BIFURCATIONS AND EXACT TRAVELING WAVE SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH FOURTH-ORDER DISPERSION AND DUAL POWER LAW NONLINEARITY

JIBIN LI
School of Mathematical Science, Huaqiao University
Quanzhou, Fujian 362021, China
and
Department of Mathematics, Zhejiang Normal University
Jinhua, Zhejiang 321004, China

YAN ZHOU
School of Mathematical Science, Huaqiao University
Quanzhou, Fujian 362021, China

Abstract. For the nonlinear Schrödinger (NLS) equation with fourth-order dispersion and dual power law nonlinearity, by using the method of dynamical systems, we investigate the bifurcations and exact traveling wave solutions. Because obtained traveling wave system is an integrable singular traveling wave system having a singular straight line and the origin in the phase plane is a high-order equilibrium point. We need to use the theory of singular systems to analyze the dynamics and bifurcation behavior of solutions of system. For $m > 1$ and $0 < m = \frac{1}{2} < \frac{1}{4}$, corresponding to the level curves given by $H(\psi, y) = 0$, the exact explicit bounded traveling wave solutions can be given. For $m = 1$, corresponding all bounded phase orbits and depending on the changes of system’s parameters, all exact traveling wave solutions of the equation can be obtain.

1. Introduction. In 2018, [7] studied the exact solutions for the following nonlinear Schrödinger (NLS) equation with fourth-order dispersion and dual power law nonlinearity:

$$i\psi + a\psi_{xx} - b\psi_{xxxx} + c(|\psi|^{2m} + k_1|\psi|^{4m})\psi = 0,$$

(1)

by using the generalized projective Riccati equations method. This equation describes the propagation of optical pulse in a medium, and $\psi(x, t)$ is the slowly varying envelope of the electromagnetic field, where $a, b, c$ are real numbers. If $b = 0$, equation (1) reduces to the NLS. In addition if $m = 1$, equation (1) reduces to parabolic law nonlinearity, for which the exact solutions have been discussed in [Xu, 2011] using two direct algebraic methods. The coefficient of $a$ represents the group velocity dispersion, while the coefficient of $c$ represents the self-phase modulation.
with dual power law nonlinearity. The constant $k_1$ binds the two nonlinear terms and the exponent $m$ governs the power law. Also, the coefficient $b$ is the fourth-order dispersion term. For finding many new exact solutions, equation (1) has been studied in [10] using five different techniques, namely, the $\left(\frac{G'}{G}\right)_n$-expansion method, the improved Sub-ODE method, the extended auxiliary equation method, the new mapping method, and the Jacobi elliptic function method. In [7], the authors applied the generalized projective Riccati equations method to find some new soliton and periodic solutions.

Let

$$q(x, t) = \phi(\xi)e^{iQ(x, t)},$$

where $i = \sqrt{-1}$, $\xi = x - vt$, $Q(x, t) = -kx + \omega t$. Substituting (2) into (1) and separating the real and imaginary parts, we have

$$-(\omega + ak^2 + bk^4)\phi + (a + 6bk^2)\phi'' - b\phi'''' + c(\phi^{2m+1} + k_1\phi^{4m+1}) = 0,$$

$$-(v + 2ak + 4bk^3)\phi' + 4b\phi'' = 0.$$  

Differentiating (4) and substituting the resulting equation in (3), we get the following nonlinear equation:

$$\phi'' - \beta\phi + \gamma(\phi^{2m+1} + k_1\phi^{4m+1}) = 0,$$

where we assume that $2ak + 20bk^3 - v \neq 0$ and $\beta = \frac{-4k(\omega + ak^2 + bk^4)}{2ak + 20bk^3 - v}$, $\gamma = \frac{4kc}{2ak + 20bk^3 - v}$. Making the transformation

$$\phi(\xi) = (\psi(\xi))^{\frac{1}{2m}},$$

we obtain

$$2m\psi\psi'' + (1 - 2m)(\psi')^2 + 4m^2\psi^2(\beta + \gamma\psi + k_1\gamma\psi^2) = 0.$$  

Equation (7) is equivalent to the following planar integral system:

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{(1 - 2m)y^2 + 4m^2\psi^2(\beta + \gamma\psi + k_1\gamma\psi^2)}{2m\psi},$$

with the first integral

$$H(\psi, y) = y^2\psi^{-2} + \frac{1}{2}$$

$$+ \frac{4m^2\psi^2}{m + 1} \left[(m + 1)k_1\gamma\psi^2 + (2m + 1)\gamma\psi + \beta(2m^2 + 3m + 1)\right]$$

$$= h.$$  

Clearly, when $m \neq \frac{1}{2}$, system (8) is a singular traveling wave system of the first class defined in [3], [5], [2],[4] and [6] with a singular straight line $\psi = 0$.

Because the three papers [9], [10] and [7] did not study the dynamical behavior of system (8). In this paper, it is different from these references that we use the method of dynamical systems to investigate the bifurcations of the phase portraits of system (8) and find all possible exact solutions of system (8) depending on the changes of the parameter group ($\beta, \gamma, k_1$) and $m$. The results of this paper provide more complete understanding for the traveling wave solutions of equation (1).

This paper is organized as follows. In section 2, we consider the bifurcations of phase portraits of system (8). In section 3, corresponding to the level curves defined by $H(\psi, y) = 0$, we give all possible exact solutions of system (8) and equation (1) for all $m \geq 1$. In section 4, for the case $m = 1$, corresponding to all real level curves defined by $H(\psi, y) = h$, we calculate all possible exact explicit bounded solutions for system (8) and equation (1). In section 5, for $0 < m = \frac{1}{n} < \frac{1}{2}$, we discuss the exact solutions of system (8). As special case, corresponding to
the homoclinic orbit defined by \( H(\psi, y) = 0 \), we obtain the solution of the form 
\[
\phi(\xi) = \frac{A}{\cosh^n(\omega \xi)} = \text{Asech}^n(\omega \xi)
\]
of the equation (5), which give rise to the geometric explanation of solutions found in [8].

2. The bifurcations of phase portraits of system (8) for \( m \geq 1 \). We first consider the associated regular system of system (8) as follows:

\[
\frac{d\psi}{d\zeta} = 2m\psi y, \quad \frac{dy}{d\zeta} = -[(1 - 2m)y^2 + 4m^2\psi^2(\beta + \gamma \psi + k_1 \gamma \psi^2)],
\]

where \( m > 0, m \neq \frac{1}{2} \) and \( d\zeta = 2m\psi d\xi \), for \( \psi \neq 0 \). By the transformation (6), we only consider the positive solutions of \( \psi(\xi) \) in the right phase plane.

Obviously, for \( m \neq \frac{1}{2}, \beta \neq 0 \), the origin \( O(0, 0) \) is a two-order equilibrium point of system (10). To consider the directions that the orbits of system (10) tend to the origin when \( \zeta \to -\infty \) (or \( \infty \)), we know from \( G(\theta) = -\cos \theta(\sin^2 \theta + 4m^2 \beta \cos^2 \theta) = 0 \) that for \( \beta < 0 \), we have \( \theta_0 = \frac{1}{2} \pi, \theta_{1,2} = \mp \arctan \left(2m\sqrt{-\beta}\right) \). When \( \beta > 0 \), we only have \( \theta_0 = \frac{1}{2} \pi \). Therefore, when \( \beta < 0 \), in a neighborhood of the origin, there are six sectors in which lie the different types of orbits of system (10). When \( \beta > 0 \), there exist two areas (left phase plane and right phase plane) laying different orbits of system (10).

To find the other equilibrium points of system (10), we write that \( f(\psi) = k_1 \gamma \psi^2 + \gamma \psi + \beta, f'(\psi) = 2k_1 \gamma \psi + \gamma \). When \( k_1 < 0 \), \( f'(\psi) \) has a positive real zero at \( \psi = \psi_0 = -\frac{1}{2k_1} \). The function \( f(\psi) \) has two real zeros \( \psi_{1,2} = \frac{1}{2k_1} \left[ -1 \pm \sqrt{\Delta_1} \right] \), where \( \Delta_1 = \gamma(\gamma - 4k_1\beta) \).

Clearly, We have the following conclusions:

(1) Assume that \( k_1 = 0 \). In this case, system (10) has two equilibrium points \( O(0, 0) \) and \( E_0(\psi, 0) \), where \( \psi = -\frac{\beta}{\gamma} \). We have \( h_a = H(\psi, 0) = \frac{4\beta m^3 \psi_3^5}{(m+1)} \).

(2) Assume that \( k_1 < 0 \).

(i) When \( \gamma > 4k_1\beta \) and \( \beta < 0 \), \( f(\psi) \) has two positive real zeros: \( \psi_{1,2} = \frac{1}{2k_1} \left[ -1 \pm \sqrt{1 - \frac{4k_1\beta}{\gamma}} \right] \). When \( \gamma > 0, \beta > 0 \), \( f(\psi) \) has a positive real zero \( \psi_1 = \frac{1}{2k_1} \left[ 1 + \sqrt{1 + \frac{4k_1\beta}{\gamma}} \right] \).

(ii) When \( \gamma < 0, \gamma > 4|k_1|\beta \) and \( \beta > 0 \), \( f(\psi) \) has two positive real zeros: \( \psi_{1,2} = \frac{1}{2k_1} \left[ -1 \pm \sqrt{1 - \frac{4|k_1|\beta}{|\gamma|}} \right] \). When \( \gamma < 0, \beta < 0 \), \( f(\psi) \) has a positive real zero \( \psi_1 = \frac{1}{2k_1} \left[ -1 + \sqrt{\Delta_1} \right] \).

(3) Assume that \( k_1 > 0 \).

(i) When \( \gamma > 0, \beta < 0 \), \( f(\psi) \) has a positive real zero \( \psi_1 = \frac{1}{2k_1} \left[ -1 + \sqrt{1 + \frac{4k_1\beta}{\gamma}} \right] \).

(ii) When \( \gamma < 0, \beta > 0 \), \( f(\psi) \) has a positive real zero \( \psi_1 = \frac{1}{2k_1} \left[ -1 + \sqrt{1 + \frac{4k_1\beta}{|\gamma|}} \right] \).

Let \( M(\psi, 0) \) be the coefficient matrix of the linearized system of system (10) at an equilibrium point \( E_j(\psi_j, 0) \) and \( J(\psi, 0) = \det M(\psi, 0) \). We have

\[
J(\psi_j, 0) = 8\gamma m^3 \psi_3^3 (2k_1 \psi_j + 1).
\]

Specially, for \( k_1 = 0 \), \( J(\psi_a, 0) = 8\gamma m^3 \psi_3^3 = -\frac{8m^2 \beta^3}{\gamma} \).

By the theory of planar dynamical systems (see [Li, 2013]), for an equilibrium point of a planar integrable system, if \( J < 0 \) (or \( J > 0 \)), then the equilibrium point is
a saddle point (a center point); if \( J = 0 \) and the Poincaré index of the equilibrium point is 0, then this equilibrium point is a cusp.

Write that \( h_j = H(\psi_j, 0) \) where \((\psi_j, 0)\) is a equilibrium point of system (10). We have

\[
\begin{align*}
  h_1 &= H(\psi_1, 0) = 2^{1 - \frac{m}{2}} m^3 \psi_1^\frac{3}{2} \left[ (\sqrt{\Delta_1} + 4(m + 1)\beta k_1 - \gamma) \right] \left( \frac{2m + 1}{(m + 1)(m + 1)k_1} \right), \\
  h_2 &= H(\psi_2, 0) = -2m^3 \psi_2^\frac{3}{2} \left[ (\sqrt{\Delta_1} - 4(m + 1)\beta k_1 + \gamma) \right] \left( \frac{2m + 1}{(m + 1)(m + 1)k_1} \right).
\end{align*}
\]

Especially, when \( k_1 = 0, h_a = H(\psi_a, 0) = 4^{m^3} \beta \psi_a^{\frac{3}{2}} \left( \frac{m + 1}{m + 1} \right)^2 \). When \( m \geq 1 \), we see from (9) that \( h_0 = H(0, 0) \) is unbounded.

Notice that when \( \gamma = \gamma_0 = \frac{4(m + 1)^2 k_1 \beta}{2m + 1} \), we have \( h_2 = 0 \).

For the case \( m \geq 1 \), by using the above information to do qualitative analysis, corresponding to the cases of there is only one equilibrium point \( O(0, 0) \) and there exist two or three equilibrium points, respectively, we have the bifurcations of phase portraits of system (10) shown in Fig.1, Fig.2 and Fig.3, Fig.4.

2.1. The case of system (10) has only one equilibrium point \( O(0, 0) \). The parameter condition is \( \Delta_1 < 0 \). When \( \beta > 0 \), in the right phase plane, there is an elliptic area laying infinitely many periodic orbits which contact to the origin \( O(0, 0) \). When \( \beta < 0 \), in the right phase plane, there exist two parabolic areas near the origin \( O(0, 0) \).

![Fig.1 The bifurcations of phase portraits of system (8) when \( \Delta_1 < 0 \)](image)

2.2. The case of system (10) has two equilibrium points \( O(0, 0) \) and \( E(\psi_j, 0), \phi_j > 0 \). The parameter condition is \( k_1 \beta \gamma < 0 \) or \( k_1 = 0, \beta \gamma < 0 \).

![Fig.2 The bifurcations of phase portraits of system (8) when \( k_1 \beta \gamma < 0 \)](image)
2.3. The case of system (10) has three equilibrium points \( O(0,0) \) and \( E_j(\psi_j,0), j = 1, 2 \) with \( 0 < \psi_1 < \psi_2 \). The parameter condition is \( k_1 \beta \gamma > 0, \Delta_1 > 0 \).

\[
\begin{align*}
(a) & \ h_1 < h_2 < 0 \\
(b) & \ h_1 = 0 = h_2 \\
(c) & \ h_1 < 0 < h_2
\end{align*}
\]

Fig.3 The bifurcations of phase portraits of system (8) when \( \beta < 0, k_1 < 0, \Delta_1 > 0 \)

Parameters: (a) \( \gamma_0 > \gamma > 4k_1 \beta > 0 \). (b) \( \gamma = \gamma_0 = \frac{2(m+1)^2k_1\beta}{2m+1} \). (c) \( \gamma > \gamma_0 \).

\[
\begin{align*}
(a) & \ 0 < h_2 < h_1 \\
(b) & \ h_2 = 0 < h_1 \\
(c) & \ h_2 < 0 < h_1
\end{align*}
\]

Fig.4 The bifurcations of phase portraits of system (8) when \( \beta > 0, k_1 < 0, \Delta_1 > 0 \)

Parameters: (a) \( \gamma_0 < \gamma < 4k_1 \beta \). (b) \( \gamma = \gamma_0 \). (c) \( \gamma < \gamma_0 < 0 \).

3. The exact traveling wave solutions of equation (1) with \( m > 1 \) given by the level curves \( H(\psi, y) = 0 \). We know from (9) that for any positive integer \( m > 1 \),

\[
y^2 = h\psi^2 - \frac{4m^2\psi^2}{(2m+1)(m+1)}[(2m+1)(m+1)\beta + (2m+1)\gamma\psi + k_1\gamma(m+1)\psi^2].
\]

It is easy to see that for \( m > 1 \), if and only if \( h = 0 \), by using the first equation of system (8), we have

\[
\frac{\sqrt{(2m+1)(m+1)}}{2m} \xi = \int_{\psi_0}^{\psi} \frac{d\psi}{\psi^2 -(2m+1)(m+1)\beta - (2m+1)\gamma\psi - k_1\gamma(m+1)\psi^2}.
\]

It means that for \( m > 1 \), we only can obtain the exact solutions of system (8) defined by the real level curves \( H(\psi, y) = 0 \). All real level curves of given by \( H(\psi, y) = 0 \) are shown in Fig.5.
Fig. 5 The level curves of $H(\psi, y) = 0$

Parameters: (a) $\beta < 0, \Delta_1 < 0$. (b) $k_1 < 0, \beta < 0, \Delta_1 > 0, \gamma = \gamma_0 > 0, b_2 = 0$. (c) $k_1 < 0, \beta < 0, \Delta_1 > 0, \gamma > \gamma_0 > 0$. (d) $k_1 < 0, \beta > 0, \Delta_1 > 0, \gamma_0 < \gamma < 0$. (e) $\beta \gamma k_1 < 0, \beta < 0$. (f) $\beta \gamma k_1 < 0, \beta > 0$.

We notice that if

$$\Delta_2 = (2m + 1)\gamma[(2m + 1)\gamma - 4(m + 1)^2k_1\beta] = (2m + 1)^2\gamma(\gamma - \gamma_0) > 0,$$

then, the quadratic polynomial $F(\psi) = (2m+1)(m+1)\beta+(2m+1)\gamma\psi+k_1\gamma(m+1)\psi^2$ has two roots as follows:

$$\tilde{\psi}_{1,2} = \frac{2m + 1}{2(m + 1)k_1} \left[ -1 \pm \sqrt{\gamma(\gamma - \gamma_0)} \right].$$

We next consider the bounded solutions of equation (8).

(i) We assume that $k_1 < 0, \beta < 0, \gamma = \gamma_0 > 4k_1\beta$. In this case, the level curves defined by $H(\psi, y) = 0$ are two heteroclinic orbits connecting the equilibrium points $O(0,0)$ and $E_2(\psi_2, 0)$ shown in Fig.5 (b). Now, we have $F(\psi) = -[(m+1)|k_1|\gamma_0(\psi_D - \psi)^2]$, where $\psi_D = \frac{2m+1}{2(2m+1)|k_1|\gamma_0}$. Hence, (11) becomes $\sqrt{\frac{2m+1}{2(m+1)|k_1|\gamma_0}}$.

We have:

$$\psi(\xi) = \pm \frac{\psi_2}{1 + e^{\omega_1\xi}} = \pm \frac{1}{2} \psi_2 \left( 1 + \tanh \left( \frac{1}{2} \omega_1 \xi \right) \right),$$

(14)

where $\omega_1 = \frac{1}{2m}(2m + 1)(m + 1)\sqrt{|\beta|}$. Thus, by using (2), we obtain the following exact solution of equation (1):

$$q(x, t) = \pm \left[ \frac{1}{2} \psi_2 \left( 1 + \tanh \left( \frac{1}{2} \omega_1 (x - vt) \right) \right) \right]^\frac{1}{2m} e^{(-kx+\omega t)}.$$
(ii) We assume that \( k_1 < 0, \beta < 0, \Delta_1 > 0, \gamma > \gamma_0 > 0 \). In this case, the level curves defined by \( H(\psi, y) = 0 \) are a homoclinic orbit to the origin \( O(0,0) \) passing through the point \((\hat{\psi}_M, 0) (\hat{\psi}_M = \hat{\psi}_2)\) and an open curve passing through the point \((\psi_L, 0) (\psi_L = \hat{\psi}_1)\) (see Fig.5 (c)).

Corresponding to the homoclinic orbit, (11) becomes that
\[
\sqrt{(2m+1)(m+1)^2k_1\gamma} = \int_\psi^{\psi_M} \frac{d\psi}{\psi \sqrt{(\psi_L - \psi)(\psi_M - \psi)}}.
\]

Therefore, we have the following parametric representation of this homoclinic orbit:
\[
\psi(\xi) = \frac{2(m+1)\beta}{\sqrt{\gamma(\gamma - \gamma_0)\cosh(\Omega_0\xi) - \gamma}}.
\]
(16)

where \( \Omega_0 = \frac{(2m+1)(m+1)}{2m} \sqrt{\beta(m+1)k_1\gamma} \). (16) gives rise to the solution of equation (1):
\[
q(x,t) = \left[ \frac{2(m+1)\beta}{\sqrt{\gamma(\gamma - \gamma_0)\cosh(\Omega_0(x-\omega t)) - \gamma}} \right] \frac{\Omega_0}{\pi} e^{i(-kx+\omega t)}.
\]
(17)

(iii) We assume that \( k_1 < 0, \beta > 0, \Delta_1 > 0, \gamma < \gamma_0 < 0 \). In this case, the level curves defined by \( H(\psi, y) = 0 \) are a periodic orbit enclosing the equilibrium point \( E_2(\hat{\psi}_2,0) \) (see Fig.5 (d)). Integral (11) becomes \( \sqrt{(2m+1)(m+1)^2k_1\gamma} = \int_\psi^{\psi} \frac{d\psi}{\psi \sqrt{(\psi_L - \psi)(\psi_M - \psi)}} \).

Hence, we obtain the following periodic solution:
\[
\psi(\xi) = \frac{2(m+1)\beta}{\gamma - \sqrt{\gamma(\gamma - \gamma_0)\sin(\Omega_1\xi - \xi_0)}}.
\]
(18)

where \( \Omega_1 = \frac{(2m+1)(m+1)}{2m} \sqrt{\beta(m+1)k_1\gamma}, \xi_0 = \arcsin \left( \frac{2\hat{\psi}_1 - 2(m+1)\beta}{\sqrt{\gamma(\gamma - \gamma_0)\hat{\psi}_1}} \right) \). (18) implies the parametric representation of the equation (1) as follows:
\[
q(x,t) = \left[ \frac{2(m+1)\beta}{|\gamma| - \sqrt{\gamma(\gamma - \gamma_0)\sin(\Omega_1(x-\omega t)) - \xi_0}} \right] \frac{\Omega_1}{\pi} e^{i(-kx+\omega t)}.
\]
(19)

(iv) We assume that \( k_1 = 0, \beta < 0, \gamma > 0 \). In this case, the level curves defined by \( H(\psi, y) = 0 \) are a homoclinic orbit to the origin \( O(0,0) \) passing through the point \((\psi_M, 0)\), where \( \psi_M = \frac{(m+1)\beta}{\gamma} \). Corresponding to this homoclinic orbit, (11) becomes that \( \sqrt{(2m+1)(m+1)^2k_1\gamma} = \int_\psi^{\psi_M} \frac{d\psi}{\psi \sqrt{\psi_M - \psi}} \).

It gives rise to the following solution of system (10):
\[
\psi(\xi) = \psi_M \text{sech}^2(\omega\xi),
\]
(20)

where \( \omega = \frac{(2m+1)(m+1)\sqrt{\beta}}{4m} \). (20) follows the exact solution of equation (1):
\[
q(x,t) = \left[ \sqrt{\psi_M \text{sech}(\omega(x-\omega t))} \right] \frac{\Omega_1}{\pi} e^{i(-kx+\omega t)}.
\]
(21)

To sum up, we have the following conclusion.

**Theorem 1.** For \( m > 1 \), corresponding to the three different level curves defined by \( H(\psi, y) = 0 \) in (9), equation (1) has the exact explicit bounded solutions given by (15), (17), (19) and (21).
4. The exact traveling wave solutions of equation (1) with \( m = 1 \) corresponding to the level curves \( H(\psi, y) = h \). When \( m = 1, H(\psi, y) = h \) defined by (9) becomes that

\[
y^2 = h\psi - \frac{2\psi^2}{3}[6\beta + 3\gamma\psi + 2k_1\gamma\psi^2],
\]

namely,

\[
y^2 = \frac{4k_1\gamma}{3}\psi \left[ \frac{3\beta}{4k_1\gamma} - \frac{3\beta}{k_1\gamma} - \frac{3}{2k_1}\psi^2 - \psi^3 \right], \quad \text{for } k_1\gamma > 0,
\]

\[
y^2 = \frac{4k_1\gamma}{3}\psi \left[ \frac{3\beta}{4k_1\gamma} + \frac{3\beta}{k_1\gamma} + \frac{3}{2k_1}\psi^2 + \psi^3 \right], \quad \text{for } k_1\gamma < 0.
\]

(22)

Therefore, by using the first equation of system (8) and (22), we can get all exact solutions for equation (1) with \( m = 1 \).

(i) Consider the case of system (8) has only one equilibrium point \( O(0,0) \). The parameter condition is \( \Delta_1 < 0 \). When \( \beta > 0 \), in the right phase plane, there is an elliptic area laying infinitely many periodic orbits which contact to the origin \( O(0,0) \) (see Fig.1 (a)). Assume that \( k_1\gamma > 0 \). We see from (22) that \( \sqrt{\frac{4k_1\gamma}{3}} \xi = \int_0^\psi \frac{d\psi}{\sqrt{(\psi_1 - \psi)(\psi_1 - \psi + a_1^2 + a_2^2)}} \). Hence, we have the following family of periodic solutions:

\[
\psi(\xi) = \frac{\psi_1 B_1(1 - \text{cn}(\Omega_2 \xi, k))}{(A_1 + B_1) + (A_1 - B_1)\text{cn}(\Omega_2 \xi, k)},
\]

where

\[
A_1^2 = (\psi_1 - b_1)^2 + a_1^2, \quad B_1^2 = a_1^2 + b_1^2, \quad k^2 = \frac{\psi_1^2 - (A_1 - B_1)^2}{4A_1B_1},
\]

\[
\Omega_2 = 2\sqrt{\frac{k_1\gamma}{3}}A_1B_1, \quad \text{cn}(\cdot, k), \quad \text{dn}(\cdot, k), \quad \text{sn}(\cdot, k)
\]

are Jacobian elliptic functions (see [Byrd & Fridman, 1973]).

It gives rise to the exact solution family of equation (1):

\[
q(x, t) = \left[ \frac{\psi_1 B_1(1 - \text{cn}(\Omega_2(x - vt), k))}{(A_1 + B_1) + (A_1 - B_1)\text{cn}(\Omega_2(x - vt), k)} \right] \frac{1}{2} e^{i(-kx + \omega t)}.
\]

(24)

(ii) Consider the case of system (8) has two equilibrium points \( O(0,0) \) and \( E(\psi_j, 0) \). First, we assume that \( \beta > 0, \gamma > 0, k_1 < 0 \). We have phase portrait of system (8) shown in Fig.2 (a).

For \( h \in (0, h_1) \), the level curves defined by \( H(\psi, y) = h \) has a family of closed branch contacting to the singular straight line \( \psi = 0 \) at the origin \( O(0,0) \). We have \( \sqrt{\frac{4k_1\gamma}{3}} \xi = \int_0^\psi \sqrt{(\psi_1 - \psi)(\psi_1 - \psi + \psi_M^2)} \frac{d\psi}{\psi_1 \psi(\psi_1 - \psi)} \), where \( -\psi_1 < 0 < \psi_M < \psi_1 < \psi_L \). Thus, we obtain the following periodic solution family of system (8):

\[
\psi(\xi) = \psi_L - \frac{\psi_L - \psi_M}{1 - \delta_2^2\text{sn}^2(\Omega_3 \xi, k)},
\]

where \( \delta_2^2 = \frac{\psi_M}{\psi_L}, \quad k^2 = \frac{\psi_M(\psi_L + \psi_M)}{\psi_L(\psi_M + \psi_M)}, \quad \Omega_3 = \sqrt{\frac{k_1\gamma}{3}}\psi_L(\psi_M + \psi_1) \). (25) follows the exact solution family of equation (1):

\[
q(x, t) = \left[ \psi_L - \frac{\psi_L - \psi_M}{1 - \delta_2^2\text{sn}^2(\Omega_3(x - vt), k)} \right] \frac{1}{2} e^{i(-kx + \omega t)}.
\]

(26)

A branch of the level curves defined by \( H(\psi, y) = h_1 \) is a homoclinic orbit contacting to the singular straight line \( \psi = 0 \) at the origin \( O(0,0) \), for which, we
have \( \sqrt{-\frac{4k_2\gamma}{3}} \xi = \int_{\psi_1}^{\psi} \frac{d\psi}{(\psi_1-\psi)\sqrt{\psi_1+\psi_1}} \). This gives to the following exact solution of system (8):

\[
\psi(\xi) = \psi_1 - \frac{2\psi_1(\psi_1 + \psi_{10})}{\psi_{10} \cosh(\omega_2 \xi) + 2\psi_1 + \psi_{10}},
\]  

(27)

where \( \omega_2 = \sqrt{\frac{4k_2\gamma}{3}} \psi_1(\psi_1 + \psi_{10}) \). (27) follows an exact solution of equation (1):

\[
g(x,t) = \left[ \psi_1 - \frac{2\psi_1(\psi_1 + \psi_{10})}{\psi_{10} \cosh(\omega_2(x-vt)) + 2\psi_1 + \psi_{10}} \right]^2 e^{i(-kx+\omega t)}. \]

(28)

Second, we assume that \( \beta < 0, \gamma > 0, k_1 > 0 \) and \( h_2 < 0 \). We have phase portrait of system (8) shown in Fig.2 (b). For \( h \in (h_2, 0) \), the level curves defined by \( H(\psi, y) = h \) has a family of closed branch enclosing the equilibrium point \( E_2(\psi_2, 0) \). Now, we have \( \sqrt{\frac{4k_1\gamma}{3}} \xi = \int_{\psi_0}^{\psi} \frac{d\psi}{(\psi_1-\psi)\sqrt{\psi_1+\psi_1}} \), where \( -\psi_d < \psi_b < \psi_2 < \psi_a \). It gives rise to the following exact periodic solution family of system (8):

\[
\psi(\xi) = \frac{\psi_b}{1 - \hat{\alpha}_b^2 \text{sn}^2(\Omega_4 \xi, k)},
\]

(29)

where \( \hat{\alpha}_b^2 = 1 - \frac{\psi_b}{\psi_a}, k^2 = \frac{(\psi_a - \psi_b)(\psi_a + \psi_d)}{(\psi_a + \psi_y)(\psi_a + \psi_d)} \), \( \Omega_4 = \sqrt{\frac{k_1\gamma}{3}} \psi_a(\psi_b + \psi_d) \). (29) follows the exact solution family of equation (1):

\[
g(x,t) = \left[ \psi_b \frac{1 - \hat{\alpha}_b^2 \text{sn}^2(\Omega_4(x-vt), k)}{1 - \hat{\alpha}_b^2 \text{sn}^2(\Omega_4(x-vt), k)} \right]^2 e^{i(-kx+\omega t)}. \]

(30)

A branch of the level curves defined by \( H(\psi, y) = 0 \) is a homoclinic orbit to the origin \( O(0, 0) \), for which we have \( \sqrt{\frac{4k_1\gamma}{3}} \xi = \int_{\psi_0}^{\psi_M} \frac{d\psi}{\psi_1 \sqrt{\psi_1+\psi_1}} \). This gives to the following exact solution of system (8):

\[
\psi(\xi) = \frac{2\psi_1 \psi_M}{(\psi_1 + \psi_M) \cosh(\omega_3 \xi) - (\psi_1 - \psi_M)},
\]

(31)

where \( \omega_3 = \sqrt{\frac{4k_1\gamma}{3}} \psi_1 \psi_M \). (31) follows an exact solution of equation (1):

\[
g(x,t) = \left[ \frac{2\psi_1 \psi_M}{(\psi_1 + \psi_M) \cosh(\omega_3(x-vt)) - (\psi_1 - \psi_M)} \right]^2 e^{i(-kx+\omega t)}. \]

(32)

For \( h \in (0, \infty) \), the level curves defined by \( H(\psi, y) = h \) has a family of closed branch enclosing the homoclinic orbit defined by \( H(\psi, y) = 0 \). We have \( \sqrt{\frac{4k_1\gamma}{3}} \xi = \int_{\psi_0}^{\psi} \frac{d\psi}{(\psi_1-\psi)\sqrt{\psi_1+\psi_1}} \), where \( -\psi_d < -\psi_c < 0 < \psi_2 < \psi_a \). It gives rise to the following exact periodic solution family of system (8):

\[
\psi(\xi) = \psi_1 \hat{\alpha}_b^2 \text{sn}^2(\Omega_5 \xi, k),
\]

(33)

where \( \hat{\alpha}_b^2 = \frac{\psi_b}{\psi_a + \psi_c}, k^2 = \frac{(\psi_a - \psi_c)(\psi_a + \psi_c)}{(\psi_a + \psi_c)(\psi_a + \psi_c)} \), \( \Omega_5 = \sqrt{\frac{k_1\gamma}{3}} \psi_a(\psi_a + \psi_c) \). (33) follows the exact solution family of equation (1):

\[
g(x,t) = \left[ \psi_1 \hat{\alpha}_b^2 \text{sn}^2(\Omega_5(x-vt), k) \right]^2 e^{i(-kx+\omega t)}. \]

(34)
(iii) Consider the case of system (8) has three equilibrium points $O(0,0)$, $E_1(\psi_1,0)$ and $E_2(\psi_2,0)$, $0 < \psi_1 < \psi_2$. First, we assume that $\beta < 0$, $k_1 < 0$, $\Delta_1 > 0$.

(1) When $\beta < 0$, $\gamma_0 = \frac{16}{3}k_1 \beta > \gamma > 4k_1 \beta$, $h_1 < h_2 < 0$, system (8) has the phase portrait Fig.3 (a).

Corresponding to the closed branch defined by $H(\psi, y) = h, h \in (h_1, h_2)$, there exists a family of periodic orbits of system (8), for which we have $\sqrt{\frac{4k_1}{3} \gamma} \xi = \int_{\psi_c}^{\psi} \frac{d\psi}{\sqrt{(\psi_u-\psi)(\psi_v-\psi)(\psi-\psi_v)}}$, where $0 < \psi_c < \psi_1 < \psi_h < \psi_2 < \psi_u$. It gives rise to the following exact periodic solution family of system (8):

$$\psi(\xi) = \frac{\psi_c}{1 - \hat{\alpha}_3^2 \text{sn}^2(\Omega_0 \xi, k)}, \quad (35)$$

where $\hat{\alpha}_3^2 = 1 - \frac{\psi_1}{\psi_c}$, $k^2 = \frac{\psi_u(\psi_u - \psi_1)}{\psi_u(\psi_u - \psi_c)}$, $\Omega_0 = \sqrt{\frac{4k_1}{3}\gamma} \psi_u(\psi_u - \psi_c)$. (35) gives the exact solution family of equation (1):

$$q(x, t) = \left[ \frac{\psi_c}{1 - \hat{\alpha}_3^2 \text{sn}^2(\Omega_0(x - vt), k)} \right]^{\frac{3}{2}} e^{i(-kx + \omega t)}. \quad (36)$$

Corresponding to the homoclinic orbit of system (8) defined by $H(\psi, y) = h_2$ to the equilibrium point $E_2(\psi_2,0)$, we have $\sqrt{\frac{4k_1}{3} \gamma} \xi = \int_{\psi_2}^{\psi} \frac{d\psi}{\psi_m(\psi_2 - \psi)(\psi - \psi_m)}$. It gives rise to the following exact solution of system (8):

$$\psi(\xi) = \psi_2 - \frac{2\psi_2(\psi_2 - \psi_m)}{\psi_m \cosh(\omega_4 \xi)} + 2\psi_2 - \psi_m, \quad (37)$$

where $\omega_4 = \sqrt{\frac{4k_1}{3}\gamma} \psi_2(\psi_2 - \psi_m)$. Thus, we have the following exact solution of equation (1):

$$q(x, t) = \left[ \frac{\psi_2 - \frac{2\psi_2(\psi_2 - \psi_m)}{\psi_m \cosh(\omega_4(x - vt))} + 2\psi_2 - \psi_m}{\psi_m \cosh(\omega_4(x - vt))} \right]^{\frac{3}{2}} e^{i(-kx + \omega t)}. \quad (38)$$

(2) When $\beta < 0$, $\gamma = \gamma_0 = \frac{16}{3}k_1 \beta$, $h_1 < h_2 = 0$, system (8) has the phase portrait Fig.3 (b). In this case, the level curves defined by $H(\psi, y) = 0$ are two heteroclinic orbits connecting the equilibrium points $O(0,0)$ and $E_2(\psi_2,0)$, which have the same parametric representations as (14) with $m = 1$. Equation (1) has the exact solutions as (17) with $m = 1$.

The closed branch family of the level curves defined by $H(\psi, y) = h, h \in (h_1, 0)$ is a periodic solution family of system (8), which has the same parametric representation as (35). Equation (1) has the exact solutions as (36).

(3) When $\beta < 0$, $\gamma > \gamma_0 = \frac{16}{3}k_1 \beta$, $h_1 < 0 < h_2$, system (8) has the phase portrait Fig.3 (c).

For $h \in (h_1, 0)$, the level curves defined by $H(\psi, y) = h$ has a family of closed branch enclosing the equilibrium point $E_1(\psi_1,0)$. Corresponding to this family of periodic orbits of system (8), it has the same parametric representation as (35). So that, equation (1) has the exact solutions as (36).

Corresponding to the homoclinic orbit of system (8) defined by $H(\psi, y) = 0$ to the origin $O(0,0)$, we have $\sqrt{\frac{4k_1}{3} \gamma} \xi = \int_{\psi_1}^{\psi_M} \frac{d\psi}{\psi(\psi_u - \psi)(\psi_v - \psi)}$, where $0 < \psi_1 < \psi_M$. \quad (39)
\( \psi_M < \psi_2 < \psi_a \). It gives rise to the following exact solution of system (8):

\[
\psi(\xi) = \frac{2\psi_a \psi_M}{(\psi_a - \psi_M)\cosh(\omega_5 \xi) + (\psi_a + \psi_M)},
\]

where \( \omega_5 = \sqrt{\frac{4k_1}{3}\psi_a \psi_M} \). Hence, we have the following exact solution of equation (1):

\[
q(x, t) = \left[ \frac{2\psi_a \psi_M}{(\psi_a - \psi_M)\cosh(\omega_5(x - vt)) + (\psi_a + \psi_M)} \right]^{\frac{1}{2}} e^{(-kx + \omega t)}.
\]

Corresponding to the family of periodic orbits of system (8) defined by \( H(\psi, y) = h, h \in (0, h_2) \) contacting to the origin \( O(0, 0) \), we have \( \sqrt{\frac{4k_1}{3}} \xi = \int_0^\psi \frac{d\psi}{\sqrt{(\psi_a - \psi)(\psi_b - \psi)(\psi_c + \psi)}} \), where \( -\psi_d < 0 < \psi_1 < \psi_2 < \psi_a \). It gives rise to the following exact solution of system (8):

\[
\psi(\xi) = \frac{\psi_d \tilde{a}_5^2 \text{sn}^2(\Omega_7 \xi, k)}{1 - \tilde{a}_5^2 \text{sn}^2(\Omega_7 \xi, k)},
\]

where \( \tilde{a}_5^2 = \frac{\psi_a}{\psi_b + \psi_d}, k^2 = \frac{\psi_a(\psi_a + \psi_d)}{\psi_a(\psi_b + \psi_d)} \), \( \Omega_7 = \sqrt{\frac{4k_1}{3}} \psi_a(\psi_b + \psi_d) \). Therefore, we obtain the exact solution family of equation (1):

\[
q(x, t) = \left[ \frac{\psi_d \tilde{a}_5^2 \text{sn}^2(\Omega_7 \xi, k)}{1 - \tilde{a}_5^2 \text{sn}^2(\Omega_7 \xi, k)} \right]^{\frac{1}{2}} e^{(-kx + \omega t)}.
\]

The level curves defined by \( H(\psi, y) = h_2 \) contain two heteroclinic orbits connecting the equilibrium points \( O(0, 0), E_1(\psi_1, 0) \) and \( E_2(\psi_2, 0), 0 < \psi_1 < \psi_2 \). Now we assume that \( \beta > 0, k_1 < 0, \Delta_1 > 0 \).

(iv) Consider the case of system (8) has three equilibrium points \( O(0, 0), E_1(\psi_1, 0), E_2(\psi_2, 0), 0 < \psi_1 < \psi_2 \); system (8) has the phase portrait Fig.4 (a). As \( h \) is varied, the level curves defined by \( H(\psi, y) = h \) are changed which are shown in Fig.6 (a)-(d).

(a) \( h \in (0, h_2) \)  \hspace{1cm} (b) \( h \in (h_2, h_1) \)  \hspace{1cm} (c) \( h = h_1 \)  \hspace{1cm} (d) \( h_1 < h < \infty \)

Fig.6 The changes of the level curves defined by \( H(\psi, y) = h \) of system (8)

Corresponding to the level curves defined by \( H(\psi, y) = h, h \in (0, h_2) \) (see Fig.6 (a)), we have \( \sqrt{\frac{4k_1}{3}} \xi = \int_0^\psi \frac{d\psi}{\sqrt{(\psi_a - \psi)(\psi_b - \psi)(\psi_c + \psi)}} \), where \( 0 < \psi_m < \psi_1 \). Thus, we have the same parametric representation of the periodic orbit family of system (8) as (23) and the exact solution of equation (1) as (24).
Corresponding to the two closed branches of the level curves defined by \( H(\psi, y) = h, h \in (h_2, h_1) \) (see Fig. 6 (b)), we have
\[
\sqrt{\frac{4k_1 \gamma}{3}} \xi = \int_0^\psi \frac{d\psi}{\sqrt{(\psi - \psi_h)(\psi - \psi_h)}}
\]
and
\[
\sqrt{\frac{4k_1 \gamma}{3}} \xi = \int_0^\psi \frac{d\psi}{\sqrt{(\psi - \psi_h)(\psi - \psi_h)}}
\]
respectively. Thus, we obtain the following two families of periodic solutions of system (8):
\[
\psi(\xi) = \psi_c + \frac{\psi_b - \psi_c}{1 - \hat{\alpha}_b^2 \text{sn}^2(\Omega_8 \xi, k)}
\]
and
\[
\psi(\xi) = -\psi_a \hat{\alpha}_a^2 \text{sn}^2(\Omega_8 \xi, k)
\]
where \( \hat{\alpha}_b^2 = \frac{\psi_b - \psi_c}{\psi_a - \psi_c} \), \( \hat{\alpha}_b^2 = \frac{-\psi_c}{\psi_a - \psi_c} < 0, k^2 = \frac{(\psi_a - \psi_c) \psi_c}{(\psi_a - \psi_c) \psi_b} \). (43) and (44) give rise to two exact solution families of equation (1) as follows:
\[
q(x, t) = \left[ \psi_c + \frac{\psi_b - \psi_c}{1 - \hat{\alpha}_b^2 \text{sn}^2(\Omega_8 (x - vt), k)} \right]^{\frac{1}{2}} e^{i(-kx + \omega t)}
\]
and
\[
q(x, t) = \left[ -\psi_a \hat{\alpha}_a^2 \text{sn}^2(\Omega_8 (x - vt), k) \right]^{\frac{1}{2}} e^{i(-kx + \omega t)}.
\]

Corresponding to the two homoclinic orbits of system (8) defined by \( H(\psi, y) = h_1 \) (see Fig. 6 (c)), we have
\[
\sqrt{\frac{4k_1 \gamma}{3}} \xi = \int_0^\psi \frac{d\psi}{(\psi - \psi_1)(\psi - \psi_1)}
\]
and
\[
\sqrt{\frac{4k_1 \gamma}{3}} \xi = \int_0^\psi \frac{d\psi}{(\psi - \psi_M)(\psi - \psi_M)}
\]
respectively. Hence, the two homoclinic orbits of system (8) have the following parametric representations:
\[
\psi(\xi) = \psi_1 + \frac{2(\psi_M - \psi_1) \psi_1}{\psi_M \cosh(\omega_0 \xi) - (\psi_M - 2 \psi_1)}
\]
and
\[
\psi(\xi) = \psi_1 - \frac{2(\psi_M - \psi_1) \psi_1}{\psi_M \cosh(\omega_0 \xi) + (\psi_M - 2 \psi_1)},
\]
where \( \omega_0 = \sqrt{\frac{4k_1 \gamma}{3}} \psi_1 (\psi_M - \psi_1) \). (47) and (48) give rise to two exact solution families of equation (1) as follows:
\[
q(x, t) = \left[ \psi_1 + \frac{2(\psi_M - \psi_1) \psi_1}{\psi_M \cosh(\omega_0 (x - vt)) - (\psi_M - 2 \psi_1)} \right]^{\frac{1}{2}} e^{i(-kx + \omega t)}
\]
and
\[
q(x, t) = \left[ \psi_1 - \frac{2(\psi_M - \psi_1) \psi_1}{\psi_M \cosh(\omega_0 (x - vt)) + (\psi_M - 2 \psi_1)} \right]^{\frac{1}{2}} e^{i(-kx + \omega t)}.
\]

Corresponding to the level curves defined by \( H(\psi, y) = h, h \in (h_1, \infty) \) (see Fig. 6 (d)), we have
\[
\sqrt{\frac{4k_1 \gamma}{3}} \xi = \int_0^\psi \frac{d\psi}{(\psi - \psi_h)(\psi - \psi_h) + a^2}
\]
where \( 0 < \psi_1 < \psi_2 < \phi_b \). It follows the same parametric representation of the periodic orbit family of system (8) as (23) and the exact solution of equation (1) as (24).

(2) When \( \beta > 0, \gamma = \gamma_0 = \frac{16}{3} k_1 \beta, 0 = h_2 < h_1 \), system (8) has the phase portrait Fig. 4 (b).

Corresponding to the two closed branches of the level curves defined by \( H(\psi, y) = h, h \in (0, h_1) \) (see Fig. 6 (b)), we have the same parametric representations of the
Bifurcations and exact traveling wave solutions for the NSE

[Extracted content]

two periodic orbit families of system (8) as (43) and (44). Equation (1) has the two exact solutions as (45) and (46).

Corresponding to the two homoclinic orbits defined by $H(\psi, y) = h_1$ (see Fig.6 (c)), we have the same parametric representations of the two homoclinic orbits of system (8) as (47) and (48). Equation (1) has the two exact solutions as (49) and (50).

Corresponding to the family of level curves defined by $H(\psi, y) = h, h \in (h_1, \infty)$ (see Fig.6 (d)), it has the same parametric representation of the periodic orbit family of system (8) as (23) and the exact solution of equation (1) as (24).

When $\beta > 0, \gamma < \gamma_0 = \frac{16}{3} k_1 \beta < 0, h_2 < 0 < h_1$, system (8) has the phase portrait Fig.4 (c).

Corresponding to the closed branch of the level curves defined by $H(\psi, y) = h, h \in (h_2, 0]$, enclosing the equilibrium point $E_2(\psi_2, 0)$, we have $\sqrt{4k_1 \gamma} \xi = \int_{\psi_b}^{\psi} \frac{dy}{\sqrt{(\psi_a - \psi)(\psi_b - \psi)(\psi + \psi_d)}}$, where $-\psi_d < 0 < \psi_1 < \psi_b < \psi_2 < \psi_a$. Thus, we have the same parametric representation of the periodic orbit family of system (8) as (29). Equation (1) has the exact solution as (30).

Other cases are the same as in the above subsection (2).

We see from the above discussion that the following conclusion holds.

**Theorem 2.** For $m = 1$, under different parameter conditions, corresponding to the bounded real level curves defined by $H(\psi, y) = h$, equation (1) has the exact explicit solutions given by (24), (26), (28), (30), (32), (34), (36), (38), (40), (42), (45), (46), (49) and (50).

5. The bifurcations of phase portraits of system (8) and exact solutions for $0 < m = \frac{1}{n} < \frac{1}{2}$. In this section, we consider the case $0 < m = \frac{1}{n} < \frac{1}{2}$. Now, the first integral (9) of system (8) becomes

$$H(\psi, y) = y^2 \psi^{n-2} + \frac{4 \psi^n [n(n+1)k_1 \gamma \psi^2 + n(n+2)\gamma \psi + \beta(n+1)(n+2)]}{n^2(n+1)(n+2)} = h. \quad (51)$$

Clearly, it is different from (9) that we now have $H(0, 0) = 0$ and when system (8) has three equilibrium points (i.e., $k_1 < 0, \Delta_1 > 0$), by using the results in section 2, we have the bifurcations of phase portraits shown in Fig.7 (a)-(d) and Fig.8 (a)-(b).

![Fig.7 The bifurcations of phase portraits of system (8) for $m = \frac{1}{n}$, $\beta < 0$](image)

(a) $\gamma > \gamma_0$ (b) $\gamma = \gamma_0 = \frac{4(n+1)^2 k_1 \beta}{n(n+2)}$ (c) $4\beta k_1 < \gamma < \gamma_0$ (d) $\gamma = 4\beta k_1$

(a) $h_1 < 0 < h_2$. (b) $h_1 < h_2 = 0$. (c) $h_1 < h_2 < 0$. (d) $h_1 = h_2 < 0$. 


Fig.8 The bifurcations of phase portraits of system (8) for $m = \frac{1}{n}, \beta > 0$

(a) $\gamma < \gamma_0 < 0, h_2 < 0 < h_1; \gamma = \gamma_0, 0 = h_2 < h_1; \gamma_0 < \gamma < 4k_1\beta, 0 < h_2 < h_1.$ (b) $\gamma = 4k_1\beta, 0 < h_1 = h_2.$

By using the results in section 3, we have the following conclusions.

(i) When $\beta < 0, \gamma > \gamma_0$, corresponding to the homoclinic orbit defined by $H(\psi, y) = 0$ in Fig.7 (a), we have

$$\psi(\xi) = \frac{2(n + 1)\beta}{n(\sqrt{n(\gamma - \gamma_0)} \cosh(\Omega_0\xi) - \gamma)},$$ (52)

where $\Omega_0 = \frac{(n+2)(n+1)}{2n^2} \sqrt{\beta(n + 1)k_1\gamma}$.

(ii) When $\beta < 0, \gamma = \gamma_0$, corresponding to the heteroclinic orbits defined by $H(\psi, y) = 0$ in Fig.7 (b), we have

$$\psi(\xi) = \pm \frac{\psi_2}{1 + e^{-\omega_1\xi}} = \pm \frac{1}{2} \psi_2 \left( 1 + \tanh \left( \frac{1}{2} \omega_1\xi \right) \right),$$ (53)

where $\omega_1 = \frac{1}{2\pi} (n + 2)(n + 1) \sqrt{\beta}, \psi_2 = \frac{1}{2 \pi k_1} \left( 1 + \frac{2\psi_1}{\gamma_0} \right)$.

(iii) When $\beta > 0, \gamma < \gamma_0 < 0$, corresponding to the periodic orbit defined by $H(\psi, y) = 0$ in Fig.8 (a), we have

$$\psi(\xi) = \frac{2(n + 1)\beta}{n(\gamma - \gamma_0) \sin(\Omega_1\xi - \xi_0)},$$ (54)

where $\Omega_1 = \frac{(n+2)(n+1)}{2n^2} \sqrt{\beta(n + 1)k_1\gamma}, \xi_0 = \arcsin \left( \frac{n(\gamma - \gamma_0)}{\gamma \sqrt{n(\gamma - \gamma_0)\psi_1}} \right)$.

(iv) When $k_1 = 0, \beta < 0, \gamma > 0$, corresponding to the homoclinic orbit defined by $H(\psi, y) = 0$ in Fig.5 (e), we have

$$\psi(\xi) = \psi_M \text{sech}^2(\tilde{\omega}\xi),$$ (55)

where $\tilde{\omega} = \frac{(n+2)(n+1)}{4n\sqrt{\beta}}, \psi_M = \frac{(n+1)\beta}{\gamma \psi_1}$.

These results give rise to the following theorem.

**Theorem 3.** Suppose that $0 < m = \frac{1}{n} < \frac{1}{2}$. Then, under the above parametric conditions, equation (1) has the following exact solutions:

$$q(x, t) = \left[ \frac{2(n + 1)\beta}{n(\sqrt{n(\gamma - \gamma_0)} \cosh(\Omega_0(x - vt)) - \gamma)} \right]^{\frac{1}{2n}} e^{(-kx + \omega t)};$$ (56)
\[ q(x, t) = \pm \left[ \frac{1}{2} \psi_2 \left( 1 + \tanh \left( \frac{1}{2} \omega_1 (x - vt) \right) \right) \right] \frac{1}{2^n} e^{i(-kx + \omega t)}; \quad (57) \]

\[ q(x, t) = \left[ \frac{2(n + 1)\beta}{n(|\gamma| - \sqrt{\gamma(\gamma - \gamma_0)} \sin(\Omega_1 (x - vt) - \xi_0))} \right] \frac{1}{2^n} e^{i(-kx + \omega t)}; \quad (58) \]

\[ q(x, t) = \psi_M^2 \text{sech}^n(\omega(x - vt)) e^{i(-kx + \omega t)}. \quad (59) \]

Specially, we see from (55) that when \( k_1 = 0 \), corresponding to the homoclinic orbit defined by \( H(\psi, y) = 0 \), equation (5), i.e., \( \phi'' - \beta \phi + \gamma \phi^{1 + \frac{2}{\alpha}} = 0 \) has the exact solution

\[ \phi(\xi) = \psi_M^2 \text{sech}^n(\omega \xi). \quad (60) \]

[8] discussed the solution of the form \( \frac{1}{\cosh^{n(a+x+\beta t)}} \) for some nonlinear wave equations. Our result (60) gives the geometric explanation for this solution.

REFERENCES

[1] P. F. Byrd and M. D. Fridman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Second edition, revised. Die Grundlehren der mathematischen Wissenschaften, Band 67 Springer-Verlag, New York-Heidelberg, 1971.

[2] J. Li, J. Wu and H. Zhu, Travelling waves for an integrable higher order KdV type wave equations, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 16 (2006), 2235–2260.

[3] J. Li and G. Chen, On a class of singular nonlinear traveling wave equations, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 17 (2007), 4049–4065.

[4] J. Li and H. Dai, On the Study of Singular Nonlinear Travelling Wave Equations: Dynamical Approach, Science Press, Beijing, 2007.

[5] J. Li, *Singular Nonlinear Travelling Wave Equations: Bifurcations and Exact solutions*, Science Press, Beijing, 2013.

[6] J. Li, W. Zhou and G. Chen, Understanding peakons, periodic peakons and compactons via a shallow water wave equation, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 26 (2016), 1650207, 27 pp.

[7] A. M. Shahoot, K. A. E. Alurrfi, I. M. Hassan and A. M. Almari, Solitons and other exact solutions for two nonlinear PDEs in mathematical physics using the generalized projective Riccati equations method, *Adv. Math. Phys.*, 2018, Art. ID 6870310, 11 pp.

[8] N. K. Vitanov, Z. D. Dimitrova and T. I. Ivanova, On solitary wave solutions of a class of nonlinear partial differential equations based on the function \( \frac{1}{\cosh^{n(a+x+\beta t)}} \), *Appl. Math. Comput.*, 315 (2017), 372–380.

[9] G. Q. Xu, New types of exact solutions for the fourth-order dispersive cubic-quintic nonlinear Schrödinger equation, *Appl. Math. Comput.*, 217 (2011), 5967–5971.

[10] E. M. Zayed, A. G. Al-Nowehy and M. E. Elshater, Solitons and other solutions to nonlinear schrödinger equation with fourth-order dispersion and dual power law nonlinearity, *Ric. Mat.*, 66 (2017), 531–552.

Received for publication August 2018.

E-mail address: lijb@zjnu.cn, lijb@hqu.edu.cn
E-mail address: zy4233@hqu.edu.cn