SHILNIKOV HOMOCLINIC BIFURCATION OF MIXED-MODE OSCILLATIONS

JOHN GUCKENHEIMER∗ AND IAN LIZARRAGA†

Abstract. The Koper model is a three-dimensional vector field that was developed to study complex electrochemical oscillations arising in a diffusion process. Koper and Gaspard described paradoxical dynamics in the model: they discovered complicated, chaotic behavior consistent with a homoclinic orbit of Shil’nikov type, but were unable to locate the orbit itself. The Koper model has since served as a prototype to study the emergence of mixed-mode oscillations (MMOs) in slow-fast systems, but only in this paper is the existence of these elusive homoclinic orbits established. They are found first in a larger family that has been used to study singular Hopf bifurcation in multiple time scale systems with two slow variables and one fast variable. A curve of parameters with homoclinic orbits in this larger family is obtained by continuation and shown to cross the submanifold of the Koper system. The strategy used to compute the homoclinic orbits is based upon systematic investigation of intersections of invariant manifolds in this system with multiple time scales. Both canards and folded nodes are multiple time scale phenomena encountered in the analysis. Suitably chosen cross-sections and return maps illustrate the complexity of the resulting MMOs and yield a modified geometric model from the one Shil’nikov used to study spiraling homoclinic bifurcations.

Key words. Koper model, mixed mode oscillations, Shilnikov homoclinic bifurcation

AMS subject classifications.

1. Introduction. In 1992, Marc Koper and Pierre Gaspard introduced a three-dimensional model to analyze an electrochemical diffusion problem, in which layer concentrations of electrolytic solutions fluctuate nonlinearly at an electrode [24, 25]. They sought to model mixed-mode oscillations (hereafter MMOs) arising in a wide variety of electrochemical systems. Their analysis revealed a host of complicated dynamics, including windows of period-doubling bifurcations, Hopf bifurcations, and complex Farey sequences of MMO signatures.

They also found regions in the parameter space where the equilibrium point of the system satisfies the Shilnikov condition. Within these regions, they observed that trajectories repeatedly come close to the fixed point, and return maps strongly suggest chaotic motion consistent with a Shilnikov homoclinic bifurcation. However, they were unable to locate a genuine homoclinic orbit to account for this behavior, so it was catalogued as a near homoclinic scenario. In such a scenario, complex and chaotic MMOs could suddenly arise—as if from a homoclinic bifurcation—but without the existence of the homoclinic orbit to serve as an organizing center.

Nevertheless, the Koper model has emerged as a paradigm in studies of slow-fast systems containing MMOs [5, 7]. As its four parameters are varied, local mechanisms such as folded nodes [16, 30] and singular Hopf bifurcation [2, 5, 17, 20, 21] generate small-amplitude oscillations. A global return mechanism allows for repeated reinjection into the regions containing these local objects. The interplay of these local and global mechanisms gives rise to sequences of large and small oscillations characterized by signatures that count the numbers of consecutive small and large oscillations. Shilnikov homoclinic orbits are limits of families of MMOs with an unbounded number of small oscillations in their signatures. When the Shilnikov condition is satisfied, they

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1Let the linearization of a three-dimensional vector field at an equilibrium point p have eigenvalues ρ ± iω and λ, where ρ, ω, and λ are all real. Then p satisfies the Shilnikov condition if |ρ/λ| < 1.
also guarantee the existence of chaotic invariant sets whose presence in the Belousov-Zhabotinsky reaction was controversial for several years.

The homoclinic orbits that could explain Koper’s original observations have remained elusive. This paper describes their first successful detection. Multiple timescales make numerical study of these homoclinic orbits quite delicate; on the other hand, the presence of slow manifolds, allows us to analyze many aspects of the system with low-dimensional maps. We find the homoclinic orbits by exhibiting multiple time scale phenomena such as canards and folded nodes. We first locate such orbits in a five-parameter family of vector fields used to explore the dynamics of singular Hopf bifurcation. After an affine coordinate change, the Koper model is a four-parameter subfamily. Shooting methods that compute trajectories between carefully chosen cross-sections cope with the numerical instability resulting from the singular behavior of the equations. Following identification of a homoclinic orbit in the larger family, a continuation algorithm is used to track a curve on the codimension-one manifold of spiraling homoclinic orbits in parameter space. This manifold intersects the parametric submanifold corresponding to the Koper model, locating the homoclinic orbit that is the target of our search.

2. Theory of slow-fast systems. We consider vector fields of the form

\begin{align}
\varepsilon \dot{x} &= f(x,y,\varepsilon), \\
\dot{y} &= g(x,y,\varepsilon),
\end{align}

(2.1)

where \( x \in \mathbb{R}^m \), \( y \in \mathbb{R}^n \), and the functions \( f \) and \( g \) are smooth. In this paper, \( f \) and \( g \) are polynomials. Slow-fast vector fields are those where \( \varepsilon \ll 1 \). In this case, \( x \) is the fast variable and \( y \) is the slow-variable. The set of points defined by \( C = \{ f = 0 \} \) is called the critical manifold.

Fenichel [13] proved the existence of locally invariant slow manifolds near regions of \( C \) where \( D_x f \) is hyperbolic. Trajectories on the slow manifolds are approximated by trajectories of the slow flow defined by

\begin{align}
\dot{y} &= g(h(y),y,0),
\end{align}

(2.2)

where \( h \) is defined implicitly by \( f(h(y),y,0) = 0 \). While the slow manifolds are not unique, compact portions are exponentially close to each other: their distances from each other are \( O(\exp(-c/\varepsilon)) \) as \( \varepsilon \to 0 \). We often refer to ‘the’ slow manifold in statements where the choice of slow manifold does not matter. The points \( x \in C \) where \( D_x f \) is singular are called fold points. Rescaling time of the slow flow produces a desingularized slow flow that extends to the fold curve. In systems with two slow variables, folded nodes, saddles, and foci are equilibrium points of node, saddle, and focus type of the desingularized slow flow on the fold curve of the critical manifold. Trajectory segments in a repelling slow manifold of a slow-fast systems are canards.

3. The singular Hopf extension of the Koper model. We now introduce a family of vector fields with two slow variables and one fast variable that contains a singular Hopf bifurcation [2, 5, 17, 20, 21]. It is given by

\begin{align}
\varepsilon \dot{x} &= y - x^3 - x^2, \\
\dot{y} &= z - x, \\
\dot{z} &= -\nu - ax - by - cz,
\end{align}

(3.1)
where $\varepsilon, \nu, a, b,$ and $c$ are parameters, $x$ is the fast variable, and $y$ and $z$ are the slow variables. We denote $\alpha = (\varepsilon, \nu, a, b, c)$ and by $P$ the five dimensional space of parameters $\alpha$. The critical manifold is an $S$-shaped cubic surface \( \{ y = x^3 + x^2 \} \) with two fold lines at $L_0$ defined as $\{ x = 0 \}$ and $L_{-2/3}$ defined as $\{ x = -2/3 \}$. In all following applications, we set $\varepsilon = 0.01$.

“The" slow manifold $S$ has sheets $S_{\varepsilon}^-$, $S_{\varepsilon}^a$ and $S_{\varepsilon}^{\alpha+}$ that lie close to the sheets on $C$ defined by $C \cap \{ x < -2/3 \}$, $C \cap \{ -2/3 < x < 0 \}$ and $C \cap \{ 0 < x \}$. Away from the fold lines, forward trajectories are attracted to $S_{\varepsilon}^\pm$ and repelled from $S_{\varepsilon}^\alpha$ at fast exponential rates (see for eg. [22] for a derivation of estimates using the Fenichel normal form).

The Koper model is defined by

\[ (3.1) \]

\[
\begin{align*}
\varepsilon_1 \dot{u} &= kv - u^3 + 3u - \lambda, \\
\dot{v} &= u - 2v + w, \\
\dot{w} &= \varepsilon_2(v - w),
\end{align*}
\]

where $\varepsilon_1, \varepsilon_2, k,$ and $\lambda$ are parameters. If we set $\varepsilon_1 \ll 1$ and $\varepsilon_2 = 1$, then the system has the two slow variables $v$ and $w$ and one fast variable $u$.

After the affine coordinate change defined by $(x, y, z) = ((u - 1)/3, (kv - \lambda + 2)/27, 2v - w - 1)/3$ [7], scaling time by $-k/9$ and the substitutions

\[ (\varepsilon, a, b, c, \nu) = (-k\varepsilon_1/81, 18/k, 81\varepsilon_2/k^2, -9(\varepsilon_2 + 2)/k, (3\lambda - 6 - 3k)\varepsilon_2/k^2), \]

the Koper model becomes a parametric subfamily of (3.1), with parameters satisfying the equation

\[ (3.3) \]

\[ 2b + a(a + c) = 0. \]

Note that the above parametric equation corrects a sign error in Desroches et al.[7]. We work henceforth with the Koper model in the form given by (3.1).

We consider only parameter sets of (3.1) satisfying the following conditions: (i) there is a folded node $n = (0, 0, 0) \in L_0$, (ii) exactly one equilibrium point $p_{eq}$ exists in the full system with $\nu = O(\varepsilon)$, with a pair of complex conjugate eigenvalues $\rho \pm i\omega$ and one real eigenvalue $\lambda$, (iii) the stable manifold $W^s$ of $p_{eq}$ is one-dimensional and the unstable manifold $W^u$ of $p_{eq}$ is two-dimensional ($\lambda < 0 < \rho$). Our notation for $W^s$ and $W^u$ hides their dependence on the parameter values.

We comment on the $\nu = O(\varepsilon)$ requirement in (ii) above. In this regime, small-amplitude oscillations may be due to intersections of the attracting and repelling slow manifolds as they twist around each other in the vicinity of the folded node [7, 30] or to the spiraling of trajectories near the unstable manifold of the equilibrium or both. We find that the homoclinic orbits we seek pass through the folded node region, so that the interactions of $W^s$ and $W^u$ with the slow manifolds of the system play a significant role in their existence. In particular, the homoclinic orbits we locate contain segments that lie close to the intersection of $W^u$ with the repelling slow manifold $S_{\varepsilon}^\alpha$, similar to the homoclinic orbits that form the traveling wave profiles for the FitzHugh-Nagumo equation [19], a system with one slow variable and two fast variables. The homoclinic orbits also contain segments where $W^s$ lies close to $S_{\varepsilon}^{\alpha+}$.

The folded node of (3.1) occurs at the origin, independent of the parameters. To focus upon trajectories that pass through a folded node region before encountering the equilibrium, we use the position of the equilibrium as an alternative to the parameter.
Fig. 1: The image of a square of initial conditions in $a$ and $b$ under the shooting function $\psi_{p_{eq}}$ in the surface $\Sigma_0 = \{z = 0\}$. The location of the Shilnikov homoclinic orbit is found to be at $\tilde{\alpha} \approx (0.01, -0.03, -0.2515348, -1.6508230, 1)$. Integrations were performed for a square grid of initial conditions specified by $a \in [\tilde{a} - 2 \times 10^{-6}, \tilde{a} + 2 \times 10^{-6}]$ and $b \in [\tilde{b} - 2 \times 10^{-6}, \tilde{b} + 2 \times 10^{-6}]$.

This position is given by $p_{eq} = (x_{eq}, x_{eq}^2 + x_{eq}^3, x_{eq})$, so $\nu = -x_{eq}[a + bx_{eq}(x_{eq} + 1) + c]$ and the family can be parameterized by $(x_{eq}, a, b, c)$ instead of $(\nu, a, b, c)$. With this new parameterization, the equilibrium point remains fixed as the parameters $a$, $b$, and $c$ are varied, while the folded node remains at the origin. In the rest of this paper, we set $x_{eq} = -0.03$.

4. The shooting procedure. The boundary value algorithm HOMCONT [12] was created within the package AUTO [9] to compute homoclinic orbits with a collocation procedure. Nonetheless, we have been unsuccessful in using AUTO or MATCONT [8] to locate a Shilnikov homoclinic orbit in (3.1) when $\varepsilon \ll 1$. The stiffness of the vector field appears to prevent convergence to a homoclinic solution even when very large numbers of collocation points are used. Shooting algorithms are also problematic since trajectories of $W^u$ diverge rapidly from each other near a canard segment of the homoclinic orbit. Thus, parameterizing trajectories in $W^u$ by an angular variable and varying this angle of an initial point is not well-suited to locating a Shilnikov homoclinic orbit because trajectories in $W^u$ are extremely sensitive to the angle of an initial condition near $p_{eq}$ (see Fig. 7). This issue suggests a different shooting strategy than varying this angle. Instead, we define an angular variable, $\theta$ that parameterizes trajectories in $W^u$ smoothly (including as a function of the parameters) and regard it as an additional parameter for the system. So $W^u = \bigcup_{\theta} W^u_{\theta}$. The shooting procedure can then fix $\theta$ and use another parameter in the search for a homoclinic orbit that contains $W^u_{\theta}$.

Our extended family has the six dimensional parameter space $\bar{P}$ with coordinates $(\alpha, \theta)$. The homoclinic submanifold of the extended parameter space persists as a codimension-one object [18]. To obtain defining equations $\psi : \bar{P} \to \Sigma$ for this manifold, we choose the surface of section $\Sigma$ defined by $z = 0$, set $s^u$ to be the first
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Fig. 2: Homoclinic orbit (blue curve) to $p_{eq}$ specified by the parameters $\theta = 0$ and $\tilde{\alpha}$ in Fig. 1. The critical manifold $C = \{y = x^2 + x^3\}$ is given by the light green manifold in (a) and its $xy$-projection is given by the black dashed curve in (b).

intersection (in backward time) of $W^s$ with $\Sigma$, $u^0_\theta$ to be the first intersection (in forward time) of $W^u_\theta$ with $\Sigma$ and $\psi(\alpha, \theta) = s^\alpha - u^0_\theta$. The relation $\psi(\alpha, \theta) = 0$ defines a four dimensional submanifold $\bar{H}$ of $\bar{P}$. The projection of $\bar{H}$ to $P$ is the homoclinic manifold $H$ consisting of parameters for which (3.1) has a homoclinic orbit.

Approximations to $s^\alpha$ and $u^0_\theta$ are obtained by numerically integrating trajectories with initial conditions in the linear stable and unstable subspaces of $p_{eq}$. Denoting these approximations by $\tilde{s}^\alpha$ and $\tilde{u}^0_\theta$, the formula $\tilde{\psi}(\alpha, \theta) = s^\alpha - \tilde{u}^0_\theta$ approximates the defining equations. Previous studies [3, 4, 27, 28] analyze the convergence of the solutions $\tilde{\alpha}$ of $\tilde{\psi} = 0$ as the distance of the initial conditions to $p_{eq}$ tends to 0. Hyperbolicity of the fixed point, which is satisfied by $p_{eq}$, is required for these estimates.

We now reduce the number of active parameters by fixing $\varepsilon = 0.01$, $\theta = 0$, $c = 1$, and $x_{eq} = -0.03$. Note that our choice of $\Sigma$ as the hyperplane $z = 0$ is motivated by the complicated dependence of $W^s$ on the parameters [16]. This section contains the folded node and is close enough to the equilibrium point that small changes in $\alpha$ do not produce large jumps in $s^\alpha$. These choices leave $a$ and $b$ as active parameters to vary in $P$ to locate an approximate homoclinic orbit by solving the approximate defining equations $\tilde{\psi} = 0$.

Fig. 1 illustrates the regularity of the defining equations $\tilde{\psi} = 0$ on a small rectangle $A \subset R^2$ of initial conditions in the space of active parameters $(a, b)$. Blue dots represent the images of a $5 \times 5$ lattice of points in $A$ under the shooting function $\tilde{\psi}$. The data indicate that $\tilde{\psi}$ is close to affine and regular on $A$ and that its image contains the point $(0, 0)$, implying that there exists $(\tilde{a}, \tilde{b}) \in A$ with $\tilde{\psi}(\tilde{a}) = 0$, where $\tilde{\alpha}$ is the parameter set with second and third components given by $(\tilde{a}, \tilde{b})$. This in turn implies the existence of a Shilnikov homoclinic orbit, depicted by the blue curve in figure 2.

5. Continuation of the homoclinic orbit. Continuation algorithms [26] are widely used to find curves of bifurcations in multi-dimensional parameter spaces. Here, the goal is to find an intersection of the homoclinic manifold $H$ with the subfamily of
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Fig. 3: A portion of $H$ created with continuation. The portion was projected from $(\theta, a, b, c)$ space to $(a, b, c)$ space. The apparent curves show the results of continuation in the direction of one of the two nullvectors of $J = D\xi$.

(3.1) that yields the Koper model. The tangent space to $H$ is the null space of $D\psi$. We estimate $D\psi$ with a central finite difference method to provide starting data for continuation calculation of curves on $H$. These are iterative calculations that use a predictor-corrector algorithm to compute a sequence of parameter values $\alpha_j$ on $H$:

1. **Prediction step:** Compute $z^0 = \alpha^j + hv^j$, where $v^j \in \text{null}(D\psi)$ and $h$ is our chosen stepsize.

2. **Correction step:** Choose a tolerance $\delta$ and iteratively compute $z^{k+1} = z^k - (D\psi)^+(z^k)\psi(z^k)$, where $J^+$ is the Moore-Penrose pseudoinverse matrix of $J$ defined by $J^+ = J^T(JJ^T)^{-1}$.

3. **Stopping criterion:** Stop when $\|z^{k+1} - z^k\| < \delta$ and let $\alpha^{j+1} = z^{k+1}$.

We fix $\varepsilon = 0.01$, $x_{eq} = -0.03$ and $\theta$, and then use $c$ as the third active parameter in addition to $a$ and $b$ in the continuation calculation of a curve on $H$. Fig. 3 shows a computation of a patch of $H$ in $(a, b, c)$ space with different curves corresponding to different values of $\theta$. Fig. (4a) shows a transversal intersection of one of these curves on $H$ with the Koper manifold given by Eq. 3.3. The affine transformations relating (3.1) to the Koper model will rescale and shift the homoclinic orbit (shown in Fig. (4b)) while preserving its topological structure.

**6. Invariant Manifolds.** As $\varepsilon \to 0$, trajectories for (3.1) have singular limits consisting of concatenations of fast segments ("jumps") parallel to the $x$-axis and segments that are trajectories of the slow flow on the critical manifold. Transitions from slow to fast segments in the singular limit trajectories can occur at folds or anywhere along a slow segment on the repelling sheet of the critical manifold. The slow trajectory segments are contained in invariant manifolds that are approximated
Fig. 4: (a) Continuation of a curve (blue) on $H'_0$ through a local patch of the Koper manifold (yellow) defined by Eq. 3.3. (b) A comparison of the homoclinic orbit defined by the parameter set $\tilde{\alpha}$ (blue curve) and the homoclinic orbit lying in the Koper subfamily (red curve) obtained via the continuation in (a), with parameter set $\beta = (\varepsilon, a_\beta, b_\beta, c_\beta) \approx (0.01, -4.16165, 2.891404, 5.725663)$. The eigenvalues of $p_{eq}$ for the Koper homoclinic are $(\rho \pm i\omega, \lambda) \approx (0.790204 \pm 8.482321i, -1.576071)$, so $p_{eq}$ satisfies the Shilnikov condition.

by invariant manifolds appearing in the singular limit $\varepsilon \to 0$ of the system. These invariant manifolds provide a substrate for theoretical analysis of the homoclinic orbits of (3.1).

Starting at the equilibrium point, the singular limit of the homoclinic orbits can be decomposed as follows:

- An initial segment which lies in the intersection of $W^u(p_{eq})$ and the repelling sheet $S'_0$ of the critical manifold.
- A jump from $S'_0$ to the attracting sheet $S'^{-}_0$ of the critical manifold.
- A slow trajectory on $S'^{-}_0$ that ends at the fold at $x = -2/3$.
- A jump from the fold of $S'^{-}_0$ to $S'^{+}_0$.
- A slow trajectory that follows $S'^{+}_0$ to the folded node point at the origin.
- A slow trajectory that connects the origin to $p_{eq}$ on $S'_0$.

We want to investigate the persistence of each of the transitions between items on this list for the Shilnikov homoclinic orbit defined by the parameter set $\beta$ (Fig. (4)). In the next section, we further discuss geometric models for the flow maps from one transition to the next.

The unstable manifold $W^u(p_{eq})$ and the repelling slow manifold $S'^+_0$ are each two dimensional, so they can intersect transversely along a trajectory. Moreover, $W^u(p_{eq})$ satisfies the conditions for it to be a sheet of the strong unstable foliation of $S'_0$. Tiny variations in initial conditions on $W^u(p_{eq})$ yield trajectories that turn abruptly at different heights, as illustrated in Fig. (7). This fast portion of $W^u(p_{eq})$ turns again to follow $S'^{-}_0$ exponentially closely, then jumps to $S'^{+}_0$ and follows $S'^{+}_0$ to the folded node. The singular limits of these transitions are given by smooth one dimensional maps whose composition with one another maps a segment of $W^u(p_{eq})$ to a section of $S'^{-}_0$ that passes through the folded node.
The folded node point is an equilibrium of the desingularized slow flow ((7.1))
, which reverses the orientation of trajectories on the repelling sheet of the critical
manifold. Therefore, some slow trajectories that arrive at the folded node are limits of
trajectories that pass through this region and continue as canards along the repelling
sheet. This creates a lack of uniqueness in the singular limit: as shown in Fig. (6), a
sector of trajectories on the attracting sheet arrives at the folded node and a sector
of trajectories on the repelling sheet leaves the folded node. One of these proceeds
to the equilibrium point and is the limit of the stable manifold of the equilibrium.
However, when $\varepsilon > 0$, the geometry at the folded node becomes much more compli-
cated. Continuations of the attracting and repelling slow manifolds $S_{\varepsilon}^{a+}$ and $S_{\varepsilon}^{a-}$ form
spiral surfaces that wrap around a maximal canard and intersect each other in a set
of secondary canards.

The homoclinic orbit exists when the one dimensional stable manifold $W^s(p_{eq})$
meets trajectories that jump from $S_{\varepsilon}^r$ to $S_{\varepsilon}^{a-}$, flow along $S_{\varepsilon}^{a-}$ to its fold, jump again to
$S_{\varepsilon}^{a+}$, and finally flow along this manifold to the folded node region. This only happens
when the parameter values lie in the homoclinic submanifold $H$ of the parameter
space. We can visualize how this happens by looking at intersections of $W^s(p_{eq})$ and
$W^u(p_{eq})$ in the cross-section $z = 0$ that passes through the folded node. On this
cross-section, $W^s$ sweeps out a curve $C$ and $S_{\varepsilon}^a$ sweeps out a two-dimensional surface
$S$ as the parameter $a$ is varied. Fig. ((8)) shows this intersection in $(x, y, n)$ space
where the local coordinate $n$ is defined via $(x, y, a) \cdot \eta = n$ and $\eta$ is a unit vector normal
to a small patch of $S$. This choice of coordinates increases the angle of intersection
that occurs in $(x, y, a)$ space. As $a$ varies, the surface swept out by $W^u(p_{eq})$ intersects
the curve swept out by $W^s(p_{eq})$ transversely, demonstrating that the solution of the
defining equation for the homoclinic orbit is regular.

It is difficult to directly compute the relevant portion of $W^u(p_{eq})$ since it contains
canards. The two-dimensional subset of $S_{\varepsilon}^r$ we want to compute consists of trajectories
which leave the fixed point $p_{eq}$ along its unstable manifold and follow the repelling
slow manifold $S_{\varepsilon}^r$ up to some height $y$, before finally jumping across to $S_{\varepsilon}^{a-}$. In order
to locate this part of $S_{\varepsilon}^r$, we first integrate backwards a line of initial conditions at a
particular height lying midway between the jump from $S_{\varepsilon}^r$ to $S_{\varepsilon}^{a-}$. These backward
Fig. 6: Nonunique singular cycles of the desingularized slow flow ((7.1)). Recall that stability is reversed on $S_r^0$ because of the time reparametrization used to desingularize the slow flow. (a) The stable manifold (red curve) of the equilibrium point (blue square) extended up $S_r^0$ to the jump curve $x = -2/3$ (dashed purple line). (b) The projection of the stable manifold (black curve) onto $S_{a-}^0$. Forward trajectories (red region) having initial conditions on the projection flow forward to the jump curve. (c) Forward trajectories after jumping to $S_{a+}^0$. at $x = 1/3$ (green dashed line). These lie inside the wedge of trajectories, defined by the two black curves, that flow toward the folded node (orange square) without jumping at $x = 0$. The upper black curve is the trajectory with initial condition $(1/3, 1/3)$. The lower black curve is the strong stable manifold of the folded node.
trajectories jump to \( S_r^\varepsilon \) and flow along it before turning along one of the two branches of \( W^s \) as they approach the equilibrium point. Since trajectories lying in \( W^u \) separate those trajectories which follow the two branches of \( W^s \), we can locate the trajectory \( \gamma \) in \( S_r^\varepsilon \cap W^u \) with a bisection method. We then compute an approximation to the strong unstable manifold of \( \gamma \) by integrating forward points on either side of \( W^u \) that lie close to \( \gamma \). This strategy relies on the fact that \( W^u \) approximates a leaf of the strong unstable foliation of \( S_r^\varepsilon \) for the canard trajectory in \( S_r^\varepsilon \cap W^u \). As discussed in Sec. 4, the resulting heights of the trajectories are extremely sensitive to the angle as illustrated in Fig. (7). Since the calculation of \( W^u \) is lengthy and indirect, we located the homoclinic orbit parameters by instead finding intersections of \( W^s(p_{eq}) \) with \( S_r^{a+} \). Fig. (5) shows that as we vary the parameter \( a \), the intersections of \( W^s(p_{eq}) \) with \( S_r^{a+} \) are similar to those of \( W^u \) and \( W^s \) shown in Fig. (8). As before, this figure locates the transversal intersections of \( W^u \) and \( W^s \) on the surface of section specified by \( \{ z = 0 \} \). Trajectories lying on \( S_r^\varepsilon \cap W^u \) sweep out a two-dimensional surface \( S' \) and the stable manifold sweeps out a curve \( C \). The objects \( S' \) and \( C \) intersect transversely (Fig. 8).

7. Return maps. We now turn to a study of the return map to a suitably chosen cross-section near the Shilnikov homoclinic orbit we have found. The classical analysis of Shilnikov begins with a homoclinic orbit of a three dimensional vector field that has a very special form. First, it is assumed that the vector field is linear in a neighborhood \( U \) of an equilibrium \( p \) and that the eigenvalues \( \lambda, \rho \pm i\omega \) at \( p \) satisfy \( |\rho/\lambda| < 1 \). Cross-sections \( \Sigma_1 \) and \( \Sigma_2 \) are chosen in \( U \), and the flow map from \( \Sigma_1 \) to \( \Sigma_2 \) is computed explicitly. The second assumption is that the “global return” from \( \Sigma_2 \) back to \( \Sigma_1 \) is an affine map. The return map obtained by composing these two flow maps is then proved to have hyperbolic invariant sets. Since hyperbolic invariant sets persist under perturbation of the vector field, homoclinic orbits of vector fields that do not have this special form still have nearby hyperbolic invariant sets. In particular, the Shilnikov analysis applies to the homoclinic orbits of (3.1) with the parameter set \( \beta \).
Fig. 8: Transversal intersection in \( \{z = 0\} \) of the surface \( S' \) (green) swept out by \( S'_r \cap W^u \) and the curve \( C \) (red) swept out by \( W^s \) as the parameter \( a \) is varied. The intersection corresponds to the homoclinic orbit defined by \( \beta \) as given in Fig. 4. Integrations were performed for five equally spaced values of \( a \in [a_{\beta} - 3 \times 10^{-5}, a_{\beta} + 3 \times 10^{-5}] \). Intersections of \( W^u \) (blue curves) for different values of \( a \) are shown.

Fig. 9: A segment of \( S^{a^+} \) on \( \{z = 0\} \) (blue curve) and its first return (red points). The first intersection of \( W^s \) with \( \{z = 0\} \) is depicted by the green square. Parameter set is \( \beta \).

(As in Fig. 4). However, we expect that two aspects of the slow-fast structure of (3.1) may significantly distort the “standard” Shilnikov return map: (1) the folded node region may introduce additional twisting of the flow near the homoclinic orbit, and (2) the strong attraction and repulsion to the slow manifolds might make the global return map from \( \Sigma_2 \) to \( \Sigma_1 \) almost singular. We investigate these issues, producing modifications of the Shilnikov return map suitable for the homoclinic orbits we have located in (3.1).

We consider the parameter set \( \beta \) (as in Fig. 4) and analyze the returns of a thin strip \( \Sigma \) near \( S^{a^+} \cap \{z = 0\} \). Exponential contraction of the flow onto \( S^{a^+} \) suggests that \( \Sigma \) may be mapped into itself in the vicinity of its intersection with
the homoclinic orbit. Numerical computations suggest that this does happen (Fig. (9)). Approximating $\Sigma$ by a small segment $I$ parametrized by the $x$-coordinate, the corresponding one-dimensional approximation to the return map $R : I \to I$ reveals complicated dynamics (Fig. (10a)). In particular, we find a sequence of fixed points in steep portions of the map $R$ that accumulate at the homoclinic orbit intersection. This agrees with previous analyses of homoclinic orbits to spiraling equilibrium points that identified a countable number of periodic orbits of decreasing Hausdorff distance to the Shilnikov homoclinic orbit [29, 14]. The novel behavior here is that these fixed points lie in steep portions of the return map whose trajectories contain canard segments (Fig. (10b)). These periodic orbits may have additional twists (small-amplitude oscillations) associated with reinjection into the folded node region, in addition to the spiraling local to the equilibrium point. We explain this distinction with a closer examination of the slow flow equations.

The desingularized two-dimensional slow flow of (3.1) is defined by

$$
\begin{align*}
\dot{x} &= z - x, \\
\dot{z} &= -(2x + 3x^2)(\nu + ax + b(x^2 + x^3) + cx).
\end{align*}
$$

At the parameter set $\beta$, the folded node has eigenvalues $w_1 \approx -0.920102$ and $w_2 \approx -0.0798982$. Results of Wechselberger [30] relate the ratio of eigenvalues $\mu = w_1/w_2$ to the number of intersections of the (extended) attracting and repelling slow manifolds near the folded node. Thus we estimate the number of intersections to be $1 + [(\mu - 1)/2] = 6$. However, since the equilibrium point lies in the intersection of the extended slow manifolds, $S_r^{\epsilon}$ has an infinite number of turns that yield a countable number of intersections (Fig. (11)). The issue of concern here is whether the twisting at the folded node contributes significantly to the geometry of the return map near the homoclinic orbit.

Since trajectories beginning in $I$ have canard segments when $I$ intersects $S_r^{\epsilon}$, we examine the resulting distortion by focusing on a section closer to the equilibrium point. Fig. (12) shows not only that $I$ intersects $S_r^{\epsilon}$ countably many times near to the equilibrium point, but also that canard lengths of forward trajectories are organized smoothly in neighborhoods of $S_r^{\epsilon}$. The property of countable intersections is explained by the “local” Shilnikov map (in reverse time) applied to initial points in $S_r^{\epsilon}$. Backwards trajectories flowing past the equilibrium point spiral very close to $W^s$ by the time the trajectory exits a neighborhood of the equilibrium. The distribution of canard segment lengths implies that the return map of $I$ stretches and folds subsets depending on how the subsets straddle the spiral of the repelling slow manifold.

We get additional insight into the return map from the Exchange Lemma [23]. This result analyzes the Jacobian of a flow map for trajectories that jump from a repelling slow manifold. The simplest example is the system

$$
\begin{align*}
\varepsilon \dot{x} &= 1 \\
\dot{y} &= \lambda y.
\end{align*}
$$

The flow map of this system from the section $x = 0$ to the section $y = 1$ is given by $x(1) = \frac{1}{\lambda} \log y(0)$, with derivative $\left(\frac{1}{\lambda y(0)}\right) = \frac{-1}{\lambda} \exp(-\lambda x(1))$. Thus the derivative of the flow map grows exponentially with the distance a trajectory flows along the repelling manifold. For us, this is a source of stretching in the global return map of the system (3.1). So long as the projections of trajectories on $S_r^{\epsilon}$ to $S_r^{a-\epsilon}$ and $S_r^{a+\epsilon}$ along
Fig. 10: (a) The return map $R$ of points in $I$, where both $(x, y) \in I$ and $R(x, y) \in R(I)$ are parametrized by their $x$-coordinates. Points $(x, R(x))$ lie on the solid blue curve, and the fixed points that also lie on the line $x = R(x)$ (dotted black) belong to periodic orbits that intersect the cross section $z = 0$ just once. (b) Periodic orbit corresponding to the fixed point $p \approx -5.18996 \times 10^{-4}$ of the map $R$. Parameter set is $\beta$.

Fig. 11: Intersection of the extensions of $S^a_{\varepsilon}$ (red) and $S^r_{\varepsilon}$ (blue) in the section $\{z = 0\}$. Inset shows magnification of spiraling of $S^r_{\varepsilon}$ near the first intersection of $W^s$ with $\{z = 0\}$. Parameter set is $\beta$. 
the fast direction are transverse to the trajectories on these attracting manifolds, the stretching is maintained as part of the global return.

Of course, there is also fast contraction to the attracting manifolds as well. Unless contraction of the flow along $S^a_{\varepsilon}$ at the folded node dominates this stretching, we can expect that the global return map to be approximately a rank one map of large norm for the trajectories that have longer canards. This is apparent in the spikes of Figure 10.

We now verify the claim that stretching is maintained in the global return map. Define the vectors $u_1, u_2,$ and $s$ so that $E^u = \text{span}\{u_1, u_2\}$ and $E^s = \text{span}\{s\}$. Fix two compact, planar cross-sections $\Sigma_0$ and $\Sigma_1$, transverse to $W^u$ and $W^s$, respectively. In particular, $\Sigma_0$ is spanned by $u_1$ and $s$, and $\Sigma_1$ lies above $p_{eq}$ and is spanned by $u_1$ and $u_2$. The global return map $R : \Sigma_0 \to \Sigma_0$ can then be decomposed into the two maps $\varphi : \Sigma_0 \to \Sigma_1$ and $\psi : \Sigma_1 \to \Sigma_0$, so that $R = \psi \circ \varphi$.

We compute $D\varphi(p)$ with central differences, where $p$ is a $1^{-10}$ perturbation of the point where the homoclinic orbit intersects $\Sigma_0$. The matrix $D\varphi(p)$ has singular values $\sigma_1 \approx 1.53064$ and $\sigma_2 \approx 2.66223 \cdot 10^{-7}$, indicating that the global part of $R$ is close to rank one due to strong contraction onto the attracting slow manifolds.

The Jacobian $D\psi$ of the local part of the return map is approximated analytically. First, transform coordinates with the real Jordan form $P^{-1}JP = J'$, where $J$ is the Jacobian of (3.1) at $p_{eq}$. We denote transformations of variables $x$ and maps $\zeta$ by...
primes $x'$ and $\zeta'$. Thus, $u_1'$ and $u_2'$ are parallel to the $x'y'$-plane and $s'$ is parallel to the $z'$-axis. Following Shilnikov, $\psi': \Sigma_1' \to \Sigma_0'$ and its derivative $D\psi'$ are approximated explicitly from the normal form of the spiraling equilibrium point:

\begin{equation}
\psi'(x',y') = (r e^{\rho \theta / \omega}, d e^{-\lambda \theta / \omega}),
\end{equation}

where $r = \sqrt{x'^2 + y'^2}$, $\tan \theta = y'/x'$, and $d$ is the height of $\Sigma_1'$ above $(0, 0, 0)$.

We recover the Jacobian $D\psi$ by transforming $D\psi'$ to the original coordinates on the cross-sections $\Sigma_{0,1}$. By the chain rule, we have $DR(p) = D\psi(\phi(p)) \circ D\phi(p)$, with eigenvalue magnitudes $|\lambda_1| \approx 80166$ and $|\lambda_2| \approx 2 \times 10^{-16}$. We have not tried to confirm the relative accuracy of the small eigenvalue, but clearly it is very small. Note also that the stretching factor in the local map can be shown to become unbounded by picking points approaching the stable manifold on $\Sigma_1$ and a sequence of cross-sections $\Sigma_0$ with decreasing heights. These points spiral out along the unstable manifold. The number of turns $n$ that the trajectory makes before intersecting $\Sigma_0$ determines the appropriate solution of the multivalued function $\theta = \arctan(y'/x') + 2n\pi$ in (7.2). The effect on the resulting Jacobian matrix $D\psi'$ is multiplication by a diagonal matrix with entries $e^{2n\pi \rho / \omega}$ and $e^{-2n\pi \lambda / \omega}$.

Summarizing, the main difference between the geometry of the homoclinic orbit of this system and homoclinic orbits to spiral equilibria in systems with a single time scale is that the global return is almost singular while still stretching along its image. Combined with the local flow past the equilibrium point, we approximate the return map by an almost rank one map that has an infinite number of horseshoes.

8. Concluding Remark. We have given a fairly complete description of a Shilnikov homoclinic orbit in the Koper model, and we have verified numerically its structural stability by locating transversal intersections of the slow manifolds appearing in the model. The model is only moderately stiff in the regime in which we did this, raising the question as to whether the qualitative structure of the homoclinic orbits remains unchanged as one approaches the singular limit of the system. We continued the homoclinic orbit in the direction of decreasing $\varepsilon_1$ to approach the singular limit. We found no significant changes in the homoclinic orbits in this continuation, but did not get very far. As $\varepsilon_1$ decreases, the scales of the structures appearing near the folded node quickly diminish to the point that the intersections of $W^s$ and $W^u$ cannot be resolved with double precision floating point arithmetic. Moreover, when the scales of the oscillations shrink, plots of the homoclinic orbits reveal less of their structure.

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