Optimality and duality for nonsmooth semi-infinite E-convex multi-objective programming with support functions

Tarek Emam*1,2

1 Department of Mathematics, Faculty of Science, Jouf University, P.O. Box 2014, Sakaka, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Suez University, P.O. Box 43533, Suez, Egypt

Abstract. In this paper, we study a nonsmooth semi-infinite multi-objective E-convex programming problem involving support functions. We derive sufficient optimality conditions for the primal problem. We formulate Mond-Weir type dual for the primal problem and establish weak and strong duality theorems under various generalized E-convexity assumptions.

Keywords: Nonsmooth semi-infinite multi-objective optimization / generalized e-convexity / duality

1 Introduction

Semi-infinite multi-objective programming consider several conflicting objective functions have to be optimized over a feasible set described by infinite number of inequality constraints. Semi-infinite programming problems have occupied the attention of a number of mathematicians due to their applications in many areas such as in engineering, robotics, and transportation problems, see [1]. Optimality conditions and duality results for semi-infinite programming problems have been studied see, [2–10]. Optimality and duality results for semi-infinite multi-objective programming problems that involved differentiable functions were obtained by Caristi et al. [11]. Several kinds of constraints qualifications were defined by Kanzi and Nobakhtian [12] and they obtained necessary and sufficient optimality conditions for nonsmooth semi-infinite multi-objective programming problems. Mishra et al. [13] proved necessary and sufficient optimality conditions for nondifferential semi-infinite programming problems involving square root of quadratic functions, for more details see [14]. Mond and Schechter [15] have constructed symmetric duality of both Wolfe and Mond-Weir types for nonlinear programming problems where the objective contains the support function. Optimality and duality for a nondifferentiable nonlinear programming problem involving support function have been obtained by Husain et al. [16], see for more details [17–20]. In other hand, convexity and their generalizations play an important role in optimization theory. Youness point of view of convexity is based on the effecting of an operator E on the domain on which functions are defined [21,22]. This kind of convexity is called E-convexity and can be viewed in many fields such as in differential geometry when a manifold is deformed by an operator E. In the field of physical chemistry an E-convexity can be occurred when the binding force f between elements construct a crystal effect by a solution E. In mathematical programming, the notion of E-convexity of functions plays an important role in solving the problem of type composite model problem [23] such as the problem

\[
\min ||F||_{s.t.x\in M}, \text{where } f = ||.||_E(x) \\
= F(x) \text{ and } (f \circ E)x = ||F(x)||.
\]

This paper is organized as follows: In Section 2, we mention some definitions and preliminaries. In Section 3, the sufficient optimality conditions for multi-objective semi-infinite E-convex programming problems involving support functions are established. In Section 4, we formulate Mond-Weir type dual for multi-objective semi-infinite E-convex programming problems involving support functions and establish weak, strong and strict-converse duality theorems under generalized E-convexity assumptions.

2 Definitions and preliminaries

In this section, we present some definitions and results, which will be needed in this article. Let \( R^n \) be the n-dimensional Euclidean space and \( R^+_n \) be the nonnegative orthant of \( R^n \). Let \( \langle , \rangle \) denotes the Euclidean inner product and \( ||.|| \) be Euclidean norm in \( R^n \). Given a nonempty

*e-mail: drtemam@yahoo.com
set $D \subseteq \mathbb{R}^n$, we denote the closure of $D$ by $\overline{D}$ and convex cone (containing origin) by $cone(D).$ The native derivative and the strictly negative polar cone are defined respectively by

$$D^\leq := \{d \in \mathbb{R}^n | (x, d) \leq 0, \forall x \in D\},$$

$$D^\prec := \{d \in \mathbb{R}^n | (x, d) < 0, \forall x \in D\}.$$ 

**Definition 1** [24] Let $D \subseteq \mathbb{R}^n.$ The contingent cone $T(D, x)$ at $x \in D$ is defined by

$$T(D, x) := \{d \in \mathbb{R}^n | \exists t_k \to 0, \forall k \in \mathbb{N}, \exists d_k \in D \in D, \forall d \in D\}.$$ 

**Definition 2** [24] A function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be Lipschitz near $x \in \mathbb{R}^n$, if there exist a positive constant $K$ and a neighborhood $N$ of $x$ such that for any $y, z \in N$, we have

$$|f(y) - f(z)| \leq K||y - z||.$$ 

The function $f$ is said to be locally Lipschitz on $\mathbb{R}^n$ if it is Lipschitz near $x$ for every $x \in \mathbb{R}^n$.

**Definition 3** [24] The Clarke generalized directional derivative of a locally Lipschitz function $f$ at $x \in \mathbb{R}^n$ is defined by $f^*(x, d)$, denoted by $f^*(x, d)$, is defined as

$$f^*(x, d) = \lim_{t \to 0^+} \frac{f(y + td) - f(y)}{t},$$

where $y \in \mathbb{R}^n$.

**Definition 4** [24] The Clarke generalized subdifferential of $f$ at $x \in \mathbb{R}^n$ is denoted by $\partial f(x)$, defined as

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f^*(x, d) \geq 0, \forall d \in \mathbb{R}^n\}.$$ 

**Definition 5** [21] A set $M \subseteq \mathbb{R}^n$ is said to be $E$-convex set with respect to an operator $E: \mathbb{R}^n \to \mathbb{R}^n$ if $\lambda E(x) + (1 - \lambda)E(y) \in M$ for each $x, y \in M$ and $0 \leq \lambda \leq 1$.

Every $E$-convex set with respect to an operator $E: \mathbb{R}^n \to \mathbb{R}^n$ is a convex set when $E = I$. If $M_1$ and $M_2$ are $E$-convex sets, then $M_1 \cap M_2$ is $E$-convex set but $M_1 \cup M_2$ is not necessarily $E$-convex set. If $E: \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, and $M_1, M_2 \subseteq \mathbb{R}^n$ are $E$-convex sets, then $M_1 + M_2$ is $E$-convex set.

**Example 1** Let $E: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $E(x, y) = (y, 0)$. The set $M = \{(x, y) \in \mathbb{R}^2 | (x, y) = (0, 0) + \lambda_2 (1, 2) + \lambda_3 (-1, -2)\}$ is an $E$-convex set with respect to the operator $E$.

**Definition 6** A locally Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be $E$-convex with respect to an operator $E: \mathbb{R}^n \to \mathbb{R}^n$ at $x^* \in \mathbb{R}^n$ if

$$(f \circ E)x - (f \circ E)x^* \geq \langle \xi, Ex - Ex^* \rangle$$

for each $x \in \mathbb{R}^n$ and every $\xi \in \partial f(Ex^*)$.

The function $f$ is said to be $E$-convex near $x^* \in \mathbb{R}^n$ if it is $E$-convex at each point of neighborhood of $x^* \in \mathbb{R}^n$.

**Definition 7** A locally Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be strictly $E$-convex with respect to an operator $E: \mathbb{R}^n \to \mathbb{R}^n$ at $x^* \in \mathbb{R}^n$ if

$$(f \circ E)x - (f \circ E)x^* > \langle \xi, Ex - Ex^* \rangle$$

for each $x \in \mathbb{R}^n$, $x \neq x^*$ and every $\xi \in \partial f(Ex^*)$.

The function $f$ is said to be strictly $E$-convex near $x^* \in \mathbb{R}^n$ if it is strictly $E$-convex at each point of neighborhood of $x^* \in \mathbb{R}^n$.

**Proposition 1** [21] If $g_i: \mathbb{R}^n \to \mathbb{R}, i = 1, 2, ..., m$ is $E$-convex with respect to $E: \mathbb{R}^n \to \mathbb{R}^n$ then the set $M = \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, 2, ..., m\}$ is $E$-convex set.

**Definition 8** A locally Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be pseudo $E$-convex with respect to $E: \mathbb{R}^n \to \mathbb{R}^n$ at $x^* \in \mathbb{R}^n$ if

$$(f \circ E)x \geq (f \circ E)x^*,$$

for each $x \in \mathbb{R}^n$ and every $\xi \in \partial f(Ex^*)$.

**Definition 9** A locally Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be strictly pseudo $E$-convex with respect to $E: \mathbb{R}^n \to \mathbb{R}^n$ at $x^* \in \mathbb{R}^n$ if

$$(f \circ E)x > (f \circ E)x^*,$$

for each $x \in \mathbb{R}^n$, $x \neq x^*$ and every $\xi \in \partial f(Ex^*)$.

**Definition 10** A locally Lipschitz function $f: \mathbb{R}^n \to \mathbb{R}$ is said to be quasi $E$-convex with respect to $E: \mathbb{R}^n \to \mathbb{R}^n$ at $x^* \in \mathbb{R}^n$ if

$$(f \circ E)x \leq (f \circ E)x^* \Rightarrow \langle \xi, Ex - Ex^* \rangle \leq 0,$$

for each $x \in \mathbb{R}^n$ and every $\xi \in \partial f(Ex^*)$.

The function $f$ is said to be quasi $E$-convex near $x^* \in \mathbb{R}^n$ if it is quasi $E$-convex at each point of neighborhood of $x^* \in \mathbb{R}^n$.

**Proposition 2** [21] If $g_i: \mathbb{R}^n \to \mathbb{R}, i = 1, 2, ..., m$ is quasi $E$-convex with respect to $E: \mathbb{R}^n \to \mathbb{R}^n$ then the set $M = \{x \in \mathbb{R}^n | g_i(x) \leq 0, i = 1, 2, ..., m\}$ is $E$-convex set.

**Remark 1**

- Every $E$-convex function is also quasi $E$-convex with respect to same $E: \mathbb{R}^n \to \mathbb{R}^n$, but not conversely.
- Every $E$-convex function is also pseudo $E$-convex with respect to same $E: \mathbb{R}^n \to \mathbb{R}^n$, but not conversely.
- Every strictly $E$-convex function is also strictly pseudo $E$-convex with respect to same $E: \mathbb{R}^n \to \mathbb{R}^n$, but not conversely.

Let $C$ be a nonempty compact $E$-convex set in $\mathbb{R}^n$. The support function $S(\cdot, C): \mathbb{R}^n \to [0, +\infty]$ is given by

$$S(x, C) = \max \{\langle x, E_z \rangle : z \in C\}.$$ 

**Example 2** Let $E: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $E(y_1, y_2) = (0, y_2)$. If $C = \{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) = \lambda_1 (0, 0) + \lambda_2 (1, 2) + \lambda_3 (0, 3)\}$, then the support function $S(x, C)$ is...
functions \( f_j(x) = \max \{x, E(y_1, y_2) \} \) \( (y_1, y_2) \in C \).

The support function, being convex and everywhere finite, has a Clark subdifferential [24], in the sense of convex analysis. Its subdifferential is given by

\[
\partial S(x|C) = \{ z \in C \mid \langle x, Ez \rangle = S(x|C) \}.
\]

In this paper, we consider the following nonsmooth semi-infinite multi-objective E-convex programming problem:

\[
(P) \quad \min_{j=1, \ldots, p} f_j(x) + S(x|C_j),
\]

subject to

\[
g_i(x) \leq 0, \quad i \in I,
\]

\[
x \in \mathbb{R}^n.
\]

where \( I \) is an index set which is possibly infinite, \( f_j(x), j = 1, \ldots, p \) and \( g_i(x), i \in I \) are locally Lipschitz E-convex functions from \( \mathbb{R}^n \) to \( \mathbb{R} \cup \{ +\infty \} \). Let \( M \) denote the E-convex feasible set of (P).

\[
M := \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I \}.
\]

Let \( x^* \in M \). We denote \( I(x^*) = \{ i \in I \mid (g_i \circ E)x^* - 0\} \), the index set of active constraints and let

\[
F(Ex^*) := \bigcup_{j=1}^p \partial (f_j(Ex^*) + S(Ex^*|C_j))
\]

\[
G(Ex^*) := \bigcup_{i \in I(x^*)} \partial g_i(Ex^*).
\]

The following constraint qualifications are generalization of constraint qualifications from [12] for multi-objective E-convex programming problem with support functions (P).

**Definition 11** We say that:
- The Abedie constraint qualification (ACQ) holds at \( \hat{x} \in M \) if \( G^\Delta(\hat{x}) \subseteq T(M, \hat{x}) \).
- The Basic constraint qualification (BCQ) holds at \( \hat{x} \in M \) if \( T^\Delta(M, \hat{x}) \subseteq \text{cone}(G(x)) \).
- The Regular constraint qualification (RCQ) holds at \( x \in M \) if \( F^\Delta(\hat{x}) \cap G^\Delta(\hat{x}) \subseteq T(M, \hat{x}) \).

**Definition 12** A feasible point \( x^* \in M \) is said to be weakly efficient solution for (P) if there is no \( x \in M \) such that

\[
f_j(x) + S(x|C_j) < f_j(x^*) + S(x^*|C_j), \quad \text{for all } j = 1, \ldots, p.
\]

**3 Optimality conditions**

In this section, we prove the sufficient optimality conditions for considered nonsmooth semi-infinite multi-objective E-convex programming problem (P) as follows:

**Theorem 1** (Necessary optimality conditions)

Let \( E : \mathbb{R}^n \to \mathbb{R}^n \) and \( x^* \) be a feasible solution of (P). Assume that \( Ex^* \) be a weakly efficient solution of (P) and a suitable constraints qualification from Definition (11) holds at \( E(x^*) \). If \( \text{cone}(G(Ex^*)) \) is closed, then there exist \( \tau_j \geq 0, \quad z_j \in C_j \) \( (j = 1, 2, \ldots, p) \) and \( \lambda_i \geq 0 \) \( (i \in I(x^*)) \) with \( \lambda_i \neq 0 \) for finitely many indices \( i \), such that

\[
0 \in \sum_{j=1}^p \tau_j [\partial f_j(Ex^*)] + z_j + \sum_{i \in I(x^*)} \lambda_i \partial g_i(Ex^*),
\]

\[
\sum_{j=1}^p \tau_j = 1,
\]

\[
\langle z_j, Ex^* \rangle = S(x^*|C_j), j = 1, 2, \ldots, p.
\]

**Proof:** See Theorem 3.4 (ii) of Kanzi and Nobakhtian [12].

**Theorem 2** (Sufficient optimality conditions)

Let \( E : \mathbb{R}^n \to \mathbb{R}^n \) and \( x^* \) be a feasible solution of (P). Assume that there exist \( \tau_j \geq 0, \quad z_j \in C_j \) \( (j = 1, 2, \ldots, p) \) and \( \lambda_i \geq 0 \) \( (i \in I(x^*)) \) with \( \lambda_i \neq 0 \) for finitely many indices \( i \), such that necessary optimality conditions (1)–(3) hold at \( x^* \). If \( \tau_j (f_j(\cdot) + \langle z_j, \cdot \rangle) \) \( (j = 1, 2, \ldots, p) \) are pseudo E-convex at \( x^* \) and \( \lambda_i g_i(\cdot), i \in I(x^*) \) are quasi E-convex at \( x^* \) with respect to the same \( E \) and \( f_i(Ex) \leq f_i(x), j = 1, 2, \ldots, p, \forall x \in M \). Then, \( Ex^* \) is a weakly efficient solution for (P).

**Proof:** Suppose, contrary to the result, that \( Ex^* \in M \), is not a weakly efficient solution for (P). Then, there exists a feasible point \( x \in M \) for (P) such that

\[
f_j(x) + S(x|C_j) < f_j(Ex^*) + S(Ex^*|C_j), \quad \text{for all } j = 1, \ldots, p,
\]

but \( f_j(Ex) \leq f_j(x) \) and \( \tau_j \geq 0 \), for \( j = 1, 2, \ldots, p \), so we have

\[
\sum_{j=1}^p \tau_j [f_j(Ex) + S(x|C_j)] < \sum_{j=1}^p \tau_j [f_j(Ex^*) + S(Ex^*|C_j)].
\]

Since \( \langle z_j, Ex \rangle \leq S(x|C_j), j = 1, 2, \ldots, p \) and the assumption \( \langle z_j, Ex^* \rangle = S(x^*|C_j), j = 1, 2, \ldots, p \), we have

\[
\sum_{j=1}^p \tau_j [f_j(Ex) + \langle z_j, Ex \rangle] < \sum_{j=1}^p \tau_j [f_j(Ex^*) + \langle z_j, Ex^* \rangle].
\]

Now, from equation (1), there exist \( \xi_j \in \partial cf_j(Ex^*) \) and \( \xi_i \in \partial cg_i(Ex^*) \) such that

\[
\sum_{j=1}^p \tau_j (\xi_j + z_j) + \sum_{i \in I(x^*)} \lambda_i \xi_i = 0.
\]
Since \( Ex \) is a feasible point for \((P)\) where \( M \) is E-convex set and \( \lambda, g_i(Ex^*) = 0, \ i \in I(x^*) \), we have
\[
\sum_{i \in I(x^*)} \lambda_i g_i(Ex^*) \leq \sum_{i \in I(x^*)} \lambda_i g_i(Ex^*) , \tag{7}
\]
and from quasi E-convexity of \( g_i, \ i \in I(x^*) \), we get
\[
\left\langle Ex - Ex^*, \sum_{i \in I(x^*)} \lambda_i \xi_i \right\rangle \leq 0 ,
\]
by using (6), we have
\[
\left\langle Ex - Ex^*, \sum_{j = 1}^p \tau_j (\xi_j + z_j) \right\rangle \geq 0 .
\]
Thus, from pseudo E-convexity of \( \tau_j (f_j(\cdot) + \langle z_j, \cdot \rangle) \), for \( j = 1, 2, ..., p \), we get
\[
\sum_{j = 1}^p \tau_j [f_j(Ex) + \langle z_j, Ex \rangle] \geq \sum_{j = 1}^p \tau_j [f_j(Ex^*) + \langle z_j, Ex^* \rangle] ,
\]
which contradicts (5). Thus \( Ex^* \) is a weakly efficient solution for \((P)\).

The following corollary is a direct consequence of Remark 1 and Theorem 2.

**Corollary 1** Let \( E : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( x^* \) be a feasible solution of \((P)\). Assume that there exist \( \tau_j \geq 0, z_j \in C_j \) for \( j = 1, 2, ..., p \) and \( \lambda_i \geq 0 \) (for \( i \in I(x^*) \)) with \( \lambda_i \neq 0 \) for finitely many indices \( i \), such that necessary optimality conditions (1)–(3) hold at \( x^* \). If \( \tau_j (f_j(\cdot) + \langle z_j, \cdot \rangle) \) for \( j = 1, 2, ..., p \) and \( \lambda_i g_i(\cdot), \ i \in I(x^*) \) are E-convex at \( x^* \) with respect to the same \( E \) and \( f_j(Ex) \leq f_j(x), \ j = 1, 2, ..., p, \ \forall x \in M \). Then, \( Ex^* \) is a weakly efficient solution for \((P)\).

**Example 3** Let \( E : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined as \( E(x_1, x_2) = (\frac{1}{2} x_1, x_2) \) and let \( M \) be given by
\[
\begin{align*}
M &= \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 - 6 \leq 0, x_1 - 4x_2 \leq 0, x_2 \geq 0, x_1 \geq 0 \} .
\end{align*}
\]
Consider the bicriteria E-convex programming problem
\[
(P) \ \min (f_1(x) + S(x|C_1), f_2(x) + S(x|C_2))
\]
Subject to \( x \in M \),
where \( f_1(x_1, x_2) = x_1^3 \) and \( f_2(x_1, x_2) = (x_2 - x_1)^3 \) and \( S(x|C_1) = S(x|C_2) = \frac{1}{2} |x_2| \) where \( x = (x_1, x_2) \) for \( C_1 = C_2 = \{ (0, x_2) : -12 \leq x_2 \leq 0 \} \). It is clear that \( M \) is E-convex with respect to \( E \) and \( E(M) = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \leq 3 \leq 0, x_1 - 4x_2 \leq 0, x_2 \geq 0, x_1 \geq 0 \} \).

By choosing \( (g \circ E)x^* = x_1^* - 2x_2^* \) as the active constraint of \((P)\) then \( I(x^*) = 1 \). It is clear that all defined functions are locally Lipschitz functions at \( Ex^* \) and \( \partial f_1(Ex^*) = (12x_1^2, 0), \ \partial f_2(Ex^*) = (-3x_2^2, 3x_1^2), \ \partial g_1(Ex^*) = (1, -2) \) where \( \alpha \in [0, 1] \). Since \( \tau_j (f_j(x) + \langle z_j, x \rangle) \) for \( j = 1, 2 \) are pseudo E-convex and \( \lambda_i g_i(x) \) are quasi E-convex at \( x^* \) with respect to same \( E \) and conditions (1)–(3) of theorem (1) holds at \( x^* \in M \) as there exist \( \tau_1 = \tau_2 = \lambda = \frac{1}{2}, z_1 = 0, z_2 = -\alpha(9\alpha + 2, 3\alpha + 1), \xi_1 = (12\alpha^2, 0), \xi_2 = (-3\alpha^2, 3\alpha^2), \zeta_1 = (2\alpha, \alpha) \), where \( \alpha \in [0, 1] \) such that
\[
\sum_{j = 1}^2 \tau_j (\xi_j + z_j) + \sum_{i \in I} \lambda_i \xi_i = 0 .
\]

Then there is no \( x \in M \) such that
\[
f_j(x) + S(x|C_j) < f_j(Ex^*) + S(Ex^*|C_j), \ j = 1, 2,
\]
and hence \( Ex^* = (x_1^*, x_2^*) \) where \( x_1^* = 2x_2^* \) and \( x_2^* \in [0, 1] \) are weakly efficient solutions for \((P)\).

**4 Duality criteria**

Many authors have formulated Mond-Weir type dual and established duality results in various optimization problems with support functions; see [10,15,17,20,21,25] and the references therein. Following the above mentioned works, we formulate Mond-Weir type dual for nonsmooth semi-infinite E-convex programming problem with support function \((P)\) and establish duality theorems.

\[
\max (f_1(Ex) + \langle z_1, Ey \rangle, ..., f_p(Ex) + \langle z_p, Ey \rangle) \quad \text{(D)},
\]
subject to
\[
0 \in \sum_{j = 1}^p \tau_j [\partial f_j(Ex) + \langle z_j, Ex \rangle] + \sum_{i \in I} \lambda_i \partial g_i(Ex), \tag{8}
\]
\[
\sum_{i \in I} \lambda_i g_i(Ex) \geq 0 . \tag{9}
\]

We now discuss the weak, strong and strict converse duality for \((P)\) and \((D)\).

**Theorem 3 (Weak Duality)** Let \( x \) be feasible for \((P)\) and \( (y, \tau, \lambda, z_1, ..., z_p) \) be feasible for \((D)\). If \( \tau_j (f_j(\cdot) + \langle z_j, \cdot \rangle) \) for \( j = 1, 2, ..., p \) are pseudo E-convex at \( y \) and \( \lambda_i g_i(\cdot), i \in I \) are E-convex at \( y \) with respect to the same \( E \) and \( f_j(Ex) \leq f_j(x), \ j = 1, 2, ..., p, \ \forall x \in M \). Then the following cannot hold:
\[
f_j(x) + S(x|C_j) < f_j(Ex^*) + \langle z_j, Ey \rangle , \quad \text{for all} \quad j = 1, ..., p .
\]

**Proof:** Let \( x \) be feasible for \((P)\) and \( (y, \tau, \lambda, z_1, ..., z_p) \) be feasible for \((D)\), then from (8), there exist \( \xi_j \in \partial c_f_j(Ex) \) and \( \xi_i \in \partial c_g_i(Ex) \) such that
\[
\sum_{j = 1}^p \tau_j (\xi_j + z_j) + \sum_{i \in I} \lambda_i \xi_i = 0 . \tag{10}
\]

We proceed to the result of the theorem by contradiction. Assume that
\[
f_j(x) + S(x|C_j) < f_j(Ex^*) + \langle z_j, Ey \rangle , \quad \text{for all} \quad j = 1, ..., p .
\]
But \( f_j (Ex) \leq f_j (x) \) and \( r_j \geq 0 \), for \( j = 1, 2, ..., p \), so we have
\[
\sum_{j=1}^{p} \tau_j [f_j (Ex) + S(x[C_j])] < \sum_{j=1}^{p} \tau_j [f_j (Ey) + \langle z_j, Ey \rangle],
\]
and by using the inequality \( \langle z, Ex \rangle \leq S(x[C]) \), we get
\[
\sum_{j=1}^{p} \tau_j [f_j (Ex) + \langle z_j, Ex \rangle] < \sum_{j=1}^{p} \tau_j [f_j (Ey) + \langle z_j, Ey \rangle].
\]

(11)

Now, since \( Ex \) is feasible for (P) where \( M \) is E-convex set and \( (y, \tau, \lambda, z_1, ..., z_p) \) is feasible for (D), we have
\[
\sum_{i \in I} \lambda_i g_i (Ex) \leq 0 \leq \sum_{i \in I} \lambda_i g_i (Ey),
\]
and from definition of quasi E-convexity of \( g_i (x), i \in I \) at \( y \), we have
\[
\left\langle Ex - Ey, \sum_{i \in I} \lambda_i \xi_j \right\rangle \geq 0,
\]
for each \( x \in M \) and every \( \xi_j \in \partial c g_i (Ex) \). From (10) in (12), we get
\[
\left\langle Ex - Ey, \sum_{j=1}^{p} \tau_j [f_j (Ex) + \langle z_j, Ex \rangle] \right\rangle \geq 0,
\]
for each \( x \in M \) and some \( \xi_j \in \partial c f_j (Ey) \). Thus, from the definition of pseudo E-convexity of \( \tau_j (f_j (.)) \) for \( j = 1, 2, ..., p \), we have
\[
\sum_{j=1}^{p} \tau_j [f_j (Ex) + \langle z_j, Ex \rangle] \geq \sum_{j=1}^{p} \tau_j [f_j (Ey) + \langle z_j, Ey \rangle],
\]
which contradicts (12). Hence,
\[
f_j (x) + S(x[C_j]) < f_j (Ey) + \langle z_j, Ey \rangle, \forall j = 1, ..., p,
\]
cannot hold.

The following corollary is a direct consequence of Remark 1 and Theorem 3.

**Corollary 2** Let \( x \) be feasible for (P) and \( (y, \tau, \lambda, z_1, ..., z_p) \) be feasible for (D). If \( \tau_j (f_j (.)) \) and \( \lambda g_i (.), i \in I \) are E-convex at \( y \) and \( \lambda g_i (.), i \in I \) are E-convex at \( y \) with respect to the same \( E \) and \( f_j (Ex) \leq f_j (x), j = 1, ..., p, \forall x \in M \). Then the following cannot hold:
\[
f_j (x) + S(x[C_j]) < f_j (Ey) + \langle z_j, Ey \rangle, \forall j = 1, ..., p.
\]

The following example shows that the generalized B-invexity imposed in the above theorem is essential.

**Example 4** [26] We consider the following problem:
\[
(P) \min (f_1 (x) + S(x[C_1], f_2 (x) + S(x[C_2]))
\]
Subject to
\[
g_i (x) \leq 0, i \in I
\]
\[
x \in R^n,
\]
where \( f_1 (x) = -2x, f_2 (x) = x^2 \), \( S(x[C_1] = S(x[C_2] = |x| for \( C_1 = C_2 = [-1, 1] \) and \( g_1 (x) = -|x|, i \in I = N \). It is clear that the feasible set of (P) is \( M := R \) and for \( y = 1 \in M \), \( f_i (y) = 1 \). Let us formulate Mond-Weir dual of (P) as follow:
\[
(D) \max \{(f_1 \circ E)y + z_1, (f_2 \circ E)y + z_2\}
\]
Subject to
\[
g_i (Ey) \leq 0, i \in I
\]
\[
\frac{1}{0} \sum_{j=1}^{2} \tau_j [f_j (Ey) + z_j] + \sum_{i \in I} \lambda_i g_i (Ey) \geq 0,
\]
where \( y \in R, \tau_j \geq 0, \sum_{i=1}^{2} \tau_j = 1, \lambda_i \geq 0, \lambda = (\lambda_i)_{i \in I} \neq 0 \), for finitely many indices \( i \in N \), and \( z_i \in C_1 \) for \( j = 1, 2 \). By choosing \( y^* = 0, \tau_1 = \tau_2 = \frac{1}{2}, \lambda = (1, 0, 0, ...), z_1 = 1, z_2 = 0 \). We have \((y^*, \tau, \lambda, z_1, z_2)\) be feasible for (D).

Note that \( \lambda g_i (.0 \) is not quasi E-convex at \( y \) with respect to \( E (y) = y \) and that \( f_1 (x) + S(x[C_1]) = -1 < f_1 (Ey) + \langle z_1, Ey \rangle = 0 \) holds. This means that pseudo E-convexity and quasi E-convexity assumptions are essential for weak duality.

The following theorem gives strong duality relation between the primal problem (P) and the dual problem (D).

**Theorem 4 (Strong Duality)** Let \( E : R^n \rightarrow R^n \) and \( x^* \) be a feasible solution of (P). Assume that \( Ex^* \) be a weakly efficient solution of (P) and a suitable constraints qualification from Definition (11) holds at \( x^* \) and cone \( (G(x^*)) \) is closed. If the pseudo E-convexity and quasi E-convexity assumptions of the weak duality theorem are satisfied, and \( f_j (Ex) \leq f_j (x), j = 1, ..., p, \forall x \in M \). Then there exists \((\tau, \lambda, z_1, ..., z_p)\) such that \((x^*, \tau, \lambda, z_1, ..., z_p)\) is a weakly efficient solution for (D) and the respective objective values are equal.

**Proof:** Since \( Ex^* \) is a weakly efficient solution for (P) at which the suitable constraints qualification holds and cone \( (G(x^*)) \) is closed, from the Kuhn-Tucker necessary conditions, there exists \((\tau, \lambda, z_1, ..., z_p)\) such that \((x^*, \tau, \lambda, z_1, ..., z_p)\) is feasible for (D).

From weak duality theorem (3), the following cannot hold:
\[
f_j (x) + S(x[C_j]) < f_j (Ex^*) + \langle z_j, Ex^* \rangle, \forall j = 1, ..., p.
\]
Since \( \langle z, Ex \rangle \leq S(x[C]) \), and \( f_j (Ex) \leq f_j (x), j = 1, ..., p \), we have
\[
f_j (Ex) + \langle z_j, Ex \rangle < f_j (Ex^*) + \langle z_j, Ex^* \rangle, \forall j = 1, ..., p.
\]
cannot hold, and hence \((x^*, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((D)\) and the objective values of \((P)\) and \((D)\) are equal at \(x\).

The following corollary is a direct consequence of Remark 1 and Theorem 4.

**Corollary 3** Let \(E: R^n \rightarrow R^n\) and \(x^*\) be a feasible solution of \((P)\). Assume that \(E(x^*)\) be a weakly efficient solution of \((P)\) and a suitable constraints qualification from Definition 11 holds at \(x^*\) and \(\text{cone}(G(x^*))\) is closed. If the \(E\)-convexity assumptions of the weak duality theorem are satisfied, and \(\tilde{f}_j(E(x)) \leq \tilde{f}_j(x)\), \(j = 1, 2, \ldots, p\), \(\forall x \in M\). Then there exists \((\tau, \lambda, z_1, \ldots, z_p)\) such that \((x^*, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((D)\) and the respective objective values are equal.

The following theorem gives strict converse duality relation between the primal problem \((P)\) and the dual problem \((D)\).

**Theorem 5 (Strict converse duality)** Let \(E(x^*)\) be a weakly efficient solution for \((P)\) at which a suitable constraints qualification from Definition 11 holds at \(x^*\) and \(\text{cone}(G(x^*))\) is closed. Let \(\tilde{r}_j(f_j(\cdot) + \langle z_j, \cdot \rangle)\) for \(j = 1, 2, \ldots, p\) be pseudo \(E\)-convex and \(\lambda_i g_i(\cdot), i \in I\) be quasi \(E\)-convex with respect to the same \(E\). If \((\tilde{x}, \tau, \lambda, z_1, \ldots, z_p)\) is a weak efficient solution for \((D)\) and \(\tilde{r}_j(f_j(\cdot) + \langle z_j, \cdot \rangle)\) for \(j = 1, 2, \ldots, p\) are strictly pseudo \(E\)-convex at \(\tilde{x}\), then \(\tilde{x} = x^*\).

**Proof:** We prove the result of theorem by contradiction. Assume that \(\tilde{x} \neq x^*\). Then by strong duality Theorem (4) there exists \((\tau, \lambda, z_1, \ldots, z_p)\) such that \((E\tilde{x}^*, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((P)\) and the inequality
\[
\tilde{f}_j(E\tilde{x}) + \langle z_j, E\tilde{x} \rangle \leq \tilde{f}_j(Ex^*) + \langle z_j, Ex^* \rangle, \\
\text{for } j = 1, \ldots, p,
\]
cannot be hold. i.e.
\[
\sum_{j=1}^{p} \tilde{f}_j(Ex^*) + \langle z_j, Ex^* \rangle < \sum_{j=1}^{p} \tilde{f}_j(E\tilde{x}) + \langle z_j, E\tilde{x} \rangle. \tag{14}
\]

Now, since \(E(x^*)\) is a weakly efficient solution for \((P)\), \(\lambda_i \geq 0\) and \((\tilde{x}, \tau, \lambda, z_1, \ldots, z_p)\) is a weakly efficient solution for \((D)\), we have
\[
\sum_{i \in I} \lambda_i g_i(Ex^*) \leq \sum_{i \in I} \lambda_i g_i(E\tilde{x}),
\]
and from the definition of quasi \(E\)-convexity of \(\lambda_i g_i(\cdot), i \in I\)
\[
\langle Ex^* - E\tilde{x}, \sum_{i \in I} \lambda_i \xi_i \rangle \leq 0, \tag{15}
\]
for every \(x^* \in M\) and every \(\xi \in \partial g_i(E\tilde{x})\). By substituting from \((10)\) in \((15)\), we get
\[
\langle Ex^* - E\tilde{x}, \sum_{j=1}^{p} \tau_j (\xi_j + z_j) \rangle \geq 0.
\]
for each \(x^* \in M\) and some \(\xi \in \partial f_j(E\tilde{x})\). Thus from strict pseudo \(E\)-convexity of \(\tilde{r}_j(f_j(\cdot) + \langle z_j, \cdot \rangle)\) for \(j = 1, 2, \ldots, p\) at \(\tilde{x}\), we get
\[
\sum_{j=1}^{p} \tilde{f}_j(E\tilde{x}) + \langle z_j, E\tilde{x} \rangle > \sum_{j=1}^{p} \tilde{f}_j(E\tilde{x}) + \langle z_j, E\tilde{x} \rangle. \tag{16}
\]
which contradicts (1). Therefore, \(\tilde{x} = x^*\).

**Some Applications**

Let us briefly review a few interesting applications. Nonsmooth semi-infinite multi-objective programming problems very naturally lead to a highly disaggregated formulation. Computation of economic equilibria is a very promising area of application for nonsmooth semi-infinite multi-objective programming problems. A paper [27] gives example evidence of the solving-power of ACCPM (analytical center cutting plane method) on these reputedly difficult problems. At the end, we would like to mention applications to nonsmooth semi-infinite multi-objective programming problems. In the first application [28], ACCPM (analytical center cutting plane method) is used to solve a Lagrangian relaxation of the capacitated multi-item lot sizing problem with set-up times. A full integration of ACCPM in a column generation, or Lagrangian relaxation, framework, for structured nonsmooth semi-infinite multi-objective programming problems, shows that the reliability and robustness of ACCPM in applications where a non-differentiable problem must be solved repeatedly makes it a very powerful alternative to subgradient optimization [29].

**5 Conclusions**

This paper investigates the optimality conditions and duality for nonsmooth semi-infinite E-convex multi-objective programming with support functions. The obtained results extended and improved corresponding results of [26,21] to nonsmooth E-convex case. By applying the obtained results, one can study fractional programming, set-valued optimization and variational inequalities and so on. For instance, we can apply the obtained Kuhn-Tucker necessary and sufficient conditions to study the optimality conditions and duality for nonsmooth multiobjective programming problems with generalized E-convexity. We can also apply the Kuhn-Tucker sufficient conditions to consider the solvability of some vector variational inequalities.

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