HOPF TYPE LEMMAS FOR SUBSOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS. SUPPLEMENT: DISCUSSION ON THE APPLICATIONS OF THE RESULTS

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Abstract. In the paper we prove a lower bound for subsolutions of the integro-differential equation: 

\[-Au + cu = 0\]

in a domain \(D\). It states that there exists a Borel function \(\psi\), strictly positive on \(D\), depending only on the coefficients of the operator \(A\), \(c\) and \(D\) such that for any subsolution \(u(\cdot)\), that satisfies \(\sup_{D_x} u(y) \geq 0\), one can find a constant \(a > 0\) (that in general depends on \(u\)), for which \(\sup_{D_x} u(y) - u(x) \geq a\psi(x)\), \(x \in D\). The bound is valid for a wide class of Lévy type integro-differential operators \(A\), non-negative, bounded and measurable function \(c\) and a quite general domain \(D \subset \mathbb{R}^d\). Here \(D_x\) is a certain set containing the closure of \(D\) and determined by the support of the Levy jump measure associated with \(A\). The main assumptions made about \(A\) are that: there exists a strong Markov solution to the martingale problem associated with the operator and its resolvent satisfies some minorization condition. This type of a result we call the generalized Hopf lemma.

For certain classes of operators the constant \(a\) could be taken to be equal to \(\sup_{D_x} u(y)\). We refer to such a result as a quantitative version of the Hopf lemma. In some cases a non-negative eigenfunction corresponding to the operator in \(D\) can be admitted as the function \(\psi\) appearing in the lower bound. In particular, this occurs when the transition probability semigroup associated with \(A\) is ultracontractive.

1. Introduction

Let \(m\) be a \(\sigma\)-finite Borel measure, let \(D \subset \mathbb{R}^d\) be a domain, i.e. an open and connected set, and let \(A\) be an integro-differential operator of the form

\[
Au(x) = \frac{1}{2} \text{Tr}(Q(x)\nabla^2 u(x)) + b(x) \cdot \nabla u(x) + \int_{\mathbb{R}^d} \left( u(x+y) - u(x) - y \cdot \nabla u(x) \right) \frac{1}{1 + |y|^2} N(x,dy), \quad x \in D
\]

(1.1)

for any \(u \in C^2(D) \cap C_b(\mathbb{R}^d)\). Consider the following condition.

Minoration condition: there exist \(\alpha \geq 0\), a Borel measurable function \(\psi_D^\alpha : D \to [0, +\infty)\) and \(\nu_D^\alpha\) - a \(\sigma\)-finite Borel measure on \(D\) - such that

\[
R_\alpha^D f(x) \geq \psi_D^\alpha(x) \int_D f d\nu_D^\alpha, \quad x \in D, \ f \in B_b(D).
\]

Given particular \(\alpha, \psi_D^\alpha\) and \(\nu_D^\alpha\) we shall refer to the above hypothesis as condition \(M(\alpha, \psi_D^\alpha, \nu_D^\alpha)\).

Mathematics Subject Classification: Primary 35B50; Secondary 35J15 , 35J08

Keywords: Integro-differential elliptic equation, weak subsolution, maximum principle, the Hopf lemma

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Here $R^D_\alpha$ is the resolvent of the operator $A$ on $D$, see (2.8) below for the precise definition, and $B^*_\epsilon(D)$ is the space of all non-negative, bounded, Borel measurable functions.

The main result of the present paper, see Theorem 3.2, states that if the minorization condition $M(\alpha, \psi^\alpha_D, \nu^\alpha_D)$ holds with strictly positive $\psi^\alpha_D$, the measure $m$ is absolutely continuous with respect to $\nu^\alpha_D$ and some additional, rather natural, assumptions about $A$ (stated below) are satisfied, then there exists a strictly positive function $\psi_D$ on $D$ such that for any subsolution $u$ to

$$(-A + c)u(x) = 0, \quad x \in D$$

(1.2)
satisfying $\bar{u}_{D_S} := \sup_{D_S} u \geq 0$ there exists $a > 0$ for which

$$\bar{u}_{D_S} - u(x) \geq a\psi_D(x), \quad x \in D.$$  

(1.3)

Moreover, if $\alpha \geq \sup_D c$, then we may take $\psi_D = \psi^\alpha_D$. The set $D_S$ appearing above is the extended closure of $D$. In case $A$ is local it is the usual closure of $D$. For non-local operators it contains the closure of the domain and is determined by the Lévy jump measure, see (1.5) below for the precise definition.

Concerning the hypotheses on the operator $A$, we suppose that:

A1) the entries of the symmetric matrix valued function $Q(x) = [q_{i,j}(x)]_{i,j=1}^d$ and coordinates of the vector valued $b(x) = (b_1(x), \ldots, b_d(x))$, $x \in \mathbb{R}^d$ are bounded and Borel measurable. In addition, $Q(x)$ is non-negative definite, i.e.

$$\sum_{i,j=1}^d q_{i,j}(x)\xi_i\xi_j \geq 0, \quad x \in \mathbb{R}^d, \quad (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d,$$

A2) the (Lévy) kernel $N(x, dy)$ is a $\sigma$-finite Borel measure on $\mathbb{R}^d \setminus \{0\}$ for each $x \in \mathbb{R}^d$ and

$$N_x := \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \min\{1, |y|^2\} N(x, dy) < +\infty.$$  

We assume also that:

A3) $c$ is non-negative, bounded and Borel measurable on $D$.

Besides the hypotheses listed above, we shall also assume the existence of a strong Markov solution of the martingale problem associated with $A$, see hypothesis A4) formulated in Section 2.3.

When $N$ is non-trivial, then $A$ is non-local, i.e. the evaluation of $Au(x)$ no longer depends only on the values of $u$ in an arbitrarily small neighborhood of $x$. To describe this dependence define the range of non-locality of $A$ at $x$ as:

$$S_x := \left\{ z \in \mathbb{R}^d : z = x + y, \text{ where } y \in \text{supp } N(x, \cdot) \right\}.$$  

Here supp $N(x, \cdot)$ denotes the support of the measure, i.e. the smallest closed set whose complement is of null measure. The value of $Au(x)$ depends therefore on the values of $u$ in an arbitrarily small neighborhood of $x$ and on its values in $S_x$. Furthermore, for an open set $D \subset \mathbb{R}^d$ we define the range of non-locality of $A$ over $D$ as

$$S(D) = \bigcup_{x \in D} S_x.$$  

(1.4)

For example if $N(x, dy) \equiv 0$, i.e. $A$ is local, then $S(D) = \emptyset$. On the other hand, in the case of the fractional power of the free Laplacian $A = \Delta^{s/2} := (-\Delta)^{s/2}$, with $s \in (0, 2)$, see (5.3) below, we have $S(D) = \mathbb{R}^d$.

It is well known that in the case of a local operator $A$ and a sufficiently regular $D$ the solution of (1.2) (recall that $c \geq 0$) is uniquely determined by its values
on the boundary of the domain $\partial D$. However, this, in general, is no longer true for non-local operators, see e.g. [24]. For this reason, we introduce the extended boundary and extended closure

\[ \partial S \!D := \partial D \cup (\mathcal{S}(D) \setminus D), \quad D_S := \partial S \!D \cup D, \]

respectively. The above notions appear naturally when we want to formulate a counterpart of the weak maximum principle for operators of the form (1.1). Namely, if $u$ is a subsolution of (1.2) in $D$, then

\[ \bar{u}_D \leq \bar{u}_{\partial S} \]

(see Proposition 3.4). Here and in what follows, for a given subset $B \subset \mathbb{R}^d$, we denote $\bar{u}_B := \sup_{y \in B} u(y)$ and $u^+ := \max\{u, 0\}$.

Concerning the subsolution to (1.2), we consider a quite general notion of the weak subsolution, see Definition 2.6 below. It is probabilistically motivated and includes, besides the classical subsolutions also viscosity and Sobolev subsolutions, see Remark 2.6. We denote by $U_c$ the class of all weak subsolutions to (1.2) and by $U^+_c$ the set of $u \in U_c$ that satisfy $\bar{u}_{\partial S} \geq 0$.

In the local case, i.e. when $N \equiv 0$, uniform ellipticity condition holds on compact sets and the boundary is sufficiently regular (1.3) is valid with $\psi_D(x) = \delta_D(x) := \text{dist}(x, \partial D)$ - the distance of $x$ from $\partial D$, see e.g. [44, Section 2.3]. This is the contents of the Hopf lemma valid for second order elliptic operators, see also [27] for the result for a fractional laplacian. An estimate of the form (1.3) can be therefore considered as a generalization of the classical Hopf lemma and we shall refer to $\psi_D$ appearing there as a bottom function. In the present paper, we take a comprehensive look at the validity of (1.3) from a probabilistic viewpoint and propose a unified approach to the problem for a wide class of integro-differential operators.

The main result of the paper, sketched in the foregoing, is proved in Section 3, see Theorem 3.2. Some complementary results, concerning the relationships between the generalized Hopf lemma, irreducibility, minorization condition and the strong maximum principle, are shown in Section 4. The diagram presented below illustrates the relations between them. The symbol “$\psi > 0$ in $D$” means $\psi(x) > 0$ for all $x \in D$.

In general, the constant $a$ appearing in (1.3) may depend on a subsolution in some implicit and complicated way and the bottom function $\psi_D$ is not given explicitly. In Sections 5–7 we formulate several results that yield additional information about these objects. From Proposition 5.1 it follows that if the transition probability semigorup $(P^D_t)$, corresponding to the Markov solution of the martingale problem, is intrinsically ultracontractive, then for any $\alpha > 0$, the minorization condition $M(\alpha, \psi_\alpha^D, \nu_\alpha^D)$ holds with

\[ \psi_\alpha^D = \varphi_D, \quad \nu_\alpha^D(dx) = c_\alpha \varphi_D(x) m(dx), \]

where $c_\alpha > 0$ is some constant, depending on $\alpha$, and $\varphi_D$ and $\varphi_D$ are the principal eigenfunctions for the semigroup and its dual, respectively. Therefore, in particular, the Hopf lemma, as formulated in (1.3), is valid in this case with $\psi_D = \varphi_D$. In fact, as it can be seen from Theorem 6.10, the aforementioned bound holds if and only if the $\varphi_D$-Doob transform of the canonical process has a uniformly ergodic resolvent. In Theorem 7.3, we formulate sufficient conditions under which the constant $\alpha$ appearing in (1.3) can be written in the form

\[ a = a' \bar{u}_{\partial S} + \int_D (Au - cu) \, d\nu_\alpha^D, \]

(1.8)
for some constant $a' > 0$ independent of $u$. In particular, this, combined with (1.7) shows that if $(P_t^D)$ is intrinsically ultracontractive, then there exists $a'' > 0$ such that

$$
\bar{u}_{D^c} - u(x) \geq a' \varphi_D(x) \left( \bar{u}_{D^c} + \int_D \varphi_D(y)(A - c)u(y) \, dy \right)
$$

for any subsolution $u$ to (1.2) satisfying $\bar{u}_{D^c} \geq 0$. A particular example when (1.3) holds, with $\varphi_D(x) = \varphi_D(x) = \delta_{D^c}^s(x)$ and the constant $a$ given by (1.8), is furnished by the fractional laplacian $\Delta^{s/2}$, in case $D$ is bounded and of $C^{1,1}$ class, see Remark 5.4.

Concerning the existing literature, our results are related to the current research dealing with the boundary regularity of solutions to integro-differential equations, see e.g. [6, 9, 10, 15, 16, 27, 45, 46] and references therein. Most of the existing results deal with the fractional Laplacian $\Delta^{s/2}$, i.e. the operator of the form (1.1) with $Q \equiv 0$, $b \equiv 0$ and

$$
N(x, dy) = c|y|^{-d-s}\, dy, \quad x, y \in \mathbb{R}^d
$$

(1.9)

for some $s \in (0, 2)$ and $c > 0$. Although the equations with the fractional Laplacian are fundamental and their analysis is important in understanding the nature of non-local equations, this class of operators is not sufficient for many applications. One should keep in mind that even a small modification of the non-local part of an operator, especially when its local part is degenerate, may profoundly change the regularity properties of solutions. For example, it is known that for positive bounded and continuous $\Delta^{s/2}$-subharmonic functions, with $s \in (0, 2)$, on bounded smooth domains $D$ - i.e. satisfying $\Delta^{s/2}u(x) \leq 0$, $x \in D$ - the difference $\bar{u}_{D^c} - u(x)$ behaves near the boundary like $\delta_{D^c}^{s/2}(x)$, if $\bar{u}_{D^c} = u(\hat{x})$ for some $\hat{x} \in \partial D$. However, if an innocent looking (as one might be tempted to think) modification of $\Delta^{s/2}$ is made by replacing constant $c$ in (1.9) by a function $d(y)$ which is bounded from below and above by positive constants, then the differences $\bar{u}_{D^c} - u(x)$ for the subharmonic functions corresponding to the respective operators, in general, are not comparable with each other near the boundary, see [46, Section 2.3]. As we have already mentioned, in the present paper we strive for estimates of the form (1.3) that are not related to some special features (such as e.g. scaling properties of the Lévy kernel) of the non-local operator under the consideration.
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The maximum principles both weak and strong, together with the Hopf lemma in the case when the local part of \( A \) satisfies uniform ellipticity condition on compact sets can be found e.g. in [12, Section I.4], [51, Section 10.2] and [52, Appendix C]. This is also the topic of our paper [33] and an interested reader can find some additional references therein. Finally, we mention also that there exists a substantial literature concerning the Hopf lemma and maximum principles on non-local partial differential equations with \( S(D) \subset \text{cl} \, D \) (then \( D_S = \text{cl} \, D \) and \( \partial_S D = \partial D \)), see e.g. [51, Appendix C] and the references therein.

Concerning the organization of the present paper, in Section 2 we recall the notion of a strong Markov solution of the martingale problem associated with the operator \( A \), see the hypothesis A4) formulated in Section 2.3. Its existence (uniqueness is not required) is assumed throughout the paper. Under fairly mild assumptions on the coefficients of \( A \), see Theorem A.1 (which summarizes some of the existing results on the subject) one can guarantee the existence of such a solution. In addition, throughout the paper we shall suppose that the canonical process exits \( D \) a.s. starting from any point of the domain, see condition ET) expressed by formula (2.5). Some useful conditions guaranteeing ET) are presented in Section A.2. This hypothesis holds e.g. if we assume that \( D \) is bounded and the differential part of the operator is uniformly elliptic in \( D \) (the latter however is not assumed in the present paper). Finally, using a strong Markov solution of the martingale problem we define in Section 2.5 the notion of a weak subsolution (supersolution and solution) of (1.2).

In Sections 3 and 4 we formulate and prove the main results of the paper. In Sections 5 and 6 various relationships between estimate (1.3), the ultracontractivity and some ergodic properties of the canonical process are discussed. Section 7 is devoted to the formulation and the proof of the quantitative version of the Hopf lemma. Finally, in Section 7.1 we present some auxiliaries concerning the validity of various hypotheses made throughout the paper: such as e.g. the existence of a strong Markov solution of the martingale problem and the existence of the principal eigenfunction corresponding to \( A \).

2. Preliminaries

2.1. Basic notation. Suppose that \( B \) is an arbitrary set. For functions \( f, g: B \to [0, +\infty) \) we write \( f \leq g \) on \( B \) if there exists number \( C > 0 \), i.e., constant, such that

\[
 f(x) \leq Cg(x), \quad x \in B.
\]

Furthermore, we write \( f \sim g \) if \( f \leq g \) and \( g \leq f \). Throughout the paper we denote

\[
 \overline{f}_B := \sup_{x \in B} f(x) \quad \text{and} \quad \underline{f}_B := \inf_{x \in B} f(x).
\]

For a metric space \( E \) we denote by \( \mathcal{B}(E) \) its Borel \( \sigma \)-algebra. Given a subset \( D \subset E \) we let \( D^c := E \setminus D \) be its complement and \( \text{cl} \, D \) be its closure. Let \( \mathcal{B}_b(E) \) \( (\mathcal{B}^*_b(E)) \) be the space of all (non-negative) bounded Borel measurable functions and let \( C_b(E) \) \( (C^*_b(E)) \) be the space of all bounded continuous (compactly supported) functions on \( E \). Furthermore by \( \mathcal{M}(E) \) we denote the set of all Borel positive measures on \( E \). Suppose that \( \mu, \nu \in \mathcal{M}(E) \). We say that \( \mu \) dominates \( \nu \) (\( \nu \) is absolutely continuous with respect \( \mu \)) and write \( \nu \ll \mu \) if all null sets for \( \mu \) are also null for \( \nu \). The measures are equivalent and write \( \mu \sim \nu \) if \( \mu \ll \nu \) and \( \nu \ll \mu \).

Given a point \( x \in E \) and \( r > 0 \) we let \( B(x, r) \) be the open ball of radius \( r \) centered at \( x \) and \( \overline{B}(x, r) \) its closure. As it is customary for a given function \( f: E \to \mathbb{R} \) we denote \( \|f\|_\infty = \sup_{x \in E} |f(x)| \). For \( \nu \) - a signed Borel measure on \( E \) - we define its
total variation norm as
\[
\|\nu\|_{TV} := \sup_{|f| \leq 1} \left| \int_{E} f d\nu \right|.}
\] (2.1)

If \( D \subset \mathbb{R}^d \) is open we let \( C^k(D) \), \( k \geq 1 \) be the class of \( k \)-times continuously differentiable functions in \( D \). By \( C_0(D) \) we denote the subset of \( C(D) \) that consists of functions extending continuously to \( \text{cl} D \) by letting \( f(x) \equiv 0 \), \( x \in \partial D \) - the boundary of \( D \).

Throughout the paper, if it is not stated otherwise, \( m \) shall denote any non-trivial positive \( \sigma \)-finite Borel measure on \( \mathbb{R}^d \). We let \( \ell_d \) be the \( d \)-dimensional Lebesgue measure on \( \mathbb{R}^d \) and \( \ell^d \) be the respective volume element.

For \( p \in [1, +\infty) \) we denote by \( L^p(D) \) (\( L^p_{\text{loc}}(D) \)) the space of functions that are integrable with their \( p \)-th power on \( D \) (any compact subset of \( D \) with respect to \( \ell_d \). As usual \( L^\infty(D) \) is the space of functions with a finite essential supremum norm. By \( W^{k,p}(D) \) (\( W^{k,p}_{\text{loc}}(D) \)) we denote the Sobolev space of functions whose \( k \) generalized derivatives belong to \( L^p(D) \) (\( L^p_{\text{loc}}(D) \)). Finally, let \( W^{k,p}_0(D) \) be the closure of \( C_0^\infty(D) \) in the \( W^{k,p}(D) \) - norm.

### 2.2. Second-order, elliptic integro-differential operators.
Suppose that \( D \) is an open set and \( A \), defined by (1.1), satisfies A1) and A2). We let
\[
M_A := \sum_{i,j=1}^{d} \| q_{i,j} \|_\infty + \sum_{i=1}^{d} \| b_i \|_\infty + N_\ast < +\infty.
\] (2.2)

Obviously in order to give meaning to \( Au(x) \) for \( x \in D \) it suffices only to assume that \( u \in C^2(D) \cap C_0(\mathbb{R}^d) \). In fact, we can define \( Au \) as an element of \( L^p(D) \) even if \( u \in W^{2,p}_{\text{loc}}(D) \cap C_0(\mathbb{R}^d) \) when \( p > d \). This is possible due to a well known estimate, see [11, Lemme 1, p. 361]: for any \( p > d \) there exists \( C > 0 \) such that
\[
\| U[u] \|_{L^p(\mathbb{R}^d)} \leq C \| \nabla^2 u \|_{L^p(\mathbb{R}^d)}, \quad u \in W^{2,p}(\mathbb{R}^d),
\]
where
\[
U[u](x) := \sup_{|y| > 0} \left| u(x + y) - u(x) - \sum_{i=1}^{d} y_i \partial_{x_i} u(x) \right|.
\]

Given a bounded and measurable function \( c: D \to \mathbb{R} \) we let
\[
\bar{c}_D = \sup_{x \in D} c(x), \quad \underline{c}_D := \inf_{x \in D} c(x) \text{ and } (c)_{D,m} := \int_{D} c(x) \, m(dx).
\] (2.3)

Concerning hypotheses made about \( c \). In some of the results we shall require a stronger condition than A3). Namely, we suppose either
\[
A3') \quad c \in B^*_m(D) \text{ and } (c)_{D,m} > 0, \text{ (i.e. } c \not\equiv 0, m\text{-a.e.)}
\]
or an even stronger assumption
\[
A3^{\prime\prime}) \quad c \in B^*_m(D) \text{ and } \underline{c}_D > 0.
\]

### 2.3. Strong Markovian solution to a martingale problem associated with operator \( A \).
Suppose that \( \partial \not\subset \mathbb{R}^d \). Consider the space \( \mathbb{R}^d := (\partial) \cup \mathbb{R}^d \) with the topology of the one point compactification of \( \mathbb{R}^d \) by \( \partial \). Any function \( f: \mathbb{R}^d \to \mathbb{R} \) can be extended to \( \mathbb{R}^d \) by letting \( f(\partial) = 0 \). Let \( \mathcal{D} \) be the space consisting of all functions \( \omega: [0, +\infty) \to \mathbb{R}^d \), that are right continuous and possess the left limits for all \( t \geq 0 \) (cadlag), equipped with the Skorohod topology, see e.g. Section 12 of [5]. Define the canonical process \( X_t(\omega) := \omega(t), \omega \in \mathcal{D} \) and its natural filtration \( (\mathcal{F}_t) \), with \( \mathcal{F}_s := \sigma(X_s, 0 \leq s \leq t) \). Point \( \partial \) is called the cemetery state of the process. Let
$\mathcal{D} := D([0, +\infty); \mathbb{R}^d)$ be the subset of $\bar{\mathcal{D}}$ consisting of all $\omega: [0, +\infty) \rightarrow \mathbb{R}^d$. Given $t \geq 0$ define the shift operator $\theta_t: \mathcal{D} \rightarrow \mathcal{D}$ by $\theta_t(\omega)(s) := \omega(t + s)$, $s \geq 0$.

**Definition 2.1** (A solution of the martingale problem associated with $A$). Suppose that $\mu$ is a Borel probability measure on $\mathbb{R}^d$. A Borel probability measure $P_\mu$ on $\mathcal{D}$ is called a solution of the martingale problem associated with $A$ with the initial distribution $\mu$ if

i) $P_\mu[Z_0 \in Z] = \mu[Z]$ for any Borel measurable $Z \subset \mathbb{R}^d$.

ii) For every $f \in C^2_b(\mathbb{R}^d)$ - a $C^2$-smooth function bounded with its two derivatives on $\mathbb{R}^d$ - the process $$ M_t[f] := f(X_t) - f(X_0) - \int_0^t A f(X_r) \, dr, \ t \geq 0 $$

is a (càdlàg) martingale under measure $P_\mu$ with respect to natural filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by the canonical process.

iii) $P_\mu[\mathcal{D}] = 1$.

As usual we write $P_x := P_{\delta_x}$, $x \in \mathbb{R}^d$ and say that $x$ is the initial condition. We also let $P_0 = \delta_0$. The expectations with respect to $P_\mu$ and $P_x$ shall be denoted by $E_\mu$ and $E_x$, respectively.

**Definition 2.2** (A strong Markovian solution of the martingale problem). We say that a family of Borel probability measures $(P_x)_{x \in \mathbb{R}^d}$ on $\mathcal{D}$ is a strong Markovian solution to the martingale problem associated with $A$ if:

i) each $P_x$ is a solution of the martingale problem associated with $A$, corresponding to the initial condition at $x$,

ii) the canonical process $(X_t)$ is strongly Markovian with respect to the natural filtration $(\mathcal{F}_t)$ and the family $(P_x)_{x \in \mathbb{R}^d}$,

iii) the mapping $x \mapsto P_x[C]$ is measurable for any Borel $C \subset \mathcal{D}$,

iv) for any Borel probability measure $\mu$ on $\mathbb{R}^d$ the probability measure $$ P_\mu(.) := \int_{\mathbb{R}^d} P_x(.) \mu(dx) $$

is a solution to the martingale problem associated with $A$ with the initial distribution $\mu$.

Our hypothesis concerning the martingale problem can be formulated as follows.

A4) The martingale problem associated with the operator $A$ admits a strong Markovian solution.

We discuss conditions sufficient for the validity of A4) in Section A.1.

### 2.4. Analytic description of the canonical process.

#### 2.4.1. Exit time, gauge function, transition probability semigroup and resolvent operator

For a given domain $D$ define the exit time $\tau_D: \mathcal{D} \rightarrow [0, +\infty]$ of the canonical process $(X_t)_{t \geq 0}$ from $D$ as $$ \tau_D := \inf\{t > 0 : X_t \notin D\}. $$

It is a stopping time, i.e., for any $t \geq 0$ we have $[\tau_D \leq t] \in \mathcal{F}_t$, see Theorem I.10.7, p. 54 of [8] and Theorem IV.3.12, p. 181 of [26].

We formulate the following hypothesis:
ET) the exit time from $D$ is a.s. finite, i.e.

$$P_x[\tau_D < +\infty] = 1, \quad x \in D.$$  \hspace{1cm} (2.5)

**Remark 2.3.** It turns out that uniform ellipticity of $Q$ (cf. (A.4)) implies ET for a bounded domain $D$, see e.g. [41, Lemma 4]. In many cases however we can also verify it without assuming uniform ellipticity condition, see Section A.2 below for a more detailed discussion.

**Definition 2.4 (The gauge function for $c(\cdot)$ and domain $D$).** The function

$$v_{c,D}(x) = \mathbb{E}_x c(\tau_D), \quad x \in \mathbb{R}^d,$$

where

$$c(t) := \exp\left\{ - \int_0^t c(X_r) \, dr \right\}, \quad t \geq 0,$$

is called the **gauge function** corresponding to $c(\cdot)$ and domain $D$, cf Section 4.3 of [20].

Obviously, if $c$ satisfies $A3)$, then $0 \leq v_{c,D} \leq 1$. Let us denote then

$$w_{c,D}(x) := 1 - v_{c,D}(x), \quad x \in \mathbb{R}^d.$$  \hspace{1cm} (2.6)

Obviously $w_{c,D}(x) \geq 0$, $x \in \mathbb{R}^d$. We postpone a more detailed discussion of properties of $w_{c,D}$ until Section 7.2 below.

Define the transition semigroup generated by operator (1.1) on $D$ with the null exterior condition

$$P^D_t f(x) := \mathbb{E}_x [f(X_t), t < \tau_D] \quad t \geq 0, \quad f \in B_b(D).$$  \hspace{1cm} (2.7)

**Definition 2.5.** Suppose that $\alpha \geq 0$. A Borel function $f : E \to [0, +\infty)$ is called **$\alpha$-excessive**, if $P^D_t f(x) \leq e^{\alpha t} f(x)$, $t \geq 0$ and $\lim_{t \to 0} P^D_t f(x) = f(x)$ for all $x \in E$.

When $\alpha = 0$, the function is simply called **excessive**.

We define the resolvent of $A$ on $D$ for any non-negative $f \in B(D)$ and $\alpha \geq 0$ by letting

$$R^D_\alpha f(x) := \int_0^\infty e^{-\alpha t} P^D_t f(x) \, dt = \mathbb{E}_x \left[ \int_0^{\tau_D} e^{-\alpha t} f(X_t) \, dt \right], \quad x \in D.$$  \hspace{1cm} (2.8)

Set $R^D := R^D_0$. The definition of $R^D_\alpha$ obviously extends to $f \in B_b(D)$, when $\alpha > 0$.

The operators $(R^D_\alpha)_{\alpha \geq 0}$ satisfy the resolvent identity

$$R^D_\alpha - R^D_\beta = (\beta - \alpha)R^D_\alpha R^D_\beta, \quad \alpha, \beta \geq 0.$$  \hspace{1cm} (2.9)

We shall denote by $R^D_\alpha(x, \cdot)$ the Borel measure on $D$, defined by

$$R^D_\alpha(x, B) = R^D_\alpha 1_B(x), \quad B \in \mathcal{B}(D), \quad x \in D.$$

Suppose that $m$ is a Borel measure on $D$ and $t \geq 0$. We define the transfer measure $mP_t(B) := \int_D P_t 1_B \, dm$, $B \in \mathcal{B}(D)$. An analogous notation shall be used in the case of the resolvent family $(R^D_\alpha)$. Measure $m$ is called **excessive** if $mP_t \leq m$ for all $t \geq 0$.

It will be convenient for us to work sometimes with measures $P^c_x$, $x \in D$ defined on the path space $\mathcal{D}$, cf Section 2.3, that correspond to the process $(X_t)$ killed at rate $c(X_t)$, see [8, Section III]. We shall denote by $(P^c_\alpha)_{\alpha \geq 0}$ and $(R^c_\alpha)_{\alpha \geq 0}$ the corresponding semigroup and the resolvent family, determined by formulas analogous to (2.7) and (2.8), with respect to $P^c_x$.
2.5. Weak subsolution of \((-A+c)v=g\).

**Definition 2.6** (Weak subsolution, supersolution and solution). Suppose that \(c(\cdot)\) satisfies the hypothesis A3, \(D\) is open and \(g(\cdot)\) is a Borel measurable function on \(D\) such that \(g \leq 0\). A function \(u \in B_b(\mathbb{R}^d)\) is called a weak subsolution of the equation
\[
(-A+c)v(x) = g(x), \quad x \in D, \tag{2.10}
\]
if
\[
u(x) \leq \mathbb{E}_x [c(\tau_{D^+} \wedge t)u(X_{\tau_{D^+} \wedge t})] + \mathbb{E}_x \left[ \int_0^{\tau_{D^+} \wedge t} c(r) g(X_r) \, dr \right] \quad \text{for all } x \in D, \quad t \geq 0
\]
for some solution of the martingale problem associated with \(A\). Throughout the remainder of the paper we assume that the probability measure appearing in (2.11) (thus also the respective semigroup and the resolvent family) is fixed.

Denote by \(\mathcal{U}_c(g)\) the set of all weak subsolutions to (2.10) and by \(\mathcal{U}_c^+(g)\) its subset consisting of those \(u\), for which \(\check{u} \geq 0\). We let \(\mathcal{U}_c = \mathcal{U}_c(0)\) and \(\mathcal{U}_c^+ = \mathcal{U}_c^+(0)\).

We say that \(u\) is a weak supersolution of (2.10) if the sign \(\leq\) in the above inequality is replaced by \(\geq\). Furthermore \(u\) is a weak solution if it is both weak subsolution and supersolution. Observe that the regularity of a subsolution to (2.10) is subject to no restriction (besides boundedness).

**Remark 2.7.** The above definition of a weak subsolution coincides with that of [7, Definition 3.2, p. 1551]. It is used there in the formulation of the Alexandrov-Bakelman-Pucci type estimates and maximum principles for a certain class of generators of Lévy processes.

**Remark 2.8.** Note that, if \(u \in B_b(D)\) is a subsolution of (2.10), then \(-R^{c,D}g(x) \geq 0\) is finite for each \(x \in D\). In fact \(R^{c,D}g \in B_b(D)\) and
\[
u(x) := u(x) - R^{c,D}g(x), \quad x \in D \tag{2.12}
\]
is a subsolution of the homogeneous equation \((-A+c)v(x) = 0\).

Condition (2.11) can be rewritten using the path measure \(P_x^c\) of the killed process. It reads
\[
u(x) \leq \mathbb{E}_x^c u(X_{\tau_{D^+} \wedge t}) + \mathbb{E}_x^c \left[ \int_0^{\tau_{D^+} \wedge t} g(X_r) \, dr \right], \quad \text{for all } x \in D, \quad t \geq 0
\]
Thus,
\[
u(x) \leq P_x^{c,D}u(x) + \mathbb{E}_x^c [u(X_{\tau_{D^+}}) \mathbf{1}_{\{\tau_{D^+} \leq t\}}], \quad x \in D. \tag{2.14}
\]

2.6. Relation between weak and other types of subsolutions. We briefly discuss the relationship between the notion of a weak subsolution, introduced in the foregoing, and some other notions of subsolutions that appear throughout the literature, such as: the classical, Sobolev and viscosity subsolutions. Analogous statements can be made about supersolutions and solutions.

Let us recall the definition of a classical subsolution.

**Definition 2.9.** We say that a function \(u\) is a classical subsolution of (2.10) in \(D\) if \(u \in C^2(D) \cap C_b(\mathbb{R}^d)\) and
\[-Au(x) + c(x)u(x) \leq g(x), \quad x \in D. \tag{2.15}
\]

By the definition of the martingale problem and Itô’s formula (applied to the product \(u(X_t)e_{c,\mathbf{1}}(t)\)) we can easily deduce that any classical subsolution is a weak subsolution with respect to any solution of the martingale problem associated with operator \(A\).
Definition 2.10. A Sobolev subsolution to (2.10) is a function $u \in W^{2,p}_{\text{loc}}(D) \cap C_0(\mathbb{R}^d)$ $(p > d)$ such that (2.15) holds $\ell_d$-a.e.

According to the remark following (2.2) $Au$ is well defined as an element of $L^p(D)$ for any $u \in W^{2,p}_{\text{loc}}(D) \cap C_0(\mathbb{R}^d)$ $(p > d)$. If we assume additionally that the matrix $Q(x)$ is uniformly elliptic on compacts, then by [33, Proposition 4.6], (2.11) holds. Therefore, under uniform ellipticity condition, each Sobolev subsolution is a weak subsolution of (2.10).

Definition 2.11. A viscosity subsolution to (2.10) is an upper semi-continuous function $u \in B_{\text{b}}(\mathbb{R}^d)$ satisfying: for any $x \in D$ and $\phi \in C^2_{\text{b}}(\mathbb{R}^d)$ such that $\phi(x) = u(x)$ and $\phi(y) \geq u(y)$, $y \in \mathbb{R}^d$ we have

$$-A\phi(x) + c(x)\phi(x) \leq g(x).$$

It goes beyond the scope of this paper to examine in depth the relation between the weak subsolutions to (2.10) and the viscosity subsolution. However, some sufficient conditions for viscosity solutions to be also weak subsolutions can be formulated. We postpone a more detailed discussion until Section A.4.

3. The main result - Hopf’s lemma

Throughout the remainder of the paper we shall always assume hypotheses A1) - A4) and ET) without further mentioning them in the subsequent formulations of the results.

We start with the formulation of the minorization condition.

Definition 3.1 (Minorization condition $M(\alpha, \psi^\alpha_D, \nu^\alpha_D)$. Suppose that $\alpha \geq 0$, $\psi^\alpha_D : D \to [0, +\infty]$ and $\nu^\alpha_D$ is a $\sigma$-finite Borel measure on $D$. We say the minorization condition $M(\alpha, \psi^\alpha_D, \nu^\alpha_D)$ holds if

$$R_{\alpha}^D f(x) \geq \psi^\alpha_D(x) \int_D f d\nu^\alpha_D, \quad x \in D, \quad f \in B_{\text{b}}^+(D). \quad (3.1)$$

The following version of the Hopf lemma holds.

Theorem 3.2 (The Hopf lemma). Recall that $\bar{c}_D = \sup_{x \in D} c(x)$. Assume that the operator $A$ satisfies $M(\bar{c}_D, \psi^\bar{c}_D, \nu^\bar{c}_D)$ for some strictly positive function $\psi^\bar{c}_D : D \to (0, +\infty)$ and measure $\nu^\bar{c}_D$.

1) Suppose that $\text{supp} \nu^\bar{c}_D = D$ and $u \in U^+_c$ is non-constant and continuous in $D$. Then, there exists $a > 0$ for which

$$\bar{u}_{D_{\bar{c}}} - u(x) \geq a\psi^\bar{c}_D(x), \quad x \in D. \quad (3.2)$$

2) Suppose that $m \ll \nu^\bar{c}_D$ and $u \in U^+_c$ is non-constant $m$-a.e. in $D$. Then, the conclusion of part 1) is in force.

The proof of this result is presented below. Due to its length we divide it into three parts. First, in Section 3.1 we introduce some preliminaries needed in the argument. The main step of the proof is made in Section 3.2, modulo some technical estimate given in (3.15). The latter is shown in Section 3.3.

3.1. Preliminaries.
3.1.1. On the weak maximum principle.

**Lemma 3.3.** We have

\[ P_x(X_{\tau_D} \in \partial S D) = 1, \quad x \in D. \]  

(3.3)

**Proof.** Using the Ikeda-Watanabe formula, see [13, Remark 2.46, page 65], we obtain

\[
\begin{align*}
\mathbb{E}_x \left[ \sum_{0 < s \leq \tau_D} 1_D(X_{s-}) 1_{S(D) \setminus D}(X_s) \right] \\
= \mathbb{E}_x \left[ \int_0^{\tau_D} ds \int_{\mathbb{R}^d} 1_D(X_{s-}) 1_{S(D) \setminus D}(y) N(X_{s-}, dy - X_{s-}) \right].
\end{align*}
\]

From the definition of the range of non-locality \( S(D) \), see (1.4), we conclude that the right-hand side of the above equation vanishes. Thus, \( X_{\tau_D} \in S(D) \setminus D \), if \( X_{\tau_D} \not\in \partial D \), or \( X_{\tau_D} \in \partial D \), if otherwise. Hence (3.3) follows from the definition of \( \partial S D \), see (1.5).  \( \boxdot \)

The following version of the weak maximum principle is a direct consequence of Lemma 3.3.

**Proposition 3.4** (Weak maximum principle for \( A \)). If \( u \) is a weak subsolution to (2.10), then

\[ \bar{u}^*_D \leq \bar{u}^{*}_{\partial S D}. \]  

(3.4)

When \( c \equiv 0 \) we get (3.4) with \( \bar{u}^{*}_{\partial S D} \) replaced by \( \bar{u}^{*}_{\partial D} \).

As a corollary to the above proposition we conclude that for any weak subsolution \( u \) to (2.10) with \( \bar{u}^*_{\partial S} \geq 0 \), we have

\[ \bar{u}^*_{\partial S} = \bar{u}^{*}_{\partial S D}. \]  

(3.5)

3.1.2. Some semimartingales associated with the canonical process. Define

\[ w := \bar{u}^*_{\partial S} - u \geq 0 \quad \text{in } D, \]

where \( u \) is a subsolution to (2.10) such that \( \bar{u}^*_{\partial S} \geq 0 \). From inequality (2.14) we infer that

\[ P^{c,D}_t w \leq w + \mathbb{E}^c_{\tau_D} \left[ (-\bar{u}^*_{\partial S} + u(X_{\tau_D})) 1_{\{t \geq \tau_D\}} \right]. \]

Therefore, by Lemma 3.3, we have

\[ P^{c,D}_t w \leq w \quad \text{in } D \quad \text{for any } t \geq 0. \]  

(3.6)

By [8, Exercise II.2.17]

\[ \bar{w} := \sup_{t > 0} P^{c,D}_t w = \lim_{t \to 0^+} P^{c,D}_t w \]

is an excessive function with respect to \( (P^{c,D}_t)_{t \geq 0} \). Thanks to (3.6) we have

\[ \bar{w} \leq w \quad \text{in } D. \]  

(3.7)

Define

\[ \hat{u} := \lim_{t \to 0^+} P^{c,D}_t u, \quad \text{in } D \]  

(3.8)

(the limit exists, since \( \lim_{t \to 0^+} P^{c,D}_t w \) exists). Clearly,

\[ \hat{w} = \bar{u}^*_{\partial S} - \hat{u} \quad \text{in } D. \]  

(3.9)

Therefore, from (3.7) and the fact that \( \bar{w} \geq 0 \), we have

\[ \bar{u}(x) \geq u(x), \quad x \in D. \]  

(3.10)

From this and (3.8)

\[ \sup D \hat{u} = \sup D u. \]
We get also, see the comment preceding [8, Theorem II.3.6], that
\[ R_0^{c,D} \bar{u}(x) = R_0^{c,D} u(x), \quad x \in D, \alpha \geq 0. \] (3.11)

A process \( (Z_t)_{t \geq 0} \) is called additive, with respect to \( (P^c_x) \), if for any \( s, t \geq 0 \),
\( Z_{t+s} = Z_s + Z_t \circ \theta_s \) a.s., where \( \theta_s \) is the shift operator.

**Proposition 3.5.** Suppose that \( u \in \mathcal{B}(D) \) is a weak subsolution of (2.10). Then, there exists a càdlàg, increasing, predictable process \( (A_t)_{t \geq 0} \) and uniformly integrable \( P^c_x \)-martingale \( (M_t)_{t \geq 0} \) such that for each \( x \in D \) we have
\[ \bar{u}(X_t) = \hat{u}(X_0) + A_t - \mathbf{1}_{(t \geq \tau_D)} \bar{u}_{D,t} - \int_0^t g(X_r) \, dr + M_t, \quad t \geq 0, \quad P^c_x \text{-}a.s. \] (3.12)

Both processes \( (A_t)_{t \geq 0} \) and \( (M_t)_{t \geq 0} \) are additive with respect to \( (P^c_x) \).

**Proof.** We can assume with no loss of generality that \( g \equiv 0 \). Otherwise we would consider \( \bar{u} \), given by (2.12). By [8, Theorem III.5.7], the process \( (\hat{u}(X_t))_{t \geq 0} \) is a right-continuous bounded supermartingale under measure \( P^c_x \) for any \( x \in D \). Let \( \hat{Y}_t := \hat{u}(X_t) - \hat{u}(X_0), \quad t \geq 0. \) By the version of Doob-Meyer decomposition, see [21, Theorem 3.18], we conclude that there exist a càdlàg increasing predictable process \( (A_t)_{t \geq 0} \) and uniformly integrable martingale \( (M_t)_{t \geq 0} \) under measure \( P^c_x \) such that \( \hat{Y}_t = M_t - A_t, \quad t \geq 0. \) According to ibid. both processes \( (A_t)_{t \geq 0} \) and \( (M_t)_{t \geq 0} \) are additive with respect to \( (P^c_x) \). Therefore their choices do not depend on \( x \in D \).

Formula (3.12) follows directly from (3.9).

Let
\[ Y_t := \hat{u}(X_t) + \mathbf{1}_{(t \geq \tau_D)} \bar{u}_{D,t}, \quad t \geq 0. \]

Formula (3.12) can be rewritten as follows
\[ Y_t = \hat{u}(X_0) + A_t - \int_0^t g(X_r) \, dr + M_t, \quad t \geq 0, \quad P^c_x \text{-}a.s. \] (3.13)

Since \( (M_t)_{t \geq 0} \) are additive, with respect to \( (P^c_x) \), so is the process \( (Y_t)_{t \geq 0} \).

Since \( \hat{u}(\theta) \equiv 0 \) we have \( A_t = A_{t \wedge \tau_D}, \quad t \geq 0, \quad P^c_x \) a.s. The same also holds for \( (M_t)_{t \geq 0} \) and \( (Y_t)_{t \geq 0} \).

3.1.3. **Some properties of \( \hat{u} \).** Recall that \( \underline{u}_D := \inf_D u \) and \( \overline{u}_D := \sup_D u \). If \( m \) is a \( \sigma \)-finite Borel measure on \( D \) we let
\[ \underline{u}_{D,m} := \text{ess inf}_{D} u, \quad \overline{u}_{D,m} := \text{ess sup}_{D} u, \]
where the respective essential supremum and infimum correspond to the measure \( m \).

If \( \hat{u} \) is defined by (3.8) we define
\[ \underline{\hat{u}}_{D,m} := \text{ess inf}_{D} \hat{u}, \quad \overline{\hat{u}}_{D,m} := \text{ess sup}_{D} \hat{u}. \]

The following result shall be useful in the proof of Theorem 3.2.

**Proposition 3.6.** Suppose that \( u \) is a weak subsolution of (1.2). If \( u \in C_0(\mathbb{R}^d) \),
then \( \hat{u} = u \). Furthermore, assume that \( m \) is a \( \sigma \)-finite Borel measure and \( u \) is a bounded measurable subsolution such that \( \underline{u}_{D,m} < \overline{u}_{D,m} \). Suppose that the operator \( A \) satisfies the minorization condition \( M(\mathbf{c}_D, \mathbf{\psi}_D^{\mathbf{c}_D}, \mathbf{\nu}_D^{\mathbf{c}_D}) \) for some \( \mathbf{\psi}_D^{\mathbf{c}_D} : D \to [0, +\infty) \),
that is not identically equal 0, \( m \)-a.e. and \( m \ll \nu_D^{\mathbf{c}_D} \). Then
\[ \underline{\hat{u}}_{D,m} < \overline{\hat{u}}_{D,m}. \] (3.14)
Proof. The first part of the proposition concerning a continuous subsolution is obvious, thanks to the continuity of \( X_t \) at \( t = 0 \). To prove (3.14), we show that \( \hat{u} = u \), \( m \)-a.e. on \( D \). Suppose that \( m(\{\hat{u} > u\}) > 0 \). Then \( \nu_D^\varphi(\{\hat{u} > u\}) > 0 \). Applying condition \( M(c_D, \psi_D^\varphi, \nu_D^\varphi) \) we obtain
\[
R_{\varepsilon_D}^c(D)(\hat{u} - u)(x) \geq \psi_D^\varphi(x) \int_D (\hat{u} - u) d\nu_D^\varphi, \quad x \in D,
\]
which contradicts (3.11).

3.2. Main step in the proof of Theorem 3.2. We carry out the proof of both parts of the theorem simultaneously. The main point of the proof is to show that there exists an excessive function \( v \), with respect to \( (F_t^{c,D}) \), such that
\[
\hat{u}_{D^c} - u(x) \geq v(x), \quad x \in D \quad \text{and} \quad \nu_D^\varphi(x : v(x) > 0) > 0,
\]
where \( \nu_D^\varphi \) is the measure appearing in condition \( M(c_D, \psi_D^\varphi, \nu_D^\varphi) \). Once this is done, we construct an appropriate approximation of \( v(x) \), see (3.16) below, which can be bounded from below using the minorization condition (3.1) and the conclusion of the theorem follows.

**Conclusion of the proof of the theorem under (3.15).** Let us suppose that (3.15) holds and we show then how to conclude the proof of the theorem. We construct an approximating sequence of non-negative functions \( (f_k)_{k \geq 1} \) such that \( (R_{c,D} f_k(x))_{k \geq 1} \) is increasing and
\[
\lim_{k \to + \infty} R_{c,D} f_k(x) = v(x), \quad x \in D. \tag{3.16}
\]
The sequence \( (f_k) \) is constructed by taking the Yosida approximation
\[
f_k(x) := k(v(x) - kR_{c,D}^k v(x)), \quad x \in D, \quad k \geq 1.
\]
By [8, (2.6) of Chapter II, p. 73] we have \( R_{c,D} f_k(x) = k R_{c,D}^k v(x), x \in D \). Therefore, by virtue of [8, (2.4) of Chapter II, p. 73], the sequence \( (R_{c,D} f_k(x))_{k \geq 1} \) is monotone increasing for each \( x \in D \) fixed and by Proposition II.2.6 of ibid. (3.16) holds. Using the monotone convergence theorem, we conclude from (3.16) that for each \( n \geq 1 \),
\[
\lim_{k \to + \infty} \frac{\mathbb{E}_x^c}{E_x^c} \left[ \int_0^{\tau_{D}^{n+1}} f_k(X_s) \, ds \right] = \lim_{k \to + \infty} R_{c,D} f_k(x) - P_{c,D}^n R_{c,D} f_k(x)
\]
\[
= v(x) - P_{c,D}^n v(x) =: v_n(x), \quad x \in D. \tag{3.17}
\]
Obviously \( (v_n(x))_{n \geq 1} \) is increasing for each \( x \in D \) fixed and
\[
\lim_{n \to + \infty} v_n(x) = v(x), \quad x \in D.
\]
Therefore, by the second inequality in (3.15), there exists \( N \geq 1 \) such that \( \int_D v_N d\nu_D^\varphi > 0 \).

Substituting \( R_{c,D}^n f \) in place of \( f \) in (3.1), with \( \alpha = c_D \), we get, by the resolvent identity (2.9), that
\[
R_{c,D} f(x) \geq R_{c,D} R_{c,D}^n f(x) \geq \psi_D^\varphi(x) \int_D R_{c,D}^n f d\nu_D^\varphi, \quad x \in D, \quad f \in B_1^\circ(D).
\]
From (3.18) we conclude that
\[
R_{c,D} f_k(x) \geq R_{c,D} R_{c,D}^n f_k(x) \geq \psi_D^\varphi(x) \int_D R_{c,D}^n f_k d\nu_D^\varphi, \quad x \in D. \tag{3.19}
\]
Moreover, for each $N \geq 1$

$$R^D_{n+1} f_k(x) = \mathbb{E}_x \left[ \int_0^{\tau_D} e^{-(\varepsilon + 1)} f_k(X_s) \, ds \right]$$

$$\geq \mathbb{E}_x \left[ \int_0^{\tau_D \wedge N} e^{-\varepsilon} f_k(X_s) \, ds \right] \geq e^{-N(\varepsilon + 1)} \mathbb{E}_x \left[ \int_0^{\tau_D} f_k(X_s) \, ds \right].$$

Substituting into (3.19) we get

$$R^D f_k(x) \geq e^{-N(\varepsilon + 1)} \nu^D_D(x) \int_D \mathbb{E}^D_x \left[ \int_0^{\tau_D \wedge N} f_k(X_s) \, ds \right] \nu^D_D(dy), \quad x \in D.$$

Letting $k \to \infty$, we get, cf (3.17),

$$v(x) \geq b_N \nu^D_D(x), \quad x \in D,$$

where

$$b_N := e^{-N(\varepsilon + 1)} \int_D \nu^D_D(dy) > 0.$$

From (3.20) and (3.15) we conclude that (3.2) holds.

3.3. The proof of (3.15). It suffices to show the existence of an excessive function $v$ such that

$$\bar{u}_{D,x} - \hat{u}(x) \geq v(x), \quad x \in D \quad \text{and} \quad \nu^D_D(x : v(x) > 0) > 0$$

(note $\hat{u}$ in the first inequality). Estimate (3.15) then follows from (3.10).

Observe that by (3.12), (2.5) and the Fatou lemma

$$\bar{u}_{D,x} - \hat{u}(x) \geq \mathbb{E}^D_x A_{\tau_D}, \quad x \in D.$$ (3.22)

Since $(A_t)_{t \geq 0}$ is additive, increasing, right-continuous and $A_0 = 0$, the function

$$v^A(x) := \mathbb{E}^D_x A_{\tau_D}$$

is excessive with respect to $(P^D_t)$. If

$$\nu^D_D(x : v^A(x) > 0) > 0,$$

then we get (3.21), with $v(x) := v^A(x)$, as a conclusion from (3.22).

Suppose now that

$$\nu^D_D(x : v^A(x) > 0) = 0.$$ (3.24)

Since $\hat{u}$ is non-constant in $D$ in part 1) of the theorem and non-constant in $D, m$-a.e. in part 2), we have $\bar{u}_{D,m} < \bar{u}_{D,m}$ in both cases (see Proposition 3.6). Suppose that $\varepsilon > 0$ and $a \in \mathbb{R}$ are such that

$$\bar{u}_D < \bar{u}_{D,m} < a - 2\varepsilon < a \leq a + 2\varepsilon < \bar{u}_{D,m} \leq \bar{u}_D = \bar{u}_D.$$ (3.25)

The last equality follows from (3.10). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be smooth, nondecreasing, convex and such that

$$\varphi(x) = x, \quad x \geq a + \varepsilon.$$ (3.26)

By the Itô formula, see [43, III Theorem 7.32, p.78] applied to (3.13), for any smooth, nondecreasing, convex $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\varphi(Y_{t\wedge \tau_D}) = \varphi(\hat{u}(X_0)) + \int_0^{t \wedge \tau_D} \varphi'(Y_s) \, dA_s + \int_0^{t \wedge \tau_D} \varphi'(Y_s) \, dM_s + \frac{1}{2} \int_0^{t \wedge \tau_D} \varphi''(Y_s) \, [M]_s + J_{t \wedge \tau_D}, \quad t \geq 0, P^D_x \text{-a.s.}$$ (3.27)
Here \( \{M_t^c\}_{t \geq 0} \) is the continuous part of the quadratic variation process \( \{M_t\}_{t \geq 0} \) associated with martingale \( (M_t)_{t \geq 0} \). The jump part of \( \{M_t\}_{t \geq 0} \) is given by

\[
J_t := \sum_{0 \leq s \leq t} \left\{ \varphi(Y_s) - \varphi(Y_{s-}) - \varphi'(Y_{s-})(Y_s - Y_{s-}) \right\}, \quad t \geq 0, \ P^c_{x,D}-\text{a.s.} \tag{3.28}
\]

By convexity of \( \varphi \) each term in the above sum is non-negative. Thus, \( (J_t)_{t \geq 0} \) is an increasing càdlàg process satisfying \( J_0 = 0 \). Since all the processes appearing in the right-hand side of (3.28) are additive (see Section 3.1.2), the process \( (J_t)_{t \geq 0} \) is also additive.

Let \( (J^p_t)_{t \geq 0} \) be the dual predictable projection of \( (J_t)_{t \geq 0} \), with respect to \( P^c_{x,D} \), i.e. a predictable increasing càdlàg process such that

\[
N_t := J_t - J^p_t, \quad t \geq 0 \tag{3.29}
\]

is a martingale. By [21, Theorem 3.18], \( J^p \) exists and is additive. Define the process

\[
K_t := \frac{1}{2} \int_0^t \varphi''(Y_s) \, d[M]^c_s + J^p_t, \quad t \geq 0.
\]

Since all the processes appearing in the definition are additive, so is \( (K_t)_{t \geq 0} \). In addition, it is càdlàg, increasing and \( K_0 = 0 \). We infer therefore that \( v^K(x) := E^c_{x,D} K_{\tau_D} \) is an excessive function with respect to \( (P^c_{x,D}) \). Applying expectation \( E^c_{x,D} \) to both sides of (3.27) and taking the limit \( t \to +\infty \) we obtain

\[
\varphi(\tilde{u}_{D.x}) \geq \varphi(\tilde{u}(x)) + E^c_{x,D} \left[ \int_0^{\tau_D} \varphi'(Y_s) \, dA_s \right] + v^K(x), \quad x \in D. \tag{3.30}
\]

By the definition of function \( \varphi \), cf (3.26), we have

\[
\varphi(y) \geq y, \quad y \in \mathbb{R} \quad \text{and} \quad \varphi(\tilde{u}_{D.x}) = \tilde{u}_{D.x}.
\]

This combined with (3.30) gives the first inequality of (3.21), with \( v := v^K \). We shall prove the second inequality of (3.15) arguing by contradiction. Suppose that

\[
v^K(x) = 0 \quad \text{for} \ \nu^c_D\text{-a.e.} \ x \in D. \tag{3.31}
\]

Since we have also assumed that \( \nu^c_D(v^A > 0) = 0 \), therefore (cf (3.23))

\[
P^c_{x,D}(K_{\tau_D} = 0) = 1 \quad \text{and} \quad P^c_{x,D}(A_{\tau_D} = 0) = 1, \quad \text{for} \ \nu^c_D\text{-a.e.} \ x \in D. \tag{3.32}
\]

The first equality implies that

\[
\frac{1}{2} \int_0^{\tau_{\alpha_D}} \varphi''(Y_s) \, d[M]^c_s = -J^p_{\tau_{\alpha_D}}, \quad t \geq 0, \ P^c_{x,D}-\text{a.s.} \tag{3.33}
\]

Combining (3.27), (3.32) and (3.33) we conclude that (cf (3.29))

\[
\varphi(Y_{\alpha_D}) = \varphi(\tilde{u}(X_0)) + \int_0^{\tau_{\alpha_D}} \varphi'(Y_r) \, dM_r + N_{\alpha_D}, \quad t \geq 0, \ P^c_{x,D}-\text{a.s.} \tag{3.34}
\]

for \( \nu^c_D\text{-a.e.} \ x \in D \).

Let

\[
\mathcal{N} := \{ x \in D : v^K(x) > 0, \text{ or } v^A(x) > 0 \}.
\]

We have

\[
\nu^c_D(\mathcal{N}) = 0, \tag{3.35}
\]

by virtue of (3.31) and (3.24). As we have already shown above, thanks to (3.34) and the fact that \( \tilde{u}(\cdot) \) is bounded, the process \( M^c_t := \varphi(Y_{\alpha_D} \cdot) \), \( t \geq 0 \) is a bounded \((\mathcal{F}_t)\)-martingale under measure \( P^c_{x,D} \) for each \( x \in D \setminus \mathcal{N} \). Obviously, from (3.26),

\[
M^c_t \geq a - \varepsilon, \quad t \geq 0. \tag{3.36}
\]
By (3.25), (3.35), and either the continuity of \( \hat{u} \) and the fact that \( \nu_{D}^{\hat{u}} \) has full support (in the case of part 1) of the theorem) or the fact that \( m \ll \nu_{D}^{\hat{u}} \) (in the case of part 2) of the theorem, we infer

\[
\nu_{D}^{\hat{u}} \left( (D \setminus \mathcal{N}) \cap \{ \hat{u} < a - \varepsilon \} \right) > 0.
\]

Suppose now that \( \bar{x} \in \{ \hat{u} < a - \varepsilon \} \cap (D \setminus \mathcal{N}) \). Then, \( M_{t}^{\bar{x}} = a - \varepsilon, \ P_{\bar{x}}^{c,D} \)-a.s. Since for this particular choice of \( \bar{x} \) the process \( (M_{t}^{\bar{x}}) \) is a martingale under measure \( P_{\bar{x}}^{c,D} \), thanks to (3.36) we have, see e.g. [43, VI Theorem 3.13, page 371],

\[
M_{t}^{\bar{x}} = a - \varepsilon, \quad t \geq 0, \ P_{\bar{x}}^{c,D} \)-a.s.
\]

From this and (3.26) we get

\[
\hat{u}(X_{t}) \leq a + \varepsilon, \quad t \in [0, \tau_{D}), \ P_{\bar{x}}^{c,D} \)-a.s. \quad \text{(3.37)}
\]

On the other hand, by the minorization condition \( M(c_{D}, \psi_{D}^{\hat{u}}, \nu_{D}^{\hat{u}}) \), the assumptions made on \( \psi_{D}^{\hat{u}}, \nu_{D}^{\hat{u}} \), and (3.25), we conclude that for any \( x \in D \) we have

\[
E_{\bar{x}}^{c,D} \left[ \int_{0}^{\tau_{D}} 1_{\{\hat{u} > a + 2\varepsilon\}}(X_{t}) \, dt \right] = R_{c,D}^{\hat{u}} 1_{\{\hat{u} > a + 2\varepsilon\}}(x) \geq R_{c,D}^{\hat{u}} 1_{\{\hat{u} > a + 2\varepsilon\}}(x) \geq \nu_{D}^{\hat{u}}(x) \int_{D} 1_{\{\hat{u} > a + 2\varepsilon\}}(x) \, d\nu_{D}^{\hat{u}} > 0.
\]

In the case of part 1) of the theorem the positivity of the last integral in the utmost right hand side follows from \( \text{supp} \nu_{D}^{\hat{u}} = D \) and the continuity of \( u \). In the case of part 2) it is a consequence of the facts that \( m(\hat{u} > a + 2\varepsilon) > 0 \) and \( m \ll \nu_{D}^{\hat{u}} \). Hence, for every \( x \in D \), we have

\[
P_{\bar{x}}^{c,D} \left( \exists t \in [0, \tau_{D}): \hat{u}(X_{t}) > a + 2\varepsilon \right) > 0, \quad x \in D,
\]

which obviously contradicts (3.37). Hence, (3.31) cannot be true and therefore \( \nu_{D}^{\hat{u}}(x: \nu^{K}(x) > 0) > 0 \). This ends the proof of (3.21).

4. Minorization condition, irreducibility of the resolvent and strong maximum principle

Suppose that \( m \) is a non-trivial, \( \sigma \)-finite Borel measure in \( D \). We say that the resolvent of \( A \) is \( m \)-irreducible if

\[
IR_{m}) \text{ for any } f \in B_{+}(D) \text{ such that } \int_{D} f \, dm > 0 \text{ we have}
\]

\[
R_{1}^{\alpha} f(x) > 0 \quad \text{for all } x \in D. \quad \text{(4.1)}
\]

Remark 4.1. Observe that for any \( \sigma \)-finite Borel measure \( \nu \) on \( D \) and non-negative Borel function \( f \) on \( D \) if \( \int_{D} R_{\alpha}^{\nu} f \, d\nu > 0 \) for some \( \alpha \geq 0 \), then \( \int_{D} R_{\alpha}^{D} f \, d\nu > 0 \) for any \( \alpha \geq 0 \). Indeed, it follows easily that the assumption made is equivalent to

\[
\ell_{1}\left( \{ t \geq 0: \int_{D} P_{t}^{\nu} f \, d\nu > 0 \} \right) > 0. \quad \text{(4.2)}
\]

In particular if (4.1) holds, then \( R_{\alpha}^{D} f > 0 \) for all \( \alpha \geq 0 \).

Remark 4.2. Note that if the minorization condition \( M(\alpha, \psi_{D}^{\hat{u}}, \nu_{D}^{\hat{u}}) \) holds for some \( \alpha \geq 0 \), strictly positive \( \psi_{D}^{\hat{u}} \) and \( \nu_{D}^{\hat{u}} \) satisfying \( m \ll \nu_{D}^{\hat{u}} \), then the \( m \)-irreducibility of \( A \) follows.

Recall the definition of an irreducibility measure for a non-negative kernel. Given \( \alpha > 0 \) we let \( K_{\alpha} : D \times B(D) \to [0, +\infty) \) be a substochastic kernel defined by

\[
K_{\alpha}(x, B) := \alpha R_{\alpha}^{D} 1_{B}(x) \quad \text{for } (x, B) \in D \times B(D).
\]
Then, for each $a$ measure

Theorem 4.4. Assume that $\text{IR}_m$ holds and

for some (thus all) $\alpha \geq 0$ we have $m R^D_\alpha \ll m$. (4.3)

Then, for each $\alpha \geq 0$ there exist an $(\alpha + 1)$-excessive function $\psi^n_\alpha : D \to (0, +\infty)$ and a measure $\nu^n_\alpha$ satisfying $m \ll \nu^n_\alpha$, such that $M(\alpha, \psi^n_\alpha, \nu^n_\alpha)$ is in force.

Proof. Suppose that $\alpha > 0$. Condition $\text{IR}_m$ implies that the kernel $K_\alpha(\cdot, \cdot)$ is $m$-irreducible. By [40, part ii) of Proposition 2.4]

$$
\mu_\alpha(\cdot) = \sum_{n=1}^{\infty} \frac{1}{2^n} \int_D K^n_\alpha(x, \cdot) m(dx).
$$

is a maximal irreducibility measure for $K_\alpha$, and so $m \ll \mu_\alpha$. Clearly $\mu_\alpha \sim \mu_\beta$ for any $\alpha, \beta > 0$, i.e. both measures have the same null sets. We let $\mu := \mu_1$, so $m \ll \mu$. Recall that

$$(R^D_\alpha)^n f(x) = \int_0^\infty \frac{t^n}{n!} e^{-\alpha t} P^D_\alpha f(x) dt, \quad x \in D, \; n \geq 1.
$$

From the above, assumption $m R^D_\alpha \ll m$ and characterization (4.2) we infer that $m(R^D_\alpha)^n \ll m$ for any $n \geq 1$, $\alpha \geq 0$. Thus, using (4.4) we obtain $\mu \ll m$. Thus $\mu \sim m$ and $m$ is a maximal irreducibility measure.

By [40, Theorem 2.1] there exist a non-trivial with respect to $m$ (i.e. $\int_D \hat{s}_\alpha dm > 0$) function $\hat{s}_\alpha : D \to [0, \infty)$ and a non-trivial (i.e. $\hat{\nu}_\alpha(D) > 0$) $\sigma$-finite Borel measure $\hat{\nu}_\alpha$ on $D$, the so called small function and small measure, such that

$$
(R^{D+}_\alpha)^n f(x) \geq \hat{s}_\alpha(x) \int_D f d\hat{\nu}_\alpha, \quad f \in B^+(D), \quad x \in D.
$$

We claim that in fact from (4.5) it follows that

$$
R^D_\alpha f(x) \geq R^D_{\alpha+1} f(x) \geq \hat{s}_\alpha(x) \int_D f d\hat{\nu}_\alpha, \quad f \in B^+(D), \quad x \in D.
$$

Indeed, we obviously have $R^D_{\alpha+1} f \leq R^D_\alpha f$ for any $f \in B^+(D)$. By the resolvent identity, see (2.9),

$$
R^D_{\alpha} R^D_{\alpha+1} f = R^D_{\alpha+1} R^D_\alpha f = R^D_\alpha f - R^D_{\alpha+1} f \leq R^D_\alpha f \quad \text{for any } f \in B^+(D).
$$

If $n \geq 2$ in (4.5), then we can write

$$
R^D_\alpha \left( R^D_{\alpha+1} \right)^{n-2} f(x) \geq \left( R^D_\alpha R^D_{\alpha+1} \right) \left( R^D_{\alpha+1} \right)^{n-2} f(x)
$$

$$
\geq \left( R^D_{\alpha+1} \right)^{n-2} f(x) \geq \hat{s}_\alpha(x) \int_D f d\hat{\nu}_\alpha, \quad f \in B^+(D), \quad x \in D.
$$

Iterating this procedure, we conclude (4.6). By (4.7), $R^D_\alpha R^D_{\alpha+1} f \leq R^D_\alpha f$. Therefore, applying (4.6) for $R^D_{\alpha+1} f$ in place of $f$ we get

$$
R^D_\alpha f(x) \geq \hat{s}_\alpha(x) \int_D f d\hat{\nu}_\alpha, \quad f \in B^+(D), \quad x \in D,
$$

(4.8)
where \( \nu^\alpha_D(B) := \int_B R^D_{\alpha+1} 1_B d\tilde{\nu}_\alpha \), \( B \in \mathcal{B}(D) \). Since \( \tilde{\nu}_\alpha \) is non-trivial, we get by \( \text{IR}_m \) that \( m \ll \nu^\alpha_D \). Applying \( R^D_{\alpha+1} \) to both sides of (4.8) we get (3.1) with \( \psi^\alpha_D := R^D_{\alpha+1} \tilde{\nu}_\alpha \). Note that \( \psi^\alpha_D \) is an \((\alpha + 1)\)-excessive function with respect to \( (P^D_t) \). By hypothesis \( \text{IR}_m \) we conclude that \( \psi^\alpha_D \) is strictly positive on \( D \).

From Theorems 4.4 and 3.2 we immediately conclude the following.

**Corollary 4.5.** Suppose that the operator \( A \) satisfies the assumptions of Theorem 4.4. Then, there exists a function \( \psi : D \to (0, \infty) \) such that for any \( u \in U_c^+ \) non-constant \( m \)-a.e. in \( D \), we can find \( a > 0 \), for which

\[
\bar{u}_{D_S} - u(x) \geq a\psi(x), \quad x \in D. \tag{4.9}
\]

**Remark 4.6.** Observe that \( \text{IR}_m \) and (4.3) together imply that \( m \sim mR^D_\alpha \) for any \( \alpha \geq 0 \).

**Remark 4.7.** Consider the following condition

for some (thus all) \( \alpha \geq 0 \) we have \( m \ll mR^D_\alpha \). \( \tag{4.10} \)

Condition (4.10) holds in particular when the measure \( m \) is excessive. Indeed, then \( \alpha m R^D_\alpha \leq m \) for all \( \alpha > 0 \) and, according to [23, Section 37b], we can write

\[
\lim_{\alpha \to +\infty} \alpha \int_D R^D_\alpha f dm = \int_D f dm, \quad \text{for any } f \in B^+_c(D), \tag{4.11}
\]

which implies that \( m \ll mR^D_\alpha \). Obviously, by the excessiveness of \( m \), we have \( mR^D_\alpha \ll m \). Thus, in fact, the excessiveness of \( m \) implies that \( m \sim mR^D_\alpha \) for any \( \alpha \geq 0 \).

**Remark 4.8.** It is easy to see that (4.10) also holds if

\[
\liminf_{t \to 0^+} \int_D P^D_t f dm > 0, \quad \text{for any } f \in B^+_c(D), \text{ such that } \int_D f dm > 0. \tag{4.12}
\]

In our next result we show a form of the converse to the result of Theorem 4.4. Namely, we have the following.

**Theorem 4.9.** Assume that operator \( A \), domain \( D \) and an excessive measure \( m \) are fixed. Suppose furthermore that the conclusion of Corollary 4.5 holds for some \( c \geq 0 \) and any \( u \in U_c^+ \). Then, the operator \( A \) satisfies condition \( \text{IR}_m \).

**Proof.** Suppose that \( f \in B^+_c(D) \) and \( \int_D f dm > 0 \). By ET) (see [4, Proposition 2.2]), there exists a strictly positive \( h \in B^+_c(D) \) such that \( R^D f \) is bounded by a constant \( m \)-a.e. Set \( u_n := -R^D(f \wedge (nh)) \) for \( n \geq 1 \). Directly from the definition one can see that \( u_n \) is a weak subsolution to (1.2), with the given \( c \). If for some \( n \geq 1 \) we have \( \bar{u}_{n,D_S} < 0 \), then, clearly, \( R^D f(x) > 0, \ x \in D \).

Suppose, therefore that \( \bar{u}_{n,D_S} = 0 \) for all \( n \geq 1 \). Note that then \( R^D(f \wedge (nh)) \) is non-constant \( m \)-a.e. for some sufficiently large \( n \geq 1 \). Indeed, if otherwise we would have \( R^D(f \wedge (nh)) \equiv 0, \ m \text{ a.e. for all } n \geq 1 \). By virtue of Remark 4.1 then \( R^D(f \wedge (nh)) \equiv 0 \) for all \( \alpha \geq 0 \) and \( n \geq 1 \). From the fact that \( m \) is excessive we conclude, using (4.11), that

\[
0 = \lim_{\alpha \to +\infty} \alpha \int_D R^D_\alpha(f \wedge (nh)) dm = \int_D (f \wedge (nh)) dm, \quad n \geq 1.
\]

Letting \( n \to +\infty \) leads to a contradiction with the assumption that \( f \) is non-trivial.

Since \( u_n \) is non-constant for some \( n \) we can use estimate (4.9) (remember that \( \bar{u}_{n,D_S} = 0 \)). There exists \( a_n > 0 \) such that

\[
R^D(f \wedge nh)(x) = -u_n(x) \geq a_n \psi_D(x), \quad x \in D.
\]
Thus, again $R^D_f(x) > 0$, $x \in D$. By Remark 4.1, we conclude that the latter implies in fact (4.1).

**Remark 4.10.** The conclusion of Theorem 4.9 is still in force, if we assume that estimate (4.9) holds for any $u \in \mathcal{U}_1^*$ and, instead of the excessiveness of $m$, we require the weaker condition (4.10). Indeed, suppose that $f \in B_2^e(D)$ is such that $\int_D f \, dm > 0$. Then $u := -R^D_1 f$ is a weak subsolution to (1.2) with $c \equiv 1$. If $\tilde{u}_{D_x} < 0$, then obviously $R^D_1 f(x) > 0$ in $D$. Thanks to (4.9) the same conclusion holds, if $R^D_1 f$ is not constant. If the latter does not hold, then the only possibility for $R^D_1 f(x)$ to admit zero value is if it identically vanishes, but this would contradict (4.10).

**Definition 4.11** (Strong maximum principle). We say that the operator $-\Delta + c$, with $A$ given by (1.1), satisfies the strong maximum principle with respect to measure $m$ (SMP$_m$) if for any weak subsolution $u$ to (1.2) on $D$, for which $u(\hat{x}) = \tilde{u}_{D_x} \geq 0$ at some $\hat{x} \in D$ we have $u \equiv \tilde{u}_{D_x}$, $m$-a.e. in $D$.

**Theorem 4.12.** The hypothesis IR$_m$ implies the validity of the SMP$_m$ for the operator $-\Delta + c$ with any $c$ satisfying A3).

Conversely, if we assume that (4.10) and the SMP$_m$ for $-\Delta + c$, with $c \equiv 1$ are in force, then IR$_m$ holds.

**Proof.** Assume condition IR$_m$). Let $u$ be a weak subsolution to (1.2) in $D$ such that $M := \tilde{u}_{D_x} \geq 0$. Let $u_M := M - u$. From the fact that $M \geq 0$ and $u$ is a weak subsolution we conclude, via (2.11), that $u_M$ satisfies

$$u_M(x) \geq E_x \left[ e_c(t \wedge \tau_D) u_M(X_{t \wedge \tau_D}) \right], \quad x \in D, \ t \geq 0.$$  

Since $c$ is bounded, $e_c(t \wedge \tau_D)$ is strictly positive. Suppose that there exists $x_0 \in D$ such that $u(x_0) \equiv M$. Then $u_M(x_0) = 0$. Therefore, by the above inequality and strict positivity of $e_c(t \wedge \tau_D)$ we have $E_{x_0} u_M(X_{t \wedge \tau_D}) = 0$ for any $t \geq 0$. Thus,

$$P^D_t u_M(x_0) \leq E_{x_0} u_M(X_{t \wedge \tau_D}) = 0, \quad t \geq 0$$

and, as a result, $R^D_1 u_M(x_0) \equiv 0$. Thanks to the irreducibility assumption IR$_m$ we have $u_M \equiv 0$, $m$-a.e. on $D$. Therefore $u \equiv M$, $m$-a.e. on $D$ and the SMP$_m$ holds (cf Definition 4.11).

Conversely, assume the SMP$_m$ and (4.10). We show that IR$_m$ holds. We shall argue by contradiction. Let

$$f \in B_2^e(D) \quad \text{and} \quad \int_D f \, dm > 0. \quad (4.13)$$

Then $u := -R^D_1 f$ is a weak subsolution to (2.10), with $c \equiv 1$.

Assume that for some $x_0 \in D$ we have $u(x_0) = 0$. Thus, $\tilde{u}_{D_x} \equiv 0$ and by the SMP$_m$ we have $u \equiv 0$, $m$-a.e., which in turn implies that $R^D_1 f \equiv 0$, $m$-a.e. for all $\alpha \geq 0$. In light of (4.13) this clearly contradicts (4.10). We conclude therefore that $R^D_1 f > 0$ in $D$, thus IR$_m$ holds.

**Remark 4.13.** If we assume in Theorem 4.12 a stronger condition that $m$ is excessive instead of (4.10), then, by the argument used in the proof of Theorem 4.9, we conclude that the fact that SMP$_m$ holds for $-\Delta + c$ with some $c$ satisfying A3) implies IR$_m$)

5. Relationship between the minorization condition and intrinsic ultracontractivity

The concept of the intrinsic ultracontractivity has been introduced for symmetric operators by Davies and Simon in [22] and later extended to the non-symmetric...
We normalize Theorem 5.1. This estimate immediately implies the following. For example, if \( e.g. [29] \). This in turn implies that some sufficient regularity of the domain is usually required, e.g. of that state in particular that each Observe also that ultracontractivity of \( N \) and a symmetric Lévy semigroup, i.e. for all \( x \in \mathbb{R}^d \) and sub-Markovian semigroups \( \phi \). Recall that if \( \alpha \) exists of a continuous function \( \phi \), \( \hat{\phi}_D : D \to (0, +\infty) \) belonging to \( L^\infty(D, m) \) for both \( P_t^{D_0} \) and \( \hat{P}_t^{D_0} \), i.e.

\[
e^{-\lambda x t} \phi_D(x) = \int_D p_D(t, x, y) \phi_D(y) m(dy), \quad t \geq 0, x \in D
\]

and

\[
e^{-\lambda x t} \hat{\phi}_D(x) = \int_D p_D(t, y, x) \hat{\phi}_D(y) m(dy), \quad t \geq 0, x \in D.
\]

We normalize \( \phi_D \) and \( \hat{\phi}_D \) by letting \( \int_D \phi_D dm = \int_D \hat{\phi}_D dm = 1 \).

We say that \((P_t^{D_0})_{t \geq 0}\) is intrinsically ultracontractive, if for each \( t > 0 \) there exists constant \( c_t > 0 \) such that

\[
p_D(t, x, y) \leq c_t \phi_D(x) \hat{\phi}_D(y) \quad \text{for any } (x, y) \in D \times D. \tag{5.1}
\]

It is well known (see e.g. [32, Proposition 2.5]), that condition (5.1) implies the existence of a continuous function \( \tilde{c} : [0, +\infty) \to (0, +\infty) \) such that

\[
p_D(t, x, y) \geq \tilde{c}(t) \phi_D(x) \hat{\phi}_D(y) \quad \text{for any } (x, y) \in D \times D.
\]

This estimate immediately implies the following.

**Theorem 5.1.** Assume that \((P_t^{D_0})_{t \geq 0}\) is intrinsically ultracontractive. Then for any \( \alpha > 0 \) condition \( M(\alpha, \psi_D^\alpha, \nu_D^\alpha) \) holds, with

\[
\psi_D^\alpha = \phi_D, \quad \nu_D^\alpha(dx) := \left( \int_0^{\infty} \tilde{c}(s)e^{-\alpha s} ds \right) \phi_D(x) m(dx).
\]

There exists a vast literature on the intrinsic ultracontractivity of Lévy type operators, see e.g. [18, 32] and the references therein. Recall that if \((P_t)_{t \geq 0}\) is a symmetric Lévy semigroup, i.e. for all \( x \in D \) we have \( Q(x) \equiv Q(0) \), \( b \equiv 0 \) and \( N(x, dy) \equiv N(0, dy) \) is symmetric, then the full support of \( N(0, dy) \) implies ultracontractivity of \((P_t^{D_0})_{t \geq 0}\) for any bounded domain \( D \subset \mathbb{R}^d \) (see e.g. [28]). Observe also that ultracontractivity of \((P_t^{D_0})_{t \geq 0}\) implies that \( P_t^{D_0} 1 \sim \phi_D \) on \( D \) for each \( t > 0 \). There are many results concerning the behavior of \( P_t^{D_0} 1 \) (see e.g. [17, 29]) that state in particular that \( P_t^{D_0} 1 \sim \phi(\delta_D) \) on \( D \) for some function \( \phi \), which can be determined from the coefficients of \( A \) and the domain \( D \). For this type of result some sufficient regularity of the domain is usually required, e.g. of \( C^{1,1} \) class, see e.g. [29]. This in turn implies that

\[
\phi_D \sim \phi(\delta_D) \quad \text{on } D. \tag{5.2}
\]

For example, if \( A = \Delta^{s/2} \) for some \( s \in (0, 2) \), i.e.

\[
Au(x) = \lim_{h \to 0} \int_{B(x,h)} \left[ u(x+y) - u(x) \right] \frac{dy}{|y|^{s+2}}, \quad u \in C^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d), \quad x \in \mathbb{R}^d, \tag{5.3}
\]

then \( \phi(t) = t^{s/2}, \quad t \geq 0 \) for a sufficiently regular domain \( D \), see e.g. [35].

Therefore, directly from Theorems 3.2 and 5.1, we conclude the following version of the Hopf lemma.
Corollary 5.2. Suppose that (5.2) holds and the assumptions of Theorem 5.1 are in force. Let \( u \) be a non-constant \( \mathcal{D} \)-a.e. weak subsolution to (1.2). Then there exists \( a > 0 \) such that for any \( \hat{x} \in \partial D \) with \( u(\hat{x}) = \bar{u}_{D_s} \geq 0 \) we have
\[
\liminf_{x \to \hat{x}} \frac{u(\hat{x}) - u(x)}{\phi(\delta_D(x))} \geq a. \tag{5.4}
\]

Remark 5.3. In particular, if \( D \) satisfies the interior ball condition and \( n(\hat{x}) \) is the normal outward vector at \( \hat{x} \), then for sufficiently small \( h > 0 \),
\[
\frac{u(\hat{x}) - u(\hat{x} - h n(\hat{x}))}{\phi(h)} = \frac{u(\hat{x}) - u(\hat{x} - h n(\hat{x}))}{\phi(\delta_D(\hat{x} - h n(\hat{x}))}.
\]
So, by Corollary 5.2, the \( \phi \)-normal derivative at a maximal point \( \hat{x} \), defined as
\[
\partial^n_{\phi} u(\hat{x}) := \liminf_{h \to 0^+} \frac{u(\hat{x}) - u(\hat{x} - h n(\hat{x}))}{\phi(h)},
\]
is strictly positive.

Remark 5.4. Estimate (5.4) is an analogue of the classical Hopf lemma. As we have mentioned in the foregoing, it does require, as in the classical case, regularity of the domain. Recall that the classical Hopf lemma for the Laplace operator does not hold in general, even for \( C^1 \)-regular domains. However, in many applications estimates of the type (3.2) with \( \psi_D = \varphi_D \) are quite sufficient. In such a situation there is no need for a strong regularity assumption on \( D \), as in the case of estimates (5.4). For many operators \( A \) regularity assumptions on \( D \) are only required to characterize the behaviour of the principal eigenfunction near the boundary of the domain. More specifically, if \( A = \Delta^s \) for some \( s \in (0, 2) \), then for any bounded domain we have (3.2) with \( \psi_D^s = \varphi_D \), since it is well known (see e.g. [18, Theorem 1]) that \( (P_t^D)_{t \geq 0} \) is intrinsic ultracontractive, regardless of the regularity of \( D \). On the other hand even for Lipschitz domains \( D \) it is not true that \( \varphi_D \sim \delta_D \). So, we cannot expect (5.4) to hold for a non-regular \( D \).

6. Minorization condition with bottom function as an eigenfunction and ergodic properties of the canonical process

In the previous section, we have proved that intrinsic ultracontractivity of the semigroup \( (P^D_t) \) implies \( M(\alpha, \psi_D^m, \nu_D^m) \) with \( \psi_D^m = \varphi_D \), where \( \varphi_D \) is the respective principal eigenfunction. Here we discuss the relationship between the minorization condition and ergodic properties of the canonical process. It turns out, see Theorem 6.5 below, that under the hypotheses of irreducibility (condition IR) and finiteness of the exit time of the canonical process (condition ET), the validity of \( M(\alpha, \varphi_D, \nu_D^m) \) is equivalent with the property of uniform ergodicity of the resolvent family corresponding to the \( \varphi_D \)-process obtained from the canonical process via the \( \varphi_D \)-Doob transform, see Definition 6.4. We also discuss a relation between uniform conditional ergodicity of the canonical process and the minorization condition. Recall that \( m \) is some non-trivial, \( \sigma \)-finite Borel measure on \( D \).

Definition 6.1. A function \( \varphi_D \) and \( \lambda_D \) are called a non-negative eigenfunction and eigenvalue for \( A \), see e.g. [3], if \( \varphi_D : D \to [0, +\infty) \), \( \lambda_D \geq 0 \),
\[
e^{-\lambda_D t} \varphi_D(x) = P^D_t \varphi_D(x), \quad t \geq 0, x \in D \tag{6.1}
\]
and \( \int_D \varphi_D dm > 0 \).

In some of our results we shall assume the condition.
Eig) The operator $A$ possesses a non-negative eigenfunction $\varphi_D$ and eigenvalue $\lambda_D$.

Observe that $\varphi_D$ is an excessive function with respect to $(P_t^D)$.

**Remark 6.2.** Obviously hypothesis $IR_m$) implies that $\varphi_D(x) > 0$ for all $x \in D$. Under some additional assumptions $\lambda_D$ could be a unique eigenvalue with a positive eigenfunction. If, it is indeed simple, then $\varphi_D$ can be determined, up to a multiplicative constant. In such a situation the pair $(\lambda_D, \varphi_D)$ is called the principal pair while $\lambda_D, \varphi_D$ are referred to as the principal eigenvalue and eigenfunction for $A$, respectively. A discussion on the validity of Eig) is carried out in Section A.3 below.

Throughout this section we suppose the irreducibility condition $IR_m$) and the existence of a non-negative eigenfunction $\varphi_D$ (condition Eig)). Then, as we have already mentioned, $\varphi_D > 0$ in $D$. Consider $(P_t^{D,\varphi_D})$ - the transition probability semigroup of the conservative $\varphi_D$-process (see [19, Chapter 11]), defined by

$$P_t^{D,\varphi_D} f(x) := e^{\lambda_D t} \frac{P^D_t(f\varphi_D)(x)}{\varphi_D(x)}, \quad f \in B_b(D), x \in D. \quad (6.2)$$

Let $(R_{a,D,\varphi_D})_{a>0}$ be the respective resolvent family.

6.1. **Uniform ergodicity of a Markov process and its resolvent.** Let us recall the notion of the uniform ergodicity of a Markov process, see [25, p. 1675].

**Definition 6.3.** Suppose that $(X_t)_{t \geq 0}$ is an $E$-valued Markov process with the transition probabilities $P_t(x, \cdot), t \geq 0, x \in E$. We say that its transition semigroup $(P_t)_{t \geq 0}$ is uniformly ergodic if there exists $\Pi_{\text{inv}}$ - an invariant Borel probability measure on $E$, constants $C > 0$ and $\rho \in (0,1)$ such that, see (2.1),

$$\sup_{x \in E} \|P_t(x, \cdot) - \Pi_{\text{inv}}\|_{TV} \leq C \rho^t, \quad t \geq 0.$$

An analogous definition can be formulated for a discrete time Markov chain.

Given $\alpha > 0$ and a Markov process $(X_t)_{t \geq 0}$, with the respective resolvent family $(R_{\alpha})_{\alpha > 0}$, consider the transition probability kernel

$$K_{\alpha}(x, B) := \alpha R_{\alpha} 1_B(x), \quad x \in E, B \in \mathcal{B}(E). \quad (6.3)$$

We suppose that a Markov chain $(\tilde{X}_{n}^{(\alpha)})_{n \geq 0}$ has the transition probability given by (6.3).

**Definition 6.4.** We say that the resolvent family $(R_{\alpha})_{\alpha > 0}$ is uniformly ergodic if the chain $(\tilde{X}_{n}^{(\alpha)})_{n \geq 0}$ is uniformly ergodic for any $\alpha > 0$.

6.2. **Minoration condition and uniform ergodicity of the resolvent.** It turns out that, if $(X_t)_{t \geq 0}$ is irreducible and $\varphi_D$ is bounded, then the minoration condition $M(\alpha, \varphi_D, \nu_D^\alpha)$ is equivalent with the uniform ergodicity of the resolvent of the $\varphi_D$-process.

**Theorem 6.5.** Assume that $IR_m$) and Eig) hold, the eigenfunction $\varphi_D$ of Eig) is bounded, and $mR^D_{\alpha} \ll m$ for some $\alpha \geq 0$. Then, condition $M(\alpha, \varphi_D, \nu_D^\alpha)$ is satisfied with non-trivial measure $\nu_D^\alpha$ for any $\alpha \geq 0$ if and only if $(R_{\alpha,D,\varphi_D})_{\alpha>0}$ is uniformly ergodic. Moreover, if $(P_{\alpha,D,\varphi_D})_{t \geq 0}$ is uniformly ergodic, then $M(\alpha, \varphi_D, \nu_D^\alpha)$ holds for any $\alpha \geq 0$ with $m \ll \nu_D^\alpha$. 

Proof. Suppose first that the \( \varphi_D \)-process \( (X_t)_{t \geq 0} \) is uniformly ergodic with \( \Pi \) the unique invariant probability measure. Then for every \( f \in B_0^* (D) \) there exists \( t_f > 0 \) such that
\[
P_t^{D, \varphi_D} f \geq \frac{1}{2} \int_D f \, d\Pi, \quad t \geq t_f.
\]
From the definition (6.2) we conclude
\[
P_t^{D} f \geq \frac{\varphi_D e^{-\lambda_D t}}{2 \parallel \varphi_D \parallel} \int_D f \, d\Pi, \quad t \geq t_f.
\]
If \( w \) is a bounded, \( \alpha \)-excessive function, with respect to \( (P^D_t)_{t \geq 0} \), then there exists \( t_w > 0 \) such that
\[
w \geq e^{-\alpha t} P^D_t w \geq \frac{\varphi_D e^{-(\lambda_D + \alpha) t}}{2 \parallel \varphi_D \parallel} \int_D w \, d\Pi, \quad t \geq t_w.
\]
Therefore, for any \( \alpha \)-excessive non-trivial \( m \)-a.e. function \( w \) one can find a constant \( c_w > 0 \) such that
\[
w \geq c_w \varphi_D.
\] (6.4)

Using Theorem 4.4, we conclude the existence of an \( \alpha + 1 \)-excessive function \( \psi_D^\alpha \) and measure \( \nu_D^\alpha \), that dominates the Lebesgue measure \( m \) such that (3.1) holds. Combining this fact with the argument presented in the foregoing we conclude that there exists \( c > 0 \) such that \( \psi_D^\alpha \geq \varphi_D \), and \( M(\alpha, \varphi_D, \nu_D^\alpha) \) follows.

Suppose now that \( (R^D_{\beta, \varphi_D})_{\beta \geq 0} \) is uniformly ergodic. Fix any \( \alpha_0 > \lambda_D + 1 \). We establish (3.1) with \( \psi_D^\alpha = \varphi_D \) for any \( \alpha \in [0, \alpha_0 - \lambda_D - 1] \). This obviously would imply the validity of the condition for any \( \alpha \geq 0 \).

Given \( \alpha \in [0, \alpha_0 - \lambda_D - 1] \) choose \( \psi_D^\alpha \) and \( \nu_D^\alpha \) satisfying the conclusion of Theorem 4.4. In particular, \( \psi_D^\alpha \) is \( \alpha + 1 \)-excessive and \( m \ll \nu_D^\alpha \). Observe that
\[
R^D_{\beta, \varphi_D} f = \varphi_D^{-1} R^D_{\beta - \lambda_D} (f \varphi_D), \quad \beta \geq \lambda_D, \ f \in B^*(D).
\] (6.5)

Using the uniform ergodicity of the Markov chain corresponding to the kernel \( \alpha_0 R^D_{\alpha_0} \) and (6.5) for \( \beta = \alpha_0 \) we conclude, as in the foregoing, that there exists \( \tilde{\Pi}^{(\alpha_0)} \) - a probability measure - such that for any \( f \in B^*_0 (D) \) one can find \( n_f > 0 \) for which
\[
(R^D_{\alpha_0 - \lambda_D})^n f \geq \frac{\varphi_D}{2 \alpha_0 \parallel \varphi_D \parallel} \int_D f \, d\tilde{\Pi}^{(\alpha_0)}, \quad n \geq n_f.
\]
Hence, if \( w \) is any non-trivial \( m \)-a.e. \( \alpha + 1 \)-excessive function (w.r.t. \( (P^D_t) \)) we get that there exists \( n_w > 0 \) such that
\[
w \geq \frac{\varphi_D}{2 \parallel \varphi_D \parallel} \left( 1 - \frac{\lambda_D + \alpha + 1}{\alpha_0} \right)^n \int_D w \, d\tilde{\Pi}^{(\alpha_0)}, \quad n \geq n_w
\]
and (6.4) is in force. Thus, \( \psi_D^\alpha \geq c \varphi_D \) for some constant \( c > 0 \). Theorem 4.4 allows us to conclude then \( M(\alpha, \varphi_D, \nu_D^\alpha) \).

Suppose now that \( M(\alpha, \varphi_D, \nu_D^\alpha) \) holds. Then, by virtue of (6.5), the measure \( d\tilde{\nu}_D^\alpha := \varphi_D d\nu_D^\alpha \) satisfies
\[
R^D_{\alpha, \varphi_D} f \geq R^D_{\alpha + \lambda_D} f \geq \int_D f d\tilde{\nu}_D^\alpha, \quad x \in D, \ f \in B^*_0 (D), \ \alpha > 0.
\]
Using the Dobrushin theorem, see [36, Theorem 2.3.1, p. 33], we conclude the uniform ergodicity in the total variation norm of the Markov chain \( (\tilde{X}^{(\alpha)}_n)_{n \geq 0} \) and the proof of Theorem 6.5 is therefore concluded. \( \Box \)
6.3. Uniform conditional ergodicity and minorization condition $M'(\alpha, \nu_D')$.

**Definition 6.6.** The canonical process $(X_t)_{t \geq 0}$ is called uniformly conditionally ergodic, cf [34, Section 2.2], if there exists a (unique) probability Borel measure $\Pi$ on $D$, called quasi-stationary distribution, such that

$$\lim_{t \to +\infty} \sup_{B \in \mathcal{B}(D), x \in D} \left| P_t^D(X_t \in B | t < \tau_D) - \Pi(B) \right| = 0. \quad (6.6)$$

It turns out that the minorization condition $M'(\alpha, \nu_D')$ is a consequence of the uniform conditional ergodicity of $(X_t)_{t \geq 0}$. This is a consequence of Theorem 6.5 and the following result.

**Proposition 6.7.** Suppose that the assumptions of Theorem 6.5 are in force. If the canonical process $(X_t)_{t \geq 0}$ is uniformly conditionally ergodic, then $(P_t^{D, \varphi_D})_{t \geq 0}$ is uniformly ergodic.

**Proof.** Condition (6.6) is equivalent with

$$\lim_{t \to +\infty} \sup_{B \in \mathcal{B}(D), x \in D} \left| \frac{P_t^D(x, B)}{P_t^D(x, D)} - \Pi(B) \right| = 0. \quad (6.7)$$

Here $P_t^D(x, B) := P_t^D 1_B(x)$. From (6.7) we obtain

$$\lim_{t \to +\infty} \sup_{x \in D, \|f\|_1 \leq 1} \left| \frac{P_t^D(\varphi_D f)(x)}{P_t^D 1(x)} - \int_D f \varphi_D d\Pi \right| = 0.$$

Equivalently,

$$\lim_{t \to +\infty} \sup_{x \in D, \|f\|_1 \leq 1} \left| P_t^D \varphi_D f(x) P_t^D 1(x) - \int_D f \varphi_D d\Pi \right| = 0.$$

Condition (6.6) implies that $P_t^D \varphi_D f / P_t^D 1$ converges uniformly to $a := \int_D \varphi_D d\Pi$, as $t \to +\infty$. Therefore,

$$\lim_{t \to +\infty} \sup_{x \in D, \|f\|_1 \leq 1} \left| P_t^D \varphi_D f(x) - a \int_D f \varphi_D d\Pi \right| = 0$$

and the conclusion of the proposition follows. \hfill \Box

6.4. $\varphi_D$-Hopf lemmas and uniform ergodicity of the resolvent. We close this section with an interesting corollary to the results presented above. It states essentially that the Hopf lemma with the bottom function $\varphi_D$ (a positive eigenfunction) is equivalent with the uniform ergodicity of the resolvent family corresponding to the $\varphi_D$-Doob transform of the canonical process.

**Definition 6.8.** Assume that condition Eig) is in force. We say that a $\varphi_D$-Hopf lemma holds for subsolutions to (1.2) if for any $u \in \mathcal{U}_+^*$, that is non-constant $m$-a.e. in $D$, there exists $a > 0$ such that

$$\tilde{u}_D - u(x) \geq a \varphi_D(x), \quad x \in D. \quad (6.8)$$

**Proposition 6.9.** Assume that $m R_\alpha^D \ll m$ for some $\alpha \geq 0$, and conditions IRm) and Eig) hold. Suppose that for any $\alpha \geq 0$ and for any non-trivial function $f \in B^*(D)$ there exists $c(f, \alpha) > 0$ such that

$$R_\alpha^D f \geq c(f, \alpha) \varphi_D, \quad \text{in } D. \quad (6.9)$$

Then, $M(\alpha, \varphi_D, \nu_D')$ holds for any $\alpha \geq 0$, with $m \ll \nu_D'$. 
Proof. Since $\text{IR}_m$ and $mR^D_\alpha \ll m$ are assumed, by Theorem 4.4, there exist a measure $\nu^D_\alpha$, satisfying $m \ll \nu^D_\alpha$, and a strictly positive, $\alpha + 1$-excessive function $\psi^D_\alpha$ such that estimate (3.1) holds. Thus, by (6.9) applied for $\alpha + 2$ and $f = \psi^D_\alpha$, we have

$$\psi^D_\alpha \geq R^D_{\alpha+2}\psi^D_\alpha \geq c(\psi^D_\alpha, \alpha + 2)\varphi_D.$$  \hfill (6.10)

The conclusion of the proposition follows from an application of (3.1) and then (6.10). \hfill \Box

**Theorem 6.10.** Suppose that the assumptions of Theorem 6.5 are in force. The $\varphi_D$-Hopf lemma, as formulated in Definition 6.8, holds for any $c(\cdot)$ satisfying the hypothesis A3) if and only if the resolvent family $(R^D_{\alpha, \varphi_D})_{\alpha > 0}$ is uniformly ergodic.

**Proof.** Assume the validity of the $\varphi_D$-Hopf lemma. Suppose that $\alpha \geq 0$ and that $f \in B^+_0 (D)$ satisfies $\int_D f dm > 0$. Then $u := -R^D_\alpha f$ is a weak subsolution to $-Av + \alpha v = 0$ and, as a result, $u \in \mathcal{U}_\alpha$. Clearly $\bar{u}_D \leq 0$. If $\bar{u}_D = 0$, then obviously (6.9) holds (as $\varphi_D$ is assumed to be bounded). Suppose therefore that $\bar{u}_D > 0$. It cannot be constant, as then we would have $\int_D R^D_\alpha f dm = 0$, which in turn, by the assumption that $mR^D_\alpha \ll m$, would imply that $\int_D f dm = 0$.

The latter contradicts the fact that $f$ is non-trivial $m$ a.e. We conclude from (6.8) that there exists $a > 0$ such that

$$R^D_\alpha f(x) = R^D_\alpha f(x) + \bar{u}_D = \bar{u}_D - u(x) \geq a\varphi_D(x).$$

Thus (6.9) follows again. From Proposition 6.9 and Theorem 6.5 we infer therefore the uniform ergodicity of $(R^D_{\alpha, \varphi_D})_{\alpha > 0}$.

Conversely, suppose that the resolvent family is uniformly ergodic. Then, by Theorem 6.5 for any $\alpha \geq 0$ condition $M(\alpha, \varphi_D, \nu^D_\alpha)$ holds with non-trivial $\nu^D_\alpha$ satisfying $m \ll \nu^D_\alpha$. The $\varphi_D$-Hopf lemma is then a direct consequence of Theorem 3.2. \hfill \Box

7. Quantitative Hopf Lemma

Recall that $\mathcal{U}_\alpha (g)$ denotes the set of all weak subsolutions to (2.10) and $\mathcal{U}^+_\alpha (g)$ consists of those $u \in \mathcal{U}_\alpha (g)$, for which $\bar{u}_D \geq 0$. Given a function $f : D \to \mathbb{R}$ we let $f^* = \max\{-f, 0\}$. Throughout this section we assume that $\check{g}_D \leq 0$, therefore $g^* = -g$.

A starting point of our discussion is the following simple result.

**Proposition 7.1.** Suppose that $u \in \mathcal{U}_\alpha (g)$. Then, cf (2.6),

$$\bar{u}_D - u(x) \geq \bar{u}_D w_cD(x), \quad x \in D. \hfill (7.1)$$

Note that $w_cD$ is entirely determined by the operator $A$ and function $c$. Clearly, (7.1) is non-trivial as long as function $w_cD$ does not vanish. The latter is guaranteed e.g. by the hypothesis $\text{IR}_m$ and A3') as can be seen from Proposition 7.7 below.

We see that in case $\bar{u}_D > 0$ and $c$ is in some sense non-trivial, then the Hopf lemma holds with $a = \bar{u}_D$ and the bottom function $w_cD$. A result of this kind, where the dependence of the constant $a > 0$ on $\bar{u}_D$ is explicit, e.g. $a = C(\bar{u}_D + 1)$, and the constant $C > 0$ depends only on the coefficients $c, g$, or operator $A$ shall be called a quantitative Hopf lemma.

**Proof of Proposition 7.1.** With no loss of generality we may and shall assume that $\bar{u}_D \geq 0$. Otherwise the estimate (7.1) is trivial. By the definition of a weak
subsolution and Lemma 3.3, we can write
\[ \bar{u}_{D_s} - u(x) \geq \bar{u}_{D_s} - \mathbb{E}_x [e_c(\tau_D \land t)u(X_{\tau_D \land t})] \]
\[ \geq \bar{u}_{D_s} - \mathbb{E}_x [e_c(\tau_D \land t)] \geq \bar{u}_{D_s} \{ 1 - \mathbb{E}_x [e_c(\tau_D \land t)] \} \]
\[ \geq w_{c,D}(x)\bar{u}_{D_s}, \quad x \in D \]
and estimate (7.1) follows. \[ \square \]

In what follows we formulate a number of refinements of Proposition 7.1.

7.1. Quantitative Hopf lemmas. We start with the following.

**Theorem 7.2** (Quantitative \( \varphi_D \)-Hopf lemma). Suppose that \( \text{Eig} \) holds. Then, (see (2.3)) for any \( u \in \mathcal{U}_c^\lambda(g) \),
\[ \bar{u}_{D_s} - u(x) \geq \frac{\varphi_D(x)}{2\|\varphi_D\|_{\infty}} \left( \frac{\xi_D\bar{u}_{D_s}}{\lambda_D + \xi_D} + \frac{g_D}{\lambda_D + \lambda_D} \right), \quad x \in D. \]  
(7.2)

**Proof.** By the definition of a weak subsolution, Lemma 3.3 and (3.5),
\[ \bar{u}_{D_s} - u(x) \geq \left( 1 - \mathbb{E}_x e_c(\tau_D \land t) \right) \bar{u}_{D_s} - \mathbb{E}_x \left[ \int_0^{\tau_D \land t} e_c(s)g(X_s) ds \right], \quad t > 0. \]  
(7.3)

Observe that (here \( \xi_D = \inf_D c \))
\[ 1 - \mathbb{E}_x e_c(\tau_D \land t) \geq (1 - e^{-\xi_D t}) P_x(\tau_D > t). \]  
(7.4)

From (6.1) we have
\[ e^{-\lambda_D t}\varphi_D(x) = \mathbb{E}_x [\varphi_D(X_t), \tau_D > t] \leq \|\varphi_D\|_{\infty} P_x(\tau_D > t). \]  
(7.5)

Substituting into (7.4) the lower bound on \( P_x(\tau_D > t) \) obtained from (7.5) and maximizing over \( t > 0 \) we conclude that
\[ \bar{u}_{D_s} - u(x) \geq \frac{\bar{u}_{D_s}\varphi_D(x)}{\|\varphi_D\|_{\infty}} \frac{\xi_D/\lambda_D}{(1 + \xi_D/\lambda_D)^{1 + \xi_D/\lambda_D}} \]  
(7.6)

In the last inequality we use an elementary bound \( (1 + s)^\lambda \leq e \) valid for \( s > 0 \). We also have (here, as we recall, \( \bar{c}_D = \sup_D c \) and \( \bar{g}_D = \inf_D g^- \))
\[ -\mathbb{E}_x \left[ \int_0^{\tau_D \land t} e_c(r)g(X_r) dr \right] \geq \mathbb{E}_x \left[ \int_0^{\tau_D \land t} e^{-\bar{c}_D r}\bar{g}_D^d dr \right] \geq \frac{1}{\bar{c}_D} (1 - e^{-\bar{c}_D t}) P_x(\tau_D > t) \bar{g}_D^d. \]

Using again (7.5) to estimate \( P_x(\tau_D > t) \) from below and maximizing over \( t > 0 \) we conclude that
\[ \bar{u}_{D_s} - u(x) \geq \frac{\varphi_D(x)}{\|\varphi_D\|_{\infty}\bar{c}_D} \frac{\bar{g}_D\bar{c}_D/\lambda_D}{(1 + \bar{c}_D/\lambda_D)^{1 + \bar{c}_D/\lambda_D}} \geq \frac{\varphi_D(x)}{\|\varphi_D\|_{\infty}} \frac{\bar{g}_D}{\lambda_D + \bar{c}_D}. \]  
(7.7)

Estimate (7.2) follows easily from (7.6) and (7.7). \[ \square \]

**Theorem 7.3.** Assume the minorization condition \( M(\bar{c}_D, \psi_D^\lambda, \nu_D^\lambda) \) (see Definition 3.1) for some non-negative function \( \psi_D^\lambda \) and Borel measure \( \nu_D^\lambda \). Suppose furthermore that \( u \in \mathcal{U}_c^\lambda(g) \). Then,
\[ \bar{u}_{D_s} - u(x) \geq \psi_D^\lambda(x) \left\{ \bar{u}_{D_s} \int_D c e_{c,D} dv_{D}^\lambda + \int_D g^d dv_{D}^\lambda \right\}, \quad x \in D. \]  
(7.8)
Proof. We start with the estimate (7.3). Letting \( t \to +\infty \) we conclude, upon an application of the Fatou lemma, that
\[
\bar{u}_{D_s} - u(x) \geq w_{c,D}(x) \bar{u}_{D_s} - \mathbb{E}_x \left[ \int_0^T e_c(s) g(X_s) \, ds \right], \quad t > 0.
\]
To estimate the first term in the right hand side we use (7.11) and \( M(\bar{c}_D, \bar{\psi}_D, \bar{\nu}_D) \).
Then, for any \( x \in D \),
\[
w_{c,D}(x) \bar{u}_{D_s} = \mathbb{E}_x \left[ \int_0^T v_{c,D}(X_t) \psi(X_t) \, dt \right] \bar{u}_{D_s} \geq \bar{u}_{D_s} \bar{\psi}_D(x) \int_D c_{v_{c,D}} d\nu_{D,A}
\]
and
\[
-\mathbb{E}_x \left[ \int_0^T e_c(s) g(X_s) \, ds \right] \geq -\mathbb{E}_x \left[ \int_0^T e^{-\bar{c}_D} g(X_s) \, ds \right] \geq -\bar{\psi}_D(x) \int_D g \, d\nu_{D,A}.
\]
Estimate (7.8) thus follows.

Theorem 7.3 combined with the results implying the minorization condition, obtained in the foregoing, allow us to formulate various quantitative Hopf lemmas. We formulate two such results, which may be of special interests in the theory of P.D.E-s

**Theorem 7.4.** 1) Assume that \( IR_{m,\nu} \) holds and \( mR_{\alpha}D \ll m \) for some \( \alpha \geq 0 \). Then, there exist a function \( \psi_{D,A} : D \to (0, +\infty) \) and a Borel measure \( \nu_{D,A} \) on \( D \) such that \( m \ll \nu_{D,A} \) and any \( u \in \mathcal{U}_c(g) \) satisfies
\[
\bar{u}_{D_s} - u(x) \geq \psi_{D,A}(x) \left\{ \bar{u}_{D_s} \int_D c_{v_{c,D}} d\nu_{D,A} + \int_D g^- d\nu_{D,A} \right\}, \quad x \in D. \tag{7.9}
\]
Moreover, if coefficient \( c(\cdot) \) satisfies \( A3' \), then \( \int_D c_{v_{c,D}} d\nu_{D,A} > 0 \).

2) Assume that \( (P) \) is intrinsically ultracontractive. Then there exists \( a > 0 \) such that
\[
\bar{u}_{D_s} - u(x) \geq a \varphi_D(x) \left\{ \bar{u}_{D_s} \int_D c_{\bar{\varphi}_D} d\nu_{D,A} + \int_D \varphi_D g^- d\nu_{D,A} \right\}, \quad x \in D. \tag{7.10}
\]
for any \( u \in \mathcal{U}_c(g) \). Moreover, if coefficient \( c(\cdot) \) satisfies \( A3' \), then \( \int_D c_{v_{c,D}} d\nu_{D,A} > 0 \).

**Proof.** Estimate (7.9) follows from Theorems 7.3 and 4.4, while (7.10) follows from Theorems 7.3 and 5.1. From condition ET) we conclude that \( v_{c,D}(x) > 0 \) for each \( x \in D \). Therefore, thanks to assumption \( A3' \), we obtain that \( \int_D c_{v_{c,D}} d\nu_{D,A} > 0 \). From this and the fact that \( m \ll \nu_{D,A} \), we further conclude that \( \int_D c_{v_{c,D}} d\nu_{D,A} > 0 \).

**Corollary 7.5.** Assume that \( (P) \) is intrinsically ultracontractive. Suppose that \( (5.2) \) holds and \( c \) satisfies \( A3' \). Then, there exists a constant \( a > 0 \) such that for any \( u(\cdot) - \text{non-constant} \ ell_{D-a.e.} \text{ weak subsolution to (2.10)} \) and \( \hat{x} \in \partial D \) with \( u(\hat{x}) = \bar{u}_{D_s} \geq 0 \) we have
\[
u(\hat{x}) - u(x) \geq a \varphi(\delta(x)) \left( u(\hat{x}) + \int_D \varphi(y) g^- (y) \, dy \right).
\]

**Remark 7.6.** Theorem 7.4 is a far reaching generalization of the Morel-Oswald formulation of the Hopf-Lemma (see [14]) as can be seen from Corollary 7.5. The result of ibid. states that: if \( D \) is a smooth bounded domain, then there exists \( c > 0 \) such that for any \( u \in \mathcal{W}^{1,2}(D) \cap \mathcal{W}^{1,2}_0(D) \) satisfying
\[
-\Delta u = f \quad \text{in } D,
\]
with \( f \in L^\infty(D) \), such that \( f \geq 0 \) we have
\[
u(x) \geq c \delta(x) \int_D \delta(y) f(y) \, dy, \quad x \in D.
\]
7.2. Properties of $w_{c,D}$. Here we take a closer look at the function $w_{c,D}$ appearing in Proposition 7.1. Recall that for its definition, cf (2.6), we require only conditions A1) - A4). Obviously $0 \leq w_{c,D}(x) \leq 1$, $x \in D$. Since $w_{c,D}$ is bounded, a simple calculation shows that

$$w_{c,D}(x) = R^D(cv_{c,D})(x) = R^D(c - cw_{c,D})(x), \quad x \in D. \quad (7.11)$$

Indeed, let $C_t = \int_0^t c(X_r) \, dr$. Clearly, $v_{c,D}(x) = \mathbb{E}_x e^{-C_{\tau_D}}, \ x \in \mathbb{R}^d$. Then,

$$1 - v_{c,D}(x) = 1 - \mathbb{E}_x e^{-C_{\tau_D}} = \mathbb{E}_x \left[ \int_0^{\tau_D} e^{-C_t} \, dC_t \right] = \mathbb{E}_x \left[ \int_0^{\tau_D} c(X_t)e^{-C_t} \, dt \right]. \quad (7.12)$$

On the other hand, by the Markov property,

$$1_{(t < \tau_D)}v_{c,D}(X_t) = 1_{(t < \tau_D)}\mathbb{E}_x \left[ e^{-\int_0^t c(X_r) \, dr} \, \mathcal{F}_t \right].$$

Thus,

$$R^D(cv_{c,D})(x) = \mathbb{E}_x \left[ \int_0^{\tau_D} c(X_t)v_{c,D}(X_t) \, dt \right] = \mathbb{E}_x \left[ \int_0^{\tau_D} c(X_t)e^{-\int_0^t c(X_r) \, dr} \, dt \right]$$

$$= \mathbb{E}_x \left[ \int_0^{\tau_D} c(X_t)e^{-\int_0^t c(X_r) \, dr} \, dt \right].$$

This combined with (7.12) yields (7.11).

From the definition of $w_{c,D}$ we conclude that

$$w_{c,D}(x) > 0 \quad \text{iff} \quad P_x \left( \int_0^{\tau_D} c(X_r) \, dr > 0 \right) = 1. \quad (7.11)$$

Thanks to (7.11) we conclude the following.

**Proposition 7.7.** Suppose that $c$ satisfies assumption A3'). Then, condition IR$_m$ implies that $w_{c,D}(x) > 0$ for all $x \in D$.

**Remark 7.8.** In particular, the assumption of uniform ellipticity of the local part of $A$ implies condition IR$_m$), see [33, Section 7.7.1]. Then, $w_{c,D}$ is strictly positive. It turns out that it is also continuous on $D$, see Lemma 7.6 of ibid.

APPENDIX A. Supplement: Some additional comments on the hypotheses and applications of the results

A.1. Remarks on the existence of a strong Markovian solution of the martingale problem. Fairly easy to verify conditions on the coefficients of the operator $A$ implying A4) appear in the literature, see e.g. [50, 39], Chapter 4 of [26], [38, 37], [1, 30, 31]. Below, we review some of the existing results. First note that without any loss of generality we may and shall assume that

$$N(x, B^c(0, 1)) = 0, \quad x \in \mathbb{R}^d. \quad (A.1)$$

Indeed, let $(P_x)_{x \in \mathbb{R}^d}$ be a strong Markov process solving the martingale problem for an operator $A_0$ given by (1.1), with $N$ replaced by

$$\tilde{N}(x, dy) = 1_{B^c(0, 1)}(y) N(x, dy).$$

By [26, Proposition 10.2, Section 4] there exists then a strong Markov solution to the problem associated with $A = A_0 + A_1$, where

$$A_1 f(x) := \int_{B^c(0, 1)} (f(x + y) - f(x)) N(x, dy), \quad x \in \mathbb{R}^d.$$  

From condition (A.1) and assumption (2.2) we infer

$$\lim_{R \to \infty} \sup_{|x| \leq R} \sup_{|\xi| \leq 1/R} |p(x, \xi)| = 0, \quad (A.2)$$
where $p(x, \xi)$ is the Fourier symbol associated with the operator $A$, defined as

$$p(x, \xi) = \frac{1}{2} \sum_{k, \ell=1}^{d} q_{k, \ell}(x) \xi_{k} \xi_{\ell} - i \sum_{k=1}^{d} b_{k}(x) \xi_{k} + \int_{\mathbb{R}^{d}} \left(1 - e^{i \xi \cdot y} + \frac{iy \cdot \xi}{1 + |y|^{2}} \right) N(x, dy) \quad (A.3)$$

for any $x, \xi \in \mathbb{R}^{d}$. Obviously $\text{Re} p(x, \xi) \geq 0$.

By [38, Theorem 4.1] condition (A.2) implies the existence of a strong Markovian solution associated with the operator $A$, provided that we can prove that there exists a solution to the martingale problem for $A$ and an arbitrary initial Borel probability distribution $\mu$.

To state the result concerning the existence of a strong Markovian solution we formulate some additional hypotheses.

SE) In addition to the assumptions made in A1) suppose that the matrix $Q(x)$ is uniformly positive definite on compact sets, i.e. for any compact set $K \subset \mathbb{R}^{d}$ there exists $\lambda_{K} > 0$ such that

$$\lambda_{K} |\xi|^{2} \leq \sum_{i, j=1}^{d} q_{i, j}(x) \xi_{i} \xi_{j}, \quad x \in K, \xi = (\xi_{1}, \ldots, \xi_{d}) \in \mathbb{R}^{d}. \quad (A.4)$$

C) the mapping $\mathbb{R}^{d} \ni x \mapsto Q(x) \in \mathbb{R}^{d \times d}$ is continuous together with

$$\mathbb{R}^{d} \ni x \mapsto N_{B}(x) := \int_{B} \min\{|y|^{2}, 1\} N(x, dy)$$

for any Borel $B \subset B(0, 1)$.

The following summarizes a few of sufficient conditions for the validity of A4).

**Theorem A.1.** Hypothesis A4) is satisfied, provided that one of the following conditions are fulfilled:

a) (2.2) and (A.4) are in force, or

b) (2.2), C) hold and $Q(x)$ is invertible for every $x \in \mathbb{R}^{d}$, or

c) (2.2), C) are satisfied and the mapping $x \mapsto b(x)$ is continuous.

**Proof.** The fact that a) implies A4) follows from [1], see also [39]. The result concerning condition b) is a consequence of [31, Theorem III.2.34, p. 159], see also [50]. The implication for c) follows from [30], see also [13, Theorem 3.24], respectively. □

**A.2. Finiteness of the exit time.** Clearly, for the validity of the weak maximum principle (WMP), see (1.6), we need, some additional condition besides A1) - A4), to exclude at least the case $A = 0$, in which the principle obviously fails. Recall that in the case of the Laplace operator we have the so called infinite propagation speed of disturbances. Heuristically, the necessary condition for the validity of the WMP is the communication (via the canonical process $(X_{t})_{t \geq 0}$) between the interior and the exterior of the domain $D$, or in other words propagation of the disturbance in the entire $\mathbb{R}^{d}$. Observe that if $P_{x}(\tau_{D} = +\infty) = 1$ for all $x \in D$ (no communication), then $u = 1_{D}$ is a weak subsolution of $Av = 0$ (here $c \equiv 0$). Obviously, (1.6) does not hold for $u$. So, condition ET), used in Proposition 3.4, seems to be fairly close to optimal for the validity of the weak maximum principle. Many sufficient conditions can be found in the literature implying ET). Below we review a few of them.

As we have already mentioned in Remark 2.3 the uniform ellipticity on compact sets, see (A.4), suffices for the validity of ET). It is also possible to formulate
sufficient conditions without assuming SE). Using (A.3) the operator $A$ can be written as

$$Au(x) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi$$

for any $u$ belonging to $C_c^\infty(\mathbb{R}^d)$ - the set of $C^\infty$ smooth and compactly supported functions.

(i) By [48, p. 3275] there exist constants $C, c > 0$, depending only on $d$, such that

$$E_x \left[ \int_0^\infty 1_{B(0, r)}(X_t) dt \right] \leq C r^d \int_{\{ |\xi| \leq r \}} \frac{d\xi}{\inf_{z \in \mathbb{R}^d} \text{Re} p(z, \xi)}, \quad x \in D.$$

for all $r > 0$. This in particular shows that ET) holds, provided that

$$\int_{\{ |\xi| \leq r \}} \frac{d\xi}{\inf_{z \in \mathbb{R}^d} \text{Re} p(z, \xi)} < +\infty \quad \text{for some } r > 0.$$

(ii) Let $r = \text{diam } D$. Suppose that for every $x \in \text{cl } D$ there exists $k(x) \geq 1$ such that

$$2r |\xi| \text{Im } p(z, \xi)| \leq \text{Re } p(z, \xi) \quad \text{for } |\xi| \leq \frac{1}{k(x) r}, \quad z \in B(x, r). \quad \text{(A.5)}$$

Then, by [13, Theorem 5.5], there exists $c > 0$ such that

$$P_x(\tau_D > t) \leq \frac{c}{t} \left( \sup_{|\xi| \leq 1/(rk(x))} \inf_{z \in B(x, r)} \text{Re } p(z, \xi) \right)^{-1}. \quad \text{(A.6)}$$

So, if

$$\sup_{|\xi| \leq 1/(rk(x))} \inf_{z \in B(x, r)} \text{Re } p(z, \xi) < +\infty, \quad x \in \mathbb{R}^d,$$

then letting $t \to \infty$, we conclude ET), in fact $E_x \tau_D^\rho < +\infty$ for any $\rho \in (0, 1)$. For example consider the operators

$$A_1 u(x) = \frac{1}{2} \sum_{j,s=1}^d q_{j,s}(x) u_{x_j x_s}(x), \quad A_2 u = \Delta^s(x) u(x),$$

$$A_3 u = -\sum_{j=1}^d |\partial_{x_j}^2|^{s_j} u(x), \quad u \in W^{2,p}(D) \cap C_b(\mathbb{R}^d),$$

with $p > d$, $s : \mathbb{R}^d \to (0, 1)$, and $s_1, \ldots, s_d \in (0, 1)$ fixed. Symbol $p_1$ for $A_1$ is given by (A.3) with $b = 0, N = 0$, and symbols $p_2, p_3$, for $A_2$ and $A_3$ respectively, are given by

$$p_2(x, \xi) = |\xi|^{2s(x)}, \quad p_3(\xi) = \sum_{j=1}^d |\xi_j|^{2s_j}.$$

In all the cases listed above the imaginary part of the symbols vanishes, so (A.5) trivially holds. Now, we see that by (A.6), ET) holds for $p_2$ and $p_3$, and if for some $j, \ell \in \{1, \ldots, d\}$, $q_{j,\ell}$ is strictly positive on compacts, then ET) holds for $p_1$ as well.

(iii) Let $r = \text{diam } D$. Then, by [49, Proposition 3.7] ET) holds, provided that for every $x \in D$ we have

$$\inf_{z \in B(x, r)} \int_{|y| \geq 3r} N(z, dy) > 0.$$
A.3. Existence of the principal eigenvalue and eigenfunction. Frequently, in applications we have extra information about the structure of semigroup \((P_t^D)_{t \geq 0}\) in particular consider the case when

\[ P_t^D f(x) = \int_D p_D(t,x,y) f(y) \, m(dy), \quad f \in B_c^*(D), \quad t > 0 \]

for some \(p_D : (0, \infty) \times D \times D \to (0, \infty)\) and \(m\) a finite Borel measure on \(D\). In this case some assumption about integrability of \(p_D\), sufficing for compactness of \((P_t^D)_{t \geq 0}\), implies \(\text{Eig}\). For example if

\[ \int_D \int_D p_{2t}^D(t,x,y) \, m(dx)m(dy) < \infty, \quad t > 0, \]

then \(P_t^D\) is a Hilbert-Schmidt operator on \(L^2(D;m)\) for each \(t > 0\), hence it is compact. By Jentzsch’s theorem (see [47, Theorem V.6.6, p. 337]) condition \(\text{Eig}\) holds. If we know that for some \(1 < p < q\) the hypercontractivity condition holds, i.e. for any \(t > 0\) there exists \(c_t > 0\) such that

\[ \|P_t^D f\|_{L^q(D;m)} \leq c_t \|f\|_{L^p(D;m)}, \quad f \in L^p(D;m), \]

then \(P_t^D\) is compact on \(L^p(D;m)\) for each \(t > 0\) and, again by Jentzsch’s theorem, \(\text{Eig}\) holds. Moreover \(\lambda_D > 0\) is a unique simple eigenvalue of \((P_t^D)_{t \geq 0}\). The corresponding eigenfunction \(\varphi_D\) can be chosen to be strictly positive. Normalizing it by letting \(\int_D \varphi_D \, dm = 1\) we can uniquely determine its choice. The proof of compactness is analogous to the argument presented in [33, Section 7.1.1].

In some cases however the aforementioned properties of the semigroup are not so easy to come by. Then one can try to apply the Jentzsch theorem having sufficient information about the resolvent operator \(R_0^D\). In this way one can e.g. conclude \(\text{Eig}\) from hypotheses A2) and SE), see [33, Theorem 5.1].

A.4. Viscosity subsolutions. Suppose that the transition probability semigroup \((P_t)_{t \geq 0}\) associated with the martingale problem is strongly Feller, i.e. \(P_t(B_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d), \quad t > 0\). Assume that \(D\) is Dirichlet regular, i.e. \(P_x(\tau_D > 0) = 0, \quad x \in \partial D\). Furthermore, suppose that \(c,g,q_{i,j},b_i \in C_b(\mathbb{R}^d), \quad i,j = 1,\ldots,d\), the Levy kernel \(N\) satisfies condition C) (see Section A.1) and the family \(\mathcal{O}_R\) of open Dirichlet regular subsets of \(\mathbb{R}^d\) forms a base for the Euclidean metric.

Let \(u \in C_b(\mathbb{R}^d)\) be a viscosity subsolution to (2.10). Using a suitable comparison principle for viscosity subsolutions one could prove that, under the assumptions made in the foregoing, \(u\) is also a weak subsolution to (2.10). This can be seen as follows. With no loss of generality we may assume that

\[ u(x) \geq 0, \quad x \in D^c := \mathbb{R}^d \setminus D. \quad (A.7) \]

Indeed, otherwise we would consider \(\tilde{u}(x) := u(x) + h(x)\), where \(h(x) := \bar{u}_{2\varepsilon}\mathbb{E}_x[e_c(\tau_D)]\). The latter is a weak solution to (1.2) satisfying \(h(x) = \bar{u}_{2\varepsilon}\), \(x \in D^c\), due the fact that \(D \in \mathcal{O}_R\). By virtue of e.g. [42, Theorem 2.2] it is also a viscosity solution. Therefore \(\tilde{u}(x)\) is a viscosity subsolution to (2.10) that satisfies (A.7). The fact that \(u\) is also a weak subsolution would follow, if we can prove that \(\tilde{u}\) has this property.

Let \(V \subset D\) and \(V \in \mathcal{O}_R\). By the assumptions made in the foregoing

\[ w_V(x) := \mathbb{E}_x[e_c(\tau_V)u(X_{\tau_V})] + \mathbb{E}_x \left[ \int_0^{\tau_V} e_c(r)g(X_r) \, dr \right], \quad x \in \mathbb{R}^d \quad (A.8) \]

is continuous on \(\mathbb{R}^d\). Thanks to the aforementioned result of [42] it is a viscosity solution to (2.10) with \(D\) replaced by \(V\). Since \(V\) is regular, it satisfies the exterior condition \(w(y) = u(y), \quad y \in V^c\). Under appropriate assumptions, see e.g. [2, Theorem
1.2], we can use a comparison principle for viscosity solutions to the exterior Dirichlet problem for (2.10) with \( D \) replaced by \( V \). It allows us to conclude that
\[
u(x) \leq w_V(x), \quad x \in \mathbb{R}^d.
\] (A.9)
The above inequality holds for arbitrary \( V \subset D \), with \( V \in \mathcal{O}_R \). To conclude that \( u \) is a weak subsolution, cf (2.13), we need to replace the family of stopping times \((\tau_V)_{V \subset \mathcal{O}_R}\) by the family \((\tau_D \wedge t)_{t \geq 0}\). Let
\[ v := u - R^{c: D} g. \]
From (A.8) and (A.9) we can write
\[
u(x) - R^{c: V} g(x) \leq w_V(x) - R^{c: V} g(x) = \mathbb{E}_x^c u(X_{\tau_V}), \quad x \in \mathbb{R}^d.
\] (A.10)
Since
\[ R^{c: D} g(x) = R^{c: V} g(x) + \mathbb{E}_x^c R^{c: D} g(X_{\tau_V}), \]
we conclude from (A.10) that
\[
\mathbb{E}_x^c v(X_{\tau_V}) \geq v(x), \quad x \in V.
\] (A.11)
Let \( \tilde{v} := \mathbb{E}^c_{\infty} - v \). Obviously \( \tilde{v} \geq 0 \) in \( D \). From (A.11) we have
\[
\tilde{v}(x) = \mathbb{E}^c_{\infty} - v(x) \geq \mathbb{E}_x^c \tilde{v}(X_{\tau_V}) \geq \mathbb{E}^c_{x: D} \tilde{v}(X_{\tau_V}), \quad x \in V.
\]
Since the sets \( V \) belonging to \( \mathcal{O}_R \) and contained in \( D \) form a base for the Euclidean metric in \( D \), it follows from [8, Theorem II.5.1] that \( \tilde{v} \) is an excessive function with respect to \( (P^{c: D}_t)_{t \geq 0} \). Thus, \( \left( \tilde{v}(X_{t \wedge \tau_D}) \right)_{t \geq 0} \) is a supermartingale with respect to \( P^{c: D}_x \).
The above implies in turn that \( (v(X_{t \wedge \tau_D}))_{t \geq 0} \) is a submartingale with respect to this measure. Since \( v(X_{\tau_D}) = u(X_{\tau_D}) \) is non-negative, by the optional sampling theorem, we have
\[
v(x) \leq \mathbb{E}^c_x \left[ v(X_t), t < \tau_D \right] \leq \mathbb{E}^c_x \left[ v(X_t), t < \tau_D \right] + \mathbb{E}^c_x \left[ v(X_{\tau_D}), t \geq \tau_D \right]
\]
\[= \mathbb{E}^c_x \tilde{v}(X_{t \wedge \tau_D}), \quad x \in D
\]
and (2.13) follows.

Acknowledgements. T. Klimsiak is supported by Polish National Science Centre: Grant No. 2017/25/B/ST1/00878. Both T. Klimsiak and T. Komorowski acknowledge the support of the Polish National Science Centre: Grant No. 2020/37/B/ST1/00426.

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