WEIGHTED BEREZIN AND BERGMAN ESTIMATES ON THE UNIT BALL IN $\mathbb{C}^n$

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Abstract. Using modern techniques of dyadic harmonic analysis, we are able to prove sharp estimates for the Bergman projection and Berezin transform and more general operators in weighted Bergman spaces on the unit ball. The estimates are in terms of the Bekolle-Bonami constant of the weight.

1. Introduction and main results

Recall that the Bergman space $A^p_t(\mathbb{B}_n) := A^p_t$ is defined to be the space of holomorphic functions on $\mathbb{B}_n$ with finite $L^p_t(\mathbb{B}_n) := L^p_t$ norm. That is $f \in A^p_t$ if it is holomorphic and the following norm is finite:

$$\|f\|_{A^p_t}^p := c_t \int_{\mathbb{B}_n} |f(z)|^p \left(1 - |z|^2\right)^t dV(z).$$

Above, $dV(z)$ is the standard Lebesgue measure on $\mathbb{B}_n$ and for $t > -1$, the constant $c_t$ is chosen so that $\int_{\mathbb{B}_n} c_t(1 - |z|^2)^t dV(z) = 1$. When $t \leq -1$, we set $c_t = 1$. We will let $dV_t = c_t(1 - |z|^2)^t dV(z)$.

The purpose of this paper is to prove one–weight inequalities for the operators given by:

$$S_{a,b} f(z) := (1 - |z|^2)^a \int_{\mathbb{B}_n} \frac{f(w)}{(1 - z\overline{w})^{n+1+a+b}} dV_b(w)$$

and

$$S^+_{a,b} f(z) := (1 - |z|^2)^a \int_{\mathbb{B}_n} \frac{f(w)}{|1 - z\overline{w}|^{n+1+a+b}} dV_b(w),$$

where $-a < b + 1$ and $z\overline{w} = \sum_{i=1}^n z_i \overline{w}_i$. That is, we want to know for which weights, i.e. positive locally integrable functions $u$, we have the following norm inequality:

$$\|S_{a,b} : L^p_b(u) \to L^p_b(u)\| < \infty$$

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where \( L^p_\sigma(u) \) denotes the set of functions that are \( p \)th power integrable with respect to \( u(z)dv_b(z) \).

The operators \( S_{a,b} \) and \( S^+_{a,b} \) are important in the study of function-theoretic operator theory on the Bergman spaces (see for example, [31]) and so are interesting in their own right. However, our main motivation comes from the operators \( S_{0,b} \) and \( S^+_{n+1,b} \) which are the Bergman projection and Berezin transform respectively.

Before we state our main result, we need to give some definitions. Recall that for \( b > -1 \), the Carleson tent over \( z \in B_n \) is defined to be the set:

\[
T_z := \left\{ w \in B_n : \left| 1 - \frac{w}{z} \right| < 1 - |z| \right\}
\]

and the Carleson tent over \( 0 \) is \( B_n \). For \( b > -1 \), we define the \( D_{p,a,b} \) characteristic of two weights \( u, \sigma \) by:

\[
[u, \sigma]_{D_{p,a,b}} := \sup_{z \in \mathbb{B}_n} \left( \int_{T_z} \sigma dv_b \right)^{p-1} \frac{\int_{T_z} udv_{pa+b}}{\int_{T_z} dv_{pa+b}}
\]

\[
\simeq \sup_{z \in \mathbb{B}_n} \left( \int_{T_z} \sigma dv_b \right)^{p-1} \frac{\int_{T_z} \tilde{u}dv_b}{\int_{T_z} dv_b} \text{vol}_b(T_z)^{\frac{pa}{n+pa}}.
\]

where \( \tilde{u}(z) := u(z)(1 - |z|^2)^{pa} \). Using the notation we will use in this paper (defined below), we can write this more compactly as:

\[
[u, \sigma]_{D_{p,a,b}} = \sup_{z \in \mathbb{B}_n} \left( \int_{T_z} \sigma dv_b \right)^{p-1} \langle u \rangle_{T_z}^{dv_{pa+b}} \simeq \sup_{z \in \mathbb{B}_n} \left( \int_{T_z} \sigma dv_b \right)^{p-1} \langle \tilde{u} \rangle_{T_z}^{dv_b} \text{vol}_b(T_z)^{\frac{pa}{n+pa}}.
\]

Our main theorem is:

**Theorem 1.** Let \( 1 < p < \infty \) and let \( u \) be a weight and let \( \sigma = u^{-\frac{1}{p'}} \) be the dual weight. If \( b > -1 \) there holds:

\[
[u, \sigma]_{D_{p,a,b}}^{\frac{1}{p'}} \leq \|S_{a,b} : L^p_b(u) \to L^p_b(u)\| \leq \|S^+_{a,b} : L^p_b(u) \to L^p_b(u)\| \preceq [u, \sigma]_{D_{p,a,b}}^{\frac{1}{p'}}
\]

If \( b \leq -1 \), let \( \psi(z) = \frac{1}{u(z)^{1/p}}(1 - |z|^2)^{\frac{1}{p}(p'+pa)} \) and \( \nu(z) = \psi(z)^{\frac{1}{p'}} \). There holds:

\[
[u, \sigma]_{D_{p',b,a}}^{\frac{1}{p'}} \simeq \|S_{a,b} : L^p_b(u) \to L^p_b(u)\| \leq \|S^+_{a,b} : L^p_b(u) \to L^p_b(u)\| \preceq [u, \sigma]_{D_{p',b,a}}^{\frac{1}{p'}}
\]

The classical \( B_p \) characteristic of a weight is \( [u]_{B_p,b} = [u, \sigma]_{D_{p,0,b}} \) where \( \sigma = u^{-\frac{1}{p'}} \). Therefore, as a corollary of Theorem 1 we have the following theorem, which is new for \( n \geq 2 \):
Theorem 2. Let $1 < p < \infty$ let $u$ be a weight and let $P_b = S_{0,b}$ be the Bergman projection. There holds:

$$[u]_{Bp,b}^\frac{1}{p} \leq \|P_b : L_p^b(u) \to L_p^b(u)\| \leq [u]_{Bp,b}^{\max\left\{1, \frac{1}{p-1}\right\}}.$$ 

Restricting attention to $B_b := S_{n+1+b,b}$, we define $[u]_{Cp,b} := [u, \sigma]_{Dp,n+1+b,b}$. As a corollary of Theorem 1 we have:

Theorem 3. Let $1 < p < \infty$ let $u$ be a weight and let $B_b := S_{n+1+b,b}$ be the Berezin transform. There holds:

$$[u]_{Cp,b}^\frac{1}{p} \leq \|B_b : L_p^b(u) \to L_p^b(u)\| \leq [u]_{Cp,b}^{\max\left\{1, \frac{1}{p-1}\right\}}.$$ 

The following corollary of Theorem 1 is well–known. See for example, [9, 31].

Corollary 4. For $1 < p < \infty$ the operator $S_{a,b}$ is bounded from $L_t^p$ to itself if and only if $-pa < t + 1 < p(b + 1)$.

The proof is to take $u(z) = (1 - |z|^2)^{t-b}$ and to note that the integrals in the definition of the $D_{p,a,b}$ condition are finite if and only if $-pa < t + 1 < p(b + 1)$. The details are left to the reader.

It is also well–known by now that $P_b$ is bounded from $L_t^p(u)$ to itself if any only if $u$ is a $B_{p,b}$ weight. This was proven for the disc in [4] and for the ball in [3]. The sharp dependence of the operator norm on the $B_{p,b}$ characteristic was given by S. Pott– M.C. Reguera in [27] for the Bergman space on the disc; namely the case when $n = 1$.

Our technique is that of dyadic operators. We show that the operators of interest can be dominated by positive dyadic operators and we use the techniques of modern dyadic harmonic analysis to deduce the desired estimates. This is the approach that S. Pott– M. C. Reguera took in [27], though we use a recent but similar approach of Lacey [14] that avoids an extrapolation argument.

The outline of the paper is as follows. In Section 2 we briefly give requisite background information and we recall a dyadic structure for $B_n$ given in, for example, [2]. In Section 3, we show that $S_{a,b}$ is equivalent to a finite sum of dyadic operators and in Section 4, we prove Theorem 1. Section 5 contains an example showing that the upper bound in Theorem 1 is sharp. Finally, Section 6 contains concluding remarks.

2. Background Information and Notation

The following notation will be used throughout the paper. For a weight $u$ and a subset $E \subset B_n$, we set $u_t(E) = \int_E u(z) dv_t(z)$ and $vol_t(E) = \int_E dv_t$. For a measure, $\mu$, and a subset $E \subset B_n$ we define $\langle f \rangle_{E}^{\mu} := \frac{1}{\mu(E)} \int_E f(z) d\mu(z)$.

We begin by recalling some geometric facts on the ball $B_n$. Let $\varphi_z$ be the involutive automorphism of $B_n$ that interchanges $z$ and $0$. That is, $\varphi_z$ is a holomorphic function
from $\mathbb{B}_n$ to itself that satisfies $\varphi_z \circ \varphi_z = \text{id}$, $\varphi_z(0) = z$, and $\varphi_z(z) = 0$. Using the maps $\varphi_z$ we can define the so-called Bergman metric, $\beta$ on $\mathbb{B}_n$, by:

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$ 

Let $B_\beta(z, r)$ be the ball in the Bergman metric of radius $r$ centered at $z$. It is well-known (see for example, [31]) that for $w \in B_\beta(z, r)$ there holds:

$$\text{vol}_t(B_\beta(z, w)) \approx |1 - z\omega|^{n+1+t} \approx (1 - |z|^2)^{n+1+t} \approx (1 - |w|^2)^{n+1+t}.$$ 

It is worthwhile to note that we will make heavy use of this and similar estimates.

We next introduce a dyadic structure on the ball. The construction we use is the one given in, for example, [2]. We start by fixing two parameters, $\theta, \lambda > 0$. These parameters will roughly correspond to the "sizes" of Carleson boxes.

For $N \in \mathbb{N}$, let $S_{N\theta}$ be the sphere of radius $N\theta$ in the Bergman metric. We can find a sequence of points, $E_N = \{w_j\}_{j=1}^N$ and a corresponding sequence of Borel subsets, $\{Q_j^N\}_{j=1}^N$ of $S_{N\theta}$ that satisfy:

$$\text{(i) } S_{N\theta} = \bigcup_{j=1}^N Q_j^N,$$

$$\text{(ii) } Q_i^N \cap Q_j^N = \emptyset \text{ when } i \neq j,$$

$$\text{(iii) } S_{N\theta} \cap B_\beta(w_j, \lambda) \subset Q_j^N \subset S_{N\theta} \cap B_\beta(w_j, \lambda \lambda).$$

Let $P_{N\theta}z$ be the radial projection of $z$ onto the sphere $S_{N\theta}$. Define subsets, $K_j^N$ of $\mathbb{B}_n$ by:

$$K_1^0 := \{z \in \mathbb{B}_n : \beta(0, z) < \theta\}$$

$$K_j^N := \{z \in \mathbb{B}_n : N\theta \leq \beta(0, z) < (N + 1)\theta \text{ and } P_{N\theta}z \in Q_j^N\}, N \geq 1, j \geq 1.$$ 

Now, let $c_j^N \in K_j^N$ be defined by $P_{(N+\frac{1}{2})\theta} w_j^N$. The sets $K_j^N$ are referred to as kubes and the points $c_j^N$ are the centers of the kubes.

Now we define a tree structure $\mathcal{T} := \{c_j^N\}$ on the centers of the kubes. We say that $c_j^{N+1}$ is a child of $c_j^N$ if $P_{N\theta}c_j^{N+1} \in Q_j^N$.

We will denote elements of the tree by the letters $\alpha$ and $\beta$ and $K_\alpha$ will be the kube with center $\alpha$. We will also abuse notation and use, for example, $\alpha$ to denote both an element of a tree $\mathcal{T}$ and the center of the corresponding kube or, in fact, any convenient element of the kube. There is the usual partial order on the tree: if $\alpha, \beta \in \mathcal{T}$ we say that $\beta \geq \alpha$ if $\beta$ is a descendant of $\alpha$. We will use $\overline{K}_\alpha$ to be the dyadic tent under $K_\alpha$. That is:

$$\overline{K}_\alpha := \bigcup_{\beta \in \mathcal{T} : \beta \geq \alpha} K_\beta.$$ 

We will also use $d(\alpha)$ to denote the "generation" of $\alpha$, or the distance in the tree from $\alpha$ to the root. Thus, if $N\theta < \beta(0, \alpha) < (N + 1)\theta$, then $d(\alpha) = N$. 

\[\text{Figure 1: Tree structure on the centers of the kubes.}\]
We have the following lemma proven in [2].

**Lemma 7.** Let $t > -1$ and let $T$ be a tree constructed with positive parameters $\lambda$ and $\theta$. Then the tree satisfies the following properties:

1. $B_0 = (\alpha \varsigma T_0\varsigma, 0) \cup \{B_\alpha\}$ and the kubs $K_\alpha$ are pairwise disjoint. Furthermore, there are constants, $C_1$ and $C_2$ depending on $\lambda$ and $\theta$ such that for all $\alpha \in T$ there holds:
$$B_\beta(\alpha, C_1) \subset K_\alpha \subset B_\beta(\alpha, C_2),$$

2. $\exists N\theta \subset \Theta(\theta)$ and $\exists N\theta \subset \Theta(\theta)$.

Note that if $z \in S_{\Theta(\theta)}$, then $\Theta(\theta) = \frac{1}{2}\log \frac{1 - |z|}{1 + |z|}$ and therefore, $|z| = \frac{e^{\Theta(\theta)} - 1}{e^{\Theta(\theta)} + 1}$. For the rest of the paper, set $\tau_{\Theta(\theta)} := \frac{e^{\Theta(\theta)} - 1}{e^{\Theta(\theta)} + 1}$. Note that we have $1 - \tau_{\Theta(\theta)} \approx e^{-2\Theta(\theta)}$. Therefore, if $\Theta(\alpha, \theta) = (N + \frac{1}{2})\Theta$ (that is, $d(\alpha) = N$), we have:

$$1 - |\alpha|^2)^{n+1+t} = \left(1 - \left(\frac{e^{2(N+\frac{1}{2})\Theta} - 1}{e^{2(N+\frac{1}{2})\Theta} + 1}\right)^n + t \right)^{n+1+t} \approx e^{-2(N+\frac{1}{2})\Theta(n+1+t)} \approx e^{-2\Theta(n+1+t)}.$$

Therefore, if $d(\alpha) = N$, there holds:

$$\rho(T_\alpha) \approx \rho(K_\alpha) \approx \rho(K_\alpha) \approx (1 - |\alpha|^2)^{n+1+t} \approx e^{-2\Theta(n+1+t)}.$$

We now show that every Carleson tent is well-approximated by a dyadic tent. To do this, we start with a special case of a lemma from [11].

Let $\rho$ be the pseudo-metric on $\partial B_n$ given by $\rho(z, w) = |1 - zw|$. As usual, $D(z, r) := \{w \in \partial B_n : \rho(z, w) < r\}$. A system of dyadic cubes of calibre $\delta$ is a collection of Borel subsets $D := \{Q^k\}_{i,k \in \mathbb{Z}}$ and points $\{z^k\}_{i,k \in \mathbb{Z}}$ in $\partial B_n$ that satisfy:

1. There are constants $c_1, c_2$ such that for every $k$, $i \in \mathbb{Z}$ there holds:
$$D(z^k_i, \delta^k) \subset Q^k_i \subset D(z^k_i, C_2\delta^k).$$

2. For all $k \in \mathbb{Z}$ there holds $\partial B_n = \bigcup_{i \in \mathbb{Z}} Q^k_i$ and the sets are disjoint.

3. If $Q, R \in D$ and $Q \cap R \neq \emptyset$, then either $Q \subset R$ or $R \subset Q$.

We have the following lemma which is a special case of [11, Theorem 4.1].

**Lemma 8 (Hytönen and Kairema).** For every $\delta > 0$ there is an $M \in \mathbb{N}$ such that there is a collection of dyadic systems of cubes $(\{Q^k_i\}_{i=1}^M)$ with the following property: For every disc $D(z, r) := \{w \in \partial B_n : \rho(z, w) < r\}$, there is a $1 \leq t \leq M$ such that there is a dyadic cube $Q^k_i \in D_t$ with $D(z, r) \subset Q^k_i$ and $\delta^k \approx r$ where the implied constants are independent of $z$ and $r$.

**Lemma 9.** There is a finite collection of Bergman trees $\{\tau_i\}_{i=1}^N$ such that for all $z \in \mathbb{B}_n$, there is a tree $T$ from the finite collection and an $\alpha \in T$ such that the dyadic tent $\hat{K}_\alpha := \cup_{\beta \geq \alpha} K_\beta$ contains the tent $T_z$ and $\rho(K_\alpha) \approx \rho(T_z)$. 
Proof. Let \( D = \{ Q_k^i \}_{k \in \mathbb{N}} \) be a dyadic system of calibre \( \delta \). We will use this dyadic system to create a Bergman tree with parameters \( \theta \) and \( \lambda \) where \( \delta = e^{-20} \) and \( \lambda \) will be chosen below. For each \( k \in \mathbb{N} \), we project the sets \( \{ Q_k^i \}_{i \in \mathbb{Z}} \) radially onto the sphere \( S_{k \theta} \). Let \( P_{k \theta} \) be the radial projection onto the sphere \( S_{k \theta} \). We will now show that these sets \( \{ P_{k \theta} Q_k^i \}_{k \in \mathbb{N}} \) satisfy the three properties in (6) which means a Bergman tree can be constructed from them according to the construction in [2].

Clearly the sets \( \{ P_{k \theta} Q_k^i \}_{k \in \mathbb{N}} \) satisfy Properties (i) and (ii) in (6). For the third property, observe that it is enough to show that there are two positive constants \( \lambda_1 \) and \( \lambda_2 \) independent of \( i, k \) that:

\[
S_{k \theta} \cap B_{\beta}(r_{k \theta} z_k^i, \lambda_1) \subset P_{k \theta} Q_k^i \subset S_{k \theta} \cap B_{\beta}(r_{k \theta} z_k^i, \lambda_2)
\]

where \( z_k^i \) is the centre of \( Q_k^i \) and \( P_{k \theta} Q_k^i \) is the projection of \( Q_k^i \) onto \( S_{k \theta} \). Now, recall that \( \tanh \beta(z, w) = |\varphi_z(w)| \). Therefore, \( \beta(z, w) \leq R \) if and only if \( |\varphi_z(w)| \leq \tanh R \) if and only if \( 1 - |\varphi_z(w)|^2 \geq 1 - (\tanh R)^2 \). Now, \( 1 - |\varphi_z(w)|^2 = (1 - |z|^2)(1 - |w|^2) |1 - z\overline{w}|^{-2} = (1 - r_{k \theta}^2)^2 |1 - z\overline{w}|^{-2} \). Thus, for \( z, w \in S_{k \theta} \), \( \beta(z, w) \leq R \) if and only if \( |1 - z\overline{w}| \leq (1 - r_{k \theta}^2)(1 - (\tanh R)^2)^{-1/2} \). Now, if \( \xi \in Q_k^i \) then there holds:

\[
|1 - P_{k \theta} \overline{\xi} P_{k \theta} z_k^i| = |1 - r_{k \theta} \overline{\xi} r_{k \theta} z_k^i| \leq |1 - r_{k \theta}^2| + r_{k \theta}^2 |1 - \xi \overline{z_k^i}| \leq e^{-2k \theta}.
\]

On the other hand, if \( \xi \in S_{k \theta} \cap B_{\beta}(P_{k \theta} z_k^i, R) \), then \( |1 - r_{k \theta} \overline{\xi} r_{k \theta} z_k^i| \leq e^{-2k \theta} (1 - (\tanh R)^2)^{-1/2} \) and so there holds:

\[
|1 - \overline{\xi} z_k^i| \leq |1 - r_{k \theta} \overline{\xi} r_{k \theta} z_k^i| + |\overline{\xi} r_{k \theta}^2 z_k^i - \xi | \leq e^{-2k \theta} = \delta^k.
\]

Clearly, (11) and (12) together imply the existence of \( \lambda_1 \) and \( \lambda_2 \) such that (10) is satisfied.

Let \( T_z \) be a Carleson tent and note that the "base" of \( T_z \) is the disc \( D(P_z, 1 - |z|) \). By Lemma 8, there is a finite number of dyadic systems, \( \{ D_i \}_{i=1}^M \), such that every disc \( D \) is contained in a dyadic cube of comparable radius. Then this disc is contained in some element \( Q \) of one of the dyadic systems and the dyadic tent over \( Q \) contains the tent \( T_z \). This completes the proof. \( \Box \)

Of course, maximal functions with respect to this dyadic structure will play a role. Thus, for a weight, \( u \), and a Bergman tree, \( \mathcal{T} \), define the following maximal function:

\[
M_{\mathcal{T}, u}(w) := \sup_{a \in \mathcal{T}} \frac{1}{u_t(K_a)} \int_{K_a} |f(z)| u(z) dv_t(z).
\]

The following lemma is well-known:

**Lemma 13.** Let \(-1 < t \) and \( u \) be a weight, then \( M_{\mathcal{T}, u} \) is bounded on \( L^p_t(u) \) for \( 1 < p \leq \infty \) and is bounded from \( L^1_t(u) \to L^1_t(\infty)(u) \).
Finally, we make an observation. The norm inequality \( \|S_{a,b}f\|_{L^p_B(u)} \leq \|f\|_{L^p_B(u)} \) is the same as the norm inequality \( \|Q_{a,b}f\|_{L^p_B(\bar{u})} \leq \|f\|_{L^p_B(u)} \) where \( \bar{u}(z) := u(z)(1-|z|^2)^p \) and

\[
Q_{a,b}f(z) := \int_{\mathbb{B}_n} \frac{1}{(1-z\bar{w})^{n+1+a+b}} f(w) dv_b(w).
\]

A similar remark is true for \( S_{a,b}^+ \) and the similarly defined operator \( Q_{a,b}^+ \). Thus for \( b > -1 \) the claim in Theorem 1 is equivalent to:

\[
[u,\sigma]|_{D_{p,a,b}} \leq \|Q_{a,b} : L^p_B(u) \to L^p_B(\bar{u})\| \leq \|Q_{a,b}^+ : L^p_B(u) \to L^p_B(\bar{u})\| \leq [u,\sigma]|_{D_{p,a,b}}^{\max\{1,p-1\}}.
\]

3. Equivalence to a Dyadic Operator

In this section, we will show that when \( b+1 > 0 \), \( Q_{a,b}^+ \) is pointwise equivalent to a finite sum of simple operators of the form:

\[
T_{\mathcal{T}}f := \sum_{\alpha \in \mathcal{T}} \operatorname{vol}_b\left(K_\alpha\right) \frac{\alpha}{\alpha+n+1+b} (f)_{K_\alpha} B^0\left|K_\alpha\right|
\]

where \( \mathcal{T} \) is a Bergman tree.

We first show that \( Q_{a,b}^+ \) is dominated by a finite sum of operators of the desired form. Assume that for every \( z, w \in \mathbb{B}_n \), there is a Carleson tent, \( T \), containing \( z \) and \( w \) such that \( \operatorname{vol}_b(T) \approx |1-zw|^{n+1+b} \). Then we can use Lemma 9 to deduce that

\[
Q_{a,b}^+f(z) \leq \sum_{l=1}^M \sum_{\alpha \in \mathcal{T}_l} \operatorname{vol}_b\left(K_\alpha\right) \frac{\alpha}{\alpha+n+1+b} (f)_{K_\alpha} B^0\left|K_\alpha\right|(z).
\]

Therefore, we only need to show that for every \( z, w \in \mathbb{B}_n \), there is a Carleson tent \( T \) containing \( z, w \) such that \( \operatorname{vol}_b(T) \approx |1-zw|^{n+1+b} \). We now turn to that.

Note that there is an \( N \in -1, 0, 1, 2, \ldots \) such that \( |1-zw| \approx e^{-2N\theta} \) (that is, \( e^{-2(N+1)\theta} \leq |1-zw| \leq e^{-2N\theta} \)). We will show that there is an \( k \in \mathbb{N} \) so that \( P_{N\theta}w \) is on the same ray as \( z \) or \( w \). Such that \( z, w \in T_{P_{N\theta}w} \). Since \( \operatorname{vol}_b(T_{P_{N\theta}w}) \approx e^{-2(N-k)\theta(n+1+b)} \approx e^{-2N\theta(n+1+b)} \approx |1-zw|^{n+1+b} \), this will prove the claim.

We first show that \( w \in T_{P_{N\theta}w} \). Indeed, we will show that \( w \in T_{P_{N\theta}w} \). Since \( w \) and \( P_{N\theta}w \) are on the same ray, we need to show that \( |P_{N\theta}w| \leq |w| \). But this is not difficult:

\[
|P_{N\theta}w| = \frac{e^{2N\theta} - 1}{e^{2N\theta} + 1} \leq 1 - e^{-2N\theta} \leq 1 - |1-zw| \leq 1 - |zw| = |zw| \leq |w|.
\]

We next show that \( z \in T_{1-P_{N\theta}w} \). To do this, it is enough to show that \( |1-zPw| \leq e^{-2N\theta} \approx 1 - |P_{N\theta}w| \). (The \( k \) will essentially be the logarithm of the implied constant, but since the exact value is not important, we do not attempt to calculate it.) First, note that \( |P_{N\theta}w| \leq |w| \) and so there holds:

\[
|Pw - w| \leq |Pw - P_{N\theta}w| = \left| \frac{w}{|w|} - \frac{w}{|w|} \right| = 1 - r_{N\theta} \approx e^{-2N\theta}.
\]
And we then have:

\[ |1 - zPw| \leq |1 - zw| + |zPw - zw| \leq e^{-2N_0} + |Pw - w| \leq e^{-2N_0}. \]

Therefore, we have shown that \( Q_{a,b}^+ \) is dominated by a finite sum of operators of the form \( T_T \).

We now show that \( Q_{a,b}^+ \) dominates every dyadic operator as we have defined above. That is, for every Bergman tree \( T \) we will show that for all \( z \in B_n \) there holds \( |T_T f(z)| \leq Q_{a,b}^+ |f| (z) \). The proof here is similar to the one in, for example, [5] for the fractional integral operator.

We first make some computations and fix some notation. First, we may assume that \( f \) is non-negative. For \( z \in B_n \), let \( \alpha = \alpha(z) \) be the unique element of \( T \) such that \( z \in K_{\alpha} \). For \( \beta \in T \) with \( \beta \geq \alpha \), let \( s(\alpha, \beta) \) denote the unique element of \( B_n \) that satisfies \( \beta \leq s(\alpha, \beta) \leq \alpha \) and \( d(s(\alpha, \beta)) = d(\beta) + 1 \). That is, \( s(\alpha, \beta) \) is the child of \( \beta \) that is “in-between” \( \beta \) and \( \alpha \). Let \( E_{\alpha, \beta} := K_\beta \setminus K_{s(\alpha, \beta)} \) and observe that for fixed \( \alpha \), these sets are pairwise disjoint. Note that \( \text{vol}_b(K_\beta) \approx e^{2\theta (n+1+b)} \text{vol}_b(K_{s(\alpha, \beta)}) \). Also, note that for \( z, w \in K_\beta \), there holds \( |1 - zw| \leq 1 - |\beta|^2 \). This can be seen by, for example, noting that \( |1 - zw| \leq |1 - \beta \bar{w}| + |\beta - z| \); since \( z, w \in K_\beta \), both of these terms are dominated by \( 1 - |\beta| < 1 - |\beta|^2 \). Thus, for fixed \( \alpha \in T \) and \( z \in K_\alpha \), there holds:

\[
\sum_{\beta \in T; \beta \leq \alpha} \int_{E_{\alpha, \beta}} \frac{f(w) \, dv_b(w)}{\text{vol}_b(K_\beta)^{1+\frac{a}{n+1+b}}} \leq \sum_{\beta \in T; \beta \leq \alpha} \int_{E_{\alpha, \beta}} \frac{f(w) \, dv_b(w)}{|1 - zw|^{n+1+a+b}} \leq Q_{a,b}^+ f(z).
\]

Also, there holds (again for fixed \( \alpha \) and \( z \in K_\alpha \)):

\[
\sum_{\beta \in T; \beta \leq \alpha} \left( \text{vol}_b(K_\beta) \right)^{-\frac{a}{n+1+b}} \int_{K_{s(\alpha, \beta)}} f(w) \, dv_b(w)
\]

is controlled by

\[
e^{-2\theta (n+1+b)} \sum_{\beta \in T; \beta \leq \alpha} \left( \text{vol}_b(K_{s(\alpha, \beta)}) \right)^{-\frac{a}{n+1+b}} \int_{K_{s(\alpha, \beta)}} f(w) \, dv_b(w).
\]

But this is just:

\[
e^{-2\theta (n+1+b)} \sum_{\alpha \in T} \text{vol}_b \left( K_\alpha \right)^{-\frac{a}{n+1+b}} \langle f \rangle_{K_\alpha} \mathbb{1}_{K_\alpha} (z) = e^{-2\theta (n+1+b)} T_T f(z).
\]

Therefore, for fixed \( z \in B_n \) and \( \alpha = \alpha(z) \)

\[
T_T f(z) = \sum_{\gamma \in T} \text{vol}_b \left( K_\gamma \right)^{-\frac{a}{n+1+b}} \langle f \rangle_{K_\gamma} \mathbb{1}_{K_\gamma} (z)
\]

is equal to

\[
\sum_{\beta \in T; \beta \leq \alpha} \int_{E_{\alpha, \beta}} \frac{f(w) \, dv_b(w)}{\text{vol}_b(K_\beta)^{1+\frac{a}{n+1+b}}} + \sum_{\beta \in T; \beta \leq \alpha} \left( \text{vol}_b(K_\beta) \right)^{-\frac{a}{n+1+b}} \int_{K_{s(\alpha, \beta)}} f(w) \, dv_b(w).
\]
Therefore, by the above, we have:

\[ T_T f(z) \leq C Q_{a,b}^+ f(z) + e^{-2b|n+1+a+b|} T_T f(z). \]

Since \( n + 1 + a + b > 0 \), rearranging the above completes the proof.

Thus, we have proven the following lemma:

**Lemma 15.** There is a finite collection of Bergman trees, \( \{T_i\}_{i=1}^M \) such that for \( b > -1 \) there holds:

\[ Q_{a,b}^+ f(z) \sim \sum_{i=1}^M \sum_{a \in T_i} \text{vol}_b \left( \mathcal{K}_a \right)^{\frac{n+1}{n+1+b}} \langle f \rangle^{\text{div}}_{\mathcal{K}_a} \mathbf{1}_{\mathcal{K}_a}(z). \]

### 4. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. The proof requires that either \( b + 1 > 0 \) or \( a + 1 > 0 \). Since \(-a < b + 1\) then this holds. The proofs given in this section are for the case \( b + 1 > 0 \) and the case \( b + 1 \leq 0 \) and \( a > 0 \) is obtained by the following argument using duality.

For a weight, \( \omega \), the dual space of \( L^p_\omega(\omega) \) under the unweighted inner product on \( L^2_\omega \) is \( L^p_\omega(\omega \omega^p) \). It is also easy to see that the \( L^2_\omega \) adjoint of \( S_{a,b} \) is \( S_{b,a} \). Let \( \rho(z) = u(z)(1 - |z|^2)^b \) and let \( \psi(z) = u(z) \frac{p-a}{p} (1 - |z|^2)^{\frac{a}{p}} (p^b + p^a) \). Note that \( \psi(z)(1 - |z|^2)^a = u(z) \frac{p-a}{p} (1 - |z|^2)^{\frac{a}{p}} \) is the dual weight of \( \rho(z) \). There holds:

\[ \|S_{a,b} : L^p_\omega(\omega) \rightarrow L^p_\omega(\omega)\| = \|S_{a,b} : L^p_\omega(\rho) \rightarrow L^p_\omega(\rho)\| = \|S_{b,a} : L^p_\omega(\psi) \rightarrow L^p_\omega(\psi)\|, \]

and a similar statement holds for \( S_{a,b}^+ \) and \( S_{b,a}^+ \). Letting \( \nu(z) = \psi(z) \frac{p-a}{p} \) we may use the results of this section to deduce:

\[ [\psi, \nu]_{D_{p',b,a}}^{\frac{1}{p'}} \leq \|S_{a,b} : L^p_\omega(\omega) \rightarrow L^p_\omega(\omega)\| \leq \|S_{b,a}^+ : L^p_\omega(\omega) \rightarrow L^p_\omega(\omega)\| \leq [\psi, \nu]_{D_{p',b,a}}^{\frac{1}{p'}} \]

and this is exactly what is claimed in Theorem 1 for the case \( b > -1 \).

Recall that \( \tilde{u}(z) = u(z) \left(1 - |z|^2\right)^{\frac{a}{p}} \) and \( \sigma(z) = u(z) \frac{p-a}{p} \). We will use the following fact:

\[ \|S_{a,b} : L^p_\omega(\omega) \rightarrow L^p_\omega(\omega)\| = \|Q_{a,b} : L^p_\omega(\omega) \rightarrow L^p_\omega(\tilde{u})\| = \|Q_{a,b}(\sigma) : L^p_\omega(\sigma) \rightarrow L^p_\omega(\tilde{u})\|. \]

A similar statement of course also holds for \( Q_{a,b}^+ \) and \( S_{a,b}^+ \).
4.1. Proof of Lower Bound in Theorem 1 when $b + 1 > 0$. In this subsection we prove the lower bound in Theorem 1 under the assumption that $b + 1 > 0$. That is, we will show
\[
\mathcal{A} := \|Q_{a,b}(\sigma \cdot) : L^p_b(\sigma) \to L^p_b(\tilde{u})\| < \infty \quad \Rightarrow \quad [u, \sigma]_{D_p,a,b} \leq \mathcal{A}^p.
\]

We first give a familiar property of weights.

**Lemma 16.** There holds:
\[
\langle \tilde{u} \rangle_{K_{\alpha}}^{d\nu_b} \left( \langle \sigma \rangle_{K_{\alpha}}^{d\nu_b} \right)^{p-1} \nu_b \left( K_{\alpha} \right)^{\frac{a}{n+1+b}} \geq 1.
\]

**Proof.** Recall that $\tilde{u}(z) = u(z)(1 - |z|^2)^{pa}$ and $\tilde{u}_b(K_{\alpha}) = \int_{K_{\alpha}} \tilde{u}(z)d\nu_b(z)$. To prove the claim, we will prove the equivalent inequality:
\[
\nu_b(K_{\alpha})\nu_b(K_{\alpha})^{\frac{a}{n+1+b}} \leq \tilde{u}_b(K_{\alpha})^{\frac{1}{p}} \left( \sigma_b(K_{\alpha}) \right)^{\frac{1}{p'}}.
\]

Indeed, there holds:
\[
\nu_b(K_{\alpha})\nu_b(K_{\alpha})^{\frac{a}{n+1+b}} \simeq (1 - |\alpha|^2)^{n+1+b+a} \simeq \int_{K_{\alpha}} (1 - |z|^2)^a d\nu_b(z).
\]

So by Hölder’s Inequality we have:
\[
\nu_b(K_{\alpha})\nu_b(K_{\alpha})^{\frac{a}{n+1+b}} \simeq \int_{K_{\alpha}} \sigma(z)^{\frac{1}{p'}} \tilde{u}(z)^{\frac{1}{p'}} d\nu_b(z) \leq \tilde{u}_b(K_{\alpha})^{\frac{1}{p'}} \left( \sigma_b(K_{\alpha}) \right)^{\frac{1}{p'}}.
\]

If $\mathcal{A} < \infty$, then in particular the following weak–type inequality holds:
\[
\tilde{u}_b \left( \{|w| \in \mathbb{B}_n : |Q_{a,b}(\sigma f)(w)| > \lambda \} \right) \leq \frac{\mathcal{A}^p}{\lambda^p} \int_{\mathbb{B}_n} |f(z)|^p \sigma(z)d\nu_b(z).
\]

Since $n + 1 + a + b > 0$, by [3, Lemma 5] there is an $N = N(n, a, b) > 0$ so that if $\alpha \in \mathcal{T}$ and $d(\alpha) > N$, then there is a $\beta \in \mathcal{T}$ with $d(\beta) = d(\alpha)$ such that for all $z \in \mathcal{K}_{\beta}$ there holds: $|Q_{a,b}(\sigma \mathbb{1}_{\mathcal{K}_{\alpha}})(z)| \gtrsim \langle \sigma \rangle_{K_{\alpha}}^{d\nu_b} \nu_b \left( K_{\alpha} \right)^{\frac{a}{n+1+b}}$ . Therefore,
\[
\mathcal{K}_{\beta} \subset \left\{ w \in \mathbb{B}_n : |Q_{a,b}(\sigma \mathbb{1}_{\mathcal{K}_{\alpha}})(w)| \gtrsim \langle \sigma \rangle_{K_{\alpha}}^{d\nu_b} \nu_b \left( K_{\alpha} \right)^{\frac{a}{n+1+b}} \right\}.
\]

By the weak–type inequality, there holds:
\[
\tilde{u}_b(\mathcal{K}_{\beta}) \leq \mathcal{A}^p \frac{\nu_b \left( K_{\alpha} \right)^{\frac{a}{n+1+b}} \nu_b \left( K_{\alpha} \right)^{p}}{\sigma_b(K_{\alpha})^p} \int_{K_{\alpha}} \sigma(z)d\nu_b(z).
\]

Rearranging this we find:
\[
\langle \tilde{u} \rangle_{\mathcal{K}_{\beta}}^{d\nu_b} \left( \langle \sigma \rangle_{K_{\alpha}}^{d\nu_b} \right)^{p-1} \nu_b \left( K_{\alpha} \right)^{\frac{a}{n+1+b}} \leq \mathcal{A}^p,
\]
and interchanging the roles of $\alpha$ and $\beta$ yields:
\[
\langle \tilde{u} \rangle_{K_\alpha}^{dv_b} \langle (\sigma) \rangle_{K_\alpha}^{dv_b} \overset{p-1}{\ vol_b} \left( \frac{2p}{n+1+q} \right) \leq A^p.
\]
Thus, using Lemma 16 there holds:
\[
\sup_{\alpha \in T:d(\alpha) > N} \langle \tilde{u} \rangle_{K_\alpha}^{dv_b} \langle (\sigma) \rangle_{K_\alpha}^{dv_b} \overset{p-1}{\ vol_b} \left( \frac{2p}{n+1+q} \right) \leq A^{2p}.
\]

This proves the lower bound in Theorem 1 when the supremum is taken over small tents. We now show that it holds when the supremum is taken over big tents. Define:
\[
\mathbb{I}_{B_p(0,N)}(w) = \left( 1 - |w|^2 \right)^{-b}
\]
and note that $f$ is bounded. Then since $(1 - \mathbb{I}w)^{-1}$ is analytic as a function of $w$ and has no zeros, it follows that $Q_{a,b}f(z) = C_N$. Using again the weak–type inequality this implies:
\[
\bar{u}_b(B_N) = \bar{u}_b \left( \left\{ w \in B_N : |Q_{a,b}f(w)| > \frac{C_N}{2} \right\} \right) \leq \frac{2pA^p}{C_N} \int_{B_p(0,N)} |f(z)|^p \ dv_b(z) \simeq A^p.
\]

On the other hand, if $Q_{a,b}$ is well–defined for $f \in L_b^p(u)$, then $\sigma(B_N) < \infty$ by [3, Lemma 4]. Therefore, by (17) and this observation there holds:
\[
\sup_{\alpha \in T} \langle \tilde{u} \rangle_{K_\alpha}^{dv_b} \langle (\sigma) \rangle_{K_\alpha}^{dv_b} \overset{p-1}{\ vol_b} \left( \frac{2p}{n+1+q} \right) \leq A^{2p},
\]
as desired.

4.2. **Proof of Upper Bound in Theorem 1 when $b + 1 > 0$.** In this subsection we prove the upper bound in Theorem 1 under the assumption that $b + 1 > 0$. That is, we will show:
\[
\|Q_{a,b}^+(\sigma \cdot) : L_b^p(\sigma) \rightarrow L_b^p(\bar{u})\| \leq [u, \sigma]_{D_{p,a,b}}^{\max\{1, \frac{1}{1-p} \}}.
\]
We first handle the case $1 < p \leq 2$; the other case will follow from a duality argument. Fix a Bergman tree $T$ and let $T = T$ where $T$ is the operator in (14). It will be enough to estimate $\|T(\sigma \cdot) : L_b^p(\sigma) \rightarrow L_b^p(\bar{u})\|$. That is, we will show:
\[
\|T(\sigma f)\|_{L_b^p(\bar{u})} \leq [u, \sigma]_{D_{p,a,b}}^{\frac{1}{1-p}} \|f\|_{L_b^p(\sigma)}.
\]
It is more convenient to prove the equivalent inequality:
\[
\|T(\sigma f)^{p-1}\|_{L_b^p(\bar{u})} \leq [u, \sigma]_{D_{p,a,b}} \|f\|^{p-1}_{L_b^p(\sigma)}.
\]
We will use duality and prove the following estimate, for all non–negative $f \in L_b^p(\sigma)$ and $g \in L_b^p(\bar{u})$:
\[
\langle T(\sigma f)^{p-1}, \bar{u}g \rangle_{L_b^2} \leq [u]_{D_{p,a,b}} \|f\|^{p-1}_{L_b^p(\sigma)} \|g\|_{L_b^p(\bar{u})}.
\]
Before proving (18), we discuss some facts that will be used. First, using (5) there holds:

$$\text{vol}_b(\widehat{K}_\alpha)^{1+\frac{q}{n+1+b}} \simeq \text{vol}_b(K_\alpha)^{1+\frac{q}{n+1+b}} \simeq (1 - |\alpha|^2)^{n+1+a+b} \int_{\widehat{K}_\alpha} (1 - |z|^2)^{a} \, dv_b(z).$$

Therefore using the fact that $\sigma = u^{-\frac{1}{p'}}$ we have:

$$\text{vol}_b(\widehat{K}_\alpha)^{1+\frac{q}{n+1+b}} \simeq \int_{\widehat{K}_\alpha} \sigma(z) \frac{1}{p} u(z) \frac{1}{p} (1 - |z|^2)^{a} \, dv_b(z) \leq (\sigma_b(K_\alpha))^\frac{1}{p} (\tilde{u}_b(K_\alpha))^\frac{1}{p}.$$

Recall also that $\langle \sigma f \rangle_{K_\alpha}^{dv_b} = \langle f \rangle_{K_\alpha}^{\sigma dv_b}$. Finally, since $h(x) = x^r$ is subadditive for $0 < r \leq 1$, using this applied with $r = p - 1$, recall that $1 < p \leq 2$, gives:

$$(T(\sigma f)(z))^{p-1} \leq \sum_{\alpha \in T} \text{vol}_b(\widehat{K}_\alpha)^{\frac{(n+1+b)}{n+1}} \left( \langle \sigma f \rangle_{K_\alpha}^{dv_b} \right)^{p-1} \mathbb{1}_{\widehat{K}_\alpha}(z).$$

Using these facts, we now prove (18). Indeed, we have

$$\left\langle (T(\sigma f))^{p-1}, \tilde{u} g \right\rangle_{L^2_b} \leq \sum_{\alpha \in T} \left( \langle f \rangle_{K_\alpha}^{\sigma dv_b} \langle \sigma f \rangle_{K_\alpha}^{dv_b} \text{vol}_b(K_\alpha)^{\frac{1}{n+1+b}} \right)^{p-1} \int_{\widehat{K}_\alpha} g(z) \tilde{u}(z) dv_b(z)$$

$$\leq \sum_{\alpha \in T} \left( \langle f \rangle_{K_\alpha}^{\sigma dv_b} \langle \sigma f \rangle_{K_\alpha}^{dv_b} \text{vol}_b(K_\alpha)^{\frac{1}{n+1+b}} \right)^{p-1} \left( \langle \sigma b(K_\alpha) \rangle_{K_\alpha}^{\frac{1}{p}} \langle g \rangle_{K_\alpha}^{\tilde{d} dv_b} \langle \tilde{u} \rangle_{K_\alpha}^{dv_b} \right)^{p-1} \text{vol}_b(\widehat{K}_\alpha)^{1+\frac{q}{n+1+b}}$$

$$\leq [u, \sigma]_{D_{p,a,b}} \sum_{\alpha \in T} \left( \langle f \rangle_{K_\alpha}^{\sigma dv_b} \langle \sigma f \rangle_{K_\alpha}^{dv_b} \text{vol}_b(K_\alpha)^{\frac{1}{n+1+b}} \right)^{p-1} \left( \langle \sigma b(K_\alpha) \rangle_{K_\alpha}^{\frac{1}{p}} \langle g \rangle_{K_\alpha}^{\tilde{d} dv_b} \langle \tilde{u} \rangle_{K_\alpha}^{dv_b} \right)^{p-1} \text{vol}_b(\widehat{K}_\alpha)^{1+\frac{q}{n+1+b}}.$$

By Hölder’s Inequality, the sum above is dominated by:

$$\left\{ \sum_{\alpha \in T} \left( \langle f \rangle_{K_\alpha}^{\sigma dv_b} \langle \sigma f \rangle_{K_\alpha}^{dv_b} \text{vol}_b(K_\alpha)^{\frac{1}{n+1+b}} \right)^{p-1} \right\}^{\frac{1}{p}} \left\{ \sum_{\alpha \in T} \left( \langle g \rangle_{K_\alpha}^{\tilde{d} dv_b} \langle \tilde{u} \rangle_{K_\alpha}^{dv_b} \text{vol}_b(K_\alpha)^{\frac{1}{n+1+b}} \right)^{p-1} \right\}^{\frac{1}{p}}.$$

Using the disjointness of the sets $K_\alpha$ we estimate the first factor above using Lemma 13:

$$\sum_{\alpha \in T} \left( \langle f \rangle_{K_\alpha}^{\sigma dv_b} \langle \sigma f \rangle_{K_\alpha}^{dv_b} \text{vol}_b(K_\alpha)^{\frac{1}{n+1+b}} \right)^{p} \sigma_b(K_\alpha) \leq \int_{\mathbb{B}_n} (M_{T, \sigma dv_b} f(z))^p \sigma(z) dv_b(z) \leq \|f\|_{L^p_b(\sigma)}^p.$$

A similar estimate holds for the second factor, completing the proof in the case $1 < p \leq 2$. We now handle the case $2 < p < \infty$. That is we want to show:

$$\left\langle (T(\sigma f), \tilde{u} g) \right\rangle_{L^2_b} \leq [u, \sigma]_{D_{p,a,b}} \|f\|_{L^p_b(\sigma)} \|g\|_{L^p_b'(\tilde{u})},$$

for all non-negative $f \in L^p_b(\sigma)$ and $g \in L^{p'}_b'(\tilde{u})$. Now, define:

$$\psi(z) := \sigma(z) \left(1 - |z|^2\right)^{-p'a} \quad \text{and} \quad \bar{\psi}(z) := \psi(z) \left(1 - |z|^2\right)^{p'a}.$$
Clearly, $\tilde{\psi} = \sigma$. Set $\rho(z) := \psi(z) \tilde{\psi}^{p'}/p$. There holds:

$$\tilde{u}(z) = \sigma(z) \tilde{\psi}^{p'}(1 - |z|^2)^{pa} = \left(\sigma(z)(1 - |z|^2)^{-p'a}\right) \tilde{\psi}^{p'} = \rho(z).$$

Now, it is easy to see that:

$$[\psi, \rho]_{D_{p'}^{a,b}} := \sup_{\alpha \in \mathcal{T}} \left(\int_{K_{\alpha}} |\tilde{\psi}|^{p'} \frac{dv}{|vol|_{B}}\right) \left(\int_{K_{\alpha}} |\psi^{a} - \sigma^{a}| \frac{dv}{|vol|_{B}}\right) \frac{|vol|_{B}}{|\psi|^{p'a}}$$

$$= \sup_{\alpha \in \mathcal{T}} \left(\int_{K_{\alpha}} |\tilde{\psi}|^{p'} \frac{dv}{|vol|_{B}}\right) \left(\int_{K_{\alpha}} |\psi^{a} - \sigma^{a}| \frac{dv}{|vol|_{B}}\right)$$

$$= [u, \sigma]^{p'-1}_{D_{p'}^{a,b}}.$$

Therefore, using the fact that $T$ is self-adjoint we have $\langle T(\sigma f), \tilde{u} g \rangle_{L^2_{B}} = \langle T(\rho g), \tilde{\psi} f \rangle_{L^2_{B}}$. Using the fact that $p' < 2$, yields:

$$\langle T(\rho g), \tilde{\psi} f \rangle_{L^2_{B}} \leq [\psi, \rho]_{D_{p'}^{a,b}} \|f\|_{L^p_{B}(\psi)} \|g\|_{L^p_{B}(\tilde{\psi})} = [u, \sigma]_{D_{p'}^{a,b}} \|f\|_{L^p_{B}(\sigma)} \|g\|_{L^p_{B}(\tilde{\psi})}.$$

This completes the proof of the upper bound in Theorem 1.

5. A Sharp Example

In this section, we give a weight, $u$ and a function $f$ such that:

$$\|Pf\|_{L^2_{B}(u)} \geq [u]_{B_{2}} \|f\|_{L^2_{B}(u)},$$

which implies that the upper bound in Theorem 1 is sharp. The idea is to reduce to the one-dimensional case and to use what is essentially the sharp example in [27].

Let $u(z) = u(z_1) = |1 - z_1|^{(n+1+b)/(1-\delta)} |1 + z_1|^{(n+1+b)/(\delta-1)}$. We want to compute the $B_2$ characteristic of $u$. For $r_0 > 0$, abuse notation and let $r_0$ denote the vector $(r_0, 0, \ldots, 0)$ and similarly for $-r_0$. Let $z = (z_1, z')$, that is $z' = (z_2, \ldots, z_n)$. There holds:

$$\int_{B_{r_0}} u(z)(1 - |z|^2)^{b} dV(z)$$

is equal to

$$\int_{|z_1| < 1 - r_0} \int_{|z_1^{(n+1+b)(1-\delta)}|} |\frac{1 - z_1^{(n+1+b)(1-\delta)}|}{1 + z_1^{(n+1+b)(\delta-1)}}|^{z':|z'|^2 < 1 - |z_1|^2} (1 - |z|^2)^{b} dV_{n-1}(z') dA(z_1),$$

where above $dV_{n-1}$ is Lebesgue measure on $\mathbb{C}^{n-1}$ and $dA(z)$ is Lebesgue measure on $\mathbb{C}$. For the inner integral, let $w = z'/\sqrt{1 - |z_1|^2}$. Using this change of variables, the inner integral becomes:

$$\int_{B_{n-1}} (1 - |z_1|^2)^{b} (1 - |w|^2)^{b} \left(\sqrt{1 - |z_1|^2}\right)^{2(n-1)} dV_{n-1}(w) \simeq (1 - |z_1|^2)^{n+b-1}.$$
Inserting this into (20), we see that (19) is comparable to:

$$\int_{\{z_1 \in \mathbb{D} : |1 - z_1| < 1 - r_0\}} u(z_1) \left(1 - |z_1|^2\right)^{n+b-1} dA(z_1).$$

To estimate this integral, it is easiest to make the conformal change of variables $w = \frac{i - z_1}{1 + z_1}$ so that when $r_0$ is bounded away from 0 (as is the case here) this integral is comparable to:

$$\int_{\{w \in \mathbb{H} : |w| < R(\tau_0)\}} |w|^{(n+1+b)(1-\delta)} (\mathcal{F}w)^{n+b-1} dA(w) \simeq \frac{R_0^{(n+1+b)(2-\delta)}}{(n + 1 + b)(2 - \delta)},$$

where above $R_0 = R(\tau_0)$.

Using similar reasoning, there holds:

$$\int_{\tau_0} u^{-1}(z)(1 - |z|^2)\,dV(z) \simeq \int_{\{w \in \mathbb{H} : |w| < R(\tau_0)\}} |w|^{(n+1+b)(\delta-1)} (\mathcal{F}w)^{n+b-1} dA(w) \simeq \frac{R_0^{(n+1+b)\delta}}{(n + 1 + b)\delta}.$$

With $R = R(\tau_0)$ there holds $v_b(\tau_0) \simeq R^{n+1+b}$. Thus, there holds $\langle u \rangle_{\tau_0}^{d_0} (u^{-1})_{\tau_0}^{d_0} \simeq \delta^{-1}$. Similarly, $\langle u \rangle_{\tau_0}^{d_0} (u^{-1})_{\tau_0}^{d_0} \simeq \delta^{-1}$. Now, the singularities of $u$ and $u^{-1}$ are at $(-1, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$ so the argument above implies that if we take, say, $r_0 > \frac{1}{2}$ (so that $R_0 \leq 1$), then $u$ is a $B_2$ weight with $\|u\|_{B_2} \simeq \delta^{-1}$.

Now, let $f(w) = u^{-1}(w) \mathbb{1}_{\tau_{1/2}}$. Then $\int_{\mathbb{B}_n} |f(w)|^2 \, udv_b(w) = \int_{\tau_{1/2}} u^{-1}(w)dv_b(w) \simeq \delta^{-1}$. Now, we give a pointwise estimate of $P_b f(z)$. To do this, we may use the idea that we used to obtain the lower bound in Theorem 1 from [3]. That is, for $z \in T_{\tau_{1/2}}$ there holds $|P_b f(z)| \geq \langle f \rangle_{\tau_{1/2}} \simeq \delta^{-1}$. Therefore, making the change of variables $w' = -w$ there holds:

$$\|Pf\|_{L_b^2(u)}^2 = \int_{\mathbb{B}_n} |Pf(w)|^2 \, udv_b(w) \geq \delta^{-2} \int_{T_{\tau_{1/2}}} u(w) dv_b(w) = \delta^{-2} \int_{T_{\tau_{1/2}}} u^{-1}(w') dv_b(w') = [u]_{B_2}^2 \|f\|_{L_b^2(u)}^2.$$

6. Conclusion

The subject of this paper has been one weight inequalities for operators acting on function spaces defined on $\mathbb{B}_n$. There are at least two additional directions in which one may continue this line of research. The first is proving results like the ones in this paper for more general domains. The second is proving two-weight inequalities. That is, if
T is one of the operators discussed in this paper, for which weights \( w, \sigma \) do we have 
\[
\| T : L^p_w \rightarrow L^p_\sigma \| \text{ finite?}
\]

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