On possible issues of Backus average

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Abstract

In this paper, we continue the study of Bos et al. (2017b) regarding statistical and numerical considerations of the Backus (1962) product approximation. While the approximation is typically quite good for seismological scenarios, Bos et al. (2017b) demonstrate a physical scenario that could, in spite of the stability conditions for isotropic media, lead to an issue within the Backus average. Using the Preliminary Reference Earth Model of Dziewoński and Anderson (1981), we investigate whether this issue is likely to occur in the context of seismology.

1 Introduction

The Backus average is a method that produces a homogenous medium that is long-wave equivalent to an inhomogeneous stack of thin layers. Notwithstanding the ubiquitous acceptance of the Backus average, it has been the topic of recent study for Adamus et al. (2018) and Bos et al. (2017a,b, 2018) as well as Dalton and Slawinski (2016) and Dalton et al. (2018). While the mathematical underpinnings of the Backus approach are analyzed by Bos et al. (2017a), there may exist a possible issue with the sole mathematical approximation used by Backus (Bos et al., 2017b).

In spite of stability conditions, Bos et al. (2017b) demonstrate that it is mathematically and physically possible for the relative error of the Backus product approximation to equal 100%. Herein, we show that it is unlikely for such an error to occur within the context of seismology at a regional scale.

2 Product approximation

Let us consider the product approximation of Backus (1962), which states that

\[ \text{the only approximation that he makes in the present paper is the following: if } f(x_3) \text{ is nearly constant when } x_3 \text{ changes by no more than } \epsilon, \text{ while } g(x_3) \text{ may vary by a large fraction of this distance, then, approximately,} \]

\[ \overline{fg} \approx \overline{f} \overline{g}, \]  

(1)

Using the formulation of Bos et al. (2017b), which states that

the difference between the average of the product and the product of the averages is

\[ E(f, g) := \overline{fg} - \overline{f} \overline{g}, \]  

(2)

where, for any vector \( \mathbf{x} \in \mathbb{R}^n \), [they] set

\[ \mathbf{x} := \sum_{k=1}^{n} w_k x_k. \]

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The relative error is

\[ R(f, g) = \frac{E(f, g)}{f g} \times 100\% . \]  

(3)

It follows that if \( g = 0 \) then \( R(f, g) = 100\% \). To examine the consequences of \( g = 0 \), in the context of layers composed of isotropic Hookean solids, expressions for \( f \) and \( g \) may be obtained from the isotropic stress-strain relations (Bos et al., 2017b, Section 3.6). We find that \( f \) corresponds to lateral-strain-tensor components that are assumed to be nearly constant, whereas

\[ g = \frac{c_{1111} - 2c_{2323}}{c_{1111}} \]  

(4)

corresponds to elasticity parameters that rapidly vary from layer to layer.

Let us examine the stability conditions for isotropic media to determine the range of physically possible values of \( g \). Herein, the stability conditions may derived from the stress-strain relations for isotropy, which are

\[ \sigma_{ij} = \lambda \delta_{ij} \sum_{k=1}^{3} \varepsilon_{kk} + 2 \mu \varepsilon_{ij} , \quad i,j \in \{1, 2, 3\} , \]  

(5)

where \( \lambda \) and \( \mu \) are the Lamé parameters and are defined as \( \lambda := c_{1111} - 2c_{2323} \) and \( \mu := c_{2323} \). As it is a requirement for all symmetric positive-definite matrices, all of its eigenvalues must be positive (see e.g. Slawinski, 2015, Theorem 4.3.2). Thus, for isotropy, the stability conditions are

\[ c_{1111} > 0 , \quad c_{2323} > 0 , \quad c_{1111} > \frac{4}{3} c_{2323} . \]  

(6)

By applying expressions (6) to expression (4), we deduce that \( g \) is positive when \( c_{1111} > 2c_{2323} \) and that \( g \) is negative when \( \frac{4}{3} c_{2323} < c_{1111} < 2c_{2323} \); the range of \( g \) is illustrated in Figure 1.

![Figure 1: The value of \( g \) approaches a maximum of 1 when \( c_{2323} \) is at a minimum. Conversely, the value of \( g \) approaches a minimum of \(-\frac{1}{2}\) when \( c_{2323} \) is at a maximum; thus, \( g \in (-\frac{1}{2}, 1) \).](image)

Figure 1: The value of \( g \) approaches a maximum of 1 when \( c_{2323} \) is at a minimum. Conversely, the value of \( g \) approaches a minimum of \(-\frac{1}{2}\) when \( c_{2323} \) is at a maximum; thus, \( g \in (-\frac{1}{2}, 1) \).

Since \( g \) can be either negative or positive, and the elasticity parameters—by the stability conditions—are continuous and positive, we conclude that it is possible for \( g \) to equal zero.

Considering Slawinski (2015, Exercise 5.13), we might obtain Poisson’s ratio in terms of the Lamé parameters,

\[ \nu = \frac{\lambda}{2(\lambda + \mu)} , \]  

(7)

which is the desired expression. Alternatively, we might obtain expression (7) by using the relations among Poisson’s ratio, Young’s modulus and the Lamé parameters (see e.g. Slawinski, 2015, Remark 5.14.7).

For a two-dimensional case, expression (7) becomes

\[ \nu = \frac{\lambda}{\lambda + 2\mu} , \]  

(8)

which is equivalent to expression (4). Notably, the properties of a transversely isotropic medium are captured by two-dimensional model that contains the rotation-symmetry axis. Expression (8) might be useful considering the fact that the Backus average produces a homogeneous transversely isotropic medium that is long-wave equivalent to a stack of thin isotropic layers.

The range of possible values of \( \nu \) in expression (7) are determined by the stability conditions, which are determined from the eigenvalues of the positive-definite elasticity tensor used therein. Thus, the stability conditions for isotropy, in terms of \( \lambda \) and \( \mu \), are

\[ \lambda > \frac{2}{3} \mu \quad \text{and} \quad \mu > 0 . \]  

(9)
Figure 2: The value of $\nu$ approaches a maximum of $\frac{1}{2}$ when $\mu$ is at a minimum. Conversely, the value of $\nu$ approaches a minimum of $-1$ when $\mu$ is at a maximum; thus, $\nu \in (-1, \frac{1}{2})$.

Considering the ranges of values of expressions (4) and (7), we reckon that if (a) $g > 0$ then $\nu > 0$, (b) $g = 0$ then $\nu = 0$, and (c) $g < 0$ then $\nu < 0$. For naturally occurring solids, $\nu > 0$; the ratio being positive means that the diminishing of a cylinder’s length is being accompanied by the extension of its radius (e.g. Slawinski, 2015, p. 203). Hence, the range illustrated in Figure 2 reduces to $\nu \in (0, \frac{1}{2})$.

3 Preliminary Reference Earth Model

To gain insight into whether or not we might encounter $g = 0$, let us consider a seismological example. The Preliminary Reference Earth Model (PREM) of Dziewoński and Anderson (1981) is a one-dimensional model that presents the properties of the Earth as a function of depth. The PREM is a mathematical analogy that serves as a background model for the planet as a whole; it assumes spherical symmetry in order to subdivide the interior of the Earth into nine principal regions. This model establishes Earth-specific properties that include density, $\rho$, and $P$- and $S$-wave speeds, which are

$$v_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad \text{and} \quad v_S = \sqrt{\frac{\mu}{\rho}}$$

(10)

These nine principal regions are distinguished from one another by a rapid change in speeds of $P$ and $S$ waves along interfaces, which indicates a diverse range of elastic properties within the medium. The speeds of the irrotational and equivoluminal waves are functions of the different elasticity parameters and, hence, propagate at different speeds within the model.

Let us consider data from Bormann (2012, Table 1), which lists 84 samples of—among other parameters—$v_P$, $v_S$, and $\rho$ as functions of depth ranging from 0 to 6371 km for an isotropic PREM. In view of the relationship between Lamé and elasticity parameters, we may compute

$$c_{1111} = \rho v_P^2 \quad \text{and} \quad c_{2323} = \rho v_S^2$$

(11)

for each of the 84 samples. Using expressions (11), we plot the values of expression (4) as a function of depth in Figure 3. Therein, the resultant points of discontinuity arise from the rapid change in speed of $P$ and $S$ waves across the interfaces of the principal regions.

Figure 3: $g$ as a function of depth (km)

For samples between 2891 km and 5150 km, $v_S = 0$. Recalling that $S$ waves do not propagate in liquids, as their resistance to change of shape vanishes (see e.g. Slawinski, 2015, p. 217), we interpret this range
of samples to correspond to the outer core. Since the $S$-wave speed equals zero, $c_{2323} = 0$; consequently, expression (4) equals 1.

From Figure 3, we observe that $g > 0$ throughout and, thus, deduce that $g$ cannot equal zero. Therefore, following the conclusions of Section 2, our results support that $g > 0$ for naturally occurring solids within an isotropic PREM. Hence, we may conclude that it is improbable for the relative error of the Backus average approximation to equal 100% for such a model.

4 Conclusions and future work

In this paper, we continue the work of Bos et al. (2017b) to investigate the sole mathematical approximation made by Backus (1962). Although it is mathematically possible to achieve a relative error of 100% for the Backus product approximation, each of the evaluated samples of Bormann (2012) result in $g > 0$. Thus, in the context of seismology at a regional scale, potential issues are unlikely to occur.

With that being said, the Backus average is often said to be error-laden (Sams and Williamson, 1994), which might lead to issues when using a less-idealized model. Thus, we cannot exclude the possibility of such issues occurring in shallow-data acquisition, i.e. within the upper lithosphere. We suspect that such issues are unlikely occur but future work using measurements from the wellbore is required to verify this supposition. Additionally, we presume that the Backus average might encounter issues when used on synthetic materials with, say, $\nu \approx 0$ or $\nu < 0$.

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