Fractal Curves and Rugs of Prescribed Conformal Dimension

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Abstract

We construct Jordan arcs of prescribed conformal dimension which are minimal for conformal dimension. These curves are used to design fractal rugs, similar to Rickman’s rug, that are also minimal for conformal dimension. These fractal rugs could potentially settle a standing conjecture regarding the existence of metric spaces of prescribed topological conformal dimension.

Keywords: metric space, Cantor sets, Hausdorff dimension, conformal dimension, topological dimension, quasisymmetric map.

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1. Introduction

Let $(X, d)$ be a metric space. The subscripts of $\text{dim}$ indicate the type of dimension, and we set $\text{dim} \emptyset = -1$ for every dimension.

Quasisymmetric maps form an interesting intermediate class lying between homeomorphisms and bi-Lipschitz maps [7, 9]. Topological dimension is invariant under homeomorphisms, and Hausdorff dimension is bi-Lipschitz invariant. Conformal dimension classifies metric spaces up to quasisymmetric equivalence [12].

Definition 1.1. The conformal dimension of $X$ is

$$\text{dim}_C X = \inf \{ \text{dim}_H f(X) : f \text{ is quasisymmetric} \}.$$ 

It is clear from the definition that conformal dimension is invariant under quasisymmetric maps, and hence under bi-Lipschitz maps.

Pansu introduced conformal dimension in 1989 [13], and the concept has been widely studied since. The primary applications of the theory of conformal dimension are in the study of Gromov hyperbolic spaces and their boundaries. The boundary of a Gromov hyperbolic space admits a family of metrics which are not bi-Lipschitz equivalent, but quasisymmetrically equivalent. Consequently,
the conformal dimension of the boundary is well-defined, unlike its Hausdorff dimension \[12\]. Recent advancements involving applications of conformal dimension are exposed in \[3\] and \[4\]. Determining the conformal dimension of the Sierpinski carpet (denoted \(\text{dim}_C SC\)) is an open problem, but in \[10\] Keith and Laakso proved that \(\text{dim}_C SC < \text{dim}_H SC\). Kovalev proved a conjecture of Tyson: conformal dimension does not take values strictly between 0 and 1 \[11\].

In \[8\] Hakobyan proved that if \(E \subset \mathbb{R}\) is a uniformly perfect middle-interval Cantor set, then \(\text{dim}_H E = \text{dim}_C E\) if and only if \(\text{dim}_H E = 1\).

**Definition 1.2.** A metric space \(X\) is called **minimal for conformal dimension** if \(\text{dim}_C X = \text{dim}_H X\).

In \[5\] topological conformal dimension was defined; it is an adaptation of **topological Hausdorff dimension** which was defined in \[1\] as
\[
\text{dim}_{tH} X = \inf \{d : X\text{ has a basis }\mathcal{U}\text{ such that }\text{dim}_H \partial U \leq d - 1 \text{ for all } U \in \mathcal{U}\}.
\]

**Definition 1.3.** The topological conformal dimension of \(X\) is
\[
\text{dim}_{tC} X = \inf \{d : X\text{ has basis }\mathcal{U}\text{ such that }\text{dim}_C \partial U \leq d - 1 \text{ for all } U \in \mathcal{U}\}.
\]

There is a key difference between conformal dimension and \(tC\)-dimension. Lower bounds for the former can be obtained through the presence of "diffuse" families of curves, while diffuse families of surfaces provide lower bounds for the latter. For precise statements, see Theorem 4.5 in \[5\] and Proposition 4.1.3 in \[12\]. While Fact 4.1 in \[5\] shows \(\text{dim}_C X \in \{-1, 0, 1\} \cup [2, \infty]\), it is unknown whether \(tC\)-dimension attains all values in \([2, \infty]\).

The following conjecture was posed in \[5\]:

**Conjecture 1.4.** For every \(d \in [2, \infty]\) there is a metric space \(X\) with \(\text{dim}_{tC} X = d\). 

In this paper we provide examples of fractal spaces that could potentially settle Conjecture 1.4. To this end, it seems appropriate to consider topological squares that are not quasisymmetrically equivalent to \([0, 1]^2\). A classical fractal of this kind is Rickman’s rug, which is the cartesian product of the von Koch snowflake with the standard unit interval. In general, a fractal rug is a product space of the form \(R_d = V_d \times [0, 1]\), where \(V_d\) is a Jordan arc homeomorphic to \([0, 1]\) with \(d = \text{dim}_C V_d\). At present, we do not have the tools necessary to determine the \(tC\)-dimensions of these fractals, but we suspect that \(\text{dim}_H R_d = \text{dim}_{tC} R_d\).

This would be consistent with the fact that \(R_d\) is minimal for conformal dimension, which follows from a result of Bishop and Tyson \[12\]. Since spaces of prescribed Hausdorff dimension are easily obtained (see e.g. Remark 4.1 below), the equality \(\text{dim}_H R_d = \text{dim}_{tC} R_d\) would provide an affirmative answer to the question of existence in Conjecture 1.4.

In Section 3 we discuss fractal rugs and their dimensions in the context of Conjecture 1.4. In Section 4 we construct the Jordan arcs that are discussed in Section 3, which is the main result of the paper:

**Theorem 1.5.** For every \(c \geq 1\) there is a Jordan arc \(\Lambda\) with \(\text{dim}_C \Lambda = c\).
2. Preliminaries

The symbol $B(x, \varepsilon)$ denotes the open ball centered at $x$ of radius $\varepsilon$. For $x \in \mathbb{R}^n$, $|x|$ is the Euclidean modulus of $x$. Unless otherwise stated, distance in the metric space $Y$ is denoted $d_Y$. To discuss conformal dimension, we need the notion of quasisymmetry. A quasisymmetric map allows for rescaling with aspect ratio control:

**Definition 2.1.** An embedding $f : X \to Y$ is quasisymmetric if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$\frac{d_Y(f(x), f(a))}{d_Y(f(x), f(b))} \leq \eta\left(\frac{d_X(x, a)}{d_X(x, b)}\right)$$

for all triples $a, b, x$ of points in $X$ with $x \neq b$ [12].

Conformal dimension is defined via Hausdorff dimension. For the latter, recall the following definition.

**Definition 2.2.** The $p$-dimensional Hausdorff measure of $X$ is

$$\mathcal{H}^p(X) = \lim_{\delta \to 0} \inf \left\{ \sum (\text{diam } E_j)^p : X \subset \bigcup E_j \text{ and } \text{diam } E_j \leq \delta \forall j \right\}.$$ 

The Hausdorff dimension of $X$ is $\dim_H X = \inf \{ p : \mathcal{H}^p(X) = 0 \}$.

An interesting combination of the Hausdorff and topological dimensions called **topological Hausdorff dimension** was introduced in [1]:

$$\dim_{tH} X = \inf \{ d : X \text{ has a basis } U \text{ such that } \dim_H \partial U \leq d - 1 \forall U \in U \}.$$ 

In certain favorable circumstances, both of these dimensions are additive under products. For sake of completeness, we include Theorem 4.21 from [1].

**Theorem 2.3.** If $X$ is a nonempty separable metric space, then

$$\dim_{tH}(X \times [0, 1]) = \dim_{tH}(X \times [0, 1]) = \dim_H X + 1.$$  \hspace{1cm} (2.1)

In particular, $\dim_{tH} \mathbb{R}$ can attain any value greater than 2 [2].

The first inequality in (2.1) is due to Balka, Buczolich, and Elekes [1]. The second inequality is a generalization of Product Formula 7.3 in [6], which is a well-known result. Let $\dim\overline{H} X$ be the upper box-counting dimension of $X$ (see e.g. [6]). If $X, Y \subset \mathbb{R}^n$ are Borel sets with $\dim_H X = \dim_B X$, then Corollary 7.4 in [6] yields $\dim_H(X \times Y) = \dim_H X + \dim_H Y$. The condition $\dim_H X = \dim_B X$ holds for a wide variety of spaces, including uniform Cantor sets (see Example 4.5 in [6]).

Hausdorff dimension is invariant under bi-Lipschitz maps.

**Definition 2.4.** An embedding $f$ is $L$-bi-Lipschitz if both $f$ and $f^{-1}$ are $L$-Lipschitz, and we say $f$ is bi-Lipschitz if it is $L$-bi-Lipschitz for some $L$. 

\[3\]
Every bi-Lipschitz map is quasisymmetric, but not every quasisymmetric map is bi-Lipschitz.

We are now prepared to define conformal dimension, which measures the distortion of Hausdorff dimension by quasisymmetric maps.

**Definition 2.5.** The conformal dimension of $X$ is
\[
\dim_C X = \inf \{ \dim_H f(X) : f \text{ is quasisymmetric} \}.
\]

In case $\dim_C X = \dim_H X$ we say that $X$ is minimal for conformal dimension. Bishop and Tyson proved that for every compact set $Y \subset \mathbb{R}^n$, the space $Z = Y \times [0,1]$ is minimal for conformal dimension [12]. The following string of inequalities is a useful tool for determining dimensions. The first two comprise Proposition 2.2 in [5], while the third is evident considering Definition 2.5.

**Proposition 2.6.** If $X$ is a metric space, then
\[
\dim_t X \leq \dim_{tC} X \leq \dim_C X \leq \dim_H X.
\]

For any product space $X \times Y$, we use the metric
\[
d((x_1, y_1), (x_2, y_2)) = \max(d_X(x_1, x_2), d_Y(y_1, y_2)).
\]

A Jordan arc is an arc of a Jordan curve; that is, a homeomorphic image of $[0,1]$ with the usual topology.

### 3. Rickman’s Rug

Let $\varepsilon \in (0,1)$. The snowflake mapping $([0,1], | \cdot |) \to ([0,1], | \cdot |^\varepsilon)$ is quasisymmetric [9], and we write $[0,1]^\varepsilon$ for the target space. It is readily seen that $\dim_H ([0,1]^\varepsilon) = \varepsilon^{-1}$. Regardless of the choice of $\varepsilon \in (0,1)$, one has $\dim_C ([0,1]^\varepsilon) = 1$ since the inverse of a quasisymmetric map is again quasisymmetric. Equivalently, one can obtain the metric space $[0,1]^\varepsilon$ by choosing an appropriate scaling factor and following the construction of the classical von Koch snowflake. From this point forward, when the value $\varepsilon \in (0,1)$ is unimportant for our discussion, we will write $V = [0,1]^\varepsilon$ and refer to $R = V \times [0,1]$ as Rickman’s rug. We use the term fractal rug for a product space of the form $R_d = V_d \times [0,1]$, where $V_d$ is a Jordan arc with $d = \dim_C V_d$. As usual, this product is equipped with the metric
\[
d((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|^{\varepsilon}, |y_1 - y_2|).
\]

The case $\varepsilon = \frac{\ln(3)}{\ln(4)}$ corresponds to the aforementioned von Koch snowflake curve.

Since $R$ is homeomorphic to $[0,1]^2$, $\dim_H R = 2$. Tukia proved that $R$ is not quasisymmetrically equivalent to $[0,1]^2$ [12]. In fact, Example 4.1.9 in [12] shows that $R$ is minimal for conformal dimension, meaning $\dim_C R = \dim_H R = 1+\varepsilon^{-1}$, where the last equality follows from Theorem 4.2 in [12]. We can compute the $tH$ and $tC$ dimensions of $R$. Here is a simple way to compute the $tC$-dimension of $R$. 

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Fact 3.1. \( \dim_{tC} R = 2 \).

**Proof.** Since \( V \) is a Jordan arc, Theorem 3.7 in [5] implies \( \dim_{tC} R \leq 2 \). The reverse inequality holds since \( 2 = \dim_t R \leq \dim_{tC} R \) by Proposition 2.6.

It is not clear how to compute the topological conformal dimension of more general fractal rugs. The difficulty in determining \( \dim_{tC} R \) lies in giving a non-trivial lower bound. Theorem 3.7 in [5] yields the upper bound \( \dim_{tC} R \leq d + 1 \), but a lower bound takes into account the conformal dimension of the boundary of an arbitrary open subset of \( R_d \), which can be quite bizarre.

In view of Fact 3.1, Rickman’s rug cannot be used to answer Conjecture 1.4. To accomplish that goal, one needs a more general construction. One approach is to try to compute \( \dim_{tC} R_d \) for \( d > 1 \), but in order to do this, one first needs to construct \( V_d \) with \( d > 1 \).

The idea of the following conjecture is to prescribe a number \( c \geq 1 \), then use Theorem 4.2 to obtain \( V_{c-1} \) and ultimately show that \( \dim_{tC} R_{c-1} = c \).

**Conjecture 3.2.** For any \( c \geq 1 \) there is a Jordan arc \( V_{c-1} \) such that \( \dim_{tC}(V_{c-1} \times [0,1]) = c \).

This conjecture seems reasonable if one hopes to prove it by showing that \( \dim_H R_{c-1} = \dim_{tC} R_{c-1} \). In particular, it would follow from Proposition 2.6 that \( \dim_{tC} R_{c-1} = \dim_H R_{c-1} = c \).

4. Jordan Arcs of Prescribed Conformal Dimension

In this section we show that for any number \( c \geq 1 \) there is a Jordan arc with conformal dimension \( c \). The following is a modest yet useful remark on Cantor sets that will help us accomplish this task.

**Remark 4.1.** For any \( a \in [0, \infty) \) there is a Cantor type set \( K_a \subset [0,1]^n \) with \( \dim_H K_a = a \) for large enough \( n \). For instance, if \( N \) is the least positive integer such that \( b = \frac{a}{N} < 1 \), let \( K_b \subset [0,1] \) be the Cantor set with \( \dim_H K_b = b \) obtained by the usual construction with scaling factor \( 0 < r < \frac{1}{2} \) defined by \( b = \frac{\ln(2)}{\ln(r)} \). Then \( K_a = \prod_{i=1}^{N} K_b \) is a self-similar Cantor set, and \( \dim_H K_a = \sum_{i=1}^{N} \dim_H K_b = a \) by Corollary 7.4 in [6].

**Theorem 4.2.** For every \( c \geq 1 \) there is a Jordan arc \( \Lambda \) with \( \dim_{tC} \Lambda = c \).

For \( c = 1 \) put \( \Lambda = [0,1] \). We will need several lemmas to verify the case \( c > 1 \) in Theorem 4.2. The result will be shown for any number \( c = 1 + d \), \( d > 0 \).

**Lemma 4.3.** Suppose \( 0 < d < \infty \). Let \( E \) be the Cantor set constructed from the sequence of ratios \( \{c_i\}_{i=1}^{\infty} \) where \( c_i \rightarrow 0 \) as \( i \rightarrow \infty \), and let \( Y = K_d \) be the self-similar Cantor set with \( \dim_H Y = d \) as in Remark 4.1. Then \( \dim_{tC}(E \times Y) = 1 + d \).
Proof. We will show that $E$ satisfies the conditions of Corollary 5.6 in [8] and the result will follow. First let us show that $E$ is uniformly perfect. Since $\dim_H E = 1$, Corollary 3.3 in [8] will then imply $\dim_C E = 1$. To this end, let $x \in E$ and $r > 0$. Write $B(x,r) \cap E = B(x,r)$ for the open ball. Then for large enough $k$ there is a $k$th generation interval $I_{k,j}$, for some $j \in \{1, \ldots, 2^k\}$, such that $x \in I_{k,j} \subset B(x,r)$. Choose the smallest such $k$. Then the length of $I_{k,j}$ is

$$m(I_{k,j}) = s_k = \frac{\prod_{i=1}^k (1 - c_i)}{2^k},$$  \hspace{1cm} (4.1)$$

and $s_k < r \leq s_{k-1}$. Say $I_{k,j} = [a,b]$ so that $a, b \in E$. Then at least one of $|x - a| \geq \frac{s_k}{2}$ and $|x - b| \geq \frac{s_k}{2}$ holds. Say $|x - a| \geq \frac{s_k}{2}$. By (4.1),

$$\frac{s_{k-1}}{s_k} = \frac{2^k \prod_{i=1}^{k-1} (1 - c_i)}{2^{k-1} \prod_{i=1}^k (1 - c_i)} = \frac{2}{1 - c_k} \leq \frac{2}{1 - \sup_i c_i} = K.$$  \hspace{1cm} (4.2)$$

Inequality (4.2) yields

$$|x - a| \geq \frac{s_k}{2} \geq \frac{s_{k-1}}{2K} \geq \frac{r}{2K}.$$  \hspace{1cm} (4.3)$$

By (4.3) we have $a \in B(x,r) \setminus B(x,\frac{r}{2K}) \neq \emptyset$, and hence $E$ is uniformly perfect. Since $\dim_H E = 1$, Corollary 3.3 in [8] gives $\dim_C E = 1$. That is, $E$ is minimal for conformal dimension.

To satisfy Corollary 5.6 in [8] it remains to show that $E$ supports a measure $\mu$ such that for every $\varepsilon > 0$ there is a constant $C$ so that whenever $x \in E$ and $r < \text{diam}E$,

$$\frac{r^{1+\varepsilon}}{C} \leq \mu(B(x,r)) \leq Cr^{1-\varepsilon}.$$  

Write $E = \bigcap_k E_k$ where $E_k = \bigcup_{j=1}^{2^k} I_{k,j}$ are the intervals used to construct $E$. Let $\mu_k$ be the probability measure supported on $E_k$ that gives equal weight to each $I_{k,j}, j = 1, \ldots, 2^k$. Since $E$ is compact there is a subsequence $\mu_{k_i} \to \mu$ where $\mu$ is a probability measure supported on $E$. In particular $\mu(I_{k,j}) = \mu_k(I_{k,j}) = \frac{1}{2^k}$ for all $k,j$. Let $\varepsilon > 0, x \in E$ and $0 < r < \text{diam}E$. Choose $k$ in the same manner as in the proof of uniform perfectness of $E$. For some $j \in \{1, \ldots, 2^k\}$ we have $x \in I_{k,j} \subset B(x,r)$. Since $s_{k-1} \geq r$,

$$2^{k-1} = \frac{\prod_{i=1}^{k-1} (1 - c_i)}{s_{k-1}} \leq \frac{1}{r},$$  \hspace{1cm} (4.4)$$

so by (4.4)

$$\mu(B(x,r)) \geq \mu(I_{k,j}) = \mu_k(I_{k,j}) = \frac{1}{2^k} \geq \frac{r}{2}.$$  \hspace{1cm} (4.5)$$

By choice of $k$ it follows from (4.5) that at most three intervals of generation $k-1$ intersect $B(x,r)$, each with $\mu(I_{k-1,j}) = \frac{1}{2^{k-1}}$. Therefore $\mu(B(x,r)) \leq$
$3\mu(I_{k-1,j}) = \frac{3}{2^{k-1}}$. Since $s_k < r$ it suffices to show that there is a constant $C$ such

$$\frac{3}{2^{k-1}} \leq C s_k^{1-\varepsilon}$$

That is, we must show that there is $C$ such that $a_k \leq C$, where

$$a_k = \frac{6(2^{-k\varepsilon})}{\left(\prod_{i=1}^{k}(1-c_i)\right)^{1-\varepsilon}}.$$  \hspace{1cm} (4.6)

Note that (4.6) implies $\frac{a_{n+1}}{a_n} = \frac{2^{-\varepsilon}}{(1-c_{n+1})^{1-\varepsilon}} \to 2^{-\varepsilon} < 1$ so that $\sum a_n < \infty$ and hence $a_n \to 0$. In particular, $a_n$ is bounded so say $a_k \leq C$ for all $k$. Finally

$$\frac{r^{1+\varepsilon}}{2} \leq \mu(B(x,r)) \leq Cr^{1-\varepsilon},$$

and by (4.7) there is a constant $K$ such that $\frac{1}{K}r^{1+\varepsilon} \leq \mu(B(x,r) \cap E) \leq Kr^{1-\varepsilon}$. This shows that Corollary 5.6 in [8] is satisfied so that

$$\dim_C(E \times Y) \geq 1 + \dim_H Y = 1 + d.$$  \hspace{1cm} \Box

Since $E$ is a self-similar Cantor set, Example 4.5 and Corollary 7.4 in [6] yield

$$\dim_C(E \times Y) \leq \dim_H (E \times Y) = \dim_H E + \dim_H Y = 1 + d.$$

Therefore $\dim_C(E \times Y) = 1 + d$. \hspace{1cm} \Box

In [7], Gehring and Väisälä constructed a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ which maps one $n$-dimensional Cantor set onto another. Their construction involves a sequence of piecewise linear mappings, and we use that idea to produce a Jordan arc containing a (sufficiently large) product of Cantor sets.

**Lemma 4.4.** Let $E$ and $Y \subset [0,1]^n$ be as in Lemma 4.3. There is a Jordan arc $\Lambda \subset [0,1]^{n+1}$ such that $\Lambda \supset (E \times Y)$.

**Proof.** For each $k \in \mathbb{N}$ we will construct curves $\Gamma_k$ such that $\Gamma = \bigcup_k \Gamma_k$ and $\Lambda = \Gamma$. Since $Y \subset [0,1]^n$ is a product of $n$ copies of the same Cantor set, we see that $F_1 = E_1 \times Y_1$ is the first generation of $E \times Y$, where $E_1 = I_{1,1} \cup I_{1,2}$ and $Y_1 = \bigcup_{i_1,\ldots,i_n = 1,2} (J_{1,i_1} \times \cdots \times J_{1,i_n})$. Then

$$F_1 = \bigcup_{j,i_1,\ldots,i_n = 1,2} (J_{1,j} \times (J_{1,i_1} \times \cdots \times J_{1,i_n}))$$  \hspace{1cm} (4.8)

is a union of $t_1 = 2^{n+1}$ disjoint products whose sides are rectangles. Let us say $F_1 = \bigcup_{s = 1}^{t_1} Q_s^1$ where $\text{dist}(Q_1^1,0) < \text{dist}(Q_2^1,0) \leq \cdots \leq \text{dist}(Q_{t_1-1},0) < \text{dist}(Q_{t_1}^1,0)$. For each $s$ there are unique points $x_s^1, y_s^1 \in Q_s^1$ with $|x_s^1| = \text{dist}(Q_s^1,0)$ and $|y_s^1| = \max\{|z| : z \in Q_s^1\}$. For $s = 1,\ldots,t_1-1$ there is a simple curve $\gamma_s^1$ in
Let $[0,1]^{n+1}$ from $y_n$ to $x_{n+1}^j$. Since $n + 1 \geq 2$ we may choose these $2^{n+1} - 1$ curves to be disjoint.

Parametrize these curves by first dividing the interval $[0, 1]$ into $2(2^{n+1} - 1) + 1 = 2^{n+2} - 1$ subintervals of equal length. Call them

$$P_j^1 = \left[\frac{j}{2^{n+2} - 1}, \frac{j + 1}{2^{n+2} - 1}\right], \quad 0 \leq j \leq 2^{n+2} - 2.$$

Choose smooth parametrized curves $\gamma_1^1, \ldots, \gamma_{t_1}^1 \subset F_1^c$ for odd $j$:

$$\hat{\gamma}_j^1: P_j^1 \to \gamma_j^1$$

(see Figure 1 for the case $n = 1$).

For this we call $\{P_j^1 : j \text{ odd}\}$ used and $\{P_j^1 : j \text{ even}\}$ neglected. Put $\Gamma_1 = \bigcup_{j=1}^{t_1 - 1} \gamma_j^1$. Note that there are $2^{n+2} - 1 - (2^{n+1} - 1) = 2^{n+1} = t_1$ neglected subintervals of $[0, 1]$ after this parametrization, which is the number of products in $F_1$. Reindex $\{P_j^1 : j \text{ even}\} = \{P_j^1\}_{t_1}$ in increasing order of distance from 0.

For each integer $1 \leq s \leq t_1$ we repeat the above path construction process for the pair $P_s^1, Q_s^1$. This gives $2^{2(n+1)} - 1$ curves whose union we call $\Gamma_2$. Continuing in this fashion, we obtain for each $k \in \mathbb{N}$, the subintervals $P_j^k$, along with $2^k(n+1) - 1$ curves, and their union $\Gamma_k$.

Let $\Gamma = \bigcup_k \Gamma_k$. It remains to show that $\Gamma$ is a Jordan arc and that $(E \times Y) \subset \Gamma$. The construction of $\Gamma$ defines a function $f : D \to \Gamma$ where $D$ is dense in $[0,1]$. We will show that $f$ is uniformly continuous so that it extends to a continuous function $\hat{f} : [0,1] \to \Gamma$. Call $x \in D k$-used if $x \in \bigcup_j P_j^k$, and call $x$ $k$-neglected if $x \in \bigcap_j (P_j^k)^c$.

Let $\delta_k = m(P_j^k)$ and $\varepsilon > 0$. Take $K$ to be the smallest integer such that $\text{diam}(Q_1^K) = \cdots = \text{diam}(Q_{t_k}^K) < \varepsilon$. Note that $\bigcup_j (\Gamma_{K+1} \cap Q_i^K)$ is composed of $t_{K+1} = 2^{k+1}(n+1) - 1$ disjoint paths. Let $\delta' = \frac{\delta_{K+1}}{2K}$ and

$$L_K = \max\{L_{j,k} | \gamma_j^k \text{ is \, } L_{j,k}\text{-Lipschitz, } j \text{ even, } 1 \leq k \leq K\} \quad (4.9)$$

Put $\delta = \min\{\delta', \frac{\varepsilon}{2L_K}\}$. If $x, y \in D$ are such that $|x - y| < \delta$ then there are three possibilities. In any case, we must show $|f(x) - f(y)| < \varepsilon$.

1. **Both $x$ and $y$ are $(K + 1)$-used.** Then $x, y \in Q_j^K$ for some $j$, hence $|f(x) - f(y)| \leq \text{diam}(Q_j^K) = \text{diam}(Q_1^K) < \varepsilon$.

2. **Both $x$ and $y$ are $(K + 1)$-neglected.** Since $|x - y| < \frac{\delta_{K+1}}{2K}$ it follows that either (a), (b), or (c) holds.
Figure 1: For \( n = 1 \) and \( \dim H Y = \frac{\ln(2)}{\ln(3)} \), these line segments are examples of smooth curves that might comprise the first two generations of \( \Gamma \). Taking the closure of the union of all such segments results in a Jordan arc with the desired conformal dimension.

(a) \( x, y \in P^{K+1}_s \) for some \( s \). Then \( f(x), f(y) \in f(D \cap P^{K+1}_s) \subset Q^{K+1}_s \) so that \( |f(x) - f(y)| \leq \text{diam}(Q^K) < \epsilon \).

(b) \( x, y \in P^K_j \) for some \( k \leq K \). By (4.9) we have
\[
|f(x) - f(y)| = |\hat{\gamma}^k_j(x) - \hat{\gamma}^k_j(y)| \leq L_{j,k}|x - y| \leq L_K \frac{\epsilon}{2L_K} = \epsilon/2.
\]

(c) \( x \) is \( k \)-used for some \( k \leq K \) and \( y \in P^{K+1}_s \) for some \( s \). Say \( x \in P^K_j, j \leq K \) and \( P^K_j = [a,b] \). Then \( f(a) = x^{K+1}_j \) is the corner of \( Q^{K+1}_{j+1} \) closest to 0. By construction \( f(a) \) is also the corner of \( Q^{K+1}_r \) closest to 0 for some \( r \). Since \( |y - a| \leq |y - x| < \frac{1}{2} \delta_{K+1} \), there are no \((K + 1)\)-used intervals between \( y \) and \( a \). Therefore \( f(P^{K+1}_s) \subset Q^{K+1}_r \) so that
\[
|f(y) - f(a)| \leq \text{diam}(Q^{K+1}_r) < \frac{1}{2} \text{diam}(Q^K_1) < \epsilon/2.
\]
Also (b) implies \(|f(a) - f(x)| \leq \frac{ε}{2}\), so
\[
|f(y) - f(x)| \leq |f(y) - f(a)| + |f(a) - f(x)| < ε.
\]

3. **x is (K+1)-used and y is (K+1)-neglected.** Without loss of generality we assume \(y < x\). Say \(x \in P^K_{j+1}\). Then \(y \in P^K_{j+1}\). Note that \(f(x)\) lies on a curve connecting two products \(Q^K_{s+1}, Q^K_{j+1}\) for some \(s\). By construction \(f(y) \in Q^K_{s+1}\) so that \(f(x), f(y) \in Q^K_i\) for some \(i\). Thus \(|f(x) - f(y)| \leq \text{diam}(Q^K_i) < ε\).

So \(f\) is uniformly continuous on \(D\), and a continuous extension \(\hat{f} : [0,1] \to \Gamma\) exists. We show that \(\hat{f}\) is injective. Let \(x \neq y\) for \(x, y \in [0,1]\). If \(x, y \in D\) then either \(f(x)\) and \(f(y)\) lie on disjoint arcs so that \(f(x) \neq f(y)\), or they lie on the same curve \(\gamma^k_i\) in which case \(f(x) \neq f(y)\) because \(\gamma^k_i\) is injective. If \(x, y \in D^c\) then there is a used interval \(P^k_j\) between \(x\) and \(y\). By construction, \(f(x) \in Q^k_i\) and \(f(y) \in Q^k_i\) for some \(i \neq j\), so \(f(x) \neq f(y)\). If \(x \in D\) and \(y \in D^c\), then there is a used interval \(P^k_j\) strictly between \(x\) and \(y\) and the above argument implies \(f(x) \neq f(y)\). Then \(\hat{f}\) is a continuous bijection whose domain is compact, so it is a homeomorphism and hence \(\Gamma\) is a Jordan arc. To see that \((E \times Y) \subset \Gamma\), let \(z \in E \times Y\) and note that \(z \in Q^k_{i,k}\) for infinitely many \(k\) and \(\Gamma \cap Q^k_j \neq \emptyset\) for all \(k\). Choose \(z_k \in Q^k_{i,k} \cap \Gamma\) for each \(k\). Then \(|z - z_k| \leq \text{diam}(Q^k_{i,k}) \to 0\) as \(k \to \infty\).

Since \(\Gamma\) is compact, \(z \in \Gamma\), so \((E \times Y) \subset \Gamma = \Lambda\).

We now prove Theorem 4.2 with \(\Lambda = \overline{\Gamma}\).

**Proof of Theorem 4.2.** By Lemmas 4.3 and 4.4 we have \(\dimc \Gamma \geq \dimc (E \times Y) = 1 + d\). Note that \(\Gamma \setminus (E \times Y)\) is a countable union of disjoint smooth curves of Hausdorff dimension 1 so that \(\dimh(\Gamma \setminus (E \times Y)) = 1\). Also \(\partial \Gamma \subset (E \times Y)\) so that \(\Gamma \setminus (E \times Y) = \Gamma \setminus (E \times Y)\). The stability and additivity properties of Hausdorff dimension yield
\[
\dimh \Gamma = \max\{\dimh(\Gamma \setminus (E \times Y)), \dimh(E \times Y)\}
= \max\{1, 1 + d\}
= 1 + d.
\]

It follows from (4.10) and Proposition 2.6 that \(\dimc \Gamma \leq 1 + d\), and hence \(\dimc \Lambda = \dimc \Gamma = 1 + d = c\).

**Corollary 4.5.** Let \(V_{c-1}\) be as in Conjecture 3.2. Then
\[
\dimc R_{c-1} = \dimh R_{c-1} = c.
\]

Theorem 4.2 guarantees the existence of the spaces \(R_d\), but the value \(\dimc R_d\) remains unknown. Since \(\dimc R_d \leq \dimc R_d\) by Proposition 2.6 Corollary 4.3 provides a crude upper bound on \(\dimc R_d\). We do not know any non-trivial lower bounds. Indeed, without the presence of a diffuse family of surfaces, it is difficult to determine any nontrivial lower bound on \(\dimc R_d\).
Question 4.6. Determine $\dim_{tC} R_d$.

Topological conformal dimension and topological Hausdorff dimension are related in the following way. For every metric space $X$,

$$\dim_{tC} X \leq \inf \{ \dim_{tH} f(X) : f \text{ quasisymmetric} \}. \quad (4.11)$$

Question 6.4 in [5] asks whether equality holds in (4.11) for every $X$. It is not clear whether the $tH$-dimension of $R_d$ can be lowered by quasisymmetric maps.

Question 4.7. Given $0 < d < \infty$, is there a quasisymmetric mapping $f$ such that $\dim_{tH} f(R_d) < \dim_{tH} R_d$?

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