THE EXCITATION SPECTRUM FOR
WEAKLY INTERACTING BOSONS

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Abstract. We investigate the low energy excitation spectrum of a Bose gas with weak, long range repulsive interactions. In particular, we prove that the Bogoliubov spectrum of elementary excitations with linear dispersion relation for small momentum becomes exact in the mean-field limit.

1. Introduction and Main Results

Bogoliubov’s seminal 1947 paper [1] contains several important results concerning the low energy behavior of bosonic systems. Among its striking predictions is the fact that the excitation spectrum is made up of elementary excitations whose energy is linear in the momentum for small momentum. Bogoliubov’s method is based on various approximations and crucially uses a formalism on Fock space that does not conserve particle number. Mathematically, the validity of his method has so far only been established for the ground state energy of certain systems, see [12, 13, 14, 4, 8, 16]. In particular, there are no rigorous results on the low energy excitation spectrum of interacting Bose gases, with the notable exception of exactly solvable models in one dimension [7, 9, 10, 2, 15].

In this article, we shall prove the validity of Bogoliubov’s approximation scheme for a Bose gas in arbitrary dimension in the mean-field (Hartree) limit, where the interaction strength is proportional to the inverse particle number, and its range extends over the whole system. In particular, we verify that the low energy excitation spectrum for such a system equals the sum of elementary excitations, as predicted by Bogoliubov. As a corollary, we observe that the lowest energy in the sector of total momentum \( P \) depends linearly on \(|P|\), a property that is crucial for the superfluid behavior of the system. The mean-field limit has served as a convenient and instructive toy model for several aspects of bosonic systems over the years. We refer to [5, 6] for a review and further references.

We consider a homogeneous system of \( N \geq 2 \) bosons on the flat unit torus \( \mathbb{T}^d \), \( d \geq 1 \). The bosons interact with a weak two-body interaction which we write for convenience as \( (N - 1)^{-1} v(x) \). We assume that \( v \) is positive, bounded, periodic (with period one), and \( v(-x) = v(x) \). We also assume that \( v \) is of positive type, i.e., it has only non-negative Fourier coefficients. With \( \Delta \) denoting the usual Laplacian on \( \mathbb{T}^d \), the Hamiltonian equals

\[
H_N = -\sum_{i=1}^{N} \Delta_i + \frac{1}{N - 1} \sum_{i<j} v(x_i - x_j)
\]
in suitable units. It acts on $L^2_{\text{sym}}(\mathbb{T}^d)$, the permutation-symmetric square integrable functions of $N$ variables $x_i \in \mathbb{T}^d$. Let $E_0(N)$ denote the ground state energy of $H_N$. The Bogoliubov approximation [11] predicts that $E_0(N)$ is close to $\frac{1}{2}N\tilde{v}(0) + E_{\text{Bog}}$, where

$$E_{\text{Bog}} = -\frac{1}{2} \sum_{\mathbf{p} \neq 0} \left( |\mathbf{p}|^2 + \tilde{v}(\mathbf{p}) - \sqrt{|\mathbf{p}|^4 + 2|\mathbf{p}|^2\tilde{v}(\mathbf{p})} \right).$$

The sum runs over $\mathbf{p} \in (2\pi \mathbb{Z})^d$, and

$$\tilde{v}(\mathbf{p}) = \int_{\mathbb{T}^d} v(x) e^{-i\mathbf{p}x} dx$$

are the Fourier coefficients of $v$. Note that the sum above converges, since the summands behave like $\tilde{v}(\mathbf{p})^2/|\mathbf{p}|^2$ for large $\mathbf{p}$.

More importantly, the Bogoliubov approximation predicts that the excitation spectrum of $H_N$ is made up of elementary excitations of momentum $\mathbf{p}$ with corresponding energy

$$e_p = \sqrt{|\mathbf{p}|^4 + 2|\mathbf{p}|^2\tilde{v}(\mathbf{p})}.$$  

One noteworthy feature of (2) is that it is linear in $\xi$ for small $p$, in contrast to the case when interactions are absent.

Our main results can be summarized as follows.

**THEOREM 1.** The ground state energy $E_0(N)$ of $H_N$ equals

$$E_0(N) = \frac{N}{2} \tilde{v}(0) + E_{\text{Bog}} + O(N^{-1/2}),$$

with $E_{\text{Bog}}$ defined in (1). Moreover, the spectrum of $H_N - E_0(N)$ below an energy $\xi$ is equal to finite sums of the form

$$\sum_{\mathbf{p} \in (2\pi \mathbb{Z})^d \setminus \{0\}} e_p n_p + O\left(\xi^{3/2} N^{-1/2}\right),$$

where $e_p$ is given in (2) and $n_p \in \{0, 1, 2, \ldots\}$ for all $p \neq 0$.

The error term $O(N^{-1/2})$ in (3) refers to an expression that is bounded, in absolute value, by a constant times $N^{-1/2}$ for large $N$, where the constant depends only on the interaction potential $v$; likewise for the error term $O(\xi^{3/2} N^{-1/2})$ in (4). The dependence on $v$ is rather complicated but can be deduced from our proof, which gives explicit bounds. Keeping track of this dependence allows to draw conclusions about the spectrum for large $N$ even if $v$ depends on $N$.

In the case of a fixed, $N$-independent $v$, Theorem 1 implies that the Bogoliubov approximation becomes exact in the mean-field (Hartree) limit. As long as $\xi \ll N^{1/3}$, each individual excitation energy $\xi$ is of the form $\sum_p e_p n_p$ with error $o(1)$. Moreover, as long as $\xi \ll N$, it is of the form $\sum e_p n_p(1 + o(1))$, i.e., the error is small relative to the magnitude of the excitation energy. In other words, in the mean field limit the whole excitation spectrum with energy $\xi \ll N$ is given in terms of sums of elementary excitations. The condition $\xi \ll N$ can be expected to be optimal, since only under this condition a large fraction of the particles are guaranteed to occupy the zero momentum mode, one of the key assumptions in the Bogoliubov approximation. For excitation energies of the order $N$ and larger, the spectrum will not be composed of elementary excitations anymore but has a more complicated structure.
Our proof shows that for each value of the \(\{n_p\}\) there exists exactly one eigenvalue of the form (4). Moreover, the eigenfunction corresponding to an eigenvalue with given \(\{n_p\}\) has total momentum \(\sum_p p n_p\). Given this fact we readily deduce the following corollary from Theorem 1.

**Corollary 1.** Let \(E_P(N)\) denote the ground state energy of \(H_N\) in the sector of total momentum \(P\). We have

\[
E_P(N) - E_0(N) = \min_{\{n_p\}} \sum_{p=p_0} e_p n_p + O \left( |P|^{3/2} N^{-1/2} \right).
\]

In particular, \(E_P(N) - E_0(N) \geq |P| \min_p \sqrt{2\hat{v}(p)} + |p|^2 + O(|P|^{3/2} N^{-1/2})\).

The linear dependence of \(E_P(N)\) on \(|P|\) is of crucial importance for the superfluid behavior of the Bose gas; see, e.g., the detailed discussion in [3]. It is a result of the interactions among the particles. The expression (5) differs markedly from the corresponding result for an ideal, non-interacting gas, especially if \(\hat{v}(0)\) is large.

Finally, we note that under the unitary transformation \(U = \exp(-i q \sum_{j=1}^N x_j)\), \(q \in (2\pi \mathbb{Z})^d\), the Hamiltonian \(H_N\) transforms as

\[U^\dagger H_N U = H_N + N|q|^2 - 2qP,\]

where \(P = -i \sum_{j=1}^N \nabla_j\) denotes the total momentum operator. Hence our results apply equally also to the parts of the spectrum of \(H_N\) with excitation energies close to \(N|q|^2\), corresponding to collective excitations where the particles move uniformly with momentum \(q\).

The remainder of this paper is devoted to the proof of Theorem 1. The main strategy is to compare \(H_N\) with Bogoliubov’s approximate Hamiltonian. The latter has to be suitably modified to take particle number conservation into account.

## 2. Preliminaries

We denote by \(P\) the projection onto the constant function in \(L^2(\mathbb{T}^d)\), and \(Q = 1 - P\). The operator that counts the number of particles outside the zero momentum mode will be denoted by \(N^>\), i.e.,

\[N^> = \sum_{i=1}^N Q_i.\]

We shall also use the symbol \(T\) for the kinetic energy \(T = -\sum_{i=1}^N \Delta_i\).

Lemma 1 below gives simple upper and lower bounds on the ground state energy \(E_0(N)\) of \(H_N\), as well as an upper bound on the expectation value of \(T\) in a low energy state.

**Lemma 1.** The ground state energy of \(H_N\) satisfies the bounds

\[0 \geq E_0(N) - \frac{N}{2} \hat{v}(0) \geq - \frac{N}{2(N-1)} (v(0) - \hat{v}(0)).\]

Moreover, in any \(N\)-particle state \(\Psi\) with \(\langle \Psi | H_N | \Psi \rangle \leq \frac{N}{2} \hat{v}(0) + \mu\) we have

\[(2\pi)^2 \langle \Psi | N^>^2 | \Psi \rangle \leq \langle \Psi | T | \Psi \rangle \leq \frac{N}{2(N-1)} (v(0) - \hat{v}(0)) + \mu.\]
Proof. The upper bound to the ground state energy follows from using a constant trial function. Since \( \hat{v} \geq 0 \), we have
\[
\sum_{p \in (2\pi Z)^d \setminus \{0\}} \hat{v}(p) \left| \sum_{j=1}^N e^{ipx_j} \right|^2 \geq 0.
\]
This inequality can be rewritten as
\[
\sum_{1 \leq i < j \leq N} v(x_i - x_j) \geq \frac{N^2}{2} \hat{v}(0) - \frac{N}{2} v(0) \tag{8}
\]
and thus
\[
H_N \geq \frac{N}{2} \hat{v}(0) + T - \frac{N}{2(N-1)} \left( v(0) - \hat{v}(0) \right).
\]
The rest follows easily. \( \square \)

In the following, we shall also need a bound on the expectation value of higher powers of \( N \). More precisely, we shall use the following lemma.

**Lemma 2.** Let \( \Psi \) be an \( N \)-particle wave function in the spectral subspace of \( H_N \) corresponding to energy \( E \leq E_0(N) + \mu \). Then
\[
(2\pi)^2 \langle \Psi \left| N^> T \right| \Psi \rangle \leq \left( v(0) + \frac{\mu}{2} \right)^2 + \frac{N}{2(N-1)} (\mu + 3v(0) + \hat{v}(0)) (2\mu + v(0) - \hat{v}(0)) .
\]
In particular, \( \langle \Psi \left| N^> T \right| \Psi \rangle \) is bounded above by an expression depending only on \( \mu \) and \( v \) but not on \( N \).

**Proof.** Since \( \Psi \) is permutation symmetric,
\[
\langle \Psi \left| N^> T \right| \Psi \rangle = N \langle \Psi \left| Q_1 S \right| \Psi \rangle + \langle \Psi \left| N^> \left( H_N - E_0(N) - \frac{1}{2} \mu \right) \right| \Psi \rangle
\]
where \( S = E_0(N) + \frac{1}{2} \mu - (N - 1)^{-1} \sum_{i<j} v(x_i - x_j) \). Using Schwarz’s inequality, the last term can be bounded as
\[
\langle \Psi \left| N^> \left( H_N - E_0(N) - \frac{1}{2} \mu \right) \right| \Psi \rangle \leq \frac{\mu}{2} \langle \Psi \left| (N^>)^2 \right| \Psi \rangle^{1/2} .
\]
We split \( S \) into two parts, \( S = S_a + S_b \), with
\[
S_a = E_0(N) + \frac{\mu}{2} - \frac{1}{N-1} \sum_{2 \leq i < j \leq N} v(x_i - x_j)
\]
and
\[
S_b = -\frac{1}{N-1} \sum_{j=2}^N v(x_1 - x_j) .
\]
Note that \( S_a \) does not depend on \( x_1 \). Using the positivity of \( \hat{v}(p) \) as in \( [8] \), but with \( N \) replaced by \( N - 1 \), as well as the upper bound on \( E_0(N) \) in \( [10] \), we see that
\[
S_a \leq \frac{1}{2} (\mu + \hat{v}(0) + v(0)) .
\]
In particular, this implies that
\[
N \langle \Psi \left| Q_1 S_a \right| \Psi \rangle \leq \frac{1}{2} (\mu + \hat{v}(0) + v(0)) \langle \Psi \left| N^> \right| \Psi \rangle .
\]
To bound the contribution of $S_b$, we use

$$-\langle \Psi | Q_1 S_b | \Psi \rangle = \langle \Psi | Q_1 v(x_1 - x_2) | \Psi \rangle + \langle \Psi | Q_1 P_2 v(x_1 - x_2) P_2 | \Psi \rangle + \langle \Psi | Q_1 P_2 v(x_1 - x_2) Q_2 | \Psi \rangle.$$  

The second term on the right side is positive. For the first and the third, we use Schwarz’s inequality and $\|v\|_\infty = v(0)$ to conclude

$$\langle \Psi | Q_1 S_b | \Psi \rangle \leq v(0) \langle \Psi | Q_1 Q_2 | \Psi \rangle^{1/2} + v(0) \langle \Psi | Q_1 | \Psi \rangle.$$

Since

$$N^2 \langle \Psi | Q_1 Q_2 | \Psi \rangle \leq \langle \Psi | (N^>)^2 | \Psi \rangle$$
we have thus shown that

$$\langle \Psi | N^> T | \Psi \rangle \leq \frac{1}{2} \left( \mu + \hat{v}(0) + 3v(0) \right) \langle \Psi | N^> | \Psi \rangle + \left( v(0) + \frac{1}{2} \mu \right) \langle \Psi | (N^>)^2 | \Psi \rangle^{1/2}.$$

Using $N^> \leq (2\pi)^{-2} T$ this yields

$$\langle \Psi | N^> T | \Psi \rangle \leq \left( \frac{v(0) + \frac{1}{2} \mu}{2\pi} \right)^2 + \left( \mu + 3v(0) + \hat{v}(0) \right) \langle \Psi | N^> | \Psi \rangle.$$  

The result then follows from Lemma 1.

3. The Bogoliubov Hamiltonian

The main strategy in the proof of Theorem 1 is to compare the Hamiltonian $H_N$ with the Bogoliubov Hamiltonian. We will use a slightly modified version of it, which is, in particular, particle number conserving.

Let $a_p$ and $a_p^\dagger$ denote the usual creation and annihilation operators on Fock space for a particle with momentum $p \in (2\pi \mathbb{Z})^d$, satisfying the canonical commutation relations $[a_p, a_q^\dagger] = \delta_{pq}$. For $p \neq 0$, let

$$b_p = \frac{a_p a_p^\dagger}{\sqrt{N - 1}}$$

and define the Bogoliubov Hamiltonian

$$H_{\text{Bog}} = \sum_{p \neq 0} \left[ |p|^2 b_p^\dagger b_p + \frac{1}{2} \hat{v}(p) \left( 2b_p^\dagger b_p + b_{-p}^\dagger b_{-p} + b_p b_{-p} \right) \right].$$

Note that $H_{\text{Bog}}$ conserves particle number. We are interested in $H_{\text{Bog}}$ in the sector of exactly $N$ particles. Note also that $\hat{v}(-p) = \hat{v}(p)$ for all $p \in (2\pi \mathbb{Z})^d$.

Let

$$A_p = |p|^2 + \hat{v}(p)$$

and

$$B_p = \hat{v}(p).$$
A simple computation (compare with [12, Thm. 6.3]) shows that
\[ A_p \left( b_p^+b_p + b_p^+\, p \right) + B_p \left( b_{-p}^+b_{-p} + b_{-p}^+\, p \right) = \sqrt{A_p^2 - B_p^2} \left( \frac{b_{-p}^+ + \alpha_{-p}b_{-p}}{1 - \alpha_{-p}^2} \right) + \frac{\left( b_{p}^+ + \alpha_{p}b_{-p} \right) \left( b_p - \alpha_{p}b_{-p} \right)}{1 - \alpha_{p}^2} \]
\[ - \frac{1}{2} \left( A_p - \sqrt{A_p^2 - B_p^2} \right) \left( [b_p, b_{p}^+] + [b_{-p}^+, b_{-p}] \right), \]
where
\[ \alpha_p = \frac{1}{B_p} \left( A_p - \sqrt{A_p^2 - B_p^2} \right) \quad \text{if} \; B_p > 0, \quad \alpha_p = 0 \quad \text{if} \; B_p = 0. \]

Note that \( 0 \leq \alpha_p \leq \hat{\varphi}(p)/(|p|^2 + \hat{\varphi}(p)) \). In particular, \( \sup_{p \neq 0} \alpha_p < 1 \), and \( \alpha(p) \sim \hat{\varphi}(p)/|p|^2 \) for large \( p \).

Define further
\[ c_p = \frac{b_p + \alpha_{p}b_{-p}}{\sqrt{1 - \alpha_{p}^2}} \]
for \( p \neq 0 \). What the above calculation shows is that
\[ H_{\text{Bog}} = -\frac{1}{2} \sum_{p \neq 0} \left( A_p - \sqrt{A_p^2 - B_p^2} \right) \frac{[b_p, b_{p}^+] + [b_{-p}^+, b_{-p}]}{2} + \sum_{p \neq 0} e_p c_p^+ c_p \]  
(11)

with \( e_p \) defined in (2). The commutators equal
\[ [b_p, b_{p}^+] = \frac{a_p^+a_0 - a_0^+a_p}{N - 1} \leq \frac{N}{N - 1}. \]  
(12)

4. PROOF OF THEOREM 1 - LOWER BOUND

Recall that \( P \) denotes the projection onto the constant function in \( L^2(\mathbb{R}^d) \), and \( Q = 1 - P \). Denote the two-particle multiplication operator \( v(x_1 - x_2) \) by \( v \) for short. Using translation invariance and the Schwarz inequality
\[ (P \otimes Q + Q \otimes P)vQ \otimes Q + Q \otimes Qv(P \otimes Q + Q \otimes P) \geq -\epsilon (P \otimes Q + Q \otimes P)v(P \otimes Q + Q \otimes P) - \epsilon^{-1}Q \otimes QvQ \otimes Q \]
(which follows from positivity of \( v \)) we conclude that
\[ v \geq P \otimes P_{\text{env}}P \otimes P + P \otimes P_{\text{env}}Q \otimes Q + Q \otimes Q_{\text{env}}P \otimes P \]
\[ + (1 - \epsilon)(P \otimes Q + Q \otimes P)v(P \otimes Q + Q \otimes P) - \epsilon^{-1}Q \otimes QvQ \otimes Q \]  
(13)

for any \( \epsilon > 0 \). The last term can be bounded by \( v(0)Q \otimes Q \). In second quantized language, this means that \( H_N \) is bounded from below by the restriction of
\[ \sum_{p} |p|^2 a_p^+ a_p + \frac{\hat{\varphi}(0)}{2(N - 1)} \left( N(N - 1) - 2\epsilon(N - N^-)N^+ - N^+ \left( N^+ - 1 \right) \right) \]
\[ + \sum_{p \neq 0} \frac{\hat{\varphi}(p)}{2} \left( 2(1 - \epsilon)b_p^+ b_{-p} + b_{-p}^+ b_p + b_p b_{-p} \right) - \frac{N^+ (N^+ - 1)v(0)}{2\epsilon(N - 1)} \]
to the $N$-particle sector. Here, we use again the definition (9) of $b_p$. From now on, we shall work with operators on Fock space, but it is always understood that we are only concerned with the sector of $N$ particles.

Next, we observe that

$$a_p^\dagger a_p \geq \frac{N - 1}{N} b_p^\dagger b_p.$$  

Moreover,

$$\sum_{p \neq 0} \tilde{\nu}(p) b_p^\dagger b_p \leq \frac{N}{N - 1} N^>,$$

and hence

$$H_N \geq \frac{N}{2} \tilde{\nu}(0) + H^{\text{Bog}} - E_\varepsilon$$

where the Bogoliubov Hamiltonian $H^{\text{Bog}}$ was defined in (11) and

$$E_\varepsilon = \frac{1}{N - 1} T + \frac{N^>(N^> - 1)}{2(N - 1)} \left( \tilde{\nu}(0) + \frac{\nu(0)}{\varepsilon} \right) + \varepsilon \nu(0) \frac{2N - 1}{N - 1} N^>.$$  

(14)

In particular, using (11) and (12),

$$H_N + E_\varepsilon \geq \frac{N}{2} \tilde{\nu}(0) + \frac{N}{N - 1} E^{\text{Bog}} + \sum_{p \neq 0} e_p c_p^\dagger c_p,$$  

(15)

where $E^{\text{Bog}}$ is the Bogoliubov energy defined in (1).

The last term on the right side of (15) is positive and can be dropped for a lower bound on the ground state energy of $H_N$. For the choice $\varepsilon = O(N^{-1/2})$, the expected value of $E_\varepsilon$ in the ground state of $H_N$ is bounded above by $O(N^{-1/2})$, as the bounds in Lemma 1 and 2 show. This proves the desired lower bound on $E_0(N)$.

To obtain lower bounds on excited eigenvalues, it remains to investigate the positive last term in (15). We do this via a unitary transformation. Let $U = e^X$, where

$$X = \sum_{p \neq 0} \beta_p \left( b_p^\dagger b_{-p}^\dagger - b_p^\dagger b_{-p} \right)$$

with $\beta_p \geq 0$ determined by

$$\tanh(2\beta_p) = \alpha_p.$$  

Note that $X$ is anti-hermitian and hence $U$ is unitary. If $a_0$ and $a_0^\dagger$ were replaced by $\sqrt{N - 1}$, $U$ would be the usual Bogoliubov transformation. Our modified $U$ has the advantage of being particle number conserving, however. The price to pay for this modification is that $U^\dagger a_q U$ can not be calculated anymore so easily.

A second order Taylor expansion yields

$$e^{-tX}a_q e^{tX} = a_q - t[X, a_q] + \int_0^t (t - s) e^{-sX} [X, [X, a_q]] e^{sX} ds$$

for any $t > 0$. We compute

$$[X, a_q] = -\frac{2}{N - 1} \beta_q a_{-q}^\dagger a_0^2$$
and
\[
[X, [X, a_q]] = 4\beta_q^2 a_q a_0^2 a_q^2 (N - 1)^2 - 4\beta_q \left( \sum_{p \neq 0} \beta_p a_p a_{-p} \right) a_0 \frac{2a_0^2 a_0 + 1}{(N - 1)^2} =: 4\beta_q^2 a_q + J_q.
\]

For any \( t > 0 \) we thus have
\[
e^{-tX} a_q e^{tX} = a_q + \frac{2t}{N - 1} \beta_q a_q^2 a_0^2 + \int_0^t (t - s)e^{-sX} (4\beta_q^2 a_q + J_q) e^{sX} ds.
\]

Iterating this identity leads to
\[
U^j a_q U = \cosh(2\beta_q) a_q + \sinh(2\beta_q) a_q \frac{a_0^2}{N - 1} + K_q =: d_q + K_q
\]
with
\[
K_q = \int_0^1 e^{-sX} J_q e^{sX} \frac{\sinh(2\beta_q(1 - s))}{2\beta_q} ds.
\]

In particular, we see that
\[
U^j a_q U = d_q + K_q + \sum_{j = 0}^{N - 1} (1 + \lambda) d_q + (1 + \lambda^{-1}) K_q
\]
for any \( \lambda > 0 \). We further have
\[
d_q^j d_p = c_q^j c_p \frac{a_q^j a_p^j}{1 - a_p^2} \left( a_q^0 a_0^0 \frac{a_0^2}{N - 1} - \frac{\alpha_p^2 a_{-p}^2}{1 - a_p^2} \right) \leq c_q^j c_p \frac{a_q^j a_p^j}{1 - a_p^2} \frac{N - 1}{N},
\]

Using Schwarz’s inequality, we can bound \( K_q^j K_q \) as
\[
K_q^j K_q \leq \frac{\cosh(2\beta_q) - 1}{(2\beta_q)^2} \int_0^1 e^{-sX} J_q^j J_q e^{sX} \frac{\sinh(2\beta_q(1 - s))}{2\beta_q} ds.
\]

To get an upper bound on \( J_q^j J_q \), we write \( J_q \) as the sum of two terms, \( J_q^{(1)} + J_q^{(2)} \), where \( J_q^{(2)} \) is the second term on the right side in the first line of (19), and \( J_q^{(1)} = 4\beta_q^2 a_q ((N - 1)^{-2} a_0^2 a_q^2 - 1) \). Using \( J_q^{(1)} J_q \leq 2J_q^{(1)} J_q^{(1)} + 2J_q^{(2)} J_q^{(2)} \) as well as
\[
\left( \sum_{p \neq 0} \beta_p a_p a_{-p} \right) \left( \sum_{q \neq 0} \beta_q a_q a_{-q} \right) \leq \left( \sum_{p \neq 0} \beta_p^2 \right)^2 N^>(N^> - 1),
\]
we obtain the bound
\[
J_q^j J_q \leq C_0 \beta_q \frac{(N^> + 1)^2}{N - 1}, \quad C_0 = \frac{64}{N - 1} \left( \sup_{q \neq 0} \beta_q^2 + \sum_{p \neq 0} \beta_p^2 \right).
\]

To proceed, we need an upper bound on \( e^{-sX} (N^> + 1)^2 e^{sX} \) for \( 0 \leq s \leq 1 \). For this purpose, let us compute
\[
[X, N^>] = -2 \sum_{q \neq 0} \beta_q \left( b_q b_{-q} + b_q b_{-q} \right).
\]
We have
\[
[X, N>]^2 \leq 8 \left( \sum_{p \neq 0} \beta_p b_p^\dagger b_{-p}^\dagger \right) \left( \sum_{q \neq 0} \beta_q b_q b_{-q} \right) + 8 \left( \sum_{p \neq 0} \beta_p b_p b_{-p} \right) \left( \sum_{q \neq 0} \beta_q b_{-q}^\dagger b_{-p}^\dagger \right)
\leq \frac{8}{(N-1)^2} \left( \sum_p \beta_p^2 \right) \left[ N> (N> - 1)(N + 1 - N>)(N + 2 - N>) + (N> + 1)(N> + 2)(N - N>)(N - 1 - N>) \right]
\leq 16 \left( \sum_p \beta_p^2 \right) \frac{N}{N - 1} (N> + 1)^2.
\] (21)

With the aid of Schwarz’s inequality, we obtain
\[
[X, (N> + 1)^2] = (N> + 1)[X, N>] + [X, N>](N> + 1)
\geq -\eta(N> + 1)^2 - \eta^{-1}[X, N>^2]
\]
for any \(\eta > 0\). In particular,
\[
[X, (N> + 1)^2] \geq -C_2(N> + 1)^2
\]
with
\[
C_2 = 8 \sqrt{\frac{N}{N - 1} \sum_{q \neq 0} \beta_q^2}.
\]

We conclude that
\[
e^{-tX}(N> + 1)^2 e^{tX} \leq (N> + 1)^2 + C_2 \int_0^t e^{-sX}(N> + 1)^2 e^{sX} ds
\]
for any \(t > 0\). Iterating this bound gives
\[
e^{-tX}(N> + 1)^2 e^{tX} \leq e^{tC_2}(N> + 1)^2.
\] (22)

In combination, (19), (20) and (22) yield the bound
\[
K_q^\dagger K_q \leq \frac{(N> + 1)^2}{N - 1} C_1 e^{C_2} \left( \frac{\cosh(2\beta_q) - 1}{16\beta_q^2} \right).
\] (23)

By combining (17) with (18) and (23), we obtain
\[
\sum_p e_p c_p^\dagger c_p \geq \frac{1}{1 + \lambda} U^\dagger \left( \sum_p e_p a_p^\dagger a_p \right) U - \frac{N> T}{N} \left( \sup_{p \neq 0} \frac{e_p}{|p|^2(1 - \alpha_p^2)} \right)
\]
\[- (1 + \lambda^{-1}) \frac{(N> + 1)^2}{N - 1} C_1 e^{C_2} \left( \sum_q e_q \frac{(\cosh(2\beta_q) - 1)^2}{16\beta_q^2} \right).
\] (24)

Note that \(|q|^2 \beta_q^2 \sim \hat{v}(q)^2/|q|^2\) for large \(q\), hence the last sum is finite. Applying this bound to (15), we have thus shown that
\[
H_N + \tilde{E}_{\varepsilon, \lambda} \geq \frac{N}{2} \hat{v}(0) + \frac{N}{N - 1} E_{\text{Bog}} + \frac{1}{1 + \lambda} U^\dagger \left( \sum_p e_p a_p^\dagger a_p \right) U
\]
where

\[
\tilde{E}_{\varepsilon, \lambda} = E_c + \frac{N^> T}{N} \left( \sup_{p \neq 0} \frac{e_p}{|p|^2 \left( 1 - \alpha_p^2 \right)} \right) \\
+ (1 + \lambda^{-1}) \frac{(N^> + 1)^2}{N - 1} C_1 e^{C_2 \left( \sum_q \frac{e_q \left( \cosh(2 \beta_q) - 1 \right)^2}{16 \beta_q^2} \right)},
\]

with \( E_c \) defined in (14).

The desired lower bound now follows easily from the min-max principle, and the fact that the spectrum of \( \sum_p e_p a_p^\dagger a_p \) equals \( \sum_p e_p n_p \), with \( n_p \in \{0, 1, 2, \ldots\} \) for all \( p \in (2\pi \mathbb{Z})^d \). In fact, for any function \( \Psi \) in the spectral subspace of \( H_N \) corresponding to energy \( E \leq E_0(N) + \xi \), we have

\[
\langle \Psi | \tilde{E}_{\varepsilon, \lambda} | \Psi \rangle \leq O \left( \left( \varepsilon + N^{-1} \right) \xi + \xi^2 N^{-1} \left( \varepsilon^{-1} + \lambda^{-1} \right) \right)
\]

according to Lemmas 1 and 2. The choice \( \varepsilon = O \left( \sqrt{\xi/N} \right) = \lambda \) then leads to the conclusion that the spectrum \( H_N \) below an energy \( E_0(N) + \xi \) is bounded from below by the corresponding spectrum of \( \hat{v}(0) + H_{\text{Bog}} + \sum_{p \neq 0} e_p a_p^\dagger a_p - O \left( \xi^{3/2} N^{-1/2} \right) \).

This completes the proof of the lower bound.

5. **Proof of Theorem 1: Upper Bound**

We proceed in essentially the same way as in the lower bound. In analogy to (14), we have

\[
v \leq P \otimes P v P \otimes P + P \otimes P v Q \otimes Q + Q \otimes Q v P \otimes P \\
+ (1 + \varepsilon)(P \otimes Q + Q \otimes P) v (P \otimes Q + Q \otimes P) + (1 + \varepsilon^{-1}) Q \otimes Q v Q \otimes Q
\]

for any \( \varepsilon > 0 \). Together with

\[
b_p^\dagger b_p \geq a_p^\dagger a_p \left( 1 - \frac{N^>}{N} \right)
\]

this implies the upper bound

\[
H_N \leq \frac{N}{2} \hat{v}(0) + H_{\text{Bog}} + F_\varepsilon
\]

where

\[
F_\varepsilon = \frac{N^>}{N} T + \varepsilon \hat{v}(0) \left( \frac{2 N - 1}{N - 1} N^> + (1 + \varepsilon^{-1}) \frac{N^> (N^> - 1) v(0)}{2(N - 1)} \right).
\]

From a lower bound to the commutator (12), namely

\[
[b_p, b_p^\dagger] + [b_{-p}, b_{-p}^\dagger] = \left( 2 a_{0}^\dagger a_0 - a_p^\dagger a_p - a_{-p}^\dagger a_{-p} \right) \geq 2 - \frac{3 N^>}{N - 1},
\]

we get

\[
H_{\text{Bog}} \leq E_{\text{Bog}} \left( 1 - \frac{3 N^>}{2(N - 1)} \right) + \sum_{p \neq 0} e_p c_p^\dagger c_p.
\]
Finally, to investigate the last term on the right side of this expression, we proceed as in (17)–(24), with the obvious modifications to get an upper bound instead of a lower bound. In replacement of (18) we use
\[ d_p^\dagger p d_p \geq c_p^\dagger p c_p - \frac{1}{N - 1} \frac{\alpha_p^2 N (N^\dagger + 1)}{1 - \alpha_p^2 N - 1}. \]
The result is
\[ \sum_p e_p c_p^\dagger p c_p \leq \frac{1}{1 - \lambda} U^\dagger \left( \sum_p e_p a_p^\dagger a_p \right) U + \frac{N (N^\dagger + 1)}{N - 1} \left( \sum_{p \neq 0} \frac{e_p \alpha_p^2}{1 - \alpha_p^2} \right) \]
\[ + \frac{T}{N - 1} \left( \sup_{p \neq 0} |e_p| (1 - |e_p|) \right) \]
\[ + \lambda^{-1} \frac{(N^\dagger + 1)^2}{N - 1} C_1 e^{C_2} \left( \sum_q e_q \left( \frac{\cosh(2\beta_q) - 1}{16\beta_q^2} \right) \right) \]
(27)
for any \( \lambda > 0. \)

Altogether, this shows that
\[ H_N \leq \frac{N}{2} \tilde{\sigma}(0) + E_{\text{Bog}} + \frac{1}{1 - \lambda} U^\dagger \left( \sum_p e_p a_p^\dagger a_p \right) U + \bar{F}_{\varepsilon, \lambda}, \]
with \( \bar{F}_{\varepsilon, \lambda} \) given by the sum of \( F_\varepsilon \) in (23), \( \frac{1}{2} N^\dagger (N - 1)^{-1} |E_{\text{Bog}}| \) from (26) and the last three terms in (27). To complete the upper bound, we need a bound on \( U \bar{F}_{\varepsilon, \lambda} U^\dagger. \) For this purpose, we find it convenient to first bound \( \bar{F}_{\varepsilon, \lambda} \) by
\[ F_{\varepsilon, \lambda} \leq C_3 \left( (\varepsilon^{-1} + \lambda^{-1}) \frac{(T + 1)^2}{N} + (\varepsilon + N^{-1}) T \right) \]
for an appropriate constant \( C_3 > 0. \) What remains to be shown is that
\[ U(T + 1)^2 U^\dagger \leq e^{C_4} (T + 1)^2 \]
(28)
for some constant \( C_4 > 0. \) Given (28), we obtain
\[ U H_N U^\dagger \leq \frac{N}{2} \tilde{\sigma}(0) + E_{\text{Bog}} + \frac{1}{1 - \lambda} \sum_{p \neq 0} e_p a_p^\dagger a_p \]
\[ + C_4 e^{C_4} \left( (\varepsilon^{-1} + \lambda^{-1}) \frac{(T + 1)^2}{N} + (\varepsilon + N^{-1}) T \right). \]
(29)
The spectrum of the operator on the right side of this inequality has exactly the desired form. Given an eigenvalue of \( \sum_{p \neq 0} e_p a_p^\dagger a_p \) with value \( \varepsilon, \) we choose \( \varepsilon = O(\sqrt{\xi/N}) = \lambda \) to obtain \( \frac{N}{2} \tilde{\sigma}(0) + E_{\text{Bog}} + \xi + O(\xi^{3/2} N^{-1} / 2) \) for the right side of (29). This gives the desired upper bound.

It remains to prove (28). This can be done in essentially the same way as in the proof of (22). In fact,
\[ [X, T] = -2 \sum_q \beta_q |q|^2 \left( b_q^\dagger b_{-q}^\dagger + b_q b_{-q} \right) \]
and hence, similarly to (21),
\[ [X, T]^2 \leq 16 \left( \sum_q |q|^4 \beta_q^2 \right) \frac{N}{N - 1} (N^\dagger + 1)^2. \]
Recall that $\beta_q^2 \sim \hat{v}(q)^2/|q|^4$ for large $q$, which implies the finiteness of the sum since $v$ is bounded by assumption (and hence, in particular, square integrable on $\mathbb{T}^d$).

By Schwarz,

$$[X, (T + 1)^2] \leq \eta(T + 1)^2 + \eta^{-1}[X, T]^2$$

for any $\eta > 0$. In particular,

$$[X, (T + 1)^2] \leq C_4(T + 1)^2$$

for some $C_4 > 0$. We conclude that

$$e^{tX} (T + 1)^2 e^{-tX} \leq (T + 1)^2 + C_4 \int_0^t e^{sX} (T + 1)^2 e^{-sX} ds$$

for any $t > 0$. Iterating this bound yields (25). This completes the proof of the upper bound, and hence the proof of Theorem 1.

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