Gauss-Bonnet theorems in the affine group and the group of rigid motions of the Minkowski plane

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Abstract In this paper, we compute sub-Riemannian limits of Gaussian curvature for a Euclidean $C^2$-smooth surface in the affine group and the group of rigid motions of the Minkowski plane away from characteristic points and signed geodesic curvature for Euclidean $C^2$-smooth curves on surfaces. We get Gauss-Bonnet theorems in the affine group and the group of rigid motions of the Minkowski plane.

Keywords affine group, group of rigid motions of the Minkowski plane, Gauss-Bonnet theorem, sub-Riemannian limit

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1 Introduction

In [4], Gaussian curvature for non-horizontal surfaces in sub-Riemannian Heisenberg space $\mathbb{H}^1$ was defined and a Gauss-Bonnet theorem was proved. The definition was analogous to Gaussian curvature of surfaces in $\mathbb{R}^3$ with a particular vector which is normal to the surface and Hausdorff measure of area. The image of the Gauss map was in the cylinder of radius one. In [1], Balogh et al. used a Riemannian approximation scheme to define a notion of intrinsic Gaussian curvature for a Euclidean $C^2$-smooth surface in the Heisenberg group $\mathbb{H}^1$ away from characteristic points, and a notion of intrinsic signed geodesic curvature for Euclidean $C^2$-smooth curves on surfaces. These results were then used to prove a Heisenberg version of the Gauss-Bonnet theorem. In [5], Veloso verified that Gaussian curvature of surfaces and normal curvature of curves in surfaces introduced by Diniz and Veloso [4] and by Balogh et al. [1] to prove Gauss-Bonnet theorems in Heisenberg space $\mathbb{H}^1$ were unequal and he applied the same formalism of [4] to get the curvature of [1]. With the obtained formulas, it is possible to prove the Gauss-Bonnet theorem in [1] as a straightforward application of the Stokes theorem.

The Riemannian approximation scheme used in [1] can in general depend on the choice of the complement to the horizontal distribution. In the context of $\mathbb{H}^1$ the choice which they have adopted is rather natural. The existence of the limit defining the intrinsic curvature of a surface depends crucially on the cancellation of certain divergent quantities in the limit. Such cancellation stems from the specific

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choice of the adapted frame bundle on the surface, and symmetries of the underlying left-invariant group structure on the Heisenberg group. In [1], Balogh et al. proposed an interesting question to understand to what extent similar phenomena hold in other sub-Riemannian geometric structures. In this paper, we solve this problem for the affine group and the group of rigid motions of the Minkowski plane. In the case of the affine group, the cancellation of certain divergent quantities in the limit does not happen and the limit of the Riemannian Gaussian curvature is divergent. In the case of the group of rigid motions of the Minkowski plane, similar to the Heisenberg group, the cancellation of certain divergent quantities in the limit happens and the limit of the Riemannian Gaussian curvature exists. We also get Gauss-Bonnet theorems in the affine group and the group of rigid motions of the Minkowski plane.

The rest of this paper is organized as follows. In Section 2, we compute the sub-Riemannian limit of curvature of curves in the affine group. In Section 3, we compute sub-Riemannian limits of geodesic curvature of curves on surfaces and the Riemannian Gaussian curvature of surfaces in the affine group. In Section 4, we prove the Gauss-Bonnet theorem in the affine group. In Section 5, we compute the sub-Riemannian limit of curvature of curves in the affine group and the group of rigid motions of the Minkowski plane. In Section 6, we compute sub-Riemannian limits of geodesic curvature of curves on surfaces and the Riemannian Gaussian curvature of surfaces in the group of rigid motions of the Minkowski plane and a Gauss-Bonnet theorem in the group of rigid motions of the Minkowski plane is also obtained.

2 The sub-Riemannian limit of curvature of curves in the affine group

Firstly we introduce some notations on the affine group. Let $G$ be the affine group $(0, \infty) \times \mathbb{R}^2$, where the non-commutative group law is given by

$$(a, b, c) \star (x, y, z) = (ax, ay + b, z + c).$$

Then $(1, 0, 0)$ is a unit element. Let

$$X_1 = x_1 \partial_{x_1}, \quad X_2 = x_1 \partial_{x_2} + \partial_{x_3}, \quad X_3 = x_1 \partial_{x_2},$$

(2.1)

Then

$$\partial_{x_1} = \frac{1}{x_1} X_1, \quad \partial_{x_2} = \frac{1}{x_1} X_3, \quad \partial_{x_3} = X_2 - X_3$$

(2.2)

and

$$\text{span}\{X_1, X_2, X_3\} = TG.$$

Let $H = \text{span}\{X_1, X_2\}$ be the horizontal distribution on $G$. Let

$$\omega_1 = \frac{1}{x_1} dx_1, \quad \omega_2 = dx_3, \quad \omega = \frac{1}{x_1} dx_2 - dx_3.$$  

Then $H = \text{Ker}\omega$. For the constant $L > 0$, let $g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L \omega \otimes \omega$, and $g = g_1$ be the Riemannian metric on $G$. Then $X_1, X_2, \bar{X}_3 := L^{-\frac{1}{2}} X_3$ are the orthonormal basis on $TG$ with respect to $g_L$. We have

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = X_3.$$  

(2.3)

Let $\nabla^L$ be the Levi-Civita connection on $G$ with respect to $g_L$. By the Koszul formula, we have

$$2\langle \nabla^L_{X_i} X_j, X_k \rangle_L = \langle [X_i, X_j], X_k \rangle_L - \langle [X_j, X_k], X_i \rangle_L + \langle [X_k, X_i], X_j \rangle_L,$$  

(2.4)

where $i, j, k = 1, 2, 3$. By (2.3) and (2.4), we have the following lemma.

Lemma 2.1. Let $G$ be the affine group. Then

\begin{align*}
\nabla^L_{X_1} X_j &= \begin{cases} 
0, & 1 \leq j \leq 2, \\
\frac{1}{2} X_3, & j = 1, \\
-\frac{1}{2} X_3, & j = 2 
\end{cases}, \\
\nabla^L_{X_1} X_3 &= \frac{L}{2} X_2, \\
\nabla^L_{X_2} X_3 &= \frac{L}{2} X_1, \\
\nabla^L_{X_3} X_1 &= -\frac{L}{2} X_2 - X_3, \\
\nabla^L_{X_3} X_2 &= \frac{3}{2} X_1, \\
\nabla^L_{X_3} X_3 &= LX_1.
\end{align*}  

(2.5)
Definition 2.2. Let $\gamma : [a, b] \to (G, g_L)$ be a Euclidean $C^1$-smooth curve. We say that $\gamma$ is regular if $\dot{\gamma} \neq 0$ for every $t \in [a, b]$. Moreover, we say that $\gamma(t)$ is a horizontal point of $\gamma$ if

$$\omega(\dot{\gamma}(t)) = \frac{\dot{\gamma}_2(t)}{\gamma_1(t)} - \dot{\gamma}_3(t) = 0.$$  

Definition 2.3. Let $\gamma : [a, b] \to (G, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(G, g_L)$. The curvature $k^L_\gamma$ of $\gamma$ at $\gamma(t)$ is defined as

$$k^L_\gamma := \sqrt{\frac{\|\nabla^L_{\dot{\gamma}}\|^2_{\hat{L}}}{\|\dot{\gamma}\|^2_{\hat{L}}}} \frac{(\nabla^L_{\dot{\gamma}} \dot{\gamma})^2_{\hat{L}}}{\|\dot{\gamma}\|^2_{\hat{L}}}. \quad (2.6)$$

Lemma 2.4. Let $\gamma : [a, b] \to (G, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(G, g_L)$. Then

$$k^L_\gamma = \left\{ \left( \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\dot{\gamma}_1^2} + L \omega(\dot{\gamma}(t)) + \frac{\gamma_1 - L \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{\gamma_1}^2}{\gamma_1^2} \cdot \left( \frac{\dot{\gamma}_1^2}{\gamma_1^2} + \gamma_3^2 + L \omega(\dot{\gamma}(t))^2 \right) \right)^2 \cdot \left( \frac{\dot{\gamma}_1}{\gamma_1} \right)^2 + \gamma_3^2 + L \omega(\dot{\gamma}(t))^2 \right)^{-3} \frac{1}{2} \right\}. \quad (2.7)$$

In particular, if $\gamma(t)$ is a horizontal point of $\gamma$,

$$k^L_\gamma = \left\{ \left( \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\dot{\gamma}_1^2} + \gamma_3^2 \right)^2 + \gamma_3 \gamma_3 \right\} \cdot \left( \frac{\dot{\gamma}_1^2}{\gamma_1^2} + \gamma_3^2 \right)^{-3} \frac{1}{2}. \quad (2.8)$$

Proof. By (2.2), we have

$$\dot{\gamma}(t) = \frac{\dot{\gamma}_1}{\gamma_1} X_1 + \frac{\dot{\gamma}_3}{\gamma_1} X_2 + \omega(\dot{\gamma}(t)) X_3. \quad (2.9)$$

By Lemma 2.1 and (2.9), we have

$$\nabla^L_{\dot{\gamma}} X_1 = -\frac{L}{2} \left( \frac{\dot{\gamma}_2}{\gamma_1} - \frac{\dot{\gamma}_3}{\gamma_1} \right) X_2 + \left( \frac{\dot{\gamma}_1}{2} - \frac{\dot{\gamma}_2}{\gamma_1} \right) X_3, \quad (2.10)$$

$$\nabla^L_{\dot{\gamma}} X_2 = \frac{L}{2} \left( \frac{\dot{\gamma}_2}{\gamma_1} - \frac{\dot{\gamma}_3}{\gamma_1} \right) X_1 + \frac{\dot{\gamma}_1}{2} X_3,$nabla^L_{\dot{\gamma}} X_3 = -\frac{\dot{\gamma}_2}{2} + \frac{\dot{\gamma}_3}{\gamma_1} L X_1 - \frac{L \dot{\gamma}_1}{2 \gamma_1} X_2.$$

By (2.9) and (2.10), we have

$$\nabla^L_{\dot{\gamma}} \dot{\gamma} = \left[ \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\dot{\gamma}_1^2} + L \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_2}{\gamma_1} \right] X_1 + \left[ \frac{\dot{\gamma}_3 - L \omega(\dot{\gamma}(t)) \frac{\dot{\gamma}_1}{\gamma_1}^2}{\gamma_1^2} \right] X_2 + \left[ -\frac{\dot{\gamma}_2}{2} + \frac{\dot{\gamma}_3}{\gamma_1} L X_1 - \frac{L \dot{\gamma}_1}{2 \gamma_1} X_2. \right. \quad (2.11)$$

By (2.6), (2.9) and (2.11), we get Lemma 2.4. \qed
Definition 2.5. Let $\gamma : [a, b] \to (G, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(G, g_L)$. We define the intrinsic curvature $k^\gamma_\infty$ of $\gamma$ at $\gamma(t)$ to be

$$k^\gamma_\infty := \lim_{L \to +\infty} k^L_\gamma,$$

if the limit exists.

We introduce the following notation: for continuous functions $f_1, f_2 : (0, +\infty) \to \mathbb{R}$,

$$f_1(L) \sim f_2(L) \quad \text{as} \quad L \to +\infty \iff \lim_{L \to +\infty} \frac{f_1(L)}{f_2(L)} = 1. \quad (2.12)$$

Lemma 2.6. Let $\gamma : [a, b] \to (G, g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(G, g_L)$. Then

$$k^\gamma_\infty = \frac{\sqrt{\gamma_1^2 + \gamma_2^2}}{|\gamma_1| \omega(\gamma(t))}, \quad \text{if} \quad \omega(\gamma(t)) \neq 0, \quad (2.13)$$

$$k^\gamma_\infty = \left\{ \begin{array}{ll}
\left\{ \left[ \frac{\gamma_1}{\gamma_1} \right]^2 + \lambda \right\} \cdot \left[ \left( \frac{\gamma_1}{\gamma_1} \right)^2 + \gamma_3^2 \right]^{-3} & \\
- \left\{ \left[ \frac{\gamma_1}{\gamma_1} \right]^2 + \lambda \right\} \cdot \left[ \left( \frac{\gamma_1}{\gamma_1} \right)^2 + \gamma_3^2 \right]^{-3} & 
\end{array} \right.,
$$

if $\omega(\gamma(t)) = 0$ and $\frac{d}{dt}(\omega(\gamma(t))) = 0$, \quad (2.14)

$$\lim_{L \to +\infty} k^L_\gamma \sqrt{L} = \frac{|d^2(\omega(\gamma(t)))|}{(\frac{\gamma_1}{\gamma_1})^2 + \gamma_2^2}, \quad \text{if} \quad \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0. \quad (2.15)$$

Proof. Using the notation introduced in (2.12), when $\omega(\gamma(t)) \neq 0$, we have

$$\|\nabla^L_\gamma \gamma\|_L^2 \sim \left( \frac{\omega(\gamma(t))}{\gamma_1} \right)^2 \left( \frac{\gamma_1^2 + \gamma_2^2}{\gamma_1^2} \right) L^2 \quad \text{as} \quad L \to +\infty,$$

$$\|\gamma\|_L^2 \sim L \omega(\gamma(t))^2 \quad \text{as} \quad L \to +\infty,$$

$$\langle \nabla^L_\gamma \gamma, \gamma \rangle_\gamma^2 \sim O(L^2) \quad \text{as} \quad L \to +\infty.$$

Therefore,

$$\frac{\|\nabla^L_\gamma \gamma\|_L^2}{\|\gamma\|_L^2} \to \frac{\gamma_1^2 + \gamma_2^2}{\gamma_1^2 \omega(\gamma(t))^2} \quad \text{as} \quad L \to +\infty,$$

$$\frac{\langle \nabla^L_\gamma \gamma, \gamma \rangle_\gamma^2}{\|\gamma\|_L^2} \to 0 \quad \text{as} \quad L \to +\infty.$$

So by (2.6), we have (2.13). (2.14) comes from (2.8) and

$$\frac{d}{dt}(\omega(\gamma(t))) = 0.$$

When

$$\omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0,$$

we have

$$\|\nabla^L_\gamma \gamma\|_L^2 \sim L \left[ \frac{d}{dt}(\omega(\gamma(t))) \right]^2 \quad \text{as} \quad L \to +\infty,$$

$$\|\gamma\|_L^2 = \left( \frac{\gamma_1}{\gamma_1} \right)^2 + \gamma_2^2,$$

$$\langle \nabla^L_\gamma \gamma, \gamma \rangle_\gamma^2 \sim O(1) \quad \text{as} \quad L \to +\infty.$$

By (2.6), we get (2.15).
3 The sub-Riemannian limit of geodesic curvature of curves on surfaces in the affine group

We will say that a surface $\Sigma \subset (G, g_L)$ is regular if $\Sigma$ is a Euclidean $C^2$-smooth compact and oriented surface. In particular, we will assume that there exists a Euclidean $C^2$-smooth function $u : G \rightarrow \mathbb{R}$ such that

$$\Sigma = \{(x_1, x_2, x_3) \in G : u(x_1, x_2, x_3) = 0\}$$

and

$$u_{x_1} \partial_{x_1} + u_{x_2} \partial_{x_2} + u_{x_3} \partial_{x_3} \neq 0.$$ 

Let

$$\nabla_H u = X_1(u)X_1 + X_2(u)X_2.$$ 

A point $x \in \Sigma$ is called characteristic if $\nabla_H u(x) = 0$. We define the characteristic set

$$C(\Sigma) := \{x \in \Sigma \mid \nabla_H u(x) = 0\}.$$ 

Our computations will be local and away from characteristic points of $\Sigma$. Let us define first

$$p := X_1 u, \quad q := X_2 u \quad \text{and} \quad r := \bar{X}_3 u.$$ 

We then define

$$l := \sqrt{p^2 + q^2}, \quad l_L := \sqrt{p^2 + q^2 + r^2}, \quad \bar{p} := \frac{p}{l}, \quad \bar{q} := \frac{q}{l}, \quad \bar{r} := \frac{r}{l_L}.$$ 

In particular, $\bar{p}^2 + \bar{q}^2 = 1$. These functions are well defined at every non-characteristic point. Let

$$v_L = \bar{p}X_1 + \bar{q}X_2 + \bar{r}X_3, \quad e_1 = \bar{q}X_1 - \bar{p}X_2, \quad e_2 = \bar{r}X_1 + \bar{p}X_2 - \frac{l}{l_L} \bar{X}_3.$$ 

Then $v_L$ is the Riemannian unit normal vector to $\Sigma$ and $e_1$ and $e_2$ are the orthonormal basis of $\Sigma$. On $T\Sigma$ we define a linear transformation $J_L : T\Sigma \rightarrow T\Sigma$ such that

$$J_L(e_1) = e_2, \quad J_L(e_2) = -e_1.$$ 

For every $U, V \in T\Sigma$, we define

$$\nabla^\Sigma_{U} V = \pi \nabla^l_{U} V,$$

where $\pi : T G \rightarrow T \Sigma$ is the projection. Then $\nabla^\Sigma_{\gamma}$ is the Levi-Civita connection on $\Sigma$ with respect to the metric $g_L$. By (2.11), (3.2) and

$$\nabla^\Sigma_{\gamma} \dot{\gamma} = (\nabla^l_{\gamma} \dot{\gamma}, e_1)_L e_1 + (\nabla^l_{\gamma} \dot{\gamma}, e_2)_L e_2,$$

we have

$$\nabla^\Sigma_{\gamma} \dot{\gamma} = \left\{ \begin{align*}
\bar{q} \left[ \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\gamma_1^2} + L \omega(\gamma(t)) \frac{\dot{\gamma}_2}{\gamma_1} \right] - \bar{p} \left[ \frac{\dot{\gamma}_3 - L \omega(\gamma(t)) \frac{\dot{\gamma}_1}{\gamma_1}}{\gamma_1} \right] \\
+ \left\{ \bar{r} \bar{p} \left[ \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\gamma_1^2} + L \omega(\gamma(t)) \frac{\dot{\gamma}_2}{\gamma_1} \right] + \bar{r} \bar{q} \left[ \frac{\dot{\gamma}_3 - L \omega(\gamma(t)) \frac{\dot{\gamma}_1}{\gamma_1}}{\gamma_1} \right] \\
- \frac{l}{l_L} L^2 \left[ \frac{d}{dt} (\omega(\gamma(t))) - \omega(\gamma(t)) \frac{\dot{\gamma}_1}{\gamma_1} \right] \right\} e_1 \\
+ \left\{ \bar{r} \bar{p} \left[ \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\gamma_1^2} \right] + \bar{r} \bar{q} \gamma_3 - \frac{l}{l_L} L^2 \left[ \frac{d}{dt} (\omega(\gamma(t))) \right] \right\} e_2.
\right\}$$ 

Moreover, if $\omega(\gamma(t)) = 0$, then

$$\nabla^\Sigma_{\gamma} \dot{\gamma} = \left\{ \begin{align*}
\bar{q} \left[ \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\gamma_1^2} \right] - \bar{p} \gamma_3 \\
+ \left\{ \bar{r} \bar{p} \left[ \frac{\dot{\gamma}_1 \gamma_1 - (\dot{\gamma}_1)^2}{\gamma_1^2} \right] + \bar{r} \bar{q} \gamma_3 - \frac{l}{l_L} L^2 \left[ \frac{d}{dt} (\omega(\gamma(t))) \right] \right\} e_2.
\right\}$$
Definition 3.1. Let $\Sigma \subset (\mathbb{G}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. The geodesic curvature $k^L_{\gamma, \Sigma}$ of $\gamma$ at $\gamma(t)$ is defined as
\[
k^L_{\gamma, \Sigma} := \sqrt{\frac{\left\|\nabla^{\Sigma, L}_\gamma \frac{\dot{\gamma}}{||\dot{\gamma}||_L} \right\|^2_{L^2, \Sigma} - \left\langle \nabla^{\Sigma, L}_\gamma \frac{\dot{\gamma}}{||\dot{\gamma}||_L}, \frac{\ddot{\gamma}}{||\dot{\gamma}||_L} \right\rangle_{L^2, \Sigma}}.
\]

Definition 3.2. Let $\Sigma \subset (\mathbb{G}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. We define the intrinsic geodesic curvature $k^\infty_{\gamma, \Sigma}$ of $\gamma$ at $\gamma(t)$ to be
\[
k^\infty_{\gamma, \Sigma} := \lim_{L \to +\infty} k^L_{\gamma, \Sigma},
\]
if the limit exists.

Lemma 3.3. Let $\Sigma \subset (\mathbb{G}, g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma$ be a Euclidean $C^2$-smooth regular curve. Then
\[
k^\infty_{\gamma, \Sigma} = \frac{|p_1 - p_2|}{\gamma_1 ||\omega(\gamma(t))||}, \quad \text{if} \quad \omega(\gamma(t)) \neq 0,
\]
\[
k^\infty_{\gamma, \Sigma} = 0, \quad \text{if} \quad \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) = 0,
\]
\[
\lim_{L \to +\infty} \frac{k^L_{\gamma, \Sigma}}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\gamma(t)))|}{(\frac{\gamma_1}{\gamma})^2 - p_3^2}, \quad \text{if} \quad \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0.
\]

Proof. By (2.9) and $\gamma \in TS\Sigma$, we have
\[
\dot{\gamma} = \left(\frac{p_1}{\gamma_1} - p_3\right) e_1 - \frac{L}{T} L^2 \omega(\gamma(t)) e_2.
\]
By (3.6), we have
\[
\left\|\nabla^{\Sigma, L}_\gamma \frac{\dot{\gamma}}{||\dot{\gamma}||_L} \right\|^2_{L^2, \Sigma} \sim \left(\frac{p_1}{\gamma_1} - p_3\right)^2 + \frac{L}{T} L^2 \omega(\gamma(t))^2 \sim L^2 |\omega(\gamma(t))|^2 \quad \text{as} \quad L \to +\infty.
\]
Similarly, we have that when $\omega(\gamma(t)) \neq 0$,
\[
||\dot{\gamma}||_{L^2, \Sigma} \sim \sqrt{\left(\frac{p_1}{\gamma_1} - p_3\right)^2 + \left(\frac{L}{T} \omega(\gamma(t))^2 \right)^2} \sim \frac{L}{T} \omega(\gamma(t))^2 \sim L^\frac{1}{2} |\omega(\gamma(t))| \quad \text{as} \quad L \to +\infty.
\]
By (3.6) and (3.10), we have
\[
\left\langle \nabla^{\Sigma, L}_\gamma \frac{\dot{\gamma}}{||\dot{\gamma}||_L}, \frac{\ddot{\gamma}}{||\dot{\gamma}||_L} \right\rangle_{L^2, \Sigma} \sim \left(\frac{p_1}{\gamma_1} - p_3\right) \omega(\gamma(t)) \frac{\dot{\gamma}}{||\dot{\gamma}||_L} - \frac{L}{T} L^2 \omega(\gamma(t))^2 = \frac{L}{T} L^2 \omega(\gamma(t))^2 \sim L^\frac{1}{2} |\omega(\gamma(t))| \quad \text{as} \quad L \to +\infty.
\]
where \( M_0 \) does not depend on \( L \). By (3.7) and (3.11)–(3.13), we get (3.8). When
\[
\omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) = 0,
\]
we have
\[
\|\nabla_{\gamma}^\Sigma L \gamma\|_{L, \Sigma}^2 = \left[ \frac{\gamma_1 \gamma_1 - (\gamma_1)^2}{\gamma_1^2} - \mathcal{P}\gamma_3 \right]^2 + \left[ \frac{\mathcal{P}_{L \gamma} \gamma_1 \gamma_1 - (\gamma_1)^2}{\gamma_1^2} + \mathcal{P}_{L \gamma} \gamma_3 \right]^2
\]
\[
\sim \left[ \frac{\gamma_1 \gamma_1 - (\gamma_1)^2}{\gamma_1^2} - \mathcal{P}\gamma_3 \right]^2 \quad \text{as} \quad L \to +\infty \tag{3.14}
\]
and
\[
\|\dot{\gamma}\|_{\Sigma, L} = \left| \frac{\dot{\gamma}_1}{\gamma_1} - \mathcal{P}\gamma_3 \right|,
\]
\[
\langle \nabla_{\gamma}^\Sigma L \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma, L} = \left( \frac{\dot{\gamma}_1}{\gamma_1} - \mathcal{P}\gamma_3 \right) \cdot \left( \frac{\gamma_1 \gamma_1 - (\gamma_1)^2}{\gamma_1^2} - \mathcal{P}\gamma_3 \right). \tag{3.16}
\]
By (3.14)–(3.16) and (3.7), we get \( k_{\gamma, \Sigma}^\infty = 0 \). When
\[
\omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0,
\]
we have
\[
\|\nabla_{\gamma}^\Sigma L \dot{\gamma}\|_{L, \Sigma}^2 \sim L \left[ \frac{d}{dt}(\omega(\gamma(t))) \right]^2,
\]
\[
\langle \nabla_{\gamma}^\Sigma L \dot{\gamma}, \dot{\gamma} \rangle_{\Sigma, L} = O(1),
\]
so we get (3.9). \( \square \)

**Definition 3.4.** Let \( \Sigma \subset (\mathcal{G}, g_L) \) be a regular surface. Let \( \gamma : [a, b] \to \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. The signed geodesic curvature \( k_{\gamma, \Sigma}^L \) of \( \gamma \) at \( \gamma(t) \) is defined as
\[
k_{\gamma, \Sigma}^L := \frac{\langle \nabla_{\gamma}^\Sigma L \dot{\gamma}, J_L(\gamma) \rangle_{\Sigma, L}}{\|\dot{\gamma}\|_{\Sigma, L}^3}, \tag{3.17}
\]
where \( J_L \) is defined by (3.3).

**Definition 3.5.** Let \( \Sigma \subset (\mathcal{G}, g_L) \) be a regular surface. Let \( \gamma : [a, b] \to \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. We define the intrinsic geodesic curvature \( k_{\gamma, \Sigma}^\infty \) of \( \gamma \) at the non-characteristic point \( \gamma(t) \) to be
\[
k_{\gamma, \Sigma}^\infty := \lim_{L \to +\infty} k_{\gamma, \Sigma}^L,
\]
if the limit exists.

**Lemma 3.6.** Let \( \Sigma \subset (\mathcal{G}, g_L) \) be a regular surface. Let \( \gamma : [a, b] \to \Sigma \) be a Euclidean \( C^2 \)-smooth regular curve. Then
\[
k_{\gamma, \Sigma}^\infty, k_{\gamma, \Sigma}^L = \begin{cases} 
\frac{\mathcal{P}\gamma_1 + \mathcal{P}\gamma_2}{\gamma_1 |\omega(\gamma(t))|}, & \text{if } \omega(\gamma(t)) \neq 0,
0, & \text{if } \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) = 0,
\end{cases}
\]
\[
\lim_{L \to +\infty} \frac{k_{\gamma, \Sigma}^L}{\sqrt{L}} = \frac{(-\mathcal{P}\gamma_1 + \mathcal{P}\gamma_3) \frac{d}{dt}(\omega(\gamma(t)))}{|\mathcal{P}\gamma_1 - \mathcal{P}\gamma_3|^3}, \quad \text{if } \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0. \tag{3.19}
\]
Proof. By (3.3) and (3.10), we have
\[ J_L(\gamma) = \frac{1}{L^2} \frac{d}{dt} \omega(\gamma(t)) e_1 + \left( \frac{q_{\gamma_1}}{\gamma_1} - p_{\gamma_3} \right) e_2. \] (3.20)

By (3.5) and (3.20), we have
\[ \langle \nabla^{E,L}_\gamma \gamma, J_L(\gamma) \rangle_{L, \Sigma} = \frac{1}{L^2} \omega(\gamma(t)) \left\{ \frac{\gamma_{\gamma_1}}{\gamma_1} - \frac{q_{\gamma_1}}{\gamma_1} \right\} + \frac{\tau L}{\tau E} \left\{ \frac{\gamma_{\gamma_1}}{\gamma_1} - \frac{q_{\gamma_1}}{\gamma_1} \right\} + L \omega(\gamma(t)) \frac{\gamma_{\gamma_1}}{\gamma_1} \]
\[ \sim L^2 \omega(\gamma(t)) \frac{p_{\gamma_1}}{\gamma_1} \] as \( L \to +\infty. \] (3.21)

So we get (3.18). When
\[ \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) = 0, \]
we get
\[ \langle \nabla^{E,L}_\gamma \gamma, J_L(\gamma) \rangle_{L, \Sigma} = \left( \frac{q_{\gamma_1}}{\gamma_1} - p_{\gamma_3} \right) \left( \frac{\tau L}{\tau E} \frac{\gamma_{\gamma_1}}{\gamma_1} \right) \sim M_0 L^{-\frac{1}{2}} \quad \text{as} \quad L \to +\infty. \] (3.22)

So \( k_{\gamma_1, \Sigma}^\infty = 0. \) When
\[ \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0, \]
we have
\[ \langle \nabla^{E,L}_\gamma \gamma, J_L(\gamma) \rangle_{L, \Sigma} \sim L^2 \left( -\frac{q_{\gamma_1}}{\gamma_1} + p_{\gamma_3} \right) \frac{d}{dt}(\omega(\gamma(t))) \quad \text{as} \quad L \to +\infty. \] (3.23)

So we get (3.19). \( \square \)

In the following, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the affine group. We define the second fundamental form \( II_L \) of the embedding of \( \Sigma \) into \((G, g_L)\):
\[ II_L = \left( \frac{\langle X_1(\gamma) + X_2(\gamma) \rangle L}{\tau E} \frac{\langle X_1(\gamma) + X_2(\gamma) \rangle L}{\tau E} - \frac{\langle e_1, \nabla_H (\tau E) \rangle L}{\tau E} - \frac{\langle e_2, \nabla_H (\tau E) \rangle L}{\tau E} \right). \] (3.24)

Similar to [3, Theorem 4.3], we have the following theorem.

**Theorem 3.7.** The second fundamental form \( II_L \) of the embedding of \( \Sigma \) into \((G, g_L)\) is given by
\[ II_L = \left( \frac{L}{\tau E} \frac{\langle X_1(\gamma) + X_2(\gamma) \rangle L}{\tau E} \frac{\langle X_1(\gamma) + X_2(\gamma) \rangle L}{\tau E} - \frac{\langle e_1, \nabla_H (\tau E) \rangle L}{\tau E} - \frac{\langle e_2, \nabla_H (\tau E) \rangle L}{\tau E} \right). \] (3.25)

The Riemannian mean curvature \( H_L \) of \( \Sigma \) is defined by
\[ H_L := \text{tr}(II_L). \]

Define the curvature of a connection \( \nabla \) by
\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z. \] (3.26)

Let
\[ K^{E,L}(e_1, e_2) = -\langle R^{E,L}(e_1, e_2)e_1, e_2 \rangle_{\Sigma, L}, \quad K^L(e_1, e_2) = -\langle R^L(e_1, e_2)e_1, e_2 \rangle_{L}. \] (3.27)

By the Gauss equation, we have
\[ K^{E,L}(e_1, e_2) = K^L(e_1, e_2) + \text{det}(II_L). \] (3.28)
Proposition 3.8. Away from characteristic points, the horizontal mean curvature $\mathcal{H}_\infty$ of $\Sigma \subset \mathbb{G}$ is given by

$$\mathcal{H}_\infty = \lim_{L \to +\infty} \mathcal{H}_L = X_1(\overline{p}) + X_2(\overline{\eta}) - \overline{p}.$$ 

Proof. By

$$\frac{l^2}{\mathcal{L}} \left< e_2, \nabla_H \left( \frac{r}{1} \right) \right>_L = \frac{\overline{p}^2}{L} X_1(\overline{\mathcal{L}}) + \frac{\overline{\mathcal{L}}}{L} X_2(\overline{\mathcal{L}}) = O(L^{-1}),$$

$$\frac{l}{\mathcal{L}} [X_1(\overline{p}) + X_2(\overline{\eta})] \rightarrow X_1(\overline{p}) + X_2(\overline{\eta}), \quad \overline{X}_3(\overline{\mathcal{L}}) \rightarrow 0, \quad \overline{\mathcal{L}} \rightarrow \overline{p},$$

we get (3.29). \qed

By Lemma 2.1 and (3.26), we have the following lemma.

Lemma 3.9. Let $\mathbb{G}$ be the affine group. Then

$$R^L(X_1, X_2)X_1 = \frac{3}{4} LX_2 + X_3, \quad R^L(X_1, X_2)X_2 = -\frac{3}{4} LX_1, \quad R^L(X_1, X_2)X_3 = -LX_3,$$

$$R^L(X_1, X_3)X_1 = LX_2 + \frac{3}{4} LX_3, \quad R^L(X_1, X_3)X_2 = -LX_1, \quad R^L(X_1, X_3)X_3 = \left( \frac{L^2}{4} - L \right) X_1,$$

$$R^L(X_2, X_3)X_1 = 0, \quad R^L(X_2, X_3)X_2 = -\frac{L}{4} X_3, \quad R^L(X_2, X_3)X_3 = \frac{L^2}{4} X_2.$$

Proposition 3.10. Away from characteristic points, we have

$$K^{\Sigma, L}(e_1, e_2) \rightarrow -\overline{\eta}^2 L + A + O \left( \frac{1}{\sqrt{L}} \right) \quad \text{as} \quad L \to +\infty,$$

where

$$A := \left< e_1, \nabla_H \left( \frac{X_3 u}{\sqrt{\nabla_H u}} \right) \right> - \overline{p}[X_1(\overline{p}) + X_2(\overline{\eta})] - \frac{\overline{p}^2}{L} \frac{(X_3 u)^2}{L} + 2\overline{\eta} \frac{X_3 u}{L}.$$ 

Proof. By (3.2), we have

$$\langle R^L(e_1, e_2)e_1, e_2 \rangle_L$$

$$= \mathcal{L}^2 \left< R^L(X_1, X_2)X_1, X_2 \right>_L - 2 \frac{l}{L} \overline{\mathcal{L}}^{-\frac{1}{2}} \mathcal{L} \left< R^L(X_1, X_2)X_2, X_3 \right>_L$$

$$+ 2 \frac{l}{L} \overline{\mathcal{L}}^{-\frac{1}{2}} \mathcal{L} \left< R^L(X_1, X_2)X_1, X_3 \right>_L$$

$$- 2 \left( \frac{l}{L} \right)^2 \overline{\mathcal{L}} \left< R^L(X_1, X_3)X_1, X_2 \right>_L + \left( \frac{l}{L} \right)^2 \mathcal{L} \left< R^L(X_1, X_3)X_2, X_3 \right>_L.$$

By Lemma 3.9, we have

$$K^L(e_1, e_2) = \left( \frac{1}{4} \overline{\mathcal{L}} + \frac{1}{4} \overline{\mathcal{L}} \right)^2 - \frac{3}{4} \left( \frac{1}{4} \overline{\mathcal{L}} \right)^2 \left( L - \frac{3}{4} L \overline{\mathcal{L}}^2 + 2 \frac{l}{L} \overline{\mathcal{L}}^2 \right).$$

By (3.25) and

$$\nabla_H(\mathcal{L}) = \mathcal{L}^{-\frac{1}{2}} \nabla_H \left( \frac{X_3 u}{\sqrt{\nabla_H u}} \right) + O(L^{-1}) \quad \text{as} \quad L \to +\infty,$$

we get

$$\det(\mathcal{L}) = \frac{L}{4} - \langle e_1, \nabla_H \left( \frac{X_3 u}{\sqrt{\nabla_H u}} \right) \rangle - \overline{\eta}[X_1(\overline{p}) + X_2(\overline{\eta})] + O(L^{-\frac{1}{2}}) \quad \text{as} \quad L \to +\infty.$$

By (3.28), (3.34), (3.35) and

$$\left[ \frac{1}{4} \left( \frac{1}{4} \frac{1}{L} \right)^2 - \frac{3}{4} \left( \frac{1}{4} \frac{1}{L} \right)^2 \right] \left( \frac{1}{4} \overline{\mathcal{L}}^2 - \frac{3}{4} \overline{\mathcal{L}}^2 \right) \left( \frac{(X_3 u)^2}{L} \right) + O(L^{-\frac{1}{2}}) \quad \text{as} \quad L \to +\infty,$$

we get (3.31). \qed
4 A Gauss-Bonnet theorem in the affine group

Let us first consider the case of a regular curve $\gamma : [a, b] \to (\mathbb{G}, g_L)$. We define the Riemannian length measure

$$ds_L = ||\dot{\gamma}||_L dt.$$ 

Lemma 4.1. Let $\gamma : [a, b] \to (\mathbb{G}, g_L)$ be a Euclidean $C^2$-smooth regular curve. Let

$$ds := |\omega(\dot{\gamma}(t))| dt, \quad d\bar{\sigma} := \frac{1}{2} |\omega(\dot{\gamma}(t))| \left( \frac{\dot{\gamma}_1^2}{\gamma_1^2} + \frac{\dot{\gamma}_3^2}{\dot{\gamma}_3^2} \right) dt.$$ 

Then

$$\lim_{L \to +\infty} \frac{1}{\sqrt{L}} \int_{\gamma} ds_L = \int_a^b ds. \quad (4.2)$$

When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}} ds_L = ds + \bar{\sigma} L^{-1} + O(L^{-2}) \quad \text{as} \quad L \to +\infty. \quad (4.3)$$

When $\omega(\dot{\gamma}(t)) = 0$, we have

$$\frac{1}{\sqrt{L}} ds_L = \frac{1}{\sqrt{L}} \sqrt{\frac{\dot{\gamma}_1^2}{\gamma_1^2} + \frac{\dot{\gamma}_3^2}{\dot{\gamma}_3^2}} dt. \quad (4.4)$$

Proof. We know that

$$||\dot{\gamma}(t)||_L = \sqrt{\left( \frac{\dot{\gamma}_1}{\gamma_1} \right)^2 + \dot{\gamma}_3^2} + L\omega(\dot{\gamma}(t))^2.$$

Similar to the proof of [1, Lemma 6.1], we can prove (4.2). When $\omega(\dot{\gamma}(t)) \neq 0$, we have

$$\frac{1}{\sqrt{L}} ds_L = \sqrt{L^{-1}\left( \frac{\dot{\gamma}_1}{\gamma_1} \right)^2 + \dot{\gamma}_3^2 + \omega(\dot{\gamma}(t))^2} dt.$$

Using the Taylor expansion, we can prove (4.3). From the definition of $ds_L$ and $\omega(\dot{\gamma}(t)) = 0$, we get (4.4). \qed

Proposition 4.2. Let $\Sigma \subset (\mathbb{G}, g_L)$ be a Euclidean $C^2$-smooth surface and $\Sigma = \{u = 0\}$. Let $d\sigma_{\Sigma,L}$ denote the surface measure on $\Sigma$ with respect to the Riemannian metric $g_L$. Let

$$d\sigma_{\Sigma} := (\overline{\omega}_2 - \overline{\omega}_1) \wedge \omega, \quad d\bar{\sigma}_{\Sigma} := \frac{X_3 u}{f_1} \omega_1 \wedge \omega_2 - \frac{(X_3 u)^2}{2f_1^2} (\overline{\omega}_2 - \overline{\omega}_1) \wedge \omega.$$ 

Then

$$\frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} = d\sigma_{\Sigma} + d\bar{\sigma}_{\Sigma,L}^{-1} + O(L^{-2}) \quad \text{as} \quad L \to +\infty. \quad (4.6)$$

If $\Sigma = f(D)$ with

$$f = f(u_1, u_2) = (f_1, f_2, f_3) : D \subset \mathbb{R}^2 \to \mathbb{G},$$

then

$$\lim_{L \to +\infty} \frac{1}{\sqrt{L}} \int_{\Sigma} d\sigma_{\Sigma,L}$$

$$= \int_D \left\{ \frac{(f_3 u_1 (f_2) u_2 - (f_3) u_2 (f_2) u_1)}{f_1} - 2(f_3) u_1 (f_3) u_2 \right\}^2$$

$$+ \frac{(f_1 u_1 (f_2) u_2 - (f_1) u_2 (f_2) u_1)}{f_1} + \frac{(f_1) u_2 (f_3) u_1 - (f_1) u_1 (f_3) u_2}{f_1} \right\}^2 \frac{1}{2} du_1 du_2. \quad (4.7)$$
Proof. We know that
\[ g_L(X_1, \cdot) = \omega_1, \quad g_L(X_2, \cdot) = \omega_2, \quad g_L(X_3, \cdot) = L\omega. \]
We define
\[ e_1^* := g_L(e_1, \cdot), \quad e_2^* := g_L(e_2, \cdot). \]
Then
\[ e_1^* = \bar{p}\omega_1 - \bar{p}\omega_2, \quad e_2^* = \frac{l}{t_L}L\bar{p}\omega_1 + \frac{l}{t_L}L\bar{q}\omega_2 - \frac{l}{t_L}L^{\frac{3}{2}}\omega. \]
Then we have
\[ \frac{1}{\sqrt{L}}d\sigma_{\Sigma, L} = \frac{1}{\sqrt{L}}e_1^* \wedge e_2^* = \frac{l}{t_L}(\bar{p}\omega_2 - \bar{q}\omega_1) \wedge \omega + \frac{1}{\sqrt{L}}\tau_L\omega_1 \wedge \omega. \quad (4.8) \]
By
\[ \tau_L = \frac{(X_3u)L^{-\frac{3}{2}}}{\sqrt{p^2 + q^2 + L^{-1}(X_3u)^2}} \]
and the Taylor expansion
\[ \frac{1}{t_L} = \frac{1}{l} - \frac{1}{2l^3}(X_3u)^2L^{-1} + O(L^{-2}) \quad \text{as} \quad L \to +\infty, \]
we get (4.6). By (2.2), we have
\[ f_{u_1} = (f_1)_{u_1}\partial_{x_1} + (f_2)_{u_1}\partial_{x_2} + (f_3)_{u_1}\partial_{x_3} = \left(\frac{f_1}{f_1}\right)_{u_1}X_1 + (f_3)_{u_1}X_2 + \sqrt{L}\left[\left(\frac{f_2}{f_1}\right)_{u_1} - (f_3)_{u_1}\right]X_3 \quad (4.10) \]
and
\[ f_{u_2} = \left(\frac{f_1}{f_1}\right)_{u_2}X_1 + (f_3)_{u_2}X_2 + \sqrt{L}\left[\left(\frac{f_2}{f_1}\right)_{u_2} - (f_3)_{u_2}\right]X_3. \quad (4.11) \]
Let
\[ \tau_L = \left| \begin{array}{ccc} X_1 & X_2 & \bar{X}_3 \\ \left(\frac{f_1}{f_1}\right)_{u_1} & \sqrt{L}\left(\frac{f_2}{f_1}\right)_{u_1} - (f_3)_{u_1} \\ \left(\frac{f_1}{f_1}\right)_{u_2} & \sqrt{L}\left(\frac{f_2}{f_1}\right)_{u_2} - (f_3)_{u_2} \end{array} \right|. \quad (4.12) \]
We know that
\[ d\sigma_{\Sigma, L} = \sqrt{\det(g_{ij})}du_1du_2, \quad g_{ij} = g_L(f_{u_i}, f_{u_j}), \quad \det(g_{ij}) = ||\tau_L||_L^2, \]
so by the dominated convergence theorem, we get (4.7).
\[ \square \]
**Theorem 4.3.** Let \( \Sigma \subset (\mathbb{G}, g_L) \) be a regular surface with finitely many boundary components \( (\partial \Sigma)_i \), \( i \in \{1, \ldots, n\} \), given by the Euclidean \( C^2 \)-smooth regular and closed curves \( \gamma_i : [0, 2\pi] \to (\partial \Sigma)_i \). Let \( A \) be defined by (3.32), \( d\sigma_{\Sigma} \) and \( d\sigma_{\Sigma^*} \) be defined by (4.5), \( ds \) be defined by (4.1) and \( k_{\infty, \Sigma}^{\gamma, \gamma} \) be the sub-Riemannian signed geodesic curvature of \( \gamma \) relative to \( \Sigma \). Suppose that the characteristic set \( C(\Sigma) \) satisfies \( \mathcal{H}^1(C(\Sigma)) = 0 \), where \( \mathcal{H}^1(C(\Sigma)) \) denotes the Euclidean 1-dimensional Hausdorff measure of \( C(\Sigma) \) and that \( ||\nabla H u||_{L}^{-1} \) is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set \( C(\Sigma) \). Then
\[ \int_{\Sigma} q^2d\sigma_{\Sigma} = 0, \quad (4.13) \]
\[ -\int_{\Sigma} \bar{q}^2d\sigma_{\Sigma} + \int_{\Sigma} A d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k_{\infty, \Sigma}^{\gamma, \gamma} ds = 0. \quad (4.14) \]
Proof. Using the discussions in [2], we know that the number of points satisfying
\[ \omega(\gamma_i(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma_i(t))) \neq 0 \]
on \gamma_i \text{ is finite. Since our proof of Theorem 4.3 is based on an approximation argument relying on the Lebesgue dominated convergence theorem, in the application of this theorem a set of finitely many points can be ignored as a null set. Then by Lemma 3.6, we have}
\[ k^{L,s}_{\gamma,\Sigma} = k^{\infty,s}_{\gamma,\Sigma} + O(L^{-\frac{1}{2}}). \] (4.15)
We assume firstly that \( C(\Sigma) \) is the empty set. By the Gauss-Bonnet theorem, we have
\[ \int_{\Sigma} k_{\Sigma,L} \frac{1}{\sqrt{L}} d\sigma_{\Sigma,L} + \sum_{i=1}^{n} \int_{\gamma_i} k^{L,s}_{\gamma,\Sigma} \frac{1}{\sqrt{L}} ds_{L} = 2\pi \chi(\Sigma) \sqrt{\frac{1}{L}}. \] (4.16)
So by (4.15), (4.16), (4.6), (3.31), (4.3) and (4.4), we get
\[ - \left( \int_{\Sigma} q^2 d\sigma_{\Sigma} \right)_L - \left( - \int_{\Sigma} q^2 d\sigma_{\Sigma} + \int_{\Sigma} A d\sigma_{\Sigma} + \sum_{i=1}^{n} \int_{\gamma_i} k^{\infty,s}_{\gamma,\Sigma} d\sigma \right) + O(L^{-\frac{1}{2}}) = 2\pi \chi(\Sigma) \sqrt{\frac{1}{L}}. \] (4.17)
We multiply (4.17) by a factor \( \frac{1}{2} \), let \( L \) go to the infinity, and use the dominated convergence theorem. Then we get (4.13). Using (4.13) and (4.17), we get (4.14). Using the similar discussions of [1, p.27], we can relax the condition that the characteristic set \( C(\Sigma) \) is the empty set and only suppose that the characteristic set \( C(\Sigma) \) satisfies \( \mathcal{H}^1(C(\Sigma)) = 0 \) and that \( \| \nabla_H u \|^{-1}_H \) is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set \( C(\Sigma) \).

5 The sub-Riemannian limit of curvature of curves in the group of rigid motions of the Minkowski plane

We consider the group of rigid motions of the Minkowski plane \( E(1,1) \), a unimodular Lie group with a natural sub-Riemannian structure. As a model of \( E(1,1) \) we choose the underlying manifold \( \mathbb{R}^3 \). On \( \mathbb{R}^3 \), we let
\[ X_1 = \partial_{x_3}, \quad X_2 = \frac{1}{\sqrt{2}}(-e^{x_3} \partial_{x_1} + e^{-x_3} \partial_{x_2}), \quad X_3 = -\frac{1}{\sqrt{2}}(e^{x_3} \partial_{x_1} + e^{-x_3} \partial_{x_2}). \] (5.1)
Then
\[ \partial_{x_1} = -\frac{\sqrt{2}}{2} e^{-x_3} (X_2 + X_3), \quad \partial_{x_2} = \frac{\sqrt{2}}{2} e^{x_3} (X_2 - X_3), \quad \partial_{x_3} = X_1 \] (5.2)
and
\[ \text{span}\{X_1, X_2, X_3\} = T(E(1,1)). \]
Let \( H = \text{span}\{X_1, X_2\} \) be the horizontal distribution on \( E(1,1) \). Let
\[ \omega_1 = dx_3, \quad \omega_2 = \frac{1}{\sqrt{2}}(-e^{-x_3} dx_1 + e^{x_3} dx_2), \quad \omega = -\frac{1}{\sqrt{2}}(e^{-x_3} dx_1 + e^{x_3} dx_2). \]
Then \( H = \text{Ker}\omega \). For the constant \( L > 0 \), let \( g_L = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2 + L \omega \otimes \omega \), and \( g = g_1 \) be the Riemannian metric on \( E(1,1) \). Then \( X_1, X_2, X_3 := L^{-\frac{1}{2}} X_3 \) are the orthonormal basis on \( T(E(1,1)) \) with respect to \( g_L \). We have
\[ [X_1, X_2] = X_3, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = X_2. \] (5.3)
Let \( \nabla_L \) be the Levi-Civita connection on \( E(1,1) \) with respect to \( g_L \). By the Koszul formula and (5.3), similar to Lemma 2.1, we have the following lemma.
Lemma 5.1. Let $E(1,1)$ be the group of rigid motions of the Minkowski plane. Then
\begin{align}
\nabla^L_{X_i} X_j &= 0, \quad 1 \leq j \leq 3, \quad \nabla^L_{X_i} X_2 = -\frac{L-1}{2L} X_3, \quad \nabla^L_{X_2} X_1 = -\frac{L-1}{2L} X_3, \\
\nabla^L_{X_3} X_3 &= \frac{1-L}{2} X_2, \quad \nabla^L_{X_3} X_2 = -\frac{1-L}{2} X_2, \quad \nabla^L_{X_2} X_3 = \frac{1-L}{2} X_2.
\end{align}

Lemma 5.3. Let $E(1,1)$ be the group of rigid motions of the Minkowski plane. Then
\begin{align}
\nabla^L_{X_i} X_j &= 0, \quad 1 \leq j \leq 3, \quad \nabla^L_{X_i} X_2 = -\frac{L-1}{2L} X_3, \quad \nabla^L_{X_2} X_1 = -\frac{L-1}{2L} X_3, \\
\nabla^L_{X_3} X_3 &= \frac{1-L}{2} X_2, \quad \nabla^L_{X_3} X_2 = -\frac{1-L}{2} X_2, \quad \nabla^L_{X_2} X_3 = \frac{1-L}{2} X_2.
\end{align}

Definition 5.2. Let $\gamma : [a, b] \to (E(1,1), g_L)$ be a Euclidean $C^1$-smooth curve. We say that $\gamma(t)$ is a horizontal point of $\gamma$ if
\begin{align}
\omega(\dot{\gamma}(t)) = -\frac{\sqrt{2}}{2}(e^{-\gamma_2}+e^{\gamma_2}) = 0.
\end{align}

Similar to Definitions 2.3 and 2.5, we can define $k^L_\gamma$ and $k^\infty_\gamma$ for the group of rigid motions of the Minkowski plane. We have the following lemma.

Lemma 5.3. Let $\gamma : [a, b] \to (E(1,1), g_L)$ be a Euclidean $C^2$-smooth regular curve in the Riemannian manifold $(E(1,1), g_L)$. Then
\begin{align}
k^\infty_\gamma &= \sqrt{\frac{2}{3}(e^{-\gamma_2}+e^{\gamma_2})^2 + \gamma_3^2}, \quad \text{if} \quad \omega(\dot{\gamma}(t)) \neq 0, \\
k^\infty_\gamma &= \left\{ \begin{array}{ll}
\sqrt{2} \left\{ (e^{-\gamma_2}+e^{\gamma_2})^2 + \gamma_3^2 \right\} & \text{if} \quad \omega(\dot{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) = 0, \\
\sqrt{2} \left\{ (e^{-\gamma_2}+e^{\gamma_2})^2 + \gamma_3^2 \right\} & \text{if} \quad \omega(\dot{\gamma}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\dot{\gamma}(t))) \neq 0.
\end{array} \right.
\end{align}

Proof. By (5.2), we have
\begin{align}
\dot{\gamma}(t) = \left[ \frac{\sqrt{2}}{L}(e^{-\gamma_2}+e^{\gamma_2}) \right]_t X_2 + \omega(\dot{\gamma}(t)) X_3.
\end{align}

By Lemma 5.1 and (5.8), we have
\begin{align}
\nabla^L_{X_i} X_3 &= \frac{L+1}{2} \omega(\dot{\gamma}(t)) X_2 - \frac{\sqrt{2}}{2}(e^{-\gamma_2}+e^{\gamma_2}) \frac{L+1}{2L} X_3, \\
\nabla^L_{X_2} X_3 &= \frac{L+1}{2} \omega(\dot{\gamma}(t)) X_1 + \frac{L-1}{2L} \gamma_3 X_3, \\
\nabla^L_{X_3} X_3 &= \frac{\sqrt{2}}{4} (L+1)(e^{-\gamma_2}+e^{\gamma_2}) X_1 + \frac{1-L}{2} \gamma_3 X_2.
\end{align}

By (5.8) and (5.9), we have
\begin{align}
\nabla^L_{X_i} \dot{\gamma} &= \left[ \dot{\gamma}_3 + \left( \frac{\sqrt{2}}{2} (L+1)(e^{-\gamma_2}+e^{\gamma_2}) \omega(\dot{\gamma}(t)) \right) X_1 \\
&\quad + \left( \frac{\sqrt{2}}{2} (e^{-\gamma_2}+e^{\gamma_2}) X_2 - \frac{L-1}{2L} \gamma_3 X_3 \right) X_2 \\
&\quad + \frac{d}{dt}(\omega(\dot{\gamma}(t))) - \frac{\sqrt{2}}{2L} (e^{-\gamma_2}+e^{\gamma_2}) \gamma_3 X_3.\right.
\end{align}

By (5.8) and (5.10), when $\omega(\dot{\gamma}(t)) \neq 0$, we have
\begin{align}
\|\nabla^L_{X_i} \dot{\gamma}\|^2_{L^2} = \frac{1}{2}(e^{-\gamma_2}+e^{\gamma_2})^2 + \gamma_3^2 \omega(\dot{\gamma}(t))^2 L^2 \quad \text{as} \quad L \to +\infty.
\end{align}
Similar to Section 3, we define
\begin{align*}
\|
\frac{\partial}{\partial t} \bar{L} \|
\|_{\bar{L}} \sim L \omega(\bar{t})(t)^2 \quad \text{as} \quad L \to +\infty,
\langle \nabla_{\bar{t}} L, \bar{t} \rangle \sim O(L^2) \quad \text{as} \quad L \to +\infty.
\end{align*}
Therefore,
\begin{align*}
\| \nabla_{\bar{t}} L \bar{t} \|_{\bar{L}} \to \frac{1}{2}(-e^{-\gamma} \bar{t} + e^{\gamma} \bar{t}^2)^2 + \bar{t}^3 & \quad \text{as} \quad L \to +\infty,
\end{align*}
\begin{align*}
\langle \nabla_{\bar{t}} L, \bar{t} \rangle \to 0 & \quad \text{as} \quad L \to +\infty.
\end{align*}
So by (2.6), we have (5.5). (5.6) comes from (5.8), (5.10), (2.6) and
\begin{align*}
\omega(\bar{t}(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\bar{t}(t))) = 0.
\end{align*}
When \( \omega(\bar{t}(t)) = 0 \) and \( \frac{d}{dt}(\omega(\bar{t}(t))) \neq 0 \), we have
\begin{align*}
\| \nabla_{\bar{t}} L \bar{t} \|_{\bar{L}} \sim L \left[ \frac{d}{dt}(\omega(\bar{t}(t))) \right]^2 & \quad \text{as} \quad L \to +\infty,
\end{align*}
\begin{align*}
\| \bar{t} \|_{\bar{L}} = \frac{1}{2}(-e^{-\gamma} \bar{t} + e^{\gamma} \bar{t}^2)^2 + \bar{t}^3,
\end{align*}
\begin{align*}
\langle \nabla_{\bar{t}} L, \bar{t} \rangle \sim O(1) & \quad \text{as} \quad L \to +\infty.
\end{align*}
By (2.6), we get (5.7).

6 The sub-Riemannian limit of geodesic curvature of curves on surfaces in the group of rigid motions of the Minkowski plane

We consider a regular surface \( \Sigma_1 \subset (E(1,1), g_L) \) and a regular curve \( \gamma \subset \Sigma_1 \). We assume that there exists a Euclidean \( C^2 \)-smooth function \( u : E(1,1) \to \mathbb{R} \) such that
\begin{align*}
\Sigma_1 = \{ (x_1, x_2, x_3) \in E(1,1) : u(x_1, x_2, x_3) = 0 \}.
\end{align*}
Similar to Section 3, we define \( p, q, r, l, l_L, \varphi, \varphi_L, \varphi_L, \Sigma_L, e_1, e_2, J_L, k_L, k_{\gamma_1}, k_{\gamma_2}, k_{\gamma_3}, k_{\gamma_4}, k_{\gamma_5}, k_{\gamma_6}, k_{\gamma_7}, k_{\gamma_8}, k_{\gamma_9} \) and \( k_{\gamma_{10}} \). By (3.4) and (5.10), we have
\begin{align*}
\nabla_{\gamma_L} \bar{t} \gamma & = \left\{ \nabla_\gamma \left[ \bar{t}_3 + \frac{\sqrt{2}}{2} (L + 1)(-e^{-\gamma} \bar{t}_1 + e^{\gamma} \bar{t}_2) \omega(\bar{t}(t)) \right] 
\right.
\left. - \varphi \left[ \frac{\sqrt{2}}{2} (\bar{t}_2 e^{\gamma_3} + \bar{t}_2 e^{\gamma_3} - \bar{t}_1 e^{-\gamma_3} + \bar{t}_1 e^{-\gamma_3}) - L \omega(\bar{t}(t)) \bar{t}_3 \right] \right\} e_1 
\left. + \left\{ \nabla_\gamma \left[ \bar{t}_3 + \frac{\sqrt{2}}{2} (L + 1)(-e^{-\gamma} \bar{t}_1 + e^{\gamma} \bar{t}_2) \omega(\bar{t}(t)) \right] 
\right. 
\left. + \varphi \left[ \frac{\sqrt{2}}{2} (\bar{t}_2 e^{\gamma_3} + \bar{t}_2 e^{\gamma_3} - \bar{t}_1 e^{-\gamma_3} + \bar{t}_1 e^{-\gamma_3}) - L \omega(\bar{t}(t)) \bar{t}_3 \right] 
\right. 
\left. - \frac{l}{l_L} \frac{d}{dt}(\omega(\bar{t}(t))) \right\} e_2.
\end{align*}
By (5.8) and \( \gamma(t) \in T \Sigma_1 \), we have
\begin{align*}
\gamma(t) = \left[ \bar{t}_3 - \frac{\sqrt{2}}{2} \varphi(-e^{-\gamma} \bar{t}_1 + e^{\gamma} \bar{t}_2) \right] e_1 - \frac{l}{l_L} \frac{d}{dt}(\omega(\bar{t}(t))) e_2.
\end{align*}
We have the following lemma.
Lemma 6.1. Let $\Sigma_1 \subset (E(1,1), g_L)$ be a regular curve. Let $\gamma: [a, b] \to \Sigma_1$ be a Euclidean $C^2$-smooth regular curve. Then

$$k_{\gamma, \Sigma_1}^* = \sqrt{\frac{1}{2} \gamma^2 ( -e^{-\gamma_1} + e^{\gamma_2})^2 + p^2 \gamma_1^2}, \quad \text{if } \omega(\gamma(t)) \neq 0,$$

$$k_{\gamma, \Sigma_1}^* = 0, \quad \text{if } \omega(\gamma(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\gamma(t))) = 0,$$

$$\lim_{L \to +\infty} \frac{k_{\gamma, \Sigma_1}^*}{\sqrt{L}} = \frac{|\frac{d}{dt}(\omega(\gamma(t)))|}{|\gamma \gamma_3 - \frac{\sqrt{2}}{2} p(-e^{-\gamma_1} + e^{\gamma_2})|^2}, \quad \text{if } \omega(\gamma(t)) = 0 \text{ and } \frac{d}{dt}(\omega(\gamma(t))) \neq 0. \quad (6.4)$$

Proof. By (6.1), we have

$$\|\nabla_{\gamma}^{\Sigma_1} \gamma \|^2_{L^2, \Sigma_1} \sim L^2 \omega(\gamma(t))^2 \bigg[ \frac{1}{2} \gamma^2 ( -e^{-\gamma_1} + e^{\gamma_2})^2 + p^2 \gamma_1^2 \bigg] \quad \text{as } L \to +\infty. \quad (6.5)$$

By (6.2), we have that when $\omega(\gamma(t)) \neq 0$,

$$\|\gamma\|_{\Sigma_1, L} \sim L^{\frac{1}{2}} \omega(\gamma(t)) \quad \text{as } L \to +\infty. \quad (6.6)$$

By (6.1) and (6.2), we have

$$\langle \nabla_{\gamma}^{\Sigma_1} \gamma, \gamma \rangle_{\Sigma_1, L} \sim M_0 L, \quad (6.7)$$

where $M_0$ does not depend on $L$. By (3.7) and (6.5)–(6.7), we get (6.3). When

$$\omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) = 0,$$

we have

$$\|\nabla_{\gamma}^{\Sigma_1} \gamma \|^2_{L^2, \Sigma_1} \sim \left[ \frac{\sqrt{2}}{2} \right] \omega(\gamma(t))^2 \bigg[ \gamma_3 - \frac{\sqrt{2}}{2} p(-e^{-\gamma_1} + e^{\gamma_2}) \bigg] \quad \text{as } L \to +\infty \quad (6.8)$$

and

$$\|\gamma\|^2_{L^2, \Sigma_1} = \left[ \frac{\sqrt{2}}{2} \right] \omega(\gamma(t))^2 \bigg[ \gamma_3 - \frac{\sqrt{2}}{2} p(-e^{-\gamma_1} + e^{\gamma_2}) \bigg] \quad (6.9)$$

$$\langle \nabla_{\gamma}^{\Sigma_1} \gamma, \gamma \rangle_{\Sigma_1, L} = \left[ \gamma_3 - \frac{\sqrt{2}}{2} p(-e^{-\gamma_1} + e^{\gamma_2}) \right] \times \left[ \gamma_3 - \frac{\sqrt{2}}{2} p(-e^{-\gamma_1} + e^{\gamma_2}) \right]. \quad (6.10)$$

By (6.8)–(6.10) and (3.7), we get $k_{\gamma, \Sigma_1}^* = 0$. When

$$\omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0,$$

we have

$$\|\nabla_{\gamma}^{\Sigma_1} \gamma \|^2_{L^2, \Sigma_1} \sim L \left[ \frac{d}{dt}(\omega(\gamma(t))) \right]^2,$$

$$\langle \nabla_{\gamma}^{\Sigma_1} \gamma, \gamma \rangle_{\Sigma_1, L} = O(1),$$

so we get (6.4).
Lemma 6.2. Let $\Sigma_1 \subset (E(1,1), g_L)$ be a regular surface. Let $\gamma : [a, b] \to \Sigma_1$ be a Euclidean $C^2$-smooth regular curve. Then

$$k_\gamma^{\infty,s} = \frac{\tilde{p}j_3 + \frac{\sqrt{2}}{2}q(-e^{-\gamma_3}\bar{\gamma}_1 + e^{\gamma_3}\bar{\gamma}_2)}{|\omega(\gamma(t))|}, \quad \text{if } \omega(\gamma(t)) \neq 0,$$

(6.11)

$$k_\gamma^{\infty,s} = 0, \quad \text{if } \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) = 0,$$

(6.12)

$$\lim_{L \to +\infty} \frac{k^L_{\gamma, \Sigma_1}}{L} = \frac{[-\tilde{p}j_3 + \frac{\sqrt{2}}{2}q(-e^{-\gamma_3}\bar{\gamma}_1 + e^{\gamma_3}\bar{\gamma}_2)]^3}{|\tilde{p}j_3 - \frac{\sqrt{2}}{2}q(-e^{-\gamma_3}\bar{\gamma}_1 + e^{\gamma_3}\bar{\gamma}_2)|^3},$$

(6.13)

$$\text{if } \omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0.$$

Proof. By (3.3) and (6.2), we have

$$J_L(\gamma) = \frac{l}{L} L^\frac{3}{2} \omega(\gamma(t))e_1 + \left[\tilde{p}j_3 - \frac{\sqrt{2}}{2}q(-e^{-\gamma_3}\bar{\gamma}_1 + e^{\gamma_3}\bar{\gamma}_2)\right]e_2.$$ 

(6.14)

By (6.1) and (6.13), we have

$$\langle \nabla_{\gamma}^{\Sigma_1, L} \tilde{j}_L(\gamma) \rangle_{L, \Sigma_1} \sim L^\frac{3}{2} \omega(\gamma(t))^2 \left[\tilde{p}j_3 + \frac{\sqrt{2}}{2}q(-e^{-\gamma_3}\bar{\gamma}_1 + e^{\gamma_3}\bar{\gamma}_2)\right]$$

as $L \to +\infty.$

(6.15)

So by (3.17), (6.6) and (6.14), we get (6.11). When

$$\omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) = 0,$$

we get

$$\langle \nabla_{\gamma}^{\Sigma_1, L} \tilde{j}_L(\gamma) \rangle_{L, \Sigma_1} \sim M_0 L^{-\frac{1}{2}}$$

as $L \to +\infty.$

(6.16)

So $k_\gamma^{\infty,s} = 0.$ When

$$\omega(\gamma(t)) = 0 \quad \text{and} \quad \frac{d}{dt}(\omega(\gamma(t))) \neq 0,$$

we have

$$\langle \nabla_{\gamma}^{\Sigma_1, L} \tilde{j}_L(\gamma) \rangle_{L, \Sigma_1} \sim L^\frac{3}{2} \left[-\tilde{p}j_3 + \frac{\sqrt{2}}{2}q(-e^{-\gamma_3}\bar{\gamma}_1 + e^{\gamma_3}\bar{\gamma}_2)\right] \frac{d}{dt}(\omega(\gamma(t)))$$

as $L \to +\infty.$

(6.17)

So we get (6.12). \qed

In the following, we compute the sub-Riemannian limit of the Riemannian Gaussian curvature of surfaces in the group of rigid motions of the Minkowski plane. Similar to [3, Theorem 4.3], we have the following theorem.

Theorem 6.3. The second fundamental form $II_1^L$ of the embedding of $\Sigma_1$ into $(E(1,1), g_L)$ is given by

$$II_1^L = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

(6.18)

where

$$h_{11} = \frac{l}{L} [X_1(p) + X_2(q)] - \frac{p \cdot \nabla_H r_L}{L} L^{-\frac{1}{2}},$$

$$h_{12} = h_{21} = -\frac{l}{L} \left< e_1, \nabla_H (r_L) \right> L - \frac{\sqrt{L}}{2} + \frac{1}{2} \left( \frac{L}{\sqrt{L}} - \sqrt{L}^2 \right) + \frac{1}{2} \left( \frac{L}{\sqrt{L}} - \sqrt{L} \right)^2,$$

$$h_{22} = -\frac{l^2}{2L} \left< e_2, \nabla_H \left( \frac{r}{L} \right) \right> _L + \frac{1}{2} \left( \frac{L}{\sqrt{L}} - \sqrt{L} \right)^2 L^{-\frac{1}{2}} + \frac{1}{2} \left( \frac{L}{\sqrt{L}} - \sqrt{L} \right)^3 L^{-\frac{1}{2}}.$$
Similar to Proposition 3.8, we have the following proposition.

**Proposition 6.4.** Away from characteristic points, the horizontal mean curvature $H_{\infty}^1$ of $\Sigma_1 \subset E(1,1)$ is given by

$$H_{\infty}^1 = X_1(\overline{p}) + X_2(\overline{q}).$$

By (6.20) and (6.22), we have

$$\mathcal{K}^{\Sigma_1,\infty}(e_1, e_2) = \left\langle e_1, \nabla_H \left( \frac{X_3 u}{\nabla_H u} \right) \right\rangle - \frac{(X_3 u)^2}{l^2}. \tag{6.20}$$

**Proof.** By (3.33) and Lemma 6.5, we have

$$K^{E_{1,1},L}(e_1, e_2) = -\frac{l^2}{L^2} \left( \frac{1}{2} - \frac{1}{4L} + \frac{3L}{4} \right) - \left( \frac{l}{L \overline{p}} \right)^2 \left( \frac{1}{2} - \frac{L}{4} + \frac{3}{4L} \right)$$

$$+ \left( \frac{l}{L \overline{p}} \right)^2 \left( \frac{1}{2} + \frac{1}{4L} + \frac{L}{4} \right)$$

$$\sim \left( \frac{l}{L \overline{p}} \right)^2 \frac{L}{4} - \frac{3}{4} \frac{(X_3u)^2}{l^2} - \frac{q^2}{2} + \frac{1}{2} \text{ as } L \to +\infty. \tag{6.21}$$

Similar to (3.35), we have

$$\det(IH_{\Sigma_1}^1) = -\frac{L}{4} - \left\langle e_1, \nabla_H \left( \frac{X_3 u}{\nabla_H u} \right) \right\rangle + \frac{1}{2} (\overline{q}^2 - \overline{p}^2) + O(L^{-\frac{1}{2}}) \text{ as } L \to +\infty. \tag{6.22}$$

By (6.21) and (6.22), we have (6.20). \(\square\)

Similar to (4.3) and (4.6), for the group of rigid motions of the Minkowski plane, we have

$$\lim_{L \to +\infty} \frac{1}{\sqrt{L}} ds_{L} = ds, \lim_{L \to +\infty} \frac{1}{\sqrt{L}} d\sigma_{\Sigma_1, L} = d\sigma_{\Sigma_1}. \tag{6.23}$$

By (6.20), (6.23) and Lemma 6.2, similar to the proof of [1, Theorem 1], we have the following theorem.

**Theorem 6.7.** Let $\Sigma_1 \subset E(1,1), g_L$ be a regular surface with finitely many boundary components $(\partial \Sigma_1), i \in \{1, \ldots, n\},$ given by the Euclidean $C^2$-smooth regular and closed curves

$$\gamma_i : [0, 2\pi] \to (\partial \Sigma_1).$$

Suppose that the characteristic set $C(\Sigma_1)$ satisfies $H_{\infty}^1(C(\Sigma_1)) = 0$ and that $\|\nabla_H u\|_{H^{-1}}$ is locally summable with respect to the Euclidean 2-dimensional Hausdorff measure near the characteristic set $C(\Sigma_1).$ Then

$$\int_{\Sigma_1} K^{\Sigma_1,\infty} d\sigma_{\Sigma_1} + \sum_{i=1}^{n} \int_{\gamma_i} K^{\gamma_i,\infty} d\sigma = 0. \tag{6.24}$$

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