STABILITY OF SMOOTH PERIODIC TRAVELING WAVES IN THE CAMASSA-HOLM EQUATION

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Abstract. Smooth periodic travelling waves in the Camassa–Holm (CH) equation are revisited. We show that these periodic waves can be characterized in two different ways by using two different Hamiltonian structures. The standard formulation, common to the Korteweg–de Vries (KdV) equation, has several disadvantages, e.g., the period function is not monotone and the quadratic energy form may have two rather than one negative eigenvalues. We explore the nonstandard formulation and prove that the period function is monotone and the quadratic energy form has only one simple negative eigenvalue. We deduce a precise condition for the spectral and orbital stability of the smooth periodic travelling waves and show numerically that this condition is satisfied in the open region where the smooth periodic waves exist.

1. Introduction

The Camassa–Holm (CH) equation

\[ u_t - u_{txx} + 3u u_x = 2u_x u_{xx} + uu_{xxx} \]  

was derived in [3, 4] and justified in [10, 27] as a model for the propagation of unidirectional shallow water waves. A generalized version of this equation also models propagation of nonlinear waves inside a cylindrical hyper-elastic rod with a small diameter [14]. The CH equation can be interpreted geometrically in terms of geodesic flows on the diffeomorphism group [29, 39].

We consider the CH equation (1.1) on the periodic domain \( \mathbb{T}_L := [0, L] \) of length \( L > 0 \). For notational simplicity, we write \( H^s_{\text{per}} \) instead of \( H^s_{\text{per}}(\mathbb{T}_L) \). The CH equation (1.1) on \( \mathbb{T}_L \) conserves formally the mass, momentum, and energy given respectively by

\[ M(u) = \int_0^L u \, dx, \]  

\[ E(u) = \frac{1}{2} \int_0^L (u_x^2 + u^2) \, dx, \]  

and

\[ F(u) = \frac{1}{2} \int_0^L (u^3 + uu_{xx}^2) \, dx. \]
Many results are available for the CH equation (1.1) in the periodic domain $\mathbb{T}_L$. The initial-value problem is locally well-posed in the space $H^3_{\text{per}}$, $H^s_{\text{per}}$ with $s > \frac{3}{2}$, $H^1_{\text{per}} \cap L^1$, where $L^1$ stands for Lipshitz continuous functions suitable for peaked periodic waves.

Smooth, peaked, and cusped periodic travelling waves were classified in [33]. Cusped periodic waves were constructed in [2, 24] to show that local solutions in $H^1_{\text{per}}$ are not uniformly continuous with respect to the initial data. More recently, the period function for the smooth periodic waves was analyzed in [19].

Orbital stability of the smooth periodic travelling waves in $H^1_{\text{per}}$ was obtained in [34] with the inverse scattering transform for initial data $u_0$ in $H^3_{\text{per}}$ such that $m_0 := u_0 - u_0''$ is strictly positive. Thanks to the Lax representation, the orbital stability of the smooth periodic waves in the time evolution of the CH equation (1.1) follows from the structural stability of the Floquet spectrum in the associated Lax equations.

Orbital stability of peaked periodic waves in $H^1_{\text{per}}$ was proven in [30, 32] by using two different variational methods, each uses the three conserved quantities (1.2), (1.3), and (1.4). The recent work [37] shows that the perturbations to the peaked periodic waves grow exponentially in $H^1_{\text{per}}$ and may blow up in finite time.

Stability of cusped periodic waves is an open problem due to the lack of continuity with respect to initial data in $H^1_{\text{per}}$.

Regarding the limit to the solitary waves, orbital stability of smooth solitary waves in $H^1(\mathbb{R})$ was obtained in [13] for a modified version of the CH equation, where the standard orbital stability for solitary waves hold. Orbital stability of peaked solitary waves in $H^1(\mathbb{R})$ was obtained in [11, 12] but the recent work [42] showed that the perturbations to the peaked solitary waves actually grow in $W^{1,\infty}(\mathbb{R})$.

The main purpose of this paper is to address the spectral and orbital stability of the smooth periodic waves by using the analytic theory used for stability of periodic waves in other nonlinear evolution equations of KdV type [1, 17, 21, 25, 26, 41].

In the standard spectral stability theory, we identify the smooth periodic travelling wave as a critical point of the action functional and compute the number of negative eigenvalues in the linearized operator which represents the quadratic energy form. The number of negative eigenvalues is typically controlled by the monotonicity of the period function for the smooth periodic waves [20]. If the action functional is a linear combination of the mass, momentum, and energy, the spectral stability is determined by positivity of the linearized operator under the constraints of fixed mass and momentum. If the periodic waves of a fixed period are extended smoothly with respect to parameters, derivatives of the mass and momentum computed at the periodic waves with respect to their parameters determine the number of negative eigenvalues of the linearized operator under the constraints.

The technique is straightforward in the case when the linearized operator has only one simple negative eigenvalue, e.g. in [17]. Moreover, orbital stability of such periodic waves in the energy space can be easily concluded from its spectral stability [11]. However, computations become messy when the linearized operator has two negative eigenvalues and the periodic wave has limited smoothness with respect to its parameters [25, 41].
**The main novelty of this paper** is to show that stability of the smooth periodic waves of the CH equation (1.1) can be characterized in two different ways by using two different Hamiltonian structures [6]. The two equivalent formulations are related to two different parameters $a$ and $b$ of the periodic travelling wave solutions in addition to the wave speed $c$. The standard formulation common to evolution equations of KdV type has many disadvantages, whereas the nonstandard formulation common to the evolution equations of CH type is suitable for the proof of spectral and orbital stability of the smooth periodic waves. The results obtained here for the CH equation (1.1) could be applicable to other evolution equations of CH type such as the Degasperis–Procesi equation or the $b$-family of the CH equations [22].

We shall now describe the main results of this paper.

Smooth traveling waves of the form $u(x, t) = \phi(x - ct)$ with speed $c$ and profile $\phi$ satisfy the third-order differential equation

\[ - (c - \phi)(\phi''' - \phi') - 2\phi\phi' + 2\phi'\phi'' = 0. \] (1.5)

This equation can be integrated in two different ways. The standard integration of (1.5) in $x$ gives the second-order equation:

\[ - (c - \phi)\phi'' + c\phi - \frac{3}{2}\phi^2 + \frac{1}{2}\phi'^2 = b, \] (1.6)

where $b$ is the integration constant. However, another integration is obtained after multiplying (1.5) by $(c - \phi)$, which yields the second-order equation:

\[ - (c - \phi)^2(\phi'' - \phi) = a, \] (1.7)

where $a$ is another integration constant. Both second-order equations (1.6) and (1.7) are compatible if and only if $\phi$ satisfies the first-order invariant:

\[ (c - \phi)(\phi'^2 - \phi^2 - 2b) + 2a = 0. \] (1.8)

It is easy to verify that one of the three equations (1.6), (1.7), and (1.8) is satisfied if and only if the other two equations are satisfied. We consider the smooth $L$-periodic travelling wave solutions of (1.1), which means that we are looking for solutions $\phi \in H^\infty_{\text{per}}$ of the system (1.6), (1.7), and (1.8).

Let us connect the second-order equations (1.6) and (1.7) with two different Hamiltonian structures of the CH equation (1.1).

The standard Hamiltonian structure for the CH equation (1.1) is given by

\[ \frac{du}{dt} = J\frac{\delta F}{\delta u}, \quad J = -(1 - \partial_x^2)^{-1}\partial_x, \quad \frac{\delta F}{\delta u} = \frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}u_x^2, \] (1.9)

where $J$ is a well-defined operator from $H^s_{\text{per}}$ to $H^{s+1}_{\text{per}}$ for every $s \in \mathbb{R}$ and $\frac{\delta F}{\delta u}$ is an operator from $H^s_{\text{per}}$ to $H^{s-2}_{\text{per}}$ for $s > \frac{3}{2}$ thanks to Sobolev’s embedding of $H^s_{\text{per}}$ into $C^1_{\text{per}}$. The evolution problem (1.9) is well-defined for local solutions $u \in C((-t_0, t_0), H^s_{\text{per}}) \cap C^1((-t_0, t_0), H^{s-1}_{\text{per}})$ with some $t_0 > 0$ and $s > \frac{3}{2}$, see [7, 20, 23].

The second-order equation (1.6) is the Euler–Lagrange equation for the action functional

\[ \Lambda_{c,b}(u) := cE(u) - F(u) - bM(u). \] (1.10)
The linearized operator for the second-order equation (1.6) is given by
\[ L := -\partial_x(c - \phi)\partial_x + (c - 3\phi + \phi''), \]
which is related to the action functional (1.10) as \( L = \Lambda''_c(\phi). \) The linearized operator \( L : H^2_{\text{per}} \subset L^2_{\text{per}} \mapsto L^2_{\text{per}} \) is a self-adjoint, unbounded operator in \( L^2_{\text{per}} \) equipped with the standard inner product \( \langle \cdot, \cdot \rangle. \)

The alternative Hamiltonian structure for the CH equation (1.1) is given by
\[ \frac{dm}{dt} = J_m \delta E \delta m, \quad J_m = - \left( m\partial_x + \partial_x m \right), \quad \delta E \delta m = u, \]
where \( m := u - u_{xx} \) and \( E(u) \) can be written equivalently as
\[ E(u) = \frac{1}{2} \int_0^L (u_x^2 + u^2) dx = \frac{1}{2} \int_0^L um dx = \frac{1}{2} \int_0^L m(1 - \partial_x^2)^{-1} m dx. \]
By using (1.12), the CH equation (1.1) is rewritten in the local differential form
\[ m_t + um_x + 2mu_x = 0. \]

The second-order equation (1.7) is related to the action functional
\[ \Lambda_c(m) := E(u) - cM(u), \quad u := (1 - \partial_x^2)^{-1} m. \]
Indeed, the Euler–Lagrange equation \( J_m \frac{\delta \Lambda_c}{\delta m} = 0 \) gives the differential equation
\[ \mu'(\phi - c) + 2\mu\phi'' = 0, \]
where \( \mu = \phi - \phi'' \) or \( \phi = (1 - \partial_x^2)^{-1} \mu. \) Integration of (1.16) multiplied by \( \phi - c \) yields \((c - \phi)^2\mu = a\) which is equivalent to (1.7). The linearized operator for \((c - \phi)^2\mu = a(c - \phi)\) acting on \( \mu \) is given by
\[ K := (c - \phi)^3 - 2a(1 - \partial_x^2)^{-1}, \]
The linearized operator \( K : L^2_{\text{per}} \mapsto L^2_{\text{per}} \) is the sum of a bounded and a compact self-adjoint operator in \( L^2_{\text{per}}. \) The relation of this operator to the action functional (1.15) is not obvious and will be shown in Lemma 3.1.

Let us now give the definitions of spectral and orbital stability of the smooth periodic travelling waves in the CH equation (1.1).

**Definition 1.1.** We say that the smooth periodic travelling wave \( \phi \in H^\infty_{\text{per}} \) is spectrally stable in the evolution problem (1.1) if the spectrum of \( JL \) in \( L^2_{\text{per}} \) is located on the imaginary axis.

**Definition 1.2.** We say that the smooth periodic travelling wave \( \phi \in H^\infty_{\text{per}} \) is orbitally stable in the evolution problem (1.1) in \( H^1_{\text{per}} \) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( u_0 \in H^s_{\text{per}} \) with \( s > \frac{3}{2} \) satisfying
\[ \|u_0 - \phi\|_{H^s_{\text{per}}} < \delta, \]
the global solution \( u \in C(\mathbb{R}, H^s_{\text{per}}) \) with the initial data \( u_0 \) satisfies
\[ \inf_{r \in \mathbb{R}} \|u(t, \cdot) - \phi(\cdot + r)\|_{H^1_{\text{per}}} < \varepsilon \]
for all \( t \geq 0. \)
The following two theorems represent the main results of this paper.

**Theorem 1.3.** For a fixed \( c > 0 \), smooth periodic solutions of the system \((1.6), (1.7),\) and \((1.8)\) exist in an open, simply connected region on the \((a,b)\) plane closed by three boundaries:

- \( a = 0 \) and \( b \in (-\frac{1}{2}c^2, 0) \) (where the periodic solutions are peaked),
- \( a = a_+(b) \) and \( b \in (0, \frac{1}{6}c^2) \) (where the solutions have infinite period),
- \( a = a_-(b) \) and \( b \in (-\frac{1}{2}c^2, \frac{1}{6}c^2) \) (where the solutions are constant),

where \( a_+(b) \) and \( a_-(b) \) are smooth functions of \( b \) specified in Lemmas 2.2 and 2.3. For every point inside the region, the periodic solutions are smooth functions of \((a,b,c)\) and their period is strictly increasing in \( b \) for every fixed \( a \in (0, \frac{4}{27}c^3) \) and \( c > 0 \). Moreover, \( K \) has exactly one simple negative eigenvalue, a simple zero eigenvalue, and the rest of its spectrum in \( L^2_{\text{per}} \) is strictly positive and bounded away from zero.

**Theorem 1.4.** For a fixed \( c > 0 \) and a fixed period \( L > 0 \), there exists a \( C^1 \) mapping \( a \mapsto b = B_L(a) \) for \( a \in (0, a_L) \) with some \( a_L \in (0, \frac{4}{27}c^3) \) and a \( C^1 \) mapping \( a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}} \) of smooth \( L \)-periodic solutions along the curve \( b = B_L(a) \). Let

\[
\begin{align*}
\mathcal{M}_L(a) &:= M(\Phi_L(\cdot, a)) \\
\mathcal{E}_L(a) &:= E(\Phi_L(\cdot, a)).
\end{align*}
\]

The \( L \)-periodic wave with profile \( \Phi_L(\cdot, a) \) is spectrally and orbitally stable in the sense of Definitions 1.1 and 1.2 respectively, if the mapping

\[
a \mapsto \frac{\mathcal{E}_L(a)}{\mathcal{M}_L(a)^2}
\]

is strictly decreasing.

**Remark 1.5.** In comparison with Theorem 1.3, it follows from the results in \([19]\) that the period of the periodic solutions of the system \((1.6), (1.7),\) and \((1.8)\)

- is monotonically increasing in \( a \) if \( b \in (-\frac{1}{2}c^2, -(1 - \frac{\sqrt{2}}{\sqrt{3}})c^2] \);
- has a single maximum point in \( a \) if \( b \in (-1 - \frac{\sqrt{2}}{\sqrt{3}}c^2, 0) \);
- is monotonically decreasing in \( a \) if \( b \in [0, \frac{1}{6}c^2) \).

We will show in Theorem 3.6 that \( L \) has two simple negative eigenvalues if the period is increasing in \( a \) and one simple negative eigenvalue if the period is decreasing in \( a \), in addition to the simple zero eigenvalue in \( L^2_{\text{per}} \).

**Remark 1.6.** The spectrum of \( L \) in \( L^2_{\text{per}} \) is purely discrete, whereas the spectrum of \( K \) in \( L^2_{\text{per}} \) includes both continuous and discrete parts, see Lemma 3.2. However, the continuous part of \( K \) is strictly positive and does not contribute to the count of negative eigenvalues.

**Remark 1.7.** In the context of Theorem 1.4, we show numerically that the stability criterion \((1.18)\) is satisfied for every periodic solution of Theorem 1.3, see Figure 4.1.

**Remark 1.8.** We also show in Lemma 5.5 and Remark 5.6 without appealing to the numerical verification of the stability criterion \((1.18)\) that the smooth periodic traveling waves of Theorem 1.3 are orbitally stable for \( b \leq 0 \) and also in a neighborhood of the boundary \( a = a_-(b) \) where the solutions are constant.
Remark 1.9. The following transformation
\[ \phi(x) = c\varphi(x), \quad b = c^2\beta, \quad a = c^3\alpha \] (1.19)
normalizes the parameter $c$ to unity, so that $\varphi$, $\beta$, and $\alpha$ satisfy the same system (1.6), (1.7), and (1.8) but with $c = 1$. Hence, the smooth periodic waves are uniquely determined by the free parameters $(a, b)$ and $c = 1$ can be used everywhere. For clarity of presentation, we will keep the parameter $c$ in all equations until Lemma 4.6, where we will use the scaling transformation (1.19).

Remark 1.10. We only consider the case of right-propagating waves with $c > 0$; however, all results are extended to the left-propagating waves with $c < 0$ by simply flipping the signs in the scaling transformation (1.19).

The paper is organized as follows. In Section 2, we study the existence of the smooth periodic solutions and give the proof of the first two assertions in Theorem 1.3. In Section 3, we study the eigenvalues of the linearized operators $\mathcal{L}$ and $\mathcal{K}$ and give the proof of the last assertion of Theorem 1.3. In parallel, we also show the assertions in Remarks 1.5 and 1.6. In Section 4, we study the linearized evolution under two constraints and give the proof of spectral stability in Theorem 1.4 and Remark 1.7. In Section 5, we prove the orbital stability in Theorem 1.4 and also obtain orbital stability directly for $b \leq 0$ as in Remark 1.8. Section 6 concludes the paper with a summary and a discussion of open directions. Appendix A describes the approach used for numerical approximations of the smooth periodic waves.

2. Existence of smooth periodic traveling waves

Here we study existence of smooth periodic solutions of the system (1.6), (1.7), and (1.8) and provide the proof of the first two assertions of Theorem 1.3.

Let us rewrite (1.8) as the total energy $b$ of Newton’s particle of unit mass with the coordinate $\phi$ in “time” $x$ with the potential energy $U(\phi)$:
\[ b = \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 + U(\phi), \quad U(\phi) = -\frac{1}{2} \phi^2 + \frac{a}{c - \phi} \] (2.1)
Critical points of $U$ are given by roots of the cubic equation $a = \phi(c - \phi)^2$. The local maximum of $\phi \mapsto \phi(c - \phi)^2$ occurs at $\phi = \frac{c}{3}$, from which we define $a_c := \frac{4c^3}{27}$. For $a \in (-\infty, 0) \cup (a_c, \infty)$, there exists only one critical (maximum) point of $U$, whereas for $a \in (0, a_c)$ there exist three critical points of $U$, two are local maximum and one is local minimum. In addition, $\phi = c$ is the pole singularity of $U(\phi)$ if $a \neq 0$. See Figure 2.1 for illustration of the three different cases of $U$.

It follows from dynamics of the Newton particle with the total energy in (2.1) that all smooth mappings $x \mapsto \phi(x)$ for $a \in (-\infty, 0] \cup [a_c, \infty)$ are unbounded. Although peaked and cusped periodic solutions exist in this case [33, 37], we are only concerned with the smooth periodic solutions here.

For $a \in (0, a_c)$, we shall label the critical points of $U$ as $\phi_1 < \phi_2 < \phi_3$. The following order is obtained from the graphical analysis of the cubic equation $a = \phi(c - \phi)^2$ on Fig.
\[0 < \phi_1 < \frac{c}{3} < \phi_2 < c < \phi_3.\] (2.2)

The local minimum of \(U\) at \(\phi_2\) gives the center of the second-order equation (1.7) at \((\phi_2, 0)\). This implies that the smooth periodic solutions form a period annulus, that is, a punctured neighbourhood of the center \((\phi_2, 0)\) enclosed by the homoclinic orbit connecting the saddle \((\phi_1, 0)\). The phase portrait \((\phi, \phi')\) with the period annulus around the center \((\phi_2, 0)\) is illustrated on Figure 2.2.

**Figure 2.1.** The graph of \(U\) versus \(\phi\) in (2.1) for \(a \in (-\infty, 0)\) (left), \(a \in (0, a_c)\) (middle), and \(a \in (a_c, \infty)\) (right) for \(c = 2\).

**Figure 2.2.** Phase portrait of the second-order equation (1.7) constructed from the level curves of the first-order invariant (2.1) for \(a = 0.4\) and \(c = 2\).

The smooth periodic solutions for fixed \(c > 0\) and \(a \in (0, a_c)\) are parameterized by the parameter \(b\) in \((b_-, b_+),\) where \(b_- = U(\phi_2)\) and \(b_+ = U(\phi_1).\) The following result summarizes the existence of smooth periodic solutions.
Lemma 2.1. Fix $c > 0$. For a fixed $a \in (0, a_c)$ with $a_c := \frac{4c^3}{27}$, there exists a family of smooth periodic solutions $\phi$ of the second-order equation (1.7) closed with the first-order invariant (2.1) parameterized by $b \in (b_-, b_+)$, where $b_- = U(\phi_2)$ and $b_+ = U(\phi_1)$, such that $\phi \in (0, c)$. The solution $\phi$ is smooth with respect to parameters $a$, $b$, and $c$.

Proof. Every periodic solution $\phi$ of the second-order equation (1.7) corresponds to a periodic orbit of the planar system with the first integral given by (2.1). Its level set parametrized by $b \in (b_-, b_+)$ defines the periodic orbits inside the period annulus around the center $(\phi_2, 0)$, which exists if $a \in (0, a_c)$ as shown above. Due to the ordering (2.2), the periodic solutions satisfy $\phi \in (\phi_1, c)$ which implies that $\phi \in (0, c)$. Since the first-order invariant (2.1) is smooth with respect to parameters $a$, $b$, and $c$, the periodic orbits inside the period annulus are also smooth with respect to parameters. □

Let us now define the period function $\mathcal{L}(a, b, c)$ for the smooth $L$-periodic solutions of Lemma 2.1. For fixed $a \in (0, a_c)$, $b \in (b_-, b_+)$, and $c > 0$, let $\phi_+$ and $\phi_+$ be turning points of the Newton’s particle satisfying the ordering

$$0 < \phi_1 < \phi_- < \phi_2 < \phi_+ < c < \phi_3.$$  

(2.3)

The turning points are roots of the algebraic equation

$$(c - \phi_+)(2b + \phi_+^2) = 2a. \quad \text{(2.4)}$$

Without loss of generality, we place the maximum of $\phi$ at $x = 0$ and the minimum of $\phi$ at $x = \pm L/2$ so that $\phi(0) = \phi_+$ and $\phi(\pm L/2) = \phi_$. Since the extremal values of $\phi$ are non-degenerate if $b \in (b_-, b_+)$, then $\phi''(0) < 0$ and $\phi''(\pm L/2) > 0$. It follows from (2.4) that

$$(c\phi_+ - \frac{3}{2}\phi_+^2 - b)b_\phi = 1, \quad \text{ (2.5)}$$

and since $c - \phi_+ > 0$ and

$$\phi''(0) = \frac{c\phi_+ - \frac{3}{2}\phi_+^2 - b}{c - \phi_+} < 0, \quad \phi''(\pm L/2) = \frac{c\phi_+ - \frac{3}{2}\phi_+^2 - b}{c - \phi_+} > 0, \quad \text{ (2.7)}$$

we have $\partial_a \phi_\pm, \partial_b \phi_\pm \neq 0$ with

$$\text{sign}(\partial_a \phi_\pm) = -\text{sign}(\partial_b \phi_\pm). \quad \text{ (2.8)}$$

The period function $\mathcal{L}(a, b, c)$ is defined by integrating the quadrature (2.1)

$$\mathcal{L}(a, b, c) := \int_{\phi_-}^{\phi_+} \frac{2\sqrt{c - 2b + \phi^2}}{\sqrt{(c - \phi)(2b + \phi^2) - 2a}}. \quad \text{ (2.9)}$$

Figure 2.3 shows the existence region of the smooth periodic solutions on the $(a, b)$ plane for a fixed $c > 0$. Roots of the cubic equation $a = \phi(c - \phi)^2$ satisfying the ordering (2.2) are computed numerically for every $a \in (0, a_c)$, from which we compute the values $b_- = U(\phi_2)$ and $b_+ = U(\phi_1)$. Plotting $b_-$ and $b_+$ versus $a$ gives the existence region enclosed by three boundaries.
The following three lemma clarify how the smooth periodic solutions transform when \((a, b)\) approach each boundary for a fixed \(c > 0\).

**Lemma 2.2.** Fix \(c > 0\) and \(a \in (0, a_c)\). The smooth periodic solutions of Lemma 2.1 transform as \(b \to b_-(a)\) to the constant solutions. The limiting period function 
\[
\mathcal{L}_-(a) := \mathcal{L}(a, b_-(a), c)
\]
satisfies \(\mathcal{L}_-'(a) > 0\) with \(\mathcal{L}_-(a) \to 0\) as \(a \to 0\) and \(\mathcal{L}_-(a) \to \infty\) as \(a \to a_c\). The mapping \(a \mapsto b_-(a)\) is \(C^1\) and invertible with the inverse \(a = a_-(b)\) for \(b \in (-\frac{1}{2}c^2, \frac{1}{6}c^2)\).

**Proof.** It follows from the ordering (2.3) that the boundary \(b = b_-(a)\) corresponds to the center \(\phi_- = \phi_+ = \phi_2\). Hence, \(\phi(x) = \phi_2\) is constant in \(x\). Linearization of the second-order equation (1.7) at the center point \((\phi_2, 0)\) determines the period \(\mathcal{L}_-(a) := \mathcal{L}(a, b_-(a), c)\) in the form:
\[
\mathcal{L}_-(a) = \frac{2\pi}{\omega}, \quad \omega = \sqrt{\frac{2a}{(c - \phi_2)^3} - 1}.
\tag{2.10}
\]
Along the curve \(b = b_-(a)\), \(a\) and \(b\) can be parametrized by \(\phi_2\) as
\[
\begin{cases}
  b = c\phi_2 - \frac{3}{2}\phi_2^2, \\
  a = \phi_2(c - \phi_2)^2,
\end{cases}
\tag{2.11}
\]
which follow from equations (1.6) and (1.7) using that \(\phi = \phi_2\) is constant.
Solving the first (quadratic) equation in (2.11) for $\phi_2$ as

$$\phi_2 = \frac{c}{3} + \frac{\sqrt{c^2 - 6b}}{3}$$

and substituting the second (cubic) equation in (2.11) for $a$ into (2.10) yields

$$\omega^2 = \frac{2a}{(c - \phi_2)^3} - 1 = \frac{3\phi_2 - c}{c - \phi_2} = \frac{3\sqrt{c^2 - 6b}}{2c - \sqrt{c^2 - 6b}}.$$  

This allows us to express $b$ explicitly in terms of $L_-(a)$ by

$$b = \frac{c^2}{6} \left[ 1 - \frac{64\pi^4}{(4\pi^2 + 3L_-(a)^2)^2} \right].$$ (2.12)

It follows from (2.12) that $L_-(a)$ increases in $b$ along the curve $b = b_-(a)$ and satisfies $L_-(a) \to 0$ as $b \to -\frac{1}{2}c^2$ (or equivalently, $a \to 0$) and $L_-(a) \to \infty$ as $b \to \frac{1}{6}c^2$ (or equivalently, $a \to a_c$). Since the parametrization (2.11) implies that $\frac{db}{d\phi_2} = c - 3\phi_2$, $\frac{da}{d\phi_2} = (c - \phi_2)(c - 3\phi_2)$, it follows from ordering (2.3) that the boundary $b = b_+(a)$ corresponds to $\phi = \phi_1$. Hence, $\phi(x)$ is the solitary wave solution satisfying $\phi(x) \to \phi_1$ as $x \to \pm\infty$, so that $L_+(a) := L(a, b_+(a), c) = \infty$. Along the curve $b = b_+(a)$, $a$ and $b$ can be parametrized by $\phi_1$ as

$$\begin{cases}
  b = c\phi_1 - \frac{3}{2}\phi_1^2, \\
  a = \phi_1(c - \phi_1)^2,
\end{cases}$$ (2.14)

which follow from equations (1.6) and (1.7) using that $\phi = \phi_1$ is a constant solution if $b = U(\phi_1)$. By the same argument as in (2.13) but with $\phi_2$ replaced by $\phi_1$, the mapping $a \mapsto b_+(a)$ is $C^1$, invertible, and monotonically increasing from $(a, b) = (0, 0)$ to $(a, b) = (\frac{1}{2\pi^2}c^3, \frac{1}{6}c^2)$.

**Lemma 2.3.** Fix $c > 0$ and $a \in (0, a_c)$. The smooth periodic solutions of Lemma 2.1 transform as $b \to b_+(a)$ to the solitary wave solutions with

$$L_+(a) := L(a, b_+(a), c) = \infty.$$  

The mapping $a \mapsto b_+(a)$ is $C^1$ and invertible with the inverse $a = a_+(b)$ for $b \in (0, \frac{1}{6}c^2)$.

**Proof.** It follows from ordering (2.3) that the boundary $b = b_+(a)$ corresponds to $\phi = \phi_1$. Hence, $\phi(x)$ is the solitary wave solution satisfying $\phi(x) \to \phi_1$ as $x \to \pm\infty$, so that $L_+(a) := L(a, b_+(a), c) = \infty$. Along the curve $b = b_+(a)$, $a$ and $b$ can be parametrized by $\phi_1$ as

$$\begin{cases}
  b = c\phi_1 - \frac{3}{2}\phi_1^2, \\
  a = \phi_1(c - \phi_1)^2,
\end{cases}$$ (2.14)

which follow from equations (1.6) and (1.7) using that $\phi = \phi_1$ is a constant solution if $b = U(\phi_1)$. By the same argument as in (2.13) but with $\phi_2$ replaced by $\phi_1$, the mapping $a \mapsto b_+(a)$ is $C^1$, invertible, and monotonically increasing from $(a, b) = (0, 0)$ to $(a, b) = (\frac{1}{2\pi^2}c^3, \frac{1}{6}c^2)$.

**Lemma 2.4.** Fix $c > 0$ and $b \in (-\frac{1}{2}c^2, 0)$. The smooth periodic solutions of Lemma 2.1 transform as $a \to 0$ to the peaked periodic solutions and the period function

$$L_0(b) := L(0, b, c)$$  

satisfies $L_0'(b) > 0$ with $L_0(b) \to 0$ as $b \to -\frac{1}{2}c^2$ and $L_0(b) \to \infty$ as $b \to 0$. 


Proof. If \( a = 0 \), then \( \phi \) satisfies the equation \( \phi'' - \phi = 0 \) with
\[
\max_{x \in [-\frac{L}{2}, \frac{L}{2}]} \phi(x) = \phi(0) = c
\]
since \( \phi_+ = c \) and \( \phi_- = \sqrt{2|b|} \). This equation can be solved explicitly
\[
\phi(x) = \frac{c}{2} \cosh \left( -\frac{L}{2} - \frac{|x|}{2} \right), \quad x \in \left[ -\frac{L}{2}, \frac{L}{2} \right].
\] (2.15)
The periodic wave is peaked at \( x = 0 \) and smooth at \( x = \pm \frac{L}{2} \) with \( \phi'(\pm \frac{L}{2}) = 0 \). It follows from (1.8) and (2.15) that
\[
b = \frac{1}{2} \left[ (\phi')^2 - \phi^2 \right] = -\frac{c^2}{2 \cosh^2 \left( \frac{L}{2} \right)},
\] (2.16)
in agreement with \( \phi_- = \sqrt{2|b|} \). Hence \( b \in (-\frac{1}{2}c^2, 0) \) and it follows from (2.16) that \( L = \mathcal{L}_0(b) \) increases in \( b \) and satisfies \( \mathcal{L}_0(b) \to 0 \) as \( b \to -\frac{1}{2}c^2 \) and \( \mathcal{L}_0(b) \to \infty \) as \( b \to 0 \). \( \square \)

Remark 2.5. The two boundaries of Lemmas 2.2 and 2.3 intersect at \( a = a_c = \frac{4c^3}{27} \), where the two critical points coalesce: \( \phi_1 = \phi_2 = \frac{c}{3} \). This corresponds to \( b = b_c = c^2/6 \). The two boundaries intersect with the third boundary \( a = 0 \) of Lemma 2.4 at \( b = 0 \) and \( b = -\frac{1}{2}c^2 \) respectively.

Finally, we prove the main result of this section that the period function \( \mathcal{L}(a, b, c) \) is a strictly increasing function of \( b \) for any fixed \( a \in (0, a_c) \) and \( c > 0 \).

Theorem 2.6. Fix \( c > 0 \) and \( a \in (0, a_c) \), where \( a_c := \frac{4c^3}{27} \). The period function \( \mathcal{L}(a, b, c) \) is strictly increasing in \( b \).

Proof. Let \( a = \phi_2(c - \phi_2)^2 \), where \( \phi_2 \) is the second root in the ordering (2.2). Using the transformation \( \{ x = \frac{\phi - \phi_2}{\phi_2}, y = \frac{\phi}{\phi_2} \} \), we can write the second-order equation (1.7) as the planar system
\[
\begin{align*}
x' &= y, \\
y' &= 1 + x - \frac{\eta^2}{(\eta - x)^2},
\end{align*}
\] (2.17)
associated with the Hamiltonian
\[
H(x, y) = \frac{y^2}{2} + V(x), \quad V(x) := -\frac{x^2}{2} - x - \eta + \frac{\eta^2}{\eta - x},
\] (2.18)
where \( \eta = \frac{c - \phi_2}{\phi_2} \in (0, 2) \).

The potential \( V \) is smooth away from the singular line \( x = \eta \), has a local minimum at \( x = 0 \) and two maxima at \( x_1 := \eta - \frac{1}{2} - \frac{\sqrt{1 + 2\eta}}{2} \) and \( x_3 := \eta - \frac{1}{2} + \frac{\sqrt{1 + 2\eta}}{2} \). The center at the origin is surrounded by periodic orbits \( \gamma_h \), which lie inside the level curves \( H(x, y) = h \) with \( h \in (0, h^*) \) and \( h^* = V(x_1) \). Denote by \( x_2 \) the unique solution of \( V(x_1) = V(x) \) such
that \( x_1 < 0 < x_2 < \eta < x_3 \), see Figure 2.4. Finally, define the period function of the center \((0, 0)\) of system (2.17) by

\[
\ell(h) = \int_{\gamma_h} \frac{dx}{y} \quad \text{for each } h \in (0, h^*).
\]  

(2.19)

Note that \( b = \phi_2^2 (h + \eta - \frac{1}{2}) \) and \( \mathcal{L}(a, b, c) = \ell(h) \) for fixed \( a \in (0, a_c) \) and \( c > 0 \). Since \( \phi_2 \) is fixed, we have \( \partial_b \mathcal{L}(a, b, c) > 0 \) if and only if \( \ell'(h) > 0 \).

![Figure 2.4. The potential function \( V(x) \) plotted for \( \eta = \frac{1}{4} \).](image)

To prove that \( \ell'(h) > 0 \), we shall use a monotonicity criterion by Chicone [5] for planar systems with Hamiltonians of the form (2.18), where \( V \) is a smooth function on \((x_1, x_2)\) with a nondegenerate relative minimum at the origin. The period function \( \ell(h) \) is monotonically increasing in \( h \) if the function

\[
W(x) := \frac{V(x)}{(V'(x))^2}
\]

is convex in \((x_1, x_2)\). Hence, we have to prove that \( W''(x) > 0 \) for every \( x \in (x_1, x_2) \). A straightforward computation shows that

\[
W''(x) = \frac{-3(\eta - x)R(x)}{(x - x_1)^2(x - x_3)^4},
\]  

(2.20)

where

\[
R(x) = (-2 \eta + 1) x^3 + \eta (6 \eta - 7) x^2 - 3 \eta^2 (2 \eta - 3) x + \eta^2 (2 \eta + 1) (\eta - 2).
\]

Since \( x_2 < \eta \), we need to show that \( R(x) < 0 \) for \( x \in [x_1, x_2] \) and \( \eta \in (0, 2) \). Note that \( R(0) = \eta^2 (2 \eta + 1)(\eta - 2) < 0 \) for \( \eta \in (0, 2) \).
The discriminant of $R$ with respect to $x$ is given by

$$\text{Disc}_x(R) = -4(4\eta + 1)(4\eta^2 - 16\eta + 27)\eta^4,$$  \hspace{1cm} (2.21)

which is strictly negative for $\eta \in (0, 2)$. Hence, for $\eta \neq \frac{1}{2}$ the cubic polynomial $R$ has exactly one real root, say $x_0$.

For $\eta < \frac{1}{2}$, it follows from the dominant behavior of $R$ that $R(x) \to -\infty$ as $x \to -\infty$. Since $R(\eta) = -2\eta^2 < 0$, it is clear that the only real root $x_0$ is located for $x_0 > \eta$. Therefore, $R(x) < 0$ for $x \in [x_1, x_2]$ with $x_2 < \eta$.

For $\eta > \frac{1}{2}$, we have $R(x) \to -\infty$ as $x \to +\infty$. We claim that

$$R(x_1) = \frac{1}{2}((\eta - 1)\sqrt{4\eta + 1} - \eta - 1)(4\eta + 1) < 0$$  \hspace{1cm} (2.22)

for $\eta \in (0, 2)$. Therefore, the only real root $x_0$ is located for $x_0 < x_1$ and $R(x) < 0$ for $x \in [x_1, x_2]$. In order to prove (2.22), we substitute $\eta = (w^2 - 1)/4$ into $R(x_1)$ and obtain $R(x_1) = \frac{1}{4}(w - 3)(w + 1)^2w^2$ which is negative for $w \in (1, 3)$.

Finally, for $\eta = \frac{1}{2}$ we have that $R(x) = -2x^2 + 3x^2/2 - 3/4$ which is strictly negative for all $x$.

Hence $R(x) < 0$ for $x \in [x_1, x_2]$ if $\eta \in (0, 2)$. Therefore, $W''(x) > 0$ for $x \in (x_1, x_2)$ and $\ell'(h) > 0$ follows by theorem proven in [3].

**Remark 2.7.** The result of Theorem 2.6 can also be verified using the tools from [38] and [18], where Hamiltonian systems with Hamiltonian in the form $H(x, y) = \frac{1}{2}y^2 + V(x)$ are considered with $V(x) = \frac{1}{2m}x^{2m} + o(x^{2m})$, which is analytic in a neighborhood of $x = 0$.

**Remark 2.8.** As claimed in Remark 1.5, the period function $L(a, b, c)$ has different monotonicity properties in $a$ for fixed $b \in (-\frac{1}{2}c^2, \frac{1}{6}c^2)$ and $c > 0$. To be precise, the period function

- is monotonically increasing in $a$ if $b \in (-\frac{1}{2}c^2, -(1 - \frac{2}{\sqrt{3}})c^2]$;
- has a single maximum point in $a$ if $b \in (-1 - \frac{2}{\sqrt{3}}c^2, 0)$;
- is monotonically decreasing in $a$ if $b \in [0, \frac{1}{6}c^2)$.

This result was obtained in [19, Theorem 2.5], where the second-order equation (1.6) with the first-order invariant (1.8) was reformulated into the system

$$\dot{x} = y, \quad \dot{y} = -\frac{y^2 + x - 3x^2}{2(x + \nu)},$$  \hspace{1cm} (2.23)

where

$$\nu := \frac{1}{6}\left[\frac{2c}{\sqrt{c^2 - 6b}} - 1\right].$$

The value $b = -\frac{1}{2}c^2$ corresponds to $\nu = 0$, the value $b = -(1 - \frac{2}{\sqrt{3}})c^2$ corresponds to $\nu = -\frac{1}{10} + \frac{\sqrt{6}}{15}$, the value $b = 0$ corresponds to $\nu = \frac{1}{6}$, and the value $b = \frac{1}{6}c^2$ corresponds to the limit $\nu \to \infty$. 

Figure 2.5 show the graphs of $L(a, b, c)$ versus $b$ for three cases of $a$. The period function is monotonically increasing in $b$ in agreement with Theorem 2.6. Figure 2.6 shows the graphs of $L(a, b, c)$ versus $a$ for three representative cases of $b$. The period function is increasing in $a$ for $b \in \left(-\frac{1}{2}c^2, -(1 - \sqrt{\frac{2}{3}})c^2\right)$ (left), has a single maximum point in $a$ if $b \in \left(-\left(1 - \sqrt{\frac{2}{3}}\right)c^2, 0\right)$ (middle) and is monotonically decreasing in $a$ if $b \in \left[0, \frac{1}{6}c^2\right)$ (right), in agreement with Remark 2.8.

**Figure 2.5.** The period function $L(a, b, c)$ versus parameter $b$ for the smooth periodic solutions with $c = 2$ and three values of $a$: (left) $a = 0.3$, (middle) $a = 0.6$, and (right) $a = 0.9$.

**Figure 2.6.** The period function $L(a, b, c)$ versus parameter $a$ for the smooth periodic solutions with $c = 2$ and three values of $b$: (left) $b = -1.2$, (middle) $b = -0.6$, and (right) $b = 0$.

For the study of spectral stability of periodic solutions in Section 4, it is important to fix the period $L$ and consider the family of $L$-periodic solutions along a curve in the $(a, b)$ plane for a fixed $c > 0$. The following result provides this characterization of the $L$-periodic solutions.

**Lemma 2.9.** Fix $c > 0$ and $L > 0$. There exists a $C^1$ mapping $a \mapsto b = B_L(a)$ for $a \in (0, a_L)$ with some $a_L \in \left(0, \frac{1}{27}c^3\right)$ and a $C^1$ mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^{\infty}_{\text{per}}$ of smooth $L$-periodic solutions along the curve $b = B_L(a)$.

**Proof.** It follows from the monotonicity results in Lemmas 2.2 and 2.4 that for every $c > 0$ and $L > 0$ there exists exactly one $L$-periodic solution on the left and right boundaries of
the existence domain on the \((a,b)\)-plane. The left boundary corresponds to \(a = 0\) and the right boundary corresponds to \(a = a_L\), where \(a_L\) is uniquely defined from the equation \(\mathcal{L}(a_L, b_-(a_L), c) = L\) with \(b_-(a)\) defined in Lemma 2.1.

Since \(\mathcal{L}(a, b, c)\) is smooth in \((a, b, c)\) and it is strictly increasing in \(b\) by Theorem 2.6 the existence of the \(C^1\) mapping \(a \mapsto b = \mathcal{B}_L(a)\) for \(a \in (0, a_L)\) follows by the implicit function theorem for \(\mathcal{L}(a, b, c) = L\) with fixed \(c > 0\) and \(L > 0\). Indeed, \(\partial_a \mathcal{L} + \mathcal{B}_L'(a) \partial_b \mathcal{L} = 0\) and since \(\partial_b \mathcal{L} > 0\), \(\mathcal{B}_L'(a)\) is uniquely defined for every \(a \in (0, a_L)\). Since \(\phi\) is smooth with respect to parameters by Lemma 2.1 the mapping \(a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}}\) is \(C^1\) along the curve \(b = \mathcal{B}_L(a)\).

**Remark 2.10.** The mapping \(b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}}\) may not be \(C^1\) along the curve \(b = \mathcal{B}_L(a)\) because of the non-monotonicity of \(\mathcal{L}(a, b, c)\) with respect to \(a\). It follows from Remark 2.8 that there exists at most one point where \(\mathcal{B}_L'(a) = 0\) and this is the minimum of the mapping \(a \mapsto b = \mathcal{B}_L(a)\). The mapping \(b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}}\) is not \(C^1\) at the minimum point.

The blue curve in Fig. 2.3 shows the location of the single maximum of the period function \(\mathcal{L}(a, b, c)\) in the \((a,b)\) plane for fixed \(c = 2\) according to Remark 2.8. The location of the maximum moves to the right boundary as \(b \to -\left(1 - \frac{\sqrt{2}}{\sqrt{3}}\right)c^2\) and to the left boundary as \(b \to 0\).

The black, green, and cyan curves in Figure 2.3 also show the curves in the \((a,b)\) plane where smooth \(L\)-periodic solutions exist for three different values of \(L\). The black curve with smaller \(L\) does not intersect the blue curve and the family of \(L\)-periodic solutions remains smooth both in \(a\) and \(b\). However, the green and cyan curves with larger periods intersect the blue curve and the family of \(L\)-periodic solutions is smooth in \(a\) but is not smooth in \(b\) at the minimum point of \(b = \mathcal{B}_L(a)\).

The numerical method used to generate Figures 2.3, 2.5, and 2.6 is described in Appendix A.

### 3. Spectral properties of the linearized operator

Here we study the linearization of the CH equation (1.1) at the smooth periodic solutions of the system (1.6), (1.7), and (1.8) and provide the proof of the last assertion of Theorem 1.3.

Adding a perturbation \(v\) to the smooth travelling wave \(\phi\) propagating with the same fixed speed \(c\) in

\[
u(x, t) = \phi(x - ct) + v(x - ct, t)\tag{3.1}
\]
gives the perturbation equation derived from the CH equation (1.1):

\[
(1 - \partial_x^2)(v_t - cv_x) + 3\partial_x(\phi v) + 2vv_x = \partial_x(\phi v_{xx} + \phi' v_x + \phi''v) + 2v_x v_{xx} + vv_{xxx}.\tag{3.2}
\]

Dropping the quadratic terms in \(v\) yields the linearized evolution equation

\[
v_t = -J\mathcal{L}v,\tag{3.3}
\]

where \(J\) is defined in (1.9) and \(\mathcal{L}\) is the linearized operator given by (1.11).
Recall that the travelling periodic wave is spectrally stable in the sense of Definition 1.1 if the spectrum of the linearized operator $JL$ in $L^2_{\text{per}}$ is located on $i\mathbb{R}$. The following lemma reformulates the spectral stability criterion in terms of the linearized operator $K$ introduced in (1.17).

**Lemma 3.1.** The spectrum of $JL$ in $L^2_{\text{per}}$ is located on the imaginary axis if and only if the spectrum of $(c - \phi)^{-1}\partial_x(c - \phi)^{-1}K$ in $L^2_{\text{per}}$ is located on the imaginary axis.

**Proof.** Consider the time evolution of the CH equation in the form (1.14), where $m := u - u_{xx}$. We add a perturbation $p$ to the smooth travelling wave $\mu$ in $m(x, t) = \mu(x - ct) + p(x - ct, t)$. (3.4)

It follows from the decompositions (3.1) and (3.4) that $\mu = \phi - \phi''$ and $p = v - v_{xx}$. Substituting (3.1) and (3.4) into (1.14) gives the perturbation equation

$$p_t + (\phi - c)p_x + 2p\phi' + v\mu' + 2\mu v_x + vp_x + 2pv_x = 0.$$ (3.5)

Dropping the quadratic terms in $p$ and $v$ yields the linearized evolution equation

$$p_t + (\phi - c)p_x + 2p\phi' + v\mu' + 2\mu v_x = 0.$$ (3.6)

It follows from (1.7) that $(c - \phi)^2\mu = a$ and hence

$$v\mu' + 2\mu v_x = \frac{2ac\phi'}{(c - \phi)^3}v + \frac{2a}{(c - \phi)^2}v_x.$$ (3.7)

Multiplying (3.6) by $c - \phi$ and using (3.7) yield the equivalent evolution form:

$$\frac{\partial}{\partial t}[(c - \phi)p] = \frac{\partial}{\partial x}[(c - \phi)^2p - \frac{2a}{c - \phi}v],$$ (3.8)

which can be written in the form

$$p_t = (c - \phi)^{-1}\partial_x(c - \phi)^{-1}Kp,$$ (3.9)

where $K$ is the linearized operator given by (1.17). It follows from the equivalence of (3.3) and (3.9) under the transformation $v = (1 - \partial_x^2)^{-1}p$ that $\lambda \in \sigma(JL)$ in $L^2_{\text{per}}$ if and only if $\lambda \in \sigma((c - \phi)^{-1}\partial_x(c - \phi)^{-1}K)$ in $L^2_{\text{per}}$, where $\sigma(A)$ denotes the spectrum of a linear operator $A$ in $L^2_{\text{per}}$. $\square$

In what follows we study the spectra of the linearized operators $L$ and $K$ in $L^2_{\text{per}}$. The following lemma shows that the spectra of these operators are different.

**Lemma 3.2.** The spectrum of $L$ in $L^2_{\text{per}}$ is purely discrete. The spectrum of $K$ in $L^2_{\text{per}}$ consists of the strictly positive continuous spectrum at

$$\text{image}[\{(c - \phi)^3\}] = [(c - \phi_+)^3, (c - \phi_-)^3]$$

and the discrete spectrum outside $\text{image}[\{(c - \phi)^3\}]$, where $\phi_\pm$ are the turning points defined in (2.4).
Proof. Since \( c - \phi > 0 \) and \( \phi \in H^\infty_{\text{per}} \), the linearized operator \( \mathcal{L} \) with the dense domain \( H^2_{\text{per}} \subset L^2_{\text{per}} \) is a self-adjoint, unbounded operator in \( L^2_{\text{per}} \). Consequently, \( \sigma(\mathcal{L}) \subset \mathbb{R} \) is purely discrete in \( L^2_{\text{per}} \) due to the compact embedding of \( H^2_{\text{per}} \) into \( L^2_{\text{per}} \).

Since \( c - \phi > 0 \), the linearized operator \( \mathcal{K} \) is a self-adjoint, bounded operator in \( L^2_{\text{per}} \), which is the sum of a bounded and a compact operator in \( L^2_{\text{per}} \). Consequently, \( \sigma(\mathcal{K}) \subset \mathbb{R} \) includes both the continuous and discrete spectra in \( L^2_{\text{per}} \) denoted by \( \sigma_c \) and \( \sigma_d \) respectively.

Since the compact operator \( -2a(1 - \partial^2_x) - 1 \) is in the trace class in \( L^2_{\text{per}} \), Kato’s theorem (Theorem 4.4 in [28]) gives

\[
\sigma_c(\mathcal{K}) = \sigma_c((c - \phi)^3) = \text{image}[(c - \phi)^3] = [(c - \phi_+)^3, (c - \phi_-)^3].
\]

Since \( \phi_+ < c \), \( \sigma_c(\mathcal{K}) \) is strictly positive. \( \square \)

The following two theorems describe the non-positive part of the spectrum of \( \mathcal{L} \) and \( \mathcal{K} \) in \( L^2_{\text{per}} \). The proofs rely on Theorem 3.1 in [43] (see also the classical Floquet theory in [16, 36]) and on Sylvester’s inertial law theorem (see [35, Theorem 2.2]). These auxiliary results are formulated in the following two propositions.

**Proposition 3.3.** [43] Let \( \mathcal{M} := -\partial^2_x + Q(x) \) be the Schrödinger operator with the even, \( L \)-periodic, smooth potential \( Q \). Assume that \( \mathcal{M}w = 0 \) is satisfied by a linear combination of two solutions \( \varphi_1 \) and \( \varphi_2 \) satisfying

\[
\varphi_1(x + L) = \varphi_1(x) + \theta \varphi_2(x)
\]

and

\[
\varphi_2(x + L) = \varphi_2(x)
\]

with some \( \theta \in \mathbb{R} \). Assume that \( \varphi_2 \) has two zeros on the period of \( Q \). The zero eigenvalue of \( \mathcal{M} \) in \( L^2_{\text{per}} \) is simple if \( \theta \neq 0 \) and double if \( \theta = 0 \). It is the second eigenvalue of \( \mathcal{M} \) if \( \theta \geq 0 \) and the third eigenvalue of \( \mathcal{M} \) if \( \theta < 0 \).

**Remark 3.4.** Compared to [43], we have interchanged the order of non-periodic \( \varphi_1 \) and periodic \( \varphi_2 \) so that our \( \theta \) is negative relative to \( \theta \) used in [43].

**Proposition 3.5.** [35] Let \( L \) be a self-adjoint operator in a Hilbert space \( H \) and \( S \) be a bounded invertible operator in \( H \). Then, \( SLS^* \) and \( L \) have the same inertia, that is the dimension of the negative, null, and positive invariant subspaces of \( H \).

We can now formulate and prove two theorems on the non-positive part of the spectrum of \( \mathcal{L} \) and \( \mathcal{K} \) in \( L^2_{\text{per}} \).

**Theorem 3.6.** The linearized operator \( \mathcal{L} : H^2_{\text{per}} \subset L^2_{\text{per}} \rightarrow L^2_{\text{per}} \) admits

- two negative eigenvalues and a simple zero eigenvalue if \( \partial_a \mathcal{L} > 0 \);
- one negative eigenvalue and a double zero eigenvalue if \( \partial_a \mathcal{L} = 0 \);
- one negative eigenvalue and a simple zero eigenvalue if \( \partial_a \mathcal{L} < 0 \),

where \( \mathcal{L}(a,b,c) \) is the period function for the smooth periodic wave \( \phi \) of Lemma 2.1. The rest of the spectrum of \( \mathcal{L} \) in \( L^2_{\text{per}} \) is strictly positive and bounded away from zero.
Proof. Due to the invariance of the CH equation \ref{CH} with respect to spatial translations, the third-order equation \ref{THF} is equivalent to $L \phi' = 0$, which means that $\phi' \in \text{Ker}(L) \subset H^2_{per}$. On the other hand, differentiating of the second-order equation \ref{L} in $a$ is equivalent to $L \partial_a \phi = 0$, which means that $\partial_a \phi$ is the second, linearly independent solution of $L v = 0$. Note that $\partial_a \phi$ is well-defined by Lemma \ref{L} and may not be $L$-periodic in $x$.

Let $\{y_1, y_2\}$ be the fundamental set of solutions associated to the equation $L v = 0$ in $H^2(0, L)$ such that

\[
\begin{cases}
y_1(0) = 1, \\
y'_1(0) = 0,
\end{cases}
\begin{cases}
y_2(0) = 0, \\
y'_2(0) = 1,
\end{cases}
\tag{3.10}
\]

As previously, we set $\phi(0) = \phi(L) = \phi_+$ for the smooth $L$-periodic solution of Lemma \ref{L} where $\phi_+$ is the turning point for the maximum of $\phi$ in $x$. Hence, we have $\phi'(0) = \phi'(L) = 0$ so that we define

\[
y_1(x) := \frac{\partial_a \phi(x)}{\partial_a \phi_+}, \quad y_2(x) := \frac{\phi'(x)}{\phi''(0)},
\tag{3.11}
\]

where $\partial_a \phi_+ \neq 0$ and $\phi''(0) \neq 0$ as follows from \ref{L} and \ref{L}. Differentiating of the boundary conditions $\phi(L) = \phi_+$ and $\phi'(L) = 0$ for $L = \Sigma(a, b, c)$ in $a$ yields $y_1(L) = y_1(0) = 1$ and

\[
y'_1(L) = -\frac{\partial_a \Sigma}{\partial_a \phi_+} \phi''(0),
\]

which implies that

\[
y_1(x + L) = y_1(x) + \theta y_2(x),
\tag{3.12}
\]

where

\[
\theta = y'_1(L) = -\frac{\partial_a \Sigma}{\partial_a \phi_+} \phi''(0).
\tag{3.13}
\]

Since $c - \phi_+ > 0$, it follows from \ref{L} and \ref{L} that $\text{sign}(\theta) = -\text{sign}(\partial_a \Sigma)$.

In order to transform the spectral problem $L v = \lambda v$ to the spectral problem $M w = \lambda w$ for the Schrödinger operator $M$ in Proposition \ref{prop}, we write $L v = \lambda v$ as the second-order differential equation

\[
p(x)v'' + q(x)v' + (r(x) + \lambda)v = 0,
\tag{3.14}
\]

with $p(x) := c - \phi(x)$, $q(x) := -\phi'(x)$, and $r(x) := -\phi''(x) + 3\phi(x) - c$. The Liouville transformation

\[
D(x) = -\int_0^x \frac{\phi'(s)}{c - \phi(s)} ds = \ln \left( \frac{c - \phi(x)}{c - \phi(0)} \right)
\tag{3.15}
\]

is nonsingular since $c - \phi > 0$. Substituting the change of variables

\[
v(x) = w(x)e^{-\frac{1}{2}D(x)} = w(x)\sqrt{\frac{c - \phi(0)}{c - \phi(x)}}.
\tag{3.16}
\]

into the second-order equation \ref{3.14}, we obtain the equivalent equation

\[-w''(x) + Q(x)w(x) = \lambda(c - \phi(x))^{-1} w(x),
\tag{3.17}
\]

where

\[
Q(x) := \frac{c - 3\phi(x)}{c - \phi(x)} + \frac{\phi''(x)}{2(c - \phi(x))} \frac{1}{4} \left( \frac{\phi'(x)}{c - \phi(x)} \right)^2.
\]
With the transformation \( w = (c - \phi)^{1/2} v \), the spectral problem (3.17) is equivalent to the spectral problem for the operator \( SM_S \), where \( M := -\partial_x^2 + Q(x) \) is self-adjoint in \( L^2_{\text{per}} \) and \( S = (c - \phi)^{1/2} \) is a bounded and invertible multiplication operator in \( L^2_{\text{per}} \). By Proposition 3.5, the numbers of negative and zero eigenvalues of the spectral problem (3.17) coincides with those of the operator \( M \).

The operator \( M \) satisfies the condition of Proposition 3.3 since \( Q \) is even, \( L \)-periodic, and smooth. Since the set \( \{ w_1, w_2 \} \) is a fundamental set for the equation \( \mathcal{L} v = 0 \) and the initial conditions \( v(0) = w(0) \) and \( v'(0) = w'(0) \) are preserved in the transformation (3.16), it follows that

\[
\{ \varphi_1, \varphi_2 \} := \left\{ \left( \begin{array}{c} c - \phi(0) \\ c - \phi \end{array} \right)^{-1/2} y_1, \left( \begin{array}{c} c - \phi(0) \\ c - \phi \end{array} \right)^{-1/2} y_2 \right\}
\]

is the fundamental set of solutions to \( Mw = 0 \). It follows from (3.12) and (3.18) that

\[
\varphi_1(x + L) = \varphi_1(x) + \theta \varphi_2(x).
\]

where \( \theta \) is given by the same expression (3.13). Furthermore, since \( \phi' \) has two zeros in \( \mathbb{T} \), the same is true for \( y_2 \) and \( \varphi_2 \). By the standard Floquet theory in [16, 36], it follows that \( \lambda = 0 \) is the second or third eigenvalue of \( \mathcal{M} \) in \( L^2_{\text{per}} \). If \( \theta = 0 \), then \( \lambda = 0 \) is the double eigenvalue so that it is the second eigenvalue of \( \mathcal{M} \). If \( \theta \neq 0 \), then \( \lambda = 0 \) is a simple eigenvalue of \( \mathcal{M} \). By Proposition 3.3, it is the second eigenvalue if \( \theta \geq 0 \) and the third eigenvalue if \( \theta < 0 \). Due to the equivalence provided by the nonsingular transformation (3.16), the same is true for the operator \( \mathcal{L} \) in \( L^2_{\text{per}} \), which yields the assertion of the theorem since \( \text{sign}(\theta) = -\text{sign}(\partial_a \mathbb{L}) \).

**Theorem 3.7.** The linearized operator \( \mathcal{K} : L^2_{\text{per}} \to L^2_{\text{per}} \) admits

- two negative eigenvalues and a simple zero eigenvalue if \( \partial_a \mathbb{L} < 0 \);
- one negative eigenvalue and a double zero eigenvalue if \( \partial_a \mathbb{L} = 0 \);
- one negative eigenvalue and a simple zero eigenvalue if \( \partial_a \mathbb{L} > 0 \),

where \( \mathbb{L}(a, b, c) \) is the period function for the smooth periodic wave \( \phi \) of Lemma 2.1. The rest of the spectrum of \( \mathcal{K} \) is strictly positive and bounded away from zero.

**Proof.** The linear operator \( \mathcal{K} \) is congruent to another operator \( \mathcal{K}_0 \) by the transformation

\[
\mathcal{K} = (1 - \partial_x^2)^{-1/2} \mathcal{K}_0 (1 - \partial_x^2)^{-1/2},
\]

where

\[
\mathcal{K}_0 = (1 - \partial_x^2)^{1/2} (c - \phi)^3 (1 - \partial_x^2)^{1/2} - 2a.
\]

Since \( S := (1 - \partial_x^2)^{-1/2} \) is a bounded and invertible operator in \( L^2_{\text{per}} \) and \( \mathcal{K}_0 \) is self-adjoint in \( L^2_{\text{per}} \), it follows by Proposition 3.5 that \( \mathcal{K} \) and \( \mathcal{K}_0 \) in (3.20) have the same inertia, that is, the dimension of the negative, null, and positive invariant subspaces of \( L^2_{\text{per}} \). By Lemma 3.2 the positive invariant subspace of \( \mathcal{K} \) is infinite-dimensional. Hence, we study the non-positive spectrum of \( \mathcal{K}_0 \).

It follows from (3.21) that \( \mathcal{K}_0 \) is an unbounded self-adjoint operator defined in \( L^2_{\text{per}} \) with densely defined domain \( H^2_{\text{per}} \subseteq L^2_{\text{per}} \). The spectrum of \( \sigma(\mathcal{K}_0) \) is given by the union of the continuous and discrete spectra. However, since the embedding of \( H^2_{\text{per}} \) into \( L^2_{\text{per}} \)
is compact, the continuous spectrum is an empty set. Hence, we consider the spectral problem for the discrete spectrum:

\[ K_0 w = \lambda w, \quad w \in H^2_{\text{per}}, \]  

where \( \lambda \in \mathbb{R} \) is an isolated eigenvalue of \( K_0 \) and \( w \neq 0 \) is the corresponding eigenfunction. Considering the change of variables \( w := (1 - \partial_x^2)^{1/2} v \), it follows from (3.22) that

\[ (1 - \partial_x^2)^{1/2} [(c - \phi)^3(1 - \partial_x^2)v - (2a + \lambda)v] = 0. \]  

Since \( (1 - \partial_x^2)^{1/2} \) is invertible in \( L^2_{\text{per}} \), the spectral problem (3.23) is equivalent to the spectral problem

\[ Mv = \lambda(c - \phi)^{-3} v, \]  

where \( M \) is the Schrödinger operator given by

\[ M := -\partial_x^2 + 1 - \frac{2a}{(c - \phi)^3}. \]  

With the transformation \( v = (c - \phi)^{3/2} \tilde{v} \), the spectral problem (3.24) is equivalent to that for the operator \( S \mathcal{M} \mathcal{S} \), where \( S := (c - \phi)^{3/2} \) is a bounded and invertible operator in \( L^2_{\text{per}} \) and \( M \) is a self-adjoint operator in \( L^2_{\text{per}} \). By Proposition 3.5 operators \( M \) and \( S \mathcal{M} \mathcal{S} \) have the same inertia in \( L^2_{\text{per}} \).

Finally, we study the non-positive spectrum of \( M \). It follows from the differential equation (1.7) that

\[ M\phi' = 0, \quad M\partial_b\phi = 0. \]  

Therefore, the general solution of \( Mv = 0 \) is given by a linear combination of two linearly independent solutions

\[ y_1(x) := \frac{\partial_b \phi(x)}{\partial_b \phi_+}, \quad y_2(x) := \frac{\phi'(x)}{\phi''(0)}, \]  

where \( \partial_b \phi_+ \neq 0 \) and \( \phi''(0) \neq 0 \) as follows from (2.5) and (2.7). Differentiating of the boundary conditions \( \phi(L) = \phi_+ \), and \( \phi'(L) = 0 \) for \( L = \mathcal{L}(a,b,c) \) in \( b \) yields \( y_1(L) = y_1(0) = 1 \) and

\[ y_1'(L) = -\frac{\partial_b \mathcal{L}}{\partial_b \phi_+} \phi''(0), \]  

so that

\[ y_1(x + L) = y_1(x) + \theta y_2(x), \quad \theta := y_1'(L) = -\frac{\partial_b \mathcal{L}}{\partial_b \phi_+} \phi''(0). \]  

Since \( c - \phi_+ > 0 \), \( \phi''(0) < 0 \) and \( \phi''(0) \partial_b \phi_+ = -1 \), as follows from (2.5) and (2.7), we obtain \( \text{sign}(\theta) = \text{sign}(\partial_b \mathcal{L}) \). The assertion of the theorem follows by Proposition 3.3 due to equivalence of the negative and null subspaces of \( K_0 \) and \( M \) and the inertial law between \( K \) and \( K_0 \) and between \( M \) and \( S \mathcal{M} \mathcal{S} \).

**Remark 3.8.** By Remark 2.8, we have \( \partial_a \mathcal{L} > 0 \) for every point \( (a,b) \) below the blue curve in the existence region of Fig. 2.3 and \( \partial_a \mathcal{L} < 0 \) for every point \( (a,b) \) above the blue curve. Therefore, the count of negative eigenvalues of the linearized operator \( \mathcal{L} \) in Theorem 3.6 changes depending on the point \( (a,b) \). However, by Theorem 2.6, \( \partial_a \mathcal{L} > 0 \) for every point \( (a,b) \) inside the existence region, hence the linearized operator \( K \) in Theorem 3.7 admits
a simple negative eigenvalue and a simple zero eigenvalue for every \((a, b)\) in the existence region.

4. Spectral stability of periodic waves

Here we study the linearized CH equations (3.3) and (3.9) and prove the spectral stability of periodic waves stated in Theorem 1.4. We start by deducing the constraints on the perturbations \(v \in H^2_{\text{per}}\) and \(p \in L^2_{\text{per}}\) satisfying these linearized equations.

**Lemma 4.1.** Let \(v_0 \in X_0 \subset H^2_{\text{per}}\), where \(X_0\) is given by

\[
X_0 := \{v \in H^2_{\text{per}} : \langle 1, v \rangle = 0, \quad \langle \phi - \phi'', v \rangle = 0 \}.
\] (4.1)

If \(v \in C^0(\mathbb{R}, H^2_{\text{per}}) \cap C^1(\mathbb{R}, H^1_{\text{per}})\) is a solution to the linearized CH equation (3.3) with initial data \(v_0\), then \(v(t, \cdot) \in X_0 \subset H^2_{\text{per}}\) for all \(t \in \mathbb{R}\).

**Proof.** Conservation of the two orthogonality conditions in \(X_0\) in the time evolution of the linearized CH equation (3.3) are checked directly using integration by parts:

\[
\frac{d}{dt} \langle 1, v \rangle = \langle 1, \partial_x(1 - \partial_x^2)^{-1}L v \rangle = 0
\]

and

\[
\frac{d}{dt} \langle \phi - \phi'', v \rangle = \langle \phi - \phi'', (1 - \partial_x^2)^{-1}\partial_x L v \rangle = \langle \phi', L v \rangle = \langle L \phi', v \rangle = 0,
\]

where we recall that \(L \phi' = 0\). Integrations by parts are justified since \(\phi \in H^\infty_{\text{per}}\) and \(v(t, \cdot) \in H^2_{\text{per}}\) is in the domain of \(L\).

**Corollary 4.2.** Let \(p_0 \in Y_0 \subset L^2_{\text{per}}\), where \(Y_0\) is given by

\[
Y_0 := \{p \in L^2_{\text{per}} : \langle 1, p \rangle = 0, \quad \langle \phi, p \rangle = 0 \}.
\] (4.2)

If \(p \in C^0(\mathbb{R}, L^2_{\text{per}}) \cap C^1(\mathbb{R}, H^1_{\text{per}})\) is a solution to the linearized CH equation (3.9) with initial data \(p_0\), then \(p(t, \cdot) \in Y_0 \subset H^2_{\text{per}}\) for all \(t \in \mathbb{R}\).

**Proof.** Orthogonality conditions in (4.2) follow from those in (4.1) by using the relation \(v = (1 - \partial_x^2)^{-1}p\) between solution \(v \in C^0(\mathbb{R}, H^2_{\text{per}}) \cap C^1(\mathbb{R}, H^1_{\text{per}})\) of (3.3) and the corresponding solution \(p \in C^0(\mathbb{R}, L^2_{\text{per}}) \cap C^1(\mathbb{R}, H^1_{\text{per}})\) of (3.9). In particular,

\[
\langle 1, v \rangle = \langle 1, v - v_{xx} \rangle = \langle 1, p \rangle
\]

and

\[
\langle \phi - \phi'', v \rangle = \langle \phi, v - v_{xx} \rangle = \langle \phi, p \rangle.
\]

The linearized equations (3.3) and (3.9) are equivalent by Lemma 3.1.

**Remark 4.3.** The two orthogonality conditions in (4.1) are related to the conservation of mass (1.2) and energy (1.3) by adding a perturbation of \(v\) to the smooth periodic wave \(\phi\) and truncating the quadratic terms in \(v\). The third orthogonality condition related to the higher-order energy (1.4) is redundant due to the other two conditions:

\[
\frac{3}{2} \phi^2 - \phi''^2 - \frac{1}{2}(\phi')^2, v \rangle = c \langle \phi - \phi'', v \rangle - b \langle 1, v \rangle = 0,
\] (4.3)
where the second-order equation (1.6) has been used.

The following lemma together with Lemma 3.1 gives the sufficient condition for spectral stability of the periodic wave in Definition 1.1.

**Lemma 4.4.** Let $\mathcal{K}|_{Y_0}$ be the restriction of $\mathcal{K}$ on $Y_0 \subset L^2_{\text{per}}$. If

$$\mathcal{K}|_{Y_0} \geq 0 \quad \text{and} \quad \ker(\mathcal{K}|_{Y_0}) = \ker(\mathcal{K}),$$

then the spectrum of $(c - \phi)^{-1}\partial_x(c - \phi)^{-1}\mathcal{K}$ in $L^2_{\text{per}}$ is located on the imaginary axis.

**Proof.** Consider the spectral problem

$$J_\phi \mathcal{K} p = \lambda p, \quad p \in H^1_{\text{per}},$$

where $J_\phi := (c - \phi)^{-1}\partial_x(c - \phi)^{-1}$ satisfies $J_\phi^* = -J_\phi$ in $L^2_{\text{per}}$. The spectrum of $J_\phi \mathcal{K}$ is purely discrete due to compact embedding of $H^1_{\text{per}}$ into $L^2_{\text{per}}$.

By Corollary 4.2 if $\lambda_0$ is an eigenvalue of $J_\phi \mathcal{K}$ in $L^2_{\text{per}}$ and $\lambda_0 \neq 0$, then the corresponding eigenfunction $p_0$ satisfies $p_0 \in H^1_{\text{per}} \cap Y_0$. A simple computation shows that for this $p_0 \in H^1_{\text{per}} \cap Y_0$

$$\lambda_0 \langle \mathcal{K}p_0, p_0 \rangle = \langle \mathcal{K}J_\phi \mathcal{K}p_0, p_0 \rangle = -\langle \mathcal{K}p_0, J_\phi \mathcal{K}p_0 \rangle = -\bar{\lambda}_0 \langle \mathcal{K}p_0, p_0 \rangle,$$

so that

$$(\lambda_0 + \bar{\lambda}_0) \langle \mathcal{K}p_0, p_0 \rangle = 0.$$

Since $p_0 \in H^1_{\text{per}} \cap Y_0$, then $\langle \mathcal{K}p_0, p_0 \rangle = 0$ if and only if $p_0 \in \ker(\mathcal{K})$ due to assumptions of the lemma. However, this is a contradiction with $\lambda_0 \neq 0$. Hence, $\langle \mathcal{K}p_0, p_0 \rangle > 0$, which implies that $\lambda_0 \in i\mathbb{R}$. This proves the assertion of the lemma. $\square$

For the proof of spectral stability in Theorem 1.4 it remains to justify the sufficient condition (4.4) for the operator $\mathcal{K}$. The following proposition from Theorem 4.1 in [44] formulates the useful result.

**Proposition 4.5.** [44] Let $L$ be a self-adjoint operator in a Hilbert space $H$ with the inner product $\langle \cdot, \cdot \rangle$ such that $L$ has $n(L)$ negative eigenvalues (counting their multiplicities) and $z(L)$ multiplicity of the zero eigenvalue bounded away from the positive spectrum of $L$. Let $\{v_j\}_{j=1}^N$ be a linearly independent set in $H$ and define

$$H_0 := \{ f \in H : \langle f, v_1 \rangle = \langle f, v_2 \rangle = \cdots = \langle f, v_N \rangle = 0 \}.$$

Let $A(\lambda)$ be the matrix-valued function defined by its elements

$$A_{ij}(\lambda) := \langle (L - \lambda I)^{-1}v_i, v_j \rangle, \quad 1 \leq i, j \leq N, \quad \lambda \notin \sigma(L).$$

Then,

$$\begin{cases} n(L|_{H_0}) = n(L) - n_0 - z_0, \\ z(L|_{H_0}) = z(L) + z_0 - z_\infty. \end{cases}$$

(4.6)

where $n_0$, $z_0$, and $p_0$ are the numbers of negative, zero, and positive eigenvalues of $\lim_{\lambda \uparrow 0} A(\lambda)$ (counting their multiplicities) and $z_\infty = N - n_0 - z_0 - p_0$ is the number of eigenvalues of $A(\lambda)$ diverging in the limit $\lambda \uparrow 0$.\]
By Lemma 2.9 for a fixed $c > 0$ and $L > 0$, there exists a $C^1$ mapping $a \mapsto b = B_L(a)$ and a $C^1$ mapping $a \mapsto \phi = \Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ of smooth $L$-periodic solutions along the curve $b = B_L(a)$. Along this curve we define
\begin{equation}
\mathcal{M}_L(a) := M(\Phi_L(\cdot, a)) \quad \text{and} \quad \mathcal{E}_L(a) := E(\Phi_L(\cdot, a)),
\end{equation}
where $M(u)$ and $E(u)$ are given by (1.2) and (1.3). In order to include the dependence on $c$, we will now write $\mathcal{M}_L(a, c)$ and $\mathcal{E}_L(a, c)$. The following lemma provides the criterion for positivity of $\mathcal{K}|_{Y_0}$ based on Proposition 4.5.

**Lemma 4.6.** For fixed $c > 0$ and $L > 0$, the condition (4.4) is satisfied if and only if
\begin{equation}
\frac{d}{da} \mathcal{E}_L(a) \mathcal{M}_L(a)^2 < 0
\end{equation}
along the curve $b = B_L(a)$.

**Proof.** Since $\Phi_L(\cdot, a) \in H^\infty_{\text{per}}$ is also $C^1$ with respect to $c$ as follows from the scaling transformation (1.19), we are allowed to differentiate the second-order equation (1.7) in $a$ and $c$. Writing this equation as $(\mathcal{M}_L(a, c))^2 \cdot \phi = a(c - \phi)^3$ for $\mu = \phi - \phi''$ and differentiating it in $a$ and $c$, we obtain
\begin{equation}
\mathcal{K}\partial_a \mu = c - \phi, \quad \mathcal{K}\partial_c \mu = -2a.
\end{equation}

Since $a > 0$, we express
\begin{equation}
\mathcal{K}^{-1} = -\frac{1}{2a} \partial_c \mu, \quad \mathcal{K}^{-1} \phi = -\partial_a \mu - \frac{c}{2a} \partial_c \mu,
\end{equation}

By Proposition 4.5 we construct the bounded 2-by-2 matrix $P := \lim_{\lambda \to 0} A(\lambda)$ in
\begin{equation}
P = \left[\begin{array}{cc}
\langle \mathcal{K}^{-1}, 1 \rangle & \langle \mathcal{K}^{-1} \phi, 1 \rangle \\
\langle \mathcal{K}^{-1}, \phi \rangle & \langle \mathcal{K}^{-1} \phi, \phi \rangle
\end{array}\right] = \left[\begin{array}{cc}
-\frac{1}{2a} \partial_c \mathcal{M}_L & -\partial_a \mathcal{M}_L - \frac{c}{2a} \partial_c \mathcal{M}_L \\
-\frac{1}{2a} \partial_c \mathcal{E}_L & -\partial_a \mathcal{E}_L - \frac{c}{2a} \partial_c \mathcal{E}_L
\end{array}\right],
\end{equation}
where $\mathcal{M}_L$ and $\mathcal{E}_L$ in (4.7) are $C^1$ functions in $a$ and $c$. It follows from (4.10) that
\begin{equation}
det(P) = \frac{1}{2a} \left[\partial_c \mathcal{M}_L \partial_a \mathcal{E}_L - \partial_a \mathcal{M}_L \partial_c \mathcal{E}_L\right].
\end{equation}

By using the scaling transformation (4.19), we write
\begin{equation}
\hat{\mathcal{M}}_L(a, c) = c \hat{\mathcal{M}}_L(\alpha), \quad \hat{\mathcal{E}}_L(a, c) = c^3 \hat{\mathcal{E}}_L(\alpha), \quad a = c^3 \alpha,
\end{equation}
where $\hat{\mathcal{M}}_L$ and $\hat{\mathcal{E}}_L$ can be computed by formally setting $c = 1$. Substituting the transformation (4.12) into (4.11), we obtain
\begin{equation}
det(P) = \frac{1}{2ac^4} \left[\hat{\mathcal{M}}_L(\alpha) \hat{\mathcal{E}}'_L(\alpha) - 2\hat{\mathcal{E}}_L(\alpha) \hat{\mathcal{M}}'_L(\alpha)\right]
\end{equation}
(4.13)
Thus, $\det(P) < 0$ if and only if the condition (4.8) is satisfied for a given $c > 0$. Since $n(\mathcal{K}) = 1$ and $z(\mathcal{K}) = 1$ by Theorem 1.3 independently of $a$ and $c$, we use the count
formulas (4.6) to get \( n(\mathcal{K}|_{\gamma_0}) = 0 \) and \( z(\mathcal{K}|_{\gamma_0}) = 1 \) since \( n_0 = 1, z_0 = z_\infty = 0 \). Hence, the conditions (4.4) are satisfied if and only if the condition (4.8) is satisfied. \( \square \)

Numerical results show that the condition (4.8) is satisfied for every \( c > 0 \) and \( L > 0 \) along the curve \( b = B_L(a) \) for \( a \in (0, a_L) \) in Lemma 2.9. Figure 4.1 shows that the mapping \( a \mapsto \frac{\mathcal{E}_L(a)}{\mathcal{M}_L(a)} \) is monotonically decreasing for four values of \( L \). The numerical method used to generate Figure 4.1 is described in Appendix A.

![Figure 4.1](image)

**Figure 4.1.** The dependence of \( \mathcal{E}_L/\mathcal{M}_L^2 \) versus \( a \) along the curve \( b = B_L(a) \) for \( c = 2 \) and five values of period: \( L = \pi/2 \) (black), \( L = 3\pi/4 \) (green), \( L = \pi \) (cyan), and \( L = 2\pi \) (red).

In the rest of this section, we will explain why the linearized CH equation (3.3) associated with the operator \( JL \) is not convenient for the proof of spectral stability of the smooth periodic waves. The following lemma gives the necessary and sufficient condition for the \( C^1 \) continuation of the smooth periodic waves with respect to parameter \( b \).

**Lemma 4.7.** For fixed \( c > 0 \) and \( L > 0 \), there exists a \( C^1 \) mapping \( b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}} \) of smooth \( L \)-periodic solutions of Lemma 2.1 if and only if \( \partial_a \Sigma \neq 0 \), where \( \Sigma(a, b, c) \) is the period function.

**Proof.** If \( \partial_a \Sigma \neq 0 \), then arguments of the proof of Lemma 2.9 based on the implicit function theorem and smoothness of smooth periodic solutions of Lemma 2.1 with respect to parameters gives existence of the \( C^1 \) mapping \( b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}} \).

In the converse direction, we assume existence of the \( C^1 \) mapping \( b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}} \) and prove that \( \partial_a \Sigma \neq 0 \). Due to the \( C^1 \) smoothness, it follows by differentiating the second-order equation (1.6) in \( c \) and \( b \) that

\[
\mathcal{L} \partial_c \phi = \phi'' - \phi \quad \text{and} \quad \mathcal{L} \partial_b \phi = 1,
\]

(4.14)
Let \( \{y_1, y_2\} \) be the fundamental set of solutions associated to the equation \( L v = 0 \) in \( H^2(0, L) \) as in (3.10) and (3.11). By Liouville’s theorem, the associated Wronskian is given by

\[
W(y_1, y_2)(x) = e^{\int_0^x \frac{\phi'(s)}{\phi(s)} \, ds} = \frac{c - \phi(0)}{c - \phi(x)} > 0. \tag{4.15}
\]

Now, since \( W(y_1, y_2)(x) = y_1(x)y_2^*(x) - y_1^*(x)y_2(x) \) for all \( x \in \mathbb{R} \) and \( y_2(x) = \phi'(x)/\phi''(0) \), we obtain by (4.15) that

\[
\phi''(0) \int_0^L \frac{c - \phi(0)}{c - \phi(x)} \, dx = \int_0^L [y_1(x)\phi''(x) - y_1'(x)\phi'(x)] \, dx \tag{4.16}
\]

By contradiction, assume that \( \partial_a \mathcal{L} = 0 \), then \( y_1 \) is \( L \)-periodic similar to \( y_2 \). Integration by parts in (4.16) yields

\[
\phi''(0) \int_0^L \frac{c - \phi(0)}{c - \phi(x)} \, dx = 2\langle \phi'', y_1 \rangle. \tag{4.17}
\]

It follows from (4.14) that

\[
\langle \phi'', y_1 \rangle = \langle \phi, y_1 \rangle, \quad \langle 1, y_1 \rangle = 0. \tag{4.18}
\]

On the other hand, we also have \( \mathcal{L}1 = c - 3\phi + \phi'' \), hence

\[
\langle \phi'', y_1 \rangle - 3\langle \phi, y_1 \rangle + c\langle 1, y_1 \rangle = 0. \tag{4.19}
\]

Substituting (4.18) into (4.19) yields \( \langle \phi'', y_1 \rangle = 0 \), which is a contradiction with the non-zero left-hand side in (4.17). Hence, \( \partial_a \mathcal{L} = 0 \) leads to the contradiction with the \( C^1 \) smoothness of the mapping \( b \mapsto \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}} \).

**Remark 4.8.** By Lemma 4.7, the relation \( \mathcal{L}\partial_b \phi = 1 \) in (4.14) cannot be used at the points where \( \partial_a \mathcal{L} = 0 \). Away from these points, the 2-by-2 matrix of projections in Proposition 4.5 can be constructed and evaluated for the operator \( \mathcal{L} \) under the two orthogonality conditions in \( X_0 \) given by (4.1) as follows:

\[
S := \begin{bmatrix}
\langle \mathcal{L}^{-1}1, 1 \rangle & \langle \mathcal{L}^{-1}(\phi - \phi''), 1 \rangle \\
\langle \mathcal{L}^{-1}(\phi - \phi''), (\phi - \phi'') \rangle & \langle \mathcal{L}^{-1}(\phi - \phi''), (\phi - \phi'') \rangle
\end{bmatrix} = \begin{bmatrix}
\partial_b \mathcal{M}_L & -\partial_b \mathcal{E}_L \\
\partial_c \mathcal{M}_L & -\partial_c \mathcal{E}_L
\end{bmatrix}, \tag{4.20}
\]

where \( \mathcal{M}_L \) and \( \mathcal{E}_L \) are computed at \( \phi = \Psi_L(\cdot, b) \in H^\infty_{\text{per}} \) and extended in both \( b \) and \( c \). By using the scaling transformation (4.19), we write

\[
\mathcal{M}_L(c, b) = c\hat{\mathcal{M}}_L(\beta), \quad \mathcal{E}_L(c, b) = c^2\hat{\mathcal{E}}_L(\beta), \quad b = c^2\beta. \tag{4.21}
\]

Substituting the transformation (4.21) into (4.20) yields

\[
det(S) = \hat{\mathcal{M}}_L(\beta)\hat{\mathcal{E}}_L'(\beta) - 2\hat{\mathcal{E}}_L(\beta)\hat{\mathcal{M}}_L'(\beta) = \hat{\mathcal{M}}_L^3(\beta) \frac{d}{d\beta} \left( \frac{\hat{\mathcal{E}}_L(\beta)}{\hat{\mathcal{M}}_L(\beta)^2} \right). \tag{4.22}
\]

Here the derivative is computed along the curve \( b = B_L(a) \), where \( B_L'(a) > 0 \) if \( n(\mathcal{L}) = 1 \) and \( B_L'(a) < 0 \) if \( n(\mathcal{L}) = 2 \), see Figure 2.3. In the former case, our numerical results show that \( \det(S) < 0 \) so that \( n_0 = 1, z_0 = z_\infty = 0 \) and by Proposition 4.3, we have \( n(\mathcal{L}|_{X_0}) = 0 \) and \( z(\mathcal{L}|_{X_0}) = 1 \). In the latter case, our numerical results give \( \det(S) > 0 \) and \( \hat{\mathcal{M}}_L'(\beta) < 0 \).
so that the 2-by-2 matrix $S$ is negative with $n_0 = 2$, $z_0 = z_{\infty} = 0$ and by Proposition 4.3, we still have $n(L|_{X_0}) = 0$ and $z(L|_{X_0}) = 1$. At the points where $\partial_x \mathcal{L} = 0$, $S$ is unbounded with $n_0 = 1$, $z_0 = 1$ and $z_{\infty} = 1$ which still gives $n(L|_{X_0}) = 0$ and $z(L|_{X_0}) = 1$ since $n(L) = 1$ and $z(L) = 2$. Thus, we obtain conditions
\[
L|_{X_0} \geq 0 \quad \text{and} \quad \ker(L|_{X_0}) = \ker(L)
\]
for the linearized operator $L$ but with three different computations depending on whether $\partial_x \mathcal{L} < 0$, $\partial_x \mathcal{L} = 0$, and $\partial_x \mathcal{L} > 0$.

5. Orbital stability of periodic waves

Here we prove the orbital stability of periodic waves stated in Theorem 1.4. We follow the approach in [1], where the following useful result was proven in Proposition 3.8 and Theorem 4.2.

**Proposition 5.1.** [1] Let $V(u)$ be a conserved quantity in the time evolution of the Hamiltonian system (1.9). Assume that the linearized operator $L$ at the periodic travelling wave with profile $\phi$ admits a simple negative and a simple zero eigenvalue with $\ker(L) = \text{span}(\phi')$ satisfying $\langle V'(\phi), \phi' \rangle = 0$. Assume that there exists $\mathcal{Y} \in H^2_{\text{per}}$ such that $\langle \mathcal{L} \mathcal{Y}, v \rangle = 0$ for every $v \in L^2_{\text{per}}$ such that $\langle V'(\phi), v \rangle = 0$. If $\langle \mathcal{L} \mathcal{Y}, \mathcal{Y} \rangle < 0$, then the periodic travelling wave is orbitally stable in the time evolution of (1.9) in $H^1_{\text{per}}$.

**Remark 5.2.** The notion of orbital stability in Definition 1.2 prescribes the existence of global solutions $u \in C(\mathbb{R}, H^s_{\text{per}})$ for $s > \frac{3}{2}$. The local solutions $u \in C((-t_0, t_0), H^s_{\text{per}})$ for some $t_0 > 0$ exist due to the local well-posedness theory in [7, 20, 23]. Since $M$, $E$, and $F$ are conserved quantities, one can combine the local solution with the standard a priori estimates $M(u(t)) = M(u_0)$, $E(u(t)) = E(u_0)$, and $F(u(t)) = F(u_0)$ for all $t \geq 0$ in order to extend the local to global solutions near the smooth periodic waves in the case when they are stable by Proposition 5.1.

**Remark 5.3.** The result of Proposition 5.1 can be equivalently written for the Hamiltonian system (1.12) with the conserved quantity $V(m)$ written in variable $m = u - u_{xx}$ and with the linearized operator $\mathcal{L}$ at the periodic travelling wave $\mu = \phi - \phi''$.

The following lemma transfers the spectral stability criterion in Lemma 4.6 to the orbital stability criterion.

**Lemma 5.4.** For fixed $c > 0$ and $L > 0$, the smooth $L$-periodic wave with profile $\phi = \Phi_L(\cdot, a) \in H_{\text{per}}^\infty$ is orbitally stable in $H^1_{\text{per}}$ if the mapping
\[
a \mapsto \frac{\mathcal{E}_L(a)}{M_L(a)^2}
\]
is strictly decreasing along the curve $b = \mathcal{B}_L(a)$.

**Proof.** For $\mu = \phi - \phi''$, let us rewrite (4.9) in the form:
\[
\mathcal{K} \left( \frac{1}{2a} \partial_x \mu \right) = -1, \quad \mathcal{K} \left( \partial_a \mu + \frac{c}{2a} \partial_x \mu \right) = -\phi.
\]
For \( m = u - u_{xx} \), we define a linear superposition of the two conserved quantities (1.2) and (1.3):
\[
V(m) := rM(u) + sE(u), \quad u = (1 - \partial_x^2)^{-1}m,
\]
where \( r \) and \( s \) are real coefficients, \( M(u) \) is given by (1.2), and \( E(u) \) is given by (1.13). Since \( V'(\mu) = r + s\phi \) and \( \text{Ker}(K) = \text{span}(\mu') \), we check that \( \langle V'(\mu), \mu' \rangle = 0 \) since \( \phi \in H^\infty_{\text{per}} \).

By Theorem 1.3, the linearized operator \( K \) satisfies the assumption of Proposition 5.1.

We then proceed by constructing \( \Upsilon \). Letting
\[
\Upsilon := \frac{r}{2a} \partial_c \mu + s \left( \frac{c}{2a} \partial_c \mu \right),
\]
it follows from (5.2) that
\[
\langle K\Upsilon, \Upsilon \rangle = r^2 \langle K^{-1}1, 1 \rangle + 2rs \langle K^{-1}\phi, 1 \rangle + s^2 \langle K^{-1}\phi, \phi \rangle.
\]

The quadratic form (5.4) in \( r \) and \( s \) is defined by the same 2-by-2 symmetric matrix \( P \) as in (4.10). If condition (5.1) is satisfied, we have that \( \text{det}(P) < 0 \) and there exists a choice of real coefficients \( r \) and \( s \) such that \( \langle K\Upsilon, \Upsilon \rangle < 0 \). Hence, the orbital stability of the periodic waves in the time evolution of the Hamiltonian system (1.12) follows from Proposition 5.1 and Remark 5.3.

In what follows, we show that the orbital stability condition is satisfied for every \( b \leq 0 \).

The main advantage of this result is that we do not need to verify the criterion (5.1) by using numerical computations. The following lemma reports the relevant result.

**Lemma 5.5.** For fixed \( c > 0 \) and \( b \leq 0 \), the smooth periodic wave with profile \( \phi \in H^\infty_{\text{per}} \) is orbitally stable in \( H^1_{\text{per}} \).

**Proof.** It follows from (1.7) and (1.17) that
\[
K\mu = (c - \phi)^3\mu - 2a(1 - \partial_x^2)^{-1}\mu = a(c - 3\phi).
\]
We can define \( V(m) := cM(u) - 3E(u) \) so that if \( \langle V'(\mu), p \rangle = 0 \), then \( \langle K\mu, p \rangle = 0 \). Thus, we can take \( \Upsilon := \mu \) and compute
\[
\langle K\mu, \mu \rangle = a \left[ cM(\phi) - 6E(\phi) \right].
\]
Since \( a > 0 \), we check the sign of \( cM(\phi) - 6E(\phi) \):
\[
cM(\phi) - 6E(\phi) = \int_0^L \left[ c\phi - 3\phi^2 - 3(\phi')^2 \right] dx
= \int_0^L \left[ 2b + c\phi'' - c\phi - 2(\phi')^2 \right] dx
= 2bL - cM(\phi) - 2 \int_0^L (\phi')^2 dx,
\]
where we have used \( L1 = c - 3\phi + \phi'' \) and \( L\phi = 2b + c(\phi'' - \phi) \) and \( \langle L1, \phi \rangle = \langle 1, L\phi \rangle \). If \( b \leq 0 \) and \( c > 0 \), then \( \langle K\mu, \mu \rangle < 0 \) since \( M(\phi) > 0 \), so that the periodic waves are
orbitally in the time evolution of the Hamiltonian system (1.12) by Proposition 5.1 and Remark 5.3.

Remark 5.6. The criterion \( \langle K_\mu, \mu \rangle < 0 \) of the orbital stability is satisfied near the boundary \( a = a_-(b) \) in Lemma 2.3 both for \( b \leq 0 \) and \( b > 0 \). Indeed, we can write

\[
\langle K_\mu, \mu \rangle = a^2 \int_0^L \frac{c - 3\phi}{(c - \phi)^2} dx,
\]

where \( c - 3\phi < 0 \) because the constant solution \( \phi = \phi_2 \) satisfies the ordering (2.2).

Remark 5.7. The criterion \( \langle K_\mu, \mu \rangle < 0 \) is not satisfied near the boundary \( a = a_+(b) \) in Lemma 2.3 for \( b > 0 \). Indeed, since \( \phi = \phi_1 + \hat{\phi} \), where \( \hat{\phi}(x) \to 0 \) as \( |x| \to \infty \), we derive

\[
cM(\phi) - 6E(\phi) = L\phi_1(c - 3\phi_1) + O(1),
\]

where \( O(1) \) denotes bounded terms in the limit \( L \to \infty \). Since \( c - 3\phi_1 > 0 \) by the ordering (2.2) and \( a > 0 \), we have \( \langle K_\mu, \mu \rangle > 0 \) near the boundary \( a = a_+(b) \). Thus, the criterion for orbital stability in Lemma 5.5 is not as sharp as the criterion in Lemma 5.4.

Remark 5.8. The result of Lemma 5.5 can be established directly for the linearized operator \( L \). Since \( L\phi = 2b + c(\phi'' - \phi) \), we can define \( V(u) := 2bM(u) - cE(u) \) so that if \( \langle V'(u), v \rangle = 0 \), then \( \langle L\phi, v \rangle = 0 \). Thus, we can take \( \Upsilon := \phi \) and compute

\[
\langle L\phi, \phi \rangle = 2bM(\phi) - 2cE(\phi).
\]

(5.7)

If \( b \leq 0 \) and \( c > 0 \), then \( \langle L\phi, \phi \rangle < 0 \) since \( M(\phi) > 0 \). However, Proposition 5.1 can only be used if \( L \) has a simple negative eigenvalue, which is only true in a subset of \( b \leq 0 \), where \( \partial_a L < 0 \). Similar to Remark 4.8, we can see that the linearized operator \( K \) provides wider region for orbital stability compared to the linearized operator \( L \).

6. Conclusion

We have studied spectral and orbital stability of smooth periodic travelling waves in the Camassa–Holm (CH) equation by using functional-analytic tools. We showed that the standard Hamiltonian formulation of the CH equation has several shortcomings, e.g. the number of negative eigenvalues in the linearized operator is either one or two depending on the parameters of the periodic travelling wave. On the other hand, the nonstandard Hamiltonian formulation based on the momentum quantity \( m := u - u_{xx} \) provides a better framework for analysis with only one simple negative eigenvalue of the associated linearized operator.

The criterion for spectral and orbital stability has been derived by using the nonstandard Hamiltonian formulation. The stability criterion has been checked numerically and it is an open problem to prove analytically that this criterion is satisfied in the entire existence region for the smooth periodic travelling waves. We proved analytically that the stability criterion is satisfied in a subset of the existence region.

Since the CH equation is a prototypical example of a more general class of nonlinear evolution equations, it is expected that our methods will be useful for the analysis of spectral and orbital stability in the systems where other methods based on the inverse scattering transform are not applicable, e.g., for the \( b \)-family of the CH equations.
Appendix A. Smooth periodic waves in the explicit form

Here we derive the explicit expressions for the smooth periodic wave with the profile $\phi$ satisfying the system (1.6), (1.7), and (1.8). Since $\phi < c$, we can transform the variables

$$\phi(x) = \psi(z(x)), \quad z(x) = \int_0^x ds \frac{d}{\sqrt{c - \phi(s)}}$$  \hspace{1cm} (A.1)

and rewrite the second-order equation (1.6) with the chain rule to the form

$$-\psi'' + c\psi - \frac{3}{2} \psi^2 = b,$$  \hspace{1cm} (A.2)

which is the stationary KdV equation. Note that this reduction of the travelling periodic waves of the CH equation (1.1) to the travelling periodic waves of the KdV equation is different from the previously explored connection between the CH and KdV equations in [31]. A similar transformation was used in [13] in the context of the solitary waves at the zero background.

The second-order equation (A.2) has the following explicit periodic solution (see, e.g., [41]):

$$\psi(z) = \frac{1}{3} c + \frac{4}{3} \gamma^2 [1 - 2k^2 + 3k^2 \text{cn}^2(\gamma z; k)],$$  \hspace{1cm} (A.3)

where $\gamma > 0$ and $k \in (0, 1)$ are arbitrary parameters, and $\text{cn}$ is the Jacobian elliptic function. The period of the periodic solution $\psi$ in $z$ is $P = 2\gamma^{-1} K(k)$. The free parameters $\gamma$ and $k$ parametrize the turning points $\phi_-$ and $\phi_+$ in (2.3) and (2.4):

$$\begin{cases} 
\phi_+ = \frac{1}{3} c + \frac{4}{3} \gamma^2 (1 + k^2), \\
\phi_- = \frac{1}{3} c + \frac{4}{3} \gamma^2 (1 - 2k^2),
\end{cases}$$  \hspace{1cm} (A.4)

which can be inverted as follows

$$\gamma^2 = \frac{1}{4} (2\phi_+ + \phi_- - c), \quad k^2 = \frac{\phi_+ - \phi_-}{2\phi_+ + \phi_- - c}.$$  \hspace{1cm} (A.5)

Parameters $a$ and $b$ are related to parameters $\gamma$ and $k$ by

$$a = \frac{1}{2} (\phi_+ + \phi_-)(c - \phi_+)(c - \phi_-)$$
$$= \frac{4}{27} (c + 2\gamma^2 (2 - k^2))(c - 2\gamma^2 (1 + k^2))(c - 2\gamma^2 (1 - 2k^2))$$  \hspace{1cm} (A.6)

and

$$b = \frac{1}{2} c (\phi_+ + \phi_-) - \frac{1}{2} (\phi_+^2 + \phi_+ \phi_- + \phi_-^2)$$
$$= \frac{1}{6} c^2 - \frac{8}{3} \gamma^4 (1 - k^2 + k^4),$$  \hspace{1cm} (A.7)

where equations (2.4) have been used. When $a$ or $b$ are fixed, e.g., for numerical results obtained on Figs. 2.5 and 2.6, we express $\gamma$ from the roots of either (A.6) or (A.7) and parameterize the family by the only parameter $k$ in a subset of $(0, 1)$. 
The period function $L = L(a, b, c)$ can be expressed by (A.1) and (A.3) in the form:

$$L = \frac{\sqrt{2}}{\gamma \sqrt{3}} \int_0^{2K(k)} \sqrt{c - 2\gamma^2(1 - 2k^2 + 3k^2cn^2(z; k))} \, dz.$$

(A.8)

If $L$ is fixed, e.g., for numerical results obtained on Figs. 2.3 and 4.1, then $\gamma$ can be found from a root finding algorithm for equation (A.8), after which the periodic solutions are parameterized by the only parameter $k$ in a subset of $(0, 1)$. The mass and energy integrals in (1.2) and (1.3) are evaluated at the periodic wave (A.1) with the chain rule:

$$M(\phi) = \int_0^{2\gamma^{-1}K(k)} \psi(z) \sqrt{c - \psi(z)} \, dz$$

(A.9)

and

$$E(\phi) = \int_0^{2\gamma^{-1}K(k)} \left[ b + \psi(z)^2 - \frac{a}{c - \psi(z)} \right] \sqrt{c - \psi(z)} \, dz,$$  

(A.10)

where we have used $(\phi')^2 = \phi^2 + 2b - 2a/(c - \phi)$.

The limit $k \to 0$ corresponds to the constant solution

$$\psi = \frac{1}{3}c + \frac{4}{3}\gamma^2$$

in Lemma 2.2. The limit $k \to 1$ corresponds to the solitary wave solution

$$\psi(z) = \frac{1}{3}c + \frac{4}{3}\gamma^2 \left[ -1 + 3\text{sech}^2(\gamma z) \right]$$

(A.11)

in Lemma 2.3. The curve $c = 2\gamma^2(1 + k^2)$ corresponds to the peaked periodic wave

$$\psi(z) = 2\gamma^2 \left[ 1 - k^2 + 2k^2cn^2(\gamma z; k) \right],$$

(A.12)

in Lemma 2.4.

REFERENCES

[1] G. Alves, F. Natali and A. Pastor, “Sufficient conditions for orbital stability of periodic traveling waves”, J. Diff. Equat. 267 (2019), 879-901.
[2] P. Byers, “Existence time for the Camassa-Holm equation and the critical Sobolev index”, Indiana Univ. Math. J. 55 (2006), 941–954.
[3] R. Camassa and D.D. Holm, “An integrable shallow water equation with peaked solitons”, Phys. Rev. Lett. 71 (1993), 1661–1664.
[4] R. Camassa, D. Holm, and J. Hyman, “A new integrable shallow water equation”, Adv. Appl. Math. 31 (1994), 1–33.
[5] C. Chicone, “The monotonicity of the period function for planar Hamiltonian vector fields”, J. Diff. Equat. 69 (1987), 310—321.
[6] A. Constantin, “The Hamiltonian structure of the Camassa–Holm equation”, Expo. Math. 15 (1997), 53–85.
[7] A. Constantin and J. Escher, “Wave breaking for nonlinear nonlocal shallow water equations”, Acta Math. 181 (1998), 229–243.
[8] A. Constantin and J. Escher, “Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation”, Comm. Pure Appl. Math. 51 (1998), 475–504.
[9] A. Constantin and J. Escher, “On the blow-up rate and the blow-up set of breaking waves for a shallow water equation”, Math. Z. 233 (2000), 75–91.
[10] A. Constantin and D. Lannes, “The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations”, *Arch. Ration. Mech. Anal.* **192** (2009), 165–186.
[11] A. Constantin and L. Molinet, “Orbital stability of solitary waves for a shallow water equation”, *Physica D* **157** (2001), 75–89.
[12] A. Constantin and W.A. Strauss, “Stability of peakons”, *Comm. Pure Appl. Math.* **53** (2000), 603–610.
[13] A. Constantin and W.A. Strauss, “Stability of the Camassa–Holm solitons”, *J. Nonlinear Sci.* **12** (2002), 415–422.
[14] H.H. Dai, “Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod”, *Acta Mech.* **127** (1998), 193–207.
[15] C. De Lellis, T. Kappeler, and P. Topalov, “Low-regularity solutions of the periodic Camassa–Holm equation”, *Comm. PDEs* **32** (2007), 87–126.
[16] M.S.P. Eastham, *The spectral theory of periodic differential equations*, (Scottish Academic Press, London, 1973).
[17] A. Geyer and D.E. Pelinovsky, “Spectral stability of periodic waves in the generalized reduced Ostrovsky equation”, *Lett. Math. Phys.* **107** (2017), 1293–1314.
[18] A. Garijo and J. Villadelprat. “Algebraic and analytical tools for the study of the period function”, *J. Differ. Equ.* **257** (2014), 2464–2484.
[19] A. Geyer and J. Villadelprat, “On the wave length of smooth periodic traveling waves of the Camassa–Holm equation”, *J. Diff. Equat.* **259** (2015), 2317–2332.
[20] S. Hakkaev, I.D. Iliev and K. Kirchev, “Stability of periodic travelling shallow-water waves determined by Newton’s equation”, *J. Phys. A: Math. Theor.* **41** (2008), 085203.
[21] M. Hărăguş and T. Kapitula, “On the spectra of periodic waves for infinite-dimensional Hamiltonian systems”, *Physica D* **237** (2008), 2649–2671.
[22] A. Himonas, K. Grayshan, and C. Holliman, “Ill-posedness for the b-family of equations”, *J. Nonlin. Sci.* **26** (2016), 1175–1190.
[23] A. Himonas and G. Misiolek, “The Cauchy problem for an integrable shallow water equation”, *Diff. Int. Eqs.* **14** (2001), 821–831.
[24] A. Himonas, G. Misiolek, and G. Ponce, “Non-uniform continuity in $H^1$ of the solution map of the CH equation”, *Asian J. Math.* **11** (2007), 141–150.
[25] V.M. Hur and M. Johnson, “Stability of periodic traveling waves for nonlinear dispersive equations”, *SIAM J. Math. Anal.* **47** (2015), 3528–3554.
[26] M. Johnson, “Nonlinear stability of periodic traveling wave solutions of the generalized Korteweg-de Vries equation”, *SIAM J. Math. Anal.* **41** (2009), 1921–1947.
[27] R.S. Johnson, “Camassa–Holm, Korteweg-de Vries and related models for water waves”, *J. Fluid Mech.* **455** (2002), 63–82.
[28] T. Kato, *Perturbation Theory for Linear Operators*, (Springer-Verlag, Berlin, 1995).
[29] S. Kouranbaeva, “The Camassa–Holm equation as geodesic flow on the diffeomorphism group”, *J. Math. Phys.* **40** (1999), 857–868.
[30] J. Lenells, “Stability of periodic peakons”, *Int. Math. Res. Not.* **2004** (2004), 485–499.
[31] J. Lenells, “The correspondence between KdV and Camassa-Holm”, *Int. Math. Res. Not.* **2004** (2004), 3797–3811.
[32] J. Lenells, “A variational approach to the stability of periodic peakons”, *J. Nonlinear Math. Phys.* **11** (2004) 151–163.
[33] J. Lenells, “Traveling wave solutions of the Camassa–Holm equation”, *J. Diff. Equat.* **217** (2005) 393–430.
[34] J. Lenells, “Stability for the periodic Camassa–Holm equation”, *Math. Scand.* **97** (2005) 188–200.
[35] O. Lopes, “A class of isoinertial one parameter families of selfadjoint operators”, in *Nonlinear Differential Equations: Methods, Models and Applications* (Birkhäuser, Basel, Boston 2003).
[36] W. Magnus and S. Winkler, *Hill’s equation* (New York, Wiley 1966).
[37] A. Madiyeva and D.E. Pelinovsky, “Growth of perturbations to the peaked periodic waves in the Camassa-Holm equation”, *SIAM J. Math. Anal.* (2021), in print.

[38] F. Mañosas and J. Villadelprat, “Criteria to bound the number of critical periods”, *J. Dif. Equat.* 246 (2009), 2415–2433.

[39] G. Misiolek, “Shallow water equation as a geodesic flow on the Bott–Virasoro group”, *J. Geom. Phys.* 24 (1998) 203–208.

[40] G. Misiolek, “Classical solutions of the periodic Camassa–Holm equation”, *Geom. Funct. Anal.* 12 (2002) 1080–1104.

[41] F. Natali, U. Le, and D.E. Pelinovsky, “New variational characterization of periodic waves in the fractional Korteweg-de Vries equation”, *Nonlinearity* 33 (2020), 1956–1986.

[42] F. Natali and D.E. Pelinovsky, “Instability of $H^1$-stable peakons in the Camassa–Holm equation”, *J. Diff. Eqs.* 268 (2020), 7342–7363.

[43] A. Neves, “Floquet’s Theorem and the stability of periodic waves”. *J. Dyn. Diff. Equat.* 21 (2009), 555–565.

[44] D.E. Pelinovsky, *Localization in Periodic Potentials: from Schrödinger Operators to the Gross–Pitaevskii Equation*, LMS Lecture Note Series 390 (Cambridge University Press, Cambridge, 2011).