Dynamical systems made of many coupled units with long-range or global coupling are models of numerous important situations in physics and beyond, ranging from the synchronization of oscillators and neural networks to gravitational systems, plasma, and hydrodynamics [1, 2]. Their properties can be quite remarkable: for instance, globally-coupled dissipative systems can give rise to collective chaos, where macroscopic variables show incessant irregular behavior due to nontrivial correlations between local units [3]. Their Hamiltonian counterparts, in the microcanonical ensemble, are now well-known to show negative specific heat, long-lived quasi-stationary states, all features ultimately related to their non-additivity [1]. Their unusual properties make these systems deceptively simple, at least in many ways, less well understood than systems with short-range interactions.

The status of the chaos which may be present in these dynamical systems is particularly unclear: whereas systems with short-range interactions are now well-known to exhibit extensive chaos [4], at least in the absence of non-trivial collective behavior [3], there is, in our view, no solid evidence or argument for or against the extensivity of chaos in long-range or globally-coupled systems.

In most dynamical systems, chaos is customarily quantified by Lyapunov exponents (LEs), which measure the average rate of divergence of nearby trajectories, and thus chaotic local maps [4].

Eq. (1) is then simplified as

$$\frac{1}{N} \sum_{j=1}^{N} x_j^t = f(x_j^t), \quad y_j^t = (1 - \varepsilon)x_j^t + \frac{\varepsilon}{N} \sum_{j'=1}^{N} x_{j'}^t,$$

with $j = 1, \ldots, N$, time $t$, coupling constant $\varepsilon$, and a chaotic local map $f(x)$, which is chosen here to be one-dimensional for the sake of simplicity. If $f(x)$ shows sufficiently strong mixing, its Jacobian may be approximated by a random multiplier. The tangent-space dynamics of Eq. (1) is then simplified as

$$v_j^t = \mu_j^t \left[ (1 - \varepsilon)v_j^t + \frac{\varepsilon}{N} \sum_{j'=1}^{N} v_{j'}^t \right],$$

where $\mu_j^t$ are the Lyapunov modes of $f(x)$. Using a combination of analytical and numerical techniques, we show that chaos in globally-coupled identical dynamical systems, be they dissipative or Hamiltonian, is both extensive and sub-extensive: their spectrum of Lyapunov exponents is asymptotically flat (thus extensive) at the value $\lambda_0$ given by a single unit forced by the mean-field, but sandwiched between sub-extensive bands containing typically $O(\log N)$ exponents whose values vary as $\lambda \simeq \lambda_\infty + c/\log N$ with $\lambda_\infty \neq \lambda_0$. In most dynamical systems, chaos is customarily quantified by Lyapunov exponents (LEs), which measure the average rate of divergence of nearby trajectories, and thus chaotic local maps [4].
with iid random numbers $\mu^i$, unless the coupling $\varepsilon$ is too strong to regard $f'(y^i_j)$ as independent. The mean-field forcing argument amounts to ignoring the global-coupling term in Eq. (2), which is then reduced to the biased Brownian motion of a particle of coordinate $\log (|v^i_j|)$ with average velocity $\lambda_0 \equiv \langle \log (|1 - \varepsilon|\mu^i) \rangle$ and diffusion coefficient $D \equiv \langle (\log (|1 - \varepsilon|\mu^i - \lambda_0)^2 \rangle$, where $\lambda_0$ is the mean-field LE. From this viewpoint, the full system can be seen as $N$ interacting Brownian particles. Assume now that the Lyapunov vector $[v^1_M, \ldots, v^N_M]$ is sufficiently localized, which is indeed the case except when it is associated with collective behavior [3]. In this case, its largest component $v^i_M$ dominates the coupling term in Eq. (2).

Thus, the Brownian particles log $|v^i_j|$ diffuse freely as long as $|\langle 1 - \varepsilon \rangle v^i_j| \gg |\varepsilon/N|v^i_M|$, otherwise the coupling term takes effect, keeping any $|\langle 1 - \varepsilon \rangle v^i_j|$ larger than $|\langle \varepsilon/N \rangle v^i_M|$. In other words, the $N$ Brownian particles log $|v^i_j|$ diffuse within a box of size $\log [N(1 - \varepsilon)/\varepsilon]$, whose right end corresponds to the rightmost particle, while the other end pulls all the particles left behind. The first LE $\lambda^i(1)$ is then simply given as the average velocity of this box. This process is described by the following Fokker-Planck equation in a frame moving at velocity $\lambda^i(1)$:

$$\frac{\partial}{\partial t} P(u, t) = -\frac{\partial}{\partial u} [(\lambda_0 - \lambda^i(1)) P] + \frac{D}{2} \frac{\partial^2 P}{\partial u^2},$$

(3)

where $u$ is the coordinate in this frame and the particle distribution function $P(u, t)$ is confined, roughly in $0 \leq u \leq u_{\max} \equiv \log [N(1 - \varepsilon)/\varepsilon]$. For large $N$, its stationary solution can be approximated by the one in the limit $u_{\max} \to \infty$, $P_s(u) = (2\Delta \lambda^i(1)/D) \exp(-2\Delta \lambda^i(1)u/D)$ with $\Delta \lambda^i(1) \equiv \lambda^i(1) - \lambda_0$. Further, by the definition of the box, there should be $O(1)$ particles near its right end $u_{\max}$, which implies $\int_{u_{\max}}^\infty P_s(u) du = c_1/N$ with a constant $c_1 \sim O(1)$. This yields our central result for the first LE:

$$\Delta \lambda^i(1) = \lambda^i(1) - \lambda_0 = \frac{D}{2} \left( 1 + \frac{c_2}{\log N} \right) + O \left( \frac{1}{\log^2 N} \right),$$

(4)

with $c_2 \equiv \log \varepsilon/((1 - \varepsilon)c_1)$. The probability distribution $P(v)$ for the vector components $v_j = P(v) = P_s(\log v)(dv/du) \sim u^{-2 - c_2/\log N}$, whose exponent is smaller than $-1$ and thus consistent with our assumption of localization of the Lyapunov vector. A similar result holds for the last LE [11]: $\Delta \lambda^i(N) \equiv \lambda^i(N) - \lambda_0 \simeq -(D/2)(1 + c'_2/\log N)$ with another coefficient $c'_2$.

These results are confirmed in Fig. 1(b) by direct simulations of the random multiplier (RM) model (2) and of globally-coupled maps (GCM) (1) [12].

For the RM model, we used $\varepsilon = 0.1$ and $\mu^i = \pm \exp \xi^i_j/(1 - \varepsilon)$ with random signs (here “+” with probability 0.5) and $\xi^i_j$ drawn from the centered Gaussian with variance $a^2$, which gives $\lambda_0$ and $D = a^2$. Quantitative agreement is found with Eq. (4) for the first LE and its counterpart for the last LE [Fig. 1(a,c)].

For our GCM system, we chose skewed tent maps $f(x) = bx$ (resp. $b(x - 1)/(1 - b)$) if $0 \leq x \leq 1/b$ (resp. $1/b < x \leq 1$) coupled with strength $\varepsilon = 0.02$. The results in Fig. 1(b) demonstrate again the logarithmic size-dependence of $\Delta \lambda^i(1)$ and $\Delta \lambda^i(N)$. Their asymptotic values are not symmetric anymore [Fig. 1(b)], but the difference from $D/2$ remains small [Fig. 1(c)]. In addition, we note that here $\lambda_0$ depends residually on $N$ through changes in the invariant measure. This effect is however so weak that in practice we observe the same logarithmic law for the first and last LEs, $\lambda^i(1)$ and $\lambda^i(N)$.

Let us summarize our results so far: The first and last LEs remain distinct from the mean-field forcing LE $\lambda_0$ in the $N \to \infty$ limit. They are shifted from $\lambda_0$ by an amount $\Delta \lambda$ controlled by $D$, the amplitude of the fluctuations in the Jacobian, and the coupling strength $\varepsilon$ is only involved in the logarithmic finite-size corrections [13].

We now investigate the full Lyapunov spectrum of our systems. As seen above, it cannot be entirely flat at $\lambda_0$ asymptotically. However, for finite-size systems, Lyapunov spectra become flatter for larger sizes under the conventional rescaling $\lambda^i(1)$ vs $h \equiv (i - \frac{1}{2})/N$ [Fig. 2(a)]. In the RM model, a closer look at the “bulk” LEs with fixed $h$ reveals an asymptotic power-law decay $\Delta \lambda(h) \equiv \lambda^i(1) - \lambda_0 \sim 1/\sqrt{N}$ toward the mean-field forcing value $\lambda_0 = 0$ [Fig. 2(b)]. This scaling is only reached for large-enough sizes and sooner near the middle of the spectrum, as shown clearly by rescaled spectra $\Delta \lambda/\sqrt{N}$ [Fig. 2(c)]: they collapse very well within a central region $[h_c(N), 1 - h_c(N)]$, with $h_c(N)$ decreasing toward zero as $1/N$ [Fig. 2(c) arrows and Fig. 2(d)]. Thus, in the infinite-size limit, the Lyapunov spectrum of the RM model is indeed flat at $\lambda_0$, but sandwiched between two sub-extensive bands of LE taking different values [14].

That $h_c \sim 1/N$ [Fig. 2(d)] implies that the number of non-extensive LEs increases slower than any power of $N$.\[FIG. 1: (color online). Size-dependence of the first and last LEs [12]. (a,b): $\Delta \lambda^i(1)$ and $|\Delta \lambda(N)|$ against $1/\log N$ for the RM model with $a = 1$ (a) and for the skewed-tent GCM with $b = 4$ (b). Dashed lines indicate linear fits to the data. (c) Estimated value of $\Delta \lambda^i(1) = \lim_{N \to \infty} \Delta \lambda^i(1)$ for the RM model and our GCM with varying $a$ and $b$, respectively, plotted against $D/2$. The diffusion constant $D$ is obtained by $D = a^2$ for the RM model and numerically measured for the GCM. Dashed line: $\Delta \lambda^i(2) = D/2$ as predicted in Eq. (4).]
We now show that it actually grows logarithmically with $N$. With fixed indices $i$, these LEs at size $N$ seem to obey Eq. (4), $\lambda_N^{(i)} \simeq \lambda_{\infty}^{(i)} + c^{(i)}/\log N$ [Fig. 2(a)], but the estimated $\lambda_{\infty}^{(i)}$ increase with $i$ (dashed lines), at odds with the monotonicity of the Lyapunov spectrum. This is better seen when plotting $(\lambda_{2N}^{(i)} \log 2N − \lambda_{\infty}^{(i)} \log N)/\log 2$ as estimates for $\lambda_{\infty}^{(i)}$ [inset of Fig. 2(a)], where $\lambda_{\infty}^{(i)}$ is found to be larger than $\lambda_{\infty}^{(1)}$ within the non-extensive region $1 \leq i \lesssim i_0 \equiv h_c N$. Instead, if we rescale the index logarithmically as $h' \equiv (i − 1)/(i_0 + \log N)$, with $i_0$ adjusted here for the LEs to show the $1/\log N$ law, the asymptotic LEs $\lambda_{\infty}^{(i)}(h')$ do not increase with $h'$ anymore, but stay constant in the non-extensive region except near the threshold (Fig. 2b, left of the dashed line). This indicates that all the non-extensive LEs converge to the same value as the first LE and that their number increases logarithmically with $N$. The same conclusion is reached for our GCM system [Fig. 2(c,d)], though we could not compute all the non-extensive LEs within reasonable time $\lambda_{\infty}^{(1)}$. In short, we find that $O(N)$ extensive LEs are sandwiched by two sub-extensive bands at both ends of the spectrum, each of which consists of $O(\log N)$ LEs with asymptotic values shifted approximately by $D/2$ from $\lambda_0$.

We now show that our results also extend to the HMF model, and thus probably also to other globally-coupled Hamiltonian models. Defined by the Hamiltonian $H = \frac{1}{2} \sum_j p_j^2 + \frac{1}{2N} \sum_{j,j'} [1 - \cos(\theta_j - \theta_j')]$, the HMF model is intensely studied mostly because its infinite-size limit displays an abundance of non-trivial solutions which appear as so-called quasi-stationary states at finite $N$.

Contradictory results exist about the nature of chaos in this model [7,10], even in its reference “equilibrium” state. The motion of a single particle is given by $\dot{\theta}_j = -M \sin(\theta_j - \Theta)$, where $Me^{i\Theta} \equiv \frac{1}{N} \sum_j e^{i\theta_j}$ is the mean field which is non-zero in the (equilibrium) ferromagnetic phase present for energy density $U < \frac{2}{3}$. Here the naïve argument yields $\lambda_0 = 0$ because a single particle forced by a constant mean-field cannot be chaotic.

We were able to extend the argument leading to Eq. (4) to the HMF model (details will appear elsewhere [11]): At finite $N$, the mean field fluctuates and the energy of a particle diffuses, so that it eventually visits the surroundings of $U_M = 1 + M$, the unstable maximum of the mean-field potential. There, it experiences a chaotic kick, and this produces a finite diffusion coefficient $D$ for the logarithm of the tangent-space amplitudes. Taking these effects into account, we obtain a strictly positive asymptotic first LE with, again, $1/\log N$ corrections, which we numerically confirm in Fig. 3(a). The full Lyapunov spectrum, on the other hand, gets flatter for larger $N$ and we could observe the emergence of the $\Delta h(h) \sim 1/\sqrt{N}$ scaling [Fig. 3(b)], but we are currently unable to study the larger systems, in order to overcome finite-size effects to obtain clear evidence of $O(\log N)$ sub-extensive LEs. Nevertheless, it is already clear from Fig. 3(b) that the $h$-domain where the $1/\sqrt{N}$ scaling holds widens with $N$, suggesting a flat (zero-valued) extensive part with a sub-extensive, possibly logarithmic, band of positive LEs.

We finally examine the influence of collective chaos
First 3 LEs for globally-coupled Hopf oscillators (see text).

In summary, we have shown that microscopic chaos in systems made of $N$ globally-coupled dynamical units exhibits a rather peculiar form of extensivity: their Lyapunov spectrum $\lambda(h)$ is asymptotically flat, thus “trivially” extensive, but sandwiched between sub-extensive bands with LEs taking different values. In presence of macroscopic dynamics, the corresponding collective Lyapunov modes are just superimposed on this structure. The bulk LEs converge as $\lambda(h) \simeq \lambda_0 + \text{ct.}/\sqrt{N}$ to the value $\lambda_0$ given by a single dynamical unit forced by the mean-field. In contrast, the sub-extensive layers contain $O(\log N)$ LEs whose values vary as $\lambda \simeq \lambda_{\infty} + \text{ct.}/\log N$ with $\lambda_\infty \neq \lambda_0$. Investigating further the genericity of our results and providing a theoretical basis to the $1/\sqrt{N}$ scaling of bulk LEs and the $\log N$ size of sub-extensive bands are important tasks left for future study.

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FIG. 4: (color online) (a-c): HMF model with energy density $U=0.7$. (a) First LE, maximum size $N=10^5$. Dashed line: linear fit in the $1/\log N$ regime. (b) $\Delta \lambda (= \lambda)$ vs $N$ at fixed rescaled indices $h$. Dashed lines: $\Delta \lambda \sim 1/\sqrt{N}$. (c) Positions of the oscillator most contributing to the 1st and 256th Lyapunov vector (black dots and brown crosses, respectively) at $N=512$. Red dashed line indicates the separatrix. (d) First 3 LEs for globally-coupled Hopf oscillators (see text).