SIGN-CHANGING BLOW-UP FOR THE YAMABE EQUATION
AT THE LOWEST ENERGY LEVEL

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ABSTRACT. We investigate the blow-up behavior of sequences of sign-changing solutions for the Yamabe equation on a Riemannian manifold \((M, g)\) of positive Yamabe type. For each dimension \(n \geq 11\), we describe the value of the minimal energy threshold at which blow-up occurs. In dimensions \(11 \leq n \leq 24\), where the set of positive solutions is known to be compact, we show that the set of sign-changing solutions is not compact and that blow-up already occurs at the lowest possible energy level. We prove this result by constructing a smooth, non-locally conformally flat metric on space forms \(S^n/\Gamma, \Gamma \neq \{1\}\), whose Yamabe equation admits a family of sign-changing blowing-up solutions. As a counterpart of this result, we also prove a sharp compactness result for sign-changing solutions at the lowest energy level, in small dimensions or under strong geometric assumptions.

1. INTRODUCTION

1.1. Introduction and statements of the main results. Let \((M, g)\) be a smooth, closed (i.e. compact and without boundary) Riemannian manifold of dimension \(n \geq 3\). In this paper, we are interested in the existence of sequences of sign-changing blowing-up solutions \((u_k)_{k \in \mathbb{N}}\) in \(C^2(M)\) to the nodal (or sign-changing) Yamabe equation

\[
\Delta_g u_k + c_n \text{Scal}_g u_k = |u_k|^{2^*-2} u_k \quad \text{in } M,
\]

where \(\Delta_g := -\text{div}_g \nabla\) is the Laplace–Beltrami operator, \(c_n := \frac{n-2}{4(n-1)}\), \(\text{Scal}_g\) is the scalar curvature of the manifold and \(2^* = \frac{2n}{n-2}\) is the critical exponent for the embeddings of the Sobolev space \(H^1(M)\) into Lebesgue’s spaces. Solutions of (1.1) are in \(C^{3,\alpha}(M)\) for \(0 < \alpha < \min(2^* - 2, 1)\) by Trudinger’s result \[57\] and standard elliptic theory. We recall that \((u_k)_{k \in \mathbb{N}}\) is said to blow up as \(k \to \infty\) if \(\|u_k\|_{L^\infty(M)} \to \infty\) as \(k \to \infty\).

The Yamabe invariant of the conformal class \([g]\) is defined as

\[
Y(M, [g]) := \inf_{\tilde{g} \in [g]} \left( \text{Vol}_{\tilde{g}}(M) \frac{2-n}{n-2} \int_M \text{Scal}_{\tilde{g}} \, dv_{\tilde{g}} \right)
= \frac{4(n-1)}{n-2} \inf_{u \in C^\infty(M) \setminus \{0\}} \frac{\int_M (|\nabla u|^2_g + c_n \text{Scal}_g u^2) \, dv_g}{\left(\int_M |u|^{2^*} \, dv_g\right)^{\frac{2}{2^*}}},
\]

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where Vol$(M)$ is the volume of $(M, \hat{g})$. Letting $(S^n, g_{\text{std}})$ be the standard unit sphere of dimension $n \geq 3$, by conformal invariance, we also have

$$Y(S^n, [g_{\text{std}}]) = \inf_{u \in C_c^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} \left| \nabla u \right|^2 \, dx}{\left( \int_{\mathbb{R}^n} |u|^{2^*} \, dx \right)^{\frac{n}{2n-2}}}.$$

so that $Y(S^n, [g_{\text{std}}])^{-\frac{1}{2}}$ is the optimal constant for the Sobolev inequality in $\mathbb{R}^n$. We say that $(M, g)$ is of positive Yamabe type if $Y(M, [g]) > 0$, i.e. $\Delta_g + c_n \text{Scal}_g$ is coercive. In this case, when $(M, g)$ is not conformally diffeomorphic to the standard sphere $(S^n, g_{\text{std}})$ (which we denote in what follows by $(M, g) \not\approx (S^n, g_{\text{std}})$), we have $Y(M, [g]) < Y(S^n, [g_{\text{std}}])$, and the existence of a positive solution $u_0$ to the Yamabe equation

$$\Delta_g u_0 + c_n \text{Scal}_g u_0 = u_0^{2^*-1} \text{ in } M$$

attaining the Yamabe invariant $Y(M, [g])$ is known since the work of Trudinger [57], Aubin [3] and Schoen [51].

In this paper, we are interested in determining the value of the minimal energy level at which sign-changing blow-up occurs for (1.1). For every $u \in H^1(M)$, we define the energy of $u$ as

$$E(u) := \int_M |u|^2 \ast dv_g.$$

We define $I(M, [g]) \subset (0, \infty]$ as the set of numbers $E \in (0, +\infty]$ such that (1.1) admits a blowing-up sequence of solutions $(u_k)_{k \in \mathbb{N}}$ with $\limsup_{k \to \infty} E(u_k) = E$. We then define

$$E(M, [g]) := \inf I(M, [g]).$$

Similarly, we let $I_+(M, [g]) \subset (0, +\infty]$ be the set of numbers $E \in (0, +\infty]$ such that (1.1) admits a blowing-up sequence of positive solutions $(u_k)_{k \in \mathbb{N}}$ satisfying $\limsup_{k \to \infty} E(u_k) = E$, and we define

$$E_+(M, [g]) := \inf I_+(M, [g]).$$

The value $+\infty$ is allowed in the definitions of $E(M, [g])$ and $E_+(M, [g])$ and corresponds to sequences of solutions of (1.1) with diverging energies. When $I(M, [g])$ (resp. $I_+(M, [g])$) is empty, it means that (1.1) does not admit any blowing-up solutions (resp. positive blowing-up solutions), and thus the set of solutions (resp. positive solutions) of (1.1) is compact in $C^2(M)$ by standard elliptic theory. In this case, we let $E(M, [g]) := -\infty$ (resp $E_+(M, [g]) := -\infty$). If $I(M, [g]) \neq \emptyset$ and $(u_k)_{k \in \mathbb{N}}$ is a sequence of solutions of (1.1) satisfying $\limsup_{k \to \infty} E(u_k) < E(M, [g])$, then by definition $(u_k)_{k \in \mathbb{N}}$ does not blow-up, and is thus precompact in $C^2(M)$.

As long as positive solutions are considered, $E_+(M, [g])$ is well understood: in the proof of the compactness of positive solutions of the Yamabe equation, assuming the validity of the Positive Mass Theorem when $n \geq 8$, Khuri–Marques–Schoen [20] (with previous contributions by Schoen [52] [53], Li–Zhu [30], Druet [19], Marques [31] and Li–Zhang [28] [29]) proved that

$$E_+(M, [g]) = -\infty \text{ when }$$

$$\begin{cases} 3 \leq n \leq 24 \text{ and } (M, g) \not\approx (S^n, g_{\text{std}}), \\
(M, g) \text{ is l.c.f. and } (M, g) \not\approx (S^n, g_{\text{std}}), \text{ or} \\
n \geq 6 \text{ and } \sum_{k=0}^{\left\lfloor \frac{n-6}{2} \right\rfloor} |\nabla^k \text{Weyl}_g(x)|^2 > 0 \; \forall x \in M, \end{cases}$$

(1.3)
where l.c.f. stands for locally conformally flat and Weyl, is the Weyl curvature tensor of the manifold. On the other side, Brendle [7] and Brendle–Marques [8] proved that there exists a non-locally conformally flat metric \( g \) on \( S^n \) such that

\[
E_+(S^n, [g]) = Y(S^n, [g_{std}])^{\frac{2}{n}} \quad \text{when } n \geq 25. \tag{1.4}
\]

The blow-up behavior of sign-changing solutions of (1.1) is far less understood. A simple application of Struwe’s celebrated \( H^1 \)-compactness result [54] (see Proposition 5.1 below for a proof) shows that if \((M, g)\) is of positive Yamabe type, then any sign-changing blowing-up sequence \((u_k)_{k \in \mathbb{N}}\) of solutions of (1.1) satisfies

\[
\liminf_{k \to \infty} E(u_k) \geq Y(S^n, [g_{std}])^{\frac{2}{n}} + Y(M, [g])^{\frac{2}{n}}. \tag{1.5}
\]

As a consequence, if \( g \) is the Brendle [7] and Brendle–Marques [8] metric, then, by (1.4), we obtain

\[
E(S^n, [g]) = E_+(S^n, [g]) = Y(S^n, [g_{std}])^{\frac{2}{n}} \quad \text{when } n \geq 25.
\]

In dimensions \( n \geq 25 \), blow-up for (1.1) thus occurs at the lowest energy level for positive solutions. Our aim in this paper is to investigate the situation for sign-changing solutions and in particular, the value of \( E(M, [g]) \) in dimensions \( n \leq 24 \), where the set of positive solutions of (1.1) is compact, i.e. \( E_+(M, [g]) = -\infty \). As follows from (1.5), a lower bound on \( E(M, [g]) \) is given by \( Y(S^n, [g_{std}])^{\frac{2}{n}} + Y(M, [g])^{\frac{2}{n}} \). Our main result shows that this lower bound is attained in dimensions \( 11 \leq n \leq 24 \):

**Theorem 1.1.** Assume that \( 11 \leq n \leq 24 \) and let \( \Gamma \) be a finite subgroup of isometries of \((S^n, g_0), \Gamma \neq \{Id\}\), acting freely and smoothly on \( S^n \). There exists a smooth, non-locally conformally flat Riemannian metric \( g \) on \( S^n / \Gamma \) of positive Yamabe type and such that

\[
E(M, [g]) = Y(S^n, [g_{std}])^{\frac{2}{n}} + Y(S^n / \Gamma, [g])^{\frac{2}{n}}.
\]

The metric \( g \) in Theorem 1.1 is not the quotient metric \( q \), but can be chosen arbitrarily close to it. Theorem 1.1 is a special case of a more general result, Theorem 2.2 below, which holds true in any dimension \( n \geq 11 \) and for a larger class of manifolds than the spherical space forms \( S^n / \Gamma \). We refer to Section 2 for more details on this regard.

By (1.3), the set of positive solutions of (1.1) on \((S^n / \Gamma, g)\), where \( g \) is given by Theorem 1.1, is compact in \( C^2(M) \). As Theorem 1.1 shows, however, in this case, the set of sign-changing solutions is not compact and blow-up already occurs at the minimal energy level. This phenomenon of loss of compactness for sign-changing solutions in situations where the set of positive solutions is compact was recently highlighted in Premoselli–Vétois [43] for critical Schrödinger-type equations in \( M \).

We prove Theorem 1.1 by constructing a sign-changing blowing-up sequence \((u_k)_{k \in \mathbb{N}}\) of solutions of (1.1) such that

\[
\lim_{k \to \infty} E(u_k) = Y(S^n, [g_{std}])^{\frac{2}{n}} + Y(M, [g])^{\frac{2}{n}}. \tag{1.6}
\]

As a counterpart of Theorem 1.1, we also prove the following result, which provides a lower bound for \( E(M, [g]) \) in smaller dimensions or under strong geometric assumptions:

**Theorem 1.2.** Let \((M, g)\) be a smooth, closed Riemannian manifold of dimension \( n \geq 3 \) and positive Yamabe type which is not conformally diffeomorphic to the standard sphere \((S^n, [g_{std}])\). Assume that one of the following conditions is satisfied:
• $(M, g)$ is locally conformally flat,
• $n \leq 9$,
• $n = 10$ and $u_0 \not\equiv \frac{5}{8\pi} |\text{W}y\tilde{g}|^2$ for all points in $M$ and all solutions $u_0$ of \((1.2)\) attaining $Y(M, [g])$, or
• $n \geq 11$ and $\text{W}y\tilde{g} \neq 0$ for all points in $M$.

Then
\[
E(M, [g]) > Y(S^n, [g_{\text{std}}])^{\frac{2}{n}} + Y(M, [g])^{\frac{2}{n}}.
\]

Under the assumptions of Theorem \ref{thm:1.2} and by \ref{thm:1.3}, the set of positive solutions of \((1.1)\) is compact in $C^2(M)$, and thus blow-up for \((1.1)\) can only occur for sign-changing solutions. If sign-changing blow-up occurs it happens at a possibly infinite energy level that is, at least, strictly higher than the minimal one given by \ref{eq:1.5}. In particular, if the assumptions of Theorem \ref{thm:1.2} are satisfied, then there exists a constant $\varepsilon_0 > 0$ such that the set of all solutions $u \in C^2(M)$ of \((1.1)\) satisfying
\[
E(u) \leq Y(M, [g])^{\frac{2}{n}} + Y(S^n, [g_{\text{std}}])^{\frac{2}{n}} + \varepsilon_0
\]
is compact in $C^2(M)$. The contrapositive of Theorem \ref{thm:1.2} provides necessary conditions for sign-changing blowing-up solutions to exist at the minimal energy level. In dimensions $n \geq 11$, this condition is that Weyl$_g$ vanishes at some (but not all) points in $M$. This is consistent with Theorem \ref{thm:1.1} since the sequence $(u_k)_{k \in \mathbb{N}}$ that we construct to prove Theorem \ref{thm:1.1} blows up at a point where Weyl$_g$ vanishes. Hence, Theorems \ref{thm:1.1} and \ref{thm:1.2} are sharp in every dimension $11 \leq n \leq 24$. The condition arising when $n = 10$ is purely analytical (see \ref{equation:5.44} below). The exact value of $E(M, [g])$ when $3 \leq n \leq 10$ is not yet known: as Theorem \ref{thm:1.2} shows, at least when $n \leq 9$, it is strictly larger than $Y(M, [g])^{\frac{2}{n}} + Y(S^n, [g_{\text{std}}])^{\frac{2}{n}}$. Computing $E(M, [g])$ when $3 \leq n \leq 10$ will be the focus of forthcoming work.

We conclude this subsection by mentioning an important additional motivation for investigating the value of $E(M, [g])$ and, more generally, (non-)compactness issues for sign-changing solutions of \((1.1)\). Let $g$ be a Riemannian metric in $M$ of positive Yamabe type. For $k \in \mathbb{N}$ and $\tilde{g} \in [g]$, we denote by $\lambda_k(\tilde{g})$ the $k$-th eigenvalue (counted with multiplicity) of the conformal Laplacian $\Delta_{\tilde{g}} + c_n \text{Scal}_{\tilde{g}}$ in $M$. Ammann–Humbert introduced in \cite{2} the so-called $k$-th Yamabe invariant:
\[
\mu_k(M, [g]) = \inf_{\tilde{g} \in [g]} \lambda_k(\tilde{g}) \text{Vol}_{\tilde{g}}(M)^{\frac{2}{n}}.
\]

In the case where $k = 2$, test functions computations show that
\[
\mu_2(M, [g])^{\frac{2}{n}} \leq Y(M, [g])^{\frac{2}{n}} + Y(S^n, [g_{\text{std}}])^{\frac{2}{n}}
\]
holds. For $k \geq 2$, extremal metrics attaining $\mu_k(M, [g])$, when they exist, are not smooth in general. When $(M, g)$ is of positive Yamabe type, Amman–Humbert \cite{2} established the existence of extremal metrics attaining $\mu_2(M, [g])$ provided $(M, g)$ is not locally conformally flat and $n \geq 11$. Moreover, Amman–Humbert \cite{2} obtained that if $\mu_2(M, [g])$ is attained by a generalized metric $\tilde{g} = u^{\frac{4}{n-2}}g$ with $u \geq 0$ and $\|u\|_{L^{\frac{n}{n-2}}(M)} = 1$, then there exists a generalized eigenvector $\tilde{w}$ associated to $\mu_2(M, [g])$ such that $u = |\tilde{w}|$ and $w := \mu_2(M, [g])^{\frac{n-2}{2}} \tilde{w}$ is a sign-changing solution of \((1.1)\) satisfying $E(w) = \mu_2(M, [g])^{\frac{2}{n}}$. With \ref{equation:1.7}, this shows that extremal metrics for $\mu_2(M, [g])$ give rise to sign-changing solutions of \((1.1)\) whose energies lie below the minimal blow-up level given by \ref{equation:1.5}. Theorem \ref{thm:1.2} thus provides a compactness result for a range of energy levels including $\mu_2(M, [g])$. Such a result is generally
perceived as a strong indication that $\mu_2(M, [g])$ is attained (at least for analogous problems in the two-dimensional case, see for example Matthiesen–Siffert [32] or Pétrides [38]): Theorem 1.2 can therefore also be seen as a first step in a more systematic investigation of $\mu_k(M, [g])$.

1.2. Review of the literature and outline of the paper. Existence results for the nodal Yamabe equation (1.1) have been the subject of several works in the last decades. On the standard sphere $(\mathbb{S}^n, g_{std})$, existence results of large-energy sign-changing solutions of (1.1) are in Ding [18], del Pino–Musso–Pacard–Pistoia [15, 16], Musso–Wei [36], Medina–Musso–Wei [34] and Medina–Musso [33]; other existence results at lower energy levels are in Clapp [11] and Fernandez–Petean [22]. For more general manifolds, existence and multiplicity results of sign-changing solutions of (1.1) have been obtained by Ammann–Humbert [2], Vétos [58], Clapp–Fernández [12], Clapp–Pistoia–Tavares [13] and Gursky–Pérez-Ayala [23]. Multiplicity results of sign-changing solutions of Yamabe–Schrödinger-type equations with more general potential functions can also be found in Vétos [58] and Clapp–Fernández [12].

Theorem 1.1 is both a non-compactness and an existence result: it shows in particular the existence of infinitely many solutions of (1.1) on $(\mathbb{S}^n/\Gamma, g)$. Theorem 1.2, on the contrary, is a compactness result for (1.1) below the energy level $Y(\mathbb{S}^n, [g_{std}]) + Y(M, [g])$. Compactness and non-compactness results for sign-changing solutions of Yamabe–Schrödinger-type equations have been obtained by Vétos [58] and recently by Premoselli–Vétos [42, 43] (see also Robert–Vétos [47, 49], Pistoia–Vétos [39] and Deng–Musso–Wei [17] for existence results of sign-changing blowing-up solutions to equations of type (1.1) with asymptotically critical nonlinearities). A general pointwise description of finite-energy blowing-up sequences of solutions of such equations, including the geometric case of (1.1), has recently been obtained by Premoselli [40]. To the best of the authors’ knowledge, Theorem 1.1 is the first constructive result of sign-changing blowing-up solutions for the geometric equation (1.1) on a different manifold than the standard sphere.

The paper is organised as follows. In Section 2, we state Theorem 2.2 which is a generalization of Theorem 1.1 in dimensions $n \geq 11$. We then prove Theorem 2.2 in Sections 3 and 4. The proof relies on a constructive Lyapunov–Schmidt reduction method. Our approach is inspired from the constructions on the sphere by Brendle [7] and Brendle–Marques [8] (see also Ambrosetti–Malchiodi [1] and Berti–Malchiodi [5]). In Section 3, we perform the Lyapunov–Schmidt reduction and construct a blowing-up sequence of approximate solutions of (1.1) of the form $u_k = -u_0 + B_k + \text{lower order terms}$, where $u_0 > 0$ solves (1.1) and $B_k$ is a positive bubbling profile modeled on the positive standard bubble (see (3.18) below). In the course of our construction we are able to control the lower-order terms in strong spaces so that the solutions $u_k$ that we construct are pointwise equal, to first-order, to $-u_0 + B_k$. They are therefore sign-changing, see Remark 3.6 below. In Section 4, we reduce the proof of Theorem 2.2 to finding a critical point of an energy function in $\mathbb{R}^{n+1}$ (see (4.3) below). The main difference with the constructions of Brendle [7] and Brendle–Marques [8] for positive solutions is that the critical point of $F$ that we find is of saddle-type: this allows us to conclude up to dimension 11 but in turn forces us to work with greater precision and expand the reduced energy to the fourth order (see (4.23) below). Finally, we prove Theorem 1.2 in Section 5 by using a Pohozaev-type identity together with the pointwise blow-up description for
sign-changing solutions of (1.1) recently obtained by Premoselli [30], which we refine here by using an approach based on iterated estimates, in the spirit of the method introduced by Chen–Lin [10] in the case of positive solutions (see also Marques [31] for applications to the Yamabe equation).

2. \textit{Y}-non-degenerate metrics and a refined version of Theorem 1.1

Let $(M, g)$ be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and positive Yamabe type. By the resolution of the Yamabe problem (see Trudinger [57], Aubin [9] and Schoen [51]) there exists a smooth positive function $u_0$ in $M$ that attains $Y(M, [g])$ and solves (1.2). In particular $\int_M u_0^2 \, dv_g = Y(M, [g])^{\frac{2}{n}}$. Let $\varphi \in C^\infty(M), \varphi > 0$. By the conformal invariance property of the conformal Laplacian, $u_0 = u_0/\varphi$ solves

$$\triangle_g \hat{u}_0 + c_n \text{Scal}_g \hat{u}_0 = u_0^{2^{*} - 1} \quad \text{in } M,$$

where we have let $g_0 := \varphi^{2^{*} - 2}g$. Assume that one of the positive minimizers $u_0$ achieving $Y(M, [g])$ is non-degenerate as a solution of the Yamabe equation. This means that

$$\text{Ker}(\triangle_g + c_n \text{Scal}_g - (2^{*} - 1)u_0^{2^{*} - 2}) = \{0\},$$

where this kernel is regarded as a subset of $H^1(M)$. A simple application of the Implicit Function Theorem shows that for any metric $g_\varepsilon$ close enough to $g$ in some $C^p$ topology, $p \geq 3$, there exists a unique positive $u_* \in C^2(M)$ close to $u_0$ in $C^2(M)$ that solves

$$\triangle_g, u_* + c_n \text{Scal}_g, u_* = u_*^{2^{*} - 1} \quad \text{in } M \quad \text{(2.2)}$$

and that is also non-degenerate. Generically with respect to perturbations of the metric, at least in dimensions $n \leq 24$ (see Theorem 10.3 of Khuri–Marques–Schoen [26]), all positive solutions of (1.2) are non-degenerate. In the locally conformally flat case, concrete examples of situations where $u_0$ is non-degenerate are given by $u_0 := ((n - 2)/2) \frac{\sqrt[n-2]}{\sqrt[n-2]}$ on $S^1(r) \times S^{n-1}$, where $S^1(r)$ is the circle of radius $r \in (0, \infty) \setminus \{i/\sqrt{n-2} : i \in \mathbb{N}\}$ and $S^{n-1}$ is the unit $(n - 1)$-sphere, both equipped with their standard metrics (see Proposition 3.4 of Robert–Vétois [77]).

We introduce the following definition:

\textbf{Definition 2.1.} Let $(M, g)$ be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and positive Yamabe type. We say that $g$ is \textit{Y}-non-degenerate if

- one of the positive minimizers $u_0$ achieving $Y(M, [g])$ is non-degenerate
- and there exists a constant $\varepsilon_g > 0$ such that for any metric $g_\varepsilon$ satisfying $\|g_\varepsilon - g\|_{C^0(M)} \leq \varepsilon_g$, the unique function $u_*$ close to $u_0$ in $C^2(M)$ satisfying (2.2) still attains $Y(M, [g_\varepsilon])$, i.e. satisfies

$$\int_M u_*^{2^{*}} \, dv_{g_\varepsilon} = Y(M, [g_\varepsilon])^{\frac{2}{n}}.$$

By the conformal invariance of $\triangle_g + c_n \text{Scal}_g$, if $g$ is Y-non-degenerate in $M$ then any metric in the conformal class of $g$ is still Y-non-degenerate. This notion allows us to state a generalization of Theorem 1.1.

\textbf{Theorem 2.2.} Let $(M, \tilde{g})$ be a smooth, closed, locally conformally flat Riemannian manifold of dimension $n \geq 11$ which is Y-non-degenerate in the sense of Definition 2.1. Then there exist a smooth, non-locally conformally flat metric $g$ in $M$ and a
sequence of blowing-up sign-changing solutions \((u_k)_{k \in \mathbb{N}}, u_k \in C^{3, \alpha}(M)\) for some \(0 < \alpha < 1\), to the nodal Yamabe equation for \(g\):
\[
\Delta_g u_k + \frac{n-2}{4(n-1)} \text{Scal}_g u_k = |u_k|^{2^*-2} u_k \quad \text{in } M,
\]
which satisfies
\[
\int_M |u_k|^{2^*} \, dv_g \nearrow Y(M, [g])^\frac{2}{2^*} + Y(S^n, [g_{std}])^\frac{2}{2^*}
\]
as \(k \to \infty\).

The metric \(g\) in Theorem 2.2 can be chosen arbitrarily close to the original metric \(\hat{g}\) in \(C^p(M)\) for any \(p \geq 3\). Theorem 2.2 is proven in Sections 3 and 4. In the rest of this section, we prove that spherical space forms and their locally conformally flat perturbations are \(Y\)-non-degenerate and that Theorem 1.1 follows from Theorem 2.2. The first result is as follows:

**Proposition 2.3.** Let \(n \geq 3\) and let \(\Gamma\) be a finite subgroup of isometries of \((S^n, g_{std})\), \(\Gamma \neq \{Id\}\) acting freely and smoothly on \(S^n\). The quotient manifold \(M_\Gamma := S^n/\Gamma\) endowed with the quotient metric \(g_\Gamma\) is locally conformally flat and \(Y\)-non-degenerate in the sense of Definition 2.1.

**Proof.** By definition, \((M_\Gamma, g_\Gamma)\) is locally isometric to \((S^n, g_{std})\), hence it is locally conformally flat, has constant scalar curvature equal to \(n(n-1)\) and is thus of positive Yamabe type.

The constant function \(u_0 \equiv (n(n-2)/4)^{\frac{n-2}{2}}\) is a solution of the Yamabe equation on \((S^n, g_{std})\), so \(u_0\) descends to \(M_\Gamma\) as a constant positive solution, that we still denote by \(u_0\), of the Yamabe equation
\[
\Delta_{g_\Gamma} u_0 + \frac{n(n-2)}{4} u_0 = u_0^{2^*-1} \quad \text{in } M_\Gamma. \tag{2.3}
\]
We claim that the linearized operator at \(u_0\) in \(M_\Gamma\), which is given by \(L = \Delta_{g_\Gamma} - n\), has zero kernel. Indeed, if \(L \varphi = 0\) for some \(\varphi \in H^1(M_\Gamma)\) then \(\varphi\) lifts to \(S^n\) as a function \(\tilde{\varphi} \in H^1(S^n)\) which is \(\Gamma\)-invariant and satisfies \(\Delta_{g_{std}} \tilde{\varphi} - n \tilde{\varphi} = 0\). But the kernel of \(\Delta_{g_{std}} - n\) in \(S^n\) consists of the restrictions of the coordinate functions \((x_i)_{0 \leq i \leq n}\) of \(\mathbb{R}^{n+1}\) to \(S^n\), that are not \(\Gamma\)-invariant. Hence \(\varphi \equiv 0\) and \(u_0\) is a non-degenerate solution of the Yamabe equation in \(M_\Gamma\). This proves the first point in Definition 2.1.

We now prove the second point. Since \((M_\Gamma, g_\Gamma)\) is Einstein, a celebrated theorem of Obata 37 shows that \(u_0\) is the only positive solution of (2.3), so in particular \(u_0\) satisfies
\[
\int_{M_\Gamma} u_0^{2^*} \, dv_{g_\Gamma} = Y(M_\Gamma, [g_{\Gamma}])^\frac{2}{2^*} = \frac{1}{|\Gamma|} Y(S^n, [g_{std}])^\frac{2}{2^*}.
\]
By the Implicit Function Theorem there is \(\eta_0 > 0\) such that, for any \(\varepsilon\) small enough and any metric \(g\) on \(M_\Gamma\) such that \(\|g - g_{\Gamma}\|_{C^2(M_\Gamma)} \leq \varepsilon\), there is a unique positive function \(u_g \in C^2(M_\Gamma) \cap B_{H^1(M_\Gamma)}(u_0, \eta_0)\) satisfying
\[
\Delta_g u_g + c_n \text{Scal}_g u_g = u_g^{2^*-1} \quad \text{in } M. \tag{2.4}
\]
We also have \(\|u_g - u_0\|_{C^2(M_\Gamma)} \leq C \varepsilon\) for some \(C > 0\) independent of \(\varepsilon\), so \(u_g\) is still non-degenerate for \(\varepsilon\) small enough. We claim that for \(\varepsilon\) small enough, we again have
\[
\int_{M_\Gamma} u_g^{2^*} \, dv_g = Y(M_\Gamma, [g])^\frac{2}{2^*}
\]
for any \( g \) with \( \|g - g_{\Gamma}\|_{C^3(M_\Gamma)} \leq \varepsilon \). Assume by contradiction that, for a sequence of real numbers \((\varepsilon_j)_{j \in \mathbb{N}}\) such that \(\varepsilon_j \to 0\) as \(j \to \infty\) and a sequence of metrics \((g_j)_{j \in \mathbb{N}}\) with \(\|g_j - g_{\Gamma}\|_{C^3(M_\Gamma)} \leq \varepsilon_j\), the latter equality does not hold. Then

\[
\int_{M_\Gamma} u_j^2 v_{g_j} > Y(M_\Gamma, [g_j])^\frac{2}{n} \quad \forall j \in \mathbb{N},
\]

where we have let \(u_j := u_{g_j}\). Since \(g_j \to g_{\Gamma}\) in \(C^3(M_\Gamma)\), we have \(Y(M_\Gamma, [g_j]) \to Y(M_\Gamma, [g_{\Gamma}]) < Y(S^n, [g_{std}])\) and thus, by the resolution of the Yamabe problem, there exists a sequence \((v_j)_{j \in \mathbb{N}}\) of positive solutions of (2.4) with \(g = g_j\) satisfying

\[
\int_{M_\Gamma} v_j^2 v_{g_j} = Y(M_\Gamma, [g_j])^\frac{2}{n} \leq \frac{1}{|\Gamma|} Y(S^n, [g_{std}])^\frac{2}{n} + o(1)
\]

as \(j \to \infty\). In particular, \(u_j \neq v_j\) for all \(j \in \mathbb{N}\). Struwe’s \(H^1\)-compactness result \([54]\) together with (2.5) show that the sequence \((v_j)_{j \in \mathbb{N}}\) strongly converges in \(H^1(M_\Gamma)\) as \(j \to \infty\) towards a positive solution of (2.3). By the uniqueness result of Obata \([37]\), \(v_j \to v_0\), and hence the local uniqueness shows that \(v_j = u_j\) for large \(j\), a contradiction.

**Proof of Theorem 1.1.** Thanks to Proposition 2.3 we can apply Theorem 2.2 to \((M, \hat{g}) = (M_\Gamma, g_{\Gamma})\) for some finite subgroup \(\Gamma \neq \{Id\}\) of isometries of \((S^n, g_{std})\) acting freely and smoothly on \(S^n\). We can then let \(g\) be as in the statement of Theorem 2.2. By definition of \(E(M, [g])\), Theorem 2.2 then gives

\[
E(M, [g]) \leq Y(M, [g])^\frac{2}{n} + Y(S^n, [g_{std}])^\frac{2}{n}.
\]

The other inequality follows from Proposition 5.1 below, which concludes the proof of Theorem 1.1. \(\square\)

The arguments developed in the proof of Proposition 2.3 similarly show that if a metric \(g\) in \(M\) attains \(Y(M, [g])\) at a unique positive minimizer \(u_0\), that is also non-degenerate in the sense of (2.1), then \((M, g)\) is \(Y\)-non-degenerate. With this observation, we can prove the following result that provides additional \(Y\)-non-degenerate examples to which Theorem 2.2 applies:

**Proposition 2.4.** Let \(n \geq 3\), let \(\Gamma\) be a finite subgroup of \(G\) of isometries of \((S^n, g_{std})\), \(\Gamma \neq \{Id\}\) acting freely and smoothly on \(S^n\) and let \(g\) be a locally conformally flat metric in \(M_\Gamma\), \(\Gamma \neq \{Id\}\), that is close to \(g_{\Gamma}\) in \(C^p(M_\Gamma)\) for some \(p\) large enough. Then \(g\) is \(Y\)-non-degenerate.

**Proof.** This is a consequence of the uniqueness result of de Lima–Piccione–Zedda \([14]\) (Theorem 5). We provide some additional details here since the result of \([14]\) is not stated in this way. The analysis in \([14]\) leading to Theorem 5 applies as long as sequences of Yamabe metrics close to \(g_{\Gamma}\) can be made to converge strongly up to a subsequence. This is the case for sequences of locally conformally flat metrics \((g_k)_{k \in \mathbb{N}}\) on \(M_\Gamma\) close to \(g_{\Gamma}\) in \(C^p(M_\Gamma)\) for some \(p\) large enough. Local arguments indeed show that any metric \(g_k\) has positive Riemannian mass, with a positive uniform bound from below, at every point of \(M_\Gamma\). This is a purely local argument that works for any dimension \(n \geq 3\) and does not rely on the positive mass theorem. The convergence of \((g_k)_{k \in \mathbb{N}}\) up to a subsequence then follows from the arguments of Schoen \([53]\) and Li–Zhu \([30]\). Hence for any metric \(g\) on \(M\) with \(|\Gamma| \geq 2\) that is locally conformally flat and \(C^p\)-close to \(g_{\Gamma}\) for \(p\) large enough, the equation (2.4)
has a unique positive solution $u_\sigma$. This solution $u_\sigma$ remains non-degenerate for $\sigma$ close enough to $g$. \hfill \Box

**Remark 2.5.** A natural question connected with Definition 2.1, and with the observation before Proposition 2.4, is whether it is possible that the Yamabe equation admits multiple minimizers. This is indeed the case. Examples of such situations can be obtained by considering manifolds with a nontrivial isometry group. In particular, Schoen [53] gave a detailed study of the multiplicity of positive solutions to the Yamabe equation in the case of the product manifold $S^1 \times \Sigma^{n-1}$ with $r > 0$. In this case, if $r$ is chosen large enough, then there exists a family of distinct (degenerate) minimizers parametrized by $S^1$. This example can be extended to more general manifolds with different isometry groups (see Hebey–Vaugon [25]).

3. Proof of Theorem 2.2—Part 1: A Lyapunov–Schmidt reduction

3.1. The geometric setting. In this section and the next, we prove Theorem 2.2. Throughout the paper, we denote by $\delta_0$ the Euclidean metric in $\mathbb{R}^n$.

Let $(M, \hat{g})$ be a smooth, closed, locally conformally flat Riemannian manifold of positive Yamabe type that is $Y$-non-degenerate in the sense of Definition 2.1. In this section and the next, we always assume that $n := \dim(M) \geq 11$. Fix $x_0 \in M$ once and for all, and let $\delta > 0$ and $\varphi \in C^\infty(B_{\delta}(x_0, 8\delta))$, $\varphi > 0$, be such that $g_0 = \varphi^2 - 2\hat{g}$ is flat in $B_{\delta}(x_0, 8\delta)$. By decreasing $\delta$ if necessary and picking a local chart $\Phi$ that sends $x_0$ to 0, we can assume that $B_{\delta}(x_0, 6\delta)$ contains $\Phi^{-1}(B(0, 4\delta))$ and that $\Phi_* g_0$ is the Euclidean metric in $B(0, 4\delta) \subset \mathbb{R}^n$, where $B(0, 4\delta)$ is an Euclidean ball.

For any $k \in \mathbb{N}$, we let $y_k = (\delta/k, 0, \ldots, 0) \in \mathbb{R}^n$ and we let $(r_k)_{k \in \mathbb{N}}$ be a decreasing sequence of positive numbers converging to 0 such that $r_0 \leq \delta$ and $4r_k \leq |y_k - y_{k+1}|$ for all $k \in \mathbb{N}$. Any two Euclidean balls $B(y_k, 2r_k)$ and $B(y_l, 2r_l)$ are thus disjoint for $k \neq l$. Let $(\varepsilon_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers converging to 0 such that $\varepsilon_k = o(r_k^n)$ for any $p \geq 1$. Define, for any $k \in \mathbb{N}$,

$$\mu_k = \varepsilon_k^{\frac{4}{n-10}}. \quad (3.1)$$

Since $n \geq 11$ and $\varepsilon_k = o(r_k^n)$, we have $\mu_k = o(r_k^n)$ as $k \to \infty$ for any $p \geq 1$. Let $h$ be a smooth, symmetric bilinear form in $\mathbb{R}^n$ that satisfies

$$\text{tr } h(x) = 0, \quad \text{div } h(x) = 0 \quad \text{and} \quad \sum_{j=1}^n x_j h_{ij}(x) = 0 \quad (3.2)$$

for all $x \in \mathbb{R}^n$ and $1 \leq i \leq n$, where we have let $\text{tr } h := \sum_{j=1}^n h_{jj}$ and $\text{div } h_i := \sum_{j=1}^n \partial_j h_{ij}$. These are respectively the trace and the divergence of $h$ with respect to the Euclidean metric $\delta_0$, but the subscript $\delta_0$ will be omitted for clarity. We assume that for any $1 \leq i, j \leq n$, $x \mapsto h_{ij}(x)$ is a homogeneous polynomial of second-order in $\mathbb{R}^n$. Examples of such $h$ satisfying (3.2) are given in (4.2) below. Let $\chi \in C^\infty_c(\mathbb{R})$ be such that $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ on $\mathbb{R} \setminus [0, 2]$. We define a new metric in $B(0, 4\delta)$ by

$$\tilde{g}(x) := \exp \left( \sum_{k=1}^\infty \varepsilon_k \chi \left( \frac{|x - y_k|}{r_k} \right) h \left( \frac{x - y_k}{r_k} \right) \right). \quad (3.3)$$

We assume in addition that $\sum_{k \in \mathbb{N}} \varepsilon_k r_k^{-p} < +\infty$ for all $p \geq 0$. Since the components of $h$ are homogeneous polynomials of second order, $\tilde{g}$ is thus a smooth metric in
\(B(0, 4\delta)\), is flat in \(B(0, 4\delta) \setminus B(0, 3\delta)\) and satisfies

\[
\hat{g}(x) = \exp \left( \varepsilon_k \chi \left( \frac{|x - y_k|}{r_k} \right) h(x - y_k) \right)
\]

in \(B(y_k, 2r_k)\) for any \(k \in \mathbb{N}\). When extended and pulled back to \(M\), \(\Phi^* \hat{g}\) defines a metric in \(B_\delta(x_0, 8\delta)\), equal to \(g_0\) in \(B_\delta(x_0, 8\delta) \setminus B_\delta(x_0, 6\delta)\). The metric \(g = \varphi^{2-2r} \Phi^* \hat{g}\) thus defines a smooth metric in \(B_\delta(x_0, 8\delta)\), equal to the original metric \(\hat{g}\) in \(B_\delta(x_0, 8\delta) \setminus B_\delta(x_0, 6\delta)\), that we extend to be equal to \(\hat{g}\) on \(M \setminus B_\delta(x_0, 8\delta)\). We still call this new metric \(g\). We now define \(\tilde{g} = \varphi^{2-2g}\), which is a smooth metric in \(M\) such that \(\Phi_\ast \tilde{g} = \hat{g}\) as in (3.3) in \(B(0, 4\delta)\). By (3.2), we have \(\det \tilde{g} \equiv 1\) in \(B(0, 4\delta)\). Note that \(\tilde{g}\) can be chosen to be arbitrarily close to the Euclidean metric \(\hat{g}\) in \(C^p(B(0, 4\delta))\) for any \(p \geq 3\) by assuming that \(\sum_{k \in \mathbb{N}} \varepsilon_k r_k^{-p}\) is small enough. Hence \(g\) can be chosen arbitrarily close to the initial metric \(\hat{g}\) in \(C^p(M)\) for any \(p \geq 3\).

Since \(\hat{g}\) is \(Y\)-non-degenerate, so is \(g_0 = \varphi^{2-2r} \hat{g}\), and we can let \(\hat{u}_0\) be a non-degenerate positive solution of

\[
\triangle_{g_0} \hat{u}_0 + c_n \text{Scal}_{g_0} \hat{u}_0 = \hat{u}_0^{2^* - 1} \quad \text{in} \ M,
\]

that also satisfies

\[
\int_M \hat{u}_0^2 \, dv_{g_0} = Y(M, [g_0])^\frac{2}{2} = Y(M, [\hat{g}])^\frac{2}{2}.
\]

Definition 2.1 then yields the existence of a unique positive function \(\hat{u}_0 \in C^2(M)\) solving

\[
\triangle_{\hat{g}} \hat{u}_0 + c_n \text{Scal}_{\hat{g}} \hat{u}_0 = \hat{u}_0^{2^* - 1} \quad \text{in} \ M,
\]

that is still non-degenerate in the sense of 2.1 and satisfies

\[
\int_M \hat{u}_0^2 \, dv_{\hat{g}} = Y(M, [\hat{g}])^\frac{2}{2} = Y(M, [g])^\frac{2}{2}.
\]

In the rest of this section and in the following one, we construct, when \(n \geq 11\), a sequence of sign-changing solutions \((u_k)_{k \in \mathbb{N}}\) of class \(C^{3, \alpha}(M)\), \(0 < \alpha \leq 2^* - 2\), to

\[
\triangle_{\hat{g}} u_k + c_n \text{Scal}_{\hat{g}} u_k = |u_k|^{2^*-2} u_k \quad \text{in} \ M,
\]

where \(c_n := \frac{n-2}{4(n-1)}\), that satisfies

\[
\int_M |u_k|^2 \, dv_{\hat{g}} \leq Y(M, [\hat{g}])^\frac{2}{2} + Y(S^n, [g_{std}])^\frac{2}{2}.
\]

Since \(\hat{g} = \varphi^{2r-2} g\), and by the conformal invariance of the conformal Laplacian, by replacing \(u_k\) with \(\varphi u_k\), this will prove Theorem 2.2.

3.2. The ansatz of the construction. Define \(D^{1, 2}(\mathbb{R}^n)\) to be the completion of \(C^\infty_c(\mathbb{R}^n)\) for the norm \(u \mapsto \|
abla u\|_{L^2(\mathbb{R}^n)}\), that we endow with the associated scalar product. We fix \(A > 0\) to be chosen later, and for \((t, z) \in [1/A, A] \times B(0, 1)\) we let

\[
\mu_k(t) := \mu_k t \quad \text{and} \quad \xi_k(z) := y_k + \mu_k z,
\]

where \(y_k\) is as in the previous subsection, and for \(x \in \mathbb{R}^n\),

\[
B_{k, t, z}(x) = \left( \frac{\mu_k(t)^{2^* - 2}}{\left( \mu_k(t)^2 + \frac{|x - \xi_k(z)|^2}{m(n-2)} \right)^\frac{n-2}{2}} \right)^{\frac{n-2}{2}}.
\]
For any \((t, z) \in [1/A, A] \times B(0, 1)\), \(B_{k, t, z}\) solves

\[
\triangle_{\delta_0} B_{k, t, z} = B_{k, t, z}^{2^* - 1} \quad \text{in } \mathbb{R}^n,
\]

where \(\delta_0\) is the Euclidean metric in \(\mathbb{R}^n\) and \(\triangle_{\delta_0} := -\text{div}_{\delta_0} \nabla\). For \(x \in \mathbb{R}^n\), we let

\[
V_0(x) := \frac{\|x\|^2}{n(n-2)} - 1 \quad \text{and, for } 1 \leq j \leq n, \quad V_j(x) := \frac{x_j}{(1 + \frac{|x|^2}{n(n-2)})^{\frac{n}{2}}}. \quad (3.8)
\]

We then let

\[
Z_{0, k, t, z}(x) := \mu_k(t)^{1-\frac{2}{n}} V_0 \left( \frac{x - \xi_k(z)}{\mu_k(t)} \right) = \frac{2}{n-2} t\partial_t B_{k, t, z}
\]

and, for \(1 \leq j \leq n\),

\[
Z_{j, k, t, z}(x) := \mu_k(t)^{1-\frac{2}{n}} V_j \left( \frac{x - \xi_k(z)}{\mu_k(t)} \right) = -nt\partial_{z_j} B_{k, t, z}, \quad (3.10)
\]

and we let

\[
K_{k, t, z} := \text{span} \{ Z_{j, k, t, z} : 0 \leq j \leq n \},
\]

which is a finite-dimensional subspace of \(D^{1,2}(\mathbb{R}^n)\). We denote by \(K_{k, t, z}^\perp\) its orthogonal complement in \(D^{1,2}(\mathbb{R}^n)\). The functions \(Z_{j, k, t, z}\) satisfy

\[
\triangle_{\delta_0} Z_{j, k, t, z} = (2^*-1) B_{k, t, z}^{2^*-2} Z_{j, k, t, z} \quad \text{in } \mathbb{R}^n
\]

for all \(0 \leq j \leq n\), and by a result of Rey [45] and Bianchi-Egnell [9], they form an orthogonal basis of the set of solutions of this equation in \(D^{1,2}(\mathbb{R}^n)\). Letting \(h\) be as in the previous subsection, we define, for \(k \in \mathbb{N}\) and \(x \in \mathbb{R}^n\),

\[
h_k(x) := h(x - y_k).
\]

By (3.2), \(h_k\) is trace-free and divergence-free in \(\mathbb{R}^n\). As a first result, we obtain the following lemma:

**Lemma 3.1.** For any \((t, z) \in [1/A, A] \times B(0, 1)\) and \(k \in \mathbb{N}\), there exists a unique \(R_{k, t, z} \in K_{k, t, z}^\perp\) that satisfies

\[
\triangle_{\delta_0} R_{k, t, z} = (2^*-1) B_{k, t, z}^{2^*-2} R_{k, t, z} = -\varepsilon_k \sum_{p,q=1}^n (h_k)_{pq} \partial_{pq} B_{k, t, z} \quad \text{in } \mathbb{R}^n. \quad (3.12)
\]

This function \(R_{k, t, z}\) also satisfies, for \(i \in \{0, 1, 2\}\),

\[
|\nabla^i R_{k, t, z}(x)| \leq C \varepsilon_k \mu_k(t)^{\frac{n+i}{n+2}} \ln \left( \frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t) + |x - \xi_k(z)|} \right) \quad \text{for all } x \in \mathbb{R}^n, \text{ for some } C > 0 \text{ independent of } k, t, z. \quad (3.13)
\]

for all \(x \in \mathbb{R}^n\), for some \(C > 0\) independent of \(k, t, z\).

**Proof.** Let \(u \in C^2(\mathbb{R}^n)\) be such that \(u \in L^2(\mathbb{R}^n)\), \((1 + |x|)|\nabla u(x)| \in L^2(\mathbb{R}^n)\) and \((1 + |x|)^2 |\nabla^2 u(x)| \in L^2(\mathbb{R}^n)\). Define then

\[
G_k(u) := \frac{1}{2} \int_{\mathbb{R}^n} \sum_{p,q=1}^n (h_k)_{pq} \partial_p u \partial_q u \, dx.
\]

Integrating by parts and since \(h_k\) is divergence free, we get that

\[
G_k(u) = -\frac{1}{2} \int_{\mathbb{R}^n} \sum_{p,q=1}^n (h_k)_{pq} u \partial_{pq} u \, dx.
\]
Choose now $u = B_{k,t,z}$ for $t > 0$ and $z \in B(0,1)$, which is admissible for $G_k$ since $n \geq 11$. Explicit computations show that
\[
B_{k,t,z} \partial_{pq}^2 B_{k,t,z} = -\frac{1}{n} \frac{\mu_k(t)^{n-2} \delta_{pq}}{\left(\frac{|x - \xi(z)|^2}{n(n-2)}\right)^{n-1}} + \frac{n}{n-2} \partial_p B_{k,t,z} \partial_q B_{k,t,z},
\]
which gives, since $h_k$ is trace-free, $G_k(B_{k,t,z}) = -\frac{n}{n-2} G_k(B_{k,t,z})$, and so $G_k(B_{k,t,z}) = 0$. Differentiating with respect to $t$ and $z$, using (3.9) and (3.10) and integrating by parts shows that for any $j \in \{0, \ldots, n\}$,
\[
\int_{\mathbb{R}^n} Z_{j,k,t,z} \sum_{p,q=1}^n (h_{k})_{pq} \partial_{pq}^2 B_{k,t,z} \, dx = 0
\]
holds. Since, by Bianchi–Egnell [6], $\triangle_{h_k} - (2^* - 1)B_{k,t,z}^{2^*-2}$ is Fredholm and injective on $K_{k,t,z}^+$, the existence of a unique $R_{k,t,z} \in K_{k,t,z}^+$ satisfying (3.12) follows. By standard elliptic theory $R_{k,t,z}$ is smooth.

By (3.2), $h_k$ is trace-free and satisfies $\sum_{j=1}^n (x - y_k)(h_{k})_{ij}(x) = 0$ for all $1 \leq j \leq n$ and $x \in \mathbb{R}^n$. Direct computations then show that
\[
\sum_{p,q=1}^n (h_{k})_{pq} \partial_{pq}^2 B_{k,t,z} = -\frac{\mu_k^2}{n(n-2)} \sum_{p,q=1}^n (h_{k})_{pq} z_p z_q \frac{\mu_k(t)^{n-2}}{\left(\frac{|x - \xi(z)|^2}{n(n-2)}\right)^{n-2}}.
\]
It is then easily seen that, for all $x \in \mathbb{R}^n$,
\[
R_{k,t,z}(x) = \frac{\varepsilon_k n^{3/2}}{\mu_k} R_{t,z}(\frac{x - y_k}{\mu_k})
\]
holds, where $R_{t,z}$ is the unique solution in $K_{t,z}^+$ of
\[
\triangle_{h_k} R_{t,z} - (2^* - 1)B_{t,z}^{2^*-2} R_{t,z} = -\frac{t^{n-2}}{n(n-2)} \sum_{p,q=1}^n h_{pq} z_p z_q \left( t^2 + \frac{|x - z|^2}{n(n-2)} \right)^{n-2}
\]
in $\mathbb{R}^n$, (3.15)
where we have let
\[
B_{t,z}(x) := t^{\frac{n-2}{2}} \left( t^2 + \frac{|x - z|^2}{n(n-2)} \right)^{-\frac{n-2}{2}}
\]
and
\[
K_{t,z} := \text{span} \left\{ \partial_i B_{t,z}, \partial_{z_i} B_{t,z}, \ldots, \partial_{z_n} B_{t,z} \right\}.
\]
Since $t \in [1/A, A]$ and $z \in B(0,1)$ and since the $h_{ij}$ are homogeneous of degree 2, we can again write
\[
R_{t,z}(x) = t^{1/2} S_{0,z}(\frac{x - z}{t})
\]
for some smooth function $S_{0,z} \in K_{t,0}^+$ that satisfies
\[
|\triangle_{h_0} S_{0,z}(y) - (2^* - 1)B_{1,0}(y)^{2^*-2} S_{0,z}(y)| \leq C |z|^2 (1 + |y|)^{-n}
\]
for all $y \in \mathbb{R}^n$, for some $C > 0$ independent of $t$ and $z$. We can now write a representation formula for $\triangle_{h_0} - (2^* - 1)B_{1,0}^{2^*-2}$ (see for example Lemma 3.3 of Premoselli [11]) that shows that, for any $x \in \mathbb{R}^n$,
\[
|S_{0,z}(x)| \leq C |z|^2 \int_{\mathbb{R}^n} |x - y|^{2-n}(1 + |y|)^{-n} \, dy \leq C |z|^2 \ln(2 + |x|) \frac{\ln(2 + |x|)}{(1 + |x|)^{n-2}}
\]
for some constant \( C > 0 \) independent of \( z \). Differentiating the representation formula similarly yields
\[
|\nabla S_{0, z}(x)| \leq C|x|^2 \frac{\ln(2 + |x|)}{(1 + |x|)^{n+1}} \quad \text{and} \quad |\nabla S_{0, z}(x)|^2 \leq C|x|^2 \frac{\ln(2 + |x|)}{(1 + |x|)^{n+1}}
\]
for any \( x \in \mathbb{R}^n \). Going back to \( R_{k, t, z} \), this proves (3.13). \( \square \)

Let again \( \chi \in C^\infty_c(\mathbb{R}) \) be such that \( \chi \equiv 1 \) on \( [0, 1] \), \( 0 \leq \chi \leq 1 \) and \( \chi \equiv 0 \) on \( \mathbb{R}\setminus[0, 2] \). For \( t \in [1/A, A] \) and \( z \in B(0, 1) \), we let
\[
U_{k, t, z}(x) := \chi \left( \frac{x - y_k}{\epsilon_k} \right) (B_{k, t, z}(x) + R_{k, t, z}(x))
\] (3.17)
for any \( x \in \mathbb{R}^n \), and
\[
W_{k, t, z}(x) := U_{k, t, z}(\Phi(x)) - \tilde{u}_0(x)
\] (3.18)
for any \( x \in M \), where \( \Phi \) is the chart around \( x_0 \) introduced in the previous subsection, and where \( \tilde{u}_0 \) is as in (3.5). We let
\[
f(s) := |s|^{2^*-2} \quad \forall s \in \mathbb{R}.
\] (3.19)
We also let for \( k \in \mathbb{N} \) and \( (t, z) \in [1/A, A] \times B(0, 1) \),
\[
E_{k, t, z} := (\Delta \tilde{g} + c_n \text{Scal}_\tilde{g}) W_{k, t, z} - f(W_{k, t, z}).
\] (3.20)
Until the end of this section, \( C \) will denote a positive constant independent of \( k, t \) and \( z \), that might change from one line to the other.

**Lemma 3.2.** For any \( k \in \mathbb{N} \) and \( (t, z) \in [1/A, A] \times B(0, 1) \), we have
\[
\|E_{k, t, z}\|_{L^{2^*} \left( \mathcal{M} \right)} \leq C \mu_k^{\frac{n+2}{n}}.
\] (3.21)

**Proof.** By (3.1) and (3.13), we have \( \|R_{k, t, z}/B_{k, t, z}\|_{L^\infty(\overline{B(y_k, 2r_k)})} \to 0 \) as \( k \to \infty \) uniformly with respect to \( t, z \). As a consequence, and since \( \mu_k^2 = o(r_k) \),
\[
|f(W_{k, t, z}) - f(U_{k, t, z} \circ \Phi) + f(\tilde{u}_0)| \leq C \begin{cases} (B_{k, t, z} \circ \Phi)^{2^*-2} & \text{in} \ \Phi(B(\xi_k(z), \sqrt{\mu_k(t)}/2)) \\ (B_{k, t, z} \circ \Phi)^{2^*-2} & \text{otherwise} \end{cases}
\]
Since \( \tilde{u}_0 \) satisfies (3.5) and \( \tilde{g} = \Phi_{*} \tilde{g} \) in \( B(0, 4\delta) \), we obtain
\[
\|E_{k, t, z}\|_{L^{2^*} \left( \mathcal{M} \right)} \leq ||(\Delta \tilde{g} + c_n \text{Scal}_\tilde{g}) U_{k, t, z} - f(U_{k, t, z})||_{L^{2^*}(B(y_k, 2r_k))} + C \mu_k^{\frac{n+2}{n}},
\] (3.22)
where \( \tilde{g} \) is given by (3.3). First, by using (3.13) together with straightforward computations, we obtain
\[
(\Delta \tilde{g} + c_n \text{Scal}_\tilde{g}) U_{k, t, z} - f(U_{k, t, z}) = O \left( \frac{n-2}{\mu_k^n} r_k^{-n} \right)
\] (3.23)
in \( B(y_k, 2r_k) \setminus B(y_k, r_k) \). In \( B(y_k, r_k) \), by (3.4), we have
\[
\tilde{g} = \exp(\varepsilon_k h_k(x)),
\]
where \( h_k \) is defined in (3.11). Since the components of \( h \) are homogeneous of degree two, \( |\varepsilon_k h_k(x)| \leq C \varepsilon_k r_k^2 = o(1) \) holds for all \( x \in B(y_k, r_k) \). The definition of \( \tilde{g} \) therefore allows to expand its inverse as
\[
\tilde{g}^{ij}(x) = \delta_{ij} - \varepsilon_k (h_k)^{ij}(x) + \frac{\varepsilon_k^2}{2} \sum_{p=1}^n (h_k)^{ip}(x)(h_k)^{pj}(x) + O \left( \varepsilon_k^3 |x - y_k|^2 \right)
\] (3.24)
for $i,j \in \{1, \ldots, n\}$. Similarly, the Christoffel symbols of $\tilde{g}$ expand as

$$\Gamma_{ij}^{\ell}(\tilde{g})(x) = \frac{\varepsilon_k}{2} (\partial_i(h_k)_{j\ell}(x) + \partial_j(h_k)_{i\ell}(x) - \partial_\ell(h_k)_{ij}(x)) + O \left( \varepsilon_k^2 |x - y_k|^3 \right) \quad (3.25)$$

for $i,j,\ell \in \{1, \ldots, n\}$. Using Proposition 26 of Brendle [7], and since $h_k$ is trace-free and divergence-free, the scalar curvature of $\tilde{g}$ expands as

$$\text{Scal}_{\tilde{g}}(x) = -\frac{1}{4} \varepsilon_k^2 \sum_{i,j,\ell=1}^n (\partial_i(h_k)_{j\ell}(x))^2 + O \left( \varepsilon_k^4 |x - y_k|^4 \right). \quad (3.26)$$

Remark finally that, by definition of $\xi_k(z)$ in (3.7), there exists $C > 1$ such that for any $x \in B(y_k, r_k)$,

$$\frac{1}{C} \leq \frac{\mu_k + |x - y_k|}{\mu_k + |x - \xi_k(z)|} \leq C$$

holds true. Using (3.24), (3.25) and (3.26), we thus have, for $x \in B(y_k, r_k)$,

$$(\Delta_{\tilde{g}} + c_n \text{Scal}_{\tilde{g}})B_{k,t,z} - f(B_{k,t,z}) = (\Delta_{\tilde{g}} - \Delta_{\delta_0})B_{k,t,z} + O \left( \varepsilon_k^2 \mu_k^{\frac{n+2}{2}} (\mu_k + |x - y_k|)^{4-n} \right)$$

$$= \varepsilon_k \sum_{p,q=1}^n (h_k)_{pq} \alpha_{pq}^2 B_{k,t,z} + O \left( \varepsilon_k^2 \mu_k^{\frac{n-2}{2}} (\mu_k + |x - y_k|)^{4-n} \right). \quad (3.27)$$

Since $U_{k,t,z}(x) = B_{k,t,z}(x) + R_{k,t,z}(x)$, by (3.13),

$$|f(U_{k,t,z}(x)) - f(B_{k,t,z}(x)) - f'(B_{k,t,z}(x))R_{k,t,z}(x)| \leq CB_{k,t,z}(x)^{2r-3}R_{k,t,z}(x)^2$$

holds for any $x \in B(y_k, r_k)$. With (3.1), (3.13), (3.24), (3.25), (3.26) and (3.27), we obtain that, in $B(y_k, r_k)$,

$$(\Delta_{\tilde{g}} + c_n \text{Scal}_{\tilde{g}})U_{k,t,z} - f(U_{k,t,z}) = (\Delta_{\tilde{g}} + c_n \text{Scal}_{\tilde{g}})B_{k,t,z} - f(B_{k,t,z}) + \Delta_{\delta_0}R_{k,t,z} - f'(B_{k,t,z}(x))R_{k,t,z}$$

$$+ (\Delta_{\tilde{g}} - \Delta_{\delta_0})R_{k,t,z} + O \left( \varepsilon_k^2 (\mu_k + |x - y_k|)^2 |R_{k,t,z}| \right) + O \left( B_{k,t,z}^{2r-3} R_{k,t,z}^2 \right)$$

$$= O \left( \varepsilon_k^2 \mu_k^{\frac{n+2}{2}} (\mu_k + |x - y_k|)^{4-n} \right)$$

holds. With (3.1), (3.22) and (3.23), this finally shows that

$$\|E_{k,t,z}\|_{L^{\frac{2n}{n+2}}(M)} \leq C \left( \mu_k^{n+2} + \mu_k^{n-2} r_k^{1-\frac{2}{n}} + \varepsilon_k^2 \mu_k^{4} \right) \leq C \mu_k^{n+2},$$

where the last inequality follows since $n \geq 11$ and $\mu_k = o(r_k^p)$ for any $p \geq 1$. \hfill \Box

3.3. The Lyapunov–Schmidt reduction. We endow $H^1(M)$ with the norm

$$\|u\|_{H^1(M)} := \sqrt{\int_M \left( \|\nabla u\|^2_{\tilde{g}} + c_n \text{Scal}_{\tilde{g}} u^2 \right) dv_{\tilde{g}},}$$

and for $u \in H^1(M)$, we let

$$I(u) := \frac{1}{2} \int_M \left( \|\nabla u\|^2_{\tilde{g}} + c_n \text{Scal}_{\tilde{g}} u^2 \right) dv_{\tilde{g}} - \frac{1}{2} \int_M |u|^2 dv_{\tilde{g}}.$$

For $(t, z) \in [1/A, A] \times B(0,1)$, $1 \leq j \leq n$ and $x \in M$, we let

$$\tilde{Z}_{0,k,t,z}(x) := \frac{2}{n-2} t \partial_j W_{k,t,z}(x) \quad \text{and} \quad \tilde{Z}_{j,k,t,z}(x) := -n t \partial_j W_{k,t,z}(x). \quad (3.28)$$
and we let
\[ \tilde{K}_{k,t,z} := \text{span}\{ \tilde{Z}_{j,k,t,z} : 0 \leq j \leq n \} \],
which is regarded as a subset of \( H^1(M) \). We denote by \( \tilde{K}_{k,t,z}^+ \) its orthogonal complement in \( H^1(M) \). The following proposition shows the existence of a canonical solution of (3.6) in \( \tilde{K}_{k,t,z}^+ \):

**Proposition 3.3.** There exists \( C > 0 \) and, for large \( k \in \mathbb{N} \), a function \( \varphi_k : [1/A, A] \times B(0, 1) \rightarrow H^1(M) \) of class \( C^1 \) which is the only solution of

\[ \Pi_{\tilde{K}_{k,t,z}^+} \left( u_{k,t,z} - (\Delta g + c_n \text{Scal}_g)^{-1} (f(u_{k,t,z})) \right) = 0 \]

in the set

\[ \left\{ \varphi \in \tilde{K}_{k,t,z}^+ : \| \varphi \|_{H^1(M)} \leq C \| E_{k,t,z} \|_{L^{2n/(n+2)}(M)} \right\} \],

where we have let \( u_{k,t,z} := W_{k,t,z} + \varphi_k(t,z) \). In particular,

\[ \| \varphi_k(t,z) \|_{H^1(M)} \leq C \| E_{k,t,z} \|_{L^{2n/(n+2)}(M)} \] \hspace{1cm} (3.29)

for some \( C > 0 \) independent of \( k,t,z \). In addition, for large \( k \in \mathbb{N} \), the function \( u_{k,t,z} \) is a critical point of \( I \) (hence a solution of (3.6)) if and only if \( (t,z) \) is a critical point of the mapping \((t,z) \mapsto I(u_{k,t,z})\).

**Proof.** By (3.18), we have, for \( 0 \leq j \leq n \) and \( x \in B(y_k, 2r_k) \),

\[ \hat{Z}_{j,k,t,z} \left( \Phi^{-1}(x) \right) = \chi \left( \frac{x - y_k}{r_k} \right) \left( Z_{j,k,t,z}(x) + Q_{j,k,t,z}(x) \right), \] \hspace{1cm} (3.30)

where we have let \( Q_{0,k,t,z} := \frac{2}{n-2} t \partial_t R_{k,t,z} \) and \( Q_{j,k,t,z} := -nt \partial_z R_{k,t,z} \). By differentiating (3.12) and using (3.13), one gets that, for \( 0 \leq j \leq n, k \in \mathbb{N}, x \in B(y_k, 2r_k) \) and any \((t,z) \in [1/A, A] \times B(0,1), Q_{j,k,t,z} \) satifies

\[ |\Delta g_{k,t,z} Q_{j,k,t,z} - (2^* - 1) B_{k,t,z}^{2*-2} Q_{j,k,t,z}| \leq C_\varepsilon k \mu_k^{n+2} \mu_k + |\xi_k(z) - |z||^{-n}. \] \hspace{1cm} (3.31)

As in the proof of (3.13), a representation formula yields once again that, for \( i \in \{0, 1, 2\} \),

\[ |\nabla^i Q_{j,k,t,z}(x)| \leq C_\varepsilon \mu_k(t)^{n+2} \ln \left( \frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t) + |x - \xi_k(z)|^{n-2}} \right)^{n-2+i} \] \hspace{1cm} (3.32)

for \( x \in \mathbb{R}^n \). With (3.9), (3.10) and (3.28), we then obtain, for \( 0 \leq j \leq n \) and \( x \in B(y_k, 2r_k) \), that

\[ \hat{Z}_{j,k,t,z} \left( \Phi^{-1}(x) \right) = \chi \left( \frac{x - y_k}{r_k} \right) Z_{j,k,t,z}(x) \]

\[ + O \left( \varepsilon_k \mu_k(t)^{n+2} \ln \left( \frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t) + |x - \xi_k(z)|^{n-2}} \right)^{n-2} \right), \] \hspace{1cm} (3.33)

\[ \nabla \left( \hat{Z}_{j,k,t,z} \circ \Phi^{-1} \right)(x) = \nabla \left( \chi \left( \frac{y_j - y_k}{r_k} \right) Z_{j,k,t,z} \right)(x) \]

\[ + O \left( \varepsilon_k \mu_k(t)^{n+2} \ln \left( \frac{2\mu_k(t) + |x - \xi_k(z)|}{\mu_k(t) + |x - \xi_k(z)|^{n-2}} \right)^{n-1} \right). \] \hspace{1cm} (3.34)
With these estimates, we can easily adapt the proof of Proposition 5.1 of Robert–Vétois [18] (see also Esposito–Pistoia–Vétois [21]), and Proposition 3.3 follows. □

We let $K_n$ be the optimal constant of the Sobolev embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$. As explained in the introduction, $K_n^{-2} = Y(S^n, \{g_{std}\})$. Since the latter is attained by the stereographic projection of the bubbles $B_{t,z}$ defined in (3.16), for all $(t, z) \in [1/A, A] \times B(0, 1)$, we have

$$
\int_{\mathbb{R}^n} |\nabla B_{t,z}|^2 dx = K_n^{-n}.
$$

The explicit value of $K_n$ is known (see Aubin [3] and Talenti [55]). The next result is an expansion of $(t, z) \mapsto I(W_{k,t,z})$ as $k \to \infty$.

**Proposition 3.4.** We have, as $k \to \infty$:

$$
I(W_{k,t,z}) = \frac{1}{n} K_n^{-n} + I(\tilde{u}_0) + \mu_k \frac{n-2}{2} \left( \Lambda(n)\tilde{u}_0(x_0)t \right)^{\frac{n-2}{2}} - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla R_{t,z}|^2 dx + \frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n h_{ip} h_{pj} \partial_i B_{t,z} \partial_j B_{t,z} dx - \frac{n-2}{32(n-1)} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n (\partial_i h_{ip})^2 B_{t,z}^2 dx + o(1),
$$

where $\tilde{u}_0$ is as in (3.35), $R_{t,z}$ is as in (3.14) and where $\Lambda(n)$ is a positive dimensional constant given by (3.38) below. This expansion holds true in $C^1([1/A, A] \times B(0, 1))$.

**Proof.** First, by (3.13) and (3.17), it is easily seen that $0 \leq U_{k,t,z} \leq C B_{k,t,z}$ in $B(y_k, 2r_k)$. By (3.18), we can then write, for some $0 < \theta < 2^* - 2$, that

$$
\left| |W_{k,t,z}|^{2^*} - (U_{k,t,z} \circ \Phi)^{2^*} \tilde{u}_0 + 2(U_{k,t,z} \circ \Phi)^{2^*-1} \tilde{u}_0 \right| \leq C \left( (B_{k,t,z} \circ \Phi)^{2^*-1-\theta} + (B_{k,t,z} \circ \Phi)^{1+\theta} \right)
$$

holds in $M$. As a consequence, and since $\tilde{u}_0$ solves (3.35), straightforward computations show that, for $0 < \theta < 2^* - 2$,

$$
I(W_{k,t,z}) = I(U_{k,t,z} \circ \Phi) + I(\tilde{u}_0) + \int_M (U_{k,t,z} \circ \Phi)^{2^*-1} \tilde{u}_0 dv_{\tilde{g}} + O \left( \int_M (B_{k,t,z} \circ \Phi)^{2^*-1-\theta} + (B_{k,t,z} \circ \Phi)^{1+\theta} dv_{\tilde{g}} \right)
$$

$$
= I(U_{k,t,z} \circ \Phi) + I(\tilde{u}_0) + \int_M (U_{k,t,z} \circ \Phi)^{2^*-1} \tilde{u}_0 dv_{\tilde{g}} + O \left( \mu_k^{\frac{n-2}{2}(1+\theta)} \right). \quad (3.36)
$$

Independently, by (3.13) and (3.17), and since $\tilde{u}_0$ is of class $C^2$, we have

$$
\int_M (U_{k,t,z} \circ \Phi)^{2^*-1} \tilde{u}_0 dv_{\tilde{g}} = \Lambda(n)\tilde{u}_0(x_0)\mu_k(t) \frac{n-2}{2} + o \left( \mu_k^{\frac{n-2}{2}} \right), \quad (3.37)
$$

where $\mu_k(t) := \mu_k t$ is as in (3.7) and where we have let

$$
\Lambda(n) := \int_{\mathbb{R}^n} \left( 1 + \frac{|x|^2}{n(n-2)} \right)^{-\frac{n+2}{2}} dx. \quad (3.38)
$$

We have thus proven that

$$
I(W_{k,t,z}) = I(U_{k,t,z} \circ \Phi) + I(\tilde{u}_0) + \Lambda(n)\tilde{u}_0(x_0)\mu_k(t) \frac{n-2}{2} + o \left( \mu_k^{\frac{n-2}{2}} \right). \quad (3.39)
$$
It remains to expand $I(\Phi)$. By definition, we have $\bar{g} = \Phi^* \tilde{g}$ in $B_2(x_0, 6\delta)$, where $\bar{g}$ is given by (3.3), so by (3.17) and since $\det \bar{g} = 1$, we have

$$I(\Phi) = \frac{1}{2} \int_{\mathbb{R}^n} \left( |\nabla (B_k + R_k R_k)|^2 + c_n \operatorname{Scal}_{\bar{g}} (B_k + R_k R_k)^2 \right) dx$$

$$- \frac{1}{2\varepsilon_k} \int_{\mathbb{R}^n} |B_k + R_k R_k|^{2^*} dx + o \left( \mu_k^{\frac{n-2}{2}} \right), \quad (3.40)$$

where $K_k$ is given by Lemma 3.1. In the latter equality, we implicitly assume, in accordance with (3.3), that $\bar{g}$ has been extended as a metric in $\mathbb{R}^n$ that coincides with the Euclidean metric in $\mathbb{R}^n \setminus B(0, 4\delta)$. First, by (3.13) and (3.24), we have

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla (B_k + R_k R_k)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla B_k|^2 dx + \int_{\mathbb{R}^n} \langle \nabla B_k, \nabla R_k R_k \rangle dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^n} |\nabla R_k R_k|^2 dx - \frac{\varepsilon_k}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^n (h_k)_{ij} \partial_i B_k \partial_j B_k \partial_i \partial_j B_k R_k R_k dx$$

$$+ \frac{\varepsilon_k}{4} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n (h_k)_{ip}(h_k)_{pj} \partial_i B_k \partial_j B_k \partial_i B_k \partial_j B_k dx$$

$$- \varepsilon_k \int_{\mathbb{R}^n} \sum_{i,j=1}^n (h_k)_{ij} \partial_i B_k \partial_j R_k R_k dx + o \left( \mu_k^{\frac{n-2}{2}} \right), \quad (3.41)$$

where $h_k$ is as in (3.11). Similarly, using (3.1), (3.13) and (3.26), we have

$$\frac{c_n}{2} \int_{\mathbb{R}^n} \operatorname{Scal}_{\bar{g}} (B_k + R_k R_k)^2 dx = -\frac{c_n}{8} \varepsilon_k^2 \int_{\mathbb{R}^n} \sum_{i,j,l=1}^n (\partial_i (h_k)_{j,l})^2 B_k^2 R_k R_k dx$$

$$+ o \left( \mu_k^{\frac{n-2}{2}} \right). \quad (3.42)$$

Finally, using (3.13), we have

$$\frac{1}{2} \int_{\mathbb{R}^n} |B_k + R_k R_k|^{2^*} dx = \frac{1}{2^*} \int_{\mathbb{R}^n} B_k^{2^*} R_k R_k dx + \int_{\mathbb{R}^n} B_k^{2^* - 1} R_k R_k dx$$

$$+ (2^* - 1) \int_{\mathbb{R}^n} B_k^{2^* - 2} R_k^{2^* - 2} R_k R_k dx + o \left( \mu_k^{\frac{n-2}{2}} \right). \quad (3.43)$$

As shown in the proof of Lemma 3.1, we have

$$\int_{\mathbb{R}^n} \sum_{i,j=1}^n (h_k)_{ij} \partial_i B_k \partial_j B_k \partial_i B_k \partial_j B_k R_k R_k dx = 0 \quad (3.44)$$

for any $(t, z) \in [1/A, A] \times B(0, 1)$. We now integrate (3.12) against $R_k R_k$ after integrating by parts and using that $h$ is divergence-free, we find that

$$\int_{\mathbb{R}^n} |\nabla R_k R_k|^2 dx - (2^* - 1) \int_{\mathbb{R}^n} B_k^{2^* - 2} R_k^{2^* - 2} R_k R_k R_k R_k dx$$

$$= \varepsilon_k \int_{\mathbb{R}^n} \sum_{i,j=1}^n (h_k)_{ij} \partial_i B_k \partial_j R_k R_k R_k R_k dx. \quad (3.45)$$
It remains to plug \((3.41), (3.42), (3.43), (3.44)\) and \((3.45)\) into \((3.40)\). This gives
\[
I(U_{k,t,z} \circ \Phi) = \frac{1}{n} K_n^{-n} - \frac{1}{2} \int_{\mathbb{R}^n} |\nabla R_{k,t,z}|^2 \, dx \\
+ \frac{\varepsilon_k^2}{4} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n (h_k)_{ip} (h_k)_{pj} \partial_i B_{k,t,z} \partial_j B_{k,t,z} \, dx \\
- \frac{n-2}{32(n-1)} \varepsilon_k^2 \int_{\mathbb{R}^n} \sum_{i,j,t=1}^n (\partial_t (h_k)_{jt}) B_{k,t,z}^2 \, dx + o\left(\mu_k^{-2}\right).
\]
Simple changes of variables using \((3.11)\) and \((3.15)\) now yield
\[
\frac{\varepsilon_k^2}{4} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n (h_k)_{ip} (h_k)_{pj} \partial_i B_{k,t,z} \partial_j B_{k,t,z} \, dx \\
- \frac{n-2}{32(n-1)} \varepsilon_k^2 \int_{\mathbb{R}^n} \sum_{i,j,t=1}^n (\partial_t (h_k)_{jt}) B_{k,t,z}^2 \, dx \\
= \frac{\varepsilon_k^2}{4} \left(\frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n h_{ip} h_{pj} \partial_i B_{t,z} \partial_j B_{t,z} \, dx - \frac{n-2}{32(n-1)} \int_{\mathbb{R}^n} \sum_{i,j,t=1}^n (\partial_t h_{jt}) B_{t,z}^2 \, dx\right)
\]
and
\[
\frac{1}{2} \int_{\mathbb{R}^n} |\nabla R_{t,z}|^2 \, dx = \frac{1}{2} \varepsilon_k^2 \mu_k^4 \int_{\mathbb{R}^n} |\nabla R_{t,z}|^2 \, dx,
\]
where \(R_{t,z}\) and \(B_{t,z}\) are as in \((3.14)\) and \((3.16)\). Together with \((3.1), (3.36), (3.37)\) and \((3.39)\), this concludes the proof of \((3.35)\). The expansions of the derivatives with respect to \(t\) and \(z\) follow from similar estimates. \(\square\)

To conclude this section, we show that \(I(U_{k,t,z})\) expands at first-order as \(I(W_{k,t,z})\) and that this expansion is \(C^1\) in \(t\):

**Proposition 3.5.** We have
\[
I(U_{k,t,z}) = I(W_{k,t,z}) + o\left(\mu_k^{-2}\right) \quad \text{and} \quad \partial_t (I(U_{k,t,z})) = \partial_t (I(W_{k,t,z})) + o\left(\mu_k^{-2}\right)
\]
as \(k \to \infty\), uniformly with respect to \((t,z) \in [1/A, A] \times \overline{B(0,1)}\), where \(U_{k,t,z}\) is as in Proposition 3.3.

\(C^1\)-expansions in the \(z\) variable also hold true, but we will not need them here.

**Proof.** By the mean value theorem, \((3.21)\) and \((3.29)\), we have
\[
I(U_{k,t,z}) = I(W_{k,t,z}) + DI(W_{k,t,z})(\phi_k(t,z)) + O\left(\|\phi_k(t,z)\|_{H^1(M)}^2\right) \\
= I(W_{k,t,z}) + \int_M E_{k,t,z} \phi_k(t,z) \, dv_g + o\left(\mu_k^{-2}\right) = I(W_{k,t,z}) + o\left(\mu_k^{-2}\right),
\]
which proves the first equality. We now show the \(C^1\)-estimate in \(t\). By Proposition 3.3 there exist, for any \((t,z) \in [1/A, A] \times B(0,1)\), real numbers \((\lambda_k^j(t,z))_{0 \leq j \leq n}\) such that \(u_{k,t,z}\) satisfies
\[
(\triangle_g + c_n \text{Scal}_g) u_{k,t,z} - f(u_{k,t,z}) = \sum_{j=0}^n \lambda_k^j(t,z) (\triangle_g + c_n \text{Scal}_g) \tilde{Z}_{j,k,t,z} \text{ in } M. \ (3.46)
\]
By definition of $u_{k,t,z}$, we have independently
\[
(\Delta g + c_n \text{Scal}_g) u_{k,t,z} - f(u_{k,t,z}) = E_{k,t,z} + (\Delta g + c_n \text{Scal}_g - f'(W_{k,t,z})) \varphi_k(t,z) - f(u_{k,t,z}) - f(W_{k,t,z}) \varphi_k(t,z), \tag{3.47}
\]
where $E_{k,t,z}$ is as in (3.20). By (3.21) and (3.29), Theorem 4.3 of Premoselli \[40\] applies and shows the existence of a sequence $\sigma_k$ of positive numbers with $\sigma_k \to 0$ as $k \to \infty$ such that for any $x \in M$ and $(t, z) \in [1/A, A] \times B(0, 1)$,
\[
|\varphi_k(t,z)(x)| \leq \sigma_k \left( 1 + U_{k,t,z}(\Phi(x)) \right) \tag{3.48}
\]
holds. By (3.13) and (3.18), we can then let $R > 0$ be large enough so that
\[
W_{k,t,z}(\Phi^{-1}(x)) \geq \frac{1}{2} B_{k,t,z}(x)
\]
holds for any $x \in B(\xi_k(z), \sqrt{\mu_k}/R) \subset \mathbb{R}^n$ and for any $k \in \mathbb{N}$ and $(t, z) \in [1/A, A] \times B(0, 1)$. In $M \setminus \Phi^{-1}(B(\xi_k(z), \sqrt{\mu_k}/R))$, we then have
\[
|f(u_{k,t,z}) - f(W_{k,t,z}) - f'(W_{k,t,z}) \varphi_k(t,z)| \leq C(U_{k,t,z} \circ \Phi)^{2^*-3} \varphi_k(t,z)^2.
\]
In $M \setminus \Phi^{-1}(B(\xi_k(z), \sqrt{\mu_k}/R))$, (3.48) similarly shows that
\[
|f(u_{k,t,z}) - f(W_{k,t,z}) - f'(W_{k,t,z}) \varphi_k(t,z)| \leq C|\varphi_k(t,z)|.
\]
We let $j \in \{0, \ldots, n\}$ and integrate (3.47) against $\bar{Z}_{j,k,t,z}$. Since $|\bar{Z}_{j,k,t,z}| \leq C(U_{k,t,z} \circ \Phi)$ in $M$, by using the latter inequality together with straightforward computations, we obtain
\[
\int_M (f(u_{k,t,z}) - f(W_{k,t,z}) - f'(W_{k,t,z}) \varphi_k(t,z)) \bar{Z}_{j,k,t,z} \, dv_g = O\left( \int_{\Phi^{-1}(B(\xi_k(z), \sqrt{\mu_k}/R))} (U_{k,t,z} \circ \Phi)^{2^*-2} |\varphi_k(t,z)|^2 \, dv_g \right)
\]
\[
+ O\left( \int_{M \setminus \Phi^{-1}(B(\xi_k(z), \sqrt{\mu_k}/R))} (U_{k,t,z} \circ \Phi) |\varphi_k(t,z)| \, dv_g \right)
\]
\[
= O\left( \|\varphi_k(t,z)\|_{H^1(M)}^2 + \|\varphi_k(t,z)\|_{H^1(M)} \|B_{k,t,z}\|_{L^\infty(\mathbb{R}^n \setminus B(\xi_k(z), \sqrt{\mu_k}/R))} \right)
\]
\[
= o\left( \mu_k^{-\frac{2}{n}} \right)
\]
holds, where we have used (3.21) and (3.29) in the last line. With (3.33), (3.34), (3.46) and (3.47), the latter shows that
\[
(1 + o(1))\|\nabla V_j\|_{L^2(\mathbb{R}^n)}^2 \lambda_1^j(t,z) = \int_M E_{k,t,z} \bar{Z}_{j,k,t,z} \, dv_g + o\left( \mu_k^{-\frac{2}{n}} \right)
\]
\[
+ \int_M \varphi_k(t,z) (\Delta g + c_n \text{Scal}_g - f'(W_{k,t,z})) \bar{Z}_{j,k,t,z} \, dv_g. \tag{3.49}
\]
By (3.28) and Proposition 3.4, we have
\[
\int_M E_{k,t,z} \bar{Z}_{0,k,t,z} \, dv_g = \begin{cases} 
\frac{2}{n-2} t \partial_t (I(W_{k,t,z})) = O\left( \mu_k^{-\frac{2}{n}} \right) & \text{if } j = 0 \\
- nt \partial_{z_j} (I(W_{k,t,z})) = O\left( \mu_k^{-\frac{2}{n}} \right) & \text{if } 1 \leq j \leq n.
\end{cases} \tag{3.50}
\]
By (3.13), (3.18), (3.28) and (3.30), we have, for $0 \leq j \leq n$ and $x \in B(y_k, 2r_k)$,

$$
(\Delta \bar{g} + c_n \text{Scal}_{\bar{g}} - f'(W_{k,t,z})) \nabla_j (\Phi^{-1}(x))
$$

$$
= (\Delta \bar{g} + c_n \text{Scal}_{\bar{g}} - f'(B_{k,t,z})) (Z_{j,k,t,z} + Q_{j,k,t,z})(x) + O \left( r_k^{-n} \frac{\mu_k^{n+2}}{\mu_k + |\xi_k(z) - x|} \right).
$$

Using then (3.24), (3.25), (3.26), (3.31) and (3.32), we find that, for $x \in B(y_k, 2r_k)$,

$$
(\Delta \bar{g} + c_n \text{Scal}_{\bar{g}} - f'(B_{k,t,z})) (Z_{j,k,t,z} + Q_{j,k,t,z})(x) = O \left( r_k^{-n} \frac{\mu_k^{n+2}}{\mu_k + |\xi_k(z) - x|} \right).
$$

holds. Thus,

$$
\left| (\Delta \bar{g} + c_n \text{Scal}_{\bar{g}} - f'(W_{k,t,z})) \nabla_j (\Phi^{-1}(x)) \right|
$$

$$
\leq C \left( r_k^{-n} \frac{\mu_k^{n+2}}{\mu_k + |\xi_k(z) - x|} + \varepsilon_k B_{k,t,z}(x) + \left\{ \begin{array}{ll}
B_{k,t,z}(x) & \text{if } |x - \xi_k(z)| \leq \sqrt{\mu_k(t)} \\
B_{k,t,z}(x) & \text{otherwise}
\end{array} \right. \right).
$$

(3.51)

holds for any $x \in B(y_k, 2r_k)$. As a consequence, with (3.1),

$$
\left\| (\Delta \bar{g} + c_n \text{Scal}_{\bar{g}} - f'(W_{k,t,z})) \nabla_j (\Phi^{-1}(x)) \right\|_{L^{\frac{2n}{n+2}}(M)} \leq C \left( r_k^{-n} \frac{\mu_k^{n+2}}{\mu_k + |\xi_k(z) - x|} + \varepsilon_k B_{k,t,z}(x) + \left\{ \begin{array}{ll}
B_{k,t,z}(x) & \text{if } |x - \xi_k(z)| \leq \sqrt{\mu_k(t)} \\
B_{k,t,z}(x) & \text{otherwise}
\end{array} \right. \right).
$$

$$
\leq C \mu_k^{\frac{n-2}{2}}.
$$

Hence, with (3.31) and (3.29), Hölder’s inequality shows that

$$
\left| \int_M \varphi_{k}(t, z) (\Delta \bar{g} + c_n \text{Scal}_{\bar{g}} - f'(W_{k,t,z})) \nabla_j (\Phi^{-1}(x)) \right| \leq o \left( \mu_k^{\frac{n-2}{2}} \right).
$$

(3.52)

Together with (3.49) and (3.50), this shows that

$$
\sum_{j=0}^{n} |\lambda_j^k(t, z)| = O \left( \mu_k^{\frac{n+2}{2}} \right).
$$

(3.53)

We can now use (3.46) to write that

$$
\partial_t (I(u_{k,t,z})) = DI(u_{k,t,z}) (\partial_t u_{k,t,z})
$$

$$
= \sum_{j=0}^{n} \lambda_j^k(t, z) \partial_t W_{j,k,t,z}(x)
$$

where $W_{j,k,t,z}$ is as in (3.18). On the one side, with (3.9), (3.10), (3.33) and (3.34), we have

$$
(\partial_t W_{j,k,t,z})_{H^1(M)} = \frac{n-2}{2t} \delta_{j0} \| \nabla V \|_{L^2(R^n)}^2 + o(1)
$$

as $k \to \infty$, where $\delta_{j0}$ is the Kronecker symbol. On the other side, since $\varphi_{k}(t, z) \in H^1_{k,t,z}$, we have $(\varphi_{k}(t, z), \dot{Z}_{j,k,t,z})_{H^1(M)} = 0$ for all $0 \leq j \leq n$, hence

$$
(\partial_t W_{j,k,t,z}, \partial_t \varphi_{k}(t, z))_{H^1(M)} = -(\partial_t \dot{Z}_{j,k,t,z}, \varphi_{k}(t, z))_{H^1(M)} = o(1),
$$

as $k \to \infty$. The proof is complete.
where we have used (3.21), (3.29), (3.33) and (3.34). Together with (3.53), we thus obtain that
\[ \partial_t (I (u_{k,t,z})) = \frac{n-2}{2t} \bar{\delta}_{j0} \| \nabla V_0 \|_{L^2 (\mathbb{R}^n)}^2 \Lambda_k^0 (t, z) + o \left( \frac{t_0^2}{\Lambda_k^0} \right). \]
Together with (3.49), (3.50) and (3.52), this becomes
\[ \partial_t (I (u_{k,t,z})) = \partial_t (I (W_{k,t,z})) + o \left( \frac{t_0^2}{\Lambda_k^0} \right), \]
which concludes the proof of Proposition 3.5. □

**Remark 3.6.** Estimate (3.48), together with the definition of \( u_{k,t,z} \) in Proposition 3.3 and with (3.17) and (3.18), shows that there exists a sequence \( (\sigma_k)_{k \geq 0} \) of positive numbers converging to 0 such that, for any \( x \in M \) and for any \( k \geq 0 \) and \( (t, z) \in [1/A, A] \times B(0, 1) \),
\[
\left| u_{k,t,z}(x) + \bar{u}_0(x) - U_{k,t,z}(\Phi(x)) \right| \leq \sigma_k \left( \bar{u}_0(x) + U_{k,t,z}(\Phi(x)) \right),
\]
holds. Let \( ((t_k, z_k))_{k \geq 0} \) be a sequence in \( [1/A, A] \times B(0, 1) \) converging to some \( (t, z) \) and let \( u_k = u_{k,t_k,z_k} \) and \( w_k = \Phi^{-1} (\xi_k(z_k)) \) where \( \xi_k(z_k) \) is given by (3.7). Let \( w = \lim_{k \to +\infty} w_k \in B_\delta (x_0) \subset M \). It is easily seen with (3.54) that \( u_k \to -\bar{u}_0 < 0 \) in \( C^2_{loc} (M \setminus \{w\}) \) as \( k \to +\infty \) and that \( u_k(y_k) \to +\infty \). This shows that all the functions \( u_{k,t,z} \) that we constructed in Proposition 3.3 are sign-changing.

4. **Proof of Theorem 2.2 – Part 2: Finding critical points of the reduced energy**

Let \( n \geq 11 \) and let \( W : (\mathbb{R}^n)^4 \to \mathbb{R} \) be a four-linear form in \( \mathbb{R}^n \) possessing the same symmetries as a Weyl tensor, that is \( W_{ijkl} = -W_{jikl} = -W_{ijlk} = W_{klij} \), \( \sum_{i=1}^n W_{ijit} = 0 \) and
\[ W_{ijkl} + W_{jikl} + W_{ikjl} = 0 \]
for any \( i, j, k, \ell \in \{1, \ldots, n\} \). Assume that \( W \) is not identically zero, that is
\[ |W|^2 = \sum_{i,j,k,\ell=1}^n (W_{ijkl})^2 > 0. \]
For any \( k, \ell \in \{1, \ldots, n\} \), let
\[ T_{k\ell} := \sum_{p,q,r=1}^n (W_{kpqr} + W_{krqp}) (W_{\ell pqr} + W_{\ell rqp}). \]
Straightforward computations using the symmetries of \( W \) show that
\[ T_{k\ell} = 3 \sum_{p,q,r=1}^n W_{kpqr}W_{\ell pqr} \quad \text{and} \quad \sum_{k=1}^n T_{kk} = 3|W|^2 > 0. \]
We let, for \( x \in \mathbb{R}^n \) and \( 1 \leq i, j \leq n \),
\[ h(x)_{ij} := \frac{1}{3} \sum_{p,q=1}^n W_{ijpq} x_p x_q. \]
Using the symmetries of $W$, it is easily seen that $h$ satisfies (3.2). We define, for $t > 0$ and $z \in B(0, 1)$,

$$F(t, z) := \frac{1}{4} \int_{\mathbb{R}^n} \sum_{i,j,p=1}^n h_{ip} h_{pj} \partial_i B_{t,z} \partial_j B_{t,z} \, dx - \frac{n-2}{32(n-1)} \int_{\mathbb{R}^n} \sum_{i,j,t=1}^n (\partial_t h_{ij})^2 B_{t,z}^2 \, dx$$

$$- \frac{1}{2} \int_{\mathbb{R}^n} |\nabla R_{t,z}|^2 \, dx + \Lambda(n) \bar{u}_0(x_0) t^{n-2}$$

$$=: F_1(t, z) + F_2(t, z) + F_3(t, z) + \Lambda(n) \bar{u}_0(x_0) t^{n-2}, \quad (4.3)$$

where $\bar{u}_0$ is as in (3.5) and $x_0$ is as in the beginning of Section 3. $R_{t,z}$ is defined by (3.14) for $h$ given by (4.2), $B_{t,z}$ is as in (3.16) and $\Lambda(n)$ is the positive numerical constant given by (3.38). The function $F$ is smooth in $(0, \infty) \times B(0, 1)$. By (3.15), the uniqueness of $R_{t,z}$ and since $h(-x) = h(x)$ for all $x \in \mathbb{R}^n$, we have, for any $t > 0, z \in B(0, 1)$ and $x \in \mathbb{R}^n$, $R_{t,z}(x) = R_{t,z}(-x)$ and thus $F(t, -z) = F(t, z)$. This shows that, for all $t > 0$ and $1 \leq i, j, k \leq n$,

$$\partial_i \partial_j F(t, z)|_{z=0} = 0 \quad \text{and} \quad \partial_i \partial_j \partial_k F(t, z)|_{z=0} = 0. \quad (4.4)$$

For real numbers $p, q \geq 0$ with $p > q + 2$, we define

$$I_p^q := \int_0^\infty \frac{r^q}{(1+r)^p} \, dr. \quad (4.5)$$

From (4.5) the following expressions are easily obtained:

$$I_n^1 = \frac{n}{n-2} I_{n-2}^1, \quad I_n^{n-2} = \frac{n+2}{n-4} I_{n-2}^{n-2}, \quad I_n^{n+4} = \frac{(n+4)(n+2)}{4(n-2)(n+1)} I_{n-2}^{n-2},$$

$$I_{n+1}^{n+2} = \frac{n+2}{2(n-2)} I_n^{n-2}, \quad I_{n-1}^{n-2} = \frac{4(n-1)}{n-4} I_n^{n-2}, \quad I_n^2 = \frac{4n(n-1)}{(n-4)(n-6)} I_{n-2}^{n-2}. \quad (4.6)$$

Recall that, for any $t > 0$ and $z \in \mathbb{R}^n$, we have $\int_{\mathbb{R}^n} B_{t,z}^{2n} \, dx = K_n^{-n}$. A simple change of variables then yields

$$I_{n-2}^{n-2} = \frac{2}{n(n-2)} (n(n-2))^{-\frac{n}{2}} K_n^{-n}, \quad (4.7)$$

where $\omega_{n-1}$ is the volume of the standard unit sphere of dimension $n - 1$. This allows us to compute the integrals in (4.6).

We first prove the following lemma which shows the existence of a strict global minimum of $t \mapsto F(t, 0)$:

**Lemma 4.1.** Assume that $n \geq 11$. Then $t \in [0, \infty) \mapsto F(t, 0)$ possesses a unique global minimum $t_0 > 0$ such that $F(t_0, 0) < 0$. This minimum is also non-degenerate, meaning that $\partial_t^2 F(t_0, 0) > 0$.

**Proof.** First, by (3.15) and the uniqueness of $R_{t,z}$, we have $R_{t,0} = 0$, hence $F_t(t, 0) = 0$ for all $t > 0$. Let $t > 0$ and $z \in B(0, 1)$. Since $\sum_{j=1}^n x_j h(x)_{ij} = 0$ for all $1 \leq i \leq n$
and by definition of $B_{t,z}$, we have
\[
\frac{1}{4} \sum_{i,j,p=1}^{n} h(x)_{ip} h(x)_{pj} \partial_i B_{t,z}(x) \partial_j B_{t,z}(x) = \frac{1}{4n^2} \frac{t_{n-2}}{t^2 + \frac{|x-z|^2}{n(n-2)}} \sum_{i,j,p=1}^{n} h(x)_{ip} h(x)_{pj} z_i z_j. \tag{4.8}
\]

Thus $F_1(t,0) = 0$ and $F(t,0) = F_2(t,0) + \Lambda(n) \tilde{u}_0(x_0)t^{\frac{n-2}{2}}$ for any $t > 0$. By (4.2), we have, for all $x \in \mathbb{R}^n$,
\[
\partial_t h_{jt}(x) = \frac{1}{3} \sum_{p=1}^{n} (W_{ji} p + W_{jp} i) x_p. \tag{4.9}
\]

Hence, by (4.3), we have
\[
F_2(t,0) = -\frac{n-2}{32(n-1)} t^4 \int_{\mathbb{R}^n-1} \sum_{i,j,\ell=1}^{n} (\partial_i h_{jt}(x))^2 d\sigma(x) \cdot \int_{0}^{\infty} \frac{r^{n+1} dr}{(1 + \frac{r^2}{n(n-2)})^{n-2}}.
\]

Recall that
\[
\int_{\mathbb{R}^n-1} x_i x_j d\sigma(x) = \frac{\omega_{n-1}}{n} \delta_{ij}, \tag{4.10}
\]

for all $i, j \in \{1, \ldots, n\}$. As a consequence, we have
\[
\int_{\mathbb{R}^n-1} \sum_{i,j,\ell=1}^{n} (\partial_i h_{jt}(x))^2 d\sigma(x) = \frac{\omega_{n-1}}{9n} \sum_{k=1}^{n} T_{kk},
\]

where $T_{kk}$ is as in (4.1). Independently, straightforward computations with (4.6) and (4.7) give that
\[
\int_{0}^{\infty} \frac{r^{n+1} dr}{(1 + \frac{r^2}{n(n-2)})^{n-2}} = \frac{4n^2(n-1)(n-2)}{(n-4)(n-6)} \frac{K_n^{-n}}{\omega_{n-1}}.
\]

This shows that
\[
F(t,0) = -\left( \frac{n(n-2)^2}{72(n-4)(n-6)} K_n^{-n} \sum_{k=1}^{n} T_{kk} \right) t^4 + \Lambda(n) \tilde{u}_0(x_0)t^{\frac{n-2}{2}}. \tag{4.11}
\]

Since $\Lambda(n) > 0$, $\tilde{u}_0 > 0$ in $M$, $\sum_{k=1}^{n} T_{kk} > 0$ and $n \geq 11$, Lemma 4.1 follows. \hfill \square

The value of $t_0$ only depends on $n, W$ and $\tilde{u}_0$, that have been fixed from the beginning. From now on, we will let $A > 0$ be chosen large enough so that $t_0 \in [2/A, A/2]$. We now show that the second derivatives of $F$ at $(t,0)$ in $z$ vanish for every $t > 0$:

**Proposition 4.2.** Let $n \geq 11$ and let $t > 0$ be fixed. Then, for any $1 \leq k, \ell \leq n$,
\[
\partial_{zz}^2 F(t,z)|_{z=0} = 0.
\]

**Proof.** Let $t > 0$ and $k, \ell \in \{1, \ldots, n\}$ be fixed throughout this proof. First, by (3.15) and the uniqueness of $R_{t,z}$, we have $R_{t,0} = 0$ and $\partial_z (R_{t,z})|_{z=0} = 0$ for all
1 \leq i \leq n$. As a consequence, \( \partial^2_{z_i z_i} F_3(t, z)_{|z=0} = 0 \). On the one hand, with (4.8) and differentiating under the integral, we get that
\[
\partial^2_{z_i z_i} F_1(t, z)_{|z=0} = \frac{1}{2n^2} \int_{\mathbb{R}^n} \sum_{p=1}^{n} h(x)_{kp} h(x)_{pl} \frac{t^{n-2} dx}{(t^2 + \frac{|x|^2}{n(n-2)})^n}.
\]
\[
= \frac{t^2}{2n^2} \int_{\mathbb{R}^n} \sum_{p=1}^{n} h(x)_{kp} h(x)_{pl} \, d\sigma(x) \cdot \int_{0}^{\infty} \frac{x^{n+3} dx}{(1 + \frac{r^2}{n(n-2)})^n}.
\]
By using the symmetries of \( W \), it is easily seen that
\[
\sum_{p,i,j=1}^{n} W_{kpqj} (W_{ipj} + W_{ijpq}) = \frac{1}{2} T_{kl},
\]
where \( T_{kl} \) is as in (4.1). Using (4.2) and Corollary 29 of Brendle [7], we thus obtain
\[
\int_{\mathbb{R}^n} \sum_{p=1}^{n} h(x)_{kp} h(x)_{pl} \, d\sigma(x) = \frac{\omega_{n-1}}{18n(n+2)} T_{kl}.
\]
Together with (4.6) and (4.7), we then obtain
\[
\partial^2_{z_i z_i} F_1(t, z)_{|z=0} = \frac{t^2}{36} \frac{n-2}{n-4} K^{-n}_n T_{kl}.
\]
By using (4.9) together with change of variables, we obtain
\[
F_2(t, z) = -\frac{n-2}{288(n-1)} \int_{\mathbb{R}^n} \sum_{p,q=1}^{n} T_{pq}(x+z)^{p}(x+z)^{q} \frac{t^{n-2} dx}{(t^2 + \frac{|x|^2}{n(n-2)})^n},
\]
so that differentiating and using (4.6), (4.7) and (4.10) yields
\[
\partial^2_{z_i z_i} F_2(t, z)_{|z=0} = -\frac{n-2}{144(n-1)} T_{kl} \int_{\mathbb{R}^n} \frac{t^2 dx}{(1 + \frac{|x|^2}{n(n-2)})^n} = -\frac{t^2}{36} \frac{n-2}{n-4} K^{-n}_n T_{kl}.
\]
Together with (4.12), this concludes the proof of Proposition 4.2.

In order to prove that \((t_0, 0)\) is a non-degenerate critical point of \( F \), we now expand, for a fixed value of \( t, z \mapsto F(t,z) \) to the fourth order as \( z \to 0 \). A first, obvious, remark is that for \( i, j, k, l \in \{1, \ldots, n\} \), we have
\[
\partial^4_{z_i z_j z_k z_l} F_2(t, z)_{|z=0} = 0.
\]
This easily follows from the expression of \( F_2 \) given by (4.3). Indeed, by (4.2), \( \sum_{i,j,l=1}^{n} (\partial_i h_{jlp}(x))^2 \) is a second-order polynomial in \( x \), hence \( F_2(t, z) \) is a second-order polynomial in \( z \) by a change of variables. We now expand \( F_1 + F_2 \):

**Proposition 4.3.** Let \( t > 0 \). We have, as \( z \to 0 \),
\[
F_1(t, z) + F_2(t, z) = F_2(t, 0) + \frac{K^{-n}_n}{4n} \sum_{p,q=1}^{n} (h_{pq}(z))^2 + O ([z]^5).
\]

**Proof.** By definition, \( F_1 \) and \( F_2 \) are even in \( z \). The expansions to second-order thus follow from \( F_1(t, 0) = 0 \), (4.12) and (4.13). By (4.14), we only have to expand \( F_1(t, z) \) to fourth order. By (4.3), (4.8) and a change of variables, we have
\[
F_1(t, z) = \frac{1}{4n^2} \int_{\mathbb{R}^n} \frac{t^{n-2} dx}{(t^2 + \frac{|x|^2}{n(n-2)})^n} \sum_{i,j,p=1}^{n} h(x+z)_{ip} h(x+z)_{pj} z_i z_j dx.
\]
It is easily seen that, for any \( x, z \in \mathbb{R}^n \) and \( 1 \leq a, b, c, d \leq n \),
\[
(x + z)_a (x + z)_b (x + z)_c (x + z)_d = x_a x_b x_c x_d + x_a x_b x_c z_d + x_a x_b x_d z_c + x_a x_c x_d z_b + x_b x_c x_d z_a + x_a x_b z_c z_d + x_a x_c z_b z_d + x_b x_c z_a z_d + x_a x_d z_b z_c + x_b x_d z_a z_c + x_c x_d z_a z_b + O \left( |x|^4 |x| + |z|^4 \right).
\]
As a consequence and by (4.2), we have, as \( z \to 0 \),
\[
F_1(t, z) + F_2(t, z) = F_2(t, 0) + G(z) + O \left( |z|^5 \right),
\]
where
\[
G(z) := \frac{1}{36n^2} \int_{\mathbb{R}^n} \left( 1 + \frac{|x|^2}{n(n-2)} \right)^{-n} \sum_{i,j,p,a,b,c,d=1}^n W_{iapb} W_{pcjd} z_i z_j (x_a x_b z_c z_d + x_a x_c z_b z_d + x_a x_d z_b z_c + x_b x_c z_a z_d + x_b x_d z_a z_c + x_c x_d z_a z_b) \, dx.
\]
By (4.6), (4.7) and (4.10), we have, for \( 1 \leq a, b \leq n \),
\[
\int_{\mathbb{R}^n} \frac{x_a x_b \, dx}{(1 + \frac{|x|^2}{n(n-2)})^n} = n K_n^{-n} \delta_{ab}.
\]
Since \( W \) is totally traceless, we thus obtain
\[
G(z) = \frac{1}{36n} K_n^{-n} \sum_{i,j,p,a,b,c=1}^n \left( W_{iapb} W_{pajc} z_b z_c z_i z_j + W_{iapb} W_{pajc} z_b z_c z_i z_j \right) + W_{iapb} W_{pajc} z_b z_c z_i z_j + W_{iapb} W_{pajc} z_b z_c z_i z_j \right).
\]
Since \( W_{pajc} = -W_{pajc} \), we have \( \sum_{i,j,p,a,b,c=1}^n W_{iapb} W_{pajc} z_b z_c z_i z_j = 0 \). Similarly, we obtain that the last two terms in the right-hand side of (4.15) vanish. Since moreover \( W_{pajc} = W_{pajc} \), we then obtain
\[
G(z) = \frac{1}{36n} K_n^{-n} \sum_{a,b,p=1}^n W_{iapb} W_{japc} z_b z_c z_i z_j
\]
\[
= \frac{1}{36n} K_n^{-n} \sum_{a,b,p=1}^n \left( - \sum_{i,j=1}^n W_{iapb} z_i z_j \right)^2 = \frac{1}{4n} K_n^{-n} \sum_{a,b,p=1}^n (h_{ap}(z))^2
\]
by (4.2), which concludes the proof of Proposition 4.3.

We next expand \( F_3(t, z) \) at fourth-order in \( z \) at \( (t, 0) \) for some fixed \( t > 0 \):

**Proposition 4.4.** Let \( t > 0 \). As \( z \to 0 \), we have
\[
F_3(t, z) = -\frac{n + 4}{48(n + 1)} K_n^{-n} \sum_{p,q=1}^n (h_{pq}(z))^2 + O \left( |z|^5 \right).
\]

**Proof.** By (3.15), for \( (t, z) \in (0, \infty) \times \mathbb{R}^n \), we have \( R_{t,z} = \tilde{R}_{t,z}(x - z) \), where \( \tilde{R}_{t,z} \) is the unique solution in \( K_{t,\delta}^+ \) of
\[
\triangle_{\delta_0} \tilde{R}_{t,z}(x) - (2^* - 1) B_{t,0}(x) 2^{*-2} \tilde{R}_{t,z}(x) = -\frac{t^{\frac{n+2}{2}}}{n(n-2)} \sum_{p,q=1}^n h_{pq}(x + z) z_p z_q \left( t^2 + \frac{|x|^2}{n(n-2)} \right)^{\frac{n-2}{2}} ,
\]
(4.16)
where $h$ is given by (4.2). Hence, for fixed $t > 0$, all the derivatives of $R_{t,z}$ with respect to $z$, taken at $z = 0$, belong to $K_{t,0}^{1}$. We let in what follows, for $a, b \in \{1, \ldots, n\}$,

$$L_{ab} := \partial_{z_a z_b}^2 (R_{t,z})|_{z=0} \in K_{t,0}^{1}.$$ 

Since $R_{t,0} = 0$ and $\partial_{z_i} (R_{t,z})|_{z=0} = 0$ for all $1 \leq i \leq n$, it is easily seen that, for $1 \leq a, b, c, d \leq n$, we have

$$\partial_{z_a} F_3(t, z)|_{z=0} = 0, \quad \partial_{z_a z_b}^2 F_3(t, z)|_{z=0} = 0, \quad \partial_{z_a z_b z_c}^3 F_3(t, z)|_{z=0} = 0 \quad \text{and}$$

$$\partial_{z_a z_b z_c z_d}^4 F_3(t, z)|_{z=0} = -\int_{\mathbb{R}^n} \left( \langle \nabla L_{ab}, \nabla L_{cd} \rangle + \langle \nabla L_{ac}, \nabla L_{bd} \rangle + \langle \nabla L_{ad}, \nabla L_{bd} \rangle \right) dx.$$ 

Differentiating (4.16) shows that $L_{ab}$ satisfies

$$\Delta_{\mathbb{R}^n} L_{ab}(x) - (2^* - 1) B_{1,0}^{2^* - 2} L_{ab}(x) = -\frac{2^* - 2}{n(n-2)} \frac{h_{ab}(x)}{t}.$$ 

In particular, for any $x \in \mathbb{R}^n$, we have $L_{ab}(x) = t^{1-\frac{2}{n}} L_{ab}^0(x/t)$, where $L_{ab}^0 \in K_{1,0}^{1}$ satisfies

$$\Delta_{\mathbb{R}^n} L_{ab}^0(x) - (2^* - 1) B_{1,0}^{2^* - 2} L_{ab}^0(x) = -\frac{2}{n(n-2)} \frac{h_{ab}(x)}{1 + \frac{|x|^2}{n(n-2)}}.$$ 

Coming back to (4.17), we also have, with a change of variables,

$$\partial_{z_a z_b z_c z_d}^4 F_3(t, z)|_{z=0} = -\int_{\mathbb{R}^n} \left( \langle \nabla L_{ab}^0, \nabla L_{cd}^0 \rangle + \langle \nabla L_{ac}^0, \nabla L_{bd}^0 \rangle + \langle \nabla L_{ad}^0, \nabla L_{bd}^0 \rangle \right) dx.$$ 

We first find an explicit expression for $L_{ab}^0$. Since $W$ is totally traceless, $h$ is a harmonic polynomial in $\mathbb{R}^n$. As a consequence, we can look for a solution of (4.18) under the form

$$L_{ab}^0(x) = h_{ab}(x) f(|x|^2).$$

Simple computations show that $f(t) = -\frac{1}{n} \left( 1 + \frac{t}{n(n-2)} \right)^{-n/2}$ satisfies (4.18) for all $t \in \mathbb{R}$. By using (4.20), we thus obtain

$$L_{ab}^0(x) = -\frac{1}{n} \frac{h_{ab}(x)}{1 + \frac{|x|^2}{n(n-2)}} = -\frac{1}{3n} \sum_{p,q=1}^n W_{apbq} x_p x_q \left( 1 + \frac{|x|^2}{n(n-2)} \right)^{-2}.$$ 

for $1 \leq a, b \leq n$ and all $x \in \mathbb{R}^n$. Note that $L_{ab}^0$ given by (4.20) belongs to $K_{1,0}^{1}$, which follows again from the symmetries of $W$, and is thus the unique solution of (4.18) in $K_{1,0}^{1}$. Straightforward computations with (4.20) then show that

$$\int_{\mathbb{R}^n} \langle \nabla L_{ab}^0, \nabla L_{cd}^0 \rangle dx = K_1 + K_2 + K_3.$$ 

(4.21)
where

\[ A_1 := \frac{1}{9n^2} \int_{\mathbb{R}^n} \left( 1 + \frac{|x|^2}{n(n-2)} \right)^{-n} \sum_{i,p,q=1}^n (W_{aibp} + W_{apbi})(W_{cidq} + W_{cqdi})x_p x_q \, dx, \]

\[ A_2 := \frac{1}{n^2(n-2)^2} \int_{\mathbb{R}^n} \left( 1 + \frac{|x|^2}{n(n-2)} \right)^{-n-2} |x|^2 h_{ab}(x) h_{cd}(x) \, dx, \]

\[ A_3 := -\frac{4}{n^2(n-2)^2} \int_{\mathbb{R}^n} \left( 1 + \frac{|x|^2}{n(n-2)} \right)^{-n-1} h_{ab}(x) h_{cd}(x) \, dx. \]

By (4.6), (4.7) and (4.10), we have

\[ A_1 = \frac{1}{9n^2} K_n^{-n} \sum_{p,q=1}^n (W_{apbq} + W_{aqbp})(W_{cpdq} + W_{cqdp}), \]

\[ = \frac{2}{9n} K_n^{-n} \sum_{p,q=1}^n W_{apbq}(W_{cpdq} + W_{cqdp}). \]

Using again Corollary 29 of Brendle [7], we have

\[ \int_{\mathbb{S}^{n-1}} h_{ab}(x) h_{cd}(x) \, d\sigma(x) = \frac{1}{9n(n+2)^n} \sum_{p,q=1}^n W_{apbq}(W_{cpdq} + W_{cqdp}). \]

Together with (4.6) and (4.7), we thus obtain

\[ A_2 = \frac{1}{n^2(n-2)^2} \int_0^\infty \frac{r^{n+5}}{(1 + \frac{r^2}{n(n-2)})^{n+2}} dr \cdot \int_{\mathbb{S}^{n-1}} h_{ab}(x) h_{cd}(x) \, d\sigma(x) \]

\[ = \frac{n + 4}{36(n+1)} K_n^{-n} \sum_{p,q=1}^n W_{apbq}(W_{cpdq} + W_{cqdp}). \]

Similarly, we get

\[ A_3 = -\frac{2}{9n} K_n^{-n} \sum_{p,q=1}^n W_{apbq}(W_{cpdq} + W_{cqdp}). \]

Combining the latter equalities with (4.21) finally shows that

\[ \int_{\mathbb{R}^n} \langle \nabla L_0^{ab}, \nabla L_0^{cd} \rangle \, dx = \frac{n + 4}{36(n+1)} K_n^{-n} \sum_{p,q=1}^n W_{apbq}(W_{cpdq} + W_{cqdp}). \]

Going back to (4.19), we have thus proven that

\[ \partial_t^4 z_{a,b,c,d} F_3(t, z)|_{z=0} = -\frac{n + 4}{36(n+1)} K_n^{-n} \sum_{p,q=1}^n (W_{apbq}(W_{cpdq} + W_{cqdp}) \]

\[ + W_{apcq}(W_{bdpq} + W_{bdqp}) + W_{apdq}(W_{bpcq} + W_{bqcp})). \] (4.22)

Using the symmetries of \( W \), it is now easily seen that

\[ \sum_{a,b,c,d,p,q=1}^n W_{apbq}(W_{cpdq} + W_{cqdp}) z_{a,b,c,d} = 18 \sum_{p,q=1}^n (h_{pq}(z))^2. \]
A Taylor formula for $F_3$ at fourth order in $(t,0)$ thus shows with (4.22) that

$$F_3(t,z) = \frac{1}{24} \sum_{a,b,c,d=1}^n \partial_{a z_b z_c z_d}^4 F_3(t,z) \big|_{z=0} z_a z_b z_c z_d + O \left( |z|^5 \right)$$

$$= - \frac{n+4}{48(n+1)} K_n^{-n} \sum_{p,q=1}^n (h_{pq}(z))^2 + O \left( |z|^5 \right),$$

which concludes the proof of Proposition 4.4. \qed

Combining Propositions 4.2, 4.3 and 4.4 and since $F$ is even in $z$, we have therefore proven that, for any $t > 0$ fixed,

$$F(t,z) = F(t,0) - \frac{n^2 - 8n - 12}{48(n+1)} K_n^{-n} \sum_{p,q=1}^n (h_{pq}(z))^2 + O \left( |z|^5 \right) \quad (4.23)$$

as $z \to 0$. It is easily checked that $n^2 - 8n - 12 > 0$ for $n \geq 11$. We are now in position to conclude the proof of Theorem 2.2.

**Proof of Theorem 2.2.** Let $(M, \hat{g})$ be a locally conformally flat manifold that is $Y$-non-degenerate in the sense of Definition 2.1. Let $W : (\mathbb{R}^n)^4 \to \mathbb{R}$ be a four-linear form as in Section 4 and $h$ be defined by (4.22). By the symmetries of $W$, $h$ satisfies (4.2). We assume that $W$ is chosen so that

$$\sum_{p,q=1}^n (h_{pq}(z))^2 \geq C |z|^4 \quad \text{for all } z \in \mathbb{R}^n \quad (4.24)$$

for some constant $C > 0$ independent of $z$. Assumption (4.24) is for example satisfied when $W$ has the following diagonal form:

$$W_{ijkl} = \frac{A_{ij}}{2} (\delta_{ik}\delta_{j\ell} - \delta_{ij}\delta_{k\ell}),$$

where $A$ is a nonzero symmetric matrix satisfying $A_{ii} = 0$ and $\sum_{j=1}^n A_{ij} = 0$ for any $1 \leq i \leq n$ and $A_{ij} \neq 0$ whenever $i \neq j$. Let $(\varepsilon_k)_k$, $(\mu_k)_k$ and $(\tau_k)_k$ be sequences as in Section 3. Let $\tilde{g}$ and $g$ be defined as in (4.3) and $\tilde{g}$ and $\tilde{u}_0$ be defined as in (3.5). The analysis of Section 3 applies. For $(t,z) \in [1/A, A] \times B(0,1)$, we let $u_{k,t,z} := W_{k,t,z} + \varphi_k(t,z)$, where $W_{k,t,z}$ is as in (3.18) and $\varphi_k(t,z)$ is given by Proposition 3.3. For $k \in \mathbb{N}$, we let $F_k : [1/A, A] \times B(0,1) \to \mathbb{R}$ be defined by

$$F_k(t,z) := \mu_k^{1-\frac{2}{n}} \left( I(u_{k,t,z}) - \frac{1}{n} K_n^{-n} - I(\tilde{u}_0) \right).$$

By Propositions 3.4 and 3.5 we have

$$F_k(t,z) = F(t,z) + \Lambda_k^0(t,z) \quad \text{and} \quad \partial_t F_k(t,z) = \partial_t F(t,z) + \Lambda_k^1(t,z) \quad (4.25)$$

uniformly with respect to $(t,z) \in [1/A, A] \times B(0,1)$, where $\Lambda_k^0$ and $\Lambda_k^1$ converge uniformly to 0 in compact subsets of $[1/A, A] \times B(0,1)$ and $F$ is as in (4.3). By Lemma 4.1, (4.23) and (4.24), $F$ has a saddle-type geometry on $[1/A, A] \times B(0,1)$, so that we can let $\varepsilon$ and $\eta$ be small enough so that

$$\min_{t \in [t_0-\eta,t_0+\eta]} F(t,0) = F(t_0,0) > \max_{t \in [t_0-\eta,t_0+\eta], |z| = \varepsilon} F(t,z)$$
and $\partial_t F(t_0 - \eta, z) < 0$ and $\partial_t F(t_0 + \eta, z) > 0$ for any $z \in B(0, \varepsilon)$. Since $F(t_0, 0) < 0$, by decreasing $\varepsilon$ and $\eta$ if necessary, we can also assume that

$$\max_{(t, z) \in [t_0 - \eta, t_0 + \eta] \times B(0, \varepsilon)} F(t, z) < 0.$$ 

By (4.25), choosing $k$ large enough, we again have

$$\min_{t \in [t_0 - \eta, t_0 + \eta]} F_k(t, 0) > \max_{t \in [t_0 - \eta, t_0 + \eta], |z| = \varepsilon} F_k(t, z),$$

and $\partial_t F_k(t_0 - \eta, z) < 0$ and $\partial_t F_k(t_0 + \eta, z) > 0$ for any $z \in B(0, \varepsilon)$. Thus $F_k$ also has a saddle-type geometry for $k$ large enough. Lemma A.1 of Thizy–Vétois [56] then applies and shows that, for $k$ large enough, $F_k$ admits a critical point $(t_{k, z_k})$ in $(t_0 - \eta, t_0 + \eta) \times B(0, \varepsilon)$. By Proposition 3.3, $u_{k, t_k, z_k}$ is then a solution of (3.6). Remark 3.5 shows that, for any $k \geq 0$ large enough, $u_{k, t_k, z_k}$ changes sign in $M$. Standard arguments show that $u_k$ blows-up as $k \to \infty$. Finally, by Propositions 3.4 and 3.5 and since $u_{k, t_k, z_k}$ solves (3.6), we have

$$\int_M |u_{k, t_k, z_k}|^2 \, dv_{\tilde{g}} = n I(u_{k, t_k, z_k}) = K_n^{-n} + \int_M \tilde{u}_0^2 \, dv_{\tilde{g}} + n \mu_k^{\frac{n-2}{2}} (F(t_k, z_k) + o(1))$$

$$< K_n^{-n} + \int_M \tilde{u}_0^2 \, dv_{\tilde{g}}$$

since $F(t_k, z_k) < 0$. Since $\int_M \tilde{u}_0^2 \, dv_{\tilde{g}} = Y(M, [g])^{\frac{n}{2}}$ by the $Y$-non-degeneracy assumption, this concludes the proof of Theorem 2.2.

**Remark 4.5.** If $\tilde{g}$ is defined as in (3.3), it follows from an expansion at first-order in $x - y_k$ that $\tilde{g}$ satisfies

$$\text{Weyl}_{\tilde{g}}(y_k) = -\varepsilon_k W$$

for any $k \in \mathbb{N}$. In view of (4.11), Proposition 3.4 then shows that

$$I(W_{k, t, 0}) = \frac{1}{n} K_n^{-n} + I(\tilde{u}_0) + \Lambda(n) \tilde{u}_0(x_0) \mu_k(t)^{\frac{n-2}{2}} - C(n) |\text{Weyl}_{\tilde{g}}(y_k)|_2^2 \mu_k(t)^{\frac{1}{2}}$$

$$+ o \left( \mu_k^{\frac{n-2}{2}} \right),$$

an expansion that is reminiscent of those in Esposito–Pistoia–Vétois [21] and Premoselli–Wei [41].

5. Asymptotic Analysis at the Lowest Sign-Changing Energy Level and Proof of Theorem 1.2

We begin this section with a simple result showing that $Y(S^n, [g_0])^{\frac{n}{2}} + Y(M, [g])^{\frac{n}{2}}$ is the minimal energy level for the blow-up of sign-changing solutions of (1.1).

**Proposition 5.1.** Let $(M, g)$ be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and positive Yamabe type. Let $(u_k)_{k \in \mathbb{N}}$ be a blowing-up sequence of solutions to (1.1). If $u_k$ changes sign for large $k$ and blows-up as $k \to \infty$, then

$$E(u_k) \geq Y(M, [g])^{\frac{n}{2}} + Y(S^n, [g_{std}])^{\frac{n}{2}} + o(1)$$

as $k \to \infty$. Moreover, if equality holds true in (5.1) and $(M, g)$ is not conformally diffeomorphic to $(S^n, g_{std})$, then up to a subsequence and replacing $u_k$ by $-u_k$ if
necessary, \( u_k \) is of the form

\[
    u_k = u_0 - \left( \frac{\mu_k}{\mu_k^2 + d_g(\xi_k)^2} \right)^{\frac{n-2}{2}} + o(1) \text{ in } H^1(M) \text{ as } k \to \infty, \tag{5.2}
\]

where \( u_0 \) is a positive solution to the Yamabe equation \((1.2)\) that attains \( Y(M,[g]) \), \( d_g \) is the geodesic distance with respect to the metric \( g \) and \( (\mu_k)_k \) and \( (\xi_k)_k \) are two families of positive numbers and points in \( M \), respectively, such that \( \mu_k \to 0 \) and \( \xi_k \to \xi_0 \) as \( k \to \infty \) for some point \( \xi_0 \in M \).

**Proof of Proposition 5.1.** We let

\[
    E := \liminf_{k \to \infty} E(u_k) \quad \text{and} \quad E_0 := Y(M,[g])^{\frac{n}{2}} + Y(S^n,[g_{std}])^{\frac{n}{2}}.
\]

On the one hand, \( Y(M,[g]) < Y(S^n,[g_{std}]) \) when \((M,g)\) is not conformally diffeomorphic to \((S^n,g_{std})\) by the results of Trudinger [51], Aubin [3] and Schoen [51]. On the other hand, it is easy to see that the energy of any sign-changing solutions to the Yamabe equation on \((S^n,g_{std})\) is greater than \( 2Y(S^n,[g_{std}])^{\frac{n}{2}} \). Struwe’s decomposition [34] then shows that up to a subsequence and replacing \( u_k \) by \(-u_k\) if necessary,

\[
    \{5.2\} \text{ holds true with } u_0 = 0 \text{ in } M \text{ if } E < E_0,
\]

\[
    \{5.2\} \text{ holds true for some energy-minimizing solution } u_0 \text{ of } (1.2) \text{ if } E = E_0 \text{ and}
\]

\( (M,g) \) is not conformally diffeomorphic to \((S^n,g_0)\).

We are left to show that \((5.2)\) cannot hold true with \( u_0 = 0 \) in \( M \). Assume by contradiction that this is the case. For large \( k \), by testing \((1.1)\) with \( u_k^\pm := \max(u_k,0) \), we obtain

\[
    \int_M \left( |\nabla u_k^\pm|^2 + c_n \text{Scal}_g(u_k^\pm)^2 \right) dv_g = \int_M (u_k^\pm)^{2^*} dv_g. \tag{5.3}
\]

Since \( u_k \) changes sign the right-hand side in \((5.3)\) is nonzero. An easy density argument then gives

\[
    \int_M \left( |\nabla u_k^\pm|^2 + c_n \text{Scal}_g(u_k^\pm)^2 \right) dv_g \geq \inf_{u \in C^\infty(M), u > 0} \int_M \left( |\nabla u|^2 + c_n \text{Scal}_g(u)^2 \right) dv_g
\]

\[
    \left( \int_M (u_k^\pm)^{2^*} dv_g \right)^{2/2^*} = c_n Y(M,[g]), \tag{5.4}
\]

where \( c_n := \frac{n-2}{4(n-1)} \). By using \((5.3)\) and \((5.4)\), we then obtain

\[
    c_n Y(M,[g]) \leq \left( \int_M (u_k^\pm)^{2^*} dv_g \right)^{2/n}. \tag{5.5}
\]

Since \( Y(M,[g]) > 0 \), it then follows from \((5.5)\) that \( u_k^\pm \not\to 0 \) in \( L^{2^*}(M) \) as \( k \to \infty \). This is in contradiction with \((5.2)\) when \( u_0 = 0 \) in \( M \).

The rest of this subsection is devoted to the proof of Theorem 1.2. By Lee and Parker’s construction of conformal normal coordinates [27] there exists a smooth family of metrics \( (g_\xi)_{\xi \in \mathcal{M}}, g_\xi = \Lambda_\xi^{2^*-2} g \), such that for every point \( \xi \in \mathcal{M} \), the function \( \Lambda_\xi \) is smooth, positive and satisfies

\[
    \Lambda_\xi(\xi) = 1, \quad \nabla \Lambda_\xi(\xi) = 0 \quad \text{and} \quad dv_{g_\xi}(x) = (1 + O(|x|^N)) dv_{g_0}(x). \tag{5.6}
\]
in geodesic normal coordinates, where $dv_{g_k}$ and $dv_{\delta_0}$ are the volume elements of the metric $g_\xi$ and the Euclidean metric $\delta_0$, respectively, and $N \in \mathbb{N}$ can be chosen arbitrarily large. In particular (see [27]), it follows from (5.6) that

$$\text{Scal}_{g_\xi}(\xi) = 0, \quad \nabla \text{Scal}_{g_\xi}(\xi) = 0 \quad \text{and} \quad \Delta_g \text{Scal}_{g_\xi}(\xi) = \frac{1}{6} |\text{Weyl}_g(\xi)|^2_g. \quad (5.7)$$

Also, if $g$ is locally conformally flat in a neighbourhood of $\xi \in M$, then $g_\xi$ can be chosen flat in a neighbourhood of $\xi$. We first state some preliminary lemmas that apply to the more general case where the weak limit $u_0$ is a possibly sign-changing solution of

$$\Delta_g u_0 + c_n \text{Scal}_g u_0 = |u_0|^{2^* - 2} u_0 \quad \text{in} \ M. \quad (5.8)$$

The following lemma is essentially contained in Premoselli [40] (see also Druet–Hebey–Robert [20] and Hebey [24] in the case of positive solutions):

**Lemma 5.2.** Let $(M, g)$ be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and $u_0$ be a solution to (5.8). Assume that there exists a sequence of solutions $(u_k)_{k \in \mathbb{N}}$ of type (5.2) to (1.1). Let $(g_\xi)_{\xi \in M}$, $g_\xi = \Lambda_{\xi_k}^{2^* - 2}g$, be the Lee–Parker family of conformal metrics to $g$. Let $\Lambda_{\xi_k}$ and $(\xi_k)_{k \in \mathbb{N}}$ be families of positive numbers and points in $M$, respectively, such that

$$u_k (\xi_k) = \min_M u_k = -\bar{\mu}_k^{1-\frac{2}{n}}. \quad (5.9)$$

Then

$$u_k - u_0 + B_k = o (B_k + 1) \quad (5.10)$$

as $k \to \infty$, uniformly in $M$, where $B_k$ is defined as

$$B_k (x) := \Lambda_{\xi_k} (x) \frac{n-2}{(\bar{\mu}_k + d_{\xi_k}(x,\xi_k)^2)^{\frac{n-2}{2}}} \quad \forall x \in M.$$

**Proof of Lemma 5.2.** Since $(u_k)$ is of the form (5.2), the main result in Premoselli [40] shows that

$$u_k = u_0 - \frac{\mu_k^{\frac{n-2}{2}}}{(\mu_k^2 + d_{\xi_k}(x,\xi_k)^2)^{\frac{n-2}{2}}} + o \left( 1 + \frac{\mu_k}{(\mu_k^2 + d_{\xi_k}(x,\xi_k)^2)^{\frac{n-2}{2}}} \right) \quad (5.11)$$

as $k \to \infty$, uniformly in $M$, where $(\mu_k)_k$ and $(\xi_k)_k$ are as in (5.2). Let $(\bar{\mu}_k)_k$ and $(\bar{\xi}_k)_k$ be such that (5.9) holds true. It then follows from (5.11) that

$$\bar{\mu}_k \sim \mu_k \quad \text{and} \quad d_{\xi_k}(\bar{\xi}_k, \xi_k) = o (\mu_k) \quad (5.12)$$

as $k \to \infty$. Moreover, since $\Lambda_{\bar{\xi}_k} (\bar{\xi}_k) = 1$, we obtain

$$d_{\bar{\xi}_k}(x,\bar{\xi}_k)^2 = d_g (x,\bar{\xi}_k)^2 + O \left( d_g (x,\bar{\xi}_k)^3 \right) \quad (5.13)$$

uniformly with respect to $k \in \mathbb{N}$ and $x \in M$. By using again that $\Lambda_{\bar{\xi}_k} (\bar{\xi}_k) = 1$ together with (5.12) and (5.13), we obtain

$$\frac{\mu_k^{\frac{n-2}{2}}}{(\mu_k^2 + d_{\xi_k}(x,\xi_k)^2)^{\frac{n-2}{2}}} = B_k + o (B_k + 1) \quad (5.14)$$

as $k \to \infty$, uniformly in $M$, so that (5.10) follows from (5.11) and (5.14). \qed
We then state the following result from Premoselli–Vétois [33], which essentially follows from a Pohozaev-type identity together with the estimate (5.11):

**Lemma 5.3.** (Lemma 3.2 in [33]) Let \((M,g)\) be a smooth, closed Riemannian manifold of dimension \(n \geq 3\) and \(u_0\) be a solution to (5.8). Assume that there exists a sequence of solutions \((u_k)_{k \in \mathbb{N}}\) to (1.1) satisfying (5.2). Let \((\overline{M}_k), (\overline{\xi}_k)\), \((B_k)\) and \((g_k)\) be as in Lemma 5.2. Then

\[
\int_{B(0,1/\sqrt{\mu_k})} \left((\nabla \hat{u}_k, \cdot)_{\hat{h}_k} + \frac{n-2}{2} \hat{u}_k \right) \left( (\Delta \hat{g}_k - \Delta_{\hat{h}_k}) \hat{u}_k + \hat{h}_k \hat{u}_k \right) dv_{\hat{g}_k} = \frac{1}{2} n^{n-2} \frac{n+2}{n} \omega_{n-2} u_0 (\xi_0) \mu_k^{\frac{n-2}{2}} + o \left( \mu_k^{\frac{n-2}{2}} \right)
\]

as \(k \to \infty\), where

\[
\hat{u}_k(y) := \mu_k^{\frac{n-2}{2}} (\Lambda_{\overline{\xi}_k}^{-1} u_k)(\exp_{\overline{\xi}_k}(\hat{g}_k y)), \quad \hat{g}_k(y) := \exp_{\overline{\xi}_k}^* g_{\overline{\xi}_k} (\mu_k y)
\]

and

\[
\hat{\pi}_k(y) := c_n \text{ Scal}_{\hat{g}_k}(\exp_{\overline{\xi}_k}(\mu_k y))
\]

for all points \(y \in B(0,1/\sqrt{\mu_k})\), \(\omega_{n-2}\) is the volume of the standard unit sphere of dimension \(n - 1\), \(\delta_0\) is the Euclidean metric in \(\mathbb{R}^n\), \(\exp_{\overline{\xi}_k}\) is the exponential map at the point \(\overline{\xi}_k\) with respect to the metric \(g_{\overline{\xi}_k}\) and we identify \(\overline{\xi}_k M\) with \(\mathbb{R}^n\).

Finally, to prove Theorem 1.2 in dimensions \(n \geq 7\), we need the following refinement of Lemma 5.2.

**Lemma 5.4.** Let \((M,g)\) be a smooth, closed Riemannian manifold of dimension \(n \geq 7\) and \(u_0\) be a solution to (5.8). Let \((u_k)_{k \in \mathbb{N}}, (\overline{M}_k), (\overline{\xi}_k)\), \((B_k)\) and \((g_k)\) be as in Lemma 5.2. Then, for \(i \in \{0,1,2\}\),

\[
|\nabla^i (u_k - u_0 + B_k)(x)| = O \left\{ \begin{array}{ll}
1 & \text{if } (M,g) \text{ is l.c.f.} \\
\frac{1}{(\overline{\pi}_k + d_{\pi_{\overline{\xi}_k}}(x,\overline{\xi}_k))^{n-i}} & \text{otherwise}
\end{array} \right.
\]

uniformly with respect to \(k \in \mathbb{N}\) and \(x \in M\), where l.c.f. stands for locally conformally flat.

**Proof of Lemma 5.4.** We only need to prove (5.18) when \(i = 0\), since the cases \(i = 1\) and \(i = 2\) follow by standard interior estimates. Remark that when \(x\) satisfies \(d_{\pi_{\overline{\xi}_k}}(x,\overline{\xi}_k) \geq \sqrt{\mu_k}\), the estimate (5.18) when \(i = 0\) follows from (5.10). Therefore, it suffices to consider the case where \(d_{\pi_{\overline{\xi}_k}}(x,\overline{\xi}_k) < \sqrt{\mu_k}\) as \(k \to \infty\). In this case, we use an approach inspired from the work of Chen–Lin [10] in the case of positive solutions that was applied to the Yamabe equation by Marques [31]. By letting \(x = \exp_{\overline{\xi}_k}(\mu_k y)\) for \(y \in B(0,1/\sqrt{\mu_k})\), it is easily seen that (5.18) is implied by the
following estimate:

\[ |\nabla^i (\hat{u}_k + B_0) (y)| = O \left( \begin{cases} \frac{n^{2i}}{\mu_k^{n-2}} \frac{1}{(1 + |y|)^i} & \text{if } (M, g) \text{ is l.c.f.} \\ \frac{n^{2i}}{\mu_k^{n-2}} + \frac{\mu_k^{4 \lambda_k}}{(1 + |y|)^{n-6+\epsilon}} & \text{otherwise} \end{cases} \right), \]

(5.19)

where \( \hat{u}_k \) is as in (5.16) and we have let

\[ B_0 (y) := \left( 1 + \frac{|y|^2}{n(n-2)} \right)^{-\frac{n-2}{2}} \quad \forall y \in \mathbb{R}^n. \]

As mentioned before, we only have to prove (5.19) for \( i = 0 \). By conformal invariance of the conformal Laplacian, the equation (4.11) can be rewritten as

\[ \Delta_{\hat{g}_k} \hat{u}_k + \mu_k^2 \hat{h}_k \hat{u}_k = |\hat{u}_k|^{2^* - 2} \hat{u}_k \quad \text{in } B(0, 1/\mu_k), \]

(5.21)

where \( \hat{h}_k \) and \( \hat{g}_k \) are as in (5.16) and (5.17). We let \( y_k \in B(0, 1/\mu_k) \) be such that

\[ |(\hat{u}_k + B_0) (y_k)| = \max_{B(0, 1/\mu_k)} |(\hat{u}_k + B_0)| =: \lambda_k, \]

(5.22)

and we define

\[ \psi_k := \lambda_k^{-1} (\hat{u}_k + B_0). \]

By (5.10), \( \lambda_k = o(1) \) as \( k \to \infty \). By using (5.21) together with the equation \( \Delta_{\hat{g}_0} B_0 = B_0^{2^* - 1} \), we obtain

\[ \Delta_{\hat{g}_k} \psi_k = (2^* - 1) B_0^{2^* - 2} \psi_k + \lambda_k^{-1} f_k \quad \text{in } B(0, 1/\mu_k), \]

(5.23)

where

\[ f_k := (\Delta_{\hat{g}_k} - \Delta_{\hat{g}_0}) B_0 - \mu_k^2 \hat{h}_k \hat{u}_k + |\hat{u}_k|^{2^* - 2} \hat{u}_k + B_0^{2^* - 1} - (2^* - 1) \lambda_k B_0^{2^* - 2} \psi_k. \]

(5.24)

We now estimate the terms in the right-hand side of (5.24). Since \( B_0 \) is radially symmetric, it follows from (5.6) that

\[ (\Delta_{\hat{g}_k} - \Delta_{\hat{g}_0}) B_0 (y) = O \left( \frac{n^N |y|^{N-1}}{(1 + |y|)^n} \right) \]

(5.25)

uniformly with respect to \( k \in \mathbb{N} \) and \( y \in B(0, 1/\mu_k) \). By using (5.7) and (5.10), we obtain

\[ \hat{h}_k (y) \hat{u}_k (y) = \begin{cases} 0 & \text{if } (M, g) \text{ is l.c.f.} \\ O \left( \frac{\mu_k^2 |y|^2}{(1 + |y|)^{n-2}} \right) & \text{otherwise} \end{cases} \]

(5.26)

uniformly with respect to \( k \in \mathbb{N} \) and \( y \in B(0, 1/\mu_k) \). Since \( 2^* - 2 < 2 \) when \( n \geq 7 \), by using (5.10) together with straightforward estimates, we obtain

\[ |\hat{u}_k|^{2^* - 2} \hat{u}_k + B_0^{2^* - 1} - (2^* - 1) \lambda_k B_0^{2^* - 2} \psi_k = o \left( \lambda_k B_0^{2^* - 2} |\psi_k| \right) \]

(5.27)
as \( k \to \infty \), uniformly with respect to \( y \in B(0, 1/\sqrt{\mu_k}) \). Let \( G_k \) be the Green’s function of the operator \( \Delta_{g_k} \) in \( B(0, 1/\sqrt{\mu_k}) \) with zero Dirichlet boundary condition. Then

\[
\psi_k(y) = \int_{B(0, 1/\sqrt{\mu_k})} G_k(y, \cdot) \left( (2^* - 1) B_0^{2^* - 2} \psi_k + \lambda_k^{-1} f_k \right) dv_{g_k} - \int_{\partial B(0, 1/\sqrt{\mu_k})} \partial_{v} G_k(y, \cdot) \psi_k d\sigma_{g_k} \quad (5.28)
\]

for all points \( x \in B(0, 1/\sqrt{\mu_k}) \). Standard estimates on Green’s functions (see e.g. Robert [46]) give that for large \( k \),

\[
G_k(y, z) = O \left( |y - z|^{2-n} \right) \text{ for } z \in B(0, 1/\sqrt{\mu_k}) \quad (5.29)
\]

and

\[
\partial_v G_k(y, z) = O \left( |y - z|^{1-n} \right) \text{ for } z \in \partial B(0, 1/\sqrt{\mu_k}) \quad (5.30)
\]

uniformly with respect to \( k \in \mathbb{N} \) and \( y \in B(0, 1/\sqrt{\mu_k}) \). Remark also that (5.10) gives

\[
\psi_k = O \left( \frac{1}{\mu_k^{\frac{n-4}{4}}} \right) \text{ as } k \to \infty, \text{ uniformly in } B(0, 1/\sqrt{\mu_k}) \backslash B(0, 1/2\sqrt{\mu_k}) \quad (5.31)
\]

By using (5.29), (5.26), (5.27) and (5.28)-(5.31) together with straightforward integral estimates and the fact that \( n \geq 7 \), we obtain

\[
\psi_k(y) = O \left( \int_{B(0, 1/\sqrt{\mu_k})} \left| \frac{\psi_k(z)}{|y - z|^{n-4}} \right| dv_{g_k} + \lambda_k^{-1} \mu_k^{\frac{n-2}{2}} \left\{ + \frac{\lambda_k^{-1} \mu_k^{\frac{n-2}{2}}}{(1 + |y|^{n-6})} \text{ if } (M, g) \text{ is not l.c.f.} \right\} \right) \quad (5.32)
\]

uniformly with respect to \( k \in \mathbb{N} \) and \( y \in B(0, 1/\sqrt{\mu_k}) \). We now claim that

\[
\lambda_k = O \left( \frac{n-2}{\mu_k^{\frac{n-2}{2}} + \mu_k^{1/2}} \right) \text{ if } (M, g) \text{ is not l.c.f.} \quad (5.33)
\]

uniformly with respect to \( k \in \mathbb{N} \). Assume by contradiction that (5.33) does not hold true, i.e. there exists a subsequence \( (k_j)_{j \in \mathbb{N}} \) of positive numbers such that

\[
k_j \to 0 \quad \text{and} \quad \mu_k^{\frac{n-2}{2}} \left\{ + \mu_k^{1/2} \text{ if } (M, g) \text{ is not l.c.f.} \right\} = o \left( \lambda_{k_j} \right) \quad (5.34)
\]

as \( j \to \infty \). Since \( |\psi_{k_j}| \leq 1 \) in \( B(0, 1/\sqrt{\mu_k}) \) and \( g_{k_j} \to \delta_0 \) as \( j \to \infty \), uniformly in compact subsets of \( \mathbb{R}^n \), it follows from (5.23)-(5.27) together with standard elliptic estimates that up to a subsequence, \( (\psi_{k_j})_j \) converges in \( C^1_{loc}(\mathbb{R}^n) \) as \( j \to \infty \) to a solution \( \psi_0 \in C^2(\mathbb{R}^n) \) of the equation

\[
\Delta_{\delta_0} \psi_0 = (2^* - 1) B_0^{2^* - 2} \psi_0 \quad \text{in } \mathbb{R}^n. \quad (5.35)
\]

Independently, it follows from (5.32) and (5.34) that

\[
\psi_{k_j}(y) = O \left( (1 + |y|^{-2}) + o(1) \right) \quad (5.36)
\]

as \( j \to \infty \), uniformly with respect to \( y \in B(0, 1/\sqrt{\mu_k}) \). Passing to the limit into (5.36) as \( j \to \infty \), we then obtain

\[
\psi_0(y) = O \left( (1 + |y|^{-2}) \right) \quad (5.37)
\]
uniformly with respect to \( y \in \mathbb{R}^n \). By applying Lemma 2.4 of Chen–Lin [9], it follows from (5.35) and (5.37) that

\[
\psi_0 (y) = c_0 \frac{1 - |y|^2}{(1 + |y|^2)^{\frac{n}{n(n-2)}}} + \sum_{i=1}^n c_i \frac{y_i}{(1 + |y|^2)^{\frac{n}{n(n-2)}}} \quad \forall y \in \mathbb{R}^n
\]

for some numbers \( c_0, \ldots, c_n \in \mathbb{R} \). On the other hand, it follows from (5.9) that \( \psi_k (0) = |\nabla \psi_k (0)| = 0 \), which gives \( \psi_0 (0) = |\nabla \psi_0 (0)| = 0 \). Therefore, we obtain \( c_0 = \cdots = c_n = 0 \), and so \( \psi_0 = 0 \) in \( \mathbb{R}^n \). Since \( \psi_{k_j} (y_{k_j}) = 1 \) for all \( j \in \mathbb{N} \), where \( y_{k_j} \) is as in (5.22), we then obtain that \( |y_{k_j}| \to \infty \) as \( j \to \infty \), which is in contradiction with (5.36). This proves that (5.33) holds true. Finally, by using (5.33) together with successive applications of (5.32), we obtain that

\[
\psi_k (y) = O \left( \frac{\ln (2 + |y|)}{(1 + |y|)^{n-2}} + \lambda_k^{-1} \frac{n-2}{n} \right) \quad \text{if (} M, g \text{) is not l.c.f.}
\]

uniformly with respect to \( k \in \mathbb{N} \) and \( y \in B(0, 1/\sqrt{\mu_k}) \). The estimate (5.19) with \( i = 0 \) then follows from (5.33) and (5.38).

We can now use Lemmas 5.3 and 5.4 to prove Theorem 1.2.

**Proof of Theorem 1.2** Assume by contradiction that

\[ E(M, [g]) \leq Y(S^n, [g_{std}])^\frac{2}{n} + Y(M, [g])^\frac{2}{n}. \]

By using the definition of \( E(M, [g]) \) together with a diagonal argument, it follows that there exists a sequence of blowing-up solutions \( (u_k)_{k \in \mathbb{N}} \) to (1.1) such that

\[ E(u_k) \leq Y(M, [g])^\frac{2}{n} + Y(S^n, [g_{std}])^\frac{2}{n} + o(1) \]

as \( k \to \infty \). In the case where the functions \( u_k \) do not change sign, we obtain a contradiction with the assumptions of Theorem 1.2 by applying the compactness results for positive solutions of the Yamabe equation (see Schoen [52, 53], Li–Zhu [30, Druet [19], Marques [31], Li–Zhang [28, 29] and Khuri–Marques–Schoen [26]). Therefore, in what follows, we assume that the functions \( u_k \) change sign. It then follows from Proposition 5.1 that up to a subsequence and replacing \( u_k \) by \(-u_k\) if necessary, the functions \( u_k \) are of the form (5.2) for some energy-minimizing, positive solution \( u_0 \) to (1.2). In the case where \( n \leq 6 \), we can apply a more general compactness result that we obtained in Premoselli–Vétois [43] (Theorem 1.2 in [43]). Therefore, in what follows, we assume that \( n \geq 7 \). By using (5.18) (see also (5.19)) and (5.25), we then obtain

\[
(\Delta \phi_k - \Delta \delta_0) \hat{u}_k = O \left( \frac{\mu_k^n |y|^N}{(1 + |y|)^n} + |(\hat{\phi}_k - \delta_0) (y)| |\nabla^2 (\hat{u}_k + B_0) (y)| \right.
\]

\[
+ \left| \nabla (\hat{\phi}_k - \delta_0) (y) | \nabla (\hat{u}_k + B_0) (y)| \right|
\]

\[
= O \left( \frac{\mu_k^{n+2}}{\mu_k^{n}} \left\{ \frac{\mu_k^n |y|}{(1 + |y|)^{n-2}} \right\} \right) \quad \text{if (} M, g \text{) is not l.c.f.} \tag{5.39}
\]
uniformly with respect to $k \in \mathbb{N}$ and $y \in B(0, 1/\sqrt{p_k})$. It then follows from (5.18), (5.26) and (5.39) that
\[
\int_{B(0, 1/\sqrt{p_k})} \left( (\nabla \hat{u}_k, \cdot)_{\delta_0} + \frac{n-2}{2} \hat{u}_k \right) \left( (\Delta_{\bar{g}_k} - \Delta_{\delta_0}) \hat{u}_k + p_k^2 \hat{h}_k \hat{u}_k \right) dv_{\delta_0}
- \frac{p_k^2}{2} \int_{B(0, 1/\sqrt{p_k})} \left( (\nabla B_0, \cdot)_{\delta_0} + \frac{n-2}{2} B_0 \right) \hat{h}_k B_0 dv_{\delta_0}
= O \left( \begin{cases} p_k^2 & \text{if } (M, g) \text{ is l.c.f.} \\ p_k^2 + \bar{p}_k^2 & \text{otherwise} \end{cases} \right) \] 
for large $k$. On the other hand, by using (5.6) and (5.7) together with straightforward computations and symmetry arguments, we obtain
\[
\int_{B(0, 1/\sqrt{p_k})} \left( (\nabla B_0, \cdot)_{\delta_0} + \frac{n-2}{2} B_0 \right) \hat{h}_k B_0 dv_{\delta_0}
= \begin{cases} 0 & \text{if } (M, g) \text{ is l.c.f.} \\ a_n |\text{Weyl}_g(\xi_0)|^2_{g} p_k^2 + o(\bar{p}_k^2) & \text{otherwise} \end{cases}
\] 
as $k \to \infty$, where
\[
a_n := \frac{c_n}{24} (n-2)^2 \int_{\mathbb{R}^n} \left( 1 + \frac{|y|^2}{n(n-2)} \right)^{1-n} \left( \frac{|y|^2}{n(n-2)} - 1 \right) \frac{|y|^2 dy}{n(n-2)}. \] 

The constant $a_n$ is computed by using (4.15) and (4.7), and we obtain
\[
a_n = \frac{c_n n^7 (n-2)^{n+4}}{6(n-6)} \left( \frac{n-2}{n-4} \right)^{n-1} I_{n-1}^{n-2} = \frac{n-2}{12(n-6)(n-4)} \omega_{n-1} I_{n-1}^{n-2} = \frac{n(n-2)^2}{6(n-6)(n-4)} K_n^{-n}.
\] 

By putting together (5.15), (5.40) and (5.41), we then obtain
\[
\frac{1}{2} n \frac{n-2}{4} (n-2)^{n-4} \omega_{n-1} u_0(\xi_0) p_k^{n-6} - a_n |\text{Weyl}_g(\xi_0)|^2_{g} \bar{p}_k^2
= o \left( \begin{cases} \frac{n-6}{p_k^{n-6}} & \text{if } (M, g) \text{ is l.c.f.} \\ \frac{n-6}{\bar{p}_k^{n-6}} + \frac{n-6}{\bar{p}_k^{n-6}} & \text{otherwise} \end{cases} \right)
\] 
as $k \to \infty$. In the case where $n = 10$, we obtain
\[
2 \cdot 10^{-4} 8^{-6} a_{10} = \frac{5}{567} \omega_9.
\] 

Finally, by using (5.44) and (5.45), we obtain a contradiction with the assumptions of Theorem 1.2. \hfill \Box

Remark 5.5. The approach used in the proof of Theorem 1.2 can be extended to the case of sequences of solutions $(u_k)_{k \in \mathbb{N}}$ of type (5.2) to linear perturbations of the Yamabe equation of the form
\[
\Delta_g u_k + (c_n \text{Scal}_g + \varepsilon_k h) u_k = |u_k|^{2^*-2} u_k \quad \text{in } M,
\] 
\[\tag{5.46} \]
where \( h \in C^{0, \vartheta}(M) \), \( \vartheta \in (0, 1) \), and \((\varepsilon_k)_k\) is a sequence of positive real numbers such that \( \varepsilon_k \to 0 \) as \( k \to \infty \). In the case of positive solutions, perturbed equations of the form \((5.2)\) have been studied for example by Esposito–Pistoia–Vétois \([21]\), Morabito–Pistoia–Vaira \([35]\), Premoselli \([11]\) and Robert–Vétois \([50]\). In this case, in place of \((5.18)\), we obtain, for \( i \in \{0, 1, 2\} \),

\[
|\nabla^i (u_k - u_0 + B_k)(x)| = O \left\{ \begin{array}{ll}
\frac{1}{n^{\frac{n-2}{2}} (n-2)^{\frac{n-2}{2}} \omega_{n-1} u_0(\xi_0) \pi_n^{-\frac{n-6}{2}}} & 
\text{if } (M, g) \text{ is l.c.f.} \\
\frac{\varepsilon_k \pi_n^{-\frac{n-2}{2}}}{(\mu_k + d_\gamma(\xi_k))^i} & 
\text{otherwise}
\end{array} \right.
\]

uniformly with respect to \( k \in \mathbb{N} \) and \( x \in M \), and in place of \((5.44)\), we obtain

\[
\frac{1}{2} n^{\frac{n-2}{2}} (n-2)^{\frac{n-2}{2}} \omega_{n-1} u_0(\xi_0) \pi_n^{-\frac{n-6}{2}} - a_n \left| \text{Weyl}_g(\xi_0) \right|^2_g \pi_k^2 + b_n \varepsilon_k h(\xi_0)
= O \left\{ \begin{array}{ll}
\varepsilon_k + \pi_k^{-\frac{n-6}{2}} & 
\text{if } (M, g) \text{ is l.c.f.} \\
\varepsilon_k + \pi_k^2 + \pi_k^{n-6} & 
\text{otherwise}
\end{array} \right.
\]

as \( k \to \infty \), where \( a_n \) as in \((5.42)\) and \( b_n \) is another positive constant depending only on \( n \). This yields that there does not exist any sign-changing blowing-up sequences of solutions of type \((5.2)\) to \((5.46)\) in each of the following situations:

- \( 3 \leq n \leq 6 \),
- \( h \geq 0 \) and \( 7 \leq n \leq 9 \),
- \( h \geq 0 \), \( n = 10 \) and \( u_0 > \frac{5}{567} \) \text{Weyl}_g^2 \) for all points in \( M \),
- \( h \leq 0 \), \( n = 10 \) and \( u_0 < \frac{5}{567} \) \text{Weyl}_g^2 \) for all points in \( M \),
- \( h \geq 0 \), \( n \geq 11 \) and \((M, g)\) is locally conformally flat,
- \( h \leq 0 \), \( n \geq 11 \) and \text{Weyl}_g \neq 0 \) for all points in \( M \).

Conversely, it is not difficult to adopt the constructive proofs in Esposito–Pistoia–Vétois \([21]\) and Robert–Vétois \([50]\) to prove that if \( u_0 \) is a nondegenerate positive solution to \((1.2)\), then there exists a blowing-up sequence of solutions of type \((5.2)\) to \((5.46)\) in each of the following situations:

- \( \min_M h < 0 \) and \( 7 \leq n \leq 9 \),
- \( \min_M h < 0 \), \( n = 10 \) and \( u_0 > \frac{5}{567} \) \text{Weyl}_g^2 \) for all points in \( M \),
- \( \max_M h > 0 \), \( n = 10 \) and \( u_0 < \frac{5}{567} \) \text{Weyl}_g^2 \) for all points in \( M \),
- \( \min_M h < 0 \), \( n \geq 11 \) and \((M, g)\) is locally conformally flat,
- \( \max_M h > 0 \), \( n \geq 11 \) and \text{Weyl}_g \neq 0 \) for all points in \( M \).

This is again in sharp contrast with the case of positive solutions (see Esposito–Pistoia–Vétois \([21]\)), where blowing-up sequences of solutions to \((5.46)\) exist when \( \max_M h > 0 \) in all dimensions \( n \geq 4 \).
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