Boundary element methods for Dirichlet boundary control problems

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In this article we discuss the application of boundary element methods for the solution of Dirichlet boundary control problems subject to the Poisson equation with box constraints on the control. The solutions of both the primal and adjoint boundary value problems are given by representation formulae, where the state enters the adjoint problem as volume density. To avoid the related volume potential we apply integration by parts to the representation formula of the adjoint problem. This results in a system of boundary integral equations which is related to the Bi-Laplacian. For the related Dirichlet to Neumann map, we analyse two different boundary integral representations. The first one is based on the use of single and double layer potentials only, but requires some additional assumptions to ensure stability of the discrete scheme. As a second approach, we consider the symmetric formulation which is based on the use of the Calderon projector and which is stable for standard boundary element discretizations. For both methods, we prove stability and related error estimates which are confirmed by numerical examples.

Keywords: optimal control; Dirichlet control; boundary element methods

1. Introduction

Optimal control problems of elliptic or parabolic partial differential equations with a Dirichlet boundary control play an important role [1]. In particular in fluid mechanics, the cost functional can be the integral over the strain tensor of the velocity field, where the Dirichlet control describes the inflow velocity [2]. A similar minimization problem is considered in [3], where the cost functional is the boundary integral to describe the work needed to overcome the drag exerted on a given body. A difficulty in handling the Dirichlet control problems is the choice of the control space, where the Sobolev trace space, \( H^{1/2}(\Gamma) \), appears as a natural choice [4]. To obtain smoother optimal solutions one may even consider \( H^2(\Gamma) \) as control space, where the Sobolev norm for sufficient regular boundaries can be realized by using the Laplace–Beltrami operator [5]. The most popular choice is to consider \( L_2(\Gamma) \) as control space. Although this choice allows the use of a piecewise constant control function, the associated partial differential equation has to be considered within an ultra-weak variational formulation, see, for example, [6] and [7] for an appropriate finite element approximation using standard piecewise linear basis functions. The use of the ultra-weak variational formulation of the primal Dirichlet boundary value problem in the context of an optimal control problem requires the adjoint variable \( p \) to be sufficiently regular, i.e. \( p \in H^2(\Omega) \cap H^1_{\text{div}}(\Omega) \). As the adjoint variable \( p \) itself is the unique solution of the adjoint partial differential equation with homogeneous Dirichlet boundary conditions, either a smooth boundary \( \Gamma \), or a polygonal or polyhedral but convex domain \( \Omega \) has to be assumed. For related finite element approximations, see, e.g., [2, 8–11] or [12] in the case of a finite dimensional Dirichlet control. To include a Dirichlet boundary condition, \( u = z \in L_2(\Gamma) \), in a standard variational formulation, alternatively one may consider a penalty approximation of the Dirichlet boundary condition by using a Robin boundary condition [4, 13–15]. Again, sufficient smoothness of the boundary \( \Gamma \) has to be assumed.

In [16], a finite element approach was considered, where the energy norm was realized by using some hypersingular boundary integral operator which links the Dirichlet control with the normal derivative of the adjoint variable. The related optimality condition results in a higher regularity of the control and requires weaker assumptions on the smoothness of the adjoint variable, in fact, one may even consider general Lipschitz domains \( \Omega \). Moreover, for polygonal- or polyhedral-bounded domains \( \Omega \) one also obtains higher order convergence results for the approximate finite element solution.
2. Dirichlet control problems

In boundary control problems, a Dirichlet boundary data \( z \) is to be determined such that the state \( u \) as the solution of the related Dirichlet boundary value problem matches a desired target \( \mathbf{u} \) taking into account the costs of the control. As a model problem, we consider, for a bounded domain \( \Omega \subset \mathbb{R}^n \), \( n = 2, 3 \), with Lipschitz boundary \( \Gamma = \partial \Omega \), the Dirichlet boundary control problem to minimize the cost functional

\[
J(u,z) = \frac{1}{2} \int_{\Omega} [u(x) - \mathbf{u}(x)]^2 \, dx + \frac{1}{2} \varphi(Dz, z)_\Gamma
\]

subject to the constraint

\[
-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = z(x) \quad \text{for } x \in \Gamma,
\]

and where the control \( z \) satisfies the box constraints

\[
z \in \mathcal{X} := \{ w \in H^{1/2}(\Gamma) : g_a(x) \leq w(x) \leq g_b(x) \text{ for } x \in \Gamma \}.
\]

We assume \( f, \mathbf{u} \in L_2(\Omega) \), \( g_a, g_b \in \mathbb{R}_+ \), and \( g_a, g_b \in H^{1/2}(\Gamma) \) satisfying \( g_a \leq g_b \) on \( \Gamma \). Moreover, we use the hypersingular boundary integral operator \( D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) to describe the cost, or some regularization term, via a semi-norm in \( H^{1/2}(\Gamma) \). In particular, for \( z \in H^{1/2}(\Gamma) \) we have

\[
(Dz)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma} \frac{\partial}{\partial n_y} U^n(x, y) z(y) \, ds_y \quad \text{for } x \in \Gamma,
\]

where

\[
U^n(x, y) = \begin{cases} 
\frac{1}{2\pi} \log |x - y| & \text{for } n = 2 \\
1 & \text{for } n = 3 
\end{cases}
\]

is the fundamental solution of the Laplace operator. Note that the hypersingular integral operator \( D \) is spectrally equivalent to the Steklov–Poincaré operator, which describes the energy of the harmonic extension of the Dirichlet control.

Let \( u_f \in H^1_0(\Omega) \) be the weak solution of the homogeneous Dirichlet boundary value problem

\[
-\Delta u_f(x) = f(x) \quad \text{for } x \in \Omega, \quad u_f(x) = 0 \quad \text{for } x \in \Gamma.
\]

The solution of the Dirichlet boundary value problem (2) is then given by \( u = u_z + u_f \), where \( u_z \in H^1(\Omega) \) is the unique solution of the Dirichlet boundary value problem

\[
-\Delta u_z(x) = 0 \quad \text{for } x \in \Omega, \quad u_z(x) = z(x) \quad \text{for } x \in \Gamma.
\]
Note that the solution of the Dirichlet boundary value problem (5) defines a linear map \( u_z = \mathcal{Y} z \) with \( \mathcal{Y} : H^{1/2}(\Gamma) \to H^1(\Omega) \subset L^2(\Omega) \). Then, by using \( u = \mathcal{Y} z + u_f \), we consider the problem to find the minimizer, \( z \in \mathcal{W} \subset H^{1/2}(\Gamma) \), of the reduced cost functional
\[
\tilde{J}(z) = \frac{1}{2} \int_{\Omega} \left[ (\mathcal{Y} z(x) + u_f(x) - \overline{u})^2 \right] dx + \frac{1}{2} \delta_0(Dz, z)_{\Gamma}. 
\] (6)
To characterize the minimizer, \( z \in \mathcal{W} \), of the reduced cost functional (6) we introduce the self-adjoint, bounded and \( H^{1/2}(\Gamma) \)-elliptic operator
\[
T_v := gD + \mathcal{Y}^* \mathcal{Y} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) 
\] (7)
satisfying [16]
\[
(T_v z, z)_{\Gamma} \geq c_1^T \| z \|^2_{H^{1/2}(\Gamma)} \quad \| T_v z \|_{H^{-1/2}(\Gamma)} \leq c_2^T \| z \|^2_{H^{1/2}(\Gamma)} \quad \text{for all} \ z \in H^{1/2}(\Gamma),
\]
where \( \mathcal{Y}^*: L^2(\Omega) \to H^{-1/2}(\Gamma) \) is the adjoint operator of \( \mathcal{Y} : H^{1/2}(\Gamma) \to L^2(\Omega) \), i.e.
\[
(\mathcal{Y}^* \psi, \phi)_{\Gamma} = (\psi, \mathcal{Y} \phi)_{\Omega} = \int_{\Omega} \psi(x) \mathcal{Y} \phi(x) \ dx \quad \text{for all} \ \phi \in H^{1/2}(\Gamma), \psi \in L^2(\Omega).
\]
Moreover, we define
\[
g := \mathcal{Y}^*(\overline{u} - u_f) \in H^{-1/2}(\Gamma). 
\] (8)
Hence we can rewrite the reduced cost functional (6) as
\[
\tilde{J}(z) = \frac{1}{2} (T_v z, z)_{\Gamma} - (g, z)_{\Gamma} + \frac{1}{2} \| u_f - \overline{u} \|^2_{L^2(\Omega)}.
\]
As \( \mathcal{W} \subset H^{1/2}(\Gamma) \) is convex and closed, and since \( T_v \) is self-adjoint and \( H^{1/2}(\Gamma) \)-elliptic, the minimization of the reduced cost functional, \( \tilde{J}(z) \), is equivalent to solving the variational inequality to find \( z \in \mathcal{W} \) such that
\[
(T_v z, w - z)_{\Gamma} \geq (g, w - z)_{\Gamma} \quad \text{for all} \ w \in \mathcal{W}. 
\] (9)
As (9) is an elliptic variational inequality of the first kind, we can use standard arguments as given, for example in [6, 25–27], to establish unique solvability of the variational inequality (9).

By using the primal variable \( u = \mathcal{Y} z + u_f \), and by introducing the adjoint variable \( \tau = \mathcal{Y}^* (u - \overline{u}) \in H^{-1/2}(\Gamma) \), we can rewrite the variational inequality (9) as
\[
(gDz + \tau, w - z)_{\Gamma} \geq 0 \quad \text{for all} \ w \in \mathcal{W}. 
\] (10)
Note that for given \( z \in H^{1/2}(\Gamma) \) and \( f \in L^2(\Omega) \) the application of \( u = \mathcal{Y} z + u_f \) corresponds to the solution of the Dirichlet boundary value problem (2). The application of the adjoint operator, \( \tau = \mathcal{Y}^* (u - \overline{u}) \), is characterized by the Neumann datum
\[
\tau = -\frac{\partial}{\partial n} p \quad \text{in the sense of} \ H^{-1/2}(\Gamma), 
\] (11)
where \( p \in H^1_0(\Omega) \) is the unique solution of the adjoint Dirichlet boundary value problem
\[
-\Delta p(x) = u(x) - \overline{u}(x) \quad \text{for} \ x \in \Omega, \quad p(x) = 0 \quad \text{for} \ x \in \Gamma. 
\] (12)
As the unknown control \( z \in \mathcal{W} \subset H^{1/2}(\Gamma) \) is considered on the boundary \( \Gamma = \bar{\Omega} \), the use of boundary integral equations to solve both the primal boundary value problem (2) and the adjoint boundary value problem (12) seems to be a natural choice. In what follows, we will describe and analyse two different boundary element methods to solve the variational inequality (9) numerically. This will be based on the use of appropriate boundary integral operator representations of \( T_v \) and \( g \) as introduced in (7) and (8), respectively.

### 3. Laplace boundary integral equations

#### 3.1. Primal boundary value problem

The solution of the Dirichlet boundary value problem (2),
\[
-\Delta u(x) = f(x) \quad \text{for} \ x \in \Omega, \quad u(x) = z(x) \quad \text{for} \ x \in \Gamma,
\]
is given by the representation formula for \( x \in \Omega \),
\[
\begin{align*}
\begin{split}
\notag u(x) &= \int_{\Gamma} U^+ (\bar{x}, y) t(y) \ dy - \int_{\Gamma} \frac{\partial}{\partial n_y} U^+ (\bar{x}, y) z(y) \ dy + \int_{\Omega} U^+ (\bar{x}, y) f(y) \ dy, 
\end{split}
\end{align*}
\] (13)
where \( t = (\partial / \partial n) u \in H^{-1/2}(\Gamma) \) is the unique solution of the boundary integral equation
\[
(Vt)(x) = \left( \frac{1}{2} I + K \right) z(x) - (N_0 f)(x) \quad \text{for} \ x \in \Gamma. 
\] (14)
Note that

\[(Vt)(x) = \int_{\Gamma} U^*(x, y)t(y)\,ds_y \quad \text{for} \quad x \in \Gamma\]

is the Laplace single layer integral operator \(V: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\), and

\[(Kz)(x) = \int_{\Gamma} \frac{\partial}{\partial n_y} U^*(x, y)z(y)\,ds_y \quad \text{for} \quad x \in \Gamma\]

is the Laplace double layer integral operator \(K: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)\). Moreover,

\[(N_0 f)(x) = \int_{\Omega} U^*(x, y)f(y)\,dy \quad \text{for} \quad x \in \Gamma\]

is the related Newton potential. The single layer integral operator \(V\) is \(H^{-1/2}(\Gamma)\)-elliptic; for \(n = 2\) we assume the scaling condition \(\text{diam} \Omega < 1\) to ensure this. For the solution of the boundary integral equation (14) we therefore obtain

\[t = V^{-1}(\frac{1}{2}I + K)z - V^{-1}N_0 f.\]  

(15)

3.2. Adjoint boundary value problem

The solution of the adjoint Dirichlet boundary value problem (12),

\[-\Delta p(x) = u(x) - \overline{u}(x) \quad \text{for} \quad x \in \Omega, \quad p(x) = 0 \quad \text{for} \quad x \in \Gamma ,\]

is given correspondingly by the representation formula for \(\tilde{x} \in \Omega ,\)

\[p(\tilde{x}) = \int_{\Gamma} U^*(\tilde{x}, y)q(y)\,ds_y + \int_{\Omega} U^*(\tilde{x}, y)[u(y) - \overline{u}(y)]\,dy ,\]  

(16)

where \(q = (\partial/\partial n)p \in H^{-1/2}(\Gamma)\) is the unique solution of the boundary integral equation

\[(Vq)(x) = (N_0 \overline{u})(x) - (N_0 u)(x) \quad \text{for} \quad x \in \Gamma .\]  

(17)

Remark 3.1

While the boundary integral equation (14) can be used to determine the unknown Neummann datum, \(t \in H^{-1/2}(\Gamma)\), of the primal Dirichlet boundary value problem (2), the unknown Neumann datum, \(q \in H^{-1/2}(\Gamma)\), of the adjoint Dirichlet boundary value problem (12) is given as the solution of the boundary integral equation (17). Then, by using \(r = -q\), the control \(z \in H^{1/2}(\Gamma)\) is determined by the variational inequality (10). However, since the solution \(u\) of the primal Dirichlet boundary value problem (2) enters the volume potential \(N_0 u\) in the boundary integral equation (17), it seems to be necessary to include the representation formula (13). In this case, we would have to solve a coupled system of boundary and domain integral equations, which still would require some domain mesh. Instead, we will now describe a system of only boundary integral equations to solve the adjoint boundary value problem (12).

To end up with a system of boundary integral equations only, instead of (16), we will introduce a modified representation formula for the adjoint state \(p\) as follows. First we note that

\[V^*(x, y) = \begin{cases} -\frac{1}{8\pi}|x - y|^2(\log|x - y| - 1) & \text{for} \quad n = 2 \\ \frac{1}{8\pi}|x - y| & \text{for} \quad n = 3 \end{cases}\]  

(18)

is a solution of the Poisson equation

\[\Delta_p V^*(x, y) = U^*(x, y) \quad \text{for} \quad x \neq y,\]  

(19)

i.e., \(V^*(x, y)\) corresponds to the fundamental solution of the Bi-Laplacian. Hence we can rewrite the volume integral for \(u\) in (16), by using Green’s second formula, as follows:

\[\int_{\Omega} U^*(\tilde{x}, y)u(y)\,dy = \int_{\Omega} [\Delta_p V^*(\tilde{x}, y)]u(y)\,dy = \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(\tilde{x}, y)u(y)\,ds_y - \int_{\Gamma} V^*(\tilde{x}, y)\frac{\partial}{\partial n_y}u(y)\,ds_y + \int_{\Omega} V^*(\tilde{x}, y)[\Delta u(y)]\,dy = \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(\tilde{x}, y)z(y)\,ds_y - \int_{\Gamma} V^*(\tilde{x}, y)t(y)\,ds_y - \int_{\Omega} V^*(\tilde{x}, y)f(y)\,dy.\]
Therefore, we now obtain from (16) the modified representation formula for \( \tilde{x} \in \Omega \),

\[
p(\tilde{x}) = \int_{\Gamma} U^*(\tilde{x}, y)q(y)\,ds_y + \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(\tilde{x}, y)z(y)\,ds_y - \int_{\Omega} V^*(\tilde{x}, y)t(y)\,dy - \int_{\Omega} V^*(\tilde{x}, y)\bar{u}(y)\,dy - \int_{\Omega} V^*(\tilde{x}, y)f(y)\,dy,
\]

where the volume potentials involve given data only.

The representation formula (20) results, when taking the limit \( \Omega \ni \tilde{x} \to x \in \Gamma \), in the boundary integral equation for \( x \in \Gamma \),

\[
0 = p(x) = \int_{\Gamma} U^*(x, y)q(y)\,ds_y + \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(x, y)z(y)\,ds_y - \int_{\Gamma} V^*(x, y)t(y)\,ds_y - \int_{\Omega} U^*(x, y)\bar{u}(y)\,dy - \int_{\Omega} V^*(x, y)f(y)\,dy,
\]

which can be written as

\[
(Vq)(x) = (V_1 t)(x) - (K_1 z)(x) + (N_0 \bar{u})(x) + (M_0 f)(x) \quad \text{for} \ x \in \Gamma. \tag{21}
\]

Note that

\[
(V_1 t)(x) = \int_{\Gamma} V^*(x, y)t(y)\,ds_y \quad \text{for} \ x \in \Gamma
\]

is the Bi-Laplace single layer integral operator \( V_1 : H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma) \), see, for example, [19, Theorem 5.7.3]. Moreover,

\[
(K_1 z)(x) = \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(x, y)z(y)\,ds_y \quad \text{for} \ x \in \Gamma
\]

is the Bi-Laplace double layer potential \( K_1 : H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \). In addition, we have introduced a second Newton potential, which is related to the fundamental solution of the Bi-Laplace operator,

\[
(M_0 f)(x) = \int_{\Omega} V^*(x, y)f(y)\,dy \quad \text{for} \ x \in \Gamma.
\]

With (15), we conclude from (21) the boundary integral equation

\[
Vq = V_1 V^{-1}\left(\frac{1}{2}I + K\right)z - K_1 z + N_0 \bar{u} + M_0 f - V_1 V^{-1}N_0 f,
\]

and therefore

\[
q = V^{-1}V_1 V^{-1}\left(\frac{1}{2}I + K\right)z - V^{-1}K_1 z + V^{-1}N_0 \bar{u} + V^{-1}M_0 f - V^{-1}V_1 V^{-1}N_0 f. \tag{22}
\]

By replacing \( \tau = -q \) in (10) we, therefore, obtain a boundary integral representation of the operator \( T_\varepsilon \) as defined in (7),

\[
T_\varepsilon := VD + V^{-1}K_1 - V^{-1}V_1 V^{-1}\left(\frac{1}{2}I + K\right), \tag{23}
\]

and a related representation for \( g \) as defined in (8),

\[
g := V^{-1}N_0 \bar{u} + V^{-1}M_0 f - V^{-1}V_1 V^{-1}N_0 f. \tag{24}
\]

To investigate the unique solvability of the variational inequality (9) based on the boundary integral representations (23) and (24), we will first recall some properties of boundary integral operators which are related to the Bi-Laplace partial differential equation, see also [19, 21, 22].

4. Bi-Laplace boundary integral equations and properties of \( T_\varepsilon \)

In this section, we consider a representation formula and related boundary integral equations for the Bi-Laplace equation

\[
\Delta^2 u(x) = 0 \quad \text{for} \ x \in \Omega, \tag{25}
\]

which can be written as a system,

\[
\Delta u_{\Delta}(x) = 0, \quad \Delta u(x) = u_{\Delta}(x) \quad \text{for} \ x \in \Omega. \tag{26}
\]

As for the Laplace equation we can first write the boundary integral equations

\[
u_{\Delta}(x) = (V_\Delta t)(x) + \frac{1}{2}u_{\Delta}(x) - (K u_{\Delta})(x) \quad \text{for} \ x \in \Gamma \tag{27}
\]

and

\[
t_{\Delta}(x) = \frac{1}{2} t_{\Delta}(x) + (K' t_{\Delta})(x) + (D u_{\Delta})(x) \quad \text{for} \ x \in \Gamma, \tag{28}
\]
Hence we obtain the boundary integral equation

\[(K't_A)(x) = \int_{\Gamma} \frac{\partial}{\partial n_x} U^n(x, y)t_A(y) \, ds_y \quad \text{for} \ x \in \Gamma \]

is the adjoint Laplace double layer integral operator \(K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)\). Note that

\[u_{A} = \Delta u \quad \text{and} \quad t_A = \frac{\partial}{\partial n} u_\Lambda = n \cdot \nabla u_\Lambda \]

are the associated Cauchy data on \(\Gamma\).

To obtain a representation formula for the solution \(u\) of the Bi-Laplace Equation (25), we first consider the related Green’s first formula

\[\int_{\Omega} \Delta u(y) \Delta v(y) \, dy = \int_{\Gamma} \frac{\partial}{\partial n_y} u(y) \Delta v(y) \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} \Delta v(y) u(y) \, ds_y + \int_{\Omega} [\Delta^2 v(y)] u(y) \, dy, \quad (29)\]

and in the sequel Green’s second formula,

\[\int_{\Gamma} \frac{\partial}{\partial n_y} u(y) \Delta v(y) \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} \Delta v(y) u(y) \, ds_y + \int_{\Omega} [\Delta^2 v(y)] u(y) \, dy = \int_{\Gamma} \frac{\partial}{\partial n_y} v(y) \Delta u(y) \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} u(y) \Delta v(y) \, ds_y + \int_{\Omega} [\Delta^2 u(y)] v(y) \, dy. \]

When choosing \(v(y) = V^*(\tilde{x}, y)\) for \(\tilde{x} \in \Omega\), i.e. the fundamental solution (18) of the Bi-Laplace operator, the solution of the Bi-Laplace partial differential equation (25) is given by the representation formula for \(\tilde{x} \in \Omega\) by

\[u(\tilde{x}) = \int_{\Gamma} \frac{\partial}{\partial n_y} u(y) \Delta y V^*(\tilde{x}, y) \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} \Delta y V^*(\tilde{x}, y) u(y) \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(\tilde{x}, y) \Delta u(y) \, ds_y + \int_{\Gamma} \frac{\partial}{\partial n_y} \Delta u(y) V^*(\tilde{x}, y) \, ds_y. \]

By using (19), this can be written as

\[u(\tilde{x}) = \int_{\Gamma} U^n(\tilde{x}, y) t(y) \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} U^n(\tilde{x}, y) u(y) \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(\tilde{x}, y) u_A(y) \, ds_y + \int_{\Gamma} \frac{\partial}{\partial n_y} U^n(\tilde{x}, y) t_A(y) \, ds_y. \quad (30)\]

Hence we obtain the boundary integral equation

\[u(x) = (Vt)(x) + \frac{1}{2} u(x) - (Ku)(x) - (K_1 u_A)(x) + (V_1 t_A)(x) \quad \text{for} \ x \in \Gamma. \quad (31)\]

Moreover, when taking the normal derivative of the representation formula (30), this gives another boundary integral equation for \(x \in \Gamma\),

\[t(x) = \frac{1}{2} t(x) + (K't)(x) + (Du)(x) + (D_1 u_A)(x) + (K'_1 t_A)(x), \quad (32)\]

where

\[(K'_1 t_A)(x) = \int_{\Gamma} \frac{\partial}{\partial n_x} V^*(x, y) t_A(y) \, ds_y \quad \text{for} \ x \in \Gamma \]

is the adjoint Bi-Laplace double layer integral operator \(K'_1 : H^{-3/2}(\Gamma) \to H^{1/2}(\Gamma)\), and

\[(D_1 u_A)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma} \frac{\partial}{\partial n_y} V^*(x, y) u_A(y) \, ds_y \quad \text{for} \ x \in \Gamma \]

with \(D_1 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)\).

The boundary integral Equations (27), (28), (31), and (32) can now be written as a system, including the so-called Calderon projection \(\mathcal{C}\),

\[
\begin{pmatrix}
  u \\
  t \\
  u_A \\
  t_A
\end{pmatrix} =
\begin{pmatrix}
  I - K & V & -K_1 & V_1 \\
  D & 1/2I + K' & D_1 & K'_1 \\
  1/2I - K & V & D & 1/2I + K' \\
  D & 1/2I + K'
\end{pmatrix}
\begin{pmatrix}
  u \\
  t \\
  u_A \\
  t_A
\end{pmatrix}. \quad (33)
\]

Lemma 4.1

The Calderon projection \(\mathcal{C}\) as defined in (33) is a projection, i.e. \(\mathcal{C}^2 = \mathcal{C}\).

Proof

The proof follows as in the case of the Laplace equation [20, 24], for the Bi-Laplace equation, see also [22].

From the projection property as stated in Lemma 4.1 we obtain some well-known relations of all boundary integral operators, which were introduced for both the Laplace and the Bi-Laplace equations.
Lemma 4.2
For the boundary integral operators as introduced above there hold the relations
\[
KV = VK', \quad DK = K'D, \quad VD = \frac{1}{2}l - K^2, \quad DV = \frac{1}{2}l - K'^2
\] (34)
and
\[
K_1 V - VK'_1 = V_1 K' - KV_1, \quad K'_1D - DK_1 = D_1 K - K'D_1, \quad VD_1 + V_1 D + K K_1 + K_1 K = 0, \quad DV_1 + D_1 V + K' K'_1 + K'_1 K' = 0.
\]

Proof
The relations of (34) for the Laplace operator are well known, see, e.g., [24], for the Bi-Laplace operator, see also [22]. □

To prove the ellipticity of the boundary integral operator \( T_\Gamma \) as defined in (23), we use the following result.

Lemma 4.3
For any \( t \in H^{-1/2}(\Gamma) \) there holds the equality
\[
\| \tilde{V}t \|_{L^2(\Omega)}^2 = (K_1 V_t, t)_{\Gamma} - (V_1 (\frac{1}{2}l + K') t, t)_{\Gamma}
\] (39)
where
\[
(\tilde{V}t)(x) = \int_{\Gamma} U^s(x, y) t(y) \, ds_y \quad \text{for} \ x \in \Omega.
\]

Proof
For \( x \in \Omega \) and \( t \in H^{-1/2}(\Gamma) \), we define the Bi-Laplace single layer potential
\[
\begin{align*}
\forall t(x) := \tilde{V}_1 t(x) &= \int_{\Gamma} V^s(x, y) t(y) \, ds_y,
\end{align*}
\]
which is a solution of the Bi-Laplace differential equation (25). Then, the related Cauchy data are given by
\[
\begin{align*}
\forall t(x) &= (V_1 t)(x), \quad \frac{\partial}{\partial n_x} \forall t(x) = (K'_1 t)(x) \quad \text{for} \ x \in \Gamma.
\end{align*}
\]
On the other hand, for \( x \in \Omega \),
\[
\forall t(x) := \Delta x \forall t(x) = \Delta x \int_{\Gamma} V^s(x, y) t(y) \, ds_y = \int_{\Gamma} U^s(x, y) t(y) \, ds_y = (\tilde{V}t)(x)
\]
is a solution of the Laplace equation. Hence, the related Cauchy data are given by
\[
\begin{align*}
\forall t(x) &= (V_1 t)(x), \quad \frac{\partial}{\partial n_x} \forall t(x) = \frac{1}{2} t(x) + (K'_1 t)(x) \quad \text{for} \ x \in \Gamma.
\end{align*}
\]
Now, for \( u = v = \forall t \), Green’s first formula (29) reads as
\[
\int_{\Omega} [\Delta u(x)]^2 \, dx = \int_{\Gamma} \frac{\partial}{\partial n_x} u(x) \Delta u(x) \, ds_x - \int_{\Gamma} \frac{\partial}{\partial n_x} \Delta u(x) u(x) \, ds_x,
\]
and therefore we conclude
\[
\begin{align*}
\int_{\Omega} [\forall t(x)]^2 \, dx &= \int_{\Gamma} \frac{\partial}{\partial n_x} u(x) \forall t(x) \, ds_x - \int_{\Gamma} \frac{\partial}{\partial n_x} \forall t(x) u(x) \, ds_x
\end{align*}
\]
\[
\begin{align*}
&= \int_{\Gamma} (K'_1 t)(x) (V_1 t)(x) \, ds_x - \int_{\Gamma} (\frac{1}{2} t(x) + (K'_1 t)(x)) (V_1 t)(x) \, ds_x
\end{align*}
\]
\[
\begin{align*}
&= (K'_1 t, V_1 t)_{\Gamma} - (\frac{1}{2} t + K'_1 t, V_1 t)_{\Gamma}
\end{align*}
\]
\[
\begin{align*}
&= (t, K'_1 V_1 t)_{\Gamma} - (V_1 (\frac{1}{2} l + K')) t, t)_{\Gamma}.
\end{align*}
\]
The assertion now follows with \( \forall t = \tilde{V}t \). □
Now we are able to state the mapping properties of the boundary integral operator $T_\varphi$ as defined in (23), see also the properties of $T_\varphi$ as introduced in (7).

**Theorem 4.4**

The composed boundary integral operator

$$T_\varphi := gD + V^{-1}K_1 - V^{-1}V_1 V^{-1}(\frac{1}{2}I + K): H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$$

is self-adjoint, bounded and $H^{1/2}(\Gamma)$-elliptic, i.e.

$$\langle T_\varphi z, z \rangle_{\Gamma} \geq \frac{T_\varphi}{\ell^2} \|z\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma).$$

**Proof**

The mapping properties of $T_\varphi: H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$ follow from the boundedness of all used boundary integral operators [20, 23, 24]. In addition, we use the compact embedding of $H^{3/2}(\Gamma)$ in $H^{1/2}(\Gamma)$.

Next we will show the self-adjointness of $T_\varphi$. For $z, w \in H^{1/2}(\Gamma)$ we have

$$\langle T_\varphi z, w \rangle_{\Gamma} = \langle gDz, w \rangle_{\Gamma} + \langle V^{-1}K_1 z, w \rangle_{\Gamma} - \frac{1}{2} \langle V^{-1}V_1 V^{-1}z, w \rangle_{\Gamma} - \langle V^{-1}V_1 V^{-1}Kz, w \rangle_{\Gamma}$$

$$= \langle z, gDw \rangle_{\Gamma} + \langle z, K_1'V^{-1}w \rangle_{\Gamma} - \frac{1}{2} \langle z, V^{-1}V_1 V^{-1}w \rangle_{\Gamma} - \langle z, K'V^{-1}V_1 V^{-1}w \rangle_{\Gamma}$$

Now, we conclude, by using the relations (34) and (35),

$$K_1'V^{-1} - K'V^{-1}V_1 V^{-1} = K_1'V^{-1} - V^{-1}KV_1 V^{-1} = V^{-1}(K_1 - V_1 K_1')V^{-1} - V^{-1}K_1 - V^{-1}V_1 K'V^{-1}$$

Hence we have

$$\langle T_\varphi z, w \rangle_{\Gamma} = \langle z, gDw \rangle_{\Gamma} - \frac{1}{2} \langle z, V^{-1}V_1 V^{-1}w \rangle_{\Gamma} + \langle z, [V^{-1}K_1 - V^{-1}V_1 V^{-1}K]w \rangle_{\Gamma}$$

$$= \langle z, [gD + V^{-1}K_1 - V^{-1}V_1 V^{-1}(\frac{1}{2}I + K)]w \rangle_{\Gamma} = \langle z, T_\varphi w \rangle_{\Gamma},$$

i.e. $T_\varphi$ is self-adjoint.

Moreover, for $z \in H^{1/2}(\Gamma)$ we have, by using (34), $t = V^{-1}z$ and by Lemma 4.3

$$\langle T_\varphi z, t \rangle_{\Gamma} = \langle gDz, t \rangle_{\Gamma} + \langle V^{-1}K_1 z, z \rangle_{\Gamma} - \langle V^{-1}V_1 V^{-1}(\frac{1}{2}I + K)z, z \rangle_{\Gamma}$$

$$= \langle gDz, t \rangle_{\Gamma} + \langle K_1 V^{-1}z, V^{-1}z \rangle_{\Gamma} - \langle V_1 (\frac{1}{2}I + K)V^{-1}z, V^{-1}z \rangle_{\Gamma}$$

$$= \langle gDz, t \rangle_{\Gamma} + \langle K_1 z, t \rangle_{\Gamma} - \langle V_1 (\frac{1}{2}I + K)z, t \rangle_{\Gamma}$$

$$= \langle gDz, t \rangle_{\Gamma} + \|Vt\|^2_{L^2(\Omega)}.$$  

As the last expression defines an equivalent norm in $H^{1/2}(\Gamma)$, the $H^{1/2}(\Gamma)$-ellipticity of $T_\varphi$ follows. $\square$

### 5. A non-symmetric boundary element method

Let

$$S_\varphi^1(\Gamma) = \text{span}\{\varphi_i\}_{i=1}^M \subset H^{1/2}(\Gamma)$$

be a boundary element space of piecewise linear and continuous basis functions $\varphi_i$, which are defined with respect to a globally quasi-uniform and shape regular boundary mesh $\Gamma_H$ of mesh size $H$. Define the discrete convex set

$$\mathcal{W}_H := \{w_H \in S_\varphi^1(\Gamma) : g_0(x_i) \leq w_H(x_i) \leq g_0(x_i) \text{ for all nodes } x_i \in \Gamma\}.$$

Then the Galerkin discretization of the variational inequality (9) is to find $z_H \in \mathcal{W}_H$ such that

$$\langle T_\varphi z_H, w_H - z_H \rangle_{\Gamma} \geq \langle g, w_H - z_H \rangle_{\Gamma} \quad \text{for all } w_H \in \mathcal{W}_H.$$  

(41)
Theorem 5.1
Let \( z \in \mathcal{V} \) and \( z_H \in \mathcal{W}_H \) be the unique solutions of the variational inequalities (9) and (41), respectively. If we assume \( z, g_\alpha, g_b \in H^2(\Gamma) \) for some \( s \in \{2, 2\} \), then there hold the error estimates
\[
\|z - z_H\|_{H^{1/2}(\Gamma)} \leq c_H H^{-1/2} \|z\|_{H^2(\Gamma)}
\]
(42)
and
\[
\|z - z_H\|_{L_2(\Gamma)} \leq c_H \|z\|_{H^2(\Gamma)}.
\]
(43)

Proof
The error estimate (42) in the energy norm follows from the general abstract theory as presented, e.g. in [28, 29], see also [26]. The error estimate (43) follows from the Aubin–Nitsche trick for variational inequalities, see [30] for the case \( \mathcal{W}_H \subset \mathcal{V} \) and [31] for the more general case \( \mathcal{W}_H \subset \mathcal{V} \).

Although the error estimates (42) and (43) seem to be optimal, the operator \( T_\varepsilon \) as considered in the variational inequality (41) does not allow a practical implementation, since this would require the discretization of the operator \( T_\varepsilon \) as defined in (23), which is not possible in general. Hence, instead of (41) we need to consider a perturbed variational inequality to find \( z_H \in \mathcal{W}_H \) such that
\[
(\overline{T}_\varepsilon z_H, w_H - z_H)_\Gamma \geq (\overline{g}_H, w_H - z_H)_\Gamma \quad \text{for all } w_H \in \mathcal{W}_H,
\]
(44)
where \( \overline{T}_\varepsilon \) and \( \overline{g}_H \) are appropriate approximations of \( T_\varepsilon \) and \( g \), respectively. The following theorem, see, e.g., [16], presents an abstract consistency result, which will later be used to analyse the boundary element approximation of both the primal and adjoint boundary value problems.

Theorem 5.2
Let \( T_\varepsilon : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) be a bounded and \( S^1_H(\Gamma) \)-elliptic approximation of \( T_\varepsilon \) satisfying
\[
(\overline{T}_\varepsilon z_H, z_H)_\Gamma \geq c_T^1 \|z_H\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } z_H \in S^1_H(\Gamma)
\]
and
\[
\|\overline{T}_\varepsilon z\|_{H^{-1/2}(\Gamma)} \leq c_T^2 \|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma).
\]

Let \( \overline{g} \in H^{-1/2}(\Gamma) \) be some approximation of \( g \). For the unique solution, \( \overline{z}_H \in \mathcal{W}_H \), of the perturbed variational inequality (44) there holds the error estimate
\[
\|z - \overline{z}_H\|_{H^{1/2}(\Gamma)} \leq c_1 \|z - z_H\|_{H^{1/2}(\Gamma)} + c_2 \|T_\varepsilon - \overline{T}_\varepsilon\|_{H^{-1/2}(\Gamma)} + \|g - \overline{g}\|_{H^{-1/2}(\Gamma)},
\]
(45)
where \( z_H \in \mathcal{W}_H \) is the unique solution of the discrete variational inequality (41).

In the remaining part of this paper we will analyse approximations \( \overline{T}_\varepsilon \) and \( \overline{g}_H \) which are based on the use of boundary element methods. This is related to the boundary element approximation of Steklov–Poincaré operators [32–35].

5.1. Boundary element approximation of \( T_\varepsilon \)
For an arbitrary but fixed \( \varepsilon \in H^{1/2}(\Gamma) \), the application of \( T_\varepsilon z \) reads as
\[
T_\varepsilon z = \varepsilon Dz + V^{-1} K \varepsilon z - V^{-1} V_1 V^{-1} (1/2 + K) z = \varepsilon Dz - q_\varepsilon,
\]
where \( q_\varepsilon \in H^{-1/2}(\Gamma) \) is the unique solution of the boundary integral equation
\[
(Vq_\varepsilon)(x) = (V_1 t_2)(x) - (K_1 z)(x) \quad \text{for } x \in \Gamma,
\]
and \( t_2 \in H^{-1/2}(\Gamma) \) solves
\[
(Vt_2)(x) = (1/2 + K) z(x) \quad \text{for } x \in \Gamma.
\]

For a Galerkin approximation of the above boundary integral equations, let
\[
S^0_H(\Gamma) = \text{span}(\psi^N_{k=1} H^{-1/2}(\Gamma))
\]
be another boundary element space of, e.g., piecewise constant basis functions \( \psi_k \), which are defined with respect to a second globally quasi-uniform and shape regular boundary element mesh of mesh size \( h \). Now, \( q_{\varepsilon,h} \in S^0_H(\Gamma) \) is the unique solution of the Galerkin formulation
\[
(Vq_{\varepsilon,h})(\tau_h) = (V_1 t_2 - K_1 z, \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_H(\Gamma).
\]
and \( t_{z,h} \in S^0_h(\Gamma) \) solves

\[
(Vt_{z,h}, \tau_h)_\Gamma = ((\frac{1}{2} I + K)z, \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_h(\Gamma).
\]

Hence we can define an approximation \( \tilde{T}_0 \) of the operator \( T_0 \) by

\[
\tilde{T}_0 z := gDz - \tilde{q}_{z,h}.
\]

Lemma 5.3

The approximate operator \( \tilde{T}_0 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) as defined in (46) is bounded, i.e.

\[
\| \tilde{T}_0 z \|_{H^{-1/2}(\Gamma)} \leq C_2 \| z \|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma).
\]

Proof

The assertion is a direct consequence of the mapping properties of all boundary integral operators involved, we skip the details. \( \Box \)

Lemma 5.4

Let \( T_0 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \) be given by (23), and let \( \tilde{T}_0 \) be defined as in (46). Then there holds the error estimate

\[
\| T_0 z - \tilde{T}_0 z \|_{H^{-1/2}(\Gamma)} \leq c_1 \inf_{\tau_h \in S^0_h(\Gamma)} \| q_z - \tau_h \|_{H^{-1/2}(\Gamma)} + c_2 \| t_z - t_{z,h} \|_{H^{-3/2}(\Gamma)}.
\]

Proof

For an arbitrary chosen but fixed \( z \in H^{1/2}(\Gamma) \) we have, by definition,

\[
\tilde{T}_0 z = gDz - q_{z,h} \quad q_z = V^{-1} [V_1 t_z - K_1 z], \quad t_z = V^{-1} (\frac{1}{2} I + K)z.
\]

By using (46), we also have

\[
\tilde{T}_0 z = gDz - \tilde{q}_{z,h}
\]

and therefore

\[
T_0 z - \tilde{T}_0 z = \tilde{q}_{z,h} - q_z.
\]

Let us further define \( q_{z,h} \in S^0_h(\Gamma) \) as the unique solution of the variational problem

\[
(Vq_{z,h}, \tau_h)_\Gamma = (V t_z - K_1 z, \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_h(\Gamma).
\]

(48)

Then the perturbed Galerkin orthogonality

\[
(V(q_{z,h} - \tilde{q}_{z,h}), \tau_h)_\Gamma = (V(t_z - t_{z,h}), \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_h(\Gamma)
\]

follows. From this we further conclude

\[
\| q_{z,h} - \tilde{q}_{z,h} \|_{H^{-1/2}(\Gamma)} \leq c \| t_z - t_{z,h} \|_{H^{-3/2}(\Gamma)}.
\]

The assertion now follows from the triangle inequality, and by applying Cea’s lemma. \( \Box \)

By using the approximation property of the trial space \( S^0_h(\Gamma) \) and the Aubin–Nitsche trick, we conclude an error estimate from (47) when assuming some regularity of \( q_z \) and \( t_z \), respectively.

Corollary 5.5

Assume \( q_z \) and \( t_z \) to be piecewise smooth, i.e. \( q_z, t_z \in H^s_{pw}(\Gamma) \) for some \( s \in [0, 1] \). Then there holds the error estimate

\[
\| T_0 z - \tilde{T}_0 z \|_{H^{-1/2}(\Gamma)} \leq c_1 h^{s+\frac{1}{2}} \| q_z \|_{H^s_{pw}(\Gamma)} + c_2 h^{s+\frac{3}{2}} \| t_z \|_{H^s_{pw}(\Gamma)}.
\]

(49)

5.2. Boundary element approximation of \( g \)

As in (46), we may also define a boundary element approximation of the right-hand side \( g \) as defined in (24),

\[
g = V^{-1} N_0 T + V^{-1} M_0 f - V^{-1} V_1 V^{-1} N_0 f.
\]

In particular, \( g \in H^{-1/2}(\Gamma) \) is the unique solution of the variational problem

\[
(VG, \tau)_\Gamma = (N_0 T + M_0 f, \tau)_\Gamma - (V_1 V_1 V^{-1} N_0 f, \tau)_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).
\]
where \( tf = V^{-1} N_0 f \in H^{-1/2}(\Gamma) \) solves the variational problem
\[
(Vf_t, \tau)_\Gamma = (N_0 f, \tau)_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).
\]

Hence we can define a boundary element approximation, \( \tilde{g}_h \in S^0_h(\Gamma) \), as the unique solution of the Galerkin variational problem
\[
(V\tilde{g}_h \cdot \tau_h)_\Gamma = (N_0 \tilde{g}_h + M_0 f_h \cdot \tau_h)_\Gamma - (V_t f_h, \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_h(\Gamma),
\]
where \( t_{f,h} \in S^0_h(\Gamma) \) is the unique solution of the Galerkin problem
\[
(Vt_{f,h} \cdot \tau_h)_\Gamma = (N_0 f, \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_h(\Gamma).
\]

**Lemma 5.6**

Let \( g \) be the right-hand side as defined in (24), and let \( \tilde{g}_h \) be the boundary element approximation as defined in (50). Then there holds the error estimate
\[
\| g - \tilde{g}_h \|_{H^{-1/2}(\Gamma)} \leq c_1 \| g \|_{H^0(\Gamma)} + c_2 \| t_f \|_{H^{-3/2}(\Gamma)}.
\]

**Proof**

The assertion follows as in the proof of Lemma 5.4, we skip the details.

By using the approximation property of the trial space \( S^0_h(\Gamma) \) and the Aubin–Nitsche trick, we conclude an error estimate from (52) when assuming some regularity of \( g \) and \( t_f \), respectively.

**Corollary 5.7**

Assume \( g, t_f \in H^s_{pw}(\Gamma) \) for some \( s \in [0,1] \). Then there holds the error estimate
\[
\| g - \tilde{g}_h \|_{H^{-1/2}(\Gamma)} \leq c_1 h^{s+\frac{1}{2}} \| g \|_{H^s(\Gamma)} + 2 c_2 h^{s+\frac{1}{2}} \| t_f \|_{H^{-3/2}(\Gamma)}.
\]

### 5.3. Approximate variational inequality

We consider the variational inequality (10) with \( \tau = -q \) to find \( z \in \mathcal{W} \) such that
\[
(Vq)z = (V_1 t)(x) - (K_1 z)(x) + (N_0 \tilde{g})(x) + (M_0 f)(x) \quad \text{for } x \in \Gamma,
\]
and \( t \in H^{-1/2}(\Gamma) \) is the unique solution of
\[
(Vt)z = (\frac{1}{2} I + K)z(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma.
\]

The Galerkin boundary element approximation of the variational inequality (54), and therefore the boundary element discretization of the perturbed variational inequality (44), is to find \( \tilde{z}_H \in \mathcal{W}_H \) such that
\[
(Vq)\tilde{z}_h = q_h w_H - \tilde{z}_h \|_{H^{-1/2}(\Gamma)} \geq 0 \quad \text{for all } w_H \in \mathcal{W}_H,
\]
where \( q_h \in S^0_h(\Gamma) \) is the unique solution of the Galerkin formulation
\[
(Vq_h, \tau_h)_\Gamma = (V_1 t_h - K_1 \tilde{z}_h + N_0 \tilde{g} + M_0 f_h \cdot \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_h(\Gamma),
\]
and \( t_h \in S^0_h(\Gamma) \) solves
\[
(Vt_h, \tau_h)_\Gamma = (\frac{1}{2} I + K)\tilde{z}_h - N_0 f_h \cdot \tau_h)_\Gamma \quad \text{for all } \tau_h \in S^0_h(\Gamma).
\]

The Galerkin formulation (58) is equivalent to the linear system
\[
V_h \tilde{q} = V_1 h t - K_1 h \tilde{z} + f_{1},
\]
and (59) is equivalent to
\[
V_h \tilde{t} = (\frac{1}{2} M_h + K_h)\tilde{z} - f_{2}.
\]
where
\[
V_h \varphi = (V \varphi, \psi)_\Gamma, \quad K_h \varphi = (K \varphi, \psi)_\Gamma,
\]
\[
V_1_h \varphi = (V_1 \varphi, \psi)_\Gamma, \quad K_1_h \varphi = (K_1 \varphi, \psi)_\Gamma,
\]
\[
M_h \varphi = (M \varphi, \psi)_\Gamma.
\]
and

\[ f_{1,\ell} = \langle N_0 f + M_0 f, \psi_\ell \rangle_1, \quad f_{2,\ell} = \langle N_0 f, \psi_\ell \rangle_1 \]

for \( k, \ell = 1, \ldots, N \) and \( i = 1, \ldots, M \). Recall that we use piecewise linear basis functions \( \phi_i \), and piecewise constant basis functions \( \psi_k \). Moreover, let \( D_H \) be the Galerkin matrix of the hypersingular boundary integral operator \( D \), i.e.

\[ D_H[i,j] = \langle D \phi_i, \phi_j \rangle_1 \quad \text{for} \ i,j = 1, \ldots, M. \]

The matrix representation of the variational inequality (57) is then given by the discrete variational inequality with the Euclidian inner product

\[ (q D_H \bar{z} - M_H^{-1} q, w - \bar{z}) \geq 0 \quad \text{for all} \ w \in \mathbb{R}^M \leftrightarrow w_H \in \mathcal{W}_H \]

or

\[ (T_{\phi,H} \bar{z} - \bar{q}, w - \bar{z}) \geq 0 \quad \text{for all} \ w \in \mathbb{R}^M \leftrightarrow w_H \in \mathcal{W}_H \]

where

\[ \bar{q} = M_H^{-1} \left[ (1 - V_1 K_1 h) f_1 - V_1 K_1 h f_2 \right] \]

as defined in (23). Moreover,

\[ \bar{t}_z = M_H^{-1} \left[ (1 - V_1 K_1 h) f_1 - V_1 K_1 h f_2 \right] \]

is a boundary element approximation of \( g \) as defined in (24).

*Theorem 5.8*

The approximation \( \bar{t}_z \) of the Schur complement as defined in (63) is positive definite, i.e.

\[ (T_{\phi,H} \bar{z}, \bar{z}) \geq \frac{1}{2} c_1^r \| z_H \|^2_{H^1/2(\Gamma)} \quad \text{for all} \ z \in \mathbb{R}^M \leftrightarrow z_H \in S_H^1(\Gamma). \]

if \( h \leq c_0 h \) is sufficiently small.

*Proof*

For an arbitrary chosen but fixed \( z \in \mathbb{R}^M \) let \( z_H \in S_H^1(\Gamma) \) be the associated boundary element function. Then we have

\[ \bar{t}_z = (T_{\phi,H} z_H, z_H)_1 = (T_{\phi,H} z_H, z_H)_1 - (T_{\phi} z_H, z_H)_1 \]

\[ \geq c_1^r \| z_H \|^2_{H^1/2(\Gamma)} - \| T_{\phi} z_H \|_{H^{-1/2}(\Gamma)} \| z_H \|_{H^{1/2}(\Gamma)} \]

As \( z_H \in S_H^1(\Gamma) \) is a continuous function, we have \( z_H \in H^1(\Gamma) \). Hence we find

\[ t_{2,\ell} = V^{-1} \left[ (\frac{1}{2} I + K) z_H \right]_{L^2(\Gamma)}, \quad q_{2,\ell} = V^{-1} \left[ V_1 t_{2,\ell} - K_1 z_H \right]_{L^2(\Gamma)}. \]

Therefore, we can apply the error estimate (49) for \( s = 0 \) to obtain

\[ \| T_{\phi,H} z_H - \bar{t}_z \|_{H^{-1/2}(\Gamma)} \leq c_1^r \| q_{2,\ell} \|_{L^2(\Gamma)} + c_2^2 \| t_{2,\ell} \|_{L^2(\Gamma)} \leq c_3^r \| z_H \|_{H^1(\Gamma)}. \]

Now, by applying the inverse inequality for \( S_H^1(\Gamma) \),

\[ \| z_H \|_{H^1(\Gamma)} \leq c_4^r \| z_H \|_{H^1(\Gamma)} \]

we obtain

\[ \| T_{\phi,H} z_H - \bar{t}_z \|_{H^{-1/2}(\Gamma)} \leq c_3^r c_4^r \| z_H \|_{H^1(\Gamma)}. \]

Hence we finally obtain

\[ (T_{\phi,H} \bar{z}, \bar{z}) \geq \frac{1}{2} c_1^r \| z_H \|^2_{H^1(\Gamma)} \geq \frac{1}{2} c_1^r \| z_H \|^2_{H^1(\Gamma)} \]

if

\[ c_3^r c_4^r \leq \frac{1}{2} c_1^r \]

is satisfied.

Now we are in a position to apply Theorem 5.2 to ensure unique solvability of the perturbed Galerkin variational inequality (44), and to derive related error estimates.
Corollary 5.9
When combining the error estimate (45) with the approximation property of the ansatz space \( S_1^1(\Gamma) \), and with the error estimates (49) and (53), we finally obtain the error estimate

\[
\|z - \tilde{z}_h\|_{H^{1/2}(\Gamma)} \leq c_1 h^{s+1/2} |z|_{H^{s+1}(\Gamma)} + c_2 h^{s+1/2} \|q_2\|_{H^s(\Gamma)} + c_3 h^{s+3/2} \|\tau\|_{H^s(\Gamma)}
\]

\[
+ c_4 h^{s+1/2} \|g\|_{H^s(\Gamma)} + c_5 h^{s+3/2} \|\tau\|_{H^s(\Gamma)}
\]

when assuming \( z \in H^{s+1}(\Gamma) \), and \( q_2, \tau, g, \tau \in H^s(\Gamma) \) for some \( s \in [0, 1] \). For \( h \leq c_0 h \) we therefore obtain the error estimate

\[
\|z - \tilde{z}_h\|_{H^{1/2}(\Gamma)} \leq c |z, \tau, f|_{H^{s+1}}^1.
\]

Moreover, we are also able to derive an error estimate in \( L_2(\Gamma) \), i.e.

\[
\|z - \tilde{z}_h\|_{L_2(\Gamma)} \leq c |z, \tau, f|_{H^{s+1}},
\]

when applying the Aubin–Nitsche trick.

In the particular case of a non-constrained minimization problem, instead of the discrete variational inequality (62) we have to solve

\[
(\mathcal{M} - \mathcal{K})z = q + \mathcal{K}^* \mathcal{K} z,
\]

where \( \mathcal{M} \) is symmetric and \( \mathcal{K} \) is skew-symmetric. Inserting the ansatz (15) and (22), we obtain the error estimate

\[
\|z - \tilde{z}_h\|_{H^{1/2}(\Gamma)} \leq c |z, \tau, f|_{H^{s+1}}^1.
\]

Remark 5.10
The error estimates (64) and (65) provide optimal convergence rates when approximating the control \( z \) by using piecewise linear basis functions. However, we have to assume \( h \leq c_0 h \) to ensure the unique solvability of the perturbed Galerkin variational inequality (44), where the constant \( c_0 \) is in general unknown. Moreover, the matrix \( \mathcal{T}_e \) as given in (63) defines a non-symmetric approximation of the self-adjoint operator \( \mathcal{T}_e \). Hence we are interested in deriving a symmetric boundary element method which is stable without any additional constraints in the choice of the boundary element trial spaces.

6. A symmetric boundary element method

The boundary integral formulation of the primal boundary value problem (2) is given by (14), while the adjoint boundary value problem (12) corresponds to the modified boundary integral equation (21). In what follows, we will use a second boundary integral equation of the adjoint boundary value problem to obtain an alternative representation for \( q \) and therefore of the adjoint operator \( \mathcal{S}^* \). In particular, when computing the normal derivative of the representation formula, (20), this gives

\[
q(x) = \left( \frac{1}{2} I + K^* \right) q(x) - (D_1 z)(x) - (K^*_f t)(x) - (N_1 \mathcal{U})(x) - (M_1 f)(x) \quad \text{for } x \in \Gamma,
\]

where

\[
(N_1 \mathcal{U})(x) = \lim_{\Omega \ni x \to \Gamma} n_x \cdot \nabla \int_{\Omega} \mathcal{U}^*(\vec{x}, y) \mathcal{U}(y) \, dy \quad \text{for } x \in \Gamma
\]

and

\[
(M_1 f)(x) = \lim_{\Omega \ni x \to \Gamma} n_x \cdot \nabla \int_{\Omega} \mathcal{V}^*(\vec{x}, y) f(y) \, dy \quad \text{for } x \in \Gamma.
\]

Hence, from (11) we obtain

\[
\tau = \mathcal{S}^*(u - \mathcal{U}) = -q = -\left( \frac{1}{2} I + K^* \right) q + D_1 z + K^*_t t + N_1 \mathcal{U} + M_1 f
\]

and, by using (15) and (22), we conclude the alternative representations

\[
\mathcal{T}_e = \mathcal{D} + D_1 - \left( \frac{1}{2} I + K^* \right) V^{-1} V^{-1} \left( \frac{1}{2} I + K^* \right) + \left( \frac{1}{2} I + K^* \right) V^{-1} \left( \frac{1}{2} I + K^* \right)
\]

and

\[
g = K^*_f V^{-1} N_0 f - N_1 \mathcal{U} - M_1 f + \left( \frac{1}{2} I + K^* \right) V^{-1} (N_0 \mathcal{U} + M_0 f - V^{-1} N_0 f).
\]
The boundary integral operator $T_0: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as defined in (68) is self-adjoint, bounded and $H^{1/2}(\Gamma)$-elliptic.

**Proof**

While the self-adjointness of $T_0$ in the symmetric representation (68) is obvious, the boundedness and ellipticity estimates follow as in the proof of Theorem 4.4. In particular, the operators $T_0$ in the symmetric representation (68) and in the non-symmetric representation (23) coincide. Indeed, by using (34) and (35) we obtain

$$T_0 = qD + D_1 + [K'_1 - (\frac{1}{2} I + K')V^{-1}V_1]V^{-1}(\frac{1}{2} I + K) + (\frac{1}{2} I + K')V^{-1}K_1$$

Due to the representation of the Laplace Steklov–Poincaré operator, see, e.g., [24], we further conclude

$$(\frac{1}{2} I + K')V^{-1}(\frac{1}{2} I + K) = V^{-1}(\frac{1}{2} I + K) - D.$$ 

Therefore, by using (34) and (37) we have

$$T_0 = qD + D_1 + V^{-1}(\frac{1}{2} I + K) - V^{-1}V_1[V^{-1}(\frac{1}{2} I + K) - D] + V^{-1}(\frac{1}{2} I + K)K_1$$

where $q_z \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(Vq_z)(x) = (V_1 t_z)(x) - (K_1 z)(x) \quad \text{for } x \in \Gamma,$$

and $t_z \in H^{-1/2}(\Gamma)$ solves

$$(Vt_z)(x) = (\frac{1}{2} I + K)z(x) \quad \text{for } x \in \Gamma.$$ 

As for the non-symmetric representation of $T_0$, we can define approximate Galerkin solutions $t_{zh}, \hat{q}_{zh} \in S_h^{(1)}(\Gamma)$, and therefore we can introduce the approximation

$$\hat{T}_0 z := qD + D_1 z + K'_1 t_{zh} - (\frac{1}{2} I + K')\hat{q}_{zh}. \quad (70)$$

**Lemma 6.2**

The approximate operator, $\hat{T}_0: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, as defined in (70) is bounded, i.e.

$$\|\hat{T}_0 z\|_{H^{-1/2}(\Gamma)} \leq C_2 \|z\|_{H^{1/2}(\Gamma)} \quad \text{for all } z \in H^{1/2}(\Gamma).$$

Moreover, there holds the error estimate

$$\|T_0 z - \hat{T}_0 z\|_{H^{-1/2}(\Gamma)} \leq C_1 \inf_{\tau_h \in S_h^{(1)}(\Gamma)} \|q_z - \tau_h\|_{H^{-1/2}(\Gamma)} + C_2 \|t_z - t_{zh}\|_{H^{-1/2}(\Gamma)} + C_4 \|z - t_{zh}\|_{H^{-3/2}(\Gamma)}. \quad (71)$$

**Proof**

The proof follows as for the boundary element approximation of the non-symmetric formulation, see Lemmas 5.3 and 5.4. 

By using the approximation property of the trial space $S_h^{(1)}(\Gamma)$ and the Aubin–Nitsche trick, we then conclude an error estimate from (71) when assuming some regularity of $q_z$ and $t_z$, respectively.
Assume $q_z, t_z \in H^s_{pw}(\Gamma)$ for some $s \in [0, 1]$. Then there holds the error estimate
\[ \|T_z - \hat{T} \|_{H^{-1/2}(\Gamma)} \leq c_1 h^{s+\frac{1}{2}} \|q_z\|_{H^s_{pw}(\Gamma)} + c_2 h^{s+\frac{3}{2}} \|t_z\|_{H^s_{pw}(\Gamma)}. \]  

6.2. Boundary element approximation of $g$

As in the approximation (70), we can define a boundary element approximation of $g$ as defined in (69),
\[ g = K_1 t_f - N_1 u - M_1 f + (\frac{1}{2} I + K') q_f, \]
where $q_f \in H^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation
\[ (V q_f)(x) = (N_0 \overline{u})(x) + (M_0 f)(x) - (V_1 t_f)(x) \quad \text{for } x \in \Gamma, \]
and $t_f \in H^{-1/2}(\Gamma)$ solves
\[ (V t_f)(x) = (N_0 f)(x) \quad \text{for } x \in \Gamma. \]

Hence we can define approximate Galerkin solutions $\hat{q}_{t_f}, t_{t_f} \in S^s_{\hat{H}}(\Gamma)$, and therefore, we can introduce the approximation
\[ \hat{g} = K_1 t_{t_f} - N_1 u - M_1 f + (\frac{1}{2} I + K') \hat{q}_{t_f}. \]  

As in (53) we conclude the error estimate
\[ \|g - \hat{g}\|_{H^{-1/2}(\Gamma)} \leq c_1 h^{s+\frac{1}{2}} \|q_f\|_{H^s_{pw}(\Gamma)} + c_2 h^{s+\frac{3}{2}} \|t_f\|_{H^s_{pw}(\Gamma)} \]
when assuming $q_f, t_f \in H^s_{pw}(\Gamma)$ for some $s \in [0, 1]$.

6.3. Approximate variational inequality

The use of the symmetric approximations (70) and (73) results in the approximate variational inequality
\[ \langle \hat{T}_{f, \mu} z - \hat{g}, w - z \rangle \geq 0 \quad \text{for all } w \in \mathbb{R}^M \Leftrightarrow w_\mu \in \mathbb{V}_\mu, \]
where
\[ \hat{T}_{f, \mu} = \hat{D}_H + D_{1, \mu} - (1/2) M_h^T + K_h^T) V_h^{-1} V_1 h V_h^{-1} (1/2) M_h + M_h + K_h - K_1 h V_h^{-1} (1/2) M_h + M_h + K_h + (1/2) M_h^T + K_h^T) V_h^{-1} K_1 h \]
defines a symmetric Galerkin boundary element approximation of the self-adjoint operator $T_f$, and
\[ \hat{g} = K_1 t_{t_f} V_h^{-1} V_2 - f_3 + (1/2) M_h^T + K_h^T) V_h^{-1} (1/2) M_h + M_h + K_h \]
is the related boundary element approximation of $g$ as defined in (69). Note that in addition to those entries of the non-symmetric approximation, we use
\[ D_{1, \mu}[j, j] = (D_1 \phi_j, \phi_j)_\Gamma, \quad f_{3, j} = (N_1 \overline{u} + M_1 f, \phi_j)_\Gamma \quad \text{for } i, j = 1, \ldots, M. \]

Lemma 6.4

The symmetric matrix
\[ \hat{T}_H := \hat{T}_{f, \mu} = \hat{D}_H + D_{1, \mu} - K_1 h V_h^{-1} (1/2) M_h + M_h + K_h - (1/2) M_h^T + K_h^T) V_h^{-1} K_1 h \]
is positive semi-definite, i.e.
\[ \langle \hat{T}_H z, z \rangle \geq 0 \quad \text{for all } z \in \mathbb{R}^M. \]

Proof

We consider the generalized eigenvalue problem
\[ \hat{T}_H \xi = \mu \hat{S}_H + (1/2) M_h^T + K_h^T) V_h^{-1} (1/2) M_h + M_h + K_h \xi \]
where the stabilized discrete Steklov–Poincaré operator
\[ \hat{S}_H = D_H + (1/2) M_h^T + K_h^T) V_h^{-1} (1/2) M_h + M_h + K_h \]

is symmetric and positive definite. Note that the vector \( \varphi \) is given by
\[
a_i = \int_I \varphi_i(x) \, dx \quad \text{for } i = 1, \ldots, M.
\]

Since the eigenvalue problem (77) can be written as
\[
\left( \left( \frac{1}{2} M_h^T + K_h^T \right) V_h^{-1} \right) \begin{pmatrix} -V_{1,h} & K_{1,h} \\ K_{1,h}^T & I \end{pmatrix} \begin{pmatrix} V_h^{-1} \left( \frac{1}{2} M_h + K_h \right) \end{pmatrix} \begin{pmatrix} Z \\ Z \end{pmatrix} = \mu \left( \left( \frac{1}{2} M_h^T + K_h^T \right) V_h^{-1} \right) \begin{pmatrix} V_h \ \bar{Z} \end{pmatrix} \begin{pmatrix} V_h^{-1} \left( \frac{1}{2} M_h + K_h \right) \end{pmatrix} \begin{pmatrix} Z \\ \bar{Z} \end{pmatrix},
\]

it is sufficient to consider the generalized eigenvalue problem
\[
\begin{pmatrix} -V_{1,h} & K_{1,h} \\ K_{1,h}^T & I \end{pmatrix} \begin{pmatrix} w \\ Z \end{pmatrix} = \mu \begin{pmatrix} V_h \bar{Z} \end{pmatrix} \begin{pmatrix} V_h^{-1} \left( \frac{1}{2} M_h + K_h \right) \end{pmatrix} \begin{pmatrix} Z \\ \bar{Z} \end{pmatrix},
\]

where
\[
w = V_h^{-1} \left( \frac{1}{2} M_h + K_h \right) Z.
\]

From (78) we conclude
\[
\langle K_{1,h} Z, w \rangle - \langle V_{1,h} w, w \rangle = \mu \langle V_h w, w \rangle,
\]
and by taking the difference we obtain
\[
\langle V_{1,h} w, w \rangle = \mu \langle \bar{S}_H Z, Z \rangle - \langle V_h w, w \rangle = \mu \langle \bar{S}_H Z, Z \rangle - \mu \langle \bar{S}_H Z, Z \rangle.
\]

Hence, \( \mu \geq 0 \) follows, which implies the assertion. \( \square \)

As a corollary of Lemma 6.4, we find the positive definiteness of the symmetric Schur complement matrix \( \bar{T}_{0,H} \) as defined in (76).

**Corollary 6.5**
The approximate Schur complement \( \bar{T}_{0,H} \) as defined in (76) is positive definite, i.e.
\[
\langle \bar{T}_{0,H} Z, Z \rangle \geq 0 \quad \text{for all } Z \in R^M \leftrightarrow z_H \in S^1_H(\Gamma),
\]
for all \( Z \in R^M \leftrightarrow z_H \in S^1_H(\Gamma) \), since \( gD + D_1 \) implies an equivalent norm in \( H^{1/2}(\Gamma) \).

Hence we can apply Theorem 5.2 to ensure unique solvability of the perturbed variational inequality to find \( \bar{z} \in R^M \leftrightarrow \bar{z}_H \in \mathcal{W}_H \) such that
\[
\langle \bar{T}_{0,H} Z - \bar{g}, W - Z \rangle \geq 0 \quad \text{for all } W \in R^M \leftrightarrow W_H \in \mathcal{W}. \tag{79}
\]

**Corollary 6.6**
When combining the general error estimate (45) with the approximation property of the ansatz space \( S^1_H(\Gamma) \), and with the error estimates (72) and (74), we finally obtain the error estimate
\[
\| z - \bar{z}_H \|_{H^{1/2}(\Gamma)} \leq c_1 h^{s+1/2} \| z \|_{H^{1+}(\Gamma)} + c_2 h^{s+1/2} \| q_z \|_{H^{s+2}(\Gamma)} + c_3 h^{s+3/2} \| f_z \|_{H^{s+2}(\Gamma)}
\]
\[
+ c_4 h^{s+1/2} \| g \|_{H^{s+2}(\Gamma)} + c_5 h^{s+3/2} \| f \|_{H^{s+2}(\Gamma)}
\]
when assuming \( z \in H^{1+}(\Gamma) \), and \( q_z, f_z, g, f \in H^{s+2}(\Gamma) \) for some \( s \in [0, 1] \). In particular for \( h = H \) we therefore obtain the error estimate
\[
\| z - \bar{z}_H \|_{H^{1/2}(\Gamma)} \leq c(z, \bar{z}, f) H^{s+1/2}. \tag{80}
\]

Moreover, we are also able to derive an error estimate in \( L_2(\Gamma) \), i.e.
\[
\| z - \bar{z}_H \|_{L_2(\Gamma)} \leq c(z, \bar{z}, f) H^{s+1}, \tag{81}
\]
when applying the Aubin–Nitsche trick.
In the particular case of a non-constrained minimization problem, instead of the discrete variational inequality (79) we have to solve the linear system

$$\tilde{t}_{e,H} = \tilde{q},$$

which is equivalent to a system of linear equations,

$$\begin{pmatrix} -V_{1,h} & V_h & K_{1,h} \\ V_h & -\left(\frac{1}{2}M_h + K_h\right) & 0 \\ K_{1,h}^T & 0 & -\frac{1}{2}M_h^T + K_h^T \end{pmatrix} \begin{pmatrix} \tilde{t} \\ q \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{pmatrix}.$$ (82)

**Remark 6.7**

The symmetric boundary element approximation $\tilde{t}_{e,H}$ is positive definite for any choice of conformal boundary element spaces $S_{h1}^0(\Gamma) \subset H^{1/2}(\Gamma)$ and $S_{h2}^0(\Gamma) \subset H^{-1/2}(\Gamma)$. In particular we may use the same boundary element mesh with mesh size $h = H$ to define the basis functions $\tilde{\psi}_1$ and $\tilde{\psi}_k$, respectively. From a theoretical point of view, this is not possible in general when using the non-symmetric approximation $\tilde{t}_{e,H}$.

### 7. Numerical results

As numerical example we first consider as in [8, 11], see also [16], the unconstrained Dirichlet boundary control problem (1)–(2) for the domain $\Omega = (0, 1]^2 \subset \mathbb{R}^2$ where

$$\Pi(x) = (x_1^2 + x_2^2)^{-1/5}, \quad f(x) = 0, \quad g = 1.$$ In this case we have to solve the coupled linear system (66) in the case of the non-symmetric boundary element approach, and (82) for the symmetric approach.

For the boundary element discretization, we introduce a uniform triangulation of the boundary $\Gamma = \partial \Omega$ on several levels where the mesh size is $h = 2^{-l+1}$. As the minimizer of (1) is not known in this case, we use the boundary element solution $z_{h_0}$ of the ninth refinement level as reference solution. The boundary element discretization is done by using the trial space of piecewise linear and continuous functions. In particular, we use the same boundary element mesh to approximate the control $z$ by a piecewise linear approximation, and piecewise constant approximations for the fluxes $t$ and $q$. Note that we have $h = H$ in this case, and therefore we can not ensure the $S_{h1}^0(\Gamma)$-ellipticity of the non-symmetric boundary element approximation, see Theorem 5.8. However, the numerical example shows stability in this case.

In Table I, we present the errors for the control $z$ in the $L_2(\Gamma)$ norm and the estimated order of convergence (eoc). These results correspond to the error estimate (65) of the non-symmetric boundary element approximation, and to the error estimate (81) of the symmetric boundary element approximation. Note that in this example we have $\Pi \in H^{3}(\Omega)$ for $s < 2/3$ implying $p \in H^{2+s}(\Omega)$, and $(\tilde{g}/\sqrt{n})p \in H^{1/2+s}(\Gamma)$. As $D : H^{1/2+s}(\Gamma) \rightarrow H^{-1/2+s}(\Gamma)$ for $s < 2/3$, see [36], we further conclude $z \in H^{7/6}(\Gamma)$, and therefore we can expect $7/6$ as order of convergence. For comparison, we also give the error of the related finite element solution, see [16]. From the numerical results we conclude, that all three different approaches behave almost similar, as predicted by the theory.

As a second example, we consider in addition the box constraints (3) with $g_a = -1$, which is inactive, and $g_b = 2.23$. In Figure 1 we give a comparison of the unconstrained and constrained solutions, and in Figure 2 we plot the related controls for $x_1 \in (0, 0.5), x_2 = 0$.

### 8. Concluding remarks

In this paper, we have shown that we can use boundary element methods to solve Dirichlet boundary control problems. The first approach is based on the first Bi-Laplace boundary integral equation only, but a stable boundary element discretization requires
the use of appropriate boundary element spaces. However, the symmetric formulation is stable for all standard boundary element spaces without any further condition. Moreover, the Galerkin discretization of the symmetric formulation results in a symmetric approximation, while the first approach does not. Hence, the symmetric formulation seems to be the method of choice. The numerical results coincide with those of a comparable finite element approach. The advantage of using boundary element methods lies in the fact that only a discretization of the boundary is required. In the case of smooth data we can prove, with respect to the used lowest order trial spaces, the best possible order of convergence for the boundary element approximation of the control $z$, whereas for a finite element approximation we are only able to prove some reduced order, see [16]. Moreover, optimal control problems subject to partial differential equations in unbounded exterior domains can be handled analogously.

While this paper is on the stability and error analysis of boundary element methods for optimal control problems only, further research will be done for an efficient solution of the resulting discrete systems. Hereby, special focus will be on appropriate solution methods to solve the discrete variational inequalities. This also involves the construction of efficient preconditioners, as well as the use of fast boundary element methods.

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