DOMAIN PERTURBATIONS FOR ELLIPTIC PROBLEMS WITH ROBIN BOUNDARY CONDITIONS OF OPPOSITE SIGN

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Abstract. The energy of the torsion problem with Robin boundary conditions is considered in the case where the solution is not a minimizer. Its dependence on the volume of the domain and the surface area of the boundary is discussed. In contrast to the case of positive elasticity constants, the ball does not provide a minimum. For nearly spherical domains and elasticity constants close to zero, the energy is the largest for the ball. This result is true for general domains in the plane under an additional condition on the first nontrivial Steklov eigenvalue. For more negative elasticity constants the situation is more involved and is strongly related to the particular domain perturbation. The methods used in the paper are the series representation of the solution in terms of Steklov eigenfunctions, the first and second shape derivatives, and an isoperimetric inequality of Payne and Weinberger for the torsional rigidity.

§1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let $\nu$ denote its outer normal. In this paper we study the Poisson problem

\[ \Delta u + 1 = 0 \quad \text{in} \; \Omega, \quad \partial_\nu u = \alpha u \quad \text{on} \; \partial \Omega. \]

It is the Euler–Lagrange equation corresponding to the energy functional

\[ E(V, \Omega) := \int_\Omega |\nabla V|^2 \; dy - \alpha \oint_{\partial \Omega} V^2 \; dS - 2 \int_\Omega V \; dy. \]

If $\alpha < 0$, there exists a unique solution $u(x)$ that minimizes the energy among all functions in $W^{1,2}(\Omega) \cap L^2(\partial \Omega)$. In this case, Bucur and Giacomini [7] showed that among all domains of given volume the ball has the smallest energy. This property is well known if $u$ satisfies Dirichlet conditions and follows immediately by symmetrization. The presence of Robin boundary conditions requires completely new arguments.

In this study we are interested in the case where $\alpha > 0$. The motivation comes from the eigenvalue problem $\Delta \varphi + \lambda \varphi = 0$ in $\Omega$, $\partial_\nu \varphi = \alpha \varphi$ on $\partial \Omega$, considered for the first time by Bareket [5]. She observed that for nearly circular domains of a given area the disk has the largest first eigenvalue. Recently this result was extended in [11] to higher dimensions for nearly spherical domains by Ferone, Nitsch, and Trombetti (cf. also [4]). The question whether or not the ball is optimal for all domains of the same volume remained open until recently when Freitas and Krejčiřík [12] showed that for large $\alpha$ the annuli have a larger eigenvalue than the ball with the same volume.

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If $\alpha > 0$, Problem (1.1) is not always solvable. In fact, if $\alpha$ coincides with an eigenvalue $0 = \mu_1 < \mu_2 \leq \ldots$ of the Steklov problem
\[ \Delta \phi = 0 \quad \text{in } \Omega, \quad \partial_\nu \phi = \mu \phi \quad \text{on } \partial \Omega, \]
then problem (1.1) has a solution if and only if the compatibility condition
\[ \int_\Omega \phi_i \, dy = 0 \]
is satisfied for all eigenfunctions corresponding to $\mu_i = \alpha$. If (1.3) is true, then (1.1) is solvable, but a solution is not unique.

If $\alpha \neq \mu_i$, then there exists a unique solution. It is a critical point of $E(V, \Omega)$ in $W^{1,2}(\Omega) \cap L^2(\partial \Omega)$ in the sense that the Fréchet derivative vanishes. However, in contrast to the case of $\alpha < 0$, the nature of the critical point highly depends on the dimension. For $0 < \alpha R < 1$, the ball of radius $R$ is a strict local maximizer for all dimensions.

Our goal in this paper is to investigate $E(u, \Omega)$ for all domains with given volume. In contrast to the case where $\alpha$ is negative, the ball has in general not the smallest energy. By using the shape derivative and a result of Serrin [18] for overdetermined boundary-value problems, it can be shown that the ball is the only critical domain. The analysis of the second shape derivative reveals that for nearly spherical domains and for $\alpha$ sufficiently small the energy is larger or smaller than that of the ball, depending on the perturbation. The most surprising result in this context is that, for $\alpha$ close to zero, the ball has the largest energy among all domains of given volume. This phenomenon is related in a wider sense to the anti-maximum principles, see [8].

This paper is organized as follows. First, we use the Steklov eigenfunctions to derive a series representation of the energy, which will be used for deriving global estimates. This is the content of §3. Then we discuss the first shape derivative for general domains and the second shape derivative for nearly spherical domains. At the end, we prove the optimality of the disk in two dimensions.

\section*{§2. Preliminaries}

The Steklov eigenvalues and eigenfunctions will play a crucial role in our considerations. They belong to the Sobolev space $W^{1,2}(\Omega)$ and since $\partial \Omega$ is Lipschitz continuous, they have a trace in $L^2(\partial \Omega)$. The eigenfunctions can be chosen so that
\[\int_{\partial \Omega} \phi_i \phi_j \, dS = \delta_{ij}, \quad \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \, dy = 0 \quad \text{if } i \neq j, \quad \text{and} \quad \int_\Omega |\nabla \phi_i|^2 \, dy = \mu_i. \tag{2.1}\]
Moreover, every harmonic function $h$ in $\Omega$ with a trace in $L^2(\partial \Omega)$ can be expanded in a series of Steklov eigenfunctions, which converges in $W^{1,2}(\Omega)$. It should be mentioned that, by a result of Mazya [15], in a Lipschitz domain the norms corresponding to the inner products $\langle u, v \rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx$ and $\langle u, v \rangle_{\partial \Omega} = \int_\Omega \nabla u \cdot \nabla v \, dx + \oint_{\partial \Omega} uv \, dS$ are equivalent.

In the next lemma we show how to expand harmonic functions in a series of Steklov eigenfunctions.

\begin{lemma}
(i) Suppose that $\alpha \in \mathbb{R}$ does not coincide with a Steklov eigenvalue $\mu_i$. Let $h$ be the solution of the problem
\[ \Delta h = 0 \quad \text{in } \Omega, \quad \partial_\nu h = \alpha h + g(x) \quad \text{on } \partial \Omega. \tag{2.2}\]
\end{lemma}
Then $h = \sum_1^\infty h_i \phi_i$, where

$$h_i = \frac{\oint_{\partial \Omega} \phi_i g dS}{\mu_i - \alpha}.$$  

This series converges in $W^{1,2}(\Omega) \cap L^2(\partial \Omega)$.

(ii) Assume that $\alpha = \mu_k$ and denote by $\mathcal{L}_k$ the linear space generated by the Steklov eigenfunctions belonging to the eigenvalue $\mu_k$. A solution exists if and only if the compatibility condition

$$\oint_{\partial \Omega} g \phi_k dS = 0 \quad (2.3)$$

is satisfied for all $\phi_k \in \mathcal{L}_k$. In this case (2.2) has infinitely many solutions, which are expressed as

$$h = \sum_{\phi_i \notin \mathcal{L}_k} h_i \phi_i + \mathcal{L}_k$$

with $h_i$ given as in (i).

Proof. The completeness of the Steklov eigenfunctions allows us to write $h = \sum_1^\infty h_i \phi_i$. Testing (2.2) with $\phi_j$ and using (2.1), we get

$$0 = \int_\Omega \phi_j \Delta h dy = \oint_{\partial \Omega} (\phi_j \partial_\nu h - h \mu_j \phi_j) dS = (\alpha - \mu_j) h_j + \oint_{\partial \Omega} \phi_j g dS.$$

This proves the first assertion. Convergence follows from the results of [15, 1]. The second statement is a consequence of the classical theory of inhomogeneous linear problems. \[\square\]

For the next considerations we decompose the solution $u$ of (1.1) as $u = h + s$, where $s$ is the solution of the Dirichlet problem

$$(2.4) \quad \Delta s + 1 = 0 \text{ in } \Omega, \quad s = 0 \text{ on } \partial \Omega.$$

Then $h$ is a solution of (2.2) with $g = -\partial_\nu s$.

A straightforward computation shows that

$$E(u, \Omega) = -\int_\Omega u dy = -\int_\Omega (h + s) dy = -\sum_1^\infty h_i \int_\Omega \phi_i dy - \int_\Omega s dy.$$\[\text{Observe that}\]

$$-\int_\Omega s dy = \min_{W_0^{1,2}(\Omega)} \int_\Omega (|\nabla V|^2 - 2V) dy =: T(\Omega).$$

Moreover, we have

$$-\sum_1^\infty h_i \int_\Omega \phi_i dy = \sum_1^\infty \frac{\oint_{\partial \Omega} \phi_i \partial_\nu s dS}{\mu_i - \alpha} \int_\Omega \phi_i dy$$

and

$$\int_\Omega \phi_i dy = -\int_\Omega \phi_i \Delta s dy = -\oint_{\partial \Omega} \phi_i \partial_\nu s dS.$$

Hence,

$$E(u, \Omega) = T(\Omega) + \sum_1^\infty \frac{(\oint_{\partial \Omega} \phi_i \partial_\nu s dS)^2}{\alpha - \mu_i}. \quad (2.5)$$

Theorem 1. Assume that $\mu_p < \alpha < \mu_{p+1}$. Set

$$\mathcal{E}^+ := \sum_1^p \frac{(\oint_{\partial \Omega} \phi_i \partial_\nu s dS)^2}{\alpha - \mu_i} \geq 0 \quad \text{and} \quad \mathcal{E}^- := \sum_{p+1}^\infty \frac{(\oint_{\partial \Omega} \phi_i \partial_\nu s dS)^2}{\alpha - \mu_i} \leq 0.$$

Then the following statements hold true.
Remark. The series expansion (2.5) is also valid for negative $\alpha$. $\Omega$

Proof. The first assertion follows from (2.5). Replacing $v \in L_p$ by its series $\sum_i v_i \phi_i$, we find

$$H(v) = \sum_i v_i^2(\mu_i - \alpha) + 2 \sum_i v_i \int_{\partial\Omega} \phi_i \partial_{\nu}s dS.$$ 

By assumption, we have $\mu_i - \alpha < 0$. This implies that the positive maximum is attained

$$E^+ = \max_{L_p} H(v) \quad \text{and} \quad E^- = \min_{L_p^\infty} H(v) = \min_v H(v),$$

where $\int_{\partial\Omega} v \phi_i dS = 0$ for $i = 1, 2, \ldots, p$.

3. If $\alpha = \mu_i$, then (1.1) has a solution if and only if

$$\int_{\Omega} \phi_i dx = -\int_{\partial\Omega} \phi_i \partial_{\nu}s dS = 0$$

for all eigenfunctions belonging to $\mu_i$.

In the sequel $h_i$ stands for the Fourier coefficient of $h$ in the decomposition $u = s + h$.

Remark. The series expansion (2.5) is also valid for negative $\alpha$. In this case $E^+ = 0$ and, therefore, $E(u, \Omega) = T(\Omega) + E^-$.

Examples. 1. Let $\Omega = B_R$ be the ball of radius $R$ centered at the origin. The Steklov eigenfunctions for the ball are of the form $r^k X_k(\theta)$, where $\theta \in \partial B_1$ and $X_k(\theta)$ are the spherical harmonics of degree $k$. The eigenvalues are $\mu = \frac{k}{R}$, $k \in \mathbb{N}$, and their multiplicities are $\frac{(k+n-1)!}{k!(n-1)!}$. By the maximum principle for harmonic functions, $\phi_1 = \text{const}$ is the only radial eigenfunction. Here $s = -\frac{R^2}{2n} - \frac{r^2}{2n}$ and, thus,

$$h_1(\mu_i - \alpha) = -\int_{\partial B_R} \phi_i \partial_{\nu}s dS = 0 \quad \text{for all} \quad i > 1.$$ 

Consequently, (1.1) has a solution for all $\alpha \neq 0$. It is of the form

$$u = \begin{cases} \frac{R^2}{2n} - \frac{r^2}{2n} \quad \text{if} \quad \alpha \neq \mu_j, \\ \frac{R^2}{2n} - \frac{r^2}{2n} + w \quad \text{if} \quad \alpha = \mu_j, \end{cases}$$

where $w$ is any function in the eigenspace of $\mu_j$. In both cases we get

$$E(u, B_R) = T(B_R) + \frac{|B_R|^2}{\alpha|\partial B_R|} = |B_R| \left( -\frac{R^2}{n(n+2)} + \frac{R}{\alpha n} \right).$$

2. Let $\Omega = \{y : r_0 < |y| < R\}$ be an annulus and set $r_0 = \kappa R$. Suppose for simplicity that $\Omega \subset \mathbb{R}^n$, $n > 2$. The radial solutions of (1.1) are of the form

$$u = \frac{r^2}{2n} + c_1 + \frac{c_2}{r^{n-2}}.$$
The boundary conditions lead to the linear system

\begin{equation}
\begin{aligned}
c_1 \alpha + c_2 \left( \frac{\alpha}{R_{n-2}} + \frac{n-2}{R_{n-1}} \right) &= \frac{\alpha R^2}{2n} - \frac{R}{n}, \\
c_1 \alpha + c_2 \left( \frac{\alpha}{(kR)^{n-2}} - \frac{n-2}{(kR)^{n-1}} \right) &= \alpha (kR)^2 \frac{2n}{n} + \frac{\kappa R}{n}.
\end{aligned}
\end{equation}

This system has a unique solution if the determinant is different from zero. The determinant vanishes if

\[ \alpha = \alpha_1 = 0 \quad \text{and} \quad \alpha = \alpha_2 = \frac{n-2}{R} \left( \frac{k^{1-n} + 1}{k^{2-n} - 1} \right). \]

The eigenfunctions of the Steklov problem in the annulus are similar to those for the ball, namely, they are \((c_1 r^k + c_2 r^{-k-n+2})X_k(\theta), k = 1, \ldots\) In addition to \(\phi_1 = \text{const}\), there is a radial eigenfunction \(\phi_r = c_1 + c_2 r^{-n} \) with \(c_1 \mu_r + c_2 \left( \frac{\mu_r}{n-2} + \frac{n-2}{R^2} \right) = 0\) and \(c_1 \mu_r + c_2 \left( \frac{\mu_r}{(kR)^2} - \frac{n-2}{(kR)^{n-1}} \right) = 0\). Notice that \(\alpha_1\) and \(\alpha_2\) correspond to the Steklov eigenvalues \(\mu_1\) and \(\mu_r\) of the radial eigenfunctions. For \(\kappa \neq 1\) the inhomogeneous linear system (2.7) is not solvable if \(\alpha = \mu_r\). Hence, the Fourier coefficient \(h_r\) is not defined. From the symmetry of the annulus it follows that \(h_k = 0\) for all \(k \neq 1, r\).

The same argument as for the ball shows that, for problem (1.1) in an annulus,

- there exists a unique solution if \(\alpha \neq \mu_i\);
- there is no solution if \(\alpha = \mu_r\);
- there is a family of solutions of the form \(-\frac{c_1}{2n} + c_1 + c_2 r^{-n} + w\) where \(w\) is in the eigenspace of \(\mu_i\) if \(\alpha = \mu_i\) and \(\mu_i \neq \mu_r\).

Therefore, by Theorem 1 for the annulus we obtain

\begin{equation}
E(u, \Omega) = T(\Omega) + \frac{|\Omega|^2}{\alpha |\partial \Omega|} + \frac{h_r^2}{\alpha - \mu_r} \quad \text{for all} \quad \alpha \neq 0
\end{equation}

and \(\alpha \neq \mu_r = \frac{n-2}{R^{n-2}} \left( \frac{k^{1-n} + 1}{k^{2-n} - 1} \right)\).

§3. Global estimates

3.1. General estimates. Theorem 1 shows that for all \(\alpha \neq \mu_i\) we have \(E(u, \Omega) = T(\Omega) + \mathcal{E}^+ + \mathcal{E}^\pm\). Many estimates are known for \(T(\Omega)\), which is related to the torsion. The expressions \(\mathcal{E}^\pm\) are less known and more difficult to estimate. We start with the observation that

\[ a_i := \oint_{\partial \Omega} \phi_i \partial_{\nu}s \, dS \]

is the Fourier coefficient of \(\partial_{\nu}s\) with respect to the Steklov eigenfunction \(\phi_i\). We write

\[ \partial_{\nu}s = \sum_{1}^{p} a_i \phi_i + \sum_{p+1}^{\infty} a_i \phi_i \]

and set for short \(\|v\| := \|v\|_{L^2(\partial \Omega)}\). Then \(\|\partial_{\nu}s^+\|^2 = \sum_{1}^{p} a_i^2\) and \(\|\partial_{\nu}s^-\|^2 = \sum_{p+1}^{\infty} a_i^2\). Under the assumption \(0 \leq \mu_p < \alpha < \mu_{p+1}\) it follows immediately that

\[ \alpha^{-1} \|\partial_{\nu}s^+\|^2 \leq \mathcal{E}^+ \leq (\alpha - \mu_p)^{-1} \|\partial_{\nu}s^+\|^2, \]

\[ (\alpha - \mu_{p+1})^{-1} \|\partial_{\nu}s^-\|^2 \leq \mathcal{E}^- \leq (\alpha - \mu_m)^{-1} \sum_{p+1}^{m} a_i^2. \]
Application. If $\alpha = -c^2$ is negative, we have $E^+ = 0$, and therefore,

$$E^- \geq \alpha^{-1} \|\partial_\nu s\|^2_{L^2(\partial\Omega)}.$$  

Hence,

$$E(u, \Omega) \geq T(\Omega) - c^{-2} \|\partial_\nu s\|^2.$$  

Equality occurs for balls. Schwarz symmetrization shows immediately that $T(\Omega) \geq T(B_R)$, where $B_R$ is a ball with the same volume as $\Omega$. Moreover, $\int_\Omega |\nabla s_\Omega|^2 \, dx \leq \int_{B_R} |\nabla s_{B_R}|^2 \, dx$. However, it is not clear whether $\|\partial_\nu s\|_{L^2(\partial\Omega)} \leq \|\partial_\nu s\|_{L^2(\partial B_R)}$, which would prove that the ball has the smallest energy. Pointwise estimates for $|\nabla s|^2$ are well known in the literature, see [13, 17].

3.2. Let $0 < \alpha < \mu_2(\Omega)$.

In this case, by Theorem 1 we have

$$E^+ = \alpha^{-1} \left( \int_{\partial\Omega} \phi_1 \partial_\nu s \, dS \right)^2.$$  

Since $\phi_1 = \frac{1}{\sqrt{|\partial\Omega|}}$, we find

$$E^+ = \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}.$$  

Together with Theorem 1, this leads to the next statement.

**Lemma 2.** Assume $0 < \alpha < \mu_2(\Omega)$. Then

$$E(u, \Omega) \leq T(\Omega) + \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}.$$  

Equality occurs for the ball.

If $a_i = 0$ for $i = 1, \ldots, r$, like, e.g., for the annulus, then the estimate above is true for $0 < \alpha < \mu_r(\Omega)$.

An interesting question is to find an isoperimetric upper bound for

$$J(\Omega) := T(\Omega) + \frac{|\Omega|^2}{\alpha|\partial\Omega|^2}.$$  

If the volume $|\Omega| = |B_R|$ is fixed then, as was mentioned before, Schwarz symmetrization implies that $T(\Omega) \geq T(B_R)$, whereas $\frac{|\Omega|^2}{\alpha|\partial\Omega|^2} \leq \frac{|B_R|^2}{\alpha|\partial B_R|^2}$. The question arises which inequality prevails.

**Proposition 1.** Let $\Omega \neq B_R$ be a fixed domain in $\mathbb{R}^n$ such that $|\Omega| = |B_R|$. Then there exists a positive number $\alpha_0(\Omega) > 0$ such that

$$J(\Omega) \begin{cases} < J(B_R) & \text{if } \alpha < \alpha_0, \\ > J(B_R) & \text{if } \alpha > \alpha_0. \end{cases}$$  

**Proof.** It is well known that $T(\Omega) - T(B_R) = \epsilon_0 > 0$ for any domain different from a ball. Define $\alpha_0 = \frac{|B_R|^2}{\epsilon_0} \left( \frac{1}{|\partial B_R|} - \frac{1}{|\partial\Omega|} \right)$. Then the assertion follows. \qed

**Remarks.**

1. An estimate sharper than in Lemma 2 can be derived from Theorem 1 (2). In fact,

$$E(u, \Omega) \leq T(\Omega) + \frac{|\Omega|^2}{\alpha|\partial\Omega|} + H(V),$$  

where $V$ is any trial function such that $\int_{\partial \Omega} V \, dS = 0$. Observe that if $V$ is an admissible trial function, then so is $tV$ for any $t \in \mathbb{R}$. Thus,

$$\mathcal{E}^- \leq \min_{\mathbb{R}} H(tV) = -\frac{\left(\int_{\Omega} V \partial_\nu s \, dS\right)^2}{\int_{\Omega} |\nabla V|^2 \, dy - \alpha \int_{\partial \Omega} V^2 \, dS}.$$ 

Suppose that the origin is the barycenter with respect to $\partial \Omega$, i.e., $\int_{\partial \Omega} x_i \, dS = 0$ for $i = 1, \ldots, n$. Then $x_i$ is admissible for the variational characterization of $\mathcal{E}^-$. By our assumption, $\int_{\Omega} |\nabla x_i|^2 \, dy \geq \mu_2 \int_{\partial \Omega} x_i^2 \, dS > \alpha \int_{\partial \Omega} x_i^2 \, dS$. Consequently,

$$\mathcal{E}^- \leq -\frac{\sum_{i=1}^{n} (\int_{\Omega} x_i \, dy)^2}{n|\Omega| - \alpha \int_{\partial \Omega} |x|^2 \, dS}.

2. By the Brock–Weinstock inequality (see [19, 6]), we have $\mu_2(\Omega) \leq \mu_2(B_\rho)$, where $B_\rho$ is a ball of the same volume as $\Omega$. Thus, if $|\Omega|$ is large, then $\mu_2(\Omega)$ is small.

3.3. Let $\mu_p(\Omega) < \alpha < \mu_{p+1}(\Omega)$.

This case is more involved. From Theorem\[1\] it follows that $E(u, \Omega) = T(\Omega) + \mathcal{E}^+ + \mathcal{E}^-$. Rough estimates are obtained from (3.1).

Observe that if the Fourier coefficient $a_p = \int_{\partial \Omega} \phi_p \partial_\nu s \, dS$ is nonzero, then $\mathcal{E}^+$ is positive and becomes arbitrarily large as $\alpha$ tends to $\mu_p$ from above. Similarly, if the Fourier coefficient $\int_{\partial \Omega} \phi_{p+1} \partial_\nu s \, dS$ is nonzero, then $\mathcal{E}^-$ is negative and becomes arbitrarily small if $\alpha$ tends to $\mu_{p+1}$ from below.

Examples.

1. In a ball, $E(u, \Omega; \alpha)$ has only one pole $\alpha = \mu_1 = 0$. Hence,

$$\lim_{\alpha \searrow 0} E(u, \Omega; \alpha) = \infty \quad \text{and} \quad \lim_{\alpha \nearrow 0} E(u, \Omega; \alpha) = -\infty.$$

2. In an annulus, $E(u, \Omega; \alpha)$ has two poles $\alpha = \mu_1 = 0$ and $\alpha = \mu_2$, see (2.8).

§4. Domain variations

4.1. First domain variation.

4.1.1. General remarks. Let a family of perturbations of the domain $\Omega$ be given by

$$\Omega_t = \left\{ y : y = x + tv(x) + t^2 w(x) + o(t^2) : x \in \Omega \right\},$$

where $v$ and $w$ are smooth vector fields $v, w : \Omega \to \mathbb{R}^n$ belonging to $C^{2,\epsilon}(\Omega)$.

We assume that, on $\partial \Omega$, the vector $v$ points in the normal direction, i.e., $v = (\nu \cdot v)\nu$. The parameter $t$ belongs to $(-t_0, t_0)$, where $t_0$ is chosen so small that $y : \Omega \to \Omega_t$ is a diffeomorphism. We consider the family of problems

$$\Delta_y u(y, t) + 1 = 0 \quad \text{in} \quad \Omega_t, \quad \partial_\nu u(y, t) = \alpha u(y, t) \quad \text{on} \quad \partial \Omega_t,$$

where $\nu_t$ is the outer unit normal of $\Omega_t$. For short, we set

$$\tilde{u}(t) := u(y(x, t), t) \quad \text{for} \quad x \in \Omega, \quad |t| < t_0.$$

Now we map this problem via $y(x, t)$ to $\Omega$; after the change of variable $y \to x$, we get the following problem:

$$\partial_j \left( A_{ij}(x) \partial_i \tilde{u}(t) \right) + J(t) = 0 \quad \text{in} \quad \Omega, \quad \partial_\nu \tilde{u}(t) = \alpha m(x, t) \tilde{u}(t) \quad \text{on} \quad \partial \Omega,$$
where
\[ \partial_t = \frac{\partial}{\partial x_t}, \quad dy = J(t)dx, \quad dS_y = m(t)dS_x, \]

and
\[ A_{ij}(t) := \frac{\partial x_i}{\partial y_k} \frac{\partial x_j}{\partial y_k} J(t), \quad \partial_{\nu_A} = \nu_i A_{ij} \partial_j. \]

In [3] it was shown that for small \(|t|\) we have
\[ J(t) := \det \left( I + t D_v + \frac{t^2}{2} D_w \right) \]
(4.3)
\[ = 1 + t \text{div} v + \frac{t^2}{2} ((\text{div} v)^2 - D_v : D_v + \text{div} w) + o(t^2). \]

Here we have used the notation
\[ D_v : D_v := \partial_i v_j \partial_j v_i, \]
where summation over repeated indices is understood. Furthermore,
\[ m(t) = 1 + t(n - 1)(v \cdot \nu)H + o(t), \]
where \(H\) is the mean curvature of \(\partial \Omega\). In [3] we also showed that
\[ A_{ij}(0) = \delta_{ij}; \]
\[ \dot{A}_{ij}(0) = \text{div} v \delta_{ij} - \partial_j v_i - \partial_i v_j; \]
\[ \ddot{A}_{ij}(0) = \left( (\text{div} v)^2 - D_v : D_v \right) \delta_{ij} + 2(\partial_k v_i \partial_j v_k + \partial_k v_j \partial_i v_k) \]
\[ + 2 \partial_k v_i \partial_k v_j - 2 \text{div} v (\partial_j v_i + \partial_i v_j) + \text{div} w \delta_{ij} - \partial_i w_j - \partial_j w_i. \]

Similarly we can transform the Steklov problem. In terms of the x-coordinates it reads as
\[ (4.4) \quad L_A \phi(t) = 0 \quad \text{in} \quad \Omega, \quad \partial_{\nu_A} \phi(t) = \mu(t)m(t)\phi(t) \quad \text{on} \quad \partial \Omega, \quad L_A := \partial_j (A_{ij} \partial_i). \]

The next lemma is well known, see, e.g., [14, IV, 3.5] or [9, VI, 6].

**Lemma 3.** Suppose that \(\mu_p(\Omega) < \alpha < \mu_{p+1}(\Omega)\). Then there exists \(t_0 > 0\) such that \(\mu_p(\Omega_t) < \alpha < \mu_{p+1}(\Omega_t)\) for all \(t \in (-t_0, t_0)\).

**Proof.** By the min-max principle, we have
\[ \mu_p(\Omega_t) = \min_{L_p} \max_{V \in L_p} \frac{\int_{\Omega} \nabla V \cdot A(t) \nabla V dy}{\int_{\partial \Omega} V^2 m(t) dS_y}, \]
where \(L_p\) is an \(n\)-dimensional linear space in \(W^{1,2}(\Omega)\).

We have the estimates
\[ |\nabla V|^2 (1 - c_1 |t|) < \nabla V \cdot A(t) \nabla V \leq |\nabla V|^2 (1 + c_1 |t|) \]
and
\[ V^2 (1 - c_2 |t|) < m(t)V^2 \leq V^2 (1 + c_2 |t|). \]

From the min-max principle it follows that \(|\mu_p(\Omega) - \mu_p(\Omega_t)| \leq tC\), where \(C\) depends on \(v\) and \(w\). \(\square\)

In the sequel we shall always assume that, for all \(t \in (-t_0, t_0)\), the number \(\alpha\) does not coincide with an eigenvalue of \(\Omega_t\).

Suppose that \(\Omega \in C^{2,\epsilon}, A_{ij}(t) \in C^{1,\epsilon}, J(t) \in C^{0,\epsilon}, \) and \(m(t) \in C^{1,\epsilon}\). We also assume that all the data are at least twice continuously differentiable in \(t\). Then by Schauder’s regularity theory [13] it follows that \(\hat{u}(t) \to \hat{u}(0) =: u(x) \in C^{2,\epsilon'}, \epsilon' < \epsilon\). Our assumptions imply that \(|\partial \Omega_t| \to |\partial \Omega|\), which is crucial for the convergence of the
eigenvalues. A general study of domain perturbations for elliptic problems with Robin boundary conditions was carried out by Dancer and Daners in [10].

4.1.2. First variation of the energy. Consider problem (1.1) in a class of domains \( \Omega \) described in (4.1). As before, the solutions of (1.1) in \( \Omega \) will be denoted by \( u(x) \). We shall use the abbreviation \( E(t) \) for \( E(\tilde{u}(y,t), \Omega_t) \). Under the conditions stated above, the solution \( \tilde{u}(y,t) \) of (1.1) is continuous and continuously differentiable in \( t \).

In [3] it was shown that the first domain variation \( \frac{d}{dt}E(t)|_{t=0} \) is given by

\[
\dot{E}(0) = \int_{\partial \Omega} (v \cdot \nu) \left[ |\nabla u|^2 - 2u - 2\alpha^2 u^2 - \alpha(n-1)u^2 H \right] dS.
\]

Example. If \( \Omega = B_R \), then

\[
\dot{E}(0) = \left( \frac{n+1}{\alpha n^2} - \frac{R^2}{n^2} \right) \int_{\partial B_R} (v \cdot \nu) dS.
\]

This leads to the following statement.

**Corollary 1.** Let \( \Omega_t \) be a family of nearly spherical domains with prescribed volume \( |\Omega_t| = |B_R| \). Then \( \dot{E}(0) = 0 \).

**Proof.** From (4.3) it follows that for volume preserving transformations we have

\[
\int_{\partial \Omega} (v \cdot \nu) dS = 0.
\]

Combined with (4.5), this establishes the assertion. \( \square \)

A further consequence of (4.5) is the local monotonicity property.

**Corollary 2.** If \( 0 < \alpha R < n+1 \) and \( |\Omega_t| > |B_R| \), then \( \dot{E}(0) > 0 \); otherwise, if \( \alpha R > n+1 \), then \( \dot{E}(0) < 0 \).

**Proof.** By our assumption we have \( \int_{\partial B_R} (v \cdot \nu) dS > 0 \). By (4.5), the sign of \( \dot{E}(0) \) depends on the sign of \( (n+1)\alpha R - (\alpha R)^2 \). \( \square \)

4.1.3. First variation of \( J(\Omega_t) \). In the case where \( 0 < \alpha < \mu_2(\Omega) \) (see Subsection 3.2), the energy \( E(\tilde{u}, \Omega_t) \) is bounded from above by \( J(\Omega) = T(\Omega) + \frac{|\Omega|^2}{\alpha |\partial \Omega|^2} \). Let \( S(t) = |\partial \Omega_t| \). If \( |\Omega_t| = |\Omega| \), then the first variation is given by

\[
\dot{J}(0) = \dot{T}(0) - \frac{|\Omega|^2}{\alpha |\partial \Omega|^2} \dot{S}(0),
\]

where

\[
\dot{T}(0) = -\int_{\partial \Omega} |\nabla s|^2 (v \cdot \nu) dS,
\]

\[
\dot{S}(0) = (n-1) \int_{\partial \Omega} (v \cdot \nu) H dS.
\]

Thus, for all critical domains, the solution \( s \) of (2.4) satisfies the additional boundary condition

\[
\frac{|\Omega|^2}{\alpha |\partial \Omega|^2} (n-1) H + |\nabla s|^2 = \text{const on } \partial \Omega.
\]

This is a direct consequence of (4.6). By Theorem 3 in [18] concerning overdetermined boundary value problems, the ball is the only domain for which \( s \) is constant on \( \partial \Omega \), and \( |\nabla s| = c(H) \) for a monotone nonincreasing function \( c \). Consequently, we get the following.
Theorem 2. For \( \alpha > 0 \), the ball is the only critical domain for the functional \( J(\Omega) \) among all domains of equal volume.

4.2. Second domain variation for nearly spherical domains.

4.2.1. Second variation for the energy. Corollary 1 gives rise to the following question: is \( E(u, B_R) \) a local extremum among the family \( \Omega_t, t \in (-t_0, t_0) \), of perturbed domains with the same volume as \( B_R \)? The answer will be obtained from the second variation.

Consider the family of nearly spherical domains \( \Omega_t := \{ y = x + tv(x) + t^2 w(x) : x \in B_R \} \). Let \( \tilde{u}(t) := u(y(x), t) \) be the solution of the problem \( \Delta u + 1 = 0 \) in \( \Omega_t \), \( \partial_\nu u = \alpha u \) on \( \partial \Omega_t \), transformed onto \( \Omega \). If \( \tilde{u}(t) \) is differentiable — this is the case when the data are Hölder continuous as described in the previous section and \( \alpha \neq \mu_i(\Omega_t) \) for all \( t \in (-t_0, t_0) \) — then

\[
\left. \frac{d}{dt} \tilde{u}(t) \right|_{t=0} = u'(x) + v \cdot \nabla u_0,
\]

where \( u = u_0 \) is the solution of (1.1) in \( B_R \).

It was shown in [3] that the shape derivative \( u' \) solves the inhomogeneous boundary value problem

\[
\begin{align*}
\Delta u' &= 0 \quad \text{in } B_R, \\
\partial_\nu u' &= \alpha u' + \left( \frac{1 - \alpha R}{n} \right) v \cdot \nu \quad \text{on } \partial B_R.
\end{align*}
\]

Let us assume that such a solution \( u' \) exists. This is certainly the case unless \( \alpha \) coincides with a Steklov eigenvalue \( \mu_i(B_R) \).

For the next result we consider perturbations which, in addition to condition (4.6), satisfy the volume preservation condition of the second order, namely,

\[
\int_{B_R} (\text{div} \ v)^2 - D_v : D_v + \text{div} w \ dx = 0.
\]

This formula can be simplified if \( v \) points to the normal direction only. It takes the form

\[
(n - 1) \int_{\partial \Omega} H(v \cdot \nu)^2 \, dS + \int_{\partial \Omega} (w \cdot \nu) \, dS = 0.
\]

Set

\[
Q(u') := \int_{B_R} |\nabla u'|^2 \, dx - \alpha \int_{\partial B_R} u'^2 \, dS.
\]

The following formula was derived in [3]. Recall that for nearly spherical domains we have \( \mathcal{E}(0) = E(u, B_R) \). Moreover, whenever \( \alpha \neq \mu_i(B_R) \), Lemma 3 implies that if \( t \) is sufficiently small, then \( \alpha \) never coincides with an eigenvalue \( \mu_j(\Omega_t) \).

Lemma 4. Assume that \( \alpha \neq \mu_i(B_R) \), and let the volume preservation conditions (4.6) and (4.13) be satisfied. Put \( S(t) := |\partial \Omega| \). Then

\[
\dot{\mathcal{E}}(0) = -2Q(u') + \frac{2R}{n^2} (1 - \alpha R) \int_{\partial B_R} (v \cdot \nu)^2 \, dS - \frac{R^2}{\alpha n^2} \dot{S}(0).
\]

For a ball, the second variation of the surface area is of the form

\[
\dot{S}(0) = \int_{\partial B_R} \left( |\nabla^* (v \cdot \nu)|^2 - \frac{(n - 1)}{R^2} (v \cdot \nu)^2 \right) \, dS,
\]

where \( \nabla^* \) stands for the tangential gradient on \( \partial B_R \).
4.2.2. Discussion of the sign of $\ddot{S}(0)$. For brevity, we write

$$F := -2Q(u') + \frac{2R}{n^2} (1 - \alpha R) \int_{\partial B_R} (v \cdot \nu)^2 dS.$$  

In order to estimate $F$, we consider the Steklov eigenvalue problem (1.2). An elementary computation yields $\mu_1 = 0$, and $\mu_k = \frac{k}{R}$ (for $k \geq 2$ and counted without multiplicity). The second eigenvalue $\mu_2 = 1/R$ has multiplicity $n$ and its eigenfunctions are $\frac{x_1}{R}, \ldots, \frac{x_n}{R}$.

From now on we shall count the eigenvalues $\mu_i$ with their multiplicity, i.e., $\mu_2 = \mu_3 = \ldots = \mu_{n+1} = 1/R$ and $\mu_{n+2} = 2/R$ etc.

Let $\{\phi_i\}_{i \geq 1}$ be the system of normalized Steklov eigenfunctions introduced in §2. The function $u'$ solves (2.2) with $g = \left(\frac{1-\alpha R}{n}\right)v \cdot \nu$. Hence, by Lemma 1 we have

$$u'(x) = \sum_{i=1}^{\infty} c_i \phi_i \quad \text{and} \quad (v \cdot \nu) = \sum_{i=1}^{\infty} b_i \phi_i.$$ 

Note that, since the first eigenfunction $\phi_1$ is a constant, the condition

$$0 = \oint_{\partial B_R} (v \cdot \nu) dS = \oint_{\partial B_R} \phi_1 (v \cdot \nu) dS$$

implies that $b_1 = 0$. From (4.12) we also have $c_1 = 0$. The coefficients $b_i$ for $i \geq 2$ are determined from the boundary value problem (4.11), (4.12). In fact,

$$b_i = \frac{n c_i (\mu_i - \alpha)}{1 - \alpha R} \quad \text{for} \quad i = 2, 3, \ldots$$

The orthonormality conditions (2.1) for the eigenfunctions imply

$$Q(u') = \sum_{i=2}^{\infty} c_i^2 (\mu_i - \alpha).$$

If we insert this in (4.16), we get

$$F = 2 \sum_{i=2}^{\infty} c_i^2 (\mu_i - \alpha)^2 \left[ \frac{R}{1 - \alpha R} - \frac{1}{\mu_i - \alpha} \right].$$

Since $\mu_2 = \ldots = \mu_{n+1} = \frac{1}{R}$, it follows that

$$F = 2 \sum_{i=n+2}^{\infty} c_i^2 (\mu_i - \alpha)^2 \left[ \frac{R}{1 - \alpha R} - \frac{1}{\mu_i - \alpha} \right]$$

$$= 2 \frac{(1 - \alpha R)^2}{n^2} \sum_{i=n+2}^{\infty} b_i^2 \left[ \frac{R}{1 - \alpha R} - \frac{1}{\mu_i - \alpha} \right].$$

Next we shall discuss the sign of $\dot{S}(0)$. Observe that

$$\mathcal{R} = \frac{\int_{\partial B_R} |\nabla^* \chi|^2 dS}{\int_{\partial B_R} \chi^2 dS}$$

is the Rayleigh quotient of the Laplace–Beltrami operator on $\partial B_R$. Its eigenvalues $\Lambda_i$ are $k(n - 2 + k)/R^2$, $k \in \mathbb{N}^+$. Observe that the multiplicity of this eigenvalue is the same as for the Steklov eigenvalue corresponding to $k/R$. Remember that for the volume preserving perturbations of the first order we have $\oint_{\partial B_R} (v \cdot \nu) dS = 0$ and, therefore, $(v \cdot \nu)$ is orthogonal to the first eigenfunction, which is a constant. Thus,

$$\mathcal{R}[(v \cdot \nu)] \geq \frac{n-1}{R^2}. $$
Equality occurs if and only if \((v \cdot \nu)\) belongs to the eigenspace spanned by \(\{\xi_i\}_{i=1}^n\). This does not occur if we exclude small translations. The rotations are excluded by the second order volume preservation condition \((4.13)\) (cf. Remark 1 in [3] for a more detailed discussion). Consequently, \(\tilde{S}(0) > 0\). This is consistent with the isoperimetric inequality.

If in \(\tilde{S}(0)\) we replace \((v \cdot \nu)\) by \(\sum_1^\infty b_i\phi_i\), we obtain

\[
(4.19) \quad \tilde{S}(0) = \sum_2^\infty b_i^2 \left( \Lambda_i - \frac{n - 1}{R^2} \right) > 0,
\]

where the \(\Lambda_i\) are counted in accordance with their multiplicity. From \((4.18)\) and \((4.19)\) we then get

\[
(4.20) \quad \tilde{\xi}(0) = \sum_{i=n+2}^\infty \frac{b_i^2}{\alpha n^2} \left\{ 2(1 - \alpha R)^2 \left( \frac{\alpha R}{1 - \alpha R} - \frac{\alpha}{\mu_i - \alpha} \right) - R^2 \Lambda_i + n-1 \right\}.
\]

Since the multiplicity of the Steklov eigenvalues and \(\Lambda_i\) depending on \(k\) is the same, we can replace \(\mu_i\) by \(k_i/R\) for a suitable integer \(k_i\) and \(\Lambda_i\) by \(k_i(k_i+n-2)/R^2\). Consequently,

\[
(4.21) \quad d_i = \frac{2\xi(1-\xi)(k_i-1)}{k_i-\xi} - k_i(k_i + n - 2) + n - 1,
\]

where \(\xi := \alpha R\) and \(k_i = 2, 3, 4, \ldots\).

Next we shall discuss the sign of \(d_i = d_i(\xi)\). It is easily seen that \(d_i(\xi)\) has two zeros

\[
Z_1(i) := \frac{1}{4} \left( k_i + n + 1 + \sqrt{-7k_i^2 - 6nk_i + 10k_i + n^2 + 2n + 1} \right),
\]

\[
Z_2(i) := \frac{1}{4} \left( k_i + n + 1 - \sqrt{-7k_i^2 - 6nk_i + 10k_i + n^2 + 2n + 1} \right).
\]

The root is real if and only if \(-7k_i^2 - 6nk_i + 10k_i + n^2 + 2n + 1 > 0\), and this is the case if and only if \(2 \leq k_i \leq R_1\), where

\[
R_1 := \frac{1}{7} \left( -3n + 5 + 4\sqrt{n^2 - n + 2} \right).
\]

It is easy to check that \(R_1 \geq 2\) if and only if \(n \geq 11\). Thus, for \(n \leq 10\) the sign of \(d_i(\xi)\) is constant in each component \(\{0 \leq \xi < k_i\}\) and \(\{\xi > k_i\}\). Since \(d_i(0) < 0\), and since \(d_i(\xi) > 0\) for \(\xi = k_i + \delta\) with small \(\delta\), we get the following lemma.

**Lemma 5.** For \(n \leq 10\) we have \(\tilde{\xi}(0) > 0\) for \(\xi > k_i\) and \(\tilde{\xi}(0) < 0\) for \(\xi < k_i\). In particular, the ball is a strict local maximum if and only if \(\xi < 2\).

We assume that \(n \geq 11\).

1. **The case where** \(\xi < k_i\). In this case \(d_i(0) < 0\) and \(\lim_{\xi \to k_i} d_i(\xi) = -\infty\). Moreover, \(d_i(\xi)\) has exactly one critical point \(\tilde{\xi} = k_i - \sqrt{k_i(k_i-1)}\). It is a strict maximum point for \(d_i\). A straightforward computation yields

\[
d_i(\tilde{\xi}) = -4(k_i-1)\sqrt{k_i(k_i-1)} + 3k_i^2 - (n+4)k_i + (n+1) < -k_i^2 - n + 9 < 0
\]

for \(n \geq 11\). (See Figure 1 for \(k_i = 2\) and \(n = 11\).)

2. **Lemma 6.** For \(n \geq 11\) and \(\xi < k_i\) we have \(d_i < 0\). Thus, the ball is a strict local maximum for the energy if \(\xi < 2\).
2. The case where $\xi > k_i$. In this case $\lim_{\xi \to \infty} d_i(\xi) = \infty$ and $\lim_{\xi \to k_i} d_i(\xi) = \infty$. Moreover, $d_i(\xi)$ has exactly one critical point $\xi = k_i + \sqrt{k_i(k_i - 1)}$. It is a strict minimum point for $d_i$. A straightforward computation yields 

$$d_i(\xi) = 4(k_i - 1)\sqrt{k_i(k_i - 1)} + 3k_i^2 - (n + 4)k_i + (n + 1).$$

There are values of $k_i$ for which this minimum is negative. Thus, $d_i$ changes its sign and the zeros are given by $Z_1(i)$ and $Z_2(i)$ (see, e.g., Figure 2 for $k_i = 2$ and $n = 11$, so that $Z_1(i) = 3$ and $Z_2(i) = 4$).

**Lemma 7.** For $n \geq 11$ and $\xi > k_i$, the coefficient $d_i$ changes its sign, depending on $\xi$.

(i) If $Z_1(i) \leq \xi \leq Z_2(i)$, then $d_i \leq 0$, and $d_i = 0$ if and only if $\xi = Z_1(i)$ or $\xi = Z_2(i)$.

(ii) If $k_i < \xi < Z_1(i)$ or $Z_2(i) < \xi$, then $d_i > 0$.

4.2.3. Example. Let $\Omega \subset \mathbb{R}^2$ be the ellipse whose boundary $\partial \Omega$ is given by

$$\{(R\cos(\theta), (1 + t)R\sin(\theta))\},$$

where $(r, \theta)$ are the polar coordinates in the plane. This ellipse has the same area as the circle $B_R$ and can be interpreted as a perturbation described in (4.1). We have $y = x + t(-x_1, x_2) + \frac{t^2}{2}(x_1, 0) + o(t^2)$. The eigenvalues and eigenfunctions of the Steklov eigenvalue problem (1.2) in $B_R$ are

$$\mu = \frac{k}{R} \quad \text{and} \quad \phi = r^k \left\{ \begin{array}{ll} a \cos(k\theta), \\ a \sin(k\theta), \end{array} \right.$$ 

where $a = \frac{1}{\sqrt{\pi R^{2k+1}}}$ is the normalization constant. We have

$$\langle v \cdot \nu \rangle = -R\cos(2\theta) = b_4 \phi_3,$$

and

$$\tilde{S}(0) = \int_{\partial B_R} \left( |\nabla^* (v \cdot \nu)|^2 - \frac{(v \cdot \nu)^2}{R^2} \right) dS = 3\pi R.$$

A straightforward computation yields

$$\tilde{E}(0) = \left[ -\frac{3}{4\alpha} + \frac{R(1 - \alpha R)}{2(2 - \alpha R)} \right] \int_{\partial B_R} (v \cdot \nu)^2 dS.$$
with $\int_{\partial B_R} (v \cdot \nu)^2 \, dS = \pi R^3$. This expression shows immediately that

$$\ddot{\mathcal{E}}(0) \begin{cases} > 0 & \text{if } \alpha R > 2, \\ < 0 & \text{if } \alpha R < 2. \end{cases}$$

This result is in agreement with Lemma 5. In this example, $b_i = 0$ for all $i \neq 4$. The sign of $\ddot{\mathcal{E}}(0)$ depends therefore on $d_4$. It changes the sign at $\alpha R = 2$.

As has already been mentioned, $b_1 = 0$ for all volume preserving perturbations. The coefficients $b_2, \ldots, b_{n+1}$ belong to the Steklov eigenvalue $\mu_2 = \ldots = \mu_{n+1} = 1/R$ and give no contribution to $\ddot{\mathcal{E}}(0)$. This is due to the fact that on $\partial B_R$ we have

$$\sum_{i=2}^{n+1} b_i \phi_i = \sum_{i=1}^{n} b_{i+1} c x_i = \vec{b} \cdot \nu,$$

where $\vec{b}$ is a constant vector. The presence of $b_i$ for $i = 2, \ldots, n + 1$ means that the perturbed domain $\Omega$ has been shifted by a vector $t\vec{b}$. Notice that such a shift does not affect the higher coefficients $b_k$, $k \geq n + 2$. Obviously, it leaves the energy invariant. Therefore, there is no loss in generality in assuming that

$$(4.22) \quad b_2 = b_3 = \ldots = b_{n+1} = 0.$$

This condition also implies that $c_2 = \ldots = c_{n+1} = 0$. Hence, problem (4.11), (4.12) is solvable for $\alpha R = 1$. This observation together with (4.20) implies the following result about the perturbations that are not pure translations or rotations.

**Theorem 3.**

1. Assume that $0 < \alpha R < 1$. Then

$$\ddot{\mathcal{E}}(0) \leq -\frac{n-0.5}{\alpha n^2} \int_{\partial B_R} (v \cdot \nu)^2 \, dS < 0.$$

2. Assume that $1 < \alpha R < 2$. Then

$$\ddot{\mathcal{E}}(0) \leq \frac{1}{\alpha^2} \left( \frac{2\alpha R(1-\alpha R)}{2-\alpha R} - n - 1 \right) \int_{\partial B_R} (v \cdot \nu)^2 \, dS < 0.$$

In both cases the energy is maximal for the ball among all nearly spherical domains of given volume.

In general, if $\alpha R > 2$, the energy $\mathcal{E}(t)$ has a saddle in $t = 0$.

**Theorem 4.** Assume that $n = 2, 3, 4$ and $k_p < \alpha R < k_{p+1}$. Let $\mathcal{L}_p$ be the linear space generated by the eigenfunctions $\phi_i$ belonging to the eigenvalues $\mu_i = 1/R, \ldots, k_p/R$, and let $\mathcal{L}_p^\perp$ be its complement generated by $\phi_i$ belonging to the remaining eigenvalues $\mu_i = k_{p+1}/R, \ldots, \infty$. Then

$$\ddot{\mathcal{E}}(0) \begin{cases} > 0 & \text{if } (v \cdot \nu) \in \mathcal{L}_p, \\ < 0 & \text{if } (v \cdot \nu) \in \mathcal{L}_p^\perp. \end{cases}$$

**4.2.4. The second variation of $\mathcal{J}(\Omega_t)$.** Like for $\ddot{\mathcal{E}}(0)$, for the functional $\mathcal{J}$ we can derive a formula for the second volume preserving domain variation. For the notation, see Subsection 4.1.3. Applying the rules of differentiation, we get

$$(4.23) \quad \ddot{\mathcal{J}}(0) = \ddot{\mathcal{F}}(0) + \frac{2|\Omega|^2}{\alpha|\partial\Omega|^2} \dddot{S}(0) - \frac{|\Omega|^2}{\alpha|\partial\Omega|^2} \dddot{S}(0).$$
By analogy with formulas (4.7)–(4.9), and using (4.14), we get

\[ \ddot{T}(0) = \int_{\partial\Omega} |\nabla s|^2 \left( (n-1)H(v \cdot \nu)^2 - (w \cdot \nu) \right) dS + 2 \int_{\Omega} |\nabla s'|^2 \, dx + 2 \int_{\partial\Omega} (v \cdot \nu)^2 \partial_s s \, dS, \]

where the shape derivative \( s' \) satisfies

\[ \Delta s' = 0 \text{ in } \Omega, \quad s' = -v \cdot \nabla s = v \cdot \nu |\nabla s| \text{ in } \partial\Omega. \]

Moreover, by formula (19) in [3] we have

\[ \ddot{S}(0) = \int_{\partial\Omega} |\nabla^* (v \cdot \nu)|^2 dS - \int_{\partial\Omega} (|A|^2 - (n-1)H^2) (v \cdot \nu)^2 dS + (n-1) \int_{\partial\Omega} (w \cdot \nu) H dS, \]

where \( |A|^2 = \sum_{i,j} (\partial_i^* \nu \cdot x_i \xi_j)(\partial_j^* \nu \cdot x_j \xi_i) \)

denotes the second fundamental form of \( \partial\Omega \).

From Subsection 4.1.3 we know that the ball is the only critical point of \( J \). For the ball \( B_R \) we have

\[ \ddot{S}(0) = 0, \]

\[ \ddot{S}(0) = \oint_{\partial B_R} \left( |\nabla^* (v \cdot \nu)|^2 - \frac{n-1}{R^2} (v \cdot \nu)^2 \right) dS \geq 0, \]

and

\[ s(x) = \frac{1}{2n} \left( R^2 - |x|^2 \right). \]

If \( R \) is chosen so that \( |\Omega_t| = |B_R| + o(t^2) \) for all \( t \in (-t_0, t_0) \), then

\[ \ddot{J}(0) = \frac{R^2}{n^2} \int_{\partial B_R} (H(v \cdot \nu)^2 - (w \cdot \nu)) dS + 2 \int_{B_R} |\nabla s'|^2 \, dx - \frac{2R}{n} \int_{\partial B_R} (v \cdot \nu)^2 dS - \frac{R^2}{\alpha n^2} \dot{S}(0). \]

Thus, the volume constraint (4.14) implies

\[ \ddot{J}(0) = -2 \frac{R}{n^2} \int_{\partial B_R} (v \cdot \nu)^2 dS + 2 \int_{B_R} |\nabla s'|^2 \, dx - \frac{R^2}{\alpha n^2} \dot{S}(0). \]

If we use (4.25) to eliminate \( (v \cdot \nu) \), we can write \( \ddot{J}(0) \) as a functional of \( s' \) alone:

\[ \ddot{J}(0) = I(s') := 2 \int_{B_R} |\nabla s'|^2 \, dx - \frac{2}{R} \int_{\partial B_R} s'^2 dS - \frac{1}{\alpha} \int_{\partial B_R} \left( |\nabla^* s'|^2 - \frac{n-1}{R^2} s'^2 \right) dS. \]

4.2.5. The sign of \( \ddot{J}(0) \). We want to find the sign of \( I \). For the ball, from the volume constraint it follows that \( \int_{\partial B_R} s' \, dS = 0 \). Hence,

\[ \int_{B_R} |\nabla s'|^2 \, dx \geq \mu_2(B_R) \int_{\partial B_R} s'^2 dS. \]

Since \( \mu_2 = 1/R \), we get the lower estimate

\[ I(s') \geq -\frac{1}{\alpha} \dot{S}(0). \]
Keeping in mind that \( s' \) is harmonic, we get
\[
(4.28) \quad \int_{B_R} |\nabla s'|^2 \, dx = \oint_{\partial B_R} s' \partial_{s'} dS \leq \frac{1}{2R} \oint_{\partial B_R} s^2 \, dS + \frac{R}{2} \oint_{\partial B_R} (\partial_{s'} s')^2 \, dS.
\]
Next we multiply \(-\Delta s = 1\) by \(x \cdot \nabla s\) and integrate over \(\Omega\). Since \(s = 0\) on \(\partial \Omega\), this gives
\[
\int_{B_R} |\nabla s'|^2 \, dx = \frac{R}{n-2} \oint_{\partial B_R} |\nabla^* s'|^2 \, dS - \frac{R}{n-2} \oint_{\partial B_R} (\partial_{s'} s')^2 \, dS.
\]
Combining this with estimate (4.28), we get
\[
\int_{B_R} |\nabla s'|^2 \, dx \leq \frac{R}{n} \oint_{\partial B_R} s^2 \, dS + \frac{R}{2} \oint_{\partial B_R} |\nabla^* s'|^2 \, dS.
\]
This results in the following upper bound:
\[
\mathcal{I}(s') \leq \left( \frac{2R}{n} - \frac{1}{\alpha} \right) \tilde{S}(0).
\]
Thus, we have proved the next claim.

**Lemma 8.** For \(\alpha < \frac{n}{2R}\), the ball is a local maximizer of \(\mathcal{J}(\Omega)\) among nearly circular domains of equal volume.

### 4.3. Optimality of the ball in two dimensions

From Proposition [1] it follows that, for all domains of given area \(A := |B_R|\), the functional \(\mathcal{J}(\Omega)\) is smaller than the corresponding expression for the disk, provided \(\alpha < \alpha_0\) with
\[
\alpha_0 = \frac{A^2}{T(\Omega) - T(B_R)} (|\partial B_R|^{-1} - L^{-1}) \quad \text{where} \quad L = |\partial \Omega|.
\]
If we replace \(T(\Omega)\) by an upper bound \(T^*\), then
\[
\alpha_0 \geq \frac{A^2}{T^* - T(B_R)} (|\partial B_R|^{-1} - L^{-1}).
\]
Observe that
\[
T(B_R) = -\frac{A^2}{8\pi} \quad \text{and} \quad |\partial B_R| = \sqrt{4\pi A}.
\]
We are interested in estimates for \(T(\Omega)\) that depend only on \(L\) and \(A\). Such an inequality was derived in [10] by Payne and Weinberger with the help of the method of parallel lines.

We introduce the notation
\[
2\pi \tilde{R} := L, \quad A =: \pi(\tilde{R}^2 - \tilde{r}^2) \quad \text{and} \quad \tilde{r} = y\tilde{R}.
\]
Then
\[
y^2 = 1 - \frac{4\pi A}{L^2}, \quad L^2 = \frac{4\pi A}{1 - y^2} = \frac{4\pi^2 R^2}{1 - y^2} \quad \text{and} \quad \tilde{R}^2 = \frac{A}{\pi(1 - y^2)} = \frac{R^2}{1 - y^2}.
\]
Payne–Weinberger’s inequality says that
\[
T(\Omega) \leq \frac{\pi}{2} \left( \tilde{r}^4 \log \frac{\tilde{r}}{\tilde{R}} - \frac{3}{4} \tilde{r}^4 + \tilde{R}^2 \tilde{r}^2 - \frac{\tilde{R}^4}{4} \right).
\]
The expression on the right-hand side is the energy corresponding to the boundary value problem \(\Delta U + 1 = 0\) in \(B_R \setminus B_{\tilde{r}}\) with \(U = 0\) on \(\partial B_R\) and \(\partial_{s'} U = 0\) on \(\partial B_{\tilde{r}}\). Consequently, equality occurs for the disk.
This inequality implies that
\[
\epsilon_0 = T(\Omega) - T(B_R) \leq \frac{\pi}{4} \tilde{R}^4 y^2 \left[ 1 + y^2 \log y^2 - y^2 \right]
\]
\[
= \frac{\pi R^4}{4(1-y^2)^2} y^2 \left[ 1 + y^2 \log y^2 - y^2 \right].
\]
Moreover,
\[
|\partial B_R|^{-1} - L^{-1} = \frac{1}{\sqrt{4\pi A}} \left( 1 - \frac{\sqrt{4\pi A}}{L} \right)
\]
\[
= \frac{y^2}{\sqrt{4\pi A} \left( 1 + \frac{\sqrt{4\pi A}}{L} \right)} = \frac{y^2}{2\pi R(1 + \sqrt{1 - y^2})}.
\]
Collecting all the terms, we obtain the estimate
\[
(4.29) \quad \alpha_0 \geq \frac{2(1 - y^2)^2}{R(1 + \sqrt{1 - y^2}) (1 + y^2 \log y^2 - y^2)} =: \frac{2}{R} g(y^2).
\]
The function \( g(t) \) is monotone increasing for \( t \in (0, 1) \), with \( \lim_{t \to 1} g(t) = 2 \) and \( g(0) = 1/2 \). The number \( y^2 \) measures the defect of \( \Omega \) with respect to the disk. Estimate (4.29) together with the monotonicity of \( g \) implies the following.

**Theorem 5.** (i) Let \( \Omega \subset \mathbb{R}^2 \) be a domain with fixed area \( A \), and let \( B_R \) be a disk with the same area. Then

\[
\mathcal{J}(\Omega) \leq \mathcal{J}(B_R) \quad \text{for all} \quad \alpha \leq \frac{2}{R} g(y^2).
\]

(ii) In particular, \( \mathcal{J}(\Omega) \) is smaller than the corresponding quantity for the disk if \( \alpha \leq 1/R \).

Observe that the second statement is consistent with Lemma 8.

**Corollary 3.** Under the same assumptions, \( E(\tilde{u}, \Omega) \) attains its maximum for the disk provided \( \alpha < \mu_2(\Omega) \).

**Proof.** From Lemma 2 and Theorem 5 (ii) it follows that \( E(\tilde{u}, \Omega) < \min\{ \frac{1}{R}, \mu_2(\Omega) \} \).

Note that by Weinstock’s result we have \( \mu_2(\Omega) \leq \mu_2(B_R) \leq \frac{1}{R} \), so that \( \min\{ \mu_2(\Omega), \frac{1}{R} \} = \mu_2(\Omega) \). \( \square \)

**Open problem.** In order to extend Corollary 3 a generalization of Payne–Weinberger’s inequality to higher dimensions would be helpful. This inequality is based on estimates for the length of parallel curves, which to our knowledge are not available in higher dimensions.

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