Genuine multipartite entanglement measure

Yu Guo¹,∗, Yanping Jia¹, Xinping Li² and Lizhong Huang¹

¹ Institute of Quantum Information Science, School of Mathematics and Statistics, Shanxi Datong University, Datong, Shanxi 037009, People’s Republic of China
² Institute of Theoretical Physics, Shanxi Datong University, Datong, Shanxi 037009, People’s Republic of China

E-mail: guoyu3@aliyun.com

Received 11 November 2021, revised 24 January 2022
Accepted for publication 17 February 2022
Published 9 March 2022

Abstract
Quantifying genuine entanglement is a crucial task in quantum information theory. In this work, we give an approach of constituting genuine \(m\)-partite entanglement measures from any bipartite entanglement and any \(k\)-partite entanglement measure, \(3 \leq k < m\). In addition, as a complement to the three-qubit concurrence triangle proposed in (Phys. Rev. Lett. 127 040403), we show that the triangle relation is also valid for any continuous entanglement measure and system with any dimension. We also discuss the tetrahedron structure for the four-partite system via the triangle relation associated with tripartite and bipartite entanglement respectively. For multipartite system that contains more than four parties, there is no symmetric geometric structure as that of tri- and four-partite cases.

Keywords: multipartite state, genuine multipartite entanglement measure, triangle relation

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement as one of the most puzzling features of quantum mechanics has been recognized as a crucial resource in many quantum information processing protocols in the last three decades [1–6]. Numerous entanglement measures, especially for the bipartite entanglement, have been proposed so far [3, 4, 7–15, 15]. Yet, the situation is much more complicated for multipartite systems. How to quantify the genuine entanglement contained in the multipartite state is still lacking in research.

∗Author to whom any correspondence should be addressed.
The first effort in this connection is the ‘residual tangle’ which reports the genuine three-qubit entanglement by Coffman et al in 2000 [16]. In 2011, Ma et al [12] established some conditions for a quantity to be a genuine multipartite entanglement measure (GMEM) and proposed a GMEM, called genuinely multipartite concurrence (GMC), according to the origin bipartite concurrence. In such a sense, the ‘residual tangle’, the generalizations of concurrence [15, 17, 18], the generalization of negativity [19] the SL-invariant multipartite measure of entanglement [20–25], and the α-entanglement entropy [13] are not GMEMs since all these quantities do not obey all the genuine entanglement conditions in reference [12]. Very recently, a new genuine three-qubit entanglement measure, called concurrence triangle, was put forward in reference [28]. It is quantified as the square root of the area of concurrence triangle. Under this scenario, the GHZ state is more entangled than the W state.

Apart from GMC and the concurrence triangle, there are very few GMEM by now. We list here for reader’s convenience. The GMC is further explored in reference [29], the generalized geometric measure is introduced in references [30, 31], the average of ‘residual tangle’ and GMC [32] are shown to be GMEMs. Another one is the divergence-based GMEM presented in [26, 27]. The aim of this work is to construct (genuine) m-partite entanglement measures from bipartite entanglement measures and k-partite entanglement measures, 3 ≤ k < m, and, in addition, generalize the concurrence triangle in reference [28] to any other continuous entanglement measure and system with any dimension.

The rest of this article is organized as follows. In section 2 we review some necessary concepts. Section 3 discusses the triangle relation and present genuine tripartite entanglement measure via bipartite entanglement measure. In section 4 we establish genuine four-partite entanglement measures via the tripartite entanglement measure, bipartite entanglement measure and the genuine tripartite entanglement measure defined in section 3, respectively. The approach for extending to general multipartite case is also discussed. In section 5, we investigate the tetrahedron structure for any four-partite state by tripartite and bipartite entanglement measures. In section 4 we establish genuine four-partite entanglement measure via bipartite entanglement measure. In section 4 we establish genuine four-partite entanglement measures via the tripartite entanglement measure, bipartite entanglement measure and the genuine tripartite entanglement measure defined in section 3, respectively. The approach for extending to general multipartite case is also discussed. In section 5, we investigate the tetrahedron structure for any four-partite state by tripartite and bipartite entanglement measures, respectively. A summary is conclude in the last section. For simplicity, we fix some notations. Throughout this paper, we denote by \( H^{A_1A_2 \cdots A_m} \), the set of density operators acting on \( H^X \). We denote the set of density operators acting on \( H^X \).

2. Preliminary

For convenience, in this section, we recall the concepts of genuine entanglement, the bipartite entanglement measure, the complete multipartite entanglement measure and the GMEM in detail.

2.1. Multipartite entanglement

An \( m \)-partite pure state \( |\psi\rangle \in H^{A_1A_2 \cdots A_m} \) is called biseparable if it can be written as \( |\psi\rangle = |\psi\rangle^X \otimes |\psi\rangle^Y \) for some bipartition of \( A_1A_2 \cdots A_m \) (for example, for a four-partite state, \( A_1A_3|A_2A_4 \) is a bipartition of \( A_1A_2A_3A_4 \)). |\psi\rangle is said to be \( k \)-separable if \( |\psi\rangle = |\psi\rangle^{X_1} |\psi\rangle^{X_2} \cdots |\psi\rangle^{X_k} \) for some \( k \)-partition of \( A_1A_2 \cdots A_m \) (for example, for a five-partite state, \( A_1A_2A_4|A_3A_5 \) is a three-partition of \( A_1A_2A_3A_4A_5 \) with \( X_1 = A_1A_3, X_2 = A_2 \) and \( X_3 = A_4A_5 \)). |\psi\rangle is called fully separable if it is \( m \)-separable. An \( m \)-partite mixed state \( \rho \) is biseparable if it can be written as a convex combination of biseparable pure states \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \), wherein the contained \( \{|\psi_i\rangle\} \) can be biseparable with respect to different bipartitions (i.e., a mixed biseparable state does not need to be separable with respect to any particular bipartition). If \( \rho \) is not biseparable, then it is called genuinely entangled. \( \rho \) is said to be \( k \)-separable with
respect to partition $X_1|X_2|\ldots|X_k$ if $\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|$ with $|\psi_i\rangle$ is $k$-separable with respect to partition $X_1|X_2|\ldots|X_i$ for any $i$. Throughout this paper, for any $\rho \in S_{A_1A_2\ldots A_m}$ and any given $k$-partition $X_1|X_2|\ldots|X_k$ of $A_1A_2\ldots A_m$, we denote by $\rho^{X_1|X_2|\ldots|X_k}$ the state for which we consider it as a $k$-partite state with respect to the partition $X_1|X_2|\ldots|X_k$.

It is clear that whenever a state is $k$-separable, it is automatically also $l$-separable for all $1 < l < k$. We denote the set of all $k$-separable states by $S_k$ ($k=2,3,\ldots,m$), then $S_m \subseteq S_{m-1} \subseteq \ldots \subseteq S_2 \subseteq S$. $S\setminus S_2$ is the set of all $k$-entangled states. $S\setminus S_2$ is just the set of all genuinely entangled (or equivalently, two-entangled) states (here we denote $S_{A_1A_2\ldots A_m}$ by $S$ for simplicity). If a state is $k$-entangled, it is automatically also $j$-entangled for all $k < j \leq m$ but not vice versa.

### 2.2. Bipartite entanglement measure

A function $E : S_{AB} \to \mathbb{R}_+$ is called an entanglement measure if it satisfies [33]:

- **(E-1)** $E(\rho) = 0$ if $\rho$ is separable;
- **(E-2)** $E$ cannot increase under local operation and classical communications (LOCC), i.e., $E(\varepsilon(\rho)) \leq E(\rho)$ for any LOCC $\varepsilon \in \{I\}$ implies that $E$ is invariant under local unitary operations, i.e., $E(\rho) = E(U_A^\dagger \otimes U_B^\dagger \rho U_A \otimes U_B)$ for any local unitaries $U_A$ and $U_B$. The map $\varepsilon$ is completely positive and trace preserving (CPTP).

In general, LOCC can be stochastic, in the sense that $\rho$ can be converted to $\sigma_j$ with some probability $p_j$. In this case, the map from $\rho$ to $\sigma_j$ cannot be described in general by a CPTP map. However, by introducing a ‘flag’ system $A'$, we can view the ensemble $\{\sigma_j, p_j\}$ as a classical quantum state $\sigma' := \sum_j p_j |j\rangle \langle j| \otimes \sigma_j$. Hence, if $\rho$ can be converted by LOCC to $\sigma_j$ with probability $p_j$, there exists a CPTP LOCC map $\Phi$ such that $\Phi(\rho) = \sigma'$. Therefore, the definition above of a measure of entanglement captures also probabilistic transformations. In particular, $E$ must satisfy $E(\sigma') \leq E(\rho)$.

Any measure of entanglement studied in literature satisfy

$$E(\sigma) = \sum_j p_j E(\sigma_j),$$

which is very intuitive since $A'$ is just a classical system encoding the value of $j$. In this case the condition $E(\sigma') \leq E(\rho)$ becomes

$$\sum_j p_j E(\sigma_j) \leq E(\rho).$$  \hspace{1cm} (1)

That is, LOCC cannot increase entanglement on average (it is possible that $E(\sigma) > E(\rho)$ for some $j_0$). An entanglement measure $E$ is said to be an entanglement monotone [35] if it satisfies equation (1) and is convex additionally (almost all entanglement measures are entanglement monotones although not all, see in reference [34] for an example, the logarithmic negativity is not convex).

### 2.3. Complete multipartite entanglement measure

A function $E^{(m)} : S_{A_1A_2\ldots A_m} \to \mathbb{R}_+$ is called an $m$-partite entanglement measure [3, 15, 17] if it satisfies:

- **(E1)** $E^{(m)}(\rho) = 0$ if $\rho$ is fully separable;
- **(E2)** $E^{(m)}$ cannot increase under $m$-partite LOCC.
An \( m \)-partite entanglement measure \( E^{(m)} \) is said to be an \( m \)-partite entanglement monotone if it is convex and does not increase on average under \( m \)-partite stochastic LOCC. \( E^{(m)} \) is called a unified multipartite entanglement measure if it also satisfies the following condition [14]:

- **(E3) the unification condition**, i.e., \( E^{(m)} \) is consistent with \( E^{(k)} \) for any \( 2 \leq k < m \).

The unification condition should be comprehended in the following sense [14] (we take \( m = 3 \) for example). Let \( |\psi\rangle^{ABC} \) be a bi-separable pure state in \( \mathcal{H}^{ABC} \), e.g., \( |\psi\rangle^{ABC} = |\psi\rangle^{AB} |\psi\rangle^{C} \) then

\[
E^{(3)}(|\psi\rangle^{AB} |\psi\rangle^{C}) = E^{(2)}(|\psi\rangle^{AB}),
\]

In this way, the link between \( E^{(2)} \) and \( E^{(3)} \) can be established. The unification condition also requires the measure must be invariant under the permutations of the subsystems, i.e.,

\[
E^{(3)}(\rho^{ABC}) = E^{(3)}(\rho^{\pi(ABC)}), \quad \text{for any } \rho^{ABC} \in \mathcal{S}^{ABC},
\]

where \( \pi \) is a permutation of the subsystems. In addition,

\[
E^{(3)}(\rho^{ABC}) \geq E^{(2)}(\rho^{XY}), \quad \rho^{XY} = \text{Tr}_{Z}(\rho^{ABC}), \quad X, Y, \in \{A, B, C\}
\]

for any \( \rho^{ABC} \in \mathcal{S}^{ABC} \). That is, a unified multipartite entanglement measure \( E^{(m)} \) is indeed a series of measures \( \{E^{(m)}, E^{(m-1)}, \ldots, E^{(2)}\} \).

\( E^{(m)} \) is called a complete multipartite entanglement measure if it satisfies both (E3) above and the following [14]:

- **(E4) \( E^{(m)}(\rho^{A_1A_2\ldots A_m}) \geq E^{(k)}(\rho^{A_1A_2\ldots A_k}) \)** holds for all \( \rho^{A_1A_2\ldots A_m} \in \mathcal{S}^{A_1A_2\ldots A_m}, A'_1A'_2\ldots A'_k \) is any \( k \)-partition of \( A_1A_2\ldots A_m \).

For instance, for any pure state \( |\psi\rangle \in \mathcal{H}^{AB} \), the entanglement of formation (EoF) [36, 37], \( E_f \), is defined as

\[
E^{(2)}_f(|\psi\rangle) = E_f(|\psi\rangle) = S(\rho^A) = S(\rho^B) = \frac{1}{2} [S(\rho^A) + S(\rho^B)]
\]

and the tangle is defined by [38]

\[
\tau(|\psi\rangle) = 2 \left( 1 - \text{Tr}(\rho^A)^2 \right) = 2 - \text{Tr}(\rho^A)^2 - \text{Tr}(\rho^B)^2.
\]

These measures are extended into three-partite case by [14]

\[
E^{(3)}_f(|\psi\rangle) := \frac{1}{2} \left[ S(\rho^A) + S(\rho^B) + S(\rho^C) \right],
\]

\[
\tau^{(3)}(|\psi\rangle) := 3 - \text{Tr}(\rho^A)^2 - \text{Tr}(\rho^B)^2 - \text{Tr}(\rho^C)^2,
\]

for pure state \( |\psi\rangle \in \mathcal{H}^{ABC} \). By the convex-roof extension, the measures can be defined for mixes states. In reference [14], we show that \( \{E^{(3)}_f, E^{(2)}_f\} \) and \( \{\tau^{(3)}, \tau^{(2)}\} \) are complete tripartite entanglement measures (one can see other complete tripartite entanglement measures in reference [14] for detail).

### 2.4. Genuine entanglement measure

A function \( E_g : \mathcal{S}^{A_1A_2\ldots A_m} \to \mathbb{R}_+ \) is defined to be a GMEM if it admits the following conditions [12]:
J. Phys. A: Math. Theor. 55 (2022) 145303

Y Guo et al

- (GE1) $E_{\phi}(\rho) = 0$ for any biseparable $\rho \in S^{A_1A_2\ldots A_n}$.
- (GE2) $E_{\phi}(\rho) > 0$ for any genuinely entangled state $\rho \in S^{A_1A_2\ldots A_n}$. (This item can be weakened as: $E_{\phi}(\rho) \geq 0$ for any genuinely entangled state $\rho \in S^{A_1A_2\ldots A_n}$. That is, maybe there exists some state which is genuinely entangled such that $E_{\phi}(\rho) = 0$. In such a case, the measure is called not faithful. Otherwise, it is called faithful. For example, the `residual triangle’ is not faithful since it is vanished for the $W$ state.)
- (GE3) $E_{\phi}(\sum_i p_i \rho_i) \leq \sum_i p_i E_{\phi}(\rho_i)$ for any $\{p_i, \rho_i\}, \rho_i \in S^{A_1A_2\ldots A_n}, p_i > 0, \sum_i p_i = 1$.
- (GE4) $E_{\phi}(\rho) \geq E_{\phi}(\rho')$ for any $m$-partite LOCC $\varepsilon, \varepsilon(\rho) = \rho'$.

Note that (GE4) implies $E_{\phi}$ is invariant under local unitary transformations. $E_{\phi}$ is said to be a genuine multipartite entanglement monotone if it does not increase on average under $m$-partite stochastic LOCC.

We recall a genuine measure, concurrence fill, defined in reference [28]. The concurrence triangle for three-qubit state is constructed from the following triangle relation [28, 39, 40]:

\[
C_{ABC}^2 \leq C_{A|BC}^2 + C_{B|AC}^2.
\]

That is, for any three-qubit pure state $|\psi\rangle^{ABC}$, edges with $C_{A|BC}^2$, $C_{B|AC}^2$, and $C_{C|AB}^2$ make up a triangle. Then the square root of the area of this triangle, i.e., the so-called concurrence fill, quantifies the amount of genuine entanglement contained in the state. It can then be defined for mixed state via the convex-roof extension. (We note here that, whether the concurrence fill is monotonic under LOCC is not proved in reference [28]. We call it genuine entanglement measure throughout this paper.)

Similarly, we call a function $E_{m(\varepsilon)} : S^{A_1A_2\ldots A_n} \to \mathbb{R}_+$ a $k$-entanglement measure if it admits the following conditions:

- (k-E1) $E_{m(\varepsilon)}(\rho) = 0$ for any $k$-separable $\rho \in S^{A_1A_2\ldots A_n}$.
- (k-E2) $E_{m(\varepsilon)}(\rho) > 0$ for any $k$-entangled state $\rho \in S^{A_1A_2\ldots A_n}$. (This item can be weakened as: $E_{m(\varepsilon)}(\rho) \geq 0$ for any $k$-entangled state $\rho \in S^{A_1A_2\ldots A_n}$. That is, maybe there exists some $k$-entangled state such that $E_{m(\varepsilon)}(\rho) = 0$. In such a case, the measure is called not faithful. Otherwise, it is called faithful.)
- (k-E3) $E_{m(\varepsilon)}(\sum_i p_i \rho_i) \leq \sum_i p_i E_{m(\varepsilon)}(\rho_i)$ for any $\{p_i, \rho_i\}, \rho_i \in S^{A_1A_2\ldots A_n}, p_i > 0, \sum_i p_i = 1$.
- (k-E4) $E_{m(\varepsilon)}(\rho) \geq E_{m(\varepsilon)}(\rho')$ for any $m$-partite LOCC $\varepsilon, \varepsilon(\rho) = \rho'$.

For simplicity, throughout this paper, if $E$ is an entanglement measure (bipartite, or multipartite, or $k$-entanglement) for pure states, we define

\[
E_{\phi}(\rho) := \min \sum_i p_i E^{m_i}(|\psi_i\rangle)
\]

and call it the EoF associated with $E$, where the minimum is taken over all pure-state decomposition $\{p_i, |\psi_i\rangle\}$ of $\rho$ (namely, the convex-roof extension of $E$. Sometimes, we use $E_{\phi}$ to denote $E_{\phi}$ hereafter).

3. Triangle measure

Motivated by the concurrence triangle, we firstly consider whether there exists a triangle for any tripartite pure state $|\psi\rangle^{ABC} \in H^{ABC}$ with arbitrarily given bipartite entanglement measure...
\( E \). That is, for any given bipartite entanglement measure \( E \), whether \( E(|\psi\rangle^{A|BC}) \), \( E(|\psi\rangle^{B|AC}) \) and \( E(|\psi\rangle^{AB|C}) \) can represent the lengths of the three edges of a triangle.

**Theorem 1.** Let \( E \) be a continuous bipartite entanglement measure. Then there exists \( 0 < \alpha < \infty \) such that

\[
E^\gamma(|\psi\rangle^{A|BC}) \leq E^\gamma(|\psi\rangle^{B|AC}) + E^\gamma(|\psi\rangle^{AB|C})
\]

for all pure states \(|\psi\rangle^{ABC} \in \mathcal{H}^{ABC}\) with fixed \( \dim \mathcal{H}^{ABC} = d < \infty \).

**Proof.** For any given \(|\psi\rangle^{ABC}\), we assume that \( E(|\psi\rangle^{A|BC}) = x \), \( E(|\psi\rangle^{B|AC}) = y \) and \( E(|\psi\rangle^{AB|C}) = z \). If \( x < y \) or \( x < z \), equation (4) is obvious. If \( x > \max\{y, z\} \), we claim that \( y > 0 \) and \( z > 0 \) (note that, if \( x > \max\{y, z\} \), then \( \max\{y, z\} > 0 \)). In order to see this, we assume with no loss of generality that \( x > y > z > 0 \). We now suppose \( z = 0 \), then we let

\[
|\psi\rangle^{ABC} = |\psi\rangle^{AB}|\psi\rangle^C
\]

for some \(|\psi\rangle^{AB} \in \mathcal{H}^{AB}\) and \(|\psi\rangle^C \in \mathcal{H}^C\). It follows that

\[
E^{\gamma}_{ABC}(|\psi\rangle^{ABC}) = E^{\gamma}_{BAC}(|\psi\rangle^{ABC}).
\]

Here \( E^{\gamma}_{ABC} \) denotes the bipartite measure with respect to bi-partition \( A|BC \). In such a case, equation (4) is clear with equality. If \( z > 0 \), then there always exists \( \gamma > 0 \) such that

\[
1 \leq \left( \frac{x}{y} \right)^\gamma + \left( \frac{z}{y} \right)^\gamma
\]

since \( \left( \frac{x}{y} \right)^\gamma \to 1 \) and \( \left( \frac{z}{y} \right)^\gamma \to 1 \) when \( \gamma \) decreases. Let \( f(\rho^{ABC}) \) be the largest value of \( \gamma \) for which the inequality (5) is saturated. \( E \) is continuous leads to \( f \) is continuous. Since \( f \) is continuous and the set of all pure states in \( S^{ABC} \) is compact, we get

\[
\alpha \equiv \inf_{\rho^{ABC} \in S^{ABC}} f(\rho^{ABC}) < \infty,
\]

which satisfies equation (4). The proof is completed. \( \square \)

Note that almost all entanglement measures so far are continuous [41] and that \( E^\gamma(|\psi\rangle^{A|BC}) \leq E^\gamma(|\psi\rangle^{B|AC}) + E^\gamma(|\psi\rangle^{AB|C}) \) implies \( E^\gamma(|\psi\rangle^{A|BC}) \leq E^\gamma(|\psi\rangle^{B|AC}) + E^\gamma(|\psi\rangle^{AB|C}) \) for any \( \gamma \in [0, \alpha] \). We denote the exponent \( \alpha \) associated with \( E \) by \( \alpha(E) \). According to theorem 1 and the arguments in the proof we know that \( E^\gamma(|\psi\rangle^{A|BC}) \), \( E^\gamma(|\psi\rangle^{B|AC}) \) and \( E^\gamma(|\psi\rangle^{AB|C}) \) can induce a triangle (the case of \( x = y, z = 0 \) is reduced to a line segment which is regarded as a trivial triangle hereafter), which we call it \( E \)-triangle (see in figure 1). Going further, we denote the circumradius and the inscribed radius of the \( E \)-triangle by \( R_E \) and \( r_E \), respectively. We call the triangle, induced by \( E^\gamma(|\psi\rangle^{A|BC}) \), \( E^\gamma(|\psi\rangle^{B|AC}) \) and \( E^\gamma(|\psi\rangle^{AB|C}) \) with \( 0 < \gamma < \alpha(E) \), \( E \)-triangle with exponent \( \gamma \). And we denote the circumradius and the inscribed radius of the \( E \)-triangle with exponent \( \gamma \) by \( R_E^{(\gamma)} \) and \( r_E^{(\gamma)} \), respectively.

For any \(|\psi\rangle \in \mathcal{H}^{ABC}\), we write \( E_{\gamma}^{A|BC} = x \), \( E_{\gamma}^{AB|C} = y \), \( E_{\gamma}^{B|AC} = z \) and denote the area of the \( E \)-triangle of \(|\psi\rangle \) by \( E_{\text{in}}(|\psi\rangle) \). It is worth mentioning that we cannot know whether \( E_{\text{in}} \) is nonincreasing under LOCC (the area of a triangle could increase even though the edges are decreased...
Figure 1. The $E$-triangle for a tripartite system. The lengths of the three edges are set equal to the $\alpha$th power of the three one-to-other bipartite entanglement measured by $E$.

We only know $E_{\text{tri}}(\ket{\psi}) > 0$ if and only if $\ket{\psi}$ is genuinely entangled. We thus turn to consider another quantity $R_E(\ket{\psi}) \cdot r_E(\ket{\psi}) = xyz$ whenever $xyz > 0$. We can verify that $R_E(\ket{\psi}) \cdot r_E(\ket{\psi})$ cannot increase when the edges decreasing (in fact, $\prod_i x_i$ decreases whenever $x_i$'s decrease for any $m, \sum_i x_i > 0$). In addition, in the expression $xyz$, whether $E_{A|BC}, E_{A|BC},$ and $E_{B|AC}$ make up a triangle is not necessary. We thus define

$$F_{123}(\rho) \equiv \begin{cases} \frac{xyz}{x+y+z}, & x+y+z > 0, \\ 0, & x+y+z = 0 \end{cases} \text{ or } F_{123}(\rho) \equiv xyz,$$

for any $\rho \in S_{ABC}$ with $E_{A|BC}(\rho) = x, E_{A|BC}(\rho) = y, E_{B|AC}(\rho) = z$. For any LOCC acting on $\rho$, we assume the output state is $\rho'$. Then

$$F_{123}(\rho') \leq F_{123}(\rho)$$

since $E$ is an entanglement measure which implies that $E^{X|YZ}(\rho') \leq E^{X|YX}(\rho)$ for any possible $\{X, Y, Z\} = \{A, B, C\}$. It is clear that $F_{123}(\rho) = 0$ whenever $\rho$ is fully separable (i.e., three-separable), and that there exists two-separable but not fully separable state $\sigma$ such that $F_{123}(\sigma) = 0$. We thus get the following theorem.

**Theorem 2.** Let $E$ be a bipartite entanglement measure. Then $F_{123}$ is a three-entanglement measure but not a faithful three-entanglement measure.

**Remark 1.** $F_{123}$ is not a genuine entanglement measure in general since there exists two-separable but not fully separable mixed state $\sigma$ such that $F_{123}(\sigma) > 0$ in high-dimension system.
It is immediate to realize that $\mathcal{F}_{123}^F$ is not an entanglement measure any more in general since we cannot guarantee that it is nonincreasing under LOCC even for pure state. When we take

$$\mathcal{E}_{x-123}(\rho) = \delta(xyz)(x + y + z),$$

$$\mathcal{E}_{123}(\rho) = x + y + z,$$

with $x, y, z$ as above and define $\mathcal{E}_{x-123}^F$ and $\mathcal{E}_{123}^F$ as in equation (3), it turns out that $\mathcal{E}_{x-123}(\ket{\psi}) > 0$ if and only if $\ket{\psi}$ is genuinely entangled, and $\mathcal{E}_{123}(\rho) > 0$ if and only if $\rho$ is three-entangled. Here $\delta(x) = 0$ whenever $x = 0$, otherwise $\delta(x) = 1$. On the other hand, one can easily check that $\mathcal{E}_{x-123}^F$, $\mathcal{E}_{123}$ and $\mathcal{E}_{123}^F$ are nonincreasing under LOCC and do not increase on average under stochastic LOCC whenever $E$ is an entanglement monotone. We thus obtain a genuine tripartite entanglement measure.

**Theorem 3.** Let $E$ be a bipartite entanglement measure (resp. monotone). Then $\mathcal{E}_{x-123}^F$ is a genuine tripartite entanglement measure (resp. monotone), $\mathcal{E}_{x-123}$, $\mathcal{E}_{123}$ and $\mathcal{E}_{123}^F$ are three-entanglement measures (resp. monotones) but not genuine entanglement measures (resp. monotones).

**Proof.** The case of $\mathcal{E}_{x-123}^F$ is obvious according to the argument above. For $\mathcal{E}_{x-123}$, $\mathcal{E}_{123}$ and $\mathcal{E}_{123}^F$, if $\rho$ is fully separable, then $\mathcal{E}_{123}(\rho) = \mathcal{E}_{123}^F(\rho) = 0$. In addition, if $\sigma$ is biseparable but not fully separable, then $\mathcal{E}_{123}(\sigma) > 0$, $\mathcal{E}_{123}^F(\sigma) > 0$. The proof is completed. □

**Remark 2.** When we take

$$\sigma = p_1|\psi\rangle\langle\psi|^A \otimes |\psi\rangle\langle\psi|^{BC} + p_2|\phi\rangle\langle\phi|^A \otimes |\phi\rangle\langle\phi|^C + p_3|\xi\rangle\langle\xi|^B \otimes |\xi\rangle\langle\xi|^C$$

with $|\psi\rangle^{BC}$, $|\phi\rangle^{AB}$ and $|\xi\rangle^{AC}$ being entangled, $p_1 + p_2 + p_3 = 1$, $p_1p_2p_3 > 0$, then it is possible that $\mathcal{E}_{x-123}(\sigma) > 0$ for some choice of $|\psi\rangle^A$, $|\psi\rangle^{BC}$, $|\phi\rangle^C$, $|\phi\rangle^{AB}$, $|\xi\rangle^B$, $|\xi\rangle^{AC}$ in high-dimensional system.

We take tangle for example to explain further. We can easily get

$$\tau(|\psi\rangle^{ABC}) \leq \tau(|\psi\rangle^{AB}C) + \tau(|\psi\rangle^{BAC})$$

holds for any $|\psi\rangle^{ABC} \in \mathcal{H}^{ABC}$ since [43, theorem 2]

$$1 + \text{Tr}(\rho^{AB})^2 \geq \text{Tr}(\rho^A)^2 + \text{Tr}(\rho^B)^2.$$ 

It is straightforward that the $\tau$-triangle for the GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is a regular triangle with side length is 1. Thus

$$\mathcal{F}_{123}(\ket{\text{GHZ}}) = \frac{1}{3}, \quad \text{or} \quad \mathcal{F}_{123}^F(\ket{\text{GHZ}}) = 1,$$

$$\tau_{x-123}(\ket{\text{GHZ}}) = 1,$$

$$\tau_{123}(\ket{\text{GHZ}}) = 3.$$  

Here, by $\mathcal{F}_{123}$ we denote the measure $\mathcal{F}_{123}$ induce by $\tau$. For the W state $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, the $\tau$-triangle is a regular triangle with side length is $\frac{1}{3}$ and

$$\mathcal{F}_{123}(\ket{W}) = \frac{64}{243}, \quad \text{or} \quad \mathcal{F}_{123}^F(\ket{W}) = \frac{512}{729},$$

$$\tau_{x-123}(\ket{W}) = \frac{8}{3},$$

$$\tau_{123}(\ket{W}) = \frac{8}{3}.$$
Further, since $\tau_{123}(|GHZ\rangle) > \tau_{123}(|W\rangle)$, and thus $\tau^{F}_{123}$ is also a ‘proper’ genuine tripartite entanglement measure in the sense of reference [28].

4. Four-partite entanglement

In the previous section, we derive $F_{123}$, $E_{g-123}$ and $E_{123}$ from the bipartite entanglement measure $E$. With this spirit in mind, moving to four-partite case, there are two ways to derive entanglement measures: from bipartite entanglement measure $E$ or three-partite entanglement measure $E^{(3)}$.

4.1. From bipartite entanglement measure

For any given $\rho \in S^{ABCD}$ and any bipartite entanglement measure $E$, we let

\[
E(\rho^{ABCD}) = x^{(2)}_{1},
\]

\[
E(\rho^{A|BCD}) = x^{(2)}_{2},
\]

\[
E(\rho^{AC|BD}) = x^{(2)}_{3},
\]

\[
E(\rho^{ABC|D}) = x^{(2)}_{4},
\]

\[
E(\rho^{AD|BC}) = x^{(2)}_{5},
\]

\[
E(\rho^{B|ACD}) = x^{(2)}_{6},
\]

\[
E(\rho^{C|ABD}) = x^{(2)}_{7},
\]

and define

\[
F^{1234}_{1234}(\rho) \equiv \begin{cases}
\prod_{i} x^{(2)}_{i} \sum_{i} x^{(2)}_{i} > 0, & \text{or } F^{1234}_{1234}(\rho) \equiv \prod_{i} x^{(2)}_{i}.
\end{cases}
\tag{17}
\]

Then $F^{1234}_{1234}$ is a four-entanglement measure straightforwardly. Moreover, we let

\[
E^{1234}_{1234}(\rho) \equiv \sum_{i} x^{(2)}_{i}
\tag{18}
\]

and

\[
E^{g-1234}_{1234}(\rho) \equiv \delta \left( \prod_{i} x^{(2)}_{i} \right) \sum_{i} x^{(2)}_{i}.
\tag{19}
\]

Using the similar argument as that of theorem 3, we can check that $E^{1234}_{1234}$, $E^{F}_{1234}_{1234}$ and $E^{g-1234}_{1234}$ are four-entanglement measure while $E^{F}_{g-1234}_{1234}$ is a genuine four-partite entanglement measure. We conclude this argument with the following theorem.

Theorem 4. Let $E$ be a bipartite entanglement measure (resp. monotone). Then

- $F^{1234}_{1234}, E^{1234}_{1234}, E^{F}_{1234}_{1234}$ and $E^{g-1234}_{1234}$ are four-entanglement measures (resp. monotones), but not genuine entanglement measures (resp. monotones).
- $E^{F}_{g-1234}_{1234}$ is a genuine faithful four-partite entanglement measure (resp. monotone).
We illustrate the measures here with $|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ and $|W_4\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$ and the associated bipartite entanglement measure $\tau$. For $|GHZ_4\rangle$, we have $x_i^{(2)} = 1, i = 1, 2, \ldots, 7$, and for $|W_4\rangle, x_1^{(2)} = x_2^{(2)} = x_3^{(2)} = x_4^{(2)} = \frac{3}{4}$. It turns out that 

$$F_{1234(2)}(|GHZ_4\rangle) = \frac{1}{7} \quad \text{or} \quad F_{1234(2)}(|GHZ_4\rangle) = 1,$$

$$E_{1234(2)}(|GHZ_4\rangle) = E_{g - 1234(2)}(|GHZ_4\rangle) = 7,$$

and 

$$F_{1234(2)}(|W_4\rangle) = \frac{27}{512} \quad \text{or} \quad F_{1234(2)}(|W_4\rangle) = \frac{81}{256},$$

$$E_{1234(2)}(|W_4\rangle) = E_{g - 1234(2)}(|W_4\rangle) = 6.$$ 

Obviously, $|GHZ_4\rangle$ is more entangled than $|W_4\rangle$ under these measures.

### 4.2. From tripartite entanglement

For any given $\rho \in S_{ABCD}$ and any tripartite entanglement measure $E^{(3)}$ (e.g., the unified tripartite entanglement measure as in reference [14], or other tripartite entanglement measure as $F_{123}, F_{123}, E_{g - 123}, E_{F - 123}$ discussed in the previous section), we let 

$$E^{(3)}(\rho_{AB|CD}) = x_1^{(3)},$$

$$E^{(3)}(\rho_{A|BCD}) = x_2^{(3)},$$

$$E^{(3)}(\rho_{AC|BD}) = x_3^{(3)},$$

$$E^{(3)}(\rho_{A|BC}) = x_4^{(3)},$$

$$E^{(3)}(\rho_{A|CD}) = x_5^{(3)},$$

and define 

$$F_{1234(3)}(\rho) \equiv \begin{cases} \prod_i x_i^{(3)} & \text{if} \sum_i x_i^{(3)} > 0, \\ \sum_i x_i^{(3)} = 0, \end{cases}$$

or 

$$F_{1234(3)}(\rho) \equiv \prod_i x_i^{(3)}. \quad (24)$$

In addition, let 

$$E^{(3)}(\rho^{PQR}) = \delta(\cdot|\cdot) E^{(3)}(\rho^{PQR})$$

for any three-partition $PQR$ of $ABCD$, $\delta(\cdot|\cdot) = 0$ if $\rho^{ABCD}$ is biseparable up to some bi-partition and $\delta(\cdot|\cdot) = 1$ otherwise. Define 

$$\tilde{E}^{(3)}(\rho_{A|BCD}) = x_1^{(3)},$$

$$\tilde{E}^{(3)}(\rho_{A|BCD}) = x_2^{(3)},$$

$$\tilde{E}^{(3)}(\rho_{A|CD}) = x_3^{(3)},$$

and define 

$$\tilde{F}_{1234(3)}(\rho) \equiv \begin{cases} \prod_i x_i^{(3)} & \text{if} \sum_i x_i^{(3)} > 0, \\ \sum_i x_i^{(3)} = 0, \end{cases}$$

or 

$$\tilde{F}_{1234(3)}(\rho) \equiv \prod_i x_i^{(3)}. \quad (25)$$
\[ \tilde{E}^{(3)}(\rho^{AB|CD}) = x_4^{(3)}, \]
\[ \tilde{E}^{(3)}(\rho^{AD|BC}) = x_5^{(3)}, \]
\[ \tilde{E}^{(3)}(\rho^{AB|DC}) = x_6^{(3)}, \]

and

\[ \tilde{F}_{1234}(\rho) \equiv \begin{cases} \prod_i \tilde{x}_i^{(3)}, & \sum_i \tilde{x}_i^{(3)} > 0, \\ 0, & \sum_i \tilde{x}_i^{(3)} = 0, \end{cases} \]
\[ \tilde{F}_{1234}(\rho) \equiv \prod_i \tilde{x}_i^{(3)}. \] (26)

Analogous to \( E_{-123} \) and \( E_{123}(\rho) \), we define

\[ E_{-1234}(\rho) = \delta(\sum_i x_i^{(3)}) \] (27)

and

\[ E_{1234}(\rho) = \sum_i x_i^{(3)}. \] (28)

It is clear that for any fully separable state \( \rho \in S_{ABCD} \), \( F_{1234}(\rho) = \tilde{F}_{1234}(\rho) = E_{-1234}(\rho) = E_{1234}(\rho) = E_{F1234}(\rho) = 0 \). For any pure state \( |\psi\rangle \in H_{ABCD} \), \( E_{-1234}(|\psi\rangle) > 0 \) if and only if \( |\psi\rangle \) is genuinely entangled. In addition, all these quantities do not increase under any four-partite LOCC since \( E^{(3)} \) is non-increasing under any three-partite LOCC. \( F_{1234}, \tilde{F}_{1234}, E_{1234} \) and \( E_{F1234} \) are not GMEMs in general since these measures are defined with respect to the fixed partition. For instance, \( \rho^{X'YZW} \) is not separable with respect to partition \( X'Y'ZW \) cannot guarantee \( \rho^{ABCD} \) is genuinely entangled. We can now conclude the following.

**Theorem 5.** Let \( E^{(3)} \) be a tripartite entanglement measure (resp. monotone). Then

- \( F_{1234}, \tilde{F}_{1234}, E_{-1234}, \tilde{E}_{1234} \) and \( E^{(3)}_{F1234} \) are four-entanglement measures (resp. monotones).
- \( E^{(3)}_{-1234} \) is a faithful genuine four-partite entanglement measures (resp. monotones).

Simple calculation reveals that, \( \tau^{(3)}(X'|Y'ZW) \) are \( \frac{3}{2} \) and \( \frac{5}{4} \) for \( |GHZ_4\rangle \) and \( |W_4\rangle \) respectively for any tripartition of \( ABCD \). It follows that

\[ F^*_{1234}(|GHZ_4\rangle) = \tilde{F}^*_{1234}(|GHZ_4\rangle) = \frac{81}{64}, \]
\[ F^*_{1234}(|W_4\rangle) = \tilde{F}^*_{1234}(|W_4\rangle) = \frac{3125}{1536} \]

or

\[ F^*_{1234}(|GHZ_4\rangle) = \tilde{F}^*_{1234}(|GHZ_4\rangle) = \frac{27}{8}, \]
\[ F^*_{1234}(|W_4\rangle) = \tilde{F}^*_{1234}(|W_4\rangle) = \frac{125}{64}. \]
any function \( f \) that satisfies the monotonic condition, i.e.,

\[
f(x'_1, x'_2, \ldots, x'_n) \leq f(x_1, x_2, \ldots, x_n)
\]

whenever \( x'_i \leq x_i, i = 1, \ldots, n, n \geq m \), can educe the corresponding entanglement measures. Moreover, if \( f \) satisfies

\[
f(x_1, x_2, \ldots, x_n) \geq \sum_j p_j f(x^{(j)}_1, \ldots, x^{(j)}_n)
\]

for any probability distribution \( \{p_j\} \) and \( \{x^{(j)}_i\} \) such that \( x_i \geq \sum_j p_j x^{(j)}_i \), \( f \) can educe an entanglement measure for pure states and then extending to mixed states via the convex-roof extension. For the genuine measure, it requires

\[
f(x_1, x_2, \ldots, x_n) = 0
\]

whenever the state is biseparable. In general, any \( k \)-partite entanglement measure \( E^{(k)} \) can educe \( m \)-partite entanglement measures, \( 2 \leq k < m \). All of these multipartite measures can be easily followed analogous to that of four-partite case.

5. The tetrahedron structure for the four partite state

For any four-partite state \( \rho^{ABCD} \in S^{ABCD} \), there are two classes of quantities: (i) \( E^{(3)}(A|B|CD) \), \( E^{(3)}(A|BC|D) \), \( E^{(3)}(AB|C|D) \), \( E^{(3)}(AC|B|D) \), \( E^{(3)}(AD|B|C) \) and \( E^{(3)}(A|C|BD) \) derived from any given tripartite entanglement measure \( E^{(3)} \); (ii) \( E(AB|CD), E(AC|BD), E(AD|BC) \) for any given bipartite entanglement measure \( E \). In the following, with the same spirit as that of \( E \)-triangle, we discuss whether these two kinds of quantities can constitute a tetrahedron or triangle respectively.

5.1. Tetrahedron based on tripartition

When it comes to identifying whether six given quantities can constitute a tetrahedron, we firstly need to check the following triangle relation which can also be regarded as a polygamy relation of entanglement.

**Proposition 6.** Let \( E^{(3)} \) be a continuous unified tripartite entanglement measure. Then there exists \( 0 < \alpha < \infty \) such that

\[
E^\alpha(|\psi\rangle^{X|Y|Z}) \leq E^\alpha(|\psi\rangle^{X|Y|W}) + E^\alpha(|\psi\rangle^{X|W|Y})
\]

for all \( |\psi\rangle^{ABCD} \in \mathcal{H}^{ABCD} \) with fixed dim \( \mathcal{H}^{ABCD} = d < \infty \), where \( \{X|Y|ZW\} \) runs over all possible tripartite partition of \( A|B|C|D \), \( \{X, Y, Z, W\} = \{A, B, C, D\} \). Here we omit the superscript \( (3) \) of \( E^{(3)} \) for brevity.
With no loss of generality, we only need to check the three cases above. Hence, we denote by $S_{ABC}$ the set of all states for this subcase in $S_{p}^{ABC}$, and denote the set of all pure states for this subcase in $S_{p}^{ABC}$ by $S_{p}^{ABC}$. Clearly, $S_{ABC} \subseteq S_{p}^{ABC}$. For the case (b), there are three different subcases:

(a) $S_{XYZ} + S_{XYW} + S_{XZW} > S_{YZW}$ for any $\{ X, Y, Z, W \} = \{ A, B, C, D \}$ (see in figure 2).

(b) $S_{XYZ} + S_{XYW} + S_{XZW} = S_{YZW}$ for some $\{ X, Y, Z, W \} = \{ A, B, C, D \}$ (see in figures 3 and 4).

(c) $S_{XYZ} + S_{XYW} + S_{XZW} < S_{YZW}$ for some $\{ X, Y, Z, W \} = \{ A, B, C, D \}$ (see in figure 5).

That is, for the case (a), $S_{RST} > 0$ for any three-partition $R|S|T$ of $ABCD$. $E^{(3)}(\ket{\psi}^{Y|ZW})$ can make up a tetrahedron, $\{X, Y, Z, W\} = \{A, B, C, D\}$ (see in (a) of figure 2). We denote by $S_{ABC}^{p}$ the set of all pure states in $S_{ABC}^{RST}$, and denote the set of all pure states for this subcase in $S_{p}^{ABC}$ by $S_{p}^{ABC}$. Clearly, $S_{ABC}^{RST} \subseteq S_{p}^{ABC}$. For the case (b), there are three different subcases:

(b1) $S_{RST} > 0$ for any three-partition $R|S|T$ of $ABCD$. We denote the set of all states for this subcase in $S_{p}^{ABC}$ by $S_{p}^{ABC}$. (b2) $E^{(3)}(\ket{\psi}^{X|YZ}) = 0$ and $E^{(2)}(\ket{\psi}^{Z|W}) > 0$ for some $\{X, Y, Z, W\} = \{A, B, C, D\}$. In such a situation, $E^{(3)}(\ket{\psi}^{X|YZ})$ reduces to a regular triangle with edge $E^{(2)}(\ket{\psi}^{Z|W})$. (b3) $E^{(3)}(\ket{\psi}^{X|YZ}) = E^{(2)}(\ket{\psi}^{Z|W}) = 0$ for some $\{X, Y, Z, W\} = \{A, B, C, D\}$. In such a situation, $E^{(3)}(\ket{\psi}^{X|YZ})$ reduces to a point, i.e., the state is fully separable. For the case (c), we have $S_{RST} > 0$ for any three-partition $R|S|T$ of $ABCD$. We denote the set of all states for
Figure 3. The triangle structure for a four-partite system in the case of (b1). (Assume that $S_{BCD} = S_{ABC} + S_{ACD} + S_{ABD}$.)

Figure 4. The regular triangle structure for a four-partite system in the case of (b2). (Assume that $E^{(3)}(\langle \psi | \psi \rangle_{A|B|CD}) = 0$ and $E^{(2)}(\langle \psi | \psi \rangle_{CD}) > 0$.)

Figure 5. The triangle structure for a four-partite system in the case of (c). (Assume that $E^{(3)}(\langle \psi | \psi \rangle_{ABCD}) > 0$ for any tripartite partition of $ABCD$ and $S_{BCD} > S_{ABC} + S_{ACD} + S_{ABD}$.)

this subcase in $S^{ABC} \psi_{p}$ by $S^{ABC} \Delta_{-3}$. We conjecture that, both $S^{ABCD}_{\Delta_{-1}}$ and $S^{ABC} \Delta_{-3}$ are not empty sets.

As one may expect, the superficial area or the volume of the tetrahedron above really represents the entanglement contained in the state. However, whether the superficial area or the volume of the tetrahedron can represent as a measures of four partite entanglement is unknown since proving or disproving theses quantities are monotonic under LOCC seems a hard work.
5.2. Triangle based on bipartition

Observing that, almost all bipartite entanglement measure are related to a function of the reduced state for pure state [14, 35, 41]. For example, \( \tau(|\psi\rangle_{AB}) = 2 \left( 1 - \text{Tr}(\rho^2 A) \right) \), \( E_f(|\psi\rangle_{AB}) = S(\rho) \). These measures are in fact determined by the reduced state for pure states.

We start our with the following proposition, which also reports the polygamy relation for the four-partite entanglement.

**Proposition 7.** Let \( E \) be a continuous bipartite entanglement measure that is determined by the eigenvalues of the reduced state. Then there exists \( 0 < \alpha < \infty \) such that

\[
E^\alpha(|\psi\rangle_{XY}^{'ZW}) \leq E^\alpha(|\psi\rangle_{XZ}^{'YW}) + E^\alpha(|\psi\rangle_{XW}^{'YZ})
\]
for all \( |\psi\rangle_{ABCD} \in \mathcal{H}_{ABCD} \) with fixed dim \( \mathcal{H}_{ABCD} = d < \infty \), where \( \{X,Y,Z,W\} \) runs over all possible tripartite partition of \( A B C D \), \( \{X,Y,Z,W\} = \{A,B,C,D\} \).

**Proof.** With no loss of generality, we only need to verify \( E(|\psi\rangle_{ABCD}) = 0 \) implying \( E(|\psi\rangle_{ACBD}) = E(|\psi\rangle_{ADBC}) \). The other cases are similar. If \( E(|\psi\rangle_{ABCD}) = 0 \), then \( |\psi\rangle = |\psi\rangle_{AB} |\psi\rangle_{CD} \). Thus, \( E(|\psi\rangle_{ACBD}) = h(\rho^C) \) and \( E(|\psi\rangle_{ADBC}) = h(\rho^D) \) for some function \( h \). On the other hand, \( \rho_{AC} = \rho^A \otimes \rho^C \), \( \rho_{AD} = \rho^A \otimes \rho^D \), and the eigenvalues of \( \rho^C \) coincides with that of \( \rho^D \) since \( |\psi\rangle = |\psi\rangle_{AB} |\psi\rangle_{CD} \). It follows that \( h(\rho^C) = h(\rho^D) \) since \( h(\rho) \) is only determined by the eigenvalues of \( \rho \). The proof is completed. \( \square \)
By proposition 7, any pure state in $\mathcal{H}^{ABCD}$ can induce a triangle for any bipartite entanglement measure that determined by the reduced state for the pure states (see in figure 6). In addition, $E(AB|CD)$, $E(AC|BD)$, $E(AD|BC)$ can be regarded as 6 quantities $\{E(AB|CD), E(AC|BD), E(AD|BC), E(BC|AD), E(AC|BD), E(AD|BC)\}$ due to $E(XY|ZW) = E(ZW|XY)$, and therefore we can consider whether these 6 quantities can build a tetrahedron. It depends obviously since in such a case it is equivalent to the fact that whether four same triangles can constitute a tetrahedron (namely, four copies of the same triangle can not necessarily constitute a tetrahedron). When it does, the tetrahedron is illustrated in figure 7.

Similar to that of $E$-triangle for tripartite case, whether the area of such a triangle (or, superficial area or volume of tetrahedron if the tetrahedron exists) is a four-partite entanglement measure is unknown. Going further, it is hard to find a symmetric geometric structure for $m$-partite system whenever $m > 4$. That is, the method in reference [28] is hard to be extended into $m$-partite system whenever $m > 3$.

6. Conclusion and discussion

In summary, we have established a scenario for establishing genuine entanglement measures and $m$-entanglement measures for tripartite and four-partite quantum system with any finite dimension of the state space. The method is feasible for $m$-partite system with $m > 4$. The concurrence triangle for three-qubit system in reference [28] was generalized to any higher dimensional tripartite system and shown to be valid for any bipartite entanglement measure. Although the area of the triangle and the superficial area of the tetrahedron can increase whenever the lengths of the edges decrease, whether such area is nonincreasing under LOCC is unknown. If it is nonincreasing under LOCC, then the area is a well-defined GMEM. In addition, equations (4), (32) and (33) indeed report the polygamy relation of the bipartite entanglement in the multipartite state. This relation combined with the monogamy relation reveals the distribution of multipartite entanglement more efficiently.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grants No. 11971277 and No. 11901421, the Program for the Outstanding Innovative Teams of Higher Learning Institutions of Shanxi, the Scientific Innovation Foundation of the Higher Education Institutions of Shanxi Province under Grants No. 2019J034 and No. 2019L0742, and the Science Technology Plan Project of Datong City under Grant No. 2020155.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Yu Guo https://orcid.org/0000-0002-5698-9167
References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)

[2] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Mixed-state entanglement and quantum error correction Phys. Rev. A 54 3824

[3] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Quantum entanglement Rev. Mod. Phys. 81 865

[4] Gühne O and Töth G 2009 Entanglement detection Phys. Rep. 474 1

[5] Burkhart L D et al 2021 Error-detected state transfer and entanglement in a superconducting quantum network PRX Quantum 2 030321

[6] Yu X D, Imai S and Gühne O 2021 Optimal entanglement certification from moments of the partial transpose Phys. Rev. Lett. 127 060504

[7] Plenio M B and Virmani S 2007 An introduction to entanglement measures Quantum Inf. Comput. 7 1

[8] Yang D, Horodecki M and Wang Z D 2008 An additive and operational entanglement measure: conditional entanglement of mutual information Phys. Rev. Lett. 101 140501

[9] Gao T and Hong Y 2010 Detection of genuinely entangled and nonseparable n-partite quantum states Phys. Rev. A 82 062311

[10] Gao T, Yan F and van Enk S J 2014 Permutationally invariant part of a density matrix and nonseparability of N-qubit states Phys. Rev. Lett. 112 180501

[11] Li M, Jia L, Wang J, Shen S and Fei S M 2017 Measure and detection of genuine multipartite entanglement for tripartite systems Phys. Rev. A 96 052314

[12] Ma Z H, Chen Z H, Chen J L, Spengler C, Gabriel A and Huber M 2011 Measure of genuine multipartite entanglement with computable lower bounds Phys. Rev. A 83 062325

[13] Szalay S 2015 Multipartite entanglement measures Phys. Rev. A 92 042329

[14] Guo Y and Zhang L 2020 Multipartite entanglement measure and complete monogamy relation Phys. Rev. A 101 032301

[15] Hong Y, Gao T and Yan F 2012 Measure of multipartite entanglement with computable lower bounds Phys. Rev. A 86 062323

[16] Coffman V, Kundu J and Wootters W K 2000 Distributed entanglement Phys. Rev. A 61 052306

[17] Hiesmayr B C and Huber M 2008 Multipartite entanglement measure for all discrete systems Phys. Rev. A 78 012342

[18] Carvalho A R R, Mintert F and Buchleitner A 2004 Decoherence and multipartite entanglement Phys. Rev. Lett. 93 230501

[19] Jungnitsch B, Moroder T and Gühne O 2011 Taming multiparticle entanglement Phys. Rev. Lett. 106 190502

[20] Verstraete F, Dehaene J and De Moor B 2003 Normal forms and entanglement measures for multipartite quantum states Phys. Rev. A 68 012103

[21] Luque J G and Thibon J Y 2003 Polynomial invariants of four qubits Phys. Rev. A 67 042303

[22] Osterloh A and Siewert J 2005 Constructing N-qubit entanglement monotones from antilinear operators Phys. Rev. A 72 012337

[23] Gour G 2010 Evolution and symmetry of multipartite entanglement Phys. Rev. Lett. 105 190504

[24] Viehmann O, Ettschka C and Siewert J 2011 Polynomial invariants for discrimination and classification of four-qubit entanglement Phys. Rev. A 83 052330

[25] Osterloh A 2009 On polynomial invariants of several qubits J. Math. Phys. 50 033509

[26] Contreras-Tejada P, Palazuelos C and de Vicente J I 2019 Resource theory of entanglement with a unique multipartite maximally entangled state Phys. Rev. Lett. 122 120503

[27] Das S, Bäuml S, Winczewski M and Horodecki K 2020 Universal limitations on quantum key distribution over a network (arXiv: quant-ph/1912.03646v3)

[28] Hashemi Rafsanjani S M, Huber M, Broadbent C J and Eberly J H 2012 Genuinely multipartite concurrence of N-qubit X matrices Phys. Rev. A 86 062303

[29] Sen(De) A and Sen U 2010 Channel capacities versus entanglement measures in multiparty quantum states Phys. Rev. A 81 012308

[30] Sadhukhan D, Roy S S, Pal A K, Rakshit D, Sen(De) A and Sen U 2017 Multipartite entanglement accumulation in quantum states: localizable generalized geometric measure Phys. Rev. A 95 022301
[32] Emary C and Beenakker C W J 2004 Relation between entanglement measures and Bell inequalities for three qubits Phys. Rev. A 69 032317
[33] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 Quantifying entanglement Phys. Rev. Lett. 78 2275
[34] Plenio M B 2005 Logarithmic negativity: a full entanglement monotone that is not convex Phys. Rev. Lett. 95 090503
[35] Vidal G 2000 Entanglement monotones J. Mod. Opt. 47 355
[36] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 Concentrating partial entanglement by local operations Phys. Rev. A 53 2046
[37] Horodecki M 2001 Entanglement measures Quantum Inf. Comput. 1 3
[38] Rungta P and Caves C M 2003 Concurrence-based entanglement measures for isotropic states Phys. Rev. A 67 012307
[39] Qian X-F, Alonso M A and Eberly J H 2018 Entanglement polygon inequality in qubit systems New J. Phys. 20 063012
[40] Zhu X N and Fei S M 2015 Generalized monogamy relations of concurrence for N-qubit systems Phys. Rev. A 92 062345
[41] Gour G and Guo Y 2018 Monogamy of entanglement without inequalities Quantum 2 81
[42] Cheng S M 2021 private communication. For example, for two triangles with edges lengths are 8, 8, 8 and 10, 10, 19.9 respectively, it is easy to see that the area of the former one is bigger than the later one although edges lengths of the former one is smaller than the later one.
[43] Audenaert K M R 2007 Sub additivity of q entropies for q > 1 J. Math. Phys. 48 083507