Central differences, Euler numbers
and symbolic methods

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I relate some coefficients encountered when computing the functional
determinants on spheres to the central differentials of nothing. In
doing this I use some historic works, in particular, transcribing the el-
egant symbolic formalism of Jeffery (1861) into central difference form
which has computational advantages for Euler numbers, as discovered
by Shovelton (1915). I derive sum rules for these, and for the central
differentials, the proof of which involves an interesting expression for
powers of sech \(x\) as multiple derivatives. I present a more general,
symbolic treatment of central difference calculus which allows known,
and unknown, things to be obtained in an elegant and compact fashion
gaining, at no cost, the expansion of the powers of the inverse sinh,
a basic central function. Systematic use is made of the operator 2
\(\text{asinh}(D/2)\). Umbral calculus is employed to compress the operator
formalism. For example the orthogonality/completeness of the facto-
rial numbers, of the first and second kinds, translates, umbrally, to
\(T(t(x))=x\). The classic expansions of multiple angle cosh and sinh in
terms of powers of sinh are thence obtained with minimal effort.

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1. Introduction

In a work, [1], involving quantum field theory calculations of the effective action on spheres (as the logarithm of the operator determinant of the field propagation operator), in order to show the equivalence of two explicit expressions for this ‘logdet’, I was obliged to look at an expansion of the powers of the sech function. Some details are in the following section.

As background, the power series expansion of sec \(x\) goes back to Euler, the coefficients being named Euler numbers in the 1820’s to 1840’s, e.g. by Scherk and Raabe. They were, and still are, called secant numbers. Their evaluation is distinct from those of tan \(x\) which are related to the Bernoulli numbers. The series expansion of powers of sech \(x\) yields generalised Euler numbers.

The study of Bernoulli numbers, their relatives and their generalisations remains strong after nearly three hundred years. For this reason, it is difficult to know whether any displayed result is really new and, not unexpectedly, the expansion in [1] turns out to be known and I wish here to go into more detail about this interesting topic which has involved me in an analysis of the existing, and sometimes ancient, literature. The Dilcher online bibliography contains most (not all) of these references. Whether anyone other than the historically minded has actually used, or even examined, this old material is open to question.

The relevant calculus turns out to be that of central differences and in section 5 I transcribe some of the symbolic formalism given by Jeffery, [2], who uses forward differences, into central calculus. This allows various sum rules and identities to be derived, which might be new, as well as some (known) series expansions.

In sections 6 and 7 I further develop the symbolic approach to central calculus in a more general way presenting some standard and non-standard applications in a (possibly) novel fashion. In section 8, I combine this approach with the umbral notation, generating functions, and other things, emerging effortlessly.

2. The expansion and central factorials

The integral needed in [1] is,

\[
\int_0^\infty dx \frac{1}{(x^2 + 1) \cosh^{2r+1} \pi x/2},
\]

and, in order to allow partial integration, one requires the expansion,

\[
\sech^{2r+1} x = \frac{1}{(2r)!} \sum_{\rho=0}^r (-1)^\rho \mathcal{G}_\rho \frac{d^{2\rho}}{dx^{2\rho}} \sech x,
\]
for odd powers in terms of even derivatives.\(^2\) The coefficients satisfy the recursion, (difference equation),
\[
G_r^\rho = (2r-1)^2 G_{r-1}^\rho + G_{r-1}^{r-1},
\]
(3)
together with \(G_{-1}^r = 0 = G_{r+1}^0\) and the initial value \(G_0^0 = 1\). Special values are \(G_r^r = 1\). They are directly related to central difference factorial numbers (e.g. Steffensen, [3]) as is easily seen by repeated action of the basic, and old, relation,

\[
\text{sech}^n x = -\frac{1}{(n-1)(n-2)}(D^2 - (n-2)^2)\text{sech}^{n-2} x, \quad D = \frac{d}{dx},
\]
which yields,
\[
\text{sech}^{2r+1} x = \frac{(-1)^r}{(2r)!} (D^2 - (2r-1)^2)(D^2 - (2r-3)^2)\ldots(D^2 - 1) \text{sech} x
\equiv 2^{2r} \left(\frac{D}{2}\right)^{2r+1-1} \text{sech} x,
\]
(4)
in Steffensen's notation. The coefficients of the expansion of the central factorial in ordinary powers of \(D\) are, by definition, the central factorial numbers of the first kind (or, up to a factor, the 'central differentials of nothing')\(^3\). The complete iteration, (4), shows that these are equal to the sum of the squares of the first \(r\) odd integers taken a certain number of times (cf Ely [4], Jeffery [2]) but I will not continue with this combinatorial view just now.

The precise connection between the coefficients in (2) and the central differentials of nothing is,\(^4\)
\[
G_r^\rho = (-1)^{r-\rho} 2^{2r} \frac{(2r)!}{(2\rho + 1)!} D^{2\rho+1} 0^{2r+1}
\]
(5)
where I have introduced the notation of Riordan, [6], for the central factorial numbers, \(t\). The recursion, (3), is then equivalent to those in Steffensen, [3], and in [6] and [7].

Writing (2) this way round was the most convenient for the purposes in [1]. One could, of course, invert it and, in this form, a related expansion is given by

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\(^2\) I have improved the definition of the coefficients in order to make them integers.

\(^3\) One might call them ‘central Stirling numbers of the first kind’.

\(^4\) A slightly more complicated treatment, but essentially the same, is given by Zhang, [5].
Stern, [8], which I wish to relate to (2). He actually gives two expansions, one for even derivatives,
\[
\frac{d^{2\rho}}{dx^{2\rho}} \text{sech } x = (-1)^\rho \text{sech } x \sum_{k=0}^{\rho} (-1)^k a_k^{(2\rho)} \tanh^{2k} x,
\]
and one for odd,
\[
\frac{d^{2\rho+1}}{dx^{2\rho+1}} \text{sech } x = (-1)^{\rho+1} \text{cosech } x \sum_{k=0}^{\rho} (-1)^k a_k^{(2\rho+1)} \tanh^{2k+2} x
\]
\[
= (-1)^{\rho+1} \text{sech } x \tanh x \sum_{k=0}^{\rho} (-1)^k a_k^{(2\rho+1)} \tanh^{2k} x.
\]
One sees the appearance of ‘derivative polynomials’ which have been subject to more recent investigations e.g. [9].

Stern gives the coupled recursions,
\[
a_k^{(2n+1)} = (2k+1) a_k^{(2n)} + (2k+2) a_{k+1}^{(2n)}
\]
\[
a_k^{(2n+2)} = 2k a_{k-1}^{(2n+1)} + (2k+1) a_k^{(2n+1)}.
\]

The coefficients \( G_\rho^r \) can be connected to the \( a_k^{(2\rho)} \) by substituting (6) into (2) giving,
\[
(1 - \tanh^2 x)^r = \sum_{\rho=0}^{r} G_\rho^r \sum_{k=0}^{\rho} (-1)^k a_k^{(2\rho)} \tanh^{2k} x,
\]
whence,
\[
\sum_{\rho=s}^{r} G_\rho^s a_s^{(2\rho)} = (2r)! \binom{r}{s}.
\]
Similarly, substituting (7) into the derivative of (2) produces
\[
\sum_{\rho=s}^{r} G_\rho^s a_s^{(2\rho+1)} = (2r+1)! \binom{r}{s}.
\]

The second recurrence in (8) is equivalent, directly, to the binomial recurrence
\[
\binom{r}{s+1} = \frac{r-s}{s+1} \binom{r}{s}, \quad \binom{r}{0} = 1.
\]
3. Generalised Euler numbers

Generalised Euler numbers can be defined by the power series (I use Nörlund’s notation, [10,11]),

$$\text{sech}^r x = \sum_{\nu} \frac{x^{2\nu}}{(2\nu)!} E^{(r)}_{2\nu}.$$  \hspace{1cm} (11)

The original Euler numbers correspond to \( n = 1 \), \( E_{2\nu} \equiv E^{(1)}_{2\nu} \).

Setting \( x \) to zero in (6) and using the MacLaurin expansion, yields

$$E_{2\nu} = a_0^{(2\nu)} = a_0^{(2\nu-1)},$$

where the second equality follows on differentiating (7), and so Stern refers to the other coefficients, \( a_k \), as Euler numbers of higher order but this name is more often used for the numbers defined by (11). Stern’s term reappears in some recent works, e.g. [9].

By differentiating (2) \( 2\nu \) times, one can easily obtain an expression for the odd generalised numbers in terms of the ordinary ones,

$$E_{2\nu}^{(2r+1)} = \frac{1}{(2r)!} \sum_{\rho=0}^{r} (-1)^\rho G^r_{\rho} E_{2\nu+2\rho}, \quad \forall \nu.$$  \hspace{1cm} (12)

This relation can be found in Ely, [4], where the coefficients, \( G \), are expressed in the combinatorial fashion.

Schwatt, [12], pp.68–71, has an extensive algebraic treatment of expansions of powers of sech \( x \) and also of the derivatives of these in the Stern form, (6), (7).

One could look upon (12) as a means of computing the (odd) generalised numbers since the recursion (3) for \( G \) is easily solved and the first order Euler numbers are fairly well studied. In particular an efficient recursion method is adopted by Knuth and Buckholtz, [13], which it is worth recalling and then comparing with earlier work. (See also Brent and Harvey, [14].)

4. Computing Euler numbers

Knuth and Buckholtz use trigonometric functions and, by differentiation, deduce the relation,

$$\frac{d^n}{dz^n} \sec z = \sec z (E_{n0} + E_{n1} \tan z + E_{n2} \tan^2 z + \ldots),$$  \hspace{1cm} (13)
and the recursion,
\[ E_{n+1,k} = kE_{n,k-1} + (k + 1)E_{n,k+1}, \quad E_{0k} = \delta_{0k}. \quad (14) \]

Since \( E_n = E_{n0} \), this allows an efficient computation of the Euler numbers. 5

The earlier Stern recursions are the same as (13) which is easily seen by splitting (13) into two by the parity behaviour under \( z \rightarrow -z \). Then one sees that,
\[ E_{2\rho,2k} = a_2^{2\rho}, \quad E_{2\rho,2k+1} = 0, \]
and
\[ E_{2\rho+1,2k+1} = a_k^{(2\rho+1)}, \quad E_{2\rho+1,2k} = 0. \]

Stern uses the recursions to derive many arithmetical properties, as do Knuth and Buckholtz, who were unaware of Stern’s work.

Historical computations of the secant numbers were performed by Euler and by Scherk (in 1825). I do not give these continental references as they can be found in the 1893 text by Saalschütz, [15], as well as a list of the first 14 numbers. Glaisher extended the list to 27 and Joffe to 50. The effort seems to be more of a computational challenge than necessity.

Herschel, [16,17], (1816 and 1820) gave finite difference expressions suitable for numerical computation (in terms of forward differences of nothing) and also related Euler numbers to Bernoulli numbers via tangent numbers.

Herschel’s theorem applied to \( \text{sech} \) \( x \) yields,
\[ E_{2n} = (-1)^n \frac{2}{(1 + \Delta) + (1 + \Delta)^{-1}} 0^{2n}, \quad (15) \]
which is manipulated in various ways. However, a more elegant, and practical, formulation was found by Shovelton, [18], in terms of the central difference operator, \( \delta \), as, 6
\[ E_{2n} = (-1)^n \frac{1}{1 + \delta^2/2} 0^{2n} \equiv (-1)^n \left( 1 - \frac{1}{2} \delta^2 + \frac{1}{2^2} \delta^4 + \ldots + \frac{(-1)^n}{2^n} \delta^{2n} \right) 0^{2n}, \quad (16) \]

5 A simple algorithm implementing (13) is given by Brent and Harvey. Programming this in DERIVE (not a fast CAS) I calculated the first 1000 Euler numbers in 24 seconds using an Athlon 610e processor. The values could be used in a look–up table for fast access.
6 Shovelton develops his results via the Bernoulli polynomials (the ‘Bernoullian function’) and introduces central differences as a simplifying tactic.
which can be obtained directly from (15) by noting that $\delta = \Delta E^{-1/2}$ ($E$ being the translation operator, $E = 1 + \Delta$) so that (e.g. [18], [3] pp.10,11, [19]),

$$\delta = E^{1/2} - E^{-1/2}, \quad i.e. \quad 2 + \delta^2 = E + E^{-1},$$

and also,

$$(E^{1/2} + E^{-1/2})^2 = 4 + \delta^2. \quad (17)$$

After use of the equivalence $f(E^n) \mathbf{0}^s = n^s f(E) \mathbf{0}^s, \forall n$ (Herschel, [20] VII §(8).) equations (17) and (15) allow the alternative compact expression, also given by Shovelton, [18],

$$E_{2n} = \frac{(-1)^n}{2^{2n}} \frac{1}{(1 + \delta^2/4)^{1/2}} \mathbf{0}^{2n} \equiv \frac{(-1)^n}{2^{2n}} \left[ 1 + \left( -\frac{1}{2} \right) \delta^2 + \left( -\frac{1}{2} \right)^2 \delta^4 + \ldots + \left( -\frac{1}{2} \right)^n \frac{\delta^{2n}}{2^{2n}} \right] \mathbf{0}^{2n}. \quad (18)$$

(See Butzer et al, [7].)

Joffe, [21], undertook the numerical evaluation of $E_{2n}$ using (16) in terms of the central differences of nothing which were computed using the recursion formula. An early table of values is provided by Shovelton, [18].

The expansion of the powers of sec is formally no harder and by the same process one has the coefficients, [18],

$$E_{2n}^{(r)} = (-1)^n \frac{1}{(1 + \delta^2/2)^r} \mathbf{0}^{2n} \equiv (-1)^n \left[ 1 + \frac{1}{2} \left( -\frac{r}{1} \right) \delta^2 + \frac{1}{2^2} \left( -\frac{r}{2} \right) \delta^4 + \ldots + \frac{1}{2^n} \left( -\frac{r}{n} \right) \delta^{2n} \right] \mathbf{0}^{2n}, \quad (19)$$

which has the advantage over (12) of displaying the polynomial nature of $E_{2n}^{(r)}$ in $r$.

The form corresponding to (18) is

$$E_{2n}^{(r)} = \frac{(-1)^n}{2^{2n}} \frac{1}{(1 + \delta^2/4)^{r/2}} \mathbf{0}^{2n} \equiv \frac{(-1)^n}{2^{2n}} \left[ 1 + \left( -\frac{r}{2} \right) \delta^2 + \left( -\frac{r}{2} \right)^2 \delta^4 + \ldots + \left( -\frac{r}{2} \right)^n \frac{\delta^{2n}}{2^{2n}} \right] \mathbf{0}^{2n}. \quad (20)$$

The central differences of nothing are, to a factor, the central factorial numbers of the second kind, e.g. Steffensen, [3], and [6] and [7].
Earlier expansions of sech\(^n\) \(x\) exist but are generally expressed in terms of the forward differences of nothing. For example Jeffery, [22], gives the neat looking, but computationally somewhat awkward, form,

\[
\sec^n x = \left( \frac{2}{2 + \Delta} \right)^n \cos(2.0 + n)x.
\]

An expression, essentially equivalent to (19), has been given more recently by Liu and Zhang, [23].

5. Jeffery’s theorem, central differences and Euler relations

The early works of Herschel, Jeffery, Boole and others are characterised by the extensive use of symbolic methods. Shovelton does likewise, but in terms, partly, of central differences. The 1909 textbook by Thiele, [24], is a rare example of their application by a continental worker. Another is the classic text by Steffensen, [3], both geared to interpolation questions. There is also an 1865 work by Hansen that I have not been able to access. Bickley, [19], gives a useful summary, without references, of standard operator relations. Chapter 6 in Riordan, [6], is a good, modern pedagogic treatment.

The elegant papers by Jeffery contain applications of four basic theorems concerning the transformation of symbolic expressions. (Herschel’s much earlier work also involves a lot of these manipulations.) The formalism is expressed in terms of forward differences. Shovelton’s results show that considerable simplification ensues if central differences are brought into play and I wish here to transcribe some of Jeffery’s formulae into central form and apply them to obtain Euler number relations as an illustration.

For this purpose, I return to the alternative formula, (18), for the Euler numbers, and write it equivalently as,

\[
E_{2n} = \frac{(-1)^n}{2^{2n}} \frac{2}{E^{1/2} + E^{-1/2}} 0^{2n}
\]

\[
= \frac{(-1)^n}{2^{2n}} \text{sech} \frac{D}{2} 0^{2n}.
\]

Rather than expanding the operator in powers of \(\delta^2\) directly (as in (18)), I substitute for the \(0^{2n}\) in central factorials using,

\[
x^{2n} = \sum_{\nu=1}^{n} x^{[2\nu]} \frac{\delta^{2\nu} 0^{2n}}{(2\nu)!},
\]
to give the intermediate formula,
\[ E_{2n} = \frac{(-1)^n}{2^{2n}} \sum_{\nu=1}^{n} \text{sech} \frac{D}{2} 0^{[2\nu]} \cdot \frac{\delta^{2\nu} 0^{2n}}{(2\nu)!}. \tag{21} \]

I am following Jeffery, [22] §3, and, as there, a theorem is employed to re-express the action of the pre-operator, here, sech \( D/2 \). The theorem is the central difference form of Theorem \( D \) in Jeffery, [2], and reads (I prove this in section 6),
\[ f(D) 0^{[n]} = \frac{d f(D)}{d\delta} 0^{[n-1]} = \ldots = \frac{d^n f(D)}{d\delta^n} 1, \tag{22} \]
which can be run left to right or right to left. Applied to powers of sech \( D/2 \), taking it all the way to the right trivially gives the expansion (20), and so nothing new.

More interesting is to work right to left. For example, set \( n = 2\nu + 1 \) and choose,
\[ f(D) = \int d\delta \text{sech} \frac{D}{2} = \int dD \cosh \frac{D}{2} \text{sech} \frac{D}{2} = D, \tag{23} \]
(since \( \delta = 2 \sinh D/2 \)) so that (21) becomes,
\[ E_{2n} = \frac{(-1)^n}{2^{2n}} \sum_{\nu=1}^{n} D 0^{[2\nu+1]} \cdot \frac{\delta^{2\nu} 0^{2n}}{(2\nu)!}, \tag{24} \]
which is identical to (18) as the value of the particular central differential in (24) is standard, [3] (see below). This result is given in [7] but only by comparing the actual values.

Another conclusion can be drawn from the identity,
\[ \text{sech} \frac{D}{2} 0^{[2\nu]} = D 0^{[2\nu+1]}, \tag{25} \]
by comparing with the expansion (11) to give the sum rule for Euler numbers,
\[ \sum_{n=1}^{s} \frac{E_{2n}}{(2n)!} \frac{D^{2n} 0^{[2s]}}{2^{2n}} = D 0^{[2s+1]} = \frac{(2s)!}{2^{2s}} \left( -\frac{1}{2} \right)^s \]
\[ = (-1)^s \left( \frac{1.3\ldots(2s-1)}{2^s} \right)^2, \tag{26} \]
which might be novel. Given the central differentials, it could be used to compute the Euler numbers, but there is no need.
Related manipulations can be performed for the generalised Euler numbers, (20), although the simple relation, (25), becomes more complicated. As an example, in order to further illustrate the theorem, (22), I look at the $E^{(3)}_{2n}$ numbers. Instead of (23), $f(D)$ in (22) is given by,

$$f(D) = \int dD \cosh \frac{D}{2} \sech^3 \frac{D}{2} = 2 \tanh D/2,$$

which results in,

$$\sech^3 \frac{D}{2} 0^{[2\nu]} = 2 \tanh \frac{D}{2} 0^{[2\nu+1]}.$$  \hspace{1cm} (28)\hspace{1cm}

The tanh could be expanded in order to evaluate the right-hand side but it is better to apply (22) for a second time by choosing,

$$f(D) = 2 \int dD \cosh \frac{D}{2} \tanh \frac{D}{2}$$

$$= 4 \cosh D/2,$$

yielding the successive equivalences,

$$\sech^3 \frac{D}{2} 0^{[2\nu]} = 2 \tanh \frac{D}{2} 0^{[2\nu+1]} = 4 \cosh \frac{D}{2} 0^{[2\nu+2]},$$

the last of which can now easily be evaluated as a finite sum of central differentials,

$$4 \cosh \frac{D}{2} 0^{[2\nu+2]} = 4 \sum_{k=1}^{\nu+1} \frac{D^{2k} 0^{[2\nu+2]}}{2^{2k}(2k)!},$$

generalising (25).

Spelling out the details, the resulting expression for the generalised Euler numbers follows as (21),

$$E^{(3)}_{2n} = \frac{(-1)^n}{2^{2n}} \sum_{\nu=1}^n \sech^3 \frac{D}{2} 0^{[2\nu]} \cdot \frac{\delta^{2\nu} 0^{2n}}{(2\nu)!},$$

$$= \frac{(-1)^n}{2^{2n-2}} \sum_{\nu=1}^{\nu+1} \sum_{k=1}^{\nu+1} \frac{D^{2k} 0^{[2\nu+2]}}{2^{2k}(2k)!} \cdot \frac{\delta^{2\nu} 0^{2n}}{(2\nu)!},$$

which is considerably more complicated than the direct expansion, (20), but allows us, therefrom, to infer the (even) sum rules,

$$\sum_{k=1}^{\nu+1} \frac{D^{2k} 0^{[2\nu+2]}}{2^{2k-2}(2k)!} = \frac{(2\nu)!}{2^{2\nu}} \left( -\frac{3}{2} \right) \left( \begin{array}{c} 2\nu \\ \nu \end{array} \right),$$

(32)
which can be confirmed numerically.

As another example, I give the (odd) sum rule that results from looking at $E_{2n}^{(5)}$,

$$\frac{4}{3} \sum_{k=0}^{\nu+1} \frac{D^{2k+1}0^{[2\nu+3]}}{(2k+1)!} = \frac{(2\nu)!}{2^{2\nu}} \left( -\frac{5}{2} \right)^\nu, \tag{33}$$

the formula corresponding to (30) being, after three integrations,

$$\text{sech}^5 \frac{D}{2} 0^{[2\nu]} = \frac{4}{3} \sinh D 0^{[2\nu+3]} \tag{34}.$$

The general case depends on integrating the relation,\(^7\)

$$\text{sech}^{2r+1} x = \frac{(\pm 1)^{r+1}}{r \left( 2r - 1 \right)!!} \left( \frac{d}{d \sinh x} \right)^{r+1} e^{\pm rx}, \quad r = 1, 2, \ldots, \tag{35}$$

which I cannot find listed. I need not write out the corresponding sum rules as examples (32) and (33) are sufficient.

It is clear that there are a large number of sum rules derived by using the different transformations, such as (30) and (34), arising from (22) and by computing higher Euler numbers, $E_{2n}^{(2r+1)}$. They would seem to have only a curiosity or checking value. I have not seen them elsewhere.

With (35) I have essentially returned to the direct expansion, (20), since the multiple–angle cosh and sinh can be expanded in powers of sinh $x$ in a familiar way, the coefficients usually being derived by recursion or by schoolbook trigonometry (e.g. Loney, [25], Bromwich, [26], Workman, [27], Schwatt, [12] p.124).

The expansions can actually be combined neatly to the unfamiliar form,

$$e^{rx} = \sum_{s=0}^{\infty} 2^s \frac{(r/2)^s}{s!} \sinh^s x, \quad r \in \mathbb{Z}, \tag{36}$$

where I have written the coefficients in terms of central factorials, for future use in section 8. (See [7], §3.1.)

Eq.(35), after some brief algebra, coincides with the direct expansion, (cf (20)),\(^8\)

$$\text{sech}^{2r+1} x = \sum_{s=0}^{\infty} \left( -r - \frac{1}{2} \right) \sinh^{2s} x, \quad \sinh x < 1,$$

cf also Schwatt, [12] p.70.\(^9\) Questions of convergence have been mostly ignored, and, in any case, for the finite series involved in (20) are irrelevant.

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\(^7\) Constants of integration give zero.

\(^8\) This could be considered a proof of (35), given the expansions, (36), or these could be derived, by continuation.

\(^9\) Schwatt also refers to Shovelton.
6. Central difference theorems

I develop some basic concepts in central difference calculus using a mainly symbolic approach. I do this by first proving the theorem (22) and then passing on to other notions and some expansion formulae. Of the few discussions of this topic, that by Riordan, [6], is most valuable. The more recent review by Butzer et al, [7], is also useful.

The proof of (22) is not obvious so I expose it here, following the discussion by Jeffery, [2], for forward difference calculus. For comparison I write out the theorems as given by him,

\begin{align}
A: & \quad f(\Delta) 0^n = n \frac{f(\Delta)}{D} 0^{n-1} = \ldots = n! \frac{f(\Delta)}{D^n} 1 \\
B: & \quad f(\Delta) 0^n = \frac{df(\Delta)}{dD} 0^{n-1} = \ldots = \frac{d^n f(\Delta)}{dD^n} 1 \\
C: & \quad f(D) 0^{(n)} = n \frac{f(D)}{\Delta} 0^{(n-1)} = \ldots = n! \frac{f(D)}{\Delta^n} 1 \\
D: & \quad f(D) 0^{(n)} = \frac{df(D)}{d\Delta} 0^{(n-1)} = \ldots = \frac{d^n f(D)}{d\Delta^n} 1.
\end{align}

(37)

I will not prove them but only say that A and C express the fact that the symbols \( \Delta \) and \( D \) equally commute when acting upon \( x \) at \( x = 0 \) as they do when \( x \) is current. \(^{10}\)

In proving D, Jeffery makes essential use of the operator \( \zeta \equiv \log(1 + D) \) which is a sort of dual, or functional inverse to the relation \( D = e^\Delta - 1 \) in the sense that,

\[ e^{\log(1 + D)} - 1 = D. \]

I start from the very basic property of differences as the action on factorials,

\[ F(\delta) 0^{[r]} = F(D) 0^r, \]

expressed in terms of central quantities for which a basic relation is,

\[ \delta = 2 \sinh \frac{D}{2}, \quad \text{i.e.} \quad D = 2 \sinh^{-1} \frac{\delta}{2}. \]

(39)

I can therefore define a new function \( f \) by the functional relation,

\[ F(\delta) = F(2 \sinh D/2) \equiv f(D) = f(2 \sinh^{-1} \delta/2), \]

\(^{10}\) As noted by Jeffery, A and B appear as Misc. Ex. (1) on p.242 of Boole, [28]. Since this was a Senate House Examination problem, it seems likely it was due to Herschel.
and just replace \( F(\delta) \) by \( f(D) \) on the left–hand side of (38) while, on the right–hand side, replace \( \delta \) by \( D \), regarded as a function of \( \delta \). So, corresponding to \( \zeta \), if I define the operator \( \xi \), by

\[
\xi = 2\sinh^{-1} \frac{D}{2}.
\]

I obtain the handy reciprocal result,

\[
f(D) 0^{[r]} = f(\xi) 0^{r}.
\]

I have not seen this approach in the literature.

The proof of the central version of Theorem \( D \) now follows more or less directly. Applying Theorem \( B \), which holds generally, on appropriate interpretation of \( f(\Delta) \), one finds,

\[
f(D) 0^{[r]} = f(\xi) 0^{r} = \frac{f'(\xi)}{(1 + D^{2}/4)^{1/2}} 0^{r-1}
\]

\[
= f'(\xi) \left[ 1 + \left( -\frac{1}{2} \right) \frac{D^{2}}{2^{2}} + \left( -\frac{1}{2} \right) \frac{D^{4}}{2^{4}} + \ldots \right] 0^{r-1}
\]

\[
= f'(\xi) \left[ 0^{r-1} + \left( -\frac{1}{2} \right) \frac{0^{r-3}}{2^{2}} + \left( -\frac{1}{2} \right) \frac{0^{r-5}}{2^{4}} + \ldots \right]
\]

\[
= f'(D) \left[ 0^{[r-1]} + \left( -\frac{1}{2} \right) \frac{0^{[r-3]}}{2^{2}} + \left( -\frac{1}{2} \right) \frac{0^{[r-5]}}{2^{4}} + \ldots \right]
\]

\[
= \frac{f'(D)}{(1 + \delta^{2}/4)^{1/2}} 0^{[r-1]} = \frac{df(D)}{d\delta} 0^{[r-1]}.
\]

This is equation (22).

Continuing with some central formalism, I display the expansion of the central factorial,

\[
x^{[n]} = \sum_{\nu=1}^{n} x^{\nu} \frac{D^{\nu} 0^{[n]}}{\nu!} = \sum_{\nu=1}^{n} x^{\nu} \frac{\xi^{\nu} 0^{n}}{\nu!}
\]

\[
= e^{x} \xi 0^{n},
\]

using (41) to rewrite the coefficients and also formally summing the series \((1.0^{n} = 0 \text{ for } n \neq 0)\). (Compare Jeffery, [2], §5.)

Furthermore, from the explicit definition of \( \xi \), (40), the expansion of the inverse sinh function is contained in,

\[
(2\sinh^{-1} \frac{x}{2})^{n} = \xi^{n} e^{0x},
\]

\[11\text{Jeffery derives his formula in a different way, but I find the present route easier.}\]
showing rapidly that the coefficients are just the central factorial numbers of the first kind (cf. [7] §4.2 for a more expansive treatment.)

The same expansion is derived by Butzer et al, [7] §4.1, but less attractively in my view as the proof involves manipulating several series. See also Charalambides, [29] and the classic Schwatt, [12], pp.6,7,125, who credits Cauchy.

As a further illustration of the symbolic formalism, I derive an intrinsic relation between the first and second kind central factorial numbers, i.e. between the $\xi^m 0^n$ and $\delta^r 0^s$.

I start from the definition of the second kind,

$$x^n = \sum_{\nu=1}^{n} x^{[\nu]} \frac{\delta^\nu n^n}{\nu!} = \sum_{\nu=1}^{n} e^{x\xi} 0^\nu \frac{\delta^\nu n^n}{\nu!},$$

and equate powers of $x$ to get the compact statement of ‘orthogonality’,

$$\xi^m e^0 0^n = n! \delta^m_n. \quad (45)$$

The $\delta$ on the right–hand side is a Kronecker delta. Equation (45) is really a statement about inverses, and tantamount to completeness (of polynomial bases), the reciprocal relation being,

$$\delta^n e^0 0^m = m! \delta^n_m. \quad (46)$$

cf. Riordan, [6] p.213.

A more formal expression of completeness is,

$$e^0 e^0 = 1_{\xi}$$
$$e^0 e^0 = 1_{\delta}, \quad (47)$$

with,

$$f(\xi) 1_{\xi} = f(\xi), \quad f(\delta) 1_{\delta} = f(\delta).$$

Two further useful relations are

$$f(D) 0^r = f(\xi) e^0 0^r \quad (48)$$

and

$$f(\xi) = f(D) e^0 0^r \quad (49)$$

The first is a corollary of (45) and the second a trivial consequence of the basic derivative, $D^m 0^n = m! \delta^n_m$. 

13
Corresponding to (45) and (46) there are two important expansions connecting multiple derivatives and multiple differences. I derive these for central differences following the forward difference treatment of Boole, [28] p.24, which is an application of Maclaurin’s theorem in its secondary form,

\[ \phi(t) = \phi(D) e^{0,t} , \]  

(50)

where \( D \) acts on \( 0 \). From the identities, (39), \( \delta \) is a function of \( D \) and so, setting \( t = D \) and \( \phi = \delta^n \),

\[ \delta^n = \delta^n e^{0,D} , \]  

(51)

where the \( \delta \) on the right–hand side acts on \( 0 \). This is the required expression in symbolic form. Expansion of the exponential gives a sum of powers of \( D \) and, acting on a function of \( x \), yields the concrete representation,\(^{12}\)

\[ \delta^n u(x) = \sum_{m=n}^{\infty} \frac{\delta^n 0^m}{m!} D^m u(x) . \]  

(52)

Inversely, consider \( \phi = D^m \) as a function of \( \delta \), and so set \( t = \delta \). According to (50), \( \delta \) in \( D \) then has to be replaced by \( D \). This gives the \( \xi \) operator, (40), and so one has the required symbolic expression,

\[ D^m = \xi^m e^{0,\delta} , \]  

(53)

yielding the concrete, Newton series form,

\[ D^m u(x) = \sum_{n=m}^{\infty} \frac{\xi^m 0^n}{n!} \delta^m u(x) , \]

which could be derived from (52) using (45). These expansions are standard, see, e.g., [7] §6.1.

I note in passing that the explicit forms of the operators, (39), (40), are not required when deriving the symbolic equations. Writing the relation, (39), as \( \delta = \Psi(D) \), \( \xi \) is defined by the functional inverse, \( \xi = \Psi^{-1}(D) = \Psi^{-2}(\delta) \). A choice of \( \Psi(*) \) other than (40), or \( \log(1 + \ast) \), would entail a factorial different from the usual ones.

\(^{12}\) I ignore questions of convergence and assume that the function, \( f \), possesses the qualities required for all the equations to exist.
7. Expansions

As Jeffery points out, knowledge of the factorial numbers is useful in many expansions. For example,

\[
f \left( 2 \sinh^{-1} \left( \frac{\sinh^{-1} x}{2} \right) \right) = f \left( 2 \sinh^{-1} \frac{\xi}{2} \right) e^{0.\xi} e^{0.\xi} \\
= f \left( 2 \sinh^{-1} \frac{D}{2} \right) e^{0.\xi} e^{0.\xi} \\
= f(\xi) e^{0.\xi} e^{0.\xi} \\
= f(\xi) \left( 0^{[1]} \frac{x}{1!} + 0^{[2]} \frac{x^2}{2!} \cdots \right),
\]

where (49) and (43) have been used to give the second and fourth lines. This process can be continued.

As a calculated example, consider the simplest case of (54) which is

\[
2 \sinh^{-1} \left( \frac{\sinh^{-1} x}{2} \right) = \xi 0^{[1]} \cdot \frac{x}{1!} + \xi 0^{[2]} \cdot \frac{x^2}{2!} \ldots 
\]

and require the coefficient of \(x^r/r!\),

\[
\xi 0^{[r]} = \sum_{s=0}^{r} \frac{\xi 0^{s}}{s!} \cdot \xi^s 0^r.
\]

(This is really going back to the third line of (54). A simple iteration of (44) also yields the same result.) The summation index \(s\) must be odd, and so therefore must \(r\) (as one knows). Hence the coefficient of \(x^{2k+1}/(2k+1)!\) is

\[
\xi 0^{[2k+1]} = \sum_{j=0}^{k} \frac{\xi 0^{2j+1}}{(2j+1)!} \cdot 2^{2j+1} 0^{2k+1} \\
= \sum_{j=0}^{k} \left( \frac{-1}{j} \right) \frac{1}{2^{2j}(2j+1)} \cdot \xi^{2j+1} 0^{2k+1}
\]

which can be evaluated straightforwardly.
8. Using Representative Notation

A simpler case to (54) follows either as an extension of (44) or directly,

\[ f(2 \sinh^{-1} \frac{x}{2}) = f(\xi) e^{0.x} = \sum_{r=0}^{\infty} \frac{f(\xi) 0^r}{r!} x^r, \]  

and allows me to bring in Blissard’s seductive ‘Representative Notation’ (or classical umbral calculus, [30]) by writing the primary form of Maclaurin’s theorem as,

\[ f(\xi) = \sum_{s=0}^{\infty} \frac{f^{(s)}(0)}{s!} \xi^s = e^{f \xi}, \]

where the power \( f^n \) is to be replaced by the derivative, \( f^{(n)}(0) \), after the expansion of the exponential. Then the elegant result,

\[ f(2 \sinh^{-1} \frac{x}{2}) = \sum_{r=0}^{\infty} \frac{e^{f \xi} 0^r}{r!} x^r = \sum_{r=0}^{\infty} \frac{f^{[r]}}{r!} x^r, \]

follows on using (43). One could write this result more enigmatically as,

\[ f(\xi) e^{0.x} = e^{f x}, \]  

where now the power, \( f^r \), first represents the central factorial, \( f^{[r]} \), in which the powers of \( f \) are still to be replaced by the derivatives, \( f^{(n)}(0) \), as above.

In the form,

\[ f(y) = \sum_{r=0}^{\infty} \frac{f^{[r]}}{r!} (2 \sinh \frac{y}{2})^r, \]

this expansion is derived, in a complicated way, by Teixeira, [31], using existing trigonometric expansions, cf (36). He sets \( y = i\theta \) and splits the formula into real and imaginary parts. The derivation here, in contrast, is rapid and from finite difference fundamentals, which is how the trigonometric expansions can be, and are, more logically obtained, [32]. A more elaborate, conventional discussion can be found in [7] §5.

Riordan employs the umbral ‘method’ and I use his notation, (see Eq.(5)), for the factorial numbers, which has operational advantages). My object is to combine umbral techniques with the Jeffery operator formalism (in central form).
I have already introduced the representative, \( t \), by,

\[
t^n(x) = t_n(x) \equiv \sum_{\nu=0}^{n} t(n, \nu) x^\nu = x^{[n]} = e^{x\xi} 0^n,
\]

so that, by definition,

\[
x^n = e^{t\delta} 0^n, \quad t = t(x).
\]

I now define its reciprocal, \( T \), by, cf \([6]\),

\[
T^n(x) = T_n(x) \equiv \sum_{\nu=0}^{n} T(n, \nu) x^\nu = e^{x\delta} 0^n.
\]

On the right I have given the Jeffery central equivalents which enable me to obtain, by summation, the umbral expressions,

\[
e^{yt} = e^{x\xi} e^{0y}
\]

\[
e^{yT} = e^{x\delta} e^{0y},
\]

where \( t = t(x) \) and \( T = T(x) \).

From the definitions of the operators \( \xi \) and \( \delta \), the generating functions, \((61)\) and \((62)\), take the explicit forms, cf \((56)\),

\[
e^{yt}(x) = e^{2x \sinh^{-1} y/2},
\]

and

\[
e^{yT}(x) = e^{2x \sinh y/2},
\]

both of which Riordan (see also Carlitz and Riordan, \([33]\)) derives by series combinatorial manipulation but which here follow almost painlessly.

The two expressions are related by reciprocity as can be shown in the following way.

I first derive the umbral statement of reciprocity. Umbrally, from Eq.\((62)\) by setting \( x \to t(x) \) one arrives at the formally neat expression of orthogonality/completeness,

\[
T(t(\ast)) = \ast.
\]

(This also easily results from \((60)\) followed by \((58)\), which gives \( T^n(t(x)) = x^n.\))

The reciprocal relation follows on iteration of \((65)\),

\[
t(T(t(\ast))) = t(\ast),
\]

\[
(66)
\]
or, invoking completeness,
\[ t(T(\ast)) = \ast . \] \hspace{1cm} (67)

One might therefore formally set \( T = t^{-1} \), etc. Eqs. (65) and (67) are the umbral equivalents of the explicit (45) and (46).

Now, making the replacement \( x \to T(x) \) and employing (67) turns (63) into (64) as promised.

I emphasise the rather interesting fact that Eq. (63) is identical with the classical trigonometric expansion, (36), which was expressed in a relevant way for this reason. Adjusting notation, (63), under \( y \to 2 \sinh x \) and \( x \to r/2 \), becomes, neatly,

\[ e^{rx} = e^{2t \sinh x}, \quad t = t(r/2), \] \hspace{1cm} (68)

which is the umbral representation of (36). A trigonometric expansion has therefore been obtained with very little effort. \(^{13}\)

Finally I make a formal functional generalisation in accordance with the remarks at the end of section 6. If \( \xi = \Psi^{-1}(D) \), the generating functions are

\[ e^{yt(x)} = e^{x\Psi^{-1}(y)}, \] \hspace{1cm} (69)

and

\[ e^{yT(x)} = e^{x\Psi(y)}, \] \hspace{1cm} (70)

with \( t \) and \( T \) now standing for mutually inverse generic ‘factorial’ numbers.

9. Conclusion

The compactness and rapidity of the central difference symbolic calculus has, I think, been demonstrated. I hope also to have shown that there is a considerable body of ignored historical work that could usefully be exploited. Whether the particular results are interesting is, of course, a matter of personal taste. Certainly, the symbolic approach has an intrinsic elegance and power. I draw attention to the convenience of the operator, \( \xi \). As shown in the last section it allows the double generating functions of Carlitz and Riordan to be obtained with ease and thence some classic trigonometric expansions, likewise.

\(^{13}\) Eq.(36) is to be found as Eq.(28) on p.215 of [6] derived in a combinatorial way. (Riordan refers to Pólya and Szegő.) The approach could also be considered as an alternative derivation of the expansions (36). Riordan also exhibits (68). (To make the connection, set \( x \to z/2, r \to 2x \) and \( t(r/2) \to t(x) \).)
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