Nonlinear Modal Decoupling of Multi-Oscillator Systems with Applications to Power Systems

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Abstract—Many natural and manmade dynamical systems that are modeled as large nonlinear multi-oscillator systems like power systems are hard to analyze. For such a system, we propose a nonlinear modal decoupling (NMD) approach inverting constructively as many decoupled nonlinear oscillators as the system’s oscillation modes so that individual decoupled oscillators can easily be analyzed to infer dynamics and stability of the original system. The NMD follows a similar idea to the normal form except that we eliminate inter-modal terms but allow intra-modal terms of desired nonlinearities in decoupled systems, so decoupled systems can flexibly be shaped into desired forms of nonlinear oscillators. The NMB is then applied to power systems towards two types of nonlinear oscillators, i.e. the single-machine-infinite-bus (SMIB) systems and a proposed non-SMIB oscillator. Numerical studies on a 3-machine 9-bus system and New England 10-machine 39-bus system show that (i) decoupled oscillators keep a majority of the original system’s modal nonlinearities and the NMB provides a bigger validity region than the normal form, and (ii) decoupled non-SMIB oscillators may keep more authentic dynamics of the original system than decoupled SMIB systems.

Index Terms—Nonlinear modal decoupling, inter-modal terms, intra-modal terms, oscillator systems, normal form, power systems, nonlinear dynamics.

I. INTRODUCTION

Oscillator systems, i.e. a system with a number of oscillators interacting with each other, are ubiquitous in both natural systems and manmade systems. In biological systems, low-frequency oscillations in metabolic processes can be observed at intracellular, tissue and entire organism levels and they have a deterministic nonlinear causality [1]. In electric power grids, which are among the largest manmade physical networks, oscillations are continuously presented during both normal operating conditions and disturbed conditions [2]. In some fields of both natural science and social science, the Kuramoto model is built upon a large set of coupled oscillators modeling periodic, self-oscillating phenomena in, e.g., reaction-diffusion systems in ecology [3] and opinion formation in sociophysics [4]. For all these oscillator systems, the common underlying mathematical model is actually a set of interactive governing differential equations, linear or nonlinear. An ideal way to study dynamics of a multi-oscillator system from an initial state is to find an analytical solution of its differential equation models and use the solution for further prediction and control. However, even finding an approximate solution of a high-dimensional nonlinear multi-oscillator system has been a challenge for a long time to mathematicians, physicists and engineers [5]. Analytical efforts have been made in broader topics, like dynamical systems [6], [7], nonlinear oscillations [8] and complex networks [9], to better understand, predict and even control the oscillator systems, and some well-known theories are such as the perturbation theory and Kolmogorov-Arnold-Moser theory. Most of these efforts attempt to directly analyze an oscillator system as a whole and extract desired information, e.g. approximate solutions and stability criteria, from the governing differential equations. Especially, extensive attentions recently have been paid to using the theory of synchronization to analyze the interactions among oscillators in a system [10]-[14]. In addition, numerical studies can provide dynamical behaviors of high-dimensional oscillation systems with desired accuracy. However, simulating a high-dimensional oscillator system like a power grid could be very slow if oscillators are coupled through a complex network and interact nonlinearly [15].

In this paper, we aim at inversely constructing a set of decoupled, independent oscillators from a given high-dimensional multi-oscillator system. Each of those decoupled oscillators is a fictitious 2nd order nonlinear system that corresponds to a single oscillation mode of the original system. For some real-life oscillator networks such as a power grid networking synchronous generators, those real oscillators themselves often have strong couplings and interactions in dynamics. However, the modal dynamics with respect to different oscillation modes may have relatively weak couplings or interferences unless significant resonances happen between oscillation modes. Thus, the fictitious oscillators that are inversely constructed to represent different oscillation modes are independent, or in other words naturally decoupled, to some extend and hence can be more easily understood and analyzed to gain insights on the dynamical behaviors, stability and control of the whole original system. In this paper, we define such a process as nonlinear modal decoupling, i.e. inverse

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construction from the original nonlinear multi-oscillator system to a set of decoupled fictitious nonlinear oscillators.

Finding the modal decoupling transformation even for general linear dynamical systems has been studied for more than two hundred years, and massive papers aimed at decoupling linear dynamical systems with non-classical damping. In the 1960s, Caughey and O’Kelly [16] found the necessary and sufficient conditions for a set of damped second-order linear differential equations to be transformed into decoupled linear differential equations based on early mathematical works by Weirestras in the 1850s [17]. In just the last decade, the decoupling of linear dynamical systems with non-classical damping was achieved [18]-[20].

For nonlinear oscillator systems, the modal decoupling has not been studied well despite its importance in simplification of stability analysis and control on such complex systems. Some related efforts have focused on transformation of a given nonlinear oscillator system towards an equivalent linear system. One approach is feedback linearization that introduces additional controllers to decouple the relationship between and outputs and inputs in order to control one or some specific outputs of an oscillator system [21]-[24]. Another approach is normal form [25]-[29] that applies a series of coordinate transformations to eliminate nonlinear terms starting from the 2nd order until the simplest possible form. If regardless of resonances, such a simplest form is usually approximated by a linear oscillator system, whose explicit solution together with the involved series of transformations are then used to study the behavior of the original nonlinear oscillator system. To summarize, to analyze the high dimensional nonlinear oscillator systems, the efforts in the present literature tend to achieve an approximate or equivalent linear system so as to utilize available linear analysis methods. Based on these efforts, it is quite intuitive to move one step forward to achieve a set of decoupled nonlinear oscillator systems where each are simple enough for analysis of dynamics and stability. That is the objective of this work.

For real-life nonlinear oscillator systems such as a multi-machine power system, a linear decoupling, if exists, can transform the system into its modal space, which may help improve the modal estimation [30] and assess the transient stability of the system [31]. The normal form method was introduced to power systems in 1992 by paper [32] for analyzing stressed power systems and enables the design of controllers considering partial nonlinearities of the systems. Since nonlinearities are considered, like the 2nd order nonlinearity in [33] and the 3rd order nonlinearity in [34], the approximated solution from normal form may have a larger validity region than the linearized system [35]. Among these attempts, including [31] on a nonlinear modal decoupling, the results do not provide the corresponding nonlinear transformation that links the original oscillator system to the nonlinear modal decoupled systems.

In this paper, the proposed nonlinear modal decoupling approach is derived adopting an idea similar to the Poincaré normal form in generating a set of nonlinear homogeneous polynomial transformations [36]. However, different from the classic theory of normal forms, we eliminate only the inter-modal terms and allow decoupled systems to have intra-modal terms of desired nonlinearities for nonlinear modal decoupling.

The rest of the paper is organized as follows: in Section II, the definitions, derivations and error estimation indices on the nonlinear modal decoupling are presented. In Section III, the nonlinear modal decoupling approach is applied to multi-machine power systems and its application in first-integral based stability analysis. Section IV shows numerical studies on the IEEE 3-machine 9-bus power system and IEEE 10-machine 39-bus power system. Conclusions are drawn in Section V.

II. NONLINEAR MODAL DECOUPLING

We will first introduce several definitions and one lemma before presenting Theorem 1 on nonlinear modal decoupling.

Given a nonlinear dynamical system described by a set of ordinary differential equations:

\[ \dot{X} = F(X) \]

where \( X \) is the vector containing \( N \) state variables and \( F \) is a smooth vector field. The origin is assumed to be an equilibrium point (if not, it can be easily moved to the origin by a coordinate transformation).

In this paper, the given dynamical system in (1) is called a multi-oscillator system if and only if all eigenvalues of its Jacobian matrix, say \( A \), appear as conjugate pairs of complex numbers. Each conjugate pair defines a unique mode of the system. Let \( \Lambda = \{\lambda_1, \lambda_2, ..., \lambda_N\} \) represent the matrix of \( A \)’s eigenvalues, where \( N \) is an even number. Without loss of generosity, let \( \lambda_{2i-1} \) and \( \lambda_{2i} \) be conjugate pair corresponding to the mode \( i \).

**Definition 1 (Desired modal nonlinearity)** If the multi-oscillator system (1) can mathematically be transformed into the form (2) and the two governing differential equations in (2) regarding mode \( i \) have \( \mu \)-coefficients of desired values, then the \( i \)-th mode is said to have the desired modal nonlinearity.

\[
\begin{align*}
\dot{z}_{2i-1} & = \lambda_{2i-1} z_{2i-1} + \sum_{a=1}^{N} \sum_{\beta=a}^{N} \mu_{2i-1,2i-1,\beta} z_{\beta} z_{\beta} + \cdots \\
& + \sum_{a=1}^{N} \sum_{\beta=a}^{N} \sum_{\varphi=0}^{\mu} \mu_{2i-1,2i-1,\beta,\varphi} z_{\beta} z_{\beta} z_{\varphi} + \cdots \\
\dot{z}_{2i} & = \bar{\lambda}_{2i} z_{2i} 
\end{align*}
\]

where \( \bar{Z} = \{z_1, ..., z_N\} \) is the vector of state variables and \( k > 1 \).

In the traditional normal form method, only the modal nonlinearities that cannot be eliminated due to resonance are retained, which is equivalent to making as many \( \mu \)-coefficients be zero as possible in (2). If regardless of the resonance, the advantage of the standard normal form is that the resulting truncated system will be a linear dynamical system having an analytical solution.

However, it is not always true that a linear system is the most desired. For instance, in power systems, power engineers and researchers prefer to assume that the underlying low-dimensional system dominating each nonlinear oscillatory mode follows the nonlinearity of a single-machine-infinite-bus (SMIB) power system [31][46][47], i.e. the simplest single-
degree-of-freedom power system. Thus, this paper is motivated
to keep specific nonlinear terms for the desired modal
nonlinearity by following either the SMIB assumption, as
shown in Section III-B, or another assumption proposed in
Section III-C.

For the normal form method, the truncated linear system
cannot be used for estimating the boundary of stability, which
is meaningful for a nonlinear system. As a comparison, the
nonlinear modal decoupling to be proposed provides the
possibility to estimate the boundary of stability using the
nonlinearities intentionally kept in the model, although
estimation of the stability boundary of a truncated system model
is a long standing problem. An approximation of the stability
boundary will be presented in Section III-D.

For the convenience of statements, the following definitions
are adopted to introduce which nonlinear terms should be kept
or eliminated.

**Definition 2 (Intra-modal term and inter-modal term)** Given
the desired modal nonlinearity (2) for mode \(i\) of the multi-
oscillator system (1), the *intra-modal terms* are the nonlinear
terms in the form of \(\mu_{j,\alpha\beta...\rho} z_\alpha z_\beta \cdots z_\rho\) (for \(k = 2, 3, \ldots\)) which
involve state variable(s) only corresponding to mode \(i\), i.e.,
indices \(j, \alpha, \beta, \ldots, \rho \in \{2i-1, 2i\}\). All the other nonlinear terms
are called the *inter-modal terms*, which involve state variables
corresponding to other modes.

**Definition 3 (Mode-decoupled system)** If the form (2) with
desired modal nonlinearity regarding the \(i\)-th mode also makes
(3) satisfied, then (2) is called a *mode-decoupled system* for
mode \(i\).

\[
\mu_{j,\alpha\beta...\rho} = \begin{cases} \text{desired value} & \text{if } j, \alpha, \beta\ldots, \rho \in \{2i-1, 2i\} \\ 0 & \text{otherwise} \end{cases} \quad (3)
\]

Based on the concept of resonance [36], the \(n\)-triple \(A = \{\lambda_1,
\lambda_2, \ldots, \lambda_N\}\) of eigenvalues is said to be *resonant* if among
the eigenvalues there exists an integral relation \(\lambda_i = \sum m_k \lambda_k\),
where \(s = 1, \ldots, N\), \(m_k \geq 0\) are integers and \(\sum m_k \geq 2\). Such a
relation is called a *resonance*. The number \(\sum m_k\) is called the
order of the resonance. Now, we present the Theorem on
nonlinear modal decoupling.

**Theorem 1 (Nonlinear modal decoupling)** Given a multi-
oscillator system in (1) and a desired modal nonlinearity having
inter-modal terms eliminated, if resonance does not exist for
any order, then (1) can be transformed into (2) by a certain
nonlinear transformation, denoted as \(H\).

**Remark** The rest of this section will focus on giving a
constructive proof of the theorem, in which we derive the
transformation \(H\) and its inverse that can be numerically
computed. This is different from the normal form theory where
the focus is on the existence of the normal form. Unlike the
normal form, the nonlinear modal decoupling requires
elimination of only inter-modal terms so as to decouple the
dynamics regarding different modes while leaving room for
intra-modal terms to be designed for desired characteristics
with each mode-decoupled system. For simplicity, we use \(\mu_{\text{intra}}\)
and \(\mu_{\text{inter}}\) to respectively call the intra- and inter-modal term
coefficients. For this section, we assume that the desired modal
nonlinearity for each mode to be known. Nonlinear modal
decoupling on a real-life high-dimensional multi-oscillator
system like a power system might intentionally make each
mode-decoupled system have the same modal nonlinearity as a
single-oscillator system of the same type, e.g., an SMIB system
for power systems, for the convenience of using the existing
methods on the same type of systems. However, for the purpose
of stability analysis and control, decoupling a real-life system
into a different type of oscillators might also be an option. In
the next section, two ways to choose the desired modal
nonlinearity will be illustrated on power systems.

The detailed derivation of the transformation \(H\) used for
nonlinear modal decoupling will be presented in the
constructive proof of Theorem 1, where \(H\) will be a composition
of a sequence of transformations, denoted as \(H_1, H_2, \ldots, H_{\text{last}}\),
where \(H_{\text{last}}\) is a homogeneous polynomial. The relationship
between the state variables of the mode-decoupled system, say
\(Z\), and the state variables after the \(k\)-th transformation are shown
in (4) based on \(H_1, H_2, \ldots, H_k, \ldots\), where we use the "\(Z^{(k)}\"
replace the vector of state variables after the \(k\)-th
transformation.

\[
X = H(Z) = \cdots = (H_{k-1} \circ H_{k-2} \circ \cdots \circ H_2 \circ H_1)(Z)
\]
\[
Z^{(1)} = (H_2 \circ \cdots \circ H_k \circ \cdots \circ H_1)(Z)
\]
\[
Z^{(2)} = (H_{k+1} \circ \cdots \circ H_k)(Z)
\]
\[
\vdots
\]

We first introduce a lemma before presenting the proof of
Theorem 1.

**Lemma 1.** Given one transformed form (5) of a multi-oscillator
system, where \(D_j\) only contains intra-modal terms and \(C_j\) only
contains inter-modal terms and they are vector polynomial
functions of degree \(j\) in \(Z^{(p)}\). If resonance does not exist up to
the order \(p+1\), then in a certain neighborhood of the origin of
\(Z^{(p+1)}\), denoted as \(\Omega_{p+1}\), the inter-modal terms of degree \(p+1\)
can be completely eliminated to give (6) by a polynomial
transformation of degree \(p+1\) in (7), i.e., \(H_{p+1}\).

\[
Z^{(p)} = A \cdot Z^{(p)} + \sum_{j=2}^{n} D_j(Z^{(p)}) + \sum_{j=p+1}^{n} C_j(Z^{(p)})
\]
(5)

\[
Z^{(p+1)} = A \cdot Z^{(p+1)} + \sum_{j=2}^{p+1} D_j'(Z^{(p+1)}) + \sum_{j=p+2}^{n} C_j'(Z^{(p+1)})
\]
(6)

\[
Z^{(p)} = H_{p+1}(Z^{(p+1)}) = Z^{(p+1)} + h_{p+1}(Z^{(p+1)})
\]
(7)

**Proof of Lemma 1** Consider the transformation in (7), where
\(h_{p+1}\) is a column vector whose elements are the homogeneous
polynomials of degree \(p+1\) in \(Z^{(p+1)}\). The \((2i-1)\)-th and \(2i\)-th
elements of \(h_{p+1}\) are shown in (8).
Taylor expansion of (12) can be written as
\[ Z^{(i)} = A \cdot Z^{(i)} + \sum_{j=2}^{\infty} D_j^{(i)} (Z^{(i)}) + C_j^{(i)} (Z^{(i)}) \]  
(14)

Apply Lemma 1 with p=1, then we can transform (14) into
\[ Z^{(2)} = A \cdot Z^{(2)} + D_2^{(2)} (Z^{(2)}) + \sum_{j=5}^{\infty} D_j^{(2)} (Z^{(2)}) + C_j^{(2)} (Z^{(2)}) \]  
(15)

Apply Lemma 1 for k-2 times respectively with p= 2, ..., k-1, then we can transform (15) into (16).
\[ Z^{(k)} = A \cdot Z^{(k)} + \sum_{j=2}^{\infty} D_j^{(k)} (Z^{(k)}) + \sum_{j=p+1}^{\infty} D_j^{(k)} (Z^{(k)}) + C_j^{(k)} (Z^{(k)}) \]  
(16)

When the order k approaches infinity, the convergence has to be considered. Since investigating the convergence issue is not a trivial task and it is not the focus of this paper, we assume the convergence of this process holds when k approaches infinity. Then, (2) will be achieved eventually, i.e. \( Z=Z^{(k)} \), and the transformation \( H \) in (4) is composed by \( H_1 \) in (13) and \( H_{p+1} \) in (7) with \( p=1, 2, \ldots \).

In practice, it is hard to deal with an infinite number of transformations. Still, for any expected order k, we can use the truncated system as an approximation for practical applications. The following gives three corollaries of the nonlinear modal decoupling for any expected order k with the help of the k-jet concept. Then, the decoupled k-jet system is introduced.

**Definition 4 (k-jet equivalence [21])** Assume \( F(X) \) and \( G(X) \) are two vector functions of the same dimension. We say that \( F(X) \) and \( G(X) \) are k-jet equivalent at \( X_0 \), or \( F(X) \) is a k-jet equivalence of \( G(X) \) and vice versa, iff corresponding terms in the Taylor expansions of \( F(X) \) and \( G(X) \) at \( X_0 \) are identical up to order \( k \).

Then, a k-jet system of (1) can be rewritten in (17). The errors between the solutions of these two systems (1) and (17) are totally due to the truncation of high-order terms, whose impact will be investigated in the case studies. The following will start from (17) and analyze the nonlinear modal decoupling.

\[ \dot{X} = AX + \sum_{j=2}^{k} A_j (X) \]  
(17)

where \( A_j(X) \) is a vector function, each of whose elements is a weighted sum of all homogeneous polynomials of degree \( j \) in \( X \).

**Corollary 1 (k-th order nonlinear modal decoupling)**. Given a multi-oscillator system in (17), if the resonance does not exist up to the given order \( k \), then the \( k \)-th order nonlinearly mode-decoupled system can be achieved as (18) by the \( k \)-th order decoupling transformation \( H^{(k)} \) in (19).

\[ \dot{Z}^{(k)} = A \cdot Z^{(k)} + \sum_{j=2}^{k} D_j (Z^{(k)}) + \sum_{j=p+1}^{\infty} D_j (Z^{(k)}) + C_j (Z^{(k)}) \]  
(18)

\[ X = H^{(k)} (Z^{(k)}) = (H_{c}, H_{c}, \ldots, H_{k}) (Z^{(k)}) \]  
(19)
where $\mathbf{Z}^{(k)}$ is the vector of state variables in the $k$-th order mode-
decoupled space, $D_t$ and $C_t$ are vector functions where each of
their elements is a weighted sum of the terms of degree $j$ in $\mathbf{Z}^{(k)}$. $D_t$
only contains intra-modal terms, while $C_t$ only contains inter-
modal terms.

**Corollary 2.** The validity region for the transformation $H^{(k)}$,
denoted as $\Omega^{(k)}$, is

$$
\Omega^{(k)} = \bigcap_{p=2}^k \Omega_p
$$

(20)

**Corollary 3.** The inverse coordinate transformation, i.e. the
inverse of the transformation $H_{p+1}$ in (7), can be approximated by a power series

$$
z_i^{(p+1)} = z_i^{(p)} + \sum_{a=1}^N \sum_{\beta_1=0}^N s_{i,a\beta} a_i^{(p)} + \cdots
+ \sum_{a=1}^N \sum_{\beta_1=0}^N s_{i,a\beta\cdots\beta} a_i^{(p)} + \cdots
$$

(21)

**Remark** Based on existing literature, it is difficult to obtain
such an inverse transformation in an explicit form or even a reliably transformation of a single point from the coordinates of
$\mathbf{Z}^0$ to that of $\mathbf{Z}^{(p+1)}$. It was reported that the effectiveness of
solving the nonlinear algebraic equation (7) by a certain iterative algorithm with an initial guess of $\mathbf{Z}^{(p+1)}$ largely depends on
that initial guess. The iterative algorithm may either diverge or converge to a different point in $\mathbf{Z}^{(p+1)}$ [39][40]. Actually,
the inverse of (7) can be written as (22). An approximate analytical expression to (22) is provided by Corollary 3. The proof is
omitted while the idea is quite straightforward: (i) First, assume that the inverse transformation (22) follows the polynomial form,
as shown in (21) which is the $i$-th equation of (22) where the coefficient $s$ of each term is an unknown. Substitute (21)
into the right side of (7) and equate both sides term by term in
$\mathbf{Z}^0$ to formulate equations in $s$. Solving those equations for $s$
and substituting them back to (21) will give the inverse
transformation. Note that those formulated equations in $s$
can always be solved due to the characteristic of (7), i.e. the function
$h$ only contains homogeneous polynomials of degree $p+1$ in $\mathbf{Z}^{(p)}$
and the formulated equations can always be solved order by
order from low to high.

$$
\mathbf{Z}^{(p+1)} = \mathbf{Z}^{(p)} + h_{p+1}^{-1}(\mathbf{Z}^{(p)})
$$

(22)

**Definition 5. (Decoupled $k$-jet system)** By ignoring terms with
orders higher than $k$ in (18), we obtain a special $k$-jet system of
(18), called a decoupled $k$-jet system.

$$
\dot{\mathbf{Z}}_{\text{jet}}^{(k)} = \mathbf{A} \cdot \mathbf{Z}_{\text{jet}}^{(k)} + \sum_{j=2}^k D_T^j \mathbf{Z}_{\text{jet}}^{(j)}
$$

(23)

**Remark** Generally speaking, the nonlinearities maintained in the
decoupled systems (18) by the intra-modal terms, i.e. $D_t$, can
follow any pre-designed form and then defines a corresponding
$k$-th order nonlinear transformation $H^{(k)}$. If without any a priori
knowledge about the nonlinear characteristics of the original
system (17), there could be an infinite number of ways to
intentionally design the intra-modal terms, such that the
resulting decoupled $k$-jet system (23) by different ways will
differ from each other in terms of the size and shape of their
validity regions. Also note that the equations in (23) about one
mode are completely independent with those about any other
mode, while the nonlinearities up to order $k$ within each
individual mode are still maintained. Next, two theorems about
the decoupled $k$-jet are introduced.

**Theorem 2 (Real-valued decoupled $k$-jet).** The decoupled
$k$-jet in (23) is equivalent to a real-valued system, called a
real-valued decoupled $k$-jet.

**Proof of Theorem 2** Since there may be multiple ways to
construct a real-valued decoupled $k$-jet, we only provide the
construction leading to two coordinates respectively having
physical meanings similar to displacement and velocity.

The differential equations for mode $i$ in the decoupled $k$-jet
are shown in (24), which are the $(2i-1)$-th and $2i$-th equations
of (24). Note that the two state variables in (24) are complex-
valued and those $\mu$-coefficients are determined in (7).

$$
\begin{align*}
\dot{z}_i^{(k)} & = 2z_i^{(1)} + \sum_{\alpha=1}^N \sum_{\beta=0}^N \mu_{zi,\alpha\beta} z_i^{(\alpha)} + \cdots \\
& + \sum_{\alpha=1}^N \sum_{\beta=0}^N \sum_{\gamma=0}^N \mu_{zi,\alpha\beta\gamma} z_i^{(\alpha)} \cdot \cdots \cdot z_i^{(\gamma)} \\
& \quad \text{in terms of total}
\end{align*}
$$

(24)

Because the two state variables in (24) are a conjugate pair, the
right-hand sides of the first and second equations can be
denoted respectively as $a+jb$ and $a-jb$, where $a$ and $b$ are real-valued functions in $\mathbf{Z}^{(k)}$, $\lambda$ and $\mu$. Then, apply the coordinate transformation in (25) and (26) to yield (27), where all
parameters and variables are real-valued.

$$
\begin{align*}
\begin{bmatrix} z_1^{(k)} \\ z_2^{(k)} \end{bmatrix} & = U_{\text{mode}}^{-1} \begin{bmatrix} w_{21-1} \\ w_{21-1} \end{bmatrix} \\
U_{\text{mode}} & = \begin{bmatrix} \lambda_{21-1} & \lambda_{21} \\ 1 & 1 \end{bmatrix}
\end{align*}
$$

(25)

$$
\begin{align*}
\dot{w}_{21-1} & = \sum_{l=1}^{2i} v_{l0} w_{21-1}^{l0} + \sum_{j=1}^{2i} \sum_{l=0}^{j-1} v_{jl} w_{21-1}^{lij} \\
\dot{w}_{21} & = w_{21-1} + \sum_{j=0}^{2i} \sum_{l=0}^{j} v_{jl} w_{21-1}^{lij}
\end{align*}
$$

(27)

**Remark** Note that the transformation in (13) also need to be
normalized. The purpose of a normalization is to make the new
coordinates of (27) have a scale comparable to that of (23). The
normalization introduced in [31] is adopted here:
(i) Classify the elements in the left eigenvector (complex-valued) related to the displacement into two opposing groups based on their angles;
(ii) Calculate the sum of the coefficients in one group;
(iii) Divide the left eigenvector by that sum;
(iv) Do such normalization for all left eigenvectors.

Theorem 3 (Error estimation) Given the multi-oscillator system in (17), its k-th order nonlinearly mode-decoupled system follows (18) and the corresponding decoupled k-jet follows (23). Assume the convergence of the decoupling transformation [38] and also assume all eigenvalues \( \lambda \) in matrix \( A \) to satisfy \( \text{Re}[\lambda] < \alpha < 0 \). Let \( X(t), Z^{(0)}(t) \) and \( Z^{(k)}(t) \) be the solutions of (17), (18) and (23), respectively, under an identical initial condition \( X_0 = Z^{(0)}_0 = Z^{(k)}_{jet0} \). Then, there exists constants \( c_1 > 0 \) and \( c_2 > 0 \) which are independent of \( k \) such that for any \( \varepsilon \in 0 \leq \varepsilon \leq c_1, |X(0)| \leq \varepsilon \) implies
\[
\varepsilon_k(t, X_0) = \|X(t) - H^{(k)}C_{jet}(t)\| \leq c e^{\alpha k} e^{-\varepsilon/2}
\]
where \( l \)-1 represents a type of norm. Note that this theorem indicates that a better convergency can be achieved for any \( t \) when \( k \) increases.

Proof of Theorem 3 It is easy to see that the two systems in (17) and (18) are equivalent in \( \Omega^{(k)} \), over the transformation \( H^{(k)} \) in (19). To show the error \( \varepsilon_k(t, X_0) \) approaches zero, we only need to show that error defined in (29), or equivalently (30), approaches zero:
\[
\varepsilon_k(t, X_0) = \|X(t) - H^{(k)}C_{jet}(t)\|
\]
\[
\widetilde{\varepsilon}_k(t, X_0) = \|Z^{(k)}(t) - Z^{(k)}_{jet}(t)\|
\]
The rest of the proof is omitted since it is similar to the Theorem 5.3.4 in [41].

Given the k-th order nonlinearly mode-decoupled system in (18) and the corresponding decoupled k-jet system in (24), the k-th order nonlinear modal decoupling is \( H^{(k)} \) in (19), whose inverse is assumed to be \( (H^{(k)})^{-1} \), which is the composition of inverse transformations in (22) with different \( p \). Denote \( X(t, X_0) \) as the solution of (23) under the initial condition \( X_0 \) and denote \( Z^{(k)}_{jet}(t, Z^{(k)}_{jet0}) \) as the solution of (24) under the initial condition \( Z^{(k)}_{jet0} \) where \( Z^{(k)}_{jet0} = (H^{(k)})^{-1}(X_0) \). For a certain given \( \varepsilon > 0 \), the validity region \( \Omega \) of the decoupled k-jet system is defined in (31).
\[
\Omega_{\varepsilon} = \{ X_0 \mid \|X(t, X_0) - H^{(k)}C_{jet}(t, Z^{(k)}_{jet0})\| < \varepsilon \}
\]

Note that the validity region \( \Omega \) relies on the selection of \( \varepsilon \). A larger \( \varepsilon \) will lead to a larger validity region.

III. NONLINEAR MODAL DECOUPLING OF POWER SYSTEMS

This section will apply the proposed nonlinear modal decoupling analysis to power systems. Firstly, the nonlinear differential equations of a multi-machine power system is introduced. Then, two forms of desired modal nonlinearity for the decoupled k-jet system are proposed. Finally, the first-integral based method is applied to the decoupled k-jet systems for stability analysis.

A. Power system model

An m-machine power system is modeled by (32) and (33), where each generator is represented by a 2nd order classic model:
\[
\dot{\delta}_i + \frac{D_i}{2H_i} \delta_i + \frac{\omega_0}{2H_i} (P_{\alpha i} - P_{\alpha i}^*) = 0
\]
\[
P_{\alpha i} = E_i^2 G_i + \sum_{j=1,\neq i} C_{ij} \sin(\delta_i - \delta_j) + D_i \cos(\delta_i - \delta_j)
\]
where \( i \in \{1, 2, \ldots, m\} \), \( \delta_i \), \( P_{\alpha i} \), \( P_{\alpha i}^* \), \( E_i \), \( H_i \) and \( D_i \) respectively represent the absolute rotor angle, mechanical power, electrical power, electromotive force, the inertia constant and damping constant of machine \( i \), and \( G_{ij}, C_{ij}, D_{ij} \) represent network parameters including loads modeled by constant impedances. The system state vector \( X \) has a dimension of 2m.

\[
X = [\delta_1, \delta_2, \delta_3, \ldots, \delta_m, \dot{\delta}_1, \dot{\delta}_2, \ldots, \dot{\delta}_m, \dot{\delta}_m]^{T}
\]

Remark The m-machine power system modeled by (32) and (33) has \( m-1 \) pairs of conjugate complex eigenvalues, which respectively correspond to \( m-1 \) oscillatory modes, and two real eigenvalues (including one zero eigenvalue) [45]. We will focus on \( m-1 \) oscillatory models, which mainly determine rotor angle dynamics and stability of the power system. Thus, after the first coordinate transformation towards \( Z^{(k)} \), only 2m-2 differential equations corresponding to those oscillatory modes are kept for further analysis, i.e. \( N=2m-2 \). Denote the eigenvalues of these oscillatory modes as \( \lambda_{11}, \lambda_{12}, \ldots, \lambda_{2m-2} \), where \( \lambda_{2i-1} \) and \( \lambda_{2i} \) \( (i=1, 2, \ldots, m-1) \) belong to one conjugate pair.

Next, we present two ways to choose the desired modal nonlinearity respectively under the SMIB assumption and another proposed small transfer (ST) assumption.

B. Nonlinear modal decoupling with the SMIB assumption

SMIB assumption [31][42][44][46][47] The nonlinearity associated with each oscillatory mode has the same form as an SMIB system.

In practice, this assumption is widely made by scholars and engineers in power systems. For instance, a power system that consists of two areas being weakly interconnected is often simplified to an SMIB system for stability studies regarding the inter-area oscillation mode. In the following, we study a general multi-machine power system. The goal is to find a way to intentionally transform the nonlinear terms of each decoupled system into the form of an SMIB system.

Desired decoupled system for mode i The desired decoupled system about mode \( i \), i.e. eigenvalues \( \lambda_{2i-1} \) and \( \lambda_{2i} \), can be written as (35) [48].
\[
\ddot{y}_i + \alpha_i \dot{y}_i + \beta_i \left( \sin(y_i + y_\alpha) - \sin y_\alpha \right) = 0
\]
where \( y_i \) is the generalized angle coordinate of mode \( i \), while \( \alpha_i, \beta_i \) and \( y_\alpha \) are constants that can be uniquely determined by
\[
\begin{aligned}
\begin{cases}
y_u = \sum_{j=1}^n \tau_j \delta_j \\
\alpha_i = -2 \text{Re}\{\lambda_{2,1-i}\} \\
\beta_i = \frac{\lambda_{2,1-i} \lambda_{2,1}}{\cos \gamma_i}
\end{cases}
\end{aligned}
\]

where \( \tau_j \) is the \( i \)-th row \( j \)-th column element from the matrix of the left eigenvectors defined using the state matrix of the linearization of (35) [31] and \( \delta_j \) is the steady-state value of \( \delta_j \).

**Nonlinear modal decoupling transformation** Assume each complex-valued decoupled system to be

\[
\begin{aligned}
\dot{z}_{2,1-i} &= \lambda_{2,1-i} z_{2,1-i} + \sum_{j=1}^l \sum_{t=0}^\infty \mu_{i,j,t} z_{2,1-i}^{l-t} \\
\dot{z}_{2,1} &= \frac{\tau}{\lambda_{2,1-i}}
\end{aligned}
\]

Toward the real-valued desired form of the decoupled system in (35), \( \mu \)-coefficients of intra-modal terms have yet to be determined. Apply the following coordinate transformation to (37) to obtain (40).

\[
\begin{aligned}
\begin{cases}
\dot{y}_i = V_{\text{mode } i} \cdot \dot{y}_i \\
y_i = V_{\text{mode } i} \cdot y_i
\end{cases}
\end{aligned}
\]

where

\[
V_{\text{mode } i} = \frac{2}{\lambda_{2,1-i} - \lambda_{2,1}} \left[ \begin{array}{c} 1 & -\lambda_{2,1} \\
-1 & \lambda_{2,1-i}
\end{array} \right]
\]

\[
\begin{aligned}
\frac{d\dot{y}_i}{dt} + \alpha_i \dot{y}_i + \sum_{n=4}^\infty \tau_n y_i^n = 0 \\
\frac{dy_i}{dt} = \dot{y}_i
\end{aligned}
\]

Coefficient \( r_{\text{intra}} \) is determined by (41) to make (35) and (40) have identical nonlinearities up to the \( k \)-th order.

\[
r_{\text{intra}} = \beta_i \cos \left( \frac{y_u + (n-1)\pi}{2} \right)
\]

**C. Nonlinear modal decoupling with the ST assumption**

We also propose the following alternative assumption for each desired mode-decoupled system and compare its result with that from the SMIB assumption.

**Small transfer (ST) assumption** Assume the second equation of (11), i.e. \( H_{i,1,1,\text{intra},a\gamma\gamma\gamma} \), to be zero.

**Nonlinear modal decoupling transformation** The desired modal nonlinearity, i.e. \( \mu \)-coefficients, is chosen based on

\[
\mu_{i,\text{intra},a\gamma\gamma\gamma} = c_{i,1,1,\text{intra},a\gamma\gamma\gamma}
\]

**Remark** The desired decoupled system with the ST assumption might not be available before finishing the entire nonlinear modal decoupling process. However, the implementation is quite convenient, since we just need to let the second equation in (11) be zero. The physical insight behind this ST assumption is that we want to limit the propagation of nonlinear terms to higher orders over a transformation in (7), which can be seen in the example below.

Consider a system of two first-order differential equations with polynomial nonlinearities up to the \( 2^\text{nd} \) order in (43), which is a special case of (17) with \( N=2 \) and \( k=2 \).

\[
\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + b_{1,11} z_1^2 + b_{1,12} z_1 z_2 + b_{1,22} z_2^2 \\
\dot{z}_2 &= \lambda_2 z_2 + b_{2,11} z_1^2 + b_{2,12} z_1 z_2 + b_{2,22} z_2^2
\end{aligned}
\]

Note that (43) only gives one of the differential equations on each mode, so \( \lambda_1 \) and \( \lambda_2 \) are in fact the eigenvalues on two different modes, not the conjugate pair on one mode. This is for only simplicity of description. The idea is also applicable to other values of \( N \) and \( k \).

Intra-modal terms and inter-modal terms for these two equations in (43) are respectively listed in (44) and (45).

\[
\begin{aligned}
\{ h_{1,11} z_1^2, b_{1,2,2} z_2^2 \} \\
\{ h_{1,12} z_1^2, b_{1,2,2} z_1^2, b_{2,1,1} z_1^2, b_{2,1,2} z_2^2 \}
\end{aligned}
\]

Then, consider a coordinate transformation by a \( 2^\text{nd} \) order polynomial in (46).

\[
\begin{aligned}
\dot{z}_1 &= u_1 + h_{1,1} u_1^2 + h_{1,2} u_1 u_2 + h_{2,2} u_2^2 \\
\dot{z}_2 &= u_2 + h_{2,1} u_1^2 + h_{2,2} u_1 u_2 + h_{2,2} u_2^2
\end{aligned}
\]

Substitute (46) into (43) and obtain a new system about \( u \), where intra-modal terms and inter-modal terms are similar to (44) and (45) but defined about \( u \) instead of \( z \). In the new system, utilize the first equation of (11) to find coefficients \( h \) to cancel its inter-modal terms as shown in (47) and obtain (48) where \( P, Q, R \) and \( S \) are polynomial functions, \( S \) satisfies (50), and the coefficients of the intra-modal terms \( h_{1,11} \) and \( h_{2,22} \), denoted by \( h_{\text{intra}} \), are yet to be determined.

\[
\begin{aligned}
\begin{cases}
h_{1,12} &= b_{1,12} \\
h_{2,12} &= \frac{b_{1,2,2}}{2 \lambda_2 - \lambda_1} \\
h_{1,22} &= \frac{b_{1,2,2}}{2 \lambda_2 - \lambda_1} \\
h_{2,21} &= \frac{b_{2,2,1}}{2 \lambda_1 - \lambda_2} \\
h_{2,22} &= \frac{b_{2,2,2}}{2 \lambda_2 - \lambda_1}
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
\dot{u}_1 = \lambda_1 u_1 + (b_{1,11} h_1 - h_{1,11} \lambda_1) u_1^2 + \sum_{i=3}^\infty \sum_{j=0}^{\infty} T_{1i,j} u_1^i u_2^j \\
\dot{u}_2 = \lambda_2 u_2 + (b_{2,22} h_2 - h_{2,22} \lambda_2) u_2^2 + \sum_{i=3}^\infty \sum_{j=0}^{\infty} T_{2i,j} u_1^i u_2^j
\end{cases}
\end{aligned}
\]

where

\[
\begin{aligned}
T_{1i,j} &= \frac{P_{1i,j}(\lambda) Q_{1j}(\lambda)}{Q_{1j}(\lambda)} + \frac{P_{1i,j}(\lambda) R_{1j}(\lambda)}{Q_{1j}(\lambda)} S_{1j}(h_{\text{intra}}) \\
T_{2i,j} &= \frac{P_{2i,j}(\lambda) R_{2j}(\lambda)}{Q_{2j}(\lambda)} + \frac{P_{2i,j}(\lambda) R_{2j}(\lambda)}{Q_{2j}(\lambda)} S_{2j}(h_{\text{intra}})
\end{aligned}
\]

Note that the information corresponding to the \( 2^\text{nd} \) order nonlinearities in system (43) spreads out to all higher order nonlinear terms in system (48) through the transformation in (46). Theoretically speaking, such a transfer of nonlinearities...
does not impact the process of the nonlinear modal decoupling if keeping all nonlinear terms up to the infinite order in (48). However, the implementation of the nonlinear modal decoupling in practice can only keep nonlinear terms up to a finite order, say $k$, such that it will always be desired to keep the nonlinearities transferred to as few higher-order terms as possible. For specific systems, it might be possible that there exists a way to determine those non-zero $h_{\text{intra}}$ which can guarantee a minimum transfer, e.g. (51) where $D$ is a set of points $(u_1,u_2)$ containing all concerned dynamics of system (48). In general, without any a priori knowledge about the system, it might be preferred to let $h_{\text{intra}}=0$ in order to limit the transfer of nonlinearities, which is called the ST assumption.

$$\begin{align*}
\min_{\mathbf{w}_i \in D} \max_{j \in D} \left[ \frac{1}{2} \sum_{j \in D} \sum_{i \in D} T_{ij} u_i u_j'^2 + \sum_{i \in D} \sum_{i \in D} T_{ji} u_i u_j'^2 \right]
\end{align*}$$

(D) Nonlinear modal decoupling based stability analysis

The following assumption is adopted to create an explicit transient energy function for stability analysis: (52) and (53) holds for coefficients in (27).

$$\begin{align*}
\nu_{ij} &= 0 \quad \text{for all } j \geq 1, i \geq 1, j + l \leq k, (j,l) \neq (1,0) \\
\nu_{ii} &= 0 \\
\nu_{11} &= 0
\end{align*}$$

Then, (27) becomes (54) with the above assumption.

$$\begin{align*}
\mathbf{w}_{2i-1} &= \sum_{j=1}^{k} \mathbf{v}_j w_i^j \\
\mathbf{w}_2 &= w_{2i-1}
\end{align*}$$

With the assumption in (52) and (53), a transient energy function of the real-valued decoupled k-jet about mode $i$, shown in (27), is

$$V_i(w_{2i-1},w_2) = \frac{w_{2i-1}^2}{2} + \sum_{j=1}^{k} \int_0^1 \mathbf{v}_j s^j ds = \frac{w_{2i-1}^2}{2} + \sum_{j=1}^{k} \mathbf{v}_j (j+1) w_{2i}^{j+1}$$

Given the system in (54) and its energy function in (55), the unstable equilibrium point (UEP) around the origin of (54) can be obtained by letting the right hand side of (54) be zero and solving the resulting algebraic equations. Denote the smallest positive real solution by $w_{2i,\text{UEP}}$. Here is Theorem 4:

**Theorem 4 (Stability criterion)** Given an initial condition of (54), e.g. $(w_{2i-1}(0), w_2(0))$, if $V(0,w_{2i,\text{UEP}})$ is greater than $V(w_{2i-1}(0), w_2(0))$, then the system (54) is stable; otherwise, the system (54) is unstable.

IV. NUMERICAL STUDY

This section will present the numerical studies of the proposed nonlinear modal decoupling on two test power systems: the IEEE 3-machine, 9-bus system [49] and the New England 10-machine, 39-bus system [50]. Each system is modeled by (32) and (33).

In the IEEE 9-bus power system, the detailed results from the proposed nonlinear modal decoupling will be presented: 1) two sets of decoupled system equations are respectively derived under the SMIB and ST assumptions; 2) numerical simulation results on the decoupled systems are created and compared to that from the normal form method; 3) stability on the original system is analyzed by means of analysis on decoupled systems. The New England 39-bus power system is then used to demonstrate the applicability of the proposed nonlinear modal decoupling method on a high-dimensional dynamical system.

A. Test on the IEEE 9-bus system

The following disturbance is considered: a temporary three-phase fault is added on bus 5 and cleared by tripping line 5-7 after a fault duration time. The critical clearing time (CCT) of this disturbance, i.e. the longest fault duration without causing instability, is found to be 0.17s. The post-disturbance system is represented by differential equations in (56) and has a stable equilibrium $x_{\text{UEP}} = (3.12, 0, 3.12, 0, 3.12, 0)$, which is not the origin but is normal for a stable power system that has all generators operate coherently at one common speed after the disturbance. A 3$^{rd}$ order Taylor expansion of (56) gives an estimate of CCT equal to $0.16s$, which has been very close to the accurate $0.17s$, so the 3$^{rd}$ order Taylor expansion can credibly keep the stability information of the original system and is used below as the basis for deriving decoupled systems as well as the benchmark for comparison.

$$\begin{align*}
\dot{x}_1 &= x_2 + 3.12 \\
\dot{x}_2 &= -0.5x_2 - 1.14\cos(x_3 - 0.728) - 6.25\sin(x_3 - 0.728) - 1.56\cos(x_3 - 0.463) - 9.11\sin(x_3 - 0.463) - 5.98 \\
\dot{x}_3 &= x_4 + 3.12 \\
\dot{x}_4 &= -0.5x_4 - 4.22\cos(x_3 - 0.728) + 23.1\sin(x_3 - 0.728) - 6.04\cos(x_3 + 0.265) - 38.0\sin(x_3 + 0.265) - 5.98 \\
\dot{x}_5 &= x_6 + 3.12 \\
\dot{x}_6 &= -0.5x_6 - 12.3\cos(x_5 - 0.463) + 71.6\sin(x_5 - 0.463) - 12.8\cos(x_5 + 0.265) + 80.7\sin(x_5 + 0.265) - 5.98
\end{align*}$$

where $x_0 = x_i - x_j$
where the transformations are omitted. As a comparison, the
counterpart from the 3rd order normal form gives (59).

\[
\begin{align*}
\dot{z}_1^{(3)} &= (-0.25 + j12.9)z_1^{(3)} - j2.83z_2^{(3)}z_3^{(3)}z_1^{(3)}z_2^{(3)} \\
- j1.08(z_1^{(3)})^3 - j1.42(z_2^{(3)})^2 - j1.08(z_3^{(3)})^3 - j1.42(z_2^{(3)})^2 \\
- j3.23(z_1^{(3)})^2 z_2^{(3)} - j3.23z_3^{(3)}(z_2^{(3)})^2 + O(z^{(3)})^4 \\
z_2^{(3)} &= \bar{T}_1^{(3)}(z_1^{(3)}) \\
z_3^{(3)} &= (-0.25 + j6.08)z_3^{(3)} - j1.52(z_3^{(3)})^2 z_4^{(3)} \\
- j0.51(z_3^{(3)})^3 - j0.86(z_3^{(3)})^2 - j0.51(z_3^{(3)})^3 - j0.86(z_4^{(3)})^2 \\
- j1.72z_3^{(3)}z_4^{(3)} - j1.52z_3^{(3)}(z_4^{(3)})^2 + O(z^{(3)})^4 \\
z_4^{(3)} &= \bar{T}_3^{(3)}(z_1^{(3)}) \\
\end{align*}
\]

(57)

\[
\begin{align*}
\dot{z}_1^{(4)} &= (-0.25 + j12.9)z_1^{(4)} + O(z^{(3)})^4 \\
- (0.0019 + j0.0975)(z_1^{(4)})^2 + (0.0057 - j0.097)(z_2^{(4)})^2 \\
+ (0.0038 - j0.195)z_1^{(4)}z_2^{(4)} \\
- (0.0144 + j0.372)(z_1^{(4)})^3 - (1.3e-4 + j0.99)(z_1^{(4)})^2 z_2^{(4)} \\
+ (0.048 - j1.11)z_1^{(4)}(z_3^{(4)})^2 + (0.02 - j0.262)(z_1^{(4)})^3 \\
z_2^{(4)} &= \bar{T}_1^{(4)}(z_1^{(4)}) \\
z_3^{(4)} &= (-0.25 + j6.08)z_3^{(4)} + O(z^{(3)})^4 \\
- (0.023 + j0.57)(z_3^{(4)})^2 + (0.07 - j0.566)(z_4^{(4)})^2 \\
+ (0.047 - j1.14)z_3^{(4)}z_4^{(4)} \\
- (0.008 + j0.098)(z_3^{(4)})^3 + (0.002 - j0.365)(z_3^{(4)})^2 z_4^{(4)} \\
+ (0.025 - j0.293)z_3^{(4)}(z_4^{(4)})^2 + (0.017 - j0.092)(z_3^{(4)})^3 \\
z_4^{(4)} &= \bar{T}_3^{(4)}(z_1^{(4)}) \\
\end{align*}
\]

(58)

Systems (57), (58) and (59) are respectively named ND- SMIB, ND-ST and NF. Their dynamical performances are compared with the same initial conditions under the post disturbance condition. The error of each simulated system response is calculated compared to the “true” system response, which is simulated from the 3rd Taylor expansion (56).

The errors \( e(t) \) of these trajectories in the time domain are calculated by (29) and shown in Table I for four disturbances with increasing fault duration times from 0.01s to 0.15s. The last one gives a marginally stable case. The simulated trajectories from these systems and their time domain errors are shown in Fig. 2 to Fig. 5. From those figures and Table I, the ND-ST has the smallest error and the ND-SMIB has the largest error.

Then, the stability of the original system is studied using the ND-ST system (58). Transform (58) into real-valued equations by (25) to give (60).

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\hline
FD & \text{ND-SMIB} & \text{ND-ST} & \text{NF} \\
\hline
0.01s & 0.95 & 1.06 & 0.07 & 0.08 & 0.17 & 0.18 \\
0.05s & 1.19 & 1.36 & 0.12 & 0.14 & 0.44 & 0.50 \\
0.10s & 3.93 & 4.36 & 0.40 & 0.46 & 2.41 & 2.74 \\
0.15s & 15.31 & 14.01 & 1.82 & 2.21 & 16.47 & 18.67 \\
\hline
\end{array}
\]

Table I: Time domain errors of simulated responses

\[
\begin{align*}
\text{E}[e(t)] \text{ and Std}[e(t)] & \text{ are the expectation and standard deviation of the error signal } e(t), \text{ which is in degrees.}
\end{align*}
\]

Fig. 2. Simulated system responses under the disturbance (fault duration = 0.01s) respectively by 3rd-order Taylor expansion, ND-SMIB, ND-ST and NF.

Fig. 3. Time domain error of the simulated system responses under the disturbance (fault duration = 0.01s) respectively by ND-SMIB, ND-ST and NF.

Fig. 4. Simulated system responses under the disturbance (fault duration = 0.15s) respectively by 3rd-order Taylor expansion, ND-SMIB, ND-ST and NF.
\[
\begin{align*}
\dot{w}_1 &= -0.5 w_1 - 166.0 w_2 + 5.0 w_2 + 35.3 w_3 + 0.015 w_1 w_2 \\
&\quad -0.139 w_1^2 + 0.016 w_1^2 w_2 + (1e-5) \cdot w_1^2 - (1e-4) \cdot w_1^3 \\
\dot{w}_2 &= w_1 + (2e-8) \cdot w_1^2 + 0.008 w_1^2 - (3e-5) w_2 w_2 \\
&\quad + (1e-4) \cdot w_2^3 + 0.05 w_3^3 + (3e-4) w_2^3 w_2 - 0.016 w_2 w_2^2 \\
\dot{w}_3 &= -0.5 w_3 - 37.1 w_3 + 13.8 w_3^2 + 5.13 w_3^2 - 0.186 w_3 w_4 \\
&\quad - 0.087 w_3^2 w_3 + 0.015 w_3^2 w_4 + (6e-4) \cdot w_3^3 + (1e-4) \cdot w_3^4 \\
\dot{w}_4 &= w_4 + (4e-6) \cdot w_1^2 + 0.093 w_3^2 - 0.0013 w_3 w_4 \\
&\quad - (3e-4) \cdot w_3^3 + 0.03 w_3^3 - (2e-4) w_3^2 w_4 - 0.015 w_3 w_4^2 \\
\end{align*}
\]

Simplify (60) to (61) using assumption in (52) and (53):
\[
\begin{align*}
\dot{w}_1 &= -166.0 w_2 + 5.0 w_2^2 + 35.3 w_3^3 \\
\dot{w}_2 &= w_1 \\
\dot{w}_3 &= -37.1 w_4 + 13.8 w_3^2 + 5.13 w_4^3 \\
\dot{w}_4 &= w_3 \\
\end{align*}
\]

Compare (61) with (60), the terms ignored according to (53) are actually either small or related to the damping effects. Thus, the stability analysis results on (61) may be conservative for systems in (60). Then, the first-integral based energy functions for the two modes are calculated to be (62).
\[
\begin{align*}
V_1(w_1, w_2) &= \frac{w_1^2}{2} - 83w_2^2 + 1.6667 w_3^3 + 8.825 w_2^4 \\
V_2(w_3, w_4) &= \frac{w_3^2}{2} - 18.55 w_4^2 + 4.6 w_3^3 + 1.2825 w_4^4 \\
\end{align*}
\]

Let the right hand side of (61) be zeros and solve for the UEPs and get \(w_{2,\text{UEP}} = 2.0986\) and \(w_{4,\text{UEP}} = 1.6618\). The critical energy for the two modes are \(V_1(0, w_{2,\text{UEP}}) = 178.9041\) and \(V_2(0, w_{4,\text{UEP}}) = 20.3363\). Under different fault durations, the initial energy of the decoupled systems is shown in Table II, which tells that the initial energy of the system corresponding to the second mode first exceeds its critical energy when the fault duration reaches 0.16s while the initial energy corresponding to the first mode is always much smaller than its critical energy. Table II also shows that the CCT found by this analysis is 0.15s, which is fairly accurate when compared to 0.16s, the “true” CCT of the 3rd order Taylor expansion system.

### Table II

| Initial Energy of NDST Systems Under Different Fault Durations | \(V_1(0, w_{2,\text{UEP}})\) | \(V_2(0, w_{4,\text{UEP}})\) |
|-------------------------------------------------------------|--------------------------|--------------------------|
| 0.01                                                       | 0.0022                   | 3.2072                   |
| 0.05                                                       | 0.0128                   | 4.6495                   |
| 0.10                                                       | 0.1173                   | 9.6173                   |
| 0.15                                                       | 0.6785                   | 19.402                   |
| 0.16                                                       | 0.8643                   | 22.092                   |

Another benefit of the nonlinear modal decoupling is that the trajectory of each decoupled system can be drawn in the corresponding coordinates as a trajectory only about one mode. In that sense, the original system’s trajectories regarding different modes are also nonlinearly decoupled. For the marginally stable case with fault duration \(= 0.15s\), Fig. 6 plots the trajectories of the original system in different coordinates while Fig. 7 visualizes the modal trajectories in the coordinates about each decoupled mode. In this case, both oscillatory modes of the system are excited, so the original trajectories are tangled. However, the trajectory on each decoupled system is clean and easier to analyze.

**B. Test on the New England 39-bus system**

This subsection will test the proposed nonlinear modal decoupling on the New England 10-machine, 39-bus power system [50]. Using the 2nd order Taylor expansion of the 20 nonlinear differential equations and the two assumptions in Section III, two sets of decoupled 2-jets can be obtained. A three-phase fault is added on bus 16 and cleared after 0.2 second by tripping the line 15-16. With the same initial condition under this fault, the 2nd order Taylor expansion of the original system, the two decoupled 2-jets, and the 2nd order normal form are simulated and compared in the original space, as shown in Fig. 9 and Fig. 10. Similar to the case study on the IEEE 9-bus system, the error of ND-ST is the smallest among the three.
step in deriving the nonlinear modal decoupling is the elimination of the inter-modal terms and the retention of non-linearities only related to the intra-modal terms. The elimination of inter-modal terms can be achieved uniquely, while the intra-modal terms could be maintained in an infinite number of ways such that a desired form has to be specified.

Then, the nonlinear modal decoupling analysis is applied to power systems toward two forms of decoupled systems: (i) the single-machine-infinite-bus (SMIB) assumption; (ii) the small transfer (ST) assumption. Note that the ST assumption does not limit mode-decoupled systems to the power system models; rather, they can be any other type of oscillator systems if a priori knowledge or preference on the form of mode-decoupled systems is not available. Numerical studies on both a small IEEE 3-machine, 9-bus system, and a larger New England 10-machine, 39-bus system, show that the decoupled system under the ST assumption has a larger validity region than the decoupled systems under the SMIB assumption and the transformed linear system from the normal form method. It is also demonstrated that the decoupled systems can enable easier and fairly accurate analyses, e.g. on stability of the original system.

Future work includes calculation of validity region $\Omega^{(k)}$, finding the best way to maintain the intra-modal terms to achieve the largest validity region, the stability analysis and controller design based on the nonlinear modal decoupling.

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