Gravitational deformation and elasticity of magnetic force-lines

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Abstract

Magnetic fields are a very special form of elastic medium. Within astrophysical environments (magnetised stars and protogalaxies) they counteract shear and rotational distortions as well as gravitational collapse. Their vector nature allows for their extraordinary coupling with spacetime curvature in the framework of general relativity. This particular coupling points out the way to study magnetic elasticity under gravitational deformation. In this context, we reveal their law of elasticity, calculate their fracture limit and argue that they potentially impede gravitational contraction before being broken.

1 Introduction

It is well known (mainly from astrophysical studies of magnetised fluids, e.g. see [1] and [2], but from relativistic as well [3]) that magnetic force-lines behave like an elastic medium under their kinematic (shear or rotational) deformation. Namely, in analogy with a spring under pressure they develop tension stresses resisting their deflection. However, it is less known how to achieve a theoretical description of (elastic) magnetic distortion due to gravity [4]-[8] (namely spacetime curvature within general relativity). In particular, the aforementioned studies have shown that the elastic behaviour in question is expressed through a magneto-curvature tension stress coming from the Ricci identities. In fact, due to their vector nature, magnetic fields present a double coupling with spacetime curvature, not only via Einstein’s equations but via the Ricci identities as well. Thus, from a relativistic point of view it has been found out that interestingly magnetic force-lines do not self-gravitate [3], [7], [10]. Moreover, they counteract gravitational implosion of a highly conducting fluid and potentially hold it up [5]-[9]. In reference to this problem and given the elastic behaviour of magnetic fields, one can raise the question regarding the existence of a possible elastic magnetic limit. Also, a more crucial associated question is whether they manage to disrupt gravitational collapse before reaching their fracture limit [8].

Addressing the above question through an insightful introduction to magnetic elasticity, basically forms the object of the present piece of work, motivated by [8]. In detail, we begin with a brief presentation and mathematical description of the kinematically induced magnetic tension stresses. Then, we focus our attention on the magneto-curvature tension stress and reveal the law of magnetic elasticity under gravitational distortions. Subsequently, we move on to our principal task which consists of a theoretical calculation of the magnetic fracture limit during the gravitational collapse of magnetised matter. This limit is ultimately used, in
combination with some recent results [8], to argue that magnetic fields are potentially able to
impede gravitational contraction before being broken.

2 Kinematically induced magnetic tension stresses

To begin with, let us consider the decomposition of the magnetic 3-D gradient $D_b B_a$ into its
symmetric (trace-free), antisymmetric and trace part. In other words,

$$D_b B_a = D_{(b} B_{a)} + D_{[b} B_{a]} + \frac{1}{3} (D^c B_c) h_{ab},$$

which reveals the individual tension components triggered by, and resisting to shape (i.e. $\sigma_{ab}^{(B)} = D_{(b} B_{a)}$), rotational (i.e. $\omega_{ab}^{(B)} = D_{[b} B_{a]}$) and volume distortions (i.e. $\Theta^{(B)} = D^a B_a$) of the magnetic forcelines respectively. Besides, at the magnetohydrodynamic limit (MHD) the tension component opposing to volume expansion/contraction (last term) vanishes (i.e. $D^c B_c = 0$ from Gauss’s law). In the above $D_a = h_{ab} \nabla_b$ is the projected (3-D) covariant derivative operator and $h_{ab} = g_{ab} + u_a u_b$ (with $g_{ab}$ being the spacetime metric and $u^a$ being a timelike 4-velocity vector) an operator projecting upon the observer’s (3-D) rest-space. The covariant kinematics of the magnetic tension stresses are monitored by the Ricci identities for the magnetic field

$$2\nabla_{[a} \nabla_{b]} B_c = R_{abcd} B^d,$$

where $R_{abcd}$ is the Riemann spacetime tensor. In particular, the timelike part of the above leads to propagation equations for the magnetic shear $\sigma_{ab}^{(B)}$ and vorticity $\omega_{ab}^{(B)} = \epsilon_{abc} \omega^{bc}$. On the other hand, its spacelike part leads to divergence conditions (constraints) for the aforementioned quantities. The equations in question could prove useful when studying the kinematics of magnetised fluids in various contexts. However, as we do not make any use of those in the present manuscript, we have chosen to place them in a brief appendix.

3 Gravitationally induced magnetic tension stresses

In analogy with their deflection due to kinematic effects associated with the fluid’s motion, magnetic forcelines counteract their gravitational distortion. Where does the corresponding magneto-curvature tension stress come from? The answer lies in the direct coupling of magnetic fields (as vectors) with spatial curvature via the (3-D projected) Ricci identities,

$$2D_{[a} D_{b]} B_c = -2 \omega_{ab} \dot{B}_{(c)} + \mathcal{R}_{dab} B^d,$$

where $\mathcal{R}_{abcd}$ represents the 3-D counterpart of the Riemann tensor. Note that the aforementioned coupling manifests itself at the second differentiation order.

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$^1$Actually the magnetic tension force vector refers to the directional derivative along the field itself. See the following analysis.
3.1 Describing the magneto-curvature tension stress

Let us consider the 3-gradient of the magnetic tension force vector \( \tau_a = B^b D_b B_a \) (the non-zero tension force implies that the magnetic fieldlines are not spacelike geodesics). Employing the 3-D Ricci identities \(^3\) we arrive at

\[
D_c \tau_a = D_c B^b D_b B_a + B^b D_b D_c B_a + 2 \omega_{bc} B^b \dot{B}_c + R_{dabc} B^b B^d. \tag{4}
\]

The first three terms involve kinematic effects through eq \( \square \) whilst the last one can be envisaged as the magneto-curvature tension stress (or the gradient of the magneto-curvature tension component). If \( n^a \) is the magnetic field direction (i.e. \( B^a = B n^a \)), the term in question can alternatively be written as

\[
s_{ac} = R_{dabc} B^b B^d = B^2 R_{dabc} n^b n^d = B^2 u_{ac}, \tag{5}
\]

where \( u_{ac} = R_{dabc} n^b n^d \) can be envisaged as the strain tensor\(^2\) describing spatial distortions of the magnetic fieldlines. As for the stress tensor \( s_{ab} \), it includes those forces which act against (see the following discussion on the problem of gravitational collapse) spatial curvature and tend to restore the fieldlines to their initial state. Overall, the meaning of \( \square \) is the following. Due to spatial curvature the magnetic fieldlines are bent and twisted. In analogy with an elastic rod under pressure they react via the restoring stress \( u_{ab} \) (Hooke’s law of elasticity) and the magnetic density. In fact, when appearing in the kinematic equations for a magnetised fluid, it turns out that the magneto-curvature tension stress depends on the ratio of the magnetic density over the total system’s density (i.e. matter and magnetic fields—see eq. (12) in the following).

Let us recall that any kind of deformation can be reduced into a sum of a pure shear \( u_{(ab)} = R_{d(ab)c} n^c n^d \), a torsional one (or twisting \(^3\) \( u_{[ab]} = R_{d(ab)c} n^c n^d \)) and a hydrostatic compression \( ((u^c / 3) h_{ab} = (1/3) R_{cd} n^c n^d h_{ab} \). Hence, on splitting the magneto-curvature tension stress into its symmetric trace-free \( (s_{(ac)}) \), antisymmetric \( (s_{[ac]} \) and trace part \( s = s^c c \), we receive its associated component counteracting shape, rotational and volume changes respectively, due to gravity. The aforementioned components are\(^4\)

\[
s_{(ac)} = B^2 R_{d(ac)b} n^b n^d = B^2 \left[ \epsilon_{[a|q|c]} \mathcal{E}^{qs} + \frac{1}{2} \left( 2 \Pi n_{(a} n_{c)} + 2 \Pi_{(a} n_{c)} - \pi_{ac} \right) - \frac{\Theta}{3} (\Sigma n_{(a} n_{c)} + 4 \Sigma_{(a} n_{c)}) \right], \tag{6}
\]

\[
s_{[ac]} = B^2 R_{d[ac]b} n^b n^d = B^2 \left( \Pi_{[a} n_{c]} - \frac{4 \Theta}{3} \Sigma_{[a} n_{c]} \right) \tag{7}
\]

and

\[
s = s^c c = B^2 R_{bd} n^b n^d = B^2 \left( \frac{2}{3} \rho + \mathcal{E} + \Pi \right), \tag{8}
\]

\(^2\)Typically, within conventional elastic mechanics the strain tensor is defined to be a symmetric quantity \( \square \). However, here we allow for a non-vanishing anti-symmetric part taking into account any torsional deformation.

\(^3\)For the sake of accuracy, vorticity or rotational deformations are included in the shear/shape type of distortions as well.

\(^4\)On deriving eqs \( \square \)–\( \square \) we make use of the so-called Gauss-Codacci formula (e.g. see eq. 1.3.39 in \( \square \)).
where \( \pi_{ab} \) and \( E_{ab} \) are the anisotropic stress and the tidal (or electric Weyl) tensors respectively. Moreover, we have \( \Pi \equiv \pi_{ab} n^a n^b, \Pi_a \equiv \tilde{h}_a^b n^c \pi_{bc}, \mathcal{E} \equiv E_{ab} n^a n^b, \mathcal{E}_{ab} \equiv (\tilde{h}_a^c \tilde{h}_b^d - (1/2)\tilde{h}_{ab} \tilde{h}^{cd}) E_{cd}, \Sigma \equiv \sigma_{ab} n^a n^b = -\Theta/3 \) and \( \Sigma_a \equiv \tilde{h}_a^b n^c \sigma_{bc} = -\epsilon_{abc} \Omega^b \) (see [8] for the last two expressions), with \( \epsilon_{abc} \equiv \epsilon_{abc} n^c \) being the 2-D counterpart of the Levi-Civita pseudotensor, \( \Omega_a \equiv \tilde{h}_a^b \omega_b \) and \( \tilde{h}_{ab} \equiv \tilde{h}_{ab} - n_a n_b \) an operator projecting orthogonal to the magnetic field direction \( n^a \). We observe that tidal effects (electric Weyl components) are associated with shape and volume magnetic distortions only. Assuming an ideal fluid model, the anisotropic stress terms in the above vanish. Then, of particular interest is that the deformation due to gravitational compression/expansion in (8) is determined by the density of matter and the tidal tensor projected along the magnetic fieldlines. Note that there is no magnetic input, coming from anisotropic stresses (recall that \( \pi_{ab}^{\text{magn}} = -B_{(a} B_{b)} \), in (7) and (8) whilst there is in (5). In fact, \( \Pi_a^{(\text{magn})} = 0 \) (for \( n^a \parallel B^a \)) and the magnetic tension exactly cancels out the gravitational magnetic contribution in (8) (e.g. refer to [7]).

### 3.2 Magneto-curvature tension and gravitational collapse

Now having in hand the expression for the magneto-curvature tension stress, how can we reveal its competitive behaviour towards the corresponding cause of magnetic deformation? An illustrative description can be achieved by making use of the shear, vorticity and volume scalar propagation equations (e.g. see [12]). In each case the associated tension term turns out to have opposite sign to its triggering source. Here being especially interested in the problem of magnetised collapse, we focus on the volume scalar propagation equation, known as Raychaudhuri’s equation, in combination with Euler’s equation for a magnetised fluid. More specifically, for an ideal and barotropic fluid \((P = w\rho)\) the latter reads

\[
(\rho + P + B^2)\ddot{u}_a = -w D_a \rho - \frac{1}{2} D_a B^2 + B_b^b D_b A_a + \dot{\omega}^b B_b B_a, \tag{9}
\]

where the second and third term in the right hand side split the magnetic Lorentz force into its pressure and tension component respectively. Taking the divergence of the above under the assumption of near homogeneity (i.e. \( D_a \rho = 0 \sim D_a B^2 \) but \( D_a B_b \neq 0 \)), and taking into account eq (3), it turns out that

\[
\mathcal{D}^a \dot{u}_a = c_A^2 \mathcal{R} \pi_{ab} n^a n^b + 2(\sigma^a_B - \omega^a_B). \tag{10}
\]

Note that \( s^a \equiv c_A^2 (\mathcal{R} \pi_{ab} n^a n^b) \) (with \( c_A^2 \equiv B^2 / (\rho + P + B^2) \) being the Alfvén speed) actually comes from (3) by contracting the indices \( a \) and \( c \), and dividing by \( \rho + P + B^2 \). In other words, it represents the magneto-curvature tension component opposing to volume distortions of the magnetic force lines (or of the magnetised fluid). On the other hand, \( \sigma^2_B \equiv D_{[a} B_{b]} D^b B^a / 2(\rho + P + B^2) \) and \( \omega^2_B \equiv D_{[a} B_{b]} D^b B^a / 2(\rho + P + B^2) \) refer to the norms of magnetic tension counteracting shape (shear) and rotational distortions respectively.

Finally, the substitution of (10) into the well known Raychaudhuri formula, monitoring the magnetised fluid’s volume expansion/contraction, brings the latter into the intuitive form

\[
\dot{\Theta} + \frac{1}{3} \Theta^2 = -(\mathcal{R}_{ab} u^a u^b - c_A^2 \mathcal{R}_{ab} \pi_{ab}) - 2(\sigma^2_B - \omega^2_B) + 2(\omega^2_B - \omega^2_B) + \dot{\omega}^a \dot{u}_a. \tag{11}
\]
Each parenthesis in the above includes two opposite sign terms (negative sign terms in the right hand side favour volume contraction whilst positive ones favour expansion), the cause of the magnetic deformation and the associated tension (presented above) opposing to it.

4 The law of magnetic elasticity under (volume) gravitational distortions

In equation (11) we particularly observe that

\[ s^* = c_A^2 (\mathcal{R}_{ab} n^a n^b) \]

resists the magnetised fluid’s gravity, \( R_{ab} u^a u^b = (1/2)(\rho + 3P + B^2) > 0 \), in accordance with our initial claim. In complete analogy with (11), the symmetric-trace-free and the antisymmetric counterparts of (12) can be obtained via the shear and the vorticity propagation formulae respectively, along with Euler’s equation of motion. Finally, it should be noted that although (11) has been derived under the assumption of near homogeneity, expression (12) for the magneto-curvature tension stress remains exactly the same under the consideration of inhomogeneous gravitational contraction.

4.1 Insight into the law of magnetic elasticity

The meaning of (12)\(^5\) is that the tension stress \( s^* \), tending to restore the magnetic field into its initial (undeformed) state, is proportional to the distortion of the magnetic force lines \( \mathcal{R}_{ab} n^a n^b = (2/3)\rho + \mathcal{E} \). It follows that the condition \( \mathcal{E} = -(2/3)\rho \) corresponds to the natural (undeformed) volume state of the magnetic field, where \( s^* = 0 \). The proportionality factor \( 0 < c_A^2 < 1 \) (note the difference to (5)) is always positive and its definition implies that the greater the magnetic density contribution to the total fluid’s density, the more rigid the magnetic field lines are (or the more they resist to their deformation). In other words, eq (12)\(^6\) is a relativistic expression of Hooke’s law of elasticity for a magnetic field frozen into a highly conducting fluid. Nevertheless, in contrast to an elastic spring, the proportionality factor \( c_A^2 \) is not a constant but a variable quantity (a function of \( B^2/\rho \)). Moreover, although Hooke’s law is an approximate relation valid for sufficiently small deformations, eq (12)\(^7\) seems to be valid for any deformation, given that the Ricci identities (3) hold. Therefore, from our point of view, magnetic fields appear to keep their elastic behaviour as well as to satisfy Hooke’s law no matter how big their deformation is.

Even if magnetic force lines do not present an elastic limit under their gravitational bending, one expects that they can support a finite amount of distortion. Thus, we expect that there must be at least a fracture limit of the magnetic field lines. The significance of such a limit becomes clear on considering for instance the astrophysical/cosmological phenomenon of

\(^5\)The expression in question has appeared several times in past works (e.g. see [5]-[8]) but it was not recognised as an expression of Hooke’s law of elasticity and therefore was not given its full interpretation presented here.

\(^6\)Written here for an ideal (magnetised) fluid.

\(^7\)The elastic limit refers to that value of distortion beyond which the elastic medium is unable to return to its initial state. Mathematically speaking, on setting the external forces equal to zero, the deformation becomes zero as well. Of course we do not know any such example of material in nature.
magnetised gravitational collapse. In particular, magnetic fields are known not to self-gravitate as well as to have the potential to impede the gravitational implosion from reaching a spacetime singularity. In the following we suggest a definition and theoretical calculation of a potential magnetic fracture limit under gravitational distortion.

4.2 Magnetic fracture limit and impedance of gravitational contraction

From our knowledge in conventional elastic mechanics, as well as from our aforementioned data/results, we understand that as the deformation of a medium increases, so do the internal tension stresses acting against it. Hence, we claim that the fracture limit must correspond to a maximum of the tension stress as a function of the deformation. Given that \( (12) \) is not an approximate law, the simplest (and maybe the only) theoretical way to achieve a maximum, is to assume that the proportionality factor \( c_A^2 \) depends on the system’s distortion. As a matter of fact, we assume that the magnetic distortion is a monotonically increasing function of the fluid’s density \( \rho \) and vice versa.

Subsequently, we proceed to the differentiation of \( (12) \) with respect to \( u \equiv R_{ab}n^a n^b \), leading to

\[
\frac{ds^*}{du} = \frac{\partial c_A^2}{\partial \rho} \frac{d\rho}{du} + c_A^2 = c_A^2 \left[ 1 - \frac{(1 + w)c_A^2}{B^2} \left( \frac{d\rho}{du} \right) u \right].
\]

Setting the above equal to zero, we find out that the function’s critical point is

\[
u_{\text{crit}} = C \left[ (1 + w)\rho_{\text{crit}} + B_{\text{crit}}^2 \right] \quad \text{and} \quad s_{\text{crit}} = CB_{\text{crit}}^2,
\]

where \( C = ((2/3)\rho_{\text{crit}} + E_{\text{crit}})/(1 + w)\rho_{\text{crit}} + B_{\text{crit}}^2 \) is a constant determined by the condition \( u_{\text{crit}} = (2/3)\rho_{\text{crit}} + E_{\text{crit}} \). We observe that the critical value of the magneto-curvature tension stress sensibly depends on the corresponding value of the magnetic density. The aforementioned critical point will be a maximum of our function if and only if

\[
\left( \frac{d^2 s^*}{du^2} \right)_{u=u_{\text{crit}}} < 0 \iff \left( \frac{d^2 \rho}{du^2} \right)_{u=u_{\text{crit}}} > \frac{B^4}{(1 + w)^2 c_A^2}.
\]

Alternatively, \( u_{\text{crit}} \) will be a maximum if \( s^* \) is a monotonically increasing/decreasing function of \( u \) for \( u < u_{\text{crit}} \) and \( u > u_{\text{crit}} \) respectively. In the former case our condition reads

\[
\frac{ds^*}{du} > 0 \iff \frac{d\rho}{du} < \frac{B^2}{(1 + w)c_A^2} \quad \text{and} \quad \frac{ds^*}{du} < 0 \iff \frac{d\rho}{du} > \frac{B^2}{(1 + w)c_A^2}.
\]

Although the above conditions are imposed\(^9\), it appears that they are a necessary consequence of requiring the existence of a magnetic fracture limit. Now in reference to the problem of magnetised gravitational collapse we face the following question. What happens with the magnetic fieldlines at an advanced stage of the collapse? Will they manage to impede gravitational contraction?

\(^{9}\) For simplicity we put aside the dependence of \( s^* \) on \( B^2 \), as it is weak and it lasts for a small period of time. In fact, the two parts of the fraction \( c_A^2 = B^2 / ((1 + w)\rho + B^2) \) increase in almost the same rate during contraction, so that \( c_A^2 \) tends rapidly to a nearly constant value (we have taken into account that \( B^2 \propto a^{-6} \) and \( \rho \propto a^{-3(1+w)} \), see \(^8\)).

\(^{9}\) A direct verification of those entails knowledge of \( \rho \) as a function of the magnetic deformation \( u \).
contraction towards a singularity or will they be inevitably broken beforehand? Following \[6\] a plausible relativistic non-collapse criterion reads:

\[ c^2 A R_{ab} u^a u^b > R_{ab} u^a u^b, \]  

(17)
namely contraction will be held up if at some time the magneto-curvature tension stress overwhelms the magnetised fluid’s gravity (refer also to (11)). At the magnetic fracture limit (14) the above condition transforms into

\[ CB_{\text{frac}}^2 > \frac{1}{2} \left[ (1 + 3w) \rho_{\text{frac}} + B_{\text{frac}}^2 \right] \Leftrightarrow \frac{B_{\text{frac}}^2}{\rho_{\text{frac}}} > \frac{1 + 3w}{2C - 1}, \]  

(18)
where we shall be mindful of \(2C > 1\), which comes from (17), implying that \(E > B^2/2\)-see \[8\].

Given that \(B^2 \propto a^{-6}, \rho \propto a^{-3(1+w)}, 0 \leq w \leq 1, 2C > 1\), \((a\) is the scale factor associated with the fluid’s volume expansion/contraction) and allowing sufficient time for the collapse to evolve, condition (18) is generally valid for \(0 \leq w < 1\). Therefore, we expect that magnetic fieldlines manage to hold up the worldlines’ focussing before reaching their fracture limit. Nevertheless, in the special case of stiff matter \((w = 1)\), the magnetic and fluid energy densities increase with the same rate. Hence, the validity of (18) is a matter of initial conditions in that particular case.

5 Discussion

Overall, the core of our reasoning is that gravitational deformation of magnetic forcelines is governed by Hooke’s law of elasticity, originating from Ricci identities. However, there are two basic features distinguishing gravitational distortions of magnetic forcelines from mechanical distortions of elastic materials. Firstly, unlike mechanical distortions of elastic materials, Hooke’s law in the form of (12) is not an approximate expression only valid for small magnetic deformations (thus magnetic forcelines do not seem to have an elastic limit). In contrast, as long as Ricci identities are an appropriate definition of spatial curvature for large values of the latter (advanced stages of gravitational collapse), the law in question consists of an exact expression, valid for any size of distortion. Secondly, the proportionality factor in the elasticity law (12) is a variable instead of a constant quantity.

Based on the aforementioned law we have calculated the magnetic fracture limit under gravitational volume distortions, and imposed the conditions of its existence. Ideally, those conditions would have not been imposed but verified. Such a possibility presupposes actual knowledge of the fluid density as a function of the magnetic deformation. Most crucially, a relevant fundamental problem regards the experimental (and further theoretical) verification of the magnetic elasticity law. Although magnetic elasticity under great gravitational distortions is practically not a subject offered for study in earthly laboratories, progress towards the experimental path could alternatively and in the first place be achieved by examining magnetic distortions under progressively increasing rotations. Ultimately, using a relativistic non-collapse criterion (17) (which has appeared in past papers), the present work peaks with the argument that magnetic forcelines not only are they able to impede the implosion process but also they most probably manage to do it before being broken. The importance of this result essentially comes from
considering that many stars or protogalactic clouds are associated with (even small) magnetic fields which are rapidly increasing during the collapse.

Wherever they appear in our universe, either in astrophysical or in cosmological environments, magnetic fields are impelled by gravity to manifest their extraordinary elastic features. The phenomena arising from those properties await our exploration.

Appendix: Propagation equations and constraints for the kinematically induced magnetic tension stresses

In order to arrive at the propagation equation for \( \sigma_{ab}^{(B)} \), we make the following steps. First, project eq. (2) along the timelike 4-velocity \( u^a \); second, project orthogonal to \( u^a \) with the aid of \( h_{ab} \) and with respect to both indices (removing thus timelike terms); third, take the symmetric and trace-free part of the resulting relation. The equation in question finally reads

\[
\dot{\sigma}_{ab}^{(B)} = -\Theta u_{(a} \omega_{b)} + 2 \dot{u}_{(a} \omega_{b)} B_c + D_{(a} \dot{B}_{b)} - \left( \sigma_{c(a} + \omega_{c(a} + \frac{1}{3} \Theta h_{c(a} \right) \sigma_{(B) b}^{(B)} + \frac{1}{2} B_{(a} q_{b)}, \tag{19}
\]

where \( \dot{u}_a = u^b \nabla_b u_a \) is the fluid’s acceleration and \( q_a \) its flux vector. On deriving the above we have taken into account eq 1.3.1 of [12], as well as that

\[
\nabla_a B_b = D_a B_b - u_a \dot{B}_b + u_b (\nabla_a u^d) B_d + u_a u_b \dot{u}_c B_c \quad \text{and} \quad R_{a(bc)d} u^a B^d = \frac{1}{2} B_{(b} q_{c)}, \tag{20}
\]

where eqs 1.2.6, 1.2.8 and 1.2.11 of [12] have been used on finding the latter of the above. Following a similar procedure but taking the antisymmetric part of (2) (via contraction with the 3-D Levi-Civita pseudotensor \( \epsilon_{abc} \)) this time, we arrive at the propagation equation for the magnetic tension induced by twisting effects

\[
\omega_{(a}^{(B)} = -3 \epsilon_{abc} \dot{u}^b \sigma_{d}^{(B) c} B^d - \epsilon_{abc} D^b B^c - \epsilon_{abc} \left( \sigma_{bd}^{(B)} + \omega_{bd}^{(B)} + \frac{1}{3} \Theta h_{bd}^{(B)} \right) D^c D^d - \frac{1}{2} \epsilon_{abc} B^{b} q^c, \tag{21}
\]

where \( H_{ab} \) is the magnetic Weyl component and we have taken into account that

\[
\dot{B}_a = \frac{2}{3} \Theta B_a + (\sigma_{ab} + \epsilon_{abc} \omega^c) B^b \quad \text{and} \quad \epsilon_{abc} R^{c} e_{bed} u_c B_d = H_{ab} B^b - \frac{1}{2} \epsilon_{abc} B^b q^c. \tag{22}
\]

Note that eq (22a) is an expression of Faraday’s law at the MHD limit. On the other hand, the spacelike part of (2) leads to the divergence conditions for the aforementioned quantities. In

\[10\] An index with bar denotes that the associated component has been projected orthogonal to \( u^a \).
detail, we start from the 3-D Ricci identities (3). Subsequently, we take either its trace or its contraction with $\epsilon_{abc}$. The former case leads to
\[
\mathbf{D}^{b}\sigma_{ab}^{(B)} = \text{curl}\omega_{a}^{(B)} + 2\omega_{ab}\left(-\frac{2}{3}\Theta B^{b} + \sigma_{c}^{b}B^{c}\right) + \mu\omega_{a} - 2\omega^{2}B_{a} + \mathcal{R}_{ba}B^{b}
\]
whilst the latter to
\[
\mathbf{D}^{a}\omega_{a}^{(B)} = \frac{1}{6}\Theta\mu - 2\sigma_{ab}\omega^{a}B^{b}.
\]
In (23) $\mathcal{R}_{ab}$ represents the 3-D Ricci tensor. On deriving the above we have made use of (22a) as well as of
\[
\omega_{a}B_{a} = \mu/2 \quad \text{and} \quad \epsilon_{abc}\mathcal{R}_{dcba}B^{d} = \frac{2}{3}\Theta\mu - 4\sigma_{ab}\omega^{a}B^{b},
\]
where $\mu$ is the charge density and (25a) is an expression of Gauss’s law at the MHD limit. It is worth noting that for zero rotational distortions (i.e. $\omega_{ab} = 0$) of the magnetic field eqs (19) and (23) significantly simplify to
\[
\dot{\sigma}_{ab}^{(B)} = -\Theta\dot{\mathbf{u}}_{(a}B_{b)} + \mathbf{D}_{(a}\dot{B}_{b)} - \left(\sigma_{c(a} + \frac{1}{3}\Theta h_{c(a}\right)\sigma^{c(B)}_{b)} \quad \text{and} \quad \mathbf{D}^{b}\sigma_{ab}^{(B)} = \mathcal{R}_{ba}B^{b}.
\]
Overall, equations (19), (21), (23) and (24) determine the kinematics of the magnetic tension stresses triggered by shear and vorticity effects.

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\footnote{An ideal fluid (i.e. $q_{a} = 0$) has been assumed in the first equation.}
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