MODULI OF OPEN STABLE MAPS TO A HOMOGENEOUS SPACE

AMITAI NETSER ZERNIK

Abstract. For \( L \hookrightarrow X \) a Lagrangian embedding associated with a real homogeneous variety, we construct the moduli space of stable holomorphic discs mapping to \((X, L)\) as an orbifold with corners equipped with a group action. Some essential constructions involving orbifolds with corners are also discussed, including the existence of fibered products and pushforward and pullback of differential forms with values in a local system.

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1. Introduction

Let \((X, \omega)\) be a closed symplectic manifold and \( L \subset X \) be a Lagrangian submanifold. We are interested in invariants derived from stable maps of discs to \( X \), whose boundary is required to lie on \( L \). The first step in producing such invariants is to construct something akin to a singular chain from the moduli spaces of such maps, and requires introducing perturbations that make the Cauchy-Riemann operator regular. Proving the existence of the desired perturbation data and keeping track of the choices involved is a highly non-trivial task, which has been tackled using

Hebrew University, amitai.zernik@mail.huji.ac.il
different approaches, each with contributions by numerous authors (we will not attempt to list them here). In contrast, we will show that if the pair \((X, L)\) is a real homogeneous pair, no perturbations are necessary, and the moduli space of stable disc-maps \(\overline{M}_{0,k,l}(X, L, \beta)\) can be obtained from the moduli space of closed maps \(\overline{M}_{0,n}(X, A)\) through a sequence of simple geometric constructions.

This is an open analog of a result, due to Fulton and Pandharipande \[3\], that the moduli space of stable maps to a convex non-singular projective variety, such as an algebraic homogeneous space, is a smooth Deligne-Mumford stack whose associated coarse moduli space is projective. The starting point for our construction is the corresponding analytical statement, due to Robbin, Ruan and Salamon \[13, 
Remark 3.13\]: if a Kähler manifold \(X\) is endowed with a transitive action of a compact Lie group, then the moduli space \(\overline{M}_{0,n}(X, A)\) is a compact, complex orbifold (without boundary).

We turn to a precise statement of our main result.

**Definition 1.** A real homogeneous variety is a tuple 
\((X, \omega, J, G_X, \alpha, c_G, c_X)\)
where:

- \(X\) is a compact Kähler manifold, with symplectic form \(\omega\) and an \(\omega\)-compatible integrable complex structure \(J\),
- \(G_X\) is a compact Lie group,
- \(\alpha : G_X \times X \to X\) defines a transitive action of \(G_X\) on \(X\) which preserves \(\omega\) and \(J\), and
- \(c_G : G_X \to G_X\) and \(c_X : X \to X\) are a pair of involutions such that \(c_G\) is a group homomorphism, \(c_X\) is anti-symplectic and anti-holomorphic, and \(c_X \alpha (g, x) = \alpha (c_G g, c_X x)\).

A real homogeneous pair \((X, L = X^{\mathbb{Z}/2})\) is a pair where \(X\) is a real homogeneous variety and \(L = X^{\mathbb{Z}/2}\) denotes the submanifold of real, or \(c_X\)-fixed, points of \(L\).

If \((X, L)\) is a real homogeneous pair,

\[i_L : L \hookrightarrow X\]

is a Lagrangian embedding and the action of the \(c_G\)-invariant subgroup \(G = G_X^{\mathbb{Z}/2}\) on \(X\) preserves \(L\). It is not hard to check that the induced map \(g := T_0 G \to T_x L\) is surjective for every \(x \in L\).

If \(k, l\) are non-negative integers, we write

\[G_{k,l} = G \times \text{Sym}(k) \times \text{Sym}(l)\]

for the product of \(G\) with the symmetric groups on \(k\) and on \(l\) elements. Section 2 is devoted to proving the following theorem.

**Theorem 2.** Let \((X, L)\) be a real homogeneous pair. Let \(k, l\) be non-negative integers and \(\beta \in H_2(X, L)\). Suppose \(\beta \neq 0\) or \(k + 2l \geq 3\). Then the moduli space

\[\overline{M}_{0,k,l}(\beta) = \overline{M}_{0,k,l}(X, L, \beta)\]

parameterizing families of stable holomorphic disc-maps of degree \(\beta\) with \(k\) boundary marked points and \(l\) interior marked points, is a compact \(G_{k,l}\)-orbifold with corners,
admitting a $G_{k,l}$-equivariant map

$$\overline{\mathcal{M}}_{0,k,l} (\beta) \xrightarrow{f} \overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta}).$$

There’s a unique $G_{k,l}$-equivariant map

$$\overline{\mathcal{M}}_{0,k,l} (\beta) \xrightarrow{\text{ev}} L^k \times X^l$$

such that

$$\text{ev}_c \circ f = (i_L^k \times (i_X^l \circ \Delta_X^l)) \circ \text{ev}. $$

Here

$$\overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta}) = \overline{\mathcal{M}}_{0,k+2l} (X, \beta + \overline{\beta})$$
denotes the moduli space of stable genus zero maps of class $\beta + \overline{\beta}$ (see (3)) with $k+2l$ marked points, and $\text{ev}_c : \overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta}) \to X^{k+2l}$ is the associated evaluation map. The precise meaning of “a compact $G_{k,l}$-orbifold with corners” and “a $G_{k,l}$-equivariant map” is discussed in Section 3.

**Example 3.** For every positive integer $n$, $(X, L) = (\mathbb{CP}^n, \mathbb{RP}^n)$ is a real homogeneous pair with $G_X = U(n + 1)$ the group of $(n + 1) \times (n + 1)$ unitary matrices, acting by the restriction of the standard $GL_{n+1} (\mathbb{C})$ group action on $\mathbb{CP}^n$, and with $c_G$ and $c_X$ given by

$$[c_G (A)]_{i,j} := A_{i,j}$$

and

$$c_X ([z_0 : \cdots : z_n]) := [\frac{z_0}{z_1} : \cdots : \frac{z_n}{z_1}].$$

By Theorem 2 the moduli spaces $\overline{\mathcal{M}}_{0,k,l} (\mathbb{CP}^n, \mathbb{RP}^n, \beta)$ are compact orbifolds with corners.

Doing away with perturbations has several technical advantages. For instance, one can given an elementary construction of the Fukaya $A_\infty$ algebra of $L \subset X$ using pullback and pushforward of differential forms, see Solomon and Tukachinsky [13]. Moreover, the $G$-equivariant extension of various constructions is considerably simpler than if perturbations have to be taken into account, see [19].

The close relationship between the moduli spaces of discs and the moduli space of curves is also quite useful - see for example the computation of the torus fixed points and their tubular neighborhoods in [17].

We can summarize the construction of $\overline{\mathcal{M}}_{0,k,l} (\beta)$ by the following $G$-equivariant diagram

$$\overline{\mathcal{M}}_{0,k,l} (\beta) \xrightarrow{\delta} \mathcal{M}^{\mathbb{C}} \xrightarrow{\text{ev}} \overline{\mathcal{M}} \xrightarrow{\overline{B}_{\text{ev}}} \overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta}) \xrightarrow{\overline{\text{ev}}} \overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta}).$$

The moduli space $\overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta})$ of closed maps is a complex orbifold of the expected dimension, since the associated Cauchy-Riemann operator is regular at every point. We denote by $\overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta})^{\mathbb{R}}$ the stacky fixed-points of this space, with respect to an anti-holomorphic involution which conjugates the map and swaps some of the markings. One should think of $\overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta})^{\mathbb{R}}$ as parametrizing the double of the disc-map; points of $\overline{\mathcal{M}}_{0,k+2l} (\beta + \overline{\beta})^{\mathbb{R}}$ are represented by a stable map $(\Sigma \xrightarrow{b} X, ...)$ together with an anti-holomorphic involution of the domain $b : \Sigma \to \Sigma$, that swaps some of the marked points and fixes others (see Lemma 4).
Following Liu \cite{Liu} §2.3 we call the data \(\left(\left(\Sigma \twoheadrightarrow X, \ldots \right), b\right)\) a symmetric configuration. To halve the double and recover the disc map we’re interested in, we must choose a fundamental domain \(\Sigma^{1/2} \subseteq \Sigma\) for the involution \(b\), whose boundary \(\Sigma^b\) is the \(b\)-fixed points of \(\Sigma\):

\[
\Sigma^{1/2} \cap b(\Sigma^{1/2}) = \Sigma^b,
\]

see Definition \cite{Liu}. Indeed, the space \(\mathcal{M}^\circ\) will parameterize all possible fundamental configurations \(\left(\left(\Sigma \twoheadrightarrow X, \ldots \right), b, \Sigma^{1/2}\right)\), and the map \(\mathcal{M}^\circ \xrightarrow{B \circ o} \overline{\mathcal{M}}_{0, k+2l} \left(\beta + \overline{\beta}\right)^{2/2}\) will be the forgetful map

\[
\left(\left(\Sigma \twoheadrightarrow X, \ldots \right), b, \Sigma^{1/2}\right) \mapsto \left(\left(\Sigma \twoheadrightarrow X, \ldots \right), b\right).
\]

To construct \(\mathcal{M}^\circ\), we must first cut up \(\overline{\mathcal{M}}_{0, k+2l} \left(\beta + \overline{\beta}\right)^{2/2}\) along the locus of configurations with real, or \(b\)-invariant, nodes. We call this process a hyperplane blowup (see \S3.3). It produces the map

\[
\overline{\mathcal{M}} \xrightarrow{B} \overline{\mathcal{M}}_{0, k+2l} \left(\beta + \overline{\beta}\right)^{2/2}.
\]

The choice of fundamental domain then corresponds to choosing an element of the map

\[
\mathcal{M}^\circ \sim \overline{\mathcal{M}},
\]

which is a 2-sheeted cover of its image.

There are two ways to see why we need to introduce the hyperplane blowup \(B\). First, note that sometimes \(\Sigma^b = \emptyset\), as in the following example adapted from Liu, \cite{Liu} Example 3.6]. For \(-1 \leq \epsilon \leq 1\) the maps

\[
\Sigma = \left\{\left\{x : y : z\right\} \in \mathbb{CP}^2 \mid x^2 + y^2 + \epsilon z^2 = 0\right\} \to X = \mathbb{CP}^1
\]

by

\[
[x : y : z] \mapsto [x : y]
\]

with the standard conjugation action on the domain and range, define a path

\[
\gamma : [-1, 1] \to \overline{\mathcal{M}}_{0, 0, 0} \left(\mathbb{CP}^1, 2\right)^{2/2}.
\]

For \(\epsilon > 0\) we have \(\Sigma^b = \emptyset\), which are configurations we’d like to discard since they’re not the double of any disc map. For \(\epsilon < 0\) we have a valid configuration, obtained as the double of a degree two map \(\left(D^2, \partial D^2\right) \to \left(\mathbb{CP}^1, \mathbb{RP}^1\right)\). At \(\epsilon = 0\) the boundary of the disc map degenerates to a real node of type \(E\) (see Definition \cite{Liu} and \cite{Liu} Definition 3.4). We find that \(\gamma|_{[-1, 0]}\) admits a (non-unique) lift to a path in \(\overline{\mathcal{M}}_{0, 0, 0} \left(\mathbb{CP}^1, \mathbb{RP}^1, (1, 1)\right)\), where the boundary of the disc-map shrinks to \([0 : 0 : 1] \in L\). In other words, the first reason to blow up is so we can discard unwanted symmetric topologies (see \cite{Liu} §2.3) - such configurations cannot be in the image of the map \(\mathcal{M}^\circ \xrightarrow{B \circ o} \overline{\mathcal{M}}_{0, k+2l} \left(\beta + \overline{\beta}\right)^{2/2}\).

We turn to the second reason to introduce the hyperplane blowup \(B\). Consider some configurations with some number \(r \geq 1\) of type \(H\) nodes, so \(\Sigma^b/\nu\) consists of \(r + 1\) circles glued at \(r\) pairs of points, and the number of possible fundamental domains (i.e., the size of the fiber of the map \(B \circ o\)) is \(2^{r+1}\). This also suggests we should cut up the moduli space. Indeed, the map \(B\) is locally modeled on the gluing of orthants: \(2^r \times \mathbb{R}^{n-r} \times [0, \infty)^r \to \mathbb{R}^n\). Picking one of the \(2^r\) inverse images of \(0 \in \mathbb{R}^n\) amounts to picking a smoothing direction for the \(r\) nodes, and there are

\[\text{More precisely, we have } \partial \Sigma^{1/2} = \Sigma^b \text{ only at configurations which do not have an E-type node. Such a node represents the degeneration of } \partial \Sigma \text{ to a single point, see more about this below.}\]
precisely two fundamental domains compatible with every such choice of smoothing directions.

With $\tilde{M}$ in place, the map $\mathcal{M}^c \xrightarrow{\sim} \tilde{M}$ is the composition of a 2-sheeted cover (forgetting the choice of fundamental domain $\Sigma^{1/2} \subset \Sigma$) with the inclusion of a clopen component (omitting configurations with $\Sigma^b = \emptyset$).

Finally, we restrict to a clopen component $\overline{\mathcal{M}}_{\beta} \xrightarrow{\sim} \mathcal{M}^c$, corresponding to those fundamental domains $\Sigma^{1/2}$ such that (i) $u_\ast ([\Sigma^{1/2}, \Sigma^b]) = \beta$ and (ii) $\Sigma^{1/2}$ contains a specific subset of the $k + 2l$ marked points. Put another way, once we restrict to the clopen component

$$\mathcal{M}_\beta \hookrightarrow \mathcal{M}^c$$

of those fundamental domains such that (i) holds, we need to take the quotient

$$\mathcal{M}_\beta \to \overline{\mathcal{M}}_{0, k, l} (\beta)$$

by the free $(\mathbb{Z}/2)^i$ group action where the $i$'th $\mathbb{Z}/2$ factor acts by swapping the labels of the $k + i$ and $k + l + i$ markings (these two markings should be indistinguishable, representing the same interior marked point). Instead, we take a section of this quotient

$$\overline{\mathcal{M}}_{0, k, l} (\beta) \to \mathcal{M}_\beta$$

by restricting further the clopen component to include only those configurations where the fundamental domain contains a particular representative of each pair \{k + i, k + l + i\}.

Section 3 contains essential constructions and results related to orbifolds with corners. This section is written with a view towards subsequent applications, and thus covers significantly more than is strictly necessary for the proof of Theorem 2.

Most notably, we discuss

- the existence of fibered products in the category of orbifolds with corners, relying on the work of Joyce \cite{Joyce} on manifolds with corners.
- Differential forms, local systems, pushforward and pullback operations.
- The notion of a hyperplane blowup.
- Group actions on orbifolds with corners.

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2. Proof of Theorem 2

2.1. The groupoid of disc-map configurations. We begin by reviewing the notion of a stable disc-map and isomorphisms of such maps. In Liu \cite[§5.1]{Liu}, the general notion of a stable map from a Riemann surfaces with boundary to an arbitrary pair $(X, L)$, is given. In general, the moduli spaces of such maps admit only a Kuranishi structure with corners, see \cite[Theorem 1.2]{Liu}. In contrast, we will see that the moduli spaces of disc maps to a real homogeneous pair can be given the structure of an orbifold with corners.

Let $(X, L)$ be a real homogeneous pair as in Definition 1. A tuple $(k, l, \beta)$, where $k$ and $l$ are non-negative integers and $\beta \in H_2 (X, L)$, is called a moduli specification if either $\beta \neq 0$ or $k + 2l \geq 3$. Fix some moduli specification $\mathfrak{b} = (k, l, \beta)$. Consider
the groupoid $D_b$ of $(k,l,\beta)$-disc-map configurations, whose objects consist of tuples $(\Sigma, \kappa, \lambda, \nu, u)$, where

- $\Sigma$ is a possibly disconnected compact Riemann surface with boundary and $\nu : (\Sigma, \partial \Sigma) \to (\Sigma, \partial \Sigma)$ is an involution with finitely many orbits of size 2. The configuration is connected and has genus zero. That is to say, each connected component of $C \subset \Sigma$ is diffeomorphic to $D^2$ or to $\mathbb{C}\mathbb{P}^1$, and the graph $(\pi_0 (\Sigma), \nu)$ is in fact a tree.

Here

\[(\pi_0 (\Sigma), \nu)\]

is the incidence graph, whose vertices are the connected components of $\Sigma$, and where each orbit of size two $o$ of $\nu$ determines an edge, incident to the connected components $o$ intersects. We denote by $\Sigma^\nu$ and $\partial \Sigma^\nu$ the fixed points of $\nu$ in the interior and boundary of $\Sigma$, respectively. These define the smooth points of the orbit space $\Sigma/\nu$. The orbits of size two are called nodes.

- $\kappa : [k] \to \partial \Sigma^\nu$ and $\lambda : [l] \to \Sigma^\nu$ are injective maps.

Note that we do not assume the points $\text{Im} \kappa$ appear in any particular order around $\partial \Sigma$.

- $u : (\Sigma, \partial \Sigma) \to (X, L)$ is a $\nu$-invariant holomorphic map with $u_* [\Sigma, \partial \Sigma] = \beta \in H_2 (X, L)$.
- For each connected component $C \subset \Sigma$, we have $u_* [C, \partial C] \neq 0$ or

\[\left| (\kappa^{-1} (\partial C)) + |\partial C \setminus \Sigma^\nu| \right| + 2 \left| (\lambda^{-1} (C)) + |\partial C \setminus \Sigma^\nu| \right| \geq 3.\]

There’s an arrow in $D_b$ connecting two objects $(\Sigma, \kappa, \lambda, \nu, u)$ and $(\Sigma', \kappa', \lambda', \nu', u')$ for every biholomorphism $\phi : \Sigma \to \Sigma'$ preserving all of the additional structure. The product $G_b = G \times \text{Sym} (k) \times \text{Sym} (l)$ of the compact lie group $G$ with the permutation groups acts on $D_b$ by translating maps and relabeling markings.

We will show that the groupoid $D_b$ is equivalent to a $G_b$-orbifold with corners $\overline{M}_{0,k,l} (\beta)$. The construction proceeds from right to left, along the diagram (1).

### 2.2. Complex moduli of closed maps.

Let $n$ be a non-negative integer and $A \in H_2 (X)$. Following [13] we consider $(n, A)$-rational configurations. An $(n, A)$-rational configuration is a tuple $(\Sigma, \nu, \lambda, \mu, u)$ where $\Sigma$ is a disjoint union of $\mathbb{C}\mathbb{P}^1$s, $\nu$ is an involution with finitely many orbits of size two, and the incidence graph $(\pi_0 (\Sigma), \nu)$ is required to be a tree. $\lambda$ is an injective map $[n] \to \Sigma^\nu$, and $u : \Sigma \to X$ is a $\nu$-invariant holomorphic map with $u_* [\Sigma] = A \in H_2 (X)$. Finally, for every connected component $C \subset \Sigma$ if $u_* [C] = 0$ then $|C \setminus \Sigma^\nu| \geq 3$. The collection of configurations are the objects of a groupoid, in which there’s an arrow $(\Sigma, \nu, \lambda, \mu, u) \to (\Sigma', \nu', \lambda', \mu', u')$ for every biholomorphism $\phi : \Sigma \to \Sigma'$ respecting all of the structure.

By [13] Remark 3.13 if $X$ is a real homogeneous variety then for any $A \in H_2 (X)$ the moduli space of stable holomorphic maps $\overline{M}_{0,n} (A)$ is a compact complex orbifold.

The compact group $G_X = G_X \times \text{Sym} (n)$ acts on $\overline{M}_{0,n} (A)$ by translating maps and relabeling markings. This action can be constructed by direct analogy with the construction of the $\mathbb{Z}/2$-action given in [12] below.

In (1), we’ve set $n = k + 2l$, and taken $A = \beta + \bar{\beta} \in H_2 (X)$, which is defined as follows. There’s a map

\[(3) \quad H_2 (X, L) \ni \beta \mapsto \beta + \bar{\beta} \in H_2 (X),\]
which takes a singular chain \( \sigma \in C_2(X) \) with \( \partial \sigma \in C_1(L) \subset C_1(X) \) representing \( \beta \in H_2(X,L) \) to the homology class represented by the cycle \( \sigma + (c_X)_* \sigma \) (recall \( c_X : X \to X \) is an antiholomorphic involution which fixes \( L \)).

2.3. Orbifolds as Stacks. We briefly review how one can think about orbifolds in terms of stacks, following Metzler [10] and Pronk [12]. See also Lerman [7] for a very pleasant explanation of this approach. A smooth stack \( \mathcal{C} \) is a category together with a functor \( \mathcal{C} \to \text{Man} \) to the category of smooth manifolds, such that (i) \( \mathcal{C} \) is a category fibered in groupoids over \( \text{Man} \) and (ii) a certain descent condition is satisfied. Given an orbifold without boundary \( \mathcal{M} \) (which we take to mean an object of the bicategory of fractions \( \text{Orb}_{\partial = 0} \) given by a groupoid in \( \text{Man} \); see Remark [24]), we obtain a stack by taking \( \mathcal{C} \) to be the category whose objects are 1-cells \( T \to \mathcal{M} \) where \( T \) is a manifold (without boundary) and morphisms between \( T_1 \xrightarrow{f_1} \mathcal{M} \) to \( T_2 \xrightarrow{f_2} \mathcal{M} \) are smooth maps \( T_1 \xrightarrow{g} T_2 \) together with a 2-cell \( f_1 \Rightarrow f_2 \circ g \). The functor \( \pi \) sends \( T \to \mathcal{M} \) to \( T \).

The category of smooth maps \( \mathcal{M} \to \mathcal{M} \) is equivalent to the full subcategory of the category of functors \( \mathcal{C} \to \mathcal{C} \), consisting of those functors that commute with \( \pi \). In particular, a 2-cell \( \alpha : c^2 \to \text{id} \) is given simply by a natural transformation between functors \( \mathcal{C} \to \mathcal{C} \).

This perspective is especially useful for thinking about \( \overline{\mathcal{M}}_{0,n} (A) \). The category of holomorphic maps \( T \to \overline{\mathcal{M}}_{0,n} (A) \) is equivalent to the category of stable families of maps over \( T \), i.e. stable families of maps of type \((0,n,A)\) in the sense of [13, 3.1], which are of the form

\[
(\pi : Q \to T, S_1, \ldots, S_n : T \to Q, H : Q \to X).
\]

We recall that this means: (i) \( \pi, S_*, H \) are holomorphic maps, (ii) \( \pi \) is only allowed certain nodal singularities, (iii) for each \( t \in T \) the fiber \( \pi^{-1}(t) \) is a nodal Riemann surface of genus zero, and \( H|_{\pi^{-1}(t)} \) represents \( A \), (iv) \( S_1, \ldots, S_n \) are sections of \( \pi \), and (v) \((\pi^{-1}(t), S_1(t), \ldots, S_n(t), H|_{\pi^{-1}(t)})\) is a stable map. An arrow

\[
(\pi : Q \to T, S_* : T \to Q, H : Q \to X) \to (\pi' : Q' \to T, S'_* : T \to Q', H' : Q' \to X)
\]

is a holomorphic morphism \( Q \to Q' \) which preserves the additional data (cf. [13 §3.1]).

This allows us to define the 1-cells \( \overline{\mathcal{M}}_{0,n} (A) \to \overline{\mathcal{M}}_{0,n} (A) \) and the 2-cells between them simply by describing functorial manipulations of stable families of maps. We can extend the discussion to include anti-holomorphic maps (an anti-holomorphic map \( T \to \overline{\mathcal{M}}_{0,n} (A) \) is just a holomorphic map \( T \xrightarrow{\overline{\mathcal{M}}_{0,n} (A)} \), arrows between such maps are still given by holomorphic morphisms of families \( Q \to Q' \) as above), so we can also consider \( \overline{\mathcal{M}}_{0,n} (A) \) as a stack over the category of complex manifolds with both holomorphic and antiholomorphic maps.

2.4. A \( \mathbb{Z}/2 \)-action on \( \mathcal{M} = \overline{\mathcal{M}}_{0,n} (\beta + \bar{\beta}) \). Consider \( \mathcal{M} = \overline{\mathcal{M}}_{0,n} (\beta + \bar{\beta}) \) as a complex stack. There’s an antiholomorphic map

\[
c_M : \overline{\mathcal{M}}_{0,n} (\beta + \bar{\beta}) \to \overline{\mathcal{M}}_{0,n} (\beta + \bar{\beta})
\]

which sends a point \( p \) represented by the \((n, \beta + \bar{\beta})\)-rational configuration \((\Sigma, \nu, \lambda, u)\) to a point represented by

\[
(\Sigma, \nu, \lambda, u) := \left( \Sigma, c_\Sigma \circ \nu \circ c_\Sigma^{-1}, c_\Sigma \circ \lambda \circ c_\Sigma^{-1}, c_X \circ u \circ c_X^{-1} \right).
\]
with $c_\Sigma : \Sigma \to \Sigma$ the map which replaces the almost complex structure $j$ on $\Sigma$ by $-j$, and $c_n : [n] \to [n]$ the involution

$$
\begin{pmatrix}
1 & \cdots & k & k + 1 & \cdots & k + l & k + l + 1 & \cdots & k + 2l \\
1 & \cdots & k & k + l + 1 & \cdots & k + 2l & k + 1 & \cdots & k + l \\
\end{pmatrix}.
$$

To construct $\mathfrak{c}$ as a map of stacks, start from any family of maps

$$(\pi_B : Q \to B, S_1, \ldots, S_n : B \to Q, H_B : Q \to X)$$

and map it to the conjugate family of maps

$$
\left(\overline{Q} \xrightarrow{\overline{\pi}_B} \overline{B}, (S_{c_n(i)} : \overline{B} \to \overline{Q})_{i=1}^n, c_X \circ \overline{H}_B\right).
$$

This is functorial and thus defines the desired map of stacks $c = c_M : \mathcal{M} \to \mathcal{M}$. There’s a 2-cell

$$
\alpha : c^2 \Rightarrow \text{id}
$$

which is given by the obvious arrow $\overline{Q} \to Q$, considered as a natural transformation between functors-of-families. It satisfies a coherence condition, which is an equality of two 2-cells $c^2 \Rightarrow c$. Thus, $\mathfrak{c}$ and $\mathfrak{c}$ define a $\mathbb{Z}/2$-action on $\mathcal{M}$.

In fact, we will need a bit more than this, namely that the $G_X^+$ action extends to a $G_X^+ \times \mathbb{Z}/2$ action on $\mathcal{M}$, where the semidirect product is defined using the homomorphism $\mathbb{Z}/2 \to \text{Aut}(G_X^+)$ in which the generator acts by

$$
G_X^+ = G_X \rtimes \text{Sym}(n) \ni (\phi, \sigma) \mapsto (c_G(\phi), c_n \circ \sigma \circ c_n^{-1}).
$$

It is straightforward to construct the data of this group action in terms of functors and natural transformations of families, as above. We will focus our attention on the $\mathbb{Z}/2$ action to keep the notation palatable.

2.5. The $\mathbb{Z}/2$-fixed points of $\mathcal{M}$. Let $\mathcal{M}^{\mathbb{Z}/2} = \overline{\mathcal{M}}_{0,n} (\beta + \overline{\beta})^{\mathbb{Z}/2}$ denote the $\mathbb{Z}/2 < G_X^+$-stacky fixed points of $\mathcal{M}$, and let $G_+$ denote the group of elements of $G_X^+$ that are fixed under $\mathfrak{c}$:

$$
G_+ = (G_X^+)^{\mathbb{Z}/2} = G \times \text{Sym}(k) \times \text{Sym}(l) \times (\mathbb{Z}/2)^1.
$$

**Lemma 4.** (a) $\mathcal{M}^{\mathbb{Z}/2}$ is a $G_+$-orbifold (without boundary) with

$$
\dim \mathcal{M}^{\mathbb{Z}/2} = \frac{1}{2} \dim \mathcal{M}
$$

and the map $\mathcal{M}^{\mathbb{Z}/2} \to \mathcal{M}$ is a proper closed immersion. In particular, $\mathcal{M}^{\mathbb{Z}/2}$ is compact.

(b) As a groupoid $\mathcal{M}^{\mathbb{Z}/2}$ is equivalent to the groupoid whose points are represented by pairs $((\Sigma, \nu, \lambda, u), b)$, where $(\Sigma, \nu, \lambda, u)$ is a rational $(n, \beta + \overline{\beta})$-configuration and $b : (\Sigma, \nu, \lambda, u) \to (\Sigma, \nu, \lambda, u)$ is an arrow in $\mathcal{M}$ such that $\overline{b} \circ b$ is the identity map, considered as a biholomorphism $(\Sigma, \nu, \lambda, u) \to (\Sigma, \nu, \lambda, u)$.

---

2See Definition 25. We warn the reader that this is not the usual meaning in algebraic geometry; rather it is a generalization of the notion of closed immersion of manifolds without boundary in differential geometry. In particular, $\iota$ will not be injective in general.
Proof. We prove part (a). Note that \( \mathcal{M} \) is in fact represented by a proper étale groupoid \( M_1 \to M_0 \) where \( M_i \) are complex manifolds and the structure maps \( s, t, e, i, m \) are local biholomorphisms. This implies we can consider \( \mathcal{M} \) as a holomorphic stack, or stack over the category of complex manifolds with holomorphic maps (cf. [10, Definition 28]). As a map between holomorphic stacks, \( M = M_0 \overset{a}{\to} \mathcal{M} \) is a holomorphic atlas. That is, a map whose domain is equivalent to a complex manifold, and such that the weak fibered product \( W = T \times_\mathcal{M} M_0 \) with any other map of holomorphic stacks from a complex manifold \( T \to \mathcal{M} \) is a complex manifold, and the pullback map \( W \to T \) is a surjective local biholomorphism.

We will now use a trick to construct a manifold \( P' \) with \( \dim P' = \frac{1}{2} \dim \mathcal{M} \) and an étale covering \( P' \overset{a'}{\to} \mathcal{M}^{\mathbb{Z}/2} \) for \( \mathcal{M}^{\mathbb{Z}/2} \), in the sense of [10, Proposition 75]. Since the diagonal \( \mathcal{M}^{\mathbb{Z}/2} \to \mathcal{M}^{\mathbb{Z}/2} \times \mathcal{M}^{\mathbb{Z}/2} \) is proper, this will imply that the conditions of [10, Proposition 75] are met and \( \mathcal{M}^{\mathbb{Z}/2} \) is a smooth orbifold as claimed.

To construct \( P' \), let \( \overline{\mathcal{M}} \) be the complex conjugate of \( \mathcal{M} \), so that \( c \circ \overline{\pi} : \overline{\mathcal{M}} \to \mathcal{M} \) is also a holomorphic atlas. A holomorphic map \( \mathcal{M} \to P \), or \( \mathcal{S}-\text{point} \), is given by

\[
\left( S \overset{x}{\to} \mathcal{M}, S \overset{y}{\to} \overline{\mathcal{M}}, a \circ x \right)
\]

Here \( \overset{b}{\Rightarrow} \) is a morphism of the category \( \mathcal{M} \), between the objects \( a \circ x \) and \( c \circ \overline{\pi} \circ y \).

We define a strict anti holomorphic involution on \( P \) by specifying its action on the \( S \)-points for all \( S \). Namely the involution sends \( (10) \) to

\[
\left( S \overset{y}{\to} \mathcal{M}, S \overset{x}{\to} \overline{\mathcal{M}}, a \circ y \overset{c(b)^{-1} \alpha^{-1}}{\Rightarrow} c \circ \overline{\pi} \circ x \right)
\]

This is an involution since

\[
c \circ (\alpha y) c^2 \circ (b) \alpha_x^{-1} = c \circ (\alpha_y) c^2 \circ (b) \alpha_x^{-1} = b \alpha x \alpha_x^{-1} = b.
\]

The first equality is the coherence condition, and the second is naturality of the transformation \( \alpha \). Now let \( P' \) denote the fixed-points of the involution. Using the slice theorem and linear algebra, the fixed-points of an anti holomorphic involution form a real submanifold \( P' \subset P \) with \( \dim P' = \frac{1}{2} \dim P \). It is easy to see that the \( S \)-points of \( P' \) are pairs, consisting of a map \( S \overset{x}{\to} \mathcal{M} \) together with an arrow \( a \circ x \overset{b}{\Rightarrow} c \circ a \circ x \), such that

\[
(11) \quad b = c(b)^{-1} \alpha^{-1}
\]

(we do not care about complex structures from this point inwards, so we write \( a \) and not \( \overline{a} \), etc.). Eq (11) is the defining relation for a \( \mathbb{Z}/2 \)-fixed point, cf. [14, Proposition 2.5]. In other words, we have a map of smooth stacks \( P' \overset{a'}{\to} \mathcal{M}^{\mathbb{Z}/2} \). To prove this is an étale covering, it suffices to show that \( a' \) is the pullback of the étale
covering \( a \). More precisely, we claim that the square

\[
\begin{array}{ccc}
P' & \rightarrow & M \\
\downarrow & & \downarrow \\
\mathcal{M}^\mathbb{Z}/2_{a} & \rightarrow & \mathcal{M}
\end{array}
\]

is 2-cartesian, where \( i' \) is the composition \( P' \rightarrow P = M \times_{\mathcal{M}} M \rightarrow M \). Indeed, an \( S \)-point of \( \mathcal{M}^\mathbb{Z}/2_{a} \) is represented by \((\pi, r, g)\) where \( S \pi \rightarrow \mathcal{M}^\mathbb{Z}/2 \) is an \( S \)-fixed-point, \( S r \rightarrow M \) is an \( S \)-point of \( M \), and \( a g \Rightarrow i \pi \) is an arrow of \( \mathcal{M} \). Let \( f := i \pi \), so \( \pi \) is represented by some arrow \( f \Rightarrow c \circ f \) in \( \mathcal{M} \) satisfying (11). Now it is easy to check that

\[
ag \Rightarrow c(r)^{-1} \circ c(a g)
\]
satisfies (11). Since this works for any \( S \), we’ve constructed an arrow

\[
\mathcal{M}^\mathbb{Z}/2_{a} \rightarrow P'.
\]

It is now straightforward to construct the reverse arrow \( P' \rightarrow \mathcal{M}^\mathbb{Z}/2_{a} \) and show these form an equivalence of stacks.

The map \( P' \rightarrow M \), hence the map \( i \), is seen to be a closed immersion. Since the fiber of \( i \) is finite, it is proper, and so \( \mathcal{M}^\mathbb{Z}/2 \) is compact. Since \( \mathcal{M} \) was equipped with a \( G^+ \times X \) action, \( \mathcal{M}^\mathbb{Z}/2 \) is equipped with a residual \( G^+ \)-action (see [14, Remark 2.4]).

Part (b) is straightforward. \(\square\)

**Definition 5.** We call \( ((\Sigma, \nu, \lambda, u), b) \) as in part (b) of Lemma 4 a symmetric configuration.

Note a fixed configuration \( ((\Sigma, \nu, \lambda, u), b) \) defines a “symmetric Riemann surface with \((l, k)\) marked points” in the sense of [8, §2.2.2].

2.6. **The blowup** \( \widetilde{\mathcal{M}} \) of \( \mathcal{M}^\mathbb{Z}/2 \). The next step is to blow up a hyper subset in \( \mathcal{M}^\mathbb{Z}/2 \) (the motivation for this was discussed in the introduction). Let \( ((\Sigma, \nu, \lambda, u), b) \) be a \( \mathbb{Z}/2 \)-fixed configuration. Let \( o = \{o_1, o_2\} \) be a node of the configuration, that is, an orbit of \( \nu \) of size 2. We say \( o \) is a real node if \( b(o) = o \).

**Definition 6.** A real node \( o = \{o_1, o_2\} \), \( b(o) = o \), is of one of two types (cf. [8 Definition 3.4]):

1. If \( b(o_1) = o_1 \) we say \( o \) is a type \( H \) node.
2. If \( b(o_1) = o_2 \) we say \( o \) is a type \( E \) node.

Nodes of type \( H \) correspond to strip breaking in Floer theory. Nodes of type \( E \) correspond to disc configurations where the boundary has degenerated to a point. Since we’re considering only genus zero configurations, if \( o \) is a type \( E \) node it is the only real node.

In [§3.3.2] we construct the hyperplane blowup of an orbifold \( X = X_1 \overset{s, t}{\rightarrow} X_0 \) along a hyper subset \( E \subset X_0 \), \( s^{-1}E = t^{-1}E \). In our case, we take \( E \) to be the subset of all points represented by configurations with a real node.
We now explain why this is a hyper subset. First, we reformulate some well-known properties of the divisor of nodal configuration.\footnote{In algebraic geometry this is often called the “boundary” divisor. This terminology becomes especially confusing in our context, so we avoid it.}

Let $B_{n,\beta,\overline{\beta}}$ denote set of all 4-tuples $(l_1, A_1, l_2, A_2)$ with $l_i \subset [n]$ and $A_i \in H_2(X)$, such that: (i) $A_1 + A_2 = \beta + \overline{\beta}$, (ii) $l_1 \sqcup l_2 = [n]$, and (iii) $A_i = 0 \Rightarrow |l_i| \geq 3$ for $i = 1, 2$. We denote

$$H_b = \{(l_1, A_1, l_2, A_2) \in B_{n,\beta,\overline{\beta}}| c_n(l_i) = l_i \text{ and } c_{X^*}A_i = A_i \text{ for } i = 1, 2\}$$

$$E_b = \{(l_1, A_1, l_2, A_2) \in B_{n,\beta,\overline{\beta}}| c_n(l_i) = l_{2-i} \text{ and } c_{X^*}A_i = A_{2-i} \text{ for } i = 1, 2\}.$$

Remark 7. Note that $E_b = \emptyset$ unless $k = 0$ and $\beta + \overline{\beta} \in \text{Im}(\text{id} + c_{X^*})$.

We have maps between orbifolds without boundary

\[ D^C_{H_b} : \left( \bigsqcup \overline{\mathcal{M}}_{0,l_i,\{*\}}(A_1) \times X \overline{\mathcal{M}}_{0,l_2,\{*\}}(A_2) \right)_{\mathbb{Z}/2} \to \overline{\mathcal{M}}_{0,n}(\beta + \overline{\beta}) \]

and

\[ D^C_{E_b} : \left( \bigsqcup \overline{\mathcal{M}}_{0,l_i,\{*\}}(A_1) \times X \overline{\mathcal{M}}_{0,l_2,\{*\}}(A_2) \right)_{\mathbb{Z}/2} \to \overline{\mathcal{M}}_{0,n}(\beta + \overline{\beta}) \]

defined by gluing. Henceforth, a subscript such as $l_i \sqcup \{*\}$ means we label the markings by the indicated finite set instead of $\{1, \ldots, n\}$. The fibered products are over the evaluation maps at $*_1, *_2$, which are transverse (cf. \cite{9}). The $\mathbb{Z}/2$ subscript denotes the stacky quotient by the action which swaps the two factors. The map

\[ D^C_{k,l,\beta} = D^C_{H_b} \bigsqcup D^C_{E_b} \]

is a faithful, proper, closed immersion with transversal self-intersection (cf. Definition \footnote{In algebraic geometry this is often called the “boundary” divisor. This terminology becomes especially confusing in our context, so we avoid it.}). Indeed, the essential fiber $F = (D^C_{k,l,\beta})^{-1}(p)$ over $p = [\Sigma, \kappa, \lambda, \nu, u]$ is in natural bijection with the subset of nodes $\{\{a_i, b_i\}\}_{i=1}^\nu$ of $\nu$ that partition the markings $[n]$ and the degree $\beta + \overline{\beta}$ into 4-tuples as in (12) or in (13). For any subset of nodes the associated map of conormal bundles

\[ \bigoplus_{i=1}^r (L_{a_i}^{\nu} \otimes L_{b_i}^{\nu})^* \to T^*\overline{\mathcal{M}}_{0,n}(\beta + \overline{\beta}) \]

is injective. Here $L_{a_i}^{\nu}$ denotes the tangent line to the universal curve at $a_i$.

We make $D^C_{H_b}, D^C_{E_b}$; and thus also $D^C_{k,l,\beta}$, into $\mathbb{Z}/2$-equivariant maps. $\mathbb{Z}/2$ acts on the domain of $D^C_{H_b}$ by the product of conjugation maps, where we treat $*_1$ and $*_2$ as fixed markings (i.e. we identify them with one of the first $k$ markings in (5)). The $\mathbb{Z}/2$ action on the domain of $D^C_{E_b}$ swaps the two moduli factors, sending $*_1$ to $*_2$ and vice-versa. Passing to $\mathbb{Z}/2$ invariants we obtain a hyper map (cf. Definition \footnote{In algebraic geometry this is often called the “boundary” divisor. This terminology becomes especially confusing in our context, so we avoid it.})

\[ W_{k,\beta} = \left( \bigsqcup \overline{\mathcal{M}}_{0,l_i,\{*\}}(A_1)^{\mathbb{Z}/2} \times L \overline{\mathcal{M}}_{0,l_2,\{*\}}(A_2) \right)_{\mathbb{Z}/2} \xrightarrow{D_{k,\beta}} \overline{\mathcal{M}}_{0,n}(\beta + \overline{\beta})^{\mathbb{Z}/2}, \]

with $\text{Im} D_{k,\beta} = E$. We let $\beta : \overline{\mathcal{M}} \to \overline{\mathcal{M}}_{0,n}(\beta + \overline{\beta})^{\mathbb{Z}/2}$ denote the hyperplane blowup of $\overline{\mathcal{M}}^{\mathbb{Z}/2}$ along $E$. It is a compact $G_\nu$-orbifold. $\overline{\mathcal{M}}$ is equivalent to the
2.7. The forgetful map. Let \( b = (k, l, \beta) \) be a basic moduli specification, let \( b_+ = (k + 1, l, \beta) \), and write \( n = k + 2l \). The forgetful map

\[
\pi^C: \overline{M}_{0, n+1} \left( \beta + \bar{\beta} \right) \to \overline{M}_{0, n} \left( \beta + \bar{\beta} \right)
\]

induces a map of fixed points \( \pi: \overline{M}_{0, n+1} \left( \beta + \bar{\beta} \right)^{2/2} \to \overline{M}_{0, n} \left( \beta + \bar{\beta} \right)^{2/2} \). Let \( \overline{M}_{0, n} \left( \beta + \bar{\beta} \right)^{2/2} \) denote the open suborbifold consisting of symmetric configurations with no nodes of type E, let \( \overline{M}_{0, n} \left( \beta + \bar{\beta} \right)^{2/2} = \pi^{-1} \left( \overline{M}_{0, n} \left( \beta + \bar{\beta} \right)^{2/2} \right) \), and let \( \overline{M}_{0, n} E, \overline{M}_{0, n} E,+ \) denote the corresponding hyperplane blowups at the loci of configurations with real nodes.

**Lemma 8.** \( \pi \) lifts to a b-fibration (see Definition 25)

\[
\tilde{\pi}: \overline{M}_{0, n} E, + \to \overline{M}_{0, n} E.
\]

**Proof.** The claim can be checked locally on the domain. Let \( \sigma_+ = \left( (\Sigma_+, \nu_+, \lambda_+, u_+), b_+ \right) \) and \( \sigma = \left( (\Sigma, \nu, \lambda, u), b \right) \) be symmetric configurations so that \( \pi \) maps

\[
p = [\sigma_+] \in \overline{M}_{0, n+1} \left( \beta + \bar{\beta} \right)^{2/2}
\]

to

\[
q = [\sigma] \in \overline{M}_{0, n} \left( \beta + \bar{\beta} \right)^{2/2}.
\]

This means \( \sigma \) is obtained from \( \sigma_+ \) by erasing the marked point \( \lambda_+ (k + 1) \) and contracting the incident component \( C \subset \Sigma_+ \) if it becomes unstable. Let \( x_1, ..., x_{N+1} \) be local coordinates for \( \overline{M}_{0, n+1} \left( \beta + \bar{\beta} \right)^{2/2} \) centered around \( p \) and \( y_1, ..., y_N \) be local coordinates for \( \overline{M}_{0, n} \left( \beta + \bar{\beta} \right)^{2/2} \) centered around \( q \). There are three cases to consider.

1. No component is contracted. In this case we can choose the coordinates so that the germ \( \pi_p \) is given by

\[
(y_1, ..., y_N) = (x_2, ..., x_N),
\]

with \( x_2, ..., x_{r+1} \) and \( y_1, ..., y_r \) the smoothing parameters for the real nodes of \( p \) and of \( q \), respectively. Clearly this map lifts to the hyperplane blowup and is a b-fibration.

2. The component \( C \) is contracted to a node of type H in \( \sigma \). In this case, we can choose the coordinates so \( \pi_{p} \) is given by

\[
(y_1, ..., y_N) = (x_1 \cdot x_2, x_3, ..., x_N),
\]

with \( x_1, x_2 \) the smoothing parameters of the two nodes of \( \sigma_+ \) incident to \( C \), \( y_1 \) the image H-node, and \( x_3, ..., x_{r+1} \) smoothing parameters for the other nodes of \( \sigma_+ \), with \( y_2, ..., y_r \) smoothing parameters for the corresponding nodes of \( \sigma \). Again, this map lifts to the hyperplane blowup. It is a b-fibration (cf. [4, Example 4.4(ii)])

3. The component \( C \) is contracted to a node of type E in \( \sigma \). In this case, we can choose the coordinates so \( \pi_p \) is given by

\[
(y_1, ..., y_N) = (\sqrt{x_1^2 + x_2^2}, x_3, ..., x_N).
\]
Here $x_1 + ix_2, x_1 - ix_2$ are the smoothing parameters for the complex-conjugate nodes incident to $C$. $\gamma_1$ is a smoothing parameter for the $E$-node. $\sigma, \sigma$, have no other real nodes in this case. The map $\mathbb{R}^2 \to [0, \infty)$, $(x_1, x_2) \mapsto (x_1^2 + x_2^2)$ is not smooth (as a map of manifolds with corners it is only weakly smooth; cf. Example 2.3(i) for a closely related example). This is why we discard such configurations from the codomain.

\begin{remark}
In studying the open Gromov-Witten theory of $(\mathbb{C}P^{2m}, \mathbb{R}P^{2m})$ $E$-nodes are excluded since the moduli spaces $\overline{\mathcal{M}}_{0,k,l}(\beta)$ there either have $k > 0$ or else $\beta \in H_2(X, L) = \mathbb{Z}$ is odd, which implies $\beta + \overline{\beta} \notin \text{Im}(1 + c_{X, \ast})$. Either way there are no $E$-type nodes by Remark 7. In other applications, one may be content to consider the forgetful map as a weakly smooth map only, though this requires a revision of the category of orbifolds with corners as defined in Section 3, which we will not pursue here. Finally, an $S^1$-blowup may be used to resolve the problem: the map $[0, \infty) \times S^1 \to [0, \infty)$

\[(r, \theta) \mapsto (r \cos \theta, r \sin \theta) \mapsto r^2\]

is a (smooth) b-fibration.
\end{remark}

2.8. Picking a fundamental domain and the map $\mathcal{M}^\circ \to \overline{\mathcal{M}}$. If $\mathcal{X}$ is an orbifold with corners, we denote by $\mathcal{X}^o$ the interior, or depth zero, points of $\mathcal{X}$. By construction, $\overline{\mathcal{M}}^o$ is an open suborbifold of $\mathcal{M}^{2/2}$, and its points are represented by symmetric configurations $\sigma = (\Sigma, \ldots, b)$ with no additional data.

Let $\overline{\mathcal{M}}^\circ_{s\emptyset} \subset \overline{\mathcal{M}}$ denote the clopen component which is the closure of those points $\overline{\mathcal{M}}^\circ$ represented by a symmetric configuration $((\Sigma, \nu, \lambda, u), b)$ with $\Sigma^b \neq \emptyset$. In fact, since there are no real nodes, we conclude that for $q \in \overline{\mathcal{M}}^\circ_{s\emptyset} \subset \mathcal{M}^{2/2}$ we must have

\[\tilde{\pi}^{-1}(q) = \Sigma^b \simeq S^1,\]

which is orientable. Let $\mathcal{L}$ denote the extension to the boundary (cf. Lemma 30 (a)) of the fiber orientation local system

\[\left[\tilde{\pi}^{-1}(\text{Or}(T\overline{\mathcal{M}}^\circ)) \otimes \text{Or}(T\overline{\mathcal{M}})^\nu\right]|_{\overline{\mathcal{M}}^\circ_{s\emptyset}}.\]

The unit sections of $\mathcal{L}$ form a 2-sheeted cover $\mathcal{M}^\circ \to \overline{\mathcal{M}}^\circ_{s\emptyset}$, and we define $o$ to be the composition

\[\mathcal{M}^\circ \to \overline{\mathcal{M}}^\circ_{s\emptyset} \to \overline{\mathcal{M}}.\]

We now describe $\mathcal{M}^\circ$ and $B \circ o$ in terms of fundamental domains.

\begin{definition}
(a) Let $\sigma = ((\Sigma, \nu, \lambda, u), b)$ be a symmetric configuration. A fundamental domain $\Sigma^{1/2}$ for $\sigma$ is a subset $\Sigma^{1/2} \subset \Sigma$ biholomorphic to a disjoint union of $\mathbb{C}P^1$s and $D^2$s such that:

(i) $\Sigma = \Sigma^{1/2} \cup b(\Sigma^{1/2})$

(ii) $\Sigma^{1/2} \cap b(\Sigma^{1/2}) = \Sigma^b$

(iii) $\nu|_{\Sigma \setminus \Sigma^b}(\Sigma^{1/2}) \subset \Sigma^{1/2}$ (in other words, $\Sigma^{1/2}$ either contains both sides of each non-real node, or none of them).

(b) A fundamental configuration consists of $((\Sigma, \nu, \lambda, u), b, \Sigma^{1/2})$ where $\sigma = ((\Sigma, \nu, \lambda, u), b)$ is a symmetric configuration and $\Sigma^{1/2}$ is a fundamental domain for $\sigma$.
\end{definition}
(c) For $i = 1, 2$ let $\phi_i = \left( (\Sigma_i, \nu_i, \lambda_i, u_i), b_i, \Sigma_i^{1/2} \right)$ be fundamental configurations. A morphism of fundamental configurations $\phi_1 \to \phi_2$ is a biholomorphism $\Sigma_1 \to \Sigma_2$ that respects all of the additional data.

**Lemma 11.** $\mathcal{M}^\circ$ is equivalent to the groupoid of fundamental configurations. The map $B \circ o$ corresponds to the map

$$ \left( (\Sigma, \nu, \lambda, u), b, \Sigma^{1/2} \right) \mapsto \left( (\Sigma, \nu, \lambda, u), b \right). $$

**Proof.** Consider first an interior point $\tilde{\mathcal{M}}_{\sigma_2}$ represented by a symmetric configuration $\sigma = ((\Sigma, \nu, \lambda, u), b)$. It is clear that there are two fundamental domains for $\sigma$ in this case, and they induce opposite orientations on $\partial \Sigma = \Sigma^b = S^1$ by the outward normal orientation convention. Clearly, $B \circ o$ is the map that forgets the chosen orientation and corresponding fundamental domain.

Heuristically, we want to extend this correspondence continuously to the boundary. To achieve this, we consider how the fundamental domain changes in continuous families. Recall we constructed a holomorphic atlas

$$ P \to \mathcal{M} = \tilde{\mathcal{M}}_{0,n} \left( \beta + \beta^\ast \right), \quad P = M_c \times_{\text{coa}} \tilde{\mathcal{M}}, $$

equipped with an antiholomorphic involution $P \to P$ whose fixed points form an atlas $P^{2/2} \to \mathcal{M}^{2/2}$. The pullback $\tilde{P} \to \tilde{\mathcal{M}}_{\sigma_2}$ along $\tilde{\mathcal{M}}_{\sigma_2} \to \mathcal{M}^{2/2}$ is an atlas for $\tilde{\mathcal{M}}_{\sigma_2}$ ($\tilde{P}$ can also be obtained by taking a clopen component of a hyperplane blowup of $P^{2/2}$).

Consider a stable family of maps $(Q \to P, S_x, H)$ associated with $P \to \mathcal{M}$ (cf. 2.3). Let $\tilde{Q} \to \tilde{P}$ denote the topological pullback along $\tilde{P} \to P$. There’s an involution $\tilde{Q} \to \tilde{Q}$ over $\text{id}_P$. The fiber over each $p \in \tilde{P}$ is a nodal Riemann surface with an antiholomorphic involution, whose normalization is a symmetric configuration representing the image of $p$ in $\mathcal{M}^{2/2}$. We abuse notation and treat the local system $\mathcal{L}$ as defined over $\tilde{P}$, by pulling it back. By definition (cf. Lemma 30) there’s a sufficiently small open neighborhood $p \in U \subset \tilde{P}$ such that the stalk $\mathcal{L}_p$ is in bijection with relative orientations for the circle bundle $\tilde{Q}^{1/2}|_{U^o} \to U^o$.

Fix a germ $g \in \mathcal{L}_p$. As we discussed in the first paragraph, the corresponding relative orientation determines a family of nodal fundamental domains $Q^{1/2} \to U^o$. That is, $Q^{1/2}$ is the closure of connected component of $\tilde{Q}|_{U^o \setminus (\tilde{Q}^b)|_{U^o}}$ such that the normalization $Q^{1/2}_q$ of the fiber $Q^{1/2}_q$ over every point $q \in U^o$ is the fundamental domain associated with the specified orientation of $\tilde{Q}^{1/2}_q$. We define the fundamental domain at $p \in \tilde{P}$ corresponding to $g$ to be the normalization of

$$ \overline{Q^{1/2}_p \cap Q_p}, $$

where the closure is taken in $\tilde{Q}$. We need to check that this is indeed a fundamental domain for a symmetric configuration $\sigma = ((\Sigma, \nu, \lambda, u), b)$ representing $B(p)$. Let us assume $\sigma$ has a single node which is a real node of type $H$, the other cases are similar. In this case the blowup over $B(p)$ is modeled on the gluing of half-planes $\{+, -\} \times [0, \infty) \times \mathbb{R}^{N-1} \to \mathbb{R}^N$. The choice of $p \in \{+, -\} \times \{0\}$ determines the direction the parameter $y_1$ approaches zero, where $y_1 = x_1 \cdot x_2$ is the real smoothing parameter as in (16). It is easy to see that a consistent orientation for the hyperbolas for values of $y_1$ approaching zero, determines a pair of orientations for the branches $x_1 = 0$ and $x_2 = 0$ over $y_1 = 0$, and that the family of fundamental domains converges to a pair.
of connected components of $\Sigma\setminus\Sigma^b$ near the real node that specify a fundamental domain for $\sigma$ inducing the given orientations on the boundary.

2.9. **Admissible disc-configurations and $\overline{\mathcal{M}}_{0,k,l}(\beta) \to \mathcal{M}^\times$.** Note that up until now, the construction only “knew” about $\beta + \beta' \in H_2(X)$. In general we may have $\beta + \beta' = \beta' + \beta$ with $\beta \neq \beta'$. For an example of this, take $\beta = (1, 0) \in H_2(C\mathbb{P}^1, \mathbb{R}P^1)$ and $\beta' = \overline{\beta} = (0, 1)$.

Thus, we first restrict to a clopen component $\mathcal{M}^\times_{\beta} \subset \mathcal{M}^\times$ of points represented by fundamental configurations $((\Sigma, \nu, \lambda, \gamma), \nu, \lambda, \Sigma \setminus \Sigma^b)$ with $u_*[\Sigma \setminus \Sigma^b, \partial \Sigma \setminus \Sigma^b] = \beta$.

Next, for each $1 \leq i \leq l$, we want to identify the markings $k + i$ and $k + l + i$; in other words, we have $\overline{\mathcal{M}}_{0,k,l}(\beta) = (\mathcal{M}^\times_{\beta})^{(2\mathbb{Z})^l}$, the stacky quotient by the $(2\mathbb{Z})^l \subset G_*$ action on $\mathcal{M}^\times_{\beta}$. It is naturally a $G_{k,l} = G_*/(2\mathbb{Z})^l$ orbifold with corners (see [14] Remark 2.4), and is clearly equivalent to the groupoid of $(k, l, \beta)$-disc configurations.

In fact, it is not hard to see that the map $q$ admits a smooth $G_{k,l}$-equivariant section $\mathcal{M}_P \rightarrow \mathcal{M}^\times_{\beta}$ which we take to be the inclusion of the clopen component corresponding to fundamental configurations where

$$\lambda((k + 1, \ldots, k + 2l)) \cap \Sigma \setminus \Sigma^b = (k + 1, \ldots, k + l).$$

It is easy to check that there’s a unique $G_{k,l}$-equivariant map $\text{ev}$ such that

$$\text{ev} \circ f = (i^k \times (i_d^X \times i_d^X) \circ \Delta_X) \circ \text{ev}$$

This completes the proof of Theorem 2.

3. **Orbifolds with Corners**

In this section we fix our notion of orbifold with corners, and introduce some related constructions. Some care is required, since fiber products in the category of manifolds with corners are somewhat elusive. We will define the category of orbifolds with corners as the Pronk 2-localization [12] of the bicategory of proper étale groupoids in a category of manifolds with corners, obtained by formally inverting étale equivalences. Our setup of the category of manifolds with corners follows Joyce’s work [4] closely.

3.1. **Manifolds with corners.** We refer the reader to [4] §2 for the terminology we use regarding manifolds with corners. The manifolds we’ll consider have “ordinary” corners (as opposed to generalized corners), which are modeled on $\mathbb{R}^n_+ := [0, \infty)^k \times \mathbb{R}^{n-k}$.

A weakly smooth map $f : U \to V$ between open subsets $U \subset \mathbb{R}^m_k$ and $V \subset \mathbb{R}^n_l$ is a continuous map $f = (f_1, \ldots, f_n)$ such that all the partial derivatives $\frac{\partial^{(n_1, \ldots, n_m)}}{\partial x_1^{n_1} \cdots \partial x_m^{n_m}} f_j : U \to \mathbb{R}$ exist and are continuous (including one-sided derivatives where applicable).

An $n$-dimensional manifold with corners $X$ is a second countable Hausdorff space equipped with a maximal $n$-dimensional atlas of charts $(U, \phi)$ where $U \subset X$ is open.
and $\phi : U \to \mathbb{R}^n_k$ is a homeomorphism ($n$ is fixed, $k$ may vary), with weakly smooth transitions. A weakly smooth map $f : X \to Y$ between manifolds with corners is a continuous map which is of this form in every coordinate patch. A weakly smooth map $f : X \to Y$ is said to be smooth, strongly smooth, interior, $b$-normal, simple, or a $b$-fibrations as in [4, Definitions 2.1, 4.3]. “A map” between manifolds with corners will always be assumed to be smooth unless specifically stated otherwise, and we denote by $\text{Man}^i$ the category of manifolds with corners with smooth maps.

The depth of a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_k$ is defined by depth $(x) = \# \{1 \leq i \leq k | x_i = 0 \}$. It is easy to see that the transitions preserve the depth, so we can speak of the depth of a point $x \in X$. We define $S^k(X) = \{x \in X | \text{depth}(x) = k\}$. A local $k$-corner component $\gamma$ of $X$ at $x$ is a local choice of connected component of $S^k(X)$ near $x$ (cf. [4 Definition 2.7]); a local 1-corner component is also called a local boundary component.

We have manifolds with corners

$$\partial X = C_1(X) = \{(x, \beta) | x \in X, \beta \text{ is a local boundary component of } X \text{ at } x\}$$

and, for every $k \geq 0$,

$$C_k(X) = \{(x, \gamma) | x \in X, \gamma \text{ is a local } k \text{-corner component of } X \text{ at } x\}.$$

Letting $\partial^k X$ denote the iterated boundary, we note that $C_k(X) \cong \partial^k X / \text{Sym}(k)$ where $\text{Sym}(k)$ acts by permuting the local boundary components.

We can consider $C(X) = \coprod_{k \geq 0} C_k(X)$ as a local manifold with corners (or “manifold with corners of mixed dimension”, in Joyce’s terms). These form a category and the various properties of maps can be used to describe maps between local manifolds with corners. If $f : X \to Y$ is a smooth map of manifolds with corners, there’s an induced interior map

$$C(f) : C(X) \to C(Y)$$

We denote by $i^\partial_X : \partial X \to X$ the map defined by $i^\partial_X((x, \beta)) = x$. Even if $X$ is connected, $\partial X$ may be disconnected and $i^\partial_X$ may not be injective. Sometimes we abbreviate $i^\partial = i^\partial_X$.

A strongly smooth map $f : X \to Y$ between manifolds with corners is a submersion if, whenever $x$ of depth $k$ maps to $y = f(x)$ of depth $l$, both $df|_x : T_x X \to T_y Y$ and $df|_x : T_x S^k(X) \to T_y S^l(Y)$ are surjective (see [5, Definition 3.2]; beware that a “smooth map” there is what we call a strongly smooth map, see [4] Remark 2.4,(iii))). We say a map $f : X \to Y$ is perfectly simple if it is simple and maps points of depth $k$ to points of depth $k$, and is étale if it is a local diffeomorphism.

If $X$ is a manifold with corners its tangent bundle $TX$ is defined in the obvious way. In addition, one can consider the $b$-tangent bundle $bTX$. It is a vector bundle on $X$ whose sections can be identified with sections $v \in C^\infty(TX)$ such that $v|_{S^k(X)}$ is tangent to $S^k(X)$ for all $k$ (cf. [4 Definition 2.15]). If $f : X \to Y$ is an interior map of orbifolds with corners, there’s an induced map $bdf : bTX \to bTY$. Two interior maps $f : X \to Z$ and $g : Y \to Z$ are called $b$-transverse if for any $x \in S^j(X), y \in S^k(Y)$ such that $f(x) = g(y) = z$, the map

$$bdf \oplus^{b} dg : bT_x X \oplus^{b} T_y Y \to bT_z Z$$

is surjective.
Remark 12. In case \( \partial Z = \emptyset \), \( f, g \) are b-transverse if and only if for every \( x \in S^j (X), y \in S^k (Y) \) with \( f(x) = g(y) = z \) the map

\[
df|_{TS^j(X)} \oplus dg|_{TS^k(Y)} : TS^j(X) \oplus TS^k(Y) \to T_z Z
\]

is surjective.

Lemma 13. Let \( X, Y, Z \) be manifolds with corners and let \( f : X \to Z \) and \( g : Y \to Z \) be continuous. Consider the topological fiber product

\[
P = X_f \times_g Y = \{(x, y) \in X \times Y | f(x) = g(y) \}.
\]

Suppose at least one of the following conditions holds.

(i) \( f \) is a b-normal submersion and \( g \) is strongly smooth and interior,
(ii) \( f \) is étale, \( g \) is a smooth map,
(iii) \( f \) is a b-submersion, \( g \) is perfectly simple, or
(iv) \( \partial Z = \emptyset \), \( f, g \) are b-transverse and smooth.

Then \( P \) admits a unique structure of a manifold with corners making it the fiber product in \( \text{Man}^c \), and we have

\[
C_i(W) = \coprod_{j,k,l \geq 0, i = j + k - l} C^j(X) \times_{C_i(Z)} C^k(Y)
\]

where \( C^j(X) = C_j(X) \cap C(f)^{-1}(C_i(Z)) \) and \( C^k(Y) = C_k(Y) \cap C(g)^{-1}(C_i(Z)) \), and the fiber product is taken over \( C(f), C(g) \).

Moreover, if \( X \overset{f}{\to} Z \) (respectively, \( Y \overset{g}{\to} Z \)) is b-normal then so is \( P \overset{f'}{\to} Y \) (resp., \( P \overset{g'}{\to} X \)).

Proof. Let \( \text{Man}^{gc} \) denote the category of manifolds with generalized corners with smooth maps (cf. [3]). This category contains \( \text{Man}^c \) as a full subcategory. In cases (i), (iii) and (iv) the fiber product exists in \( \text{Man}^{gc} \) as an embedded submanifold of \( X \times Y \), and \( (18) \) holds, by [3] Proposition 4.25, Theorem 4.28]. Since the structure of an embedded submanifold is unique if it exists [3 Corollary 4.12], it suffices to check that the fiber product is in fact a manifold with ordinary corners. In cases (i) and (iv) this follows from [5] Theorem 6.4], and in case (iii) this follows from [6] Theorem B].

Case (ii) is proven by a simple direct argument.

The last statement follows from [3] Proposition 2.11(c): if \( f \) is b-normal then \( l \leq j \) in \( (18) \), which implies \( i \leq k \), so \( f' \) is b-normal. \( \square \)

In what follows the discussion diverges from [3] (see more specifically §4.2 there). More precisely we introduce a stronger notion of a closed immersion, that has the implicit function theorem built into it. This is the only kind of closed immersion that we need to consider, and makes the discussion considerably simpler.

Definition 14. A map \( f : X \to Y \) of manifolds with corners is called a closed immersion if for every \( p \in X \) there exists an open neighborhood \( p \in U \subset X \), an open neighborhood \( f(U) \subset V \subset Y \), and a strongly smooth submersion \( h : V \to \mathbb{R}^N \) for some integer \( N \geq 0 \) such that the following square is cartesian

\[
\begin{array}{ccc}
U & \overset{f|_U}{\longrightarrow} & V \\
\downarrow & & \downarrow h \\
0 & \longrightarrow & \mathbb{R}^N
\end{array}
\]
Lemma 18. If it is étale and injective.

Remark 15. Any b-submersion to a manifold without boundary is automatically a strongly smooth submersion, so in Definition 14 it suffices to assume that \( h \) is a b-submersion.

Definition 16. A map \( f : X \to Y \) of manifolds with corners is called a closed embedding if it is a closed immersion, has a closed image, and induces a homeomorphism on its image.

Definition 17. A map \( f : X \to Y \) of manifolds with corners is an open embedding if it is étale and injective.

Lemma 18. If \( i : X \to Y \) and \( f : W \to Y \) are smooth maps of manifolds with corners, with \( i \) either a closed or an open embedding, and if \( f(W) \subset i(X) \), then there is a unique smooth map \( g : W \to X \) with \( f = i \circ g \). If we assume in addition that \( f \) is also a closed or an open embedding, and that \( f(W) = i(X) \), then \( g \) is a diffeomorphism.

Proof. A closed or open embedding is an embedding in the sense of [4], so this is a special case of Corollary 4.11 *ibid.* The last statement is immediate.

Definition 19. (a) Let \( f : X \to Y \) be a map of manifolds with corners. We say \( f \) is horizontally submersive if for every \( \hat{x} \in X \) the germ \( \tilde{f}_{\hat{x}} \) is isomorphic to the projection \( \mathbb{R}^k_+ \to \mathbb{R}^k \),

\[
(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{k'}, x_{k'+1}, \ldots, x_{k+n'-k'}).
\]

(b) Let \( f : X \to Y \) be a b-normal map. We call

\[
C_{k}^{\text{hor}}(X) := \left(C(f)^{-1}(C_0(Y)) \cap C_k(X)\right)
\]

the horizontal \( k \)-corners of \( X \) with respect to \( f \).

Lemma 20. A map \( f : X \to Y \) is horizontally submersive if and only if it is b-normal and the induced map \( C_{k}^{\text{hor}}(X) \xrightarrow{\text{C}(f)} Y \) is a submersion for every \( k \); that is,

\[
T_xC_{k}^{\text{hor}}(X) \xrightarrow{d\text{C}(f)} T_yY
\]

is surjective for all \( x \in C_{k}^{\text{hor}}(X) \).

Proof. The “only if” part is straightforward. Suppose \( f \) is b-normal and \( C(f)|_{C_0^{\text{hor}}(X)} \) is a submersion for all \( k \), and let us prove it is horizontally submersive. We have \( C_{0}^{\text{hor}}(X) = C_0(X) \). Fix some \( \hat{x} \) of depth \( k \) and suppose \( \tilde{y} = f(\hat{x}) \) has depth \( k' \). Since \( f \) is b-normal and the induced map \( f : C_0(X) \to Y \) is a submersion, we find that there’s an injective map \( \{1, \ldots, k'\} \to \{1, \ldots, k\} \) such that the germ \( \tilde{f}_{\hat{x}} \) is isomorphic to the map

\[
(x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n')
\]

where for each \( 1 \leq j \leq k' \)

\[
y_j = Y_j(x_1, \ldots, x_n) \cdot x_i(j)
\]

for some \((0, \infty)\)-valued smooth functions \( Y_j \) defined in a neighborhood of \( \hat{x} \).

We also assumed that \( \tilde{f}_{\hat{x}|_{\{x_1=\ldots=x_k=0\}}} \) is a submersion. It follows that there’s some subset of the coordinates \((x_{a_1}, \ldots, x_{a''})\) such that
Suppose now $X,Y$ are manifolds with corners, $f$ is horizontally submersive with oriented fibers, and let $\omega$ be a compactly supported differential form on $X$. In this case we can define $f_*\omega$ by integration along the fiber.

3.2. Orbifolds with corners.

**Definition 21.** A groupoid $(G_0,G_1,s,t,e,i,m)$ is a category where every arrow is invertible. Namely, $G_0$ is a class of points and $G_1$ is a class of arrows. The maps $s,t : G_1 \to G_0$ take an arrow to its source and target objects, respectively. The composition map $m : \{(f,g) \in G_1 \times G_1 \mid t(f) = s(g)\} \to G_1$ takes a pair of composable arrows to their composition. The identity map $e : G_0 \to G_1$ takes an object to the identity arrow and the inverse map $i : G_1 \to G_1$ takes an arrow to its inverse.

The equivalence classes of the equivalence relation $\text{Im}(s \times t) \subset G_0 \times G_0$ are called the orbits of the groupoid; the class of all orbits is denoted $G_0/G_1$. We will use different notations for groupoids, depending on how much of the structure we want to label:

$$(G_0,G_1,s,t,e,i,m) = G_* = G_1 \rightrightarrows G_0.$$

**Definition 22.** A groupoid $(X_0,X_1,s,t,e,i,m)$ will be called \textit{étale} if $X_0,X_1$ are objects of $\text{Man}^\text{t}$, and the maps $s,t,e,i,m$ are all étale (in fact, it suffices to require that $s:X_1 \to X_0$ is \textit{étale}). An étale groupoid will be called \textit{proper} if the map $s \times t : X_1 \to X_0 \times X_0$ is proper. We will mostly be interested in proper \textit{étale} groupoids, or PEG’s for short.

Let $X_\ast$ be a PEG. The set of orbits $X_0/X_1$, taken with the quotient topology, forms a locally compact Hausdorff space. $X_\ast$ is called \textit{compact} if $X_0/X_1$ is compact.

Let $X_\ast,Y_\ast$ be two PEG’s. A smooth functor $X_\ast \xrightarrow{F_\ast} Y_\ast$ consists of a pair of smooth maps $F_0:X_0 \to Y_0$ and $F_1:X_1 \to Y_1$ which is a functor between the underlying categories. If $F_\ast,G_\ast:X_\ast \to Y_\ast$ are two functors a smooth transformation $\alpha : F_\ast \Rightarrow G_\ast$ is a smooth map $X_0 \to Y_1$ which is a natural transformation between the underlying functors. In this way we obtain a bicategory (see [2]) PEG, whose objects, or 0-cells, are proper \textit{étale} groupoids, morphisms (or 1-cells) are smooth functors, and 2-cells are natural transformations. A refinement $R_\ast : X_\ast \to X'_\ast$ is a smooth functor which is an equivalence of categories and such that $R_0$ (hence also $R_1$) is an \textit{étale} map.

**Lemma 23.** As a subset of the 1-cells of PEG the refinements admit a right calculus of fractions, in the sense of [12] §2.1.

**Proof.** We use the notation \textit{ibid}. BF1, BF2 and BF5 are straightforward. To establish BF3, use the weak fiber product (the construction of the weak fibered product in PEG is reviewed in Lemma 26 below). We prove BF4. Suppose $f_\ast,g_\ast$:
\( X_\bullet \to Y_\bullet \) are smooth functors and \( w_\bullet : \left( Y_1 \overset{s \times t}{\Rightarrow} Y_0 \right) \to \left( Y'_1 \overset{s' \times t'}{\Rightarrow} Y'_0 \right) \) is a refinement, and \( \alpha : w_\bullet \circ f_\bullet \Rightarrow g_\bullet \) is a smooth transformation (here we use lowercase letters to denote functors, to keep close to the notation in [12]). Since the following square is cartesian

\[
\begin{array}{ccc}
Y_1 & \overset{w_1}{\longrightarrow} & Y'_1 \\
\downarrow^{s \times t} & & \downarrow^{s' \times t'} \\
Y_0 \times Y_0 & \overset{w_0 \times w_0}{\longrightarrow} & Y'_0 \times Y'_0
\end{array}
\]

the maps \( \alpha : X_0 \to Y_1 \) and \( f_0 \times g_0 : X_0 \to Y_0 \times Y_0 \) define the desired \( \beta : X_0 \to Y_1 \), with \( v = id_{X_\bullet} \). The second requirement holds since all 2-cells are invertible. For the final requirement, take \( u = v', u' = id \) and \( \epsilon = id \). \( \square \)

We define the category \( \text{Orb} \) of orbifolds (always with corners, unless specifically mentioned otherwise) to be the 2-localization of \( \text{PEG} \) by the refinements. We usually denote orbifolds by calligraphic letters \( \mathcal{X}, \mathcal{Y}, \mathcal{M} \ldots \) They are given by proper \( \text{étale} \) groupoids. Maps \( \mathcal{X} \to \mathcal{Y} \) are given by fractions \( F_\bullet / R_\bullet \) with \( X_\bullet \overset{F_\bullet}{\rightarrow} X'_\bullet \) a refinement and \( \mathcal{X}' \overset{F_\bullet}{\rightarrow} \mathcal{Y} \) a smooth functor. We refer the reader to [12] for further details, including the definition of the 2-cells, the composition operations, etc.

Remark 24. We will occasionally consider other categories of orbifolds. First, there’s the category of orbifolds without boundary \( \text{Orb}_{\partial = \emptyset} \). This can be realized simply as the 2-full bicategory of \( \text{Orb} \) spanned by all objects \( \mathcal{X} \) with \( \partial \mathcal{X} = \emptyset \). We will also encounter the category \( \text{Orb}_C \) of complex orbifolds. To construct it, we begin with the bicategory \( \text{PEG}_C \) whose objects are groupoids \( M_1 \rightrightarrows M_0 \) where \( M_i, i = 0, 1 \), is a complex manifold, the structure maps \( s, t, e, i, m \) are local biholomorphisms, and \( s \times t \) is proper. 1-cells and 2-cells in \( \text{PEG}_C \) are given by holomorphic functors and holomorphic natural transformations, respectively. To obtain \( \text{Orb}_C \) we invert \textit{holomorphic refinements}, that is, equivalences \( (R_0, R_1) \) where \( R_0 \) is a local biholomorphism. There’s an obvious way to extend \( \text{Orb}_C \) to allow also \textit{anti-holomorphic morphisms}, where the category of \textit{antiholomorphic} morphisms \( (\mathcal{X} = X_1 \rightrightarrows X_0) \to \mathcal{Y} \) is, by definition, equal to the category of morphisms \( \mathcal{X} \to \mathcal{Y} \) in \( \text{Orb}_C \) where \( \mathcal{X} = X_1 \rightrightarrows X_0 \).

There are obvious bifunctors

\[ \text{Orb}_C \to \text{Orb}_C^\circ \to \text{Orb}_{\partial = \emptyset} \to \text{Orb}. \]

Definition 25. We say \( f \) is strong-smooth, \( \text{étale} \), \( \text{interior} \), \( \text{b-normal} \), submersive, \( b\text{-submersive, horizontally submersive, simple or perfectly simple} \) if \( F_0 \) has the corresponding property as a map of manifolds with corners. It is easy to check that these properties are preserved by 2-cells (and thus are properties of the homotopy class of \( f \)). The map \( f \) is called a \textit{\( b \)-fibration} if it is \( b \)-normal and \( b \)-submersive (cf. [1] Definition 4.3).

For \( i = 1, 2 \) let \( f^i = F^i|R^i : \mathcal{X}^i \to \mathcal{Y} \) be an interior map. We say \( f^1 \) and \( f^2 \) are \( b \)-transverse if \( F^1_0, F^2_0 \) are \( b \)-transverse (as maps of manifolds with corners).

An equivalence in \( \text{Orb} \) is called a \textit{diffeomorphism}. We say \( f = F|R : \mathcal{X} \to \mathcal{Y} \) is full, \textit{essentially surjective}, or \textit{faithful} if \( F \) is full, essentially surjective, or faithful, respectively.
If $\mathcal{X} = X_1 \rightrightarrows X_0$ is an orbifold with corners, $\partial \mathcal{X} = \partial X_1 \rightrightarrows \partial X_0$ is naturally an orbifold with corners and the smooth functor $(i_{X_1}^\partial, i_{X_0}^\partial)$ induces a map $i_\mathcal{X}^\partial : \partial \mathcal{X} \to \mathcal{X}$. We denote

$$i_\mathcal{X}^\partial := i_X^\partial \circ i_{1,X}^\partial \circ \cdots \circ i_{i-1,X}^\partial : \partial \mathcal{X} \to \mathcal{X}.$$  

Since the maps $s, t, e, i, m$ are étale, they preserve the depth and we obtain orbifolds with corners

$$C_k(\mathcal{X}) = C_k(X_1) \Rightarrow C_k(X_0)$$

for all $k$. A local orbifold with corners $\mathcal{X} = \coprod X_n$ (or just an l-orbifold) is a disjoint union of orbifolds with corners with $\dim X_n = n$. It is obvious how to turn this into a category and extend the definitions of various types of maps to this situation. If $\mathcal{X}$ is an orbifold with corners, we can consider $C(\mathcal{X}) = \coprod_{k\geq 0} C_k(\mathcal{X})$ as an l-orbifold.

A smooth map $f : \mathcal{X} \to \mathcal{Y}$ induces an interior map $C(f) : C(\mathcal{X}) \to C(\mathcal{Y})$.

We turn to a discussion of the weak fibered product in $\text{Orb}$.

**Lemma 26.** Let

$$f : \mathcal{X} \leftarrow X^f \xrightarrow{F} Z$$

be two 1-cells in $\text{Orb}$. Suppose at least one of the following conditions holds.

(i) $F$ is a b-normal submersion and $G$ is strongly smooth and interior,  
(ii) $F$ is étale, $G$ is a smooth map,  
(iii) $F$ is a b-submersion, $G$ is perfectly simple, or  
(iv) $\partial Z = \emptyset$, $F$ and $G$ are $b$-transverse (see Remark 12 for an equivalent condition) and smooth.

Then

(a) The weak fiber product $\mathcal{P} = \mathcal{X} \times_g \mathcal{Y}$ exists in $\text{Orb}$. In fact, we can take

$$\mathcal{P} = \mathcal{X}^f \times_G \mathcal{Y}$$

the weak fiber product in $\text{PEG}$, given by the groupoid $P_1 \rightrightarrows P_0$ where

$$P_0 = X'_0 \times_{s} Z_1 \times_{G_0} Y'_0,$$

$$P_1 = X'_1 \times_{s} Z_1 \times_{G_1} Y'_1.$$

Here an element of $P_1$ specifies the three solid arrows in the diagram below,

\[ \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\uparrow & \uparrow & \uparrow \\
\bullet & \longrightarrow & \bullet \\
\downarrow & \downarrow & \downarrow \\
\bullet & \longrightarrow & \bullet \\
\end{array} \]

The horizontal dashed arrow is uniquely determined by requiring the square to be commutative; $s, t : P_1 \to P_0$ are the projections on the top and bottom rows of the diagram, respectively, and the other structure maps are computed similarly.

(b) We have

$$C_i(\mathcal{P}) = \coprod_{j,k,l\geq i+j+k-1} C_j^i(\mathcal{X}) \times_{C_i(\mathcal{Z})} C_k^i(\mathcal{Y})$$

where $C_j^i(\mathcal{X}) = C_j(\mathcal{X}) \cap C(f)^{-1}(C_i(\mathcal{Z}))$ and $C_j^i(\mathcal{Y}) = C_j(\mathcal{Y}) \cap C(g)^{-1}(C_i(\mathcal{Z}))$, and the weak fiber product is taken over $C(f), C(g)$.

(c) If we assume, in addition, that $\mathcal{X} \xrightarrow{f} Z$ (respectively, $\mathcal{Y} \xrightarrow{g} Z$) is b-normal, then so is $\mathcal{P} \xrightarrow{f'} Z$ (resp., $\mathcal{P} \xrightarrow{g'} Z$).
Proof. It is well-known (and easy to verify) that the weak fibered product
\[ \mathcal{P} = \mathcal{X}' \times_{\mathcal{F}} \mathcal{Y}' \]
in the bicategory of proper étale groupoids in topological spaces is represented by
\[ P_1 \Rightarrow P_0 \] as described above (see [11]). Using Proposition 13 it is not hard to show
that \( P_1, P_0 \) are smooth manifolds with corners, that the structure maps are étale
(i.e., a local diffeomorphism), and that \( \mathcal{P} \) represents the weak fibered product in
\( \text{PEG} \), the category of proper étale groupoids in manifolds with corners.
It then follows from a result of Tommasini [16, Corollary 0.3] that
\[ \mathcal{P} = \mathcal{X} \times_{f,g} \mathcal{Y}, \]
the weak fiber product of \( f \) and \( g \) in \( \text{Orb} \).
Claims (b) and (c) are straightforward, again using Proposition 13.

\[ \square \]

Definition 27. A map \( F[R : \mathcal{X} \rightarrow \mathcal{Y} \) of orbifolds with corners is a closed immersion
if \( F_0 \) is a closed immersion. In this case, the same holds for any map homotopic to
\( F[R \).
A manifold with corners \( M \) specifies an orbifold \( \overline{M} = M \Rightarrow M \) with only iden-
tity morphisms, and this extends to a 2-fully-faithful pseudofunctor \( \text{Man}^c \rightarrow \text{Orb} \)
(namely, it restricts to an equivalence \( \text{Man}^c(\mathcal{X}, \mathcal{Y}) \cong \text{Orb}(\overline{X}, \overline{Y}) \) for any pair \( X, Y \)
of objects of \( \text{Man}^c \). We say an orbifold “is” a manifold with corners if it is in the
essential image of this functor.

Definition 28. Let \( \mathcal{X} \) be an orbifold with corners. An atlas for \( \mathcal{X} \) is a map
\( p : M \rightarrow \mathcal{X} \) where \( M \) is some manifold with corners, such that for any other map
\( f : N \rightarrow \mathcal{X} \) from a manifold with corners, \( M \times_{\mathcal{X}} N \) is a manifold with corners and
the projection \( M \times_{\mathcal{X}} N \rightarrow N \) is étale and surjective (as a map of \( \text{Man}^c \)).
The obvious map \( X_0 \rightarrow (X_1 \Rightarrow X_0) \) is an atlas. Conversely, any atlas \( M \rightarrow \mathcal{X} \) defines an orbifold equivalent to \( \mathcal{X} \), whose objects are \( M \) and morphisms are
\( M \times_{\mathcal{X}} M \).

Definition 29. A map \( f : \mathcal{X} \rightarrow \mathcal{Y} \) of orbifolds with corners is a closed
(respectively, open) embedding if for some (hence any) atlas \( p : M \rightarrow \mathcal{Y} \), the 2-pullback
\( M \times_f \mathcal{X} \) is a manifold with corners and the map \( M \times_f \mathcal{X} \rightarrow M \) is a closed (resp. open)
embedding of manifolds with corners.
If \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is a closed embedding we may refer to \( \mathcal{X} \) as a suborbifold of \( \mathcal{Y} \).

The notion of a sheaf on an orbifold \( \mathcal{X} \) is the same as the notion of a sheaf on the
underlying topological orbifold, see Moerdijk and Pronk [11, 12] for a comprehensive
treatment. A vector bundle \( E \) on an orbifold with corners \( \mathcal{X} = X_1 \Rightarrow X_0 \) is given
by \( (E_0, \phi) \) where \( E_0 \) is a smooth vector bundle on \( X_0 \) and
\[ \phi : s^* E_0 \rightarrow t^* E_0 \]
is an isomorphism satisfying some obvious compatibility requirements with the
groupoid structure. The sections of \( (E_0, \phi) \) form a sheaf over \( \mathcal{X} \). An important
elementary example of a vector bundle is the tangent bundle, \( T\mathcal{X} = (TX_0, dt \circ ds^{-1}) \), whose
sections are vector fields on \( \mathcal{X} \). A local system on an orbifold \( \mathcal{X} \) is a sheaf which
is locally isomorphic to the constant sheaf \( \mathbb{Z} \). We extend the conventions set forth
We can work with \( \mathbb{Z} \)-differential forms on \( \mathcal{X} \), lying over \( i_{\mathcal{X}}^\partial : \partial \mathcal{X} \to \mathcal{X} \), defined by appending the outward normal to the boundary at the beginning of the oriented base for \( T \partial \mathcal{X} \). Given a short exact sequence of vector bundles

\[
0 \to E_1 \to E \xrightarrow{q} E_2 \to 0
\]
on \( \mathcal{X} \), we obtain a local system isomorphism

\[
\text{Or} (E_1) \otimes \text{Or} (E_2) \to \text{Or} (E),
\]

which, using oriented bases to represent orientation, can be expressed by

\[
[e_1, ..., e_1^n] \otimes [e_2, ..., e_2^n] \mapsto [f(e_1^1), ..., f(e_1^n), g(e_2^1), ..., g(e_2^n)]
\]

where \( g : E_2 \to E \) is any local section of \( q \).

Maps of local systems are always assumed to be cartesian, so to specify a local system map \( L_1 \xrightarrow{\varphi} L_2 \) over \( X_1 \xrightarrow{f} X_2 \) is equivalent to giving an isomorphism \( L_1 \to f^{-1} L_2 \).

**Lemma 30.** Let \( \mathcal{X} \) be an orbifold with corners. We denote by \( \hat{\mathcal{X}} := S^0(\mathcal{X}) \) the orbifold (without boundary or corners) consisting of points of depth zero, and by \( j : \hat{\mathcal{X}} \to \mathcal{X} \) the inclusion.

(a) The pushforward and inverse image functors \( j_*, j^{-1} \) form an adjoint equivalence of groupoids between local systems on \( \hat{\mathcal{X}} \) and local systems on \( \mathcal{X} \).

(b) \( \text{Or}(dj) : \text{Or} (T \hat{\mathcal{X}}) \to j^{-1} \text{Or} (T \mathcal{X}) \) is an isomorphism.

Let \( f : \mathcal{X} \to \mathcal{Y} \) be a b-normal map of orbifolds with corners.

(c) There exists a unique map \( \hat{f} : \hat{\mathcal{X}} \to \hat{\mathcal{Y}} \) with \( f \circ j_{\mathcal{X}} = j_{\mathcal{Y}} \circ \hat{f} \).

Let \( \mathcal{L} \) be a local system on \( \mathcal{X} \) and let \( \mathcal{L}' \) be a local system on \( \mathcal{Y} \), and denote by \( \check{\mathcal{L}} = j_{\mathcal{X}}^* \mathcal{L} \), \( \check{\mathcal{L}}' = j_{\mathcal{Y}}^* \mathcal{L}' \) their restrictions to \( \mathcal{X}, \mathcal{Y} \), respectively. Define a map taking a map of sheaves \( \mathcal{F} : \mathcal{L} \to \mathcal{L}' \) over \( f \) to the map \( \check{\mathcal{F}} : \check{\mathcal{L}} \to \check{\mathcal{L}}' \) over \( \hat{f} \) given by the composition

\[
j_{\mathcal{X}}^\partial \mathcal{L} \xrightarrow{j_{\mathcal{X}}^\partial F} j_{\mathcal{X}}^{-1} \mathcal{L}' \xrightarrow{f^{-1} j_{\mathcal{Y}}^{-1}} \mathcal{L}'.
\]

(d) \( \mathcal{F} \mapsto \check{\mathcal{F}} \) is a bijection

\[\{\text{maps } \mathcal{F} : \mathcal{L} \to \mathcal{L}' \text{ over } f\} \simeq \{\text{maps } \check{\mathcal{F}} : \check{\mathcal{L}} \to \check{\mathcal{L}}' \text{ over } \hat{f}\}.\]

and together with \( \mathcal{L} \mapsto \check{\mathcal{L}} \) forms a functor from the category of sheaves (respectively, local systems) over orbifolds with corners with b-normal maps to the category of sheaves (resp. local systems) over orbifolds without boundary.

**Proof.** Straightforward. \( \square \)

Let \( \mathcal{X} \) be an orbifold with corners and \( \mathcal{L} \) a local system on \( \mathcal{X} \). We define the complex of differential forms on \( \mathcal{X} \) with values in \( \mathcal{L} \)

\[
\Omega(\mathcal{X}; \mathcal{L}) = \Gamma(\mathcal{L}) = \Gamma(C^\infty(\mathcal{T} \mathcal{X}) \otimes \mathcal{L})
\]

4Note there we had to work with \( \mathbb{C} \)-valued local systems, but for the purposes of this paper we can work with \( \mathbb{Z} \)-valued local systems.
as the global sections of the sheaf of sections of the vector bundle $\wedge T\mathcal{X}$, twisted by $\mathcal{L}$.

Suppose $\mathcal{X}, \mathcal{Y}$ are compact orbifolds with corners, $\mathcal{K}, \mathcal{L}$ are local systems on $\mathcal{X}$ and on $\mathcal{L}$, respectively, and $f : (\mathcal{X}, \mathcal{K}) \to (\mathcal{Y}, \mathcal{L})$ is an oriented map, which means it is a map of local systems $\mathcal{K} \to \mathcal{L}$ lying over a smooth map of orbifolds with corners $\mathcal{X} \to \mathcal{Y}$. We have a pullback operation

$$\Omega \left( \mathcal{Y}; \mathcal{L} \right) \xrightarrow{f^*} \Omega \left( \mathcal{X}; \mathcal{K} \right).$$

If, in addition, we assume that $f$ is horizontally submersive, then there’s a pushforward operation

$$\Omega \left( \mathcal{X}; \mathcal{K} \otimes \text{Or} (T\mathcal{X})^\vee \right) \xrightarrow{f_*} \Omega \left( \mathcal{Y}; \mathcal{L} \otimes \text{Or} (T\mathcal{Y})^\vee \right).$$

We now sketch how these operations are constructed. Define the complex of compactly supported differential forms on $\mathcal{X}$ by

$$\Omega_c \left( \mathcal{X}; \mathcal{L} \right) := \text{coker} \left( t_* - s_* : \Omega_c \left( X_1; s^* \mathcal{L}_0 \right) \to \Omega_c \left( X_0; \mathcal{L}_0 \right) \right),$$

where on the right hand side, $\Omega_c$ denotes the usual complex of compactly supported forms on a manifold with corners. In case $f = (F_0, F_1) : \mathcal{X} \to \mathcal{Y}$ is a smooth functor, $F_0^*$ induces a pullback map (23) and (if $f$ is a horizontally submersive) $(F_0)_*$ induces a pushforward map of compactly supported forms,

$$\Omega_c \left( \mathcal{X}; \mathcal{K} \otimes \text{Or} (T\mathcal{X})^\vee \right) \xrightarrow{f^*} \Omega_c \left( \mathcal{Y}; \mathcal{L} \otimes \text{Or} (T\mathcal{Y})^\vee \right).$$

In defining the operations $F_0^*$ and $(F_0)_*$ (for forms on manifolds with corners) we follow the orientation conventions in [18]. A partition of unity for $\mathcal{X}$ is a smooth map $\rho : X_0 \to [0, 1]$ such that $\text{supp}(s^* \rho) \cap t^{-1} (K)$ is compact for every compact subset $K \subset X_0$ and $t_* s^* \rho \equiv 1$ (the fiber of $t$ is discrete, hence canonically oriented). Partitions of unity always exist; since $\mathcal{X}$ is assumed to be compact we can require that $\rho$ has compact support in $X_0$, and use this to construct an isomorphism

$$\Omega \left( \mathcal{X}; \mathcal{L} \right) \simeq \Omega_c \left( \mathcal{X}; \mathcal{L} \right),$$

see Behrend [1]. The isomorphism (26) allows us to define (24) using (25). Now if $f = \mathcal{X} \xrightarrow{R} \mathcal{X}' \xrightarrow{F} \mathcal{Y}$ is a general oriented map, we define (23) by

$$f^* = R_* F^*,$$

pulling back along the smooth functor $F$ and then pushing forward along the refinement $R$ (note $R$ is horizontally submersive since it is étale; moreover, any refinement defines an equivalence between the categories of local systems on $\mathcal{X}$ and on $\mathcal{X}'$, so orientations for $f$ are in natural bijection with orientations for $F$). If $f$ is oriented and horizontally submersive we define the pushforward (24) by

$$f_* = F_* R_*.$$

By construction, the operations (23, 24) extend the operations defined in [18] for the case $\mathcal{X}, \mathcal{Y}$ are manifolds, and they satisfy the same relations.

To make the paper more readable, outside of this appendix we will sometimes abuse notation and refer to maps which have a specified isomorphism as being equal. For example, if $G$ acts on $\mathcal{X}$ (see §3.4 below) we may write

$$g.h. = (gh).$$
even though in general the two sides differ by a (specified) 2-cell. The same goes for orbifolds which are canonically equivalent (that is, with a given equivalence, or with an equivalence which is specified up to a unique 2-cell). For example we may write

$$ (M_1 \times M_2) \times M_3 = M_1 \times (M_2 \times M_3). $$

When we write \( p \in X \) we mean \( p \in X_0 \), where \( X \equiv X_1 \simeq X_0 \).

3.3. **Hyperplane Blowup.** In this subsection we explain how to blow up an orbifold along a nice codimension one locus, to obtain an orbifold with corners. This is an important step in the construction of the moduli spaces of discs from the moduli spaces of curves (see §2.6 and the motivating discussion in the introduction). The construction is carried out in two steps: first, we discuss the hyperplane blowup of manifolds, and then we extend this to orbifolds.

3.3.1. **Hyperplane blowup of manifolds.**

**Definition 31.** (a) Let \( h : W \to X \) be a proper closed immersion between manifolds without boundary. Write \( h^{-1}(x) = \{w_1, \ldots, w_r\} \) (this is finite since \( h \) is proper), and let \( N^\vee_{w_i} = \ker \left( T^\vee_{x} X \xrightarrow{dh^\vee_{w_i}} T^\vee_{w_i} W \right) \) denote the conormal bundle to \( h \). We say \( h \) has transversal self-intersection at \( x \in X \) if the induced map

$$ \bigoplus_{i=1} N^\vee_{w_i} \to T^\vee_x X $$

is injective. We say \( h \) has transversal self-intersection if it has transversal self intersection at every \( x \in X \).

(b) Let \( h : W \to X \) be a proper closed immersion which has transversal self intersection. Suppose further that \( h \) is codimension one, i.e. \( \dim X - \dim W = 1 \). In this case we call \( E = \Im h \) a hyper subset, and call \( h \) a hyper map. Note since the conditions on \( h \) can be checked locally on the codomain \( X \), being a hyper subset is a local property. Moreover, it follows from Proposition 32 below that the map \( h \) is essentially unique: if \( W \xrightarrow{\phi} X, W' \xrightarrow{h'} X \) are two hyper maps with \( \Im h = \Im h' \) then there’s a unique diffeomorphism \( W \xrightarrow{\phi} W' \) such that \( h = h' \circ \phi \).

(c) Let \( Y \to X \) be a smooth map of manifolds without boundary, and let \( E \subset X \) be a hyper subset. We say \( f \) is multi-transverse to \( E \) if for some (hence any) hyper map \( h \) such that \( E = \Im h \), \( f \) is transverse to \( h \) and the pullback \( f^{-1}W \xrightarrow{f^{-1}h} Y \) has transversal self intersection (so in fact, since \( f^{-1}h \) is necessarily a codimension one proper closed immersion, \( f^{-1}E \subset Y \) is a hyper subset).

The following proposition explains the usefulness of these notions and prepares the ground for the construction of the hyperplane blow up.

**Proposition 32.** Let \( h : W \to X \) be a hyper map between manifolds without boundary.

(a) For every \( x \in X \) with \( h^{-1}(x) = \{w_1, \ldots, w_r\} \) there exists an open neighborhood \( x \in V \subset X \) such that \( h^{-1}(V) = U_1 \coprod \cdots \coprod U_r \) where \( w_i \in U_i \subset W \) is an open neighborhood for \( 1 \leq i \leq r \), together with charts \( V \xrightarrow{\varphi_i} \mathbb{R}^n \) and \( U_i \xrightarrow{\varphi} \mathbb{R}^{n-1} \) so that \( h|_{U_i} \) corresponds to the map \( \mathbb{R}^{n-1} \to \mathbb{R}^n \) given by

$$ (t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}). $$
We call the coordinate chart \( V \xrightarrow{\varphi} \mathbb{R}^n \) an orthant chart for \( h \) at \( x \).

(b) Suppose \( V \xrightarrow{\psi} X \) is multi-transverse to \( \text{Im} \ h \) and let \( x \in X \) with \( |h^{-1}(x)| = r \). Then there exists an open neighborhood \( x \in V \subset X \) together with orthant charts \( f^{-1}(V) \xrightarrow{\varphi} \mathbb{R}^m \) and \( V \xrightarrow{\psi} \mathbb{R}^n \) for \( f^{-1}h \) and \( h \), respectively, so that \( \psi \circ f \circ \varphi^{-1} \) is given by

\[
(t_1, \ldots, t_m) \rightarrow (t_1, \ldots, t_r, \phi_{r+1}(t_1, \ldots, t_m), \ldots, \phi_n(t_1, \ldots, t_m))
\]

for some smooth functions \( \phi_{r+1}, \ldots, \phi_n \).

Proof. We prove part (a). Since \( h \) is a codimension one closed immersion for every \( 1 \leq i \leq r \) there exist open neighborhoods \( w_i \in U''_i \in W \) and \( x \in V_i \subset X \) and a submersion \( h_i : V_i \rightarrow \mathbb{R} \) such that the following square is cartesian

\[
\begin{array}{ccc}
U''_i & \xrightarrow{h_i''} & V_i \\
\downarrow & & \downarrow \\
0 & \xrightarrow{v_i} & \mathbb{R}
\end{array}
\]

Since \( h \) has transversal self-intersection, \( dv_{1}, \ldots, dv_{r} \) are linearly independent at \( x \) and therefore, in a perhaps smaller open neighborhood \( x \in V' \subset \bigcap_{i=1}^{r} V_i \) they extend to a coordinate chart \( (v_1, \ldots, v_r, v_{r+1}, \ldots, v_n) : V' \rightarrow \mathbb{R}^n \). Set \( U''_i = h^{-1}(V') \cap U''_i \).

Uniqueness of the pullback implies that for \( 1 \leq i \leq r \),

\[
\left( v_1 \circ h|_{U''_i}, \ldots, v_i \circ h|_{U''_i}, \ldots, v_n \circ h|_{U''_i} \right) : U''_i \rightarrow \mathbb{R}^{n-1}
\]

is a coordinate chart for \( W \) and \( h|_{U''_i} \) obtains the desired form in these coordinate systems. Since \( h \) is proper, there’s an open neighborhood \( x \in V \subset V' \) such that, setting \( U_i = U''_i \cap D^{-1}(V) \) we have

\[
h^{-1}(V) = U_1 \coprod \cdots \coprod U_r,
\]

completing the proof of part (a).

The proof of part (b) is similar. \( \square \)

Definition 33. (a) Let \( X \) be a manifold without boundary, let \( U \subset X \) be an open subset. Consider the set of germs of connected components,

\[
I(X,U) = \bigcup_{x \in X} \{ x \} \times \lim_{x \in V \subset X} \pi_0^X(V \cap U)
\]

where for \( V \) an open neighborhood of \( x \in X \), \( \pi_0^X(V \cap U) \) denotes the set of connected components \( C \subset V \cap U \) with \( x \in \bar{C} \) in the closure. If \( V_1 \subset V_2 \) are two such neighborhoods, there’s an induced map \( \pi_0^X(V_1 \cap U) \rightarrow \pi_0^X(V_2 \cap U) \), and \( \lim_{x \in V \subset X} \pi_0^X(V \cap U) \) denotes the inverse limit of this system of sets.

(b) If \( (X_1, U_1) \rightarrow (X_2, U_2) \) is a map of pairs there’s an induced map \( I(X_1, U_1) \rightarrow I(X_2, U_2) \) making \( I \) a functor; there’s an obvious natural transformation \( I(X, U) \rightarrow X \).

(c) Let \( E \subset X \) be a hyper subset. As a set, the blow up of \( X \) along \( E \) is given by

\[
B(X,E) = I(X, X \setminus E).
\]

The associated natural transformation is denoted \( B(X, E) \xrightarrow{\beta(X,E)} X \), and if \( Y \xrightarrow{f} X \) is multi-transverse to \( E \) write

\[
B(Y, f^{-1}E) \xrightarrow{B(f)} B(X, E)
\]
Proposition 34. Let $\text{Man}^+$ denote the category of marked manifolds, whose objects are pairs $(X,E)$ where $X$ is a manifold without boundary and $E$ is a hyper subset of $X$, and where an arrow $(X_1,E_1) \to (X_2,E_2)$ is given by a map $X_1 \xrightarrow{f} X_2$ which is multi-transverse to $E_2$ and such that $f^{-1}E_2 = E_1$. Let $\text{Man}_{ps}^+$ denote the category of manifolds with corners with perfectly simple maps. Then blowing up gives a faithful functor

$$B : \text{Man}^+ \to \text{Man}_{ps}^+$$

together with a natural transformation $B(X,E) \xrightarrow{\beta(X,E)} X$.

Moreover, if $(X_1,E_1), (X_2,E_2)$ are any two objects of $\text{Man}^+$, any étale map $f : X_1 \to X_2$ is a morphism of $\text{Man}^+$ and $B(f)$ is also étale in this case.

The proof of this proposition appears below. The following definition and lemma characterize the manifold with corners structure on the blow up. More precisely, $B(X,E)$ will be equipped with the unique manifold with corners structure on the set $B(X,E)$ making the map $\beta(X,E)$ rectilinear:

Definition 35. Let $C$ be a manifold with corners, $M$ a manifold without boundary. A map $f : C \to M$ will be called rectilinear if the restriction of $f$ to interior points is an injective map $\overline{C} \to M$, and for every $c \in C$ there exist a non-negative integer $k$ and coordinate charts $U \xrightarrow{\varphi} \mathbb{R}^n, c \in U, \varphi(c) = 0$ and $V \xrightarrow{\psi} \mathbb{R}^n, f(c) \in V, \psi(f(c)) = 0$ such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1}$ is the standard embedding of $\mathbb{R}^k$ to $\mathbb{R}^n$, restricted to $\varphi(U)$.

Lemma 36. (a) Let $C$ be a set, $M$ a manifold without boundary, and $f : C \to M$ a map of sets. A structure of a manifold with corners on $C$ making $f$ a rectilinear map is unique if it exists.

(b) If there’s an open cover $M = \bigcup V_i$ such that $f^{-1}(V_i)$ admits a manifold with corners structure making $f|_{f^{-1}(V_i)}$ rectilinear, then $C$ admits a structure of a manifold with corners making $f$ rectilinear.

(c) For $i = 1, 2$, let $C_i$ be a manifold with corners, $M_i$ a manifold without boundary, and $C_i \xrightarrow{f_i} M_i$ a rectilinear map. Let $M_1 \xrightarrow{g} M_2$ be a smooth map. An interior map $C_1 \xrightarrow{g} C_2$ making the square

$$\begin{array}{ccc}
C_1 & \xrightarrow{g} & C_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{g} & M_2
\end{array}$$

commute is unique if it exists.

Proof. We prove (a). Suppose $C_1, C_2$ are two manifolds with corners with the same underlying set $C$, making $C \xrightarrow{f} M$ rectilinear. It suffices to show that the identity map $C_1 \to C_2$ is weakly smooth. Consider any $c \in C$. For $i = 1, 2$ let

$$\left( c \in U_i \xrightarrow{\varphi_i} \mathbb{R}^n, f(c) \in V_i \xrightarrow{\psi_i} \mathbb{R}^n \right)$$

be a pair of coordinate charts satisfying the conditions of Definition 35 with respect to the manifolds with corners structure $C_i$ on $C$.\[\text{That is, a Hausdorff second countable topology on } C \text{ together with a suitable maximal atlas.}\]
$U_1 \cap U_2 = f^{-1}(V_1 \cap V_2)$ must be open in both topologies on $C$. $\psi_1 (V_1 \cap V_2) \xrightarrow{\psi_2 \psi_1^{-1}} \psi_2 (V_1 \cap V_2)$ is a smooth map between open neighborhoods of $0 \in \mathbb{R}^n$, so its restriction $\phi_1 (U_1 \cap U_2) \xrightarrow{\phi_2}(U_1 \cap U_2)$ is weakly smooth map.

We prove (b). For each pair of indices $f^{-1}(V_i)$ and of $f^{-1}(V_j)$ (since $V_i \cap V_j$ is an open subset of $V_i$ and of $V_j$) and inherits a manifold with corners structure from both. By part (a), these structure must be equal. In particular the maps $f^{-1}(V_i \cap V_j) \rightarrow f^{-1}(V_i)$ are continuous (in fact, open topological embeddings) and we can equip $C$ with the colimit topology, which is clearly second countable. It remains only to check that this topology is Hausdorff: consider two distinct points $x, y \in C$. If $f(x) \neq f(y)$ there are open neighborhoods $U_x, U_y \subseteq M$ of $x$ and $y$ respectively with $U_x \cap U_y = \emptyset$, and we may assume without loss of generality that $U_x, U_y$ are contained in some $V_i, V_j$ so that their inverse images are open. If $f(x) = f(y) \in V_i$ then we can use the assumption that $f^{-1}(V_i)$ is Hausdorff to separate $x$ from $y$.

We prove (c). We can consider the interior points $C_i$ as subsets of $M_i$. If $\tilde{g}$ is an interior map then $\tilde{g}(C_i) \subseteq C_2$ so the restriction of $\tilde{g}$ to interior points is determined by $g$, $\tilde{g}|_{C_i} = g|_{C_i}$. Since the interior points are dense in $C_1$, a continuous extension of $g|_{C_i}$ to $C_1$ is unique if it exists. 

\textbf{Proof (of Proposition 34).} Let $h_0 : [k] \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$ denote the standard immersion of the first $k$ coordinate hyperplanes, and write $E_0 = \text{Im} \ h_0$. It is a hyper map and there’s an obvious identification $B(\mathbb{R}^n, E_0) = 2^k \times \mathbb{R}_k^n$ (a choice of germ of connected component of $\mathbb{R}^n \setminus E_0$ amounts to choosing a point of $\mathbb{R}^n$ together with an incident orthant). Clearly, the manifold with corners structure on $2^k \times \mathbb{R}_k^n$ makes the map $B(\mathbb{R}^n, E_0) \rightarrow \mathbb{R}^n$ rectilinear. Now suppose $(X, E)$ is an object of $\text{Man}^+$,

and let $W \xrightarrow{h} X$ be a hyper map with $E = \text{Im} \ h$. We can cover $X$ by orthant charts, $X = \bigcup V_i$ with $V_i \xrightarrow{\psi_i} \mathbb{R}^n$ as in Proposition 32 so $B(V_i, E \cap V_i) \subseteq B(X, E)$ inherits the structure of a manifold with corners from $B(V_i, E) \subseteq B(\mathbb{R}^n, E_0) = 2^k \times \mathbb{R}_k^n$ making the map $B(V_i, E) \rightarrow V_i$ rectilinear. It follows that $B(X, E)$ admits a unique manifold with corners structure making the map to $X$ rectilinear, and this defines the action of the functor $B$ on objects. Its action on multi-transverse arrows is defined similarly, using part (b) of Proposition 32 and the uniqueness of lifts, part (c) of Lemma 36. This uniqueness also implies that the functor is faithful.

Clearly, a local diffeomorphism is multi-transverse to any $(X, E)$ and its lift admits local inverses, proving the last statement. 

3.3.2. Hyperplane blowup of orbifolds. We consider the bicategory $\text{PEG}_{\partial = \emptyset}$ of marked proper étale groupoids (without boundary). The objects of $\text{PEG}_{\partial = \emptyset}$ are pairs $(X_1 \xrightarrow{s} X_0, E)$ where $X_1 \xrightarrow{s} X_0$ is a proper étale groupoid without boundary, and $E \subset X_0$ is a hyper subset which is a union of orbits, $s^{-1} E = t^{-1} E$.

If $(X_1^{(1)}, E^{(1)}), (X_2^{(2)}, E^{(2)})$ are two objects, a 1-cell of $\text{PEG}_{\partial = \emptyset}$ consists of a smooth functor

$$X_1^{(1)} \xrightarrow{F = (F_0, F_1)} X_2^{(2)}$$

such that $F_0$ is multi-transverse to $E^{(2)}$ and $E^{(1)} = F_0^{-1} E^{(2)}$. Note that every étale map, and in particular every refinement, satisfies this condition. The 2-cells in $\text{PEG}_{\partial = \emptyset}$ are all the 2-cells of $\text{PEG}$ spanned by the 1-cells specified above.
For emphasis, in this subsection we denote the bicategory of proper étale groupoids and orbifolds in the category Man by Peg and Orb, respectively. We denote by Pegps, Orbps the subcategories whose maps are perfectly simple maps.

If $X_1 \xrightarrow{a} X_0$ is a groupoid, we write

$$X_2 = X_1 \times_s X_1$$

for the manifold with corners parameterizing composable arrows

$$x_1 \xrightarrow{a} x_2 \xrightarrow{b} x_3$$

and, for $i = 1, 2, 3$, we denote by $p_i : X_2 \rightarrow X_0$ the map sending a composable arrow as above to $x_i$.

**Theorem 37.** The functor $B$ extends to a strict 2-functor

$$B : \text{PEG}^+_\partial \rightarrow \text{PEG}^c_{\partial}$$

which takes

$$\left( \mathcal{X} = X_2 \xrightarrow{m} X_1 \xrightarrow{e} X_0, E \right)$$

to

$$B(\mathcal{X}) = B\left( X_2, p_1^{-1}E \right) \xrightarrow{B(m)} B\left( X_1, s^{-1}E \right) \xrightarrow{B(e)} B\left( X_0, E \right)$$

together with the obvious strict natural transformation $B(\mathcal{X}) \xrightarrow{\beta_{(X,E)}} \mathcal{X}$.

This functor takes refinements to refinements, and thus there’s an induced functor between the 2-localization of these categories

$$B : \text{Orb}_{\partial} \rightarrow \text{Orb}^c_{\partial}.$$  

**Proof.** The verification that $B(\mathcal{X})$ is a groupoid with étale structure maps is straightforward. We write $B_0 = B\left( X_0, E \right)$, $B_1 = B\left( X_1, s^{-1}E \right)$, $B_2 = B\left( X_2, p_1^{-1}E \right)$

and, for $i = 0, 1, 2$, $B_i \xrightarrow{\beta_i} X_i$ for the associated map. We need to check that

$$B(s) \times B(t) : B_1 \rightarrow B_0 \times B_0$$

is proper. Note $\beta_1$ is proper, since it is closed (as any continuous map from a locally compact space to a Hausdorff space is closed) and the fiber over a point is compact (in fact, finite). It follows that $(s \times t) \circ \beta_1$ is proper and so, if $K \subset B_0 \times B_0$ is any compact subset,

$$(B(s) \times B(t))^{-1}(K) \subset ((s \times t) \circ \beta_1)^{-1}((\beta_0 \times \beta_0)(K))$$

is the inclusion of a closed subset into a compact subset; therefore $(B(s) \times B(t))^{-1}(K)$ must be compact.

The other statements are also straightforward. □

3.3.3. A hyper map between orbifolds. There’s a natural way to construct objects and arrows in Orb\textsuperscript{\#}. Let $h : \mathcal{W} \rightarrow \mathcal{X}$ be a map of orbifolds without boundary, given by a pair of smooth functors

$$\mathcal{W} \xrightarrow{S} \left( \mathcal{W} = \overline{W}_1 \rightarrow \overline{W}_0 \xrightarrow{H = (H_1, H_0)} (\mathcal{X} = X_1 \rightarrow X_0) \right)$$

with $S$ a refinement.

Let $\overline{W}_0 = X_1 \times_{H_0} \overline{W}_0$. Since $t$ is étale this fiber product exists. We let $H'_{\overline{W}_0 : W_0 \rightarrow X_0}$ denote the composition $X_1 \times_{H_0} \overline{W}_0 \rightarrow X_1 \xrightarrow{a} X_0$. We call the image of $H'$
the essential image of \( h \), and denote it \( \text{Im} \, h \). Fix some point \( x \in X_0 \). The essential fiber of \( h \) over \( x \) is a topological groupoid, with object space \((H_0^x)^{-1}(x)\) and with arrows between \( (x \xrightarrow{\alpha} H_0(w), w) \) and \((x \xrightarrow{\alpha'} H_0(w'), w')\) consisting of the arrows in \( W_1 \) between \( w \) and \( w' \) (this is a special case of the weak fiber product, see Lemma \ref{lem:weak-fiber-product}). If \( R \) is a refinement, the essential fiber of \( h \) over \( x \) and of \( R \circ h \) over \( R(x) \) are equivalent. The essential image and, up to equivalence the essential fiber, depend only on the homotopy class of \( H \) (in particular, they do not depend on \( S \)).

**Definition 38.** We say that \( h \) is hyper if the following five conditions are met (cf. Definition \ref{def:hyper-map})

- \( h \) is faithful, which means \( H_0 \) is faithful. This implies the essential fiber over every point is equivalent to a set (with a topology).
- \( h \) is a closed immersion, which means \( H_0 \) is a closed immersion. This implies the orbit space of each essential fiber has the discrete topology.
- \( h \) is proper, which means the essential fibers have compact orbit spaces. Given our previous assumptions this means the essential fiber is equivalent to a finite set, and and we fix representatives \( \{ q_i = x \xrightarrow{\alpha_i} H_0(w_i) \}_{i=1}^r \).
- We require that \( h \) has transversal self-intersection, that is, we require the map

\[
\bigoplus_{i=1}^r N^v_{w_i} \to T^v_x X_0
\]

be injective, where \( N^v_{w_i} = \ker \left( T^v_{w_i} W_0 \to T^v_x X_0 \right) \); this is independent of the choice of representatives.
- \( h \) has codimension one, meaning \( \dim X - \dim W = 1 \).

If \( Y \) is another orbifold without boundary, we say a map \( Y \xrightarrow{f} X \) is multi-transverse to \( h \) if \( h \) is (b-)transverse to \( f \) and the 2-pullback \( f^{-1}h \) has transversal self-intersection (it is automatically a proper, faithful closed immersion).

**Lemma 39.** (a) Let \( W \xrightarrow{h} X \) be a hyper map. Then \((X, \text{Im} \, h)\) is an object of \( \text{Orb}_{\partial = \emptyset}^b \).

(b) If \( f \) is multi-transverse to \( h \), \((Y, \text{Im} \, f^{-1}h) \xrightarrow{f} (X, \text{Im} \, h)\) is an arrow in \( \text{Orb}_{\partial = \emptyset}^b \).

**Proof.** Straightforward. \hfill \qed

### 3.4. Group actions

Let \( G \) be a compact lie group, with multiplication \( m : G \times G \to G \) and identity \( e : pt \to G \). Given a bicategory of spaces \( C \) such as \( ^c \text{Man} \), \( \text{Orb}, \text{Orb}_{\partial = \emptyset}^b \), we construct a category \( G^{-C} \) of \( G \)-equivariant objects following Romagny \cite{Romagny}. We briefly explain how to translate his definitions to our setup, and refer the reader to \cite{Romagny} for more details. A 0-cell of \( G^{-C} \) is given by a 4-tuple \((X, \mu, \alpha, a)\) where \( X \) is a 0-cell of \( C \), \( \mu : G \times X \to X \) is a 1-cell, and \( \alpha \) and \( a \) are 2-cells filling in, respectively,

\[\text{(X, E) } \times \text{(Y, F) } = \text{(X} \times \text{Y, E} \times \text{Y, \bigcup} X \times F) \]
the following square and triangle:

\[
\begin{array}{ccc}
G \times G \times \mathcal{X} & \xrightarrow{\mu \times \text{id}_G} & G \times \mathcal{X} \\
\downarrow \text{id}_G \times \mu & & \downarrow \mu \times \text{id}_\mathcal{X} \\
G \times \mathcal{X} & \xrightarrow{\mu} & \mathcal{X} \\
\end{array}
\]

A 1-cell (or $G$-equivariant map)

\[(\mathcal{X}, \mu, \alpha, a) \rightarrow (\mathcal{X}', \mu', \alpha', a') \]

is given by a pair \((\mathcal{X} \xrightarrow{f} \mathcal{X}', \sigma)\) where $\sigma$ is a 2-cell filling in the square

\[
\begin{array}{ccc}
G \times \mathcal{X} & \xrightarrow{\mu} & \mathcal{X} \\
\downarrow \text{id}_G \times f & & \downarrow f \\
G \times \mathcal{X}' & \xrightarrow{\mu'} & \mathcal{X}' \\
\end{array}
\]

A 2-cell \((f, \sigma) \Rightarrow (f', \sigma')\) is given by a 2-cell \(\mathcal{X} \xrightarrow{\beta} \mathcal{X}'\). As usual, the 2-cells $\alpha, a, \sigma, \beta$ are required to satisfy some coherence conditions, cf. [14, Definition 2.1].

3.4.1. The fixed-points of a hyperplane blowup. There’s an obvious pair of pseudo-functors $\iota : \mathcal{C} \rightarrow G-\mathcal{C}$ and $f : G-\mathcal{C} \rightarrow \mathcal{C}$; $\iota$ equips every 0-cell with the trivial $G$-action and every 1-cell with the trivial $G$-equivariant structure, $f$ forgets the extra structure. Let $\mathcal{X}$ be a 0-cell of $G-\mathcal{C}$. Consider the pseudofunctor $\mathcal{X}^G_! : \mathcal{C}^{op} \rightarrow \mathbf{Cat}$

\[\mathcal{X}^G_! (-) = \text{Hom}_{G-\mathcal{C}} \left( \iota (-), \mathcal{X} \right)\]

together with the obvious pseudonatural transformation $\mathcal{X}^G_! \rightarrow \text{Hom}_{\mathcal{C}} (-, f(\mathcal{X}))$. $\mathcal{X}^G_!$ may or may not be represented by an object of $\mathcal{C}$; if it is, we denote the representing object by $\mathcal{X}^G$ and say “the fixed-point locus $\mathcal{X}^G$ exists (as an object of $\mathcal{C}$)” (cf. [14 Definition 2.3]). Here are a few examples.

- $\mathcal{C}$ is the category of stacks: the fixed-point locus always exist, see [14 Proposition 2.5] (though for a general stack, this may be quite ill-behaved).
- $\mathcal{C} = \mathbf{Orb}$: it follows from the slice theorem that the fixed point locus $\mathcal{X}^G$ of any $G$-orbifold without boundary $\mathcal{X}$ exists, and the map $\mathcal{X}^G \rightarrow \mathcal{X}$ is a closed embedding.
- $\mathcal{C} = \mathbf{Man}^c$ (considered as a bicategory with only identity 2-cells), $G = \mathbb{T}$ a torus group: we do not know whether the fixed-point set exists in general.

The last example motivates restricting our attention to manifolds (and orbifolds) with corners which are obtained as a hyperplane blowup. We will see that if $\mathcal{X}$ is obtained as a hyperplane blowup of $\mathcal{X}$ at an invariant hyper subset, then the fixed-points of $\mathcal{X}$ exist and the map $\mathcal{X}^G \rightarrow \mathcal{X}$ is a closed embedding. The remainder of this subsection is devoted to formulating and proving this result.

Any pseudofunctor $\mathcal{C}_1 \rightarrow \mathcal{C}_2$, where $\mathcal{C}_i$ is a bicategory of spaces as above for $i = 1, 2$, induces a functor between the corresponding $G$-categories. In particular, the blow up $B : \text{Orb}^+ \rightarrow \text{Orb}_{ps}^c$ induces a functor

\[G - \text{Orb}^+ \xrightarrow{B} G - \text{Orb}_{ps}^c.\]
Proposition 40. Let $G = U(1)^m$ be a torus group, and let $\tilde{X} = B(X, E)$ denote the blowup of an object of $G\text{-}\text{Orb}^+$ (we’re supressing the $G$-action data from the notation). Let $i : \mathcal{X}^G \to \mathcal{X}$ be the closed embedding of the fixed-points of $\mathcal{X}$. Then

(a) the fixed points $\left(\tilde{X}\right)^G$ exist as an object of $\text{Orb}^+$, and the map $\left(\tilde{X}\right)^G \to \tilde{X}$ is a closed embedding.

(b) $i$ is multi-transverse to $h$ and there’s a natural isomorphism $\left(\tilde{X}\right)^G \cong \left(\mathcal{X}^G\right)^{-} = B\left(\mathcal{X}^G,\text{Im}i^{-1}h\right)$, and

(c) there’s a 2-cell making the following square cartesian

$$
\begin{array}{ccc}
\left(\mathcal{X}^G\right)^- & \cong & \left(\tilde{X}\right)^G \\
\downarrow & & \downarrow \\
\mathcal{X}^G & \to & \mathcal{X}
\end{array}
$$

where the horizontal maps are the structure maps of the fixed points and the vertical maps are the maps associated with the hyperplane blowup.

Proof. We describe the argument when $\tilde{X} = B(X, E)$ is the blowup of a marked manifold with corners, and leave the general case to the reader. Note that the statement is local, so we fix some $p \in X^G$ and work throughout with germs of functions around $p$ and its images (i.e., we will not specify the open subsets on which each map is defined). Using Proposition 32 we reduce to the following setup.

Let $W = [k] \times \mathbb{R}^{n-1} \xrightarrow{h} X = \mathbb{R}^n$ denote the standard hyper map, so for $1 \leq i \leq k$, $\{i\} \times \mathbb{R}^{n-1} \xrightarrow{h_i} \mathbb{R}^n$ is the map

$$
(t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_i, 0, t_{i+1}, \ldots, t_{n-1}).
$$

we assume that $G = \mathbb{T}$ acts compatibly on the domain and range of $h$; since $G$ is connected this means each $h_i$ is $G$-equivariant. We take $p$ to be the origin $0 \in \mathbb{R}^n$, and write $\{q_i = \{i\} \times 0\}_{i=1}^k$ for its preimages under $h$.

Although a-priori, the action in these coordinates is not linear, it does preserve the zeros of the first $k$ components, and thus also their signs. More precisely, if we define the signature $\sigma(x) \in \{-, 0, +\}^k$ of a point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ by recording the sign of the first $k$ coordinates, then $\sigma(g, x) = \sigma(x)$ for all $g, x$. We define a new orthant chart $(x'_1(q), \ldots, x'_n(q))$ (in a perhaps smaller neighborhood of the origin) by

$$
x'_i(q) = \begin{cases} 
\int_{g \in G} x_i(g, q) \, dH(g) & 1 \leq i \leq k \\
x_i(q) & k + 1 \leq i \leq n
\end{cases}
$$

where $dH$ denotes the Haar measure on $G$. Clearly, $x'_i$ are $G$-invariant and $x'_i \circ h_i \equiv 0$ for $1 \leq i \leq k$; the key point is that, since $G$ admits no non-trivial rank one representations, the line $\ker(T^*_p X \to T^*_q W)$ are fixed, and thus $dx'_1|_0 = dx|_0$ for $1 \leq i \leq n$; in particular $(dx'_1|_0)^n_{i=1}$ form a basis for $T^*_p X$.

We can choose a basis $v_1^*, \ldots, v_n^*$ to $T^*_p X$ so that $v_i^* = dx'_i|_0$ for $1 \leq i \leq k$ and $v_{k+1}^*, \ldots, v_n^*$ are common eigenvectors for the infinitesimal generators $\xi_1, \ldots, \xi_m \in \text{End}(T^*_p X)$ of the linearized action. Fix a $G$-invariant metric $R$, let $v_1, \ldots, v_n$ be a basis for
$T_pX$ dual to $v_1^*, ..., v_n^*$, and consider exponential coordinates $(y_1, ..., y_n)$ for $X = \mathbb{R}^n$ around $p = 0$ so that

$$(y_1, ..., y_n) \mapsto \exp_R \left( \sum y_i v_i \right).$$

Now consider the coordinates

$$(x'_1, ..., x'_k, y_{k+1}, ..., y_n).$$

These define an orthant chart. Moreover, in these coordinates the action is the linear action on

$$\mathbb{R}^n = \mathbb{R}^s \oplus \bigoplus_{i=1}^r \mathbb{C}_{\lambda_i}$$

fixing $\mathbb{R}^s$, $s \geq k$ and acting on $\mathbb{C}_{\lambda_i}$ by a character $\lambda_i : \mathbb{T} \to U(1)$.

Working with such orthant charts, the verification of properties (a)-(c) is straightforward. □

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