On covariant phase space methods

Bernard Julia\textsuperscript{a} and Sebastián Silva\textsuperscript{b}

\textsuperscript{a}Laboratoire de Physique Théorique CNRS-ENS
24 rue Lhomond, F-75231 Paris Cedex 05, France

\textsuperscript{b}Max Planck Institut für Gravitationsphysik, Albert Einstein Institut,
Am Mühlenberg 5, D-14476 Golm, Germany

ABSTRACT

It is well known that the Lagrangian and the Hamiltonian formalisms can be combined
and lead to “covariant symplectic” methods. For that purpose a “pre-symplectic
form” has been constructed from the Lagrangian using the so-called Noether form.
However, analogously to the standard Noether currents, this symplectic form is only
determined up to total divergences which are however essential ingredients in gauge
theories.

We propose a new definition of the symplectic form which is covariant and free of
ambiguities in a general first order formulation. Indeed, our construction depends on
the equations of motion but not on the Lagrangian. We then define a generalized
Hamiltonian which generates the equations of motions in a covariant way. Applica-
tions to Yang-Mills, general relativity, Chern-Simons and supergravity theories are
given. We also consider nice sets of possible boundary conditions that imply the
conservation and of the total symplectic form.

We finally revisit the construction of conserved charges associated with gauge symme-
tries, from both the “covariant symplectic” and the “covariantized Regge-Teitelboim”
points of view. We find that both constructions coincide when the ambiguity in the
Noetherian pre-symplectic form is fixed using our new prescription. We also present
a condition of integrability of the equations that lead to these quantities.

\textsuperscript{1}UMR 8549 du CNRS et de l’École Normale Supérieure. This work has been partly
supported by the EU TMR contract ERBFMRXCT96-0012 and HPRN-CT-2000-122.
1 Introduction

Field theories are usually described by choosing between either a Lagrangian or an Hamiltonian formalism. In one case, Lorentz covariance is manifest and the charges are computed via the Noether procedure. On the other side, the Hamiltonian mechanics provides a symplectic structure, with an associated bracket. However, this framework requires a full machinery involving a space/time splitting, followed by a Legendre transform and eventually by the identification of first and second class constraints.

It is naturally tempting to look for a formalism which could combine both, the explicit covariance and symplectic geometry. A decisive step in that direction was introduced in references [1, 2, 3, 4] where the covariant symplectic method was developed. Using the so-called Noether construction, a symplectic structure was computed in an explicitly covariant way without the need for canonical momenta.

As long as the boundary contributions can be neglected, the Noether method proposed in references [1, 2, 3, 4] is perfectly suitable and leads to a well-defined symplectic form. However, in most physically relevant field theories, boundary terms are quite important. In these cases, the Noetherian approach fails to produce a non-ambiguous symplectic structure without an additional prescription. Indeed, the boundary contributions to the symplectic structure cannot be computed in a canonical way, in complete analogy with the Noether symmetry currents which are only defined up to total derivatives. We then recall in section 2.1 the Noether construction, we pay special attention to the ambiguities, we separate the on-shell from off-shell statements and explain the role played by boundary terms (we treat the general case where spacetime may have several boundary components).

In section 2.2 we propose a new method that is free of ambiguities to construct the symplectic structure. The first step is to introduce auxiliary fields such that that the equations of motion of the theory become first order (no more than one derivative). These auxiliary fields are the covariant generalization of the canonical momenta of Hamiltonian mechanics. We then give an explicit formula for the symplectic density which is covariant by construction and depends only on the equations of motion. We compare our proposal with the usual Noether construction in section 2.3. In the next subsection 2.4, we follow the analogy with the ordinary Hamiltonian formalism by defining a natural Hamiltonian density which generates the dynamics. In other words, we give a covariant generalization of \( \partial_t q^i = \{ H, q^i \} \). In section 2.5, we compute the pre-symplectic structure of Yang-Mills, gravity,
(non-Abelian) Chern-Simons and eleven dimensional supergravity theories. A careful comparison between our proposal and the Noether construction of the symplectic density is carried out on these examples. Finally, global considerations as the definition, the conservation and the closure of the symplectic form are studied in section 2.6. The relations with the boundary conditions and the variational principle are also explained.

The construction of conserved charges associated with gauge symmetries using the covariant symplectic methods is discussed in section 3. In section 3.1 we recall the standard method used for instance in references [5, 6, 7, 8]. The alternative proposal of reference [11] is explained in section 3.2. We then show in section 3.3 that both constructions are equivalent provided our new symplectic density is used. Finally, we comment on the integrability of the “covariantized” Regge-Teitelboim equation in section 3.4, this was a completely open question at least until some result in [8].

In the body of this manuscript, we use a compact differential form notation. However, the important formulas and many (technical) proofs can be found in the appendix in components.

## 2 Some new results in the covariant phase space formalism

We denote by \( \mathcal{F} \) the space of smooth field configurations (off-shell) which satisfy some given boundary conditions. The exterior derivative on this infinite dimensional manifold is called \( \delta \). The spacetime manifold of dimension \( D \) and its exterior derivative are respectively denoted by \( \mathcal{M} \) and \( d \) (for more details on the geometry of \( \mathcal{M} \), see section 2.6). We take the usual convention that the two exterior derivatives commute.

In order to remove some ambiguities in the notation, one may distinguish field configuration Forms with a capital \( F \) and spacetime forms with a lower case one. Moreover, all spacetime differential forms are denoted with bold characters. A \( (p,q) \)-Form will be a \( p \)-form over \( \mathcal{M} \) and a \( q \)-Form on \( \mathcal{F} \). Finally, the wedge-product \( \wedge \) antisymmetrizes both, the spacetime indices \( (\mu, \nu, \rho, \ldots) \) and the field configurations ones \( (i, j, k, \ldots) \).

### 2.1 The (off-shell) Noetherian pre-symplectic forms

We denote by \( \varphi^i \) the fields of a given theory. Generically, the index \( i \) will be a Lie-valued index or will label different types of fields, even auxiliary
ones. For instance, in the first order vielbein formulation of gravity, the index $i$ encompasses the Lorentz indices ($a$, $b$) associated respectively with the vielbein and the spin connection. The spacetime indices are taken into account by the use of differential forms, the degree of $\varphi^i$ is denoted by $p_i$.

The dynamics is encoded in a $D$-form Lagrangian $L$, which is a local functional of the fields $\varphi^i$. Under an arbitrary variation $\varphi^i \to \varphi^i + \delta \varphi^i$, the Lagrangian transforms into the equations of motion plus a boundary term:

$$\delta L = \delta \varphi^i \wedge E_i + d\theta. \quad (1)$$

The Euler-Lagrange variations (alias the variational derivatives of the action, that is the equations of motion) associated with the fields $\varphi^i$ are the $(D-p_i)$ spacetime forms $E_i := \frac{\delta L}{\delta \varphi^i}$. The (D-1,1) Form $\theta$ is called a Noether-Form, it is a 1-Form in the space of field configurations $\mathcal{F}$ and a $(D-1)$-form over spacetime $M$.

Let us first assume that our Lagrangian $L$ depends on the fields $\varphi^i$ and at most on their first derivatives $d\varphi^i$. Then, the equations of motion are explicitly given by:

$$E_i = \frac{\partial L}{\partial \varphi^i} - (-)^{p_i} d \frac{\partial L}{\partial d\varphi^i}. \quad (2)$$

This allows us to give a closed formula for a particular Noether-Form $\hat{\theta}$ associated to a given Lagrangian $L$:

$$\hat{\theta} := \delta \varphi^i \wedge \frac{\partial L}{\partial d\varphi^i}. \quad (3)$$

However, the general Noether-Form $\theta$ is afflicted by two ambiguities $[3, 5]$:

- From equation (4), we can add a total derivative $dY$ to $\hat{\theta}$:

$$\theta = \hat{\theta} + dY, \quad (4)$$

for any $(D - 2, 1)$-Form $Y$.

- The Lagrangian $L' = L + dK$ generates the same equations of motion as $L$, that is, the bulk term in the decomposition (1) remains unchanged.

\[^{2}\text{We shall assume trivial cohomology in this work: otherwise the arbitrariness of } \theta \text{ includes an arbitrary } d\text{-closed Form.}\]
unchanged. The associated Noether-Form $\hat{\theta}'$ (derived from equation 3) however differs from $\hat{\theta}$ by $\delta$-exact and $d$-exact terms,

$$\hat{\theta}' = \hat{\theta} + \delta K + dY(K),$$

(5)

where we emphasized that in general the total derivative $dY(K)$ depends on the choice of $K$ see for instance equation (2.15) of [3].

The relevant object is not the Noether-Form, but its exterior derivative in $\mathcal{F}$, the so-called (off-shell) pre-symplectic $(D-1,2)$-Form

$$\omega_{\text{No}} := \delta \hat{\theta}.$$  

(6)

By definition (6), the pre-symplectic form $\omega_{\text{No}}$ is $\delta$-closed. From the above ambiguities (4) and (5), it is itself determined only up to a $d\delta$-exact term:

$$\omega_{\text{No}} = \hat{\omega}_{\text{No}} + d\delta Y,$$

(7)

where (see equation (3))

$$\hat{\omega}_{\text{No}} := \hat{\delta} \hat{\theta} = -\delta \varphi^i \wedge \delta \left( \frac{\partial L}{\partial d\varphi^i} \right).$$

(8)

When boundary terms are important, the above ambiguity in $Y$ has to be resolved in some way. For instance, $Y$ was fixed in lemma 3.1 of [5] using a “gauge covariance” criterion. We shall propose a more general definition of the pre-symplectic Form (henceforth called symplectic density) in the following subsection. Basically we shall relax the $\delta$-closure condition and replace it by $\delta$-closure up to a total divergence. This will allow us to solve the ambiguity problem in a general way.

2.2 The covariant (off-shell) symplectic two-Form densities in first order theories

Equation (4) is nothing but the $\delta$-exterior derivative in $\mathcal{F}$ (see previous definitions) of the Lagrangian $L$. Let us now take a (second) $\delta$-exterior derivative of equation (4):$^3$

$$\delta \varphi^i \wedge \delta E_i = d\delta \theta =: d\omega.$$  

(9)

$^3$Recall that $\delta^2 = 0$, $\delta(A\wedge B) = \delta A \wedge B + (-)^{p_A} A \wedge \delta B$, with $A$ a $p_A$-Form on $\mathcal{F}$.  

5
We shall define a *symplectic density* \( \omega \) by requiring \( \omega \) to be any \((D-1,2)\)-form which satisfies equation (9). This defines \( \omega \) up to a d-exact form \( dX \):

\[
\omega = \hat{\omega} + dX.
\]

(10)

Note that a pre-symplectic Noether form \( \omega_{\text{No}} \) is nothing but a symplectic density which is \( \delta \)-closed.

Let us now assume that we are working with a *first order theory*, this being defined by the property that the equations of motion \( E_i \) depend on \( \varphi^i \) and/or on \( d\varphi^i \) but not on higher derivatives of the fields. This is not really a limitation: it is in fact always possible to add auxiliary fields to the Lagrangian to reduce second order equations of motion to first order ones while preserving the gauge symmetries [12, 13].

The point in going to these first order formulations is that we can find an explicit formula for one symplectic density. In fact, in appendix A, equation (A.23), we exhibit the special representative:

\[
\hat{\omega} := \delta \varphi^i \wedge \delta \varphi^j \wedge \frac{1}{2} \hat{\omega}_{ij},
\]

(11)

with

\[
\hat{\omega}_{ij} := (-)^{p_i p_j} \frac{\partial E_i}{\partial d\varphi^j}.
\]

(12)

The form \( \hat{\omega} \) gives a "bulk" contribution common to all symplectic densities, more correctly any symplectic density differs from it by a total divergence \( dX \). In particular, if we assume that \( X \) vanishes at the boundaries, the choice of \( X \) will be irrelevant and equation (12) gives an alternative definition for the symplectic density which may or may not be equal to \( \hat{\omega}_{\text{No}} \).

The symplectic density \( \hat{\omega} \) depends only on the equations of motion and satisfies some important local properties:

(i) \( d\hat{\omega} = \delta \varphi^i \wedge \delta E_i \);

(ii) it is antisymmetric in \( i \) and \( j \), that is

\[
\hat{\omega}_{ij} = -(-)^{p_i p_j} \hat{\omega}_{ji},
\]

(13)

(iii) \( \hat{\omega} \) is generally covariant.

---

4For instance in reference [3], boundary terms are set to zero assuming either appropriate boundary conditions or that the fields are of compact support.
Property (i) is a direct consequence of equations (9) and (10) but it can also be proven from definition (12). The antisymmetry property (13) also follows after a simple calculation given in equation (A.26) in component notation. There the first order formulation is crucial. The third point (iii) follows from definition (12). In fact, the $d$-derivative can be replaced by a covariant derivative in first order theories. The covariance of $\hat{\omega}$ then follows from the covariance of the equations of motion.

As opposed to the Noetherian pre-symplectic Forms $\omega_{\text{No}}$ given by equation (6), our more general $\omega$’s indeed are not automatically $\delta$-closed. In fact, a direct calculation shows that (see the result (A.33))

$$\delta \omega = d\left( \frac{1}{3} \delta \varphi^i \wedge \frac{\partial \hat{\omega}}{\partial d\varphi^i} + \delta X \right).$$ (14)

We have thus extended the consideration of $\delta$-closed Noetherian pre-symplectic Forms to the more general definition of symplectic densities, which are $\delta$-closed only up to $d$-exact terms. Consequently, we increased the ambiguity from an arbitrary $(D-2,1)$-Form $Y$ (equation (1)) to an arbitrary $(D-2,2)$-Form $X$ (equation (10)). On the other hand, we are now able to give a formula for the “bulk” symplectic density as a functional of the equations of motion without direct dependence on the Lagrangian. The later usually depends on the boundary conditions considered while the Euler-Lagrange variation does not. After these observations it is natural to choose $X = 0$, independently of the boundary conditions. That is, we should define the symplectic density of a first order theory by equation (12). We will justify in the following this very special way of fixing the ambiguity (10).

### 2.3 Comparison between $\omega_{\text{No}}$ and $\hat{\omega}$

The two-Form $\omega_{\text{No}}$ is explicitly $\delta$-closed but depends on a choice of Lagrangian, see equation (6). On the other hand, $\hat{\omega}$ given by equation (12) is independent of the Lagrangian (and explicitly covariant) but it is not in general $\delta$-closed, see equation (14). These two symplectic densities are however closely related to each other. In fact a straightforward calculation given in (A.34) shows that:

$$\hat{\omega} = \omega_{\text{No}} + \frac{(-)^p_i}{2} d \left( \delta \varphi^i \wedge \delta \varphi^j \wedge \frac{\partial^2 L}{\partial d\varphi^j \wedge \partial d\varphi^i} \right)$$
\[
\hat{\omega} = \hat{\omega}_{\text{No}} - \frac{1}{2} d \left( \delta \varphi_i \wedge \frac{\partial \hat{\theta}}{\partial d\varphi^i} \right),
\]

(15)

where we used the definition (3) in the second line.

Note that while each term on the right hand side of equation (15) depends in general on the choice of the Lagrangian, the sum does not. The differences between \( \hat{\omega} \) and \( \hat{\omega}_{\text{No}} \) will be analyzed on specific examples in section 2.5. Note that even when they coincide each formulation exhibits different properties.

Finally, equation (15) also provides an alternative formula for \( \delta \hat{\omega} \) (other than (14)) which is explicitly \( d\delta \)-exact:

\[
\delta \hat{\omega} = \frac{1}{2} d \left( \delta \varphi_i \wedge \delta \frac{\partial \hat{\theta}}{\partial d\varphi^i} \right)
\]

(16)

2.4 The covariant “Hamiltonian” equations

In the previous subsections, we derived two “bulk” symplectic densities \( \hat{\omega} \) and \( \hat{\omega}_{\text{No}} \). Each of these two objects can be used to rewrite the equations of motion in a form familiar from Hamiltonian mechanics. Let us define a “Hamiltonian” \( D \)-form by:

\[
H = d\varphi^i \wedge \frac{\partial L}{\partial d\varphi_i} - L.
\]

(17)

This formula generalizes in an obvious way the well-known definition of Hamilton. Note however that \( H \) depends on the fields and on their first derivatives as well but that we do not Legendre transform to conjugate momenta here as earlier advocates of this approach like De Donder and Weyl did (see for instance [14, 15] and references therein). A straightforward calculation gives the Euler-Lagrange variation (also called variational derivative) of the Hamiltonian (17) with respect to \( \varphi^i \):

\[
\frac{\delta H}{\delta \varphi^i} = (-)^{p_i} d\varphi^i \wedge \hat{\omega}_{ij} - E_i.
\]

(18)

The proof is provided in the appendix, equation (A.38), in component notation. Note that it is precisely \( \hat{\omega} \) defined by equation (12) which appears on the right hand-side of the identity (18). That is, the choice \( d\mathbf{X} = 0 \) is automatically selected. Moreover, the final equation (18) does not depend on the ambiguity of adding a surface term to the Lagrangian or to
the Hamiltonian. It is independent of the necessary choice of boundary conditions.

Another related and direct calculation shows\(\text{[6]}\) that (see the result (A.44))
\[\delta H + i_{d\varphi} \hat{\omega}_{N_0} = -\delta \varphi^i \wedge E_i\]  
(19)
where \(i_{d\varphi}\) denotes the interior product along the vector \(d\varphi^i\) (tangent to \(\mathcal{F}\)). More explicitly we have,
\[i_{d\varphi} \left( \frac{1}{2} \delta \varphi^i \wedge \delta \varphi^j \wedge A_{ij} \right) = d\varphi^i \wedge \delta \varphi^j \wedge A_{ij} \]  
(20)
\[i_{d\varphi} d \left( \frac{1}{2} \delta \varphi^i \wedge \delta \varphi^j \wedge B_{ij} \right) = -d \left( d\varphi^i \wedge \delta \varphi^j \wedge B_{ij} \right) \]  
(21)
for any \((p,0)\)-forms \(A_{ij}\) and \(B_{ij}\) (the minus sign in equation (21) comes from usual “differential-form gymnastics”).

Now it is precisely the pre-symplectic form \(\hat{\omega}_{N_0}\) (see equation (8)) which appears in equation (19). Although both terms on the left-hand-side of this formula depend on the choice of Lagrangian surface term their sum is independent on it and is given by the equations of motion.

In summary, the dynamics of a field theory can be rewritten equivalently in the following ways:
\[0 \approx -\delta \varphi^i \wedge E_i = \delta \varphi^i \wedge \frac{\delta H}{\delta \varphi^i} + i_{d\varphi} \hat{\omega} \]  
(22)
\[= \delta H + i_{d\varphi} \hat{\omega}_{N_0}. \]  
(23)

2.5 Examples

The Yang-Mills and gravity cases

Let us take the simplest but rather general example of first order Lagrangian,
\[L_1 = d\varphi^i \wedge f_i + L_0, \]  
(24)
with \(f_i\) and \(L_0\) some \((D - p, 1)\)-form and \(D\)-form respectively which depend on the fields \(\varphi^i\) but not on their derivatives. This kind of Lagrangian is sufficiently general to cover (4d super)gravities, Yang-Mills or p-form theories.

\[\text{Note that } \delta H = \delta \varphi^i \wedge \frac{\delta H}{\delta \varphi^i} + d(\delta \varphi^i \wedge \frac{\delta H}{\delta \varphi^i}). \]
Since \( \frac{\partial^2 L_1}{\partial d\varphi_i \partial d\varphi_j} = 0 \), the symplectic densities \( \hat{\omega} \) and \( \hat{\omega}_{No} \) should coincide (see equation (15)). For illustrative purposes, let us check this explicitly:

Following our definition (12), a simple calculation (for the Lagrangian (24)) yields

\[
\hat{\omega}_{ij} = (-)^{p_{ij}} \frac{\partial E_i}{\partial d\varphi_j} = (-)^{p_{ij}} \frac{\partial f_j}{\partial \varphi^i} - \frac{\partial f_i}{\partial \varphi^j},
\]

which obviously satisfy (13).

It is easy to compare with the result given by the Noether-Form method (equation (8)): \( \hat{\omega}_{No} = \delta (\delta \varphi^i \wedge \frac{\partial L_1}{\partial d\varphi_i}) = -\delta \varphi^i \wedge \delta f_i = -\delta \varphi^i \wedge \delta \varphi^j \wedge \frac{\partial f_i}{\partial \varphi^j} = \hat{\omega} \), (26)

where we used equation (11) together with equation (25) for the last equality.

As we argued in section 2.2, the result (26) will be automatically gauge invariant. Let us consider more explicit examples: For Yang-Mills theories, formula (26) gives the well-known result [1, 3, 4]:

\[
\hat{\omega}_{YM} = Tr (\delta A \wedge * \delta F)
\]

For the affine-\( GL(D; \mathbb{R}) \) gravity (see definitions and notations in [10, 17]) we find:

\[
\hat{\omega}_{GL} = \frac{1}{16\pi G} \delta \omega^a \wedge \delta \left( \sqrt{|g|} g^{ab} \Sigma_{ab} \right)
\]

with \( \omega^a \) and \( g^{ab} \) the \( GL(D; \mathbb{R}) \) connection and metric respectively, \( \Sigma_{ab} = \frac{1}{(D-2)!} \varepsilon_{abc_1...c_D} \theta^{c_1} \wedge ... \wedge \theta^{c_D} \) and \( \theta^a \) is the canonical one-form.

From formula (28), we can extract the pre-symplectic Form in the Palatini formalism by fixing \( \theta^a_{\mu} = \delta^a_{\mu} \) (this breaks all the \( GL(D; \mathbb{R}) \) gauge symmetry). In components the result is [1, 2, 3, 4, 5]:

\[
\hat{\omega}_{\nu a} = \frac{1}{8\pi G} \delta \Gamma^\mu_{\nu \rho} \wedge \delta \left( \sqrt{|g|} g^{\nu \rho} \right)
\]

with \( \Gamma^\mu_{\nu \rho} = \Gamma^\mu_{(\nu \rho)} \) the torsionless connection. Note that we recover ordinary second order Einstein gravity by using the metricity condition \( \nabla_{\mu} g_{\rho \sigma} = 0 \) which implies \( \delta \Gamma^\mu_{\nu \rho} = \frac{1}{2} g^{\rho \sigma} (\nabla_{\nu} \delta g_{\rho \sigma} + \nabla_{\mu} \delta g_{\nu \sigma} - \nabla_{\sigma} \delta g_{\nu \rho}) \).

The result (28) also contains the symplectic density of the first order orthonormal frame (vielbein) formalism. Indeed, after implementing the
gauge choice \( g^{ab} = \eta^{ab} \) which breaks \( GL(D; \mathbb{R}) \) down to \( SO(D - 1, 1; \mathbb{R}) \), equation (28) reduces to

\[
\hat{\omega}^\mu_{SO} = \frac{1}{8\pi G} \delta \omega^{ab}_\nu \wedge \delta \left( |e| e_\mu^a e_\nu^b \right),
\]  

(30)

with \( e_\mu^a \) the inverse of the orthonormal frame and \( \omega^{ab}_\mu = \omega^{|ab|}_\mu \) the \( SO(D - 1, 1; \mathbb{R}) \) spin connection.

**Chern-Simons theories**

Let us start with the five dimensional Abelian Chern-Simons theory. It is the simplest first order theory whose Lagrangian is not of the form (24):

\[
L_{CS}^5 := \frac{1}{3} A \wedge dA \wedge dA.
\]  

(31)

The equations of motion are then simply:

\[
E_{CS}^5 := \frac{\delta L_{CS}^5}{\delta A} = dA \wedge dA.
\]  

(32)

Following equation (11), the symplectic density computed from equation (32) gives

\[
\hat{\omega}_{CS}^5 = -\delta A \wedge \delta A \wedge dA
\]  

(33)

which is gauge invariant as promised.

If we follow the Noether-Form construction recalled in section 2.1, we get:

\[
\hat{\omega}_{CS}^{5_{No}} = \delta \left( \frac{\delta L_{CS}^5}{\delta dA} \right) = -\frac{2}{3} (\delta A \wedge \delta A \wedge dA + \delta A \wedge A \wedge d\delta A).
\]  

(34)

Note that \( \hat{\omega}_{CS}^{5_{No}} \) is not anymore explicitly gauge invariant. A straightforward calculation shows that it differs from (33) by an exact term,

\[
\hat{\omega}_{CS}^5 - \hat{\omega}_{CS}^{5_{No}} = -\frac{1}{3} d(\delta A \wedge \delta A \wedge A),
\]  

(35)

in agreement with the general formula (15).

Therefore, the gauge invariant result (33) cannot be found using the Noether-Form procedure because there is no \( Y \) which could cancel the right-hand side of equation (35). On the other hand, the symplectic density (33)
is not $\delta$-closed if no appropriate boundary conditions are implemented (see subsection 2.6).

The higher-dimensional ($D = 2n + 1$) non-Abelian Chern-Simons case is very similar. The equations of motion are generally given by:

$$E^{\text{CS}_{2n+1}} = F^n$$

(36)

where $F := dA + A \wedge A$ is the curvature of a Lie-valued gauge field and $F^n := F \wedge \ldots \wedge F$ ($n$ times).

From equation (31), we find the covariant symplectic density of the theory

$$\hat{\omega}^{\text{CS}_{2n+1}} = -\frac{n}{2} \text{Tr} \left( \delta A \wedge \delta A \wedge F^{n-1} \right).$$

(37)

On the other hand, the pre-symplectic Form given by the “Noether method” is not explicitly covariant as in the previous case. It is moreover far more complicated to compute it because of the involved structure of the non-Abelian Chern-Simons Lagrangian (see for instance [11] for explicit formulas and references therein).

**Eleven-dimensional supergravity**

As a last example which nicely combines the structures of gravity and higher dimensional Chern-Simons theories, we compute the symplectic structure of eleven dimensional supergravity [18]. Moreover, it gives a simple example with fermionic fields.

The complete first order formulation of eleven dimensional supergravity was derived in [13]. Partial results for the spin connection or for the four-form gauge field treated as independent fields can be found in [19] and [20] respectively.

We follow the differential forms notation of eleven dimensional supergravity extensively detailed in [13]. The independent fields are the elfbein $e^a$, the three form $A$, the Majorana gravitino $\psi$ and the two auxiliary fields, namely the spin connection $\omega^a_{\beta \gamma}$ and the four-form field strength $F$. Using again equation (31), we find the following symplectic density:

$$\hat{\omega}^{11} = \frac{1}{4\kappa^2} \delta \omega^{ab} \wedge \delta e^c \wedge \Sigma_{abc} - \frac{i}{2} \delta \bar{\psi} \wedge \gamma_{(8)} \wedge \delta \psi - \delta A \wedge \delta \ast F$$

$$- \frac{i}{4} \delta e^a \wedge \delta \bar{\psi} \wedge (2 \gamma_{(7)a} - \gamma_{(6)} \wedge e_a) \wedge \psi - \frac{i \kappa}{4} \delta A \wedge \delta \left( \bar{\psi} \wedge \gamma_{(5)} \wedge \psi \right)$$

$$+ \kappa \delta A \wedge \delta A \wedge dA - \frac{i}{8} \delta e^a \wedge \delta e^b \wedge \bar{\psi} \wedge \left( \gamma_{(6)} \eta_{ab} + \gamma_{(5)} e^b \right) \wedge \psi$$

(38)
This symplectic density is explicitly covariant and differs from the Noether one in the Chern-Simons term (first term of the last line), in analogy with equation (35).

Note that there is a non vanishing symplectic coefficient between pairs of three form components (proportional to $\delta A \wedge \delta A$) as well as pairs of elfein's (proportional to $\delta e^a \wedge \delta e^b$). The first one arises because of the Chern-Simons term in the action and the second reflects the special torsion term present in the Lagrangian of first order higher dimensional supergravities (see the third term of equation (6.1) of [13]). It is interesting that this torsion term is needed for any supergravity of dimension five and more because of $\gamma$-matrix gymnastics. The Chern-Simons term is required for supersymmetry, also in five dimensions and more. This suggests some deep relation between torsion and Chern-Simons effects.

2.6 Boundary conditions, conservation and closure of the symplectic Form

The geometrical data

We denote by $\mathcal{M}$ a $D$-dimensional piece of spacetime between two (partial) Cauchy hypersurfaces $\Sigma_{t_1}$ and $\Sigma_{t_2}$. We suppose that $\mathcal{M}$ admits a foliation and thus is topologically $\Sigma_t \times \mathbb{R}$.

Let $\mathcal{M}$ be also bounded by a set of $n$ $(D-1)$-dimensional time-like or null hypersurfaces, denoted by $\mathcal{H}_r$, $r = \{1, \ldots, n\}$. Depending on our specific examples, these $\mathcal{H}_r$ can be for example pieces of future (or past) null infinity after some compactification [21], of spatial infinity [22] or even of the horizon of a (locally) isolated black hole [6].

We can define at each time, a set of $n$ $(D-2)$-dimensional closed manifolds embedded in $\mathcal{M}$ by $\mathcal{B}_r = \Sigma_t \cap \mathcal{H}_r$, $r = \{1, \ldots, n\}$. These $\mathcal{B}_r$ are simply all the disconnected boundary components of $\Sigma_t$, that is $\partial \Sigma_t = \sum_{r=1}^{n} \mathcal{B}_r$. We suppressed an explicit time dependence for notational simplicity. When needed, we will however write $\mathcal{B}_{t_1}$ or $\mathcal{B}_{t_2}$ for $\mathcal{B}_r$ at time $t_1$ or $t_2$ respectively.

We denote by $s_t : \Sigma_t \to \mathcal{M}$, $h_r : \mathcal{H}_r \to \mathcal{M}$ and $b_r : \mathcal{B}_r \to \mathcal{M}$ the natural embeddings in $\mathcal{M}$ of the submanifolds previously described. We will use the shorthand notation $\int_{\Sigma_t} \omega$ and $\omega|_{\Sigma_t}$ for respectively $\int_{\Sigma_t} s_t^* \omega$ and $s_t^* \omega$ (for any $\omega$), and so on for $\mathcal{H}_r, h_r$ and $\mathcal{B}_r, b_r$. 
The symplectic structure, off-shell and on-shell

Let us first define the off-shell symplectic Form by

\[ \Omega_t := \int_{\Sigma_t} \omega. \] (39)

A priori, this (0, 2)-Form \( \Omega_t \) depends on the choice of the (partial) Cauchy hypersurface \( \Sigma_t \) and on the ambiguity on \( \omega \) (namely, the choice of \( dX \) in equation (10)).

Let us denote by \( \bar{F} \) (subspace of \( F \)) the space of smooth fields that satisfy the equations of motion \( E_i = 0 \). Since we have a natural embedding \( \epsilon \) of \( \bar{F} \) in \( F \), the two-Form \( \omega \) pulls back to a two-Form on \( \bar{F} \), namely \( \bar{\omega} = \epsilon^* \omega \). Note that now the exterior derivative of \( \bar{F} \), denoted by \( \bar{\delta} \) defines the linearized equations of motion by \( \bar{\delta} \epsilon E_i = 0 \) on \( \bar{F} \).

We can now pull back equation (9) (which is defined on \( F \)) to \( \bar{F} \) to verify that

\[ d\bar{\omega} = 0. \] (40)

By integrating this equation on \( \mathcal{M} \), we find

\[ \bar{\Omega}_{t_1} - \bar{\Omega}_{t_2} = \sum_{r=1}^{n} \int_{\mathcal{H}_r} \bar{\omega} \] (41)

with the definition (39) naturally pulled back on \( \bar{F} \).

Then, the symplectic two-Form \( \Omega_t \) will be on-shell conserved only if the right-hand side of equation (11) vanishes. This condition must be satisfied in order to define a time-independent symplectic structure.

A natural but stronger requirement for the possible boundary conditions on \( \mathcal{H}_r \) would be:

\[ \omega \bigg|_{\mathcal{H}_r} = 0 \quad \forall r. \] (42)

This constraint can give in principle some indication on how to fix the ambiguity (10). Concretely, given a set of boundary conditions on \( \mathcal{H}_r \), we may want to find a Form \( X \) such that equation (12) is satisfied. We shall argue momentarily that the choice \( X = 0 \) (and then \( \omega = \bar{\omega} \)) is appropriate if the boundary conditions on \( \mathcal{H}_r \) are compatible with a variational principle.

---

\(^6\)We abusively use the word “symplectic” as it is common in the literature although the quantity (39) is usually degenerate due to gauge symmetries.

\(^7\)We adopt the “bar notation” of reference \( \text{3} \) for on-shell quantities. We therefore denote by a bar the pulled-back quantities on \( \bar{F} \).
Condition (42) also implies the (off-shell) closure of the symplectic form (39),
\[ \delta \Omega = 0. \] (43)

This can be verified by the following argument. First, equation (42) together with (14) imply that:
\[ 0 = \int_{H_r} \delta \omega = \int_{B_{r,1}} \left( \frac{1}{3} \delta \phi^i \wedge \frac{\partial \hat{\omega}}{\partial \phi^i} + \delta X \right) - \int_{B_{r,2}} \left( \text{"same"} \right), \] (44)
for all times \( t_1 \) and \( t_2 \) and for all \( r \)'s independently. Moreover, since each piece of the right hand-side of equation (44) depends on arbitrary \( \delta \phi^i \)'s, each of these two term has to vanish separately, that is:
\[ \int_{B_r} \left( \frac{1}{3} \delta \phi^i \wedge \frac{\partial \hat{\omega}}{\partial \phi^i} + \delta X \right) = 0, \] (45)
where we again suppressed the time dependence for simplicity.

Using this result together with equation (14) again, we straightforwardly find that the symplectic Form is \( \delta \)-closed:
\[ \delta \Omega = \sum_{r=1}^{n} \int_{B_r} \left( \frac{1}{3} \delta \phi^i \wedge \frac{\partial \hat{\omega}}{\partial \phi^i} + \delta X \right) = 0. \] (46)

In summary, we have shown that:
\[ \omega |_{H_r} = 0 \quad \forall r \quad \Rightarrow \quad \partial_t \Omega = 0 \]
\[ \Rightarrow \quad \delta \Omega = 0. \] (47)

The boundary conditions and the variational principle

We shall now relate the condition \( \hat{\omega} |_{H_r} = 0 \) with the boundary conditions imposed on \( H_r \) by the choice of space \( \mathcal{F} \). Let us consider the boundary component \( H_r \), between two times \( t_1 \) and \( t_2 \). The boundary conditions on \( H_r \) lead to a well-defined variational principle for an appropriate Lagrangian on our space \( \mathcal{F} \) provided
\[ \exists \hat{\theta} \text{ in the class (5) such that } \hat{\theta} |_{H_r} = 0. \] (48)

\[ \text{In fact, it is enough to suppose that we can find one } \delta \phi^i \text{ which vanishes at some } t_1 \text{ but not at } t_2. \text{ Note that } \delta \phi^i \text{ should be compatible with the boundary conditions but does not need to satisfy the linear equations of motion. In other words, } \delta \phi^i \text{ belongs to } T^* \mathcal{F} \text{ but not necessarily to } T^* \bar{\mathcal{F}}. \]
Note that in order to satisfy the variational principle, that is

$$\delta S = \delta \int_{\mathcal{M}} L = 0 \iff E_i = 0 \quad (\text{without any surface term}),$$

the condition (48) must hold on each boundary $\mathcal{H}_r$ ($r = 1, \ldots, n$) of the spacetime $\mathcal{M}$ as well as on the Cauchy hypersurfaces $\Sigma_{t_1}$ and $\Sigma_{t_2}$. This is the usual way to fix the boundary conditions of a variational problem.

A direct consequence of the statement (48) is that if the variational principle is satisfied on the boundary $\mathcal{H}_r$, there will exist a pre-symplectic form whose pullback vanishes on this $\mathcal{H}_r$:

$$(48) \implies \exists \hat{\omega}_{\mathcal{H}_r} := \delta \hat{\theta}, \text{ such that } \hat{\omega}_{\mathcal{H}_r} = 0.$$  

(50)

It is then possible to analyze the pullback of the symplectic density $\hat{\omega}$ (12) on the boundary $\mathcal{H}_r$. From the identity (15), we find that:

$$\hat{\omega}|_{\mathcal{H}_r} = -\frac{1}{2} d \left( \delta \varphi^j \wedge \frac{\partial \hat{\theta}}{\partial \varphi^i} \right)\bigg|_{\mathcal{H}_r}. \quad (51)$$

It is hard to prove that the righthand side of this equation will vanish for all boundary conditions compatible with a variational principle. We can however give a quite general argument: let us assume that our set of boundary conditions are denoted by:

$$F_A(\varphi)\bigg|_{\mathcal{H}_r} = 0,$$  

(52)

for $A = 1, \ldots, N$, with $N$ a given number of boundary conditions.

We assume that $F_A(\varphi)$ is a functional of the fields but not of their derivatives. This is quite natural in first order theories where the dynamical fields and the momenta (auxiliary fields) are independent. We basically suppose that the boundary conditions are functions of the canonical variables $(q, p)$ but not of their derivatives.

Now, since $\hat{\theta}$ vanishes on $\mathcal{H}_r$ by hypothesis, it should take the general form:

$$\hat{\theta} = F_A \wedge \Theta_A^0 + dF_A \wedge \Theta_A^1 + \delta F_A \wedge \Theta_A^2 + d\delta F_A \wedge \Theta_A^3 + O(F^2), \quad (53)$$

for some given functionals $\Theta_A^0, \Theta_A^1, \Theta_A^2$ and $\Theta_A^3$.

We can then check that the right hand-side of equation (51) vanishes on $\mathcal{H}_r$ for the general expression (53) by making use of equation (52) and its $\delta$-derivative.
It is therefore tempting to extend our analysis to boundary conditions more general than (52), and therefore to conjecture that:

\[(48) \Rightarrow \hat{\omega}|_{H_r} = 0.\] (54)

However, we were not able to prove this statement in complete generality.

3 Gauge symmetries and conserved charges

Let us now suppose that the Lagrangian \(L\) is invariant under the following gauge symmetry that preserves the boundary conditions off shell:

\[
\delta_{\xi} \phi^i = d\xi^\alpha \Delta^i_\alpha + \xi^\alpha \tilde{\Delta}^i_\alpha,
\] (55)

with \(\phi^i\) some \(p_i\)-form field, \(\Delta^i_\alpha\) and \(\tilde{\Delta}^i_\alpha\) some field-dependent \((p_i - 1)\) and \(p_i\) space-time forms respectively and \(\xi^\alpha(x)\) the infinitesimal parameter.

The Noether current associated with the gauge symmetry (55) is given by (see [16, 11] and references therein)

\[
J_{\xi} = dU_{\xi} + W_{\xi},
\] (56)

with

\[
W_{\xi} := \xi^\alpha \Delta^i_\alpha \wedge E_i \quad \text{and} \quad E_i := \frac{\delta L}{\delta \phi^i}.
\] (57)

Equation (56) means that the Noether current \(J_{\xi}\) is on-shell the divergence of a “superpotential” \(U_{\xi}\), in other words it is weakly equal to a topological current. However, this superpotential cannot be computed without a “case by case” prescription using the usual Noether method. In fact Noether’s theorem allows us to compute \(dJ_{\xi}\) but not \(J_{\xi}\) itself (see also the recent work [23] and references therein).

Given a boundary \(B_r\) (a closed \((D-2)\)-dimensional manifold, see section 2.6) our goal is to compute \(U_{\xi}\) in an unambiguous way. This would allow us to define an associated charge by:

\[
Q_{\xi}^{(r)} := \int_{B_r} U_{\xi}.
\] (58)

An important point is that we can construct charges on each boundary component \(B_r, r = \{1, \ldots, n\}\), completely independent from each other.

The symplectic methods described in the previous section can be used to derive the superpotential \(U_{\xi}\), in analogy with the Hamiltonian result of
Regge and Teitelboim [24, 25]. Let us first recall the construction developed by Ashtekar, Wald and collaborators and the alternative method proposed in reference [11]. In the third subsection, we clarify in which cases they are equivalent. Finally in subsection 3.4, we shall give a necessary condition for the existence of the charge \( Q_{\xi}^{(r)} \).

3.1 The charges from the symplectic method

The off-shell symmetry \( \delta_{\xi} \varphi^i \) given by equation (55) restricts naturally to a vector field on \( \bar{F} \) since \( \delta_{\xi} E_i \) vanishes on-shell. We can contract the symplectic density (which is a two-Form on \( F \)) along this vector:

\[
\omega_{\xi} := \omega(\delta_{\xi} \varphi, \delta \varphi) = i_{\delta_{\xi}} \omega,
\]

with \( i_{\delta_{\xi}} \) the interior product (in \( F \)) with respect to the vector \( \delta_{\xi} \varphi^i \) (see also equations (20) and (21)).

The total charge associated with the symmetry (55) obeys [2, 3, 4]

\[
\delta Q_{\xi} = i_{\delta_{\xi}} \Omega = \int_{\Sigma_t} \omega_{\xi}. \tag{60}
\]

The charge \( Q_{\xi} \) will be conserved (on-shell) only if the symplectic Form is also time-independent.

For a gauge symmetry (55), \( \omega_{\xi} \) (defined by equations (10) and (59)) is on-shell a total derivative (see the Appendix, equation (A.49) for a proof):

\[
\omega_{\xi} \approx d \left( \xi^\alpha \Delta^i_{\alpha} \wedge \delta_{\xi} \varphi^j \wedge \hat{\omega}_{ij} + \delta_{\xi} \varphi^i \wedge \delta_{\xi} \varphi^j \wedge X_{ij} \right) =: d \nu_{\xi}. \tag{61}
\]

This implies that the total charge (60) will be on-shell a boundary term. Equivalently, \( Q_{\xi} \) is given by the integral of the Noether current on a Cauchy hypersurface:

\[
Q_{\xi} = \int_{\Sigma_t} J_{\xi} \approx \int_{\partial \Sigma_t} U_{\xi} \tag{62}
\]

by virtue of equation (56). Then, using equations (60) and (61), we obtain an on-shell equation for the boundary charges,

\[
\delta Q_{\xi}^{(r)} := \int_{B_r} \delta U_{\xi} \approx \int_{B_r} \nu_{\xi} \quad \text{for all } B_r \text{ 's}. \tag{63}
\]

At each boundary \( B_r \), the equation (63) gives an equation for the variation of the superpotential. As in the Regge-Teitelboim procedure, the last
step is to “integrate” the equations (63) using the boundary conditions imposed on \( B_r \) in order to determine \( U_\xi \). Moreover, from equations (61) and (63), we see that it is extremely important to fix the tensor \( X \) in a correct way if we want to compute appropriate charges.

As a final comment, in principle the equation (63) has to be integrated (and therefore has to be integrable!) on all the boundaries \( B_r \) to ensure consistency of equation (60). But we will see in the following subsections that one can work in each boundary component separately.

3.2 The superpotential from the “differentiability” of the Noether current

We now recall the construction of the charges given in [11] and compare it with the symplectic method described above.

Without giving a prescription, the superpotential is completely ambiguous. In other words, both \( J_\xi \) and \( U_\xi \) of equation (56) are unknown. This equation then only gives the Noether current in term of the superpotential, but no more. However, we can still take the \( \delta \)-exterior derivative of equation (56). We then obtain a (D-1,1)-Form equation in \( \mathcal{F} \) (off-shell):

\[
\delta J_\xi = \delta \phi^i \wedge \frac{\delta W_\xi}{\delta \phi^i} + d \left( \delta U_\xi + \delta \phi^i \wedge \frac{\partial W_\xi}{\partial d \phi^i} \right),
\]

where we assumed for the last identity that \( W_\xi \) depends at most on first derivatives of the fields (first order theories).

Let us now choose one \( \mathcal{H}_r \) and integrate equation (64) on this hypersurface, between two times \( t_1 \) and \( t_2 \):

\[
\int_{\mathcal{H}_{t_12}} \delta J_\xi - \int_{\mathcal{H}_{t_12}} \delta \phi^i \wedge \frac{\delta W_\xi}{\delta \phi^i} = \int_{B_{t_1}} \left( \delta U_\xi + \delta \phi^i \wedge \frac{\partial W_\xi}{\partial d \phi^i} \right) - \int_{B_{t_2}} \text{ ("same")}. \tag{65}
\]

In analogy with the Hamiltonian formalism [24, 25], the next idea is to impose a (local) “differentiability condition” for the Noether current \( J_\xi \) on the boundary \( \mathcal{H}_r \), namely

\[
\int_{\mathcal{H}_{t_12}} \delta J_\xi = \int_{\mathcal{H}_{t_12}} \delta \phi^i \wedge \frac{\delta W_\xi}{\delta \phi^i}. \tag{66}
\]

Note that this equation is evaluated on a fixed boundary component \( \mathcal{H}_r \) and not on a Cauchy hypersurface \( \Sigma_t \). Therefore, the knowledge of the data on one boundary is enough for what follows.

\[\ldots\text{ or } Y \text{ if we restrict ourselves to } \delta\text{-closed pre-symplectic Forms, see section 2.1.}\]
We will justify equation (66) in the next subsection. For the time being let us assume that it is satisfied. Then, the sum of the two terms on the right hand-side of equation (65) vanishes. However, since $\delta \phi^i$ is arbitrary, each term has to vanish separately (see the analogous argument after equation (44)), that is:

$$\delta Q^{(r)}_\xi = \int_{B_r} \delta U_\xi = - \int_{B_r} \delta \phi^i \wedge \frac{\partial W_\xi}{\partial \phi^i} \tag{67}$$

which is the main equation given in [11]. Examples for which this equation was used successfully can be found in [11, 26, 27, 17, 28, 29, 30].

As opposed to equation (63), the right hand-side of equation (67) is non ambiguous since it does not depend on any specific choice of $X$ or $Y$.

### 3.3 Comparison between methods 3.1 and 3.2

Using the following on-shell identity,

$$(-)^{(p_i+1)(p_j+1)} \xi^\alpha \Delta^i_\alpha \wedge \frac{\partial E_i}{\partial \phi^j} \approx \frac{\partial \left( \xi^\alpha \Delta^i_\alpha \wedge E_i \right)}{\partial \phi^j} = \frac{\partial W_\xi}{\partial \phi^j} \tag{68}$$

and using equations (12) and (61), we find that

$$v_\xi \approx - \delta \phi^i \wedge \frac{\partial W_\xi}{\partial \phi^j} + \delta_\xi \phi^j \wedge \delta \phi^i \wedge X_{ij}, \tag{69}$$

up to some $d$-exact terms which does not contribute to the charge since $B_r$ is a closed manifold.

Therefore, comparing equations (63), (67) and (69), the “differentiability” condition (namely equation (66)) is on-shell equivalent to the criterion $X = 0$ in the choice of our symplectic density. In other words, this criterion implies that the symplectic density of a first order theory is $\hat{\omega}$, defined by formula (12).

Note finally that this result could also be recovered using the (off-shell) identity

$$\delta W_\xi = \frac{\partial}{\partial \phi^j} \left( \delta_\xi \phi^j \wedge E_j \right). \tag{70}$$

in the integrand of equation (66)

$$\delta J_\xi = \delta \phi^i \wedge \frac{\partial W_\xi}{\partial \phi^i} \approx \delta_\xi \phi^j \wedge \delta \phi^i \wedge (-)^{(p_i)} \frac{\partial E_i}{\partial \phi^j} = \hat{\omega}_\xi, \tag{71}$$

10Equation (71) was also used in reference [13] to compute the transformation laws of the auxiliary fields in first order supergravities.
where we used the definition (12) in the last equality.

If we compare the result (71) with equations (60) and (62), we again conclude that the differentiability condition singles out the non-ambiguous (and covariant) symplectic density $\hat{\omega}_\xi$.

Up to now, we have proven that equation (67) is equivalent to the differentiability condition (65), which on-shell, is simply

$$\delta \int_{\mathcal{H}_r} J_\xi = \int_{\mathcal{H}_r} \hat{\omega}_\xi.$$  \hspace{1cm} (72)

We then see that nothing is required on the other boundary components $\mathcal{H}_s, s \neq r$. This is the mathematical realization of the physical idea that the ADM mass can be defined at spatial infinity no matter what “happens” in the bulk or on the other internal boundaries as for instance on a horizon.

Now, equation (72) can be trivially justified if:

1. The charge $Q^{(r)}_\xi = \int_{\mathcal{B}_r} U_\xi$ is conserved.

2. The boundary conditions on $\mathcal{H}_r$ are compatible with the variational principle.

The point 1 implies the vanishing of the left-hand side of equation (72) while the point 2 implies the vanishing of the right-hand side: First, if the charge $Q^{(r)}_\xi$ is on-shell conserved, then the flux of the Noether current vanishes on $\mathcal{H}_r$ due to equation (56):

$$\int_{\mathcal{H}_r} J_\xi = 0.$$  \hspace{1cm} (73)

This implies in particular that $\int_{\mathcal{H}_r} \delta J_\xi = 0$ and therefore the left hand-side of equation (72) is zero.

Second, we argued in section 2.6 that boundary conditions adapted to a variational principle imply that $\hat{\omega} |_{\mathcal{H}_r} = 0$. This condition contracted along the symmetry $\delta_{\xi} \phi^i$ (a vector of $\mathcal{F}$) forces the vanishing of the right-hand side of the equation (72).

In summary, we have first shown that equation (67) is equivalent to the method of section 3.1 if the used Noetherian pre-symplectic form is equal to the symplectic density given by equation (12). Second, we have derived equation (72) from the natural assumptions 1 and 2 described above.
3.4 Integrability

One can analyze the integrability of equations (60) or (67) by taking their exterior δ-derivative. This gives a necessary condition for the existence of the charge $Q^{(r)}_\xi$.

As shown by equation (A.56) of the appendix, equation (67) will be integrable in $\tilde{F}$ if and only if

$$\int_{B_r} \left( \delta \varphi^k \wedge \frac{\partial \delta \varphi^i}{\partial d \varphi^k} \wedge \delta \varphi^j \wedge \hat{\omega}_{ij} - \frac{(-)^{(p_i+1)(p_j+p_k)}}{2} \delta \xi \varphi^k \wedge \delta \varphi^i \wedge \delta \varphi^j \wedge \frac{\partial \hat{\omega}_{ij}}{\partial d \varphi^k} \right) \approx 0.$$  \hspace{1cm} (74)

The second term of equation (74) comes from the failure of $\hat{\omega}$ to be locally $\delta$-closed. However, this term will vanish for suitable boundary conditions when integrated on $B_r$. This follows after contracting the equation (45) along a symmetry $\delta \xi \varphi^i$ (remember also that $X = 0$).

Therefore, for boundary conditions compatible with the variational principle, the conserved charge $Q^{(r)}_\xi$ will exist only if

$$\int_{B_r} \delta \varphi^k \wedge \frac{\partial \delta \varphi^i}{\partial d \varphi^k} \wedge \delta \varphi^j \wedge \hat{\omega}_{ij} \approx 0.$$  \hspace{1cm} (75)

This condition is trivially satisfied for any Lie-type gauge symmetry. In fact, in that case the symmetry transformation laws do not depend on the derivatives of the fields. The example of a diffeomorphism invariant theory is more interesting. In that case, $\delta \xi \varphi^i = di \xi \varphi^i + i \xi d \varphi^i$, with $i \xi$ the interior product along the vector $\xi^\mu$. Then it is easy to check that equation (75) reduces to

$$\int_{B_r} i \xi \hat{\omega} \approx 0.$$  \hspace{1cm} (76)

This result was found in reference [8] using a direct calculation in general relativity. We have just proven that this results indeed remains valid for any diffeomorphism-invariant theory.

We can also comment on the integrability condition of the symplectic equation (60):

$$\delta i_{\delta \xi} \Omega = \mathcal{L}_{\delta \xi} \Omega = \delta \xi \Omega = 0$$  \hspace{1cm} (77)

11This can be proven using components. If fact, $\delta \xi \varphi^i = \xi^\mu \partial_{\mu} \varphi^i + \text{“more”}$, where “more” does not contain derivatives of the fields. We can then plug this transformation law into equation (A.56), and Hodge-dualize to get the result (76).
where $\mathcal{L}_{\delta\xi} = \delta^i i_{\delta\xi} + i_{\delta\xi} \delta$ is the Lie derivative in $\mathcal{F}$ with respect to the vector $\delta\xi \varphi^i$. In the second equality of equation (77), we used the $\delta$-closure of $\Omega$ (for appropriate boundary conditions, see equation (47)). The third equality follows using that $[\mathcal{L}_{\delta\xi}, \delta] = [\mathcal{L}_{\delta\xi}, d] = [\delta\xi, \delta] = [\delta\xi, d] = 0$. We then see from equation (74) that if the symplectic structure $\Omega$ is gauge invariant, the integrability is guaranteed.

Moreover, for a theory invariant under diffeomorphisms we can recover the result (76). In fact, if $\delta\xi$ is the Lie derivative $\mathcal{L}_{\xi}$ along the vector field $\xi^\mu(x)$, the equation (77) implies that $\int_{\partial\Sigma_t} i_{\xi}\omega \approx 0$ since $d\omega \approx 0$ by equation (40).

4 Conclusion

We have proposed a new definition for the covariant symplectic form in the Lagrangian framework. The associated symplectic density is covariant and boundary condition independent since it is constructed using only the equations of motion of the theory. For consistency we required boundary conditions compatible with the variational principle. We have also shown that our proposal coincides with the Noetherian pre-symplectic form of Yang-Mills and general relativity theories. However differences were found for higher dimensional Chern-Simons theories and eleven dimensional supergravity. We finally also defined a generalized Hamilton functional which together with our symplectic density generates the dynamics.

In the second part of the paper, we have revisited the construction of the conserved charges associated with gauge symmetries. We first recalled the standard covariant symplectic method paying special attention to ambiguities, on-shell and off-shell statements and to boundary terms. We then concluded that this construction coincides with the one proposed in [11] if the ambiguity in the symplectic density is fixed according to our new prescription.

Acknowledgments.

We would like to thank C. Beetle for discussions on Isolated Horizons.

Appendix: The symplectic density in components

The purpose of this appendix is to give rigorous proofs of several statements made in the main part of the manuscript. We work in component nota-
tion, in the sense that spacetime forms are replaced by their duals, and the spacetime indices, namely $\mu, \nu, \rho, \ldots$, are written explicitly. Therefore, no bold characters (which denote in the main text a spacetime differential form, see section 2) are used. This gives a “component translation” of the main results.

**Basic definitions and notations**

Let us start with some notations. We define the variational and partial derivatives:

$$\delta_i := \frac{\delta}{\delta \varphi^i}, \quad \partial_i := \frac{\partial}{\partial \varphi^i}, \quad \partial_i^\mu := \frac{\partial}{\partial \varphi^\mu}.$$  \hfill (A.1)

The operators (A.1) act on functionals $A$ which depend on the fields $\varphi^i$ and their first derivatives only, that is $A = A(\varphi, \partial_\mu \varphi)$. In that case, the Euler-Lagrange variation can be rewritten as:

$$\delta_i A = \partial_i A - \partial_\mu \partial_i^\mu A.$$

(A.2)

Using the notation (A.1), the spacetime derivative operator acting on a functional $A$ is simply:

$$\partial_\mu A = \partial_\mu \varphi^i \partial_i A + \partial_\mu \partial_\nu \varphi^i \partial_\nu^i A.$$ \hfill (A.3)

The variation of a functional $A$ (that is, its exterior derivative in $F$) under an arbitrary variation of the fields $\delta \varphi^i$ can be rewritten in two natural ways:

$$\delta A = \delta \varphi^i \partial_i A + \partial_\mu \delta \varphi^i \partial_i^\mu A$$

(A.4)

$$= \delta \varphi^i \partial_i A + \partial_\mu \left( \delta \varphi^i \partial_i^\mu A \right).$$ \hfill (A.5)

Note that $\delta$ (as $\partial_i$, $\partial_i^\mu$ and $\partial_\mu$) is a derivation, in the sense that $\delta(AB) = \delta(A)B + A\delta(B)$ (with the usual sign correction if $A$ is an odd $p_A$-Form of $F$). This is not true for $\delta_i$. A straightforward calculation (using the definition (A.2)) shows that

$$\delta_i (AB) = \delta_i (A) B + A \delta_i (B) - \partial_\mu A \partial_i^\mu B - \partial_i^\mu A \partial_\mu B.$$ \hfill (A.6)

A well-known result from variational calculus is that the Euler-Lagrange variation of a derivative vanishes identically:

$$\delta_i \partial_\mu A = 0 \quad \forall A.$$

(A.7)
Note that in this last equation, \( A \) can carry any spacetime index.

In general, the basic operators \( \partial_i, \partial_i^\mu, \delta_i \) and \( \partial_\mu \) do not commute (they however all commute with \( \delta \)). A straightforward calculation using the above definitions shows that:

\[
\begin{align*}
[\partial_\mu, \partial_i^\nu] A &= -\delta^\nu_\mu \partial_i A \\
[\partial_\mu, \delta_i] A &= \partial_\mu (\delta_i A) \\
[\partial_i^\mu, \delta_j] A &= -\partial_i \partial_j^\mu A \\
[\delta_i, \delta_j] A &= \delta_i \partial_j^\mu (\delta_i A) + \partial_i \partial_\nu (\partial^\mu_i \partial^\nu_j A) \\
&= \frac{1}{2} \partial_\mu \left( \partial^\mu_i (\delta_i A) - \partial^\mu_j (\delta_j A) \right) \\
\text{and } [\cdot, \cdot] A &= 0 \text{ for the others.} \quad (A.13)
\end{align*}
\]

Another useful formula for the following calculations is that

\( \delta_j (\delta_i A) = \delta_j (\partial_i A) = \partial_i (\delta_j A) \quad (A.14) \)

which comes from the definition \((A.2)\), the property \((A.7)\) and the commutation relations \((A.13)\).

We will intensively use equations \((A.1)\)-\((A.14)\) in the following subsections.

“Cascade equations” for the symplectic density 2-Form; proof of equations \((12)\)

Let us start with the (off-shell) equation \((9)\) which defines a symplectic density:

\[
\partial_\mu \omega^\mu = \delta \varphi_i \wedge \delta E_i. \quad (A.15)
\]

The symplectic density takes the general following form (we work with first order theories):

\[
\omega^\mu = \frac{1}{2} \omega^\mu_{ij} \delta \varphi^i \wedge \delta \varphi^j + X^\mu_{ij} \delta \varphi^i \wedge \partial_j \delta \varphi^j. \quad (A.16)
\]

Therefore the equation \((A.15)\) together with \((A.16)\) and \((A.4)\) can be rewritten as:

\[
\partial_\mu \left( \frac{1}{2} \omega^\mu_{ij} \delta \varphi^i \wedge \delta \varphi^j + X^\mu_{ij} \delta \varphi^i \wedge \partial_j \delta \varphi^j \right) = \delta \varphi^i \wedge \left( \delta \varphi^j \partial_j E_i + \partial_\mu \delta \varphi^j \partial^\mu_j E_i \right). \quad (A.17)
\]
The equation (A.17) contains a lot of information due to the arbitrariness of $\delta \varphi^i$ and of its derivatives. Then, analogously to the “cascade equations” presented in the reference [16], the terms proportional to $\delta \varphi^i \wedge \delta \varphi^j$, $\delta \varphi^i \wedge \partial_\mu \delta \varphi^j$, etc., give independent equations:

\[ X_{\mu \nu}^{ij} = 0 \Rightarrow X_{ij}^{\mu \nu} = -X_{ij}^{\nu \mu} \quad (A.18) \]
\[ \partial_\mu \delta \varphi^i \wedge \partial_\nu \delta \varphi^j \Rightarrow X_{ij}^{\mu \nu} = X_{ij}^{\nu \mu} \quad (A.19) \]
\[ \delta \varphi^i \wedge \partial_\mu \delta \varphi^j \left( \omega_{ij}^\mu + \partial_\nu X_{ij}^{\nu \mu} = \partial_\mu E_i \right) \quad (A.20) \]
\[ \delta \varphi^i \wedge \delta \varphi^j \left( \frac{1}{2} \partial_\mu \omega_{ij}^\mu = \partial_\nu E_i \right) = \delta_{[i} E_{j]} \quad (A.21) \]

The first two equations simply show that $X_{ij}^{\mu \nu} = -X_{ij}^{\nu \mu}$. The third equation (A.20), together with equation (A.16), shows that any symplectic density which satisfies the equation (A.15) can be rewritten as:

\[ \omega^\mu = \hat{\omega}^\mu + \partial_\mu X^{\mu \nu} \quad (A.22) \]

for some arbitrary $X^{\mu \nu} = \frac{1}{2} X_{ij}^{\mu \nu} \delta \varphi^i \wedge \delta \varphi^j$ satisfying the above antisymmetry properties. We also denoted by

\[ \hat{\omega}^\mu = \frac{1}{2} \omega_{ij}^\mu \delta \varphi^i \wedge \delta \varphi^j \quad (A.23) \]

the spacetime-Hodge-dual of equation (12).

Finally, it is straightforward to check that the last equation (A.21) follows after taking the divergence of (A.20) (and using equations (A.2) and (A.14)).

**Antisymmetry of $\hat{\omega}_{ij}^\mu$**

In this subsection we explicitly prove the antisymmetry property (13) of the symplectic density $\hat{\omega}^\mu$.

We assume that the equations of motion of our theory are derived from a Lagrangian $L$, in the class $L \sim L + \partial_\mu K^\mu$, by $E_i = \delta_i L$. The definition of the symplectic density (A.23) only depends on the equations of motion, but not on a specific given Lagrangian. We then choose in the above equivalence class one Lagrangian which depends at most on first derivatives of the fields $L = L(\varphi, \partial_\mu \varphi)$ (for first order theories). For that kind of Lagrangian, the equations of motion are generally given by (see (A.2-A.3)):

\[ E_i = \delta_i L = \partial_i L - \partial_\alpha \varphi^\alpha \partial_j \partial_\mu \varphi^j \partial_\mu L - \partial_\mu \partial_\nu \varphi^\alpha \partial_\nu \partial_j \partial_\mu L. \quad (A.24) \]
Now, the restriction to first order theories, \( E_i = E_i(\varphi, \partial_\mu \varphi) \) implies that the tensor
\[
\partial^\nu j \partial^\mu i L
\]
is antisymmetric in \( \mu \) and \( \nu \), (A.25)
and then \( E_i = \partial_i L - \partial_\mu \varphi^j \partial_j \partial^\mu L \). Moreover, this condition implies that the tensor (A.25) is also antisymmetric in \( i \) and \( j \).

We can then check the antisymmetry in \( i \) and \( j \) of the symplectic density (A.23) by a straightforward calculation:
\[
\partial^\mu j E_i = \partial^\mu j \partial_i L - \partial_\mu \varphi^j \partial_j L - \partial_\nu \varphi^k \partial_k \partial^\mu j L = -\partial^\mu i E_j
\]
where the last equality follows from (A.25).

**Proof of equation (14)**

Due to equations (A.15) and (A.22), the symplectic density \( \hat{\omega}^\mu \) (A.23) satisfies the (off-shell) identity:
\[
\partial^\mu \delta \hat{\omega}^\mu = 0,
\]
that is from equation (A.4),
\[
\frac{1}{2} \partial^\mu \left( \delta \varphi^j \land \delta \varphi^j \land \left( \delta \varphi^k \partial_k \hat{\omega}^\mu_{ij} + \delta \partial_\nu \varphi^k \partial^\mu \hat{\omega}^\nu_{ij} \right) \right) = 0.
\]

We can then use the “Abelian cascade trick” given in the reference [14] to extract all the information contained in equation (A.28). This basically consists in making the replacement \( \delta \varphi \rightarrow \epsilon(x) \delta \varphi \) in equation (A.28) (because of the arbitrariness of \( \delta \varphi \)). The result is:
\[
\partial^\mu \left( \epsilon^3 \delta \hat{\omega}^\mu + \epsilon^2 \partial_\nu \epsilon \delta \varphi^j \land \delta \varphi^j \land \delta \varphi^k \partial_k \hat{\omega}^\nu_{ij} \right) = 0.
\]

Using the arbitrary of \( \epsilon(x) \) and its derivatives, we find a cascade of equations:
\[
\epsilon^3 \partial^\mu \delta \hat{\omega}^\mu = 0 \quad \text{(A.30)}
\]
\[
\epsilon^2 \partial^\mu \left( 3 \delta \hat{\omega}^\mu + \partial_\nu \left( \delta \varphi^j \land \delta \varphi^j \land \delta \varphi^k \partial_k \hat{\omega}^\nu_{ij} \right) \right) = 0 \quad \text{(A.31)}
\]
\[
\left( 2 \epsilon \partial^\mu \partial^\nu \epsilon + \epsilon^2 \partial^\mu \partial_\nu \epsilon \right) \left( \varphi^j \land \delta \varphi^j \land \delta \varphi^k \partial_k \hat{\omega}^\nu_{ij} \right) = 0. \quad \text{(A.32)}
\]

The first result (A.30) obviously reproduces the equation (A.27) we started with. The third equation (A.32), together with (A.23) and (A.26),
implies that \( \partial^\mu_k \hat{\omega}^\mu_{ij} = \partial_k [\hat{\omega}^\mu_{ij}] = \partial^\mu_k [\hat{\omega}^\mu_{ij}] \). Finally the second equations simply says that the \( \delta \)-exterior derivative of \( \hat{\omega}^\mu_{ij} \) is a total derivative:
\[
\delta \hat{\omega}^\mu_{ij} = \frac{1}{3} \partial^\nu (\delta \varphi^k \land \partial^\nu_k \hat{\omega}^\mu_{ij}) .
\] (A.33)

Note that these results can also be obtained by a direct (but tedious) calculation from the definition (A.23).

Relation between \( \hat{\omega}^\mu_{ij} \) and \( \hat{\omega}^\mu_{i_0} \)

We now prove equation (15) by a straightforward calculation:
\[
\hat{\omega}^\mu_{i_0} = \delta \hat{\theta}^\mu_{ij} = -\delta \varphi^i \land \delta \varphi^j \partial^\nu_{ij} L - \delta \varphi^i \land \partial^\nu_{ij} \delta \varphi^j \partial^\mu_{ij} L
= \hat{\omega}^\mu - \frac{1}{2} \delta \varphi^i \land \delta \varphi^j \partial^\nu_i \partial^\mu_j L + \frac{1}{2} \delta \varphi^i \land \delta \varphi^j \partial^\nu_i \partial^\mu_j L
- \partial^\nu \left( \delta \varphi^i \land \delta \varphi^j \right) \frac{1}{2} \partial^\nu_i \partial^\mu_j L
= \hat{\omega}^\mu + \frac{1}{2} \partial^\nu \left( \delta \varphi^i \land \delta \varphi^j \partial^\nu_i \partial^\mu_j \right)
, \tag{A.34}
\]
where we respectively used definitions (3) and (8) and equations (A.4), (A.26), (A.23), (A.3) and (A.25).

The covariant “Hamiltonian” equations

Let us now define the covariant Hamiltonian by:
\[
H := \partial_\mu \varphi^i \partial^\mu_i L - L . \tag{A.35}
\]

The purpose is then to prove that the equations of motion \( E_i \) are equivalent to the “covariant Hamilton equation”:
\[
\delta_i H \approx \hat{\omega}^\mu_{ij} \partial_\mu \varphi^j , \tag{A.36}
\]
where the symplectic density is given by our proposal (A.23) and \( \approx \) means on-shell.
To proceed, we can first check (using equations (A.1-A.14)) the following identity, valid on any functional $A(\varphi, \partial_\mu \varphi)$:

$$\left[ \delta_i, \partial_\mu \varphi^j \partial_\nu^j \right] A = -2 \partial_\mu \partial_\nu \varphi^i \partial_\tau^\mu \partial_\tau^\nu A.$$  \hfill (A.37)

We then take the Euler-Lagrange variation $\delta_i$ of equation (A.35), which together with the equations (A.37), (A.24) and (A.25), simplifies to:

$$\delta_i H = \partial_\mu \varphi^j \partial_\nu^j \left( \delta_i L \right) - \delta L. \hfill (A.38)$$

Therefore, using the proposal (A.23), the equation (A.38) reduces to:

$$\delta_i H = \hat{\omega}_{ij}^\mu \partial_\mu \varphi^j - E_i. \hfill (A.39)$$

This then proves that the Euler-Lagrange equations $E_i \approx 0$ can be rewritten in an Hamiltonian form using our symplectic density (A.23). Note finally that we can add an arbitrary total derivative to the Hamiltonian (A.35) (and of course to the Lagrangian) without changing the equations of motion (A.36), again due to the identity (A.7).

As explained in section 2.4, the Noether pre-symplectic structure $\hat{\omega}_{ij}^\mu_{\text{No}}$ can also be used in order to rewrite the equations of motion in an Hamiltonian form. Let us contract equation (A.39) with $\delta \varphi^i$ and then use equation (A.5) on the left hand-side:

$$\delta H - \partial_\mu \left( \delta \varphi^i \partial_\mu^i H \right) = -i \partial_\mu \hat{\omega}^\mu - \delta \varphi^i E_i, \hfill (A.40)$$

where the interior product (in $\mathcal{F}$) is defined as usual (see also equations (20) and (21)):

$$i_{\partial_\mu \varphi^i} \left( \frac{1}{2} \delta \varphi^j \wedge \delta \varphi^j A_{ij} \right) = \partial_\mu \varphi^i \delta \varphi^j A_{ij} \hfill (A.41)$$

$$i_{\partial_\mu \varphi^i} \partial_\nu \left( \frac{1}{2} \delta \varphi^j \wedge \delta \varphi^j B_{ij} \right) = \partial_\nu \left( \partial_\mu \varphi^i \delta \varphi^j B_{ij} \right). \hfill (A.42)$$

Now, if we contract equation (A.34) along the vector (of $\mathcal{F}$) $\partial_\mu \varphi^i$, we get that (see equation (A.42)):

$$i_{\partial_\mu \varphi^i} \hat{\omega}^\mu = i_{\partial_\mu \varphi^i} \hat{\omega}_{\text{No}}^\mu + \partial_\nu \left( \partial_\mu \varphi^i \wedge \delta \varphi^j \partial_\tau^\mu \partial_\tau^\nu L \right). \hfill (A.43)$$
Plugging this result into (A.40) and using the explicit definition of $H$ (A.35) in the second term of the left hand, we verify that this equation simplifies to:

$$
\delta H = -i \partial_{\nu} \hat{\omega}_{\alpha}^{\mu} - \delta \varphi^\imath E_\imath. \tag{A.44}
$$

The equation (A.44) gives an alternative way to rewrite the equations of motion. As opposed to equation (A.38), this result depends on the used Lagrangian, both in the Hamiltonian and in the Noetherian pre-symplectic structure.

The symplectic density along a gauge symmetry

Let us assume that our theory is invariant under the following gauge symmetry (see equation (55)):

$$
\delta \xi \varphi^i = \partial_{\mu} \xi^\alpha \Delta^{\mu}_\alpha + \xi^\alpha \tilde{\Delta}^i_\alpha.	ag{A.45}
$$

If we contract equation (A.15) (which is satisfied by $\hat{\omega}^\mu$) along the symmetry $\delta \xi \varphi^i$, we get that on-shell:

$$
\partial_{\mu} \left( \hat{\omega}_{\xi}^{\mu} \delta \varphi^i \delta \varphi^j \right) \approx 0. \tag{A.46}
$$

We can again use the “Abelian cascade equations” technique (equations (41-46) of reference [16]) to extract the information hidden in equation (A.46), due to the arbitrariness of the gauge parameter $\xi^\alpha(x)$. The first step is to replace $\xi^\alpha(x)$ by $\epsilon(x)\xi^\alpha(x)$ in equation (A.46):

$$
\partial_{\mu} \left( \epsilon \hat{\omega}_{\xi}^{\mu} + \partial_{\nu} \epsilon \Delta^{i\mu\nu}_\alpha \omega_{\imath j}^{\alpha} \delta \varphi^j \right) \approx 0, \tag{A.47}
$$

where $\hat{\omega}_{\xi}^{\mu} := i \delta_{\xi} \hat{\omega}^{\mu} = \hat{\omega}_{\imath j}^{\mu} \delta \varphi^i \delta \varphi^j$ and we used equation (A.47).

Then, from the arbitrary of $\epsilon(x)$ and its derivatives, we get a “cascade” of three equations:

$$
\epsilon \partial_{\mu} \hat{\omega}_{\xi}^{\mu} \approx 0 \tag{A.48}
$$

$$
\partial_{\mu} \epsilon \left( \hat{\omega}_{\xi}^{\mu} + \partial_{\nu} \left( \Delta^{i\mu\nu\alpha}_\alpha \omega_{\imath j}^{\alpha} \delta \varphi^j \right) \right) \approx 0 \tag{A.49}
$$

$$
\partial_{\mu} \partial_{\nu} \epsilon \left( \Delta^{i\mu\nu\alpha}_\alpha \omega_{\imath j}^{\alpha} \delta \varphi^j \right) \approx 0. \tag{A.50}
$$

As usual, the first equation (A.48) reproduces the equation (A.46) we started with. The second equation (A.49) shows that on-shell, $\hat{\omega}_{\xi}^{\mu}$ is a total derivative. The last equation (A.50) states that this total derivative is antisymmetric in $\mu$ and $\nu$. 
Integrability

As recalled in section 3.2, the conserved charges associated with the gauge symmetry (A.43) are computed on a boundary component $B_r$ by integrating the following equation:

$$\delta Q^{(r)}_\xi = - \int_{B_r} \delta \varphi^i \partial^\mu \varphi^j W^\mu_\xi d\Sigma_{\mu\nu}, \quad \text{(A.51)}$$

where

$$W^\mu_\xi := \xi^\alpha \Delta^i_\alpha E_i. \quad \text{(A.52)}$$

Let us then analyze the integrability of equation (A.51). Taking the $\delta$-derivative, we find:

$$\delta \left( \delta \varphi^i \partial^\mu \varphi^j W^\mu_\xi \right) = - \delta \varphi^i \wedge \partial^\mu \delta W^\mu_\xi$$

where we used equation (A.8) in the second line.

Now the second term in the right hand-side of equation (A.53) can be rewritten in the following way:

$$- \delta \varphi^i \wedge \partial^\mu \partial^\rho \left( \delta \varphi^j \varphi^j W^\mu_\xi \right) = \delta \varphi^i \wedge \left( \partial^\mu \varphi^j \delta \varphi^j \varphi^j W^\mu_\xi \right) + \delta \varphi^i \wedge \partial^\rho \delta \varphi^j \varphi^j W^\mu_\xi$$

where we used equations (A.8) and (A.4) and the antisymmetry properties $\partial^\mu \partial^\rho W^\mu_\xi = \partial^\nu \partial^\rho W^\mu_\xi$ which follow from the identity $\partial^\mu W^\mu_\xi = \partial^\nu W^\mu_\xi$ proven in the reference [11].

We can now plug the equation (A.54) into equation (A.53) dropping the total derivative (first term of equation (A.54)) since $B_r$ is a closed manifold. The integrability condition then becomes:

$$\frac{1}{2} \int_{B_r} \delta \varphi^i \partial^\mu \left( \delta \varphi^j \varphi^j W^\mu_\xi \right) d\Sigma_{\mu\nu} = 0. \quad \text{(A.55)}$$

We can now use the identity $\delta \varphi^j \varphi^j W^\mu_\xi = \partial^\mu \left( \delta \varphi^k E_k \right)$ (see equation (70)) in the result (A.55), together with the definition (A.23) to verify that the
integrability condition on-shell becomes,

\[
\int_{B_r} \left( \delta \varphi^i \partial'_i \left( \delta \xi \varphi^j \right) \wedge \omega^\mu_{jk} \delta \varphi^k - \frac{1}{2} \delta \varphi^i \wedge \delta \varphi^j \delta \xi \varphi^k \partial'_i \omega^\mu_{jk} \right) d\Sigma_{\mu\nu} \approx 0. \tag{A.56}
\]

The consequences of this result are analyzed in the subsection 3.4.

References

[1] E. Witten, Nucl. Phys. **B276** (1986) 291; C. Crnkovic and E. Witten, (1987) in 300 Years of Gravitation, eds. S.W. Hawking and W. Israel (Cambridge: Cambridge University Press) 676; C. Crnkovic, Nucl. Phys. **B288** (1987) 431, Class. Quantum Grav. **5** (1988) 1557.

[2] A. Ashtekar, L. Bombelli and O. Reula (Cordoba U.) The covariant phase space of asymptotically flat gravitational fields in 'Analysis, Geometry and Mechanics: 200 Years After Lagrange' 417, Ed. by M. Francaviglia, D. Holm, North-Holland, Amsterdam (1990).

[3] J. Lee and R.M. Wald, J. Math. Phys. **31** (1990) 725.

[4] J. Jezierski and J. Kijowski, Gen. Rel. Grav. **22** (1990) 1283; J. Kijowski, Gen. Rel. Grav. **29** (1997) 307.

[5] R.M. Wald, Phys. Rev **D48** (1993) 3427, gr-qc/9307038. V. Iyer, R.M. Wald, Phys. Rev **D50** (1994) 846, gr-qc/9403028.

[6] A. Ashtekar, A. Corichi and K. Krasnov, gr-qc/9905089. A. Ashtekar, C. Beetle and S. Fairhurst, Class.Quant.Grav. **17** (2000) 253, gr-qc/9907068.

[7] S. Carlip, Class. Quant. Grav. **16** (1999) 3327, gr-qc/9906126.

[8] R.M. Wald and A. Zoupas, Phys. Rev. **D61** (2000) 084027, gr-qc/9911095.

[9] C-M. Chen and J. Nester, A symplectic Hamiltonian Derivation of Quasilocal Energy Momentum for GR, gr-qc/0001088.

[10] M. Francaviglia and M. Raiteri, Class. Quant. Grav. **19** (2002) 237, gr-qc/0107074. G. Allemandi, M. Francaviglia and M. Raiteri, “The first law of isolated horizons via Noether theorem”, gr-qc/0110104.
[11] S. Silva, Nucl. Phys. B558 (1999) 391, hep-th/9809109.
[12] M. Henneaux, Phys. Lett. B 238 (1990) 299.
[13] B. Julia and S. Silva, JHEP 00001:026 (2000), hep-th/9911037.
[14] H.A. Kastrup, Phys. Rept. 101 (1983) 1.
[15] I.V. Kanatchikov, Int. J. Theor. Phys. 40 (2001) 1121, gr-qc/0012074.
[16] B. Julia and S. Silva, Class. Qu. Grav., 15 (1998) 2173, gr-qc/9804029.
[17] B. Julia and S. Silva, Class. Qu. Grav., 17 (2000) 4733, gr-qc/0005127.
[18] E. Cremmer, B. Julia and J. Scherck, Phys.Lett. B76 (1978) 409.
[19] L. Castellani, P. Fre, F. Giani, K. Pilch and P. van Nieuwenhuizen, Ann. Phys. 146 (1983) 35.
[20] H. Nastase, D. Vaman and P. van Nieuwenhuizen, Phys. Let. B 469 (1999) 96, hep-th/9905075, Nucl. Phys. B581 (2000) 179, hep-th/9911238.
[21] R. Penrose, Proc. Roy. Soc. (London) A284 (1965) 159.
[22] A. Ashtekar and J.D. Romano, Class. Quant. Grav. 9 (1992) 1069.
[23] G. Barnich and F. Brandt, “Covariant theory of asymptotic symmetries, conservation laws and central charges”, hep-th/0111246.
[24] T. Regge and C. Teitelboim, Ann. of Phys. 88 (1974) 286.
[25] J.D. Brown and M. Henneaux, J. Math. Phys. 27 (1986) 489.
[26] M. Henneaux, B. Julia and S. Silva, Nucl. Phys. B563 (1999) 448, hep-th/9904003.
[27] S. Silva, Charges et Algèbres liées aux symétries de jauge: Construction Lagrangienne en (Super)Gravités et Mécanique des fluides, Thèse de Doctorat, Septembre 99, LPT-ENS.
[28] S. Silva, Class. Qu. Grav., 18 (2001) 1577, hep-th/0010093, PRHEP-tmr2000/012, Talk presented at TMR-conference “Nonperturbative Quantum Effects 2000”, hep-th/0010099.

33
[29] B. Julia Proceedings of International Conference on Supersymmetry and Quantum Field Theory: D.V. Volkov Memorial Conference (SSQFT 2000), Kharkov, Ukraine, [hep-th/0104231].

[30] S. Silva, “Black hole entropy and thermodynamics from symmetries”, [hep-th/0204179].