A study on generalized hesitant intuitionistic Fuzzy soft sets

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Abstract. By combining the concept of hesitant intuitionistic fuzzy sets, fuzzy soft sets and fuzzy sets, we extend hesitant intuitionistic fuzzy soft sets to a generalized hesitant intuitionistic fuzzy soft sets. Some operations on generalized hesitant intuitionistic fuzzy soft sets, such as union, complement, operations "AND" and "OR", and intersection are defined. From such operations the authors obtain related properties such as commutative, associative and De Morgan's laws. The authors also get an algebraic structure of the collection of all generalized hesitant intuitionistic fuzzy soft sets over a set.

1. Introduction
The concept on fuzzy sets is one of mathematical fields which is rather popular. Almost all mathematical fields can be studied in point of view fuzzy sets. One of such research is what is done by Nazra [1] who studied on fuzzy algebra. The studies on fuzzy sets continue to grow from year to year with various types of fuzzy sets such as Fuzzy Soft Sets and Intuitionistic Fuzzy Soft Sets. Such research either related to decision making problems or not, it is not regardless from the algebraic structure. A study by Karaaslan [2] is one of the new research on fuzzy sets that studies on Intuitionistic Fuzzy Parameterized Intuitionistic Fuzzy Soft Sets. However, the types of research that is mentioned above is assumed that the membership (or non-membership) value of an element in a set is a unique value in closed interval [0,1]. Then since 2009, it was introduced the concept on Hesitant Fuzzy Sets (HFSs) by Torra and Narukawa [3] where in this concept it is assume that the membership (or non-membership) value of an element in a set is a subset of [0,1]. If a decision maker hesitates in determining the choice of membership (or non-membership) value of an element in a set in the framework of a decision-making process then the concept on HFSs is very beneficial in handling a problem.

Babitha and John [4] introduced the Hesitant Fuzzy Soft Sets (HFSSs) concept as a combination between HFSs and Fuzzy Soft Sets, and analyzed some of its basic mathematical operations. Recent studies on HFSSs are in [5] and [6] in which it is defined other operations on HFSSs and its application to decision making problems. Since it is assumed that the non-membership value is an empty set on HFSs and HFSSs, then to develop these results, it was introduced the concept on Intuitionistic Hesitant Fuzzy Sets (IHFSs) as a merge the concept on Intuitionistic Fuzzy Sets and HFSs [7]. Bin [8] has done the similar research which is called Generalized Hesitant Fuzzy Sets (GHFSs). However, the algebraic properties of IHFSs or GHFSs are not investigated in Beg and Rashid [7] and Bin [8]. To study the algebraic properties of IHFSs, Nazra, et al. [9] studied the similar concept of IHFSs which is called Hesitant Intuitionistic Fuzzy Soft Sets (HIFSSs). On the other hand, Dinda, et al. [10] studied on Generalized Intuitionistic Fuzzy Soft Sets (GIFSSs) and its applications. Inspired from the study by Dinda, et al. [10], in this paper we generalize the concept of HIFSSs
introduced by Nazra, et al. [9]. We give the definition of a Generalized Hesitant Intuitionistic Fuzzy Soft Set (GHIFSS) and its some operations. Then we prove some properties of GHIFSSs such as commutative, associative and De Morgan’s laws and also, we get an algebraic structure of the collection of all GHIFSSs.

2. Preliminaries
To study on generalized hesitant fuzzy soft sets, one needs to know some concepts on fuzzy sets, fuzzy soft sets, and intuitionistic fuzzy sets and hesitant intuitionistic fuzzy sets. In this section we review definitions and some properties on such sets.

First of all we recall the definition and some properties on the fuzzy sets. The fuzzy set (FS) X over a set U is a set X={ (u,µ(u)) | u ∈ U } where µ: U → [0,1] which is called the membership function of X and µ(u) is called the membership value of u in X. For simplification, we use µ to denote a fuzzy set X. The complement of a fuzzy set µ is defined by µc = { (u,1-µ(u)) | u ∈ U } with µc(u)=1-µ(u). The union and the intersection of two fuzzy sets µ and ϑ denoted by µ ∪ ϑ and µ ∩ ϑ respectively are fuzzy sets defined by µ ∪ ϑ ={ (u,µ(u)∪ϑ(u)) | u ∈ U }, with µ(u)∪ϑ(u)=max{ µ(u), ϑ(u) } and µ ∩ ϑ ={ (u,µ(u)∩ϑ(u)) | u ∈ U }, with µ(u)∩ϑ(u)= min{ µ(u), ϑ(u) } respectively. A fuzzy set µ over U defined by Ø := { (u,0) | u ∈ U } which is called the null FS and defined by 1 := { (u,1) | u ∈ U } which is called the universal FS.

**Proposition 1.** Given two FSs µ, ϑ and ω over U. Then the following hold.

i) (µc)c = µ.

ii) µ ∪ ϑ = ϑ ∪ µ.

iii) µ ∩ ϑ = ϑ ∩ µ.

iv) (µ ∪ ϑ)c = ϑc ∩ µc.

v) (µ ∩ ϑ)c = ϑc ∪ µc.

vi) (µ ∩ ϑ) ∩ ω = µ ∩ ( ϑ ∩ ω ).

vii) (µ ∪ ϑ) ∪ ω = µ ∪ ( ϑ ∪ ω ).

**Definition 2.** [11] Suppose that U is a universal set and E is a parameter set, A ⊆ E. A pair (fA, A) is called a fuzzy soft set over U if fA: A → I where I is the collection of all fuzzy sets over U.

Suppose that U be a set. An intuitionistic fuzzy set Y over the set U is a set which is defined as

\[ Y = \{ (z, µ_Y(z), γ_Y(z)) | z ∈ U \} \]

where \( µ_Y : U → [0,1] \) and \( γ_Y : U → [0,1] \), and \( 0 ≤ µ_Y(z) + γ_Y(z) ≤ 1 \) for all \( z ∈ U \). Here \( µ_Y(z) \) and \( γ_Y(z) \) are called the membership and the non-membership values of \( z \) in \( Y \) respectively.

3. Hesitant Intuitionistic Fuzzy Sets
The concept on Hesitant Intuitionistic Fuzzy Sets (HIFSs) is needed to study the concept on Generalized Hesitant Intuitionistic Fuzzy Soft Sets (GHIFSSs), and some results on GHIFSSs are very related to the HIFSs. In this section, we review the definition of Hesitant Intuitionistic Fuzzy Sets (HIFSs) and Hesitant Intuitionistic Fuzzy Elements (HIFEs) [7]. Based on these definitions, we recall some operations and results on HIFEs and HIFSs [9].

**Definition 3.** [7] Given a set U. Suppose µ and µ’ are functions applied to U return subsets of [0,1] where for any \( u ∈ U \), µ(u) and µ’(u) are sets of some values in [0,1]. A hesitant intuitionistic fuzzy set (HIFS) on U is a set

\[ X = \{ (u, µ(u), µ’(u)) | u ∈ U \}. \]

Values µ(u) and µ’(u) denote the possible membership and non-membership values of the element \( u ∈ U \) to the set \( X \) respectively, which satisfy

\[ \max \{ µ(u) \} + \min \{ µ’(u) \} ≤ 1 \]
and 
\[ \min \{ \mu(u) \} + \max \{ \mu'(u) \} \leq 1. \]

Nazra, et al. [9] defined \( \bar{\mu}(u)=(\mu(u),\mu'(u)) \) as a hesitant intuitionistic fuzzy element (HIFE) on \( u \in U \), where \( \mu(u)=\{a_1, \ldots, a_m \ | \ a_i \in [0,1] \} \) and \( \mu'(u)=\{a'_1, \ldots, a'_n \ | \ a'_i \in [0,1] \} \). Then the HIFS \( \bar{X} \) can be written as \( \bar{X} = \{<u, \bar{\mu}(u)> \mid u \in U \} \). They denote \( \bar{H}(U) \) as the set of all HIFSs on \( U \). A HIFS \( \bar{X} = \{<u, \mu(u), \mu'(u)> \mid u \in U \} \) is called the null HIFS, denoted by \( \emptyset \), if \( \bar{\mu}(u)=(\mu(u),\mu'(u))=(\{0\},\{1\}) \), and it is called the universal HIFS, denoted by \( \top \), if \( \bar{\mu}(u)=(\mu(u),\mu'(u))=(\{1\},\{0\}) \).

Further, Nazra, et al. [9] defined some operations on HIFEs and the corresponding HIFSs stated in Definition 4 and Definition 5 below.

Definition 4. [9] Let \( \bar{\mu}_i(u)=(\mu_i(u),\mu'_i(u)) \), \( i=1,2 \) be HIFEs on \( u \in U \) with 
\[ \mu_i(u)=[a_1, \ldots, a_m \ | \ a_i \in [0,1]] \] 
\[ \mu'_i(u)=[a'_1, \ldots, a'_n \ | \ a'_i \in [0,1]] \] 
\[ \mu_2(u)=[\beta_1, \ldots, \beta_p \ | \ \beta_i \in [0,1]] \] 
\[ \mu'_2(u)=[\beta'_1, \ldots, \beta'_q \ | \ \beta'_i \in [0,1]] \] 
Then the complement of \( \bar{\mu}_i(u) \), the union and the intersection of \( \bar{\mu}_1(u) \) and \( \bar{\mu}_2(u) \) are defined as follows respectively.

\begin{enumerate}
  \item \( \bar{\mu}_i^c(u) = (\mu_i(u), \mu'_i(u)) \).
  \item \( \bar{\mu}_1(u) \cup \bar{\mu}_2(u) = (\cup_{\alpha_i \in \mu_1(u)} \max \{ \alpha_i, \beta_i \}) \cup (\cup_{\alpha'_i \in \mu'_1(u)} \min \{ \alpha'_i, \beta'_i \}) \).
  \item \( \bar{\mu}_1(u) \cap \bar{\mu}_2(u) = (\cap_{\alpha_i \in \mu_1(u)} \min \{ \alpha_i, \beta_i \}) \cup (\cap_{\alpha'_i \in \mu'_1(u)} \max \{ \alpha'_i, \beta'_i \}) \).
\end{enumerate}

Definition 5. [9] Given two HIFSs \( \bar{X} = \{<u, \bar{\mu}(u)> \mid u \in U \} \), \( \bar{Y} = \{<u, \bar{\nu}(u)> \mid u \in U \} \) on \( U \). The complement of \( \bar{X} \), the union and the intersection of \( \bar{X} \) and \( \bar{Y} \) are defined as follows respectively.

\begin{enumerate}
  \item \( \bar{X}^c = \{<u, \bar{\mu}(u)> \mid u \in U \} \).
  \item \( \bar{X} \cup \bar{Y} = \{<u, \bar{\mu}(u) \cup \bar{\nu}(u)> \mid u \in U \} \).
  \item \( \bar{X} \cap \bar{Y} = \{<u, \bar{\mu}(u) \cap \bar{\nu}(u)> \mid u \in U \} \).
\end{enumerate}

Refer to Definition 4 and Definition 5, Nazra et al. [9] proved the following properties of HIFEs and HIFSs.

Proposition 6. [9] Let \( \bar{\mu}_i(u) \) be HIFEs on \( u \in U \). Then the following hold.

\begin{enumerate}
  \item \( (\bar{\mu}_i^c(u))^c = \bar{\mu}_i(u) \).
  \item \( \bar{\mu}_1(u) \cup \bar{\mu}_2(u) = \bar{\mu}_2(u) \cup \bar{\mu}_1(u) \).
  \item \( \bar{\mu}_1(u) \cap \bar{\mu}_2(u) = \bar{\mu}_2(u) \cap \bar{\mu}_1(u) \).
  \item \( (\bar{\mu}_1(u) \cup \bar{\mu}_2(u))^c = \bar{\mu}_1^c(u) \cap \bar{\mu}_2^c(u) \).
  \item \( (\bar{\mu}_1(u) \cap \bar{\mu}_2(u))^c = \bar{\mu}_1^c(u) \cup \bar{\mu}_2^c(u) \).
  \item \( \bar{\mu}_1(u) \cap (\bar{\mu}_2(u) \cap \bar{\mu}_3(u)) = (\bar{\mu}_1(u) \cap \bar{\mu}_2(u)) \cap \bar{\mu}_3(u) \).
  \item \( \bar{\mu}_1(u) \cup (\bar{\mu}_2(u) \cup \bar{\mu}_3(u)) = (\bar{\mu}_1(u) \cup \bar{\mu}_2(u)) \cup \bar{\mu}_3(u) \).
  \item \( \bar{\mu}_1(u) \cup (\bar{\mu}_2(u) \cap \bar{\mu}_3(u)) = (\bar{\mu}_1(u) \cup \bar{\mu}_2(u)) \cap (\bar{\mu}_1(u) \cup \bar{\mu}_3(u)) \).
  \item \( \bar{\mu}_1(u) \cap (\bar{\mu}_2(u) \cup \bar{\mu}_3(u)) = (\bar{\mu}_1(u) \cap \bar{\mu}_2(u)) \cup (\bar{\mu}_1(u) \cap \bar{\mu}_3(u)) \).
\end{enumerate}

Proposition 7. [9] Let \( \bar{X}, \bar{Y} \) and \( \bar{Z} \) be HIFSs on \( U \). Then the following hold.

\begin{enumerate}
  \item \( (\bar{X}^c)^c = \bar{X} \).
  \item \( \bar{X} \cup \bar{Y} = \bar{Y} \cup \bar{X} \).
  \item \( \bar{X} \cap \bar{Y} = \bar{Y} \cap \bar{X} \).
  \item \( (\bar{X} \cup \bar{Y})^c = \bar{X}^c \cup \bar{Y}^c \).
  \item \( (\bar{X} \cap \bar{Y}^c) = \bar{X}^c \cup \bar{Y} \).
  \item \( \bar{X} \cap (\bar{Y} \cup \bar{Z}) = (\bar{X} \cap \bar{Y}) \cup \bar{Z} \).
\end{enumerate}
\[\begin{align*}
\text{vii) } & \tilde{X} \cup (\tilde{Y} \cap \tilde{Z}) = (\tilde{X} \cup \tilde{Y}) \cup \tilde{Z}.
\text{viii) } & \tilde{X} \cup (\tilde{Y} \cap \tilde{Z}) = (\tilde{X} \cap \tilde{Y}) \cup (\tilde{X} \cap \tilde{Z}).
\text{ix) } & \tilde{X} \cap (\tilde{Y} \cup \tilde{Z}) = (\tilde{X} \cap \tilde{Y}) \cup (\tilde{X} \cap \tilde{Z}).
\end{align*}\]

4. Generalized Hesitant Intuitionistic Fuzzy Soft Sets

Now, we extend the concept on HIFSs to Generalized Hesitant Intuitionistic Fuzzy Soft Sets (GHIFSSs) as a combination of HIFSs, fuzzy soft sets and fuzzy sets. First of all, we introduce the definition and some operations on GHIFSSs. Further, we obtain some results related to some operations on such GHIFSSs.

**Definition 8.** Let \( \overline{IH}(U) \) be the collection of all HIFSs on a set \( U \), \( \alpha \) is a fuzzy set over a set \( E \) and given a map \( \overline{F}_\alpha : E \to \overline{IH}(U) \). A pair \((\overline{F}_\alpha, E)\) is called a Generalized Hesitant Intuitionistic Fuzzy Soft Set over \( U \), where \( \overline{F}_\alpha : E \to \overline{IH}(U) \times [0,1] \), defined by \( \overline{F}_\alpha(e) = (\overline{F}(e), \alpha(e)) = \{ \langle u, \mu_e(u), \nu_e(u) \rangle | u \in U \}, \alpha(e) \in \overline{IH}(U) \times [0,1] \).

For simplification, we write \((\overline{F}_\alpha, E) = \{<e, \overline{F}(e), \alpha(e)>\}\). Here, we define the complement of a GHIFSS by \((\overline{F}_\alpha, E)^\prime = \{<e, \overline{F}(e), \alpha(e)^\prime>\}\).

Now, we introduce some operations on GHIFSSs such as union, intersection and special elements of GHIFSSs which are called null and universal GHIFSSs.

**Definition 9.** The Union of two GHIFSSs \((\overline{F}_\alpha, A)\) and \((\tilde{G}_\beta, B)\) is a GHIFSS, denoted by \((\overline{F}_\alpha, A) \cup (\tilde{G}_\beta, B) =: (\tilde{F}_\gamma, C)\), where \( C = A \cup B \), and \( \forall e \in C \),

\[\tilde{f}(c) = \overline{F}(c) \quad \text{if } c \in A \setminus B, \quad \gamma(c) = \alpha(c) \quad \text{if } c \in A \setminus B,\]

\[\tilde{g}(c) = \beta(c) \quad \text{if } c \in B \setminus A, \quad \gamma(c) = \alpha(c) \beta(c) \quad \text{if } c \in A \setminus B.\]

**Definition 10.** The Intersection of two GHIFSSs \((\overline{F}_\alpha, A)\) and \((\tilde{G}_\beta, B)\) is a GHIFSS, denoted by \((\overline{F}_\alpha, A) \cap (\tilde{G}_\beta, B) =: (\tilde{f}, C)\), where \( C = A \cap B \neq \emptyset \), and \( \forall e \in C \), \( \tilde{f}(c) = \overline{F}(c) \cap \tilde{G}(c) \) and \( \gamma(c) = \alpha(c)^\prime \beta(c)\).

A GHIFSS \((\overline{F}_\alpha, A) = \{<e, \overline{F}(e), \alpha(e)>\}\) is called the null GHIFSS, denoted by \((\overline{G}_\emptyset, A)\), if \( \overline{F}(e) : = \overline{G}, \alpha(e) : = 0 \) and it is called the universal GHIFSS, denoted by \((\tilde{1}, A)\), if \( \overline{F}(e) : = \overline{1}, \alpha(e) : = 1 \).

**Theorem 11.** Given two GHIFSSs over \( U \), \((\overline{F}_\alpha, A)\) and \((\tilde{G}_\beta, B)\). Then the following De Morgan's laws hold.

\[\text{i) } (\overline{F}_\alpha, A) \cup (\tilde{G}_\beta, B)^\prime = (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, B)^\prime, \]

\[\text{ii) } (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, B)^\prime = (\overline{F}_\alpha, A)^\prime \cup (\tilde{G}_\beta, B)^\prime.\]

**Proof.** Let \((\overline{F}_\alpha, A) \cup (\tilde{G}_\beta, A)^\prime = (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, A)^\prime\) where \( \forall e \in A \), \( \overline{G}(e) = \overline{F}(e) \cup \tilde{G}(e) \) and \( \gamma(e) = \alpha(e)^\prime \beta(e) \). Therefore by Proposition 7 iv) and the definition complement of FS, \( \overline{G}(e) = \overline{F}(e) \cap \tilde{G}(e) \) and \( \gamma(e) = 1 - \gamma(e) = 1 - \alpha(e)^\prime \beta(e) \).

Note that \( 1 - \max\{\alpha(e), \beta(e)\} = \min\{1 - \alpha(e), 1 - \beta(e)\} \). On the other hand, by the definition and complement of a GHIFSS, \( (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, A)^\prime = (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, A)^\prime = (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, A)^\prime = (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, A)^\prime = (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, A)^\prime = (\overline{F}_\alpha, A)^\prime \cap (\tilde{G}_\beta, A)^\prime \).

Therefore Theorem 11 i) is proved. By similar argument to i) Theorem 11 ii) can be proved.

It is clear that the De Morgan’s law is not satisfied for the union and the intersection of GHIFSSs for \( A \neq B \).
Theorem 12. Given three GHIFSSs over $U$, $(\tilde{F}_\alpha, A)$, $(\tilde{G}_\beta, B)$ and $(\tilde{K}_\gamma, C)$. Then the following associative laws hold.

i) $((\tilde{F}_\alpha, A) \cup (\tilde{G}_\beta, B)) \cup (\tilde{K}_\gamma, C) = (\tilde{F}_\alpha, A) \cup ((\tilde{G}_\beta, B) \cup (\tilde{K}_\gamma, C))$.

ii) $((\tilde{F}_\alpha, A) \cap (\tilde{G}_\beta, B)) \cap (\tilde{K}_\gamma, C) = (\tilde{F}_\alpha, A) \cap ((\tilde{G}_\beta, B) \cap (\tilde{K}_\gamma, C))$.

Proof. By using Proposition 1, Definition 9 and 10, and Proposition 7 vi) and vii), one can prove the theorem.

Now we give the other operations on GHIFSS called operations "AND" and "OR" denoted by "$\Delta$" and "$\forall$" respectively.

Definition 13. Given two GHIFSSs $(\tilde{F}_\alpha, A)$ and $(\tilde{G}_\beta, B)$ over $U$. It is defined two operations "$\Delta$" and "$\forall$" on such GHIFSSs as follows.

1) $(\tilde{F}_\alpha, A) \Delta (\tilde{G}_\beta, B) := (\tilde{T}_\gamma, A \times B)$ where $\forall (a,b) \in A \times B$, $\tilde{T}_\gamma : A \times B \rightarrow \tilde{IH}(U) \times [0,1]$ and $\gamma : A \times B \rightarrow [0,1]$ which are defined by $\tilde{T}(a,b) := (\tilde{F}(a) \cap \tilde{G}(b), \gamma(a,b)) = (\alpha(a) \cap \beta(b), \gamma(a,b))$.

2) $(\tilde{F}_\alpha, A) \forall (\tilde{G}_\beta, B) := (\tilde{N}_\psi, A \times B)$ where $\forall (a,b) \in A \times B$, $\tilde{N}_\psi : A \times B \rightarrow \tilde{IH}(U) \times [0,1]$ and $\psi : A \times B \rightarrow [0,1]$ which are defined by $\tilde{N}(a,b) := (\tilde{F}(a) \cup \tilde{G}(b), \psi(a,b)) = (\alpha(a) \cup \beta(b), \psi(a,b))$.

Related to the above definition, we obtain the following theorems.

Theorem 14. Given two GHIFSSs $(\tilde{F}_\alpha, A)$ and $(\tilde{G}_\beta, B)$ over $U$. Then the following hold.

i) $((\tilde{F}_\alpha, A) \Delta (\tilde{G}_\beta, B))^\vee = (\tilde{F}_\alpha, A)^\vee \forall (\tilde{G}_\beta, B)^\vee$.

ii) $((\tilde{F}_\alpha, A) \forall (\tilde{G}_\beta, B))^\vee = (\tilde{F}_\alpha, A)^\vee \Delta (\tilde{G}_\beta, B)^\vee$.

Proof. By using definition on complement of a GHIFSS and on Fuzzy sets, Definition 13, Proposition 7 vi), we prove Theorem 14 i) as follows. $((\tilde{F}_\alpha, A) \Delta (\tilde{G}_\beta, B))^\vee = (\tilde{F}_\alpha, A)^\vee \forall (\tilde{G}_\beta, B)^\vee$. Let $(a,b) \in A \times B$, with $\tilde{F}(a,b) := (\tilde{F}(a) \cap \tilde{G}(b), \gamma(a,b)) = (\alpha(a) \cap \beta(b), \gamma(a,b))$, and $\forall (a,b) := (\alpha(a) \cup \beta(b), \psi(a,b))$. On the other hand, $(\tilde{F}_\alpha, A)^\vee \forall (\tilde{G}_\beta, B)^\vee = (\tilde{F}_\alpha, A)^\vee \forall (\tilde{G}_\beta, B)^\vee$. Since $\tilde{T}(a,b) = \tilde{N}(a,b)$ and $\psi(a,b) = \psi(a,b)$, then Theorem 14 i) is proved.

The proof of Theorem 14 ii) is similar to i).

Theorem 15. Suppose that $(\tilde{F}_\alpha, A)$, $(\tilde{K}_\beta, B)$ and $(\tilde{K}_\gamma, C)$ are GHIFSSs over $U$. Then the following associative laws satisfy.

i) $(\tilde{F}_\alpha, A) \Delta ((\tilde{K}_\beta, B)) \Delta (\tilde{K}_\gamma, C) = (\tilde{F}_\alpha, A) \Delta ((\tilde{K}_\beta, B)) \Delta (\tilde{K}_\gamma, C)$.

ii) $(\tilde{F}_\alpha, A) \forall ((\tilde{K}_\beta, B)) \forall (\tilde{K}_\gamma, C) = (\tilde{F}_\alpha, A) \forall ((\tilde{K}_\beta, B)) \forall (\tilde{K}_\gamma, C)$.

Proof. Here we prove ii). To prove i) we do the similar way with ii).

Using Definition 13 and Proposition 7 vii), we get $(\tilde{F}_\alpha, A) \forall ((\tilde{K}_\beta, B)) \forall (\tilde{K}_\gamma, C) = (\tilde{F}_\alpha, A) \forall (\tilde{K}_\beta, B)$.

$\forall (b,c) \in B \times C$, $\forall (b,c) \in \tilde{G}(b) \cap \tilde{G}(c)$, and $\forall (b,c) \in \tilde{G}(b) \cup \tilde{G}(c)$, with $\tilde{F}(a,b,c) := (\tilde{F}(a) \cup \tilde{K}(b) \cup \tilde{G}(c)) = (\tilde{F}(a) \cup \tilde{K}(b) \cup \tilde{G}(c))$. Therefore the proof of the theorem is complete.
Since $\Delta((B \times C) \neq (A \times B)) \times (A \times C)$, the distributive laws of GHIFSSs do not hold. This means $(\tilde{F}_\alpha, A) \Delta ((\tilde{F}_\beta, B) \cup (\tilde{F}_\gamma, C)) \neq ((\tilde{F}_\alpha, A) \Delta (\tilde{F}_\beta, B)) \cup ((\tilde{F}_\alpha, A) \Delta (\tilde{F}_\gamma, C))$, and $(\tilde{F}_\alpha, A) \Delta ((\tilde{F}_\beta, B)) \Delta ((\tilde{F}_\gamma, C)) \neq ((\tilde{F}_\alpha, A) \Delta (\tilde{F}_\beta, B)) \Delta ((\tilde{F}_\alpha, A) \Delta (\tilde{F}_\gamma, C)).$

In the end of this section, we present the following result on an algebraic structure of the collection of all GHIFSSs over $U$ as a consequence of Definition 9 and 10 and Theorem 12.

**Corollary 16.** Suppose that $\tilde{G}H$ is the collection of all GHIFSSs $(\tilde{F}_\alpha, A)$ over $U$ for any $\alpha$. Then we obtain the following.

1. $\tilde{G}H$ is closed over operations $\cup$ and $\cap$.
2. $\tilde{G}H$ satisfies the associative law over operations $\cup$ and $\cap$.
3. There are GHIFSSs $(\tilde{F}_\alpha, A)$ and $(\tilde{F}_\beta, A)$ such that $(\tilde{F}_\alpha, A) \cap (\tilde{F}_\beta, A) = (\tilde{F}_\alpha, A)$ and $(\tilde{F}_\alpha, A) \cup (\tilde{F}_\beta, A) \in \tilde{G}H$.

Therefore, $\tilde{G}H$ is a monoid (semi group with the identity element) over binary operations $\cup$ and $\cap$.

5. Conclusion
The Generalized Hesitant Intuitionistic Fuzzy Soft Set (GHIFSS) is a generalization of the Hesitant Intuitionistic Fuzzy Soft Set. By defining some operations on GHIFSS such as union, intersection, and some binary operations which are called “AND” and “OR” we obtain some related properties, such as commutative, associative and De Morgan’ laws, and an algebraic structure of the collection of all GHIFSSs.

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