds, Trans. Amer. Math. Soc., 286 (1984), pp. 629-649.

[18] W. Goldman and M. Hirsch, Affine manifolds and orbits of algebraic groups, Trans. Amer. Math. Soc., 295 (1986), pp. 175-198.

[19] R. Godement, Topologie algébrique et théorie des faisceaux Hermann., (1958).

[20] J. L. Koszul, Variétés localement plates et convexité, Osaka J. Math., (1965), pp. 285-290.

[21] A.-O. Kuku, Ranks of $K_n$ and $G_n$ of orders and groups rings of finite groups over integer in number fields, Journal of Pure and Applied Algebra., 138 (1999), 39-44.

[22] S. MacLane, Homology, Springer-Verlag., (1963).

[23] G. Margulis, Complete affine locally flat manifolds with a free fundamental group, J. Soviet. Math., 134 (1987), pp. 129-134.

[24] J. W. Milnor, On fundamental groups of complete affinely flat manifolds, Advances in Math., 25 (1977), pp. 178-187.

[25] M. Nguifo Boyom, Algebres a associateur symetrique et algebres reductives, These Universite de Grenoble., (1968).

[26] A. Tsemo, Dynamique des variétés affines, J. London Math. Soc., 63 (2001) pp. 469-487.

[27] A. Tsemo, Fibrés affines, Michigan J. Math., 49 (2001) pp. 459-484.

[28] A. Tsemo, Composition series of affine manifolds and $n$-gerbes Adv. Math. Research 3, Nova Sci. Publ., (2003), pp. 1-37.