Dissipative Visco-plastic Deformation in Dynamic Fracture:
Tip Blunting and Velocity Selection

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Dynamic fracture in a wide class of materials reveals “fracture energy” $\Gamma$ much larger than the expected nominal surface energy due to the formation of two fresh surfaces. Moreover, the fracture energy depends on the crack velocity, $\Gamma = \Gamma(v)$. We show that a simple dynamical theory of visco-plasticity coupled to asymptotic pure linear-elasticity provides a possible explanation to the above phenomena. The theory predicts tip blunting characterized by a dynamically determined crack tip radius of curvature. In addition, we demonstrate velocity selection for cracks in fixed-grip strip geometry accompanied by the identification of $\Gamma$ and its velocity dependence.

**Introduction:** The dynamics of rapid fracture are still posing a great challenge to theoretical physics \cite{1}. A major difficulty in understanding the rich experimental phenomenology is the coupling between very different scales; while fracture is ultimately driven by the release of the large scales elastic energy, the detailed deformation on the small scales near the moving crack edge determines both the stress conditions and energy dissipation that control the rate of growth and its direction. Since the near crack edge deformation is extremely complicated, it is common to lump all the near crack edge physics into a phenomenological unknown fracture energy, leaving the theory with a crucial missing ingredient. To better understand these issues, consider a semi-infinite crack forming in an infinite strip under fixed-grip boundary conditions. As the crack proceeds at velocity $v$, a constant energy per unit advance per unit time, $J$, is released from the elastic fields. In the classical theory of Linear Elasticity Fracture Mechanics one assumes that $J = v \Gamma$ where $\Gamma$ is the fracture energy, putatively a constant, equal to the energy 2$\gamma_s$ of forming two fresh surfaces. In fact, in a large variety of materials, from steel to brittle plastics, the measured fracture energy $\Gamma$ can be orders of magnitude larger than 2$\gamma_s$. Moreover, in many instances the measured fracture energy depends on the velocity, $\Gamma = \Gamma(v)$. This is important; cracks tend to reach a stationary velocity $v$ that grows as a function of the loading and this is possible only if $\Gamma$ depends on $v$. It is tempting to assert that some kind of plastic processes may be responsible for this kind of findings. Straightforward plasticity theory assumes that the stress fields near the tip of a crack are tamed by plastic deformations and remain at a value close to the yield stress $s_y$ of the material. Alas, this value of stress is assumed to be smaller than the level of stress necessary for overcoming the cohesive forces that bind material together. Without a proper theory of the dynamical processes occurring near the crack tip it remains very difficult to explain all these observations in a coherent and self-consistent manner.

**Theoretical framework:** a promising formulation of visco-plasticity was proposed recently by Falk and Langer \cite{2}. As in all approaches to plasticity, one adds to the usual stress and elastic strain tensors $\sigma_{ij}$ and $\epsilon_{ij}^{el}$, a plastic component of the strain tensor, denoted as a trace-less symmetric tensor $\epsilon_{ij}^{pl}$. Essential to this theory are the highly localized regions called the “shear transformation zones” (STZ). These dominate the plastic events and their presence is carried by two “internal state” fields \cite{3}. One is a trace-less symmetric tensor field $\Delta$ that is related to the orientations of these zones and acts as a “back stress”; the second is a scalar field $\Lambda$ which stands for the density of these STZ. We employ here the tensorial quasi-linear version of this model, which is given by the following three coupled equations \cite{4}:

\[
epsilon_{ij}^{pl} = \frac{1}{\tau} (\lambda s_{ij} - \Delta_{ij}) \quad \text{ (1)}
\]

\[
\Delta_{ij} = \epsilon_{ik}^{pl} \left( \delta_{il} \delta_{jk} - \frac{s_{lk}}{2\lambda s_y^2} \Delta_{ij} \right) \quad \text{ (2)}
\]

\[
\Lambda = \frac{s_{lk} \epsilon_{ik}^{pl}}{2\lambda s_y^2} (1 - \Lambda) \quad \text{ (3)}
\]

Here $s_{ij}$ is the 2D deviatoric stress tensor, $s_{ij} \equiv \sigma_{ij} - \frac{1}{3} \text{Tr} \sigma \delta_{ij}$, $\tau$ is a typical time scale and $\lambda$ is a parameter of dimension inverse stress that measures the plastic strain rate sensitivity to stress. This set of equations and further developments were studied in considerable detail in other contexts \cite{5,6}. In this Letter we study the implications of this model for dynamic fracture. The reader should note that in principle the time scale $\tau$ may depend strongly on $s_{ij}$, maybe even diverging for $|s_{ij}| \ll s_y$. With such divergence one can connect smoothly to elastic behavior far away from the crack tip. In our analysis below we take $\tau$ to be constant, but on the other hand allow the density $\Lambda$ of STZ to decay to its pre-deformation small value far away from the crack tip. In this we differ from previous treatments \cite{6} where $\Lambda$ was taken to be unity. We will show below that the dynamics of $\Lambda$ are interesting and important, in addition to allowing us a smooth coupling to elasticity.

Consider then a semi-infinite crack in a 2D infinite strip of a given width, loaded in fixed-grip boundary conditions such that the energy release rate $G = J/v$ is experimentally controlled \cite{3}. The crack advances steadily at a velocity $v$ (to be found self-consistently). The effect of the plastic response should be strongest near the tip of the crack, while far away from the tip linear elasticity can be safely employed. Like in \cite{3} we expect the crack
tip radius of curvature $R$ to adjust itself dynamically, allowing the crack tip to blunt. The degree of blunting will be connected self-consistently to the steady velocity $v$. While we adopt below a number of approximations, we insist on the conservation of energy in the sense that the stored elastic energy is mainly dissipated near the blunted tip (a feature that appears missing in [8]), though part of it can be locked in the crack wake. Sensitive to the experimental finding that $\Gamma \gg 2\gamma_s$, we will assert that the dominant contribution to $\Gamma$ comes from the plastic work which we assume to include both the deviatoric stress to the elastic part of the strain tensor, and $\Lambda$ approaches the plastic strain rate dominance.

We thus expect $\Lambda$ to deviate from the usual linear-elastic behavior. To quantify this deviation we introduce the visco-plastic analog of an inverse shear modulus. To proceed to our results, we recall a different analysis, namely that $\Lambda_s \ll 1$. Observe Eq. (1), and note that the LHS is of $\lambda$-dependence in $s$ and velocity in units of $\tau$.

An analytically tractable parameter regime is $\lambda_s \ll 1$. We observe here the LHS is of $\mathcal{O}(1)$. The fact that $\lambda_s \ll 1$ renders $s_{ij}/s_y \gg 1$, sending $\Delta s_{ij}$ to be $\ll 1$, according to Eq. (2). It also follows that $\Lambda$ is very rapidly. In this limit the set of equations (1)-(3) are approximated to be equated to the energy release rate $\Gamma$. The two approaches, ours and the one presented in [8], are approximate in nature and even share some of the underlying assumptions, but in fact are very different. Indeed, we use elastic stress solutions inside the plastic zone as in [8] and also demand explicitly energy conservation, but the similarity ends here. We consider the sharp tip assumption unjustified and by asserting that the crack tip advances mainly by plastic flow, arrive at Eq. (3) that has no analog in [8]: as a result the unrealistic stress singularity is cut off. Moreover, in contrast to [8] we assert that the plastic dissipation is dominant. Finally, we incorporate a dynamical rate-and-state formulation of visco-plasticity that is fundamentally different from the phenomenological constitutive law employed in [8].

**Analysis:** To analyze the various regimes of solutions we non-dimensionalize the equations by measuring stress in units of $s_y$, time in units of $\tau$ and velocity in units of $c_s$. Having only one typical time scale, we expect that

$$\frac{vt}{R} \sim \mathcal{O}(1).$$

In light of Eq. (4), sufficiently near the tip the dimensionless plastic strain rate is also of $\mathcal{O}(1)$. In seeking solutions we keep this physical constraint in mind.

- First regime of solutions, apparently not physical

Before proceeding to our results, we recall a different line of investigation taken in [8]. There the near sharp tip stress distribution of Eq. (6), with $h_{ij} = 1$, was used to compute the plastic strain rate using a phenomenological visco-plastic constitutive law. Then the dissipation rate associated with the plastic strain rate was computed and added to an unknown velocity-independent dissipation to be equated to the energy release rate $\Gamma$. The two approaches, ours and the one presented in [8], are approximate in nature and even share some of the underlying assumptions, but in fact are very different. Indeed, we use elastic stress solutions inside the plastic zone as in [8] and also demand explicitly energy conservation, but the similarity ends here. We consider the sharp tip assumption unjustified and by asserting that the crack tip advances mainly by plastic flow, arrive at Eq. (3) that has no analog in [8]: as a result the unrealistic stress singularity is cut off. Moreover, in contrast to [8] we assert that the plastic dissipation is dominant. Finally, we incorporate a dynamical rate-and-state formulation of visco-plasticity that is fundamentally different from the phenomenological constitutive law employed in [8].

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$$\dot{\epsilon}_{ij}^p \sim \frac{\lambda}{\tau} s_{ij}.$$  

Therefore the equations for the components of $\dot{\epsilon}_{ij}^p$ decouple in this limit, identifying $\lambda$, as expected, with the visco-plastic analog of an inverse shear modulus. To proceed, we assert that the predominant $v$-dependence in Eq. (8) is carried by $K_{ij}(v)$. This is justified since $\bar{\Sigma}_{ij}$ are qualitatively the same for velocities below the Yoffe threshold and the $v$-dependence in $h_{ij}$ can be neglected according to [8]. We can substitute now Eq. (8) in Eqs. (4) and (5), and use the approximate $s_{ij}$ of (8) we end up with

$$v \sim \frac{\Lambda G}{\tau}.$$  

$$R \sim \frac{\tau v}{\Lambda E} A_{ij}(v/c_s, \nu) \sim \frac{G}{E} A_{ij}(v(G)/c_s, \nu).$$

Here $A_i$ is a known universal function, and $c_s$, $E$ and $\nu$ are the shear wave speed, Young’s modulus and Poisson’s ratio respectively. Below we approximate the functions $h_{ij}$ in order to integrate Eqs. (1)-(3) and find self-consistently $v$ and $R$ according to Eqs. (10) and (11), respectively.
Note that this solution has interesting aspects: we find a self-consistent steady velocity that increases linearly upon increasing $G$. This implies that $\Gamma(v) \sim v$. This solution is realized self-consistently by having also $R$ increasing with $G$. Since $A_t \rightarrow 1$ with zero slope when $v \rightarrow 0^+$. $R$ starts linearly with $G$ and later turns nonlinear according to the dependence of $A_t$ on $G$. Unfortunately, this solution allows very large stress concentrations near the tip, i.e. $s_{ij}/s_g \gg 1$ and as we are unaware of experimental observations indicating such high stress levels, we propose that the limit $\lambda s_g \ll 1$ is not physically relevant.

- Second regime of solutions, apparently physical

The range of parameters that furnishes an interesting solution is $\lambda s_g \gtrsim 1$. In this range the constitutive equations 1-3 are fully coupled and the “internal state” fields $\Delta$ and $\Lambda$ play an important role in the dynamics. To see this we first note that in order to satisfy Eq. 2 in the near tip region the combination $\Lambda s_{ij}$ should be of $O(1)$. According to Eqs. 1 and 3 this condition can be realized with $\Lambda < 1$ and $s_{ij} \gtrsim s_g$. In that case, according to Eq. 2, the “back stress” $\Delta_{ij}$ is comparable to $\lambda s_{ij}$ and indeed resists plastic deformations. Thus for $\lambda s_g \gtrsim 1$ we expect a solution with $s_{ij} \gtrsim s_g$, where $\Lambda$ does not saturate rapidly and the dynamics of $\Delta$ are important. To shed more light on the nature of this solution and to substantiate the first solution, we consider the equations numerically.

**Numerics:** For the sake of numerical solutions we approximate the solution for $s_{ij}$ by allowing the visco-plastic processes to blunt the tip and then solve approximately for the stress field up to the blunted tip via quasi-static linear elasticity. This approach inevitably violates compatibility, but is expected to yield reasonable approximations as shown in 4. Thus the main effect of the visco-plastic processes on the stress is via the introduction of the length scale $R$ near the tip, hence neglecting all the $v$-dependence except for $K_l(v)$, as was explained above. In that case, outside the crack we solve for the stress field with boundary conditions

$$\sigma_{rr}(r=R, \theta) = \sigma_{\theta \theta}(r=R, \theta) = 0,$$  

for a finite range of angles (say, $-\pi/3 < \theta < \pi/3$). Since $r$ cannot go to zero, we can add to the usual $r^{-1/2}$ solution more “divergent” powers; explicitly, we add terms to $\sigma_{ij}$ of the form

$$K_l(v) \frac{R}{\sqrt{2\pi r^3}} f_{ij}(\theta), \quad K_l(v) \frac{R}{\sqrt{2\pi r^2}} g_{ij}(\theta).$$  

The pre-factors of the new terms can be selected such as to best approximate Eq. 12 in the sense that relative to the non-vanishing component $\sigma_{\theta \theta}$ the ratios $|\sigma_{rr}/\sigma_{\theta \theta}|$ and $|\sigma_{\theta \theta}/\sigma_{\theta \theta}|$ are bounded below 1%. This solution is naturally matched to the usual linear-elastic asymptotic solution far from the tip, while capturing qualitatively the elimination of the common sharp tip singularity near the tip.

At this point we solve Eqs. 1-3 along the line $\theta = 0$. The calculation of the integrals in Eq. 1 should be implemented with care. In contrast to classical theories of plasticity, here there is no sharp boundary between the visco-plastic and elastic solutions: the deviatoric stress $s_{ij}$ and therefore $\epsilon_{ij}^{pl}$ decay algebraically. Thus we should choose a reasonable cutoff for which the integral in our model converges. For that purpose, we compute the integrand of 1 along the line $\theta = 0$ and in particular determine the position $x = w$ at which the plastic work rate $s_{ij} \epsilon_{ij}^{pl}$ reaches 2% of its maximal value at the tip. Using this upper limit of integration the value of the integral becomes insensitive to the long tail of the plastic rate of work. We then estimate the 2D integral as the 1D integral times $\alpha w$, with $\alpha$ being a dimensionless factor representing the full contribution from the loading and unloading regions (where in the latter the model is inapplicable). The model equations exhibit a solution when $\alpha$ is chosen in the range $0.17 - 0.25$. Finally, the 1D integral can be computed over time rather than space (since all the functions depend on $x - vt$). This allows us to integrate Eqs. 3 in time, capturing the essential history dependence of $\epsilon_{ij}^{pl}$ on the internal state fields $\Lambda$ and $\Delta$. At this point, Eqs. 1-3 can be regarded as defining two surfaces in an abstract $v-R$ space. We search graphically for the zero of the intersection line of these surfaces as parametrized by $G$. 

FIG. 1: Solutions for $\lambda s_g = 1$. Upper panel: The tip radius $R$ normalized by the first data point as a function of the normalized velocity $v/c_s$. Lower panel: The fracture energy $\Gamma_y$ normalized by the first data point as a function of the normalized velocity $v/c_s$. 




Following this procedure we solved for $R$ and $v$ for the two ranges of $\lambda s_y$ considered above. First, we analyzed the model with $\lambda s_y = 0.01$, $\nu = 0.3$, $E/s_y = 50$, $\alpha = 0.25$, $\eta^{ij}_y(t=0) = \Delta^{ij}_y(t=0) = 0$, and $\Lambda(t=0) = 0.01$ and compared the numerical solution to the analytic counterpart. We verified that for similar small initial values of $\Lambda$ the results are rather insensitive to the exact value. The numerical solution matches Eqs. (10)-(11) extremely well. Nevertheless, the dimensionless combination $G/s_y R$ in the numerical solution equals $\approx 3 \times 10^4$. Taking PMMA [10] as a representative for the relevant class of materials, this number implies $R \approx 5 \mu m$. This length scale is much smaller than expected for PMMA, thus supporting our proposition that the range $\lambda s_y \lesssim 1$ is not physically relevant. On the other hand, in the parameter range $\lambda s_y \gtrsim 1$ we verified that both “internal state” fields $\Delta$ and $\Lambda$ play an important role in the dynamics and the dependence of both $v$ and $R$ on $G$ is non-linear. The dependence of $R$ and $\Gamma$ on $v$ for $\lambda s_y = 1$ is shown in Fig. 1. Numerically we have found that to a very good approximation

$$\Gamma \propto v A_I(v), \quad R \propto v A_I(v). \quad (14)$$

Since $A_I$ appears in the theory always in the combination $A_I/E$ (cf. Eq. (8)), we can write $\Gamma \propto v s_y A_I(v)/E$, $R \propto v E/s_y A_I(v)$ where we non-dimensionalized $E$ by $s_y$. Rearranging these equations and keeping dimensions right we end up with the implicit relations

$$v = \frac{E G}{\tau s_y^2 A_I(v)} f_v(\lambda s_y), \quad R = \frac{E^2 G}{s_y^3 A_I^2(v)} f_R(\lambda s_y), \quad (15)$$

where $f_v$ and $f_R$ are two dimensionless nonlinear functions computed from the numerical solution, reflecting the importance of the internal state variables $\Lambda$ and $\Delta$ in the solution. Note that the solution for $R$ in (15) differs from the one provided in [8] even in the quasistatic limit, reflecting the difference between the two approaches, mainly in the use here of Eq. (14). In this equation $G$ appears twice, once on the RHS and once through the stress intensity factor on the LHS, thus changing the way that $E$ appears in the final solution.

Finally, the dimensionless combination $G/s_y R$ for this class of solutions is typically of $O(1)$. Applying the analysis to PMMA we estimate $R \approx 15 \mu m$, which is an acceptable estimate for this class of materials. We propose therefore that the solutions in the range $\lambda s_y \gtrsim 1$ are physically relevant for materials in which the main dissipation mechanism is visco-plasticity. The stresses transmitted to the crack tip itself are moderately larger than $s_y$, supposedly sufficient to overcome the cohesive forces binding the material together, and the plastic zones are relatively thin and changing smoothly towards the linear elastic asymptotic fields.

**Summary**: We have employed a theoretical framework of visco-plasticity to propose a direct measure of the fracture energy in dynamic fracture for a class of materials where the fracture energy exceeds significantly the bare surface energy. The fracture energy is a function of the crack velocity, rationalizing the experimental observations of velocity selection for cracks in a fixed-grip strip geometry. This selection is accompanied by the blunting of the tip, with a radius that is self-consistent with the velocity. The theory, within the stated approximations, predicts that both $\Gamma$ and $R$ are monotonically increasing functions of the velocity $v$ in the considered range. In the physical range of parameters ($\lambda s_y \gtrsim 1$) the stress at the tip exceeds the yield-stress $s_y$, affording rupture of the cohesive bonds. In this range of parameters the plastic zone is relatively thin with respect to the tip radius.

The STZ model as used here cannot be considered a final theory of visco-plasticity [3]. A rigorous coupling to elasticity and a consistent description of the wake of the crack are still missing here. Nevertheless we believe that the encouraging approximate results presented here are generic to any reasonable model that addresses the complex state of deformation near a moving crack tip. In future work one should invest effort in improved models where the approach can be tested in some quantitative detail. In such future work the questions of crack tip instabilities that were not addressed here at all should be directly addressed and explained.

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[1] J. Fineberg and M. Marder, Phys. Rep. 313, 1 (1999).
[2] M. L. Falk and J. S. Langer, Phys. Rev. E. 57, 7192 (1998).
[3] J. Lubliner, Plasticity Theory, (Macmillan, New York, 1990).
[4] M. L. Falk and J. S. Langer, MRS Bull. 25, 40 (2000).
[5] J. S. Langer and L. Pechenik, Phys. Rev. E. 68, 061507 (2003). J. S. Langer, Phys. Rev. E. 64, 011504 (2001). L. O. Eastgate, J. S. Langer and L. Pechenik, Phys. Rev. Lett. 90, 045506 (2003).
[6] J. S. Langer, Phys. Rev. E. 62, 1351 (2000).
[7] L. B. Freund, Dynamic Fracture Mechanics, (Cambridge, 1998).
[8] L. B. Freund and J. W. Hutchinson, J. Mech. Phys. Solids 33, 169 (1985).
[9] E. A. Brener and R. Spatschek, Phys. Rev. E. 67, 016112 (2003).
[10] The material parameters used are $E = 2.3 \text{GPa}$, $s_y = 60 \text{MPa}$ and $G \approx 10^3 \text{J/m}^2$. 
