INDEX THEORY FOR PARTIAL-BIJections

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Abstract. We offer streamlined proofs of fundamental theorems regarding the index theory for partial self-maps of an infinite set that are bijective between cofinite subsets.

0. Introduction

In a recent paper [1] we called the self-map $f$ of the infinite set $\Omega$ a \textit{near-bijection} precisely when $f$ restricts to a true bijection from a cofinite subset $A \subseteq \Omega$ to a cofinite subset $B \subseteq \Omega$. Along with the range $f(\Omega)$ of $f$ we introduced its ‘monoset’

$$\Omega_f = \{ \omega \in \Omega: f(f(\omega)) = \{\omega\}\};$$

in these terms, $f$ is a near-bijection precisely when $f(\Omega)$ and $\Omega_f$ are cofinite. The \textit{index} of the near-bijection $f$ is then defined by

$$\text{ind}(f) = |\Omega_f' - |f(\Omega)'| - |f(\Omega)| | \in \mathbb{Z}$$

where if $C \subseteq \Omega$ then $|C|$ is its cardinality and $C' = \Omega \setminus C$ is its complement. We showed in [1] that the index is insensitive to changes on a finite set and that $\text{ind}(f)$ is zero precisely when $f$ differs from a bijection on a finite set. We also showed that when near-bijections that differ on a finite set are identified, their equivalence classes constitute a group on which the $\mathbb{Z}$-valued index is a homomorphism.

In [1] considerable effort was devoted to the careful handling of $\Omega_f'$ and $f(\Omega)'$: for example, when the value of a near-bijection $f$ is changed at $\omega \in \Omega_f'$ it is important to know whether the number of points at which $f(\omega)$ was formerly the value is two or is greater than two; it is also important to know whether the new value of $f$ at $\omega$ was or was not formerly a value of $f$. These circumstances cause technical complications: for example, in the verification that the index is insensitive to changes on a finite set and in the verification that the index is a homomorphism. Our primary purpose in this paper is to reformulate the notion of near-bijection in a way that circumvents these complications and facilitates streamlined proofs of the fundamental results.

1. Index theory

Let $\Omega$ be an infinite set.

\textbf{Definition:} A \textit{partial-bijection} is a (true) bijection $f : A_f \to B_f$ from a cofinite subset $A_f \subseteq \Omega$ to a cofinite subset $B_f \subseteq \Omega$.

This is our reformulation of the notion of near-bijection: as $f(\Omega)'$ and (especially) $\Omega_f'$ were the source of complications in [1] we simply eliminate them; much of the focus in [1] was on properties defined only up to changes on finite sets, so this reformulation is eminently reasonable. More strictly, we should perhaps speak of a bijective partial self-map; but the convenient abuse ‘partial-bijection’ is also reasonable.
**Definition:** The *index* of the partial-bijection $f$ is defined by

$$\text{ind}(f) = |A'_f| - |B'_f| \in \mathbb{Z}.$$  

We should of course verify that this notion of index agrees with the notion in [1]. To this end, let $f: \Omega \to \Omega$ be a near-bijection in the sense of [1]: that is, a map for which the complements $\Omega'_f$ (see the Introduction) and $f(\Omega)'$ are finite. Restricting $f$ (but using the same symbol for convenience) yields a partial-bijection $f: \Omega_f \to f(\Omega_f)$: in fact,

$$\Omega \setminus f(\Omega_f) = (\Omega \setminus f(\Omega)) \cup (f(\Omega) \setminus f(\Omega_f))$$

where $f(\Omega) \setminus f(\Omega_f) = f(\Omega \setminus f(\Omega))$ by definition of $\Omega_f$; accordingly,

$$|f(\Omega_f)'| = |f(\Omega)'| + |f(\Omega_f)|.$$  

It follows that the index of $f: \Omega \to \Omega$ as defined in [1] is

$$|\Omega_f| - |f(\Omega_f)'| = |f(\Omega)'| = |\Omega'_f| - |f(\Omega_f)'|$$

and so agrees with the index of $f: \Omega_f \to f(\Omega_f)$ as presently defined.

With the current definitions, the following is immediate.

**Theorem 1.** The partial-bijection $f$ extends to a true bijection $\Omega \to \Omega$ precisely when $\text{ind}(f)$ vanishes.

*Proof.* The bijection $f: A_f \to B_f$ extends to a bijection from $\Omega$ to itself precisely when the finite complements $A'_f$ and $B'_f$ have the same cardinality. \hfill \Box

In [1] we identified near-bijections when they differed on a finite set. Here, the corresponding identification results from the following definition.

**Definition:** The partial-bijections $f$ and $g$ are *almost equal* (notation: $f \equiv g$) precisely when $f$ and $g$ agree on a cofinite subset of $A_f \cap A_g$.

It is readily verified that almost equality is an equivalence relation; transitivity would fail were we simply to insist that $f$ and $g$ agree on their common domain $A_f \cap A_g$.

As expected, the index is insensitive to changes on a finite set.

**Theorem 2.** Let $f$ and $g$ be partial-bijections. If $f \equiv g$ then $\text{ind}(f) = \text{ind}(g)$.

*Proof.* Let $f$ and $g$ agree on the cofinite set $E \subseteq A_f \cap A_g$ and write $e: E \to e(E)$ for the common restriction $f|_E = g|_E$. Now $f$ restricts to a bijection $A_f \setminus E \to B_f \setminus e(E)$ where

$$A_f \setminus E = A_f \cap E' = E' \setminus A_f'$$

and

$$B_f \setminus e(E) = B_f \cap e(E)' = e(E)' \setminus B_f'$$

whence

$$|E'| - |A_f'| = |e(E)'| - |B_f'|$$

and therefore

$$\text{ind}(f) = |A_f'| - |B_f'| = |E'| - |e(E)'| = \text{ind}(e).$$

The symmetric observation that $\text{ind}(e)$ equals $\text{ind}(g)$ completes the proof. \hfill \Box

**Remark:** Comparison with the proof of the corresponding result (Theorem 19) in [1] amply demonstrates the virtue of the approach adopted here.

Let us now consider the composition of partial-bijections. The natural approach to composition of partial maps suggests the following.
**Definition:** The *composite* of the partial-bijections $f$ and (then) $g$ is the map $g \circ f : A_{gof} \to B_{gof}$ with domain

$$A_{gof} := \overrightarrow{f}(B_f \cap A_g)$$

with codomain

$$B_{gof} := \overrightarrow{g}(B_f \cap A_g)$$

and with rule

$$(\forall \omega \in \Omega) \ (g \circ f)(\omega) := g(f(\omega)).$$

**Theorem 3.** If $f$ and $g$ are partial-bijections, then so is $g \circ f$.

**Proof.** That $g \circ f : A_{gof} \to B_{gof}$ is a bijection is clear. To see that $A_{gof}$ is cofinite, observe that

$$A'_{gof} = \overrightarrow{f}(B_f \cap A_g)' = (\Omega \setminus A_f) \cup (A_f \setminus \overrightarrow{f}(B_f \cap A_g))$$

where $|\Omega \setminus A_f| = |A_f'|$ and where (as $f$ is a bijection from $A_f$ to $B_f$)

$$|A_f \setminus \overrightarrow{f}(B_f \cap A_g)| = |B_f \setminus (B_f \cap A_g)| = |B_f \setminus A_g| = |A'_f \setminus B'_g|.$$ 

Thus

$$|\overrightarrow{f}(B_f \cap A_g)| = |A_f'| + |A_g' \setminus B_f'|$$

is finite, as likewise is

$$|\overrightarrow{g}(B_f \cap A_g)| = |B_g'| + |B_f' \setminus A_g'|.$$

We now propose to prove that when partial-bijections are composed their indices add. For this purpose, it is convenient first to record the following triviality.

**Theorem 4.** If $X$ and $Y$ are finite sets then $|X \setminus Y| + |Y| = |Y \setminus X| + |X|.$

**Proof.** Each side of the equation is precisely $|X \cup Y|.$

Verification of our claim regarding the index of a composite is now quite straightforward.

**Theorem 5.** If $f$ and $g$ are partial-bijections then $\text{ind}(g \circ f) = \text{ind}(g) + \text{ind}(f)$.

**Proof.** We continue from the close of the proof for Theorem 3. Thus

$$\text{ind}(g \circ f) = |\overrightarrow{f}(B_f \cap A_g)| - |\overrightarrow{g}(B_f \cap A_g)'| = |A_f'| + |A'_g \setminus B_f'| - |B_g'| - |A_f ' \setminus B_g'|$$

while

$$\text{ind}(g) + \text{ind}(f) = |A_g'| - |B_g'| + |A_f'| - |B_f'|$$

whence

$$\text{ind}(g \circ f) - \text{ind}(g) - \text{ind}(f) = |A_g' \setminus B_f'| + |B_g'| - |B_f' \setminus A_g'| - |A_f'|$$

and an application of Theorem 4 with $X = A_g'$ and $Y = B_f'$ ends the argument.

**Remark:** In [1] the corresponding result is Theorem 27; once again, comparison highlights the virtue of the approach taken in the present paper.

Let $f : A_f \to B_f$ be a partial bijection: as a bijection, $f$ has an inverse map $f^{-1} : B_f \to A_f$ which is also a partial-bijection; the composites $f^{-1} \circ f = \text{Id}_{A_f}$ and $f \circ f^{-1} = \text{Id}_{B_f}$ imply that $f^{-1} \circ f \equiv \text{Id}_{\Omega} \equiv f \circ f^{-1}$. As a companion to the last theorem, we have the next.
Theorem 6. If \( f \) is a partial-bijection then \( \text{ind}(f^{-1}) = -\text{ind}(f) \).

Proof. Immediate: passage from \( f \) to its inverse switches \( A_f \) and \( B_f \).

The permutations of \( \Omega \) make up the symmetric group \( S_\Omega \); Theorem 4 says that the partial-bijections having index zero are exactly the restrictions of these permutations to cofinite subsets of \( \Omega \). It is clear that composing a partial-bijection with a permutation (on either side, left or right) does not affect the index: indeed, if \( f \) is a partial-bijection and \( \pi \) is a permutation, then \( A'_{\pi \circ f} = A'_f \) and \( B'_{\pi \circ f} = \pi(B_f)' = \pi(B'_f) \) so that

\[
\text{ind}(\pi \circ f) = |A'_{\pi \circ f}| - |B'_{\pi \circ f}| = |A'_f| - |\pi(B'_f)| = |A'_f| - |B'_f| = \text{ind}(f)
\]

while \( A'_{f \circ \pi} = \pi(A_f)' = \pi(A'_f) \) and \( B'_{f \circ \pi} = B'_f \) so that

\[
\text{ind}(f \circ \pi) = |A'_{f \circ \pi}| - |B'_{f \circ \pi}| = |\pi(A'_f)| - |B'_f| = |A'_f| - |B'_f| = \text{ind}(f)
\]

In fact, this essentially covers all cases of equal index: any two partial-bijections having the same index are related by permutations in this way, up to almost equality.

Theorem 7. Let \( f \) and \( g \) be partial-bijections. If \( \text{ind}(f) = \text{ind}(g) \) then there exist permutations \( \lambda \in S_\Omega \) and \( \rho \in S_\Omega \) such that \( \lambda \circ f \equiv g \equiv f \circ \rho \).

Proof. The composite partial-bijection \( g \circ f^{-1} \) after \( f^{-1} \) is a true bijection

\[
g \circ f^{-1} : \overrightarrow{f}(A_g) \rightarrow \overrightarrow{g}(A_f).
\]

Theorem 5 and Theorem 6 show that \( \text{ind}(f) \) extends to a permutation \( \lambda \) of \( \Omega \); the almost equality \( \lambda \circ f \equiv g \) is clear. Similarly, the composite partial-bijection

\[
f^{-1} \circ g : \overrightarrow{g}(B_f) \rightarrow \overrightarrow{f}(B_g)
\]

has vanishing index and extends to a permutation \( \rho \) of \( \Omega \) such that \( g \equiv f \circ \rho \).

Remark: This demonstrates quite strikingly the virtue of the present approach when dealing with matters that allow indeterminacy on finite sets. In [1] the corresponding result is a combination of Theorem 21, Theorem 22 and Theorem 23; there, complications involving range and monoset necessitate the separate handling of \( \lambda \) and \( \rho \) as well as the separate handling of negative index and positive index.

As in [1] it is of interest to view these results from a group-theoretic perspective. Almost equality defines an equivalence relation on the set of all partial-bijections of \( \Omega \); we denote by \( G_\Omega \) the set comprising all such \( \equiv \)-classes, denoting the \( \equiv \)-class of \( f \) by \([f]\) as usual.

Theorem 8. Let \( f_1, f_2, g_1, g_2 \) be partial-bijections. If \( f_1 \equiv f_2 \) and \( g_1 \equiv g_2 \) then \( g_1 \circ f_1 \equiv g_2 \circ f_2 \).

Proof. Note from Theorem 5 that \( g_1 \circ f_1 \) and \( g_2 \circ f_2 \) are partial-bijections. Let \( F \subseteq A_{f_1} \cap A_{f_2} \) and \( G \subseteq A_{g_1} \cap A_{g_2} \) be cofinite sets on which \( f_1|_F = f_2|_F =: f \) and \( g_1|_G = g_2|_G =: g \). Verification that \( g_1 \circ f_1 \) and \( g_2 \circ f_2 \) agree on \( \overrightarrow{f}(G) \) is immediate; verification that \( \overrightarrow{f}(G) \) is cofinite presents no difficulties.

It follows that composition descends to a well-defined (associative) binary operation on \( G_\Omega \); this makes \( G_\Omega \) into a group, the inverse of \([f]\) being \([f^{-1}]\). Theorem 6 guarantees that the index map \( \text{ind} \) descends to a well-defined map

\[
\text{Ind} : G_\Omega \rightarrow \mathbb{Z}
\]

which is a group homomorphism by Theorem 3. By Theorem 4 the kernel \( S_\Omega \) of \( \text{Ind} \) comprises precisely all \( \equiv \)-classes containing permutations. The image of \( \text{Ind} \) is of course \( \mathbb{Z} \): note that if \( \omega_1 \in \Omega \) then any bijection \( u : \Omega \rightarrow \Omega \setminus \{\omega_0\} \) has index \(-1\) and if \( n \in \mathbb{Z} \) then \([u]^n \) has index \(-n\). The cosets of \( S_\Omega \subseteq G_\Omega \) are labelled by \( \text{Ind} \): this is clear from the fundamental isomorphism theorem, but is also explicit in Theorem 7 and the discussion leading up to it.
Thus, we have constructed a short exact sequence of groups
\[ \text{Id} \rightarrow \mathbb{S}_\Omega \rightarrow \mathbb{G}_\Omega \rightarrow \mathbb{Z} \rightarrow 0. \]
This sequence splits, as \( \mathbb{Z} \) is infinite cyclic: with \( u \) as above, a splitting homomorphism is
\[ \mathbb{Z} \rightarrow \mathbb{G}_\Omega : n \mapsto [u]^{-n}. \]

In summary, the approach taken in this paper, based on partial-bijections in place of near-bijections, offers a significantly streamlined route to those results of [1] pertaining to properties that are unaffected by changes on finite subsets of \( \Omega \); in particular, it is well-suited to handling the group \( \mathbb{G}_\Omega \) and the index.

REFERENCES

[1] P.L. Robinson, *Fredholm theory for cofinite sets*, arXiv 1509.08039 (2015).

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