K-STABILITY OF CONSTANT SCALAR CURVATURE POLARIZATION

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Abstract. In this paper, we shall show that a polarized algebraic manifold is K-stable if the polarization class admits a Kähler metric of constant scalar curvature. This generalizes the results of Chen-Tian [1], Donaldson [5] and Stoppa [22].

1. Introduction

Yau’s conjecture† ([24], pp.49–50) suggests a strong correlation between stability of polarized algebraic manifolds and the existence of extremal metrics in the polarization class (cf. [23], [2], [9]). Especially, for Kähler metrics of constant scalar curvature, the following is still open as an interesting conjecture:

Conjecture (Tian [23], Donaldson [3]). A polarized algebraic manifold \((M, L)\) is K-stable if and only if the polarization class \(c_1(L)_{\mathbb{R}}\) admits a Kähler metric of constant scalar curvature.

For “if” part of this conjecture, Chen-Tian [1] and Donaldson [3] showed that a polarized algebraic manifold \((M, L)\) is K-semistable when the class \(c_1(L)_{\mathbb{R}}\) admits a Kähler metric of constant scalar curvature. Very recently, by an effective use of moduli spaces, Stoppa [22] proved a stronger result showing that a polarized algebraic manifold \((M, L)\) with a Kähler metric of constant scalar curvature in \(c_1(L)_{\mathbb{R}}\) is K-stable if the group \(\text{Aut}(M, L)\) of holomorphic automorphisms of \((M, L)\) is discrete. The purpose of this paper is to extend Stoppa’s result to the following general case without assuming such discreteness:

†Its vector bundle counterpart known also as the Hitchin-Kobayashi correspondence is affirmative by the works of Kobayashi, Lübke, Donaldson, Uhlenbeck and Yau.
**Main Theorem.** A polarized algebraic manifold \((M, L)\) is K-stable if the class \(c_1(L)_\mathbb{R}\) admits a Kähler metric of constant scalar curvature.

It should be emphasized that one of the main ingredients of this paper is the energy-theoretic approach to K-stability as in Tian [23] (see also [1]), where in our actual proof of Main Theorem, the logarithm of the Chow norm is used in place of the K-energy. To see this clearly, for a connected Fano manifold \(X\), we consider a special degeneration

\[
\text{pr} : \mathcal{X} \to \Delta_{1+\varepsilon} (= \{ z \in \mathbb{C} ; |z| < 1 + \varepsilon \})
\]

of \(X\) as in [23], so that the fiber \(\mathcal{X}_1 := \text{pr}^{-1}(1)\) is just \(X\). Then we take a holomorphic embedding

\[
\mathcal{X} \subset \Delta \times \mathbb{P}^{N-1}(\mathbb{C}),
\]

with \(\text{pr} = \pi_1\) such that \(\pi_1^*\mathcal{O}_{\mathbb{P}^N}(1)\) coincides with the relative canonical sheaf \(\mathcal{K}_{\mathcal{X}/\Delta}\) on the regular part \(\mathcal{X}_{\text{reg}}\) of \(\mathcal{X}\), where \(\pi_1\) (resp. \(\pi_2\)) denotes the restriction to \(\mathcal{X}\) of the projection of the product \(\Delta \times \mathbb{P}^{N-1}(\mathbb{C})\) to the first (resp. second) factor. Let

\[
\psi : \mathbb{C}^* \to \text{SL}(N, \mathbb{C})
\]

be an algebraic group homomorphism chosen in such a way that the induced action of \(t \in \mathbb{C}^*\), \(|t| \leq 1\), on \(\Delta \times \mathbb{P}^{N-1}(\mathbb{C})\) defined by

\[
\Delta \times \mathbb{P}^N(\mathbb{C}) \ni (z, p) \mapsto (tz, \psi(t) \cdot p) \in \Delta \times \mathbb{P}^{N-1}(\mathbb{C})
\]

maps \(\mathcal{X}\) into \(\mathcal{X}\). Then for the Fubini-Study form \(\omega_{\text{FS}}\) on \(\mathbb{P}^n\), let \(f = f(s)\) be the real-valued function defined by

\[
f(s) := \kappa(\psi(\exp(s))^*\omega_{\text{FS}}),
\]

where \(\kappa\) denotes the K-energy map. Recall that \(\lim_{s \to -\infty} \dot{f}(s)\) is just the real part of the generalized Futaki invariant of the central fiber \(\mathcal{X}_0\), and if \(X\) admits a Kähler-Einstein metric, then \(X\) is weakly K-stable in the sense of Tian (denoted energy-theoretically K-stable in this paper), i.e., the limit is always negative for all nontrivial special degenerations. Lemma 4.8 in this paper shows that Donaldson’s K-stability for a general \(L\) can be characterized also energy-theoretically, where in the definition of \(f_m\) in (4.7), the K-energy appearing in the expression (1.1) for \(f\) is replaced by the logarithm of the Chow norm.
This paper is organized as follows: In Section 2, we fix notation by
defining K-stability (by Donaldson). In Section 3, by [11], we describe
the asymptotic behavior of the $k$-th weighted balanced metric $\omega_k$, as
$k \to \infty$. In Section 4, we study the relationship between the Futaki
invariant of a test configuration and the asymptotic behavior of the
Chow norm of fibers (cf. Lemma 4.8). Finally in Section 5, based on
the preceding sections, a proof for Main Theorem will be given by
developing the Chow norm method in [9] and [10], where the results
of Phong and Sturm [19] are used to estimate the second derivative of
the Chow norm.

2. K-stability

In this paper, by a polarized algebraic manifold $(M, L)$, we mean a
pair of a smooth projective algebraic variety $M$, defined over $\mathbb{C}$, and
a very ample line bundle $L$ over $M$. Let $H$ be the maximal connected
linear algebraic subgroup of the identity component $\text{Aut}^0(M)$ of the
group of all holomorphic automorphisms of $M$, so that $\text{Aut}^0(M)/H$ is
an Abelian variety (cf. [6]). Replacing $L$ by its suitable positive integral
multiple if necessary, we can choose an $H$-linearization of $L$ (cf. [16]).
Fix the natural action of the group $T := \mathbb{C}^\ast$ on the complex affine line
$\mathbb{A}^1 := \{ z ; z \in \mathbb{C} \}$ by multiplication of complex numbers,

$$T \times \mathbb{A}^1 \to \mathbb{A}^1, \quad (t, z) \mapsto tz.$$

Let $\pi : \mathcal{M} \to \mathbb{A}^1$ be a $T$-equivariant projective morphism between
complex varieties with an invertible sheaf $\mathcal{L}$ on $\mathcal{M}$, relatively very ample over $\mathbb{A}^1$, where the algebraic group $T$ acts on $\mathcal{L}$, linearly on fibers, lifting the $T$-action on $\mathcal{M}$. For each $z \in \mathbb{A}^1$, we put

$$\mathcal{L}_z := \mathcal{L}|_{\mathcal{M}_z},$$

where $\mathcal{M}_z := \pi^{-1}(z)$ denotes the scheme-theoretic fiber of $\pi$ over $z$.
Then the following notion of a test configuration is defined by Don-
aldson [3] (see special degenerations by Tian [23]). Actually, the pair
$(\mathcal{M}, \mathcal{L})$ with a flat family

$$\pi : \mathcal{M} \to \mathbb{A}^1$$
is called a test configuration for \((M, L)\), if for some positive integer \(\ell\), there exist the following isomorphisms of polarized algebraic manifolds
\[(\mathcal{M}_z, \mathcal{L}_z) \cong (M, \mathcal{O}_M(L^\ell)), \quad 0 \neq z \in \mathbb{A}^1.\]

In the special case when \(\mathcal{M} = M \times \mathbb{A}^1\), a test configuration is called a product configuration, where for such a configuration, \(T\) does not necessarily act on the first factor \(M\) trivially.

Given a test configuration \(\pi : \mathcal{M} \to \mathbb{A}^1\) for \((M, L)\), we consider the vector bundles \(E_m\) over \(\mathbb{A}^1\) by
\[\mathcal{O}_{\mathbb{A}^1}(E_m) = \pi_* \mathcal{L}^m, \quad m = 1, 2, \ldots,\]
associated to the direct image sheaves \(\pi_* \mathcal{L}^m\). Then \(E_m\) admits a natural \(T\)-action \(\rho_m : T \times E_m \to E_m\) induced by the \(T\)-action on \(\mathcal{L}\). Consider the fibers \((E_m)_z, z \in \mathbb{A}^1\), of the bundle \(E_m\) over \(z\). Since the fiber \((E_m)_0 = (\pi_* \mathcal{L}^m)_0 \otimes \mathbb{C}\) over the origin is preserved by the \(T\)-action \(\rho_m\), we can talk about the weight \(w_m\) of the \(T\)-action on \(\det (E_m)_0\). Put \(n := \dim_{\mathbb{C}} M\), and we consider the degree \(d_m\) of the image of the Kodaira embedding
\[(2.2) \quad \Phi_{[L^m]} : M \hookrightarrow \mathbb{P}^s(V_m),\]
where \(\mathbb{P}^s(V_m)\) is the set of all hyperplanes in \(V_m := H^0(M, \mathcal{O}_M(L^m))\) through the origin. Put \(N_m := \dim (E_m)_0 = \dim V_m\). Then for \(m \gg 1\),
\[(2.3) \quad \left\{\begin{array}{l} N_m = a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0, \\
                  w_m = b_{n+1} m^{n+1} + b_n m^n + \ldots + b_1 m + b_0,
\end{array}\right.\]
for some rational numbers \(a_i, b_j \in \mathbb{Q}\) independent of the choice of \(m\). Note here that \(a_n = \ell^n c_1(L)^n [M]/n! \geq 0\). If \(m \gg 1\), then we have
\[(2.4) \quad \frac{w_m}{m^N_m} = F_0 + F_1 m^{-1} + F_2 m^{-2} + \ldots,\]
with coefficients \(F_i = F_i(\mathcal{M}, \mathcal{L}) \in \mathbb{Q}\) independent of the choice of \(m\). In particular \(F_1 = F_1(\mathcal{M}, \mathcal{L})\) is called the Futaki invariant for the test configuration. In contrast to the energy-theoretic one by Tian, the following K-stability is given by Donaldson [3]:
Definition 2.5. (i) \((M, L)\) is said to be \(K\)-semistable, if the inequality 
\[ F_1(M, L) \leq 0 \]
holds for all test configurations \((\mathcal{M}, \mathcal{L})\) for \((M, L)\).

(ii) Let \((M, L)\) be \(K\)-semistable. Then \((M, L)\) is said to be \(K\)-stable, if for every test configuration \((\mathcal{M}, \mathcal{L})\) for \((M, L)\), it reduces to a product configuration if and only if \(F_1(\mathcal{M}, \mathcal{L})\) vanishes.

In this paper, we fix once for all a test configuration \((M, L)\) of a polarized algebraic manifold \((M, L)\) which admits a Kähler metric \(\omega_{\infty}\) in \(c_1(L)_R\) of constant scalar curvature. Obviously \(F_i(\mathcal{M}, \mathcal{L})\) coincides with \(F_i(M, L)\) for all positive integers \(i\) and \(j\), and hence to discuss \(K\)-stability of \((M, L)\), replacing \(L\) by its suitable positive multiple if necessary, we may assume that \(\dim H^0(M_z, \mathcal{L}_z^m) = \dim H^0(M_0, \mathcal{L}_0^m)\) and that the natural homomorphisms
\[
\otimes^m H^0(M_z, \mathcal{L}_z) \rightarrow H^0(M_z, \mathcal{L}_z^m), \quad m = 1, 2, \ldots,
\]
are surjective for all \(z \in \mathbb{A}^1\). We can see this easily by the fact that, if \(L\) is replaced by its very high multiple while \(L\) is fixed, then \(\ell\) becomes large so that the assumptions above are automatically satisfied (cf. [14]; see also [12], Remark 4.6).

Finally, as remarked in [5], Lemma 2, we have the following theorem of equivariant trivialization for \(E_m\):

**Fact 2.6.** Let \(H_1\) be a Hermitian metric on the vector space \((E_m)_z\) at \(z = 1\). Then there is a \(T\)-equivariant trivialization
\[
(2.7) \quad E_m \cong \mathbb{A}^1 \times (E_m)_0
\]
taking \(H_1\) to a Hermitian metric, denoted by \(H_0\), on the central fiber \((E_m)_0\) which is preserved by the action of \(S^1 \subset \mathbb{C}^* (= T)\) on \((E_m)_0\).

3. **Asymptotic behavior of weighted balanced metrics**

Now choose a Hermitian metric \(h_{\infty}\) for \(L\) such that \(\omega_{\infty} = c_1(L, h_{\infty})\). Let \(\ell\) be as in (2.1). Following [11], Section 2, we here study the asymptotic behavior of the weighted balanced metrics for polarized algebraic manifolds \((M, L^m\ell)\) as \(m \to \infty\). For the linear algebraic group \(H\) in the previous section, choose the maximal compact subgroup \(K\) of \(H\) such that \(\omega_{\infty}\) is \(K\)-invariant (cf. [7]). Then for the identity
component $Z$ of the center of $K$, take its complexification $Z^C$ in $H$. For the $H$-linearization of $L$ in the previous section, there exist mutually distinct characters $\chi_{m;1}, \chi_{m;2}, \ldots, \chi_{m;\nu_m} \in \text{Hom}(Z^C, \mathbb{C}^*)$ such that the vector space $V_m$ is written as a direct sum

$$V_m = \bigoplus_{i=1}^{\nu_m} V(\chi_{m;i}),$$

where $V(\chi) := \{ \sigma \in V_m : g \cdot \sigma = \chi(g)\sigma \text{ for all } g \in Z^C \}$ for all $\chi \in \text{Hom}(Z^C, \mathbb{C}^*)$. Let $\mathfrak{h}$ be the real Lie subalgebra of $H^0(M, \mathcal{O}(T^{1,0}M))$ corresponding to the real Lie subgroup $Z$ of Aut($M$). Put $\mathfrak{h} := \sqrt{-1} \mathfrak{h}$. Let $h$ be a Hermitian for $L$ such that $\omega = c_1(L; h)$ is a $K$-invariant Kähler form. Define a $K$-invariant Hermitian pairing $(\cdot, \cdot)_h$ for $V_m$ by

$$(\sigma, \sigma')_h := \int_M (\sigma, \sigma')_h \omega^n, \quad \sigma, \sigma' \in V_m,$$

where $(\sigma, \sigma')_h$ denotes the pointwise Hermitian inner product of $\sigma, \sigma'$ by the $m$-multiple of $h$. Then by this Hermitian pairing $(\cdot, \cdot)_h$, we have

$$V(\chi_{m;i}) \perp V(\chi_{m;j}), \quad i \neq j.$$

Put $n_{m;i} := \dim_{\mathbb{C}} V(\chi_{m;i})$. Let $P_m$ be the set of all pairs $(i, \alpha)$ of integers such that $1 \leq i \leq \nu_m$ and $1 \leq \alpha \leq n_{m;i}$. For the Hermitian pairing in (3.1), we say that an orthonormal basis $\{ \sigma_{i,\alpha} : (i, \alpha) \in P_m \}$ for $V_m$ is admissible if $\sigma_{i,\alpha} \in V(\chi_{m;i})$ for all $(i, \alpha) \in P_m$. Fixing an admissible orthonormal basis $\{ \sigma_{i,\alpha} : (i, \alpha) \in P_m \}$ of $V_m$ with $(\cdot, \cdot)_h$, we now define $Z_m(\omega, \mathcal{Y}, x)$ to be

$$(n!/m^n) \sum_{i=1}^{\nu_m} \sum_{\alpha=1}^{n_{m;i}} \exp\{-\langle \chi_{m;i} \rangle_\mathcal{Y} + 2x_i\} |\sigma_{i,\alpha}|^2_h,$$

for each $\mathcal{Y} \in \mathfrak{h}$ and $x = (x_1, x_2, \ldots, x_{\nu_m}) \in \mathbb{R}^{\nu_m}$, where we put $|\sigma|^2_h := (\sigma, \sigma)_h$ for all $\sigma \in V_m$, and $\langle \chi_{m;i} \rangle_\mathcal{Y} : \mathfrak{h} \to \mathbb{R}$, $i = 1, 2, \ldots$, denote the differentials at $g = 1$ of the restriction to $\mathfrak{h}$ of the characters $\chi_{m;i} : Z^C \to \mathbb{C}^*$. Put $r_0 := n(2c_1(L)^n[M])^{-1} \{c_1(L)^{n-1} c_1(M)[M]\}$, and consider

$$B_m := \{ x = (x_1, x_2, \ldots, x_{\nu_m}) \in \mathbb{R}^{\nu_m} : \|x\| \leq q^2 \},$$

where $q := m^{-1}$ and $\|x\| := (\sum_{i=1}^{\nu_m} n_{m;i} x_i^2)^{1/2}$. Then fixing a sufficiently large positive integer $k$, we see from $[\mathbb{S}]$, Theorem B, that there exist
vector fields \( Y_j \in \mathfrak{z} \), real numbers \( r_j \in \mathbb{R}, \ j = 1, 2, \ldots, k \), and a \( K \)-invariant Hermitian metric \( u_m \) for \( L \) such that

\[
Z_m(v_m, Y, 0) = (1 + \Sigma_{j=0}^k r_j q^{j+1}) + O(q^{k+2}),
\]

(3.3)

\[
u_m \to h_\infty \text{ in } C^\infty, \quad \text{as } m \to \infty,
\]

(3.4)

where \( v_m := c_1(L; u_m) \) and \( Y := \Sigma_{j=1}^k q^{j+2} Y_j \). In view of the definition of \( \delta_0 \) in [10], Step 5, the proof of Lemma 3.4 in [9] allows us to make a perturbation of \( h_m \) via the action of \( \exp(p''_m) \) (see [10] for the definition of \( p''_m \)) to obtain a critical point for the Chow norm. Then by (3.3) and (3.4), we obtain from [8], pp. 574–576, a \( K \)-invariant Hermitian metric \( h_m \) for \( L \) such that, for some \( b_m = (b_{m;1}, b_{m;2}, \ldots, b_{m;\nu_m}) \in B_m \),

\[
Z_m(\omega_m, Y, b_m) = 1 + \Sigma_{j=0}^k r_j q^{j+1}, \quad k \gg 1,
\]

(3.5)

\[
h_m \to h_\infty \text{ in } C^\infty, \quad \text{as } m \to \infty,
\]

(3.6)

where we set \( \omega_m := c_1(L; h_m) \). For an admissible orthonormal basis \( \{ \sigma_{i,\alpha}; (i, \alpha) \in P_m \} \) for \( V_m \) with the pairing \( \langle \ , \rangle_{h_m} \), by setting

\[
\beta_{m;i} := \exp\{\langle \chi_{m;i} \rangle (Y) + 2b_{m;i} \} - 1,
\]

(3.7)

we see from (3.5) and (3.7) the following:

\[
\frac{(n! / m^n)}{n！} \Sigma_{i=1}^{\nu_m} \Sigma_{\alpha=1}^{\nu_n} (1 + \beta_{m;i}) |\sigma_{i,\alpha}|^2_{h_m} = 1 + \Sigma_{j=0}^k r_j q^{j+1},
\]

(3.8)

where, in view of [10], Lemma 2.6, there exists a positive constant \( C_1 \) independent of the choice of \( m \gg 1 \) and \( i \) such that

\[
|\beta_{m;i}| \leq C_1 q^2 \quad \text{for all } m \gg 1 \text{ and } i.
\]

(3.9)

Then by (3.8) and (3.9), we obtain

\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \langle \Sigma_{i=1}^{\nu_m} \Sigma_{\alpha=1}^{\nu_n} |\sigma_{i,\alpha}|^2_{h_m} \rangle - m \omega_m = O(q^2).
\]

(3.10)

4. THE CHOW NORM AND THE FUTAKI INVARIANT

In this section, we fix a Hermitian metric \( H_1 \) on \( V_m \), where \( (E_m)_s \) at \( s = 1 \), denoted by \( (E_m)_1 \), is identified with \( V_m \). By the trivialization (2.7), \( H_1 \) induces a Hermitian metric \( H_0 \) on \( (E_m)_0 \). Then

\[
W_m := \{ \text{Sym}^n((E_m)_0) \}^{\otimes n+1}
\]

(4.1)
admits the Chow norm (cf. Zhang [25], 1.5; see also §4 in [8])

\[ W_m^* \ni w \mapsto \|w\|_{\text{CH}(H_0)} \in \mathbb{R}_{\geq 0}. \]

Choose an element \( \hat{M}_m \) of \( W_m^* \) such that the corresponding point \([\hat{M}_m]\) in \( \mathbb{P}^*(W_m) \) is the Chow point for the reduced effective algebraic cycle

\[ \gamma_1 := \Phi_{|L^e_m|}(M) \]

on \( \mathbb{P}^*((E_m)_0) \). Here each \((E_m)_s, s \neq 0\), is identified with \((E_m)_0\) via the trivialization (2.7), and by letting \( s = 1 \), we regard \( \Phi_{|L^e_m|}(M) \) on \( \mathbb{P}^*(V_m) \) as the algebraic cycle \( \gamma_1 \) on \( \mathbb{P}^*((E_m)_0) \). Since the \( T \)-action on \( E_m \) preserves \((E_m)_0\), we have a representation

\[ \psi_m : T \to \text{GL}((E_m)_0) \]

induced by the \( T \)-action on \( E_m \). Note that this \( T \)-action on \((E_m)_0\) naturally induces a \( T \)-action on \( \mathbb{P}^*((E_m)_0) \). By the complete linear systems \(|L^s_m|\), \( s \in \mathbb{A}^1 \), we have the relative Kodaira embedding

\[ \mathcal{M} \hookrightarrow \mathbb{P}^*(E_m), \]

over \( \mathbb{A}^1 \), where by (2.6) the projective bundle \( \mathbb{P}^*(E_m) \) over \( \mathbb{A}^1 \) is viewed as product bundle \( \mathbb{A}^1 \times \mathbb{P}^*((E_m)_0) \). Then each fiber \( \mathbb{P}^*(E_m)_s \) of \( \mathbb{P}^*(E_m) \) over \( s \in \mathbb{A}^1 \) is naturally identified with \( \mathbb{P}^*((E_m)_0) \), so that all \( \mathcal{M}_z, z \in \mathbb{A}^1 \), are regarded as subschemes of \( \mathbb{P}^*((E_m)_0) \). Then

\[ \mathcal{M}_z = \psi_m(z) \cdot \mathcal{M}_1, \quad z \in \mathbb{C}^*, \]

where on the right-hand side, the element \( \psi_m(s) \) in \( \text{GL}((E_m)_0) \) acts naturally on \( \mathbb{P}^*((E_m)_0) \) as a projective linear transformation. Note that \( \mathcal{M}_1 \) is nothing but \( \gamma_1 \) as an algebraic cycle, and that \( \mathcal{M}_0 \) is preserved by the \( T \)-action on \( \mathbb{P}^*((E_m)_0) \).

Let us now consider the \( N_m \)-fold covering \( \hat{T} := \{ \hat{t} \in \mathbb{C}^* \} \) of the algebraic torus \( T := \{ t \in \mathbb{C}^* \} \) by setting

\[ t = \hat{t}^{N_m} \]

for \( t \) and \( \hat{t} \). Then the mapping \( \psi_m^{\text{SL}} : \hat{T} \to \text{SL}((E_m)_0) \) defined by

\[ \psi_m^{\text{SL}}(\hat{t}) := \frac{\psi_m(\hat{t}^{N_m})}{\det(\psi_m(\hat{t}))} = \frac{\psi_m(t)}{\det(\psi_m(t))}, \quad \hat{t} \in \hat{T}, \]

\[ \text{SL}((E_m)_0) \]
is also an algebraic group homomorphism. Consider the quotient group
\[ G_m := \text{SL}(E_0)/\Pi_m, \]
where \( \Pi_m := \{ \zeta^\alpha \text{id} \mid \alpha = 1, 2, \ldots, N_m \} \) for a
primitive \( N_m \)-th root \( \zeta \) of unity. We then define an algebraic group
homomorphism \( \hat{\psi}_m : T \rightarrow G_m \) by sending each \( t \in T \) to
\[ \hat{\psi}_m(t) : \text{natural image of } \psi_{\text{SL}}(\hat{t}) \text{ in } G_m. \]

Consider \( \psi_m(t), \hat{\psi}_m(t), \psi_{\text{SL}}(\hat{t}) \) above. Then these all induce exactly
the same projective linear transformation on \( \mathbb{P}^*((E_0)_0) \). Let \( \gamma_t \) be the
algebraic cycle on \( \mathbb{P}^*((E_0)_0) \) obtained as the image of \( \gamma_1 \) by this pro-
jective linear transformation. Then as \( t \rightarrow 0 \), we have a limit algebraic
cycle
\[ (4.5) \quad \gamma_0 := \lim_{t \rightarrow 0} \gamma_t \]
on \( \mathbb{P}^*((E_0)_0) \). To have another understanding of \( \gamma_z, z \in \mathbb{A}^1 \), recall that
we can regard each \( M_z \) as a subscheme
\[ M_z \hookrightarrow \mathbb{P}^*((E_0)_0), \quad z \in \mathbb{A}^1. \]
Then by (4.3), the algebraic cycle \( \gamma_z \) is nothing but \( M_z \) viewed just as
an algebraic cycle on \( \mathbb{P}^*((E_0)_0) \) counted with multiplicities. In particular, \( \gamma_0 \) is the \( T \)-invariant algebraic cycle on \( \mathbb{P}^*((E_0)_0) \) associated to
the subscheme \( M_0 \) counted with multiplicities.

By \( \hat{M}_m^{(0)} \in W_m^* \), we denote the element in \( W_m^* \) such that the associ-
ated element \( [\hat{M}_m^{(0)}] \in \mathbb{P}^*(W_m) \) is the Chow point for the cycle \( \gamma_0 \) on
\( \mathbb{P}^*((E_0)_0) \). Then (4.5) is interpreted as
\[ (4.6) \quad \lim_{t \rightarrow 0} [\psi_{\text{SL}}(\hat{t}) \cdot \hat{M}_m] = [\hat{M}_m^{(0)}] \]
in \( \mathbb{P}^*(W_m) \). Here by (4.1), the group \( \text{GL}(E_0)_0 \) acts naturally on \( W_m^* \),
and hence acts also on \( \mathbb{P}^*(W_m) \). We now consider the function
\[ (4.7) \quad f_m(s) := \log \| \hat{\psi}_m(\exp(s)) \cdot \hat{M}_m \|_{\text{CH}(H_0)}, \quad s \in \mathbb{R}. \]
Put \( \hat{f}_m(s) := (df_m/ds)(s) \). The purpose of this section is to show the
following (see Phong and Sturm [20], eqn 7.29, for the leading term; see also [5], p.464–467):
Lemma 4.8. Let $a_n$ be as in (2.3). Then the function $\hat{f}_m(s)$ has a limit, as $s \to -\infty$, written in the following form for $m \gg 1$:

\[
\lim_{s \to -\infty} \hat{f}_m(s) = (n+1)! a_n \left(\frac{w_m}{mN_m} - F_0\right) m^{n+1}.
\]

Proof: Since $\gamma_0$ is preserved by the $\hat{T}$-action on $(E_m)_0$, the Chow point $[\hat{M}(0)]$ for $\gamma_0$ is fixed by the $\hat{T}$-action on $\mathbb{P}^*(W_m)$, i.e., for some $q_m \in \mathbb{Z}$,

\[
\psi_m^\text{SL}(\hat{t}) \cdot \hat{M}(0) = \hat{t}^{q_m} \hat{M}(0), \quad t \in \mathbb{C}^*,
\]

where the left-hand side is $\hat{\psi}_m(t) \cdot \hat{M}(0)$ modulo the action of $\Pi_m$. Since the $\hat{T}$-action on $W_m^*$ is diagonalizable, we can write $\hat{M}_m$ in the form

\[
\hat{M}_m = \sum_{\nu=1}^{N_m} w_\nu,
\]

where $0 \neq w_\nu \in W_m^*$, $\nu = 1, 2, \ldots, N$, are such that, for an increasing sequence of integers $e_1 < e_2 < \cdots < e_N$, the equality

\[
\psi_m^\text{SL}(\hat{t}) \cdot w_\nu = \hat{t}^{e_\nu} w_\nu
\]

holds for all $\nu \in \{1, 2, \ldots, N\}$ and $\hat{t} \in \hat{T}$. In particular, in view of (4.6), we can find a complex number $c \neq 0$ such that

\[
\hat{M}_m = cw_1,
\]

and hence $q_m$ coincides with $e_1$. Then by (4.10) and (4.11), it is easy to check that

\[
\lim_{s \to -\infty} \hat{f}_m(s) \left(= \frac{e_1}{N_m}\right) = \frac{q_m}{N_m}.
\]

Hence it suffices to show that $q_m/N_m$ admits the asymptotic expansion as in the right-hand side of (4.9) above. Consider the graded algebra

\[
\bigoplus_{k=0}^\infty (E_{km})_0,
\]

where via $\psi_m^\text{SL}$, the group $\hat{T}$ acts on $(E_m)_0$ and hence on $(E_{km})_0$. Then by [15], Proposition 2.11, the weight $p_k$ for the $\hat{T}$-action on $\det(E_{km})_0$ satisfies the following:

\[
p_k + \frac{q_m}{(n+1)!} k^{n+1} = O(k^n), \quad k \gg 1,
\]
i.e., there exists a constant \( C > 0 \) independent of \( k \), possibly depending on \( m \), such that the left-hand side of (4.13) has absolute value bounded by \( Ck^n \) for positive integers \( k \). Recall the definition of \( w_{km} \) and \( w_m \) in Section 2. Then by the expression of \( \psi_{SL}^m \) in (4.4), the weight \( p_k \) for \( \det(E_{km})_0 \) induced by the \( \hat{T} \)-action on \( (E_m)_0 \) via \( \psi_{SL}^m \) is expressible as

\[
p_k = N_m w_{km} - k w_m N_{km}.
\]

Here the term \( N_m w_{mk} \) in the right-hand side of (4.14) is the weight in \( \hat{t} \) for \( \det(E_{km})_0 \) induced from the action on \( (E_m)_0 \) by the numerator \( \psi_m(t) \) in (4.4), since it is nothing but the weight in \( \hat{t} \) for the action of \( \psi_{mk}(t) \) on \( \det(E_{km})_0 \), while in view of the natural surjective homomorphism \( S^k((E_m)_0) \to (E_{km})_0 \), the term \( kw_m N_{km} \) is just the weight in \( \hat{t} \) induced from the scalar action on \( (E_m)_0 \) by the denominator of (4.4). Then for \( k \gg 1 \), by (4.14) and (2.4), we obtain

\[
p_k = (km) N_m N_{km} \left\{ \frac{w_{km}}{(km)N_{km}} - \frac{w_m}{mN_m} \right\}
= - (km) N_m N_{km} \left\{ (F_1m^{-1} + F_2m^{-2} + F_3m^{-3} + \ldots) + O(k^{-1}) \right\}
= - k^{n+1} a_n N_m \left\{ (F_1m^n + F_2m^{n-1} + F_3m^{n-2} + \ldots) + O(k^{-1}) \right\},
\]

where the last equality follows from (2.3) applied to \( km \). Then by comparing this with (4.13), and then by (2.4), we obtain

\[
\frac{q_m}{N_m} = (n+1)! a_n (F_1m^n + F_2m^{n-1} + F_3m^{n-2} + \ldots)
= (n+1)! a_n \left( \frac{w_m}{mN_m} - F_0 \right) m^{n+1}.
\]

as required. \( \square \)

5. Proof of Main Theorem

In this section, by using the notation in (3.1), we choose \( \langle \cdot, \cdot \rangle_{h_m} \) as the Hermitian metric \( H_1 \) for \( V_m \) in Section 4, where \( h_m \) is as in (3.6). For the corresponding Chow norm

\[
W^*_m \ni w \mapsto \|w\|_{\text{CH}(H_0)} \in \mathbb{R}_{\geq 0},
\]
we consider the real-valued function \( f_m \) on \( \mathbb{R} \) as in (4.7). For the one-parameter group \( \hat{\psi}_m : \hat{T} \to G_m \), the vector space \((E_m)_0\) admits an orthonormal basis \( T := \{ \tau_1, \tau_2, \ldots, \tau_{N_m} \} \) such that, for some \( e_i \in \mathbb{Q} \) with \( \Sigma_{\alpha=1}^{N_m} e_\alpha = 0 \), we have

\[
(5.1) \quad \hat{\psi}_m(t) \tau_\alpha \equiv t^{e_\alpha} \tau_\alpha, \quad t \in T,
\]

modulo the action of \( \Pi_m \). By the associated Kodaira embedding \( \mathcal{M}_0 \hookrightarrow \mathbb{P}^*((E_m)_0) \), we regard \( \mathcal{M}_0 \) as a subscheme

\[
\mathcal{M}_0 \hookrightarrow \mathbb{P}^{N_m-1}(\mathbb{C}), \quad p \mapsto (\tau_1(p) : \tau_2(p) : \cdots : \tau_{N_m}(p)),
\]

where we identify \( \mathbb{P}^*((E_m)_0) \) with \( \mathbb{P}^{N_m-1}(\mathbb{C}) \) by the basis \( T \). Since we regard \((E_m)_1\) just as \( V_m \), the identification (2.7) allows us to obtain a basis \( T' := \{ \tau'_1, \tau'_2, \ldots, \tau'_{N_m} \} \) for \( V_m \) corresponding to the basis \( T \) for \((E_m)_0\). Note that this basis \( T' \) is orthonormal with respect to the Hermitian metric \( H_1 = \langle \cdot, \cdot \rangle_{h_m} \). Then the Kodaira embedding \( \Phi_{|L^{\ell_m}} : M (= \mathcal{M}_1) \hookrightarrow \mathbb{P}^*(V_m) \) is given by

\[
(5.2) \quad M \hookrightarrow \mathbb{P}^{N_m-1}(\mathbb{C}), \quad p \mapsto (\tau'_1(p) : \tau'_2(p) : \cdots : \tau'_{N_m}(p)),
\]

where we identify \( \mathbb{P}^*(V_m) = \mathbb{P}^{N_m-1}(\mathbb{C}) = \mathbb{P}^*((E_m)_0) \) by the bases \( T' \) and \( T \). Then the Fubini-Study form \( \omega_{FS} \) on \( \mathbb{P}^{N_m-1}(\mathbb{C}) (= \mathbb{P}^*(V_m)) \) is

\[
\omega_{FS} := \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\Sigma_{\alpha=1}^{N_m} |z_\alpha|^2).
\]

By (3.10), we here observe that

\[
(5.3) \quad \omega_{FS} - m\omega_m = O(q^2),
\]

on \( M \). For the function \( f_m(s) \) in (4.7), we first give an estimate of the first derivative \( \dot{f}_m(0) \). In view of [25] (see also [5]),

\[
(5.4) \quad \dot{f}_m(0) = (n + 1) \int_M \frac{\Sigma_{\alpha=1}^{N_m} e_\alpha |\tau'_\alpha|_{h_m}^2}{\Sigma_{\alpha=1}^{N_m} |\tau'_\alpha|_{h_m}} \omega_{FS}^n,
\]
where by (3.8) and (3.9), we observe that

\[
\sum_{\alpha=1}^{N_m} |\tau_{\alpha,h_m}'|^2 = \sum_{i=1}^{\nu_m} \sum_{\alpha=1}^{n_m,i} |\sigma_{i,\alpha}|^2_{h_m} = (m^n/n!) (1 + \sum_{j=0}^{k} r_j q^{j+1}) - \sum_{i=1}^{\nu_m} \sum_{\alpha=1}^{n_m,i} \beta_{m,i} |\sigma_{i,\alpha}|^2_{h_m} = \sum_{i=1}^{\nu_m} \sum_{\alpha=1}^{n_m,i} |\sigma_{i,\alpha}|^2_{h_m} (5.5)
\]

= \left( \frac{m^n}{n!} \right) \left( 1 + \sum_{j=0}^{k} r_j q^{j+1} \right) \{1 + O(q^2)\}.

Now, we can rewrite (5.4) in the form

\[
\dot{f}_m(0) = (n + 1)! \int_M \sum_{\alpha=1}^{N_m} e_{\alpha} |\tau_{\alpha,h_m}'|^2 \left\{1 + O(q^2)\right\} \omega^n_m
\]

= \int_M O(q^2) (\sum_{\alpha=1}^{N_m} e_{\alpha} |\tau_{\alpha,h_m}'|^2) \omega^n_m,

where the last equality follows from

\[
\int_M \sum_{\alpha=1}^{N_m} e_{\alpha} |\tau_{\alpha,h_m}'|^2 \omega^n_m = \sum_{\alpha=1}^{N_m} e_{\alpha} = 0.
\]

All weights \(e_{\alpha}\) for \(\hat{\psi}_m\) have absolute value bounded by \(C_2m\) for some constant \(C_2 > 0\) independent of both \(\alpha\) and \(m\), i.e.,

\[
|e_{\alpha}| \leq C_2m, \quad \alpha = 1, 2, \ldots, N_m.
\]

In view of (5.5), \(\sum_{\alpha=1}^{N_m} |\tau_{\alpha,h_m}'|^2 = O(m^n)\). Then by (5.6) and (5.7),

\[
\dot{f}_m(0) = O(m^{n-1}).
\]

By [25] (see also [8], 4.5), \(\ddot{f}_m(s) \geq 0\) for all \(s \in \mathbb{R}\). In (4.9), let \(m \to \infty\). Then by (5.8), we obtain \(F_1 \leq 0\), i.e., \(K\)-semistability of \((M, L)\) follows.

To show \(K\)-stability, we now assume that \(F_1 = 0\) for the test configuration above. Then by Lemma 4.8,

\[
\lim_{s \to -\infty} \dot{f}_m(s) = O(m^{n-1}).
\]

Here we consider the second derivative \(\ddot{f}_m(s)\). From now on, by setting \(\delta := C_3(\log m)q\), we require the real number \(s\) to satisfy

\[
|s| \leq \delta,
\]

where \(C_3\) is a positive real number independent of the choice of \(m\). For local one-parameter group

\[
\mu_{m,s} := \hat{\psi}_m(\exp(s)) \in G_m, \quad -\delta \leq s \leq \delta,
\]
we regard each \( \mu_{m,s} \) as a linear isomorphism of \( \mathbb{C}^{N_{m-1}} (= V_m) \), modulo the action by \( \Pi_m \), via the identification of \( (E_m)_0 \) with \( (E_m)_1 (= V_m) \). Note also that \( G_m \) acts on \( \mathbb{P}^*((E_m)_0) (= \mathbb{P}^*(V_m)) \) via the projection

\[
\pi_m : G_m (= \text{SL}(N_m; \mathbb{C})/\Pi_m) \to \text{PGL}((E_m)_0) (= \text{PGL}(N_m; \mathbb{C})).
\]

Now by Appendix, the family of Kähler manifolds

\[(5.11) \quad (M, q(\mu_{m,s}^*\omega_{FS}|_M), \quad -\delta \leq s \leq \delta, \ m = 1, 2, \ldots,\]

has bounded geometry. Let us now consider the holomorphic vector field \( \mathcal{V}^m := (\pi_m \circ \hat{\psi}_m)_*(\partial/\partial s) \) on \( \mathbb{P}^{N_{m-1}}(\mathbb{C}) \) which generates the local one-parameter group \( \pi_m(\mu_{m,s}), -\delta \leq s \leq \delta \). For each \( s \), consider the holomorphic tangent bundle \( TM_s \) of \( M_s := \mu_{m,s}(M) \), where \( M \) is viewed as a subvariety of \( \mathbb{P}^{N_{m-1}}(\mathbb{C}) \) by (5.2). Metrically, for the orthogonal complement \( TM_s^\perp \) of \( TM_s \) in \( T\mathbb{P}^{N_{m-1}}(\mathbb{C})|_{M_s} \) by the metric \( \omega_{FS} \), we can regard the normal bundle of \( M_s \) in \( \mathbb{P}^{N_{m-1}}(\mathbb{C}) \) as the subbundle \( TM_s^\perp \) of \( T\mathbb{P}^{N_{m-1}}(\mathbb{C})|_{M_s} \). Hence \( T\mathbb{P}^{N_{m-1}}(\mathbb{C})|_{M_s} \) is differentiably a direct sum \( TM_s \oplus TM_s^\perp \), and we can uniquely write

\[(5.12) \quad \mathcal{V}^m|_{M_s} = \mathcal{V}^m_{TM_s} + \mathcal{V}^m_{TM_s^\perp},\]

where \( \mathcal{V}_{TM} \) and \( \mathcal{V}_{TM^\perp} \) are smooth sections of \( TM_s \) and \( TM_s^\perp \), respectively. Consider the exact sequence of holomorphic vector bundles

\[
0 \to TM_s \to T\mathbb{P}^{N_{m-1}}(\mathbb{C})|_{M_s} \to TM_s^\perp \to 0
\]

over \( M_s \). Then the pointwise estimate (cf. [19], (5.16)) for the second fundamental form for this exact sequence is valid also in our case (cf. [8], Step 2), and as in [19], (5.15), we obtain the inequality

\[(5.13) \quad \int_{M_s} |\mathcal{V}_{TM_s^\perp}^m|^2 \omega_{FS}^n \geq C_4 \int_{M_s} |\bar{\partial}\mathcal{V}_{TM_s^\perp}^m|^2 \omega_{FS}^n \omega_{FS}^n,
\]

where \( C_4 \) is a positive real constant independent of the choice of \( m \).

The second derivative \( \ddot{f}_m(s) \) is (see for instance [8], [19]) given by

\[(5.14) \quad \ddot{f}_m(s) = \int_{M_s} |\mathcal{V}_{TM_s^\perp}^m|^2 |\omega_{FS}^n \geq 0.
\]
Put \( \varphi_m := \frac{\left( \sum_{\alpha=1}^{N_m} e_\alpha |z_\alpha|^2 \right) / (m \sum_{\alpha=1}^{N_m} |z_\alpha|^2)}{\left( \sum_{\alpha=1}^{N_m} |z_\alpha|^2 \right)} \) on \( \mathbb{P}^{N_m-1}(\mathbb{C}) \). Then by (5.7) \( \varphi_m, m = 1, 2, \ldots, \) are uniformly bounded satisfying (cf. [9], (4.5))

\[
(5.15) \quad iV_m(q \omega_{FS}) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \varphi_m.
\]

For \( M \) as a submanifold of \( \mathbb{P}^{N_m-1}(\mathbb{C}) \) in (5.2), we consider the sheaves \( A^0_j(TM), j = 0, 1, \ldots, \) of germs of smooth \((0,j)\)-forms on \( M \) with values in the holomorphic tangent bundle \( TM \) of \( M \), and endow \( M \) with the Kähler metric \( q\mu_{m,s} \omega_{FS} |_M \) for \( s \) as in (5.10). Now on \( A^{(0,0)}(TM) \), we consider the operator

\[
\Delta_{TM,s} := -\bar{\partial}^\# \bar{\partial},
\]

where \( \bar{\partial}^\# \) is the formal adjoint of \( \bar{\partial} : A^{(0,0)}(TM) \rightarrow A^{(0,1)}(TM) \). For \( \Gamma := H^0(M, A^{(0,0)}(TM)) \), we consider the Hermitian \( L^2 \)-pairing

\[
\langle V_1, V_2 \rangle_s := \int_M (V_1, V_2) q\mu_{m,s}^* \omega_{FS} (q\mu_{m,s}^* \omega_{FS})^n, \quad V_1, V_2 \in \Gamma,
\]

where \( (V_1, V_2) q\mu_{m,s}^* \omega_{FS} \) is the pointwise Hermitian pairing of \( V_1 \) and \( V_2 \) by the Kähler metric \( q\mu_{m,s}^* \omega_{FS} \). For the subspace \( g := H^0(M, \mathcal{O}(TM)) \) of \( \Gamma \), we consider its orthogonal complement \( g_s^\perp \) in \( \Gamma \) by the pairing \( \langle \ , \ \rangle_s \). Then \( V_{TM,s}^m \) in (5.12) is expressible as

\[
V_{TM,s}^m = V_{m,s}^\circ + V_{m,s}^\bullet,
\]

where \( V_{m,s}^\circ \) and \( V_{m,s}^\bullet \) belong to \( (\mu_{m,s})_* g \) and \( (\mu_{m,s})_* g_s^\perp \), respectively. Since the left-hand side of (5.12) is holomorphic,

\[
(5.16) \quad \bar{\partial} V_{TM,s}^m = -\bar{\partial} V_{m,s}^m = -\bar{\partial} V_{m,s}^\bullet.
\]

Since the family (5.11) has bounded geometry, the first positive eigenvalue \( \lambda_1 \) of the operator \( \Delta_{TM} \) on \( A^{(0,0)}(TM) \) is bounded from below by some positive constant \( C_5 \) independent of the choice of \( m \). Hence

\[
(5.17) \quad \int_{M_s} |\bar{\partial} V_{m,s}^\bullet(q \omega_{FS})|^2 (q \omega_{FS})^n \geq C_5 \int_{M_s} |V_{m,s}^\bullet|^2 (q \omega_{FS})^n.
\]

From (5.13), (5.16) and (5.17), it now follows that

\[
(5.18) \quad \int_{M_s} |V_{TM,s}^m|^2 \omega_{FS}^n \geq C_4 C_5 q \int_{M_s} |V_{m,s}^\bullet|^2 \omega_{FS}^n.
\]
In view of (5.14), \( \dot{f}_m(0) - \dot{f}_m(-\delta) = \int_{-\delta}^{0} \ddot{f}_m(s) \, ds \geq 0 \), and it follows from (5.8) and (5.9) that
\[
O(m^{n-1}) = \dot{f}_m(0) - \lim_{s \to -\infty} \dot{f}_m(s) \geq \dot{f}_m(0) - \dot{f}_m(-\delta)
\]
where \( s_m, m = 1, 2, \ldots, \) are real numbers at which the functions \( \ddot{f}_m(s) \), \( -\delta \leq s \leq 0 \), attain their minima, i.e.,
\[
\ddot{f}_m(s_m) = \min_{-\delta \leq s \leq 0} \ddot{f}_m(s).
\]
Therefore, in view of (5.14) and \( \delta = O(q \log m) \), we obtain
\[
\int_{M_{s_m}} |\nabla_{TM_{s_m}}^\perp q \omega_{FS} \rangle^n = O \left( \frac{q}{\log m} \right),
\]
infinitesimally
\[
(5.19)
\]
when restricted to \( M \subset P^*(V_m) \). Note that \( \tilde{\varphi}_m, m = 1, 2, \ldots, \) is a bounded sequence of real numbers. Then we can write the uniformly bounded real-valued functions \( \eta_m \) on \( M \) as
\[
\eta_m := \left( \int_{M_{s_m}} (q \omega_{FS} \rangle^n \right)^{-1} \int_{M_{s_m}} \varphi_m(q \omega_{FS} \rangle^n,
\]
where \( e_{a,m} := e_a - m \tilde{\varphi}_m \). Put \( \omega_m := q \mu_{s_m}^* \omega_{FS}|_M \). Hereafter, replace the sequence \( s_m, m \gg 1, \) by its suitable subsequence \( s_m, j = 1, 2, \ldots, \) if necessary. We write \( m_j, m_j^{-1}, N_{m_j}, s_{m_j}, \omega_{m_j}, \eta_{m_j}, e_{a,m_j} \) as \( m(j), q(j) \),
where \( \mu \) assume that \( \omega \) converges to \( \omega_\infty \) in \( C^\infty \) as \( j \to \infty \). Moreover, we set

\[
\begin{align*}
\mathcal{V}(j) & := \mathcal{V}^m(j) = (\mu_j^{-1})_* \mathcal{V}^m(j), \\
\mathcal{V}_{TM}(j) & := (\mu_j^{-1})_* \mathcal{V}_{TM}^m(j), \\
\mathcal{V}^o(j) & := (\mu_j^{-1})_* \mathcal{V}^o_{m(j),s(j)}, \\
\mathcal{V}^\bullet(j) & := (\mu_j^{-1})_* \mathcal{V}^\bullet_{m(j),s(j)},
\end{align*}
\]

where \( \mu_j := \mu_{m(j),s(j)} \). Then the following cases 1 and 2 are possible:

Case 1: \( I_j^0 := \int_M |\mathcal{V}^o(j)|^2_{\omega(j)} \omega(j)^n \), \( j = 1, 2, \ldots \), are bounded. In this case, by \( |\mathcal{V}_{TM}^o(j)|^2_{\omega(j)} = |\mathcal{V}^o(j)|^2_{\omega(j)} + |\mathcal{V}^\bullet(j)|^2_{\omega(j)} \), this boundedness together with (5.20) implies that

\[
(5.22) \quad \int_M |\mathcal{V}_{TM}(j)|^2_{\omega(j)} \omega(j)^n \), \( j = 1, 2, \ldots \), are bounded.
\]

Note that \( \omega(j) \to \omega_\infty \) in \( C^\infty \), as \( j \to \infty \). Hence in view of (5.22), since \( |\mathcal{V}_{TM}(j)|^2_{\omega(j)} = |\partial\eta(j)|^2_{\omega(j)} \) by (5.15), the sequence of integrals \( \int_M |\partial\eta(j)|^2_{\omega_\infty} \omega_\infty^n \), \( j = 1, 2, \ldots \), is bounded, so that \( \eta(j), \ j = 1, 2, \ldots \), is a bounded sequence in the Sobolev space \( L^{1,2}(M, \omega_\infty) \). Now by [21], we may assume that \( n \geq 2 \). Then replacing \( \eta(j), \ j = 1, 2, \ldots \), by its subsequence if necessary, we may further assume the convergence

\[
(5.23) \quad \eta(j) \to \eta_\infty \text{ in } L^2(M, \omega_\infty^n), \quad \text{as } j \to \infty,
\]

where \( \eta_\infty \) is a real-valued function in \( L^2(M, \omega_\infty) \). Recall that the Lichnerowich operator \( \Lambda_j : C^\infty(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}} \) for the Kähler manifold \( (M, \omega(j)) \) is an elliptic operator, of order 4, with kernel consisting of all Hamiltonian functions for the holomorphic Hamiltonian vector fields on \( M \). Now, to each smooth function \( f \in C^\infty(M)_{\mathbb{C}} \), we associate a complex vector field \( \mathfrak{g}_{f,j} \) of type \((1,0)\) on \( M \) such that

\[
i(\mathfrak{g}_{f,j}) \omega(j) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} f.
\]

Note that \( \mathfrak{g}_{\eta(j),j} = \mathcal{V}_{TM}(j) \) by (5.15) and (5.21). Then for the formal adjoint \( \Lambda_j^\# : C^\infty(M)_{\mathbb{C}} \to C^\infty(M)_{\mathbb{C}} \) of the operator \( \Lambda_j \), we have

\[
\int_M \eta(j) \{ \Lambda_j^\# f \} \omega(j)^n = \int_M \{ \Lambda_j \eta(j) \} f \omega(j)^n = \langle \partial \mathfrak{g}_{\eta(j),j}, \partial \mathfrak{g}_{f,j} \rangle_{s(j)}

= \langle \partial \{ \mathcal{V}_{TM}(j) \}, \partial \mathfrak{g}_{f,j} \rangle_{s(j)} = \langle \partial \{ \mathcal{V}^\bullet(j) \}, \partial \mathfrak{g}_{f,j} \rangle_{s(j)},
\]
for all $f \in C^\infty(M)_\mathbb{C}$. Here the last equality follows from the identities $\mathcal{V}_{TM}(j) = \mathcal{V}^0(j) + \mathcal{V}^\bullet(j)$ and $\partial\mathcal{V}^0(j) = 0$. Hence, for each fixed $f$ in $C^\infty(M)_\mathbb{C}$, we obtain

\begin{equation}
(5.24) \quad \left\{ \begin{array}{l}
\left| \int_M \eta(j) \{ \Lambda^\#_j f \} \omega(j)^n \right| = \left| \langle \mathcal{V}^\bullet(j), \Delta_j \mathcal{G}_{f,j} \rangle_{s(j)} \rangle \right| \\
\leq \left\{ \int_M |\Delta_j \mathcal{G}_{f,j}|^2 \omega(j)^n \right\}^{1/2} I_j^\bullet,
\end{array} \right.
\end{equation}

where $I_j^\bullet := \{ \int_M |\mathcal{V}^\bullet(j)|^2 \omega(j)^n \}^{1/2}$ and $\Delta_j := \Delta_{TM,s(j)}$. Let $j \to \infty$ in (5.24). Since $I_j^\bullet \to 0$ by (5.20), and since $\omega(j) \to \omega_\infty$ in $C^\infty$, by passing to the limit, we see from (5.23) and (5.24) that

$$
\int_M \eta_\infty \{ \Lambda^\#_\infty f \} \omega_\infty^n = 0 \quad \text{for all } f \in C^\infty(M)_\mathbb{C},
$$

where $\Lambda_\infty : C^\infty(M)_\mathbb{C} \to C^\infty(M)_\mathbb{C}$ is the Lichnerowich operator for the Kähler manifold $(M, \omega_\infty)$, and $\Lambda^\#_\infty$ is its formal adjoint. Since $\Lambda_\infty$ is elliptic, any weak solution $\eta = \eta_\infty$ for the equation

$$
\Lambda_\infty \eta = 0
$$

is always a strong solution. In particular $\eta_\infty$ is a real-valued smooth function on $M$ such that the complex vector field $W$ of type $(1, 0)$ on $M$ defined by $i(W)\omega_\infty = \bar{\partial}\eta_\infty$ is holomorphic. Then by [13], the test configuration $\pi : \mathcal{M} \to \mathbb{A}^1$ is a product configuration.

Case 2: $I_j^\bullet \to +\infty$ as $j \to \infty$. In this case, we put $\hat{\mathcal{V}}_{TM}(j) := \mathcal{V}_{TM}(j)/\sqrt{T_j}$, $\hat{\mathcal{V}}^0(j) := \mathcal{V}^0(j)/\sqrt{T_j}$, and $\hat{\mathcal{V}}^\bullet(j) := \mathcal{V}^\bullet(j)/\sqrt{T_j}$. Then

\begin{equation}
(5.25) \quad \int_M |\hat{\mathcal{V}}^0(j)|^2 \omega(j)^n = 1, \quad j = 1, 2, \ldots,
\end{equation}

where by setting $\hat{\eta}(j) := \eta(j)/\sqrt{T_j}$, we see from $\mathcal{G}_{\eta(j),j} = \mathcal{V}_{TM}(j)$ that the complex vector field $\hat{\mathcal{V}}_{TM}(j)$ of type $(1, 0)$ on $M$ satisfies

\begin{equation}
(5.26) \quad i(\hat{\mathcal{V}}_{TM}(j)) \omega(j) = \sqrt{-1} \bar{\partial}(\hat{\eta}(j)).
\end{equation}

Since the functions $\eta_m, m \gg 1$, are uniformly bounded on $M$, and since $\omega(j)$ converges to $\omega_\infty$ as $j \to \infty$, we obtain the convergence

\begin{equation}
(5.27) \quad \hat{\eta}(j) \to 0 \text{ in } C^0(M), \quad \text{as } j \to \infty.
\end{equation}
By (5.25), replacing $\hat{V}^0(j)$, $j = 1, 2, \ldots$, by its subsequence if necessary, we may assume that

\begin{equation}
(5.28) \quad \hat{V}^0(j) \to \hat{V}^0_\infty \text{ in } g, \quad \text{as } j \to \infty,
\end{equation}

for some $0 \neq \hat{V}^0_\infty \in g$. Let $\hat{\eta}^0(j)$ and $\hat{\eta}^*(j)$ be the Hamiltonian functions associated to the vector fields $\hat{V}^0(j)$ and $\hat{V}^*(j)$ by

\begin{equation}
\begin{cases}
i(\hat{V}^0(j)) \omega(j) = \sqrt{-1} \bar{\partial}(\hat{\eta}^0(j)), \\
i(\hat{V}^*(j)) \omega(j) = \sqrt{-1} \bar{\partial}(\hat{\eta}^*(j)),
\end{cases}
\end{equation}

where the functions $\hat{\eta}^0(j)$ and $\hat{\eta}^*(j)$ are normalized by the vanishing of the integrals $\int_M \hat{\eta}^0(j) \omega(j)^n$ and $\int_M \hat{\eta}^*(j) \omega(j)^n$. Then

\begin{equation}
(5.29) \quad \hat{\eta}(j) = \hat{\eta}^0(j) + \hat{\eta}^*(j).
\end{equation}

In view of (5.28), there exists a non-constant real-valued $C^\infty$ function $\rho$ on $M$ such that $i(\hat{V}^0_\infty) \omega_\infty = \sqrt{-1} \bar{\partial} \rho$ and that

\begin{equation}
\hat{\eta}^0(j) \to \rho \text{ in } C^\infty(M), \quad \text{as } j \to \infty.
\end{equation}

Hence by (5.29), it follows from (5.27) that

\begin{equation}
(5.30) \quad \hat{\eta}^*(j) \to -\rho \text{ in } C^0(M), \quad \text{as } j \to \infty.
\end{equation}

On the other hand, by (5.20), $\int_M |\bar{\partial} \hat{\eta}^*(j)|^2_{\omega(j)} \omega(j)^n \to 0$ as $j \to \infty$, and hence for each fixed smooth $(0,1)$-form $\theta$ on $M$, we have

\begin{align*}
\left| (\hat{\eta}^*(j), \bar{\partial}(j)^* \theta)_{L^2(M, \omega(j)^n)} \right| &= \left| \int_M (\bar{\partial} \hat{\eta}^*(j), \theta)_{\omega(j)} \omega(j)^n \right| \\
&\leq \left\{ \int_M |\bar{\partial} \hat{\eta}^*(j)|^2_{\omega(j)} \omega(j)^n \right\}^{1/2} \left\{ \int_M |\theta|^2_{\omega(j)} \omega(j)^n \right\}^{1/2} \to 0,
\end{align*}

as $j \to \infty$, where $\bar{\partial}(j)^*$ and $\bar{\partial}^*_\infty$ are the formal adjoints of the operator $\bar{\partial}$ on functions for the Kähler manifolds $(M, \omega(j))$ and $(M, \omega_\infty)$, respectively. Then by letting to $j \to \infty$, we obtain the vanishing for the Hermitian $L^2$-inner product of functions $\rho$ and $\bar{\partial}^* \theta$,

\begin{equation}
(\rho, \bar{\partial}^* \theta)_{L^2(M, \omega_\infty)} = 0,
\end{equation}

for every smooth $(0,1)$-form $\theta$ on $M$, i.e., $\bar{\partial} \rho = 0$ in a weak sense, and therefore in a strong sense. Hence we conclude that $\rho$ is constant on $M$ in contradiction to $\hat{V}^0_\infty \neq 0$. 

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In this appendix, we shall show that the family of Kähler manifolds

\[(M, q (\mu_{m,s}^* \omega_{FS})_{|M})], \quad -\delta \leq s \leq \delta, \ m = 1, 2, \ldots,\]

has bounded geometry. By Fact 2.6 applied to \(m = 1\), we identify \(\mathbb{P}^*(E_1)\) with \(\mathbb{A}^1 \times \mathbb{P}^*((E_1)_0)\), and let \(\text{pr}_2 : \mathbb{P}^*(E_1) \to \mathbb{P}^*((E_1)_0)\) be the projection to the second factor. As in Section 4, we have

\[(\mathcal{M} \hookrightarrow \mathbb{P}^*(E_1)),\]

where the pullback \(\mathcal{H} := \text{pr}_2^* \mathcal{O}_{\mathbb{P}^*((E_1)_0)}(1)\) to \(\mathbb{P}^*(E_1)\) of the the hyperplane bundle \(\mathcal{O}_{\mathbb{P}^*((E_1)_0)}(1)\) on \(\mathbb{P}^*((E_1)_0)\) is written as

\[\mathcal{H}_{|\mathcal{M}} = \mathcal{L} (= L^f).\]

Recall that the action of \(T = \mathbb{C}^*\) on \(\mathcal{M}\) lifts to an action of \(T\) on \(\mathcal{L}\), and hence \(T\) acts on \(E_1 = \mathbb{A}^1 \times (E_1)_0\) by

\[T \times (\mathbb{A}^1 \times (E_1)_0) \to \mathbb{A}^1 \times (E_1)_0, \quad (t, (z, e)) \mapsto (tz, \psi_1(t) \cdot e).\]

This induces a \(T\)-action on \(\mathbb{P}^*(E_1) (= \mathbb{A}^1 \times \mathbb{P}^*((E_1)_0))\), and for (6.2), \(\mathcal{M}\) is preserved by the \(T\)-action. Note that the \(T\)-action on \(\mathcal{L}\) lifts the \(T\)-action on \(\mathcal{M}\). By

\[T \times \mathcal{M} \to \mathcal{M}, \quad (t, p) \mapsto g_{\mathcal{M}}(t) \cdot p,\]

we mean the \(T\)-action on \(\mathcal{M}\), and the corresponding \(T\)-action on \(\mathcal{L} \otimes \tilde{\mathcal{L}}\) upstairs will be denoted by

\[T \times (\mathcal{L} \otimes \tilde{\mathcal{L}}) \to \mathcal{L} \otimes \tilde{\mathcal{L}}, \quad (t, h) \mapsto g_{|\mathcal{L} \otimes \tilde{\mathcal{L}}}(t) \cdot h.\]

Since \(\text{GL}((E_m)_0)\) acts on \(\mathbb{P}^*((E_m)_0)\) via the projection of \(\text{GL}((E_m)_0)\) onto \(\text{PGL}((E_m)_0)\), by setting \(\tilde{\mu}_{m,s} := \psi_m(\exp(s))\), we have

\[q \mu_{m,s}^* \omega_{FS}^q = q \tilde{\mu}_{m,s}^* \omega_{FS}.\]

In view of \(\delta = C_3(\log m)q, \ m \gg 1\), we estimate \(\exp(s)\) in the form

\[1 - \epsilon \leq e^{-C_3(\log m)/m} \leq \exp(s) \leq e^{C_3(\log m)/m} \leq 1 + \epsilon\]

for some \(0 < \epsilon \ll 1\). As in Section 5, by the bases \(\{\tau_1, \tau_2, \ldots, \tau_{N_m}\}\) and \(\{\tau'_1, \tau'_2, \ldots, \tau'_{N_m}\}\) for \((E_m)_0\) and \((E_m)_1 (= V_m)\), respectively, the spaces
$\mathbb{P}^*(\mathcal{E}_m) \text{ and } \mathbb{P}^*(\mathcal{E}_m)_1 (= \mathbb{P}^*(\mathcal{V}_m))$ are identified with

$\mathbb{P}^{N_m-1}(\mathbb{C}) = \{(z_1 : z_2 : \cdots : z_{N_m})\}.$

Note that $q_{\omega_{FS}} = (\sqrt{-1}/2\pi)\partial\bar{\partial} \log \Omega_{FS,m}$, where $\Omega_{FS,m}$ denotes the positive real smooth section $\{(n!/m^n)^{\sum_{\alpha=1}^{N_m} |z_\alpha|^2}\}$ of $\mathcal{H} \otimes \bar{\mathcal{H}}$. In view of (6.3), identifying $M$ with $\mathcal{M}_1$, we easily see that $q \mu_{m,s}^{\omega_{FS}}$ is

\begin{equation}
(\sqrt{-1}/2\pi) g_M(\exp(s))^* \partial\bar{\partial} \log \{g_{[\mathcal{L}]^2}(\exp(s)) \cdot \Omega_{FS,m}\},
\end{equation}

when restricted to $M$. By (3.6) and (5.3), we now conclude from (6.4), (6.5) and (6.6) that the family of Kähler manifolds in (6.1) has bounded geometry, as required.

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