SUBDIFFERENTIAL OF THE JOINT NUMERICAL RADIUS

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Abstract. An expression for the subdifferential of the joint numerical radius is obtained. Its applications to the best approximation problems in the joint numerical radius are discussed.

1. Introduction

Let $M_{p,q}(\mathbb{C})$ be the set of $p \times q$ matrices over $\mathbb{C}$ with a given norm. Let $f : M_{p,q}(\mathbb{C}) \to \mathbb{R}$ be a continuous convex function. Let $A \in M_{p,q}(\mathbb{C})$. The subdifferential of $f$ at $A$, denoted by $\partial f(A)$, is defined as

$$\partial f(A) = \{ C \in M_{p,q}(\mathbb{C}) : f(B) \geq f(A) + \text{Re trace}((B - A)^*C) \text{ for all } B \in M_{p,q}(\mathbb{C}) \}.$$

The right hand derivative of $f$ and the subdifferential of $f$ are related as follows. For $B \in M_{p,q}(\mathbb{C})$,

$$\lim_{t \to 0^+} \frac{f(A + tB) - f(A)}{t} = \max \{ \text{Re trace}(C^*A) : C \in \partial f(A) \}. \quad (1.1)$$

Characterizations of subdifferentials of matrix norms has been of interest to many mathematicians. Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices over $\mathbb{C}$. For $A \in M_n(\mathbb{C})$, let $s_1(A) \geq \cdots \geq s_n(A)$ be the singular values of $A$. Let $\| \cdot \|$ denote a unitarily invariant norm on $M_n(\mathbb{C})$ (that is, for any unitary matrices $U$ and $U'$, we have $\| UAU' \| = \| A \|$). Then there is a unique symmetric gauge function $\Phi$ on $\mathbb{R}^n$ such that $\| A \| = \Phi((s_1(A), \ldots, s_n(A))$ for every $A \in M_n(\mathbb{C})$. In [21 Theorem 3.1, Theorem 3.2], it was shown that for $A \in M_n(\mathbb{C})$,

$$\partial \| A \| = \{ U \text{diag}(d_1, \ldots, d_n)U^* : A = U\Sigma U^* \text{ is a singular value decomposition of } A, \sum s_i(A)d_i = \| A \| = \Phi((s_1, \ldots, s_n)), \Phi^*((d_1, \ldots, d_n)) = 1 \}. \quad (1.2)$$

This was an improvement of Theorem 2 of [17], where an expression of $\partial \| \cdot \|$ was given in $M_n(\mathbb{R})$. In [18, Theorem 1], the above result was proved using a different approach. Let $\| \cdot \|$ be the operator norm (or the spectral norm) on $M_n(\mathbb{C})$, defined as:

$$\| A \| = \max \{ \| Au \| : \| u \| = 1 \}. \quad (1.3)$$

The operator norm is a unitarily invariant norm and we have the following. For $A \in M_n(\mathbb{C})$, 

$$\partial \| A \| = \text{co} \{ uv^* : \| u \| = \| v \| = 1, Av = \| A \| u \}, \quad (1.4)$$

where $\text{co}(S)$ denotes the convex hull of a set $S$. For $1 \leq k \leq n$, the Ky Fan $k$-norm $\| \cdot \|_{(k)}$ is defined as

$$\| A \|_{(k)} = s_1(A) + \cdots + s_k(A). \quad (1.5)$$

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The subdifferential set of the Ky Fan $k$-norms on $\mathbb{M}_n(\mathbb{C})$ was obtained in Theorem 2.7 of [5]. Another useful norm on $\mathbb{M}_n(\mathbb{C})$ is the *numerical radius*, defined as

$$w(A) = \max_{||x||=1} |\langle x, Ax \rangle|.$$  

More generally, we consider the *joint numerical radius* of a tuple of matrices defined as follows. Let $A_1, \ldots, A_d \in \mathbb{M}_n(\mathbb{C})$. Let $A = (A_1, \ldots, A_d) : \mathbb{C}^n \to (\mathbb{C}^n)^d$ be defined as $Ax = (A_1x, \ldots, A_dx)$ for all $x \in \mathbb{C}^n$. The joint numerical radius of $A$ is defined as

$$\omega(A) = \max_{x \in \mathbb{C}^n, ||x||=1} \left( \sum_{k=1}^d |\langle x|A_kx \rangle|^2 \right)^{1/2}.$$  

For $x \in \mathbb{C}^n$, let $x \otimes x$ be the rank one operator on $\mathbb{C}^n$ defined as $x \otimes x(y) = \langle y|x \rangle x$ for all $y \in \mathbb{C}^n$. We will use the same symbol $x \otimes x$ for the rank one operator as well as its matrix representation. Let $0 = (0, \ldots, 0) \in \mathbb{M}_n(\mathbb{C})^d$. The main result of this paper is as follows.

**Theorem 1.1.** Let $A \in \mathbb{M}_n(\mathbb{C})^d \setminus \{0\}$. Then

(a) the subdifferential of $\omega(\cdot)$ at $A$ is given by

$$\partial \omega(A) = \text{co} \left\{ \frac{1}{\omega(A)} \left( \langle x|A_1x \rangle x \otimes x, \ldots, \langle x|A_dx \rangle x \otimes x \right): ||x|| = 1, \right\}$$

(1.2)

$$\omega(A) = \left( \sum_{k=1}^d |\langle x|A_kx \rangle|^2 \right)^{1/2},$$

and

(b) for $B \in \mathbb{M}_n(\mathbb{C})^d$,

$$\lim_{t \to 0^+} \frac{\omega(A + tB) - \omega(A)}{t} = \frac{1}{\omega(A)} \max_{||x||=1, \omega(A) = \left( \sum_{k=1}^d |\langle x|A_kx \rangle|^2 \right)^{1/2}} \text{Re} \left( \sum_{k=1}^d \langle x|A_kx \rangle \langle x|B_kx \rangle \right).$$

For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ and $B = (B_1, \ldots, B_d) \in \mathbb{M}_n(\mathbb{C})^d$, let $\lambda B = (\lambda_1B_1, \ldots, \lambda_dB_d)$. As a consequence of Theorem 1.1, we obtain the following result.

**Corollary 1.1.** Let $A = (A_1, \ldots, A_d)$, $B = (B_1, \ldots, B_d) \in \mathbb{M}_n(\mathbb{C})^d$. Then

(1.3)

$$\omega(A + \lambda B) \geq \omega(A)$$

if and only if there exist $h$ unit vectors $x_1, \ldots, x_h \in \mathbb{C}^n$ with $\omega(A) = \left( \sum_{k=1}^d |\langle x_1|A_kx_1 \rangle|^2 \right)^{1/2}$ for all $1 \leq i \leq h$ and there exist $h$ positive numbers $t_1, \ldots, t_h > 0$ with $t_1 + \cdots + t_h = 1$ such that

$$\sum_{i=1}^h t_i \langle x_i|A_kx_i \rangle \overline{\langle x_i|B_kx_i \rangle} = 0$$

for all $1 \leq k \leq d$.

When $d = 1$, the sufficiency of the above condition was given in [11, Theorem 2.11]. In Section 3, we give proofs of Theorem 1.1 and Corollary 1.1. We also obtain analogous results for the *joint operator norm*. Finally, we end with some remarks in Section 3.
To prove Theorem 1.1 we will need the following propositions from the subdifferential calculus.

**Proposition 2.1.** Let $T_1 : M_{p,q}(\mathbb{C}) \to M_{r,s}(\mathbb{C})$ be a linear map. Let $B \in M_{r,s}(\mathbb{C})$. Let $T_2 : M_{p,q}(\mathbb{C}) \to M_{r,s}(\mathbb{C})$ be the affine map defined as $T_2(A) = T_1(A) + B$. Let $g : M_{r,s}(\mathbb{C}) \to \mathbb{R}$ be a continuous convex function. Then for $A \in M_{p,q}(\mathbb{C})$,

$$\partial (g \circ T_2)(A) = T_1^* \partial g(T_2(A)).$$

**Proposition 2.2.** Let $J$ be a compact set in some metric space. Let $\{f_j\}_{j \in J}$ be a collection of continuous convex functions from $M_{p,q}(\mathbb{C})$ to $\mathbb{R}$ such that for $A \in M_{p,q}(\mathbb{C})$, the maps $j \to f_j(A)$ are upper semi-continuous. Let $f : M_{p,q}(\mathbb{C}) \to \mathbb{R}$ be defined as $f(A) = \sup\{f_j(A) : j \in J\}$. Let $J(A) = \{j \in J : f_j(A) = f(A)\}$. Then

$$\partial f(A) = \text{co} (\cup \{\partial f_j(A) : j \in J(A)\}).$$

The proofs of these can be found in Theorem 4.2.1 and Theorem 4.4.2 of [9]. In this book the author deals with real valued convex functions on Euclidean space $\mathbb{R}^n$. The same proofs can be extended to real valued continuous convex functions on a normed space also (see [22] for more detail). Now we prove Theorem 1.1.

**Proof of Theorem 1.1.**

(a) In [11], it was shown that $\omega(A)$ can also be expressed as

\begin{equation}
\omega(A) = \max_{x \in \mathbb{C}^n, \|x\|=1} \max_{(\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d, \|(\lambda_1, \ldots, \lambda_d)\|=1} \left| \sum_{k=1}^d \lambda_k \langle x | A_k x \rangle \right|.
\end{equation}

Let $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$ and let $x \in \mathbb{C}^n$. Let $C = (C_1, \ldots, C_d) \in M_n(\mathbb{C})^d$. Let $T_{x,\lambda} : M_n(\mathbb{C})^d \to \mathbb{C}$ be the linear map defined as

$$T_{x,\lambda}(C) = \sum_{k=1}^d \lambda_k \langle x | C_k x \rangle.$$  

Let $z \in \mathbb{C}$. Let $g : \mathbb{C} \to \mathbb{R}$ be the map defined as $g(z) = |z|$. Let $f_{x,\lambda} : M_n(\mathbb{C})^d \to \mathbb{R}$ be the map defined as $f_{x,\lambda} = g \circ T_{x,\lambda}$. Let $J$ be the compact set $\{(x, \lambda) \in \mathbb{C}^n \times \mathbb{C}^d : \|x\| = 1, \|\lambda\| = 1\}$. Note that for $C \in M_n(\mathbb{C})^d$, the map $(x, \lambda) \to f_{x,\lambda}(C)$ is continuous. Now (2.1) can be rewritten as

$$\omega(A) = \max \{ f_{x,\lambda}(A) : (x, \lambda) \in J \}.$$  

Let $J(A) = \{(x, \lambda) \in J : f_{x,\lambda}(A) = \omega(A)\}$. By Proposition 2.2

$$\partial \omega(A) = \text{co} (\cup \{ \partial f_{x,\lambda}(A) : (x, \lambda) \in J(A)\}).$$
Let \((x, \lambda) \in J(A)\). Then \(\sum_{k=1}^{d} \lambda_k \langle x | A_k x \rangle \neq 0\). By Proposition 2.1 we get

\[
\partial f_{x, \lambda}(A) = T_{x, \lambda}^* \partial g(T_{x, \lambda}(A))
\]

\[
= T_{x, \lambda}^* \partial g \left( \sum_{k=1}^{d} \lambda_k \langle x | A_k x \rangle \right)
\]

\[
= \left\{ T_{x, \lambda} \left( \sum_{k=1}^{d} \lambda_k \langle x | A_k x \rangle \right) \right\} .
\]

Now \(T_{x, \lambda}^* : \mathbb{C} \to M_n(\mathbb{C})^d\) is the unique map satisfying

\[
(2.2) \quad \text{trace}[(T_{x, \lambda}^*(z))^*C] = \sum_z T_{x, \lambda}(C).
\]

If \(T_{x, \lambda}^*(z) = (T_1, \ldots, T_d)\), then (2.2) gives

\[
\sum_{k=1}^{d} \text{trace}(T_k^* C_k) = \sum_{k=1}^{d} \sum \lambda_k \langle x | C_k x \rangle.
\]

This implies that for \(z \in \mathbb{C}\), \(T_{x, \lambda}^*(z) = z \left( \overline{\lambda}_1 x \otimes x, \overline{\lambda}_2 x \otimes x, \ldots, \overline{\lambda}_d x \otimes x \right)\). So

\[
\partial f_{x, \lambda}(A) = \left\{ \sum_{k=1}^{d} \lambda_k \langle x | A_k x \rangle \left( \overline{\lambda}_1 x \otimes x, \overline{\lambda}_2 x \otimes x, \ldots, \overline{\lambda}_d x \otimes x \right) \right\} .
\]

This gives

\[
(2.3) \quad \partial \omega(A) = \co \left\{ \sum_{k=1}^{d} \lambda_k \langle x | A_k x \rangle \overline{\lambda}_1 x \otimes x, \ldots, \overline{\lambda}_d x \otimes x) : (x, \lambda) \in J(A) \right\} .
\]

For each \((x, \lambda) \in J(A)\), we have

\[
\left( \sum_{k=1}^{d} |\langle x | A_k x \rangle|^2 \right)^{1/2} \leq \omega(A) = \left( \sum_{k=1}^{d} \lambda_k \langle x | A_k x \rangle \right) \leq \left( \sum_{k=1}^{d} |\langle x | A_k x \rangle|^2 \right)^{1/2} .
\]

The last inequality follows by the Cauchy-Schwarz inequality. Hence

\[
\left| \sum_{k=1}^{d} \lambda_k \langle x | A_k x \rangle \right| = \left( \sum_{k=1}^{d} |\langle x | A_k x \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{d} |\lambda_k|^2 \right)^{1/2} .
\]

By the condition of equality in the Cauchy-Schwarz inequality, there exists \(\alpha \in \mathbb{C}\) such that \((\overline{\lambda}_1, \ldots, \overline{\lambda}_d) = \alpha (\langle x | A_1 x \rangle, \ldots, \langle x | A_d x \rangle)\). This gives \(\alpha = \left( \sum_{k=1}^{d} |\langle x | A_k x \rangle|^2 \right)^{-1/2} \). Substituting the value of \(\overline{\lambda}_k\) in (2.3), we get (1.2).
Using Theorem [1.1] we give the proof of Corollary [1.1]. The idea is similar to [3, Theorem 2.6] and [6, Theorem 1].

**Proof of Corollary [1.1].** Without loss of generality, let $A \neq 0$. Let $T_1 : \mathbb{C}^d \to M_n(\mathbb{C})^d$ be the linear map defined as $T_1(\lambda) = \lambda B$. Let $T_2 : \mathbb{C}^2 \to M_n(\mathbb{C})^2$ be defined as the affine map $L(\lambda) = T_1(\lambda) + A$ for all $\lambda \in \mathbb{C}^d$. It is easy to see that

$$\omega(A + \lambda B) \geq \omega(A)$$

for all $\lambda \in \mathbb{C}^d$ if and only if $\lambda \in \mathbb{C}^d$. By Proposition [2.1], we get

$$\omega(A + \lambda B) \geq \omega(A)$$

if and only if $0 \in T_1^* \partial \omega(A)$.

The map $T_1^* : M_n(\mathbb{C})^d \to \mathbb{C}^d$ is given by $T_1^*(C) = (\text{trace}(C_1B_1), \ldots, \text{trace}(C_dB_d))$ for all $C = (C_1, \ldots, C_d) \in M_n(\mathbb{C})^d$. Therefore

$$T_1^* \partial \omega(A) = \text{co} \left\{ \frac{1}{\omega(A)} \left( \langle x|B_ix\rangle \langle x|A_1x\rangle, \ldots, \langle x|B_dx\rangle \langle x|A_dx\rangle \right) : \|x\| = 1, \right\}$$

(2.5)

The result follows by substituting (2.5) in (2.4).

Let $(X, \| \cdot \|)$ be a normed space. An element $x \in X$ is said to be *Birkhoff-James* orthogonal to a subspace $W$ in $\| \cdot \|$ if

$$\|x + y\| \geq \|x\|$$

for all $y \in W$.

If $W$ is a one-dimensional subspace generated by $z$ and (2.6) is satisfied, then we say that $x$ is orthogonal to $z$. For $(M_n(\mathbb{C})^d, \omega(\cdot))$, (1.3) is equivalent to saying that $A$ is orthogonal to the subspace $\{ \lambda B : \lambda \in \mathbb{C}^d \}$ in $\omega(\cdot)$. In the proof of Corollary [1.1] if we take $T_1 : \mathbb{C} \to H^d$ to be the linear map defined as $T_1(\lambda) = \lambda B$ and $T_2 : \mathbb{C} \to H^d$ to be the affine map $T_2(\lambda) = T_1(\lambda) + A$, then we get the following characterization of orthogonality in $(M_n(\mathbb{C})^d, \omega(\cdot))$.

**Theorem 2.1.** Let $A = (A_1, \ldots, A_d), B = (B_1, \ldots, B_d) \in M_n(\mathbb{C})^d$. Then $A$ is orthogonal to $B$ if and only if there exist $h$ unit vectors $x_1, \ldots, x_h \in H$ with $\omega(A) = \left( \sum_{k=1}^{d} \langle x_i|A_kx_i\rangle^2 \right)^{1/2}$ for all $1 \leq i \leq h$ and there exist $h$ positive numbers $t_1, \ldots, t_h > 0$ with $t_1 + \cdots + t_h = 1$ such that

$$\sum_{k=1}^{d} \sum_{i=1}^{h} t_i \langle x_i|A_kx_i\rangle \langle x_i|B_kx_i\rangle = 0.$$

The joint operator norm of $A$ is equal to sup

$$\left\{ \left( \sum_{k=1}^{d} \|A_kx\|^2 \right)^{1/2} : x \in \mathbb{C}^n, \|x\| = 1 \right\}.$$

For the joint operator norm, an analogous result to Theorem [2.1] was proved in [3, Corollary 3.4]. A bounded linear map $T$ from a finite dimensional space $X$ to a Banach space $Y$ can...
be identified with the continuous function from the unit sphere $S_X$ of $X$ to $Y$, defined by $\tilde{T}(x) = T(x)$ for all $x \in S_X$. Let $C(S_X, Y)$ denote the space of continuous functions from $S_X$ to $Y$ with the supremum norm $\| \cdot \|_{\infty}$. Then we have $\|T\| = \|\tilde{T}\|_{\infty}$. In particular, the space $M_n(\mathbb{C})$ equipped with the joint operator norm is isometrically isomorphic to a closed subspace of $C(S_{\mathbb{C}^n} \times (\mathbb{C}^n)^d)$. In [14], this identification was used to give an alternate proof of orthogonality to one dimensional subspaces in $M_n(\mathbb{C})$ given in [2] Theorem 1. We use this identification to prove the following result for the joint operator norm, analogous to Corollary 1.1.

**Theorem 2.2.** Let $A, B \in M_n(\mathbb{C})$. Then

$$\|A + \lambda B\| \geq \|A\| \quad \text{for all } \lambda \in \mathbb{C}^d$$

if and only if there exist $h$ unit vectors $x_1, \ldots, x_h \in \mathbb{C}^n$ with $\|Ax_i\| = \|A\|$ for all $1 \leq i \leq h$ and there exist $h$ positive numbers $t_1, \ldots, t_h > 0$ with $t_1 + \cdots + t_h = 1$ such that

$$\sum_{i=1}^{h} t_i \langle A_k x_i | B_k x_i \rangle = 0 \quad \text{for all } 1 \leq k \leq d.$$

Moreover, we have $1 \leq h \leq 2d + 1$.

**Proof.** If $A \in \mathbb{C}^d B$, then the theorem holds trivially and $A = 0$ if it satisfies any of the conditions stated. So, without loss of generality, $A \notin \mathbb{C}^d B$. By [13, Theorem 1.6, p. 201], $A$ is orthogonal to $\mathbb{C}^d B$ if and only if there exist $h$ functionals $f_1, f_2, \ldots, f_h \in ((\mathbb{C}^n)^d)^*$ of unit norm with $1 \leq h \leq 2d + 1$, $h$ unit vectors $x_1, \ldots, x_h \in \mathbb{C}^n$ and $t_1, \ldots, t_h > 0$ with $\sum_{i=1}^{h} t_i = 1$ such that

$$f_i(Ax_i) = \|A\| \quad \text{for all } 1 \leq i \leq h$$

and

$$\sum_{i=1}^{n} t_i f_i(\lambda B x_i) = 0 \quad \text{for all } \lambda \in \mathbb{C}^d.$$

By the Riesz Representation Theorem, there exist unit vectors $y_1, \ldots, y_h \in \mathcal{H}$ such that for $1 \leq i \leq h$, $f_i(x) = \langle y_i | x \rangle$ for all $x \in \mathcal{H}$. So (2.8) is equivalent to the condition $\langle y_i | Ax_i \rangle = \|A\|$. By the condition of equality in the Cauchy-Schwarz inequality, this is equivalent to $y_i = \frac{1}{\|A\|} Ax_i$. So $\|Ax_i\| = \|A\|$. Thus (2.9) is equivalent to $\sum_{i=1}^{h} t_i \langle Ax_i | \lambda B x_i \rangle = 0$ for all $\lambda \in \mathbb{C}^d$, that is, for $1 \leq k \leq d$, $\sum_{i=1}^{h} t_i \langle A_k x_i | B_k x_i \rangle = 0$. \hfill $\square$

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the space of bounded operators from $\mathcal{H}$ to $\mathcal{K}$. The notation $\mathcal{B}(\mathcal{H})$ stands for $\mathcal{B}(\mathcal{H}, \mathcal{H})$. In Theorem 2.8 of [11], the following characterization is obtained. Let $A \in \mathcal{B}(\mathcal{H})$ be such that $\|A\| = 1$, the set $\{ x \in \mathcal{H} : \|Ax\| = \|A\| \}$ is the unit ball of a finite dimensional subspace $\mathcal{H}_1$ of $\mathcal{H}$ and $\|A\|_{\mathcal{H}_1} < \|A\|$. Then for any subspace $\mathcal{W}$ of $\mathcal{B}(\mathcal{H})$, $A$ is orthogonal to $\mathcal{W}$ if and only if there exist unit vectors $x_1, \ldots, x_h \in \mathcal{H}_1$ with $\|Ax_i\| = \|A\|$ for all $1 \leq i \leq h$ and there exist $t_1, \ldots, t_h > 0$ with $\sum_{i=1}^{h} t_i = 1$ such that $\sum_{i=1}^{h} t_i \langle Ax_i | Bx_i \rangle = 0$ for all $B \in \mathcal{W}$. Along the lines of the proof of Theorem 2.2 above, we get the following generalization of this.
Theorem 2.3. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be such that the set $\{x \in \mathcal{H} : \|Ax\| = \|A\|\}$ is the unit ball of a finite dimensional subspace $\mathcal{H}_1$ of $\mathcal{H}$ and $\|A\|_{\mathcal{H}_1} < \|A\|$. Then for any subspace $\mathcal{W} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$, $A$ is orthogonal to $\mathcal{W}$ if and only if there exist unit vectors $x_1, \ldots, x_h \in \mathcal{H}_1$ with $\|Ax_i\| = \|A\|$ for all $1 \leq i \leq h$ and there exist $t_1, \ldots, t_h > 0$ with $\sum_{i=1}^{h} t_i = 1$ such that

$$\sum_{i=1}^{h} t_i \langle Ax_i, Bx_i \rangle = 0 \text{ for all } B \in \mathcal{W}. \text{ Moreover, } 1 \leq h \leq 2 \dim(\mathcal{W}) + 1.$$ 

Since the vectors $x_1, \ldots, x_h$ can be chosen to be linearly independent, $\dim(\mathcal{H})$ is also a bound on $h$. Theorem 1 of [6] and Theorem 8.4 of [20] are special cases of Theorem 2.3. In both the papers, the bound on $h$ was shown to be $\dim(\mathcal{H})$ and we have been able to find a better bound on $h$. A generalization of the above theorem without any condition on $A$ can be found in [16, Theorem 1.3]. When $\mathcal{W}$ is a one dimensional subspace, a characterization of orthogonality was first proved in [10, Lemma 2.2]. It was motivated by the proof of [4, Lemma 9.14]. An alternate proof of this can be found in [2, Remark 3.1]. For a detailed survey on orthogonality to subspaces and its applications, see [7, 8] and the references therein.

3. Remarks

Remark 1. Let $X$ be a reflexive Banach space and $Y$ be a Banach space. Let $\mathcal{K}(X, Y)$ be the space of compact operators from $X$ to $Y$ with the operator norm. For $x \in X$ and a subspace $\mathcal{W}$ of $X$, let $\text{dist}(x, \mathcal{W}) = \inf \{\|x - w\| : w \in \mathcal{W}\}$. Theorem 2.2 also holds for $A \in \mathcal{K}(X, Y)$ such that $\text{dist}(A, \mathcal{K}(X, Y)) < \|A\|$. This can be seen from [19, Lemma 3.1] and the proof of Corollary [13]. An expression for the subdifferential set of the norm function in $\mathcal{B}(X, Y)$ for a reflexive Banach space $X$ was also obtained in [19, Theorem 3.2].

Remark 2. Birkhoff-James orthogonality is closely related to the notion of norm parallelism. In a normed space, an element $x$ is said to be norm parallel to another element $y$ if there exists $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\|x + \lambda y\| = \|x\| + \|y\|$. Let $A, B \in M_n(\mathbb{C})^d$. Then by [13, Theorem 2.4] and Theorem 2.1, we get that $A$ is norm parallel to $B$ in the joint numerical radius if and only if there exists a unit vector $x \in \mathbb{C}^n$ such that $|\sum_{k=1}^{d} \langle x | B_k x \rangle \langle x | A_k x \rangle| = \omega(A)\omega(B)$. The same characterization also holds for $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{H}^d)$ for a Hilbert space $\mathcal{H}$. The proof can be done along the lines of the proof of [12, Theorem 2.2].

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