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CANONICAL BASES AND AFFINE
HECKE ALGEBRAS OF TYPE D

P. SHAN, M. VARAGNOLO, E. VASSEROT

Abstract. We prove a conjecture of Miemietz and Kashiwara on canonical bases
and branching rules of affine Hecke algebras of type D. The proof is similar to the
proof of the type B case in [VV].

INTRODUCTION

Let $f$ be the negative part of the quantized enveloping algebra of type $A^{(1)}$.
Lusztig’s description of the canonical basis of $f$ implies that this basis can be natu-
really identified with the set of isomorphism classes of simple objects of a category of
modules of the affine Hecke algebras of type $A$. This identification was mentioned
in [G], and was used in [A]. More precisely, there is a linear isomorphism between
$f$ and the Grothendieck group of finite dimensional modules of the affine Hecke
algebras of type $A$, and it is proved in [A] that the induction/restriction functors
for affine Hecke algebras are given by the action of the Chevalley generators and
their transposed operators with respect to some symmetric bilinear form on $f$.

The branching rules for affine Hecke algebras of type $B$ have been investigated
quite recently, see [E], [EK1,2,3], [M] and [VV]. In particular, in [E], [EK1,2,3] an
analogue of Ariki’s construction is conjectured and studied for affine Hecke algebras
of type $B$. Here $f$ is replaced by a module $\theta V(\lambda)$ over an algebra $\theta B$. More precisely it
is conjectured there that $\theta V(\lambda)$ admits a canonical basis which is naturally identified
with the set of isomorphism classes of simple objects of a category of modules of
the affine Hecke algebras of type $B$. Further, in this identification the branching
rules of the affine Hecke algebras of type $B$ should be given by the $\theta B$-action on
$\theta V(\lambda)$. This conjecture has been proved [VV]. It uses both the geometric picture
introduced in [E] (to prove part of the conjecture) and a new kind of graded algebras
similar to the KLR algebras from [KL], [R].

A similar description of the branching rules for affine Hecke algebras of type $D$
has also been conjectured in [KM]. In this case $f$ is replaced by another module $\vartheta V$
over the algebra $\theta B$ (the same algebra as in the type $B$ case). The purpose of this
paper is to prove the type $D$ case. The method of the proof is the same as in [VV].
First we introduce a family of graded algebras $\vartheta R_m$ for $m$ a non negative integer.
They can be viewed as the Ext-algebras of some complex of constructible sheaves
naturally attached to the Lie algebra of the group $SO(2m)$, see Remark 2.8. This
complex enters in the Kazhdan-Lusztig classification of the simple modules of the
affine Hecke algebra of the group $Spin(2m)$. Then we identify $\vartheta V$ with the sum of
the Grothendieck groups of the graded algebras $\vartheta R_m$.

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The plan of the paper is the following. In Section 1 we introduce a graded algebra \( \mathcal{R}(\Gamma)_{\nu} \). It is associated with a quiver \( \Gamma \) with an involution \( \theta \) and with a dimension vector \( \nu \). In Section 2 we consider a particular choice of pair \( (\Gamma, \theta) \). The graded algebras \( \mathcal{R}(\Gamma)_{\nu} \) associated with this pair \( (\Gamma, \theta) \) are denoted by the symbol \( \mathcal{R}_{m} \).

Next we introduce the affine Hecke algebra of type D, more precisely the affine Hecke algebra associated with the group \( SO(2m) \), and we prove that it is Morita equivalent to \( \mathcal{R}_{m} \). In Section 3 we categorify the module \( \mathcal{V} \) from [KM] using the graded algebras \( \mathcal{R}_{m} \), see Theorem 3.28. The main result of the paper is Theorem 3.33.

0. Notation

0.1. Graded modules over graded algebras. Let \( k \) be an algebraically closed field of characteristic 0. By a graded \( k \)-algebra \( R = \bigoplus_d R_d \) we'll always mean a \( \mathbb{Z} \)-graded associative \( k \)-algebra. Let \( R\text{-mod} \) be the category of finitely generated graded \( R \)-modules, \( R\text{-fmod} \) be the full subcategory of finite-dimensional graded modules and \( R\text{-proj} \) be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We'll abbreviate

\[
K(R) = [R\text{-proj}], \quad G(R) = [R\text{-fmod}].
\]

Here \( [\mathcal{C}] \) denotes the Grothendieck group of an exact category \( \mathcal{C} \). Assume that the \( k \)-vector spaces \( R_d \) are finite dimensional for each \( d \). Then \( K(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in \( R\text{-proj} \), and \( G(R) \) is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in \( R\text{-fmod} \). Given an object \( M \) of \( R\text{-proj} \) or \( R\text{-fmod} \) let \([M]\) denote its class in \( K(R) \), \( G(R) \) respectively. When there is no risk of confusion we abbreviate \( M = [M] \). We'll write \([M : N]\) for the composition multiplicity of the \( R \)-module \( N \) in the \( R \)-module \( M \). Consider the ring \( A = \mathbb{Z}[v, v^{-1}] \). If the grading of \( R \) is bounded below then the \( A \)-modules \( K(R), G(R) \) are free. Here \( A \) acts on \( G(R) \), \( K(R) \) as follows

\[
vM = M[1], \quad v^{-1}M = M[-1].
\]

For any \( M, N \) in \( R\text{-mod} \) let

\[
\text{hom}_R(M, N) = \bigoplus_d \text{Hom}_R(M, N[d])
\]

be the \( \mathbb{Z} \)-graded \( k \)-vector space of all \( R \)-module homomorphisms. If \( R = k \) we'll omit the subscript \( R \) in hom's and in tensor products. For any graded \( k \)-vector space \( M = \bigoplus_d M_d \) we'll write

\[
g\text{dim}(M) = \sum_d v^d \text{dim}(M_d),
\]

where \( \text{dim} \) is the dimension over \( k \).
0.2. Quivers with involutions. Recall that a quiver $Γ$ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where $I$ is the set of vertices, $H$ is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of $h$ respectively. Note that the set $I$ may be infinite. We’ll assume that no arrow may join a vertex to itself. For each $i, j \in I$ we write

$$H_{i,j} = \{h \in H; h' = i, h'' = j\}.$$  

We’ll abbreviate $i \to j$ if $H_{i,j} \neq \emptyset$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$  

An involution $θ$ on $Γ$ is a pair of involutions on $I$ and $H$, both denoted by $θ$, such that the following properties hold for each $h \in H$:

- $θ(h)' = θ(h''')$ and $θ(h)'' = θ(h')$,
- $θ(h') = h''$ iff $θ(h) = h$.

We’ll always assume that $θ$ has no fixed points in $I$, i.e., there is no $i ∈ I$ such that $θ(i) = i$. To simplify we’ll say that $θ$ has no fixed point. Let

$$θ_{HI} = \{ν = \sum_{i} ν_i ∈ NI : νθ(i) = ν_i, \forall i\}.$$  

For any $ν ∈ θ_{HI}$ set $|ν| = \sum_ν ν_i$. It is an even integer. Write $|ν| = 2m$ with $m ∈ N$. We’ll denote by $θ_{Iν}$ the set of sequences

$$i = (i_{1-m}, \ldots, i_{m-1}, i_m)$$

of elements in $I$ such that $θ(i_j) = i_{j-1}$ and $\sum_k i_k = ν$. For any such sequence $i$ we’ll abbreviate $θ(i) = (θ(i_{1-m}), \ldots, θ(i_{m-1}), θ(i_m))$. Finally, we set

$$θ_{Iν} = \bigcup_ν θ_{Iν}, \quad ν ∈ θ_{HI}, \quad |ν| = 2m.$$  

0.3. The wreath product. Given a positive integer $m$, let $Σ_m$ be the symmetric group, and $Z_2 = \{-1, 1\}$. Consider the wreath product $W_m = Σ_m \wr Z_2$. Write $s_1, \ldots, s_{m-1}$ for the simple reflections in $Σ_m$. For each $l = 1, 2, \ldots, m$ let $ε_l \in (Z_2)^m$ be $-1$ placed at the $l$-th position. There is a unique action of $W_m$ on the set $\{1−m, \ldots, m−1, m\}$ such that $Σ_m$ permutes $1, 2, \ldots, m$ and such that $ε_l$ fixes $k$ if $k \neq l, 1−l$ and switches $l$ and $1−l$. The group $W_m$ acts also on $θ_{Iν}$. Indeed, view a sequence $i$ as the map

$$\{1 − m, \ldots, m − 1, m\} \to I, \quad l \mapsto i_l.$$  

Then we set $w(i) = i \circ w^{-1}$ for $w ∈ W_m$. For each $ν$ we fix once for all a sequence $ι_ν = (ι_{1-m}, \ldots, ι_m) ∈ θ_{Iν}$. Let $W_e$ be the centralizer of $ι_ν$ in $W_m$. Then there is a bijection

$$W_e \backslash W_m \to θ_{Iν}, \quad W_e w \to w^{-1}(ι_ν).$$  

Now, assume that $m > 1$. We set $S_0 = ε_1 s_1 ε_1$. Let $σW_m$ be the subgroup of $W_m$ generated by $s_0, \ldots, s_{m-1}$. We’ll regard it as a Weyl group of type $D_m$ such that $s_0, \ldots, s_{m-1}$ are the simple reflections. Note that $W_e$ is a subgroup of $σW_m$. Indeed, if $W_e \not\subset σW_m$ there should exist $l$ such that $ε_l$ belongs to $W_e$. This would imply that $ι_l = θ(ι_l)$, contradicting the fact that $θ$ has no fixed point. Therefore $θ_{Iν}$ decomposes into two $σW_m$-orbits. We’ll denote them by $θ_{Iν}^+$ and $θ_{Iν}^-$. For $m = 1$ we set $σW_1 = \{ε\}$ and we choose again $θ_{Iν}^+$ and $θ_{Iν}^-$ in an obvious way.
1. The graded $k$-algebra $\mathcal{R}(\Gamma)_\nu$

Fix a quiver $\Gamma$ with set of vertices $I$ and set of arrows $H$. Fix an involution $\theta$ on $\Gamma$. Assume that $\Gamma$ has no 1-loops and that $\theta$ has no fixed points. Fix a dimension vector $\nu \neq 0$ in $\mathbb{N}I$. Set $|\nu| = 2m$.

1.1. Definition of the graded $k$-algebra $\mathcal{R}(\Gamma)_\nu$. Assume that $m > 1$. We define a graded $k$-algebra $\mathcal{R}(\Gamma)_\nu$ with 1 generated by $1$, $\tau_i$, $\sigma_k$, with $i \in I^\nu$, $l = 1, 2, \ldots, m$, $k = 0, 1, \ldots, m - 1$ modulo the following defining relations

(a) $1_i 1_V = \delta_{iV} 1_i$, \quad $\sigma_k 1_i = 1_{s_k k} \sigma_k$, \quad $\tau_i 1_i = 1_i \tau_i$,

(b) $\tau_i \tau_i' = \tau_i' \tau_i$,

(c) $\sigma_k^2 1_i = Q_{i s_k \theta(k)}(\sigma_{s_k(k)} \neg\sigma_k) 1_i$,

(d) $\sigma_k \sigma_k' = \sigma_k' \sigma_k$ if $1 \leq k < k' - 1 < m - 1$ or $0 = k < k' \neq 2$,

(e) $(\sigma_{s_k(k)} \sigma_k \sigma_{s_k(k)} - \sigma_k \sigma_{s_k(k)} \sigma_k) 1_i = \begin{cases} Q_{i s_k \theta(k)}(\sigma_{s_k(k)} \neg\sigma_k) - Q_{i s_k \theta(k)}(\sigma_{s_k(k)} \neg\sigma_k + 1) & \text{if } i_k = i_{s_k(k)} + 1, \\ 0 & \text{else,} \end{cases}$

(f) $(\sigma_k \sigma_i - \sigma_{s_k(k)} \sigma_{s_k(k)}) 1_i = \begin{cases} -1_i & \text{if } l = k, i_k = i_{s_k(k)}, \\ 1_i & \text{if } l = s_k(k), i_k = i_{s_k(k)}, \\ 0 & \text{else.} \end{cases}$

Here we have set $\tau_{1-i} = -\tau_i$ and

\[
Q_{i,j}(u, v) = \begin{cases} (-1)^{h_{i,j}}(u - v)^{-i - j} & \text{if } i \neq j, \\ 0 & \text{else.} \end{cases}
\]

If $m = 0$ we set $\mathcal{R}(\Gamma)_0 = k \oplus k$. If $m = 1$ then we have $\nu = i + \theta(i)$ for some $i \in I$. Write $i = i \theta(i)$, and

$\mathcal{R}(\Gamma)_\nu = k[\tau_i]l_1 \oplus k[\tau_i]l_{\theta(i)}$.

We’ll abbreviate $\sigma_l k = \sigma_{l+1} l_1$ and $\tau_{l+1} = \tau_l l_1$. The grading on $\mathcal{R}(\Gamma)_0$ is the trivial one. For $m \geq 1$ the grading on $\mathcal{R}(\Gamma)_\nu$ is given by the following rules:

\[
\begin{align*}
\deg(l_1) &= 0, \\
\deg(\tau_{l+1}) &= 2, \\
\deg(\sigma_l k) &= -i_k \tau_{s_k(k)}.
\end{align*}
\]

We define $\omega$ to be the unique involution of the graded $k$-algebra $\mathcal{R}(\Gamma)_\nu$ which fixes $1$, $\tau_i$, $\sigma_k$. We set $\omega$ to be identity on $\mathcal{R}(\Gamma)_0$.

1.2. Relation with the graded $k$-algebra $\mathcal{R}(\Gamma)_{\lambda, \nu}$. A family of graded $k$-algebra $\mathcal{R}(\Gamma)_{\lambda, \nu}$ was introduced in [VV, sec. 5.1], for $\lambda$ an arbitrary dimension vector in $N I$. Here we’ll only consider the special case $\lambda = 0$, and we abbreviate $\mathcal{R}(\Gamma)_\nu = \mathcal{R}(\Gamma)_{0, \nu}$. Recall that if $\nu \neq 0$ then $\mathcal{R}(\Gamma)_\nu$ is the graded $k$-algebra with 1 generated
by \( l_1, x_1, \sigma_k, \pi_1 \), with \( i \in \mathcal{I}^\nu, l = 1, 2, \ldots, m, k = 1, \ldots, m - 1 \) such that \( l_1, x_1 \) and \( \sigma_k \) satisfy the same relations as before and
\[
\begin{align*}
\sigma_1^2 &= 1, \quad \pi_1 l_1 \pi_1 = \varepsilon_{i_1}, \quad \pi_1 x_1 \pi_1 = \varepsilon_{x_1(l)}, \\
(\pi_1 \sigma_1)^2 &= (\sigma_1 \pi_1)^2, \quad \pi_1 \sigma_k \pi_1 = \sigma_k \text{ if } k \neq 1.
\end{align*}
\]

If \( \nu = 0 \) then \( \theta \mathcal{R}(\Gamma)_0 = k \). The grading is given by setting \( \deg(l_1), \deg(x_1), \\deg(\sigma_1 ) \) to be as before and \( \deg(\pi_1 l_1) = 0 \). In the rest of Section 1 we'll assume \( m > 0 \). Then there is a canonical inclusion of graded \( k \)-algebras
\[
\tag{1.2} \circ \mathcal{R}(\Gamma)_\nu \subset \theta \mathcal{R}(\Gamma)_\nu
\]
such that \( l_1, x_1, \sigma_k \mapsto l_1, x_1, \sigma_k \) for \( i \in \mathcal{I}^\nu, l = 1, \ldots, m, k = 1, \ldots, m - 1 \) and such that \( \sigma_0 \mapsto \pi_1 \sigma_1 \pi_1 \). From now on we'll write \( \sigma_0 = \pi_1 \sigma_1 \pi_1 \) whenever \( m > 1 \). The assignment \( x \mapsto \pi_1 x \pi_1 \) defines an involution of the graded \( k \)-algebra \( \theta \mathcal{R}(\Gamma)_\nu \) which normalizes \( \circ \mathcal{R}(\Gamma)_\nu \). Thus it yields an involution
\[
\gamma : \circ \mathcal{R}(\Gamma)_\nu \to \circ \mathcal{R}(\Gamma)_\nu.
\]

Let \( \langle \gamma \rangle \) be the group of two elements generated by \( \gamma \). The smash product \( \circ \mathcal{R}(\Gamma)_\nu \rtimes \langle \gamma \rangle \) is a graded \( k \)-algebra such that \( \deg(\gamma) = 0 \). There is an unique isomorphism of graded \( k \)-algebras
\[
\tag{1.3} \circ \mathcal{R}(\Gamma)_\nu \rtimes \langle \gamma \rangle \to \theta \mathcal{R}(\Gamma)_\nu
\]
which is identity on \( \circ \mathcal{R}(\Gamma)_\nu \) and which takes \( \gamma \) to \( \pi_1 \).

1.3. The polynomial representation and the PBW theorem. For any \( i \) in \( \mathcal{I}^\nu \) let \( \theta F_i \) be the subalgebra of \( \circ \mathcal{R}(\Gamma)_\nu \) generated by \( l_1 \) and \( x_1 \) with \( l = 1, 2, \ldots, m \).

It is a polynomial algebra. Let
\[
\theta F_\nu = \bigoplus_{i \in \mathcal{I}^\nu} \theta F_i.
\]
The group \( W_m \) acts on \( \theta F_\nu \) via \( w(x_1) = \varepsilon_{w(l), w(l)} \) for any \( w \in W_m \). Consider the fixed points set
\[
\circ S_\nu = (\theta F_\nu)^{W_m}.
\]

Regard \( \circ \mathcal{R}(\Gamma)_\nu \) and \( \text{End}(\theta F_\nu) \) as \( \theta F_\nu \)-algebras via the left multiplication. In [VV, prop. 5.4] is given an injective graded \( \theta F_\nu \)-algebra morphism \( \circ \mathcal{R}(\Gamma)_\nu \to \text{End}(\theta F_\nu) \).

It restricts via (1.2) to an injective graded \( \theta F_\nu \)-algebra morphism
\[
\circ \mathcal{R}(\Gamma)_\nu \to \text{End}(\theta F_\nu).
\]

Next, recall that \( \circ W_m \) is the Weyl group of type D_m with simple reflections \( s_0, \ldots, s_{m-1} \).

For each \( w \) in \( \circ W_m \) we choose a reduced decomposition \( \check{w} \) of \( w \). It has the following form
\[
w = s_{k_1} s_{k_2} \cdots s_{k_r}, \quad 0 \leq k_1, k_2, \ldots, k_r \leq m - 1.
\]

We define an element \( \sigma_{\check{w}} \) in \( \circ \mathcal{R}(\Gamma)_\nu \) by
\[
(1.4) \quad \sigma_{\check{w}} = \sum_i 1_i \sigma_{\check{w}}, \quad 1_i \sigma_{\check{w}} = \begin{cases} 1_i & \text{if } r = 0 \\ 1_i \sigma_{s_{k_1}} \sigma_{s_{k_2}} \cdots \sigma_{s_{k_r}} & \text{else}, \end{cases}
\]

Observe that the element \( \sigma_{\check{w}} \) may depend on the choice of the reduced decomposition \( \check{w} \).
1.4. Proposition. The $\mathbf{k}$-algebra $\mathfrak{g}R(\Gamma)_\nu$ is a free (left or right) $\mathfrak{g}F_\nu$-module with basis $\{\sigma_w; w \in \mathfrak{g}W_m\}$. Its rank is $2^{m-1}m!$. The operator $1_\mathfrak{g}\sigma_w$ is homogeneous and its degree is independent of the choice of the reduced decomposition $\tilde{w}$.

Proof: The proof is the same as in [VV, prop. 5.5]. First, we filter the algebra $\mathfrak{g}R(\Gamma)_\nu$ with $1_\mathfrak{g}$, $\sigma_i$ in degree 0 and $\sigma_j$ in degree 1. The Nil Hecke algebra of type $D_m$ is the $\mathbf{k}$-algebra $\mathfrak{g}NH_m$ generated by $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ with relations

$$\sigma_k\sigma_l = \sigma_l\sigma_k = \sigma_l\sigma_k\sigma_l, \quad \sigma_k^2 = 0.$$ 

We can form the semidirect product $\mathfrak{g}F_\nu \rtimes \mathfrak{g}NH_m$, which is generated by $1_\mathfrak{g}$, $\tilde{\sigma}_i$, $\tilde{\sigma}_k$ with the relations above and

$$\sigma_k\sigma_l = \sigma_l\sigma_k = \sigma_l\sigma_k\sigma_l, \quad \sigma_k^2 = 0.$$ 

One proves as in [VV, prop. 5.5] that the map

$$\mathfrak{g}F_\nu \rtimes \mathfrak{g}NH_m \to \text{gr}(\mathfrak{g}R(\Gamma)_\nu), \quad 1_\mathfrak{g} \mapsto 1_\mathfrak{g}, \quad \tilde{\sigma}_i \mapsto \tilde{\sigma}_i, \quad \tilde{\sigma}_k \mapsto \sigma_k.$$ 

is an isomorphism of $\mathbf{k}$-algebras.

Let $\mathfrak{g}F'_\nu = \bigoplus_i \mathfrak{g}F_i'$, where $\mathfrak{g}F_i'$ is the localization of the ring $\mathfrak{g}F_i$ with respect to the multiplicative system generated by

$$\{\sigma_i, \sigma_i; 1 \leq i \neq l \leq m\} \cup \{\sigma_i; l = 1, 2, \ldots, m\}.$$ 

1.5. Corollary. The inclusion $\mathfrak{g}R(\Gamma)_\nu \subset \text{End}(\mathfrak{g}F_\nu)$ yields an isomorphism of $\mathfrak{g}F_\nu'$-algebras $\mathfrak{g}F'_\nu \otimes_{\mathfrak{g}F_\nu} \mathfrak{g}R(\Gamma)_\nu \to \mathfrak{g}F'_\nu \rtimes \mathfrak{g}W_m$, such that for each $i$ and each $l = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots, m - 1$ we have

$$1_\mathfrak{g} \mapsto 1_\mathfrak{g}, \quad \tilde{\sigma}_i \mapsto \tilde{\sigma}_i, \quad \sigma_k \mapsto \sigma_k.$$ 

Proof: Follows from [VV, cor. 5.6] and Proposition 1.4.

Restricting the $\mathfrak{g}F_\nu'$-action on $\mathfrak{g}R(\Gamma)_\nu$ to the $\mathbf{k}$-subalgebra $\mathfrak{g}S_\nu$ we get a structure of graded $\mathfrak{g}S_\nu$-algebra on $\mathfrak{g}R(\Gamma)_\nu$.

1.6. Proposition. (a) $\mathfrak{g}S_\nu$ is isomorphic to the center of $\mathfrak{g}R(\Gamma)_\nu$.

(b) $\mathfrak{g}R(\Gamma)_\nu$ is a free graded module over $\mathfrak{g}S_\nu$ of rank $(2^{m-1}m!)^2$.

Proof: Part (a) follows from Corollary 1.5. Part (b) follows from (a) and Proposition 1.4.
2. Affine Hecke algebras of type D

2.1. Affine Hecke algebras of type D. Fix \( p \in \mathbb{k}^\times \). For any integer \( m \geq 0 \) we define the extended affine Hecke algebra \( H_m \) of type \( D_m \) as follows. If \( m > 1 \) then \( H_m \) is the \( \mathbb{k} \)-algebra with 1 generated by

\[
T_k, \quad X^\pm_1, \quad k = 0, 1, \ldots, m - 1, \quad l = 1, 2, \ldots, m
\]

satisfying the following defining relations:

(a) \( X_lX_{l'} = X_{l'}X_l \),
(b) \( T_kT_{s_k(k)}T_k = T_{s_k(k)}T_kT_{s_k(k)}, T_kT_{k'} = T_{k'}T_k \) if \( 1 \leq k < k' - 1 \) or \( k = 0, k' \neq 2 \),
(c) \( (T_k - p)(T_k + p^{-1}) = 0 \),
(d) \( T_0X_{l}^{-1}T_0 = X_2, \quad T_kX_kT_k = X_{s_k(k)} \) if \( k \neq 0, T_kX_l = X_lT_k \) if \( k \neq 0, l, l - 1 \) or \( k = 0, l \neq 1, 2 \).

Finally, we set \( H_0 = \mathbb{k} \oplus \mathbb{k} \) and \( H_1 = \mathbb{k}[X^\pm_1] \).

2.2. Remarks. (a) The extended affine Hecke algebra \( H^B_m \) of type \( B_m \) with parameters \( p, q \in \mathbb{k}^\times \) such that \( q = 1 \) is generated by \( P, T_k, X^\pm_1, k = 1, \ldots, m - 1, l = 1, \ldots, m \) such that \( T_k, X^\pm_1 \) satisfy the relations as above and

\[
P^2 = 1, \quad (PT_1)^2 = (T_1P)^2, \quad PT_k = T_kP \quad \text{if} \quad k \neq 1, \quad PX_1^{-1}P = X_1, \quad PX_l = X_lP \quad \text{if} \quad l \neq 1.
\]

See e.g., [VV, sec. 6.1]. There is an obvious \( \mathbb{k} \)-algebra embedding \( H_m \subset H^B_m \). Let \( \gamma \) denote also the involution \( H_m \rightarrow H_m, a \mapsto PaP \). We have a canonical isomorphism of \( \mathbb{k} \)-algebras

\[
H_m \rtimes \langle \gamma \rangle \simeq H^B_m.
\]

Compare Section 1.2.

(b) Given a connected reductive group \( G \) we call affine Hecke algebra of \( G \) the Hecke algebra of the extended affine Weyl group \( W \times P \), where \( W \) is the Weyl group of \( (G, T) \), \( P \) is the group of characters of \( T \), and \( T \) is a maximal torus of \( G \). Then \( H_m \) is the affine Hecke algebra of the group \( SO(2m) \). Let \( H^B_m \) be the affine Hecke algebra of the group \( Spin(2m) \). It is generated by \( H_m \) and an element \( X_0 \) such that

\[
X_0^2 = X_1X_2 \ldots X_m, \quad T_kX_0 = X_0T_k \quad \text{if} \quad k \neq 0, \quad T_0X_0X_1^{-1}X_2^{-1}T_0 = X_0.
\]

Thus \( H_m \) is the fixed point subset of the \( \mathbb{k} \)-algebra automorphism of \( H^c_m \) taking \( T_k, X_l \) to \( T_k, (-1)^{l(a)}X_l \) for all \( k, l \). Therefore, if \( p \) is not a root of 1 the simple \( H_m \)-modules can be recovered from the Kazhdan-Lusztig classification of the simple \( H^c_m \)-modules via Clifford theory, see e.g., [Re].

2.3. Intertwiners and blocks of \( H_m \). We define

\[
A = \mathbb{k}[X^\pm_1, X^\pm_2, \ldots, X^\pm_m], \quad A' = A[\Sigma^{-1}], \quad H'_m = A' \otimes_A H_m,
\]
where $\Sigma$ is the multiplicative set generated by 
\[
1 - X_i X_i^{\pm 1}, \quad 1 - p^2 X_i X_i^{\pm 1}, \quad l \neq l'.
\]

For $k = 0, \ldots, m - 1$ the intertwiner $\varphi_k$ is the element of $H'_m$ given by the following formulas
\[
(2.1) \quad \varphi_k - 1 = \frac{X_k - X_s(k)}{pX_k - p^{-1}X_s(k)} (T_k - p).
\]
The group $W_m$ acts on $A'$ as follows
\[
(s_{k0}) (X_1, \ldots, X_m) = a(X_1, \ldots, X_k+1, X_k, \ldots, X_m) \text{ if } k \neq 0,
\]
\[
(s_{00}) (X_1, \ldots, X_m) = a(X_1^{-1}, X_1^{-1}, \ldots, X_m).
\]

There is an isomorphism of
\[
A' \times W_m \to H'_m, \quad s_k \mapsto \varphi_k.
\]
The semi-direct product group $Z \times Z_2 = Z \times \{-1, 1\}$ acts on $k^\times$ by $(n, z) \mapsto z^n p^m$. Given a $Z \times Z_2$-invariant subset $I$ of $k^\times$ we denote by $H_{m,I}$ the category of all $H_m$-modules such that the action of $X_1, X_2, \ldots, X_m$ is locally finite with eigenvalues in $I$. We associate to the set $I$ and to the element $p \in k^\times$ a quiver $\Gamma$ as follows. The set of vertices is $I$, and there is one arrow $p^2 i \to i$ whenever $i$ lies in $I$. We equip $\Gamma$ with an involution $\theta$ such that $\theta(i) = i^{-1}$ for each vertex $i$ and such that $\theta$ takes the arrow $p^2 i \to i$ to the arrow $i^{-1} \to p^{-2} i^{-1}$. We'll assume that the set $I$ does not contain 1 nor $-1$ and that $p \neq 1, -1$. Thus the involution $\theta$ has no fixed points and no arrow may join a vertex of $\Gamma$ to itself.

2.4. Remark. We may assume that $I = \pm \{p^n; n \in \Z_{\text{odd}}\}$. See the discussion in [KM]. Then $\Gamma$ is of type $\Lambda_{\infty}$ if $p$ has infinite order and $\Gamma$ is of type $\Lambda^{(1)}_r$ if $p^2$ is a primitive $r$-th root of unity.

2.5. $H_m$-modules versus $R_m$-modules. Assume that $m \geq 1$. We define the graded $k$-algebra 
\[
\mathcal{R}_{I,m} = \bigoplus_\nu \mathcal{R}_{I,\nu}, \quad \mathcal{R}_{I,\nu} = \mathcal{R}(I)_\nu, \quad \mathcal{R}_{I,\nu} = \mathcal{R}(I)_\nu,
\]
\[
\mathcal{R}_m = \bigcup_\nu \mathcal{R}_{I,\nu},
\]
where $\nu$ runs over the set of all dimension vectors in $\mathcal{R}_{I,m}$ such that $|\nu| = 2m$. When there is no risk of confusion we abbreviate
\[
\mathcal{R}_{\nu} = \mathcal{R}_{I,\nu}, \quad \mathcal{R}_m = \mathcal{R}_{I,m}, \quad \mathcal{R}_0 = \mathcal{R}_{I,0}, \quad \mathcal{R}_m = \mathcal{R}_{I,m}.
\]

Note that $\mathcal{R}_{\nu}$ and $\mathcal{R}_m$ are the same as in [VV, sec. 6.4], with $\lambda = 0$. Note also that the $k$-algebra $\mathcal{R}_m$ may not have 1, because the set $I$ may be infinite. We define $\mathcal{R}_{m,-\text{Mod}_0}$ as the category of all (non-graded) $\mathcal{R}_m$-modules such that the elements $x_1, x_2, \ldots, x_m$ act locally nilpotently. Let $\mathcal{R}_m^{-\text{ftMod}_0}$ and $H_m^{-\text{ftMod}}_m$ be the full subcategories of finite dimensional modules in $\mathcal{R}_m^{-\text{Mod}_0}$ and $H_m^{-\text{Mod}}_m$ respectively. Fix a formal series $f(\varpi)$ in $k[[\varpi]]$ such that $f(\varpi) = 1 + \varpi$ modulo $(\varpi^2)$. 
2.6. Theorem. We have an equivalence of categories

\[ \mathcal{O}R_m \text{-} \text{Mod}_0 \rightarrow \mathcal{H}_m \text{-} \text{Mod}_I, \quad M \mapsto M \]

which is given by

(a) \( X_l \) acts on \( 1_k M \) by \( \iota_l^{-1} f(x_l) \) for each \( l = 1, 2, \ldots, m \),
(b) if \( m > 1 \) then \( T_k \) acts on \( 1_k M \) as follows for each \( k = 0, 1, \ldots, m-1 \),

\[
\begin{align*}
\frac{(pf(x_k) - p^{-1}f(x_{ss(k)}))(x_k - x_{ss(k)})}{f(x_k) - f(x_{ss(k)})} \sigma_k + p & \quad \text{if } i_{ss(k)} = i_k, \\
\frac{f(x_k) - f(x_{ss(k)})}{(p^{-1}f(x_k) - p_f(x_{ss(k)}))(x_k - x_{ss(k)})} \sigma_k + \frac{(p^{-2} - 1)f(x_{ss(k)})}{pf(x_k) - p^{-1}f(x_{ss(k)})} & \quad \text{if } i_{ss(k)} = p^2 i_k, \\
\frac{p_if(x_k) - p^{-1}i_{ss(k)}f(x_{ss(k)})}{i_k f(x_k) - i_{ss(k)}f(x_{ss(k)})} \sigma_k + \frac{(p^{-1} - p)i_k f(x_{ss(k)})}{i_k f(x_k) - i_{ss(k)}f(x_{ss(k)})} & \quad \text{if } i_{ss(k)} \neq i_k, p^2 i_k.
\end{align*}
\]

Proof: This follows from [VV, thm. 6.5] by Section 1.2 and Remark 2.2(a). One can also prove it by reproducing the arguments in loc. cit. by using (1.5) and (2.1).

2.7. Corollary. There is an equivalence of categories

\[ \Psi : \mathcal{O}R_m \text{-} \text{fMod}_0 \rightarrow \mathcal{H}_m \text{-} \text{fMod}_I, \quad M \mapsto M. \]

2.8. Remarks. (a) Let \( g \) be the Lie algebra of \( G = SO(2m) \). Fix a maximal torus \( T \subset G \). The group of characters of \( T \) is the lattice \( \bigoplus_{l=1}^{m} \mathbb{Z} \epsilon_l \), with Bourbaki’s notation. Fix a dimension vector \( \nu \in \mathfrak{g} \text{-} \mathbb{N} \). Recall the sequence \( \mathbf{i}_c = (i_{1-m}, \ldots, i_{m-1}, i_m) \) from Section 0.3. Let \( g \in T \) be the element such that \( \epsilon_l(g) = i_l^{-1} \) for each \( l = 1, 2, \ldots, m \). Recall also the notation \( \mathfrak{g}_\nu, \mathfrak{V}, \mathfrak{V}^+, \mathfrak{V}^- \), and \( \mathfrak{g}_\nu \text{-} \mathfrak{g}_\nu \) from [VV]. Then \( \mathfrak{V} \) is an object of \( \mathfrak{g}_\nu, \mathfrak{g}_\nu \text{-} \mathfrak{g}_\nu = G_g \) is the centralizer of \( g \) in \( G \), and

\[ \mathfrak{g}_\nu \text{-} \mathfrak{V} = \mathfrak{g}_g, \quad \mathfrak{g}_g = \{ x \in \mathfrak{g} : \text{ad}_g(x) = p^2 x \}. \]

Let \( F_g \) be the set of all Borel Lie subalgebras of \( g \) fixed by the adjoint action of \( g \). It is a non connected manifold whose connected components are labelled by \( \mathfrak{g}_\nu^+ \). In Section 3.14 we’ll introduce two central idempotents \( 1_{\nu^+}, 1_{\nu^-} \) of \( \mathcal{O}R \). This yields a graded \( k \)-algebra decomposition

\[ \mathcal{O}R_{\nu} = \mathcal{O}R_p 1_{\nu^+} \oplus \mathcal{O}R_p 1_{\nu^-}. \]

By [VV, thm. 5.8] the graded \( k \)-algebra \( \mathcal{O}R_p 1_{\nu^+} \) is isomorphic to

\[ \text{Ext}_{\mathcal{O}R_p}(\mathcal{L}_{g,p}, \mathcal{L}_{g,p}), \]

where \( \mathcal{L}_{g,p} \) is the direct image of the constant perverse sheaf by the projection

\[ \{(b, x) \in F_g \times \mathfrak{g}_g : x \in b\} \to \mathfrak{g}_g, \quad (b, x) \mapsto x. \]

The complex \( \mathcal{L}_{g,p} \) has been extensively studied by Lusztig, see e.g., [L1], [L2]. We hope to come back to this elsewhere.

(b) The results in Section 2.5 hold true if \( k \) is any field. Set \( f(\mathfrak{x}) = 1 + \mathfrak{x} \) for instance.
2.9. Induction and restriction of $\text{H}_m$-modules. For $i \in I$ we define functors

$$E_i : \text{H}_{m+1}\text{-fMod}_I \to \text{H}_m\text{-fMod}_I,$$

$$F_i : \text{H}_m\text{-fMod}_I \to \text{H}_{m+1}\text{-fMod}_I,$$

where $E_i M \subset M$ is the generalized $i^{-1}$-eigenspace of the $X_{m+1}$-action, and where

$$F_i M = \text{Ind}_{\text{H}_m \otimes k[X_{m+1}]}^{\text{H}_{m+1}}(M \otimes k_i).$$

Here $k_i$ is the 1-dimensional representation of $k[X_{m+1}]$ defined by $X_{m+1} \mapsto i^{-1}$.

3. Global bases of $^\circ \text{V}$ and projective graded $^\circ \text{R}$-modules

3.1. The Grothendieck groups of $^\circ \text{R}_m$. The graded $k$-algebra $^\circ \text{R}_m$ is free of finite rank over its center by Proposition 1.6, a commutative graded $k$-subalgebra. Therefore any simple object of $^\circ \text{R}_m$-$\text{mod}$ is finite-dimensional and there is a finite number of isomorphism classes of simple modules in $^\circ \text{R}_m$-$\text{mod}$. The Abelian group $G(^\circ \text{R}_m)$ is free with a basis formed by the classes of the simple objects of $^\circ \text{R}_m$-$\text{mod}$. The Abelian group $K(^\circ \text{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both $G(^\circ \text{R}_m)$ and $K(^\circ \text{R}_m)$ are free $A$-modules, where $v$ shifts the grading by 1. We consider the following $A$-modules

$$^\circ \text{K}_I = \bigoplus_{m \geq 0} ^\circ \text{K}_I,m, \quad ^\circ \text{K}_I,m = K(^\circ \text{R}_m),$$

$$^\circ \text{G}_I = \bigoplus_{m \geq 0} ^\circ \text{G}_I,m, \quad ^\circ \text{G}_I,m = G(^\circ \text{R}_m).$$

We’ll also abbreviate

$$^\circ \text{K}_I,* = \bigoplus_{m > 0} ^\circ \text{K}_I,m, \quad ^\circ \text{G}_I,* = \bigoplus_{m > 0} ^\circ \text{G}_I,m.$$

From now on, to unburden the notation we may abbreviate $^\circ \text{R} = ^\circ \text{R}_m$, hoping it will not create any confusion. For any $M,N$ in $^\circ \text{R}$-$\text{mod}$ we set

$$(M : N) = \text{gdim}(M^\omega \otimes_R N), \quad \langle M : N \rangle = \text{gdim} \text{hom}_R(M,N),$$

where $\omega$ is the involution defined in Section 1.1. The Cartan pairing is the perfect $A$-bilinear form

$$^\circ \text{K}_I \times ^\circ \text{G}_I \to A, \quad (P,M) \mapsto \langle P : M \rangle.$$

First, we concentrate on the $A$-module $^\circ \text{G}_I$. Consider the duality

$$^\circ \text{R}$-$\text{fmod} \to ^\circ \text{R}$-$\text{fmod}, \quad M \mapsto M^\flat = \text{hom}(M,k),$$

with the action and the grading given by

$$(xf)(m) = f(\omega(x)m), \quad (M^\flat)_d = \text{Hom}(M_{-d},k).$$

This duality functor yields an $A$-antilinear map

$$^\circ \text{G}_I \to ^\circ \text{G}_I, \quad M \mapsto M^\flat.$$

Let $^\circ \text{B}$ denote the set of isomorphism classes of simple objects of $^\circ \text{R}$-$\text{fMod}_0$. We can now define the upper global basis of $^\circ \text{G}_I$ as follows. The proof is given in Section 3.21.
3.2. Proposition/Definition. For each \( b \) in \( \mathcal{O}B \) there is a unique selfdual irreducible graded \( \mathcal{O}R \)-module \( \mathcal{O}G^{\text{up}}(b) \) which is isomorphic to \( b \) as a (non graded) \( \mathcal{O}R \)-module. We set \( \mathcal{O}G^{\text{up}}(0) = 0 \) and \( \mathcal{O}G^{\text{up}} = \{ \mathcal{O}G^{\text{up}}(b); b \in \mathcal{O}B \} \). Hence \( \mathcal{O}G^{\text{up}} \) is a \( \mathcal{O}A \)-basis of \( \mathcal{O}G \).

Now, we concentrate on the \( \mathcal{O}A \)-module \( \mathcal{O}K \). We equip \( \mathcal{O}K \) with the symmetric \( \mathcal{O}A \)-bilinear form

\[
(3.1) \quad \mathcal{O}K \times \mathcal{O}K \to \mathcal{O}A, \quad (M, N) \mapsto (M : N).
\]

Consider the duality

\[
\mathcal{O}R\text{-proj} \to \mathcal{O}R\text{-proj}, \quad P \mapsto P^\sharp = \text{hom}_\mathcal{O}R(P, \mathcal{O}R),
\]

with the action and the grading given by

\[
(xf)(p) = f(p)\omega(x), \quad (P^\sharp)_d = \text{Hom}_\mathcal{O}R(P[-d], \mathcal{O}R).
\]

This duality functor yields an \( \mathcal{O}A \)-antilinear map

\[
\mathcal{O}K \to \mathcal{O}K, \quad P \mapsto P^\sharp.
\]

Set \( \mathcal{O}K = \mathbb{Q}(v) \). Let \( \mathcal{K} \to \mathcal{K}, f \mapsto \bar{f} \) be the unique involution such that \( \bar{v} = v^{-1} \).

3.3. Definition. For each \( b \) in \( \mathcal{O}B \) let \( \mathcal{O}G^{\text{low}}(b) \) be the unique indecomposable graded module in \( \mathcal{O}R\text{-proj} \) whose top is isomorphic to \( \mathcal{O}G^{\text{up}}(b) \). We set \( \mathcal{O}G^{\text{low}}(0) = 0 \) and \( \mathcal{O}G^{\text{low}} = \{ \mathcal{O}G^{\text{low}}(b); b \in \mathcal{O}B \} \), a \( \mathcal{O}A \)-basis of \( \mathcal{O}K \).

3.4. Proposition. (a) We have \( \langle \mathcal{O}G^{\text{low}}(b) : \mathcal{O}G^{\text{up}}(b') \rangle = \delta_{b,b'} \) for each \( b,b' \) in \( \mathcal{O}B \).

(b) We have \( (P^\sharp : M) = (P : M^\sharp) \) for each \( P, M \).

(c) We have \( \mathcal{O}G^{\text{low}}(b)^\sharp = \mathcal{O}G^{\text{low}}(b) \) for each \( b \) in \( \mathcal{O}B \).

The proof is the same as in \([\text{VV}, \text{prop. 8.4}]\).

3.5. Example. Set \( \nu = i + \theta(i) \) and \( i = i\theta(i) \). Consider the graded \( \mathcal{O}R_i \)-modules

\[
\mathcal{O}R_i = \mathcal{O}R_l = 1_i, \quad \mathcal{O}L_i = \text{top(\mathcal{O}R_i)}.
\]

The global bases are given by

\[
\mathcal{O}G^{\text{low}}_\nu = \{ \mathcal{O}R_l, \mathcal{O}R_{\theta(l)} \}, \quad \mathcal{O}G^{\text{up}}_\nu = \{ \mathcal{O}L_l, \mathcal{O}L_{\theta(l)} \}.
\]

For \( m = 0 \) we have \( \mathcal{O}R_0 = \mathbb{k} \oplus \mathbb{k} \). Set \( \phi_+ = \mathbb{k} \oplus 0 \) and \( \phi_- = 0 \oplus \mathbb{k} \). We have

\[
\mathcal{O}G^{\text{low}}_0 = \mathcal{O}G^{\text{up}}_0 = \{ \phi_+, \phi_- \}.
\]
3.6. Definition of the operators $e_i, f_i, e_i', f_i'$. In this section we'll always assume $m > 0$ unless specified otherwise. First, let us introduce the following notation. Let $D_{m,1}$ be the set of minimal representative in $^\circ W_{m+1}$ of the cosets in $^\circ W_m \setminus ^\circ W_{m+1}$. Write

$$D_{m,1;m,1} = D_{m,1} \cap (D_{m,1})^{-1}.$$ 

For each element $w$ of $D_{m,1;m,1}$ we set

$$W'(w) = ^\circ W_m \cap w(^\circ W_m)w^{-1}.$$ 

Let $R_1$ be the $k$-algebra generated by elements $1_i, \kappa_i, i \in I$, satisfying the defining relations $1_i 1_{i'} = \delta_{i,i'} 1_i$ and $\kappa_i = 1, \kappa_i 1_i$. We equip $R_1$ with the grading such that $\deg(1_i) = 0$ and $\deg(\kappa_i) = 2$. Let

$$R_i = 1_i R_1 = R_1 1_i, \quad L_i = \text{top}(R_i) = R_i/(\kappa_i).$$ 

Then $R_i$ is a graded projective $R_1$-module and $L_i$ is simple. We abbreviate

$$^\theta R_{m,1} = ^\theta R_m \otimes R_1, \quad ^\circ R_{m,1} = ^\circ R_m \otimes R_1.$$ 

There is an unique inclusion of graded $k$-algebras

$$^\theta R_{m,1} \hookrightarrow ^\theta R_{m+1},$$

$$1_i \otimes 1_i \mapsto 1_{i'},$$

$$1_i \otimes \kappa_{i',I} \mapsto \kappa_{i,m+1},$$

$$\kappa_{i',I} \otimes 1_i \mapsto \kappa_{i',I},$$

$$\pi_{i,1} \otimes 1_i \mapsto \pi_{i,1},$$

$$\sigma_{i,k} \otimes 1_i \mapsto \sigma_{i,k},$$

(3.2)

where, given $i \in ^\theta I^m$ and $i \in I$, we have set $i' = \theta(i) i$, a sequence in $^\theta I^{m+1}$. This inclusion restricts to an inclusion $^\circ R_{m,1} \subset ^\circ R_{m+1}$.

3.7. Lemma. The graded $^\circ R_{m,1}$-module $^\circ R_{m+1}$ is free of rank $2(m + 1)$.

Proof: For each $w$ in $D_{m,1}$ we have the element $\sigma_w$ in $^\circ R_{m+1}$ defined in (1.5). Using filtered/graded arguments it is easy to see that

$$^\circ R_{m+1} = \bigoplus_{w \in D_{m,1}} ^\circ R_{m,1} \sigma_w.$$ 

We define a triple of adjoint functors $(\psi_!, \psi^*, \psi_\ast)$ where

$$\psi^* : ^\circ R_{m+1}\text{-mod} \rightarrow ^\circ R_m\text{-mod} \times R_1\text{-mod}$$

\[\square\]
is the restriction and $\psi_i, \psi_*$ are given by

$$\psi_i : \begin{cases} \mathfrak{m}_R\text{-mod} \times \mathfrak{r}_I\text{-mod} \to \mathfrak{m}_{R_{m+1}}\text{-mod}, \\ (M, M') \mapsto \mathfrak{m}_{R_{m+1}} \otimes_{\mathfrak{r}_{m+1}} (M \otimes M'), \end{cases}$$

$$\psi_* : \begin{cases} \mathfrak{m}_R\text{-mod} \times \mathfrak{r}_I\text{-mod} \to \mathfrak{m}_{R_{m+1}}\text{-mod}, \\ (M, M') \mapsto \text{hom}_{\mathfrak{r}_{m+1}}(\mathfrak{m}_{R_{m+1}}, M \otimes M'). \end{cases}$$

First, note that the functors $\psi_i, \psi_*$ commute with the shift of the grading. Next, the functor $\psi_*$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. The right graded $\mathfrak{m}_{R_{m+1}}\text{-module} \mathfrak{m}_{R_{m+1}}$ is free of finite rank. Thus $\psi_i$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. The left graded $\mathfrak{m}_{R_{m+1}}\text{-module} \mathfrak{m}_{R_{m+1}}$ is also free of finite rank. Thus the functor $\psi_*$ is exact, and it takes finite dimensional graded modules to finite dimensional ones. Further $\psi_i$ and $\psi_*$ take projective graded modules to projective ones, because they are left adjoin to the exact functors $\psi_i, \psi_*$ respectively. To summarize, the functors $\psi_i, \psi_*$, $\psi_*$ are exact and take finite dimensional graded modules to finite dimensional ones, and the functors $\psi_i, \psi_*$ take projective graded modules to projective ones.

For any graded $\mathfrak{m}_{R_m}\text{-module} M$ we write

$$f_i(M) = \mathfrak{m}_{R_{m+1}} \otimes \mathfrak{m}_M,$$

$$e_i(M) = \mathfrak{m}_{R_{m-1}} \otimes \mathfrak{m}_{m-1,i} \mathfrak{m}_M.$$  

Let us explain these formulas. The symbols $1_{m,i}$ and $1_{m-1,i}$ are given by

$$1_{m-1,i} = \bigoplus_i 1_{\theta(i) i} M, \quad i \in \theta T^{m-1}.$$  

Note that $f_i(M)$ is a graded $\mathfrak{m}_{R_{m+1}}\text{-module}$, while $e_i(M)$ is a graded $\mathfrak{m}_{R_{m-1}}\text{-module}$. The tensor product in the definition of $e_i(M)$ is relative to the graded $k$-algebra homomorphism

$$\mathfrak{m}_{m-1,1} \to \mathfrak{m}_{m-1} \otimes \mathfrak{r}_I \to \mathfrak{m}_{m-1} \otimes \mathfrak{r}_i \to \mathfrak{m}_{m-1} \otimes \mathfrak{l}_i = \mathfrak{m}_{R_{m-1}}.$$  

In other words, let $e_i'(M)$ be the graded $\mathfrak{m}_{R_{m-1}}\text{-module}$ obtained by taking the direct summand $1_{m-1,i} M$ and restricting it to $\mathfrak{m}_{R_{m-1}}$. Observe that if $M$ is finitely generated then $e_i'(M)$ may not lie in $\mathfrak{m}_{R_{m-1}}\text{-mod}$. To remedy this, since $e_i'(M)$ affords a $\mathfrak{m}_{R_{m-1}} \otimes \mathfrak{l}_i$-action we consider the graded $\mathfrak{m}_{R_{m-1}}\text{-module}$

$$e_i(M) = e_i'(M)/\mathfrak{c}_0 e_i'(M).$$

### 3.8 Definition. The functors $e_i, f_i$ preserve the category $\mathfrak{m}_{R\text{-proj}}$, yielding $A$-linear operators on $\mathfrak{m}_I$ which act on $\mathfrak{m}_i$ by the formula (3.3) and satisfy also

$$f_i(\phi_+) = \mathfrak{m}_{R_{\theta(i)}}, \quad f_i(\phi_-) = \mathfrak{m}_{R_{\theta(i)}}, \quad e_i(\mathfrak{m}_{R_{\theta(j)b}}) = \delta_{i,j} \phi_+ + \delta_{i,j} \phi_-.$$  

Let $e_i, f_i$ denote also the $A$-linear operators on $\mathfrak{m}_I$ which are the transpose of $f_i, e_i$ with respect to the Cartan pairing.

Note that the symbols $e_i(M), f_i(M)$ have a different meaning if $M$ is viewed as an element of $\mathfrak{m}_{K_I}$ or if $M$ is viewed as an element of $\mathfrak{m}_I$. We hope this will not create any confusion. The proof of the following lemma is the same as in [VV, lem. 8.9].
3.9. Lemma. (a) The operators $e_i$, $f_i$ on $^G_I$ are given by

$$e_i(M) = 1_{m-1,i}M \quad f_i(M) = \text{hom}_{^R_m}(^R_{m+1,M} \otimes L_i), \quad M \in \mathcal{R}_m$$

(b) For each $M \in \mathcal{R}_m$, $M' \in \mathcal{R}_{m+1}$ we have

$$(e_i(M') : M) = (M' : f_i(M)).$$

(c) We have $f_i(P) = f_i(P')$ for each $P \in \mathcal{R}_m$.

(d) We have $e_i(M) = e_i(M')$ for each $M \in \mathcal{R}_m$.

3.10. Induction of $H_m$-modules versus induction of $^R_m$-modules. Recall the functors $E_i$, $F_i$ on $H_m$-mod defined in (2.2). We have also the functors

$$\Psi : \mathcal{R}_m \rightarrow \mathcal{R}_m, \quad \Psi : \mathcal{R}_m \rightarrow H_m$$

where $\Psi$ is the forgetting of the grading. Finally we define functors

$$E_i : \mathcal{R}_m \rightarrow \mathcal{R}_m, \quad E_i = 1_{m-1,i}M,$$

$$F_i : \mathcal{R}_m \rightarrow \mathcal{R}_m, \quad F_i = \psi(M, L_i).$$

3.11. Proposition. There are canonical isomorphisms of functors

$$E_i \circ \Psi = \Psi \circ E_i, \quad F_i \circ \Psi = \Psi \circ F_i, \quad E_i \circ \text{for} = \text{for} \circ e_i, \quad F_i \circ \text{for} = \text{for} \circ f_\theta(i).$$

Proof: The proof is the same as in [VV, prop. 8.17].

3.12. Proposition. (a) The functor $\Psi$ yields an isomorphism of Abelian groups

$$\bigoplus_{m \geq 0} [\mathcal{R}_m] = \bigoplus_{m \geq 0} [H_m].$$

The functors $E_i$, $F_i$ yield endomorphisms of both sides which are intertwined by $\Psi$.

(b) The functor for factors to a group isomorphism

$$^G_I/(v-1) = \bigoplus_{m \geq 0} [\mathcal{R}_m].$$

Proof: Claim (a) follows from Corollary 2.7 and Proposition 3.11. Claim (b) follows from Proposition 3.2.
3.13. Type D versus type B. We can compare the previous constructions with their analogues in type B. Let $^{\theta}K$, $^{\theta}B$, $^{\theta}G^{\text{low}}$, etc, denote the type B analogues of $^\theta K$, $^\theta B$, $^\theta G^{\text{low}}$, etc. See [VV] for details. We’ll use the same notation for the functors $\psi^*$, $\psi_*$, $e_i$, $f_i$, etc, on the type B side and on the type D side. Fix $m > 0$ and $\nu \in ^\theta N$ such that $|\nu| = 2m$. The inclusion of graded $k$-algebras $^\theta R_\nu \subset ^\theta R_\nu$ in (1.2) yields a restriction functor

$$\text{res} : ^\theta R_\nu \text{-mod} \to ^\theta R_\nu \text{-mod}$$

and an induction functor

$$\text{ind} : ^\theta R_\nu \text{-mod} \to ^\theta R_\nu \text{-mod}, \quad M \mapsto ^\theta R_\nu \otimes _R M.$$ 

Both functors are exact, they map finite dimensional graded modules to finite dimensional ones, and they map projective graded modules to projective ones. Thus, they yield morphisms of $A$-modules

$$\text{res} : ^\theta K_{I,m} \to ^\theta K_{I,m}, \quad \text{res} : ^\theta G_{I,m} \to ^\theta G_{I,m},$$

$$\text{ind} : ^\theta K_{I,m} \to ^\theta K_{I,m}, \quad \text{ind} : ^\theta G_{I,m} \to ^\theta G_{I,m}.$$ 

Moreover, for any $P \in ^\theta K_{I,m}$ and any $L \in ^\theta G_{I,m}$ we have

$$\text{res}(P^\gamma) = (\text{res} P)^\gamma, \quad \text{ind}(P^\gamma) = (\text{ind} P)^\gamma,$$

$$\text{res}(L^\gamma) = (\text{res} L)^\gamma, \quad \text{ind}(L^\gamma) = (\text{ind} L)^\gamma.$$ 

Note also that ind and res are left and right adjoint functors, because

$$^\theta R_\nu \otimes _R M = \text{hom}_R(^\theta R_\nu,M)$$

as graded $^\theta R_\nu$-modules.

3.14. Definition. For any graded $^\theta R_\nu$-module $M$ we define the graded $^\theta R_\nu$-module $M^\gamma$ with the same underlying graded $k$-vector space as $M$ such that the action of $^\theta R_\nu$ is twisted by $\gamma$, i.e., the graded $k$-algebra $^\theta R_\nu$ acts on $M^\gamma$ by $a m = \gamma(a)m$ for $a \in ^\theta R_\nu$ and $m \in M$. Note that $(M^\gamma)^\gamma = M$, and that $M^\gamma$ is simple (resp. projective, indecomposable) if $M$ has the same property.

For any graded $^\theta R_m$-module $M$ we have canonical isomorphisms of $^\theta R$-modules

$$(f_i(M))^\gamma = f_i(M^\gamma), \quad (e_i(M))^\gamma = e_i(M^\gamma).$$

The first isomorphism is given by

$$^\theta R_{m+1,i} \otimes _R M \to ^\theta R_{m+1,i} \otimes _R M, \quad a \otimes m \mapsto \gamma(a) \otimes m.$$ 

The second one is the identity map on the vector space $1_{m,i} M$.

Recall that $^\theta I^\nu$ is the disjoint union of $^\theta I^\nu_+$ and $^\theta I^\nu_-$. We set

$$1_{\nu,+} = \sum_{i \in ^\theta I^\nu_+} 1_i, \quad 1_{\nu,-} = \sum_{i \in ^\theta I^\nu_-} 1_i.$$
3.15. Lemma. Let $M$ be a graded $\mathcal{T}_\nu$-module.

(a) Both $1_{\nu,+}$ and $1_{\nu,-}$ are central idempotents in $\mathcal{T}_\nu$. We have $1_{\nu,+} = \gamma(1_{\nu,-})$.
(b) There is a decomposition of graded $\mathcal{T}_\nu$-modules $M = 1_{\nu,+}M \oplus 1_{\nu,-}M$.
(c) We have a canonical isomorphism of $\mathcal{T}_\nu$-modules $\text{res} \circ \text{ind}(M) = M \oplus M^\gamma$.
(d) If there exists $a \in \{+,-\}$ such that $1_{\nu,-a}M = 0$, then there are canonical isomorphisms of graded $\mathcal{T}_\nu$-modules

$$M = 1_{\nu,a}M, \quad 0 = 1_{\nu,a}M^\gamma, \quad M^\gamma = 1_{\nu,-a}M^\gamma.$$  

Proof: Part (a) follows from Proposition 1.6 and the equality $\varepsilon_1(\theta_I^\nu) = \theta_I^\nu$. Part (b) follows from (a), (c) is given by definition, and (d) follows from (a), (b).

Now, let $m$ and $\nu$ be as before. Given $i \in I$, we set $\nu' = \nu + i + \theta(i)$. There is an obvious inclusion $W_m \subset W_{m+1}$. Thus the group $W_m$ acts on $\theta^\nu\nu'$, and the map

$$\theta^\nu \nu' \to \theta^\nu \nu', \quad i \mapsto \theta(i)i$$

is $W_m$-equivariant. Thus there is $a_i \in \{+, -\}$ such that the image of $\theta^\nu \nu'$ is contained in $\theta^\nu I_{a_i}$, and the image of $\theta^\nu \nu'$ is contained in $\theta^\nu I_{-a_i}$.

3.16. Lemma. Let $M$ be a graded $\mathcal{T}_\nu$-module such that $1_{\nu,-a}M = 0$, with $a = +, -$. Then we have

$$1_{\nu',-a}f_i(M) = 0, \quad 1_{\nu',a}f_{\theta(i)}(M) = 0.$$  

Proof: We have

$$1_{\nu',-a}f_i(M) = 1_{\nu',-a_1}^\mathcal{T}_\nu 1_{\nu',i} \mathcal{T}_\nu M$$

$$= \mathcal{T}_\nu 1_{\nu',-a_1}^\mathcal{T}_\nu 1_{\nu',i} \mathcal{T}_\nu M.$$  

Here we have identified $1_{\nu,a}$ with the image of $(1_{\nu,a},1)$ via the inclusion (3.2). The definition of this inclusion and that of $a_i$ yield that

$$1_{\nu',a_1}1_{\nu,a} = 1_{\nu,a}, \quad 1_{\nu',-a_1}1_{\nu,a} = 0.$$  

The first equality follows. Next, note that for any $i \in \theta^\nu \nu'$, the sequences $\theta(i)i$ and $i\theta(i) = \varepsilon_{m+1}(\theta(i)i)$ always belong to different $\mathcal{T}_W$-orbits. Thus, we have $a_\theta(i) = -a_i$. So the second equality follows from the first.

Consider the following diagram

$$\begin{array}{ccc}
\mathcal{T}_\nu\text{-mod} \times \mathcal{T}_\nu\text{-mod} & \xrightarrow{\psi} & \mathcal{T}_\nu\text{-mod} \\
\text{res} \times \text{id} & = & \text{res} \\
\varepsilon \times \text{id} & & \varepsilon \\
\mathcal{T}_{\nu'}\text{-mod} \times \mathcal{T}_{\nu'}\text{-mod} & \xrightarrow{\psi} & \mathcal{T}_{\nu'}\text{-mod}
\end{array}$$
3.17. Lemma. There are canonical isomorphisms of functors

\[ \text{ind} \circ \psi_h = \psi_h \circ (\text{id} \times \text{id}), \quad \psi^* \circ \text{ind} = (\text{id} \times \text{id}) \circ \psi^*, \quad \text{ind} \circ \psi_* = \psi_* \circ (\text{id} \times \text{id}), \]

\[ \text{res} \circ \psi_l = \psi_l \circ (\text{res} \times \text{id}), \quad \psi^* \circ \text{res} = (\text{res} \times \text{id}) \circ \psi^*, \quad \text{res} \circ \psi_* = \psi_* \circ (\text{res} \times \text{id}). \]

Proof: The functor \( \text{ind} \) is left and right adjoint to \( \text{res} \). Therefore it is enough to prove the first two isomorphisms in the first line. The isomorphism

\[ \text{ind} \circ \psi_h = \psi_h \circ (\text{id} \times \text{id}) \]

comes from the associativity of the induction. Let us prove that

\[ \psi^* \circ \text{ind} = (\text{ind} \times \text{id}) \circ \psi^*. \]

For any \( M \) in \( ^\nu \mathcal{R} \)-mod, the obvious inclusion \( \theta^{^\nu \mathcal{R}_i} \subset \theta^{^\nu \mathcal{R}_i} \) yields a map

\[ (\text{id} \times \text{id}) \psi^*(M) = (\theta^{^\nu \mathcal{R}_i} \otimes \theta^{^\nu \mathcal{R}_i}) \rightarrow \psi^* (\theta^{^\nu \mathcal{R}_i} \otimes \theta^{^\nu \mathcal{R}_i}, M). \]

Combining it with the obvious map

\[ \theta^{^\nu \mathcal{R}_i} \rightarrow \theta^{^\nu \mathcal{R}_i} \]

we get a morphism of \( \theta^{^\nu \mathcal{R}_i} \)-modules

\[ (\text{id} \times \text{id}) \psi^*(M) \rightarrow \psi^* \text{ind}(M). \]

We need to show that it is bijective. This is obvious because at the level of vector spaces, the map above is given by

\[ M \oplus (\pi_{1,\nu} \otimes M) \rightarrow M \oplus (\pi_{1,\nu} \otimes M), \quad m + \pi_{1,\nu} \otimes n \mapsto m + \pi_{1,\nu} \otimes n. \]

Here \( \pi_{1,\nu} \) and \( \pi_{1,\nu'} \) denote the element \( \pi_1 \) in \( ^\nu \mathcal{R}_i \) and \( ^\nu \mathcal{R}_i \) respectively.

3.18. Corollary. (a) The operators \( e_i, f_i \) on \( ^\nu \mathcal{K}_{\nu,*} \) and on \( ^\nu \mathcal{K}_{\nu,*} \) are intertwined by the maps \( \text{ind}, \text{res} \), i.e., we have

\[ e_i \circ \text{ind} = \text{ind} \circ e_i, \quad f_i \circ \text{ind} = \text{ind} \circ f_i, \quad e_i \circ \text{res} = \text{res} \circ e_i, \quad f_i \circ \text{res} = \text{res} \circ f_i. \]

(b) The same result holds for the operators \( e_i, f_i \) on \( ^\nu \mathcal{G}_{\nu,*} \) and on \( ^\nu \mathcal{G}_{\nu,*} \).

3.19. Now, we concentrate on non graded irreducible modules. First, let

\[ \text{Res} : ^\nu \mathcal{R}_i \text{-Mod} \rightarrow ^\nu \mathcal{R}_i \text{-Mod}, \quad \text{Ind} : ^\nu \mathcal{R}_i \text{-Mod} \rightarrow ^\nu \mathcal{R}_i \text{-Mod} \]

be the (non graded) restriction and induction functors. We have

\[ \text{for} \circ \text{res} = \text{Res} \circ \text{for}, \quad \text{for} \circ \text{ind} = \text{Ind} \circ \text{for}. \]
3.20. Lemma. Let $L, L'$ be irreducible $\mathfrak{R}_v$-modules.

(a) The $\mathfrak{R}_v$-modules $L$ and $L'$ are not isomorphic.

(b) $\text{Ind}(L)$ is an irreducible $\mathfrak{R}_v$-module, and every irreducible $\mathfrak{R}_v$-module is obtained in this way.

(c) $\text{Ind}(L) \simeq \text{Ind}(L')$ iff $L' \simeq L$ or $L'$.

Proof: For any $\mathfrak{R}_v$-module $M \neq 0$, both $1_{v,+} M$ and $1_{v,-} M$ are nonzero. Indeed, we have $M = 1_{v,+} M \oplus 1_{v,-} M$, and we may suppose that $1_{v,+} M \neq 0$. The automorphism $M \to M, m \mapsto \pi_1 m$ takes $1_{v,+} M$ to $1_{v,-} M$ by Lemma 3.15(a). Hence $1_{v,-} M \neq 0$.

Now, to prove part (a), suppose that $\phi : L \to L'$ is an isomorphism of $\mathfrak{R}_v$-modules. We can regard $\phi$ as a $\gamma$-antilinear map $L \to L$. Since $L$ is irreducible, by Schur's lemma we may assume that $\phi^2 = \text{id}_L$. Then $L$ admits a $\mathfrak{R}_v$-module structure such that the $\mathfrak{R}_v$-action is as before and $\pi_1$ acts as $\phi$. Thus, by the discussion above, neither $1_{v,+} L$ nor $1_{v,-} L$ is zero. This contradicts the fact that $L$ is an irreducible $\mathfrak{R}_v$-module.

Parts (b), (c) follow from (a) by Clifford theory, see e.g., [RR, appendix].

We can now prove Proposition 3.2.

3.21. Proof of Proposition 3.2. Let $b \in \mathcal{B}$. We may suppose that $b = 1_{v,+} b$. By Lemma 3.20(b) the module $\mathfrak{g} = \text{Ind}(b)$ lies in $\mathfrak{B}$. By [VV, prop. 8.2] there exists a unique selfdual irreducible graded $\mathfrak{R}$-module $\mathcal{G}^{\text{up}}(\mathfrak{g})$ which is isomorphic to $\mathfrak{g}$ as a non graded module. Set

$$\mathcal{G}^{\text{up}}(b) = 1_{v,+} \text{res}(\mathcal{G}^{\text{up}}(\mathfrak{g})).$$

By Lemma 3.15(d) we have $\mathcal{G}^{\text{up}}(b) = b$ as a non graded $\mathfrak{R}$-module, and by (3.5) it is selfdual. This proves existence part of the proposition. To prove the uniqueness, suppose that $M$ is another module with the same properties. Then $\text{ind}(M)$ is a selfdual graded $\mathfrak{R}$-module which is isomorphic to $\mathfrak{g}$ as a non graded $\mathfrak{R}$-module. Thus we have $\text{ind}(M) = \mathcal{G}^{\text{up}}(\mathfrak{g})$ by loc. cit. By Lemma 3.15(d) we have also

$$M = 1_{v,+} \text{res}(\mathcal{G}^{\text{up}}(\mathfrak{g})).$$

So $M$ is isomorphic to $\mathcal{G}^{\text{up}}(b)$.

3.22. The crystal operators on $\mathfrak{G}_I$ and $\mathfrak{B}$. Fix a vertex $i$ in $I$. For each irreducible graded $\mathfrak{R}_{v_i}$-module $M$ we define

$$\bar{e}_i(M) = \text{soc}(e_i(M)), \quad \bar{f}_i(M) = \text{top}\psi_i(M, L_i), \quad e_i(M) = \max\{n \geq 0; e_i^n(M) \neq 0\}.$$

3.23. Lemma. Let $M$ be an irreducible graded $\mathfrak{R}$-module such that $e_i(M), f_i(M)$ belong to $\mathfrak{G}_I, \mathfrak{B}$. We have

$$\text{ind}(\bar{e}_i(M)) = e_i(\text{ind}(M)), \quad \text{ind}(\bar{f}_i(M)) = f_i(\text{ind}(M)), \quad e_i(M) = e_i(\text{ind}(M)).$$

In particular, $\bar{e}_i(M)$ is irreducible or zero and $\bar{f}_i(M)$ is irreducible.
Proof: By Corollary 3.18 we have \( \text{ind}(e_i(M)) = e_i(\text{ind}(M)) \). Thus, since \( \text{ind} \) is an exact functor we have \( \text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M)) \). Since \( \text{ind} \) is an additive functor, by Lemma 3.20(b) we have indeed
\[
\text{ind}(\tilde{e}_i(M)) \subset e_i(\text{ind}(M)).
\]
Note that \( \text{ind}(M) \) is irreducible by Lemma 3.20(b), thus \( \tilde{e}_i(\text{ind}(M)) \) is irreducible by [VV, prop. 8.21]. Since \( \text{ind}(\tilde{e}_i(M)) \) is nonzero, the inclusion is an isomorphism. The fact that \( \text{ind}(\tilde{e}_i(M)) \) is irreducible implies in particular that \( \tilde{e}_i(M) \) is simple. The proof of the second isomorphism is similar. The third equality is obvious.

Similarly, for each irreducible \( \mathcal{O}_R \)-module \( b \) in \( \mathcal{O}_B \) we define
\[
\tilde{E}_i(b) = \text{soc}(E_i(b)), \quad \tilde{F}_i(b) = \text{top}(F_i(b)), \quad \varepsilon_i(b) = \max\{ n \geq 0; E_i^n(b) \neq 0 \}.
\]
Hence we have
\[
\text{for } \circ \tilde{e}_i = \tilde{E}_i \circ \text{for}, \quad \text{for } \circ \tilde{F}_i = \tilde{F}_i \circ \text{for}, \quad \varepsilon_i = \varepsilon_i \circ \text{for}.
\]

3.24. Proposition. For each \( b, b' \) in \( \mathcal{O}_B \) we have
(a) \( \tilde{F}_i(b) \subset \mathcal{O}_B \),
(b) \( \tilde{E}_i(b) \subset \mathcal{O}_B \cup \{0\} \),
(c) \( \tilde{F}_i(b) = b' \iff \tilde{E}_i(b') = b \),
(d) \( \varepsilon_i(b) = \max\{ n \geq 0; \tilde{E}_i^n(b) \neq 0 \} \),
(e) \( \varepsilon_i(\tilde{F}_i(b)) = \varepsilon_i(b) + 1 \),
(f) \( \text{if } \tilde{E}_i(b) = 0 \text{ for all } i \text{ then } b = \phi_\perp \).

Proof: Part (c) follows from adjunction. The other parts follow from [VV, prop. 3.14] and Lemma 3.23.

3.25. Remark. For any \( b \in \mathcal{O}_B \) and any \( i \neq j \) we have \( \tilde{F}_i(b) \neq \tilde{F}_j(b) \). This is obvious if \( j \neq \theta(i) \). Because in this case \( \tilde{F}_i(b) \) and \( \tilde{F}_j(b) \) are \( \mathcal{O}_R \)-modules for different \( \nu \). Now, consider the case \( j = \theta(i) \). We may suppose that \( \tilde{F}_i(b) = 1_{\nu, +} \tilde{F}_i(b) \) for certain \( \nu \). Then by Lemma 3.16 we have \( 1_{\nu, +} \tilde{F}_{\theta(i)}(b) = 0 \). In particular \( \tilde{F}_i(b) \) is not isomorphic to \( \tilde{F}_{\theta(i)}(b) \).

3.26. The algebra \( \mathcal{O}_B \) and the \( \mathcal{O}_B \)-module \( \mathcal{O}_V \). Following [EK1,2,3] we define a \( \mathcal{K} \)-algebra \( \mathcal{O}_B \) as follows.

3.27. Definition. Let \( \mathcal{O}_B \) be the \( \mathcal{K} \)-algebra generated by \( e_i, f_i \) and invertible elements \( t_i, i \in I \), satisfying the following defining relations
(a) \( t_i t_j = t_j t_i \) and \( t_{\theta(i)} = t_i \) for all \( i, j \),
(b) \( t_i e_i f_i^{-1} = v^{i+j+\theta(i) \cdot j} e_j \) and \( t_i f_j t_i^{-1} = v^{-i+j-\theta(i) \cdot j} f_j \) for all \( i, j \),
(c) \( e_i f_j = v^{i+j} f_j e_i + \delta_{i,j} + \delta_{\theta(i), 3} t_i \) for all \( i, j \),
(d) \[ \sum_{a+b=1-i-j} (-1)^{a} e^{(a)}_{i} f^{(b)}_{j} = \sum_{a+b=1-i-j} (-1)^{a} f^{(a)}_{j} f^{(b)}_{i} = 0 \text{ if } i \neq j. \]

Here and below we use the following notation

\[ \theta^{(a)} = \theta^{a}/(a)! , \quad (a) = \sum_{l=1}^{a} q^{a+1-2l}, \quad \langle a \rangle! = \prod_{l=1}^{m} (l). \]

We can now construct a representation of \( \mathcal{B} \) as follows. By base change, the operators \( e_{i}, f_{i} \) in Definition 3.8 yield \( K \)-linear operators on the \( K \)-vector space

\[ ^{0}V = \mathcal{K} \otimes_{A} ^{0}Kf. \]

We equip \( ^{0}V \) with the \( K \)-bilinear form given by

\[ (M : N)_{K,m} = (1 - v^{2})^{m} (M : N), \quad \forall M, N \in \mathcal{R}_{m}^{\text{proj}}. \]

### 3.28. Theorem

(a) The operators \( e_{i}, f_{i} \) define a representation of \( \mathcal{B} \) on \( ^{0}V \). The \( \mathcal{B} \)-module \( ^{0}V \) is generated by linearly independent vectors \( \phi_{+} \) and \( \phi_{-} \) such that for each \( i \) in \( I \) we have

\[ e_{i} \phi_{\pm} = 0, \quad t_{i} \phi_{\pm} = \phi_{\mp}, \quad \{ x \in ^{0}V; e_{j}x = 0, \forall j \} = k \phi_{+} \oplus k \phi_{-}. \]

(b) The symmetric bilinear form on \( ^{0}V \) is non-degenerate. We have \( (\phi_{a} : \phi_{a'})_{K,m} = \delta_{a,a'} \) for \( a, a' = +, -, \) and \( (e_{i}x : y) = (x : f_{j}y)_{K,m} \) for \( i \in I \) and \( x, y \in ^{0}V \).

**Proof:** For each \( i \) in \( I \) we define the \( A \)-linear operator \( t_{i} \) on \( ^{0}Kf \) by setting

\[ t_{i} \phi_{\pm} = \phi_{\mp} \quad \text{and} \quad t_{i}P = v^{-\nu(i+\theta(i))} P^{\gamma}, \quad \forall P \in \mathcal{R}_{\nu}^{\text{proj}}. \]

We must prove that the operators \( e_{i}, f_{i}, \) and \( t_{i} \) satisfy the relations of \( \mathcal{B} \). The relations (a), (b) are obvious. The relation (d) is standard. It remains to check (c). For this we need a version of the Mackey’s induction-restriction theorem. Note that for \( m = 1 \) we have

\[ D_{m,1,m,1} = \{ e, \varepsilon_{m}, \varepsilon_{m+1} \varepsilon_{1} \}, \]

\[ W(e) = ^{0}W_{m}, \quad W(\varepsilon_{m}) = ^{0}W_{m-1}, \quad W(\varepsilon_{m+1} \varepsilon_{1}) = ^{0}W_{m}. \]

Recall also that for \( m = 1 \) we have set \( ^{0}W_{1} = \{ e \} \).

### 3.29. Lemma

Fix \( i, j \in I \). Let \( \nu, \theta \) in \( \mathbb{N}^{I} \) be such that \( \nu + i + \theta(i) = \mu + j + \theta(j) \).

Put \( |\nu| = |\mu| = 2m. \) The graded \( \mathcal{R}_{m,1}^{\nu}, \mathcal{R}_{m,1}^{\nu} \)-bimodule \( 1_{\nu,i} \mathcal{R}_{m+1}^{\nu} \mathcal{R}_{m+1}^{\mu,j} \) has a filtration by graded bimodules whose associated graded is isomorphic to

\[ \delta_{i,j} \left( ^{0}R_{v} \otimes R_{s} \right) \oplus \delta_{\theta(i),j} \left( \left( ^{0}R_{v} \right)^{\gamma} \otimes R_{\theta(i)} \right) [d'] \oplus A[d], \]

where \( A \) is equal to

\[ \left( ^{0}R_{m} 1_{\nu,i} \otimes R_{s} \right) \otimes_{R^{c}} \left( 1_{\nu,i} ^{0}R_{m} \otimes R_{s} \right) \quad \text{if } m > 1, \]

\[ \left( ^{0}R_{\theta(j)} \otimes R_{s} \otimes R_{s} \otimes R_{s} \right) \left( ^{0}R_{\theta(i)} \otimes R_{j} \right) \oplus \left( ^{0}R_{j} \otimes R_{i} \otimes_{R_{s}} R_{i} \otimes R_{j} \right) \quad \text{if } m = 1. \]
Here we have set $\nu = \nu - j - \theta(j)$, $R' = \mathfrak{g}R_{m-1,1} \otimes R_1$, $i = i\theta(i)$, $j = j\theta(j)$, $d = -i \cdot j$, and $d' = -\nu \cdot (i + \theta(i))/2$.

The proof is standard and is left to the reader. Now, recall that for $m > 1$ we have
\[ f_j(P) = \mathfrak{g}R_{m+1,m-1} \otimes \mathfrak{g}R_{m-1,1} (P \otimes R_1), \quad e'_i(P) = 1_{m-1,i}P, \]
where $1_{m-1,i}P$ is regarded as a $\mathfrak{g}R_{m-1,1}$-module. Therefore we have
\[ e'_i f_j(P) = 1_{m,i} \mathfrak{g}R_{m+1,m-1} \otimes \mathfrak{g}R_{m-1,1} (P \otimes R_1), \]
\[ f_j e'_i(P) = \mathfrak{g}R_m 1_{m-1,i} \otimes \mathfrak{g}R_{m-1,1} (1_{m-1,i}P \otimes R_1). \]

Therefore, up to some filtration we have the following identities
\begin{itemize}
  \item $e'_i f_i(P) = P \otimes R_i + f_i e'_i(P)[-2],$
  \item $e'_i f_{\theta(i)}(P) = P \otimes R_{\theta(i)}[-\nu \cdot (i + \theta(i))/2] + f_{\theta(i)} e'_i(P)[-i \cdot \theta(i)],$
  \item $e'_i f_j(P) = f_j e'_i(P)[-i \cdot j]$ if $i \neq j, \theta(j).$
\end{itemize}

These identities also hold for $m = 1$ and $P = \mathfrak{g}R_{\theta(i)}$ for any $i \in I$. The first claim of part (a) follows because $R_i = \mathfrak{k} \oplus R_i[2]$. The fact that $\mathfrak{g}V$ is generated by $\phi_{\pm}$ is a corollary of Proposition 3.31 below. Part (b) of the theorem follows from [KM, prop. 2.2(ii)] and Lemma 3.9(b).

3.30. Remarks. (a) The $\mathfrak{B}$-module $\mathfrak{g}V$ is the same as the $\mathfrak{B}$-module $V_0$ from [KM, prop. 2.2]. The involution $\sigma : \mathfrak{g}V \to \mathfrak{g}V$ in [KM, rem. 2.5(ii)] is given by $\sigma(P) = P^\sigma$. It yields an involution of $\mathfrak{g}B$ in the obvious way. Note that Lemma 3.20(a) yields $\sigma(b) \neq b$ for any $b \in \mathfrak{g}B$.

(b) Let $\mathfrak{g}V$ be the $\mathfrak{B}$-module $\mathcal{K} \otimes \mathfrak{g}K$ and let $\phi$ be the class of the trivial $\mathfrak{g}R_0$-module $\mathfrak{k}$, see [VV, thm. 8.30]. We have an inclusion of $\mathfrak{B}$-modules
\[ \mathfrak{g}V \to \mathfrak{g}V, \quad \phi \mapsto \phi_+ \oplus \phi_-, \quad P \mapsto \text{res}(P). \]

3.31. Proposition. For any $b \in \mathfrak{g}B$ the following holds
\begin{align*}
\begin{cases}
  f_i(\mathfrak{g}^\text{low}(b)) = (\varepsilon_i(b) + 1) \mathfrak{g}^\text{low}((\tilde{F}_i)b) + \sum_{b'} f_{b,b'} \mathfrak{g}^\text{low}(b'), \\
  b' \in \mathfrak{g}B, \quad \varepsilon_i(b') > \varepsilon_i(b) + 1, \quad f_{b,b'} \in v^{2-\varepsilon_i(b')}\mathbb{Z}[v],
\end{cases} \\
\begin{cases}
  e_i(\mathfrak{g}^\text{low}(b)) = v^{1-\varepsilon_i(b)} \mathfrak{g}^\text{low}((\tilde{E}_i)b) + \sum_{b'} e_{b,b'} \mathfrak{g}^\text{low}(b'), \\
  b' \in \mathfrak{g}B, \quad \varepsilon_i(b') \geq \varepsilon_i(b), \quad e_{b,b'} \in v^{1-\varepsilon_i(b')}\mathbb{Z}[v].
\end{cases}
\end{align*}

Proof: We prove part (a), the proof for (b) is similar. If $\mathfrak{g}^\text{low}(b) = \phi_{\pm}$ this is obvious. So we assume that $\mathfrak{g}^\text{low}(b)$ is a $\mathfrak{g}R_m$-module for $m \geq 1$. Fix $\nu \in \mathfrak{g}M$
such that \( f_i(\mathcal{G}^{\text{low}}(b)) \) is a \( ^\circ R_\alpha \)-module. We’ll abbreviate \( l_{\nu,a} = 1_a \) for \( a \in \{ +, - \} \).
Since \( \mathcal{G}^{\text{low}}(b) \) is indecomposable, it fulfills the condition of Lemma 3.16. So there exists \( a \in \{ +, - \} \) such that \( l_{-a} f_i(\mathcal{G}^{\text{low}}(b)) = 0 \). Thus, by Lemma 3.15(c), (d) and Corollary 3.18 we have
\[
f_i(\mathcal{G}^{\text{low}}(b)) = 1_a \text{res } \text{ind } f_i(\mathcal{G}^{\text{low}}(b)) = 1_a \text{res } \text{ind}(\mathcal{G}^{\text{low}}(b)).
\]
Note that \( \theta b = \text{Ind}(b) \) belongs to \( \theta B \) by Lemma 3.20(b). Hence (3.5) yields
\[
\text{ind}(\mathcal{G}^{\text{low}}(b)) = \mathcal{G}^{\text{low}}(\theta b).
\]
We deduce that
\[
f_i(\mathcal{G}^{\text{low}}(b)) = 1_a \text{res } f_i(\theta b).
\]
Now, write
\[
f_i(\theta b) = \sum f_{\theta b_{1}, \theta b_{1}'} \mathcal{G}^{\text{low}}(\theta b'), \quad \theta b' \in \theta B.
\]
Then we have
\[
f_i(\mathcal{G}^{\text{low}}(b)) = \sum f_{\theta b_{1}, \theta b_{1}'} l_{a} \text{res}(\mathcal{G}^{\text{low}}(\theta b')).
\]
For any \( \theta b' \in \theta B \) the \( ^\circ R \)-module \( l_{a} \text{Res}(\theta b') \) belongs to \( \theta B \). Thus we have
\[
l_{a} \text{res}(\mathcal{G}^{\text{low}}(\theta b')) = \mathcal{G}^{\text{low}}(l_{a} \text{Res}(\theta b')).
\]
If \( \theta b' \neq \theta b'' \) then \( l_{a} \text{Res}(\theta b') \neq l_{a} \text{Res}(\theta b'') \), because \( \theta b' = \text{Ind}(l_{a} \text{Res}(\theta b')) \). Thus
\[
f_i(\mathcal{G}^{\text{low}}(b)) = \sum f_{\theta b_{1}, \theta b_{1}'} \mathcal{G}^{\text{low}}(l_{a} \text{Res}(\theta b')),
\]
and this is the expansion of the lhs in the lower global basis. Finally, we have
\[
\varepsilon_i(1_{a} \text{Res}(\theta b')) = \varepsilon_i(\theta b')
\]
by Lemma 3.23. So part (a) follows from [VV, prop. 10.11(b), 10.16].

3.32. The global bases of \( ^\circ V \). Since the operators \( e_i, f_i \) on \( ^\circ V \) satisfy the relations \( e_i f_i = v^{-2} f_i e_i + 1 \), we can define the modified root operators \( \tilde{e}_i, \tilde{f}_i \) on the \( ^\circ B \)-module \( ^\circ V \) as follows. For each \( u \in ^\circ V \) we write
\[
u = \sum_{n \geq 0} f_i^{(n)} u_n \text{ with } e_i u_n = 0,
\]
\[
\tilde{e}_i(u) = \sum_{n \geq 1} f_i^{(n-1)} u_n, \quad \tilde{f}_i(u) = \sum_{n \geq 0} f_i^{(n+1)} u_n.
\]
Let \( R \subset \mathcal{K} \) be the set of functions which are regular at \( v = 0 \). Let \( ^\circ L \) be the \( R \)-submodule of \( ^\circ V \) spanned by the elements \( \tilde{f}_i \cdots \tilde{f}_i(\phi_{l}) \) with \( l \geq 0, i_1, \ldots, i_l \in I \). The following is the main result of the paper.
3.33. Theorem. (a) We have

$$\mathcal{L} = \bigoplus_{b \in B} \mathcal{R} \mathcal{G}^\text{low} (b), \quad \mathcal{E}_i (\mathcal{L}) \subset \mathcal{L}, \quad \mathcal{F}_i (\mathcal{L}) \subset \mathcal{L},$$

$$\mathcal{E}_i (\mathcal{G}^\text{low} (b)) = \mathcal{G}^\text{low} (\hat{E}_i (b)) \mod v \mathcal{L}, \quad \mathcal{F}_i (\mathcal{G}^\text{low} (b)) = \mathcal{G}^\text{low} (\hat{F}_i (b)) \mod v \mathcal{L}.$$

(b) The assignment \( b \mapsto \mathcal{G}^\text{low} (b) \mod v \mathcal{L} \) yields a bijection from \( \mathcal{B} \) to the subset of \( \mathcal{L}/v \mathcal{L} \) consisting of the \( \hat{E}_i, \ldots, \hat{E}_i (\delta_B) \)'s. Further \( \mathcal{G}^\text{low} (b) \) is the unique element \( x \in \mathcal{V} \) such that \( x^2 = x \) and \( x = \mathcal{G}^\text{low} (b) \mod v \mathcal{L} \).

(c) For each \( b, b' \) in \( \mathcal{B} \) let \( E_{i, b, b'}, F_{i, b, b'} \in \mathcal{A} \) be the coefficients of \( \mathcal{G}^\text{low} (b') \) in \( e_{b(i)} (\mathcal{G}^\text{low} (b)), f_i (\mathcal{G}^\text{low} (b)) \) respectively. Then we have

$$E_{i, b, b'} |_{v=1} = [F_i \Psi \mathbf{for} (\mathcal{G}^\text{op} (b')) : \Psi \mathbf{for} (\mathcal{G}^\text{op} (b))],$$

$$F_{i, b, b'} |_{v=1} = [E_i \Psi \mathbf{for} (\mathcal{G}^\text{op} (b')) : \Psi \mathbf{for} (\mathcal{G}^\text{op} (b))].$$

Proof: Part (a) follows from \([E,K3, \text{thm. 4.1, cor. 4.4}], [E, \text{Section 2.3}], \) and Proposition 3.11. The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 3.11. More precisely, by duality we can regard \( E_{i, b, b'}, F_{i, b, b'} \) as the coefficients of \( \mathcal{G}^\text{op} (b) \) in \( f_i (\mathcal{G}^\text{low} (b')) \) and \( e_{b(i)} (\mathcal{G}^\text{low} (b)) \) respectively. Therefore, by Proposition 3.11 we can regard \( E_{i, b, b'} |_{v=1}, F_{i, b, b'} |_{v=1} \) as the coefficients of \( \Psi \mathbf{for} (\mathcal{G}^\text{op} (b')) \) in \( F_i \Psi \mathbf{for} (\mathcal{G}^\text{op} (b)) \) and \( E_i \Psi \mathbf{for} (\mathcal{G}^\text{op} (b')) \) respectively. \( \square \)

References

[A] Ariki, S., On the decomposition numbers of the Hecke algebra of \( \text{G}(m, 1, n) \), J. Math. Kyoto Univ. 36 (1996), 789-808.

[E] Enomoto, N., A quiver construction of symmetric crystals, Int. Math. Res. Notices 12 (2009), 2200-2247.

[EK1] Enomoto, N., Kashiwara, M., Symmetric Crystals and the affine Hecke algebras of type \( B \), Proc. Japan Acad. Ser. A Math. Sci. 83 (2007), 135-139.

[EK2] Enomoto, N., Kashiwara, M., Symmetric Crystals and LLT-Araki type conjectures for the affine Hecke algebras of type \( B \), Combinatorial representation theory and related topics, RIMS Köyōroku Bessatsu, B8, Res. Inst. Math. Sci. (RIMS), Kyoto, 2008, pp. 1-20.

[EK3] Enomoto, N., Kashiwara, M., Symmetric Crystals for \( \text{gl}_n \), Publications of the Research Institute for Mathematical Sciences, Kyoto University 44 (2008), 837-891.

[G] Grojnowski, I., Representations of affine Hecke algebras (and affine quantum \( \text{GL}_n \)) at roots of unity, Internat. Math. Res. Notices 5 (1994), 215.

[KL] Khovanov, M., Lauda, A. D., A diagrammatic approach to categorification of quantum groups I, Represent. Theory 13 (2009), 309-347.

[KM] Kashiwara, M., Miemietz, V., Crystals and affine Hecke algebras of type D, Proc. Japan Acad. Ser. A Math. Sci. 83 (2007), 135-139.

[L1] Lusztig, G., Study of perverse sheaves arising from graded Lie algebras, Advances in Math 112 (1995), 147-217.

[L2] Lusztig, G., Graded Lie algebras and intersection cohomology, arXiv:0604.535.

[M] Miemietz, V., On representations of affine Hecke algebras of type \( B \), Algebr. Represent. Theory 11 (2008), 369-405.
[Re] Reeder, M., *Isogenies of Hecke algebras and a Langlands correspondence for ramified principal series representations*, Representation Theory 6 (2002), 101-126.

[R] Rouquier, R., *2-Kac-Moody algebras*, arXiv:0812.5023.

[RR] Ram, A., Ramagge, J., *Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory*, A tribute to C. S. Seshadri (Chennai, 2002), Trends Math., Birkhäuser, Basel, 2003, pp. 428-466.

[VV] Varagnolo, M., Vasserot, E., *Canonical bases and affine Hecke algebras of type B*, arXiv:0911.5209.