Numerical radius of operators on Hilbert modules

Saleh Omran\textsuperscript{a,b,*}, A. EL-S. Ahmed\textsuperscript{a,c}

\textsuperscript{a}Department of Mathematics, Faculty of Science, Taif University, Saudi Arabia.
\textsuperscript{b}Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt.
\textsuperscript{c}Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt.

Abstract

In the present paper we will introduce a special numerical range and numerical radius on operator in Hilbert $\mathcal{C}^*$-modules. This definition will generalize the classical numerical range and numerical radius on operator in Hilbert spaces.

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1. Introduction

Hilbert modules are generalize the notion of a Hilbert space. The Hilbert $\mathcal{C}^*$-- modules were first introduced in the work of Kaplansky [7] in 1953, In the 1970s the theory was extended to non-commutative $\mathcal{C}^*$--algebras independently by Paschke [11] and Rieffel [13] Hilbert $\mathcal{C}^*$- modules are an often used tool in operator theory and in operator algebra theory. They serve as a major class of examples in operator module theory. Beside this, the theory of Hilbert $\mathcal{C}^*$-- modules is very interesting on its own. Interacting with the theory of operator algebras and including ideas from non-commutative geometry it progresses and produces results and new problems attracting attention. During the last decade many interesting applications of Hilbert $\mathcal{C}^*$-- module theory have been found. At the contrary, the pieces of Hilbert module theory are still rather scattered through the literature. There are some papers, chapters in monographs and lecture notes that give comprehensive representations of parts of the theory (see [1, 2, 9, 7, 14] and others). Now, we need the following definitions . Now, we give the following definitions:

Definition 1.1. A $\mathcal{C}^*$-algebra $\mathcal{A}$ is an involutive Banach algebra over $\mathbb{C}$ such that for each $a \in \mathcal{A}$ the following relation is satisfied

$$||a^*a|| = ||a||^2.$$
**Definition 1.2.** Let $\mathcal{A}$ be a $C^*$-algebra. The pre(left)-Hilbert $C^*$-module is a left $\mathcal{A}$-module $\mathcal{E}$ over $\mathcal{A}$-valued inner product $< , > : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ satisfy the following conditions:

1. $< x , y > \geq 0$ and $< x , x > = 0$ iff $x = 0$, $x \in \mathcal{E}$.
2. $< x + y , z > = < x , z > + < y , z >$, where $x, y, z \in \mathcal{E}$.
3. $< ax , y > = a < x , y >$, where $a \in \mathcal{A}$.
4. $< x , y >^* = < y , x >$.
5. $< \lambda x , y > = \lambda < x , y >$, $\lambda \in \mathbb{C}$.

**Definition 1.3.** The norm of an element $x \in \mathcal{E}$ is defined by

$$\|x\|_{\mathcal{E}} = \sqrt{\|< x , x >\|_{\mathcal{A}}}.$$  

(1)

**Definition 1.4.** The pre-Hilbert module $\mathcal{E}$ over $\mathcal{A}$ which is complete with the norm (1) is called Hilbert $\mathcal{A}$-module.

**Remark 1.5.** (1) The Hilbert module generalizes the Hilbert space. When $\mathcal{A} = \mathbb{C}$, then the Hilbert module coincides with the Hilbert space.

(2) If $\mathcal{E}$ is a left Hilbert $C^*$-module over $\mathcal{A}$, We denote the left action of an element $a \in \mathcal{A}$ on $\mathcal{E}$ by $a \cdot x$, $x \in \mathcal{E}$ such that $\lambda (a \cdot x) = (\lambda a) \cdot x = a \cdot (\lambda x)$, $\lambda \in \mathbb{C}$.

(3) For any $x, y \in \mathcal{E}$ and $a \in \mathcal{A}$, we have

$$< x , ay > = < ay , x >^* = (a < y , x >)^* = < y , x >^* a^* = < x , y > a^*.$$  

(2)

(4) One can define the (right)-Hilbert $C^*$-module as a natural analogous definition of the left Hilbert $C^*$-module with (2) become linear in the second variable and conjugate linear in the first, and replace (3) by the condition $< x , ya > = < x , y > a$.

In the rest of paper, we will consider only the left-Hilbert $C^*$-module over $\mathcal{A}$.

**Example 1.6.** $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with the inner product given by

$$< x , y > = xy^*, \forall x, y \in \mathcal{A}.$$  

Let $\mathcal{E}$, $\mathcal{M}$ be two Hilbert modules over the $C^*$-algebra $\mathcal{A}$. We will denote by $\mathcal{L}(\mathcal{E}, \mathcal{M})$ the set of all bounded operators from $\mathcal{E}$ to $\mathcal{M}$ and $\mathcal{B}(\mathcal{E})$ denotes the set of all bounded operators from $\mathcal{E}$ to itself.

**Definition 1.7.** The operator $T : \mathcal{E} \rightarrow \mathcal{E}$ is called adjointable if

$$< x , Ty > = < T^* x , y >, \ x, y \in \mathcal{E}.$$  

Let $\mathcal{B}(\mathcal{E})$ be the set of all bounded operators on the left Hilbert $\mathcal{A}$-module. Then, $\mathcal{B}(\mathcal{E})$ define a $C^*$-algebra with the norm

$$\|T\| = \sup_{\|x\| \leq 1} (\|Tx\| \in \mathcal{E})$$  

and the involution is adjointable mapping.

**2. Certain numerical radius on Hilbert $C^*$-module**

In this section, we define a certain numerical range and numerical radius on Hilbert $C^*$-module over $\mathcal{A}$.

**Definition 2.1.** Let $T$ be an operator in $\mathcal{B}(\mathcal{E})$ then the numerical range of an operator $T$ is defined by:

$$W_{\mathcal{A}}(T) = \{ < Tx , x >, \ x \in \mathcal{E}, \|x\| = 1\}$$  

(3)

and the numerical radius is given by

$$w_{\mathcal{A}}(T) = \sup (\| < Tx , x > \|_{\mathcal{A}}, \|x\| = 1).$$  

(4)
Remark 2.2. (1) The elements of the numerical range $W_{\mathcal{A}}(T)$ in eq. (3) are in the set of $C^*$-algebras $\mathcal{A}$, and the elements of the numerical radius $w_{\mathcal{A}}(T)$ in the eq. (4) are in $\mathbb{R}^+$.

(2) In the case $\mathcal{A} = \mathbb{C}$, $\mathcal{E}$ is become a Hilbert space over $\mathbb{C}$ and both definitions in (3) and (4) coincide with the definition of the numerical range and numerical radius, respectively of operator on Hilbert space.

(3) $W_{\mathcal{A}}(T)$ is not empty because it contain $W_{\mathcal{C}}(T)$ and the last one is not empty.

**Proposition 2.3.** Let $T$ be an operator in $\mathcal{B}(\mathcal{E})$, then $W_{\mathcal{A}}(T^*) = W_{\mathcal{A}}(T)^*$.

**Proof.** It is immediately, we have $< T^*x, x > = < x, Tx > = < Tx, x >^*$.

**Proposition 2.4.**

(i) Let $T, S$ be an operators in $\mathcal{B}(\mathcal{E})$ and $\alpha, \beta \in \mathbb{C}$ then $W_{\mathcal{A}}(\alpha T + \beta S) \subseteq \alpha W_{\mathcal{A}}(T) + \beta W_{\mathcal{A}}(S)$.

(ii) $W_{\mathcal{A}}(I) = 1$.

**Proof.** The proof of item (i) is immediately from applying the Definition 2.1 (3).

For (ii) we note that $< Ix, x > = < x, x > = \|x\| = 1$.

**Proposition 2.5.** Let $T$ be an operators in $\mathcal{B}(\mathcal{E})$, for any unitary $U$, we have $W_{\mathcal{A}}(UTU^*) = W(U)$.

**Proof.** $< UTUx, x > = < T(Ux), Ux >$. There exist $y \in \mathcal{E}$ such that $Ux = y$ and $\|y\| = 1$ whenever $\|x\| = 1$. Therefore $W_{\mathcal{A}}(UTU^*) = \{ < Ty, y > , y \in \mathcal{E}, \|y\| = 1 \} = W(U)$.

**Theorem 2.6.** $w_{\mathcal{A}}(T)$ satisfy the following:

1. $w_{\mathcal{A}}(T) \leq \|T\|$.
2. $w_{\mathcal{A}}(T) = w(T^*)$.
3. $w_{\mathcal{A}}(T^*T) \leq \|T\|^2$.

**Proof.** For (1) it’s obvious from the Definition 2.1 that

$$\|(T^*x, x)\|^2 \leq \sup \{\|T\| \|x\|^2 \mid \|x\| = 1 \} \leq \|T\|.$$ 

To prove the second claim we apply the eq. (4) in the 2.1

$$w(T) = \sup \{\|T^*x, x\| \mid x \in \mathcal{E}, \|x\| = 1 \}$$
$$= \sup \{\|T^*x, x\| \mid x \in \mathcal{E}, \|x\| = 1 \}$$
$$= \sup \{\|T^*x, x\| \mid x \in \mathcal{E}, \|x\| = 1 \}.$$

Since $\|a\| = \|a^*\|$, for any $a \in \mathcal{A}$, therefore

$$w(T) = \inf \{\|T^*x, x\| \mid x \in \mathcal{E}, \|x\| = 1 \} = w(T^*).$$

Now we prove the third statement,

$$w(T^*T) = \inf \{\|T^*Tx, x\| \mid x \in \mathcal{E}, \|x\| = 1 \}$$
$$= \inf \{\|Tx, Tx\| \mid x \in \mathcal{E}, \|x\| = 1 \} = \|T\|^2.$$ 

**Lemma 2.7.** If $T$ is a self-adjoint element in the $C^*$-algebra, then $S = T^2$ is a self-adjoint.

**Proof.** Clearly

$$S^* = (T^2)^* = (T^*T)^2 = (T^*)^*T^* = TT^* = T^2 = S.$$ 

Note that: The existence of the operator $S$ come from the positivity of $T$. 

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Theorem 2.8. If $T$ is self adjoint element in $\mathcal{L}(E)$, then $w(T) = ||T||$.

Proof. To prove the first claim. For a self adjoint operator $T$, we have

$$w(T^2) = \text{sub}(||T^2 x, x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= \text{sub}(||Tx, T^* x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

since $||T|| = ||T^*||$, so

$$= \text{sub}(||Tx, Tx||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= \text{sub}(TY||^2_{\mathcal{E}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= ||T||^2 = ||T^* T|| = ||T^2||.$$

Put $S = T^2$ as in Lemma 2.7, we get $w(S) = ||S||$, in particular $w(T) = ||T||$.

Theorem 2.9. For any operator $T$ in $\mathcal{L}(E)$, we have $w(T) \geq \frac{1}{2}||T||$.

Proof. Since $\mathcal{L}(E)$ is a $C^*$-algebra, then any operator $T$ in $\mathcal{L}(E)$ can be represent in the form $T = T_1 + iT_2$ where $T_1, T_2$ are self adjoint elements given by

$$T_1 = \frac{T + T^*}{2} \text{ and } T_2 = \frac{T - T^*}{2i}.$$

Now,

$$w(T_1) = \text{sub}(||\left(\frac{T + T^*}{2}\right)x, x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= \text{sub}(||x, \left(\frac{T + T^*}{2}\right)x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= 1/2 \text{sub}(||x, (T^* + T)x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= 1/2 \text{sub}(||x, T^*x + x, Tx||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= 1/2 \text{sub}(||Tx, x + T^*x, x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$\leq 1/2 \text{sub}(||Tx, x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1) + 1/2 \text{sub}(||T^* x, x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$= 1/2 w(T) + 1/2 w(T^*)$$

since $w(T) = w(T^*)$ from Theorem 2.6. So, we get $w(T_1) \leq w(T)$, and similarly $w(T_2) \leq w(T)$. Thus, $||T|| = ||T_1 + iT_2|| \leq ||T_1|| + ||T_2|| = w(T_1) + w(T_2) \leq 2w(T)$, and this complete the proof.

Proposition 2.10. If $T, S$ are any operators in $\mathcal{L}(E)$, then

(1) $w(TS) \leq 4w(T)w(S)$.

In particular $w(T^2) \leq 4w(T)^2$.

Proof. Clearly,

$$w(TS) = \text{sub}(||TSx, x||_{\mathcal{A}}, x \in \mathcal{E}, ||x|| = 1)$$

$$\leq ||T|| ||S||.$$

Using Theorem 2.9 above, we get $w(TS) \leq 4w(T)w(S)$, and in particular when $T = S$, we get $w(T^2) \leq 4w(T)^2$.

Theorem 2.11. The numerical radius defined above in eq. (4) defined a norm in $\mathcal{L}(E)$.

Proof. It is clear from Definition 2.1 (4) that $w(T) \geq 0$. If $w(T) = 0$, then from Theorem 2.9 $||T|| = 2w(T) = 0$, therefore $||T|| = 0$ and this yield that $T = 0$. Moreover, one can easy to check that $w(aT) = |a| w(T)$ and $w(T + S) \leq w(T) + w(S)$. Hence, $w(T)$ is a norm in $\mathcal{L}(E)$.
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