A NON-VARIATIONAL SYSTEM INVOLVING THE CRITICAL SOBOLEV EXPONENT. THE RADIAL CASE

By

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Abstract. In this paper, we consider the non-variational system

\[
\begin{align*}
-\Delta u_i &= \sum_{j=1}^{k} a_{ij} u_j^{2^* - 1} \quad \text{in } \mathbb{R}^N, \\
u_i &> 0 \quad \text{in } \mathbb{R}^N, \\
u_i &\in D^{1,2}(\mathbb{R}^N)
\end{align*}
\]

and give some sufficient conditions on the matrix \((a_{ij})_{i,j=1,...,k}\) which ensure the existence of solutions bifurcating from the bubble of the critical Sobolev equation.

1 Introduction

1.1 Setting of the problem. In this paper we consider the \(k \times k\) system

\[
\begin{align*}
-\Delta u_i &= \sum_{j=1}^{k} a_{ij} u_j^{2^* - 1} \quad \text{in } \mathbb{R}^N, \\
u_i &> 0 \quad \text{in } \mathbb{R}^N, \\
u_i &\in D^{1,2}(\mathbb{R}^N)
\end{align*}
\]

for \(i = 1, \ldots, k\), where \(N \geq 3\),

\[D^{1,2}(\mathbb{R}^N) = \{ u \in L^{2^*}(\mathbb{R}^N) \text{ such that } |\nabla u| \in L^2(\mathbb{R}^N) \},\]

\[2^* = \frac{2N}{N-2}\] and the matrix \(A := (a_{ij})_{i,j=1,...,k}\) satisfies

\[
\sum_{j=1}^{k} a_{ij} = 1 \quad \text{for any } i = 1, \ldots, k.
\]

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Assumptions (1.2) and (1.3) imply that $A$ possesses $(k - 1)k/2$ free parameters, i.e., the set of matrices satisfying (1.2)–(1.3) form an affine subspace of dimension $(k - 1)k/2$. For example, for $k = 3$, such a matrix can be written

$$A = \begin{pmatrix}
1 - a_1 - a_2 & a_1 & a_2 \\
 a_1 & 1 - a_1 - a_3 & a_3 \\
 a_2 & a_3 & 1 - a_2 - a_3
\end{pmatrix} \quad \text{for } a_1, a_2, a_3 \in \mathbb{R}.$$

If $u_1 = u_2 = \cdots = u_k$ then (1.1) reduces to the classical critical Sobolev equation

$$\begin{cases}
-\Delta u = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u \in D^{1,2}(\mathbb{R}^N).
\end{cases}$$

(1.4)

Actually, our problem can be seen as a straightforward generalization of equation (1.4) to the case of systems.

It is well-known (see [CGS]) that (1.4) admits the $(N + 1)$-parameter family of solutions given by the standard bubbles

$$U_{\delta, y}(x) := \left[\frac{N(N - 2)}{(\delta^2 + |x - y|^2)^{\frac{N-2}{2}}}\right]^{\frac{N-2}{4}}$$

where $\delta > 0$ and $y \in \mathbb{R}^N$. For simplicity we will denote

$$U(x) := U_{1,0}(x) = \left[\frac{N(N - 2)}{(1 + |x|^2)^{\frac{N-2}{2}}}\right]^{\frac{N-2}{4}}.$$  

(1.5)

In this paper we want to prove the existence of solutions to Problem (1.1) that are different from the trivial solution $(U_{\delta, y}, \ldots, U_{\delta, y})$.

We now show how our system, choosing particular matrices $A$, extends some known cases in the literature. The first example corresponds to the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that we have

$$\begin{cases}
-\Delta u = v^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\
-\Delta v = u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\
u, v > 0 & \text{in } \mathbb{R}^N, \\
u, v \in D^{1,2}(\mathbb{R}^N).
\end{cases}$$

(1.6)

This is a case in which the powers of the non-linearity belong to the so-called critical hyperbola introduced by Mitidieri in [M1] and [M2] (see also [CDM].
and [PV]). It is an interesting open problem to determine whether the system (1.6) admits non-trivial solutions (this corresponds to the case $\alpha = 0$ in Theorem 1.4 and is outside the scope of this paper). 

Among the other results, we will show that the trivial solution $(U, U)$ to (1.6) is non-degenerate, up to translations and dilations (see Proposition 1.2 and the ensuing discussion on page 648).

Another interesting system which has a lot of similarities with (1.1) is the well-known Toda system, namely

\begin{align*}
\begin{cases}
-\Delta u_i = \sum_{j=1}^{k} c_{ij} e^{u_j} \quad \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{u_j} < +\infty,
\end{cases}
\end{align*}

for $i = 1, \ldots, k$, where $C = (c_{ij})_{i,j=1,\ldots,k}$ is the Cartan matrix given by

\[
C = \begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 \\
0 & -1 & 2 & -1 & 0 & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2
\end{pmatrix}.
\]

Since we are considering a problem in the plane, the exponential non-linearity is the natural equivalent. There is a huge literature about the Toda system; we just recall the classification result of [JW] and the papers [BGG, GGW] where the matrix $C = (\frac{\mu}{2})_{i,j=1,\ldots,k}$ was considered for $\mu \in (-2, 0)$. Observe that if we consider the Toda system with a general symmetric, invertible, irreducible matrix $C$ and assume that all the entries $c_{ij}$ are positive, then (see [CK, CSW, LZ1, LZ2]) all solutions are radial. In analogy with this result, we believe that if all entries of $A$ are positive then all solutions to (1.1) are radial. This conjecture relies on the fact that the assertion is true for some cooperative systems involving the critical exponent as the one considered in [DH] (see Proposition 3.1) and the one considered in [CSW], see Theorem 1.6. We do not investigate this problem in this paper (even if we think that this problem deserves to be studied in the future) and allow some coefficients $a_{ij}$ to be negative. We will see that this causes the solution set to have a richer structure.

### 1.2 Main results and idea of the proof.

A basic remark is that the system (1.1) does not have a variational structure and so we cannot apply variational methods. Moreover, the critical powers induce a lack of compactness.
Our basic tool will be the bifurcation theory. Solutions to (1.1) (necessarily positive) are zeros of the functional \( F = (F_1, \ldots, F_k) : (D^{1,2}(\mathbb{R}^N))^k \to (D^{1,2}(\mathbb{R}^N))^k \) defined by

\[
F_i(u_1, \ldots, u_k) = u_i - (-\Delta)^{-1} \left( \sum_{j=1}^{k} a_{ij}(u_j^+) \frac{N-2}{N} \right)
\]

for \( i = 1, \ldots, k \). Of course we have that \( F_i(U, \ldots, U) = 0 \) and our aim is to find solutions close to \((U, \ldots, U)\) for suitable values of \((a_{ij})\). We will use the classical Crandall–Rabinowitz theorem \([CR]\). Its application requires three basic ingredients:

(i) a good functional setting for the operator \( F \),
(ii) a 1-dimensional kernel for the linearized operator \( F' \),
(iii) a transversality condition.

The lack of compactness and the rich structure of the kernel of the linearized operator (see Proposition 1.2 below) make conditions (i) and (ii) not easy to check. (Condition (iii) will be a straightforward computation involving the Jacobi polynomials.) Now we discuss the main points and the difficulties to be overcome in (i) and (ii).

Let us start with the functional setting.

First of all, let us note that it is not immediate to derive that our solutions are positive. Indeed, since some of the entries \( a_{ij} \) are not necessarily positive, we cannot apply the maximum principle. And even if they were, \( F \) defined in (1.8) is not smooth enough because \( D^{1,2} \to D^{1,2} : u \mapsto u^+ \) is not differentiable. This problem will be solved by restricting our operator to the subspace \( X \) of \( D^{1,2}(\mathbb{R}^N) \) of functions decaying at least as \( |x|^{2-N} \) at infinity (see Section 3.1 for a precise definition of \( X \)). This choice, if from one side it will allow us to establish the positivity of the solution, on the other hand it creates problems to prove the compactness of the linearized operator. This will be discussed in Section 2.2.

Now we discuss the point (ii), i.e., the linearization of \( F \) around the trivial solution \((U, \ldots, U)\). This leads to study the problem,

\[
\begin{cases}
-\Delta v_i = \frac{N(N+2)}{(1+|x|^2)^2} \sum_{j=1}^{k} a_{ij} v_j & \text{in } \mathbb{R}^N, \\
v_i \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]

for \( i = 1, \ldots, k \). It will be shown that (1.9) can be reduced to the classification of eigenvalues and eigenfunctions of the linearized problem associated to the critical
equation (1.4) at the standard bubble $U$, namely,

$$
\begin{cases}
-\Delta w = \lambda \frac{N(N+2)}{(1+|x|^2)^{\frac{N}{2}}} w & \text{in } \mathbb{R}^N, \\
w \in D^{1,2}(\mathbb{R}^N).
\end{cases}
$$

(1.10)

It is known that $\lambda_0 = \frac{N-2}{N+2}$ and $\lambda_1 = 1$ but nothing is known about the other eigenvalues. Our first result completely describes problem (1.10). We believe that this result has its own independent interest.

**Theorem 1.1.** The eigenvalues of Problem (1.10) are the numbers

$$
\lambda_n = \frac{(2n + N - 2)(2n + N)}{N(N + 2)}, \quad n \geq 0.
$$

(1.11)

Each eigenvalue $\lambda_n$ has multiplicity

$$
m(\lambda_n) = \frac{(N + 2n - 1)(N + n - 2)!}{(N - 1)! n!}
$$

(1.12)

and the corresponding eigenfunctions are, in spherical coordinates, linear combinations of

$$
w_{n,h}(r, \theta) = \frac{r^h}{(1 + r^2)^{h + \frac{N-2}{2}}} P^{(h+\frac{N-2}{2}, h+\frac{N+2}{2})}_{n-h} \left( \frac{1 - r^2}{1 + r^2} \right) Y_h(\theta)
$$

(1.13)

for $h = 0, \ldots, n$, where $Y_h(\theta)$ are spherical harmonics related to the eigenvalue $h(h + N - 2)$ and $P_j^{(\beta, \gamma)}$ are the Jacobi polynomials.

This result extends Theorem 11.1 in [GG] where the linearized problem of the Liouville equation in $\mathbb{R}^2$ at the standard bubble was considered and highlights the role of the Jacobi polynomials as extension of the Legendre polynomials.

Theorem 1.1 will be used to describe all solutions to (1.9) thanks to a change of variables to diagonalize $A$. Since $A$ is symmetric, it possesses $k$ real eigenvalues $\Lambda_1, \ldots, \Lambda_k$, counting algebraic multiplicity. Assumption (1.3) implies that 1 is always an eigenvalue of $A$ with (at least) the eigenvectors spanned by $(1, \ldots, 1)$. So, without loss of generality, we can set $\Lambda_1 = 1$. We have the following result,

**Proposition 1.2.** Equation (1.9) possesses a solution

$$v = (v_1, \ldots, v_k) \neq (0, \ldots, 0)
$$

if and only if

$$
\Lambda_i = \lambda_n := \frac{(2n + N - 2)(2n + N)}{N(N + 2)}
$$

(1.14)
for some \( i \in \{1, \ldots, k\} \) and \( n \in \mathbb{N} \). The solutions coming from (1.14) are given by
\[
\nu = \sum_{h=0}^{\infty} c_h w_{n,h},
\]
where \( c_h \in \mathbb{R}^k \) satisfy \( A c_h = \lambda_n c_h \) and \( w_{n,h} \) are defined by (1.13). If several equalities of the form (1.14) hold at the same time, the set of solutions to (1.9) is the linear span of the associated solutions of the form (1.15).

In particular, one always has \( \Lambda_1 = 1 = \lambda_1 \) and the corresponding solutions are given by
\[
\nu = c_0 \left( x \cdot \nabla U + \frac{N - 2}{2} U \right) + \sum_{i=1}^{N} c_i \frac{\partial U}{\partial x_i}
\]
for some \( c_0, c_1, \ldots, c_N \in \mathbb{R}^k \) such that \( A c_i = c_i \) for all \( i \in \{0, 1, \ldots, N\} \).

To apply the Crandall–Rabinowitz theorem, let us consider a \( C^1 \)-path of matrices
\[
I \subseteq \mathbb{R} \to \mathbb{R}^{k \times k} : \alpha \mapsto A(\alpha)
\]
such that, for all \( \alpha \in I \), \( A(\alpha) \) satisfies (1.2)–(1.3). How to deal with more general situations involving two or more parameters can be found, for example, in the book [K]. Let \( \Lambda_1(\alpha) = 1 \) and \( \Lambda_2(\alpha), \ldots, \Lambda_k(\alpha) \) be the eigenvalues of \( A(\alpha) \). The previous result shows that the linearized system (1.9) has two types of degeneracies. The first one, which holds for every value of \( \alpha \), is due to the invariance of the problem (1.1) under dilations and translations, while the second one appears only at the special values \( \alpha \) that satisfy (1.14).

Note that, if \( n = 0 \) (resp. \( n = 1 \)) in (1.14), i.e., if \( \Lambda_i(\alpha) = \frac{N - 2}{N + 2} \) (resp. \( \Lambda_i(\alpha) = 1 \)), then \( \nu = c U \) (resp. \( \nu = c_0(x \cdot \nabla U + \frac{N - 2}{2} U) + \sum c_j \frac{\partial U}{\partial x_j} \)) are solutions. These are the trivial values of \( \alpha \) and they do not provide new solutions to problem (1.1).

Thus (1.14) with \( n \geq 2 \) is a necessary condition to guarantee bifurcating branches of solutions. A similar phenomenon was previously observed in [GGW] for a general \( 2 \times 2 \) Toda system in \( \mathbb{R}^2 \). Coming back to Problem (1.6), we have \( \Lambda_1 = 1 \) and \( \Lambda_2 = -1 \), so (1.14) is never satisfied if \( n \neq 1 \). Hence the solution \( (U, U) \) is non-degenerate (up to dilation and translation).

Proposition 1.2 says that the kernel of the linearized operator is composed by both radial and non-radial eigenfunctions. Of course this is a great obstruction to applying the Crandall–Rabinowitz Theorem for which a one-dimensional kernel is required. For this reason we restrict the problem to the case of radial solutions in \( D^{1,2}(\mathbb{R}^N) \), i.e., we work in the space \( D^{1,2}_{rad}(\mathbb{R}^N) \). However, the existence of non-radial eigenfunctions implies a non-radial bifurcation from the trivial solution. This problem has been investigated later in the paper [GGT].
In the radial setting (see Corollary 2.2), the kernel of the linearized operator (1.9) has a lower dimension. Its dimension however depends on $A$. Here we summarize some sufficient conditions on the matrix $A$ such that, in the radial setting, the kernel of the linearized operator is two-dimensional.

**Assumptions on the matrix $A$.** Let us suppose that $A$ satisfies (1.2) and (1.3). Moreover, assume that there exist $\bar{\alpha}$ and $\bar{\mathfrak{i}} \in \{2, \ldots, k\}$ such that

$$
\Lambda_{i}(\bar{\alpha}) = \lambda_n \quad \text{for some } n \geq 2 \text{ (see (1.14) for the definition of } \lambda_n),
$$

$$
\Lambda_{j}(\bar{\alpha}) \neq \lambda_n \quad \text{for any } j \notin \{1, \bar{\mathfrak{i}}\} \text{ and any } n \in \mathbb{N},
$$

$$
\frac{d\Lambda_{\bar{\mathfrak{i}}}}{d\alpha}(\bar{\alpha}) \neq 0.
$$

Some comments on the previous assumptions: (1.18) implies that $\Lambda_{\bar{\mathfrak{i}}}(\bar{\alpha})$ is simple and then the function $\alpha \mapsto \Lambda_{\mathfrak{i}}(\alpha)$ is smooth in a neighborhood of $\bar{\alpha}$ (see [S] for example). Assumption (1.19) can be reformulated in terms of $\frac{dA}{d\alpha}$. Indeed, if $e_n \neq 0$ is a unit eigenvector of $A(\bar{\alpha})$ for the eigenvalue $\lambda_n$, it is not difficult to show that

$$
eq n \cdot \frac{dA}{d\alpha}(\bar{\alpha})e_n = \frac{d\Lambda_{\bar{\mathfrak{i}}}}{d\alpha}(\bar{\alpha}).
$$

Assumption (1.18) also implies that the kernel of the linearized operator, in this radial setting, is two-dimensional and (1.19) gives the transversality condition of Crandall and Rabinowitz; see [CR]. Finally, constraining further our operator to the orthogonal space with the radial function

$$
W(|x|) = \frac{1 - |x|^2}{(1 + |x|^2)^{\mathfrak{n}/2}},
$$

we get a one-dimensional kernel and so Crandall–Rabinowitz Theorem applies. Then the point $(\bar{\alpha}, U, \ldots, U)$ is a bifurcation point when $\bar{\alpha}$ satisfies $\Lambda_{\bar{\mathfrak{i}}}(\bar{\alpha}) = \lambda_n$ for some $\bar{\mathfrak{i}} = 2, \ldots, k$ and some $n \geq 2$. However, this construction produces a Lagrange multiplier for the equations satisfied by the zeros of $F$.

The last step is to show that this Lagrange multiplier is 0. In our opinion this is one of the interesting points of the paper and it will be done using a suitable version of the Pohozaev identity.

The Pohozaev identity was used by many authors dealing with systems of just two equations. The extension to the case of more equations is not straightforward and requires the additional assumption of the invertibility of the matrix $A$ (see Section 3.3). Now we are in position to state our bifurcation result.
\textbf{Theorem 1.3.} If \( A \) satisfies (1.2) and (1.3) and if \( \bar{a} \) and \( \bar{r} \) verify the assumptions (1.17)--(1.19) and if \( \bar{\alpha} \) and \( \bar{u} \) verify the assumptions (1.17)--(1.19) and if \( \text{the matrix } A(\bar{\alpha}) \text{ is invertible}, \)

then the point \( (\bar{\alpha}, U, \ldots, U) \) is a radial bifurcation point for the curve of trivial solutions \( \alpha \mapsto (\alpha, U, \ldots, U) \) to equation (1.1). More precisely, there exists a continuously differentiable curve defined for \( \varepsilon \) small enough,

\[(\varepsilon_0, \varepsilon_0) \to \mathbb{R} \times (D^{1,2}_{\text{rad}}(\mathbb{R}^N))^k : \varepsilon \mapsto (\alpha(\varepsilon), u_1(\varepsilon), \ldots, u_k(\varepsilon)),\]

emanating from \( (\bar{\alpha}, U, \ldots, U), \) i.e., \( (\alpha(0), u_1(0), \ldots, u_k(0)) = (\bar{\alpha}, U, \ldots, U) \), such that, for every \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \), \( u(\varepsilon) = (u_1(\varepsilon), \ldots, u_k(\varepsilon)) \) is a radial solution to

\[
\begin{cases}
-\Delta u_i = \sum_{j=1}^{k} a_{ij}(\alpha) \frac{u_j}{\bar{u}^2} & \text{in } \mathbb{R}^N, \\
 u_i > 0 & \text{in } \mathbb{R}^N, \\
 u_i \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\]

with \( \alpha = \alpha(\varepsilon) \). Moreover

\[(1.24) \quad u(\varepsilon) = (1, \ldots, 1) U + \varepsilon e_n W_n(|x|) + \varepsilon \varphi(\varepsilon|x|),\]

where \( e_n \) is an eigenvector of \( A \) for the eigenvalue \( \Lambda_1(\bar{\alpha}) = \lambda_n \), \( W_n \) is the function defined in (2.10), and \( \varphi(\varepsilon|x|) \) is a uniformly bounded function in \( (D^{1,2}(\mathbb{R}^N))^k \) such that \( \varphi_0 = 0 \).

Now let us discuss the case \( k = 2 \). Here the number of degrees of freedom is \((k-1)k/2 = 1\) and the matrix \( A \) depends on a single parameter: \( A = \left( \begin{array}{cc} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{array} \right) \). Its eigenvalues are given by \( \Lambda_1 = 1, \Lambda_2 = 2\alpha - 1 \). It is easily seen that (1.17)--(1.19) are verified with \( \bar{r} = 2 \) and, in view of (1.14), with \( \alpha = \bar{\alpha}_n \) satisfying

\[(1.25) \quad \alpha_n = \frac{2n^2 + 2Nn - 2n + N^2}{N(N + 2)},\]

so that the degeneracy occurs at a sequence of values \( \alpha_n \) such that \( \alpha_n \to +\infty \) as \( n \to +\infty \). Note that \( \bar{\alpha}_n \neq \frac{1}{2} \) for any \( n \in \mathbb{N} \), which implies (1.21). Hence Theorem 1.3 holds at the values \( \alpha_n \) without additional assumptions and it becomes

\textbf{Theorem 1.4.} If \( A = \left( \begin{array}{cc} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{array} \right) \), then, for any \( n \geq 2 \), the points \( (\bar{\alpha}_n, U, U) \) are radial bifurcation points for the curve of trivial solutions \( (\alpha, U, U) \) to equation (1.1). More precisely, there exists a continuously differentiable curve defined for \( \varepsilon \) small enough,

\[(-\varepsilon_0, \varepsilon_0) \to \mathbb{R} \times (D^{1,2}_{\text{rad}}(\mathbb{R}^N))^2 : \varepsilon \mapsto (\alpha(\varepsilon), u_1(\varepsilon), u_2(\varepsilon)),\]
passing through $(\bar{\alpha}_n, U, U)$, i.e., $(\alpha(0), u_1(0), u_2(0)) = (\bar{\alpha}_n, U, U)$, such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $u_i(\varepsilon)$ is a radial solution to

$$(1.27) \begin{cases}
-\Delta u_1 = au_1^{\frac{N+2}{N-2}} + (1 - \alpha)u_2^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\
-\Delta u_2 = (1 - \alpha)u_1^{\frac{N+2}{N-2}} + au_2^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N, \\
u_1, u_2 > 0 & \text{in } \mathbb{R}^N, \\
u_1, u_2 \in D^{1,2}(\mathbb{R}^N),
\end{cases}$$

with $\alpha = \alpha(\varepsilon)$. Moreover,

$$(1.28) u_1(\varepsilon) = U + \varepsilon W_n(|x|) + \varepsilon \varphi_{1,\varepsilon}(|x|),$$

$$u_2(\varepsilon) = U - \varepsilon W_n(|x|) + \varepsilon \varphi_{2,\varepsilon}(|x|),$$

where $W_n$ is the function defined in (2.10), and $\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon}$ are functions uniformly bounded in $D^{1,2}(\mathbb{R}^N)$ and such that $\varphi_{s,0} = 0$ for $s = 1, 2$.

The same type of result was proved in [GGW] for the Toda system in $\mathbb{R}^2$.

1.3 Extensions and related problems A first interesting question that arises from Proposition 1.2 concerns the existence of non-radial solutions.

**Question 1.** Are there any non-radial solutions to (1.1) bifurcating from the values $\alpha$ which verify (1.14)? How many?

In analogy with the classification result of Jost–Wang ([JW]), it is possible to think that the number of solutions of (1.1) coincides with that of the linearized operator (see also [WZZ]). In our case, if for example $k = 2$, we would have the existence of at least $\sum_{h=0}^{n} \frac{(N+2h-2)(N+3)}{(N-2)!h!} \frac{(N+2h-2)(N+3)}{(N-2)!h!} n$ distinct solutions.

Another interesting question concerns the shape of the branch of our solutions.

**Question 2.** What about the bifurcation diagram for $\alpha(\varepsilon)$ close to $\alpha$?

This question is quite delicate and the answer seems to depend strongly on $A$. In Appendix A, we carry out the computation of the first derivative of $\alpha(\varepsilon)$ with respect to $\varepsilon$. It is worth noting that if $k = 2$, then $\frac{d\alpha}{d\varepsilon}(0) = 0$ (then nothing can be said about the behaviour of the branch), but if $k > 2$, we can have that $\frac{d\alpha}{d\varepsilon}(0) \neq 0$. In this case, the bifurcation is transcritical.

**Question 3.** What possible extensions may be considered?
Another interesting problem to which one can apply our techniques is given by the Gross–Pitaevskii type systems, namely,

\[
\begin{cases}
-\Delta u_i = (\sum_{j=1}^k a_{ij}u_j^2)^{\frac{2}{N-2}}u_i & \text{in } \mathbb{R}^N, \\
u_i > 0 & \text{in } \mathbb{R}^N, \\
u_i \in D^{1,2}(\mathbb{R}^N).
\end{cases}
\]

When \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), this problem was studied in [DH] as the limit problem for the blowing up solution on Riemannian manifolds. The authors proved that only the trivial solution \((U, \ldots, U)\) exists.

The paper is organized as follows. In Section 2, we prove Theorem 1.1 and we study the linearization of our system at the trivial solution. In Section 3 we define the functional setting, we apply the Crandall–Rabinowitz result, and we prove the Pohozaev identity getting our bifurcation result, Theorem 1.3. Finally in the appendix, we give some examples of possible behaviour of the branches.

## 2 Linearization at the standard bubble

### 2.1 The case of a single equation.

Let us consider the critical equation (1.4) and the associated eigenvalue problem (1.10). It is well-known that the first eigenvalue to (1.10) is given by \( \lambda_0 = \frac{N-2}{N+2} < 1 \) and the corresponding eigenfunction is \( U \), while the second eigenvalue is \( \lambda_1 = 1 \) and it has an \((N+1)\)-dimensional kernel spanned by \( \frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_N}, x \cdot \nabla U + \frac{N-2}{2} U \) (see for example [BE] or [AGAP]). In this section, we compute all eigenvalues and corresponding eigenfunctions of (1.10).

**Proof of Theorem 1.1.** Let us consider the stereographic projection \( \Pi : S^N \rightarrow \mathbb{R}^N \) and define \( \Phi : S^N \rightarrow \mathbb{R}^N : y \mapsto \Phi(y) \) as

\[
\Phi(y) = \left( \frac{\omega(\Pi(y))}{(2 + |\Pi(y)|^2)^{\frac{N}{2}}} \right).
\]

Using the fact that, if \( \omega \) satisfies \( -\Delta_{\mathbb{R}^N} \omega = F(x, \omega) \), then

\[
-\Delta_{S^N} \Phi + \frac{N(N-2)}{4} \Phi = -\left( \frac{2}{1 + |\Pi(y)|^2} \right)^{\frac{N-2}{4}} F \left( \Pi(y), \Phi(y) \left( \frac{2}{1 + |\Pi(y)|^2} \right)^{\frac{N-2}{4}} \right),
\]

one deduces that, if \( \omega \) is a non-trivial solution to (1.10), then

\[
-\Delta_{S^N} \Phi = \left( \lambda \frac{N(N+2)}{4} - \frac{N(N-2)}{4} \right) \Phi,
\]

i.e., \( \Phi \) is an eigenfunction of the Laplace–Beltrami operator on \( S^N \). Consequently the eigenvalue must be \( \mu_n := n(N-1+n) \) for some \( n \in \mathbb{N} \), that is

\[
\lambda \frac{N(N+2)}{4} - \frac{N(N-2)}{4} = n(N-1+n),
\]
which proves (1.11). Our next aim is to show that a basis of the eigenfunctions is given by (1.13). From (2.1), we get

$$w(x) = \left(\frac{2}{1 + |x|^2}\right)^{\frac{n-2}{2}} \Phi(\Pi^{-1}(x)), \quad x \in \mathbb{R}^N.$$  

Let us parametrize $S^N$ with cylindrical coordinates: consider the map

$$S^{N-1} \times [-1, 1] \to S^N : (\theta, z) \mapsto (\theta \sqrt{1 - z^2}, z) \in \mathbb{R}^{N+1}.$$  

In these coordinates, the Laplace–Beltrami operator is given by

$$\Delta_{S^N} \Phi = (1 - z^2)\partial_z^2 \Phi - Nz \partial_z \Phi + \frac{1}{1 - z^2} \Delta_{S^{N-1}} \Phi.$$  

Now let us try the ansatz $\Phi(y) = (1 - z^2)^{h/2} \varphi(z) Y_h(\theta)$, where $Y_h$ is a spherical harmonic of degree $h$ on $S^{N-1}$. Thus $-\Delta_{S^{N-1}} Y_h = h(h + N - 2) Y_h$. Using (2.5), a tedious but simple computation shows that $\Delta_{S^N} \Phi = -n(N - 1 + n) \Phi$ holds if and only if $\varphi$ satisfies

$$(1 - z^2)\varphi'' + [\gamma - \beta - (2 + \beta + \gamma)z] \varphi' + m(1 + \beta + \gamma + m) \varphi = 0,$$

This is a particular case of the Jacobi equation

$$(1 - z^2)\varphi'' + [\gamma - \beta - (2 + \beta + \gamma)z] \varphi' + m(1 + \beta + \gamma + m) \varphi = 0$$

with $\beta = \gamma = h + \frac{N-2}{2}$ and $m = n - h$. Thus

$$\Phi(y) = (1 - z^2)^{h/2} P^{(h+\frac{N-2}{2}, h+\frac{N-2}{2})}_{\beta, \gamma}(z) Y_h(\theta), \quad h = 0, \ldots, n,$$

are eigenfunctions of $-\Delta_{S^N}$ with eigenvalue $n(N - 1 + n)$. It is easy to see that they are all linearly independent. Denote $\mathcal{Y}^N_h$ the space of spherical harmonics on $S^{N-1}$ of degree $h$. A simple proof by induction shows that

$$\frac{(N + 2n - 1)(N + n - 2)!}{(N - 1)! n!} = \dim \mathcal{Y}^{N+1}_n = \sum_{h=0}^n \dim \mathcal{Y}^N_h = \sum_{h=0}^n \frac{(N + 2h - 2)(N + h - 3)!}{(N - 2)! h!},$$

which implies that the functions given by (2.6) (varying $h$ and $Y_h$) form a basis of the eigenfunctions of $-\Delta_{S^N}$. To deduce Formula (1.13), it suffices to use (2.4), (2.6), to recall that the Jacobi polynomials $P^{(\beta, \gamma)}_j$ are even or odd (depending on $j$), and to remark that $\Pi(\theta \sqrt{1 - z^2}, z) = \theta \sqrt{\frac{1 + z}{1 - z}}$, so $\Pi^{-1}(r \theta) = (\theta \sqrt{1 - z^2}, z)$ implies $z = \frac{r^2 - 1}{r^2 + 1}$ and $\sqrt{1 - z^2} = \frac{2r}{r^2 + 1}$. \hfill \Box
Remark 2.1. Note that Theorem 1.1 also holds when \( N = 2 \). In this case we have that \( \lambda_n = \frac{n(n+1)}{2} \). These are exactly the eigenvalues associated to the linearization of the classical Liouville problem
\[
\begin{aligned}
\begin{cases}
-\Delta U = e^U & \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} e^U \, dx < \infty
\end{cases}
\end{aligned}
\]
at the standard bubble
\[
U_{\mathbb{R}^2}(x) = \log \frac{64}{(8+|x|^2)^2}.
\]
This result was proved in [GG] (see Theorem 11.1) and the corresponding eigenfunctions are spanned by \((P_L)_{n-h}^{h}(\frac{8-2r^2}{8+r^2})Y_h(\theta)\), where \((P_L)\) are the Legendre polynomials and \(Y_h(\theta)\) are the spherical harmonics in \(\mathbb{R}^2\). Observe that for \( N = 2 \) the Jacobi polynomials \(P_{n-h}^{\frac{N-2}{2},\frac{N+2}{2}}(1-r^2)\) in (1.13) become the Legendre polynomials and so Theorem 1.1 contains also the result of [GG] for \(\mathbb{R}^2\).

2.2 The case of the system. Now we are in position to prove Proposition 1.2 in the Introduction.

Proof of Proposition 1.2. Because \( A \) is symmetric, there exists an orthogonal matrix \( B = (b_{ij}) \) such that
\[
B^{-1}AB = \Lambda,
\]
where \( \Lambda \) is the diagonal matrix with the eigenvalues \((\Lambda_1, \ldots, \Lambda_k)\) as diagonal elements. Let \( w_i := \sum_{j=1}^{k} b_{ij}^{-1} v_j = \sum_{j=1}^{k} b_{ji} v_j \). Then \( w = (w_1, \ldots, w_k) \) is a solution to
\[
\begin{aligned}
\forall i = 1, \ldots, k, \quad \begin{cases}
-\Delta w_i = \frac{N(N+2)}{(1+|x|^2)^2} \Lambda_i w_i & \text{in } \mathbb{R}^N, \\
w_i \in D^{1,2}(\mathbb{R}^N).
\end{cases}
\end{aligned}
\]
By our choice of the matrix \( B \) in (2.7) the \( k \) equations in (2.8) are decoupled and so we can solve them independently. Remember that we have set \( \Lambda_1 = 1 \). So the first equation of (2.8) reads
\[
-\Delta w_1 = \frac{N(N+2)}{(1+|x|^2)^2} w_1 \quad \text{in } \mathbb{R}^N
\]
and it is well-known that it admits a non-trivial solution which is a linear combination of \( \frac{\partial U}{\partial x_i} \) for \( i = 1, \ldots, N \) and \( x \cdot \nabla U + \frac{N-2}{2} U \). It is important to observe that equation (2.9) does not depend on \( \alpha \) and so system (2.8) has the solution in (1.16) for every value of \( \alpha \).
The other equations in (2.8) have a non-trivial solution if and only if \( \Lambda_i \) is an eigenvalue of problem (1.10), i.e., using Theorem 1.1 if and only if (1.14) is satisfied for some \( i = 2, \ldots, k \), for some \( n \in \mathbb{N} \) and for some value of \( \alpha \in \mathbb{R} \). When (1.14) is satisfied the \( i \)-th equation in system (2.8) has as a solution a linear combination of the eigenfunctions of (1.10) related to the eigenvalue \( \lambda_n \), and so (1.15) follows. □

One of the main hypotheses of the bifurcation result is that the kernel of the linearized operator has to be one-dimensional. From the previous result, we know that the linearized operator has instead a very rich kernel. To overcome this problem we consider only the case of radial solutions in \( D^{1,2}(\mathbb{R}^N) \), that is, we will work in the space \( D^{1,2}_{\text{rad}}(\mathbb{R}^N) \). First we state the result of Proposition 1.2 in this radial setting.

**Corollary 2.2.** Equation (1.9) possesses a radial solution

\[
\nu = (v_1, \ldots, v_k) \neq (0, \ldots, 0)
\]

if and only if \( \Lambda_i = \lambda_n \) for some \( i \in \{1, \ldots, k\} \) and \( n \in \mathbb{N} \), in which case the associated radial solutions are given by \( \nu = cW_n \), where \( c \in \mathbb{R}^k \) is an eigenvector associated to the eigenvalue \( \Lambda_i \) and

\[
W_n(|x|) := \frac{1}{(1 + |x|^2)^{\frac{N-2}{2}}} P_n^{\frac{N-2}{2}, \frac{N-2}{2}}(\frac{1 - |x|^2}{1 + |x|^2}).
\]

Here

\[
P_n^{\frac{N-2}{2}, \frac{N-2}{2}}(\xi) = \sum_{s=0}^{n} \binom{n + \frac{N-2}{2}}{s} \binom{n + \frac{N-2}{2}}{n - s} \left( \frac{\xi - 1}{2} \right)^{n-s} \left( \frac{\xi + 1}{2} \right)^{s}.
\]

If several equalities of the form \( \Lambda_i = \lambda_n \) hold at the same time, the set of radial solutions to (1.9) is the linear span of the associated solutions of the form (2.10).

In particular, if \( \Lambda_1 = 1 \) is a simple eigenvalue of \( A \) (which is the case under assumption (1.18)), all corresponding radial solutions to equation (1.9) are given by multiples of

\[
(1, \ldots, 1) W \quad \text{where} \quad W(|x|) := x \cdot \nabla U + \frac{N - 2}{2} U = d \frac{1 - |x|^2}{(1 + |x|^2)^{N/2}}
\]

and \( d = \frac{1}{2} N^{(N-2)/4} (N - 2)^{(N+2)/4} \).

**Proof.** Restricting to the radial setting we have that equation (2.9) admits only the solution in (2.11). From Theorem 1.1 it instead follows that any eigenvalue \( \lambda_n \) of the critical problem admits one radial solution which is the one in (1.13) that corresponds to \( h = 0 \). Then (2.10) follows from (1.15). □
3 The bifurcation result

3.1 The functional setting. As mentioned in the introduction, the proof of our bifurcation result requires an appropriate functional setting which is a delicate part of the proof.

Both for the lack of differentiability of \( u \mapsto u^+ \) and the difficulty of proving that the solution \((u_1, \ldots, u_k)\) is positive, we need to restrict to a subset of \( D^{1,2}(\mathbb{R}^N) \) with a stronger topology. As before let \( D^{1,2}_{\text{rad}}(\mathbb{R}^N) = \{ u \in D^{1,2}(\mathbb{R}^N) | u = u(|x|) \} \) and set
\[
D := \left\{ u \in L^\infty(\mathbb{R}^N) \mid \sup_{x \in \mathbb{R}^N} \frac{|u(x)|}{U(x)} < +\infty \right\}
\]
endowed with the norm \( \|u\|_D := \sup_{x \in \mathbb{R}^N} \frac{|u(x)|}{U(x)} \) and we define
\[
X = D^{1,2}_{\text{rad}}(\mathbb{R}^N) \cap D.
\]
Then \( X \) is a Banach space equipped with the norm \( \|u\|_X := \max\{ \|u\|_{1,2}, \|u\|_D \} \) where \( \|u\|_{1,2} \) is the classical norm on \( D^{1,2}(\mathbb{R}^N) \).

To rule out the degeneracy due to the invariance under dilations of Problem (1.1), we will solve the linearized equation in the subspace of functions that are orthogonal in \( (D^{1,2}_{\text{rad}}(\mathbb{R}^N))^k \) to \((1, \ldots, 1)W(|x|)\) defined in (2.11). Let \( P_K \) be the orthogonal projection (with respect to the inner product of \( (D^{1,2}_{\text{rad}}(\mathbb{R}^N))^k \)) from \( X^k \) onto the subspace \( K \) given by
\[
K := \left\{ g \in X^k \mid \sum_{i=1}^k \int_{\mathbb{R}^N} \nabla W \cdot \nabla g_i(x) \, dx = 0 \right\}.
\]

**Definition 3.1.** Let us denote \( B := \{ u \in X | \|u - U\|_X < \frac{1}{2} \} \) and define the operator
\[
T : \mathbb{R} \times (K \cap B^k) \to K
\]
as
\[
T(\alpha, u_1, \ldots, u_k) := P_K \begin{pmatrix}
(1 + \Delta)^{-1} \sum_{j=1}^k a_{ij}(\alpha) u_j^{2^* - 1} \\
\vdots \\
(1 + \Delta)^{-1} \sum_{j=1}^k a_{kj}(\alpha) u_j^{2^* - 1}
\end{pmatrix}.
\]
Note that, since \( u_i \in B, u_i = U + (u_i - U) > \frac{1}{2}U \) is positive so that \( u_i^{2^* - 1} \) is well-defined for any \( N \geq 3 \).

The zeros of the operator \( T \) satisfy
\[
-\Delta u_i = \sum_{j=1}^k a_{ij}(\alpha) u_j^{2^* - 1} + L \frac{N(N+2)}{(1+|x|^2)^N} W \quad \text{in } \mathbb{R}^N, \text{ for } i = 1, \ldots, k,
\]
\[
\text{for } u = (u_1, u_2, \ldots, u_k) \in K \cap B^k,
\]
where $L = L(u) \in \mathbb{R}$ is a Lagrange multiplier. Once we prove the existence of 
$(u_1, \ldots, u_k)$ that satisfies (3.3), the final step will be to show that $L = 0$ so that $u$ is
indeed a solution to (1.1) and this will be done in the next section using a Pohozaev
identity.

First we prove some properties of the operator $T$.

**Lemma 3.2.** The projector $P_K : X \to X$ is well-defined and continuous.

**Proof.** Let $W^* := W/\|W\|_{1,2}$. One can write

$$P_K u = u - (1, \ldots, 1) W^* \sum_{i=1}^k (u_i | W^* )_{D^{1,2}}.$$ 

From (2.11), one easily shows that there exists a $C \in \mathbb{R}$ such that $|W^*| \leq CU$ and
so $\|W^*\|_X < +\infty$. The statement readily follows from these facts. \hfill \Box

**Lemma 3.3.** The operator $T$ in (3.2) is continuous from $\mathbb{R} \times (K \cap B^k)$ into $K$
and its derivatives $\partial_\alpha T$, $\partial_\alpha T$ and $\partial_{\alpha u} T$ exist and are continuous.

**Proof.** Since $u_j$ belongs to $X$, we have that $u_j^{2^* - 1} \in L^{2^{*\prime}}_{\text{rad}}(\mathbb{R}^N)$ for any
$j = 1, \ldots, k$. As a consequence, there exists a unique $g_i \in D^{1,2}_{\text{rad}}(\mathbb{R}^N)$ for
$i = 1, \ldots, k$ such that $g_i$ is a weak solution to $-\Delta g_i = f_i$ in $\mathbb{R}^N$ where

$$f_i := \sum_{j=1}^k a_{ij}(\alpha) u_j^{2^* - 1}. \quad (3.4)$$

The solution $g_i$ enjoys the following representation:

$$g_i(x) = \frac{1}{\omega_N(N-2)} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} f_i(y) \, dy,$$

where $\omega_N$ is the area of the unit sphere in $\mathbb{R}^N$. By assumption $u_i \in B$ and this
implies that $|f_i(x)| \leq CU^{2^* - 1}(x)$, so that

$$|g_i(x)| \leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} U^{2^* - 1}(y) \, dy = CU(x)$$

and $g_i \in X$. (Different occurrences of $C$ may denote different constants.) To prove
the continuity of $T$ in $K \cap B^k$, let $\alpha_n \to \alpha$ in $\mathbb{R}$ and $u_{i,n} \to u_i$ in $X$ (for $i = 1, \ldots, k$)
as $n \to \infty$, and set

$$g_{i,n} := (-\Delta)^{-1} f_{i,n} \quad \text{where} \quad f_{i,n} := \sum_{j=1}^k a_{ij}(\alpha_n) u_j^{2^* - 1}. \quad \text{Since} \quad u_{i,n} \to u_i \in D^{1,2}(\mathbb{R}^N), \quad \text{the convergence also holds in} \quad L^{2^*}(\mathbb{R}^N). \quad \text{Using}
\text{Lebesgue’s dominated convergence theorem and its converse, one deduces that} \quad f_{i,n} \to f_i \in L^{2^{*\prime}} \quad \text{where} \quad f_i \text{ is defined as in (3.4). Therefore} \quad g_{i,n} \to g_i \in D^{1,2} \text{ and} \quad T(\alpha_n, u_n) \to T(\alpha, u) \text{ in} \ D^{1,2}.
Now let us show the convergence in $D$. We have that
\[
\frac{|g_{i,n}(x) - g_i(x)|}{U(x)} \leq \frac{1}{\omega_N(N - 2)U(x)} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}} \frac{|f_{i,n}(y) - f_i(y)|}{U(y)^{2'}} dy
\]
\[
\leq C \sup_{y \in \mathbb{R}^N} \frac{|f_{i,n}(y) - f_i(y)|}{U(y)^{2'}}.
\]

Moreover,
\[
\sup_{y \in \mathbb{R}^N} \frac{|f_{i,n}(y) - f_i(y)|}{U(y)^{2'}} \leq \sum_{j=1}^k |a_{ij}(\alpha_n) - a_{ij}(\alpha)| \sup_{y \in \mathbb{R}^N} \left( \frac{|u_j(y)|}{U(y)} \right)^{2' - 1}
\]
\[
+ \sum_{j=1}^k |a_{ij}(\alpha_n)| \sup_{y \in \mathbb{R}^N} \left( \frac{|u_j(y)|}{U(y)} \right)^{2' - 1} - \left( \frac{|u_j(y)|}{U(y)} \right)^{2' - 1}.
\]

The first term goes to 0 because $a_{ij}(\alpha_n) \to a_{ij}(\alpha)$ and $|u_j| \leq CU$. As $(a_{ij}(\alpha_n))_n$ are bounded sequences, it is enough to show that the last factor goes to 0. This is the case because, thanks to the convergence in $D$, $u_{j,n}/U \to u_j/U$ uniformly for all $j$ and the map $\zeta \mapsto \zeta^{2' - 1}$ is continuous.

The existence of $\partial_{\alpha} T$, $\partial_{u} T$ and $\partial_{aa} T$ (for the topology of $X$) and their continuity follows in a very similar way and we omit it.

Next we show a compactness result for the operator $w \mapsto (-\Delta)^{-1}\left( \frac{w}{1 + |x|^2} \right)$.

**Lemma 3.4.** If $0 < p < N$ and $h$ is a non-negative, radial function belonging to $L^1(\mathbb{R}^N)$, then
\[
\int_{\mathbb{R}^N} \frac{h(y)}{|x - y|^p} dy = O\left( \frac{1}{|x|^p} \right) \quad \text{as } |x| \to +\infty.
\]

The general statement of this lemma, which also applies in a non-radial setting, can be found in [ST]. Here we report only the radial version.

Now we can prove our compactness result:

**Lemma 3.5.** The operator
\[
M(w) := (-\Delta)^{-1}\left( \frac{w}{1 + |x|^2} \right)
\]
is compact from $X$ to $X$.

**Proof.** First of all, let us show that $M$ is well-defined. If $w \in X$, then $|w| \leq \|w\|_{D} U$ so that
\[
\frac{|w|}{(1 + |x|^2)^2} \leq C \|w\|_{D} U^{n+2} \in L^{n+2}(\mathbb{R}^N)
\]
and, using the fact that \((-\Delta)^{-1} : L^{\frac{2N}{N+2}} \to D^{1,2}\), one gets \(M(w) \in D^{1,2}\). Moreover

\[
|M(w)| \leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{|w(y)|}{(1+|y|^2)^2} \, dy \leq C\|w\|_D \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{2}}(y)}{|x-y|^{N-2}}
\]

\[
= C\|w\|_D U(x),
\]

and so \(M(w) \in D\). This argument incidentally shows that \(M : X \to X\) is continuous.

Now let \((w_n)\) be a bounded sequence in \(X\) and let us prove that, up to a subsequence, \(g_n := M(w_n)\) converges strongly to some \(g \in X\). On one hand, since \((w_n)\) is bounded in \(D^{1,2}\), going if necessary to a subsequence, one can assume that \((w_n)\) converges weakly to some \(w\) in \(D^{1,2}\) and \(w_n \to w\) almost everywhere. On the other hand, \((\|w_n\|_D)\) is also bounded, which means that \(|w_n| \leq CU\) where \(C\) is independent of \(n\) and so

\[
\frac{|w_n|}{(1+|x|)^2} \leq CU^{\frac{N+2}{2}}.
\]

Lebesgue’s dominated convergence theorem then implies that \(\frac{w_n}{(1+|x|)^2}\) converges strongly to \(\frac{w}{(1+|x|)^2}\) in \(L^{\frac{2N}{N+2}}\). From the continuity of

\[
(-\Delta)^{-1} : L^{\frac{2N}{N+2}} \to D^{1,2},
\]

one concludes that \(g_n \to g\) in \(D^{1,2}\). Moreover, passing to the limit on \(|w_n| \leq CU\) yields \(w \in D\), and passing to the limit on the inequality (3.6) for \(w = w_n\) yields \(g \in D\).

It remains to show that \(\|g_n - g\|_D \to 0\). This is somewhat similar to the argument used in Lemma 3.3. First, the Hölder inequality allows us to get the estimate

\[
|g_n(x) - g(x)| \leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{|w_n(y) - w(y)|}{(1+|y|^2)^2}
\]

\[
= C \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{2} - \varepsilon}(y)}{|x-y|^{N-2}} \frac{|w_n(y) - w(y)|}{U^{1-\varepsilon}(y)}
\]

\[
\leq C \left( \int_{\mathbb{R}^N} \frac{U^{\frac{N+2}{2} - \varepsilon}(y)}{|x-y|^{N-2}} \right)^{\frac{q-1}{q}} \cdot \left( \int_{\mathbb{R}^N} \frac{|w_n(y) - w(y)|}{U^{1-\varepsilon}(y)} \right)^{\frac{1}{q}},
\]

where \(\varepsilon > 0\) will be chosen small and \(q\) large and satisfying \(\varepsilon q \leq \frac{2N}{N-2}\). Because of this latter constraint, the integrand of the right integral is bounded by

\[
\|w_n - w\|_D^q U^{\varepsilon q}(y) \leq CU^{\varepsilon q}(y) \in L^1
\]

where \(C\) is independent of \(n\). Lebesgue’s dominated convergence theorem then implies that this integral converges to 0 as \(n \to \infty\).
The proof will be complete if we show

\[
\int_{\mathbb{R}^N} \frac{|U^{\frac{N+2}{N-2}}(y)|^{\frac{q}{q-1}}}{|x-y|^{N-2}} \, dy \leq \frac{C}{(1+|x|)^{(N-2)\frac{q}{q-1}}} = C U^{\frac{q}{q-1}}.
\]

This inequality follows from Lemma 3.4 because \( h(y) = U^{\frac{N+2}{N-2} - \varepsilon} \) if and only if \( (\frac{N+2}{N-2} - \varepsilon) \frac{q}{q-1} > \frac{N}{N-2} \), which is possible if \( \varepsilon \) is small enough and \( q \) is large enough. \( \square \)

3.2 Application of the Crandall–Rabinowitz Theorem. In this section we will verify the assumptions of the Crandall–Rabinowitz Theorem. Let us recall that by Corollary 2.2, the linearized system (1.9) has the following radial solutions:

(i) \((1, \ldots, 1) W \) (due to the dilation invariance of the problem), for every \( \alpha \),

(ii) \( \eta := e_n W_n(|x|) \), where \( e_n \neq 0 \) satisfies \( A(\bar{\alpha}) e_n = \lambda_n e_n \) (see (2.10)) for \( \bar{\alpha} \) satisfying (1.14).

Notice that \((1, \ldots, 1) \perp e_n \) and so \((1, \ldots, 1) W \perp \eta \) in \( (D^{1,2})^k \), i.e., \( \eta \in K \).

To apply Rabinowitz’ result, we need to verify the assumptions of Theorem 1.7 in [CR]. This is the purpose of the following lemmas.

**Lemma 3.6.** Let \( T \) be as defined in (3.2) and assume that \( \bar{\alpha} \) satisfies (1.17)–(1.18). Then \( \ker(\partial_u T(\bar{\alpha}, U, \ldots, U)) \) is one-dimensional and it is given by

\[
\ker(\partial_u T(\bar{\alpha}, U, \ldots, U)) = \text{span}\{ \eta \} \quad \text{where } \eta = e_n W_n;
\]

\( W_n \) is defined in (2.10), \( e_n \neq 0 \), and \( A(\bar{\alpha}) e_n = \lambda_n e_n \).

**Proof.** Let us consider the Fréchet derivative of \( T \) at \((\alpha, U, \ldots, U)\). We have that

\[
\partial_u T(\alpha, U, \ldots, U) \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = P_K \left( w_i - (-\Delta)^{-1} \left( \sum_{j=1}^k a_{ij}(\alpha) \frac{N(N+2)}{1+|x|^2} w_j \right) \right)_{i=1}^k
\]

so that \( \partial_u T(\bar{\alpha}, U, \ldots, U) \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \) if and only if \((w_1, \ldots, w_k) \in K \) is a solution to

\[
\forall i = 1, \ldots, k, \quad -\Delta w_i = \frac{N(N+2)}{1+|x|^2} \sum_{j=1}^k a_{ij}(\bar{\alpha}) w_j = -L \Delta W \quad \text{in } \mathbb{R}^N,
\]
for some $L = L(\omega) \in \mathbb{R}$. Multiplying by $W$, integrating, and summing up yields

$$\sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} \nabla w_i \cdot \nabla W \, dx - \int_{\mathbb{R}^N} \frac{N(N + 2)}{(1 + |x|^2)^2} \sum_{j=1}^{k} a_{ij}(\tilde{\alpha}) w_j(x)W(x) \, dx \right) = kL \int_{\mathbb{R}^N} |\nabla W|^2 \, dx.$$

Recalling that $-\Delta W = \frac{N(N+2)}{(1+|x|^2)^2} W$ (see Corollary 2.2) and $\sum_i a_{ij} = 1$, one sees that the left-hand side of the equation vanishes and so $L = 0$. Thus, $w = (w_1, \ldots, w_k)$ is a solution to (1.9) and, using again Corollary 2.2 and assumptions (1.17)–(1.18), this is the case if and only if

$$w \in \text{span}\{ (1, \ldots, 1)W, \eta \}.$$

Recalling that $w \in K$, which means that $w$ is orthogonal to $(1, \ldots, 1)W$, and that $\eta \perp (1, \ldots, 1)W$, one concludes that $w$ is a multiple of $\eta$. □

**Lemma 3.7.** Under the assumptions of Lemma 3.6 the range

$$\text{Ran}(\partial_{\alpha}T(\tilde{\alpha}, U, \ldots, U)) \subseteq K$$

has codimension one. It is the set of functions $f = (f_1, \ldots, f_k) \in K$ that are orthogonal to $\eta$ in $(D^{1,2}(\mathbb{R}^N))^k$, that is,

$$(f|\eta) := \sum_{i=1}^{k} e_{n,i} \int_{\mathbb{R}^N} \nabla f_i \cdot \nabla W_n \, dx = 0,$$

where $e_n = (e_{n,i})_{i=1}^{k}$. Hence a complement of $\text{Ran}(\partial_{\alpha}T(\tilde{\alpha}, U, \ldots, U))$ in $K$ is spanned by the vector $\eta$ defined in Lemma 3.6.

**Proof.** This is a consequence of Lemma 3.5. Indeed one can write the operator $\partial_{\alpha}T(\tilde{\alpha}, U, \ldots, U)$ as

$$\partial_{\alpha}T(\tilde{\alpha}, U, \ldots, U)[w] = w - P_K \left( (-\Delta)^{-1} \sum_{j=1}^{k} a_{ij}(\tilde{\alpha}) \frac{N(N + 2)}{(1 + |x|^2)^2} w_j \right)_{i=1}^{k},$$

because $w \in K$, and so is a compact perturbation of the identity. Thus (3.12) follows from the Fredholm Alternative. □

**Lemma 3.8.** Under the assumptions of Lemma 3.6 and (1.19), the operator $T$ satisfies

$$(3.13) \quad \partial_{\alpha\alpha}T(\tilde{\alpha}, U, \ldots, U)[\eta] \notin \text{Ran}(\partial_{\alpha}T(\tilde{\alpha}, U, \ldots, U))$$

where $\eta$ is as defined in (3.9).
**Proof.** The derivative $\partial_u T(\alpha, U, \ldots, U)$ is given by (3.10). Differentiating with respect to $\alpha$ yields

$$\partial_{\alpha u} T(\bar{\alpha}, U, \ldots, U)[\eta] = P_K g,$$

where $g = (g_1, \ldots, g_k)$ and

$$g_i := -(-\Delta)^{-1} \left( \sum_{j=1}^k \partial_{a a_{ij}}(\bar{\alpha}) \frac{N(N + 2)}{(1 + |x|^2)^2} \eta_j \right), \quad i = 1, \ldots, k.$$

In view of Lemma 3.7, we have to show that $(P_K g|\eta) \neq 0.$ Since $\eta \in K,$ $(P_K g|\eta) = (g|\eta).$ Thus, we have to show that

$$(g|\eta) = \sum_{i=1}^k e_{n,i} \int_{\mathbb{R}^N} \nabla \left[ -(-\Delta)^{-1} \left( \sum_{j=1}^k \partial_{a a_{ij}}(\bar{\alpha}) \frac{N(N + 2)}{(1 + |x|^2)^2} \eta_j \right) \right] \cdot \nabla W_n \, dx \neq 0,$$

that is, recalling that $\eta_j = e_{n,j} W_n,$

$$e_n \cdot \frac{dA}{d\alpha}(\bar{\alpha}) e_n \int_{\mathbb{R}^N} \frac{N(N + 2)}{(1 + |x|^2)^2} W_n^2(x) \, dx \neq 0.$$

The proof is complete thanks to assumption (1.19) (see also (1.20)). $\square$

Now we are in position to apply the bifurcation result of [CR]:

**Proposition 3.9.** Assume that $A$ satisfies (1.2) and (1.3). Assume further that there exist $\bar{\alpha}$ and $\bar{T}$ such that (1.17)–(1.19) are satisfied. Then the point $(\bar{\alpha}, U, \ldots, U)$ is a radial bifurcation point for the curve $\alpha \mapsto (\alpha, U, \ldots, U)$ of solutions to (1.1). More precisely, there exist continuous curves $\varepsilon \mapsto \alpha_{\varepsilon}$ and $\varepsilon \mapsto (u_{1,\varepsilon}, \ldots, u_{k,\varepsilon}),$ defined for $\varepsilon \in \mathbb{R}$ small enough, such that $\alpha_0 = \bar{\alpha}, \ u_{0,\varepsilon} = U,$ and

$$-\Delta u_{i,\varepsilon} = \sum_{j=1}^k a_{ij}(\alpha_{\varepsilon}) u_{j,\varepsilon}^{2^* - 1} + L_{\varepsilon} \frac{N(N+2)}{(1+|x|^2)^2} W \quad \text{in } \mathbb{R}^N,$$

$$u_{i,\varepsilon} > 0,$$

for some Lagrange multiplier $L_{\varepsilon}.$ Moreover, for $\varepsilon$ small enough,

$$\left( u_{1,\varepsilon}, \ldots, u_{k,\varepsilon} \right) = (1, \ldots, 1) \ U + \varepsilon e_n W_n(|x|) + \varepsilon \varphi_{\varepsilon}(|x|)$$

where $e_n$ is an eigenvector of $A$ for the eigenvalue $\Lambda_1(\bar{\alpha}) = \lambda_n$ and $\varphi_{\varepsilon}$ is a uniformly bounded function in $(D^{1,2}(\mathbb{R}^N))^k$ and such that $\varphi_0 = 0.$

**Proof.** We apply Theorem 1.7 in [CR] at the operator $T$ defined in (3.2). It is easy to see that $T(\alpha, U, \ldots, U) = 0$ for any $\alpha.$ By Lemma 3.3 the operators $\partial_{a} T,$ $\partial_{u} T$ and $\partial_{a, u} T$ are well-defined and continuous from $\mathbb{R} \times (K \cap B^k)$ to $K.$
Lemma 3.6 says that the kernel of $\partial u_T(\bar{\alpha}, U, \ldots, U)$ is one-dimensional while Lemma 3.7 implies that its range has codimension one. Finally, Lemma 3.8 guarantees that the transversality condition holds. Therefore all assumptions of Theorem 1.7 in [CR] are satisfied. As a consequence, there exists a neighborhood $V$ of $(\bar{\alpha}, U, \ldots, U) \in \mathbb{R} \times (K \cap B^k)$, an interval $(-\varepsilon_0, \varepsilon_0)$, and continuous functions $(-\varepsilon_0, \varepsilon_0) \rightarrow \mathbb{R}: \varepsilon \mapsto \alpha_{\varepsilon}$ and $(-\varepsilon_0, \varepsilon_0) \rightarrow B : \varepsilon \mapsto \phi_{i,\varepsilon}$ for $i = 1, \ldots, k$ such that $\alpha_0 = \bar{\alpha}$, $\phi_{i,0} = 0$ for $i = 1, \ldots, k$ and $T^{-1}(\{0\}) \cap V = \{(\alpha, U, \ldots, U) \in V \mid (\alpha, U, \ldots, U) \in V \} \cup \{(\alpha_\varepsilon, u_{1,\varepsilon}, \ldots, u_{k,\varepsilon}) \mid |\varepsilon| < \varepsilon_0\}$.

where $(u_{1,\varepsilon}, \ldots, u_{k,\varepsilon})$ is defined by (3.15). In particular, $T(\alpha_{\varepsilon}, u_{1,\varepsilon}, \ldots, u_{k,\varepsilon}) = 0$ which means that $(u_{1,\varepsilon}, \ldots, u_{k,\varepsilon})$ solves (3.14). This concludes the proof. □

**Lemma 3.10.** Let $L_\varepsilon$ be the Lagrange multiplier of Proposition 3.9. Then

(3.16) \[ |L_\varepsilon| \leq C. \]

**Proof.** Let us use the function $W$ as test function in the first equation to (3.14). We get

\[ N(N + 2)L_\varepsilon \int_{\mathbb{R}^N} \frac{W^2}{(1 + |x|^2)^2} \, dx = \int_{\mathbb{R}^N} \nabla u_{1,\varepsilon} \cdot \nabla W \, dx - \sum_{s=1}^k a_{1s}(\alpha) \int_{\mathbb{R}^N} u_{s,\varepsilon}^{2^* - 1} W(x) \, dx. \]

The fact that $u_{s,\varepsilon}$ are uniformly bounded in $D^{1,2}(\mathbb{R}^N)$ then implies the claim. □

### 3.3 The Pohozaev identity.

**Proposition 3.11.** Suppose that $(u_i)_{i=1,\ldots,k}$ are positive solutions in $D^{1,2}(\mathbb{R}^N)$ to

(3.17) \[ -\Delta u_i = \sum_{j=1}^k a_{ij} u_j^{2^* - 1} + H_i(x) \quad \text{in} \ \mathbb{R}^N, \]

where $H_i$ are smooth functions satisfying

(3.18) \[ H_i \in L^{2^*}(\mathbb{R}^N), \quad |x|H_i \in L^2(\mathbb{R}^N) \]

and $A$ is an invertible symmetric matrix. Let us write $A^{-1} = (a_{ij}^{-1})_{i,j=1,\ldots,k}$. Then, the following Pohozaev identity holds:

(3.19) \[ 0 = \sum_{i,j=1}^k a_{ij}^{-1} \int_{\mathbb{R}^N} H_i(x) \left( x \cdot \nabla u_j + \frac{N - 2}{2} u_j \right) \, dx. \]
Proof. We will denote by $I_{i,R}$ various boundary terms on $\partial B_R$ such that, for any integer $i$,

$$
|I_{i,R}| \leq C(N)R \int_{\partial B_R} \left( \sum_{i,h=1}^k u_i u_h^{\gamma - 1} + |\nabla u_i \cdot \nabla u_h| \right).
$$

Set

$$
\sum_{i,h=1}^k u_i u_h^{\gamma - 1} + |\nabla u_i \cdot \nabla u_h| =: G(u_1, \ldots, u_k)
$$

and, as in [BL], let us show that there exists a sequence $R_n \to +\infty$ such that $I_{i,R_n} \to 0$. Indeed since $u_i \in D^{1,2}(\mathbb{R}^N)$ we know that $G(u_1, \ldots, u_k) \in L^1(\mathbb{R}^N)$, so that

$$
\int_0^{+\infty} \int_{\partial B_R} G(u_1, \ldots, u_k) \, d\sigma \, dR < +\infty.
$$

Hence, there exists a sequence $R_n \to +\infty$ such that

$$
R_n \int_{\partial B_{R_n}} G(u_1, \ldots, u_k) \, d\sigma \to 0 \quad \text{as } n \to +\infty,
$$

and this shows that $I_{i,R_n} \to 0$. From now, to simplify the notation, we agree that $R = R_n$ and we denote by $C(R)$ a linear combination of $I_{i,R}$.

Let us now start our main argument with the identity:

$$
-\int_{B_R} \Delta u_i(x \cdot \nabla u_i) = \left(1 - \frac{N}{2}\right) \int_{B_R} |\nabla u_i|^2 \, dx - \int_{\partial B_R} (x \cdot \nabla u_i) \frac{\partial u_i}{\partial \nu} \bigg|_{\partial B_R} + \frac{1}{2} \int_{\partial B_R} |\nabla u_i|^2 (x \cdot \nu)
$$

Using the $i$-th equation in (3.17), we get

$$
\left(1 - \frac{N}{2}\right) \int_{B_R} |\nabla u_i|^2 \, dx + C(R) = \sum_{j=1}^k a_{ij} \int_{B_R} u_j^{\frac{N}{N-2}}(x \cdot \nabla u_i) \, dx + \int_{B_R} H_i(x)(x \cdot \nabla u_i) \, dx.
$$

Next we estimate

$$
\int_{B_R} u_j^{\frac{N}{N-2}}(x \cdot \nabla u_i)
$$

$$
= -N \int_{B_R} u_i u_j^{\frac{N}{N-2}} - \frac{N + 2}{N - 2} \int_{B_R} u_i u_j^{\frac{N}{N-2}}(x \cdot \nabla u_j) + \int_{\partial B_R} u_i u_j^{\frac{N}{N-2}}(x \cdot \nu)
$$

which, when $i = j$, simplifies to

$$
\int_{B_R} u_j^{\frac{N}{N-2}}(x \cdot \nabla u_i) = -\frac{N - 2}{2} \int_{B_R} u_i^{\frac{2N}{N-2}} + \frac{N - 2}{2N} R \int_{\partial B_R} u_i^{\frac{N}{N-2}}
$$
Using (3.22) and (3.23) we get

\[
\left(1 - \frac{N}{2}\right) \int_{B_R} |\nabla u_i|^2 \, dx + C(R) + \sum_{j \neq i} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}}
\]

\[
= - a_{ii} \frac{N - 2}{2} \int_{B_R} u_i^{\frac{N+2}{N-2}} - N \sum_{j \neq i} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}}
\]

\[
- \frac{N + 2}{N - 2} \sum_{j \neq i} a_{ij} \int_{B_R} (x \cdot \nabla u_j)u_i u_j^{\frac{N-2}{N-2}} + \int_{B_R} H_i(x)(x \cdot \nabla u_i) \, dx.
\]  
(3.24)

On the other hand, multiplying equation (3.17) by \(u_i\) and integrating yields

\[
\int_{B_R} |\nabla u_i|^2 + C(R) = a_{ii} \int_{B_R} u_i^{\frac{N+2}{N-2}} + \sum_{j \neq i} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} + \int_{B_R} H_i(x)u_i \, dx.
\]  
(3.25)

Summing (3.24) and (3.25) multiplied by \(\frac{N-2}{2}\) gives

\[
\frac{N + 2}{2} \sum_{j \neq i} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} + \frac{N + 2}{N - 2} \sum_{j \neq i} a_{ij} \int_{B_R} (x \cdot \nabla u_j)u_i u_j^{\frac{N-2}{N-2}} + C(R)
\]

\[
= \int_{B_R} H_i(x)(x \cdot \nabla u_i + \frac{N - 2}{2} u_i).
\]  
(3.26)

Setting

\[
A_{ij} := \frac{N + 2}{2} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} + \frac{N + 2}{N - 2} \int_{B_R} (x \cdot \nabla u_j)u_i u_j^{\frac{N-2}{N-2}}
\]

and

\[
B_{ij} := \int_{B_R} H_i(x)(x \cdot \nabla u_j + \frac{N - 2}{2} u_j),
\]

the previous identity becomes

\[
\sum_{j \neq i} a_{ij} A_{ij} = B_{ii} + C(R).
\]  
(3.28)

Now let us use the factor \(x \cdot \nabla u_h\) against \(u_i\). We start with the identity

\[
- \int_{B_R} \Delta u_i(x \cdot \nabla u_h) = \int_{B_R} \nabla u_i \cdot \nabla u_h + \sum_{\ell, m=1}^N \int_{B_R} x_{\ell} \frac{\partial u_i}{\partial x_m} \frac{\partial^2 u_h}{\partial x_{\ell} \partial x_m} - \int_{\partial B_R} \frac{\partial u_i}{\partial \nu}(x \cdot \nabla u_h) \cdot
\]

Using (3.17), one gets

\[
\int_{B_R} \nabla u_i \cdot \nabla u_h + \sum_{\ell, m=1}^N \int_{B_R} x_{\ell} \frac{\partial u_i}{\partial x_m} \frac{\partial^2 u_h}{\partial x_{\ell} \partial x_m} + C(R)
\]

\[
= - N \sum_{j \neq h} a_{ij} \int_{B_R} u_i u_j^{\frac{N+2}{N-2}} - \frac{N + 2}{N - 2} \sum_{j \neq h} a_{ij} \int_{B_R} (x \cdot \nabla u_j)u_h u_j^{\frac{N-2}{N-2}}
\]

\[
- a_{ih} \frac{N - 2}{2} \int_{B_R} u_h^{\frac{N+2}{N-2}} + \int_{B_R} H_i(x)(x \cdot \nabla u_h) \, dx.
\]  
(3.29)
Our intention is to sum (3.29) and the same expression with the indices $i$ and $h$ swapped. Let us start by remarking that

$$\sum_{\ell,m=1}^{N} \int_{B_{R}} x_{\ell} \left( \frac{\partial u_i}{\partial x_{m}} \frac{\partial^2 u_h}{\partial x_{\ell} \partial x_{m}} + \frac{\partial u_h}{\partial x_{m}} \frac{\partial^2 u_i}{\partial x_{\ell} \partial x_{m}} \right)$$

(3.30)

$$= -N \int_{B_{R}} \nabla u_i \cdot \nabla u_h + \int_{\partial B_{R}} \nabla u_i \cdot \nabla u_h \quad \text{for} \quad i = h, R.$$

Using (3.30), the sum of (3.29) and its symmetric expression reads

$$(2 - N) \int_{B_{R}} \nabla u_i \cdot \nabla u_h + C(R)$$

$$= - N \sum_{j \neq h} a_{ij} \int_{B_{R}} u_h u_j^{N-2} - \frac{N + 2}{N - 2} \sum_{j \neq h} a_{ij} \int_{B_{R}} (x \cdot \nabla u_j) u_h u_j^{N-2}$$

$$- a_{ih} \frac{N - 2}{2} \int_{B_{R}} u_h^{N-2} - N \sum_{j \neq i} a_{hj} \int_{B_{R}} u_i u_j^{N-2}$$

$$- \frac{N + 2}{N - 2} \sum_{j \neq i} a_{hj} \int_{B_{R}} (x \cdot \nabla u_j) u_i u_j^{N-2} - a_{hi} \frac{N - 2}{2} \int_{B_{R}} u_i^{N-2}$$

$$+ \int_{B_{R}} H_i(x)(x \cdot \nabla u_h) + \int_{B_{R}} H_h(x)(x \cdot \nabla u_i).$$

(3.31)

Using again the $i$-th equation of (3.17), but this time multiplying by $u_h$, yields

$$\int_{B_{R}} \nabla u_i \cdot \nabla u_h + C(R) = \sum_{j \neq h} a_{ij} \int_{B_{R}} u_h u_j^{N-2} + a_{ih} \int_{B_{R}} u_i^{N-2} + \int_{B_{R}} H_i(x) u_h \, dx.$$  

(3.32)

Now, let us write $2 \int_{B_{R}} \nabla u_i \cdot \nabla u_h = \int_{B_{R}} \nabla u_i \cdot \nabla u_h + \int_{B_{R}} \nabla u_i \cdot \nabla u_i$ and substitute the first term using (3.32) and the second term using (3.32) with $i$ and $h$ swapped. Let us then multiply the resulting expression by $\frac{N-2}{2}$ and add it to (3.31). This gives the following equality:

$$\frac{N + 2}{2} \sum_{j \neq h} a_{ij} \int_{B_{R}} u_h u_j^{N-2} + \frac{N + 2}{N - 2} \sum_{j \neq h} a_{ij} \int_{B_{R}} (x \cdot \nabla u_j) u_h u_j^{N-2}$$

$$+ \frac{N + 2}{2} \sum_{j \neq i} a_{hj} \int_{B_{R}} u_i u_j^{N-2} + \frac{N + 2}{N - 2} \sum_{j \neq i} a_{hj} \int_{B_{R}} (x \cdot \nabla u_j) u_i u_j^{N-2}$$

$$= \int_{B_{R}} H_i(x)(x \cdot \nabla u_h + \frac{N - 2}{2} u_h) + \int_{B_{R}} H_h(x)(x \cdot \nabla u_i + \frac{N - 2}{2} u_i)$$

$$+ C(R).$$

(3.33)

Recalling the definition of $A_{ij}$ and $B_{ij}$, one can write (3.33) as

$$\sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj} A_{ij} = B_{ih} + B_{hi} + C(R).$$

(3.34)
Now let us multiply (3.28) by $a_{ii}^{-1}$ for $i = 1, \ldots, k$ and sum on $i$. We get that (3.28) becomes

$$\sum_{i=1}^{k} \sum_{j \neq i} a_{ii}^{-1} a_{ij} A_{ij} = \sum_{i=1}^{k} a_{ii}^{-1} B_{ii} + C(R).$$

Multiplying (3.34) by $a_{ih}^{-1}$ and summing on the triangular bloc of the indices $(i, h)$ satisfying $1 \leq i < h \leq k$, we get

$$\sum_{i=1}^{k-1} \sum_{h=i+1}^{k} a_{ih}^{-1} \left( \sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj} A_{ij} \right) = \sum_{i=1}^{k-1} \sum_{h=i+1}^{k} a_{ih}^{-1} (B_{ih} + B_{hi}) + C(R).$$

Finally summing up the previous two relations, we get

$$\sum_{i=1}^{k-1} \sum_{h=i+1}^{k} a_{ih}^{-1} \left( \sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj} A_{ij} \right) + \sum_{i=1}^{k} \sum_{j \neq i} a_{ii}^{-1} a_{ij} A_{ij} = \sum_{i=1}^{k-1} \sum_{h=i+1}^{k} a_{ih}^{-1} (B_{ih} + B_{hi}) + \sum_{i=1}^{k} a_{ii}^{-1} B_{ii} + C(R).$$

(3.35)

Let us consider the RHS of (3.35) and observe that, using the symmetry of the matrix $A^{-1}$, we have

$$\sum_{i=1}^{k-1} \sum_{h=i+1}^{k} a_{ih}^{-1} (B_{ih} + B_{hi}) + \sum_{i=1}^{k} a_{ii}^{-1} B_{ii} = \sum_{i=1}^{k-1} \sum_{h=i+1}^{k} a_{ih}^{-1} B_{ih} + \sum_{i=1}^{k} a_{ii}^{-1} B_{ii} = \sum_{i=1}^{k} \sum_{h=1}^{i} a_{ih}^{-1} B_{ih},$$

where the last equality results from the fact that all $(i, h) \in \{1, \ldots, k\}^2$ are present in the previous terms: all $i < h$ in the first double sum, $i > h$ in the second one, and $i = h$ in the third sum.

Doing the same kind of computation for the LHS, we have

$$\sum_{i=1}^{k-1} \sum_{h=i+1}^{k} a_{ih}^{-1} \left( \sum_{j \neq h} a_{ij} A_{hj} + \sum_{j \neq i} a_{hj} A_{ij} \right) + \sum_{i=1}^{k} \sum_{j \neq i} a_{ii}^{-1} a_{ij} A_{ij} = \sum_{i=1}^{k} \sum_{j \neq i} A_{ij} \left( \sum_{h=1}^{k} a_{ih}^{-1} a_{hj} \right) = \sum_{i=1}^{k} \sum_{j \neq i} A_{ij} \delta_{i}^{j} = 0.$$

Therefore, (3.35) reads

$$\sum_{i=1}^{k} \sum_{h=1}^{k} a_{ih}^{-1} B_{ih} + C(R) = 0.$$
From the summability assumptions on $u_i$ and $H$, we can pass to the limit along the sequence $R_n \to +\infty$ chosen at the beginning of this proof and get
\[
\sum_{i=1}^k \sum_{h=1}^k a_{ih}^{-1} \int_{\mathbb{R}^N} H_i(x) \left( x \cdot \nabla u_h + \frac{N-2}{2} u_h \right) = 0.
\]
\[\square\]

**Lemma 3.12.** Let $A$ be an invertible matrix satisfying (1.2), (1.3) and denote $a_{ij}^{-1}$ the entries of $A^{-1}$. Then
\[
\sum_{i,j=1}^k a_{ij}^{-1} = k.
\]

**Proof.** Assumption (1.3) can be written $A(1, \ldots, 1) = (1, \ldots, 1)$. Multiplying both sides by $(1, \ldots, 1)^\top A^{-1}$ one gets
\[
\sum_{i,j=1}^k a_{ij}^{-1} = (1, \ldots, 1)^\top A^{-1}(1, \ldots, 1) = (1, \ldots, 1)^\top (1, \ldots, 1) = k.
\]
\[\square\]

We are in position to prove our main result:

**Proof of Theorem 1.3.** Proposition 3.9 says that there exists $u_{i,\varepsilon}$ satisfying (3.14) for $\varepsilon$ small enough. Assumption (1.21) says that the matrix $A$ is invertible at $\bar{\alpha}$ and then, from Proposition 3.11, we get
\[
(3.36) \quad L_\varepsilon \sum_{i,j=1}^k a_{ij}^{-1} \int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} W(x) \left( x \cdot \nabla u_{i,\varepsilon} + \frac{N-2}{2} u_{i,\varepsilon} \right) \, dx = 0.
\]
Recalling that $u_{i,\varepsilon} \to U$ in $D^{1,2}(\mathbb{R}^N)$ when $\varepsilon \to 0$, one can pass to the limit and get
\[
\int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} W(x) \left( x \cdot \nabla u_{i,\varepsilon} + \frac{N-2}{2} u_{i,\varepsilon} \right) \, dx \to \int_{\mathbb{R}^N} \frac{N(N+2)}{(1+|x|^2)^2} W^2(x) \, dx \neq 0.
\]
Thanks to Lemma 3.12, $\sum_{i,j=1}^k a_{ij}^{-1} \neq 0$ and then (3.36) implies that $L_\varepsilon = 0$ for $\varepsilon$ small enough, concluding the proof. $\square$

A Computation of the first derivative of the parameter

In this appendix we give some information on the behavior of a branch of solutions of Theorem 1.3. Let us recall that the bifurcation is called **transcritical** if
\[
(\text{A.1}) \quad \frac{d\alpha_\varepsilon}{d\varepsilon} \bigg|_{\varepsilon=0} \neq 0.
\]
Although in the literature formulas for the calculation of the derivative of $\alpha_\varepsilon$ are present (see, for example, [K]), it seems difficult to provide a complete characterization of the bifurcation diagram. In the next proposition we give a sufficient condition to have a transcritical bifurcation.

**Proposition A.1.** Let us suppose that

$$\sum_{j=1}^k e_{n,j}^3 \int_{\mathbb{R}^N} U^{\frac{6-N}{N-2}} W_n^3 \, dx \neq 0 \quad (A.2)$$

(see (1.20) for the definition of $e_n = (e_{n,i})_{i=1}^k \in \mathbb{R}^k$ and (2.10) for the definition of $W_n$). Then the bifurcation given in Theorem 1.3 is transcritical.

**Remark A.2.** If $k = 2$ we have $e_n = (1, -1)$, and so (A.2) is never satisfied. In this case we need refined estimates involving higher order derivatives. We do not investigate this situation. On the other hand, if $k \geq 3$ it is easy to find matrices $A$ verifying $\sum_{j=1}^k e_{n,j}^3 \neq 0$. Finally, in the special case $N = 4$ and $n = 2$ we get

$$\int_{\mathbb{R}^N} U^{\frac{6-N}{N-2}} W_n^3 \, dx = \int_{\mathbb{R}^4} U W_2^3 \, dx = \int_{-1}^1 (1 - \xi^2)(P_2^{(1,1)}(\xi))^3 \, d\xi$$

$$= \frac{27}{64} \int_{-1}^1 (1 - \xi^2)(5\xi^2 - 1)^3 \neq 0 \quad (A.3)$$

**Proof.** Using the formula (1.6.3) on page 21 in [K] we get

$$\frac{d\alpha_\varepsilon}{d\varepsilon} \bigg|_{\varepsilon = 0} = -\frac{1}{2} \frac{(\partial^2 T(\vec{\alpha}, U, \ldots, U)[\eta, \eta][\eta]}{(\partial^2 T(\alpha, U, \ldots, U)[\eta, \eta])},$$

where $\eta$ is defined in (3.9) and $[\cdot, \cdot]$ denotes the inner product in $(D^{1,2}(\mathbb{R}^N))^k$. Let us compute the numerator. Given the definition (3.2) of $T$, one easily gets

$$\partial^2 T(\alpha, U, \ldots, U)[w, w'] = P_K \left( -(-\Delta)^{-1} \left( \sum_{j=1}^k a_{ij}(\alpha) U_2 w_j w'_j \right) \right)_{i=1}^k,$$

where $U_2 := \frac{4(N+2)}{(N-2)^2} U^{\frac{6-N}{N-2}}$. In particular, in view of the definition of $\eta$ (see (3.9)),

$$\partial^2 T(\vec{\alpha}, U, \ldots, U)[\eta, \eta] = P_K \left( -(-\Delta)^{-1} \left( \sum_{j=1}^k a_{ij}(\vec{\alpha}) e_{n,j}^2 U_2 W_n^2 \right) \right)_{i=1}^k.$$

Since $\eta \in K$, we can drop $P_K$ when performing the inner product. Thus the numerator reads

$$\sum_{i=1}^k e_{n,i} \int_{\mathbb{R}^N} \nabla \left( -(-\Delta)^{-1} \left( \sum_{j=1}^k a_{ij}(\vec{\alpha}) e_{n,j}^2 U_2 W_n^2 \right) \right) \cdot \nabla W_n \, dx$$

$$= -e_{n,i} a_{ij}(\vec{\alpha}) e_{n,j}^2 \int_{\mathbb{R}^N} U_2 W_n^3 \, dx. \quad (A.5)$$
Recalling that $e_n$ is an eigenvector of $A(\alpha)$ for the eigenvalue $\Lambda_1(\alpha)$, one gets from (A.5)
\[
\left. \frac{d\alpha}{d\epsilon} \right|_{\epsilon=0} = 0 \iff -\Lambda_1(\alpha) \sum_{j=1}^{k} e_{n,j}^2 \int_\mathbb{R}^N U_2 W_3^n \, dx = 0,
\]
which gives the claim. □

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