New conservation laws for electromagnetic fields in gravity

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Abstract
A recently found \cite{5} 2-index, symmetric, trace-free, \textit{divergence-free} tensor is introduced for arbitrary source-free electromagnetic fields. The tensor can be constructed for any test Maxwell field in Einstein spaces (including proper vacuum), and more importantly for any Einstein-Maxwell spacetime. The tensor is explicitly given and analyzed in some special situations, such as general null electromagnetic fields, Reissner-Nordström solution, or classical electrodynamics. We present an explicit example where the conserved currents derived from the energy-momentum tensor using symmetries are trivial, but those derived from the new tensor are not.

1 Introduction: electromagnetic fields
Let $(V, g)$ be a 4-dimensional spacetime with metric $g$ (signature $-++,++)$. An electromagnetic field is any 2-form $F$ satisfying the Maxwell equations (e.g. \cite{17,23})

\begin{align*}
  dF &= 0, \\
  \delta F &= -j \\
  \iff\ & \nabla_{[\mu}F_{\nu\rho]} = 0, \\
  \nabla_\sigma F^{\sigma\nu} &= j^\nu
\end{align*}

(1)

where $\vec{j}$ is the electromagnetic current 4-vector. As usual, one can define the dual of $F$ and the associated complex self-dual 2-form $\mathcal{F}$ by means of

\begin{align*}
  F^\star_{\mu\nu} &= \frac{1}{2}\eta_{\mu\nu\rho\tau}F^{\rho\tau}, \\
  \mathcal{F} &= \frac{1}{2}(F - i F^\star)
\end{align*}

where $\eta$ is the canonical volume element 4-form. A duality rotation (e.g. \cite{25}) is any transformation of the type

$\mathcal{F}' = e^{i\theta} \mathcal{F}$.

∗Dedicated to Professor José A. Azcárraga on the occasion of his 60th birthday.
The source-free Maxwell equations—that is, $\mathbf{j} = \mathbf{0}$—, which can be written simply as $\nabla_\sigma F^{\sigma\nu} = 0$, remain clearly invariant under duality rotations.

The energy-momentum tensor of $\mathbf{F}$ can be written in any of the following forms

$$T_{\mu\nu} = F_{\mu\rho} F^\rho_\nu - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} = \frac{1}{2} \left( F_{\mu\rho} F^\rho_\nu + F_{\mu\rho} F^\rho_\nu \right) = 2 F_{\mu\rho} \mathbf{T}^{\nu}_\rho$$

where an overbar denotes complex conjugation. Obviously, $T_{\mu\nu}$ is symmetric, traceless, and invariant against duality rotations. Furthermore, it satisfies the dominant energy condition, that is to say

$$T_{00} \geq |T_{\mu\nu}|$$

or equivalently, see e.g. [15, 31]

$$T(\vec{u}, \vec{v}) \geq 0$$

for all future pointing (ergo non-spacelike) vectors $\vec{u}$ and $\vec{v}$. Hence, if the energy density with respect to an observer vanishes, then the whole energy-momentum tensor is zero, and the electromagnetic field is absent. Another important relation satisfied by this tensor is the algebraic Rainich identity [26, 24, 25, 32, 6]

$$T_{\mu\rho} T^\rho_\nu = \frac{1}{4} (T_{\sigma\rho} T^{\sigma\rho}) g_{\mu\nu}$$

which fully characterizes the explicit form (2) if $T_{\mu\mu} = 0$ is taken into account.

A trivial calculation using (1) and (2) leads to

$$\nabla_\mu T^{\mu\nu} = F^\nu_\rho j^\rho \text{ whence: } j = \mathbf{0} \implies \nabla_\mu T^{\mu\nu} = 0. \quad (4)$$

This encodes the “covariant” conservation of energy and momentum for electromagnetic fields in the absence of charges and electric currents. It also implies that, for any conformal Killing vector $\xi$ [32], the corresponding Killing current $\mathbf{J}$ is divergence-free:

$$J^\mu (\xi) \equiv T^{\mu\nu} \xi_\nu \implies \nabla_\mu J^\mu = 0 \quad (5)$$

providing conserved quantities via Gauss’ theorem [15, 23], see e.g. [18, 31] for some explicit examples. On the other hand, if there are electric charges and currents on the spacetime, one can still restore conservation of the total energy and momentum by adding the corresponding energy-momentum tensors of those sources. A typical example can be found in [17] for classical electrodynamics.

2 The main result: $H_{\mu\nu}$

For any electromagnetic field, there exists another tensor quadratic in $\mathbf{F}$ and with the same properties as $T_{\mu\nu}$ except for the dominant energy condition. This tensor can be written in any of the following equivalent forms

$$H_{\mu\nu} = \nabla_\tau F_{\mu\rho} \nabla^\tau F^\rho_\nu - \frac{1}{4} g_{\mu\nu} \nabla_\tau F_{\rho\sigma} \nabla^\tau F^{\rho\sigma} =$$

$$= \frac{1}{2} \left( \nabla_\tau F_{\mu\rho} \nabla^\tau F^\rho_\nu + \nabla_\tau F_{\mu\rho} \nabla^\tau \mathbf{F}^\rho_\nu \right) = 2 \nabla_\tau F_{\mu\rho} \nabla^\tau \mathbf{F}^\rho_\nu \quad (6)$$
from where immediately follows that $H_{\mu\nu}$ is symmetric, traceless, and invariant under duality rotations (with constant $\theta$!). More importantly, it is also divergence-free if the source-free Maxwell equations hold

$$\nabla_\mu H^{\mu\nu} = \nabla^\tau F^{\nu\rho} \nabla_\tau j_\rho \quad \text{whence:} \quad \vec{j} = \vec{0} \implies \nabla_\mu H^{\mu\nu} = 0.$$ (7)

This is valid (i) if the electromagnetic field is “test”, that is to say, it does not contribute to the righthand side of the Einstein field equations, so that the spacetime is vacuum with a possible cosmological constant. And (ii) in the full non-linear theory such that the Einstein-Maxwell field equations with a possible cosmological constant are satisfied. The proof of these results is far from obvious using tensor calculations. Actually, its derivation was achieved using spinors in [5]. For a full proof involving only tensor techniques one can consult [12] (see also [11], where related results for a “Lanczos potential” $L_{\mu\nu\tau}$ in place of $\nabla_\tau F_{\mu\nu}$ were found).

Analogously to the case of $T_{\mu\nu}$, using (7) one can construct divergence-free vector fields associated to any conformal Killing vector $\vec{\xi}$ by means of

$$J^\mu(\vec{\xi}) \equiv H^{\mu\nu} \xi_\nu \Rightarrow \nabla_\mu J^\mu = 0$$ (8)

providing new conserved quantities in any Einstein-Maxwell spacetime (or any Einstein space with a test $\mathbf{F}$) having such $\vec{\xi}$.

Note finally that, generically, the dominant property (i.e. that $H(\vec{u}, \vec{v}) \geq 0$ for all future-pointing $\vec{u}, \vec{v}$) need not be satisfied.

An obvious question arises: why was $H_{\mu\nu}$ unknown until very recently? One reason may be, as previously stated, that the proof of (7) involves difficult calculations. Another one may be the circumstance that its physical units are not those of energy density. Actually, $H_{\mu\nu}$ has units of “superenergy” in analogy with the well-known Bel-Robinson tensor, see e.g. [2, 3, 4, 8, 25, 31], which means units of energy density per unit surface. One can further prove (see [5]) that $H_{\mu\nu}$ in the unique trace of the 4-index superenergy tensor of the electromagnetic field, as given originally by Chevreton [8] or in the unified general treatment of [31, 30]. Thus, if the new tensor has any physical relevance at all, this could help in understanding the concept of gravitational superenergy as well.

3 Uniqueness of $H_{\mu\nu}$

The tensor (6) is unique and solely constructable for the source-free electromagnetic field. I would like to stress these properties by presenting a list of more detailed points:

1 In Special Relativity, that is to say, in flat spacetime, there exist many tensors which are quadratic in $\mathbf{F}$ and with the previous properties. One example is the so-called zilch tensor [21, 16], but there are many other. For a full list, see [11, 13]. Several examples pertinent here are

$$H_{\mu\nu}^{(n)} = \nabla_\tau_1 \cdots \nabla_\tau_n F_{\mu\rho} \nabla_\tau_1 \cdots \nabla_\tau_n F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} \nabla_\tau_1 \cdots \nabla_\tau_n F_{\rho\sigma} \nabla_\tau_1 \cdots \nabla_\tau_n F^{\rho\sigma}$$

3
for any natural number $n$. Observe that $H^{(0)}_{\mu\nu} = T_{\mu\nu}$ and $H^{(1)}_{\mu\nu} = H_{\mu\nu}$. Obviously, these tensors are traceless and symmetric, and furthermore they are divergence-free for any $n$ if the electromagnetic field has no sources. This is obvious from the commutation of the covariant derivatives in flat spacetime. However, the only tensors of the previous list which keep the divergence-free property in curved spacetimes are $T_{\mu\nu}$ and $H_{\mu\nu}$.

2 The divergence-free property of $H_{\mu\nu}$ in generic spacetimes holds in four dimensions exclusively. Thus, (7) is not valid, not even for test electromagnetic fields, in higher dimensions.

3 The construction of $H_{\mu\nu}$ is not translatable to other physical fields. Take, for instance, a minimally coupled massless scalar field $\phi$. The tensor analogous to $H_{\mu\nu}$ can be seen to be

$$D_{\mu\nu} = \nabla_\rho \nabla_\mu \phi \nabla^\rho \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \phi \nabla^\sigma \phi.$$  \hspace{1cm} (9)

This can be derived in several independent ways, such as by taking the trace of the 4-index superenergy tensor of the scalar field [31, 30, 33], or in direct analogy with the expressions (6) and (2) if one recalls the energy-momentum tensor of $\phi$: $T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \phi \nabla^\sigma \phi$. Nevertheless, in non-flat spacetimes, and despite assuming the massless Klein-Gordon equation $\square \phi = 0$, $D_{\mu\nu}$ is not divergence-free. This happens even for test scalar fields, that is to say, in Ricci-flat spacetimes.

4 The existence of $H_{\mu\nu}$ in the full Einstein-Maxwell theory, specially as given in its form (5) not involving any curvature terms—just first derivatives of $F$—may seem a little surprising. For source-free test electromagnetic fields in Einstein spaces—including the Ricci-flat case—, the existence of $H_{\mu\nu}$ can be understood as a “second derivative” of the energy-momentum tensor. To see this, let us start with the simplest case of flat spacetimes, where clearly $2H_{\mu\nu} = \square T_{\mu\nu}$—if the source-free Maxwell equations (1) hold—due to the commutativity of the covariant derivatives. In curved spacetimes things are not so simple. However, in Einstein spaces, there is a D’Alembertian-like operator $\square_L$ introduced by Lichnerowicz [20] which commutes with the divergence. Thus, after some manipulations and using some tensor identities one can actually prove that $2H_{\mu\nu} = \square_L T_{\mu\nu}$ in Einstein spaces. This result has been pointed out in [9] for the Ricci-flat case.

One can thus ask whether a similar but more involved calculation permits a relation between the energy-momentum tensor and the new tensor via a second order operator in the full non-linear Einstein-Maxwell case (allowing for an additional cosmological constant). This has been solved by Edgar [10] recently in an elegant manner. He shows that one can define in general an operator acting on divergence-free rank-2 tensor fields such that the result is divergence-free too. Nonetheless, this operator involves curvature terms in addition to second order derivatives. Thus, the path to prove the relation of this with $H_{\mu\nu}$ is still complicated. As a matter of fact, in order to rederive the expression (6) one has to (i) restrict the discussion to 4 dimensions, (ii) use the Einstein field equations, (iii) then utilize the source-free Maxwell equations (1) explicitly, (iv) take into account...
the special Rainich identity \([3]\), and (v) manipulate the result employing a 4-dimensional identity \([11, 12]\) involving the curvature tensor and the electromagnetic field\(^1\).

One could argue, as for instance in \([9]\), that this deprives \(H_{\mu\nu}\) of relevant “independent content” from \([2]\). This is not clear to me, though, for there are cases in which the divergence-free currents \([5]\) constructed from the energy-momentum tensor are trivial while those built from the new tensor, as in \([3]\), are not. An explicit example is presented in \(\S 4.1.2\). The point here is that the commutativity of the divergence and the second order operator does not immediately translates to the currents, as they involve (conformal) Killing vectors. Thus, the curved space derivatives of the infinitesimal symmetry are also involved here.

### 4 Applications and explicit examples

In this final section we present some explicit expressions for the new tensor \(H_{\mu\nu}\) in some cases of physical interest. The results may shed some light into its interpretation and its potential usefulness.

#### 4.1 Null electromagnetic fields

Null electromagnetic fields can be characterized by any one of the following equivalent conditions:

\[
F_{\mu\nu}F^{\mu\nu} = 0 \quad \text{or equivalently} \quad F_{\mu\nu}F^{\mu\nu} = \ast F_{\mu\nu} F^{\mu\nu} = 0,
\]

\[
\exists \ell: \quad \ell^\mu F_{\mu\nu} = \ell^\mu \ast F_{\mu\nu} = 0 \quad \text{(then necessarily} \quad \ell^\mu \ell_\mu = 0),
\]

\[
F = \ell \wedge p, \quad \ast F = \ell \wedge q \tag{10}
\]

where the unit spacelike vectors \(\vec{p}\) and \(\vec{q}\) are orthogonal to each other and orthogonal to \(\vec{\ell}\), the latter being intrinsically defined (up to a proportionality factor) by the null field and sometimes called “wave vector”—it defines the null direction of propagation of the electromagnetic field—. It must be noted that \(\vec{p}\) and \(\vec{q}\) are not intrinsically defined by \(F\) and are subject to redefinitions of the form \(\vec{p} \rightarrow \vec{p} + a \vec{\ell}, \vec{q} \rightarrow \vec{q} + b \vec{\ell}\) for arbitrary functions \(a\) and \(b\). In other words, only the direction of the wave vector \(\vec{\ell}\) is intrinsically defined and therefore, its orthogonal spacelike 2-plane, which is spanned by \(\vec{p}\) and \(\vec{q}\) and sometimes called “polarization plane”, is also intrinsic to the null \(F\).

Due to the above remarks, the energy momentum tensor should only depend on the wave propagation direction, and this is certainly so, as is well known, because using \([2]\) and \([10]\) one immediately gets

\[
T_{\mu\nu} = \ell_\mu \ell_\nu.
\]

\(^1\) Points (i) to (iv) could be analogously followed for the scalar field without problems—substituting the Klein-Gordon equation for the Maxwell equations—, because there is an identity similar to \([3]\) in that case too, see e.g.\([6]\). However, the final point (v) has no translation to the scalar field, and thus an expression without curvature terms, such as \([9]\), is impossible for divergence-free tensors in this case. This holds also for test scalar fields, as remarked in point 3.
Hence, the full Einstein-Maxwell system of equations is given by \( R_{\mu\nu} \) is the Ricci tensor, units with \( 8\pi G = c = 1 \) and the cosmological constant \( \Lambda \) is kept

\[
R_{\mu\nu} = \ell_{\mu} \ell_{\nu} + \Lambda g_{\mu\nu}
\]

together with (11) appropriately specialized on using (10). An old result due to Mariot and Robinson \([22, 28]\) states that this last part requires a null vector field \( \ell \) which is geodesic and shear-free (see e.g. \([32]\) for these definitions), i.e.

\[
\ell^\mu \nabla_\mu \ell^\nu \propto \ell^\nu, \quad \nabla_{(\mu} \ell_{\nu)} \nabla^\mu \ell^\nu = \frac{1}{2} (\nabla_\mu \ell^\mu)^2.
\]

But then a series of results eventually summarized in a paper by Goldberg and Sachs \([14]\) and references therein) imply that actually the spacetime must be “algebraically special” in the sense that the Weyl tensor \( C^\alpha_{\beta\mu\nu} \) has a repeated principal direction given by \( \ell \), see e.g. \([32]\) p. 88, that is to say

\[
\ell^\beta \ell^\mu C^\alpha_{\beta\mu\nu} = 0.
\]

For the Petrov classification and the nomenclature about principal null directions, see e.g. \([4, 25, 32]\). Using the Newman-Penrose notation (e.g. \([25, 32]\)) these can be written as

\[
\Psi_0 = \Psi_1 = 0, \quad \sigma = \kappa = 0
\]

in a null tetrad with \( \ell \) as first null vector. The first two here express that \( \ell \) is a multiple principal null direction of the Weyl tensor, and the second pair that it is geodesic and shear-free.

Let us turn our attention to \( H_{\mu\nu} \) for these null fields. If one plugs (10) into (6), a na"ive calculation leads to expressions involving derivatives of \( \vec{p} \) and/or \( \vec{q} \). But this is undesirable, as we know that only \( \vec{\ell} \) is intrinsically defined, so that an expression involving only \( \vec{\ell} \) must exist. This can certainly be achieved, after a very long and not easy calculation where all the previous results and other identities must be used. The result is

\[
H_{\mu\nu} = \nabla_\rho \left[ \ell_{(\mu} \nabla^\rho \ell_{\nu)} - \ell^\rho \nabla_{(\mu} \ell_{\nu)} - \ell_{(\mu} \nabla^\rho \ell_{\nu)\ell^\rho} \right].
\]  

(11)

This is the general expression of \( H_{\mu\nu} \) for null electromagnetic fields. Several alternative but equivalent formulae can be given, such as

\[
H_{\mu\nu} = \frac{1}{2} \Box (\ell_{\mu} \ell_{\nu}) + \ell^\alpha \ell^\beta C_{\alpha\mu\beta\nu} - \frac{4}{3} \Lambda \ell_{\mu} \ell_{\nu}
\]

which may also be useful in some particular situations. Nonetheless, this last expression depends explicitly on the curvature of the spacetime, and thus (11) is preferable in principle, and physically more interesting.

Some particular relevant examples of null electromagnetic fields are examined next.

\[2\] An analysis of the 4-index Chevreton tensor for null electromagnetic fields can be found in \([34]\).
4.1.1 Maxwell pp-waves

The plane-fronted waves with parallel rays (pp-waves in short) are defined as those spacetimes containing a constant null vector field $\vec{\ell}$, see e.g. [32] and references therein, that is

$$\nabla_\mu \ell_\nu = 0.$$  

They contain a large subclass of solutions depending on two arbitrary functions which satisfy the Einstein-Maxwell equations. As is obvious from the previous formula and (11) one has for all these solutions $H_{\mu\nu} = 0$. Thus, the new tensor is identically zero for the usual plane waves, including here the classical case of plane waves in Special Relativity.

4.1.2 Robinson-Trautman type-D solution for a null Maxwell field

In 1962, Robinson and Trautman [29] found, among many other algebraically special solutions, a particular spacetime which solves the Einstein-Maxwell system with a possible $\Lambda$ for a null electromagnetic field. The solution is of Petrov type D and has a 3-parameter group of isometries acting transitively on spacelike 2-surfaces. In local coordinates $\{u, r, x, y\}$, the line-element takes the explicit form

$$ds^2 = r^2(dx^2 + dy^2) - 2dudr + \left(\frac{2m(u)}{r} + \frac{\Lambda}{3} r^2\right) du^2$$

where $m(u)$—the “mass”—is an arbitrary but non-increasing function of the retarded time $u$: $\dot{m} \leq 0$ (this solution is reminiscent of the more popular Vaidya’s spherically symmetric radiating solution). The null electromagnetic field is given (up to duality rotations) by

$$F = \sqrt{-2\dot{m}} \, du \wedge dx$$

and its wave vector field is

$$\ell = \frac{\sqrt{-2\dot{m}}}{r} \, du$$

so that the only non-zero component of the energy-momentum tensor is

$$T_{uu} = -\frac{2\dot{m}}{r^2}.$$  

Using (11), a direct computation leads in this spacetime to

$$H^\mu_\nu = \dot{H}^\mu_\nu + \left(\frac{\dot{m}}{mr} - \frac{2m}{r^3} - \frac{4}{3} \Lambda\right) \ell^\mu \ell_\nu$$

where $\dot{H}^\mu_\nu$ takes the simple form

$$\left(\dot{H}^\mu_\nu\right) = -\frac{2\dot{m}}{r^4} \times \text{diag}(-1, -1, 1, 1)$$
which is a structure analogous to that of typical “Coulombian” fields (see subsection 4.2 below). Thus, the form of $H_{\mu\nu}$ splits as the sum of a “Coulombian-like” term, due to a “charge” represented by $\sqrt{-2m} \dot{m}$ (the “mass loss” or radiation rate) plus a typical null term with contributions from $m$, $\ddot{m}/\dot{m}$ and the cosmological constant.

Using now the three Killing vectors

$$\vec{\xi}_i = \{\partial_x, \partial_y, y\partial_x - x\partial_y\}$$

one can construct conserved currents in the way given in $\mathbb{5}$ and $\mathbb{8}$. It is remarkable that, as is obvious, all the currents $\mathbb{4}$ constructed from the energy-momentum tensor are identically vanishing,

$$J^\mu(\vec{\xi}_i) = 0,$$

whence the corresponding conserved quantities are trivial. On the other hand, the divergence-free currents $\mathbb{11}$ constructed from the new tensor are non-vanishing in general. For any one of the three previous $\vec{\xi}_i$ one has

$$\tilde{J}(\vec{\xi}_i) = \frac{-2\dot{m}}{r^4} \vec{\xi}_i,$$

leading to non-trivial conserved quantities, as can be easily checked. Thus, in this particular spacetime, the new tensor provides useful information, and conservation laws, involving the physically relevant magnitude $\dot{m}$.

### 4.2 Non-null electromagnetic fields

In this case, it is not possible to give a general formula such as $\mathbb{11}$ because there are two null directions defined by $\vec{F}$, and they need not be shear-free nor multiple null directions of the Weyl tensor. We consider briefly two simple examples.

### 4.2.1 The Reissner-Nordström solution

The more general spherically symmetric solution of the Einstein-Maxwell equations is given by the following line-element in typical spherical coordinates $\{t, r, \theta, \varphi\}$

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

where $m$ and $q$ are arbitrary constants representing the total mass and electric charge of the body creating the gravitational field. Only the exterior asymptotically flat region defined by $1 - 2m/r + q^2/r^2 > 0$ will be considered. This restricts the range of $r$ to $r > 0$ if $m^2 < q^2$, or to $r > r_+ \equiv m + \sqrt{m^2 - q^2}$ if $m^2 \geq q^2$. Then, the metric is static ($\partial_t$ is a hypersurface-orthogonal timelike Killing vector) and spherically symmetric.

The electromagnetic field is given by

$$\vec{F} = \frac{q}{r^2} dt \wedge dr$$
so that its energy-momentum tensor in the coordinate basis is

\[(T^\mu_\nu) = \frac{q^2}{r^4} \times \text{diag}(-1, -1, 1, 1).\]

A straightforward calculation leads to

\[(H^\mu_\nu) = \frac{q^2}{r^6} \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) \times \text{diag}(-3, -1, 2, 2).\]

Notice the physical units. This particular tensor satisfies the dominant property as well. The conserved currents that can be constructed from here are similar to that of the energy-momentum tensor, see also [18].

### 4.2.2 The “Bertotti-Robinson” solution

As early as 1917, Levi-Civita [19] gave the only Einstein-Maxwell spacetime which is homogeneous with homogeneous electromagnetic field. However, this line-element is commonly known as the “Bertotti-Robinson” solution, as it was rediscovered in [27] and, with the cosmological constant, in [7]. Locally, it can be given in adequate coordinates by

\[ds^2 = -dt^2 + dx^2 + \cos^2(ax)dy^2 + \cos^2(at)dz^2\]

and the electromagnetic field is

\[F = \sqrt{2}a \cos(at)dt \wedge dz \implies \nabla F = 0\]

so that \(H_{\mu\nu}\) is identically zero. The energy-momentum tensor is in this case

\[(T^\mu_\nu) = 2a^2 \times \text{diag}(-1, -1, 1, 1).\]

### 4.3 Classical electrodynamics

In flat spacetime of Special Relativity, and using an inertial reference system, one can define the associated electric \(\vec{E}\) and magnetic \(\vec{B}\) fields. Then, using dots for time derivatives and \(\cdot_i\) for derivatives with respect to the spacelike rectilinear coordinates \(x^i\), one can obtain (0 is the time-component and \(\cdot\) is the usual scalar product of 4-vectors)

\[
\begin{align*}
H_{00} &= \frac{1}{2} \left[ \dot{E} \cdot \vec{E} + \vec{E},_i \cdot \dot{E}^i + \dot{B} \cdot \vec{B} + \vec{B},_i \cdot \dot{B}^i \right] = \frac{1}{2} \Box (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) \\
H_{0i} &= \dot{\vec{E}} \times \vec{B} + \vec{E},_i \times \dot{B}^i = \Box (\vec{E} \times \vec{B}) \\
H_{ij} &= \frac{1}{2} \Box T_{ij}.
\end{align*}
\]
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