Entanglement generation through an open quantum dot: an exact approach

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We analytically study entanglement generation through an open quantum dot system described by the two-lead Anderson model. We exactly obtain the transition rate between the non-entangled incident state in one lead and the outgoing spin-singlet state in the other lead. In the cotunneling process, only the spin-singlet state can transmit. To discuss such an entanglement property in the open quantum system, we construct the exact two-electron scattering state of the Anderson model. It is striking that the scattering state contains spin-singlet bound states induced by the Coulomb interaction. The bound state describes the scattering process in which the set of momenta is not conserved and hence it is not in the form of a Bethe eigenstate.

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Introduction: We present a new exact approach to electron entanglement generation in an open quantum system. Entanglement has attracted much attention in wide range of physics: it is a resource for quantum information processing and provides a new insight into quantum phase transitions in statistical physics [1, 2]. In most studies, the entanglement properties are discussed in closed systems in equilibrium. In order to study manipulation of entanglement, however, we need to consider an open system out of equilibrium. Entanglement generation using electrons in mesoscopic structures has been proposed recently [3–8]. In Refs. [3, 4, 8], in particular, devices are connected to reservoirs, electrons enter the device from the reservoirs, and interactions (the Coulomb interaction as well as the interaction between electron spin and nuclear spin) are essential for the entanglement generation. It is our purpose to discuss the entanglement generation in such a situation through an exact solution of scattering theory.

In this Letter, we obtain an exact result for entanglement property of transported electrons of the two-lead Anderson model. The Anderson model is a fundamental model describing the electron transport through a quantum dot as illustrated in Fig. 1(a). It consists of one energy level (the quantum dot) in which two occupying electrons with opposite spins interact with each other (the Hubbard interaction), and two leads of noninteracting electrons each of which lead is coupled to the dot. We calculate the transition rate from the non-entangled incident state with momenta \( k_1 \) and \( k_2 \) on the lead 1 to the singlet and triplet states with momenta \( q_1 \) and \( q_2 \) on the lead 2. In the scattering process which conserves the set of momenta as in Fig. 1(b), both the triplet and the singlet components of the incident state can be transmitted to the lead 2. On the other hand, in the cotunneling process which conserves the total energy but not the set of momenta as in Fig. 1(c), only the singlet component can be transmitted and the triplet component is filtered out. We clarify this mechanism by calculating the transition rates exactly, which is the main achievement of our approach. The mechanism of the entanglement generation was first proposed in Ref. [3]; the lowest order of our result reproduces their perturbative result. An interaction-induced orbital entanglement property has been also discussed in a quantum-dot system [9].

For the above purpose, we obtain the exact solution of the two-electron scattering state. A remarkable point of the state is that it contains a two-body singlet bound state. The bound state is induced in the cotunneling process (Fig. 1(c)) by the Hubbard interaction on the quantum dot. A many-body eigenstate of the closed Anderson model can be constructed by the Bethe Ansatz [10–13]. In contrast, our exact scattering state is a many-body eigenstate of the open Anderson model and essentially different from the Bethe eigenstate. A similar bound state is also discussed in Ref. [14], where the two-electron scattering matrix has been constructed exactly in the Anderson model. While their study is focused on the asymptotic states of electrons which lie far from the quantum dot, our exact solution describes electron states both inside and around the quantum dot.

Model and Result: The Hamiltonian of the Anderson model can be constructed by the Bethe Ansatz [10–13]. A cotunneling process.

FIG. 1: (a) A schematic diagram of the Anderson model. (b) A scattering process which conserves the set of momenta. (c) A cotunneling process.
model is defined as $H = H_0 + H_1$ where

$$H_0 = \sum_{\sigma=\uparrow, \downarrow} \sum_{\ell=1,2} \int dx c_{\ell \sigma}^\dagger(x) \frac{1}{2} \frac{d^2}{dx^2} c_{\ell \sigma}(x) + \sum_{\sigma=\uparrow, \downarrow} \epsilon_\sigma n_{d\sigma} + U n_{d\uparrow} n_{d\downarrow},$$

\begin{equation}
H_1 = \frac{t}{\sqrt{2}} \sum_{\sigma=\uparrow, \downarrow} \sum_{\ell=1,2} \left( c_{\ell \sigma}^\dagger(0) d_{\sigma} + d_{\sigma}^\dagger c_{\ell \sigma}(0) \right).
\end{equation}

Here $c_{\ell \sigma}(x)$ (or $c_{\ell \sigma}^\dagger(x)$) denotes the creation (annihilation) operator of an electron with spin $\sigma = \uparrow, \downarrow$ on the lead $\ell = (1, 2)$. The dispersion relation is linearized to be $\epsilon(k) = v_F k$ in each lead. Hereafter we set $v_F = 1$. The operator $d_{\sigma}^\dagger$ ($d_{\sigma}$) represents the creation (annihilation) operator on the energy level of a quantum dot with energy $\epsilon_d$, and $n_{d\sigma} = d_{\sigma}^\dagger d_{\sigma}$. The parameter $t$ represents the coupling between each lead and the dot. When the energy level is occupied by two electrons with opposite spins, they feel the Coulomb repulsion energy $U > 0$.

We consider the situation studied in Ref. [3]. Let $|k_1, k_2; 1\rangle$ be the non-entangled incident state with momenta $k_1$ and $k_2$ on the lead 1 defined by

$$|k_1, k_2; 1\rangle = c_{1k_1}^\dagger c_{1k_2}^\dagger |0\rangle,$$

\begin{equation}
\end{equation}

where $c_{\ell k \sigma}$ ($c_{\ell k \sigma}^\dagger$) denotes the creation (annihilation) operator of an electron with momentum $k$ and spin $\sigma$ on the lead $\ell$, and $|0\rangle$ denotes the vacuum state. We also define a triplet state $|q_1, q_2; 2, +\rangle$ and a singlet state $|q_1, q_2; 2, -\rangle$ with momenta $q_1$ and $q_2$ on the lead 2,

$$|q_1, q_2, 2, \pm\rangle = \frac{1}{\sqrt{2}} \left( c_{2q_1}^\dagger c_{2q_1}^\dagger \pm c_{2q_1}^\dagger c_{2q_1}^\dagger \right) |0\rangle,$$

\begin{equation}
\end{equation}

which are used as outgoing states. We calculate the transition amplitude between these states:

$$\langle q_1, q_2; 2, \pm | T(E_k) | k_1, k_2; 1 \rangle \delta(E_k - E_q),$$

\begin{equation}
\end{equation}

where $E_k = k_1 + k_2$, $E_q = q_1 + q_2$ and $T(E)$ represents the transition matrix recursively defined by

$$T(E) = H_1 + H_1 \frac{1}{E - H_0 + i0} T(E).$$

We obtain new exact results for the transition rate:

$$\langle q_1, q_2; 2, + | T(E_k) | k_1, k_2; 1 \rangle \delta(E_k - E_q)$$

$$= \frac{t^2 \epsilon_{k_1} \epsilon_{k_2}}{4 \sqrt{2}} \left( \delta(k_1 - q_1) \delta(k_2 - q_2) - \delta(k_1 - q_2) \delta(k_2 - q_1) \right),$$

\begin{equation}
\end{equation}

$$\langle q_1, q_2; 2, - | T(E_k) | k_1, k_2; 1 \rangle \delta(E_k - E_q)$$

$$= \frac{t^2 \epsilon_{k_1} \epsilon_{k_2}}{4 \sqrt{2}} \left( \delta(k_1 - q_1) \delta(k_2 - q_2) + \delta(k_1 - q_2) \delta(k_2 - q_1) \right)$$

$$+ \frac{U}{2 \sqrt{2} 2 \epsilon_d} - \frac{t^2}{2} \epsilon_{k_1} \epsilon_{k_2} \epsilon_{q_1} \epsilon_{q_2} \delta(E_k - E_q),$$

\begin{equation}
\end{equation}

where $\epsilon_k$ is defined in Eq. (19) below. Expanding them in the lowest order of $t$, we reproduce the perturbative result obtained in Ref. [3].

Equation (7) and the first term in Eq. (8) represent the contributions from the scattering process which conserves the set of momenta $\{k_1, k_2\} = \{q_1, q_2\}$ as shown in Fig. 1(b). Note that they vanish if $k_i$ and $q_j$ satisfy the condition of the cotunneling process (Fig. 1(c)).

$$E_k = E_q, \quad \{k_1, k_2\} \neq \{q_1, q_2\}. $$

\begin{equation}
\end{equation}

In contrast, Eq. (8) has an additional term which remains non-zero for $U > 0$ under the condition (9). The contribution appears only in the transition into the singlet state (Eq. (8)), not in the transition into the triplet state (Eq. (7)). In other words, we will observe only singlet states if we wait for outgoing electrons on the lead 2 under the condition (9).

Figure 2 shows the dependence of the transition rate $2\pi \langle q_1, q_2; 2, - | T(E_k) | k_1, k_2; 1 \rangle |^2$ on the interaction energy $U$ for $t < E_k$ (Fig. 2(a)) and $t > E_k$ (Fig. 2(b)). The solid lines indicate our exact result and the dashed lines represent the perturbative result in the lowest order of $t$ obtained in Ref. [3]. The perturbative one, being divergent, fails when $U \approx E_k$ even for $t < E_k$.

**Exact two-electron scattering state:** In order to discuss the entanglement generation in such an open quantum dot system, what kind of quantum states should we focus on? What is the origin of the second term in Eq. (8)?

We find that the two-electron scattering state $|\phi\rangle$ given by the solution of the Lippmann-Schwinger equation,

$$|\phi\rangle = |k_1, k_2; 1\rangle + \frac{1}{E_k - H_0 + i0} H_1 |\phi\rangle$$

\begin{equation}
\end{equation}

plays an important role. In fact we have the relation

$$T(E_k) |k_1, k_2; 1\rangle = H_1 |\phi\rangle,$$

\begin{equation}
\end{equation}

which enables us to calculate the transition amplitude (5).

In the remaining part of this paper, we construct the exact solution of $|\phi\rangle$. In Ref. [15], we obtain the exact solution of many-electron scattering scattering states in a spinless model. Here we apply the same technique to the two-lead Anderson model. The importance of $|\phi\rangle$ results from the fact that the system is an open quantum system. It is suitable for the study of transport properties in the open quantum system, but the exact solution of such a state has not been discussed very much. As we will explain later, our exact solution has a remarkable property that the Bethe eigenstate does not have.

By applying the transformation $c_{\ell \sigma}(x) = (c_{1 \sigma}(x) + c_{2 \sigma}(x))/\sqrt{2}$, $c_{\ell \sigma}(x) = (c_{1 \sigma}(x) - c_{2 \sigma}(x))/\sqrt{2}$, we decompose the Hamiltonian into $H = H_e + H_o$, where $H_e$ is the effective Hamiltonian of the electron system, and $H_o$ is the interaction Hamiltonian between the dot and the leads.
where

\[ H_e = \sum_{\sigma=\uparrow, \downarrow} \left( \int dx c_{\sigma \alpha}^\dagger(x) \frac{1}{i} \frac{d}{dx} c_{\sigma \alpha}(x) + \epsilon_\alpha n_{d\sigma} \right) + U n_{d\uparrow} n_{d\downarrow} \]

\[ + \sum_{\sigma=\uparrow, \downarrow} t \left( c_{\sigma \alpha}(0) d_{\sigma \alpha} + d_{\sigma \alpha}^\dagger c_{\sigma \alpha}(0) \right), \tag{12} \]

\[ H_\alpha = \sum_{\sigma=\uparrow, \downarrow} \int dx c_{\sigma \alpha}^\dagger(x) \frac{1}{i} \frac{d}{dx} c_{\sigma \alpha}(x). \tag{13} \]

Note that the odd part \( H_\alpha \) is completely decoupled from the even part \( H_e \). Thus the two-lead Anderson model (1) and (2) can be transformed to the one-lead Anderson model (12) with the free part (13). Let \( |k_1, k_2; s\rangle_{\alpha\beta} \), \((\alpha, \beta) = (e, e), (e, o), (o, o)\), \( s = +, - \) denote the scattering eigenstate with the incident state being a two-electron plain-wave state

\[ |k_1, k_2; \pm\rangle_{\alpha\beta}^{(i)} = \frac{1}{\sqrt{2}} \left( c_{\alpha k_1 \pm}^\dagger c_{\beta k_2 \mp} + c_{\alpha k_1 \mp}^\dagger c_{\beta k_2 \pm} \right) |0\rangle. \tag{14} \]

The scattering state \( |\phi\rangle \) is expressed as \([15]\)

\[ |\phi\rangle = \frac{1}{2\sqrt{2}} \left( \sum_{s=\pm} (|k_1, k_2; s\rangle_{ee} + |k_1, k_2; s\rangle_{eo} + |k_1, k_2; s\rangle_{eo}) \right. \]

\[ - |k_2, k_1; +\rangle_{eo} + |k_2, k_1; -\rangle_{eo} \right), \tag{15} \]

because substituting the incident states (14) for all states on the right-hand side gives the incident state (3).

Before giving the two-electron scattering eigenstate, we first mention the one-electron eigenstate \( |k\sigma\rangle_{e/o} \) defined as a solution of the Schrödinger equation \( H_{e/o} |k\sigma\rangle_{e/o} = k |k\sigma\rangle_{e/o} \). The one-electron eigenstate can be expressed as follows:

\[ |k\sigma\rangle_e = \left( \int dx g_k(x) c_{e\sigma}^\dagger(x) + e_k d_{e\sigma}^\dagger \right) |0\rangle, \]

\[ |k\sigma\rangle_o = \left( \int dx h_k(x) c_{o\sigma}^\dagger(x) |0\rangle, \right. \tag{16} \]

where the eigenfunctions are given by

\[ g_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \left( \theta(-x) + \theta(x) \frac{e_k}{\epsilon_k} \right), \tag{18} \]

\[ e_k = \frac{1}{\sqrt{2\pi}} \frac{t}{k - \epsilon_d + i\hbar^2/2}, \tag{19} \]

\[ h_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}. \tag{20} \]

Here \( \theta(x) \) is the step function and \( e_k/\epsilon_k \) in Eq. (18) represents the phase factor due to the scattering by the dot.

Next we consider the two-electron scattering state, which is written in the form

\[ |k_1, k_2; s = \pm\rangle_{\alpha\beta} \]

\[ = \left( \int dx_1 dx_2 g_{\alpha\beta}^s(x_1, x_2; k_1, k_2) c_{\alpha \uparrow}^\dagger(x_1) c_{\beta \downarrow}^\dagger(x_2) \right. \]

\[ + \left. \int dx_1 c_{\alpha \downarrow}^\dagger(x; k_1, k_2) \left( c_{\beta \uparrow}^\dagger(x) d_{\downarrow}^\dagger + c_{\beta \downarrow}^\dagger(x) d_{\uparrow}^\dagger \right) \right) |0\rangle, \tag{21} \]

Here \( \gamma = e \) for \((\alpha, \beta) = (e, e)\), and \( \gamma = o \) otherwise. From Eqs. (11) and (15), we find that the transition rate (5) is expressed in terms of the eigenfunctions:

\[ \langle q_1, q_2; 2, -|T(E_k)|k_1, k_2; 1\rangle \delta(E_k - E_q) \]

\[ = \delta(E_k - E_q) \frac{t}{8\sqrt{2\pi}} \int_{-\infty}^{\infty} dx (h_{q_1}(x) \mp h_{q_2}(x)) \]

\[ \times \left( e_{ee}^s(x; k_1, k_2) - e_{eo}^s(x; k_1, k_2) \pm e_{oo}^s(x; k_2, k_1) \right). \tag{22} \]

We construct the eigenfunctions specifically \( e_{\alpha\beta}^s(x) \) for all cases of \((\alpha, \beta)\) to calculate the transition rate (22). First of all, let us consider the case \( \alpha = \beta = e, s = - \). Only in this case, the eigenfunctions depend on the
Coulomb interaction $U$. The functions with total energy $E_k = k_1 + k_2$ are obtained by solving the two-electron Schrödinger equation

$$\left( \frac{1}{2} \left( \partial_1 + \partial_2 \right) - E_k \right) g_{ee}(x_1, x_2) + t \left( \delta(x_1) \epsilon_{ee}(x_2) + \delta(x_2) \epsilon_{ee}(x_1) \right) = 0,$$

$$\left( \frac{1}{2} \frac{d}{dx} + \epsilon_d - E_k \right) \epsilon_{ee}(x) + t g_{ee}(x, 0) + t \delta(x) f_{ee} = 0,$$

$$2\epsilon_d + U - E_k) f_{ee} + 2t \epsilon_d(0) = 0$$

under the condition

$$g_{ee}^{-}(x_1, x_2) = \frac{1}{2\sqrt{\pi}} \left( e^{i(k_1x_1 + k_2x_2)} + e^{i(k_1x_2 + k_2x_1)} \right)$$

for $x_1, x_2 < 0$. The result is as follows:

$$g_{ee}^{-}(x_1, x_2) = \sum_Q \left( \frac{1}{2\sqrt{\pi}} e^{iA} \right) g_k(\alpha_1, \alpha_2) g_k(\beta_1, \beta_2) \left( \delta^{\alpha_1}_1 x_1 + \delta^{\beta_1}_Q \right) Z(\alpha_1 Q, \beta_1 Q, \alpha) \phi(x),$$

$$e^{-}_{ee}(x) = \frac{1}{\sqrt{2}} \left( e^{iA} \right) g_k(\alpha_1, \alpha_2) + e^{iA} g_k(\beta_1, \beta_2) \right),$$

$$f_{ee} = \frac{2\epsilon_d}{2\epsilon_d + U - E_k - i\hbar^2} e^{-}_{ee}(x),$$

where $Q = (\alpha_1, \beta_1)$ is a permutation of (1,2), $x_i = x_j$ and

$$Z(x) = e^{i\delta^{\alpha_1}_1 x_1 + i\delta^{\beta_1}_Q \theta(x)}.$$

It is obvious that Eq. (27) satisfies the condition (26). Note that the condition (26) corresponds to the plane-wave incident state (14). We confirm that our solution (27)–(29) of the Schrödinger equation (23)–(25) satisfies the Lippmann-Schwinger equation,

$$|k_1, k_2; -\rangle_{ee} = |k_1, k_2; -\rangle_{ee} + \frac{1}{E_k - H_0 + i\hbar} H |k_1, k_2; -\rangle_{ee}.$$

Remarkable is that Eqs. (27) and (28) contain the two-body bound state $Z(x)$ in Eq. (30). The range of binding is $\tau^2$, which itself is independent of $U$. Note the following properties: (i) It describes a scattering process that does not conserve the momentum set as the cotunneling process (9). In contrast, the first term of Eq. (27) represents the direct ($k_1 = q_0, k_2 = q_2$) and exchange ($k_1 = q_2, k_2 = q_1$) processes. (ii) It is induced by the Coulomb interaction $U$; It vanishes for $U = 0$. (iii) It only appears in the case $\alpha = \beta = e$ and $s = -$ since $f_{\alpha\beta}^{s}$ in Eq. (21) vanishes in the other cases and thereby the eigenfunctions do not depend on $U$. From these properties, we find that it is this bound state that produces the second term of Eq. (8).

This solution is essentially different from the Bethe eigenstate [10–13]. In the Bethe ansatz, we would suppose that $f_{ee}(x_1, x_2)$ is in the following form:

$$g_{ee}^{-}(x_1, x_2) = \sum_Q A Q g_k(\alpha_1 Q, \beta_1 Q, \alpha) \phi(x),$$

$$e^{-}_{ee}(x) = \frac{1}{\sqrt{2}} \left( e^{iA} \right) g_k(\alpha_1, \alpha_2) + e^{iA} g_k(\beta_1, \beta_2) \right),$$

$$e^{\pm}_{ee}(x) = \mp \frac{1}{\sqrt{2}} e^{iA} h_k(\alpha) \phi(x),$$

The scattering state $|\phi\rangle$ constructed from Eq. (15) with these eigenfunctions is shown to satisfy the Lippmann-Schwinger equation (10). Using all these exact solutions in Eq. (22), we finally arrive at the desired results (7) and (8).

**Conclusion:** In this Letter we have constructed the exact two-electron scattering state and discussed its entanglement property. For the exact calculation of the quantities (7) and (8), our solution is essential. We have clarified that the electron transport through the quantum dot has a potential advantage for entanglement generation. The construction of many-electron scattering states in other more complicated models and the understanding of their entanglement property are interesting issues in the future.

Our exact scattering state could also be a powerful tool to understand non-equilibrium electron transport with finite bias voltage. In a spinless model, the analytic approach for this topic has been proposed [16–19] and the importance of many-electron scattering states on the nonequilibrium current has been pointed out recently [15, 20, 21]. In particular, our approach in Ref. [15] succeeded in obtaining nonperturbative result on nonlinear current-voltage characteristics. This suggests that our exact many-electron scattering state yields important information about the Kondo effect out of equilibrium [22–25].

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