INDECOMPOSABLE POLYNOMIALS AND THEIR SPECTRUM

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Abstract. We address some questions concerning indecomposable polynomials and their spectrum. How does the spectrum behave via reduction or specialisation, or via a more general ring morphism? Are the indecomposability properties equivalent over a field and over its algebraic closure? How many polynomials are decomposable over a finite field?

1. Introduction

Fix an integer \( n \geq 2 \) and a \( n \)-tuple of indeterminates \( \overline{x} = (x_1, \ldots, x_n) \). A non-constant polynomial \( F(\overline{x}) \in \mathbb{k}[\overline{x}] \) with coefficients in an algebraically closed field \( \mathbb{k} \) is said to be indecomposable in \( \mathbb{k}[\overline{x}] \) if it is not of the form \( u(H(\overline{x})) \) with \( H(\overline{x}) \in \mathbb{k}[\overline{x}] \) and \( u \in \mathbb{k}[t] \) with \( \deg(u) \geq 2 \).

An element \( \lambda^* \in \mathbb{k} \) is called a spectral value of \( F(\overline{x}) \) if \( F(\overline{x}) - \lambda^* \) is reducible in \( \mathbb{k}[\overline{x}] \). It is well-known that

1. \( F(\overline{x}) \in \mathbb{k}[\overline{x}] \) is indecomposable if and only if \( F(\overline{x}) - \lambda \) is irreducible in \( \mathbb{k}(\lambda)[\overline{x}] \) (where \( \lambda \) is an indeterminate),

2. if \( F(\overline{x}) \in \mathbb{k}[\overline{x}] \) is indecomposable, then the subset \( \text{sp}(F) \subset \mathbb{k} \) of all spectral values of \( F(\overline{x}) \) — the spectrum of \( F(\overline{x}) \) — is finite; and in the opposite case, \( \text{sp}(F) = \mathbb{k} \),

3. more precisely, if \( F(\overline{x}) \in \mathbb{k}[\overline{x}] \) is indecomposable and for every \( \lambda^* \in \mathbb{k} \), \( n(\lambda^*) \) is the number of irreducible factors of \( F(\overline{x}) - \lambda^* \) in \( \mathbb{k}[\overline{x}] \), then we have \( \rho(F) := \sum_{\lambda^* \in \mathbb{k}} (n(\lambda^*) - 1) \leq \deg(F) - 1 \). In particular \( \text{card}(\text{sp}(F)) \leq \deg(F) - 1 \).

Statement (3), which is known as Stein’s inequality, is due to Stein [St] in characteristic 0 and Lorenzini [Lo] in arbitrary characteristic (but for 2 variables); see [Na] for the general case.

This paper offers some new results in this context.

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In §2, given an indecomposable polynomial $F(x)$ with coefficients in an integral domain $A$ and a ring morphism $\sigma : A \to k$ with $k$ an algebraically closed field, we investigate the connection between the spectrum of $F(x)$ and that of the polynomial $F^\sigma(x)$ obtained by applying $\sigma$ to the coefficients of $F(x)$. Theorem 2.1 provides a conclusion à la Bertini-Noether, which, despite its basic nature, does not seem to be available in the literature: under minimal assumptions on $A$, the connection is the expected one generically. For example if $A = \mathbb{Z}$, “spectrum” and “reduction modulo a prime $p$” commute if $p$ is suitably large (depending on $F$). We give other typical applications, notably for a specialization morphism $\sigma$. Related results are given in [BCN].

For two variables, we can give in §3 an indecomposability criterion for a reduced polynomial modulo some prime $p$ (theorem 3.1) that is more precise than theorem 2.1: the condition “for suitably large $p$” is replaced by some explicit condition on $F(x, y)$ and $p$, possibly satisfied for small primes. This criterion uses some results on good reduction of curves and covers due to Grothendieck, Fulton et al; we will follow here Zannier’s version [Za]. Another criterion based on the Newton polygon of a polynomial is given in [ChNa].

§4 is devoted to the connection between the indecomposability properties over a field $K$ and over its algebraic closure $\overline{K}$. While it was known they are equivalent in many circumstances, for example in characteristic 0, it remained to handle the inseparable case to obtain a definitive conclusion. That is the purpose of proposition 4.1, which, conjoined with previous works, shows that the only polynomials $F(x)$ indecomposable in $K[x]$ but decomposable in $\overline{K[x]}$ are $p$-th powers in $\overline{K[x]}$, where $p > 0$ is the characteristic of $K$ (theorem 4.2).

§5 is aimed at counting the number of indecomposable polynomials of a given degree $d$ with coefficients in the finite field $\mathbb{F}_q$. We show that most polynomials are indecomposable: the ratio $I_d/N_d$ of indecomposables of degree $d$ tends to 1 (as $d \to \infty$ or as $q \to \infty$), and we give some estimate for the error term $1 - I_d/N_d$. The constants involved in our estimates are explicit (as in [vzG1] for irreducible polynomials). For simplicity we mostly restrict to polynomials in two variables as calculations become more intricate when $n > 2$. We also consider the one variable situation (for which the definition of indecomposability is slightly different, see §4.3) with the restriction that $q$ and $d$ are relatively prime. The cases $(n > 2)$ and $(n = 1$ with $(q, d) \neq 1$) are considered by J. von zur Gathen in a parallel work [vzG2] [vzG3].

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2. Spectrum and morphisms

Notation: If $\sigma : A \to B$ is a ring morphism, we denote the image of elements $a \in A$ by $a^\sigma$. For $P(x) \in A[x]$, we denote the polynomial obtained by applying $\sigma$ to the coefficients of $P$ by $P^\sigma(x)$. If $V \subset A^n$ is the Zariski closed subset associated with a family of polynomials $P_i(x) \in A[x]$, we denote by $V^\sigma$ the Zariski closed subset of $A^n_B$ associated with the family of polynomials $P_i^\sigma(x) \in B[x]$.

If $S \subset A$ is a multiplicative subset such that all elements from $S^\sigma$ are invertible in $B$, we still denote by $\sigma$ the natural extension $S^{-1}_A \to B$ of the original morphism $\sigma$.

Fix an integrally closed ring $A$, with a perfect fraction field $K$.

An effective divisor $D = \sum_{i=1}^r n_i a_i$ of $\overline{K}$ is said to be $K$-rational if the coefficients of the polynomial $P(T) = \prod_{i=1}^r (T-a_i)^{n_i}$ are in $K^1$. A morphism $\sigma : A \to k$ in an algebraically closed field $k$ is then said to be defined at $D$ if the coefficients of $P(T)$ have a common denominator $d \in A$ such that $d^\sigma$ is non-zero in $k^2$. In this case we denote by $P^\sigma(T) \in k[T]$ the image polynomial of $P(T)$ by the morphism $\sigma$ (extended to the fraction field of $A$ with denominators a power of $d$) and by $D^\sigma$ the effective divisor of $k$ whose support is the set of roots of $P^\sigma(T)$ and coefficients are the corresponding multiplicities.

2.1. Statement. For more precision, we use the spectral divisor rather than the spectrum: it is the divisor $\text{spdiv}(F) = \sum_{\lambda^* \in k^1} (n(\lambda^*) - 1) \lambda^*$ of the affine line $A^1_k$. Its support is the spectrum of $F$ and Stein’s inequality rewrites: $\deg(\text{spdiv}(F)) \leq \deg(F) - 1$.

Theorem 2.1. Let $F(x) \in A[x]$ be indecomposable in $\overline{K}[x]$. Then there exists a non-zero element $h_F \in A$ such that the following holds. For every morphism $\sigma : A \to k$ in an algebraically closed field $k$, if $h_F^\sigma \neq 0$, then $F^\sigma(x)$ is indecomposable in $k[x]$, the morphism $\sigma : A \to k$ is defined at the divisor $\text{spdiv}(F)$ and we have $\text{spdiv}(F^\sigma) = (\text{spdiv}(F))^\sigma$; in particular $\rho(F^\sigma) = \rho(F)$ and $\text{sp}(F^\sigma) = (\text{sp}(F))^\sigma$.

The first stage of the proof will produce the spectrum as a Zariski closed subset of the affine line $A^1_A$ over the ring $A$. Specifically the

1which, under our hypothesis “$K$ perfect”, amounts to the invariance of $P(T)$, or of $D$, under $\text{Gal}(\overline{K}/K)$.

2which, under our hypothesis “$A$ integrally closed”, amounts to saying the elements $a_i$ themselves have a common denominator $d \in A$ (that is, $da_i$ integral over $A$, $i = 1, \ldots, r$) such that $d^\sigma \neq 0$. 
following can be drawn from the proof: there is a proper\(^3\) Zariski closed subset \(V_F \subset \mathbb{A}_A^1\) such that for every morphism \(\sigma : A \rightarrow k\) as above, 

\((*)\) the polynomial \(F^\sigma(x)\), if it is of degree \(d\), is indecomposable in \(k[x]\) if and only if the Zariski closed subset \(V_F^\sigma \subset \mathbb{A}_k^1\) is proper, and in this case, we have \(\text{sp}(F) = V_F^\sigma(k)\).

When applied to the inclusion morphism \(A \rightarrow \overline{K}\), theorem 2.1 yields that the spectrum of \(F(x)\) is equal to the Zariski closed subset \(V_F(\overline{K})\). In particular, it is \(K\)-rational. The same is true for the spectral divisor of \(F(x)\) as \(n(\lambda^\tau) = n(\lambda)\) for each \(\lambda \in \overline{K}\) and each \(\tau \in \text{Gal}(\overline{K}/K)\).

2.2. Typical applications.

2.2.1. Situation 1. For \(A = \mathbb{Z}\), then \(h_F \in \mathbb{Z}\), \(h_F \neq 0\). Theorem 2.1, applied with \(\sigma : \mathbb{Z} \rightarrow \mathbb{F}_p\) the reduction morphism modulo a prime number \(p\) yields:

for all suitably large \(p\), the reduced polynomial \(\overline{F}(x)\) modulo \(p\) is indecomposable in \(\mathbb{F}_p[x]\) and its spectral divisor is obtained by reducing that of \(F(x)\), that is: \(\text{spdiv}(\overline{F}) = \text{spdiv}(F)\).

2.2.2. Situation 2. Take \(A = k[t]\) with \(k\) an algebraically closed field and \(\underline{t} = (t_1, \ldots, t_r)\) some indeterminates. Denote in this situation by \(\overline{F}(\underline{t}, x)\) the polynomial \(F(x)\) of the general statement. Theorem 2.1, applied with \(\sigma\) the specialisation morphism \(k[t] \rightarrow k\) that maps \(\underline{t} = (t_1, \ldots, t_r)\) on an \(r\)-tuple \(\underline{t}^* = (t_1^*, \ldots, t_r^*) \in k^r\) yields:

for all \(\underline{t}^*\) off a proper Zariski closed subset of \(k^r\), the specialized polynomial \(F(\underline{t}^*, x)\) is indecomposable in \(k[x]\) and its spectral divisor is obtained by specializing that of \(F(\underline{t}, x)\).

2.2.3. Situation 3. \(F(x)\) is the generic polynomial in \(n\) variables and of degree \(d\). Take for \(A\) the ring \(\mathbb{Z}[a_i]\) generated by the indeterminates \(a_i\) corresponding to the coefficients of \(F(x)\); the multi-index \(i = (i_1, \ldots, i_n)\) ranges over the set \(I_{n,d}\) of all \(n\)-tuples of integers \(\geq 0\) such that \(i_1 + \cdots + i_n \leq d\).

Classically the polynomial \(F(x)\) is irreducible in \(\mathbb{Q}(a_i)[x]\), hence it is indecomposable. Theorem 2.1, applied with \(\sigma : A \rightarrow k\) a specialization morphism of the \(a_i\), yields that all polynomials \(f(x) \in k[x]\) of degree \(d\) are indecomposable but possibly those from the proper Zariski closed subset corresponding to the equation \(h_F = 0\) (with \(h_F\) viewed in \(k[a_i]\)).

For polynomials \(f(x)\) outside the closed subset \(h_F = 0\), the spectrum of \(f\) is obtained by specializing the generic spectrum. However we have:

\(^3\)that is, distinct from the whole surrounding space (here the affine line \(\mathbb{A}^1_A\) over the ring \(A\)); equivalently, there exists a non-zero polynomial in the associated ideal.
Proposition 2.2. For \( d > 2 \) or \( n > 2 \), the generic spectrum is empty. For \( d = 2 \), it contains a single element, given by

\[
a_{00} - \frac{a_{02}a_{10}^2 + a_{20}a_{01}^2 - a_{01}a_{10}a_{11}}{4a_{02}a_{20} - a_{11}^2}
\]

For \( d > 2 \) or \( n > 2 \), polynomials with a non-empty spectrum lie in the Zariski closed subset \( h_F = 0 \).

Proof. Assume that the generic spectrum is not empty. If \( k \) is an algebraically closed field and \( R_{n,d} \) (resp. \( P_{n,d} \)) denotes the set of polynomials \( P(x) \in k[x] \) of degree \( \leq d \) that are reducible in \( k[x] \) (resp. whose constant term is zero), the correspondence \( P(x) \to P(x) - P(0) \) induces an algebraic morphism \( R_{n,d} \to P_{n,d} \) which is generically surjective (that is, surjective above a non-empty Zariski open subset of \( P_{n,d} \)). It follows that \( R_{n,d} \) is of codimension \( \leq 1 \) in the space \( P_{n,d} \) of all polynomials in \( k[x] \) of degree \( \leq d \). This observation provides the desired conclusion in the case \( n = 2 \) and \( d > 2 \): indeed we have \( \text{codim}_{P_{2,d}}(R_{2,d}) = d - 1 \) [vzG1, theorem 2].

For \( d = 2 \), the equation \(((ux + ay + b)(vx + cy + d)) = F(x) \) modulo the constant term” with unknowns \( u, a, b, v, c, d \) is readily solved: reduce to the case \( a_{20} = u = v = 1 \), find the unique solution for the 4-tuple \((a, b, c, d)\) and compute \( bd \); the generic spectral value is then \( a_{00} - bd \).

Finally assume that for \( d \geq 2 \) and \( n > 2 \), there exists a generic spectral value \( \lambda \in K \) (with \( K = \mathbb{Q}(a_i) \)). Let \( F(x) - \lambda = Q(x)R(x) \) be a non trivial factorization in \( K[x] \). Specializing \( x_3, \ldots, x_n \) to 0 gives a non trivial factorization in \( K[x_1, x_2] \) of the generic polynomial of degree \( d \) in 2 variables. From the first part of the proof, we have \( d = 2 \). Furthermore, the above case provides the necessary value of \( \lambda \). Now specializing \( x_2 \) and \( x_4, \ldots, x_n \) to 0 leads to a different value. Whence a contradiction. \( \square \)

2.3. Proof of theorem 2.1.

2.3.1. 1st stage: elimination theory. This stage is aimed at showing proposition 2.3 below, which generalizes the Bertini-Noether theorem [FrJa, prop.9.4.3]. It is proved in the general situation

(Hyp) a polynomial \( F(\lambda, x) \in A[\lambda, x] \) is irreducible in \( K(\lambda)[x] \), where \( \lambda = (\lambda_1, \ldots, \lambda_s) \) is an \( s \)-tuple of indeterminates \( s \geq 0 \).

We will use it in the special case \( F(\lambda, x) = F(x) - \lambda \). The hypotheses “\( A \) integrally closed” and “\( K \) perfect” are not necessary for this stage.

As in situation 3, consider some indeterminates \( (a_i)_{i \in I_{n,d}} \) corresponding to the coefficients of a polynomial of degree \( d \) in \( n \) variables. A
polynomial with coefficients in a ring $R$ corresponds to a morphism
\( \varphi : \mathbb{Z}[a] \to R \); denote by \( F(a_\varphi^\Sigma)(x) \in \mathbb{A}[x] \) the corresponding polynomial. Let \( \varphi_\Delta : \mathbb{Z}[a] \to \mathbb{A}[\Delta] \) be the morphism corresponding to the polynomial from statement (Hyp): \( \mathcal{F}(\Delta, x) = F(a_\varphi^\Sigma)(x) \).

From Noether’s theorem [Sc, §3.1 theorem 32], there exist finitely many universal homogeneous forms \( N_h(a_i) \) (1 \( \leq h \leq D = D(n, d) \)) in the \( a_i \) and with coefficients in \( \mathbb{Z} \) such that:

1. For every morphism \( \phi : \mathbb{Z}[a] \to k \) in an algebraically closed field \( k \), the polynomial \( F(a_\varphi^\Sigma)(x) \), if it is of degree \( d \), is reducible in \( k[x] \) if and only if \( N_h(a_i^\phi) = 0 \) for \( h = 1, \ldots, D \).

For \( \phi \) taken to be the morphism \( \varphi_\Delta : \mathbb{Z}[a] \to \mathbb{A}[\Delta] \subset \overline{K(\Delta)} \), the elements \( N_h(a_i^\varphi_\Delta) \in \mathbb{A}[\Delta] \) are polynomials \( N_h(\Delta) \). Let \( V_\Sigma \subset \Delta^* \) be the Zariski closed subset corresponding to the ideal they generate; it is a proper closed subset. Indeed, as \( \mathcal{F}(\Delta, x) \) is irreducible in \( \overline{K(\Delta)}[x] \), from (4), at least one of the polynomials \( N_h(\Delta) \), say \( N_{ho}(\Delta) \), is non-zero. Denote by \( a_\Sigma \in A \) the product of a non-zero coefficient of \( N_{ho}(\Delta) \) and the non-zero coefficient of some monomial of \( \mathcal{F}(\Delta, x) \) of degree \( d \) in \( x \).

If \( R \) is an integral domain and \( \Sigma : A[\Delta] \to R \) a morphism, then (4), with \( \phi \) taken to be \( \Sigma \circ \varphi_\Delta : \mathbb{Z}[a] \to R \); and \( \Sigma = \text{Frac}(R) \), yields that the polynomial \( \mathcal{F}(\Sigma) \in R[x] \), if of degree \( d \), is irreducible in \( \kappa[x] \) if and only if at least one of the elements \( N_h^\Sigma \in R \) is non-zero (note that \( \mathcal{F}(\Sigma) = F(a_\Sigma^\varphi_\Delta)(x) \) and \( N_h(a_i^\varphi_\Delta) = N_h(a_i^\varphi_\Delta)^\Sigma \), or, equivalently, if the corresponding Zariski closed subset of \( \text{Spec}(R) \) is proper.

Let \( \sigma : A \to k \) be a morphism with \( k \) algebraically closed. Apply the above first to the morphism \( \sigma \circ \varphi_\Delta : \mathbb{Z}[a] \to k[\Delta] \) and then, for \( \Delta^* \in k^* \), to the morphism \( s_\Delta \circ \sigma \circ \varphi_\Delta : \mathbb{Z}[a] \to k \) obtained by composing \( \sigma \circ \varphi_\Delta \) with the specialization morphism \( s_\Delta^* : k[\Delta] \to k \).

\textbf{Proposition 2.3} (Bertini-Noether generalized).

(a) The polynomial \( \mathcal{F}^\sigma(\Delta, x) \), if it is of degree \( d \) in \( x \), is irreducible in \( k[\Delta][x] \) if and only if the Zariski closed subset \( V_{\Sigma}^\sigma \subset \Delta^*_k \) is proper. All these conditions are satisfied if \( a_{\Sigma}^\sigma \) in non-zero in \( k \).

(b) If the polynomial \( \mathcal{F}^\sigma(\Delta^*, x) \) is of degree \( d \), then it is reducible in \( k[x] \) if and only if \( \Delta^* \) is in the set \( V_{\Sigma}^\sigma(k) \).

2.3.2. 2nd stage: implications for the spectrum of \( F(x) \). We return to the situation where \( \mathcal{F}(\lambda, x) = F(x) - \lambda \). Denote the Zariski closed subset \( V_\Sigma \) from §2.3.1 by \( V_{\Sigma} \); it is a Zariski closed subset of the affine line \( A_\lambda^* \). The preceding conclusions, conjoined with the connection, recalled in §1, between indecomposability of \( F(x) \) and irreducibility of \( F(x) - \lambda \), yield statement (*) from §2.1.
Denote by $s_F(\lambda)$ the g.c.d. of the polynomials $N_h(\lambda)$ in the ring $K[\lambda]$. Write it as $s_F(\lambda) = S_F(\lambda)/c_1$ with $S_F(\lambda) \in A[\lambda]$ and $c_1 \in A$ non-zero. The polynomial $S_F(\lambda)$ is non-zero and its distinct roots in $\overline{K}$, say $\lambda_1, \ldots, \lambda_s$, which are the common roots in $\overline{K}$ of the polynomials $N_h(\lambda)$, are the spectral values of $F(x)$ (note that $F(x) - \lambda^*$ is of degree $d$ for all $\lambda^* \in \overline{K}$). Thus we have $S_F(\lambda) = c_2 \prod_{i=1}^s (\lambda - \lambda_i)^{n_i} \in A[\lambda]$ for some exponents $n_i > 0$ and $c_2 \in A$, $c_2 \neq 0$. It follows that the set $\text{sp}(F) = \{\lambda_1, \ldots, \lambda_s\}$ is $K$-rational. As already noted, the same is then true for the spectral divisor $\text{spdiv}(F)$.

### 2.3.3. 3rd stage: invariance of the spectrum of $F$ via morphisms.

Fix a morphism $\sigma : A \to k$ with $k$ algebraically closed. Denote by $a_F$ the element $a_F$ from §2.3.1 for $\mathcal{F} = F(x) - \lambda$. If $a_F^\sigma \neq 0$, $F^\sigma(x)$ is of degree $d$ and indecomposable in $k[\lambda]$. Furthermore, its spectral values are the roots in $k$ of the g.c.d. of the polynomials $N_h^\sigma(\lambda)$.

Note that the element $c_2$ above is a common denominator of $\lambda_1, \ldots, \lambda_s$; if $c_2^\sigma \neq 0$, the morphism $\sigma : A \to k$ is defined at $\text{spdiv}(F)$.

**Lemma 2.4.** There exists $c_3 \in A$, $c_3 \neq 0$ such that, if $a_F^\sigma c_1^\sigma c_2^\sigma c_3^\sigma \neq 0$, the polynomial $S_F^\sigma(\lambda) \in k[\lambda]$ equals (up to some non-zero multiplicative constant in $k$) the g.c.d. in $k[\lambda]$ of polynomials $N_h^\sigma(\lambda)$ ($1 \leq h \leq D$). In particular $\text{sp}(F^\sigma) = (\text{sp}(F))^\sigma$.

**Proof.** The problem is whether the g.c.d. commutes with $\sigma$. The Euclidean algorithm provides the g.c.d. as the last non-zero remainder. To reach our goal, it suffices to guarantee that for each division $a = bq + r$ in $K[\lambda]$ involved in the algorithm, the identity $a^\sigma = b^\sigma q^\sigma + r^\sigma$, with $\sigma$ suitably extended, be the division of $a^\sigma$ by $b^\sigma$ in $k[\lambda]$. For this, write $a, b, q$ as $r$ in the form $n(\lambda)/m$ with $n(\lambda) \in A[\lambda]$ and $m \in A$, consider the product $\beta$ of denominators $m$ of $a, b, q$ and $r$ with the coefficients of highest degree monomials in the numerators $n(\lambda)$ of $b$ and $r$ and request that $\beta^\sigma \neq 0$. Multiplying all elements $\beta$ for all divisions leading to the g.c.d. of two, then of all polynomials in question, leads to a non-zero element $c_3 \in A$ which satisfies the desired statement. \(\square\)

**Remark 2.5.** Morphisms and g.c.d. do not commute in general: for example $\gcd(\lambda, \lambda + a)$ is 1 generically, but equals $\lambda$ if $a = 0$.

### 2.3.4. 4th stage: invariance of $\text{spdiv}(F)$ via morphisms.

It remains to extend the conclusion “$\text{sp}(F^\sigma) = (\text{sp}(F))^\sigma$” to the spectral divisor $\text{spdiv}(F)$. We will show how to guarantee that, via the morphism $\sigma$, ...
the spectral values remain distinct and the associated decompositions of \( F(x) - \lambda \) have the same numbers of distinct irreducible factors\(^4\).

Consider the discriminant of the polynomial \( \prod_{i=1}^{s}(\lambda - \lambda_i) \); it is a non-zero element of \( K \). Write it as \( c_4/c_5 \) with \( c_4, c_5 \in A \), non-zero. If \( c_4^6 c_5^7 \neq 0 \), the polynomials \( S_F(\lambda) \) and \( S'_F(\lambda) \) have the same number of distinct roots, whence \( \text{card}(\text{sp}(F^{\sigma})) = \text{card}((\text{sp}(F))^{\sigma}) = \text{card}(\text{sp}(F)) \).

For \( i = 1, \ldots, s \), let \( F(x) - \lambda_i = \prod_{j=1}^{n(\lambda_i)} Q_{ij}(x)^{\delta_{ij}} \) be a factorization (into distinct irreducible polynomials) in \( K[x] \). Let \( E/K \) be a finite Galois extension that contains the finite set \( C \) of all coefficients involved in all above factorizations, \( c_6 \) be a non-zero element of \( A \) such that \( c_6 c \) is integral over \( A \) for all \( c \in C \) and \( c_7 \) be the discriminant of a basis \( E \) over \( K \) the elements of which are integral over \( A \). Denote by \( B \) the fraction ring of \( A \) with denominator a power of \( c_6 c_7 \) and by \( B'_E \) the integral closure of \( B \) in \( E \). The ring \( B'_E \) is a free \( B \)-module of rank \( [E : K] \). Assume that \( c_6^5 c_7^6 \neq 0 \). The morphism \( \sigma : A \rightarrow k \) extends to a morphism \( B \rightarrow k \), and, as \( k \) is algebraically closed, this morphism \( \sigma : B \rightarrow k \) can in turn be extended to a morphism \( \tilde{\sigma} : B'_E \rightarrow k \).

The polynomials \( Q_{ij}(x) \) are in the ring \( B'_E[x] \) and are absolutely irreducible. The (classical) Bertini-Noether theorem provides a non-zero element \( \beta \in B'_E \) such that, if \( \beta \tilde{\sigma} \neq 0 \), then each of the polynomials \( \tilde{Q}_{ij}(x) \) is absolutely irreducible. Therefore the decomposition \( F^{\sigma}(x) - \lambda_i^6 = \prod_{j=1}^{n(\lambda_i)} \tilde{Q}_{ij}(x) \) obtained from the preceding one by applying \( \tilde{\sigma} \), is the factorization of \( F^{\sigma}(x) - \lambda_i^6 \) into irreducible polynomials in \( k[x] \).

It remains to assure that for \( i \) fixed, the polynomials \( \tilde{Q}_{ij}(x) \) are different, even up to non-zero multiplicative constants. For any two (distinct) polynomials \( Q_{ij}(x), Q_{i'j'}(x) \), the matrix with rows the tuples of coefficients of the two polynomials has a \( 2 \times 2 \)-block with a non-zero determinant. Denote the product of all such determinants for all possible couples \( (Q_{ij}(x), Q_{i'j'}(x)) \) by \( \delta \); it is a non-zero element of \( B'_E \). Denote then by \( \nu \) the norm of \( \beta \tilde{\sigma} \) relative to the extension \( E/K \). As \( A \) is integrally closed, so is \( B \) and \( \nu \in B \). Write it as \( \nu = c_8/(c_6 c_7 \gamma) \) with \( c_8 \in A \) and \( \gamma \in \mathbb{N} \). Condition \( c_6^5 c_7^6 \neq 0 \) implies \( \beta \tilde{\sigma} \delta \neq 0 \). Theorem 2.1 is finally established for \( h_F = a_F \prod_{i=1}^{s} c_i \).

**Remark 2.6.** The same proof, with the polynomial \( F(x) \) from §2.3.1 of the form \( F(x) - \lambda G(x) \) with \( F(x), G(x) \in A[x] \) and \( \deg G \leq \deg F \), leads to the more general form of theorem 2.1 for which indecomposable polynomials are replaced by indecomposable rational functions (in this

\(^4\)The argument will also show the degrees of these irreducible factors, say \( Q_{\lambda,j} \), remain the same and thus so does the quantity \( \min_{\lambda \in \text{sp}(F)} (\sum_{j} \deg(Q_{\lambda,j}) - 1) \) which replaces \( \deg(F) - 1 \) in Lorenzini’s refined version [Lo] of Stein’s inequality.
case, “indecomposable” means not of the form \( u(H(x)) \) with \( H(x) \) and \( u(t) \) rational functions and \( \deg(u) \geq 2 \). A spectral value of a rational function \( F(x)/G(x) \) is an element \( \lambda \) such that the polynomial \( F(x) - \lambda G(x) \) is reducible. Statements (1), (2) and (3) from §1 remain true, except that the bound in Stein’s inequality should be replaced by \( (\deg(F))^2 - 1 \) [Bo] [Lo]. More generally one can take \( F(\lambda, x) \) of the form \( F(x) - \lambda G_1(x) - \cdots - \lambda_s G_s(x) \) with \( F(x), G_1(x), \ldots, G_s(x) \in A[x] \) and handle other situations studied in the literature. In this context, some effective results are given in [BCN].

3. An indecomposability criterion modulo \( p \)

In this section \( n = 2 \), \( A \) is a Dedekind domain and its fraction field \( K \) is assumed to be of characteristic 0. Fix also a non-zero prime ideal \( \mathfrak{p} \) of \( A \) and assume its residue field \( k = A/\mathfrak{p} \) is of characteristic \( p > 0 \). Denote by \( \bar{x} \) the image of an element \( x \) by the reduction morphism \( A \to k \). The situation “\( A = \mathbb{Z} \) and \( \mathfrak{p} = p\mathbb{Z} \)” is typical.

Let \( F(x, y) \in A[x, y] \) be an indecomposable polynomial in \( \overline{K}[x, y] \) of degree \( d \geq 1 \), monic in \( y \).

Here is our strategy to guarantee indecomposability of \( F(x, y) \) modulo \( \mathfrak{p} \). Pick \( \lambda^* \in A \setminus \text{sp}(F) \) (using Stein’s theorem, this can be done with \( \lambda^* \) not too big). Thus \( F(x, y) - \lambda^* \) is irreducible in \( \overline{K}[x, y] \). It follows from the classical Bertini-Noether theorem that if “\( \mathfrak{p} \) is big enough”, then the reduced polynomial \( F(x, y) - \lambda^* \) modulo \( \mathfrak{p} \) is absolutely irreducible. Therefore \( F(x, y) \) is indecomposable modulo \( \mathfrak{p} \) (as there is at least one non spectral value). However the constants involved in the condition “\( \mathfrak{p} \) big enough” are too big for a practical algorithmic use. We will follow an alternate approach, based on good reduction criteria for covers, and more precisely Zannier’s criterion [Za].

Consider the discriminant with respect to \( y \) of \( F(x, y) - \lambda \):

\[
\Delta_F(x, \lambda) = \text{disc}_y(F(x, y) - \lambda)
\]

Denote then the product of all distinct irreducible factors of \( \Delta_F(x, \lambda) \) in \( K(\lambda)[x] \) by \( \Delta_F^{\text{red}}(x, \lambda) \); more precisely, \( \Delta_F^{\text{red}}(x, \lambda) \) is defined by the following formula, which is also algorithmically more practical:

\[
\Delta_F^{\text{red}}(x, \lambda) = c(\lambda) \frac{\Delta_F(x, \lambda)}{\gcd(\Delta_F(x, \lambda), (\Delta_F)_x'(x, \lambda))}
\]

\(^5\) the degree of a rational function is the maximum of the degrees of its numerator and denominator.
where the g.c.d. is calculated in the ring \( K(\lambda)[x] \) (using the Euclidean algorithm for example) and \( c(\lambda) \in K(\lambda) \) is the rational function, defined up to some invertible element in \( A \), that makes \( \Delta_F^{\text{red}}(x, \lambda) \) a primitive polynomial in \( A[\lambda][x] \). Consider next the polynomial:

\[
\Delta_F(\lambda) = \text{disc}_x(\Delta_F^{\text{red}}(x, \lambda)).
\]

We have \( \Delta_F(\lambda) \in A[\lambda] \) and \( \Delta_F(\lambda) \neq 0 \). Finally let \( \Delta_0(\lambda) \in A[\lambda] \) be the coefficient of the highest monomial in \( \Delta_F(x, \lambda) \) (viewed in \( A[\lambda][x] \)).

**Theorem 3.1.** Assume, in addition to \( F(x, y) \) being indecomposable in \( K[x, y] \), that the reduced polynomial \( \widetilde{\Delta}_0(\lambda)\widetilde{\Delta}_F(\lambda) \) is non-zero in \( k[\lambda] \) and that \( p > \deg_Y(F) \). Then \( \widetilde{F}(x, y) \) is indecomposable in \( \overline{K}[x, y] \).

The assumption \( p > \deg_Y(F) \) can be replaced by the weaker condition that \( p \) does not divide the order of the Galois group of \( F(x, y) - \lambda \), viewed as a polynomial in \( K(\lambda)(x) \) (see footnote 8).

**Remark 3.2.** Theorem 3.1 can be combined with preceding results. If \( V_F \subset \mathbb{A}_K^1 \) is the Zariski closed subset from §2.1, then, under the above hypotheses, the reduced Zariski closed subset \( \widetilde{V}_F \subset \mathbb{A}_k^1 \) is proper and its points are the spectral values of \( \widetilde{F} \): \( \text{sp}(\widetilde{F}) = \widetilde{V}_F(\overline{k}) \). However the assumptions on \( p \) and \( F(x, y) \) may not be sufficient to guarantee the extra conclusions \( \text{sp}(\widetilde{F}) = \text{sp}(F) \) and \( \text{spdiv}(\widetilde{F}) = \text{spdiv}(F) \) from theorem 2.1 (which may not even be well-defined).

**Proof of theorem 3.1.** The prime ideal \( p \subset A \) determines a discrete valuation \( v \) of \( K \) whose valuation ring is the localized ring \( A_p \); the fraction field of \( A_p \) and its residue field remain equal to \( K \) and \( k \) respectively. Hypotheses and conclusions from theorem 3.1 are unchanged if \( A \) is replaced by \( A_p \). The valued field \( (K, v) \) can then also be replaced by any finite extension of the completion \( K_v \) and \( A \) by the new valuation ring; the discrete valuation \( v \) uniquely extends, the residue field is replaced by some (finite) extension of \( k \), the indecomposability properties of \( F(x, y) \) over \( K \) or over \( K_v \) are equivalent.

Thus we may and will assume that \( (K, v) \) is a complete discretely valued field, that \( A \) is its valuation ring (which is integrally closed) and that the field \( K \) and the residue field \( k \) contain as many (finitely many) algebraic elements over the original fields as necessary.

The polynomial \( \Delta_F(x, \lambda) \) is in \( A[x, \lambda] \) and its factorization into irreducible polynomials in \( K(\lambda)[x] \) can be written

\[
\Delta_F(x, \lambda) = \delta_0(\lambda) \prod_{i=1}^{s} \Delta_i(x, \lambda)^{\alpha_i}
\]
where the polynomials $\Delta_i(x, \lambda)$ are in $A[x, \lambda]$, irreducible in $K(\lambda)[x]$, pairwise distinct (even up to some constant in $K$) and are primitive in $A[\lambda][x]$, where $\delta_i(\lambda) \in A[\lambda]$ and where the $\alpha_i$ are positive integers. Then, up to some invertible element in $A$, we have

$$\Delta_F^{\text{red}}(x, \lambda) = \prod_{i=1}^s \Delta_i(x, \lambda)$$

Also note that the polynomial $\Delta_0(\lambda)$ is a multiple in $A[\lambda]$ of the product of $\delta_0(\lambda)$ with the highest monomial coefficients $\delta_1(\lambda), \ldots, \delta_s(\lambda)$ of the polynomials $\Delta_1(x, \lambda), \ldots, \Delta_s(x, \lambda)$ (viewed in $A(\lambda)[x]$).

Pick next $\lambda^* \in k$ such that $\tilde{\Delta}_0(\lambda^*)\tilde{\Delta}_F(\lambda^*) \neq 0$ in $k$, then lift it to some element $\lambda^* \in A$ such that $\lambda^* \notin \text{sp}(F)$. This is possible in view of the preliminary remark.

The set of roots of $\Delta_F(x, \lambda^*)$ contains the set of finite\(^6\) branch points of the cover of $\mathbb{P}^1_x$ determined by the (absolutely irreducible) polynomial $F(x, y) - \lambda^*$. The preliminary remark makes it possible to assume that these roots are in $K$. Furthermore as $\tilde{\delta}_i(\lambda^*) \neq 0$, we have $\delta_i(\lambda^*) \in A \setminus p$, $i = 1, \ldots, s$; therefore these roots are integral over $A$ and so are in $A$.

As $\Delta_F(\lambda^*) \neq 0$, the roots of $\Delta_F^{\text{red}}(x, \lambda^*)$ in $\overline{K}$ are distinct and as $\delta_0(\lambda^*) \neq 0$, they are the roots of $\Delta_F(x, \lambda^*)$. As $\tilde{\Delta}_0(\lambda^*) \neq 0$, $\tilde{\Delta}_F(\lambda^*)$ is not the zero polynomial. As $\tilde{\Delta}_F(\lambda^*) \neq 0$, the roots of $\tilde{\Delta}_F^{\text{red}}(x, \lambda^*)$, which are those of the polynomial $\Delta_F(x, \lambda^*)$, are distinct. Thus we obtain that the distinct roots of the polynomial $\Delta_F(x, \lambda^*)$, and a fortiori the branch points of the cover considered above, have distinct reductions modulo the ideal $p$.

It follows from standard results on good reduction of covers, and more precisely here, from the main theorem of [Za] that, under the assumption $p > \text{deg}_{\mathbb{Y}}(F)$\(^8\), $\tilde{F}(x, y) - \lambda^*$ is absolutely irreducible. Hence $\tilde{F}(x, y)$ is indecomposable in $\overline{k}[x, y]$. \qed

4. INDECOMPOSABILITY OVER $K$ VERSUS $\overline{K}$

4.1. Statements (for $n \geq 2$ variables). The indecomposability property which we recalled the definition of in §1 over an algebraically closed

\(^6\)i.e., distinct from the point at infinity.

\(^7\)The subscript “$x$” indicates that the cover is induced by the correspondence $(x, y) \rightarrow x$. In fact the problem is symmetric in the variables $x$ and $y$ which can be switched in our statement.

\(^8\)It suffices to assume that $p$ does not divide the order of the Galois group of $F(x, y) - \lambda^*$, which divides the order of the Galois group of $F(x, y) - \lambda$, which itself divides $(\text{deg}_{\mathbb{Y}}(F))!$.
field can in fact be defined over an arbitrary field: just require that
the polynomials $u(t)$ and $H(x)$ involved have their coefficients in the field
in question. The results below identify the only cases where the prop-
erty is not the same over some field $K$ and over some extension $E$.
The following result handles the case that $E/K$ is purely inseparable,
which was missing in the literature.

**Proposition 4.1.** Let $E/K$ be a purely inseparable algebraic field
extension of characteristic $p > 0$ and $F(x) \in K[x]$. Assume $F(x)$ is not
of the form $bG(x)^p + c$ with $G(x) \in E[x]$ and $b, c \in K$. Then $F(x)$ is
indecomposable in $K[x]$ if and only if it is indecomposable in $E[x]$.

If $E = K$, the assumption on $F(x)$ rewrites to merely say that $F(x)$
is not a $p$-th power in $K[x]$, which in turn is equivalent to at least one
exponent in $F(x)$ not being a multiple of $p$. Clearly this assumption
cannot be removed: for example, if $\alpha \in K \setminus K$ but $\alpha^p = a \in K$ then
$x^p + ay^p$ is indecomposable in $K[x]$ but decomposable in $K[x]$.

In [AP, proposition 1], Arzhantsev and Petravchuk show the equiva-
lence from proposition 4.1 without any assumption on $F(x)$, but in the
case of a separable extension $E/K$ (possibly of positive transcendence
degree). As any extension is a purely inseparable algebraic extension of
some separable extension, conjoining their result with ours yields that,
under the assumption on $F(x)$ from proposition 4.1, the equivalence
holds for an arbitrary extension $E/K$. We can be more precise.

**Theorem 4.2.** Let $E/K$ be a field extension and $F(x) \in K[x]$ be a
non-constant polynomial. Then the following are equivalent:

(i) $F(x)$ is indecomposable in $K[x]$ but decomposable in $E[x]$.

(ii) (a) $K$ is of characteristic $p > 0$ and $E/K$ is inseparable,
    (b) $F(x) = bG(x)^p + c$ for some $G(x) \in E[x]$ and $b, c \in K$, and
    (c) $G(x)^p$ is indecomposable in $K[x]$.

Condition (ii) (c) implies that $G(x)$ is not of the form $u(H(x))$ with
$u \in E[t], H(x) \in E[x], \deg(u) \geq 2$ and both $u(t)^p \in K[t]$ and $H(x)^p \in K[x]$. But there are other possible polynomials that should be excluded
whose description is more intricate.

4.2. **Proofs.**

Proof of proposition 4.1. The converse part is obvious. For the di-
rect part, assume $F(x)$ is decomposable in $E[x]$. Then it is decom-
posable over some finite extension of $K$ contained in $E$, which admits
a finite system of generators $\alpha_1, \ldots, \alpha_s$ with irreducible polynomial
over $K$ of the form $x^{\alpha_1} - a$ with $a \in K$. The multiplicativity
of the degree and of the separable degree imply that the extensions $K(\alpha_1, \ldots, \alpha_{j+1})/K(\alpha_1, \ldots, \alpha_j)$ are purely inseparable, $j = 1, \ldots, s - 1$. By induction one reduces to the case $s = 1$, and then a new induction reduces to the case $E = K(\alpha)$ with $\alpha^p = a \in K \setminus K^p$.

Assume $F(x) = h(G(x))$ with $h(t) \in K(\alpha)[t]$ such that $\deg(h) \geq 2$ and $G(x) \in K(\alpha)[x]$. We deduce

$$F(x)^p = p h(G(x)^p)$$

where, if $h(t) = \sum_{i=0}^{\deg(h)} k_i t^i$, we set $p h(t) = \sum_{i=0}^{\deg(h)} p^i k_i t^i$. As $p h(t) \in K[t]$ and $G(x)^p \in K[x]$ (since $y^p \in K$ for all $y \in K(\alpha)$), this shows that the field $K(F(x), G(x)^p)$ is of transcendence degree 1 over $K$. From Gordan’s theorem [Sc, §1.2, th.3], there exists $\theta(x) \in K(x)$ such that

$$K(F(x), G(x)^p) = K(\theta(x))$$

Furthermore from [Sc, §1.2, th.4], one may assume that $\theta(x) \in K[x]$. Thus we have

$$\begin{cases} F(x) = u(\theta(x)) \text{ with } u(t) \in K(t) \\ G(x)^p = v(\theta(x)) \text{ with } v(t) \in K(t) \end{cases}$$

As $F(x)$ and $G(x)^p$ are polynomials, $u(t), v(t)$ are necessarily in $K[t]$. It follows from the indecomposability of $F(x)$ over $K$ that $\deg(u) = 1$, which gives $G(x)^p = w(F(x))$ for some polynomial $w \in K[t]$. But then we obtain $G(x)^p = w \circ h(G(x))$, which, since $G(x)$ is non constant, amounts to $T^p = w \circ h(T)$ where $T$ is an indeterminate. As $\deg(h) \geq 2$ and $p$ is a prime, we have $\deg(w) = 1$ and $\deg(h) = p$, which gives $F(x) = b G(x)^p + c$ for some $b, c \in K$.

Note that because of the inductive process, conclusion “$b, c \in K$” should really be that $b, c$ are in the first subfield of the initial reduction. But $F(x)$ being in $K[x]$ then implies that $b \gamma^p \in K$ for some non-zero $\gamma \in E$ and $b G(0)^p + c \in K$. Up to changing $G(x)$ to $\gamma^{-1} G(x) \gamma^{-1} G(0)$, one can indeed conclude that $b, c \in K$ in the general situation.

**Proof of theorem 4.2.** (i) $\Rightarrow$ (ii): If $K_\alpha/K$ is the maximal separable extension contained in $E$, then, from the Arzhantsev-Petravchuk result, $F(x)$ is indecomposable in $K_\alpha[x]$. In particular $E \neq K_\alpha$, which gives (ii) (a). Proposition 4.1 then provides condition (ii) (b) except that $b$ and $c$ are *a priori* in $K_\alpha$, but using again the final note of the proof of Proposition 4.1, one can indeed choose $b, c \in K$. Condition (ii) (c) then readily follows from (ii) (b) and the indecomposability of $F(x)$ in $K[x]$. The other implication (ii) $\Rightarrow$ (i) is clear. □
4.3. One variable. In proposition 4.1, \( F(x) \) is a polynomial in two variables or more. In one variable, the indecomposability definition should be modified (for otherwise it is trivial): a polynomial \( F(x) \in k[x] \) is said to be indecomposable in \( k[x] \) if it is not of the form \( u(H(x)) \) with \( H(x) \in k[x] \) and \( u \in k[t] \) with \( \deg(u) \geq 2 \) and \( \deg(H) \geq 2 \).

**Proposition 4.3.** Proposition 4.1 holds for one variable polynomials.

**Proof.** The same proof can be used as for proposition 4.1. It leads to

\[
\begin{align*}
  F(x) &= u(\theta(x)) \text{ with } u(t) \in K[t] \\
  G(x)^p &= v(\theta(x)) \text{ with } v(t) \in K[t]
\end{align*}
\]

But from the indecomposability of \( F(x) \) over \( K \), we now deduce that \( \deg(u) = 1 \) or \( \deg(\theta) = 1 \).

The case \( \deg(u) = 1 \) is handled as before. In the other case, we deduce from \( \deg(\theta) = 1 \) that \( K(F(x), G(x)^p) = K(x) \), which implies that \( K(\alpha)(h(G(x)), G(x)^p) = K(\alpha)(x) \) and so that

\[ K(\alpha)(x) \subset K(\alpha)(G(x)) \]

which forces \( \deg(G) = 1 \) and contradicts the decomposability assumption in one variable made at the beginning of the proof. \( \square \)

5. Counting indecomposable polynomials over finite fields

For each integer \( d \geq 1 \), denote the number of polynomials in \( \mathbb{F}_q[x] \) \((x = (x_1, \ldots, x_n))\) of degree \( d \) by \( N_d \). We have

\[
\begin{align*}
  N_d &= \left( q^{\binom{n+\frac{d-1}{2}}{n-1}} - 1 \right) \cdot q^{\binom{n+d-1}{2}} \text{ (for general } n) \\
  N_d &= q^{\frac{1}{2}(d+1)(d+2)}(1 - q^{-d-1}) \text{ (for } n = 2) \\
  N_d &= (q - 1)q^d \text{ (for } n = 1)
\end{align*}
\]

Denote the number of those polynomials which are indecomposable (resp. decomposable) by \( I_d \) (resp. \( D_d \)). We have \( N_d = I_d + D_d \).

We will study separately the case of \( n \geq 2 \) variables (§5.1 - §5.4) and the case \( n = 1 \) (§5.5).

5.1. Main result. From §5.1 to §5.4, we assume \( n \geq 2 \).

**Theorem 5.1.** (a) \( I_d/N_d \) tends to 1 in the two situations where \( d \to \infty \) with \( q \) fixed, and where \( q \to \infty \) with \( d \) fixed.

(b) If \( d \) is a product of at most 2 prime numbers \( p \leq p' \), then

- \( d = p \) and \( D_d = q^d(q^n - 1) \), or
- \( d = p^2 \) and \( D_d = q^{p-1}N_p + (q^d - q^{2p-1})(q^n - 1) \), or
\[ D_d = q^p - N_p + q^p - 1 = (q^d - 2q^{-1} + 1)(q^n - 1). \]

(c) Assume \( n = 2 \). If \( d \) is the product of at least 3 prime numbers, then

\[
\left| \frac{D_d}{N_d} - \alpha_d \right| \leq \alpha_d \beta_d \quad \text{where} \quad \begin{cases} 
\alpha_d = \frac{q^{\ell - 1} + \frac{1}{2}(\frac{d}{q} + 1)(\frac{d}{q} + 2)}{q^{\frac{d}{q}(d+1)(d+2)}} \\
\beta_d = \frac{d}{q^\ell}
\end{cases}
\]

\( \ell > 1 \) is the first (hence prime) divisor of \( d \).

5.2. An induction formula. Let \( K \) be an arbitrary field. Let \( F = u \circ H \) be a decomposition of \( F \in K[x] \) with \( u \in K[t] \), \( \deg u \geq 2 \), and \( H \in K[x] \). We say that \( F = u \circ H \) is a normalized decomposition if \( H \) is indecomposable, monic (i.e. the coefficient of the leading term of a chosen order is 1) and its constant term equals zero. Given a decomposition \( F = u \circ H \), there exists an associated normalized decomposition \( F = u' \circ H' \). The following lemma shows it is unique.

**Lemma 5.2.** Let \( F = u \circ H = u' \circ H' \) be two normalized decompositions of \( F \in K[x] \). Then \( u = u' \) and \( H = H' \).

**Proof.** It follows from \( u(H) - u'(H') = 0 \) that \( H \) and \( H' \) are algebraically dependent over \( K \). By Gordan’s theorem [Sc, §1.2, theorems 3 and 4] (already used in §4.2), there exists a polynomial \( \theta(x) \in K[x] \) such that \( K[\theta] = K[H, H'] \). That is, there exist \( v, v' \in K[t] \) such that \( H = v(\theta) \) and \( H' = v'(\theta) \). As the two decompositions of \( F \) are normalized, \( H \) and \( H' \) are indecomposable, so \( \deg v = \deg v' = 1 \), and so using the other normalization conditions, we obtain \( H = H' \). Finally it follows from \( u(H) = u'(H) \) that \( u = u' \). \( \square \)

**Corollary 5.3** (induction formula). With notation as in §5.1, we have

\[
I_d = N_d - \sum_{d' | d, \ d' < d} q^{\frac{d}{d'} - 1} \times I_{d'}
\]

**Proof.** Let \( d' \geq 1 \) be a divisor or \( d \). There are \((q - 1)q^{d/d'}\) polynomials \( u \in F_q[t] \) of degree \( d/d' \) and \( I_{d'}/q(q - 1) \) normalized indecomposable polynomials \( H \in F_q[x] \) of degree \( d' \). The formula follows as from lemma 5.2, every polynomial \( F \) counted by \( D_d \) can be uniquely written \( F = u \circ H \) with \( u \) and \( H \) as above for some integer \( d' \) such that \( d'|d, d' < d \). \( \square \)

Conjoined with \( I_1 = N_1 = q(q^n - 1) \) this formula provides an algorithm to compute \( I_d \) and \( D_d \), which is convenient for small \( d \).
5.3. **Proof of theorem 5.1** (a) and (b). The formulas in (b) straightforwardly follow from corollary 5.3. If \( d = p \) is a prime number, we have \( D_p = q^{p-1}I_1 = q^{p-1}N_1 = q^p(q^n - 1) \). If \( d = p^2 \) then

\[
D_d = q^{p-1}I_p + q^{p^2-1}I_1 = q^{p-1}(N_p - q^p(q^n - 1)) + q^{p^2}(q^n - 1).
\]

Computations are similar for \( d = pp' \). To prove (a) we write

\[
N_d - I_d = D_d = \sum_{d'|d, d' < d} q^{d/d'} I_{d'} \leq \sum_{d'|d, d' < d} q^{d/d'} N_{d'}
\]

The sum has at most \( d \) terms and each is \( \leq q^d N_{d/2} \), whence

\[
1 - \frac{I_d}{N_d} \leq d q^d \frac{N_{d/2}}{N_d}
\]

and the announced result as the right-hand side term tends to 0 in the two situations considered in the statement of theorem 5.1 (a). □

5.4. **Proof of theorem 5.1** (c). In this subsection we assume that \( n = 2 \) and that \( d \) has at least three prime divisors.

5.4.1. *A technical lemma.*

**Lemma 5.4.** Let \( b(d) = \frac{1}{2}(d+1)(d+2) \). Let \( \ell > 1 \) be the first divisor of \( d \) and \( \ell' > \ell \) be the second divisor of \( d \). Let \( \lambda \geq \ell' \) be a divisor of \( d \) and \( \ell'' > 1 \) be the first divisor of \( d/\ell \). Then we have

1. \( b(d/\ell') + \ell' \geq b(d/\lambda) + \lambda \).
2. \( b(d/\ell) + \ell - d/\ell \geq b(d/\ell') + \ell' \).
3. \( b(d/\ell) + 1 - d/\ell \geq b(d/\ell\ell'') + \ell'' \).

**Proof.** (1) We have

\[
b(d/\ell') + \ell' - b(d/\lambda) - \lambda = \frac{1}{2} \left( \frac{d}{\ell'} - \frac{d}{\lambda} \right) \left( \frac{d}{\ell'} + \frac{d}{\lambda} + 3 - 2 \frac{\ell'\lambda}{d} \right) \geq 0
\]

because \( d/\ell' - d/\lambda \geq 0 \) and \( \frac{d}{\ell'} + \frac{d}{\lambda} + 3 - 2 \frac{\ell'\lambda}{d} \geq \frac{d}{\ell'} + 4 - 2\ell' \geq 0 \) as \( d \) has at least 3 prime divisors.

(2) We have \( \ell\ell' \leq d \) so \( \ell' - \ell \leq \frac{d}{\ell} \). Moreover we have \( \frac{d}{\ell'} \leq \frac{d}{\ell} - 2 \) and for all \( d \geq 6 \) we have \( b(d/\ell') \leq b(d/\ell - 2) \). Hence

\[
b(d/\ell) - b(d/\ell') + \ell - \ell' - \frac{d}{\ell} \geq b(d/\ell) - b(d/\ell - 2) - 2d/\ell = 1
\]
(3) If we set $\delta = d/\ell$ then

$$b(\delta) + 1 - \delta - b(\delta/\ell'') - \ell'' = \frac{1}{2} \left( \delta - \frac{\delta}{\ell''} \right) \left( \delta + \frac{\delta}{\ell''} - 2 \right) + \frac{1}{2} \left( 3\delta - 5\frac{\delta}{\ell''} - 2\ell'' + 2 \right)$$

Now $\delta - \frac{\delta}{\ell''} \geq 0$, $\delta + \frac{\delta}{\ell''} - 2 \geq 0$ and as $\delta$ has at least 2 prime divisors, then $u(\ell'') = 3\delta - 5\frac{\delta}{\ell''} - 2\ell'' + 2 \geq u(2) = \frac{\delta}{2} - 2 \geq 0$. □

5.4.2. An upper bound for $D_d$. Using the notations of Lemma 5.4, we have

$$D_d = q^{e-1}I_{d/\ell} + \sum_{\lambda | d, \lambda > \ell} q^{\lambda-1}I_{d/\lambda} \quad \text{(corollary 5.3)}$$

$$\leq q^{e-1}N_{d/\ell} + \sum_{\lambda | d, \lambda > \ell} q^{\lambda-1}N_{d/\lambda}$$

$$\leq q^{b(d/\ell)+e-1} \left( 1 - \frac{1}{q^{e+1}} \right) + \sum_{\lambda | d, \lambda > \ell} q^{\lambda-1}q^{b(d/\lambda)} \quad \text{(explicit formula for } N_{d/\lambda})$$

$$\leq q^{b(d/\ell)+e-1} \left( 1 - \frac{1}{q^{e+1}} \right) + (d-1)q^{b(d/\ell')+e'1-1} \quad \text{(lemma 5.4 (1))}$$

$$\leq q^{b(d/\ell)+e-1} \left( 1 - \frac{1}{q^{e+1}} \right) \left( 1 + \frac{d}{q^{b(d/\ell)-b(d/\ell')+e''-e'}} \right) \quad \text{(because } \frac{d-1}{1 - q^{-e-1}} \leq d)$$

$$\leq q^{b(d/\ell)+e-1} \left( 1 - \frac{1}{q^{e+1}} \right) \left( 1 + \frac{d}{q^{e}} \right) \quad \text{(lemma 5.4 (2))}$$

5.4.3. A lower bound for $D_d$. Start from $D_d \geq q^{e-1}I_{d/\ell}$. Then use §5.4.2 right above (or the formulas already proved from theorem 5.1 (b)) to bound $I_{d/\ell} = N_{d/\ell} - D_{d/\ell}$ from below. We obtain

$$D_d \geq q^{e-1} \times \left( q^{b(d/\ell)} \left( 1 - \frac{1}{q^{e+1}} \right) - q^{b(d/\ell')+e''-1} \left( 1 - \frac{1}{q^{e''+1}} \right) \left( 1 + \frac{d/\ell}{q^{e''}} \right) \right)$$

$$\geq q^{e-1} \left( 1 - \frac{1}{q^{e+1}} \right) \left( q^{b(d/\ell)} - 2q^{b(d/\ell')+e''-1} \right) \quad \text{(because } \frac{d}{q^{e+1}} \leq 1)$$

$$= q^{e-1} \left( 1 - \frac{1}{q^{e+1}} \right) q^{b(d/\ell)} \left( 1 - \frac{2}{q^{b(d/\ell)-b(d/\ell')+e''-e''}} \right)$$

$$\geq q^{b(d/\ell)+e'-1} \left( 1 - \frac{1}{q^{e+1}} \right) \left( 1 - \frac{2}{q^{e}} \right) \quad \text{(lemma 5.4 (3))}$$
5.4.4. Final estimate for $D_d/N_d$. The upper and lower bounds for $D_d$ yield the following inequalities

$$\frac{q^{b(d)} q^{(d\ell - \ell - 1)}}{q^{\frac{d}{2}} - 1 - q^{-\frac{d}{2} - 1}} \left(1 - \frac{2}{q^d}\right) \leq \frac{D_d}{N_d} \leq \frac{q^{b(d)} q^{(d\ell - \ell - 1)}}{q^{\frac{d}{2}} - 1 - q^{-\frac{d}{2} - 1}} \left(1 + \frac{d}{q^d}\right)$$

which are a little more precise than the announced statement. □

5.5. One variable. Here we assume $n = 1$. For polynomials in one variable, we use the definition of indecomposability given in §4.3.

5.5.1. Main result.

**Theorem 5.5.** Assume $q$ and $d$ are relatively prime.

(a) If $d$ is a product of at most 2 prime numbers $p \leq p'$, then

- $d = p$ and $D_d = 0$,
- $d = p^2$ and $D_d = \frac{q^2}{q} q^2$, or
- $d = pp'$ with $p < p'$ and
  $$2q - \frac{1}{q} q^{p + p'} - q^5 \leq D_d \leq 2q - \frac{1}{q} q^{p + p'}$$

(b) Assume $d$ is the product of at least 3 prime numbers. Let $\ell > 1$ be the first divisor of $d$ and $\ell' > \ell$ be its second divisor. Then we have

$$\frac{d}{2\ell q^{\frac{d}{2}} - \frac{d}{2}} - \alpha_d \leq \frac{d - 2}{2q^{\ell + \frac{d}{2}} - \frac{d}{2}}$$

where $\alpha_d = \frac{2}{q^{d - \ell + \frac{d}{2}} + 1}$.

As a consequence we have that $I_d/N_d$ tends to 1 in the two situations where $d \to \infty$ with $q$ fixed, and $q \to \infty$ with $d$ fixed.

Theorem 5.5 fails if the assumption $(q, d) = 1$ is removed. For example for $q = 2$ and $d$ even one can compute that $D_d/N_d \sim 3.2^{-d/2}$ while $\alpha_d = 4.2^{-d/2}$ in this case.

From now on we assume $q$ and $d$ are relatively prime. The rest of the paper is devoted to the proof of theorem 5.5. Our strategy is similar to the one used for $n \geq 2$. We view the set $D_d$ of all decomposable polynomials $f(x) \in \mathbb{F}_q[x]$ of degree $d$ as the union of smaller sets which we will estimate. More specifically we write

$$D_d = \bigcup_{\lambda | d, \ell \leq \lambda \leq d/\ell} D_{\lambda, d/\lambda}$$

where $D_{\lambda, d/\lambda} \subset D_d$ is the subset of all $f(x)$ which admit a decomposition $f = u \circ v$ with $u, v \in \mathbb{F}_q[x]$, $\deg u = \lambda \geq 2$, $\deg v = d/\lambda \geq 2$, $v$ monic and of constant term equal to 0. A difference with the case $n \geq 2$ is that we do not have a partition.
5.5.2. 1st stage: upper bounds. (Assumption \((q,d) = 1\) is not used in this paragraph). For every divisor \(\lambda \geq 1\) of \(d\), denote the cardinality of \(D_{\lambda,\frac{d}{\lambda}}\) by \(D_{\lambda,\frac{d}{\lambda}}\). We have

\[
D_{\lambda,\frac{d}{\lambda}} \leq N_{\lambda} \frac{N_{d/\lambda}}{q(q-1)} = \frac{q-1}{q} q^{\lambda + \frac{d}{\lambda}}
\]

If \(\ell > 1\) is the first divisor of \(d\) and \(\ell' > \ell\) the second divisor, we have

\[
D_d \leq \sum_{\lambda | d, \ell \leq \lambda \leq d/\ell} D_{\lambda,\frac{d}{\lambda}} \leq \frac{q-1}{q} \sum_{\lambda | d, \ell \leq \lambda \leq d/\ell} q^{\lambda + \frac{d}{\lambda}}
\]

The idea is that the main contribution comes from \(D_{\ell,\frac{d}{\ell}}\) and \(D_{\frac{d}{\ell},\ell}\).

If \(d\) is the product of exactly 2 prime numbers \(\ell\) and \(d/\ell\), then these are the only contributions and we have the desired upper bound. Otherwise we write \(\lambda + \frac{d}{\lambda} \leq \ell' + \frac{d}{\ell'}\) to bound the extra terms and obtain

\[
D_d \leq \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} \left( 2 + \frac{d-2}{q^{\ell} - \ell' - \frac{d}{\ell'}} \right)
\]

which yields all announced upper bounds in theorem 5.5. We also deduce this practical bound: \(D_d \leq d \frac{q-1}{q} q^{\ell + \frac{d}{\ell}} (\text{as } \ell + \frac{d}{\ell} - \ell' - \frac{d}{\ell'} \geq 1)\).

5.5.3. 2nd stage: uniqueness results. We will use Ritt’s theorems to control the number of possible decompositions of a given polynomial.

**Proposition 5.6.** Let \(K\) be a field and \(f \in K[x]\) be a polynomial of degree \(d > 0\). Assume the characteristic \(p\) of \(K\) does not divide \(d\). Suppose we have two decompositions \(f = u \circ v = u' \circ v'\) of \(f\) with

- \(u, v, u', v'\) indecomposable,
- \(\deg u = \deg u' \geq 2\), \(\deg v = \deg v' \geq 2\),
- with \(v, v'\) monic with a zero constant term.

Then \(u = u'\) and \(v = v'\).

**Proof.** This follows from the first Ritt theorem [Sc, §1.3 theorem 7] which more generally describes in which cases an equality \(G_1 \circ \cdots \circ G_s = H_1 \circ \cdots \circ H_s\) with \(G_i, H_j\) indecomposable of degree \(> 1\) may hold. \(\square\)

As an immediate consequence, we obtain the case \(d = p^2\) of theorem 5.5 (a): namely we have \(D_{p^2} = D_{p,p} = \frac{q-1}{q} q^{2p}\).
5.5.4. 3rd stage: lower bounds for $D_{\ell, \ell}$ and $D_{\ell, \ell}$.  

**Lemma 5.7.** Assume $d$ is not a prime number. Then we have

$$D_{\ell, \ell} \geq \frac{q - 1}{q} q^{\ell + d} \left(1 - \frac{d/\ell}{q^{\ell+1} - q^{d/\ell}}\right).$$

And the same inequality holds for $D_{\ell, \ell}$ replaced by $D_{\ell, \ell}$.  

**Proof.** We only give the proof for $D_{\ell, \ell}$ as computations for $D_{\ell, \ell}$ are the same. In $D_{\ell, \ell}$ we will only count those polynomials $f$ which decompose as $f = u \circ v$ with $u$ and $v$ as in proposition 5.6. Then we obtain 

$$D_{\ell, \ell} \geq \frac{1}{q(q-1)} I_{\ell} \cdot I_{\ell}$$

$$\geq \frac{1}{q(q-1)} N_{\ell} (N_{\ell} - D_{\ell}) \quad (D_{\ell} = 0 \text{ as } \ell \text{ is prime})$$

$$= \frac{1}{q(q-1)} (q-1)q^{\ell} \left((q-1)q^{\ell} - D_{\ell}\right)$$

$$= \frac{q - 1}{q} q^{\ell + \frac{d}{\ell}} \left(1 - \frac{D_{\ell}}{(q-1)q^{\ell+1}}\right)$$

If $d$ is the product of exactly 2 primes then $D_{\ell, \ell} = 0$ and

$$(*) \quad D_{\ell, \ell} \geq \frac{q - 1}{q} q^{\ell + \frac{d}{\ell}}$$

which in this case is better than the announced result.

If $d$ is the product of at least 3 primes, use the practical upper bound for $D_{d}$ obtained in §5.5.2 to write $D_{\ell, \ell} \leq \frac{q - 1}{q} q^{\ell + \frac{d}{\ell}}$ and deduce

$$D_{\ell, \ell} \geq \frac{q - 1}{q} q^{\ell + \frac{d}{\ell}} \left(1 - \frac{(d/\ell)q^{\ell + \frac{d}{\ell}}}{(q-1)q^{\ell+1}}\right) = \frac{q - 1}{q} q^{\ell + \frac{d}{\ell}} \left(1 - \frac{d/\ell}{q^{\ell+1} - q^{d/\ell}}\right)$$

$\square$

5.5.5. Estimating the multiple decompositions. Next we write

$$D_{d} \geq \text{card}(D_{\ell, \ell} \cup D_{d, \ell}) = D_{\ell, \ell} + D_{d, \ell} - \text{card}(D_{\ell, \ell} \cap D_{d, \ell})$$

In order to estimate $D_{d}$ we need to estimate the intersection.

**Lemma 5.8.** We have
\[
\begin{dcases}
\text{card}(D_{\ell, 4} \cap D_{4, \ell}) \leq \frac{d}{\ell} q^{\frac{d}{2} + 2\ell - 1} \\
D_d \geq 2 \left( q^{\frac{d}{2}} + 4 \right) \left( 1 - \frac{2d}{d} \frac{1}{q^{\frac{d}{2}} - \frac{d}{\ell} - \ell + 1} \right)
\end{dcases}
\]

The lower bound for \(D_d\) is the remaining inequality to be proved in theorem 5.5 (b). The more precise inequality (***) in the proof below will complete the proof of theorem 5.5 (a) in the special case \(d = pp'\).

**Proof of lemma 5.8.** (a) If \(\gcd(\ell, d/\ell) = 1\) then \(\text{card}(D_{\ell, 4} \cap D_{4, \ell}) \leq q^5\).

Indeed let \(f \in D_{\ell, 4} \cap D_{4, \ell}\) and let \(f = u \circ v - \ell\) be a decomposition with \(\deg u = \ell\) and \(\deg v = d/\ell\). We follow Ritt’s second theorem (see [Sc, §1.4, theorem 8] and the notation there). The hypotheses of that result are satisfied because the derivative \(u'\) of \(u\) is non zero; otherwise \(f' = 0\), and so \(f \in \mathbb{F}_q[x^\ell]\) and the characteristic \(p\) of \(\mathbb{F}_q\) divides \(d = \deg f\). In first case of Ritt’s second theorem we have \(L_1 \circ u = x^\ell P(x)^n\) and \(v \circ L_2 = x^n\) (where \(r \geq 0\), \(P \in \mathbb{F}_q[x]\) and \(L_1, L_2\) are linear functions). In our situation we get \(n = \frac{d}{\ell}\) and \(\ell = r + \frac{d}{\ell}\deg P\). Then \(\deg P = \frac{\ell - 1}{d} \leq \frac{\ell - 1}{\ell} < 1\) so \(\deg P = 0\), \(L_1 \circ u = x^\ell\) and \(v \circ L_2 = x^\ell\). Considering all possible linear functions yield at most \((q - 1)^2q^2\) such decompositions. In second case of Ritt’s second theorem we have \(L_1 \circ u = D_m(x, a^n)\) and \(v \circ L_2 = D_n(x, a)\), \(a \in \mathbb{F}_q\) (where \(D_n(x, a)\) denote Dickson’s polynomials). We here obtain \(m = \ell\) and \(n = \frac{d}{\ell}\). Considering all possible linear functions and all \(a \in \mathbb{F}_q\) yield at most \((q - 1)^2q^2\) such decompositions. Finally we obtain

\[
\text{card}(D_{\ell, 4} \cap D_{4, \ell}) \leq (q - 1)^2q^2 + (q - 1)^2q^3 \leq q^5
\]

(b) If \(\gcd(\ell, d/\ell) \neq 1\) then we have \(\text{card}(D_{\ell, 4} \cap D_{4, \ell}) \leq \frac{d}{\ell} q^{\frac{d}{2} + 2\ell - 1}\).

Indeed let \(f \in D_{\ell, r} \cap D_{r, \ell}\) and let \(f = u \circ v\) be a decomposition with \(\deg u = \ell\) and \(\deg v = d/\ell\). By Ritt’s first theorem and because \(\gcd(\ell, d/\ell) \neq 1\) either \(u\) or \(v\) is decomposable. But as \(\ell\) is a prime \(D_{\ell}\) is empty and so \(v \in D_{r, \ell}\). Thus we obtain

\[
\text{card}(D_{\ell, 4} \cap D_{4, \ell}) \leq \frac{1}{q(q - 1)} D_{r, 4} \leq \frac{1}{q(q - 1)} (q - 1) q^{\frac{d}{\ell}} q^{\frac{d}{\ell} + \frac{d}{\ell}} \leq \frac{d}{\ell} q^{\frac{d}{\ell} + 2\ell - 1} \quad \text{end of §5.5.2}
\]
The proof follows as for all \( d > 6 \) we have \( \frac{d}{\ell^2} + 2\ell - 1 \geq 5 \). \( \square \)

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