On the irregular coloring of bipartite graph and tree graph families

Q A’yun, Dafik, R Adawiyah, Ika Hesti Agustin, and E R Albirri

1CGANT-Research Group, University of Jember, Indonesia
2Department of Mathematics Education, University of Jember, Indonesia
3Department of Mathematics, University of Jember, Indonesia

E-mail: 170210101086@students.unej.ac.id

Abstract. This article discusses irregular coloring. Irregular coloring was first introduced by Mary Radcliffe and Ping Zhang in 2007. The coloring $c$ is called irregular coloring if distinct vertices of $G$ have distinct codes. The color code of a vertex $v$ of $G$ with respect to $c$ is $\text{code}(v) = (a_0, a_1, \ldots, a_k)$, where $a_0 = c(v)$ and $a_i$ ($1 \leq i \leq k$) is the number of vertices that are adjacent to $v$ and colored $i$. The minimum $k$-color used in irregular coloring is called the irregular chromatic number and is denoted by $\chi_{ir}$. Irregular coloring is included in proper coloring, where each vertex that is the neighbors must not be the same color. The graphs used in this article are a family of bipartite graphs and a family of tree graphs, including complete bipartite graphs, crown graphs, star graphs, centipede graphs, and double star graphs.

1. Introduction
A pair $G = (V, E)$ with $E \subseteq E(V)$ is called a graph. The elements of $V$ are the vertices of $G$, and those of $E$ the edges of $G$. The vertex set of a graph $G$ is denoted by $V_G$ and its edge set by $E_G$. Therefore $G = (V_G, E_G)$ [7]. The degree of graph is the number of vertices adjacent to vertex $v$ and is denoted $\text{deg}(v)$.

The coloring of the graph is done by coloring the elements on the graph, be it vertices, sides, or regions. Coloring on a graph is called vertex coloring if all vertices on a graph with each neighboring vertex cannot have the same color. Coloring on a graph is called side coloring if the sides on a graph are colored with each adjacent side that cannot have the same color. Likewise with regional coloring, this coloring is done by coloring all areas on the graph, with neighboring areas not being the same color[8]. An edge is called adjacent if the same knot connects two adjacent sides. Meanwhile, when two regions are related to the same side, the area on a graph is called neighboring.

There are many kinds of graph coloring, including local irregularity vertex coloring. Define a condition $f$ if for every $uv \in E(G)$, $w(u) \neq w(v)$ and $\text{max}(l) = \min\{\text{max}\{l_i\}; l_i \text{ vertex irregular labelling}\}$. The chromatic number of local irregularity vertex coloring of $G$, denoted by $\chi_{lis}(G)$, is the minimum cardinality of the largest label over all such local irregularity vertex coloring[8]. Kristiana et. al [8] has researched local irregularity vertex coloring on path graphs, cycle graphs, complete graphs, Friendship graphs, wheel graphs, and complete bipartite graphs. Apart from that, Azahra [5] researched ladder graph and $H$ graph.
The next coloring type is gracefull coloring, A graceful coloring of a graph is a proper vertex coloring \( f : V(G) \rightarrow \{1, 2, \ldots, k\}\), where \( k \geq 2 \) which induces a proper edge coloring \( f' : E(G) \rightarrow \{1, 2, \ldots, k - 1\}\) defined by \( f'(uv) = |f(u) - f(v)|\). A vertex coloring \( f \) of graph is a graceful coloring if \( f \) is a graceful \( k \)-coloring for \( k \geq 2 \). Alfarisi et. al [4] has discussed gracefull coloring in tadepole graph and sun graph. Bell [6] has discussed gracefull graph in bipartite graph.

In addition, the topic of coloring on graphs is irregular coloring. Irregular coloring was first introduced by Mary Radcliffe and Ping Zhang in 2007. Irregular coloring is included in proper coloring of graph. A proper coloring of graph \( G \) is a function \( c : V(G) \rightarrow N \) having the property that \( c(u) \neq c(v) \) for every pair \( u, v \) of adjacent vertices of \( G \)[1]. The coloring \( c \) is called irregular if distinct vertices of \( G \) have distinct color codes. For a graph \( G \) and a positive integer \( k \), let \( c: V(G) \rightarrow [k] = \{1, 2, \ldots, k\} \) be a proper \( k \)-coloring of the vertices of \( G \). Here, the color code (or simply the code) of a vertex \( v \) of \( G \) with respect to \( c \) is the ordered \((k + 1)\)-tuple

\[
\text{code}(v) = (a_0, a_1, a_2, \ldots, a_k) = a_0a_1a_2\ldots a_k.
\]

Where \( a_0 = c(v) \) and \( a_i, (1 \leq i \leq k) \) is the number of vertices that are adjacent to \( v \) and colored \( i \). Consequently, if \( c(v) = i \), then \( a_i = 0 \), and \( \Sigma a_i = \deg Gv \). The irregular chromatic number \( \chi_{ir} \) of \( G \) is the minimum positive integer \( k \) for which \( G \) has an irregular \( k \)-coloring. Irregular coloring is included proper coloring of \( G \), it follows that [13]

\[
\chi(G) \leq \chi_{ir}(G)
\]

**Theorem 1** [13] For every pair \( a, b \) of integers with \( 2 \leq a \leq b \), there is a connected graph \( G \) with \( \chi(G) = a \) and \( \chi_{ir} = b \).

**Corollary 1** [13] For every graph \( G \), \( \omega(G) \leq \chi(G) \). The clique number \( \omega(G) \) of a graph \( G \) is the maximum order of a complete subgraph of \( G \).

**Observation 1** [13] Let \( c \) be a (proper) vertex coloring of a nontrivial graph \( G \) and let \( u \) and \( v \) be two distinct vertices of \( G \).

a. if \( c(u) \neq c(v) \), then \( \text{code}(u) \neq \text{code}(v) \).

b. if \( \deg G u \neq \deg G v \), then \( \text{code}(u) \neq \text{code}(v) \).

c. if \( c \) is irregular coloring and \( N(u) = N(v) \), then \( c(u) \neq c(v) \)

A. Rohoni and M. Venkatachalam have discussed irregular coloring in the triple star graph family [11], double wheels graph family [9], and fan graph family[12]. A. Rohini M. Venkatachalam and R. Sangamithra has discussed the irregular coloring of the graph flower [10], Avudainayaki et. al [3] have discussed irregular coloring on central and middle graphs of double star graphs, and Zhang[7] has discussed irregular coloring in petersen and unconnected graphs. Anderson et.al [2] has discussed irregular coloring in voltage graphs to construct graphs.

In this paper, we will discuss irregular coloring in several graphs. The topic of irregular coloring will be extended to the bipartite graphs family and tree graphs family. Complete bipartite graph, crown graph, and star graph are the bipartite graph family that will be studied. What will be studied in the tree graph family, meanwhile, are centipede and double star graphs.

### 2. Irregular Chromatic Number of Some Graphs

This paper focuses on some graphs to find the irregular chromatic number. We begin this section with the result of the following theorem, an irregular chromatic number of the complete bipartite graph, crown graph, star graph, centipede graph, and double star graph.

**Theorem 2** Irregular chromatic number of complete bipartite graph \( (K_{m,n}) \) is \( m+n \), for \( m, n \geq 2 \).
Proof. Let $V = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and $E = \{x_iy_{i+1}; 1 \leq i \leq n, 1 \leq j \leq n - 1\} \cup \{x_{i+1}y_i; 1 \leq i \leq n - 1, 1 \leq j \leq n\} \cup \{x_iy_j; 1 \leq i \leq n, 1 \leq j \leq n\}$. The degree of vertex are $deg(x_i) = n$, and $deg(y_i) = m$.

In order to prove $\chi_{ir}(K_{m,n}) = m + n$, we will show $\chi_{ir}(K_{m,n}) \leq m + n$ and $\chi_{ir}(K_{m,n}) \geq m + n$.

First, we will show that $\chi_{ir}(K_{m,n}) \leq m + n$. The first step is to label each vertex on the graph, where neighbors $x_i$ and $y_j$ will be colored differently. The obtained pattern is as follows:

$\chi_{ir}(K_{m,n}) = m + n$.

Based on the color function and color code, a different color is obtained for each vertex. Based on Observation 1.3(a), that if there is a different color on the vertex, a different color code will be obtained. So that, $\chi_{ir}(K_{m,n}) \leq m + n$.

Next, we prove that $\chi_{ir}(K_{m,n}) \geq m + n$. This proof uses contradiction, assume that $\chi_{ir}(K_{m,n}) = m + n - 1$, we will get two of the same color on the vertices. It is divided into three cases.

Case 1. For $c(x_i) = c(x_j)$

If $c(x_i)$ and $c(x_j)$ are the same color, we will get the same color code. So this contradicts the definition of irregular coloring where each vertex on the graph must have a different color.

Case 2. For $c(y_i) = c(y_j)$

If $c(y_i)$ and $c(y_j)$ are the same color, we will get the same color code. So this contradicts the definition of irregular coloring where each vertex on the graph must have a different color.

Case 3. For $c(x_i) = c(y_j)$

If $c(x_i)$ and $c(y_j)$ have the same color, while $c(x_i)$ and $c(y_j)$ are adjacent. We will get the same vertex color on the adjacent vertex. Thus, this contradicts the definition of proper coloring where each adjacent vertex must have a different color.

Based on the proof that has been explained, it is found that $\chi_{ir}(K_{m,n}) \leq m + n$ and $\chi_{ir}(K_{m,n}) \geq m + n$. So, it can be concluded that $\chi_{ir}(K_{m,n}) = m + n$.

![Figure 1](https://example.com/figure.png)

Figure 1. (a)$\chi_{ir}(K_{3,3}) = 6$ (b)$\chi_{ir}(B_n) = 4$

**Theorem 3** Irregular chromatic number of crown graph $(B_n)$ is $n + 1$, for $n \geq 3$.

**Proof.** Let $V = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\}$ and $E = \{x_iy_{i+1}; 1 \leq i \leq n, 1 \leq j \leq n - 1\} \cup \{x_{i+1}y_i; 1 \leq i \leq n - 1, 1 \leq j \leq n\}$. The degree of vertex are $deg(x_i) = n - 1$, and $deg(y_i) = n - 1$. The cardinality of the crown graph is $|V(B_n)| = 2n$ and $|E(B_n)| = n^2 - n$. 
In order to prove $\chi_{ir}(B_n) = n+1$, we will show that $\chi_{ir}(B_n) \leq n+1$ and $\chi_{ir}(B_n) \geq n+1$. First, we will show that $\chi_{ir}(B_n) \leq n+1$. The first step is label each vertex on the graph, provided that $x_i$ and $y_j$, where $i \neq j$, will have different colors. The obtained pattern is $c(x_i) = i, 1 \leq i \leq n-1; c(x_i) = n+1, i = n; c(y_j) = i, 1 \leq i \leq n$. Code representation of irregular coloring on a crown graph is as follows.

$$
col(x_i) = \begin{cases} 
(i, 0, 1, 1, ..., 1, 0), & \text{for } i = 1 \\
(i, 1, 0, 1, ..., 1, 1), & \text{for } i = 2 \\
(i, 1, 0, 1, ..., 1, 0), & \text{for } i = 3 \\
(i, 1, 1, 1, ..., 0, 0), & \text{for } i = n-1 \\
(i, 1, 1, 1, ..., 1, 0), & \text{for } i = n+1
\end{cases}
$$

Based on the color function and color code, there is the same color at vertices $x_i$ and $y_j$, but vertices adjacent to $x_i$ and $y_j$ are different, so we get a different color code. It can be concluded that $\chi_{ir}(B_n) \leq n+1$.

Next, we show that $\chi_{ir}(B_n) \geq n+1$. The proof uses contradiction, assume that $\chi_{ir} \geq n$, we will get two of the same color on the vertices. It is divided into three cases.

**Case 1.** $c(x_i) = c(y_j)$ for $i \neq j$

If $c(x_i)$ and $c(y_j)$ are the same color, we will get the same color code. So this contradicts the definition of irregular coloring where each vertex on the graph must have a different code.

**Case 2.** $c(x_i) = c(x_j)$

If $c(x_i)$ and $c(x_j)$ are the same color, we will get the same color code. So this contradicts the definition of irregular coloring where each vertex on the graph must have a different code.

**Case 3.** $c(y_i) = c(y_j)$

If $c(y_i)$ and $c(y_j)$ are the same color, we will get the same color code. So this contradicts the definition of irregular coloring where each vertex on the graph must have a different code.

Based on the proof that has been explained, it is found that $\chi_{ir}(B_n) \leq n+1$. and $\chi_{ir}(B_n) \geq n+1$. So, it can be concluded that $\chi_{ir}(B_n) = n+1$.

**Theorem 4** Irregular chromatic number of star graph $(S_n)$ is $n+1$, for $n \geq 3$.

**Proof.** Let $V = Ax_1, x_2, x_3, ..., x_n$ and $E = Ax_1, Ax_2, Ax_3, ..., Ax_n$. The degree of vertex are $\deg(A) = n$, and $\deg(x_i) = 1, 1 \leq i \leq n$. The cardinality of the star graph is $|V| = n+1$ and $|E| = 3n+1$.

In order to prove $\chi_{ir}(S_n) = n+1$, we will show that $\chi_{ir}(S_n) \geq n+1$ and $\chi_{ir}(S_n) \leq n+1$. First, we will show that $\chi_{ir}(S_n) \geq n+1$. The first step is label each vertex on the graph. The obtained pattern is $c(x) = 1; x_i = 2, 3, 4, \cdots, n$.

Code representation of irregular coloring on a crown graph is as follows.

$Code(x) = (1, 0, 1, 1, 1, ..., 1, 0)$

In order to prove $\chi_{ir}(B_n) = n+1$, we will show that $\chi_{ir}(B_n) \leq n+1$ and $\chi_{ir}(B_n) \geq n+1$. First, we will show that $\chi_{ir}(S_n) \geq n+1$. The first step is label each vertex on the graph. The obtained pattern is $c(x) = 1; x_i = 2, 3, 4, \cdots, n$.

Code representation of irregular coloring on a crown graph is as follows.

$Code(x) = (1, 0, 1, 1, 1, ..., 1, 0)$

In order to prove $\chi_{ir}(B_n) = n+1$, we will show that $\chi_{ir}(B_n) \leq n+1$ and $\chi_{ir}(B_n) \geq n+1$. First, we will show that $\chi_{ir}(S_n) \geq n+1$. The first step is label each vertex on the graph. The obtained pattern is $c(x) = 1; x_i = 2, 3, 4, \cdots, n$.

Code representation of irregular coloring on a crown graph is as follows.
\begin{equation*}
\text{Code}(x_i) = (i, 0, 0, 0, 0, \cdots, 0, 0)
\end{equation*}

Based on the color function obtained, all vertices on the graph have a different color, so each vertex will have a different color code. So, $\chi_{ir}(S_n) \leq n + 1$.

Next, we will show that $\chi_{ir}(S_n) \geq n + 1$. The proof uses contradiction, assume that $\chi_{ir}(S_n) \geq n$, we will get two of the same color on the vertices. It is divided into two cases.

**Case 1.** $c(x_i) = c(x_i)$

If $c(x_i)$ and $c(x_i)$ are the same color, while $x$ and $c(x_i)$ are adjacent, we will get the same color in the adjacent vertices. So this contradicts the definition of proper coloring where each adjacent vertex must have a different color.

**Case 2.** $c(x_i) = c(x_j)$

If $c(x_i)$ and $c(x_j)$ are the same color, while $c(x_i)$ and $c(x_j)$ are adjacent to $x$. So this contradicts the definition of irregular coloring where each vertex on the graph must have a different code.

Based on the proof that has been explained, it is found that $\chi_{ir}(S_n) \leq n + 1$ dan $\chi_{ir}(S_n) \geq n + 1$. So, it can be concluded that $\chi_{ir}(S_n) = n + 1$.

![Figure 2. Irregular Coloring](a) $\chi_{ir}(S_n) = 5$ (b) $\chi_{ir}(S_4) = 3$

**Theorem 5** For $k \geq 1, n \geq 3, k^2 + k + 1 \leq n \leq k^2 + 3k + 2$, irregular chromatic number of centipede graph is

\begin{equation*}
\chi_{ir}(C_n) = \begin{cases} 
2, & \text{for } n = 2 \\
2k + 2, & \text{for } n \geq 3
\end{cases}
\end{equation*}

**Proof.** Let $V = \{x_i, 1 \leq i \leq n\} \cup \{y_i, 1 \leq i \leq n\}$ and $E = \{x_1y_i, 1 \leq i \leq n\} \cup \{x_ix_{i+1}, 1 \leq i \leq n - 1\}$. The degree of each vertex are $\deg(x_1) = 2$, $\deg(x_i) = 3$, for $2 \leq i \leq n$, $\deg(x_n) = 2$, dan $\deg(y) = 1$. The cardinality of the double star graph are $|V| = 2n$ and $|E| = 2n - 1$.

In a centipede graph, it is divided into two cases. That is, when $n = 2$ and $n \geq 3$. First we prove it for $n = 2$. We will prove the upper bound $\chi_{ir}(C_2) \leq 2$ and lower bounds $\chi_{ir}(C_2) \geq 2$. To prove of upper bound, we use the color function. The color function obtained is $c(x_1) = 1, 2; c(y_j) = 1, 2$ where $i = j$ will be colored differently. The color code that we get is as follows.

- $\text{code}(x_1) = (1, 0, 2)$
- $\text{code}(x_2) = (2, 2, 0)$
- $\text{code}(y_1) = (2, 1, 0)$
- $\text{code}(y_2) = (1, 0, 1)$
based on the color function and color code obtained, we can conclude that $\chi_{ir}(C_n) = \chi + 1$. We will get the same color in the adjacent vertices. So, we can conclude that $\chi_{ir}(C_2) \geq 2$.

Second, we will prove that $\chi_{ir}(C_n) = k + 2$. We have to prove that upper bound $\chi_{ir}(C_n) \leq k + 2$ and $\chi_{ir}(C_n) \geq k + 2$. To prove upper bound, we use the following color function. Coloring the $x_i$ vertices using repeated colors between 1 and $\chi$, where the colors in $x_i$ and $y_i$, $x_i \in E(C_n)$ cannot have the same color.

Based on the color function obtained, each vertex has adjacent vertices which have a different color. Therefore, different color codes are obtained. So it can be concluded that $\chi_{ir}(C_n) \leq k + 2$.

Next, to prove the lower bound $\chi_{ir}(C_n) \geq k + 2$, we assume that $\chi_{ir}(C_n) < k + 2$, for example $\chi_{ir}(C_n) = k + 1$. If we use $k + 1$ color, we will get the same color on neighboring vertices, namely $x_i$ and $y_j$. Because the color of the vertex $y_i$ must have a color of $k + 1$.

**Theorem 6** Irregular chromatic number of double star graph is $n+1$, for $n \geq 3$.

**Proof.** Let $V = \{x\} \cup \{y\} \cup \{x_i, 1 \leq i \leq n\} \cup \{y_i, 1 \leq i \leq n\}$ and $E = \{xy\} \cup \{xx_i, 1 \leq i \leq n\} \cup \{yy_i, 1 \leq i \leq n\}$. The degree of vertex are $\deg(x) = \deg(y) = n + 1$, and $\deg(x_i) = \deg(y_i) = 1$. The cardinality of the crown graph is $|V| = 2n + 2$ and $|E| = 2n + 1$.

In order to prove $\chi_{ir}(K_{1,n,n}) = n + 1$ we will show that $\chi_{ir}(K_{1,n,n}) \leq n + 1$ and $\chi_{ir}(K_{1,n,n}) \geq n + 1$. First, we will show that $\chi_{ir} \leq n + 1$. The first step is labe each vertex on the graph. The obtain pattern is as follows. $c(x) = 1$

$c(y) = 2$;
$c(y_1) = 1$;
$c(x_1) = i + 1$;
$c(y_i) = i + 1$

Code representation of irregular coloring on a double star graph is as follows.

$code(x) = (1, 0, 2, 1, 1 \cdots , 1)$

$code(y) = (2, 2, 0, 1, 1 \cdots , 1)$

$code(y_1) = (1, 0, 1, 0, 0 \cdots , 0)$

$code(x_i) = (i + 1, 1, 0, 0, \cdots , 0)$ for $1 \leq i \leq n$

$code(y_i) = (i + 1, 1, 0, 1, 0, \cdots , 0)$ for $1 \leq i \leq n$

Based on the color pattern obtained, there will be the same color on adjacent vertices, namely the $x$ and $x_1$, as well as $y$ and $y_2$. This contradicts the definition of the proper coloring, where each neighboring vertex must have a different color.

Next, we will show that $\chi_{ir} \geq n + 1$. The proof uses contradiction, assume that $\chi_{ir} = n$. we will get two of the same color on the vertices. It is devided into two cases.

**Case 1.** $c(x) = c(x_i)$
If $c(x)$ and $c(x_i)$ are the same color, while $x$ and $x_i$ are adjacent, we will get the same color in the adjacent vertices. So this contradicts the definition of proper coloring where each adjacent vertex must have a different color.

**Case 2.** $c(y) = c(y_i)$
If $c(y)$ and $c(y_i)$ are the same color, while $y$ and $y_i$ are adjacent, we will get the same color in
the adjacent vertices. So this contradicts the definition of proper coloring where each adjacent
vertex must have a different color.

Based on the proof of the upper and lower bound that has been obtained, it is obtained
\( \chi_{ir} \leq n + 1 \) and \( \chi_{ir} \geq n + 1 \). So, \( \chi_{ir} = n + 1 \).

\[ \chi_{ir}(K_{m,n}) = m + n, \chi_{ir}(B_n) = n + 1, \chi_{ir}(S_n) = n + 1, \chi_{ir}(C_n) = k + 2 \]
and \( \chi_{ir}(K_{1,n,n}) = n + 1 \).

\[ \text{Figure 3. Irregular Coloring } \chi_{ir}(S_n) = 5 \]

3. Concluding Remarks
In some graphs, we have obtained the irregular chromatic numbers, namely the complete
bipartite graph \( (K_{m,n}) \), crown graph \( (B_n) \), star graph \( (S_n) \), centipede graph \( (C_n) \), and double
star graph \( (K_{1,n,n}) \). \( \chi_{ir}(K_{m,n}) = m + n, \chi_{ir}(B_n) = n + 1, \chi_{ir}(S_n) = n + 1, \chi_{ir}(C_n) = k + 2 \)
and \( \chi_{ir}(K_{1,n,n}) = n + 1 \).

4. Open Problem
Determine the lower and upper boundary of the graph on specific graph families, especially for
unicyclic graph families, book graph families, and graph operation of any simple graphs.

Acknowledgments
We gratefully acknowledge the support from CGANT - University of Jember of year 2021.

References
[1] Anitha K, Selvam B, and Thirusangu K 2019 New Irregular Coloring of Graphs Journal of Applied Science
and Computations VI V 269-279.
[2] Anderson M, Vitray R P, and Yellen J 2012 Irregular colorings of regular graphs Discrete Mathematics 312
2329-2336.
[3] Avudainayaki R, Selvam B, and Thirusangu K 2016 Irregular Coloring of Some Classes of Graphs International
Journal of Pure and Applied Mathematics 100 (10) 119-127.
[4] Alfarisi R, Dafik, Prihandini R M, Adawiyah R and Albirri E R 2019 Graceful Chromatic Number of Unicyclic
Graphs Journal of Physics 1306 1-8.
[5] Azahra N, Kristiana A I, Dafik, and Alfarisi R 2020 On the Local Irregularity Vertex Coloring of Related Grid
Graph International Journal of Academic and Applied Research (IJAAR) 4 (2) 1-4.
[6] Bell J R 2019 Products of graceful bipartite graphs Aequationes Mathematicae 93 1245-1280.
[7] Harju T 2011 Graph Theory (Departement of Mathematics).
[8] Kristiana A I, Dafik, Utomo M I, Slamir, Alfarisi R, Agustin I H, and Venkatachalam M 2019 Local Irregularity
Vertex Coloring of Graphs International Journal of Civil Engineering and Technology 10 (03) 1606-1616.
[9] Rohini A and Venkatachalam M 2019 On Irregular Colorings Of Double Wheel Graph Families. International
Conference on Current Scenario in Pure and Applied Mathematics 68 (1) 944-949.
[10] Rohini A, Venkatachalam M and Sangamithra R 2019 Irregular colorings of derived graphs of flower graph
ScMA Journal 10.1007/s40324-019-00201-1.
[11] Rohini A and Venkatachalam M 2019 On irregular colorings of triple star graph families *Journal of Discrete Mathematical Sciences and Cryptography* 10.1080/09720529.2019.1645394.

[12] Rohini A and Venkatachalam M 2018 On irregular colorings of fan graph families *International Conference on Applied and Computational Mathematics* 10.1088/1742-6596/1139/1/012061.

[13] Zhang P 2016 *A Kaleidoscopic View of Graph Colorings* (Springer Nature).