Beyond the Worst Case: Structured Convergence of High Dimensional Random Walks

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Abstract

One of the most important properties of high dimensional expanders is that high dimensional random walks converge rapidly. This property has proven to be extremely useful in variety of fields in the theory of computer science from agreement testing to sampling, coding theory and more. In this paper we improve upon the result of [KO18, AL20] regarding the convergence of random walks by presenting a structured version of their result. While previous works examined the expansion in the viewpoint of the worst possible eigenvalue, in this work we relate the expansion of a function to the entire spectrum of the random walk operator using the structure of the function. Note that in some cases this finer result can be much better than the worst case.

In order to prove our structured version of the convergence of random walks, we present a general framework that allows us to relate the convergence of random walks to the trickling down theorem for the first time. Concretely, we show that both the state of the art results for convergence of random walks and the tricking down theorem can be derived using the same argument that we present here.

This new, unified, way of looking at the convergence of high dimensional random walks and the trickling down theorem gives us a new understanding of pseudorandom functions that allows us to consider pseudorandom functions in one-sided local spectral expanders for the first time.

1 Introduction

In recent years much attention has been given to the field of high dimensional expanders which are high dimensional analogues of expander graphs. One extremely useful property of high dimensional expanders is that higher dimensional random walks (which are higher dimensional analogues of random walks on graphs) converge rapidly (For example, this property was used in [DK17, KM20, ALGV18, DHK+19] and more). Consequently here has been some work studying the convergence of higher dimensional random walks [DK17, KM17, KO18, AL20]. In this paper we aim to improve upon these convergence results by relating the structure of the function to its expansion. Specifically we present the following improvements:

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Random walk decomposition  Prior to this paper, the state of the art analysis of high dimensional random walks was done by Alev and Lau in [AL20] following [KO18]. Their work analyzed the eigenvalues of a central random walk called the non-lazy up-down random walk. Their result, however, was only useful for the worst case analysis as it did not relate the structure of the function to its expansion and thus was forced to consider the worst possible function. In this paper we present an improvement upon Alev and Lau’s result by finding a connection between the structure of a function and how well it expands and can therefore yield better results on cochains that possess a nice structure.

Our framework  In order to decompose the higher dimensional random walk operators we present a new framework for analyzing the non-lazy up-down random walk. This new approach is fairly generic and seems to be of independent interest as it can be used to prove another central theorem in the theory of high dimensional expansion, namely the trickling down theorem [Opp17].

Pseudorandom cochains  Another implication of our new approach is that it provides us with a better understanding of pseudorandom cochains in a much larger class of simplicial complexes than was previously known. Specifically, we prove small set expansion of pseudorandom cochains in one-sided local spectral expanders for the first time, while previous works could only study small set expansion of pseudorandom cochains in two-sided local spectral expanders, which is much more restrictive family of high dimensional expanders. We note however, that in order to derive optimal expansion of small sets in one-sided local spectral expanders, our result requires the cochains to be extremely pseudorandom, whereas previous work that assumed the more restrictive assumption of two-sided local spectral expansion [BHKL20] requires considerably weaker pseudorandom assumptions towards optimal expansion of small sets.

Before we can state our results more formally, we have to define the high dimensional analogs of expander graphs. As graphs do not possess high dimension it is necessary to find an object that generalizes graphs into higher dimensions. This high dimensional object is called “simplicial complex” and is defined as:

**Definition 1.1 (Simplicial complex).** A set $X$ is a simplicial complex if it is closed downwards meaning that if $\sigma \in X$ and $\tau \subseteq \sigma$ then $\tau \in X$. We call members of $X$ the faces of $X$.

Simplicial complexes can be thought of as hyper-graphs with closure property (i.e. every subset of a hyper-edge is a hyper-edge). We are interested in higher dimensions and therefore it would be useful to define the dimension of these higher dimensional objects:

**Definition 1.2 (Dimension).** Let $X$ be a simplicial complex and let $\sigma \in X$ be a face of $X$. Define the dimension of $\sigma$ to be:

$$\dim(\sigma) = |\sigma| - 1$$

Denote the set of all faces of dimension $i$ in $X$ as $X(i)$. Also define the dimension of the complex $X$ as:

$$\dim(X) = \max_{\sigma \in X} \{\dim(\sigma)\}$$

Note that there is a single $(-1)$-dimensional face - the empty face.

Of particular interest are simplicial complexes whose maximal are of the same dimension, defined below:
Definition 1.3 (Pure simplicial complex). A simplicial complex $X$ is a pure simplicial complex if every face $\sigma \in X$ is contained in some $(\dim(X))$-dimensional face.

Throughout this paper we will assume that every simplicial complex is pure. In most cases we will be interested in weighted pure simplicial complexes. In weighted pure simplicial complexes the top dimensional faces are weighted and the weight of the rest of the faces follows from there as described here:

Definition 1.4 (Weight). Let $X$ be a pure $d$-dimensional simplicial complex. Define its weight function $w : X \to [0,1]$ to be a function such that:

- $\sum_{\sigma \in X(d)} w(\sigma) = 1$
- For every face $\tau$ of dimension $i < d$ it holds that:
  $$w(\tau) = \frac{\sum_{\sigma \in X(d) : \tau \subseteq \sigma} w(\sigma)}{\binom{d+1}{i+1}}$$

It is important to note that we think of unweighted complex as complexes that satisfy $\forall \sigma \in X(d) : w(\sigma) = \frac{1}{|X(d)|}$. While the top dimensional faces of unweighted complexes all have the same weight, the same cannot be said for lower dimensional faces.

It is also important to note that the sum of weights in every dimension is exactly 1 and therefore for every $k$ the weight function can, and at times will, be thought of as a distribution on $X(k)$.

One key property of high dimensional expanders is that they exhibit local to global phenomena. These phenomena are at the main interest of this paper. It is therefore useful to consider local views of the simplicial complex which we define as follows:

Definition 1.5 (Link). Let $X$ be a simplicial complex and $\sigma \in X$ be a face of $X$. Define the link of $\sigma$ in $X$ as:

$$X_\sigma = \{ \tau \setminus \sigma | \sigma \subseteq \tau \}$$

It is easy to see that the link of any face is a simplicial complex.

The weight of faces in the links is induced by the weights of the faces in the original complex. Specifically, we denote by $w_\sigma$ the weight function in the link of $\sigma \in X(i)$ and it holds that:

$$\forall \tau \in X_\sigma(j) : w_\sigma(\tau) = \frac{w(\tau \cup \sigma)}{\binom{i+j+2}{i+1}} w(\sigma)$$

Generally speaking, the local to global phenomena are ways to derive properties of the entire complex by only looking at local views.

Another important substructure of a simplicial complex is its skeletons

Definition 1.6 (Skeleton). Let $(X, w(k))$ be a weighted pure $d$-dimensional simplicial complex and let $i \leq d$. Define the $i$-skeleton of $X$ as the following weighted simplicial complex:

$$X^{(i)} = \{ \sigma \in X | \dim \sigma \leq i \}$$

With the original weight function.
In many cases we will think of the 1-skeleton of a simplicial complex as a graph. In addition, it is important to note that even if the original complex is unweighted, the skeletons of said complex might be.

We are now ready to define a high dimensional expander

**Definition 1.7** (Local spectral expander). A pure $d$-dimensional simplicial complex $X$ is a $\lambda$-local spectral expander if for every face $\sigma$ of dimension at most $d-2$ it holds that $X^{(1)}_{\sigma}$ is a $\lambda$-spectral expander. Note that this includes $\sigma = \emptyset$, i.e. the entire complex.

Much like graphs, simplicial complexes also support random walks. In graphs, the random walks are of the form vertex-edge-vertex - the walk might move between two vertices if they are connected by an edge. The high dimensional analogue of these random walks travel between two $k$-dimensional faces if they are part of a common $(k+1)$-dimensional face. Our particular random walk of interest is the higher dimensional analogue of the non-lazy random walk, defined as follows:

**Definition 1.8** (Non-lazy up-down operator informal, for formal see 4.1). Define the $k$-dimensional non-lazy up-down random walk, $(M')^k_{\mathbb{R}}$ as the $k$ dimensional analogue of the non-lazy random walk on the vertices of a graph: A walk that moves between two $k$-dimensional faces if they are contained in a $(k+1)$-dimensional face and does not stay in place.

We are going to characterize the eigenspaces of this higher dimensional random walks. These eigenspaces are correlated to another natural structure on high dimensional expanders called cochains that is defined as follows:

**Definition 1.9** (Cochains). Let $X$ be a pure $d$-dimensional simplicial complex. For $-1 \leq k \leq d$ define a $k$-dimensional cochain $F$ to be any function from $X(k)$ to $\mathbb{R}$. We also denote by $C^k(X;\mathbb{R})$ the set of all $k$-dimensional cochains.

We are going to be interested in ways of viewing the cochains in the links of the complex. For now we will only introduce one such way. Namely localization:

**Definition 1.10** (Localization). Let $X$ be a pure $d$-dimensional simplicial complex, $k, i$ be dimensions such that $i < k$ and $F \in C^k(X;\mathbb{R})$. Also let $\sigma \in X(i)$. Define the localization of $F$ to $\sigma$ to be:

$$F_{\sigma}(\tau) = F(\sigma \cup \tau)$$

We note that there is a very natural inner product defined on the cochains of a simplicial complex, defined as follows:

**Definition 1.11** (Inner product). Let $X$ be a pure $d$-dimensional simplicial complex and let $F, G \in C^k(X;\mathbb{R})$. Define the inner product of $F$ and $G$ to be:

$$\langle F, G \rangle = \sum_{\sigma \in X(k)} w(\sigma) F(\sigma)G(\sigma)$$

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1 There is no singular definition of high dimensional expander but in this paper we only use the algebraic definition - local spectral expansion.

2 Much like one dimensional expanders, in high dimensions there is also notion of one-sided vs. two-sided local spectral expansion. The definition we use throughout the paper is that of one-sided local spectral expander. The difference being that in two-sided local spectral expander the underlying graph of every link is a two-sided expander rather than a one-sided expander.

3 The complexes are weighted and therefore their expansion property is defined as the second largest eigenvalue of the non-lazy random walk. A random walk that walks from a face to one of its neighbours with probability equal to the proportion between the weight of the edge that connects them and the sum of the weights of the edges that include said vertex.
1.1 Main Results

In this paper we study the high dimensional analogue of the non-lazy random walk. We are specifically interested in going beyond Alev and Lau’s worst case result [AL20, Theorem 1.5] and relate the structure of the cochain to its expansion:

**Theorem 1.12** (Informal, for formal see Theorem 7.9). Let $F \in C^k(X; \mathbb{R})$ and let $F_0, \ldots, F_k$ be an orthogonal decomposition of $F$ such that $F_i$ corresponds to a cochain in the $i$th dimension (an $i$-level cochain) and $F = \sum_{i=0}^k F_i$. In addition, let $\gamma_i = \max_{\sigma \in X(i)} \{\lambda_2(X_{\sigma})\}$ then:

$$\langle (M')^k F, F \rangle \leq \sum_{i=0}^k \left(1 - \frac{1}{k-i+1} \prod_{j=i-1}^{k-1} (1 - \gamma_j) \right) \|F_i\|^2$$

Note that in the worst case (i.e. no assumption is made on the structure of $F$) this result matches that of Alev and Lau. In cases where the cochain is structured this Theorem yields strictly better results then what was previously know.

Our main tool for proving the decomposition of high dimensional random walks is a bootstrapping theorem that reduces decomposition of higher dimensional random walk to an understanding of highly structured cochains (for example cochains that correlate with 0-dimensional cochains). The bootstrapping theorem is fairly general and thus we bring a special case of it here:

**Theorem 1.13** (Bootstrapping theorem, informal. For formal see Theorem 5.8). Let $X$ be a simplicial complex and $k$ be a dimension. Every $F \in C^k(X; \mathbb{R})$ that is orthogonal to the constants can be decomposed into $F = \sum_{i=0}^k F_i$ such that the cochains $F_i$ are both:

1. $i$-level cochains: For every $\sigma \in X(i-1)$ it holds that $\langle (F_i)_\sigma, 1 \rangle = 0$.
2. Orthogonal: For every $i \neq j$ it holds that $F_i$ is orthogonal to $F_j$.

and the problem of finding $\{\lambda_i\}_{i=0}^k$ such that:

$$\langle (M')^k F, F \rangle = \sum_{i=0}^k \lambda_i \|F_i\|^2$$

can be reduced to solving some recursive formula on 0-level cochains.

We note that while this decomposition seems similar to the decomposition of Kaufman and Oppenheim presented in [KO18], it is different in one crucial aspect: The $F_i$s are not approximate eigenfunctions. Moreover, applying the non-lazy up-down random walk operator to any one of them yields a cochain that is not orthogonal to many of the other $F_j$s.

Solving the recursive formula in the theorem requires us to gain some “advantage” for highly structured cochains in the complex. We can therefore view this Theorem as a tool that allows us to bootstrap an advantage we have to a decomposition of the non-lazy random walks.

As we said, our bootstrapping theorem is fairly generic and can also yield the celebrated Oppenheim’s trickling down theorem [Opp17, Theorem 4.1]:

**Theorem 1.14** (Trickling Down, [Opp17]). Let $X$ be a pure $d$-dimensional simplicial complex. If it holds that:
• for every vertex v: $X^{(1)}_v$ is a $\lambda$ spectral expander.
• $X$ is connected.

Then it holds that $X^{(1)}$ is a $\frac{1}{\lambda}$ spectral expander.

In order to prove the trickling down we use Theorem 1.13 while defining the $i$-level cochains differently. For more detail, see Section 6.

Turn your mind back to our new, stronger version, of the random walk convergence Theorem that allows us to relate the structure of the cochain to how well it expands. This new perspective naturally yields the following question:

**Question 1.15.** What can be said about the expansion of a cochain such that, in the decomposition, $\|F_0\|, \ldots, \|F_i\| \leq \epsilon$?

This question initiates the study of pseudorandom cochains.

**Definition 1.16 (pseudorandom cochains).** Let $X$ be a pure $d$-dimensional simplicial complex and let $F \in C^k(X; \mathbb{R})$. Assume that $F = \sum_{i=0}^k F_i$ as in Theorem 1.13. We call $F$ an $(\epsilon, i)$-pseudorandom cochain if for every $j$ such that $0 \leq j \leq i$ it holds that:

$$\|F_j\| \leq \epsilon$$

Pseudorandom cochains were recently studied in [BHKL20] for a slightly different definition of high dimensional expansion, namely two-sided local spectral expanders. We prove that their definition of pseudorandom cochains coincides with the notion described above. Using this we can begin to consider pseudorandom functions in one-sided high dimensional expanders (i.e. the expanders we presented in this paper) for the first time. Using this definition we prove a weaker version of their result to one-sided expanders. Consider the following definition of the expansion of a set:

**Definition 1.17 (Expansion of a set).** Let $S$ be a set of faces of dimension $k$. Define the expansion of the set $S$ as:

$$\Phi(S) = 1 - \frac{\left(\langle M' \rangle^+_k 1_S, 1_S \right)}{\|1_S\|^2}$$

We show that small (extremely) pseudorandom sets expand almost optimally:

**Theorem 1.18 (Small set expansion, informal. For formal see Lemma 8.6).** Any small set of faces $S$ of $k$-dimensional faces that is sufficiently pseudorandom expands optimally in the following way:

$$\Phi(S) \geq 1 - \|S\| - \gamma_{k-1} (1 - \|S\|) - \epsilon^2$$

Where $\epsilon$ is a measurement of how pseudorandom the set $S$ is and $\gamma_{k-1}$ is a bound on the second eigenvalue of the non-lazy random walks on $k-2$ dimensional links.

This result shows small set expansion in one-sided local spectral expanders which was previously only known for two-sided local spectral expanders. However, in order to derive optimal expansion of small sets our result requires the cochains to be extremely pseudorandom, whereas [BHKL20] requires considerably weaker pseudorandom assumptions.

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4These expanders are a generalization of two sided spectral expander graphs to higher dimensions, much like the generalization we already presented.
1.2 Proof Layout

As we mentioned before, we analyze the non-lazy up-down random walk - a high dimensional analogue of the non-lazy random walk in graphs. We would like to get a decomposition of the non-lazy random walk operator. In order to do that we decompose the space of cochains to spaces which we term level cochains. Our proof is then comprised of two steps: A bootstrapping Lemma that reduces the problem of decomposing the non-lazy up-down random walk operator to a simple recursive condition about the 0-level cochains and an advantage that allows us to solve said recursive condition. We note that solving the recursive condition is considerably simpler than proving the decomposition directly as the 0-level cochains are very structured objects.

Link viewers We start by asking in what ways can the non-lazy up-down random walk can be viewed from the links. We are specifically interested in ways to view the non-lazy up-down random walk such that the global walk is the average of local walks over the links of the complex. We define an object called a link viewer that describes a way of restricting a cochain $F$ to a cochain $\Lambda_\sigma F$ in the link of $\sigma$. The running example we will use throughout this layout is the localization link viewer. i.e. the link viewer that sets $\Lambda_\sigma F = F_\sigma$. We note that this link viewer satisfies all the conditions listed above. Any link viewer decomposes the space of cochains to subspaces called $i$-level cochains. These spaces are the set of cochains whose expected value is 0 when viewed from any $i$-dimensional face. For example, under the localization link viewer the $i$-level cochains are cochains $G$ that satisfy: $\forall \sigma \in X(i) : \mathbb{E}_{\tau \in X_\sigma(k-i-1)}[G_\sigma(\tau)] = 0$. These spaces contain each other (i.e. any $i$-level cochain is an $(i-1)$-level cochain) and every cochain $F$ can be decomposed to $F = \sum_{\sigma}^i F_\sigma$ such that all the $F_\sigma$s are orthogonal to each other and every $F_i$ is an $i$-level cochain.

The bootstrapping argument We can now present our proof for the bootstrapping argument. We assume that the non-lazy up-down random walk expands locally and show that this yields that the non-lazy up-down random walk expands globally. We note that the largest eigenvalue of this random walk is 1 and that it corresponds to the constants and that the rest of its eigenvalues are strictly smaller than 1. And therefore we turn our attention to the space of cochains that are orthogonal to the constants i.e. the 0-level cochains. Let $F = \sum_{\sigma}^i F_\sigma$ be such a cochain. Note that for every vertex $v$ it holds that $\mathbb{E}[\sum_{i=1}^1 \Lambda_v F_i] = 0$. In addition, it holds that $\mathbb{E}[\Lambda_v F_0 - \mathbb{E}[\Lambda_v F]] = 0$ as well. We would then like to apply our decomposition theorem to $\sum_{i=0}^1 \Lambda_v F_i - \mathbb{E}[\Lambda_v F]$ in the link of every vertex. Note that if $F_i$ is an $i$-level cochain in the original complex then $\Lambda_v F_i$ is an $(i-1)$-level cochain in the link of $v$. It therefore holds that $F_0$ is a $-1$-level cochain. We therefore decompose it into two parts - its projection to the space of 0-level cochains and the remainder. We can thus apply our Theorem to the sum of the following level functions: $\Lambda_v F_k, \ldots, \Lambda_v F_2, \Lambda_v F_1 + \Lambda_v F_0 - \mathbb{E}[\Lambda_v F_0]$. Applying the decomposition Theorem leaves us with the task of bounding the components that originated in the 0-level cochain which we can do using the advantage.

Note that a key step in this decomposition is that despite the fact that these new localized cochains are not orthogonal to each other, they are all orthogonal to the constants. This allows us to separate them from the constant part of every localization.

The Advantage Gaining the advantage for the random walk decomposition Theorem is done by considering the localization link viewer that we described earlier. It involves representing the 0-level cochains using 0-dimensional cochains and is based on the following key observations:
1. For every vertex \( v \) it holds that \( \mathbb{E}[X_v^i F] \) is only dependent on the 0-level cochains. We also claim that the part of the cochain that originates in the vertices is orthogonal to the 1-level cochains and is therefore contained in \( F_0 \). Thus we can turn our attention to understanding \( \mathbb{E}[X_v^i F_0] \).

2. If a cochain \( F_0 \) is originated in the vertices then there exists a cochain \( G \) of vertices such that \( \|F_0\| = \|G\| \) and that \( \mathbb{E}[X_v^i F_0] \) is exactly \( \|M_k^0 G\| \) where \( M_k^0 \) is the random walk the applied the up step \( k \) times followed by applying the down step \( k \) times.

3. There is a connection between the random walk on the underlying graph of the complex and \( M_k^0 \). The key claim we use is that once the random walk has performed its first up step it can already “see” all the vertices it will ever see (this is due to the structure of simplicial complexes). Going further up only decreases the probability of staying in place. We can therefore describe \( M_k^0 \) using the underlying graph’s non-lazy random walk transition matrix and a lazy component (which corresponds to \( I \)). We note that this observation can be generalized to any random walk on \( k \)-dimensional faces that “walks through” faces of dimension higher than 2\( k \).

Using these three observations as well as our understanding of the 0-dimensional random walk we obtain the advantage we seek.

1.3 Related Work

One of the most important properties of high dimensional expanders is that many of its natural random walks converge rapidly. This was first proved by Kaufman and Mass [KM17] and was later improved by Kaufman and Oppenheim [KO18]. Following the work of Kaufman and Oppenheim, Alev and Lau [AL20, Theorem 1.5] have shown a bound on the second eigenvalue of the high dimensional random walk that is meaningful regardless of how small the spectral gap of the complex is:

**Theorem 1.19 (Convergence of Random Walks [AL20]).** Let \( X \) be a pure \( d \)-dimensional simplicial complex and let

\[
\gamma_i = \max \{ \lambda_2 ((X_\sigma(0), X_\sigma(1))) | \sigma \in X(i) \}
\]

Then:

\[
\lambda_2 \left( (M_k^0)^+ \right) \leq 1 - \frac{1}{k+1} \prod_{j=-1}^{k-1} (1 - \gamma_j)
\]

This line of work had various applications. For example, Dinur and Kaufman’s proof of existence of agreement expanders [DK17], Anari et al.’s algorithm for sampling bases of a matroid [ALGV18] and many more. We present a structured version of this theorem which uses tools that are fundamentally different to those used by Alev and Lau. In their proof, Alev and Lau use an inductive averaging argument in order to derive the bound on the second eigenvalue of the walk operator. In our proof, however, we use *local to global* arguments. While the proof of [AL20] does characterize the eigenvalues of the walk operator, it does not yield a result that can relate the structure of a cochain to its expansion properties. We stress that this connection is of particular interest as it improves our understanding of pseudorandom cochains as cochains that are close to relating only to small eigenvalues of the walk operator. This understanding of pseudorandom cochains allows us to tackle pseudorandom cochains in one-sided local spectral expanders for the first time.
Another important property of high dimensional expanders is Oppenheim’s trickling down Theorem [Opp17, Theorem 4.1] (Also, Theorem [1.14]). The trickling down theorem is a good example of a local to global property. In this theorem one assumes that locally the complex expands and, using the fact that the underlying graph of the complex is connected, one derives that the entire complex expands. This theorem plays a key part in the construction of local spectral expanders.

In this work we show an argument that, for the first time, captures both the trickling down theorem and the convergence of random walks.

2 The Signless Differential and Its Adjoint Operator

One of the key operators induced by any simplicial complex is its signless differential operator. The signless differential operator is an averaging operator that accepts a $k$-dimensional cochain and returns a $(k+1)$-dimensional cochain. We adopt the terminology of [KO18] and consider repeated application of the signless differential and its adjoint operator.

**Definition 2.1 (Signless differential).** The signless differential operator $d_k : C^k(X; \mathbb{R}) \to C^{k+1}(X; \mathbb{R})$ in the following way:

$$d_k F(\sigma) = \mathbb{E}_{\tau \in X(k)} [F(\tau)|\tau \subseteq \sigma] = \sum_{\tau \in (\sigma)_{k+2}} \frac{1}{k+2} F(\tau)$$

When the dimension is clear from context it will be omitted from the notation. Also define $d_k^* : C^{k+1}(X; \mathbb{R}) \to C^k(X; \mathbb{R})$ to be the adjoint operator of $d_k$.

**Note.** The signless differential does not meet the definition of a differential as $d_k d_{k-1} \neq 0$.

**Lemma 2.2.** It holds that:

$$d_k^* F(\tau) = \mathbb{E}_{\sigma \in X(k+1)} [F(\sigma)|\tau \subseteq \sigma] = \mathbb{E}_{\sigma \in X(0)} [F(\sigma)]$$
Lemma 2.3.

Proof. Consider, now, the following:

\[ (dF, G) = \sum_{\sigma \in X(k+1)} w(\sigma) dF(\sigma) G(\sigma) = \sum_{\sigma \in X(k+1)} w(\sigma) \mathbb{E}_{\tau \in X(k)} [F(\tau) | \tau \subset \sigma] G(\sigma) = \]

\[ = \sum_{\sigma \in X(k+1)} w(\sigma) \left( \sum_{\tau \in X(k) | \tau \subset \sigma} \frac{1}{k+2} F(\tau) \right) G(\sigma) = \]

\[ = \sum_{\sigma \in X(k+1)} \sum_{\tau \in X(k) | \tau \subset \sigma} \frac{w(\sigma)}{k+2} F(\tau) G(\sigma) = \]

\[ = \sum_{\tau \in X(k)} \sum_{\sigma \in X(k+1) | \tau \subset \sigma} \frac{w(\tau) w(\sigma)}{(k+2) w(\tau)} F(\tau) G(\sigma) = \]

\[ = \sum_{\tau \in X(k)} w(\tau) F(\tau) \sum_{\sigma \in X(k+1) | \tau \subset \sigma} \frac{w(\sigma)}{(k+2) w(\tau)} G(\sigma) = \]

\[ = \sum_{\tau \in X(k)} w(\tau) F(\tau) \sum_{\sigma \in X(k+1) | \tau \subset \sigma} w(\sigma) G(\sigma) = \]

\[ = \sum_{\tau \in X(k)} w(\tau) F(\tau) \mathbb{E}_{\sigma \in X(k+1)} [G(\sigma) | \tau \subset \sigma] \]

\[ \square \]

We will be interested in repeated application of the signless differential and its adjoint operator. To that effect it will be useful to present them explicitly. Lemmas 2.3 and 2.5 will present repeated applications of these operators explicitly.

**Lemma 2.3.** Let \( F \) be an \( i \)-dimensional cochain. Then:

\[ d_{j-1} \cdots d_i F(\sigma) = \mathbb{E}_{\tau \in X(i)} [F(\tau) | \tau \subset \sigma] \]

Proof. By induction, the case of \( j = i + 1 \) follows from the definition of \( d_i \).

Induction step:

\[ d_j \cdots d_i F(\sigma) = \mathbb{E}_{\tau \in X(j)} [d_j \cdots d_i F(\tau) | \tau \subset \sigma] = \mathbb{E}_{\tau \in X(j)} [\mathbb{E}_{\tau' \in X(i)} [F(\tau') | \tau' \subset \tau] | \tau \subset \sigma] = \]

\[ = \sum_{\tau \in (\sigma)} \sum_{\tau' \in (\sigma_i'' \sigma_{i+1})} \frac{1}{j+2} \frac{1}{(j+1)!} F(\tau') = \sum_{\tau \in (\sigma)} \sum_{\tau' \in (\sigma_i'' \sigma_{i+1})} \frac{1}{j+2} \frac{1}{(j+1)!} \frac{1}{(j+2)!} F(\tau') \]

\[ = \sum_{\tau \in (\sigma)} \sum_{\tau' \in (\sigma_i'' \sigma_{i+1})} \frac{(j-i)!(i+1)!}{(j+2)!} F(\tau') = \sum_{\tau \in (\sigma_i'' \sigma_{i+1})} \frac{(j-i)!(i+1)!}{(j+2)!} F(\tau) \]

\[ = \sum_{\tau \in (\sigma_i'' \sigma_{i+1})} \frac{(j-i)!(i+1)!}{(j+2)!} F(\tau) = \sum_{\tau \in (\sigma_i'' \sigma_{i+1})} \frac{1}{(j+2)!} F(\tau) \]

\[ = \mathbb{E}_{\tau \in X(i)} [F(\tau) | \tau \subset \sigma] \]
We turn our attention to showing that the adjoint operator of the signless differential localizes to the links in the following way:

**Lemma 2.4.** For every cochain $F$ and every face $\tau$ it holds that:

$$d^*_\tau F = (d^* F)_\tau$$

**Proof.** The Lemma holds due to:

$$d^*_\tau F_\tau (\sigma) = \sum_{\tau' \in \mathcal{X}_\sigma} w_\sigma(\tau') F_\sigma(\tau') = \sum_{\tau' \in \mathcal{X}_\sigma} w_\sigma(\tau') F_\sigma(\tau') = d^*_\tau F_\tau (\sigma)$$

**Lemma 2.5.** Let $F$ be an $i$-dimensional cochain. Then:

$$d^*_i \cdots d^*_{j-1} F(\sigma) = \mathbb{E}_{\tau \in \mathcal{X}_{\sigma, (j-i-1)}} [F_{\sigma}(\tau)]$$

**Proof.** By induction, the case where $j = i + 1$ holds due to Lemma 2.2. The induction step follows the following argument:

$$d^*_i \cdots d^*_{j-1} F(\sigma) = \mathbb{E}_{v \in \mathcal{X}_\sigma} [d^*_i \cdots d^*_j F(\sigma)(v)] = \mathbb{E}_{v \in \mathcal{X}_\sigma} [d^*_i \cdots d^*_{j-1} F(\sigma)(v)]$$

$$= \sum_{v \in \mathcal{X}_\sigma} w_\sigma(v) \sum_{\tau \in \mathcal{X}_{\sigma, (j-i-1)}} F_{\sigma}(\tau)$$

The second equality is due to Lemma 2.4.

### 3 Up-Down and Down-Up Operators

In this section we will present two objects of interest - the up-down and down-up walks. These are natural operators that result from considering a standard walk on the $k$-dimensional faces of a simplicial complex using the following two steps: A down step where, given a $k$-dimensional face, the walk moves to a $(k-1)$-dimensional face that is contained in it with equal probability. And an up step in which, given a $k$-dimensional face, the walk moves to a $(k+1)$-dimensional face with probability proportional to its weight. Applying the up step followed by the down step yields the up-down walk while applying the down step followed by the up step yields the down-up walk.
**Definition 3.1** (Up-down and down-up operators). Let $X$ be a simplicial complex. Then define the up-down random walk to be:

$$[M_k^+]_{\sigma,\tau} = \begin{cases} \frac{1}{k+2} \sum_{\tau' \in \binom{\tau}{k}} w_{\tau'} (\sigma \setminus \tau') & \sigma = \tau \\ \frac{k+2}{k+2} w_{\sigma \cap \tau} (\tau \setminus \sigma) & \sigma \cap \tau \in X(k-1) \\ 0 & \text{Otherwise} \end{cases}$$

And the down-up random walk to be:

$$[M_k^-]_{\sigma,\tau} = \begin{cases} \frac{1}{k+1} \sum_{\tau' \in \binom{\tau}{k}} w_{\tau'} (\sigma \setminus \tau') & \sigma = \tau \\ \frac{k+1}{k+1} w_{\sigma \cap \tau} (\tau \setminus \sigma) & \sigma \cap \tau \in X(k-1) \\ 0 & \text{Otherwise} \end{cases}$$

We would now like to characterize the up-down random walk and the down-up random walk using the signless differential. In order to do that, consider the following Lemma:

**Lemma 3.2.** Let $X$ be a simplicial complex, it holds that:

$$M_k^+ = d_k^* d_k \quad M_k^- = d_{k-1}^* d_{k-1}$$
Proof. Let \( F \in C^k (X; \mathbb{R}) \) and \( \sigma \in X(k) \), then:

\[
\begin{align*}
    d^* dF(\sigma) &= \mathbb{E}_{\tau \in X_\sigma(0)} [(dF)_\sigma(\tau)] = \sum_{\tau \in X_\sigma(0)} w_\sigma(\tau) (dF)_\sigma(\tau) = \sum_{\tau \in X_\sigma(0)} w_\sigma(\tau) dF(\tau \cup \sigma) \\
    &= \sum_{\tau \in X_\sigma(0)} w_\sigma(\tau) \frac{1}{k + 2} \sum_{\tau' \in X(k)} F(\tau') \\
    &= \frac{1}{k + 2} \sum_{\tau \in X_\sigma(0)} \sum_{\tau' \in X(k) \atop \tau' \subseteq \sigma \cup \tau} w_\sigma(\tau) F(\tau') \\
    &= \frac{1}{k + 2} \sum_{\tau \in X_\sigma(0)} \left( w_\sigma(\tau) F(\sigma) + \sum_{\tau' \in X(k) \atop \tau' \subseteq \sigma \cup \tau} w_\sigma(\tau) F(\tau') \right) \\
    &= \frac{1}{k + 2} \left( F(\sigma) + \sum_{\tau \in X_\sigma(0)} \sum_{\tau' \in X(k) \atop \tau' \subseteq \sigma \cup \tau} w_\sigma(\tau) F(\tau') \right) \\
    &= \frac{1}{k + 2} \left( F(\sigma) + \sum_{\tau' \in X(k) \atop \sigma \cup \tau' \in X(k+1)} w_\sigma(\tau' \setminus \sigma) F(\tau') \right) \\
    &= M_k^+ F
\end{align*}
\]
In addition:

\[
\dd^* F(\sigma) = \frac{1}{k+1} \sum_{\tau \in X(k-1)} \sum_{\tau \subseteq \sigma} d^* F(\tau) = \frac{1}{k+1} \sum_{\tau \in X(k-1)} \sum_{\tau \subseteq \sigma} \mathbb{E}_{\tau' \in X(0)} [F_{\tau'}(\tau')]
\]

\[
= \frac{1}{k+1} \sum_{\tau \in X(k-1)} \sum_{\tau \subseteq \sigma} F_\tau (\tau) \sum_{\tau' \in X(0)} w_\tau (\tau') F_{\tau'}(\tau')
\]

\[
= \frac{1}{k+1} \left( \sum_{\tau \in X(k-1)} F_\tau (\tau) \sum_{\tau \subseteq \sigma} \sum_{\tau' \in X(0)} w_\tau (\tau') F_{\tau'}(\tau') \right)
\]

\[
= \frac{1}{k+1} \left( \sum_{\tau \in X(k-1)} F_\tau (\tau) \sum_{\tau \subseteq \sigma} \sum_{\tau' \in X(0)} w_\tau (\tau') F_{\tau'}(\tau') \right)
\]

\[
= \frac{1}{k+1} \left( \sum_{\tau \in X(k-1)} F_\tau (\tau) \sum_{\tau \subseteq \sigma} \sum_{\tau' \in X(0)} w_\tau (\tau') F_{\tau'}(\tau') \right)
\]

\[
= \frac{1}{k+1} \left( \sum_{\tau \in X(k-1)} F_\tau (\tau) \sum_{\tau \subseteq \sigma} \sum_{\tau' \in X(0)} w_\tau (\tau') F_{\tau'}(\tau') \right)
\]

\[
= M_k^{-i} F
\]

We are also going to be interested in applying the signless differential and its adjoint operator multiple times in a row. We will therefore define the \(k\)-dimensional \(i\)-up-down operator and the \(k\)-dimensional \(i\)-down-up operator in the following way:

**Definition 3.3.** Let \(M_k^{+i}\) be the \(k\)-dimensional \(i\)-up-down operator and \(M_k^{-i}\) be the \(k\)-dimensional \(i\)-down-up operator defined as follows:

\[
M_k^{+i} = d_k^i \cdots d_{k+i-1} \cdots d_k \\
M_k^{-i} = d_{k+i} \cdots d_{k+i-1} \cdots d_k
\]

Recall that the \(i\)-up-down operator and the \(i\)-down-up operators can be presented explicitly using Lemma 2.3 and Lemma 2.5. In addition, note that the \(i\)-up-down operator corresponds to applying the up step \(i\) times and then applying the down step \(i\) times. Likewise the \(i\)-down-up operator corresponds to applying the down step \(i\) times and then applying the up step \(i\) times.

### 4 The Non-Lazy Walk Operator

Our main object of study is going to be the non-lazy \(k\)-dimensional random walk operator. This operator is a generalization of the non-lazy random walk operator in graphs to higher dimensions.
In graphs the non-lazy random walk operator moves between two vertices if they have an edge connecting them. The higher dimensional version of this operator is going to be something very similar: It is going to move between two $k$-faces if there is a $(k + 1)$-face that contains both faces.

**Definition 4.1 (The Non-Lazy $k$-dimensional Random Walk Operator).** Let $X$ be a pure $d$-dimensional simplicial complex define the $k$-dimensional random walk operator to be the following operator:

$$\left[(M')_k^+\right]_{\sigma,\tau} = \begin{cases} \frac{w_\sigma(\tau \setminus \sigma)}{k + 1} & \sigma \cup \tau \in X(k + 1) \\ 0 & \text{Otherwise} \end{cases}$$

One can also think of the non-lazy random walk operator as the regular up-down operator with the lazy part removed. Formally:

**Observation 4.2.** For every dimension $k$ it holds that:

$$(M')_k^+ = \frac{k + 2}{k + 1} M_k^+ - \frac{1}{k + 1} I$$

Of specific interest is the $0$-dimensional non-lazy up-down operator as it can be used to describe every one of the $0$-dimensional $i$-up-down operators. This is because every one of these operators ultimately move between a vertex and its neighbours. The more steps one takes up the complex the more mixed the result is. By that we mean that the non-lazy walk operator has a larger effect on the result. Quantitatively, this can be formulated via the following Lemma.

**Lemma 4.3.**

$$(M)_0^+ = \frac{i + 1}{i} M_0^+ - \frac{1}{i} I$$
Proof. Consider the up operator:

\[
M_0^+ F(v) = d_0^i \cdots d_{i-1}^i d_{i-1}^i \cdots d_0^i F(v) = \mathbb{E}_{\tau \in X_v(i-1)} [(d_{i-1}^i \cdots d_0^i) (\tau v)] = \\
= \mathbb{E}_{\tau \in X_v(i-1)} [d_{i-1}^i \cdots d_0^i (\tau v)] = \mathbb{E}_{\tau \in X_v(i-1)} \left[ \mathbb{E}_{u \in X(0)} [F(u) | u \subseteq \tau v] \right] \\
= \sum_{\tau \in X_v(i-1)} w_v(\tau) \sum_{u \in \tau} \frac{1}{i+1} F(u) = \sum_{\tau \in X_v(i-1)} \sum_{u \in \tau v} \frac{w_v(\tau)}{i+1} F(u) \\
= \sum_{\tau \in X_v(i-1)} \frac{w_v(\tau)}{i+1} F(v) + \sum_{\tau \in X_v(i-1)} \sum_{u \in \tau} \frac{w_v(\tau)}{i+1} F(u) \\
= \frac{1}{i+1} F(v) + \sum_{u \in X(0)} \sum_{\tau \in X(i) \setminus u \not\in v} \frac{w(\tau)}{(i+1)^2 w(v)} F(u) \\
= \frac{1}{i+1} F(v) + \sum_{u \in X(0)} \sum_{\tau \in X(i) \setminus u, v \in \tau} \frac{w(\tau)}{(i+1)^2 w(v)} F(u) \\
= \frac{1}{i+1} F(v) + \sum_{u \in X(0) \setminus \tau \in X(i) \setminus u, v \in \tau} \frac{1}{(i+1)^2 w(v)} F(u) \\
= \frac{1}{i+1} F(v) + \sum_{u \in X(0) \setminus \tau \in X(i) \setminus u, v \in \tau} \frac{1}{(i+1)^2 w(v)} F(u) \\
= \frac{1}{i+1} F(v) + \sum_{u \in X(0) \setminus \tau \in X(i) \setminus u, v \in \tau} \frac{1}{(i+1)^2 w(v)} F(u) \\
= \frac{1}{i+1} F(v) + \frac{i}{i+1} \sum_{u \in X(0)} w_v(u) F(u) \\
= \frac{1}{i+1} F(v) + \frac{i}{i+1} (M^+_0)^+ F(v) \\
\]

Therefore:

\[
(M^+_0)^+ = \frac{i+1}{i} M_0^+ - \frac{1}{i} I
\]

5 Analyzing the Non-Lazy Random Walk Operator

We are now ready to start analyzing the random walk operators. In order to do so we are going to use a local to global argument. In order to apply a local to global argument we must understand
how to view a cochain through the links. Specifically we will be interested in viewing methods that satisfy the following:

**Definition 5.1** (Link viewer). A link viewer is any transformation $\Lambda$ that accepts a face $\sigma$ and a cochain in $X$. It then returns a cochain in $X\sigma$. In addition, a link viewer satisfies the following:

- For every face $\sigma$ it holds that $\Lambda_{\sigma}$ is linear.
- For every face $\sigma$ it holds that $\Lambda_{\sigma}1 = 1$.
- For every two cochains $F, G$ and any two faces of the same dimension $\sigma, \tau$ it holds that:
  \[
  \dim(F) - \dim(\Lambda_{\sigma}F) = \dim(G) - \dim(\Lambda_{\tau}F)
  \]

  In addition, denote the dimensional difference for vertices by:
  \[
  \Delta(\Lambda) = \dim(F) - \dim(\Lambda_{v}F)
  \]

- Viewing of a cochain in a link is only determined by the cochain and the face for which it is the link (and not the path taken to achieve said view). Formally: For every $\tau \subseteq \sigma \in X$ it holds that:
  \[
  \Lambda_{\sigma} = \Lambda_{\sigma\setminus\tau}\Lambda_{\tau}
  \]

- For every dimension $i$ it holds that:
  \[
  \langle F, G \rangle = \mathbb{E}_{\sigma \in X(i)}[\langle \Lambda_{\sigma}F, \Lambda_{\sigma}G \rangle]
  \]

Of particular interest are link viewers that view the non-lazy random walk operator in “the right way”:

**Definition 5.2.** A link viewer respects the non-lazy up-down random walk if for every $k$:

\[
\left\langle (M')^{+}_{k}F, F \right\rangle = \mathbb{E}_{\sigma \in X(0)}[\left\langle (M')^{+}_{k-\Delta(\Lambda)}\Lambda_{v}F, \Lambda_{v}F \right\rangle]
\]

**Definition 5.3** (i-level cochain). A cochain $F$ is an $i$-level cochain with respect to $\Lambda$ if it holds that:

\[
\forall \sigma \in X(i-1) : \langle \Lambda_{\sigma}F, 1 \rangle = 0
\]

When the link viewer is clear from context we will simply refer to them as i-level cochains. Denote the set of i-level $j$-dimensional cochains in $X$ by $C^{i}_{\Lambda,i}(X; \mathbb{R})$.

**Lemma 5.4.** For every dimension $j \leq i$:

\[
C^{k}_{\Lambda,i}(X; \mathbb{R}) \subseteq C^{k}_{\Lambda,j}(X; \mathbb{R})
\]

**Proof.** Let $F \in C^{k}_{\Lambda,j}(X; \mathbb{R})$ and let $\sigma$ be a $(j-1)$-dimensional face and note the following:

\[
\langle \Lambda_{\sigma}F, 1 \rangle = \mathbb{E}_{\tau \in X_{\sigma}(i-j)}[\langle \Lambda_{\tau}\Lambda_{\sigma}F, \Lambda_{\tau}1 \rangle] = \mathbb{E}_{\tau \in X_{\sigma}(i-j)}[\langle \Lambda_{\tau}\Lambda_{\sigma}F, 1 \rangle] = 0
\]

Where the last equality is due to the fact that $\sigma \cup \tau$ is an $(i-1)$-dimensional face. \qed
Of particular interest are $i$-level cochains that are orthogonal to the $(i+1)$-level cochains. We will therefore define the following:

**Definition 5.5** (Proper $i$-level cochain). A cochain $F$ is a proper $i$-level cochain with respect to $\Lambda$ if it holds that:

$$F \in C^j_{\Lambda,i} (X; \mathbb{R}) \cap \left( C^j_{\Lambda,i+1} (X; \mathbb{R}) \right)^\perp$$

When the link viewer is clear from context we will simply refer to them as proper $i$-level cochains.

Denote the set of proper $i$-level $j$-dimensional cochains in $X$ by $C^j_{\Lambda,i} (X; \mathbb{R})$.

Consider the following key property of proper level cochains:

**Lemma 5.6.** Let $i < j$ and let $F$ be a proper $i$-level cochain and $G$ be a proper $j$-level cochain then:

$$\langle F, G \rangle = 0$$

**Proof.** Due to Lemma 5.4 it holds that $G$ is also a $i + 1$ level cochain and thus is orthogonal to $G$ by definition.

Note that considering proper level cochains is one of the key ingredients of this paper. In previous results (for example [KO18]) instead of considering proper $i$-level cochains the authors essentially considered cochains in $\left( C^k_{\Lambda,i} (X; \mathbb{R}) \right)^\perp$. Using proper level cochains allows us to separate the different levels completely and achieve a proper decomposition. Since analyzing the decomposition with only pure level cochains in mind is hard (as even if $F$ is a level cochain the same cannot be said about $\Lambda_\sigma F$) we resort to first show the following technical theorem.

**Theorem 5.7** (Bootstrapping Theorem, Technical Version). Let $X$ be a pure $d$-dimensional simplicial complex, $\Lambda$ be a link viewer that respects the non-lazy up-down random walk and for every $i$ let $F_i \in C^k_{\Lambda,i} (X; \mathbb{R})$ and $F = \sum_{i=0}^{p} F_i$. Denote $r = k - \Delta (\Lambda)$ and suppose that there are values of $\{\lambda_{\sigma,i,j}\}_{\sigma \in X,i \in [d], j \in [d]}$ such that for every $\sigma \in X$ and $G_0$ a 0-level cochain:

$$\begin{cases} \lambda_{\sigma,1,k} \|G_0\|^2 + \mathbb{E}_{v \in X_\sigma(0)} \left[(1 - \lambda_{\sigma,\cup\{v\},1,r}) \|M_v^{-r} \Lambda_v G_0\|^2\right] \leq \lambda_{\sigma,0,k} \|G_0\| \\ \max_{v \in X_\sigma(0)} \{\lambda_{\sigma,\cup\{v\},i-1,r}\} \leq \lambda_{\sigma,i,k} \end{cases}$$

Then:

$$\langle (M')^+_k F, F \rangle \leq \sum_{i=0}^{p} \lambda_{\emptyset,i,k} \|F_i\|^2 + \sum_{i=0}^{p} \sum_{j=1}^{p} c_{i,j} \langle F_i, F_j \rangle$$

For some constants $\{c_{i,j}\}$ and where $p$ is the number of level functions that span the space orthogonal to the constants.
Proof. Let $r$ be the dimension of $\Lambda_\sigma F$. Consider the following:

$$\langle (M')^+_k F, F \rangle = \left\langle (M')^+_k \sum_{i=0}^{p} F_i, \sum_{i=0}^{p} F_i \right\rangle = \mathbb{E}_{v \in X(0)} \left[ \left\langle \Lambda_v \left( (M')^+_r \sum_{i=0}^{p} F_i \right), \Lambda_\sigma \left( \sum_{i=0}^{p} F_i \right) \right\rangle \right]$$

$$= \mathbb{E}_{v \in X(0)} \left[ \left\langle (M')^+_r \sum_{i=0}^{p} \Lambda_v F_i, \sum_{i=0}^{p} \Lambda_v F_i \right\rangle \right]$$

$$= \mathbb{E}_{v \in X(0)} \left[ \left\langle (M')^+_r (I - M'^-r) \sum_{i=0}^{p} \Lambda_v F_i, (I - M'^-r) \sum_{i=0}^{p} \Lambda_v F_i \right\rangle \right]$$

$$+ \mathbb{E}_{v \in X(0)} \left[ \left\langle (M')^+_r M'^-r \sum_{i=0}^{p} \Lambda_v F_i, M'^-r \sum_{i=0}^{p} \Lambda_v F_i \right\rangle \right]$$

$$= \mathbb{E}_{v \in X(0)} \left[ \left\langle (M')^+_r (I - M'^-r) \sum_{i=0}^{p} \Lambda_v F_i, (I - M'^-r) \sum_{i=0}^{p} \Lambda_v F_i \right\rangle \right]$$

$$+ \mathbb{E}_{v \in X(0)} \left[ \left\| M'^-r \Lambda_v F_0 \right\|^2 \right]$$

(1)

We will now like to apply our Theorem to the localized cochains in the links. For that, consider the following level cochains:

| level | cochain       | square of norm |
|-------|---------------|----------------|
| $k-1$ | $F^{v}_{k-1} := \Lambda_v F_k$ | $\|\Lambda_v F_k\|^2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $1$   | $F^{v}_{1} := \Lambda_v F_2$ | $\|\Lambda_v F_2\|^2$ |
| $0$   | $F^{v}_{0} := \Lambda_v F_1 + (I - M'^-r) \Lambda_v F_0$ | $\|\Lambda_v F_1\|^2 + \| (I - M'^-r) \Lambda_v F_0 \|^2 + 2 \langle \Lambda_v F_1, \Lambda_v F_0 \rangle$ |

Note that, by definition, for every $i \geq 2$ it holds that $\Lambda_v F_i \in C^\nu_{\Lambda, i-1}(X_v; \mathbb{R})$. In addition $\Lambda_v F_1 + (I - M'^-r) \Lambda_v F_0 \in C^\nu_{\Lambda, 0}(X_v; \mathbb{R})$. We can therefore apply the Theorem to every link which (after some manipulation of the mixed terms that, for completion, is presented in Lemma 5.10) yields that, for every link $v$, it holds that:

$$\langle (M')^+_{v,k} (I - M'^-r) \Lambda_v F, (I - M'^-r) \Lambda_v F \rangle \leq \sum_{i=0}^{k-1} \lambda_{u,i,r} \left\| F^{v}_{i} \right\|^2 + \sum_{i=0}^{p} \sum_{j=1}^{p} c_{i,j} \langle F^{v}_{i}, F^{v}_{j} \rangle =$$

$$= \sum_{i=0}^{k-1} \lambda_{u,i,r} \left\| \Lambda_v F_i \right\|^2 + \lambda_{u,0,r} \left\| (I - M'^-r) \Lambda_v F_0 \right\|^2 + \sum_{i=0}^{p} \sum_{j=1}^{p} c'_{i,j} \langle \Lambda_v F_i, \Lambda_v F_j \rangle$$

Combining this with (1) and noting that $\lambda_{\emptyset,i,k} \geq \max_{v \in X(0)} \{ \lambda_{u,i-1,r} \}$ yields that:

$$\langle (M')^+_k F, F \rangle \leq \sum_{i=1}^{k} \lambda_{\emptyset,i,k} \left\| F_i \right\|^2 + \mathbb{E}_{v \in X(0)} \left[ \left\| M'^-r \Lambda_v F_0 \right\|^2 + \lambda_{u,0,r} \left\| (I - M'^-r) \Lambda_v F_0 \right\|^2 \right] +$$

$$+ \sum_{i=0}^{p} \sum_{j=1}^{p} c'_{i,j} \langle F_i, F_j \rangle$$

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Therefore all we have to do is prove that:

$$\mathbb{E}_{v \in \mathcal{X}(0)} \left[ \|M_{r}^{-r} \Lambda_{v} F_{0}\|^{2} + \lambda_{v,0,r} \| (I - M_{r}^{-r}) \Lambda_{v} F_{0}\|^{2} \right] \leq \lambda_{\emptyset,0,k} \|F_{0}\|^{2}$$

This follows directly from our choice of \(\lambda\):

$$\mathbb{E}_{v \in \mathcal{X}(0)} \left[ \|M_{r}^{-r} \Lambda_{v} F_{0}\|^{2} + \lambda_{v,0,r} \| (I - M_{r}^{-r}) \Lambda_{v} F_{0}\|^{2} \right] =$$

$$= \mathbb{E}_{v \in \mathcal{X}(0)} \left[ \left(1 - \lambda_{v,0,r}\right) \|M_{r}^{-r} \Lambda_{v} F_{0}\|^{2} + \lambda_{v,0,r} \left(\|M_{r}^{-r} \Lambda_{v} F_{0}\|^{2} + \| (I - M_{r}^{-r}) \Lambda_{v} F_{0}\|^{2}\right) \right] =$$

$$\leq \mathbb{E}_{v \in \mathcal{X}(0)} \left[ \left(1 - \lambda_{v,0,r}\right) \|M_{r}^{-r} \Lambda_{v} F_{0}\|^{2} \right] + \lambda_{\emptyset,1,k} \|F_{0}\|^{2} \leq \lambda_{\emptyset,0,k} \|F_{0}\|^{2}$$

\[\square\]

**Theorem 5.8** (Bootstrapping Theorem). Let \(X\) be a pure \(d\)-dimensional simplicial complex, \(\Lambda\) be a link viewer that respects the non-lazy up-down random walk and for every \(i\) let \(F_{i} \in C_{\Lambda,i}^{k}(X; \mathbb{R})\) be a proper \(i\)-level cochain and \(F = \sum_{i=0}^{p} F_{i}\). Denote \(r = k - \Delta(\Lambda)\) and suppose that there are values of \(\{\lambda_{\sigma,i,j}\}_{\sigma \in \mathcal{X}, i \in [d], j \in [d]}\) such that for every \(\sigma \in \mathcal{X}\) and \(G_{0}\) a 0-level cochain:

\[
\begin{align*}
\lambda_{\sigma,1,k} \|G_{0}\|^{2} + \mathbb{E}_{v \in \mathcal{X}_{\sigma}(0)} \left[ \left(1 - \lambda_{\sigma,\cup\{v\},1,r}\right) \|M_{r}^{-r} \Lambda_{v} G_{0}\|^{2} \right] &\leq \lambda_{\sigma,0,k} \|G_{0}\| \\
\max_{v \in \mathcal{X}_{\sigma}(0)} \{\lambda_{\sigma,\cup\{v\},i-1,r}\} &\leq \lambda_{\sigma,i,k}
\end{align*}
\]

Then:

\[
\langle (M')^{+}_{k} F, F \rangle \leq \sum_{i=0}^{p} \lambda_{\emptyset,i,k} \|F_{i}\|^{2}
\]

For some constants \(\{c_{i,j}\}\) and where \(p\) is the number of level functions that span the space orthogonal to the constants.

**Proof.** Note that the difference between this Theorem and Theorem 5.7 is the choice of \(F_{i}\)s. Specifically, in this Theorem the the cochains \(F_{i}\) are chosen to be proper \(i\)-level cochains. Therefore, due to Lemma 5.6 they are orthogonal to each other. This allows us to apply Theorem 5.7 and note that:

\[
\langle (M')^{+}_{k} F, F \rangle \leq \sum_{i=0}^{p} \lambda_{\emptyset,i,k} \|F_{i}\|^{2} + \sum_{i=0}^{p} \sum_{j=1}^{p} \sum_{i<j} c'_{i,j} \langle F_{i}, F_{j} \rangle = \sum_{i=0}^{p} \lambda_{\emptyset,i,k} \|F_{i}\|^{2}
\]

\[\square\]

It is important to note that, unlike the decomposition known in the two-sided case, this decomposition is not a decomposition to approximate eigenfunctions. When applying the walk operator to a level function the result might be spread over multiple levels. Theorem 5.8 also yields a decomposition to the up-down operator:

**Corollary 5.9.** With the same assumptions as Theorem 5.8 it holds that

\[
\langle M_{k}^{+} F, F \rangle \leq \sum_{i=0}^{p} \left(\frac{k+1}{k+2} \lambda_{\emptyset,i,k} - \frac{1}{k+1}\right) \|F_{i}\|^{2}
\]
Proof. The following holds:

\[ \langle M_k^+ F, F \rangle = \frac{k+2}{k+1} \langle M' k^+ F, F \rangle - \frac{1}{k+1} \langle F, F \rangle \leq \frac{k+2}{k+1} \langle \lambda_{\emptyset, i, k} F, F \rangle - \frac{1}{k+1} \| F \|^2 = \]

\[ = \frac{k+2}{k+1} \lambda_{\emptyset, i, k} \| F \|^2 - \frac{1}{k+1} \| F \|^2 = \sum_{i=0}^{p} \left( \frac{k+1}{k+2} \lambda_{\emptyset, i, k} - \frac{1}{k+1} \right) \| F_i \|^2 \]

Before we finish this section, we prove the following (fairly technical) Lemma for completion

**Lemma 5.10.** With the notations in Theorem 5.7 and \( L = \{ F_i^p \mid 1 \leq i \leq p \} \) it holds that for every set of constants \( \{ c_{G, G'} \}_{G \neq G'} \) there exists constants \( \{ c'_{G, G'} \}_{G \neq G'} \) such that:

\[ \sum_{G, G' \in L \backslash G \neq G'} c_{G, G'} \langle G, G' \rangle = \sum_{i=0}^{p} \sum_{j=1, i < j}^{p} c'_{i, j} \langle \Lambda_{v} F_i, \Lambda_{v} F_j \rangle \]

Proof. Consider the following:

\[ \sum_{G, G' \in L \backslash G \neq G'} c_{G, G'} \langle G, G' \rangle = \sum_{i=2}^{p} \sum_{j=2, i < j}^{p} c_{i, j} \langle \Lambda_{v} F_i, \Lambda_{v} F_j \rangle + \sum_{i=2}^{p} c_{0, i} \langle \Lambda_{v} F_i, \Lambda_{v} F_1 + (I - M^{-r}) \Lambda_{v} F_0 \rangle \]

\[ = \sum_{i=2}^{p} \sum_{j=2, i < j}^{p} c_{i, j} \langle \Lambda_{v} F_i, \Lambda_{v} F_j \rangle + \sum_{i=2}^{p} c_{0, i} \langle \Lambda_{v} F_i, \Lambda_{v} F_1 + \Lambda_{v} F_0 \rangle - \sum_{i=2}^{p} c_{0, i} \langle \Lambda_{v} F_i, M^{-r} \Lambda_{v} F_0 \rangle \]

\[ = \sum_{i=2}^{p} \sum_{j=2, i < j}^{p} c_{i, j} \langle \Lambda_{v} F_i, \Lambda_{v} F_j \rangle + \sum_{i=2}^{p} c_{0, i} \langle \Lambda_{v} F_i, \Lambda_{v} F_1 \rangle + \sum_{i=2}^{p} c_{0, i} \langle \Lambda_{v} F_i, \Lambda_{v} F_0 \rangle \]

\[ = \sum_{i=0}^{p} \sum_{j=1, i < j}^{p} c'_{i, j} \langle \Lambda_{v} F_i, \Lambda_{v} F_j \rangle \]

6 Trickling Down

Before we present our random walk decomposition Theorem, let us start with a “warm up”: An alternative proof for the trickling down theorem [Opp17] that is based on Theorem 5.8. We believe that the fact that the main tool presented here can be used to prove the trickling down theorem is of independent interest: it shows that there is a single, local to global argument at the heart of both claims. In the trickling down theorem we are interested in the connection between the 0-dimensional non-lazy random walk on the vertices of a complex and the 0-dimensional non-lazy up-down random walk on the links of the vertices of the complex. It will, therefore, be natural to consider a link viewer that does not incur a decrease in dimension. One such link viewer is the restriction link viewer defined as:

\[ 2 \]
**Definition 6.1** (Restriction). Given a simplicial complex $X$ and a cochain $F \in C^k(X; \mathbb{R})$ define the restriction link viewer in the following way:

$$\forall \sigma \in X : \Lambda^r_\sigma F(\tau) = F(\tau)$$

This link viewer maps the 0-dimensional non-lazy up-down random walk to the 0-dimensional non-lazy up-down on the links of the vertices. We will show that applying Theorem 5.8 to the restriction link viewer yields the trickling down theorem. However, before applying Theorem 5.8, we must first show that restriction is indeed a link viewer that respects the non-lazy up-down random walk.

**Lemma 6.2.** The restriction link viewer is a link viewer.

**Proof.** We will prove the properties point by point:

- $\Lambda^r_\sigma (F + c \cdot G)(\tau) = (F + c \cdot G)(\tau) = F(\tau) + c \cdot G(\tau) = \Lambda^r_\sigma F(\tau) + c \cdot \Lambda^r_\sigma G(\tau)$
- $\Lambda^r_\sigma 1 = 1$
- It holds that for every cochain $F$ and every face $\sigma$:
  $$\dim(F) - \dim(\Lambda^r_\sigma F) = \dim(F) - \dim(F) = 0$$
- Let $\tau \subseteq \sigma \in X$ then:
  $$\Lambda^r_\sigma \setminus \tau \Lambda^r_\tau F(\tau') = \Lambda^r_\tau F(\tau') = F(\tau') = \Lambda^r_\sigma F(\tau')$$
- For every dimension $i$ and $j = \dim(\Lambda^r_\sigma F)$ it holds that:

  $$\mathbb{E}_{\sigma \in X(i)} \left[ \langle \Lambda^r_\sigma F, \Lambda^r_\sigma G \rangle \right] = \sum_{\sigma \in X(i)} \sum_{\tau \in X(j)} \sum_{\tau \in X(i+j+1)} w(\sigma) \Lambda^r_\sigma F(\tau) \Lambda^r_\sigma G(\tau) = \sum_{\tau \in X(j)} \sum_{\tau \in X(i+j+1)} \tau \Lambda^r_\tau F(\tau) \Lambda^r_\tau G(\tau) = \sum_{\tau \in X(j)} F(\tau) G(\tau) \sum_{\sigma \in X(i)} w(\sigma) \Lambda^r_\sigma F(\tau) \Lambda^r_\sigma G(\tau) = \sum_{\tau \in X(j)} F(\tau) G(\tau) \sum_{\sigma \in X(i)} w(\sigma) \Lambda^r_\sigma F(\tau) \Lambda^r_\sigma G(\tau) = \sum_{\tau \in X(j)} F(\tau) G(\tau) = \langle F, G \rangle$$

**Lemma 6.3.** The restriction link viewer respects the non-lazy random walk operator.
Proof. Note that, for every $i$-dimensional cochain $G$ it holds that:

$$d\Lambda^r_s G(\tau) = \sum_{\tau' \in (\tau)} \frac{1}{i+1} \Lambda^r_s G(\tau') = \sum_{\tau' \in (\tau)} \frac{1}{i+1} G(\tau') = dG(\tau) = \Lambda^s_r (dG)(\tau)$$

It therefore holds, for every dimension $k$:

$$\langle M_k^+ F, G \rangle = \langle d_k^* d_k F, G \rangle = \langle d_k F, d_k G \rangle = \mathbb{E}_{\sigma \in X(i)} \left[ \langle \Lambda^r_k \Lambda^r_k F, \Lambda^r_k d_k G \rangle \right]$$

$$= \mathbb{E}_{\sigma \in X(i)} \left[ \langle d_k \Lambda^r_k F, d_k \Lambda^r_k G \rangle \right] = \mathbb{E}_{\sigma \in X(i)} \left[ \langle d_k d_k \Lambda^r_k F, \Lambda^r_k G \rangle \right] = \mathbb{E}_{\sigma \in X(i)} \left[ \langle M_k^+ \Lambda^r_k F, \Lambda^r_k G \rangle \right]$$

And thus:

$$\mathbb{E}_{\sigma \in X(i)} \left[ \langle M_k^+ \Lambda^r_k F, \Lambda^r_k G \rangle \right] = \mathbb{E}_{\sigma \in X(i)} \left[ \langle M_k^+ \Lambda^r_k F, \Lambda^r_k G \rangle \right] = \mathbb{E}_{\sigma \in X(i)} \left[ \langle M_k^+ \Lambda^r_k F, \Lambda^r_k G \rangle \right]$$

$$= \mathbb{E}_{\sigma \in X(i)} \left[ \langle M_k^+ \Lambda^r_k F, \Lambda^r_k G \rangle \right]$$

**Lemma 6.4.** It holds for every vertex $v$ and every cochain $F \in C^0(X; \mathbb{R})$ that:

$$M^+_{v,0} \Lambda^r_v F_0 = \langle \Lambda^r_v F, 1 \rangle_v = \mathbb{E}_{u \in X_v(0)} [\Lambda^r_v F(u)] = \langle M^+_{v,0} F \rangle_v$$

**Proof.**

$$\langle M^+_{v,0} F \rangle_v = \sum_{u \in X(0)} \left[ \langle M^+_{v,0} F \rangle \right] = \sum_{u \in X(0)} w_v(u \setminus v) F(u)$$

$$= \sum_{u \in X_v(0)} w_v(u \setminus v) \Lambda^r_v F(u) = \mathbb{E}_{u \in X_v(0)} [\Lambda^r_v F(u)]$$

**Corollary 6.5** (The advantage). If $X$’s 1-skeleton is a $\lambda$-spectral expander it holds for every vertex $v$ and every cochain $F \in C^0(X; \mathbb{R})$ that:

$$\|d_0^* \Lambda^r_v F_0\| = \|M^+_{v,0} \Lambda^r_v F_0\| \leq \lambda^2 \|F\|^2$$

**Proof.** Note that the 1-skeleton of $X$ is a $\lambda$-spectral expander and $F \in C^0(X; \mathbb{R})$ therefore $\|\langle M^+_{v,0} F \rangle\|^2 \leq \lambda^2 \|F\|^2$. Combining this with Lemma 6.4 yields:

$$\|M^+_{v,0} \Lambda^r_v F_0\|^2 = \|\langle M^+_{v,0} F \rangle\|^2 \leq \|\langle M^+_{v,0} F \rangle\|^2 \leq \lambda^2 \|F_0\|^2$$
We can now show how applying Theorem 5.8 to the restriction link viewer yields Oppenheim’s trickling down theorem [Opp17, Theorem 4.1].

**Theorem 6.6** (Trickling Down, restated Theorem 1.14). If it holds that:

- For every vertex $v$: $X_v$ is a $\lambda_{v,0,k}$ spectral expander.
- $X$ is connected.

Then it holds that $\lambda_{\emptyset,0,k} = \frac{\lambda_{\emptyset,1,k}}{1-\lambda_{\emptyset,1,k}}$.

*Proof.* Consider the following:

$$\lambda_{\emptyset,1,k} \|F_0\|^2 + (1 - \lambda_{\emptyset,1,k}) \mathbb{E}_{v \in X(0)} \left[ \left\| M^{-}_v \Lambda^*_v F_0 \right\|^2 \right] \leq \lambda_{\emptyset,0,k} \|F_0\|$$

Using Corollary 6.5 it suffices to find values of $\lambda_{\sigma,i,j}$ such that:

$$\lambda_{\emptyset,1,k} \|F_0\|^2 + (1 - \lambda_{\emptyset,1,k}) \lambda_{\emptyset,0,k}^2 \|F_0\|^2 \leq \lambda_{\emptyset,0,k} \|F_0\|$$

Therefore solving the following inequality would bound $\lambda_{\emptyset,0,k}$:

$$\lambda_{\emptyset,1,k} + (1 - \lambda_{\emptyset,1,k}) \lambda_{\emptyset,0,k}^2 \leq \lambda_{\emptyset,0,k}$$

Note that picking $\lambda_{\emptyset,0,k} = \frac{\lambda_{\emptyset,1,k}}{1-\lambda_{\emptyset,1,k}}$ satisfies the inequality and thus proves the Theorem. \hfill \Box

Before we end this section we would like to give a few words on why this proof does not yield a decomposition of the 0-dimensional random walk operator that improves upon the Theorem’s original formulation. Consider how the 1-level cochains behave when applying the non-lazy random walk to them: Due to Lemma 6.4 the following holds for every 1-level cochain $F$:

$$(M'^\dagger_0) F(v) = \langle \Lambda^*_v F, 1 \rangle = 0$$

Where the second inequality is due to the very definition of 1-level cochains. Therefore, while the decomposition that Theorem 5.8 guarantees does exist, it is also meaningless as $(M'^\dagger_0) F = (M'^\dagger_0) F_0$.

### 7 Decomposition of the Random Walk Operators

We are now ready to present the random walk decomposition theorem based on Theorem 5.8. Unlike the trickling down theorem, here the assumption we have is only on the expansion of the no-lazy up-down random walk on the vertices. We will, therefore, be interested in a link viewer that decreases the dimension of the cochain. Namely, the localization link viewer, defined as follows:

**Definition 7.1** (Localization). Given a simplicial complex $X$ and a cochain $F \in C^k(X; \mathbb{R})$ define the localization link viewer in the following way:

$$\forall \sigma \in X : \Lambda^*_\sigma F = F_\sigma$$

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As with the trickling down theorem, we will be interested in applying Theorem 5.8 to the localization link viewer. We will start by proving that the localization link viewer is indeed a link viewer that respects the non-lazy up-down random walk.

**Lemma 7.2.** The localization link viewer is a link viewer.

**Proof.** We will prove the properties point by point:

- \( \Lambda^\ell_{\sigma} (F + c \cdot G) (\tau) = (F + c \cdot G) (\sigma \cup \tau) = F(\sigma \cup \tau) + c \cdot G(\sigma \cup \tau) = \Lambda^\ell_{\sigma} F(\tau) + c \cdot \Lambda^\ell_{\sigma} G(\tau) \)

- \( \Lambda^\ell_{\sigma} 1(\tau) = 1(\sigma \cup \tau) = 1 \)

- For every cochain \( F \) it holds that \( \dim(\Lambda^\ell_{\sigma} F) = \dim(F) - \dim(\sigma) \) therefore for every two cochains \( G, G' \) and any two \( i \)-dimensional faces \( \tau, \tau' \) it holds that:

\[
\dim(G) - \dim(\Lambda^\ell_{\tau} G) = \dim(\tau) = \dim(\tau') = \dim(G') - \dim(\Lambda^\ell_{\tau'} G')
\]

And \( \Delta(\Lambda^\ell) = 1 \).

- Let \( \tau \subseteq \sigma \in X \) then:

\[
\Lambda^\ell_{\sigma \setminus \tau} \Lambda^\ell_{\tau} F(\tau') = \Lambda^\ell_{\tau} F(\tau' \cup (\sigma \setminus \tau)) = F(\tau' \cup (\sigma \setminus \tau) \cup \tau) = F(\tau' \cup \sigma) = \Lambda^\ell_{\sigma} F(\tau')
\]

- For every dimension \( i \) and \( j = \dim(\Lambda^\ell_{\sigma} F) \) it holds that:

\[
\mathbb{E}_{\sigma \in X(i)} \left[ \left( \Lambda^\ell_{\sigma} F, \Lambda^\ell_{\sigma} G \right) \right] = \sum_{\sigma \in X(i)} w(\sigma) \left( \Lambda^\ell_{\sigma} F, \Lambda^\ell_{\sigma} G \right)
\]

\[
= \sum_{\sigma \in X(i)} w(\sigma) \sum_{\tau \in X_{\sigma}(j)} w(\tau) \Lambda^\ell_{\sigma} F(\tau) \Lambda^\ell_{\sigma} G(\tau) = \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j)} \frac{w(\tau \cup \sigma)}{(i+j+2)(i+1)} \Lambda^\ell_{\sigma} F(\tau) \Lambda^\ell_{\sigma} G(\tau)
\]

\[
= \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j)} \frac{w(\tau \cup \sigma)}{(i+j+2)(i+1)} F(\sigma \cup \tau) G(\sigma \cup \tau) = \sum_{\tau \in X(i+j+1)} w(\tau) F(\tau) G(\tau) = \langle F, G \rangle
\]

**Lemma 7.3.** The localization link viewer respects the non-lazy random walk operator.

**Proof.** Note that, by definition, it holds that

\[
(M')_k^+ F(\sigma) = \frac{1}{k+1} \sum_{\tau \in X(k)} w(\tau \setminus \sigma) F(\tau)
\]
The Lemma follows from the following calculation:

\[
\mathbb{E}_{\sigma \in X(i)} \left[ \left\langle (M')^+_{\sigma,j-i-1} \Lambda^\ell_{\sigma} F, \Lambda^\ell_{\sigma} G \right\rangle \right] = \\
= \frac{1}{j-i} \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j-i-1)} w(\sigma) w_\sigma(\tau) \sum_{\tau' \in X_{\sigma}(j-i-1) \tau \cup \tau' \in X_{\sigma}(j-i)} w_{\sigma \cup \tau'} (\tau' \setminus \tau) \Lambda^\ell_{\sigma} F(\tau') \Lambda^\ell_{\sigma} G(\tau) = \\
= \frac{1}{j-i} \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j-i-1)} \frac{w(\sigma \cup \tau)}{(j+1)} \sum_{\tau' \in X_{\sigma}(j-i-1) \tau \cup \tau' \in X_{\sigma}(j-i)} w_{\sigma \cup \tau'} (\tau' \setminus \tau) \Lambda^\ell_{\sigma} F(\tau') \Lambda^\ell_{\sigma} G(\tau) = \\
= \frac{1}{j-i} \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j-i-1)} \frac{1}{(j+1)} \sum_{\tau' \in X_{\sigma}(j-i-1) \tau \cup \tau' \in X_{\sigma}(j-i)} w(\sigma) w_{\sigma \cup \tau'} (\tau' \setminus \tau) F(\sigma \cup \tau') G(\sigma \cup \tau) = \\
= \frac{1}{j-i} \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j-i-1)} \frac{1}{(j+1)} \sum_{\tau' \in X_{\sigma}(j-i-1) \tau \cup \tau' \in X_{\sigma}(j-i)} \left( \sum_{\tau' \subseteq \tau} w(\tau) w_{\tau}(\tau' \setminus \tau) F(\tau') G(\tau) \right) = \\
= \frac{1}{j-i} \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j-i-1)} \frac{1}{(j+1)} \sum_{\tau' \in X(j) \tau \cup \tau' \in X_{\sigma}(j-i) \sigma \subseteq \tau'} \left( \sum_{\tau' \subseteq \tau} w(\tau) w_{\tau}(\tau' \setminus \tau) F(\tau') G(\tau) \right) = \\
= \frac{1}{j-i} \sum_{\sigma \in X(i)} \sum_{\tau \in X_{\sigma}(j-i-1)} \frac{1}{(j+1)} \sum_{\tau' \in X_{\sigma}(j-i-1) \tau \cup \tau' \in X_{\sigma}(j-i) \sigma \subseteq \tau'} \left( \sum_{\tau' \subseteq \tau} w(\tau) w_{\tau}(\tau' \setminus \tau) F(\tau') G(\tau) \right) = \\
= \frac{1}{j-i} \sum_{\tau \in X(j)} \left( \sum_{\tau' \subseteq \tau} w(\tau) G(\tau) \sum_{\tau' \in X(j) \tau \cup \tau' \in X_{\sigma}(j-i) \sigma \subseteq \tau'} w_{\tau}(\tau' \setminus \tau) F(\tau') \right) = \\
= \frac{1}{j-i} \sum_{\tau \in X(j)} \left( \sum_{\tau' \subseteq \tau} w(\tau) G(\tau) \left( (M')^+_{\sigma,j-i-1} F, G \right) \right) = \left\langle (M')^+_{\sigma,j-i-1} F, G \right\rangle
\]

Where (*) follows from:

\[
\frac{1}{j-i} \frac{1}{(j+1)!} = \frac{1}{j-i} \frac{(j-i)!}{(j+1)!} = \frac{1}{j+1} \frac{(j-i)!}{j!} = \frac{1}{j+1} \frac{1}{(j+1)!}
\]

7.1 Gaining the Advantage

Before we present the exact Lemma we use as the advantage step we should expand our understanding of \( \Lambda^\ell \)'s level functions. We will begin by characterising the space of cochains that are orthogonal
to the eigenspace of 1 (i.e. the constants).

**Lemma 7.4.** Let $X$ be a pure $d$-dimensional simplicial complex and let $F \in C^k (F; \mathbb{R})$ be a cochain. Then:

$$\langle F, 1 \rangle = 0 \iff F \in \ker (d_{k-1}^* \cdots d_{k}^*)$$

**Proof.** Note that, due to Lemma 2.5 it holds that:

$$\langle F, 1 \rangle = \mathbb{E}_{\sigma \in X(k)} [F(\sigma)] = d_{k-1}^* \cdots d_{k}^*(\emptyset)$$

Which proves the lemma.

Now that we understand the constant part of a cochain we are ready to move on to understanding cochains of a higher level.

**Lemma 7.5.** Let $X$ be a pure $d$-dimensional simplicial complex, $i$ be a dimension and $F \in C^k (X; \mathbb{R})$ be a cochain. Then:

$$\forall \sigma \in X(i) : \left< \Lambda_{\sigma}^i F, 1 \right>_{\sigma} = 0 \iff F \in \ker (d_i^* \cdots d_{k-1}^*)$$

**Proof.** Using Lemma 2.5 we prove that:

$$\forall \sigma \in X(i) : \left< \Lambda_{\sigma}^i F, 1 \right>_{\sigma} = \sum_{\tau \in X_{\sigma}(0)} w_v (\tau) \Lambda_{\sigma}^i F(\tau) = d_{i-1}^* \cdots d_{k-2}^* F_v(\emptyset) = (d_i^* \cdots d_{k-1}^* F)_{v}(\emptyset)$$

Which proves the Lemma.

**Corollary 7.6.** It holds that $F$ is a proper $i$-level cochain iff $F \in \text{Im} (d_i \cdots d_0) \cap \ker (d_{i-1}^* \cdots d_{i}^*)$.

**Proof.** The Corollary holds due the definition of $i$-level cochains and Lemma 7.5.

Note that cochains that pure $i$-level cochains can be thought of as originating in the $i$-dimensional faces. For example, for every pure 0-level cochain there is a 0-dimensional cochain $G$ such that $F = d \cdots d F$. We also note that any cochain that is not originated in the vertices can be distributed along the links in the sense that they remain orthogonal to the constants when applying the localization link viewer. We will therefore be interested in the cochains that originated in the vertices (as these are exactly the cochains which the local perspective seems to miss). Consider the following:

**Lemma 7.7.** Let $F$ be a $k$-dimensional proper 0-level cochain then there exists $F=0 \in C^0 (X; \mathbb{R})$ such that:

1. $d_{k-1}^* F=0 = 0$
2. $\|F\|^2 = \|F=0\|^2$
3. $\|d_0^* \cdots d_{k-1}^* F\|^2 = \|d_{k-1} \cdots d_0 F=0\|^2$
Proof. \( F \in \text{Im}(d_{k-1} \cdots d_0) \) therefore let \( G \in C^0(X; \mathbb{R}) \) such that \( F = d_{k-1} \cdots d_0 G \). Note that \( M_0^{+k} \) is a self adjoint positive semidefinite operator and thus its square root can be defined. \( \sqrt{M_0^{+k}} \) is defined as the operator whose eigenvectors are the same as \( M_0^{+k} \) and whose eigenvalue are the positive square root of the eigenvalues of \( M_0^{+k} \). Note that \( \sqrt{M_0^{+k}} \) is also self adjoint. We will now show that \( F = 0 = \sqrt{M_0^{+k}} G \) satisfies all of the conditions of the lemma.

1. Note that this condition is equivalent to showing that \( \langle F = 0, 1 \rangle = 0 \). Also note that \( 1 \) is an eigenvector of \( \sqrt{M_0^{+k}} \) with eigenvalue 1. It is therefore not hard to see that:

\[
\langle F = 0, 1 \rangle = \langle \sqrt{M_0^{+k}} G, \sqrt{M_0^{+k}} 1 \rangle = \langle M_0^{+k} G, 1 \rangle = \langle d_{k-1} \cdots d_0 G, d_0^* \cdots d_{k-1}^* 1 \rangle = \langle F, 1 \rangle = 0
\]

2. Consider the following:

\[
\|F = 0\|^2 = \langle F = 0, F = 0 \rangle = \langle \sqrt{M_0^{+k}} G, \sqrt{M_0^{+k}} G \rangle = \langle M_0^{+k} G, G \rangle = \langle d_{k-1} \cdots d_0 G, d_{k-1} \cdots d_0 G \rangle = \langle F, F \rangle = \|F\|^2
\]

3. To conclude, note that:

\[
\|d_0^* \cdots d_{k-1}^* F\|^2 = \langle d_0^* \cdots d_{k-1}^* F, d_0^* \cdots d_{k-1}^* F \rangle = \langle d_0^* d_k^* \cdots d_0^* d_{k-1}^* d_0 \cdots d_{k-1} \cdots d_0 F, d_0^* d_k^* \cdots d_0^* d_{k-1}^* d_0 \cdots d_{k-1} \cdots d_0 G \rangle = \langle M_0^{+k} G, M_0^{+k} G \rangle = \langle \sqrt{M_0^{+k}} \sqrt{M_0^{+k}} G, \sqrt{M_0^{+k}} \sqrt{M_0^{+k}} G \rangle = \langle \sqrt{M_0^{+k}} F = 0, \sqrt{M_0^{+k}} F = 0 \rangle = \langle M_0^{+k} F = 0, F = 0 \rangle = \langle d_{k-1} \cdots d_0 F = 0, d_{k-1} \cdots d_0 F = 0 \rangle = \|d_{k-1} \cdots d_0 F = 0\|^2
\]

We are now ready to present the advantage we use:

**Lemma 7.8** (The advantage). Let \( X \) be a \( d \) dimensional simplicial complex whose 1-skeleton is a \( \gamma \) spectral expander. Also let \( F \in C^k_{\Lambda_1, 0}(X; \mathbb{R}) \) then:

\[
\|d_0^* \cdots d_{k-1}^* F\|^2 \leq \left( 1 - \frac{k}{k+1} (1 - \gamma) \right) \|F\|^2
\]
Proof. Let $F_0$ be the projection of $F$ into $C^k_{\alpha,0}(X; \mathbb{R}) \cap \left( C^k_{\alpha,1}(X; \mathbb{R}) \right)^\perp$. Due to Lemma 7.5, it holds that for every dimension $i$ that $C^k_{\alpha,i}(X; \mathbb{R}) = \ker (d^*_{k-1} \cdots d^*_{k-1})$ and therefore $F_0 \in \text{Im} (d_{k-1} \cdots d_0) \cap \ker (d^*_{k-1} \cdots d^*_{k-1})$. We can therefore use Lemma 7.7 to find $F^=0$ such that:

1. $d^*_{k-1}F^=0 = 0$
2. $\|F_0\|^2 = \|F^=0\|^2$
3. $\|d^*_0 \cdots d^*_{k-1}F_0\|^2 = \|d_{k-1} \cdots d_0F^=0\|^2$

Due to Lemma 7.8, it holds that:

$$M^+_{0,k} = \frac{k}{k+1} (M^+_0)^+ + \frac{1}{k+1} I$$

And therefore:

$$\|d_{k-1} \cdots d_0F^=0\|^2 = \langle d_{k-1} \cdots d_0F^=0, d_{k-1} \cdots d_0F^=0 \rangle = \langle M^+_{0,k}F^=0, F^=0 \rangle =$$

$$= \left\langle \left( \frac{k}{k+1} (M^+_0)^+ - \frac{1}{k+1} I \right) F^=0, F^=0 \right\rangle = \frac{k}{k+1} \left\langle (M^+_0)^+ F^=0, F^=0 \right\rangle + \frac{1}{k+1} \langle F^=0, F^=0 \rangle \leq$$

$$\leq \left( \frac{k}{k+1} \gamma + \frac{1}{k+1} \right) \|F^=0\|^2 = \left( \frac{k}{k+1} \gamma - \frac{k}{k+1} + 1 \right) \|F^=0\|^2 =$$

$$= \left( 1 - \frac{k}{k+1} (1 - \gamma) \right) \|F^=0\|^2 = \left( 1 - \frac{k}{k+1} (1 - \gamma) \right) \|F_0\|^2$$

And thus:

$$\|d^*_0 \cdots d^*_{k-1}F\|^2 = \|d^*_0 \cdots d^*_{k-1}F_0\|^2 \leq \left( 1 - \frac{k}{k+1} (1 - \gamma) \right) \|F_0\|^2 \leq \left( 1 - \frac{k}{k+1} (1 - \gamma) \right) \|F\|^2$$

\[\square\]

7.2 Decomposing the Random Walk Operators

Now that we have developed the tools we need, we can move on to strengthening the result of Alev and Lau [AL20] by showing a decomposition of the random walk operators. We will do so by applying Theorem 5.8 to the localization link viewer:

**Theorem 7.9 (Random walk decomposition).** Let $X$ be a $d$-dimensional pure simplicial complex. Also, assume that for every face $\sigma$ of dimension $d-2$ it holds that $\lambda_2 \left( (M^+_\sigma)^+ \right) \leq \chi_\sigma$. Denote by $\gamma_{\tau, i} = \max_{\sigma \in X_{i+}} (\lambda_\sigma)$. For every set of proper level cochains $F_i \in C^k_{\alpha, i}(X; \mathbb{R})$ it holds that:

$$\left\langle (M^+_k)^k \sum_{i=0}^k F_i, \sum_{i=0}^k F_i \right\rangle \leq \sum_{i=0}^k \left( 1 - \frac{1}{k-i+1} \prod_{j=i-1}^{k-1} (1 - \gamma_j) \right) \|F_i\|^2$$
Proof. We will prove this theorem by applying Theorem \[5.8\] to the \( k \)-dimensional non-lazy random walk operator. We start by noting that the space of \( k \)-dimensional cochains that are orthogonal to the constants is comprised of exactly \( k \) level functions as the space orthogonal to the constants is exactly \( \ker (d_0^* \cdots d_k^*) \).

We prove the rest of this theorem using a recursive argument. First note that for \( k = 0 \) the claim holds trivially as:

\[
\left\langle (M')^+ F, F \right\rangle \leq \lambda_\sigma \| F \|^2 = \gamma_{-1} \| F \|^2
\]

Assume that for every non-empty face \( \sigma \) it holds that:

\[
\lambda_{\sigma, i, k} \leq 1 - \frac{1}{k - i + 1} \prod_{j=i-1}^{k-1} (1 - \gamma_{\sigma, j})
\]

Note that for every \( i \geq 1 \):

\[
\lambda_{\emptyset, i, k} = \max_{\nu \in X(0)} \{ \lambda_{\nu, i-1, k-1} \} \leq \max_{\nu \in X(0)} \left\{ 1 - \frac{1}{k - i + 1} \prod_{j=i-2}^{k-2} (1 - \gamma_{\sigma, j}) \right\} \leq 1 - \frac{1}{k - i + 1} \prod_{j=i-1}^{k-1} (1 - \gamma_{\emptyset, j})
\]

Consider the left hand side of the recursive formula:

\[
\lambda_{\emptyset, 1, k} \| F_0 \|^2 + \mathbb{E}_{\nu \in X(0)} \left[ (1 - \lambda_{\emptyset, 1, k}) \left\| M_{(k-1)}^{-\ell} \Lambda_\nu^\ell F_0 \right\|^2 \right] \leq \lambda_{\emptyset, 0, k} \| F_0 \| \quad (2)
\]

And note that:

\[
\lambda_{\emptyset, 1, k} \| F_0 \|^2 + \mathbb{E}_{\nu \in X(0)} \left[ (1 - \lambda_{\emptyset, 1, k}) \left\| M_{(k-1)}^{-\ell} \Lambda_\nu^\ell F_0 \right\|^2 \right] = \lambda_{\emptyset, 1, k} \| F_0 \|^2 + \mathbb{E}_{\nu \in X(0)} \left[ (1 - \lambda_{\emptyset, 1, k}) \left\| d_0^* \cdots d_{k-2}^* \Lambda_\nu^\ell F_0 \right\|^2 \right] = \lambda_{\emptyset, 1, k} \| F_0 \|^2 + (1 - \lambda_{\emptyset, 1, k}) \| d_0^* \cdots d_{k-1}^* F_0 \|^2
\]

Consider, again, the left hand side of inequality \[2\] and note that due to Lemma \[7.8\] it suffices to solve the following:

\[
\lambda_{\emptyset, 1, k} \| F_{=0} \|^2 + (1 - \lambda_{\emptyset, 1, k}) \left( 1 - \frac{k}{k+1} (1 - \gamma_{-1}) \right) \| F_{=0} \|^2 \leq \lambda_{\emptyset, 0, k} \| F_{=0} \|^2
\]

\[
\lambda_{\emptyset, 1, k} + (1 - \lambda_{\emptyset, 1, k}) \left( 1 - \frac{k}{k+1} (1 - \gamma_{-1}) \right) \leq \lambda_{\emptyset, 0, k}
\]

\[
\frac{k}{k+1} (1 - \gamma_{-1}) \lambda_{\emptyset, 1, k} + \left( 1 - \frac{k}{k+1} (1 - \gamma_{-1}) \right) \leq \lambda_{\emptyset, 0, k}
\]

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Consider the following:

\[
\frac{k}{k + 1} (1 - \gamma_{-1}) \lambda_{0,1,k} + \left(1 - \frac{k}{k + 1} (1 - \gamma_{-1})\right)
\leq \frac{k}{k + 1} (1 - \gamma_{-1}) \left(1 - \frac{1}{k} \prod_{j=0}^{k-1} (1 - \gamma_{0,j})\right) + 1 - \frac{k}{k + 1} (1 - \gamma_{-1})
\]

\[
= \frac{k}{k + 1} (1 - \gamma_{-1}) - \frac{1}{k + 1} \prod_{j=-1}^{k-1} (1 - \gamma_{0,j}) + 1 - \frac{k}{k + 1} (1 - \gamma_{-1})
\]

\[
= 1 - \frac{1}{k + 1} \prod_{j=-1}^{k-1} (1 - \gamma_{0,j})
\]

Thus if we set:

\[
\lambda_{0,0,k} = 1 - \frac{1}{k + 1} \prod_{j=-1}^{k-1} (1 - \gamma_{0,j})
\]

We get that for every face \(\sigma\) and dimensions \(i, k\):

\[
\lambda_{\sigma,i,k} \leq 1 - \frac{1}{k - i + 1} \prod_{j=i-1}^{k-1} (1 - \gamma_{\sigma,j})
\]

Applying Theorem 5.8 proves the decomposition.

\[\square\]

8 Pseudorandom Cochains

As we have explored in Section 5 we are interested in examining cochains that originate, in a sense, from high dimensions (as these cochains are correlated to smaller eigenvalues of the walk operator). In this section we will be interested in examining cochains that are close to originating in the \(i\)th dimension. Specifically, we will consider cochains that satisfy the following definition:

**Definition 8.1 (\(\Lambda\)-pseudorandom cochain).** Given a link viewer \(\Lambda\), a cochain \(F\) is said to be \((\epsilon_0, \cdots, \epsilon_l)\)-\(\Lambda\)-pseudorandom if for all \(0 \leq i \leq l\) if:

\[
\forall \sigma \in X(i) : |\langle \Lambda_\sigma F - \mathbb{E}[F], 1 \rangle_\sigma| \leq \epsilon_i
\]

When \(\epsilon = \epsilon_0 = \cdots = \epsilon_l\) we would call the cochain \((\epsilon, l)\)-pseudorandom.

It is clear that if a cochain satisfies the above definition then it is indeed close to originating in the \(i\)th dimension. One key property of \(\Lambda\)-pseudorandom cochains is that they remain pseudorandom even when a constant is added to them.

**Lemma 8.2.** If \(F\) is a \((\epsilon_0, \cdots, \epsilon_l)\)-\(\Lambda\)-pseudorandom cochain then so is \(F - c\) for any constant \(c\).

**Proof.** The proof follows by noting that that the localization of a constant cochain is constant:

\[
\forall \sigma \in X(i) : |\langle \Lambda_\sigma (F - c) - \mathbb{E}[F - c], 1 \rangle_\sigma| = |\langle \Lambda_\sigma F - \mathbb{E}[F] + c, 1 \rangle_\sigma| = |\langle \Lambda_\sigma F - \mathbb{E}[F], 1 \rangle_\sigma| \leq \epsilon_i
\]

\[\square\]
We are now ready to prove that $\Lambda^\ell$-pseudorandomness is equivalent to the definition of $l_\infty$-pseudorandomness that was defined in [BHKL20].

**Lemma 8.3.** A cochain $F \in C^k(X; \mathbb{R})$ is $(\varepsilon_0, \ldots, \varepsilon_l)$-$\Lambda^\ell$-pseudorandom iff for all $0 \leq i \leq l$ it holds that:

\[ \forall \sigma \in X(i) : |d_i^\sigma \cdots d_{k-1}^\sigma F(\sigma) - \mathbb{E} [F]| \leq \varepsilon_i \]

**Proof.** Let $\sigma \in X(i)$. Consider the following:

\[ d_i^\sigma \cdots d_{k-1}^\sigma F(\sigma) = \mathbb{E}_{\tau \in X_{\sigma(k-i)}} \left[ \Lambda^\ell_\sigma F(\tau) \right] = \left\langle \Lambda^\ell_\sigma F, 1 \right\rangle_\sigma \]

And thus:

\[ d_i^\sigma \cdots d_{k-1}^\sigma F(\sigma) - \mathbb{E} [F] = \left\langle \Lambda^\ell_\sigma F - \mathbb{E} [F], 1 \right\rangle_\sigma \]

Proving the Lemma. \(\square\)

**Theorem 8.4.** Let $F$ be a $k$-dimensional $(\varepsilon, i)$-$\Lambda$-pseudorandom cochain such that $\mathbb{E}_{\sigma \in X(k)} [F(\sigma)] = 0$ then:

\[ \left\langle (M')^+_k F, F \right\rangle \leq \lambda_{\theta, i, k} \|F\|^2 + \varepsilon^2 \]

**Proof.** Consider the following:

\[
\left\langle (M')^+_k F, F \right\rangle = \mathbb{E}_{\sigma \in X(i)} \left[ \left\langle (M')^+_k \Lambda_\sigma F, \Lambda_\sigma F \right\rangle \right]
\]

\[
= \mathbb{E}_{\sigma \in X(i)} \left[ \left\langle (M')^+_\sigma,k \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F, \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F \right\rangle + \left\langle (M')^+_\sigma,k M_{k-i-1}^{-}(k-i-1) \Lambda_\sigma F, M_{k-i-1}^{-}(k-i-1) \Lambda_\sigma F \right\rangle \right]
\]

\[
= \mathbb{E}_{\sigma \in X(i)} \left[ \left\langle (M')^+_\sigma,k \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F, \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F \right\rangle + \left\langle M_{k-i-1}^{-}(k-i-1) \Lambda_\sigma F, M_{k-i-1}^{-}(k-i-1) \Lambda_\sigma F \right\rangle \right]
\]

\[
\leq \mathbb{E}_{\sigma \in X(i)} \left[ \left\langle (M')^+_\sigma,k \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F, \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F \right\rangle + \left\langle \mathbb{E}_{\tau \in X_{\sigma(k-i)}} [\Lambda_\sigma F(\tau)] 1, \mathbb{E}_{\tau \in X_{\sigma(k-i)}} [\Lambda_\sigma F(\tau)] 1 \right\rangle \right]
\]

\[
\leq \mathbb{E}_{\sigma \in X(i)} \left[ \left\langle (M')^+_\sigma,k \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F, \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F \right\rangle + \varepsilon^2 \right]
\]

We use this observation together with Theorem [7.9] to conclude that:

\[ \left\langle (M')^+_\sigma,k \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F, \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F \right\rangle \leq \lambda_{\sigma,0,k-i} \left\| \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F \right\|^2 \]

We note that:

\[ \left\| \left( I - M_{k-i-1}^{-}(k-i-1) \right) \Lambda_\sigma F \right\|^2 \leq \| \Lambda_\sigma F \|^2 \]

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Therefore:
\[
\left( (M')^+_k F, F \right) \leq \mathbb{E}_{\sigma \in X(i)} \left[ \left( (M')^+_{\sigma,k} \left( I - M_{k-i-1}^{-1} \right) \Lambda_{\sigma} F, \left( I - M_{k-i-1}^{-1} \right) \Lambda_{\sigma} F \right) + \epsilon^2 \right]
\]
\[
= \mathbb{E}_{\sigma \in X(i)} \left[ \left( (M')^+_{\sigma,k} \left( I - M_{k-i-1}^{-1} \right) \Lambda_{\sigma} F, \left( I - M_{k-i-1}^{-1} \right) \Lambda_{\sigma} F \right) + \epsilon^2 \right]
\]
\[
\leq \mathbb{E}_{\sigma \in X(i)} \left[ \left\| (I - M_{k-i-1}^{-1}) \Lambda_{\sigma} F \right\|^2 + \epsilon^2 \right]
\]
\[
\leq \lambda_{\emptyset,i,k} \mathbb{E}_{\sigma \in X(i)} \left[ \left\| \Lambda_{\sigma} F \right\|^2 + \epsilon^2 \right]
\]
\[
\leq \lambda_{\emptyset,i,k} \left\| F \right\|^2 + \epsilon^2
\]
\[
\square
\]

**Lemma 8.5.** It holds, for every cochain \( F \), that:
\[
\| F \|_\infty \leq \| F \|_2
\]

**Proof.** Consider the following:
\[
\| F \|_\infty = \max_{\sigma \in X(k)} \{ w(\sigma) | F(\sigma) | \} = \max_{\sigma \in X(k)} \left\{ w(\sigma) \sqrt{(F(\sigma))^2} \right\} = \max_{\sigma \in X(k)} \left\{ \sqrt{w(\sigma)^2 (F(\sigma))^2} \right\}
\]
\[
\leq \max_{\sigma \in X(k)} \left\{ \sqrt{w(\sigma) (F(\sigma))^2} \right\} \leq \sqrt{\sum_{\sigma \in X(k)} w(\sigma) (F(\sigma))^2} = \sqrt{\langle F, F \rangle} = \| F \|_2
\]
\[
\square
\]

This gives us a tool with which we can describe the expansion of small pseudorandom sets:

**Lemma 8.6.** Let \( S \) be a set of \( k \)-dimensional faces such that \( 1_S \) is \( (\epsilon \| 1_S \|_\infty, i) \)-\( \Lambda \)-pseudorandom cochain then it holds that:
\[
\Phi(S) \geq 1 - \mathbb{E}_{\sigma \in X(k)} [1_S(\sigma)] - \lambda_{\emptyset,i,k} (1 - \mathbb{E}_{\sigma \in X(k)} [1_S(\sigma)]) - \epsilon^2
\]

**Proof.** Denote by \( \alpha = \mathbb{E}_{\sigma \in X(k)} [1_S(\sigma)] \). Using Lemma 8.4 we note that:
\[
\left( (M')^+_k (1_S - \alpha), 1_S - \alpha \right) \leq \lambda_{\emptyset,i,k} \| 1_S - \alpha \|^2 + \epsilon^2 \| 1_S \|^2 \leq
\]
\[
\leq \lambda_{\emptyset,i,k} \left( \| 1_S \|^2 - \| \alpha \|^2 \right) + \epsilon^2 \| 1_S \|^2 \| 1_S \|^2 = \lambda_{\emptyset,i,k} (\alpha - \alpha^2) + \epsilon^2 \| 1_S \|^2 \| 1_S \|^2
\]

Also note that:
\[
\left( (M')^+_k 1_S, 1_S \right) = \left( (M')^+_k (1_S - \alpha + \alpha), 1_S - \alpha + \alpha \right) = \left( (M')^+_k (1_S - \alpha), 1_S - \alpha \right) + \langle \alpha, \alpha \rangle
\]
Therefore:

\[
\frac{1}{\alpha} \left\langle (M'_{k})^{+} 1_{S}, 1_{S} \right\rangle = \frac{1}{\alpha} \left( \left\langle (M'_{k})^{+} (1_{S} - \alpha), 1_{S} - \alpha \right\rangle + \langle \alpha, \alpha \rangle \right) \\
= \frac{1}{\alpha} \left\langle (M'_{k})^{+} (1_{S} - \alpha), 1_{S} - \alpha \right\rangle + \alpha \\
\leq \lambda_{\emptyset, i, k} \left( \frac{\alpha - \alpha^2}{\alpha} + \epsilon^2 \|1_{S}\|_{\infty}^2 \right) + \alpha \\
= \lambda_{\emptyset, i, k} (1 - \alpha) + \frac{\epsilon^2 \|1_{S}\|_{\infty}^2}{\alpha} + \alpha \\
\leq \lambda_{\emptyset, i, k} (1 - \alpha) + \frac{\epsilon^2 \alpha}{\alpha} + \alpha \\
= \lambda_{\emptyset, i, k} (1 - \alpha) + \epsilon^2 + \alpha
\]

And thus:

\[
\Phi(S) = 1 - \frac{1}{\alpha} \left\langle (M'_{k})^{+} 1_{S}, 1_{S} \right\rangle \geq 1 - \alpha - \lambda_{\emptyset, i, k} (1 - \alpha) - \epsilon^2
\]

\[\square\]

**Corollary 8.7.** Let \( S \) be a set of \( k \)-dimensional faces such that \( 1_{S} \) is \((\epsilon \|1_{S}\|_{\infty}, i)\)-\( \Lambda^\ell \)-pseudorandom cochain then it holds that:

\[
\Phi(S) \geq 1 - \mathbb{E}_{\sigma \in X(k)} [1_{S}(\sigma)] - \left( 1 - \frac{1}{k - i + 1} \prod_{j=i-1}^{k-1} (1 - \gamma_j) \right) \left( 1 - \mathbb{E}_{\sigma \in X(k)} [1_{S}(\sigma)] \right) - \epsilon^2
\]

**Proof.** Applying Lemma 8.6 to \( \Lambda^\ell \) proves the Corollary. \[\square\]

**References**

[AL20] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. *CoRR*, abs/2001.02827, 2020.

[ALGV18] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials ii: High-dimensional walks and an fpras for counting bases of a matroid, 2018.

[BHKL20] Mitali Bafna, Max Hopkins, Tali Kaufman, and Shachar Lovett. High dimensional expanders: Eigenstripping, pseudorandomness, and unique games, 2020.

[DHK+19] Irit Dinur, Prahladh Harsha, Tali Kaufman, Inbal Livni Navon, and Amnon Ta-Shma. List decoding with double samplers. In Timothy M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 2134–2153. SIAM, 2019.
Irit Dinur and Tali Kaufman. High dimensional expanders imply agreement expanders.
In Chris Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 974–985. IEEE Computer Society, 2017.

Tali Kaufman and David Mass. High dimensional random walks and colorful expansion.
In Christos H. Papadimitriou, editor, *8th Innovations in Theoretical Computer Science Conference, ITCS 2017, January 9-11, 2017, Berkeley, CA, USA*, volume 67 of *LIPIcs*, pages 4:1–4:27. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.

Tali Kaufman and David Mass. Local-to-global agreement expansion via the variance method.
In Thomas Vidick, editor, *11th Innovations in Theoretical Computer Science Conference, ITCS 2020, January 12-14, 2020, Seattle, Washington, USA*, volume 151 of *LIPIcs*, pages 74:1–74:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

Tali Kaufman and Izhar Oppenheim. High order random walks: Beyond spectral gap.
In Eric Blais, Klaus Jansen, José D. P. Rolim, and David Steurer, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2018, August 20-22, 2018 - Princeton, NJ, USA*, volume 116 of *LIPIcs*, pages 47:1–47:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

Izhar Oppenheim. Local spectral expansion approach to high dimensional expanders part i: Descent of spectral gaps, 2017.