On a theorem of Wiegerinck

Róbert Szőke
Department of Analysis, Institute of Mathematics
ELTE Eötvös Loránd University
Pázmány Péter sétány 1/c, Budapest 1117, Hungary
E-mail: rszoke@caesar.elte.hu
ORCiD:0000-0002-8723-1068

Abstract

A theorem of Wiegerinck says that the Bergman space over any domain in \( \mathbb{C} \) is either trivial or infinite dimensional. We generalize this theorem in the following form. Let \( E \) be a Hermitian, holomorphic vector bundle over \( \mathbb{P}^1 \), the later equipped with a volume form and \( D \) an arbitrary domain in \( \mathbb{P}^1 \). Then the space of holomorphic \( L^2 \) sections of \( E \) over \( D \) is either equal to \( H^0(M,E) \) or it has infinite dimension.

1 Introduction

Let \( D \) be a domain in \( \mathbb{C}^n \) and \( \mathcal{O}L^2(D) \) the Bergman space, that is the space of those holomorphic functions on \( D \) that also belong to \( L^2(D) \), with respect to the Lebesgue measure. Wiegerinck proved in [W84], that for \( n = 1 \), and \( D \) any domain in \( \mathbb{C} \), either \( \mathcal{O}L^2(D) \) is trivial (= \{0\}) or has infinite dimension. In the same paper for any \( k > 0 \) he constructed a Reinhardt domain in \( \mathbb{C}^2 \) with \( k \)-dimensional Bergman space. These examples were not logarithmically convex, hence they were not Stein domains. These results give rise to the following natural conjecture.

Conjecture 1.1. (Wiegerinck) Let \( D \) be a Stein domain in \( \mathbb{C}^n \), then its Bergman space is either trivial or infinite dimensional.
Despite of a lot of work and partial results (cf. [BZ20, D07, GH17, J12, PW07, PW17], as far as we can tell, the conjecture is still open.

The purpose of this paper is to show that there is another very natural direction in which Wiegerinck’s result could be generalized. More precisely we mean to consider the following situation. Let $M$ be a compact, connected Riemann surface, $E \to M$ a holomorphic vector bundle equipped with a smooth Hermitian metric $h$, and $dV$ a smooth volume form on $M$. For a domain $D \subset M$, denote by $\mathcal{O}L^2(E|_D)$ the set of those holomorphic sections of $E$ over $D$ for which

$$
\|s\|^2 := \int_D h(s, s) dV < \infty.
$$

As is well known $\mathcal{O}L^2(E|_D)$ is a Hilbert space and the vector space $H^0(M, E)$ of global holomorphic sections of $E$ is finite dimensional ([GH]). Clearly $H^0(M, E) \subset \mathcal{O}L^2(E|_D)$.

**Conjecture 1.2.** $\mathcal{O}L^2(E|_D)$ is either equal to $H^0(M, E)$ or it has infinite dimension.

Supporting this conjecture we have the following observations. First of all Wiegerinck’s result in [W84] implies that Conjecture 1.2 is true when $M = \mathbb{P}^1$ and $E$ is the canonical bundle of $\mathbb{P}^1$. It is also true when $M$ and $E$ are arbitrary and $K$ is either small, i.e., locally polar (cf. Proposition 2.1) or $K$ is big, i.e., has nonempty interior (cf. Proposition 2.2).

Our main result is that for $M = \mathbb{P}^1$, Conjecture 1.2 holds.

**Theorem 1.3.** Let $M = \mathbb{P}^1$ and $D$ and $E$ be arbitrary. Then Conjecture 1.2 is true.

## 2 Proofs

Suppose $M$ is an arbitrary Riemann surface. Recall that a subset $K \subset M$ is called locally polar, if for each point $p \in K$, there exists a connected neighborhood $U$ of $p$ and a subharmonic function $\varphi \not\equiv -\infty$ on $U$ such that
$K \cap U \subset \{ y \in U | \varphi(y) = -\infty \}$. Now back to our original setting, let us assume that $M$ is a compact Riemann surface, $(E, h) \to M$ a holomorphic Hermitian vector bundle over $M$, $dV$ a smooth volume form on $M$, $D \subset M$ a domain and $K = M \setminus D$. When $K$ is small, every element of $\mathcal{O}L^2(E|_D)$ extends holomorphically to a global section of $E$ yielding:

**Proposition 2.1.** Suppose $K$ is locally polar. Then $\mathcal{O}L^2(E|_D) = H^0(M, E)$.

The special case when $M$ is $\mathbb{P}^1$ and $E$ is its canonical bundle, is a reformulation of a result of Carleson in [Ca67]. Indeed Carleson proves that in this case the Bergman space over $D = \mathbb{P}^1 \setminus K$ is trivial. On the other hand the canonical bundle of $\mathbb{P}^1$ is negative so its only holomorphic section is the zero section, i.e., $H^0(M, E)$ is trivial as well. The general case is also essentially known, but for the readers’ sake we include a proof here.

**Proof.** We need to show that each element of $\mathcal{O}L^2(E|_D)$ extends holomorphically to yield a global holomorphic section of $E$. It suffices to prove local extension. But locally the bundle $E$ is trivial and the volume form of $M$ is equivalent to the Euclidean volume form (in a coordinate neighborhood). So the problem reduces to an extension problem on holomorphic functions.

Let $W \subset \mathbb{C}$ be open, $X \subset W$ a closed, locally polar set in $W$ and $f$ a holomorphic $L^2$ function on $W \setminus X$. We need to show that for each point $p \in X$, $f$ extends holomorphically to a neighborhood of $p$. According to [Co95] Ch. 21, Proposition 5.5], $X$ is a polar subset of $\mathbb{C}$. In light of [Co95, Ch. 21, Theorem 9.5 (c)], it suffices to find an open set $V$, that contains $p$, so that $K = V \cap X$ is compact. To find such a set, one can argue as follows. Since $X$ is a polar subset of $\mathbb{C}$, by [R95 Corollary 3.8.5], $X$ is a totally disconnected space. Now $X$ is closed in $W$, hence it is also locally compact. Let $F \subset X$ be a compact neighborhood of $p$ and $U \subset X$ open with $p \in U \subset F$. A general result on totally disconnected, locally compact spaces ([E89 Theorem 6.2.9]) tells us that there exists a set $K \subset U$ that is both open and closed in $X$ and $p \in K$. Let $V \subset \mathbb{C}$ be open with $V \cap X = K$ (can assume $V \subset W$). Since $K$ is a closed subset of a compact space (namely $F$), it is also compact.

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1 We thank László Fehér for discussions on totally disconnected spaces.
The other extreme case is when $K$ is big in the sense, that its interior is nonempty.

**Proposition 2.2.** Suppose the interior of $K$ is nonempty. Then $\mathcal{O}L^2(E|_D)$ is infinite dimensional.

**Proof.** Let $p$ be an interior point of $K$ and $l$ be the rank of $E$. Let $G = M\setminus \{p\}$. Since $G$ is an open Riemann surface, $E|_G$ is holomorphically trivial ([F81, Theorem 30.4]), hence $H^0(G, E|_G) = Hol(G, \mathbb{C}^l)$, the space of holomorphic maps from $G$ to $\mathbb{C}^l$. $G$ being Stein, the latter space is infinite dimensional. But $\overline{D}$ is a compact subset in $G$, hence all these holomorphic maps (sections), via restriction to $D$, belong to $L^2(E|_D)$.

Denote by $H^\infty(D)$ the set of bounded holomorphic functions on $D$. We shall say that $H^\infty(D)$ is nontrivial, if it contains an element not identically constant.

**Proposition 2.3.** Suppose $H^\infty(D)$ is nontrivial. Then Conjecture 1.2 holds.

**Proof.** Suppose $\exists s \in \mathcal{O}L^2(E|_D)$, with $s \notin H^0(M, E)$. Let $g \in H^\infty(D)$ be a nonconstant element and $w \in D$ be arbitrary. Then $f := g - g(w) \in H^\infty(D)$ vanishes at $w$ with finite, nonzero multiplicity. This yields

$$f^m s \in \mathcal{O}L^2(L|_D), \quad m \in \mathbb{N},$$

implying the infinite dimensionality of $\mathcal{O}L^2(L|_D)$.

For an arbitrary holomorphic line bundle $L \to M$, denote by $\mathcal{M}(L)$ the set of meromorphic sections. Whereas for an open subset $D \subset M$ the set of meromorphic functions on $D$ is denoted by $\mathcal{M}(D)$.

**Proof of Theorem 1.3.** Let $E \to \mathbb{P}^1$ be a holomorphic vector bundle of rank $r$. According to Grothendieck’s theorem ([G57]) $E$ holomorphically splits as a direct sum of holomorphic line bundles:

$$E = L_1 \oplus \cdots \oplus L_r.$$
Therefore it is enough to prove Conjecture 1.2 when \( E = L \) is a holomorphic line bundle. The proof then follows similar reasonings as [W84].

As is well known ([V11, Prop. 4.6.2]), there exists an integer \( k \), so that \( L = O(k) \), where \( O(k) \) denotes the line bundle associated to the divisor \( kp, p \) being an (arbitrary) point in \( \mathbb{P}^1 \). Suppose \( \exists s \in \mathcal{O}L^2(L|_D), \) with \( s \notin H^0(\mathbb{P}^1, L) \).

**Case I.** \( s \in \mathcal{M}(L) \).

Since isolated \( L^2 \) singularities are removable and \( s \in \mathcal{M}(L) \setminus H^0(\mathbb{P}^1, L) \), we get

\[
\int_{\mathbb{P}^1} h(s, s) dV = \infty. \tag{2.1}
\]

Now \( s|_D \) is in \( L^2 \), hence

\[
\lambda(\mathbb{P}^1 \setminus D) > 0, \tag{2.2}
\]

where \( \lambda \) denotes Lebesgue measure on \( \mathbb{P}^1 \). (Having Lebesgue measure zero or positive is well defined on any smooth manifold and does not depend on any choice of a Riemannian metric or a volume form.) From (2.2) and a well known result about the Cauchy transform (cf. [G72, p.2]) we get that \( H^\infty(D) \) is nontrivial, so in light of Proposition 2.3 we are done.

**Case II.** \( s \notin \mathcal{M}(L) \) and \( k \geq 0 \).

Pick an arbitrary point \( \infty \neq z_0 \in D \). Since \( L = O(k) \), \( L \) admits a holomorphic section \( s_L : \mathbb{P}^1 \to L \) with the only possible zero at \( z_0 \), the multiplicity at \( z_0 \) being \( k \). Then

\[
s = f s_L,
\]

with some \( f \in \mathcal{M}(D) \). Here \( f \) must be holomorphic in \( D \) except perhaps at \( z_0 \), where \( f \) may have a pole at most of order \( k \). Since \( s \) is not in \( \mathcal{M}(L) \), \( f \) cannot be in \( \mathcal{M}(\mathbb{P}^1) \). Let \( z_1 \in D \setminus \{ z_0, \infty \} \) be another arbitrary point. Define the function \( g_1 \) by

\[
g_1(z) = (f(z) - f(z_1)) \frac{(z - z_0)}{(z - z_1)}. \tag{2.3}
\]
Claim 2.4. \( s_1 := g_1 s_L \in \mathcal{O}L^2(L|_D) \)

Proof. \( g_1 \) is holomorphic in \( D \setminus \{ z_0 \} \) and in \( z_0 \) it has either a removable singularity or a pole of order at worst \( k - 1 \). Hence \( g_1 s_L \) is a holomorphic section of \( L \) over \( D \). Let \( U \subset D \) be an open subset with \( z_0, z_1 \in U \) and \( \bar{U} \subset D \) compact. Holomorphicity of \( g_1 s_L \) implies that \( g_1 s_L \) is an \( L^2 \) section of \( L \) over \( U \).

Since \((f - f(z_1))s_L\) is an \( L^2 \) section over \( D \) and

\[
\frac{z - z_0}{z - z_1}
\]

is bounded on \( \mathbb{P}^1 \setminus U \), we get that \( g_1 s_L \) is in \( L^2 \) over \( D \setminus U \) as well. Since \( g_1 s_L \) is holomorphic, it is also in \( L^2 \) over the compact set \( \bar{U} \) which proves Claim 2.4.

Back to the proof of our theorem, let now \( N \) be an arbitrary positive integer and \( z_1, z_2, \ldots, z_N \in D \setminus \{ z_0, \infty \} \) be arbitrary different points. Denote the corresponding functions defined by formula (2.3) by \( g_j \) and let \( s_j := g_j s_L \), where \( j = 1, \ldots, N \). Due to Claim 2.4 they all belong to \( \mathcal{O}L^2(L|_D) \).

Claim 2.5. \( s_1, \ldots, s_N \) are linearly independent.

Proof. Indeed if they were linearly dependent, that would yield that \( f \) is a meromorphic function on \( \mathbb{P}^1 \), a contradiction.

Claim 2.5 shows that \( \mathcal{O}L^2(L|_D) \) has infinite dimension when \( k \geq 0 \), finishing the proof of Case II.

Case III. \( L = \mathcal{O}(k), k < 0, s \in \mathcal{O}L^2(D, L), \) and \( s \notin \mathcal{M}(L) \)

Similarly to Case II, pick a point \( \infty \neq z_0 \in D \). \( L = \mathcal{O}(k) \) implies that \( L \) admits a meromorphic section \( m_L \) with \( z_0 \) being the only singularity, a pole of order \( k \) and that \( m_L \) does not vanish on \( \mathbb{P}^1 \setminus \{ z_0 \} \).

Because \( s \in \mathcal{O}L^2(L|_D) \), we can write

\[
s = fm_L,
\]
where \( f \) now is holomorphic in \( D \) but is not in \( \mathcal{M}(\mathbb{P}^1) \). Since \( s \) is holomorphic, \( f \) must vanish at \( z_0 \). Denote by \( l \) its multiplicity (necessarily \( l \geq k \)). Then \( s \) vanishes at \( z_0 \) with multiplicity \( l \).

Let \( N > l \) be an arbitrary positive integer and \( z_1, z_2, \ldots, z_N \in D \setminus \{z_0, \infty\} \) arbitrary but different points. Define the corresponding functions \( g_j \) by formula (2.3). Then all \( g_j \) are holomorphic in \( D \), but \( g_jm_L \) has a pole at \( z_0 \) of order \( k-1 \) if \( f(z_j) \neq 0 \). Let \( \lambda_j \in \mathbb{C} \), \( j = 1, \ldots, N \), \( \lambda = (\lambda_1, \ldots, \lambda_N) \) and define \( g_\lambda \) by

\[
g_\lambda(z) := \sum_{j=1}^{N} \lambda_j g_j(z) = f(z) \left( \sum_{j=1}^{N} \frac{\lambda_j}{z - z_j} \right) (z - z_0) - \left( \sum_{j=1}^{N} \frac{\lambda_j f(z_j)}{z - z_j} \right) (z - z_0).
\]

(2.4)

Since \( f \notin \mathcal{M}(\mathbb{P}^1) \), \( g_\lambda \notin \mathcal{M}(\mathbb{P}^1) \) except if all the \( \lambda_j \)'s are zero. In (2.3) the first term is holomorphic near \( z_0 \) and vanishes at \( z_0 \) with multiplicity at least \( l + 1 \). The function

\[
f_\lambda := \sum_{j=1}^{N} \frac{\lambda_j f(z_j)}{z - z_j}
\]

(2.5)

is also holomorphic near \( z_0 \) and the coefficients of its Taylor series around \( z_0 \) depend linearly on the \( \lambda_j \)'s. Therefore to impose the condition on \( f_\lambda \) to vanish at \( z_0 \) with multiplicity \( l \) means solving a homogeneous system of linear equations in \( N \) unkowns and \( l \) equations. Since \( N > l \), this system will have a nontrivial solution. Let \( \lambda \) be such a solution. Then the corresponding \( g_\lambda \) will vanish at \( z_0 \) with multiplicity at least \( l + 1 \). Hence the holomorphic section \( s_\lambda := g_\lambda m_L \) vanishes at \( z_0 \) with higher multiplicity than \( s \) does. A similar argument as in the proof of Claim (2.4) shows that \( s_\lambda \in \mathcal{O}L^2(L|_D) \). Moreover \( g_\lambda \notin \mathcal{M}(\mathbb{P}^1) \) implies that \( s_\lambda \notin \mathcal{M}(L) \). Hence we can now repeat the whole process to \( s_\lambda \). That yields indeed that \( \mathcal{O}L^2(L|_D) \) has infinite dimension. \( \square \)

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Note added in proof. During the refereeing process, a closely related paper appeared in the arxiv: A-K. Gallagher, P. Gupta and L. Vivas: On the dimension of Bergman spaces on $\mathbb{P}^1$, arXiv:2110.02324v2. Here the authors give potential theoretic characterizations of the dimension of the Bergman space. Furthermore, by generalizing Wiegerinck’s example, they show that in higher dimensions the dichotomy we considered in our paper, does not hold in general for certain line bundles over $\mathbb{P}^2$. 