On F-theory Quiver Models
and
Kac-Moody Algebras

Rachid Ahl Laamara\textsuperscript{1}\ast, Adil Belhaj\textsuperscript{1,2}\dagger, Luis J. Boya\textsuperscript{3}\ddagger, Leila Medari\textsuperscript{4}\§, Antonio Segui\textsuperscript{3}\¶

\textsuperscript{1}UFR- Lab de Physique des Hautes Energies and Groupement National de PHE GNPHE, Siège focal: Faculté des Sciences, Rabat, Morocco
\textsuperscript{2}Centre National de l’Energie, des Sciences et des Techniques Nucléaires, Rabat, Morocco
\textsuperscript{3}Departamento de Física Teórica, Universidad de Zaragoza, E-50009-Zaragoza, Spain
\textsuperscript{4}LPHEA, Physics Department, Faculty of Science Semlalia, Marrakesh, Morocco

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Abstract

We discuss quiver gauge models with bi-fundamental and fundamental matter obtained from F-theory compactified on ALE spaces over a four dimensional base space. We focus on the base geometry which consists of intersecting \( F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1 \) Hirzebruch complex surfaces arranged as Dynkin graphs classified by three kinds of Kac-Moody (KM) algebras: ordinary, i.e finite dimensional, affine and indefinite, in particular hyperbolic. We interpret the equations defining these three classes of generalized Lie algebras as the anomaly cancelation condition of the corresponding \( N = 1 \) F-theory quivers in four dimensions. We analyze in some detail hyperbolic geometries obtained from the affine \( \hat{A} \) base geometry by adding a node, and we find that it can be used to incorporate fundamental fields to a product of \( SU \)-type gauge groups and fields.

KEYWORDS: F-theory, quiver gauge theories, Kac-Moody algebras.

\ast\texttt{doctorants.lphe@fsr.ac.ma}
\dagger\texttt{belhaj@unizar.es}
\ddagger\texttt{luisjo@unizar.es}
\§\texttt{l.medari@ucam.ac.ma}
\¶\texttt{segui@unizar.es}
1 Introduction

Four dimensional gauge theories with only four supercharges ($4D, N = 1$), obtained from local F-theory models have been originated by Vafa, and are very interesting in connections with Grand Unification Theories (GUT), the Standard Model (SM) and the Large Hadron Collider Physics (LHC) [1]-[10]. In the study of F-theory GUT models, it involves considerations of geometric singularities which are related to seven-branes extant in type IIB superstring theory, and are localized in two dimensional transverse space. It has been shown that the gauge fields similar to those Minimal Supersymmetric Standard Model (MSSM) can be gotten from the eight-dimensional worldvolume of the seven-brane wrapping del Pezzo surfaces with a GUT gauge group. The latter can be broken to $SU(3)_C \times SU(2)_L \times U(1)_Y$ via an internal flux through the seven-brane in the $U(1)_Y$ direction of the GUT group. Further details on building of F-theory models can be found in [11]-[24].

Motivated by these large activities, we have recently manipulated some quivers with only bi-fundamental matter from F-theory compactified on Asymptotically Locally Euclidean (ALE) spaces over a four-dimensional base space [25]. The geometry of the base has been obtained by blowing up the extended $ADE$-type singularities of eight dimensional hyper-Kähler manifolds, considered as sigma model target spaces with eight supercharges [26]. We have shown that the base can be identified with intersecting $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ according to the affine $ADE$ Dynkin diagrams required by the anomaly cancelation condition. The corresponding seven-brane configurations lead to a $\prod_i SU(s_i N)$ gauge group with only bi-fundamental chiral matter transforming in $(s_i N, s_j N)$ representations, where $N$ is an unspecified positive integer. Besides, the positive integers $s_i$ appearing in the gauge group are the Coxeter labels (sometimes referred to as marks or Dynkin numbers). They form a positive-definite integer vector $(s_i)$ satisfying $\sum_i K_{ij} s_i = 0$ where $K$ is the Cartan matrix of the affine Lie algebras.

The aim of this paper is to discuss some seven-brane configurations leading to F-theory quivers involving both bi-fundamental and fundamental matter. In particular, the main result is the building of a base geometry in F-theory compactification producing such quivers. More precisely, we consider different geometries which consist of intersecting $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ according to Dynkin diagrams of the three types of Kac-Moody (KM) algebras, ordinary finite dimensional, affine (with Cartan matrix determinant $= 0$) and indefinite, in particular ($det < 0$) [27, 28, 29]. The algebraic equations defining these Lie algebras can be interpreted as the anomaly cancellation condition of $N = 1$ F-theory quivers in four dimensions. For a particular example of hyperbolic geometries obtained from the $\hat{A}$ geometry, we implement
fundamental fields to a linear chain of gauge groups $\prod_i SU$ with bi-fundamental matter. But, first we discuss the geometric engineering procedure to obtain bi-fundamental matter in F-theory quivers. We then extend this analysis to geometric engineering of fundamental matter in F-theory compactification, with particular emphasis on the two-dimensional complex base geometry classified by KM algebras.

## 2 Bi-fundamental Matter from Affine Lie Algebras

In this section, we will discuss how the bi-fundamental matter can be obtained from the intersecting seven-branes in F-theory compactifications [25]. We will start with a general construction of the base of local F-theory geometries. The base is obtained by resolving the so-called extended ADE singularities of eight dimensional hyper-Kähler manifolds considered as a target space of $N = 4$ sigma model [26]. The theory contains $U(1)^r$ gauge symmetry and $r + 2$ hypermultiplets with a matrix charge $Q^a_i, \ a = 1, \ldots , r; \ i = 1, \ldots , r + 2$. Up to some details, this matrix can be identified with the Cartan matrices of the ADE Lie algebras of rank $r$. In this way, the eight dimensional manifolds are solutions of the $N = 4$ D-flatness equations which are generally given by

$$
\sum_{i=1}^{r+2} Q^a_i [\phi^\alpha_i \bar{\phi}^\beta_i + \phi^\beta_i \bar{\phi}^\alpha_i] = \xi^a \bar{\sigma}^a, \quad (1)
$$

where $\phi^\alpha_i$’s ($\alpha = 1, 2$) denote $r + 2$ component field doublets of hypermultiplets, $\xi^a$ are Fayet-Iliopolos (FI) 3-vector couplings rotated by $SU(2)$ symmetry, and $\bar{\sigma}^a$ are the traceless $2 \times 2$ Pauli matrices. Equations (1) deal with the hypermultiplet branch and give a gauge invariant hyper-Kähler target space. For each $U(1)$ factor of the $U(1)^r$ gauge group, they involve a triplet of FI parameters. Using the $SU(2)$ R-symmetry transformations $\phi^\alpha = \epsilon^{\alpha\beta} \phi^\beta$, $\bar{\phi}^\alpha = \bar{\phi}_\alpha$, $\epsilon_{12} = \epsilon_{21} = 1$ and replacing the Pauli matrices by their expressions, the identities (1) can be split as follows

$$
\sum_{i=1}^{r+2} Q^a_i (|\phi^1_i|^2 - |\phi^2_i|^2) = \xi^3_a
$$

$$
\sum_{i=1}^{r+2} Q^a_i \bar{\phi}^1_i \phi^2_i = \xi^1_a + i \xi^2_a \quad (2)
$$

$$
\sum_{i=1}^{r+2} Q^a_i \phi^1_i \bar{\phi}^2_i = \xi^1_a - i \xi^2_a.
$$
As in the case of the $N = 2$ realization of the K3 surface, the solution of these constraints involves intersecting 4-cycles according to the Dynkin graphs of the $ADE$ Lie algebras, allowing us to produce a product of $SU$-type gauge groups with bi-fundamental matter in F-theory compactification with the ALE space fibration. These models can be encoded in quiver diagrams similar to the Dynkin graphs. In this graph, each node represents a $SU$ gauge group factor and the link is associated with matter. Usually, the matter is in the bi-fundamental representation of the gauge group. In this paper, we extend this techniques to allow fundamental fields. Roughly, we have shown that the base solution of $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ according to the $ADE$ Dynking geometries [25]. This is a particular example of del Pezzo complex surfaces which is needed for the decoupling limit of the supergravity. This is in agreement with the flux condition studied in [9]. From the obvious similarity with the $N = 2$ scenario given in [30] it is not surprising to see some analogs between the $ADE$ diagrams in eight-dimensional hyper-Kähler manifolds and the ones considered in the Kähler case. In fact, the intersection theory assigns the intersection number to complex surfaces inside of these eight dimensional manifolds. The self-intersection of the zero section in the cotangent bundle of $F_0$ is equal to its minus Euler characteristic, i.e. $-4$. Consider now a lattice of compact 4-cycles generated by $F_0^i$ and assume that $F_0^i$ intersects $F_0^{i+1}$ at two points. This can be supported by the fact that each $\mathbb{CP}^1$ inside $F_0^i$ intersects just one $\mathbb{CP}^1$ in the next $F_0^{i+1}$. In this way, the intersection numbers of the $F_0$’s reads

$$
[F_0^i] \cdot [F_0^j] = -4 \\
[F_0^i] \cdot [F_0^{i+1}] = 2,
$$

with others vanishing. This means that $F_0^i$ does not intersect $F_0^j$ if $|j - i| > 1$. Endowed with this intersection form, the lattice of these compact 4-cycles can be identified with the root lattice of the $ADE$ Lie algebras, up to a multiplication factor. More precisely, the relation takes the form

$$
[F_0^i] \cdot [F_0^j] = -2K^{ij}(ADE).
$$

To describe the physics on seven-branes wrapped on such intersecting geometries, we will use the result of the geometric engineering method used in type II superstrings [30]. We take different stacks of seven-branes. If we wrap each stack of seven-branes on one $F_0$ complex surface, the total gauge group takes the following form

$$
G = \prod_i SU(N_i),
$$

3
where $i$ runs through the set of vertices of the Dynkin diagram of $ADE$ type. The matter fields are bi-fundamentals transforming in the $(N_i, N_j)$ representations. However, it has been shown in [25] that the anomaly cancellation condition

$$\sum_i [F^i_0] \cdot [F^j_0] N_i = 0$$

is translated into a condition on the $\tilde{ADE}$ affine Lie algebra on the base of local F-theory compactification. In this way, $N_i$ should be given in terms of Coxeter labels $s_i$ of the associated affine Lie algebra, namely

$$N_i = s_i N.$$  \hspace{1cm} (6)

where $N$ is an unspecified positive integer.

At the end of this section, we would like to list some results for $\tilde{AD}$ Dynkin geometries. The result for $\tilde{E}$ can be obtained easily using the same method. For $\tilde{A}_r$ for instance, the base of local F-theory compactification consists of $r + 1$ intersecting $F_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ arranged as shown here

\[\tilde{A}_r: \]

\[\begin{array}{c}
\circ \\
\circ \circ \\
\circ \\
\end{array}\]

In this graph each node represents a $F_0$ complex surface and the link is associated with the intersection. In this case, we have $s_i = 1$ and this geometry describes a $SU(N)^{r+1}$ quiver gauge theory with bi-fundamental field representations.

In the case where the base is a $\tilde{D}_r$ Dynkin geometry

\[\tilde{D}_r: \]

\[\begin{array}{c}
\circ \\
\circ \circ \\
\circ \\
\end{array}\]

the $s_i$ vector appearing in the gauge group is given by $s_i = (1, 1, 2, \ldots, 2, 1, 1)$. The number 1 corresponds to the monovalent nodes and 2 is associated with the remaining ones. The gauge
group takes the form \( SU(N)^4 \times SU(2N)^r-3 \).
For the exceptional base geometries, the vector \( s_i \) is as follows
\[
\begin{align*}
\hat{E}_6 & : s_i = (1,1,2,2,3,2,1) \\
\hat{E}_7 & : s_i = (1,2,2,3,4,3,2,1) \\
\hat{E}_8 & : s_i = (1,2,3,4,5,6,4,3,2).
\end{align*}
\] (8)

Having discussed F-theory quivers with only bi-fundamental matter, in what follows we would like to incorporate fundamental fields. In particular, we consider a special class of hyperbolic Dynkin base geometries in the compactification of F-theory. Other possibilities will be addressed in future work.

3 Fundamental Matter Fields in F-theory Quivers

We will focus in this section on a particular base geometry producing F-theory quivers with fundamental matter. More precisely, we will discuss how the fundamental fields can be obtained from intersecting seven-branes wrapped on geometries classified by three kinds of the Kac-Moody algebras. There are in general three kinds of the \( K_{ij} \)'s Cartan matrices \([27, 28, 29]\). These involve three classes defining the finite, affine and the indefinite sectors of the Kac-Moody algebras. They are classified by the following Cartan matrices satisfying
\[
\begin{align*}
\sum_j K_{ij}^{(+)} N_j & = M_i \\
\sum_j K_{ij}^{(0)} N_j & = 0 \\
\sum_j K_{ij}^{(-)} N_j & = -M_i,
\end{align*}
\] (9)

where \( N_j \) and \( M_i \) are positive integers specified later on. In these equations, \( K_{ij}^{(+)} \), \( K_{ij}^{(0)} \) and \( K_{ij}^{(-)} \) refer respectively to the Cartan matrices of finite, affine and indefinite classes. They can be put into one equation with the following form
\[
\sum_j K_{ij}^{(q)} N_j = q M_i,
\] (10)

with \( q = 0, \pm 1 \). In this way, \( K_{ij}^{(q)} \) denotes the generalized Cartan matrix. These equations will play a nice role in the study of \( N = 1 \) F-theory quivers in four dimensions based on Dynkin
geometries. For instance, the equation
\begin{equation}
\sum_j K^{(0)}_{ij} N_j = 0 \tag{11}
\end{equation}
has been interpreted as the anomaly cancellation condition in the affine $\widehat{ADE}$ supersymmetric quivers without fundamental matter \cite{25}. We expect that $\sum_j K^{(\pm)}_{ij} N_j = \pm M_i$ should play a quite similar role to the relation $\sum_j K^{(0)}_{ij} N_j = 0$. In fact, we will interpret the quantities $\pm M_i$ as fundamental matter incorporated in F-theory quivers encoded in Dynkin geometries. This may be seen by modifying the anomaly cancellation condition (11) as
\begin{equation}
\sum_j K^{(q)}_{ij} N_j - qM_i = 0. \tag{12}
\end{equation}
These equations describe now the anomaly cancellation condition for the three classes of F-theory quivers where the base geometries are classified by the equation (9), while $M_i$ denote the contributions of fundamental matter multiplets and where $N_j$ denote the colors. These numbers cannot be arbitrary and can be fixed by (12). It follows that one may distinguish between two models of F-theory quivers in four dimensions according to whether or not the model contains fundamental fields. The first model involves a quiver theory with a gauge group as $\prod_i SU(s_i N)$ without fundamental fields. The absence of fundamental matter can be related the zero appearing in the second equation of (9). This model, which has been studied in \cite{25}, has no flavor symmetry. The second model, which we are interested in here, has an extra flavor symmetry. In this case, we can have fundamental matter transforming under the flavor group.

However, the general study is a highly non-trivial task as it requires solving the above complicated matter equations (13). The general solution is beyond the scope of the present work, though we will consider in some details the case of a special base geometry obtained from the affine $\widehat{A}$ quiver. A simple situation is to add one extra node to the Dynkin graph of the affine $\widehat{A}$ Lie algebra. In geometric Dynkin language, this corresponds to a particular hyperbolic Dynkin diagram. The derivation of such a hyperbolic geometry is based on the same philosophy one uses in the building of the affine Dynkin diagrams from the finite ones by adding a node. In other words, by cutting this node in such a hyperbolic Dynkin diagram, the resulting sub-diagram coincides with the Dynkin graph of the affine $\widehat{A}$ Lie algebras. The new Dynkin graph looks like
and contains now \( r + 2 \) nodes. It can be viewed as an extension of the bivalent geometry describing a linear chain of gauge groups \( \prod_j SU \) with only bi-fundamental matter we have considered earlier. Recall that, in the sigma model language the bivalent geometry means that each vector charge takes the form

\[
Q_i = (0, \ldots, 0, 1, -2, 1, 0, \ldots, 0).
\]  

The extra node leads to a new ingredient in F-theory quivers by introducing the so-called trivalent geometry. In sigma model language, this geometry contains a vector charge of the form

\[
Q_i = (-2, 1, 1, 0, \ldots, 0).
\]  

Note that one can add an extra auxiliary field with contribution \((-1)\) in order to fulfill the Calabi-Yau condition \( \sum_i Q_i = 0 \) involved in local ADE geometries.

Roughly, the corresponding base geometry in F-theory compactification contains now a central \( F_0^3 \) (which can be identified with the affine node) with self intersection \((-4)\) intersecting three other \( F_0 \)'s with contribution \((+2)\). One of them is associated with the hyperbolic node denoted by \( F_0^{-1} \). Therefore, the new \( F_0^{-1} \) satisfies the following intersection numbers

\[
\begin{align*}
[F_0^{-1}] \cdot [F_0^{-1}] & = -4, \\
[F_0^{-1}] \cdot [F_0^0] & = 2, \\
[F_0^{-1}] \cdot [F_0^i] & = 0 \quad i \neq 0.
\end{align*}
\]  

In what follows, we will see that this trivalent geometry can been used to incorporate fundamental matter in a product of \( SU \)-type gauge groups with bi-fundamental fields. This can be seen from the fact that any Cartan matrix \( K_{ij}^{(-1)} \) of hyperbolic class can be split as

\[
K_{ij}^{(-)} = K_{ij}^{(0)}(\widehat{A}) + \Delta_{ij}^{(-)},
\]  

where \( K_{ij}^{(0)}(\widehat{A}) \) denotes the Cartan matrix of the affine \( \widehat{A} \) Lie algebras. \( \Delta_{ij} \) can be viewed as matter contributions which can be interpreted as fundamental fields.
Now let us give a concrete example of the overextended version of $\hat{A}_2 (\hat{H}A_2)$. Using the trivalent geometry on the affine node, the generalized Dynkin diagram is

$$\hat{H}A_2 :$$

and its Cartan matrix reads as

$$K^{-1}_{ij}(\hat{H}A_2) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}, \quad det = -3. \quad (19)$$

For this example, the (12) reduces to

$$2N_1 - N_2 = -M_1$$
$$-N_1 + 2N_2 - N_3 - N_4 = -M_2$$
$$-N_2 + 2N_3 - N_4 = -M_3$$
$$-N_2 - N_3 + 2N_4 = -M_4. \quad (20)$$

The general solution of the previous equation is

$$N_1 = M_2 + M_3 + M_4$$
$$N_2 = M_1 + 2M_2 + 2M_3 + 2M_4$$
$$N_3 = M_1 + 2M_2 + \frac{4}{3}M_3 + \frac{5}{3}M_4$$
$$N_4 = M_1 + 2M_2 + \frac{5}{3}M_3 + \frac{4}{3}M_4. \quad (21)$$

If we put $M_3 = M_4 = 0$, the solution becomes

$$N_1 = M_2,$$
$$N_2 = N_3 = N_4 = M_1 + 2M_2. \quad (22)$$

A simple seven-brane configuration can be given by the following representation

$$N_1 = 0, \quad N_2 = N_3 = N_4 = N$$
$$M_1 = N, \quad M_2 = M_3 = M_4 = 0.$$
This brane representation engineers a $SU(N)$ flavor symmetry on the hyperbolic node. In the geometric language, this solution could be also obtained from an extra bi-fundamental matter associated with the hyperbolic node by assuming that the volume of $F^{-1}_0$ is very large. The Yang-Mills gauge coupling $g_{-1}$ associated with this node tends to zero. The corresponding $SU(N)$ dynamics becomes very weak, so it will be considered as a spectator flavor symmetry group. For $N = 3$, we get a $SU_C(3) \times SU(3)_L \times SU(3)$ with a $SU(3)$ flavor symmetry. In this way, $SU_L(3)$ contains $SU_L(2)$; while the electric charge can have contributions from the Cartan sub-algebras of $SU(3)$ and $SU(3)_L$. Recall that $SU_C(3) \times SU(3)_L \times SU(3)$ is a maximal subgroup of the exceptional $E_6$ gauge symmetry appearing as a possible gauge group in GUT. In fact, the adjoint 78 representation of $E_6$ can be broken as

$$78 \rightarrow (8, 1, 1) + (1, 8, 1) + (1, 1, 8) + (3, \bar{3}, 3) + (3, 3, 3).$$

(23)

4 Discussions

In this work, we have discussed F-theory quivers with bi-fundamental and fundamental matter. Our focus has been on the building of the base geometry of the F-theory compactification with ALE space fibrations. The base consists of intersecting $F_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ arranged as Dynkin graphs classified by three kinds of Kac-Moody algebras: ordinary finite dimensional, affine and indefinite ones. We have interpreted the equations defining these three classes of Lie algebras as the anomaly cancelation condition of the corresponding F-theory quivers. We have analyzed in detail hyperbolic geometries obtained from the affine $\hat{A}$ geometry and we have found that it can be used to incorporate fundamental fields to a product of gauge groups $\prod_j SU$ with bi-fundamental matter. We would like to note that this analysis can be extended by replacing each bivalent geometry of $\hat{A}$ Dynkin graph by a trivalent one. This could give a chain of $SU(N)$ flavor symmetries. We also intend to discuss elsewhere the extension of this explicit study to the other $\hat{D}\hat{E}$ geometries. Our approach can be adapted to a broad variety of geometries whose Dynkin extensions may then be obtained by adding a node to any affine Dynkin diagram. The extension from one node to more is straightforward. We anticipate that this will introduce non trivial fundamental matter in F-theory quivers based on indefinite Dynkin geometries. This will be addressed elsewhere, where we also intend to discuss the implementation of polyvalent geometry where the bivalent and trivalent ones appear just as the leading terms of a more general case.

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