SPECTRAL ALGEBRAIC CURVES AND DECOMPOSABLE OPERATOR TUPLES

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Abstract. Joint spectra of tuples of operators are subsets in complex projective space. The corresponding tuple of operators can be viewed as an infinite dimensional analog of a determinantal representation of the joint spectrum. We investigate the relationship between the geometry of the spectrum and the properties of the operators in the tuple when these operators are self-adjoint. In the case when the spectrum contains an algebraic curve passing through an isolated spectral point of one of the operators we give necessary and sufficient geometric conditions for the operators in the tuple to have a common reducing subspace. We also address spectral continuity and obtain a norm estimate for the commutant of a pair of self-adjoint matrices in terms of the Hausdorff distance of their joint spectrum to a family of lines.

1. Introduction

When \( A_1, \ldots, A_n \) are \( N \times N \) complex matrices, the determinant

\[
S(x_1, \ldots, x_n) = \det(x_1 A_1 + \cdots + x_n A_n)
\]

is a homogeneous polynomial of degree \( N \) in the variables \( x_1, \ldots, x_n \), and zeros of this polynomial determine a hypersurface in the projective space \( \mathbb{C}P^{n-1} \). Conversely, given a hypersurface \( \Gamma \) of degree \( N \) in \( \mathbb{C}P^{n-1} \), if there are \( N \times N \) matrices \( A_1, \ldots, A_n \) such that

\[
\Gamma = \{ \det(x_1 A_1 + \cdots + x_n A_n) = 0 \}
\]

then the tuple \( (A_1, \ldots, A_n) \) is called a determinantal representation of \( \Gamma \). Of course, if a determinantal representation exists, then for every pair of invertible \( N \times N \) matrices \( M \) and \( N \), the tuple of matrices \( (MA_1N, \ldots, MA_nN) \) determines the same projective hypersurface, and these two determinantal representations are called equivalent.

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A classical avenue of research in algebraic geometry with a long history, see e.g. [1, 9, 10, 18, 30, 25], is determining when a given hypersurface admits a determinantal representation, and classifying all such representations. If the hypersurface is real (that is, the corresponding polynomial has real coefficients), when is there a determinantal representation with self-adjoint matrices? When is there a determinantal representation with triangular, or decomposable matrices, etc.? We would like to mention specifically self-adjoint determinantal representations of real curves, and decomposable representations of reducible curves, since they are close to the subject of this paper. The former produce hyperbolic polynomials and are important in relation to the Lax conjecture, cf. [19, 20, 15]. The latter have special meaning in operator theory. These representations were considered in [18].

The point of view from operator theory leads us to a second natural avenue of research, that seems to have attracted less attention in algebraic geometry: given that a hypersurface has a determinantal representation (or self-adjoint representation), what does the geometry of the hypersurface say about mutual relationships between the matrices $A_1, \ldots, A_n$? In this direction we would like to mention the result of Motzkin and Taussky [23], which states that a real curve in $\mathbb{CP}^2$ with a self-adjoint determinantal representation satisfies the condition: the matrices of the corresponding tuple commute if and only if this curve is a union of projective lines (in [23] the result is stated in equivalent but different terms).

In 2009 R. Yang [31] started an investigation of what can be called infinite dimensional determinantal representations. Since well-known definitions of spectra of a tuple of operators such as Taylor spectrum, cf. [28, 11], are defined for commuting tuples, Yang was looking for a good definition of joint spectrum for non-commuting operators and introduced the notion of joint spectrum of a tuple $(A_1, \ldots, A_n)$ of operators acting on a Hilbert space $H$:

**Definition.** The joint spectrum $\sigma(A_1, \ldots, A_n)$ of $A_1, \ldots, A_n$ consists of all $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that the operator $x_1A_1 + \cdots + x_nA_n$ is not invertible on $H$. If $A_n = I$, the identity operator, the proper part of the joint spectrum of $A_1, \ldots, A_{n-1}$ is defined as $\sigma_p(A_1, \ldots, A_{n-1}) = \sigma(A_1, \ldots, A_{n-1}, I) \cap \{x_n = -1\}$

Joint spectra were further investigated in [8, 11, 27]. It is easily seen that if $(x_1, \ldots, x_n) \in \sigma(A_1, \ldots, A_n)$, then the whole complex line $\{(cx_1, \ldots, cx_n) : c \in \mathbb{C}\}$ lies in $\sigma(A_1, \ldots, A_n)$, and, therefore, $\sigma(A_1, \ldots, A_n)$ determines a set in $\mathbb{CP}^{n-1}$. By analogy with the finite dimensional case, given a set $\Gamma$ in $\mathbb{CP}^{n-1}$, if there are operators $A_1, \ldots, A_n$
acting on a Hilbert space $H$ such that
\[ \Gamma = \{ [x_1 : \ldots : x_n] \in \mathbb{C}P^{n-1} \mid x_1 A_1 + \cdots + x_n A_n \text{ is not invertible} \}, \]
then it is natural to call the tuple $(A_1, \ldots, A_n)$ a spectral representation of $\Gamma$. The main difference compared to the classical matrix case is that this set is not necessarily an analytic set. For example, if $A_1$ and $A_2$ are compact and of infinite rank, and $A_3$ is invertible, the whole line $\{ [x_1 : x_2 : 0] \}$ in $\mathbb{C}P^2$ is contained in the joint spectrum and the spectrum is not an analytic set near each point of this line. It was shown in [27] that if $A_1, \ldots, A_{n-1}$ are compact and $A_n$ is invertible (and, therefore, can be considered to be identity) the part of the joint spectrum that lies in the chart $\{ x_n \neq 0 \}$ is an analytic set. When the operators $A_1, \ldots, A_{n-1}$ are trace class, that part of the joint spectrum is given by the equation
\[ S(x_1, \ldots, x_{n-1}) = \det(x_1 A_1 + \cdots + x_{n-1} A_{n-1} - I) = 0, \]
and we obtain that in this case the spectral representation is a “true” determinantal representation. In particular, when all the operators are of finite rank, the joint spectrum is a classical determinantal hypersurface in $\mathbb{C}P^{n-1}$. Of course, for infinite rank operators the analyticity holds only on an open subset of $\mathbb{C}P^{n-1}$ and that moves the problem of describing properties of the joint spectrum from the area of projective geometry to analytic geometry.

We are unaware of any results addressing the question of when a spectral representation exists. We will see in the next section that if the operators $A_1, \ldots, A_{n-1}$ are compact, then the proper part of the spectrum $\sigma_p(A_1, \ldots, A_{n-1})$ is a globally determined analytic set of codimension one in every compact subset of the chart $\{ x_n \neq 0 \}$. It would be interesting to know for which analytic sets that are zeros of entire functions spectral representations exist. We will also show that if $(x_0, y_0)$ is a point in the joint spectrum of $A_1$ and $A_2$ such that $x_0 A_1 + y_0 A_2$ is normal and 1 is an isolated spectral point of $x_0 A_1 + y_0 A_2$ of finite multiplicity, then the joint spectrum is an analytic set in a neighborhood of $(x_0, y_0)$.

A main goal of our paper is to address the relationship between the geometry of the spectrum and the mutual behavior of the operators in the pencil. There is a recent result [3] which is a generalization of the Motzkin-Taussky theorem mentioned above to the infinite dimensional case. This result states that a tuple $(A_1, \ldots, A_n)$ of self-adjoint compact operators acting on a separable Hilbert space commute pairwise if and only if their proper joint spectrum $\sigma_p(A_1, \ldots, A_n)$ is a locally finite
union of affine hyperplanes (of course, local finiteness is not a condition but just a property coming from compactness).

Clearly, if the operators $A_1, \ldots, A_n$ have a common eigenvector with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, then the proper joint spectrum $\sigma_p(A_1, \ldots, A_n)$ contains a hyperplane $\{\lambda_1 x_1 + \cdots + \lambda_n x_n = 1\}$. More generally, if these operators have a common invariant subspace $L$ of dimension $k$ (so the corresponding tuple is decomposable with one block having dimension $k$), then the proper spectrum contains an algebraic hypersurface of degree $k$ given by

$$\det (x_1(A_1|_L) + \cdots + x_n(A_n|_L)) = 0.$$  

It is natural to ask whether the converse is true: if the proper spectrum of self-adjoint operators $A_1, \ldots, A_n$ contains an algebraic hypersurface of degree $k$, does this imply that those operators have a common invariant $k$-dimensional subspace, so that the tuple is decomposable? In general the answer is negative as shown by the following simple example. Consider

$$\Gamma = \{(x, y) \in \mathbb{C}^2 : (x + y - 1)(5xy + 5y^2 - 15y - 10x + 2) = 0\}$$

and let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 1/2 \end{bmatrix}.$$ 

It is easy to check that $\Gamma$ is the set of zeros of the determinant of the monic pencil $xA_1 + yA_2 - I$, and, therefore, is the proper joint spectrum of $A_1$ and $A_2$. Note that $\Gamma$ has two irreducible components: the line $\{x + y = 1\}$ and the quadratic $\{5xy + 5y^2 - 15y - 10x + 2 = 0\}$; but $A_1$ and $A_2$ have neither a common eigenvector nor a common two-dimensional invariant subspace (in fact $\Gamma$ has no decomposable determinantal representation as it can also be shown using one of the results in [18]).

Our first main task is to give necessary and sufficient geometric conditions for the presence of an algebraic curve in the proper joint spectrum to indicate decomposability, that is the existence of a common invariant subspace of dimension equal to the degree of the curve. Our consideration will be concentrated on a local analysis near an isolated spectral point of one of the operators in the tuple. In particular, when one of the operators in the tuple is self-adjoint and is a sum of compact and real multiple of the identity, the spectrum of such an operator has at most one accumulation point and our consideration is valid near every spectral point of this operator excluding the accumulation one.
We also address the issue of spectral continuity, that is, if two curves are close in a neighborhood of a point, both have self-adjoint spectral representation of which one is decomposable, how far from being decomposable is the other? The specific question we are considering is: given that the proper joint spectrum is close to a line in a neighborhood of one of its points, does this mean that the operators have a common “almost eigenvector” (common “almost invariant” subspace)? Results in Sections 7 and 8 present conditions that guarantee that this is true, and give some norm estimates.

Let us now describe in more detail the main results of the paper. The first important case of the problem when an algebraic curve in the proper joint spectrum is associated with a common invariant subspace is the case of a spectral line. This case turns out to be crucial for higher order spectral curves. The following result is proved in Section 4 (here, as well as in the rest of the paper, we denote by $\Delta_\rho(x)$ the polydisk of radius $\rho$ centered at $x \in \mathbb{C}^n$).

**Theorem A.** Let $A_1, \ldots, A_n$ be self-adjoint, $\lambda \neq 0$ be an isolated point of $\sigma(A_1)$, and there exists $\rho > 0$ such that, up to multiplicity,

$$
\Delta_\rho(1/\lambda, 0, \ldots, 0) \cap \{\lambda x_1 + a_2 x_2 + \cdots + a_n x_n = 1\}
= \Delta_\rho(1/\lambda, 0, \ldots, 0) \cap \sigma_p(A_1, \ldots, A_n).
$$

The following are equivalent:

1. The eigensubspace of $A_1$ corresponding to eigenvalue $\lambda$ is an eigensubspace for each of the operators $A_2, \ldots, A_n$;
2. There exist an $\epsilon \in \mathbb{R}$, $\epsilon \neq 1$, and $\rho' > 0$ such that $A_1(\epsilon, \lambda)$ is invertible and, up to multiplicity,

$$
\Delta_{\rho'}(\lambda, 0, \ldots, 0) \cap \{(1/\lambda) x_1 + a_2 x_2 + \cdots + a_n x_n = 1\}
= \Delta_{\rho'}(\lambda, 0, \ldots, 0) \cap \sigma_p(A_1(\epsilon, \lambda)^{-1}, A_2(\epsilon, a_2), \ldots, A_n(\epsilon, a_n)),
$$

where $A(\epsilon, b) = (1 + \epsilon)A - bI$.

The most important case here is the one of two operators. Theorem B below is obtained from this case by passing to tensor powers of operators and considering their action on the exterior power of the corresponding Hilbert space. Here for an operator $A$ acting on a Hilbert space $H$ we write $\wedge^n A$ to indicate that we consider the action of $\otimes^n A$ on $\wedge^n H$. We say that a self-adjoint operator $A$ on a separable Hilbert space $H$ belongs to the class $\mathcal{E}(H)$ if $A = K + aI$ for some compact self-adjoint operator $K$ and some $a \in \mathbb{R}$. It is shown in Section 5 that for operators $A$ and $B$ in $\mathcal{E}(H)$ one can always use an appropriate change of coordinates to reduce the search for common invariant subspaces to the “general position” setting considered in our next main result:
Theorem B. Let $A = K_1 + aI$ and $B = K_2 + bI$ be self-adjoint operators in the class $\mathcal{E}(H)$, with $A$ invertible. Let $\Gamma$ be an algebraic curve of degree $k$ which is a union of components of the proper joint spectrum $\sigma_p(A, B)$, and which does not have the line $\{ax + by = 1\}$ as a reduced component. Suppose that the $x$-axis (resp. the $y$-axis) intersects $\Gamma$ in the $k$ points (counted with multiplicity) $1/\lambda_1, \ldots, 1/\lambda_k$ (resp. $1/\mu_1, \ldots, 1/\mu_k$) such that each point $(1/\lambda_i, 0)$ belongs only to components of $\sigma_p(A, B)$ contained in $\Gamma$, Set $\lambda = \lambda_1 \ldots \lambda_k$ and $\mu = \mu_1 \ldots \mu_k$, and suppose that $\lambda$ is an isolated eigenvalue of multiplicity 1 in the spectrum of $\wedge^k A$. The following are equivalent:

1. The eigenspace for $A$ corresponding to $\lambda_1, \ldots, \lambda_k$ is invariant for $B$.
2. There exists $\rho > 0$ such that the line segments $\{\lambda x + \mu y = 1\} \cap \Delta_\rho(1/\lambda, 0)$ and $\{(1/\lambda)x + \mu y = 1\} \cap \Delta_\rho(\lambda, 0)$ are contained in $\sigma_p(\wedge^k A, \wedge^k B)$ and $\sigma_p(\wedge^k A^{-1}, \wedge^k B)$, respectively.
3. The lines $\{\lambda x + \mu y = 1\}$ and $\{(1/\lambda)x + \mu y = 1\}$ are contained in $\sigma_p(\wedge^k A, \wedge^k B)$ and $\sigma_p(\wedge^k A^{-1}, \wedge^k B)$, respectively.

An extension of this result to a tuple of arbitrary length holds here as well, and is derived from Theorem B exactly the same way as the result of Theorem A is derived from the corresponding result for two operators. For this reason its precise statement is omitted. Application to the classical setting when $A_1, \ldots, A_n$ are self-adjoint operators on $\mathbb{C}^N$ yields the following theorem (here, as explained in Section 5, for an invertible self-adjoint operator $A$ we consider $\wedge^{N-k} A$ in a natural way as an operator on $\wedge^k \mathbb{C}^N$, and again, we can always reduce to the “general position” situation considered below).

Theorem C. Let $C$ be a reducible real algebraic hypersurface of degree $N$ in $\mathbb{C}^n$, and let $\Gamma$ be a degree $k$ hypersurface that is a union of components of $C$, such that for each $i$ the $x_i$-axis intersects $\Gamma$ in the $k$ points $1/\alpha_{i1}, \ldots, 1/\alpha_{ik}$, counted with multiplicities. Let $\alpha_i = \alpha_{i1} \ldots \alpha_{ik}$ and suppose also that each point $1/\alpha_{ij}$ belongs only to components of $C$ contained in $\Gamma$. Let a tuple $(A_1, \ldots, A_n)$ consisting of self-adjoint operators on $\mathbb{C}^N$, with $A_1$ invertible, be a determinantal representation of $C$, and suppose that $\alpha_1$ is an eigenvalue of multiplicity 1 for $\wedge^k A_1$. 


This representation induces a determinantal representation of $\Gamma$ if and only if the hypersurface
\[
\{ \det(x_1 \wedge^k A_1 + \cdots + x_n \wedge^k A_n - I) = 0 \}
\]
contains the hyperplane \{ $a_1 x_1 + \cdots + a_n x_n = 1$ \}, and the hypersurface
\[
\{ \det(x_1 \wedge^{N-k} A_1 + x_2 \wedge^k A_2 + \cdots + x_n \wedge^k A_n - I) = 0 \}
\]
contains the hyperplane \{ $(\det A_1/a_1)x_1 + a_2 x_2 + \cdots + a_n x_n = 1$ \}.

Now we turn to spectral continuity. For a positive $\epsilon$ we say that a vector $\xi$ is an $\epsilon$-eigenvector of an operator $A$ (almost eigenvector) if there exists $\lambda$ such that
\[
\| A\xi - \lambda \xi \| < \epsilon \| \xi \|.
\]
Our first result regarding spectral continuity, Theorem 7.3, states that, under some natural assumptions, if the joint spectrum of a pair $(A_1, A_2)$ of self-adjoint operators, with $A_1$ invertible, is $\epsilon$-close in the Hausdorff metric to a line \{ $a x + \beta y = 1$ \} in a neighborhood of an isolated spectral point of $A_1$, and the same is true for the joint spectrum of the pair $(A_1^{-1}, A_2)$, then they have a common almost eigenvector of order $\sqrt{\epsilon}$. If $|\beta| = \| A_2 \|$, the condition on the joint spectrum of $A_1^{-1}$ and $A_2$ can be omitted. As a corollary to this result we obtain the following estimate for the commutant of two self-adjoint matrices.

**Theorem D.** Let $A_1$ and $A_2$ be two self-adjoint $N \times N$ matrices with eigenvalues $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$ respectively, satisfying $|\alpha_1| > \cdots > |\alpha_N| > 0$ and $|\beta_1| > \cdots > |\beta_N| > 0$. Suppose that $\ell_1, \ldots, \ell_N$ is a family of lines,
\[
\ell_j = \{ \alpha_{n(j)} x + \beta_j y = 1 \}, \ 1 \leq j, n(j) \leq N,
\]
such that:

1. each of the points $(\frac{1}{\alpha_k}, 0), \ 1 \leq k \leq N$ belongs to one of these lines;
2. there exist $0 < \rho < 1$ and $0 < \epsilon \ll \rho$ such that conditions (1) and (2) of Theorem 7.22 are true for $\sigma_p(A_1, A_2)$ and each $\ell_j$.

Then if $\epsilon$ is small enough, the norm of the commutant of $A_1$ and $A_2$ is at most of order $\epsilon^{1/2^N}$.

The structure of this paper is as follows. Section 2 is devoted to determining functions. In Section 3 we find conditions for an algebraic curve to be a part of the proper joint spectrum. These conditions are expressed in terms of holomorphy of a sequence of certain operator-valued functions. In Section 4 we prove Theorem A. Theorems B and C are proved in Section 5. In Section 6 we give an example of a circle
in the spectrum, where the existence of a common reducing subspace is derived not from exterior products, but directly from the vanishing residues conditions found in Section 3. Section 7 is devoted to spectral continuity. Theorem D is proved in Section 8. Finally, Section 9 contains several concluding remarks and open questions.

We would like to thank R. Yang for useful comments.

2. Determining functions

Let $A_1$ and $A_2$ be bounded operators on a Hilbert space $H$. Recall that the proper part of the projective joint spectrum $\sigma(A_1, A_2, I)$, or just the proper projective spectrum $\sigma_p(A_1, A_2)$ is the following set:

$$\sigma_p(A_1, A_2) = \{(x, y) \in \mathbb{C}^2 : (x, y, -1) \in \sigma(A_1, A_2, I)\}.$$ 

It was shown in [27] that if $A_1$ and $A_2$ are compact, then $\sigma_p(A_1, A_2)$ is an analytic set of codimension one in $\mathbb{C}^2$. The following explicit construction of the analytic function locally determining this set was given there. We present it for tuples of self-adjoint operators: the setting we consider in this paper.

Suppose that $A_1, \ldots, A_m$ is a tuple of compact self-adjoint operators on a Hilbert space $H$. Choose a small $\epsilon > 0$ and finite rank self-adjoint operators $K_1, \ldots, K_m$ such that $\|A_j - K_j\| < \epsilon$. If $(w_1, \ldots, w_m) \in \mathbb{C}^m$ satisfy $\sum_{j=1}^m |w_j| < \frac{1}{\epsilon}$, then the operator $I - \sum_{j=1}^m w_j(A_j - K_j)$ is invertible and we have

$$\sum_{j=1}^m w_jA_j - I = \sum_{j=1}^m w_jK_j - I + \sum_{j=1}^m w_j(A_j - K_j)$$

$$= \left( I - \sum_{j=1}^m w_j(A_j - K_j) \right) \left( \sum_{l=1}^m w_l \left( I - \sum_{j=1}^m w_j(A_j - K_j) \right)^{-1} K_l - I \right).$$

Thus, $(w_1, \ldots, w_m) \in \sigma_p(A_1, \ldots, A_m)$ if and only if the operator

$$\sum_{l=1}^m w_l \left( I - \sum_{j=1}^m w_j(A_j - K_j) \right)^{-1} K_l - I$$

is not invertible. Since $\sum_{l=1}^m w_l \left( I - \sum_{j=1}^m w_j(A_j - K_j) \right)^{-1} K_l$ is of finite rank, there is a finite dimensional subspace $L$ of $H$ such that this operator vanishes on the complement to this subspace and is represented by an $n \times n$ matrix on this subspace. Therefore, this operator is not invertible if and
only if
\[
(2.1) \quad \det \left( \sum_{l=1}^{m} w_l \left( I - \sum_{j=1}^{m} w_j (A_j - K_j) \right)^{-1} K_l - I \right) = 0.
\]

The left-hand side of (2.1) is clearly an analytic function of \( w_1, \ldots, w_m \) in the domain \( \{ \sum_{j=1}^{m} |w_j| < \frac{1}{r} \} \) and (2.1) determines \( \sigma_p(A_1, \ldots, A_m) \) in this domain. We call this function a determining function of the proper projective spectrum. It follows that a different choice of the finite rank approximations leads to a determining function with the same divisor of zeros in \( \{ (w_1, \ldots, w_m) \in \mathbb{C}^m : \sum_{j=1}^{m} |w_j| < \frac{1}{r} \} \).

If \( A_1 \) and \( A_2 \) are not compact, the joint spectrum is not necessarily an analytic set. For example, if \( A_1 = I \) is the identity operator, the joint spectrum is a cone with vertex at \( (1, 0) \) that consists of lines \( \{ x + \lambda y = 1 \} \), \( \lambda \in \sigma(A_2) \). Thus, if the cardinality of \( \sigma(A_2) \) is infinite, the joint spectrum is not analytic at \( (1, 0) \). Nevertheless, essentially the same argument we used above to show the analyticity of the joint spectrum in the compact case, establishes the following local result.

**Theorem 2.2.** Let \( A_1 \) and \( A_2 \) be bounded operators on \( H \), with \( A_1 \) normal, and \( \lambda \neq 0 \) be an isolated point of \( \sigma(A_1) \) of finite multiplicity. Then \( \sigma_p(A_1, A_2) \) is an analytic set in a neighborhood of \( (\frac{1}{\lambda}, 0) \).

**Proof.** The spectral decomposition of \( A_1 \) is in the form
\[
A_1 = \lambda P_1 + \int_{\sigma(A_1) \setminus \{\lambda\}} zdE(z),
\]
where the operator \( P_1 \) is the finite rank orthogonal projection of \( H \) onto the eigenspace of \( A_1 \) with eigenvalue \( \lambda \) and \( dE \) is the spectral measure on the rest of \( \sigma(A_1) \). Since \( \lambda \) is an isolated spectral point of \( A_1 \), if \( (x, y) \) is close to \( (\frac{1}{\lambda}, 0) \), the operator
\[
\tilde{A}(x, y) = x \int_{\sigma(A_1) \setminus \{\lambda\}} zdE(z) + yA_2 - I
\]
is invertible. Therefore, such a point \( (x, y) \) belongs to the joint spectrum if and only if the operator
\[
B(x, y) = xP_1 \tilde{A}(x, y)^{-1} - I
\]
is not invertible. Since \( xP_1 \tilde{A}(x, y) \) has finite rank \( n \) which is equal to the rank of \( P_1 \), the pairs \( (x, y) \) for which \( B(x, y) \) is not invertible are zeros of a determinant of an \( n \times n \) matrix, whose coefficients are analytic functions of \( (x, y) \), and the result follows. \( \square \)

If the multiplicity of an isolated spectral point \( \lambda \in \sigma(A_1) \) is equal to one, the local analyticity of the joint spectrum holds even without \( A_1 \) being normal.
Let $A$ be a bounded operator acting on $H$, and let $\lambda$ be an isolated spectral point of $A$. Recall that $\lambda$ is said to have multiplicity $k$ if for a contour $\gamma$ in the resolvent set that contains $\lambda$ as the only spectral point of $A$

$$P_\lambda = \frac{1}{2\pi i} \int_\gamma (wI - A)^{-1}dw,$$

is a rank $k$ projection (not necessarily orthogonal).

**Theorem 2.4.** Let $A_1$ and $A_2$ be operators on $H$ and $\lambda \neq 0$ be an isolated spectral point of $A_1$ of multiplicity one. Then there exists $\rho > 0$ such that in $\Delta_\rho(\frac{1}{x}, 0)$ the proper joint spectrum $\sigma_\rho(A_1, A_2)$ is a nonsingular analytic set.

**Proof.** If $\rho$ is small enough and $(x, y) \in \Delta_\rho(\frac{1}{x}, 0)$, the operator $A(x, y) = xA_1 + yA_2$ has an isolated spectral point $\lambda(x, y)$ close to 1 that also has multiplicity one, so the projection

$$P(x, y) = \frac{1}{2\pi i} \int_\gamma (wI - A(x, y))^{-1}dw$$

has rank one, cf. [13, p. 13], and the range of $P(x, y)$ consists of eigenvectors of $A(x, y)$ with eigenvalue $\lambda(x, y)$. The joint spectrum of $A_1$ and $A_2$ consists of those pairs $(x, y)$ for which $\lambda(x, y) = 1$. Let $e$ be the unit eigenvector of $A_1$ with eigenvalue $\lambda$. Then for $(x, y) \in \Delta_\rho(\frac{1}{x}, 0)$ we have that $P(x, y)e$ is close to $e$, and, therefore, $P(x, y)e \neq 0$. Now, $\lambda(x, y) = 1$ if and only if $A(x, y)P(x, y)e = P(x, y)e$ and that happens if and only if

$$\langle (A(x, y)P(x, y) - P(x, y))e, e \rangle = 0.$$ 

Equation (2.5) determines $\sigma_\rho(A_1, A_2)$ near the point $(\frac{1}{x}, 0)$ and it is easily seen that the left-hand side is analytic in $x$ and $y$. Now we write down explicitly the Taylor decomposition of this function in terms of $\Delta x = x - \frac{1}{x}$ and $y$. We have

$$A(x, y)P(x, y) - P(x, y) = \frac{1}{2\pi i} \int_\gamma (w - 1)(wI - A(x, y))^{-1}dw$$

$$= \frac{1}{2\pi i} \int_\gamma (w - 1)(wI - \frac{1}{\lambda}A_1)^{-1} \left( I - (\Delta x A_1 + yA_2)(wI - \frac{1}{\lambda}A_1)^{-1} \right)^{-1}dw$$

$$= \sum_{j=0}^{\infty} \frac{1}{2\pi i} \int_\gamma (w - 1)(wI - \frac{1}{\lambda}A_1)^{-1} \left[ (\Delta x A_1 + yA_2)(wI - \frac{1}{\lambda}A_1)^{-1} \right]^j dw$$

$$= \sum_{k, m=0}^{\infty} \Delta x^k y^m \left( \frac{1}{2\pi i} \int_\gamma (w - 1)(wI - \frac{1}{\lambda}A_1)^{-1}D_{k, m}(w)dw \right),$$

where

$$D_{k, m}(w) = \sum_{\alpha} \prod_{l=1}^{k+m} S_{\alpha l}(w),$$
with summation taken over all sequences \( \alpha = (\alpha_1, \ldots, \alpha_{k+m}) \) of zeros and ones of length \((k + m)\) having \(k\) zeros and \(m\) ones, and

\[
S_0 = A_1(wI - \frac{1}{\lambda} A_1)^{-1}, \quad S_1 = A_2(wI - \frac{1}{\lambda} A_1)^{-1}.
\]

Thus we have

\[
\sigma_p(A_1, A_2) \cap \Delta_p \left( \frac{1}{\lambda}, 0 \right) = \left\{ (x, y) \in \Delta_p \left( \frac{1}{\lambda}, 0 \right) : \mathcal{F}(x, y) = 0 \right\},
\]

where

\[
\mathcal{F}(x, y) = \sum_{k,m=0}^{\infty} (x - \frac{1}{\lambda})^k y^m \left( \frac{1}{2\pi i} \int_\gamma (w - 1) ((wI - \frac{1}{\lambda} A_1)^{-1} D_{k,m}(w)e, e) dw \right).
\]

Obviously, \( \mathcal{F}(x, y) \) is a nontrivial analytic function, so the joint spectrum is an analytic set in \( \Delta_p \left( \frac{1}{\lambda}, 0 \right) \). Further, it follows directly from the Taylor decomposition above that

\[
\left. \frac{\partial \mathcal{F}}{\partial x} \right|_{x=\frac{1}{\lambda}, y=0} = \frac{1}{2\pi i} \int_\gamma (w - 1) ((wI - \frac{1}{\lambda} A_1)^{-1} A_1(wI - \frac{1}{\lambda} A_1)^{-1} e, e) dw = \frac{\lambda}{2\pi i} \int_\gamma \frac{dw}{w - 1} = \lambda \neq 0,
\]

and, therefore, the zero set of \( \mathcal{F} \) is nonsingular near \( \left( \frac{1}{\lambda}, 0 \right) \). \( \square \)

3. NECESSARY CONDITIONS FOR AN ALGEBRAIC CURVE IN THE JOINT SPECTRUM

Let \( A_1 \) and \( A_2 \) be self-adjoint operators and \( \lambda \neq 0 \) be an isolated point of \( \sigma(A_1) \) such that

(a) \( \sigma_p(A_1, A_2) \) in a neighborhood \( \Delta_p \left( \frac{1}{\lambda}, 0 \right) \) of \( \left( \frac{1}{\lambda}, 0 \right) \) is an algebraic curve given by a polynomial equation \( \mathcal{R}(x, y) = 0 \) of degree \( k \), where \( \mathcal{R} \) is a polynomial with real coefficients, that is

\[
\sigma_p(A_1, A_2) \cap \Delta_p \left( \frac{1}{\lambda}, 0 \right) = \left\{ (x, y) \in \Delta_p \left( \frac{1}{\lambda}, 0 \right) : \mathcal{R}(x, y) = 0 \right\};
\]

(b) \( \left( \frac{1}{\lambda}, 0 \right) \) belongs to only one reduced component of this curve and is a nonsingular point on this reduced component;

(c) the axis \( \{ y = 0 \} \) is not tangent to this reduced component of the curve at \( \left( \frac{1}{\lambda}, 0 \right) \).

Here for a curve, or more generally, for a hypersurface defined by a polynomial \( G = G_1^{r_1} \ldots G_m^{r_m} \) (with each polynomial \( G_i \) irreducible and \( G_i \) not associate with \( G_j \) for \( i \neq j \)), the components of that hypersurface are defined by the polynomials \( G_i^{r_i} \) (thus they are irreducible but not necessarily
reduced), and the reduced components are defined by the polynomials \( G_t \).

Write

\[
R(x, y) = \sum_{j=0}^{k} R_j(x, y), \text{ where } R_j = \sum_{m=0}^{j} r_m^j x^m y^{j-m}, \text{ and } R_0 = -1.
\]

Passing to a smaller neighborhood if necessary, we may assume that

(i) the reduced component containing \((1/\lambda, 0)\) of the curve \( \{R(x, y) = 0\} \) has no singular points in \( \Delta_\rho(\frac{1}{\lambda}, 0) \);

(ii) there is \( 0 < \rho' < \rho \) such that for \((x, y) \in \Delta_\rho(\frac{1}{\lambda}, 0) \) the complex line \( \{(\tau x, \tau y) : \tau \in \mathbb{C}\} \) has (up to multiplicity) exactly one point of intersection with \( \{R(x, y) = 0\} \) that lies in \( \Delta_{\rho'}(\frac{1}{\lambda}, 0) \).

Let \((x, y) \in \Delta_\rho(\frac{1}{\lambda}, 0) \) and \((\tau x, \tau y) \in \{R(x, y) = 0\} \). Then \( R(\tau x, \tau y) = 0 \), and the equation in \( \tau \)

\[
\tau^k R_k(x, y) + \tau^{k-1} R_{k-1}(x, y) + \cdots + \tau R_1(x, y) - 1 = 0
\]

has exactly one root, \( \tau(x, y) \), in a neighborhood of 1. The corresponding eigenvalue \( \mu(x, y) = 1/\tau(x, y) \) of the operator \( xA_1 + yA_2 \) satisfies the equation

\[
\mu^k - \mu^{k-1} R_1(x, y) - \cdots - R_k(x, y) = 0.
\]

Of course, \( \mu(x, y) \) is the only eigenvalue of \( xA_1 + yA_2 \) which lies at distance of order \( \rho \) from 1 and is an isolated point of the spectrum \( \sigma(xA_1 + yA_2) \). It is also clear that if \( \lambda \) is a multiple spectral point of \( A_1 \), then \( \mu(x, y) \) has the same multiplicity.

If both \( x \) and \( y \) are real, \( xA_1 + yA_2 \) is self-adjoint. Let \( \zeta(x, y) \) be an eigenvector of \( xA_1 + yA_2 \) with eigenvalue \( \mu(x, y) \). Then equation (3.2) implies

\[
[(xA_1 + yA_2)^k - R_1(x, y)(xA_1 + yA_2)^{k-1} - \cdots - R_k(x, y)] \zeta(x, y) = 0.
\]

Let \( L(x, y) \subset H \) be the eigensubspace of \( xA_1 + yA_2 \) corresponding to \( \mu(x, y) \), and let \( P(x, y) : H \to L(x, y) \) be the orthogonal projection. For \( 0 < \delta < \tau \) write \( \gamma = \{z \in \mathbb{C} : |z - 1| < \delta\} \). We have

\[
P(x, y) = \frac{1}{2\pi i} \int_{\gamma} (wI - (xA_1 + yA_2))^{-1} dw.
\]

It is readily seen that for \( m = 0, 1, 2, \ldots \)

\[
(xA_1 + yA_2)^m P(x, y) = \frac{1}{2\pi i} \int_{\gamma} w^m (wI - (xA_1 + yA_2))^{-1} dw.
\]

Equations (3.3) and (3.5) imply that for every \((x, y)\) sufficiently close to \((\frac{1}{\lambda}, 0)\) the following identity holds:

\[
\frac{1}{2\pi i} \int_{\gamma} \left[ w^k - \sum_{j=1}^{k} R_j(x, y) w^{k-j} \right] (wI - (xA_1 + yA_2))^{-1} dw = 0.
\]
Write $\Delta x = x - \frac{1}{x}$. If $\Delta x$ and $y$ are sufficiently small, the last relation implies

$$
\left(3.6\right) \frac{1}{2\pi i} \int_{\gamma} \left[ w^k - \sum_{j=1}^{k} R_j(x,y)w^{k-j} \right] (wI - \frac{1}{\lambda}A_1)^{-1} \times \sum_{m=0}^{\infty} \left[ (\Delta xA_1 + yA_2)(wI - \frac{1}{\lambda}A_1)^{-1} \right]^m dw = 0.
$$

If $\Delta x = 0$, the last relation turns into the following:

$$
\frac{1}{2\pi i} \int_{\gamma} \left[ w^k - \sum_{j=1}^{k} w^{k-j} \sum_{n=0}^{j} r_{j-n}^j y^n x^{j-n}_1 \right] (wI - x_1 A_1)^{-1} \times \sum_{m=0}^{\infty} y^m \left[ A_2(wI - \frac{1}{\lambda}A_1)^{-1} \right]^m dw = 0.
$$

Rearranging terms in the last equation, we obtain

$$
\sum_{m=0}^{k-1} \frac{y^m}{2\pi i} \int_{\gamma} \left\{ \left( wI - \frac{1}{\lambda}A_1 \right)^{-1} \times \left( \left[ w^k \sum_{j=1}^{k} \frac{w^{k-j}r_j^j}{\lambda^j} \right] A_2(wI - \frac{1}{\lambda}A_1)^{-1} \right)^m \right. \\
- \left. \sum_{n=1}^{m} \left[ \sum_{j=n}^{k} \frac{w^{k-j}r_j^j}{\lambda^{j-n}} \right] A_2(wI - \frac{1}{\lambda}A_1)^{-1} \right)^{m-n} \right\} dw
$$

$$
\left(3.7\right) + \sum_{m=k}^{\infty} \frac{y^m}{2\pi i} \int_{\gamma} \left\{ \left( wI - \frac{1}{\lambda}A_1 \right)^{-1} \times \left( \left[ w^k \sum_{j=1}^{k} \frac{w^{k-j}r_j^j}{\lambda^j} \right] A_2(wI - \frac{1}{\lambda}A_1)^{-1} \right)^k \right. \\
- \left. \sum_{n=1}^{k} \left[ \sum_{j=n}^{k} \frac{w^{k-j}r_j^j}{\lambda^{j-n}} \right] A_2(wI - x_1 A_1)^{-1} \right)^{k-n} \right) A_2(wI - \frac{1}{\lambda}A_1)^{-1} \left. \right\}^m dw = 0.
$$
Since (3.7) holds for every $y$ in a neighborhood of the origin, it implies

$$
(3.8) \quad \frac{1}{2\pi i} \int_{\gamma} \left\{ (wI - x_1 A_1)^{-1} \left( w^k - \sum_{j=1}^{k} \frac{w^{k-j}r_j^j}{\lambda^j} \right) [A_2(wI - \frac{1}{\lambda} A_1)^{-1}]^m 
- \sum_{n=1}^{m} \left( \sum_{j=n}^{k} \frac{w^{k-j}r_j^j - \lambda^{j-n}}{\lambda^{j-n}} \right) [A_2(wI - \frac{1}{\lambda} A_1)^{-1}]^{m-n} \right\} \, dw = 0,
$$

for $1 \leq m \leq k - 1$, and

$$
(3.9) \quad \frac{1}{2\pi i} \int_{\gamma} \left\{ \left( \frac{wI - 1}{\lambda} A_1 \right)^{-1} \times \left( w^k - \sum_{j=1}^{k} \frac{w^{k-j}r_j^j}{\lambda^j} \right) [A_2(wI - \frac{1}{\lambda} A_1)^{-1}]^k 
- \sum_{n=1}^{k} \left( \sum_{j=n}^{k} \frac{w^{k-j}r_j^j - \lambda^{j-n}}{\lambda^{j-n}} \right) [A_2(wI - \frac{1}{\lambda} A_1)^{-1}]^{k-n} \right) 
\times [A_2(wI - \frac{1}{\lambda} A_1)^{-1}]^{m-k} \right\} \, dw = 0
$$

for $m \geq k$. The integrands in (3.8) and (3.9) are operator-valued holomorphic functions in the punctured disk $\{ w \in \mathbb{C} : 0 < |w - 1| < \delta \}$ with poles at one. We denote these integrands by $\Psi_m(w)$, $m \geq 1$. Thus, (3.8) and (3.9) imply the following result:

**Theorem 3.10.** Suppose that $A_1$ and $A_2$ are self-adjoint operators acting on a separable Hilbert space $H$, with $\lambda \in \sigma(A_1)$ an isolated point, and an algebraic curve determined by a polynomial equation (3.1) lies in $\sigma_p(A_1,A_2)$ and satisfies conditions (a) - (c) above. Then the integrands $\Psi_m(w)$ of (3.8) and (3.9) satisfy the equation

$$
\frac{1}{2\pi i} \int_{\gamma} \Psi_m(w) \, dw = 0, \quad m \geq 1,
$$

which is equivalent to

$$
(3.11) \quad \text{Res}_{w=1}(\Psi_m) = 0, \quad m \geq 1.
$$

**Remark 3.12.** If the operators $A_1$ and $A_2$ are not self-adjoint, in general, the result of Theorem 3.10 does not hold since the range of the operator (3.3) does not necessarily consist of eigenvectors of $x A_1 + y A_2$. However, if $\lambda$ is an isolated spectral point of $A_1$ of multiplicity one, then for $(x, y)$ sufficiently close to $(\frac{1}{\lambda}, 0)$, the operator $P(x, y)$ is a rank one projection (not necessarily orthogonal), and its range is an eigensubspace of $x A_1 + y A_2$, so the result of Theorem 3.10 is valid in this case too.
It follows directly from (3.8) and (3.9) that $\Psi_m$ has a pole of order at most $m + 1$ at $w = 1$. We will now obtain the expression of the residue of $\Psi_m$ at 1. Let

$$A_1 = \lambda P_1 + \int_{\sigma(A_1)\setminus\{\lambda\}} zdE(z)$$

be the spectral decomposition of $A_1$ with $P_1$ being the orthogonal projection on the eigenspace of $A_1$ corresponding to eigenvalues $\lambda$. If $\delta$ is small enough, we have

$$(wI - \frac{1}{\lambda} A_1)^{-1} = \frac{1}{w - 1} P_1 + \int_{\sigma(A_1)\setminus\{\lambda\}} \frac{dE(z)}{w - \frac{1}{\lambda}}$$

$$= \frac{1}{w - 1} P_1 - \int_{\sigma(A_1)\setminus\{\lambda\}} \left( \sum_{m=0}^{\infty} \left( \frac{\lambda}{z - \lambda} \right)^{m+1} (w - 1)^m \right) dE(z)$$

$$= \frac{1}{w - 1} P_1 - \sum_{m=0}^{\infty} (w - 1)^m \left( \int_{\sigma(A_1)\setminus\{\lambda\}} \left( \frac{\lambda}{z - \lambda} \right)^{m+1} dE(z) \right).$$

Write

$$T(A_1) = T = \int_{\sigma(A_1)\setminus\{\lambda\}} \frac{\lambda}{z - \lambda} dE(z),$$

then

$$\int_{\sigma(A_1)\setminus\{\lambda\}} \left( \frac{\lambda}{z - \lambda} \right)^{m+1} dE(z) = T^{m+1},$$

so (3.14) can be written as

$$(wI - \frac{1}{\lambda} A_1)^{-1} = \frac{1}{w - 1} P_1 - \sum_{m=0}^{\infty} (w - 1)^m T^{m+1},$$

and

$$(3.17) \quad A_2(wI - \frac{1}{\lambda} A_1)^{-1} = \frac{1}{w - 1} A_2 P_1 - \sum_{m=0}^{\infty} (w - 1)^m A_2 T^{m+1}.$$

The following result follows from Theorem 3.10 and equations (3.16) and (3.17).

**Theorem 3.18.** Under the conditions of Theorem 3.10 the associated integrands $\Psi_m(\lambda)$ determined by (3.8) and (3.9) are holomorphic in $\{w \in \mathbb{C} : |w - 1| < \delta\}$. 
Proof. It follows from (3.8) and (3.9) that
\[
\Psi_m(w) = \Psi_{m-1}(w)[A_2(w) - \frac{1}{\lambda}A_1]^{-1}
\]
(3.19)
\[
- \left( \sum_{j=m}^{k} w^{k-j} \frac{r_j}{j!} \right) (wI - \frac{1}{\lambda}A_1)^{-1}, \quad 2 \leq m \leq k,
\]
\[
\Psi_m(w) = \Psi_{m-1}(w)[A_2(w) - \frac{1}{\lambda}A_1]^{-1}, \quad m \geq k + 1.
\]

Relations (3.16), (3.17), and (3.19) imply that if \( \Psi_{m-1} \) is holomorphic, then \( \Psi_m \) has pole of order at most one at \( \lambda = 1 \), and, therefore, by (3.11) \( \Psi_m \) is holomorphic. Thus, it suffices to show that \( \Psi_1(w) \) is holomorphic at \( w = 1 \).

We have
\[
\Psi_1(w) = (wI - \frac{1}{\lambda}A_1)^{-1} \left( w^k - \sum_{j=1}^{k} w^{k-j} \frac{r_j}{j!} \right) [A_2(wI - \frac{1}{\lambda}A_1)^{-1}]
\]
(3.20)
\[
- \left( \sum_{j=1}^{k} w^{k-j} \frac{r_j}{j!} \right) (wI - \frac{1}{\lambda}A_1)^{-1} = \tilde{\Psi}_1(w) + \tilde{\Psi}_2(w).
\]

Observe that the polynomial \( P(w) = w^k - \sum_{j=1}^{k} w^{k-j} \frac{r_j}{j!} \) satisfies \( P(1) = -\mathcal{F}(\frac{1}{\lambda}, 0) = 0 \), and, therefore, \( P(w) = (w-1)Q(w) \), where \( Q \) is a polynomial of degree \( k-1 \). Now relations (3.16) and (3.17) show that both \( \tilde{\Psi}_1 \) and \( \tilde{\Psi}_2 \) have poles of order at most one at \( w = 1 \), and relation (3.11) implies that \( \Psi_1 \) is holomorphic at \( w = 1 \). \( \square \)

4. Line in the spectrum

Now, suppose that, as in the previous section, \( A_1 \) and \( A_2 \) are self-adjoint, that \( \lambda \neq 0 \) is an isolated spectral point of \( A_1 \), and that \( \sigma_p(A_1, A_2) \cap \Delta(\frac{1}{\lambda}, 0) \) coincides, up to multiplicity, with a line segment \( \{ (x, y) \in \Delta(\frac{1}{\lambda}, 0) : \lambda x + ay = 1 \} \) where \( a \neq 0 \). Passing to \( A_1/\lambda \) and \( A_2/a \) if necessary, we may assume that \( \lambda = a = 1 \), that is,
\[
\{ x + y = 1 \} \cap \Delta_p(1, 0) = \sigma_p(A_1, A_2) \cap \Delta_p(1, 0)
\]
up to multiplicity. Coming back to relation (3.11), here we have \( k = 1 \), \( r_1^0 = r_1^1 = 1 \), \( \mu_1 = 1 \), \( x_1 = 1 \). Let us write down the functions \( \Psi_1 \) and \( \Psi_2 \) in this particular case. Equations (3.9), (3.16), and (3.17) imply
\[
\Psi_1(w) = \left( \frac{1}{w-1} P_1 - \sum_{m=0}^{\infty} (w-1)^m T^{m+1} \right) \left( (w-1) \left[ \frac{1}{w-1} A_2 P_1 \right] - \sum_{m=0}^{\infty} (w-1)^m A_2 T^{m+1} \right) - I
\]
(4.1)
\[
\Psi_2(w) = \Psi_1(w) \left[ A_2(wI - A_1)^{-1} \right]
\]
\[
(4.2) \quad = \left( \frac{1}{w-1} P_1 - \sum_{m=0}^{\infty} (w - 1)^m T^{m+1} \right) \left( (w - 1) \left[ \frac{1}{w-1} A_2 P_1 - \sum_{m=0}^{\infty} (w - 1)^m A_2 T^{m+1} \right] - I \right)
\]

It follows from (3.11), (4.1), and (4.2) that
\[
\text{Res}_{\lambda} = 1 (\Psi_1) = P_1 A_2 P_1 - P_1 = 0
\]
\[
(4.3)
\]
\[
\text{Res}_{\lambda} = 1 (\Psi_2) = P_1 A_2 T A_2 P_1 - P_1 (A_2 P_1 - I) A_2 T
\]
\[
- T (A_2 P_1 - I) A_2 P_1 = 0
\]

The last two equations imply
\[
(4.5) \quad P_1 A_2 T A_2 P_1 = 0.
\]

**Remark 4.6.** Coming back to the beginning of this section, suppose that
\[
\{ \lambda x + ay = 1 \} \cap \Delta_\rho \left( \frac{1}{\lambda}, 0 \right) = \sigma_\rho(A_1, A_2) \cap \Delta_\rho \left( \frac{1}{\lambda}, 0 \right)
\]
up to multiplicity. Then the operators \( A_1/\lambda \) and \( A_2/a \) satisfy (4.3) and (4.4). Since the projections \( P_j \) and the operator \( T \) for \( A_1 \) and \( A_1/a \) are the same, we obtain
\[
(4.7) \quad P_1 A_2 P_1 = a P_1.
\]

Equation (4.5) stays the same.

Now we use (4.5) to establish necessary and sufficient conditions for a common eigenvector in the case when at least one of the operators \( A_1, A_2 \) is invertible.

**Lemma 4.8.** Let \( A_1, A_2 \) be self-adjoint, \( 1 \) be an isolated spectral point of \( A_1 \), and assume that \( A_1 \) is invertible. If there is \( \rho > 0 \) such that
\[
\{ x + y = 1 \} \cap \Delta_\rho (1, 0) = \sigma_\rho(A_1, A_2) \cap \Delta_\rho (1, 0)
\]
up to multiplicity, then the following are equivalent:

1. \( A_1 \) and \( A_2 \) have an \( n \)-dimensional common eigensubspace, where \( n = \text{Rank}(P_1) \), and the whole line \( \{ x + y = 1 \} \) is in \( \sigma_\rho(A_1, A_2) \);
2. There is \( \rho' > 0 \) such that the line segment \( \{ x + y = 1 \} \cap \Delta_\rho'(1, 0) \) agrees with \( \sigma_\rho(A_1^{-1}, A_2) \cap \Delta_\rho'(1, 0) \) up to multiplicity;
3. There is \( \rho'' \) such that the plane segment \( \{ x + y + z = 1 \} \cap \Delta_\rho''(1, 1, 0) \) agrees with \( \sigma_\rho(A_1, A_1^{-1}, A_2) \cap \Delta_\rho''(1, 1, 0) \) up to multiplicity.

**Proof.** The implications (1) \( \implies \) (2), (1) \( \implies \) (3), and (3) \( \implies \) (2) are obvious. Thus, it suffices to prove (2) \( \implies \) (1).
Suppose that (2) holds. Let $L_1$ be the eigensubspace of $A_1$ with eigenvalue one. Choose an orthonormal basis of $L_1$: $e_1, \ldots$. Equation (3.15) implies that in our case for every $\xi \in H$

$$T(A_1)\xi = T\xi = \int \sigma(A_1)\setminus\{1\} \frac{1}{z-1}dE(z)(\xi).$$

Therefore, using that $A_2$ is self-adjoint we have for every $j$

$$P_1A_2T_A_2P_1e_j = \sum_m \left( \int \sigma(A_1)\setminus\{1\} \frac{1}{z-1} \langle dE(z)A_2e_j, A_2e_m \rangle \right) e_m = \sum_m \left( \int \sigma(A_1)\setminus\{1\} \frac{1}{z-1} \langle dE(z)A_2e_j, dE(z)A_2e_m \rangle \right) e_m.$$

Equation (4.5) implies that for every pair $j, m$

$$\int \sigma(A_1)\setminus\{1\} \frac{1}{z-1} \langle dE(z)A_2e_j, dE(z)A_2e_m \rangle = 0.$$

In particular, when $j = m$ we obtain

$$\int \sigma(A_1)\setminus\{1\} \frac{1}{z-1} \|dE(z)A_2e_j\|^2 = 0. \tag{4.9}$$

We now apply all preceding considerations to the pair $(A^{-1}_1, A_2)$. First we observe that

$$A^{-1}_1 = P_1 + \int \sigma(A_1)\setminus\{1\} \frac{1}{z}dE(z).$$

Hence,

$$P_1A_2\tilde{T}A_2P_1 = 0,$$

where

$$\tilde{T} = T(A^{-1}_1) = \int \sigma(A_1)\setminus\{1\} \frac{z}{1-z}dE(z). \tag{4.10}$$

In a similar way the last two relations yield

$$\int \sigma(A_1)\setminus\{1\} \frac{z}{1-z} \|dE(z)A_2e_j\|^2 = 0. \tag{4.11}$$

Adding (4.9) and (4.11) we obtain

$$\int \sigma(A_1)\setminus\{1\} \|dE(z)A_2e_j\|^2 = 0.$$

This means that $A_2e_j \in L_1$ for every $j$. Thus, $L_1$ is invariant under $A_2$. Since the restriction of $A_1$ to $L_1$ is the identity operator, the joint spectrum of $A_2|_{L_1}$ and the identity of $L_1$ contains a cone with vertex at $(1,0)$ that contains every line of the family $\{x + \frac{a}{n} = 1 : a \in \sigma(A_2|_{L_1})\}$, and, of course, this cone lies in $\sigma_p(A_1, A_2)$. Since the intersection of $\sigma_p(A_1, A_2)$ with a neighborhood of $(1,0)$ is a line segment, we conclude that the spectrum of
$A_2|_{L_1}$ consists of a single point, and, since $A_2$ is self-adjoint, this means that $L_1$ is an eigenspace for $A_2$. \hfill \Box$

Since eigenvectors of an operator $A$ and its scalar multiple are the same, the following result is a straightforward corollary to Lemma 4.8.

**Lemma 4.12.** Let $A_1, A_2$ be self-adjoint, $\lambda \neq 0$ be an isolated point of $\sigma(A_1)$, and $A_1$ be invertible. If there exist $a \neq 0$ and $\rho > 0$ such that, up to multiplicity, $\{\lambda x + ay = 1\} \cap \Delta_\rho(\frac{1}{\lambda}, 0) = \sigma_\rho(A_1, A_2) \cap \Delta_\rho(1, 0)$, then the following are equivalent:

1. $A_1$ and $A_2$ have a common eigensubspace of dimension equal to the rank of $P_1$;
2. there is $\rho'$ such that, up to multiplicity, $\sigma_\rho(A_1^{-1}, A_2) \cap \Delta_{\rho'}(\lambda, 0) = \{\frac{x}{\lambda} + ay = 1\} \cap \Delta_{\rho'}(\lambda, 0)$;
3. There is $\rho'' > 0$ such that, up to multiplicity, $\{\lambda x + \frac{x}{\lambda} + az = 1\} \cap \Delta_{\rho''}(\lambda, \frac{1}{\lambda}, 0) = \sigma_\rho(A_1, A_1^{-1}, A_2) \cap \Delta_{\rho''}(\lambda, \frac{1}{\lambda}, 0)$.

We will use the result of Lemma 4.12 to give a necessary and sufficient condition for a common eigenvector for an arbitrary pair of self-adjoint operators. To this end, for any self-adjoint operator $A$ we consider the following family of perturbations:

$$A(\epsilon, \lambda) = (1 + \epsilon)A - \lambda \epsilon I, \ \epsilon \in \mathbb{R}, \ \epsilon \neq -1.$$  

**Remark 4.14.** It is easily seen that for every $\epsilon, \lambda \in \mathbb{R}$ the operator $A(\epsilon, \lambda)$ is self-adjoint. Furthermore, if $\lambda$ is an isolated spectral point of $A$, then it is an isolated spectral point of $A(\epsilon, \lambda)$ for every $\epsilon \neq -1$; and the line segment $\{\lambda x + ay = 1\} \cap \Delta_\rho(\frac{1}{\lambda}, 0)$ is in $\sigma_\rho(A_1, A_2)$ if and only if it is in $\sigma_\rho(A_1(\epsilon, \lambda), A_2(\epsilon, a))$. It is also straightforward that the eigensubspace of $A(\epsilon, \lambda)$ corresponding to eigenvalue $\lambda$ is either empty or is the same for all $\epsilon \neq -1$. We further remark that if $\lambda \neq 0$, then there exists $\epsilon$ such that $A(\epsilon, \lambda)$ is invertible. Indeed, the spectral mapping theorem, cf. [8, Chapter 7, Section 3, Thm. 11], implies that $\sigma(A(\epsilon, \lambda)) = (1 + \epsilon)\sigma(A) - \lambda \epsilon$. Thus $0 \in \sigma(A(\epsilon, \lambda))$ if and only if $\frac{\lambda}{1 + \epsilon} \in \sigma(A)$. Since $\lambda$ is an isolated point of $\sigma(A)$, if $\epsilon \in \mathbb{R}$ and $|\epsilon|$ is big enough, zero is not in the spectrum of $A(\epsilon, \lambda)$, that is $A(\epsilon, \lambda)$ is invertible.

Since by making a linear change of coordinates (which amounts to replacing $A_2$ by $A_2 + \delta A_1$ for a sufficiently small real $\delta$) we can always reduce to the case when $a \neq 0$, the following result is an immediate corollary to Lemma 4.12.

**Theorem 4.15.** Let $A_1, A_2$ be self-adjoint operators on $H$, let $\lambda$ be an isolated spectral point of $\sigma(A_1)$, and suppose that in some neighborhood $\Delta_\rho(\frac{1}{\lambda}, 0)$ of $(\frac{1}{\lambda}, 0)$ the joint spectrum $\sigma_\rho(A_1, A_2)$ coincides up to multiplicity with a line segment $\{(x, y) \in \Delta_\rho(\frac{1}{\lambda}, 0) : \lambda x + ay = 1\}$. The following are equivalent:

1. The eigenspace of $A_1$ corresponding to the eigenvalue $\lambda$ is also an eigensubspace for $A_2$;
There exist \( \epsilon \in \mathbb{R} \), \( \epsilon \neq -1 \) and \( \rho' > 0 \) such that \( A_1(\epsilon, \lambda) \) is invertible and the line segment \( \{(x, y) \in \Delta_{\rho'}(\lambda, 0) : \frac{1}{\lambda}x + ay = 1\} \) coincides up to multiplicity with \( \sigma_p((A_1(\epsilon, \lambda))^{-1}, A_2(\epsilon, a)) \cap \Delta_{\rho'}(\lambda, 0) \);

There exist \( \epsilon \in \mathbb{R} \), \( \epsilon \neq -1 \) and \( \rho'' > 0 \) such that \( A_1(\epsilon, \lambda) \) is invertible and the plane segment \( \{(x, y, z) \in \Delta_{\rho''}(\frac{1}{\lambda'}, \lambda, 0) : \lambda x + \frac{1}{\lambda'}y + az = 1\} \) coincides with \( \sigma_p(A_1(\epsilon, \lambda), A_1(\epsilon, \lambda)^{-1}, A_2(\epsilon, a)) \cap \Delta_{\rho''}(\frac{1}{\lambda'}, \lambda, 0) \) up to multiplicity.

The following are equivalent:

1. The eigensubspace of \( A_1 \) corresponding to eigenvalue \( \lambda \) is an eigensubspace for each of the operators \( A_2, \ldots, A_n \);
2. There exist an \( \epsilon \in \mathbb{R} \), \( \epsilon \neq -1 \) and \( \rho' > 0 \) such that \( A_1(\epsilon, \lambda) \) is invertible and

\[
\{(1/\lambda)x_1 + a_2x_2 + \cdots + a_nx_n = 1\} \cap \Delta_{\rho}(\lambda, 0, \ldots, 0)
\]

\[
= \sigma_p(A_1(\epsilon, \lambda)^{-1}, A_2(\epsilon, a_2), \ldots, A_n(\epsilon, a_n)) \cap \Delta_{\rho'}(\lambda, 0, \ldots, 0),
\]

up to multiplicity.

\begin{proof}
Obviously (1) implies (2).

Suppose that (2) holds. Since the line segments \( \{(x_1, x_j) \in \Delta_{\rho}(\frac{1}{\lambda}, 0) : \lambda x_1 + a_jx_j = 1\} \) and \( \{(x_1, x_j) \in \Delta_{\rho}(\lambda, 0) : \frac{1}{\lambda}x_1 + a_jx_j = 1\} \) coincide with \( \sigma_p(A_1(\epsilon, \lambda), A_j(\epsilon, a_j)) \cap \Delta_{\rho}(\frac{1}{\lambda}, 0) \) and \( \sigma_p(A_1(\epsilon, \lambda)^{-1}, A_j(\epsilon, a_j)) \cap \Delta_{\rho'}(\lambda, 0) \) respectively for all \( j = 2, \ldots, n \), it follows from Theorem 4.15 that the eigenspace of \( A_1 \) that corresponds to the eigenvalue \( \lambda \) is an eigenspace of \( A_j \) for all \( j = 2, \ldots, n \), and (1) holds.
\end{proof}

5. Spectral algebraic curves, exterior powers, and common reducing subspaces

It is clear that if operators \( A_1 \) and \( A_2 \) have a common reducing subspace of dimension \( n \), the joint spectrum \( \sigma_p(A_1, A_2) \) contains an algebraic curve of order \( n \). Our example from the Introduction shows that in general the converse is not true. In this section we use results of the previous section to establish necessary and sufficient conditions under which the presence of an algebraic curve in the joint spectrum implies the existence of a common reducing subspace. As our conditions are expressed in terms of joint spectra of exterior products, we begin by recalling some basic facts about exterior
products of Hilbert spaces. For more details we refer the reader to [29, Chapter V.1] and [24, Chapter X.7].

For any \( n \geq 1 \) the \( n \)th tensor power \( \otimes^n H \) of a Hilbert space \( H \) has inner product given by

\[
\langle x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n \rangle = \langle x_1, y_1 \rangle \cdots \langle x_n, y_n \rangle
\]

The \( n \)th exterior power \( \bigwedge^n H \) of \( H \) is defined as the quotient of \( \otimes^n H \) modulo the subspace generated by all elements of the form

\[
x_1 \otimes \cdots \otimes x \otimes x \cdots \otimes x_n
\]

The image of a simple tensor \( x_1 \otimes \cdots \otimes x_n \) in \( \bigwedge^n H \) is denoted by \( x_1 \wedge \cdots \wedge x_n \).

We consider \( \bigwedge^n H \) as a subspace of \( \otimes^n H \) via the canonical “antisymmetrizing map”

\[
x_1 \wedge \cdots \wedge x_n \mapsto \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}
\]

In particular, \( \bigwedge^n H \) inherits via this map a Hilbert space structure from \( \otimes^n H \), and it is straightforward to compute that its inner product satisfies

\[
\langle x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n \rangle = \det \begin{bmatrix}
\langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\
\vdots & \ddots & \vdots \\
\langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle
\end{bmatrix}.
\]

Therefore if \( \{e_1, \ldots, e_n, \ldots\} \) is an orthonormal basis of \( H \) we obtain that the set \( \{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \} \) is an orthonormal basis of \( \bigwedge^n H \).

When \( A \) is a linear operator on \( H \) it induces a linear operator \( \bigwedge^n A \) on \( \bigwedge^n H \) via the formula \( [\bigwedge A](v_1 \wedge \cdots \wedge v_n) = Av_1 \wedge \cdots \wedge Av_n \). Thus \( \bigwedge^n A \) is just the restriction of \( \otimes^n A \) to \( \bigwedge^n H \) considered as a subspace of \( \otimes^n H \). It is immediate that if \( A \) is of finite rank or self-adjoint then so is \( \bigwedge^n A \); and when \( A \) is a bounded we get \( \|\bigwedge^n A\| \leq \|A\|^n \). In particular, compactness of \( A \) implies compactness of \( \bigwedge^n A \).

When \( A \) is self-adjoint and compact and \( \lambda_1, \ldots \) are the eigenvalues of \( A \), with \( e_1, \ldots \) being a corresponding eigenbasis, then the orthogonal set \( \{e_{i_1} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \} \) is an eigenbasis with \( \{\lambda_1 \ldots \lambda_n\} \) as the corresponding multiset of eigenvalues for the compact self-adjoint \( \bigwedge^n A \).

**Definition 5.1.** Let \( A \) be a self-adjoint operator. We say that a finite multiset \( L = \{\lambda_1, \ldots, \lambda_n\} \) is a spectral multiset for \( A \) if each \( \lambda_i \) is an isolated point in the spectrum of \( A \) of finite multiplicity, and the number of times it occurs in \( L \) is at most its multiplicity as an eigenvalue of \( A \). We say that a spectral multiset for \( A \) is generic if \( \lambda = \lambda_1 \ldots \lambda_n \) is isolated and of multiplicity 1 in the spectrum of \( \bigwedge^n A \).

**Remark 5.2.** A straightforward consequence of the definition is that if \( L = \{\lambda_1, \ldots, \lambda_n\} \) is a generic spectral multiset for the self-adjoint operator \( A \), then each eigenvalue \( \lambda_i \) of \( A \) has multiplicity equal to the number of times it occurs in \( L \).
Remark 5.3. Suppose $A$ is a self-adjoint operator with countable spectrum, and $L = \{\lambda_1, \ldots, \lambda_n\}$ is a spectral multiset for $A$ such that each eigenvalue $\lambda_i$ of $A$ has multiplicity equal to the number of times it occurs in $L$. Then there is an open subset $U \subset \mathbb{R}$ such that $\mathbb{R} \setminus U$ is countable, and such that for every $\delta \in U$ the operator $A + \delta I$ is invertible and has $L + \delta = \{\lambda_1 + \delta, \ldots, \lambda_n + \delta\}$ as a generic spectral multiset. Indeed, since $A + \delta I$ is invertible for $\delta \in \mathbb{R} \setminus -\sigma(A)$, and since the spectrum of $\wedge^n (A + \delta)$ is a subset of the spectrum of $\otimes^n (A + \delta)$, it suffices to show that for every $\delta$ in the set

$$\{\delta \in \mathbb{R} \mid (\lambda_1 + \delta)\ldots(\lambda_n + \delta) \neq (\mu_1 + \delta)\ldots(\mu_n + \delta)$$

for $\mu_i \in \sigma(A)$ such that $\{\mu_1, \ldots, \mu_n\} \neq L$ as multisets\}

the point $(\lambda_1 + \delta)\ldots(\lambda_n + \delta)$ is isolated in the spectrum of $\otimes^n (A + \delta I)$. But, as the spectrum of a tensor product of operators is the product of their spectra, cf. [2], and each $\lambda_i + \delta$ is isolated in the spectrum of $A + \delta I$, this follows from the compactness of spectra by a standard argument.

The assumptions in the following theorem describe what we consider to be a “general position” setting.

Theorem 5.4. Let $A_1, A_2$ be self-adjoint operators with $A_1$ invertible. Consider a generic spectral multiset $L = \{\lambda_1, \ldots, \lambda_n\}$ for $A_1$, and let $\lambda = \lambda_1 \ldots \lambda_n$. Suppose also that for some $a \neq 0$ and $\rho > 0$ the line segments

$$\{\lambda x + ay = 1\} \cap \Delta_\rho(1/\lambda, 0) \quad \text{and} \quad \{1/(\lambda)x + ay = 1\} \cap \Delta_\rho(\lambda, 0)$$

are inside $\sigma_p(\wedge^n A_1, \wedge^n A_2)$ and $\sigma_p(\wedge^n A_1^{-1}, \wedge^n A_2)$ respectively.

Then the eigenspace of $A_1$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ is invariant under $A_2$.

Proof. Since $\lambda$ is isolated and simple in the spectrum of $\wedge^n A_1$ it follows that the joint spectrum $\sigma_p(\wedge^n A_1, \wedge^n A_2)$ is nonsingular at the point $(1/\lambda, 0)$, in particular this point belongs to no component other than the line $\lambda x + ay = 1$. By Lemma 4.12 the operators $\wedge^n A_1$ and $\wedge^n A_2$ have a common unit eigenvector $v$ (of eigenvalue $\lambda$ for $\wedge^n A_1$ and eigenvalue $a$ for $\wedge^n A_2$). Since $L$ is generic $v$ must be of the form $v = e_1 \wedge \cdots \wedge e_n$, where each $e_i$ is a unit eigenvector for $A_1$ of eigenvalue $\lambda_i$. Now we show that $span(e_1, \ldots, e_n)$ is invariant under $A_2$. Indeed, let $e$ be any other eigenvector for $A_1$, and consider the column vector

$$w = \begin{bmatrix} \langle A_2 e_1, e \rangle \\ \vdots \\ \langle A_2 e_n, e \rangle \end{bmatrix}.$$

For $i = 1, \ldots, n$ set $v_i = e_1 \wedge \cdots \wedge e_{i-1} \wedge e \wedge e_{i+1} \wedge \cdots \wedge e_n$ and note that each $v_i$ is orthogonal to $v$. As $\wedge^n A_2(v) = av$, it follows that $a = \langle \wedge^n A_2(v), v \rangle = \langle A_2 e_1 \wedge \cdots \wedge A_2 e_n, e_1 \wedge \cdots \wedge e_n \rangle$. 


and therefore $\det Y = a \neq 0$, where $Y$ is the matrix

$$Y = \begin{bmatrix}
    \langle A_2 e_1, e_1 \rangle & \cdots & \langle A_2 e_1, e_n \rangle \\
    \vdots & \ddots & \vdots \\
    \langle A_2 e_n, e_1 \rangle & \cdots & \langle A_2 e_n, e_n \rangle
\end{bmatrix}. $$

In particular the linear system of equations

$$Y \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = w$$

has a unique solution given, according to Cramer’s Rule, by the formula

$$x_i = \frac{\det Y_i(w)}{a}$$

for each $i = 1, \ldots, n$, where $Y_i(w)$ is obtained by replacing with $w$ the $i$th column of the matrix $Y$. But since

$$\det Y_i(w) = \langle \wedge^n A_2(v), e_1 \wedge \cdots \wedge e \wedge \cdots \wedge e_n \rangle = a\langle v, v_i \rangle = 0$$

we see that $x_i = 0$ for each $i$, and therefore $w = 0$. 

Recall that a self-adjoint operator $A$ on an infinite dimensional separable Hilbert space $H$ belongs to the class $E(H)$ when $A = K + aI$ with $K$ a compact self-adjoint operator on $H$. In this case every point in $\sigma(A) \setminus a$ is isolated of finite multiplicity, and the point $a$ is either an accumulation point, or isolated of infinite multiplicity.

For operators in the class $E(H)$ we are now ready to address the question of when the presence of an algebraic curve in the proper joint spectrum indicates the existence of a common reducing subspace. Consider two self-adjoint operators $A = K_1 + aI$ and $B = K_2 + bI$ in $E(H)$, and suppose that $\Gamma$ is an algebraic curve of degree $k$ which is a union of components of the proper joint spectrum $\sigma_p(A, B)$. Note that the line $\{ax + by = 1\}$ is always in $\sigma_p(A, B)$ and therefore carries no information about common reducing subspaces. We will refer to this line as the accumulation line of the joint spectrum. Thus, without loss of generality we can assume that it is not a reduced component of $\Gamma$. Therefore, by making a linear change of coordinates if necessary (which amounts to replacing $A$ and $B$ by appropriate linear combinations of $A$ and $B$ with real coefficients, hence does not affect the presence and degrees of algebraic curves or common reducing subspaces and their dimensions) we may also assume that $\Gamma$ intersects the $x$-axis in $k$ points (counted with multiplicities) $1/\lambda_1, \ldots, 1/\lambda_k$ and the $y$-axis in $k$ points $1/\mu_1, \ldots, 1/\mu_k$, and that each point $(1/\lambda_i, 0)$ belongs only to components of $\sigma_p(A, B)$ contained in $\Gamma$; in particular, the multiset $L = \{\lambda_1, \ldots, \lambda_k\}$ is a spectral mutiset for $A$ and each eigenvalue $\lambda_i$ of $A$ has multiplicity equal to the number of times it occurs in $L$. Therefore, by Remark 5.3 if needed,
we can make an additional suitable linear fractional change of coordinates of the form

\[ u = \frac{x}{1 + \delta x}, \quad v = \frac{y}{1 + \delta x} \]

(which amounts to replacing \( A \) by \( A + \delta I \)) and also assume that \( A \) is invertible, and that \( L \) is a generic spectral multiset for \( A \). Thus, for operators in the class \( \mathcal{E}(H) \) we can always reduce the search for a common invariant subspace to the “general position” case considered in the following theorem, which is one of the main results in this paper.

**Theorem 5.5.** Let \( A = K_1 + aI \) and \( B = K_2 + bI \) be self-adjoint operators in the class \( \mathcal{E}(H) \), with \( A \) invertible. Let \( \Gamma \) be an algebraic curve of degree \( k \) which is a union of components of the proper joint spectrum \( \sigma_p(A, B) \), and which does not have the accumulation line \( \{ax + by = 1\} \) as a reduced component. Suppose that the \( x \)-axis (resp. the \( y \)-axis) intersects \( \Gamma \) in the \( k \) points (counted with multiplicity) \( 1/\lambda_1, \ldots, 1/\lambda_k \) (resp. \( 1/\mu_1, \ldots, 1/\mu_k \)) such that each point \( (1/\lambda, 0) \) belongs only to components of \( \sigma_p(A, B) \) contained in \( \Gamma \), and the multiset \( L = \{\lambda_1, \ldots, \lambda_k\} \) is a generic spectral multiset for \( A \). Set \( \lambda = \lambda_1 \cdots \lambda_k \) and \( \mu = \mu_1 \cdots \mu_k \). The following are equivalent:

1. The eigenspace for \( A \) corresponding to \( \lambda_1, \ldots, \lambda_k \) is invariant for \( B \).
2. There exists \( \rho > 0 \) such that the line segments
   \[ \{\lambda x + \mu y = 1\} \cap \Delta_\rho(1/\lambda, 0) \]
   \[ \{(1/\lambda)x + \mu y = 1\} \cap \Delta_\rho(\lambda, 0) \]
   are contained in \( \sigma_p(\wedge^k A, \wedge^k B) \) and \( \sigma_p(\wedge^k A^{-1}, \wedge^k B) \), respectively.
3. The lines
   \[ \{\lambda x + \mu y = 1\} \quad \text{and} \quad \{(1/\lambda)x + \mu y = 1\} \]
   are contained in \( \sigma_p(\wedge^k A, \wedge^k B) \) and \( \sigma_p(\wedge^k A^{-1}, \wedge^k B) \), respectively.

**Proof.** The implications (1) \( \Rightarrow \) (3) \( \Rightarrow \) (2) are straightforward, and (2) implies (1) by Theorem 5.4. \( \square \)

The case when \( H \) is a finite-dimensional Hilbert space is of classical interest in algebraic geometry. In that setting every linear operator is bounded and of finite rank, hence we can take \( \mathcal{E}(H) \) to be the space of all linear operators on \( H \). Now the statement of our last theorem can be somewhat simplified and slightly expanded as follows.

Suppose that \( \dim CH = N \), and let \( H^* \) denote the space dual to \( H \). Then we have the canonical isomorphism

\[ \wedge^k H^* \rightarrow \wedge^{N-k} H \otimes \wedge^N H^* , \]

which, together with the inner product induced isomorphism \( \wedge^k H \rightarrow \wedge^k H^* \), allows us to consider in a natural way \( \wedge^{N-k} A \) as acting on the space \( \wedge^k H \) for any linear operator \( A \) on \( H \). In particular, we can consider the proper joint spectrum \( \sigma_p(\wedge^{N-k} A, \wedge^k B) \). When \( A \) is self-adjoint, a standard exercise in multilinear algebra shows that the Sylvester expansion formula for
the determinant transforms into the equality $(\wedge^{N-k}A)(\wedge^k A) = (\det A)I$.
Thus, for an invertible self-adjoint $A$ we have $\wedge^{N-k}A = \det(A) \wedge^k A^{-1}$, and therefore Theorem 5.5 implies immediately the following result.

**Theorem 5.6.** With assumptions and notation as in Theorem 5.5, suppose in addition that $a = b = 0$ and $\dim C H = N$. The following are equivalent:

1. The eigenspace for $A$ corresponding to $\lambda_1, \ldots, \lambda_k$ is invariant for $B$.
2. For some $\rho > 0$ the line segments
   \[ \{\lambda x + \mu y = 1\} \cap \Delta_\rho(1/\lambda, 0) \]
   \[ \{(1/\lambda)x + \mu y = 1\} \cap \Delta_\rho(\lambda, 0) \]
   are contained in $\sigma_p(\wedge^k A, \wedge^k B)$ and $\sigma_p(\wedge^k A^{-1}, \wedge^k B)$, respectively.
3. For some $\rho > 0$ the line segments
   \[ \{\lambda x + \mu y = 1\} \cap \Delta_\rho(1/\lambda, 0) \]
   \[ \{(\det A/\lambda)x + \mu y = 1\} \cap \Delta_\rho(\lambda/\det A, 0) \]
   are contained in $\sigma_p(\wedge^k A, \wedge^k B)$ and $\sigma_p(\wedge^{N-k} A, \wedge^k B)$, respectively.
4. The lines
   \[ \{\lambda x + \mu y = 1\} \quad \text{and} \quad \{(\det A/\lambda)x + \mu y = 1\} \]
   are contained in $\sigma_p(\wedge^k A, \wedge^k B)$ and $\sigma_p(\wedge^{N-k} A, \wedge^k B)$, respectively.

Finally, we note that Theorem C from the Introduction is obtained from the result above in the same way that Theorem 4.16 is obtained from Theorem 4.15.

6. Example: unit circle in the spectrum

Here we present an example of a setting where the existence of a common reducing subspace is established without exterior products but rather in a way similar to Theorem 4.15.

We start with an elementary result from [5] about spectra of linear transforms. Let
\[
C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}
\]
be a complex-valued matrix. Write
\[
B_1 = c_{11}A_1 + c_{12}A_2, \quad B_2 = c_{21}A_1 + c_{22}A_2.
\]

**Proposition 6.2.** $\sigma_p(A_1, A_2) \supset C^T \sigma_p(B_1, B_2)$.

The proof is straightforward.

**Corollary 6.3.** If $C$ is invertible, then
\[
\sigma_p(A_1, A_2) = C^T \sigma_p(B_1, B_2).
\]

The following result shows that under some conditions the presence of a circle in the joint spectrum implies the existence of a common reducing subspace.
Theorem 6.5. Let $A_1, A_2$ be self-adjoint and suppose that $A_1$ is invertible, $\|A_2\| = 1$, and both $\pm 1$ be isolated spectral points for $A_1$ of finite multiplicities $k_1$ and $l_1$ and be eigenvalues of $A_2$ of finite multiplicities $k_2$ and $l_2$ respectively with $k_2 + l_2 \leq k_1 + l_1$. If there is $\rho > 0$ such that

$$\{x^2 + y^2 = 1\} \cap \Delta_\rho(z, 0) = \sigma_\rho(A_1, A_2) \cap \Delta_\rho(z, 0)$$

$$= \sigma_\rho(A_1^{-1}, A_2) \cap \Delta_\rho(z, 0); \quad z = \pm 1.$$

Then

1. $k_1 = l_1 = k_2 = l_2$;
2. $A_1$ and $A_2$ have a common $2n$-dimensional reducing subspace $L$, where $n = k_1 = l_1 = k_2 = l_2$;
3. the pair of restrictions of $A_1|_L$ and $A_2|_L$ is unitary equivalent to a pair of $2n \times 2n$ involutions $C_1$ and $C_2$, where

$$C_1 = \begin{bmatrix} I_n & 0_n \\ 0_n & -I_n \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0_n & D_n \\ D_n^* & 0_n \end{bmatrix},$$

$0_n$ and $I_n$ are zero and identity $n \times n$ matrices and $D$ is a unitary $n \times n$ matrix;
4. The group generated by $C_1$ and $C_2$ represents Coxeter’s group $BC_2$.

Proof. Let $e_1, \ldots, e_{k_1}$, and $e_{k_1+1}, \ldots, e_{k_1+l_1}$ be a pair of orthonormal eigenbases for the eigenspaces of $A_1$ with eigenvalues $1$ and $-1$ respectively and $\xi_1, \ldots, \xi_{k_2}$ and $\xi_{k_2+1}, \ldots, \xi_{k_2+l_2}$ be a similar pair of orthonormal eigenbases for $A_2$. Write down the spectral decomposition of $A_1$:

$$A_1 = P_1 - P_2 + \int_{\sigma(A_1) \setminus \{\pm 1\}} zdE(z),$$

where $P_1$ and $P_2$ are the original projections on the spaces $span\{e_1, \ldots, e_{k_1}\}$ and $span\{e_{k_1+1}, \ldots, e_{k_1+l_1}\}$ respectively. Applied to this particular situation equation (3.11) for $m = 1$ and $m = 2$ gives

$$P_1 A_2 P_1 = 0$$

(6.7)

$$P_1 A_2 T A_2 P_1 = -\frac{1}{2} P_1$$

(6.8)

Equation (6.8) implies that for every $j = 1, \ldots, k_1$ we have

$$\sum_{m=k_1+1}^{k_1+l_1} \frac{|\langle A_2 e_j, e_m \rangle|^2}{2} + \int_{\sigma(A_1) \setminus \{\pm 1\}} \frac{|\langle A_2 e_j, dE(z) \rangle|^2}{1 - z} = \frac{1}{2}. \quad (6.9)$$

The same computation applied to the unit circle in $\sigma_\rho(A_1^{-1}, A_2)$ yields

$$P_1 A_2 T(A_1^{-1}) A_2 P_1 = -\frac{1}{2} P_1,$$

and equation (4.10) now gives

$$\sum_{m=k_1+1}^{k_1+l_1} \frac{|\langle A_2 e_j, e_m \rangle|^2}{2} + \int_{\sigma(A_1) \setminus \{\pm 1\}} \frac{z |\langle A_2 e_j, dE(z) \rangle|^2}{1 - z} = -\frac{1}{2}. \quad (6.10)$$
Subtracting (6.10) from (6.9) we obtain

\[ (6.11) \]

\[
\sum_{m=k_1+1}^{k_1+l_1} |\langle A_2 e_j, e_m \rangle|^2 + \int_{\sigma(A_1) \setminus \{\pm 1\}} |\langle A_2 e_j, dE(z) \rangle|^2 = 1.
\]

Since (6.7) means \( P_1(A_2 e_j) = 0 \), the relation (6.11) implies

\[ (6.12) \]

\[ |A_2 e_j| = 1. \]

Since \( \|A_2\| = 1 \), it follows from (6.12) that for every \( 1 \leq j \leq k_1 \), \( e_j \in \text{span}\{\xi_1, \ldots, \xi_{k_2+l_2}\} \). A similar analysis applied to the decomposition near the spectral point \((-1,0)\) shows that for each \( k_1 + 1 \leq j \leq k_1 + l_1 \), \( e_j \in \text{span}\{\xi_1, \ldots, \xi_{k_2+l_2}\} \). Thus, \( \text{span}\{e_1, \ldots, e_{k_1+l_1}\} \subset \text{span}\{\xi_1, \ldots, \xi_{k_2+l_2}\} \) and, since \( k_2 + l_2 \leq k_1 + l_1 \), we have \( k_1 + l_1 = k_2 + l_2 \) and \( L = \text{span}\{\xi_1, \ldots, \xi_{k_2+l_2}\} = \text{span}\{e_1, \ldots, e_{k_1+l_1}\} \) is a common reducing subspace of \( A_1 \) and \( A_2 \). We also remark that the restrictions of \( A_1 \) and \( A_2 \) to \( L \) are involutions.

To show that \( k_1 = l_1 = k_2 = l_2 \) observe that condition (6.6), spectral continuity and Corollary 6.3 imply that for every real \( \theta \) close to zero the operator \( A_0 = (\cos \theta) A_1 + (\sin \theta) A_2 \) has \( L \) as an invariant subspace, is self-adjoint and the spectrum of the restriction of \( A_0 \) to \( L \) consists of \( \pm 1 \). Hence, \( A_0|_L \) is an involution, and we have

\[ I = A_0^2 = (\cos^2 \theta) A_1^2 + (\sin^2 \theta) A_2^2 + (\cos \theta)(\sin \theta)(A_1 A_2 + A_2 A_1) \]

\[ = I + (\cos \theta)(\sin \theta)(A_1 A_2 + A_2 A_1). \]

If \( \sin 2\theta \neq 0 \), the last relation implies

\[ (6.13) \]

\[ A_1 A_2 = -A_2 A_1. \]

In the basis \( e_1, \ldots, e_{k_1+l_1} \) the restriction of \( A_1 \) to \( L \) has the form

\[
\begin{bmatrix}
I_{k_1} & 0_{k_1 \times l_1} \\
0_{l_1 \times k_1} & I_{l_1}
\end{bmatrix}.
\]

Since \( A_2 \) is self-adjoint, it follows from (6.13) that the restriction of \( A_2 \) to \( L \) in the same basis has the form

\[ (6.14) \]

\[
\begin{bmatrix}
0_{k_1} & D_{k_1 \times l_1} \\
D_{l_1 \times k_1} & 0_{l_1}
\end{bmatrix}.
\]

If \( k_1 \neq l_1 \), the determinant of the matrix (6.14) is equal to zero contradicting the fact that it is an involution. Thus, \( k_1 = l_1 = 1/2(k_1 + l_1) \). Also, the trace of matrix (6.14) is equal to zero, which implies that \( k_2 = l_2 = k_1 = l_1 \), and both (2) and (3) are proved.

Since the restrictions of \( A_1 \) and \( A_2 \) to \( L \) are involutions, to prove (4) it suffices to verify that

\[ (A_1|_L A_2|_L)^4 = I. \]

This can be done by a direct computation. \(\square\)
Remark 6.15. As we saw above, the joint spectrum of a pair of compact operators is an analytic set, and, of course, away from the accumulation line, the same is true for operators in $\mathcal{E}(H)$. In this setting it is easy to show using spectral continuity and the fact that the set of singularities of an analytic set has higher codimension, that the spectrum is a divisor, that is the multiplicity is the same at every pair of points in the joint spectrum belonging to the same irreducible component of the spectrum where this component is smooth. Thus, in such setting the statement (1) of Theorem 6.5 does not require any special proof.

7. Spectral continuity and common “almost eigenvectors”

Spectral continuity is well-known in the classical spectral theory, see [7]. In our case it implies that if $A_{1,n} \to A_1$ and $A_{2,n} \to A_2$ in operator norm topology as $n \to \infty$, then $\sigma_p(A_{1,n}, A_{2,n})$ converges to $\sigma_p(A_1, A_2)$ in Hausdorff topology uniformly on compact subsets of $\mathbb{C}^2$. In particular, this implies that if $A_1$ and $A_2$ have a common eigenvector, $A_{1,n}$ and $A_{2,n}$ have a common “almost eigenvector” (we define it below) and $\sigma_p(A_{1,n}, A_{2,n})$ contains an irreducible component that converges to a line in Hausdorff topology uniformly on compacts as $n \to \infty$. In this section we prove results that establish the converse: under certain natural assumptions local closeness of $\sigma_p(A_1, A_2)$ to a line implies existence of a common almost eigenvector.

Definition 7.1. We say that a non-zero vector $\xi$ is an $\epsilon$-eigenvector (almost eigenvector) of an operator $A$ if there exists $\lambda \in \mathbb{C}$ such that $\|A\xi - \lambda\xi\| \leq \epsilon\|\xi\|$. Since the distance from $A\xi$ to the line $\{\lambda\xi : \lambda \in \mathbb{C}\}$ is equal to $\|A\xi - \frac{\langle A\xi, \xi \rangle}{\|\xi\|^2}\xi\|$, we come to an equivalent definition of an $\epsilon$-eigenvector: $\xi$ is an $\epsilon$-eigenvector of $A$ if

\begin{equation}
\|A\xi - \frac{\langle A\xi, \xi \rangle}{\|\xi\|^2}\xi\| \leq \epsilon\|\xi\|.
\end{equation}

It immediately follows from (7.2) that $\xi$ is an eigenvector of $A$ in the traditional sense, if and only if it is an $\epsilon$-eigenvector for all $\epsilon > 0$. More generally, if $\lambda \in \sigma(A)$, then for every $\epsilon > 0$ there exists an $\epsilon$-eigenvector $\xi$ such that $\|A\xi - \lambda\xi\| \leq \epsilon\|\xi\|$. Of course, every vector $\xi$ is an $\epsilon$-eigenvector with the appropriate choice of $\epsilon$ to be equal to the lefthand side of (7.2), but this is quite meaningless. The notion of an $\epsilon$-eigenvector is meaningful when $\epsilon$ is small. In this case being an $\epsilon$-eigenvector means that $A\xi$ lies in a small aperture cone that has the line $\{\lambda\xi : \lambda \in \mathbb{C}\}$ as the symmetry axis.

Our next result is a generalization of Theorem 6.15 to the case of common almost eigenvectors for compact operators. Let $\Gamma$ be an analytic curve that passes through $(x, y) \in \mathbb{C}^2$ and let $\rho > 0$. We will use the following notation:

$$\Gamma_\rho(x, y) = \Gamma \cap \Delta_\rho(x, y).$$
If \( \epsilon \) is close to zero and \( A_1(\epsilon) \) and \( A_2(\epsilon) \) are close to \( A_1 \) and \( A_2 \) respectively, then spectral continuity implies that locally \( \sigma_p(A_1, A_2) \) is close to \( \sigma_p(A_1(\epsilon), A_2(\epsilon)) \). For this reason in the next theorem without loss of generality we may assume that \( A_1 \) is invertible. To simplify the notation we will also use rescaling, if necessary, so that the point \((1, 0)\) is in the joint spectrum, of \( A_1 \) and \( A_2 \) and \( A_1 \) is invertible. Finally, recall that the operator \( A \) belongs to the class \( \mathcal{E}(H) \), if it is represented as \( A = K + \alpha I \) where \( K \) is compact and \( \alpha \in \mathbb{R} \).

**Theorem 7.3.** Let \( A_1, A_2 \in \mathcal{E}(H) \) such that \( 1 \in \sigma(A_1) \), and, therefore, the point \((1, 0)\) belongs to \( \sigma_p(A_1, A_2) \) and to \( \sigma_p(A_1^{-1}, A_2) \). Suppose that \( 1 \) is not an accumulation point of \( \sigma(A_1) \) and \((1, 0)\) is not a singular point of either \( \sigma_p(A_1, A_2) \), or \( \sigma_p(A_1^{-1}, A_2) \). Let \( \sigma_p(A_1, A_2) \) and \( \sigma_p(A_1^{-1}, A_2) \) near \((1, 0)\) be zeros of analytic functions \( f_1(x, y) \) and \( f_2(x, y) \) respectively. If there exist \( 0 < \rho < 1 \) and \( 0 < \epsilon < \rho \) such that

1. \( d = 1 - (1 - \rho)\|A_1\| - \epsilon \sqrt{2}\|A_2\| > 0; \)
2. the Hausdorff distances from \( \sigma_p(A_1, A_2)_\rho(1, 0) \) and \( \sigma_p(A_1, A_2)_\rho(1, 0) \) to the line \( \{x + \beta y = 1\} \) are less than \( \epsilon \) (\( \beta \) is a real number);

then \( A_1 \) and \( A_2 \) have a common \( \delta \)-eigenvector, where \( \delta = D\sqrt{\epsilon} \), and \( D \) is a constant independent of \( \beta \).

**Proof.** First we observe that conditions (1) and (2) imply that \( |\beta| \) has an upper bound expressed in terms of \( \rho, \epsilon \) and the norms of \( A_1 \) and \( A_2 \). Indeed, suppose that \( |\beta| > 1 \). Without loss of generality we may assume that \( \beta > 0 \). It is shown below that there is a point \((1 - \rho, \tau) \in \sigma_p(A_1, A_2) \). The distance from this point to \( \{x + \beta y = 1\} \) is equal to \( \frac{|\beta \tau - \rho|}{\sqrt{1 + \beta^2}} \). Condition (2) implies

\[
\frac{\rho}{\beta} - \epsilon \sqrt{1 + \frac{1}{\beta^2}} \leq \tau \leq \frac{\rho}{\beta} + \epsilon \sqrt{1 + \frac{1}{\beta^2}},
\]

so that

\[
|\tau| \leq \frac{\rho}{\beta} + \epsilon \sqrt{2}.
\]

Since the operator \((1 - \rho)A_1 + \tau A_2 - I\) is not invertible, we have

\[
1 \leq \|(1 - \rho)A_1 + \tau A_2\| \leq (1 - \rho)\|A_1\| + |\tau|\|A_2\|
\]

\[
\leq (1 - \rho)\|A_1\| + (\frac{\rho}{\beta} + \epsilon \sqrt{2})\|A_2\|.
\]

This implies

\[
\beta \leq \frac{\rho\|A_2\|}{1 - (1 - \rho)\|A_1\| - \epsilon \sqrt{2}\|A_2\|} = \frac{\rho\|A_2\|}{d}.
\]

Now we approximate the compact parts of \( A_1 \) and \( A_2 \) by finite rank operators with simple spectra (that is every non-zero eigenvector has multiplicity one) resulting in operators \( \tilde{A}_1 \) and \( \tilde{A}_2 \). We can find \( \tau \in \mathbb{R} \) as close to zero as we want, such that \( \tilde{A}_1 = \tilde{A}_1 + \tau I \) is invertible. Since \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are close
to $A_1$ and $A_2$ respectively, the spectral continuity implies that $\sigma_p(\hat{A}_1, \hat{A}_2)$ is close to $\sigma_p(A_1, A_2)$ in the bidisk $\{ |x - 1| \leq \rho, |y| \leq \rho \}$. Also if $\hat{A}_1$ and $\hat{A}_2$ are close enough to $A_1$ and $A_2$ respectively, then $\delta$-eigenvectors for $\hat{A}_1$ and $\hat{A}_2$ are $2\delta$-eigenvectors for $A_1$ and $A_2$. It is clear that $\sigma_p(\hat{A}_1, \hat{A}_2)$ and $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ are algebraic sets. Finally, the points of intersection of $\sigma_p(\hat{A}_1, \hat{A}_2)$ and $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ with the $x$-axis that are close to one, are regular points, and the distances between the line $\{ x + \beta y = 1 \}$ and $\sigma_p(\hat{A}_1, \hat{A}_2)$ and $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ is less than $2\epsilon$. We denote the polynomials that determine $\sigma_p(\hat{A}_1, \hat{A}_2)$ and $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ by $R(x, y)$ and $S(x, y)$. As the rank of the approximating operators increases and $\tau$ approaches zero, the polynomials $R$ and $S$ approach $f_1$ and $f_2$ respectively. It follows from the direct expression of the defining function given by equation (2.1). Thus, the condition (2) holds for these polynomials with $\epsilon_1 = 2\epsilon$. Again rescaling with a coefficient close to one we may assume that both $\sigma_p(\hat{A}_1, \hat{A}_2)$ and $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ pass through $(1, 0)$.

Next we note that since $\epsilon \ll \rho$, the orthogonal projection in $\mathbb{C}^2$ onto $\{ x + \beta y = 1 \}$ of each curve $\sigma_p(\hat{A}_1, \hat{A}_2)$ and $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ contains the disk of radius $\frac{\rho}{\sqrt{2}}$ centered at $(1, 0)$. Indeed, again we may assume $\beta \geq 0$. Let us change the coordinates to

$$
(7.5) \quad u = \frac{x - 1}{\sqrt{1 + \beta^2}} + \frac{\beta y}{\sqrt{1 + \beta^2}}, \quad v = \frac{\beta(x - 1)}{\sqrt{1 + \beta^2}} - \frac{y}{\sqrt{1 + \beta^2}}.
$$

It is easily seen that the the bidisk $\{ |x - 1| \leq \rho, |y| \leq \rho \}$ contains the bidisk $\Delta_p(\beta) = \{ |u| \leq \frac{\rho\sqrt{1 + \beta^2}}{1 + \beta}, |v| \leq \frac{\rho\sqrt{1 + \beta^2}}{1 + \beta} \}$. Since $\frac{\rho\sqrt{1 + \beta^2}}{1 + \beta} \geq \frac{1}{\sqrt{2}}$, the bidisk $\Delta_p = \{ |u| \leq \frac{\rho}{\sqrt{2}}, |v| \leq \frac{\rho}{\sqrt{2}} \}$ is in $\{ |x - 1| \leq \rho, |y| \leq \rho \}$. In the $(u, v)$-coordinates $\sigma_p(\hat{A}_1, \hat{A}_2)$ and $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ are zeros of the polynomials $\tilde{R}(u, v) = R(\frac{u + \beta v}{\sqrt{1 + \beta^2}} + 1, \frac{\beta u - v}{\sqrt{1 + \beta^2}})$ and $\tilde{S}(u, v) = S(\frac{u + \beta v}{\sqrt{1 + \beta^2}} + 1, \frac{\beta u - v}{\sqrt{1 + \beta^2}})$ respectively. Suppose that $(u_0, v_0) \in \Delta_p$ and $\tilde{R}(u_0, v_0) = 0$, that is $(u_0, v_0) \in \sigma_p(\hat{A}_1, \hat{A}_2)$. The distance from this point to the line $\{ x + \beta y = 1 \} = \{ u = 0 \}$ is equal to $|u_0|$. Hence, $|u_0| \leq 2\epsilon$. Consider the functions $\phi_v(u) = \tilde{R}(u, v)$. Since $\phi_{v_0}(u_0) = 0$, Hurwitz’s theorem (see, for example, [12], page 231) implies that if $v_1$ is close to $v_0$, $\phi_{v_1}$ has zero $u_1$ close to $u_0$. Since $|u_0| \leq 2\epsilon \ll \frac{\rho}{\sqrt{2}}$, we conclude that $|u_1| < \frac{\rho}{\sqrt{2}}$, and, therefore, $(u_1, v_1) \in \Delta_p$, and $(0, v_1)$ belongs to the projection of $\sigma_p(\hat{A}_1, \hat{A}_2)$ onto $\{ u = 0 \}$. A similar argument applied to $\sigma_p(\hat{A}_1^{-1}, \hat{A}_2)$ finishes the proof of the claim.

We now return to relations (3.19) and (3.20) to express residues of $\Psi_1$ and $\Psi_2$ explicitly in terms of derivatives of the determining polynomial.
It is easy to check that in our case these relations for \( \sigma_p(\hat{A}_1, \tilde{A}_2) \) yield

\begin{equation}
\text{Res } \Psi_1(\lambda) \mid_{\lambda=1} = \frac{\partial R}{\partial x} \bigg|_{(1,0)} P_1 \hat{A}_2 P_1 - \frac{\partial R}{\partial y} \bigg|_{(1,0)} P_1 = 0
\end{equation}

\begin{equation}
\text{Res } \Psi_2(\lambda) \mid_{\lambda=1} = \frac{\partial R}{\partial x} \bigg|_{(1,0)} P_1 \hat{A}_2 T \tilde{A}_2 P_1 - \frac{1}{2} \left( \frac{\partial^2 R}{\partial x^2} \left( \frac{\partial R}{\partial y} \right)^2 \right)
-2 \frac{\partial^2 R}{\partial x \partial y} \frac{\partial R}{\partial x} + \frac{\partial^2 R}{\partial y^2} \right) \bigg|_{(1,0)} P_1 = 0,
\end{equation}

where, as in Section 3, \( P_1 \) is the orthogonal projection on the eigenspace of \( \hat{A}_1 \) corresponding to the eigenvalue one, and \( T \) is defined by (3.15).

Equation (7.6) together with condition (3) of this theorem imply that \( \frac{\partial R}{\partial x} \bigg|_{(1,0)} \neq 0 \), and, hence, this derivative does not vanish in a neighborhood of \((1,0)\). By the implicit function theorem the relation \( \hat{R}(x, y) = 0 \) determines \( x \) as an analytic function of \( y \) in a neighborhood of \((1,0)\). In terms of this function \( x(y) \) equations (7.6) and (7.7) can be written as

\begin{equation}
P_1 \hat{A}_2 P_1 = -x'(0) P_1,
\end{equation}

\begin{equation}
P_1 \hat{A}_2 T \tilde{A}_2 P_1 = -\frac{x''(0)}{2} P_1.
\end{equation}

As it was done before, we denote by \( e_1, e_2, \ldots \) an orthonormal eigenbasis for \( \hat{A}_1 \) with \( \hat{A}_1(e_1) = e_1, \hat{A}_1(e_j) = \mu_j e_j, \mu_j \neq 1, j = 2, \ldots \) and \( P_j \) being the orthogonal projection onto a subspace spanned by \( e_j \). In this basis relation (7.9) can be written as

\begin{equation}
\sum_{j=2}^{\infty} |\langle \tilde{A}_2 e_1, e_j \rangle|^2 \mu_j - 1 = \frac{x''(0)}{2}.
\end{equation}

Our next step is to show that \( x''(0) \) is small. Let us once again pass to the coordinates (7.5). We have \( \frac{\partial \hat{R}}{\partial u} = \frac{1}{\sqrt{1+\beta^2}} \frac{\partial \hat{R}}{\partial x} + \frac{\beta}{\sqrt{1+\beta^2}} \frac{\partial \hat{R}}{\partial y} \neq 0 \) for every \((u, v) \in \{|u| \leq \frac{\rho}{\sqrt{2}}, |v| \leq \frac{\rho}{\sqrt{2}}\}\). Applying the implicit function theorem we see that equation \( \hat{R}(u, v) = 0 \) determines \( u \) as an analytic function of \( v \) in a neighborhood of every point \( v \in \{|v| \leq \frac{\rho}{\sqrt{2}}\} \). Since this function is globally continuous in \( \{|v| \leq \frac{\rho}{\sqrt{2}}\} \), by the monodromy theorem, see [12, p.161], \( u \) is holomorphic in the whole disk \( \{|v| \leq \frac{\rho}{\sqrt{2}}\} \). The above argument showed that \( |u(v)| \leq 2\epsilon \) for every \( v \in \{|v| \leq \frac{\rho}{\sqrt{2}}\} \). Now the Cauchy theorem implies that

\begin{equation}
|u'(0)| \leq \frac{4\epsilon}{\rho},
\end{equation}

\begin{equation}
|u''(0)| \leq \frac{4\sqrt{2}\epsilon}{\rho^2}.
\end{equation}
A straightforward computation shows that

\[
\frac{d^2 x}{dy^2} = \frac{(1 + \beta^2)^{3/2} \frac{d^2 y}{dy^2}}{(1 - \beta \frac{du}{dv})^3}.
\]

Equations (7.4), (7.11), (7.12), and (7.13) yield

\[
|x''(0)| \leq C\epsilon
\]

where

\[
C = \frac{4\sqrt{2}}{\left( \rho - \frac{4\rho \|A_2\|}{d} \epsilon \right)^3}
\]

is a constant independent of \(\beta\). Now equations (7.10) and (7.14) give

\[
\left| \sum_{j=2}^{\infty} \frac{|\langle \tilde{A}_2 e_1, e_j \rangle|^2}{\mu_j - 1} \right| \leq \frac{C}{2}\epsilon.
\]

Applying a similar argument to the pair \(\hat{A}_1^{-1}, \tilde{A}_2\) and using (4.10) we obtain

\[
\left| \sum_{j=2}^{\infty} \frac{\mu_j |\langle \tilde{A}_2 e_1, e_j \rangle|^2}{\mu_j - 1} \right| \leq \frac{C}{2}\epsilon.
\]

Equations (7.16) and (7.17) result in

\[
\sum_{j=2}^{\infty} |\langle \tilde{A}_2 e_1, e_j \rangle|^2 \leq C\epsilon.
\]

Set \(\lambda = \langle \tilde{A}_2 e_1, e_1 \rangle\). The last relation implies

\[
\|\tilde{A}_2 e_1 - \lambda e_1\| \leq \sqrt{C}\epsilon,
\]

which means that \(e_1\) is \(\tilde{\delta}\)-eigenvector of \(\tilde{A}_2\), where \(\tilde{\delta} = \sqrt{C}\epsilon\). Thus, \(e_1\) is a common \(\tilde{\delta}\)-eigenvector of \(\hat{A}_1\) and \(\tilde{A}_2\). Therefore, as it was mention above, \(e_1\) is a \(\delta\)-eigenvector of \(A_1\) and \(A_2\) with \(\delta = 2\tilde{\delta}\). We are done. \(\square\)

It was mentioned in the proof of Theorem 7.3 that the polynomial \(R\) determining the spectrum of \(A_1\) and \(\tilde{A}_2\) converges uniformly on compacts in a neighborhood of \((1, 0)\) to the function \(f\) determining \(\sigma_p(A_1, A_2)\) as the finite rank approximations of compact parts converge to those of \(A_1\) and \(A_2\). Thus, we have the following corollary to the proof of Theorem 7.3

**Corollary 7.19.** Suppose that \(A_1, A_2 \in \mathcal{E}(H)\), with \((1, 0) \in \sigma_p(A_1, A_2)\). Suppose further that \(f(x, y)\) is an analytic function that determines \(\sigma_p(A_1, A_2)\)
Assume that eigenvalue \( \lambda_1 \). Also, as in Theorem 7.3 using an arbitrary small perturbation we may \( \sigma \) and \( \rho \) implies that in the bidisk

\[ f(x, y) = 0, \quad x(0) = 1. \]

If in Theorem 7.3 the norm of \( A_2 \) is equal to \( |\beta| \), then no condition on \( \sigma_p(A_1^{-1}, A_2) \) is necessary for the existence of a common almost eigenvector.

**Theorem 7.22.** Let \( A_1, A_2 \) be compact and \( \|A_2\| = |\beta| \). Suppose that \( \alpha \neq 0 \) and \( (1/\alpha, 0) \) belongs to \( \sigma_p(A_1, A_2) \). If there exist \( \rho > 0 \) and \( 0 < \epsilon < \rho \) such that

1. the Hausdorff distance from \( \sigma(A_1, A_2)|_{(1/\alpha, 0)} \) to the line \( \{\alpha x + \beta y = 1\} \) does not exceed \( \epsilon \);
2. \( \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} \neq 0 \) in the bidisk \( \{|x-1/\alpha| \leq \rho, |y| \leq \rho\} \), where \( f \) is an analytic function that determines \( \sigma_p(A_1, A_2) \),

then there is an eigenvector of \( A_1 \) that is \( (\sqrt{\frac{8|\beta|(1+\beta^2)}{\rho - 4\beta\epsilon}})\)-eigenvector of \( A_2 \).

**Proof.** As we mentioned before, we can replace \( A_1 \) with \( A_1/\alpha \), so that \( \alpha = 1 \). Also, as in Theorem 7.3 using an arbitrary small perturbation we may assume that eigenvalue \( \lambda = 1 \) of \( A_1 \) has multiplicity one. Condition (2) implies that in the bidisk \( \{|x-1| \leq \rho, |y| \leq \rho\} \) the joint spectrum \( \sigma_p(A_1, A_2) \) is nonsingular and, therefore, is a smooth analytic curve \( \Gamma \). Using condition (1) and the argument with passing to the coordinates (7.5) similar to the one that was used in the proof of Theorem 7.3 and the fact that \( \frac{dx}{dy} = \frac{dx}{dy} - \frac{\beta}{\alpha} \)

we show that \( | - x'(0) - \beta | \leq \frac{4(1+\beta^2)\epsilon}{\rho - 4\beta\epsilon} \). Thus, if \( e_1 \) is a unit eigenvector of \( A_1 \) with eigenvalue \( \lambda = 1 \), the relation (7.20) implies

\[ |(A_2 e_1, e_1)| = |x'(0)| \geq |\beta| - \frac{4(1+\beta^2)\epsilon}{\rho - 4\beta\epsilon}. \]

Hence,

\[ \|A_2 e_1 - (A_2 e_1, e_1) e_1\|^2 = \|A_2 e_1\|^2 - |(A_2 e_1, e_1)|^2 \leq \beta^2 - \left( |\beta| - \frac{4(1+\beta^2)\epsilon}{\rho - 4\beta\epsilon} \right)^2 \leq \frac{8|\beta|(1+\beta^2)\epsilon}{\rho - 4\beta\epsilon}. \]

We are done.

**Remark 7.24.** The condition \( \|A_2\| = |\beta| \) can obviously be replaced with \( \|A_2\| - |\beta| < \delta \). In this case there is a common \( \sqrt{2\beta\delta + \delta^2} + \frac{8\beta(1+\beta^2)\epsilon}{\rho - 4\beta\epsilon} \)-eigenvector.
8. Norm estimates for the commutant of a pair of matrices

Under the assumptions of Theorem 7.22, we will now define a new operator close to \( A_2 \) that has a common eigenvector with \( A_1 \). Let as above \( e_1 \) be an eigenvector of \( A_1 \) with \( \lambda = 1 \). Write
\[
\hat{A}_2 = P_1 A_2 P_1 + (I - P_1) A_2 (I - P_1).
\]
Of course, \( e_1 \) is a common eigenvector of \( A_1 \) and \( \hat{A}_2 \).

Let \( \xi \) be a unit vector orthogonal to \( e_1 \), that is \( \|\xi\| = 1 \), \( (I - P_1) \xi = \xi \).

We have
\[
\|A_2 \xi - \hat{A}_2 \xi\| = \|P_1 A_2 \xi\| = \|\langle A_2 e_1, \xi \rangle\| = \|\langle (A_2 e_1 - (A_2 e_1, e_1) e_1), \xi \rangle\| \leq C \sqrt{\epsilon},
\]
where \( C = \sqrt{\frac{8|A_2||(1 + \|A_2\|)^2}{\rho - 4\|A_2\|}} \).

For \( \zeta = ce_1 + \sqrt{1 - |c|^2} \xi \) with \( \|\xi\| = 1 \), \( (I - P_1) \xi = \xi \) the last relation yields
\[
\|A_2 - \hat{A}_2\| \leq \sqrt{2} C \sqrt{\epsilon}.
\]
This gives us the following estimate for the norm of the commutant \([A_1, A_2]\):
\[
\|\{[A_1, A_2]\} \| \leq \|\|A_1\|, (A_2 - \hat{A}_2)\| + \|\|A_1, \hat{A}_2\|\|
\leq \sqrt{2} C \sqrt{\epsilon} \|A_1\| + \|\|A_1^{(1)}, A_2^{(1)}\|\|,
\]
where \( A_1^{(1)} = (I - P_1) A_1 (I - P_1) \), \( A_2^{(1)} = (I - P_1) A_2 (I - P_1) \) are the compressions of \( A_1 \) and \( A_2 \) to the orthocomplement to \( e_1 \).

**Remark 8.3.** If the the point \( (\frac{1}{\alpha}, 0) \) is not a singular point of the proper joint spectrum of \( A_1 \) and \( A_2 \) with \( \|A_2\| = |\beta| \), and \( \sigma_p(A_1, A_2, \rho, (\frac{1}{\alpha}, 0)) \) is at less than \( \epsilon \) Hausdorff distance from the line \( \{\alpha x + \beta y = 1\} \), the inequality (8.2) still holds. This follows from the fact that the pair \( (A_1/\alpha, A_2) \) satisfies the conditions of Theorem 7.22.

Now we will use the relation (8.2) to estimate the norm of the commutant of a pair of self-adjoint \( N \times N \) matrices in terms of the Hausdorff distance from the joint spectrum to a set of lines that imitates a joint spectrum of a pair of commuting matrices. Since our result is stable with respect to small perturbations, in the next theorem we will assume that both matrices have simple spectra and the absolute values of their eigenvalues are different. Since the commutant of \( A_1 \) and \( A_2 \) is the same as the commutant of \( A_1 + \alpha I \) and \( A_2 + \beta I \) for every \( \alpha, \beta \), we may assume that \( A_1 \) and \( A_2 \) are invertible, that is all their eigenvalues are nontrivial.

Let \( f(z) \) be analytic in the closed disk \( \Delta_{\rho}(a) = \{|z - a| \leq \rho\} \) and its derivative does not vanish there. Then \( f \) is locally univalent in \( \Delta_{\rho}(a) \). Write
\[
\delta(w) = \sup\{r : f \text{ is univalent in } \Delta_r(w)\}.
\]
(8.4)
\[
\delta(f, \rho, a) = \min\{\delta(w) : w \in \Delta_{\rho}(a)\}.
\]
Remark 8.5. The above definition of $\delta(f, \rho, a)$ is slightly reminiscent of Bloch’s constant $B$, cf. [22], but, of course, they are very different.

Theorem 8.6. Let $A_1, A_2$ be $N \times N$ self-adjoint matrices, and $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$ be the eigenvalues of $A_1$ and $A_2$ respectively with $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_N| > 0$ and $|\beta_1| > |\beta_2| > \cdots > |\beta_N| > 0$. Suppose that $\ell_1, \ldots, \ell_N$ is a family of complex lines of the form $\ell_j = \{\alpha_{n(j)}x + \beta_{j}y = 1\}$, $1 \leq j, n(j) \leq N$ such that

1. Each of the points $(\frac{1}{\alpha_{n(k)}}, 0)$, $1 \leq k \leq N$ belongs to one of these lines;
2. There exist $0 < \rho < 1$ and $0 < \epsilon \ll \rho$ such that conditions (1) and (2) of Theorem 7.22 are satisfied by $\sigma_p(A_1, A_2)$ and each line $\{\alpha_{n(j)}x + \beta_{j}y = 1\}$.

Then if $\epsilon$ is small enough, the norm of the commutant of $A_1$ and $A_2$ is at most of order $\epsilon^{1/2N}$.

Proof. By Theorem 7.22 the eigenvector $e_n(1)$ of $A_1$ that corresponds to eigenvalues $\alpha_{n(1)}$ is a $\sqrt{\frac{8|\beta_1|(1+|\beta_1|^2)}{\rho - 4|\beta_1|^2}}$-eigenvector of $A_2$, and relation (8.2) holds with $P_1$ being replaced with $P_{n(1)}$. Write $\epsilon_1 = \epsilon$. We want to estimate $\epsilon_2$ such that the compression of $A_1^{(1)}$ and $A_2^{(1)}$ to $\text{span}\{e_k, k = 1, \ldots, N, k \neq n(1)\}$ satisfy conditions (1) and (2) of the present Theorem with $\epsilon_2, \rho/2$.

It follows from [7.23] that in the eigenbasis $e_1, \ldots, e_N$ of the matrix $A_1$ every entry of the $n(1)$-th row (and column) of the matrix $A_2$ except for the one on the main diagonal has absolute value that does not exceed $C_1\sqrt{\epsilon_1}$, where $C_1 = \sqrt{\frac{8|\beta_1|(1+|\beta_1|^2)}{\rho - 4|\beta_1|^2}}$. Let

$P(x, y) = \det [xA_1 + yA_2 - I], \quad P_1(x, y) = \det[xA_1^{(1)} + yA_2^{(1)} - I],$

and let

$$(8.7) \quad d = \min \left\{ \left| \alpha_{n(j)} \frac{\partial P}{\partial x} + \beta_j \frac{\partial P}{\partial y} \right| : \left| x - \frac{1}{\alpha_{n(j)}} \right| \leq \epsilon, \left| y \right| \leq \epsilon, \quad 1 \leq j \leq N \right\}.$$

Of course, the determining polynomials $P$ and $P_1$ satisfy

$$(8.8) \quad P(x, y) = (\alpha_{n(1)}x + \beta_1 y - 1)P_1(x, y) + Q(x, y),$$

where $Q$ is a polynomial of degree $N$ whose coefficients in absolute values do not exceed $NC_1|\beta_1|^{N-1}\sqrt{\epsilon_1}$. Write

$M = \max\{|1/\alpha_j| + \rho + 1\}$.

We obviously have

$$|P(x, y)| \leq (|\alpha_{n(j)}| + |\beta_1| + 1)M|P_1(x, y)| + NC_1|\beta_1|^{N-1}M^\ell\sqrt{\epsilon_1}.$$

Now, let $(x, y) \in \sigma_p(A_1^{(1)}, A_2^{(1)}) \cap \{|x - 1/\alpha_{n(m)}| \leq \rho/2, |y| \leq \rho/2\}$ for some $2 \leq m \leq N$. Then $|P(x, y)| \leq NC_1|\beta_1|^{N-1}M^\ell\sqrt{\epsilon_1}$. Write $f(t) = P(x + t\alpha_{n(m)}, y + t\beta_m)$. Equation (8.7) implies that

$$(8.9) \quad |f'(t)| \geq d > 0$$
for \(|t| \leq \rho/2\), and, therefore, \(f(t)\) is locally univalent in the disk \(\{|t| \leq \rho/2\}\). By (8.4) \(f\) is univalent in the disk of radius \(\tau = \min\{\delta(F_{x,y}, \rho, 0) : |x - 1/\alpha_n| \leq \rho/2, |y| \leq \rho/2\}\), where \(F_{x,y}(w) = P(x + w\alpha_n, y + w\beta_m)\), so that this radius does not depend on the point \((x, y)\). By (8.9) \(|f'(0)| \geq d\), so Koebe's 1/4 theorem, cf. [17, p.150], implies that if \(\epsilon_1\) is small enough so that \(NC_1|\beta_1|^{N-1}M^N \sqrt{\epsilon_1} < \frac{d\tau}{4}\), the function \(f\) has a zero in \(\{|t| \leq 4NC_1|\beta_1|^{N-1}M^N \sqrt{\epsilon_1}\}\), and, hence, the distance from 

\[\sigma_p(A_1^{(1)}, A_2^{(1)}) \cap \{|x - 1/\alpha_n| \leq \rho/2, |y| \leq \rho/2\}\]

to \(\sigma(A_1, A_2)\) does not exceed \(4NC_1|\beta_1|^{N-1}M^N \sqrt{\epsilon_1}\), and, therefore, the distance from 

\[\sigma_p(A_1^{(1)}, A_2^{(1)}) \cap \{|x - 1/\alpha_n| \leq \rho/2, |y| \leq \rho/2\}\]

to the line \(\{\alpha_n x + \beta_n y = 1\}\) does not exceed \(\epsilon_2 = 5NC_1|\beta_1|^{N-1}M^N \sqrt{\epsilon_1}\).

The fact that the eigenvalues of \(A_2^{(1)}\) differ from those of \(A_2\) by a magnitude of order \(\sqrt{\epsilon_1}\) follows directly from (8.1) and the fact that for two compact normal operators the distance between their spectra does not exceed the distance between them in the operator norm topology, cf. [14, Proposition 1].

Finally, it follows from (8.8) that the difference between \(\alpha_n(k) \frac{\partial P_1}{\partial x} + \beta_k \frac{\partial P_1}{\partial y}\), \(k \neq 1\) and \(\alpha_n(k) \frac{\partial P}{\partial x} + \beta_k \frac{\partial P}{\partial y}\) is of order of \(\epsilon_2\), and, therefore, the \((N - 1)\times(N - 1)\)-dimensional matrices \(A_1^{(1)}\) and \(A_2^{(1)}\) satisfy the conditions of this theorem with \(\epsilon_2\) and \(\rho/2\). Continuing inductively we arrive at the claimed estimate. \(\Box\)

9. CONCLUDING REMARKS

Here we want to state and discuss four problems immediately related to the material of the preceding sections. The first one was essentially stated in the Introduction. It was mentioned there that, according to [27], if \(A_1, \ldots, A_n\) are compact, then on every compact subset of \(\mathbb{C}^n\) the joint spectrum \(\sigma_p(A_1, \ldots, A_n)\) has a global defining function. Our first problem is:

Problem 9.1. Describe those globally defined in \(\mathbb{C}^n\) analytic sets that have spectral representation. In particular, which real globally defined analytic sets (that is, zeros of entire functions with real Taylor coefficients) have self-adjoint spectral representation?

In the case of matrices the similar problem for general matrices was solved by Dickson [9] and for self-adjoint matrices by Helton and Vinnikov [16].

The second problem is related to the example in Section 6 and Theorem B. It was required in Theorem 6.5 that \(\|A_2\| = 1\) and this condition substantially differentiates this result from Theorem B. We wonder whether this condition is redundant and the result holds without it. More generally,
it is natural to ask whether a complete analog of Theorem B for algebraic curves of order higher than 1 is valid. Thus, we are compelled to pose the following problem.

**Problem 9.2.** Suppose that $A$ and $B$ are self-adjoint, $A$ is invertible and $\sigma_p(A, B)$ contains a real algebraic curve $\Gamma$ of order $k$ that meets the $x$- and $y$-axes at points $(\alpha_1, 0), \ldots, (\alpha_k, 0)$ and $(\beta_1, 0), \ldots, (\beta_k, 0)$ respectively. Also suppose that all these points of intersection of $\Gamma$ with the coordinate axes are isolated spectral points of the corresponding operators. If $\sigma_p(A^{-1}, B)$ contains an algebraic curve $\Gamma_1$ of the same order $k$ that meets the coordinate axes at points $(1/\alpha_1, 0), \ldots, (1/\alpha_k, 0)$ and $(\beta_1, 0), \ldots, (\beta_k, 0)$ respectively, does this imply that $A$ and $B$ have a common $k$-dimensional reducing subspace?

We saw that the appearance of circular arcs in the joint spectrum of $A$ and $B$ under the assumptions of Theorem 6.5 implies that the group generated by $A$ and $B$ “contains a representation” of Coxeter’s group $BC_2$ in the sense that the the pair $(A, B)$ is decomposable having a finite dimensional reducing subspace and the restriction of the group to this subspace represents $BC_2$. It would be interesting to find out whether there are other curves or surfaces in joint spectra of operator tuples that are associated with other symmetry groups. For instance, Theorem 4.15 shows that this is true for lines appearing in joint spectra of $A$ and $B$, when the corresponding lines appear in $\sigma_p(A^{-1}, B)$. In this case the represented group is a product of copies of the Coxeter group $A_1$.

**Problem 9.3.** Which curves (or surfaces) appearing in the joint spectrum of a pair of operators (tuple of operators) indicate that the group generated by this pair (tuple) ”contains a representation” of a corresponding symmetry group?

The last problem we would like to mention is related to the norm estimate of the commutant of two matrices, or more generally two compact operators. The estimate given by Theorem 8.6 seems to be rather rough. Besides, this estimate does not allow any extension of the result of Theorem 8.6 to an infinite dimensional case. One possible way of improving the estimate is to consider that the proper spectrum is close to a set of lines not locally, but on a big compact subset of $\mathbb{C}^2$. Alternatively, we might impose the condition that the joint projective spectrum in $\mathbb{CP}^2$ is close to a set of projective lines in Fubini study metric. This leads us to the following problem.

**Problem 9.4.** Let $A$ and $B$ be self-adjoint compact operators acting on a separable Hilbert space $H$. Suppose that the distance from $\sigma(A, B, I)$ to a set of projective lines that contains $\{[x : y : 0] \}$ and satisfies the following condition:

(C) the intersection of this set of lines with the lines $\{[0 : y : z]\}$ and $\{[x : 0 : z]\}$ coincides with the inverse spectra $\sigma(A)^{-1}$ and $\sigma(B)^{-1}$, counting multiplicities;
does not exceed $\epsilon$ in the Fubini study metric. Estimate the norm of the commutant $[A,B]$ in terms of $\epsilon$.

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