GENERICITY OF CONTRACTING ELEMENTS IN GROUPS

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Abstract. In this paper, we establish that, for statistically convex-cocompact actions, contracting elements are exponentially generic in counting measure. Among others, the following exponential genericity results are obtained as corollaries for the set of hyperbolic elements in relatively hyperbolic groups, the set of rank-1 elements in CAT(0) groups, and the set of pseudo-Anosov elements in mapping class groups.

Regarding a proper action, the set of non-contracting elements is proven to be growth-negligible. In particular, for mapping class groups, the set of pseudo-Anosov elements is generic in a sufficiently large subgroup, provided that the subgroup has purely exponential growth. By Roblin’s work, we obtain that the set of hyperbolic elements is generic in any discrete group action on CAT(-1) space with finite BMS measure.

Applications to the number of conjugacy classes of non-contracting elements are given for non-rank-1 geodesics in CAT(0) groups with rank-1 elements.

Contents
1. Introduction 1
2. Preliminary 7
3. Conjugacy classes of non-contracting elements 12
4. Genericity of contracting elements 15
5. More preliminary: Projection complex and a quasi-tree of spaces 17
6. Projecting in the quasi-tree of spaces 18
7. Almost geodesic decomposition 21
References 25

1. Introduction

1.1. Main results on genericity. In recent years, the notion of a contracting element is receiving a great deal of interests in studying various classes of groups with negative curvature. The prototype of this notion is a hyperbolic isometry on hyperbolic spaces, but more interesting examples are furnished by the following:

- hyperbolic elements in relatively hyperbolic groups, cf. [25], [24];
- rank-1 elements in CAT(0) groups, cf. [3], [7];
- certain infinite order elements in graphical small cancellation groups, cf. [2];
- pseudo-Anosov elements in mapping class groups, cf. [33].

Usually, the existence of a contracting element represents a situation where a certain negative curvature along geodesics exists in ambient spaces although with which non-negatively curved parts could coexist. One of the findings of the present study is that under natural assumptions, the negatively curved portion dominates the remaining part, once one contracting element is supplied. This fits in the rapidly developing research scheme where the statistical and random properties are

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studied in counting measure (compared with harmonic measure in random walk). To be precise, we are aiming to address the following question in the present paper:

**Question.** Suppose that a countable group $G$ admits a proper and isometric action on a proper geodesic metric space $(Y,d)$ with a contracting element. Fix a basepoint $o \in Y$. Denote

$$N(o,n) := \{g \in G : d(o,go) \leq n\}.$$

Is the set of contracting elements generic in counting measure, i.e.

$$\frac{\# \{g \in N(o,n) : g \text{ is contracting} \}}{\# N(o,n)} \to 1,$$

as $n \to \infty$? It is called exponentially generic if the rate of convergence happen exponentially fast.

Let us clarify the question by introducing a few more notions. The critical exponent $\omega(\Gamma)$ for a subset $\Gamma \subset G$ is defined by

$$\omega(\Gamma) = \limsup_{n \to \infty} \frac{\log \#(N(o,n) \cap \Gamma)}{n},$$

which is independent of the choice of $o \in Y$. A subset $X$ in $G$ is called growth-tight if $\omega(X) < \omega(G)$; growth-negligible if $\# X = o(\exp(\omega(G)n))$, where $o$ is the Landau little-o notation.

Assuming that $G$ has the following purely exponential growth property:

$$\# N(o,n) \sim \exp(\omega(G)n),$$

the above question is rephrased as saying that whether the set of non-contracting elements is growth-negligible/growth-tight.

In [40], we investigated the asymptotic geometry of a class of actions called statistically convex-cocompact actions (SCC) (cf. §2.4). Among other things, we proved that SCC actions have purely exponential growth. A class of barrier-free sets were introduced and proved to be growth-tight for SCC actions, and growth-negligible for a general proper action. These results therefore constitute the basis for the present study.

The group $G$ is always assumed to be non-elementary: there is no finite-index cyclic subgroup. The main theorem of this paper is the following.

**Theorem A.** Suppose that a non-elementary group $G$ admits a proper (resp. SCC) action on a geodesic metric space $(Y,d)$ with a contracting element. Then the set of non-contracting elements in $G$ has growth-negligible (resp. growth-tight).

In particular, for SCC actions, contracting elements are exponentially generic.

The distinction between SCC actions and proper actions is subtle in this study, in that for proper actions even with purely exponential growth, our methods only allow to obtain the genericity of contracting elements, while SCC actions get the exponential genericity in one shot. It is a natural problem to determine when the exponential genericity holds as well for a proper action.

Consider first the applications to the weak form of genericity, before turning to the exponential genericity.

**Corollary 1.1.** Assume that a non-elementary group $G$ admits a proper action on a geodesic metric space $(Y,d)$ with a contracting element. If $G$ has purely exponential growth, then the set of contracting elements is generic.

We now explain two implications in some specific classes of groups. In the context of mapping class groups, a sufficiently large subgroup was first studied by McCarthy and Papadopoulos [32], as an analog of non-elementary subgroups of Kleinian groups. By definition, a sufficiently large subgroup is one with at least two independent pseudo-Anosov elements. We refer the reader to [32] for a detailed discussion. The interesting examples include convex-cocompact subgroups [21], the handlebody group, among many others. The following result reduces the genericity question to the problem whether the subgroup has purely exponential growth.
Theorem 1.2 (Mod). In mapping class groups, the set of non-pseudo-Anosov elements in a sufficiently large subgroup \( \Gamma \) has growth-negligible. In particular, if \( \Gamma \) has purely exponential growth, then the set of pseudo-Anosov elements is generic.

An obvious instance with purely exponential growth is a subclass of sufficiently large subgroups, whose action on \( Y \) is itself SCC. This was studied in [40] as SCC subgroups, a dynamical generalization of convex-cocompact subgroups. We emphasize that there exists indeed non-convex-cocompact non-free and free subgroups admitting SCC actions on Teichmüller space. Hence, for these groups, the set of contracting elements is (exponentially) generic.

In terms of Theorem 1.2, it would be desirable to investigate which sufficiently large subgroups have purely exponential growth. The similar problem has been completely answered in the setting of Riemannian manifolds with variable negative curvature by work of T. Robin [36]. In fact, his results were obtained in a more general setting. For any proper action on a CAT(-1) space, Robin showed that the growth of the action is of purely exponential if and only if the corresponding Bowen-Margulis-Sullivan measure is finite on the geodesic flow. From this, we obtain the following application to discrete groups on a CAT(-1) space.

Theorem 1.3 (CAT(-1)). Suppose that \( G \) acts properly on a CAT(-1) space such that the BMS measure is finite. Then the set of hyperbolic elements is generic.

As suggested in Riemannian case, it seems interesting to develop an analogue of Roblin’s result for sufficiently large subgroups in mapping class groups.

We are now turning to the class of SCC actions, which admits stronger consequence by Theorem A. This class of groups encompasses many interesting classes of groups, listed as in [40]. Again, we first consider the instance of mapping class groups. They are known to act on Teichmüller space by SCC actions a result of by Eskin, Mirzakhani and Rafi [18, Theorem 1.7], as observed in [1, Section 10]. This is the starting point of our approach to mapping class groups, and modulo this input, our argument is completely general and is working for any SCC action on a metric space.

Using dynamical methods, Maher [30] has proved the genericity of pseudo-Anosov elements for the Teichmüller metric. Strengthening this result, Theorem A allows to obtain the exponential genericity of pseudo-Anosov elements.

Theorem 1.4 (Mod). In mapping class groups, the set of pseudo-Anosov elements is exponentially generic in Teichmüller metric. More generally, given any convex-cocompact subgroup \( \Gamma \), the set of pseudo-Anosov elements not conjugated into \( \Gamma \) are exponentially generic.

After the first version was posted in Arxiv [40], we learnt that this result was independently obtained by Spencer Dowdall in joint work with Howard Masur in a conference “Geometry of mapping class groups and Out(\( F_n \))”, October 25 - October 28, 2016.

Remark. In fact, Maher proved in [30] that the set of elements (containing possibly pseudo-Anosov ones) with bounded translation length on curve graphs is negligible. Our last statement strengthens it as well to the exponential negligibility, and even more: since all conjugates of a convex-cocompact subgroup can have arbitrary large translation length on curve graphs. See Lemma 4.3.

Our study also sheds some light on a conjecture of Farb [20], which predicts genericity for a class of combinatorial metrics (in contract with Teichmüller metric in Maher’s theorem).

Conjecture 1.5. The set of pseudo-Anosov elements is generic with respect to the word metric.

Since any group acts properly and cocompactly on its Cayley graph, the action of mapping class groups on Cayley graphs is SCC. Farb’s conjecture can be reduced by Theorem A to the following:

Conjecture 1.6. Mapping class groups act on their Cayley graphs with contracting elements.

\(^{1}\)The author is indebted to I. Gehktman and S. Taylor for telling him of Maher’s result at MSRI in Aug. 2016 when this manuscript was in its final stage of preparation.
Remark. Note that a contracting element in any proper action of $\text{Mod}$ on a geodesic metric space has to be pseudo-Anosov, since a reducible element is of infinite index in its centralizer, while a contracting element is of finite index. So, if this conjecture is true, it will imply the stronger version of Farb’s conjecture that pseudo-Anosov elements is exponentially generic.

To give more results on genericity, we mention some applications of Theorem A to another two classes of relatively hyperbolic groups and CAT(0) groups with rank-1 elements.

The class of relatively hyperbolic groups, introduced by Gromov [27] as generalization of hyperbolic groups, has been developed by many people [19], [5], [34], [14], [17], [23] and so on. In last twenty years, their intensive study achieves a huge success and inspires as well many research works on other classes of groups with negative curvature. For instance, the recent work of Dahmani, Guirardel and Osin [15] on hyperbolically embedded subgroups with applications to mapping class groups.

It is widely believed that hyperbolic elements are generic in a relatively hyperbolic group, which by definition are infinite order elements not conjugated into maximal parabolic subgroups. Our next result confirms this for the action on Cayley graphs with respect to word metric.

**Theorem 1.7 (RelHyp).** The set of hyperbolic elements in a relatively hyperbolic group is exponentially generic with respect to the word metric.

**Remark.** Note that this is not a direct consequence of Theorem A, since contracting elements might be parabolic. The proof indeed follows from a more general result in §4.

We remark that this same conclusion holds for the cusp-uniform action with parabolic gap property, or a more general property introduced by Dal’bo, Otal and Peigné [16]. Since, under these properties, the corresponding action has purely exponential growth, see [43].

It is well-known that a free product of any two groups (or generally, a graph of groups with finite edge groups) is hyperbolic relative to factors, by an equivalent definition in [8]. We thus obtain the following corollary.

**Corollary 1.8 (Free product).** The set of elements in a free product of two groups not conjugated into both factors is exponentially generic with respect to the word metric, if the free product is not elementary (i.e., $\neq \mathbb{Z}_2 \ast \mathbb{Z}_2$).

Lastly, we consider the implications for the class of CAT(0) groups with rank-1 elements, which admits a geometric (and thus SCC) action with a contracting element. There are two important subclasses, right-angled Artin groups (RAAG) and right-angled Coxeter groups (RACG), which are receiving a great deal of interests in last years. For stating the next result, we shall explain which RAAGs and RACGs contain contracting elements.

It is well-known that an RAAG acts properly and cocompactly on a non-positively curved cube complex called the *Salvetti complex*. The defining graph of a RAAG is a join if and only if the RAAG is a direct product of non-trivial groups. In [4, Theorem 5.2], Behrstock and Charney proved that any subgroup of an RAAG $G$ that is not conjugated into a join subgroup (i.e., obtained from a join subgraph) contains a contracting element.

An RACG also acts properly and cocompactly on a CAT(0) cube complex called the *Davis complex*. In [5, Proposition 2.11], an RACG of linear divergence was characterized as virtually a direct product of groups. Hence, an RACG which is not virtually a direct product of non-trivial groups contains a rank-1 element, for its existence is equivalent to a superlinear divergence by [11, Theorem 2.14].

By the above discussion, we have the following.

**Theorem 1.9 (CAT1).** The set of rank-1 elements in a CAT(0) group with a rank-1 element is exponentially generic with respect to the CAT(0) metric. This includes, in particular, the following two subclass of groups:

1. The action on the Salvetti complex of right-angled Artin groups that are not direct products. As a consequence, any subgroup conjugated into a join subgroup is growth-tight.
The action on the Davis complex of right-angled Coxeter groups that are not virtually a direct product of non-trivial groups.

1.2. Conjugacy classes of non-contracting elements. We consider an implication of Theorem A on counting conjugacy classes of non-contracting elements. To state the result, we need introduce the length of a conjugacy class, which is motivated by the length of a geodesic in a geometric setting below, for CAT(0) spaces.

Recall that a hyperbolic isometry on CAT(0) spaces preserves a geodesic called the axis, on which it acts by translation (cf. [9]). Among the hyperbolic ones, a rank-1 isometry requires in addition that the axis does not bound on a half-Euclidean plane. Therefore, in a compact rank-1 manifold, each geodesic corresponds to a (conjugacy class of) hyperbolic isometry, among which we can distinguish rank-1 and non-rank-1 geodesics. An example presented in Kneiper [29] shows that non-rank-1 geodesics can grow exponentially fast. However, as a corollary to his solution of a conjecture of Katok [29, Corollary 1.2], Kneiper showed that the number of them is strictly less than that of rank-1 geodesics. The second main result we are stating can be viewed as a generalization of this result in the setting of a SCC action with contracting elements.

Let $[g]$ denote the conjugacy class of $g \in G$. With respect to a basepoint $o$, the (algebraic) length of the conjugacy class $[g]$ is defined as follows:

$$\ell^o([g]) = \min \{d(o, g'o) : g' \in [g] \}.$$

We remark that this does not agree with the geometric length of a conjugacy class (when viewed as a closed geodesic on manifolds, for example). Nevertheless, if the action of $G$ on $Y$ is cocompact, then these two lengths differ up to an additive constant. With this coarse identification, our results could be applied in a geometric setting to count the number of closed geodesics.

Similarly to that of orbital points, the growth rate of conjugacy classes of an infinite set $X$ in $G$ is defined as follows

$$\omega^c(X) := \limsup_{n \to \infty} \frac{\log \# \{ [g] : g \in X, \ell^o([g]) \leq n \}}{n}.$$

We are ready to state the next main result, which is proved in §3 as an initial step in the proof of Theorem A.

**Theorem B** (Conjugacy classes of non-contracting elements). Assume that $G$ admits a SCC action on a geodesic metric space $(Y, d)$ with a contracting element. Then the growth rate of conjugacy classes of non-contracting elements is strictly less than $\omega(G)$.

The first application is given to the class of groups with rank-1 elements, generalizing the above Kneiper’s result on compact rank-1 manifolds to the setting of singular spaces:

**Theorem 1.10 (CAT(0)).** Suppose that $G$ acts properly and cocompactly on a CAT(0) space with a rank-1 element. Then the growth rate of conjugacy classes of non-rank-1 elements is strictly less than $\omega(G)$.

Kneiper’s proof in the smooth case makes essential use of conformal densities on the boundary, while our proof replies on a growth-tightness result for a class of barrier-free set we are describing now.

With a basepoint $o$ fixed, an element $h \in G$ is called $(\epsilon, M, g)$-barrier-free if there exists an $(\epsilon, g)$-barrier-free geodesic $\gamma$ with $\gamma_+ \in B(o, M)$ and $\gamma_- \in B(ho, M)$: there exists no $t \in G$ such that $d(t \cdot o, \gamma), d(t \cdot ho, \gamma) \leq \epsilon$. Denote below by $V(\epsilon, M, g)$ the set of $(\epsilon, M, g)$-barrier-free elements. One of main results proven in [30] is that $V(\epsilon, M, g)$ is growth-tight, stated here in Theorem 2.10.

We are now sketching the proof in the CAT(0) case, which illustrates the basic idea of the more complex Theorem B.

**Sketch of the proof of Theorem 1.10.** Let $NC$ denote the set of non-rank-1 elements. We consider only hyperbolic non-rank-1 elements $g \in G$, since there are only finitely many conjugacy classes of
elliptic elements [9]. The axis of such an element $g$ bounds a Euclidean half-plane. Since the action is cocompact, we can translate the axis of $g$ (and the bounding half-plane) to a compact neighborhood of diameter $M$ of a basepoint $o$. To bound $\omega'(Nc)$, it suffices to bound the cardinality of the set $X$ of non-rank-1 elements $g \in G$ with $go$ on the boundary of half-planes.

We now fix a rank-1 element $c$ (of sufficiently high power). Since $\langle c \rangle \cdot o$ is contracting, a segment $[o, c \cdot o]$ is not likely to be near any Euclidean half-plane. This implies that $g$ are $(\epsilon, M, c)$-barrier-elements and so the set $X$ is contained in $V_{\epsilon, M, c}$. Thus, the growth-tightness theorem 2.10 implies $\omega(V_{\epsilon, M, c}) < \omega(G)$, concluding the proof. □

In a manner parallel to that for smooth manifolds, we could produce examples in the class of RAAGs with exponential growth of non-rank-1 elements. An example with infinitely many ends is the free product $\Gamma_2 \ast (\Gamma_2 \times \Gamma_2)$, whose defining graph is the disjoint union of two vertices and a square.

To conclude this discussion, we deduce the following corollary for relatively hyperbolic groups:

**Theorem 1.11** ($\mathbb{R}el\mathbb{Hyp}$). In a relatively hyperbolic group $G$, the growth rate of conjugacy classes of parabolic elements and torsion elements is strictly less than $\omega(G)$, which is computed with respect to the word metric.

By a similar trick as above, one can construct examples of relatively hyperbolic groups with an exponential number of classes of torsion elements. For instance, consider a free product of two groups with infinitely many classes of torsion elements. The theorem then implies that the number of torsion elements are exponentially small relative to that of hyperbolic elements.

### 1.3. Related works on genercity problem

Generic elements have been studied by many authors by undertaking a random walk on groups. Consider a probability measure $\mu$ on $G$ with finite support. Starting from the identity, one walks to subsequent elements according to the distribution of $\mu$. In $n$ steps, the distribution becomes the $n$th convolution $\mu^n$. In this regard, Maher [31] proved that a random element in sufficiently large subgroups in $\text{Mod}$ tends to be a pseudo-Anosov element with probability 1 as $n \to \infty$. This result was generalized to the class of groups with “weakly contracting” elements by Sisto [37] (his definition is different from ours).

As a matter of fact, the measure $\mu^n$ is far from the counting measure on $n$-spheres in groups. To be precise, consider the asymptotic entropy $h(\mu)$ and drift $\ell(\mu)$ associated with a random walk. These two quantities and the growth rate $\omega(G)$ are related by the following fundamental inequality (cf. [28]):

$$h(\mu) \leq \ell(\mu) \cdot \omega(G)$$

Equality would suggest that a random walk could approach most elements. We refer to Vershik [38] for related definitions and the background, and to Gouëzel et al. [26] for recent progress on the strictness of this inequality in hyperbolic groups.

In a sense, a counting measure could reveal more information, so the generic elements in counting measure is usually quite different from that in a random walk. To our best knowledge, there are very few results in counting measures arising from word metrics. A progress made by Caruso and Wiest [10] in braid groups showed that generic elements are pseudo-Anosov in the word metric with respect to Garside’s generating set. Some of the ingredients were generalized later by Wiest to treat other classes of groups in [39]. For instance, partial cases of Theorems 1.7 and 1.9 was obtained there under some automatic hypothesis. Recently, Gehktman, Tylor and Tiozzo [22] established for word metrics the generic elements in a non-elementary hyperbolic group action.

We emphasize that all these works assume a non-elementary action on $\delta$-hyperbolic spaces and the existence of certain automatic structures, which are not needed in the present work. In contrast, by assuming the existence of a contracting element, our methods presented here seems to be effective in treating many genericity problems in a unified way. Except the ones stated in §1.1, we mention another result in [40, Proposition 2.21] which was proved by using similar technics.
Proposition 1.12. Assume that a finitely generated group $G$ acts properly on a geodesic metric space $(Y,d)$ with a contracting element. Then for any finite generating set $S$, 
\[ \# \{ g \in N(1,n) : g \text{ is contracting} \} > 0 \]
where $N(1,n) = \{ g \in G : d_S(1,g) \leq n \}$ where $d_S$ the corresponding word metric.

This generalizes the recent result of Cumplido and Wiest [13] in mapping class groups that a positive proportion of elements are pseudo-Anosov. See also Cumplido [12] for a similar result in Artin-Tits groups.

The structure of the rest of this paper is as follows. The preliminary §2 recalls necessary results proved in [40]. Theorem B is first proved in section §3. A general theorem §4 is stated in §4. Its proof is given by assuming an almost geodesic decomposition in Proposition §4.4, which is the goal of the following three sections §§5, 6, 7. More preliminary is recalled in §5 to give a brief introduction of projection complex and quasi-tree of spaces. They are used in the following §6 to prove Lemma §6.6, which is the starting point of the proof of Proposition §4.4 occupying the final §7.

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2. Preliminary

Most of materials are taken from the paper [40], to which we refer for more details and complete proofs.

2.1. Notations and conventions. Let $(Y,d)$ be a proper geodesic metric space. Given a point $y \in Y$ and a subset $X \subset Y$, let $\pi_X(y)$ be the set of points $x$ in $X$ such that $d(y, x) = d(y, X)$. The projection of a subset $A \subset Y$ to $X$ is then $\pi_X(A) := \cup_{a \in A} \pi_X(a)$.

Denote $d_X^*(Z_1, Z_2) := \diam(\pi_X(Z_1 \cup Z_2))$, which is the diameter of the projection of the union $Z_1 \cup Z_2$ to $X$. So $d_X^*(\cdot,\cdot)$ satisfies the triangle inequality

\[ d_X^*(A,C) \leq d_X^*(A,B) + d_X^*(B,C). \]

We always consider a rectifiable path $\alpha$ in $Y$ with arc-length parameterization. Denote by $\ell(\alpha)$ the length of $\alpha$, and by $\alpha_-$, $\alpha_+$ the initial and terminal points of $\alpha$ respectively. Let $x, y \in \alpha$ be two points which are given by parameterization. Then $[x,y]_\alpha$ denotes the parameterized subpath of $\alpha$ going from $x$ to $y$. We also denote by $[x,y]$ a choice of a geodesic in $Y$ between $x, y \in Y$.

Entry and exit points. Given a property (P), a point $z$ on $\alpha$ is called the entry point satisfying (P) if $\ell([\alpha_-z]_\alpha)$ is minimal among the points $z$ on $\alpha$ with the property (P). The exit point satisfying (P) is defined similarly so that $\ell([w,\alpha_+]_\alpha)$ is minimal.

A path $\alpha$ is called a $c$-quasi-geodesic for a constant $c \geq 1$ if the following holds

\[ \ell(\beta) \leq c \cdot d(\beta_-,\beta_+) + c \]

for any rectifiable subpath $\beta$ of $\alpha$.

Let $\alpha, \beta$ be two paths in $Y$. Denote by $\alpha \cdot \beta$ (or simply $\alpha \beta$) the concatenated path provided that $\alpha_- = \beta_+$.

Let $f, g$ be real-valued functions with domain understood in the context. Then $f <_{c_i} g$ means that there is a constant $C > 0$ depending on parameters $c_i$ such that $f < Cg$. The symbols $>_{c_i}$ and $\asymp_{c_i}$ are defined analogously. For simplicity, we shall omit $c_i$ if they are universal constants.
2.2. Contracting elements.

Definition 2.1 (Contracting subset). For given \( C \geq 1 \), a subset \( X \) in \( Y \) is called \( C \)-contracting if for any geodesic \( \gamma \) with \( d(\gamma, X) \geq C \), we have

\[
d_X^C(\gamma) \leq C.
\]

A collection of \( C \)-contracting subsets is referred to as a \( C \)-contracting system.

We collect a few properties that will be used often later on.

Proposition 2.2. Let \( X \) be a contracting set.

(1) (Quasi-convexity) \( X \) is \( \sigma \)-quasi-convex for a function \( \sigma : \mathbb{R}_+ \to \mathbb{R}_+ \): given \( c \geq 1 \), any \( c \)-quasi-geodesic with endpoints in \( X \) lies in the neighborhood \( N_{\sigma(\gamma)}(X) \).

(2) (Finite neighborhood) Let \( Z \) be a set with finite Hausdorff distance to \( X \). Then \( Z \) is contracting.

(3) (Subpaths) If \( X \) is a quasi-geodesic, then any subpath of \( X \) is contracting with contraction constant depending only on \( X \).

There exists \( C > 0 \) such that the following hold:

(4) (1-Lipschitz projection) \( d_X^C((y, z)) \leq d(y, z) + C \).

Proof. Except Assertion (3), the others are straightforward applications of contracting property. The (3) for geodesics in CAT(0) spaces can be found in [7, Lemma 3.2]; here we provide a proof in this general setting.

Assume that \( \gamma := X \) is a \( C \)-contracting \( c \)-quasi-geodesic for some \( c, C > 0 \). We first observe the following.

Claim. There exists \( D = D(c, C) > 0 \) such that any subpath \( \alpha \) of \( \gamma \) has at most a Hausdorff distance \( D \) to a geodesic \( \tilde{\alpha} \) with \( d(\alpha, \tilde{\alpha}_-) \leq 2C \).

Proof of the claim. Indeed, by the quasi-convexity [1], there exists \( \sigma = \sigma(C) > 0 \) such that \( \tilde{\alpha} \subset N_\sigma(\gamma) \). We shall only prove \( \tilde{\alpha} \subset N_D(\alpha) \); the other inclusion follows from this one by a standard argument using the connectedness of \( \tilde{\alpha} \).

Without loss of generality, we prove \( \tilde{\alpha} \subset N_\sigma(\alpha) \) by assuming that \( \ell(\alpha) > c(2\sigma + 1) \). Argue by contradiction. If \( \tilde{\alpha} \not\subset N_\sigma(\alpha) \), there exists a non-empty open (connected) interval \( I \) in \( \tilde{\alpha} \) such that \( I \cap N_\sigma(\alpha) = \emptyset \). By the fact \( \tilde{\alpha} \subset N_\sigma(\gamma) \), we must have \( I \subset N_\sigma([\gamma_-, \alpha_-]_\gamma) \cup N_\sigma([\alpha_-, \gamma_+]_\gamma) \). The connectivity of \( I \) then implies that \( I \cap N_\sigma([\gamma_-, \alpha_-]_\gamma) \cap N_\sigma([\alpha_-, \gamma_+]_\gamma) \neq \emptyset \); there exists \( x \in I \) such that \( d(x, [\gamma_-, \alpha_-]_\gamma), d(x, [\alpha_-, \gamma_+]_\gamma) \leq \sigma \). As a consequence, \( \alpha \) is contained in a \( \sigma \)-quasi-geodesic with two endpoints within a \( 2\sigma \)-distance, so gives that \( \ell(\alpha) \leq c(2\sigma + 1) \). This is a contradiction with \( \ell(\alpha) > c(2\sigma + 1) \), so \( \tilde{\alpha} \subset N_\sigma(\alpha) \) is proved. The proof of the claim is thus finished. \( \square \)

We now prove that \( \alpha \) is \( 2(D + C) \)-contracting. Consider a geodesic \( \beta \) such that \( \beta \cap N_D(\alpha) = \emptyset \). Let \( x, y \in \pi_\alpha(\beta) \) be projection points of \( \tilde{x}, \tilde{y} \in \beta \) respectively such that \( d(x, y) = d_\alpha^C(\beta) \). The goal is to show that \( d(x, y) \leq 2(D + C) \).

Without loss of generality, assume that the projection \( \pi_\alpha(\beta) \) is not entirely contained in \( \alpha \); otherwise, the contracting property of \( \alpha \) would follow from the one of \( \gamma \). For definiteness, assume that there exists a point \( z \in \pi_\alpha(\beta) \) lies on the left side of \( \alpha_- : z \in [\gamma_-, \alpha_-]_\gamma \); and by switching \( x, y \) if necessary, assume further that \( x \) is on the left of \( y \).

Let \( w \in \pi_\alpha(\tilde{y}) \) be a project point of \( \tilde{y} \) to \( \gamma \). The contracting property of \( \gamma \) implies \( d(w, [\tilde{y}, y]) \leq 2C \). Noticing that \( w, y \in \alpha \), we deduce from the Claim above that \( [w, y]_\gamma \) is contained in a \( D \)-neighborhood of \( [y, \tilde{y}] \).

We observe that \( w \in N_{2c}(\gamma) \). Indeed, to derive a contradiction, assume that \( w \) lies on \( [\alpha_-, \gamma_+] \), and \( d(w, \alpha_-) > C \). Noting that \( z \) lies on \( [\alpha_, \gamma_+]_\gamma \) and \( z \in \pi_\alpha(\beta) \), we have \( d(z, w) \geq d(w, \alpha_-) > C \). By the contracting property, the points \( z, w \in \pi_\alpha(\beta) \) with \( d(z, w) > C \) implies \( N_{2c}(\gamma) \cap \beta = \emptyset \). Moreover, a projection argument shows that \( d(z, \beta), d(w, \beta) \leq 2C \). So by the Claim
above, \([z, w]_\gamma\) lies within the \(D\)-neighborhood of \(\beta\). Since \(\alpha_- \in [z, w]_\gamma\), we get \(N_D(\alpha) \cap \beta \neq \emptyset\), a contradiction with the hypothesis.

It is therefore proved that either \(w\) is on the left of \(\alpha_-\), or \(d(w, \alpha_-) \leq C\). Recall that \(x \in [\alpha_-, y]_\alpha\), and \([w, y]_\gamma \subset N_D([y, \tilde{y}]_\gamma)\) as proved above. Examining the aforementioned relative position of \(w\) to \(\alpha_-\), we then obtain that in the former case, \(d(x, [\tilde{y}, y]) \leq D\) and in the later case, \(d(x, [\tilde{y}, y]) \leq D + C\). In both cases, let \(x' \in [\tilde{y}, y]\) so that \(d(x, x') \leq D + C\).

To conclude the proof, we verify that \(d(x, y) \leq 2(D + C)\): since \(y\) is a shortest point on \(\alpha\) to \(\tilde{y}\), we have \(d(x, x') \geq d(y, x')\). Hence, \(d(x, y) \leq d(x, x') + d(x', y) \leq 2(D + C)\), proving that \(\alpha\) is \(2(D + C)\)-contracting.

A contracting system has a \(R\)-bounded intersection property for a function \(R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) if the following holds

\[
\forall X \neq X' \in \mathcal{X} : \text{diam}(N_r(X) \cap N_r(X')) \leq R(r)
\]

for any \(r \geq 0\). This property is, in fact, equivalent to a bounded intersection property of \(\mathcal{X}\): there exists a constant \(B > 0\) such that the following holds

\[
d_{\mathcal{X}}(X) \leq B
\]

for \(X \neq X' \in \mathcal{X}\). See [32] for further discussions.

Recall that \(G\) acts properly by isometry on a geodesic metric space \((Y, d)\). An element \(h \in G\) is called contracting if the orbit \(\langle h \rangle \cdot o\) is contracting, and the orbital map

\[
n \in \mathbb{Z} \to h^n o \in Y
\]

is a quasi-isometric embedding. The set of contracting elements is preserved under conjugacy.

Given a contracting element \(h\), there exists a maximal elementary group \(E(h)\) containing \(\langle h \rangle\) as a finite index subgroup. Precisely,

\[
E(h) = \{g \in G : \exists n > 0, (gh^n g^{-1} = h^n) \lor (gh^n g^{-1} = h^{-n})\}.
\]

In what follows, the contracting subset

\[
Ax(h) = \{ f \cdot o : f \in E(h) \}
\]

shall be called the axis of \(h\). Hence, the collection \(\{gAx(h) : g \in G\}\) is a contracting system with bounded intersection.

Two contracting elements \(h_1, h_2 \in G\) are called independent if the collection \(\{gAx(h_i) : g \in G; i = 1, 2\}\) is a contracting system with bounded intersection.

2.3. Admissible paths. The notion of an admissible path is defined relative to a contracting system \(\mathcal{X}\) in \(Y\). Roughly speaking, an admissible path can be thought of as a concatenation of quasi-geodesics which travels alternatively near contracting subsets and leave them in an orthogonal way.

**Definition 2.3 (Admissible Path).** Given \(D, \tau \geq 0\) and a function \(R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\), a path \(\gamma\) is called \((D, \tau)\)-admissible in \(Y\), if the path \(\gamma\) contains a sequence of disjoint geodesic subpaths \(p_i\) \((0 \leq i \leq n)\) in this order, each associated to a contracting subset \(X_i \in \mathcal{X}\), with the following called Long Local and Bounded Projection properties:

- **(LL1)** Each \(p_i\) has length bigger than \(D\), except that \((p_i)_- = \gamma_-\) or \((p_i)_+ = \gamma_+\);
- **(BP)** For each \(X_i\), we have
  \[
  d_{\mathcal{X}}^\tau((p_i)_+, (p_{i+1})_-) \leq \tau
  \]
  and
  \[
  d_{\mathcal{X}}^\tau((p_{i-1})_+, (p_i)_-) \leq \tau
  \]
  when \((p_{i-1})_+ := \gamma_-\) and \((p_{n+1})_- := \gamma_+\) by convention.
- **(LL2)** Either \(X_i, X_{i+1}\) has \(R\)-bounded intersection or \(d((p_i)_+, (p_{i+1})_-) > D\).
Saturation. The collection of \( X_i \in \mathcal{X} \) indexed as above, denoted by \( \mathcal{X}(\gamma) \), will be referred to as contracting subsets for \( \gamma \). The union of all \( X_i \in \mathcal{X}(\gamma) \) is called the saturation of \( \gamma \).

The set of endpoints of \( p_i \) shall be referred to as the vertex set of \( \gamma \). We call \((p_i)_-\) and \((p_i)_+\) the corresponding entry vertex and exit vertex of \( \gamma \) in \( X_i \). (compare with entry and exit points in subSection 2.1)

By definition, a sequence of points \( x_i \) in a path \( \alpha \) is called linearly ordered if \( x_{i+1} \in [x_i, \alpha_+] \alpha \) for each \( i \).

Definition 2.4 (Fellow travel). Assume that \( \gamma = p_0 q_1 p_1 \ldots q_n p_n \) is a \((D, \tau)\)-admissible path, where each \( p_i \) has two endpoints in \( X_i \in \mathcal{X} \). The paths \( p_0, p_n \) could be trivial.

Let \( \alpha \) be a path such that \( \alpha_- = \gamma_- , \alpha_+ = \gamma_+ \). Given \( \epsilon > 0 \), the path \( \alpha \) \( \epsilon \)-fellow travels \( \gamma \) if there exists a sequence of linearly ordered points \( z_i, w_i \) \((0 \leq i \leq n)\) on \( \alpha \) such that \( d(z_i, (p_i)_-) \leq \epsilon \), \( d(w_i, (p_i)_+) \leq \epsilon \).

The basic fact is that a “long” admissible path is a quasi-geodesic.

Proposition 2.5. Let \( C \) be the contraction constant of \( \mathcal{X} \). For any \( \tau > 0 \), there are constants \( B = B(\tau) , D = D(\tau) , c = c(\tau) , \epsilon = \epsilon(\tau) , c = c(\tau) > 0 \) such that the following holds.

Let \( \gamma \) be a \((D, \tau)-\)admissible path and \( \alpha \) a geodesic between \( \gamma_- \) and \( \gamma_+ \). Then

(1) For a contracting subset \( X_i \in \mathcal{X}(\gamma) \) with \( 0 \leq i \leq n \),

\[
d_X(\beta_1) \leq B, \quad d_X(\beta_2) \leq B
\]

where \( \beta_1 = [\gamma_-, (p_i)_-], \beta_2 = [(p_i)_+, \gamma_+] \gamma \).

(2) \( \alpha \cap N_C(X) \neq \emptyset \) for every \( X \in \mathcal{X}(\gamma) \).

(3) \( \alpha \) \( \epsilon \)-fellow travels \( \gamma \). In particular, \( \gamma \) is a \( c \)-quasi-geodesic.

The main use of this lemma (the second statement) is to construct the following type of paths in verifying that an element is contracting.

Definition 2.6. Let \( L, \Delta > 0 \). With notations in definition of a \((D, \tau)-\)admissible path \( \gamma \), if the following holds

\[
|d((p_{i+1})_-(p_i)_+) - L| \leq \Delta
\]

for each \( i \), we say that \( \gamma \) is a \((D, \tau, L, \Delta)-\)admissible path.

Proposition 2.7. Assume that \( \mathcal{X} \) has bounded intersection in admissible paths considered in the following statements. For any \( \tau > 0 \) there exists \( D = D(\tau) > 0 \) with the following properties.

(1) For any \( L, \Delta > 0 \), there exists \( C = C(L, \Delta) > 0 \) such that the saturation of a \((D, \tau, L, \Delta)-\)admissible path is \( C \)-contracting.

(2) For any \( L, \Delta, K > 0 \), there exists \( C = C(L, \Delta, K) > 0 \) such that if the entry and exit vertices of a \((D, \tau, L, \Delta)-\)admissible path \( \gamma \) in each \( X \in \mathcal{X}(\gamma) \) has distance bounded above by \( K \), then \( \gamma \) is \( C \)-contracting.

2.4. Statistically convex-cocompact actions and growth-tightness theorem. In this subsection, we recall the definition of statistically convex-cocompact actions, which is understood as a statistical version of convex-cocompact actions. By abuse of language, a geodesic between two sets \( A \) and \( B \) is a geodesic \([a, b]\) between \( a \in A \) and \( b \in B \).

Given constants \( 0 \leq M_1 \leq M_2 \), the a concave region \( \mathcal{O}_{M_1, M_2} \) consists of the set of elements \( g \in G \) such that there exists some geodesic \( \gamma \) between \( B(o, M_2) \) and \( B(go, M_2) \) with the property that the interior of \( \gamma \) lies outside \( N_{M_1}(Go) \).

Definition 2.8 (statistically convex-cocompact action). If there exist two positive constants \( M_1, M_2 > 0 \) such that \( \omega(\mathcal{O}_{M_1, M_2}) < \omega(G) \), then the action of \( G \) on \( Y \) is called statistically convex-cocompact (SCC).

We are interested in the following important examples of SCC actions:
Examples. (1) Any proper and cocompact group action on a geodesic metric space.
(2) The action of relatively hyperbolic groups with parabolic gap property on a hyperbolic space (cf. [10]).
(3) The action of mapping class groups on Teichmüller spaces is SCC (cf. [1]).

In applications, since $\mathcal{O}_{M_2, M_2} \subset \mathcal{O}_{M_1, M_2}$, we can assume that $M_1 = M_2$ and henceforth, denote $\mathcal{O}_M := \mathcal{O}_{M, M}$ for easy notations.

We remark that the definition of a SCC action is independent of the choice of basepoint $o$, when there exists a contracting element. Namely, for any basepoint $o$, there exist $M_1, M_2 > 0$ such that $\omega(\mathcal{O}_{M_1, M_2}) < \omega(G)$. See Lemma 6.1 in [10].

By definition, the union of two growth-tight (resp. growth-negligible) sets is growth-tight (resp. growth-negligible). The main result of this section shall provide a class of growth-tight sets. These growth-tight sets are closely related to a notion of a barrier we are going to introduce now.

Definition 2.9. Fix constants $\epsilon, M > 0$ and a set $P$ in $G$.

1. (Barrier/Barrier-free geodesic) Given $\epsilon > 0$ and $f \in P$, we say that a geodesic $\gamma$ contains an $(\epsilon, f)$-barrier if there exists an element $h \in G$ so that
   \[ \max\{d(h \cdot o, \gamma), d(h \cdot f \cdot o, \gamma)\} \leq \epsilon. \]

   If no such $h \in G$ exists so that (3) holds, then $\gamma$ is called $(\epsilon, f)$-barrier-free.

   Generally, we say $\gamma$ is $(\epsilon, P)$-barrier-free if it is $(\epsilon, f)$-barrier-free for some $f \in P$. An obvious fact is that any subsegment of $\gamma$ is also $(\epsilon, P)$-barrier-free.

2. (Barrier-free element) An element $g \in G$ is $(\epsilon, M, P)$-barrier-free if there exists an $(\epsilon, P)$-barrier-free geodesic between $B(o, M)$ and $B(go, M)$.

The following result is proved in [40, Theorem C]. We remark that the constant $M$ can be chosen as big as necessary, which is guaranteed by Lemma 6.1 in [40].

Theorem 2.10 (Growth tightness). Suppose that $G$ has an SCC action on a geodesic space $(Y, d)$ with a contracting element. Then there exist constants $\epsilon, M > 0$ such that for any given $g \in G$, we have $\mathcal{V}_{\epsilon, M, g}$ is growth-tight. If the action is proper, then $\mathcal{V}_{\epsilon, M, g}$ is growth-negligible.

We sketch the proof at the convenience of the reader, and refer to [40, Section 4] for complete details.

Sketch of the proof. Let $\mathcal{B}$ be a maximal $R$-separated subset in $\mathcal{A} := \mathcal{V}_{\epsilon, M, g}$ so that

- for any distinct $a, a' \in \mathcal{B}$, $d(aa, aa') > R$, and
- for any $x \in \mathcal{V}_{\epsilon, M, g}$, there exists $a \in \mathcal{B}$ such that $d(xo, ao) \leq R$.

Denote by $\mathcal{W}(\mathcal{A})$ the set of all (finite) words over $\mathcal{A}$. We defined an extension map $\Phi : \mathcal{W}(\mathcal{A}) \to G$ as follows: given a word $W = a_1 a_2 \cdots a_n \in \mathcal{W}(\mathcal{A})$, set

\[ \Phi(W) = a_1 \cdot f_1 g f'_1 \cdot a_2 \cdot f_2 g f'_2 \cdots a_{n-1} \cdot f_{n-1} g f'_{n-1} \cdot a_n \in G, \]

where $f_i, f'_i \in F$ are supplied by the extension lemma in [40] such that $\Phi(W)$ labels a $(D, \tau)$-admissible path. Consider $X := \Phi(\mathcal{W}(\mathcal{B}))$. The key fact is that $\Phi : \mathcal{W}(\mathcal{B}) \to G$ is injective.

Consider the Poincaré series

\[ P_\Gamma(s, o) = \sum_{n \geq 1} \exp(-sd(o, go)), \quad s \geq 0, \]

which diverges for $s < \omega(\Gamma)$ and converges for $s > \omega(\Gamma)$.

Note that $P_B(s, o) \asymp P_A(s, o)$, whenever they are finite, and so $\omega(B) = \omega(A)$.

If the action is assumed to only be proper, then with the critical gap criterion in [10, Lemma 2.23], the injectivity assertion of $\Phi$ implies that $P_B(s, o)$ converges at $s = \omega(G)$: $\mathcal{B}$ is a growth-negligible set.
If the action is assumed to be SCC, we were able to prove that \( \mathcal{P}_A(s,o) \) and thus \( \mathcal{P}_B(s,o) \) are divergent at \( s = \omega(A) \). Again, by critical gap criterion in [40], we proved that \( \omega(X) > \omega(B) \) and so \( \omega(G) \geq \omega(X) > \omega(A) \): \( A \) is a growth-tight set. \( \square \)

3. Conjugacy classes of non-contracting elements

Recall that the growth rate of conjugacy classes of an infinite set \( X \) is defined as

\[
\omega^c(X) = \limsup_{n \to \infty} \frac{\log \# \{ [g] : g \in X, t^c([g]) \leq n \}}{n}
\]

where \([g] \) denotes the conjugacy class of \( g \) in \( G \). An element \( h \) in \([g]\) is called minimal if \( d(o, ho) = t^c([g]) \). Denote by \( N_C \) the set of non-contracting elements in \( G \) for the action of \( G \) on a geodesic metric space \((Y,d)\).

The goal of this section is the following result.

**Theorem 3.1.** Assume that \( G \) admits a proper action on \( Y \). Fix a constant \( \epsilon > 0 \) and a contracting element \( f \in G \). For any sufficiently large \( M > 0 \), there exists an integer \( n > 0 \) such that each element \( g \in N_C \) admits a minimal conjugacy representative in \( \mathcal{V}_{\epsilon,M,f^n} \).

Assuming SCC action, Theorem 3.1 follows immediately from it.

**Proof of Theorem 3.1.** Let \( \epsilon, M > 0 \) be the constants given by the growth-tightness Theorem 2.10 and the constant \( M \) is also assumed to satisfy Theorem 3.1. It follows immediately from Theorems 2.10 and 3.1 that \( \omega^c(N_C) < \omega(G) \). \( \square \)

The rest of this section is devoted to the proof of Theorem 3.1. In the next three lemmas, we would like to first identify two subsets of \( N_C \), and prove that they belong to \( \mathcal{V}_{\epsilon,M,f^n} \). In the last ingredient, we show in Proposition 3.3 that, up to finitely many exceptions, these two subsets comprises the entire \( N_C \).

For any \( M,D > 0 \), let \( K_{M,D} \) be the set of elements \( h \) in \( G \) such that any subpath of length \( D \) in a geodesic \([o, ho]\) is not contained in \( N_M(Go) \). Noting the similarity between \( K_{M,0} \) and \( O_M \), the following fact is not surprising.

**Lemma 3.2.** For any sufficiently large \( M \gg 0 \) and any \( D > 0 \), there exists \( n > 0 \) such that \( K_{M,D} \subset \mathcal{V}_{\epsilon,M,f^n} \).

**Proof.** Since the axis \( Ax(f) \) is contracting, it follows by Proposition 2.2.1 that \( Ax(f) \) is \( \sigma \)-quasi-convex for some function \( \sigma : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \). Suppose, to the contrary, that some \( h \in K_{M,D} \) contains an \((\epsilon,f^n)\)-barrier \( g \in G \), so \( d(go, [o, ho]) \leq \epsilon \). As a consequence, there exists a subsegment \( \alpha \) of \([o, ho]\) with their endpoints \( \alpha \in \mathcal{V}_{\epsilon,M,f^n} \) such that

\[
\ell(\alpha) > d(o, f^n o) - 2\epsilon.
\]

From the \( \sigma \)-quasi-convexity of \( Ax(f) \), we obtain

\[
\alpha \subset N_{\sigma}(Ax(f)) \subset N_\sigma(Go).
\]

We choose now \( M > \sigma \) and \( n \) large enough such that \( d(o, f^n o) > D + 2\epsilon \). Thus, the subpath \( \alpha \) with length \( \ell(\alpha) > D \) is contained inside \( N_M(Go) \). This gives a contradiction to the definition of \( h \in K_{M,D} \). Hence, \( K_{M,D} \subset \mathcal{V}_{\epsilon,M,f^n} \). \( \square \)

Given \( D,C > 0 \), a geodesic \( \gamma \) is called \( D \)-local \( C \)-non-contracting if any connected subsegment of \( \gamma \) with length \( D \) contained in \( N_M(Go) \) is not \( C \)-contracting. Denote by \( \mathcal{P}_{D,C} \) the set of \( h \in G \) such that \([o, ho]\) is \( D \)-local \( C \)-non-contracting.

The following lemma makes essential use of the hypothesis [1]. However, it is noteworthy that most results in this paper do not require it.

**Lemma 3.3.** There exists \( C > 0 \) depending on \( \epsilon \) and \( f \) such that any geodesic between \( B(o, \epsilon) \) and \( B(f^n o, \epsilon) \) is \( C \)-contracting for \( n \gg 0 \).
Proof. By the hypothesis \([1]\), we have that \(n \in \mathbb{Z} \rightarrow f^n o \in Y\) is a quasi-isometric embedding with a contracting image. That is to say, the path \(\gamma\) labeled by \(\{f^n : n \in \mathbb{Z}\}\) is a contracting quasi-geodesic. By Assertion \((3)\) of Proposition \(2.2\), a subpath of a contracting quasi-geodesic is uniformly contracting. So for any geodesic \(\alpha\) between \(B(o, \epsilon)\) and \(B(f^n o, \epsilon)\), it has finite Hausdorff distance to a subpath of \(\gamma\) which is contracting. Because the contracting property is preserved up to a finite Hausdorff distance by \((2)\) of Proposition \(2.2\), we conclude that \(\alpha\) is contracting as well. \(\square\)

Lemma 3.4. For any \(D > 0\), there exists \(n\) such that \(P_{D,C} \subset V_{\epsilon,M,f^n}\), where \(C > 0\) is given by Lemma \(3.3\).

Proof. The proof is similar to that of Lemma \(3.2\). We give it for completeness.

Lemma 3.3. \(\prooftext\)

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Lemma 3.3. \(\prooftext\)
from which we infer
\[(5) \quad d(m, hm) \leq d(m, \alpha-) + d(\alpha-, h\alpha) + d(h\alpha, hm) \leq 3D,\]
for \(d(\alpha+, h\alpha-) \leq D\) was assumed in \([4]\).

Now, for \(\alpha \in N_M(GO)\), let \(k \in G\) such that \(d(ko, m) \leq M\). By \([4]\), \(d(ko, hko) \leq 2M + 3D\). However, \(d(o, ho) \leq d(ko, hko)\) holds by the minimal choice of \(h\). By the choice of \(D > 2M\), we get a contradiction with \(d(o, ho) > 4D\). So the Condition (LL2) is fulfilled.

\[\text{Figure 1. Proof of Proposition 3.5}\]

**Condition (BP):** Up to a translation, it suffices to prove
\[d^+_\gamma(\{\alpha_+, h\alpha_\}) \leq \tau\]
and
\[d^+_\gamma(\{h^{-1}\alpha_+, \alpha_-\}) \leq \tau\]
for some constant \(\tau > 0\) given below. We only prove the first inequality: the second one is symmetric.

To derive a upper bound of \(d^+_\gamma(\{\alpha_+, h\alpha_-\})\), we will do it for \(d^+_\gamma(\{\alpha_+, ho\})\) and \(d^+_\gamma(\{ho, h\alpha_-\})\). First of all, since \(\alpha\) is a contracting subsegment of the geodesic \([o, ho]\), we see \(d^+_\gamma(\{\alpha_+, ho\}) \leq 2C\) by a projection argument: project \(ho\) to a point \(z \in \alpha\) then by \(C\)-contracting property of \(\alpha\), we have \(d(z, [\alpha_+, ho]) \leq 2C\) so \(d(z, \alpha_+) \leq 2C\).

Note that \([ho, h\alpha_-] \cap N_C(\alpha) \neq \emptyset\) implies \(d^+_\gamma(\{ho, h\alpha_-\}) \leq C\) by the contracting property. Henceforth, it remains to consider the case \([ho, h\alpha_-] \cap N_C(\alpha) \neq \emptyset\).

We prove now that \(\alpha_+ \in [m, ho]\). Indeed, otherwise we have \(\alpha_+ \in [o, m]\), then \(d(o, m) < d(\alpha_+, ho)\).

It then follows:
\[d(ho, h\alpha_-) = d(o, \alpha-) = d(o, m) - \ell(\alpha) - d(\alpha_+, m) < d(ho, ho) - D.\]

On the other hand, since \([ho, h\alpha_-] \cap N_C(\alpha) \neq \emptyset\) is assumed, we see \(d(\alpha_+, ho) \leq C + d(ho, ho\alpha_-)\). By the choice of the constant \(D > C\), this contradicts to the above inequality. Hence, \(\alpha_+ \in [m, ho]\) is proved.

Before proceeding further, we note the following two additional facts.

1. We project \(ho\alpha_-\) to a point \(x \in \alpha\). By the assumption that \(\alpha\) is \(C\)-contracting and \([ho, ho\alpha_-] \cap N_C(\alpha) \neq \emptyset\), we see that there exists \(y \in [ho, ho\alpha_-]\) such that \(d(x, y) \leq 2C\).
2. Taking \(\alpha \in N_M(GO)\) into account, there exist \(M > 0\) and \(f \in G\) such that \(d(x, fo) \leq M\). Thus,
\[(6) \quad d(fo, hfo) \leq d(fo, x) + d(x, y) + d(y, hx) + d(hx, hfo) \leq 2M + 2C + d(y, hx).\]

By the observation above that \(\alpha_+ \in [m, ho]\), we examine the following two cases, the second case of which will be proven impossible.

**Case 1.** Assume that \(\alpha_- \in [m, ho]\). By the position of \(y \in [ho, ho\alpha_-]\) and \(hx \in [ho\alpha_-, h^2o]\), we see
\[(7) \quad d(y, hx) = d(ho, h^2o) - d(hx, h^2o) - d(y, ho) \leq d(o, ho) - 2d(y, ho) + 2C,\]
where we used \(d(x, y) \leq 2C\). By the minimal choice of \(h\), we have \(d(o, ho) \leq d(o, hfo)\) so by combining (4) and (7), \(d(y, ho) \leq M + 2C\). Since \(d(x, y) \leq 2C\) and \(\alpha, \in [x, ho]\), we have \(d(x, \alpha) \leq d(x, ho) \leq M + 4C\). Because \(\alpha\) is a subsegment of a geodesic \([o, ho]\), the endpoint \(\alpha_+\) of \(\alpha\) must be a projection point of \(ho\). Hence, \(d(x, \alpha) \leq M + 4C\).

**Case 2.** Otherwise, we have \(m \in \alpha\) and then \(d(x, m) \leq D\) for \(x \in \alpha\). Since \(d(x, y) \leq 2C\) and \(d(o, m) = d(ho, m)\), we have

\[
|d(ho, y) - d(ho, hm)| = |d(ho, y) - d(ho, m)| \leq d(y, m) \leq 2C + D
\]
yielding

\[
d(y, hx) = |d(ho, hx) - d(ho, y)| \leq |d(ho, y) - d(ho, hm)| + d(x, m) \leq 2D + 2C.
\]

We thus derive from (6) that

\[
d(fo, hfo) \leq D + 2M + 4C.
\]

However, the minimal choice of \(h\) gives the following

\[
4D \leq d(o, ho) \leq d(fo, hfo).
\]

Since \(D > 2M + 2C\), we get a contradiction and the **Case 2** is impossible.

Setting \(\tau = 2M + 8C\), this completes the verification of the Condition (BP). In a word, we have proved that \(\gamma\) is a \((D, \tau)\)-admissible path. Moreover, \(\gamma\) is a \((D, \tau, L, 0)\)-admissible path where \(L := d(\alpha_+, ho_+)\).

We now choose the constant \(D = D(\tau) > 0\) by Proposition 2.7(2), so there exists a constant \(C' = C'(L, D)\) such that \(\gamma\) is a \(C'\)-contracting quasi-geodesic.

Recall that \(\gamma = \cup_{n \in \mathbb{Z}} h^n(\alpha \cdot \{\alpha_+, ho_+\})\) so \(\gamma\) has a finite Hausdorff distance to an orbit of \(h\). Therefore, \(h\) is a contracting element: this contradicts to the assumption that \(g \in \mathcal{NC}\). The proposition is thus proved.

We are now ready to complete the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Let \(D = D(C)\) be the constant given by Proposition 3.5 where \(C\) is given by Lemma 3.3. Let \(M\) be the constant satisfying Lemma 3.2 so that \(K_{M, D} \subset \mathcal{V}_{\epsilon, M, f^n}\). Meanwhile, \(\mathcal{P}_{D, C} \subset \mathcal{V}_{\epsilon, M, f^n}\) by Lemma 3.4.

Every element \(g \in \mathcal{NC}\) has conjugating representative \(h \in G\) in one of the three categories in Proposition 3.5. In the first and third cases, the set of such \(h\) has been proved to be contained in the barrier-free set \(\mathcal{V}_{\epsilon, M, f^n}\). Note there are only finitely many possibilities that \(d(o, ho) \leq 4D\). Clearly, we can raise \(n\) such that \(d(o, f^n)\) is sufficiently large so these \(h\) lie in \(\mathcal{V}_{\epsilon, M, f^n}\) as well. So the proof is finished.

4. Genericity of contracting elements

4.1. **Statements and corollaries.** This subsection is to derive various genericity results, including Theorem A from a more general technical theorem as follows.

Let \(\epsilon, M > 0\) be given by Theorem 2.10. Given a contracting element \(f \in G\), denote by \(\mathcal{BF}\) the set of elements \(g \in G\) admitting conjugacy representatives in a barrier-free set \(\mathcal{V}_{\epsilon, M, f^n}\) (for simplicity, the constant \(M\) will be omitted below).

**Theorem 4.1.** Assume that the action of \(G\) on \(Y\) is SCC. Then the set \(\mathcal{BF}\) is growth-tight: there exists \(\epsilon > 0\) such that

\[
\#(N(o, n) \cap \mathcal{BF}) \leq \exp(-\epsilon n) \cdot \#(N(o, n))
\]

for all \(n \gg 0\).

If the action is only assumed to be proper, then the set \(\mathcal{BF}\) is growth-negligible.

Therefore, Theorem A follows as a direct consequence.
Proof of Theorem A. By Theorem 3.1, the set \( N \) of non-contracting elements admits minimal conjugacy representatives in a barrier-free set \( \mathcal{V}_{\epsilon,M,f} \) for some contracting \( f \). Hence, \( N \) is either growth-tight or growth-negligible by the nature of the action, proving Theorem A.

To give further corollaries of Theorem 4.1, we consider a weakly quasi-convex subgroup defined in [40]. A subset \( X \) in \( Y \) is called weakly \( M \)-quasi-convex for a constant \( M > 0 \) if for any two points \( x, y \) in \( X \) there exists a geodesic \( \gamma \) between \( x \) and \( y \) such that \( \gamma \subseteq N_M(X) \). Then a subgroup \( \Gamma \) is weakly quasi-convex if \( \Gamma \) is weakly \( M \)-quasi-convex for some \( M > 0 \) and \( \epsilon \in Y \). The following result is proven in [40, Theorem 4.8].

Lemma 4.2. Suppose \( G \) admits a proper action on \( (Y,d) \) with a contracting element. Then every infinite index weakly quasi-convex subgroup \( \Gamma \) of \( G \) is contained in a barrier-free set \( \mathcal{V}_{\epsilon,M,f} \) for some contracting \( f \).

We now prove that hyperbolic elements are exponentially generic in relatively hyperbolic groups, i.e.:

Proof of Theorem 1.7. By Theorem A it remains to show that all parabolic elements is growth-tight. Since there are finitely many conjugacy classes, we only need to consider one maximal parabolic subgroup \( P \) and their conjugates. Since \( P \) is quasi-convex (cf. [24, Lemma 3.3]), we see by Lemma 4.2 that \( P \) is contained in a barrier-free set \( \mathcal{V}_{\epsilon,M,f} \) for some contracting \( f \). So Theorem 4.1 implies that all elements conjugated into \( P \) is growth-tight. Therefore, the set of hyperbolic elements is exponentially generic.

By Lemma 4.2 the following corollary generalizes the previous result in [40] that convex-cocompact subgroups are growth-tight.

Lemma 4.3. The set of elements conjugated into a fixed convex-cocompact subgroup \( \Gamma \) of \( G \in \text{Mod} \) is growth-tight.

In light of this result, the last statement of Theorem 1.4 is proved.

4.2. Proof of Theorem 4.1. By the definition of \( g \in \mathcal{BF} \), write

\[ g = kkhk^{-1} \]

for some \( k \in G \), where \( h \) is an \((\epsilon,f)\)-barrier-free element. Denote \( \gamma = [o, go] \).

The core of the proof is the following proposition whose proof will be given in next three sections. It says that we can write the element \( g = k'gk'^{-1} \) as an “almost geodesic” product, where the elements \( k', \hat{g} \) have the properties stated in the following.

Proposition 4.4 (Almost geodesic form). There exist a constant \( \Delta > 0 \) and an element \( \hat{f} \in E(f) \) with the following property. Denote \( Z := E(f) \cup \mathcal{V}_{\epsilon,\Delta,f} \). For each \( g \in \mathcal{BF} \), there exist \( k' \in G \) and \( \hat{g} = k'^{-1}gk' \in Z \) and two points \( s, t \in \gamma \) such that

\[ \max\{d(k' \cdot o, s), d(k' \cdot \hat{g}o, t)\} \leq \Delta. \]

Assuming Proposition 4.4, we complete the proof of Theorem 4.1. For \( \Delta > 0 \), we consider the annulus

\[ A(o, n, \Delta) = \{g \in G : \text{divides.alt0} d(o, go) - n < \text{divides.alt0} \Delta\}, \]

and the following holds for any \( \Gamma \subset G \),

\[ \limsup_{n \to \infty} \frac{\log \| A(o, n, \Delta) \cap \Gamma \|}{n} = \omega(\Gamma). \]

Denote the distance \( l := d(s, t) \). We thus have

\[ |d(o, go) - l| \leq 2\Delta, \]
where $\hat{g} \in A(o,l,2\Delta) \cap Z$ and $|2d(o,k'o) - n + l| \leq 4\Delta$.

so $k' \in A(o,(n-l)/2,2\Delta)$.

Consequently, for each $n \geq 1$, the number of elements $g \in A(o,n,\Delta) \cap BF$ is upper bounded by

\[
\sum_{1 \leq l \leq n} \| (A(o,(n-l)/2,2\Delta)) \cdot \| (A(o,l,2\Delta) \cap Z) ).
\]

Recall that $E(f)$ is elementary, i.e.: virtually $Z$.

If the action is SCC, then the set $Z := \{ E(f) \cup \mathcal{V}_{r,\Delta,f} \}$ is a growth-tight set with exponent $\omega_1 < \omega(G)$ such that

\[
\| (A(o,l,2\Delta) \cap Z) < \exp(\omega_1 l)
\]

for $l > 0$. Otherwise, if the action is proper, then $Z$ is growth-negligible:

\[
\frac{\| (A(o,l,2\Delta) \cap Z) }{\exp(\omega(G)l)} \to 0.
\]

By definition of $\omega(G)$ in \([8]\), there exist a constant $\omega_2 < 2\cdot \omega(G)$ such that

\[
\| (A(o,(n-l)/2,2\Delta)) \cdot \| \exp(\omega_2(n-l)/2).
\]

1). For SCC actions, the above sum \([9]\) is upper bounded by

\[
\sum_{1 \leq l \leq n} c \cdot \exp(\omega(G)(n-l)/2) \cdot \exp(\omega_1 l) \leq c \cdot n \cdot \exp(\omega_0 n)
\]

where $\omega_0 := \max\{\omega_2/2, \omega_1\} < \omega(G)$. A direct computation shows that $BF$ is growth-tight:

\[
\omega(A(o,n,\Delta) \cap \mathcal{N}C) \leq \lim_{n \to \infty} \frac{\log(c \cdot n \cdot \exp(\omega_0 n))}{n} \leq \omega_0 < \omega(G).
\]

2). For proper actions, the above sum \([9]\) is bounded by

\[
\sum_{1 \leq l \leq n} c \cdot \exp(\omega(G)(n-l)/2) \cdot \| (A(o,l,2\Delta) \cap Z) \]

\[
\leq \exp(\omega_2(n-l)/2) \cdot o(\omega(G)l) \leq o(\omega(G)n)
\]

so $BF$ is growth-negligible.

Therefore, the theorem is proved, modulo Proposition \([4.4]\) whose proof will take up the next three sections.

5. More preliminary: Projection complex and a quasi-tree of spaces

The purpose of this section is to recall a construction, due to Bestvina, Bromberg and Fujiwara, of projection complex and correspondingly, a quasi-tree of spaces, a blown-up of it. We assume certain familiarity with their construction and refer to \([8]\) for details.

In our concrete setting, let us just point out that a contracting system $\mathcal{K}$ with bounded intersection satisfies the axioms in \([6]\) (cf. \([11]\) Appendix) for this fact. In what follows, we examine their construction and derive a few consequences in this specific setting.

We first introduce a notion of interval in $\mathcal{K}$ (cf. \([9]\) Theorem 2.3.G]). For $K > 0$, denote $\mathcal{K}_K(Y,Z)$ by the set of $W \in \mathcal{K} \setminus \{Y,Z\}$ such that $d^\mathcal{K}_W(Y,Z) > K$. For two points $y, z \in \mathcal{K}$, the set $\mathcal{K}_K(y,z)$ is defined as the collection of $W \in \mathcal{K}$ such that $d^\mathcal{K}_W(y,z) > K$.

**Projection complex.** The projection complex $P_K(\mathcal{K})$ is a graph such that the vertex set is $\mathcal{K}$, and two vertices $Y \neq Z \in \mathcal{K}$ is adjacent if $\mathcal{K}_K(Y,Z) = \emptyset$.

In fact, their construction requires a slightly variant, $d_\mathcal{X}(Y,Z)$, of the distance-like function $d^\mathcal{K}_\mathcal{X}(Y,Z)$. However, this does not matter in the interest of the present paper since $d^\mathcal{K}_\mathcal{X}(Y,Z) \sim d_\mathcal{X}(Y,Z)$ by \([6]\) Theorem 2.3.B].

Furthermore, assume that $\mathcal{K}$ is preserved by a group action of $G$ on $Y$. Their fundamental result is then stated as follows.
Theorem 5.1. \cite{6} Theorem D] There exists $K > 0$ such that $P_K(X)$ is a quasi-tree on which $G$ acts co-boundedly.

Quasi-tree of spaces. Following the adjacency in $P_K(X)$, a quasi-tree of spaces $X$ is constructed to recover the geometry of each $X \in X$ in $P_K(X)$ where they were condensed to be one point.

For a constant $N > 0$, a quasi-tree $C_N(X)$ of spaces is obtained by taking the disjoint union of $X$ with edges of length $N$ connecting each pair of points $(y, z)$ in $(\pi_Y(Z), \pi_Z(Y))$ if $X_K(Y, Z) = \emptyset$. We denote by $d_C$ the induced length metric.

A technical issue is that $X \in X$ might not be connected or even so, the induced metric on $X$ may differ from the one on the ambient space $Y$. Since $X$ is quasi-convex, this could be overcome by taking a $C$-neighborhood of $X$ such that any geodesic with endpoints in $X$ lies in $N_C(X)$. It is readily seen that $N_C(X)$ is connected and its induced metric agrees with $d_Y$ up to a uniform additive error (for instance $4C$). For convenience, each $X \in X$ is assumed to be a metric graph by its the Vietoris-Rips complex. See discussion \cite[Section 3.1]{6}.

Theorem 5.2. \cite{6} Theorem E] For $N \gg K$, $C_N(X)$ contains $X$ as totally geodesic subspaces and for any two $Y, Z \in C_N(X)$, the shortest projection of $Y$ to $Z$ in $C_N(X)$ is uniformly close to the set $\pi_Z(Y)$. Moreover, if every $X \in X$ are uniformly hyperbolic spaces, then $C_N(X)$ is hyperbolic.

In a hyperbolic space, a totally geodesic subspace is quasiconvex so it is contracting. Recall that the bounded intersection property is equivalent to the bounded projection property in \cite[Lemma 2.3]{42}. Hence, by Theorem 5.2 $X$ is a contracting system with bounded intersection in $C_N(X)$.

At last, the following result shall be important in next section.

Lemma 5.3. \cite{6} Lemma 3.11] There are $\tilde{K} > K, R > 0$ so that for any $y$ and $z$ in $C_N(X)$, any geodesic $[y, z]$ passes within $R$-neighborhood of $\pi_X(y)$ and $\pi_X(z)$ for each $X \in X_K(y, z)$.

6. Projecting in the quasi-tree of spaces

Let $X := \{g \cdot Ax(f) : g \in G\}$ be the collection of contracting sets with bounded intersection. Denote by the same $C > 0$ the contraction and bounded intersection constants for $X$. According to Section 5 we consider a projection complex $P_K(X)$ and its quasi-tree of spaces $C_N(X)$ endowed with length metric $d_C$.

Consider the quadrangle $\triangle_{g=khk^{-1}}$ by four (oriented) geodesics $\gamma = [o, go], p = [o, ko], \alpha = [ko, kho]$ and $q = [go, kho]$ as depicted in Figure 2. The important relation $q = g \cdot p$ will be implicit in our discussion.

We now give an overview of this section. After some preliminary observations, we project the quadrangle $\triangle_{g=khk^{-1}}$ into $C_N(X)$ and show that the top geodesic $\alpha$ becomes uniformly bounded.

Since the goal of Theorem 5.3 is to prove that $BF$ is a growth-tight (resp. growth-negligible) set, without loss of generality, we can assume $g \in \mathcal{V}_{\epsilon, f}$ for some $\tilde{f} \in E(f)$: the bottom geodesic $\gamma$ contains an $(\epsilon, f)$-barrier. The element $\tilde{f}$ will be made sufficiently “long” in a quantitative sense.

Note that any $(\epsilon, f)$-barrier in $\gamma$ gives rise to a “sufficiently long” barrier in the left side $p$, for $\tilde{f}$ is relatively longer. By looking at the projected quadrangle in $C_N(X)$, the hyperbolicity of $C_N(X)$ (cf. Lemma 5.3) allows to argue that this long barrier in the left side $p$ has to intersect boundedly with the right side $q$. This is the goal of this section, Lemma 6.6 which provides the base of the further analysis in next Section.

6.1. Some auxiliary lemmas. We begin with an elementary observation facilitating some computations.

Lemma 6.1. For any $X \in X$ and any geodesic $p$, the following holds:

$$d_C(p) \leq 4C + \text{diam}(N_C(X) \cap p).$$
Proof. Denote by $p_1$ the part of $p$ before entering into $N_C(X)$, and by $p_2$ the part of $p$ after exiting $N_C(X)$. It is possible that $p_1, p_2$ may be trivial paths. Hence, the proof is completed by a projection argument as follows:

$$d_X^\pi(p) \leq d_X^\pi(p_1 \cup p_2) + d((p_1)_+, X) + d((p_2)_-, X) + \text{diam}(N_C(X) \cap p) \leq 4C + \text{diam}(N_C(X) \cap p),$$

where $d_X^\pi(p_1 \cup p_2) \leq 2C$ follows by contracting property, and

$$d((p_1)_+, X), d((p_2)_-, X) \leq C$$

since $(p_1)_+$ and $(p_2)_-$ both belong to $N_C(X)$.

\[ \square \]

Lemma 6.2 (Long intersection $\Rightarrow$ Existence of barrier). There exist $\epsilon > 0$ and $L = L(f) > 0$ such that if $\alpha$ is a geodesic satisfying

$$\text{diam}(\alpha \cap N_C(\text{Ax}(f))) > L,$$

then it contains an $(\epsilon, f)$-barrier.

Proof. Recall that $\gamma := \text{Ax}(f)$ is a contracting quasi-geodesic. By hypothesis, we see that $\alpha$ contains a subpath $\bar{\alpha}$ of length at least $L$ which has finite Hausdorff distance to a subpath $\bar{\gamma}$ of $\gamma$. By Assertions (9) and (11) of Proposition 2.2, we have that $\bar{\alpha}$ is also contracting and thus is $\epsilon$-quasi-convex for some $\epsilon = \epsilon(C)$. A priori, we can choose $L$ large enough such that $\bar{\gamma}$ contains a subpath labeled by $f$. From the $\epsilon$-quasi-convexity of $\bar{\alpha}$, this subsegment stays in the $\epsilon$-neighborhood of $\bar{\alpha}$, so produces an $(\epsilon, f)$-barrier as required.

Let $L > 0$ be the constant supplied by Lemma 6.2.

Lemma 6.3. Let $\alpha$ be a geodesic side in the quadrangle $\Box = kk'$. Then

$$d_X^\pi(\alpha) \leq 4C + L$$

for any $X \in \mathcal{K}$.

Proof. By Lemma 6.1, it suffices to prove that $\text{diam}(N_C(X) \cap \alpha) \leq L$. If the inequality does not hold, then $\alpha = k \cdot [a, ho]$ contains an $(\epsilon, f)$-barrier by Lemma 6.2, this contradicts to the first reduction of $h$ that $[\alpha, ho]$ is $(\epsilon, f)$-barrier-free. The proof is thereby complete.

\[ \square \]

The point of the next lemma is to determine the appropriate constants $K, N$ for these spaces.

Lemma 6.4. There exists a constant $N > 0$ with the following property.

Let $\alpha$ be an $(\epsilon, f)$-barrier-free geodesic between $Y$ and $Z$ in $\mathcal{K}$. Then the endpoints $\alpha_-, \alpha_+$ of $\alpha$ is uniformly bounded in $C_N(\mathcal{K})$ as follows:

$$d_C(\alpha_-, \alpha_+) \leq 2L + N.$$

Proof. Consider the geodesic $\bar{\alpha} = [\alpha_-, \alpha_+]$ in $C_N(\mathcal{K})$ so the goal is to estimate the length of $\bar{\alpha}$.

We first consider the projection complex $\mathcal{P}_K(\mathcal{K})$ where the constant $K > L + 2C$ is given by Theorem 5.1. Observe that $\mathcal{K}_K(Y, Z)$ is empty. Indeed, since $\alpha$ is $(\epsilon, f)$-barrier-free, it follows by Lemma 6.2 that $\text{diam}(\alpha \cap N_C(X)) < L$ for any $X \in \mathcal{K}$. Note that $d_X^\pi(Y), d_X^\pi(Z) \leq C$. Then $d_X^\pi(Y, Z) \leq \text{diam}(\alpha \cap N_C(X)) + d_X^\pi(Y) + d_X^\pi(Z) \leq L + 2C$. Hence, $\mathcal{K}_K(Y, Z) = \emptyset$ so $Y, Z$ are adjacent in $\mathcal{P}_K(\mathcal{K})$.

We then construct a quasi-tree of spaces $\mathcal{C}_N(\mathcal{K})$ where $N = N(K)$ is given by Theorem 5.2. Since $Y, Z$ are adjacent in $\mathcal{P}_K(\mathcal{K})$, by construction, the subsets $\pi_Y(Z), \pi_Z(Y)$ in $\mathcal{C}_N(\mathcal{K})$ are connected by edges of length $N$.

By a similar reasoning, we obtain $d(\alpha_-, d_X^\pi(\alpha_+)), d(\alpha_+, d_X^\pi(\alpha_-)) \leq L$. Recalling in Section 5 we make the assumption that the induced metric on $Y, Z$ is identical to the ambient metric on $Y$, up to a uniform error (for simplicity we ignore it here). Moreover, since $Y, Z$ are isometrically embedded in $\mathcal{C}_N(\mathcal{K})$, there exist two paths $\alpha_1, \alpha_2$ of length at most $L$ respectively in $Y, Z$ such that $\alpha_1$ connects $\alpha_-$ and $d_X^\pi(\alpha_+)$, and $\alpha_2$ connects $\alpha_+$ and $d_X^\pi(\alpha_-)$. 

\[ \square \]
Moreover, the endpoints of \( \bar{\alpha} \) are connected by a path composing \( \alpha_i \) with such an edge of length \( N \), hence \( d_C(ko, kho) \leq 2L + N \). The proof is then complete.

6.2. **Constants.** Let \( R > 0 \) be a constant given by Lemma 5.3. Since \( X \) has bounded intersection in \( C_N(X) \), there exists \( D = D(R) \) such that

\[
\forall X \neq X' \in \mathcal{X}: \text{diam}_C\left(N_R(X) \cap N_R(X')\right) \leq D
\]

where the diameter \( \text{diam}_C \) is computed using the metric \( d_C \).

We also choose the following constant

\[
(10) \quad \tilde{K} > 4C + 2R + 3L + N + D.
\]

satisfying the conclusion of Lemma 5.3, a constant \( A \) as follows

\[
(11) \quad A = L + \tilde{K} + 116C.
\]

Choose a “long” element \( \tilde{f} \in E(f) \) such that

\[
(12) \quad d(o, \tilde{f}o) > 2(25C + A + L + \tilde{K} + \epsilon),
\]

By Theorem 2.10, the set \( \mathcal{V}_{\epsilon, f} \) is growth-tight (resp. growth-negligible). Since the goal is to prove that \( \mathcal{BF} \) is a growth-tight (resp. growth-negligible) set, without loss of generality, that we can assume \( g \notin \mathcal{V}_{\epsilon, f} \). By definition 2.9, there exists an element \( b \in G \) such that

\[
\max\{d(b \cdot o, \gamma), d(b \cdot \tilde{f}o, \gamma)\} \leq \epsilon,
\]

whence by (12) this gives

\[
(13) \quad \text{diam}(N_C(X) \cap \gamma) \geq 2(25C + A + L + \tilde{K}),
\]

where \( X = b \cdot Ax(f) \).

By abuse of language, we will say hereafter that the element \( b \) is an \((\epsilon, \tilde{f})\)-barrier of \( \gamma \).

**Lemma 6.5.** For each \((\epsilon, \tilde{f})\)-barrier \( b \) of \( \gamma \), the following holds

\[
\text{diam}(N_C(X) \cap p) + \text{diam}(N_C(X) \cap q) > 2A
\]

where \( X = b \cdot Ax(f) \).

**Proof.** Let \( x, y \) denote the entry and exit points of \( \gamma \) in \( N_C(X) \) respectively. If we had

\[
\text{diam}(N_C(X) \cap p) + \text{diam}(N_C(X) \cap q) \leq 2A,
\]

then by Lemma 6.3 we would obtain the following:

\[
d(x, y) \leq d(x, X) + d_X^\gamma(p) + d_X^\gamma(\alpha) + d_X^\gamma(q) + d(y, X) \\
\leq 14C + \text{diam}(N_C(X) \cap p) + \text{diam}(N_C(X) \cap \alpha) + \text{diam}(N_C(X) \cap q) \\
\leq 18C + 2A + L,
\]

where \( d(x, X), d(y, X) \leq C \) for \( x, y \in N_C(X) \). This results a contradiction with (13), so the lemma is proved.

6.3. **Bounded intersection in quadrangle.**

**Lemma 6.6.** Let \( X \in \mathcal{X} \) such that \( gX \neq X \) and \( \text{diam}(N_C(X) \cap p) > A \). Then \( \text{diam}(N_C(X) \cap q) \leq \tilde{K} \) and \( \text{diam}(N_C(X) \cap \gamma) > 100C \).

**Proof.** The idea of proof is to project the quadrangle \( \square_{g=khk^{-1}} \) to a quadrangle in \( C_N(X) \) with the corresponding geodesics \( \tilde{\gamma} = [o, go], \tilde{p} = [o, ko], \tilde{\alpha} = [ko, kho] \) and \( \tilde{q} = [go, kho] \).

Suppose by way of contradiciton that \( \text{diam}(N_C(X) \cap q) > \tilde{K} \) so \( X \in \mathcal{X}_K(go, kho) \) by definition. For any \( X \in \mathcal{X}_K(o, ko) \), let \( v, w \) denote the corresponding exit points of \( \tilde{p} \) and \( \tilde{q} \) in \( N_R(X) \), where the constant \( R > 0 \) is given by Lemma 5.3. Hence,

\[
d_C(v, \pi_X(ko)), \quad d_C(w, \pi_X(kho)) \leq R.
\]
By Theorem 5.2, the subset $X$ is totally geodesic in $C_N(X)$ so
\[ d_C(\pi_X(\kappa_0), \pi_X(\kappa_0)) = d_X^\pi(\pi_X(\kappa_0), \pi_X(\kappa_0)) \]
where the right-hand side is the projection distance measured in $Y$. Noting also that
\[ d_X^\pi(\pi_X(\kappa_0), \pi_X(\kappa_0)) \leq d_X^\pi(\alpha) \leq 4C + L \]
where the second inequality follows by Lemma 6.3. As a consequence of the above three estimates, we obtain the following
\[ d_C(v, w) \leq d_C(v, \pi_X(\kappa_0)) + d_C(\pi_X(\kappa_0), \pi_X(\kappa_0)) + d_C(w, \pi_X(\kappa_0)) \leq 4C + 2R + L. \]

One needs $d_C(\kappa_0, \kappa_0) \leq N + 2L$ by Lemma 6.3. Therefore,
\[ |\ell_C([v, \kappa_0]) - \ell_C([w, \kappa_0])| \leq d_C(v, w) + d_C(\kappa_0, \kappa_0) \leq 4C + 2R + 3L + N, \]
where $\ell_C(\cdot)$ stands for the length of a path in $C_N(X)$.

On the other hand, $gX \in \mathcal{K}(go, \kappa_0)$, and $gv$ is the exit point of $\bar{\rho}$ in $N_R(gX)$ so
\[ \ell_C([v, \kappa_0]) = \ell_C([gv, \kappa_0]). \]

Recall that $\mathcal{X}$ has bounded intersection in $C_N(X)$ so for $gX \neq X \in \mathcal{X}$,
\[ \text{diam}_C(N_R(X) \cap N_R(gX)) \leq D. \]

Since $q = gp$ then $\bar{q} = g\bar{p}$, we obtain that $\text{diam}_C(N_R(X) \cap \bar{p}) = \text{diam}_C(N_R(gX) \cap \bar{q}) > \bar{K}$. Consequently,
\[ |\ell_C([w, \kappa_0]) - \ell_C([v, \kappa_0])| > \min\{\text{diam}_C(N_R(X) \cap q), \text{diam}_C(N_R(gX) \cap q)\} - D > \bar{K} - D. \]

Via (15), this yields a contradiction to (14) since it was assumed in (10) that
\[ \bar{K} > 4C + 2R + 3L + N + D. \]

Hence, $\text{diam}(q \cap N_C(X)) \leq \bar{K}$ is proved.

Lastly, let us prove that $\text{diam}(N_C(X) \cap \gamma) > 100C$. If not, then $d_X^\pi(\gamma) \leq 104C$ by Lemma 6.1. Let $x, y$ be the entry and exit points of $p$ in $N_C(X)$ respectively so
\[ d_X^\pi(p \cup [x, y]) \leq 2C \]
by contracting property. Since $d_X^\pi(q) \leq 4C + \bar{K}$ follows by Lemma 6.1 and $d_X^\pi(\alpha) \leq 4C + L$ by Lemma 6.3, we see by a projection argument that
\[ \text{diam}(N_C(X) \cap p) = d(x, y) \leq d(x, X) + d_X^\pi(\alpha \cup \gamma) + d_X^\pi(q) + d_X^\pi(p \cup [x, y]) + d(w, X) \leq L + \bar{K} + 116C < \bar{A}, \]
which is a contradiction. The lemma is thus proved.

\[ \square \]

7. Almost geodesic decomposition

We are now ready to prove Proposition 4.4, the last ingredient in the proof of Theorem 4.1.

Let’s first outline the proof: The consequence of Lemma 6.3 in the previous section provides a contracting set $X \in \mathcal{X}$ which has a large $A$-intersection with $p$, but intersect $q$ in a bounded amount by $\bar{K}$. The constant $A$ (11) is chosen sufficiently large relative to $\bar{K}$ (10).

We then focus on the collection of such $X$ with this property, of which the last one intersecting $p$ is given a particular focus on. For the relation $q = gp$, $gX$ is also the last for $q$. In Proposition 7.2 we establish, case by case, that the intersection of $\gamma$ with the pair of $(X, gX)$ provides an almost geodesic product of $g$. 
In the sequel, the following fact is frequently used, whose proof is straightforward by the contracting property and left to the reader.

Lemma 7.1 (Fellow entry/exit). Let $X$ be a $C$-contracting subset in $Y$. Consider two geodesics $\alpha, \beta$ issuing from the same point and both intersecting $N_C(X)$. Then their corresponding entry points of $\alpha, \beta$ in $N_C(X)$ have a distance at most $4C$.

Let us repeat Proposition 4.4 with some additional quantifiers.

Proposition 7.2 (Almost geodesic form). There exists $\Delta = \Delta(C,L) > 0$ with the following property. Denote $Z := E(f) \cup \cup_{s,\Delta, X}$. For each $g \in BF$, there exist $k' \in G$ and $\hat{g} = k^{-1}gk' \in Z$ and two points $s, t \in \gamma$ such that

$$\max\{d(k' \cdot s), d(k' \cdot t)\} \leq \Delta.$$

Proof. By Lemma 6.6, we consider the set of $X \in \mathbb{X}$ such that $\text{diam}(N_C(X) \cap p) > A$ and $\text{diam}(N_C(X) \cap g) \leq K$. The proof shall treat two mutually exclusive configurations.

**Configuration I.** Assume that there exists a contracting set $X$ such that $gX = X$. Let $x, y$ be the corresponding entry and exit points of $\gamma$ in $N_C(X)$.

If $z, w$ denote the entry and exit points of $p$ in $N_C(X)$ respectively then do $gz, gw$ for $q$ in $N_C(gX)$. By Lemma 7.1, we have $d(z, x) \leq 4C$ and $d(gz, y) \leq 4C$, which implies

$$d(gz, y) \leq 8C.$$

For concreteness, assume that $X = bAx(f)$ for some $b \in G$. In addition, let $M$ be diameter of fundamental domain of the action of $E(f)$ on $Ax(f)$, so there exists $k' \in bE(f)$ such that

$$d(k' \cdot x, x) \leq M + C$$

and

$$d(gk' \cdot y, y) \leq d(gk' \cdot x, x) + d(gx, y) \leq M + 9C.$$

Since $b \cdot Ax(f) = gb \cdot Ax(f)$, it follows by definition of $E(f)$ that $g \in bE(f) b^{-1}$. Thus, the element $\hat{g} := k^{-1}gk'$ lies in $E(f) \subset Z$ for the above $k' \in bE(f)$.

Setting $\Delta := M + 5C$, $s = x$ and $t = y$ the desired points on $\gamma$, the proof in Configuration I is finished.

![Figure 2. Case 1 (left) and Case 2 (right) in Proposition 7.2](image)

**Configuration II.** Let $X$ be a last contracting set $X \in \mathbb{X}$ for $p$ with the following defining property:

$$\text{diam}(N_C(X) \cap p) > A$$

and

$$\text{diam}(N_C(X) \cap \cup_{\gamma, p_+}) \leq A,$$
where \( w \) is the exit point of \( p \) in \( N_C(X) \). By the symmetry of \( q = gp \), it then follows that \( gX \) is the last for \( q \) as well. Keep in mind that \( gX \neq X \) through out the discussion of this Configuration.

Denoting by \( x', y' \) the entry and exit points respectively in \( N_C(gX) \), we are led to deal with the following two cases.

**Case 1.** \([x, y]\) \( \gamma \) appears before \([x', y']\) \( \gamma \). Noting \( \text{diam}(N_C(X) \cap q) \leq \tilde{K} \), we obtain \( d_X(q) \leq \tilde{K} + 4C \) from Lemma 6.1. Together, \( d_X(q) \leq L + 4C \) by Lemma 6.3 we see that

\[
\begin{align*}
d(w, y) & \leq d(w, X) + d_X([w, p], p) + d_X([q, u \cup q]) + d(y, X) \\
& \leq 12C + L + \tilde{K},
\end{align*}
\]

where \( d_X([w, p], p), d_X([y, \gamma], \gamma) \leq C \) by contracting property, and \( d(w, X), d(y, X) \leq C \).

The same argument shows

\[
\begin{align*}
d(gw, x') & \leq 12C + L + \tilde{K}.
\end{align*}
\]

Combining (16) and (17), we thus obtain

\[
d(gy, x') \leq d(gy, gw) + d(gw, x') \leq 2(12C + L + \tilde{K}).
\]

Again, the cocompact action of \( E(f) \) on \( Ax(f) \) provides an element \( k' \in bE(f) \) such that

\[
d(k' o, y) \leq C + M
\]

and

\[
d(gk' o, x') \leq d(gk' o, gy) + d(gy, x') \\
\leq 2(12C + L + \tilde{K}) + M + C.
\]

Denote \( s = y, t = x' \). Setting \( \tilde{g} := k'^{-1} gk' \) and \( \Delta = 2(13C + L + \tilde{K}) + M \), it suffices to establish the following claim which then implies \( \tilde{g} \in \mathcal{V}_{\epsilon, \Delta, f} \), completing the proof of the Case 1.

**Claim.** \([y, x']\) \( \gamma \) is \((\epsilon, \tilde{f})\)-barrier-free.

**Proof of the claim.** If not, then \([y, x']\) \( \gamma \) contains an \((\epsilon, \tilde{f})\)-barrier \( b \), so for \( X := b, Ax(f) \), we have

\[
\begin{align*}
d_X(\gamma \setminus [u, v], \gamma) & \leq 2C
\end{align*}
\]

by the contracting property, where \( u, v \) are the entry and exit points of \( \gamma \) respectively in \( N_C(X) \).

Denoting \( p_1 := [w, p], q_1 := [q, gw] \), we deduce

\[
\max\{\text{diam}(p_1 \cap N_C(X)), \text{diam}(q_1 \cap N_C(X))\} \leq A,
\]

from the assumption that \( X \) and \( gX \) are last. This implies, by Lemma 6.1 that

\[
d_X(p_1 \cup p_2) \leq 2A + 8C.
\]

On the other hand, applying Proposition 2.2.5, we obtain from (16) and (17) that

\[
\begin{align*}
\max\{d_X([w, y]), d_X([gw, x'])\} & \leq 13C + L + \tilde{K}.
\end{align*}
\]

Finally, consider the hexagon formed by \([y, w], p_1, \alpha, q_1, [gw, x'] \) and \([y, x']\). Recall that \( d_X(\alpha) \leq 4C + L \) by Lemma 6.3. Using the above estimates from (18), (19) and (20), we obtain

\[
\begin{align*}
\text{diam}(N_C(X) \cap \gamma) & \leq d(w, X) + d_X([w, p_1 \cup \alpha q_1]) + d_X([w, v], \gamma) \\
& + d_X([w, y]) + d_X([gw, x']) + d(v, X) \\
& \leq 50C + 2A + 2L + 2\tilde{K},
\end{align*}
\]

a contradiction with the inequality (13). The claim is thus proved. \(\square\)

**Case 2.** \([x', y']\) \( \gamma \) appears before \([x, y]\) \( \gamma \). We obtain \( d(z, x) \leq 4C \) and \( d(gz, y') \leq 4C \) by Lemma 7.1 so

\[
\begin{align*}
d(gx, y') & \leq 8C
\end{align*}
\]

which gives by Proposition 2.2.5:

\[
\begin{align*}
d_X([gx, y')] & \leq 9C.
\end{align*}
\]
Let us look at two geodesics \([y', go]_\gamma\) and \([gx, go]_\gamma\) with at the same terminal point but their initial points \(8C\)-apart. First of all, observe that

\[ N_C(X) \cap [gx, go]_\gamma \neq \emptyset. \]

Indeed, if not, we would obtain from the contracting property that

\[ d_X^\gamma([gx, go]) \leq C, \quad d_X^\gamma([y', go] \setminus [x, y]_\gamma) \leq 2C \]

and hence

\[
\text{diam}(N_C(X) \cap \gamma) \leq d_X^\gamma([gx, go]) + d_X^\gamma([y', go]) + d_X^\gamma([y', go] \setminus [x, y]_\gamma) + d(x, X) + d(y, X) \\
\leq 14C,
\]

where \(d_X^\gamma([y', go]) \leq 9C\) by (22) and \(d(x, X), d(y, X) \leq C\). However, this contradicts to the conclusion of Lemma \(\ref{lem:9}\). Hence, \(N_C(X) \cap [gx, go]_\gamma \neq \emptyset\).

So, consider the entry point \(u\) of \([gx, go]_\gamma\) in \(N_C(X)\). We next need to determine the points \(s, t\) on \(\gamma\). By a projection argument, it follows that:

\[ d(x, u) \leq d(x, X) + d_X^\gamma([y', x]) + d_X^\gamma([y, x]) + d(x, U) \leq 13C, \]

where (22) is used and \(d_X^\gamma([gx, u]), d_X^\gamma([y', x]) \leq C\) by contracting property. Since \(d(gx, y') \leq 8C\) in (21) we have

\[ d(gy, u) = |d(gx, u) - d(gx, gy)| \leq d(gx, y') + d(x, u) \leq 21C, \]

which in turn shows

\[ d(gy, x) \leq d(gy, u) + d(u, x) \leq 34C. \]

Denoting \(s = y', t = x\) gives our desired points. As in Case (1), it remains to prove the following.

Claim. \([y', x]_\gamma\) is \((\epsilon, \hat{f})\)-barrier-free as well.

Proof of the claim. Indeed, suppose to the contrary that \(b_\ast\) is a barrier of \([y', x]_\gamma\) so for \(X_\ast := b_\ast Ax(f)\), the inequality \(\ref{eq:26}\) implies

\[
\text{diam}(N_C(X_\ast) \cap \gamma) > A + 13C.
\]

Since \(d(z, x) \leq 4C\) by Lemma \(\ref{lem:8}\), we get \(d_X^\gamma([z, x]) \leq 5C\) by Proposition \(\ref{prop:2.2.2}\).

Denote \(\beta = [p_\ast, x]_p\). We look at the triangle at left corner (cf. Figure \(\ref{fig:2}\)). We shall prove that \(\text{diam}(N_C(X_\ast) \cap \beta) > A\). If \(\text{diam}(N_C(X_\ast) \cap \beta) \leq A\) so \(d_X^\gamma(\beta) \leq A + 4C\) by Lemma \(\ref{lem:6.1}\), then we obtain

\[
\text{diam}(N_C(X_\ast) \cap \gamma) \leq d_X^\gamma(\beta) + d_X^\gamma([z, x]) + d_X^\gamma(\beta \setminus N_C(X_\ast)) + 2C \\
\leq A + 13C
\]

where \(d_X^\gamma(\beta \setminus N_C(X_\ast)) \leq 2C\) by contracting property. This contradicts to the above inequality \(\ref{eq:23}\) so it follows as desired:

\[
\text{diam}(N_C(X_\ast) \cap p) > \text{diam}(N_C(X_\ast) \cap \beta) > A.
\]

Similarly, we obtain \(\text{diam}(N_C(X_\ast) \cap q) > A > \hat{K}\) by looking at the triangle at right corner (cf. Figure \(\ref{fig:2}\)). By Lemma \(\ref{lem:6.6}\) we must have \(gX_\ast = X_\ast\). This contradicts to the assumption made in Configuration II. Therefore, such a barrier \(b_\ast\) does not exist so the claim is established. \(\square\)

With \(s = x', t = x\), we can follow the same line in the Case (1) using the action of \(E(f)\) on \(Ax(f)\). So the element \(\hat{g} := k^{-1} g k'\) is \((\epsilon, \Delta, \hat{f})\)-barrier-free for \(\Delta = M + 35C\). This concludes the proof of Configuration II and thereby the proposition. \(\square\)
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