Abstract
Discretely-constrained Nash-Cournot games have attracted attention as they arise in various competitive energy production settings in which players must make one or more discrete decisions. Gabriel et al. (Netw Spat Econ 13(3):307–326 2013) claim that the set of equilibria to a discretely-constrained Nash-Cournot game coincides with the set of solutions to a corresponding discretely-constrained mixed complementarity problem. We show that one direction of this claim is false by providing counterexamples to show that there exist solutions to the discretely-constrained Nash-Cournot game that do not coincide with solutions to the discretely-constrained mixed complementarity problem. The updated theorem in this note formally states that every solution to the discretely-constrained mixed complementarity problem is a solution to the discretely-constrained Nash-Cournot game, but not vice versa.

Keywords
Complementarity · Equilibria · Integrality · Nash-Cournot games · Relaxation

1 Introduction
A Nash-Cournot game is a game-theoretical framework of imperfect competition in which multiple producers/players compete to optimize their individual objective functions, which also depend on other players’ production decisions. Traditional
(i.e., purely continuous) Nash-Cournot problems have been extensively studied and it is well known that they can be expressed either as nonlinear complementarity or variational inequality problems (Facchinei and Pang 2007). Discretely-constrained Nash-Cournot (DC-NC) games arise when a subset of a player’s decisions are required to be discrete, for example, when a player must make a binary on/off decision. Such problems are becoming increasingly popular in day-ahead electricity markets where certain players, namely conventional generators like nuclear, coal, and natural gas plants, must submit bids that involve on/off decisions with substantial costs (Gabriel and Leuthold 2010; Garcia-Bertrand et al. 2005; Guo et al. 2020; Huppmann and Siddiqui 2018; Leuthold et al. 2012; Weinhold and Gabriel 2020). More general applications and theoretical investigations are discussed in (Fuller and Çelebi 2017; Gabriel et al. 2013; Guo et al. 2021; O’Neill et al. 2005; Pedroso and Smees 2014).

Gabriel et al. (2013) approached discretely-constrained Nash-Cournot games by framing the problem as a discretely-constrained mixed complementarity problem (DC-MCP). We consider the same setup and, to the extent possible, the same notation as Gabriel et al. (2013). There are $N$ players indexed by $p \in P = \{1, \ldots, N\}$. Player $p$ optimizes her cost function $f_p : \mathbb{R}^{n_p} \mapsto \mathbb{R}$ that depends on her decision vector $x_p \in \mathbb{R}^{n_p}$ and the vector $x_{-p} = (x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_N)$ denoting the decisions of all other players besides player $p$. Here, $n = \sum_{p \in P} n_p$. Specifically, we assume that player $p$ solves the following discretely-constrained optimization problem parameterized by $x_{-p}$:

$$f^*_p(x_{-p}) = \min_{x_p} \quad f_p(x_p, x_{-p}) \quad [\text{dual vars}]$$  \hspace{1cm} (1a)

subject to

$$g_{pj}(x_p) \leq 0 \quad [\lambda_{pj} \geq 0] \quad \forall j \in I_p$$  \hspace{1cm} (1b)

$$h_{pk}(x_p) = 0 \quad [\gamma_{pk} \in \mathbb{R}] \quad \forall k \in E_p$$  \hspace{1cm} (1c)

$$x_p \geq 0$$  \hspace{1cm} (1d)

$$x_{pr} \in \mathbb{Z}_+ \quad \forall r \in D_p ,$$  \hspace{1cm} (1e)

where $I_p$, $E_p$, and $D_p$ denote the set of inequalities, equalities, and integer variables for player $p \in P$. Let $\mathcal{X}_p = \{x_p \in \mathbb{R}^{n_p} : \text{Eq. 1b, 1c, 1d, 1e} \}$ denote the discretely-constrained feasible region for player $p \in P$ and let $\mathcal{C}_p = \{x_p \in \mathbb{R}^{n_p} : \text{Eq. 1b, 1c, 1d} \}$ denote the continuous relaxation of $\mathcal{X}_p$. A vector $\hat{x}$ is called a Nash equilibrium of this DC-NC game if $\hat{x}_p \in \mathcal{X}_p$ for all $p \in P$ and

$$f_p(\hat{x}_p, \hat{x}_{-p}) \leq f_p(x_p, \hat{x}_{-p}), \quad \forall p \in P, x_p \in \mathcal{X}_p .$$  \hspace{1cm} (2)

Gabriel et al. (2013) approach convex DC-NC games, i.e., games in which the continuous relaxation of each player’s optimization problem is a convex optimization problem, by applying the following four-step procedure: 1) relax the integrality constraints for each player; 2) write the KKT conditions for each player; 3) re-impose the integrality constraints; 4) solve the resulting DC-MCP. More concretely, since KKT conditions are neither necessary nor sufficient for a discrete optimization problem,
Gabriel et al. (2013) attempt to find the set of Nash equilibria to Eq. 2 by appealing to the continuous relaxation of each player’s parametric optimization problem:

$$\min \left\{ f_p(x_p, x_{-p}) : x_p \in C_p \right\}.$$  \hspace{1cm} (3)

Assume that the functions $f_p(\cdot, x_{-p})$ are convex and a constraint qualification for the continuous relaxation $C_p$ holds. Then, the KKT conditions for player $p$’s relaxed problem (3) are to find $x_p \in \mathbb{R}^{n_p}$, $\lambda_p \in \mathbb{R}^{|I_p|}$, $\gamma_p \in \mathbb{R}^{|E_p|}$ such that

$$0 \leq \nabla x_p f_p(x_p, x_{-p}) + \sum_{j \in I_p} \lambda_{pj} \nabla g_{pj}(x_p) + \sum_{k \in E_p} \gamma_{pk} \nabla h_{pk}(x_p) \perp x_p \geq 0 \hspace{1cm} (4a)$$

$$0 \leq -g_{pj}(x_p, x_{-p}) \perp \lambda_{pj} \geq 0 \hspace{1cm} \forall j \in I_p \hspace{1cm} (4b)$$

$$h_{pk}(x_p, x_{-p}) = 0, \hspace{1cm} \forall k \in E_p \hspace{1cm} (4c)$$

Gabriel et al. (2013) (p.313) then write:

“An interesting question is whether the set of $x_p$ that solves (4a), but with the discrete restrictions for $x_{pr} \in \mathbb{Z}_+$ for $r \in D_p$, corresponds to the solution set of the original problem (2). The next result shows that this correspondence is correct.”

**Theorem 1** (Theorem 3 in Gabriel et al. (2013)) Let $S^{DC-Nash}$ be the set of solutions to the discretely-constrained Nash-Cournot game (2) and $S^{DC-MCP}$ be the set of solutions to Eq. 4a for which $x_{pr} \in \mathbb{Z}_+$ for $r \in D_p$. Then, $S^{DC-Nash} = S^{DC-MCP}.$

### 2 Counterexamples

We now provide two simple discretely-constrained Nash-Cournot duopoly games (i.e., $\mathcal{P} = \{1, 2\}$) for which one or more equilibria exist to Eq. 2, but the complementarity conditions coupled with integrality restrictions are either 1) empty, or 2) non-empty, but a strict subset of the true set of equilibria. In both examples, because each player controls a single decision variable, we index player $p$’s decision variable as $x_p$ rather than $x_{p1}$.

#### 2.1 “Linear” Players with Weak Continuous Relaxations

Consider the simple Nash-Cournot duopoly game with the following symmetric cost matrix:

\[
\begin{array}{c|cc}
   & x_2 = 0 & x_2 = 1 \\
\hline
x_1 = 0 & 0 & -1 \\
\hline
x_1 = 1 & -1 & -2
\end{array}
\]

Here each player can take a discrete (binary) action with the unique equilibrium being $x_1 = x_2 = 1$, i.e., each player chooses action 1 for a (minimum) payoff of
-2, which is obviously a dominant strategy for each player. We now translate this DC-NC game into an optimization framework. Suppose player $p \in \{1, 2\}$ solves the following problem:

$$f_p^*(x_p) = \min \left\{ -x_p - x_{-p} : x_p \in [0, 1 + \epsilon] \cap \mathbb{Z} \right\},$$

(5)

where $\epsilon > 0$ and $\mathbb{Z}$ is the set of integers. Note that $f_p(x_p, x_{-p})$ are linear (hence, convex) functions and a constraint qualification holds. The corresponding KKT optimality conditions to the continuous problem are

1. $0 \leq \lambda_p - 1 \perp x_p \geq 0 \quad \forall p$ (6a)
2. $0 \leq 1 + \epsilon - x_p \perp \lambda_p \geq 0 \quad \forall p$ (6b)

We now plug in the unique equilibrium solution $x_1 = x_2 = 1$. Complementarity conditions (6a) imply that $\lambda_p = 1$, while conditions (6b) imply that $\lambda_p = 0$. This contradiction reveals that the unique equilibrium solution $x_1 = x_2 = 1$ is not in $S_{DC-MCP}$, i.e., $\emptyset = S_{DC-MCP} \subset S_{DC-Nash} \neq \emptyset$.

### 2.2 “Quadratic” Players with Tight Continuous Relaxations

In this example, the continuous relaxation for each player is tight. Consider the payoff matrix

|       | $x_2 = 0$ | $x_2 = 1$ |
|-------|-----------|-----------|
| $x_1 = 0$ | (0, 0)    | (9, 9)    |
| $x_1 = 1$ | (4, 4)    | (1, 1 - $\delta$) |

For $\delta > -3$, there are two equilibria in pure strategies: $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 1)$.

This corresponds to player 1 solving the following convex quadratic problem (as a function of $x_2$):

$$f_1^*(x_2) = \min_{x_1} (2x_1 - 3x_2)^2$$

s.t.

$$x_1 - 1 \leq 0$$

(7a)

$$-x_1 \leq 0$$

(7b)

Meanwhile, player 2 solves a similar convex quadratic problem (as a function of $x_1$):

$$f_2^*(x_1) = \min_{x_2} (2x_1 - 3x_2)^2 - \delta x_1 x_2$$

s.t.

$$x_2 - 1 \leq 0$$

(8a)

$$-x_2 \leq 0$$

(8b)

Note that $f_p(\cdot, x_{-p})$ are convex functions and a constraint qualification holds.
The KKT conditions (4a) become

\begin{align}
0 & \leq 4(2x_1 - 3x_2) + \lambda_1 \quad \downarrow x_1 \geq 0 \quad (9a) \\
0 & \leq -6(2x_1 - 3x_2) - \delta x_1 + \lambda_2 \quad \downarrow x_2 \geq 0 \quad (9b) \\
0 & \leq 1 - x_p \quad \downarrow \lambda_p \geq 0 \quad \forall p \quad (9c)
\end{align}

Assume \( \delta > -3 \). It is straightforward to verify that \( x_p = \lambda_p = 0 \) for all \( p \) satisfy the complementarity conditions (9a). The situation is different for \( (x_1, x_2) = (1, 1) \). Condition (9a) implies that \( \lambda_1 = 4 \), while condition (9c) implies that \( \lambda_p \geq 0 \) for all \( p \). However, condition (9b) implies that \( \lambda_2 = -6 + \delta \). Thus, for \( \delta \in (-3, 6) \), the complementarity approach fails to recognize \( (x_1, x_2) = (1, 1) \) as an equilibrium. It it tempting to argue that when \( \delta \in (-3, 1] \), this omission is not a concern because \( (x_1, x_2) = (0, 0) \) is the preferred equilibrium (i.e., the global minimizer for both players). However, for \( \delta > 1 \), player 2’s global minimizer is \( (x_1, x_2) = (1, 1) \) with a payoff of \( 1 - \delta \) and, for \( \delta \in (1, 6) \), the complementarity approach does not “see” this solution as an equilibrium. In short, this example shows that, not only can the complementarity approach fail to find all equilibria to a DC-NC game, it is not guaranteed to find global optima for each player when it does return an equilibrium.

Note that one can obtain a similar result (counterexample) by replacing the \( L_2 \) term \( (2x_1 - 3x_2)^2 \) with the \( L_1 \) term \( |2x_1 - 3x_2| \) so that each player solves a linear optimization problem instead of a convex quadratic one.

\section{Resolution}

For completeness, the correct version of Theorem 3 in Gabriel et al. is

\begin{theorem}
Let \( S^{DC-Nash} \) be the set of solutions to the discretely-constrained Nash-Cournot game (2) and \( S^{DC-MCP} \) be the set of solutions to Eq. 4a for which \( x_p \in \mathbb{Z}_+ \) for \( r \in D_p \). Then, \( S^{DC-MCP} \subseteq S^{DC-Nash} \) and there exist cases when \( S^{DC-MCP} \nsubseteq S^{DC-Nash} \).
\end{theorem}

\begin{proof}
See the proof of Theorem 3 in Gabriel et al. (2013), excluding the last sentence.
\end{proof}

The flaw in Theorem 3 of Gabriel et al. is in attempting to reverse the very first statement of the proof. That is, the fact that \( \hat{x}_p \) solves (1a) does not imply that \( \hat{x}_p \in S^{DC-MCP} \). Why? KKT conditions require differentiable functions. Thus, Gabriel et al. chose to relax integrality to apply the theory that works so well with convex differentiable functions and then re-introduce integrality. This sequence of operations is practical, but does not come with theoretical guarantees. As shown in counterexample 1, coupling the KKT conditions of the continuous relaxation \( \mathcal{C}_p \) with integrality restrictions yielded an empty set. This could have been corrected in counterexample 1 if \( \mathcal{C}_p \) were the integer hull of each player’s feasible region \( \mathcal{X}_p \). Counterexample 2 shows that even if \( \mathcal{C}_p \) corresponds to the integer hull of \( \mathcal{X}_p \), the complementarity approach may still fail to identify all equilibria.
Finally, note that the heuristic proposed by Gabriel et al. using model (15) in Gabriel et al. (2013) to solve the DC-NC game (2) is still valid.

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