Classifying the non-Markovian, non-time-local, and entangling
dynamics of an open qubit system

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We study the dynamical map and master equation describing the evolution of one qubit coupled to another via a family of Hamiltonians with a parity symmetry. We classify the structure of the dynamical map and the non-Markovian and non-time-local aspects of the single-qubit dynamics according to the degree of symmetry and coupling strength in the full Hamiltonian, as well as the initial state of the environment qubit. We demonstrate the relationship between strong-coupling, perfect entanglers, non-Markovian features, and non-time-locality in this simple system. We show that by perturbing the initial environment state, effective time-local descriptions can be obtained that are non-singular however capture essential non-unitary features of the reduced dynamics. These results can inform the construction of effective theories of open systems when the larger system dynamics is unknown.

CONTENTS

I. Introduction \hspace{3cm} 3
A. Master equations for open systems \hspace{3cm} 4

II. The system-environment Hamiltonian and the dynamical map \hspace{3cm} 6
A. Exact, closed dynamics \hspace{3cm} 6
B. Entanglement generation \hspace{3cm} 8
C. The reduced dynamics \hspace{3cm} 9
D. Dynamical map families \hspace{3cm} 13

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III. Features of the reduced dynamics 15
   A. Initial environment dependence 15
   B. Invertibility 16

IV. Master equations 19
   A. Standard master equations 19
   B. Effective master equation 22

V. Conclusions 23

VI. Acknowledgements 25

References 26

A. Derivation of the maximally entangling parameter space 27

B. Divisibility in Channels 28
I. INTRODUCTION

The evolution of quantum systems coupled to unobserved or unobservable degrees of freedom can be much more complex than the evolution of closed systems [1]. Information may flow back and forth between the observed and unobserved parts of the system, leading to equations of motion that may not be local in time, and that give rise to non-unitary, non-Markovian evolution [2]. While formal expressions for the evolution of open systems exist, and exact expressions can be derived in particular cases where the unobserved physics is known, there is not yet a procedure for systematically constructing effective theories of open systems that can encapsulate the full range of possible phenomena.

We use an exactly solvable example of coupled qubits to explore how non-Markovianity and the non-time-local aspects of the master equation for one qubit depend on the initial state of the unobserved qubit and on the symmetries and coupling constants of the full Hamiltonian. We use these relationships to clarify which assumptions about the unobserved environment and its coupling to the observed system would underlie various choices for an effective theory construction of a qubit master equation. The Hamiltonian we use has (1) a conserved quantity, and (2) a block-diagonal structure of two equal-size pieces, allowing additional symmetry structures at special points in parameter space. This example is nearly the simplest non-trivial open system, however it allows us to study the features of the reduced dynamics exhaustively. In particular, we can avoid any limitation of perturbation theory and construct exact, non-perturbative, non-local master equations. Since non-time-local master equations can be particularly difficult to work with, we examine how much of the parameter space requires a time non-local equation, and the perturbations or approximations in both the Hamiltonian or the state of the unobserved qubit that will generate time-local equations of motion.

In the rest of the introduction we briefly review the formalism for open system dynamics and master equations in the context of our goals and model. Then, in Section II, we introduce the two-qubit system used in this paper and derive the dynamical map describing the evolution of one of the qubits when the other is traced out. We also introduce several features that are used to classify the dynamical map including the non-Markovianity, divisibility, and symmetries of the Hamiltonian. Section III derives the conditions for non-time-local dynamics via the non-invertibility of the dynamical map. We determine how this
depends on Hamiltonian parameters, in particular whether it is strongly coupled or not, and
what role the initial environment state plays in non-time-locality. We synthesize many of
the elements of the previous sections to examine exact and approximate master equations
in Section [IV] and conclude in Section [V].

A. Master equations for open systems

For some open systems, the master equation governing the evolution of the density matrix
for the observed system, $\rho_S(t)$, is [3–6]

$$
\partial_t \rho_S(t) = -i[H_{\text{free}}(t) + H_{\text{open}}(t), \rho_S(t)] + \sum_k \gamma_k(t) \left( L_k \rho_S(t) L_k^\dagger - \frac{1}{2} \{L_k L_k^\dagger, \rho_S(t)\} \right).
$$

(1)

Here $H_{\text{free}}(t) + H_{\text{open}}(t) = H_{\text{eff}}(t)$ is the effective Hamiltonian of the system, containing both
the original system Hamiltonian $H_{\text{free}}(t)$ and a piece, $H_{\text{open}}(t)$, generated by the coupling
to an environment. The $L_k$ are operators acting on the system, and the $\gamma_k(t)$ are functions
describing the flow of information between the system and environment. The $\gamma_k(t)$, which
control the subset of all possible operators $L_k$ that appear with non-zero coefficients, depend
on the system-environment coupling and the state of the environment. $H_{\text{eff}}(t)$ and $\gamma_k(t)$ are
given by environment correlation functions calculated using the initial environment state
$\rho_E(0)$.

If the environment and full Hamiltonian are unknown, one might begin constructing an
effective theory for the system by writing all possible $L_k$ and a generic $H_{\text{eff}}$ from the com-
plete set of operators that act on the system. Then, the work in the effective theory comes
in specifying some structure for the dissipation functions $\gamma_k(t)$, determining any approx-
imations that may allow some possible terms in $H_{\text{eff}}$ to be discarded, and in determining
consistency between effects captured in $H_{\text{open}}(t)$ and in the non-unitary part of Eq. (1). Some
broad guidelines for this process are known: the simplest choice would be all $\gamma_k \geq 0$ and
constant, restricting the system to non-unitary however time-independent, Markovian evo-
lution. Time-dependent Markovian dynamics would be described by $\gamma_k(t) \geq 0$ at all times.
Finally a restricted set of non-Markovian dynamics would be captured by considering generic
functions $\gamma(t)$.

However, the most general case allows the master equation to be non-local in time. Then,
in addition to the time-local part one adds an integral term. This is the Nakajima-Zwanzig
\[ \partial_t \rho_S(t) = K_{TL}(t) \rho_S(t) + \int K_{NZ}(t, \tau) \rho_S(\tau) d\tau, \]  

(2)

where the time-local piece, \( K_{TL}(t) \), generates the same action on \( \rho_S(t) \) as given in Eq. (1). The integral is over the history of the evolution \([t_0, t]\) where \( t_0 \) is a time where the system and environment are uncorrelated and \( t \) is the time where one is interested in calculating ensemble averages. In constructing a parameterized effective theory for the open system dynamics, one would like to know how to systematically address whether a non-local equation is necessary. In addition, which qualitative aspects of information flow can be captured in either the \( \gamma(t) \) or the time non-local kernel \( K_{NZ}(t, \tau) \), and how they should be implemented?

To address these questions in a simple case, we explore the relationship between the full Hamiltonian for system and environment, together with the initial state of the environment, to several features of the reduced dynamics. We consider measures of non-Markovianity and the conditions under which time non-local evolution is required. We do this by first computing the exact reduced dynamics via the dynamical map. This is a non-unitary generalization of the time evolution operator, a completely positive and trace preserving (CPTP) map from the initial density matrix to the density matrix at a later time \( t \),

\[ \rho_S(t) = \Lambda(t, 0) \circ \rho_S(0). \]  

(3)

Dynamical maps do not have to be invertible; maps with \( \text{Det}\Lambda(\tau_i) = 0 \) for some times \( \tau_i \) require either a time-local description that diverges at each \( \tau_i \) or a non-time-local integral kernel, as written in Eq. (2). For a simple system, we will use the non-invertibility of the dynamical map to derive the conditions on the full Hamiltonian and the environment that make a non-time-local master equation necessary. We find that, independent of the Hamiltonian, there are always a set of initial environment states which support time-local reduced dynamics. This allows a time-local, approximate, master equation to be constructed by shifting the initial state of the environment. For an initial environment state \( (\rho_{E;\text{NTL}}(t_0)) \) and dynamics that requires a non-time-local piece, there exist states nearby in trace distance norm, \( \{\rho_{E;\text{TL}}(t_0)\} \), which can be used to define a master equation of the form

\[ \partial_t \rho_S(t; \rho_{E;\text{NTL}}(t_0)) = K_{TL}(t; \rho_{E;\text{TL}}(t_0)) \rho_S(t; \rho_{E;\text{NTL}}(t_0)) \]

\[ + \int K_{NTL}(t, \tau; \Delta \rho_E) \rho_S(\tau; \rho_{E;\text{NTL}}(t_0)) d\tau. \]  

(4)
Here the non-time-local component is linear in \( \Delta \rho_E = \rho_{E;\text{NTL}}(t_0) - \rho_{E;\text{TL}}(t_0) \), and the integral is over the evolution history \([t_0, t]\).

II. THE SYSTEM-ENVIRONMENT HAMILTONIAN AND THE DYNAMICAL MAP

This section introduces the family of two-qubit Hamiltonians we will consider. The model has an associated parity symmetry, which splits the Hamiltonian into two equal-sized blocks. Such models are interesting as they have an intermediate level of symmetry: more than the class with no non-trivial symmetries, however not as much as Hamiltonians that preserve the total angular momentum of the two qubits. Physically, dynamics of the type we use here describe a pair of non-interacting qubits, most clearly seen through a change of meronomic frame \[10\]. We also characterize features of the non-Markovianity of the single-qubit evolution, according to parameter choices in the full Hamiltonian.

A. Exact, closed dynamics

The Hamiltonian that we study is

\[
H = H_{\text{free}} + H_{\text{int}} = \omega_S (\sigma_S^z \otimes 1_E) + \omega_E (1_S \otimes \sigma_E^z) + \kappa_{SE} (\sigma_S^y \otimes \sigma_E^z) + \kappa_{ES} (\sigma_S^z \otimes \sigma_E^y). \tag{5}
\]

The free parameters \( \omega_S \) and \( \omega_E \) provide the time scales associated to the free dynamics of each individual qubit (with \( \hbar = 1 \)), and the parameters \( \kappa_{SE} \) and \( \kappa_{ES} \) are coupling strengths. \( H \) has a symmetry, \([H, \sigma_S^z \otimes \sigma_E^z] = 0\), so the eigenstates have definite parity associated to \( P_{zz} = \sigma_S^z \otimes \sigma_E^z \). This \( Z_2 \) symmetry allows \( H \) to be split into even and odd blocks, where the even block is spanned by states with correlated spins (e.g. \(|\uparrow\uparrow\rangle\)) and the odd block is spanned by states with anti-correlated spins (e.g. \(|\uparrow\downarrow\rangle\)).

The block diagonalization is achieved by splitting \( H \) into symmetric and anti-symmetric parts under exchange of system and environment operators. Defining new parameters,

\[
2\Delta_\pm = \omega_S \pm \omega_E \\
2\kappa_\pm = \kappa_{SE} \pm \kappa_{ES}, \tag{6}
\]
the Hamiltonian can be written as
\[
H = \Delta_+ (\sigma^z_S \otimes 1_E + 1_S \otimes \sigma^z_E) + \kappa_+ [\sigma^y_S \otimes \sigma^x_E + \sigma^x_S \otimes \sigma^y_E] \\
+ \Delta_- (\sigma^z_S \otimes 1_E - 1_S \otimes \sigma^z_E) + \kappa_- [\sigma^y_S \otimes \sigma^x_E - \sigma^x_S \otimes \sigma^y_E] \\
\equiv H_+ + H_-
\]
where the ± labels correspond to the $\mathbb{Z}_2$ (parity) eigenvalues of each block. The block diagonalization of $H$ introduces a subspace decomposition $\mathcal{H} = Q_+ \oplus Q_-$, where the spaces $Q_\pm$ are spanned by the eigenstates of $H_\pm$. As we show below, there is also a tensor product decomposition for which the two subsystems decouple.

Using $|0\rangle$ and $|1\rangle$ to label the eigenstates of $\sigma^z_S$ and $\sigma^z_E$, the stationary states of $H$ are
\[
|0_+\rangle = \cos \frac{\phi_+}{2} |0_S, 0_E\rangle + i \sin \frac{\phi_+}{2} |1_S, 1_E\rangle \\
|1_+\rangle = \sin \frac{\phi_+}{2} |0_S, 0_E\rangle - i \cos \frac{\phi_+}{2} |1_S, 1_E\rangle \\
|0_-\rangle = \cos \frac{\phi_-}{2} |0_S, 1_E\rangle + i \sin \frac{\phi_-}{2} |1_S, 0_E\rangle \\
|1_-\rangle = \sin \frac{\phi_-}{2} |0_S, 1_E\rangle - i \cos \frac{\phi_-}{2} |1_S, 0_E\rangle,
\]
with eigenvalues
\[
\pm \omega_\pm = \pm \sqrt{\Delta^2_\pm + \kappa^2_\pm}.
\]
In Equation (8), the angles
\[
\phi_\pm = \arctan \frac{\kappa_\pm}{\Delta_\pm}
\]
indicate the relative size of the interaction and free Hamiltonian parameters, and if either provide a dominate contribution to the energy eigenvalues $\omega_\pm$. As long as at least one of the blocks is interacting i.e. $\kappa_\pm \neq 0$, the stationary states in the subsystem decomposition $\mathcal{H} = Q_S \otimes Q_E$ are entangled, and maximally entangled as $\frac{\kappa_\pm}{\Delta_\pm} \to \pm \infty$.

We will see below that the reduced dynamics may acquire a non-time-local component if $\phi_+ + \phi_- \geq \frac{\pi}{2}$. Since this condition requires that one or both of $\kappa_\pm \geq \Delta_\pm$, the part of parameter space where non-time-local master equations can be required coincides with strong coupling, although not all strongly coupled Hamiltonians will have non-time-local dynamics.

So far, we have defined subsystems assuming a laboratory-based notion of locality for operations on qubits, established by the system/environment labels. However, the block-diagonal structure of the Hamiltonian suggests that we also consider a re-organization of
the Hilbert space into degrees of freedom that decouple. That is, we can define qubits $A$ and $B$, with orthonormal basis states $\{1_A, 0_A\}$ and $\{1_B, 0_B\}$, so that $\mathcal{H} = \mathcal{Q}_A \otimes \mathcal{Q}_B$. This is a change of meronomic frame [10]. Explicitly, a (non-unique) mapping between these bases is given by

$$
\begin{align*}
|0_+\rangle &= |0_A, 0_B\rangle \\
|1_+\rangle &= |1_A, 1_B\rangle \\
|0_-\rangle &= |0_A, 1_B\rangle \\
|1_-\rangle &= |1_A, 0_B\rangle.
\end{align*}
$$

One finds $H = \omega_+ \langle 0_+ |0_+ \rangle - |1_+\rangle \langle 1_+ |\rangle + \omega_- \langle 0_- |0_- \rangle - |1_-\rangle \langle 1_- |\rangle$, it is straightforward to change to the frame given in Eq.(11), where

$$
H_{AB} = \omega_A \sigma_A^z \otimes 1_B + \omega_B 1_A \otimes \sigma_B^z. 
$$

Here $\omega_A = \frac{1}{2}(\omega_+ + \omega_-)$, $\omega_B = \frac{1}{2}(\omega_+ - \omega_-)$, and $\sigma_A^z = |0_A\rangle \langle 0_A| - |1_A\rangle \langle 1_A|$, etc. Comparing to Eq.(7), the two terms in Eq.(12) are just the A/B frame expressions for $H_+$ and $H_-$. An advantage of this frame is that it is easy to characterize the regions of parameter space with extra symmetry. A particularly useful region of parameter space is

$$
\omega_+ = \omega_- \Leftrightarrow \omega_B = 0 \ (\text{the degenerate family})
$$

which is symmetric under local rotations on the B qubit subsystem. Since symmetries are clearest in the A/B frame, we will continue to use it for that purpose in Table II D below, where we classify the reduced system dynamics possible with this Hamiltonian, Eq.(5).

### B. Entanglement generation

Returning to the system/environment frame, we characterize the entangling properties of the time evolution. Using the computational basis for system and environment qubits, the time evolution operator, $U(t)$, is

$$
U(t) = 
\begin{bmatrix}
\alpha_+(t) & 0 & 0 & -\beta_+(t) \\
0 & \alpha_-(t) & -\beta_-(t) & 0 \\
0 & \beta_-(t) & \bar{\alpha}_-(t) & 0 \\
\beta_+(t) & 0 & 0 & \bar{\alpha}_+(t)
\end{bmatrix},
$$

(14)
where
\begin{align}
\alpha_\pm(t) &= \cos \omega_\pm t - i \cos \phi_\pm \sin \omega_\pm t, \\
\beta_\pm(t) &= \sin \phi_\pm \sin \omega_\pm t,
\end{align}
(15)
and $\bar{\alpha}_\pm(t)$ is the complex conjugate of $\alpha_\pm(t)$. The pair of functions from each doublet satisfy
\[|\alpha_\pm(t)|^2 + |\beta_\pm(t)|^2 = 1.\]

The functions $\beta_\pm(t)$ are generated by the interaction between system and environment, so they determine both the entanglement and, as we see below, the invertibility of the reduced dynamics for the system. The parameters required for the time-evolution to be (periodically) perfectly entangling can be found using the criteria of Makhlin [11, 12], that the convex hull of the eigenvalues of the matrix $m(U) = (Q^\dagger U Q)^T Q^\dagger U Q$ contains zero, where $Q$ is the operator that changes to the Bell basis.

Evaluating the eigenvalues of $m(U)$, the convex hull condition becomes (see details in Appendix A)
\[\cos^2 \varphi \beta_\pm^2(t) + \sin^2 \varphi \beta_\pm^2(t) = \frac{1}{2},\]
(16)
where $\varphi \in [0, 2\pi)$ parameterizes the remaining convex combinations. This can be satisfied iff the largest of $\beta_\pm^2(t)$ and $\beta_-^2(t)$ is greater than or equal to $\frac{1}{2}$, which will hold at some times as long as $\text{Max}(\phi_+, \phi_-) \geq \frac{\pi}{4}$. This condition on the parameter space of the Hamiltonian is shown in Figure 2 compared with other conditions we derive below related to properties of the reduced dynamics.

C. The reduced dynamics

The reduced dynamics of the system is obtained from the full dynamics by tracing out the environment. Restricting to factorized initial states ensures that the reduced dynamics is completely positive [3]. In that case, the density matrix of the system qubit alone, at time $t$, is
\[\rho_S(t) = tr_E[U(t) \rho_S(0) \otimes \rho_E(0) U^\dagger(t)].\]
(17)

The state of the system qubit at any time can be found using the dynamical map, represented by a matrix acting on the vectorization of the reduced density matrix [2]. For a qubit density matrix this is the Bloch representation,
\[\rho_S(0) = \frac{1}{2}(\mathbb{1}_S + \vec{r}_S(0) \cdot \vec{\sigma}_S),\]
(18)
where \( \vec{r}_S(0) \) is a real, 3-dimensional vector with Euclidean norm \( ||\vec{r}_S(0)|| \leq 1. \) The evolved density matrix can then be written as

\[
\rho_S(t) = \Lambda(t, 0) \circ \rho_S(0).
\] (19)

where \( \Lambda(t, 0) \) is a 4 \( \times \) 4 matrix. Note that as the Hamiltonian is time independent, the dynamical map depends on the initial and final times only through the difference \( t - t_0 \), so for what follows we can simply write \( \Lambda(t) \) and suppress the initial time dependence. It is helpful to further define a 3 \( \times \) 3 matrix with components \( T_{ij} = \Lambda_{ij} \) and a 3-vector \( d^i = \Lambda_{0i} \), so that the action of the dynamical map can be written

\[
\rho_S(t) = \frac{1}{2}(1_S + \vec{r}_S(t) \cdot \vec{\sigma}_S)
= \frac{1}{2} \left( 1_S + \left( \bar{T}(t) \circ \vec{r}_S(0) + \bar{d}(t) \right) \cdot \vec{\sigma}_S \right).
\] (20)

This presentation has the advantage that the trace fixing requirement is immediate. For \( \Lambda \) to be physical, \( \vec{T} \) and \( \vec{d} \) must be real so that \( \rho_S(t) \) is Hermitian. Unlike unitary maps, dynamical maps need not be divisible. For example, one is generally unable to split the time evolution as \( \Lambda(t + \tau) = \Lambda(\tau) \Lambda(t) \). The reduced dynamics further differs from unitary dynamics by exhibiting hallmark features of open systems, including purity change and decoherence.

As with ordinary operators, we can define components of the dynamical map with respect to an operator basis on \( Q_S \),

\[
\Lambda^{ab}(t) = \frac{1}{2} tr_S[\sigma_S^a \Lambda(t) \circ \sigma_S^b] = \frac{1}{2} tr[(\sigma_S^a \otimes 1_E)U(t)(\sigma_S^b \otimes \rho_E(0))U^\dagger(t)]
\] (21)

where \( a \) and \( b \) can take the values 0, \( x, y, z \) where \( \sigma_S^0 = 1_S \). The components can be efficiently computed by also expanding \( U(t) = U_{ab}(t)\sigma_S^a \otimes \sigma_E^b \). Then

\[
\Lambda^{ab}(t) = \frac{1}{2} tr[(\sigma_S^a \otimes 1_E)(U_{cd}\sigma_E^c \otimes \sigma_S^d)(\sigma_S^b \otimes \rho_E(0))(U_{ef}\sigma_E^e \otimes \sigma_S^f)]
= \frac{1}{2} tr[(\sigma_S^a \sigma_S^b \sigma_E^c \sigma_S^d) \otimes (\sigma_E^e \sigma_E^f \sigma_S^g)]U_{cd}U_{ef}r_g(0)
= \frac{1}{2} tr_S[(\sigma_S^a \sigma_S^b \sigma_E^c \sigma_S^d)]tr_E[(\sigma_E^e \sigma_E^f \sigma_S^g)]U_{cd}U_{ef}r_g(0).
\] (22)

Carrying out this calculation, we find the non-zero dynamical map components, in terms of the environment qubit’s initial state, \( r_E(0) \), and the functions \( \alpha_\pm(t) \) and \( \beta_\pm(t) \) appearing in the time evolution operator, Eq.(15), to be
\[
\Lambda^{z_0}(t) = \frac{1}{2} \left[ |\alpha_+(t)|^2 - |\beta_+(t)|^2 - |\alpha_-(t)|^2 \right] r^+_E(0) \\
\Lambda^{xx}(t) = \Re[\alpha_+(t)\alpha_-(t) - \beta_+(t)\beta_- (t)] \\
\Lambda^{xy}(t) = \Im[\alpha_+(t)\alpha_-(t) + \beta_+(t)\beta_- (t)] \\
\Lambda^{xz}(t) = \Re[\alpha_+(t)\beta_-(t) + \alpha_-(t)\beta_+(t)] r^+_E(0) - \Im[\alpha_+(t)\beta_-(t) - \alpha_-(t)\beta_+(t)] r^u_E(0) \\
\Lambda^{yz}(t) = -\Im[\alpha_+(t)\alpha_-(t) - \beta_+(t)\beta_- (t)] \\
\Lambda^{zy}(t) = \Re[\alpha_+(t)\alpha_-(t) + \beta_+(t)\beta_- (t)] \\
\Lambda^{yy}(t) = \Im[\alpha_+(t)\alpha_-(t) - \beta_+(t)\beta_- (t)] \\
\Lambda^{zz}(t) = \frac{1}{2} \left[ |\alpha_+(t)|^2 - |\beta_+(t)|^2 + |\alpha_-(t)|^2 - |\beta_-(t)|^2 \right].
\]

(23)

Note the difference in how the components of the initial environment state enter, with \( r^+_E(0) \) appearing separately from \( r^u_E(0) \) and \( r^u_E(0) \). In the next section we show how the \( P_{zz} \) symmetry of the Hamiltonian affects the dependence of the map components on \( \tilde{r}_E(0) \).

Further, the shift vector \( \vec{d} \) may only have a non-zero \( z \)-component. From the work done in [13], the matrix \( \vec{T} \) may be factored as a diagonal real matrix comprised of the singular values of \( \vec{T} \) and a rotation about the \( \vec{d} \) axis. Given a dynamical map represented using Eq [20] one may freely perform rotations such that \( \vec{T} \) becomes a diagonal real matrix. For our model the decomposition \( \vec{T} = RS \) is achieved with \( \vec{d} \) oriented along the \( z \)-direction and \( R \) being a rotation about the \( z \)-axis. We see the connection that the free system Hamiltonian \( (H_{S,\text{free}}) \) sets the direction of \( \vec{d} \).

\( H_{E,\text{free}} \) also plays a role in the unitality of the reduced dynamics. The dynamical maps generated are unital i.e.,

\[
\Lambda(t)\mathbb{1}_S = \mathbb{1}_S,
\]

(24)

iff \( \vec{d} = \vec{0} \). \( \Lambda(t) \) is always unital when \( r^+_E(0) = 0 \) (see Eq.23), or equivalently the state \( \Omega_E = \frac{1}{2} (\mathbb{1}_S \otimes \rho_E(0)) \) and \( H_{\text{free}} \) are orthogonal operators i.e. \( \text{Tr}(\Omega_E H_{\text{free}}) = 0 \). One sees that the direction and magnitude of \( \vec{d} \) are controlled by the components of \( H_{\text{free}} \). In the next section we show that special sub-regions exist in the parameter space where \( \vec{d}(t) = \vec{0} \) independent of the initial environment state.
For nearly all parameter values in the Hamiltonian, Eq.(7), and initial states, the dynamical map is non-Markovian. This is expected since the system and environment are the same (small) size. The non-Markovianity is diagnosed by information back-flow into the system from the environment [14], with standard indicators being non-monotonicity in the evolution of trace distance and fidelity. For any two states on the reduced system, $\vec{r}_1(t)$ and $\vec{r}_2(t)$, the trace distance ($D$) and fidelity ($F$) are [15],

\[ 2D(\vec{r}_1(t), \vec{r}_2(t)) = ||\vec{r}_1(t) - \vec{r}_2(t)|| \]
\[ 2F(\vec{r}_1(t), \vec{r}_2(t)) = 1 + \vec{r}_1(t) \cdot \vec{r}_2(t) + \sqrt{(1 - r_1^2(t))(1 - r_2^2(t))}. \]

(25)

Generic these measures are oscillatory, and aperiodic unless $\omega_+ = q\omega_-$ for some $q \in \mathbb{Q}$.

The degenerate family, Eq.(13), always has periodic measures of non-Markovianity and is a useful case to look at in more detail. Figure 1 shows the trace distance and fidelity in the degenerate family, demonstrating the non-Markovian character of the reduced dynamics. The dimensionless parameter $\chi = \frac{\omega t}{\pi}$ ($\omega = \omega_+ = \omega_-$) is used to construct these plots, and the environment memory time-scale can be read off as $\tau_{NM} \sim O\left(\frac{\pi}{2\omega}\right)$.

![Figure 1](image-url)

**FIG. 1**: The fidelity and trace distance between time evolved reduced states using the initial conditions $\vec{r}_1(0) = \hat{x}$ and $\vec{r}_2(0) = \cos \theta \hat{x} + \sin \theta \hat{y}$ are plotted. The parameters used are $\omega = \omega_+ = \omega_- = \frac{\sqrt{5}}{2}$, $\tan \phi_+ = \tan \phi_- = 2$, and $\vec{r}_E(0) = \vec{0}$. The non-Markovian nature of the reduced dynamics is evident from the oscillations of $D$ and $F$.

Figure 1 is generated with $\phi_+ = \phi_-$ allowing for interesting features to appear in $F$. If $\phi_\pm$ are perturbed so that $\phi_+ \neq \phi_-$, these interesting features also vanish. First note the plateaus, which indicate there is a unitary phase (i.e. $F=$constant) if we further require that $\vec{r}_E(0) = \hat{x}$. Additionally, preceding the plateaus are discontinuities in $\dot{F}$. In some other finite open systems, such discontinuities indicate dynamical phase transitions [16].

Non-Markovianity of the dynamics has been equated to the indivisibility of the dynamical map into channels (CPTP maps) [14], however the actual relationship is more complicated.
as non-Markovian dynamical maps can be CP divisible \cite{17,19}. To study the divisibility of the dynamical map one looks at the interweaving maps (\(\Phi\)), defined using two times \(\tau_2 > \tau_1 \geq 0\),

\[
\Lambda(\tau_2) = \Phi(\tau_2, \tau_1)\Lambda(\tau_1) .
\] (26)

For invertible reduced dynamics the interweaving map is computed as \(\Phi(\tau_2, \tau_1) = \Lambda(\tau_2)\Lambda^{-1}(\tau_1)\), although \(\Phi\) may still be defined even when \(\Lambda^{-1}\) does not exist \cite{18}.

The positivity of \(\Phi(\tau_2, \tau_1)\) determines the divisibility class of the map \(\Lambda(\tau_2)\), as \(\Lambda(\tau_1)\) is completely positive by construction. It is important to note that divisibility is meant in a holistic sense i.e. the reduced dynamics is considered CP divisible only if \(\Phi(\tau_2, \tau_1)\) is completely positive for all \(\tau_1 < \tau_2\). We demonstrate later that \(\Lambda(\tau_2)\) is always P divisible by showing that Det\(\Lambda(t)\) \(\geq 0\) for all \(t \geq 0\) \cite{19}. The question of CP divisibility is more subtle. In Appendix \[B\] we study the CP divisibility of a unital family of dynamical maps, where we find that there typically exist time intervals s.t. for \(\tau \in [\tau_a, \tau_b]\) the map \(\Phi(\tau_2, \tau)\) fails to be CP.

\textbf{D. Dynamical map families}

We can use properties of the non-Markovianity and symmetry to classify all the dynamics possible with the Hamiltonian in Eq.(5). Table \[ID\] lists the various dynamical map families contained in this model. The largest family is the non-commensurate family (\(\mathcal{N}\)), with two independent frequencies. If we consider the time evolution in the Bloch ball, the trajectories generated are dense for any initial state \(\vec{r}_S(0)\). As mentioned previously, we use the (AB) frame to discuss the symmetries as they are simplest in this frame. For example the symmetries present in the entire family of Hamiltonians are \(\sigma^z_A \otimes 1_B\) and \(1_A \otimes \sigma^z_B\). Of course these are equivalent to \(H\) and \(P_{zz}\), however we find that this frame compresses the discussion of Hamiltonian families that have additional symmetries beyond these two.

The Hamiltonians which generate the family \(\mathcal{D}\) having more symmetries than those corresponding to \(\mathcal{N}\); these Hamiltonians commute with all rotations performed in the B subsystem are symmetries. (The apparent symmetry between A and B in Eq.(12) is broken by the relationship to the S/E frame.) The presence of more symmetries simplifies the time-dependence of the dynamical map, thus less complicated trajectories are generated and the non-Markovian measures are periodic. \(\Lambda \in \mathcal{D}\) are not structurally different than
those in \( \mathcal{N} \) i.e. no additional dynamical maps components are forced to zero. \( \mathcal{D} \) does not have conserved quantities at the level of reduced dynamics.

There are additional subfamilies of \( \mathcal{D} \) that have more restricted structures. These families \( \mathcal{D}_\pm \) satisfy the additional condition that \( \phi_+ = \pm \phi_- \). These conditions are equivalent to setting \( \kappa_+ = \kappa_- \) and either \( \omega_\text{E} = 0 \ (\mathcal{D}_+) \) or \( \omega_\text{S} = 0 \ (\mathcal{D}_-) \). While the Hamiltonians generating \( \mathcal{D}_\pm \) do not have additional symmetries, one is able to swap stationary states between the even and odd blocks performing only a local environment transformations. In \( \mathcal{D}_+ \) the local environment rotation \( \mathbb{1}_\text{S} \otimes \sigma_\text{E}^x \) acts to exchange,

\[
(\mathbb{1}_\text{S} \otimes \sigma_\text{E}^x)|0_+\rangle = |0_-\rangle \tag{27}
\]
\[
(\mathbb{1}_\text{S} \otimes \sigma_\text{E}^x)|1_+\rangle = |1_-\rangle.
\]

\( \Lambda \in \mathcal{D}_+ \) have fewer non-zero components as the conditions \( \omega_+ = \omega_- \) and \( \phi_+ = \phi_- \) imply that \( \alpha_+(t) = \alpha_-(t) \) and \( \beta_+(t) = \beta_-(t) \). These dynamical maps only depend on \( r^E_\text{E}(0) \), thus \( \vec{d} \) is set to zero as it is linear in \( r^E_\text{E}(0) \), thus \( \Lambda \in \mathcal{D}_+ \) are unital. An exchange symmetry also exists for \( \mathcal{D}_- \), which leaves only the \( r^\text{E}(0) \) dependence of the dynamical map. The singular value decomposition of the dynamical map show that these families have an instantaneous conserved quantity, \( \vec{Q}_{\pm}(t) \). That is we can find time-dependent reduced state(s),

\[
\rho_\pm(t) = \frac{1}{2}(\mathbb{1}_\text{S} + \vec{Q}_\pm(t) \cdot \vec{\sigma}_\text{S}), \tag{28}
\]

such that for dynamical maps in \( \mathcal{D}_\pm \) we have that,

\[
\Lambda(t)\rho_\pm(t) = \rho_\pm(t). \tag{29}
\]

Finally there are two remaining Markovian families \( \mathcal{M}_\pm \), containing the maps in \( \mathcal{D}_\pm \) generated using \( \vec{r}_\text{E}(0) = \hat{x} \ (\mathcal{M}_+) \) and \( \vec{r}_\text{E}(0) = \hat{y} \ (\mathcal{M}_-) \). Markovian is meant in the sense that the master equation is of time-dependent Lindblad type [19], which implies that the map \( \Phi(t + \epsilon, t) \) is CP for all \( t \geq 0 \) and for all \( \epsilon \geq 0 \). Note that the dissipator is zero, thus \( \Lambda \in \mathcal{M}_\pm \) are rotations about a time dependent axis. Unitary dynamics is the only kind of Markovian dynamics possible the environment Hilbert space is not big compared to the system Hilbert space. Note that like other Markovian open systems, the initial environment state remains fixed under the time evolution in \( \mathcal{M}_\pm \). As the dynamical map is a rotation in these families, there is an instantaneously fixed frame we denote as \( \vec{Q}^i_\pm(t) \).
| Family                  | Time Evolution | Extra Symmetries       | Unital                        |
|------------------------|----------------|------------------------|-------------------------------|
| Non-Commensurate (N)   | Aperiodic      | None                   | If \( \vec{r}_E(0) = 0 \)  |
| \( \omega_+ \neq q\omega_- \) |                |                        |                               |
| Commensurate (C)       | Periodic       | None                   | If \( \vec{r}_E(0) = 0 \)  |
| \( \omega_+ = q\omega_- \) |                |                        |                               |
| Degenerate (D)         | Periodic       | \( \{ \mathbb{1}_A \otimes \sigma_B^x, \mathbb{1}_A \otimes \sigma_B^y \} \) | If \( \vec{r}_E(0) = 0 \)  |
| \( \omega_+ = \omega_- \) |                |                        |                               |
| Aligned-Degenerate (D±) | Periodic      | \( \{ \mathbb{1}_A \otimes \sigma_B^x, \mathbb{1}_A \otimes \sigma_B^y \} \) | Any \( \vec{r}_E(0) \)     |
| \( \phi_+ = \pm \phi_- \) |                |                        |                               |
| Markovian (M±)         | Markovian      | \( \{ \mathbb{1}_A \otimes \sigma_B^x, \mathbb{1}_A \otimes \sigma_B^y \} \) | Always                       |
| \( \vec{r}_E(0) = \hat{x}(+) \) or \( \hat{y}(-) \) |                |                        |                               |

### III. FEATURES OF THE REDUCED DYNAMICS

In this section we study the invertibility of the dynamical map for the system qubit, and how it depends on the Hamiltonian and the state of the environment. We find that there are in fact three independent ingredients that play a role in defining the invertibility: the strength of the system/environment coupling, the ratio between eigenenergies, and the initial environment state. The fact that these are distinct criteria will be seen from the form of the condition for non-invertibility of the reduced dynamics,

\[
\text{Det}(\Lambda(t)) = 0 \iff \vec{T}_z = \vec{0} \tag{30}
\]

where \( \vec{T}_z \) is the third row of \( \vec{T} \). The three independent components of \( T_z \) that must be zero clarify the separate roles played by the Hamiltonian parameters and initial environment state in determining the presence of non-time-locality.

#### A. Initial environment dependence

Before finding the non-invertibility criteria, we first clarify how the symmetry of the full Hamiltonian simplifies the dynamical map dependence on the initial state of the environment
qubit, $\vec{r}_E(0)$. To that end, define the partial components of the dynamical map by

$$
\Lambda^{ab} = \frac{1}{2} \Lambda^{abc} r_c E(0) \tag{31}
$$

$$
\Lambda^{abc} = \text{tr}[(\sigma^a_S \otimes \mathbb{1}_E)U(t)(\sigma^b_S \otimes \sigma^c_E)U^\dagger(t)].
$$

The parity symmetry $P_{zz} = \sigma^z_S \otimes \sigma^z_E$ essentially halves the number of non-zero $\Lambda^{abc}$, and separates the $xy$ and $z$ components of initial environment state in the dynamical map.

Then, since $P_{zz}$ commutes with $U(t)$ and satisfies $P_{zz}^2 = \mathbb{1}$, we see that

$$
\Lambda^{abc} = \text{tr}[P_{zz}^2(\sigma^a_S \otimes \mathbb{1}_E)U(t)(\sigma^b_S \otimes \sigma^c_E)U^\dagger(t)] = (-1)^{\pi_a + \pi_b + \pi_c} \Lambda^{abc}, \tag{32}
$$

where the $\pi_d$ are defined such that,

$$
P_{zz}(\sigma^e_S \otimes \sigma^f_E) = (-1)^{\pi_e + \pi_f} (\sigma^e_S \otimes \sigma^f_E) P_{zz}. \tag{33}
$$

Considering the cases where either $e=0$ or $f=0$, allows one to speak of the parity of local system and environment operators. For example the parity of the operator $\mathbb{1} \otimes \sigma^f_E$ is determined by $\pi_f$. The condition $\pi_a + \pi_b + \pi_c = 1 \pmod{2}$ if satisfied implies that $\Lambda^{abc}$ vanishes. This is equivalent to the following: given $\Lambda^{ab}$ if the parity of $(\pi_a + \pi_b)$ is even (odd) then the only contributing partial components have $\pi_c$ even (odd). $\Lambda^{zzz}$ also vanishes as a consequence of the parity symmetry. However this is obviously not a consequence of the previous argument. Instead it follows using the $P_{zz}$ symmetry and that $\text{tr}_S(\sigma^z_S) = 0$.

Several additional partial components vanish, although not enforced by Eq.(32). For example,

$$
\Lambda^{xtx} = \Lambda^{xty} = 0 \tag{34}
$$

$$
\Lambda^{ytz} = \Lambda^{yty} = 0
$$

which imposes that the shift $\vec{d}$ must be parallel to the $z$ axis. Finally,

$$
\Lambda^{xzz} = \Lambda^{xyz} = \Lambda^{yzz} = \Lambda^{yyz} = 0. \tag{35}
$$

### B. Invertibility

The invertibility structure of the dynamical map clearly displays the independent roles of the Hamiltonian parameters and the initial environment state in the appearance of time-non-local reduced dynamics. The invertibility is compactly determined by special row structure
of this model, where the third row of $\vec{T}$ satisfies $\vec{T}_z = \vec{T}_x \times \vec{T}_y$. The determinant of the dynamical map is

$$ \text{Det } \Lambda(\tau) = \text{Det } \vec{T}(\tau) = |\vec{T}_z(\tau)|^2 $$

which implies,

$$ \text{Det } \Lambda(\tau) = 0 \iff \vec{T}_z(\tau) = \vec{0}. $$

Starting with the $z$-component, $\Lambda_{zz}(\tau) = 0$ implies

$$ \sin^2 \phi_+ \sin^2(\omega_+ \tau) + \sin^2 \phi_- \sin^2(\omega_- \tau) = 1. \quad (38) $$

Eq.(38) may only be satisfied at a discrete set of times, and only if $\sin^2 \phi_+ + \sin^2 \phi_- \geq 1$. The set of $\phi_\pm$ that satisfy this condition constitute the strong coupling regime of parameter space. Figure 2 shows that this region of parameter space partially overlaps with the space where $U(t)$ is a perfect entangler.

The $(\phi_-, \phi_+)$ plane

![The $(\phi_-, \phi_+)$ plane split into regions based on the non-local properties of the dynamics. The light gray region contains Hamiltonians that are not perfect entanglers, whereas the other regions (black+dark grey) are perfect entanglers. The entire gray region supports time local dynamics. The black region contains the only Hamiltonians that may have Det$\Lambda = 0$.]

However, strong coupling between system and environment is not a sufficient for the map to be non-invertible. Eq.(38) also depends on the energy eigenvalues. For example consider the boundary between the dark gray and black regions of Figure 2 where $\sin^2 \phi_+ + \sin^2 \phi_- = 1$ (excluding the corners where either $\phi_+ = 0$ or $\phi_- = 0$). In this region Eq.(38) is satisfied iff,

$$ \sin^2 \omega_+ \tau = \sin^2 \omega_- \tau = 1. \quad (39) $$
In order for this to be possible integers $k$ and $l$ must exist such that,

$$\frac{\omega_+}{\omega_-} = \frac{2k + 1}{2l + 1}. \tag{40}$$

Defining $\nu$ such that $\omega_+ = (2k + 1)\nu$ and $\omega_- = (2l + 1)\nu$; for any positive integer $n$ a non-invertibility can appear in the reduced dynamics at times given by,

$$\tau_n = \frac{(2n + 1)\pi}{2\nu}. \tag{41}$$

So on this boundary only Hamiltonians with commensurate eigenenergies require time-non-local dynamics (although incommensurate Hamiltonians on the boundary can generate dynamical maps with small determinant). On the other hand, away from the boundary, farther into the blue region, most frequency pairs will generate dynamics with non-time-locality for some initial environment states.

So far, we have considered only one of the conditions for non-invertibility, $\Lambda_{zz}(\tau) = 0$, which imposes conditions on the Hamiltonian that must be satisfied for the map to be non-invertible. However, constraining the Hamiltonian to be in the strong-coupling regime is not sufficient: the initial environment state also plays a role. The class of initial states of the environment that lead to non-time-local dynamics is found by setting the remaining components of $\vec{T}_z(\tau)$ to zero,

$$
\begin{align*}
\Lambda_{zx}(\tau) &= \Lambda_{zxx}(\tau)\hat{x} + \Lambda_{zxy}(\tau)\hat{y} = 0, \\
\Lambda_{zy}(\tau) &= \Lambda_{zyx}(\tau)\hat{x} + \Lambda_{zyy}(\tau)\hat{y} = 0.
\end{align*}
\tag{42}
$$

We can characterize the set of initial states for which non-invertibility will occur by finding the vector $\vec{\eta}(\tau)$ in the $xy$ plane associated to the environment qubit state for which both

$$V(\tau)\vec{\eta}(\tau) = \begin{pmatrix}
\Lambda_{zxx}(\tau) & \Lambda_{zxy}(\tau) \\
\Lambda_{zyx}(\tau) & \Lambda_{zyy}(\tau)
\end{pmatrix} \vec{\eta}(\tau) = 0 \tag{43}
$$

and $\text{Det } V(\tau) = 0$ at some fixed $\tau$. The solution is

$$\vec{\eta}(\tau) = \Lambda_{zyy}(\tau)\hat{x} - \Lambda_{zyx}(\tau)\hat{y}, \tag{44}$$

and the orthogonal direction is

$$\vec{\eta}^\perp(\tau) = \Lambda_{zxx}(\tau)\hat{x} + \Lambda_{zxy}(\tau)\hat{y}. \tag{45}$$
A non-invertibility at time $\tau$ can be removed by shifting the initial state to contain a component in the direction $\vec{\eta}^\perp(\tau)$. Furthermore, there can only be a discrete set of times, $\tau_i < T$, where the condition $\Lambda_{zz}(\tau_i) = 0$ can be satisfied. Assume there are $N$ non-invertible times,

$$0 < \tau_1 < \ldots < \tau_N < T,$$  \hspace{1cm} (46)

with the associated initial environment states that preserve the non-invertibility $\{\hat{\vec{\eta}}_1, \ldots, \hat{\vec{\eta}}_N\}$. Single out $\hat{\vec{\eta}}_1$ and note that so long as $\hat{\vec{\eta}}_1^\perp \cdot \hat{\vec{\eta}}_k^\perp \neq 0$, then the non-invertibility at $\tau_k$ is eliminated by the presence of $\hat{\vec{\eta}}_1^\perp$ in $\vec{r}_E(0)$. The remaining directions all must satisfy $\hat{\vec{\eta}}_1^\perp \cdot \hat{\vec{\eta}}_i^\perp = 0$. Since these initial environment states lie in a two dimensional space, all the remaining $\hat{\vec{\eta}}_i^\perp$ must equal $\hat{\vec{\eta}}_1$. The question of removing all non-invertibilities up to $T$ is then equivalent to finding a $\theta$ such that,

$$\eta_1^\perp \cdot (\cos \theta \eta_1^\perp + \sin \theta \eta_1) \neq 0,$$  \hspace{1cm} (47)

for all $i \in \{1, N\}$.

We have demonstrated that certain off diagonal components (coherences) of the initial environment state in the eigenbasis of the free environment Hamiltonian ($H_{E;\text{free}}$) control the appearance of time-non-locality in the reduced dynamics. In the next section we use this knowledge of the time-local environment states in order to construct exact non-local master equations as well as approximate time-local master equations.

\section*{IV. MASTER EQUATIONS}

We present the standard master equations that can be associated to any $\Lambda(t, 0)$, both local and non-local in time. We find that by changing $\vec{r}_E(0)$, different partitions of the singularity between non-local and local terms can be achieved. We show that we can capture some non-unitarity features of the reduced dynamics just using the local term.

\subsection*{A. Standard master equations}

The dynamical map can be used to construct the time-local, although possibly singular, generator for the master equation

$$\partial_t \rho_S(t) = K_{TL}(t) \rho_S(t) = \dot{\Lambda}(t) \Lambda^{-1}(t) \rho_S(t).$$  \hspace{1cm} (48)
Since the inverse dynamical map is
\[
\Lambda^{-1}(t) = \frac{1}{T_z^2} \begin{bmatrix} T_z^2 & 0 & 0 & 0 \\
-d_z T_z & T_y & T_z \times T_x & T_z \times T_x 
\end{bmatrix},
\]
the time-local generator is expressed in terms of the dynamical map components as
\[
K_{TL}(t) = \frac{1}{T_z^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\
-d_z (\hat{T}_x \cdot \hat{T}_z) & \hat{T}_x \cdot (\hat{T}_y \times \hat{T}_z) & \hat{T}_x \cdot (\hat{T}_z \times \hat{T}_x) & \hat{T}_x \cdot (\hat{T}_z \times \hat{T}_x) 
-d_z (\hat{T}_y \cdot \hat{T}_z) & \hat{T}_y \cdot (\hat{T}_x \times \hat{T}_z) & \hat{T}_y \cdot (\hat{T}_z \times \hat{T}_x) & \hat{T}_y \cdot (\hat{T}_z \times \hat{T}_x) 
-d_z (\hat{T}_z \cdot \hat{T}_z) & \hat{T}_z \cdot (\hat{T}_y \times \hat{T}_z) & \hat{T}_z \cdot (\hat{T}_z \times \hat{T}_x) & \hat{T}_z \cdot (\hat{T}_z \times \hat{T}_x) 
\end{bmatrix}.
\]

Fig. 3 contains plots of Det\(\Lambda\) and Tr\(K_{TL}\) using \(\Lambda(t) \in \mathcal{D}_+\), which demonstrate that singularities in \(K_{TL}\) occur where Det\(\Lambda = 0\).

The generator \(K_{TL}\) can be put into Lindblad form,
\[
\partial_t \rho_S(t) = -i[(\tilde{H}_{\text{eff}}(t) \cdot \hat{\sigma}_S), \rho_S(t)] + \sum_{i=1}^3 \sum_{j=1}^3 \gamma_{ij}(t)(\sigma^i_S \rho_S(t) \sigma^j_S - \frac{1}{2} \{\sigma^i_S \sigma^j_S, \rho_S(t)\}) ,
\]
where \(\tilde{H}_{\text{eff}}(t) = \omega_S \hat{z} + \tilde{H}_{\text{open}}(t)\) generates unitary evolution on \(Q_S\) with a contribution from internal parameters and a portion that knows about the environment and interaction, \(\tilde{H}_{\text{open}}(t)\). The coefficients \(\gamma_{ij}(t)\) are the Lindblad coefficients which generate the non-unitarity that appears in the reduced dynamics.
(a) $\Re \gamma_{xy} = \Re \gamma_{yz}$

(b) $\Im \gamma_{xy} = \Im \gamma_{yz}$

FIG. 4: Plotted are the real and imaginary parts of the off diagonal Lindblad coefficients. Plots are generated using the parameters $\omega_+ = \omega_- = \frac{\sqrt{5}}{2}$ and $\tan \phi_+ = \tan \phi_- = 2$. Included in the plots is the trace distance evaluated using the same parameters and initial states $\vec{r}_1(0) = -\vec{r}_2(0) = \hat{x}$. We include the trace distance to demonstrate that experimentally accessible quantum information measures can constrain the parameters that appear in effective master equations; in this case oscillation frequencies of the Lindblad coefficients.

Obtaining the Lindblad form of the master equation amounts to a change of basis, for example the effective Hamiltonian has components

$$H_{\text{eff}}(t) = \frac{1}{2} \epsilon^{ij} K_{jk}(t),$$

and similar expression exist for the $\gamma_{ij}(t)$, though we do not include them here. Figure 4 shows how we can use the quantities such as the $\mathcal{D}(t)$ to determine the oscillation timescales that appear in the Lindblad coefficients, which is useful to set a scale for environment memory timescales that will inevitably appear in effective master equations.

In the strong coupling region where the time-local description has singular behavior, we instead use an integral master equation known as the Nakajima-Zwanzig equation. The Hamiltonian being time independent allows the NZ generator to take the form $K_{NZ}(t, \tau) = K_{NZ}(t - \tau)$. Thus the NZ equation is a convolution,

$$\partial_t \rho_S(t) = \int_0^t K_{NZ}(t - \tau) \rho_S(\tau) d\tau.$$

The convolution kernel may be obtained from the dynamical map using the Laplace transf-
form of the dynamical map $\Phi(s)$ (as in [20]),

$$\hat{K}_{NZ}(s) = s^{2/4} - \Phi^{-1}(s)$$

(54)

The form of the functions $\alpha_{\pm}(t)$ and $\beta_{\pm}(t)$ restrict the dynamical map components to having few Fourier components. That is each $\Lambda_{ab}$ has a Fourier decomposition of the form,

$$\Lambda_{ab}(t) = C_{ab} + F_{1ab} e^{-i2\omega_+ t} + F_{2ab} e^{-i2\omega_- t} + F_{3ab} e^{-i(\omega_+ + \omega_-) t} + F_{4ab} e^{-i(\omega_+ - \omega_-) t} + G_{1ab} e^{i2\omega_+ t} + G_{2ab} e^{i2\omega_- t} + G_{3ab} e^{i(\omega_+ + \omega_-) t} + G_{4ab} e^{i(\omega_+ - \omega_-) t}.$$  (55)

If $ab$ is even (odd) then the component $\Lambda_{ab}$ involves only even (odd) time dependent functions. It is now simple to go to the frequency domain, where we can write the generic Laplace transformed components,

$$\Phi_{ab}(s) = \frac{C_{ab}}{s} + \frac{F_{1ab}}{s + i2\omega_+} + \frac{F_{2ab}}{s + i2\omega_-} + \frac{F_{3ab}}{s + i(\omega_+ + \omega_-)} + \frac{F_{4ab}}{s + i(\omega_+ - \omega_-)} + \frac{G_{1ab}}{s - i2\omega_+} + \frac{G_{2ab}}{s - i2\omega_-} + \frac{G_{3ab}}{s - i(\omega_+ + \omega_-)} + \frac{G_{4ab}}{s - i(\omega_+ - \omega_-)}.$$  (56)

In principle one can use these formulae to construct the exact non-time-local master equation associated to the reduced dynamics, although the expressions involved are cumbersome. Instead for what follows we devise a method to expand the NZ kernel,

$$K_{NZ}(t - \tau) = \delta(t - \tau) K_{TL}(t) + K_{NTL}(t, \tau).$$  (57)

Such an expansion is available for each time-local initial environment state ($\vec{r}_{TL}$), where the relative importance of the time-local vs non-time-local component is controlled by the magnitude of $\vec{r}_{TL} - \vec{r}_E(0)$.

B. Effective master equation

An exact time local master equation without singularities exists as long as either of the following conditions are met

1. The Hamiltonian parameters are in the weak coupling region, i.e. $\sin^2 \phi_+ + \sin^2 \phi_- < 1$.

2. $\vec{r}_E(0)$ has off diagonal elements in the eigenbasis of $H_{E;\text{free}}$ along certain directions described in Section IIIB.
However, when neither of the above holds only a non-local equation can capture the exact time evolution of the reduced state. Such a master equation can be found using reduced dynamics from the time-local region.

Consider the case where the Hamiltonian parameters are in the strong coupling region and $\vec{r}_E(0) = \vec{r}_{NTL}$, yielding non-invertibilities in the reduced dynamics. By choosing a shifted initial environment state ($\tilde{\vec{r}}_E$), we can engineer the time local component of a non-time local master equation. By using the exact non-time local dynamical map we can also determine the corresponding non-time-local component for the master equation. As the non-time-local piece is linear in $\delta \vec{r} = \vec{r}_{NTL} - \tilde{\vec{r}}_E$, the choice of shifted initial state controls its relevance.

We can choose $\tilde{\vec{r}}_E$ by considering states near to $\vec{r}_E(0)$ that are shifted along the direction(s) $\eta^\perp$ as defined in Eq. (45). The time derivative of $\rho_S(t; \vec{r}_{TNL})$ may be expanded as,

$$
\dot{\rho}_S(t; \vec{r}_{NTL}) = \partial_t \rho_S(t; \vec{r}_E + \delta \vec{r}) = \partial_t \left[ \Lambda(t; \vec{r}_E) + (\Lambda(t; \delta \vec{r}) - \Lambda(t; \vec{0})) \right] \rho_S(0)
$$

$$= K_{TL}(t; \vec{r}_E) \rho_S(t; \vec{r}_E) + \delta \vec{r} \cdot \left[ \frac{\partial \Lambda(t; \delta \vec{r})}{\partial \delta \vec{r}} \right] \rho_S(0) \tag{58}
$$

Note that since the dynamical map is linear in the initial environment state, we can replace $\frac{\partial \Lambda(t; \delta \vec{r}_E)}{\partial \delta \vec{r}}$ in the above equation with $\frac{\partial \Lambda(t; \vec{r}_E)}{\partial \vec{r}_E}$

$$
\dot{\rho}_S(t; \vec{r}_{NTL}) = K_{TL}(t; \vec{r}_E) \rho_S(t; \vec{r}_{NTL}) + \delta \vec{r} \cdot \left[ \frac{\partial \Lambda(t; \vec{r}_E)}{\partial \vec{r}_E} - K_{TL}(t; \vec{r}_E) \frac{\partial \Lambda(t; \delta \vec{r}_E)}{\partial \delta \vec{r}_E} \right] \rho_S(0) \tag{59}
$$

Which can be further simplified noting that $\dot{\Lambda}(t; \vec{r}_E) - K_{TL}(t; \vec{r}_E) \Lambda(t; \vec{r}_E) = 0$. Setting $\Lambda(t, \vec{r}_E) = \tilde{\Lambda}(t)$ we have that,

$$
\dot{\rho}_S(t; \vec{r}_{NTL}) = \tilde{K}_{TL}(t) \rho_S(t; \vec{r}_{NTL}) + \delta \vec{r} \cdot \left[ \frac{\partial \tilde{K}_{TL}(t)}{\partial \vec{r}_E} \tilde{\Lambda}(t) \right] \rho_S(0). \tag{60}
$$

We have now explicitly isolated a time-local component which is evaluated at the same Hamiltonian parameters as $\Lambda(t; \vec{r}_{TNL})$, but uses the initial environment state $\vec{r}_E$ instead of $\vec{r}_{NTL}$. How large $\delta \vec{r}$ is compared with $\vec{r}_{TNL}$ determines the maximum value of Tr$\tilde{K}_{TL}(t)$, which in turn determines how relevant the integral term is in capturing the non-invertibility.

Now the non-local component can be expressed in terms of an integral kernel, expanded
as a power series in $\delta \vec{r}$,

$$\dot{\rho}_S(t; \vec{r}_{\text{NTL}}) = \dot{K}_{\text{TL}}(t)\rho_S(t; \vec{r}_{\text{NTL}})$$

$$+ \delta \vec{r} \cdot \left[ \frac{\partial \dot{K}_{\text{TL}}(0)}{\partial \vec{r}_E} \rho_S(t; \vec{r}_{\text{NTL}}) - \int_0^t \frac{\partial}{\partial \tau} \left[ \frac{\partial \dot{K}_{\text{TL}}(t-\tau)}{\partial \vec{r}_E} \dot{\Lambda}(t-\tau) \rho_S(\tau; \vec{r}_{\text{NTL}}) \right] d\tau \right]$$

(61)

From this we can obtain an approximate non-time-local master equation by isolating the $O(\delta \vec{r})$ term in the above equation. Plugging the master equation into itself and dropping the term $O(\delta \vec{r}^2)$ yields the memory kernel,

$$\tilde{K}_{\text{NTL}}^{(1)}(t, \tau) = \frac{\partial \dot{K}_{\text{TL}}(t-\tau)}{\partial \vec{r}_E} \tilde{\Lambda}(t-\tau) \left[ \delta(t-\tau) \mathbb{1} - \dot{K}_{\text{TL}}(\tau) \right] + \frac{\partial}{\partial t} \frac{\partial \dot{K}_{\text{TL}}(t-\tau)}{\partial \vec{r}_E} \tilde{\Lambda}(t-\tau)$$

(62)

Which leads to an effective non-time-local master equation,

$$\partial_t \rho_S(t; \vec{r}_{\text{NTL}}) \approx \dot{K}_{\text{TL}}(t)\rho_S(t; \vec{r}_{\text{NTL}}) + \delta \vec{r} \cdot \left[ \int_0^t \tilde{K}_{\text{NTL}}^{(1)}(t, \tau) \rho_S(\tau; \vec{r}_{\text{NTL}}) d\tau \right]$$

(63)

An effective time local description is obtained keeping only the zeroth order term,

$$\partial_t \rho_S(t; \vec{r}_{\text{NTL}}) \approx \dot{K}_{\text{TL}}(t)\rho_S(t; \vec{r}_{\text{NTL}})$$

(64)

Using this equation to generate the reduced dynamics is equivalent to exchanging the dynamical map $\Lambda(t, \vec{r}_{\text{NTL}})$ for the dynamical map $\tilde{\Lambda}(t)$, so obviously defines a completely positivity master equation.

V. CONCLUSIONS

We have classified how two key aspects of open-system dynamics, non-Markovianity and time non-locality, depend on the strength and symmetry properties of the system/environment coupling and on the initial state of the environment in this simple, two-qubit system with symmetry.

We found that the information flow back and forth between system and environment is generically aperiodic. A key feature of such dynamics being that the reduced state remains partially entangled to the environment for all $t > 0$. however, when the energy eigenvalues of the full Hamiltonian are commensurate, the non-Markovian measures become periodic. In such cases the system and environment become uncorrelated infinitely often as $t \to \infty$. We find that when the most symmetry is present in $H$, including an exchange symmetry between
the even and odd blocks, special initial environment states support Markovian reduced dynamics. This occurs if ρ₀ is a fixed point of U(t), allowing the system and environment to remain uncorrelated under time evolution. This hints at a connection between the integrability of H (along the lines of [21]) and the non-Markovianity of the reduced dynamics, but further investigation in other open contexts is needed to verify the connection and quantify exactly the degree of symmetry needed.

The non-invertibility of the reduced dynamics also depends both on the Hamiltonian and the initial environment state. Strong coupling is a necessary condition for non-invertibility, however so is a rather restricted set of initial environment states. Those state are defined largely by the dynamics of the free Hamiltonian for the environment. Diagonal initial environment states in the eigenbasis of Hₑ;free are the most non-local, as their dynamical maps are non-invertible whenever allowed.

These results give some helpful perspective on how one might construct effective master equations for open qubit systems without reliance on some standard assumptions about perturbative or Markovian dynamics. For example, for systems like the one studied here, a time-local master equation can be written without assuming weak coupling, as long as one assumes a generic initial environment state among a large, symmetry-protected set of possible initial states. However, mixed states, including thermal states, are not likely to satisfy this assumption. The importance of the non-local kernel in the master equation, for environment states that do not satisfy those conditions, can be approximated as proportional to the difference between a generic state used to evaluate the time-local kernel, and the exact environment state. Although non-Markovian information flow will be generically non-periodic, this choice can be understood as more nearly accurate if the full Hamiltonian has essentially one fundamental frequency, with all eigenfrequencies being nearly multiples of that scale. It will be interesting to explore how these conclusions generalize to more complex systems and environments.

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Appendix A: Derivation of the maximally entangling parameter space

Following [11] [12], we find the essential non-local properties of $U$ by changing to the Bell basis using the unitary operator,

$$ Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{bmatrix}. \quad (A1) $$

The choice of Bell states is made so that the entanglement quadratic form $\hat{\text{Ent}}$ maps to the identity matrix. The operator $\hat{\text{Ent}}$ is defined as,

$$ \text{Det}I_C(|\psi\rangle) = \psi_{00}\psi_{11} - \psi_{01}\psi_{10} = |\psi^T\hat{\text{Ent}}|\psi\rangle, \quad (A2) $$

where $I_C$ is the Choi isomorphism $I_C : \mathcal{H} \rightarrow \text{Mat}(2, \mathbb{C})$, and one sees upon further inspection that $\hat{\text{Ent}} = \frac{1}{2} \sigma_S^{xx} \otimes \sigma_E^{xx}$ and $Q^\dagger(\hat{\text{Ent}})Q = \frac{1}{2} \mathbb{1}_S \otimes \mathbb{1}_E$. At the level of observables $Q$ takes the
subalgebra of local observables \( \mathfrak{su}(2)_S \oplus \mathfrak{su}(2)_E \) to the generators of 4D rotations on the Bell space \( \mathfrak{so}(4)_{\text{Bell}} \).

The time evolution operator is expressed in the Bell basis as,

\[
U_{\text{Bell}}(t) = \begin{bmatrix}
\cos \omega_+ t & 0 & 0 & e^{i\phi_+} \sin \omega_+ t \\
0 & \cos \omega_- t & -e^{i\phi_-} \sin \omega_- t & 0 \\
0 & e^{-i\phi_-} \sin \omega_- t & \cos \omega_- t & 0 \\
-e^{-i\phi_+} \sin \omega_+ t & 0 & 0 & \cos \omega_+ t
\end{bmatrix} \tag{A3}
\]

The non-local properties of the time evolution in the reduction frame are determined by the eigenvalues of \( U_{\text{Bell}}^T U_{\text{Bell}} \) found to be,

\[
u_+ = \cos^2 \omega_+ t + (\cos 2\phi_+ + i \sqrt{\sin^2 2\phi_+ + \sin^2 \phi_+}) \sin^2 \omega_+ t \\

v_- = \cos^2 \omega_- t + (\cos 2\phi_- - i \sqrt{\sin^2 2\phi_- + \sin^2 \phi_-}) \sin^2 \omega_- t \\
v_+ = \cos^2 \omega_+ t + (\cos 2\phi_+ - i \sqrt{\sin^2 2\phi_+ + \sin^2 \phi_+}) \sin^2 \omega_+ t \\
v_- = \cos^2 \omega_- t + (\cos 2\phi_- + i \sqrt{\sin^2 2\phi_- + \sin^2 \phi_-}) \sin^2 \omega_- t
\]

Assume we are given a linear combination of these eigenvalues \( au_+ + bu_- + cv_+ + dv_- \) such that \( a, b, c, d \geq 0 \) and \( a + b + c + d = 1 \). For this combination to be real we must have \( a = b \) and \( c = d \). Setting \( a = \cos^2 \varphi \) and \( c = \sin^2 \varphi \) we have,

\[
0 = \cos^2 \varphi (\cos^2 \omega_+ t + \cos 2\phi_+ \sin^2 \omega_+ t) + \sin^2 \varphi (\cos^2 \omega_- t + \cos 2\phi_- \sin^2 \omega_- t) \\
= \cos^2 \varphi (|\alpha_+|^2(t) - \beta_+^2(t)) + \sin^2 \varphi (|\alpha_-|^2(t) - \beta_-^2(t)) \tag{A5}
\]

A little bit of algebra and we find,

\[
\cos^2 \varphi \beta_+^2(t) + \sin^2 \varphi \beta_-^2(t) = \frac{1}{2}. \tag{A6}
\]

This condition may only be satisfied iff the largest of \( \beta_+^2(t) \) and \( \beta_-^2(t) \) is greater than or equal to \( \frac{1}{2} \), or equivalently \( \max(\phi_+, \phi_-) \geq \frac{\pi}{4} \). That is maximally entangled states may only be generated when \( U \) has large enough off diagonal components, which we see becomes one of the conditions that non-invertibilities appear in the reduced dynamics.

**Appendix B: Divisibility in Channels**

In this appendix we establish the divisibility of the dynamical map family \( \mathcal{D}_+ \). For simplicity we assume that \( \vec{r}_E(0) = \vec{0} \) although the results derived apply even if \( \vec{r}_E(0) = \)
We are interested in when the map $\Phi(\tau_2, \tau_1)$ fails to be completely positive, and how this depends on $\tau_2$ and $\phi$. To that end, the dynamical map has the structure,

$$\Lambda(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \Lambda_{xx}(t) & \Lambda_{xy}(t) & 0 \\
0 & \Lambda_{yx}(t) & \Lambda_{yy}(t) & 0 \\
0 & 0 & 0 & \Lambda_{zz}(t)
\end{bmatrix},$$

with determinant $\text{Det}\Lambda = \Lambda_{zz}^2$ and inverse,

$$\Lambda^{-1}(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\Lambda_{yz}(t)}{\Lambda_{zz}(t)} & -\frac{\Lambda_{xy}(t)}{\Lambda_{zz}(t)} & 0 \\
0 & -\frac{\Lambda_{yz}(t)}{\Lambda_{zz}(t)} & \frac{\Lambda_{xx}(t)}{\Lambda_{zz}(t)} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

and the (possibly singular) interweaving map is,

$$\Phi(\tau_2, \tau_1) = \begin{bmatrix}
\Lambda_{xx}(\tau_2)\Lambda_{yy}(\tau_1) - \Lambda_{xy}(\tau_2)\Lambda_{yx}(\tau_1) & 0 & 0 & 0 \\
0 & \Lambda_{zz}(\tau_1) & -\Lambda_{xx}(\tau_2)\Lambda_{yy}(\tau_1) + \Lambda_{xy}(\tau_2)\Lambda_{yx}(\tau_1) & 0 \\
0 & \Lambda_{yz}(\tau_2)\Lambda_{xx}(\tau_1) - \Lambda_{yy}(\tau_2)\Lambda_{yx}(\tau_1) & \Lambda_{yy}(\tau_2)\Lambda_{xx}(\tau_1) - \Lambda_{yx}(\tau_2)\Lambda_{xy}(\tau_1) & 0 \\
0 & 0 & 0 & \Lambda_{zz}(\tau_1)
\end{bmatrix}. $$

N.B. that the map $\Phi(\tau_2, \tau_1)$ can have a restricted domain, where instead of the Bloch ball the interweaving map only acts on the image of $\Lambda(\tau_1)$.

The criterion in [13] asserts that $\Phi(\tau_2, \tau_1)$ is completely positive if,

$$\left[ \frac{\Lambda_{xx}(\tau_2)\Lambda_{yy}(\tau_1) - \Lambda_{xy}(\tau_2)\Lambda_{yx}(\tau_1)}{\Lambda_{zz}(\tau_1)} + \frac{\Lambda_{yy}(\tau_2)\Lambda_{xx}(\tau_1) - \Lambda_{yx}(\tau_2)\Lambda_{xy}(\tau_1)}{\Lambda_{zz}(\tau_1)} \right]^2 \leq \left[ 1 \pm \frac{\Lambda_{zz}(\tau_2)}{\Lambda_{zz}(\tau_1)} \right]^2. $$

The $+$ inequality is saturated at all times, so the components of $\Lambda$ satisfy the relation (recall $\Lambda_{yx}(t) = -\Lambda_{xy}(t)$),

$$\Lambda_{xx}(\tau_2)\Lambda_{yy}(\tau_1) + 2\Lambda_{xy}(\tau_2)\Lambda_{yx}(\tau_1) + \Lambda_{yy}(\tau_2)\Lambda_{xx}(\tau_1) = \Lambda_{zz}(\tau_2) + \Lambda_{zz}(\tau_1). $$

Thus $\Phi$ is completely positive if,

$$(\Lambda_{xx}(\tau_2)\Lambda_{yy}(\tau_1) - \Lambda_{yy}(\tau_2)\Lambda_{xx}(\tau_1))^2 \leq (\Lambda_{zz}(\tau_2) - \Lambda_{zz}(\tau_1))^2.$$
We already see that if \( \tau_n = \frac{n\pi}{2\omega} \), then \( \Phi(\tau_n, \tau_1) \) is CP as the above inequality reduces to a CP condition satisfied by \( \Lambda(\tau_1) \). Therefore, special times \( \tau_2 \) exist where the dynamics is CP divisible. This condition is not dependent on what particular values are chosen for \( \omega \) and \( \phi \). However for other values of \( \tau_2 \), the CP inequality will fail to be satisfied for certain values of \( \tau_1 \). The size of this interval is not dependent on \( \omega \), but depends on \( \phi \) and \( \tau_2 \).