POLYGONS IN THREE-DIMENSIONAL SPACE

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Abstract. Let \( P = A_1 \ldots A_n \) be a generic polygon in three-dimensional space and let \( v_1, v_2, \ldots, v_n \) be vectors \( \overrightarrow{A_1A_2}, \overrightarrow{A_2A_3}, \ldots, \overrightarrow{A_nA_1} \), respectively. \( P \) will be called regular, if there exist vectors \( u_1, \ldots, u_n \) such that cross products \([u_1, u_2], [u_2, u_3], \ldots, [u_n, u_1]\) are equal to vectors \( v_2, v_3, \ldots, v_1 \), respectively. In this case the polygon \( P' \), defined by vectors \( u_2 - u_1, u_3 - u_2, \ldots, u_1 - u_n \) will be called the derived polygon or the derivative of the polygon \( P \). In this work we formulate conditions for regularity and discuss geometric properties of derived polygons for \( n = 4, 5, 6 \).

1. Introduction

In this work we consider duality problems in the three-dimensional space. The duality for plane polygons is discussed in work [2] (in particular, the duality of plane quadrangles is the subject of work [1]). However, the use of complex numbers is the main tool in this approach. In three-dimensional space the natural "multiplication" is the cross product. So we try to construct dual objects for space polygons using cross product, as main tool. Spaces of three-dimensional polygons are considered in [3] from points of view of differential and algebraic geometry. However, in this work we use only elementary properties of cross and dot products in three-dimensional space, so it can be understood by undergraduates. The standard reference here is [4].

We work in the standard space \( \mathbb{R}^3 \). In what follows \((a, b)\) will be the dot product of vectors \( a \) and \( b \), and \([a, b]\) will be the cross product. Let us remind that

\[
(a, [b, c]) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}
\]

for any three vectors \( a = (a_1, a_2, a_3) \), \( b = (b_1, b_2, b_3) \) and \( c = (c_1, c_2, c_3) \).

Let \( P = A_1, \ldots, A_n \) be a generic polygon in three-dimensional space, i.e. any pair of its consecutive edges are not collinear and any triple of its consecutive edges are not coplanar. Let \( v_1 = \overrightarrow{A_1A_2}, v_2 = \overrightarrow{A_2A_3}, \ldots, v_n = \overrightarrow{A_nA_1} \).

We want to construct a system of vectors \( u_1, \ldots, u_n \) such that

\[
[u_1, u_2] = v_2, \ [u_2, u_3] = v_3, \ldots, [u_n, u_1] = v_1.
\]

Definition 1.1. A generic polygon \( P \) is called regular, if such system exists and the system itself will be called a support system for the polygon \( P \). If \( P \) is regular and \( u_1 = \overrightarrow{OB_1}, \ldots, u_n = \overrightarrow{OB_n} \) — its support system, then the polygon \( P' = B_1 \ldots B_n \) will be called the derived polygon (or the derivative) of \( P \).

Example 1.1. Let \( P = A_1A_2A_3A_4 \), where \( A_1 = (0, 0, 0), A_2 = (1, 1, 2), A_3 = (2, 3, 1), A_4 = (-1, 2, -2) \), then \( v_1 = (1, 1, 2), v_2 = (1, 2, -1), v_3 = (-3, -1, -3) \) and \( v_4 = (1, -2, 2) \). As \( u_1 \perp v_1 \) and \( u_1 \perp v_2 \), then vector \( u_1 \) is a multiple of the cross product \([v_1, v_2]\). Analogously, \( u_2 \) is a multiple of \([v_2, v_3]\), \( u_3 \) is a multiple of \([v_3, v_4]\) and \( u_4 \) is a multiple of \([v_4, v_1]\).

Let \( u_1 = [v_1, v_2] = (-5, 3, 1) \). As \( [u_1, [v_2, v_3]] = (9, 18, -9) \), than \( u_2 = \frac{1}{9} [v_2, v_3] = (-\frac{2}{9}, \frac{2}{3}, \frac{5}{9}) \). Analogously, as \( [u_2, [v_3, v_4]] = (3, 1, 3) \), then \( u_3 = -[v_3, v_4] = (8, -3, -7) \). And as \( [u_3, [v_4, v_1]] = (-9, 18, -18) \), then \( u_4 = \ldots \).
We will work with determinants \( \Delta_1 = ([v_2, v_1] = (\frac{2}{3}, 0, -\frac{1}{3}) \). At last we have, that

\[
[u_4, u_1] = \left( \frac{2}{3}, 0, -\frac{1}{3} \right), (-5, 3, 1) = (1, 1, 2)
\]

— support system is constructed and our polygon is regular.

Now let us study the derived polygon \( P' = B_1B_2B_3B_4 \). As

\[
(u_2 - u_1, [u_3 - u_1, u_4 - u_1]) = \begin{vmatrix} 38/9 & -21/9 & -4/9 \\ 13 & -6 & -8 \\ 17/3 & -3 & -4/3 \end{vmatrix} = 0,
\]

then the polygon \( P' \) is a plane quadrangle. Moreover, as

\[
[u_2 - u_1, u_3 - u_1] + [u_3 - u_1, u_4 - u_1] = (16, 28, 5) + (-16, -28, -5) = (0, 0, 0),
\]

then the oriented area of \( B_1B_2B_3B_4 \) is zero. Thus, \( P' \) is self-intersecting:

\[\begin{array}{c}
\text{(x}, y, z) \text{ are coordinates of the vector } v_j. \text{ It must be noted, that as our polygon is generic, then these determinants are nonzero.}
\end{array}\]

Here is the summary of obtained results.

- A generic polygon \( P = A_1 \ldots A_n \) for even \( n \) is regular, if \( \Delta_1 \cdot \Delta_3 \ldots \Delta_{n-1} = \Delta_2 \cdot \Delta_4 \ldots \Delta_n \). In this case its support systems constitute an infinite family. If \( n \) is odd, then \( P \) is regular, if \( \Delta_1 \cdot \Delta_2 \ldots \Delta_n > 0 \), then its support system is unique up to the sign (Theorem 2.1).
- If \( n = 4 \) then \( P \) is always regular. Each its derivative is a plane self-intersecting quadrangle with oriented area 0 (Theorem 3.1).
- If \( n = 5 \), then the derivative of a regular polygon is a plane pentagon with oriented area 0 (Theorem 4.1).
- A regular hexagon is called strongly-regular, if \( \Delta_1 = \Delta_4, \Delta_2 = \Delta_5 \) and \( \Delta_3 = \Delta_6 \). The type of a strongly-regular hexagon is the cyclic ratio \( \Delta_1 : \Delta_2 : \Delta_3 \). All derivatives of a regular hexagon \( P \) are strongly-regular and of the same type. The types of derived hexagon \( P' \) and its derivative \( P'' \) are the same. If \( P' = B_1B_2B_3B_4B_5B_6 \), then vertices of \( P' \) belong to two parallel planes \( \Pi_1 \) and \( \Pi_2 \): points \( B_1, B_3 \) and \( B_5 \) belong to \( \Pi_1 \), and points \( B_2, B_4 \) and \( B_6 \) belong to \( \Pi_2 \). Moreover, the oriented area of the plane hexagon \( B_1B_2'B_3B_4B_5B_6 \) is 0 (here points \( B_2', B_4' \) and \( B_6' \) are projections of points \( B_2, B_4, B_6 \) to the plane \( \Pi_1 \) (Theorems 5.1, 5.2 and 5.3).
In the last section the question of the regularity of a knotted 6-gon is discussed.

2. General remarks

We use notations from the previous section.

Theorem 2.1. Let \( P \) be a generic \( n \)-gon in three-dimensional space. If \( n = 2m \) is even, then \( P \) is regular if and only if

\[
\Delta_1 \cdot \Delta_3 \cdot \ldots \cdot \Delta_{2m-1} = \Delta_2 \cdot \Delta_4 \cdot \ldots \cdot \Delta_{2m}.
\]

In this case there is an infinite family of its support systems. If \( n \) is odd then \( P \) is regular if and only if \( \Delta_1 \cdot \Delta_2 \cdot \ldots \cdot \Delta_n > 0 \). In this case its support system is unique up to the sign.

Proof. To prove the regularity we must construct a support system. As \([u_{i-1}, u_i] = v_i \) and \([u_i, u_{i+1}] = v_{i+1} \), then \( u_i \) is orthogonal to \( v_i \) and \( v_{i+1} \), hence, it is proportional to their cross product: \( u_i = \alpha [v_i, v_{i+1}] \).

Let \( u_1 = [v_1, v_2] \). As

\[
[u_1, u_2] = [v_1, v_2], \quad \frac{1}{\Delta_1} \cdot [v_2, v_3] = \frac{1}{\Delta_1} \cdot [v_1, v_2], v_2 = \frac{1}{\Delta_1} \cdot [v_1, v_2], v_3 = v_2.
\]

Analogously,

\[
u_3 = \frac{\Delta_1 \cdot [v_3, v_4]}{\Delta_2}, \quad u_4 = \frac{\Delta_2 \cdot [v_4, v_5]}{\Delta_1 \cdot \Delta_3}, \ldots
\]

Let \( n = 2m \) be even, then

\[
u_{2m} = \frac{\Delta_2 \cdot \Delta_4 \cdot \ldots \cdot \Delta_{2m-2} \cdot [v_n, v_1]}{\Delta_1 \cdot \Delta_3 \cdot \ldots \cdot \Delta_{2m-1}}.
\]

This choice of vectors \( u_1, \ldots, u_n \) allows one to satisfy conditions \( [u_1, u_2] = v_2, \ldots, [u_{n-1}, u_n] = v_n \). But the condition \( [u_n, u_1] = v_1 \) can be satisfied only when

\[
u_1 = \frac{\Delta_1 \cdot \Delta_3 \cdot \ldots \cdot \Delta_{2m-1} \cdot [v_1, v_2]}{\Delta_2 \cdot \Delta_4 \cdot \ldots \cdot \Delta_{2m}}.
\]

But \( u_1 = [v_1, v_2] \), hence,

\[
\Delta_1 \cdot \Delta_3 \cdot \ldots \cdot \Delta_{n-1} = \Delta_2 \cdot \Delta_4 \cdot \ldots \cdot \Delta_n.
\]

Let \( n = 2m \) and \( u_1, \ldots, u_n \) be a constructed above support system. Then vectors \( u'_1, \ldots, u'_n \), where \( u'_i = \alpha u_i \) for even \( i \), and \( u'_i = u_i / \alpha \) for odd \( i \) also constitute a support system for all nonzero \( \alpha \). Hence, there is an infinite family of support systems (of derived polygons) for a regular \( 2m \)-gon.

Let now \( n \) be odd. We construct vectors \( u_1, \ldots, u_n \), as above. The key moment is the satisfaction of the condition

\[
[u_n, u_1] = v_1 \Leftrightarrow \Delta_1 \cdot \Delta_3 \cdot \ldots \cdot \Delta_{n-2} \cdot [v_n, v_1], [v_1, v_2] = v_1.
\]

Now (2) gives us a sufficient condition: \( P \) is regular if \( \Delta_1 \cdot \Delta_3 \cdot \ldots \cdot \Delta_n = \Delta_2 \cdot \Delta_4 \cdot \ldots \cdot \Delta_{n-1} \). However, in the odd case this condition is not necessary.

As above we define vectors \( u'_i \): \( u'_i = \alpha \cdot u_i \), if \( i \) is even, and \( u'_i = u_i / \alpha \), if \( i \) is odd. Then \( [u'_i, u'_{i+1}] = v_{i+1} \) for \( i = 1, \ldots, n-1 \). But

\[
[u'_n, u'_1] = \frac{\Delta_1 \cdot \Delta_3 \cdot \ldots \cdot \Delta_n}{\alpha^2 \cdot \Delta_2 \cdot \Delta_4 \cdot \ldots \cdot \Delta_{n-1}} \cdot v_1.
\]

Thus, if \( \Delta_1 \cdot \Delta_2 \cdot \ldots \cdot \Delta_n > 0 \), then there exists the unique (up to the sign) number \( \alpha \) such, that vectors \( u'_1, \ldots, u'_n \) constitute a support system of \( P \).
Remark 2.1. Let $n$ be odd. If an $n$-gon $P$ is not regular, then its mirror-symmetric is regular.

Example 2.1. Let us consider a pentagon $P$:

\[
\begin{align*}
v_1 &= (1, 0, 0) & \Delta_1 &= 1 & \{v_1, v_2\} &= (0, 0, 1) & u_1 &= (0, 0, 1) \\
v_2 &= (0, 1, 0) & \Delta_2 &= 2 & \{v_2, v_3\} &= (1, 0, 0) & u_2 &= (1, 0, 0) \\
v_3 &= (0, 0, 1) & \Delta_3 &= -4 & \{v_3, v_4\} &= [2, 2, 0] & u_3 &= (1, 1, 0) \\
v_4 &= (2, -2, 3) & \Delta_4 &= 5 & \{v_4, v_5\} &= (5, -1, -4) & u_4 &= (-5/2, 1/2, 2) \\
v_5 &= (-3, 1, -4) & \Delta_5 &= -4 & \{v_5, v_1\} &= (0, -4, -1) & u_5 &= (0, 8/5, 2/5)
\end{align*}
\]

As $\Delta_1 \cdot \Delta_2 \cdot \Delta_3 \cdot \Delta_4 \cdot \Delta_5 > 0$, then $P$ is regular, and as $[u_5, u_1] = (8/5, 0, 0)$, then $\alpha = 2\sqrt{2}/\sqrt{5}$. Thus, we have the following support system:

\[
\begin{align*}
u'_1 &= \left(0, \frac{\sqrt{5}}{2\sqrt{2}}, 0\right) & u'_2 &= \left(\frac{2\sqrt{2}}{\sqrt{5}}, 0, 0\right) & u'_3 &= \left(\frac{\sqrt{5}}{2\sqrt{2}}, \frac{\sqrt{5}}{2\sqrt{2}}, 0\right), \\
u'_4 &= \left(-\sqrt{10}, \frac{\sqrt{2}}{\sqrt{5}}, 4\sqrt{2}\right) & u'_5 &= \left(0, \frac{2\sqrt{2}}{\sqrt{5}}, \frac{1}{\sqrt{10}}\right).
\end{align*}
\]

Remark 2.2. Let $Q$ be an $n$-gon defined by vectors $v_1, \ldots, v_n$, $v_1 + \ldots + v_n = 0$. $Q$ is a derived polygon of some polygon $P$ if there exists a vector $u$ such that

$[u, u + v_1] + [u + v_1, u + v_1 + v_2] + \ldots + [u + v_1 + \ldots + v_{n-2}, u + v_1 + \ldots + v_{n-1}] + [u + v_1 + \ldots + v_{n-1}, u] = 0$.

The lefthand part of this condition is equal to $\sum_{0 < i < j < n} [v_i, v_j]$ and does not depend on $u$ at all.

3. Quadrangles

Theorem 3.1. Each quadrangle is regular and each its derivative is a plane quadrangle with oriented area $0$.

Proof. We use notations of the previous section. Let $P$ be a quadrangle and $v_i = (a_i, b_i, c_i)$, $i = 1, 2, 3, 4$. Then

\[
\Delta_2 = \begin{vmatrix}
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
a_4 & b_4 & c_4
\end{vmatrix} = \begin{vmatrix}
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
a_4 & b_4 & c_4
\end{vmatrix} = -\Delta_1.
\]

Analogously, $\Delta_3 = \Delta_1$ and $\Delta_4 = -\Delta_1$, i.e. $P$ is regular.

Let us consider a support system $u'_1, u'_2, u'_3, u'_4$, where $u'_1 = u_1/\alpha$, $u'_2 = \alpha \cdot u_2$, $u'_3 = u_3/\alpha$, $u'_4 = \alpha \cdot u_4$ and $u_1 = [v_1, v_2]$, $u_2 = [v_2, v_3]/\Delta_1$, $u_3 = -[v_3, v_4]$, $u_4 = -[v_4, v_1]/\Delta_1$. We must prove that the mixed product of vectors $u'_2 - u'_1$, $u'_3 - u'_1$ and $u'_4 - u'_1$ is zero. We have

\[
\begin{align*}
(u'_2 - u'_1, [u'_3 - u'_1, u'_4 - u'_1]) &= (u'_2 - u'_1, [u'_3 - u'_1, u'_4 - u'_1]) = (u'_2, [u'_3, u'_4]) - (u'_2, [u'_1, u'_4]) = \alpha \cdot (\alpha - u_2, v_4) + \alpha \cdot (u_2, v_1) - (u_1, v_3)/\alpha - (u_1, v_4)/\alpha = \\
&= \frac{\alpha}{\Delta_1} \cdot (\alpha \cdot (\alpha - (v_2, v_3), v_4) + (v_2, v_3), v_1)) - \frac{1}{\alpha} \cdot (\alpha \cdot ((v_1, v_2), v_3) + ((v_1, v_2), v_4)) = \\
&= \frac{\alpha}{\Delta_1} \cdot (\Delta_2 + \Delta_1) - \frac{1}{\alpha} \cdot (\Delta_1 + \Delta_4) = 0.
\end{align*}
\]

Let $u'_i = \overrightarrow{OB}_i$, $i = 1, 2, 3, 4$, and $\Pi$ be the plane of points $B_1, B_2, B_3, B_4$. In a coordinate system, where $\Pi$ is parallel to the plane $xy$, the $z$-coordinate of the sum $[u'_1, u'_2] + [u'_3, u'_4] + [u'_4, u'_1]$ (which is zero) is the oriented area of the quadrangle $B_1B_2B_3B_4$, multiplied by 2. In particular $B_1B_2B_3B_4$ is a self-intersecting quadrangle. \[\square\]
Theorem 4.1. Let $P$ be a generic regular pentagon. Then its derivative is a plane pentagon with oriented area zero.

Proof. Let $u_1, \ldots, u_5$ be a support system for $P$. We must prove that $(u_2 - u_1, [u_3 - u_1, u_4 - u_1]) = 0$ and $(u_3 - u_1, [u_4 - u_1, u_5 - u_1]) = 0$. As $[u_1, u_2] + [u_2, u_3] + [u_3, u_4] + [u_4, u_5] + [u_5, u_1] = 0$, then

$$(u_2 - u_1, [u_3 - u_1, u_4 - u_1]) = (u_2, [u_3, u_4]) - (u_2, [u_3, u_1]) - (u_2, [u_1, u_4]) - (u_1, [u_3, u_4]) =$$

$$= (u_4, [u_2, u_3]) + (u_4, [u_1, u_2]) - (u_1, [u_2, u_3]) - (u_1, [u_3, u_4]) = -(u_4, [u_5, u_1]) + (u_1, [u_4, u_5]) = 0.$$

The second equality can be proved analogously.

The second statement can be proved in the same way, as the analogous statement in Theorem 3.1.

Remark 4.1. If vectors $u_1, u_2, u_3, u_4, u_5$ have the same z-coordinate, then $x$- and $y$-coordinates of the sum $[u_1, u_2] + [u_2, u_3] + [u_3, u_4] + [u_4, u_5] + [u_5, u_1]$ are zero.

Remark 4.2. Up to rotation, dilation and mirror symmetry each generic pentagon can be recovered from a pentagon in the plane $z = 1$, with oriented area zero and with centroid positioned at positive $x$ half-axis.

Example 4.1. Let $u_1 = (2, 2, 1), u_2 = (3, -1, 1), u_3 = (-3, 1, 1), u_4 = (-4, 0, 1), u_5 = (-1, -1, 1)$.

Endpoints $A, B, C, D, E$ of these vectors belong to the plane $z = 1$ and define at this plane the self-intersecting pentagon with oriented area zero:

Thus, vectors $u_1, \ldots, u_5$ constitute as a support system:

$v_2 = [u_1, u_2] = (3, 1, -8), v_3 = [u_2, u_3] = (-2, -6, 0), v_4 = [u_3, u_4] = (1, -1, 4), v_5 = [u_4, u_5] = (1, 3, 4), v_1 = [u_5, u_1] = (-3, 3, 0)$.

Now let us perform the inverse computation:

$\Delta_1 = 192, \Delta_2 = -128, \Delta_3 = 32, \Delta_4 = 48, \Delta_5 = -144$

and

$w_1 = [v_1, v_2] = (-24, -24, -12)$

$w_2 = \frac{[v_2, v_3]}{\Delta_1} = \left(-\frac{1}{4}, -\frac{1}{12}, -\frac{1}{12}\right)$

$w_3 = \frac{\Delta_1 \cdot [v_3, v_4]}{\Delta_2} = (36, -12, -12)$

$w_4 = \frac{\Delta_2 \cdot [v_4, v_5]}{\Delta_1 \cdot \Delta_3} = \left(\frac{1}{3}, 0, -\frac{1}{12}\right)$

$w_5 = \frac{\Delta_1 \cdot \Delta_3 \cdot [v_4, v_5]}{\Delta_2 \cdot \Delta_4} = (12, 12, -12)$

As $[w_5, w_1] = 144 \cdot v_1$, then $\alpha = 12$ and we return to the initial set $u_1, \ldots, u_5$. 
5. Hexagons

Example 5.1. Let vectors $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$, $v_4 = (2, -1, 3)$, $v_5 = (-1, 5, 2)$ and $v_6 = (-2, -5, -6)$ define the hexagon $P$. As

$$\Delta_1 = 1, \Delta_2 = 2, \Delta_3 = 9, \Delta_4 = 15, \Delta_5 = 20, \Delta_6 = -6$$

and $\Delta_1 \Delta_3 \Delta_5 = \Delta_2 \Delta_4 \Delta_6 = -180$, then $P$ is regular. Vectors

$$u_1 = (0, 0, 1), u_2 = (1, 0, 0), u_3 = \left(\frac{1}{2}, 1, 0\right), u_4 = \left(-\frac{34}{9}, -\frac{14}{9}, 2\right), u_5 = \left(-6, -\frac{3}{2}, 9\right), u_6 = \left(0, 1, \frac{5}{9}\right)$$

constitute a support system, and vectors

$$u_2 - u_1 = (1, 0, -1), u_3 - u_2 = \left(-\frac{1}{2}, 1, 0\right), u_4 - u_3 = \left(-\frac{77}{18}, -\frac{23}{9}, 2\right), u_5 - u_4 = \left(-\frac{20}{9}, -\frac{13}{9}, 2\right),$$

$$u_6 - u_5 = \left(6, 4, -\frac{16}{3}\right), u_1 - u_6 = \left(0, -1, \frac{11}{6}\right)$$

define the derived hexagon $P'$. The corresponding determinants are:

$$\Delta_1' = -\frac{32}{9}, \Delta_2' = 8, \Delta_3' = \frac{4}{3}, \Delta_4' = -\frac{32}{9}, \Delta_5' = 8, \Delta_6' = \frac{4}{3}.$$  

We see that $P'$ satisfies a stronger condition, than regularity.

Definition 5.1. A regular hexagon is called strongly-regular, if $\Delta_1 = \Delta_4, \Delta_2 = \Delta_5$ and $\Delta_3 = \Delta_6$. The type of a strongly-regular hexagon is the cyclic ratio $\Delta_1 : \Delta_2 : \Delta_3$.

Theorem 5.1. All derivatives of a regular hexagon $P$ are strongly-regular and have the same type.

Proof. We use notations from Section 2. A derived hexagon $P'$ is defined by vectors $u_2' - u_1', u_3' - u_2', \ldots, u_6' - u_5'$, where $u_i' = \alpha \cdot u_i$ for even $i$, and $u_i' = u_i/\alpha$ for odd $i$. We must prove that $\Delta_1' = \Delta_4', \Delta_2' = \Delta_5'$ and $\Delta_3' = \Delta_6'$ (determinants $\Delta_i'$ are defined in the same way, as determinants $\Delta_i$, but for polygon $P'$). We will prove the first equality. It can be rewritten in the form

$$(\alpha \cdot u_2 - u_1/\alpha, [u_3/\alpha - \alpha \cdot u_2, \alpha \cdot u_4 - u_3/\alpha]) = (u_5/\alpha - \alpha \cdot u_4, [\alpha \cdot u_6 - u_5/\alpha, u_1/\alpha - \alpha \cdot u_6]).$$

Here coefficients at $\alpha^3$ and $\alpha^{-3}$ are zero. Let compare coefficients at $\alpha$ and at $\alpha^{-1}$ in the left and in the right hand sides of this relation. Coefficient at $\alpha^{-1}$ in the left is

$$-(u_1, [u_3, u_4]) - (u_1, [u_2, u_3]) = -(u_1, v_4) - (u_1, v_3) = -([v_1, v_2], v_4) - ([v_1, v_2], v_3) = -([v_1, v_2], v_4) - \Delta_1.$$

Coefficient at $\alpha^{-1}$ in the right is

$$(u_5, [u_6, u_1]) + (u_4, [u_5, u_1]) = (u_1, [u_5, u_6]) + (u_1, [u_4, u_5]) = (u_1, v_6) + (u_1, v_5) =$$

$$= ([v_1, v_2], v_6) + ([v_1, v_2], v_5) = \Delta_6 - ([v_1, v_2], v_3 + v_4 + v_6) = -([v_1, v_2], v_4) - \Delta_1.$$

Then, coefficient at $\alpha$ in the left is

$$(u_2, [u_3, u_4]) + (u_1, [u_2, u_4]) = (u_2, v_4) + (u_2, [u_1, u_2]) = (u_2, v_4) + (u_4, v_2) =$$

$$= \frac{1}{\Delta_1} \cdot ([v_2, v_3], v_4) + \frac{\Delta_2}{\Delta_1 \Delta_3} \cdot ([v_4, v_5], v_2) = \frac{\Delta_2}{\Delta_1} - \frac{\Delta_2}{\Delta_1 \Delta_3} \cdot ([v_4, v_5], v_1 + v_3 + v_6) =$$

$$= -\frac{\Delta_2}{\Delta_1 \Delta_3} ([v_4, v_5], v_1) - \frac{\Delta_2 \Delta_4}{\Delta_1 \Delta_3}.$$
Coefficient at $\alpha$ in the right is

$$-(u_4, [u_6, u_1]) - (u_4, [u_5, u_6]) = -(u_4, v_1) - (u_4, v_6) = -\frac{\Delta_2}{\Delta_1 \Delta_3} \cdot ([v_4, v_5], v_1) -$$

$$-\frac{\Delta_2}{\Delta_1 \Delta_3} \cdot ([v_4, v_5], v_6) = -\frac{\Delta_2}{\Delta_1 \Delta_3} \cdot ([v_4, v_5], v_1) - \frac{\Delta_2 \Delta_4}{\Delta_1 \Delta_3} .$$

Conditions $\Delta'_2 = \Delta'_5$ and $\Delta'_3 = \Delta'_6$ can be proved analogously.

Now let us turn to the second part of the theorem. We need to prove that ratios

$$\frac{(\alpha \cdot u_2 - u_1/\alpha, [u_3/\alpha - \alpha \cdot u_2, \alpha \cdot u_4 - u_3/\alpha])}{(u_3/\alpha - \alpha \cdot u_2, [\alpha \cdot u_4 - u_3/\alpha, u_5/\alpha - \alpha \cdot u_4])} \quad \text{and} \quad \frac{(u_3/\alpha - u_3/\alpha, [\alpha \cdot u_4 - u_3/\alpha, u_5/\alpha - \alpha \cdot u_4])}{(\alpha \cdot u_4 - u_3/\alpha)}$$

are constants, as functions of $\alpha$. We will prove it for the first ratio, which can be rewritten in the following way:

$$\frac{\alpha \cdot [(u_2, v_4) + (u_4, v_2)] - \frac{1}{\alpha} \cdot [(u_1, v_4) + (u_1, v_3)]}{[(u_3, v_5) + (u_5, v_3)] - \alpha \cdot [(u_2, v_5) + (u_2, v_4)]}$$

It is enough to prove that the ratio of coefficients at $\alpha$ and the ratio of coefficients at $\alpha^{-1}$ are the same, i.e. to prove that

$$(u_2, v_4) + (u_4, v_2) = (u_1, v_3) \quad \frac{(u_1, v_3) + (u_1, v_4)}{(u_3, v_5) + (u_5, v_3)} \Leftrightarrow$$

$$\frac{\Delta_1}{\Delta_1} + \frac{\Delta_2}{\Delta_1} \cdot v_2, [v_4, v_5]) \Leftrightarrow$$

$$\frac{\Delta_4}{\Delta_2} + \frac{\Delta_2}{\Delta_1} \cdot v_3, [v_5, v_6]) \Leftrightarrow$$

$$\frac{\Delta_4}{\Delta_2} + \frac{\Delta_2}{\Delta_1} \cdot v_3, [v_5, v_6]) \Leftrightarrow$$

Let

$$(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3), (e_1, e_2, e_3), (f_1, f_2, f_3)$$

be coordinates of vectors $u_1, u_2, u_3, u_4, u_5, u_6$, respectively. And let $M$ be the matrix of coordinates of vectors $v_1, \ldots, v_6$:

$$M = \begin{pmatrix}
f_2a_3 - f_3a_2 & f_3a_1 - f_1a_3 & f_1a_2 - f_2a_1 \\
a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\
b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \\
c_2d_3 - c_3d_2 & c_3d_1 - c_1d_3 & c_1d_2 - c_2d_1 \\
d_2e_3 - d_3e_2 & d_3e_1 - d_1e_3 & d_1e_2 - d_2e_1 \\
e_2f_3 - e_3f_2 & e_3f_1 - e_1f_3 & e_1f_2 - e_2f_1
\end{pmatrix}$$

Let us denote by $\Delta_{ijk}$ the third order determinant of the submatrix, composed of rows of $M$ with numbers $i, j$ and $k$. Thus, in these notation, $\Delta_1$ and $\Delta_{123}$ are the same, $\Delta_2$ and $\Delta_{234}$ are the same, and so on. The equality, we need to prove, can be rewritten, as:

$$\Delta_1 \cdot (\Delta_3 + \Delta_{245}) \cdot (\Delta_4 + \Delta_{356}) = \Delta_4 \cdot (\Delta_1 + \Delta_{124}) \cdot (\Delta_2 + \Delta_{235}).$$

Each factor in this equality, as a polynomial in variables $a_i, b_i, \ldots, f_i$, is a product of two irreducible polynomials. Thus, we have the product of six polynomials in the lefthand side of the equality, and the product of six polynomials in the righthand side. However, four of them can be cancelled. Now, the lefthand side is the product of two polynomials $p_1p_2$, where $p_1$ depends only on $a_i, b_i, c_i, d_i$ and $p_2$ — on $b_i, c_i, d_i, e_i$. The righthand side is the product of two polynomials $q_1q_2$ with the same properties.
An easy computation demonstrates, that for any choice of vectors \( u_1, \ldots, u_6 \) the equality \( \Delta_1 \Delta_3 \Delta_5 = \Delta_2 \Delta_4 \Delta_6 \) is automatically satisfied. But these vectors constitute a support system only if the sum of matrix \( M \) rows is zero, i.e. if the following conditions are satisfied

\[
\begin{align*}
    f_2 a_3 - f_3 a_2 + a_2 b_3 - a_3 b_2 + b_2 c_3 + b_3 c_2 + c_2 d_3 - c_3 d_2 + d_2 e_3 - d_3 e_2 + e_2 f_3 - e_3 f_2 &= 0 \\
    f_3 a_1 - f_1 a_3 + a_3 b_1 - a_1 b_3 + b_3 c_1 - b_1 c_3 + c_1 d_3 - c_3 d_1 + d_3 e_1 - d_1 e_3 + e_3 f_1 - e_1 f_3 &= 0 \\
    f_1 a_2 - f_2 a_1 + a_1 b_2 - a_2 b_1 + b_1 c_2 - b_2 c_1 + c_2 d_1 - c_1 d_2 + d_1 e_2 - d_2 e_1 + e_1 f_2 - e_2 f_1 &= 0
\end{align*}
\] (3)

If we eliminate from these three relations variables \( f_3 \) and \( f_2 \), then variable \( f_1 \) will be eliminated also and we will obtain the equation \( r = 0 \), where \( r \) is a polynomial that depends on variables \( a_i, b_i, c_i, d_i, e_i \).

Let us transform the relation \( p_1 p_2 = q_1 q_2 \) into relation \( \tilde{p}_1 p_2 = \tilde{q}_1 q_2 \), where

\[
\tilde{p}_1 = \text{resultant}(p_1, r, a_3), \quad \tilde{q}_1 = \text{resultant}(q_1, r, a_3).
\]

Polynomials \( \tilde{p}_1 \) and \( \tilde{q}_1 \) each are products of two irreducible polynomials. In result, three factors in the lefthand side are cancelled with three factors in the righthand side.

\[\square\]

**Theorem 5.2.** Let \( P \) be a regular hexagon, \( P' \) be its derivative and \( P'' \) be a derivative of \( P' \). Then types of \( P' \) and \( P'' \) are the same.

**Proof.** Let vectors \( u_1 = (a_1, a_2, a_3), \ldots, u_6 = (f_1, f_2, f_3) \) constitute a support system of hexagon \( P \) and vectors \( w_1, \ldots, w_6 \) constitute a support system of hexagon \( P' \). We will work with the matrix \( M' \) of coordinates of vectors \( u_2 - u_1, \ldots, u_1 - u_6 \) and with the matrix \( M'' \) of coordinates of vectors \( w_2 - w_1, \ldots, w_1 - w_6 \). Let \( \Delta_i' \) and \( \Delta_i'' \) be corresponding determinants. Let us prove, that

\[
\frac{\Delta_i''}{\Delta_i'} = \frac{\Delta_2''}{\Delta_3'}, \quad \frac{\Delta_2''}{\Delta_3'} = \frac{\Delta_2}{\Delta_3}, \quad \frac{\Delta_2}{\Delta_3} = \frac{\Delta_1''}{\Delta_1'}.
\]

We will prove the first equality. As

\[
\frac{\Delta_i''}{\Delta_i'} = \frac{(w_2 - w_1, [w_3 - w_2, w_4 - w_3])}{(w_3 - w_2, [w_4 - w_3, w_5 - w_4])} = \frac{\Delta_i' / \Delta_i' - \Delta_i 241 + \Delta_i 245 / \Delta_i' \Delta_i' - \Delta_i'}{\Delta_i' / \Delta_i' - \Delta_i 245 / \Delta_i' + \Delta_i 356 / \Delta_i' \Delta_i' - \Delta_i 245 / \Delta_i'},
\]

then

\[
\Delta_i'' / \Delta_i' = \Delta_i' / \Delta_i' \Leftrightarrow
\]

\[
\Leftrightarrow \Delta_i' / \Delta_i' - \Delta_i 241 + \Delta_i 245 / \Delta_i' - \Delta_i 356 / \Delta_i' + \Delta_i 245 / \Delta_i' - (\Delta_i' / \Delta_i')^2 \Delta_i'.
\]

(We remind that \( \Delta_i' = \Delta_i' \)). Now we separate summands of the first degree in \( \Delta_i' \) from summands of the second degree in \( \Delta_i' \) and will prove that

\[
\frac{\Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 + \Delta_6}{2 \Delta_i' + \Delta_i 245 + \Delta_i 356} = 0
\]

Let us remind that variables \( a_i, b_i, \ldots, f_i \) satisfy conditions (3).

The lefthand side of the first equation in (4) is a polynomial \( g \) of degree 3 in variables \( b_i, c_i, d_i, e_i, f_i \) and this polynomial is exactly the result of the elimination of variables \( a_i \) from relations (3).

The lefthand side of the second equation in (4) is a polynomial \( h \) in all variables \( a_i, \ldots, f_i \). If we eliminate from \( h \) variables \( a_i \), using relations (3), then we will obtain \( g \).

\[\square\]

**Example 5.2.** If a hexagon \( P \) is strongly-regular and \( P' \) is the derived hexagon, then they can be of different types. Here is the example: let \( P \) be defined by vectors

\[
v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1), \quad v_4 = (2, -1, 3), \quad v_5 = \left(\frac{-3}{2}, \frac{-1}{2}, \frac{-3}{2}\right), \quad v_6 = \left(\frac{-3}{2}, \frac{1}{2}, \frac{-5}{2}\right).
\]
The derived hexagon is defined by vectors

\[ u_1 = (0, 0, 1), \quad u_2 = (1, 0, 0), \quad u_3 = \left( \frac{1}{2}, 1, 0 \right), \quad u_4 = \left( -\frac{12}{5}, \frac{6}{5}, 2 \right), \quad u_5 = \left( -\frac{5}{2}, \frac{15}{8}, \frac{15}{8} \right), \quad u_6 = (0, 1, \frac{1}{5}). \]

The derived hexagon is defined by vectors

\[
\begin{align*}
    u_2 - u_1 &= (1, 0, -1) \\
u_3 - u_2 &= \left( -\frac{1}{2}, 1, 0 \right) \\
u_4 - u_3 &= \left( -\frac{29}{10}, \frac{1}{5}, 2 \right) \\
u_5 - u_4 &= \left( -\frac{1}{10}, \frac{27}{40}, -\frac{1}{8} \right) \\
u_6 - u_5 &= \left( \frac{5}{2}, -\frac{7}{8}, -\frac{67}{40} \right) \\
u_1 - u_6 &= \left( 0, -1, \frac{4}{5} \right)
\end{align*}
\]

The corresponding determinants are as follows.

\[
\begin{align*}
    \Delta'_4 &= -\frac{4}{5} \quad \Delta'_2 = \frac{1}{8} \quad \Delta'_3 = \frac{3}{10} \quad \Delta'_4 = -\frac{4}{5} \quad \Delta'_5 = \frac{1}{8} \quad \Delta'_6 = \frac{3}{10}.
\end{align*}
\]

We see that types of \( P \) and \( P' \) are different. The reason here is that \( P \) is not a derivative of some regular hexagon, because \( \sum_{0<i<j<6}[v_i,v_j] \neq 0 \) (see Remark 2.2).

6. THE GEOMETRY OF A DERIVED HEXAGON

Vectors \( u_1 = (a_1, a_2, a_3), \ldots, u_6 = (f_1, f_2, f_3) \) constitute a support system, if conditions (3) are satisfied. The elimination of variables \( f_1, f_2, f_3 \) from these conditions gives us the following relation:

\[
\begin{align*}
    (a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1) - \\
    - (a_1 b_2 e_3 - a_1 b_3 e_2 - a_2 b_1 e_3 + a_2 b_3 e_1 + a_3 b_1 e_2 - a_3 b_2 e_1) + \\
    + (a_1 c_2 d_3 - a_1 c_3 d_2 + a_2 c_3 d_1 - a_2 c_1 d_3 + a_3 c_1 d_2 - a_3 c_2 d_1) + \\
    + (a_1 d_2 e_3 - a_1 d_3 e_2 + a_2 d_3 e_1 - a_2 d_1 e_3 + a_3 d_1 e_2 - a_3 d_2 e_1) - \\
    - (b_1 e_2 c_3 - b_1 e_3 c_2 + b_2 e_3 c_1 - b_2 e_1 c_3 + b_3 e_1 c_2 - b_3 e_2 c_1) - \\
    - (c_1 d_2 e_3 - c_1 d_3 e_2 + c_2 d_3 e_1 - c_2 d_1 e_3 + c_3 d_1 e_2 - c_3 d_2 e_1) &= 0
\end{align*}
\]

The left-hand side of this relation is the, multiplied by 6, oriented volume of the following polyhedron \( Q \): let \( B_1, B_2, B_3, B_4, B_5 \) be endpoints of vectors \( u_1, u_2, u_3, u_4, u_5 \), respectively. Then \( Q \) is the result of pasting together tetrahedrons \( B_1 B_2 B_3 B_4 \) and \( B_1 B_2 B_5 B_4 \) along the face \( B_1 B_3 B_5 \). But the volume of \( Q \) is zero, hence, vertices \( B_2 \) and \( B_4 \) are in one half-space with respect to the plane \( B_1 B_3 B_5 \) and at the same distance from it.

If we start our numeration with the vector \( u_3 \), then in the same way we will have that vertices \( B_4 \) and \( B_6 \) are in one half-space with respect to \( B_1 B_3 B_5 \) and at the same distance from it. Thus, we have the following configuration of points \( B_1, \ldots, B_6 \): points \( B_1, B_3, B_5 \) belong to plane \( \Pi_1 \) and points \( B_2, B_4, B_6 \) belong to a parallel plane \( \Pi_2 \). We can assume that these planes are parallel to the plane \( xy \). Let \( B'_2, B'_4 \) and \( B'_6 \) be projections of points \( B_2, B_4 \) and \( B_6 \) to plane \( \Pi_1 \), then the third coordinate of the sum

\[
[u_1, u_2] + [u_2, u_3] + [u_3, u_4] + [u_4, u_5] + [u_5, u_6] + [u_6, u_1]
\]

(i.e. zero) is the oriented area of the plane hexagon \( B_1 B'_2 B_3 B'_4 B_5 B'_6 \).

Thus, up to rotation and dilation, we have the following construction of a regular hexagon: we draw a zero-area hexagon \( B_1 B_2 B_3 B_4 B_5 B_6 \) at the plane \( xy \). Then vertices with even numbers we lift to the plane \( z = 1 \). Then we choose a point \( O \) at the \( z \)-axis: vectors \( OB_1, \ldots, OB_6 \) are vectors \( u_1, \ldots, u_6 \). It must be noted that in terms of this construction \( x \)- and \( y \)-coordinates of the sum \( [u_1, u_2] + \ldots + [u_6, u_1] \) are zero automatically.
7. Knots

Proposition 1. A pentagon in space cannot be knotted.

Sketch of proof. Let a pentagon $ABCDE$ be knotted as a trefoil:

Let vertices $A$, $B$ and $C$ belong to the plane $xy$. Then the point $D$ is above this plane and the point $E$ — below. But then all segment $[EA]$ is below $xy$ and cannot pass above the segment $CD$. □

A hexagon in space can be knotted, however:

Proposition 2. A knotted hexagon is not regular.

Sketch of proof. Let $ABCDEF$ be a hexagon knotted as a trefoil:

As above we will assume that points $A$, $B$ and $C$ belong to the plane $xy$. As $D$ is above $xy$, then the vector $\overrightarrow{CD}$ is directed "up", thus, the triple $\{AB, BC, CD\}$ is right and $\Delta_1 > 0$.

The point $E$ is below $xy$, hence, from $E$'s point of view, the rotation from the vector $\overrightarrow{BC}$ to the vector $\overrightarrow{CD}$ is clockwise, i.e. the triple $\{BC, CD, DE\}$ is left and $\Delta_2 < 0$.

As the point $F$ is above the plane $CDE$, then the triple $\{CD, DE, EF\}$ is right and $\Delta_3 > 0$.

Points $D$ and $F$ are above $xy$ and the point $E$ — below, hence, from $A$'s point of view, the rotation from the vector $\overrightarrow{DE}$ to the vector $\overrightarrow{EF}$ is clockwise, i.e. the triple $\{DE, EF, FA\}$ is left and $\Delta_4 < 0$.

Analogously, the triple $\{EF, FA, AB\}$ is right and $\Delta_5 > 0$. At last, the triple $\{FA, AB, BC\}$ is left and $\Delta_6 < 0$.

We have that $\Delta_1 \Delta_3 \Delta_5 > 0$ and $\Delta_2 \Delta_4 \Delta_6 < 0$, i.e. the hexagon $ABCDEF$ is irregular. □
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