Quantum decay law: critical times
and the equivalence of approaches

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Abstract
Methods based on the use of Green’s functions or Jost functions and
the Fock–Krylov method are apparently very different approaches to understand
the time evolution of unstable states. We show that the two former methods
are equivalent up to some constants and as an outcome find an analytic expression
for the energy density of states in the Fock–Krylov amplitude in terms of
the coefficients introduced in the Green’s functions and the Jost functions
methods. This model-independent density is further used to obtain an analytical
expression for the survival amplitude and study its behaviour at large times.
Using these expressions, we investigate the origin of the oscillatory behaviour
of the decay law in the region of the transition from the exponential to the
non-exponential at large times. With the objective of understanding the failure
of nuclear and particle physics experiments in observing the non-exponential
decay law predicted by quantum mechanics for large times, we derive analytical
formulae for the critical transition time, \( t_c \), from the exponential to the inverse
power law behaviour at large times. Evaluating \( t_c = \Gamma t_\nu \) for some particle
resonances and narrow nuclear states which have been tested experimentally
to verify the exponential decay law, we conclude that the large time power law
in particle and nuclear decay is hard to find experimentally.

Keywords: non-exponential decay law, analytical survival amplitude,
critical transition time

(Some figures may appear in colour only in the online journal)

1. Introduction

The decay law of an unstable system can be shown classically to be of an exponential nature
but in a quantum mechanical analysis, this law is an approximation which fails for short and

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large times [1–4]. The former case is described by a quadratic function in $t$ [5] and the latter by an inverse power law in $t$ [3]. The non-exponential behaviour predicted by quantum mechanics has intrigued experimental nuclear and particle physicists who performed experiments (see [6] and references therein) with nuclei such as $^{222}$Rn, $^{60}$Co, $^{56}$Mn and measured the decay law for several half-lives to disappointingly find only an exponential decay law. Even though the non-exponential behaviour at large times was confirmed in an experiment measuring the luminescence decays of many dissolved organic materials after pulsed laser excitation [7], the failure of the nuclear and particle physics experiments raised questions about observation such as: (i) how long should one wait or in other words, what is the critical transition time ($\tau_C$) from the exponential to a power law behaviour, (ii) does the interaction with the environment affect the measurement and (iii) could it be possible that every measurement resets the decay to an exponential one, thus making the non-exponential behaviour not observable. There exist different points of view [3, 8] and the above questions still seem to be open for discussions. Apart from all this, there exist different theoretical formalisms for the quantum mechanical treatment of the time evolution and decay of an unstable state [9–16]. In order to at least partly answer the above questions, it is essential to investigate if the different formalisms agree only globally on the short and large time behaviour or also in details such as the prediction of the critical transition times from the exponential to the power law as well as the exponent in the power law behaviour. With this objective, in the present work, we investigate some of the most popularly known approaches for the calculation of survival probabilities, namely, the method of García-Calderón (GC) [10, 16, 17] and collaborators [11, 15, 18–20] which uses the Green’s functions, the method of van Dijk and Nogami (DN) [13, 14] and the Fock–Krylov (FK) method [9] which involves the Fourier transform of an energy density. We show that the seemingly different methods are indeed equivalent and derive analytical expressions for the survival amplitudes as well as their large time behaviour. A natural continuation of this investigation is to study the critical time for the transition from the exponential to the non-exponential power law behaviour. Here, we also obtain an analytical expression for the critical time and apply it to study the decay of nuclear and particle resonances.

The paper is organized as follows: in section 2, we present the basic results of the GC, DN and FK approaches without entering into the details of the derivations. In section 3, we shall show how the GC and DN approaches lead to the survival amplitude as written in the Fock–Krylov method (with a density based on a relation from statistical physics) and we shall obtain an expression for the energy density of the initial state. In section 4, we shall derive an expression for the survival amplitude in terms of incomplete gamma functions and thereby study its behaviour for large $t$. Finally, we present the analysis of an isolated resonance and obtain an analytic expression to find the critical transition time from the exponential to the power law behaviour, in section 5. Here, we compare our results with an existing work on an isolated resonance [18]. Application of the results to realistic resonances in section 6, unveils some reasons for the non-observability of non-exponential decay in nuclear and particle physics. In section 7, we present an analysis of the oscillatory transition region produced by the interference of the exponential and power law decay at large times. We discuss the origin of the oscillatory term and present an expression for the modulating function which describes it. In section 8, we summarize our findings.

2. Survival probabilities

In general terms, if $H$ is the Hamiltonian of a system and its initial state is $|\Psi(0)\rangle$, the state of system $|\Psi(t)\rangle$ at a time $t > 0$ is given as a solution of the Schrödinger equation
\[ i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle. \] (1)

The quantum decay law is the probability (called non-decay or survival probability) that the state at time \( t \) is in its initial state and is given by, \[ |\langle \Psi(0)|\Psi(t)\rangle|^2 = |\langle \Psi(0)|e^{-iHt}|\Psi(0)\rangle|^2. \]

Starting with the survival amplitude \( A(t) \), one can write it as the projection of the state \( |\Psi(t)\rangle \) on the state \( |\Psi(0)\rangle \):
\[ A(t) = \langle \Psi(0)|\Psi(t)\rangle = \langle \Psi(0)|e^{-iHt}|\Psi(0)\rangle, \] (2)

and the survival probability mentioned above is simply
\[ P(t) = |A(t)|^2. \] (3)

Both the survival amplitude and the survival probability are equal to one when \( t = 0 \) because \( A(0) = \langle \Psi(0)|\Psi(0)\rangle = 1 \) and from (3), \( P(0) = |A(0)|^2 = 1 \).

We note here that in the context of the time evolution of an unstable state, a widely discussed quantity is also the so called ‘nonescape probability’ which is essentially the probability that the particle remains confined inside the interaction or potential region after a time \( t \). Though the two concepts of survival and nonescape probabilities are closely related, there are instances when they could be significantly different [21]. We refer the reader to [11, 19, 21–25] for some interesting discussions on this topic.

2.1. Fock–Krylov method

A popular method for computing the survival amplitude is the Fock–Krylov (FK) approach [9]. In general, this method involves expanding the initial state in eigenstates of a complete set of observables which commute with the Hamiltonian. If we let \( \{ H, \beta \} \) to be this set of observables and let \( |E, b\rangle \) be an eigenstate of them:
\[ H|E, b\rangle = E|E, b\rangle, \] (4)
\[ \beta|E, b\rangle = b|E, b\rangle, \] (5)

then the normalized initial state can be expanded in this basis as
\[ |\Psi(0)\rangle = \int_{E_{\text{min}}}^E dE \int db |E, b\rangle \langle E, b|\Psi(0)\rangle. \] (6)

Let us now consider an intermediate unstable state (resonance) formed in a scattering process such as \( A + a \rightarrow R^* \rightarrow A + a \). Since the initial unstable state, \( |\Psi(0)\rangle \), cannot be an eigenstate of the (hermitian) Hamiltonian, an expansion as in (6) (assuming a continuous spectrum) in terms of the energy eigenstates of the decay products \( A \) and \( a \) can be considered to express \( |\Psi(0)\rangle \) as
\[ |\Psi(0)\rangle = \int dE a(E) |E\rangle \] (7)

where \( |E\rangle \) is the eigenstate and \( E \) the total energy of the system \( A + a \). Substituting now for \( |\Psi(0)\rangle \) in (2), we get,
\[ A(t) = \int dE' \int dE \ a^*(E') \ a(E) \ \langle E' e^{-iHt} | E \rangle \]
\[ = \int dE' \int dE \ a^*(E') \ a(E) \ e^{(-i\Delta E)} \ \delta(E - E') \]
\[ = \int dE \ |a(E)|^2 \ e^{-iEt} \] (8)

The proper normalization of \( \Psi \) tells us that \( |a(E)|^2 \) should have the dimension of \( (1/E) \) and hence can be associated with an energy density of states. Thus, in the Fock–Krylov method,

\[ A(t) = \int_{E_{th}}^\infty dE \ \rho(E) \ e^{-iEt} \] (9)

where \( E_{th} \) is the minimum sum of the masses of the decay products.

2.2. Statistical physics based approach

The advantage of the FK method is that it is not necessary to solve the Schrödinger equation and in cases where one does not know \( |\Psi(0)\rangle \), one can proceed to evaluate \( A(t) \) if the ‘spectral function’ or the energy distribution of the resonant state is known. Such an approach was given in [26, 27] in order to analyze realistic cases of nuclear and particle resonances. The authors noted that theoretically, many different forms of \( \rho(E) \) are available but they may not necessarily have a connection with experiments. One of the experimental signatures for the existence of a resonance is the sharp jump in the phase shift \( \delta(E) \), as a function of energy. The energy derivative of the phase shift displays the typical Lorentzian form associated with a resonance [28] and has different interpretations. One of its first appearances in relation with resonances was in the definition of Wigner’s time delay [29]. Though Wigner’s work dealt with the single channel case, the energy derivative, \( d\delta_l(E)/dE \), for a resonance occurring in the 4th partial wave in scattering can be shown to be the difference between the time spent by the interacting particles with and without interaction in a given region of space [30–32]. This interpretation led the authors in [26, 27] to find the connection between \( d\delta_l(E)/dE \) and \( \rho_l(E) \) for the density of states in a resonance.

In calculating the second virial coefficients \( B \) and \( C \) for the equation of states in a gas,

\[ pV = RT \left[ 1 + B/V + C/V^2 + \ldots \right] \]

Beth and Uhlenbeck [33] (the derivation of their result is reproduced in [34], see also [35, 36]) found that the difference between the density of states with interaction, \( n_l \), and without, \( n_l^{(0)} \), is given by the derivative of the scattering phase shift \( \delta_l \) as,

\[ \rho_l^{BU}(E) = n_l(k) - n_l^{(0)}(k) = \frac{2l + 1}{\pi} \frac{d\delta_l(E)}{dE} \] (11)

where \( k \) and \( E \) are the momentum and energy in the centre-of-mass system of the scattering particles. If a resonance is formed during the scattering process, \( E \) becomes the energy of the resonance in its rest frame. In the absence of interaction, since no resonance can be produced, one would expect the density of states \( \rho(E) \) to be zero. If the interaction is switched off, \( n_l \) will tend to \( n_l^{(0)} \) from above. Therefore, the authors concluded that, as long as \( n_l - n_l^{(0)} \geq 0 \) (which is at least the case for an isolated resonance), one can write for the continuum probability density of states of the decay products in a resonance,
Finally, using the phase shift values extracted from scattering experiments, the authors calculated the survival amplitude, equation (10) with the substitution of equation (12) as

$$A_i(t) = \int_{E_{th}}^{\infty} dE \frac{d\delta_i(E)}{dE} e^{-iEt}.$$  

Analytical expressions for the above have been provided in a recent work [28] with the use of the Mittag-Leffler theorem.

Before we proceed to the next subsections, let us clarify the notation used in this work. For mathematical simplicity, we set, 

$$2m = \hbar^2 = 1$$ and hence 

$$k^2 = E.$$ For a resonance pole given by 

$$E_r - i\Gamma_r/2$$ in the complex energy plane, 

$$k^2_r = \epsilon_r - i\Gamma_r/2$$ where 

$$\epsilon_r = E_r - E_{th},$$ with 

$$E_{th}$$ being the threshold energy (or the sum of the masses of the decay products of the resonance).

Having shifted the energies by an amount 

$$E_{th},$$ the lower limit on the integral for the survival amplitude will be 0 instead of the threshold energy 

$$E_{th}.$$
to every pole \( k_n = a_n - ib_n \) \((a_n, b_n > 0)\), there exists due to time reversal invariance, a complex pole, \( k_{-n} \), situated symmetrically with respect to the imaginary axis, i.e. \( k_{-n} = -k_n^* \). In [15, 16], the authors considered examples with potentials having no bound states so that all poles were located only in the lower half of the complex \( k \) plane.

Coming back to (15), \( M(k_n, t) \) is the integral

\[
M(k_n, t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ix^2}}{x - k_n} \, dx,
\]

(16)

\( u_n(r, k_n) \) is the resonant state associated with the pole \( k = k_n \) and is the solution of the differential equation [39]

\[
\frac{d^2 u_n(r, k_n)}{dr^2} + [V(r) - k_n^2] u_n(r, k_n) = 0,
\]

(17)

with boundary conditions

\[
u_n(0, k_n) = 0,
\]

(18)

\[
\left. \frac{du_n(r, k_n)}{dr} \right|_{r=R} = ik_n u_n(R, k_n),
\]

(19)

and satisfies the normalization condition\(^2\)

\[
\int_0^R u_n^2(r, k_n) \, dr + \frac{i}{2k_n} u_n^2(R, k_n) = 1
\]

(20)

which as given in [39] follows from the residue at a complex pole \( k_n \) of the outgoing Green’s function to the problem. The coefficient \( C_n(k_n) \) is given by

\[
C_n(k_n) = \int_0^R \psi(r, 0) u_n(r, k_n) \, dr.
\]

(21)

The survival amplitude, in this case is given by,

\[
A(t) = \langle \Psi(r, 0) | \Psi(r, t) \rangle = \int_0^R \psi^*(r, 0) \psi(r, t) \, dr = \sum_n C_n(k_n) \overline{C}_n(k_n) M(k_n, t),
\]

(22)

where the coefficient \( \overline{C}_n(k_n) \) is:

\[
\overline{C}_n(k_n) = \int_0^R \psi^*(r, 0) u_n(r, k_n) \, dr.
\]

(23)

Each pair of coefficients \( C_n(k_n) \) and \( \overline{C}_n(k_n) \) for a given \( n \) satisfy certain properties (see [15] for details). In the appendix A, we use the steepest descent method for showing that, for large \( t \) the survival amplitude given by (22) is equal to:

\[
A(t) = -\frac{1}{\sqrt{4\pi}} k_n^{3/4} \text{Im} \left[ \sum_p \frac{C_p(k_p) \overline{C}_p(k_p)}{k_p^3} \right] t^{-3/2} + O(t^{-5/2}).
\]

(24)

\(^2\)This normalization is equivalent to the one originally proposed by Zel’dovich [41]. A similar form was also found in [42]. We refer the reader to [39] for more references.
where \( \{ k_p \}, \ p = 1, 2, \ldots \) are the fourth-quadrant poles of the S-matrix in the complex \( k \) plane. For large \( t \), the survival amplitude is proportional to \( t^{-3/2} \) and the survival probability is proportional to \( t^{-3} \). This result for \( l = 0 \) is consistent with the expectation of \( t^{2l+3} \) (for the \( l \)th partial wave) in literature \([3, 26, 27, 40]\). It is also consistent with the density given by 

\[
\rho_l(E) \propto \frac{d \delta_l(E)}{dE}
\]

since one expects the phase shift to behave as \( \delta_l \sim k^{2l+1} \) near threshold which eventually leads to the above power law at large times (see sections 4.3 and 5.2 in \([28]\)). In the experimental observation of the non-exponential decay \([7]\), however, the exponent was found to vary between \(-2\) to \(-4\). The experimental observation was made with complex organic systems which are not spherically symmetric and hence it is not surprising that other powers of time are exhibited. We also note that in \([43]\), within a model of a two level system coupled to the continuum, the authors found an exponent of \(-4\).

A small note regarding the steepest descents method is in order here before closing this subsection. This method has been used earlier in \([10, 16, 19]\) in the context of arriving at the above result but in a somewhat different manner as compared to the present work where it is used to directly evaluate \( A(t) \). The authors in \([19]\) for example, use this method in order to obtain the retarded Green’s function, \( g(r, r'; t) \), entering into the definition of the time evolved wave function, namely, \( \psi(r, t) = \int_0^R g(r, r'; t)\psi(r', 0)dr' \), which eventually defines the survival amplitude. The contours of integration in \([19]\) and in the present work are hence also different.

### 2.4. Jost and Moshinsky functions method

In an attempt to obtain the expression for the wave function of a decaying quantum system, the authors van Dijk and Nogami (DN) in \([13]\), proposed a method which involved the description of the wave function as a linear combination of the the Moshinsky functions, \( M(k, r, t) \) \([44]\), each of which is associated with a pole of the scattering matrix, \( S \). In a follow-up work \([14]\), the authors used this formalism to study the survival and nonescape probabilities of decaying quantum systems. In this subsection, we shall describe the DN approach for the evaluation of survival probabilities in some detail, in order to later compare it with the GC and FK approaches discussed before.

The authors in \([14]\) begin by considering the case of \( S \)-wave unstable states and attempt to find a solution of the time dependent Schrödinger equation with a central potential \( V(r) \) of finite range \( R \) and an initially normalized wave function \( \psi(r, 0) \) confined to \( r < R \), i.e. \( \psi(r, 0) = 0 \) for \( r > R \). The scattering solutions are expressed in terms of Jost functions such that for the case of no bound states,

\[
\psi(r, t) = \frac{2}{\pi} \int_0^\infty \frac{k^2}{f(k)} c(k)u(k, r)e^{-ikr}dk,
\]

where \( k > 0 \) and \( k^2 \) is the corresponding energy. Here, \( \psi(r, t) \) is the wave function, \( \Psi \), times \( r \). \( c(k) \) is given as,

\[
c(k) = \int_0^R \psi(r, 0)u(k, r)dr,
\]

with \( u(k, r) \) being the function defined as

\[
u(k, r) = \frac{1}{2ik} \left[ f(k)f(-k, r) - f(-k)f(k, r) \right].
\]
$f(k, r)$ is the Jost solution of the time independent Schrödinger equation \[45\] (with potential $V(r)$) and $f(k)$ is the Jost function related to it as $f(\pm k) = f(\pm k, 0)$. The function $u(k, r)$ is real and $u(k, r)$ and $c(k)$ are both entire and even in the parameter $k$. This function is normalized such that

$$
\int_0^\infty u(k, r)u(k', r)\, dr = \frac{\pi}{2k^2}|f(k)|^2 \delta(k - k').
$$

(28)

Using (25) the survival amplitude is written as,

$$
A(t) = \int_0^\infty \psi^*(r, 0)\psi(r, t)\, dr = \frac{2}{\pi} \int_0^\infty \frac{k^2}{|f(k)|^2} c(k)e^{-ik^2t} \left[ \int_0^R \psi^*(r, 0)u(k, r)\, dr \right].
$$

(29)

Before proceeding to the comparison of approaches reviewed in this section, we recall an older work \[46\] on the complex energy eigenfunctions (as those given by equations (17)–(19)) where the effectiveness of this method, in spite of several shortcomings has been discussed. Another work worth mentioning in the context of the present investigations is \[47\] where the authors performed a comparison of the Hermitian and non-Hermitian formulation for the time evolution of quantum decay and showed that they lead to an identical description for a large class of well-behaved potentials.

3. Energy density of the initial state

Having introduced the different approaches for the calculation of survival amplitudes, we shall now examine the expressions, equations (13), (22) and (29) to obtain a definition of the energy density of states in the GC and DN formalisms and compare the survival probabilities in these two approaches with that of the frequently used Fock–Krylov method.

3.1. GC formalism

We begin by writing the integral $M(k_n, t)$ as

$$M(k_n, t) = \frac{i}{2\pi} \int_0^\infty \frac{e^{-ix^2}}{x - k_n}\, dx - \frac{i}{2\pi} \int_0^\infty \frac{e^{-ix^2}}{x + k_n}\, dx$$

$$= \frac{i}{2\pi} \int_0^\infty \left( \frac{1}{x - k_n} - \frac{1}{x + k_n} \right) e^{-ix^2}\, dx$$

$$= \frac{1}{2\pi i} \int_0^\infty \frac{2k_n}{k_n^2 - x^2} e^{-ix^2}\, dx,$$

(30)

and making the change of variable $E = x^2$, we have

$$M(k_n, t) = \frac{1}{2\pi i} \int_0^\infty \frac{k_n}{\sqrt{E(k_n^2 - E)}} e^{-iE}\, dE.$$

(31)

Since

$$\frac{k_n}{k_n^2 - E} = \frac{1}{k_n} \left( 1 + \frac{E}{k_n^2 - E} \right),$$

$$\frac{k_n}{k_n^2 - E} = \frac{1}{k_n} \left( 1 + \frac{E}{k_n^2 - E} \right),$$



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the integral takes the form:

\[ M(k_n, t) = \frac{1}{2\pi i} \int_0^\infty \frac{\sqrt{E}}{k_n(k_n^2 - E)} e^{-iEt} dE + \frac{1}{2\pi i k_n} \int_0^\infty \frac{e^{-iEt}}{\sqrt{E}} dE. \]  (32)

Substituting (32) in (22), we obtain:

\[ A(t) = \int_0^\infty \left[ \frac{1}{2\pi i} \sum_n C_n(k_n) \bar{C}_n(k_n) \frac{\sqrt{E}}{k_n(k_n^2 - E)} \right] e^{-iEt} dE + \frac{1}{2\pi i} \left[ \sum_n C_n(k_n) \bar{C}_n(k_n) \right] \int_0^\infty \frac{e^{-iEt}}{\sqrt{E}} dE. \]  (33)

The last term is zero because of the properties of the coefficients \( C_n \). The final form of the survival amplitude is:

\[ A(t) = \int_0^\infty \left[ \frac{1}{2\pi i} \sum_n C_n(k_n) \bar{C}_n(k_n) \frac{\sqrt{E}}{k_n(k_n^2 - E)} \right] e^{-iEt} dE. \]  (34)

We can see that \( A(t) \) is the Fourier transform of the series given in the square brackets. In other words, the GC approach leads to a survival amplitude which is very similar in form to that of the Fock–Krylov method. Comparing equation (34) with the FK amplitude given in equation (10), we consider identifying the quantity in square brackets with the energy density \( \rho(E) \) of the initial state and write

\[ \rho^{GC}(E) = \frac{1}{2\pi i} \sum_n C_n(k_n) \bar{C}_n(k_n) \frac{\sqrt{E}}{k_n(k_n^2 - E)}. \]  (35)

If we perform the last sum with poles of the fourth quadrant only, this energy density can be written as

\[ \rho^{GC}(E) = \frac{1}{\pi} \text{Im} \sum_p C_p(k_p) \bar{C}_p(k_p) \frac{\sqrt{E}}{k_p(k_p^2 - E)}. \]  (36)

3.2. DN formalism

Let us start by considering the integral in the square brackets in (29). It is the complex conjugate of \( c(k) \) given by (26). Thus

\[ A(t) = \frac{2}{\pi} \int_0^\infty k^2 |c(k)|^2 \frac{e^{-iEt}}{|f(k)|^2} dk. \]  (37)

Performing a change of variable \( k^2 = E \), we have:

\[ A(t) = \int_0^\infty \frac{\sqrt{E}}{\pi} \left| \frac{c(\sqrt{E})}{f(\sqrt{E})} \right|^2 e^{-iEt} dE. \]  (38)

Comparing the above expression with the Fock–Krylov amplitude, the energy density in the DN formalism is given by,

\[ \rho^{DN}(E) = \frac{\sqrt{E}}{\pi} \left| \frac{c(\sqrt{E})}{f(\sqrt{E})} \right|^2. \]  (39)

If we consider the integrand in (37) without the exponential part \( e^{-iEt} \), then using (26) and the property of the Jost function \( f'(k) = f(-k) \) for real \( k \), we get:
\( \varphi(k) = \frac{2}{\pi} k^2 \frac{\text{Im}(k)^2}{|f(k)|^2} = \frac{2}{\pi} \int_0^R \int_0^R \psi(r,0) \psi^*(r',0) \left[ k^2 \frac{u(k,r)u(k,r')}{f(k)f(-k)} \right] \, dr' \, dr. \)  

Now, taking the function in square brackets (let us call it \( I(k,r,r') \)) and considering the definition of the \( S \)-matrix in terms of the Jost functions,

\[
S(k) = \frac{f(k)}{f(-k)},
\]

we get,

\[
I(k,r,r') = -\frac{1}{4} [S(k)f(-k,r) - f(k,r)][f(-k,r') - f(k,r')/S(k)].
\]

Taking into account that \{ \( k_p \} \), \( p = 1, 2, \ldots \) are the poles of \( S \)-matrix in the fourth-quadrant of the complex \( k \) plane, the \( S \)-matrix has additional poles \{ \( -k_p^* \) \} and zeros \{ \( -k_p \) \} and \{ \( k_p^* \) \}, where all these zeros and poles are simple [48]. Thus, the poles of the function \( I(k,r,r') \) correspond to the poles and zeros of the \( S \)-matrix. If \( b_p \) are the residues of the \( S \)-matrix in the fourth-quadrant, the residues corresponding to its poles of the third-quadrant are \(-b_p^*\) [28] while the residues of the inverse of the \( S \)-matrix, in terms of \( b_p \), are (see appendix B):

\[
\text{Res} \left[ 1/S(k), k = -k_p \right] = -b_p,
\]

\[
\text{Res} \left[ 1/S(k), k = k_p^* \right] = b_p^*.
\]

Thus, the residues of the function \( I(k,r,r') \) can be expressed in terms of the residues of the \( S \)-matrix. If we call \( \lambda(k_p, r, r') \) the residues of this function corresponding to the poles of the fourth-quadrant, then:

\[
\text{Res} \left[ I(k,r,r'), k = k_p \right] = -\frac{1}{4} b_p f(-k_p,r)f(-k_p,r') \equiv \lambda(k_p,r,r'),
\]

\[
\text{Res} \left[ I(k,r,r'), k = -k_p^* \right] = \frac{1}{4} b_p^* f^*(-k_p^*,r)f^*(-k_p^*,r') = -\lambda^*(k_p,r,r'),
\]

\[
\text{Res} \left[ I(k,r,r'), k = -k_p \right] = \frac{1}{4} b_p f(-k_p,r)f(-k_p,r') = -\lambda(k_p,r,r'),
\]

\[
\text{Res} \left[ I(k,r,r'), k = k_p^* \right] = \frac{1}{4} b_p^* f^*(-k_p^*,r)f^*(-k_p^*,r') = \lambda^*(k_p,r,r').
\]

In the calculation of these residues, we used the property \( f^*(-k^*,r) = f(k,r) \) for complex \( k \) [49] (see also [50] for a discussion on the use of Jost functions in bound and resonant state problems). From the Mittag-Leffler theorem\(^3\) and taking into account that \( I(0, r, r') = 0 \), we have:

\(^3\)If the only singularities of a meromorphic function \( f(z) \) are the simple poles \( z = a_1, a_2, \ldots \) such that \( |a_1| \leq |a_2| \leq \ldots \), with residues \( b_1, b_2, \ldots \) respectively, and if \( C_0 \) is a circumference of radius \( R_0 \) which contains \( N \) poles of the function \( f(z) \) (and does not pass through any of the remaining poles), i.e. \( |a_N| < R_0 < |a_{N+1}| \), and on \( C_N, |f(z)| < M \), where \( M \) is not dependent on \( N \), then

\[
f(z) = f(0) + \lim_{N \to \infty} \sum_{n=1}^{N} b_n \left( \frac{1}{z-a_n} + \frac{1}{a_n} \right) + \lim_{N \to \infty} \frac{z}{2\pi i} \oint_{C_N} \frac{f(\zeta)}{\zeta(z-\zeta)} \, d\zeta = f(0) + \sum_{n=1}^{\infty} \frac{b_n z}{a_n(z-a_n)}.
\]

This theorem is known as the Mittag-Leffler theorem [51].
\[ I(k, r, r') = 4k^2 \text{ Re } \left[ \sum_p \frac{i \zeta(k_p, r, r')}{k_p(k_p^2 - k^2)} \right]. \quad (49) \]

Finally, after some lengthy algebra (see appendix C), it is possible to write \( \rho(k) \) in the following form:

\[ \rho(k) = \frac{2}{\pi} k^2 \text{ Re } \sum_p \frac{i a_p(k_p)}{k_p(k_p^2 - k^2)}, \quad (50) \]

where the coefficients \( a_p(k_p) \) are given by,

\[ a_p(k_p) \equiv 4i \int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) \zeta(k_p, r, r') \, dr \, dr'. \quad (51) \]

Noting the definition of \( \rho(k) \) in (40) and substituting (50) in (37), the survival amplitude in the DN formalism becomes,

\[ A(t) = \int_0^\infty \frac{2}{\pi} k^2 \text{ Re } \left[ k^2 \sum_p \frac{i a_p(k_p)}{k_p(k_p^2 - k^2)} \right] e^{-i k^2 t} \, dk, \quad (52) \]

which, after a change of variable \( E = k^2 \), can be expressed as,

\[ A(t) = \int_0^\infty \frac{1}{\pi} \text{ Re } \left[ \sqrt{E} \sum_p \frac{i a_p(k_p)}{k_p(k_p^2 - E)} \right] e^{-i E t} \, dE = \int_0^\infty \frac{1}{\pi} \text{ Im } \left[ \sum_p \frac{a_p(k_p)}{k_p} \sqrt{E} \frac{1}{k_p^2 - E} \right] e^{-i E t} \, dE, \quad (53) \]

so that

\[ \rho_{\text{DN}}(E) = \frac{1}{\pi} \text{ Im } \left[ \sum_p \frac{a_p(k_p)}{k_p} \frac{\sqrt{E}}{k_p^2 - E} \right]. \quad (54) \]

The above expression for the survival amplitude is the same as that in equation (36) up to the constants \( C_p(k_p) \bar{C}_p(k_p) \) and \( a_p(k_p) \). Note however that there is a subtle difference between the constants of the GC and DN formalism. \( C_p(k_p) \bar{C}_p(k_p) \) of the GC formalism depend solely on the resonant poles \( k_p \). However, the constants \( a_p(k_p) \) which apparently depend only on \( k_p \), in principle depend on all other existing poles through their dependence on the residues \( \zeta(k_p, r, r') \) (see equations (45) and (B.5)). Though in practice such a calculation may not be feasible, under certain conditions, it is possible to use an approximate solution as given in appendix B.

3.3. Comparison of the GC and DN coefficients

The coefficients in the GC formalism written as a double integral:

\[ C_p(k_p) \bar{C}_p(k_p) = \int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) u_p(r, k_p) u_p(r', k_p) \, dr \, dr', \quad (55) \]

where \( R \) is the range of the potential, are not equal to the coefficients \( a_p(k_p) \) unless

\[ u_p(r, k_p) \equiv \sqrt{\frac{1}{P_p}} f(-k_p, r) . \quad (56) \]
The above expression is deduced by substituting the definition of $\ell(k_n, r, r')$ in the integral (51) and comparing with (55). In principle, this result shows that the resonant state associated with the fourth-quadrant pole $k = k_p$ which is also a pole of the $S$-matrix may be computed in terms of the residues of the $S$-matrix at the corresponding pole and the Jost function.

From the Riemann–Lebesgue theorem we know that $\varrho(k) \to 0$ when $k \to \infty$. This implies that

$$\lim_{k \to \infty} \varrho(k) = \lim_{k \to \infty} \frac{2}{\pi} k^2 \Re \sum_{\rho} \frac{i a_\rho(k_p)}{k_p(k_p^2 - k^2)} = 0 \quad \Rightarrow \quad \Im \sum_{\rho} \frac{a_\rho(k_p)}{k_p} = 0.$$  

(57)

Since $A(0) = 1$, from (53) we have that:

$$\int_0^\infty \frac{1}{\pi} \Im \left[ \sum_{\rho} \frac{a_\rho(k_p)}{k_p} \frac{\sqrt{E}}{k_p^2 - E} \right] dE = 1. \tag{58}$$

However,

$$\frac{\sqrt{E}}{k_p^2 - E} = \frac{1}{\sqrt{E}} \left( \frac{k_p^2}{k_p^2 - E} - 1 \right).$$  

(59)

Using the condition (57), equation (58) takes the form:

$$\frac{1}{\pi} \Im \sum_{\rho} \frac{a_\rho(k_p)}{k_p} \int_0^\infty \frac{1}{\sqrt{E}} \left( \frac{k_p^2}{k_p^2 - E} - 1 \right) dE = \frac{1}{\pi} \Im \sum_{\rho} k_p a_\rho(k_p) \int_0^\infty \frac{dE}{\sqrt{E(k_p^2 - E)}} = 1. \tag{60}$$

Since the integral in the last equation is equal to $i \pi / k_p$, (60) reduces to

$$\Im \left[ i \sum_{\rho} a_\rho(k_p) \right] = \Re \sum_{\rho} a_\rho(k_p) = 1. \tag{61}$$

(61)

The properties (57) and (61) satisfied by the coefficients $a_\rho(k_p)$ are the same as those satisfied by $C_\rho(k_p) C_\rho(k_p)$.

Since the energy density and hence the survival amplitude in the GC and DN formalisms have been shown in the previous section to be equivalent up to the constants appearing in equations (34) and (53), it is convenient to write both equations in one compact expression before computing the survival amplitude and other quantities of interest. Thus, if we define the coefficient $\gamma_\rho(k_p)$ as:

$$\gamma_\rho(k_p) = \begin{cases} C_\rho(k_p) \tilde{C}_\rho(k_p), & \text{for GC formalism,} \\ a_\rho(k_p), & \text{for DN formalism,} \end{cases} \tag{62}$$

The integral was calculated following this theorem: if $f(z)$ is a single-valued analytic function in the domain $0 < \Arg z < 2\pi$, except for a finite number of singularities $z_k, k = 1, \ldots, n$ not lying on the positive real axis and let $z = \infty$ be a zero of order not lower than first of the function $f(z)$, then

$$\int_0^\infty x^{\alpha-1} f(x) dx = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_{k=1}^n \Res \left[ x^{\alpha-1} f(z), z = z_k \right],$$

where $0 < \alpha < 1$ [52].
then both the energy densities can be written in a common form as:

$$\rho(E) = \frac{1}{\pi} \text{Im} \left[ \sum_p \frac{\gamma_p(k_p)}{k_p} \sqrt{E - k^2_p} \right].$$  \hspace{1cm} (63)

The coefficients $\gamma_p(k_p)$ satisfy the same properties as $C_p(k_p)C_p(k_p)$ and $a_p(k_p)$, i.e.

$$\text{Re} \sum_p \gamma_p(k_p) = 1, \hspace{1cm} (64)$$

$$\text{Im} \sum_p \gamma_p(k_p) = 0. \hspace{1cm} (65)$$

### 3.4. Energy density of an isolated resonance

In the case of an isolated resonance, with a pole at say $k_r$, the conditions on the coefficients $\gamma_r$ given by equations (64) and (65) are reduced to

$$\text{Re} \gamma_r(k_r) = 1, \hspace{1cm} (66)$$

$$\text{Im} \gamma_r(k_r) = 0 \hspace{1cm} (67)$$

leading to

$$\gamma_r(k_r) = 1 + i \frac{\text{Im} (k_r)}{\text{Re} (k_r)} = \frac{k_r}{\text{Re} (k_r)}. \hspace{1cm} (68)$$

This equation was first deduced in [18] for the GC formalism and is also valid for the DN formalism. Replacing the above in (36), we obtain the energy density for an isolated resonance,

$$\rho_{iso}^{GC}(E) = \frac{1}{\pi} \text{Im} \left[ \frac{\sqrt{E}}{k_r(k_r^2 - E)} \right] + \frac{1}{\pi} \text{Im} \left[ \frac{\text{Im} (k_r)}{\text{Re} (k_r)} \frac{\sqrt{E}}{k_r(k_r^2 - E)} \right]. \hspace{1cm} (69)$$

In order to confirm our identification of the quantity in square brackets in (34) with the density of states, we note as mentioned earlier, that (a) the energy derivative of the scattering phase shift, $d\delta_0/dE$, in the vicinity of a resonance, can be derived analytically by making use of the properties of the $S$-matrix and a theorem of Mittag-Leffler. For the case of an $s$-wave resonance, it is given by [28],

$$\frac{d\delta_0(E)}{dE} = \text{Im} \left[ \frac{\sqrt{E}}{k_r(k_r^2 - E)} \right] \hspace{1cm} (70)$$

and (b) the Beth and Uhlenbeck formula (11) allows us to relate the energy derivative of the phase shift with the density of states in an $s$-wave resonance as:

$$\rho_0^{BU}(E) = \frac{1}{\pi} \frac{d\delta_0(E)}{dE}, \hspace{1cm} (71)$$

so that

$$\rho_{iso}^{GC}(E) = \rho_0^{BU}(E) + \frac{1}{\pi} \text{Im} \left[ \frac{\text{Im} (k_r)}{\text{Re} (k_r)} \frac{\sqrt{E}}{k_r(k_r^2 - E)} \right] \hspace{1cm} (72)$$
The density of states as given by the Beth–Uhlenbeck formula is the same as the first term in (69). The second term in (69) can be seen to be a small correction to the first term for narrow resonances. The reason for the correction term not appearing in the Beth–Uhlenbeck (BU) formula could be due to the approximations made in the derivation of the BU formula and remains to be investigated. With the above confirmation, we conclude that the GC and DN formalisms (taken for the case of an isolated s-wave resonance), and the Fock–Krylov method with the density given using the Beth–Uhlenbeck formula, are equivalent.

4. Analytical expression for the survival amplitude

Analytical expressions for the survival amplitude, \(A(t)\), of a resonance given by a Breit–Wigner form for the energy density can be found in [53, 54]. In [28], the analytical expressions for \(A(t)\) were derived using generalized expressions for the energy density (derived using the analytical properties of the \(S\)-matrix and the Mittag-Leffler theorem) within the Fock–Krylov method. The expressions were shown to reduce to those arising from the Breit–Wigner form alone plus corrections. Here, we shall find an analytical expression for the survival amplitude given in (34) and (53), study their asymptotic behaviour and analyse the transition region between the exponential and non-exponential decay law. Apart from obtaining analytical expressions for the transition time, we shall examine some nuclear and particle decays and the relevance of the results for an experimental observation of the non-exponential decay law.

4.1. Survival amplitude in terms of the incomplete gamma function

In the present section, we shall provide analytical expressions for the survival amplitudes in a combined form which is valid for both methods. Analytical formulae for the survival amplitudes evaluated in [15] within the Green’s function method, were presented in terms of the error functions. Here we present the expressions using incomplete gamma functions.

Substituting (63) in (22), the survival amplitude is given as

\[
A(t) = \int_0^\infty \rho(E) e^{-ikE} dE - \frac{1}{2\pi i} \sum_p \frac{\gamma_p(k_p)}{k_p} \int_0^\infty \frac{\sqrt{E}}{k_p^2 - E} e^{-ikE} dE - \frac{1}{2\pi i} \sum_p \frac{\gamma_p^*(k_p)}{k_p^*} \int_0^\infty \frac{\sqrt{E}}{k_p^* - E} e^{-ikE} dE. \tag{73}
\]

Using (D.4) and (D.6) (see appendix D), we get

\[
A(t) = \frac{1}{2\pi i} \sum_p \frac{\gamma_p(k_p)}{k_p} \left[ 2\pi i k_p e^{-i\gamma_p^2 t} + \frac{i}{\sqrt{2}} k_p e^{-i\gamma_p^2 t} \Gamma\left(\frac{1}{2}, -i\gamma_p^2 t\right) \right] - \frac{1}{2\pi i} \sum_p \frac{\gamma_p^*(k_p)}{k_p^*} \left[ \frac{i}{\sqrt{2}} k_p^* e^{-i\gamma_p^2 t} \Gamma\left(\frac{1}{2}, -i\gamma_p^2 t\right) \right] = \sum_p \gamma_p(k_p) e^{-i\gamma_p^2 t} + \frac{1}{4\sqrt{\pi}} \sum_p \left[ \gamma_p(k_p) e^{-i\gamma_p^2 t} \Gamma\left(\frac{1}{2}, -i\gamma_p^2 t\right) - \gamma_p^*(k_p) e^{-i\gamma_p^2 t} \Gamma\left(\frac{1}{2}, -i\gamma_p^2 t\right) \right] \tag{74}
\]

In order to ensure that the survival amplitude at \(t = 0\) is unity, using \(\Gamma\left(-\frac{1}{2}, 0\right) = -2\sqrt{\pi}\) together with the property (64) gives:

\[
A(0) = \sum_p \gamma_p(k_p) - \frac{1}{2} \sum_p \left[ \gamma_p(k_p) - \gamma_p^*(k_p) \right] = \text{Re} \sum_p \gamma_p(k_p) = 1. \tag{75}
\]
Using the properties of the incomplete gamma functions [55]:

\[ \Gamma(\alpha + 1, z) = \alpha \Gamma(\alpha, z) + z^\alpha e^{-z} \quad (76) \]

with \( \alpha = 1/2 \) and \( z = -ik_p^2 t \), we can write (74) as

\[
A(t) = \sum_p \gamma_p(k_p) e^{-\text{ik}_p^2 t} + \frac{1}{\sqrt{\pi}} \sum_p \gamma_p(k_p) \left[ 2(-i\text{ik}_p^2 t)^{-1/2} - 2e^{-i\text{ik}_p^2 t} \right]
\]

\[
- \frac{1}{\sqrt{\pi}} \sum_p \gamma_p^* (k_p) \left[ 2(-i\text{ik}_p^2 t)^{-1/2} - 2e^{-i\text{ik}_p^2 t} \right]
\]

\[
= \sum_p \gamma_p(k_p) e^{-i\text{ik}_p^2 t} - \frac{1}{2\sqrt{\pi}} \sum_p \left[ \gamma_p(k_p)e^{-i\text{ik}_p^2 t}\Gamma\left(\frac{1}{2}, -i\text{ik}_p^2 t\right) - \gamma_p^*(k_p)e^{-i\text{ik}_p^2 t}\Gamma\left(\frac{1}{2}, -i\text{ik}_p^2 t\right) \right]
\]

\[
+ \frac{1}{\sqrt{\pi}t}e^{i\pi/4} \text{Im} \sum_p \gamma_p(k_p) \frac{1}{k_p}.
\]

The last term is zero due to the property (65). Finally,

\[
A(t) = \sum_p \gamma_p(k_p) e^{-i\text{ik}_p^2 t} - \frac{1}{2\sqrt{\pi}} \sum_p \left[ \gamma_p(k_p)e^{-i\text{ik}_p^2 t}\Gamma\left(\frac{1}{2}, -i\text{ik}_p^2 t\right) - \gamma_p^*(k_p)e^{-i\text{ik}_p^2 t}\Gamma\left(\frac{1}{2}, -i\text{ik}_p^2 t\right) \right].
\quad (77)

The above expression is equivalent to equation (4.21) of [10] given in terms of the \( M \) functions. For a given \( p \), each term depends on the pole associated with the index \( p \) and it is possible to define partial survival amplitudes for the pole \( k_p \) as:

\[
A_p(t) = \gamma_p(k_p) e^{-i\text{ik}_p^2 t} - \frac{1}{2\sqrt{\pi}} \left[ \gamma_p(k_p) e^{-i\text{ik}_p^2 t}\Gamma\left(\frac{1}{2}, -i\text{ik}_p^2 t\right) - \gamma_p^*(k_p) e^{-i\text{ik}_p^2 t}\Gamma\left(\frac{1}{2}, -i\text{ik}_p^2 t\right) \right],
\quad (78)

such that the survival amplitude takes the simple form:

\[
A(t) = \sum_p A_p(t).
\quad (79)

4.2. Behaviour at large times

Using the asymptotic expansion of the incomplete gamma function [56]:

\[ e^\Gamma(\alpha, z) \sim z^{\alpha-1} + (\alpha - 1)z^{\alpha-2} + \cdots \quad (80) \]

and ignoring exponential terms, we have,

\[
A(t) \sim \frac{1}{2\sqrt{\pi}} \sum_p \left[ \frac{1}{k_p} (-i\text{t})^{-1/2} - \frac{1}{2k_p^3} (-i\text{t})^{-3/2} \right]
\]

\[
+ \frac{1}{2\sqrt{\pi}} \sum_p \left[ \frac{1}{k_p} (-i\text{t})^{-1/2} - \frac{1}{2k_p^3} (-i\text{t})^{-3/2} \right]
\]

\[
= -\frac{1}{2\sqrt{\pi}} (-i\text{t})^{-1/2} \cdot 2i \text{Im} \sum_p \frac{\gamma_p(k_p)}{k_p} + \frac{1}{4\sqrt{\pi}} (-i\text{t})^{-3/2} \cdot 2i \text{Im} \sum_p \frac{\gamma_p(k_p)}{k_p}.
\quad (81)
\]
The first term is zero due to the properties of the coefficients, $\gamma_p(k_p)$, and

$$A(t) \sim -\frac{e^{i\pi/4}}{\sqrt{4\pi}} \text{Im} \left( \sum_p \frac{\gamma_p(k_p)}{k_p^2} \right) t^{-3/2}.$$  \hfill(82)

The above result has also been obtained in appendix A by computing $A(t)$ with the steepest descent method. In both cases, the results are consistent and the survival probability is proportional to $t^{-3}$ for large $t$. We remind the reader that the above analysis has been performed for S-waves.

4.3. Survival amplitude for an isolated resonance

If the system under analysis has only one resonant pole $k_r$, the expression deduced for the survival amplitude at any time $t$ as well as that for large times can be written in a simple form:

$$A(t) = \gamma_r(k_r)e^{-i k_r^2 t} - \frac{1}{2\sqrt{\pi}} \left[ \gamma_r(k_r)e^{-i k_r^2 t} \Gamma \left( \frac{1}{2}, -i k_r^2 t \right) - \gamma^*_r(k_r)e^{-i k_r^2 t} \Gamma \left( \frac{1}{2}, -i k_r^2 t \right) \right],$$  \hfill(83)

$$A(t) \sim -\frac{e^{i\pi/4}}{\sqrt{4\pi}} \text{Im} \left( \frac{\gamma_r(k_r)}{k_r^2} \right) t^{-3/2}.$$  \hfill(84)

Substituting the expression (68), equations (83) and (84) can be alternatively written as

$$A(t) = \frac{k_r}{\text{Re} \ (k_r)} e^{-i k_r^2 t} - \frac{1}{2\sqrt{\pi} \text{Re} \ (k_r)} \left[ k_r e^{-i k_r^2 t} \Gamma \left( \frac{1}{2}, -i k_r^2 t \right) - k_r^* e^{-i k_r^2 t} \Gamma \left( \frac{1}{2}, -i k_r^2 t \right) \right],$$  \hfill(85)

$$A(t) \sim -\frac{e^{i\pi/4}}{\sqrt{4\pi} \text{Re} \ (k_r)} \text{Im} \left( \frac{1}{k_r^2} \right) t^{-3/2}.$$  \hfill(86)

An inspection of (85) reveals that, for intermediate times, the survival amplitude can be described by an exponential function, i.e.

$$A_r(t) \approx \frac{k_r}{\text{Re} \ (k_r)} e^{-i k_r^2 t}.$$  \hfill(87)

The quantum mechanical description of the decay law leads to a non-exponential behaviour at very short and very large times with the intermediate region being dominated by the exponential decay law. In what follows, we shall concentrate on the transition region from the exponential to the power law at large times. Considering the case of an isolated resonance, the critical time for the transition to the power law is investigated and its relevance for an experimental observation of the power law is discussed.

5. Critical time

It would be useful if we could find the parameters on which the critical time for the survival amplitude to go from an exponential to a power law behaviour depends. With this objective, we shall study the intersection of the intermediate and large time survival probabilities. We define the critical time between these behaviours as $t_c$, such that
\[ \left| \frac{k_x}{\text{Re} \left( k_x \right)} e^{-i \theta_k} \right|^2 = \left| -\frac{e^{i\pi/4}}{\sqrt{4\pi} \text{Re} \left( k_x \right)} \text{Im} \left( \frac{1}{k_x^2} \right) \right|^2. \]  

(88)

Since \( k_x^2 = \epsilon_x - i \Gamma / 2 \) and defining \( \tau_c = \Gamma / t_x \), it is convenient to write (88) as

\[ |k_x|^2 e^{-\tau_c} = \left( \frac{\epsilon_x}{4\pi} \text{Re} \left( k_x \right) \right)^2 \left( \frac{1}{k_x^2} \right) \left( \frac{1}{\tau_c} \right). \]  

(89)

Let \( C \) be the constant defined by

\[ C = \frac{\epsilon_x}{4\pi} \left( \frac{1}{k_x^2} \right)^2, \]  

(90)

which is always positive. The transition time, \( \tau_c = \Gamma / t_x \), shall be the zero of the function

\[ f(\tau_c) = e^{-\tau_c} - C \tau_c^{-3} = \frac{\epsilon_x^3 e^{-\tau_c}}{\tau_c^3} - C. \]  

(91)

Let the auxiliary function \( g(\tau_c) \) together with its derivative be given by

\[ g(\tau_c) = \tau_c^3 e^{-\tau_c} - C, \]  

(92)

\[ g'(\tau_c) = \tau_c^2 (3 - \tau_c) e^{-\tau_c}. \]  

(93)

\( g(\tau_c) \) has two critical points: \( \tau_c = 0 \) and \( \tau_c = 3 \). In the interval \( 0 < \tau_c < 3 \), \( g(\tau_c) > 0 \); and in the interval \( \tau_c > 3 \), \( g(\tau_c) < 0 \). This means that \( \tau_c = 3 \) is a maximum and \( \tau_c = 0 \) is a minimum. The values of \( g \) at those points are \( g(0) = -C \) and \( g(3) = 27e^{-3} - C \). When \( \tau_c \to \infty \), \( g \to -C \).

Since \( C > 0 \), \( g \) is positive on some interval or negative for all \( \tau_c > 0 \), and this depends on the sign of its maximum. If \( 27e^{-3} - C < 0 \), the maximum is negative and \( g < 0 \) in \( \tau_c > 0 \). If \( C = 27e^{-3} \), the maximum is zero and \( g \leq 0 \) in \( \tau_c > 0 \); but, if \( 27e^{-3} - C > 0 \), \( g \) will have two zeros and will be positive in the interval formed by those zeros.

Now, it is easy to find the zeros of \( f \) and this depends on the values of \( C \). We have three cases:

(i) First case: if \( C > 27e^{-3} \), \( f(\tau_c) \) has no zeros and is negative for \( \tau_c > 0 \), this means that \( e^{-\tau_c} < C \tau_c^{-3} \); there is no critical time and thus, the power law behaviour always dominates.

(ii) Second case: if \( C = 27e^{-3} \), \( f(\tau_c) \) has one zero and is negative or null in \( \tau_c > 0 \), this implies \( e^{-\tau_c} \leq C \tau_c^{-3} \); there is only one critical point and the power law behaviour dominates again.

(iii) Third case: if \( C < 27e^{-3} \), \( f(\tau_c) \) has two zeros \( \tau_{c1} \) and \( \tau_{c2} \) such that \( \tau_{c1} < \tau_{c2} \). \( f(\tau_c) > 0 \) for \( \tau_{c1} < \tau_c < \tau_{c2} \) (and \( e^{-\tau_c} > C \tau_c^{-3} \) here). \( f(\tau_c) < 0 \) for values of \( \tau_c \) out of this interval and \( e^{-\tau_c} < C \tau_c^{-3} \); the exponential behaviour is more dominant than the power law behaviour in \( \tau_{c1} < \tau < \tau_{c2} \), but, for \( \tau > \tau_{c2} \), it is the power law that dominates. We shall see that \( \tau_{c2} \) can be identified as the critical time for the transition from the exponential to the power law.

Let us now see if it is possible to write the parameter \( C \) in terms of \( \epsilon_r = \Gamma / \sqrt{4\pi} \). For a given resonance pole, \( E_r - i \Gamma / 2 \) in the complex energy plane, \( \epsilon_r \) is defined as \( E_r - E_{\text{th}} \) where \( E_{\text{th}} \) for example is the sum of the masses of the decay products of an unstable particle with mass \( E_r \). Since

\[ D F \text{ Ramirez Jiménez and N G Kelkar} \]
\[
\frac{1}{k^2_r} = \frac{1}{|k^2_r|} \exp[-i \text{Arg}(k^2_r)],
\]

we have

\[
\text{Im}\left(\frac{1}{k^2_r}\right) = -\frac{1}{|k^2_r|} \sin\left[\text{Arg}(k^2_r)\right],
\]

and \(C\) is equal to

\[
C = \frac{2}{\pi} \left(\frac{\Gamma_r}{2}\right)^3 \frac{1}{|k^2_r|^3} \sin^2\left[\text{Arg}(k^2_r)\right].
\]

But, \(|k^2_r| = \epsilon r \sqrt{1 + x_r^2}\), and \(\sin\left[\text{Arg}(k^2_r)\right] = -x_r / \sqrt{1 + x_r^2}\). Thus,

\[
C = \frac{2}{\pi} \frac{x_r^5}{(1 + x_r^2)^{5/2}} = \frac{2}{\pi} \left(\frac{x_r}{\sqrt{1 + x_r^2}}\right)^5.
\]

Thus \(C\) can be written as a function of \(x_r\). An upper bound of \(C\) can be obtained if we see that the term \(x_r(1 + x_r^2)^{-1/2}\) is always less than one for any value of \(x_r\). Thus,

\[
C < \frac{2}{\pi}.
\]

This bound is less than \(27e^{-3} = 1.34\ldots\) and the third case applies always.

Coming back to the definition of \(\tau_c\) through the zeros of the function \(f(\tau_c)\) in equation (91), we can write it as

\[
-\frac{\tau}{3} e^{-\tau/3} = -\frac{\sqrt{C}}{3}.
\]

With a change of variables, \(z = -\tau/3\), we can write

\[
z e^z = -\frac{\sqrt{C}}{3}. \tag{96}
\]

Here we note that the Lambert function \(W(x)\) is the inverse function of the function \(x = W(x)e^{W(x)}\). Although this function has infinite branches, we would be interested in the real branches: the principal one, denoted by \(W_0(x)\), which takes the values \(W_0(x) \geq -1\) for \(x \geq -1\); and the second one \(W_{-1}(x)\), which takes the values \(W_{-1}(x) \leq -1\) for \(-1 \leq x \leq 0\). Thus the solution of equation (96) would be, according the definition of the Lambert function,

\[
z = -\frac{\tau}{3} = W\left(-\frac{\sqrt{C}}{3}\right).
\]

In order to decide which branch, we recall that \(0 < C < \frac{2}{\pi}\). Thus, the argument of the Lambert function satisfies

\[
-\frac{1}{3} \sqrt{\frac{2}{\pi}} < 0,
\]

or \(-0.2867513379 < C < 0\). If we use the branch \(W_0(x)\), we will obtain a critical time that satisfies \(0 < \tau < 1.3484499515\); this interval corresponds to the solutions \(\tau_{c1}\), the smaller times, which we discard. However, if we use the other branch, the critical time satisfies
\[ \tau > 5.642\,637\,4987 \] and we associate it with the transition time \( \tau_c \). The appropriate solution is given by,

\[ \tau_c = -3W_1\left(-\frac{\sqrt{C}}{3}\right). \tag{97} \]

The above formula for \( \tau_c \) is model independent if the energy density \( \rho(E) \) entering the calculation of the survival amplitude is model independent. This density as mentioned above, is up to a factor, the same as the one obtained in [28] solely using the properties of the \( S \) matrix and a theorem of Mittag-Leffler. In [40], the authors had also obtained an equation similar to equation (96) but for a Breit–Wigner form of \( \rho(E) \) without a threshold factor. The solution of the equation in [40] can be written as

\[ \tau_{BW} = -W_1\left(-\frac{4\pi x^2}{r}\right). \tag{98} \]

In table 1, values of \( \tau_c \) (this work) and \( \tau_{BW} \) (as in [40]) for some values of \( x_r \) are compared. The absence of the threshold factor (apart from the use of a Breit–Wigner form) gives rise to smaller critical times \( \tau_{BW} \) as compared to \( \tau_c \) evaluated from the model independent form involving the correct threshold for \( \rho(E) \) as in (70). The second column displays some fitted values, \( \tau_{fit} \), to be discussed below.

Though, in principle, most real life resonances would correspond to \( x_r < 1 \), it is interesting to note that for \( C = 2/\pi \) (or \( x_r \to \infty \)), we have the lower bound of the critical time: \( \tau_c = 5.642\,6375 \).

The critical time for the transition from the exponential to the power law at large times was also studied in [18] in the context of a single isolated resonance. Determining the transition time from a numerical calculation of the survival probabilities for several values of the variable \( R = \epsilon_r/\Gamma_r \) and observing its behaviour as a function of this variable, the authors assumed a logarithmic form for the transition time as follows:

\[ \tau_{fit} = A \ln (R) + B \tag{99} \]

and obtained the values \( A = 5.41 \) and \( B = 12.25 \) from a fitting procedure. This formula is a refinement of the estimate, \( \tau_L = K \ln (R) \) of Winter [57]. In figure 1, we show a comparison of \( \tau_{fit} \) and \( \tau_c \) evaluated from the analytical expression (97) as a function of the variable \( R \). It must be noted that (i) even though \( \tau_c \) and \( \tau_{fit} \) are very similar for most values of \( R \), for small \( R \) which corresponds to the case of broad resonances (such as the sigma meson for example [58]) they can be quite different and (ii) whereas \( \tau_c \) has a finite lower limit of about 5.64 mentioned above, \( \tau_{fit} \) can even take negative values for very small \( R \). The inset in figure 1 shows that below \( R = 0.3 \), the analytical expression (97) and the fitted one, equation (99) start differing. The region of \( R < 1 \) has indeed been found to be important in literature in connection with the decay of artificial quantum structures [59]. In [59], the authors found that the decay law could have a non-exponential form at all times, for the range, \( 0 < R \leq 0.3 \). The decay of a single ultracold atom was also shown to be non-exponential in [60] below \( R = 0.3 \). The findings of [59] and [60] essentially imply that there is no transition time.

6. Non-exponential decay of particles and nuclei

We shall now apply the results obtained in this work to study some unstable states which have been investigated experimentally. In table 2, we list the critical times for the beginning of the non-exponential (power) law for the particles and nuclei which have been studied in literature.
The transition time as calculated in the present work appears many half-lives later than the number of observed half-lives. From the values given in the table, it is evident that (a) it was necessary to wait much longer to observe the power law (b) but waiting so long would also destroy most of the sample with nothing left for measurement. One could then think of observing the broader resonances such as the sigma meson with a width of a few hundred MeV leading to a very small $\tau_c \sim 9$, however, such a width corresponds to a lifetime of about $10^{-23}$ s, making the observation once again not possible.

Table 1. Critical values $\tau_c = \Gamma/\xi_c$ of the transition to the non-exponential power law behaviour of the survival probability as a function of the parameter $x_r = \Gamma_r/2\xi_r$ in the model independent (97), Breit–Wigner parametrization (98) and fitted parametrization (99) cases.

| $x_r$ | $\tau_c$ (97) | $\tau_c$ (99) | $\tau_c$ (98) |
|-------|----------------|----------------|---------------|
| 0.1   | 21.1           | 21             | 15.6          |
| 0.2   | 17.1           | 17.2           | 12.4          |
| 0.3   | 14.8           | 15             | 10.4          |
| 0.4   | 13             | 13.5           | 9             |
| 0.5   | 11.9           | 12.3           | 7.8           |

| $x_r$ | $\tau_c$ (97) | $\tau_c$ (99) | $\tau_c$ (98) |
|-------|----------------|----------------|---------------|
| 0.6   | 11             | 11.3           | 6.8           |
| 0.7   | 10.2           | 10.4           | 5.9           |
| 0.8   | 9.6            | 9.7            | 5.0           |
| 0.9   | 9.1            | 9.1            | 4.2           |
| 1     | 8.7            | 8.5            | 3.3           |

Figure 1. Comparison of critical times as a function of the variable $R = \xi_r/\Gamma_r$ using equations (97) and (99). The inset shows the same figure for a smaller range of $R$. [6, 61–63]. The transition time as calculated in the present work appears many half-lives later than the number of observed half-lives. From the values given in the table, it is evident that (a) it was necessary to wait much longer to observe the power law (b) but waiting so long would also destroy most of the sample with nothing left for measurement. One could then think of observing the broader resonances such as the sigma meson with a width of a few hundred MeV leading to a very small $\tau_c \sim 9$, however, such a width corresponds to a lifetime of about $10^{-23}$ s, making the observation once again not possible.
In one of the early works on the time evolution of unstable states, the oscillatory character of the survival probability at short and large times was demonstrated by Winter in a barrier penetration problem [64]. For narrow resonances, i.e. for \( \alpha = \frac{\Gamma_r}{2 \epsilon_r} \ll 1 \), following the non-exponential behaviour at very short times, the survival probability displays a prominent exponential decay law followed by a strong oscillatory transition region (several half-lives \( \tau_c = \Gamma_r t_c \) later) which is then followed by the power law at large times. The origin of this particular oscillation lies in the interference of the exponential and power law behaviours. In this section, we shall investigate the large time transition region and obtain an analytical expression to describe it.

### 7.1. Origin of the oscillatory term

As we have already seen, the survival amplitude can be expressed as a sum of two parts: one describing an exponential decay, \( A_e \), and another term \( A_p \) with a power law behaviour. Thus, the total amplitude \( A(t) \) is given by,

\[
A(t) = A_e(t) + A_p(t),
\]

with

\[
A_e(t) = \frac{k_r}{\text{Re} \langle k_e \rangle} e^{-i k_r^2 t},
\]

\[
A_p(t) = -\frac{1}{2 \sqrt{\pi} \text{Re} \langle k_e \rangle} \left[ k_e e^{-i k_r^2 t} \Gamma \left( \frac{1}{2}, -i k_r^2 t \right) - k_e^* e^{-i k_r^2 t} \Gamma \left( \frac{1}{2}, -i k_r^2 t \right) \right]
\]

\[
= -\frac{e^{i \pi/4}}{\sqrt{4 \pi} \text{Re} \langle k_e \rangle} \text{Im} \left( \frac{1}{k_r^2} \right) t^{-3/2} + O(t^{-5/2}), \quad t \to \infty,
\]

where \( k_r^2 = \epsilon_r - i \Gamma_r / 2 \). The survival probability is

\[
P(t) = |A(t)|^2 = |A_e(t)|^2 + |A_p(t)|^2 + 2 \text{Re} \left[ A_e(t) A_p^*(t) \right] = P_e(t) + P_p(t) + 2 \text{Re} \left[ A_e(t) A_p^*(t) \right],
\]

where \( P_e(t) = |A_e(t)|^2 \) and \( P_p(t) = |A_p(t)|^2 \). Given the fact that the oscillatory behaviour becomes evident on a logarithmic scale, we rewrite the above equation as,

### Table 2

Critical values \( \tau_c = \Gamma_r t_c \) of the transition to the non-exponential behaviour for experimentally measured particles and nuclei.

| \( ^{56}\text{Mn}(3^+) \to ^{56}\text{Fe}(2^+) + e^- + \bar{\nu}_e \) [6] | \( ^{222}\text{Rn} \to ^{218}\text{Po} + \alpha \) [61, 62] | \( K^+ \to \mu^+ \nu_\mu \) [63] | \( K^+ \to \pi^+ \pi^0 \) [63] |
|---|---|---|---|
| Lifetime | \( x_r = \Gamma_r / 2 \epsilon_r \) | \( \tau_c \) | Number of half-lives measured |
| \( ^{56}\text{Mn}(3^+) \to ^{56}\text{Fe}(2^+) + e^- + \bar{\nu}_e \) [6] | 2.5789 h | 1.2 \times 10^{-26} | 316 | 45 |
| \( ^{222}\text{Rn} \to ^{218}\text{Po} + \alpha \) [61, 62] | 3.8235 d | 1.2 \times 10^{-28} | 339 | 27, 40 |
| \( K^+ \to \mu^+ \nu_\mu \) [63] | 12.443 ns | 4.5 \times 10^{-17} | 204 | 7.3 |
| \( K^+ \to \pi^+ \pi^0 \) [63] | 12.265 ns | 8.4 \times 10^{-17} | 201 | 4 |

7. Interference region

In one of the early works on the time evolution of unstable states, the oscillatory character of the survival probability at short and large times was demonstrated by Winter in a barrier penetration problem [64]. For narrow resonances, i.e. for \( x_r = \Gamma_r / 2 \epsilon_r \ll 1 \), following the non-exponential behaviour at very short times, the survival probability displays a prominent exponential decay law followed by a strong oscillatory transition region (several half-lives \( \tau_c = \Gamma_r t_c \) later) which is then followed by the power law at large times. The origin of this particular oscillation lies in the interference of the exponential and power law behaviours. In this section, we shall investigate the large time transition region and obtain an analytical expression to describe it.
\[
P(t) = \left[ P_e(t) + P_p(t) \right] \left\{ 1 + 2 \frac{\text{Re} \left[ A_e(t)A_p^*(t) \right]}{|A_e(t)|^2 + |A_p(t)|^2} \right\} = I(t) \left[ P_e(t) + P_p(t) \right], \tag{104}
\]

where we have defined a modulating function, \( I(t) \), such that
\[
I(t) = 1 + 2 \frac{\text{Re} \left[ A_e(t)A_p^*(t) \right]}{|A_e(t)|^2 + |A_p(t)|^2}. \tag{105}
\]

Taking the logarithm on both sides of the above equation, we can write,
\[
\ln P(t) = \ln \left( P_e(t) + P_p(t) \right) + \ln I(t). \tag{106}
\]

Equation (106) hints that the modulating function \( I(t) \) must give rise to the oscillations and this is indeed confirmed in figure 2. The modulating function \( I(t) \) is shown in figure 3 on a linear scale.

7.2. Analytical expression for the modulating function

If we naively replace equations (101) and (102) (first line) in equation (105), the modulating function can be written as,
\[
I(t) = 1 + 2 \frac{\text{Re} \left[ A_e(t)A_p^*(t) \right]}{|A_e(t)|^2 + |A_p(t)|^2}
= 1 - \frac{1}{\sqrt{\pi}} \frac{\text{Re} \left[ \gamma_e(k_r)e^{-ik_r^2t} - \frac{1}{2} \gamma_e(k_r)e^{-ik_r^2t} \right] + \frac{1}{4\pi} \frac{\text{Re} \left[ \gamma_r(k_r)e^{-ik_r^2t} + \frac{1}{2} \gamma_r(k_r)e^{-ik_r^2t} \right]}{\gamma_r(k_r)e^{-ik_r^2t} - \frac{1}{2} \gamma_r(k_r)e^{-ik_r^2t}}}{\gamma_r(k_r)e^{-ik_r^2t} - \frac{1}{2} \gamma_r(k_r)e^{-ik_r^2t}}\tag{107}
\]

where \( \gamma_r(k_r) \) is given by equation (68). Thus, the above equation as such would be quite difficult to analyze and hence we consider approximating \( A_p(t) \) simply by the power law behaviour at large times. Such an approximation is quite good for small values of \( x_r \), where the critical time for the transition from the exponential to the power law behaviour (as seen in an earlier section) is quite large. Thus the expressions which will be derived below, will be valid only for resonances where \( x_r < 1 \). With \( \tau = \Gamma_r t \) and \( k_r^2 = \epsilon_r - i\tau/2 \), we now write,
\[
-ik_r^2 t = -\frac{1}{2} \tau - i\omega_r \tau, \tag{108}
\]

where \( \omega_r \) is defined as
\[
\omega_r = \frac{\epsilon_r}{\Gamma_r}. \tag{109}
\]

The expressions (101) and (102) can now be written as,
\[
A_e(t) = \frac{k_r}{\text{Re} \left( k_r \right)} e^{-ik_r^2 t} = \frac{k_r}{\text{Re} \left( k_r \right)} e^{-\tau/2} e^{-i\omega_r \tau}, \tag{110}
\]
\[
A_p(t) = -\frac{e^{i\pi/4}}{\sqrt{4\pi} \text{Re} \left( k_r \right)} \text{Im} \left( \frac{1}{k_r^2} \right) t^{3/2} = -\frac{e^{i\pi/4}}{\sqrt{4\pi} \text{Re} \left( k_r \right)} \sqrt{\frac{\Gamma_r^3}{4\pi}} \text{Im} \left( \frac{1}{k_r^2} \right) t^{3/2}, \tag{111}
\]
and the modulating function becomes

\[ I(t) = 1 + \frac{2 \text{Re} \left[ A_e(t)A_p^*(t) \right]}{|A_e(t)|^2 + |A_p(t)|^2} \]

\[ = 1 - \left[ 2|k_e| \sqrt{\frac{\Gamma^3}{4\pi}} \text{Im} \left( \frac{1}{k^2} \right) \right] \frac{e^{-\tau/2}e^{-3/2} \text{Re} \left( e^{-i\omega_0\tau} e^{-i\pi/4} e^{i \text{Arg} k_e} \right)}{|k_e|^2 e^{-\tau} + \frac{\Gamma^3}{3\pi} \text{Im} \left( \frac{1}{k^2} \right)^2} \tau^{-3} \]

(112)
Introducing the constant C given by equation (90), we get,

\[ I(\tau) = 1 + D \left( \frac{e^{-\tau/2} \tau^{-3/2}}{e^{-\tau} + C \tau^{-3}} \right) \cos \left( \omega_r \tau + \pi/4 - \text{Arg } k_r \right), \quad (113) \]

where \( D \) is given by,

\[ D = -\frac{2}{|k_r|} \sqrt{\frac{|\Gamma_3|}{4\pi}} \text{Im} \left( \frac{1}{k_r^2} \right). \quad (114) \]

The modulating function so derived allows us to infer that:

(i) \( I(\tau) \) oscillates about \( I = 1 \) with a frequency \( \omega_r \).

(ii) The function is modulated with an amplitude

\[ m(\tau) = \frac{e^{-\tau/2} \tau^{-3/2}}{e^{-\tau} + C \tau^{-3}} \quad (115) \]

which is expected to be maximum at the critical time.

(iii) Apart from the above, the function \( I(\tau) \) is expected to present problems for small values of \( \tau \) (since we approximated \( A_p(t) \) by its behaviour at large times).

(iv) Since \( \omega_r = 1/2x_r \), \( I(\tau) \) is expected to oscillate a lot if \( x_r \) is small. This will not be the case for \( x_r \) close to or bigger than unity (see for example the case of the broad \( \sigma \) meson where one observes no oscillation at all [26]).

In figure 4, we compare the modulating function calculated using equation (107) (with the complete analytical expressions for \( A_p(t) \)) and that using the approximation of the large time behaviour mentioned above. As expected, the function presents problems at small times but the approximation of using the large time behaviour instead of the exact expression is quite good. Before resolving the problem at small times, let us first study the function \( m(\tau) \).

7.3. Analysis of \( m(\tau) \)

If we write the function as follows:

\[ m(\tau) = \frac{1}{e^{-\tau/2} \tau^{-3/2} + Ce^{s/2} \tau^{-3/2}}, \]

then its derivative is given as

\[ m'(\tau) = \frac{1}{2} \frac{e^{\tau/2}(\tau - 3)(\tau^3 - Ce^\tau)}{(e^{-\tau/2} \tau^{-3/2} + Ce^{s/2} \tau^{-3/2})^2}. \]

The critical times in this function are \( \tau = 0, \tau_c, 3, \tau_{c2} \) (in increasing order), where the second and fourth ones are solutions of \( \tau^3 - Ce^\tau = 0 \) and as analyzed in section 5 has two real solutions. It is easy to see that \( \tau = 0 \) and \( \tau = 3 \) are minima, while \( \tau = \tau_c, \tau_{c2} \) are maxima. In figure 5 we show \( m(\tau) \) for \( x_r = 0.1 \). Here, \( \tau_{c1} = 0.0184942 \) and \( \tau_{c2} = 21.143362 \) with the latter corresponding to the critical time for the transition from the exponential to the power law behaviour (see table 1). The most relevant observation here is that \( m(\tau) \) does display a maximum at the critical time as expected. However, in order to have an \( m(\tau) \) that describes the modulating function correctly, we must get rid of the maximum close to \( \tau = 0 \). One way of doing this could be by constructing a function of \( \tau - \tau_c \) such that at \( \tau = \tau_c \), it is given by \( m(\tau_c) \), which is \( \frac{1}{2\sqrt{e}} \).
The best way to do this is by expanding \( \frac{1}{\sqrt{C} m(\tau)} \) in a series of \( \tau - \tau_c \), so that,

\[
\frac{1}{\sqrt{C} m(\tau)} = \sum_{n=0}^{\infty} \left( \tau - \tau_c \right)^n \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2} - n)} \left( \frac{2}{\tau_c} \right)^n + \frac{(-1)^{n+k}}{\Gamma(\frac{5}{2} - k)} \left( \frac{2}{\tau_c} \right)^k
\]

\[
= 2 + \frac{1}{4} \left( 1 - \frac{3}{\tau_c} \right)^2 \left( \tau - \tau_c \right)^2 + \frac{1}{48} \left( \frac{36}{\tau_c^2} + \frac{108}{\tau_c^2} \right) \left( \tau - \tau_c \right)^3
\]

\[
+ \frac{1}{192} \left( 1 - \frac{12}{\tau_c} + \frac{54}{\tau_c^2} - \frac{204}{\tau_c^3} + \frac{477}{\tau_c^4} \right) \left( \tau - \tau_c \right)^4 + \cdots.
\]

In order to decide on the number of relevant terms in the expansion, in figure 6 we display \( m(\tau) \) calculated by truncating the series at different number of terms. We see that already up to the fourth order, we obtain a good estimate of the exact \( m(\tau) \). There is no peak at small times.

We must mention that \( m(\tau) \) (and hence also the modulating function) is not symmetric about \( \tau = \tau_c \). Hence, if we define the constants

\[
m_2 = \frac{1}{8} \left( 1 - \frac{3}{\tau_c} \right)^2,
\]

\[
m_3 = \frac{1}{96} \left( \frac{36}{\tau_c^2} + \frac{108}{\tau_c^2} \right),
\]

\[
m_4 = \frac{1}{384} \left( 1 + \frac{12}{\tau_c} + \frac{54}{\tau_c^2} - \frac{204}{\tau_c^3} + \frac{477}{\tau_c^4} \right).
\]

then

\[
m(\tau) = \frac{1/2\sqrt{C}}{1 + m_2 (\tau - \tau_c)^2 + m_3 (\tau - \tau_c)^3 + m_4 (\tau - \tau_c)^4},
\]

Figure 4. Approximate and exact form of the modulating function.
and the modulating function can be written as

\[ I(\tau) = 1 + \left( \frac{D}{2\sqrt{C}} \right) \frac{\cos (\omega_\tau \tau + \pi/4 - \text{Arg} k_\tau)}{1 + m_2(\tau - \tau_c)^2 + m_3(\tau - \tau_c)^3 + m_4(\tau - \tau_c)^4}. \] (121)

In figure 7, we compare this expression with that of the modulating function given by equation (107). In a small region around the critical time, the two expressions coincide exactly with
Finally, in figure 8, we compare the exact survival probability with that using the approximate form of the modulating function. Apart from this, the green enveloping curves show the survival probability evaluated without the oscillatory part. These curves are evaluated by writing the modulating function in (113) with the maximum and minimum values of the cosine term, i.e.

**Figure 7.** Exact modulating function (red line) and the approximate one (blue line).

**Figure 8.** Survival probability calculated exactly (blue line) and calculated using the approximate modulating function (red line). Green lines show the curves evaluated without the oscillatory part in the modulating function (see text).
\[ I_{\pm}(\tau) = 1 + D \left( e^{-\tau/2} \tau^{-3/2} \right) (\pm 1), \] (122)

and \[ P_{\pm}(\tau) = I_{\pm}(\tau) \left[ P_c(\tau) + P_p(\tau) \right]. \] Though somewhat obvious, it is interesting to note that the two curves, \( P_+(\tau) \) and \( P_-(\tau) \) coincide in all regions except for the transition region where they separate. This implies that \( m(\tau) \) can indeed be used to define the transition region between the exponential and the non-exponential region at large times.

### 8. Summary and conclusions

Writing the survival amplitudes based on the Green’s function method (GC) as well as the Jost functions method (DN), as a Fourier transform similar to the one used in the Fock–Krylov method, it is shown that the GC and DN approaches are equivalent up to some constants. Such a rewriting allows one to define the densities \( \rho^{\text{GC}}(E) \) and \( \rho^{\text{DN}}(E) \) which are then compared with the definition of a density obtained from a statistical physics motivated expression. The latter is obtained from a relation given by Beth and Uhlenbeck which relates the density of states, \( \rho^{\text{BU}}(E) \), to the energy derivative of the scattering phase shift, \( \frac{d\delta_l}{dE} \), in the \( l \)th partial wave. A theorem of Mittag-Leffler further allows \( \rho^{\text{BU}}(E) \) to be expressed in terms of the poles of the \( S \)-matrix [28], thus making the comparison with \( \rho^{\text{GC}}(E) \) and \( \rho^{\text{DN}}(E) \) straightforward.

For the case of an isolated \( s \)-wave resonance, \( \rho^{\text{GC}}(E) \) and \( \rho^{\text{DN}}(E) \) give the same expression as \( \rho^{\text{BU}}(E) \) plus a small correction term. Noting that the coefficients appearing in the GC and DN formalism satisfy the same conditions, a general analytic form for the survival amplitude in terms of the incomplete gamma functions is also derived. Apart from this, the analysis for large times is done by applying the steepest descent method as well as using the asymptotic expansion for the incomplete gamma function. The results obtained in both cases are the same, in particular, the \( t^{-3} \) power law for \( s \)-wave resonances, \( \rho^{\text{GC}}(E) \) and \( \rho^{\text{DN}}(E) \) give the same expression as \( \rho^{\text{BU}}(E) \) plus a small correction term.

Noting that the coefficients appearing in the GC and DN formalism satisfy the same conditions, a general analytic form for the survival amplitude in terms of the incomplete gamma functions is also derived. Apart from this, the analysis for large times is done by applying the steepest descent method as well as using the asymptotic expansion for the incomplete gamma function. The results obtained in both cases are the same, in particular, the \( t^{-3} \) power law for \( s \)-wave resonances, which is consistent with most of the literature (see however, [7, 43] for systems without spherical symmetry).

The equation deduced for the survival amplitude allowed us to easily separate the exponential and power law behaviours and define a critical time at the intersection of the survival probability for intermediate and large times. A detailed analysis of the transition region reveals interesting aspects as well as the origin of the oscillatory behaviour of the survival probability in this region. An analytical expression for the critical transition time, \( \tau_c \), is obtained in terms of the Lambert W function. Calculations of \( \tau_c \) for the decays which have been measured experimentally up to several half-lives with the objective of observing the power law behaviour reveal the reason for the negative results of these experiments. The number of half-lives after which the power law starts, for example, for a narrow nuclear resonance such as \( ^{56}\text{Mn} \) is about 300, whereas the experiment was carried out only up to 45 half-lives. However, performing measurements up to 300 half-lives would be practically impossible since the exponential decay law would destroy almost all the sample by the time the narrow resonance reaches the power law. Broad resonances such as the \( \sigma \) meson reach the power law much earlier, however, the lifetime is too short making the experimental observation once again difficult. The results of the present work indicate that the non-exponential behaviour of nuclear and particle resonances at large times is hard to observe. This conclusion is in agreement with other literature such as [18], where, for the case of \( ^{56}\text{Mn} \), the authors found that the deviation
from the exponential decay law would occur around $\tau_c = 331$ and in the case of the short lived $^3\text{He}$ state with a lifetime of $\sim 10^{-22}$ s, it would occur around $\tau_c = 12$.

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Appendix A. Evaluation of survival amplitude for large times using the steepest descent method

The integrals required for the steepest descent method have the form\(^5\),

\[
\int_C f(z) e^{\phi(z)} \, dz, \tag{A.1}
\]

where $t > 0$ and $t \gg 1$, $f(z)$ and $\phi(z)$ are analytic functions in a region $D$ and $C \in D$ is the contour (not necessarily closed) of integration. In our case, the evaluation of the survival amplitude for large $t$ depends on the computation of the integral $M(k_n,t)$ when $t \gg 1$, which is identically equal to equation (A.1) if $f(z) = (z - k_n)^{-1}$, $\phi(z) = -iz^2$, $C = \{z \in C : \text{Im } z = 0\}$ and $\text{Im } k_n < 0$ (as of now, we do not include the coefficient $i/2\pi$).

The function $\phi(z)$ has one saddle-point of order $n = 2$ at $z = 0$ since $\phi'(0) = 0$ and $\phi''(0) = -2i = 2e^{-i\pi/2} = ae^{i\alpha}$. The directions of steepest descent are

$$
\theta = -\frac{\alpha}{n} + \frac{(2m + 1)\pi}{n}, \quad m = 0, 1, \ldots, n - 1.
$$

In our case,

$$
\theta = \frac{3}{4} \pi, \frac{7}{4} \pi.
$$

The contour $C$ must be deformed such that it follows these directions. In figure A1, we show how the contour $C$ is deformed.

The line $BOA$ is the contour $C$. The straight lines $OC$ and $OB$ are the steepest descent directions. For linking the integrals in those contours, we have to close them with the arcs of circumference $CD$ and $BA$, both of radius $R$. Since the integrand is analytic in the contour $OCD$, using the Cauchy theorem, we get,

\[
\int_{DO} = - \left( \int_{OC} + \int_{CD} \right). \tag{A.2}
\]

For the contour $OAB$, however, the integrand has a pole depending on the fact if $k_n$ satisfies the condition $-\frac{\pi}{2} < \text{Arg } z < 0$ or not. The residue theorem allows us to write

\[
\int_{OA} = \int_{OB} + \int_{BA} -2\pi i e^{-ik_n^2} F(k_n), \tag{A.3}
\]

\(^5\)We follow the notation from Ablowitz and Fokas’ book for steepest descent method. See [65], chapter 6.
where the function \( F(k_n) \) is defined by
\[
F(k_n) = \begin{cases} 
1 & -\frac{\pi}{4} < \text{Arg } z < 0, \\
0 & \text{i. o. c.}
\end{cases}
\] (A.4)

Adding (A.3) and (A.4), we obtain
\[
\int_{DO} + \int_{OA} = \int_{OB} + \int_{BA} - \int_{OC} - \int_{CD} - 2\pi i e^{-ik^2} F(k_n).
\] (A.5)

Since, in the limit \( R \to \infty \), both \( \int_{BA} \) and \( \int_{CD} \) tend to zero, (A.5) takes the following form:
\[
\int_{-\infty}^{\infty} e^{-ix^2} \frac{dx}{x - k_n} = \int_{\text{Arg } z = -\frac{\pi}{4}} e^{-iz^2} \frac{dz}{z - k_n} - 2\pi i e^{-ik^2} F(k_n)
\]
\[
= e^{-ir/4} \int_{0}^{\infty} e^{-r^2} \frac{dr}{re^{i\pi/4} - k_n} - e^{i3\pi/4} \int_{0}^{\infty} e^{-r^2} \frac{dr}{re^{i3\pi/4} - k_n} - 2\pi i e^{-ik^2} F(k_n).
\] (A.6)

The principal contribution to the value of the integrals for \( t \) large, comes from a neighbourhood of \( r = 0 \). Expanding \( (re^{-ir/4} - k_n)^{-1} \) and \( (re^{i3\pi/4} - k_n)^{-1} \) in a Taylor series up to the third order around \( r = 0 \) and calculating the integrals, we have:
\[
\int_{-\infty}^{\infty} e^{-ix^2} \frac{dx}{x - k_n} = \frac{i\sqrt{\pi}}{k_n} e^{i\pi/4} t^{-1/2} - \frac{i\sqrt{\pi}}{2k_n^3} e^{i3\pi/4} t^{-3/2} + O(t^{-5/2}),
\] (A.7)
and thus,
\[
M(k_n, t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-ix^2} \frac{dx}{x - k_n} = -\frac{1}{2\pi} \left[ \frac{i\sqrt{\pi}}{k_n} e^{i\pi/4} t^{-1/2} - \frac{i\sqrt{\pi}}{2k_n^3} e^{i3\pi/4} t^{-3/2} \right] + O(t^{-5/2}).
\] (A.8)

We ignore the exponential term because for large \( t \) because it is negligible with respect to the negative power of \( t \). Substituting (A.8) in (22) and using properties (2) and (3) mentioned in section 2.3, we have

![Figure A1. Contour of integration for the integral (16) and the directions of steepest descent of its integrand.](image-url)
$$A(t) = \frac{1}{4\sqrt{\pi}} e^{i\pi/4} \left( \sum_{n} \frac{C_n \bar{C}_n}{k_n^3} \right) t^{-3/2} + O(t^{-5/2}) = -\frac{1}{4\sqrt{\pi}} e^{i\pi/4} \text{Im} \left( \sum_{p} \frac{C_p \bar{C}_p}{k_p^3} \right) t^{-3/2} + O(t^{-5/2}).$$

(A.9)

**Appendix B. Residues of the inverse of the S-matrix**

In [28], the authors show that for a system under the influence of a central potential of finite range \( R \), if the S-matrix for the orbital angular momentum \( l \) is written as a product, i.e.

$$S_l(k) = e^{-2iRk} \prod_{n} \frac{(k + k_{ln})(k - k_{ln}^*)}{(k - k_{ln})(k + k_{ln}^*)} \prod_{m} \frac{i\zeta_{ln} + k}{i\zeta_{ln} - k},$$

(B.1)

where \( \{k_{ln}\} \) with \( n = 1, 2, \ldots \) corresponding to the resonant poles of the S-matrix in the fourth-quadrant and \( i\zeta_{ln} \) with \( m = \pm 1, \pm 2, \ldots \) (with the plus signs corresponding to the bound and minus to the virtual states respectively) corresponding to the poles on the imaginary axis of the complex \( k \) plane, then the residue of \( S_l(k) \) at \( k = k_{ln} \) is

$$b_{ln} = \text{Res} \left[ S_l(k), k = k_{ln} \right] = 2ik_{ln} e^{-2iR_{ln}} \tan \left( \text{Arg} \ k_{ln} \right) \prod_{p \neq n} \frac{(k_{ln} + k_p)(k_{ln} - k_p^*)}{(k_{ln} - k_p)(k_{ln} + k_p^*)} \prod_{m} \frac{i\zeta_{ln} + k_{ln}}{i\zeta_{ln} - k_{ln}},$$

(B.2)

and the residue of \( S_l(k) \) at \( k = -k_{ln}^* \) is

$$\text{Res} \left[ S_l(k), k = -k_{ln}^* \right] = -b_{ln}^*.$$  

(B.3)

If the system under consideration has no bound and virtual states, equations (B.1) and (B.2) are simplified to:

$$S_l(k) = e^{-2iRk} \prod_{n} \frac{(k + k_{ln})(k - k_{ln}^*)}{(k - k_{ln})(k + k_{ln}^*)},$$

(B.4)

$$b_{ln} = 2ik_{ln} e^{-2iR_{ln}} \tan \left( \text{Arg} \ k_{ln} \right) \prod_{p \neq n} \frac{(k_{ln} + k_p)(k_{ln} - k_p^*)}{(k_{ln} - k_p)(k_{ln} + k_p^*)},$$

(B.5)

and equation (B.3) remains the same. Here, we are interested in computing the residues of \(1/S_l(k)\) at \( k = -k_{ln} \) and \( k = k_{ln}^* \). For the former pole:

$$\text{Res} \left[ 1/S_l(k), k = -k_{ln} \right] = \lim_{k \to k_{ln}} \frac{k + k_{ln}}{S_l(k)}$$

$$= -2k_{ln} \frac{-k_{ln} + k_{ln}^*}{(k_{ln} + k_{ln}^*)} e^{-2iR_{ln}} \prod_{p \neq n} \frac{(-k_{ln} - k_p)(-k_{ln} + k_p^*)}{(-k_{ln} + k_p)(-k_{ln} - k_p^*)}$$

$$= -2ik_{ln} e^{-2iR_{ln}} \tan \left( \text{Arg} \ k_{ln} \right) \prod_{p \neq n} \frac{(k_{ln} + k_p)(k_{ln} - k_p^*)}{(k_{ln} - k_p)(k_{ln} + k_p^*)} = -b_{ln}^*.$$  

(B.6)
and the latter pole:

$$\text{Res} \left[\frac{1}{S_i(k)}, k = k_{in}^*\right] = \lim_{k \to k_{in}} \frac{k - k_{in}^*}{S_i(k)}$$

$$= 2k_{in}^{-1}k_{in} + k_{in}^* e^{2iR_k} \prod_{p \neq n} (k_{in}^* - k_{ip})(k_{in}^* + k_{ip})(k_{in}^* - k_{ip})$$

$$= 2i k_{in} e^{-2iR_k} \tan \left( \text{Arg} k_{in} \right) \prod_{p \neq n} \frac{(k_{in} + k_{ip})(k_{in}^* - k_{ip})}{(k_{in}^* - k_{ip})(k_{in} + k_{ip})} = b_{in}.$$  \hspace{1cm} (B.7)

In principle, the residues of the $S$-matrix depend on all its poles and it is difficult to compute them, however, the authors in [28] show that it is possible to write this residue in two parts such that the former depends only on the pole where we calculate the residue and the latter depends on the remaining poles. Under certain conditions, the residue can be approximated as follows:

$$b_{in} \approx 2i k_{in} \tan \text{Arg} k_{in}. \hspace{1cm} (B.8)$$

The details of the derivation of this approximation and the conditions under which it is valid can be found in [28].

**Appendix C. Detailed computation of $\varrho(k)$ from $I(k, r, r')$**

Substituting (49) in (40):

$$\varrho(k) = \frac{8}{\pi} \int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) k^2 \text{ Re} \left[ \sum_n \frac{i(k_n r, r')}{k_n(k^2 - k_n^2)} \right] dr' dr$$

$$= \frac{4}{\pi} k^2 \sum_n \frac{1}{k_n(k^2 - k_n^2)} \int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) i(k_n, r, r') dr' dr$$

$$+ \frac{4}{\pi} k^2 \sum_n \frac{1}{k_n^2(k^2 - k_n^2)} \int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) i^*(k_n, r, r') dr' dr. \hspace{1cm} (C.1)$$

If we write the double integral of the second term of the right side of (C.1) as the conjugate of some double integral and use the fact that $i(k, r, r') = i(k, r', r)$, then,

$$\int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) i^*(k_n, r, r') dr' dr = \left[ \int_0^R \int_0^R \psi^*(r, 0) \psi(r', 0) i(k_n, r, r') dr' dr \right]^*$$

$$= \left[ \int_0^R \int_0^R \psi^*(r', 0) \psi(r, 0) i(k_n, r', r) dr' dr \right]^*$$

$$= \left[ \int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) i(k_n, r, r') dr' dr \right]^*. \hspace{1cm} (C.2)$$

and substituting (C.2) in (C.1), we get:

$$\varrho(k) = k^2 \text{ Re} \left[ \sum_n \frac{1}{k_n(k^2 - k_n^2)} \left[ \int_0^R \int_0^R \psi(r, 0) \psi^*(r', 0) i(k_n, r, r') dr' dr \right] \right]. \hspace{1cm} (C.3)$$
Taking the integral in the square brackets and defining
\[ a(k_n) \equiv 4i \int_0^R \int_0^R \psi(r,0)\psi^*(r',0)\iota(k_n, r, r') \, dr \, dr', \]
allows us to write the equation (C.3) in the following form:
\[ \varrho(k) = \frac{2}{\pi} k^2 \text{Re} \sum_n \frac{i a(k_n)}{k_n (k_n^2 - k^2)}. \] (C.5)

Appendix D. Evaluation of integrals for computing the survival amplitude

For calculating the survival amplitude given by (73), we need to study the integral
\[ I(\alpha) = \oint_C \sqrt{z} \frac{e^{-iz \alpha}}{z - \alpha} \, dz, \] (D.1)
where C is the contour shown in figure D1 and \( t > 0 \).

Since the integrand has a branch point at \( z = 0 \), we take the principal branch of \( \sqrt{z} \) such that \( |\text{Arg} \, z| < \pi \). If \( \alpha \) satisfies \( \text{Arg} \, \alpha \in (-\frac{\pi}{2}, 0) \) and \( R < |\alpha| \), using the residue theorem, we have:
\[ I(\alpha) = 2\pi i \alpha^{1/2} e^{-i\alpha t}. \] (D.2)

If \( \alpha \) is not inside or on the contour, the Cauchy’s theorem leads us to,
\[ I(\alpha) = 0. \] (D.3)

Both expressions can be written together as
\[ I(\alpha) = \begin{cases} 
2\pi i \alpha^{1/2} e^{-i\alpha t}, & \text{Re } \alpha \neq 0, \text{ Im } \alpha \neq 0, \text{ Arg } \alpha \in (-\frac{\pi}{2}, 0), \\
0, & \text{Re } \alpha \neq 0, \text{ Im } \alpha \neq 0, \text{ Arg } \alpha \notin (-\frac{\pi}{2}, 0). 
\end{cases} \] (D.4)
On the other hand,

\[ I(\alpha) = \int_{AO} + \int_{OB} + \int_{BA}. \]

For the segment \( AO \), \( z = x \). When \( R \to \infty \),

\[ \int_{AO} = - \int_{\alpha}^{R} \frac{\sqrt{x}}{x - \alpha} e^{-ix} \, dx \to - \int_{0}^{\infty} \frac{\sqrt{x}}{x - \alpha} e^{-ix} \, dx. \]

For the segment \( OB \), \( z = -iy \). When \( R \to \infty \),

\[ \int_{BO} = \int_{0}^{R} \frac{\sqrt{y}}{-iy - \alpha} e^{iy} \, dy \to \frac{\sqrt{y}}{iy - \alpha} e^{-iy} \, dy. \]

For the segment \( BA \), let \( z = Re^{-i\theta} \), where \( \theta \in [0, \pi/2] \). The integral on this segment is:

\[ \int_{BA} = - \int_{0}^{\pi/2} \frac{R^{3/2} e^{-i\theta/2}}{Re^{-i\theta} - \alpha} e^{-iR\sin \theta - iR\cos \theta} \, d\theta. \]

Taking the modulus of this integral and supposing \( R > |\alpha| \), we have:

\[ \left| \int_{BA} \right| \leq \pi/2 \frac{R^{3/2}}{R - |\alpha|} e^{-2Re^{i\theta}} \, d\theta \leq \frac{\pi}{2} R^{3/2} e^{-2Re^{i\theta}} \, d\theta = \frac{\pi}{2} R^{3/2} (1 - e^{-Re^{i\theta}}). \]

Here, we used \( \sin \theta \geq 2\theta/\pi \) with \( \theta \in [0, \pi/2] \). When \( R \to \infty \),

\[ \left| \int_{BA} \right| \to 0, \]

and,

\[ \int_{BA} \to 0. \]

In the limit \( R \to \infty \), \( I(\alpha) \) is equal to

\[ I(\alpha) = - \int_{0}^{\infty} \frac{\sqrt{x}}{x - \alpha} e^{-ix} \, dx + \frac{\sqrt{y}}{y - i\alpha} e^{-iy} \, dy. \]

The integral \( [66] \)

\[ \int_{0}^{\infty} \frac{E^\nu}{E + \sigma} e^{-xE} \, dE = \Gamma(\nu + 1)e^{\sigma x} \Gamma(-\nu, \sigma) \quad \text{Re } \nu > -1, \text{ Re } x > 0, |\text{Arg } \sigma| < \pi, \]

where \( \Gamma(\alpha, z) \) is the incomplete gamma function \( [55] \), allows us to write \( I(\alpha) \) as

\[ I(\alpha) = - \int_{0}^{\infty} \frac{\sqrt{x}}{x - \alpha} e^{-ix} \, dx + e^{-ix/4} \Gamma(3/2) e^{-i\alpha/2} \Gamma(-1/2) e^{-i\alpha x} \Gamma(-1/2, -1/2, -i\alpha). \]  

Finally,

\[ \int_{0}^{\infty} \frac{\sqrt{x}}{\alpha - x} e^{-ix} \, dx = I(\alpha) + \frac{i\sqrt{\pi}}{2} \alpha^{1/2} e^{-i\alpha} \Gamma(-1/2, -1/2, -i\alpha), \quad |\text{Arg } \alpha| < \pi. \]  

(D.5)

(D.6)
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