PROPER PUSHFORWARDS ON FINITE DIMENSIONAL ADIC SPACES

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Abstract. For any separated, taut, locally of + weakly finite type morphism \( f : X \to Y \) between finite dimensional, analytic adic spaces, we construct the higher direct images with compact support \( R^q f_! \mathcal{F} \) of any abelian sheaf \( \mathcal{F} \) on \( X \). The basic approach follows that of Huber in the case of étale sheaves, and rests upon his theory of universal compactifications of adic spaces. We show that these proper pushforwards satisfy all the expected formal properties, and construct the trace map and duality pairing for any (separated, taut) smooth morphism of adic spaces, without assuming partial properness.

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Introduction

Arguably the most significant early achievement of Huber’s approach to analytic geometry, via his theory of adic spaces, was that it enabled the development of a robust theory of étale cohomology with compact support for rigid analytic varieties, and in particular the proof in [Hub96] of a Poincaré duality theorem for smooth morphisms (not necessarily assumed partially proper). The loosely analogous question of whether one might expect some form of Serre–Grothendieck duality, or (less loosely) even a form of Poincaré duality in de Rham cohomology, to hold for rigid analytic varieties, has received sporadic attention over the years, most notably in [Chi90, vdP92, Bey97], where versions of Serre duality are proved under varying hypotheses on the source, target, and morphism involved. One hypothesis in particular that is constant throughout is that the morphism is partially proper, and at least one reason (among several) for this is the lack of a reasonable definition of compactly supported cohomology for more general morphisms of rigid analytic spaces. Our main purpose in this article is to give such a definition, and to show that it exists as part of a reasonable formalism of higher direct images with compact support.

Theorem (5.1.2, 5.4.1, 5.6.4, 5.7.1, 5.8.2). There exists a unique way to define, for every separated, taut, and locally of + weakly finite type morphism \( f : X \to Y \), between finite dimensional analytic adic spaces, a functor

\[
Rf_! : D(\text{Sh}(X)) \to D(\text{Sh}(Y)),
\]

such that:

1. \( R(f \circ g)_! = Rf_! \circ Rg_! \) for composable morphisms \( f, g \);
2. if \( f \) is an open immersion, then \( Rf_! = f_! \) is the extension by zero functor;
There is a natural map $Rf_! \to Rf_*$ which is an isomorphism if $f$ is proper;
(4) if $f$ is partially proper, then $f_! := \mathcal{H}^0(Rf_!)$ is the functor of sections whose support is quasi-compact over $Y$, and $Rf_!$ is the total derived functor of $f_!$.

The construction is the same as that given by Huber in the étale case in [Hub96], and is based upon his theory of universal compactifications. One of the main theorems of [Hub96], and one of the real miracles of working in the category of adic spaces, is the fact that any separated, taut morphism, locally of $+$ weakly finite type, admits a canonical compactification. That is, there is an initial factorisation

\[
\begin{array}{ccc}
X & \xrightarrow{j} & \bar{X} \\
\downarrow f & & \downarrow \bar{f} \\
Y & & Y
\end{array}
\]

of $f$ into an open immersion $j$ followed by a partially proper morphism $\bar{f}$. (In fact, once the hard work of setting up the theory of adic spaces has been done, the basic idea behind the proof of this is rather straightforward, as we explain in §3 below.) We can then simply define $Rf_!$ as $R\bar{f}_! \circ j_!$, where $j_!$ is extension by zero, and $R\bar{f}_!$ is the total derived functor of sections with relatively quasi-compact support. As in the étale case, proving that proper pushforwards compose correctly in general boils down to showing a version of the proper base change theorem holds (see Theorem 4.1.1), and this is the only real place where we need to prove something new - everywhere else we follow the strategy of [Hub96], Chapter 5 fairly closely.

As usual, to prove the required base change theorem we can use a whole battery of formal trickery in order to reduce the general case to a very particular one, namely where:

- the base $Y = \text{Spa}(k, k^+)$ is the spectrum of an affinoid field $\kappa = (k, k^+)$ of finite height;
- the source $X = D_\kappa(0; 1^+)$ is the so-called ‘extended unit disc’ over $\kappa$, i.e. the universal compactification of the closed unit disc $D_\kappa(0; 1)$ (see §3.1.4);
- the sheaf $\mathcal{F}$ in question is a constant sheaf supported on a closed subset of $X$.

In this case we proceed by induction on the dimension of $Y$, and use an explicit topological description of $D_\kappa(0; 1^+)$ (see 3.4) in order to invoke the classical proper base change theorem (for compact Hausdorff spaces!).

Just as in [Hub96], the fact that the fibres of a morphism $f : X \to Y$ of adic spaces are not necessarily themselves adic spaces forces us to enlarge the category slightly, and we do so in a slightly different (and ultimately, less general) way than Huber does. He works with what he calls ‘pseudo-adic spaces’; roughly speaking these consist of an adic space $X$ together with a ‘reasonably nice’ constructible subset $X \subset X$. We have chosen instead to work with germs of adic spaces, i.e. pairs consisting of an adic space $X$ together with a closed subset $X \subset X$. Part of the reason for this choice is that the main application we have in mind of the formalism developed here is to the theory of rigid cohomology, where the tubes $|X|_P$ of a locally closed subset $X$ of a formal scheme $\mathfrak{P}$ are really best considered as germs of adic spaces. Armed with a reasonable formalism of higher direct images, then, we can then construct trace maps

\[
\text{Tr}_{X/Y} : Rf_! \Omega^d_{X/Y} \to \mathcal{O}_Y,
\]

with the required properties.

**Theorem** (6.1.4, 6.3.5, 6.4.1, 6.5.1). There exists a unique way to define, for any smooth, separated and taut morphism $f : X \to Y$, of relative dimension $d$, between finite dimensional, analytic adic spaces, an $\mathcal{O}_Y$-linear map

\[
\text{Tr}_{X/Y} : Rf_! \Omega^d_{X/Y} \to \mathcal{O}_Y,
\]

such that:

1. $\text{Tr}_{X/Y}$ is local on the base $Y$ and compatible with composition;
when $f$ is étale, $\text{Tr}_{X/Y}$ is the canonical map $f_!\mathcal{O}_X \to \mathcal{O}_Y$;

(3) if the base $Y = \text{Spa}(R, R^+)$ is affinoid, and $X = \mathbb{D}_Y(0; 1^{-})$ is the relative open unit disc, then via the identification $H^1_c(X/Y, \Omega^1_{X/Y}) \cong R(z^{-1})^! d\log z$

for some co-ordinate $z$ on $X/Y$, $\text{Tr}_{X/Y}$ is induced by $\sum_{i \geq 0} r_i z^i d\log z \mapsto r_0$.

The construction of $\text{Tr}_{X/Y}$ is rather involved, and broadly speaking follows the same outline as the construction given in [vdP92]. First we work with relative analytic affine space, then relative closed unit polydiscs, then closed subspaces of the same, and then finally show that the map $\text{Tr}_{X/Y}$ thus constructed doesn’t depend on any choice of embedding, and therefore globalises. The main difference is that we can use the good formal properties of $Rf_!$ in order to simplify some of the proofs, as well as being able to work with general smooth morphisms (that is, not just partially proper ones).

The question of what form of Serre–Grothendieck duality holds, and in what generality, we sadly do not address here at all. Our main motivation for developing a formalism of proper pushforwards, and for constructing trace morphisms, was to understand particular constructions in rigid cohomology and the theory of arithmetic $\mathcal{D}^!$-modules [AL]. For us, it was enough simply to have the formalism (together with an explicit computation for relative open unit polydiscs); a detailed study of duality would therefore have distracted us rather too much from our main goal. While this might deprive the current article of a satisfying dénouement, we certainly hope to be able to pursue this question in the future.

Various different definitions of ‘compactly supported cohomology’ on rigid analytic spaces have previously been given, indeed, there are such definitions in all three articles [Chi90, vdP92, Bey97], and in [vdP92] there is even a definition of higher direct images with compact support. For partially proper morphisms, our definition here coincides with all of these previous definitions (which all agree with each other), but in the non-partially proper case there are genuine differences. It is not too hard to see that the cohomology groups $H^n_c(X/Y, -)$ considered in [vdP92] are simply the right derived functors of $H^n_0(X/Y, -)$, which is not the case in general for our definition, see Example 5.5.1. We would argue that the good formal properties enjoyed by our definition are reasonably strong evidence that it is indeed the ‘correct’ one.

Let us now give a brief summary of the contents of this article. In §1 we gather together various (existing) results in general topology that will be useful in the rest of the article, in particular concerning sheaf cohomology on spectral spaces. In §2 we introduce the notion of a germ of an adic space along a closed subset, and carefully define the category in which these live. The key point is that this category is large enough to be stable under taking fibres of morphisms, which is not true for the category of adic spaces. In §3 we recall Huber’s theory of universal compactifications, which we illustrate by considering in detail the case of the closed unit disc $\mathbb{D}_\kappa(0; 1)$ over an affinoid field $\kappa = (k, k^+)$. In §4 we prove a proper base change theorem for abelian sheaves on adic spaces, by reducing to the case of the universal compactification of $\mathbb{D}_\kappa(0; 1)$. In §5 we define proper pushforwards $f_!$ and $Rf_!$ for morphisms of germs of adic spaces, and give a counter-example to show that $Rf_!$ is not the total derived functor of $f_!$ in general. We also show that these derived proper pushforwards compose correctly, and prove various expected properties: finite cohomological amplitude, Meyer-Vietoris, etcetera. Finally, in §6 we construct the trace map associated to a smooth morphism of germs of adic spaces.

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**Notation and conventions.** We will only deal with abelian sheaves. Thus if $X$ is a topological space (or more generally a topos), a sheaf on $X$ will always mean an abelian sheaf.

A Huber ring will be a topological ring admitting an open adic subring $R_0$ with finitely generated ideal of definition. For such a ring $R$, we will denote by $R^\circ \subset R$ the subset of power-bounded elements, and by $R^{\circ \circ} \subset R^\circ$ the subset of topologically nilpotent elements. A Huber pair is a pair $(R, R^+)$ consisting of a Huber ring $R$ and an open, integrally closed subring $R^+ \subset R^\circ$. A Huber ring $R$ is said to be a Tate ring if there exists some $\mathfrak{a} \in R^* \cap R^{\circ \circ}$, such an element will be called a quasi-uniformiser. A Huber pair $(R, R^+)$ is said to be a Tate pair, if $R$ is a Tate ring. An adic space isomorphic to Spa $(R, R^+)$ with $(R, R^+)$ a Tate pair will be called a Tate affinoid.

If $X$ is an adic space, and $x \in X$, we will denote by $\mathcal{O}_{X,x}$ and $\mathcal{O}^+_{X,x}$ the stalks of the structure sheaf and integral structure sheaf of $X$ at $x$ respectively. The residue field of $\mathcal{O}_{X,x}$ will be denoted $k(x)$, and the image of $\mathcal{O}^+_{X,x}$ in $k(x)$ by $k(x)^+$. The residue field of $k(x)^+$ will be denoted $\tilde{k(x)}$.

1. **General topology**

In this section we will gather together various existing results we will need in general topology, using either [FK18] or [Hub96] as references.

1.1. **Basic definitions.** For the reader’s convenience we recall several definitions that will be used in this article.

1.1.1. **Definition.** A topological space $X$ is said to be:

1. quasi-compact if every open cover has a finite sub-cover;
2. compact if it is quasi-compact and Hausdorff;
3. quasi-separated if the intersection of any two quasi-compact opens is quasi-compact;
4. coherent if it is quasi-compact, quasi-separated, and admits a basis of quasi-compact open subsets;
5. locally coherent if it admits a cover by coherent open subspaces
6. sober if every irreducible closed subset has a unique generic point;
7. spectral if it is coherent and sober;
8. locally spectral if it is locally coherent, and sober.
9. valuative if it is locally spectral, and the set of generalisations of any point $x \in X$ is totally ordered.

If $X$ is a locally spectral space, and $x, y \in X$ we will write $y \succ x$ if $x \in \overline{\{y\}}$, i.e. if $x$ is a specialisation of $y$. We will also write $G(x)$ for the set of generalisations of $x$.

1.1.2. **Definition.** Let $X$ be a locally spectral space. Then $X$ is said to be taut if it is quasi-separated and the closure of any quasi-compact open $U \subset X$ is quasi-compact.

1.1.3. **Definition.** A morphism $f : X \to Y$ of locally coherent topological spaces is said to be

1. quasi-compact if the preimage of every quasi-compact open subset $V \subset Y$ is quasi-compact;
2. quasi-separated if the preimage of every quasi-separated open subset $V \subset Y$ is quasi-separated;
3. coherent if it is quasi-compact and quasi-separated;

1.1.4. **Definition.** A morphism $f : X \to Y$ of locally spectral spaces is said to be

1. taut if the pre-image of every taut open subspace $V \subset Y$ is taut;
2. spectral if for every quasi-compact and quasi-separated open subsets $U \subset X$, $V \subset Y$ with $f(U) \subset V$, the induced map $f : U \to V$ is quasi-compact.

Any coherent morphism of locally spectral spaces is spectral.
1.2. Sheaf cohomology on spectral spaces. The following result will be used constantly.

1.2.1. **Proposition.** Let $X$ be a topological space, and $\{U_i\}_{i \in I}$ a filtered diagram of open subsets of $X$, such that:

1. each $U_i$ is spectral;
2. all transition maps $U_i \to U_j$ are quasi-compact.

Set $Z = \lim_{i \in I} U_i = \cap U_i$. Then, for any sheaf $\mathcal{F}$ on $X$, and any $q \geq 0$, the natural map

$$\text{colim}_{i \in I} \, H^q(U_i, \mathcal{F}|_{U_i}) \to H^q(Z, \mathcal{F}|_Z)$$

is an isomorphism.

**Proof.** This is a particular case of [FK18, Chapter 0, Proposition 3.1.19].

1.2.2. **Corollary.** Let $f : X \to Y$ be a coherent morphism of locally spectral spaces, $\mathcal{F}$ a sheaf on $X$, and $y \in Y$. Let $X_{G(y)} \subset X$ denote the inverse image of $G(y) \subset Y$. Then, for any $q \geq 0$, the natural map

$$(R^q(f_* \mathcal{F}))_y \to H^q(X_{G(y)}, \mathcal{F}|_{X_{G(y)}})$$

is an isomorphism.

**Proof.** We may assume that $Y$ is coherent, thus spectral. Hence $X$ is also spectral. The point $y$ admits a cofinal system of open neighbourhoods $\{U_i\}_{i \in I}$ with each $U_i$ spectral and each transition morphism $U_i \to U_j$ quasi-compact. Therefore each $f^{-1}(U_i)$ is spectral, and each $f^{-1}(U_i) \to f^{-1}(U_j)$ is quasi-compact. We therefore have

$$(R^q(f_* \mathcal{F}))_y = \text{colim}_{i \in I} \, H^q(f^{-1}(U_i), \mathcal{F}|_{f^{-1}(U_i)}) = H^q(X_{G(y)}, \mathcal{F}|_{X_{G(y)}})$$

since $\cap f^{-1}(U_i) = f^{-1}(\cap U_i) = f^{-1}(G(y)) = X_{G(y)}$.

The following two results are taken from [Stacks].

1.2.3. **Lemma.** Let $X$ be a spectral space, and $\mathcal{F}$ a sheaf on $X$. Then there exists:

1. a filtered diagram $\{\mathcal{F}_i\}_{i \in I}$ of sheaves on $X$;
2. for each $i \in I$ a finite collection of closed subsets $t_{ij} : Z_{ij} \hookrightarrow X$, a collection of abelian groups $A_{ij}$, and an injection $\mathcal{F}_i \hookrightarrow \bigoplus_j t_{ij}^* \mathcal{F}|_{A_{ij}}$;

such that $\mathcal{F} \cong \text{colim}_{i \in I} \mathcal{F}_i$.

**Proof.** This is the content of [Stacks, Tag 0CAG], adapted to the abelian case. By looking at all sections of $\mathcal{F}$ over all open subsets of $X$, we can write $\mathcal{F}$ as a filtered colimit of sheaves which are direct sums of quotients of sheaves of the form $\nu_{\mathcal{U}_i}$ for $\nu : U \hookrightarrow X$ a connected, quasi-compact open subspace. Thus we may assume that there exists such a surjection

$$\nu_{\mathcal{U}_i} : \mathcal{U}_i \to \mathcal{F}.$$ 

Applying the same argument to the kernel of this map, we can find a collection of connected, quasi-compact open subspaces $\{\mathcal{U}_j \to X\}_{j \in J}$, contained in $U$, and an exact sequence

$$\bigoplus_{j \in J} \nu_{\mathcal{U}_j} : \bigoplus_{j \in J} \nu_{\mathcal{U}_j} \mathcal{F}_j \to \mathcal{F} \to 0.$$ 

For each finite subset $I \subset J$, define $\mathcal{F}_I$ by the exact sequence

$$\bigoplus_{j \in I} \nu_{\mathcal{U}_j} : \bigoplus_{j \in I} \nu_{\mathcal{U}_j} \mathcal{F}_j \to \mathcal{F}_I \to 0,$$

then we have

$$\text{colim}_{I \subset J} \mathcal{F}_I \to \mathcal{F},$$

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so we may assume that the set \( J \) is finite. Now, applying \([\text{Hoc}69, \text{Proposition 10}]\), we can find a quasi-compact map \( p : X \to T \) to a finite, sober space, and connected open subsets \( \iota_j : V_j \hookrightarrow T \), \( \iota : V \hookrightarrow T \), such that \( U_j = p^{-1}(V_j) \) and \( U = p^{-1}(V) \). The given homomorphisms

\[
\alpha_j : \iota_j^*\mathcal{Z}_{U_j} \to \iota^*\mathcal{Z}_U
\]

correspond to sections in \( \Gamma(U_j, \mathcal{Z}_{U_j}) = \Gamma(V_j, \mathcal{Z}_{V_j}) \), thus are induced by maps

\[
\alpha_j : \iota_j^*\mathcal{Z}_{V_j} \to \iota^*\mathcal{Z}_V
\]

of sheaves on \( T \). In other words, we may assume that \( F \) is the preimage of a sheaf on \( T \). Replacing \( X \) by \( T \) we may therefore assume that \( X \) is a finite, sober space. In this case we have

\[
F \hookrightarrow \bigoplus_{x \in X} i_{x*}F_x
\]

and \( i_{x*}F_x \) is constant on the closed subset \( \{x\} \).

1.2.4. Lemma. Let \( X \) be a spectral space, and \( Z \subset X \) a closed subset such that for every \( x \in X \), the intersection \( Z \cap \{x\} \) is non-empty and connected. Then, for any sheaf \( \mathcal{F} \) on \( X \), the map

\[
\Gamma(X, \mathcal{F}) \to \Gamma(Z, \mathcal{F}|Z)
\]

is an isomorphism.

Proof. This is \([\text{Stacks}, \text{Tag 09ZG}]\). It is easy to check that the given map is injective, since every point of \( X \) specialises to a point of \( Z \). If \( \mathcal{F} \hookrightarrow \mathcal{G} \) is an injection of sheaves, and we know surjectivity of the given map for \( \mathcal{G} \), then we know it for \( \mathcal{F} \). Therefore, by Lemma 1.2.3, we can reduce to the case when \( \mathcal{F} = i_*A \) for \( \iota : T \hookrightarrow X \) a closed subspace, and \( A \) an abelian group. Thus replacing \( X \) by \( T \) we may assume that \( \mathcal{F} = A \) is a constant sheaf; it therefore suffices to show that \( \pi_0(X) = \pi_0(Z) \). But if \( Z = Z_1 \bigsqcup Z_2 \) is a disjoint union with each \( Z_i \) open and closed in \( Z \), then we can take \( X_i \) to be the set of all points of \( X \) specialising to \( Z_i \), and we obtain a disjoint union \( X = X_1 \bigsqcup X_2 \) with each \( X_i \) open and closed in \( X \).

1.3. Dimensions of spectral spaces. The dimension theory of locally spectral spaces works as in the case of schemes.

1.3.1. Definition. Let \( X \) be a locally spectral space. The dimension of \( X \) is defined to be

\[
\dim X = \sup \{ n \geq 0 \mid \exists \ x_n > x_{n-1} > \ldots > x_0, \ x_i \neq x_{i-1} \} \in \mathbb{Z}_{\geq 0} \cup \{\infty, -\infty\}.
\]

The space \( X \) is said to be finite dimensional if \( \dim X < \infty \).

We will need the following generalisation of Grothendieck vanishing.

1.3.2. Theorem. Let \( X \) be a spectral space of dimension \( d \), and \( \mathcal{F} \) a sheaf on \( X \). Then \( H^q(X, \mathcal{F}) = 0 \) for all \( q > d \).

Proof. This is the main result of \([\text{Sch}92]\). \( \square \)

1.3.3. Definition. Let \( f : X \to Y \) be a spectral morphism between locally spectral spaces. The dimension of \( f \) is defined to be

\[
\dim f = \sup \{ \dim f^{-1}(y) \mid y \in Y \} \in \mathbb{Z}_{\geq 0} \cup \{\infty, -\infty\}.
\]

The map \( f \) is said to be finite dimensional if \( \dim f < \infty \).
1.4. **Separated quotients.** Let $X$ be a valuative space, and let $[X]$ denote its set of maximal points, i.e. points such that $G(x) = \{x\}$. Then taking a point to its maximal generalisation\(^1\) induces a surjective map

$$\nu : X \to [X],$$

and we equip $[X]$ with the quotient topology.

1.4.1. **Definition.** An open (resp. closed) subset of a valuative space $X$ is said to be overconvergent if it is closed under specialisation (resp. generalisation).

Equivalently, it is the preimage of an open (resp. closed) subset of $[X]$ under the separation map.

1.4.2. **Lemma.** Let $Z \subset X$ be an overconvergent closed subset of a coherent valuative space, and $\mathcal{F}$ a sheaf on $X$. Then, for all $q \geq 0$, the natural map

$$\text{colim}_\text{open} Z \subset U \text{ } H^q(U, \mathcal{F}|_U) \to H^q(Z, \mathcal{F}|_Z)$$

is an isomorphism.

**Proof.** Follows from Proposition 1.2.1. \(\square\)

1.4.3. **Proposition.** The space $[X]$ is $T_1$, and is universal for maps from $X$ into $T_1$ topological spaces. If $X$ is coherent, then $[X]$ is compact.

**Proof.** This is [FK18, Chapter 0, Proposition 2.3.9 and Corollary 2.3.18]. \(\square\)

Note that the space $[X]$ is generally no longer valuative, since it won’t admit a basis of quasi-compact opens.

2. **Germs of adic spaces, and local adic spaces**

In this section we will introduce the category of germs of adic spaces, which will be the category that we work in for the rest of this article.

2.1. **Standing hypotheses.** We will use Huber’s theory of adic spaces, see either [Hub94] or [Hub96, Chapter 1] for an introduction. We will impose two hypotheses throughout.

1. All adic spaces will be assumed finite dimensional in the sense of Definition 1.3.1. Consequently, all valuation rings will be assumed to be of finite height.

2. All adic spaces will be assumed to be analytic in the sense of [Hub96, §1.1]. That is, each local ring $\mathcal{O}_{X,x}$ will be assumed to contain a topologically nilpotent unit. This implies that all morphisms of adic spaces are adic in the sense of [Hub96, §1.2].

We will let $\text{Ad}$ denote the category of finite dimensional, analytic adic spaces. Given the standing assumption [Hub96, (1.1.1)], it is an implicit assumption of ours that all adic spaces are locally Noetherian.

2.2. **Quasi-adic spaces.** In [Hub96, §1.10] Huber introduces the notion of a pseudo-adic space, which roughly speaking consists of an adic space $X$, together with a ‘reasonably nice’ subspace $X \subset X$. At least one reason for working with pseudo-adic spaces is that fibres of morphisms of adic spaces are no longer necessarily adic spaces. We will work instead with germs of adic spaces along closed subsets.

2.2.1. **Definition.** A germ of an adic space is a pair $(X, X)$ where $X$ is an adic space, and $X \subset X$ is a closed subset.

\(^1\)that every point of a valuative space does indeed admit a maximal generalisation is explained in [FK18, Chapter 0, Remark 2.3.2]
We can construct a category \textbf{Germ} of germs of adic spaces in the usual way. We first consider the category of pairs \((X, X)\), where morphisms are commutative squares. We then declare a morphism \(j : (X, X) \to (Y, Y)\) to be a strict neighbourhood if \(j\) is an open immersion and \(j(X) = Y\). Finally, we localise the category of pairs at the class of strict neighbourhoods (it is easy to verify that a calculus of right fractions exists).

2.2.2. Example. (1) The first basic example of a germ is any fibre of a morphism of adic spaces \(f : X \to Y\) which is locally of weakly finite type. Indeed, if \(y \in Y\), and \(G(y)\) is its set of generalisations, then \(f^{-1}(y)\) is a closed subset of the adic space

\[
\text{Spa}\left(k(y), k(y)^+\right) \times_Y X
\]

Note that if the point \(y \in Y\) in question is not maximal, this fibre \(f^{-1}(y)\) will not have any kind of ‘natural’ structure as an adic space.

(2) The second basic example we will eventually be interested in is inspired by Berthelot’s rigid cohomology. Let \(k^\circ\) be a complete DVR, with fraction field \(k\), \(P\) a formal scheme of finite type over \(\text{Spf}(k^\circ)\), and \(X \subset P\) a locally closed subset. Then there is a specialisation map

\[
sp : P \to X
\]

from the adic generic fibre of \(P\) to the formal scheme \(P\), and if we let \(Y\) denote the closure of \(X\), then the tube \(\text{sp}^{-1}(Y)\) is defined to be the interior of the inverse image of \(Y\) under \(sp\). This induces a map

\[
\text{sp}_Y : \text{sp}^{-1}(Y) \to Y,
\]

and the tube

\[
\text{sp}_Y^{-1}(X)
\]

is defined to be the closure of \(\text{sp}_Y^{-1}(X)\) inside \(\text{sp}^{-1}(Y)\). The pair of tubes \(\text{sp}_Y^{-1}(X), \text{sp}_Y^{-1}(Y)\) then defines a germ.

The assignment \(X \mapsto (X, X)\) induces a fully faithful functor \(\textbf{Ad} \to \textbf{Germ}\). We can also consider a pair \((X, X)\) as above as a pseudo-adic space in the sense of Huber [Hub96, §1.10], thus it makes sense to consider any of the following properties of morphisms of such pairs:

1. locally of finite type, locally of + weakly finite type, locally of weakly finite type;
2. quasi-compact, quasi-separated, coherent, taut;
3. an open immersion, closed immersion, locally closed immersion;
4. separated, partially proper, proper;
5. smooth, étale.

It is easily checked that these properties descend to the category \textbf{Germ} of germs. While ‘analytic’ properties of a germ \((X, X)\) are generally defined via the ambient adic space \(X\), ‘topological’ properties are generally defined using the topological space \(X\) itself. Thus, a point of a germ \((X, X)\) will simply be a point of \(X\), and a sheaf on a germ will simply be a sheaf on \(X\).

2.2.3. Definition. (1) We say that a morphism \(f : (X, X) \to (Y, Y)\) of pairs is \textit{inert} if \(X = f^{-1}(Y)\). Similarly, we say that a morphism of germs is inert if it has an inert representative as a morphism of pairs. Note that this is equivalent to having a representative such that \(X\) is open in \(f^{-1}(Y)\).

(2) A germ \((X, X)\) is said to be overconvergent if \(X \subset X\) is an overconvergent closed subset (i.e. is stable under generalisation).

Note that any morphism of quasi-adic spaces which is an open immersion, étale, or smooth, is necessarily inert. The same is not necessarily true for closed or locally closed immersions.
2.2.4. As with adic spaces, or pseudo-adic spaces, fibre products in general are not representable in \textbf{Germ}. However, they will be representable if at least one of the morphisms is locally of weakly finite type. If

\[
\begin{array}{c}
(X, X) \\
\downarrow f \\
(Y, Y) \longrightarrow (Z, Z)
\end{array}
\]

are morphisms of germs spaces, with \(f\), say, locally of weakly finite type, then their fibre product is represented by the germ

\[(p_1^{-1}(X) \cap p_2^{-1}(Y), X \times_Z Y).
\]

2.2.5. Example. Let \(k^\circ\) be a complete DVR, \(k\) its fraction field, \(\mathfrak{P}\) a formal scheme of finite type over \(k^\circ\), and \(X \subset \mathfrak{P}\) a locally closed subset. Set \(\kappa = (k, k^\circ)\). Then, for any \(n \geq 0\), we can consider \(X\) as a locally closed subset of \(\hat{A}_n^\kappa\) via the zero section, and we have

\[]X[\hat{A}_n^\kappa] \cong X[\mathfrak{P} \times_\kappa \mathbb{D}_n^\kappa(0; 1^-)]
\]
as germs over \(\kappa\).

2.3. \textbf{Valuatively Ringed Spaces}. Let \(\text{TRing}\) denote the category of linearly topologised \(\mathbb{Z}\)-algebras. That is, an object of \(\text{TRing}\) is a ring \(A\), equipped with a uniform topology, admitting a neighbourhood basis of 0 consisting of additive subgroups \(I \subset A\). The category \(\text{TRing}\) admits all limits and colimits, both of which commute with the obvious forgetful functor

\[\text{TRing} \rightarrow \text{Ring}\]
to the category of rings. A topologically ringed space \((X, \mathcal{O}_X)\) is a topological space together with a \(\text{TRing}\)-valued sheaf \(\mathcal{O}_X\). Since the functor \(\text{TRing} \rightarrow \text{Ring}\) commutes with limits, every topologically ringed space has an underlying ringed space.

2.3.1. Definition. A topologically locally ringed space is a topologically ringed space whose underlying ringed space is a locally ringed space.

The following definition is due to Huber, although the terminology is our own.

2.3.2. Definition. A valuatively ringed space is a triple

\[(X, \mathcal{O}_X, \{v_x\}_{x \in X})\]

where \((X, \mathcal{O}_X)\) is a topologically locally ringed space, and

\[v_x : k(x) \rightarrow \{0\} \cup \Gamma_x\]
is an equivalence class of valuation(s) on the residue field \(k(x)\) of \(\mathcal{O}_X, x\).

We will denote the category of valuatively ringed spaces by \(\text{VRSp}\). Huber simply calls this category \(\mathcal{V}\). The functor \((X, X) \mapsto (X, \mathcal{O}_X |_X, \{v_x\}_{x \in X})\) from pairs to \(\text{VRSp}\) clearly descends to a ‘visualisation’ functor

\[\text{Germ} \rightarrow \text{VRSp},\]

which we we abusively write as \((X, X) \mapsto X\). It is fully faithful on the subcategory \(\text{Ad} \subset \text{Germ}\), although it seems unlikely that it is fully faithful in general. It seems reasonable to guess that it might be fully faithful on the full subcategory of overconvergent germs which are locally of finite type over a fixed base.

We may therefore extend the notions of local rings, residue fields, etcetera, to germs of spaces. For example, if \(x \in X\) is a point of a germ \((X, X)\), then we may speak of the local ring \(\mathcal{O}_{X, x}\), the residue field \(k(x)\), and the residue valuation ring \(k(x)^+ \subset k(x)\).
2.4. Local adic spaces. We recall from [Hub96] the definition of an (analytic) affinoid field. In some ways, these play a role analogous to that of local rings in algebraic geometry.

2.4.1. Definition. An affinoid field is an affinoid ring $\kappa = (k, k^+)\) where $k$ is a field, $k^+ \subset k$ is a valuation ring, and the valuation topology on $k$ can be induced by a height 1 valuation. We define the height of $\kappa$ to be the height of the valuation ring $k^+$.

By our standing assumptions, our adic spaces are always of finite dimension, and consequently every affinoid field we consider will be assumed to be of finite height.

2.4.2. Definition. An adic space is called local if it is isomorphic to the spectrum $\text{Spa}(k, k^+)$ of an affinoid field. A germ is called local if it has a representative of the form $(X, x)$ with $X$ a local adic space.

The points of a local germ are totally ordered by generalisation, and any local germ has a unique closed point. Also note that if $X$ is a local germ, then there exists an essentially unique irreducible adic space $X$ such that $X$ is a closed subset of $x$, namely $X = \text{Spa}(k(x), k(x)^+)$ where $x$ is the closed point of $X$.

2.4.3. Example. (1) Let $X$ be an adic space, and $x \in X$ a point. Then $\kappa(x) := (k(x), k(x)^+)$ is an affinoid field, and $\text{Spa}(\kappa(x))$ is a local adic space, called the localisation of $X$ at $x$. There is a canonical morphism $\text{Spa}(\kappa(x)) \to X$ which induces a homeomorphism between $\text{Spa}(\kappa(x))$ and the set $G(x)$ of generalisations of $x$.

(2) We can also do the same with points of germs $(X, x)$. Namely, if $x \in X \subset X$ is such a point, then $\text{Spa}(\kappa(x)) \cap X$, which is naturally homeomorphic to the set of generalisations of $x$ within $X$, will be a closed subset of the local adic space $\text{Spa}(\kappa(x))$. It is therefore a local germ.

Let $f : X \to Y$ be a morphism of germs, locally of weakly finite type. We may therefore take the fibre product of $f$ with any other morphism $g$. For any point $y \in Y$, we define

$$X_y := X \times_Y (\text{Spa}(\kappa(y)) \cap Y),$$

which is locally of weakly finite type over the local germ $\text{Spa}(\kappa(y)) \cap Y$. Note that the underlying topological space of $X_y$ is equal to $f^{-1}(G(y))$, and it contains the space $f^{-1}(y)$ as a closed subspace, equal to the closed fibre of the natural map

$$X_y \to \text{Spa}(\kappa(y)) \cap Y.$$

The following is then just a rephrasing of Corollary 1.2.2.

2.4.4. Corollary. Let $f : X \to Y$ be a coherent morphism of germs, locally of weakly finite type, and $\mathcal{F}$ a sheaf on $X$. Then, for all $q \geq 0$, the natural map

$$(R^qf_*\mathcal{F})_y \to H^q(X_y, \mathcal{F}|_{X_y})$$

is an isomorphism.

2.4.5. Remark. Be warned that $X_y$ is in general larger than the fibre $f^{-1}(y)$.

We also have the following important consequence of Lemma 1.2.1.

2.4.6. Proposition. Let $X$ be a closed subset of a taut adic space. Then, for any sheaf $\mathcal{F}$ on $[X]$, the adjunction map

$$\mathcal{F} \to R\nu_*\nu^*\mathcal{F}$$

is an isomorphism.

Proof. The proof follows that of [Hub96, Proposition 8.1.4]. For each $x \in [X]$, it follows from [Hub96, Lemma 8.1.5] that there exists a cofinal system $\{U_i\}_{i \in I}$ of open neighbourhoods of the
closure $\overline{\{x\}}$ of $\{x\}$ in $X$ such that each $U_i$ is spectral, and all of the inclusions $U_i \to U_j$ are quasi-compact. Since $\cap_i U_i = \{x\}$, we obtain by Proposition 1.2.1 isomorphisms
\[ (R^q\nu_*\nu^*F)^x = \text{colim}_{i \in I} H^q(U_i, \nu^*F_{|U_i}) \cong H^q(\overline{\{x\}}, (\nu^*F)_{|\overline{\{x\}}}). \]

But now $\overline{\{x\}}$ is an irreducible space, and $(\nu^*F)_{|\overline{\{x\}}}$ is the constant sheaf on $\overline{\{x\}}$ with stalks $F_x$. It is therefore flasque on $\overline{\{x\}}$, and hence we have
\[ (R^q\nu_*\nu^*F_x) = \begin{cases} F_x & q = 0 \\ 0 & q > 0. \end{cases} \]

Therefore the map
\[ F \to R\nu_*\nu^*F \]

is an isomorphism on stalks, and hence an isomorphism. \hfill \Box

3. Huber’s universal compactifications

One of the key insights of the book [Hub96] is that by looking at a slightly weaker notion of what it means for a morphism to be of finite type, a very broad class of morphisms of adic spaces turn out to have canonical compactifications. The relevant condition is being “locally of $+$weakly finite type”, and in this section, we will explain how this weaker finiteness condition works, and how it relates to the problem of finding compactifications.

3.1. Morphisms of $+$weakly finite type.

3.1.1. Definition. A homomorphism $R \to S$ of complete, Tate rings is said to be topologically of finite type if there exists a continuous, open surjection $R(x_1, \ldots, x_n) \to S$ for some $n$.

3.1.2. Definition. A morphism $(R, R^+) \to (S, S^+)$ of complete Huber pairs is said to be of $+$weakly finite type if:

1. $R \to S$ is topologically of finite type;
2. there exists a finite set $T \subset S$ such that $S^+ = \overline{R^+[T \cup S^\infty]}$ is the integral closure of $R^+[T \cup S^\infty]$, that is, the smallest ring of integral elements in $S$ containing $R^+$ and $T$.

3.1.3. Definition. A morphism $f : X \to Y$ of adic spaces is said to be locally of $+$weakly finite type if it is locally of the form
\[ \text{Spa}(S, S^+) \to \text{Spa}(R, R^+) \]

with $(R, R^+) \to (S, S^+)$ of $+$weakly finite type. It is said to be of $+$weakly finite type if it is in addition quasi-compact.

3.1.4. To understand this condition a little better, suppose that $k$ is a complete, discretely and non-trivially valued field, and let $R$ be a Tate algebra over $k$, i.e. a quotient of some ring $k\langle x_1, \ldots, x_n \rangle$ of convergent power series. Then the local building blocks of adic spaces locally of finite type over $\kappa = (k, k^\circ)$ are those of the form $\text{Spa}(R, R^\circ)$, in which the ring of integral elements is canonically determined by $R$.

On the other hand, the local building blocks of spaces locally of $+$weakly finite type over $K$ are allowed to have ring of integral elements $R^+$ equal to the integral closure of $k^\circ[T \cup R^\circ]$ for any finite set $T \subset R^\circ$. This is now no longer uniquely determined by $R$. For example, the adic space $\text{Spa}(k(\langle x \rangle), k^\circ + k^\circ(\langle x \rangle))$ is of $+$weakly finite type over $\kappa$, but not of finite type over $\kappa$.

Let us describe explicitly this adic space $\text{Spa}(k(\langle x \rangle), k^\circ + k^\circ(\langle x \rangle))$. Choose a uniformiser $\varpi$ for $k$, and consider the affinoid ring $(k(\varpi x), k^\circ(\varpi x))$ consisting of the Tate algebra
\[ k(\varpi x) := \left\{ \sum_i a_i(\varpi x)^i \in k[[x]] \middle| a_i \to 0 \right\}. \]
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together with its subring of power bounded elements $k^\circ \langle x \rangle$. Thus

$$\text{Spa}(k \langle x \rangle, k^\circ \langle x \rangle) = \mathbb{D}_\kappa(0; r)$$

is the closed disc of radius $r = |x|^{-1}$ over $\kappa$. We have $k \langle x \rangle \hookrightarrow k \langle x \rangle$ as a dense subring, such that $k^\circ \langle x \rangle \subset k^\circ + k^{\circ \circ} \langle x \rangle \subset k^\circ \langle x \rangle$. We therefore obtain injections

$$\mathbb{D}_\kappa(0; 1) \hookrightarrow \text{Spa}(k \langle x \rangle, k^\circ + k^{\circ \circ} \langle x \rangle) \hookrightarrow \mathbb{D}_\kappa(0; r)$$

of adic spaces, where $\mathbb{D}_\kappa(0; 1)$ is the closed disc of radius 1 over $\kappa$. It is easy to check that the only point of the complement $\mathbb{D}_\kappa(0; r) \setminus \mathbb{D}_\kappa(0; 1)$ that lies in $\text{Spa}(k \langle x \rangle, k^\circ + k^{\circ \circ} \langle x \rangle)$ is the specialisation $\xi_+$ of the Gauss point corresponding to the rank two valuation $v_{\xi_+}$:

$$k \langle x \rangle \to \{0\} \cup \mathbb{R}_{>0} \times \gamma \mathbb{Z}$$

with $\gamma = v_{\xi_+}(x)$ satisfying $1 < \gamma < \rho$ for all $\rho > 1$. Thus $\text{Spa}(k \langle x \rangle, k^\circ + k^{\circ \circ} \langle x \rangle)$ as a set is equal to the closure of $\mathbb{D}_\kappa(0; 1)$ inside $\mathbb{D}_\kappa(0; r)$.

In fact, it is relatively straightforward to check that that $\text{Spa}(k \langle x \rangle, k^\circ + k^{\circ \circ} \langle x \rangle)$ is proper over $\kappa$ in the sense of [Hub96, Definition 1.3.2], and moreover enjoys the following universal property:

- for every adic space $X$, partially proper over $\kappa$, and every $\kappa$-morphism $\mathbb{D}_\kappa(0; 1) \to X$, there exists a unique map $\text{Spa}(k \langle x \rangle, k^\circ + k^{\circ \circ} \langle x \rangle) \to X$, such that the diagram

$$\begin{array}{ccc}
\mathbb{D}_\kappa(0; 1) & \to & \text{Spa}(k \langle x \rangle, k^\circ + k^{\circ \circ} \langle x \rangle) \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}$$

commutes.

It therefore deserves to be called ‘the’ compactification of $\mathbb{D}_\kappa(0; 1)$. It is tempting to employ the notation

$$\mathbb{D}_\kappa(0; 1^+) := \text{Spa}(k \langle x \rangle, k^\circ + k^{\circ \circ} \langle x \rangle)$$

for this compactification, and we shall do so. Note that the canonical morphism $\mathbb{D}_\kappa(0; 1^+) \to \mathbb{P}^1_\kappa$ is an example of a proper morphism of adic spaces, which is a homeomorphism onto a closed subset of its range, but which is not a closed immersion.

### 3.2. Universal compactifications

In fact, Huber showed that a very broad class of morphisms of adic spaces admit ‘canonical’ compactifications in exactly the same way as $\mathbb{D}_\kappa(0; 1)$.

#### 3.2.1. Definition

Let $f : X \to Y$ be a morphism of adic spaces, separated and locally of $^+\text{weakly finite type}$. A compactification of $f$ is a diagram

$$\begin{array}{ccc}
X & \xrightarrow{j} & X' \\
\downarrow{f} & & \downarrow{f'} \\
Y & & Y
\end{array}$$

where $j$ is an locally closed immersion and $f'$ is partially proper. A compactification is called universal if $j$ is an open immersion, and $f = f' \circ j$ is initial amongst all factorisations of $f$ into an arbitrary morphism followed by a partially proper morphism.

Clearly, a universal compactification, if it exists, is unique up to unique isomorphism. One of the main technical results of [Hub96], which appears as Theorem 5.1.5, is the existence of universal compactifications for taut morphisms of adic spaces.
3.2.2. **Theorem** (Huber). Let \( f : X \to Y \) be a morphism of adic spaces, separated, locally of + weakly finite type, and taut. Then \( f \) admits a universal compactification

\[
\begin{array}{c}
X \\
\downarrow f
\end{array}
\rightleftharpoons
\begin{array}{c}
\overline{X} \\
\downarrow \overline{f}
\end{array}
\]

Moreover, \( j \) is quasi-compact, every point of \( \overline{X} \) is a specialisation of a point of \( X \), and the map

\[
\mathcal{O}_{\overline{X}} \to j_* \mathcal{O}_X
\]

is an isomorphism. If \( f \) is quasi-compact, then \( \overline{f} \) is proper.

3.2.3. **Remark.**

1. The induced map \( \mathcal{O}_{\overline{X}}^+ \to j_* \mathcal{O}_X^+ \) on integral structure sheaves is in general not an isomorphism. It will be so if and only if \( f : X \to Y \) is already partially proper, in which case \( j = \text{id}_X \).

2. If \( \kappa = (k, k^+) \) is an affinoid field, then the universal compactification of the closed unit disc \( D_\kappa(0; 1) = \text{Spa}(k(x), k^+(x)) \) is the ‘extended’ closed unit disc that we considered above, namely \( D_\kappa(0; 1^+) : = \text{Spa}(k(x), k^+ + k^{oo}(x)) \).

3. This example essentially shows how to construct \( \overline{X} \) locally, namely if \( X = \text{Spa}(S, S^+) \) and \( Y = \text{Spa}(R, R^+) \), then \( X = \text{Spa}\left(S, R^+[S^{oo}]\right) \), see the proof of [Hub96, Theorem 5.1.5] beginning on p.275. To obtain \( \overline{X} \) in general, a rather delicate gluing procedure is required.

4. For us, adic spaces are assumed finite dimensional, thus we need to invoke [Hub96, Corollary 5.1.14] to ensure that \( X \) remains finite dimensional.

The following lemma will be useful later on.

3.2.4. **Lemma.** Let \( f : X \to Y \) be a proper morphism of adic spaces, and \( U \subset X \) a retrocompact open subset.\(^2\) Let \( \overline{f_U} : \overline{U} \to Y \) be the universal compactification of the restriction \( f_U : U \to Y \), and let \( g : \overline{U} \to X \) be the unique map such that \( f \circ g = \overline{f_U} \) and \( g|_U = f_U \). Then \( g \) is a homeomorphism onto the closure of \( U \) in \( X \).

**Proof.** First of all, since both \( \overline{f_U} \) and \( f \) are proper, so is \( g \). The image of \( g \) is therefore closed, and contains \( U \), and by combining [Hub96, Lemma 1.1.10 v)] with the fact that every point of \( \overline{U} \) is a specialisation of a point of \( U \), we deduce that the image of \( g \) is precisely the closure of \( U \). By [Hub96, Lemma 1.3.15] it therefore suffices to show that \( g \) is injective; again using the fact that every point of \( \overline{U} \) is a specialisation of a point of \( U \), this is achieved by applying the valuative criterion for properness [Hub96, Corollary 1.3.9] to the morphism \( g \). \( \square \)

3.3. **Germs of adic spaces.** There is also a notion of compactifications for morphisms of germs of adic spaces.

3.3.1. **Definition.** Let \( f : X \to Y \) be a morphism of germs of adic spaces, separated and locally of + weakly finite type. A compactification of \( f \) is a diagram

\[
\begin{array}{c}
X \\
\downarrow f
\end{array}
\rightleftharpoons
\begin{array}{c}
X' \\
\downarrow \overline{f'}
\end{array}
\]

where \( j \) is an locally closed immersion and \( f' \) is partially proper.

While germs might not admit universal compactifications, they do always admit compactifications.\(^2\) An open subset is retrocompact if the inclusion is a quasi-compact map.

---

\(^2\)An open subset is retrocompact if the inclusion is a quasi-compact map.
3.3.2. **Theorem.** Let $f : X \to Y$ be a morphism of germs, separated, locally of \( ^+ \) weakly finite type and taut. Then $f$ admits a compactification, such that $j$ is an open immersion.

**Proof.** This follows from [Hub96, Theorem 5.1.11, Corollary 5.1.14].

3.4. **The structure of $\mathbb{D}_\kappa(0; 1^+)$**. We have already explored the structure of the ‘extended’ unit disc $\mathbb{D}_\kappa(0; 1^+)$ over an affinoid field of height 1, in order to illustrate Huber’s theory of universal compactifications. We will next describe what this extended unit disc looks like over higher height affinoid fields.

So let $\kappa = (k, k^\top)$ be an affinoid field of finite height, with $k$ complete. As in Remark 3.2.3(2), we define

$$\mathbb{D}_\kappa(0; 1^+) := \text{Spa} \left( k(x), k^+ + k^\alpha(x) \right),$$

which is the universal compactification of the closed unit disc

$$\mathbb{D}_\kappa(0; 1) = \text{Spa} \left( k(x), k^+(x) \right)$$

of radius 1. Let $y_1 \succ \ldots \succ y_0$ denote the distinct points of $\text{Spa}(\kappa)$. Let $k_n = k$, and recursively set $k_i$ to be the residue field of $k_{i+1}$, with $k_i^+$ the image of $k^+$ in $k_i$. Let $\kappa_i$ denote the affinoid field $(k_i, k_i^+)$, and $\kappa_i^*$ the height one affinoid field $(k_i, k_i^*)$.

The open fibre $f^{-1}(y_n)$ is canonically homeomorphic to the extended unit disc $\mathbb{D}_{\kappa_i^*}(0; 1^+)$ over the height one affinoid field $\kappa_i^* = (k, k^\alpha)$. The usual classification of the points of this space divides them into 5 types, points of Type I, III, IV and V are all closed, whereas points of Type II are non-closed. These can also be distinguished by their residue fields: if $x$ has Type I, III, IV or V then the residue field $k(x)$ of the valuation

$$v_x : k(x) \to \{0\} \cup \Gamma$$

is a finite extension of $k_{n-1}$, and if $x$ has Type II then the residue field $\widetilde{k(x)}$ is a geometrically rational function field over a finite extension of $k_{n-1}$. In either case, let $k_{n-1}(x)$ denote this finite extension (of $k_{n-1}$), and $k_{n-1}(x)^+$ the integral closure of $k_{n-1}(x)$ inside $k_{n-1}(x)$. If $x$ is of Type II, then the closure $\overline{\{x\} \cap f^{-1}(y_n)}$ of $x$ within the open fibre $f^{-1}(y_n)$ consists of $x$, together with a set of Type V points in canonical bijection with the closed points of the unique smooth projective algebraic curve $C_x$ over $k_{n-1}(x)$ whose function field is $k(x)$.

In general, the closure $\overline{\{x\}}$ of $x$ in $\mathbb{D}_\kappa(0; 1^+)$ can be identified with the set of valuations

$$v : k(\widetilde{x}) \to \{0\} \cup \Gamma$$

which satisfy $v(k_{n-1}(x)^+) \leq 1$. If $x$ is of Type I, III, IV or V then there is one such trivial valuation, corresponding to the point $x$ itself, and then the rest are in natural bijection with the adic space $\text{Spa}(k_{n-1}(x), k_{n-1}(x)^+)\Gamma$. On the other hand, if $x$ is of Type II, then on top of the Type V points making up $\overline{\{x\} \cap f^{-1}(y_n)}$, we obtain a homeomorphism between

$$\overline{\{x\} \cap f^{-1}(\{y_{n-1}, \ldots, y_0\})}$$

and the analytification of the curve $C_x$ over the affinoid field $(k_{n-1}(x), k_{n-1}(x)^+)$.

Repeating this calculation allows us to describe the fibres of $f : \mathbb{D}_\kappa(0; 1^+) \to \text{Spa}(\kappa)$ recursively. Namely, each fibre $f^{-1}(y_{i+1})$ will consist of closed points and non-closed points. Each closed point has a unique specialisation in the fibre $f^{-1}(y_{i})$. To each non-closed point $x \in f^{-1}(y_{i+1})$ we can associate a finite extension $k_i(x)$ of the field $k_i$, and a smooth, projective, geometrically rational curve $C_x$ over $k_i(x)$, such that

- the set of specialisations of $x$ within $f^{-1}(y_{i+1})$ is in canonical bijection with the closed points of the algebraic curve $C_x$;
- the set of specialisations of $x$ within the fibre $f^{-1}(y_i)$ is in canonical bijection with the analytification $C_x^{\alpha}$ of $C_x$ over the height one affinoid field $k_i(x)^* := (k_i(x), k_i(x)^*)$.

\[3\text{note that the trivial valuation is not continuous on } k_{n-1}(x)\]
Note that in particular closed points of the algebraic curve $C_x$ give rise to Type I leaves of $C_x^{an}$. To obtain $f^{-1}(y_i)$ from $f^{-1}(y_{i+1})$, then, we remove every non-closed point $x \in f^{-1}(y_{i+1})$, and glue in a copy of $C_x^{an}$ over $\kappa^*_i$ by identifying a given Type V specialisation of $x$ within $f^{-1}(y_{i+1})$ with the corresponding Type I leaf on $C_x^{an}$. The following picture should hopefully clarify this procedure.

This description also makes clear the specialisation relationships between points of $\mathbb{D}_s(0;1^+)$. 

4. The proper base change theorem

The goal of this section is to prove the following version of the proper base change theorem.

4.1.1. **Theorem** (Proper Base Change I). Let $f : X \to Y$ be a proper morphism of germs of adic spaces, and $y \in Y$. Then, for all sheaves $\mathcal{F}$ on $X$, and all $q \geq 0$, the natural map

$$\mathcal{R}^q f_* \mathcal{F}_y \to H^q(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

is an isomorphism.

Using Corollary 2.4.4, this is equivalent to the following.

4.1.2. **Theorem** (Proper Base Change II). Let $f : X \to Y$ be a proper morphism of germs of adic spaces, with local target. Let $Z \subset X$ be the closed fibre of $f$. Then, for all sheaves $\mathcal{F}$ on $X$, and all $q \geq 0$, the natural map

$$H^q(X, \mathcal{F}) \to H^q(Z, \mathcal{F}|_Z)$$

is an isomorphism.

We will instead prove a marginally stronger version of this result, in order to be able to set up an induction (see Step 2 of the proof below).

4.1.3. **Theorem** (Proper Base Change III). Let $f : X \to Y$ be a proper morphism of germs of adic spaces, with local target. Let $S \subset Y$ be a closed subset, and $Z = f^{-1}(S) \subset X$ be the preimage of $S$ under $f$. Then, for all sheaves $\mathcal{F}$ on $X$, and all $q \geq 0$, the natural map

$$H^q(X, \mathcal{F}) \to H^q(Z, \mathcal{F}|_Z)$$

is an isomorphism.

The key step in the proof of Theorem 4.1.3 is the following very special case.

4.1.4. **Proposition.** Let $\kappa$ be an affine field, $X = \mathbb{D}_s(0;1^+)$ and $f : X \to \text{Spa}(\kappa)$ the structure morphism. Let $i : T \hookrightarrow X$ be a closed subspace, $A$ an abelian group, and $\mathcal{F} = i_* A$ the constant sheaf supported on $T$. Then Theorem 4.1.3 holds for $f$ and $\mathcal{F}$.

**Proof.** We must show that the natural map $H^q(T, A) \to H^q(T \cap Z, A)$ is an isomorphism for all $q \geq 0$. Note that if $T = T_1 \cup T_2$ is the union of two closed subsets, and we know the claim for $T_1, T_2$, and $T_1 \cap T_2$, then we know it for $T$ by Meyer-Viertoris.

Let $y_0, \ldots, y_0$ be the distinct points of $\text{Spa}(\kappa)$. Note that every point in some fibre $f^{-1}(y_i)$ specialises to some point in the fibre $f^{-1}(y_{i-1})$. Now, let $i_1$ be minimal such that $f(T) \subset \{y_{i_1}\}$, and $i_0$ maximal such that $y_{i_0} \in S$. Then we can write $T = T \cap f^{-1}(y_1) \cup T'$ with $f(T') \subset$
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Let \( f_i \leq i_0 \) there is nothing to prove, and hence by induction on \( i_1 \) we may assume that \( T = T \cap f^{-1}(y_i) \). By Proposition 2.4.6 we have

\[
H^q(T, \mathcal{A}) \cong H^q([T], \mathcal{A}) \quad \text{and} \quad H^q(T \cap Z, \mathcal{A}) \cong H^q([T \cap Z], \mathcal{A}),
\]

so we may replace \( X \) by \( X_{y_0} = X_{\kappa(y_0)} \). In other words we may assume that \( i_0 = 0 \) and \( S = \{ y_0 \} \). Now, the inclusion

\[
T \cap f^{-1}(y_i) \hookrightarrow T \cap f^{-1}(\{y_i\})
\]

induces a homeomorphism

\[
[T \cap f^{-1}(y_i)] \xrightarrow{\sim} T \cap f^{-1}(\{y_i\})
\]
on separated quotients, for all \( 0 \leq i \leq i_1 \). In this way we get a chain

\[
[T \cap Z] = [T \cap f^{-1}(y_0)] \xrightarrow{s_0} [T \cap f^{-1}(y_1)] \xrightarrow{s_1} \ldots \xrightarrow{s_{i_1-1}} [T \cap f^{-1}(y_{i_1})] = [T]
\]
of maps of compact (in particular, Hausdorff) spaces. Using (4.1.5) we may apply the Leray spectral sequence in order to reduce to showing that

\[
\mathcal{A} \rightarrow R\kappa_*\kappa^*\mathcal{A}
\]
is an isomorphism, for any \( 0 \leq i \leq i_1 - 1 \). To see this, we consider the diagram

\[
\begin{array}{ccc}
T \cap f^{-1}(\{y_i\}) & \longrightarrow & T \cap f^{-1}(\{y_{i+1}\}) \\
\downarrow & & \downarrow \\
[T \cap f^{-1}(y_i)] & \xrightarrow{s_i} & [T \cap f^{-1}(y_{i+1})]
\end{array}
\]

where the vertical maps are the separation maps, and the upper horizontal map is the natural inclusion. Since \([T \cap f^{-1}(y_i)]\) is compact, and \([T \cap f^{-1}(y_{i+1})]\) is Hausdorff, it follows that the map \( s_i \) is closed. Moreover, if \( t \in [T \cap f^{-1}(y_{i+1})] \), viewed as a point on the fibre \( f^{-1}(y_{i+1}) \), then the induced map

\[
\{t\} \cap f^{-1}(y_i) \rightarrow s_i^{-1}(t)
\]
is a continuous bijection from a compact space to a Hausdorff space, this a homeomorphism.

We therefore deduce from the explicit description of \( \mathcal{D}_n(0; 1^+) \) given in \S 3.4 above that each fibre \( s_i^{-1}(t) \) is either a singleton, or homeomorphic to the Berkovich analytification of a smooth projective conic over an affinoid field of height one. In particular, \( s_i \) is a proper (in the sense of topological spaces) morphism between compact spaces, and hence we can check the isomorphy of \( \mathcal{A} \rightarrow R\kappa_*\kappa^*\mathcal{A} \) on fibres. If \( s_i^{-1}(t) \) is a singleton, the claim then is clear, otherwise it follows from [Ber90, Corollary 4.2.2].

The proof of Theorem 4.1.3 in general is now essentially a reduction to this special case.

**Step 1:** Theorem 4.1.3 holds for \( f : \mathcal{D}_n(0; 1^+) \rightarrow \text{Spa}(\kappa) \) and any sheaf \( \mathcal{F} \) on \( \mathcal{D}_n(0; 1^+) \).

**Proof.** To show the case \( q = 0 \), we will apply Lemma 1.2.4: we need to show that for every point \( x \in X \) the space \( \overline{x} \cap Z \) is non-empty and connected. But this just follows from the concrete description given in \S 3.4.

It therefore suffices to show that, for every \( q > 0 \), the functor \( \mathcal{F} \mapsto H^q(Z, \mathcal{F}|_Z) \) is effaceable in the category of sheaves on \( X \). Since \( Z \) is coherent, cohomology commutes with filtered colimits. Appealing to Lemma 1.2.3 we may therefore assume that there exists a finite collection of closed subsets \( \iota_i : T_i \hookrightarrow X \), and abelian groups \( A_i \) such that \( \mathcal{F} \mapsto \prod_i \iota_i^* A_i \). We may therefore reduce to the case when \( \mathcal{F} = \iota_* A_i \), which was treated in Proposition 4.1.4 above.

Let \( \mathcal{D}_n^0(0; 1^+) := \text{Spa}(k(x_1, \ldots, x_n), k^+ + k^{\infty}(x_1, \ldots, x_n)) = \mathcal{D}_n(0; 1^+) \times_k \ldots \times_k \mathcal{D}_n(0; 1^+) \).

**Step 2:** Theorem 4.1.3 holds for \( f : \mathcal{D}_n^0(0; 1^+) \rightarrow \text{Spa}(\kappa) \), for any \( n \geq 0 \).
Proof. Let \( \pi : \mathbb{D}_{n}^{\kappa}(0; 1^{+}) \to \mathbb{D}_{n-1}^{\kappa}(0; 1^{+}) \) denote the natural projection, \( Z_{n} \subset \mathbb{D}_{n}^{\kappa}(0; 1^{+}) \) and \( Z_{n-1} \subset \mathbb{D}_{n-1}^{\kappa}(0; 1^{+}) \) the respective preimages of \( S \). Then \( \pi^{-1}(Z_{n-1}) = Z_{n} \), and by induction on \( n \) it suffices to show that, for all \( q \geq 0 \), the natural map

\[
(R^{q}\pi_{*}\mathcal{F})|_{Z_{n-1}} \to R^{q}\pi_{*}(\mathcal{F}|_{Z_{n}})
\]

is an isomorphism. If \( z \in Z_{n-1} \) then the stalk of the LHS at \( z \) is equal to

\[
H^{q}(\mathbb{D}_{\kappa(z)}^{\kappa}(0; 1^{+}), \mathcal{F}|_{\mathbb{D}_{\kappa(z)}^{\kappa}(0; 1^{+})})
\]

by Corollary 1.2.2. If we let \( T := G(z) \cap Z_{n-1} \) denote the set of generalisations of \( z \) inside \( Z_{n-1} \), then the stalk of the RHS at \( z \) is equal to

\[
H^{q}(\pi^{-1}(T), \mathcal{F}|_{\pi^{-1}(T)})
\]

again by Corollary 1.2.2. Since \( T \) is naturally a closed subset of \( \text{Spa}(\kappa(z)) \), the two coincide by Step 1 above. \( \square \)

**Step 3**: Theorem 4.1.3 holds whenever the base \( Y = \text{Spa}(\kappa) \) is a local adic space, and \( X \) is a closed subset of an affinoid \( X = \text{Spa}(R, R^{+}) \) such that \( X \) is proper over \( \kappa \), and \((R, R^{+}) \) is of \( + \) weakly finite type over \( \kappa \).

**Proof.** Since \( X \hookrightarrow X \) is a closed immersion, we may replace \( X \) by \( X \) and therefore assume that \( X = X = \text{Spa}(R, R^{+}) \). We have a continuous open surjection \( k(x_{1}, \ldots, x_{n}) \to R \) inducing a homomorphism of pairs

\[
(k(x_{1}, \ldots, x_{n}), k^{+} + k^{\circ}(x_{1}, \ldots, x_{n})) \to (R, R^{+})
\]

and hence a morphism of adic spaces

\[
\text{Spa}(R, R^{+}) \to \mathbb{D}_{n}^{\kappa}(0; 1^{+}).
\]

This can be checked to be a homeomorphism onto a closed subset of \( \mathbb{D}_{n}^{\kappa}(0; 1^{+}) \) exactly as in Lemma 3.2.4. Hence the claim follows by Step 2 above. \( \square \)

**Step 4**: Theorem 4.1.3 holds in general.

**Proof.** Let \( f : X \to Y \) be proper, with \( Y \) local. If we let \((Y, Y)\) be a representative for \( Y \), with \( Y \) a local adic space, then the induced map \( f : X \to Y \) is still proper, so we may assume that \( Y = \text{Spa}(\kappa) \) is a local adic space.

Now let \( f : (X, X) \to Y \) be a representative for \( f \). Since \( X \) is quasi-compact, we may assume that \( X \) is also quasi-compact. Let \( U = \text{Spa}(R, R^{+}) \subset X \) be an open affinoid, with \((R, R^{+}) \) of \( + \) weakly finite type over \( \kappa \), let \( U = \text{Spa}(R, R^{+}) \cap X \) be its intersection with \( X \), and \( \overline{U} \to Y \) its (affinoid) universal compactification over \( Y = \text{Spa}(\kappa) \). Then by Lemma 3.2.4 we know that the closure \( \overline{U} \) of \( U \) inside \( X \) can be viewed as a closed subset of \( \overline{U} \), and hence by Step 3 we know that Theorem 4.1.3 holds for the induced morphism \((\overline{U}, \overline{U}) \to Y \) of quasi-adic spaces.

Since \( f : X \to Y \) is proper, we may choose a closed hypercover \( X_{\bullet} \to X \) such that each \( X_{n} \) is a finite disjoint union of closed subspaces of \( X \), arising as the closure of \( \text{Spa}(R, R^{+}) \cap X \) for some open affinoid \( \text{Spa}(R, R^{+}) \subset X \) with \((R, R^{+}) \) of \( + \) weakly finite type over \( \kappa \) as above. We therefore know that Theorem 4.1.3 holds for each \( X_{n} \to Y \), and therefore using the Mayer-Vietoris spectral sequence for such a closed hypercover we can conclude. \( \square \)

5. **Proper pushforwards on germs of adic spaces**

Armed with the proper base change theorem, we can now proceed to define proper push-forwards for adic spaces, following Huber. As should be expected, the notion of compact supports for sections of sheaves on adic spaces needs to take into account not just quasi-compactness, but *properness* in the analytic sense.
5.1. Sections with proper support. Let $f : X \rightarrow Y$ be a separated morphism of quasi-adic spaces, locally of $+$-weakly finite type, $\mathcal{F}$ a sheaf on $X$, $V \subset Y$ an open subset and $s \in \Gamma(f^{-1}(V), \mathcal{F})$ a section. Then the support
\[
\text{supp}(s) := \{ x \in f^{-1}(V) \mid s_x \neq 0 \} \subset f^{-1}(V)
\]
is a germ of an adic space, and it therefore makes sense to ask whether or not it is proper over $V$.

5.1.1. Definition. Define
\[
f_*\mathcal{F} \subset f_*\mathcal{F}
\]
to be the subsheaf consisting of sections $s \in \Gamma(V, f_*\mathcal{F}) = \Gamma(f^{-1}(V), \mathcal{F})$ whose support is proper over $V$.

5.1.2. Example. If $f$ is an open immersion, then $f_1$ is the usual extension by zero functor. If $f$ is proper then $f_! = f_*$. We will denote $H^0(Y, f_!(\cdot))$ by either $H^0(Y, X, \cdot)$ or $\Gamma_c(Y, \cdot)$, and when $Y$ is a local germ, and $f$ is surjective, we will sometimes write $H^0(X, \cdot)$ or $\Gamma_c(X, \cdot)$ instead.\footnote{We will only use this notation where there is no possible scope for confusion.}

5.2. Comparison with van der Put’s definition. Whenever $f : X \rightarrow Y$ is a morphism of adic spaces of finite type over a height one affinoid field $(k, k^\kappa)$, and the base $Y$ is affinoid, a definition of $H^0(Y, X, \cdot)$ has already been given in [vdP92, Definitions 1.4]. In fact, van der Put worked with rigid analytic spaces rather than adic space, but since the underlying topoi are the same [Hub96, (1.1.11)] we can transport his definition to the adic context.

5.2.1. Lemma. Assume that $f : X \rightarrow Y$ is a partially proper morphism locally of finite type between adic spaces, such that $Y$ is affinoid and of finite type over a height one affinoid field. Then the above definition of $H^0(Y, X, \cdot)$ agrees with that given by van der Put.

Proof. Since any closed subset of $X$ is partially proper over $Y$, it suffices to show that such a closed subset is quasi-compact over $Y$ if and only if it can be covered by a finite family of affinoids. This is clear. \hfill \square

5.2.2. Remark. The result is false without some assumption on $f$. For example, if $Y = \text{Spa}(\kappa)$ with $\kappa = (k, k^\kappa)$ an affinoid field of height one, and $X = \mathbb{D}_{\text{an}}(0; 1)$ is the closed unit disc, then our Huber-style definition of $H^0_c(X, \cdot)$ is genuinely different from that given by van der Put. In this case, van der Put’s definition is equivalent to requiring sections to have support quasi-compact and disjoint from the closure of the Gauss point, whereas our definition only requires this support to be disjoint from the Gauss point itself.

5.3. Basic properties of proper pushforwards. The following properties of $f_!$ and $\Gamma_c(X/Y, \cdot)$ can be verified exactly as in [Hub96, Proposition 5.2.2].

5.3.1. Proposition. Let $f : X \rightarrow Y$ be a morphism of germs of adic spaces, separated and locally of $+$-weakly finite type.

(1) The functors $\Gamma_c(X/Y, \cdot)$ and $f_!$ are left exact.

(2) The functor $f_!$ commutes with filtered colimits. If $Y$ is coherent, then so does $\Gamma_c(X/Y, \cdot)$.

(3) Let $g : Y \rightarrow Z$ be a morphism of germs of adic spaces, separated and locally of $+$-weakly finite type. Then the canonical identification $(g \circ f)_* = g_* \circ f_*$ induces $(g \circ f)_! = g_! \circ f_!$.

(4) Let
\[
\begin{array}{ccc}
\text{X'} & \xrightarrow{g'} & \text{X} \\
\text{Y'} & \xrightarrow{f'} & \text{Y} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{X'} & \xrightarrow{g'} & \text{X} \\
\text{Y'} & \xrightarrow{f} & \text{Y} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{X'} & \xrightarrow{g'} & \text{X} \\
\text{Y'} & \xrightarrow{f} & \text{Y} \\
\end{array}
\]
be a Cartesian diagram of germs of adic spaces, with \( g \) a homeomorphism onto a locally closed subset of \( Y \). Then, for any sheaf \( \mathcal{F} \) on \( X \), the natural map \( g^{-1} f_* \mathcal{F} \rightarrow f'_* g^{-1} \mathcal{F} \) induces an isomorphism

\[
g^{-1} f_* \mathcal{F} \xrightarrow{\sim} f'_* g^{-1} \mathcal{F}.
\]

5.3.2. Remark. We do not have a full analogue of [Hub96, Proposition 5.2.2 iv)] essentially for the same reason as for sheaves on the Zariski site of schemes, namely that if \( g : Y' \rightarrow Y \) is an arbitrary morphism, and \( y \in Y' \), there is in general no reason for the natural map

\[
X'_y \rightarrow X_{g(y)}
\]

of adic spaces to induce a homeomorphism on closed fibres.

5.4. Derived proper pushforwards for partially proper morphisms. At least if we restrict to partially proper morphisms, we can now define \( Rf_* \) as the derived functor of \( f_* \).

5.4.1. Definition. Let \( f : X \rightarrow Y \) be a partially proper morphism of germs of adic spaces. Define

\[
Rf_* : D^+(\text{Sh}(X)) \rightarrow D^+(\text{Sh}(Y))
\]

to be the total derived functor of \( f_* \). For any \( q \geq 0 \), define \( R^q f_* = H^q(Rf_*) \).

We will also write \( R\Gamma_c(X/Y, -) \) for the total derived functor of \( H^0_c(X/Y, -) \), and \( H^q_c(X/Y, -) \) for the cohomology groups of this complex. If \( Y = \text{Spa}(\kappa) \) is local, and \( f \) is surjective, we will sometimes write \( R\Gamma_c(X, -) \) and \( H^q_c(X/Y, -) \) instead. This agrees with van der Put’s definition whenever \( \kappa \) is of height one.

5.4.2. Our next aim will be to show that these derived proper pushforwards compose correctly (for partially proper morphisms), and the key point in doing so will be is to relate them to ordinary pushforwards as in [Hub96, §5.3].

5.4.3. Lemma. Let \( f : X \rightarrow Y \) be a partially proper morphism between germs of adic spaces, and assume that \( Y \) is coherent.

1. There exists a cover of \( X \) by a cofiltered family of overconvergent open subsets \( \{U_i\}_{i \in I} \subset X \), each of which has quasi-compact closure.
2. For any such family \( \{U_i\} \), any sheaf \( \mathcal{F} \) on \( X \), and any \( q \geq 0 \), there is a canonical isomorphism

\[
\colim_{i \in I} R^q f_{*}(j_{i!}\mathcal{F}|_{U_i}) \xrightarrow{\sim} R^q f_* \mathcal{F}
\]

where \( j_i : U_i \rightarrow \overline{U}_i \) denotes the canonical open immersion, and \( f_i : \overline{U}_i \rightarrow Y \) the restriction of \( f \).

The first claim was proved in [Hub96, Lemma 5.3.3], and the second part is shown in exactly the same way as Huber does in the étale case, using the following lemma.

5.4.4. Lemma. let \( X \) be a quasi-separated germ of an adic space, \( U \subset X \) an overconvergent open subset with quasi-compact closure, and \( j : U \hookrightarrow \overline{U} \) the natural inclusion. Then, for any flasque sheaf \( \mathcal{F} \) on \( X \), and any \( q \geq 0 \), \( H^q(\overline{U}, j_*\mathcal{F}|_{U}) = 0 \).

5.4.5. Remark. Note that this does not contradict Example 5.5.1, since in that case \( U \) was not an overconvergent open subset of \( X \).

Proof. Note that if \( j' : U \hookrightarrow X \) denotes the inclusion, then \( H^q(\overline{U}, j_*\mathcal{F}|_{U}) = H^q(X, j'_*\mathcal{F}|_{U}) \), we will show that the latter vanishes if \( q > 0 \). If we let \( i : Z \rightarrow X \) denote the closed complement to \( U \), then \( Z \) is also overconvergent, and using the exact sequence

\[
0 \rightarrow j'_!\mathcal{F}|_{U} \rightarrow \mathcal{F} \rightarrow i_*\mathcal{F}|_{Z} \rightarrow 0
\]

this is equivalent to showing that:

1. \( H^0(X, \mathcal{F}) \rightarrow H^0(Z, \mathcal{F}|_{Z}) \) is surjective;
2. \( H^q(Z, \mathcal{F}|_{Z}) = 0 \) for all \( q > 0 \).
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Since $\mathcal{I}$ is flasque, and $Z$ is overconvergent, this follows from Lemma 1.4.2. □

Proof of Lemma 5.4.3. Define functors $T^q : \text{Sh}(X) \to \text{Sh}(Y)$ by

$$T^q(\mathcal{F}) = \colim_{i \in I} R^q f_* (j_{i!} \mathcal{F}|_{U_i}).$$

The $T^q$ form a $\delta$-functor, with $T^0 = f_!$. Moreover, when $q > 0$, we can deduce from Lemma 5.4.4 that $T^q$ is effaceable, hence

$$T^q(\mathcal{F}) \cong R^q f_* \mathcal{F}$$

as required. □

The following two corollaries of Lemma 5.4.3 are proved word for word as in their étale counterparts [Hub96, Propositions 5.3.7, 5.3.8].

5.4.6. Corollary. Let $f : X \to Y$ be a partially proper morphism between germs of adic spaces. Then, for each $q \geq 0$, the functor $R^q f_!$ commutes with filtered colimits. If $Y$ is coherent, then so does $H^q_\mathcal{I}(X/Y, -)$. □

Proof. We may assume that $Y$ is coherent. Let $\{U_i\}_{i \in I}$, $j_i : U_i \to Y$, and $f_i : T_i \to Y$ be as in Lemma 5.4.3, these are all coherent maps of topological spaces. Then $j_{i!}$ is a left adjoint, hence commutes with filtered colimits, and $f_i$ is coherent, hence $R^q f_* i_!$ commutes with filtered colimits. Therefore

$$R^q f_i ! \cong \colim_{i \in I} R^q f_* (j_{i!} (-)|_{U_i})$$

commutes with filtered colimits. The corresponding claim for $H^q_\mathcal{I}(X/Y, -)$ is proved similarly. □

5.4.7. Corollary. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be partially proper morphisms between germs of adic spaces. Then there is a canonical isomorphism $R(g \circ f)_! \cong Rg_! \circ Rf_!$ of functors $D^+(\text{Sh}(X)) \to D^+(\text{Sh}(Z))$. □

Proof. We may assume that $Z$ is coherent; it suffices to show that if $\mathcal{F}$ is an injective sheaf on $X$, then $H^q_\mathcal{I}(Y/Z, f_* \mathcal{F}) = 0$ for $q > 0$. Using Lemmas 5.4.3 and 5.4.4, it therefore suffices to show that the sheaf $f_* \mathcal{F}$ is flasque on $Y$, and moreover that for any open $W$, the map

$$H^q_\mathcal{I}(X/Y, \mathcal{F}) \to H^q_\mathcal{I}(X/W, \mathcal{F})$$

is surjective. So pick a section $s \in H^q_\mathcal{I}(f^{-1}(W)\setminus W, \mathcal{F})$ with proper support over $W$. Let $T$ be the closure of $\supp(s)$ in $X$, and $s' \in H^q_\mathcal{I}(f^{-1}(W)\cup (X\setminus T), \mathcal{F})$ the section with $s'|_{f^{-1}(W)} = s$ and $s'|_{X\setminus T} = 0$. Since $\mathcal{I}$ is flasque, we can pick $s'' \in \Gamma(X, \mathcal{F})$ with $s''|_{f^{-1}(W)\cup (X\setminus T)} = s'$. Then $\supp(s'') \subset T$, and, since $f : X \to Y$ is taut [Hub96, Lemma 5.1.4 ii)], $T$ is quasi-compact over $Y$. Thus $\supp(s'')$ is proper over $Y$, and gives a lift of $s$ to $H^q_\mathcal{I}(X/Y, \mathcal{F})$. □

Combining Lemma 5.4.3 with Theorem 4.1.1 gives us the following.

5.4.8. Corollary. Let $f : X \to Y$ be a partially proper morphism between quasi-adic spaces, and $y \in Y$. Then, for all sheaves $\mathcal{F}$ on $X$, and all $q \geq 0$, the natural map

$$(R^q f_* \mathcal{F})|_y \to H^0_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

is an isomorphism.

5.5. A counter-example in the non-partially proper case. Just as for étale cohomology, if $f$ and $g$ are not partially proper, the relation $(g \circ f)_! = g_! \circ f_!$ it not preserved when we pass to total derived functors.

5.5.1. Example. Let $\kappa = (k, k^\circ)$ be a height one affinoid field,

$$U = \mathbb{A}_\kappa(0; 1, 1) := \text{Spa}(k(x, x^{-1}), k^\circ(x, x^{-1}))$$

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the closed annulus of radius 1 over \( \kappa \), \( X = \text{Spa} \left( k(x, x^{-1}), k^\circ + k^\circ \langle x, x^{-1} \rangle \right) \) its universal compactification, and \( j : U \to X \) the canonical open immersion. The complement \( X \setminus U \) consists of two points which are distinct specialisations of the Gauss point \( \xi \in X \).

\[ \xi \]

Let \( Z = X \setminus U \) and \( i : Z \to X \) the inclusion. Let \( \mathcal{I} \) be any injective sheaf on \( U \), and consider the exact sequence

\[ 0 \to j_* \mathcal{I} \to j_* j^* \mathcal{I} \to i_* i^* j_* \mathcal{I} \to 0. \]

Taking the long exact sequence in cohomology, we obtain

\[ 0 \to H^0(X, j_* \mathcal{I}) \to H^0(U, \mathcal{I}) \to \mathcal{I}_\xi \oplus \mathcal{I}_\xi \to H^1(X, j_* \mathcal{I}) \to 0 \]

since the stalks of \( i^* j_* \mathcal{I} \) at both points of \( Z \) are equal to the stalk of \( \mathcal{I} \) at \( \xi \). The map \( H^0(U, \mathcal{I}) \to \mathcal{I}_\xi \oplus \mathcal{I}_\xi \) factors through the diagonal, so we deduce that \( H^1(X, j_* \mathcal{I}) \neq 0 \) if \( \mathcal{I}_\xi \neq 0 \). Therefore the total derived functor of \( H^0_c(U, -) \) does not coincide with the composite of that of \( j_* \) and \( H^0_c(X, -) \).

5.6. Higher proper pushforwards in general. To get around this problem, we use Huber’s theory of compactifications as in [Hub96]. For adic spaces, this can be done in a straightforward way, since adic spaces admit universal compactifications.

5.6.1. Definition. Let \( f : X \to Y \) be a morphism of adic spaces, separated, locally of \(+\) weakly finite type, and taut, and let \( X \xrightarrow{j} \overline{X} \xrightarrow{f} Y \) be the universal compactification of \( f \). Define

\[ \mathbf{R}f_! := \mathbf{R}f_! \circ j_* : D^+(\text{Sh}(X)) \to D^+(\text{Sh}(Y)) \]

where \( \mathbf{R}f_! \) is the total derived functor of \( f_! \).

By the universal property of universal compactifications, this only depends on \( f \) up to canonical isomorphism. Germs of adic spaces, however, do not necessarily admit universal compactifications, so we first need to prove following lemma, analogous to [Hub96, Lemma 5.4.2].

5.6.2. Lemma. Let

\[ \begin{array}{ccc}
X' & \xrightarrow{j'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{j} & Y
\end{array} \]

be a commutative square of germs of adic spaces, with \( f, f' \) partially proper and \( j, j' \) open immersions. Then there exists a canonical isomorphism

\[ j_! \circ \mathbf{R}f'_! \cong \mathbf{R}f_! \circ j'_! \]

of functors \( D^+(\text{Sh}(X')) \to D^+(\text{Sh}(Y)) \).

Proof. We may divide into two cases: first when the diagram is Cartesian, and secondly when \( j = \text{id} \). In the first case, since \( j_! \circ \mathbf{R}f'_! \) and \( \mathbf{R}f_! \circ j'_! \) are clearly isomorphic on \( Y' \), it suffices to prove that \( \mathbf{R}f_! \circ j'_! \) is supported on \( Y' \), which follows from Corollary 5.4.8. In the second case, since both \( f' \) and \( f \) are partially proper, we deduce that \( j' \) is partially proper, and hence we may apply Corollary 5.4.7. \( \square \)
5.6.3. **Theorem.** Let \( f : X \to Y \) a separated, locally of \( + \) weakly finite type and taut morphism between germs of adic spaces, and

\[
\begin{array}{ccc}
X & \xrightarrow{j} & X' \\
\downarrow & \searrow & \downarrow \\
Y & \xrightarrow{f'} & Y'
\end{array}
\]

a compactification of \( f \) provided by Theorem 3.3.2. Then the functor

\[ Rf'_! \circ j_! : D^+ (\text{Sh}(X)) \to D^+ (\text{Sh}(Y)) \]

only depends on \( f \), and not on the choice of compactification, up to canonical isomorphism.

**Proof.** Given Lemma 5.6.2, this can be proved exactly as in [SGA7, Exposé XVII, §3]. □

We can now define \( Rf'_! \) in general.

5.6.4. **Definition.** In the situation of Theorem 5.6.3, we define

\[ Rf'_! = Rf'_! \circ j_! \]

We will use the notations \( R^q f_!, R^\Gamma_c(X/Y, -), \) and \( H^q_c(X/Y, -) \) as in the partially proper case.

5.7. **Composition of proper pushforwards.** We can also prove that proper pushforwards compose correctly.

5.7.1. **Theorem.** Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \) be separated, locally of \( + \) weakly finite type, and taut morphisms between germs of adic spaces. Then there is a canonical isomorphism \( R(g \circ f)_! \to Rg_! \circ Rf'_! \) of functors \( D^+ (\text{Sh}(X)) \to D^+ (\text{Sh}(Z)) \).

**Proof.** Again, given Corollaries 5.4.7 and 5.4.8, and Lemma 5.6.2, this can be proved as in [SGA7, Exposé XVII, §3]. □

5.7.2. **Remark.** It then follows that \( Rf_! = f_! = j_! \circ i^* \) whenever \( f = j \circ i \) is a locally closed immersion.

It follows from the definition that we can identify \( Rf_* \) and \( Rf'_! \) whenever \( f \) is proper. In fact, we can do so in a few more cases than this.

5.7.3. **Lemma.** Let \( U \xrightarrow{i} X \xrightarrow{j} Y \) be open immersions of germs of adic spaces. Assume that the closure of \( U \subset Y \) is contained within \( X \). Then for any sheaf \( \mathcal{F} \) on \( U \) the canonical morphism

\[ g_i^* f^! \mathcal{F} \to Rg_i^* f_! \mathcal{F} \]

is an isomorphism.

**Proof.** The given map is clearly an isomorphism on \( X \), it therefore suffices to show that the stalks of the cohomology sheaves of the RHS are zero at every point \( x \in Y \setminus X \). These can be identified with

\[ \text{colim}_V H^q(V \cap X, f_! \mathcal{F}), \]

the colimit being taken over all open neighbourhoods of \( x \) in \( Y \). The hypothesis that \( U \subset X \) implies that there exists such a \( T \) with \( T \cap U = \emptyset \). Thus \( f_! \mathcal{F} |_{T \cap X} = 0 \) and so \( H^q(T' \cap X, f_! \mathcal{F}) = 0 \) for all \( T' \subset T \), and all \( q \geq 0 \). □

5.8. **Cohomological amplitude.** If \( f : X \to Y \) is a separated, locally of \( + \) weakly finite type, and taut morphism between germs, then we have defined the functor

\[ Rf'_! : D^+(\text{Sh}(X)) \to D^+(\text{Sh}(Y)) \]

using Huber’s compactifications. We will explain now why this extends to a functor on the unbounded derived categories. Indeed, if we take a compactification \( X \xrightarrow{j} X' \xrightarrow{f} Y \), then \( j_! \) is exact, and hence clearly extends to \( D(\text{Sh}(X)) \to D(\text{Sh}(X')) \), it therefore suffices to treat the case when \( f \) is partially proper.
5.8.1. **Proposition.** Let \( f : X \to Y \) be a partially proper morphism between germs of adic spaces. Then

\[
Rf_! : D^+(\text{Sh}(X)) \to D^+(\text{Sh}(Y))
\]

has cohomological amplitude contained in \([0, \dim f]\).

**Proof.** We may appeal to Lemma 5.4.3. Thus, there exist closed subsets \( \overline{U}_i \subset X \), and sheaves \( \mathcal{F}_i \) on \( \overline{U}_i \) such that

1. the induced map \( f_i : \overline{U}_i \to Y \) is quasi-compact, thus can be promoted to a proper morphism of germs;

2. \( Rf_i \mathcal{F} \cong \operatorname{colim}_{i \in I} Rf_{i*} \mathcal{F}_i \),

We therefore reduce to the case where \( f \) is proper, which follows from combining Theorem 4.1.1 with Theorem 1.3.2.

5.8.2. **Corollary.** Let \( f : X \to Y \) be a separated, locally of \( + \) weakly finite type, and taut morphism between germs of adic spaces (as always, assumed to be of finite dimension). Then the functor of derived pushforwards with proper support extends canonically to a functor

\[
Rf_! : D(\text{Sh}(X)) \to D(\text{Sh}(Y))
\]

on the unbounded derived categories. This functor sends \( D^- \) to \( D^- \) and \( D^b \) to \( D^b \). If \( g : Y \to Z \) is another such morphism, then there is a canonical isomorphism

\[
R(g \circ f)_! \cong Rg_! \circ Rf_!
\]

of functors

\[
D(\text{Sh}(X)) \to D(\text{Sh}(Z)).
\]

5.9. **Mayer-Vietoris for proper pushforwards.** Let \( f : X \to Y \) be a separated, locally of \( + \) weakly finite type, and taut morphism between germs of adic spaces, and \( X \xrightarrow{j} X' \xrightarrow{f'} Y \) a compactification of \( f \). Consider an open hypercover \( U_\bullet \to X \)

of \( X \), let \( j_n : U_n \to X' \) denote the induced coproduct of open immersions, and \( f_n : U_n \to Y \) the induced morphism of germs. Suppose that we have a sheaf \( \mathcal{F} \) on \( X \). Then there is a resolution

\[
\cdots \to j_{n*} \mathcal{F}|\overline{U}_i \to j_{n-1*} \mathcal{F}|\overline{U}_{i_0} \to j_i \mathcal{F} \to 0
\]

of \( j_i \mathcal{F} \), coming from the fact that \( U_{i} \to X \) is a hypercover. By Corollary 5.8.2 we can apply \( Rf_! \) to this resolution, and by Proposition 5.8.1 the cohomological dimension of \( Rf_{n*} \) is bounded independently of \( n \). Since proper pushfowards respect composition by Theorem 5.7.1, we therefore obtain a convergent second quadrant spectral sequence

\[
E'^{−n,q}_1 = R^qf_{n*} \mathcal{F}|U_n \Rightarrow R^{−n+q}f_! \mathcal{F}
\]

in the category \( \text{Sh}(Y) \) of abelian sheaves on \( Y \). The terms \( R^qf_{n*} \mathcal{F}|U_n \) can also be made slightly more explicit: if \( U_n = \bigsqcup_m U_{n,m} \) with each \( U_{n,m} \) an open subset of \( X \), and \( f_{n,m} : U_{n,m} \to Y \) is the induced morphism, then

\[
R^qf_{n,m*} \mathcal{F}|U_n = \bigoplus_m R^qf_{n,m*} \mathcal{F}|U_{n,m}
\]

by Corollary 5.4.6.

5.9.1. **Corollary.** Let \( f : X \to Y \) be a separated, locally of \( + \) weakly finite type, and taut morphism between germs of adic spaces, \( \mathcal{F} \) a sheaf on \( X \), and \( U_\bullet \to X \) an open hypercover, with \( U_n = \bigsqcup_m U_{n,m} \). Then, there exists a convergent spectral sequence

\[
E'^{−n,q}_1 = \bigoplus_m R^qf_{n,m*} \mathcal{F}|U_{n,m} \Rightarrow R^{−n+q}f_! \mathcal{F}
\]

in the category \( \text{Sh}(Y) \) of abelian sheaves on \( Y \).
5.10. **Module structures on proper pushforwards.** Let \( f : X \to Y \) be a morphism of germs, and suppose that we have sheaves of rings \( A_X \) and \( A_Y \) on \( X \) and \( Y \) respectively, together with a morphism \( A_Y \to f_* A_X \) making \( f \) into a morphism of ringed spaces.

5.10.1. **Example.** The two principal examples will be either \( A_X = \mathcal{O}_X \) and \( A_Y = \mathcal{O}_Y \), or \( A_X \) and \( A_Y \) the constant sheaves associated to some ring \( A \). We could also take \( A_X = \mathcal{O}^+_X \) and \( A_Y = \mathcal{O}^+_Y \) to be the integral structure sheaves.

5.10.2. **Lemma.** Assume that \( f : X \to Y \) is partially proper. If \( \mathcal{I} \) is an injective \( A_X \)-module, then \( \mathcal{I} \) is \( f_! \)-acyclic.

**Proof.** We may assume that \( Y \) is coherent. Since injective \( A_X \)-modules are always flasque sheaves, it suffices to prove that if

\[
0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0
\]

is an exact sequence of sheaves on \( X \) with \( \mathcal{F} \) and \( \mathcal{H} \) flasque, then the induced map

\[
H^0(X/Y, \mathcal{F}) \to H^0(X/Y, \mathcal{H})
\]

is surjective. Given \( s \in H^0(X/Y, \mathcal{H}) \) we know that there exists some \( t \in H^0(X, \mathcal{G}) \) such that \( b(t) = s \). If we let \( U \) denote \( X \setminus \text{supp}(s) \), then \( t|_U = 0 \) and hence there exists some \( r \in H^0(X, \mathcal{F}) \) such that \( a(r)|_U = t|_U \). In particular setting \( t' = t - a(r) \) we find that \( t'|_U = 0 \) and \( b(t') = s \), thus \( \text{supp}(t') \subseteq \text{supp}(s) \) and \( t' \) also has proper support. \( \square \)

For an arbitrary \( f : X \to Y \) separated, locally of \( + \)-weakly finite type and taut between germs, we therefore obtain a canonical lifting of \( Rf_! \) to a functor

\[
Rf_! : D^+(A_X) \to D^+(A_Y)
\]

on the bounded below derived categories of \( A \)-modules, by using universal compactifications. This also extends to a functor

\[
Rf_! : D(A_X) \to D(A_Y)
\]

which sends \( D^- \) to \( D^- \) and \( D^b \) to \( D^b \).

5.10.3. **Lemma.** For any locally free \( A_Y \)-module \( \mathcal{E} \) of finite rank, there exists an isomorphism

\[
\mathcal{E} \otimes_{A_Y} Rf_! \mathcal{F} \cong Rf_!(f^* \mathcal{E} \otimes_{A_X} \mathcal{F})
\]

in \( D^+(A_Y) \).

5.10.4. **Remark.** Here \( f^* \mathcal{E} \) denotes module pullback \( f^{-1} \mathcal{E} \otimes_{f^{-1} A_Y} A_Y \). Since \( \mathcal{E} \) is locally free, the functors

\[
\mathcal{E} \otimes_{A_Y} (-) : D^+(A_Y) \to D^+(A_Y)
\]

\[
f^* \mathcal{E} \otimes_{A_X} (-) : D^+(A_X) \to D^+(A_X)
\]

are well-defined.

**Proof.** Write \( f = \bar{f} \circ j \) where \( \bar{f} : \overline{X} \to Y \) is partially proper and \( j : X \to \overline{X} \) is an open immersion. We equip \( \overline{X} \) with the sheaf of rings \( \mathcal{A}_{\overline{X}} := f^* A_Y \). If we know the claim for \( \bar{f} \) and \( j \) then we have

\[
\mathcal{E} \otimes_{A_Y} Rf_! \mathcal{F} \cong \mathcal{E} \otimes_{A_Y} Rf_! j_! \mathcal{F}
\]

\[
\cong Rf_! \left( f^* \mathcal{E} \otimes_{A_{\overline{X}}} j_! \mathcal{F} \right)
\]

\[
\cong Rf_! \left( j^* f^* \mathcal{E} \otimes_{A_X} \mathcal{F} \right)
\]

\[
\cong Rf_! \left( f^* \mathcal{E} \otimes_{A_X} \mathcal{F} \right),
\]

it therefore suffices to treat separately the case when \( f \) is partially proper and the case when \( f \) is an open immersion.

If \( f \) is partially proper, we can take an injective resolution \( \mathcal{I}^\bullet \) of \( \mathcal{F} \) as an \( A_X \)-module, thus \( f^* \mathcal{E} \otimes_{A_X} \mathcal{I}^\bullet \) is an injective resolution of \( f^* \mathcal{E} \otimes_{A_X} \mathcal{F} \). Since \( Rf_! \) is simply the right derived
functor of \( f \), we can therefore reduce to the case \( \mathcal{F} \) injective, and we must produce a canonical isomorphism

\[
f_!(f^* \mathcal{E} \otimes_{A_X} \mathcal{F}) \cong \mathcal{E} \otimes_{A_Y} f_! \mathcal{F}.
\]

Note that both sides embed naturally into \( \mathcal{E} \otimes_{A_Y} f_! \mathcal{F} \equiv f_*(f^* \mathcal{E} \otimes_{A_X} \mathcal{F}) \), to check that the images are equal we may argue locally, and hence assume that \( \mathcal{E} \cong A_Y^{\text{an}} \). Since all functors in sight \((f^*, f_!, f_*, \otimes)\) commute with finite direct sums, we may therefore reduce to the trivial case \( \mathcal{E} = A_Y \).

In case \( f \) is an open immersion, since both sides are supported on \( X \) it suffices to show the isomorphism after restricting to \( X \). After doing so, the LHS becomes \( \mathcal{E}|_X \otimes_{A_Y}|_X \mathcal{F} \) and the RHS becomes \( \mathcal{E}|_X \otimes_{A_Y|x} A_X \otimes_{A_X} \mathcal{F} \).

\[\square\]

6. The trace map

Let \( f : X \to Y \) be a smooth, separated, taut morphism of germs of adic spaces, of constant relative dimension \( d \). Let \( \Omega^*_{X/Y} \) be its relative de Rham complex, and \( \omega_{X/Y} = \Omega^d_{X/Y} \) the line bundle of top degree differential forms. By Proposition 5.8.1 the complex \( R^d f_! \Omega^*_{X/Y} \) is concentrated in degrees \([0, 2d]\), and

\[
R^{2d} f_! \Omega^*_{X/Y} = \ker \left( R^d f_! \Omega^{d-1}_{X/Y} \to R^d f_! \omega_{X/Y} \right).
\]

The purpose of this section will be to construct a trace map

\[ \text{Tr} : R^d f_! \omega_{X/Y} \to \mathcal{O}_Y, \]

and study its basic properties. We generally take our inspiration from the trace map constructed in [vdP92], although we are able to avoid assuming partial properness of the morphism \( f : X \to Y \) (in fact, van der Put assumes it is a relative Stein over an affinoid). What allows us to do this is the ‘correct’ definition of compactly supported cohomology in the non-partially proper case provided by Huber’s theory of adic spaces.

6.1. Relative analytic affine space. We will first construct a trace map when \( X = \mathbb{A}^d_{Y, \text{an}} \) is the relative analytic affine space over \( Y \).

6.1.1. Of course, we will need to actually define the analytic affine space over a general (analytic) germ \( Y \), and we will begin with the case when \( Y = \text{Spa} (R, R^+) \) is a Tate affinoid adic space.

6.1.2. Definition. Suppose that \( Y = \text{Spa} (R, R^+) \) is a Tate affinoid adic space. Then we define the analytic affine space \( \mathbb{A}^d_{Y, \text{an}} \) to be the fibre product

\[
\begin{array}{ccc}
\mathbb{A}^d_{Y, \text{an}} & \longrightarrow & \text{Spv}(R[x_1, \ldots, x_d]) \\
\downarrow & & \downarrow \\
\text{Spa} (R, R^+) & \longrightarrow & \text{Spv}(R)
\end{array}
\]

in the category of topological spaces. Thus points of \( \mathbb{A}^d_{Y, \text{an}} \) are given by to valuations \( v \) on \( R[x_1, \ldots, x_d] \) whose restriction to \( R \) are continuous, and satisfy \( v(R^+) \leq 1 \).

To equip this with the structure of an adic space, we choose a topologically nilpotent unit \( \varpi \in R^\infty \cap R^+ \). For any \( n \geq 1 \), we can then consider the Tate pair

\[
(R, R^+) \langle \varpi^n z_1, \ldots, \varpi^n z_d \rangle := \left( \frac{R(z_1, \ldots, z_d, t_1, \ldots, t_d)}{(t_i - \varpi^n z_i)}, \frac{R^+(z_1, \ldots, z_d, t_1, \ldots, t_d)}{(t_i - \varpi^n z_i)} \right)
\]

and its adic spectrum

\[
B^d_n := \text{Spa} \left( (R, R^+) \langle \varpi^n z_1, \ldots, \varpi^n z_d \rangle \right),
\]

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which can be identified with the subset of $A^d_{Y,an}$ consisting of those valuations $v$ for which $v(\varpi^n z_i) \leq 1$ for all $i$. There are natural open immersions

\[ B^d_n \hookrightarrow B^d_m \]

for $m \geq n$, via which

\[ A^d_{Y,an} = \bigcup_{n \geq 1} B^d_n. \]

The resulting adic space $A^d_{Y,an}$ does not depend in the choice of pseudo-uniformiser $\varpi$. If $U \subset \text{Spa} (R, R^+)$ is an open affinoid, and $f : A^d_{Y,an} \to Y$ denotes the projection, then there is a canonical isomorphism $f^{-1}(U) \sim A^d_{Y,an}$ of adic spaces, hence the construction glues to define $A^d_{Y,an}$ for an arbitrary adic space $Y$, not necessarily Tate affinoid.

6.1.3. Remark. There are a couple of (equivalent) alternative definitions of $A^d_{Y,an}$ in the literature. Perhaps the most down to earth one simply defines $A^d_{Y,an}$ as the product of $Y$ arbi-

trarily nilpotent unit, then the closure $\overline{\mathcal{F}}_n$ of each $B^d_n$ inside $A^d_{Y,an}$ is quasi-compact, and moreover any quasi-compact subset of $A^d_{Y,an}$ has to be contained in $\overline{B}_n^d$ for some $n$. Thus, if we let $H^2_\mathcal{F}(X, -)$ denote cohomology groups with support in a closed subset $Z \subset X$, we find that

\[ H^2_\mathcal{F}(A^d_{Y,an} / Y, \mathcal{F}) = \text{colim}_n H^2_\mathcal{F}_n(A^d_{Y,an}, \mathcal{F}), \]

for any sheaf $\mathcal{F}$ on $A^d_{Y,an}$.

Using the version of Theorems A and B proved in [KL15, Theorem 2.7.7] (this can also be seen in our situation by combining [FK18, Chapter II, Theorems A.4.7 and 6.5.7]), we can translate Kiehl’s original argument [Kie67] to show that $H^q(A^d_{Y,an}, \mathcal{F}) = 0$ for any coherent $O_{A^d_{Y,an}}$-module $\mathcal{F}$ (still under the assumption that the base $Y = \text{Spa} (R, R^+)$ is a Tate affinoid), and any $q > 0$, whence we deduce isomorphisms

\[ H^d_\mathcal{F}(A^d_{Y,an} / Y, \mathcal{F}) \sim \begin{cases} \text{coker} \left( H^0(A^d_{Y,an}, \mathcal{F}) \to \text{colim}_n H^0(X \setminus \overline{B}_n^d, \mathcal{F}) \right) & d = 1 \\ \text{colim}_n H^{d-1}(A^d_{Y,an} \setminus B_n^d, \mathcal{F}) & d > 1. \end{cases} \]

We can cover $A^d_{Y,an} \setminus B_n^d$ by the Stein spaces

\[ U_{n,i} := \{ x \in A^d_{Y,an} | v_{[x]}(\varpi^n z_i) > 1 \} \]

where $v_{[x]}$ denotes the valuation corresponding to the maximal generalisation of $x$. Again applying

[KL15] to generalise [Kie67], we can show that coherent sheaves have vanishing higher cohomology groups on each $U_{n,i}$. Using this cover to compute the cohomology of $\mathcal{F}$ on $A^d_{Y,an} \setminus B_n^d$, we find that

\[ H^d_\mathcal{F}(A^d_{Y,an} / Y, \mathcal{F}) \sim \text{colim}_n \text{coker} \left( \bigoplus_{1 \leq j \leq d} H^0(\cap_{j \neq i} U_{n,j}, \mathcal{F}) \to H^0(\cap_i U_{n,i}, \mathcal{F}) \right) \]

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for all \( d \geq 1 \). If we now fix \( \mathcal{F} = \omega_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}} = \mathcal{O}_{\mathbb{A}^d_{\mathcal{Y}}},dz_1 \ldots dz_d \), then we can use this to identify \( H^d_{\mathcal{C}}(\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y},\omega_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}}) \) with the set of series

\[
\sum_{i_1,\ldots,i_d \leq 0} a_{i_1,\ldots,i_d} z_1^{i_1} \ldots z_d^{i_d} d \log z_1 \ldots d \log z_d
\]

with \( a_{i_1,\ldots,i_d} \in R \), such that there exists some \( n \geq 1 \) with \( \varpi^{-n(i_1 + \ldots + i_d)}a_{i_1,\ldots,i_d} \to 0 \) as \( i_1 + \ldots + i_d \to -\infty \). We can therefore define the trace map

\[
\text{Tr}_{z_1,\ldots,z_d} : H^d_{\mathcal{C}}(\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y},\omega_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}}) \to H^0(\mathcal{Y},\mathcal{O}_\mathcal{Y})
\]

\[
\sum_{i_1,\ldots,i_d \leq 0} a_{i_1,\ldots,i_d} z_1^{i_1} \ldots z_d^{i_d} d \log z_1 \ldots d \log z_d \mapsto a_{0,\ldots,0}
\]

in the usual way. Of course, this globalises to define

\[
\text{Tr}_{z_1,\ldots,z_d} : \mathbb{R}^d f!\omega_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}} \to \mathcal{O}_\mathcal{Y}
\]

whenever the base \( \mathcal{Y} \) is an adic space; when \( \mathcal{Y} \) is a germ we simply pullback to \( \mathcal{Y} \) from its ambient adic space \( \mathcal{Y} \) using Corollary 5.4.8. The verification of the following is straightforward.

6.1.5. **Proposition.** Let \( \mathcal{Y} \) be a germ of an adic space.

(1) The trace map

\[
\text{Tr}_{z_1,\ldots,z_d} : \mathbb{R}^d f!\omega_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}} \to \mathcal{O}_\mathcal{Y}
\]

vanishes on the image of \( \mathbb{R}^d f!\Omega^d_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}} \), and induces an isomorphism

\[
\text{Tr}_{z_1,\ldots,z_d} : \mathbb{R}^d f!\Omega^*_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}} \cong \mathcal{O}_\mathcal{Y}.
\]

(2) The trace map is compatible with composition in the following sense: let \( (z_1,\ldots,z_d) \) be co-ordinates on \( \mathbb{A}^d_{\mathcal{Y}}, \) let \( 1 \leq e \leq d, \) and let \( h : \mathbb{A}^d_{\mathcal{Y}} \rightarrow \mathbb{A}^e_{\mathcal{Y}} \) be the projection \( (z_1,\ldots,z_d) \mapsto (z_1,\ldots,z_e). \) Let \( f : \mathbb{A}^d_{\mathcal{Y}} \rightarrow \mathcal{Y} \) and \( g : \mathbb{A}^e_{\mathcal{Y}} \rightarrow \mathcal{Y} \) be the natural projections. Then via the isomorphism

\[
\omega_{\mathbb{A}^d_{\mathcal{Y}}/\mathcal{Y}} \cong h^*\omega_{\mathbb{A}^e_{\mathcal{Y}}/\mathcal{Y}} \otimes \omega_{\mathbb{A}^{d-e}_{\mathcal{Y}}/\mathbb{A}^e_{\mathcal{Y}}},
\]

and the resulting identification

\[
\mathbb{R}^d f_! \left( h^*\omega_{\mathbb{A}^e_{\mathcal{Y}}/\mathcal{Y}} \otimes \omega_{\mathbb{A}^{d-e}_{\mathcal{Y}}/\mathbb{A}^e_{\mathcal{Y}}} \right) \cong \mathbb{R}^e g_! \left( \omega_{\mathbb{A}^e_{\mathcal{Y}}/\mathcal{Y}} \otimes \mathbb{R}^{d-e} h_! \omega_{\mathbb{A}^{d-e}_{\mathcal{Y}}/\mathbb{A}^e_{\mathcal{Y}}} \right),
\]

we have

\[
\text{Tr}_{z_1,\ldots,z_d} = \text{Tr}_{z_1,\ldots,z_e} \circ \mathbb{R}^e g_!(\text{id} \otimes \text{Tr}_{z_{e+1},\ldots,z_d}).
\]

We will see later on that \( \text{Tr}_{z_1,\ldots,z_d} \) is independent of the choice of co-ordinates \( z_1,\ldots,z_d; \) for now we record a (very!) special case of this.

6.1.6. **Lemma.** Suppose \( d = 1, \) let \( z \) be a co-ordinate on \( \mathbb{A}^1_{\mathcal{Y}}, \) and \( z' \) another co-ordinate of the form \( z' = z + a \) for some \( a \in \Gamma(\mathcal{Y},\mathcal{O}_\mathcal{Y}). \) Then \( \text{Tr}_z = \text{Tr}_{z'}. \)

**Proof.** We reduce to the case when \( \mathcal{Y} = \text{Spa}(R,R^+) \) is a Tate affinoid adic space, which follows easily from a direct calculation. \( \square \)

6.1.7. As a variant, we can replace \( \pi : \mathbb{A}^d_{\mathcal{Y}} \rightarrow \mathcal{Y} \) everywhere by the relative open unit disc \( \pi : \mathbb{D}^d(0;1^-) \rightarrow \mathcal{Y} \). The definition of this open unit disc, together with the construction of the trace map

\[
\text{Tr}_{z_1,\ldots,z_d} : \mathbb{R}^d \pi_*\omega_{\mathbb{D}^d(0;1^-)/\mathcal{Y}} \to \mathcal{O}_\mathcal{Y},
\]

and the analogues of Proposition 6.1.5 and Lemma 6.1.6, are straightforward analogues of those for the relative analytic affine space, and are left to the reader (who could also consult [Bey97]).
6.2. The relative unit polydisc. Next, let $Y$ be an arbitrary germ, and $j : \mathbb{D}_Y^d(0; 1) \to \mathbb{A}^d_Y$ the open immersion from the unit polydisc over $Y$ into the analytic affine space over $Y$. We therefore have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{D}_Y^d(0; 1) & \xrightarrow{j} & \mathbb{A}^d_Y^\text{an} \\
& \searrow f & \swarrow g \\
& Y, & 
\end{array}
$$

and we define

$$
\text{Tr}_{z_1, \ldots, z_d} : \mathbb{R}^df\omega_{\mathbb{D}_Y^d(0; 1)/Y} \to \mathcal{O}_Y
$$

to simply be the composite

$$
\begin{array}{ccc}
\mathbb{R}^df\omega_{\mathbb{D}_Y^d(0; 1)/Y} & \xrightarrow{j} & \mathbb{R}^dg\omega_{\mathbb{A}_Y^d} \\
& \xrightarrow{\text{Tr}_{z_1, \ldots, z_d}} & \mathbb{R}^d \omega_{\mathcal{O}_Y} \\
& \xrightarrow{\phi} & \mathcal{O}_Y.
\end{array}
$$

The analogue of Proposition 6.1.5(2) holds, and we can also show invariance of $\text{Tr}_{z_1, \ldots, z_d}$ under certain simple changes of co-ordinate.

6.2.1. Lemma. Suppose that $Y = \text{Spa}(R, R^+)$ is affinoid, and let let $z_1', \ldots, z_d'$ be a second set of co-ordinates on $\mathbb{D}_Y^d(0; 1)$ defined by

$$
z_1' = z_1, \ldots, z_e' = z_e, z_{e+1}' = z_{e+1} + w_{e+1}, \ldots, z_d' = z_d + w_d
$$

for $w_i \in R^+(z_1, \ldots, z_e)$. Then $\text{Tr}_{z_1, \ldots, z_d} = \text{Tr}_{z_1', \ldots, z_d'}$.

Proof. By induction we may assume that $e = d - 1$, and by compatibility of the trace map with composition we may assume that $d = 1$. In this case, the given change of co-ordinate extends to a change of co-ordinate on $\mathbb{A}_Y^1$, and the claim follows from Lemma 6.1.6 above.

6.2.2. Remark. The definition of $\text{Tr}_{z_1, \ldots, z_d}$ for a relative polydisc may seem overly complicated, and it is certainly natural to ask whether or not there is a more ‘direct’ construction. The problem lies in giving an explicit description of $H^1_c(X/Y, \omega^1_{X/Y})$ when $Y = \text{Spa}(R, R^+)$ is a Tate affinoid and $X = \mathbb{D}_Y(0; 1)$. When $Y = \text{Spa}(k, k^+)$ is an affinoid field of height one, this can be done concretely, we have

$$
H^1_c(X/Y, \omega^1_{X/Y}) = \frac{\mathcal{O}_X}{K(z)}dz
$$

where $\xi \in X$ is the Gauss point and $z$ is a co-ordinate on $X$. However, it is not clear to how to ‘relativise’ this description to work for a more general base.

6.3. Closed subspaces of polydiscs. The next case we wish to deal with is smooth morphisms which admit closed immersions into relative polydiscs.\footnote{Recall that closed immersions of adic spaces are those defined by coherent ideal sheaves on the target.} In order to do this we will need to show a form of coherent duality for regular closed immersions of adic spaces, which luckily follows quite quickly from the scheme theoretic case.

Recall that on a locally ringed space $(X, \mathcal{O}_X)$ a perfect complex of $\mathcal{O}_X$-modules is one that is locally quasi-isomorphic to a bounded complex of finite free $\mathcal{O}_X$-modules. Similarly, if $A$ is a ring then a perfect complex of $A$-modules is a complex quasi-isomorphic to a bounded complex of finite projective $A$-modules. These are viewed as full subcategories of $D(\mathcal{O}_X)$ and $D(A)$ respectively.

6.3.1. Lemma. Let $X = \text{Spa}(R, R^+)$ be a Tate affinoid adic space. Then

$$
\mathbf{R}\Gamma(X, -) : D(\mathcal{O}_X) \to D(R)
$$

induces a $t$-exact equivalence of triangulated categories

$$
D^+_{\text{coh}}(\mathcal{O}_X) \xrightarrow{\sim} D^+_{\text{coh}}(R)
$$

compatible with internal homs.
6.3.2. **Remark.** The $t$-exactness here refers to the obvious $t$-structures on either side.

**Proof.** The key point is that Tate affinoids satisfy Theorems A and B for coherent sheaves [KL15, Theorem 2.7.7]. That is, $H^q(X, -)$ is an equivalence of categories between coherent $\mathcal{O}_X$-modules and coherent (i.e. finitely generated) $\mathcal{R}$-modules, and $H^q(X, \mathcal{F}) = 0$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any $q > 0$.

Indeed, it follows straight away that

$$R \Gamma : D^+_\text{coh}(\mathcal{O}_X) \xrightarrow{\sim} D^+_\text{coh}(\mathcal{R})$$

is $t$-exact; to see that it is an equivalence, we have the left adjoint

$$- \otimes^L_R \mathcal{O}_X : D^+_\text{coh}(\mathcal{R}) \to D^+_\text{coh}(\mathcal{O}_X).$$

Essential surjectivity now follows from the fact that $\mathcal{O}_X$ is $\mathcal{R}$-flat, and full faithfulness from the fact that the adjunction map

$$\mathcal{O}_X \otimes^L_R R \Gamma(X, \mathcal{F}) \to \mathcal{F}$$

is an isomorphism, for any $\mathcal{F} \in D^+_{\text{coh}}(\mathcal{O}_X)$.

Compatibility with internal homs now follows from the fact that the left adjoint $\otimes^L_R \mathcal{O}_X$ is monoidal, together with the isomorphism of functors

$$\mathcal{O}_X \otimes^L_R R \Gamma(X, -) \to \text{id}$$

as above. 

6.3.3. **Lemma.** Let $f : X \to Y$ be a regular closed immersion of affinoid adic spaces, of codimension $c$, and let $n_{X/Y}$ be the determinant of the normal bundle. Then, for any perfect complex $\mathcal{F}$ of $\mathcal{O}_X$-modules, there is a canonical isomorphism

$$\text{Tr}_f : f_* R \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, n_{X/Y}) \xrightarrow{\sim} R \text{Hom}_{\mathcal{O}_Y}(f_* \mathcal{F}, \mathcal{O}_Y)[c]$$

in $D(\mathcal{O}_Y)$. This is compatible with composition, in the sense that if $g : Y \to Z$ is a regular closed immersion of codimension $d$, and $n_{Y/Z}$ (resp. $n_{X/Z}$) the determinant of its normal bundle (resp. the normal bundle of $X$ in $Z$), then via the canonical isomorphism $n_{X/Z} \cong n_{X/Y} \otimes_{\mathcal{O}_X} f^* n_{Y/Z}$ the diagram

$$\begin{array}{ccc}
(g \circ f)_* R \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, n_{X/Z}) & \xrightarrow{\text{Tr}_f} & g_* R \text{Hom}_{\mathcal{O}_Y}(f_* \mathcal{F}, \mathcal{O}_Y)[c] \\
\downarrow \text{Tr}_{gf} & & \downarrow \text{Tr}_g \\
R \text{Hom}_{\mathcal{O}_Z}((g \circ f)_* \mathcal{F}, \mathcal{O}_Z)[c + d]
\end{array}$$

commutes.

6.3.4. **Remark.** Note that pushforward along a regular closed immersion preserves perfect complexes, which can be seen, for example by considering the Koszul complex of a regular generating sequence of the corresponding ideal sheaf.

**Proof.** By appealing (several times) to Lemma 6.3.1, we may replace our statement with its direct analogue for bounded below, coherent complexes on affine Noetherian schemes. This translation is then part of scheme theoretic coherent duality [Har66]. Indeed, if we let $f : X \to Y$ be a regular closed immersion of affine Noetherian schemes, of codimension $c$, then

$$f^* := R \text{Hom}_{\mathcal{O}_Y}(f_* \mathcal{O}_X, -) : D^+_{\text{coh}}(Y) \to D^+_{\text{coh}}(X)$$

is a right adjoint to $f_*$ by [Har66, Chapter III, Theorem 6.7], the first claim therefore reduces to constructing a natural isomorphism

$$\chi_f : f^! \mathcal{O}_Y \cong n_{X/Y}[-c].$$
Given this, and given \cite[Chapter III, Proposition 6.2]{Har66}, the second claim then boils down to showing that if \( X \xrightarrow{j} Y \xrightarrow{g} Z \) is a pair of regular closed immersions between affine Noetherian schemes, then the diagram

\[
\begin{array}{ccc}
(g \circ f)^* \mathcal{O}_Z & \xrightarrow{X^*f} & \mathcal{n}_{X/Z}[-c - d] \\
\downarrow \text{canonical} & & \downarrow \text{canonical} \\
f^*g^* \mathcal{O}_Z & \xrightarrow{X^*} & f^*\mathcal{n}_{Y/Z}[-d] \otimes_{\mathcal{O}_X} \mathcal{n}_{X/Y}[-c - d]
\end{array}
\]

commutes. Since the first claim in particular implies that the relative dualising complex is concentrated in a single degree, they may be jointly checked Zariski locally. Thus we may assume, in the first case, that \( X \) is cut out by a global regular sequence in \( Y \), and in the second case, that \( X \) is a pair of regular closed immersions between affine Noetherian schemes, and hence by applying \( R^df \) an isomorphism

\[
R^df\omega_{X/Y} \xrightarrow{\sim} R^N g_! R\text{Hom}_{\mathcal{O}_{Y^N(0,1)}}(u_! \mathcal{O}_X, \omega_{D_{Y^N(0,1)}^N}/Y)[N - d],
\]

and finally composing with \( \text{Tr}_{z_1, \ldots, z_N} \) for a choice of co-ordinates on \( \mathbb{D}_Y^n(0,1) \) gives a trace map

\[
\text{Tr}_{X/\mathcal{O}_Y} : R^d f\omega_{X/Y} \rightarrow \mathcal{O}_Y.
\]

6.3.5. **Proposition.** Suppose that \( Y \) is an adic space, and \( f : X \rightarrow Y \) is a smooth morphism factoring through a closed immersion into a unit polydisc over \( Y \).

1. The induced map \( \text{Tr}_{X/\mathcal{O}_Y} : R^d f\omega_{X/Y} \rightarrow \mathcal{O}_Y \) does not depend on the choice of embedding \( u : X \hookrightarrow \mathbb{D}_Y^n(0,1) \) over \( Y \).

2. If \( U \xrightarrow{j} X \) is an open subset which also admits a closed embedding into a unit polydisc over \( Y \), and we let \( fu \) denote the restriction of \( f \) to \( U \), then the diagram

\[
\begin{array}{ccc}
R^d f_U \omega_{U/Y} & \xrightarrow{j^*} & R^d f\omega_{X/Y} \\
\downarrow \text{Tr}_{U/X/Y} & & \downarrow \text{Tr}_{X/\mathcal{O}_Y} \\
\mathcal{O}_Y & & \mathcal{O}_Y
\end{array}
\]

commutes.

3. The trace map vanishes on the image of

\[
R^d f_! \Omega_{X/Y}^{d-1} \rightarrow R^d f\omega_{X/Y},
\]

and descends to a map

\[
\text{Tr}_{X/\mathcal{O}_Y} : R^{2d} f_! \Omega_{X/Y}^{d-1} \rightarrow \mathcal{O}_Y.
\]
Proof. Given Lemma 6.2.1, the first claim is proved verbatim as in the case of closed subspace of analytic affine space over a height one affinoid field treated in [vdP92, §3]. The second claim is local on $Y$, so we may assume it to be affinoid. Hence both $X$ and $U$ are also affinoid. Choose a closed embedding

$$u : X \hookrightarrow \mathbb{D}^N_Y(0;1)$$

and an open affinoid $V \hookrightarrow \mathbb{D}^N_Y(0;1)$ containing the image of $U$ such that the diagram

$$\begin{array}{ccc}
U & \xrightarrow{j} & X \\
\downarrow{u'} & & \downarrow{u} \\
V & \xrightarrow{j'} & \mathbb{D}^N_Y(0;1)
\end{array}$$

is Cartesian. Let $g : \mathbb{D}^N_Y(0;1) \to Y$ be the projection, and $f_U : U \to Y$, $g_V : V \to Y$ the restrictions. We therefore have a commutative diagram

$$\begin{array}{ccc}
j_!u'_*\omega_{U/Y} & \xrightarrow{Tr_u} & u_*\omega_{X/Y} \\
j_!R\text{Hom}_{O_U}(u'_*O_U, \omega_{U/Y})[N-d] & \xrightarrow{Tr_u} & R\text{Hom}_{O_{\mathbb{D}^N_Y(0;1)}}(u_*O_X, \omega_{\mathbb{D}^N_Y(0;1)/Y})[N-d].
\end{array}$$

where $Tr_u$ and $Tr_{u'}$ are the maps coming from Lemma 6.3.3, and the horizontal maps are the maps coming from adjunction. We next claim that the diagram

$$\begin{array}{ccc}
R^df_U!\omega_{U/Y} & \xrightarrow{Tr_{u'}} & R^Ng_V!R\text{Hom}_{O_V}(u'_*O_U, \omega_{U/Y}) \\
\downarrow{Tr_{u'}} & & \downarrow{Tr_{u'}^{-1}} \\
R^Ng_V!\omega_{V/Y} & \xrightarrow{j'} & Tr_{V/Y}\omega_{V/Y}
\end{array}$$

commutes. Indeed, if we choose a closed embedding $v : V \hookrightarrow \mathbb{D}^{N+M}_Y(0;1)$, and let $h : \mathbb{D}^{N+M}_Y(0;1) \to Y$ denote the projection, then the diagrams

$$\begin{array}{ccc}
R^df_U!\omega_{U/Y} & \xrightarrow{Tr_{u'}^{-1}} & R^{N+M}h_*R\text{Hom}_{O_{\mathbb{D}^{N+M}_Y(0;1)}}((v \circ u')_*O_U, \omega_{\mathbb{D}^{N+M}_Y(0;1)/Y}) \\
\downarrow{Tr_{u'}} & & \downarrow{Tr_{u'}^{-1}} \\
R^Ng_V!\omega_{V/Y} & \xrightarrow{j'} & Tr_{V/Y}\omega_{V/Y}
\end{array}$$

and

$$\begin{array}{ccc}
R^Ng_V!\omega_{V/Y} & \xrightarrow{Tr_{u'}^{-1}} & R^{N+M}h_*R\text{Hom}_{O_{\mathbb{D}^{N+M}_Y(0;1)}}(v_*O_V, \omega_{\mathbb{D}^{N+M}_Y(0;1)/Y}) \\
\downarrow{Tr_{u'}^{-1}} & & \downarrow{Tr_{u'}^{-1}} \\
R^Ng_V!\omega_{V/Y} & \xrightarrow{j'} & Tr_{V/Y}\omega_{V/Y}
\end{array}$$

commute essentially by definition; hence the fact that $Tr_{U/Y} = Tr_{V/Y} \circ Tr_{u'}$ follows from Lemma 6.3.3.

Now, let us suppose that we know that the diagram

$$\begin{array}{ccc}
R^Ng_V!\omega_{V/Y} & \xrightarrow{j'} & R^Ng_V!\omega_{\mathbb{D}^N_Y(0;1)/Y} \\
\downarrow{Tr_{V/Y}} & & \downarrow{Tr_{V/Y}} \\
O_Y & & O_Y
\end{array}$$

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commutes. Then we have
\[
\text{Tr}_{U/Y} = \text{Tr}_{V/Y} \circ \text{Tr}_{u'} = \text{Tr}_{D^N_Y(0;1)/Y} \circ j' \circ \text{Tr}_{u'} = \text{Tr}_{D^N_Y(0;1)/Y} \circ \text{Tr}_u \circ j_i = \text{Tr}_{X/Y} \circ j_i
\]
as maps \( \mathbf{R}^df_{U!}\omega_{U/Y} \to \mathcal{O}_Y \). We may therefore replace \( X \) by \( D^N_Y(0;1) \) and \( U \) by \( V \), in other words we can assume that \( X = D^N_Y(0;1) \). Now, choose a closed embedding
\[
v : U \hookrightarrow D^{N+M}_Y(0;1)
\]
over \( D^N_Y(0;1) \); by compatibility of \( \text{Tr}_{z_1, \ldots, z_{n+1}} \) with composition (Proposition 6.1.5(2)), we may replace \( Y \) by \( D^n_Y(0;1) \), in other words we may assume that \( N = 0 \). Thus, we have an open affinoid \( U \subset Y \), a factorization \( U \xrightarrow{v} D^M_Y(0;1) \xrightarrow{\delta} Y \) with \( v \) a closed immersion, and what we need to show is that the composite map
\[
\delta^*\mathcal{O}_U = g_\ast v_\ast \mathcal{O}_U = g_\ast \mathbf{Ext}_{D_Y(0;1)}^M(v_\ast \mathcal{O}_U, \omega_{D_Y(0;1)/Y}) \to \mathbf{R}^Mg_\ast \omega_{D^M_Y(0;1)/Y} \xrightarrow{\text{Tr}} \mathcal{O}_Y
\]
coincides with the natural map \( \delta^*\mathcal{O}_U \to \mathcal{O}_Y \). To prove this claim we may restrict to \( U \), thus we may assume that \( U = Y \), \( j = \text{id} \), and \( v \) is a section of \( D^M_Y(0;1) \to Y \). By making a suitable change of co-ordinate on \( D^M_Y(0;1) \) (which doesn’t change the trace map by Lemma 6.2.1), we may in fact assume that \( v \) is the zero section. Thus what we need to verify is that, in this case, the map
\[
\mathcal{O}_Y = g_\ast v_\ast \mathbf{Ext}_{D_Y(0;1)}^M(v_\ast \mathcal{O}_U, \omega_{D_Y(0;1)/Y}) \to \mathbf{R}^Mg_\ast \omega_{D^M_Y(0;1)/Y} \xrightarrow{\text{Tr}} \mathcal{O}_Y
\]
is the identity. Thus just follows from a direct calculation.

Again, the third claim is local, so we may assume that \( Y \) is affinoid, and that \( X \) admits an étale map to \( D^n_Y(0;1) \). Again, via compatibility of the trace map with composition, we reduce to the case \( X = D^M_Y(0;1) \), and therefore, by the definition of \( \text{Tr}_{z_1, \ldots, z_d} \), to the corresponding question for \( \mathbb{A}^d_{\mathcal{S}^\text{an}} \) covered in Proposition 6.1.5.

6.3.6. Corollary. If \( X = D^n_Y(0;1) \) then the trace map
\[
\text{Tr}_{X/Y} = \text{Tr}_{z_1, \ldots, z_d} : \mathbf{R}^df_{X/Y} \to \mathcal{O}_Y
\]
defined in §6.2 above doesn’t depend on the choice of co-ordinates \( z_1, \ldots, z_d \).

Note that we do not (quite) yet have the same result for the trace map on \( \mathbb{A}^d_{\mathcal{S}^\text{an}} \) or \( D^n_Y(0;1) \) (although we could easily prove this along exactly the same lines as for \( D^n_Y(0;1) \)).

6.4. Smooth morphisms of adic spaces. As a penultimate case, we now construct a trace morphism for any smooth morphism \( f : X \to Y \) of adic spaces. If \( Y \) is affinoid, then we can find a cover of \( X \) by opens \( U_i \) admitting closed embeddings \( U_i \hookrightarrow D^M_Y(0;1) \) over \( Y \). Moreover, we can then cover each \( U_i \) by opens \( U_{ij} \) admitting closed embeddings \( U_{ij} \hookrightarrow D^M_Y(0;1) \), by \( Y \). Using the spectral sequence from Corollary 5.9.1 together with Proposition 6.3.5, the trace maps
\[
\text{Tr}_{U_i/Y} : \mathbf{R}^df_{U_i/Y} \to \mathcal{O}_Y
\]
factor uniquely through a map
\[
\text{Tr}_{X/Y} : \mathbf{R}^df_{X/Y} \to \mathcal{O}_Y
\]
which can be checked not to depend on the choice of the \( U_i \), or of the \( U_{ij} \). Since the above constructed map does not depend on the choice of open cover of \( X \), it glues over an open affinoid cover of \( Y \). Also, by the corresponding fact for the \( U_i \), this morphism vanishes on the image of
\[
\mathbf{R}^df_{i_0} \omega_{X/Y}^{d-1} \to \mathbf{R}^df_{X/Y},
\]
and descends to a map
\[
\text{Tr}_{X/Y} : \mathbf{R}^{2d}f_{X/Y} \to \mathcal{O}_Y.
\]
When $X = \mathbb{A}_Y^d$ or $X = \mathbb{D}_Y^d(0;1^-)$, we have now defined two trace maps, so we’d better check that these coincide.

6.4.1. Lemma. Suppose that $X = \mathbb{A}_Y^d$ or $X = \mathbb{D}_Y^d(0;1^-)$ with co-ordinates $z_1, \ldots, z_d$. Then $\text{Tr}_{X/Y} = \text{Tr}_{z_1,\ldots,z_d}$.

Proof. First of all, we can reduce to considering $X = \mathbb{A}_Y^d$. Indeed, if we denote by $f$ and $g$ the projections from $\mathbb{D}_Y^d(0;1^-)$, and $\mathbb{A}_Y^d$ to $Y$ respectively, then it follows from the construction of $\text{Tr}_{-/Y}$ that the diagram

\[
\begin{array}{ccc}
\mathbb{R}^d f_!\omega_{\mathbb{D}_Y^d(0,1^-)/Y} & \xrightarrow{\text{Tr}_{\mathbb{D}_Y^d(0,1^-)/Y}} & \mathbb{R}^d g_!\omega_{\mathbb{A}_Y^d/Y} \\
\downarrow & & \downarrow \text{Tr}_{\mathbb{A}_Y^d/Y} \\
\mathcal{O}_Y & \xrightarrow{\text{Tr}_{z_1,\ldots,z_d}} & \mathcal{O}_Y
\end{array}
\]

commutes. On the other hand, it follows directly from the definitions that the diagram

\[
\begin{array}{ccc}
\mathbb{R}^d f_!\omega_{\mathbb{D}_Y^d(0,1^-)/Y} & \xrightarrow{\text{Tr}_{z_1,\ldots,z_d}} & \mathbb{R}^d g_!\omega_{\mathbb{A}_Y^d/Y} \\
\downarrow & & \downarrow \text{Tr}_{z_1,\ldots,z_d} \\
\mathcal{O}_Y & \xrightarrow{\text{Tr}_{z_1,\ldots,z_d}} & \mathcal{O}_Y
\end{array}
\]

commutes; thus the case $X = \mathbb{D}_Y^d(0;1^-)$ follows from the case $X = \mathbb{A}_Y^d$.

To treat the case $\mathbb{A}_Y^d$ we may argue locally, and suppose that $Y = \text{Spa}(R, R^+)$ is a Tate affinoid with $\mathfrak{o} \in R^{\omega} \cap R^*$. If we let $B_0^d \subset \mathbb{A}_Y^d$ denote the “discs of radius $|\mathfrak{o}|^{-n}$” as considered in §6.1, then $H^c_c(\mathbb{A}_Y^d/Y, \omega_{\mathbb{A}_Y^d/Y})$ is generated by the images of all of the $H^c_c(B_0^d/Y, \omega_{B_0^d/Y})$, it therefore suffices to show that the diagram

\[
\begin{array}{ccc}
H^d_c(B_n^d/Y, \omega_{B_0^d/Y}) & \xrightarrow{\alpha_n} & H^d_c(\mathbb{A}_Y^d/Y, \omega_{\mathbb{A}_Y^d/Y}) \\
\downarrow \text{Tr}_{B_0^d/Y} & & \downarrow \text{Tr}_{z_1,\ldots,z_d} \\
H^0(Y, \mathcal{O}_Y) & \xrightarrow{\text{Tr}_{z_1,\ldots,z_d}} & H^0(Y, \mathcal{O}_Y)
\end{array}
\]

commutes, for all $n$. The isomorphism $\alpha_n : \mathbb{A}_Y^d \rightarrow \mathbb{A}_Y^d$ defined by

\[(z_1, \ldots, z_d) \mapsto (\pi^n z_1, \ldots, \pi^n z_d)\]

maps $B_0^d$ isomorphically onto $\mathbb{D}_Y^d(0;1)$, by the definition of $\text{Tr}_{\mathbb{D}_Y^d(0;1)}$ it therefore suffices to show that the diagram

\[
\begin{array}{ccc}
H^d_c(\mathbb{A}_Y^d/Y) & \xrightarrow{\alpha_n} & H^d_c(\mathbb{A}_Y^d/Y) \\
\downarrow \text{Tr}_{z_1,\ldots,z_d} & & \downarrow \text{Tr}_{z_1,\ldots,z_d} \\
H^0(Y, \mathcal{O}_Y) & \xrightarrow{\text{Tr}_{z_1,\ldots,z_d}} & H^0(Y, \mathcal{O}_Y)
\end{array}
\]

commutes. Again, this now follows from a direct computation. \qed

6.4.2. Corollary. Suppose that $X = \mathbb{A}_Y^d$ or $X = \mathbb{D}_Y^d(0;1^-)$. Then $\text{Tr}_{z_1,\ldots,z_d}$ does not depend on the choice of co-ordinates $z_1, \ldots, z_d$. 33
6.5. The general case. Finally, let us consider a general smooth, separated, taut morphism \( f : X \to Y \) of germs of adic spaces, of relative dimensions \( d \). Since \( f \) is smooth, there exists a representative \( f : (X, X) \to (Y, Y) \) of \( f \) such that \( f : X \to Y \) is smooth, and \( X = f^{-1}(Y) \). We may, locally on \( Y \), cover \( X \) by open subspaces of the form \( U \cap X \), where \( U \subset X \) is taut and separated over \( Y \). Hence arguing as in §6.4 above we may reduce to the case when \( X \) itself is in taut and separated over \( Y \). In this case, by Corollary 5.4.8, there is an isomorphism
\[
R^d f_! \omega_{X/Y} \cong (R^d g_! \omega_{X/Y})|_Y,
\]
and hence we can simply restrict the trace morphism for \( X/Y \) to obtain the trace morphism for \( X/Y \). Again, when \( X = \mathbb{A}^d_{Y} \), or \( X = \mathbb{D}^d_{Y}(0; 1^-) \) this agrees with the more explicit construction given in §6.1. The following properties of \( \text{Tr}_{X/Y} \) are also easily verified.

6.5.1. Proposition. Let \( f : X \to Y \) be a smooth, separated, taut morphism of germs of adic spaces, of relative dimension \( d \).

1. \( \text{Tr}_{X/Y} \) vanishes on the image of \( R^d f_! \Omega_{X/Y}^{d-1} \), and descends to a map
\[
\text{Tr}_{X/Y} : R^d f_! \Omega_{X/Y}^d \to \mathcal{O}_Y.
\]
2. If \( g : Y \to Z \) is another smooth morphism of germs of adic spaces, of relative dimension \( d' \), then
\[
\text{Tr}_{X/Z} = \text{Tr}_{Y/Z} \circ R^e g_* \left( \text{id} \otimes \text{Tr}_{X/Y} \right) : R^{d+e} (g \circ f)_! \omega_{X/Z} = R^e g_* \left( \omega_{Y/Z} \otimes R^d \omega_{X/Y} \right) \to \mathcal{O}_Z.
\]

6.6. Duality. We can now construct the duality morphism for smooth maps between germs of adic spaces. First of all, let \( f : X \to Y \) be a morphism of (finite dimensional) germs which is separated, taut, and locally of \( + \) weakly finite type (but not necessarily smooth, yet). If \( \mathcal{I}, \mathcal{J} \) are \( \mathcal{O}_X \)-modules then the natural map
\[
H^0(X, \mathcal{I}) \times H^0(X, \mathcal{J}) \to H^0(X, \mathcal{I} \otimes \mathcal{J})
\]
induces
\[
H^0(X, \mathcal{I}) \times H^0(X/Y, \mathcal{J}) \to H^0(X/Y, \mathcal{I} \otimes \mathcal{J}),
\]
and sheafifying this gives a pairing
\[
f_* \mathcal{I} \times f_! \mathcal{J} \to f_!(\mathcal{I} \otimes \mathcal{J}).
\]
By taking suitable resolutions on a compactification of \( f \), we deduce that if \( \mathcal{E} \) and \( \mathcal{F} \) are bounded complexes of \( \mathcal{O}_X \)-modules, there is a natural pairing
\[
Rf_* \mathcal{E} \times Rf_! \mathcal{F} \to Rf_!(\mathcal{E} \otimes^L \mathcal{F})
\]
in \( D^{-}(\mathcal{O}_Y) \). In particular, if \( \mathcal{G} \) is a third bounded complex, then any pairing
\[
\mathcal{E} \times \mathcal{G} \to \mathcal{G}
\]
induces a corresponding pairing
\[
Rf_* \mathcal{E} \times Rf_! \mathcal{G} \to Rf_! \mathcal{G}
\]
in cohomology.

If we now assume, moreover, that \( f \) is smooth, and \( \mathcal{E} \) is a perfect complex on \( X \), then setting \( \mathcal{E}^\vee := R\text{Hom}(\mathcal{E}, \mathcal{O}_X) \) we have a natural evaluation pairing
\[
\mathcal{E} \times \mathcal{E}^\vee \otimes \omega_{X/Y} \to \omega_{X/Y}.
\]
Together with the trace map \( \text{Tr}_{X/Y} \) this induces a pairing
\[
Rf_* \mathcal{E} \times Rf_! (\mathcal{E}^\vee \otimes \omega_{X/Y}) \to \mathcal{O}_Y[-d],
\]
there is of course a similar pairing
\[
Rf_* \mathcal{E} \times Rf_! (\mathcal{E}^\vee \otimes \omega_{X/Y}) \to \mathcal{O}_Y[-d].
\]

6.6.1. Question. Can these complexes be endowed with natural topologies in such a way that these pairings becomes perfect?
References

[AL] T. Abe and C. Lazda, A global comparison between overconvergent isocrystals and arithmetic D-modules, in preparation.

[Ber90] V. G. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields, Mathematical Survey and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.

[Bey97] P. Beyer, On Serre-duality for coherent sheaves on rigid-analytic spaces, Manuscripta Math. 93 (1997), no. 2, 219–245. https://doi.org/10.1007/BF02677468.

[Chi90] B. Chiarellotto, Duality in rigid analysis, p-adic Analysis (F. Baldassarri, S. Bosch, and B. Dwork, eds.), Lecture Notes in Mathematics, vol. 1454, Springer, Berlin, Heidelberg, 1990. https://doi.org/10.1007/BFb0091137.

[dJ95] A. J. de Jong, Étale fundamental groups of non-Archimedean analytic spaces, Compos. Math. 97 (1995), no. 1-2, 89–118. http://www.numdam.org/item?id=CM_1995__97_1-2_89_0.

[FK18] K. Fujiwara and F. Kato, Foundations of Rigid Geometry. I, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2018. http://www.ems-ph.org/doi/10.4171/135.

[Har66] R. Hartshorne, Residues and duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966. https://doi.org/10.1007/BFb0084082.

[Hub94] R. Huber, A generalization of formal schemes and rigid analytic varieties, Math. Z. 217 (1994), no. 4, 513–551.

[Hub96] ________, Étale cohomology of rigid analytic varieties and adic spaces, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996. http://dx.doi.org/10.1007/978-3-663-09991-8.

[Kie67] R. Kiehl, Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie, Invent. Math. 2 (1967), 256–273. https://doi.org/10.1007/BF01425404.

[KL15] K. S. Kedlaya and R. Liu, Relative p-adic Hodge theory: foundations, Astérisque (2015), no. 371, 239, https://smf.emath.fr/publications/theorie-de-hodge-p-adique-dans-le-cas-relatif-fondations.

[Sche92] C. Scheiderer, Quasi-augmented simplicial spaces, with an application to cohomological dimension, J. Pure Appl. Algebra 81 (1992), no. 3, 293–311. https://doi.org/10.1016/0022-4049(92)90062-K.

[SGA73] Théorie des topos et cohomologie étale des schémas. Tome 3, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat, https://doi.org/10.1007/BFb0070714.

[Stacks] The Stacks project authors, The Stacks project, https://stacks.math.columbia.edu, 2019.

[vdP92] M. van der Put, Serre duality for rigid analytic spaces, Indag. Math. (N.S.) 3 (1992), no. 2, 219–235. https://doi.org/10.1016/0019-3577(92)90011-9.

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