NONLINEAR SCHRÖDINGER EQUATIONS WITH SYMMETRIC MULTI-POLAR POTENTIALS

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1. Introduction

Schrödinger equations with Hardy-type singular potentials have been the object of a quite large interest in the recent literature, see e.g. [1, 7, 8, 12, 15, 18, 20, 25, 26, 28]. The singularity of inverse square potentials \( V(x) \sim \lambda |x|^{-2} \) is critical both from the mathematical and the physical point of view. As it does not belong to the Kato’s class, it cannot be regarded as a lower order perturbation of the laplacian but strongly influences the properties of the associated Schrödinger operator. Moreover, from the point of view of nonrelativistic quantum mechanics, among potentials of type \( V(x) \sim \lambda |x|^{-\alpha} \), the inverse square case represent a transition threshold: for \( \lambda < 0 \) and \( \alpha > 2 \) (attractively singular potential), the energy is not lower-bounded and a particle near the origin in the presence a potential of this type “falls” to the center, whereas if \( \alpha < 2 \) the discrete spectrum has a lower bound (see [21]). Moreover inverse square singular potentials arise in many fields, such as quantum mechanics, nuclear physics, molecular physics, and quantum cosmology; we refer to [17] for further discussion and motivation.

The case of multi-polar Hardy-type potentials was considered in [14, 11]. In particular in [14] the authors studied the ground states of the following class of nonlinear elliptic equations with a critical power-nonlinearity and a potential exhibiting multiple inverse square singularities:

\[
\begin{aligned}
-\Delta v - \sum_{i=1}^{k} \frac{\lambda_i}{|x-a_i|^2} v &= v^{2^*-1}, \\
v > 0 \quad \text{in } \mathbb{R}^N \setminus \{a_1, \ldots, a_k\}.
\end{aligned}
\]

For Schrödinger operators \( -\Delta + V \) the potential term \( V \) describes the interactions of the quantum particles with the environment. Hence, multi-singular inverse-square potentials are associated with the interaction of particles with a finite number of electric dipoles. The mathematical interest in this problem rests in its criticality, for the exponent of the nonlinearity as well as the singularities share the same order of homogeneity with the laplacian.

The analysis carried out in [14] highlighted how the existence of solutions to (1) heavily depends on the strength and the location of the singularities. For the scaling properties of the problem, the mutual interaction among the poles actually depends only on the shape of their configuration. When the poles form a symmetric structure, it is natural to wonder how the symmetry affects such mutual interaction. The present paper means to study this aspect from the point of view of the existence of solutions inheriting the same symmetry properties as the set of singularities. More

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precisely we deal with a class of nonlinear elliptic equations on \( \mathbb{R}^N \), \( N \geq 3 \), involving a critical power-nonlinearity as well as a Hardy-type potential which is singular on sets exhibiting some simple kind of symmetry, as depicted in figures 1–3.

Let us start by considering a potential featuring multiple inverse square singularities located on the vertices of \( k \)-side regular concentric polygons. Let us write \( \mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2} \). For \( k \in \mathbb{N} \), we consider the group \( \mathbb{Z}_k \times \mathbb{S} \mathbb{O}(N-2) \) acting on \( D^{1,2}(\mathbb{R}^N) \) as \( u(y', z) \to v(y', z) = u(e^{2\pi \sqrt{-1}/k}y', Tz) \), \( T \) being any rotation of \( \mathbb{R}^{N-2} \).

Let us consider \( m \) regular polygons (with \( k \) sides) centered at the origin and lying on the plane \( \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N \). Let us denote as \( a^\ell_i \), \( i = 1, 2, \ldots, k \), the vertices of the \( \ell \)-th polygon, \( \ell = 1, 2, \ldots, m \). Since the polygons are regular, we have \( a^\ell_1 = e^{2\pi \sqrt{-1}/k}a^\ell_{k-1} \). We look for \( \mathbb{Z}_k \times \mathbb{S} \mathbb{O}(N-2) \)-invariant solutions to the following equation

\[
\begin{align*}
-\Delta v - \frac{\lambda_0}{|x|^2} v - \sum_{\ell=1}^m \sum_{i=1}^k \frac{\lambda_\ell}{|x-a^\ell_i|^2} v &= v^{2^*-1}, \\
v &> 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0, a^\ell_i : 1 \leq \ell \leq m, 1 \leq i \leq k\},
\end{align*}
\]

where \( 2^* = \frac{2N}{N-2} \), i.e. for solutions belonging to the space

\[
D^{1,2}_k(\mathbb{R}^N) = \{ u(z,y) \in D^{1,2}(\mathbb{R}^N) : u(e^{2\pi \sqrt{-1}/k}z, y) = u(z, |y|) \},
\]

see figures 1–3. Here \( D^{1,2}(\mathbb{R}^N) \) denotes the closure of the space \( D(\mathbb{R}^N) \) of smooth functions with compact support with respect to the Dirichlet norm

\[
\| u \|_{D^{1,2}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{1/2}.
\]

Let us denote as

\[
r_\ell = |a^\ell_1| = |a^\ell_2| = \cdots = |a^\ell_k|, \quad \text{for} \quad \ell = 1, 2, \ldots, m,
\]

the radius of the \( \ell \)-polygon and as

\[
\Lambda_\ell = k\lambda_\ell, \quad \ell = 1, \ldots, m,
\]

the total mass of poles located on the \( \ell \)-th polygon. The Rayleigh quotient associated with problem (2) in \( D^{1,2}_k(\mathbb{R}^N) \) is

\[
(3) \quad S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) = \inf_{u \in D^{1,2}_k(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \left( \frac{\lambda_0}{|x|^2} + \sum_{\ell=1}^m \sum_{i=1}^k \frac{\Lambda_\ell}{k|x-a^\ell_i|^2} \right) u^2(x) \, dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{2/2^*}}.
\]

It is well known that minimizers of (3) solve equation (2) up to a Lagrange multiplier. Theorems 1.4 and 1.5 give sufficient conditions for the attainability of \( S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) for large values of \( k \). Letting \( k \to \infty \), the Schrödinger operator converges, in the sense of distributions, to the operator associated with a continuous distribution of mass on concentric circles. We stress that the convergence of the potentials does not hold in the natural way, i.e. in \( L^p_{\text{loc}}(\mathbb{R}^N) \) for any \( p \leq \frac{N}{2} \), because of the singularity. To formulate the limiting problem, for any \( r > 0 \), we denote as

\[
S_r := \{ (x, 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : |x| = r \}.
\]
the circle of radius \( r \) lying on the plane \( \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N \) and consider the distribution \( \delta_{S_r} \in \mathcal{D}'(\mathbb{R}^N) \) supported in \( S_r \) and defined by

\[
\mathcal{D}'(\mathbb{R}^N) \left\langle \delta_{S_r}, \varphi \right\rangle_{\mathcal{D}(\mathbb{R}^N)} := \int_{S_r} \varphi(x) d\sigma(x) = \frac{1}{2\pi r} \int_{S_r} \varphi(x) d\sigma(x) \quad \text{for any } \varphi \in \mathcal{D}(\mathbb{R}^N),
\]

where \( d\sigma \) is the line element on \( S_r \). We look for solutions to the following equation

\[
\begin{aligned}
-\Delta v &- \frac{\lambda_0}{|x|^2} v - \sum_{\ell=1}^m \Lambda_{\ell} \left( \frac{1}{|x|^2} \right) v = v^{2^* - 1}, \\
v &> 0 \text{ in } \mathbb{R}^N \setminus \{0, \cup_{1 \leq \ell \leq m} S_{r\ell} \},
\end{aligned}
\]

which are invariant by the action of the group \( \mathfrak{so}(2) \times \mathfrak{so}(N-2) \). To this purpose, the natural space to set the problem is

\[ \mathcal{D}^{1,2}_{\text{circ}}(\mathbb{R}^N) = \{ u(z, y) \in \mathcal{D}^{1,2}(\mathbb{R}^N) : u(z, y) = u(|z|, |y|) \}, \]

and to consider the associated Rayleigh quotient

\[
S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) = \inf_{\mathcal{D}^{1,2}_{\text{circ}}(\mathbb{R}^N) \atop u \not \equiv 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \left( \frac{\lambda_0}{|y|^2} + \sum_{\ell=1}^m \Lambda_{\ell} \int_{S_{r\ell}} \frac{d\sigma(x)}{|x|^2} \right) u^2(y) \, dy}{\left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{2/2^*}}.
\]

The following theorem contains a Hardy type inequality for potentials which are singular at circles.

**Theorem 1.1.** Let \( N \geq 3 \) and \( r > 0 \). For any \( u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \) the map \( y \mapsto u(y) \int_{S_r} \frac{d\sigma(x)}{|x-y|^2} \) belongs to \( L^2(\mathbb{R}^N) \) and

\[
\left( \frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} |u(y)|^2 \left( \int_{S_r} \frac{d\sigma(x)}{|x-y|^2} \right) \, dy \leq \int_{\mathbb{R}^N} |\nabla u(y)|^2 \, dy.
\]

Moreover the constant \( \left( \frac{N-2}{2} \right)^2 \) is optimal and not attained.

Hardy type inequalities involving singularities at smooth compact boundaryless manifolds have been considered by several authors, see [9, 16] and references therein. In the aforementioned papers, the potentials taken into account are of the type \( |\text{dist}(x, \Sigma)|^{-2} \), where \( \text{dist}(x, \Sigma) \) denotes the distance from a smooth compact manifold \( \Sigma \). We point out that such kind of potentials are quite different from the ones we are considering. Indeed an explicit computation yields

\[
V^r(y) := \int_{S_r} \frac{d\sigma(x)}{|x-y|^2} = \frac{1}{\sqrt{(r^2 + |y|^2)^2 - 4r^2|y|^2}} \quad \text{for all } y = (y', z) \in \mathbb{R}^N = \mathbb{R}^2 \times \mathbb{R}^{N-2}.
\]

Hence

\[
V^r(y) \sim \frac{1}{|y|^2} \quad \text{as } |y| \to +\infty
\]

whereas

\[
V^r(y) \sim \frac{1}{\sqrt{r^2 + |y|^2 - 2r|y'|}} \quad \text{as } \text{dist}(y, S_r) \to 0.
\]
Hence the singularity at the circle of $V^r$ is weaker than the inverse square distance potential considered in [9, 16], but has the same behavior at $\infty$. We also remark that $V^r$ is “regular” in the sense of the classification of singular potentials given in [17].

Arguing as in [14, Proposition 1.1], it is easy to verify that solvability of equations (2) and (4) requires the positivity of the associated quadratic forms. Let us consider for example the quadratic form associated with potentials singular on circles, i.e.

$$Q_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}^{\text{circ}}(R^N) = \int_{R^N} |\nabla u|^2 \, dx - \int_{R^N} \left( \frac{\lambda_0}{|y|^2} + \sum_{\ell=1}^m \Lambda_\ell \int_{S_{r_\ell}} \frac{d\sigma(x)}{|x - y|^2} \right) u^2(y) \, dy.$$  

From (6) and Sobolev’s inequality, it follows that

$$Q_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}^{\text{circ}}(R^N) \geq \left[ 1 - \frac{4}{(N-2)^2} \left( \lambda_0^+ + \sum_{\ell=1}^m \Lambda_\ell^+ \right) \right] \int_{R^N} |\nabla u|^2 \, dx,$$

where $t^+ := \max\{t, 0\}$ denotes the positive part. Hence $Q_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}^{\text{circ}}(R^N)$ is positive definite whenever

$$\lambda_0^+ + \sum_{\ell=1}^m \Lambda_\ell^+ < \frac{(N-2)^2}{4},$$

see [14, Proposition 1.2] for further discussion on the positivity of the quadratic form. Condition (8) also ensures the positivity of the quadratic form associated to problem (2).

The attainability of $S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)$ and $S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m)$ requires a delicate balance between the contribution of positive and negative masses. In particular, if $N \geq 4$ and all the masses have the same sign, $S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)$ and $S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m)$ are never achieved.

**Theorem 1.2.** Let $N \geq 4$, $\lambda_0, \Lambda_1, \ldots, \Lambda_m \in \mathbb{R}$, $r_1, r_2, \ldots, r_m \in \mathbb{R}^+$ satisfy (8). If

- either (i) $\Lambda_\ell < 0$ for all $\ell = 1, \ldots, m$
- or (ii) $\lambda_0 > 0$, $\Lambda_\ell > 0$ for all $\ell = 1, \ldots, m$

then neither $S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)$ nor $S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m)$ are attained.

The analysis we are going to carry out in the present paper will highlight that, from the point of view of minimization of the Rayleigh quotient, spreading mass all over a continuum is more convenient than localization of mass at isolated points.

**Theorem 1.3.** Let $N \geq 4$, $\lambda_0, \Lambda_1, \ldots, \Lambda_m \in \mathbb{R}$, $r_1, r_2, \ldots, r_m \in \mathbb{R}^+$ satisfy (8) and

$$\sum_{\ell=1}^m \Lambda_\ell \leq 0, \quad \lambda_0 < \frac{(N-2)^2}{4},$$

and

$$\sum_{\ell=1}^m \frac{\Lambda_\ell}{|r_\ell|^2} > 0, \quad \text{if} \quad \lambda_0 \leq \frac{N(N-4)}{4},$$

$$\sum_{\ell=1}^m \frac{\Lambda_\ell}{|r_\ell|\sqrt{(N-2)^2 - 4\lambda_0}} > 0, \quad \text{if} \quad \frac{N(N-4)}{4} < \lambda_0 < \frac{(N-2)^2}{4}.$$

Then the infimum in (5) is achieved. In particular equation (4) admits a solution which is $\text{SO}(2) \times \text{SO}(N-2)$-invariant.
As problem (5) is the limit of (3), when $k \to \infty$, we expect the assumptions of Theorem 1.3 to ensure the existence of solutions to (3) provided $k$ is sufficiently large. Indeed the theorem below states that (9) and (10) are sufficient conditions on radii and masses of the polygons for the infimum in (3) to be achieved when $k$ is large.

**Theorem 1.4.** Let $N \geq 4$, $\lambda_0, \Lambda_1, \ldots, \Lambda_m \in \mathbb{R}$, $r_1, r_2, \ldots, r_m \in \mathbb{R}^+$ satisfy (8), (9) and (10). For any $\ell = 1, \ldots, m$ and $k \in \mathbb{N}$ let $\{a^{\ell}_i\}_{i=1}^k$ be the vertices of a regular $k$-side polygon centered at 0 of radius $r_\ell$ and let $\lambda_\ell = \Lambda_\ell/k$. Then if $k$ is sufficiently large, the infimum in (3) is achieved. In particular equation (2) admits a solution which is $\mathbb{Z}_k \times \mathbb{SO}(N-2)$-invariant.

When $N > 4$, it is possible to estimate how large $k$ must be in order to obtain the above existence result. This is the content of the following theorem.

**Theorem 1.5.** Assume that $N > 4$, $\lambda^+_0 + k \sum_{j=1}^m \lambda^+_j < (N-2)^2/4$,

\(\sum_{\ell=1}^m \lambda_\ell \leq 0,\)

\(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \frac{N(N-4)}{4},\)

\(\lambda_0 < \frac{(N-2)^2}{4},\)

\(\sum_{\ell=1}^m \frac{\lambda_\ell}{|r_\ell|^2} > 0, \quad \text{if} \quad \lambda_0 \leq \frac{N(N-4)}{4},\)

\(\sum_{\ell=1}^m \frac{\lambda_\ell}{|r_\ell|\sqrt{(N-2)^2-4\lambda_0}} > 0, \quad \text{if} \quad \frac{N(N-4)}{4} < \lambda_0 < \frac{(N-2)^2}{4},\)

\(\frac{\lambda_0}{|r_m|^2} + \lambda_m \sum_{i=1}^{k-1} \frac{1}{4r_m^2 \sin \frac{\pi i}{k}^2} + \sum_{\ell=1}^{m-1} \lambda_\ell \sum_{i=1}^{k} \frac{1}{r_m^2 + r_\ell^2 - 2r_m r_\ell \cos \left(\frac{2\pi i}{k} + \Theta_{j\ell}\right)} > 0,\)

where $\Theta_{j\ell}$ denoted the minimum angle formed by vectors $a^{\ell}_j$ and $a^{\ell}_s$. (see figure 4). Then the infimum in (3) is achieved. In particular equation (2) admits a solution.

This paper is organized as follows. In section 2 we recall some known facts about the single-polar problem and study the behavior of any solution (radial and non radial) to the one pole-equation near the singularities $0$ and $\infty$. In section 3 we prove the Hardy type inequality for potentials which are singular at circles stated in Theorem 1.1. Section 4 contains an analysis of possible reasons for lack of compactness of minimizing sequences of problems (3) and (5) and a local Palais-Smale condition below some critical thresholds. In section 5 we provide some interaction estimates which are needed in section 6 to compare the concentration levels of minimization sequences and consequently to prove Theorem 1.3. Section 7 contains the study of behavior of energy levels of minimizing sequences as $k \to \infty$ which is needed in section 8 to prove Theorem 1.4. Last section is devoted to the proof of Theorem 1.5.

Figures 1–3 are the plot of the test functions used to estimate the energy levels of Palais-Smale sequences and represent the expected shape of solutions found in Theorem 1.4 (assuming the last $N-2$ variables to be 0), which is based on the knowledge of their behavior at singularities (see
[13] and Theorem 2.2), which is known to be singular at poles with positive mass and vanishing at poles with negative mass.

Notation. We list below some notation used throughout the paper.

- \( B(a, r) \) denotes the ball \( \{ x \in \mathbb{R}^N : |x| < r \} \) in \( \mathbb{R}^N \) with center at \( a \) and radius \( r \).
- \( S_r = \{ (x, 0) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : |x| = r \} \) denotes the circle of radius \( r \) in the plane \( \mathbb{R}^2 \times \{0\} \).
- \( \delta_x \) denotes the Dirac mass located at point \( x \in \mathbb{R}^N \).
- \( D(\mathbb{R}^N) \) is the space of smooth functions with compact support in \( \mathbb{R}^N \).
- \( D^{1,2}(\mathbb{R}^N) \) is the closure of \( D(\mathbb{R}^N) \) with respect to the Dirichlet norm \((\int_{\mathbb{R}^N} |\nabla u|^2 \, dx)^{1/2}\).
- \( C_0(\mathbb{R}^N) \) denotes the closure of continuous functions with compact support in \( \mathbb{R}^N \) with respect to the uniform norm.
- \( \text{dist}(x, A) \) denotes the distance of the point \( x \in \mathbb{R}^N \) from the set \( A \subset \mathbb{R}^N \).
- \( \| \cdot \|_p \) denotes the norm in the Lebesgue space \( L^p(\mathbb{R}^N) \).
- \( A \triangle B \) denotes the symmetric difference of sets \( A \) and \( B \), i.e. \( A \triangle B = (A \setminus B) \cup (B \setminus A) \).
2. \( k = 20, m = 2, \lambda_0, \lambda_1 > 0, \lambda_2 < 0 \).

3. \( k = 6, m = 2, \lambda_1 > 0, \lambda_0, \lambda_2 < 0 \).
2. The problem with one singularity

For any \( \lambda < (N - 2)^2/4 \), the problem with one singularity

\[
\begin{align*}
-\Delta u &= \frac{\lambda}{|x|^2}u + u^{2^*-1}, \quad x \in \mathbb{R}^N, \\
u > 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \text{ and } u \in D^{1,2}(\mathbb{R}^N),
\end{align*}
\]

admits a family of positive solutions given by

\[
w^\lambda_u(x) = \mu^{-\frac{N-2}{2}} u^\mu\left(\frac{x}{\mu}\right), \quad \mu > 0,
\]

where we denote

\[
w^\lambda_u(x) = \left(\frac{N(N-2)\nu^2}{\lambda}\right)^{\frac{N-2}{4}} \left(\frac{|x|^{1-\nu} (1 + |x|^{2\nu})}{|x|^{N-2}}\right)^{-\frac{N-2}{2}}, \quad \text{and } \nu_\lambda = \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{1/2}.
\]

Moreover, when \( 0 \leq \lambda < (N - 2)^2/4 \), all \( w^\lambda_u(x) \) minimize the associated Rayleigh quotient and the minimum can be computed as:

\[
S(\lambda) := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{Q_\lambda(u)}{(\int_{\mathbb{R}^N} |u|^2 dx)^{2/2^*}} = \frac{Q_\lambda(w^\lambda_u)}{(\int_{\mathbb{R}^N} |w^\lambda_u|^2 dx)^{2/2^*}} = \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{\frac{N-4}{4}} S,
\]

where we denoted the quadratic form \( Q_\lambda(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx \), see [28], and \( S \) is the best constant in the Sobolev inequality

\[S\|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|u\|_{D^{1,2}(\mathbb{R}^N)}^2.
\]

As minimizers of problem (19), we consider

\[
z^\lambda_u(x) = \frac{w^\lambda_u(x)}{\left(\int_{\mathbb{R}^N} |w^\lambda_u|^2 dx\right)^{1/2}} = \alpha_{\lambda,N} \mu^{-\frac{N-2}{4}} \left(\frac{|x|^{1-\nu_\lambda}}{\mu} + \left|\frac{x}{\mu}\right|^{1+\nu_\lambda}\right)^{-\frac{N-2}{2}},
\]

where \( \alpha_{\lambda,N} = (N(N-2)\nu^2)^{\frac{N-2}{4}} \|w^\lambda_u\|_{L^{2^*}}^{-1} \) is a positive constant depending only on \( \lambda \) and \( N \), so that for \( 0 \leq \lambda < (N - 2)^2/4 \)

\[
S(\lambda) = Q_\lambda(z^\lambda_u) \quad \text{for all } \mu > 0.
\]

For \( -\infty < \lambda < (N - 2)^2/4 \), we also set

\[
S_\lambda := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{Q_\lambda(u)}{(\int_{\mathbb{R}^N} |u|^2 dx)^{2/2^*}}.
\]

We note that \( S(\lambda) \leq S_\lambda(\lambda) \), and equality holds whenever \( \lambda \geq 0 \). Moreover the following result has been proved in [28].

**Lemma 2.1** (see [28], Lemma 6.1). *Let \( N \geq 4 \). If \( S_\lambda(\lambda) < k^{2/N} S \) then \( S_\lambda(\lambda) \) is achieved.*

In [28] it is proved that if \( \lambda \in (0,(N - 2)^2/4) \) then all solutions to (16) are of the form (17) while if \( \lambda \ll 0 \) then also non radial solutions to (16) can exist. The behavior of any solution (radial and non radial) to problem (16) near the singularities 0 and \( \infty \) is described by the following theorem.
Theorem 2.2. If \( \lambda < (N-2)^2/4 \) and \( u \in D^{1,2}(\mathbb{R}^N) \) is a solution to problem (16), then there exist positive constant \( \kappa_0(u) \) and \( \kappa_\infty(u) \) depending on \( u \) such that

\[
\begin{align*}
(23) \quad u(x) &= |x|^{-\frac{N-2}{2}(1-\nu_\lambda)} [\kappa_0(u) + O(|x|^{\alpha})], \quad \text{as } x \to 0, \\
(24) \quad u(x) &= |x|^{-\frac{N-2}{2}(1+\nu_\lambda)} [\kappa_\infty(u) + O(|x|^{-\alpha})], \quad \text{as } |x| \to +\infty,
\end{align*}
\]

for some \( \alpha \in (0, 1) \).

Remark 2.3. Putting together (23–24) we deduce that there exists a positive constant \( \kappa(u) \) depending on \( u \) such that

\[
\frac{1}{\kappa(u)} w^1_\lambda(x) \leq u(x) \leq \kappa(u) w^1_\lambda(x).
\]

Proof of Theorem 2.2. Set

\[
a_\lambda = \frac{(N-2)(1-\nu_\lambda)}{2}, \quad v(x) = |x|^{a_\lambda} u(x).
\]

Then the function \( v \) belongs to \( D^{1,2}_{a_\lambda}(\mathbb{R}^N) \) where \( D^{1,2}_{a_\lambda}(\mathbb{R}^N) \) denotes the space obtained by completion of \( D(\mathbb{R}^N) \) with respect to the weighted Dirichlet norm

\[
\|v\|_{D^{1,2}_{a_\lambda}(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |x|^{-2a_\lambda} |\nabla v|^2 \, dx \right)^{1/2}.
\]

Moreover \( v \) solves equation

\[
-\text{div} \left( |x|^{-2a_\lambda} \nabla v \right) = \frac{v^{2^*-1}}{|x|^{2*a_\lambda}}.
\]

From [13, Theorem 1.2], it follows that \( v \) is Hölder continuous; in particular expansion (23) holds for \( \kappa_0(u) = v(0) \) and some \( \alpha \in (0, 1) \). Moreover \( v(0) \) is strictly positive in view of Harnack’s inequality for degenerate operators proved in [19], see also [10]; we mention that weights of type \( |x|^{-2a} \) with \( a < \frac{N-2}{2} \) belong to the class of quasi-conformal weights considered in [19].

To deduce (24), we perform the change of variable

\[
\tilde{v}(x) = |x|^{a_\lambda-(N-2)} u \left( \frac{x}{|x|^2} \right),
\]

and observe that the transformed function \( \tilde{v} \) solves equation (26). Hence [13, Theorem 1.2] yields that \( \tilde{v} \) is Hölder continuous and admits the following expansion

\[
\tilde{v}(x) = \tilde{v}(0) + O(|x|^\alpha), \quad \text{as } x \to 0,
\]

for some \( \alpha \in (0, 1) \), where \( \tilde{v}(0) > 0 \) in view of Harnack’s inequality in [19]. Coming back to \( u \) we obtain that \( u \) satisfies (24) with \( \kappa_\infty(u) = \tilde{v}(0) > 0 \). \( \blacksquare \)
3. HARDY’S INEQUALITY WITH SINGULARITY ON A CIRCLE

We prove now the Hardy type inequality for potentials which are singular at circles stated in Theorem 1.1.

**Proof of Theorem 1.1.** Let us consider the minimization problem

\[
I(S_r) := \inf_{u \in D^{1,2}(\mathbb{R}^N), u \not\equiv 0} \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2 dy}{\int_{\mathbb{R}^N} |u(y)|^2 \left( \int_{S_r} \frac{d\sigma(x)}{|x-y|^2} \right) dy} = \inf_{u \in D(\mathbb{R}^N \setminus \{0\})} \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2 dy}{\int_{\mathbb{R}^N} |u(y)|^2 \left( \int_{S_1} \frac{d\sigma(x)}{|x-y|^2} \right) dy},
\]

where the last equality is due to density of \( D(\mathbb{R}^N \setminus \{0\}) \) in \( D^{1,2}(\mathbb{R}^N) \) (see e.g. [5, Lemma 2.1]). An easy calculation shows that for any \( u \in D^{1,2}(\mathbb{R}^N) \)

\[
\int_{\mathbb{R}^N} |\nabla u(y)|^2 dy \left( \int_{S_r} \frac{d\sigma(x)}{|x-y|^2} \right) dy = \int_{\mathbb{R}^N} |\nabla v(y)|^2 dy \left( \int_{S_1} \frac{d\sigma(x)}{|x-y|^2} \right) dy,
\]

where \( v(y) = u(ry) \). Therefore

\[
I(S_r) = I(S_1) \quad \text{for any } r > 0.
\]

In view of (27), it is enough to prove the theorem for \( r = 1 \). The proof consists in three steps.

**Step 1:** Inequality (6) i.e. \( I(S_1) \geq \left( \frac{N-2}{2} \right)^2 \).

For any \( u \in D^{1,2}(\mathbb{R}^N), u \geq 0 \) a.e., we consider the Schwarz symmetrization \( u^* \) of \( u \) defined as

\[
u^*(x) := \inf \{ t > 0 : \{|y \in \mathbb{R}^N : u(y) > t| \leq \omega_N |x|^N \} \}
\]

where \( |\cdot| \) denotes the Lebesgue measure of \( \mathbb{R}^N \) and \( \omega_N \) is the volume of the standard unit \( N \)-ball. From [31, Theorem 21.8], it follows that for any \( x \in S_1 \)

\[
\int_{\mathbb{R}^N} \frac{|u(y)|^2}{|x-y|^2} dy \leq \int_{\mathbb{R}^N} |u^*(y)|^2 \left[ \left( \frac{1}{|x-y|} \right)^* \right]^2.
\]

Since \( \left( \frac{1}{|x-y|} \right)^* = \frac{1}{|y|} \), we deduce

\[
\int_{\mathbb{R}^N} \frac{|u(y)|^2}{|x-y|^2} dy \leq \int_{\mathbb{R}^N} \frac{|u^*(y)|^2}{|y|^2} dy.
\]

Moreover by Polya-Szego inequality

\[
\int_{\mathbb{R}^N} |\nabla u^*|^2 \leq \int_{\mathbb{R}^N} |\nabla u|^2.
\]
From (29–30) and the classical Hardy’s inequality, it follows that, for any \( u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} \), \( u \geq 0 \) a.e.,

\[
\frac{\int_{\mathbb{R}^N} \|\nabla u(y)\|^2 \, dy}{\int_{\mathbb{R}^N} |u(y)|^2 \left( \int_{S_1} \frac{d\sigma(x)}{|x-y|^2} \right) \, dy} = \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2 \, dy}{\int_{S_1} \left( \int_{\mathbb{R}^N} |u(y)|^2 \frac{dy}{|x-y|^2} \right) \, d\sigma(x)} \geq \frac{\int_{\mathbb{R}^N} |\nabla u_*(y)|^2 \, dy}{\int_{\mathbb{R}^N} |u_*(y)|^2 \, dy} \geq \left( \frac{N-2}{2} \right)^2.
\]

Due to evenness of the quotient we are minimizing, to compute \( I(S_1) \) it is enough to take the infimum over positive functions. Hence passing to the infimum in the above inequality, we obtain

\[ I(S_1) \geq \left( \frac{N-2}{2} \right)^2. \]

**Step 2: Optimality of the constant, i.e.** \( I(S_1) = \left( \frac{N-2}{2} \right)^2 \).

We fix \( u \in D(\mathbb{R}^N \setminus \{0\}) \) and let \( 0 < r < R \) be such that \( \text{supp} \, u \subset \{ x \in \mathbb{R}^N : r < |x| < R \} \). For any \( 0 < \lambda \ll 1 \), we set \( \bar{u}_\lambda(x) = u(\lambda x) \). Hence we have

\[
I(S_1) \leq \frac{\int_{\mathbb{R}^N} |\nabla \bar{u}_\lambda(y)|^2 \, dy}{\int_{\mathbb{R}^N} |\bar{u}_\lambda(y)|^2 \left( \int_{S_1} \frac{d\sigma(x)}{|x-y|^2} \right) \, dy} \geq \frac{\int_{\mathbb{R}^N} |\nabla u_*(y)|^2 \, dy}{\int_{\mathbb{R}^N} |u_*(y)|^2 \, dy} \geq \frac{\left( \frac{N-2}{2} \right)^2}{\left( \frac{N-2}{2} \right)^2}.
\]

Since \( \frac{1}{|x-y|^2} \leq \frac{1}{r^2} \) for all \( y \in \text{supp} \, u \), \( x \in S_1 \), and \( 0 < \lambda < \frac{r}{2} \), by Dominated Convergence Theorem we deduce that \( \int_{\mathbb{R}^N} |u(y)|^2 \left( \int_{S_1} \frac{d\sigma(x)}{|x-y|^2} \right) \, dy \) converges to \( \int_{\mathbb{R}^N} \frac{|u(y)|^2}{|y|^2} \, dy \) as \( \lambda \to 0 \), hence passing to the limit in (31) we obtain

\[
I(S_1) \leq \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2 \, dy}{\int_{\mathbb{R}^N} |y|^{-2}|u(y)|^2 \, dy} \quad \text{for any} \quad u \in D(\mathbb{R}^N \setminus \{0\}).
\]

By density of \( D(\mathbb{R}^N \setminus \{0\}) \) in \( D^{1,2}(\mathbb{R}^N) \) we deduce

\[
I(S_1) \leq \inf_{D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2 \, dy}{\int_{\mathbb{R}^N} |y|^{-2}|u(y)|^2 \, dy} = \left( \frac{N-2}{2} \right)^2
\]

where the last equality follows from the optimality of the constant \( \left( \frac{N-2}{2} \right)^2 \) in the classical Hardy inequality (see [18, Lemma 2.1]). Collecting the above inequality with the one proved in Step 1, we find \( I(S_1) = \left( \frac{N-2}{2} \right)^2 \).

**Step 3: The infimum \( I(S_1) \) is not attained.**

Arguing by contradiction, assume that the infimum \( I(S_1) \) is achieved by some \( \bar{u} \in D^{1,2}(\mathbb{R}^N) \). We can assume that \( \bar{u} \geq 0 \) (otherwise we consider \( |\bar{u}| \) which is also a minimizer by evenness of the
that the infimum in (32) cannot be attained (see [28, Remark 1.2]), we reach a contradiction.

\[
\left( \frac{N-2}{2} \right)^2 = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2 \, dy}{\int_{\mathbb{R}^N} \frac{|\tilde{u}(y)|^2 \, dy}{|x-y|^2} \, d\sigma(x)} \geq \int_{\mathbb{R}^N} \frac{|\tilde{u}^*(y)|^2 \, dy}{|y|^2 \, dy} \geq \left( \frac{N-2}{2} \right)^2,
\]

Therefore the above inequalities are indeed equalities and this implies that the infimum

\[
\left( \frac{N-2}{2} \right)^2 = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u(y)|^2 \, dy}{\int_{\mathbb{R}^N} \frac{|\tilde{u}^*(y)|^2 \, dy}{|y|^2 \, dy}}
\]

which yields the best constant in the classical Hardy inequality, is achieved by \( \tilde{u}^* \). Since it is known that the infimum in (32) cannot be attained (see [28, Remark 1.2]), we reach a contradiction. 

4. The Palais-Smale condition under \( \mathbb{Z}_k \times \mathbb{S}(N-2) \) and \( \mathbb{S}(2) \times \mathbb{S}(N-2) \)-invariance

Let us define the functional \( J_k : \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R} \) associated to equation (2) as

\[
J_k(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\lambda_0}{2} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2} \, dx
- \sum_{\ell=1}^m \sum_{i=1}^k \frac{\lambda_\ell}{2} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2} \, dx - \frac{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \, dx.
\]

The choice of location of the singularities ensures that \( J_k \) is \( \mathbb{Z}_k \times \mathbb{S}(N-2) \)-invariant. Since \( \mathbb{Z}_k \times \mathbb{S}(N-2) \) acts by isometries on \( \mathcal{D}^{1,2}(\mathbb{R}^N) \), we can apply the Principle of Symmetric Criticality by Palais [24] to deduce that the critical points of \( J_k \) restricted to \( \mathcal{D}_k^{1,2}(\mathbb{R}^N) \) are also critical points of \( J_k \) in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \). Therefore, if \( u \) is a critical point of \( J_k \) in \( \mathcal{D}_k^{1,2}(\mathbb{R}^N) \), \( u > 0 \) outside singularities, then \( v = S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) is a solution to equation (2).

The following theorem provides a local Palais-Smale condition for \( J_k \) restricted to \( \mathcal{D}_k^{1,2}(\mathbb{R}^N) \) below some critical threshold. We emphasize that the invariance of the problem by the action of a subgroup of orthogonal transformation allows to recover some compactness, in the sense that concentration points of invariant functions must be located in some symmetric way, thus reducing the possibility of loss of compactness. The restriction on dimension \( N \geq 4 \) is required to avoid the presence of possible concentration points on \( \{0\} \times \mathbb{R}^{N-2} \). Indeed when \( N = 3 \), \( \mathbb{S}(N-2) = \mathbb{S}(1) \) is a discrete group, making thus possible concentration at points on the axis \( \{0\} \times \mathbb{R} \).

We mention that the Concentration-Compactness method under the action of \( \mathbb{Z}_k \times \mathbb{S}(N-2) \) was used by several authors to find \( k \)-bump solutions with prescribed symmetry for different classes of nonlinear elliptic equations: nonlinear Schrödinger equation in [29], nonlinear elliptic equations in symmetric domains in [4], nonlinear elliptic equations of Caffarelli-Kohn-Nirenberg type in [6], and elliptic equations with Hardy potential and critical growth in [28].

**Theorem 4.1.** Assume \( N \geq 4 \) and \( \lambda_0^+ + k \sum_{j=1}^m \lambda_j^+ < \frac{(N-2)^2}{4} \). Let \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_k^{1,2}(\mathbb{R}^N) \) be a Palais-Smale sequence for \( J_k \) restricted to \( \mathcal{D}_k^{1,2}(\mathbb{R}^N) \), namely

\[
\lim_{n \to \infty} J_k(u_n) = c < \infty \quad \text{in } \mathbb{R} \quad \text{and} \quad \lim_{n \to \infty} J_k'(u_n) = 0 \quad \text{in the dual space } (\mathcal{D}_k^{1,2}(\mathbb{R}^N))^*.
\]
If
\[
(34) \quad c < \frac{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)^{1 - \frac{1}{m}}}{N} \min\left\{ k^{\frac{1}{N}} S, k^{\frac{1}{N}} S(\lambda_1), \ldots, k^{\frac{1}{N}} S(\lambda_m), S_k(\lambda_0, \Lambda_0), S_k\left(\lambda_0 + k \sum_{\ell=1}^{m} \lambda_\ell\right)\right\}^{\frac{1}{N}},
\]
then \( \{u_n\}_{n \in \mathbb{N}} \) has a subsequence strongly converging in \( D_k^{1,2}(\mathbb{R}^N) \).

**Proof.** Let \( \{u_n\}_{n \in \mathbb{N}} \) be a Palais-Smale sequence for \( J_k \) in \( D_k^{1,2}(\mathbb{R}^N) \), then from Hardy’s and Sobolev’s inequalities it is easy to prove that \( \{u_n\}_{n \in \mathbb{N}} \) is a bounded sequence in \( D_k^{1,2}(\mathbb{R}^N) \). Hence, up to a subsequence, \( u_n \to u_0 \) in \( D_k^{1,2}(\mathbb{R}^N) \) and \( u_n \to u_0 \) almost everywhere. Therefore, from the Concentration Compactness Principle by P. L. Lions (see [22, 23]), we deduce the existence of a subsequence, still denoted by \( \{u_n\} \), for which there exist an at most countable set \( J \), points \( x_j \in \mathbb{R}^N \setminus \{0, a_i', 1 \leq i \leq k, 1 \leq \ell \leq m\} \), real numbers \( \mu_{x_j}, \nu_{x_j}, j \in J \), and \( \mu_0, \nu_0, \mu_{a_i'}, \nu_{a_i'}, \gamma_{a_i'} \), \( i = 1, \ldots, k, \ell = 1, \ldots, m \), such that the following convergences hold in the sense of measures
\[
\begin{align*}
|\nabla u_n|^2 \to d\mu & = |\nabla u_0|^2 + \mu_0 \delta_0 + \sum_{\ell=1}^{k} \sum_{i=1}^{m} \mu_{a_i'} \delta_{a_i'} + \sum_{j \in J} \mu_{x_j} \delta_{x_j}, \\
|u_n|^2^* \to d\nu & = |u_0|^2^* + \nu_0 \delta_0 + \sum_{\ell=1}^{k} \sum_{i=1}^{m} \nu_{a_i'} \delta_{a_i'} + \sum_{j \in J} \nu_{x_j} \delta_{x_j}, \\
\lambda_0 \frac{u_n^2}{|x|^2} \to d\gamma_0 & = \lambda_0 \frac{u_0^2}{|x|^2} + \gamma_0 \delta_0, \\
\lambda_\ell \frac{u_n^2}{|x-a_i'|^2} \to d\gamma_{a_i'} & = \lambda_\ell \frac{u_0^2}{|x-a_i'|^2} + \gamma_{a_i'} \delta_{a_i'}, \quad \text{for any } i = 1, \ldots, k, \ell = 1, \ldots, m.
\end{align*}
\]
From Sobolev’s inequality it follows that
\[
(39) \quad S \nu_{x_j}^\frac{2}{\lambda_0} \leq \mu_{x_j} \text{ for all } j \in J \quad \text{and} \quad S \nu_{a_i'}^\frac{2}{\lambda_\ell} \leq \mu_{a_i'} \text{ for all } i = 1, \ldots, k.
\]
Possible concentration at infinity of the sequence can be quantified by the following numbers
\[
(40) \quad \nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} |u_n|^2^* \, dx, \quad \mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} |\nabla u_n|^2 \, dx,
\]
and
\[
\gamma_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} \left( \lambda_0 + k \sum_{\ell=1}^{m} \lambda_\ell \right) \frac{u_n^2}{|x|^2} \, dx.
\]
Let us first prove that possible concentration points are located in a symmetric fashion. Pointwise convergence of \( u_n \in D_k^{1,2}(\mathbb{R}^N) \) to \( u_0 \) implies that \( u_0 \) is invariant by the \( \mathbb{Z}_k \times \mathbb{S}(N-2) \)-action, hence \( u_0 \in D_k^{1,2}(\mathbb{R}^N) \). Moreover, for any \( \phi \in C_0(\mathbb{R}^N) \) and for any \( \tau \in \mathbb{Z}_k \times \mathbb{S}(N-2) \) we have
\[
\int_{\mathbb{R}^N} |u_n|^2^* \phi = \int_{\mathbb{R}^N} |u_n|^2^* (\phi \circ \tau^{-1})
\]
that, passing to the limit as \( n \to \infty \), yields
\[
\sum_{\ell=1}^{k} \sum_{i=1}^{m} \nu_{a_i'} \phi(a_i') + \sum_{j \in J} \nu_{x_j} \phi(x_j) = \sum_{\ell=1}^{k} \sum_{i=1}^{m} \nu_{a_i'} \phi(\tau^{-1}(a_i')) + \sum_{j \in J} \nu_{x_j} \phi(\tau^{-1}(x_j)).
\]
Choosing $\phi = \phi^j_{i_0}$ such that $\phi^j_{i_0} \equiv 1$ on $B(a_i^j, \varepsilon/2)$, $\phi^j_{i_0} \equiv 0$ on $\mathbb{R}^N \setminus B(a_i^j, \varepsilon)$, $0 \leq \phi^j_{i_0} \leq 1$, and letting $\varepsilon \to 0$, we find that $\nu_{a_i}^j = \nu_{r^{-1}a_i^j}$. Hence we deduce that for any $\ell = 1, \ldots, m$ there exists $\nu^\ell_a \in \mathbb{R}$ such that

$$\nu_{a_i}^j = \nu^\ell_a$$

for any $i = 1, \ldots, k$.

Fix $j \in J$ and let $\phi = \phi_{j, \varepsilon}$ such that $\phi_{j, \varepsilon} \equiv 1$ on $B(x_j, \varepsilon/2)$, $\phi_{j, \varepsilon} \equiv 0$ on $\mathbb{R}^N \setminus B(x_j, \varepsilon)$, $0 \leq \phi_{j, \varepsilon} \leq 1$. Letting $\varepsilon \to 0$, we find that

(i) either $\nu_{x_j} = 0$

(ii) or for any $\tau \in Z_k \times SO(N-2)$ there exists $i \in J$ such that $\tau(x_j) = x_i$.

When $N \geq 4$, $SO(N-2)$ is a continuous group, hence for any $x \not\in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$, the set $\{\tau(x) : \tau \in Z_k \times SO(N-2)\}$ is more than countable. If alternative (ii) holds, from at most countability of $J$ we deduce that $x_j \in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^N$. Moreover, arguing as above we can prove that if $\tau(x_j) = x_i$ for some $\tau \in Z_k \times SO(N-2)$ then $\nu_{x_j} = \nu_{x_i}$. Hence we can rewrite (36) as

$$|u_\alpha|^2 \to d\nu = |u_0|^2 + \nu_0 \delta_0 \sum_{\ell=1}^k \nu^\ell_a \sum_{\ell=1}^k \delta^\ell_{y_i} + \sum_{\ell \in \mathcal{L}} \nu^\ell_a \sum_{i=1}^k \delta^\ell_{y_i}$$

where $\mathcal{L}$ is an at most countable set,

$$\{y^\ell_i : i = 1, \ldots, k, \ell \in \mathcal{L}\} \subset \{x_j : j \in J\} \text{ and } y^\ell_i = e^{2\pi i / k} y^\ell_{i-1}.$$

**Concentration at non singular points.** We claim that

$$J \text{ is finite and for } j \in J \text{ either } \nu_{x_j} = 0 \text{ or } \nu_{x_j} \geq \left( \frac{S}{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)} \right)^{N/2}. \tag{41}$$

Indeed, if $\nu_{x_j} > 0$, we have $x_j = y^\ell_s$ and $\nu_{x_j} = \nu^\ell_s$ for some $1 \leq s \leq k$, $\ell \in \mathcal{L}$. For $\varepsilon > 0$, let $\phi^\ell_s$ be a smooth cut-off function in $D^{1,2}_k(\mathbb{R}^N)$, $0 \leq \phi^\ell_s(x) \leq 1$ such that

$$\phi^\ell_s(x) = 1 \text{ if } x \in \bigcup_{i=1}^k B\left(y^\ell_i, \frac{\varepsilon}{2}\right), \quad \phi^\ell_s(x) = 0 \text{ if } x \not\in \bigcup_{i=1}^k B(y^\ell_i, \varepsilon), \quad \text{and } \|
abla \phi^\ell_s\| \leq \frac{4}{\varepsilon}. \tag{39}$$

Testing $J^\ell_s(u_\alpha)$ with $u_\alpha \phi^\ell_s$ we obtain

$$0 = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \langle J^\ell_s(u_\alpha), u_\alpha \phi^\ell_s \rangle \geq \sum_{i=1}^k \mu_{y^\ell_i} - k S(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \nu^\ell_s.$$

By (39) we have that $S(\nu^\ell_s) \leq \mu_{y^\ell_s}$, then we obtain that $\nu^\ell_s \geq \left( \frac{S}{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)} \right)^{N/2}$. Hence either $\nu_{x_j} = 0$ or $\nu_{x_j} \geq \left( \frac{S}{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)} \right)^{N/2}$, which implies that $J$ is finite. In particular also $\mathcal{L}$ is finite. The claim is proved.

**Concentration at vertices of polygons.** We claim that

(i) for each $\ell = 1, \ldots, m$ either $\nu^\ell_a = 0$ or $\nu^\ell_a \geq \left( \frac{S(\lambda_\ell)}{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)} \right)^{N/2}. \tag{42}$
In order to prove claim (42), for each $i = 1, 2, \ldots, k$ and $\ell = 1, 2, \ldots, m$ we consider the smooth cut-off function $\phi_{i, \varepsilon}^\ell$ satisfying $0 \leq \phi_{i, \varepsilon}^\ell (x) \leq 1$,

$$\phi_{i, \varepsilon}^\ell (x) = 1 \quad \text{if} \quad |x - a_i^\ell| \leq \frac{\varepsilon}{2}, \quad \phi_{i, \varepsilon}^\ell (x) = 0 \quad \text{if} \quad |x - a_i^\ell| \geq \varepsilon, \quad \text{and} \quad |\nabla \phi_{i, \varepsilon}^\ell | \leq \frac{4}{\varepsilon}.$$ 

From (19) we obtain that

$$\int_{\mathbb{R}^N} |\nabla (u_n \phi_{i, \varepsilon}^\ell) |^2 dx - \lambda_\ell \int_{\mathbb{R}^N} |x - a_i^\ell|^2 |\phi_{i, \varepsilon}^\ell |^2 u_n^2 dx \geq S(\lambda_\ell)$$

hence passing to limit as $n \to \infty$ and $\varepsilon \to 0$ we obtain

(43) \quad $\mu_\alpha^\ell \geq \gamma_\alpha^\ell + S(\lambda_\ell) (\nu_\alpha^\ell)^{2/2^*}.$

For $\varepsilon > 0$, let $\psi_\varepsilon^\ell$ be a smooth cut-off function in $\mathcal{D}_{k}^{1,2}(\mathbb{R}^N)$, $0 \leq \psi_\varepsilon^\ell (x) \leq 1$ such that

$$\psi_\varepsilon^\ell (x) = 1 \quad \text{if} \quad x \in \bigcup_{i=1}^k B(a_i^\ell, \frac{\varepsilon}{2}), \quad \psi_\varepsilon^\ell (x) = 0 \quad \text{if} \quad x \notin \bigcup_{i=1}^k B(a_i^\ell, \varepsilon), \quad \text{and} \quad |\nabla \psi_\varepsilon^\ell | \leq \frac{4}{\varepsilon}.$$ 

Testing $J_k'(u_n)$ with $u_n \psi_\varepsilon^\ell$ and letting $n \to \infty$ and $\varepsilon \to 0$ we infer that

(44) \quad $\sum_{i=1}^k \mu_\alpha^\ell - \sum_{i=1}^k \gamma_\alpha^\ell \leq k S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \nu_\alpha^\ell.$

From (43) and (44) we derive (42).

**Concentration at the origin.** We claim that

(45) \quad either $\nu_0 = 0$ or $\nu_0 \geq \left( \frac{S_k(\lambda_0)}{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)} \right)^{N/2}.$

In order to prove claim (45), we consider a smooth cut-off function $\psi_0^\ell \in \mathcal{D}_{k}^{1,2}(\mathbb{R}^N)$ satisfying $0 \leq \psi_0^\ell (x) \leq 1$,

$$\psi_0^\ell (x) = 1 \quad \text{if} \quad |x| \leq \frac{\varepsilon}{2}, \quad \psi_0^\ell (x) = 0 \quad \text{if} \quad |x| \geq \varepsilon, \quad \text{and} \quad |\nabla \psi_0^\ell | \leq \frac{4}{\varepsilon}.$$ 

From (22) we obtain that

$$\int_{\mathbb{R}^N} |\nabla (u_n \psi_0^\ell) |^2 dx - \lambda_0 \int_{\mathbb{R}^N} |x|^2 |\psi_0^\ell |^2 u_n^2 dx \geq S_k(\lambda_0)$$

hence passing to limit as $n \to \infty$ and $\varepsilon \to 0$ we obtain

(46) \quad $\mu_0 \geq \gamma_0 + S_k(\lambda_0) (\nu_0)^{2/2^*}.$

On the other hand, testing $J_k'(u_n)$ with $u_n \psi_0^\ell$ and letting $n \to \infty$ and $\varepsilon \to 0$ we infer that

(47) \quad $\mu_0 - \gamma_0 \leq S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \nu_0.$

From (46) and (47) we deduce (45).
Concentration at infinity. We claim that

\[(48) \quad \text{either } \nu_\infty = 0 \text{ or } \nu_\infty \geq \left( \frac{S_k(\lambda_0 + k \sum_{\ell=1}^m \lambda_\ell)}{S_k(\lambda_0, \Lambda_1, \Lambda_2, \ldots, \Lambda_m)} \right)^{N/2}. \]

In order to prove (48), we study the possibility of concentration at \( \infty \). Let \( \psi_R \) be a regular radial cut-off function such that

\[0 \leq \psi_R(x) \leq 1, \quad \psi_R(x) = \begin{cases} 1, & \text{if } |x| > 2R \\ 0, & \text{if } |x| < R, \end{cases} \quad \text{and } |\nabla \psi_R| \leq \frac{2}{R}.\]

From (22) we obtain that

\[(49) \quad \frac{\int_{\mathbb{R}^N} |\nabla (u_n \psi_R)|^2 \, dx - \left( \lambda_0 + k \sum_{\ell=1}^m \lambda_\ell \right) \int_{\mathbb{R}^N} \frac{\psi_R^2 u_n^2}{|x|^2} \, dx}{\left( \int_{\mathbb{R}^N} |\psi_R u_n|^2 \right)^{2/2^*}} \geq S_k \left( \lambda_0 + k \sum_{\ell=1}^m \lambda_\ell \right).\]

Taking \( \lim \sup \) as \( n \to \infty \) and limit as \( R \to +\infty \), standard calculations yield

\[(50) \quad \mu_\infty - \gamma_\infty \geq S_k \left( \lambda_0 + k \sum_{\ell=1}^m \lambda_\ell \right)^{2/2^*}.\]

Testing \( J_k'(u_n) \) with \( u_n \psi_R \) and letting \( n \to \infty \) and \( R \to +\infty \), we obtain

\[(51) \quad \mu_\infty - \gamma_\infty \leq S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \nu_\infty.\]

Claim (48) follows from (50) and (51).

As a conclusion we obtain

\[(52) \quad c = J_k(u_n) - \frac{1}{2} \langle J_k'(u_n), u_n \rangle + o(1) = \frac{1}{N} S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \int_{\mathbb{R}^N} |u_n|^2 \, dx + o(1)\]

\[= \frac{S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)}{N} \left\{ \int_{\mathbb{R}^N} |u_0|^2 \, dx + \nu_0 + k \sum_{\ell=1}^m \nu_\ell^\ell + k \sum_{\ell \in \mathcal{L}} \nu_\ell^{\mathcal{L}} + \nu_\infty \right\}.\]

From (34), (41), (42), (45), (48), and (52), we deduce that \( \nu_0 = 0, \nu_\ell^\ell = 0 \) for any \( \ell \in \mathcal{L}, \nu_\ell^{\mathcal{L}} = 0 \) for any \( \ell = 1, \ldots, m \), and \( \nu_\infty = 0 \). Then, up to a subsequence, \( u_n \to u_0 \) in \( D^{1,2}_k(\mathbb{R}^N) \).

The functional \( J : D^{1,2}(\mathbb{R}^N) \to \mathbb{R} \) associated to equation (4) is

\[(53) \quad J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\lambda_0}{2} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2} \, dx - \sum_{\ell=1}^m \frac{\Lambda_\ell}{2} \int_{\mathbb{R}^N} \left( \int_{S_{\ell}} \frac{u^2(y)}{|x - y|^2} \, d\sigma(x) \right) \, dy - \frac{S_{\circ}(\lambda_0, \Lambda_1, \ldots, \Lambda_m)}{2^*} \int_{\mathbb{R}^N} |u|^2 \, dx.\]

The functional \( J \) is \( \mathcal{S}(2) \times \mathcal{S}(N - 2) \)-invariant. Since \( \mathcal{S}(2) \times \mathcal{S}(N - 2) \) acts by isometries on \( D^{1,2}(\mathbb{R}^N) \), the Principle of Symmetric Criticality by Palais [24] implies that the critical points of \( J \) restricted to \( D^{1,2}_{\circ}(\mathbb{R}^N) \) are also critical points of \( J \) in \( D^{1,2}(\mathbb{R}^N) \). Therefore, any critical point of \( J \) in \( D^{1,2}_{\circ}(\mathbb{R}^N) \) provides a solution to equation (4).

The following theorem is the analogous of Theorem 4.1 for \( J \) restricted to \( D^{1,2}_{\circ}(\mathbb{R}^N) \). However, the fact that the singularities are spread over circles instead of being concentrated at atoms reduces the possibility of lack of compactness. Indeed, according to P.L. Lions Concentration-Compactness Principle, possible forms of “non compactness” which can cause failure of Palais-Smale condition
are loss of mass at infinity and concentration at an most countable set of points. When considering $J$ restricted to $D^{1,2}_{\text{circ}}(\mathbb{R}^N)$, it turns out that if $\bar{x}$ is a concentration point of a Palais-Smale sequence, then all points of the orbit $O(\bar{x}) = \{ \tau \bar{x} : \tau \in \mathbb{S}(2) \times \mathbb{S}(N-2) \}$ must be concentration points. On the other hand, when $N \geq 4$ both groups $\mathbb{S}(2)$ and $\mathbb{S}(N-2)$ are continuous, hence the only point $\bar{x}$ for which $O(\bar{x})$ is at most countable is the origin. Hence concentration can occur only at $0$ and at $\infty$.

We mention that action of this type of groups was considered in [2] to find nonradial solutions to a Euclidean scalar field equation. We refer to [30, §1.5] for a discussion on the relation between symmetry and compactness in variational problems.

Let us define

$$
S_{\text{circ}}(\lambda_0) = \inf_{u \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2} \, dx}{\left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{2/2^*}}.
$$

The following theorem provides a threshold up to which $J$ satisfies Palais-Smale condition.

**Theorem 4.2.** Assume $N \geq 4$ and $\lambda_0^+ + \sum_{j=1}^m \Lambda_j^+ < \frac{(N-2)^2}{4}$. Let $\{u_n\}_{n \in \mathbb{N}} \subset D^{1,2}_{\text{circ}}(\mathbb{R}^N)$ be a Palais-Smale sequence for $J$ restricted to $D^{1,2}_{\text{circ}}(\mathbb{R}^N)$, namely

$$
\lim_{n \to \infty} J(u_n) = c < \infty \text{ in } \mathbb{R} \text{ and } \lim_{n \to \infty} J'(u_n) = 0 \text{ in the dual space } (D^{1,2}_{\text{circ}}(\mathbb{R}^N))^*.
$$

If

$$
c < \frac{S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m)^{1 - \frac{N}{2}}}{N} \min \left\{ S_{\text{circ}}(\lambda_0), S_{\text{circ}}(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell)^{\frac{N}{2}} \right\},
$$

then $\{u_n\}_{n \in \mathbb{N}}$ has a converging subsequence in $D^{1,2}_{\text{circ}}(\mathbb{R}^N)$.

**Proof.** Let $\{u_n\}$ be a Palais-Smale sequence for $J$ in $D^{1,2}_{\text{circ}}(\mathbb{R}^N)$, then from Sobolev’s inequality and (6), it is easy to prove that $\{u_n\}$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$. Hence, up to a subsequence, $u_n \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^N)$, $u_n \to u_0$ almost everywhere, and, from the Concentration Compactness Principle by P. L. Lions [22, 23]

$$
|\nabla u_n|^2 \to d\mu \geq |\nabla u_0|^2 + \mu_0 \delta_0 + \sum_{j \in \mathcal{J}} \mu_{x_j} \delta_{x_j},
$$

$$
|u_n|^{2^*} \to d\nu = |u_0|^{2^*} + \nu_0 \delta_0 + \sum_{j \in \mathcal{J}} \nu_{x_j} \delta_{x_j},
$$

$$
\lambda_0 \frac{|u_n|^2}{|x|^2} \to d\gamma_0 = \lambda_0 \frac{|u_0|^2}{|x|^2} + \gamma_0 \delta_0
$$

where $\mathcal{J}$ is an at most countable set, $x_j \in \mathbb{R}^N \setminus \{0\}$, $\mu_{x_j}, \nu_{x_j} \in \mathbb{R}$, $j \in \mathcal{J}$, $\mu_0, \nu_0, \gamma_0 \in \mathbb{R}$, and the above convergences hold in the sense of measures. We quantify how much the sequence concentrates at infinity by the quantities $\nu_\infty$, $\mu_\infty$ defined as in (40) and

$$
\gamma_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} \left( \lambda_0 + \sum_{\ell=1}^m \Lambda_\ell \right) \frac{|u_n|^2}{|x|^2} \, dx.
$$
From pointwise convergence of \( u_n \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) to \( u_0 \) we deduce that \( u_0 \) is invariant by the \( \text{SO}(2) \times \text{SO}(N - 2) \)-action, hence \( u_0 \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \). Moreover, for any \( \phi \in C_0(\mathbb{R}^N) \) and for any \( \tau \in \text{SO}(2) \times \text{SO}(N - 2) \) we have

\[
\int_{\mathbb{R}^N} |u_n|^2 \, \phi = \int_{\mathbb{R}^N} |u_n|^2 (\phi \circ \tau^{-1})
\]

that, passing to the limit as \( n \to \infty \), yields

\[
\sum_{j \in J} \nu_{x_j}(\phi(x_j)) = \sum_{j \in J} \nu_{x_j}(\phi(x_j)).
\]

Arguing as in the proof of Theorem 4.1, we deduce that for any \( j \in J \), either \( \nu_{x_j} = 0 \) or for any \( \tau \in \text{SO}(2) \times \text{SO}(N - 2) \) there exists \( i \in J \) such that \( \tau(x_j) = x_i \). Namely, if for some \( j \in J \), \( \nu_{x_j} \neq 0 \), then \( O(x_j) \subset \{ x_i : i \in J \} \). When \( N \geq 4 \), this is not possible since \( J \) is at most countable whereas \( O(x_j) \) is more than countable. Therefore \( \nu_{x_j} = 0 \) for all \( j \in J \) and we can rewrite (57) as

\[
|u_n|^2 \to d\nu = |u_0|^2 + \nu_0 \delta_0.
\]

**Concentration at the origin.** We claim that

(59) \quad either \( \nu_0 = 0 \) or \( \nu_0 \geq \left( \frac{S_{\text{circ}}(\lambda_0)}{S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m)} \right)^{N/2} \).

In order to prove claim (59), we consider a smooth cut-off function \( \psi_0^\varepsilon \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) satisfying \( 0 \leq \psi_0^\varepsilon(x) \leq 1 \),

\[
\psi_0^\varepsilon(x) = 1 \quad \text{if} \quad |x| \leq \frac{\varepsilon}{2} , \quad \psi_0^\varepsilon(x) = 0 \quad \text{if} \quad |x| \geq \varepsilon , \quad \text{and} \quad |\nabla \psi_0^\varepsilon| \leq \frac{4}{\varepsilon}.
\]

From (54) we obtain that

\[
\int_{\mathbb{R}^N} |\nabla (u_n \psi_0^\varepsilon)|^2 dx - \lambda_0 \int_{\mathbb{R}^N} |x|^{-2} |\psi_0^\varepsilon|^2 u_n^2 dx \geq S_{\text{circ}}(\lambda_0)
\]

hence passing to limit as \( n \to \infty \) and \( \varepsilon \to 0 \) we obtain

(60) \quad \mu_0 \geq \gamma_0 + S_{\text{circ}}(\lambda_0)(\nu_0)^{2/2^*}.

On the other hand, testing \( J'(u_n) \) with \( u_n \psi_0^\varepsilon \) we obtain

\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 \psi_0^\varepsilon + \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \psi_0^\varepsilon - \lambda_0 \int_{\mathbb{R}^N} \frac{u_n^2 \psi_0^\varepsilon}{|x|^2} dx
\]

\[
- \sum_{\ell=1}^m \Lambda_m \int_{\mathbb{R}^N} \left( \int_{S_{r_\varepsilon}} \frac{u_{n,\ell}^2(y) \psi_0^\varepsilon(y)}{|x-y|^2} d\sigma(x) \right) dy - S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \int_{\mathbb{R}^N} \psi_0^\varepsilon |u_n|^2 = o(1).
\]

In view of (7), for \( \varepsilon \) small we have

\[
\int_{\mathbb{R}^N} \left( \int_{S_{r_\varepsilon}} \frac{u_{n,\ell}^2(y) \psi_0^\varepsilon(y)}{|x-y|^2} d\sigma(x) \right) dy \geq \int_{\mathbb{R}^N} \frac{u_{n,\ell}^2(y) \psi_0^\varepsilon(y)}{\sqrt{(r^2 + |y|^2)^2 - 4r^2 |y|^2}} dx
\]

\[
\leq \int_{\mathbb{R}^N} \frac{u_{n,\ell}^2(y) \psi_0^\varepsilon(y)}{r^2 - |y|^2} dx \leq \frac{1}{r^2 - \varepsilon^2} \int_{|y| < \varepsilon} u_{n,\ell}^2(y) dy \leq \text{const} \left( \int_{\mathbb{R}^N} |u_n|^2 \right)^{2/2^*} \varepsilon^2 \to 0 ,
\]
therefore letting \( n \to \infty \) and \( \varepsilon \to 0 \) we infer that
\[
(61) \quad \mu_0 - \gamma_0 \leq S_{\text{circ}}(\lambda_0, A_1, \ldots, A_m)\nu_0.
\]
From (60) and (61) we deduce (59).

**Concentration at infinity.** We claim that
\[
(62) \quad \text{either } \nu_\infty = 0 \text{ or } \nu_\infty \geq \left( \frac{S_{\text{circ}}(\lambda_0 + \sum_{\ell=1}^m A_\ell)}{S_{\text{circ}}(\lambda_0, A_1, \ldots, A_m)} \right)^{N/2}.
\]
Indeed, let \( \psi_R \) be a smooth radial cut-off function such that \( 0 \leq \psi_R(x) \leq 1, \psi_R(x) = 1 \) if \( |x| > 2R \), \( \psi_R(x) = 0 \) if \( |x| < R \), and \( |\nabla \psi_R| \leq 2/R \). Using (5) and taking lim sup as \( n \to \infty \) and limit as \( R \to +\infty \), it is easy to show that
\[
(63) \quad \mu_\infty - \gamma_\infty \geq S_{\text{circ}}(\lambda_0 + \sum_{\ell=1}^m A_\ell)\nu_\infty^{2/2^*}.
\]
On the other hand, testing \( J'(u_n) \) with \( u_n\psi_R \) we obtain
\[
(64) \quad \int_{\mathbb{R}^N} |\nabla u_n|^2 \psi_R + \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla \psi_R - \lambda_0 \int_{\mathbb{R}^N} u_n^2 \psi_R
\]
\[
- \sum_{\ell=1}^m \Lambda_m \int_{\mathbb{R}^N} \left( \int_{S_{\ell}} \frac{2 u_n^2(y) \psi_R(y)}{|x-y|^2} d\sigma(x) \right) dy - S_{\text{circ}}(\lambda_0, A_1, \ldots, A_m) \int_{\mathbb{R}^N} \psi_R |u_n|^2 = o(1).
\]
Let \( \tilde{R} > \max\{r_\ell : \ell = 1, \ldots, m\} \). If \( R \geq \tilde{R} \), in view of (7) we have for all \( \ell = 1, \ldots, m \)
\[
\int_{S_{\ell}} \frac{2 u_n^2(y) \psi_R(y)}{|x-y|^2} d\sigma(x) - \frac{u_n^2(y) \psi_R(y)}{|y|^2} = \frac{u_n^2(y) \psi_R(y)}{|y|^2} \frac{|y|^2 - \sqrt{(r_\ell^2 + |y|^2)^2 - 4r_\ell^2 |y|^2}}{\sqrt{(r_\ell^2 + |y|^2)^2 - 4r_\ell^2 |y|^2}} \leq \frac{u_n^2(y) \psi_R(y)}{|y|^2} \left( \frac{r_\ell^2}{|y|^2} \right) \frac{2 R^4}{R^2 - r_\ell^2}.
\]
Since
\[
\int_{\mathbb{R}^N} \frac{u_n^2(y) \psi_R(y)}{|y|^4} dy \leq \frac{1}{R^2} \int_{|y| > R} \frac{u_n^2(y)}{|y|^2} dy \leq \frac{\text{const}}{R^2} \xrightarrow{R \to +\infty} 0
\]
we deduce that
\[
\int_{\mathbb{R}^N} \int_{S_{\ell}} \frac{u_n^2(y) \psi_R(y)}{|x-y|^2} d\sigma(x) dy = \int_{\mathbb{R}^N} \frac{u_n^2(y) \psi_R(y)}{|y|^2} dy + o(1) \quad \text{as } R \to +\infty.
\]
Therefore, letting \( n \to \infty \) and \( R \to +\infty \) in (64), we obtain
\[
(65) \quad \mu_\infty - \gamma_\infty \leq S_{\text{circ}}(\lambda_0, A_1, \ldots, A_m)\nu_\infty.
\]
Claim (62) follows from (63) and (65).

As a conclusion we obtain
\[
(66) \quad c = J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle + o(1) = \frac{S_{\text{circ}}(\lambda_0, A_1, \ldots, A_m)}{N} \left\{ \int_{\mathbb{R}^N} |u_0|^2 dx + \nu_0 + \nu_\infty \right\}.
\]
From (55), (59), (62), and (66), we deduce that \( \nu_0 = 0 \) and \( \nu_\infty = 0 \). Then, up to a subsequence, \( u_n \to u_0 \) in \( \mathcal{D}^{1,2}_{\text{circ}}(\mathbb{R}^N) \). 

5. Interaction Estimates

For any \( u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \), let us consider the family of functions obtained from \( u \) by dilation, i.e.

\[ u_\mu(x) = \mu^{-\frac{N-2}{2}} u(x/\mu), \quad \mu > 0. \]

The following lemma describes the behavior of \( \int |x + \xi|^2 |u_\mu^\lambda|^2 \) as \( \mu \to 0 \) for any solution \( u^\lambda \) of equation (16). We mention that estimates below were obtained in [14] for radial solutions to (16) (i.e. for functions \( w_\mu^\lambda \) in (17)).

**Lemma 5.1.** Let \( u^\lambda \in \mathcal{D}^{1,2}(\mathbb{R}^N) \) be a solution to (16). For any \( \xi \in \mathbb{R}^N \) there holds

\[
\int_{\mathbb{R}^N} \frac{|u^\lambda|^2}{|x + \xi|^2} \, dx = \begin{cases}
\frac{2}{|\xi|} \int_{\mathbb{R}^N} |u^\lambda|^2 \, dx + o(\mu^2) & \text{if } \lambda < \frac{N(N-4)}{4}, \\
\kappa_\lambda(u^\lambda)^2 \frac{\ln |\mu|}{|\xi|^2} + o(\mu^2 |\ln \mu|) & \text{if } \lambda = \frac{N(N-4)}{4}, \\
\kappa_\lambda(u^\lambda)^2 \frac{\beta_{\lambda,N} u^\lambda N^2 - 4N}{(N-2)^2 - 4N} + o(\mu \sqrt{N(N-2)^2 - 4N}) & \text{if } \lambda > \frac{N(N-4)}{4},
\end{cases}
\]

as \( \mu \to 0 \), where

\[
\beta_{\lambda,N} = \int_{\mathbb{R}^N} \frac{dx}{|x|^2|x - e_1|^{N-2+\sqrt{N(N-2)^2 - 4N}}} \quad e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N.
\]

**Proof.** We have that

\[ \int_{\mathbb{R}^N} \frac{|u^\lambda|^2}{|x + \xi|^2} \, dx = \mu^2 \int_{\mathbb{R}^N} \frac{|u^\lambda|^2}{|\mu x + \xi|^2} \, dx = \mu^2 \int_{|x|<\frac{|\xi|}{\mu}} \frac{|u^\lambda|^2}{|\mu x + \xi|^2} \, dx + \mu^2 \int_{|x|>\frac{|\xi|}{\mu}} \frac{|u^\lambda|^2}{|\mu x + \xi|^2} \, dx. \]

For \( \lambda < \frac{N(N-4)}{4} \), from (25) we have that \( u^\lambda \in L^2(\mathbb{R}^N) \). From (25) and [14, Proof of Lemma 3.4] we deduce

\[ \left| \int_{|x|<\frac{|\xi|}{\mu}} |u^\lambda(x)|^2 \left[ \frac{1}{|\mu x + \xi|^2} - \frac{1}{|\xi|^2} \right] \, dx \right| \leq \kappa(u^\lambda)^2 \int_{|x|<\frac{|\xi|}{\mu}} |w_1^\lambda(x)|^2 \left| \frac{1}{|\mu x + \xi|^2} - \frac{1}{|\xi|^2} \right| \, dx = o(1) \]

as \( \mu \to 0 \) and hence, since \( u^\lambda \in L^2(\mathbb{R}^N) \),

\[ \int_{|x|<\frac{|\xi|}{\mu}} |u^\lambda(x)|^2 \, dx = \frac{1}{|\xi|^2} \int_{|x|<\frac{|\xi|}{\mu}} |u^\lambda(x)|^2 \, dx + o(1) = \frac{1}{|\xi|^2} \int_{\mathbb{R}^N} |u^\lambda(x)|^2 \, dx + o(1). \]

On the other hand, from [14, Proof of Lemma 3.4] we have

\[ \mu^2 \int_{|x|>\frac{|\xi|}{\mu}} \frac{|u^\lambda|^2}{|\mu x + \xi|^2} \, dx \leq \kappa(u^\lambda)^2 \mu^{2-N} \int_{|x-\xi|>\frac{|\xi|}{\mu}} \left| w_1^\lambda \left( \frac{x-\xi}{\mu} \right) \right|^2 \frac{dx}{|x|^2} = O(\mu^{\nu_{\lambda}(N-2)}) = o(\mu^2). \]

From (68), (69), and (70) we deduce that

\[ \int_{\mathbb{R}^N} \frac{|u^\lambda|^2}{|x + \xi|^2} \, dx = \mu^2 \int_{\mathbb{R}^N} |u^\lambda(x)|^2 \, dx + o(\mu^2). \]
For $\lambda = \frac{N(N-4)}{4}$, from (25) and [14, Proof of Lemma 3.4] we deduce that

\begin{equation}
\int_{|x| < \frac{\mu}{\lambda^2}} |u^\lambda(x)|^2 \frac{dx}{\mu x + \xi^2} = \frac{1}{|\xi|^2} \int_{|x| < \frac{\mu}{\lambda^2}} |u^\lambda(x)|^2 \, dx + O(1).
\end{equation}

On the other hand, from (24) we obtain

\begin{equation}
\int_{|x| < \frac{\mu}{\lambda^2}} |u^\lambda(x)|^2 \, dx = \int_{1 < |x| < \frac{\mu}{\lambda^2}} |u^\lambda(x)|^2 \, dx + \int_{|x| < 1} |u^\lambda(x)|^2 \, dx
\end{equation}

\begin{align*}
&= \kappa_\infty (u^\lambda)^2 \int_{1 < |x| < \frac{\mu}{\lambda^2}} \frac{|x|^{-N} \, dx}{|\xi|^2} + O\left( \int_{1 < |x| < \frac{\mu}{\lambda^2}} |x|^{-N-2} \, dx \right) + O(1) \\
&= \kappa_\infty (u^\lambda)^2 \ln |\mu| + O(1).
\end{align*}

Arguing as above (see (70)), we obtain

\begin{equation}
\mu^2 \int_{|x| < \frac{\mu}{\lambda^2}} \frac{|u^\lambda|^2}{|\mu x + \xi|^2} \, dx = O(\mu^2).
\end{equation}

Gathering (68), (71), (72) and (73) we deduce that

\begin{equation}
\int_{\mathbb{R}^N} \frac{|u^\lambda|^2}{|x + \xi|^2} \, dx = \kappa_\infty (u^\lambda)^2 \frac{\mu^2 |\ln |\mu||}{|\xi|^2} + o(\mu^2 |\ln |\mu||).
\end{equation}

For $\lambda > \frac{N(N-4)}{4}$, in view of (24) we have that

\begin{equation}
\int_{\mathbb{R}^N} \frac{|u^\lambda|^2}{|x + \xi|^2} \, dx = \mu^{\nu_\lambda(N-2)} \left[ \kappa_\infty^2 (u^\lambda)^2 \int_{\mathbb{R}^N} \frac{1}{|x|^2 |x - \xi|^{(N-2)(1+\nu_\lambda)}} + o(1) \right] \, dx
\end{equation}

As observed in [14, Proof of Lemma 3.4], the function

\begin{equation}
\varphi(\xi) := \int_{\mathbb{R}^N} \frac{dx}{|x|^2 |x - \xi|^{(N-2)(1+\nu_\lambda)}},
\end{equation}

can be written as

\begin{equation}
\varphi(\xi) = |\xi|^{-\sqrt{(N-2)^2 - 4\lambda}} \varphi(\xi/|\xi|) = |\xi|^{-\sqrt{(N-2)^2 - 4\lambda}} \varphi(e_1).
\end{equation}

(74) and (75) yield the required estimate for $\lambda > \frac{N(N-4)}{4}$. \( \blacksquare \)

Let us now study the interaction between two minimizers of (19), i.e. functions $z^\lambda_\mu$ in (20), centered at different points as $\mu \to 0$. To this aim we note that a direct calculation yields

\begin{equation}
z^\lambda_\mu(x) = |x|^{-\frac{N-2}{2}(1+\nu_\lambda)} \left[ \alpha_\lambda N + O(|x|^{-\alpha}) \right],
\end{equation}

\begin{equation}
\nabla z^\lambda_\mu(x) = |x|^{-\frac{N-2}{2} - \nu_\lambda} \frac{N-2}{2} x \left[ - \alpha_\lambda N \frac{N-2}{2} (1 + \nu_\lambda) + O(|x|^{-\alpha}) \right],
\end{equation}

for all $0 < \alpha \leq 2\nu_\lambda$. From (20), (76), and (77), it is easy to deduce the following result.

\textbf{Lemma 5.2.} For any $\lambda \in (0, (N-2)^2/4)$ and $\xi, \zeta \in \mathbb{R}^N$, $\xi \neq 0$, there holds

\begin{equation}
\int_{\mathbb{R}^N} \frac{z^\lambda_\mu(x) \varphi_\mu(x + \xi)}{|x + \xi|^2} \, dx = \mu^{\nu_\lambda(N-2) - 4\lambda} \left[ \alpha_\lambda N \int_{\mathbb{R}^N} \frac{\varphi_\mu}{|x|^{(1+\nu_\lambda) \frac{N-2}{2}} |x + \xi|^{(1+\nu_\lambda) \frac{N-2}{2}} |x + \xi|^2} \, dx + o(1) \right]
\end{equation}
and
\[ \int_{\mathbb{R}^N} \nabla z^\lambda (x) \cdot \nabla z^\lambda (x + \xi) \, dx = \mu \sqrt{(N-2)^2 - 4\lambda} \left[ \frac{\nu^2}{4} (N-2)^2 (1+\nu\lambda)^2 \gamma_{\lambda,N} |\xi|^{-\sqrt{(N-2)^2 - 4\lambda}} + o(1) \right] \]
as \( \mu \to 0 \), where
\[ \gamma_{\lambda,N} = \int_{\mathbb{R}^N} \frac{x \cdot (x + e_1) \, dx}{|x|^{\frac{N+2}{2} + \nu\lambda} |x - e_1|^{\frac{N+2}{2} + \nu\lambda}}, \quad e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N. \]

6. Comparison between concentration levels for \( J \) and proof of Theorem 1.3

In order to compare the level \( S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) with the level \( S_{\text{circ}}(\lambda_0) \) of possible concentration at 0, we need the following lemma, which states that the infimum in (54) is achieved if \( N \geq 4 \). Such a result does not come unexpected, since it can be seen as the analogue of Lemma 2.1 when \( k = \infty \); indeed when \( k \) becomes larger and larger, assumption \( S_k(\lambda) < k^{2/N} S \) of Lemma 2.1 is weakened till it is no more needed in the limiting problem corresponding to singularities spread over circles.

**Lemma 6.1.** For any \( \lambda_0 \in (-\infty, (N-2)/4) \) and \( N \geq 4 \), the infimum in (54) is achieved.

**Proof.** Hardy’s and Sobolev’s inequalities imply that \( S_{\text{circ}}(\lambda_0) > 1 - \frac{4\lambda_0}{(N-2)^2} > 0 \). Let \( \{u_n\}_n \subset D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) be a minimizing sequence such that \( \int_{\mathbb{R}^N} |u_n|^2 = 1 \). By virtue of the Ekeland’s variational principle we can assume that \( \{u_n\}_n \) is a Palais-Smale sequence for the functional
\[ F(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\lambda_0}{2} \int_{\mathbb{R}^N} |u(x)|^2 \, dx - \frac{S_{\text{circ}}(\lambda_0)}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \, dx, \quad u \in D^{1,2}_{\text{circ}}(\mathbb{R}^N), \]
i.e.
\[ \lim_{n \to \infty} F(u_n) = \frac{S_{\text{circ}}(\lambda_0)}{N} \text{ in } \mathbb{R} \quad \text{and} \quad \lim_{n \to \infty} F'(u_n) = 0 \text{ in the dual space } (D^{1,2}_{\text{circ}}(\mathbb{R}^N))^*. \]

Let
\[ w_n(x) = \sigma_n^{-\frac{N-2}{2}} u_n(\sigma_n^{-1} x) \]
where \( \sigma_n \) is chosen in such a way that
\[ \int_{B(0,1)} |w_n(x)|^{2^*} \, dx = \int_{B(0,\sigma_n^{-1})} |u_n(x)|^{2^*} \, dx = \frac{1}{2}. \]

Scaling invariance ensures that \( \{w_n\}_n \subset D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) is also a minimizing sequence for (54),
\[ \int_{\mathbb{R}^N} |w_n(x)|^2 \, dx = 1, \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \left( |\nabla w_n(x)|^2 - \lambda_0 |w_n(x)|^2 \right) \, dx = S_{\text{circ}}(\lambda_0), \]
and
\[ F'(w_n) \to 0 \text{ in the dual space } (D^{1,2}_{\text{circ}}(\mathbb{R}^N))^*. \]

Since \( \{w_n\}_n \) is bounded in \( D^{1,2}_{\text{circ}}(\mathbb{R}^N) \), up to a subsequence \( w_n \) converges to \( w \) weakly in \( D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) and almost everywhere. Pointwise convergence implies that \( w \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \). From the Concentration Compactness Principle by P. L. Lions [22, 23] and taking into account that, when \( N \geq 4 \),
\[ \mathcal{SO}(2) \times \mathcal{SO}(N-2) \text{-invariant functions can concentrate only at } 0 \text{ and at } \infty, \text{ as already pointed out in the proof of Theorem 4.2, we have} \]

\[ |\nabla w_n|^2 \to d\mu \geq |\nabla w|^2 + \mu_0 \delta_0, \quad |w_n|^2 \to d\nu = |w|^2 + \nu_0 \delta_0, \text{ and} \]

\[ \frac{w_n^2}{|x|^2} \to d\gamma = \frac{w^2}{|x|^2} + \gamma_0 \delta_0. \]

The amount of concentration at infinity is quantified by the following numbers

\[ \nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} |w_n|^2 dx, \]

\[ \mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} |\nabla w_n|^2 dx, \]

and

\[ \gamma_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x|>R} \frac{w_n^2}{|x|^2} dx. \]

Arguing as we did to prove (60) and (63), we can easily obtain

\[ \nu_0^\frac{2}{N} (\mu_0 - \lambda_0 \gamma_0) \geq S_{\text{circ}}(\lambda_0) \nu_0 \]

and

\[ (\mu_\infty - \lambda_0 \gamma_\infty) \geq S_{\text{circ}}(\lambda_0) \nu_\infty^{\frac{2}{N}}. \]

**Step 1:** we prove that \( w \not\equiv 0 \). By contradiction, assume that \( w \equiv 0 \). Then from (79–81) we deduce

\[ 1 = \nu_0 + \nu_\infty \]

and

\[ S_{\text{circ}}(\lambda_0) = \int_{\mathbb{R}^N} d\mu - \lambda_0 \int_{\mathbb{R}^N} d\gamma + \mu_\infty - \lambda_0 \gamma_\infty. \]

Let \( \psi_R \) be a smooth radial cut-off function such that \( 0 \leq \psi_R(x) \leq 1, \psi_R(x) = 1 \text{ if } |x| > 2R, \psi_R(x) = 0 \text{ if } |x| < R \), and \( |\nabla \psi_R| \leq 2/R \). From (80), we have

\[ \sigma(1) = \langle F'(w_n), w_n \psi_R \rangle \]

\[ = \int_{\mathbb{R}^N} |\nabla w_n|^2 \psi_R + \int_{\mathbb{R}^N} w_n \nabla w_n \cdot \nabla \psi_R - \lambda_0 \int_{\mathbb{R}^N} \frac{w_n \psi_R}{|x|^2} - S_{\text{circ}}(\lambda_0) \int_{\mathbb{R}^N} \frac{w_n^2}{|x|^2} \psi_R. \]

Taking \( \limsup \) as \( n \to \infty \) and limit as \( R \to +\infty \), we find

\[ \mu_\infty - \lambda_0 \gamma_\infty = S_{\text{circ}}(\lambda_0) \nu_\infty. \]

From (85–86) it follows that

\[ \int_{\mathbb{R}^N} d\mu - \lambda_0 \int_{\mathbb{R}^N} d\gamma = S_{\text{circ}}(\lambda_0)(1 - \nu_\infty). \]

From (84), (82), and (87), it follows that

\[ 1 - \nu_\infty = \nu_0 \leq S_{\text{circ}}(\lambda_0)^{-1} \nu_0^{\frac{2}{N}} (\mu_0 - \lambda_0 \gamma_0) \leq S_{\text{circ}}(\lambda_0)^{-1} \nu_0^{\frac{2}{N}} \left( \int_{\mathbb{R}^N} d\mu - \lambda_0 \int_{\mathbb{R}^N} d\gamma \right) \]

\[ = \nu_0^{\frac{2}{N}} (1 - \nu_\infty) = (1 - \nu_\infty)^{1 + \frac{2}{N}}. \]
On the other hand, from (78) we have
\[ \int_{\mathbb{R}^N \setminus B(0,R)} |w_n|^2 \, dx = 1 - \int_{B(0,R)} |w_n|^2 \, dx \leq 1 - \int_{B(0,1)} |w_n|^2 \, dx = \frac{1}{2} \] for all \( R > 1 \), hence
\[ \nu_\infty \leq \frac{1}{2}. \tag{89} \]
From (88–89) we deduce that \( \nu_\infty = 0 \). From (84) it follows that \( \nu_0 = 1 \). Therefore (78) implies
\[ \frac{1}{2} = \int_{B(0,1)} |w_n|^2 \, dx \to_\infty d\nu(B(0,1)) = 1, \]
thus giving rise to a contradiction.

**Step 2**: we prove that
\[ \int_{\mathbb{R}^N} |w|^2 = 1 \] and \[ \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} \, dx = S_{\text{circ}}(\lambda_0). \]
Let \( \rho = \int_{\mathbb{R}^N} |w|^2 \). From (81), we have that \( 1 = \rho + \nu_0 + \nu_\infty \) and, in view of step 1, \( \rho \in (0,1] \), i.e. \( \nu_0 + \nu_\infty \in [0,1) \). Since
\[ S_{\text{circ}}(\lambda_0) = \int_{\mathbb{R}^N} d\mu - \lambda_0 \int_{\mathbb{R}^N} d\gamma + \mu_\infty - \lambda_0 \gamma_\infty \]

\[ \geq \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} \, dx + \mu_0 - \lambda_0 \gamma_0 + \mu_\infty - \lambda_0 \gamma_\infty, \]
from (82), (83), and concavity of the function \( t \mapsto t^{2/2^*} \) we deduce
\[ \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} \, dx \leq S_{\text{circ}}(\lambda_0)(1 - \nu_0^{\frac{1}{2^*}} - \nu_\infty^{\frac{1}{2^*}}) \leq S_{\text{circ}}(\lambda_0)(1 - \nu_0 - \nu_\infty)^{\frac{1}{2^*}} \]

\[ = S_{\text{circ}}(\lambda_0)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} \, dx. \]
Hence all the above inequalities are indeed equalities; in particular \( (1 - \nu_0^{\frac{1}{2^*}} - \nu_\infty^{\frac{1}{2^*}}) = (1 - \nu_0 - \nu_\infty)^{\frac{1}{2^*}} \)
which is possible only when \( \nu_0 = \nu_\infty = 0 \). Therefore \( \rho = 1 \) and
\[ \int_{\mathbb{R}^N} |\nabla w|^2 \, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{|w|^2}{|x|^2} \, dx = S_{\text{circ}}(\lambda_0), \]
i.e. \( w \) attains the infimum. \( \blacksquare \)

We now provide a sufficient condition for the infimum in (5) to stay below the level \( S_{\text{circ}}(\lambda_0) \), at which possible concentration at 0 can occur.

**Lemma 6.2.** Let \( \lambda_0, \Lambda_1, \ldots, \Lambda_m \in \mathbb{R}, r_1, r_2, \ldots, r_m \in \mathbb{R}^+ \) satisfy (10). Then
\[ S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < S_{\text{circ}}(\lambda_0). \]

**Proof.** From Lemma 6.1, we have that \( S_{\text{circ}}(\lambda_0) \) is attained by some \( u^{\lambda_0} \in D^1,2_{\text{circ}}(\mathbb{R}^N) \). By homogeneity of the Rayleigh quotient, we can assume \( \int |u^{\lambda_0}|^2 = 1 \). Moreover, the function \( v^{\lambda_0} = S_{\text{circ}}(\lambda_0)^{1/(2^* - 2)}|u^{\lambda_0}| \) is a nonnegative solution to (16), hence we can apply Lemma 5.1 to
study the behavior of the integral of \( \|u_\mu^\lambda\|^2 \) as \( \mu \to 0 \), where \( u_\mu^\lambda \) are defined in (67). Hence for some positive constant \( \tilde{\kappa} \)

\[
\int_{\mathbb{R}^N} \left( \frac{\int_{S_{r\ell}} |u_\mu^\lambda(y)|^2}{|x-y|^2} d\sigma(x) \right) dy
\]

\[
= \begin{cases}
\frac{\mu^2}{r \ell} \int_{\mathbb{R}^N} |u_1^\lambda|^2 dx + o(\mu^2) & \text{if } \lambda_0 < \frac{N(N-4)}{4}, \\
\tilde{\kappa}^2 \beta_{\lambda_0N} \sqrt{(N-2)^2 - 4\lambda_0} |r\ell|^{-\sqrt{(N-2)^2 - 4\lambda_0}} + o(\mu \sqrt{(N-2)^2 - 4\lambda_0}) & \text{if } \lambda_0 > \frac{N(N-4)}{4},
\end{cases}
\]

as \( \mu \to 0 \), where \( \beta_{\lambda_0N} \) is defined in Lemma 5.1. Therefore

\[
S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m)
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla u_\mu^\lambda|^2 \, dy - \lambda_0 \int_{\mathbb{R}^N} \frac{|u_\mu^\lambda(y)|^2}{|y|^2} \, dy - \sum_{\ell=1}^m \Lambda_\ell \int_{S_{r\ell}} \left( \frac{\int_{S_{r\ell}} |u_\mu^\lambda(y)|^2}{|x-y|^2} d\sigma(x) \right) dy
\]

\[
= S_{\text{circ}}(\lambda_0) - \begin{cases}
\mu^2 \left( \int_{\mathbb{R}^N} |u_1^\lambda|^2 \right) \left( \sum_{\ell=1}^m \frac{\Lambda_\ell}{r^2\ell} + o(1) \right) & \text{if } \lambda_0 < \frac{N(N-4)}{4}, \\
\mu^2 |\ln \mu| \tilde{\kappa}^2 \left( \sum_{\ell=1}^m \frac{\Lambda_\ell}{r^2\ell} + o(1) \right) & \text{if } \lambda_0 = \frac{N(N-4)}{4}, \\
\mu \sqrt{(N-2)^2 - 4\lambda_0} \tilde{\kappa}^2 \beta_{\lambda_0N} \left( \sum_{\ell=1}^m \frac{\Lambda_\ell}{|r\ell|^\sqrt{(N-2)^2 - 4\lambda_0}} + o(1) \right) & \text{if } \lambda_0 > \frac{N(N-4)}{4},
\end{cases}
\]

as \( \mu \to 0 \). Taking \( \mu \) sufficiently small, assumption (10) yields \( S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m) < S_{\text{circ}}(\lambda_0) \).

**Proof of Theorem 1.3.** Let \( \{u_n\}_n \subset D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) be a minimizing sequence for (5). From the homogeneity of the quotient there is no restriction requiring \( \|u_n\|_{L^\infty(\mathbb{R}^N)} = 1 \). Moreover from Ekeland’s variational principle we can assume that \( \{u_n\}_n \subset D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) is a Palais-Smale sequence, more precisely \( J'(u_n) \to 0 \) in \( (D^{1,2}_{\text{circ}}(\mathbb{R}^N))^* \) and \( J(u_n) \to \frac{1}{N} S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m) \). From assumption (9) we deduce that

\[
S_{\text{circ}}(\lambda_0) \leq S_{\text{circ}} \left( \lambda_0 + \sum_{\ell=1}^m \Lambda_\ell \right).
\]

From Lemma 6.2 and (91), it follows that the level of the minimizing Palais-Smale sequence satisfies assumption (55). Hence from Theorem 4.2, \( \{u_n\}_{n \in \mathbb{N}} \) has a subsequence strongly converging to some \( u_0 \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \) such that \( J(u_0) = \frac{1}{N} S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m) \). Hence \( u_0 \) achieves the infimum in (5). Since \( J \) is even, also \( |u_0| \) is a minimizer in (5) and then \( v_0 = S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m)^{1/(2-2)} |u_0| \) is a nonnegative solution to equation (4). The maximum principle implies the positivity outside singular circles of such a solution.
7. Limit of $S_k(\lambda)$ as $k \to \infty$.

Since $D_{\text{circ}}^{1,2}(\mathbb{R}^N) \subset D_k^{1,2}(\mathbb{R}^N)$, for any $\lambda \in (-\infty, (n-2)^2/4)$, there holds
\[0 < S(\lambda) \leq S_k(\lambda) \leq S_{\text{circ}}(\lambda).\]

From (92) and Lemma 2.1, it follows easily the following result.

**Lemma 7.1.** Let $\lambda \in (-\infty, (n-2)^2/4)$ and $N \geq 4$. Then there exists $\bar{k} = \bar{k}(\lambda, N)$ such that $S_k(\lambda)$ is achieved for all $k \geq \bar{k}$.

Let us now study the limit of $S_k(\lambda)$ as $k \to \infty$. Theorem 7.3 provides convergence of $S_k(\lambda)$ to $S_{\text{circ}}(\lambda)$. To prove it we will need the following proposition.

**Proposition 7.2.** Let $\lambda \in (-\infty, (n-2)^2/4)$ and let $\{w_k\}$ be a sequence in $D_k^{1,2}(\mathbb{R}^N)$ such that $w_k \in D_k^{1,2}(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} |w_k|^2 = 1, \quad Q_\lambda(w_k) = S_k(\lambda),
\]
and $w_k$ converges weakly to 0 in $D_k^{1,2}(\mathbb{R}^N)$ (at least along a subsequence). Then, for any $r > 0$ and $\varepsilon \in (-r, r)$, there exists $\rho$ such that $0 < |\rho| < |\varepsilon|$ and, for a subsequence,
\[
either \int_{B(0, r+\rho)} |\nabla w_k|^2 \to 0, \quad \int_{B(0, r+\rho)} |w_k|^2 \to 0, \quad \text{and} \quad \int_{B(0, r+\rho)} \frac{|w_k|^2}{|x|^2} \to 0,
\]
or \[
\int_{\mathbb{R}^N \setminus B(0, r+\rho)} |\nabla w_k|^2 \to 0, \quad \int_{\mathbb{R}^N \setminus B(0, r+\rho)} |w_k|^2 \to 0, \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B(0, r+\rho)} \frac{|w_k|^2}{|x|^2} \to 0.
\]

**Proof.** An analogous result is proved in [28] for minimizing sequences of quotient (19). Since the proof of Proposition 7.2 is similar, we will be sketchy.

Let $\varepsilon \in (0, r)$ (the proof for $\varepsilon$ negative is similar). Since
\[
\int_r^{r+\varepsilon} d\rho \int_{\partial B(0, \rho)} |\nabla w_k|^2 = \int_{B(0, r) \setminus B(0, r+\varepsilon)} |\nabla w_k|^2,
\]
we can choose $\rho \in (0, \varepsilon)$ such that, for infinitely many $k$’s (i.e. along a subsequence still denoted as $\{w_k\}$)
\[
\int_{\partial B(0, r+\rho)} |\nabla w_k|^2 \leq 2 \int_{B(0, r+\varepsilon) \setminus B(0, r)} |\nabla w_k|^2.
\]
From the uniform bound of $S_k(\lambda)$ (see (92)) and equivalence of $Q_\lambda$ to $D_k^{1,2}(\mathbb{R}^N)$-norm, it follows that
\[
\int_{B(0, r+\rho) \setminus B(0, r)} |\nabla w_k|^2 \leq \text{const} Q_\lambda(w_k) = \text{const} S_k(\lambda) \leq \text{const}.
\]
From (94–95), it follows that $\{w_k|_{\partial B(0, r+\rho)}\}$ is bounded in $H^1(\partial B(0, r+\rho))$. By compactness of the embedding $H^1(\partial B(0, r+\rho)) \hookrightarrow H_0^{1/2}(\partial B(0, r+\rho))$ and weak convergence to 0, we conclude that, up to subsequence, $\{w_k|_{\partial B(0, r+\rho)}\}$ converges strongly to 0 in $H_0^{1/2}(\partial B(0, r+\rho))$. 
Let \( w_k^1 \) (respectively \( w_k^2 \)) be the harmonic functions in \( B(0, r + \varepsilon) \setminus B(0, r + \rho) \) (respectively \( B(0, r + \rho) \setminus B(0, r - \varepsilon) \)) such that

\[
\begin{align*}
&\begin{cases}
  w_k^1 = w_k &\text{on } \partial B(0, r + \rho), \\
  w_k^1 = 0 &\text{on } \partial B(0, r + \varepsilon),
\end{cases}
&\begin{cases}
  w_k^2 = w_k &\text{on } \partial B(0, r + \rho), \\
  w_k^2 = 0 &\text{on } \partial B(0, r - \varepsilon).
\end{cases}
\end{align*}
\]

By continuity of \( \Delta^{-1} \) in contradiction with (22), unless that

\[
\text{let } \text{Theorem 7.3.}
\]

Direct computations yield

\[
Q_\lambda(u_k^1) = S_k(\lambda) \int_{B(0,r+\rho)} |w_k|^2^* \, \text{d}x + o(1),
\]

\[
Q_\lambda(u_k^2) = S_k(\lambda) \int_{\mathbb{R}^N \setminus B(0,r+\rho)} |w_k|^2^* \, \text{d}x + o(1),
\]

\[
Q_\lambda(u_k^1) + Q_\lambda(u_k^2) = S_k(\lambda) \int_{\mathbb{R}^N} |w_k|^2^* \, \text{d}x + o(1) = S_k(\lambda) + o(1) = Q_\lambda(w_k) + o(1),
\]

\[
\int_{\mathbb{R}^N} |u_k^1|^2^* \, \text{d}x + \int_{\mathbb{R}^N} |u_k^2|^2^* \, \text{d}x = \int_{\mathbb{R}^N} |w_k|^2^* \, \text{d}x + o(1).
\]

We claim that either \( Q_\lambda(u_k^1) \to 0 \) or \( Q_\lambda(u_k^2) \to 0 \) along some subsequence. Indeed, assume that \( Q_\lambda(u_k^1) \not\to 0 \) along any subsequence, i.e. \( Q_\lambda(u_k^1) \) stays bounded below away from 0. From above, (93) and (22), it follows

\[
\frac{Q_\lambda(u_k^1)}{\|u_k^1\|^2_{2^*}} = \frac{Q_\lambda(w_k) - Q_\lambda(u_k^2) + o(1)}{\|w_k\|^2_{2^*} - \|u_k^2\|^2_{2^*} + o(1)} \leq \frac{Q_\lambda(w_k)}{\|w_k\|^2_{2^*}} - \frac{Q_\lambda(u_k^2)}{\|u_k^2\|^2_{2^*} + o(1)} < S_k(\lambda)
\]

in contradiction with (22), unless \( Q_\lambda(u_k^2) \to 0 \) along some subsequence. The claim is thereby proved. The statement of the proposition follows from equivalence to norm of \( Q_\lambda \), Hardy’s and Sobolev’s inequalities.

**Theorem 7.3.** Let \( \lambda \in (-\infty, (n-2)/4) \) and \( N \geq 4 \). Then \( \lim_{k \to +\infty} S_k(\lambda) = S_{\text{circ}}(\lambda) \).

**Proof.** From Lemma 7.1 we know that, for \( \lambda \) sufficiently large, \( S_k(\lambda) \) is achieved, hence there exists some \( u_k \in D^{1,2}(\mathbb{R}^N) \) such that

\[
\int_{\mathbb{R}^N} |u_k|^2^* = 1 \quad \text{and} \quad Q_\lambda(u_k) = S_k(\lambda).
\]

From the uniform bound of \( S_k(\lambda) \) (see (92)) and equivalence of \( Q_\lambda \) to \( D^{1,2}(\mathbb{R}^N) \)-norm, it follows that \( \{u_k\}_k \) is bounded in \( D^{1,2}(\mathbb{R}^N) \). Let us set

\[
\tilde{u}_k(x) = R_k^{\frac{n-2}{2}} u_k \left( \frac{x}{R_k} \right) \quad \text{and} \quad v_k(x) = (S_k(\lambda))^{\frac{1}{2}} \tilde{u}_k(x)
\]
where $R_k$ is chosen such that
\[
\int_{B(0, R_k)} \left| \nabla u_k(x) \right|^2 - \lambda \frac{|u_k(x)|^2}{|x|^2} \, dx = \int_{\mathbb{R}^N \setminus B(0, R_k)} \left| \nabla u_k(x) \right|^2 - \lambda \frac{|u_k(x)|^2}{|x|^2} \, dx = \frac{1}{2} S_k(\lambda).
\]
Invariance by scaling yields
\[
\int_{\mathbb{R}^N} |\tilde{u}_k|^2 = 1, \quad Q_{\lambda}(\tilde{u}_k) = S_k(\lambda),
\]
and
\[
\int_{\mathbb{R}^N} |v_k|^2 = (S_k(\lambda))^\frac{N}{2}, \quad Q_{\lambda}(v_k) = (S_k(\lambda))^\frac{N}{2},
\]
and
\[
\int_{B(0,1)} \left| \nabla v_k(x) \right|^2 - \lambda \frac{|v_k(x)|^2}{|x|^2} \, dx = \int_{\mathbb{R}^N \setminus B(0,1)} \left| \nabla v_k(x) \right|^2 - \lambda \frac{|v_k(x)|^2}{|x|^2} \, dx = \frac{1}{2} (S_k(\lambda))^\frac{N}{2}.
\]
Invariance by scaling also implies that \( \{\tilde{u}_k\} \) is bounded in \( D^{1,2}(\mathbb{R}^N) \), hence there exists a subsequence (still denoted as \( \{\tilde{u}_k\} \)) weakly converging to some \( \tilde{u}_0 \) in \( D^{1,2}(\mathbb{R}^N) \).

**Claim 1.** We claim that \( \tilde{u}_0 \neq 0 \). Assume by contradiction that \( \tilde{u}_0 \equiv 0 \). Using Proposition 7.2 for sequence \( \tilde{u}_k \) with \( r = 1 \) and \( \varepsilon = \pm \frac{1}{4} \) and taking into account (99), (92), and (96), we deduce that there exist \( \rho_+ \in (0,1/4) \) and \( \rho_- \in (-1/4,0) \) such that, up to a subsequence,
\[
0 \leq \rho_+ \leq |x| \leq 1 + \rho_- \quad \text{and} \quad \eta(x) \equiv 0 \quad \text{for} \quad |x| \not\in [3/4, 5/4].
\]
Set \( \tilde{v}_k := \eta v_k \). Clearly \( \tilde{v}_k \in D^{1,2}_k(\mathbb{R}^N) \). By choice of \( \rho \) and (100)–(101) we have
\[
Q_{\lambda}(\tilde{v}_k) = Q_{\lambda}(v_k) + o(1), \quad \|\tilde{v}_k - v_k\|_{D^{1,2}(\mathbb{R}^N)} = o(1), \quad \int_{\mathbb{R}^N} |\tilde{v}_k|^2 = \int_{\mathbb{R}^N} |v_k|^2 + o(1).
\]
Let us define
\[
f(u) := \frac{1}{2} \int_{B(0,5/4) \setminus B(0,3/4)} |\nabla u|^2 - \frac{1}{2} \int_{B(0,5/4) \setminus B(0,3/4)} |u|^2, \quad u \in H^1_0(B(0,5/4) \setminus B(0,3/4)).
\]
From (100–101), (92), and (98), it is easy to verify that
\[
f'(\tilde{v}_k) \rightarrow 0 \quad \text{in} \quad (D^{1,2}(\mathbb{R}^N))^* \quad \text{and} \quad f(\tilde{v}_k) = \frac{1}{N}(S_k(\lambda))^{N/2} + o(1) \leq \frac{1}{N}(S_{\text{circ}}(\lambda))^{N/2} + o(1),
\]
i.e. \( \tilde{v}_k \) is a Palais-Smale sequence for \( f \) in \( H^1_0(B(0,5/4) \setminus B(0,3/4)) \). From Struwe’s representation lemma for diverging Palais-Smale sequences [27, Theorem III.3.1], we deduce the existence of an integer \( M \in \mathbb{N} \), \( M \) sequences of points \( \{x_i\} \subseteq B(0,5/4) \setminus B(0,3/4) \) and \( M \) sequences of radii \( \{R_i\} \), \( i = 1, \ldots, M \), such that \( \lim_k R_i = +\infty \) and
\[
\tilde{v}_k(x) = \sum_{i=1}^M (R_i^+)^{\frac{N-2}{2}} \tilde{v}_0(R_i^+(x - x_i^+)) + R_k(x) \quad \text{where} \quad R_k \rightarrow 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N),
\]
\( \tilde{v}_0 = u_0^0 \) and \( u_0^0 \) is the Talenti-Aubin function in (17). Let us consider the sequence \( \{x_k^1\}_k \); up to subsequence we can assume that it converges to some point \( x^1 \in B(0, 5/4) \setminus B(0, 3/4) \). Let us write \( x^1 = (z_1, y_1) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \) where \( z_1 = |z_1| e^{i \sqrt{-1}} \).

Let us first assume that \( |z_1| \neq 0 \). We fix \( J \in \mathbb{N} \) and for any \( i = 1, 2, \ldots, J \) we set

\[
S_i = \left\{ (z, y) = (|z| e^{i \sqrt{-1}}, y) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \bar{\theta} - \frac{(2i - 1)\pi}{J} < \theta < \bar{\theta} + \frac{(2i + 1)\pi}{J} \right\}.
\]

Note that \( x^1 \in S_1 \) and there exists \( \delta = \delta(J) > 0 \) such that

\[
B(x^1, \delta) \subset \left\{ (|z| e^{i \sqrt{-1}}, y) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \bar{\theta} - \frac{(2i - 1)\pi}{2J} < \theta < \bar{\theta} + \frac{(2i + 1)\pi}{2J} \right\}.
\]

Choose \( k = \tilde{k} = k(\delta) \) such that for all \( k \geq \tilde{k} \)

\[
x_k^1 = (z_k^1, y_k^1) \in B(x^1, \frac{\delta}{2}) \quad \text{and} \quad (R_k^1)^{-1} < \frac{\delta}{2}.
\]

Moreover, if \( \tilde{k} \) is chosen sufficiently large, for each \( i = 1, 2, \ldots, J \) it is possible to find \( \tau_i^k \in \mathbb{Z}_k \) such that \( (\tau_i^k z_1, y_1) \) stays in the middle half of \( S_i \), i.e.

\[
(\tau_i^k z_1, y_1) \in \left\{ (z, y) = (|z| e^{i \sqrt{-1}}, y) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \bar{\theta} - \frac{(2i - 1)\pi}{2J} < \theta < \bar{\theta} + \frac{(2i + 1)\pi}{2J} \right\}.
\]

Hence

\[
B((\tau_i^k z_1, y_1), \delta) \subset S_i, \quad (\tau_i^k z_1, y_1) \in B((\tau_i^k z_1, y_1), \frac{\delta}{2})
\]

and consequently

\[
B((\tau_i^k z_1, y_1), \frac{\delta}{2}) \subset S_i
\]

which yields

\[
B((\tau_i^k z_1, y_1), (R_k^1)^{-1}) \subset S_i.
\]

In particular the \( J \) balls \( B((\tau_i^k z_1, y_1), (R_k^1)^{-1}) \) are disjoint, hence, by symmetry properties of \( \tilde{v}_k \) we have that

\[
(S_k(\lambda))^{N/2} + o(1) = \int_{\mathbb{R}^N} |\tilde{v}_k|^{2^*} \geq \sum_{i=1}^{J} \int_{B((\tau_i^k z_1, y_1), (R_k^1)^{-1})} |\tilde{v}_k|^{2^*} = \sum_{i=1}^{J} \int_{B(x_k^1, (R_k^1)^{-1})} |\tilde{v}_k|^{2^*}.
\]

On the other hand from (102) we have, for \( k \) large,

\[
\left( \int_{B(x_k^1, (R_k^1)^{-1})} |\tilde{v}_k|^{2^*} \right)^{\frac{1}{2^*}} \geq \left( \int_{B(x_k^1, (R_k^1)^{-1})} (R_k^1)^N |\tilde{v}_0(R_k^1(x - x_k^1))|^{2^*} \right)^{\frac{1}{2^*}} - \left( \int_{B(x_k^1, (R_k^1)^{-1})} |R_k|^{2^*} \right)^{\frac{1}{2^*}} \geq \frac{1}{2} \left( \int_{B(0, 1)} \tilde{v}_0^{2^*} \right)^{\frac{1}{2^*}}.
\]

Therefore

\[
(S_k(\lambda))^{N/2} + o(1) \geq \frac{J}{2^{2^*}} \int_{B(0, 1)} \tilde{v}_0^{2^*}.
\]
and, in view of (92)

\[(S_{\text{circ}}(\lambda))^N/2 \geq \frac{J}{2^{2^2}} \int_{B(0,1)} \tilde{u}^2.\]

Letting \( J \to +\infty \), we find a contradiction. Claim 1 is thereby proved in the case \( |z_1| \neq 0 \). The case \( |z_1| = 0 \) can be treated exploiting the radial symmetry of functions \( \tilde{u}_k \) in the last \( N - 2 \) variables with a similar argument (even simpler due to the stronger symmetry).

**Claim 2.** We claim that \( \tilde{u}_0 \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \). We first note that \( \tilde{u}_k \) satisfy the equation \(-\Delta \tilde{u}_k - \lambda \frac{\tilde{u}_k}{|x|^2} = S_k(\lambda)\tilde{u}_k^{2^*-1} \). From (92), we can assume that \( S_k(\lambda) \to L \in (0, +\infty) \) at least for a subsequence. Hence, due to weak convergence of \( \tilde{u}_k \to \tilde{u}_0 \), we can pass to the limit in the equation to find that \( \tilde{u}_0 \) satisfies the equation \(-\Delta \tilde{u}_0 - \lambda \frac{\tilde{u}_0}{|x|^2} = L\tilde{u}_0^{2^*-1} \). By classical regularity theory for elliptic equations, we deduce that \( \tilde{u}_0 \) is a smooth function outside the origin.

Let \( R > 0 \). Assume that there exist \((z_1, y), (z_2, y) \in B(0, R) \cap (\mathbb{R}^2 \times \mathbb{R}^{N-2})\), \(|z_1| = |z_2|\), such that \( \tilde{u}_0(z_1, y) \neq \tilde{u}_0(z_2, y) \). Then there exist \( \delta > 0 \) such that \( \tilde{u}_0(x) \neq \tilde{u}_0(y) \) for any \( x \in B((z_1, y), \delta) \), \( y \in B((z_2, y), \delta) \). Let \( 0 < \varepsilon < \frac{1}{2}|B(0, \delta)| \). Since, up to a subsequence, \( \tilde{u}_k \to \tilde{u}_0 \) a.e. in \( B(0, R) \), by the Severini-Egorov Theorem, there exists a measurable set \( \Omega \subset B(0, R) \) such that \( |\Omega| \leq \varepsilon \) and \( \tilde{u}_k \to \tilde{u}_0 \) uniformly in \( B(0, R) \setminus \Omega \). Hence for \( k \) large, \( \tilde{u}_k(x) \neq \tilde{u}_k(y) \) for any \( x \in B((z_1, y), \delta) \setminus \Omega \), \( y \in B((z_2, y), \delta) \setminus \Omega \). On the other hand, if \( k \) is large enough, there exists \( \tau_k \in \mathbb{Z}_k \) such that

\[ |\tau_k(B((z_1, y), \delta)) \Delta B((z_2, y), \delta)| < \varepsilon, \]

where \( \Delta \) denotes the symmetric difference of sets. Hence

\[ |(\tau_k(B((z_1, y), \delta)) \cap B((z_2, y), \delta)) \setminus \Omega| > |B(0, \delta)| - 2\varepsilon > 0. \]

In particular the set \( (\tau_k(B((z_1, y), \delta)) \cap B((z_2, y), \delta)) \setminus \Omega \) has non-zero measure. If \((z, y) \in (\tau_k(B((z_1, y), \delta)) \cap B((z_2, y), \delta)) \setminus \Omega \), then \( z = \tau_k \tilde{z} \) with \( \tilde{z}, \tilde{y} \in B((z_1, y), \delta) \) and by symmetry of \( \tilde{u}_k \), \( \tilde{u}_k(z, y) = \tilde{u}_k(\tilde{z}, \tilde{y}) \), thus giving a contradiction. Hence \( \tilde{u}_0 \) is invariant by the \( SO(2) \)-action on the first two variables on \( B(0, R) \) for any \( R \). Invariance by the \( SO(N-2) \)-action on the last \((N-2)\) variables follows easily from pointwise convergence. Then we conclude that \( \tilde{u}_0 \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \).

Hence we have proved that, up to a subsequence, \( \tilde{u}_k \to \tilde{u}_0 \) in \( D^{1,2}(\mathbb{R}^N) \), with \( \tilde{u}_0 \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \setminus \{0\} \). Weak convergence yields

\[ Q_\lambda(\tilde{u}_k) = Q_\lambda(\tilde{u}_0) + Q_\lambda(\tilde{u}_k - \tilde{u}_0) + o(1) \]

while Brezis-Lieb Lemma implies

\[ \|\tilde{u}_k\|_{2^*}^2 = \|\tilde{u}_0\|_{2^*}^2 + \|\tilde{u}_k - \tilde{u}_0\|_{2^*}^2 + o(1). \]

From (103), (104), (97), and (22) we have

\[ S_k(\lambda) \leq \frac{Q_\lambda(\tilde{u}_0)}{\|\tilde{u}_0\|_{2^*}^2} \leq S_k(\lambda) \frac{Q_\lambda(\tilde{u}_k) - Q_\lambda(\tilde{u}_k - \tilde{u}_0) + o(1)}{(Q_\lambda(\tilde{u}_k)^{2^*/2} - Q_\lambda(\tilde{u}_k - \tilde{u}_0)^{2^*/2} + o(1))^{2/2^*}}. \]

Hence

\[ \frac{Q_\lambda(\tilde{u}_k) - Q_\lambda(\tilde{u}_k - \tilde{u}_0) + o(1)}{(Q_\lambda(\tilde{u}_k)^{2^*/2} - Q_\lambda(\tilde{u}_k - \tilde{u}_0)^{2^*/2} + o(1))^{2/2^*}} \geq 1. \]
Since $Q_{\lambda}(\tilde{u}_k)$ stay bounded away from 0, this is possible only when $Q_{\lambda}(\tilde{u}_k - \tilde{u}_0) \to 0$. Since $Q_{\lambda}^{1/2}$ is an equivalent norm, we deduce that $\tilde{u}_k \to \tilde{u}_0$ in $D^{1,2}(\mathbb{R}^N)$. In particular $\|\tilde{u}_0\|_{2^*} = \lim_k \|\tilde{u}_k\|_{2^*} = 1$. Hence, by weakly lower semi-continuity of $Q_{\lambda}$, (97), and (92)

$$S_{\text{circ}}(\lambda) \leq \frac{Q_{\lambda}(\tilde{u}_0)}{\|\tilde{u}_0\|_{2^*}^2} \leq \liminf_k Q_{\lambda}(\tilde{u}_k) = \liminf_k S_k(\lambda) \leq \limsup_k S_k(\lambda) \leq S_{\text{circ}}(\lambda).$$

Therefore all the above inequalities are indeed equalities. We have thus proved that along a subsequence, $S_k(\lambda)$ converges to $S_{\text{circ}}(\lambda)$. The Uryson’s property yields convergence of the entire sequence.

8. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 is based on Theorem 7.3 and the following lemma.

**Lemma 8.1.** $\limsup_{k \to +\infty} S_k(\lambda_0, \lambda_1, \ldots, \lambda_m) \leq S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m)$.

**Proof.** Let $\varepsilon > 0$. Then from (5) and density of $\mathcal{D}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{1,2}_0(\mathbb{R}^N)$ in $\mathcal{D}^{1,2}_0(\mathbb{R}^N)$, there exists $u \in \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{1,2}_0(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} |u|^2 = 1$ and

$$\int_{\mathbb{R}^N} |\nabla u|^2 \ dx - \lambda_0 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2} \ dx - \sum_{\ell=1}^m \Lambda_\ell \int_{\mathbb{R}^N} \left( \int_{S_{x, \ell}} \frac{u^2(y)}{|x - y|^2} \ d\sigma(x) \right) \ dy < S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m) + \varepsilon. \quad (105)$$

For any $\ell = 1, \ldots, m$, set

$$f_\ell(x) := \int_{\mathbb{R}^N} \frac{|u(y + x)|^2}{|y|^2} \ dy, \quad x \in S_{x, \ell}.$$ 

It is easy to check that $f_\ell \in C^0(S_{r, \ell})$; indeed if $x_n \in S_{r, \ell}$ converge to $x \in S_{r, \ell}$, by the Dominated Convergence Theorem we conclude that $\lim_n f_\ell(x_n) = f_\ell(x)$. Hence the Riemann sum

$$\frac{1}{k} \sum_{i=1}^k f_\ell(x_i) = \frac{1}{k} \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|u(y)|^2}{|y - x_i|^2} \ dy$$

converges to the integral

$$\frac{1}{2\pi \ell} \int_{S_{r, \ell}} f_\ell(x) \ d\sigma(x) = \int_{\mathbb{R}^N} \left( \int_{S_{x, \ell}} \frac{u^2(y)}{|x - y|^2} \ d\sigma(x) \right) \ dy.$$ 

Hence there exists $k$ such that for all $k \geq \bar{k}$

$$\int_{\mathbb{R}^N} |\nabla u|^2 \ dx - \lambda_0 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2} \ dx - \sum_{\ell=1}^m \Lambda_\ell \int_{\mathbb{R}^N} \frac{u^2(x)}{|x - a_\ell|^2} \ dx - \varepsilon \leq \int_{\mathbb{R}^N} |\nabla u|^2 \ dx - \lambda_0 \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^2} \ dx - \sum_{\ell=1}^m \Lambda_\ell \int_{\mathbb{R}^N} \left( \int_{S_{x, \ell}} \frac{u^2(y)}{|x - y|^2} \ d\sigma(x) \right) \ dy. \quad (106)$$

From (105–106), we deduce that

$$S_k(\lambda_0, \lambda_1, \ldots, \lambda_m) - \varepsilon < S_{\text{circ}}(\lambda_0, \lambda_1, \ldots, \lambda_m) + \varepsilon.$$ 

Taking $\limsup$ as $k \to +\infty$, since $\varepsilon$ is arbitrary we reach the conclusion. ■
Proof of Theorem 1.4. As in the proof of Theorem 1.3, we can find a minimizing sequence \( \{u_n\}_n \subset D^{1,2}_k(\mathbb{R}^N) \) for (3) with the Palais-Smale property. Under assumption (10), Lemma 6.2 yields

\[
S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < S_{\text{circ}}(\lambda_0),
\]

while (9) implies

\[
S_k(\lambda_0) \leq S_k\left(\lambda_0 + k \sum_{\ell=1}^m \lambda_{\ell}\right).
\]

Let \( 0 < \varepsilon < S_{\text{circ}}(\lambda_0) - S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \). From Lemma 8.1, there exists \( k_1 = k_1(\varepsilon) \) such that for all \( k \geq k_1 \)

\[
S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) + \varepsilon.
\]

From Theorem 7.3 and (107), there exists \( k_2 \) such that for all \( k \geq k_2 \)

\[
S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) + \varepsilon < S_k(\lambda_0) \leq S_{\text{circ}}(\lambda_0).
\]

Let \( k_3 \) be such that for all \( k \geq k_3 \)

\[
S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) + \varepsilon \leq \min\left\{ k^{\pm} S, k^{\pm} S(\lambda_1), \ldots, k^{\pm} S(\lambda_m) \right\}.
\]

From (108–111), we conclude that for all \( k \geq \max\{k_1, k_2, k_3\} \)

\[
S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < \min\left\{ k^{\pm} S, k^{\pm} S(\lambda_1), \ldots, k^{\pm} S(\lambda_m), S_k(\lambda_0), S_k\left(\lambda_0 + k \sum_{\ell=1}^m \lambda_{\ell}\right) \right\}.
\]

From above and the Palais-Smale condition proved in Theorem 4.1, we deduce that \( \{u_n\}_{n \in \mathbb{N}} \) has a subsequence strongly converging to some \( u_0 \in D^{1,2}_k(\mathbb{R}^N) \) such that \( J_k(u_0) = \frac{1}{N} S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \).

Hence \( u_0 \) achieves the infimum in (3). Since \( J_k \) is even, also \( |u_0| \) is a minimizer in (3) and then

\[
v_0 = S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)^{1/(2^*-2)}|u_0| \]

is a nonnegative solution to equation (2). The maximum principle implies the positivity outside singularities of such a solution. \( \blacksquare \)

9. Proof of Theorem 1.5

We now provide a sufficient condition for the infimum in (3) to stay below the level \( k^{2/N} S(\lambda_j) \), in correspondence of which possible concentration at singular points located at the \( j \)-th polygon can occur. We denote by \( \Theta_{j,\ell} \) the minimum angle formed by vectors \( a_j^i \) and \( a_j^\ell \), see figure below.
The following lemma can be proved by standard trigonometry calculus.

**Lemma 9.1.** For any $i, s = 1, 2, \ldots, k$, and $j, \ell = 1, 2, \ldots, m$, there holds

\[
|a_i^j - a_s^j| = 2r_j \left| \sin \left( \frac{(s - i)\pi}{k} \right) \right|,
\]

\[
|a_i^j - a_s^\ell|^2 = r_j^2 + r_\ell^2 - 2r_j r_\ell \cos \left( \frac{2\pi(i - s)}{k} + \Theta_{j\ell} \right).
\]

**Lemma 9.2.** Let $j \in \{1, 2, \ldots, m\}$. If

\[
0 < \lambda_j \leq \frac{N(N - 4)}{4}
\]

and

\[
\lambda_0 \frac{1}{|r_j|^2} + \lambda_j \sum_{i=1}^{k-1} \frac{1}{4r_j^2 \sin \frac{i\pi}{k}} + \sum_{\ell=1, \ell \neq j}^{m} \lambda_{\ell} \sum_{i=1}^{k} \frac{1}{r_j^2 + r_\ell^2 - 2r_j r_\ell \cos \left( \frac{2\pi(i - s)}{k} + \Theta_{j\ell} \right)} > 0,
\]

then

\[
S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < k^{2/N} S(\lambda_j).
\]

**Proof.** Let $z(x) = \sum_{i=1}^{k} z_i^{j\lambda_i}(x - a_i^j) \in D^{1,2}_k(\mathbb{R}^N)$. Then

\[
S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \leq \frac{Q_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}(z)}{\left( \int_{\mathbb{R}^N} |z|^2 \, dx \right)^{2/2^*}},
\]

where $Q_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}$ denotes the quadratic form defined by

\[
Q_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \frac{\lambda_0}{|x|^2} u^2(x) \, dx - \sum_{\ell=1}^{m} \sum_{i=1}^{k} \lambda_{\ell} \int_{\mathbb{R}^N} \frac{u^2(x)}{|x - a_i^\ell|^2} \, dx.
\]

Note that

\[
\left( \int_{\mathbb{R}^N} |z|^2 \, dx \right)^{2/2^*} \geq k^{2/2^*}.
\]
Moreover, from (19), Lemmas 5.1 and 5.2 we find that

$$Q_{\lambda_0, \lambda_1, \ldots, \lambda_m}(z) = k \int_{\mathbb{R}^N} \nabla z_{\mu}^\lambda |x|^2 \, dx - k \lambda_j \int_{\mathbb{R}^N} \frac{|z_1^\lambda|^2}{|x|^2} \, dx - \lambda_0 \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{|z_{\mu}^\lambda|^2}{|x + a_i|^2} \, dx$$

$$- \sum_{\ell=1}^m \sum_{s=1}^k \int_{\mathbb{R}^N} \frac{\lambda_{\ell} |z_\mu^\lambda|^2}{|x + a_s - a_\ell|^2} \, dx + \sum_{i=1}^k \int_{\mathbb{R}^N} \nabla z_{\mu}^\lambda (x - a_i^j) \cdot \nabla z_{\mu}^\lambda (x - a_i^j) \, dx$$

$$- m \sum_{\ell=1}^m \sum_{s \neq t}^k \int_{\mathbb{R}^N} \frac{\lambda_{\ell} z_{\mu}^\lambda (x - a_s^j) z_{\mu}^\lambda (x - a_t^j)}{|x - a_s|^2} \, dx - \lambda_0 \sum_{i=1}^k \int_{\mathbb{R}^N} \frac{z_{\mu}^\lambda (x - a_i^j) z_{\mu}^\lambda (x - a_i^j)}{|x|^2} \, dx$$

$$= \begin{cases} 
  k S(\lambda_j) - \mu^2 \left( \int_{\mathbb{R}^N} |z_1^\lambda|^2 \right) \left[ \frac{k \lambda_0}{r_j^2} + \sum_{\ell=1}^m \sum_{s=1}^k \frac{\lambda_{\ell}}{|a_s^j - a_\ell|^2} + o(1) \right] & \text{if } \lambda_j < \frac{N(N-4)}{4}, \\
  k S(\lambda_j) - \mu^2 |\ln \mu| |\alpha_j^\lambda, N \left[ \frac{k \lambda_0}{r_j^2} + \sum_{\ell=1}^m \sum_{s=1}^k \frac{\lambda_{\ell}}{|a_s^j - a_\ell|^2} + o(1) \right] & \text{if } \lambda_j = \frac{N(N-4)}{4}. 
\end{cases}$$

Therefore (114) holds provided

$$\frac{k \lambda_0}{r_j^2} + \sum_{\ell=1}^m \sum_{s=1}^k \frac{\lambda_{\ell}}{|a_s^j - a_\ell|^2} > 0.$$  
(115)

It is easy to verify that assumption (113) and Lemma 9.1 imply (115).

**Lemma 9.3.** For any $j, \ell = 1, \ldots, m$, $j \neq \ell$, there holds

$$\lim_{k \to \infty} \frac{1}{\Lambda_j} \left[ \lambda_j \sum_{i=1}^{k-1} \frac{1}{4 r_j^2 |\sin \frac{\pi t}{k}|^2} \right] = +\infty,$$

$$\lambda_{\ell} \sum_{i=1}^{k} \frac{1}{r_j^2 + r_\ell^2 - 2 r_j r_\ell \cos \left( \frac{2 \pi t}{k} + \Theta_{j\ell} \right)} = \Lambda_{\ell} O(1) \quad \text{as } k \to +\infty.$$  
(117)

**Proof.** A direct calculation yields

$$\frac{\lambda_j}{\Lambda_j} \sum_{i=1}^{k-1} \frac{1}{4 r_j^2 |\sin \frac{\pi t}{k}|^2} \geq \frac{1}{k} \int_{-\frac{k}{2}}^{k/2} ds \int_{-\frac{k}{2}}^{k/2} \frac{dt}{4 r_j^2 |\sin \frac{\pi t}{k}|^2} \geq \frac{1}{k} \int_{-\frac{k}{2}}^{k/2} \frac{dt}{4 r_j^2 |\sin t|^2} \quad \text{as } k \to +\infty.$$  
On the other hand

$$\frac{\lambda_{\ell}}{\Lambda_{\ell}} \sum_{i=1}^{k} \frac{1}{r_j^2 + r_\ell^2 - 2 r_j r_\ell \cos \left( \frac{2 \pi t}{k} + \Theta_{j\ell} \right)} \leq \frac{1}{k} \int_{0}^{k} ds \int_{0}^{2\pi} dt \quad \text{as } k \to +\infty.$$  

$$= \frac{1}{2\pi} \int_{0}^{2\pi} dt \in \mathbb{R}^+.$$
thus proving (117). ■

**Remark 9.4.** Lemma 9.3 implies that if we fix \( \lambda_0, \Lambda_1, \ldots, \Lambda_m \) and let \( k \to +\infty \), then the quantity in formula (113) tends to +\( \infty \). Hence condition (113) is satisfied for \( k \) sufficiently large.

Let us now compare levels \( S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) and \( S_k(\lambda_0) \) which is related to concentration at the origin. Two cases can occur:

(i) \( S_k(\lambda_0) \geq k^{2/N} S \),

(ii) \( S_k(\lambda_0) < k^{2/N} S \).

In case (i), since \( S = S(0) \) and \( \lambda \mapsto S(\lambda) \) is a nonincreasing function, to exclude that the infimum in (22) stays above \( S_k(\lambda_0) \) it is enough to compare \( S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) with \( k^{2/N} S(\lambda_j) \) where \( \lambda_j = \max\{\lambda_k\}_1 \leq k \leq m \), as we have done in Lemma 9.2.

The study of case (ii) is based on Lemma 2.1. Indeed, using Lemma 2.1 and estimates of Lemma 5.1, we can prove the following lemma.

**Lemma 9.5.** If \( N \geq 4 \), \( S_k(\lambda_0) < k^{2/N} S \), and one of the following assumptions is satisfied

\[
\lambda_0 \leq \frac{N(N-4)}{4} \quad \text{and} \quad \sum_{\ell=1}^{m} \frac{\lambda_{\ell}}{|r_{\ell}|^2} > 0, \quad (118)
\]

\[
\frac{N(N-4)}{4} < \lambda_0 < \frac{(N-2)^2}{4} \quad \text{and} \quad \sum_{\ell=1}^{m} \frac{\lambda_{\ell}}{|r_{\ell}|\sqrt{(N-2)^2-4\lambda_0}} > 0, \quad (119)
\]

then

\[
S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < S_k(\lambda_0). \quad (120)
\]

**Proof.** From Lemma 2.1, we have that \( S_k(\lambda_0) \) is attained by some \( u^{\lambda_0} \in D^1_k(\mathbb{R}^N) \). By homogeneity of the Rayleigh quotient, we can assume \( \int |u^{\lambda_0}|^2 = 1 \). Furthermore, the function \( v^{\lambda_0} = S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)^{1/(2^*-2)}|u^{\lambda_0}| \) is a nonnegative solution to (16), hence we can apply Lemma 5.1 to study the behavior of \( \int_{\mathbb{R}^N} \frac{|u^{\lambda_0}(x)|^2}{|x|^2} \) as \( \mu \to 0 \). Hence we obtain that there exists some positive constant \( \kappa_0 \) such that

\[
S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \quad (121)
\]

\[
\leq \int_{\mathbb{R}^N} |\nabla u^{\lambda_0}(x)|^2 \, dx - \lambda_0 \int_{\mathbb{R}^N} \frac{|u^{\lambda_0}(x)|^2}{|x|^2} \, dx - \sum_{\ell=1}^{k} \lambda_{\ell} \int_{\mathbb{R}^N} \frac{|u^{\lambda_0}(x)|^2}{|x-a_{\ell}|^2} \, dx
\]

\[
= S_k(\lambda_0) - \begin{cases} 
\mu^2 \left( \sum_{\ell=1}^{k} \frac{\lambda_{\ell}}{|r_{\ell}|^2} + o(1) \right) & \text{if } \lambda_0 < \frac{N(N-4)}{4} \\
\kappa_0^2 |\mu| \ln \mu \left( \sum_{\ell=1}^{k} \frac{\lambda_{\ell}}{|r_{\ell}|^2} + o(1) \right) & \text{if } \lambda_0 = \frac{N(N-4)}{4} \\
\kappa_0^2 \beta_{\lambda,N} \mu \sqrt{(N-2)^2-4\lambda} \left( \sum_{\ell=1}^{k} \frac{\lambda_{\ell}}{|r_{\ell}|\sqrt{(N-2)^2-4\lambda}} + o(1) \right) & \text{if } \lambda_0 > \frac{N(N-4)}{4}.
\end{cases}
\]
Taking $\mu$ sufficiently small we obtain that either assumption (118) or (119) yield (120).

**Proof of Theorem 1.5.** As in the proof of Theorem 1.3, we can find a minimizing sequence $\{u_n\}_{n \in \mathbb{N}}$ which has the Palais-Smale property, more precisely $J'_k(u_n) \to 0$ in $(D^{1,2}_k(\mathbb{R}^N))^*$ and $J_k(u_n) \to \frac{1}{\mu}S_k(\lambda_0 \Lambda_1, \ldots, \Lambda_m)$. Assumption (11) yields

$$S_k(\lambda_0 + k \sum_{\ell=1}^k \lambda_\ell) \geq S_k(\lambda_0).$$

(122)

Note also that (12) and (14) imply that $N > 4$ and $\lambda_m > 0$. Two cases can occur. If $S_k(\lambda_0) < k^2 S$, then Lemma 9.5 and assumption (14) yield

$$S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < S_k(\lambda_0).$$

If $S_k(\lambda_0) \geq k^{2/N} S$, from monotonicity we have $S_k(\lambda_0) \geq k^{2/N} S > k^{2/N} S(\lambda_m)$. In both cases from Lemma 9.2 and (15) we deduce $k^{2/N} S(\lambda_m) > S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m)$. Hence we obtain that

$$S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) < \min \{k^2 S, k^2 S(\lambda_1), \ldots, k^2 S(\lambda_m), S_k(\lambda_0), S_k(\lambda_0 + k \sum_{\ell=1}^m \lambda_\ell)\}.$$

From above and the Palais-Smale condition proved in Theorem 4.1, we deduce that $\{u_n\}_{n \in \mathbb{N}}$ has a subsequence strongly converging to some $u_0 \in D^{1,2}_k(\mathbb{R}^N)$ which achieves the infimum in (3). Moreover $v_0 = S_k(\lambda_0 \Lambda_1, \ldots, \Lambda_m)^{\frac{1}{m-1}} |u_0|$ is a solution to (2).

**Remark 9.6.** Theorem 1.5 contains an alternative proof to Theorem 1.4 in the case $N > 4$, as it follows easily gathering Theorem 1.5 and Remark 9.4. Note that the assumption $N > 4$ is needed to ensure that (12) and (14) hold. However, with respect to Theorem 1.4, it contains a more precise information on how $k$ must be large in order to solve the problem.

10. **Proof of Theorem 1.2.** Assume first (i). Let $\varepsilon > 0$. Then from (3) and density of $\mathcal{D}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{1,2}_k(\mathbb{R}^N)$ in $\mathcal{D}^{1,2}_k(\mathbb{R}^N)$, there exists $u \in \mathcal{D}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{1,2}_k(\mathbb{R}^N)$ such that $Q_{\lambda_0}(u) \leq S_k(\lambda_0) + \varepsilon$.

Let

$$u_\mu(x) = \mu^{-\frac{N-2}{2}} u(x/\mu), \quad \mu > 0.$$

By Dominated Convergence Theorem it is easy to verify that

$$\lim_{\mu \to 0} \int_{\mathbb{R}^N} \frac{|u_\mu(x)|^2}{|x-a_i|^2} = 0,$$

hence

$$S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \leq \frac{\int_{\mathbb{R}^N} |\nabla u_\mu|^2 dx - \lambda_0 \int_{\mathbb{R}^N} \frac{u_\mu^2(x)}{|x|^2} dx - \int_{\mathbb{R}^N} \sum_{\ell=1}^m \Lambda_\ell \sum_{i=1}^k \frac{u_\mu^2(x)}{|x-a_i|^{2k}} dx}{(\int_{\mathbb{R}^N} |u_\mu|^2 dx)^{2/2^*}}$$

$$= Q_{\lambda_0}(u) + o(1) \leq S_k(\lambda_0) + \varepsilon + o(1) \quad \text{as } \mu \to 0.$$
Assume by contradiction that \( S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) is attained by \( \bar{u} \in D^{1,2}_k(\mathbb{R}^N) \setminus \{0\} \), then

\[
S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) = \frac{\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx - \lambda_0 \int_{\mathbb{R}^N} \bar{u}^2(x) \frac{1}{|x|^{2\gamma}} dx - \int_{\mathbb{R}^N} \sum_{\ell=1}^m \Lambda_\ell \sum_{i=1}^k \bar{u}^2(x) \frac{\partial^2 u}{|x-\sigma_i|^2} dx}{(\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx)^{2/2^*}} > \frac{\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx - \int_{\mathbb{R}^N} \frac{\bar{u}^2(y)}{|y|^{2\gamma}} dy}{(\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx)^{2/2^*}} \geq S_k(\lambda_0),
\]

giving rise to a contradiction. The proof of non-attainability of \( S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) is analogous and is based on (90).

Assume now that (ii) holds. Then for all \( u \in D^{1,2}_{\text{circ}}(\mathbb{R}^N), u \geq 0 \), denoting by \( u^* \) the Schwarz symmetrization of \( u \) (see (28)), from (29) and (30) it follows that

\[
\frac{Q^\text{circ}_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}(u)}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{2/2^*}} \geq \frac{\int_{\mathbb{R}^N} |\nabla u^*|^2 dx - (\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell) \int_{\mathbb{R}^N} \frac{|u^*(y)|^2}{|y|^{2\gamma}} dy}{(\int_{\mathbb{R}^N} |\nabla u^*|^2 dx)^{2/2^*}} \geq S_{\text{circ}}(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell) = S(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell).
\]

Hence \( S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) is contained in \( \{14, \text{Theorem 1.3}\} \).

On the other hand, setting \( \Lambda = \lambda_0 + \sum_{\ell=1}^m \Lambda_\ell \), and using [14, Corollary 3.2] we obtain

\[
S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \leq Q^\text{circ}_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}(z^A_\mu) = Q_A(z^A_\mu) + o(1) = S(\Lambda) + o(1)
\]

\[
= S_{\text{circ}}(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell) + o(1) \quad \text{as } \mu \to \infty.
\]

Then

\[
(123) \quad S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) = S_{\text{circ}}(\lambda_0 + \sum_{\ell=1}^m \Lambda_\ell).
\]

If \( S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) was attained by some \( \bar{u} \in D^{1,2}_{\text{circ}}(\mathbb{R}^N) \setminus \{0\} \), then

\[
S_{\text{circ}}(\lambda_0, \Lambda_1, \ldots, \Lambda_m) = \frac{Q^\text{circ}_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}(\bar{u})}{(\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx)^{2/2^*}} \geq \frac{Q_A(\bar{u}^*)}{(\int_{\mathbb{R}^N} |\nabla \bar{u}^*|^2 dx)^{2/2^*}} \geq S_{\text{circ}}(\Lambda).
\]

Due to (123), all above inequalities are indeed equalities; in particular \( Q^\text{circ}_{\lambda_0, \Lambda_1, \ldots, \Lambda_m}(\bar{u}) = Q_A(\bar{u}^*) \) which (taking into account Polya-Szego inequality and (29)) yields

\[
0 \leq \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx - \int_{\mathbb{R}^N} |\nabla \bar{u}^*|^2 dx
\]

\[
= \int_{\mathbb{R}^N} \left( \frac{\lambda_0}{|y|^{2\gamma}} + \sum_{\ell=1}^m \Lambda_\ell \int_{S_{\sigma_\ell}} \frac{ds(x)}{|x-y|^{2\gamma}} \right) \bar{u}^2(y) dy - \Lambda \int_{\mathbb{R}^N} \frac{\bar{u}^*(y)^2}{|y|^{2\gamma}} dy \leq 0.
\]

Then \( \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx = \int_{\mathbb{R}^N} |\nabla \bar{u}^*|^2 dx \). From [3], it follows that \( \bar{u} \) must be spherically symmetric with respect to some point. Since \( \bar{u} \) is a solution to equation (4) (up to some Lagrange’s multiplier), the potential in equation (4) must be spherically symmetric, thus giving rise to a contradiction. The proof of non-attainability of \( S_k(\lambda_0, \Lambda_1, \ldots, \Lambda_m) \) is contained in [14, Theorem 1.3].
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