SUMS OF MULTIPLICATIVE FUNCTIONS OVER A BEATTY SEQUENCE

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Abstract

We study sums involving multiplicative functions that take values over a nonhomogenous Beatty sequence. We then apply our result in a few special cases to obtain asymptotic formulas for quantities such as the number of integers in a Beatty sequence that are representable as a sum of two squares up to a given magnitude.

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1. Introduction

Let $A \geq 1$ be an arbitrary constant, and let $\mathcal{F}_A$ be the set of multiplicative functions $f$ such that $|f(p)| \leq A$ for all primes $p$ and

$$\sum_{n \leq N} |f(n)|^2 \leq A^2 N \quad (N \in \mathbb{N}).$$

Exponential sums of the form

$$S_{\alpha, f}(N) = \sum_{n \leq N} f(n)e(n\alpha) \quad (\alpha \in \mathbb{R}, \ f \in \mathcal{F}_A),$$

where $e(z) = e^{2\pi iz}$ for $z \in \mathbb{R}$, occur frequently in analytic number theory. Montgomery and Vaughan have shown (see [8, Corollary 1]) that the upper bound

$$S_{\alpha, f}(N) \ll A \frac{N}{\log N} + \frac{N(\log R)^{3/2}}{R^{1/2}}$$

holds uniformly for all $f \in \mathcal{F}_A$, provided that $|\alpha - a/q| \leq q^{-2}$ with some reduced fraction $a/q$ for which $2 \leq R \leq q \leq N/R$. In this paper, we use the Montgomery–Vaughan result to estimate sums of the form.

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where \( \alpha, \beta \in \mathbb{R} \) with \( \alpha > 1 \), \( f \in \mathcal{F}_A \), and \( \mathcal{B}_{\alpha, \beta} \) is the nonhomogenous Beatty sequence defined by
\[
\mathcal{B}_{\alpha, \beta} = \{ n \in \mathbb{N} : n = [\alpha m + \beta] \text{ for some } m \in \mathbb{Z} \}.
\]

Our results are uniform over the family \( \mathcal{F}_A \) and nontrivial whenever
\[
\lim_{N \to \infty} \frac{\log N}{\log \log N} \left| \sum_{n \leq N} f(n) \right| = \infty,
\]
a condition which guarantees that the error term in Theorem 1 is smaller than the main term. One can remove this condition, at the expense of losing uniformity with respect to \( f \), and still obtain Theorem 1 for any bounded arithmetic function \( f \) (not necessarily multiplicative) for which the exponential sums in (2) satisfy
\[
S_{\alpha, f}(N) = o\left( \sum_{n \leq N} f(n) \right) \quad (N \to \infty).
\]

The general problem of characterizing functions for which this relation holds appears to be rather difficult; see [1] for Bachman’s conjecture and his related work on this problem.

We shall also assume that \( \alpha \) is irrational and of finite type \( \tau \). For an irrational number \( \gamma \), the type of \( \gamma \) is defined by
\[
\tau = \sup \left\{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \left\lfloor \gamma n \right\rfloor = 0 \right\},
\]
where \([·]\) denotes the distance to the nearest integer. Dirichlet’s approximation theorem implies that \( \tau \geq 1 \) for every irrational number \( \gamma \). According to theorems of Khinchin [6] and Roth [10], \( \tau = 1 \) for almost all real numbers (in the sense of the Lebesgue measure) and all irrational algebraic numbers \( \gamma \), respectively; also see [2, 11].

Our main result is the following theorem.

**Theorem 1.** Let \( \alpha, \beta \in \mathbb{R} \) with \( \alpha > 1 \), and suppose that \( \alpha \) is irrational and of finite type. Then, for all \( f \in \mathcal{F}_A \),
\[
G_{\alpha, \beta, f}(N) = \alpha^{-1} \sum_{n \leq N} f(n) + O\left( \frac{N \log \log N}{\log N} \right),
\]
where the implied constant depends only on \( \alpha \) and \( A \).

The following corollaries are immediate applications of Theorem 1.
Corollary 2. The number of integers not exceeding \( N \) that lie in the Beatty sequence \( B_{\alpha,\beta} \) and can be represented as a sum of two squares is

\[
\#\{n \leq N : n \in B_{\alpha,\beta}, n = \square + \square \} = \frac{CN}{\alpha \sqrt{\log N}} + O\left(\frac{N \log \log N}{\log N}\right),
\]

where

\[
C = 2^{-1/2} \prod_{p \equiv 3 \mod 4} (1 - p^{-2})^{-1/2} = 0.76422365 \ldots
\]

is the Landau–Ramanujan constant.

To state the next result, we recall that an integer \( n \) is said to be \( k \)-free if \( p^k \nmid n \) for every prime \( p \).

Corollary 3. For every \( k \geq 2 \), the number of \( k \)-free integers not exceeding \( N \) that lie in the Beatty sequence \( B_{\alpha,\beta} \) is

\[
\#\{n \leq N : n \in B_{\alpha,\beta}, n \text{ is } k\text{-free} \} = \alpha^{-1} \zeta^{-1}(k)N + O\left(\frac{N \log \log N}{\log N}\right),
\]

where \( \zeta(s) \) is the Riemann zeta function.

Finally, we consider the average value of the number of representations of an integer from a Beatty sequence as a sum of four squares.

Corollary 4. Let \( r_4(n) \) denote the number of representations of \( n \) as a sum of four squares. Then

\[
\sum_{\substack{n \leq N \\ n \in B_{\alpha,\beta}}} r_4(n) = \frac{\pi^2 N^2}{2\alpha} + O\left(\frac{N^2 \log \log N}{\log N}\right).
\]

where the implied constant depends only on \( \alpha \).

Any implied constants in the symbols \( O \) and \( \ll \) may depend on the parameters \( \alpha \) and \( A \) but are absolute otherwise. We recall that the notation \( X \ll Y \) is equivalent to \( X = O(Y) \).

2. Preliminaries

2.1. Discrepancy of fractional parts We define the discrepancy \( D(M) \) of a sequence of real numbers \( b_1, b_2, \ldots, b_M \in [0, 1) \) by

\[
D(M) = \sup_{\mathcal{I} \subseteq [0,1]} \frac{|\mathcal{V}(\mathcal{I}, M) - |\mathcal{I}||}{M},
\]

where the supremum is taken over all possible subintervals \( \mathcal{I} = (a, c) \) of the interval \( [0,1) \), \( \mathcal{V}(\mathcal{I}, M) \) is the number of positive integers \( m \leq M \) such that \( b_m \in \mathcal{I} \), and \( |\mathcal{I}| = c - a \) is the length of \( \mathcal{I} \).
If an irrational number $\gamma$ is of finite type, we let $D_{\gamma, \delta}(M)$ denote the discrepancy of the sequence of fractional parts $\{\gamma m + \delta\}_{m=1}^{M}$. By [7, Theorem 3.2, Ch. 2], we have the following result.

**Lemma 5.** For a fixed irrational number $\gamma$ of finite type $\tau$ and for all $\delta \in \mathbb{R}$,

$$D_{\gamma, \delta}(M) \leq M^{-1/\tau + o(1)} \quad (M \to \infty),$$

where the function defined by $o(\cdot)$ depends only on $\gamma$.

### 2.2. Numbers in a Beatty sequence

The following result is standard in characterizing the elements of the Beatty sequence $B_{\alpha, \beta}$.

**Lemma 6.** Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$, and set $\gamma = \alpha^{-1}$ and $\delta = \alpha^{-1}(1 - \beta)$. Then $n = \lfloor \alpha m + \beta \rfloor$ for some $m \in \mathbb{Z}$ if and only if $0 < \{\gamma n + \delta\} \leq \gamma$.

From Lemma 6, an integer $n$ lies in $B_{\alpha, \beta}$ if and only if $n \geq 1$ and $\psi(\gamma n + \delta) = 1$, where $\psi$ is the periodic function with period one whose values on the interval $(0, 1]$ are given by

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \leq \gamma, \\ 0 & \text{if } \gamma < x \leq 1. \end{cases}$$

We wish to approximate $\psi$ by a function whose Fourier series representation is well behaved. This will give rise to the aforementioned exponential sum $S_{\alpha, f}(N)$. To this end, we use the result of Vinogradov (see [15, Ch. I, Lemma 12]) which states that for any $\Delta$ such that

$$0 < \Delta < \frac{1}{8} \quad \text{and} \quad \frac{1}{2} \leq \frac{1}{2} \min\{\gamma, 1 - \gamma\},$$

there exists a real-valued function $\Psi$ with the following properties:

(i) $\Psi$ is periodic with period one;

(ii) $0 \leq \Psi(x) \leq 1$ for all $x \in \mathbb{R}$;

(iii) $\Psi(x) = \psi(x)$ if $\Delta \leq \{x\} \leq \gamma - \Delta$ or if $\gamma + \Delta \leq \{x\} \leq 1 - \Delta$;

(iv) $\Psi$ can be represented by a Fourier series

$$\Psi(x) = \sum_{k \in \mathbb{Z}} g(k)e(kx),$$

where $g(0) = \gamma$ and the Fourier coefficients satisfy the uniform bound

$$g(k) \ll \min\{|k|^{-1}, |k|^{-2} \Delta^{-1}\} \quad (k \neq 0). \quad (7)$$

### 3. Proofs

#### 3.1. Proof of Theorem 1

Using Lemma 6, we rewrite the sum (4) in the form

$$G_{\alpha, \beta, f}(N) = \sum_{n \leq N} f(n)\psi(\gamma n + \delta).$$
Replacing $\psi$ by $\Psi$, we obtain
\[
G_{\alpha, \beta, f}(N) = \sum_{n \leq N} f(n)\Psi(\gamma n + \delta) + O\left(\sum_{n \in V(\Delta, N)} f(n)\right),
\] (8)
where $V(\Delta, N)$ is the set of positive integers $n \leq N$ for which
\[
\{\gamma n + \delta\} \in [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).
\]
Since the length of each interval above is at most $2\Delta$, it follows from definition (6) and Lemma 5 that
\[
|V(\Delta, N)| \ll \Delta N + N^{1-1/(2\tau)},
\]
where we have used the fact that $\alpha$ and $\gamma$ have the same type $\tau$. Thus, taking (1) into account, Cauchy’s inequality gives
\[
\left|\sum_{n \in V(\Delta, N)} f(n)\right| \leq |V(I, N)|^{1/2} \left(\sum_{n \leq N} |f(n)|^2\right)^{1/2}
\ll ((\Delta N)^{1/2} + N^{1/2 - 1/(4\tau)})N^{1/2}
= \Delta^{1/2} N + N^{1-1/(4\tau)}.\] (9)
Next, let $K \geq \Delta^{-1}$ be a large real number (to be specified later), and let $\Psi_K$ be the trigonometric polynomial given by
\[
\Psi_K(x) = \sum_{|k| \leq K} g(k)e(kx) = \gamma + \sum_{0 < |k| \leq K} g(k)e(kx) \quad (x \in \mathbb{R}).\] (10)
Using (7), we see that the estimate
\[
\Psi(x) = \Psi_K(x) + O(K^{-1}\Delta^{-1})
\]
holds uniformly for all $x \in \mathbb{R}$; therefore,
\[
\sum_{n \leq N} f(n)\Psi(\gamma n + \delta) = \sum_{n \leq N} f(n)\Psi_K(\gamma n + \delta) + O(K^{-1}\Delta^{-1}N),\] (11)
where we have used the bound $\sum_{n \leq N} |f(n)| \ll N$ which follows from (1).
Combining (8), (9), (10) and (11), we derive that
\[
G_{\alpha, \beta, f}(N) = \gamma \sum_{n \leq N} f(n) + H(N) + O(K^{-1}\Delta^{-1}N + \Delta^{1/2} N + N^{1-1/(4\tau)}),
\]
where
\[
H(N) = \sum_{0 < |k| \leq K} g(k)e(k\delta)S_{k\gamma, f}(N).
\]
Put \( R = (\log N)^3 \). We claim that if \( N \) is sufficiently large, then for every \( k \) in the above sum there is a reduced fraction \( a/q \) such that \( |k\gamma - a/q| \leq q^{-2} \) and \( R \leq q \leq N/R \). Assuming this is true for the moment, (3) implies that

\[
S_{k\gamma,f}(N) \ll \frac{N}{\log N} \quad (0 < |k| \leq K);
\]

using (7), we then deduce that

\[
H(N) \ll \frac{N \log K}{\log N}.
\]

Therefore,

\[
G_{\alpha,\beta,f}(N) - \gamma \sum_{n \leq N} f(n) \ll \frac{N \log K}{\log N} + K^{-1} \Delta^{-1} N + \Delta^{1/2} N + N^{1-1/4}.\]

To balance the error terms, we choose

\[
\Delta = (\log N)^{-2} \quad \text{and} \quad K = \Delta^{-3/2} = (\log N)^3,
\]

thus obtaining the bound stated in the theorem.

To prove the claim, let \( k \) be an integer with \( 0 < |k| \leq K = (\log N)^3 \), and let \( r_i = a_i/q_i \) be the \( i \)th convergent in the continued fraction expansion of \( k\gamma \). Since \( \gamma \) is of finite type \( \tau \), for every \( \varepsilon > 0 \) there is a constant \( C = C(\gamma, \varepsilon) \) such that

\[
C(|k|q_i)^{-\left(\tau+\varepsilon\right)} < \left\lfloor \gamma |k|q_i \right\rfloor \leq |\gamma |k|q_i - a_i| \leq q_i^{-1}.
\]

Put \( \varepsilon = \tau \), and let \( j \) be the least positive integer for which \( q_j \geq R \) (note that \( j \geq 2 \)). Then

\[
R \leq q_j \ll (|k|q_j)^{2\tau} \leq (KR)^{2\tau} = (\log N)^{6\tau},
\]

and it follows that \( R \leq q_j \leq N/R \) if \( N \) is sufficiently large, depending only on \( \alpha \). This concludes the proof. \( \square \)

### 3.2. Proof of Corollary 2

Let \( f(n) \) be the characteristic function of the set of integers that can be represented as a sum of two squares. It follows from [4, Theorem 366] that \( f(n) \) is multiplicative. Hence Corollary 2 is an immediate consequence of Theorem 1 and the asymptotic formula

\[
\sum_{n \leq N} f(n) = \frac{CN}{(\log N)^{1/2}} + O\left(\frac{N}{(\log N)^{3/2}}\right)
\]

(see, for example, [12, 13]), where \( C \) is given by (5). \( \square \)

### 3.3. Proof of Corollary 3

Fix \( k \geq 2 \) and let \( f(n) \) be the characteristic function of the set of \( k \)-free integers. It is easily proved that \( f(n) \) is multiplicative. Thus Corollary 3 follows from Theorem 1 and the following estimate of Gegenbauer [3] for the number of \( k \)-free integers not exceeding \( N \):

\[
\sum_{n \leq N} f(n) = \zeta^{-1}(k) N + O(N^{1/k}).
\]

\( \square \)
3.4. Proof of Corollary 4  Put $f(n) = r_4(n)/(8n)$. From Jacobi’s formula for $r_4(n)$, namely
\[ r_4(n) = 8(2 + (-1)^n) \sum_{\substack{d \mid n \atop d \text{ odd}}} d \quad (n \geq 1), \]
it follows that $f(n)$ is multiplicative and that $f(p) \leq 3/2$ for every prime $p$. Moreover, using the formula of Ramanujan [9] (see also [14]),
\[ \sum_{n \leq N} \sigma^2(n) = \frac{5}{6} \xi(3)N^3 + O(N^2(\log N)^2) \]
where $\sigma$ is the sum of divisors function, we obtain
\[ \sum_{n \leq N} |f(n)|^2 \leq \sum_{n \leq N} \frac{\sigma^2(n)}{n^2} = \frac{5}{2} \xi(3)N + O((\log N)^3) \]
by partial summation. Therefore, $f(n) \in \mathcal{F}_A$ for some constant $A \geq 1$. Applying Theorem 1, we deduce that
\[ \sum_{n \leq N \atop n \in B_{\alpha,\beta}} r_4(n) \frac{n}{N} = \alpha^{-1} \sum_{n \leq N} \frac{r_4(n)}{n} + O\left(\frac{N \log \log N}{\log N}\right), \]
where the implied constant depends only on $\alpha$.

From the asymptotic formula
\[ \sum_{n \leq N} r_4(n) = \frac{\pi^2 N^2}{2} + O(N \log N) \]
(see for example [5, p. 22]), partial summation gives
\[ \sum_{n \leq N} \frac{r_4(n)}{n} = \pi^2 N + O((\log N)^2). \]
Consequently,
\[ \sum_{n \leq N \atop n \in B_{\alpha,\beta}} \frac{r_4(n)}{n} = \alpha^{-1} \pi^2 N + O\left(\frac{N \log \log N}{\log N}\right). \]
Using partial summation once more, we obtain the statement of Corollary 4.  \(\square\)

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