Sandwich theorems for Shioda–Inose structures

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Abstract. We give a geometric construction of three infinite families of K3-surfaces which are sandwiched by Kummer surfaces within a Shioda–Inose structure. Explicit examples are also given.

Keywords: K3-surface, Shioda–Inose structure, elliptic fibration, isogeny.

§ 1. Introduction

Shioda–Inose structures have recently featured very prominently in the arithmetic and geometry of K3-surfaces, relating specific K3-surfaces to Kummer surfaces in a natural way. Shioda showed in [1] that any Jacobian elliptic K3-surface with two singular fibres of type II* is in fact sandwiched by the Kummer surface in question (which is indeed of product type). Ma subsequently gave an abstract Hodge-theoretic proof that any Shioda–Inose structure can be extended to a sandwich [2]. However, in the generic situation of Picard number 17 there are only five explicit geometric examples, due to Kumar [3], van Geemen–Sarti [4] and Koike [5].

In this paper we use geometric means to develop three infinite series of K3-surfaces of Picard number (at least) 17 with a sandwiched Shioda–Inose structure.

Theorem 1. Let $N \in \mathbb{N}$. Assume that one of the following three alternatives holds:

1) $p \equiv 1 \pmod{4}$ for any $p | N$;
2) $N = \prod_i p_i$ or $N = 7 \prod_i p_i$, $p_i \equiv 1, 2, 4 \pmod{7}$ for any $i$;
3) $N = \prod_i p_i$ or $N = 15 \prod_i p_i$ for an odd number of primes $p_i \equiv 2, 8 \pmod{15}$, or $N = 3 \prod_i p_i$ or $N = 5 \prod_i p_i$ for an even number of primes $p_i \equiv 2, 8 \pmod{15}$.

Then for any K3 surface $X$ with a primitive embedding $T_X \hookrightarrow U^2 + \langle -2N \rangle$ there is an explicit geometric sandwiched Shioda–Inose structure.

This will be proved by exhibiting three distinct families of K3-surfaces in §§3–5; see §§3.5, 4.4, 5.4 for the precise arguments. Our construction uses specialisation from four-dimensional families of K3-surfaces via lattice enhancements and elliptic fibrations with 2-torsion sections. We point out that our construction in particular enables us to realise all five previously known examples (by inspecting the discriminants; see §2), but it does not give any transcendental lattices $U^2 + \langle -2N \rangle$ beyond those specified in Theorem 1 (cf. §3.6).

The paper is organised as follows. The next section reviews basic material on Shioda–Inose structures and sandwiches. Each of the subsequent sections is devoted to one of the families in question. We work over the field $\mathbb{C}$ of complex numbers throughout, although the equations obtained make perfect sense over any field $k$ of characteristic different from 2.

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§ 2. Shioda–Inose structures and sandwiching

A classical example of K3-surfaces consists in Kummer surfaces: starting from an Abelian surface $A$, we take the quotient by inversion with respect to the group structure. This produces 16 rational double point singularities (type $A_1$), which can be resolved into a K3-surface, which we denote by $\text{Km}(A)$. Thus there is a rational map of degree 2

$$A \dasharrow \text{Km}(A).$$

This is also reflected in the transcendental lattices, that is, the orthogonal complements $T_X$ of $\text{NS}(X)$ inside $H^2(X, \mathbb{Z})$ with respect to the cup product. Namely the transcendental lattices are similar, that is, the rank is constant while the intersection form is multiplied by 2:

$$T_{\text{Km}(A)} = T_A(2).$$

From the point of view of classification, a natural problem is to relate $\text{Km}(A)$ to a K3-surface $X$ with the original transcendental lattice:

$$T_X = T_A.$$ 

This was first achieved by Shioda and Inose in [6] in the case of a product type $A \cong E \times E'$ where $E, E'$ are elliptic curves. Their construction (geared towards K3-surfaces with Picard number 20) makes crucial use of Jacobian elliptic fibrations on $\text{Km}(E \times E')$. In fact, $X$ is shown to admit a rational map to $\text{Km}(E \times E')$ of degree 2. In other words, $X$ admits a Nikulin involution (8 isolated fixed points), the quotient by which gives rise to $\text{Km}(E \times E')$ as its resolution. Following Morrison [7], this is exactly what is usually required for a Shioda–Inose structure of an arbitrary complex Abelian surface $A$ and a complex K3-surface $X$:

\[
\begin{array}{c}
A \\
\text{Km}(A) \\
X \\
\end{array}
\]

(1)

In the situation of non-isogenous elliptic curves, one has $\text{NS}(A) = U$ and $T_A = U^2$, where $U$ denotes the hyperbolic plane $\mathbb{Z}^2$ with intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The generic situation is somewhat different as $A$ will have Picard number 1. If $A$ is endowed with a polarisation of degree $2N$ ($N \in \mathbb{N}$), then we have

$$T_A = U^2 + (-2N).$$

Morrison gave a complete answer in terms of lattice theory as to which K3-surfaces admit a Shioda–Inose structure [7].

**Theorem 2** (Morrison). An algebraic K3 surface $X$ admits a Shioda–Inose structure if and only if there is a primitive embedding

$$T_X \hookrightarrow U^2 + (-2N)$$

for some $N \in \mathbb{N}$. 
An equivalent criterion is that $X$ admits a (Nikulin) involution interchanging two orthogonal copies of $E_8$ in $\text{NS}(X)$, the unique unimodular even negative-definite lattice of rank 8. Or even more abstractly: $E_8^2 \hookrightarrow \text{NS}(X)$.

In order to discuss sandwiching, we return to the product-type situation $A = E \times E'$ in [6] mentioned above. Shioda noticed [1] that this case comes automatically with a sandwich. Namely, $\text{Km}(E \times E')$ itself possesses a Nikulin involution which gives rise to $X$, thus extending the diagram (1) as follows when $A = E \times E'$:

For general $A$, this brings us to the problem of whether every Shioda–Inose structure can be extended to a sandwich. An affirmative answer was given recently by Ma [2].

**Theorem 3** (Ma). Any Shioda–Inose structure admits a sandwich.

We emphasise that Ma’s proof is Hodge-theoretic in nature; in particular it does not give any geometric information. Indeed, there are only five $N \in \mathbb{N}$ in the notation of Theorem 2 for which an explicit geometric construction has been exhibited: the case $N = 1$ is due to Kumar [3] and corresponds to Kummer surfaces of curves of genus 2; the cases $N = 2$ [4] and $N = 3, 5, 7$ [5] correspond to elliptic K3-surfaces with MW-rank zero.

In each case the quotients are constructed using elliptic fibrations with a 2-torsion section. By means of the generic fibre, $X$ and $\text{Km}(A)$ can be interpreted as elliptic curves over the function field $k(t)$. Then sections translate into $k(t)$-rational points on the generic fibre; in particular, 2-torsion sections correspond to points of order 2, which thus induce isogenies of degree 2 of the generic fibres and rational maps of degree 2 between the associated elliptic surfaces. This will also be our preferred approach in what follows. Namely, we will construct 3-dimensional families of elliptic fibrations with MW-rank 1 and 2-torsion sections exhibiting a Shioda–Inose structure. As stated in Theorem 1, this construction enables us to realise an infinite series of families of K3-surfaces with sandwich structure, including the five families already known. For the two cases of small $N$ missing from the above list of examples ($N = 4, 8$), we provide explicit equations for the families.

§ 3. The first series

Our starting point is a 4-dimensional family $\mathcal{X}$ of K3-surfaces whose generic member $X$ has

$$\text{NS}(X) = U + E_7^2.$$
This family, which has also been recently investigated in [8], can be given as an elliptic fibration with two singular fibres of Kodaira type $\text{III}^*$ at 0 and $\infty$:

$$X: y^2 = x^3 + t^3 a(t)x + t^5 b(t),$$

where $a(t), b(t) \in k[t]$ have degree 2.

Here one can still rescale $(x, y)$ and $t$ separately to normalise two coefficients. The hyperbolic plane $U \subset \text{NS}(X)$ is spanned by the zero section $O$ and the general fibre $F$, while the $E_7$ comprise fibre components disjoint from $O$, that is, we only omit the identity component of each fibre.

This family is a natural starting point since it specialises to Kumar’s family with $\text{NS} = U + E_7 + E_8$ by setting, for example, $\deg(a) \leq 1$. The transcendental lattice of $X$ can be computed using Nikulin’s theory of the discriminant form (or as a 2-elementary lattice):

$$T_X = U^2 + A_1^2.$$  

Here we consider the dual lattice $L^\vee$ of a non-degenerate even integral lattice $L$. It gives rise to the discriminant group $L^\vee/L$, a finite Abelian group whose order is the absolute value of the discriminant of $L$. The discriminant group comes with a quadratic form induced from $L$, which is called the discriminant form and denoted by

$$q_L: L^\vee/L \to \mathbb{Q}/2\mathbb{Z}.$$  

We have isomorphisms of Abelian groups

$$E_7^\vee/E_7 \cong A_1^\vee/A_1 \cong \mathbb{Z}/2\mathbb{Z}.$$  

In fact there are generators with squares $-3/2$ (resp. $-1/2$) which give a direct identification,

$$q_{E_7} \cong -q_{A_1}.$$  

### 3.1. Lattice enhancement

A convenient way to specialise our 4-dimensional family of K3-surfaces to a subfamily of Picard number $\rho \geq 17$ is to enhance $L = \text{NS}$ by some vector $v$ of $T_X$ of negative square. In this context the only subtlety appears in the primitive closure in the K3-lattice $\Lambda$:

$$L' := \langle L, v \rangle \subset \Lambda = H^2(X, \mathbb{Z}) = U^3 + E_8^2.$$  

Then the theory of lattice-polarised K3-surfaces guarantees that $L'$ corresponds to a 3-dimensional (sub)family consisting of K3-surfaces with $\text{NS} = L'$ generically and transcendental lattice $T' = v^\perp \subset T_X$. Since we are interested in transcendental lattices containing two copies of $U$ (see Theorem 2), we will enhance $L$ with a vector from $A_1^2$ throughout.

**Example 4.** Take the vector $v = (1, 0) \in A_1^2$, that is, a generator of a single $A_1$. Via the isomorphism (5), $v/2$ corresponds to a generator $w$ of $E_7^\vee/E_7$. Thus we obtain

$$L' := \langle L, v \rangle = \langle L, v/2 + w \rangle.$$  

In fact, we have just glued together one copy each of $E_7$ and $A_1$ to the unimodular lattice $E_8$, and so $L' = U + E_7 + E_8$. 
Since lattice enhancements involve the primitive closure, we have to consider which integers \( A_1^2 \) represent primitively, that is, by a vector \( v = (v_1, v_2) \) with \( \gcd(v_1, v_2) = 1 \). The following lemma is an easy exercise in quadratic forms.

**Lemma 5.** \( A_2^2 \) represents \(-2N\) primitively if and only if \( N \) is a product of primes \( p \equiv 1 \pmod{4} \) or twice such a product.

We have already discussed the lattice enhancement for \( N = 1 \) in Example 4. The other cases correspond to \( v_1, v_2 \neq 0 \) and not both even. Let \( w_1, w_2 \) denote the generators of \( E_7^\vee / E_7 \) of square \(-3/2\), matching those for the \( A_1 \)'s via (5). Then we find that

\[
L' := \langle L, v' \rangle, \quad v' = \frac{v}{2} + \begin{cases} w_1, & v_1 \text{ odd}, \\ 0, & v_1 \text{ even} \end{cases} + \begin{cases} w_2, & v_2 \text{ odd}, \\ 0, & v_2 \text{ even} \end{cases}
\]

Note that in our set-up \( v' \) is indeed an integral even vector:

\[
2\mathbb{Z} \ni (v')^2 = -\frac{N}{2} - \begin{cases} \frac{3}{2}, & \text{if } N \text{ is odd}, \\ 3, & \text{if } N \text{ is even}. \end{cases}
\]

Recall that by the general theory of elliptic surfaces with section, NS is generated by fibre components and sections. If \( N = 1 \), then the lattice enhancement causes a degeneration of singular fibres as studied in Example 4. If \( N \neq 1 \), then the singular fibres do not degenerate, and so the additional algebraic class \( v' \) can be identified directly with a section \( P \) made orthogonal to \( O \) and \( F \) in NS while meeting one or both of the III* fibres in the non-identity component, depending on the parities of \( v_1 \) and \( v_2 \). In terms of the theory of Mordell–Weil lattices [9], the contributions to \( v'^2 \) from \( w_1, w_2 \) correspond to correction terms for those fibre components. Thus, \( P \) has height \( h(P) = N/2 \).

### 3.2. The transcendental lattice.

By construction, the subfamily with \( L' \hookrightarrow \text{NS} \) generically has transcendental lattice \( T' = U^2 + \langle -2N \rangle \).

### 3.3. An alternative elliptic fibration.

We shall now switch to an alternative elliptic fibration on \( X \) which comes with a 2-torsion section. For this purpose we identify a divisor \( D \) of Kodaira type \( I_8^* \) supported on \( O \) and on the components of the III* fibres as depicted in Fig. 1.

The linear system of this divisor \( D \) will induce another elliptic fibration \( X \to \mathbb{P}^1 \). Here the rational curves adjacent to \( D \) will serve as the zero section and 2-torsion section. The latter claim is easily verified by means of the height pairing after realising that generically the two remaining components of the original III* fibres (disjoint from \( I_8^* \)) sit in two fibres of type \( I_2 \). Apart from these three singular fibres, the new fibration generically has six fibres of type \( I_1 \).

The divisor \( D \) can be extracted explicitly from the parameter \( u = x/t^2 \) with respect to (3). Elementary transformations give the Weierstrass form

\[
X : y^2 = t(u^3t^2 + u a(t) + b(t)),
\]

an elliptic curve over \( k(u) \) with 2-torsion section \((0, 0)\). After lattice enhancement, the rational curve \( P \) defines a multisection for the alternative fibration which induces a section \( P' \), also of height \( h(P') = N/2 \).
3.4. Kummer surface. Consider the 2-isogenous elliptic surface $Y$ arising from the alternative elliptic fibration on $X$ by quotienting by translation by $(0,0)$ and resolving singularities. Generically $Y$ has singular fibres $I_4^*, \ 2 \times I_1, \ 6 \times I_2$ and a 2-torsion section. The transcendental lattice can be computed as $T_Y = (U(2))^2 + A_7^2$. We now consider the specialisation $Y'$ of $Y$ corresponding to the lattice enhancement of $X$ yielding the section $P$ (and $P'$).

Proposition 6. Let $N$ be an odd integer as in Lemma 5. Then $Y'$ is a Kummer surface with $T_{Y'} = T_X(2) = U(2)^2 + \langle -4N \rangle$.

Proof. The property of being a Kummer surface follows from the 2-divisibility of $T_{Y'}$, giving an even lattice of rank at least 17 as in Theorem 2. We shall now prove that the transcendental lattice takes exactly the shape as stated.

The enhanced section $P'$ pulls back to a section $P^*$ of $Y'$ of height $h(P^*) = 2h(P') = N$. We claim that this section is not 2-divisible in $\text{MW}(Y')$. For otherwise there would be a section $Q$ with $2Q = P^*$, so that $h(Q) = N/4$. But then the correction terms in the height pairing are all half-integers by inspection of the singular fibres present. Hence $h(Q) \in \frac{1}{2} \mathbb{Z}$, giving the required contradiction.

It follows that $\text{NS}(Y')$ is generated by fibre components, zero and 2-torsion sections together with $P^*$. In particular, we deduce from formula (22) in [10] that

$$\text{disc } \text{NS}(Y') = \frac{4 \times 2^6}{2^2} h(P^*) = 2^6 N. \quad (7)$$

Then we use that the push-forward induces an embedding

$$T_X(2) \hookrightarrow T_{Y'},$$

(which need not be primitive in general, but both lattices have the same rank by [11]; see [6], [7]). By (7) both lattices have the same discriminant, and hence they agree. \Box

Remark 7. If $N$ is even, then quite to the contrary the section $P^*$ becomes 2-divisible in $\text{MW}(Y')$ (by inspection of the 2-length; compare Remark 11). Hence...
the discriminants of $T_X(2)$ and $T_Y$, are not the same. In fact, the two cases of lattice enhancement in Lemma 5 are swapped: odd $N$ on $X$ corresponds to $2N$ on $Y$, and even $N$ on $X$ gives odd $N/2$ on $Y$.

3.5. Proof of Theorem 1, 1). Let $N$ be composed of primes $p \equiv 1 \pmod{4}$ as in Theorem 1, 1). Consider the family of K3-surfaces given by the lattice enhancement of $X$ corresponding to $N$. Then any $X'$ in this family with $\rho(X') = 17$ fits into a sandwiched Shioda–Inose structure by Proposition 6 using the 2-isogeny and its dual. Thus we only have to rule out the possibility that the construction degenerates for higher Picard number.

Note that every non-degenerate member of the family has the corresponding elliptic fibration with 2-torsion section (possibly with different singular fibres). It remains to show that the quotient always has 2-divisible transcendental lattice, regardless of the Picard number. This follows from the next lemma.

**Lemma 8.** Consider a family of K3-surfaces $(X, \iota)$ with Nikulin involution $\iota$. Assume that generically the resolution $Y$ of the quotient $X/\iota$ has transcendental lattice

$$T_Y = T_X(2).$$

Then the same applies to any non-degenerate specialisation of $(X, \iota)$.

**Proof.** Let $(X'', \iota')$ be a non-degenerate specialisation with quotient $Y'$. They have natural primitive embeddings

$$T_{X''} \hookrightarrow T_X, \quad T_{Y'} \hookrightarrow T_Y.$$

Using the push-forward via the quotient map and the hypothesis of the lemma, we obtain the commutative diagram

$$
\begin{array}{ccc}
T_{Y'} & \hookrightarrow & T_Y \\
\cup & & \| \\
T_{X''}(2) & \hookrightarrow & T_X(2)
\end{array}
$$

Considering the lower-right corner, we find that $T_{X''}(2)$ embeds primitively in $T_Y$. Arguing with the upper-left corner, we deduce the same for the inclusion $T_{X''}(2) \subset T_{Y'}$. Since these lattices have the same rank by [11], we deduce that $T_{Y'} = T_{X''}(2)$. □

This completes the proof of Theorem 1, 1).

3.6. Optimality of construction. The alert reader might wonder whether we might not be able to derive more K3-surfaces using our construction by not limiting ourselves to the summand $A_1^2$ of $T$ in Lemma 5. Indeed, the transcendental lattice $T$ can represent any integer (in many ways). However, recall that in essence we are aiming for primitive embeddings

$$U^2 + \langle -2N \rangle \hookrightarrow T. \quad (8)$$

We now explain why our construction does not give rise to any subfamilies with transcendental lattice other than those stated in Theorem 1. For this purpose, we first embed the summand $U^2$ in $T$. In general, the orthogonal complement
\( M = (U^2) \perp \subset T \) need not be unique, but the genus of \( M \) (that is, the isogeny class) is uniquely determined by \( T \). By class group theory, the genus of \( M \) consists of a single lattice. Hence, to study primitive embeddings (8) it does indeed suffice to consider representations of \(-2N\) by \( A_1^2 \). The same argument will go through verbatim for the other two constructions (see Lemmas 10, 13).

3.7. Other lattice enhancements. For the sake of completeness, we briefly comment on other lattice enhancements of the family \( \mathcal{X} \). This will also serve as a further check of the validity of the above argument. Analogous arguments apply to the K3-families in the next two sections.

Let \( X' \) be a K3-surface of Picard number 17 and discriminant \( 2N \ (N > 1) \) that arises from \( X \) by lattice enhancement. As before, \( X' \) has an additional section \( P \) of height \( N/2 \), say with respect to the original elliptic fibration. Consider the case when \( N \) is odd. Then \( P \) meets some non-identity components of the fibre, but the section \( 2P \) of height \( 2N \) does not. Let \( \varphi \) denote the orthogonal projection with respect to \( O, F \) in \( \text{NS}(X') \). By assumption, \( \varphi(2P) \) is orthogonal to the trivial lattice (that is, to the image of \( \text{NS}(X) \) in \( \text{NS}(X') \)), and we have

\[
\varphi(2P)^2 = -2N, \quad \varphi(2P)\varphi(P) = -N.
\]

In particular, we find that \( \varphi(2P)/N \in \text{NS}(X')^\vee \). In the discriminant group \( \text{NS}(X')^\vee / \text{NS}(X') \) this induces an element of order \( N \), where the discriminant form takes the value \(-2/N\).

Now assume that \( T(X') = U^2 + \langle -2N \rangle \) is one of the transcendental lattices in question. Its discriminant form also takes the value \(-2/N\) at an element of order \( N \). But then for \( \text{NS}(X') \) and \( T(X') \) to be orthogonal complements in the K3-lattice, we require that their discriminant forms have opposite signs. In particular, this implies that \(-1\) is a square modulo any prime divisor of \( N \). Thus we find exactly the conditions of Theorem 1.

For \( N \) even, we distinguish two further cases according to the parity of \( M = N/2 \). If \( N/2 \) is odd, then essentially the same argument as above goes through with the element \( \varphi(P)/M \in \text{NS}(X')^\vee \) of order \( N \) in the discriminant group. If \( N/2 \) is even, then \( P \) is in the narrow Mordell–Weil lattice, meeting all singular fibres at the identity component. Thus \( \text{NS}(X') = \text{NS}(X) + \langle \varphi(P) \rangle \), and the discriminant group has 2-length 3, which exceeds that of \( U^2 + \langle -2N \rangle \), a contradiction.

§ 4. The second series

In this section and the next, we shall argue directly with elliptic fibrations with 2-torsion section, this time semistable. Such fibrations admit an extended Weierstrass form

\[
y^2 = x(x^2 + a(t)x + b(t))
\]

with reducible fibres at the zeros of \( b(t) \). We treat two families which lend themselves directly to Shioda–Inose structures. In this section, we consider surfaces with singular fibres of Kodaira types \( I_{14} \) and \( I_2 \). The corresponding K3-surfaces form a 4-dimensional family of K3-surfaces

\[
X : y^2 = x(x^2 + a(t)x + t), \quad (9)
\]

where \( a \in k[t] \) has degree 4 and does not vanish at \( t = 0 \). Generically \( \rho(X) = 16 \).
4.1. The transcendental lattice. The transcendental lattice can be read off directly from an alternative elliptic fibration on $X$. Namely, it is easy to extract a divisor of Kodaira type $\Pi^*$ from $I_{14}$ extended by $O$ and the identity component of $I_2$. The remaining components of $I_{14}$ give rise to a section and an $A_6$-configuration of rational curves. Comparing ranks (and discriminants), we find generically that

$$\text{NS}(X) = U + A_6 + E_8, \quad T_X = U^2 + \begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}.$$  

The latter representation is easily verified using the discriminant form since $A_6^*/A_6$ has a generator of square $-6/7$.

4.2. The quotient family. Consider the quotient by translation by the 2-torsion section and denote the resulting elliptic K3-surfaces by $Y$. Then generically $Y$ has singular fibres $I_7$, $I_1$, $8 \times I_2$, and $\text{MW} \cong \mathbb{Z}/2\mathbb{Z}$.

**Lemma 9.** Generically $Y$ has transcendental lattice $T_Y \cong T_X(2)$.

**Proof.** By [11], $Y$ also has Picard number 16, and so the MW-rank is 0 by the Shioda–Tate formula. Standard formulae exclude any further torsion. Hence $\text{NS}(Y)$ is generated by fibre components and the two torsion sections; in particular, $\text{NS}(Y)$ has discriminant $2^{67}$. Then, as before, the push-forward embedding $T_X(2) \hookrightarrow T_Y$ is an isometry. □

4.3. Lattice enhancement. As before we enhance $\text{NS}(X)$ by a vector from the last summand of $T_X$. Generically we find the following possibilities.

**Lemma 10.** $\begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$ represents $-2N$ primitively if and only if $N$ is a product of primes $p \equiv 1, 2, 4 \pmod{7}$ or seven times such a product.

Now pick a vector $v$ as in Lemma 10 and enhance $\text{NS}$ by a generator of $v^\perp$ in the above lattice of rank 2. We infer the transcendental lattice

$$T' = U^2 + \mathbb{Z}v = U^2 + \langle -2N \rangle.$$  

On the elliptic fibrations, this implies an additional singular fibre ($N = 1$) or a section $P$ of height $2N/7$ ($N > 1$).

4.4. Proof of Theorem 1, 2). Let $N$ be as in Theorem 1, 2) (or equivalently Lemma 10). Let $X'$ be a member of the subfamily of K3-surfaces given by the lattice enhancement of $X$ corresponding to $N$ as above. Then by assumption, $T_{X'}$ embeds primitively in $U^2 + \langle -2N \rangle$. Moreover, the quotient surface has transcendental lattice $T_{X'}(2)$ by Lemma 8 in conjunction with Lemma 10. Using the 2-isogeny and its dual, these surfaces realise a sandwiched Shioda–Inose structure. □

**Remark** 11. The above construction implies that the image of the section $P$ on the quotient surface becomes 2-divisible. This fact can also be checked directly on the quotient surface by an argument comparing the 2-length of $\text{NS}$ to the rank of the transcendental lattice.
4.5. Example: \( N = 4 \). We need to endow the elliptic fibration (9) with a section of height \( 8/7 \). Up to translation by the 2-torsion section and inversion, this is uniquely achieved by a section \( P \) meeting \( I_{14} \) at \( \Theta_4 \) (the fourth component) and \( I_2 \) at the identity component, but not intersecting \( O \). Indeed the height pairing gives

\[
h(P) = 4 - \frac{4 \cdot 10}{14} = \frac{8}{7}.
\]

Because of the extended Weierstrass form (9), \( P \) can only take the shape \((\alpha, w)\) with \( \alpha \in k^* \) and \( w \in k[t] \) of degree 2. But then the pair \((\alpha, w)\) uniquely determines the polynomial \( a(t) \) in (9) since

\[
w^2 = \alpha^3 + \alpha^2 a(t) + \alpha t.
\]

Hence we obtain the 3-dimensional family of K3-surfaces with generically \( T = U^2 + \langle -8 \rangle \) (and the unirationality of the corresponding moduli space).

§ 5. The third series

As the third series we take elliptic K3-surfaces with 2-torsion section and singular fibres of types \( I_{10} \) and \( I_6 \). Here the Weierstrass form can be transformed into

\[
X : y^2 = x(x^2 + a(t)x + t^3),
\]

again yielding a 4-dimensional family with \( \rho = 16 \) generically. The arguments are very similar to those in the previous case, and so we only outline the main ideas.

5.1. The transcendental lattice. Using the discriminant form one finds that the transcendental lattice is generically

\[
T_X = U^2 + \begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}.
\]

The computations can be eased by switching to another elliptic fibration, for instance with fibres of types \( III^* \) and \( IV^* \) and a section of height \( 5/2 \) (elliptic parameter \( u = x/t \)). This gives a representation of \( q_{\text{NS}(X)} \) as \( \mathbb{Z}/3\mathbb{Z}(-4/3) + \mathbb{Z}/5\mathbb{Z}(-2/5) \), which agrees up to sign with \( q_{T_X} \).

5.2. The quotient family. Let \( Y \) denote the quotients of the elliptic surfaces with 2-torsion section. Then generically, \( Y \) has singular fibres \( I_5, I_3, 8 \times I_2 \), and \( \text{MW} \cong \mathbb{Z}/2\mathbb{Z} \).

Lemma 12. Generically, \( Y \) has transcendental lattice \( T_Y \cong T_X(2) \).

Proof. By inspection of the singular fibres and torsion section, \( \text{NS}(Y) \) has discriminant \( 2^615 \). Then, as before, the push-forward embedding \( T_X(2) \hookrightarrow T_Y \) is an isometry. □

5.3. Lattice enhancement. We continue by enhancing \( \text{NS}(X) \) by a vector from the last summand of \( T_X \). The analysis of the possible numbers which are represented primitively is a little more delicate. One can already observe this from the fact that each of 2, 3 and 5 is represented, but not 6, 10 or 15. This is related to the fact that the corresponding quadratic form is the 2-torsion class in the class group \( \text{Cl}(-15) \). This explains why a parity condition enters for the representations.
Lemma 13. An integer \(-2N\) is represented primitively by \((-4, -1, -1, -4)\) if and only if
1) \(N\) is a product of an odd number of primes \(p \equiv 2, 8 \pmod{15}\) or 15 times such a product, or
2) \(N\) is 3 or 5 times a product of an even number of primes \(p \equiv 2, 8 \pmod{15}\).

Enhancing NS by a primitive vector perpendicular to the vector from the lemma gives the transcendental lattice
\[ T' = U^2 + \langle -2N \rangle. \]

The elliptic fibration is thus endowed with a section \(P\) of height \(2N/15\).

5.4. Proof of Theorem 1, 3. Let \(N\) be as in Theorem 1, 3) (or equivalently Lemma 13). Let \(X'\) be a member of the subfamily of K3-surfaces given by the lattice enhancement of \(X\) corresponding to \(N\) as above. Then by assumption, \(T_{X'}\) embeds primitively in \(U^2 + \langle -2N \rangle\). Moreover, the quotient surface has transcendental lattice \(T_{X'}(2)\) by Lemma 8 in conjunction with Lemma 13. Thus these surfaces realise a sandwiched Shioda–Inose structure. □

5.5. Example: \(N = 8\). The elliptic fibration (10) must be equipped with a section \(P\) of height 16/15. By the height pairing it suffices that \(P\) meet both \(I_{10}\) and \(I_6\) at \(\Theta_2\) (the second component) while not intersecting \(O\) because
\[ h(P) = 4 - \frac{2 \cdot 8}{10} - \frac{2 \cdot 4}{6} = \frac{16}{15}. \]

The Weierstrass form (10) shows that the shape of \(P\) is \((\alpha t^2, wt^2)\) with \(\alpha \in k^*\) and \(w \in k[t]\) of degree 2. But then the pair \((\alpha, w)\) uniquely determines the polynomial \(a(t)\) in (10) since
\[ w^2 = \alpha^3 t^2 + \alpha^2 a(t) + \alpha t. \]

Hence we obtain the 3-dimensional family of K3-surfaces with generically \(T = U^2 + \langle -16 \rangle\) (and the unirationality of the corresponding moduli space).

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