Images of the Dark Soliton in a Depleted Condensate

Jacek Dziarmaga, Zbyszek P. Karkuszewski, and Krzysztof Sacha

Instytut Fizyki Uniwersytetu Jagiellońskiego,
ul. Reymonta 4, 30-059 Kraków, Poland

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The dark soliton created in a Bose-Einstein condensate becomes grey in course of time evolution because its notch fills up with depleted atoms. This is the result of quantum mechanical calculations which describes output of many experimental repetitions of creation of the stationary soliton, and its time evolution terminated by a destructive density measurement. However, such a description is not suitable to predict the outcome of a single realization of the experiment were two extreme scenarios and many combinations thereof are possible: one will see (1) a displaced dark soliton without any atoms in the notch, but with a randomly displaced position, or (2) a grey soliton with a fixed position, but a random number of atoms filling its notch. In either case the average over many realizations will reproduce the mentioned quantum mechanical result. In this paper we use $N$-particle wavefunctions, which follow from the number-conserving Bogoliubov theory, to settle this issue.

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I. INTRODUCTION

Let us start with a trivial but important remark on measurement in quantum mechanics: in general a single measurement on an $N$-particle system is neither equivalent to many measurements on the $N$-particle system, nor to $N$ measurements on $N$ one-particle systems. The only exception is the case of a product $N$-particle state where all particles are described by the same single-particle wavefunction. Not surprisingly the particles must be bosons forming a Bose-Einstein condensate, and a measurement on this $N$-particle state can be replaced with $N$ measurements on $N$ one-particle systems in the same single-particle wavefunction.

The importance of this "nonequivalence" principle can be seen in the explanation of an interference experiment Ref.[1] in an excellent paper by Javanainen and Yoo [2], and then elaborated in Refs.[3]. In Ref.[2] the authors consider a Fock state

$$\left| N \atop N \right>$$

(1)

with $N/2$ atoms in a condensate of wavefunction $\phi_0(x)$ and $N/2$ in a condensate with an orthogonal wavefunction $\phi_1(x)$. A single particle density matrix of the state $\ket{\text{1}}$ predicts that on average a density measurement will give a distribution of atoms

$$\frac{N}{2} \left( |\phi_0(x)|^2 + |\phi_1(x)|^2 \right)$$

(2)

without any interference fringes between the two condensates. This single particle distribution is an average over many density measurements on the same Fock state $\ket{\text{1}}$. Simulation of a destructive density measurement $\ket{\text{2}}$ based on the full $N$-particle wavefunction $\ket{\text{1}}$, and the actual experiment $\ket{\text{4}}$, give distributions with an interference term

$$\frac{N}{2} \left( |\phi_0(x)|^2 + |\phi_1(x)|^2 + \phi_0(x)\phi_1^*(x)e^{i\varphi} + \text{c.c.} \right).$$

(3)

For a given experimental realization the phase $\varphi$ is chosen randomly from the interval $[0, 2\pi]$. The outcome of a density measurement is the same as if before the measurement all $N$ atoms were not in the state $\ket{\text{1}}$ but in an absurd single condensate with a wavefunction

$$\frac{1}{\sqrt{2}} \left( \phi_0 + \phi_1 e^{i\varphi} \right)$$

(4)

with a relative phase $\varphi$ that is not known before we actually measure the density distribution. It turns out [5] that it is enough to measure only a small fraction of the large total $N$ to prepare the remaining atoms in one of the condensate wavefunctions $\ket{\text{1}}$, or, in other words, to establish a definite phase $\varphi$. Destructive measurement of atomic positions, which effectively annihilates measured atoms from the trap, drives the remaining atoms into a condensate state with a randomly picked condensate wavefunction. These interesting conclusions cannot be obtained without full knowledge of the $N$-particle state $\ket{\text{4}}$: the single particle density distribution $\ket{\text{4}}$ is a (misleading) average over many experiments with different outcomes $\varphi$. To predict possible outcomes of individual measurements on an $N$-particle state it is essential to know its $N$-particle wavefunction.

A perfect condensate is a state where all $N$ atoms are in the same single particle state described by a single particle wavefunction $\phi_0(\vec{x})$,

$$\left( a_0^\dagger \right)^N \ket{\text{0}}.$$

(5)

When we include interactions between atoms but neglect depletion of atoms from the condensate wavefunction, then $\phi_0$ is a solution of the celebrated Gross-Pitaevskii
equation \[4\]. Including small quantum depletion in the framework of the number-conserving Bogoliubov theory \[8\] leads to pair-correlated eigenstates \[9\]. For example, the state without any quasiparticles (the Bogoliubov vacuum) has a pair-correlated form

\[
|0\rangle = \left(\hat{a}^+_k\hat{a}_k + \sum_{k=1}^\infty \lambda_k\hat{a}^+_k\hat{a}_k\right)\frac{x}{N} |0\rangle .
\] (6)

Operators \(\hat{a}_k\) annihilate in a basis of functions \(\phi_k\) orthogonal to the condensate wavefunction \(\phi_0\). This general pair-correlated ansatz \[9\] has been known for a while, see e.g. the review by Leggett \[7\], but a general solution for the coefficients \(\lambda_k\) and corresponding wavefunctions \(\phi_k\) has been found only very recently in Ref. \[8\] (the solution in the special case of a uniform condensate was given in Ref. \[7\]). The two-particle creation operator \(d^\dagger\) has to commute with all quasiparticle annihilation operators (see Appendix A)

\[
[b_m, d^\dagger] = 0, \text{ for all } m.
\] (7)

The \(N\)-particle state \[10\] is a foundation on which one can build theory of BEC entirely in the language of \(N\)-particle wavefunctions.

In our two recent papers \[8, 9\] we studied depletion from a condensate with a dark soliton. The dark soliton state is a collectively excited condensate with a notch in the condensate wavefunction. In the Thomas-Fermi limit of strong interactions the notch behaves like a soliton with a wavefunction close to the notch proportional to

\[
\phi_0(x) \sim \tanh(x - X),
\] (8)

where \(X\) is a position of the soliton measured in units of the healing length. The anomalous Bogoliubov mode with a wavefunction

\[
\phi_1(x) \sim -\partial_X \phi_0(x) = \cosh^{-2}(x - X)
\] (9)
dominates quantum depletion from the condensate within the soliton. In Ref. \[8\] we found an average number of atoms \(dN\) depleted into the non-condensate mode \(\phi_1\) and a single particle density distribution which close to the soliton at \(X = 0\) looks like

\[
p_1(x) = (N - dN)|\phi_0(x)|^2 + dN|\phi_1(x)|^2.
\] (10)

For \(dN = 0\) the distribution has a hole at \(X = 0\), but with increasing quantum depletion \(dN\) the hole is filling with atoms. This single particle result means that on average over many experiments the notch of a depleted dark soliton appears filled with atoms (it appears grey, not dark), but it gives no clue what to expect as an outcome of a single destructive density measurement. We address this issue in the following sections.

The paper is organized as follows. In Sections IV and V we apply the number conserving Bogoliubov theory (shortly summarized in Appendix A) to the dark soliton, and simulate density measurement on different \(N\)-particle quantum states that all may pass under the same label: “depleted dark soliton”. We find that images of the “depleted dark soliton” depend qualitatively on the actual \(N\)-particle state. To simulate the measurements we adapted the numerical algorithm of Ref. \[2\] which is described in Appendix B. We conclude in Section VI.

II. STATIONARY DARK SOLITON

The dark soliton state \[10\] of a condensate in a 1D harmonic trap can be defined as an antisymmetric solution of the stationary Gross-Pitaevskii equation

\[
-\frac{1}{2}\partial_x^2 \phi_0 + \frac{1}{2}x^2 \phi_0 + g|\phi_0|^2 \phi_0 = \mu \phi_0
\] (11)

In this paper we use the dimensionless oscillator units. In the Thomas-Fermi (TF) limit of \(g \gg 1\) the dark soliton state is

\[
\phi_0(x) \approx F(x) \tanh(x/l_0),
\] (12)

with

\[
F(x) = \sqrt{\frac{2\mu - x^2}{2g}},
\] (13)

where the chemical potential \(\mu \approx (3g/2)^{2/3}/2\) and the healing length \(l_0 \approx (2/3g)^{1/3}\).

A perfect condensate with all \(N\) atoms in the dark soliton mode \(\phi_0\) is not an energy eigenstate because interactions between atoms continuously deplete atoms from the condensate wavefunction \(\phi_0\). The Bogoliubov theory is a systematic way to describe small quantum fluctuations around the Gross-Pitaevskii solution, and to find stationary states of a depleted condensate.

Solution of the Bogoliubov-de Gennes (BdG) equations (see Appendix A) reveals many different modes of the quasiparticle spectrum. However, only the negative energy (anomalous) mode \[9\] has a wavefunction localized in the soliton notch \[3, 4, 11\]. From all modes the anomalous mode will contribute the most to the density of incoherent atoms in the soliton notch. In the following, similarly as in Ref. \[8\], we truncate the Bogoliubov spectrum to the anomalous mode alone and look for the Bogoliubov vacuum state in the form

\[
|0_k : N\rangle \sim (\hat{a}^+_N + \lambda a^+_1)^N |0\rangle
\] (14)

with \(a^+_1\) creating in the mode \(\phi_1\). The anomalous mode solution of the BdG equations \[8, 9\] with \(\omega = \frac{\sqrt{3g}}{\sqrt{5}}\) is

\[
u_1 = \frac{1}{2} (f_+ + f_-), \quad v_1 = \frac{1}{2} (f_+ - f_-),
\] (15)

with functions

\[
f_+(x) = \frac{\sqrt{3g}}{\sqrt{5} \cosh^2\left(\frac{x}{l_0}\right)}, \quad f_-(x) = \sqrt{\frac{2}{3}} F(x),
\] (16)
see Ref. [3]. For $g \gg 1$ equations (15) are dominated by $f_+\sim |f_+|f_-$, and the two functions become the same, $u_1 \approx v_1 \approx \frac{1}{2} f_+$. As a result in the TF limit the anomalous mode does not mix with other modes and the dominant eigenfunction is

$$\phi_1(x) = \frac{f_+(x)}{\sqrt{|f_+|^2 + |f_-|^2}},$$

as has been verified numerically in Ref. [3]. To find the asymptotic behavior of $\lambda$ for $g \gg 1$ we have to be more careful and still keep $f_-$ for a moment. The eigenvalue can be obtained from the general prescription presented in Appendix A and it is

$$\lambda \approx \frac{\langle f_+|f_+ \rangle - \langle f_+|f_- \rangle}{\langle f_+|f_+ \rangle + \langle f_+|f_- \rangle} \approx 1 - \frac{27/6}{g^{2/3}} \lambda^{1/3}. \quad (18)$$

Knowing $\lambda$ we can find an average number of depleted atoms $dN$ in the Bogoliubov vacuum state (14). In the particle Fock representation the unnormalized vacuum state (14) is a sum

$$|0_b : N\rangle \sim \sum_{k=0}^{\infty} \frac{1}{\lambda^k} \frac{\sqrt{(N-2k)!}}{(N/2-k)!} \frac{(2k)!}{k!} |N-2k, 2k\rangle. \quad (19)$$

For depletion greater than unity but small with respect to the total number of atoms $N$, i.e. $1 \ll dN \ll N$, we may use the Stirling formula, and then replace the sum by an integral. As a result we find expressions for $dN$ and its dispersion $D(dN)$

$$dN \approx -\frac{1}{\ln \lambda} \approx \frac{g^{2/3}}{4 \sqrt{6} \lambda^{1/3}}, \quad (20)$$

$$D(dN) = dN. \quad (21)$$

The single particle density distribution in the vacuum state (14) is

$$p_1(x) = (N - dN)|\phi_0(x)|^2 + dN|\phi_1(x)|^2. \quad (22)$$

The first term on the RHS is the dark soliton density profile with a hole at $x = 0$, while the second one stands for the depleted atoms localized in the soliton notch. A good measure of depletion is a ratio of atomic density in the notch at $x = 0$ to the density near the notch

$$\frac{dN|\phi_1(0)|^2}{NF(0)^2} \sim N^{1/3}, \quad (23)$$

suggesting that for sufficiently large number of atoms, the notch will not be visible. For the parameters of the Hannover experiment [12], where $g \approx 7500$ and $N = 1.5 \times 10^5$, $dN \approx 60$ atoms and the ratio is 15%.

However, Eq. (22) predicts an average over many experimental realizations. For a single experiment one has to choose randomly positions of $N$ atoms according to the $N$-particle probability distribution. In general this is rather a formidable task, but, as shown by Javanainen and Yoo [2] (see Appendix B), for a state spanned on only two modes, and with a sequential algorithm that chooses a position of $n+1$-st atom after positions of $n$ atoms have already been “measured”, this problem is polynomial in $N$.

In general we expect that a single measurement of a depleted dark soliton will show the same atomic density distribution as if all $N$ atoms were initially prepared in a condensate with one of wavefunctions

$$\phi_0(x) - \phi_1(x) q e^{i\varphi}. \quad (24)$$

The real parameters $(q, \varphi)$ are chosen at random and fluctuate from experiment to experiment. In the special case of $\varphi = 0, \pi$ the condensate wavefunction (24) is simply a displaced dark soliton

$$\phi_0(x) - \phi_1(x) q \approx F(x) \tanh \left( \frac{x - q \sqrt{3m_0}}{l_0} \right). \quad (25)$$

In general, when $\varphi \neq 0, \pi$, the condensate wavefunction (24) gives a density distribution

$$N|\phi_0 - \phi_1 q e^{i\varphi}\rangle^2 = N|\phi_0 - \phi_1 q e^{i\varphi}\rangle^2 + N\phi_1^2 q^2 \sin^2 \varphi. \quad (26)$$

For small $q$ this distribution can be interpreted as a soliton shifted to $q \sqrt{3m_0} \cos \varphi$, but with a nonzero atomic density $N\phi_1^2 q^2 \sin^2 \varphi$ in the notch. For $\varphi \neq 0, \pi$ the soliton appears as a displaced grey soliton.
Bogoliubov vacuum state $|1415\rangle$. A generic outcome is a displaced dark soliton shown in Fig. 1. In Fig. 2 we show how definite values of $\cos(\varphi)$ and $q$ are established in the course of measurement of subsequent atomic positions. It is enough to measure only a small fraction of atoms to suppress fluctuations of $\varphi$ and $q$, see Fig. 2.

![Diagram](image)

**FIG. 2: Values of $\cos \varphi$ (upper plot) and $q$ during the density measurement performed on the state $|0_b : N\rangle$.**

A density measurement of the Bogoliubov vacuum state $|1415\rangle$ always gives a displaced dark soliton with $\varphi = 0$ in Eq. (24). At first sight this may seem obvious because the wavefunctions $\phi_0$ and $\phi_1$ are real, and all amplitudes in the state $|15\rangle$ are also real. However, this simple argument does not take into account the possibility that the measurement of a small fraction of atoms may localize the state of remaining atoms not in a single condensate $|14\rangle$, but in a superposition of several condensates $|3\rangle$ with different parameters $(q, \varphi)$. Indeed, as we can see in Eq. (26), density distributions for $(q, +\varphi)$ and $(q, -\varphi)$ are the same so the density measurement cannot distinguish $+\varphi$ from $-\varphi$. A real superposition of two condensates with opposite phases has real amplitudes, just like the state $|13\rangle$. We checked that a measurement of a Fock state $|N - n, n\rangle$, instead of the Bogoliubov vacuum state $|1415\rangle$, indeed results in a real superposition of two condensates with nonzero $+\varphi$ and $-\varphi$, which appears as a displaced grey soliton. What makes a difference is the fact the Bogoliubov Hamiltonian $|15\rangle$ is a superposition over many Fock states. It is the phase coherence between different Fock states, all with real amplitudes, that enforces the observed $\varphi = 0$.

### III. SOLITON CONDENSATE WITHOUT INITIAL DEPLETION

In current experiments $|12\rangle$ the dark solitons are generated with the help of the phase imprinting $|14\rangle$. As we argued in Ref. $|8\rangle$, when the depletion in the condensate before the imprinting is negligible, then the soliton state right after imprinting can be idealized by the state without any atoms depleted from the solitonic condensate $\phi_0$ to the anomalous mode $\phi_1$.

$$|\Psi(t = 0)\rangle = |N, 0\rangle.$$  \hspace{1cm} (27)

This state is not a stationary state. To obtain time evolution of this state we consequently truncate the Bogoliubov Hamiltonian to the anomalous mode where

$$H = -\frac{1}{\sqrt{2}} \hat{b}_1^\dagger \hat{b}_1$$ \hspace{1cm} (28)

with

$$\hat{b}_1 = \frac{\hat{a}_0^\dagger \hat{a}_1 - \lambda \hat{a}_0 \hat{a}_1^\dagger}{\sqrt{N(1 - \lambda^2)}}.$$ \hspace{1cm} (29)

With this Hamiltonian the initial perfect condensate $|\Psi(t = 0)\rangle$ evolves into a depleted state $|\Psi(t)\rangle$. In Fig. 3 we show how a single particle density evolves in time. After about 15 ms (for the parameters of the Hannover experiment $|12\rangle$) the hole in the single particle density distribution fills up with depleted atoms.

![Diagram](image)

**FIG. 3: Single particle densities of the state $|\Psi(t)\rangle$, Eq. (24), at different instants of time. The values of the parameters correspond to the Hannover experiment $|12\rangle$, where $g \approx 7500$ and $N = 1.5 \times 10^5$.**

We simulated single realizations of destructive density measurements of the state $|\Psi(t)\rangle$. A generic density distribution after evolution for 10 ms is shown in Fig. 4. We see that the soliton is dark and it is displaced with respect to the trap center. Inspection of the state after 10 ms shows that $\langle N - k, |\Psi(10\ms)\rangle \sim e^{ik\varphi}$ with a tiny $\varphi \approx 0.01$. In principle the time evolution might generate any phase $\varphi \in [0, 2\pi)$, but the parameters of the experiment $|12\rangle$ are such that there is not enough time to get any more significant $\varphi$ from the idealized initial state $|27\rangle$ before the linearized Bogoliubov theory breaks down.
FIG. 4: The same as Fig. 1 but for the state $\Psi(t = 0.15)$.

FIG. 5: The same as Fig. 1 but for the state $|\text{rand}\rangle$.

FIG. 6: The same as Fig. 1 but for the state $|\text{rand}\rangle$ after evolution over 0.7 ms.

These two examples demonstrate that phase disorder in particle Fock representation can change qualitatively possible results of density measurements.

V. CONCLUSION

In Refs. [8, 9] we have considered influence of quantum fluctuations on a density profile of a Bose-Einstein condensate excited to a solitonic state. We have shown that the soliton notch may be filled with atoms that are depleted from the condensate due to interactions between particles. The analysis has been performed by calculating a single particle density matrix. However, such a matrix provides information about the density profile that is a result of average over many experimental realizations. To obtain predictions for a single experimental
realization one has to choose randomly positions of atoms according to the multiparticle probability density.

In this paper we concentrate on the analysis of outcome of the destructive density measurement in a single experiment. Applying the number conserving Bogoliubov theory of a BEC we calculate N-particle quantum states of a depleted condensate with a dark soliton. The N-particle wavefunctions are used to see what happens after the depleted condensate is subject to an ideal depletion to the notch of the dark soliton gives a unique opportunity to detect quantum fluctuations simply by density measurement. A dark soliton in a strictly one-dimensional trap is well within the reach of present technology.

**APPENDIX**

A. The N-particle State of BEC

Here we briefly summarize main conclusions of the number-conserving Bogoliubov theory of BEC [1, 2]. A field operator $\hat{\psi}(\vec{x})$ is split into a condensate part $\hat{a}_0 \phi_0(\vec{x})$ and a non-condensate part $\delta \hat{\psi}(\vec{x})$,

$$\hat{\psi}(\vec{x}) = \phi_0(\vec{x}) \hat{a}_0 + \delta \hat{\psi}(\vec{x}).$$

To zero order in the “small” fluctuation operator $\delta \hat{\psi}(\vec{x})$ the N-particle state is a perfect condensate

$$\left( \hat{a}_0^\dagger \right)^N | 0 \rangle$$

with all N atoms in the condensate wavefunction $\phi_0(\vec{x})$. The $\phi_0$ solves a stationary Gross-Pitaevskii equation (GPE)

$$\mu \phi_0 = -\frac{1}{2} \nabla^2 \phi_0 + V(\vec{x}) \phi_0 + g | \phi_0^2 | \phi_0.$$  \hspace{1cm} (33)

To zero order the interaction affects only the shape of the condensate wavefunction $\phi_0(\vec{x})$ through the nonlinear term $g | \phi_0^2 | \phi_0$ in the GPE.

Quantum depletion from this condensate shows up in the second order of the perturbation theory where the N-particle state becomes a pair-correlated Bogoliubov vacuum state

$$| 0_k : N \rangle \sim \left( d^\dagger \right)^{ \frac{N}{2} } | 0 \rangle = \left( \hat{a}_0^\dagger \hat{a}_0^\dagger + \sum_{k=1}^{\infty} \lambda_k \hat{a}_k^\dagger \hat{a}_k^\dagger \right)^{ \frac{N}{2} } | 0 \rangle.$$  \hspace{1cm} (34)

The sum runs over an orthonormal basis of non-condensate modes $\phi_k(\vec{x})$ orthogonal to $\phi_0(\vec{x})$. The eigenvalues $\lambda_k$ and the eigenmodes $\phi_k(\vec{x})$ have been calculated in full generality only very recently [2]. Their construction runs in the following steps. Its full justification can be found in Refs. [2, 12].

- Solve the stationary GPE (34) to get $\phi_0(\vec{x})$ and $\mu$. Note that $\phi_0$ does not need to be the ground state, it can be an excited stationary state with a dark soliton or vortex.
- Solve the linear Bogoliubov-de Gennes equations for the Bogoliubov normal modes of small fluctuations around $\phi_0$,

$$-\frac{1}{2} \nabla^2 U_m + V(\vec{x}) U_m + 2 g | \phi_0 |^2 U_m + g | \phi_0 |^2 V_m =$$
\[ \mu U_m + \omega_m U_m , \]
\[ -\frac{1}{2} \nabla^2 V_m + V(\vec{x})V_m + 2g|\phi_0|^2 V_m + g(\phi_0)^2 U_m = \mu V_m - \omega_m V_m \] (35)

to get the eigenvalues \( \omega_m \) and “raw” Bogoliubov eigenmodes \( U_m(\vec{x}), V_m(\vec{x}) \).

- Project the raw \( U_m, V_m \) on the subspace orthogonal to \( \phi_0 \),
  \[ u_m = U_m - \phi_0 \langle \phi_0 | U_m \rangle , \]
  \[ v_m = V_m - \phi_0 \langle \phi_0 | V_m \rangle , \] (36)

to get the Bogoliubov eigenmodes \((u_m, v_m)\).

- Normalize the modes \((u_m, v_m)\) so that a norm
  \[ \langle u_m | v_m \rangle - \langle v_m | v_m \rangle = +1 . \] (37)
  Ignore modes with negative norm.

The normalized Bogoliubov modes define bosonic quasiparticle annihilation operators
\[ \hat{b}_m = \frac{\hat{a}_m^\dagger}{\sqrt{N}} (u_m | \delta \hat{\psi} ) - \frac{\hat{a}_m}{\sqrt{N}} (v_m^* | \delta \hat{\psi} )^\dagger \] (38)

which transfer atoms between the condensate and the non-condensate modes, but conserve the total number of atoms \( N \). For small depletion the Hamiltonian can be approximated by a sum of harmonic oscillators
\[ \hat{H} = \sum_m \omega_m \hat{b}_m^\dagger \hat{b}_m . \] (39)

The \( N \)-particle Bogoliubov vacuum \( |\Psi_0 \rangle \) is the eigenstate of \( \hat{H} \) without any quasiparticles,
\[ \hat{b}_m |\Psi_0 \rangle = 0 . \] (40)

The \( N \)-particle state \( |\Psi_N \rangle \) is annihilated by all \( \hat{b} \)'s when the \( \hat{d}^\dagger \) defined in Eq.(34) commutes with all \( \hat{b} \)'s,
\[ [\hat{b}_m, \hat{d}^\dagger] = 0 . \] (41)

In general construction of such a \( \hat{d}^\dagger \) runs along the following steps.

- Choose an orthonormal basis of states \( \hat{\phi}_k(\vec{x}) \) orthogonal to \( \phi_0 \), and then calculate matrices \( \hat{U} \) and \( \hat{V} \) with elements
  \[ \hat{U}_{mk} = \langle u_m | \hat{\phi}_k \rangle , \] (42)
  \[ \hat{V}_{mk} = \langle \hat{\phi}_k | v_m \rangle . \] (43)

- Invert \( \hat{U} \) and then calculate a matrix
  \[ \hat{Z} = \hat{U}^{-1} \hat{V} . \] (44)

- Diagonalize \( \hat{Z} \) to get its eigenvalues \( \lambda_k \) and orthonormal eigenmodes \( \phi_k \). In the basis of \( \phi_k \) the \( \hat{Z} \) becomes a \( \hat{Z} = \text{diag}\{\lambda_1, \lambda_2, \ldots\} \).

Having calculated \( \lambda_k \) and \( \phi_k \) we can construct the operator \( \hat{d}^\dagger \) and the Bogoliubov vacuum state \( |\Psi_0 \rangle \). Excited eigenstates can be obtained by repeated action of the quasiparticle creation operators \( \hat{b}_m^\dagger \) on the vacuum state.

### B. Density Measurement

In an ideal density measurement positions of all \( N \) atoms are measured. The probability distribution for the \( N \) positions in a \( N \)-particle quantum state \( |\Psi \rangle \) is given by
\[ p_N(x_1, \ldots, x_N) \sim \langle \Psi | \hat{\psi}^\dagger(x_1) \ldots \hat{\psi}^\dagger(x_N) \hat{\psi}(x_N) \ldots \hat{\psi}(x_1) |\Psi \rangle . \] (45)

We want to simulate density measurements by generating from this distribution typical outcomes \((x_1, \ldots, x_N)\). In general this is a formidable task.

The problem becomes tractable when, for some physical reasons, it is possible to truncate the single particle Hilbert space to just two modes, say, \( \phi_0(x) \) and \( \phi_1(x) \) (generalization to 2, 3, \ldots modes is straightforward). With only two modes we can adopt the algorithm of Javanainen and Yoo [2]. In their algorithm positions \((x_1, \ldots, x_N)\) are not generated all at once but one after another. To begin with, \( x_1 \) is chosen randomly with a reduced single particle probability distribution
\[ p_1(x_1) \sim \langle \Psi | \hat{\psi}^\dagger(x_1) \hat{\psi}(x_1) |\Psi \rangle . \] (46)

Once the actual \( x_1 \) is chosen we calculate a state vector
\[ |\Psi_1 \rangle = \hat{\psi}(x_1) |\Psi_0 \rangle = [\hat{a}_0 \phi_0(x_1) + \hat{a}_1 \phi_1(x_1)] |\Psi_0 \rangle , \] (47)
and store it in the memory. After \((k - 1)\) steps we know \((x_1, \ldots, x_{k-1})\) and a state vector
\[ |\Psi_{k-1} \rangle = \hat{\psi}(x_{k-1}) |\Psi_{k-2} \rangle , \] (48)
calculated for the actual \( x_{k-1} \). To generate the next coordinate \( x_k \) we use a conditional probability distribution
\[ p(x_k | x_{k-1}, \ldots, x_1) = \frac{p_k(x_k, x_{k-1}, \ldots, x_1)}{p_k-1(x_{k-1}, \ldots, x_1)} \sim \frac{\langle \Psi_{k-1} | \hat{\psi}^\dagger(x_k) \hat{\psi}(x_k) |\Psi_{k-1} \rangle}{\langle \Psi_{k-1} |\Psi_{k-1} \rangle} . \] (49)

Note that this is a function of \( x_k \) only, all coordinates \( x_{k-1}, \ldots, x_1 \) have already been fixed. Once we get the actual outcome \( x_k \) we calculate \(|\Psi_k \rangle = \hat{\psi}(x_k) |\Psi_{k-1} \rangle\) and store it in the memory.

The key simplification discovered in Ref.[2] is that all single particle probability distributions that we encounter, like Eqs.(46)_1, have the same functional form
\[ p(x) = A |\phi_0(x)|^2 + B |\phi_1(x)|^2 + \cdots \]
\[ C \phi_0^*(x) \phi_1(x) + C^* \phi_0(x) \phi_1^*(x) \quad (50) \]

To find the real \( A, B \) and the complex \( C \) we need to sample \( p(x) \) at only four values of \( x \). When \( \phi_0, \phi_1 \) are real, like for the dark soliton, then \( C \) is also real, and just 3 sampling points are enough. The total algorithm to generate \( N \) positions \((x_1, \ldots, x_N)\) scales like \( N^2 \), a slight improvement over the \( N^3 \) in Ref.\[2\].

To summarize, we provide an input \( N \)-particle state \(|\Psi\rangle\) (in a Fock representation build on the two single particle modes \( \phi_0, \phi_1 \)) and the algorithm generates a random string of coordinates \((x_1, \ldots, x_N)\) according to the \( N \)-particle probability distribution (45). Each application of the algorithm returns a random outcome \((x_1, \ldots, x_k)\) of a single realization of density measurement on the state \(|\Psi\rangle\).

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[1] M.R. Andrews et al., Science 275, 637 (1997).
[2] J. Javanainen and S.M. Yoo, Phys. Rev. Lett. 76, 161 (1996).
[3] M. Naraschewski et al., Phys. Rev. A 54, 2185 (1996); Y. Castin and J. Dalibard, Phys. Rev. A 55, 4330 (1997); R.A. Hegstrom, Chem. Phys. Lett. 288, 248 (1998); K. Mølmer, Phys. Rev. A 65, 021607 (2002); S. Ashhab and A.J. Leggett, Phys. Rev. A 65, 023604 (2002).
[4] L.P. Pitaevskii, Zh. Eksp. Teor. Fiz. 40, 646 (1961) [Sov. Phys. JETP 13, 451 (1961)]; E.P. Gross, Nuovo Cimento 20, 454 (1961); F. Dalfovo et al., Rev. Mod. Phys. 71, 463 (1999).
[5] M. D. Girardeau and R. Arnowitt, Phys. Rev. 113, 755 (1959); C. W. Gardiner, Phys. Rev. A 56, 1414 (1997); M. D. Girardeau Phys. Rev. A 58, 775 (1998); Y. Castin and R. Dum, Phys. Rev. A 57, 3008 (1998); Y. Castin, in Les Houches Session LXXII, Coherent atomic matter waves 1999, edited by R. Kaiser, C. Westbrook and F. David, (Springer-Verlag Berlin Heiderberg New York 2001).
[6] J. Dziarmaga and K. Sacha, cond-mat/0210258 [Phys. Rev. A in press].
[7] A.J. Leggett, Rev. Mod. Phys. 73, 307 (2001).
[8] J. Dziarmaga, Z.P. Karkuszewski, and K. Sacha, Phys. Rev. A 66, 043615 (2002).
[9] J. Dziarmaga and K. Sacha, Phys. Rev. A 66, 043620 (2002).
[10] J. K. Taylor, Ed., Optical Solitons Theory & Experiment (Cambridge Univ. Press, New York, 1992); A. E. Muryshev et al., Phys. Rev. A 60, R2665 (1999); Th. Bush and J. R. Anglin, Phys. Rev. Lett 84, 2298 (2000).
[11] C.K. Law, P.T. Leung, and M.C. Chu, arXiv: cond-mat/0110428.
[12] S. Burger et al., Phys. Rev. Lett. 83, 5198 (1999).
[13] J. Denschlag et al., Science 287, 97 (2000).
[14] L. Dobrak, M. Gajda, M. Lewenstein, K. Sengstock, G. Birkl, and W. Ertmer, Phys. Rev. A 60, R3381 (1999).