In most conventional theories of physics, time is not a physical entity [1–7]. Instead, time appears in dynamical models as a parameter external to the systems evolving with respect to it, and thus its appearance is left unexplained. This is particularly problematic in quantum theory, where time appears as a completely classical parameter. For instance, in quantum field theory, time is a label of the external spacetime on which fields live. However, since matter and spacetime should be able to interact according to general relativity, one may argue that spacetime itself must be quantum due to the “totalitarian property” of quantum theory, which states that any system that can interact with a quantum system must itself be quantum (see, for example, [8]).

One way to address these issues is to assume that time is not fundamental and that its emergence can be explained in a relational way (see, for example, [9] and references therein). One of the most prominent quantum versions of this “timeless approach” is the Page-Wootters (PW) model [10]. Page and Wootters proposed a quantum model where the universe is in a stationary state and recovered the usual time evolution of quantum theory from the entanglement between two (quantum) subsystems of the universe: one is the system of interest, the other acts as a clock for the first. Significantly, in the PW model, time is associated with an operator of the clock. The other systems in the model universe evolve relative to the eigenvalues of this time operator.

The PW model is usually formulated in the Schrödinger picture of quantum theory [11–15], but it would be desirable to recast it in the Heisenberg picture. There are two important reasons for this. First, even if the Schrödinger and the Heisenberg pictures are empirically equivalent, their interpretation of physical reality is different, especially with regards to locality [16–19]. Second, in this picture it may be possible to find expressions depending on the clock’s time operator and thus make the quantum nature of time more explicit than in the Schrödinger picture.

The original PW model [10] and other works in the literature [9, 20–22] recover the Heisenberg equation for the system’s observables relative to the eigenvalues of the time operator. However, they do this by tracing over the clock’s degrees of freedom, thus obfuscating the quantum nature of time in the model.

Here, I show how to formulate the PW model in the Heisenberg picture without having to trace over the clock’s degrees of freedom. This is obtained by performing a unitary transformation on the model’s observables and state vector, which transforms the entangled state of the PW model to a separable Heisenberg state and, consequently, encodes all the information about the entanglement between the clock and the system in the model’s observables. In this formulation, the observables of the system depend on the clock’s time operator and the quantum nature of time is explicit. The usual time evolution of the system’s observables can then be recovered via the relative-state construction in the Heisenberg picture [23].

This approach provides a timeless description of the universe in the Heisenberg picture and a simple interpretation of the origin of time evolution. Moreover, as I will demonstrate, this procedure applies equally well to mixed state.

This work describes an approach and interpretation that are alternative to [9, 20–22], where the focus is on the Heisenberg picture description of the system alone, and to the approach of [24], where the initial assumption is that the observables of the system must obey a generalised equation of motion expressed in terms of a time operator. Here, starting from the assumption that the universe is in a stationary state, I recover the Heisenberg picture description of the whole universe via a unitary transformation on both the observables and the state vector. The initial assumption of [24] then follows from simple algebraic relations.

In the following, I start by reviewing the assumptions of the PW model and its Schrödinger picture formulation.

**Assumptions of the PW model.** The PW model relies on the following assumptions [10, 13]:

1. The universe can be divided into two subsystems.
2. The universe is in a stationary state.
3. The observables of the universe can be expressed in terms of the clock’s time operator.
4. The entanglement between the clock and the system is preserved.
5. The observables of the universe become functions of the clock operator.
6. The generalised equation of motion expressed in terms of a time parameter is recovered.

In an elegant model, Page and Wootters (Page and Wootters, 1983) showed how time can emerge in a timeless quantum universe through the entanglement between its subsystems. The Page-Wootters model for an “evolution without evolution” is usually formulated in the Schrödinger picture or in other ways that hide the quantum nature of time. In this work, I formulate the model in the Heisenberg picture starting from its Schrödinger picture version. This is achieved by transferring all the entanglement from the state vector of the universe to the operators. In this formulation, the observables become functions of a clock operator, thus making explicit the quantum nature of time. The usual Heisenberg evolution of the observables in terms of a classical time parameter can be recovered in a relational way. I also extend the model to include mixed states of the universe and apply these results to some simple scenarios.
One is the system of interest $\mathcal{S}$, the other acts as a clock $\mathcal{C}$ for the first. The Hilbert space of the universe can be decomposed as $\mathcal{H}_U = \mathcal{H}_S \otimes \mathcal{H}_C$. In the ideal case, $\mathcal{S}$ and $\mathcal{C}$ are non-interacting, i.e. the Hamiltonian of the universe is given by:
\[
\hat{\mathcal{H}} = \hat{H} \otimes \mathbf{1}_\mathcal{C} + \mathbf{1}_\mathcal{S} \otimes \hat{h},
\]
where $\hat{H}$ is the Hamiltonian of the system and $\hat{h}$ is the Hamiltonian of the clock.

2. The universe is in an eigenstate of its Hamiltonian $\hat{\mathcal{H}}$. Usually, the universe is assumed to be in the state with eigenvalue 0, but here I will consider any eigenstate $|\Psi^e\rangle$:
\[
\hat{\mathcal{H}} |\Psi^e\rangle = \varepsilon |\Psi^e\rangle,
\]
where the double-ket notation is just a visual aid to remember that the states of the universe are defined on $\mathcal{H}_S \otimes \mathcal{H}_C$. The states $|\Psi^e\rangle$ satisfying the constraint of Eq. (2) live in the physical Hilbert space $\mathcal{H}_{phy}$, a subspace of the Hilbert space of the universe $\mathcal{H}_U$ [14].

3. The clock has an observable $\hat{t}$ conjugate to $\hat{h}$:
\[
[\hat{t}, \hat{h}] = i,
\]
where I have set $\hbar = 1$.

4. $|\Psi^e\rangle$ is an entangled state of system and clock.

**Schrödinger picture.** The usual time evolution of the system can be derived in the following way [12]. Equation (3) implies that $e^{-i\hat{H}\theta} |t\rangle = |t + \theta\rangle \forall \theta \in \mathbb{R}$, where $|t\rangle$ is the eigenstate of $\hat{t}$ with eigenvalue $t$. Using the identity decomposition in terms of $\{|t\rangle\}_t$, one can write
\[
|\Psi^e\rangle = \int dt |\psi^e(t)\rangle_{\mathcal{S}} |t\rangle_{\mathcal{C}},
\]
where it is assumed that the state of the system $|\psi^e(t)\rangle_{\mathcal{S}}$ is normalised at all times. Now, using equations (2) and (1), one can find that
\[
e^{-i\varepsilon\theta} |\Psi^e\rangle = e^{-i\hat{\mathcal{H}}\theta} |\Psi^e\rangle = \int dt e^{-i\hat{t}\theta} |\psi^e(t)\rangle_{\mathcal{S}} |t + \theta\rangle_{\mathcal{C}},
\]
and thus, equating Eqs. (4) and (5), $|\psi^e(t)\rangle_{\mathcal{S}} = e^{-(\hat{\mathcal{H}} - \varepsilon \mathbf{1}_\mathcal{S})t} |\psi(0)\rangle_{\mathcal{S}}$, where I have fixed, without loss of generality, $|\psi^e(t)\rangle = |\psi(0)\rangle \forall \varepsilon$. This means that the state of the system relative to the clock being in the state $|t\rangle$ is given by
\[
c(t | \Psi^e) = |\psi(t)\rangle_{\mathcal{S}} = e^{-i(\hat{\mathcal{H}} - \varepsilon \mathbf{1}_\mathcal{S})t} |\psi(0)\rangle_{\mathcal{S}},
\]
which shows that the relative-state $|\psi(t)\rangle_{\mathcal{S}}$ satisfies the Schrödinger equation, and where the term $-\varepsilon$ represents an unobservable shift in energy. In this model, time emerges in a relational way: the time of the system is $t$ if the clock is in the state $|t\rangle_{\mathcal{C}}$.

The assumption that the system and clock are entangled is essential to ensure that in Eq. (4) there are at least two different $|\psi^e(t)\rangle_{\mathcal{S}}$ for two different values of $t$, so that the model describes a non-trivial time evolution of the system.

**Heisenberg picture.** Here, I show how to formulate the PW model in the Heisenberg picture. For simplicity, let me focus only on the state of the universe $|\Psi^e\rangle_{\mathcal{C}}$ and omit the upper index. I will consider different eigenstates in the next section. One can rewrite $|\Psi\rangle$ in Eq. (4) in the following way [12]:
\[
|\Psi\rangle = \int dt e^{-i\hat{H}t} |\psi(0)\rangle_{\mathcal{S}} |t\rangle_{\mathcal{C}} = e^{-i\hat{\mathcal{H}}_{\mathcal{C}} \otimes i} |\psi(0)\rangle_{\mathcal{S}} |TL\rangle_{\mathcal{C}},
\]
where the “Time Line” state $|TL\rangle_{\mathcal{C}}$ [12] is defined as $|TL\rangle_{\mathcal{C}} = \int dt |t\rangle_{\mathcal{C}} = \sqrt{2\pi h} |h = 0\rangle_{\mathcal{C}}$, with $|h = 0\rangle_{\mathcal{C}}$ the 0-eigenstate of the clock’s Hamiltonian.

Now, to move to the Heisenberg picture, I transform the observables of the universe with the unitary
\[
\hat{U}_H = e^{-i\hat{H} \otimes i}.
\]
In particular, if $\hat{O}_S$ is an observable of the system in the Schrödinger picture, the corresponding observable in the Heisenberg picture is:
\[
\hat{O}_H = e^{i\hat{H} \otimes i} \left( \hat{O}_S \otimes \mathbf{1}_\mathcal{C} \right) e^{-i\hat{H} \otimes i},
\]
which is an observable depending on the clock’s time operator $\hat{t}$ and has support on $\mathcal{H}_U$. The observables $\hat{H}$, $\hat{h}$ and $\hat{t}$ transform in the following way
\[
\hat{H}_H = \hat{U}^\dagger_H \hat{H} \otimes \mathbf{1}_\mathcal{C} \hat{U}_H = \hat{H} \otimes \mathbf{1}_\mathcal{C},
\]
\[
\hat{h}_H = \hat{U}^\dagger_H \left( \mathbf{1}_\mathcal{S} \otimes \hat{h} \right) \hat{U}_H = \mathbf{1}_\mathcal{S} \otimes \hat{h} - \hat{H} \otimes \mathbf{1}_\mathcal{C},
\]
\[
\hat{t}_H = \hat{U}^\dagger_H \left( \mathbf{1}_\mathcal{S} \otimes \hat{t} \right) \hat{U}_H = \mathbf{1}_\mathcal{S} \otimes \hat{t},
\]
which show that the Hamiltonian of the system and the time operator are invariant under $\hat{U}_H$, while the Hamiltonian of the clock is not. However, the Hamiltonian of the whole system is not invariant: $\hat{H}_H = \hat{U}^\dagger_H \hat{H} \hat{U}_H = \mathbf{1}_\mathcal{S} \otimes \hat{h}$. This is because here $\hat{U}_H$ is a formal transformation between the two pictures and not a time translation generated by $\hat{H}$ (which usually defines the transformation between the two pictures).

\footnote{Since the transformation between the Schrödinger and the Heisenberg pictures is unitary, the relation $\left[ \hat{h}_H, \hat{O}_H \right] = \left[ \hat{h} - \hat{H}, \hat{O}_H \right] = 0$ holds for every observable of the system $\hat{O}_H$. This commutation relation is the initial assumption of [24].}
For consistency with the Schrödinger picture, the Heisenberg state of the universe must be
\[ |\Psi_H\rangle \equiv |\psi(0)\rangle_\Phi |TL\rangle_\zeta, \quad (12) \]
which is a separable state of clock and system. The state \(|\Psi_H\rangle\) still satisfies the stationarity constraint \(\hat{H}_H\) \(|\Psi_H\rangle = 0\) but this is now entirely due to the state in the clock’s Hilbert space: \(\hat{h} |TL\rangle_\zeta = 0\). Incidentally, this also shows that one is free to choose any initial state of the system \(|\psi(0)\rangle_\Phi\) while still preserving the constraint \(\hat{H}_H\) \(|\Psi_H\rangle = 0\).

The Heisenberg operators \(\hat{O}_H\) and the Heisenberg state \(|\Psi_H\rangle\) are everything one needs to recover the time evolution of the system’s observables and their expectation values. Let me first notice that \(\hat{O}_H\) contains the “whole history” of \(\hat{O}_S: \langle\langle \hat{O}_H \rangle\rangle = \int dt \langle\langle \hat{O}_H \rangle\rangle = \langle\langle \hat{O} \rangle\rangle \langle\langle \hat{O}_H \rangle\rangle\), with \(\langle\langle \cdot \rangle\rangle \equiv \langle\langle \Psi_H | \cdot |\Psi_H\rangle\rangle\). In order to get one time instance of \(\hat{O}_H\), consider the following “relative observables” [23]:
\[ \hat{O}_H(t) \equiv \hat{O}_H \hat{\Pi}^\dagger(\hat{t})_H, \quad (13) \]
where \(\hat{\Pi}^\dagger(\hat{t})_H \equiv \hat{U}_H^\dagger \hat{\Pi}^\dagger(\hat{t}) \hat{U}_H = \hat{\Pi}^\dagger(\hat{t})_H\), where \(\hat{\Pi}^\dagger(\hat{t})\) is the projector on the eigenstate of \(\hat{t}\) with eigenvalue \(t\). Eq. (11) implies that \(\hat{\Pi}^\dagger(\hat{t})_H = \hat{\Pi}^\dagger(\hat{t})\). Eq. (13) describes the observables of the system relative to the clock being in the eigenstate of the time operator \(t\) with eigenvalue \(t\).

Since \(\hat{O}_H, \hat{\Pi}^\dagger(\hat{t})_H = 0\) for every Heisenberg-picture observable \(\hat{O}_H\) of the system, the relative observables form a “relative sub-algebra” at each time \(t\) [23]:
\[ (\hat{O}_H \hat{\Pi}^\dagger(\hat{t})_H)^2 = \hat{O}_H^2 \hat{\Pi}^\dagger(\hat{t})_H \]
\[ [\hat{O}_H \hat{\Pi}^\dagger(\hat{t})_H, \hat{Q}_H \hat{\Pi}^\dagger(\hat{t})_H] = [\hat{O}_H, \hat{Q}_H] \hat{\Pi}^\dagger(\hat{t})_H = \]
\[ = \left[ \hat{O}_S, \hat{Q}_S \right]_{\hat{S}} \hat{\Pi}^\dagger(\hat{t})_H, \]
for all the observables \(\hat{O}_H, \hat{Q}_H\) of the system, where \([\cdot, \cdot]_{\hat{S}} \equiv \hat{U}_H^\dagger [\cdot, \cdot] \hat{U}_H\). Furthermore, the expectation value of \(\hat{O}_H\) relative to the clock being in the eigenstate with eigenvalue \(t\) is given by [23]
\[ \langle\langle \hat{O}_H(t) \rangle\rangle = \langle\langle \hat{O}_H | e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t} |\psi(0)\rangle\rangle_\Phi, \]
in agreement with what one would have obtained from the PW model in the Schrödinger picture at time \(t\).

Since the Heisenberg state of the universe is separable with respect to clock and system, one can find a “reduced description” of \(\hat{O}_H(t)\) on the Hilbert space of the system given by
\[ \left(\hat{O}_H(t)\right)_{\Phi} = \frac{\text{Tr}_C \left[ \hat{O}_H(t) (1_\Phi \otimes \hat{\rho}_\zeta) \right]}{\text{Tr}_C \left[ \hat{\Pi}^\dagger(\hat{t})_H (1_\Phi \otimes \hat{\rho}_\zeta) \right]} = e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t}, \quad (14) \]
where \(\hat{\rho}_\zeta = \text{Tr}_S |\psi\rangle\langle\psi| = |TL\rangle \langle TL|_\zeta\). These are the usual Heisenberg-picture observables of the system.

The assumption that system and clock are entangled is still crucial. If the Schrödinger-picture state of the universe \(|\Psi\rangle\) were separable the unitary \(\hat{U}_H\) such that \(\hat{U}_H |\psi(0)\rangle_\Phi |TL\rangle_\zeta = |\Psi\rangle\) would be of the form \(\hat{V} \otimes \hat{W}\) with \(\hat{V}\) and \(\hat{W}\) local unitaries. Consequently, the Heisenberg picture observables of the system would not be affected by \(\hat{W}\), and thus they would not depend on the time operator \(t\).

**Properties of \(\hat{U}_H\)**. The unitary \(\hat{U}_H\) that I have found results in a good Heisenberg picture version of the PW model. However, one may wonder if this unitary is the only possible good transformation between the Schrödinger and the Heisenberg pictures. In other words, one has to motivate the choice of the unitary \(\hat{U}_H\) and of the Heisenberg state \(|\Psi_H\rangle\) such that \(\hat{U}_H |\Psi_H\rangle = |\Psi_t\rangle\).

Firstly, one must require that the transformation between the Schrödinger and the Heisenberg pictures transfers all the “time dependence” from the state of the universe to the observables. This means that there should be no correlation between the clock and the system. Therefore, the Heisenberg state should be separable with respect to clock and system: \(|\Psi_H\rangle = |X\rangle_\Phi |Y\rangle_\zeta\). Let me also notice that different pictures of the model could be obtained by transferring only part of the entanglement between state vector and observables.

The state \(|X\rangle_\Phi\) should correspond to the state of the system one wishes to start from. For convenience, one can choose \(|X\rangle_\Phi = |\psi(0)\rangle_\Phi\), but, of course, one can change the basis by applying a unitary transformation on both the state and the observables of the system. However, this means that one must transform all the operational meanings of the theory accordingly [17].

There is a similar freedom in the choice of the state of the clock \(|Y\rangle_\zeta\). Specifically, it is always possible to find a unitary \(\hat{W}\) such that \(\hat{W} |Y\rangle_\zeta = |TL\rangle_\zeta\). The correct transformation between the Schrödinger and Heisenberg picture associated to the Heisenberg state \(|\psi(0)\rangle_\Phi |Y\rangle_\zeta\) will then be given by the unitary \(\hat{U}_H (1_\Phi \otimes \hat{W})\) with \(\hat{U}_H\) as in Eq. (8). This does not change the factual situation provided that the operational meanings are transformed accordingly, which in this case means that one should use the transformed time operator \(i\hat{t}_H \equiv \hat{W}^\dagger i\hat{t} \hat{W}\) to define the relative observables: \(\hat{O}_H(t) = \hat{O}_H \hat{\Pi}^\dagger(\hat{t}_H)\). It is easy to see that the reduced observables of the system relative to the time \(t\) are still the same \(\langle\langle \hat{O}_H(t) \rangle\rangle_\zeta = e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t}\).

The unitary \(\hat{U}_H\) of Eq. (8) and the Heisenberg state of Eq. (12) are particularly convenient since \(\hat{U}_H\) leaves
unchanged the Hamiltonian of the system and the time operator (Eqs. (10), (11)).

A different kind of freedom in the choice of the Heisenberg state is associated to the energy eigenvalue of \(|\Psi\rangle\). As I have discussed above, the universe may be in a different eigenstate of its Hamiltonian \(|\Psi^\epsilon\rangle\). In this case, one can either use a different Heisenberg state \(|\Psi_H^\epsilon\rangle \equiv |\psi(0)\rangle_\epsilon \langle TL^\epsilon| \epsilon\rangle\) with \(|TL^\epsilon| \epsilon\rangle \equiv \int dt e^{i\epsilon t} |t\rangle_\epsilon = \sqrt{2\pi} |\lambda = \epsilon\rangle_\epsilon\) and the same unitary \(\hat{U}_H\) of Eq. (8), or one can keep fixed the Heisenberg state \(|\Psi_0^H\rangle\rangle\) and use a different unitary \(\hat{U}_H^\epsilon \equiv e^{-i(\hat{H} - \epsilon \hat{1}_E)i}\) since \(|\Psi^\epsilon\rangle\rangle = \hat{U}_H^\epsilon |\Psi_0^H\rangle\rangle\). In both cases, and similarly to the Schrödinger picture, the parameter \(\epsilon\) does not give rise to observable differences.

At this point, let me notice that if one fixes the possible Heisenberg state of the universe to one of \(|\psi(0)\rangle_\epsilon \langle TL^\epsilon| \epsilon\rangle\) according to the energy eigenvalue of the universe, then the unitary \(\hat{U}\) such that \(\hat{U} |\psi(0)\rangle_\epsilon \langle TL^\epsilon| \epsilon\rangle = \langle TL^\epsilon| \epsilon\rangle\) for any energy \(\epsilon\) and for any state \(|\psi(0)\rangle_\epsilon\) is unique. This can be proved in the following way. If \(\hat{U}_1\) and \(\hat{U}_2\) are two such unitaries, then, using Eqs. (4) and (6), one finds

\[
\begin{align*}
\langle TL^\epsilon| \epsilon\rangle \langle \psi_i| \epsilon\rangle \langle \psi_j| \epsilon\rangle \langle TL^\epsilon| \epsilon\rangle_e &= \\
= \int dt dt' e^{i(\epsilon t - \epsilon' t')} \delta(t - t') \delta_{ij} = 2\pi \delta(\epsilon - \epsilon') \delta_{ij},
\end{align*}
\]

(15)

for every \(\epsilon, \epsilon', i, j\), where I have chosen the initial states of the system from an orthonormal basis of the system's Hilbert space \(|\psi_i\rangle_\epsilon\). Now, since \(|\psi_i\rangle_\epsilon \langle TL^\epsilon| \epsilon\rangle\) is a complete set in the Hilbert space of the universe (and all the states have norm \(2\pi\)), Eq. (15) implies that \(\hat{U}_1 \hat{U}_2 = \mathbf{1}_E \otimes \mathbf{1}_E\), that is \(\hat{U}_1 = \hat{U}_2\). Hence, the unitary \(\hat{U}_H = e^{-i\hat{H}\theta}i\) is unique. This does not mean that there is a unique unitary transformation for a specific energy eigenstate of the universe. The unitary is unique only when one considers all the different energy eigenvalues of the Heisenberg state of the universe.

To conclude the discussion, let me notice that due to the energy constraint on the state of the universe in Eq. (2), in the model there is what one may call a gauge freedom. In the Heisenberg picture, the unitaries \(\hat{V}(\theta) \equiv \mathbf{1}_E \otimes e^{-i\hat{H}\theta}\) are gauge transformations, since \(\hat{V}(\theta) |\Psi^\epsilon_H\rangle = e^{-i\epsilon \hat{1}_E} |\Psi^\epsilon_H\rangle\) \(\forall \epsilon, \theta \in \mathbb{R}\). Under \(\hat{V}(\theta)\), the time operator \(\hat{t}\) undergoes a time translation: \(\hat{V}^\dagger(\theta) \left( \mathbf{1}_E \otimes \hat{t} \right) \hat{V}(\theta) = \mathbf{1}_E \otimes \left( \hat{t} - \theta \mathbf{1}_E \right)\). Therefore, all the observables related to \(\hat{O}_H\) by \(\hat{V}(\theta)\)

\[
\hat{V}^\dagger(\theta) \hat{O}_H \hat{V}(\theta) = e^{i\hat{H}\theta \otimes (\hat{t} - \theta \mathbf{1}_E)} \hat{O}_S e^{-i\hat{H}\theta \otimes (\hat{t} - \theta \mathbf{1}_E)},
\]

belong to the same equivalence class\(^2\).

Apart from this gauge freedom, some other transformations could change the Heisenberg observables of the universe but not their expectation values. This is, for example, the case of any non-trivial transformation that has \(|\Psi_H\rangle\rangle\) as an eigenstate. For a general closed system, these new observables describe a different physical situation even if all the expectations values are the same [17]. The difference lies in the different dynamics described by the old and the new observables, and it can be detected by “starting” with another Heisenberg state that is not an eigenstate of \(\hat{V}(\theta)\). However, in the case of the universe, it is not clear what it means to start with a different Heisenberg state and thus whether it is possible to detect, even in principle, the difference between the new and the old observables. I leave the questions regarding this possibility open for future discussion.

**Mixed states.** In the derivation of the PW above, I have assumed that the universe is in a pure state. However, in general, the universe could be in a mixed state. This would be the case, for example, if the region of the universe we live in was entangled with other systems not accessible to us. Here, I show how to formulate the PW model in the Heisenberg picture starting from a mixed state of the universe.

In this case, the stationarity constraint of Eq. (2) for the Schrödinger state of the universe \(\hat{\rho}_S\) becomes

\[
[\hat{\rho}_S, \hat{H}] = 0,
\]

(16)

so, in general, \(\hat{\rho}_S\) can be written as

\[
\hat{\rho}_S = \sum_i p_i |\Psi_i^\epsilon\rangle \langle \Psi_i^\epsilon|,
\]

(17)

where \(\hat{H} |\Psi_i^\epsilon\rangle \langle \Psi_i^\epsilon|\). Using Eq. (7), \(\hat{\rho}_S\) can be written as:

\[
\hat{\rho}_S = \hat{U}_H \left( \sum_i p_i |\psi_i(0)\rangle \langle \psi_i(0)| \otimes |TL^\epsilon_i\rangle \langle TL^\epsilon_i| \right) \hat{U}_H^\dagger,
\]

(18)

where \(\hat{U}_H\) is the unitary defined in Eq. (8) and the \(|\psi_i(0)\rangle\rangle_i\) are assumed to be, in general, different. Therefore, the system’s observables in the Heisenberg picture can be defined as in Eq. (9) and the Heisenberg state of the universe is

\[
\hat{\rho}_H = \sum_i p_i |\psi_i(0)\rangle \langle \psi_i(0)| \otimes |TL^\epsilon_i\rangle \langle TL^\epsilon_i|,
\]

(19)

which is a separable state of system and clock. The relative observables \(\hat{O}_H\) are defined as in Eq. (13) and the expectation value of \(\hat{O}_H\) relative to the clock being in the eigenstate with eigenvalue \(t\) is given by

\[
\frac{\text{Tr}_U}{\text{Tr}_U} \left[ \hat{O}_H(t) \hat{\rho}_H \right] = \frac{\text{Tr}_S}{\text{Tr}_S} \left[ e^{i\hat{H}t} \hat{O} e^{-i\hat{H}t} \hat{\rho}_S(0) \right],
\]

(20)

---

\(^2\) The observables found in [9] are the so-called “group average” of the observables in these equivalence classes [9, 26].
where $\rho_\Theta(0) \equiv \text{Tr}_C [\hat{\rho}_H] = \sum_i p_i \ket{\psi_i(0)} \bra{\psi_i(0)}_{\Theta}$.

Eq. (20) is what one would have obtained from the PW model in the Schrödinger picture at time $t$.

Finally, since the Heisenberg state of Eq. (19) is separable and since $\rho(\text{TL}^\varepsilon) = \rho(\text{TL}^\varepsilon)$ can be obtained from the time operator $\hat{O}_H(t)$, one can define the reduced relative observables as

$$\hat{O}_H(t)_{\Theta} \equiv \frac{\text{Tr}_C \left[ \hat{O}_H(t) (1_\Theta \otimes \hat{\rho}_\varepsilon) \right]}{\text{Tr}_C \left[ 1_\Theta \otimes \hat{\rho}_\varepsilon \right]} = e^{i\varepsilon t} \hat{O}_S e^{-i\varepsilon t},$$

where $\hat{\rho}_\varepsilon \equiv \text{Tr}_S [\hat{\rho}_H] = \sum_i p_i \ket{\text{TL}^\varepsilon} \bra{\text{TL}^\varepsilon}$, where these are the usual Heisenberg-evolved observables of the system.

In conclusion, the PW model can be formulated in the Heisenberg picture also for mixed states of the universe. Furthermore, the mixed state of Eq. (17) consists of an ensemble of eigenstates of $\hat{H}$ with different eigenvalues and is thus more general than the pure state of Eq. (2).

If the mixed state $\rho$ consists only of states with the same energy eigenvalue $\varepsilon$, then the Heisenberg state of Eq. (19) becomes

$$\rho_H = \rho_\Theta(0) \otimes |\text{TL}^\varepsilon\rangle \langle \text{TL}^\varepsilon|_{\Theta},$$

which is a product state. In general, however, if the initial states of the system $\{|\psi_i(0)\rangle\}_i$ are not orthogonal, the Heisenberg state, although separable, will have some other kinds of non-classical correlations [26]. I leave the questions regarding the implications of these correlations open for future investigation.

**Some simple models.** Here I show how the results obtained above can be applied to some simple scenarios. Let me consider a simple model where the system of interest is a qubit and the clock is infinite-dimensional with two conjugate observables $\hat{t}$ and $\hat{\theta}$. Let me assume that the Hamiltonian of the clock is $\hat{H} = H \otimes 1_\varepsilon + 1_\varepsilon \otimes \hat{\theta}$ with $H = \sigma_z/2$. Here $\sigma_{x,y,z}$ denote the Pauli matrices. Let me also assume that the state of the clock is an eigenstate of $\hat{H}$ with eigenvalue 0 and that the system is in the state $|\psi(0)\rangle_{\Theta}$ at time $t_0 = 0$. Since all the qubit’s observables can be expressed in terms of two generators of its algebra, one can choose to keep only track of $(\sigma_x \otimes 1_\Theta, \sigma_y \otimes 1_\Theta)$ [27].

To move to the Heisenberg picture, one can choose the Heisenberg state $|\psi(0)\rangle_{\Theta} \otimes |\text{TL}^{\varepsilon}\rangle$ and act with the unitary $\hat{U}_H = e^{-i\hat{H} \otimes \hat{t}}$ on the generators obtaining

$$\left(\sigma_x \otimes 1_\Theta, \sigma_y \otimes 1_\Theta\right) \rightarrow \left(\sigma_x \otimes 1_\Theta, \sigma_x \otimes \cos (\hat{t}) + \sigma_y \sin (\hat{t})\right),$$

where the dependence of the system’s observables on the time operator $\hat{t}$ is explicit. Moreover, the qubit’s generators relative to the clock being in the eigenstate with eigenvalue $\varepsilon$ are given by

$$\left(\sigma_x \Pi(\hat{t}), [\sigma_x \cos (\hat{t}) + \sigma_y \sin (\hat{t})] \Pi(\hat{t})\right).$$

Using equation (14) one can get the evolved generators of the system $(\sigma_x, \sigma_y)$ that one would have obtained through the Heisenberg equation.

As a further simplification, let me consider a single qubit [28] and that the time operator $\hat{t}$ is given by $\hat{t} \equiv t_0 \otimes 0 + t_1 \otimes 1_{\Theta}$, where $0_{\Theta} = |0\rangle_{\Theta}$, $1_{\Theta} = |1\rangle_{\Theta}$, and $|0\rangle_{\Theta} = |0\rangle_{\Theta}$. In this case, to avoid a trivial evolution, one must consider only non-degenerate Hamiltonians of clock and system such that their sum has a doubly degenerate zero eigenvalue$^3$.

For example, one can choose $\hat{H} = \sigma_x/2$ and $\hat{\theta} = -\sigma_z/2$, then the Heisenberg state of Eq. (19) becomes

$$\hat{O}_H(t)_{\Theta} = \text{Tr}_S \left[ \hat{O}_H(t) (1_\Theta \otimes \hat{\rho}_\varepsilon) \right] = e^{i\varepsilon t} \hat{O}_S e^{-i\varepsilon t},$$

where $\hat{\rho}_\varepsilon \equiv \text{Tr}_S [\hat{\rho}_H] = \sum_i p_i \ket{\text{TL}^\varepsilon} \bra{\text{TL}^\varepsilon}$, where $\ket{\text{TL}^\varepsilon}$ is the +1 (-1) eigenstate of $\hat{O}_S$.

Finally, let me check that these results are consistent with the state of the system in the Schrödinger picture. To do so, let me write the density matrix of the system as

$$\rho_\Theta(t) = \frac{1}{2} \left( 1 + \sum_i \lambda_i(t) \sigma_i \right),$$

with $\lambda_i(t) = \text{Tr}_S [\rho_\Theta(t) \sigma_i] = \text{Tr}_S [\rho_\Theta(t_0) \sigma_i(t)]$, where $\{\sigma_i(t)\}_i$ are the Heisenberg-evolved Pauli matrices after a time $t$ and $\rho_\Theta(t_0) = |0\rangle \langle 0|_{\Theta}$. As I have shown above, at time $t_0$ the Pauli matrices are unchanged and thus, trivially, $\rho_\Theta(t_0) = |0\rangle \langle 0|_{\Theta}$. Using the Heisenberg-evolved Pauli matrices one finds at time $t_1$: $\lambda_i(t_1) = 0, -\sin(t), \cos(t)$. Plugging these values into Eq. (23) one gets

$$\rho_\Theta(t_1) = \frac{1}{2} \left( 1 + \cos(t) i \sin(t) \right)^2.$$

This is the density matrix that one would have obtained evolving the system’s state in the Schrödinger picture.
Conclusions. In this work, I have shown how to formulate the PW model in the Heisenberg picture. This is achieved with a formal unitary transformation that transfers the time dependence from the state of the universe to the observables. The same transformation applies to both pure and mixed states of the universe.

In this picture, the dependence of the system’s observables on the time operator of the clock is manifest, while the usual time evolution of quantum theory emerges in the relative observables.

The model can be easily extended to include also position and all the other generators of symmetry transformations [14, 29]. Furthermore, this formulation may be more suitable for future investigations on time in quantum theory as it makes the quantum nature of time more explicit and provides a timeless Heisenberg-picture description of the whole universe. For instance, this version of the model could be applied to indefinite causal order [30], to the interacting PW model [14], or to the questions on clock synchronisation raised in [23]. I leave these investigations for future work.

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