Semiclassical trace formulas in terms of phase space path integrals

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Abstract

Semiclassical trace formulas are examined using phase space path integrals. Our main concern in this paper is the Maslov index of the periodic orbit, which seems not fully understood in previous works. We show that the calculation of the Maslov index is reduced to a classification of connections on a vector bundle over $S^1$ with structure group $Sp(2n, R)$. We derive a formula for the index of the n-repetition, and show that a Bohr-Sommerfeld type quantization condition including quadratic fluctuation around the orbit is derived using this formula.
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1 Introduction

The semiclassical trace formula developed by Gutzwiller [1] is one of the most important tools to study the spectrum of non-integrable systems. However, the original derivation of this formula is very complicated, and the canonical invariance of the formula (especially the Maslov index) is unclear.

In this paper, we examine the semiclassical trace formula using phase space path integral to clarify the canonical structure of this formula. Our main concern here is the Maslov index, which is an additional phase factor appearing in the trace formula.

Although many studies have been made on the geometrical properties of this index, it doesn’t seem to be thoroughly understood. For hyperbolic orbits, Robbins [2] showed that this index is equal to twice the winding number which is defined by the invariant manifolds around the orbit. In this case, the Maslov index of the n-repetition of the orbit $\mu_n$ is equal to $n\mu_1$. However, Brack and Jain [3] investigated periodic orbits in anisotropic harmonic oscillator (these orbits are elliptic), and showed that $\mu_n$ is not equal to $n\mu_1$. This result suggests that we need new geometrical picture to understand this index.

We start with the phase space path integral of the partition function:

$$ Z(T) = \int \mathcal{D}p \mathcal{D}q \exp \left[ \frac{i}{\hbar} \oint (p dq - H dt) \right]. \quad (1) $$

The density of states is obtained as the Fourier-Laplace transformation of the partition function:

$$ \rho(E) = -\frac{1}{\pi} \text{Im} g(E + i\epsilon), \quad (2) $$

$$ g(E) = \text{Tr} \frac{1}{E - \hat{H}}. \quad (3) $$

$$ = \frac{1}{i\hbar} \int_0^\infty dT e^{iT/\hbar} Z(T). \quad (4) $$

We obtain the semiclassical trace formula by applying the stationary phase approximation to the path integral.

The stationary condition reads

$$ \delta \oint (p dq - H dt) = 0, \quad (5) $$

which leads to the Hamiltonian equation of motion. Since the periodic boundary condition of the path integral (1) has the effect of closing the paths, the solutions of (1) are periodic orbits.
Thus the partition function is approximated by a sum over the
periodic orbits:

$$Z(T) = \sum_{p.o} K \exp \left( \frac{i}{\hbar} R \right).$$  \hspace{1cm} (6)

Here, \( R = \oint p \, dq - H \, dt \) is the classical action of the periodic orbit, and \( K \) is the contribution of the quadratic fluctuation around the orbit:

$$K = \int \mathcal{D}x \exp \left[ i \delta^2 R(x(t)) \right],$$ \hspace{1cm} (7)

$$x(t) = \frac{1}{\sqrt{\hbar}} (\delta q, \delta q).$$ \hspace{1cm} (8)

The Maslov index appears as the phase factor of this quadratic path integral. Since this is a kind of Fresnel integral, the phase is determined by the signs of diagonal elements of the quadratic form \( \delta^2 R \).

Strictly speaking, we should write (1) in a discrete form to obtain well-defined continuum limit. Since the way of the discretization reflects the operator ordering of the Hamiltonian, we should treat this problem carefully [4]. In this paper, we always use the mid-point prescription, which corresponds to the Weyl-ordering of the operators. We can formulate the canonical transformations of the path integral most clearly in this prescription, as we will see in §3.

The central idea of this paper is that the Maslov index of the periodic orbit is determined by the linearized symplectic flow around the orbit. The set of displacement vectors \( \{ x(t) \} \) is considered to be a vector bundle over \( S^1 \), and the flow around the orbit define a connection (Fig. I). Mathematically speaking, our problem is the classification of connections on the fibre bundle. However, the quadratic path integral around the periodic orbit is not invariant to all canonical transformations. If the canonical transformation is topologically non-trivial, the path integral may change the sign. This is essentially the same as the global anomalies of the gauge field theories [10, 11]. Therefore we regard two connections as equivalent if they are connected by a topologically trivial canonical transformation, and classify the connections by this equivalence relation.

This paper is organized as follows. In §2 we review the derivation of the phase space path integral of the partition function and the stationary phase approximation. Note that the discussion in this section is not restricted to the time-independent Hamiltonians. §3 is the main part of this paper, and we treat the quadratic path integrals with periodic boundary conditions generally. We calculate the path integrals by reducing them to simple normal forms using canonical (gauge) transformations. We also derive a formula for the Maslov index of the
The set of displacement vectors is regarded as a fibre bundle, and Hamiltonian symplectic flow defines the connection. The structure group of this space is $Sp(2n, R)$.

n-repetition of the orbit. In §4, we discuss the trace of the resolvent $g(E)$ for time-independent systems. If the Hamiltonian of the system is time-independent, the periodic orbit in it have at least one zero-mode. Therefore we first discuss the integration with respect to zero-modes. Then we execute the Fourier transformation of the partition function and derive the semiclassical approximation to $g(E)$. We also show that the sum over repetitions of a orbit leads to a Bohr-Sommerfeld type quantization condition. Similar attempts have been made by many authors, but our result is the first one which includes the effect of the Maslov index correctly.
2 Path integral and stationary phase approximation

2.1 Path integral representation of the partition function

In this subsection, we derive phase space path integral of the partition function

\[ Z(T) = \text{Tr} \hat{U}(T). \]  

(9)

Here, \( \hat{U}(T) \) is the time evolution operator

\[ \hat{U}(T) = \text{P exp} \left[ \frac{i}{\hbar} \int_0^T dt \hat{H}(t) \right], \]  

(10)

and \( \hat{H}(t) \) is the Hamiltonian of the system. \( \text{P} \) denotes the path ordered (or time ordered) product.

\( \hat{U}(T) \) can be decomposed into a product of infinitesimal time evolution operators:

\[ \hat{U}(T) = \prod_{j=1}^N \left( 1 - \frac{i}{\hbar} \Delta t \hat{H}(t_j) \right) + O(1/N), \]  

(11)

where \( \Delta t \) denotes \( T/N \), and \( t_j \) denotes \( j \Delta t \). Therefore \( Z(T) \) can be written as

\[ Z(T) = \int dq \langle q \| \hat{U}(T) \| q \rangle, \]  

(12)

\[ = \lim_{N \to \infty} \int \prod_{j=1}^N dq_j \langle q_j | 1 - \frac{i}{\hbar} \Delta t \hat{H}(t_j) | q_{j-1} \rangle. \]  

(13)

Here, the variables \{\( q_j \)\} are cyclic and \( q_i \) is considered to be the same as \( q_j \) if \( i \equiv j (\text{mod}N) \). This notation is used throughout this paper.

The Feynmann kernel for the infinitesimal time interval can be written as

\[ \langle q_j | 1 - \frac{i}{\hbar} \Delta t \hat{H}(t_j) | q_{j-1} \rangle = \int \frac{dp_j}{(2\pi\hbar)^n} \left\{ 1 - \frac{i}{\hbar} \Delta t H_W \left( p_j, \frac{q_j + q_{j-1}}{2}, t_j \right) \right\} e^{ip_j(q_j - q_{j-1})}. \]  

(14)

\[ \simeq \int \frac{dp_j}{(2\pi\hbar)^n} \exp \left[ \frac{i}{\hbar} \left( p_j(q_j - q_{j-1}) - H_W \left( p_j, \frac{q_j + q_{j-1}}{2}, t_j \right) \Delta t \right) \right], \]  

(15)

where \( n \) is the number of degrees of freedom, and \( H_W \) is Weyl transform of \( \hat{H} \):

\[ H_W(p, q) = \int dw e^{ip \cdot w/h} \langle q - \frac{w}{2} | \hat{H} | q + \frac{w}{2} \rangle. \]  

(16)
Therefore we obtain the phase space path integral representation of the partition function:

$$Z(T) = \lim_{N \to \infty} \prod_{j=1}^{N} \int \frac{dp_j dq_j}{(2\pi \hbar)^n} \exp \left[ \frac{i}{\hbar} \sum_{j=1}^{N} \left\{ \frac{p_j (q_j - q_{j-1})}{\Delta t} - \Delta t H(p_j, \bar{q}_j, t_j) \right\} \right].$$

(17)

Here, we denote $H_W$ as simply $H$, and $q_j$ denotes $(q_j + q_{j-1})/2$ (mid-point prescription).

In the continuum limit, we denote this path integral as

$$Z(T) = \int \mathcal{D}q \mathcal{D}p \exp \left[ \frac{i}{\hbar} \oint (pdq - H dt) \right].$$

(18)

Note that the c-number Hamiltonian is defined by the Weyl transform of the q-number Hamiltonian in this paper. This c-number Hamiltonian can also be obtained by arranging the q-number Hamiltonian into Weyl ordering, and substituting c-number variables for q-number variables [4].

If the Hamiltonian is given as

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}, t),$$

(19)

integration with respect to $p$ in (17) can be done easily, and we obtain the path integral representation of $Z$ in coordinate space:

$$Z(T) = \lim_{N \to \infty} \prod_{j=1}^{N} \int \frac{m}{2\pi i \hbar \Delta t}^{n/2} dq_j \exp \left[ \frac{i}{\hbar} \sum_{j=1}^{N} \left\{ \frac{m}{2} \left( \frac{q_j - q_{j-1}}{\Delta t} \right)^2 - V(\bar{q}_j, t_j) \right\} \Delta t \right].$$

(20)

In the continuum limit, this formula can be denoted as

$$Z(T) = \int \mathcal{D}q \exp \left[ \frac{i}{\hbar} \oint L dt \right],$$

(21)

where $L$ is the Lagrangian of the system.

### 2.2 Stationary phase approximation

Let us evaluate the path integral (17) by the stationary phase approximation. We define a discretized action $R_N$ as

$$R_N(p_1, p_2, \ldots, p_N, q_1, \ldots, q_N) = \sum_{j=1}^{N} \left\{ p_j (q_j - q_{j-1}) - \Delta t H(p_j, \bar{q}_j, t_j) \right\}.$$ 

(22)
Then the stationary phase condition reads

\[ 0 = \frac{\partial R_N}{\partial p_j} = q_j - q_{j-1} - \Delta t \frac{\partial H}{\partial p}(p_j, q_j, t_j), \]  \tag{23} \\
\[ 0 = \frac{\partial R_N}{\partial q_j} = p_j - p_{j+1} - \Delta t \left\{ \frac{\partial H}{\partial q}(p_{j+1}, q_{j+1}, t_{j+1}) + \frac{\partial H}{\partial q}(p_j, q_j, t_j) \right\}, \]  \tag{24} \\

which leads to the Hamiltonian equation of motion in the limit \( N \to \infty \):

\[ \dot{q} = \frac{\partial H}{\partial p}, \]  \tag{25} \\
\[ \dot{p} = -\frac{\partial H}{\partial q}. \]  \tag{26} \\

Since variables in \( R_N \) are cyclic, the solutions of the stationary phase condition are classical periodic orbits.

The second derivatives of \( R_N \) are

\[ \frac{\partial^2 R_N}{\partial p_i \partial p_j} = -\Delta t (H_{pp})_{i,j}, \]  \tag{27} \\
\[ \frac{\partial^2 R_N}{\partial q_i \partial q_j} = -\Delta t (H_{qq})_{i,j}, \]  \tag{28} \\
\[ \frac{\partial^2 R_N}{\partial p_i \partial q_j} = \Delta t \{(\Delta)_{i,j} - (H_{pq})_{i,j}\}. \]  \tag{29} \\

Here, \( H_{pp}, H_{pq}, H_{qq} \) and \( \Delta \) are \( Nn \times Nn \) matrices, which are defined as

\[ (H_{pp})_{i,j} = \frac{\partial^2 H}{\partial p^2}(p_j, q_j, t_j)\delta_{i,j}, \]  \tag{30} \\
\[ (H_{pq})_{i,j} = \frac{1}{2} \left\{ \frac{\partial^2 H}{\partial p \partial q}(p_{j+1}, q_{j+1}, t_{j+1})\delta_{i,j+1} + \frac{\partial^2 H}{\partial p \partial q}(p_j, q_j, t_j)\delta_{i,j} \right\}, \]  \tag{31} \\
\[ (H_{qq})_{i,j} = \frac{1}{4} \left\{ \frac{\partial^2 H}{\partial q^2}(p_{j+1}, q_{j+1}, t_{j+1})(\delta_{i,j+1} + \delta_{i,j}) \right. \]  \\
\[ \left. + \frac{\partial^2 H}{\partial q^2}(p_j, q_j, t_j)(\delta_{i,j} + \delta_{i,j-1}) \right\}, \]  \tag{32} \\
\[ (\Delta)_{i,j} = (\delta_{i,j} - \delta_{i,j+1})/\Delta t. \]  \tag{33} \\

Therefore

\[ Z(T) = \sum_{p.o.} \exp \left[ \frac{i}{\hbar} R_{cl} \right] \lim_{N \to \infty} \int \frac{dX_N}{(2\pi\hbar)^{Nn}} \exp \left[ \frac{1}{2\hbar} X_N^T \mathcal{P}_N X_N \right], \]  \tag{35} \\

\[ = \sum_{p.o.} \exp \left[ \frac{i}{\hbar} R_{cl} \right] \lim_{N \to \infty} \frac{e^{i\Phi(\mu,N-\mu,-N)}}{\sqrt{\det \mathcal{P}_N}}, \]  \tag{36} 

where

\[
R_{cl} = \oint (p_{cl}\dot{q}_{cl} - H(p_{cl}, q_{cl}, t)) dt,
\]

(37)

\[
X_N = (\delta p_1, \delta p_2, ..., \delta p_N, \delta q_1, \delta q_2, ..., \delta q_N),
\]

(38)

\[
\tilde{D}/N = \Delta t \begin{pmatrix}
-H_{pp} & \Delta - H_{pq} \\
\Delta T - H_{pq}^T & -H_{qq}
\end{pmatrix},
\]

(39)

\[
\mu_{+,N} (\mu_{-,N}) \text{ is the number of positive (negative) integers. Since } \Delta \text{ becomes differential operator in the limit } N \to \infty, Z \text{ can be written as}
\]

\[
Z(T) = \sum_{\text{p.o.}} \frac{e^{-i\frac{\mu}{\hbar} \sqrt{|\text{Det} \tilde{D}/N|}}}{\text{Det} \tilde{D}/N} \exp \left[ \frac{i}{\hbar} R_{cl} \right].
\]

(41)

Here, the differential operator \( \tilde{D} \) is defined as

\[
\tilde{D} = J \frac{d}{dt} - H''(p_{cl}(t), q_{cl}(t), t).
\]

(42)

\( J \) and \( H'' \) are \( 2n \times 2n \) matrices defined as

\[
J = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix},
\]

(43)

\[
H'' = \begin{pmatrix}
\frac{\partial^2 H}{\partial p^2} & \frac{\partial^2 H}{\partial p \partial q} \\
\frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial q^2}
\end{pmatrix}.
\]

(44)

The functional determinant of \( \tilde{D} \) is defined as the limit of the determinant of \( \tilde{D}/N \):

\[
\text{Det} \tilde{D} = \lim_{N \to \infty} \det \tilde{D}/N.
\]

(45)

\( \mu \) is the Maslov index of the periodic orbit, which is defined as

\[
\mu = \lim_{N \to \infty} \frac{\mu_{-,N} - \mu_{+,N}}{2},
\]

(46)

Since \( \tilde{D}/N \) is even dimensional matrix, \( \mu_{-,N} - \mu_{+,N} \) is also even and \( \mu \) is integer if there is no zero-mode.

\[1\] Note that the definitions of \( \tilde{D} \) and \( J \) are different from those in the reference [9]. (The signs are opposite.) Hence the definition of the Maslov index (46) is also different from that in [9]. We follow the definitions in [5] in this paper.
2.3 Relation between Maslov index and Morse index

Let us consider the case in which the Hamiltonian of the system is given by

\[ H = \frac{\hat{p}^2}{2m} + V(\hat{q}, t). \]  \hfill (47)

In this case, the second variation of action around the periodic orbit is

\[ \frac{1}{2} X_N^T \mathcal{D}_N X_N = - \sum_{j=1}^{N} \left\{ \frac{\Delta t}{2m} \delta p_j^2 - \delta p_j (\delta q_j - \delta q_{j-1}) \right\} - \frac{\Delta t}{2} \sum_{i,j=1}^{N} V_{i,j} \delta q_i \delta q_j, \]  \hfill (48)

where \( V_{i,j} \) is defined as

\[ V_{i,j} = \frac{1}{4} \left\{ \frac{\partial^2 V}{\partial^2 q_i} (\bar{q}_{i,j+1}, t_{j+1}) \delta_{i,j+1} + \frac{\partial^2 V}{\partial^2 q_j} (\bar{q}_{i,j+1}, t_{j+1}) \delta_{i,j} \right\}. \]  \hfill (49)

This quadratic form can be transformed as

\[ \frac{1}{2} X_N^T \mathcal{D}_N X_N = - \sum_{j=1}^{N} \frac{\Delta t}{2m} \left( \delta p_j - \frac{m}{\Delta t} (\delta q_j - \delta q_{j-1}) \right)^2 + \Delta t \left\{ \sum_{j=1}^{N} \frac{m}{2} \left( \frac{\delta q_j - \delta q_{j-1}}{\Delta t} \right)^2 + \frac{1}{2} \sum_{i,j=1}^{N} V_{i,j} \delta q_i \delta q_j \right\}. \]  \hfill (51)

Here, \( \delta p_j' = \delta p_j - m(\delta q_j - \delta q_{j-1})/\Delta t \), and the second term of RHS is the second variation of action in coordinate space path integral \[20\]. In the limit \( N \to \infty \), \( Dq_N \) reduces to a differential operator:

\[ Dq = -m \frac{d^2}{dt^2} + V''(q_{cl}(t), t). \]  \hfill (54)

(Here we used the partial integration \((dq/dt)^2 \to -d^2q/dt^2\).) We denote the number of negative eigenvalues of \( Nn \times Nn \) quadratic form \( Dq_N \) as \( \mu_{q,-N} \). Then the Maslov index of the orbit is

\[ \mu = \lim_{N \to \infty} \frac{\mu_{-,N} - \mu_{+,N}}{2}, \]  \hfill (55)

\[ = \lim_{N \to \infty} \frac{Nn - \{(Nn - \mu_{q,-N}) - \mu_{q,-N}\}}{2}, \]  \hfill (56)

\[ = \lim_{N \to \infty} \mu_{q,-N}. \]  \hfill (57)
This is the number of negative eigenvalues of $D_q$ (Morse index), which is finite if $m$ is positive. Therefore the Maslov index is the Morse index in coordinate space path integral. We summarize the correspondence between phase space path integral and coordinate path integral in Table 1.

|                         | phase space                                      | coordinate space                                |
|-------------------------|--------------------------------------------------|-------------------------------------------------|
| partition function      | $\int DpDq \exp \left[ \frac{i}{\hbar} \oint pdq - H dt \right]$ | $\int Dq \exp \left[ \frac{i}{\hbar} \oint L dt \right]$ |
| second variation        | $\mathcal{D} = J \frac{d}{dt} - H''$             | $D_q = -m \frac{d^2}{dr^2} + V''$               |
| Maslov index            | $(\mu_- - \mu_+)/2$                               | $\mu_-$ (Morse index)                           |

Table 1: phase space v.s. coordinate space
3 General treatment of quadratic path integrals

In section 2, we evaluate path integral by the stationary phase approximation, and it is necessary to calculate quadratic path integral like (35). In this section, we study the general structure of the quadratic path integrals. First we note that quadratic path integrals are independent of \( \bar{h} \). (We can show this explicitly by changing the variable \( X_N \rightarrow \sqrt{\bar{h}}X_N \) in (35).) The period \( T \) can also be normalized by \( t \rightarrow Tt \). Therefore it is enough to consider the case in which \( \bar{h} = T = 1 \).

3.1 Quadratic path integrals

Let us consider the following quadratic Hamiltonian:

\[
\hat{H}(\hat{p}, \hat{q}, t) = \frac{1}{2} \hat{x}^T h(t) \hat{x}.
\]  

(58)

Here, \( h(t) \) is a symmetric \( 2n \times 2n \) matrix, and \( \hat{x} = (\hat{p}, \hat{q}) \). Since this Hamiltonian is already arranged into Weyl ordering, the corresponding classical Hamiltonian is

\[
H(p, q, t) = \frac{1}{2} x^T h(t) x,
\]

(59)

where \( x \) is a classical \( 2n \) dimensional vector \((p, q)\). Our concern is to calculate partition function

\[
Z = \text{Tr} \hat{U}(T = 1),
\]

(60)

\[
\hat{U}(T) = \text{P exp} \left[ -i \int_0^T \hat{H}(t) dt \right].
\]

(61)

\( Z \) can be represented by path integral as in §2:

\[
Z = \lim_{N \to \infty} \int dX_N (2\pi)^N N^n \exp \left[ i \frac{1}{2} X_N^T D_N X_N \right],
\]

(62)

\[
= \lim_{N \to \infty} \frac{e^{-i \hat{J}^\mu N}}{\sqrt{|\det D_N|}},
\]

(63)

where

\[
X_N = (p_1, p_2, ..., p_N, q_1, q_2, ..., q_N),
\]

(64)

\[
D_N = \frac{1}{N} \left( \begin{array}{cc} -H_{pp} & \Delta - H_{pq} \\ \Delta^T - H_{pq}^T & -H_{qq} \end{array} \right),
\]

(65)
\[(H_{pp})_{i,j,k,l} = h(t_j)_{k,l}\delta_{i,j}, \quad (66)\]
\[(H_{pq})_{i,j,k,l} = \frac{1}{2} \{h(t_{j+1})_{k,n+l}\delta_{i,j+1} + h(t_{j})_{k,n+l}\delta_{i,j}\}, \quad (67)\]
\[(H_{qq})_{i,j,k,l} = \frac{1}{4} \{h(t_{j+1})_{n+k,n+l}(\delta_{i,j+1} + \delta_{i,j}) + h(t_{j})_{n+k,n+l}(\delta_{i,j} + \delta_{i,j-1}) + \delta_{i,j+1}\}, \quad (68)\]
\[\Delta_{i,j,k,l} = N\delta_{k,l}(\delta_{i,j} - \delta_{i,j+1}), \quad (69)\]
\[(1 \leq i, j \leq N, 1 \leq k, l \leq n). \quad (70)\]

The quadratic path integral \((35)\) appearing in the stationary phase approximation is the special case of \((63)\) in which
\[h(t) = TH''(p_{cl}(tT), q_{cl}(tT)). \quad (71)\]

In the continuum limit, we write this path integral as
\[Z = \int D\mathbf{x} \exp \left[ \frac{i}{2} \mathbf{x}^T \mathbf{\Psi} \mathbf{x} \right], \quad (72)\]
\[= \frac{e^{-i\mathbf{\Psi} \mathbf{x}}}{\sqrt{|\text{Det}\mathbf{\Psi}|}} \quad (73)\]
\[\mathbf{\Psi} = J \frac{d}{dt} - h(t), \quad (74)\]
\[\mathbf{x} = (p, q). \quad (75)\]

Our purpose in the following subsections is to calculate \((73)\). However, it is difficult to calculate \((73)\) directly from the definition \((63)\). We will explain our strategy to calculate \((73)\) in the next subsection.

### 3.2 Outline of the discussion

Our strategy to calculate \((73)\) is to transform the Hamiltonian matrix \(h(t)\) into a simple normal form by canonical transformations.

Let us consider a time dependent symplectic matrix \(S(t)\). This matrix satisfies
\[S^T(t)JS(t) = J, \quad (76)\]
and we assume that \(S(t)\) satisfies periodic boundary condition:
\[S(0) = S(1). \quad (77)\]

\(S(t)\) defines a canonical transformation
\[\mathbf{x}'(t) = S(t)\mathbf{x}(t). \quad (78)\]
The Hamiltonian matrix $h(t)$ is transformed by $S(t)$ as

$$ h \rightarrow (S^{-1})^T h S^{-1} - JS \frac{dS^{-1}}{dt}. \quad (79) $$

We can see this transformation from a different point of view. As it was shown in [9], the set $\{x(t)|0 \leq t \leq 1, x(0) = x(1)\}$ is considered to be a vector bundle over $S^1$, and $A(t) = Jh(t)$ is a connection on this bundle. Therefore canonical transformation defined by $S(t)$ is considered to be a gauge transformation. Actually it is easy to verify that $A = Jh$ is transformed by $S$ as

$$ A \rightarrow SAS^{-1} + S \frac{dS^{-1}}{dt}. \quad (80) $$

The operator $\mathcal{D}$ can be rewritten as

$$ \mathcal{D}(t) = JD(t), \quad (81) $$

$$ D(t) = \frac{d}{dt} + A(t). \quad (82) $$

$D$ is a covariant derivative, and parallel transport defined by this covariant derivative is the same as the Hamiltonian equation of motion:

$$ \left\{ \frac{d}{dt} + A(t) \right\} x(t) = \frac{dx}{dt}(t) + Jh(t)x(t) = 0. \quad (83) $$

We can transform given $A(t)$ into simpler form by gauge transformations. However, the path integral (73) is not invariant to all gauge transformations.

Let us explain this point by a simple example, namely the time independent one dimensional harmonic oscillator:

$$ H = \frac{\alpha}{2} (p^2 + q^2). \quad (84) $$

In this case,

$$ h = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad (85) $$

and

$$ \mathcal{D} = \begin{pmatrix} -\alpha & \frac{d}{dt} \\ -\frac{d}{dt} & -\alpha \end{pmatrix}. \quad (86) $$

Eigenvalue equation

$$ \mathcal{D} x_n(t) = \epsilon_n x_n(t) \quad (87) $$

is easily solved as

$$ \epsilon_n = -\alpha + 2n\pi, \quad (88) $$

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Figure 2: Flow of eigenvalues of $D$. 

\[ x_n(t) = \begin{pmatrix} \cos 2n\pi t \\ \sin 2n\pi t \end{pmatrix}, \begin{pmatrix} -\sin 2n\pi t \\ \cos 2n\pi t \end{pmatrix}. \quad (89) \]

Note that each solution is doubly degenerate. If $\alpha = 0$, $H = 0$ and the Maslov index is considered to be 0 in this case. As $\alpha$ changes continuously, eigenvalues $\{\epsilon_n\}$ also changes continuously and two of them change their sign when $\alpha$ crosses multiples of $2\pi$ (Fig.2). Thus we obtain the formula for the Maslov index:

\[ \mu = \begin{cases} 1 + \left[ \frac{\alpha}{2\pi} \right] & (\alpha \neq 2m\pi) \\ \frac{2m}{2} & (\alpha = 2m\pi) \end{cases}. \quad (90) \]

(See also appendix B).

The canonical transformation defined by

\[ S_k(t) = \begin{pmatrix} \cos 2k\pi t & -\sin 2k\pi t \\ \sin 2k\pi t & \cos 2k\pi t \end{pmatrix}. \quad (91) \]

changes the Hamiltonian as

\[ \frac{\alpha}{2} (p^2 + q^2) \rightarrow \frac{\alpha + 2k\pi}{2} (p^2 + q^2). \quad (92) \]

Therefore the Maslov index changes as

\[ \mu \rightarrow \mu + 2k, \quad (93) \]

though the spectrum of $D$ is unchanged.

This relation means that the sign of the path integral changes if $k$ is odd. Let us discuss this result from a different point of view.

The classical time evolution operator of this system is represented by the matrix

\[ V(t) = \begin{pmatrix} \cos \alpha t & -\sin \alpha t \\ \sin \alpha t & \cos \alpha t \end{pmatrix}. \quad (94) \]
On the other hand, the quantum time evolution operator $\hat{U}(t) = \exp(-i\hat{H}t)$ acts on coherent states as

$$\hat{U}(t)|z\rangle = \exp(-i\alpha t/2)|\exp(-i\alpha t)z\rangle. \quad (95)$$

We can see that the correspondence between $\hat{U}(t)$ and $V(t)$ is not 1-to-1, but 2-to-1. For example, suppose that $t = 1$ and $\alpha$ changes to $\alpha + 2k\pi$. $V(t)$ is unchanged by this transformation, but $\hat{U}(t)$ changes to $(-1)^k\hat{U}(t)$. This is consistent with the relation $(93)$.

In general, the Maslov index changes if the gauge transformation is topologically non-trivial. (This is considered to be a global anomaly.[10, 11]) The gauge transformation $S(t)$ is an element of the fundamental group of $Sp(2n, R)$, and this group is isomorphic to $\mathbb{Z}$, namely,

$$\pi_1(\text{Sp}(2n, R)) \simeq \mathbb{Z}. \quad (96)$$

Therefore each gauge transformation has an integer $k$ as a winding number, and $\mu$ is changed by this transformation as $\mu \rightarrow \mu + 2k$. Hence we should use only topologically trivial gauge transformations when we transform $A$ into a normal form.

In the following subsections, we formulate canonical transformations for quadratic path integrals rigorously, and show normal forms of these path integrals. In §3.3.1, we introduce metaplectic group following the reference [12]. Metaplectic group is a unitary representation of symplectic group, and the 2-to-1 correspondence between the two groups is the origin of the anomaly as we saw in the Harmonic oscillator model. In §3.3.2, we formulate canonical transformation using metaplectic group. In §3.4, we discuss the classical time evolution operator $V(t)$. Since this is a symplectic matrix with a parameter $t$, we can regard $V$ as a curve on the group manifold of $Sp(2n, R)$. Canonical transformations are represented by deformations of this curve. In §3.5, we define normal forms of quadratic path integrals, and calculate them. Details of the calculations are shown in appendix B and the results are summarized in §3.6. We derive the formula for the Maslov index of the n-repetition of a orbit in §3.7.

### 3.3 Canonical transformations for quadratic path integrals

#### 3.3.1 Metaplectic group

In this section, we introduce Metaplectic group, which is a quantum counterpart of symplectic group [12]. Let us consider a linear canonical transformation specified by a symplectic matrix $S \in Sp(2n, R)$:

$$x' = Sx. \quad (97)$$
The metaplectic operator corresponding to $S$ is the unitary operator $M(S)$ specified by the relation

$$M^\dagger(S)\hat{x}M(S) = S\hat{x}. \quad (98)$$

Metaplectic group $Mp(2n, R)$ is composed of such unitary operators, and locally isomorphic to $Sp(2n, R)$. Let us consider an infinitesimal canonical transformation

$$S(\epsilon) = 1 - \epsilon A. \quad (99)$$

$A$ satisfies

$$A^T J + JA = 0, \quad (100)$$

which can be written as

$$A = Jh, \quad (101)$$

where $h$ is a symmetric matrix. Corresponding metaplectic operator is

$$M(S(\epsilon)) = 1 - i\epsilon \frac{1}{2} \hat{x}^T h \hat{x}. \quad (102)$$

It is easy to verify that (102) and (99) satisfy (98), using the commutation relation

$$[\hat{x}_\alpha, \hat{x}_\beta] = -i\alpha J_{\alpha \beta}. \quad (103)$$

Let us consider a symplectic matrix $V(t)$ specified by the equation

$$\frac{d}{dt}V(t) = -Jh(t)V(t), \quad (104)$$

$$V(0) = 1. \quad (105)$$

$V(t)$ can be represented as

$$V(t) = P \exp[-\int_0^t A(t')dt''], \quad (106)$$

where

$$A(t) = Jh(t). \quad (107)$$

The corresponding metaplectic operator $\hat{U}(t)$ is uniquely determined by the following equations:

$$\frac{d}{dt}\hat{U}(t) = -i\hat{H}(t)\hat{U}(t), \quad (108)$$

$^2$Note that we assume $\hbar = 1$ in this section. If $\hbar \neq 1$, $[\hat{x}_\alpha, \hat{x}_\beta] = -i\hbar J_{\alpha \beta}$ and $M(\epsilon) = I - \frac{i\epsilon}{2\hbar} \hat{x}^T h \hat{x}$. 

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Hamiltonian

\begin{align*}
\text{classical} & : H(t) = \frac{1}{2}x^T h(t)x \\
\text{quantum} & : \hat{H}(t) = \frac{1}{2}\hat{x}^T h(t) \hat{x}.
\end{align*}

Table 2: Correspondence between classical and quantum systems for quadratic Hamiltonian.

\begin{align*}
\hat{U}(0) &= 1, \\
\hat{H}(t) &= \frac{1}{2}x^T h(t)x.
\end{align*}

\begin{align*}
\hat{U}(t) &= P \exp[-i \int_0^t \hat{H}(t') dt'].
\end{align*}

$V(t)$ is the classical time evolution operator for the quadratic Hamiltonian $H(t) = x^T h(t)x$, and $U(t)$ is the corresponding quantum time evolution operator. $V(1)$ (classical time evolution operator for the period) is called the monodromy matrix. We summarize the correspondence between classical and quantum systems in Table 2.

Although $Mp(2n, R)$ and $Sp(2n, R)$ are isomorphic locally, they are not isomorphic globally; there is a 2-to-1 correspondence between them. Therefore if a closed curve on symplectic group $S(0) = S(1)$ has winding number $k$, the metaplectic operators corresponding to $S(0)$ and $S(1)$ are not necessarily the same, but

\begin{align*}
M(S(1)) &= (-1)^k M(S(0)).
\end{align*}

This relation is the origin of the anomaly. Note that the relation between $Sp(2n, R)$ and $Mp(2n, R)$ is very similar to the well-known relation between $SO(3)$ and $SU(2)$. The point is that both $Sp(2n, R)$ and $SO(3)$ are not simply connected and hence allow two-valued representations.

### 3.3.2 Canonical transformation

Let us consider canonical transformations of quadratic path integrals using Metaplectic operators. In continuum notation, a partition function

\begin{align*}
Z &= \int \mathcal{D}x \exp \left[ \frac{i}{2} \int_0^1 x^T(t)J \left( \frac{d}{dt} + A(t) \right) x(t) dt \right].
\end{align*}
seems invariant under the canonical (gauge) transformation by \( \{ S(t) \} \in \pi_1(\text{Sp}(2n, R)) \):

\[
\begin{align*}
x(t) & \rightarrow S(t)x(t), \\
A(t) & \rightarrow S(t)A(t)S^{-1}(t) + S(t)\frac{dS^{-1}}{dt}(t),
\end{align*}
\]

(114)

(115)

\( S(0) = S(1) \).

However, this is not correct. We must go back to the definition of the path integral to make the discussion rigorous.

\( Z \) is the trace of the time evolution operator \( \hat{U}(t = 1) \), which is represented as a product of infinitesimal time evolution operators:

\[
\hat{U}(t = 1) = \prod_{j=1}^{N} \left( 1 - \frac{i}{N} \hat{H}(t_j) \right) + O \left( \frac{1}{N} \right),
\]

(117)

where \( t_j = 1/N \). We obtain normal path integral by inserting the complete set

\[
1 = \int dq_j |q_j\rangle \langle q_j|
\]

(118)
to (117):}

\[
Z = \int \prod_{j=1}^{N} dq_j |q_j\rangle |1 - \frac{i}{N} \hat{H}(t_j)|q_{j-1}\rangle + O \left( \frac{1}{N} \right).
\]

(119)

To formulate the canonical transformation corresponding to \( S(t) \), we insert

\[
1 = \int dq_j M(S(t_j))\rangle |q_j\rangle \langle q_j|M(S(t_j))
\]

(120)

instead of (118). Then (119) is modified as

\[
Z = \int \prod_{j=1}^{N} dq_j M_j \left\{ 1 - \frac{i}{N} \hat{H}(t_j) \right\} M_{j-1} M_{j-1} \langle q_{j-1}\rangle + O \left( \frac{1}{N} \right),
\]

(121)

where \( M_j \) denotes \( M(S(t_j)) \).

\( S(t_{j-1}) \) can be expanded as

\[
S(t_{j-1}) = \left\{ 1 - \frac{1}{N} \frac{dS}{dt}(t_j)S(t_j)^{-1} \right\} S(t_j) + O \left( \frac{1}{N^2} \right).
\]

(122)

Therefore

\[
M_j M_{j-1}^\dagger = 1 - \frac{i}{2N} \left\{ JS(t_{j-1}) \frac{dS^{-1}}{dt}(t_{j-1}) \right\}_{\alpha\beta} \hat{x}_a \hat{x}_\beta + O \left( \frac{1}{N^2} \right),
\]

(123)
\[ M_j \hat{H}(t_j) M_j^\dagger = M_j \hat{H}(t_j) M_j^\dagger + O \left( \frac{1}{N} \right), \]  
(124)

\[ = \frac{1}{2} h_{\alpha \beta}(t_j) M_j \hat{x}_\alpha M_j^\dagger \hat{x}_\beta M_j^\dagger + O \left( \frac{1}{N} \right), \]  
(125)

\[ = \frac{1}{2} h_{\alpha \beta}(t_j) S^{-1}_{\alpha \gamma}(t_j) S^{-1}_{\beta \delta}(t_j) \hat{x}_\gamma \hat{x}_\delta + O \left( \frac{1}{N} \right) \]  
(126)

Here we used \( M(S) \hat{x} M(S)^{\dagger} = S^{-1} \hat{x} \) and \( \frac{dS}{dt} = S A_{S} \frac{S^{-1}}{dt}(t) \). Hence, for \( 2 \leq j \leq N \),

\[ \langle q_j | M_j \left\{ 1 - \frac{i}{N} \hat{H}(t_j) \right\} M_{j-1}^\dagger | q_{j-1} \rangle = \langle q_j | 1 - \frac{i}{N} \hat{K}(t_j) | q_{j-1} \rangle + O \left( \frac{1}{N^2} \right), \]  
(127)

\[ \hat{K}(t) = \hat{x}^T k(t) \hat{x}, \]  
(128)

\[ k(t) = (S(t)^{-1})^T h(t) S(t)^{-1} - JS(t) \frac{dS^{-1}}{dt}(t). \]  
(129)

This is the same as the classical canonical transformation. However, for \( j = 1 \),

\[ M_1 M_N^\dagger = (-1)^k + O \left( \frac{1}{N} \right), \]  
(130)

where \( k \) is the winding number of the closed path \( \{ S(t) \} \). Therefore \( Z \) is represented as

\[ Z = \int \prod_{j=1}^{N} dq_j \langle q_j | 1 - \frac{i}{N} \hat{H}(t_j) | q_{j-1} \rangle + O \left( \frac{1}{N} \right), \]  
(131)

\[ = (-1)^k \int \prod_{j=1}^{N} dq_j \langle q_j | 1 - \frac{i}{N} \hat{K}(t_j) | q_{j-1} \rangle + O \left( \frac{1}{N} \right), \]  
(132)

and the path integral (113) is transformed as a functional of \( A \) like

\[ Z[A^S(t)] = (-1)^k Z[A(t)], \]  
(133)

where \( A^S \) is the gauge field transformed by \( S \). (133) means that the Maslov index is transformed as

\[ \mu' \equiv \mu + 2k \pmod{4}, \]  
(134)

where \( \mu' \) is the Maslov index corresponding to \( A^S(t) \). We show the stronger result

\[ \mu' = \mu + 2k, \]  
(135)

in the following subsection.

\*\*Note that this simple correspondence between classical and quantum canonical transformation is due to Weyl ordering of the operators. If the Hamiltonian is written in other order, additional terms appear and the simple correspondence is lost. See, for example, chapter 3 of the reference [4].
Table 3: Hamiltonian $H$, gauge field $A$ and curve $V$. Those three have the same informations. $S(t)$ belongs to $\text{Sp}(2n, R)$ and satisfies $S(1) = S(0)$, $V$ and $S$ is continuous, but we don’t require smoothness of $V$ and $S$ in this paper. Therefore $H$ and $A$ allow discontinuities.

3.4 Curves on $\text{Sp}(2n, R)$

To investigate quadratic path integrals in detail, let us examine the classical time evolution operator $V$, which is considered to be a curve on the group manifold of $\text{Sp}(2n, R)$. \{V(t)\} is the curve which starts from the origin, and its end point is the monodromy matrix. $V$ is defined by the equations

\[ \dot{V}(t) = J \left( \frac{d}{dt} + A(t) \right) V(t) = 0, \]  
\[ V(0) = 1. \]  

(136)  
(137)

Note that the correspondence between $V$ and $\dot{V}$ is 1-to-1. $V$ is determined from $\dot{V}$ by (136) and (137), and $\dot{V}$ is determined from $V$ by the relation

\[ A(t) = -\frac{dV}{dt}(t)V(t)^{-1}. \]  

(138)

See also Table 3.

3.4.1 Gauge transformation of $V$

Let us consider how this curve is transformed by gauge transformations. If the gauge field is transformed as

\[ A(t) \rightarrow S(t)A(t)S(t)^{-1} + S(t) \frac{dS^{-1}}{dt}(t), \]  

(139)
solutions of (136) are $S(t)V(t) \times \text{constant}$. The constant should be chosen to be $S(0)^{-1}$ to satisfy (137). Thus $V$ is transformed as

\[ V(t) \rightarrow S(t)V(t)S(0)^{-1}. \]  

(140)
The gauge transformation group $G$ is considered to be a set of closed curves on $Sp(2n, R)$:

$$G = \{ S | S : [0, 1] \rightarrow Sp(2n, R), S(0) = S(1) \}. \quad (141)$$

$G$ can be decomposed into two parts,

$$G = L \times N, \quad (142)$$

where $L$ and $N$ are subgroups of $G$ defined as

$$L = \{ l | l \in G, l(0) = 1 \}, \quad (143)$$

$$N = \{ n | n: \text{time independent element in } G \} \quad (144)$$

$N$ is isomorphic to $Sp(2n, R)$. $G$ means any $S \in G$ can be written as

$$S(t) = l(t)n, \quad (145)$$

where $l \in L$ and $n \in N$. $l$ and $n$ are explicitly written as

$$n = S(0), \quad (146)$$

$$l(t) = S(t)S(0)^{-1}. \quad (147)$$

The gauge transformation by $l \in L$ is written as

$$V(t) \rightarrow l(t)V(t). \quad (148)$$

The monodromy matrix $M(=V(1))$ is unchanged by this transformation. The transformation by $n \in N$ is

$$V(t) \rightarrow nV(t)n^{-1}, \quad (149)$$

and $M$ is transformed as $nMn^{-1}$.

The elements in $G$ and $L$ are classified by winding numbers. We refer the subset of $G$ ($L$) which has winding number $k$ as $G_k$ ($L_k$). $G_0$ ($L_0$) is a subgroup of $G$ ($L$), and $N$ is a subgroup of $G_0$. It is obvious that

$$G_0 = L_0 \times N. \quad (150)$$

It is worth pointing out, in passing, that the time translation $t \rightarrow t + t_0$ is also regarded as a gauge transformation. This transformation changes $V$ as

$$V(t) \rightarrow \begin{cases} V(t + t_0)V(t_0)^{-1} & (0 \leq t \leq 1 - t_0) \\ V(t + t_0 - 1)V(t_0)^{-1} & (1 - t_0 \leq t \leq 1) \end{cases} \quad (151)$$

This is a gauge transformation given by

$$S(t) = \begin{cases} V(t + t_0)V(t)^{-1} & (0 \leq t \leq 1 - t_0) \\ V(t + t_0 - 1)V(t)^{-1} & (1 - t_0 \leq t \leq 1) \end{cases} \quad (152)$$

which can be shrunk to a point continuously as $t_0 \rightarrow 0$. Therefore the time translation is regarded as a gauge transformation in $G_0$. 

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3.4.2 Continuous deformation of $V$ and its Maslov index

Let us consider deformations of $V$. If the deformation is a gauge transformation, the monodromy matrix $M$ is transformed as $nMn^{-1}$ by a symplectic matrix $n$. The converse also holds. Let $V_1$ and $V_2$ be the curves which start from the origin. If there is a symplectic matrix $n$ which satisfies $V_1(1) = nV_2(1)n^{-1}$, the matrix $S(t)$ defined by

$$S(t) = V_1(t)nV_2(t)^{-1}$$

satisfies $S(0) = S(1) = n$ and $V_1(t) = S(t)V_2(t)S(0)^{-1}$. Therefore $V_1$ and $V_2$ are connected by the gauge transformation $S$. Thus we conclude that two curves are connected by a gauge transformation if and only if the end-points of two curves are in the same conjugacy class.

As a special case, if two curves have the same end-point (monodromy matrix), there is a gauge transformation which connects two curves. In this case, $n = 1$ and $S(t) = V_1(t)V_2(t)^{-1}$ is an element of $L$.

Since there is a 1-to-1 correspondence between $D$ and $V$, $D$ and its spectrum changes continuously as we deform $V$ continuously. We can see relative changes of Maslov index by analyzing the points where the signs of diagonal elements of $D$ changes. $D$ can be diagonalized explicitly, and what is important is that the diagonal elements are unchanged by gauge transformations. (See appendix A.) Let $V_1$ and $V_2$ be curves which are connected by a gauge transformation $S$. If $S$ belongs to $G_0$, $S$ can be shrunk to 1 continuously. Therefore $V_1$ and $V_2$ can also be united continuously, and diagonal elements of $D$ is unchanged throughout this process. Thus we conclude that if two curves are connected by $G_0$, two curves have the same Maslov index. However, if two curves are connected by $G_k$ ($k \neq 0$), two curves have different Maslov indices. Now we can prove the following important statement:

If two curves $V_1$ and $V_2$ satisfy the relation

$$V_1(t) = S(t)V_2(t)S(0)^{-1},$$

the corresponding Maslov indices $\mu_1$ and $\mu_2$ satisfy

$$\mu_2 = \mu_1 + 2k.$$  

\[4\]Strictly speaking, we should determine the Maslov index from discretized matrix $D_N$. However, when we deform $V$ (or $A$) continuously, relative changes of the Maslov index are dominated by the low-frequency part of $D_N$, which is approximated well by $D$ if $N$ is large enough. Therefore we can determine the relative changes of the Maslov index from $D$. 

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Figure 3: Deformations of curves on $Sp(2n, R)$. 

[proof]

$V_2$ can be deformed to $V_2'$, which is defined as

$$V_2'(t) = S(0)V_2(t)S(0)^{-1}. \quad (157)$$

Since the constant matrix $S(0)$ belongs to $G_0$, $V_2$ and $V_2'$ have the same Maslov index. $V_1$ and $V_2'$ have the same end-point. (See Fig. 3.)

We already know that (156) holds for some cases. For example, let us consider the Hamiltonian

$$H(p, q, \alpha) = \frac{\alpha}{2}(p_1^2 + q_1^2). \quad (158)$$

Let $W_1$ and $W_2$ be the curves corresponding to $H(p, q, \alpha)$ and $H(p, q, \alpha + 2k\pi)$. $W_1$ and $W_2$ have the same end-point, and the difference between the Maslov indices of $W_1$ and $W_2$ is $2k$. (See section 3.2 and appendix B.) Then deform $V_1$ continuously to $V_2'$ through $W_1$ and $W_2$ (Fig. 3).

The difference of the Maslov index between $V_1$ and $V_2'$ is the same as that of $W_1$ and $W_2$, because the change of the Maslov index during the deformation $V_1 \rightarrow W_2$ is canceled by that of $W_2 \rightarrow V_2'$ if the end-point go through the same path during the two processes. Therefore the difference of the Maslov index between $V_1$ and $V_2$ is also $2k$. (Q.E.D.)

(156) means that two curves considered to be equivalent if two curves are connected by $G_0$ (not $G$). We classify quadratic Hamiltonians by this equivalence relation in the next subsection.
3.4.3 Universal covering of $Sp(2n, R)$ and generalized monodromy matrices

Let us consider the set of curves on $Sp(2n, R)$ which starts from the origin. We refer this set as $F$:

$$F = \{ V| V : [0, 1] \rightarrow Sp(2n, R), V(0) = 1 \}, \quad (159)$$

which is isomorphic to the set of quadratic Hamiltonians and the set of gauge fields. Two elements $V_1$ and $V_2$ in $F$ is considered to be equivalent if they are connected by a gauge transformation which belongs to $G_0$. We express this relation as

$$V_1 \sim^{G_0} V_2. \quad (160)$$

If $V_1 \sim^{G_0} V_2$, $V_1$ and $V_2$ have the same Maslov index. Therefore the Maslov index is determined uniquely on the quotient set $F/ \sim^{G_0}$. Before turning to the division by $G_0$, let us discuss the division by $L_0$, which is a subgroup of $G_0$. If $V_1$ and $V_2$ are connected by an element in $L_0$ (i.e. $V_1 \sim^{L_0} V_2$), two curves have the same terminal points and the closed curve made by these two curves can be shrunk to a point continuously. The converse also holds. (It is obvious that $\{V_1(t)V_2(t)^{-1}\}$ belongs to $L_0$.) Therefore the quotient set $F/ \sim^{L_0}$ is the set of homotopy classes, which is simply connected by definition. Therefore this is the universal covering space of $Sp(2n, R)$:

$$F/ \sim^{L_0} = \tilde{Sp}(2n, R). \quad (161)$$

An element in $\tilde{Sp}(2n, R)$ is specified by a symplectic matrix (monodromy matrix $M$) and an integer (winding number $k$). (See Fig. [3]) This pair $\tilde{M} = (M, k)$ is regarded as a generalized monodromy matrix. We give a definition of the winding number in §3.5.2.

Let us discuss the group structure of $F$ and $\tilde{Sp}(2n, R)$. We can define the product of two elements $V$ and $W$ in $F$ as

$$(VW)(t) = V(t)W(t) \quad (0 \leq t \leq 1). \quad (162)$$

It is obvious that $F$ becomes a group by this product. Therefore we define the product in $\tilde{Sp}(2n, R) = F/ \sim^{L_0}$ as

$$[V][W] = [VW], \quad (163)$$

where $[V],[W]$ and $[VW]$ denote the homotopy classes of $V,W$ and $VW$. It is easy to verify that this product is well-defined, and $\tilde{Sp}(2n, R)$ also becomes a group by this product.
Figure 4: $Sp(2n, R)$ and $\tilde{Sp}(2n, R)$. 
We can define another product in $F$ as

$$(V * W)(t) = \begin{cases} W(2t) & (0 \leq t < 1/2), \\ V(2t - 1)W(1) & (1/2 \leq t \leq 1). \end{cases} \quad (164)$$

Note that this product doesn’t satisfy the associative law, i.e. $V * (W * X) \neq (V * W) * X$. However, $VW$ and $V * W$ is homotopic.

We can define a map $f : [0, 1] \times [0, 1] \to \tilde{Sp}(2n, R)$ which satisfies $f(t, 0) = (VW)(t)$ and $f(t, 1) = (V * W)(t)$ as

$$f(t, s) = V(t, s)W(t, s) \quad (0 \leq t \leq 1, 0 \leq s \leq 1), \quad (165)$$

$$V(t, s) = \begin{cases} 1 & (0 \leq t < s/2), \\ V \left(\frac{2t-s}{2-s}\right) & (s/2 \leq t \leq 1). \end{cases} \quad (166)$$

$$W(t, s) = \begin{cases} W \left(\frac{2}{2-s}t\right) & (0 \leq t < 1-s/2), \\ W(1) & (1-s/2 \leq t \leq 1). \end{cases} \quad (167)$$

Therefore, in $\tilde{Sp}(2n, R)$,

$$[V][W] = [VW] = [V * W]. \quad (168)$$

### 3.5 Normal forms of quadratic Hamiltonians

It was observed in the preceding subsection that

$$F/\mathcal{L} = Sp(2n, R), \quad (169)$$

$$F/\mathcal{L}_0 = \tilde{Sp}(2n, R). \quad (170)$$

We are now ready to consider $F/\mathcal{L}$ and $F/\mathcal{L}_0$. Since $G = L \times N$, $F/\mathcal{L}_0$ is the set of conjugacy classes of $Sp(2n, R)$. Therefore an element in $F/\mathcal{L}_0$ is specified by a conjugacy class of $Sp(2n, R)$ and a winding number. In the following, we show representatives of these quotient sets.

In this subsection, we change the definition of the $2n \times 2n$ matrix $J$ as

$$J = \begin{pmatrix} j & \ddots & \ddots \\ \ddots & j & \ddots \\ \vdots & \ddots & j \end{pmatrix}, \quad (171)$$

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (172)$$

for convenience.
3.5.1 Normal forms of monodromy matrices

First we consider representatives of the quotient set
\[ F/\sim = \{ \text{conjugacy class of } Sp(2n, R) \}. \] (173)

Our task here is to transform a given monodromy matrix \( M \in Sp(2n, R) \) into a simple normal form \( SMS^{-1} \), where \( S \) is also a symplectic matrix. In other words, we choose the special symplectic basis in which \( M \) has a simple form.

Let us consider the eigenvalue equation
\[ Mx = \lambda x. \] (174)

First we assume that \( M \) has no degenerate eigenvalue. If \( \lambda \) is an eigenvalue of \( M \), \( \lambda^* \) and \( 1/\lambda \) are also eigenvalues of \( M \) [13]. (Fig.5)

Therefore we can classify eigenvalues of \( M \) into four types, and \( M \) become a block diagonal matrix in the basis made of the eigen vectors. (If some of the eigenvectors are complex, we use the real parts and imaginary parts of them as the basis.)

\[ M = \begin{pmatrix} m_1 & m_2 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & m_l \\ 0 & \cdots & 0 & m_l \end{pmatrix}. \] (175)

In the following, we list up the four types of eigenvalues and the normal forms of the monodromy matrix.
1. elliptic type: $\lambda = e^{\pm \alpha}$. ($0 < \alpha < 2\pi$)

$$m = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$ \hspace{1cm} (176)

2. hyperbolic type: $\lambda = e^{\pm \beta}$.

$$m = \begin{pmatrix} e^\beta & 0 \\ 0 & e^{-\beta} \end{pmatrix}.$$ \hspace{1cm} (177)

3. inverse hyperbolic type: $\lambda = -e^{\pm \beta}$.

$$m = \begin{pmatrix} -e^\beta & 0 \\ 0 & -e^{-\beta} \end{pmatrix}.$$ \hspace{1cm} (178)

4. loxodromic type: $\lambda = e^{\pm i\alpha \pm \beta}$. ($0 < \alpha < \pi$)

$$m = \begin{pmatrix} e^\beta \cos \alpha & 0 & e^\beta \sin \alpha & 0 \\ 0 & e^{-\beta} \cos \alpha & 0 & e^{-\beta} \sin \alpha \\ -e^\beta \sin \alpha & 0 & e^\beta \cos \alpha & 0 \\ 0 & -e^{-\beta} \sin \alpha & 0 & e^{-\beta} \cos \alpha \end{pmatrix}.$$ \hspace{1cm} (179)

Here, $\alpha$ and $\beta$ are real numbers.

If $M$ have degenerate eigenvalues, we must treat such cases separately. In this paper, we treat only the most important case, parabolic type. In this case, the eigenvalue 1 is doubly degenerate and we can choose the basis $x_\alpha$ ($\alpha = 1, 2$) as

$$M x_1 = x_1, \hspace{1cm} (180)$$

$$M x_2 = x_2 - \gamma x_1. \hspace{1cm} (181)$$

The monodromy matrix of this part is

$$m = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}. \hspace{1cm} (182)$$

---

5 In the elliptic case, the normal forms corresponding to $\alpha$ and $2\pi - \alpha$ are not conjugate, though two matrices have the same eigenvalues. However, in the loxodromic case, two matrices corresponding to $\alpha$ and $2\pi - \alpha$ are conjugate. That’s why we choose $\alpha$ as $0 < \alpha < \pi$ in this case. See Appendix \ref{A.2.1} and the reference \cite{13} §42.

6 $\gamma$ can be transformed to $\pm 1$. ($x_1 \rightarrow \sqrt{|\gamma|} x_1, x_2 \rightarrow (1/\sqrt{|\gamma|}) x_2$) However, we leave $\gamma$ as a parameter for convenience.
3.5.2 Normal forms of Hamiltonians

Now we can choose representatives of $F/\mathcal{G}$ and $F/\mathcal{G}^0$. Normal forms of Hamiltonians corresponding to them are also determined.

First we show the representatives of $F/\mathcal{G}$ and the normal form of the Hamiltonian corresponding to the types of eigenvalues.

- elliptic type
  \[ V(t) = \begin{pmatrix} \cos \alpha t & -\sin \alpha t \\ \sin \alpha t & \cos \alpha t \end{pmatrix} \]
  \[ H = \frac{\alpha}{2}(p^2 + q^2). \]  

- hyperbolic type
  \[ V(t) = \begin{pmatrix} e^{\beta t} & 0 \\ 0 & e^{-\beta t} \end{pmatrix} \]
  \[ H = -\beta pq. \]

- inverse hyperbolic type
  \[
  V(t) = \begin{cases} 
  \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix} & (0 \leq t < 1/2), \\
  \begin{pmatrix} -e^{\beta(2t-1)} & 0 \\ 0 & -e^{-\beta(2t-1)} \end{pmatrix} & (1/2 \leq t \leq 1) 
  \end{cases}
  \]
  \[ H = \begin{cases} 
  \pi(p^2 + q^2) & (0 \leq t < 1/2), \\
  -2\beta pq & (1/2 \leq t \leq 1).
  \end{cases} \]

- loxodromic type
  \[
  V(t) = \begin{pmatrix}
  e^{\beta t} \cos \alpha t & 0 & e^{\beta t} \sin \alpha t & 0 \\
  0 & e^{-\beta t} \cos \alpha t & 0 & e^{-\beta t} \sin \alpha t \\
  -e^{\beta t} \sin \alpha t & 0 & e^{\beta t} \cos \alpha t & 0 \\
  0 & -e^{-\beta t} \sin \alpha t & 0 & e^{-\beta t} \cos \alpha t
  \end{pmatrix}.
  \]
  \[ H = \alpha(p_1q_2 - p_2q_1) - \beta(p_1q_1 + p_2q_2). \]

- parabolic type
  \[ V(t) = \begin{pmatrix} 1 & -\gamma t \\ 0 & 1 \end{pmatrix} \]
  \[ H = \frac{\gamma}{2}q^2. \]
Let us consider a given time-dependent quadratic Hamiltonian $H$ and the corresponding curve $V$. We can transform $V$ into $V_0$, which is a direct sum of the normal forms, using the gauge transformation in $G$:

$$V_0(t) = S(t)V(t)S(0)^{-1}. \quad (193)$$

Then we can choose a normal form of $V$ as

$$V_k = V_0 \ast U_k. \quad (194)$$

Here, $k$ denotes the winding number of $S$, and $U_k$ is a closed curve which start from the origin and have the winding number $k$. (See Fig. 6.) We can choose a normal form of $U_k$ as, for example,

$$U_k(t) = \begin{pmatrix}
cos 2\pi kt & -\sin 2\pi kt \\
\sin 2\pi kt & \cos 2\pi kt
\end{pmatrix} \begin{pmatrix} 1 \\
m_{1,2}
\end{pmatrix}. \quad (195)$$

We use this normal form $V_0 \ast U_k$ as a representative of $F/G_0$. The corresponding Hamiltonian is

$$H(p, q, t) = \begin{cases} 2k\pi(p_1^2 + q_1^2) & (0 \leq t < 1/2), \\
H_0(p, q, 2t - 1) & (1/2 \leq t \leq 1)
\end{cases} \quad (196)$$

Here, $H_0$ is the Hamiltonian corresponding to $V_0$. 

Figure 6: Normal form of $V$
### Table 4: The absolute values of the functional determinants, diagonal elements of $D$, and the Maslov indices for the normal forms. $m$ runs over all integers in elliptic and loxodromic cases. In hyperbolic and parabolic cases, $m \geq 1$, and in inverse hyperbolic case, $m \geq 0$.

| stability          | $\lambda$          | diagonal elements                                      | $|\text{Det } D|$ | index |
|--------------------|---------------------|--------------------------------------------------------|------------------|-------|
| elliptic           | $e^{\pm i\alpha}$   | $-\alpha + 2m\pi$                                      | $4\sin^2 \alpha/2$ | 1     |
| hyperbolic         | $e^{\pm \beta}$     | $\pm \beta, \pm \sqrt{\beta^2 + (2m\pi)^2}$          | $4\sinh^2 \beta/2$ | 0     |
| inverse hyperbolic | $-e^{\pm \beta}$    | $\pm \sqrt{\beta^2 + (2m + 1)^2\pi^2}$                | $4\cosh^2 \beta/2$ | 1     |
| loxodromic         | $e^{\pm i\alpha \pm \beta}$ | $\pm \sqrt{\beta^2 + (\alpha + 2m\pi)^2}$ | $4(\cosh \beta - \cos \alpha)^2$ | 0     |
| parabolic          | 1                   | $0, -\gamma, -\gamma/2 \pm \sqrt{(\gamma/2)^2 + (2m\pi)^2}$ | $0 (\text{Det } D = -\gamma)$ | $(1/2)\text{sgn }\gamma$ |

3.6 Absolute values and Maslov indices of quadratic path integrals

We can reduce a curve $V$ to the normal form $V_0 * U_k$ by means of a gauge transformation in $G_0$. The absolute value and the phase (Maslov index) of the path integral are the invariants of this transformation, and we can easily calculate the path integral of this normal form.

The absolute value of the path integral corresponding to $V_0 * U_k$ is the same as that of $V_0$, and the Maslov index of $V_0 * U_k$ is $(\text{index of } V_0) + 2k$. Since $V_0$ is a direct sum of the normal forms shown in 3.5.2, the calculation of the path integral is reduced to the calculation of the path integrals of the normal forms. We summarize the results of the calculations in Table 4. For details of the calculations, see appendix B.

The functional determinant corresponding to $V_0$ is the product of the functional determinant corresponding to the normal forms, and the Maslov index of $V_0$ is the sum of the Maslov index of the normal forms and $2k$. Therefore, if the monodromy matrix has no degenerate eigenvalue, we obtain the formula

$$|\text{Det } D| = |\det(M - I)|,$$

$$\mu = p + q + 2k;$$

where $p$ ($q$) is the number of elliptic (inverse hyperbolic) pairs of eigenvalues. The partition function become

$$Z = \int \mathcal{D}x \exp \left[ \frac{i}{2} x^T D x \right],$$

$$e^{-i\frac{\pi}{2} \mu}$$

$$\sqrt{|\det(M - I)|}.$$
If the monodromy matrix has parabolic blocks, we must add \( \sum_i \frac{1}{2} \text{sgn} \gamma_i \) to the Maslov index. In this case, \( \mathcal{D} \) has zero-modes, and \( \text{Det} \mathcal{D} = 0 \). In the space where the zero-modes are removed, the functional determinant become

\[
|\text{Det}' \mathcal{D}| = \sqrt{\left| \det(M' - I) \right|} \sqrt{\prod_i |\gamma_i|}. \quad (201)
\]

Here, \( \text{Det}' \) denotes the functional determinant in the space without zero-modes, and \( M' \) is the monodromy matrix whose parabolic parts are removed. Integration with respect to zero-modes will be discussed in §4.1.

### 3.7 Repetitions of an orbit

Let us discuss repetitions of an orbit. This is the problem of calculating the partition function

\[
Z(n) = \text{Tr} \hat{U}(T = n) = \text{Tr}\{\hat{U}(T = 1)^n\}. \quad (202)
\]

If the classical time evolution operator corresponding to \( Z(1) \) is \( V \), that corresponding to \( Z(n) \) is \( V^n = V \ast V \ast \cdots \ast V \). Let the normal form of \( V \) be \( V_0 \ast U_k \). Then we can transform \((V_0 \ast U_k)^n\) into \( V_0^n \ast U_{kn} \) continuously. (See Fig. 3.) Therefore the problem is reduced to the calculation of the repetition of the normal forms.

The Maslov index of the normal forms of hyperbolic, loxodromic and parabolic type are unchanged by the repetition. For elliptic type, the winding number increases when \( n\alpha \) goes over \( 2\pi \), where \( n \) is the number of the repetition. Therefore the Maslov index of this part is \( 1 + \left\lfloor n\alpha / 2\pi \right\rfloor \). The normal form of inverse hyperbolic type can be written as \( V_h \ast U_{\pm} \), where \( V_h \) is the normal form of hyperbolic type. Therefore \( n \)-repetition of this normal form can be transformed to \( V_h^n \ast U_{\pm} \). If \( n \) is even, this is hyperbolic, and if \( n \) is odd, this is inverse hyperbolic type. The Maslov index of this part is \( n \).

Thus we obtain the formula for the Maslov index of the \( n \)-repetition:

\[
\mu_n = \sum_{i=1}^p \left( 1 + 2 \left\lfloor \frac{n\alpha_i}{2\pi} \right\rfloor \right) + qn + 2nk. \quad (203)
\]

Here, \( \alpha_i \) denotes the stability angle of \( i \)-th elliptic block. Note that \( \mu_n \neq n\mu_1 \) if the orbit has elliptic blocks.
Figure 7: $(V_0 * U_k)^n$ can be transformed into $V_0^n * U_{2kn}$.
4 Trace of the resolvent operator

In this section, we discuss the trace of the resolvent operator

\[ g(E) = \text{Tr} \frac{1}{E - \hat{H}}, \quad (204) \]

to discuss eigenvalues of the time-independent Hamiltonian \( \hat{H} \). This is obtained by Fourier transformation of the partition function:

\[ g(E) = \frac{1}{\text{i} \hbar} \int_0^\infty e^{\text{i} E T / \hbar} Z(T). \quad (205) \]

We can find eigenvalues of \( \hat{H} \) from the poles of \( g(E) \).

\( Z(T) \) become a sum over periodic orbits, as discussed in the previous sections. In this case, each orbit has at least one zero-mode corresponding to time-translation symmetry. Therefore we first discuss the integration with respect to zero-modes in §4.1. Then we carry out Fourier transformation in §4.2, and finally we give a formula for the sum over repetitions of a orbit in §4.3.

4.1 Integration with respect to zero-modes

If the system has continuous symmetries, \( \mathcal{P} \) has zero-modes and we must execute the integration with respect to them. In our formalism based on the path integral representation of the partition function, it is easy to treat this integration because the integrand \( (-\text{i} \det' \mathcal{P})^{-1/2} \exp \left[ \frac{\text{i}}{\hbar} R \right] \) is a constant. Therefore the contribution from a continuous family of orbits is

\[ \frac{V}{\sqrt{-\text{i} \det' \mathcal{P}}} \exp \left[ \frac{\text{i}}{\hbar} R \right], \quad (206) \]

where \( V \) is a volume factor of the family of orbits.

Hereafter, we consider an isolated orbits in a time-independent Hamiltonian system. In this case, the orbit has a parabolic block corresponding to the time translation symmetry.

The parabolic block become

\[ m = \begin{pmatrix} 1 & 0 \\ \frac{\text{dT}}{\text{dE}} & 1 \end{pmatrix}, \quad (207) \]

Note that the notation of the vector in phase space is different from \[\text{§3}\]. In this paper, we note \( p \) (or \( E \) in this case) first, and \( q \) (or \( t \)) second. This is opposite of the notation in \[\text{§3}\].

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which is conjugate to
\[
\begin{pmatrix}
1 & \frac{dT}{dE} \\
0 & 1
\end{pmatrix}
\]  
(208)

Therefore the contribution from this block is
\[
\frac{T}{\sqrt{2\pi \hbar \frac{1}{2} \frac{dT}{dE}}},
\]  
(209)

(See appendix B) and the amplitude factor of this orbit is
\[
K = \frac{T e^{-i\frac{1}{2} \mu}}{\sqrt{2\pi \hbar \frac{dE}{dT} \sqrt{|\det(M' - I)|}}}. 
\]  
(210)

Here, \(M'\) denotes the monodromy matrix whose parabolic part is removed, and the Maslov index \(\mu\) become
\[
\mu = p + q + 2k - \frac{1}{2} \text{sgn} \left( \frac{dT}{dE} \right). 
\]  
(211)

### 4.2 Conversion from \(T\) to \(E\)

Let us discuss the conversion from a fixed period \(T\) to a fixed energy \(E\). The partition function for a fixed \(T\) is approximated by a sum over periodic orbits as
\[
Z(T) = \sum_{\text{p.o.}} K \exp \left[ \frac{i}{\hbar} \oint p\,dq - H\,dt \right], 
\]  
(212)

where the amplitude factor \(K\) is given in (210). The trace of the resolvent \(g(E)\) is the Fourier transformation of \(Z(T)\):
\[
g(E) = \int_0^\infty dT \ K \exp \left[ \frac{i}{\hbar} \left\{ ET + \oint (p\,dq - H\,dt) \right\} \right]. 
\]  
(213)

Let us evaluate this integral by the stationary phase approximation. The stationary phase condition
\[
\frac{d}{dT} \left\{ ET + \oint (p\,dq - H\,dt) \right\} = 0 
\]  
(214)

leads to
\[
H(p(t), q(t)) = E. 
\]  
(215)

The second derivative of the phase is
\[
\frac{d^2}{dT^2} \left\{ ET + \oint (p\,dq - H\,dt) \right\} = -\frac{dE}{dT}, 
\]  
(216)
and the contribution from this factor cancel the contribution from the parabolic block (207). Therefore $g$ is approximated as

$$g(E) = \sum_{p.o.} K' \exp \left( \frac{i}{\hbar} S \right),$$  \hspace{1cm} (217)

$$S = \oint p\,dq,$$  \hspace{1cm} (218)

$$K' = \frac{Te^{-i\frac{\pi}{2} \mu'}}{\sqrt{|\det(M' - I)|}},$$  \hspace{1cm} (219)

$$\mu' = p + q + 2k.$$  \hspace{1cm} (220)

This is the well-known Gutzwiller’s trace formula.

### 4.3 Sum over repetitions

#### 4.3.1 One dimensional case

Let us discuss the sum over repetitions of a orbit using the formula (203). First, we consider the one dimensional case. In this case, there is only one type of periodic orbit. The monodromy Matrix has only one block corresponds to time translation symmetry, and the winding number of the orbit is 1. (See Fig. 8.) Therefore the Maslov index of the prime periodic orbit is 2 and the index of the $n$-repetition is $2n$.

The trace of the resolvent $g$ is approximated as

$$g(E) = g_0(E) + \sum_{n=1}^{\infty} T \exp \left( n \left( \frac{i}{\hbar} S(E) - i\pi \right) \right),$$  \hspace{1cm} (221)

$$= g_0(E) + T \frac{\exp i[S(E)/\hbar - \pi]}{1 - \exp i[S(E)/\hbar - \pi]}.$$  \hspace{1cm} (222)

Here, $g_0(E)$ is the Thomas-Fermi term which represents the contribution of the zero-length orbit. $T$ and $S$ are the period and the action of the prime periodic orbit.

Since the poles of $g(E)$ represent the eigenvalues of the Hamiltonian, we obtain the Bohr-Sommerfeld quantization condition:

$$S(E) = \oint p\,dq = (2n + 1)\pi\hbar.$$  \hspace{1cm} (223)
Figure 8: The periodic orbit of the one-dimensional system. The local coordinate around the orbit rotate once in the phase space when the orbit goes around the course once.

4.3.2 Elliptic orbits

Let us consider the periodic orbits where all eigenvalues of the monodromy matrix is elliptic except the parabolic block corresponds to the time-translation symmetry. In this case,

\[ |\det(M' - I)| = \prod_{j=1}^{p} 2\sin\frac{\alpha_j}{2}, \]  

(224)

where \( p \) is the number of elliptic blocks, and \( \alpha_j \) is the stability angle of the \( j \)-th elliptic block. The Maslov index is

\[ \mu = p + 2k, \]

(225)

where \( k \) is the winding number. Hence the amplitude factor of this orbit is

\[ K = \frac{(-1)^k T}{\prod_{j=1}^{p} 2i \sin\frac{\alpha_j}{2}}. \]

(226)

From the formula (203), we obtain

\[ \mu_n = \sum_{j=1}^{p} \left( 1 + 2 \left[ \frac{n\alpha_j}{2\pi} \right] \right) + 2kn \]

(227)

and

\[ K_n = \frac{(-1)^k T}{\prod_{j=1}^{p} 2i \sin\frac{n\alpha_j}{2}}. \]

(228)
for \( n \)-repetition of this orbit.

Therefore the sum over repetitions of this orbit is

\[
\sum_{n=1}^{\infty} K_n \exp \left( \frac{i}{\hbar} n S(E) \right) = T \sum_{n=1}^{\infty} (-1)^{kn} \left( \prod_{j=1}^{p} \sum_{m_j=0}^{n} e^{-i(m_j + \frac{1}{2})n \alpha_j} \right) \exp \left( \frac{i}{\hbar} n S(E) \right)
\]

\[= T \sum_{n=1}^{\infty} \sum_{\{m_j\}} \exp \left( \frac{in}{\hbar} \right) S(E) - \left\{ k \pi + \sum_{j=1}^{p} \left( m_j + \frac{1}{2} \right) \alpha_j \right\} \hbar \]

\[= T \sum_{\{m_j\}} \frac{\exp \left( \frac{i}{\hbar} \right) S(E) - \left\{ k \pi + \sum_{j=1}^{p} \left( m_j + \frac{1}{2} \right) \alpha_j \right\} \hbar}{1 - \exp \left( \frac{i}{\hbar} \right) S(E) - \left\{ k \pi + \sum_{j=1}^{p} \left( m_j + \frac{1}{2} \right) \alpha_j \right\} \hbar}
\]

Here, we used

\[
\frac{1}{\sin \frac{x}{2}} = 2i \sum_{m=0}^{\infty} e^{-i(m + \frac{1}{2})x}.
\]

Therefore the quantization condition is

\[
S(E) = \left\{ (2m_0 + k)\pi + \sum_{j=1}^{n-1} \left( m_j + \frac{1}{2} \right) \alpha_j \right\} \hbar,
\]

where \( n \) is an arbitrary integer and \( m_j \) is an arbitrary integer which is not negative. Note that the quantization condition depends on if the winding number \( k \) is even or odd.

### 4.3.3 Unstable orbits

Let the monodromy matrix of a periodic orbit have \( p \) elliptic blocks, \( q \) hyperbolic blocks, \( r \) inverse hyperbolic blocks, \( s \) loxodromic blocks and a parabolic block corresponding to the time translation symmetry. Then the eigenvalues of the monodromy matrix can be written as \( \exp(\pm i\zeta) \) using the stability angle \( \zeta \), and \( \zeta \) is represented as

\[
\zeta_{e,j_e} = \alpha_{e,j_e} \quad (1 \leq j_e \leq p),
\]

\[
\zeta_{h,j_h} = i\beta_{h,j_h} \quad (1 \leq j_h \leq q),
\]

\[
\zeta_{h,j_h} = i\beta_{h,j_h} + \pi \quad (1 \leq j_h \leq r),
\]

\[
\zeta_{l,j_l} = \pm \alpha_{l,j_l} + i\beta_{l,j_l} \quad (1 \leq j_l \leq s),
\]

\[
\zeta_p = 1.
\]

Here, \( 0 < \alpha_{e,j_e} < 2\pi \) and \( 0 < \alpha_{l,j_l} < \pi \). \( \beta_{h,j_h} \), \( \beta_{h,j_l} \), and \( \beta_{l,j_l} \) are positive. \( p + q + r + 2s + 1 = n \) where \( 2n \) is the dimension of the phase space.
The amplitude factors of this orbit and its repetitions are
\[ K_n = (-1)^{kn} T K_{e,n} K_{h,n} K_{ih,n} K_{l,n}, \]
where
\[ K_{e,n} = \frac{1}{\prod_{j=1}^{p} 2i \sin \frac{n \alpha_j}{2}}, \]
\[ K_{h,n} = \frac{1}{\prod_{j=1}^{q} 2 \sinh \frac{n \beta_j}{2}}, \]
\[ K_{ih,n} = \begin{cases} \frac{(-1)^{(n-1)/2}}{\prod_{j=1}^{r} 2 \sinh \frac{n \beta_{ih,j}}{2}} & (n: \text{odd}) \\ \frac{(-1)^{n/2}}{\prod_{j=1}^{r} 2 \sinh \frac{n \beta_{ih,j}}{2}} & (n: \text{even}) \end{cases}, \]
\[ K_{l,n} = \frac{1}{\prod_{j=1}^{s} 2(\cosh n \beta_{l,j} - \cos n \alpha_{l,j})}. \]
The sum over repetitions of this orbit can be calculated in the same way as elliptic orbits, and the result is
\[ \sum_{n=1}^{\infty} K_n \exp \left[ \frac{i}{\hbar} n S(E) \right] = T \sum_{\mathbf{m}} \exp \left[ \frac{\pi}{\hbar} S_c(E, \mathbf{m}) \right] \frac{1}{1 - \exp \left[ \frac{\pi}{\hbar} S_c(E, \mathbf{m}) \right]}, \]
where
\[ S_c(E, \mathbf{m}) = S(E) - \left\{ k \pi + \sum_{j=1}^{n-1} \left( m_j + \frac{1}{2} \right) \zeta_j \right\} \hbar, \]
Here, \( \mathbf{m} = (m_1, m_2, ..., m_{n-1}) \) is the \( n-1 (= p+q+r+2s) \) dimensional vector whose components are non-negative integers. \( \zeta_j \) represents one of stability angles in Eqs. (234)-(237). Therefore the quantization condition is
\[ S(E) = \left\{ (2m_0 + k) \pi + \sum_{j=1}^{n-1} \left( m_j + \frac{1}{2} \right) \zeta_j \right\} \hbar, \]
where \( m_0 \) is an arbitrary integer. If some of stability angles have imaginary parts, RHS of (246) have imaginary parts and eigenvalues become complex numbers.
In chaotic systems, most of the periodic orbits are unstable and the eigenvalues determined by the condition (246) also have imaginary parts. Such phenomena occur even if the Hamiltonian of the system is a Hermite operator and leads to false results.
This means that, for unstable orbits, it is not enough to consider only one orbit and its repetitions. We should take into account the correlations of unstable orbits. This is a difficult but interesting future problem.
5 Conclusion

We have investigated the semiclassical trace formula using phase space path integral. We derived the semiclassical trace formula by applying the stationary approximation to the phase space path integral of the partition function.

We classified the quadratic path integrals around the periodic orbit. This is equivalent to the classification of connections on the vector bundle over $S^1$ with the structure group $Sp(2n, R)$. However, if the canonical transformation is topologically non-trivial, the Maslov index changes. Therefore we should regard two connections as equivalent only if they are connected by the topologically trivial canonical transformation. The results of the classification by this equivalence relation is shown in §3.5.2.

We defined normal forms of time-dependent quadratic Hamiltonians, and calculated quadratic path integrals. We also derived a formula for the Maslov index of the n-repetition of the orbit in §3.7, and derived the Bohr-Sommerfeld type quantization condition in §4.3.

Throughout this paper, we concentrated on the quadratic path integrals around periodic orbits. However, the analysis of higher order terms is often needed when we investigate realistic systems like nuclei and atomic clusters. Phase space path integrals used in this paper will supply powerful tools for such analysis.

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A Calculations of diagonal elements in the continuum limit

A.1 Diagonalization of $D$

First we consider the following eigenvalue equation:

$$ D x = \left( \frac{d}{dt} + A(t) \right) x = \epsilon x. \quad (247) $$

This equation is easy to solve because the time derivative has diagonal form. Let $\xi_\zeta$ be the eigenvector of the monodromy matrix $M$, and $\zeta$ be the stability angle:

$$ M \xi_\zeta = e^{-i\zeta} \xi_\zeta. \quad (248) $$

Then the solutions of (248) are written as follows:

$$ x_{\zeta,m}(t) = \exp(\epsilon_{\zeta,m} t) V(t) \xi_\zeta, \quad (249) $$

$$ D x_{\zeta,m} = \epsilon_{\zeta,m} x_{\zeta,m}, \quad (250) $$

$$ \epsilon_{\zeta,m} = (\zeta + 2m\pi)i. \quad (251) $$

Here, $V(t)$ is the classical time evolution operator which satisfies

$$ DV(t) = 0. \quad (252) $$

We can calculate $|\text{Det} D| = |\text{Det} M|$ from (251):

$$ |\text{Det} M| = \prod_{l=1}^{n} \prod_{m=-\infty}^{\infty} (2m\pi + \zeta_l)(2m\pi - \zeta_l), \quad (253) $$

$$ = \prod_{l=1}^{n} \prod_{m=-\infty}^{\infty} (2m\pi)^2 - \zeta_l^2, \quad (254) $$

$$ = \prod_{l=1}^{n} \zeta_l^2 \prod_{m=1}^{\infty} (2m\pi)^2 \left\{ 1 - \left( \frac{\zeta_l}{2m\pi} \right)^2 \right\}^2, \quad (255) $$

$$ \sim \prod_{l=1}^{n} \left( 2 \sin \frac{\zeta_l}{2} \right)^2 = |\text{det}(M - I)|. \quad (256) $$

Here, we used the formula (407) and ignored the diverging constant $\prod_{m=1}^{\infty} 2m\pi$. We can see from the calculations in Appendix B that this constant corresponds to

$$ \frac{1}{N} \prod_{m=1}^{N-1} 2 \sin \frac{\pi m}{N} = 1 \quad (257) $$

in the discrete formalism.
A.2 Diagonalization of $D$

In this subsection, we diagonalize $D$ using the basis given in A.1.

A.2.1 Elliptic type

Eigenvalues of the monodromy matrix are

$$Mx = e^{\mp i\alpha}x \quad (0 < \alpha < 2\pi),$$

($x_+ = x^*$). $\alpha$ is taken $0 < \alpha < \pi$ if $[x, Mx] > 0$, and $\pi < \alpha < 2\pi$ if $[x, Mx] < 0$. See also [13] (§42). We rewrite $x_+$ as $x_+ = x_{Re} + ix_{Im}$ where $x_{Re}$ and $x_{Im}$ are real vectors, then we obtain

$$Mx_{Re} = x_{Re}\cos\alpha + x_{Im}\sin\alpha,$$

$$Mx_{Im} = x_{Re}\sin\alpha - x_{Im}\cos\alpha.$$  

The inequality

$$[x_{Re}, x_{Im}] > 0$$

follows from the definition of $\alpha$. We use the normalization condition

$$[x_{Re}, x_{Im}] = 1,$$

$$[x_+, x_-] = -2i.$$

The eigenfunctions of $D$ are

$$x_{\pm, m}(t) = \exp(\epsilon_{\pm, m}t)V(t)x_{\pm},$$

$$\epsilon_{\pm, m} = (\pm\alpha + 2m\pi)i.$$  

We define the real functions $x_{Re,m}, x_{Im,m}$ as

$$x_{Re,m} = \frac{1}{2}(x_{+, m} + x_{-, -m}),$$

$$x_{Im,m} = \frac{1}{2i}(x_{+, m} - x_{-, -m}) \quad (n = 0, \pm1, \pm2, \ldots).$$

These functions satisfy the normalization and the orthogonalization condition:

$$\int_0^1 dt [x_{Re,m}, x_{Im,n}] = \delta_{m,n},$$

$$\int_0^1 dt [x_{Re,m}, x_{Re,n}] = \int_0^1 dt [x_{Im,m}, x_{Im,n}] = 0.$$  

8The product $[x, y]$ means $x^T J y$. 

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Matrix elements of $\mathcal{D}$ are

$$
\int_0^1 dt x^T_{Re,m} \mathcal{D} x_{Re,n} = -(\alpha + 2m\pi)\delta_{m,n},
$$

(270)

$$
\int_0^1 dt x^T_{Im,m} \mathcal{D} x_{Im,n} = -(\alpha + 2m\pi)\delta_{m,n},
$$

(271)

$$
\int_0^1 dt x^T_{Re,m} \mathcal{D} x_{Im,n} = 0.
$$

(272)

Therefore $\mathcal{D}$ is already diagonalized by this basis.

The functional determinant of this part is

$$
\prod_{m=-\infty}^{\infty} (\alpha + 2m\pi)^2 = \left\{ \prod_{m=1}^{\infty} (2m\pi)^4 \right\} \alpha^2 \prod_{m=1}^{\infty} \left\{ 1 - \left( \frac{\alpha}{2m\pi} \right)^2 \right\}
$$

$$
\sim 4\sin^2 \frac{\alpha}{2}.
$$

(273)

(274)

Here we used the formula (407).

**A.2.2 Hyperbolic type**

In this case, the eigenvectors of the monodromy matrix satisfy the equation

$$
Mx_{\pm} = e^{\pm \beta}x_{\pm}
$$

(275)

where $\beta$ is a real number. The normalization condition is taken as

$$
[x_+, x_-] = 1.
$$

(276)

We define $\epsilon_{\pm,m}, x_{\pm,m}(t)$ as before:

$$
x_{\pm,m} = \exp(\epsilon_{\pm,m}t)V(t)x_{\pm},
$$

(277)

$$
\epsilon_{\pm,m} = \mp \beta + 2m\pi i.
$$

(278)

We define real functions $x_{c,\pm,m}, x_{s,\pm,m}$ as follows:

$$
x_{c,\pm,m} = \frac{1}{\sqrt{2}}(x_{\pm,m} + x_{\pm,-m}),
$$

(279)

$$
x_{s,\pm,m} = \frac{1}{i\sqrt{2}}(x_{\pm,m} - x_{\pm,-m}) \quad (m = 1, 2, ...).
$$

(280)

Since $x_{\pm,0}$ are already real, we don’t have to redefine them. Matrix elements which have non-zero values are

$$
\int_0^1 dt x^T_{c,\pm,m} \mathcal{D} x_{c,-m} = \beta,
$$

(281)

47
\[
\int_0^1 dt \ x_{s,+m}^T D x_{c,-m} = -2m\pi, \quad (282)
\]
\[
\int_0^1 dt \ x_{c,+m}^T D x_{s,-m} = 2m\pi, \quad (283)
\]
\[
\int_0^1 dt \ x_{s,+m}^T D x_{c,-m} = \beta, \quad (284)
\]
\[
\int_0^1 dt \ x_{c,0}^T D x_{c,-0} = \beta. \quad (285)
\]

Therefore \( D \) is represented by the matrix \( d_m \) in the space spanned by \( x_{c,\pm m}, x_{s,\pm m} \):

\[
d_m = \begin{pmatrix}
0 & 0 & \beta & 2m\pi \\
0 & -2m\pi & -2m\pi & \beta \\
\beta & -2m\pi & 0 & 0 \\
2m\pi & \beta & 0 & 0
\end{pmatrix}
\quad (m \geq 1). \quad (286)
\]

In the case \( m = 0 \),

\[
d_0 = \begin{pmatrix}
0 & \beta \\
\beta & 0
\end{pmatrix}. \quad (287)
\]

The solutions of the eigenvalue equation \( \det(d_m - \lambda I) = 0 \) are

\[
\lambda = \pm \sqrt{\beta^2 + (2m\pi)^2} \quad (m \geq 1), \quad (288)
\]
\[
\lambda = \pm \beta \quad (m = 0). \quad (289)
\]

The solutions for \( m \geq 1 \) are doubly degenerate.

The functional determinant of this part is (using (408))

\[
\prod_{m=0}^{\infty} \det d_m = \beta^2 \prod_{m=1}^{\infty} \left\{ \beta^2 + (2m\pi)^2 \right\}^2, \quad (290)
\]
\[
\sim 4 \sinh^2 \frac{\beta}{2}. \quad (291)
\]

**A.2.3 Inverse hyperbolic type**

In this case, eigenvalues of the monodromy matrix are negative real numbers:

\[
M \mathbf{x}_\pm = -e^{\pm \beta} \mathbf{x}_\pm. \quad (292)
\]

The normalization condition is taken as

\[
[\mathbf{x}_+, \mathbf{x}_-] = 1. \quad (293)
\]

Eigenvalues of \( D \) are

\[
\epsilon_{\pm,m} = \mp \beta + (2m + 1)\pi i. \quad (294)
\]
We define real vectors $\mathbf{x}_{c,\pm,m}, \mathbf{x}_{s,\pm,m}$ as follows:

$$
\mathbf{x}_{c,\pm,m} = \frac{1}{\sqrt{2}}(\mathbf{x}_{\pm,m} + \mathbf{x}_{\pm,-m-1})
$$  \hspace{1cm} (295)

$$
\mathbf{x}_{s,\pm,m} = \frac{1}{i\sqrt{2}}(\mathbf{x}_{\pm,m} - \mathbf{x}_{\pm,-m-1}) \quad (m = 0, 1, \ldots)
$$  \hspace{1cm} (296)

The matrix elements of $D$ can be calculated in the same way as the hyperbolic case. In the space spanned by $\mathbf{x}_{c,\pm,m}$ and $\mathbf{x}_{s,\pm,m}$, $D$ is represented as

$$
d_m = \begin{pmatrix}
0 & 0 & \beta & (2m + 1)\pi \\
0 & 0 & - (2m + 1)\pi & \beta \\
\beta & - (2m + 1)\pi & 0 & 0 \\
(2m + 1)\pi & \beta & 0 & 0
\end{pmatrix}
$$

(297)

Eigenvalues of $d_m$ are

$$
\lambda = \pm \sqrt{\beta^2 + (2m + 1)^2\pi^2}.
$$  \hspace{1cm} (298)

Each solution is doubly degenerate. The functional determinant of this part is

$$
\prod_{m=0}^{\infty} d_m = \prod_{m=0}^{\infty} \left\{ \beta^2 + (2m + 1)^2\pi^2 \right\}^2
$$

$$
\sim \ 4 \cosh^2 \frac{\beta}{2}.
$$  \hspace{1cm} (300)

Here we used (409).

### A.2.4 Loxodromic type

Eigenvalues of the monodromy matrix are

$$
M \mathbf{x}_{\pm,\pm} = e^{\pm i\alpha \pm \beta} \mathbf{x}_{\pm,\pm} \quad (\mathbf{x}_{\pm,\pm} = \mathbf{x}_{\mp,\mp}).
$$

(301)

Normalization is taken as

$$
[\mathbf{x}_{+,+}, \mathbf{x}_{-,-}] = -2i
$$

(302)

$$
[\mathbf{x}_{+,-}, \mathbf{x}_{-,+}] = -2i
$$

(303)

The eigenvectors of $D$ are

$$
\mathbf{x}_{\pm,\pm,m}(t) = \exp(\epsilon_{\pm,\pm,m}t)V(t)\mathbf{x}_{\pm,\pm},
$$

(304)

where

$$
\epsilon_{\pm,\pm,m} = \pm i\alpha \mp \beta + 2m\pi i.
$$

(305)
We define real basis $x_{Re,\pm,m}, x_{Im,\pm,m}$ as

$$x_{Re,\pm,m} = \frac{1}{2}(x_{+,\pm,m} + x_{-,\pm,m}), \quad (m = 0, \pm 1, \pm 2, \ldots)$$

(306)

$$x_{Im,\pm,m} = \frac{1}{2i}(x_{+,\pm,m} - x_{-,\pm,m})$$

(307)

These vectors are normalized as

$$[x_{Re,+,m}, x_{Im,+,n}] = [x_{Re,+,m}, x_{Im,+,n}] = \delta_{m,n}.$$ 

(308)

Other combinations are equal to zero.

The matrix elements which have non-zero values are

$$\int_0^1 dt \, x_{Re,+}^T \rho \, x_{Re,-} = -(\alpha + 2m\pi),$$

(309)

$$\int_0^1 dt \, x_{Re,+}^T \rho \, x_{Im,-} = \beta,$$

(310)

$$\int_0^1 dt \, x_{Im,+}^T \rho \, x_{Re,-} = -\beta,$$

(311)

$$\int_0^1 dt \, x_{Im,+}^T \rho \, x_{Im,-} = -(\alpha + 2m\pi).$$

(312)

Therefore $\rho$ is represented in the space spanned by $x_{Re,\pm,m}, x_{Im,\pm,m}$ as

$$d_m = \begin{pmatrix}
0 & 0 & -(\alpha + 2m\pi) & \beta \\
0 & -\beta & -(\alpha + 2m\pi) & 0 \\
-(\alpha + 2m\pi) & -\beta & 0 & 0 \\
\beta & -(\alpha + 2m\pi) & 0 & 0
\end{pmatrix}.$$ 

(313)

Eigenvalues of $d_m$ are

$$\lambda = \pm \sqrt{\beta^2 + (\alpha + 2m\pi)^2}.$$ 

(314)

Since the determinant of $d_m$ is $\{\beta^2 + (\alpha + 2m\pi)^2\}^2$, the functional determinant of this part is

$$\prod_{m=-\infty}^{\infty} \{\beta^2 + (\alpha + 2m\pi)^2\}^2 = \prod_{m=-\infty}^{\infty} (\alpha + 2m\pi)^4 \prod_{m=-\infty}^{\infty} \left\{1 + \frac{\beta^2}{(\alpha + 2m\pi)^2}\right\}^2$$

$$\sim 4(\cosh \beta - \cos \alpha)^2.$$ 

(315)

(316)

Here, we used the formula (407) and (410).
A.2.5 Parabolic type

If there are eigenvalues of the monodromy matrix which are equal to 1, the operator $\mathcal{D}$ has zero-modes.

Here, we treat the most important case, the parabolic type. In this case, there are bases $x_1, x_2$ which satisfy
\begin{align*}
M x_1 &= x_1, \quad \text{(317)} \\
M x_2 &= x_2 - \gamma x_1. \quad \text{(318)}
\end{align*}

Here, the normalization is taken as
\[ [x_1, x_2] = 1. \quad \text{(319)} \]

The monodromy matrix is represented in this space as
\[ \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}. \quad \text{(320)} \]

We define a matrix $B$ which satisfies
\begin{align*}
B x_1 &= 0, \quad \text{(321)} \\
B x_2 &= -\gamma x_1. \quad \text{(322)}
\end{align*}

This is a generator of $M$. ($e^B = M$.)

We can diagonalize $\mathcal{D}$ using these bases. $x_{1,m}, x_{2,m}$ are defined as
\[ x_{\alpha,m}(t) = \exp(2m\pi it)V(t)\exp(-Bt)x_{\alpha} \quad (\alpha = 1, 2). \quad \text{(323)} \]

$D$ acts on these bases as
\begin{align*}
D x_{1,m} &= 2m\pi i x_{1,m}, \quad \text{(324)} \\
D x_{2,m} &= 2m\pi i x_{2,m} + \gamma x_{1,m}. \quad \text{(325)}
\end{align*}

We define real bases $x_{c,\alpha,m}, x_{s,\alpha,m}$ as
\begin{align*}
&\begin{aligned}
x_{c,\alpha,m} &= \frac{1}{\sqrt{2}}(x_{\alpha,m} + x_{\alpha,-m}) \\
x_{s,\alpha,m} &= \frac{1}{i\sqrt{2}}(x_{\alpha,m} - x_{\alpha,-m})
\end{aligned} \quad \text{(326)} \\
&\begin{aligned}
x_{1,0} \text{ and } x_{2,0} \text{ are already real.}
\end{aligned}
\end{align*}

Then the matrix element which is not equal zero are
\[ \int_0^1 dt \ x_{c,1,m}^T \mathcal{D} x_{s,2,m} = 2m\pi, \quad \text{(328)} \]
\[
\int_0^1 dt \, x_{s,1,m}^T D x_{c,2,m} = -2m\pi, \quad (329)
\]
\[
\int_0^1 dt \, x_{c,2,m}^T D x_{c,2,m} = -\gamma, \quad (330)
\]
\[
\int_0^1 dt \, x_{s,2,m}^T D x_{s,2,m} = -\gamma. \quad (331)
\]

Therefore, for \( m \geq 1 \),

\[
d_m = \begin{pmatrix}
0 & 0 & 0 & 2m\pi \\
0 & 0 & -2m\pi & 0 \\
0 & -2m\pi & -\gamma & 0 \\
2m\pi & 0 & 0 & -\gamma
\end{pmatrix},
\]

\[
det d_m = (2m\pi)^4. \quad (332)
\]

Eigenvalues are

\[
\lambda = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 + (2m\pi)^2}. \quad (333)
\]

Each solution is doubly degenerate.

For \( m = 0 \),

\[
d_0 = \begin{pmatrix}
0 & 0 \\
0 & -\gamma
\end{pmatrix}
\]

The eigenvalues of this part is

\[
\lambda = -\gamma, 0. \quad (334)
\]

Since \( D \) has a zero-mode,

\[
\text{Det} D = 0. \quad (335)
\]

The functional determinant in the space where the zero-mode is removed is

\[
det' D = -\gamma. \quad (336)
\]

The zero-mode should be integrated separately.
B Direct calculations of some path integrals

B.1 Elliptic type

Let us consider the normal form of the Hamiltonian of elliptic type:

\[ H = \frac{\alpha}{2} (p^2 + q^2). \]  

(339)

The path integral for this Hamiltonian can be calculated directly. The eigenvalue equation of \( \mathcal{D}/N \) is

\[- \frac{\alpha}{N} p_j + q_j - q_{j-1} = \epsilon p_j, \]

(340)

\[ p_j - p_{j+1} - \frac{\alpha}{4N} (q_{j-1} + 2q_j + q_{j+1}) = \epsilon q_j. \]

(341)

Since the Hamiltonian is time-independent, we assume that the solutions can be written as

\[ \begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \exp \left( i \frac{2\pi r}{N} \right) \quad (r = 0, 1, ..., N - 1). \]

(342)

Then the eigenvalue equation becomes

\[ \begin{pmatrix} - \frac{\alpha}{N} & 1 - e^{-i \frac{2\pi r}{N}} \\ 1 - e^{i \frac{2\pi r}{N}} & - \frac{\alpha}{4N} (2 + e^{i \frac{2\pi r}{N}} + e^{-i \frac{2\pi r}{N}}) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \epsilon \begin{pmatrix} p \\ q \end{pmatrix}. \]

(343)

\( \epsilon \) satisfies the following quadratic equation:

\[ \epsilon^2 + \frac{\alpha}{N} \left( 1 + \cos^2 \frac{2\pi r}{N} \right) \epsilon + \left( \frac{\alpha}{N} \right)^2 \cos^2 \frac{2\pi r}{N} - 4 \sin^2 \frac{2\pi r}{N} = 0. \]

(344)

We can find all eigenvalues by solving this equation.

Let us calculate \( \det \mathcal{D}/N \).

\[
\det \mathcal{D}/N = \prod_{r=0}^{N-1} \left\{ \left( \frac{\alpha}{N} \right)^2 \cos^2 \frac{2\pi r}{N} - 4 \sin^2 \frac{2\pi r}{N} \right\},
\]

(345)

\[
= \left( \frac{\alpha}{N} \right)^2 \prod_{r=1}^{N-1} \left( -4 \sin^2 \frac{2\pi r}{N} \right) \prod_{r=1}^{N-1} \left( 1 - \frac{\alpha^2}{4N^2} \cot^2 \frac{2\pi r}{N} \right).
\]

(346)

The first part of the product can be calculated using the formula (114):

\[
\prod_{r=1}^{N-1} \sin \frac{2\pi r}{N} = \frac{N}{2^{N-1}}.
\]

(347)
For the second part, if \( N \) is large enough,
\[
\prod_{r=1}^{N-1} \left( 1 - \frac{\alpha^2}{4N^2} \cot^2 \frac{\pi r}{N} \right) = \prod_{r=1}^{N-1} \left( 1 - \frac{\alpha^2}{4N^2} \cot^2 \frac{\pi r}{N} \right)^2 ,
\]
\( N \to \infty \implies \prod_{r=1}^{\infty} \left( 1 - \left( \frac{\alpha}{2\pi r} \right)^2 \right)^2 . \quad (349)

We can calculate this infinite product using the formula (407):
\[
\prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2} \right) = \frac{\sin \pi x}{\pi x} .
\]
Therefore
\[
\lim_{N \to \infty} |\det \mathcal{D}_N| = 4 \sin^2 \frac{\alpha}{2} . \quad (351)
\]

We can calculate the Maslov index using the quadratic equation (344). If
\[
\left( \frac{\alpha}{N} \right)^2 \cos^2 \frac{\pi r}{N} - 4 \sin^2 \frac{\pi r}{N} < 0 ,
\]
the solutions of the quadratic equation have opposite sign. Therefore this part doesn’t contribute to the Maslov index. The problem is the case where
\[
\left( \frac{\alpha}{N} \right)^2 \cos^2 \frac{\pi r}{N} - 4 \sin^2 \frac{\pi r}{N} > 0 . \quad (353)
\]
In this case, the solutions of the quadratic equation have the same sign. If \( \alpha \) is positive (negative), the coefficient of the linear term in (344) is also positive (negative) and both solutions are negative (positive). Therefore such pair contribute +1 (-1) to the Maslov index.

(354) can be rewritten as
\[
\left( \frac{\alpha}{N} \right)^2 \left( 1 - 4 \left( \frac{N}{\alpha} \right)^2 \tan^2 \frac{\pi r}{N} \right) > 0 \quad (0 \leq r \leq N - 1) . \quad (354)
\]
r = 0 always satisfies this inequality, and if \( r \neq 0 \) satisfies (354), \( N - r \) also satisfies (354). If \( N \) is large enough and \( r \leq N/2 \), (354) is reduced to
\[
1 - \left( \frac{2\pi r}{\alpha} \right)^2 > 0 . \quad (355)
\]
The number of \( r(\neq 0) \) satisfying this inequality is \( \left[ \frac{\alpha}{2\pi} \right] \). Therefore, the Maslov index is
\[
\mu = 1 + 2 \left[ \frac{\alpha}{2\pi} \right] . \quad (356)
\]

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B.2 Hyperbolic type

The normal form of the Hamiltonian of the hyperbolic type is

\[ H = -\beta pq. \]  

(357)

This case can be treated in the similar way to the elliptic type. The eigenvalue equation is

\[
\begin{pmatrix}
0 & 1 - e^{-\frac{2\pi}{N}} \sin^2 \frac{\pi r}{N} \\
1 - e^{\frac{2\pi}{N}} + \frac{\beta}{2N}(1 + e^{-\frac{2\pi}{N}}) & 0
\end{pmatrix}
\begin{pmatrix}
p \\
q
\end{pmatrix}
= \epsilon
\begin{pmatrix}
p \\
q
\end{pmatrix},
\]

\[ \epsilon^2 - \left( \frac{\beta^2}{N^2} \cos^2 \frac{\pi r}{N} + 4 \sin^2 \frac{\pi r}{N} \right) = 0. \]  

(358)

(359)

Therefore

\[
|\det D_N| = \prod_{r=0}^{N-1} \left( \frac{\beta}{N} \right)^2 \cos^2 \frac{\pi r}{N} + 4 \sin^2 \frac{\pi r}{N},
\]

\[ = \left( \frac{\beta}{N} \right)^2 \prod_{r=1}^{N-1} \left( 4 \sin^2 \frac{\pi r}{N} \right) \prod_{r=1}^{N-1} \left( 1 + \frac{\beta^2}{4N^2} \cot^2 \frac{\pi r}{N} \right).
\]

(360)

(361)

\[
\lim_{N \to \infty} |\det D_N| = \beta^2 \prod_{r=1}^{\infty} \left( 1 + \left( \frac{\beta}{2\pi r} \right)^2 \right)^2,
\]

\[ = 4 \sinh^2 \frac{\beta}{2}.
\]  

(362)

(363)

Since the two solutions of (375) have always opposite signs, the Maslov index of this part is 0.

B.3 Inverse hyperbolic type

In this case, the normal form of \( V \) is

\[
V(t) = \begin{cases}
\cos 2\pi t & -\sin 2\pi t \\
\sin 2\pi t & \cos 2\pi t \\
-e^{\beta(2t-1)} & 0 \\
0 & -e^{-\beta(2t-1)}
\end{cases}
\begin{pmatrix}
0 \leq t < 1/2, \\
1/2 \leq t \leq 1
\end{cases}
\]

(364)

We calculate the trace of the quantum time evolution operator corresponding to this \( V \):

\[ Z = \text{Tr} M (V(t = 1)), \]

\[ = \text{Tr} \{M (V(1, 1/2)) M (V(1/2, 0))\}. \]

(365)

(366)
Here, \( V(t_1, t_2) \) denotes the classical propagator from \( t = t_1 \) to \( t = t_2 \). Let us evaluate this trace using the coherent states \( |z\rangle = |q + ip\rangle \).

\[
Z = \int \frac{dz_1}{\pi} \frac{dz_2}{\pi} \langle z_1 | M(V(1, 1/2)) | z_2 \rangle \langle z_2 | M(V(1/2, 1)) | z_1 \rangle \tag{367}
\]

Since \( \langle z_2 | M(V(1/2, 1)) | z_1 \rangle = \langle -z_2 | z_1 \rangle e^{-i\pi/2} \),

\[
Z = \int \frac{dz}{\pi} \langle z | M(V(1, 1/2)) | -z \rangle. \tag{368}
\]

Therefore to calculate the path integral of the inverse hyperbolic type is equivalent to calculate the path integral of the hyperbolic type with anti-periodic boundary condition. The eigenvalue equation is

\[
q_{j+1} - q_j + \frac{\beta}{2N} (q_{j+1} + q_j) = \epsilon p_{j+1}, \tag{369}
\]

\[
p_j - p_{j+1} + \frac{\beta}{2N} (p_j + p_{j+1}) = \epsilon q_j \quad (1 \leq j \leq N - 1). \tag{370}
\]

\[
q_1 + q_N + \frac{\beta}{2N} (q_1 - q_N) = \epsilon p_1, \tag{371}
\]

\[
p_N + p_1 + \frac{\beta}{2N} (p_N - p_1) = \epsilon q_N \tag{372}
\]

We assume the solutions of eigenvalue equations as

\[
\begin{pmatrix} p_j \\ q_j \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \exp \left( \frac{i(2r+1)\pi}{N} \right) \quad (r = 0, 1, ..., N - 1). \tag{373}
\]

Then the eigenvalue equation becomes

\[
\left( 1 - e^{i\frac{\pi(2r+1)}{N}} + \frac{\beta}{2N} (1 + e^{i\frac{\pi(2r+1)}{N}}) \right) \begin{pmatrix} p \\ q \end{pmatrix} = \epsilon \begin{pmatrix} p \\ q \end{pmatrix},
\]

\[
e^2 - \left( \frac{\beta^2}{N^2} \cos^2 \frac{\pi(2r+1)}{2N} + 4 \sin^2 \frac{\pi(2r+1)}{2N} \right) = 0. \tag{374}
\]

Therefore

\[
| \det D_N | = \prod_{r=0}^{N-1} \left( \frac{\beta}{N} \right)^2 \cos^2 \frac{\pi(2r+1)}{2N} + 4 \sin^2 \frac{\pi(2r+1)}{2N} \right), \tag{375}
\]

\[
= \prod_{r=0}^{N-1} \left( 4 \sin^2 \frac{\pi(2r+1)}{2N} \right) \prod_{r=0}^{N-1} \left( 1 + \frac{\beta^2}{4N^2 \cot^2 \frac{\pi(2r+1)}{2N}} \right).
\]
\[
\lim_{N \to \infty} |\det \mathcal{D}_N| = 4 \prod_{r=0}^{\infty} \left\{ 1 + \left( \frac{\beta}{\pi (2r+1)} \right)^2 \right\}^2, \quad (378)
\]
\[
= 4 \cosh^2 \frac{\beta}{2}. \quad (379)
\]

Here we used (413) and (409). Since the inverse hyperbolic type can be deformed continuously to elliptic type ($\beta \to 0$), the Maslov index of this part is the same as the elliptic type. Therefore Maslov index of this part is 1.

### B.4 Loxodromic type

The normal Hamiltonian of this type is
\[
H = \alpha(p_1q_2 - p_2q_1) - \beta(p_1q_1 + p_2q_2). \quad (380)
\]

The eigenvalue equation is
\[
q_{1,j} - q_{1,j-1} + \frac{\beta}{2N}(q_{1,j} + q_{1,j-1}) - \frac{\alpha}{2N}(q_{2,j} + q_{2,j-1}) = \epsilon p_1, \quad (381)
\]
\[
q_{2,j} - q_{2,j-1} + \frac{\alpha}{2N}(q_{1,j} + q_{1,j-1}) + \frac{\beta}{2N}(q_{2,j} + q_{2,j-1}) = \epsilon p_2, \quad (382)
\]
\[
p_{1,j} - p_{2,j+1} + \frac{\beta}{2N}(p_{1,j} + p_{1,j+1}) + \frac{\alpha}{2N}(p_{2,j} + p_{2,j+1}) = \epsilon q_1, \quad (383)
\]
\[
p_{2,j} - p_{2,j+1} - \frac{\alpha}{2N}(p_{1,j} + p_{1,j+1}) + \frac{\beta}{2N}(p_{2,j} + p_{2,j+1}) = \epsilon q_2. \quad (384)
\]

If we assume the forms of eigenvectors of $\mathcal{D}_N$ as
\[
\begin{pmatrix}
  p_{1,j} \\
  p_{2,j} \\
  q_{1,j} \\
  q_{2,j}
\end{pmatrix} = \begin{pmatrix}
  p_1 \\
  p_2 \\
  q_1 \\
  q_2
\end{pmatrix} \exp \left( \frac{2\pi r}{N} j \right) \quad (r = 0, 1, \ldots, N-1), \quad (385)
\]

the eigenvalue equation become
\[
\begin{pmatrix}
  0 & A_r \\
  A_r^T & 0
\end{pmatrix}
\begin{pmatrix}
  p_1 \\
  p_2 \\
  q_1 \\
  q_2
\end{pmatrix} = \epsilon
\begin{pmatrix}
  p_1 \\
  p_2 \\
  q_1 \\
  q_2
\end{pmatrix}, \quad (386)
\]
\[
A_r = \begin{pmatrix}
  1 - e^{-i \frac{2\pi r}{N}} + \frac{\beta}{2N} (1 + e^{-i \frac{2\pi r}{N}}) & -\frac{\alpha}{2N} (1 + e^{-i \frac{2\pi r}{N}}) \\
  \frac{\alpha}{2N} (1 + e^{-i \frac{2\pi r}{N}}) & 1 - e^{-i \frac{2\pi r}{N}} + \frac{\beta}{2N} (1 + e^{-i \frac{2\pi r}{N}})
\end{pmatrix}, \quad (387)
\]
\[
\det (\epsilon^2 I - A_r A_r^T) = 0. \quad (388)
\]
The explicit form of $A_r A_r^\dagger$ is

$$A_r A_r^\dagger = \begin{pmatrix} \frac{\alpha^2 + \beta^2}{N^2} \cos^2 \frac{\pi r}{N} + 4 \sin^2 \frac{\pi r}{N} & \frac{4i \alpha}{N} \sin \frac{\pi r}{N} \cos \frac{\pi r}{N} \\ -4i \frac{\alpha}{N} \sin \frac{\pi r}{N} \cos \frac{\pi r}{N} & \frac{\alpha^2 + \beta^2}{N^2} \cos \frac{\pi r}{N} + 4 \sin^2 \frac{\pi r}{N} \end{pmatrix}. \tag{389}$$

Therefore the eigenvalue equation becomes

$$\epsilon^4 - 2 \left( \frac{\alpha^2 + \beta^2}{N^2} \cos^2 \frac{\pi r}{N} + 4 \sin^2 \frac{\pi r}{N} \right) \epsilon^2 + \det A_r A_r^\dagger = 0, \tag{390}$$

$$\det A_r A_r^\dagger = \left( 4 \sin^2 \frac{\pi r}{N} + \frac{\beta^2 - \alpha^2}{N^2} \cos^2 \frac{\pi r}{N} \right)^2 + \frac{4 \alpha^2 \beta^2}{N^4} \cos^4 \frac{\pi r}{N}. \tag{391}$$

Hence

$$|\det \Phi_N| = \prod_{r=0}^{N-1} \det A_r A_r^\dagger, \quad \text{(392)}$$

$$= \left( \frac{\alpha^2 + \beta^2}{N^2} \right)^2 \prod_{r=1}^{N-1} \left( 2 \sin \frac{\pi r}{N} \right)^4 \left\{ \left( 1 + \frac{\beta^2 - \alpha^2}{4N^2} \cot^2 \frac{\pi r}{N} \right)^2 + \frac{\alpha^2 \beta^2}{4N^4} \cot^4 \frac{\pi r}{N} \right\}. \tag{393}$$

$$= (\alpha^2 + \beta^2)^2 \prod_{r=1}^{N-1} \left\{ \left( 1 + \frac{\beta^2 - \alpha^2}{4N^2} \cot^2 \frac{\pi r}{N} \right)^2 + \frac{\alpha^2 \beta^2}{4N^4} \cot^4 \frac{\pi r}{N} \right\}. \tag{394}$$

$$\lim_{N \to \infty} |\det \Phi_N| = (\alpha^2 + \beta^2)^2 \prod_{r=1}^{\infty} \left\{ \left( 1 + \frac{\beta^2 - \alpha^2}{(2\pi r)^2} \right)^2 + \frac{(2\alpha \beta)^2}{(2\pi r)^4} \right\}, \tag{395}$$

$$= (2\pi)^4 (\gamma^2 + \delta^2) \prod_{r=1}^{\infty} \left( 1 - \frac{\gamma^2}{r^2} \right)^4 \prod_{r=1}^{\infty} \left( 1 + \frac{\delta^4}{(r^2 - \gamma^2)^2} \right)^2, \tag{396}$$

where

$$\gamma^2 = \frac{\alpha^2 - \beta^2}{(2\pi)^2} \tag{397},$$

$$\delta^2 = \frac{2\alpha \beta}{(2\pi)^2}. \tag{398}$$

Thus we obtain the result

$$|\det \Phi_N| = 4(\cosh \beta - \cos \alpha)^2, \tag{399}$$

using the formulas (407) and (411).

Since the eigenvalue equation (388) always has two positive and two negative eigenvalues, the Maslov index of this part is 0.

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B.5 Parabolic type

The Hamiltonian is

$$H = \frac{\gamma}{2} q^2,$$

and the eigenvalue equation is

$$\begin{pmatrix} 0 & 1 - e^{-i \frac{2\pi}{N}} \\ 1 - e^{i \frac{2\pi}{N}} & \frac{\gamma}{4N} (2 + e^{i \frac{2\pi}{N}} + e^{-i \frac{2\pi}{N}}) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \epsilon \begin{pmatrix} p \\ q \end{pmatrix},$$

$$\epsilon^2 + \frac{\gamma}{N} \cos^2 \frac{\pi r}{N} \epsilon - 4 \sin^2 \frac{\pi r}{N} = 0. \tag{402}$$

If $r \neq 0$, the eigenvalues of (402) are not equal to zero, and the determinant of this part is

$$\prod_{r=1}^{N} \left(-4 \sin^2 \frac{\pi r}{N}\right) = (-1)^{N-1} N^2. \tag{403}$$

Since the two solutions of (402) have opposite signs, these eigenvalues don’t contribute to the Maslov index.

If $r = 0$, the eigenvalues are

$$\epsilon = 0, -\frac{\gamma}{N} \tag{404}$$

and the Maslov index of this part is $\frac{1}{2} \text{sgn} \gamma$. The normalized eigenvector corresponding to these eigenvalues are

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{N}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{405}$$

The volume factor of the zero-mode measured by these vectors are $V_N = \sqrt{NV}$, where $V$ is the volume factor by the normal measure in the phase space. Therefore the contribution from this part is

$$\int \frac{dX_N}{(2\pi\hbar)^N} \exp \left[ \frac{i}{2\hbar} X^T_N \mathcal{D} X_N \right] = \frac{V_N}{\sqrt{2\pi\hbar N\gamma}} = \frac{V}{\sqrt{2\pi\hbar \gamma}}. \tag{406}$$
C Useful formulas

\[ \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x} \]  \hspace{1cm} (407)

\[ \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right) = \frac{\sinh \pi x}{\pi x} \]  \hspace{1cm} (408)

\[ \prod_{n=0}^{\infty} \left(1 + \frac{x^2}{(2n + 1)^2}\right) = \cosh \frac{\pi x}{2} \]  \hspace{1cm} (409)

\[ \prod_{n=-\infty}^{\infty} \left(1 + \frac{x^2}{(a + 2n\pi)^2}\right) = \frac{\cosh x - \cos a}{1 - \cos a} \]  \hspace{1cm} (410)

\[ \prod_{n=1}^{\infty} \left(1 + \frac{x^4}{(n^2 - a^2)^2}\right) = \frac{a^2}{\sqrt{x^4 + a^4}} \frac{\cosh 2I - \cos 2R}{2 \sin^2 \pi a} \]  \hspace{1cm} (411)

\[ R = \text{Re} \pi \sqrt{a^2 + ix^2} \]  \hspace{1cm} (412)

\[ I = \text{Im} \pi \sqrt{a^2 + ix^2} \]  \hspace{1cm} (413)

\[ \prod_{r=1}^{N-1} \sin \frac{\pi r}{N} = \frac{N}{2^{N-1}} \]  \hspace{1cm} (414)

\[ \prod_{r=0}^{N-1} \sin \frac{(2r + 1)\pi}{2N} = \frac{1}{2^{N-1}} \]  \hspace{1cm} (415)