BRAIDED QUANTUM GROUPS AND THEIR BOSONIZATIONS IN THE C*-ALGEBRAIC FRAMEWORK

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Abstract. We present a general theory of braided quantum groups in the C*-algebraic framework using the language of multiplicative unitaries. Starting with a manageable multiplicative unitary in the representation category of the quantum codouble of a regular quantum group $G$, we construct a braided C*-quantum group over $G$ as a C*-bialgebra in the monoidal category of the $G$-Yetter-Drinfeld C*-algebras. Furthermore, we establish the one to one correspondence between braided C*-quantum groups and C*-quantum groups with a projection. Consequently, we generalise the bosonization construction for braided Hopf-algebras of Radford and Majid to braided C*-quantum groups. Several examples are discussed. In particular, we show that the complex quantum plane admits a braided C*-quantum group structure over the circle group $\mathbb{T}$ and identify its bosonization with the simplified quantum $E(2)$ group.

1. Introduction

Semidirect product construction of groups is a fundamental method of extending certain homogeneous symmetries to some inhomogeneous symmetries of a given physical system. In the realm of noncommutative geometry, this would mean that the semidirect product of quantum groups may contain information about the inhomogeneous quantum symmetries of quantum spaces. Several investigations were done by several authors to understand the structure of the inhomogeneous quantum groups mostly at the algebraic level. We refer to [18] and the references therein for more details. The primary focus of those works was on the understanding of the deformations of the Poincaré group.

On the other hand, many important examples of C*-quantum groups $\mathcal{H} = (C, \Delta_C)$ were constructed by deforming semidirect product of Lie groups $K \rtimes G$ with Abelian $G$. The C*-quantum group structure is captured by a single unitary operator $\mathcal{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$, on a suitably chosen separable Hilbert space $\mathcal{H}$, with two additional properties. The first one is algebraic namely, $\mathcal{W}$ satisfies the pentagon equation

$$\mathcal{W}_{23}\mathcal{W}_{12} = \mathcal{W}_{12}\mathcal{W}_{13}\mathcal{W}_{23} \quad \text{in} \quad \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}).$$

This ensures the set $C_0 := \{(\omega \otimes \text{id}_\mathcal{H})\mathcal{W} | \omega \in \mathcal{B}(\mathcal{H}), \mathcal{W} \in \mathcal{B}(\mathcal{H})\}$ is an algebra. The analytic property of $\mathcal{W}$, namely manageability [33, Definition 1.1], implies

$$\mathcal{W}^2 = \mathcal{W}$$

for all $\omega \in \mathcal{B}(\mathcal{H})$. Here, $\mathcal{B}(\mathcal{H})$ stands for the closed linear span of $\mathcal{H}$. The C*-quantum group $\mathcal{H} = (C, \Delta_C)$ is then said to be generated by $\mathcal{W}$ in the sense of [33, Theorem 1.5].

The semidirect product group $K \rtimes G$ comes with a canonical endomorphism $p: K \rtimes G \rightarrow K \rtimes G$ defined by $p(k, g) = (1_K, g)$, where $1_K$ denotes the identity element of $K$. Clearly, $p$ is idempotent, that is $p^2 = p$, with the image $G \subset K \rtimes G$ and

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the kernel $K \subset K \rtimes G$. This fact gets translated to the deformations of $K \rtimes G$ as well. At the level of multiplicative unitaries, there exist unitaries $\mathbb{F}, \mathbb{F} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ such that $\mathbb{F}$ is again a manageable multiplicative unitary and $\mathbb{W} = \mathbb{F}\mathbb{F}$. Also, $\mathbb{F}$ generates $G$, while viewed as quantum group $G = (C_0(G), \Delta_{C_0(G)})$, as a Woronowicz closed quantum subgroup of $\mathbb{H}$, see [3] Definition 3.2. Thus $\mathbb{F}$ is the quantum analogue of the idempotent group homomorphism or projection on $\mathbb{H}$. At the $C^*$-algebra level, $C$ is identified with the crossed product $C^*$-algebra $B \rtimes \hat{G}$ for some $C^*$-algebra $B$ equipped with an action of the dual group $\hat{G}$.

However, the range of the restriction of $\Delta_C$ on $\hat{B}$ is not a $C^*$-subalgebra of the multiplier algebra of $B \otimes B$ denoted by $\mathcal{M}(B \otimes B)$ and $\mathbb{F}$ is not a multiplicative unitary. This strongly indicates that the quantum analogue of the translation group $K$ is not a $C^*$-quantum groups.

In a purely algebraic setting, when quantum groups and Hopf algebras are synonymous, Radford had discovered [19] that the Hopf algebras $C$ with a projection $p$ is equivalent to pairs consisting of a Hopf algebra $A = \text{Im}(p)$ and a braided Hopf algebra $B$ over $A$. We refer [8, Chapter 10] for a detailed discussion on it. This was further generalised in the categorical framework and extensively studied by Majid [7,9,10]. The reconstruction of the Hopf algebra $C$ and the projection $p$ starting from $A$ and $B$ is named by Majid as bosonization.

Motivated by the algebraic theory we ask the following question: does there exist a one to one correspondence between braided $C^*$-quantum groups and quantum groups with projection? A systematic investigation in this direction was initiated by the author in his thesis [21]. It was further studied in [12,16] at the level of manageable multiplicative unitaries, and in [5] at the level of von Neumann algebras. Meanwhile the braided compact quantum groups over a compact quantum group $G$ was introduced in [15]. The $C^*$-algebra version of the associated bosonization turns out to be a compact quantum group. Consequently, $q$-deformations of SU(2) group, braided analogue of the free orthogonal groups $O(n)$ (in dimension $n$) for nonzero $q \in \mathbb{C}$ with $\text{Phase}(q) \neq 1$ were constructed as braided compact quantum over $\mathbb{T}$, see [4,11]. In fact, the resulting bosonizations of braided SU$_q$(2) groups are $U_q(2)$ groups. Furthermore, the braided compact quantum groups constructed in the recent works [21,23] captures quantum symmetries of matrix algebras and graph $C^*$-algebras.

The goal of this article is twofold. First, we provide an avenue to pass from manageable braided multiplicative unitaries to braided $C^*$-quantum groups in Theorem 5.1. Consequently, we construct the duals of braided $C^*$-quantum groups as braided $C^*$-quantum groups and generalise Pontrjagin duality for braided $C^*$-quantum groups. Secondly, we establish the one to one correspondence between braided $C^*$-quantum groups and $C^*$-quantum groups with projection in Theorem 5.14. In particular, this allows to construct new examples of $C^*$-quantum groups using braided $C^*$-quantum groups over $G$ as building blocks. The resulting theory turns out to be very general and it covers the following:

1. quantum $E(2)$ groups associated to nonzero real deformation parameters [29], quantum $az+b$ groups [25,31] and quantum $az+b$ groups [50] are bosonizations of some braided $C^*$-quantum groups;
2. braided compact quantum groups over a compact quantum group [15] are braided $C^*$-quantum groups and so are their examples [21,23,11];
3. $q$-deformations of $E(2)$ group are braided $C^*$-quantum groups over $q \in \{z \in \mathbb{C} \mid 0 < |z| < 1\} \cap \mathbb{R}$, and their bosonizations provide new examples of $C^*$-quantum groups [29];
(4) the complex quantum planes associated to the real deformation parameters $0 < q < 1$ are braided C*-quantum groups over $\mathbb{T}$ with Woronowicz’s simplified $E_q(2)$ groups as their bosonization, see Section 5.

Let us briefly describe the techniques we have employed to develop this theory and give an outline of the article. We begin by fixing notations, recalling the necessary definitions, and results in Section 2.

Suppose $\mathcal{C}$ is the category of unitary representations of the quantum codouble of a C*-quantum group $\mathbb{G}$ on separable Hilbert spaces. Then $\mathcal{C}$ is a braided monoidal category and the braiding operators are unitaries, see [15] Proposition 3.4 & Section 5. In short, we call $\mathcal{C}$ a braided C*-category and the braiding operators are unitaries, see [15, Proposition 3.4 & Section 5]. In short, we call $\mathcal{C}$ a braided multiplicative unitary in $\mathcal{C}$: $\mathcal{C}$.

\begin{equation}
\mathcal{C}(L \otimes L \rightarrow L \otimes L) \ \text{is manageable} \ [16, \text{Definition 3.5}] \ \text{and define}
\end{equation}

\begin{equation}
B_0 = \{ (\omega \otimes \text{id}_\mathcal{C}) F \mid \omega \in \mathbb{B}(\mathcal{C}), \ B = B^{\text{CLS}}_0, \ \Delta_B(b) := F(b \otimes 1) F^*.
\end{equation}

Unlike unbraided situation, it is unclear whether $B_0$ is an algebra in the first place. In order to prove that $B$ is a C*-algebra, we consider the C*-quantum group $\mathbb{H} = (\mathcal{C}, \Delta_C)$, generated by the manageable multiplicative unitary $W^C$, with a projection $\mathbb{P}$ associated to $\mathbb{F}$ given by [16, Theorem 3.7]. Here we use the manageability of $\mathbb{F}$ implicitly. Next we ensure that $\mathbb{F}$ generates $\mathbb{G}$ and it is a Woronowicz closed quantum subgroup of $\mathbb{H}$ in Proposition 5.1. The “kernel” $\mathbb{P}$ corresponds to the quantum homogeneous space $\mathbb{G} \backslash \mathbb{H}$ with respect to the (left) quantum group homomorphism $\Delta_L: \mathcal{C} \rightarrow \mathbb{M}(A \otimes C)$ that corresponds to $\mathbb{P}$, see [13, Theorem 5.5]. At this point, we assume $\mathbb{G}$ is a regular quantum group [4]. Then the existence and uniqueness, up to $\mathbb{G}$-equivariant isomorphism, of the underlying C*-algebra of the quantum homogeneous space $\mathbb{G} \backslash \mathbb{H}$ inside $\mathbb{M}(C)$ follows from the Landstad-Vaes theory [23, 25]. In fact, Proposition 5.4 is an important step where we show that the underlying C*-algebra of $\mathbb{G} \backslash \mathbb{H} \subset \mathbb{M}(\mathcal{C})$ is unitarily equivalent to $B$; hence $B$ is a C*-algebra.

Consequently, we prove the first main result of this article Theorem 5.1 namely, the construction of the braided C*-quantum group $(B, \Delta_B)$ over $\mathbb{G}$ from $\mathbb{F}$. More precisely, $B \subset \mathbb{B}(\mathcal{L})$ is a $\mathbb{G}$-Yetter-Drinfeld C*-algebra and $\Delta_B: \mathbb{G} \rightarrow \mathbb{M}(B \boxtimes B)$ is a nondegenerate *-homomorphism satisfying braided analogue of coassociativity and cancellation conditions. Here $\boxtimes$ denotes the monoidal product of the category of $\mathbb{G}$-Yetter-Drinfeld C*-algebras.

Next, we discuss the bosonization construction for the braided C*-quantum group $(B, \Delta_B)$ over $\mathbb{G} = (A, \Delta_A)$ by reconstructing $\mathbb{H} = (\mathcal{C}, \Delta_C)$ and the projection $\mathbb{P}$ in Proposition 5.1. In particular, $\mathbb{G}$-Yetter-Drinfeld structure on $B$ says that there is an action $\beta$ of $\mathbb{G}$ on $B$. We identify $\mathcal{C}$ with crossed product C*-algebra $B \rtimes_{\beta} \hat{\mathbb{G}}$ and express $\Delta_C$ in terms of $\Delta_A$ and $\Delta_B$. Then we establish the desired one to one correspondence between braided C*-quantum groups and quantum groups with a projection up to isomorphism in Theorem 5.1.

Finally, in Section 4 we show that our theory applies to a large class of examples of C*-quantum groups. In particular, we apply our main results to the concretely constructed example of a manageable braided multiplicative unitary in [16, Section 4] over $\mathbb{T}$. We obtain complex quantum planes as the resulting braided C*-quantum group over $\mathbb{T}$ and the simplified quantum E(2) group coincides with the associated bosonization.
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2. Preliminaries

All Hilbert spaces and C*-algebras (which are not explicitly multiplier algebras) are assumed to be separable. For a C*-algebra $A$, let $\mathcal{M}(A)$ be its multiplier algebra and let $\mathcal{U}(A)$ be the group of unitary multipliers of $A$ and denote the identity element of $\mathcal{U}(A)$ by $1_A$. For two norm closed subsets $X$ and $Y$ of a C*-algebra $A$ and $T \in \mathcal{M}(A)$, we set

$$XY := \{xy \mid x \in X, y \in Y\}^{\mathrm{CLS}}, \quad XTY := \{xTy \mid x \in X, y \in Y\}^{\mathrm{CLS}},$$

where CLS stands for the closed linear span.

Let $\mathcal{C}^{\mathrm{alg}}$ be the category of C*-algebras with nondegenerate *-homomorphisms $\varphi: A \to \mathcal{M}(B)$ as morphisms $A \to B$; let $\operatorname{Mor}(A, B)$ denote the set of morphisms.

Let $\mathcal{H}$ be a Hilbert space. A representation of a C*-algebra $A$ is a nondegenerate *-homomorphism $\pi: A \to \mathcal{B}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H}) = \mathcal{M}(\mathcal{K}(\mathcal{H}))$ and the nondegeneracy conditions $\pi(A)\mathcal{K}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$ and $\pi(A)\mathcal{H} = \mathcal{H}$ are equivalent; hence $\pi \in \operatorname{Mor}(A, \mathcal{K}(\mathcal{H}))$. The unit element of $\mathcal{M}(\mathcal{K}(\mathcal{H}))$ is denoted by $1_H$.

We write $\Sigma$ for the tensor flip $\mathcal{H} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{H}$, $x \otimes y \mapsto y \otimes x$, where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. We write $\sigma$ for the tensor flip isomorphism $A \otimes B \to B \otimes A$ for two C*-algebras $A$ and $B$. Further we use the standard ‘leg numbering’ notation for maps acting on tensor products.

Let $\mathcal{H}$ be a Hilbert space and let $D$ be a nondegenerate C*-subalgebra of $\mathcal{B}(\mathcal{H})$. A closed and densely defined operator $T$ acting on $\mathcal{H}$ is said to be affiliated with $D$ if $\tau_T := T(I + T^*T)^{-\frac{1}{2}} \in \mathcal{M}(D)$ and $(1 - zT^T)D$ is dense in $D$ (see [32]). It is denoted by $T\eta_D$.

2.1. C*-quantum groups, their actions and representations. A C*-quantum group $G$ is a pair $(A, \Delta_A)$ consisting of a C*-algebra $A$ and an element $\Delta_A \in \operatorname{Mor}(A, A \otimes A)$ generated by a manageable multiplicative unitary $W$ in the way described in the following theorem.

**Theorem 2.1** ([26, 33]). Let $\mathcal{H}$ be a Hilbert space and let $W \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a manageable multiplicative unitary. Then

1. the sets of left and right slices of $W$, defined by
2. $A := \{\omega \otimes \operatorname{id}_\mathcal{H} \omega \mid \omega \in \mathcal{B}(\mathcal{H})\}^{\mathrm{CLS}}, \quad \hat{A} := \{(\operatorname{id}_\mathcal{H} \otimes \omega) \omega \mid \omega \in \mathcal{B}(\mathcal{H})\}^{\mathrm{CLS}},$
3. are nondegenerate C*-subalgebras of $\mathcal{B}(\mathcal{H})$;
4. $W \in \mathcal{U}(\hat{A} \otimes A) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. We write $W$ for $W$ viewed as a unitary multiplier of $\hat{A} \otimes A$ and call it reduced bicharacter.
5. $\Delta_A$ is uniquely characterised by
6. $$\Delta_A = W_{12}W_{13} \in \mathcal{U}(\hat{A} \otimes A \otimes A).$$

Moreover, $\Delta_A$ is coassociative:
7. $$\Delta_A \circ 1_A = (\Delta_A \circ 1_A) \circ \Delta_A,$$
8. and satisfies the cancellation conditions:
9. $$\Delta_A(A)(1_A \otimes A) = A \otimes A = (A \otimes 1_A)\Delta_A(A).$$
We shall not use the full power of the Haar weight approach towards C*-quantum groups developed by Kustermans and Vaes in [6]. The dual multiplicative unitary of $\mathcal{W}$ is $\mathcal{W} := \Sigma \mathcal{W}^{*} \Sigma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. If $\mathcal{W}$ is manageable so is $\mathcal{W}^{*}$. The dual C*-quantum group $\hat{G} = (\hat{A}, \hat{\Delta}_A)$ generated by $\mathcal{W}$. Its comultiplication map $\hat{\Delta}_A \in \text{Mor}(\hat{A}, \hat{A} \otimes \hat{A})$ is uniquely determined by the following equation

$$\tag{2.6} (\hat{\Delta}_A \otimes \text{id}_A)\mathcal{W} = W_{23}W_{13} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes A).$$

A C*-quantum group $G$ is regular if

$$\tag{2.7} (1_{\hat{A}} \otimes A)\mathcal{W}(\hat{A} \otimes 1_A) = \hat{A} \otimes A,$$

see [1] Proposition 3.2 (b) & Proposition 3.6 and [26] Lemma 40.

Example 2.8. Suppose $G$ is a locally compact group. Let $\mathcal{H}$ be the Hilbert space of square integrable functions with respect to the right Haar measure of $G$. We denote the right regular representation of $G$ on $\mathcal{H}$ by $\mu$. Define $(\mathcal{W}\xi)(g_1, g_2) := \xi(g_1 g_2, g_2)$ for all $\xi \in \mathcal{H} \otimes \mathcal{H}$ and $g_1, g_2 \in G$. Then $\mathcal{W}$ is a manageable multiplicative unitary and generates the C*-quantum group $G = (C_0(G), \Delta_{C_0(G)})$, where $\Delta_{C_0(G)}(f)(g_1, g_2) := f(g_1 g_2)$ for all $f \in C_0(G)$. Also $(C^*_r(G), \Delta_{C^*_r(G)})$, where $\Delta_{C^*_r(G)}(\mu_g) := \mu_g \otimes \mu_g$ for all $g \in G$, is the dual of $\hat{G}$. In fact, $G$ and $\hat{G}$ are examples of regular C*-quantum groups.

Definition 2.9. A right action of $G$ on a C*-algebra $C$ is an injective element $\gamma \in \text{Mor}(C, C \otimes A)$ with the following properties:

1. $\gamma$ is a comodule structure, that is,

$$\tag{2.10} (\text{id}_C \otimes \Delta_A) \circ \gamma = (\gamma \otimes \text{id}_A) \circ \gamma;$$

2. $\gamma$ satisfies the Podleś condition: $\gamma(C)(A \otimes 1_C) = A \otimes C$.

We call $(C, \gamma)$ a $G$-C*-algebra. We shall drop $\gamma$ from our notation whenever it is clear from the context.

Similarly, a left action of $G$ on $C$ is an injective element $\gamma \in \text{Mor}(C, A \otimes C)$ satisfying an appropriate variant of (2.10), that is $(\Delta_A \otimes \text{id}_C) \circ \gamma = (\text{id}_A \otimes \gamma) \circ \gamma$, and the Podleś condition: $\gamma_C(C)(A \otimes 1_C) = A \otimes C$. The word “action” will always mean right action throughout.

For any two $G$-C*-algebras $(C_1, \gamma_1)$ and $(C_2, \gamma_2)$ an element $f \in \text{Mor}(C_1, C_2)$ is said to be $G$-equivariant if $\gamma_2 \circ f = (f \otimes \text{id}_A) \circ \gamma_1$. The set of $G$-equivariant morphisms from $C_1$ to $C_2$ is denoted by $\text{Mor}^G(C_1, C_2)$. Let $C^*\text{alg}(G)$ be the category with $\mathbb{G}$-C*-algebras as objects and $G$-equivariant morphisms as arrows.

Definition 2.11. A (right) representation of $G$ on a Hilbert space $\mathcal{L}$ is a unitary $U \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A)$ with

$$\tag{2.12} (\text{id}_C \otimes \Delta_A)U = U_{12}U_{13} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A \otimes A).$$

The tensor product of representations $U^i \in \mathcal{U}(\mathbb{K}(\mathcal{L}_i) \otimes A)$ of $G$ on $\mathcal{L}_i$ for $i = 1, 2$ is defined by $U^1 \otimes U^2 := U^{12}_{13}U^{23}_{23} \in \mathcal{U}(\mathbb{K}(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes A)$.

Definition 2.13. A covariant representation of a $G$-C*-algebra $(C, \gamma)$ on a Hilbert space $\mathcal{L}$ is a pair $(U, \varphi)$ consisting of a representation $U \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A)$ of $G$ and an element $\varphi \in \text{Mor}(C, \mathbb{K}(\mathcal{L}))$ that satisfy the covariance condition

$$\tag{2.14} (\varphi \otimes \text{id}_A)(\gamma(c)) = U(\varphi(c) \otimes 1_A)U^* \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes A)$$

for all $c \in C$. Moreover, $(U, \varphi)$ is called faithful if $\varphi$ is faithful. Existence of faithful covariant representations is guaranteed by [14] Example 4.5.
2.2. Heisenberg pairs. Let $\mathbb{G} = (A, \Delta_A)$ be a C$^*$-quantum group, let $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta_A})$ be its dual, and let $W \in U(\hat{A} \otimes A)$ be the reduced bicharacter of $\mathbb{G}$.

For a pair of representations $(\pi, \hat{\pi})$ of $A$ and $\hat{A}$ on a Hilbert space $\mathcal{H}$ we denote $W_{1\pi} := ((|id_A \otimes \pi|)W)_{12}$ and $W_{\hat{\pi}3} := ((\hat{\pi} \otimes id_A)W)_{23}$ in $U(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes A)$.

The pair $(\pi, \hat{\pi})$ is called

1. a G-Heisenberg pair if and only if $W_{\hat{\pi}3}W_{1\pi} = W_{1\pi}W_{\hat{\pi}3}$;
2. a G-anti-Heisenberg pair if and only if $W_{1\pi}W_{\hat{\pi}3} = W_{\hat{\pi}3}W_{1\pi}$.

In fact, G-Heisenberg pairs and G-anti-Heisenberg pairs are in one to one correspondence, see [14, Lemma 3.4]. A G-Heisenberg or a G-anti-Heisenberg pair $(\pi, \hat{\pi})$ is said to be faithful if $\pi$ and $\hat{\pi}$ are faithful representations. [22, Proposition 3.2] shows that any G-Heisenberg pair is faithful.

For any G-Heisenberg or G-anti-Heisenberg pair $(\pi, \hat{\pi})$ on $\mathcal{H}$, We denote by $\hat{A}$, the set of all linear functionals on $\hat{A}$ that admit extensions to normal functionals on the weak closure of $\hat{\pi}(\hat{A})$. It turns out that $\hat{A}$ is independent of the choice of $(\pi, \hat{\pi})$.

Consider a pair of representations $(U, V)$ of $\mathbb{G}$ and $\hat{\mathbb{G}}$ on the Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. By virtue of [13, Theorem 4.1], for any G-Heisenberg pair $(\pi, \hat{\pi})$ on $\mathcal{H}$ there exists a unique $Z \in U(\mathcal{L}_1 \otimes \mathcal{L}_2)$ such that

\[(2.15) \quad U_{1\pi}V_{2\hat{\pi}}Z_{1\hat{\pi}} = V_{2\hat{\pi}}U_{1\pi} \quad \text{in} \quad U(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{H}),\]

where $U_{1\pi} := ((|id_{\mathcal{L}_1} \otimes \pi|)U)_{12}$ and $V_{2\hat{\pi}} := ((|id_{\mathcal{L}_2} \otimes \hat{\pi}|)V)_{23}$.

2.3. Landstad-Vaes theory. Let $\mathbb{G} = (A, \Delta_A)$ be a C$^*$-quantum group, let $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta_A})$ be its dual, and let $W \in U(\hat{A} \otimes A)$ be the reduced bicharacter of $\mathbb{G}$.

A G-product is a triple $(C, \gamma, i)$ consisting of a C$^*$-algebra $C$, a left action $\gamma \in \text{Mor}(C, A \otimes C)$ of $\mathbb{G}$ on $C$, and an element $i \in \text{Mor}(A, C)$ satisfying

\[(2.16) \quad \gamma \circ i = (\text{id}_A \otimes i) \circ \Delta_A.\]

Define $X := (\text{id}_A \otimes i)W \in U(\hat{A} \otimes C)$.

Theorem 2.17. Suppose $\mathbb{G} = (A, \Delta_A)$ is a regular C$^*$-quantum group and $(\pi, \hat{\pi})$ is a G-Heisenberg on a Hilbert space $\mathcal{H}_\pi$. Let $(C, \gamma, i)$ be a G-product. Define $\varphi : C \to \mathbb{K}(\mathcal{H}) \otimes C$ by $\varphi(c) := X_{c2}^\gamma \gamma(c)_{c2}X_{c2}$ for $c \in C$. There is a unique C$^*$-subalgebra $D$ of $\mathcal{M}(C)$ with the following properties:

1. $D \subseteq \{c \in \mathcal{M}(C) \mid \gamma(c) = 1_A \otimes c\}$;
2. $C = i(A)D$;
3. $\hat{A} \otimes D = (\hat{A} \otimes 1)\varphi(D)$.

More explicitly,

\[(2.18) \quad D = \{(\omega \otimes \text{id}_C)\varphi(c) \mid \omega \in B(\mathcal{H}), c \in C\}^{\text{CLS}} \subseteq \mathcal{M}(C).\]

The C$^*$-algebra $D$ is called the Landstad-Vaes algebra for the G-product $(C, \gamma, i)$. In particular, the third condition gives $\varphi \in \text{Mor}(D, \hat{A} \otimes D)$. Moreover, $\tilde{\beta} := \sigma \circ \varphi \in \text{Mor}(D, D \otimes \hat{A})$ is a (right) action of $\hat{\mathbb{G}}$ on $D$, and extends to a $\mathbb{G}$-equivariant isomorphism between $C$ and $D \times \hat{\mathbb{G}}$.

This fundamental result was first proved by Vaes [28, Theorem 6.7] (with slightly different conventions) for regular quantum groups $\mathbb{G}$ with the Haar weights and in [23, Theorem 3.6 & 3.8] in the general setting of (not necessarily regular) C$^*$-quantum groups. However, we are going to restrict our attention to the regular C$^*$-quantum groups.
2.4. Monoidal category of Yetter-Drinfeld C*-algebras.

**Definition 2.19** ([17 Definition 3.1]). A \( G \)-Yetter-Drinfeld C*-algebra is a triple \((C, \gamma, \tilde{\gamma})\) consisting of a C*-algebra \( C \) along with actions \( \gamma \in \text{Mor}(C, C \otimes A) \) and \( \tilde{\gamma} \in \text{Mor}(C, C \otimes \tilde{A}) \) of \( G \) and \( \tilde{G} \) that satisfy the Yetter-Drinfeld compatibility criterion

\[
(\tilde{\gamma} \otimes \text{id}_A)\gamma(c) = (W_{23})\sigma_{23}((\gamma \otimes \text{id}_A)\tilde{\gamma}(c))(W_{23}^*)
\]

for all \( c \in C \).

Indeed, \((C, \gamma, \tilde{\gamma})\) is a \( G \)-Yetter-Drinfeld C*-algebra if and only if \((C, \tilde{\gamma}, \gamma)\) is a \( \tilde{G} \)-Yetter-Drinfeld C*-algebra.

**Example 2.21.** Let \( G = (A, \Delta_A) \) be a regular C*-quantum group. Then \( \Theta : A \to A \otimes \tilde{A} \) defined by \( \Theta(a) := \sigma(W^*(1_A \otimes a)W) \) for \( a \in A \) is an action of \( \tilde{G} \) on \( A \), and \((A, \Delta_A, \Theta)\) is a \( \tilde{G} \)-Yetter-Drinfeld C*-algebra (see [17 Section 3]).

Let \( \mathcal{YD} C^\ast \text{alg}(G) \) be the category with \( G \)-Yetter-Drinfeld C*-algebras as objects and \( G \) and \( \tilde{G} \)-equivariant morphisms as arrows. Suppose, \((C_1, \gamma_1, \tilde{\gamma}_1)\) and \((C_2, \gamma_2, \tilde{\gamma}_2)\) are objects of \( \mathcal{YD} C^\ast \text{alg}(G) \). Without loss of generality, suppose \((U^i, \phi_i)\) are faithful covariant representation of \((C_i, \gamma_i)\) on \( L_i \) and \((V^i, \tilde{\phi}_i)\) are faithful covariant representations of \((C_i, \tilde{\gamma}_i)\) on \( L_i \) for \( i = 1, 2 \), respectively.

Define \( \xi^\ast \xi^2 : L_2 \otimes L_1 \to L_1 \otimes L_2 \) by \( \xi^\ast \xi^2 := Z \circ \Sigma \), where \( Z \in \mathcal{U}(L_1 \otimes L_2) \) is the unique solution of (2.15) for the pair of representations \((U^1, V^2)\).

**Theorem 2.22** ([13 Lemma 3.20, Theorem 4.3, Theorem 4.9]). For \( i = 1, 2 \) define \( j_i \in \text{Mor}(C_i, K(L_1 \otimes L_2)) \) by

\[
j_i(c_1) := \phi_1(c_1) \otimes 1_{L_2}, \quad j_2(c_2) := \xi^\ast \xi^2(\tilde{\phi}_2(c_2) \otimes 1_{L_1})\xi^\ast \xi^2,
\]

where \( \xi^\ast \xi^2 := \Sigma \circ Z^\ast \). Then the subspace \( C_1 \boxtimes C_2 := j_1(C_1)j_2(C_2) \) is a nondegenerate C*-subalgebra of \( \mathcal{B}(L_1 \otimes L_2) \) and the triple \((C_1 \boxtimes C_2, j_1, j_2)\), up to equivalence, does not depend on the faithful covariant representations \((U^1, \phi_1)\) and \((V^1, \phi_1)\) for \( i = 1, 2 \).

Furthermore \( C_1 \boxtimes C_2 \) becomes a \( G \)-Yetter-Drinfeld C*-algebra with respect to the diagonal actions of \( G \) and \( \tilde{G} \) defined by

\[
C_1 \boxtimes C_2 \ni x \overset{\gamma_1 \otimes \gamma_2}{\longrightarrow} (U^1 \Phi U^2)(x \otimes 1_A)(U^1 \Phi U^2)^* \in C_1 \boxtimes C_2 \otimes A,
\]

\[
C_1 \boxtimes C_2 \ni x \overset{\tilde{\gamma}_1 \otimes \tilde{\gamma}_2}{\longrightarrow} (V^1 \Phi V^2)(x \otimes 1_{\tilde{A}})(V^1 \Phi V^2)^* \in C_1 \boxtimes C_2 \otimes \tilde{A}.
\]

This following theorem has been proved in [17 Section 3] in the presence of Haar weights on \( G \) and in [15 Section 5] in the general framework of modular multiplicative unitaries.

**Theorem 2.26.** \( \mathcal{YD} C^\ast \text{alg}(G), \otimes \) is a monoidal category.

3. LANDSTAD–VAES ALGEBRA FOR THE QUANTUM GROUPS WITH A PROJECTION

Let \( \mathbb{H} = (C, \Delta_C) \) be a C*-quantum group, let \( \mathbb{H} = (\hat{C}, \hat{\Delta}_C) \) be its dual, and let \( \mathcal{W} \in U(\hat{C} \otimes C) \) be the reduced bicharacter of \( \mathbb{H} \).

An element \( P \in U(\hat{C} \otimes C) \) is a projection on \( \mathbb{H} \) if

\( P \) is a quantum group endomorphism of \( \mathbb{H} \):

\[
P(\Delta_C \otimes \text{id}_C)P = P_{23}P_{13}, \quad (\text{id}_C \otimes \Delta_C)P = P_{12}P_{13}.
\]

(2) \( P \) is idempotent: for any \( \mathbb{H} \)-Heisenberg pair \((\alpha, \check{\alpha})\) on \( \mathcal{H}_\alpha \)

\[
P_{\alpha \check{\alpha}}P_{\alpha} = P_{\alpha \check{\alpha}}P_{\alpha} \quad \text{in} \ U(\hat{C} \otimes \mathcal{K}(\mathcal{H}_\alpha) \otimes C).
\]
The condition (3.2) above can be also formulated using $\mathbb{H}$-anti-Heisenberg pairs. Suppose $(\tilde{\alpha}, \tilde{\delta})$ is an $\mathbb{H}$-anti-Heisenberg pair on $\mathcal{H}_\alpha$. Then [14] Proposition 3.9 implies that the representations $((\alpha \otimes \tilde{\delta}) \circ \Delta_C, (\tilde{\alpha} \otimes \tilde{\delta}) \circ \Delta_C)$ of $C$ and $\hat{C}$ on $\mathcal{H}_\alpha \otimes \mathcal{H}_\tilde{\alpha}$ commute. Subsequently, we have

$$
((\Delta_C \otimes (\alpha \otimes \tilde{\delta})) \Delta_C(P_{13})(\Delta_C \otimes \text{id}_{\mathcal{H}})(P_{234})\Delta_C(P_{234})(\Delta_C \otimes \text{id}_{\mathcal{H}})P_{234})_{123}
$$

in $\mathcal{U}(\hat{C} \otimes \mathbb{K}(\mathcal{H}_\alpha \otimes \mathcal{H}_\alpha) \otimes \mathcal{C})$. The conditions (3.1) simplify the last equation as follows:

$$
P_{13}P_{13}P_{\hat{4}4}P_{\hat{4}4}P_{13} = P_{\hat{4}4}P_{\hat{4}4}P_{13}P_{13}P_{\hat{4}4}.
$$

Commuting $P_{13}$ with $P_{\hat{4}4}$ and $P_{\hat{4}4}$ with $P_{13}$, the last equation becomes

$$
P_{\hat{4}4}P_{\hat{4}4}P_{13}P_{13}P_{\hat{4}4} = P_{\hat{4}4}P_{\hat{4}4}P_{13}P_{13}P_{\hat{4}4}.
$$

Hence (3.2) is equivalent to

(3.3) $$P_{13}P_{\hat{4}4}P_{\hat{4}4}P_{\hat{4}4}P_{13} = P_{\hat{4}4}P_{\hat{4}4}P_{13}P_{13}P_{\hat{4}4}$$

in $\mathcal{U}(\hat{C} \otimes \mathbb{K}(\mathcal{H}_\alpha \otimes \mathcal{C}))$,

for any $\mathbb{H}$-anti-Heisenberg pair $(\tilde{\alpha}, \tilde{\delta})$ on $\mathcal{H}_\alpha$.

Suppose $(\alpha, \delta)$ is an $\mathbb{H}$-Heisenberg pair on $\mathcal{H}_\alpha$. Now $P_\alpha := (\alpha \otimes \delta)P \in \mathcal{U}(\mathcal{H}_\alpha \otimes \mathcal{H}_\alpha)$ is a manageable multiplicative unitary, which follows from [14] Proposition 2.5.

Then the $C^*$-quantum group $\mathbb{G} = (A, \Delta_A)$ generated by $P_\alpha$, which does not depend on the choice of the $\mathbb{H}$-Heisenberg pair $(\alpha, \delta)$, and $P \in \mathcal{U}(\mathcal{A} \otimes \mathcal{A}) \subseteq \mathcal{U}(\hat{C} \otimes \mathcal{C})$. Then $\mathbb{G}$ is called the image of $P$.

In particular, we have $A \subseteq \mathcal{M}(C)$. Moreover, the inclusion $i : A \hookrightarrow \mathcal{M}(C)$ is an element of $\text{Mor}(A, C)$. To see this, once again, let us fix an $\mathbb{H}$-Heisenberg pair $(\alpha, \delta)$ on $\mathcal{H}_\alpha$. Then (2.3) for $\Delta_C$ is equivalent to $(\alpha \otimes \text{id}_{\mathcal{H}})(\Delta_C(c) = W_{\hat{A}A}^C(\alpha(c) \otimes 1_C)W_{\hat{A}A}^C)$ for all $c \in C$. Consequently, the second condition in (3.1) is equivalent to $P_{13}W_{\hat{A}A}^C P_{13} = P_{13}W_{\hat{A}A}^C$ in $\mathcal{U}(\hat{C} \otimes \mathbb{K}(\mathcal{H}_\alpha \otimes \mathcal{C}))$. This implies

$$C = \{(\omega_1 \otimes \omega_2 \otimes \text{id}_{\mathcal{H}})(P_{\hat{4}4}^*W_{\hat{A}A}^C P_{\hat{4}4}) | \omega_1 \in \hat{C}_*, \omega_2 \in \mathcal{B}(\mathcal{H}_\alpha)_\alpha\}^{\text{CLS}} \subseteq C_* = (AC)^* = C^* A^* = CA.
$$

Since, $C^* = C$ and $A^* = A$, we have $C = C^* = (AC)^* = C^* A^* = CA$.

Now [10] Proposition 2.8 shows that $\mathbb{H} = (C, \Delta_C)$ with projection $P \in \mathcal{U}(\hat{C} \otimes \mathcal{C})$ with image $\mathbb{G} = (A, \Delta_A)$ is equivalent to a quadruple $(\mathbb{G}, \mathbb{H}, i, \Delta_L)$ consisting of $C^*$-quantum groups $\mathbb{G} = (A, \Delta_A), \mathbb{H} = (C, \Delta_C)$ and morphisms $i \in \text{Mor}(A, C)$ and $\Delta_L \in \text{Mor}(C, A \otimes C)$ such that

1. $i$ is a Hopf $^*$-homomorphism: $\Delta_C \circ i = (i \otimes i) \circ \Delta_A$.
2. $\Delta_L$ is a left quantum group homomorphism:

$$
(i \Delta_A \otimes \text{id}_{\mathcal{H}}) \circ \Delta_L = (\Delta_L \otimes \text{id}_{\mathcal{H}})(\Delta_A \otimes \text{id}_{\mathcal{H}}) \circ \Delta_L = (i \Delta_A \otimes \Delta_L) \circ \Delta_L
$$

3. $(C, \Delta_L, i)$ is a $\mathbb{G}$-product, that is, $(\Delta_L, i)$ satisfy (2.19).

In the next result we describe the Landstad-Vaes algebra for this $\mathbb{G}$-product. For that matter we assume $\mathbb{G}$ to be a regular $C^*$-quantum group.

**Proposition 3.4.** Define $F := P^*W_C \in \mathcal{U}(\hat{C} \otimes \mathcal{C})$. Then

$$D := \{(\omega \otimes \text{id}_{\mathcal{H}})F | \omega \in \hat{C}_*\}^{\text{CLS}} \subseteq \mathcal{M}(C).
$$

is the Landstad-Vaes algebra for the $\mathbb{G}$-product $(C, \Delta_L, i)$.

First we prove the following technical lemma.
Lemma 3.5. Let $W \in \mathcal{U}(\hat{A} \otimes A)$ be the reduced bicharacter of $\mathbb{G}$. Define $X \in \mathcal{U}(\hat{A} \otimes C)$ by $X := (\text{id}_{\hat{A}} \otimes i)W$. Then for any $\mathbb{H}$-anti-Heisenberg pair $(\hat{a}, \hat{a})$ on a Hilbert space $\mathcal{H}_{\hat{a}}$, we have the following relation:

$$(3.6) \quad F_{\hat{a}3}X_{13}X_{1\hat{a}} = X_{13}X_{1\hat{a}}F_{\hat{a}3} \quad \text{in } \mathcal{U}(\hat{A} \otimes K(\mathcal{H}_{\hat{a}}) \otimes C).$$

Proof. Since $(\hat{a}, \hat{a})$ is an $\mathbb{H}$-anti-Heisenberg pair,

$$(3.7) \quad W_{\hat{a}3}^C W_{\hat{a}3}^C = W_{\hat{a}3}^C W_{\hat{a}3}^C \quad \text{in } \mathcal{U}(\hat{C} \otimes K(\mathcal{H}_{\hat{a}}) \otimes C).$$

Combining (2.3) and (3.7) for $\Delta_C$ we can show that

$$(3.8) \quad (\text{id}_C \otimes \hat{a})\Delta_C(c) = \sigma(W_{\hat{a}3}^C \ast (\hat{a}(c) \otimes 1_C)W_{\hat{a}3}^C) \quad \text{for } c \in C.$$

The unitary $X := (\text{id}_{\hat{A}} \otimes i)W \in \mathcal{U}(\hat{A} \otimes C)$ is a bicharacter (see [13, Definition 3.1]) because $i$ is a Hopf $\ast$-homomorphism. So, in particular, $(\text{id}_{\hat{A}} \otimes \Delta_C)X = X_{12}X_{13}$ and it is equivalent to

$$(3.9) \quad X_{1\hat{a}} W_{\hat{a}3}^C = W_{\hat{a}3}^C X_{1\hat{a}} \quad \text{in } \mathcal{U}(\hat{A} \otimes K(\mathcal{H}_{\hat{a}}) \otimes C)$$

by (3.8).

The unitary $\hat{P} := \sigma(P^\ast) \in \mathcal{U}(C \otimes \hat{C})$ is a projection on $\hat{H}$. This defines an injective Hopf $\ast$-homomorphism $\hat{i} \in \text{Mor}(\hat{A}, \hat{C})$ such that $P = (\hat{i} \otimes i)W \in \mathcal{U}(\hat{C} \otimes C)$. Recall that $P$ satisfies (3.3). Since $\hat{i}$ is injective, we may apply $\hat{i}^{-1} \otimes \text{id}_{\mathcal{H}_{\hat{a}}} \otimes \text{id}_C$ on the both sides of (3.3) and obtain

$$(3.10) \quad X_{1\hat{a}} P_{\hat{a}3} = P_{\hat{a}3} X_{1\hat{a}} \quad \text{in } \mathcal{U}(\hat{A} \otimes K(\mathcal{H}_{\hat{a}}) \otimes C).$$

Subsequently, we complete the proof below using (3.3) and (3.10):

$$F_{\hat{a}3}X_{13}X_{1\hat{a}} = P_{\hat{a}3}^C W_{\hat{a}3}^C X_{13}X_{1\hat{a}} = P_{\hat{a}3}^C X_{13} P_{\hat{a}3}^C W_{\hat{a}3}^C = X_{13}X_{1\hat{a}} P_{\hat{a}3}^C W_{\hat{a}3}^C = X_{13}X_{1\hat{a}} F_{\hat{a}3}.$$

Proof of Proposition 3.4. Let $W \in \mathcal{U}(\hat{A} \otimes A)$ be the reduced bicharacter of $\mathbb{G}$. Recall the Hopf $\ast$-homomorphisms $i : A \to \mathcal{M}(C)$, $\hat{i} : \hat{A} \to \mathcal{M}(\hat{C})$ and the bicharacter $X = (\text{id}_{\hat{A}} \otimes i)W$ from the proof of Lemma 3.5. The bicharacter $X$ corresponds to the Hopf $\ast$-homomorphism $\hat{i}$. Similarly, the bicharacter $\chi := (\hat{i} \otimes \text{id}_A)W \in \mathcal{U}(\hat{C} \otimes A)$ corresponds to the Hopf $\ast$-homomorphism $\hat{i}$. These imply

$$(3.11) \quad (\text{id}_C \otimes i)\chi = (\hat{i} \otimes i)W = P \in \mathcal{U}(\hat{C} \otimes C).$$

This shows that $P$ is the composition of bicharacters (see [13, Definition 3.5]) viewed as the quantum group homomorphisms: $\mathbb{H} \xrightarrow{\chi} \mathbb{G} \xrightarrow{\chi} \mathbb{H}$. More precisely, it is defined by

$$(3.12) \quad X_{\pi3}X_{\pi\pi} = \chi_{13}P_{13}X_{\pi3} \quad \text{in } \mathcal{U}(\hat{C} \otimes K(\mathcal{H}_{\pi}) \otimes C),$$

where $(\pi, \hat{\pi})$ is a $\mathbb{G}$-Heisenberg pair on the Hilbert space $\mathcal{H}_{\pi}$.

Suppose $\Delta_L \in \text{Mor}(C, A \otimes C)$ is the left quantum group homomorphism equivalent to $\chi$ given by [13, Theorem 5.5]

$$(3.13) \quad (\text{id}_C \otimes \Delta_L)W^C = \chi_{12}W_{13}^C \quad \text{in } \mathcal{U}(\hat{C} \otimes A \otimes C).$$

The Landstad-Vaes algebra [218] for the $\mathbb{G}$-product $(C, \Delta_L, \hat{i})$ is given by

$$(3.14) \quad D = \{ (\omega \otimes \text{id}_C)\varphi(c) | \omega \in B(\mathcal{H}_{\pi}), c \in C \}^{\text{CLS}}.$$
where $\varphi(c) = X_{x_2}^* \Delta_L(c) x_{x_2}^2 X_{x_2}$ for all $c \in C$. Using (3.13) and (3.12) we have
\[ D = \{(\omega' \otimes \omega \otimes \text{id}_C)(X_{x_3}^* (\text{id}_C \otimes \Delta_L) \mathcal{W}^C_1 x_{x_3}) \mid \omega' \in \hat{C}, \omega \in \mathcal{B}(H_\pi)\}_{\text{CLS}} \]
\[ = \{(\omega' \otimes \omega \otimes \text{id}_C)(X_{x_3}^* \text{id}_C W^C_{13} x_{x_3}) \mid \omega' \in \hat{C}, \omega \in \mathcal{B}(H_\pi)\}_{\text{CLS}} \]
\[ = \{(\omega' \otimes \omega \otimes \text{id}_C)(X_{x_3}^* \text{id}_C x_{x_3}) \mid \omega' \in \hat{C}, \omega \in \mathcal{B}(H_\pi)\}_{\text{CLS}} \]
\[ = \{(\omega' \otimes \omega \otimes \text{id}_C)(X_{x_3}^* \text{id}_C x_{x_3}) \mid \omega' \in \hat{C}, \omega \in \mathcal{B}(H_\pi)\}_{\text{CLS}} \]
\[ = \{(\omega' \otimes \omega \otimes \text{id}_C)(X_{x_3}^* \text{id}_C x_{x_3}) \mid \omega' \in \hat{C}, \omega \in \mathcal{B}(H_\pi)\}_{\text{CLS}} \]
\[ = \{(\omega' \otimes \omega \otimes \text{id}_C)(X_{x_3}^* \text{id}_C x_{x_3}) \mid \omega' \in \hat{C}, \omega \in \mathcal{B}(H_\pi)\}_{\text{CLS}} \]

where $\{\hat{\alpha}, \hat{\delta}\}$ be an $\mathbb{H}$-anti-Heisenberg pair on the Hilbert space $H_\pi$. Finally, using Lemma 3.15 in the last computation we complete the proof below:
\[ D = \{(\omega' \otimes \omega' \otimes \text{id}_C)(X_{1n}^* F_{13} x_{1n}^* x_{1n}) \mid \omega \in \hat{A}, \omega' \in \mathcal{B}(H_\pi)\}_{\text{CLS}} \]
\[ = \{(\omega' \otimes \omega' \otimes \text{id}_C) F_{\hat{A}^2} \mid \omega' \in \mathcal{B}(H_\pi)\}_{\text{CLS}} = \{(\omega' \otimes \text{id}_C) F \mid \omega' \in \hat{C}\}_{\text{CLS}}. \]

According to [12] Theorem 2.18 an isomorphism between two $C^*$-quantum groups $H_1 = (C_1, \Delta_{C_1})$ and $H_2 = (C_2, \Delta_{C_2})$ is a Hopf $^*$-isomorphism $f \in \text{Mor}(C_1, C_2)$. Let $P$ be a projection on $H_2$ and $G_2 = (A_2, \Delta_{A_2})$ be the image of $P$ for $k = 1, 2$.

**Definition 3.15.** An isomorphism between two $C^*$-quantum groups with projections $(H_1, P_1)$ and $(H_2, P_2)$ is a Hopf $^*$-isomorphism $f \in \text{Mor}(C_1, C_2)$ such that the restriction $f|_{A_1}$ is also a Hopf $^*$-isomorphism between $G_1$ and $G_2$.

Let $W^{A_1} \in U(\hat{A}_1 \otimes A_1)$ and $W^{A_2} \in U(\hat{A}_2 \otimes A_2)$ be the reduced bicharacters of $G_1$ and $G_2$, respectively. Suppose $i_k \in \text{Mor}(A_k, C_k)$ is the Hopf $^*$-homomorphism induced by $P_k$ and $i_k \in \text{Mor}(\hat{A}_k, \hat{C}_k)$ be its dual satisfying (3.11) for $k = 1, 2$. In particular, $(i_k \otimes i_k) W^{A_k} = P_k$ for $k = 1, 2$. The isomorphism $f$ in the Definition 3.15 induces the Hopf $^*$-isomorphism $f_A \in \text{Mor}(A_1, A_2)$ such that $f \circ i_1 = i_2 \circ f_A$.

Let $W^{C_1} \in U(C_1 \otimes C_1)$ and $W^{C_2} \in U(C_2 \otimes C_2)$ be the reduced bicharacters of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Then a Hopf $^*$-isomorphism $f \in \text{Mor}(C_1, C_2)$ is equivalent to the dual Hopf $^*$-isomorphism $\hat{f} \in \text{Mor}(\hat{C}_1, \hat{C}_2)$ which is characterised by the following equation: $(\hat{f} \otimes f) W^{C_1} = W^{C_2} \in U(\hat{C}_1 \otimes C_2)$. Now for any $\mathbb{H}$-Heisenberg pair $(\alpha, \delta)$ on $H$, the pair $(\alpha \circ f, \hat{\delta} \circ \hat{f})$ of representations of $C_1$ and $\hat{C}_1$ on $H$ is an $\mathbb{H}$-Heisenberg pair. Therefore $(\hat{\delta} \circ \hat{f} \otimes f) P_1$ is a manageable multiplicative unitary and generates $(f \circ i_1)(A_1), \Delta f \circ (i_1)$. By duality, the restriction $f_{\hat{A}_1}$ defines a Hopf $^*$-isomorphism between $\hat{G}_1$ and $\hat{G}_2$ (inside $\hat{H}_1$ and $\hat{H}_2$) and it is the dual of $f|_{A_1}$. Hence, we get the Hopf $^*$-isomorphism $f_{\hat{A}_2}$ in $\text{Mor}(\hat{A}_1, \hat{A}_2)$ such that $\hat{f} \circ i_1 = i_2 \circ f_{\hat{A}_2}$ and $(f_{\hat{A}_2} \otimes f_{\hat{A}_1}) W^{A_1} = W^{A_2}$. Then (3.12) gives
\[ (f \otimes f) P_1 = (f \circ i_1 \otimes f \circ i_1) W^{A_1} = (i_2 \circ f_{\hat{A}_2} \otimes i_2 \circ f_{\hat{A}_1}) W^{A_1} = (i_2 \circ i_2) W^{A_2} = P_2. \]

Therefore, $(f \otimes f) P_1 W^{C_1} = P_2 W^{C_2}$. Consequently, $f$ defines an isomorphism between the Lundstad-Vaes algebras in Proposition 3.1 for the $G_1$-product and the $G_2$-product associated to $(\mathbb{H}_1, P_1)$ and $(\mathbb{H}_2, P_2)$ are also isomorphic. We shall use these facts later in Section 5.

4. From braided multiplicative unitaries to quantum groups with projection

Let $G = (A, \Delta_A)$ be a $C^*$-quantum group, let $\hat{G} = (\hat{A}, \hat{\Delta}_A)$ be its dual, and let $W \in U(\hat{A} \otimes \hat{A})$ be the reduced bicharacter of $G$. 

Let $\mathcal{L}$ be a Hilbert space. Consider a pair of representations $(U, V)$ of $G$ and $\hat{G}$ on $\mathcal{L}$ satisfying the commutation relation
\begin{equation}
V_{12} U_{13} W_{23} = W_{23} U_{13} V_{12} \quad \text{in} \ U(\mathcal{K}(\mathcal{L}) \otimes \hat{A} \otimes A).
\end{equation}
The pair $(U, V)$ corresponds to a representation of the quantum codouble $\mathcal{D}(G)^\wedge$, the dual of the Drinfeld double $\mathcal{D}(G)$, of $G$ on $\mathcal{L}$ and it is called $\mathcal{D}(G)^\wedge$-pair on $\mathcal{L}$, see [22]. In [15, Section 5], it was observed that the representation category of $\mathcal{D}(G)^\wedge$ is a unitarily braided monoidal (tensor product of representations of $\mathcal{D}(G)^\wedge$) category. We fix a $\mathcal{D}(G)^\wedge$-pair $(U, V)$ on $\mathcal{L}$ and define $\hat{\mathcal{C}} := Z \circ \Sigma$, where $Z \in U(\mathcal{L} \otimes \mathcal{L})$ is the solution of (2.15). In fact, $\hat{\mathcal{C}}$ is the braiding isomorphism for the pair of objects $((U, V), (U, V))$ in the representation category of $\mathcal{D}(G)^\wedge$.

**Definition 4.2** (compare with [16, Definition 3.2]). A braided multiplicative unitary on $\mathcal{L}$ over $G$ relative to $(U, V)$ is a unitary $F \in U(\mathcal{L} \otimes \mathcal{L})$ such that
\begin{enumerate}
\item $F$ is invariant with respect to the tensor product representation $U \otimes U := U_{13} U_{23}$ of $G$ on $\mathcal{L} \otimes \mathcal{L}$:
\begin{equation}
U_{13} U_{23} F_{12} = F_{12} U_{13} U_{23} \quad \text{in} \ U(\mathcal{K}(\mathcal{L} \otimes \mathcal{L}) \otimes A);
\end{equation}
\item $F$ is invariant with respect to the tensor product representation $V \otimes V := V_{13} V_{23}$ of $\hat{G}$ on $\mathcal{L} \otimes \mathcal{L}$:
\begin{equation}
V_{13} V_{23} F_{12} = F_{12} V_{13} V_{23} \quad \text{in} \ U(\mathcal{K}(\mathcal{L} \otimes \mathcal{L}) \otimes \hat{A});
\end{equation}
\item $F$ satisfies the braided pentagon equation (1.2).
\end{enumerate}

Let $(\pi, \hat{\pi})$ be the $G$-Heisenberg pair on $\mathcal{H}$ coming from a manageable multiplicative unitary $\mathcal{W} \in U(\mathcal{H} \otimes \mathcal{H})$ generating $G$, that is, $(\hat{\pi} \otimes \pi) W = \mathcal{W}$. Using it, we define the unitaries $\hat{\mathcal{V}} \in U(\hat{A} \otimes \mathcal{K}(\mathcal{L}))$, $\mathcal{U}, \mathcal{V} \in U(\mathcal{L} \otimes \mathcal{H})$ and $\hat{\mathcal{V}} \in U(\mathcal{H} \otimes \mathcal{L})$ by
\begin{align*}
\hat{\mathcal{V}} := \sigma(V^*), & \quad U := (id_\mathcal{L} \otimes \pi) U, \quad V := (id_\mathcal{L} \otimes \hat{\pi}) V, \quad \hat{\mathcal{V}} := \Sigma V^* \Sigma = (\hat{\pi} \otimes id_\mathcal{L}) \hat{\mathcal{V}}.
\end{align*}
Then (2.15) and (4.1) for $U$ and $V$ are equivalent to
\begin{align}
Z_{13} &= \hat{\mathcal{V}}_{23} U_{13}^* \hat{\mathcal{V}}_{12}^* U_{12} \quad \text{in} \ U(\mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}); \tag{4.5}
W_{13} U_{23}^* \hat{\mathcal{V}}_{12} &= \hat{\mathcal{V}}_{12} W_{13} U_{23} \quad \text{in} \ U(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{L}). \tag{4.6}
\end{align}
Now $F$ gives rise to a pair of multiplicative unitaries $\mathcal{W}^C, \mathcal{P} \in U(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L})$ given by [16, Theorem 3.7]:
\begin{align}
\mathcal{W}^C := W_{13} U_{23} \hat{\mathcal{V}}_{12} F_{12} \hat{\mathcal{V}}_{12} & \quad \text{in} \ U(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}), \tag{4.7}
\mathcal{P} := W_{13} U_{23} & \quad \text{in} \ U(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L}). \tag{4.8}
\end{align}
Suppose $\mathcal{P}$ is manageable in the sense of [16, Definition 3.5]. Then we translate it to the manageability of the ordinary multiplicative unitary $\mathcal{W}^C$ using [16, Theorem 3.8]. On the other hand, manageability of $\mathcal{P}$ follows from the manageability of $\mathcal{W}$. Let $\mathbb{H} = (C, \Delta) \otimes \mathcal{C}$ be the $C^*$-quantum group generated by $\mathcal{W}^C$. Then [16, Theorem 3.7] and [13, Lemma 3.2] imply $\mathcal{P} \in U(\hat{\mathcal{C}} \otimes \mathcal{C})$ is a projection on $\mathbb{H}$. In the next lemma we ensure that the image of $\mathcal{P}$ is $G$.

**Lemma 4.9.** Let $(\pi, \hat{\pi})$ be a $G$-Heisenberg pair on $\mathcal{H}$. There is a faithful representation $\rho: A \to B(\mathcal{H} \otimes \mathcal{L})$ such that $(\hat{\rho} \otimes \pi) W = W_{13} U_{13} \in U(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H})$. Define $\rho: A \to B(\mathcal{H} \otimes \mathcal{L})$ by $\rho(a) := \pi(a) \otimes 1$. Then $\mathcal{P} = (\hat{\rho} \otimes \rho) W$ and generates the $C^*$-quantum group $G = (A, \Delta_A)$. In particular, $G$ is a Woronowicz closed quantum subgroup of $\mathbb{H}$ as in [3, Definition 3.2].
Proof. Let \((\tilde{\alpha}, \tilde{\alpha})\) be a \(\mathbb{G}\)-anti-Heisenberg pair on a Hilbert space \(\mathcal{H}_0\). By virtue of \((5.9)\), the equation \((2.12)\) for \(U\) is equivalent to
\[
U_{\tilde{\alpha}} W_{\tilde{\alpha}} = W_{\tilde{\alpha}} U_{\tilde{\alpha}} U_{\tilde{\alpha}} \quad \text{in} \ U(\mathbb{K} \otimes \mathcal{H}_0) \otimes A).
\]
Applying \(\sigma_{12}\) on both sides of the last equation and rearranging unitaries we obtain
\[
(4.10) \quad \hat{U} W_{\tilde{\alpha}} = W_{\tilde{\alpha}} U_{\tilde{\alpha}} \quad \text{in} \ U(\mathbb{K} \otimes \mathcal{L}) \otimes A).
\]
Here \(\hat{U} := \sigma(U^*) \in U(A \otimes \mathbb{K}(\mathcal{L}))\). Define a faithful representation \(\hat{\rho}' : \hat{A} \to B(\mathbb{H} \otimes \mathcal{L})\) by \(\hat{\rho}'(\hat{\alpha}) := \hat{U}_{\tilde{\alpha}}(\hat{\alpha}(\tilde{\alpha}) \otimes 1)\hat{U}_{\tilde{\alpha}}\). The right hand side of \((4.10)\) implies the first component of \(\hat{\rho}'(\hat{A})\) is inside the image of \(\hat{\alpha}\). Also the representations \(\hat{\alpha}, \hat{\pi}\) are faithful by \([22]\) Proposition 3.2. These allow to define the desired representation \(\hat{\rho} : \hat{A} \to B(\mathbb{H} \otimes \mathcal{L})\) by \(\hat{\rho}(\hat{\alpha}) := (\hat{\alpha} \circ \hat{\pi}^{-1} \otimes \text{id}_{\mathcal{L}})\hat{\rho}'(\hat{\alpha})\). Then \(\hat{\rho}\) is faithful and satisfies \((\hat{\rho} \circ \pi)W = W_{13} U_{23}\) by \((4.11)\). Since, \(\mathbb{P} = (\hat{\rho} \otimes \rho)W\) is a manageable multiplicative unitary and \(\rho\) is a faithful representation of \(A\) on \(B(\mathbb{H} \otimes \mathcal{L})\), we have \(\pi(A) \otimes 1_\mathcal{L} = \{(\omega \otimes \text{id}_{\mathcal{H} \otimes \mathcal{L}})\mathbb{P} \mid \omega \in B(\mathbb{H} \otimes \mathcal{L})_+\}\). Finally, a simple computation using Theorem \([22]\) shows that, \(\mathbb{P}\) implements the comultiplication map \(\Delta_A\) on \(\rho(A)\):
\[
(\rho \otimes \rho)\Delta_A(a) = \mathbb{P}(\rho(a) \otimes 1)\mathbb{P}^* \quad \text{for all} \ a \in A.
\]
Let us identify \(C, \hat{C}\) with their images inside \(B(\mathbb{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L})\) under the representations obtained from the \(\mathbb{H}\)-Heisenberg pair that arises from the manageable multiplicative unitary \(W_C\) in \((4.7)\). We also notice that the images of \(\rho\) and \(\hat{\rho}\) are contained inside the images of \(C\) and \(\hat{C}\) in \(B(\mathbb{H} \otimes \mathcal{L})\), respectively.

**Proposition 4.11.** The unitary \(\chi := (\hat{\rho} \otimes \text{id}_A)W \in U(\hat{C} \otimes A)\) is a bicharacter from \(\mathbb{H}\) to \(\mathbb{G}\). Suppose \(\Delta_L \in \text{Mor}(C, C \otimes A)\) is the left quantum group homomorphism associated to \(\chi\). The \(C^*\)-quantum group \(\mathbb{H} = (C, \Delta_C)\) with projection \(\mathbb{P}\) with image \(\mathbb{G}\) is equivalent to the quadruple \((G, \mathbb{H}, \rho, \Delta_L)\) described in \([16]\) Proposition 2.8.

**Proof.** Recall \(\mathbb{P} = (\hat{\rho} \otimes \rho)W \in U(\hat{C} \otimes C)\) and, in particular, \(\rho \in \text{Mor}(A, C)\) is faithful. Then the first condition in \((4.1)\) and \((2.6)\) together give
\[
(\Delta_C \circ \hat{\rho} \otimes \rho)W = (\Delta_C \circ \text{id}_C)\mathbb{P} = \mathbb{P}_{23}\mathbb{P}_{13} = ((\hat{\rho} \circ \rho) \circ \Delta_A \otimes \rho)W.
\]
Taking slices on the third leg of the last expression by \(\omega \in C'\) shows that \(\hat{\rho} \in \text{Mor}(A, \hat{C})\) is a Hopf *-homomorphism. Similarly, we can prove that \(\rho \in \text{Mor}(A, C)\) is also a Hopf *-homomorphism.

Thus \(\chi := (\hat{\rho} \otimes \text{id}_A)W \in U(\hat{C} \otimes A)\) is a bicharacter from \(\mathbb{H}\) to \(\mathbb{G}\) and the composition \(\mathbb{H} \to \mathbb{G} \to \mathbb{H}\) is the bicharacter \((\text{id}_C \otimes \rho)\chi = \mathbb{P}\).

Let \(\Delta_R \in \text{Mor}(C, C \otimes A)\) be the right quantum group homomorphism equivalent to \(\chi\). Then \([13]\) Theorem 5.3 and Lemma \([3]\) imply
\[
(\text{id}_A \otimes \Delta_R \circ \rho)W = \chi_{23} W_{13} \chi_{23} = W_{\rho_3} W_{\rho_3} = W_{13} W_{13} = (\text{id}_A \otimes (\rho \circ \text{id}_A) \circ \Delta_A)W.
\]
Taking slices on the first leg of the last expression by \(\omega \in \hat{A}\) gives \(\Delta_R \circ \rho = (\rho \otimes \text{id}_A) \circ \Delta_A\). Finally, \((\Delta_L, \rho)\) is equivalent to \((\Delta_R, \rho)\) \([16]\) Proposition 2.8] and \((\Delta_L, \rho)\) satisfies \((2.13)\).

5. The main results

Borrowing the same notations from the last section we state and prove the first main result of this article.

**Theorem 5.1.** Suppose \(F \in U(\mathbb{L} \otimes \mathbb{L})\) is a manageable braided multiplicative unitary over a regular \(C^*\)-quantum group \(\mathbb{G} = (A, \Delta_A)\) relative to \((U, V)\). Define
\[
(5.2) \quad B := \{(\omega \otimes \text{id}_L)F \mid \omega \in B(\mathcal{L})_+\}_{\text{CLS}}, \quad \Delta_B(b) := F(b \otimes 1_L)F^* \quad \text{for all} \ b \in B.
\]

Then
(1) $B$ is a nondegenerate, separable $C^*$-subalgebra of $\mathbb{B}(\mathcal{L})$;

(2) The elements $\beta \in \text{Mor}(B, \mathbb{K}(\mathcal{L}) \otimes A)$ and $\hat{\beta} \in \text{Mor}(B, \mathbb{K}(\mathcal{L}) \otimes \hat{A})$ defined by

$$\beta(b) := U(b \otimes 1)^{*}, \quad \hat{\beta}(b) := V(b \otimes 1)V^{*}$$

are actions of $G$ and $\hat{G}$ on $B$. Moreover, $(B, \beta, \hat{\beta})$ is an object of the category $\mathcal{YDC}^{*}\text{alg}(G)$.

(3) $\mathcal{F} \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes B)$;

Suppose $j_1, j_2 \in \text{Mor}(B, B \boxtimes B)$ be the canonical morphisms.

(4) Then $\hat{\Delta}_B$ is the unique arrow $B \to B \boxtimes B$ in the category $\mathcal{YDC}^{*}\text{alg}(G)$ characterised by

$$\langle \mathcal{F} \rangle (\Delta_B) = (\Delta_B \otimes 1)^{\mathcal{F}}$$

Furthermore, $\Delta_B$ is coassociative :

$$\langle \mathcal{F} \rangle (\Delta_B \otimes 1)^{\mathcal{F}} = (\Delta_B \otimes 1)^{\mathcal{F}} \circ \Delta_B$$

and satisfies the cancellation conditions:

$$j_1(B)\Delta_B(B) = B \boxtimes B = \Delta_B(B)j_2(B).$$

Proof. The image of $P$ is $G = (A, \Delta_A)$ by Lemma 13.3 and $G$ is regular by assumption. Then we apply Proposition 5.3 for the $G$-triple $(C, \Delta_L, \rho)$ constructed in Proposition 4.11.

Part (1): Since

$$D = \{(\omega' \otimes \omega \otimes \Delta_H(\mathcal{L}))^{\mathcal{F}} \omega | \omega' \in \mathbb{B}(\mathcal{H}), \omega \in \mathbb{B}(\mathcal{L})^{*}\}^{\text{CLS}}$$

is a $C^*$-algebra by Proposition 5.3 hence so is $B := \{(\omega \otimes \Delta_H(\mathcal{L}))^{\mathcal{F}} \omega | \omega \in \mathbb{B}(\mathcal{L})^{*}\}^{\text{CLS}} \subseteq \mathbb{B}(\mathcal{L})$.

Furthermore, the second condition in Theorem 2.17 gives $DC = C$. Also $C(\mathcal{H} \otimes \mathcal{L}) = \mathbb{K}(\mathcal{H} \otimes \mathcal{L})$ because $C$ is constructed from the manageable multiplicative unitary $\mathcal{W}^{C}$, and $\mathcal{V} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L})$. Therefore,

$$(1 \otimes \mathcal{H})\mathbb{K}(\mathcal{H} \otimes \mathcal{L}) = \mathcal{V}D^{*}\mathbb{K}(\mathcal{H} \otimes \mathcal{L}) = \mathcal{V}DK(\mathcal{H} \otimes \mathcal{L}) = \mathcal{V}DC(\mathcal{H} \otimes \mathcal{L})$$

Thus $B$ acts nondegenerately on $\mathcal{L}$. Separability of $B$ follows from the separability of $\mathbb{B}(\mathcal{L})$.

Part (2): Define $\hat{\beta}(b) := V(b \otimes 1A)V^{*}$ for $b \in B$. Clearly, $\hat{\beta}$ is injective. The unitary $X$ in Lemma 3.3 is $(id_{A} \otimes 1)^{W} = W_{12}$ and third condition in Theorem 2.17 becomes

$$\hat{\pi}(\hat{A}) \otimes \hat{\pi}^{*}(1_{H} \otimes B)\hat{V} = (\hat{\pi}(\hat{A}) \otimes 1_{H}(\mathcal{L}))W_{12}^{*}V_{23}^{*}(1_{H} \otimes 1_{\mathcal{L}} \otimes B)V_{23}W_{12}.$$ 

Now the condition 2.12 for the representation $V$ is equivalent to

$$\hat{V}_{23}W_{12} = W_{12}^{*}\hat{V}_{13}V_{23} \quad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{L}).$$

Using it we simplify the right hand side of (5.7):

$$\hat{\pi}(\hat{A}) \otimes \hat{\pi}^{*}(1_{H} \otimes B)\hat{V} = (\hat{\pi}(\hat{A}) \otimes 1_{H}(\mathcal{L}))V_{23}^{*}V_{13}^{*}W_{12}^{*}(1_{H} \otimes 1_{\mathcal{L}} \otimes B)V_{23}W_{12}$$

Injectivity of $\hat{\pi}$ implies

$$\hat{A} \otimes B = (\hat{A} \otimes 1_{\mathcal{L}})\hat{V}^{*}(1_{H} \otimes B)\hat{V}.$$
This is equivalent to Podleś condition (up to $\sigma$). Thus $\hat{\beta} \in \text{Mor}(B, B \otimes \hat{A})$ and the condition (2.12) for $V$ yields (2.10) for $\hat{\beta}$.

Similarly, $\beta(b) := U(b \otimes 1_A)U^*$ is injective, and it is sufficient to establish the Podleś condition for $\beta$. Then $(B, \beta, \hat{\beta})$ will become a $G$-Yetter-Drinfell $C^*$-algebra because the unitaries $\hat{U}$, $\hat{V}$ satisfy the commutation relation (4.1).

By virtue of the second condition in Theorem 2.17 $C = \rho(D) = (\pi(A) \otimes 1_C)\hat{V}^*(1_B \otimes B)\hat{V}$. Recall the right quantum group homomorphism $\Delta_R \in \text{Mor}(C, C \otimes A)$ equivalent to the bicharacter $\chi = (\hat{\rho} \otimes \pi)\hat{W} = \hat{W}_{13}U_{23}$ in Proposition 4.11. In particular, $\Delta_R$ is an action of $G$ on $C$, see [13] Lemma 5.8. Combining the Podleś condition for $\Delta_R$ and [13] Equation (33) we get

$$(\pi(A) \otimes 1_{C \otimes H})\hat{V}_{12}^*(1_H \otimes B \otimes 1_H)^{\pi(A)}\hat{V}_{12}^*(1_H \otimes B \otimes 1_H)\hat{V}_{12}((1_H \otimes C \otimes \pi(A)).$$

Multiplying $K(H)$ to the first leg from left and right of the last equation and using the nondegeneracy of $\pi$, that is $\pi(A)K(H) = K(H)$, we obtain

$$(K(H) \otimes 1_{C \otimes H})\hat{V}_{12}^*(1_H \otimes B \otimes 1_H)^{\pi(A)\otimes 1_{C \otimes H}}\hat{V}_{12}^*(1_H \otimes B \otimes 1_H)\hat{V}_{12}((K(H) \otimes 1_{C \otimes \pi(A)})$$

Similarly, the nondegeneracy of $\hat{\pi}$ and (5.8) together imply

$$(K(H) \otimes 1_{C \otimes H})U_{23}\hat{V}_{13}^*(K(H) \otimes 1_{C \otimes \pi(A)}) = (K(H) \otimes 1_{C \otimes H})U_{13}\hat{V}_{13}^*(K(H) \otimes 1_{C \otimes \pi(A)}).$$

Next we apply Theorem 2.17 (2), that is $W(K(H) \otimes \pi(A)) = K(H) \otimes \pi(A)$, to simplify the last equation

$$K(H) \otimes ((B \otimes 1_H)U^*(1_C \otimes \pi(A))) = K(H) \otimes (U^*(B \otimes \pi(A)).$$

Finally, taking slices by $\omega \in B(H)_+$ on the first leg and then multiplying the last equation by $U$ from the left, we obtain

$$U(B \otimes 1_H)U^*(1_C \otimes \pi(A)) = B \otimes \pi(A).$$

This is equivalent to the Podleś condition for $\beta$ as $\pi$ is injective.

**Part (3):** Once again, recall the second condition in the Landstad theorem (2.17)

$$C = (\pi(A) \otimes 1_C)\hat{V}^*(1_B \otimes B)\hat{V} \subset B(H \otimes L).$$

Since $W^C$ is a unitary multiplier of $K(H \otimes L) \otimes C$ we have $(K(H) \otimes K(L))W^C = K(H) \otimes K(L) \otimes C$.

Equivalently,

$$\hat{V}_{34}B_1\hat{V}_{34}K(H)K(L)\pi(A)_3\hat{V}_{34}F_{24}\hat{V}_{34} = K(H)K(L)\pi(A)_3\hat{V}_{34}B_1\hat{V}_{34}.$$

Here we have used the leg numbering for $C^*$-algebras: $K(H) = K(H) \otimes 1_{C \otimes H \otimes L}$, $K(L)_2 = 1_H \otimes K(L) \otimes 1_{C \otimes \pi(A)} \otimes 1_L$, and $B_4 = 1_{H \otimes C \otimes H \otimes B}$.

Using $K(H)K(L)\pi(A)_3\hat{V}_{34}F_{24}\hat{V}_{34}K(H)K(L)\pi(A)_3\hat{V}_{34}B_4\hat{V}_{34}$ we simplify the last equation

$$\hat{V}_{34}B_4\hat{V}_{34}K(H)K(L)\pi(A)_3\hat{V}_{34}F_{24}\hat{V}_{34} = K(H)K(L)\pi(A)_3\hat{V}_{34}B_4\hat{V}_{34}.$$
and it is equivalent to
\[ K(H)_1 K(L)_2 K(H)_3 B_4 V_{23} F_{24} = K(H)_1 K(L)_2 K(H)_3 B_4 V_{23} \]
Now \( V_{23} \) commutes with \( B_4 \). Moreover, \( K(L)_2 K(H)_3 V_{23} = K(L)_2 K(H)_3 \). Subsequently, the last equation becomes
\[ K(H)_1 K(L)_2 K(H)_3 B_4 F_{24} = K(H)_1 K(L)_2 K(H)_3 B_4. \]
Taking the slices on the first and third legs by \( \omega, \omega' \in B(H)_\ast \) give \( (K(L) \otimes B)F = K(L) \otimes B \). This shows that \( F \) is a unitary right multiplier of \( K(L) \otimes B \). Multiplying both sides of the above equation by \( F^\ast \) from the right gives \( K(L) \otimes B = (K(L) \otimes B)F^\ast \); hence \( F \) is also a unitary left multiplier of \( K(L) \otimes B \).

**Part (4):** Using the definition \( \Delta_B \) and the braided pentagon equation \([1.3]\) we verify \([5.4]\):
\[
(id_L \otimes \Delta_B)F = F_{23} b_{12}^\ast F_{23} = F_{12} c_{23} F_{12} F_{12} c_{23}. 
\]
Since \( F \in U(K(L) \otimes B) \), the right hand side of the last equation is in \( U(K(L) \otimes B) \). Hence, the image of \( \Delta_B \) lies in \( M(B \boxtimes B) \). Furthermore, taking slices on the first leg of the first equality gives \( \Delta_B(b) = F(b \otimes 1_L)F^\ast \) for all \( b \in B \). This shows that \( \Delta_B \) is the unique \(*\)-homomorphism satisfying \([1.3]\).

Next we recall \((K(L) \otimes B)F = (K(L) \otimes B)\) and use it in the following computation
\[
(K(L) \otimes j_1(B))(id_L \otimes \Delta_B)F = (K(L) \otimes j_1(B))((id_L \otimes j_1)F(id_L \otimes j_2)F)
\]
\[
= ((id_L \otimes j_1)((K(L) \otimes B)F))(id_L \otimes j_2)F
\]
\[
= (K(L) \otimes j_1(B))(id_L \otimes j_2)F. 
\]
Slicing the first leg by \( \omega \in B(L)_\ast \) on both sides gives \( j_1(B) \Delta_B(B) = j_1(B)j_2(B) = B \boxtimes B \). A similar computation yields that \( \Delta_B(j_2(B)j_1(B)) = B \boxtimes B \). Consequently,
\[
\Delta_B(B)j_1(B_1)(B) = (B \boxtimes B)j_1(B) = j_2(B_1)j_1(B) = B \boxtimes B
\]
shows that \( \Delta_B \) is non-degenerate.

Once again, the braided pentagon equation \([1.3]\) yields coassociativity of \( \Delta_B \):
\[
(\Delta_B \boxtimes id_B)\Delta_B(b) = F_{12} c_{23} F_{12} b_1^\ast F_{12} c_{23} F_{12} = F_{23} b_1^\ast F_{12} F_{23} = (id_B \otimes \Delta_B)\Delta_B(b). 
\]

Recall the diagonal action \( \beta \bowtie \beta \) of \( G \) on \( B \boxtimes B \) is described by \([2.24]\):
\[
\beta \bowtie \beta : B \boxtimes B \rightarrow B \boxtimes B \otimes A, \quad x \mapsto U_{13} U_{23} (x \otimes 1_A) U_{23}^\ast U_{13}^\ast. 
\]
The invariance \([1.8]\) of \( F \) gives
\[
\beta \bowtie \beta \Delta_B(b) = U_{13} U_{23} F_{12} (b \otimes 1_L \otimes H) F_{12} U_{23}^\ast U_{13}^\ast
\]
\[
= F_{12} U_{13} U_{23} (b \otimes 1_L \otimes H) U_{23}^\ast U_{13}^\ast = (\Delta_B \otimes id_A)\beta(b); 
\]
hence \( \Delta_B \) is \( G \)-equivariant. Similarly, we can show that \( \Delta_B \) is \( \hat{G} \)-equivariant. □

**Definition 5.9.** A braided \( C^\ast \)-quantum group over \( G \) is a pair \((B, \Delta_B)\) consisting of an object \((B, \beta, \hat{\beta})\) and a morphism \( \Delta_B \in Mor(B, B \boxtimes B) \) in the category \( \mathcal{YD}C^\ast_{\text{alg}}(G) \) constructed out of a manageable braided multiplicative unitary \( F \) over \( G \) described in the way as in Theorem 5.1. Then we say \((B, \Delta_B)\) is generated by \( F \).

Two braided \( C^\ast \)-quantum groups \((B, \Delta_B)\) and \((B', \Delta_{B'})\) over \( G \) are isomorphic if there is an isomorphism \( f \in Mor(B, B') \) in the category \( \mathcal{YD}C^\ast_{\text{alg}}(G) \) such that \((f \boxtimes f) \circ \Delta_B = \Delta_{B'} \circ f \).
5.1. Duals of braided $C^*$-quantum groups. Now we construct the reduced dual of $(B, \Delta_B)$ as a braided $C^*$-quantum group and prove the Pontrjagin duality theorem for braided quantum groups.

By [10] Definition 3.3, Proposition 3.4 & 3.6], the dual of a manageable braided multiplicative unitary $F \in U(L \otimes L)$ over $\mathbb{G}$ with respect to $(U, V)$, defined by $\hat{F} := \mathbb{C}_x \otimes \mathbb{C}_y \in U(L \otimes L)$ is again a manageable braided multiplicative unitary over $\mathbb{G}$ with respect to $(U, V)$. Also, $\hat{\mathbb{G}}$ is regular if and only if $\mathbb{G}$ is regular.

**Corollary 5.10.** $\hat{F}$ generates a braided $C^*$-quantum group $(\hat{B}, \hat{\Delta}_B)$ over $\hat{\mathbb{G}}$.

More precisely,

$$\hat{B} := \{(\omega \otimes \text{id}_L)\hat{F} \mid \omega \in \mathbb{B}(L)_*\}^{\text{CLS}}, \quad \hat{\Delta}_B(\hat{b}) := \hat{F}(\hat{b} \otimes 1_L)^{\hat{F}} \text{ for all } \hat{b} \in \hat{B}.$$

By construction $\hat{B}$ is a $\hat{\mathbb{G}}$-Yetter-Drinfeld $C^*$-algebra with respect to the actions $\hat{\delta} : \hat{B} \to \hat{B} \otimes \hat{A}$ and $\hat{\delta} : \hat{B} \to \hat{B} \otimes \hat{A}$ defined by

$$\hat{\delta}(\hat{b}) := V(\hat{b} \otimes 1_A)V^*, \quad \hat{\delta}(\hat{b}) := U(\hat{b} \otimes 1_A)U^*, \text{ for all } \hat{b} \in \hat{B}.$$  \hfill (5.11)

In particular, a variant of Theorem [2.20] shows that the monoidal product $\hat{B} \hat{\otimes} B$ in $\mathcal{YDCA}_\text{alg}(\hat{\mathbb{G}})$ is defined by $\hat{B} \hat{\otimes} B := \iota_1(B) \iota_2(B) \subset B(L \otimes L)$ where $\iota_1, \iota_2$ are faithful representations of $\hat{B}$ on $L \otimes L$ defined by $\iota_1(\hat{b}) := b \otimes 1_L$ and $\iota_2(\hat{b}) := \mathbb{C}_x \otimes \mathbb{C}_y \mathbb{C}_x \otimes \mathbb{C}_y$ for all $\hat{b} \in \hat{B}$. Consequently, $\hat{\Delta}_B : \hat{B} \to \hat{B} \hat{\otimes} B$ is an arrow in $\mathcal{YDCA}_\text{alg}(\hat{\mathbb{G}})$.

**Definition 5.12.** The braided $C^*$-quantum group $(B, \Delta_B)$ is said to be the (reduced) dual of $(B, \Delta_B)$.

Once again, [16] Definition 3.3, Proposition 3.4 & 3.6] imply the dual of $\hat{F}$ is $F$.

Consequently, we obtain the braided analogue of the Pontrjagin duality theorem:

**Corollary 5.13.** The dual of $(\hat{B}, \hat{\Delta}_B)$ is (canonically) isomorphic to $(B, \Delta_B)$ as a braided $C^*$-quantum group over $\mathbb{G}$.

5.2. The bosonization. The reconstruction of the ordinary $C^*$-quantum group $\mathbb{H} = (C, \Delta_C)$ and a projection with image $\mathbb{G}$ starting from a braided $C^*$-quantum group $(B, \Delta_B)$ over $\mathbb{G}$ is called as **bosonization**. In the compact case, that is, when $A$ and $B$ are unital, this has been already done in [13] Theorem 6.4. We extend this result for general $C^*$-quantum groups, essentially, using the same ingredients. According to Theorem [2.20] $\mathcal{YDCA}_\text{alg}(\mathbb{G})$ is a monoidal category and $(B, \beta, \beta)$ is an object of the category $\mathcal{YDCA}_\text{alg}(\mathbb{G})$. Also, regularity of $\mathbb{G}$ makes $A$ an object of $\mathcal{YDCA}_\text{alg}(\mathbb{G})$ as well, see Example 2.21.

Then $A \boxtimes B := (A \otimes 1_L)\hat{V}^*(1_H \otimes B)\hat{V}$ as shown in [15] Page 19]. Here we have suppressed the faithful representations of $A$ and $B$ on $H$ and $L$, respectively. In fact, $B \ni b \mapsto \hat{V}^*(1_H \otimes B)\hat{V} \in \hat{A} \otimes \hat{B}$ defines a left action of the co-opposite quantum group $\hat{\mathbb{G}}^{\text{cop}} := (\hat{A}, \sigma \circ \hat{\Delta}_A)$ of $\hat{\mathbb{G}}$ and $A \boxtimes B = \hat{\mathbb{G}}^{\text{cop}} \rtimes B$ ($\cong B \rtimes_{\hat{\beta}} \hat{\mathbb{G}}$).

By virtue of [15] Proposition 6.3] we get an injective morphism $\Psi : A \boxtimes B \boxtimes B \to A \boxtimes B \otimes A \boxtimes B$ defined by

$$A \boxtimes B \boxtimes B \ni x \mapsto W_{12}U_{23}V_{124}^*W_{124}U_{23}^*W_{12}^*.$$ \hfill (5.14)

**Proposition 5.15.** Let $C = A \boxtimes B$. Define $\Delta_C \in \text{Mor}(C, C \otimes C)$ by $\Delta_C := \Psi \circ (\text{id}_B \boxtimes \Delta_B)$. Then $\mathbb{H} = (C, \Delta_C)$ is the $C^*$-quantum group generated by $\mathbb{W}^C$ in $\mathbb{H}$. Moreover, $\mathbb{H}$ is a $C^*$-quantum group with a projection and $\mathbb{G}$ becomes image of the projection.

**Proof.** Let $L = \{(\omega \otimes \omega' \otimes \text{id}_N)\mathbb{W}^C \mid \omega \in \mathbb{B}(H), \omega' \in \mathbb{B}(L)\}^{\text{CLS}}$. 


Using (2.2) we get
\[ L = \{(\omega \otimes \omega') \otimes \text{id}_{\mathcal{H} \otimes \mathcal{L}})W_{13} U_{23} \hat{V}_{23}^* F_{24} \hat{V}_{34} | \ \omega, \omega' \in \mathbb{B}(\mathcal{L}), a \in \mathbb{B}(\mathcal{L}_+)^G \} \]

For \( \omega' \in \mathbb{B}(\mathcal{L}), \) and \( \xi \in \mathbb{K}(\mathcal{L}) \) define \( \omega' \cdot \xi \in \mathbb{B}(\mathcal{L}), \) by \( \omega' \cdot \xi(y) := \omega'(\xi y) \).

Replacing \( \omega' \cdot \xi \) in the last expression we get
\[ L = \{(\omega \otimes \text{id}_{\mathcal{H} \otimes \mathcal{L}})((\xi \otimes a) U_{23} \hat{V}_{23}^* F_{13} \hat{V}_{32}) | \omega' \in \mathbb{B}(\mathcal{L}), \xi \in \mathbb{K}(\mathcal{L}), a \in \mathbb{A} \} \]

We may also replace \( (\xi \otimes a) U \) by \( \xi \otimes a \) for \( \xi \in \mathbb{K}(\mathcal{L}), a \in \mathbb{A}, \) because \( U \in \mathcal{U}(\mathbb{K}(\mathcal{L}) \otimes \mathcal{A}) \)

And \( \mathbb{U} = (\text{id}_{\mathcal{H}} \otimes \pi) \mathbb{U}. \) We have
\[ L = \{(\omega \otimes \text{id}_{\mathcal{H} \otimes \mathcal{L}})((\xi \otimes a \otimes 1) \hat{V}_{23}^* F_{13} \hat{V}_{32}) | \omega' \in \mathbb{B}(\mathcal{L}), \xi \in \mathbb{K}(\mathcal{L}), a \in \mathbb{A} \} \]

Finally using (5.2) we obtain
\[ L = \{(\omega \otimes \text{id}_{\mathcal{H} \otimes \mathcal{L}})((1 \otimes 1 \otimes 1) \hat{V}_{23}^* F_{13} \hat{V}_{32}) | \xi \in \mathbb{K}(\mathcal{L}), a \in \mathbb{A}, \omega' \in \mathbb{B}(\mathcal{L}_+) \} \]

Now for any \( c \in \mathbb{C} = \mathbb{A} \otimes \mathbb{B} \subset \mathbb{H} \)
\[ \Delta_C(c) = \Psi((\text{id}_B \otimes \Delta_B)(c)) = \Psi(F_{23}(c \otimes 1_L)^2_{23}) \]
\[ = W_{13} U_{23} \hat{V}_{23}^* F_{24} \hat{V}_{34} W_{12}^* \]

Theorem 2.4 shows that \( \Delta_C \in \text{Mor}(\mathbb{C}, \mathbb{C} \otimes \mathbb{C}) \) is the unique element satisfying \( (\text{id}_C \otimes \Delta_C) \mathbb{W} = \mathbb{W} \mathbb{W} \).

The braided \( \mathbb{C} \)-quantum group \( G \) is the projection on \( \mathbb{H} \) with image \( G = (A, \Delta_A) \).

Suppose \( \mathbb{W} \in \mathcal{U}(\hat{C} \otimes \mathbb{C}) \) is the reduced bicharacter of \( \mathbb{H} = (C, \Delta_C), \mathbb{P} \in \mathcal{U}(\hat{C} \otimes \mathbb{C}) \) is the projection on \( \mathbb{H} \) and the image of \( \mathbb{P} \) is the regular \( \mathbb{C} \)-quantum group \( G = (A, \Delta_A) \).

Then we can construct a manageable braided multiplicative unitary \( \mathbb{F} \) over \( G \) described in the way as in [10] Theorem 3.9. Suppose, \( \mathbb{F} \) gives rise to the braided \( \mathbb{C} \)-quantum group \( (B, \Delta_B) \). Then the associated bosonization to projection \( \mathbb{P} \) is the regular \( \mathbb{C} \)-quantum group \( G \).

Hence, starting with a \( \mathbb{C} \)-quantum group \( \mathbb{H} \) with a projection \( \mathbb{P} \) such that \( \mathbb{W} \in \mathcal{U}(\hat{C} \otimes \mathbb{C}) \) is the regular \( \mathbb{C} \)-quantum group \( G \) we can construct a manageable braided \( \mathbb{C} \)-quantum group \( (B, \Delta_B) \) over \( G \) and reconstruct \( \mathbb{H} \) as the bosonization of \( (B, \Delta_B) \) and the projection \( \mathbb{P} \) on \( \mathbb{H} \), up to isomorphism.

Next we show that the construction \( (\mathbb{H}, \mathbb{P}) \rightarrow (B, \Delta_B) \) respects the isomorphisms. For that matter, let us recall the Drinfeld’s double \( \mathbb{D}(\mathbb{H}_i) = (\mathcal{D}_i, \Delta_{\mathbb{D}_i}) \) of \( \mathbb{H}_i \) from [22] Example 5.12 for \( i = 1, 2 \).

The isomorphism between \( \mathbb{H}_i \) and \( \mathbb{H}_j \) is the Hopf *-homomorphism. Consider the faithful representation \( \pi_i \in \text{Mor}(\mathfrak{D}_i, \mathfrak{K}(\mathfrak{H}_i)) \) for \( i = 1, 2 \). Define \( \mathbb{U}_i := (\pi_i \otimes \theta_i \otimes \text{id}_C) \mathbb{P}_i \in \mathcal{U}(\mathfrak{K}(\mathfrak{H}_i) \otimes \mathfrak{C}_i) \).

Then \[ \mathbb{F}_i := (\pi_j \otimes \theta_i \otimes \pi_j \otimes \rho_i \otimes \text{id}_C) \mathbb{P}_j \in \mathcal{U}(\mathfrak{K}(\mathfrak{H}_i) \otimes \mathfrak{C}_i) \]

Therefore, \[ \mathbb{F}_i \in \mathcal{U}(\mathfrak{K}(\mathfrak{H}_i) \otimes \mathfrak{C}_i) \]

Finally, the isomorphism \( f \) respects the isomorphisms between \( (\mathbb{H}_i, \mathbb{P}_i) \) and \( (\mathbb{H}_j, \mathbb{P}_j) \).

Consider the dual Hopf *-isomorphism \( \hat{f} \in \text{Mor}(\hat{\mathbb{C}_1}, \hat{\mathbb{C}_2}) \).

Then \( f \) induces a Hopf *-isomorphism \( h: \pi_1(D_1) \cong D_1 \rightarrow D_2 \cong \pi_2(D_2) \).
such that $h \circ \pi_1 \circ \rho_1 = \pi_2 \circ \rho_2 \circ f$ and $h \circ \pi_1 \circ \theta_1 = \pi_2 \circ \theta_2 \circ \hat{f}$. So, $(h \otimes f) U_1 = U_2$, $(h \otimes \hat{f}) V_1 = V_2$ and $(h \otimes h) \mathbb{F}_1 = \mathbb{F}_2$. Then $B_i = \{ \omega \otimes \text{id} \mathbb{F}_i \mid \omega \in \mathcal{H}(H_i) \}$ for $i = 1, 2$. Also, the Landstad-Vaes algebras in Proposition 5.4 are associated to $(\mathbb{H}_1, \mathbb{P}_1)$ and $(\mathbb{H}_2, \mathbb{P}_2)$, respectively and are isomorphic. Consequently, the restriction $h_B$ of $h$ on $B_1$ defines an isomorphism between $B_1$ and $B_2$. Since, $\mathbb{H}_1$ and $\mathbb{H}_2$ are isomorphic $C^*$-quantum groups, we may identify $A_2$ with $f(A_1)$. Let $G = \mathbb{G}_1 = \mathbb{G}_2$. Now, the $G$-action on $B_1$ is given by $B_1 \ni b_1 \rightarrow U_i(b_1 \otimes 1_A)U_i^*$ for $i = 1, 2$. Then $h_B$ is $G$-equivariant. Similarly, the $G$-action on $B_2$ is implemented by $V_i$ for $i = 1, 2$; hence $h_B$ is also $G$-equivariant. Therefore, $h_B \in \operatorname{Mor}(B_1, B_2)$ is an isomorphism in the category $\mathcal{YD}C^*\mathfrak{Alg}(G)$. Denote the embeddings $j_{1,i}, j_{2,i} : B_i \rightarrow B_i$ for $i = 1, 2$. Following 5.4 we characterise $\Delta_{B_i}$ and $\Delta_{B_3}$ as follows

$$(id_{H_i} \otimes \Delta_{B_i}) \mathbb{F}_1 = (id_{H_i} \otimes j_{1,1}) \mathbb{F}_1(id_{H_i} \otimes j_{2,1}) \mathbb{F}_1,$$

$$(id_{H_2} \otimes \Delta_{B_2}) \mathbb{F}_2 = (id_{H_2} \otimes j_{1,2}) \mathbb{F}_2(id_{H_2} \otimes j_{2,2}) \mathbb{F}_2.$$  

Then $h \otimes (h_B \boxtimes h_B)$ maps the first equation to the second equation; hence $(h_B \boxtimes h_B) \circ \Delta_{B_1} = \Delta_{B_2} \circ h_B$. Hence, $h_B$ defines an isomorphism of braided $C^*$-quantum groups between $(B_1, \Delta_{B_1})$ and $(B_2, \Delta_{B_2})$.

On the other hand, let $(B, \Delta_B)$ be a braided $C^*$-quantum group over a regular $C^*$-quantum group $G$. Suppose $\mathbb{H} = (C, \Delta_C)$ is the bosonization of $(B, \Delta_B)$ and $\mathbb{P}$ is the projection on $\mathbb{H}$ as constructed in the way as in Proposition 5.10. As before, we construct a braided $C^*$-quantum group $(B_1, \Delta_{B_1})$ to be the $G$ from $(\mathbb{H}, \mathbb{P})$ and its bosonization $\mathbb{H}_1 = (C_1, \Delta_{C_1})$ along with the projection $P_1$ on $\mathbb{H}_1$. Then $(\mathbb{H}, \mathbb{P})$ and $(\mathbb{H}_1, \mathbb{P}_1)$ are isomorphic $C^*$-quantum groups with projection. Consequently, $(B, \Delta_B)$ and $(B_1, \Delta_{B_1})$ are isomorphic braided $C^*$-quantum groups.

Finally, we are going to show that the construction $(B, \Delta_B) \rightarrow (\mathbb{H}, \mathbb{P})$ respects the isomorphisms. Suppose $(B_1, \Delta_{B_1})$ and $(B_2, \Delta_{B_2})$ are isomorphic braided $C^*$-quantum groups over a regular $C^*$-quantum group $G = (A, \Delta_A)$. Let $f \in \operatorname{Mor}(B_1, B_2)$ be the isomorphism in the category $\mathcal{YD}C^*\mathfrak{Alg}(G)$. This extends to an isomorphism $h \in \text{id}_A \boxtimes f \in \operatorname{Mor}(A \boxtimes B_1, A \boxtimes B_2)$ such that $h \circ i_A^1 = i_A^2$ and $h \circ i_{B_1} = i_{B_2} \circ f$. Here $i_A^1, i_{B_1} : A, B_1 \rightarrow C_1 = A \boxtimes B_1$ are the canonical morphisms for $k = 1, 2$. In order to keep track of the copy of $A$ inside $C_k$ we use different notations $i_A^1, i_A^2$ for their embeddings, whereas $i_A^1 = i_A^1$.

Suppose $H_k = (C_k, \Delta_{C_k})$ is the $C^*$-quantum group with the projection $P_k$ given by Proposition 5.14 for $k = 1, 2$. Then the images of $P_1$ and $P_2$ are isomorphic to the regular $C^*$-quantum group $G$. Also, $B_k$ is identified with the Landstad-Vaes algebra $i_{B_k}(B_k) \subset \mathcal{M}(C_k)$ for the $G$-product $(C_k, \Delta_{C_k}^L, i_{C_k}^1)(\Delta_{C_k}^L)$ is the left action of $G$ on $C_k$) induced by the projection $P_k$ on $H_k$ in Proposition 5.14 for $k = 1, 2$.

Recall the injective morphism $\Psi_k : A \boxtimes B_k \boxtimes B_k \rightarrow C_k \otimes C_k$ constructed in [15, Proposition 6.3] for $k = 1, 2$. On the embeddings $j_{1, k}, j_{2, k}, j_{3, k}$ they are defined by

$$\Psi_k j_{1, k}^1(a) = (i_A^1 \otimes i_A^2) \Delta_A(a), \quad \Psi_k j_{2, k}^1(b_k) = (i_{B_k} \otimes i_A^2) \beta^k(b_k),$$

where $\beta^k \in \operatorname{Mor}(B_k, B_k \otimes A)$ is the $G$-action on $B_k$ for $k = 1, 2$. Now $\Delta_{C_k} = \Psi_k \otimes (\text{id}_A \boxtimes \Delta_{B_k})$ for $k = 1, 2$.

Here $\beta^k \in \operatorname{Mor}(B_k, B_k \otimes A)$ is the $G$-action on $B_k$ for $k = 1, 2$. Now $\Delta_{C_k} = \Psi_k \otimes (\text{id}_A \boxtimes \Delta_{B_k})$ for $k = 1, 2$.

Clearly,

$$(h \otimes h) \circ \Delta_{C_1} \circ i_A^1 = (h \circ i_A^1 \otimes h \circ i_A^1) \circ \Delta_A = (i_A^2 \otimes i_A^2) \circ \Delta_A = \Delta_{C_2} \circ i_A^1 = \Delta_{C_2} \circ h \circ i_A^1.$$ 

So the restriction $h|_{A}$ is a Hopf $^*$-isomorphism. Using the fact that $f$ is $G$-equivariant and $h \circ i_{B_1} = i_{B_2} \circ f$ we verify $(h \circ f) \circ \Psi_{|A, B_1} j_{l, k}^1 = \Psi_{A, B_2} j_{l, k}^1 \circ f$ for $l = 2, 3.$
This yields
\[(h \otimes h) \circ \Delta_C \circ i_{B_1} = (h \otimes h) \Psi_{A,B_1}(\text{id}_A \boxtimes \Delta_{B_1}) \circ i_{B_1},\]
\[= \Psi_{A,B_2} \circ (\text{id}_A \boxtimes (f \boxtimes f) \circ \Delta_{B_1}) \circ i_{B_1},\]
\[= \Psi_{A,B_2} \circ (\text{id}_A \circ \Delta_{B_1}) \circ (\text{id}_A \circ f) \circ i_{B_1},\]
\[= \Delta_{C_2} \circ h \circ i_{B_1} = \Delta_{C_2} \circ h \circ i_{B_1}.\]

Therefore, h defines an isomorphism between $\mathbb{H}_1$ and $\mathbb{H}_2$. Hence, $(\mathbb{H}_1, P_1)$ and $(\mathbb{H}_2, P_2)$ are isomorphic C*-quantum groups with projection. Summarising, we have the following result.

**Theorem 5.16.** Isomorphism classes of braided C*-quantum groups over a regular C*-quantum group $G$ are in one to one correspondence with the isomorphism classes of C*-quantum groups with a projection generating $G$ as its image.

6. **Examples**

6.1. **C*-quantum groups with an idempotent Hopf *-homomorphism.** Suppose, $f \in \text{Mor}(C, C)$ is an idempotent Hopf *-homomorphism on a C*-quantum group $H = (C, \Delta_C)$. Let $W^C \in \mathcal{U}(\hat{C} \otimes C)$ be the reduced bicharacter of $H$. The unitary $P := (\text{id}_C \otimes f)W^C \in \mathcal{U}(\hat{C} \otimes C)$ is the unique bicharacter corresponding to $f$. Since $f$ is idempotent, by [13, Definition 3.5] $P$ also satisfies (3.2); hence $P$ is a projection on $\mathbb{H}$.

Clearly, $A := \text{Im}(f) = \{(\omega \otimes \text{id}_C)P \mid \omega \in \hat{C}^*\}^{\text{cls}}$ and $\Delta_A := \Delta_C|_{A} \in \text{Mor}(A, \text{id} \otimes A)$ satisfy $(\text{id}_C \otimes \Delta_A)P = P_{12}P_{13}$. So, $\mathbb{H} = (C, \Delta_C)$ is a quantum group with projection $P$ with image $G = (A, \Delta_A)$. Theorem 5.16 says that there exists a unique braided C*-quantum group $(B, \Delta_B)$ over $G$ and $\mathbb{H}$ is the associated bosonization.

Quantum $E(2)$ groups [31], quantum $\alpha z + b$ groups [25, 34] and quantum $\alpha x + b$ groups [36] are examples of C*-quantum groups with an idempotent Hopf *-homomorphism generating the multiplicative subgroups $\mathbb{T}, q^{z+iR}$ (for a suitably chosen deformation parameter $q \in \mathbb{C} \setminus \{0\}$), and $\mathbb{R}^{\times}_{>0}$ of $\mathbb{C} \setminus \{0\}$ as their images, respectively. For more details we refer [21, Section 6.2.1], [13, Section 4] and [3] Example 3.7.

6.2. **Braided compact quantum groups.** Suppose $G = (A, \Delta_A)$ is a compact quantum group. By [15, Definition 6.1], a braided compact quantum group over $G$ a pair $(B, \Delta_B)$ consisting of a unital $G$-Yetter-Drinfeld C*-algebra $(B, \beta, \hat{\beta})$ and a unital *-homomorphism $\Delta_B : B \to B \boxtimes B$ satisfying (5.5) and (5.6).

**Proposition 6.1.** Every braided compact quantum group over $G$ is a braided C*-quantum group with the underlying C*-algebra being unital.

**Proof.** Let $(B, \Delta_B)$ be a braided compact quantum group over $G$. Suppose, $H = (C, \Delta_C)$ is the bosonization, which is a compact quantum group, of $(B, \Delta_B)$ as in [15, Theorem 4.4]. Let $h$ be the Haar state of $H$ and let $\mathcal{H}_h$ be the GNS space. Then the regular representation $W^C \in \mathcal{U}(\mathcal{H}_h \otimes \mathcal{H}_h)$ of $H$ on $\mathcal{H}_h$ is a manageable multiplicative unitary and generates $H = (C, \Delta_C)$.

Moreover, there is a projection on $\mathbb{H}$ consisting of the canonical embedding $i_A : A \to A \boxtimes B = C$ and the left quantum group homomorphism $\Delta_L \in \text{Mor}(C, A \otimes C)$ given by [16, Proposition 2.8 & 2.10]. Let $P \in \mathcal{U}(C \otimes C)$ be the projection equivalent to $(i_A, \Delta_L)$. Following [16, Theorem 3.9] we may construct a manageable braided multiplicative unitary $F \in \mathcal{U}(\mathcal{H}_h \otimes \mathcal{H}_h \otimes \mathcal{H}_h \otimes \mathcal{H}_h)$ over $G$. Subsequently, the braided C*-quantum group generated by $F$ is isomorphic to $(B, \Delta_B)$. $\square$
6.3. The complex quantum planes and their bosonizations. Throughout this section, we shall consider $G = (C(T), Δ C(T))$ and $\hat{G} = (C_0(Z), Δ_c_0(Z))$. Let $\mathcal{H} = L^2(Z)$ and let $\{e_p\}_{p \in \mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$. A multiplicative unitary $W \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ generating $T$ is given by $W(e_k \otimes e_l) := e_k \otimes e_{l+k}$ for all $k, l \in Z$.

Since $T$ and $Z$ are Abelian groups, the quantum codouble of $G$ coincides with $Z \times T$, while viewed as a $C^*$-quantum group. Similarly, the category of $G$-Yetter-Drinfeld and the category of $\hat{G}$-Yetter-Drinfeld $C^*$-algebras are equivalent to the category of $Z \times T$-$C^*$-algebras and $T \times Z$-$C^*$-algebras, respectively.

Fix $L = \mathcal{H} \otimes \mathcal{H}$ and the orthonormal basis $\{e_{i,j} := e_i \otimes e_j\}_{i,j \in \mathbb{Z}}$. The canonical representations of $C(T) \cong C^*(Z)$ and $C_0(Z) \cong C^*(T)$ on $L$ through the unitary $U$ and the self adjoint operator $N$ with spectrum $Z$ and commuting with $U$. Subsequently, the right and left representations $U \in \mathcal{U}(\mathcal{L} \otimes \mathcal{H})$ and $\hat{U} := \Sigma \mathcal{V}^* \Sigma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L})$ and the resulting braiding operator $\mathcal{E}_{\mathcal{H} \otimes \mathcal{H}}$ are defined by

$$\mathcal{U} = \mathcal{W}_{23}, \quad \hat{\mathcal{U}} = \mathcal{W}_{12}, \quad \mathcal{E}_{\mathcal{H} \otimes \mathcal{H}} = Z \Sigma = \mathcal{W}_{23}^* \Sigma.$$

For a fixed $0 < q < 1$, let $C_q^T$ be the subgroup $q^Z \mathbb{R}$ of the multiplicative group $C \setminus \{0\}$ and let $\mathcal{C}_q = C_q^T \cup \{0\}$. Define $\mathcal{Y} = \Phi_T |\mathcal{Y}|$ as a closed operator acting on $L$ by

$$\Phi_T e_{i,j} := e_{i,j+1}, \quad |\mathcal{T}| e_{i,j} := q^{i+j} e_{i,j}, \quad T e_{i,j} := q^{2i+1} e_{i,j+1}.$$  

The operator $\Phi_T$ is unitary, $|\mathcal{Y}|$ is a strictly positive operator such that

$$\Phi_T |\mathcal{Y}| = q^{-1} |\mathcal{Y}|, \quad \mathcal{S} |\mathcal{Y}| = q^2 \cup \{0\}.$$  

Thus $T^{-1} e_{i,j} := q^{-2i+1} e_{i,j-1}$ and the polar decomposition $T^{-1} = \Phi_{T^{-1}} |\mathcal{Y}^{-1}|$ gives a unitary operator $\Phi_{T^{-1}}$, a strictly positive operator $|\mathcal{Y}^{-1}|$ with spectrum $q^Z \cup \{0\}$, and $\Phi_{T^{-1}}$ and $|\mathcal{Y}^{-1}|$ satisfy the following commutation relation

$$\Phi_{T^{-1}} |\mathcal{Y}^{-1}| \Phi_{T^{-1}}^* = q |\mathcal{Y}^{-1}|.$$

Proposition 6.5. Define

$$B := \left\{ \sum_{k \in \mathbb{Z}}^\text{finite} \Phi_{T^{-1}}^k f_k (|\mathcal{Y}^{-1}|) \bigg| f_k \in C_0(\mathcal{C}_q), \ f_k(0) = 0 \text{ for } k \neq 0 \right\}^\text{CLS}. $$

Then $B$ is a $C^*$-algebra, $\mathcal{Y}^{-1} \mathcal{U}$ and $\mathcal{B}$ is generated by $\mathcal{Y}^{-1}$.

Proof. For any two elements $\Phi_{T^{-1}}^k f_k (|\mathcal{Y}^{-1}|), \Phi_{T^{-1}}^l f_l (|\mathcal{Y}^{-1}|) \in B$ we observe that

$$\Phi_{T^{-1}}^k f_k (|\mathcal{Y}^{-1}|) \Phi_{T^{-1}}^l f_l (|\mathcal{Y}^{-1}|) = \Phi_{T^{-1}}^{k+l} f_k (q^2 |\mathcal{Y}^{-1}|) f_l (|\mathcal{Y}^{-1}|) \in B$$

and $B$ is $^*$-invariant; hence $B$ is a $C^*$-algebra. Rest of the proof follows using a similar line of argument used in [27 Proposition 4.1 (2-3)].

The maps $\gamma : \mathcal{Y}^{-1} \to \mathcal{U}(\mathcal{Y}^{-1} \otimes 1) \mathcal{U}^* = \mathcal{T}^{-1} \otimes u^* \eta B \otimes C(\mathcal{T})$ and $\hat{\gamma} : \mathcal{Y}^{-1} \to \mathcal{V}(\mathcal{Y}^{-1} \otimes 1) \mathcal{V}^* = \mathcal{T}^{-1} \otimes q^{-2} \eta B \otimes C_0(\mathcal{Z})$ define $\mathcal{T}$ and $\mathcal{Z}$ actions on $B$, respectively. Here $u$ and $\hat{N}$ are the generators of $C(\mathcal{T})$ and $C_0(\mathcal{Z})$ defined by $w^p := e_{p+1}$ and $\hat{N} e_p := pe_p$, respectively. Thus $B$ is a $Z \times T$-$C^*$-algebra and using the braiding unitary $\otimes$ in [24] we define $B \boxtimes B$. On the generator $\mathcal{Y}^{-1}$ the canonical $Z \times T$-equivariant embeddings $j_1, j_2 \in \text{Mor}(B, B \boxtimes B)$ are defined by

$$j_1(\mathcal{Y}^{-1}) := \mathcal{Y}^{-1} \otimes 1, \quad j_2(\mathcal{Y}^{-1}) := Z(1 \otimes \mathcal{Y}^{-1}) \mathcal{Z}^* = q^{-2} \hat{N} \otimes \mathcal{Y}^{-1},$$

where $\hat{N} e_{i,j} := je_{i,j}$. Now we recall the manageable braided multiplicative unitary over $\mathcal{T}$ relative to $(\mathcal{U}, \mathcal{V})$ constructed in [16] Theorem 4.1:

$$F := F_q(\mathcal{V}^{-2} \otimes \mathcal{Y}^{-1}) \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L}),$$

where $F_q : \mathcal{C}_g(\mathcal{T}) \to \mathcal{T}$ is the quantum exponential function [31].
Lemma 6.9. The following identity holds
\begin{equation}
F_q(T_q^{-2N} \otimes \Upsilon^{-1})(\Upsilon^{-1} \otimes 1)F_q(T_q^{-2N} \otimes \Upsilon^{-1})^* = j_1(\Upsilon^{-1}) + j_2(\Upsilon^{-1}).
\end{equation}

Proof. Suppose, \( \tilde{\Upsilon} \) is any closed operator acting on some Hilbert space \( \mathcal{L}' \) such that \( \ker(\tilde{\Upsilon}) = \{0\} \) and \( \Phi_{\tilde{\Upsilon}}(\tilde{\Upsilon}) = q^{-1}|\tilde{\Upsilon}| \), where \( \tilde{\Upsilon} = \Phi_{\tilde{\Upsilon}}(\tilde{\Upsilon}) \) is the polar decomposition of \( \tilde{\Upsilon} \). Define \( r := \tilde{\Upsilon} \otimes \Upsilon^{-1} \otimes 1 \) and \( s := \tilde{\Upsilon} \otimes q^{-2N} \otimes \Upsilon^{-1} \). A simple computation shows that the operators \( r \) and \( s \) are normal, \( \text{Sp}(r), \text{Sp}(s) \subseteq \mathbb{C}(q) \), and satisfy the commutation relations in [31] (0.1). By [31] Theorem 2.2 we get
\begin{equation}
F_q(1 \otimes \tilde{\Upsilon}q^{-2N} \otimes \Upsilon^{-1})(\tilde{\Upsilon} \otimes \Upsilon^{-1} \otimes 1)F_q(1 \otimes \tilde{\Upsilon}q^{-2N} \otimes \Upsilon^{-1})^* = \tilde{\Upsilon} \otimes \Upsilon^{-1} \otimes 1 + \tilde{\Upsilon} \otimes q^{-2N} \otimes \Upsilon^{-1}.
\end{equation}
Since \( \tilde{\Upsilon} \) is arbitrary, we have (6.10). \( \square \)

We shall prove that \((B, \Delta_B)\) with \( \Delta_B(\Upsilon^{-1}) := j_1(\Upsilon^{-1}) + j_2(\Upsilon^{-1}) \) is the braided \( C^* \)-quantum group over \( T \) generated by \( F \). For that purpose, we need to modify the techniques used by Woronowicz and Zakrzewski in [38] Theorem 4.1 and as the operator \( \Upsilon^{-1} \) is not normal. The following result is due to S. L. Woronowicz and it generalises [38] Proposition A.1.

Proposition 6.11. Let \( T_1 \) be nonzero closed densely defined operator acting on \( H_1 \) and let \( D_i \) be a nondegenerate \( C^* \)-subalgebra of \( B(H_i) \) for \( i = 1, 2 \). Then \((T_1 \otimes T_2)\eta(D_1 \otimes D_2)\) is if and only if \( T_1\eta D_1 \) and \( T_2\eta D_2 \).

Proof. The proof of reverse implication follows from [38] Theorem 6.1. For the other direction assume that \((T_1 \otimes T_2)\eta(D_1 \otimes D_2)\). Then \((T_1T_1 \otimes T_2T_2)\eta(D_1 \otimes D_2)\) and using [38] Proposition A.1 we obtain \( T_1T_1 \otimes 1 \) and \( 1 \otimes T_2T_2 \) are affiliated to \( D_1 \otimes D_2 \). Now \( z_{T_1} \otimes z_{T_2} = z_{T_1} \otimes z_{T_2} \) where \( f : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) defined by \( f(x, y) = (1 + xy)^{\frac{1}{2}}(1 + x)^{-\frac{1}{2}}(1 + y)^{-\frac{1}{2}} \). Therefore, \( z_{T_1} \otimes z_{T_2} \in \mathcal{M}(D_1 \otimes D_2) \) and taking appropriate slices give \( z_{T_i} \in \mathcal{M}(D_i) \) for \( i = 1, 2 \). Then we know that \((T_1T_1 \otimes 1)\eta(D_1 \otimes D_2)\) and \( T_1T_1 + 1 = (1 - z_{T_1}^*z_{T_1})^{-1}\eta D_1 \). This shows that the domain of \( T_1T_1 \) coincides with the range of \((1 - z_{T_1}^*z_{T_1}) \otimes 1 \) and this implies \(((1 - z_{T_1}^*z_{T_1}) \otimes 1)(D_1 \otimes D_2) \) is dense in \( D_1 \otimes D_2 \). Hence, \((1 - z_{T_1}^*z_{T_1})D_1 \) is dense in \( D_1 \). Similarly we can prove that \( T_2 \) is also affiliated to \( D_2 \). \( \square \)

In the next result, we construct the complex quantum plane as a braided \( C^* \)-quantum group \((B, \Delta_B)\) over \( T \).

Theorem 6.12. \((B, \Delta_B)\) is a braided \( C^* \)-quantum group over \( T \) generated by \( F \). Equivalently, \( B = \{(\omega \otimes \text{id}_\mathcal{L})F_q(\Upsilon^{-2N} \otimes \Upsilon^{-1}) | \omega \in \mathbb{B}(\mathcal{L})_*\}^{\text{CLS}} \) and \( \Delta_B(\Upsilon^{-1}) := j_1(\Upsilon^{-1}) + j_2(\Upsilon^{-1}) \) is the unique \( \mathbb{Z} \times T \)-equivariant element \( \Delta_B \in \text{Mor}(B, B \otimes B) \) satisfying (5.4) - (5.9).

Proof. Let \( B' := \{(\omega \otimes \text{id}_\mathcal{L})F_q(\Upsilon^{-2N} \otimes \Upsilon^{-1}) | \omega \in \mathbb{B}(\mathcal{L})_*\}^{\text{CLS}} \). Then \( B' \) is a \( C^* \)-algebra given by Theorem 5.1. Since, \( \Upsilon^{-2N} \) is a closed operator acting on \( \mathcal{L} \), it is affiliated to \( \mathbb{K}(\mathcal{L}) \). This implies that \( \Upsilon^{-2N} \otimes \Upsilon^{-1} \) is affiliated to \( \mathbb{K}(\mathcal{L}) \otimes \mathbb{K}(\mathcal{L}) \). Consequently, \( F \in \mathcal{H}(\mathbb{K}(\mathcal{L}) \otimes B) \) because of [38] Theorem 5.1]. Thus, from the definition of \( B' \), we have \( B' \subseteq \mathcal{M}(B) \).

Now \( \mathbb{F}(\mathbb{K}(\mathcal{L}) \otimes B) = \mathbb{K}(\mathcal{L}) \otimes B \) implies
\begin{equation}
B'B = \{(\omega \otimes \text{id}_\mathcal{L})F(1 \otimes b) | \omega \in \mathbb{B}(\mathcal{L})_* , b \in B\}^{\text{CLS}} = \{(\omega \otimes \text{id}_\mathcal{L})F(m \otimes b) | \omega \in \mathbb{B}(\mathcal{L})_* , m \in \mathbb{K}(\mathcal{L}) , b \in B\}^{\text{CLS}} = \{(\omega \otimes \text{id}_\mathcal{L})F | \omega \in \mathbb{B}(\mathcal{L})_* , m \in \mathbb{K}(\mathcal{L}) , b \in B\}^{\text{CLS}} = B.
\end{equation}
To prove $B = B'$, it is sufficient to show that $B'B = B'$. We shall obtain this by showing the canonical embedding $B \hookrightarrow \mathcal{B}(L)$ is an element of $\text{Mor}(B, B')$.

Define $T(\lambda) := F_q(\lambda_Y q^{-2N} \otimes Y^{-1})$ and $T'(\lambda) := F_q(\lambda_Y q^{-2N} \otimes q^{-2N} \otimes Y^{-1})$ for all $\lambda \in \mathcal{T}_q(q)$. By \cite[Theorems 2.2 & 3.1]{31}, we get $Z \otimes Y$ and $\Delta$ yield the changes in the braiding operator $\mathcal{C}$, $\mathcal{L}$, and $\mathcal{I}$ on $\mathcal{T}_q(q)$, respectively. Therefore, $(\mathcal{C}, \mathcal{L}, \mathcal{I})$ are continuous with respect to the strict topology. Therefore, $(T(\lambda) \otimes 1_{B'})_{\lambda \in \mathcal{T}_q(q)}$ is a continuous family of elements of $\text{Mor}(\mathcal{K}(L) \otimes \mathcal{K}(L) \otimes \mathcal{K}(L))$.

For a fixed $\lambda \in \mathcal{T}_q(q)$ we observe the operators

$$ R := \lambda_Y q^{-2N} \otimes Y^{-1} \otimes 1, \quad S := \lambda_Y q^{-2N} \otimes q^{-2N} \otimes Y^{-1}, $$

are normal, $\text{Sp}(R), \text{Sp}(S) \subseteq \mathcal{T}_q(q)$, and satisfy the commutation relations in \cite[(0.1)]{31}. By \cite[Theorems 2.2 & 3.1]{31} we get

$$ F_q(R^{-1}S)F_q(R)F_q(R^{-1}S)^* = F_q(F_q(R^{-1}S)RF_q(R^{-1}S)^*) = F_q(R)F_q(S) $$

and is equivalent to

$$ T(\lambda_1)_2\mathcal{T}(\lambda)_{12}\mathcal{T}_2 = F_q(\lambda_Y q^{-2N} \otimes q^{-2N} \otimes Y^{-1}). $$

Now $F \in \text{Mor}(\mathcal{K}(L) \otimes B')$ and $T(\lambda) \in \text{Mor}(\mathcal{K}(L) \otimes \mathcal{K}(L))$ implies $T'(\lambda) \in \text{Mor}(\mathcal{K}(L) \otimes \mathcal{K}(L) \otimes B')$ for all $\lambda \in \mathcal{T}_q(q)$. This shows that $\lambda \mapsto T'(\lambda) \in \text{Mor}(\mathcal{K}(L) \otimes \mathcal{K}(L) \otimes B')$ is continuous with respect to the strict topology. Therefore, $\mathcal{T}_q(q)$ is an element of $\text{Mor}(B, B')$, see \cite[Definition 3.1]{32}. Consequently, \textbf{Lemma 6.9} shows $\Delta_B(T^{-1}) = j_1(T^{-1}) + j_2(T^{-1})$. Finally, since $T^{-1}\eta_B$ and $\Delta_B \in \text{Mor}(B, B \boxtimes B)$ so is $\Delta_B(T^{-1})\eta_B \boxtimes B$.

### 6.3.1. Dual of the complex quantum plane.

Suppose $(\hat{B}, \hat{\Delta}_B)$ is the dual braided $C^*$-quantum group of $(B, \Delta_B)$ over $\mathbb{T}$ generated by the dual of $\hat{F}$ given by \cite[Definition 3.3 & Proposition 3.4]{16}. Since the roles of $\mathbb{G}$ and $\mathbb{G}$ are exchanged it yield the changes in the braiding operator $\mathcal{C}$ of $(\mathbb{C}, \mathbb{C})^* = \hat{\mathbb{C}}$ with $\hat{\mathbb{C}} := Z^* \Sigma$ and $\hat{B}$ is an object $\mathcal{YDE} \ast \mathcal{G}(\mathbb{G})$. A variant of the Proposition \textbf{6.5} shows that $\hat{B}$ is generated by $\mathcal{Y}$. Let $\mathcal{A}$ be the monoidal product of $\mathcal{YDE} \ast \mathcal{G}(\mathbb{G})$ and $i_1, i_2$ be the canonical morphisms $\mathcal{B} \mapsto \mathcal{B} \boxtimes \mathcal{B}$ defined by \cite[(2.3)]{22} (with respect to the braiding $\mathcal{C}$).

On $\mathcal{Y}$ they are defined by

$$ i_1(\mathcal{Y}) := \mathcal{Y} \otimes 1, \quad i_2(\mathcal{Y}) := \hat{Z}(1 \otimes \mathcal{Y})\hat{Z}^* = \mathcal{U} \otimes \mathcal{Y}. $$

A similar analysis describes the dual $(\hat{B}, \hat{\Delta}_B)$ of $(B, \Delta_B)$ as a braided quantum group over $\mathbb{Z}$.

\textbf{Corollary 6.15.} $\hat{B} = \{ (\omega \otimes \text{id}_\mathcal{L}) \hat{B} | \omega \in \mathcal{B}(\mathcal{L}) \}$ is a $\mathcal{G}$-Yetter-Drinfeld $C^*$-algebra with respect to the $\mathbb{G}$ and $\mathbb{G}$ actions $\hat{\delta}$ and $\delta$ defined by $\mathcal{T} \mapsto \mathcal{T} \otimes u$ and $\mathcal{T} \mapsto \mathcal{T} \otimes q^{-2N}$, respectively. The sum $i_1(\mathcal{Y}) + i_2(\mathcal{Y})$ is affiliated to $\hat{B} \boxtimes \hat{B}$.

The map $\hat{\Delta}_B(\mathcal{Y}) := i_1(\mathcal{Y}) + i_2(\mathcal{Y})$ is the unique $\mathbb{T} \times \mathbb{Z}$-equivariant element $\hat{\Delta}_B \in \text{Mor}(\hat{B}, \hat{B} \boxtimes \hat{B})$ satisfying \cite[(5.3), (5.4)]{5.3} for the dual of $\mathcal{F}$. Thus, $(\hat{B}, \hat{\Delta}_B)$ is a braided $C^*$-quantum group over $\mathbb{Z}$.

We may also realise $\hat{B}$ as a $\mathbb{T}$-Yetter-Drinfeld $C^*$-algebra. On the other hand, a simple observation shows that the polar decomposition $\gamma^* = \Phi \gamma \gamma^*$ gives a unitary operator $\Phi \gamma$, a strictly positive operator $|\gamma^*|$ with spectrum $q^2 \cup \{0\}$ and satisfy the commutation relation \cite[(6.3)]{5.4}. Since, $\hat{B}$ is also generated by $\gamma^*$, the map $\gamma : \gamma^* \mapsto \gamma$ extends to an isomorphism between $B$ and $\hat{B}$ in the category of $\mathbb{T}$-$C^*$-algebras. Thus $B$ is isomorphic to $\hat{B}$ also in the category of $\mathbb{T}$-
Z-C*-algebras. Consequently, \( f \) an isomorphism of braided C*-quantum groups between \((B, \Delta_B)\) and \((B, \Delta_B)\) over \(Z\).

6.3.2. The bosonization. Now we describe the quantum group with projection \((C, \Delta_C)\) in Proposition 5.15 associated to the quantum plane \((B, \Delta_B)\). Here \(\hat{G}\) is the compact group \(\hat{T}\) viewed as a quantum group then \(C = C(\hat{T}) \otimes B\). In fact \(C \cong B \otimes \hat{Z}\), where \(\hat{\gamma}\) is defined by \(\hat{\gamma}_m(Y^{-1}) = q^{-2m}Y^{-1}\). The embeddings of \(C(T)\) and \(B\) are given by \(u \mapsto u \otimes 1\) and \(Y^{-1} \mapsto V(Y^{-1}) = q^{2N}Y^{-1}\). Using the definitions of the unitaries \(U, \hat{W}, \hat{V}\) and \(\hat{F} \) we compute that \((\hat{W}C)(u \otimes 1 \otimes 1) (\hat{W}C)^* = u \otimes 1 \otimes u \otimes 1\)

\[
\hat{W}C(q^{-2N} \otimes Y^{-1} \otimes 1 \otimes 1)(\hat{W}C)^* = \hat{W}_{13} U_{23} \hat{V}_{14} F_{24}(q^{-2N} \otimes Y^{-1} \otimes 1 \otimes 1) F_{24} \hat{V}_{14} U_{23} \hat{W}_{13}.
\]

\[
\hat{W}_{13} U_{23} \hat{V}_{14} (q^{-2N} \otimes (Y^{-1} \otimes 1 \otimes 1 + q^{-2N} \otimes 1 \otimes Y^{-1})) \hat{V}_{14} U_{23} \hat{W}_{13}.
\]

\[
\hat{W}_{13} U_{23} (q^{-2N} \otimes (Y^{-1} \otimes 1 \otimes 1 + q^{-2N} \otimes 1 \otimes Y^{-1})) U_{23} \hat{W}_{13}.
\]

\[
\hat{W}_{13} (q^{-2N} \otimes (Y^{-1} \otimes u^* \otimes 1 + 1 \otimes q^{-2N} \otimes Y^{-1})) \hat{W}_{13}.
\]

\[
q^{-2N} \otimes Y^{-1} \otimes u^* \otimes 1 + 1 \otimes Y^{-1} \otimes q^{-2N}.
\]

Define \(\Psi := q^{-2N} \otimes Y^{-1}\) and \(V := u^* \otimes 1\). Then \(C\) is the universal C*-algebra generated by \(\Psi\) and \(V\) satisfying the following (formal) relations

\[(6.16)\]

\[V^* V = V V^* = 1, \quad \Psi^* \Psi = q^{-2} \Psi^* \Psi, \quad \text{Sp}(\Psi) = \{q^2 \cup \{0\}\}, \quad V \Psi V^* = q^{-2} \Psi, \]

and the comultiplication map \(\Delta_C \in \text{Mor}(C, C \otimes C)\) is given by

\[\Delta_C(V) = V \otimes V, \quad \Delta_C(\Psi) = \Psi \otimes V + 1 \otimes \Psi.\]

In fact, \((C, \Delta_C)\) are closely related to \(E_q(2)\) groups \([30]\). For a fixed \(0 < q < 1\) the quantum \(E(2)\) group \((C', \Delta_{C'})\) is described by a unitary operator \(v\) and a normal operator \(n\) with \(\text{Sp}(n) = q^2 \cup \{0\}\). Underlying C*-algebra \(C'\) is generated by \(v\) and \(n\) subject to the commutation relation \(v^* v = q n\) and \(\Delta_{C'} \in \text{Mor}(C', C' \otimes C')\) is defined by \(\Delta_{C'}(v) = v \otimes v\) and \(\Delta_{C'}(n) = v \otimes n + n \otimes v^*\). A simple observations show that \(V = v^*\) and \(\Psi = v^* n\) satisfy \((6.16)\) and \(\Delta_{C|C'} = \Delta_{C}\). Therefore, there exists a unique Hopf *-homomorphism \(f : C \rightarrow C'\) such that \(f(V) = v^*\) and \(f(\Psi) = v^* n\). The image of \((C, \Delta_C)\) inside \((C', \Delta_{C'})\) was constructed by Woronowicz (in an unpublished work) under the name simplified quantum \(E(2)\) groups.

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