THE FATE OF RADIATING BLACK HOLES IN NONCOMMUTATIVE GEOMETRY

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ABSTRACT

We investigate the behavior of a radiating Schwarzschild black hole toy-model in a 2D noncommutative spacetime. It is shown that coordinate noncommutativity leads to: i) the existence of a minimal non-zero mass to which black hole can shrink; ii) a finite maximum temperature that the black hole can reach before cooling down to absolute zero; iii) the absence of any curvature singularity. The proposed scenario offers a possible solution to conventional difficulties when describing terminal phase of black hole evaporation.

Key words: Black hole physics; gravitational; relativ-ity.

1. INTRODUCTION

The theoretical discovery of radiating black holes [1] disclosed the first physically relevant window on the mysteries of quantum gravity. After thirty years of intensive research in this field various aspects of the problem still remain under debate (see [2] for a recent review with an extensive reference list). For instance, a fully satisfactory description of the late stage of black hole evaporation is still missing. The string/black hole correspondence principle [3] suggests that in this extreme regime stringy effects cannot be neglected. This is just one of many examples of how the development of string theory has affected various aspects of theoretical physics. Among different outcomes of string theory, we focus on the result that target spacetime coordinates become noncommuting operators on a D-brane [4, 5]. Thus, string-brane coupling has put in evidence the necessity of spacetime quantization. This indication gave a new boost to reconsider older ideas of similar kind pioneered in a, largely ignored, paper by Snyder [6]. The noncommutativity of spacetime can be encoded in the commutator

$$[x^\mu, x^\nu] = i \theta^{\mu\nu} \quad (1)$$

where $\theta^{\mu\nu}$ is an anti-symmetric matrix which determines the fundamental cell discretization of spacetime much in the same way as the Planck constant $\hbar$ discretizes the phase space. The modifications of quantum field theory implied by (1) has been recently investigated to a large extent. Currently the approach to noncommutative quantum field theory follows two distinct paths: one is based on the Weyl-Wigner-Moyal $*$-product and the other on coordinate coherent state formalism [7, 8]. In a recent paper, following the second formulation, it has been shown that controversial questions of Lorentz invariance and unitary [9, 10], raised in the $*$-product approach, can be solved assuming $\theta^{\mu\nu} = \theta \text{diag} (\epsilon_{ij}, \epsilon_{ij}, \ldots)$ [11], where $\theta$ is a constant with dimension of length squared. Furthermore, the coordinate coherent state approach profoundly modifies the structure of the Feynman propagator rendering the theory ultraviolet finite, i.e. curing the short distance behavior of pointlike structures [12]. It is thus reasonable to believe that noncommutativity could as well cure divergences that appear, under various forms, in General Relativity. In such a framework, it would be of particular interest to study the effects of noncommutativity on the terminal phase of black hole evaporation. In the
standard formulation, the temperature diverges and an arbitrarily large curvature state is reached. One hopes that noncommutativity can have important consequences both on the black hole thermal radiation and on the curvature singularity at its center. In order to tackle this difficult problem, we are going to consider a 2D toy-model where the effects of noncommuting coordinates are implemented through a modified Schwarzschild metric [13].

2. 2D BLACK HOLE

The most direct way to study black hole radiation is by using \( t-r \) section of the Schwarzschild vacuum solution [14, 15]

\[
ds^2 = -(1 + 2\phi(r)) dt^2 + (1 + 2\phi(r))^{-1} dr^2,
\]

where, \( \phi(r) \) is the Newtonian potential

\[
\phi(r) = \frac{MG_N}{r}
\]

which solves the classical Poisson equation for a point-like source described by Dirac delta-function

\[
\nabla^2 \phi = 4\pi G_N \delta(r)
\]

In order to incorporate noncommutative effects in the analysis of black hole evaporation, at first glance, one should think of modifying the 4D Einstein action and try to solve corresponding field equations. So far, a proper modification of this kind is not known and one cannot proceed in this direction. At this point we would like to reflect on the connection between noncommutativity and curvature. Curvature on its own is a geometrical structure defined over an underlying manifold. It measures the strength of the gravitational field, i.e. is the response to the presence of a mass-energy distribution, and in this context, one can speak of strong/weak-field regime. On the other hand, noncommutativity is an intrinsic property of the manifold itself even in absence of gravity. Thus, noncommutative modification of Schwarzschild, once introduced at a given curvature, will remain valid in any other field strength regime. This is exactly the gravitational analogue of the NC modification of quantum field theory where the strength of the field is not an issue [7, 8]. Therefore, we propose to start from the Poisson equation [9] which has a precise physical meaning and guarantees physical ground for noncommutative effects. Noncommutative position operators are subject to the uncertainty principle following from the commutator [10]. The physical effect of noncommutativity is that the very concept of point-like object is no more meaningful and one has to deal with smeared objects only. These physical effects are discussed in the framework of coherent state approach in [11]. The implementation of these physically sound arguments within the mathematical formalism amounts to substitution of position Dirac-delta, characterizing point-like structures, with Gaussian function of minimal width \( \sqrt{\theta} \) describing the corresponding smeared structures.

\[
\nabla^2 \phi = 4\pi G_N \rho_\theta(\vec{x})
\]

where, \( M \) is the mass of the source, \( G_N \) is the Newton constant and \( \rho_\theta \) is the Gaussian mass density.

Our strategy to construct a NC version of [2] is summarized in the following scheme:

| GR: \( g_{00} = 1 + 2\phi \) | \( \nabla^2 \phi = 4\pi G_N \delta(r) \) |
|-----------------------------|----------------------------------|
| \( \uparrow \) \( \theta \rightarrow 0 \) | \( \theta \neq 0 \) \( \downarrow \theta \rightarrow 0 \) |
| NCGR: \( g_{00} = 1 + 2\phi \) | \( \nabla^2 \phi = 4\pi G_N \rho_\theta(r) \) |

This is the best we can do until a fully self-consistent reformulation of General Relativity over a noncommutative manifold becomes available. We obtain the NC version of [2] as

\[
ds^2 = -(1 - \frac{2M}{r\sqrt{\gamma}}) dt^2 + (1 - \frac{2M}{r\sqrt{\gamma}})^{-1} dr^2
\]

where \( \gamma \equiv \left(1/2, r^2/4\theta\right) \) is the lower incomplete Gamma function, with the definition

\[
\gamma\left(1/2, r^2/4\theta\right) \equiv \int_0^{r^2/4\theta} dt t^{-1/2} e^{-t}
\]

The line element [11] describes the geometry of a noncommutative black hole toy-model in 2D \(^1\) and

\[
\hbar = \kappa_B = 1.
\]

\(^1\)In formula \(6\) we introduced geometrical units \( G_N = c = \hbar = \kappa_B = 1.\)
should give us useful indications about possible noncommutative effects on Hawking radiation.

The classical Schwarzschild metric is obtained from (6) in the limit \( r/2\sqrt{\theta} \to \infty \).

The event horizon radius \( r_H \) is defined by the vanishing of \( g_{00} \). In our case it leads to the implicit equation

\[
r_H = \frac{2M}{\sqrt{\pi}} \gamma \left( \frac{1}{2}, r_H^2/4\theta \right)
\]

Equation (8) can be conveniently rewritten in terms of the upper incomplete Gamma function as

\[
r_H = 2M \left[ 1 - \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2}, r_H^2/4\theta \right) \right]
\]

The first term in (9) is the Schwarzschild radius, while the second term brings in \( \theta \)-corrections.

In the “large radius” regime \( r_H^2/4\theta \gg 1 \) equation (9) can be solved by iteration. At the first order approximation, we find

\[
r_H = 2M \left( 1 - \sqrt{\theta} \frac{1}{\pi M} e^{-M^2/\theta} \right)
\]

The effect of noncommutativity is exponentially small, which is reasonable to expect since at large distances spacetime can be considered as a smooth classical manifold. On the other hand, at short distance one expects significant changes due to the spacetime fuzziness, i.e. the “quantum geometry” becomes important \( r_H\sqrt{\theta} \ll 1 \). To this purpose it is convenient to invert (9) and consider the black hole mass \( M \) as a function of \( r_H \):

\[
M_0 \equiv \lim_{r_H \to 0} M (r_H) = 0.5 \sqrt{\pi \theta}
\]

which is a new and interesting result. Noncommutativity implies a minimal non-zero mass that allows the existence of an event horizon.

If the starting black hole mass is such that \( M > M_0 \), it can radiate through the Hawking process until the value \( M_0 \) is reached. At this point the horizon has totally evaporated leaving behind a massive relic. Black holes with mass \( M < M_0 \) do not exist.

The behavior of mass \( M \) as a function of horizon radius is given in Figure 1. To understand the physical nature of the mass \( M_0 \) remnant, let us also consider the black hole temperature as a function of \( r_H \). It is given by

\[
T_H (r_H) = \frac{1}{4\pi} \left[ \frac{1}{r_H} \frac{\gamma'}{(1/2 ; r_H^2/4\theta)} \right]
\]

where the “prime” denotes differentiation with respect to \( r \).

![Figure 1. Mass vs horizon relation. In the commutative case, dashed line, the mass is the linear function \( M = r_H/2 \) vanishing at the origin, while in the noncommutative case, solid line, \( M (r_H = 0) = M_0 \), i.e. for \( M < M_0 \) there is no event horizon.](image1)

For large black holes, i.e. \( r_H^2/4\theta >> 1 \), one recovers the standard result for the Hawking temperature

\[
T_H = \frac{1}{4\pi r_H}
\]

![Figure 2. Hawking temperature \( T_H \) as a function of the horizon radius \( r_H \). In the noncommutative case, solid curve, one see the temperature reaches a maximum value \( T_{Max}^\text{Max.} = 2.18 \times 10^{-2}/\sqrt{\theta} \) for \( r_H = 2.74\sqrt{\theta} \), and then decreases to zero as \( r_H \to 0 \). The commutative, divergent behavior, dashed curve, is cured.](image2)
At the initial state of radiation the black hole temperature increases while the horizon radius is decreasing. It is crucial to investigate what happens as $r_H \to 0$. In the standard (commutative) case $T_H$ diverges and this puts limit on the validity of the conventional description of Hawking radiation. Against this scenario, formula (13) leads to

$$T_H \sim \frac{r_H}{24\pi \theta}, \quad \text{as} \quad r_H \to 0 \quad (15)$$

This is another intriguing result that has two important consequences. Firstly, the emerging picture is that the black hole has reached zero temperature and the horizon has completely evaporated. Nevertheless, we are left with a frozen, massive, remnant. Secondly, passing from the regime of large radius to the regime of small radius, (13) and (15), implies the existence of a maximum temperature which is confirmed by the plot in Figure 2. The plot gives the value $T_H^{\text{Max}} = 2.18 \times 10^{-2}/\sqrt{\theta}$. The temperature behavior shows that noncommutativity plays the same role in General Relativity as in Quantum Field Theory, i.e. removes short distance divergences. The resulting picture of black hole behavior goes as follows. For $M >> M_0$ the temperature is given by (14) up to exponentially small corrections, and it increases, as the mass is radiated away. $T_H$ reaches a maximum value at $r_H = 2.74 \sqrt{\theta}$, and then decreases down to zero as $r_H$ goes to zero. At this point, important issue of Hawking radiation back-reaction should be discussed. In commutative case one expects relevant back-reaction effects during the terminal stage of evaporation because of huge increase of temperature (16). In our case, the role of noncommutativity is to cool down the black hole in the final stage. As a consequence, there is a suppression of quantum back-reaction since the black hole emits less and less energy. Eventually, back-reaction may be important during the maximum temperature phase. In order to estimate its importance in this region, let us look at the thermal energy $E = T$ and the total mass $M$ near $r_H = 2.74 \sqrt{\theta}$. From (14) one finds $M \sim \sqrt{\theta} M_{\text{Pl}}$. In order to have significant back-reaction effect $T_H^{\text{Max}}$ should be of the same order of magnitude as $\hat{M}$. This condition leads to the estimate

$$\sqrt{\theta} \sim 10^{-1} l_{\text{Pl}}, \sim 10^{-34} \text{ cm} \quad (16)$$

Expected values of $\sqrt{\theta}$ are above the Planck length $l_{\text{Pl}}$ and (16) indicates that back-reaction effects are suppressed even at $T_H^{\text{Max}} \approx 10^{18} \text{ GeV}$. For this reason we can safely use unmodified form of the metric (2) during all the evaporation process. As it appears, at the final stage of evaporation a mass $M_0$ is left behind. One would be tempted to say that the black hole evaporation has produced a naked singularity of mass $M_0$. We are going to show that this is not the case. In a 2D effective geometry all the curvature tensors can be written in terms of the Ricci scalar $R$. In case of metric (6) the scalar curvature turns out to be

$$R = \frac{2M}{\sqrt{\theta}} \left( \frac{2\gamma - \gamma' + \gamma''}{r^3} \right) \quad (17)$$

One can check that at large distances (17) reproduces the usual 2D Schwarzschild scalar curvature

$$R = \frac{4M}{r^3} \quad (18)$$

In order to establish the geometrical picture of the frozen relic we are going to describe the behavior of the curvature (17) near $r = 0$. In case of naked singularity one should obtain divergent curvature, $R \to \infty$. On the contrary, short distance behavior of $R$ is

$$R \sim -\frac{M}{3\sqrt{\pi} \theta^{3/2}} \left( 1 - \frac{9}{20} \frac{r^2}{\theta} \right) \quad (19)$$

For $r << \sqrt{\theta}$ the curvature is actually constant and negative.

Regular black holes have been introduced as ad hoc models implementing the idea of a maximum curvature (18). On the other hand we have found here an equivalent non singular 2D black hole as a solution of Einstein equations with a source suitably prescribed by coordinate noncommutativity.
3. CONCLUSIONS

As a conclusion, the results derived in this work show that the coordinate coherent state approach to noncommutative effects can cure the singularity problems at the terminal stage of black hole evaporation. We have shown that noncommutativity is an intrinsic property of the manifold itself and thus unaffected by the distribution of matter. Matter curves a noncommutative manifold in the same way as it curves a commutative one, but cannot produce singular structures. Specifically, we have shown that there is a minimal mass $M_0 = 0.5 \sqrt{\pi \theta}$ to which a black hole can decay through Hawking radiation. The reason why it does not end-up into a naked singularity is due to the finiteness of the curvature at the origin. The everywhere regular geometry and the residual mass $M_0$ are both manifestations of the Gaussian de-localization of the source in the noncommutative spacetime. On the thermodynamic side, the same kind of regularization takes place eliminating the divergent behavior of Hawking temperature. As a consequence there is a maximum temperature that the black hole can reach before cooling down to absolute zero. As already anticipated in the introduction, noncommutativity regularizes divergent quantities in the final stage of black hole evaporation in the same way it cured UV infinities in noncommutative quantum field theory. We have also estimated that back-reaction does not modify the original metric in a significant manner.

ACKNOWLEDGMENTS

One of us, P.N., thanks the “Dipartimento di Fisica Teorica dell’Università di Trieste”, the PRIN 2004 programme “Modelli della teoria cinetica matematica nello studio dei sistemi complessi nelle scienze applicate” and the CNR-NATO programme for financial support.

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