Inflation with a constant ratio of scalar and tensor perturbation amplitudes

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Abstract

The single scalar field inflationary models that lead to scalar and tensor perturbation spectra with amplitudes varying in direct proportion to one another are reconstructed by solving the Stewart–Lyth inverse problem to next-to-leading order in the slow–roll approximation. The potentials asymptote at high energies to an exponential form, corresponding to power law inflation, but diverge from this model at low energies, indicating that power law inflation is a repellor in this case. This feature implies that a fine–tuning of initial conditions is required if such models are to reproduce the observations. The required initial conditions might be set through the eternal inflation mechanism. If this is the case, it will imply that the spectral indices must be nearly constant, making the underlying model observationally indistinguishable from power law inflation.

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I. INTRODUCTION

A cosmological model with a low density of cold dark matter (\(\Omega_{\text{CDM}} \approx 0.3\)) and a dominant cosmological constant (\(\Omega_{\Lambda} \approx 0.7\)) is currently favored by recent analyses of cosmological observations, in particular those of the cosmic microwave background (CMB) power spectrum (see for instance, \([1–4]\)) and high red–shift surveys of type Ia supernovae \([5,6]\). These analyses maximize the likelihood between the observed power spectrum and that of a given theoretical model by fitting values for various model parameters. This set of parameters includes quantities that characterize the initial conditions for the evolution of the density (scalar) and gravitational wave (tensor) perturbations. If these initial conditions are described by adiabatic, gaussian fluctuations, the observations strongly constrain the present–day universe to be very nearly spatially flat \([1–3]\).

The simplest casual mechanism for producing a spatially flat universe, where large–scale structure originates from a primordial spectrum of adiabatic density perturbations, is given by the single field inflationary model \([7]\). The early universe undergoes a rapid, accelerated expansion that is driven by the self–interaction potential energy of a real scalar field – the inflaton. We consider inflation driven by a single inflaton field throughout this paper. The field undergoes quantum fluctuations as it slowly rolls down its potential and these fluctuations generate a perturbation in the spatial curvature \([8,9]\). A primordial spectrum of tensor perturbations is also generated from the scalar field fluctuations \([8,10]\).

Until recently, the majority of CMB data analyses have neglected the possible effects of the primordial gravitational wave spectrum \([1]\). (For a recent review, see, e.g., Ref. \([1]\)). In the last few years, however, there has been a growing recognition that the role of the tensor perturbations deserves more attention when determining the best–fit values of the cosmological parameters \([2,3,11]\). It has been found that if the tensor modes are included in the analysis, an extra degeneracy arises when fitting cosmological models to the CMB data \([3,11]\).

Further motivation for including the tensor modes in the analysis arises from the possibil-
ity that a measurement of the CMB polarization may indirectly determine the gravitational wave contribution to the power spectrum [11,12]. Such a contribution can be parametrized in terms of the quantity

\[ r \equiv \alpha \frac{A_T^2}{A_S^2}, \quad (1) \]

representing the relative amplitudes of the tensor \((A_T)\) and scalar \((A_S)\) perturbations, where the constant, \(\alpha\), depends on the particular normalization of the spectral amplitudes that is chosen.

Given the current accuracy of the observations, it is only possible at present to place a constant upper bound on the allowed range of \(r\) [2,3]. Looking to the future, the most optimistic expectation we have for constraining \(r\) is that the observations would favor a constant central value [11,12]. It is unlikely that a direct dependence of \(r\) on scale would ever be measurable.

Motivated by these future expectations, therefore, it is important to determine the main features of the single field inflationary models that yield perturbation spectra resulting in a tensor–to–scalar ratio, \(r\), that is precisely constant, at least over the range of scales accessible to observations. This is the purpose of the present paper. Of particular interest is the functional form of the corresponding inflaton potentials. Such a study is also relevant to the question of whether the inflaton potential can be reconstructed directly from observations and, consequently, provides important information about physics at very high energy scales [14].

It follows immediately from the definition (1) that imposing the condition \(r = \text{constant}\) is equivalent to searching for models where the scale dependences of the two spectra are identical. In general, these scale dependences are parametrized in terms of the scalar \((n_S)\) and tensor \((n_T)\) spectral indices, respectively. These are defined in terms of the logarithmic derivatives of the corresponding normalized amplitudes:

\[ \Delta \equiv \frac{n_S - 1}{2} \equiv \frac{d \ln A_S}{d \ln k}, \quad (2) \]

\[ \delta \equiv \frac{n_T}{2} \equiv \frac{d \ln A_T}{d \ln k}, \quad (3) \]
where $k = aH$ is the comoving wavenumber when the mode first crosses the Hubble radius, $d_H \equiv H^{-1}$, during inflation.

We are therefore interested in the models that produce

$$\Delta(k) = \delta(k).$$

(4)

Since exact expressions relating the perturbation spectra to the inflaton potential are presently unknown, the standard approach is to expand the power spectra in terms of a set of parameters that describe the inflationary dynamics through the slow evolution of the Hubble radius or, in the case of single scalar field models, in terms of the slow rolling of the inflaton field along a nearly flat potential (see Refs. [7,14,18] and references therein). To leading order in these expansions, the inflationary models that have precisely constant ratio, $r$, are precisely the power law inflationary models driven by an exponential potential [13]. For power law inflation the spectral indices are constant and equal to each other. However, as we discuss in the next section, to next-to-leading order, the indices can vary with scale whilst being equal to each other at each value of $k$. Specifically, we solve the next-to-leading order Stewart-Lyth inverse problem (SLIP) [15] under the condition that the spectral indices are equal at any scale. We find that for this case, power law behavior is a repellor rather than an attractor of the inflationary dynamics, in contrast to the lowest-order analysis of Hoffman and Turner [16]. This difference arises because different approximations in the slow-roll analysis are considered. Nevertheless, if the underlying inflaton potentials derived at next-to-leading order are to produce successful inflation under the condition that $r = \text{constant}$ ($\Delta = \delta$), a strong fine-tuning of the initial value for the inflaton field is required. The eternal inflation mechanism [17] provides a natural way of obtaining the required initial values, although in this case, the scale dependence of the spectral indices is strongly suppressed and, consequently, the difference between these models and the power law model is effectively erased.

The paper is organized as follows. We present the equations that determine the perturbation spectra in Section II. In Section III, we proceed to solve these equations and reconstruct
the corresponding inflationary potentials in parametric form. We conclude with a discussion in Section IV.

II. PERTURBATION SPECTRA

When determining the cosmological parameters \[1–3\], the primordial perturbation spectra are often expanded as

\[
\ln A^2_S(k) = \ln A^2_S(k_\ast) + \Delta(k_\ast) \ln \left(\frac{k}{k_\ast}\right) + \frac{1}{2} \frac{d\Delta(k)}{d\ln k} \bigg|_{k=k_\ast} \ln^2 \left(\frac{k}{k_\ast}\right) + \cdots, \\
\ln A^2_T(k) = \ln A^2_T(k_\ast) + \delta(k_\ast) \ln \left(\frac{k}{k_\ast}\right) + \frac{1}{2} \frac{d\delta(k)}{d\ln k} \bigg|_{k=k_\ast} \ln^2 \left(\frac{k}{k_\ast}\right) + \cdots, 
\]

where \(k_\ast\) is a pivotal scale. The order where these expansions are truncated is determined by the precision of the CMB observations. Only the first two terms in each expansion are usually taken into account and the ‘running’ of the indices, \(dn_i/d\ln k\), is neglected. Even so, there are still four inflationary parameters that need to be fitted to the data: \(A^2_S(k_\ast)\), \(A^2_T(k_\ast)\), \(\Delta(k_\ast)\) and \(\delta(k_\ast)\). On the other hand, the total number of cosmological parameters in the analysis can be very large \[4\], and in view of the level of accuracy expected from forthcoming observations in the near future, it is possible that higher-order terms in the expansions (5) and (6) may have to be included as well. The large number of parameters that need to be considered allows degeneracies to arise when fitting a given theoretical model to the observational data.

One way of reducing the number of degrees of freedom is to fit the ratio, \(r\), instead of the tensor quantities directly. The relevant tensor modes can then be deduced from the definition of \(r\), Eq. (1), and the next-to-leading order consistency relation for the class of single field models \[14\]:

\[
n_T = -2 \left[ \frac{r}{\alpha} - \left(\frac{r}{\alpha}\right)^2 - (n_S - 1) \left(\frac{r}{\alpha}\right) \right]. 
\]

The bounds \[3,11,12\] in the observed precision of the tensor contribution to the CMB power spectrum are such that even after a best-fit value for \(r(k_\ast)\) has been deduced, the simplest
approach to adopt when constraining the parameter space would be to assume that the value \( r(k_*) \) holds for all scales, i.e., \( r(k) = r(k_*) = \text{constant} \). As we have already seen, this corresponds formally to a model with \( \Delta(k) = \delta(k) \). Consequently, this procedure provides further motivation for analyzing the case where \( r \) is precisely constant, because this allows us in principle to gauge to what extent the accuracy of expansions (5) and (6) is sensitive to the value of \( k_* \).

In single-field inflation, the indices, \( \Delta(k) \) and \( \delta(k) \), satisfy the SLIP equations [15]:

\[
2C \epsilon_1 \dot{\epsilon}_1 - (2C + 3) \epsilon_1 \dot{\epsilon}_1 - \epsilon_1 + \epsilon_1^2 + \epsilon_1 + \Delta = 0, \tag{8}
\]

\[
2(C + 1) \epsilon_1 \dot{\epsilon}_1 - \epsilon_1^2 - \epsilon_1 - \delta = 0, \tag{9}
\]

to ‘next–to–leading order’ in the slow–roll approximation, where \( C = -0.7296 \) and a circumflex accent denotes differentiation with respect to the variable \( \tau \), defined such that \( d\tau \equiv d \ln H^2 \). The first ‘horizon flow function’ is defined as \([18,14]\)

\[
\epsilon_1 \equiv \frac{d \ln dH}{dN} = \frac{3T}{T + V}, \tag{10}
\]

where \( T \equiv \dot{\phi}^2/2 \) represents the kinetic energy of the inflaton field, and \( N \equiv \ln(a/a_i) \) is the number of e–foldings of inflationary expansion since some initial time, \( t_i \). (Note that the number of e–foldings is usually counted backwards in time. Here, we count it forward, i.e., \( N(t_i) = 0 \).) In general, Eq. (10) measures the logarithmic change of the Hubble distance per e–folding, or equivalently, the contribution of the inflaton field’s kinetic energy relative to its total energy density. Inflation proceeds for \( \epsilon_1 < 1 \) (\( \ddot{a} > 0 \)) and the weak energy condition for a spatially flat universe is satisfied for \( \epsilon_1 > 0 \). By combining Eqs. (8) and (9) we deduce that

\[
\delta - \Delta = 2C \epsilon_1 \dot{\epsilon}_1 - (\epsilon_1 + 1) \dot{\epsilon}_1. \tag{11}
\]

In the next Section, we proceed to find solutions to Eq. (11).
III. THE MODEL

When determining the single field models that lead to $r = \text{constant}$ we are interested in solving Eq. (11) when Eq. (4) is imposed. Firstly, we remark that one solution to Eq. (11) with this condition is that $\epsilon_1, \Delta$ and $\delta$ are all constants, corresponding to the power law inflationary model [13]. Moreover, power law inflation is the unique solution if second–order terms in Eq. (11) are neglected. More generally, a first integration of Eq. (11) yields

$$\dot{\epsilon}_1 = \frac{1}{2C} [\epsilon_1 + \ln (B \epsilon_1)], \quad (12)$$

when Eq. (4) is satisfied, where $B > 0$ is an integration constant. Substituting this result into Eq. (9) we find that

$$\Delta(\epsilon_1) = \delta(\epsilon_1) = \frac{C + 1}{C} \epsilon_1 + \ln (B \epsilon_1) - \epsilon_1^2 - \epsilon_1. \quad (13)$$

Eq. (13) is plotted in Fig. 1 for different values of the constant $B$.

![FIG. 1. The variation of the scalar and tensor spectral indices, $\delta = \Delta$, as a function of the first horizon flow function $\epsilon_1$ for different values of the constant $B$.](image)

Now, it has been shown in Ref. [19] that to next–to–leading order, the constraint $\delta(\epsilon_1) \leq 0$ must be satisfied for any value of $\epsilon_1$ during inflation. However, as indicated in Fig. 1, Eq. (13) is a generalized inverted parabola and always has a positive maximum. This implies that $\delta(\epsilon_1) > 0$ for some range of $\epsilon_1$. Indeed, the positive maximum is located in the interval.
\( \epsilon_1 \in (0, 1/B) \). This would seem to indicate that the condition for the spectral indices to be equal (but not constant) might not be self-consistent at this level of the slow-roll approximation. However, the point is that \( \delta < 0 \) is necessary only during inflation and, in effect, this condition further constrains the allowed range of values that \( \epsilon_1 \) may take for the analysis to be self-consistent. Indeed, we may specify the parameter, \( B \), in such a way that the values \( \delta(\epsilon_1_{(\text{max})}) \) and \( \epsilon_1_{(\text{max})} \) are negligible, where

\[
\epsilon_1_{(\text{max})} = \frac{C + 1}{2} \mathcal{L}_W \left[ \frac{2 \exp \left( -\frac{1}{C+1} \right)}{B(C+1)} \right]
\]

is the value of \( \epsilon_1 \) at the maximum value of \( \delta(\epsilon_1) \) and \( \mathcal{L}_W \) is the Lambert \( W \) function \[20\]. As we shall see later, \( 1/B \) determines the range of power law behavior of \( \Delta \). Thus, taking into account the range of \( n_S \) consistent with analyses of CMB observations \[1–3\], we can choose \( B = 100 \). In this way, \( \epsilon_1_{(\text{max})} \approx 0.00025 \) can be identified as the lower critical value of \( \epsilon_1 \) where the precision in our calculations is consistent to next-to-leading order in the slow-roll approximation, with \( \delta(\epsilon_1_{(\text{max})}) \approx 0.00009 \) then being consistently negligible.

Before proceeding, let us first analyze the quantity

\[
Q \equiv \frac{1}{\epsilon_1^2 + \epsilon_1 + \delta},
\]

since this term plays an important role in the forthcoming calculations. Substituting \( \delta \), as given by Eq. (13), into Eq. (15) implies that

\[
Q = \left( \frac{C}{C+1} \right) \frac{1}{\epsilon_1 \left[ \epsilon_1 + \ln (B \epsilon_1) \right]}.
\]

In order to obtain analytical results, it is necessary to approximate the denominator of Eq. (16) in the different limits where \( \epsilon_1 \gg |\ln (B \epsilon_1)| \) and \( \epsilon_1 \ll |\ln (B \epsilon_1)| \), respectively. In the former case, Eq. (16) can be be expanded as the series

\[
\left( \frac{C}{C+1} \right) \frac{1}{\epsilon_1^2} \left\{ 1 - \frac{\ln (B \epsilon_1)}{\epsilon_1} + \left[ \frac{\ln (B \epsilon_1)}{\epsilon_1} \right]^2 - \cdots \right\}
\]

if \( \epsilon_1 \) is sufficiently different from zero. However, since \( \epsilon_1 \) is typically small during inflation, \( B \) must be appropriately tuned if this expansion is to be valid. Consequently, the
approximation is only appropriate over a very narrow range of \( \epsilon_1 \). Indeed, a numerical investigation confirms that the relevant range of \( \epsilon_1 \) is negligible for realistic values of \( B \) and so we regard it as unphysical. For example, if \( B = 100 \), this approximation is valid only for \( \epsilon_1 \in (0.009999, 0.010001) \).

The more interesting case is the limit, \( \epsilon_1 \ll |\ln (B\epsilon_1)| \), where the following approximation for expression (16) applies:

\[
Q = \left( \frac{C}{C+1} \right) \frac{1}{\epsilon_1 \ln (B\epsilon_1)} \left\{ 1 - \frac{\epsilon_1}{\ln (B\epsilon_1)} + \left[ \frac{\epsilon_1}{\ln (B\epsilon_1)} \right]^2 - \cdots \right\} .
\]

Firstly, if \( \epsilon_1 < 1/B \), this implies that \( \epsilon_1 \) is very small and the absolute value of \( \ln(B\epsilon_1) \) will be very large\(^1\). Secondly, if \( \epsilon_1 > 1/B \), then \( \epsilon_1 \in (1/B, 1] \) implies that \( \ln(B\epsilon_1) > \epsilon_1 \) if \( B \) is sufficiently large. Note that the value of \( B \) specifies the central value of \( \epsilon_1 \) during inflation and this is expected to be less than 0.1. With this in mind, a numerical test implies that the expansion is valid if \( \epsilon_1 > 0.112 \) for \( B = 10 \) and \( \epsilon_1 > 0.0101 \) if \( B = 100 \). Furthermore, one may verify that this approximation is suitable even for \( \epsilon_1 \) close to \( 1/B \). For example, when \( B = 100 \), the reliable intervals of \( \epsilon_1 \) are \((0, 0.0095]\) and \([0.0101, 1]\), respectively.

We now employ this approximation to find semi–analytical expressions for the spectral indices. The wavenumber, \( k = aH \), at horizon crossing is evaluated as a function of \( \epsilon_1 \) by solving the first–order differential equation

\[
(C + 1)(\epsilon_1 - 1)\tilde{\epsilon}_1 - \epsilon_1^2 - \epsilon_1 - \delta = 0,
\]

where \( \tilde{\epsilon}_1 \equiv d\epsilon_1/d\ln k \). Using approximation (18) and integrating implies that

\[
\ln \frac{k}{k_0} = C \left\{ \left[ \frac{1}{2\ln^2(B\epsilon_1)} + \frac{2}{\ln(B\epsilon_1)} \right] \epsilon_1^2 - \frac{\epsilon_1}{\ln(B\epsilon_1)} + \left[ -\frac{2}{B} \mathcal{E}_i(1, \mp \ln(B\epsilon_1)) 
+ \frac{4}{B^2} \mathcal{E}_i(1, \mp 2\ln(B\epsilon_1)) - \frac{9}{2B^3} \mathcal{E}_i(1, \mp 3\ln(B\epsilon_1)) - \ln |\ln(B\epsilon_1)| \right] \right\} ,
\]

where \( \mathcal{E}_i(n, x) \) represents the exponential integral \([22]\), a \( \mp \) corresponds to \( \epsilon_1 < 1/B \) and \( \epsilon_1 > 1/B \), respectively, and a sub–dominant term has been consistently neglected. The

\(^1\)We are assuming implicitly that \( B \gg 1 \) in this discussion.
The variation of the scalar and tensor spectral indices, \( \Delta = \delta \), as a function of the Hubble radius crossing wavenumber, \( k \). The upper branch corresponds to \( \epsilon_1 < 1/B \) and the lower branch to \( \epsilon_1 > 1/B \). The power law solution, corresponding to \( \epsilon_1 = 1/B \), is represented by the dashed line.

The dependence of the spectral indices, \( \delta(k) = \Delta(k) \), on comoving wavenumber is presented in Fig. 2. It is observed that when the inflationary dynamics results in scalar and tensor perturbation spectra that satisfy \( r = \text{constant} \), the power law inflationary model, as represented by the dashed line, is a repellor rather than an attractor to next-to-leading order.

It is necessary to analyze the dynamics of the parameter, \( \epsilon_1 \), in order to understand this behavior more fully. The solution to Eq. (9) is given by

\[
\tau = \exp \left\{ 2C \left[ \frac{\epsilon_1^2}{\ln(B\epsilon_1)} - \frac{1}{B} \mathcal{E}_i(1, \mp \ln(B\epsilon_1)) \right. \\
+ \left. \frac{2}{B^2} \mathcal{E}_i(1, \mp 2 \ln(B\epsilon_1)) - \frac{9}{2B^3} \mathcal{E}_i(1, \mp 3 \ln(B\epsilon_1)) \right] \right\} + \tau_0
\]  

when the approximation (18) is valid. Eq. (21) is plotted in Fig. 3.

Integration constants are labeled by a subscript 0 in the corresponding variable in what follows.

Hereafter the plots in this paper are consistently drawn using \( \epsilon_{1(max)} \) as the lower value for \( \epsilon_1 \).
FIG. 3. The variation of $\epsilon_1$ as a function of $\tau - \tau_0 \equiv \ln H^2$. The flow of time along each branch is represented by the small arrows.

It follows from this figure that the two branches in Fig. 2 correspond to initial values of $\epsilon_1$ that are greater than or less than $1/B$, respectively. Since $d\tau/dt < 0$, the solutions diverge from the solution given by $\epsilon_1 = 1/B$ as cosmic time, $t$, increases and this is illustrated in Fig. 3 by the direction of the small arrows near to each of the branches. In order to emphasize the (inverse) asymptotic behavior in the neighborhood of the unstable fixed point $\epsilon_1 = 1/B$, the initial values for $\epsilon_1$ for this figure were chosen to be extremely close to $1/B$. However, it can be seen that any small deviation of the initial value from $1/B$ is exponentially amplified and consequently, the time interval that the solution spends inside any given neighborhood of $\epsilon_1 = 1/B$ is exponentially suppressed. This implies that, unless the initial value of $\epsilon_1$ is extremely close to $1/B$, this parameter will immediately move out of the $[0, 1)$ interval and consequently, this leads to a very rapid and undesirable end to the inflationary period. Observe that, in these cases, $\tau_0$ determines the energy scale, $H_0$, at the end of the inflationary era. Returning to Fig. 2, such a fine tuning of the initial conditions for $\epsilon_1$ is translated into the requirement that a sufficiently large value of the integration constant, $k_0$, must be chosen in order to have near power law behavior in the appropriate range of scales.

Further support for this conclusion regarding the necessary fine-tuning of the initial conditions is given by considering the variation of the $e$–foldings, $N$, with respect to $\epsilon_1$. The lower limit, $N > 60$, must be satisfied if inflation is to resolve the horizon and flatness
problems \cite{1} and the dependence of the number of e-foldings on $\epsilon_1$ is determined from the differential equation, $\dot{N} = -1/2\epsilon_1$ \cite{23} or, equivalently, by evaluating the integral

$$N = -(C + 1) \int \frac{d\epsilon_1}{\epsilon_1^2 + \epsilon_1 + \delta} + N_0. \quad (22)$$

In our case, the integration of Eq. (22) yields

$$N = -C \left\{ \left[ \frac{1}{2 \ln^2(B\epsilon_1)} - \frac{1}{\ln(B\epsilon_1)} \right] \epsilon_1^2 + \frac{\epsilon_1}{\ln(B\epsilon_1)} + \ln|\ln(B\epsilon_1)| \right. + \frac{1}{B} \mathcal{E}_i(1, \pm \ln(B\epsilon_1)) - \frac{2}{B^2} \mathcal{E}_i(1, \pm 2 \ln(B\epsilon_1)) \right\} + N_0, \quad (23)$$

and the graph corresponding to Eq. (23) is shown in Fig. 4. As expected, the relative expansion rate of the universe is larger as the initial value of $\epsilon_1$ becomes closer to $1/B$.

![FIG. 4](image)

FIG. 4. Illustrating the variation of the number of e-foldings of inflationary expansion, $N$, as a function of $\epsilon_1$. Time flow along each branch is represented by the small arrows.

The scalar field and its self-interaction potential are given in terms of $\epsilon_1$ by \cite{19},

$$V(\epsilon_1) = \frac{1}{\kappa} (3 - \epsilon_1) \exp \left[ \tau(\epsilon_1) \right], \quad (24)$$

$$\phi(\epsilon_1) = -\frac{2(C + 1)}{\sqrt{2\kappa}} \int \frac{\sqrt{\epsilon_1} d\epsilon_1}{\epsilon_1^2 + \epsilon_1 + \delta} + \phi_0. \quad (25)$$

and the potential is shown in Fig. 5, where $V_0 = \kappa^{-1} \exp(\tau_0)$, $\kappa = 8\pi/m_{Pl}^2$ is the Einstein constant and $m_{Pl}$ is the Planck mass. The right-hand branch of the potential has a minimum at $\epsilon_1^{(upper)} \approx 0.79$, not shown in the figure in order to allow for the observation of details in a more realistic range of values for $\epsilon_1$. 

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Unfortunately, even under the approximation (18), it is not possible to analytically integrate Eq. (25) for the inflaton field. Nevertheless, this integral can be performed numerically and the result is presented in Fig. 6.

**Fig. 5.** The inflaton potential as function of $\epsilon_1$.

**Fig. 6.** The inflaton field as function of $\epsilon_1$.

In Eq. (25), a sign for the square root of $\epsilon_1$ must be chosen and this specifies the signs of $\dot{\phi}$ and $dV/d\phi$. We have assumed that $\dot{\phi} > 0$ and $dV/d\phi < 0$ and so for consistency, one can only consider those intervals of $\epsilon_1$ where this is valid. With the above results it is possible to check that the necessary criteria [19].
\[
\begin{align*}
\epsilon_1 \frac{d\phi}{d\epsilon_1} &< 0, \\
\epsilon_1 \frac{dV}{d\epsilon_1} &> 0,
\end{align*}
\]

are fulfilled only for \(\epsilon_1 \in (0, 1/B)\) and \(\epsilon_1 \in (1/B, \epsilon_1^{\text{upper}}]\). To conclude this section, therefore, the potential as function of the inflaton field is shown in Fig. 7. As can be observed in this figure, the required fine–tuning of \(\epsilon_1\) is equivalent to a fine–tuning on the initial value of the inflaton field, i.e., a sufficiently high value for \(|\phi - \phi_0|\) is required. In this plot, the branch where \(\epsilon_1 \in (0, 1/B)\) is denoted by \(V_I\), and \(V_{II}\) denotes the branch where \(\epsilon_1 \in (1/B, \epsilon_1^{\text{upper}}]\).

**IV. DISCUSSION**

Motivated by future prospects for measuring any possible contribution of the primordial tensor (gravitational wave) perturbations to the CMB power spectrum, we have investigated the class of inflationary models that result in scalar and tensor fluctuation spectra with a constant ratio of amplitudes on all scales, \(r = \text{constant}\). This condition is satisfied if the spectral indices are equal. To lowest–order in the slow–roll approximation, such a condition
of equality implies that the indices must also be independent of scale and this corresponds to the power law inflationary model. However, to next-to-leading order in the slow-roll approximation, the indices can be equal but may also exhibit a non-trivial dependence on comoving wavenumber. Under a self-consistent approximation, we have determined the functional form of this dependence and reconstructed parametric solutions for the inflaton potentials that produce such spectra.

We find that there are two different possible potentials. Surprisingly, for the specific ansatz we consider, the power law inflationary model is a past, rather than a future, attractor, in the sense that at high energies (early times) both potentials converge to the exponential model, but move away from this special case at low energies. This provides a counter example to the generic, lowest-order analysis of Hoffman and Turner [16] who find that power law inflation is a future attractor for the inflationary kinematics. This difference arises because the approach of Ref. [16] employs a strong version of the slow-roll approximation to analyze the constraints on the inflationary evolution. In particular, the effects of the inflaton’s acceleration in its equation of motion and its kinetic term in the Friedmann equation were neglected. Moreover, the expressions relating the spectral indices and the ratio of the scalar and tensor amplitudes to the potential were truncated to lowest-order. In the present work, we have relaxed these restrictions and employed all of the available information on the inflaton dynamics and truncated the expressions for the spectral indices to next-to-leading order. Though the result of Ref. [16] should contain some of the essential features of the dynamics, the highly non-linear nature of the next-to-leading order expressions leads to important deviations from these lowest-order results. Indeed, if the strong slow-roll approximation of Ref. [16] had been invoked in the present analysis, power law inflation would have emerged as the unique solution. However, the next-to-leading order dynamical analysis of Ref. [15] indicates that in the reduced phase spaces for the evolution of $\epsilon_1$, there exists a saddle point in the region where $\epsilon_1$ has interesting values and $\Delta < 0$. This implies that with respect to cosmic time (recall that $d\tau/dt < 0$), attractor-like behavior will be characteristic only of those trajectories that are very close to the unstable separatrices.
Likewise, the saddle point acts as a repellor for those trajectories that are closer to the stable separatrices. Inflationary dynamics that does not correspond to a power law attractor was also found for the solution of the second case analyzed in Ref. \[23\].

One consequence of the behavior described in the present work is that a strong fine-tuning of the initial value of the inflaton field is required if this model is to produce spectra consistent with observations. This is different to what typically arises in inflationary cosmology, where any sensitivity to initial conditions is washed out by the accelerated expansion.

If we consider Figs. 1 and 2 once more, it can be seen that the upper branch of $\delta$ grows and becomes positive, thus indicating a breakdown in the next-to-leading order analysis. For the lower branch, $\delta$ begins to evolve extremely rapidly, probably indicating that the ‘running’ of the spectral index, $d\Delta/d\ln k$, becomes too large. Either way, observational constraints are difficult to satisfy. Thus, the potential in the region open to observation must be sufficiently close to the exponential (power law inflation) model and consequently the energy of the field initially stored in its potential must be sufficiently high.

One way to satisfy this requirement is through the eternal inflation mechanism, where (large) quantum fluctuations in the inflaton can cause the field to diffuse up its potential [17]. In general, the condition for eternal inflation to arise is that [17, 23],

$$\frac{H^2(\epsilon_{1(i)})}{\pi m_{\text{Pl}}^2 \epsilon_{1(i)}} \geq 1,$$

for a given value of $\epsilon_1(t_i)$. Recalling that $d\tau \equiv d\ln H^2$, we can rewrite this condition as

$$\tau - \tau_0 \geq \ln \left( \pi \epsilon_{1(i)} m_{\text{Pl}}^2 \right)$$

and consequently, $\tau - \tau_0$ must be sufficiently large for eternal inflation to proceed. On the other hand, it is clear from Fig. 3 that this is equivalent to requiring that the initial value of $\epsilon_1$ be close enough to $1/B$. In this case, however, the model would effectively be indistinguishable from that of the power law model. Moreover, it could be argued that this restriction on $\epsilon_1(t_i)$ itself represents a fine-tuning. Nevertheless, the attractive feature of eternal inflation is that only a small region of the universe need satisfy Eq. (28) for the
process of self–reproduction to start and continue indefinitely. If, as is commonly assumed, \( \epsilon_1(t_i) \) is randomly distributed, then at any given time in an inflationary universe described by this model, there is a finite probability for the existence of a region satisfying the necessary condition.

In conclusion, therefore, our analysis indicates that it is difficult, from a theoretical point of view, to obtain spectra where \( r \) is truly constant if the spectral indices have a non–trivial scale–dependence. This particularly applies in the case where the slow-roll condition is not necessarily satisfied. Moreover, the potential must be close to the exponential form over the range of inflaton values accessible to observations. This implies that the observations would not be able to discriminate between the models discussed above and power law inflation. Consequently, if a running of the scalar spectral index is eventually favored by future observations, it is likely that the underlying theory would be much more complicated and, in particular, would also result in a non–trivial scale–dependence for the ratio of amplitudes.

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