CONNECTIVITY OF THE BRANCH LOCUS OF MODULI SPACE OF RATIONAL MAPS

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Abstract. Moduli space \( M_d \) of rational maps of degree \( d \geq 2 \) has an orbifold structure, with a singular locus \( S_d \). Inside \( M_d \) there is also its branch locus \( B_d \), consisting of those equivalence classes of rational maps with non-trivial holomorphic automorphisms. Milnor observed that we may identify \( M_2 \) with \( \mathbb{C}^2 \) and, within that identification, that \( B_2 \) is a cubic curve; so \( B_2 \) is connected and \( S_2 = \emptyset \). If \( d \geq 3 \), then \( B_d = S_d \). In this paper we use simple arguments to prove the connectivity of \( B_d \).

1. Introduction

The space \( \text{Rat}_d \) of complex rational maps of degree \( d \geq 2 \) can be identified with a Zariski open set of the \((2d + 1)\)-dimensional complex projective space \( \mathbb{P}^{2d+1}_\mathbb{C} \); this is the complement of the algebraic variety defined by the resultant of two polynomials of degree at most \( d \). The group of Möbius transformations \( \text{PSL}_2(\mathbb{C}) \) acts on \( \text{Rat}_d \) by conjugation; \( \phi, \psi \in \text{Rat}_d \) are said to be equivalent if there is some \( T \in \text{PSL}_2(\mathbb{C}) \) so that \( \psi = T \circ \phi \circ T^{-1} \).

The \( \text{PSL}_2(\mathbb{C}) \)-stabilizer of a rational map \( \phi \in \text{Rat}_d \), denoted as \( \text{Aut}(\phi) \), is called the group of holomorphic automorphisms of \( \phi \). It is well known that \( \text{Aut}(\phi) \) is finite.

The quotient space \( M_d = \text{Rat}_d/\text{PSL}_2(\mathbb{C}) \) is the moduli space of rational maps of degree \( d \). Silverman [7] obtained that \( M_d \) carries the structure of an affine geometric quotient and Milnor [6] proved that it also carries the structure of a complex orbifold of dimension \( 2(d - 1) \) (Milnor also obtained that \( M_2 \cong \mathbb{C}^2 \)). Levy [4] noted that \( M_d \) is a rational variety (he also observed that the order of \( \text{Aut}(\phi) \) is bounded above by a constant depending on \( d \)).

Let us denote by \( \pi : \text{Rat}_d \to M_d \) the holomorphic branch cover obtained by the above conjugation action, that is, its deck group is \( \text{PSL}_2(\mathbb{C}) \). This map fails to be a covering exactly at those rational maps with non-trivial group of holomorphic automorphisms. Let us denote by \( B_d \subset M_d \) the locus of branch values of \( \pi \).

Using Milnor’s identification \( M_2 = \mathbb{C}^2 \), the locus \( B_2 \) corresponds to the cubic \( y^2 = x^3 \); the cuspid corresponds to the class of a rational map \( \phi(z) = 1/z^2 \) with \( \text{Aut}(\phi) \cong \text{D}_3 \) (dihedral group of order 6) and all other points in the cubic corresponds to those classes of rational maps with the cyclic group \( \Z_2 \) as group of holomorphic automorphisms. In this way, \( B_2 \) is connected. In this paper we prove the connectivity of \( B_d \) for every \( d \geq 2 \).

Theorem 1. The branch locus \( B_d \) is connected.
Let us denote by $S_d \subset M_d$ the singular locus of $M_d$, that is, the set of points over which $M_d$ fails to be a topological manifold. As already seen above, as $M_2 \cong \mathbb{C}^2$, so $S_2 = \emptyset$. But, if $d \geq 3$, $S_d = B_d$ [5], in particular, $S_d \neq \emptyset$. Theorem 1 then asserts the following.

**Corollary 1.** The singular locus $S_d$ of $M_d$ is connected.

Theorem 1 states that given any two rational maps $\phi, \psi \in \text{Rat}_d$, both with non-trivial group of holomorphic automorphisms, there a conjugated $\rho \in \text{Rat}_d$ of $\psi$ and there is a continuous family $\Theta : [0,1] \to \text{Rat}_d$ with $\Theta(0) = \phi$, $\Phi(1) = \rho$ and $\text{Aut}(\Theta(t))$ non-trivial for every $t$. Even if $\text{Aut}(\phi) \cong \text{Aut}(\psi)$, we may not ensure that $\text{Aut}(\Theta(t))$ stay in the same isomorphic class; this comes from the existence of rigid rational maps [3] (in the non-cyclic situation).

**Remark 1.** In the 80’s Sullivan provided a dictionary between dynamic of rational maps and the dynamic of Kleinian groups [8]. If we restrict our Kleinian groups to be co-compact Fuchsian groups of a fixed genus $g \geq 2$, then we will be dealing with closed Riemann surfaces of genus $g$. The moduli space of Riemann surfaces $M_g$ has the structure of an orbifold. The branch locus in $M_g$ is the set of isomorphic classes of Riemann surfaces with non-trivial holomorphic automorphisms. In [1] it was proved that in general the branch locus is non-connected; a great difference with the connectivity of brach locus for rational maps.

2. **Rational maps with non-trivial group of holomorphic automorphisms**

It is well known that the group of holomorphic automorphisms of $\phi \in \text{Rat}_d$, where $d \geq 2$, is a finite subgroup of $\text{PSL}_2(\mathbb{C})$ which are known to be (see, for instance, [2]) either the trivial group or isomorphic to a cyclic group or the dihedral group or the alternating groups $A_4, A_5$ or the symmetric group $S_4$. Moreover, for each possible group above there are rational maps admitting it as group of holomorphic automorphisms [3].

If $G$ is either a finite cyclic group, a dihedral group, $A_4, A_5$ or $S_4$, then we denote by $B_d(G) \subset M_d$ the locus of classes of rational maps $\phi$ with $\text{Aut}(\phi)$ admitting a subgroup isomorphic to $G$. We say that $G$ is admissible for $d$ if $B_d(G) \neq \emptyset$.

If $G$ is either cyclic or dihedral or $A_4$, then there may be some elements in $B_d(G)$ with group of holomorphic automorphisms non-isomorphic to $G$ (i.e., they admit more holomorphic automorphisms). If $G$ is either isomorphic to $S_4$ or $A_5$, then $B_d(G)$ may have isolated points [3], so it is not connected in general.

2.1. **Admissibility of the non-cyclic cases.** Miasnikov-Stout-Williams observed the following admissibility for the case of platonic groups.

**Theorem 2** ([5]). Let $d \geq 2$.

1. $A_4$ is admissible for $d$ if and only if $d$ is odd.
2. $A_5$ is admissible for $d$ if and only if $d$ is congruent modulo 30 to either 1, 11, 19, 21.
3. $S_4$ is admissible for $d$ if and only if $d$ is coprime to 6.
2.2. Admissibility in the cyclic case. In the case $G = C_n$, the cyclic group of order $n \geq 2$, the admissibility will depend on $d$. First, let us observe that if a rational map admits $C_n$ as a group of holomorphic automorphisms, then we may conjugate it by a suitable M"obius transformation so that we may assume $C_n$ is generated by the rotation $T(z) = \omega_n z$, where $\omega_n = e^{2\pi i/n}$. In [5] it was noticed that in this situation the rational map must have the form $\phi(z) = z\psi(z^n)$, where $\psi \in \text{Rat}_r$, for a suitable value of $r$. Since $d \geq 2$, we must have $r \geq 1$. The relations between $r$ and $d$ are provided as follows.

**Theorem 3 ([5]).** Let $d, n \geq 2$ be integers. The group $C_n$ is admissible if and only if $d$ is congruent to either $-1, 0, 1$ modulo $n$. Moreover, for such values, every rational map of degree $d$ admitting $C_n$ as a group of holomorphic automorphisms is equivalent to one of the form $\psi(z) = z\psi(z^n)$, where

$\psi(z) = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k} \in \text{Rat}_r,$

satisfies that

(a) $a_t b_0 \neq 0$, if $d = nr + 1$;
(b) $a_r \neq 0$ and $b_0 = 0$, if $d = nr$;
(c) $a_r = b_0 = 0$, if $d = nr - 1$.

**Proof.** Let $\phi$ be a rational map admitting a holomorphic automorphism of order $n$. By conjugating it by a suitable M"obius transformation, we may assume that such automorphism is the rotation $T(z) = \omega_n z$.

(1) Let us write $\phi(z) = z \rho(z)$. The equality $T \circ \phi \circ T^{-1} = \phi$ is equivalent to $\rho(\omega_n z) = \rho(z)$. Let

$\rho(z) = \frac{U(z)}{V(z)} = \frac{\sum_{k=0}^{l} \alpha_k z^k}{\sum_{k=0}^{l} \beta_k z^k},$

where either $\alpha_t \neq 0$ or $\beta_t \neq 0$ and $(U, V) = 1$.

The equality $\rho(\omega_n z) = \rho(z)$ is equivalent to the existence of some $\lambda \neq 0$ so that

$\omega_n^k \alpha_k = \lambda \alpha_k, \quad \omega_n^k \beta_k = \lambda \beta_k.$

By taking $k = l$, we obtain that $\lambda = \omega_n^l$. So the above is equivalent to have, for $k < l$,

$\omega_n^{l-k} \alpha_k = \alpha_k, \quad \omega_n^{l-k} \beta_k = \beta_k.$

So, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $l - k \equiv 0 \mod (n)$. As $(U, V) = 1$, either $\alpha_0 \neq 0$ or $\beta_0 \neq 0$; so $l \equiv 0 \mod (n)$. In this way, if $\alpha_k \neq 0$ or $\beta_k \neq 0$, then $k \equiv 0 \mod (n)$. In this way, $\rho(z) = \psi(z^n)$ for a suitable rational map $\psi(z)$.

(2) It follows from (1) that $\phi(z) = z \psi(z^n)$, for $\psi \in \text{Rat}_r$ and suitable $r$. We next provide relations between $d$ and $r$. Let us write

$\psi(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{r} a_k z^k}{\sum_{k=0}^{r} b_k z^k},$

where $(P, Q) = 1$ and either $a_r \neq 0$ or $b_r \neq 0$. In this way,

$\phi(z) = \frac{z P(z^n)}{Q(z^n)} = \frac{z \sum_{k=0}^{r} a_k z^{kn}}{\sum_{k=0}^{r} b_k z^{kn}}.$

Let us first assume that $Q(0) \neq 0$, equivalently, $\psi(0) \neq \infty$. Then $\phi(0) = 0$ and the polynomials $z P(z^n)$ and $Q(z^n)$ are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) = 0$, $\phi(\infty) = \infty$ and $\deg(\phi) = 1 + nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr$. 
Let us now assume that $Q(0) = 0$, equivalently, $\psi(0) = \infty$. Let us write $Q(u) = u^l \hat{Q}(u)$, where $l \geq 1$ and $\hat{Q}(0) \neq 0$; so $\deg(Q) = l + \deg(\hat{Q})$. In this case,

$$\phi(z) = \frac{P(z^n)}{z^{n-1} \hat{Q}(z^n)}$$

and the polynomials $P(z^n)$ (of degree $n\deg(P)$) and $z^{n-1} \hat{Q}(z^n)$ (of degree $n\deg(Q) - 1$) are relatively prime. If $\deg(P) \geq \deg(Q)$, then $r = \deg(P)$, $\psi(\infty) = 0$, $\phi(\infty) = \infty$ and $\deg(\psi) = nr$. If $\deg(P) < \deg(Q)$, then $r = \deg(Q)$, $\psi(\infty) = 0$, $\phi(\infty) = 0$ and $\deg(\phi) = nr - 1$.

Summarizing all the above, we have the following situations:

(i) If $\phi(0) = 0$ and $\phi(\infty) = \infty$, then $\psi(0) \neq \infty$ and $\psi(\infty) \neq 0$; in particular, $d = nr + 1$. This case corresponds to have $a_r b_0 \neq 0$.

(ii) If $\phi(0) = \infty = \phi(\infty)$, then $\psi(0) = \infty$ and $\psi(\infty) \neq 0$; in which case $d = nr$. This case corresponds to have $a_r \neq 0$ and $b_0 = 0$.

(iii) If $\phi(0) = 0 = \phi(\infty)$, then $\psi(0) \neq \infty$ and $\psi(\infty) = 0$; in particular, $d = nr$. This case corresponds to have $a_r = 0$ and $b_0 \neq 0$. But in this case, we may conjugate $\phi$ by $A(z) = 1/z$ (which normalizes $T$) in order to be in case (ii) above.

(iv) If $\phi(0) = \infty$ and $\phi(\infty) = 0$, then $\psi(0) = \infty$ and $\psi(\infty) = 0$; in particular, $d = nr - 1$. This case corresponds to have $a_r = b_0 = 0$.

\[ \square \]

The explicit description provided in Theorem 3 permits to obtain the connectivity of $B_d(C_n)$ and its dimension.

**Corollary 2.** If $n \geq 2$ and $C_n$ is admissible for $d$, then $B_d(C_n)$ is connected. Moreover,

$$\dim_{\mathbb{C}}(B_d(C_n)) = \begin{cases} 
2(d - 1)/n, & d \equiv 1 \mod n \\
(2d - n)/n, & d \equiv 0 \mod n \\
2(d + 1 - n)/n, & d \equiv -1 \mod n 
\end{cases}$$

**Proof.** (1) By Theorem 3, the rational maps in $\text{Rat}_d$ admitting a holomorphic automorphism of order $n \geq 2$ are conjugated those of the form $\phi(z) = z \psi(z^n) \in \text{Rat}_d$ for $\psi \in \text{Rat}_r$ as described in the same theorem.

Let us denote by $\text{Rat}_d(n,r)$ the subset of $\text{Rat}_d$ formed by all those rational maps of the $\phi(z) = z \psi(z^n)$, where $\psi$ satisfies the conditions in Theorem 3.

If $d = nr + 1$, then we may identify $\text{Rat}_d(n,r)$ with an open Zariski subset of $\text{Rat}_r$; if $d = nr$, then it is identified with an open Zariski subset of a linear hypersurface of $\text{Rat}_r$ and (iii) if $d = nr - 1$, then it is identified with an open Zariski subspace of a linear subspace of codimension two of $\text{Rat}_r$. In each case, we have that $\text{Rat}_d(n,r)$ is connected. As the projection of $\text{Rat}_d(n,r)$ to $M_d$ is exactly $B_d(C_n)$, we obtain its connectivity.

(2) The dimension counting. We may see that, (i) if $d = nr + 1$, then $\psi$ depends on $2r + 1$ complex parameters; (ii) if $d = nr$, then $\psi$ depends on $2r$ complex parameters; and (iii) if $d = nr - 1$, then $\psi$ depends on $2r - 1$ complex parameters. The normalizer in $\text{PSL}_2(\mathbb{C})$ of $\langle T \rangle$ is the $1$-complex dimensional group $N_n = \langle A_1(z) = \lambda z, B(z) = 1/z : \lambda \in \mathbb{C} - \{0\} \rangle$. If $U \in N_n$, then $U \circ \phi \circ U^{-1}$ will also have $T$ as a holomorphic automorphism. In fact,

$$A_\lambda \circ \phi \circ A_\lambda^{-1}(z) = z \psi(z^n/\lambda^n),$$

$$B \circ \phi \circ B(z) = z/\psi(1/z^n).$$
In this way, there is an action of $N_a$ over $\text{Rat}_d$, so that the orbit of $\psi(u)$ is given by the rational maps $\psi(u/t)$, where $t \in \mathbb{C} - \{0\}$, and $1/\psi(1/u)$. In this way, we obtain the desired dimensions. □

3. Proof of Theorem 1

It is clear that $\mathcal{B}_d$ is equal to the union of all $\mathcal{B}_d(G)$, where $G$ runs over the admissible finite groups for $d$.

If $G$ is admissible for $d$ and $p$ is a prime integer dividing the order of $G$ (so that the cyclic group $C_p$ is a subgroup of $G$), then $C_p$ is admissible for $d$ and $\mathcal{B}_d(G) \subset \mathcal{B}_d(C_p)$. In this way, $\mathcal{B}_d$ is equal to the union of all $\mathcal{B}_d(C_p)$, where $p$ runs over all integer primes with $C_p$ admissible for $d$. Corollary 2 asserts that each $\mathcal{B}_d(C_p)$ is connected. Now, the connectivity of $\mathcal{B}_d$ will be consequence of Lemma 1 below.

**Lemma 1.** If $p \geq 3$ is a prime and $C_p$ is admissible for $d$, then $\mathcal{B}_d(C_p) \cap \mathcal{B}_d(C_2) \neq \emptyset$.

**Proof.** We only need to check the existence of a rational map $\phi \in \text{Rat}_d$ admitting a holomorphic automorphism $T$ of order $p$ and also a holomorphic automorphism $U$ of order 2.

First, let us consider those rational maps of the form $\phi(z) = z\psi(z^p)$, where (by Theorem 3) we may assume to be of the form

$$\psi(z) = \frac{\sum_{k=0}^r a_kz^k}{\sum_{k=0}^r b_kz^k} \in \text{Rat}_r,$$

with

(a) $a_r \neq 0$, if $d = pr + 1$;

(b) $a_r \neq 0$ and $b_0 = 0$, if $d = pr$;

(c) $a_r = b_0 = 0$, if $d = pr - 1$.

In cases (a) and (c) we can find $\psi$ satisfying the relation $\psi(1/z) = 1/\psi(z)$. This is possible by considering $b_k = a_{r-k}$, for every $k = 0, 1, ..., r$. Now, we may see that $\phi$ also admits the holomorphic automorphism $U(z) = 1/z$. The automorphisms $T(z) = \omega_p z$ and $U(z) = 1/z$ generate a dihedral group of order $2p$.

In case (b), we can consider $\psi$ so that $\psi(-z) = \psi(z)$, which is possible to find if we assume that $(-1)^k a_k = (-1)^k a_k$ and $(-1)^k b_k = (-1)^k b_k$ (which means that $a_k = b_k = 0$ if $k$ and $r$ have different parity). In this case $T$ and $V(z) = -z$ are holomorphic automorphisms of $\phi$, generating the cyclic group of order $2p$.

□

**References**

[1] G. Bartolini, A. F. Costa and M. Izquierdo. On the connectivity of branch loci of moduli spaces. *Annales Academiae Scientiarum Fennicae* **38** (2013), No. 1, 245–258.

[2] A. F. Beardon. *The geometry of discrete groups*. Corrected reprint of the 1983 original. Graduate Texts in Mathematics, **91**. Springer-Verlag, New York, 1995. xii+337 pp. ISBN: 0‐387‐90788‐2.

[3] P. Doyle and C. McMullen. Solving the quintic by iteration. *Acta Mathematica* **163** (1989), 151–180.

[4] A. Levy. The space of morphisms on projective space. *Acta Arith.* **146** No. 1 (2011), 13–31.

[5] N. Miasnikov, B. Stout and Ph. Williams. Automorphism loci for the moduli space of rational maps. arXiv:1408.5655v2 [math.DS] 12 Sep 2014.

[6] J. Milnor. Geometry and dynamics of quadratic rational maps. *Experiment. Math.* **2** (1993), 37–83. With an appendix by the author and Lei Tan.
[7] J.H. Silverman. The space of rational maps on $\mathbb{P}^1$. *Duke Math. J.* **94** (1998), 41–77.

[8] D. Sullivan. Quasiconformal homeomorphisms and dynamics I. Solution of the Fatou–Julia problem on wandering domains. *Ann. of Math.* (2) **122** (1985), 401–418.

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