Projection scheme for polynomial diffusions on the unit ball

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Abstract

In this article, we consider numerical schemes for polynomial diffusions on the unit ball $B^d$, which are solutions of stochastic differential equations with a diffusion coefficient of the form $\sqrt{1 - |x|^2}$. We introduce a projection scheme on the unit ball $B^d$ based on a backward Euler–Maruyama scheme and provide the $L^2$-rate of convergence. The main idea to consider the numerical scheme is the transformation argument introduced by Swart [29] for proving the pathwise uniqueness for some stochastic differential equation with a non-Lipschitz diffusion coefficient.

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1 Introduction

Let $a = (a_{i,j})_{i,j=1}^d : \mathbb{R}^d \rightarrow \mathbb{S}^d$ and $b = (b_1, \ldots, b_d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be continuous maps with $a_{i,j} \in \text{Pol}_2$ and $b_i \in \text{Pol}_1$ for all $i, j = 1, \ldots, d$. Here $\mathbb{S}^d$ denotes the set of $d \times d$ real symmetric matrices, and $\text{Pol}_k$ denotes the vector space of polynomial functions on $\mathbb{R}^d$ of total degree less than or equal to $k$. A time-homogeneous Markov process $X = (X(t))_{t \geq 0}$ is called a polynomial diffusion if it is a solution of the stochastic differential equation (SDE) $dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$, $t \geq 0$, where $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times n}$ is a continuous map with $\sigma^\top = a$ and $B$ is a standard $n$-dimensional Brownian motion. Then the infinitesimal generator $\mathcal{L}$ of $X$, defined by $\mathcal{L}f = \langle b, \nabla f \rangle + \text{Tr}(a \nabla^2 f)/2$, preserves that $\mathcal{L}\text{Pol}_k \subset \text{Pol}_k$ for all $k \in \mathbb{N}$. Polynomial diffusions have been widely studied from both theory and practical points of view (e.g., for one-dimensional cases, Ornstein–Uhlenbeck processes, Black–Scholes models (geometric Brownian motions), Cox–Ingersoll–Ross (CIR) processes (squared Bessel processes), Wright–Fisher processes and Jacobi processes, which are applied in mathematical finance and biology (e.g., [17, 19]), and for multi-dimensional cases, affine processes ([7]) and Jacobi processes ([10, 29, 20])). For a polynomial diffusion $X = (X(t))_{t \geq 0}$ with $\mathbb{E}[|X(0)|^{2m}] < \infty$ for some $m \in \mathbb{N}$, as a conclusion of Itô’s formula, we have the following important structure:

$$
\mathbb{E}[p(X(T)) \mid X(t)] = H(X(t))^\top e^{(T-t)G} p
$$

(1)
for any polynomial \( p \in \text{Pol}_m \), where the vector valued function \( H = (h_1, \ldots, h_N) \) with \( N = \dim \text{Pol}_m \) for a given basis of polynomials \( h_1, \ldots, h_N \) of \( \text{Pol}_m \), the matrix \( G \in \mathbb{R}^{N \times N} \) and the vector \( \tilde{p} \in \mathbb{R}^N \) satisfy \( p(x) = H(x)^\top \tilde{p} \) and \( \mathcal{L} p(x) = H(x)^\top G \tilde{p} \) (see Theorem 3.1 in [8] or Theorem 2.7 in [4]). In mathematical finance, the formula \( \text{(1)} \) gives a closed form expression for the pricing of options (e.g. Section 3.2 in [1]). However, the solution \( X \) does not always have explicit forms, and thus one often approximates it by various numerical schemes.

Many of polynomial diffusions used in applications have boundary conditions. Indeed, the state spaces of CIR processes, Wright–Fisher processes and Jacobi processes are \([0, 1]\) and \([-1, 1] \), respectively. Also, multi-dimensional polynomial diffusions whose state spaces are \( \{ x \in \mathbb{R}^d; p(x) \geq 0 \} \) or \( \{ x \in \mathbb{R}^d; p(x) = 0 \} \) for some \( p \in \text{Pol}_2 \) have been studied recently. Filipović and Larsson [8], and Larsson and Pulido [20] studied some properties of the polynomial diffusion \( dX(t) = b(X(t)) \, dt + \sigma(X(t)) \, dB(t) \) on the unit ball \( \mathcal{B}^d := \{ x \in \mathbb{R}^d; \| x \| \leq 1 \} \). In particular, they proved that the equation admits a \( \mathcal{B}^d \)-valued weak solution for any initial condition in \( \mathcal{B}^d \) if and only if the coefficients \( b \) and \( a \) are of the form

\[
b(x) = b + \beta x \quad \text{and} \quad a(x) = (1 - |x|^2)\alpha + c(x), \quad x \in \mathcal{B}^d
\]

for some \( b \in \mathbb{R}^d, \beta \in \mathbb{R}^{d \times d}, \alpha \in \mathcal{S}_d^+ \), and \( c \in \mathcal{C}_+ \) such that

\[
\langle b, x \rangle + \langle \beta x, x \rangle + \frac{1}{2} \text{Tr}(c(x)) \leq 0, \quad x \in \mathcal{S}^{d-1}
\]

(see Proposition 6.1 in [8] or Theorem 2.1 in [20]). Here \( \mathcal{S}_d^+ \) denotes the set of \( d \times d \) real symmetric positive semidefinite matrices, \( \mathcal{S}^{d-1} \) is the unit sphere defined by \( \{ x \in \mathbb{R}^d; \| x \| = 1 \} \) and \( \mathcal{C}_+ := \{ c : \mathbb{R}^d \to \mathcal{S}_d^+; c_{ij} \in \text{Pol}_2 \} \) is homogeneous of degree 2 for all \( i, j \) and \( c(x)x = 0 \). Moreover, Larsson and Pulido [20] provided an SDE representation by using a \( \text{Skew}(d) \)-valued correlated Brownian motion with drift. Here \( \text{Skew}(d) \) denotes the set of \( d \times d \) real skew-symmetric matrices. More precisely, assume that \( m = \binom{d}{2} = \dim \text{Skew}(d) \), and then they proved that the map \( c \) is of the form \( c(x) = \sum_{p=1}^m A_p x^\top A_p^\top \in \text{Skew}(d) \) for some \( A_1, \ldots, A_m \in \text{Skew}(d) \) if and only if the law of the polynomial diffusion \( X \) is the same to the solution of the SDE:

\[
dX(t) = (b + \hat{\beta} X(t)) \, dt + \sqrt{1 - |X(t)|^2} \alpha^{1/2} \, dW(t) + A_0 X(t) \, dt + \sum_{p=1}^m A_p X(t) \circ d\hat{W}_p(t), \tag{2}
\]

where \( B = (W_1, \ldots, W_d, \hat{W}_1, \ldots, \hat{W}_m)^\top \) is a standard \( (d + m) \)-dimensional Brownian motion, \( A_0 = 2^{-1}(\beta - \beta^\top) \in \text{Skew}(d) \) and \( \hat{\beta} = 2^{-1}(\beta + \beta^\top) + 2^{-1} \sum_{p=1}^m A_p^\top A_p \in \mathcal{S}_d^+ \) (see Lemma 3.4 and Theorem 4.2 in [20]). One might consider the equation \( \text{(2)} \) as a multi-dimensional extension of Jacobi processes or Wright–Fisher processes. Note that the stochastic process \( A_0 t + \sum_{p=1}^m A_p \hat{W}_p(t) \) is called a \( \text{Skew}(d) \)-valued correlated Brownian motion with drift. The notation \( \circ d\hat{W}_p(t) \) means the Stratonovich integral. In other words, if one rewrite it in terms of the Itô integral, then \( A_p X(t) \circ d\hat{W}_p(t) = A_p X(t) d\hat{W}_p(t) + (1/2)A_p^2 X(t) \, dt \). Moreover, by the skew symmetry of \( A_1, \ldots, A_m \), the equation \( \text{(2)} \) coincides with the equation

\[
dX(t) = (b + \beta X(t)) \, dt + \hat{\sigma}(X(t)) \, dB(t),
\]

where \( \hat{\sigma}(x) := (\sqrt{1 - |x|^2} \alpha^{1/2}, A_1 x, \ldots, A_m x) \in \mathbb{R}^{d \times (d + m)} \) which satisfies \( \hat{\sigma} \hat{\sigma}^\top = a \). Thus the law of the SDE \( \text{(2)} \) is the same to the polynomial diffusion with \( b(x) = b + \beta x \) and \( a(x) = (1 - |x|^2)\alpha + c(x) \) (see Corollary 3.2 in [8] or Lemma 2.4 in [20]).
For one-dimensional SDEs, Yamada and Watanabe [30] proved that if the diffusion coefficient \( \sigma \) is \( 1/2 \)-Hölder continuous and the drift coefficient \( b \) is Lipschitz continuous, then the pathwise uniqueness holds. However, for multi-dimensional cases, it is difficult to apply the method in [30] to prove the pathwise uniqueness, and thus this is a significant issue in the field of stochastic calculus. On one hand, as a special case of polynomial diffusions, Swart [29] considered the SDE on the unit ball \( \mathbb{B}^d \) of the form

\[
\text{d}X(t) = -\kappa X(t) \text{d}t + \sqrt{2(1 - |X(t)|^2)} \text{d}W(t), \quad X(0) = x(0) \in \mathbb{B}^d,
\]

where \( W = (W_1, \ldots, W_d)^T \) is a standard \( d \)-dimensional Brownian motion and \( \kappa \) is a non-negative constant. It is shown in [29] that the pathwise uniqueness holds for this SDE with \( \kappa \geq 1 \) by using the transformation \( \mathbb{B}^d \ni x \mapsto (\sqrt{1 - |x|^2}, x_1, \ldots, x_d)^T \in [0,1] \times \mathbb{B}^d \). More precisely, define the \( (d + 1) \)-dimensional stochastic process \( Y = (Y_0, Y_1, \ldots, Y_d)^T := (\sqrt{1 - |X|^2}, X_1, \ldots, X_d)^T \) which takes values in the upper-half ball surface \([0,1] \times \mathbb{B}^d\), and then the pathwise uniqueness holds for the process \( Y \) (see Theorem 3 in [29]). On the other hand, DeBlassie [5] extended the results in [29] for general SDEs in which the factors \( \sqrt{2} \) and \( \kappa \) are replaced by some Lipschitz continuous functions. The idea in [5] is to use the stochastic process \( (1 - |X|^2)^p \) with suitable \( p \in (1/2, 1) \) instead of \( \sqrt{1 - |X|^2} \). Furthermore, by using the method in [5], Larsson and Pulido [20] generalized the result in [29] to the following SDE on the unit ball \( \mathbb{B}^d \) of the form

\[
\text{d}X(t) = -\kappa X(t) \text{d}t + \nu \sqrt{1 - |X(t)|^2} \text{d}W(t) + A_0 X(t) \text{d}t + \sum_{p=1}^{m} A_p X(t) \circ \text{d}W_p(t), \tag{3}
\]

where \( B = (W_1, \ldots, W_d, \tilde{W}_1, \ldots, \tilde{W}_m)^T \) is a standard \( (d + m) \)-dimensional Brownian motion and \( \kappa \) is a non-negative constant, that is, it is a special case of the SDE (2). They showed that if \( \kappa/\nu^2 > \sqrt{2} - 1 \), then the pathwise uniqueness holds for the SDE (3) (see Theorem 4.6 in [20]). In this article, we will provide a numerical scheme on the unite ball \( \mathbb{B}^d \) for the SDE (3).

As mentioned above, the solution of the SDE \( \text{d}X(t) = b(X(t)) \text{d}t + \sigma(X(t)) \text{d}B(t), \, t \in [0,T] \) does not always have explicit forms in general, and thus one often approximates it by using the Euler–Maruyama scheme \( X_{nEM}^{(k)} = (X_{EM}^{(n)}(t_k))_{k=0,\ldots,n} \) defined by

\[
X_{nEM}^{(k)}(t_{k+1}) = X_{nEM}^{(k)}(t_k) + b(X_{nEM}^{(k)}(t_k)) \Delta t + \sigma(X_{nEM}^{(k)}(t_k)) \Delta_k B
\]

for \( k = 0, 1, \ldots, n - 1 \) with the initial condition \( X^{(n)}(0) = X(0) \). Here \( \Delta t := T/n, \, t_k := k\Delta t \) and \( \Delta_k B := B(t_{k+1}) - B(t_k) \). On one hand, it is well-known that under the Lipschitz condition on the coefficients \( b \) and \( \sigma \), the \( L^p \)-rate of convergence for the Euler–Maruyama scheme \( X_{nEM}^{(n)} \) is 1/2, that is, for any \( p \geq 1 \), it holds that \( \text{E} |\max_{k=0,1,\ldots,n} |X(t_k) - X_{nEM}^{(n)}(t_k)|^p|^{1/p} \leq C_p n^{-1/2} \) (see [18]). Moreover, for SDEs with reflecting boundary conditions, the projection scheme and the penalization scheme which are based on the Euler–Maruyama scheme have been studied (see [3, 21, 23, 24, 26, 27, 28]). On the other hand, Kaneko and Nakao [15] showed that by using the arguments of Skorokhod [25], if the pathwise uniqueness holds for SDEs with continuous and linear growth coefficients, then the Euler–Maruyama scheme \( X_{nEM}^{(n)} \) converges to the solution of the corresponding SDE in the \( L^2 \) sense. Moreover, for one-dimensional SDEs with an \( \alpha \)-Hölder continuous diffusion coefficient with \( \alpha \in [1/2, 1] \), Yan [31], and Gyöngy and Rásonyi [11] provided the \( L^3 \)-rate of convergence for the Euler–Maruyama scheme by using the Itô–Tanaka formula, and Yamada and Watanabe approximation arguments, respectively. However, the Euler–Maruyama scheme \( X_{nEM}^{(n)} \) for SDEs
with boundary conditions (e.g., CIR processes, Wright–Fisher processes and Jacobi processes, and in particular (3)) does not always take values in the state space of the original processes. For this problem, the Lamperti transformation and the backward Euler–Maruyama scheme play crucial roles in the one-dimensional setting. To explain this, we consider the CIR process

\[ dy(t) = (a - by(t)) \, dt + \sigma \sqrt{y(t)} \, dW(t), \quad t \in [0, T] \]

with the initial condition \( y(0) > 0 \) and parameters \( a, b, \sigma \in \mathbb{R} \). If \( 2a \geq \sigma^2 \), then it holds that \( \mathbb{P}(y(t) \in (0, \infty), \forall t \in [0, T]) = 1 \). By using the Lamperti transformation, that is, applying Itô’s formula for \( x(t) := \sqrt{y(t)} \), \( x = (x(t))_{t \in [0, T]} \) satisfies the SDE \( dx(t) = (a/2 - \sigma^2/8)x(t)^{-1} - (b/2)x(t) \, dt + (\sigma/2) \, dW(t), t \in [0, T] \). Then the backward Euler–Maruyama scheme \( x_{\text{BEM}}^{(n)} \) for \( x \) is defined by a solution of the quadratic equation

\[
x_{\text{BEM}}^{(n)}(t_{k+1}) = x_{\text{BEM}}^{(n)}(t_k) + \left\{ \left( \frac{a}{2} - \frac{\sigma^2}{8} \right) x_{\text{BEM}}^{(n)}(t_{k+1})^{-1} - \frac{b}{2} x_{\text{BEM}}^{(n)}(t_{k+1}) \right\} \Delta t + \frac{\sigma}{2} \Delta k W
\]

for \( k = 0, 1, \ldots, n - 1 \) with the initial condition \( x_{\text{BEM}}^{(n)}(0) = x(0) \). Then the inverse transform of \( x_{\text{BEM}}^{(n)} \) approximates the original process \( y \) (for more details, see [2, 6, 22] and [12, 13]).

In this article, we will provide a numerical scheme on the unit ball \( \mathbb{B}^d \) for the solution of the multi-dimensional SDE (3) with the initial condition \( X(0) = x(0) \in \mathbb{B}^d \setminus \mathbb{S}^{d-1} \). As mentioned above, the solution of the SDE (3) takes values in the unit ball \( \mathbb{B}^d \), but the Euler–Maruyama scheme does not always. Moreover, unfortunately, for the multi-dimensional setting, it is difficult to use the Lamperti transformation, unlike the one-dimensional case. As an alternative method, we first consider the transformed process \( Y = (\sqrt{1-|X|^2}, X_1, \ldots, X_d) \), and then we approximate it by a backward Euler–Maruyama scheme. To explain the detail, we will apply the idea of Swart [29] to the general SDE (3). It can be shown that by using Itô’s formula and the skew-symmetry of \( A_0, \ldots, A_m \), \( Y = (Y_0, Y_1, \ldots, Y_d) \) satisfies the following \((d + 1)\)-dimensional SDE:

\[
\begin{align*}
\, dY_0(t) &= \left\{ \kappa - \frac{\nu^2}{2Y_0(t)} - \left( \kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2} \right)Y_0(t) \right\} \, dt - \nu X(t) \, dW(t), \\
\, dX(t) &= -\nu X(t) \, dt + \nu Y(t) \, dW(t) + A_0 X(t) \, dt + \sum_{p=1}^{m} A_p X(t) \, d\widetilde{W}_p(t) + \frac{1}{2} \sum_{p=1}^{m} A^2_p X(t) \, dt
\end{align*}
\]

(see Section 2). This shows that the process \( Y_0 \) is a “Bessel like” process. Inspired by previous studies (e.g., [2, 6, 22]), we introduce a backward type Euler–Maruyama scheme for the system of SDE (4) as follows. Let \( \Delta t := T/n, t_k := k\Delta t, \Delta k W := W(t_{k+1}) - W(t_k) \) and \( \Delta k \widetilde{W} := \widetilde{W}(t_{k+1}) - \widetilde{W}(t_k), k = 0, 1, \ldots, n - 1 \). Define \( Y^{(n)}(0) := Y(0) \) and for each \( k = 0, 1, \ldots, n - 1, Y^{(n)}(t_k) = (Y^{(n)}(t_k+1), Y_1^{(n)}(t_k), \ldots, Y_d^{(n)}(t_k+1))^{\top} := (Y_0^{(n)}(t_k), X_1^{(n)}(t_k), \ldots, X_d^{(n)}(t_k))^{\top} \) as a unique solution in \((0, \infty) \times \mathbb{R}^d\), not in \((0, 1) \times \mathbb{R}^d\), of the following equation:

\[
\begin{align*}
Y_0^{(n)}(t_{k+1}) &= Y_0^{(n)}(t_k) + \frac{\kappa - \frac{\nu^2}{2}}{Y_0^{(n)}(t_{k+1})} \Delta t - \left( \kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2} \right)Y_0^{(n)}(t_k) \Delta t - \nu X^{(n)}(t_k)^{\top} \Delta k W, \\
X^{(n)}(t_{k+1}) &= X^{(n)}(t_k) - \nu X^{(n)}(t_k) \Delta t + \nu Y^{(n)}(t_k) \Delta k W \quad + A_0 X^{(n)}(t_k) \Delta t + \sum_{p=1}^{m} A_p X^{(n)}(t_k) \Delta k \widetilde{W}_p(t) + \frac{1}{2} \sum_{p=1}^{m} A^2_p X^{(n)}(t_k) \Delta t,
\end{align*}
\]
If $\kappa/\nu^2 > 1/2$, then the quadratic equation for $Y_0^{(n)}(t_{k+1})$ has a unique solution which takes values in $(0, \infty)$ with the following explicit form:

$$Y_0^{(n)}(t_{k+1}) = \frac{1}{2} \left[ b_k + \sqrt{b_k^2 + 4 \left( \kappa - \frac{\nu^2}{2} \right) \Delta t} \right],$$

where $b_k = Y_0^{(n)}(t_k) - \nu X^{(n)}(t_k) \Delta_k W - \left( \kappa - \frac{\nu^2}{2} + \frac{d \nu^2}{2} \right) Y_0^{(n)}(t_k) \Delta t$.

We will provide the $L^2$-rate of convergence for $Y^{(n)}$ to the solution $Y$ of the system of SDEs (3) (see Theorem 2.1). This rate can theoretically be applied to the computational complexity of the multilevel Monte Carlo method, whose computational cost is much lower than that of the classical (single level) Monte Carlo method (see [9]). To the best of our knowledge, the rate of convergence for numerical schemes of multi-dimensional SDEs with a Hölder continuous and degenerate diffusion coefficient have not been obtained yet. It is worth noting that $X^{(n)}$ may still take values outside of the unit ball $B$. We solve this problem by projecting it onto the unit ball $B$. Let $\Pi$ be the projection onto the unit ball $B$ defined by $\Pi(x) := x \mathbf{1}_{B}(x) + (x/|x|) \mathbf{1}_{R \setminus B}(x)$, $x \in \mathbb{R}^d$. Then we define the projection scheme $\overline{X}^{(n)}$ on the unit ball $B$ for the SDE (3) by

$$\overline{X}^{(n)}(t_k) := \Pi(X^{(n)}(t_k)), \quad k = 0, 1, \ldots, n.$$  

We will show that the $L^2$-rate of convergence of $\overline{X}^{(n)}$ to $X$ is induced while preserving the $L^2$-rate of convergence for $Y^{(n)}$ to $Y$ (see Corollary 2.3).

This article is structured as follows. In Section 2, we provide the $L^2$-rate of convergence for the backward Euler–Maruyama scheme $Y^{(n)}$ defined in (3) (see Theorem 2.3), and for the projection scheme $\overline{X}^{(n)}$ defined in (6) (see Corollary 2.1). In Section 3, we provide some numerical results about the projection scheme $\overline{X}^{(n)}$ for the polynomial diffusion (3) and the SDE (4).

**Notations**

We give some basic notations and definitions used throughout this article. For a $d \times d$ matrix $A = (A_{i,j})_{i,j=1,\ldots,d}$, the transpose of $A$ is denoted by $A^\top$, and the Frobenius norm of $A$ is denoted by $\|A\| := (\sum_{i,j=1}^d A_{i,j}^2)^{1/2}$. Each element of $\mathbb{R}^d$ is understood as a column vector, that is, $x = (x_1, \ldots, x_d)^\top$ for $x \in \mathbb{R}^d$. We define $\mathcal{S}^d$ to be the set of $d \times d$ real symmetric matrices and $\text{Skew}(d)$ to be the set of $d \times d$ real skew-symmetric matrices, that is, $A^\top = -A$ for $A \in \text{Skew}(d)$.

We set $\mathcal{B}^d := \{ x \in \mathbb{R}^d ; |x| \leq 1 \}$ and $\mathcal{F}^{d-1} := \{ x \in \mathbb{R}^d ; |x| = 1 \}$. Let $\Pi : \mathbb{R}^d \rightarrow \mathcal{B}^d$ be the projection defined by $\Pi(x) := x \mathbf{1}_{\mathcal{B}^d}(x) + (x/|x|) \mathbf{1}_{\mathbb{R} \setminus \mathcal{B}^d}(x)$.

Let $B = (W_1, \ldots, W_d, \tilde{W}_1, \ldots, \tilde{W}_m)^\top$ be a standard $(d + m)$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}(t))_{t \geq 0}$ satisfying the usual conditions. For $T > 0$ and $n \in \mathbb{N}$, we denote $\Delta t := T/n$, $t_k := k \Delta t$, $\Delta_k W := W(t_{k+1}) - W(t_k)$ and $\Delta_k \tilde{W} := \tilde{W}(t_{k+1}) - \tilde{W}(t_k)$, $k = 0, 1, \ldots, n - 1$.

**2 Main results**

Let $d \geq 2$ and $T > 0$ be fixed, and let $X$ be a solution of the SDE (3) on the time interval $[0, T]$. In this article, we assume that $\kappa/\nu^2 \geq \sqrt{2} - 1$ and $A_1, \ldots, A_m \in \text{Skew}(d)$. Then the pathwise
must estimate the supremum for each of the martingale terms. Suppose Theorem 2.1.

The rate of convergence in Theorem 2.1 is $1/L$ does not appear in the case of constant diffusion coefficients. This is the reason that the projection scheme for $\mathbb{R}^d$ in [2, 6, 22] that the uniqueness holds for the SDE (3) (see Theorem 4.6 in [20]) and the projection scheme [6]. We first recall the transformation argument introduced by Swart [29]. We define $Y = (Y_0, Y_1, \ldots, Y_d)^\top := (\sqrt{1 - |X|^2}, X_1, \ldots, X_d)^\top$. Then $Y = (Y_0, X_1, \ldots, X_d)^\top$ satisfies [1]. Indeed, by using Itô’s formula, we obtain

$$d|X(t)|^2 = 2\nu\sqrt{1 - |X(t)|^2}X(t)^\top dW(t) + 2\sum_{p=1}^{m} \langle A_p X(t), X(t) \rangle d\tilde{W}_p(t)$$

$$- 2\kappa |X(t)|^2 dt + 2\langle A_0 X(t), X(t) \rangle dt + \sum_{p=1}^{m} |A_p X(t)|^2 dt$$

$$+ d\nu^2 (1 - |X(t)|^2) dt + \sum_{p=1}^{m} |A_p X(t)|^2 dt.$$

Then since for any $A \in \text{Skew}(d)$,

$$\langle Ax, x \rangle = 0 \text{ and } \langle A^2 x, x \rangle = -|Ax|^2, \ x \in \mathbb{R}^d,$$

we obtain

$$d|X(t)|^2 = 2\nu\sqrt{1 - |X(t)|^2}X(t)^\top dW(t) + \{d\nu^2 - (d\nu^2 + 2\kappa)|X(t)|^2\} \ dt. \ (7)$$

Hence by using Itô’s formula again, it holds that

$$dY_0(t) = \left\{ \kappa - \frac{\nu^2}{Y_0(t)} - \left( \kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2} \right) Y_0(t) \right\} dt - \nu X(t)^\top dW(t).$$

Therefore, $Y = (Y_0, X_1, \ldots, X_d)^\top$ is a solution of the $(d + 1)$-dimensional SDE [1].

Recall that $Y^{(n)}$ is the backward Euler–Maruyama scheme for $Y$ defined in [3] and $\overline{X}^{(n)}$ is the projection scheme for $X$ defined in [6]. We first provide the $L^2$-rate of convergence for $Y^{(n)}$.

**Theorem 2.1.** Suppose $\kappa/\nu^2 > 6$. Then there exists $C > 0$ such that for any $n \in \mathbb{N}$,

$$\max_{k=0,1,\ldots,n} \mathbb{E} \left[ |Y(t_k) - Y^{(n)}(t_k)|^2 \right]^{1/2} \leq C \frac{1}{n^{1/2}} \quad \text{and} \quad \mathbb{E} \left[ \max_{k=0,1,\ldots,n} |Y(t_k) - Y^{(n)}(t_k)|^2 \right]^{1/2} \leq C \frac{1}{n^{1/4}}.$$

**Remark 2.2.** Note that for one-dimensional SDEs with a constant diffusion coefficient, it is proven in [3, 6, 22] that the $L^2$-sup rate of convergence for the backward Euler–Maruyama schemes are $1/2$ or 1. However, in our case, the diffusion coefficient of the equation $Y_0$ is not constant, and thus we must estimate the supremum for each of the martingale terms $R^M$ and $S^M$ defined below, which does not appear in the case of constant diffusion coefficients. This is the reason that the $L^2$-sup rate of convergence in Theorem 2.1 is $1/4$.

By using Theorem 2.1, we provide the $L^2$-rate of convergence for $\overline{X}^{(n)}$. 

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Corollary 2.3. Suppose $\kappa/\nu^2 > 6$. Then there exists $C > 0$ such that for any $n \in \mathbb{N}$,
\[
\max_{k=0,1,\ldots,n} \mathbb{E} \left[ \frac{1}{n^{1/2}} \mathbb{E} \left[ X(t_k) - \mathbb{X}^{(n)}(t_k) \right] \right]^{1/2} \leq \frac{C}{n^{1/2}} \quad \text{and} \quad \mathbb{E} \left[ \frac{1}{n^{1/2}} \left( \max_{k=0,1,\ldots,n} \left| X(t_k) - \mathbb{X}^{(n)}(t_k) \right| \right)^{1/2} \right] \leq \frac{C}{n^{1/2}}.
\]

Proof. By Theorem 2.1 it is suffice to estimate
\[
\mathbb{E} \left[ \frac{1}{n^{1/2}} \mathbb{E} \left[ X^{(n)}(t_k) - \mathbb{X}^{(n)}(t_k) \right] \right]^{1/2} \quad \text{and} \quad \mathbb{E} \left[ \frac{1}{n^{1/2}} \left( \max_{k=0,1,\ldots,n} \left| X^{(n)}(t_k) - \mathbb{X}^{(n)}(t_k) \right| \right)^{1/2} \right].
\]

For any $x \in \mathbb{R}^d$ and $y \in \mathbb{B}^d$, it holds that $|x - \Pi(x)| \leq |x - y|$. Indeed, if $x \in \mathbb{B}^d$, then $|x - \Pi(x)| = 0 \leq |x - y|$, and if $x \in \mathbb{R}^d \setminus \mathbb{B}^d$, then we obtain $|x - \Pi(x)| = |x| - 1 \leq |x| - |y| \leq |x - y|$. Hence it holds that for any $k = 0, 1, \ldots, n$,
\[
\left| X^{(n)}(t_k) - \mathbb{X}^{(n)}(t_k) \right| \leq \left| X^{(n)}(t_k) - X(t_k) \right|.
\]
This estimate together with Theorem 2.1 yields the statements.

Before we prove Theorem 2.1, we study some properties of the solution $Y = (Y_0, X_1, \ldots, X_d)^T$ of the SDE [4]. In the following proposition, we estimate the inverse moment of $Y_0$ and the Kolmogorov type condition of $Y$.

Proposition 2.4. (i) Suppose that $\kappa/\nu^2 \geq 1$. For any $0 < q < \kappa/\nu^2$, there exists $C_1(q) > 0$ such that
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ |Y_0(t)|^{-q} \right] \leq C_1(q).
\]
(ii) Suppose that $\kappa/\nu^2 > 2$. For any $2 \leq q < \kappa/\nu^2$, there exists $C_2(q) > 0$ such that
\[
\max_{i=0,1,\ldots,d} \mathbb{E} \left[ |Y_i(t) - Y_i(s)|^q \right] \leq C_2(q)|t - s|^{q/2}.
\]

In order to prove Proposition 2.4 we consider the following one-dimensional stochastic differential equation called the Wright–Fisher diffusion with positive parameters $a$, $b$ and $\gamma$:
\[
dy(t) = ([a - by(t)]dt + \gamma \sqrt{|y(t)(1 - y(t))|} \, d\tilde{W}(t)), \quad y(0) \in [0, 1],
\]
where $\tilde{W} = (\tilde{W}(t))_{t \geq 0}$ is a standard one-dimensional Brownian motion. It is well-known that the Wright–Fisher diffusion [9] has a unique strong solution which takes values in $[0, 1]$ if and only if $2a/\gamma^2 \geq 1$ and $2(b - a)/\gamma^2 \geq 1$.

The following lemma gives the inverse moment estimates of the Wright–Fisher diffusion [9].

Lemma 2.5 (e.g. Section 3.5 in [22] and Section 4 in [14]). Let $y = (y(t))_{t \geq 0}$ be a Wright–Fisher diffusion [9] with the initial condition $y(0) \in (0, 1)$ and positive parameters $a$, $b$ and $\gamma$. Assume that $2a/\gamma^2 \geq 1$ and $2(b - a)/\gamma^2 \geq 1$. Then for any $0 < q_1 < 2a/\gamma^2$ and $0 < q_2 < 2(b - a)/\gamma^2$, there exist $C(q_1, y(0)), C(q_2, y(0)) > 0$ such that
\[
\sup_{0 \leq t \leq T} \mathbb{E}[|y(t)|^{-q_1}] \leq C(q_1, y(0)) \quad \text{and} \quad \sup_{0 \leq t \leq T} \mathbb{E}[(1 - y(t))^{-q_2}] \leq C(q_2, y(0)).
\]
By using this lemma, we prove Proposition 2.4.
Proof of Proposition 2.4. Proof of (i). We first show that \(|X|^2 = (|X(t)|^2)_{t \in [0,T]}\) and \(1 - |X|^2 = (1 - |X(t)|^2)_{t \in [0,T]}\) are solutions of the Wright–Fisher diffusion \([9]\) with some parameters. Define new one-dimensional Brownian motions \(\tilde{W}_\pm\) by

\[
\tilde{W}_\pm(t) := \pm \sum_{i=1}^{d} \int_0^t g_i(X(s)) \, dW_i(s),
\]

where \(g_i : \mathbb{R}^d \to \mathbb{R}, i = 1, \ldots, d\) are defined by

\[
g_i(x) = \begin{cases} \frac{x_i}{|x|} & \text{if } |x| \neq 0, \\ \frac{1}{\sqrt{d}} & \text{if } |x| = 0. \end{cases}
\]

It follows from Lévy’s theorem (e.g. Theorem 3.3.16 in [16]) that \(\tilde{W}_\pm\) are standard one-dimensional Brownian motions. Then by \([8]\), we see that \(|X|^2\) is a Wright–Fisher diffusion \([9]\) driven by \(\tilde{W}_+\) with the initial condition \(|x(0)|^2 \in \{0, 1\}, a = d\nu, b = d\nu^2 + 2\kappa \) and \(\gamma = 2\nu\). Furthermore, \(1 - |X|^2\) is a Wright–Fisher diffusion \([9]\) driven by \(\tilde{W}_-\) with the initial condition \(1 - |x(0)|^2 \in \{0, 1\}, a = 2\kappa, b = d\nu^2 + 2\kappa \) and \(\gamma = 2\nu\).

We first assume \(|x(0)| \in (0, 1)\). Then for any \(0 < q < \kappa/\nu^2 = (2b - a)/\gamma^2\), it holds from Lemma 2.5 that

\[
\sup_{0 \leq t \leq T} \mathbb{E}[|Y_0(t)|^{-q}] = \sup_{0 \leq t \leq T} \mathbb{E}\left[|1 - |X(t)|^2|^{-q}\right] \leq C(q, |x(0)|^2),
\]

which concludes the statement (i) for \(|x(0)| \in (0, 1)\).

Now we assume \(|x(0)| = 0\). Then by using the comparison theorem (see e.g. Proposition 5.2.18 in [16]), for any \(0 < q < \kappa/\nu^2 = (2b - a)/\gamma^2\), it holds from Lemma 2.5 that

\[
\sup_{0 \leq t \leq T} \mathbb{E}[|Y_0(t)|^{-q}] = \sup_{0 \leq t \leq T} \mathbb{E}\left[|1 - |X(t)|^2|^q\right] \leq \sup_{0 \leq t \leq T} \mathbb{E}\left[|y_{1/2}(t)|^{-q}\right] \leq C(q, 1/2),
\]

where \(y_{1/2}\) is a Wright–Fisher diffusion \([9]\) with the initial condition \(y_{1/2}(0) = 1/2, a = 2\kappa, b = d\nu^2 + 2\kappa \) and \(\gamma = 2\nu\). This concludes the statement (i) for \(|x(0)| = 0\).

Proof of (ii). Let \(0 \leq s \leq t\) be fixed. Since \(|X| \in (0, 1)\) and \(Y_0 \in (0, 1)\) a.s., by using Burkholder-Davis-Gundy’s inequality and Jensen’s inequality, we have

\[
\mathbb{E}\left[|Y_0(t) - Y_0(s)|^q\right] \leq C_q \left\{(t-s)^{q/2} + (t-s)^{q-1} \int_s^t \mathbb{E}\left[|Y_0(u)|^{-q}\right] \, du\right\}
\]

and for any \(i = 1, \ldots, d\),

\[
\mathbb{E}\left[|X_i(t) - X_i(s)|^q\right] \leq C_q \left\{(t-s)^{q/2} + (t-s)^{q-1} \int_s^t \mathbb{E}\left[|(A_0 X(u))_i|^q\right] \, du \right\}
\]

\[
+ (t-s)^{q/2 - 1} \sum_{p=1}^m \int_s^t \mathbb{E}\left[|(A_p X(u))_i|^q\right] \, du
\]

\[
+ (t-s)^{q-1} \sum_{p=1}^m \int_s^t \mathbb{E}\left[|(A_p^2 X(u))_i|^q\right] \, du
\]

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for some constant $C_q > 0$. From the statement (i), we obtain the Kolmogorov type condition for $Y_0$. Since $|X| \in [0,1]$ a.s., by using Cauchy–Schwarz’s inequality, for any $d \times d$ matrix $A$, we have

$$\mathbb{E}[|(AX(u))_i|^q] \leq \|A\|^q.$$  

This implies the Kolmogorov type condition for $X$. □

By using the estimates in Proposition 2.4, we prove Theorem 2.1.

Proof of Theorem 2.1. We first decompose $Y = (Y_0, X_1, \ldots, X_d) \top$ as follows. Fix $k = 0, 1, \ldots, n-1$ and we define $R^M(k) = (R^M_0(k), \ldots, R^M_d(k)) \top$ and $R^A(k) = (R^A_0(k), \ldots, R^A_d(k)) \top$ by

$$R^M_0(k) := -\nu \int_{t_k}^{t_{k+1}} \{X(s) - X(t_k)\} \top dW(s)$$

$$R^A_0(k) := \left(\kappa - \frac{\nu^2}{2}\right) \int_{t_k}^{t_{k+1}} \left\{\frac{1}{Y_0(s)} - \frac{1}{Y_0(t_{k+1})}\right\} ds$$

$$- \left(\kappa - \frac{\nu^2}{2} + \frac{\nu^2}{2}\right) \int_{t_k}^{t_{k+1}} \{Y_0(s) - Y_0(t_k)\} ds$$

and for any $i = 1, \ldots, d$,

$$R^M_i(k) := \nu \int_{t_k}^{t_{k+1}} \{Y_0(s) - Y_0(t_k)\} dW_i(s) + \sum_{p=1}^{m} \int_{t_k}^{t_{k+1}} (A_p \{X(s) - X(t_k)\})_i d\tilde{W}_p(s),$$

$$R^A_i(k) := -\kappa \int_{t_k}^{t_{k+1}} \{X(s) - X(t_k)\}_i ds + \int_{t_k}^{t_{k+1}} (A_0 \{X(s) - X(t_k)\})_i ds$$

$$+ \frac{1}{2} \sum_{p=1}^{m} \int_{t_k}^{t_{k+1}} (A^2_p \{X(s) - X(t_k)\})_i ds.$$  

Then we obtain the decomposition of $Y = (Y_0, X_1, \ldots, X_d) \top$ as follows:

$$Y_0(t_{k+1}) = Y_0(t_k) - \nu X(t_k) \top \Delta_k W + \frac{\kappa - \nu^2}{Y_0(t_{k+1})} \Delta t$$

$$- \left(\kappa - \frac{\nu^2}{2} + \frac{\nu^2}{2}\right) Y_0(t_k) \Delta t + R^M_0(k) + R^A_0(k)$$

and

$$X_i(t_{k+1}) = X_i(t_k) - \kappa X_i(t_k) \Delta t + \nu Y_0(t_k) \Delta_k W_i + (A_0 X(t_k))_i \Delta t$$

$$+ \sum_{p=1}^{m} (A_p X(t_k))_i \Delta_k \tilde{W}_p + \frac{1}{2} \sum_{p=1}^{m} (A^2_p X(t_k))_i \Delta t + R^M_i(k) + R^A_i(k).$$

Moreover, we define $S^M(k) = (S^M_0(k), \ldots, S^M_d(k)) \top$ and $S^A(k) = (S^A_0(k), \ldots, S^A_d(k)) \top$ by

$$S^M_0(k) := -\nu \{X(t_k) - X^{(n)}(t_k)\}_i \Delta_k W,$$
$$S_0^A(k) := -\left(\kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2}\right) (Y_0(t_k) - Y_0^{(n)}(t_k)) \Delta t$$

and for any $i = 1, \ldots, d,$

$$S_t^M(k) := \nu (Y_0(t_k) - Y_0^{(n)}(t_k)) \Delta k W_i + \sum_{p=1}^m (A_p (X(t_k) - X^{(n)}(t_k))) \Delta k \hat{W}_p,$$

$$S_t^A(k) := -\kappa (X(t_k) - X^{(n)}(t_k)) \Delta t + (A_0 (X(t_k) - X^{(n)}(t_k))) \Delta t$$

$$+ \frac{1}{2} \sum_{p=1}^m (A_p^2 (X(t_k) - X^{(n)}(t_k))) \Delta t,$$

and we set $e(k) := Y(t_k) - Y^{(n)}(t_k)$ and $r(k) := R^M(k) + R^A(k) + S^M(k) + S^A(k).$ Then by (10), (11) and the definition of $Y^{(n)} = (Y_0^{(n)}, X_1^{(n)}, \ldots, X_d^{(n)})^T,$ we have

$$e_0(k+1) = e_0(k) + r_0(k) + \left(\kappa - \frac{\nu^2}{2}\right) \left\{ \frac{1}{Y_0(t_{k+1})} - \frac{1}{Y_0^{(n)}(t_{k+1})} \right\} \Delta t$$

and for any $i = 1, \ldots, d,$

$$e_i(k+1) = e_i(k) + r_i(k).$$

By the definition of $\{e(k)\}_{k=0}^n,$ we obtain

$$|e(k+1)|^2 - 2e_0(k+1) \left(\kappa - \frac{\nu^2}{2}\right) \left\{ \frac{1}{Y_0(t_{k+1})} - \frac{1}{Y_0^{(n)}(t_{k+1})} \right\} \Delta t$$

$$\leq \left( e_0(k+1) - \left(\kappa - \frac{\nu^2}{2}\right) \left\{ \frac{1}{Y_0(t_{k+1})} - \frac{1}{Y_0^{(n)}(t_{k+1})} \right\} \Delta t \right)^2 + \sum_{i=1}^d e_i(k+1)^2$$

$$= |e(k) + r(k)|^2.$$ 

Thus by the assumption $\kappa/\nu^2 > 6,$ we have

$$|e(k+1)|^2 \leq |e(k) + r(k)|^2 + 2e_0(k+1) \left(\kappa - \frac{\nu^2}{2}\right) \left\{ \frac{1}{Y_0(t_{k+1})} - \frac{1}{Y_0^{(n)}(t_{k+1})} \right\} \Delta t$$

$$\leq |e(k) + r(k)|^2 = |e(k)|^2 + 2(e(k), r(k)) + |r(k)|^2,$$

where we used the fact that $(x-y)(1/x-1/y) = -(x-y)^2/xy \leq 0,$ $x, y > 0$ in the second inequality.

Hence, we obtain

$$|e(k)|^2 = \sum_{\ell=0}^{k-1} \left\{ |e(\ell+1)|^2 - |e(\ell)|^2 \right\} \leq \sum_{\ell=0}^{k-1} \left\{ 2(e(\ell), r(\ell)) + |r(\ell)|^2 \right\}.$$ 

Here, by using the inequality $(x,y) \leq |x|^2/(2\delta) + \delta |y|^2/2$ for $\delta > 0,$

$$\langle e(\ell), R^A(\ell) \rangle \leq \frac{1}{2} |R^A(\ell)|^2 \frac{1}{\Delta t} + \frac{1}{2} |e(\ell)|^2 \Delta t,$$
and by the assumption $\kappa/\nu^2 > 6$ and (7),
\[
\langle e(t), S^A(t) \rangle = -\left(\kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2}\right) e_0(k)^2 \Delta t - \kappa \sum_{i=1}^{d} e_i(t)^2 \Delta t \\
+ \langle A_0 \{X(t), X^{(n)}(t)\}, X(t) - X^{(n)}(t) \rangle \Delta t \\
+ \frac{1}{2} \sum_{p=1}^{m} \langle A_p^2 \{X(t), X^{(n)}(t)\}, X(t) - X^{(n)}(t) \rangle \Delta t \\
\leq 0.
\]
Therefore, it holds that
\[
|e(k)|^2 \leq 2 \sum_{\ell=0}^{k-1} \langle e(t), R^M(t) + S^M(t) \rangle + \sum_{\ell=0}^{k-1} |R^A(t)|^2 \frac{1}{\Delta t} + \frac{1}{2} \sum_{\ell=0}^{k-1} |e(t)|^2 \Delta t + \sum_{\ell=0}^{k-1} |r(t)|^2. \tag{12}
\]
Fix $\ell = 0, 1, \ldots, k-1$. We first estimate the expectations of the terms $\langle e(t), R^M(t) + S^M(t) \rangle$ and $|R^A(t)|^2$ on the right hand side of (12). By using the martingale property of the Brownian motion $B = (W_1, \ldots, W_d, \widetilde{W}_1, \ldots, \widetilde{W}_m)^\top$ and stochastic integrals, and noting that $e(t)$ is $\mathcal{F}(t)$-measurable, we have
\[
\mathbb{E} \left[ \langle e(t), R^M(t) + S^M(t) \rangle \right] = \sum_{i=0}^{d} \mathbb{E} \left[ e_i(t) \mathbb{E} \left[ R^M_i(t) + S^M_i(t) \mid \mathcal{F}(t) \right] \right] = 0. \tag{13}
\]
By using Jensen’s inequality, we obtain
\[
\mathbb{E} \left[ |R^A(t)|^2 \right] \leq 2 \left\{ \left(\kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2}\right) \int_{t_\ell}^{t_{\ell+1}} \mathbb{E} \left[ \left\{ \frac{1}{Y_0(s)} - \frac{1}{Y_0(t_{\ell+1})} \right\}^2 \right] \, ds \\
+ \left(\kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2}\right) \int_{t_\ell}^{t_{\ell+1}} \mathbb{E} \left[ (Y_0(s) - Y_0(t_{\ell+1}))^2 \right] \, ds \right\} \Delta t \\
+ (m+2) \left\{ \kappa^2 \int_{t_\ell}^{t_{\ell+1}} \mathbb{E} \left[ |X(s) - X(t_{\ell+1})|^2 \right] \, ds + \int_{t_\ell}^{t_{\ell+1}} \mathbb{E} \left[ |A_0 \{X(s) - X(t_{\ell+1})\}|^2 \right] \, ds \\
+ \frac{1}{4} \sum_{p=1}^{m} \int_{t_\ell}^{t_{\ell+1}} \mathbb{E} \left[ |A_p^2 \{X(s) - X(t_{\ell+1})\}|^2 \right] \, ds \right\} \Delta t.
\]
Here, by using Hölder’s inequality, and Proposition 2.4 (i) and (ii) with $q = 6$ for any $s \in [t_\ell, t_{\ell+1}]$,
\[
\mathbb{E} \left[ \left\{ \frac{1}{Y_0(s)} - \frac{1}{Y_0(t_{\ell+1})} \right\}^2 \right] \leq \mathbb{E} \left[ \{Y_0(t_{\ell+1}) - Y_0(s)\}^6 \right]^{1/3} \mathbb{E} \left[ Y_0(s)^{-6} \right]^{1/3} \mathbb{E} \left[ Y_0(t_{\ell+1})^{-6} \right]^{1/3} \\
\leq C_1(6)^{2/3} C_2(6)^{1/3} \Delta t,
\]
and by Cauchy–Schwarz’s inequality and Proposition 2.4 (ii), for any $d \times d$-matrix $A$,
\[
\mathbb{E} \left[ |A \{X(s) - X(t_{\ell+1})\}|^2 \right] \leq \|A\|^2 \mathbb{E} \left[ |X(s) - X(t_{\ell+1})|^2 \right] \leq \|A\|^2 d C_4(2) \Delta t. \tag{14}
\]
Thus we have
\[
\mathbb{E} \left[ |R^A(t)|^2 \right] \leq C_1 (\Delta t)^3
\]  
(15)
for some constant \( C_1 > 0 \).

We next estimate the expectation of the term \( |r(t)|^2 \) on the right hand side of (12). By Itô’s isometry of stochastic integrals, we obtain
\[
\mathbb{E} \left[ |R^M(t)|^2 \right] = \nu^2 \int_{t_{\ell}}^{t_{\ell+1}} \mathbb{E} \left[ |X(s) - X(t_{\ell})|^2 \right] ds + d\nu^2 \int_{t_{\ell}}^{t_{\ell+1}} \mathbb{E} \left[ \{Y_0(s) - Y_0(t_{\ell})\}^2 \right] ds
\]
+ \sum_{p=1}^{m} \int_{t_{\ell}}^{t_{\ell+1}} \mathbb{E} \left[ |A_p(X(s) - X(t_{\ell}))|^2 \right] ds.

Thus by Proposition 2.4 (ii), we have
\[
\mathbb{E} \left[ |R^M(t)|^2 \right] \leq C_2 (\Delta t)^2
\]  
(16)
for some constant \( C_2 > 0 \). By using the estimate (14), we obtain
\[
\mathbb{E} \left[ |S^M(t)|^2 \right] = \nu^2 \mathbb{E} \left[ |X(t_{\ell}) - X^{(n)}(t_{\ell})|^2 \right] \Delta t + d\nu^2 \mathbb{E} \left[ \{Y_0(t_{\ell}) - Y_0^{(n)}(t_{\ell})\}^2 \right] \Delta t
\]
+ \sum_{p=1}^{m} \mathbb{E} \left[ |A_p(X(t_{\ell}) - X^{(n)}(t_{\ell}))|^2 \right] \Delta t
\leq C_3 \mathbb{E} \left[ |e(t)|^2 \right] \Delta t
\]  
(17)
for some constant \( C_3 > 0 \), and by the estimate (14) again, we obtain
\[
\mathbb{E} \left[ |S^A(t)|^2 \right] \leq \left( \kappa - \frac{\nu^2}{2} + \frac{d\nu^2}{2} \right) \mathbb{E} \left[ \{Y_0(t_{\ell}) - Y_0^{(n)}(t_{\ell})\}^2 \right] (\Delta t)^2
\]
+ \( (m + 2) \left\{ \kappa^2 \mathbb{E} \left[ |X(t_{\ell}) - X^{(n)}(t_{\ell})|^2 \right] + \mathbb{E} \left[ |A_0(X(t_{\ell}) - X^{(n)}(t_{\ell}))|^2 \right] \right\} \Delta t
\]
+ \( \frac{1}{4} \sum_{p=1}^{m} \mathbb{E} \left[ |A_p(X(t_{\ell}) - X^{(n)}(t_{\ell}))|^2 \right] \Delta t \)
\leq C_4 \mathbb{E} \left[ |e(t)|^2 \right] (\Delta t)^2
\]  
(18)
for some constant \( C_4 > 0 \). By combining (15), (16), (17) and (18), it holds that
\[
\mathbb{E} \left[ |r(t)|^2 \right] \leq 4 \left\{ \mathbb{E} \left[ |R^M(t)|^2 \right] + \mathbb{E} \left[ |R^A(t)|^2 \right] + \mathbb{E} \left[ |S^M(t)|^2 \right] + \mathbb{E} \left[ |S^A(t)|^2 \right] \right\}
\leq 4 \left\{ (C_1 T + C_2) (\Delta t)^2 + (C_3 + C_4 T) \mathbb{E} \left[ |e(t)|^2 \right] \Delta t \right\}.
\]  
(19)
Therefore, if follows form (12), (13), (15) and (19) that
\[
\mathbb{E} \left[ |e(k)|^2 \right] \leq \sum_{\ell=0}^{k-1} \left\{ \mathbb{E} \left[ |R^A(t)|^2 \right] \frac{1}{\Delta t} + \mathbb{E} \left[ |e(t)|^2 \right] \Delta t + \mathbb{E} \left[ |r(t)|^2 \right] \right\}
\]
\leq (C_1 + 4(C_1 T + C_2) \Delta t + (1 + 4(C_3 + C_4 T)) \sum_{\ell=0}^{k-1} \mathbb{E} \left[ |e(t)|^2 \right] \Delta t
\]
using (8) and Gronwall’s inequality, it holds that
\[
\nu - \kappa X(t) dt + \nu \sqrt{1 - |X(t)|^2} dW(t) =: X(t)
\]
for which the pathwise uniqueness (resp. the assumption of Corollary 2.1) holds. We note that by
\[
E \left[ \left| e(t) \right|^2 \right] \leq C_5 \left( 1 + C_6 e^{C_6} \right) \Delta t =: C_7 \Delta t,
\]
which concludes the first statement.

In this section, we provide some numerical results about the projection scheme
\[
\text{3 Numerical experiments}
\]
\[
\text{of the SDE (4) through numerical experiments. Figures 1, 2, 3, 4 describe sample paths of}
\]
\[
\text{the polynomial diffusion (3) and the SDE (4).}
\]
\[
\text{First, we observe the behavior of the projection scheme with respect to the parameter } \kappa
\]
\[
\text{of the SDE (4) through numerical experiments. Figures 1, 2, 3, 4 describe sample paths of}
\]
\[
\text{the projection scheme } X^{(n)} \text{ with } n = 10000 \text{ time steps for the polynomial diffusion}
\]
\[
dX(t) = -\kappa X(t) dt + \nu \sqrt{1 - |X(t)|^2} dW(t) \text{ on the time interval } [0, T] \text{ with}
\]
\[
w = 2, T = 1, x(0) = (0.7, 0.7)^T, \nu = \sqrt{2}, \text{ and } \kappa = 2, 13, 50, 100. \text{ Here, } \kappa = 2 \text{ (resp. } \kappa = 13) \text{ means the smallest natural number for which the pathwise uniqueness (resp. the assumption of Corollary 2.1) holds. We note that by}
\]
\[
\text{using (8) and Gronwall’s inequality, it holds that}
\]
\[
E \left[ |X(t)|^2 \right] \leq (|x(0)|^2 + d\nu^2 t) e^{-(d\nu^2 + 2\kappa)t} \to 0, \ t \to \infty.
\]
\[
\text{which concludes the second statement.}
\]

3 Numerical experiments

In this section, we provide some numerical results about the projection scheme $X^{(n)}$ for the polynomial diffusion (3) and the SDE (4).

First, we observe the behavior of the projection scheme with respect to the parameter $\kappa$ of the SDE (4) through numerical experiments. Figures 1, 2, 3, 4 describe sample paths of the projection scheme $X^{(n)}$ with $n = 10000$ time steps for the polynomial diffusion $dX(t) = -\kappa X(t) dt + \nu \sqrt{1 - |X(t)|^2} dW(t)$ on the time interval $[0, T]$ with $d = 2, T = 1, x(0) = (0.7, 0.7)^T$, $\nu = \sqrt{2}$, and $\kappa = 2, 13, 50, 100$. Here, $\kappa = 2$ (resp. $\kappa = 13$) means the smallest natural number for which the pathwise uniqueness (resp. the assumption of Corollary 2.1) holds. We note that by using (8) and Gronwall’s inequality, it holds that
\[
E \left[ |X(t)|^2 \right] \leq (|x(0)|^2 + d\nu^2 t) e^{-(d\nu^2 + 2\kappa)t} \to 0, \ t \to \infty.
\]
Hence for any (small) $\varepsilon > 0$, by using Markov’s inequality, it holds that

$$P(|X(t)| < \varepsilon) \geq 1 - \varepsilon^{-2}(|x(0)|^2 + d\nu^2 t)e^{-(d\nu^2+2\kappa)t} \to 1, \ t \to \infty.$$  

Therefore if $\kappa$ and $t$ are sufficiently large, then the value of $|X(t)|$ is small with high probability. Figure 1, 2, 3, 4 express this fact.

Next, we observe the behavior of the projection scheme $\bar{X}^{(n)}$ with respect to the skew symmetric matrices $A_p, p = 0, 1, \ldots, m$ of the SDE (4) through numerical experiments. Figures 5, 6, 7, 8 describe sample paths of the projection scheme $\bar{X}^{(n)}$ with $n = 10000$ time steps for the polynomial diffusion $dX(t) = -\kappa X(t) \ dt + \nu \sqrt{1 - |X(t)|^2} \ dW(t) + A_0 X(t) \ dt + A_1 X(t) \circ d\tilde{W}_1(t)$ on the time interval $[0, T]$ with $d = 2, m = 1, T = 1, x(0) = (0.7, 0.7)^T, \nu = \sqrt{2}, \kappa = 13$ and some skew symmetric matrices $A_0$ and $A_1$. Note that the larger the value of the norm of the symmetric
matrices, the stronger the effect on the vector direction corresponding to each matrix. For example, in Figures 5, 6, 7, 8, if the norm of the matrix $A_2^1$ is sufficiently large, we can see the matrix has a strong effect on the vector direction $(1/2)A_2^1 x$ (recall $A_1^1 X(t) \circ d\hat{W}_1(t) = A_1^1 X(t) d\hat{W}_1(t) + (1/2)A_2^1 X(t) dt$).

Finally, we consider the difference of two solutions $Y^1$ and $Y^2$ of the SDE with different initial conditions, $\nu = \sqrt{2}$ and $A_p = 0$, $p = 0, 1, \ldots, m$. Swart [29] showed that the map $t \mapsto |Y^1(t) - Y^2(t)|$ is almost surely nonincreasing if $\kappa \geq 1$ (see Theorem 3 in [29]). Figure 9, 10 are behaviors of the difference by using the projection scheme for $X^1$ and $X^2$. From these figures, we can confirm that the projection scheme expresses the theoretical result in [29].
Figure 9: $\kappa = 1$, the initial conditions $X^1(0) = (0,0)^T$, $X^2(0) = (-0.7, 0.2)^T$, $n = 312500$ time steps for $X^{(n)}$.

Figure 10: $\kappa = 6/5$, the initial conditions $X^1(0) = (0,0)^T$, $X^2(0) = (-0.7, 0.2)^T$, $n = 112500$ time steps for $X^{(n)}$.

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