HEISENBERG UNCERTAINTY INEQUALITY FOR GABOR TRANSFORM

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Abstract. We discuss Heisenberg uncertainty inequality for groups of the form $K \ltimes \mathbb{R}^n$, $K$ is a separable unimodular locally compact group of type I. This inequality is also proved for Gabor transform for several classes of groups of the form $K \ltimes \mathbb{R}^n$.

1. Introduction

The uncertainty principle states that a non-zero function and its Fourier transform cannot both be sharply localized. The most precise way of formulating this principle quantitatively is the inequality known as Heisenberg uncertainty inequality. Let $f$ be any function in $L^2(\mathbb{R})$. The Fourier transform of $f$ is defined as

$$\hat{f}(\omega) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \omega x} dx.$$  

The following theorem gives the Heisenberg uncertainty inequality for the Fourier transform on $\mathbb{R}$:

**Theorem 1.1.** For any $f \in L^2(\mathbb{R})$, we have

$$\frac{\|f\|_2^2}{4\pi} \leq \left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} \omega^2 |\hat{f}(\omega)|^2 \, d\omega \right)^{1/2},$$

where $\| \cdot \|_2$ denotes the $L^2$-norm.

For proof of the theorem, refer to [5].

The representation of $f$ as a function of $x$ is usually called its *time-representation*, while the representation of $\hat{f}$ as a function of $\omega$ is called its *frequency-representation*. The Fourier transform has been the most commonly used tool for analyzing frequency properties of a given signal, but the problem with this tool is that after transformation, the information about time is lost and it is hard to tell where a certain frequency occurs. To counter this problem, we can use *joint time-frequency representation*, i.e., Gabor transform.

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Let $\psi \in L^2(\mathbb{R})$ be a fixed non-zero function usually called a window function. The Gabor transform of a function $f \in L^2(\mathbb{R})$ with respect to the window function $\psi$ is defined by

$$G_\psi f : \mathbb{R} \times \hat{\mathbb{R}} \to \mathbb{C}$$

such that

$$G_\psi f(t, \omega) = \int_{\mathbb{R}} f(x) \overline{\psi(x-t)} e^{-2\pi i \omega x} \, dx,$$

for all $(t, \omega) \in \mathbb{R} \times \hat{\mathbb{R}}$. The following uncertainty inequality of Heisenberg-type has been proved by Wilczok [13].

**Theorem 1.2.** Let $\psi$ be a window function. Then, for arbitrary $f \in L^2(\mathbb{R})$, the following inequality holds

$$\frac{\|\psi\|^2 \|f\|^2}{4\pi} \leq \left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^2} \omega^2 |G_\psi f(t, \omega)|^2 \, dt \, d\omega \right)^{1/2}.$$

(1.2)

The continuous Gabor transform for second countable, non-abelian, unimodular and type I groups has been defined by Farashahi and Kamyabi-Gol in [3].

In section 2, we shall state the Heisenberg uncertainty inequality for the groups of the form $K \ltimes \mathbb{R}^n$, where $K$ is a separable unimodular locally compact group of type I and prove it for the semi-direct product $K \ltimes \mathbb{R}^n$ (where $K$ is a compact subgroup of the group of automorphisms of $\mathbb{R}^n$). In section 3, we shall discuss continuous Gabor transform and prove Heisenberg uncertainty inequality for Gabor transform on $K \ltimes \mathbb{R}^n$ (where $K$ is a separable unimodular locally compact group of type I) that satisfy the Heisenberg uncertainty inequality for Fourier transform. The explicit forms of Heisenberg uncertainty inequality for Gabor transform are obtained for $K \ltimes \mathbb{R}^n$, $K$ is a compact subgroup of Aut($\mathbb{R}^n$); $\mathbb{R}^n \times K$, $K$ is separable unimodular locally compact group of type I; Heisenberg group $\mathbb{H}_n$; Thread-like nilpotent Lie groups; 2-NPC nilpotent Lie groups and several classes of connected, simply connected nilpotent Lie groups.

2. **Extensions of $\mathbb{R}^n$**

Let $G = K \ltimes \mathbb{R}^n$, where $K$ is a separable unimodular locally compact group of type I. For $\gamma \in \mathbb{R}^n$, let $G_\gamma, K_\gamma$ denote the stabilizer subgroup of $\gamma$ in $G$ and $K$ respectively and let

$$\hat{G}_\gamma = \{ \nu \in \hat{G}_\gamma : \nu|_{\mathbb{R}^n} \text{ is a finite multiple of } \gamma \}.$$

Then for $\nu \in \hat{G}_\gamma$, the representation $\pi_\nu = \text{ind}_{G_\gamma}^G \nu$ is irreducible and

$$\hat{G} = \bigcup_{\hat{G}_\gamma/G} \{ \pi_\nu : \nu \in \hat{G}_\gamma \}.$$
Since $\mathbb{R}^n$ is abelian, any $\nu \in \hat{G}$ is of the form $\nu = \sigma \otimes \gamma$, $\nu(kx) = \sigma(k) \gamma(x)$, $k \in K$, $x \in \mathbb{R}^n$ and $\sigma \in \hat{K}$.

We consider the induced representations $\pi_{\gamma, \sigma} = \text{ind}_{G}^{\hat{G}}(\gamma \otimes \sigma)$.

The Plancherel formula for $G$ (for details, see [7]) takes the following form:

**Proposition 2.1 (Plancherel formula).** For all $f \in L^2(G)$, we have

$$\int_{G} |f(g)|^2 \, dg = \int_{\mathbb{R}^n / G} \int_{\hat{K}} \|\pi_{\gamma, \sigma}(f)\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\mu_{\mathbb{R}^n}(\gamma).$$

We now state the Heisenberg uncertainty inequality for $G$ which has been proved, in particular cases of $\mathbb{R}^n$ (see [5]); Heisenberg group (see [12],[10] and [14]); $\mathbb{R}^n \times K$ (where $K$ is a separable unimodular locally compact group of type I), Euclidean motion group $M(n) = SO(n) \times \mathbb{R}^n$ and several general classes of nilpotent Lie groups which include thread-like nilpotent Lie groups, 2-NPC nilpotent Lie groups and several low-dimensional nilpotent Lie groups (see [2]).

**Theorem 2.2.** For any $f \in L^2(G)$ and $a, b \geq 1$, we have

$$\|f\|_2^{\frac{1}{2+a}} \leq C \left( \int_{K \times \mathbb{R}^n} \|x\|^{2a} |f(k,x)|^2 \, dx \, dk \right)^{\frac{1}{2a}}$$

$$\times \left( \int_{\mathbb{R}^n / G} \int_{\hat{K}} \|\gamma\|^{2b} \|\pi_{\gamma, \sigma}(f)\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\mu_{\mathbb{R}^n}(\gamma) \right)^{\frac{1}{2b}}, \quad \text{H},$$

where $C$ is a constant.

We do not know whether the inequality (H) is true for $K \times \mathbb{R}^n$, however we now prove the Heisenberg uncertainty inequality when $K$ is a compact subgroup of $\text{Aut}(\mathbb{R}^n)$.

Let $G$ be the semi-direct product $K \times \mathbb{R}^n$, where $K$ is a compact subgroup of $\text{Aut}(\mathbb{R}^n)$. The Haar measure on $G$ is $dg = d\nu(k) \, dx$, where $d\nu(k)$ denotes the normalized Haar measure of $K$ and $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$. We shall now give more explicit description of the unitary dual space of the group $G$ in this case which can be determined by Mackey’s theory. For more details, refer to [3].

Let $\ell$ be a non-zero real linear form on $\mathbb{R}^n$ and let $\chi_\ell$ be the unit character of $\mathbb{R}^n$ defined by $\chi_\ell(x) = e^{i\langle \ell, x \rangle}$. The natural action $g \cdot \ell$ of $G$ on the dual vector space of $\mathbb{R}^n$ is given by

$$\langle g \cdot \ell, x \rangle = \langle \ell, g^{-1} x g \rangle,$$

for $g \in G$ and $x \in \mathbb{R}^n$. Therefore, if $g$ acts on $\mathbb{R}^n$ by

$$g \cdot \chi_\ell(x) := \chi_\ell(g^{-1} x g),$$

we get $g \cdot \chi_\ell = \chi_{g \ell}$. Define

$$K_\ell = \{ k \in K : k \cdot \chi_\ell = \chi_\ell \}.$$
Then, the subgroup $K \ltimes R^n$ is the stabilizer of $\chi_\ell$ in $G$. We take the normalized Haar measure $d\nu_\ell$ on $K$ and a normalized $K$-invariant measure $d\hat{\nu}_\ell$ on $K/K$ so that

$$\int_K \xi(k) d\nu(k) = \int_{K/K} \int_K \xi(kk') d\nu_\ell(k') d\hat{\nu}_\ell(kK).$$

Regarding the action of $K$ on $\hat{R}^n$ which is isomorphic to $R^n$, we set by $d\bar{\ell}$ the image of the Lebesgue measure on $R^n/K$ by the canonical projection $R^n \ni \ell \rightarrow \bar{\ell} := K.\ell \in R^n/K$ such that

$$\int_{R^n} \phi(\ell) d\ell = \int_{R^n/K} \int_K \phi(k.\ell) d\nu(k) d\bar{\ell}.$$

Let $\sigma$ be an irreducible unitary representation of $K$ and $H_\ell,\sigma$ be the completion of the vector space of all continuous mapping $\xi : K \rightarrow H_\sigma$ which satisfies $\xi(ks) = \sigma(s)^* \xi(k)$ for $k \in K$ and $s \in K$ with respect to the norm

$$\|\xi\|_2 = \left( \int_K \|\xi(k)\|_{H_\sigma}^2 d\nu(k) \right)^{1/2}.$$

The induced representation

$$\pi_{\ell,\sigma} := \text{ind}_{K_\ell \ltimes R^n}^G (\sigma \otimes \chi_\ell),$$

realized on the Hilbert space $\mathcal{H}_{\ell,\sigma}$ by

$$\pi_{\ell,\sigma}(k,x) \xi(s) = e^{i(s^{-1}xk)} \xi(k^{-1}s) = e^{i(s.\ell,x)} \xi(k^{-1}s),$$

for $\xi \in \mathcal{H}_{\ell,\sigma}$, $(k,x) \in G$ and $s \in K$, is an irreducible representation of $G$. Furthermore, every infinite dimensional irreducible unitary representation of $G$ is equivalent to some representation $\pi_{\ell,\sigma}$.

The Plancherel formula [4, Theorem 7.44] can be stated in this particular case as follows:

**Proposition 2.3 (Plancherel formula)**. Let $f \in L^1(G) \cap L^2(G)$, then

$$\int_{K \times R^n} |f(k,x)|^2 dx dk = \int_{R^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\pi_{\ell,\sigma}(f)\|^2_{\text{HS}} d\bar{\ell}. \quad (2.1)$$

We shall now establish Heisenberg uncertainty inequality for Fourier transform on $G$. A particular case for the Euclidean motion group has been proved in [2].

**Theorem 2.4**. For any $f \in L^2(G)$ and $a, b \geq 1$, we have

$$\|f\|_{L^2(G)}^{1+b/a} \leq C \left( \int_{K \times R^n} \|x\|^{2a} |f(k,x)|^2 dx dk \right)^{1/2} \times \left( \int_{R^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\ell\|^{2b} \|\pi_{\ell,\sigma}(f)\|^2_{\text{HS}} d\bar{\ell} \right)^{1/2}, \quad (2.2)$$

where $C$ is a constant.
Proof. Define the norm $\| \cdot \|$ on $L^2(G)$ as
\[
\|f\| = \left( \int_{K \times \mathbb{R}^n} \left( 1 + \|x\|^{2a} \right)|f(k,x)|^2 \, dx \, dk \right)^{1/2} \\
+ \left( \int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\ell} \|f_\ell,\sigma(f)\|_{\text{HS}}^2 \, d\ell \right)^{1/2}.
\]
Then, the set $B = \{ f \in L^2(G) : \|f\| < \infty \}$ forms a Banach space which is contained in $L^2(G)$. If $0 \neq f \in L^2(G) \setminus B$, then the right hand side of the inequality (2.2) is always $+\infty$ and the inequality is trivially valid.

Let $S(G)$ denote the space of $C^\infty$-functions which are rapidly decreasing on $G$. It can be shown that $S(G)$ is dense in $B$. Thus it suffices to prove the inequality (2.2) for functions in $S(G)$.

Let $f \in S(G)$. Assuming that both the integrals on right hand side of (2.2) are finite, we have
\[
\int_{\mathbb{R}^n} |f(k,x)|^2 \, dx < \infty, \text{ for all } k \in K.
\]
For $k \in K$, we define $f_k(x) = f(k,x)$, for every $x \in \mathbb{R}^n$. Clearly, $f_k \in L^2(\mathbb{R}^n)$, for all $k \in K$.

Taking $x = (x_1,x_2,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n)$ and proceeding as in the case of Euclidean motion group (see [2, Theorem 2.2]), we obtain
\[
\frac{\|f\|_2^2}{4\pi} \leq \left( \int_{K \times \mathbb{R}^n} \|x\|^{2a} \left| f(k,x) \right|^2 \, dx \, dk \right)^{1/2} \left( \|f\|_2 \right)^{1/2 - \frac{1}{2a}} \\
\times \left( \int_{K \times \mathbb{R}^n} |y_1|^2 \left| \hat{f}_\ell(y) \right|^2 \, dy \, dk \right)^{1/2}. \tag{2.3}
\]
Now, using Plancherel formula on $\mathbb{R}^n$, we have
\[
\int_{K \times \mathbb{R}^n} |y_1|^2 \left| \hat{f}_\ell(y) \right|^2 \, dy \, dk \\
= \int_{K \times \mathbb{R}^n} |y_1|^2 \left| \int_{\mathbb{R}^n} f(k,x) \, e^{-2\pi i \ell \cdot x} \, dx \right|^2 \, dy \, dk \\
= \int_{K \times \mathbb{R}^n} |y_1|^2 \left| \mathcal{F}_{2,3,\ldots,n+1} f(k,y_1,y_2,\ldots,y_n) \right|^2 \, dy_1 \, dy_2 \ldots \, dy_n \\
= \int_{K \times \mathbb{R}^n} |y_1|^2 \left| \mathcal{F}_{2} f(k,y_1,x_2,\ldots,x_n) \right|^2 \, dy_1 \, dx_2 \ldots \, dx_n \, dk, \tag{2.4}
\]
where $\mathcal{F}_i$ denotes the Fourier transform in the $i^{\text{th}}$ variable.

Since $\frac{\partial f}{\partial x_1} \in S(G)$, we have
\[
\int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_1}(k,x_1,x_2,\ldots,x_n) \right|^2 \, dx_1 < \infty,
\]
for all $k \in K$ and $x_i \in \mathbb{R}$ ($i = 2, 3, \ldots, n$).
So, $y_1 \mathcal{F}_2 f(k, y_1, x_2, \ldots, x_n) \in L^2(\mathbb{R})$ and
\[
\left( \frac{\partial f}{\partial x_1}(k, x_1, x_2, \ldots, x_n) \right) (y_1) = 2\pi i y_1 \mathcal{F}_2 f(k, y_1, x_2, \ldots, x_n),
\]
for all $k \in K$ and $x_i \in \mathbb{R}$ ($i = 2, 3, \ldots, n$). Then
\[
\int_{\mathbb{R}} |y_1|^2 |\mathcal{F}_2 f(k, y_1, x_2, \ldots, x_n)|^2 \, dy_1 = \frac{1}{4\pi^2} \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_1}(k, x_1, x_2, \ldots, x_n) \right|^2 \, dx_1.
\]
Using Proposition 2.3, we have
\[
\left| \mathcal{F}_2 f(k, y_1, x_2, \ldots, x_n) \right|^2 = \left| \int_{\mathbb{R}^n} f(x_1, x_2, \ldots, x_n) e^{i \ell, \sigma} dx_1 \right|^2 = \sum_{\sigma} \left| \mathcal{F}_2 f(k, x_2, \ldots, x_n) \right|^2 \left| \left( \frac{\partial f}{\partial x_1} \right)_{\sigma} \right|^2_{HS} \, d\ell.
\]
Combining (2.3), (2.4) and (2.5), we obtain
\[
\left( \int_{\mathbb{R}^n} |x|^{2a} |f(k, x)|^2 \, dx \, dk \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^n} \left[ \left( \int_{\mathbb{R}^n} |x|^{2a} |f(k, x)|^2 \, dx \, dk \right)^{\frac{1}{2}} \right]^2 \, d\ell \right)^{\frac{1}{2}},
\]
which implies
\[
\left( \int_{\mathbb{R}^n} |x|^{2a} |f(k, x)|^2 \, dx \, dk \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^n} \left[ \left( \int_{\mathbb{R}^n} |x|^{2a} |f(k, x)|^2 \, dx \, dk \right)^{\frac{1}{2}} \right]^2 \, d\ell \right)^{\frac{1}{2}}.
\]
For each non-zero linear form $\ell$ on $\mathbb{R}^n$ and each irreducible unitary representation $\sigma$ of $K_\ell$, consider the representation $\pi_{\ell, \sigma}$ realized on the Hilbert space $\mathcal{H}_{\ell, \sigma}$ as
\[
\pi_{\ell, \sigma}(k, x) \xi(s) = e^{i(\ell, s^{-1} x) s} \xi(k^{-1} s) = e^{i(s, \ell, x)} \xi(k^{-1} s),
\]
for $\xi \in \mathcal{H}_{\ell, \sigma}$, $(k, x) \in G$ and $s \in K$. Since $f \in S(G)$, we observe that
\[
\pi_{\ell, \sigma} \left( \frac{\partial f}{\partial x_1} \right) \xi(s) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(k, x_1, x_2, \ldots, x_n) \pi_{\ell, \sigma}(k, x_1, x_2, \ldots, x_n)^* \xi(s) \, dx_1 \, dx_2 \ldots \, dx_n \, dk.
\]
Let \(e_1 = \{1, 0, 0, \ldots, 0\} \in \mathbb{R}^n\), then we can write

\[
\pi_{\ell, \sigma}(k, x_1 - h, x_2, \ldots, x_n) \xi(s) = \pi_{\ell, \sigma}(k, x - he_1) \xi(s)
\]

\[
= e^{-i(\ell, s^{-1}(x-he_1))} \xi(k^{-1}s)
\]

\[
= e^{i(\ell, s^{-1}(he_1))} e^{-i(\ell, s^{-1}xs)} \xi(k^{-1}s)
\]

\[
= e^{ih(\ell, s^{-1}e_1s)} \pi_{\ell, \sigma}(k, x, x_2, \ldots, x_n) \xi(s).
\]

Equation (2.7) can be written as

\[
\pi_{\ell, \sigma} \left( \frac{\partial f}{\partial x_1} \right) \xi(s)
\]

\[
= \lim_{h \to 0} \left[ \frac{e^{ih(\ell, s^{-1}e_1s)} - 1}{h} \right] \int_{K \times \mathbb{R}^n} f(k, x_1, x_2, \ldots, x_n) \times \pi_{\ell, \sigma}(k, x_1, x_2, \ldots, x_n) \xi(s) \, dx_1 \, dx_2 \ldots \, dx_n \, dk
\]

\[
= \lim_{h \to 0} \left[ \frac{e^{ih(\ell, s^{-1}e_1s)} - 1}{h} \right] \pi_{\ell, \sigma}(f) \xi(s)
\]

\[
= i(\ell, s^{-1}e_1s) \pi_{\ell, \sigma}(f) \xi(s).
\]

Since \(s \mapsto s^{-1}e_1s\) is a continuous map from \(K\) to \(\mathbb{R}^n\), so \(\{s^{-1}e_1s : s \in K\}\) is bounded. For any orthonormal basis \(\{\xi_j\}\) of \(\mathcal{H}_{\ell, \sigma}\), we have

\[
\left\| \pi_{\ell, \sigma} \left( \frac{\partial f}{\partial x_1} \right) \right\|_{\text{HS}}^2 = \sum_j \int_K |i(\ell, s^{-1}e_1s) \pi_{\ell, \sigma}(f) \xi_j(s)|^2 \, ds
\]

\[
\leq \text{const.} \, \|\ell\|^2 \sum_j \int_K |\pi_{\ell, \sigma}(f) \xi_j(s)|^2 \, ds = \text{const.} \, \|\ell\|^2 \|\pi_{\ell, \sigma}(f)\|_{\text{HS}}^2.
\]
So, (2.6) can be written as
\[
\|f\|_2^{1+\frac{1}{\sigma}} \leq C \left( \int_{K \times \mathbb{R}^n} |x|^{2\alpha} |f(k,x)|^2 \, dx \, dk \right)^\frac{1}{1+\frac{1}{\sigma}} \left( \int_{\mathbb{R}^n/K} \|\ell\|^2 \|\pi_{\ell,\sigma}(f)\|_{HS}^2 \, d\ell \right)^{1/2}.
\]

Using Hölder’s inequality, we have
\[
\left( \int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\ell\|^{2\beta} \|\pi_{\ell,\sigma}(f)\|_{HS}^2 \, d\ell \right)^{\frac{1}{\beta}} \left( \int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\pi_{\ell,\sigma}(f)\|_{HS}^2 \, d\ell \right)^{1-\frac{1}{\beta}}
\geq \int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\ell\|^2 \|\pi_{\ell,\sigma}(f)\|_{HS}^2 \, d\ell,
\]
which implies
\[
\int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\ell\|^2 \|\pi_{\ell,\sigma}(f)\|_{HS}^2 \, d\ell \leq \left( \int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\ell\|^{2\beta} \|\pi_{\ell,\sigma}(f)\|_{HS}^2 \, d\ell \right)^{\frac{1}{\beta}} (\|f\|_2^{2\beta})^{1-\frac{1}{\beta}}.
\]

Combining (2.8) and (2.9), we obtain
\[
\|f\|_2^{\left(\frac{1}{\sigma} + \frac{1}{\beta}\right)} \leq C \left( \int_{K \times \mathbb{R}^n} |x|^{2\alpha} |f(k,x)|^2 \, dx \, dk \right)^\frac{1}{\alpha+\frac{1}{\sigma}} \left( \int_{\mathbb{R}^n/K} \sum_{\sigma \in \hat{K}_\ell} \|\ell\|^{2\beta} \|\pi_{\ell,\sigma}(f)\|_{HS}^2 \, d\ell \right)^{\frac{1}{\beta}}.
\]

3. Continuous Gabor Transform

Let \( \mathcal{H} \) be a separable Hilbert space and let \( \mathcal{B}(\mathcal{H}) \) denotes the set of all bounded linear operators on \( \mathcal{H} \). An operator \( T \in \mathcal{B}(\mathcal{H}) \) is called Hilbert-Schmidt operator if and only if
\[
\sum_k \|Te_k\|^2 < \infty,
\]
for some, and hence for any, orthonormal basis \( \{e_k\} \) of \( \mathcal{H} \). We denote the set of all Hilbert-Schmidt operators on \( \mathcal{H} \) by \( HS(\mathcal{H}) \). For each \( T \in HS(\mathcal{H}) \), the Hilbert-Schmidt norm \( \|T\|_{HS} \) of \( T \) is defined as
\[
\|T\|_{HS}^2 := \sum_k \|Te_k\|^2.
\]
Also, \( HS(\mathcal{H}) \) forms a Hilbert space with the inner product given by
\[
\langle T, S \rangle_{HS(\mathcal{H})} = tr(S^*T).
\]

For more details, refer to \[4\].
Let $G$ be a second countable, non-abelian, unimodular and type I group. Let $dx$ be the Haar measure on $G$. Let $d\pi$ be the Plancherel measure on $\hat{G}$. For each $(x, \pi) \in G \times \hat{G}$, let

\[ \mathcal{H}_{(x, \pi)} = \pi(x) \text{HS}(\mathcal{H}_\pi), \]

where $\pi(x) \text{HS}(\mathcal{H}_\pi) = \{ \pi(x)T : T \in \text{HS}(\mathcal{H}_x) \}$. Then, $\mathcal{H}_{(x, \pi)}$ is a Hilbert space with the inner product given by

\[ \langle \pi(x)T, \pi(x)S \rangle_{\mathcal{H}_{(x, \pi)}} = \text{tr} (S^*T) = \langle T, S \rangle_{\text{HS}(\mathcal{H}_x)}. \]

One can easily verify that $\mathcal{H}_{(x, \pi)} = \text{HS}(\mathcal{H}_\pi)$ for all $(x, \pi) \in G \times \hat{G}$. The family $\{ \mathcal{H}_{(x, \pi)} \}_{(x, \pi) \in G \times \hat{G}}$ of Hilbert spaces indexed by $G \times \hat{G}$ is a field of Hilbert spaces over $G \times \hat{G}$. Let $\mathcal{H}^2(G \times \hat{G})$ denote the direct integral of $\{ \mathcal{H}_{(x, \pi)} \}_{(x, \pi) \in G \times \hat{G}}$ with respect to the product measure $dx \, d\pi$, i.e., the space of all measurable vector fields $F$ on $G \times \hat{G}$ such that

\[ \| F \|_{\mathcal{H}^2(G \times \hat{G})}^2 = \int_{G \times \hat{G}} \| F(x, \pi) \|^2_{(x, \pi)} \, dx \, d\pi < \infty. \]

$\mathcal{H}^2(G \times \hat{G})$ is a Hilbert space with the inner product given by

\[ \langle F, K \rangle_{\mathcal{H}^2(G \times \hat{G})} = \int_{G \times \hat{G}} \text{tr} [F(x, \pi)K(x, \pi)^*] \, dx \, d\pi. \]

Let $f \in C_c(G)$, the set of all continuous complex-valued functions on $G$ with compact supports and $\psi$ be a fixed non-zero function in $L^2(G)$ which is sometimes called a window function. For $(x, \pi) \in G \times \hat{G}$, the continuous Gabor Transform of $f$ with respect to the window function $\psi$ can be defined as a measurable field of operators on $G \times \hat{G}$ by

\[ G_\psi f(x, \pi) := \int_G f(y) \overline{\psi(x^{-1}y)} \pi(y)^* \, dy. \] (3.1)

The operator-valued integral (3.1) is considered in the weak-sense, i.e., for each $(x, \pi) \in G \times \hat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$, we have

\[ \langle G_\psi f(x, \pi) \xi, \eta \rangle = \int_G f(y) \overline{\psi(x^{-1}y)} \langle \pi(y)^* \xi, \eta \rangle \, dy. \]

For each $x \in G$, define $f_x^\psi : G \to \mathbb{C}$ by

\[ f_x^\psi(y) := f(y) \overline{\psi(x^{-1}y)}. \]

Since, $f \in C_c(G)$ and $\psi \in L^2(G)$, we have $f_x^\psi \in L^1(G) \cap L^2(G)$, for all $x \in G$. The Fourier transform is given by

\[ \hat{f}_x^\psi(\pi) = \int_G f_x^\psi(y) \pi(y)^* \, dy = \int_G f(y) \overline{\psi(x^{-1}y)} \pi(y)^* \, dy = G_\psi f(x, \pi). \]

Also, using Plancherel theorem [1] Theorem 7.44], we see that $\hat{f}_x^\psi(\pi)$ is a Hilbert-Schmidt operator for almost all $\pi \in \hat{G}$. Therefore, $G_\psi f(x, \pi)$ is a
Hilbert-Schmidt operator for all $x \in G$ and for almost all $\pi \in \widehat{G}$. As in \cite{3}, for $f \in C_c(G)$ and a window function $\psi \in L^2(G)$, we have

$$\|G_\psi f\|_{\mathcal{H}^2(G \times \widehat{G})} = \|\psi\|_2 \|f\|_2.$$  

The above equality shows that the continuous Gabor transform $G_\psi : C_c(G) \rightarrow \mathcal{H}^2(G \times \widehat{G})$ defined by $f \mapsto G_\psi f$ is a multiple of an isometry. So, we can extend $G_\psi$ uniquely to a bounded linear operator from $L^2(G)$ into a closed subspace $H$ of $\mathcal{H}^2(G \times \widehat{G})$ which we still denote by $G_\psi$ and this extension satisfies

$$\|G_\psi f\|_{\mathcal{H}^2(G \times \widehat{G})} = \|\psi\|_2 \|f\|_2, \quad (3.2)$$

for each $f \in L^2(G)$. We now prove an important lemma.

**Lemma 3.1.** Let $f \in L^2(G)$ and $\psi \in L^2(G)$ be a window function. Then

$$G_\psi f(x, \pi) = f_x^\psi(\pi).$$

**Proof.** Let $f \in L^2(G)$. Since $C_c(G)$ is dense in $L^2(G)$, there exists a sequence \{\phi_n\} in $C_c(G)$ such that $f = \lim_{n \to \infty} \phi_n$ in the $L^2$-norm. It follows that

$$G_\psi : L^2(G) \rightarrow H \subseteq \mathcal{H}^2(G \times \widehat{G})$$

satisfies $G_\psi f = \lim_{n \to \infty} G_\psi \phi_n$ in the $\mathcal{H}^2(G \times \widehat{G})$-norm and

$$G_\psi \phi_n(x, \pi) = (\widehat{\phi_n})_x^\psi(\pi).$$

Now,

$$\|G_\psi f - G_\psi \phi_n\|_{\mathcal{H}^2(G \times \widehat{G})}^2 = \int_G \int_{\widehat{G}} \|G_\psi f(x, \pi) - G_\psi \phi_n(x, \pi)\|_{\text{HS}}^2 \, dx \, d\pi$$

and

$$\|f\|_2 \|f - \phi_n\|_2 = \int_G |\psi(x)|^2 \int_G |(f - \phi_n)(y)|^2 \, dy$$

and

$$\int_G |(f - \phi_n)(y)|^2 \int_G |\psi(x^{-1}y)|^2 \, dx \, dy$$

and

$$\int_G |\psi(x^{-1}y) - \phi_n(y)\psi(x^{-1}y)|^2 \, dx \, dy$$

and

$$\int_G \int_G |f(y)\overline{\psi(x^{-1}y)} - \phi_n(y)\overline{\psi(x^{-1}y)}|^2 \, dy \, dx$$

and

$$\int_G \int_G |(f_x^\psi - (\phi_n)_x^\psi)(y)|^2 \, dy \, dx$$

and

$$\int_G \int_{\widehat{G}} \|f_x^\psi(\pi) - (\phi_n)_x^\psi(\pi)\|_{\text{HS}}^2 \, dx \, d\pi.$$ 

Hence, $G_\psi f(x, \pi) = f_x^\psi(\pi)$ for all $f \in L^2(G)$. \qed
We now establish Heisenberg uncertainty inequality for Gabor transform. Let $G = K \rtimes \mathbb{R}^n$, where $K$ is a separable unimodular locally compact group of type I. The continuous Gabor Transform of $f$ with respect to the window function $\psi$ can be defined as follows:

$$G\psi f(u, t, \gamma, \sigma) := \int_G f_{u,t}^\psi(k, x) \pi_{\gamma,\sigma}(k, x)^* dx dk,$$

where $f_{u,t}^\psi(k, x) = f(k, x) \psi(ku^{-1}, x-t)$, $(u, t) \in G$, $\gamma \in \hat{\mathbb{R}}^n$ and $\sigma \in \hat{K}_\gamma$. Also, the equality in Lemma 3.1 takes the following form:

$$G\psi f(u, t, \gamma, \sigma) = \pi_{\gamma,\sigma}(f_{u,t}^\psi).$$

**Theorem 3.2.** Let $G = K \rtimes \mathbb{R}^n$ satisfies the inequality (H) and $\psi$ be a window function. For $a, b \geq 1$, we have

$$\|\psi\|_2^{\frac{1}{2}} \|f\|_2^{\left(\frac{1}{a} + \frac{1}{b}\right)} \leq C \left( \int_{K \times \mathbb{R}^n} |x|^{2a} |f(k, x)|^2 dx dk \right)^{\frac{1}{2a}} \times \left( \int_{\hat{\mathbb{R}}^n/G} \left( \int_{\hat{K}_\gamma} \|\gamma\|^2 |G\psi f(u, t, \gamma, \sigma)|^2_{HS} d\mu_\gamma(\sigma) d\mathcal{H}_{\hat{R}^n}(\gamma) du dt \right)^{\frac{1}{2b}}. $$

(3.5)

**Proof.** Assume that both integrals on the right-hand side of (3.5) are finite. Since $f_{u,t}^\psi \in L^2(G)$ for all $(u, t) \in G$, so by using inequality (H) for $a = b = 1$, we have

$$\|f_{u,t}^\psi\|_2^2 \leq C \left( \int_{\mathbb{R}^n/G} \|f_{u,t}^\psi(k, x)|^2 dx dk \right)^{1/2} \times \left( \int_{\hat{\mathbb{R}}^n/G} \left( \int_{\hat{K}_\gamma} \|\gamma\|^2 |\pi_{\gamma,\sigma}(f_{u,t}^\psi)|^2_{HS} d\mu_\gamma(\sigma) d\mathcal{H}_{\hat{R}^n}(\gamma) \right) \right)^{1/2}. $$

(3.6)

Also, by Proposition 2.3 and (3.4), we have

$$\int_{\hat{\mathbb{R}}^n/G} \int_{\hat{K}_\gamma} |G\psi f(u, t, \gamma, \sigma)|^2_{HS} d\mu_\gamma(\sigma) d\mathcal{H}_{\hat{R}^n}(\gamma) = \int_{\hat{\mathbb{R}}^n/G} \left( \int_{\hat{K}_\gamma} \|\gamma\|^2 |\pi_{\gamma,\sigma}(f_{u,t}^\psi)|^2_{HS} d\mu_\gamma(\sigma) d\mathcal{H}_{\hat{R}^n}(\gamma) \right) = \|f_{u,t}^\psi\|_2^2. $$

(3.7)

On combining (3.6) and (3.7), we obtain

$$\int_{\hat{\mathbb{R}}^n/G} \int_{\hat{K}_\gamma} |G\psi f(u, t, \gamma, \sigma)|^2_{HS} d\mu_\gamma(\sigma) d\mathcal{H}_{\hat{R}^n}(\gamma) \leq C \left( \int_{K \times \mathbb{R}^n} |x|^2 |f_{u,t}^\psi(k, x)|^2 dx dk \right)^{1/2} \times \left( \int_{\hat{\mathbb{R}}^n/G} \left( \int_{\hat{K}_\gamma} \|\gamma\|^2 |\pi_{\gamma,\sigma}(f_{u,t}^\psi)|^2_{HS} d\mu_\gamma(\sigma) d\mathcal{H}_{\hat{R}^n}(\gamma) \right) \right)^{1/2}, $$

(3.8)
which holds for almost all \((u, t) \in G\). Integrating both sides with respect to \(du\,dt\) and then applying Cauchy-Schwarz inequality, we have

\[
\int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n/G} \|G_{\psi}f(u, t, \gamma, \sigma)\|^2 \, d\mu_{\gamma}(\sigma) \, d\overline{\mu}_{\mathbb{R}^n}(\gamma) \, du \, dt \\
\leq C \left( \int_{K \times \mathbb{R}^n} \int_{K \times \mathbb{R}^n} \|x\|^2 \left| f_{k,u}(k, x) \right|^2 \, dx \, dk \, du \, dt \right)^{1/2} \\
\times \left( \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n/G} \int_{K_{\gamma}} \left\| \gamma \right\|^2 \left\| \pi_{\gamma, \sigma}(f_{k,u}(k, x)) \right\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\overline{\mu}_{\mathbb{R}^n}(\gamma) \, du \, dt \right)^{1/2} \\
= C \left( \int_{K \times \mathbb{R}^n} \int_{K \times \mathbb{R}^n} \|x\|^2 \left| f(k, x) \right|^2 \, dx \, dk \, du \, dt \right)^{1/2} \\
\times \left( \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n/G} \int_{K_{\gamma}} \left\| \gamma \right\|^2 \left\| \pi_{\gamma, \sigma}(f_{k,u}(k, x)) \right\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\overline{\mu}_{\mathbb{R}^n}(\gamma) \, du \, dt \right)^{1/2} \\
= C \|\psi\|_2 \left( \int_{K \times \mathbb{R}^n} \|x\|^2 \left| f(k, x) \right|^2 \, dx \, dk \right)^{1/2} \\
\times \left( \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n/G} \int_{K_{\gamma}} \left\| \gamma \right\|^2 \left\| \pi_{\gamma, \sigma}(f_{k,u}(k, x)) \right\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\overline{\mu}_{\mathbb{R}^n}(\gamma) \, du \, dt \right)^{1/2}. \\
\tag{3.8}
\]

Using (3.2) and (3.4), we get

\[
\|\psi\|_2 \left\| f \right\|^2_{L^2} \leq C \left( \int_{K \times \mathbb{R}^n} \|x\|^2 \left| f(k, x) \right|^2 \, dx \, dk \right)^{1/2} \\
\times \left( \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n/G} \int_{K_{\gamma}} \left\| \gamma \right\|^2 \left\| \pi_{\gamma, \sigma}(f_{k,u}(k, x)) \right\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\overline{\mu}_{\mathbb{R}^n}(\gamma) \, du \, dt \right)^{1/2}. \\
\tag{3.9}
\]

Applying Hölder’s inequality, we have

\[
\left( \int_{K \times \mathbb{R}^n} \|x\|^{2a} \left| f(k, x) \right|^2 \, dx \, dk \right)^{\frac{1}{a}} \left( \int_{K \times \mathbb{R}^n} \left| f(k, x) \right|^2 \, dx \, dk \right)^{1 - \frac{1}{a}} \geq \int_{K \times \mathbb{R}^n} \|x\|^2 \left| f(k, x) \right|^2 \, dx \, dk \\
\tag{3.9}
\]

and

\[
\left( \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n/G} \int_{K_{\gamma}} \left\| \gamma \right\|^{2b} \left\| \pi_{\gamma, \sigma}(f_{k,u}(k, x)) \right\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\overline{\mu}_{\mathbb{R}^n}(\gamma) \, du \, dt \right)^{\frac{1}{b}} \\
\times \left( \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n/G} \int_{K_{\gamma}} \left\| \pi_{\gamma, \sigma}(f_{k,u}(k, x)) \right\|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\overline{\mu}_{\mathbb{R}^n}(\gamma) \, du \, dt \right)^{1 - \frac{1}{b}}. \\
\]
\[
\geq \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n} \| \gamma \|^2 \| G_\psi f(u, t, \gamma, \sigma) \|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\mu_{\mathbb{R}^n}(\gamma) \, du \, dt.
\]

(3.10)

Combining (3.8), (3.9) and (3.10), we have

\[
\| \psi \|^2 \| f \|^2 \leq C \left( \int_{K \times \mathbb{R}^n} \| x \|^2 |f(k, x)|^2 \, dk \right)^{\frac{1}{2a}} \left( \int_{\mathbb{R}^n} \| f(x) \|^2 \, dx \right)^{\frac{1}{2b}}
\]

\[
\times \left( \int_{K \times \mathbb{R}^n} \int_{\mathbb{R}^n} |\omega|^2 \| G_\psi f(u, t, \gamma, \sigma) \|^2_{\text{HS}} \, d\mu_{\gamma}(\sigma) \, d\mu_{\mathbb{R}^n}(\gamma) \, du \, dt \right)^{\frac{1}{2b}}.
\]

Thus, we have the required inequality (3.5). □

**Example 3.3.** We give the explicit expression of the Heisenberg uncertainty inequality for Gabor transform in the following cases:

1. Euclidean group \( \mathbb{R}^n \).

\[
\| \psi \|_2 \| f \| \leq C \left( \int_{\mathbb{R}^n} \| x \|^2 |f(x)|^2 \, dx \right)^{\frac{1}{2a}} \left( \int_{\mathbb{R}^n} \| f(x) \|^2 \, dx \right)^{\frac{1}{2b}}
\]

\[
\times \left( \int_{\mathbb{R}^n} \| \omega \|^2 \| G_\psi f(t, \omega) \|^2_{\text{HS}} \, dt \, d\omega \right)^{\frac{1}{2b}}.
\]

2. \( \mathbb{R}^n \times K \), where \( K \) is a separable unimodular locally compact group of type I.

\[
\| \psi \|_2 \| f \| \leq C \left( \int_{\mathbb{R}^n \times K} |x|^{2a} |f(x, k)|^2 \, dx \, dk \right)^{\frac{1}{2a}} \left( \int_{\mathbb{R}^n \times K} z \| G_\psi f(t, u, z, \gamma) \|^2_{\text{HS}} \, dz \, d\gamma \, dt \right)^{\frac{1}{2b}}.
\]

3. Heisenberg Group \( \mathbb{H}_n \) (see [12]).

\[
\| \psi \|_2 \| f \| \leq C \left( \int_{\mathbb{H}_n} |\lambda|^{2a} |f(z, t)|^2 \, dz \, dt \right)^{\frac{1}{2a}} \left( \int_{\mathbb{H}_n} \| G_\psi f(z', t', \lambda) \|^2_{\text{HS}} |\lambda|^n \, d\lambda \, dz' \, dt' \right)^{\frac{1}{2b}}.
\]
(4) $K \times \mathbb{R}^n$, where $K$ is a compact subgroup of the group of automorphisms of $\mathbb{R}^n$.

$$
\|\psi\|_2^{\frac{1}{2}} \|f\|_2^{\left(\frac{1}{2a} + \frac{1}{b}\right)} \leq C \left( \int_{K \times \mathbb{R}^n} \|x\|^{2a} |f(k,x)|^2 \, dx \, dk \right)^{\frac{1}{2a}} \times \left( \int_{K \times \mathbb{R}^n} \int_{\hat{\mathbb{R}}^n / G} \|\ell\|^{2b} \|G_\psi f(u,t,\ell,\sigma)\|_{\text{HS}}^2 \, d\ell \, du \right)^{\frac{1}{2b}}.
$$

(5) A class of connected, simply connected nilpotent Lie groups $G$ for which the Hilbert-Schmidt norm of the group Fourier transform $\pi_\xi(f)$ of $f$ attains a particular form (see [2]).

$$
\|\psi\|_2^{\frac{1}{2}} \|f\|_2^{\left(\frac{1}{2a} + \frac{1}{b}\right)} \leq C \left( \int_{G} \|x\|^{2a} |f(x)|^2 \, dx \right)^{\frac{1}{2a}} \times \left( \int_{G} \int_{W} \|\xi\|^{2b} \|G_\psi f(y,\xi)\|_{\text{HS}}^2 \frac{1}{|h(\xi)|^b |\text{Pf}(\xi)|^{b-1}} \, d\xi \, dy \right)^{\frac{1}{2b}}.
$$

(6) For thread-like nilpotent Lie groups (see [6]).

$$
\|\psi\|_2^{\frac{1}{2}} \|f\|_2^{\left(\frac{1}{2a} + \frac{1}{b}\right)} \leq C \left( \int_{G} \|x\|^{2a} |f(x)|^2 \, dx \right)^{\frac{1}{2a}} \times \left( \int_{G} \int_{W} \|\xi\|^{2b} \|G_\psi f(y,\xi)\|_{\text{HS}}^2 |\xi_1| \, d\xi \right)^{\frac{1}{2b}}.
$$

(7) For 2-NPC nilpotent Lie groups (see [11]), let $\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_m = \mathfrak{g}$ be a Jordan-Hölder sequence in $\mathfrak{g}$ such that $\mathfrak{g}_m = \mathfrak{z}(g)$ and $\mathfrak{h} = \mathfrak{g}_{m-2}$. We have the following two cases:

(a) $\dim [\mathfrak{g},\mathfrak{g}_{m+1}] = 2$.

$$
\|\psi\|_2^{\frac{1}{2}} \|f\|_2^{\left(\frac{1}{2a} + \frac{1}{b}\right)} \leq C \left( \int_{G} \|x\|^{2a} |f(x)|^2 \, dx \right)^{\frac{1}{2a}} \times \left( \int_{G} \int_{W} \|\xi\|^{2b} \|G_\psi f(y,\xi)\|_{\text{HS}}^2 \frac{1}{|h(\xi)|^b |\text{Pf}(\xi)|^{b-1}} \, d\xi \right)^{\frac{1}{2b}}.
$$

(b) $\dim [\mathfrak{g},\mathfrak{g}_{m+1}] = 1$.

$$
\|\psi\|_2^{\frac{1}{2}} \|f\|_2^{\left(\frac{1}{2a} + \frac{1}{b}\right)} \leq C \left( \int_{G} \|x\|^{2a} |f(x)|^2 \, dx \right)^{\frac{1}{2a}} \times \left( \int_{G} \int_{W} \|\xi\|^{2b} \|G_\psi f(y,\xi)\|_{\text{HS}}^2 |\text{Pf}(\xi)| \, d\xi \right)^{\frac{1}{2b}}.
$$
(8) For connected simply connected nilpotent Lie groups $G = \exp g$ such that $g(\xi) \subset [g, g]$ for all $\xi \in \mathcal{U}$ (see [11]).

\[
\|\psi\|^{\frac{1}{2}} \|f\|^{\left(\frac{1}{2} + \frac{1}{a}\right)} \leq C \left( \int_G \|x\|^{2a} |f(x)|^2 \, dx \right)^{\frac{1}{2a}} \\
\times \left( \int_G \int_{\mathcal{W}} \|\xi\|^{2b} \|G_\psi f(y, \xi)\|_{HS}^2 \frac{|Pf(\xi)|^{b+1}}{|\xi([X_{j_1}, X_{n_j}])|^b} \, d\xi \right)^{\frac{1}{2b}}.
\]

(9) For low-dimensional nilpotent Lie groups of dimension less than or equal to 6 (for details, see [9]) except for $G_{6,8}, G_{6,12}, G_{6,14}, G_{6,15}, G_{6,17}$, one can write an explicit Heisenberg uncertainty inequality for Gabor transform.

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