GLOBAL SMALL ANALYTIC SOLUTIONS OF MHD BOUNDARY LAYER EQUATIONS

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Abstract. In this paper, we prove the global existence and the large time decay estimate of solutions to the two-dimensional MHD boundary layer equations with small initial data, which is analytical in the tangential variable. The main idea of the proof is motivated by that of [27]. The additional difficulties are: 1. there appears the magnetic field; 2. the far field here depends on the tangential variable; 3. the Reynolds number is different from magnetic Reynolds number. In particular, we solved an open question in [32] concerning the large time existence of analytical solutions to the MHD boundary layer equations.

Keywords: MHD Prandtl system, Littlewood-Paley theory, analytic energy estimate

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1. Introduction

In this paper, we consider the global well-posedness of the following two-dimensional MHD boundary layer equations in the upper space \( \mathbb{R}^2_+ = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}_+\} \),

\[
\begin{align*}
\partial_t u_1 - \partial_y^2 u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 + \partial_x p &= b_1 \partial_x b_1 + b_2 \partial_y b_1, \\
\partial_t b_1 - \kappa \partial_y^2 b_1 + u_1 \partial_x b_1 + u_2 \partial_y b_1 &= b_1 \partial_x u_1 + b_2 \partial_y u_1, \\
\partial_x u_1 + \partial_y u_2 &= 0, \quad \partial_x b_1 + \partial_y b_2 = 0, \\
u_1|_{y=0} = u_2|_{y=0} = 0, \quad \partial_y b_1|_{y=0} = b_2|_{y=0} = 0, \\
\lim_{y \to +\infty} u_1 = U_1, \quad \lim_{y \to +\infty} b_1 = B_1, \\
\end{align*}
\]

(1.1)

where \((u_1, u_2)\) and \((b_1, b_2)\) represent the velocity of fluid and the magnetic field respectively, \(\kappa > 0\) is a constant which represents the difference between the Reynolds number and the magnetic Reynolds number, \((U_1, B_1, p)(t,x)\) are the traces of the tangential fields and pressure of the outflow on the boundary, which satisfies Bernoulli’s law:

\[
\begin{align*}
\partial_t U_1 + U_1 \partial_x U_1 + \partial_x p &= B_1 \partial_x B_1, \\
\partial_t B_1 + U_1 \partial_x B_1 &= B_1 \partial_x U_1.
\end{align*}
\]

(1.2)

The MHD boundary layer system (1.1) was derived in [16, 20, 21] by considering the high Reynolds number limit to the incompressible viscous MHD system (see [9, 10]) near a non-slip boundary when both the Reynolds number and the magnetic Reynolds number, \((U_1, B_1, p)(t,x)\) are the traces of the tangential fields and pressure of the outflow on the boundary, which satisfies Bernoulli’s law:

\[
\begin{align*}
\partial_t U_1 + U_1 \partial_x U_1 + \partial_x p &= B_1 \partial_x B_1, \\
\partial_t B_1 + U_1 \partial_x B_1 &= B_1 \partial_x U_1.
\end{align*}
\]

The MHD boundary layer system (1.1) was derived in [16, 20, 21] by considering the high Reynolds number limit to the incompressible viscous MHD system (see [9, 10]) near a non-slip boundary when both the Reynolds number and the magnetic Reynolds number have the same order. In particular, when \(b_1\) equals some constant in (1.1), the system reduces to the classical Prandtl equations (simplified as \((PE)\) in the sequel) which was proposed by Prandtl [30] in 1904 in order to explain the disparity between the boundary conditions verified by ideal fluid and viscous fluid with small viscosity. One may check [12, ?] and references therein for more introductions on boundary layer theory in the absence of the magnetic field. Especially we refer to [17] for a comprehensive recent survey.

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Under the monotonicity condition on the tangential velocity field in the normal direction to the boundary, Olešník [24] established the local well-posedness of \((PE)\) with initial data in Sobolev space by using the Croco transformation. Lately, this result was proved via energy method in [1] [23] independently by taking care of the cancellation property in the convection terms of \((PE)\). Under a favorable condition on the pressure, a global-in-time weak solution was proved in [33]. In general, when the monotonicity condition is violated, separation of the boundary layer is expected and observed in classical fluid. Especially when the background shear flow has a non-degenerate critical point, there are some interesting ill-posedness result to both linear and nonlinear Prandtl equations, one may check [13, 15, 17, 19] and the references therein for more details. However, surprisingly for the MHD boundary layer system \((1.1)\), Liu, Xie and Yang [20] succeeded in proving the local well-posedness in Sobolev space without any monotonicity assumption on the tangential velocity. The only essential assumption there is that the background tangential magnetic field has a positive lower bound. This result agrees with the general physical understanding that the magnetic field stabilizes the boundary layer.

On the other hand, for the data which is analytic in both \(x\) and \(y\) variables, Sammartino and Caflisch [31] established the local well-posedness result of \((PE)\). The analyticity in \(y\) variable was removed by Lombardo, Cannone and Sammartino in [22]. The main argument used in [22, 31] is to apply the abstract Cauchy-Kowalewskaya theorem. Recently, Gérard-Varet and Masmoudi [14] proved the well-posedness of \((PE)\) for a class of data with Gevrey regularity. The optimal result in this direction (with Gevrey class 2) was obtained by Dietert and Gérard-Varet in [11]. The question of the long time existence for Prandtl system with small analytic data was first addressed in [33] and an almost global existence result was provided in [18]. Finally in [27], Paicu and the second author proved the global well-posedness of \((PE)\) with small analytical data.

We mention that the main idea in [27] is to use the analytic energy estimate. This idea dates back to [4], where Chemin introduced a tool to make analytical type estimates to the Fujita-Kato solutions of 3-D Navier-Stokes system and to control the size of the analytic radius simultaneously. This idea was used in the context of anisotropic Navier-Stokes system [5] (see also [28, 29]), which implies the global well-posedness of three dimensional Navier-Stokes system with a class of “ill prepared data”, the \(B^{-1}_{\infty,\infty}(\mathbb{R}^3)\) norm of which blow up as a small parameter goes to zero.

While for the MHD boundary layer equation \((1.1)\), corresponding to the results in [18, 34], Xie and Yang [32] obtained a lower bound for the lifespan of the analytic solutions. And the authors left the following open question in [32]: “However, it is not known whether one can obtain a global or almost global in time solution like the work on Prandtl system when the background shear velocity is taken to be a Gaussian error function in [18].” The goal of this paper is to solve this problem.

In order to do it, for any constant \(\bar{B}_\kappa\), we take a cut-off function \(\chi \in C^\infty[0, \infty)\) with \(\chi(y) = \begin{cases} y & \text{if } y \geq 2, \\ 0 & \text{if } y \leq 1, \end{cases}\) and make the following change of variables:

\[
(1.3) \quad u = u_1 - \chi'(y)U \quad \text{and} \quad v = u_2 + \chi(y)\partial_x U, \\
\quad b = b_1 - \chi'(y)B - \bar{B}_\kappa \quad \text{and} \quad h = b_2 + \chi(y)\partial_x B,
\]

where \(U \equiv U_1\) and \(B \equiv B_1 - \bar{B}_\kappa\).
Then in view of (1.1) and (1.2), \((u,v,b,h)\) solves

\[
\begin{align*}
\partial_t u - \partial_y^2 u - \tilde{B}_n \partial_x b + u \partial_x u - b \partial_x b + v \partial_y u - h \partial_y b + \chi'(U \partial_x u - B \partial_x b) \\
+ \chi'(\partial_x U u - \partial_x B b) + \chi(-\partial_x U y u + \partial_x B b) + \chi''(U - B h) = m_U, \\
\partial_t b - \kappa \partial_y^2 b - \tilde{B}_n \partial_x u + u \partial_x b - b \partial_x b + v \partial_y b - h \partial_y b + \chi'(U \partial_x b - B \partial_x u) \\
+ \chi'(\partial_x B u - \partial_x B b) + \chi(-\partial_x U y b + \partial_x B \partial_y u) + \chi''(B v - U h) = m_B, \\
\end{align*}
\]

(1.4)

where \((U_0(x), B_0(x)) = (U(0,x), B(0,x))\), and the terms \((m_U, m_B)\) are given by

\[
\begin{align*}
m_U \overset{\text{def}}{=} (1 - \chi')(\partial_t U - \tilde{B}_n \partial_x B) + \chi'' U + (1 - \chi'^2 + \chi''')(U \partial_x U - B \partial_x B), \\
m_B \overset{\text{def}}{=} (1 - \chi') (\partial_t B - \tilde{B}_n \partial_x U) + \chi'' B + (1 - \chi'^2 + \chi''')(U \partial_x B - B \partial_x U). \\
\end{align*}
\]

It’s easy to observe that \((m_U, m_B)\) are supported in \(y \in [0,2]\) for each \(t > 0\).

Due to \(\partial_x u + \partial_y v = 0\) and \(\partial_x b + \partial_y h = 0\), there exist two potential functions \((\varphi, \psi)\) so that \((u, b) = \partial_y (\varphi, \psi)\) and \((v, h) = -\partial_x (\varphi, \psi)\). While it follows from the boundary conditions in (1.4) that

\[
\partial_x \int_0^\infty (u, v)(t, x, y) dy = - \int_0^\infty \partial_y (v, h)(t, x, y) dy = 0,
\]

which implies \(\int_0^\infty (u, v)(t, x, y) dy = (C_1, C_2)(t)\). Yet since \((u, b)\) decays to zero as \(|x|\) tends to \(\infty\), we have \(C_1(t) = C_2(t) = 0\). So that we can impose the following boundary conditions for the primitive functions \((\varphi, \psi)\):

\[
(\varphi, \psi)|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} (\varphi, \psi) = 0.
\]

Then by integrating \((u, b)\) equations of (1.4) with respect to \(y\) variable over \([y, \infty[\), we find

\[
\begin{align*}
\partial_t \varphi - \partial_y^2 \varphi - \tilde{B}_n \partial_x \psi + u \partial_x \varphi - b \partial_x \psi + 2 \int^\infty_y (\partial_x \varphi \partial_y u - \partial_x \psi \partial_y b) dy' \\
+ \chi'(U \partial_x \varphi - B \partial_x \psi) + 2 \int^\infty_y \chi''(U \partial_x \varphi - B \partial_x \psi) dy' + \chi(-\partial_x U u + \partial_x B b) \\
+ 2 \chi'(\partial_x U \varphi - \partial_x B \psi) + 2 \int^\infty_y \chi''(\partial_x U \varphi - \partial_x B \psi) dy' = M_U, \\
\partial_t \psi - \kappa \partial_y^2 \psi - \tilde{B}_n \partial_x \varphi + u \partial_x \psi - b \partial_x \varphi + \chi'(U \partial_x \psi - B \partial_x \varphi) \\
+ \chi(-\partial_x U b + \partial_x B u) = M_B, \\
\psi|_{y=0} = \psi|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to +\infty} \varphi = \lim_{y \to +\infty} \psi = 0;
\end{align*}
\]

(1.6)

where \((M_U, M_B) = - \int_y^\infty (m_U, m_B) dy'\) are also supported in \(y \in [0,2]\) for each \(t > 0\).

Before proceeding, let us recall from [2] that

\[
\Delta_h^a \varphi = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\tilde{a}), \quad \text{and} \quad \Delta_h^a \varphi = \mathcal{F}^{-1}(\chi(2^{-k}|\xi|)\tilde{a}),
\]

(1.7)
where \( \mathcal{F}a \) and \( \hat{a} \) denote the partial Fourier transform of the distribution \( a \) with respect to \( x \) variable, that is, \( \widehat{\alpha}(\xi, y) = \mathcal{F}_{x \to \xi}(a)(\xi, y) \), and \( \chi(\tau), \varphi(\tau) \) are smooth functions such that

\[
\text{Supp } \varphi \subset \left\{ \tau \in \mathbb{R} / \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{k \in \mathbb{Z}} \varphi(2^{-k} \tau) = 1,
\]

\[
\text{Supp } \chi \subset \left\{ \tau \in \mathbb{R} / |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) + \sum_{k \geq 0} \varphi(2^{-k} \tau) = 1.
\]

**Definition 1.1.** Let \( s \) be in \( \mathbb{R} \). For \( u \) in \( S'_h(\mathbb{R}^2_+) \), which means that \( u \) belongs to \( S'(\mathbb{R}^2_+) \) and satisfies \( \lim_{k \to -\infty} \| S_k^h u \|_{L^\infty} = 0 \), we set

\[
\| u \|_{B^s_0} \overset{\text{def}}{=} \left\| \left( 2^{ks} \| \Delta_k^h u \|_{L^2_+} \right)_{k \in \mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})}.
\]

- For \( s \leq \frac{1}{2} \), we define \( B^{s,0}(\mathbb{R}^2_+) \overset{\text{def}}{=} \left\{ u \in S'_h(\mathbb{R}^2_+) \mid \| u \|_{B^s_0} < \infty \right\} \).
- If \( j \) is a positive integer and if \( \frac{1}{2} + j < s \leq \frac{3}{2} + j \), then we define \( B^{s,j}(\mathbb{R}^2_+) \) as the subset of distributions \( u \) in \( S'_h(\mathbb{R}^2_+) \) such that \( \partial_x^j u \) belongs to \( B^{s-j,0}(\mathbb{R}^2_+) \).

In all that follows, we always denote \( B^s_+ \) to the Besov space \( \tilde{B}^s_{2,1}(\mathbb{R}_+) \).

In order to obtain a better description of the regularizing effect of the diffusion equation, we need to use Chemin-Lerner type spaces \( \tilde{L}^p_T(\mathbb{R}^2_+) \) (see [6]).

**Definition 1.2.** Let \( p \in [1, +\infty] \) and \( T_0, T \in [0, +\infty[ \). We define \( \tilde{L}^p(T_0, T; B^{s,0}(\mathbb{R}^2_+)) \) as the completion of \( C([T_0, T]; S(\mathbb{R}^2_+)) \) by the norm

\[
\| a \|_{\tilde{L}^p(T_0, T; B^{s,0})} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_{T_0}^T \| \Delta_k^h a(t) \|_{L^2_+}^p dt \right)^{\frac{1}{p}}
\]

with the usual change if \( p = \infty \). In particular, when \( T_0 = 0 \), we simplify the notation \( \| a \|_{\tilde{L}^p(0, T; B^{s,0})} \) as \( \| a \|_{\tilde{L}^p(B^{s,0})} \).

In order to globally control the evolution of the analytic band to the solution of \( (1.4) \), motivated by [18] [27], we introduce the following key quantities:

\[
(1.8) \quad G = u + \frac{y}{2(t)} \varphi \quad \text{and} \quad H = b + \frac{y}{2\kappa(t)} \psi.
\]

And as in [4] [5] [8] [28] [29] [34], for any locally bounded function \( \Phi \) on \( \mathbb{R}^+ \times \mathbb{R} \), we define

\[
(1.9) \quad u_{\Phi}(t, x, y) = \mathcal{F}^{-1}_{x \to \xi}(e^{\Phi(t, \xi)} \widehat{u}(t, \xi, y)).
\]

We also introduce a key quantity \( \theta(t) \) to describe the evolution of the analytic band of \( u \):

\[
(1.10) \quad \begin{cases} 
\dot{\theta}(t) = \langle t \rangle^\frac{1}{2} \| e^{\Psi} \partial_y(G, H) \Phi(t) \|_{B^{s,0}_+} + \varepsilon^{-\frac{1}{2}}(t)^{\frac{5}{2}} \| (U, B) \Phi(t) \|_{B^{s,0}_+}, \\
\theta|_{t=0} = 0.
\end{cases}
\]

Here the phase function \( \Phi \) is defined by

\[
(1.11) \quad \Phi(t, \xi) = (\delta - \lambda \theta(t))|\xi|,
\]

and the weighted function \( \Psi(t, y) \) is determined by

\[
(1.12) \quad \Psi(t, y) = \frac{y^2}{8(t)} \quad \text{with} \quad \langle t \rangle \overset{\text{def}}{=} 1 + t,
\]
which satisfies
\[(1.13) \quad \partial_t \Psi + 2(\partial_y \Psi)^2 = 0.\]

Our first result of this paper states as follows:

**Theorem 1.1.** Let \( \kappa \in [0, 2] \), \( \tilde{B}_\kappa = \left\{ \begin{array}{ll} 1 & \text{if } \kappa = 1, \\ 0 & \text{otherwise,} \end{array} \right. \) and \( \varepsilon, \delta > 0 \). We assume that the far field states \((U, B)\) satisfy
\[
\begin{align*}
(1.14a) & \quad \| \langle t \rangle^{\frac{2}{\kappa}} e^{\delta |D_x|} (U, B) \|_{L^\infty_x(R_+; L^2_y)} + \| \langle t \rangle^{\frac{2}{\kappa}} e^{\delta |D_x|} (\partial_t U, \partial_t B, U, B) \|_{L^2_x(R_+; B^1_{\infty} \frac{1}{\kappa})} \leq \varepsilon, \\
(1.14b) & \quad \int_0^\infty \langle t \rangle^{\frac{2}{\kappa}} \| e^{\delta |D_x|} (U, B) \|_{L^2_x} \, dt \leq \varepsilon.
\end{align*}
\]

Let the initial data \((u_0, b_0, \varphi_0, \psi_0)\) (see the definitions in (1.14) and (1.10)) satisfy the compatibility condition: \( u_0|_{y=0} = \partial_y b_0|_{y=0} = 0 \), \( \int_0^\infty u_0 \, dy = \int_0^\infty b_0 \, dy = 0 \), and
\[
\| e^{\frac{2}{\kappa}} e^{\delta |D_x|} (u_0, b_0, \varphi_0, \psi_0) \|_{B^1 \frac{1}{\kappa}} < \infty \quad \text{and} \quad \| e^{\frac{2}{\kappa}} e^{\delta |D_x|} (G_0, H_0) \|_{B^1 \frac{1}{\kappa}} \leq \sqrt{\varepsilon},
\]
where \( G_0 \stackrel{\text{def}}{=} u_0 + \frac{\varphi_0}{2 \kappa(y)} \varphi_0 \) and \( H_0 \stackrel{\text{def}}{=} b_0 + \frac{\psi_0}{2 \kappa(y)} \psi_0 \). Then there exist positive constants \( \lambda, \kappa \) and \( \varepsilon_0(\lambda, \kappa, \delta) \) so that for \( \varepsilon \leq \varepsilon_0 \) and \( l_\kappa \stackrel{\text{def}}{=} \frac{\kappa(2-\kappa)}{4} \in [0, 1/4] \), the system (1.4) has a unique global solution \((u, b)\) which satisfies \( \sup_{t \in [0, \infty]} \| \theta(t) \| \leq \frac{1}{\delta} \), and
\[
\| e^\Psi (u, b) \|_{L^\infty_x (B^1 \frac{1}{\kappa})} + \sqrt{\varepsilon} \| e^\Psi \partial_y (u, b) \|_{L^2_x (B^1 \frac{1}{\kappa})} \leq \| e^{\frac{2}{\kappa}} e^{\delta |D_x|} (u_0, b_0) \|_{B^1 \frac{1}{\kappa}} + C \sqrt{\varepsilon}.
\]
Furthermore, for any \( t > 0 \) and \( \gamma \in [0, 1] \), there hold
\[
\begin{align*}
(1.17a) & \quad \| \langle t \rangle^{\frac{1}{2} + l_\kappa - \epsilon \gamma} e^\Psi (u, b) \|_{L^\infty_x (B^1 \frac{1}{\kappa})} + \| \langle t \rangle^{\frac{1}{2} + l_\kappa - \epsilon \gamma} e^\Psi \partial_y (u, b) \|_{L^2_x (B^1 \frac{1}{\kappa})} \\
& \quad \leq C \left( \| e^{\frac{2}{\kappa}} e^{\delta |D_x|} (u_0, b_0, \varphi_0, \psi_0) \|_{B^1 \frac{1}{\kappa}} + \sqrt{\varepsilon} \right),
\end{align*}
\]
\[
\begin{align*}
(1.17b) & \quad \| \langle t \rangle^{1 + l_\kappa - \epsilon \gamma} e^\Psi (G, H) \|_{L^\infty_x (B^1 \frac{1}{\kappa})} + \| \langle t \rangle^{1 + l_\kappa - \epsilon \gamma} e^\Psi \partial_y (G, H) \|_{L^2_x (B^1 \frac{1}{\kappa})} \\
& \quad \leq C \sqrt{\varepsilon} \left( 1 + \| e^{\frac{2}{\kappa}} e^{\delta |D_x|} (u_0, b_0) \|_{B^1 \frac{1}{\kappa}} \right),
\end{align*}
\]
\[
\begin{align*}
(1.17c) & \quad \| \langle t \rangle^{1 + l_\kappa - \epsilon \gamma} e^\Phi (u, b) \|_{L^\infty_x (B^1 \frac{1}{\kappa})} + \| \langle t \rangle^{1 + l_\kappa - \epsilon \gamma} e^\Phi \partial_y (u, b) \|_{L^2_x (B^1 \frac{1}{\kappa})} \\
& \quad \leq C \sqrt{\varepsilon} \left( 1 + \| e^{\frac{2}{\kappa}} e^{\delta |D_x|} (u_0, b_0) \|_{B^1 \frac{1}{\kappa}} \right).
\end{align*}
\]

**Remark 1.1.** (1) The constants \( \epsilon \) and \( C \) in the above theorem are independent of \( \varepsilon \). Furthermore, unlike the constant \( C \), which may change from line to line, the constant \( \epsilon \) can be fixed in Sections 4, 5 and 6. So that by choosing \( \varepsilon_0 \) to be sufficiently small, we always have \( \varepsilon < l_\kappa \). This will be crucial for us to globally control the evolution of the analytic band. Moreover, the constant \( \tilde{B}_\kappa \) can be any constant if \( \kappa = 1 \), otherwise 0. This restriction is only used in the derivation of the system (1.4) for \((G, H)\).

(2) Compared with the well-posedness result for the classical Prandtl system (PE) in [27], here we allow the far field \((U, B)\) to depend on the tangential variable. And exactly due to this type of far field, there is a loss of \( \epsilon \) in the decay estimates (1.17).

(3) The main idea of the proof to the above theorem is to use analytic energy estimate, which is motivated by [27] and which originates from [4].
We remark that in the proof of Theorem 1.1, the role of \( u \) and \( b \) are the same in the process of the analytic energy estimate. By introducing the changes of variables: \((\bar{t}, \bar{x}, \bar{y}) = (t, x, \frac{y}{\sqrt{\kappa}})\), and denoting \((\bar{u}, \bar{v}, \bar{b}, \bar{h}) \equiv (u, \frac{v}{\sqrt{\kappa}}, b, \frac{h}{\sqrt{\kappa}})\), we find that (1.1) can be equivalently reformulated as

\[
\begin{cases}
\partial_{\bar{t}} \bar{u} - \frac{1}{\kappa} \partial_{\bar{y}} \bar{u} + \bar{u} \partial_{\bar{y}} \bar{u} + \bar{v} \partial_{\bar{y}} \bar{u} = \bar{b} \partial_{\bar{x}} \bar{b} + \bar{h} \partial_{\bar{y}} \bar{b}, \\
\partial_{\bar{t}} \bar{b} - \frac{1}{\kappa} \partial_{\bar{y}} \bar{b} + \bar{u} \partial_{\bar{y}} \bar{b} + \bar{v} \partial_{\bar{y}} \bar{b} = \bar{b} \partial_{\bar{x}} \bar{u} + \bar{h} \partial_{\bar{y}} \bar{u}, \\
\partial_{\bar{x}} \bar{u} + \partial_{\bar{y}} \bar{v} = 0, \quad \partial_{\bar{x}} \bar{b} + \partial_{\bar{y}} \bar{h} = 0.
\end{cases}
\]

So that the proof of Theorem 1.1 also ensures the global well-posedness result of (1.1) for \( \kappa \in ]1/2, \infty[ \). Yet for conciseness, we prefer to present the detailed result corresponding to \( \kappa \in ]1/2, \infty[ \).

In order to do so, similar to (1.10), we introduce \( \theta_\kappa(t) \) via

\[
\begin{cases}
\dot{\theta}_\kappa(t) = \langle \theta_\kappa(t), \delta \rangle = \frac{1}{\kappa} \| \kappa^{-1/2} \iota(y, H) \Phi_\kappa(t) \|_{B^{1,0}_{\kappa}} + \epsilon^{-\frac{1}{2}} \| (U, B) \Phi_\kappa(t) \|_{B^0_{\kappa}}, \\
\theta_\kappa|_{t=0} = 0.
\end{cases}
\]

Here the phase function \( \Phi_\kappa \) is defined by

\[
\Phi_\kappa(t, \xi) \equiv (\delta - \lambda \theta_\kappa(t))|\xi|,
\]

and the weighted function \( \Psi_\kappa(t, y) \) is determined by

\[
\Psi_\kappa(t, y) \equiv \frac{y^2}{\kappa \delta(t)} = \kappa^{-1} \Psi(t, y),
\]

which satisfies

\[
\partial_t \Psi_\kappa + 2\kappa (\partial_y \Psi_\kappa)^2 = 0.
\]

The second result of this paper states as follows:

**Theorem 1.2.** Let \( \kappa \) be in \( ]1/2, \infty[ \), \( \bar{B}_\kappa = \begin{cases} 1 & \text{if } \kappa = 1, \\
0 & \text{otherwise,} \end{cases} \) and \( \epsilon, \delta > 0 \). We assume that far field states \((U, B)\) satisfy (1.14a) and (1.14b). Let the initial data \((u_0, b_0, \varphi_0, \psi_0)\) satisfy the compatibility conditions listed in Theorem 1.1 and

\[
\| e^{\frac{\kappa^2}{8\kappa} e^{\|D_x\|}} (u_0, b_0, \varphi_0, \psi_0) \|_{B^{1,0}_{\kappa}} < \infty \quad \text{and} \quad \| e^{\frac{\kappa^2}{8\kappa} e^{\|D_x\|}} (G_0, H_0) \|_{B^{1,0}_{\kappa}} \leq \sqrt{\epsilon}.
\]

Then there exist positive constants \( \lambda, \epsilon \) and \( \varepsilon_0(\lambda, \kappa, \delta) \) so that for \( \varepsilon \leq \varepsilon_0 \) and \( \epsilon_\kappa \equiv \frac{\epsilon}{\kappa^{1/2}} \in \]0, 1/4[, the system (1.4) has a unique global solution \((u, b)\) which satisfies \( \sup_{t \in [0, \infty]} |\theta_\kappa(t)| \leq \frac{1}{\kappa^2} \), and

\[
\| e^{\frac{\kappa^2}{8\kappa} e^{\|D_x\|}} (u_0, b_0) \|_{L_t^{\infty}(B^{1,0}_{\kappa})} + \sqrt{\epsilon} \| e^{\frac{\kappa^2}{8\kappa} e^{\|D_x\|}} (u_0, b_0) \|_{L_t^2(B^{1,0}_{\kappa})} \leq \| e^{\frac{\kappa^2}{8\kappa} e^{\|D_x\|}} (u_0, b_0) \|_{B^{1,0}_{\kappa}} + C \sqrt{\epsilon}.
\]
Furthermore, for any \( t > 0 \) and \( \gamma \in [0, 1] \), there hold

\[
\begin{align*}
(1.24a) & \quad \| \langle \tau \rangle^{\frac{1}{2} + \varepsilon} e^{\Psi} (u, b) \Phi \|_{L^\infty_t (B^{\frac{1}{2}}_{\infty, 0})} \leq C \left( \| e^{2\varepsilon \delta |D_x|} (u_0, b_0, \varphi_0, \psi_0) \|_{B^{\frac{1}{2}}_{\infty, 0}} + \sqrt{\varepsilon} \right), \\
(1.24b) & \quad \| \langle \tau \rangle^{1 + \varepsilon} e^{\Psi} (G, H) \Phi \|_{L^\infty_t (B^{\frac{1}{2}}_{\infty, 0})} \leq C \sqrt{\varepsilon} \left( 1 + \| e^{2\varepsilon \delta |D_x|} (u_0, b_0) \|_{B^{\frac{1}{2}}_{\infty, 0}} \right), \\
(1.24c) & \quad \| \langle \tau \rangle^{1 + \varepsilon} e^{\gamma \Psi} (u, b) \Phi \|_{L^\infty_t (B^{\frac{1}{2}}_{\infty, 0})} \leq C \sqrt{\varepsilon} \left( 1 + \| e^{2\varepsilon \delta |D_x|} (u_0, b_0) \|_{B^{\frac{1}{2}}_{\infty, 0}} \right).
\end{align*}
\]

Let us end this introduction by the notations that we shall use in this context.

For \( a \lesssim b \), we mean that there is a uniform constant \( C \), which may be different on different lines, such that \( a \leq C b \). \((t) = 1 + t, (a | b) L^2_a \defeq \int_{\mathbb{R}^2} a(x, y) b(x, y) \, dx \, dy \) stands for the \( L^2 \) inner product of \( a, b \) on \( \mathbb{R}^2 \) (resp. \( \mathbb{R}_+ \)) and \( L^p_+ = L^p (\mathbb{R}^2_+) \) with \( \mathbb{R}^2_+ \defeq \mathbb{R} \times \mathbb{R}_+ \). For \( X \) a Banach space and \( I \) an interval of \( \mathbb{R} \), we denote by \( L^q (I; X) \) the set of measurable functions on \( I \) with values in \( X \), such that \( t \mapsto \| f(t) \|_X \) belongs to \( L^q (I) \). In particular, we denote by \( L^p_T (L^2_h (L^r_y)) \) the space \( L^p ([0, T]; L^q (\mathbb{R}_x; L^r (\mathbb{R}_y))) \). Finally, \((d_k)_{k \in \mathbb{Z}} \) designates a nonnegative generic element in the sphere of \( \ell^1 (\mathbb{Z}) \) so that \( \sum_{k \in \mathbb{Z}} d_k = 1 \).

2. Sketch of the proof to Theorems 1.1 and 1.2

The goal of this section is to sketch the structure of the proof to Theorems 1.1 and 1.2. Let us start with the outline of the proof to Theorem 1.1.

In what follows, we shall always assume that \( t < T^* \) with \( T^* \) being defined by

\[
T^* \defeq \sup \{ t > 0, \ \theta (t) < \delta / \lambda \} \quad \text{with} \quad \theta (t) \ \text{being determined by (1.10)}.
\]

So that by virtue of (1.10), for any \( t < T^* \), the following convex inequality holds

\[
\Phi (t, \xi) \leq \Phi (t, \xi - \eta) + \Phi (t, \eta) \quad \forall \xi, \eta \in \mathbb{R}.
\]

In section 4, we shall deal with the \textit{a priori} decay estimates for the analytic solutions of (1.6).

\textbf{Proposition 2.1.} Let \((\varphi, \psi)\) be a smooth enough solution of (1.6). Then if \( \kappa \in [0, 2[ \), there exist positive constants \( \epsilon, \lambda_0 \) and a small enough constant \( \varepsilon_0 \) so that for any \( \varepsilon \leq \varepsilon_0, \lambda \geq \lambda_0 \) and for any \( t < T^* \), we have

\[
\begin{align*}
\| \langle \tau \rangle^{1 + \varepsilon} e^{\Psi} (\varphi, \psi) \Phi \|_{L^\infty_t (B^{\frac{1}{2}}_{\infty, 0})} + \| \langle \tau \rangle^{1 + \varepsilon} \partial_y e^{\Psi} (\varphi, \psi) \Phi \|_{L^2_t (B^{\frac{1}{2}}_{\infty, 0})} & \lesssim \| e^{2\varepsilon \delta |D_x|} (\varphi_0, \psi_0) \|_{B^{\frac{1}{2}}_{\infty, 0}} + \sqrt{\varepsilon} \quad \text{with} \quad l_\kappa \defeq \frac{\kappa (2 - \kappa)}{4}.
\end{align*}
\]

In section 5, we shall deal with the \textit{a priori} decay estimates for the analytic solutions of (1.4).
Proposition 2.2. Let \((u, b)\) be a smooth enough solution of (1.4). Then if \(\kappa \in [0, 2]\), there exist positive constants \(c, \lambda_0\) and a small enough constant \(\varepsilon_0\) so that for any \(\varepsilon \leq \varepsilon_0\), \(\lambda \geq \lambda_0\) and for any \(t < T^*\), we have

\[
\|\langle \tau \rangle^{1/2 + \kappa - \kappa \varepsilon} \psi((u, b)\phi)\|_{L^2_{[\frac{T}{2}, T]}} + \|\langle \tau \rangle^{1/2 + \kappa - \kappa \varepsilon} \psi(\partial_y(u, b)\phi)\|_{L^2_{[\frac{T}{2}, T]}(B_{T/2}^0)} \\
\leq \|e^{\frac{\tau^2}{8} \delta D}\langle u_0, b_0, \varphi, \psi \rangle\|_{B_{T/2}^0} + \varepsilon \quad \text{with} \quad l_\kappa \overset{\text{def}}{=} \frac{\kappa(2 - \kappa)}{4}.
\]  

(2.4)

In section 3, we shall deal with the \textit{a priori} decay estimates of \((G, H)\) which will be the most crucial ingredient used in the control of the analytic radius.

Proposition 2.3. Let \((G, H)\) be determined by (1.8). Then if \(\kappa \in [0, 2]\), there exist positive constants \(c, \lambda_0\) and a small enough constant \(\varepsilon_0\) so that for any \(\varepsilon \leq \varepsilon_0\), \(\lambda \geq \lambda_0\) and for any \(t < T^*\), we have

\[
\|\langle \tau \rangle^{1/2 + \kappa - \kappa \varepsilon} \psi(G, H)\phi\|_{L^\infty_{[\frac{T}{2}, T]}} + \|\langle \tau \rangle^{1/2 + \kappa - \kappa \varepsilon} \psi(\partial_y(G, H)\phi)\|_{L^2_{[\frac{T}{2}, T]}(B_{T/2}^0)} \\
+ \int_0^t \langle \tau \rangle^{1/2} \|\psi(\partial_y(G, H)\phi(\tau))\|_{B_{T/2}^0} \, d\tau \leq C \sqrt{\varepsilon}\left(1 + \|e^{\frac{\tau^2}{8} \delta D}\langle u_0, b_0 \rangle\|_{B_{T/2}^0}\right),
\]  

(2.5)

where \(l_\kappa \overset{\text{def}}{=} \frac{\kappa(2 - \kappa)}{4}\).

With the above propositions, we still need the following lemma concerning the relationship between the functions \((G, H)\) given by (1.8) and the solutions of (1.4) and (1.6), which will be frequently used in the subsequent sections.

Lemma 2.1. Let \((u, b)\) and \((\varphi, \psi)\) be smooth enough solution of (1.4) and (1.6) respectively on \([0, T]\). Let \((G, H), \Phi\) and \(\Psi\) be defined respectively by (1.8), (1.11) and (1.12). Then if \(\kappa \in [0, 2]\), for any \(\gamma \in [0, 1]\) and \(t \leq T\), one has

\[
\begin{align*}
\|e^{\gamma \psi} \partial_k^h \varphi\|_{L^2} &\lesssim \langle t \rangle^{1/2} \|e^{\psi} \partial_k^h G\|_{L^2}, \\
\|e^{\gamma \psi} \partial_k^h u\|_{L^2} &\lesssim \|e^{\psi} \partial_k^h G\|_{L^2}, \\
\|e^{\gamma \psi} \partial_k^h b\|_{L^2} &\lesssim \|e^{\psi} \partial_k^h H\|_{L^2}.
\end{align*}
\]

(2.6a-b-c)

Let us postpone the proof of this lemma till the end of this section.

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The general strategy to prove the existence result for a non-linear partial differential equation is first to construct appropriate approximate solutions, then perform uniform estimates for such approximate solution sequence, and finally pass to the limit in the approximate problem. For simplicity, here we only present the \textit{a priori} estimate for smooth enough solutions of (1.4) in the analytical framework.

Indeed, let \((u, b)\) and \((\varphi, \psi)\) be smooth enough solutions of (1.4) and (1.6) respectively on \([0, T^*]\), where \(T^*\) is the maximal time of existence of the solution. Let \((G, H)\) be defined by (1.8). For any \(t < T^*\) (of course here \(T^* \leq T^*\)) with \(T^*\) being defined by (2.1), we deduce from (1.10) that

\[
\theta(t) \leq \int_0^t \langle \tau \rangle^{1/2} \|\psi(\partial_y(G, H)\phi(\tau))\|_{B_{T/2}^0} + \varepsilon^{-1/2} \langle \tau \rangle^{1/2} \|(U, B)\phi(\tau))\|_{B_{T/2}^0} \, d\tau.
\]
Notice that we can take \( \varepsilon_0 \) to be so small that \( \varepsilon_0 < \frac{1}{\lambda} \). This together with the assumption (1.14b) and Proposition 2.3 ensures that

\[
\theta(t) \leq C\sqrt{\varepsilon}(1 + \| e^{\frac{\delta}{2} Dx} (u_0, b_0) \|_{\mathcal{B}^{\frac{1}{4}, 0}}) \text{ for any } t < T^* \quad \text{and} \quad \varepsilon \leq \varepsilon_0.
\]

Then by taking \( \varepsilon_0 \) so small that \( \varepsilon_0 \leq \min \left( \left( \frac{\delta}{2C\lambda(1 + \| e^{\frac{\delta}{2} Dx} (u_0, b_0) \|_{\mathcal{B}^{\frac{1}{4}, 0}})}{\varepsilon_0} \right)^2, \frac{1}{\lambda^2} \right) \), we achieve

\[
\sup_{t \in [0, T^*]} \theta(t) \leq \frac{\delta}{2\lambda} \text{ for any } \varepsilon \leq \varepsilon_0.
\]

Then in view of (2.1), we deduce by a continuous argument that \( T^* = \infty \). And (1.11) holds for \( t = \infty \), which implies (1.16). Furthermore, Propositions 2.2 and 2.3 imply (1.17a) and (1.17b) respectively. Finally, along with Lemma 2.1, we deduce (1.17c) from Proposition 2.3. This completes the existence part of Theorem 1.1. The uniqueness part of Theorem 1.1 has been proved in [32], which can also be proved by similar a priori estimates for the difference between any two solutions of (1.4). (One may check [34] for the detailed proof to uniqueness of analytic solution to the classical Prandtl system \((PE)\).) \( \square \)

Next let us turn to the outline of the proof to Theorem 1.2. Firstly, similar to (2.1), we define \( T^*_\kappa \) via

\[
T^*_\kappa \overset{\text{def}}{=} \sup \{ t > 0, \; \theta_\kappa(t) < \delta/\lambda \} \quad \text{with} \quad \theta_\kappa(t) \text{ being determined by (1.18)}.
\]

So that by virtue of (1.19), for any \( t < T^*_\kappa \), the convex inequality (2.2) still holds for \( \Phi_\kappa \).

In section 3, we shall deal with the a priori decay estimates for the analytic solutions of (1.6) in the case when \( \kappa \in [1/2, \infty[ \).

**Proposition 2.4.** Let \((\varphi, \psi)\) be a smooth enough solution of (1.6). Then if \( \kappa \in [1/2, \infty[ \), there exist positive constants \( \lambda_0, c \) and a small enough constant \( \varepsilon_0 \) so that for any \( \varepsilon \leq \varepsilon_0, \lambda \geq \lambda_0 \) and for any \( t < T^*_\kappa \), we have

\[
\| \langle \tau \rangle^{\ell_\kappa - c} e^{\Psi_\kappa (\varphi, \psi)} \phi_\kappa \|_{\mathcal{L}^\infty(\mathcal{B}^{\frac{1}{4}, 0})} + \| \langle \tau \rangle^{\ell_\kappa - c} e^{\Psi_\kappa} \partial_y (\varphi, \psi) \phi_\kappa \|_{\mathcal{L}^{2}_{\mathcal{B}^{0, 0}}} \leq \| e^{\frac{\delta}{2} Dx} (\varphi_0, \psi_0) \|_{\mathcal{B}^{\frac{1}{4}, 0}} + \sqrt{\varepsilon} \quad \text{with} \quad \ell_\kappa \overset{\text{def}}{=} \frac{2\kappa - 1}{4\kappa^2}.
\]

In section 3, we shall deal with the following a priori decay estimates for the analytic solutions of (1.6).

**Proposition 2.5.** Let \((u, b)\) be a smooth enough solution of (1.4). Then if \( \kappa \in [1/2, \infty[ \), there exist positive constants \( \lambda_0, c \) and a small enough constant \( \varepsilon_0 \) so that for any \( \varepsilon \leq \varepsilon_0, \lambda \geq \lambda_0 \) and for any \( t < T^*_\kappa \), we have

\[
\| \langle \tau \rangle^{\frac{1}{2} + \ell_\kappa - c} e^{\Psi_\kappa (u, b)} \phi_\kappa \|_{\mathcal{L}^\infty(\mathcal{B}^{\frac{1}{4}, 0})} + \| \langle \tau \rangle^{\frac{1}{2} + \ell_\kappa - c} e^{\Psi_\kappa} \partial_y (u, b) \phi_\kappa \|_{\mathcal{L}^{2}_{\mathcal{B}^{0, 0}}} \leq \| e^{\frac{\delta}{2} Dx} (u_0, b_0, \varphi_0, \psi_0) \|_{\mathcal{B}^{\frac{1}{4}, 0}} + \sqrt{\varepsilon} \quad \text{with} \quad \ell_\kappa \overset{\text{def}}{=} \frac{2\kappa - 1}{4\kappa^2}.
\]

In section 3, we shall deal with the a priori decay estimates of \((G, H)\).

**Proposition 2.6.** Let \((G, H)\) be determined by (1.8). Then if \( \kappa \in [1/2, \infty[ \), there exist positive constants \( \lambda_0, c \) and a small enough constant \( \varepsilon_0 \) so that for any \( \varepsilon \leq \varepsilon_0, \lambda \geq \lambda_0 \) and for
any $t < T^*_\kappa$, we have
\[
\|\langle \tau \rangle^{1+\ell_\kappa-\epsilon} e^{\Psi_\kappa} (G, H) \Phi_\kappa \|_{L^1_t(G^\frac{1}{2}, 0)} + \|\langle \tau \rangle^{1+\ell_\kappa-\epsilon} e^{\Psi_\kappa} \partial_y (G, H) \Phi_\kappa \|_{L^2_{t,y}(G^\frac{1}{2}, 0)}
\]
\[
+ \int_0^t \langle \tau \rangle^{-\frac{1}{2}} \|e^{\Psi_\kappa} \partial_y (G, H) \Phi_\kappa (\tau)\|_{G^\frac{1}{2}, 0} d\tau \leq C \sqrt{\epsilon} (1 + \|e^{\frac{\sqrt{2}}{\kappa^2} e^{\delta D_x}(u_0, b_0)}\|_{G^\frac{1}{2}, 0}),
\]
where $\ell_\kappa \overset{\text{def}}{=} \frac{2\kappa - 1}{4\kappa^2}$.

We also need the following version of Lemma 2.1 for the case when $\kappa > \frac{1}{2}$.

**Lemma 2.2.** Let $(u, b)$ and $(\varphi, \psi)$ be smooth enough solution of (1.4) and (1.6) respectively on $[0, T]$. Let $(G, H), \Phi_\kappa$ and $\Psi_\kappa$ be defined respectively by (1.8), (1.19) and (1.20). Then if $\kappa \in [1/2, \infty[$, for any $\gamma \in [0, 1]$ and $t \leq T$, one has
\[
\begin{align*}
(2.11a) \quad &\|e^{\Psi_\kappa} \Delta^h_k \varphi \Phi_\kappa \|_{L^2_t} \lesssim (t) \cdot \|e^{\Psi_\kappa} \Delta^h_k G \Phi_\kappa \|_{L^2_t}, \quad \|e^{\Psi_\kappa} \Delta^h_k \psi \Phi_\kappa \|_{L^2_t} \lesssim (t) \cdot \|e^{\Psi_\kappa} \Delta^h_k H \Phi_\kappa \|_{L^2_t}, \\
(2.11b) \quad &\|e^{\Psi_\kappa} \Delta^h_k u \Phi_\kappa \|_{L^2_t} \lesssim \|e^{\Psi_\kappa} \Delta^h_k G \Phi_\kappa \|_{L^2_t}, \quad \|e^{\Psi_\kappa} \Delta^h_k b \Phi_\kappa \|_{L^2_t} \lesssim \|e^{\Psi_\kappa} \Delta^h_k H \Phi_\kappa \|_{L^2_t}, \\
(2.11c) \quad &\|e^{\Psi_\kappa} \Delta^h_k \partial_y u \Phi_\kappa \|_{L^2_t} \lesssim \|e^{\Psi_\kappa} \Delta^h_k \partial_y G \Phi_\kappa \|_{L^2_t}, \quad \|e^{\Psi_\kappa} \Delta^h_k \partial_y b \Phi_\kappa \|_{L^2_t} \lesssim \|e^{\Psi_\kappa} \Delta^h_k \partial_y H \Phi_\kappa \|_{L^2_t}.
\end{align*}
\]
We shall present the proof of this lemma at the end of this section.

Now we present the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Along the same line to the proof of Theorem 1.1 let $(u, b)$ and $(\varphi, \psi)$ be smooth enough solutions of (1.4) and (1.6) respectively on $[0, T^*_\kappa \left[ \right. \right.$, where $T^*_\kappa$ is the maximal time of existence of the solution. Let $(G, H)$ be defined by (1.8). For any $t < T^*_\kappa$ (of course here $T^*_\kappa \leq T^*_1$) with $T^*_1$ being defined by (2.7), we deduce from (1.18) that
\[
\theta_\kappa (t) \leq \int_0^t \langle \tau \rangle^{-\frac{1}{2}} \|e^{\Psi_\kappa} \partial_y (G, H) \Phi_\kappa (\tau)\|_{G^\frac{1}{2}, 0} + \epsilon^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} \|(U, B) \Phi_\kappa (\tau)\|_{G^\frac{1}{2}, 0} d\tau.
\]
We take $\epsilon_0 \leq \frac{\epsilon_1}{\kappa}$, then we deduce from the assumption (1.14b) and Proposition 2.6 that
\[
\theta_\kappa (t) \leq C \sqrt{\epsilon} (1 + \|e^{\frac{\sqrt{2}}{\kappa^2} e^{\delta D_x}(u_0, b_0)}\|_{G^\frac{1}{2}, 0}) \text{ for any } \epsilon \leq \epsilon_0 \text{ and } t < T^*_\kappa.
\]
If we take $\epsilon_0$ so small that $\epsilon_0 \leq \min \left( \frac{\delta}{2C \lambda (1 + \|e^{\frac{\sqrt{2}}{\kappa^2} e^{\delta D_x}(u_0, b_0)}\|_{G^\frac{1}{2}, 0})}, \frac{\epsilon_1}{\kappa} \right)$, then we achieve
\[
\sup_{t \in [0, T^*_\kappa]} \theta_\kappa (t) \leq \frac{\delta}{2\lambda} \text{ for any } \epsilon \leq \epsilon_0.
\]
Then in view of (2.7), we get by a continuous argument that $T^*_\kappa = \infty$. And (5.14) holds for $t = \infty$, which implies (1.23). Moreover, Propositions 2.5 and 2.6 ensure (1.24a) and (1.24b) respectively. Finally, Proposition 2.6 together with Lemma 2.2 implies (1.24d). This finishes the existence part of Theorem 1.2. The uniqueness part has already been proved in [32].

Before the proof of Lemmas 2.1 and 2.2 following [18, 27], we first introduce the following Poincaré type inequalities.
Lemma 2.3. Let $\Psi_\kappa$ be defined by (1.20). Let $f$ be a smooth enough function on $\mathbb{R}^2_+$ which decays to zero sufficiently fast as $y$ approaching to $+\infty$. Then we have

\begin{align}
\|e^{\Psi_\kappa} \partial_y f\|^2_{L^4_+} &\geq \frac{1}{2\kappa(t)} \|e^{\Psi_\kappa} f\|^2_{L^4_+}, \\
\|e^{\Psi_\kappa} \partial_y f\|^2_{L^4_+} &\geq \frac{1}{4\kappa(t)} \|e^{\Psi_\kappa} f\|^2_{L^4_+} + \frac{1}{16\kappa^2(t)^2} \|e^{\Psi_\kappa} yf\|^2_{L^4_+},
\end{align}

Especially, when $\kappa = 1$, we have

\begin{align}
\|e^\Psi \partial_y f\|^2_{L^4_+} &\geq \frac{1}{2(t)} \|e^\Psi f\|^2_{L^4_+}, \\
\|e^\Psi \partial_y f\|^2_{L^4_+} &\geq \frac{1}{4(t)} \|e^\Psi f\|^2_{L^4_+} + \frac{1}{16(t)^2} \|e^\Psi yf\|^2_{L^4_+}.
\end{align}

where (2.13a) recovers (3.1) of [27].

Proof. As in [18, 27], we get, by using integration by parts, that

\begin{align}
\int_{\mathbb{R}^2_+} f^2(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy &= \int_{\mathbb{R}^2_+} (\partial_y y)f^2(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy \\
&= -2\int_{\mathbb{R}^2_+} yf(x,y)\partial_y f(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy - \frac{1}{2\kappa(t)} \int_{\mathbb{R}^2_+} y^2 f^2(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy \\
&\leq 2\kappa(t) \int_{\mathbb{R}^2_+} (\partial_y f)^2(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy.
\end{align}

This leads to (2.12a).

We also observe from (2.14) that

\begin{align}
\int_{\mathbb{R}^2_+} f^2(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy &\leq 4\kappa(t) \int_{\mathbb{R}^2_+} (\partial_y f)^2(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy \\
&\quad - \frac{1}{4\kappa(t)} \int_{\mathbb{R}^2_+} y^2 f^2(x,y)e^{\frac{y^2}{4\kappa(t)}} dxdy,
\end{align}

which yields (2.12b). \qed

Let us end this section with the proof of Lemmas 2.1 and 2.2.

Proof of Lemma 2.7. The proof of this lemma basically follows from that of Lemma 3.2 in [27]. Due to $(u, b) = \partial_y (\varphi, \psi)$ and $\varphi|_{y=0} = \psi|_{y=0} = 0$, we deduce from (1.18) that

\begin{align}
\varphi(t, x, y) &= e^{-2\Psi(t,y)} \int_0^y e^{2\Psi(t,y')} G(t, x, y') dy', \\
\psi(t, x, y) &= e^{-2\Psi_\kappa(t,y)} \int_0^y e^{2\Psi_\kappa(t,y')} H(t, x, y') dy',
\end{align}

with $\Psi$ and $\Psi_\kappa$ being determined respectively by (1.12) and (1.20).

Thanks to (2.15), we infer

\begin{align}
u &= G - \frac{y}{2\kappa(t)} e^{-2\Psi} \int_0^y e^{2\Psi(y')} G dy', \\
b &= H - \frac{y}{2\kappa(t)} e^{-2\Psi_\kappa} \int_0^y e^{2\Psi_\kappa(y')} G dy',
\end{align}
and

\[(2.17a) \quad \partial_y u = \partial_y G - \frac{y}{2(t)} G + \frac{1}{2(t)} \left( \frac{y^2}{2(t)} - 1 \right) e^{-2\Psi} \int_0^y e^{2\Psi(y')} G dy', \]

\[(2.17b) \quad \partial_y b = \partial_y H - \frac{y}{2(t)} H + \frac{1}{2\kappa(t)} \left( \frac{y^2}{2\kappa(t)} - 1 \right) e^{-2\Psi_k a} \int_0^y e^{2\Psi_k a(y')} H dy'. \]

Let us use the relationship between \( b \) and \( H \) to prove the second inequalities of (2.6a)-(2.6c). The other inequalities follows by taking \( \kappa = 1 \) in the proof.

In view of (2.16b) and

\[(2.18) \quad \sup_{y \in [0, \infty]} \left( e^{-y^2} \int_0^y e^{z^2} dz \right) < \infty, \]

we infer that (if \( \kappa \in ]0, 2[ \Rightarrow \frac{2}{\kappa} - 1 > 0 \))

\[
\| e^{\gamma \Delta_k^b b_f} \|_{L^2_t} \lesssim \| e^{\gamma \Delta_k^b H_f} \|_{L^2_t} \\
+ \langle t \rangle^{-\frac{1}{2}} \left| ye^{(\gamma-\frac{2}{\kappa})\Psi} \left( \int_0^y e^{2\left(\frac{2}{\kappa} - 1\right)\Psi} dy' \right) \right| \left( \int_0^\infty |e^{\gamma \Delta_k^b H_f}|^2 dy \right)^{\frac{1}{2}} \|_{L^2_t} \\
\lesssim \| e^{\gamma \Delta_k^b H_f} \|_{L^2_t} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \left| ye^{(\gamma-\frac{2}{\kappa})\Psi} \left( \int_0^y e^{2\left(\frac{2}{\kappa} - 1\right)\Psi} dy' \right) \right| \right) \|_{L^2_t} \\
\lesssim \| e^{\gamma \Delta_k^b H_f} \|_{L^2_t} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \| ye^{(\gamma-\frac{2}{\kappa})\Psi} \|_{L^2_t} \right),
\]

from which and \( \gamma < 1 \), we deduce the second inequality of (2.6b).

(2.6a) follows from (2.6b) and Lemma 2.3.

Whereas due to \( \lim_{y \to +\infty} H(t, x, y) = 0 \), we write \( H = -\int_y^\infty \partial_y H dy' \), and then, by using

\[(2.19) \quad \sup_{y \in [0, \infty]} \left( e^{y^2} \int_y^\infty e^{-z^2} dz \right) < \infty, \]

we infer

\[
|\Delta_k^b H_f| \leq \left( \int_y^\infty e^{-2\Psi} dy' \right)^{\frac{1}{2}} \left( \int_0^\infty |e^{\gamma \Delta_k^b \partial_y H_f}|^2 dy \right)^{\frac{1}{2}} \\
\lesssim \langle t \rangle^{-\frac{1}{2}} e^{-\Psi} \left( \int_0^\infty |e^{\gamma \Delta_k^b \partial_y H_f}|^2 dy \right)^{\frac{1}{2}},
\]

from which, (2.17b) and (2.18), we infer that (if \( \kappa \in ]0, 2[ \Rightarrow \frac{2}{\kappa} - 1 > 0 \))

\[
\| e^{\gamma \Delta_k^b \partial_y b_f} \|_{L^2_t} \lesssim \| e^{\gamma \Delta_k^b \partial_y H_f} \|_{L^2_t} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \| ye^{(\gamma-\frac{2}{\kappa})\Psi} \|_{L^2_t} \right) \\
+ \langle t \rangle^{-\frac{1}{2}} \left( \frac{y^2}{\langle t \rangle} - 1 \right) e^{\gamma \Delta_k^b \partial_y H_f} \left( \int_0^\infty e^{2\left(\frac{2}{\kappa} - 1\right)\Psi} dy' \right) \|_{L^2_t} \\
\lesssim \| e^{\gamma \Delta_k^b \partial_y H_f} \|_{L^2_t} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \| ye^{(\gamma-\frac{2}{\kappa})\Psi} \|_{L^2_t} + \langle t \rangle^{-\frac{1}{2}} \left( \frac{y^2}{\langle t \rangle} - 1 \right) e^{\gamma \Delta_k^b \partial_y H_f} \right),
\]

from which and \( \gamma < 1 \), we deduce (2.6c). We thus conclude the proof of Lemma 2.1. \( \square \)

**Proof of Lemma 2.2.** The proof of this lemma is similar to that of Lemma 2.1. For completeness, we shall deal with the first inequalities of (2.11a)-(2.11c).
In view of (2.16a) and (2.18), we infer that (if \( \kappa > \frac{1}{2} \Rightarrow 2\kappa - 1 > 0 \))
\[
\| e^\gamma \Psi_k \Delta^b_k u_k \|_{L^2_+} \lesssim \| e^\Psi_k \Delta^b_k y G \|_{L^2_+} \\
+ \langle t \rangle^{-1} \| y e^{(\gamma - 2\kappa)} \Psi_k \left( \int_y y e^{(2\kappa - 1)\Psi_k} dy' \right) \|_{L^2_+}^{\frac{1}{2}} \| e^\Psi_k \Delta^b_k y G \|_{L^2_+}^{\frac{1}{2}} \| e^\Psi_k \Delta^b_k y G \|_{L^2_+}^{\frac{1}{2}} \\
\lesssim \| e^\Psi_k \Delta^b_k y G \|_{L^2_+} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \| y e^{(\gamma - 1)\Psi_k} \|_{L^2_+} \right) \\
\lesssim \| e^\Psi_k \Delta^b_k y G \|_{L^2_+} \left( 1 + \langle t \rangle^{-\frac{1}{4}} \| y e^{(\gamma - 1)\Psi_k} \|_{L^2_+} \right),
\]
from which and \( \gamma < 1 \), we deduce the first inequality of (2.11b).

Whereas due to \( \lim_{y \to +\infty} G(t, x, y) = 0 \), we write \( G = -\int_y^\infty \partial_y G dy' \), and then it follows from (2.19) that
\[
| \Delta^b_k y G \| \leq \left( \int_y^\infty e^{-2\Psi_k} dy' \right)^{\frac{1}{2}} \left( \int_0^\infty \| e^\Psi_k \Delta^b_k \partial_y G \|_{L^2_+}^2 \right)^{\frac{1}{2}} \\
\lesssim \langle t \rangle^{\frac{1}{4}} \| e^\Psi_k \Delta^b_k \partial_y G \|_{L^2_+} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \| y e^{(\gamma - 1)\Psi_k} \|_{L^2_+} \right)
\]
from which, (2.17a) and (2.18), we infer that (if \( \kappa > \frac{1}{2} \Rightarrow 2\kappa - 1 > 0 \))
\[
\| e^\gamma \Psi_k \Delta^b_k \partial_y u_k \|_{L^2_+} \lesssim \| e^\Psi_k \Delta^b_k \partial_y G \|_{L^2_+} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \| y e^{(\gamma - 1)\Psi_k} \|_{L^2_+} \right) \\
+ \langle t \rangle^{-\frac{3}{4}} \| y^2 (t) - 1 \| e^{(\gamma - 2\kappa)\Psi_k} \left( \int_y y e^{(2\kappa - 1)\Psi_k} dy' \right) \|_{L^2_+}^{\frac{1}{2}} \| e^\Psi_k \Delta^b_k \partial_y G \|_{L^2_+}^{\frac{1}{2}} \\
\lesssim \| e^\Psi_k \Delta^b_k \partial_y y G \|_{L^2_+} \left( 1 + \langle t \rangle^{-\frac{1}{2}} \| y e^{(\gamma - 1)\Psi_k} \|_{L^2_+} \right) \\
+ \langle t \rangle^{-\frac{3}{4}} \| y^2 (t) - 1 \| e^{(\gamma - 1)\Psi_k} \|_{L^2_+}^{\frac{1}{2}} \| e^\Psi_k \Delta^b_k \partial_y G \|_{L^2_+}^{\frac{1}{2}},
\]
from which and \( \gamma < 1 \), we deduce the first inequality of (2.11c). We thus conclude the proof of Lemma 2.2 \( \Box \)

3. Preliminaries and Some Technical Lemmas

In this section, we will present a few technical lemmas which will be frequently used in the subsequent sections.

In order to overcome the difficulty that one can not use Gronwall type argument in the framework of Chemin-Lerner space, we need to use the time-weighted Chemin-Lerner norm, which was introduced by Paicu and the second author in [20].

**Definition 3.1.** Let \( f(t) \in L^1_{lo} (\mathbb{R}^+) \) be a nonnegative function and \( t_0, t \in [0, \infty] \). We define
\[
\| a \|_{\tilde{L}_{[t_0, t]}^p (B^{s,q})} \equiv \sum_{k \in \mathbb{Z}} 2^{ks} \left( \int_{t_0}^t f(t') \| \Delta^b_k a(t') \|_{L^2_+}^p \| L^2_+ \right)^{\frac{1}{p}}.
\]
When \( t_0 = 0 \), we simplify the notation \( \| a \|_{\tilde{L}_{[0, t]}^p (B^{s,q})} \) as \( \| a \|_{\tilde{L}_{t,f}^p (B^{s,q})} \).

We also recall the following anisotropic Bernstein type lemma from [7, 25]:

**Lemma 3.1.** Let \( B_h \) be a ball of \( \mathbb{R}^n \), and \( C_h \) a ring of \( \mathbb{R}^n \); let \( 1 \leq p_2 \leq p_1 \leq \infty \) and \( 1 \leq q \leq \infty \). Then there holds:
If the support of $\hat{a}$ is included in $2^k B_h$, then
\[
\|\partial_x^2 a\|_{L^p_h(L^q)} \lesssim 2^{k [max + (\frac{1}{p_2} - \frac{1}{q_1})]} \|a\|_{L^p_2(L^q)}.
\]
If the support of $\hat{a}$ is included in $2^k C_h$, then
\[
\|a\|_{L^p_h(L^q)} \lesssim 2^{-kN} \|\partial_x^N a\|_{L^p_h(L^q)}.
\]

In the following context, we shall constantly use Bony’s decomposition (see [3]) for the horizontal variable:
\[
(f g) = T^h_f g + T^h_g f + R^h(f, g),
\]
where
\[
T^h_f g = \sum_k S^h k \Delta^h f \text{ and } R^h(f, g) = \sum_k \Delta^h f \Delta^h g \text{ with } \Delta^h_k g = \sum_{|k - k'| \leq 1} \Delta^h_{k'} g.
\]

In the rest of this paper, we always designate $\Psi$ and $\Psi_{\kappa}$ to be the functions defined respectively by (1.12) and (1.20). We present the weighted energy estimate for the linear heat equations:

**Lemma 3.2.** Let $f$ be a smooth enough function on $\mathbb{R}^2_+$, which decays to zero sufficiently fast as $y$ approaching to $+\infty$ and satisfies $f \partial_y f|_{y=0} = 0$. Then we have, if $\kappa \in ]0, 2[,]$

\[
(\partial_t f - \partial^2_y f|e^{2\Psi f})_{L^2} \geq \frac{1}{2} \frac{d}{dt} \|e^{\Psi f}\|_{L^2}^2 + \frac{1}{2} \|e^{\Psi f} \partial_y f\|_{L^2}^2,
\]

\[
(\partial_t f - \partial^2_y f|e^{2\Psi_{\kappa} f})_{L^2} \geq \frac{1}{2} \frac{d}{dt} \|e^{\Psi_{\kappa} f}\|_{L^2}^2 + \frac{\kappa(2 - \kappa)}{2} \|e^{\Psi_{\kappa} } \partial_y f\|_{L^2}^2,
\]

and if $\kappa > \frac{1}{2}$,

\[
(\partial_t f - \partial^2_y f|e^{2\Psi_{\kappa} f})_{L^2} \geq \frac{1}{2} \frac{d}{dt} \|e^{\Psi_{\kappa} f}\|_{L^2}^2 + \frac{2 - 1}{2\kappa} \|e^{\Psi_{\kappa} \partial_y f}\|_{L^2}^2,
\]

\[
(\partial_t f - \partial^2_y f|e^{2\Psi_{\kappa} f})_{L^2} \geq \frac{1}{2} \frac{d}{dt} \|e^{\Psi_{\kappa} f}\|_{L^2}^2 + \frac{\kappa}{2} \|e^{\Psi_{\kappa} \partial_y f}\|_{L^2}^2.
\]

**Proof.** Indeed it is enough to prove the following inequality for any $\alpha, \beta > 0$,

\[
(\partial_t f - \beta \partial_y^2 f|e^{2\alpha f})_{L^2} \geq \frac{1}{2} \frac{d}{dt} \|e^{2\alpha f}\|_{L^2}^2 + (\beta - \frac{\beta^2 \alpha}{2}) \|e^{2\alpha \partial_y f}\|_{L^2}^2.
\]

Then (3.3a) and (3.3b) follow by taking $\alpha \in \{1, \frac{1}{\kappa}\}$ and $\beta \in \{1, \kappa\}$ in (3.5).

We first get, by using integration by parts, that

\[
(\partial_t f|e^{2\alpha f})_{L^2} = \frac{1}{2} \frac{d}{dt} \|e^{2\alpha f}\|_{L^2}^2 - \alpha \int_{\mathbb{R}^2_+} \partial_t \Psi |e^{2\alpha f}|^2 dx dy.
\]

While due to $f \partial_y f|_{y=0} = 0$, we get, by using integration by parts and then Young’s inequality, that

\[
(-\beta \partial_y^2 f|e^{2\alpha f})_{L^2} = \beta \|e^{\alpha \partial_y f}\|_{L^2}^2 + \alpha \beta \int_{\mathbb{R}^2_+} \partial_y \Psi e^{2\alpha f} \partial_y f dx dy
\]

\[
\geq (\beta - \frac{\beta^2 \alpha}{2}) \|e^{\alpha \partial_y f}\|_{L^2}^2 - 2\alpha \int_{\mathbb{R}^2_+} (\partial_y \Psi)^2 |e^{2\alpha f}|^2 dx dy.
\]

Thanks to (1.13), summing up (3.6) and (3.7) leads to (3.5). This finishes the proof of Lemma 3.2. \qed
To handle the non-linear terms in (1.3), (1.6) and (6.4), we need the following lemmas:

**Lemma 3.3.** If $\delta - \lambda \theta(t) > 0$, then $\|\phi(D_x)[fg] \|_{L^2_\delta} \leq \|\phi(D_x)(f \phi g)\|_{L^2_\delta}$. Similar inequality holds for $\Phi_\kappa$ under the assumption that $\delta - \lambda \theta_\kappa(t) > 0$.

**Proof.** Indeed we deduce from (3.2) and Plancherel equality that

$$\|\phi(D_x)[fg] \|_{L^2_\delta} = \|\phi(\xi)e^{\phi(t,\xi)}(\hat{f} \ast \hat{g})\|_{L^2_\xi} \leq \|\phi(\xi)(e^{\phi} \hat{f}) * (e^{\phi} \hat{g})\|_{L^2_\xi} = \|\phi(D_x)(f \phi g)\|_{L^2_\delta}.$$ 

This concludes the proof of this lemma. \(\square\)

**Lemma 3.4.** Let $A, B$ and $E$ be smooth enough functions on $[0, T] \times \mathbb{R}^2_+$ with $A$ satisfying $\lim_{y \to +\infty} A = 0$. Let $f(t) \overset{\text{def}}{=} \langle t \rangle^\frac{1}{4}|e^{\phi_0} \partial_y A \Phi(t)\|_2$. Then, for any $a, b, c, d > 0$ with $a + b \geq c$ and any $t_1 < t_2 \in [0, T]$, we have

$$\int_{t_1}^{t_2} |(e^{\phi_0} \Delta^h_k[A \partial_x B] \Phi | e^{\phi_0} \Delta^h_k E \Phi)_{L^2_\kappa}| \, dt \lesssim \int_{t_1}^{t_2} \|e^{\phi_0} \partial_y B \Phi\|_{L^2_{t_1} \times \kappa} \|e^{\phi_0} E \Phi\|_{L^2_{t_1} \times \kappa}.$$ 

**Proof.** Applying Bony’s decomposition (3.2) in the horizontal variable to $A \partial_x B$ yields

$$A \partial_x B = T^h_A \partial_x B + T^h_{\partial_x B} A + R^h(A, \partial_x B).$$

By virtue of Lemma 3.3 and considering the support properties to the Fourier transform of the terms in $T^h_{\partial_x B}$, we find

$$\|e^{\phi_0} \Delta^h_k[T^h_A \partial_x B] \Phi\|_{L^2_k} \lesssim \sum_{|k' - k| \leq 4} \|e^{(c-b)\phi_0} S^h_{k'-1} A \Phi\|_{L^\infty_\kappa} \|e^{b\phi_0} \Delta^h_k \partial_x B \Phi\|_{L^2_k}.$$ 

While it follows from $\lim_{y \to +\infty} A = 0$, (2.18) and $c - a - b \leq 0$ that

$$\|e^{(c-b)\phi_0} \Delta^h_k A \Phi\|_{L^\infty_\kappa(L^2_\delta)} = \|\int_y^{\infty} \Delta^h_k \partial_y A \Phi\|_{L^\infty_\kappa(L^2_\delta)} \leq \|\int_y^{\infty} e^{-2\phi_0} \|_{L^\infty_\kappa} \|e^{a\phi_0} \Delta^h_k \partial_y A \Phi\|_{L^2_\kappa} \lesssim \langle t \rangle^{\frac{1}{2}} \sum d_k(t) \|e^{a\phi_0} \partial_y A \Phi\|_{B^\frac{1}{2}, 0}.$$ 

Here and in all that follows, we always denote $(d_k(t))_{k \in \mathbb{Z}}$ to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{k \in \mathbb{Z}} d_k(t) = 1$. (3.9) together with Lemma 3.1 ensures that

$$\|e^{(c-b)\phi_0} S^h_{k'-1} A \Phi\|_{L^\infty_\kappa} \lesssim \sum_{k \leq k' - 2} 2^{\frac{k}{2}} \|e^{(c-b)\phi_0} \Delta^h_k A \Phi\|_{L^\infty_\kappa(L^2_\delta)} \lesssim \langle t \rangle^{\frac{1}{2}} \|e^{a\phi_0} \partial_y A \Phi\|_{B^\frac{1}{2}, 0}.$$
So that by applying Lemma 3.1 and Definition 3.1 we deduce that

$$
\int_{t_1}^{t_2} \left| \left( e^{\psi \Delta^h_k[T^h_{\partial_x B}]_\Phi} e^{\psi \Delta^h_k E_\Phi} \right)_{L^2_+} \right| dt \\
\quad \lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_{t_1}^{t_2} f(t) \| e^{\psi \Delta^h_k B_\Phi(t)} \|_{L^2_+} \| e^{\psi \Delta^h_k E_\Phi(t)} \|_{L^2_+} dt
$$

(3.10)

\begin{align*}
\lesssim \sum_{|k'-k| \leq 4} & 2^{k'} \left( \int_{t_1}^{t_2} f(t) \| e^{\psi \Delta^h_k B_\Phi(t)} \|_{L^2_+}^2 dt \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} f(t) \| e^{\psi \Delta^h_k E_\Phi(t)} \|_{L^2_+}^2 dt \right)^{\frac{1}{2}} \\
\lesssim & d_2^2 2^{-k} \| e^{\psi B_\Phi} \|_{\tilde{L}^2_{|t_1-t_2|, f(B^1, 0)}} \| e^{\psi E_\Phi} \|_{\tilde{L}^2_{|t_1-t_2|, f(B^1, 0)}}.
\end{align*}

Similarly, by Lemma 3.3 and considering the support properties to the Fourier transform of the terms in $T^h_{\partial_x B} A$, we get

$$
\| e^{\psi \Delta^h_k [T^h_{\partial_x B} A]_\Phi} \|_{L^2_+} \lesssim \sum_{|k'-k| \leq 4} \int_{t_1}^{t_2} \| e^{(c-b) \psi \Delta^h_k A_\Phi} \|_{L^\infty} \| e^{b \psi S^h_{k'-1} \partial_x B_\Phi} \|_{L^2_+} dt,
$$

from which, 3.9, we infer

$$
\int_{t_1}^{t_2} \left| \left( e^{\psi \Delta^h_k[T^h_{\partial_x B} A]_\Phi} e^{\psi \Delta^h_k E_\Phi} \right)_{L^2_+} \right| dt
\quad \lesssim \sum_{|k'-k| \leq 4} \int_{t_1}^{t_2} d_{k'}(t) \| e^{\psi \partial_x B_\Phi(t)} \|_{L^2_+} \| e^{\psi \Delta^h_k E_\Phi} \|_{L^2_+} dt
$$

$$
\lesssim \sum_{|k'-k| \leq 4} d_{k'} \left( \int_{t_1}^{t_2} f(t) \| e^{b \psi \partial_x B_\Phi(t)} \|_{L^2_+}^2 dt \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} f(t) \| e^{\psi \Delta^h_k E_\Phi(t)} \|_{L^2_+}^2 dt \right)^{\frac{1}{2}}.
$$

Yet it follows from Lemma 3.1 and Definition 3.1 that

$$
\left( \int_{t_1}^{t_2} f(t) \| e^{b \psi \partial_x B_\Phi(t)} \|_{L^2_+}^2 dt \right)^{\frac{1}{2}} \lesssim \sum_{k \in \mathbb{Z}} 2^{k} \left( \int_{t_1}^{t_2} f(t) \| e^{\psi \Delta^h_k B_\Phi(t)} \|_{L^2_+}^2 dt \right)^{\frac{1}{2}}
$$

(3.11)

$$
\lesssim \| e^{b \psi B_\Phi} \|_{\tilde{L}^2_{|t_1-t_2|, f(B^1, 0)}}.
$$

As a result, we obtain

$$
\int_{t_1}^{t_2} \left| \left( e^{\psi \Delta^h_k[T^h_{\partial_x B} A]_\Phi} e^{\psi \Delta^h_k E_\Phi} \right)_{L^2_+} \right| dt
\quad \lesssim d_2^2 2^{-k} \| e^{b \psi B_\Phi} \|_{\tilde{L}^2_{|t_1-t_2|, f(B^1, 0)}} \| e^{\psi E_\Phi} \|_{\tilde{L}^2_{|t_1-t_2|, f(B^1, 0)}}.
$$

(3.12)

Finally again due to Lemma 3.3 and the support properties to the Fourier transform of the terms in $R^h(A, \partial_x B)$, we get, by applying Lemma 3.1 that

$$
\| e^{\psi \Delta^h_k [R^h(A, \partial_x B)]_\Phi} \|_{L^2_+} \lesssim 2^\frac{k}{2} \sum_{k'' \geq k-3} \| e^{(c-b) \psi \Delta^h_k A_\Phi} \|_{L^\infty} \| e^{\psi \Delta^h_{k''} \partial_x B_\Phi} \|_{L^2_+},
$$

(3.13)
Lemma 3.5. Under the assumptions of Lemma 3.4 one has

\[ \int_{t_1}^{t_2} \left( \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \sum_{|k' - k| \leq 4} \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]

Proof. By applying Bony’s decomposition (3.12) in the horizontal variable to \( \partial_y A \int_y^\infty \partial_x Bdy' \), we write

\[ \partial_y A \int_y^\infty \partial_x Bdy' = T_{\partial_y A} \int_y^\infty \partial_x Bdy' + T_{\partial_y A} \int_y^\infty \partial_x Bdy' + R_{\partial_y A} \int_y^\infty \partial_x Bdy' \]

Again thanks to Lemma 3.3 and considering the support properties to the Fourier transform of the terms in \( T_{\partial_y A} \int_y^\infty \partial_x Bdy' \), we find

\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]

While it follows from (3.13) and \( c - a - b \leq 0 \)

\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]
\[ \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \]

As a result, it comes out

\[ \int_{t_1}^{t_2} \left| \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]
\[ \left. \left\| e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right\| \right. \]

from which, (3.9) and Lemma 3.4 we deduce that

\[ \int_{t_1}^{t_2} \left( e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right) \Phi \left( e^{\partial_y A} \int_y^\infty \partial_x Bdy' \right) \left. \right| \]
\[ \left. \right| \]
\[ \left. \right| \]
\[ \left. \right| \]
\[ \left. \right| \]

By summing up the estimates (3.10), (3.12) and (3.13), we conclude the proof of (3.8). \( \square \)
from which, we get, by a similar derivation of (3.11), that

\[ \int_{t_1}^{t_2} \left| (e^{d\Psi} \Delta^h_k \Phi, \int_y^\infty \partial_y Bdy') \right|_{L^2_y} \leq d^2 2^{-k} \left\| e^{b\Psi} B \right\|_{L^2_{[t_1, t_2]; L_y^B}} \left\| e^{d\Psi} \Delta^h_k E \right\|_{L^2_{[t_1, t_2]; L_y^B}}. \]  

(3.16)

Along the same line, we get

\[ \int_{t_1}^{t_2} \left| \left( e^{d\Psi} \Delta^h_k \Phi, \int_y^\infty \partial_y Bdy' \right) \right|_{L^2_y} \leq \sum_{|k'-k| \leq 4} \left\| e^{d\Psi} \Delta^h_{k'} A \right\|_{L^2_y(L_y^B)} \times \left\| e^{(c-a)\Psi} \int_y^\infty \partial_y Bdy' \right\|_{L^\infty_y(L_y^B)} \left\| e^{d\Psi} \Delta^h_k E \Phi(t) \right\|_{L^2_y} dt \]

\[ \leq d^2 2^{-k} \left\| e^{b\Psi} B \right\|_{L^2_{[t_1, t_2]; L_y^B}} \left\| e^{d\Psi} \Delta^h_k E \Phi(t) \right\|_{L^2_y} dt \]  

Yet in view of (3.15), we get, by a similar derivation of (3.11), that

\[ \left( \int_{t_1}^{t_2} f(t) \right)^{\frac{1}{2}} \leq \left\| e^{d\Psi} \Delta^h_k E \Phi(t) \right\|_{L^2_y} dt \leq \left\| e^{b\Psi} B \right\|_{L^2_{[t_1, t_2]; L_y^B}}. \]

Hence we obtain

\[ \int_{t_1}^{t_2} \left| \left( e^{d\Psi} \Delta^h_k \Phi, \int_y^\infty \partial_y Bdy' \right) \right|_{L^2_y} \leq d^2 2^{-k} \left\| e^{b\Psi} B \right\|_{L^2_{[t_1, t_2]; L_y^B}} \left\| e^{d\Psi} \Delta^h_k E \Phi(t) \right\|_{L^2_y} dt \]  

(3.17)

Finally, we get, by applying Lemma 3.1 and (3.15), that

\[ \int_{t_1}^{t_2} \left| \left( e^{d\Psi} \Delta^h_k \Phi, \int_y^\infty \partial_y Bdy' \right) \right|_{L^2_y} \leq 2^\frac{k'}{2} \sum_{k' \leq k-3} \left( \int_{t_1}^{t_2} \left\| e^{a\Psi} \Delta^h_{k'} A \right\|_{L^2_y} \left\| e^{(c-a)\Psi} \int_y^\infty \Delta^h_{k'} \partial_y Bdy' \right\|_{L^\infty_y(L_y^B)} \left\| e^{d\Psi} \Delta^h_k E \Phi(t) \right\|_{L^2_y} dt \right) \]

\[ \leq 2^\frac{k'}{2} \sum_{k' \leq k-3} 2^{k'} \int_{t_1}^{t_2} f(t) \left\| e^{b\Psi} \Delta^h_{k'} B \Phi(t) \right\|_{L^2_y} \left\| e^{d\Psi} \Delta^h_k E \Phi(t) \right\|_{L^2_y} dt, \]

which together with a similar derivation of (3.13) gives rise to

\[ \int_{t_1}^{t_2} \left| \left( e^{d\Psi} \Delta^h_k \Phi, \int_y^\infty \partial_y Bdy' \right) \right|_{L^2_y} \leq d^2 2^{-k} \left\| e^{b\Psi} B \right\|_{L^2_{[t_1, t_2]; L_y^B}} \left\| e^{d\Psi} \Delta^h_k E \Phi(t) \right\|_{L^2_y} dt \]

Together with (3.16), (3.17), we complete the proof of (3.14).
Lemma 3.6. Under the assumptions of Lemma 3.4, if we assume moreover that \( a + b > c \). Then we have

\[
\int_{t_1}^{t_2} \left( e^{c \Psi} \int_y^\infty \Delta_k^h [\partial_y A \partial_x B] \Psi \, dy' \right) e^{d \Psi} L_k^h E_x \, dt 
\leq d_k^2 2^{-k} \| e^{b \Psi} B \Phi \| \tilde{L}_{[t_1, t_2], f(B^{1,0})} \| e^{d \Psi} E_x \| \tilde{L}^2_{[t_1, t_2], f(B^{1,0})}.
\]

Proof. Applying Bony’s decomposition (3.2) in the horizontal variable to \( \partial_y A \partial_x B \) yields

\[
\partial_y A \partial_x B = T^h_{\partial_y A \partial_x B} + T^h_{\partial_x B \partial_y A} + R^h(\partial_y A, \partial_x B)
\]

We first observe that

\[
\| e^{c \Psi} \int_y^\infty \Delta_k^h [T^h_{\partial_y A \partial_x B}] \Psi \, dy' \| L_k^h 
\lesssim \| e^{(c - a - b) \Psi} \| L_k^h \sum_{|k' - k| \leq 4} \| e^{a \Psi} S^h_{k' - 1} \partial_y A \Psi \| L^2_{[t_1, t_2], f(B^{1,0})} \| e^{b \Psi} \Delta_k^h \partial_x B \Psi \| L_k^h.
\]

Due to \( c - a - b < 0 \), one has

\[
\| e^{(c - a - b) \Psi} \| L_k^h \lesssim \langle t \rangle^{\frac{1}{4}},
\]

from which and Lemma 3.1, we infer

\[
\| e^{c \Psi} \int_y^\infty \Delta_k^h [T^h_{\partial_y A \partial_x B}] \Psi \, dy' \| L_k^h 
\lesssim \sum_{|k' - k| \leq 4} 2^{k'} f(t) \| e^{b \Psi} \Delta_k^h \partial_x B \Psi \| L_k^h.
\]

Then we get, by a similar derivation of (3.10), that

\[
\int_{t_1}^{t_2} \left( e^{c \Psi} \int_y^\infty \Delta_k^h [T^h_{\partial_x B \partial_y A}] \Psi \, dy' \right) e^{d \Psi} L_k^h E_x \, dt 
\leq d_k^2 2^{-k} \| e^{b \Psi} B \Phi \| \tilde{L}_{[t_1, t_2], f(B^{1,0})} \| e^{d \Psi} E_x \| \tilde{L}^2_{[t_1, t_2], f(B^{1,0})}.
\]

Along the same line, by applying Lemma 3.1 and (3.19), we find

\[
\| e^{c \Psi} \int_y^\infty \Delta_k^h [T^h_{\partial_x B \partial_y A}] \Psi \, dy' \| L_k^h 
\lesssim \| e^{(c - a - b) \Psi} \| L_k^h \sum_{|k' - k| \leq 4} \| e^{a \Psi} \Delta_k^h \partial_y A \Psi \| L^2_{[t_1, t_2], f(B^{1,0})} \| e^{b \Psi} S^h_{k' - 1} \partial_x B \Psi \| L_k^h
\lesssim \sum_{|k' - k| \leq 4} d_k' (t) f(t) \| e^{b \Psi} \partial_x B \Psi \| L_k^h.
\]

Then thanks to (3.11), we get, by applying a similar derivation of (3.12), that

\[
\int_{t_1}^{t_2} \left( e^{c \Psi} \int_y^\infty \Delta_k^h [T^h_{\partial_x B \partial_y A}] \Psi \, dy' \right) e^{d \Psi} L_k^h E_x \, dt 
\leq d_k^2 2^{-k} \| e^{b \Psi} B \Phi \| \tilde{L}_{[t_1, t_2], f(B^{1,0})} \| e^{d \Psi} E_x \| \tilde{L}^2_{[t_1, t_2], f(B^{1,0})}.
\]
Finally applying Lemma 3.1 and (3.19) gives rise to

\[
\|e^{c\Phi} \int_y^{\infty} \Delta_k^h[\mathcal{R}_k^b(\partial_y A, \partial_x B)]dfy\|_{L_+^2} \leq 2^{b} \left\| e^{c(a-b)\Phi} \right\|_{L_+^2} \sum_{k' \geq k-3} \left\| e^{b\Phi} \Delta_k^{h} \partial_y A_{\Phi} \right\|_{L_+^2} \left\| e^{b\Phi} \Delta_k^{h} \partial_x B_{\Phi} \right\|_{L_+^2} \
\leq 2^{b} \sum_{k' \geq k-3} \left( 2\frac{k'}{k} f(t) \right) \| e^{b\Phi} \Delta_k^{h} B_{\Phi} \|_{L_+^2}.,
\]

Then a similar derivation of (3.13) leads to

\[
(3.22) \quad \int_{t_1}^{t_2} \left( e^{c\Phi} \int_y^{\infty} \Delta_k^h[\mathcal{R}_k^b(\partial_y A, \partial_x B)]dfy \right) dt \leq 2^{b} d^2 2^{-k} \| e^{b\Phi} B_{\Phi} \|_{L_t^2} \left\| \tilde{\Phi}_{(C_\kappa, \kappa)^2} \right\|_{L_t^2} \left\| \tilde{\Phi}_{(C_\kappa, \kappa)^2} \right\|_{L_t^2}^{1/2}. \]

By summing up (3.20), (3.21) and (3.22), we finish the proof of (3.18). \(\square\)

To handle the terms in (1.4), (1.6) and (6.1) involving the far field state \((U, B)\), we need the following lemmas:

**Lemma 3.7.** Let \((m_U, m_B)\) and \((M_U, M_B)\) be determined respectively by (1.5) and (1.6). Then under the assumption of (1.14a), for any \(T < T^*\), one has

\[
(3.23) \quad \| \langle t \rangle \tilde{\Phi} e^{\Phi} (m_U, m_B, M_U, M_B) \|_{L_t^2(B^{1/2,0}_k)} \leq C \varepsilon
\]

Similar estimate holds with \(\Phi, \Psi\) and \(T^*\) being replaced respectively by \(\Phi_\kappa, \Psi_\kappa\) and \(T^*_\kappa\).

**Proof.** In view of (1.5) and the construction of \(\chi\), we observe that \((m_U, m_B, M_U, M_B)\) are supported in \([0, 2]\) for each fixed \(t > 0\). Hence we get

\[
\| \langle t \rangle \tilde{\Phi} e^{\Phi} (m_U, m_B, M_U, M_B) \|_{L_t^2(B^{1/2,0}_k)} \leq \| \langle t \rangle \tilde{\Phi} (\partial_t U, \partial_t B, U, B, U B, B B, B B) \Phi \|_{L_t^2(B_k^{1/2})}
\]

Here and all in that follows, we always denote \(B_k^\alpha\) to the Besov space \(B_{2,1}^\alpha(\mathbb{R}_k)\).

Then thanks to the following law of product in Besov spaces (see [2] for instance)

\[
(3.24) \quad \| fg \|_{L_t^2(B_k^{1/2})} \leq \| f \|_{L_t^\infty(B_k^{1/2})} \| g \|_{L_t^2(B_k^{1/2})},
\]

and Lemma 3.3 and (1.14a), we deduce that

\[
\| \langle t \rangle \tilde{\Phi} (U \partial_x U, B \partial_x B, U \partial_x B, B \partial_x U) \Phi \|_{L_t^2(B_k^{1/2})} \leq \| \langle t \rangle \tilde{\Phi} (U, B) \|_{L_t^2(B_k^{1/2})} \| (U, B) \|_{L_t^\infty(B_k^0)} \leq \varepsilon \| \langle t \rangle \tilde{\Phi} (U, B) \|_{L_t^2(B_k^{1/2})}.
\]

As a result, it comes out

\[
\| \langle t \rangle \tilde{\Phi} e^{\Phi} (m_U, m_B, M_U, M_B) \|_{L_t^2(B_k^{1/2,0})} \leq \| \langle t \rangle \tilde{\Phi} \|_{L_t^2(B_k^{1/2})} \leq C \varepsilon \| \langle t \rangle \tilde{\Phi} (U, B) \|_{L_t^2(B_k^{1/2})},
\]

which together with (1.14a) ensures (3.23). \(\square\)
Lemma 3.8. Let $F$ be smooth enough function on $[0, T] \times \mathbb{R}_+$ and $A, E$ be smooth enough function on $[0, T] \times \mathbb{R}^2_+$. We denote $g(t) \overset{\text{def}}{=} \|F \partial_x(t)\|_{L^1_h}$. Then one has

\begin{equation}
\int_{t_1}^{t_2} \left| \left( e^{\Psi_h} \Delta^h_k [F \partial_x A] \Phi | e^{\Psi_h} \Delta^h_k E \Phi \right)_{L^2_+} \right| dt 
\lesssim d_k^2 2^{-k} \| e^{\Psi_h} A \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})} \| e^{\Psi_h} E \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})}.
\end{equation}

Proof. By applying Bony’s decomposition (3.2), we write

$F \partial_x A = T^h_0 \partial_x A + T^h_\partial A F + R^h (F, \partial_x A)$.

We first observe that

\begin{equation}
\int_{t_1}^{t_2} \left| \left( e^{\Psi_h} \Delta^h_k [T^h_\partial A] \Phi | e^{\Psi_h} \Delta^h_k E \Phi \right)_{L^2_+} \right| dt 
\lesssim \sum_{|k' - k| \leq 4} \int_{t_1}^{t_2} \| \Delta^h_{k'} F \Phi \|_{L^\infty_h} \| e^{\Psi_h} \Delta^h_k \partial_x A \Phi \|_{L^2_+} \| e^{\Psi_h} \Delta^h_k E \Phi \|_{L^2_+} dt
\end{equation}

\begin{equation}
\lesssim \sum_{|k' - k| \leq 4} 2^k \int_{t_1}^{t_2} g(t) \| e^{\Psi_h} \Delta^h_k \partial_x A \Phi \|_{L^2_+} \| e^{\Psi_h} \Delta^h_k E \Phi \|_{L^2_+} dt.
\end{equation}

Then a similar derivation of (3.10) gives rise to

\begin{equation}
\int_{t_1}^{t_2} \left| \left( e^{\Psi_h} \Delta^h_k [T^h_\partial A] \Phi | e^{\Psi_h} \Delta^h_k E \Phi \right)_{L^2_+} \right| dt 
\lesssim d_k^2 2^{-k} \| e^{\Psi_h} A \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})} \| e^{\Psi_h} E \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})}.
\end{equation}

Notice that

\begin{equation}
\| e^{\Psi_h} \Delta^h_{k'} [T^h_\partial A F] \Phi \|_{L^2_+} \lesssim \sum_{|k' - k| \leq 4} \| \Delta^h_{k'} F \Phi \|_{L^\infty_h} \| e^{\Psi_h} \Delta^h_{k - 1} \partial_x A \Phi \|_{L^2_+}
\end{equation}

\begin{equation}
\lesssim \sum_{|k' - k| \leq 4} d_{k'}(t) g(t) \| e^{\Psi_h} \partial_x A \Phi \|_{L^2_+},
\end{equation}

we get, by a similar derivation of (3.12), that

\begin{equation}
\int_{t_1}^{t_2} \left| \left( e^{\Psi_h} \Delta^h_k [T^h_\partial A F] \Phi | e^{\Psi_h} \Delta^h_k E \Phi \right)_{L^2_+} \right| dt 
\lesssim d_k^2 2^{-k} \| e^{\Psi_h} A \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})} \| e^{\Psi_h} E \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})}.
\end{equation}

Whereas it follows from Lemma 3.4 that

\begin{equation}
\| e^{\Psi_h} \Delta^h_k [R^h (F, \partial_x A)] \Phi \|_{L^2_+} \lesssim 2^\frac{h}{2} \sum_{k' \geq k - 3} \| \Delta^h_{k'} F \Phi \|_{L^2_h} \| e^{\Psi_h} \Delta^h_k \partial_x A \Phi \|_{L^2_+}
\end{equation}

\begin{equation}
\lesssim 2^\frac{h}{2} \sum_{k' \geq k - 3} 2^{-\frac{k'}{4}} g(t) \| e^{\Psi_h} \Delta^h_k \partial_x A \Phi \|_{L^2_+},
\end{equation}

from which and a similar derivation of (3.13), we infer

\begin{equation}
\int_{t_1}^{t_2} \left| \left( e^{\Psi_h} \Delta^h_k [R^h (F, \partial_x A)] \Phi | e^{\Psi_h} \Delta^h_k E \Phi \right)_{L^2_+} \right| dt 
\lesssim d_k^2 2^{-k} \| e^{\Psi_h} A \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})} \| e^{\Psi_h} E \Phi \|_{\tilde{L}^2_{[t_1, t_2]}(B^{1,0})}.
\end{equation}

This proves (3.25).
Remark 3.1. It follows from a similar proof of (3.24) that
\[
\int_{t_1}^{t_2} \left| (e^{\Psi} \Delta_h^k [\partial_x FA_{\Phi}] e^{\Psi} \Delta_h^k E_{\Phi}) L^2 \right| dt \\
\lesssim d_2^2 2^{-k} \| F^0 \|_{L_{[t_1, t_2]}^2(B^0_0)} \| e^{\Psi} A_{\Phi} \|_{L_{[t_1, t_2]}^2(B^0_0)} \| e^{\Psi} E_{\Phi} \|_{L_{[t_1, t_2]}^2(B^0_0)},
\]
(3.26)

4. Analytic energy estimate to the primitive functions \((\varphi, \psi)\)

The goal of this section is to present the a priori weighted analytic energy estimate to the primitive functions \((\varphi, \psi)\) which solves (1.6). Namely, we shall present the proof of Propositions 2.1 and 2.3. The key ingredients will be the following two lemmas:

Lemma 4.1. Let \((\varphi, \psi)\) be a smooth enough solution of (1.6). Let \(\Phi(t, \xi)\) and \(\Psi(t, y)\) be given by (1.11) and (1.12) respectively. Then if \(\kappa \in ]0, 2[\), for \(l_{\kappa} \overset{\text{def}}{=} \frac{\kappa(2-\kappa)}{4}\) and for any non-negative and non-decreasing function \(h \in C^1(\mathbb{R}_+)\), there exists a positive constant \(c\) so that
\[
\| \tilde{h}^\frac{1}{2} e^{\Psi} \Delta_k^h (\varphi, \psi) \phi \|_{L_{[t_0, t]}^2(L^2_0)}^2 + 4l_{\kappa} \| \tilde{h}^\frac{1}{2} e^{\Psi} \Delta_k^h \partial_y (\varphi, \psi) \phi \|_{L_{[t_0, t]}^2(L^2_0)}^2 \\
+ 2c \lambda^2 h \| \tilde{h}^\frac{1}{2} e^{\Psi} \Delta_k^h (\varphi, \psi) \phi \|_{L_{[t_0, t]}^2(L^2_0)}^2 \leq \frac{1}{2} \| \tilde{h}^\frac{1}{2} e^{\Psi} \Delta_k^h (\varphi, \psi) \phi (t_0) \|_{L^2_0}^2
\]
+ \| \sqrt{h} e^{\Psi} \Delta_k^h (\varphi, \psi) \phi \|_{L_{[t_0, t]}^2(L^2_0)}^2 + \epsilon \varepsilon \| \tilde{h}^\frac{1}{2} e^{\Psi} \partial_y (\varphi, \psi) \phi \|_{L_{[t_0, t]}^2(B^{\frac{1}{2}}_0)}^2
\]
+ \| C \tilde{h}^\frac{1}{2} \|_{L_{[t_0, t]}^2(B^{\frac{1}{2}}_0)}^2 \| \tilde{h}^\frac{1}{2} e^{\Psi} (\varphi, \psi) \phi \|_{L_{[t_0, t]}^2(L^2_0)}^2 \|
\]
for any \(t_0 \in [0, t]\) with \(t < T^*\), which is defined by (2.1).

Proof. In view of (1.9), by applying operator \(e^{\Phi(t,D_x)}\) to (1.6), we write
\[
\partial_t \varphi_\Phi - \partial_y^2 \varphi_\Phi - \bar{B}_\kappa \partial_x \psi_\Phi + \lambda \tilde{\varphi} (D_x) \varphi_\Phi + [u \partial_x \varphi - b \partial_x \psi]_\Phi \\
+ 2 \int_y [\partial_x \varphi \partial_y u - \partial_x \psi \partial_y b]_\phi dy' + \chi' [U \partial_x \varphi - B \partial_x \psi]_\phi
\]
+ \(2 \int_y \chi'' [U \partial_x \varphi - B \partial_x \psi]_\phi dy' + \chi \delta U \partial_x \varphi - \partial_x B \psi]_\phi
\]
+ \(2 \chi' [\partial_x U \varphi - \partial_x B \psi]_\phi + 2 \int_y \chi'' [\partial_x U \varphi - \partial_x B \psi]_\phi dy' = (M_U)_\phi,
\]
and
\[
\partial_t \psi_\Phi - \kappa \partial_y^2 \psi_\Phi - \bar{B}_\kappa \partial_x \varphi_\Phi + \lambda \tilde{\varphi} (D_x) \psi_\Phi + [u \partial_x \psi - b \partial_x \varphi]_\Phi \\
+ \chi' [U \partial_x \psi - B \partial_x \varphi]_\phi + \chi \delta U \partial_x \psi - \partial_x B \varphi]_\phi = (M_B)_\phi.
\]

It is easy to observe that the terms involving the constant \(\bar{B}_\kappa\) in (4.1) and (4.2) will be canceled when performing energy estimate
\[
\bar{B}_\kappa (e^{\Psi} \Delta_h^k \partial_x \varphi_\Phi | e^{\Psi} \Delta_h^k \varphi_\Phi)_{L^2_+} + \bar{B}_\kappa (e^{\Psi} \Delta_h^k \partial_x \varphi_\Phi | e^{\Psi} \Delta_h^k \psi_\Phi)_{L^2_+}
\]
\[
= \bar{B}_\kappa \int_{\mathbb{R}_+^2} \partial_x (e^{2\Phi} \Delta_h^k \Delta_k^h \psi_\Phi) dx dy = 0
\]
(4.4)
Thanks to (4.3), by applying the dyadic operator $\Delta_{k}^{h}$ to (1.2), (1.3) and then taking the $L_{2}^{+}$ inner product of the resulting equations with $e^{2\Psi} \Delta_{k}^{h}(\varphi, \psi)_{\phi}$ respectively, we find

$$
\begin{align*}
(e^{\Psi} \Delta_{k}^{h}(\partial_{t} - \partial_{y}^{2})\varphi | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} + (e^{\Psi} \Delta_{k}^{h}(\partial_{t} - \kappa \partial_{y}^{2})\psi | e^{\Psi} \Delta_{k}^{h}\psi_{\phi})_{L_{2}^{+}} \\
+ \lambda \delta(t)(e^{\Psi} | D_{x} | \Delta_{k}^{h}\varphi_{\phi} | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} + \lambda \delta(t)(e^{\Psi} | D_{x} | \Delta_{k}^{h}\psi_{\phi} | e^{\Psi} \Delta_{k}^{h}\psi_{\phi})_{L_{2}^{+}} \\
+ (e^{\Psi} \Delta_{k}^{h}[u \partial_{x} \varphi - b \partial_{x} \psi] | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} \geq (e^{\Psi} \Delta_{k}^{h}[u \partial_{x} \psi - b \partial_{x} \varphi] | e^{\Psi} \Delta_{k}^{h}\psi_{\phi})_{L_{2}^{+}} \\
+ 2(e^{\Psi} \int_{y}^{\infty} \Delta_{k}^{h}[\partial_{y} u \partial_{x} \varphi - \partial_{y} b \partial_{x} \psi] | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} \\
+ (e^{\Psi} \Delta_{k}^{h}[U \partial_{x} \varphi - B \partial_{x} \psi] | e^{\Psi} \chi' \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} + (e^{\Psi} \Delta_{k}^{h}(U \partial_{x} \psi - B \partial_{x} \varphi) | e^{\Psi} \chi' \Delta_{k}^{h}\psi_{\phi})_{L_{2}^{+}} \\
+ 2(e^{\Psi} \int_{y}^{\infty} \chi'' \Delta_{k}^{h}(U \partial_{x} \varphi - B \partial_{x} \psi) | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} \\
+ (e^{\Psi} \Delta_{k}^{h}(U \partial_{x} \psi - \partial_{y} B \psi) | e^{\Psi} \chi' \Delta_{k}^{h}\psi_{\phi})_{L_{2}^{+}} \\
+ 2(e^{\Psi} \int_{y}^{\infty} \chi'' \Delta_{k}^{h}(\partial_{y} U \varphi - \partial_{x} B \psi) | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} \\
+ (e^{\Psi} \Delta_{k}^{h}(M \partial_{y} \varphi) | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi})_{L_{2}^{+}} + (e^{\Psi} \Delta_{k}^{h}(M \partial_{y} \psi) | e^{\Psi} \Delta_{k}^{h}\psi_{\phi})_{L_{2}^{+}}.
\end{align*}
$$

In what follows, we shall use the technical lemmas in Section 3 to handle term by term in (4.5).

Notice that for $\kappa \in [0, 2]$, $\kappa(2 - \kappa) \leq 1$, we get, by applying (3.3a) and (3.3b), that

$$
\int_{t_{0}}^{t} h(t') \bigg( \big( e^{\Psi} | D_{x} | \Delta_{k}^{h}\varphi_{\phi} | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi} \big)_{L_{2}^{+}} + \big( e^{\Psi} | D_{x} | \Delta_{k}^{h}\psi_{\phi} | e^{\Psi} \Delta_{k}^{h}\psi_{\phi} \big)_{L_{2}^{+}} \bigg) dt' \geq \frac{1}{2} \left( \| h_{x}^{\frac{1}{2}} e^{\Psi} \Delta_{k}^{h}(\varphi, \psi)_{\phi}(t) \|_{L_{2}^{+}}^{2} - \| h_{x}^{\frac{1}{2}} e^{\Psi} \Delta_{k}^{h}(\varphi, \psi)_{\phi}(t_{0}) \|_{L_{2}^{+}}^{2} \right)
$$

$$
\quad - \frac{1}{2} \| \sqrt{\nu} e^{\Psi} \Delta_{k}^{h}(\varphi, \psi)_{\phi} \|_{F_{\nu}^{2, 0, 0}(L_{2}^{+})}^{2} + 2\lambda \| h_{x}^{\frac{1}{2}} e^{\Psi} \Delta_{k}^{h}(\varphi, \psi)_{\phi} \|_{F_{\nu}^{2, 0, 0}(L_{2}^{+})}^{2}.
$$

In view of Lemma 3.1, we have

$$
\lambda \int_{t_{0}}^{t} h(t') \delta(t') \left( \big( e^{\Psi} | D_{x} | \Delta_{k}^{h}\varphi_{\phi} | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi} \big)_{L_{2}^{+}} + \big( e^{\Psi} | D_{x} | \Delta_{k}^{h}\psi_{\phi} | e^{\Psi} \Delta_{k}^{h}\psi_{\phi} \big)_{L_{2}^{+}} \right) dt' \geq \lambda \int_{t_{0}}^{t} \delta(t') h_{x}^{\frac{1}{2}} e^{\Psi} | D_{x} | \Delta_{k}^{h}\varphi_{\phi} \|_{L_{2}^{+}}^{2} dt' \geq c \lambda 2^{k} \int_{t_{0}}^{t} \delta(t') h_{x}^{\frac{1}{2}} e^{\Psi} \Delta_{k}^{h}(\varphi, \psi)_{\phi} \|_{L_{2}^{+}}^{2} dt'.
$$

Whereas due to $\lim_{y \to +\infty} u = \lim_{y \to +\infty} b = 0$, we get, by applying Lemma 3.3 with $a = \frac{1}{2}$ and $b = c = d = 1$, that

$$
\int_{t_{0}}^{t} h(t') \left( \big( e^{\Psi} \Delta_{k}^{h}[u \partial_{x} \varphi - b \partial_{x} \psi] | e^{\Psi} \Delta_{k}^{h}\varphi_{\phi} \big)_{L_{2}^{+}} + \big( e^{\Psi} \Delta_{k}^{h}[u \partial_{x} \psi - b \partial_{x} \varphi] | e^{\Psi} \Delta_{k}^{h}\psi_{\phi} \big)_{L_{2}^{+}} \right) dt' \leq d_{2}^{2} 2^{-k} \| h_{x}^{\frac{1}{2}} e^{\Psi} (\varphi, \psi)_{\phi} \|_{L_{2}^{+}}^{2} \| f(t) = (t)^{\frac{1}{2}} e^{\Psi} \partial_{y} (u, b) \phi(t) \|_{B_{\nu}^{2, 0}}.
$$

Yet it follows from Lemma 2.1 that

$$
\tag{4.6}
\int_{t_{0}}^{t} h(t') f(t) \leq (t)^{\frac{1}{2}} e^{\Psi} \partial_{y} (G, H) \phi(t) \|_{B_{\nu}^{2, 0}}.
$$
Hence according to Definition 3.1 and (1.10), we achieve
\[
\begin{aligned}
& \int_{t_0}^t h(t') \left( \left| \left( e^{\Psi} \Delta_k^h [u \partial_x \varphi - b \partial_x \psi] \phi | e^{\Psi} \Delta_k^h \varphi \phi \right) \right|_{L^2_x} \right) dt' \\
& \quad + \left| \left( e^{\Psi} \Delta_k^h [u \partial_x \psi - b \partial_x \varphi] \phi | e^{\Psi} \Delta_k^h \psi \phi \right) \right|_{L^2_x} dt' \\
& \leq d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \phi \right\|_{L^2_{[t_0, t], \theta(B^1, 0)}}^2.
\end{aligned}
\]

Similarly, by applying Lemma 3.6, we get, by a similar derivation of (4.7), that
\[
\begin{aligned}
& \int_{t_0}^t h(t') \left| \left( e^{\Psi} \Delta_k^h [\partial_y u \partial_x \varphi - \partial_y b \partial_x \psi] \phi | e^{\Psi} \Delta_k^h \varphi \phi \right) \right|_{L^2_x} dt' \\
& \quad + \left| \left( e^{\Psi} \Delta_k^h [\partial_y u \partial_x \psi - \partial_y b \partial_x \varphi] \phi | e^{\Psi} \Delta_k^h \psi \phi \right) \right|_{L^2_x} dt' \\
& \leq d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \phi \right\|_{L^2_{[t_0, t], \theta(B^1, 0)}}^2.
\end{aligned}
\]

While it follows from Lemma 3.8 that
\[
\begin{aligned}
& \int_{t_0}^t h(t') \left( \left| \left( e^{\Psi} \Delta_k^h [U \partial_x \varphi - B \partial_x \psi] \phi | e^{\Psi} \chi' \Delta_k^h \varphi \phi \right) \right|_{L^2_x} \right) dt' \\
& \quad + \left| \left( e^{\Psi} \Delta_k^h [U \partial_x \psi - B \partial_x \varphi] \phi | e^{\Psi} \chi' \Delta_k^h \psi \phi \right) \right|_{L^2_x} dt' \\
& \leq d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \phi \right\|_{L^2_{[t_0, t], \theta(B^1, 0)}}^2.
\end{aligned}
\]

Notice from the definition of \(\chi\) above (1.3) that \(\text{supp}\chi'' \subset [1, 2]\), we get, by a similar derivation of (3.25), that
\[
\begin{aligned}
& \int_{t_0}^t h(t') \left| \left( e^{\Psi} \chi'' \Delta_k^h [U \partial_x \varphi - B \partial_x \psi] \phi | e^{\Psi} \Delta_k^h \varphi \phi \right) \right|_{L^2_x} dt' \\
& \leq d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \phi \right\|_{L^2_{[t_0, t], \theta(B^1, 0)}}^2.
\end{aligned}
\]

On the other hand, it follows from (2.135) that
\[
\| e^{\Psi} y \Delta_k^h (\varphi, \psi) \phi \|_{L^2_x} \leq (t) \| e^{\Psi} \partial_y \Delta_k^h (\varphi, \psi) \phi \|_{L^2_x},
\]
which implies
\[
\| e^{\Psi} y (\varphi, \psi) \phi \|_{\tilde{L}^p_t (B^0, a)} \leq (t) \| e^{\Psi} \partial_y (\varphi, \psi) \phi \|_{\tilde{L}^p_t (B^0, a)}.
\]

So that by applying (3.26) and (1.14a), we find
\[
\begin{aligned}
& \int_{t_0}^t h(t') \left( \left| \left( e^{\Psi} \Delta_k^h [-\partial_x U u + \partial_x B b] \phi | e^{\Psi} \chi \Delta_k^h \varphi \phi \right) \right|_{L^2_x} \right) dt' \\
& \quad + \left| \left( e^{\Psi} \Delta_k^h [-\partial_x U b + \partial_x B \phi] \phi | e^{\Psi} \chi \Delta_k^h \psi \phi \right) \right|_{L^2_x} dt' \\
& \leq d_k^2 2^{-k} \left\| (\tau) (U, B) \phi \|_{L^\infty_{[t_0, t]} (\tilde{L}^2_{B_n})} \right\| h^\frac{1}{2} e^{\Psi} (u, b) \phi \|_{L^2_{[t_0, t]} (B^\frac{1}{2}, 0)} \| (\tau)^{-\frac{1}{2}} h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \phi \|_{L^2_{[t_0, t]} (B^\frac{1}{2}, 0)} \\
& \leq \varepsilon d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\Psi} \partial_y (\varphi, \psi) \phi \right\|_{L^2_{[t_0, t]} (B^\frac{1}{2}, 0)}^2.
\end{aligned}
\]

Along the same line, we infer
\[
\begin{aligned}
& \int_{t_0}^t h(t') \left| \left( e^{\Psi} \Delta_k^h [\partial_x U \varphi - \partial_x B \psi] \phi | e^{\Psi} \chi \Delta_k^h \varphi \phi \right) \right|_{L^2_x} dt' \\
& \quad + \left| \left( e^{\Psi} \Delta_k^h [\partial_x U \psi - \partial_x B \varphi] \phi | e^{\Psi} \chi \Delta_k^h \psi \phi \right) \right|_{L^2_x} dt' \\
& \leq d_k^2 2^{-k} \left\| (\tau) (U, B) \phi \|_{L^\infty_{[t_0, t]} (\tilde{L}^2_{B_n})} \right\| (\tau)^{-\frac{1}{2}} h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \phi \|_{L^2_{[t_0, t]} (B^\frac{1}{2}, 0)} \| (\tau)^{-\frac{1}{2}} h^\frac{1}{2} e^{\Psi} \varphi \phi \|_{L^2_{[t_0, t]} (B^\frac{1}{2}, 0)} \\
& \leq \varepsilon d_k^2 2^{-k} \left\| h^\frac{1}{2} e^{\Psi} \partial_y (\varphi, \psi) \phi \right\|_{L^2_{[t_0, t]} (B^\frac{1}{2}, 0)}^2.
\end{aligned}
\]
Moreover, again due to supp$'' \subset [1, 2]$, we deduce that the term $\int_{t_0}^{t} h(t') \bigl( (e^\Psi \int_{y}^{\infty} \chi'' \Delta^h_k[\partial_x U \varphi - \partial_x B\psi]) dy | e^\Psi \Delta^h_k \varphi \bigr)_L^2 \big| dt'$ shares the same estimate as above.

Finally for the remaining source term, by applying Young’s inequality, we achieve

$$
\int_{t_0}^{t} h(t') \biggl( \bigl( (e^\Psi \Delta^h_k(M_U) e^\Psi \Delta^h_k \varphi \bigr)_L^2 + \bigl( (e^\Psi \Delta^h_k(M_B) e^\Psi \Delta^h_k \psi \bigr)_L^2 \biggr) dt' \\
\leq d_2^2 2^{-k} \| (\tau_2)^{1/2} h_2^2 e^\Psi \Delta^h_k(M_U) \Phi \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2 \| (\tau_2)^{-1/2} h_2^2 e^\Psi \Phi \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2 \\
\leq d_2^2 2^{-k} \biggr( (\tau_2)^{1/2} h_2^2 e^\Psi \Delta^h_k(M_U) \Phi \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2 + (\tau_2)^{-1/2} h_2^2 e^\Psi \Phi \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2 \biggr),
$$

By multiplying (4.3) by $h(t)$ and then integrating the resulting equality over $[t_0, t]$, finally inserting the above estimates to the equality, we obtain (4.1). This completes the proof of Lemma 4.1.

**Lemma 4.2.** Let $(\varphi, \psi)$ be a smooth enough solution of (1.6). Let $\Phi_k(t, \xi)$ and $\Psi_k(t, y)$ be given by (1.19) and (1.20) respectively. Then if $\kappa \in \mathbb{R}$, for $\ell_2 \equiv 2\kappa - \ell_1$ and for any non-negative and non-decreasing function $h \in C^1(\mathbb{R})$, there exists a positive constant $\lambda$ such that

$$
\| h^{1/2} e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(L^2_+)}^2 + 4\kappa \| h^{1/2} e^{\Psi_k} \partial_y(\varphi, \psi) \|_{L^2_{[t_0,t]}(L^2_+)}^2 \\
+ 2\ell_2 \| h^{1/2} e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]; h_2^2}(L^2_+)}^2 \leq \| h^{1/2} e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(L^2_+)}^2 \\
+ \| (\tau_2)^{1/2} h_2^2 e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2 + (\tau_2)^{-1/2} h_2^2 e^{\Psi_k} \partial_y(\varphi, \psi) \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2 \\
+ 2\ell_2 \| (\tau_2)^{1/2} h_2^2 e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2 + (\tau_2)^{-1/2} h_2^2 e^{\Psi_k} \partial_y(\varphi, \psi) \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2,
$$

for any $t_0 \in [0, t]$ with $t < T_\kappa^*$, which is defined by (2.7).

**Proof.** Along the same line to the proof of Lemma 4.1 by applying the dyadic operator $\Delta^h_k$ to the modified versions of (4.2) and (4.3) (with $\Phi$ being replaced by $\Phi_k$), and then taking the $L^2_+$ inner product of the resulting equations with $e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(B^{1/2}_0)}$, we find that (4.5) holds with $\theta, \Psi$ and $\Phi$ there being replaced respectively by $\theta_k, \Psi_k$ and $\Phi_k$. Next we only present the detailed estimates to the terms in (4.5), which are different from the proof of Lemma 4.1.

Observing that for $k \in \mathbb{R}$, $2\kappa - 1 \leq \kappa$, we get, by applying (3.4a) and (3.4b), that

$$
\int_{t_0}^{t} h(t') \biggl( (e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(L^2_+)}^2 \\
+ (e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2) dt' \\
\geq \frac{1}{2} \bigl( \| h^{1/2} e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(L^2_+)}^2 - \| h^{1/2} e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(L^2_+)}^2 \bigr) \\
- \frac{1}{2} \| h^{1/2} e^{\Psi_k} \Delta^h_k(\varphi, \psi) \|_{L^2_{[t_0,t]}(L^2_+)}^2 + 2\kappa \| h^{1/2} e^{\Psi_k} \partial_y(\varphi, \psi) \|_{L^2_{[t_0,t]}(B^{1/2}_0)}^2.
$$
While due to $\lim_{y \to +\infty} u = \lim_{y \to +\infty} b = 0$, we get, by applying Lemma 3.4 with $a = \frac{1}{2\kappa}$ and $b = c = d = \frac{1}{2\kappa}$, that

$$
\int_{t_0}^{t} h(t') \left( \left| \left( e^{\psi_\kappa \Delta_b^h[tu_0 + \partial_x (\partial_x + \partial_x - b)]} \right) \Phi_\kappa \right|_{L^2_+} \right) dt' \\
\leq d_k^2 2^{-k} \| h^2 e^{\psi_\kappa (\varphi, \psi)} \|_{L^2_{[0, t]}(B^{1, 0})} \text{ with } f(t) = \langle t \rangle \frac{1}{2} \| e^{\psi_\kappa \partial_y (u, b)} \Phi_\kappa (t) \|_{B^{1, 0}}.
$$

Yet it follows from Lemma 2.2 that

$$
f(t) \lesssim \langle t \rangle \frac{1}{2} \| e^{\psi_\kappa \partial_y (G, H)} \Phi_\kappa (t) \|_{B^{1, 0}},
$$

hence we get, by a similar derivation of (4.7), that

$$
\int_{t_0}^{t} h(t') \left( \left| \left( e^{\psi_\kappa \Delta_b^h [u_0 + \partial_x \varphi - b \partial_x \psi]} \right) \Phi_\kappa \right|_{L^2_+} \right) dt' \\
\leq d_k^2 2^{-k} \| h^2 e^{\psi_\kappa (\varphi, \psi)} \|_{L^2_{[0, t]}(B^{1, 0})}.
$$

Similarly, we get, by applying Lemma 3.6 and a similar derivation of (4.10), that

$$
\int_{t_0}^{t} h(t') \left( \left| \left( e^{\psi_\kappa \Delta_b^h [u_0 + \partial_x \varphi - b \partial_x \psi]} \right) \Phi_\kappa \right| \right) \left| e^{\psi_\kappa \Delta_b^h (\varphi, \psi)} \right|_{L^2_+} dt' \\
\leq d_k^2 2^{-k} \| h^2 e^{\psi_\kappa (\varphi, \psi)} \|_{L^2_{[0, t]}(B^{1, 0})}.
$$

On the other hand, we deduce from (2.12b) that

$$
\| e^{\psi_\kappa y \Delta_b^h (\varphi, \psi)} \|_{L^2_+} \lesssim 4 \kappa \langle t \rangle \| e^{\psi_\kappa \partial_y \Delta_b^h (\varphi, \psi)} \|_{L^2_+},
$$

which implies

$$
\| e^{\psi_\kappa (\varphi, \psi)} \|_{L^p(B_{\kappa})} \lesssim \kappa \| \langle \tau \rangle e^{\psi_\kappa \partial_y (\varphi, \psi)} \|_{L^p(B_{\kappa})},
$$

So that by applying (3.26) and (1.14a), we find

$$
\int_{t_0}^{t} h(t') \left( \left| \left( e^{\psi_\kappa \Delta_b^h [-\partial_x U u + \partial_x B b]} \right) \Phi_\kappa \right|_{L^2_+} \right) dt' \\
\leq d_k^2 2^{-k} \| \langle \tau \rangle (U, B) \Phi_\kappa \|_{L^2_{[0, t]}(B^{1, 0})} \| h^2 e^{\psi_\kappa (u, b)} \|_{L^2_{[0, t]}(B^{1, 0})} \| \langle \tau \rangle e^{\psi_\kappa y (\varphi, \psi)} \Phi_\kappa \|_{L^2_{[0, t]}(B^{1, 0})} \\
\lesssim \kappa d_k^2 2^{-k} \| h^2 e^{\psi_\kappa \partial_y (\varphi, \psi)} \|_{L^2_{[0, t]}(B^{1, 0})}.
$$
Along the same line, we infer
\[
\int_{t_0}^{t} \left( e^{\Psi_s} \Delta_k^1 [\partial_x U \varphi - \partial_x B \psi] \Phi_* | e^{\Psi_s} \chi \Delta_k^1 \varphi \Phi_* \right)_{L^2_{(t_0,t)}} \ dt' 
\]
\[
\leq d_k 2^{-k} \langle \tau \rangle (U,B) \Phi_* \|_{L^{\infty}_{(t_0,t);(B)}^1} \| (\tau)^{\frac{1}{2}} h^2 e^{\Psi_s} (\varphi, \psi) \Phi_* \|_{L^2_{(t_0,t);(B^2)}} \| (\tau)^{-\frac{1}{2}} h^2 e^{\Psi_s} \varphi \Phi_* \|_{L^2_{(t_0,t);(B^2)}} 
\]
\[
\leq \kappa d_k 2^{-k} \| h^2 e^{\Psi_s} \partial_y (\varphi, \psi) \Phi_* \|_{L^2_{(t_0,t);(B^2)}}^2. 
\]
Moreover, due to \( \text{supp}(\chi'' \subset [1,2]) \), we deduce that the term \( \int_{t_0}^{t} \left( e^{\Psi_s} \int_y^\infty \chi'' \Delta_k^1 [\partial_x U \varphi - \partial_x B \psi] \Phi_* \Phi'_y d y \right)e^{\Psi_s} \Delta_k^1 \varphi \Phi_* \|_{L^2_{(t_0,t)}} \ dt' \) shares the same estimate as above.

With the above estimates, (4.8) follows by a similar derivation of (4.11). This completes the proof of Lemma 4.2. \( \square \)

An immediate corollary of Lemmas 4.1 and 4.2 gives rise to

**Corollary 4.1.** Under the assumptions of Lemma 4.1, there exist positive constants \( \varepsilon_0, \lambda_0 \) so that for any \( \lambda \geq \lambda_0 \) and \( \varepsilon \leq \varepsilon_0 \), one has
\[
\| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi \|_{L^{\infty}_{(t_0,t);(B^2)}} + \sqrt{\varepsilon} \| h^\frac{1}{2} e^{\Psi} \partial_y (\varphi, \psi) \Phi \|_{L^2_{(t_0,t);(B^2)}} 
\]
\[
\leq \| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi(t_0) \|_{B^2} + \sqrt{\varepsilon} \| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi \|_{L^2_{(t_0,t);(B^2)}} + C \sqrt{\varepsilon} \| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi \|_{L^2_{(t_0,t);(B^2)}} + C \varepsilon^{-\frac{1}{2}} \| (\tau)^{\frac{1}{2}} h^2 e^{\Psi} (M_U, M_B) \Phi \|_{L^2_{(t_0,t);(B^2)}}. 
\]

**Proof.** Indeed by taking square root of (4.11) and then multiplying the resulting inequality by \( 2^\frac{k}{2} \) and finally summing over \( k \in \mathbb{Z} \), we arrive at
\[
\| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi \|_{L^{\infty}_{(t_0,t);(B^2)}} + 2\sqrt{\varepsilon} \| h^\frac{1}{2} e^{\Psi} \partial_y (\varphi, \psi) \Phi \|_{L^2_{(t_0,t);(B^2)}} 
\]
\[
\leq \| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi(t_0) \|_{B^2} + \sqrt{\varepsilon} \| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi \|_{L^2_{(t_0,t);(B^2)}} + C \| h^\frac{1}{2} e^{\Psi} (\varphi, \psi) \Phi \|_{L^2_{(t_0,t);(B^2)}} + C \varepsilon^{-\frac{1}{2}} \| (\tau)^{\frac{1}{2}} h^2 e^{\Psi} (M_U, M_B) \Phi \|_{L^2_{(t_0,t);(B^2)}}. 
\]
Taking \( \lambda \) big enough and \( \varepsilon \leq \varepsilon_0 \) small enough in the above inequality gives rise to (4.11). \( \square \)

**Corollary 4.2.** Under the assumptions of Lemma 4.2, there exist positive constants \( \varepsilon_0, \lambda_0 \) so that for any \( \lambda \geq \lambda_0 \) and \( \varepsilon \leq \varepsilon_0 \), one has
\[
\| h^\frac{1}{2} e^{\Psi_s} \Delta_k^1 (\varphi, \psi) \Phi_* \|_{L^{\infty}_{(t_0,t);(B^2)}} + \sqrt{\varepsilon} \| h^\frac{1}{2} e^{\Psi_s} \Delta_k^1 \partial_y (\varphi, \psi) \Phi_* \|_{L^2_{(t_0,t);(B^2)}} 
\]
\[
\leq \| h^\frac{1}{2} e^{\Psi_s} \Delta_k^1 (\varphi, \psi) \Phi_* (t_0) \|_{B^2} + \sqrt{\varepsilon} \| h^\frac{1}{2} e^{\Psi_s} \Delta_k^1 (\varphi, \psi) \Phi_* \|_{L^2_{(t_0,t);(B^2)}} + C \varepsilon^{-\frac{1}{2}} \| (\tau)^{\frac{1}{2}} h^2 e^{\Psi_s} (M_U, M_B) \Phi_* \|_{L^2_{(t_0,t);(B^2)}}. 
\]

**Proof.** This proof is the same as that of Corollary 4.1, we omit the detail here. \( \square \)

Now we are in a position to prove Propositions 2.1 and 2.4.
Proof of Proposition 2.1. Let $c$ be the constant given by Lemma 4.1. We observe from Lemma 2.3 that
\[
\int_0^t \langle \tau \rangle^{2\kappa - 2\varepsilon - 1} \| e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t} \, d\tau \leq 2 \int_0^t \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t} \, d\tau.
\]
So that by taking $t_0 = 0$ and $h(t) = \langle t \rangle^{2(\kappa - \varepsilon)}$ in (4.1), we obtain
\[
\| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} + 4\varepsilon \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t} + 2c\lambda^2 \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} \\
\leq \| e^{\frac{\varepsilon^2}{2} e^{\delta |D_x|}} (\varphi_0, \psi_0) \|^2_{B^{1,0}_2} + \varepsilon \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t} + C \varepsilon \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t}.
\]
By taking square root of the above inequality and then multiplying the resulting one by $2^\frac{\pi}{2}$ and finally summing over $k \in \mathbb{Z}$, we find
\[
\| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} + \sqrt{2c\lambda} \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} \\
\leq \| e^{\frac{\varepsilon^2}{2} e^{\delta |D_x|}} (\varphi_0, \psi_0) \|^2_{B^{1,0}_2} + \sqrt{C} \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} + \sqrt{\varepsilon}.
\]
By taking $\lambda$ to be so large that $c\lambda > C$, and using (3.23), we obtain
\[
\| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} \leq \| e^{\frac{\varepsilon^2}{2} e^{\delta |D_x|}} (\varphi_0, \psi_0) \|^2_{B^{1,0}_2} + \sqrt{\varepsilon}.
\]
Thanks to (4.13), we get, by taking $t_0 = \frac{\lambda}{2}$ and $h(t) = \langle t \rangle^{2(\kappa - \varepsilon)}$ in Corollary 4.1 that
\[
\| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} \leq \| e^{\frac{\varepsilon^2}{2} e^{\delta |D_x|}} (\varphi_0, \psi_0) \|^2_{B^{1,0}_2} + \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t}
\]
\[
\leq \| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} + \| e^{\frac{\varepsilon^2}{2} e^{\delta |D_x|}} (\varphi_0, \psi_0) \|^2_{B^{1,0}_2} + \sqrt{\varepsilon}.
\]
Yet by virtue of Definition 1.1, we deduce from (4.13) that
\[
\| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} \leq \| e^{\frac{\varepsilon^2}{2} e^{\delta |D_x|}} (\varphi_0, \psi_0) \|^2_{L^2_t \cap L^{\infty}_t} + \sqrt{\varepsilon}.
\]
As a consequence, we arrive at
\[
\| \langle \tau \rangle^{\kappa - \varepsilon} e^{\varepsilon \Delta_k^h (\varphi, \psi)} \phi \|^2_{L^2_t \cap L^{\infty}_t} \leq \| e^{\frac{\varepsilon^2}{2} e^{\delta |D_x|}} (\varphi_0, \psi_0) \|^2_{L^2_t \cap L^{\infty}_t} + \sqrt{\varepsilon}.
\]
This together with (4.13) leads to (2.33).
Proof of Proposition 2.4. Similar to the proof of Proposition 2.1, we first observe from Lemma 2.3 that
\[
\int_0^t \langle \tau \rangle^{2\kappa - 2\epsilon - 1} \| e^{\Psi_k} \Delta_k^h(\varphi, \psi) \phi_n \|_{L_t^2}^2 \, d\tau \leq 2\kappa \int_0^t \| \langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} \Delta_k^h \partial_y(\varphi, \psi) \phi_n \|_{L_t^2}^2 \, d\tau.
\]

So that by taking \( t_0 = 0 \) and \( h(t) = \langle t \rangle^{2(\ell_n - \epsilon)} \) in (4.8), we obtain
\[
\| \langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} \Delta_k^h (\varphi, \psi) \phi_n \|_{L_{t,0}^\infty (B^{1/4}_2, 0)}^2 + 4\kappa \epsilon \| \langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} \partial_y \Delta_k^h(\varphi, \psi) \phi_n \|_{L_t^2 (L_x^2)}^2
\]
\[
+ 2c\lambda^k \| \langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} \Delta_k^h (\varphi, \psi) \phi_n \|_{L_{t,0}^\infty (L_x^2)}^2
\]
\[
\leq \| e^{\Psi_k} e^{\delta |D_x|} \Delta_k^h(\varphi_0, 0, \psi_0) \|_{B^{1/4}_{2,0}}^2 + \kappa \epsilon \| e^{\Psi_k} \Delta_k^h(\varphi_0, 0, \psi_0) \|_{L_t^2 (L_x^2)}^2
\]
\[
+ C \epsilon \| (\langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} (M_U, M_B) \phi_n \|_{L_t^2 (B^{1/4}_2, 0)}^2,
\]
from which we get, by a similar derivation of (4.11), that
\[
\| \langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} (\varphi, \psi) \phi_n \|_{L_{t,0}^\infty (B^{1/4}_2, 0)} \leq \| e^{\Psi_k} e^{\delta |D_x|} (\varphi_0, 0, \psi_0) \|_{B^{1/4}_{2,0}} + C \sqrt{\epsilon}
\]

On the other hand, thanks to (4.11), we get, by taking \( t_0 = \frac{t}{2} \) and \( h(t) = \langle t \rangle^{2(\ell_n - \epsilon)} \) in Corollary 4.2 that
\[
\| \langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} \partial_y (\varphi, \psi) \phi_n \|_{L_{t,\frac{t}{2},0} (B^{1/2}_2, 0)} \leq \| \langle t/2 \rangle^{\ell_n - \epsilon} e^{\Psi_k} (\varphi, \psi) \phi_n \|_{B^{1/2}_{2,0}}
\]
\[
+ \| \langle \tau \rangle^{\ell_n - \epsilon - \frac{\epsilon}{2}} e^{\Psi_k} (\varphi, \psi) \phi_n \|_{L_{t,\frac{t}{2},0} (B^{1/2}_2, 0)} + \| \langle \tau \rangle^{\ell_n - \epsilon - \frac{\epsilon}{2}} e^{\Psi_k} (M_U, M_B) \phi_n \|_{L_{t,\frac{t}{2},0} (B^{1/2}_2, 0)}
\]
\[
\leq \| \langle \tau \rangle^{\ell_n - \epsilon - \frac{\epsilon}{2}} e^{\Psi_k} (\varphi, \psi) \phi_n \|_{L_{t,\frac{t}{2},0} (B^{1/2}_2, 0)} + \| e^{\Psi_k} e^{\delta |D_x|} (\varphi_0, 0, \psi_0) \|_{B^{1/2}_2, 0} + \sqrt{\epsilon}.
\]

Whereas it follows from (4.11) that
\[
\| \langle \tau \rangle^{\ell_n - \epsilon - \frac{\epsilon}{2}} e^{\Psi_k} (\varphi, \psi) \phi_n \|_{L_{t,\frac{t}{2},0} (B^{1/2}_2, 0)} \leq \| \langle t \rangle^{\ell_n - \epsilon} e^{\Psi_k} (\varphi, \psi) \phi_n \|_{L_{t,\frac{t}{2},0} (B^{1/2}_2, 0)}
\]
\[
\leq \| e^{\Psi_k} e^{\delta |D_x|} (\varphi_0, 0, \psi_0) \|_{B^{1/2}_2, 0} + \sqrt{\epsilon}.
\]

As a consequence, we arrive at
\[
\| \langle \tau \rangle^{\ell_n - \epsilon} e^{\Psi_k} \partial_y (\varphi, \psi) \phi_n \|_{L_{t,\frac{t}{2},0} (B^{1/2}_2, 0)} \leq \| e^{\Psi_k} e^{\delta |D_x|} (\varphi_0, 0, \psi_0) \|_{B^{1/2}_2, 0} + \sqrt{\epsilon}.
\]

Together with (4.11), we conclude the proof of Proposition 2.4.
5. Analytic energy estimate of \((u, b)\)

The goal of this section is to present the \textit{a priori} weighted analytic energy estimate to the solution \((u, b)\) of \((\text{1.4})\). That is, we are going to present the proof of Propositions \((2.2)\) and \((2.5)\).

The key ingredients lie in the following lemmas:

**Lemma 5.1.** Let \((u, b)\) be a smooth enough solution of \((\text{1.4})\). Let \(\Phi(t, \xi)\) and \(\Psi(t, y)\) be given by \((\text{1.11})\) and \((\text{1.12})\) respectively. Then if \(\kappa \in [0, 2]\), for \(l_\kappa \overset{\text{def}}{=} \frac{\kappa(2-\kappa)}{4}\), and for any non-negative and non-decreasing function \(h \in C^1(\mathbb{R}_+)\), there exists a positive constant \(c\) so that

\[
\begin{align*}
&\|h^\frac{1}{2}e^{\Psi} \Delta^h_k(u, b)_\Phi\|^2_{L^2_{[0, t]}(L^1_+)} + 4l_\kappa\|h^\frac{1}{2}e^{\Psi} \Delta^h_k \partial_y (u, b)_\Phi\|^2_{L^2_{[0, t]}(L^1_+)} \\
&+ 2c\lambda^k\|h^\frac{1}{2}e^{\Psi} \Delta^h_k(u, b)_\Phi\|^2_{L^2_{[0, t]}(L^1_+)} \leq \|h^\frac{1}{2}e^{\Psi} \Delta^h_k(u, b)_\Phi(0)\|^2_{L^1_+}
\end{align*}
\]

\((5.1)\)

+ \|\sqrt{h}e^{\Psi} \Delta^h_k(u, b)_\Phi\|^2_{L^2_{[0, t]}(L^1_+)} + \epsilon \epsilon d^2_k 2^{-k} \|h^\frac{1}{2}e^{\Psi} \partial_y (u, b)_\Phi\|^2_{L^2_{[0, t]}(B_+)}

+ C d^2_k 2^{-k} \left(\epsilon^{-1}\|\tau\|^\frac{1}{2}h^\frac{1}{2}e^{\Psi}(m_U, m_B)_\Phi\|^2_{L^2_{[0, t]}(B^1_+)} + \|h^\frac{1}{2}e^{\Psi}(u, b)_\Phi\|^2_{L^2_{[0, t]}(B^1_+)\}}

\text{for any } t_0 \in [0, t] \text{ with } t < T^*, \text{ which is defined by } (2.1).

**Proof.** In view of \((\text{1.4})\), by applying operator \(e^{\Psi(t, \cdot; D_\nu)}\) to \((\text{1.4})\), we write

\[
\begin{align*}
&\partial_t u_\Phi - \partial^2_{y^2} u_\Phi - \tilde{B}_\kappa \partial_x u_\Phi + \lambda \dot{\theta}(t) D_x u_\Phi + [u \partial_x u - b \partial_x b)_\Phi \\
&+ \left[v \partial_y u - h \partial_y b\right]_\Phi + \chi'[U \partial_x u - B \partial_x b]_\Phi + \chi'[\partial_x U u - \partial_x B b]_\Phi \\
&+ \chi[-\partial_x U \partial_y u + \partial_x B \partial_y b]_\Phi + \chi''[U v - Bh]_\Phi = (m_U)_\Phi, \\
\text{and}
\end{align*}
\]

\((5.2)\)

\[
\begin{align*}
&\partial_t b_\Phi - \kappa \partial^2_{y^2} b_\Phi - \tilde{B}_\kappa \partial_x u_\Phi + \lambda \dot{\theta}(t) D_x b_\Phi + [u \partial_x b - b \partial_x u]_\Phi \\
&+ \left[v \partial_y b - h \partial_y u\right]_\Phi + \chi'[U \partial_x b - B \partial_x u]_\Phi + \chi'\partial_x B u - \partial_x U b]_\Phi \\
&+ \chi[-\partial_x U \partial_y b + \partial_x B \partial_y u]_\Phi + \chi''[Bv - Bh]_\Phi = (m_B)_\Phi.
\end{align*}
\]

\((5.3)\)

By virtue of a virtual version of \((\text{4.3})\), we get, by applying the dyadic operator \(\Delta^h_k\) to \((5.2)\), \((5.3)\) and then taking \(L^2_+\) inner product of the resulting equations with \(e^{2\Psi} \Delta^h_k (u, b)_\Phi\), that

\[
\begin{align*}
&\left(e^{\Psi} \Delta^h_k(\partial_t - \partial^2_{y^2} u_\Phi)\right)|_{L^2_+} + \left(e^{\Psi} \Delta^h_k(\partial_t - \kappa \partial^2_{y^2} b_\Phi)\right)|_{L^2_+} \\
&+ \lambda \dot{\theta}(t) \left(e^{\Psi} D_x \Delta^h_k u_\Phi\right)|_{L^2_+} + \lambda \dot{\theta}(t) \left(e^{\Psi} D_x \Delta^h_k b_\Phi\right)|_{L^2_+} \\
&+ \left(e^{\Psi} \Delta^h_k[u \partial_x u - b \partial_x b]_\Phi\right)|_{L^2_+} + \left(e^{\Psi} \Delta^h_k[u \partial_x b - b \partial_x u]_\Phi\right)|_{L^2_+} \\
&+ \left(e^{\Psi} \Delta^h_k[v \partial_y u - h \partial_y b]_\Phi\right)|_{L^2_+} + \left(e^{\Psi} \Delta^h_k[v \partial_y b - h \partial_y u]_\Phi\right)|_{L^2_+} \\
&+ \left(e^{\Psi} \Delta^h_k(U \partial_x u - B \partial_x b)\right)|_{L^2_+} + \left(e^{\Psi} \Delta^h_k[U \partial_x b - B \partial_x u]_\Phi\right)|_{L^2_+} \\
&+ \left(e^{\Psi} \Delta^h_k(U v - Bh)\right)|_{L^2_+} + \left(e^{\Psi} \chi'' \Delta^h_k u_\Phi\right)|_{L^2_+} \\
&+ \left(e^{\Psi} \Delta^h_k[-\partial_x U \partial_y u + \partial_x B \partial_y b]_\Phi\right)|_{L^2_+} \\
&+ \left(e^{\Psi} \Delta^h_k[-\partial_x U \partial_y b + \partial_x B \partial_y u]_\Phi\right)|_{L^2_+} \\
&+ \left(e^{\Psi} \Delta^h_k[\partial_x U u - \partial_x B b]_\Phi\right)|_{L^2_+} + \left(e^{\Psi} \chi' \Delta^h_k u_\Phi\right)|_{L^2_+} \\
&= \left(e^{\Psi} \Delta^h_k(m_U)_\Phi\right)|_{L^2_+} + \left(e^{\Psi} \Delta^h_k(m_B)_\Phi\right)|_{L^2_+}.
\end{align*}
\]

\((5.4)\)
In what follows, we shall use the technical lemmas in Section 3 to handle term by term in (5.4).

Notice that for $\kappa \in [0, 2[\), $\kappa(2 - \kappa) \leq 1$, we get, by applying (3.3a) and (3.3b), that

$$
\int_{t_0}^{t} h(t') \left( (e^{\Psi} \Delta^h_k (\partial_t - \partial_y^2)\phi) | e^{\Psi} \Delta^h_k u \phi) L^2_{r+} + (e^{\Psi} \Delta^h_k (\partial_t - \kappa \partial_y^2) b \phi) | e^{\Psi} \Delta^h_k b \phi) L^2_{r+} \right) dt' \\
\geq \frac{1}{2} \left( \|h^\frac{1}{2} e^{\Psi} \Delta^h_k (u, b) \phi(t)\|^2_{L^2_{r+}} - \|h^\frac{1}{2} e^{\Psi} \Delta^h_k (u, b) \phi(t_0)\|^2_{L^2_{r+}} \right) \\
- \frac{1}{2} \|\sqrt{h} e^{\Psi} \Delta^h_k (u, b) \phi\|^2_{L^2_{[t_0, t]; (L^2_{r+})}} + 2\lambda \|h^\frac{1}{2} e^{\Psi} \Delta^h_k \partial_y (u, b) \phi\|^2_{L^2_{[t_0, t]; (L^2_{r+})}}.
$$

We deduce from Lemma [3.1] that

$$
\lambda \int_{t_0}^{t} h(t') \Phi(t') \left( (e^{\Psi} |D_x| \Delta^h_k u \phi) | e^{\Psi} \Delta^h_k u \phi) L^2_{r+} + (e^{\Psi} |D_x| \Delta^h_k b \phi) | e^{\Psi} \Delta^h_k b \phi) L^2_{r+} \right) dt' \\
\geq c\lambda \int_{t_0}^{t} \Phi(t') \|h^\frac{1}{2} e^{\Psi} \Delta^h_k (u, b) \phi\|^2_{L^2_{r+}} dt'.
$$

Whereas due to $\lim_{y \to +\infty} u = \lim_{y \to +\infty} b = 0$, we get, by applying Lemma [3.4] with $a = \frac{1}{2}$ and $b = c = d = 1$, that

$$
\int_{t_0}^{t} h(t') \left( |(e^{\Psi} \Delta^h_k [u \partial_x u - b \partial_x b] \phi) | e^{\Psi} \Delta^h_k u \phi) L^2_{r+} | + |(e^{\Psi} \Delta^h_k [u \partial_x u - b \partial_x u] \phi) | e^{\Psi} \Delta^h_k b \phi) L^2_{r+} \right) dt' \\
\lesssim d_k^2 2^{-k} \|h^\frac{1}{2} e^{\Psi} (u, b) \phi\|^2_{L^2_{[t_0, t]; (B^1_{r+})}} \text{ with } f(t) = (t)^\frac{1}{2} \|e^{\Psi} \partial_y (u, b) \phi(t)\|_{B^1_{r+}},
$$

from which and (4.6), we deduce from a similar derivation of (4.7) that

$$
\int_{t_0}^{t} h(t') \left( |(e^{\Psi} \Delta^h_k [v \partial_y v - b \partial_y b] \phi) | e^{\Psi} \Delta^h_k u \phi) L^2_{r+} | + |(e^{\Psi} \Delta^h_k [v \partial_y u - b \partial_y u] \phi) | e^{\Psi} \Delta^h_k b \phi) L^2_{r+} \right) dt' \\
\lesssim d_k^2 2^{-k} \|h^\frac{1}{2} e^{\Psi} (u, b) \phi\|^2_{L^2_{[t_0, t]; (B^1_{r+})}}.
$$

Due to $\partial_x u + \partial_y v = 0 = \partial_x b + \partial_y h$, we write

$$
(v, h) = - \int_{y}^{\infty} \partial_y (v, h) dy' = \int_{y}^{\infty} \partial_x (u, b) dy'.
$$

Then by applying Lemma [3.5], we get, by a similar derivation of (5.5), that

$$
\int_{t_0}^{t} h(t') \left( |(e^{\Psi} \Delta^h_k [v \partial_y u - b \partial_y b] \phi) | e^{\Psi} \Delta^h_k u \phi) L^2_{r+} | + |(e^{\Psi} \Delta^h_k [v \partial_y u - b \partial_y u] \phi) | e^{\Psi} \Delta^h_k b \phi) L^2_{r+} \right) dt' \\
\lesssim d_k^2 2^{-k} \|h^\frac{1}{2} e^{\Psi} (u, b) \phi\|^2_{L^2_{[t_0, t]; (B^1_{r+})}}.
$$

While it follows from Lemma [3.8] that

$$
\int_{t_0}^{t} h(t') \left( |(e^{\Psi} \Delta^h_k [U \partial_x u - B \partial_x b] \phi) | e^{\Psi} \chi' \Delta^h_k u \phi) L^2_{r+} | + |(e^{\Psi} \Delta^h_k [U \partial_x b - B \partial_x u] \phi) | e^{\Psi} \chi' \Delta^h_k b \phi) L^2_{r+} \right) dt' \\
\lesssim d_k^2 2^{-k} \|h^\frac{1}{2} e^{\Psi} (u, b) \phi\|^2_{L^2_{[t_0, t]; (B^1_{r+})}}.
$$
Notice from the definition of $\chi$ above (1.3) that $\text{supp}\chi'' \subset [1, 2]$. Then in view of (5.6), we get, by a similar derivation of (3.25), that
\[
\int_{t_0}^{t} h(t') \left( \left| (e^{\Psi} \Delta_k^h [U v - Bh] \phi) u u e^{\Psi} \Delta_k^h u \phi) \right|_{L^2_x} \right) dt' \lesssim \varepsilon \int_{t_0}^{t} \left| (e^{\Psi} \Delta_k^h [U v - Bh] \phi) u u e^{\Psi} \Delta_k^h u \phi) \right|_{L^2_x} dt',
\]
so that by applying (3.26) and (1.14a), we infer
\[
\int_{t_0}^{t} h(t') \left( \left| (e^{\Psi} \Delta_k^h [\partial_x U u - \partial_x B] \phi) u e^{\Psi} \Delta_k^h \varphi \phi) \right|_{L^2_x} \right) dt' \lesssim \varepsilon \int_{t_0}^{t} \left| (e^{\Psi} \Delta_k^h [\partial_x U u - \partial_x B] \phi) u e^{\Psi} \Delta_k^h \varphi \phi) \right|_{L^2_x} dt',
\]
and
\[
\int_{t_0}^{t} h(t') \left( \left| (e^{\Psi} \Delta_k^h [m U u - m B] \phi) u e^{\Psi} \Delta_k^h u \phi) \right|_{L^2_x} \right) dt' \lesssim \varepsilon \int_{t_0}^{t} \left| (e^{\Psi} \Delta_k^h [m U u - m B] \phi) u e^{\Psi} \Delta_k^h u \phi) \right|_{L^2_x} dt'.
\]
Finally for the remained source term, we get, by using Young’s inequality, that
\[
\int_{t_0}^{t} h(t') \left( \left| (e^{\Psi} \Delta_k^h [m U u - m B] \phi) u e^{\Psi} \Delta_k^h u \phi) \right|_{L^2_x} \right) dt' \lesssim \varepsilon \int_{t_0}^{t} \left| (e^{\Psi} \Delta_k^h [m U u - m B] \phi) u e^{\Psi} \Delta_k^h u \phi) \right|_{L^2_x} dt'.
\]
non-negative and non-decreasing function \( h \in C^1(\mathbb{R}_+) \), there exists a positive constant \( c \) so that

\[
\begin{align*}
\| h^{\frac{1}{2}} e^{\Psi_\kappa} \Delta_k^h(u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 & + 4\kappa \ell_\kappa \| h^{\frac{1}{2}} e^{\Psi_\kappa} \Delta_k^h \partial_y (u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 \\
+ 2c\lambda 2^k \| h^{\frac{1}{2}} e^{\Psi_\kappa} \Delta_k^h (u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 & \leq \| h^{\frac{1}{2}} e^{\Psi_\kappa} \Delta_k^h (u, b) \Phi (t_0) \|_{L^2_+}^2 \\
+ \| \sqrt{h} e^{\Psi_\kappa} \Delta_k^h (u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 & + \kappa \varepsilon d_k^2 2^{-k} \| h^{\frac{1}{2}} e^{\Psi_\kappa} \partial_y (u, b) \Phi \|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}^2 \\
+ C d_k^2 2^{-k} \left( \| h^{\frac{1}{2}} e^{\Psi_\kappa} (u, b) \Phi \|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}^2 + \varepsilon^{-1} \| \langle \tau \rangle \frac{1}{2} h^{\frac{1}{2}} e^{\Psi_\kappa} (m_U, m_B) \Phi \|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}^2 \right),
\end{align*}
\]

for any \( t_0 \in [0, t] \) with \( t < T^*_\kappa \), which is defined by (2.27).

**Proof.** The proof of Lemma 5.2 is similar to that of Lemma 5.1. We first observe that (5.3) holds with \( \theta, \Phi \) and \( \Psi \) being replaced by \( \theta_\kappa, \Phi_\kappa \) and \( \Psi_\kappa \). In what follows, we only present the detailed estimates for terms that are different from those in the proof of Lemma 5.1.

Notice that for \( \kappa \in [1/2, \infty], 2\kappa - 1 \leq \kappa^2 \), we get, by applying (3.4a) and (3.4b), that

\[
\int_{t_0}^{t} h(t') \left( \| e^{\Psi_\kappa} \Delta_k^h \partial_t (\partial_y^2) u \Phi \|_{L^2_+} + \| e^{\Psi_\kappa} \Delta_k^h (\partial_t - \kappa \partial_y^2) b \Phi \|_{L^2_+} \right) dt' \\
\geq \frac{1}{2} \left( \| h^{\frac{1}{2}} e^{\Psi_\kappa} \Delta_k^h (u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 - \| h^{\frac{1}{2}} e^{\Psi_\kappa} \Delta_k^h (u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 \right) \\
- \frac{1}{2} \left( \| \sqrt{h} e^{\Psi_\kappa} \Delta_k^h (u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 + 2\kappa \ell_\kappa \| h^{\frac{1}{2}} e^{\Psi_\kappa} \partial_y (u, b) \Phi \|_{L^2_{[t_0, t]}(L^2_+)}^2 \right),
\]

Whereas due to \( \lim_{y \to +\infty} u = \lim_{y \to +\infty} b = 0 \), we get, by applying Lemma 3.4 with \( a = \frac{1}{2\kappa} \) and \( b = c = d = \frac{1}{\kappa} \), that

\[
\int_{t_0}^{t} h(t') \left( \| e^{\Psi_\kappa} \Delta_k^h [u \partial_x u - b \partial_x b] \Phi \|_{L^2_+} + \| e^{\Psi_\kappa} \Delta_k^h [u \partial_x u - b \partial_x u] \Phi \|_{L^2_+} \right) dt' \\
\lesssim d_k^2 2^{-k} \| h^{\frac{1}{2}} e^{\Psi_\kappa} (u, b) \Phi \|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}^2 \text{ with } f(t) = \langle \tau \rangle \frac{1}{2} \| e^{\frac{1}{2} \Psi_\kappa} \partial_y (u, b) \Phi \|_{L^2_+}^2,
\]

from which and a similar derivation of (4.10), we infer

\[
\int_{t_0}^{t} h(t') \left( \| e^{\Psi_\kappa} \Delta_k^h [u \partial_x u - b \partial_x b] \Phi \|_{L^2_+} \right) dt' \lesssim d_k^2 2^{-k} \| h^{\frac{1}{2}} e^{\Psi_\kappa} (u, b) \Phi \|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}^2.
\]

On the other hand, we deduce from (2.12b) that

\[
\| e^{\Psi_\kappa} y \Delta_k^h (u, b) \Phi \|_{L^2_+} \leq 4\kappa \langle \tau \rangle \| e^{\Psi_\kappa} \partial_y \Delta_k^h (u, b) \Phi \|_{L^2_+},
\]

which implies

\[
\| e^{\Psi_\kappa} y (u, b) \Phi \|_{L^p(B^{s_0}, 0)} \leq 4\kappa \langle \tau \rangle \| e^{\Psi_\kappa} \partial_y (u, b) \Phi \|_{L^p(B^{s_0})},
\]
So that by applying (3.26) and (1.14a), we find

\[
\int_{t_0}^{t_1} h(t') \left( \left| \left( e^{\Psi \Delta} \Delta_{\kappa}^h \left[ -\partial_x U \partial_y u + \partial_x B \partial_y b \right] \right|_{L_t^2} \right|_{L_{t_0}^2(B_{\frac{1}{2},0})} \right) dt' \leq \kappa \varepsilon d_{\kappa} 2^{-k} \left\| h_{\frac{1}{2}} e^{\Psi \Delta} \partial_y (u, b) \right\|_{L_t^2(B_{\frac{1}{2},0})}^2.
\]

Along the same line, we infer

\[
\int_{t_0}^{t_1} h(t') \left( \left| \left( e^{\Psi \Delta} \Delta_{\kappa}^h \left[ -\partial_x B u - \partial_x U b \right] \right|_{L_t^2} \right|_{L_{t_0}^2(B_{\frac{1}{2},0})} \right) dt' \leq \kappa \varepsilon d_{\kappa} 2^{-k} \left\| h_{\frac{1}{2}} e^{\Psi \Delta} \partial_y (u, b) \right\|_{L_t^2(B_{\frac{1}{2},0})}^2.
\]

With the above estimates, we can repeat the derivation of (5.1) to complete the proof of (5.7). This completes the proof of Lemma 5.2.

It is easy to observe the following corollaries from Lemmas 5.1 and 5.2:

\[\text{Corollary 5.1. Under the assumptions of Lemma 5.1, there exist positive constants } \varepsilon_0, \lambda_0 \text{ so that for any } \lambda \geq \lambda_0 \text{ and } \varepsilon \leq \varepsilon_0, \text{ one has}\]

\[
\left\| h_{\frac{1}{2}} e^{\Psi} (u, b) \right\|_{L_{t_0}^2(B_{\frac{1}{2},0})} + \sqrt{\lambda} \left\| h_{\frac{1}{2}} e^{\Psi} (u, b) \right\|_{L_{t_0}^2(B_{1,0})} \leq \left\| h_{\frac{1}{2}} e^{\Psi} (m_U, m_B) \right\|_{L_{t_0}^2(B_{\frac{1}{2},0})}.
\]

\[\text{Proof. By taking square root of (5.1) and then multiplying the resulting inequality by } 2^\frac{1}{2} \text{ and finally summing over } k \in Z, \text{ we find}\]

\[
\left\| h_{\frac{1}{2}} e^{\Psi} (u, b) \right\|_{L_{t_0}^2(B_{\frac{1}{2},0})} + 2 \sqrt{\lambda} \left\| h_{\frac{1}{2}} e^{\Psi} \partial_y (u, b) \right\|_{L_{t_0}^2(B_{1,0})} \leq \left\| h_{\frac{1}{2}} e^{\Psi} (m_U, m_B) \right\|_{L_{t_0}^2(B_{\frac{1}{2},0})}.
\]

Taking \( \lambda \) big enough and \( \varepsilon \leq \varepsilon_0 \) small enough in the above inequality gives rise to (5.9). \( \square \)
Corollary 5.2. Under the assumption of Lemma 5.2, there exist positive constants \( \varepsilon_0, \lambda_0 \) so that for any \( \lambda \geq \lambda_0 \) and \( \varepsilon \leq \varepsilon_0 \), one has

\[
\begin{align*}
& \| h^\frac{1}{2} e^{\Psi} (u, b) \Phi \|_{L^\infty_t (B^{1/2 + \delta}, 0)} + \sqrt{c \lambda} \| h^\frac{1}{2} e^{\Psi} (u, b) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} \\
& \quad + \sqrt{\kappa \phi_h} \| h^\frac{1}{2} e^{\Psi} \phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} \leq \| h^\frac{1}{2} e^{\Psi} (u, b) \phi (t_0) \|_{B^{1/2 + \delta}} \\
& \quad + \| \sqrt{\kappa} e^{\Psi} (u, b) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} + C \varepsilon^{- \frac{1}{2}} \| (\tau)^{\frac{1}{2}} h^\frac{1}{2} e^{\Psi} (m_U, m_B) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}}.
\end{align*}
\]

(5.10)

Proof. The proof is the same as that of Corollary 5.1. We omit the details here. \( \square \)

Now we are in a position to prove Propositions 2.2 and 2.5.

Proof of Proposition 2.2. Taking \( h(t) = 1 \) and \( t_0 = 0 \) in Corollary 5.1 gives rise to

\[
\| e^{\Phi} (u, b) \Phi \|_{L^\infty_t (B^{1/2 + \delta}, 0)} + \sqrt{c \lambda} \| e^{\Phi} (u, b) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} + \sqrt{\kappa \phi_h} \| \partial_y (u, b) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} \leq \| e^{\Phi} (u, b) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} + C \sqrt{\varepsilon}.
\]

(5.11)

While by taking \( t_0 = \frac{t}{2} \) and \( h(t) = (t - t_0)^{1 + 2 \kappa - 2 \varepsilon} \) in Corollary 5.1, we find

\[
\begin{align*}
& \| \sqrt{\kappa} e^{\Phi} (u, b) \Phi (t) \|_{B^{1/2 + \delta}} \lesssim \| (\tau - t_0)^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (u, b) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})} \\
& \quad \lesssim \| \tau^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (u, b) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})} + \varepsilon^{- \frac{1}{2}} \| (\tau)^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (m_U, m_B) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})} \\
& \quad \lesssim \| \tau^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (u, b) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})} + \varepsilon^{- \frac{1}{2}} \| (\tau)^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (m_U, m_B) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})}.
\end{align*}
\]

(5.12)

Due to \( (u, b) = \partial_y (\varphi, \psi) \), we get, by applying Proposition 2.1 that

\[
\| \sqrt{\kappa} e^{\Phi} (u, b) \Phi (t) \|_{B^{1/2 + \delta}} \lesssim \| e^{\sqrt{\kappa} \phi_h} (\varphi_0, \psi_0) \|_{B^{1/2 + \delta}} + \varepsilon^{- \frac{1}{2}} \| (\tau)^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (m_U, m_B) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})}.
\]

(5.13)

Thanks to Proposition 2.1 and (5.13), we get, by taking \( t_0 = \frac{t}{2} \) and \( h(t) = t^{1 + 2 \kappa - 2 \varepsilon} \) in Corollary 5.1, that

\[
\begin{align*}
& \| \sqrt{\kappa} e^{\Phi} \partial_y (u, b) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})} \lesssim \| (t/2)^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (u, b) \Phi (t/2) \|_{B^{1/2 + \delta}} \\
& \quad + \| \tau^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (u, b) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})} + \varepsilon^{- \frac{1}{2}} \| (\tau)^{1 + \frac{2 \kappa - 2 \varepsilon}{2}} e^{\Phi} (m_U, m_B) \Phi \|_{L^2_{t, \phi} (B^{1/2 + \delta})} \\
& \quad \lesssim \| e^{\sqrt{\kappa} \phi_h} (\varphi_0, \psi_0) \|_{B^{1/2 + \delta}} + C \sqrt{\varepsilon}.
\end{align*}
\]

which together with (5.11) and (5.13) ensures Proposition 2.2. \( \square \)

Proof of Proposition 2.3. The proof of this proposition is the same as that of Proposition 2.2. Indeed taking \( h(t) = 1 \) and \( t_0 = 0 \) in Corollary 5.2 gives rise to

\[
\begin{align*}
& \| e^{\Phi} (u, b) \Phi \|_{L^\infty_t (B^{1/2 + \delta}, 0)} + \sqrt{\kappa \phi_h} \| e^{\Phi} \partial_y (u, b) \Phi \|_{L^2_t (B^{1/2 + \delta})} \\
& \quad + \sqrt{c \lambda} \| h^\frac{1}{2} e^{\Phi} (u, b) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} \leq \| e^{\Phi} (u, b) \Phi \|_{L^2_t \{ \partial_y |x| \leq \varepsilon_0 \}} + C \sqrt{\varepsilon}.
\end{align*}
\]

(5.14)

We omit the other details here. \( \square \)
6. Analytic energy estimates of the quantities \((G, H)\)

In this section, we shall present the \textit{a priori} weighted analytic energy estimate to the quantities \((G, H)\) which are defined by (1.8). Those estimates will be crucial for us to globally control the analytic band of solutions to (1.4) and (1.6).

We first observe from (1.8) and the boundary conditions in (1.4) and (1.6) that

\[ G|_{y=0} = \partial_y H|_{y=0} = 0. \]

While by multiplying the \(\varphi\) equation of (1.6) by \(\frac{\varphi}{2|\varphi|}\) (resp. the \(\psi\) equation of (1.6) by \(\frac{\psi}{2|\psi|}\)) and summing up the resulting equation with the \(u\) equation of (1.4) (resp. the \(b\) equation of (1.4)), we find that \((G, H)\) verifies

\[
\begin{aligned}
&\partial_t G - \partial_y^2 G + \frac{G^2}{2(\theta)} - \tilde{B}_\kappa \partial_x H + u \partial_x G - \kappa b \partial_x H - (1 - \kappa) b \partial_y u + v \partial_y b - h \partial_y b \\
&\quad + \frac{\varphi}{(\theta)} \int_y^\infty (\partial_x \varphi \partial_y u - \partial_x \psi \partial_y b) dy' + \chi'(U \partial_x G - \kappa B \partial_x H - (1 - \kappa) B \partial_y b) \\
&\quad + \chi'(\partial_x U u - \partial_x B b) + \frac{\varphi}{(\theta)} \int_{\infty}^y \chi''(U \partial_x \varphi - B \partial_x \psi) dy' \\
&\quad + \chi(\partial_x U u + \partial_x B b) + \chi(\partial_x U u + \partial_x B b) + \chi''(U v - B h) \\
&\quad + \chi'(\partial_x U \varphi - \partial_x B \psi) + \frac{\varphi}{(\theta)} \int_{\infty}^y \chi''(\partial_x U \varphi - \partial_x B \psi) dy' \\
&= m_U + \frac{\varphi}{(\theta)} M_U, \\
&\partial_t H - \kappa \partial_y^2 H + \frac{H^2}{2(\theta)} - \tilde{B}_\kappa \partial_x G + u \partial_x H - \frac{1}{\kappa} b \partial_x G - (1 - \frac{1}{\kappa}) b \partial_y u + v \partial_y b - h \partial_y u \\
&\quad + \chi'(U \partial_x H - \frac{1}{\kappa} B \partial_x G - (1 - \frac{1}{\kappa}) B \partial_y u) + \chi'(\partial_x B u - \partial_x U b) \\
&\quad + \chi'(\partial_x U \varphi - \partial_x B \psi) + \frac{\varphi}{(\theta)} \int_{\infty}^y \chi'(\partial_x U \varphi - \partial_x B \psi) dy' \\
&= m_B + \frac{\varphi}{(\theta)} M_B, \\
&G|_{y=0} = \partial_y H|_{y=0} = 0 \quad \text{and} \quad \lim_{y \to \pm \infty} G = \lim_{y \to \pm \infty} H = 0, \\
&G|_{t=0} = G_0, \quad H|_{t=0} = H_0.
\end{aligned}
\]

We remark that the restriction for \(\tilde{B}_\kappa = 0\) except \(\kappa = 1\) is only used in the derivation of the system \((6.1)\).

The key ingredients to prove Propositions 2.3 and 2.6 lie in the following two lemmas:

**Lemma 6.1.** Let \((u, b)\) and \((\varphi, \psi)\) be smooth enough solutions of (1.4) and (1.6) respectively, and \((G, H)\) be determined by (1.3). Let \(\Phi(t, \xi)\) and \(\Psi(t, y)\) be given respectively by (1.1) and (1.12). Then if \(\kappa \in [0, 2], \) for \(\kappa = \frac{\kappa}{2(\theta)}\), and for any non-negative and non-decreasing function \(h \in C^1(\mathbb{R}_+),\) there exists a positive constant \(c\) so that

\[
\begin{aligned}
&\|\frac{h^2}{2}(\varphi) \Delta_k^h(G, H)\|_{L^2_{[t_0, t]}(L^2_2)} + 4\kappa\|\frac{h^2}{2}(\varphi) \Delta_k^h \partial_y (G, H)\|_{L^2_{[t_0, t]}(L^2_2)} \\
&+ 2\|\langle \theta \rangle^{-\frac{1}{2}} h^2 \varphi \Delta_k^h \partial_y (G, H)\|_{L^2_{[t_0, t]}(L^2_2)} + 2c \lambda 2^k \|\frac{h^2}{2}(\varphi) \Delta_k^h (G, H)\|_{L^2_{[t_0, t]}(L^2_2)} \\
&\leq \|\frac{h^2}{2}(\varphi) \Delta_k^h (G, H)\|_{L^2_{[t_0, t]}(L^2_2)} + \|\sqrt{h} \varphi \partial_y \Delta_k^h (G, H)\|_{L^2_{[t_0, t]}(L^2_2)} \\
&+ c \|\partial_y^2 \|_{L^2_{[t_0, t]}(L^2_2)} \|\frac{h^2}{2}(\varphi) \partial_y (G, H)\|_{L^2_{[t_0, t]}(L^2_2)} + C 2^k \|\frac{h^2}{2}(\varphi) (G, H)\|_{L^2_{[t_0, t]}(L^2_2)} \\
&+ \varepsilon \|\langle \theta \rangle^{-\frac{1}{2}} h^2 \varphi \partial_y (u, b)\|_{L^2_{[t_0, t]}(L^2_2)} + \varepsilon \|\langle \theta \rangle^{-\frac{1}{2}} h^2 \varphi \partial_y (u, b)\|_{L^2_{[t_0, t]}(L^2_2)} \\
&+ \varepsilon^{-1} \|\langle \theta \rangle^{-\frac{1}{2}} h^2 \varphi (m_U, m_B, M_B, M_U)\|_{L^2_{[t_0, t]}(L^2_2)},
\end{aligned}
\]

for any \(t_0 \in [0, t]\) with \(t < T^*\), which is defined by (2.1).
Proof. In view of (1.9), by applying operator $e^{\Phi(t)|D_\pm|}$ to (6.1), we write

$$
\partial_t G_\Phi - \partial_y^2 G_\Phi + \lambda \dot{\theta}(t)|D_x|G_\Phi + \frac{1}{\langle t \rangle} G_\Phi - \bar{B}_\kappa \partial_x H_\Phi + [u \partial_x G - \kappa b \partial_x H - (1 - \kappa) b \partial_x b] \Phi
$$

$$
+ [v \partial_y u - h \partial_y b] \Phi + \frac{y}{\langle t \rangle} \int_0^\infty \left( \partial_x \varphi \partial_y u - \partial_x \psi \partial_y b \right) \Phi \, dy' + \lambda \dot{\theta}(t)|D_x|H_\Phi + \frac{1}{\langle t \rangle} H_\Phi - \bar{B}_\kappa \partial_x G_\Phi
$$

$$
+ \frac{y}{\langle t \rangle} \int_0^\infty \left( \partial_x \varphi - B \partial_x \psi \right) \Phi \, dy' + \lambda \dot{\theta}(t)|D_x|H_\Phi + \frac{1}{\langle t \rangle} H_\Phi
$$

$$
= \left[ \frac{m_B}{G_\Phi} + \frac{y}{\langle t \rangle} \right] M_B \Phi
$$

(6.3)

Again thanks to a similar cancelation equality (1.4), we get, by first applying the dyadic operator $\Delta^h_\kappa$ to (6.3), (6.4) and then taking the $L^2_+$ inner product of the resulting equations

$$
\partial_t H_\Phi - \kappa \partial_y^2 H_\Phi + \lambda \dot{\theta}(t)|D_x|H_\Phi + \frac{1}{\langle t \rangle} H_\Phi - \bar{B}_\kappa \partial_x G_\Phi
$$

$$
+ [u \partial_x H - \frac{1}{\kappa} b \partial_x G - (1 - \frac{1}{\kappa}) b \partial_x u] \Phi + [v \partial_y b - h \partial_y u] \Phi
$$

$$
+ \lambda \dot{\theta}(t)|D_x|H_\Phi + \frac{1}{\langle t \rangle} H_\Phi - \bar{B}_\kappa \partial_x G_\Phi
$$

$$
+ \lambda \dot{\theta}(t)|D_x|H_\Phi + \frac{1}{\langle t \rangle} H_\Phi
$$

$$
= \left[ \frac{m_B}{G_\Phi} + \frac{y}{\langle t \rangle} \right] M_B \Phi
$$

(6.4)
with \( e^{2\Phi} \Delta^h_k(G, H)_\Phi \) respectively, that

\[
\begin{align*}
(e^\Psi \Delta^h_k(\partial_t - \partial_y^2)G|e^\Psi \Delta^h_k G)_L^2 + (e^\Psi \Delta^h_k(\partial_t - \kappa \partial_y^2)H|e^\Psi \Delta^h_k H)_L^2 \\
+ \lambda(t)(e^{\Psi} |D_x| \Delta^h_k G|e^\Psi \Delta^h_k G)_L^2 + \lambda(t)(e^{\Psi} |D_x| \Delta^h_k H|e^\Psi \Delta^h_k H)_L^2 \\
+ \frac{1}{(t)} ||e^\Psi \Delta^h_k G\Phi||^2_L + \frac{1}{(t)} ||e^\Psi \Delta^h_k H\Phi||^2_L \\
+ (e^\Psi \Delta^h_k [u\partial_x G - \kappa b\partial_x H]|e^\Psi \Delta^h_k G\Phi)_L^2 + (e^\Psi \Delta^h_k [u\partial_x H - \frac{1}{\kappa} b\partial_x G]|e^\Psi \Delta^h_k H\Phi)_L^2 \\
- (1 - \kappa)(e^\Psi \Delta^h_k [b\partial_x b]|e^\Psi \Delta^h_k G\Phi)_L^2 - (1 - \frac{1}{\kappa})(e^\Psi \Delta^h_k [b\partial_x u]|e^\Psi \Delta^h_k H\Phi)_L^2 \\
+ (e^\Psi \Delta^h_k [v\partial_y u - h\partial_y b]|e^\Psi \Delta^h_k G\Phi)_L^2 + (e^\Psi \Delta^h_k [v\partial_y b - h\partial_y u]|e^\Psi \Delta^h_k H\Phi)_L^2 \\
+ (e^\Psi \frac{y}{t} \int_y^\infty \chi\Delta^h_k[\partial_y u\partial_x \varphi - \partial_y \varphi b\partial_x \psi]|dy'|e^\Psi \Delta^h_k G\Phi)_L^2 \\
+ (e^\Psi \Delta^h_k [U\partial_x G - \kappa B\partial_x H]|e^\Psi \chi\Delta^h_k G\Phi)_L^2 \\
+ (e^\Psi \Delta^h_k [U\partial_x H - \frac{1}{\kappa} B\partial_x G]|e^\Psi \chi\Delta^h_k H\Phi)_L^2 \\
- (1 - \kappa)(e^\Psi \Delta^h_k [B\partial_x b]|e^\Psi \chi\Delta^h_k G\Phi)_L^2 - (1 - \frac{1}{\kappa})(e^\Psi \Delta^h_k [B\partial_x u]|e^\Psi \chi\Delta^h_k H\Phi)_L^2 \\
+ (e^\Psi \Delta^h_k [Uv - Bh]|e^\Psi \chi''\Delta^h_k G\Phi)_L^2 + (e^\Psi \Delta^h_k [Bv - Uh]|e^\Psi \chi''\Delta^h_k H\Phi)_L^2 \\
+ \frac{1}{(t)} (e^\Psi \chi \Delta^h_k [U\partial_x \varphi - B\partial_x \psi]|dy'|e^\Psi \Delta^h_k G\Phi)_L^2 \\
+ (e^\Psi \Delta^h_k [-\partial_x U\partial_y u + \partial_x B\partial_y b]|e^\Psi \chi\Delta^h_k G\Phi)_L^2 \\
+ (e^\Psi \Delta^h_k [-\partial_x U\partial_y b + \partial_x B\partial_y u]|e^\Psi \chi\Delta^h_k H\Phi)_L^2 \\
+ (e^\Psi \Delta^h_k [\partial_x U\varphi - \partial_x \varphi B]|e^\Psi \chi\Delta^h_k G\Phi)_L^2 + (e^\Psi \Delta^h_k [\partial_x Bu - \partial_x Ub]|e^\Psi \chi\Delta^h_k H\Phi)_L^2 \\
+ \frac{1}{2(t)} (e^\Psi \Delta^h_k [-\partial_x Uu + \partial_x Bb]|e^\Psi y\chi\Delta^h_k G\Phi)_L^2 \\
+ \frac{1}{2\kappa(t)} (e^\Psi \Delta^h_k [-\partial_x Ub + \partial_x Bu]|e^\Psi y\chi\Delta^h_k H\Phi)_L^2 \\
+ \frac{1}{(t)} (e^\Psi \Delta^h_k [\partial_x U\varphi - \partial_x \varphi B]|e^\Psi y\chi\Delta^h_k G\Phi)_L^2 \\
+ \frac{1}{(t)} (e^\Psi \Delta^h_k [U\varphi - B\psi]|e^\Psi y\chi\Delta^h_k G\Phi)_L^2 \\
= (e^\Psi \Delta^h_k [mU + \frac{y}{2(t)} M_U]|e^\Psi \Delta^h_k G\Phi)_L^2 + (e^\Psi \Delta^h_k [mB + \frac{y}{2\kappa(t)} MB]|e^\Psi \Delta^h_k H\Phi)_L^2.
\end{align*}
\]

In what follows, we shall use the technical lemmas in Section 3 to handle term by term in (6.5).
Notice that for $\kappa \in [0, 2]$, $\kappa(2 - \kappa) \leq 1$, we get, by applying (3.3a) and (3.3b), that
\[
\int_{t_0}^{t} h(t') \left( (e^\Psi \Delta_k^h \partial_t - \partial_d^2) G \phi \big| e^\Psi \Delta_k^h G \phi \big|_{L^2_+} + (e^\Psi \Delta_k^h \partial_t - \kappa \partial_y^2) H \phi \big| e^\Psi \Delta_k^h H \phi \big|_{L^2_+} \right) dt' \\
\geq \frac{1}{2} \left( \| h^\frac{1}{2} e^\Psi \Delta_k^h (G, H) \phi(t) \|_{L^2_+}^2 - \| h^\frac{1}{2} e^\Psi \Delta_k^h (G, H) \phi(t_0) \|_{L^2_+}^2 \right) \\
- \frac{1}{2} \| \sqrt{\mathcal{H}} e^\Psi \Delta_k^h (G, H) \phi \|_{L^2_{[t_0,t]}(L^2_+)}^2 + 2l_\kappa \| h^\frac{1}{2} e^\Psi \Delta_k^h \partial_y (G, H) \phi \|_{L^2_{[t_0,t]}(L^2_+)}^2.
\]

We deduce from Lemma 3.4 that
\[
\lambda \int_{t_0}^{t} h(t') \hat{\theta}(t') \left( (e^\Psi |D_x| \Delta_k^h G \phi \big| e^\Psi \Delta_k^h G \phi \big|_{L^2_+} + (e^\Psi |D_x| \Delta_k^h H \phi \big| e^\Psi \Delta_k^h H \phi \big|_{L^2_+} \right) dt' \\
\geq c\lambda 2^k \int_{t_0}^{t} \hat{\theta}(t') \| h^\frac{1}{2} e^\Psi \Delta_k^h (G, H) \phi \|_{L^2_+}^2 dt'.
\]

 Whereas due to $\lim_{y \to +\infty} u = \lim_{y \to +\infty} b = 0$, we get, by applying Lemma 3.4 with $a = \frac{1}{2}$ and $b = c = d = 1$, that
\[
\int_{t_0}^{t} h(t') \left( (e^\Psi \Delta_k^h [u \partial_x G - b \partial_x H] \phi \big| e^\Psi \Delta_k^h G \phi \big|_{L^2_+} + (e^\Psi \Delta_k^h [u \partial_x H - b \partial_x G] \phi \big| e^\Psi \Delta_k^h H \phi \big|_{L^2_+} \right) dt' \\
\lesssim d^2_k 2^{-k} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)}^2 \quad \text{with} \quad f(t) = (t) t^\frac{1}{2} \| e^\Psi \partial_y (u, b) \phi(t) \|_{B^1,0},
\]
from which, we get, by a similar derivation of (4.7), that
\[
\int_{t_0}^{t} h(t') \left( (e^\Psi \Delta_k^h [u \partial_x G - b \partial_x H] \phi \big| e^\Psi \Delta_k^h G \phi \big|_{L^2_+} \\
+ (e^\Psi \Delta_k^h [u \partial_x H - b \partial_x G] \phi \big| e^\Psi \Delta_k^h H \phi \big|_{L^2_+} \right) dt' \\
\lesssim d^2_k 2^{-k} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)}^2.
\]

Along the same line, by applying Lemma 3.4 with $a = \frac{1}{2}$, $b = \frac{3}{4}$ and $c = d = 1$, and then using (2.65), we obtain
\[
\int_{t_0}^{t} h(t') \left( (1 - \kappa) (e^\Psi \Delta_k^h [b \partial_x u] \phi \big| e^\Psi \Delta_k^h G \phi \big|_{L^2_+} + (1 - \frac{1}{\kappa}) (e^\Psi \Delta_k^h [b \partial_x u] \phi \big| e^\Psi \Delta_k^h H \phi \big|_{L^2_+} \right) dt' \\
\lesssim d^2_k 2^{-k} \| h^\frac{1}{2} e^\Psi (u, b) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)} \\
\lesssim d^2_k 2^{-k} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)}^2.
\]

Similarly, applying Lemma 3.5 and (2.6b) gives rise to
\[
\int_{t_0}^{t} h(t') \left( (e^\Psi \Delta_k^h [v \partial_y u - h \partial_y b] \phi \big| e^\Psi \Delta_k^h u \phi \big|_{L^2_+} + (e^\Psi \Delta_k^h [v \partial_y b - h \partial_y u] \phi \big| e^\Psi \Delta_k^h b \phi \big|_{L^2_+} \right) dt' \\
\lesssim d^2_k 2^{-k} \| h^\frac{1}{2} e^\Psi (u, b) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)} \\
\lesssim d^2_k 2^{-k} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}^2_{[t_0,t],\phi}(B^1,0)}^2.
\]
Whereas by applying Lemma 3.6 with \( a = \frac{3}{4}, b = \frac{3}{4}, c = \frac{5}{4} \) and \( d = 1 \), and then using (2.6a), we find

\[
\int_{t_0}^t h(t') \left| \langle e^\Psi(t')^{-1} y \int_0^\infty \Delta_h^b[\partial_y u \partial_x \varphi - \partial_y b \partial_x \psi] \phi | e^\Psi \Delta_h^b G \phi \rangle_{L_+^2} \right| dt' \\
\lesssim d_k^2 2^{-k} \| e^{-\frac{\Psi}{10}} (\tau) - \frac{\Psi}{10} y \|_{L_+^{\infty}(L^\infty)} \| \langle \tau \rangle^{-\frac{1}{2}} \frac{1}{10} e^{\frac{\Psi}{10}} (\varphi, \psi) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)} \\
\times \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)}.
\]

While it follows Lemma 3.8 that

\[
\int_{t_0}^t h(t') \left( \| e^\Psi \Delta_h^b [U \partial_x G - \kappa B \partial_x H] \phi | e^\Psi \chi' \Delta_h^b G \phi \rangle_{L_+^2} \right) \\
+ \left( \| e^\Psi \Delta_h^b [U \partial_x H - \frac{1}{\kappa} B \partial_x G] \phi | e^\Psi \chi' \Delta_h^b H \phi \rangle_{L_+^2} \right) dt' \\
\lesssim d_k^2 2^{-k} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)}.
\]

Similarly, we get, by applying Lemma 3.8 and (1.10), that

\[
\int_{t_0}^t h(t') \left( (1 - \kappa) \| e^\Psi \Delta_h^b [B \partial_x b] \phi | e^\Psi \chi' \Delta_h^b G \phi \rangle_{L_+^2} \right) \\
+ (1 - \frac{1}{\kappa}) \| e^\Psi \Delta_h^b [B \partial_x u] \phi | e^\Psi \chi' \Delta_h^b H \phi \rangle_{L_+^2} \right) dt' \\
\lesssim d_k^2 2^{-k} \| e^\Psi (u, b) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)} \\
\lesssim d_k^2 2^{-k} (\epsilon \| (\tau) - \frac{1}{2} \frac{1}{10} e^\Psi (u, b) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)} + \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)}).
\]

Notice from the definition of \( \chi \) above (1.3) that supp\( \chi'' \subset [1, 2] \). Then in view of (5.6b), we get, by a similar derivation of (3.25), that

\[
\int_{t_0}^t h(t') \left( \| e^\Psi \Delta_h^b [U v - B h] \phi | e^\Psi \chi'' \Delta_h^b G \phi \rangle_{L_+^2} \right) + \left( \| e^\Psi \Delta_h^b [B v - U h] \phi | e^\Psi \chi'' \Delta_h^b H \phi \rangle_{L_+^2} \right) dt' \\
\lesssim d_k^2 2^{-k} \| e^\Psi \|_{L^\infty_{y \leq 2}} \| h^\frac{1}{2} (u, b) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)} \\
\lesssim d_k^2 2^{-k} \| h^\frac{1}{2} e^\Psi (G, H) \phi \|_{\tilde{L}_2^{(t_0, t), \phi} G(H, \Phi)}^{(B_1, 0)}.
\]

Moreover, again due to supp\( \chi'' \subset [1, 2] \), we deduce that the term \( \int_{t_0}^t h(t') \left( \frac{1}{10} \langle e^\Psi y \int_0^\infty \chi'' \Delta_h^b [U \partial_x \varphi - B \partial_x \psi] \phi dy \right| e^\Psi \Delta_h^b G \phi \rangle_{L_+^2} \right) dt' \) shares the same estimate as above.

On the other hand, we observe from (2.13b) that

\[
\| e^\Psi y \Delta_h^b (G, H) \phi \|_{L_+^2} \leq 4(t) \| e^\Psi \Delta_h^b (G, H) \phi \|_{L_+^2},
\]

which implies

\[
\| e^\Psi y (G, H) \phi \|_{\tilde{L}_1^{(B_1, 0)}} \leq 4 \| \langle \tau \rangle e^\Psi \Delta_h^b (G, H) \phi \|_{\tilde{L}_1^{(B_1, 0)}}.
\]
So that by applying (3.26) and (1.14a), we find

\[
\int_{t_0}^t \left( \left| e^\Psi \Delta_k^h \left[ -\partial_x U \partial_y u + \partial_x B \partial_y b \right] e^\Psi \chi \Delta_k^h G \right|_{L^2_+} \right. \\
+ \left| e^\Psi \Delta_k^h \left[ -\partial_x U \partial_y b + \partial_x B \partial_y u \right] e^\Psi \chi \Delta_k^h H \right|_{L^2_+} \right) dt \\
\lesssim d_k^2 e^{-k} \| \langle \tau \rangle \frac{2}{H} (U, B) \|_{L^\infty_{[t_0, t]} (B^{3/2})} \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (u, b) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\times \| \langle \tau \rangle^{-1} \frac{1}{H} e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\lesssim e d_k^2 e^{-k} \left( \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (u, b) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 + \| e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 \right).
\]

Along the same line, we infer

\[
\int_{t_0}^t \left( \left| e^\Psi \Delta_k^h \left[ \partial_x U u - \partial_x B b \right] e^\Psi \chi \Delta_k^h G \right|_{L^2_+} \right. \\
+ \left| e^\Psi \Delta_k^h \left[ \partial_x B u - \partial_x U b \right] e^\Psi \chi \Delta_k^h H \right|_{L^2_+} \right) dt' \\
\lesssim d_k^2 e^{-k} \| \langle \tau \rangle \frac{2}{H} (U, B) \|_{L^\infty_{[t_0, t]} (B^{3/2})} \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (u, b) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\times \| \langle \tau \rangle^{-1} \frac{1}{H} e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\lesssim e d_k^2 e^{-k} \left( \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (u, b) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 + \| e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 \right).
\]

and

\[
\int_{t_0}^t \left( \left| \frac{1}{2 \langle \tau' \rangle} \left( e^\Psi \chi \Delta_k^h \left[ -\partial_x U \partial_y u + \partial_x B \partial_y b \right] e^\Psi \chi \Delta_k^h G \right) \right|_{L^2_+} \right. \\
+ \left| \frac{1}{2 \langle \tau' \rangle} \left( e^\Psi \chi \Delta_k^h \left[ -\partial_x U \partial_y b + \partial_x B \partial_y u \right] e^\Psi \chi \Delta_k^h H \right) \right|_{L^2_+} \right) dt' \\
\lesssim d_k^2 e^{-k} \| \langle \tau \rangle \frac{2}{H} (U, B) \|_{L^\infty_{[t_0, t]} (B^{3/2})} \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (u, b) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\times \| \langle \tau \rangle^{-1} \frac{1}{H} e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\lesssim e d_k^2 e^{-k} \left( \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (u, b) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 + \| e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 \right).
\]

and

\[
\int_{t_0}^t \left( \left| \frac{1}{\langle \tau' \rangle} \left( e^\Psi \chi \Delta_k^h \left[ \partial_x U \varphi - \partial_x B \psi \right] e^\Psi \chi \Delta_k^h G \right) \right|_{L^2_+} \right. \\
\lesssim d_k^2 e^{-k} \| \langle \tau \rangle \frac{2}{H} (U, B) \|_{L^\infty_{[t_0, t]} (B^{3/2})} \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (\varphi, \psi) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\times \| \langle \tau \rangle^{-1} \frac{1}{H} e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)} \\
\lesssim e d_k^2 e^{-k} \left( \| \langle \tau \rangle \frac{1}{H} e^\Psi \partial_y (\varphi, \psi) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 + \| e^\Psi \partial_y (G, H) \|_{L^2_{[t_0, t]} (B^{5/2}, a)}^2 \right).
\]
Furthermore, due to sup \chi'' \subset [1, 2[, we deduce that the term \int_{t_0}^t h(t')|\langle t'\rangle^{-1}(e^{\Psi} \int_y^\infty \chi'' \Delta^h_k[\partial_x U \varphi - \partial_x B \psi] \phi dy |e^{\Psi} y \Delta^h_k G \phi)| dt' shares the same estimate as above.

Finally for the remained source term, by Young's inequality, we achieve

\[ \int_{t_0}^t h(t') \left( |(e^{\Psi} \Delta^h_k (mU + \frac{y}{2(t')} M_U)) \phi |e^{\Psi} \Delta^h_k G \phi, L^2 \right) dt' \]

\[ + |(e^{\Psi} \Delta^h_k (mB + \frac{y}{2\kappa(t')} MB)) \phi |e^{\Psi} \Delta^h_k H \phi, L^2 \right) \]

\[ \lesssim d_0^2 \kappa^2 \|\langle \tau\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (m_U, M_U, B) \phi \parallel^2 \|\langle t\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (m_B, M_B, B) \phi \parallel^2 \|\langle t\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (m_U, M_U, B) \phi \parallel^2 \|\langle t\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (m_B, M_B, B) \phi \parallel^2 \]

By multiplying (6.3) by \( h(t) \) and then integrating the resulting equality over \([t_0, t]\), and finally inserting the above estimates to the equality, we obtain (6.2). This completes the proof of Lemma 6.1.

**Lemma 6.2.** Let \((u, b)\) and \((\varphi, \psi)\) be smooth enough solutions of (1.4) and (1.6) respectively, and \((G, H)\) be determined by (1.8). Let \(\Psi_\kappa(t, \xi)\) and \(\Psi_\kappa(t, y)\) be given by (1.19) and (1.20) respectively. Then if \(\kappa \in [1/2, \infty[, \) for \(\kappa \in [1/t^{\kappa - 1/2\kappa}, \) and for any non-negative and non-decreasing function \( h \in C^1(\mathbb{R}_+) \), there exists a positive constant \( c \) so that

\[ \|h \parallel e^{\Psi} \Delta^h_k (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ + 4\kappa \ell_\kappa \|h \parallel e^{\Psi} \Delta^h_k \partial_y (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ + 2\|\langle \tau\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ \leq \|h \parallel e^{\Psi} \Delta^h_k (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ + \sqrt{h} \parallel e^{\Psi} \Delta^h_k (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ + \ell_\kappa \|\langle \tau\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (u, b) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ + \epsilon \|\langle \tau\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (u, b) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ + \epsilon^{-1} \|\langle \tau\rangle^{-\frac{1}{2}} \parallel h \parallel \Delta^h_k e^{\Psi} \parallel (m_U, m_B, M_B, M_U) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

for any \( t_0 \in [0, t] \) with \( t < T^*_\kappa \), which is defined as by (2.7).

**Proof.** The proof of Lemma 6.2 is almost the same as that of Lemma 6.1. Indeed (6.5) holds with \( \theta, \Psi \) and \( \Phi \) there being replaced by \( \theta_\kappa, \Psi_\kappa \) and \( \Phi_\kappa \). In what follows, we just present the estimates to the terms with which are different from that of Lemma 6.1.

We first observe that due to \( \kappa \in [1/2, \infty[, \) \( 2\kappa - 1 \leq \kappa^2 \). Then we get, by applying (3.4a) and (3.4b), that

\[ \int_{t_0}^t h(t') \left( (e^{\Psi} \Delta^h_k (\partial_t - \partial_y^2) G \phi, e^{\Psi} \Delta^h_k G \phi, L^2 \right) + (e^{\Psi} \Delta^h_k (\partial_t - \kappa \partial_y^2) H \phi, e^{\Psi} \Delta^h_k H \phi, L^2 \right) \]

\[ \geq \frac{1}{2} \|h \parallel e^{\Psi} \Delta^h_k (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ - \frac{1}{2} \|\sqrt{h} \parallel e^{\Psi} \Delta^h_k (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]

\[ + 2\kappa \ell_\kappa \|h \parallel e^{\Psi} \Delta^h_k \partial_y (G, H) \phi, \parallel^2 \|L^2_{[t_0, t]} (L^2) \]
Whereas due to \(\lim_{y \to +\infty} u = \lim_{y \to +\infty} b = 0\), we get, by applying Lemma 3.4 with \(a = \frac{1}{2\kappa}\) and \(b = c = d = \frac{1}{2}\), that

\[
\int_{t_0}^{t} h(t') \left( \left| e^{\Psi_{\kappa}} \Delta_k^h [u \partial_x G - b \partial_x H] \Phi_{\kappa} \right| e^{\Psi_{\kappa}} \Delta_k^h G \Phi_{\kappa} \right)_{L^2_+} dt' 
+ \left| e^{\Psi_{\kappa}} \Delta_k^h [u \partial_x G - b \partial_x H] \Phi_{\kappa} \right| e^{\Psi_{\kappa}} \Delta_k^h H \Phi_{\kappa} \right)_{L^2_+} dt'
\leq d_2^2 2^{-k} \left\| h_{\Psi} e^{\Psi_{\kappa}} (G, H) \Phi_{\kappa} \right\|_{L^2_{[t_0, t]}; \delta_k}^2 \left( B^{1,0} \right).
\]

Similarly, by applying Lemma 3.6 with \(a = \frac{2}{3\kappa}\), \(b = \frac{3}{4\kappa}\) and \(c = d = \frac{1}{\kappa}\), we achieve

\[
\int_{t_0}^{t} h(t') \left( \left( 1 - \kappa \right) \left| e^{\Psi_{\kappa}} \Delta_k^h [b \partial_x b] \Phi_{\kappa} \right| e^{\Psi_{\kappa}} \Delta_k^h G \Phi_{\kappa} \right)_{L^2_+} dt' 
+ \left( 1 - \kappa \right) \left| e^{\Psi_{\kappa}} \Delta_k^h [b \partial_x u] \Phi_{\kappa} \right| e^{\Psi_{\kappa}} \Delta_k^h H \Phi_{\kappa} \right)_{L^2_+} dt'
\leq d_2^2 2^{-k} \left\| h_{\Psi} e^{\Psi_{\kappa}} (u, b) \Phi_{\kappa} \right\|_{L^2_{[t_0, t]}; \delta_k}^2 \left( B^{1,0} \right) \left\| h_{\Psi} e^{\Psi_{\kappa}} (G, H) \Phi_{\kappa} \right\|_{L^2_{[t_0, t]}; \delta_k}^2 \left( B^{1,0} \right).
\]

On the other hand, we deduce from (2.12b) that

\[
\| e^{\Psi_{\kappa}} y \Delta_k^h (G, H) \Phi_{\kappa} \|_{L^2_+} \leq \kappa(t) \| e^{\Psi_{\kappa}} \partial_y \Delta_k^h (G, H) \Phi_{\kappa} \|_{L^2_+},
\]

which implies

\[
\| e^{\Psi_{\kappa}} y (G, H) \Phi_{\kappa} \|_{L^p_t (B^{s,0})} \leq \kappa(t) \| e^{\Psi_{\kappa}} \partial_y (G, H) \Phi_{\kappa} \|_{L^p_t (B^{s,0})}.
\]
So that by applying (3.26) and (1.14a), we find
\[
\int_{t_0}^t h(t') \left( \left| \left( e^{\Psi} \Delta_k^h \left[ -\partial_x U \partial_y b + \partial_x B \partial_y b \right] \phi \right| e^{\Psi} \chi \Delta_k^h G \phi \right|_{L^2_k} \right)
+ \left( e^{\Psi} \Delta_k^h \left[ -\partial_x U \partial_y b + \partial_x B \partial_y u \right] \phi \right| e^{\Psi} \chi \Delta_k^h H \phi \right) dt
\leq d_k^2 2^{-k} \left( \langle \tau \rangle^{\frac{3}{2}} \left\| (U, B) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0} \cap \mathbb{R}^3)} \left\| \langle \tau \rangle^{\frac{1}{2}} \left( e^{\Psi} \partial_y (u, b) \phi \right) \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})} \right.
\times \left\| \langle \tau \rangle^{-1} h_k \left( e^{\Psi} \partial_y (G, H) \phi \right) \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}
\leq \kappa d_k^2 2^{-k} \left( \langle \tau \rangle^{\frac{1}{2}} \left( e^{\Psi} \partial_y (u, b) \phi \right) \right)^2_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})} + \left( e^{\Psi} \partial_y (G, H) \phi \right)^2_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}.
\]

With the above estimates, we can repeat the proof of (6.2) to complete the proof of (6.7). This completes the proof of Lemma 6.2.

An immediate application of Lemmas 6.1 and 6.2 leads to

Corollary 6.1. Under the assumption of Lemma 6.1, there exist positive constants \( \varepsilon_0, \lambda_0 \) so that for any \( \lambda \geq \lambda_0 \) and \( \varepsilon \leq \varepsilon_0 \), one has

\[
\left\| h \frac{1}{2} e^\Psi (G, H) \phi \right\|_{L^\infty_{[t_0, t]}(B^{\frac{1}{2}, 0})} + \sqrt{\kappa} \left\| h \frac{1}{2} e^\Psi \partial_y (G, H) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}
\leq \left\| h \frac{1}{2} e^\Psi (G, H) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})} + \left\| \sqrt{e} e^\Psi (G, H) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}
+ C \left( \varepsilon \langle \tau \rangle^\frac{1}{2} \left\| h \frac{1}{2} e^\Psi \partial_y (u, b) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})} + \varepsilon \langle \tau \rangle^\frac{1}{2} \left\| h \frac{1}{2} e^\Psi (m_U, m_B, M_B, M_U) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})} \right)
\]

Proof. By taking square root of (6.2) and then multiplying the resulting inequality by \( 2^\frac{1}{2} \) and finally summing over \( k \in \mathbb{Z} \), we find

\[
\left\| h \frac{1}{2} e^\Psi (G, H) \phi \right\|_{L^\infty_{[t_0, t]}(B^{\frac{1}{2}, 0})} + 2 \sqrt{\kappa} \left\| h \frac{1}{2} e^\Psi \partial_y (G, H) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}
+ 2 \eta \left\| \sqrt{e} e^\Psi (G, H) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})}
+ C \left( \varepsilon \langle \tau \rangle^\frac{1}{2} \left\| h \frac{1}{2} e^\Psi \partial_y (u, b) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})} + \varepsilon \langle \tau \rangle^\frac{1}{2} \left\| h \frac{1}{2} e^\Psi (m_U, m_B, M_B, M_U) \phi \right\|_{L^2_{[t_0, t]}(B^{\frac{1}{2}, 0})} \right)
\]

Then by taking \( \lambda \) big enough and \( \varepsilon \leq \varepsilon_0 \) small enough, we arrive at (6.9).
Corollary 6.2. Under the assumption of Lemma 5.2, there exist positive constants \( \varepsilon_0, \lambda_0 \) so that for any \( \lambda \geq \lambda_0 \) and \( \varepsilon \leq \varepsilon_0 \), one has
\[
\| h^{1/2} e^{\Psi} (G, H) \|_{L^\infty_t L^2_x (B^{\frac{1}{2}}_x, a)} + \sqrt{\kappa} \| h^{1/2} e^{\Psi} \partial_y (G, H) \|_{L^2_t (B^{\frac{1}{2}}_x, a)}
\leq\| h^{1/2} e^{\Psi} (G, H) \|_{L^2_t a (B^{\frac{1}{2}}_x, a)} + \sqrt{\kappa} e^{\Psi} (G, H) \|_{L^2_t a (B^{\frac{1}{2}}_x, a)}
+ C \left( \varepsilon^{1/2} \| \tau \|^{\frac{1}{2}} + \varepsilon^{1/2} \| \tau \|^{\frac{1}{2}} \right) + \varepsilon^{-1} \| \tau \|^{\frac{1}{2}} + \varepsilon^{-1} \| \tau \|^{\frac{1}{2}}
\]
(6.10)

Proof. The proof of this corollary is the same as that of Corollary 6.1; we omit the details here.

Now we are in a position to prove Proposition 2.3 and 2.6.

Proof of Proposition 2.3. We first observe from (6.11) that for \( h(t) \leq C(t) \),
\[
\| \langle \tau \rangle^{-1} \frac{1}{2} h^{1/2} e^{\Psi} \partial_y (u, b) \|_{L^2_t (B^{\frac{1}{2}}_x, a)} + \sqrt{\kappa} \| \langle \tau \rangle^{-1} \frac{1}{2} h^{1/2} e^{\Psi} (u, b) \|_{L^2_t (B^{\frac{1}{2}}_x, a)}
\leq\| \langle \tau \rangle^{1+\lambda_0} \varepsilon e^{\Psi} \partial_y (G, H) \|_{L^2_t (L^2_x)} + \| \langle \tau \rangle^{1+\lambda_0} \varepsilon e^{\Psi} (G, H) \|_{L^2_t (L^2_x)}
\]
(6.11)

Whereas it follows from Lemma 2.3 that
\[
\| e^{\Psi} \partial_y \Delta_k^h (G, H) \|_{L^2_x} \geq \frac{1}{2 \langle t \rangle} \| e^{\Psi} \Delta_k^h (G, H) \|_{L^2_x}.
\]
Then by taking \( h(t) = \langle t \rangle^{2+2\lambda_0-2\varepsilon} \) and \( t_0 = 0 \) in Lemma 6.1, we find
\[
\| \langle \tau \rangle^{1+\lambda_0} \varepsilon e^{\Psi} \partial_y \Delta_k^h (G, H) \|_{L^2_t (L^2_x)} + \| \langle \tau \rangle^{1+\lambda_0} \varepsilon e^{\Psi} (G, H) \|_{L^2_t (L^2_x)}
\leq\| \langle \tau \rangle^{1+\lambda_0} \varepsilon e^{\Psi} \partial_y (G, H) \|_{L^2_t (L^2_x)} + \| \langle \tau \rangle^{1+\lambda_0} \varepsilon e^{\Psi} (G, H) \|_{L^2_t (L^2_x)}
\]
(6.12)

Then thanks to (1.5), (3.23) and (6.11), we get, by taking \( \lambda \) large enough, that
\[
\| \langle \tau \rangle^{1+\lambda_0} \varepsilon e^{\Psi} (G, H) \|_{L^\infty _t (B^{\frac{1}{2}}_x, a)} \lesssim \varepsilon \left( \| e^{\Psi} \partial_y (u, b) \|_{B^{\frac{1}{2}}_x} + 1 \right).
\]
Along the same line, thanks to (6.23) and (6.11), we get, by taking \( h(t) = 1 \) and \( t_0 = 0 \) in Corollary 6.1, that
\[
\left\| e^{\Psi} \partial_y (G, H) \phi \right\|_{L^2_t \left( B^{1/2}_{2,0} \right)} \lesssim \sqrt{\varepsilon} \left( \left\| e^{\frac{\Psi}{8}} e^{\delta |D_z|} (u_0, b_0) \right\|_{B^{1/2}_{2,0}} + 1 \right).
\]
Similarly, by taking \( h(t) = (t)^{2+2L_- - 2\varepsilon} \) and \( t_0 = \frac{1}{t} \) in Corollary 6.1 we achieve
\[
\left\| \langle \tau \rangle^{1+L_- - \varepsilon} e^{\Psi} \partial_y (G, H) \phi \right\|_{L^2_{t} \left( B^{1/2}_{2,0} \right)} \lesssim \varepsilon^{1/2} \left\| e^{\frac{\Psi}{8}} e^{\delta |D_z|} (u_0, b_0) \right\|_{B^{1/2}_{2,0}} + \sqrt{\varepsilon} \\
+ \left\| \langle t/2 \rangle^{1+L_- - \varepsilon} e^{\Psi} (G, H) \phi \langle t/2 \rangle \right\|_{B^{1/2}_{2,0}} + \left\| \langle \tau \rangle^{1+L_- - \varepsilon} e^{\Psi} (G, H) \phi \right\|_{L^2_{t} \left( B^{1/2}_{2,0} \right)}.
\]
Notice that
\[
\left\| \langle \tau \rangle^{1+L_- - \varepsilon} e^{\Psi} (G, H) \phi \right\|_{L^2_{t} \left( B^{1/2}_{2,0} \right)} \lesssim \left\| \langle \tau \rangle^{1+L_- - \varepsilon} e^{\Psi} (G, H) \phi \right\|_{L^2_{t} \left( B^{1/2}_{2,0} \right)}.
\]
As a consequence, we deduce from (6.12) that
\[
\left\| \langle \tau \rangle^{1+L_- - \varepsilon} e^{\Psi} \partial_y (G, H) \phi \right\|_{L^2_{t} \left( B^{1/2}_{2,0} \right)} \lesssim \sqrt{\varepsilon} \left( \left\| e^{\frac{\Psi}{8}} e^{\delta |D_z|} (u_0, b_0) \right\|_{B^{1/2}_{2,0}} + 1 \right).
\]
With (6.12) and (6.14), to finish the proof of Proposition 2.3 it remains to show that for any \( t < T^* \),
\[
\int_0^t \left\| \langle \tau \rangle^{1+L_- - \varepsilon} e^{\Psi} \partial_y (G, H) \phi \right\|_{B^{1/2}_{2,0}} d\tau \lesssim \sqrt{\varepsilon} \left( \left\| e^{\frac{\Psi}{8}} e^{\delta |D_z|} (u_0, b_0) \right\|_{B^{1/2}_{2,0}} + 1 \right).
\]
Indeed taking \( \varepsilon_0 \) to be so small that \( \varepsilon_0 < L_\varepsilon \). Then for \( \varepsilon \leq \varepsilon_0 \), we deduce from (6.14) that for \( 1 \leq t < T^* \)
\[
\int_0^t \left\| \langle \tau \rangle^{1/2} e^{\Psi} \partial_y (G, H) \phi \right\|_{B^{1/2}_{2,0}} d\tau \lesssim \sqrt{\varepsilon} \left( \left\| e^{\frac{\Psi}{8}} e^{\delta |D_z|} (u_0, b_0) \right\|_{B^{1/2}_{2,0}} + 1 \right).
\]
By dividing the time interval \([0, t]\) into \([0, 1], [1, 2], [2, 4], \cdots, [2^N, t]\), and using (6.13) and (6.16), we arrive at
\[
\int_0^t \left\| \langle \tau \rangle^{1/2} e^{\Psi} \partial_y (G, H) \phi \right\|_{B^{1/2}_{2,0}} d\tau \lesssim \left( \sum_{j=1}^{N} \int_0^{2^{2j-1}} \int_0^{2^j} \left\| \langle \tau \rangle^{1/2} e^{\Psi} \partial_y (G, H) \phi \right\|_{B^{1/2}_{2,0}} d\tau \right) \lesssim \left( 1 + \sum_{j=1}^{N} \left( 2^{-j} + t^{-1/2} \right) \sqrt{\varepsilon} \right) \left( \left\| e^{\frac{\Psi}{8}} e^{\delta |D_z|} (u_0, b_0) \right\|_{B^{1/2}_{2,0}} + 1 \right) \lesssim \sqrt{\varepsilon} \left( \left\| e^{\frac{\Psi}{8}} e^{\delta |D_z|} (u_0, b_0) \right\|_{B^{1/2}_{2,0}} + 1 \right).
\]
This leads to (6.15). We thus complete the proof of Proposition 2.3. \( \square \)

**Proof of Proposition 2.4.** The proof of Proposition 2.4 is the same as that of Proposition 2.3. We omit the details here. \( \square \)
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