An algorithm to compute Born scattering amplitudes without Feynman graphs

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Abstract

In this paper we suggest an iterative algorithm to compute automatically the scattering matrix elements of any given effective lagrangian, Γ. By exploiting the relation between Γ and the connected Green function generator, Z, we provide a formula which does not require the use of the Feynman graphs and it is suitable to implement a numerical routine. By means of this algorithm we have built a relatively simple and fast fortran code which we have used to calculate, at the tree level, the rate of four fermion production at LEP II (finding a very good agreement with previous calculation) with and without the emission of one observable photon.

Introduction

The computation of the matrix element of physical processes is often a difficult task. With the increasing collider energies the complexity of the required computations has grown considerably. When the number of final particles is high, even to evaluate the corresponding tree level Feynman diagrams becomes hard and the final formula is often an intricated function of several variables, inadequate for practical use.

For this reason, many authors made a lot of work to simplify these calculations. Very good achievements have been obtained: the helicity amplitude technique \cite{1} shows how to manipulate the Feynman rules to compute by hand some extremely difficult multi-particle processes which otherwise would be impossible to handle, color decomposition, recursion

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relations and string inspired rearrangement of the diagrammatic expansion have allowed a drastic simplification in the number of graphs to be considered in some QCD processes, programs of symbolic manipulation provide a very useful tool to manage and evaluate the Feynman graphs and finally a completely automatic treatment of the Feynman rules is available.

Here we pursue a different approach. By exploiting the relation among the one particle irreducible Green Functions generator $\Gamma$ with the connected Green Functions generator $Z$ we are able to provide an explicit formula for $Z$ (for a given $\Gamma$) as a series expansion and we show that after the truncation of the series to a proper number of steps the exact scattering amplitude is recovered. We use this formula to implement a FORTRAN code and we present the computation of the rates for the processes $e^+ e^- \rightarrow \mu^+ \nu_\mu \tau^- \bar{\nu}_\tau$ and $e^+ e^- \rightarrow \mu^+ \nu_\mu \tau^- \bar{\nu}_\tau \gamma$ for some center of mass energies accessible at LEP.

### An iterative algorithm for the scattering matrix

If the one particle irreducible Green Functions generator, $\Gamma$, of a theory is known the computation of the $S$-matrix requires the evaluation of the legendre transform, $Z$, of $\Gamma$:

$$Z(j_1, \ldots, j_n) = -\Gamma(\varphi_1, \ldots, \varphi_n) + j_k(x)\varphi_k(x)$$

where $\varphi_j$ are the classical fields defined as the solution of

$$J_k = \frac{\delta\Gamma}{\delta\varphi_k}$$

and the $J_k$ play the role of classical sources.

For the sake of clarity, in the following we will stick to a specific example, namely we discuss the process

$$\phi(q_1) + \phi(q_2) \rightarrow \phi(q_3) + \phi(q_4)$$

in a $\lambda\phi^3$ theory. As a matter of convenience we use the convention that all the particles are incoming and therefore the momenta of the outcoming particle take an overall minus sign. From now on we define

$$p_1 = q_1 \quad p_2 = q_2 \quad p_3 = -q_3 \quad p_4 = -q_4$$

The lagrangian $\mathcal{L}$ of the model, is

$$\mathcal{L} = \int d^4x \left( -\frac{1}{2} \phi \partial^2 \phi + \frac{\lambda}{6} \phi^3 \right)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{1}{2} p^2 \delta(p + q) \phi(p)\phi(q) + \frac{\lambda}{6} \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \delta(p + q + k) \phi(p)\phi(q)\phi(k).$$

\footnote{In the following example we take the most simple lagrangian, but the generalization to an arbitrary effective lagrangian (see after) which also includes fermions and higher order operators (e.g. radiative corrections terms calculated by hand) is almost straightforward.}
Hence the Z functional for the theory is, at the tree level, the Legendre transform of (5)

\[ Z = -\mathcal{L} + \int d^4x \phi(x)J(x) \]  \hspace{1cm} (6)

In order to calculate the scattering matrix element for the process \( \text{(3)} \) we specify the source \( J(x) \) to be

\[ J(x) = \sum_{l=1}^{4} a_l e^{ip_l x} \]  \hspace{1cm} (7)

where \( a_l \) is a coefficient which might carry Lorentz and internal symmetry indices. The required matrix element is obtained after differentiation of \( Z \) with respect to the \( a_l \) in the limit \( a_l \to 0 \) and after having properly cut the external legs. With the source \( J(x) \) of eq. (7) the functional \( Z \) becomes

\[ Z(a_1, a_2, a_3, a_4) = -\mathcal{L} + \int d^4x \sum_{m=1,4} a_m e^{ip_m x} \phi(x) \]  \hspace{1cm} (8)

where \( \phi(x) \) is such that it minimizes (or more generally it picks up a stationary point) the right hand side of eq. (5) for any fixed set of \( a_m \), i.e. it is of the form

\[ \phi(x) = b_m e^{-iP_m x} \]  \hspace{1cm} (9)

with

\[ P_m = c^l_m p_l \quad c^l_m = 0, +1. \]  \hspace{1cm} (10)

In fact, even if the the fulfillment of the stationary condition for an arbitrary choice of the \( a_j \) would require \( c^l_m \) to take any integer value, we are interested in the Green functions which are recovered only after taking the limit \( a_j \to 0 \). In such a limit, these are the only momenta which survive and which contribute to the final result. In our specific case we have fourteen \( P_j \) (and \( b_j \))

\[ \begin{align*}
    P_1 &= p_1; & P_2 &= p_2; \\
    P_3 &= p_3; & P_4 &= p_4; \\
    P_5 &= p_1 + p_2; & P_6 &= p_1 + p_3; \\
    \ldots & \ldots & \ldots & \ldots \\
    P_{13} &= p_1 + p_3 + p_4; & P_{14} &= p_2 + p_3 + p_4;
\end{align*} \]

Replacing \( \phi(x) \) in eq. (8) we obtain, in momentum space,

\[ Z(a_1, a_2, a_3, a_4) = -\frac{1}{2} \sum_{j,l} b_j b_l \Delta_{jl} - \frac{1}{6} \sum_{j,l,m} b_j b_l b_m D_{jlm} + \sum_{m,l} a_m b_l \frac{\Delta_{ml}}{P^2_m} \]  \hspace{1cm} (11)

\footnote{Even if theoretically one should take only the real part of such \( J(x) \), in practice, for our pragmatic aim this is not significant.}

\footnote{After, when the explicit iterative solution for the stationary condition will be given, it will be manifest that other momenta will show up only with higher order \( a_j \).}
where the \( D_{ijk} \) and \( \Delta_{ml} \) take into account the momentum conservation, i.e.

\[
\begin{align*}
D_{ijk} &= \lambda \quad \text{if} \quad P_i + P_j + P_k = 0 \\
D_{ijk} &= 0 \quad \text{if} \quad P_i + P_j + P_k \neq 0.
\end{align*}
\]

\[
\begin{align*}
\Delta_{ml} &= P_m^2 \quad \text{if} \quad P_m + P_l = 0 \\
\Delta_{ml} &= 0 \quad \text{if} \quad P_m + P_l \neq 0.
\end{align*}
\]

(12)

and the \( b_j \) variables in eq. (11) satisfy the minimum equations

\[
\begin{align*}
a_i &= \frac{d\mathcal{L}}{db_i} = \Delta_i b_l + \frac{1}{2} D_{ijk} b_j b_k \quad i = 1, 4 \\
\text{and} \\
0 &= \frac{d\mathcal{L}}{db_i} = \Delta_i b_l + \frac{1}{2} D_{ijk} b_j b_k \quad i > 4
\end{align*}
\]

(13)

Here an important point should be noticed: we began to study the problem of finding the functional \( Z \) of eq. (6) which is, formally, a problem with an infinite number of degrees of freedom. Using eq. (7) with the constraint given in eq. (10) we have reduced it to the solution of eqs. (13) where the number of degrees of freedom is finite.

We now solve iteratively eq. (13) by replacing the \( b_j \) with the sum \( b_j = \sum_{r=0}^{\infty} b_{j,r} \) where the \( b_{j,r} \) are iteratively defined as functions of the \( b_{j,s} \), with \( s < r \), as follows:

\[
\begin{align*}
1 &= b_{i,0} \quad i = 1, 4 \\
0 &= b_{i,0} \quad i > 4
\end{align*}
\]

(14)

where we set \( a_i = p_i^2 \) to obtain the truncated Green function. At the second step the same eq. (13) gives

\[
\begin{align*}
b_{j,1} &= -\frac{1}{2} \Delta_{j,l}^{-1} D_{lkm} b_{k,0} b_{m,0}
\end{align*}
\]

(15)

and finally as last step

\[
\begin{align*}
b_{j,2} &= -\frac{1}{2} \Delta_{j,l}^{-1} D_{lkm} (b_{k,0} b_{m,1} + b_{k,1} b_{m,0})
\end{align*}
\]

(16)

To converge to the exact solution of eq. (13) one should continue the series of the \( b_{j,r} \) up to \( r = \infty \), however, for the purpose of computing the scattering amplitude, the terms we evaluated in (14,15,16) are sufficient. In fact the \( b_{j,r} \) series can be regarded as an expansion in the coupling \( \lambda \) and we have indeed truncated the series to the order \( \lambda^2 \).

We now substitute the obtained \( b_{j,r} \) in eq. (11). The result coincides with the required scattering amplitude

\[
\mathcal{A}_{p_1 p_2 p_3 p_4} = p_1^2 p_2^2 p_3^2 p_4^2 \lim_{a_i \to 0} \frac{\partial^4 Z(a_1, a_2, a_3, a_4)}{\partial a_1 \partial a_2 \partial a_3 \partial a_4}
\]
\[
\begin{align*}
\frac{1}{2} \sum_{s+r=2}^{14} b_{j,r} b_{l,s} \Delta_{jl} - \frac{1}{6} \sum_{s+r+t=1}^{14} D_{jlm} b_{j,r} b_{l,s} b_{m,t} \\
+ \sum_{l=1}^{4} \sum_{j=1}^{14} b_{j,2} \Delta_{jl} \\
= \lambda^2 \left( \frac{1}{(p_1 + p_2)^2} + \frac{1}{(p_1 + p_3)^2} + \frac{1}{(p_1 + p_4)^2} \right) .
\end{align*}
\]

One must notice that eq. (17) contains some redundancy and if one is wishing to use it to build a numerical routine he would immediately face the problem to handle quantities of the type 0/0. It is however possible to use eqs. (13) to simplify eq. (17) (and simultaneously overcoming the problem of dealing with 0/0 expressions). Multiplying both sides of eq. (13) by \( b_i \) and summing over the index \( i \), one obtains, from eq. (17),

\[
\frac{2}{p_1 p_2 p_3 p_4} \frac{\partial^4 Z(a_1, a_2, a_3, a_4)}{\partial a_1 \partial a_2 \partial a_3 \partial a_4} = -\frac{1}{6} \sum_{s,r,t}^{7} D_{jlm} b_{j,r} b_{l,s} b_{m,t}
\]

where the indices \( j, l, m \) are defined according to the conventional ordering given above. Notice that the evaluation of eq. (19) does not require the knowledge of the \( b_{j,2} \) of eq. (16) which are non zero only if \( j \geq 10 \). Due to this fact one does not need to evaluate the inverse of the propagator for on shell four momenta which occurs in eq. (16).

For an arbitrary lagrangian the previous iterations can be generalized in the following way

\[
\mathcal{L} = \frac{1}{2} \phi^\alpha \phi^\beta \Pi_{\alpha \beta} + \frac{1}{6} \phi^\beta \phi^\gamma \phi^\delta \mathcal{O}^{\beta \gamma \delta}
\]

\[
b_{j,0}^\alpha = \chi_j^\alpha \quad j = 1, n
\]

\[
b_{j,0}^\alpha = 0 \quad j > n \quad n = number \ of \ external \ particle
\]

\[
\ldots
\]

\[
b_{j,m}^\alpha = -\frac{1}{2} \Pi_{j,t}^{\alpha \beta} \mathcal{O}_{t,k,l}^{\beta \gamma \delta} \sum_{r+s=m-1} b_{k,r}^\gamma b_{l,s}^\delta
\]

\[
\mathcal{A}_{p_1, \ldots, p_n} = \frac{1}{2} \sum_{s+r=n-2} b_{j,l}^{\alpha \beta} \Pi_{j,t}^{\alpha \beta} b_{t,s}^\beta \left( \frac{1}{6} \sum_{s+r+t=n-3} \mathcal{O}_{j,k,l}^{\beta \gamma \delta} b_{j,r}^\gamma b_{k,s}^\delta b_{l,t}^\delta + b_{j,n-2} \chi_l^\beta \Pi_{j,l}^{\alpha \beta} \right)
\]

where greek symbols denote lorentz or internal symmetry indices, \( \phi \) denotes a generic field of the theory (a sum over distinct particles is understood), \( \chi_j^\alpha \) stand for a properly chosen source term (for instance , \( \chi_j^\alpha = u_{\alpha} \) for a scalar, \( \chi_j^\alpha = u_{\alpha} \) for a Dirac spinor with four components \( \alpha \), etc...), \( \Pi \) is the appropriate propagator, \( \Pi \) is the inverse of \( \Pi \) and \( \mathcal{O} \) is the generalization of the \( D_{jkl} \) of the eq. (13) which can depend on the spin, the momenta and any other internal quantum number. In eq. (20) we have included only trilinear

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4 In eqs. (20) we have implicitly assumed that all the fields \( \phi^\alpha \) are real. For charged fields, one can, obviously, use the more conventional (and practical) complex notation. We did not do it in order to avoid a tedious digression about the combinatorial factors.
interactions; notice, however, that any operator with a higher dimension can always be reduced to a set of trilinear and bilinear ones using a proper set of auxiliary fields.

A final remark is in order here. In eqs. (20) one has to take properly into account particle statistic. Formally this can be achieved introducing a set of Grassman variables \( \epsilon_j, \epsilon_{j+k} + \epsilon_{k-j} = 0 \), and setting \( u(p) \rightarrow \epsilon u(p) \) for the fermion sources where now \( u(p) \) is a vector of ordinary numbers. In practice this means that each term in the sums of eqs. (20) enters with a relative sign depending on the order of the \( b_j \).

Finally one should keep in mind that a simplification closely parallel to the one obtained in eq. (19) do always occur.

The advantage of the iterative eqs. (20) is that they can now be easily implemented in a fortran code.

**ALPHA, an ALgorithm to compute PHysical Amplitudes**

We have implemented a fortran code, ALPHA, based on the solution of eqs. (20). We have first studied a purely scalar theory which allows to check also processes with an high number of particles. We have then added fermions with a Yukawa type interaction and verified that the fermionic statistic is properly accounted for. Finally the complete electroweak Lagrangian, in the unitary gauge, has been added (at present the strong interactions are absent). A very important test for the code is provided by the gauge properties of the theory. The good behavior of the scattering amplitudes, calculated by ALPHA, with the increasing center of mass energies shows that delicate gauge cancellations do occur.

We have then compared our results with known one and reproduced the cross sections for many processes with two particles in the final state and the rates for the processes \( e^+e^- \rightarrow WWZ, e^+e^- \rightarrow ZZZ \) with a good accuracy.

We now want to add some technical details about the code. To stress again the simplicity of eqs. (20) we remark that the bulk of the code consists of less than 1000 FORTRAN executable statements. On top of this one has to add several peripheral subroutines, for example subroutines to read the input, to return the Dirac \( \gamma \) matrix, to perform four vector scalar products, etc. All these peripheral subroutines have two common features: they are completely trivial and they can be checked one by one in a completely independent way and therefore they do not increase the complexity of debugging.

Another important feature, which is again linked to the simplicity of eqs. (20), is the efficiency achieved by the code. As a practical demonstration of this fact we show in table II the CPU performance for various processes. It is clearly seen that the increment in CPU time is significantly smaller than the increment in the number of Feynman graphs and in their complexity. Furthermore ALPHA is only a first prototype of the algorithm and we believe that a large room for optimization is left.

**Emission of a hard photon in four fermion production at LEP II**

In this section we will use our technique to compute the cross section of four fermion production from \( e^+e^- \) collision at the LEP II center of mass energy and when an additional hard photon is found in the final state.
Table 1: Required CPU time for the computation (single precision) of 10000 matrix elements for various processes. All the Higgs couplings are set to zero. The times are given in seconds and the calculations have been performed with a DIGITAL machine ALPHA 3000/600 with 64M of memory.

| process                                      | cpu time (seconds) |
|----------------------------------------------|--------------------|
| \(e^+e^- \rightarrow \mu^+\nu_\mu \tau^-\bar{\nu}_\tau\) | 128                |
| \(e^+e^- \rightarrow \mu^+\nu_\mu \tau^-\bar{\nu}_\tau\gamma\) | 170                |
| \(e^+e^- \rightarrow \mu^+\nu_\mu \tau^-\bar{\nu}_\tau\gamma\gamma\) | 395                |
| \(e^+e^- \rightarrow e^+\nu_e e^-\bar{\nu}_e\gamma\) | 180                |
| \(e^+e^- \rightarrow e^+\nu_e e^-\bar{\nu}_e\gamma\gamma\) | 323                |
| \(e^+e^- \rightarrow e^+\nu_e e^-\bar{\nu}_e\gamma\gamma\) | 877                |

One important goal of the LEP/SLC collider program has been achieved by the precise measurements. After the first phase which has revealed an important and enlightening agreement for most [7, 8] of the experimental data and the Standard Model predictions [9], the measurement of \(W\)-boson mass at LEP II will complete the pattern of precision tests. The importance of this goal has been pointed out by several authors [10]. The most promising way to measure the charged boson mass at LEP II is to make a precise measurement of the cross section near above the threshold of \(W\)-pair production. The two body phase space is proportional to the velocity \(\beta\) of the two final \(W\)-boson; thus the total cross section near the threshold essentially behaves as \(\beta\). The strong dependence of this parameter from the boson mass means that the measurement of the total cross section can be seen as a precise measurement of the \(W\)-mass.

Experimentally the \(W\)-bosons can be detected only through their decay products. As a first approximation the rate for each process is given by the total \(W\)-pair cross section times the branching ratios of \(W\) into a pair of fermions, but the existence of a finite width and of other non-resonant diagrams introduce sizable corrections. Due to the experimental accuracy which is aimed to at LEP II the full computation of the process

\[
e^+ + e^- \longrightarrow 4 \text{ fermions}
\]  

is required[11]. In the first column of table (2) we show the results of the program ALPHA for the process

\[
e^+ + e^- \longrightarrow \mu^+ + \nu_\mu + \tau^- + \bar{\nu}_\tau
\]  

near above the \(W\)-pairs production threshold, and in the second column the set of the relevant input constants is shown. We have applied a cut over the invariant mass of \(\mu^+\), \(\nu_\mu\) and \(\nu_\tau\) of 3 GEV to eliminate the contribution from \(\tau\) pair production with subsequent \(\tau\) decay.

\[5\] However, when proper kinematical cuts are applied, there are semianalytical approximations which reproduce the correct result very accurately
The value of the $\sin^2(\theta_W)$ and of the $\alpha^{-1}_{QED}(2M_W)$ given in the table enter into the cross section through the tree level formula of the coupling describing the fermion and gauge boson SM interactions. Since the W-boson lifetime is finite, the cross section is only smoothly suppressed below the $S = 4M_W^2$ threshold; it increases up to the 200 GeV region and after decreases as $1/S$. The results are in agreement within the montecarlo error, with the existing computations for the process \((22)\).

Beyond the born level contribution, the QED virtual and real corrections are the most significant. Soft or collinear photons cannot be seen by real experimental detector, thus only a cross section that includes the emission of such photons can be compared with the experimental results \([11]\).

In the following we will not deal with the problem of infrared and collinear QED correction to the process \((22)\) and we will present the results of the study of the exclusive process

\[ e^+ + e^- \rightarrow \mu^+ + \nu_\mu + \tau^- + \bar{\nu}_\tau + \gamma. \]  

\((23)\)

The number of Feynman diagrams increases considerably and it is very difficult to handle the calculation with the ordinary methods. The matrix element is a function of eleven variables and also the numerical integration of such a big number of variables is not easy.

To achieve the numerical accuracy for each total cross section shown in table \((3)\), a set of about 1.7 millions of events has been collected. To perform the integral over the kinematical variables we have used the package VEGA \([12]\) to properly take into account the peaking behavior of scattering matrix element.

We have introduced two cut-off to keep the infrared divergences under control and to provide experimentally testable results:

- $E_\gamma > 1 G e V$ is the minimum photon energy required;

- $\theta > 200 \; m r a d \; (\cos(\theta) < 0.98)$ is the minimum angle allowed between the photon and each (both initial and final) charged fermion in the process.

The table \((3)\) shows the total cross section at various initial energies. If an integrated luminosity of about 500 $p b^{-1}$ is collected, few events will pass the above cuts with a
Table 3: $\tau^+ \nu_\tau \mu^- \bar{\nu}_\mu \gamma$ production rates: $E_\gamma > 1$ GeV and $\theta > 200$ mrad (see the text).

differential distribution as in figures (1) and (2). It is clear from figure (1) that there is a forbidden region (i.e. too low rate) and a region where some events will show up.

Conclusion

In this paper we have suggested an iterative algorithm which we have found very simple and effective to compute scattering amplitudes.

Differently from other approaches it never requires the explicit use of Feynman graphs since the truncated connected Green functions are directly derived by performing a numerical Legendre transform of the effective Lagrangian. We also have shown that the algorithm can be successfully used to implement a fortran code, that can calculate very cumbersome processes in a relatively short time. To provide an example of the performances of our code we have computed the cross sections for some relevant processes of physical interest during the LEP II phase, namely the processes $e^+e^- \rightarrow \mu^+\nu_\mu \tau^-\bar{\nu}_\tau$ and $e^+e^- \rightarrow \mu^+\nu_\mu \tau^-\bar{\nu}_\tau \gamma$.

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Figure Captions

Figure 1: differential cross section (fb/BIN) of $e^+e^- \rightarrow \mu^+\nu_\mu\tau^-\bar{\nu}_\tau\gamma$ as function of $0.42 < \cos(\theta) < 0.98$ and $1 \text{ GeV} < E_\gamma < 19.5 \text{ GeV}$ (for $E_{cm} = 200 \text{ GeV}$).

Figure 2: differential cross section (fb/BIN) of $e^+e^- \rightarrow \mu^+\nu_\mu\tau^-\bar{\nu}_\tau\gamma$ as function of $\cos(\theta)$ (for $E_{cm} = 200 \text{ GeV}$). $E_\gamma > 1 \text{ GeV}$ (see the text).
