Geometric Probabilities and Fibonacci Numbers for Maximally Random n-Qubit Quantum Information States

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Abstract

The problems of Hadamard quantum coin flipping in $n$-trials and related generalized Fibonacci sequences of numbers were introduced in [1]. It was shown that for an arbitrary number of repeated consecutive states, probabilities are determined by Fibonacci numbers for duplicated states, Tribonacci numbers for triplicated states and N-Bonacci numbers for arbitrary N-plicated states. In the present paper we generalize these results for direct product of multiple qubit states and arbitrary position of repeated states. The calculations are based on structure of Fibonacci trees in space of qubit states, growing in the left and in the right directions, and number of branches and allowed paths on the trees. By using $n$-qubit quantum coins as random $n$-qubit states with maximal Shannon entropy, we show that quantum probabilities can be calculated by means of geometric probabilities. It illustrates possible application of geometric probabilities in quantum information theory. The Golden ratio of probabilities and the limit of $n$ going to infinity are discussed.
1 Introduction

Fibonacci Numbers, 1,1,2,3,5,8,13,... satisfy recursion formula $F_n = F_{n-1} + F_{n-2}$, with initial conditions $F_1 = F_2 = 1$ and are related with Golden ratio

$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6, \quad \varphi' = -1/\varphi$, by the Binet formula

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'}.$$  

By interpreting this formula as a specific q-number in the post-quantum or pq-calculus, the quantum calculus of Fibonacci numbers (the Golden calculus) was introduced in [2], and then generalized to Fibonacci divisors in [3]. Relations of Fibonacci numbers with quantum measurement problem of qubits have been studied in [1]. It was shown that probabilities of measurement for qubit states with repeated identical states in $n$ trials are related with Fibonacci, Tribonacci and in general, $N$-Bonacci numbers. This problem, as a quantum coin tossing problem, can be regarded as a quantum analogue of the classical coin tossing problem, apparently first introduced by A. de Moivre in 18th century, in his book [4]. For relation of this problem with classical dynamical systems see [5].

In this work we generalize these results for the direct product of multiple qubit states and for arbitrary position of repeated states. The calculations are based on structure of Fibonacci trees in space of qubit states, and number of branches and allowed paths on the trees in Fibonacci garden. By using tensor product of maximally random multiple qubit states, we naturally come to problems of geometrical quantum probabilities. Applications of geometrical concepts as distance and area to calculate probability distributions are known as geometric probabilities. The oldest problem in geometric probability is Buffon’s Needle problem (1777), determining probability of needle crossing one of the lines on the page and directly related to value of number $\pi$. In quantum information theory, the unit of quantum information is represented geometrically as unit sphere, known as the Bloch sphere. By stereographic projection of Bloch sphere (considered as the Riemann sphere) to complex plain , it is possible to calculate some characteristics of qubits in terms of the plain geometry. This includes the qubit coherent states and the Apollonius representation [6]. As we show in the present paper, geometrical concepts can be used also to analyze quantum probabilities. First we show that probabilities of measurement of one qubit state can be calculated in pure geometrical way by ratio of areas of spherical cap and the Bloch
sphere itself. Then, by using quantum n-qubit coins as maximally random n-qubit states, we show that the quantum measurement probabilities can be calculated by means of geometric probabilities. This illustrates possible application of geometric probabilities in quantum information theory.

2 Geometric Quantum Probability on Bloch sphere

The qubit unit of quantum information is the state

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \in H = \mathbb{C}^2,$$

(1)

where $|c_0|^2 + |c_1|^2 = 1$ and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(2)

It can be parameterized

$$|\theta, \varphi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\varphi} |1\rangle$$

(3)

by points $(\theta, \varphi)$ on unit sphere $S^1$: $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$. Then, probabilities of the qubit measurement

$$p_0 = |c_0|^2 = \cos^2 \frac{\theta}{2}, \quad p_1 = |c_1|^2 = \sin^2 \frac{\theta}{2}, \quad p_0 + p_1 = 1,$$

(4)

are independent of angle $\varphi$ and geometrically are axial symmetry invariants. These probabilities have simple geometrical description in terms of areas on Bloch sphere. The area of spherical cap with radius $R$ (solid angle) is

$$A_\theta = \int_0^\theta \int_0^{2\pi} R^2 \sin \theta' d\theta' d\varphi = 2\pi R^2 (1 - \cos \theta)$$

(5)

or

$$A_\theta = 4\pi R^2 \sin^2 \frac{\theta}{2}.$$

(6)

Comparing with total area of the sphere $A = 4\pi R^2$, we get pure geometric form of quantum probabilities (4), as relative area of the spherical cap

$$p_0 = \sin^2 \frac{\theta}{2} = \frac{A_\theta}{A}.$$  

(7)
and the complementary area

\[ p_1 = \cos^2 \frac{\theta}{2} = \frac{A - A_\theta}{A}. \]  

(8)

### 3 n-qubit State as Random Variable

Every binary number,

\[ N = \sum_{k=0}^{n-1} a_k 2^k = a_{n-1}a_{n-2}...a_1a_0, \]

where \( a_k = 0,1 \), determines n-qubit state in binary form

\[ |N\rangle_n = |a_{n-1}a_{n-2}...a_1a_0\rangle = |a_{n-1}\rangle \otimes |a_{n-2}\rangle \otimes ... \otimes |a_1\rangle \otimes |a_0\rangle. \]

(10)

This state can be generated by one qubit flipping gate \( X = \sigma_1 \), in the form

\[ |N\rangle_n = |a_{n-1}a_{n-2}...a_1a_0\rangle = \sigma_1^{a_{n-1}} \otimes \sigma_1^{a_{n-2}} \otimes ... \otimes \sigma_1^{a_1} \otimes \sigma_1^{a_0} |0\rangle_n, \]

(11)

where \( \sigma_1|0\rangle = |1\rangle, \ \sigma_1|1\rangle = |0\rangle \). Then, for arbitrary \( n \) we have the set of \( 2^n \) orthonormal states \( |0\rangle, |1\rangle, |2\rangle, ..., |2^n - 1\rangle \), as computational basis. The normalized generic n-qubit state in this computational basis is

\[ |\psi\rangle = \sum_{k=0}^{2^n-1} c_k |k\rangle, \quad \langle \psi|\psi\rangle = \sum_{k=0}^{2^n-1} |c_k|^2 = 1. \]

(12)

According to the Born rule, the measurement probability of collapse to basis state \( |k\rangle \) is \( p_k = |c_k|^2 \) and \( \sum_{k=0}^{2^n-1} p_k = 1 \). This shows that state (12) can be considered as a random variable state, with \( 2^n \) output states, as computational basis states and the corresponding probabilities. Then, the level of randomness for this state in form of the Shannon entropy is

\[ S = -\sum_{k=0}^{2^n-1} p_k \log p_k = -\sum_{k=0}^{2^n-1} |c_k|^2 \log |c_k|^2 . \]

(13)

For maximally random n-qubit state, the entropy is maximal \( S = S_{max} \) and probabilities are equal and independent of \( k \), \( p_k = |c_k|^2 = 1/2^n \), so that \( c_k = e^{i\phi_k}/\sqrt{2^n} \). Then, up to global phase, the maximally random states have the form

\[ |\psi_{max}\rangle = \frac{1}{\sqrt{2^n}} \left( |0\rangle + e^{i\phi_1}|1\rangle + ... + e^{i\phi_{2^n-1}}|2^n - 1\rangle \right). \]

(14)
**Proposition 3.0.1** The maximal value of Shannon entropy for \( n \)-qubit state is equal to number of qubits

\[ S_{\text{max}} = n. \] (15)

For an arbitrary state, the entropy is bounded by inequality \( 0 \leq S \leq n \) and it is maximal for states (14).

**Proposition 3.0.2** For separable normalized states

\[ |\Psi\rangle = |\psi\rangle_n \otimes |\phi\rangle_m \] (16)

the Shannon entropy is additive

\[ S = S_n + S_m. \] (17)

Indeed, if

\[ |\Psi\rangle = \sum_{i_1,...,i_{n+m}=0,1} c_{i_1,...,i_{n+m}} |i_1, ..., i_n\rangle \] (18)

and

\[ |\psi\rangle_n = \sum_{i_1,...,i_n=0,1} a_{i_1,...,i_n} |i_1, ..., i_n\rangle, \quad |\phi\rangle_m = \sum_{j_1,...,j_m=0,1} b_{j_1,...,j_m} |j_1, ..., j_m\rangle, \] (19)

then \( c_{i_1,...,i_{n+m}} = a_{i_1,...,i_n} b_{i_{n+1},...,i_{n+m}} \) and probabilities

\[ p_{i_1,...,i_{n+m}} = |c_{i_1,...,i_{n+m}}|^2 = |a_{i_1,...,i_n}|^2 |b_{i_{n+1},...,i_{n+m}}|^2. \] (20)

By substituting to entropy formula (13) and using normalization conditions for states (19), we get (17).

**Corollary 3.0.3** If

\[ S \neq S_n + S_m \] (21)

the state is not separable, it is entangled state.

**Corollary 3.0.4** To every ordered partition of integer numbers

\[ n = n_1 + ... + n_N \] (22)

exists separable \( n \)-qubit state

\[ |\Psi\rangle_n = |\psi_1\rangle_{n_1} \otimes ... \otimes |\psi_N\rangle_{n_n} \] (23)
3.1 Maximally random separable states

Corollary 3.1.1 For separable n-qubit state with partition \( n = n_1 + \ldots + n_N \),
\[
|\Psi\rangle_n = |\psi_1\rangle_{n_1} \otimes \ldots \otimes |\psi_N\rangle_{n_N}
\] (24)
 entropy is the addition
\[
S_n = S_{n_1} + \ldots + S_{n_N}
\] (25)

Proposition 3.1.2 The product of maximally random states is the n-qubit state, which is maximally random and corresponding entropy \( S = n \) is given by ordered partition to the related partial entropies \( S_{n_k} = n_k, \ k = 1, \ldots, N \),
\[
n = n_1 + \ldots + n_N
\] (26)

3.1.1 Geometric quantum probability in 1D and maximally random n qubit states

Proposition 3.1.3 For maximally random state
\[
|\psi_{\text{max}}\rangle = \frac{1}{\sqrt{2^n}} \left( |0\rangle + e^{i\varphi_1} |1\rangle + \ldots + e^{i\varphi_{2^n-1}} |2^n - 1\rangle \right)
\] (27)
 probability of collapse to every basis state \( |s\rangle \) is the same and equal \( p_s = 1/2^n \).
 Then, probability to measure state \( |s\rangle \) in interval \( k \leq s < l \) is given by geometric ratio of intervals (proportional to number of states)
\[
p_{k,l} = \frac{L_{k,l}}{L} = \frac{N_{k,l}}{N} = \frac{l - k}{2^n}.
\] (28)

Proof 3.1.4 The average of projection operator \( \hat{P}_s = |s\rangle\langle s| \) gives probability \( p_s = \langle \Psi | \hat{P}_s | \Psi \rangle \) of collapse \( |\Psi\rangle \rightarrow |s\rangle \). For maximally random n-qubit state \( |\psi_{\text{max}}\rangle \) it gives
\[
p_s = \langle \Psi | \hat{P}_s | \Psi \rangle = \frac{1}{2^n}.
\] (29)

Projection operator to subspace \( H_{k,l} = \prod_{s=k}^{l-1} H_s \in H_{2^n} \) in Hilbert space is
\[
\hat{P}_{k,l} = |k\rangle\langle k| + |k+1\rangle\langle k+1| + \ldots + |l-1\rangle\langle l-1|
\] (30)
 and probability of collapse to this subspace is
\[
p_{k,l} = \langle \Psi | \hat{P}_{k,l} | \Psi \rangle = \frac{1}{2^n} + \frac{1}{2^n} + \ldots + \frac{1}{2^n} = \frac{l - k}{2^n}
\] (31)
3.1.2 Geometric quantum probability in 2D and maximally random bipartite states

Proposition 3.1.5 For maximally random bipartite state
\[ |\Psi_{\text{max}}\rangle_{n+m} = |\psi_{\text{max}}\rangle_n \otimes |\phi_{\text{max}}\rangle_m \] (32)
probability of collapse to every basis state \( |i\rangle_n |j\rangle_m \) is the same and equal \( p = \frac{1}{2^{n+m}} \). Then, probability to measure state \( |i\rangle |j\rangle \) in interval \( k_1 \leq i < l_1 \), \( k_2 \leq j < l_2 \) is given by ratio of rectangular areas
\[ p_{k_1l_1,k_2l_2} = \frac{A_{k_1l_1,k_2l_2}}{A} = \frac{N_{k_1l_1,k_2l_2}}{N} = \frac{(l_1 - k_1)(l_2 - k_2)}{2^{n+m}} \] (33)

3.1.3 Geometric quantum probability in arbitrary dimensions and maximally random multi-partite states

Proposition 3.1.6 For maximally random multi-partite state
\[ |\Psi_{\text{max}}\rangle_n = |\psi_{1,\text{max}}\rangle_{n_1} \otimes \cdots \otimes |\psi_{N,\text{max}}\rangle_{n_N} \] (34)
probability of collapse to every basis state \( |i_1\rangle_{n_1} \cdots |i_N\rangle_{n_N} \), \( n = n_1 + \cdots + n_N \), is the same and equal \( p = \frac{1}{2^{n_1+\cdots+n_N}} \). Then, probability to measure state \( |i_1\rangle \cdots |i_N\rangle \) in intervals \( k_1 \leq i_1 < l_1, \ldots, k_N \leq i_N < l_N \) is given by ratio of volumes of parallelepipeds
\[ p_{k_1l_1,\ldots,k_Nl_N} = \frac{V_{k_1l_1,\ldots,k_Nl_N}}{V} = \frac{N_{k_1l_1,\ldots,k_Nl_N}}{N} = \frac{(l_1 - k_1) \cdots (l_N - k_N)}{2^{n_1+\cdots+n_N}} \] (35)

4 Quantum Coin and Maximally Random Qubit State

For one qubit state (13), the Shannon entropy (13), as measure of uncertainty in result of measurement (7), is maximal \( S = 1 \) for \( p_0 = p_1 = \frac{1}{2} \), like for classical coin. But now it gives \( |c_0| = |c_1| = \frac{1}{\sqrt{2}} \) and the Hadamard type qubit states (the Quantum Coin States),
\[ |\varphi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\varphi}|1\rangle). \] (36)
These states are generated by Hadamard gate $H$ and the phase-shift gate $R_z(\phi)$: $|\phi\rangle = R_z(\phi)H|0\rangle$. For $\phi = 0$ and $\phi = \pi$ we have the Hadamard states $|+\rangle$ and $|-\rangle$, correspondingly. Flipping of the quantum coin is an application of $X$ gate on "heads" $|0\rangle$ and "tails" $|1\rangle$ states: $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$. Then, applying the Hadamard gate to the coin, initialized in state $|0\rangle$, quantum computer produces state $|+\rangle = H|0\rangle$. The measurement $M$ of this quantum coin state,

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$





gives states $|0\rangle$ or $|1\rangle$ with equal probabilities $p_0 = p_1 = \frac{1}{2}$.

### 5 Duplicated $|1\rangle$ States and Fibonacci Numbers

The following problem is a quantum mechanical analog of classical coin flipping problem of de Moivre [4]. The problem is to find probability of measurement quantum coin states in $n$– trials, such that repeated pattern of states $|1\rangle$ appears only in last two final measurements. The results of these measurements we can order as $n$-qubit computational basis state. Then, the first question is how many $n$-qubit states $A_n$ of following form exist:

$$|\ast\rangle \otimes |\ast\rangle \otimes \ldots |\ast\rangle \otimes |1\rangle \otimes |1\rangle \equiv |\ast\rangle |\ast\rangle \ldots |\ast\rangle |1\rangle |1\rangle \quad (37)$$

5.1 Number of coin states and Fibonacci numbers

To calculate number of allowed states, we notice that if state $|\ast\rangle = |0\rangle$, then preceding state from the left can be both, $|0\rangle$ or $|1\rangle$ state. But if $|\ast\rangle = |1\rangle$ state, then the preceding state can only be $|0\rangle$. This implies the tree of states as Fibonacci tree, where $|0\rangle$ state plays the role of adult rabbit and can produce young rabbit state. In contrast, state $|1\rangle$ corresponds to young rabbit and can become adult only. The number of states $A_n$ in Fibonacci tree for $n$ qubit quantum states is equivalent to number of different paths (of length $n$ ) in this tree. Starting from $n = 3$ and $|0\rangle$ state, number of paths satisfies the recursion formula $A_n = A_{n-1} + A_{n-2}$ and initial conditions $A_2 = A_3 = 1$. This gives number of states as Fibonacci number $A_n = F_{n-1}$, $n = 2, 3, \ldots$
5.2 Quantum coin measurement probability

The measurement of quantum coin in \( n \) trials gives states \(|0\rangle\) and \(|1\rangle\) with probabilities \( p_0 = p_1 = \frac{1}{2} \). This is equivalent to measurement of maximally random \( n \)-qubit state (14). If \( \hat{P}_F \) is projection operator to allowed duplicated states \( 3\rangle \), then probability of collapse to corresponding subspace is

\[
P_n = \langle \psi_{\text{max}} | \hat{P}_F | \psi_{\text{max}} \rangle = A_n \frac{1}{2^n} = \frac{F_{n-1}}{2^n}, \quad n = 2, 4, ... \tag{38}
\]

Probabilities \( P_n \) satisfy recursion formula for the Fibonacci polynomial numbers

\[
P_n = \frac{1}{2} P_{n-1} + \frac{1}{2} P_{n-2}, \tag{39}
\]

with initial values \( P_1 = 0, P_2 = \frac{1}{2} \). First few numbers are \( P_3 = \frac{1}{2}, P_4 = \frac{2}{2}, P_5 = \frac{3}{2}, P_6 = \frac{5}{2}, \) etc.

5.2.1 Golden Ratio and quantum probability

If we compare the number of duplicated \( n \)-qubit states \( A_n = F_{n-1} \) and \( n+1 \)-qubit states \( A_{n+1} = F_n \), then

\[
\lim_{n \to \infty} \frac{A_{n+1}}{A_n} = \lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \varphi \tag{40}
\]

is the Golden Ratio. For corresponding probabilities we have half of the Golden Ratio

\[
\lim_{n \to \infty} \frac{P_{n+1}}{P_n} = \lim_{n \to \infty} \frac{1}{2} \frac{F_n}{F_{n-1}} = \frac{1}{2} \varphi. \tag{41}
\]

6 Arbitrary position of duplicated \(|1\rangle\) states

If duplicated states appear at the end of \( n \) qubit state, the Fibonacci tree is growing to the left

\[
\leftarrow |\ast\rangle \otimes |\ast\rangle \otimes ... |\ast\rangle \otimes |1\rangle \otimes |1\rangle, \quad A_n = F_{n-1}, \tag{42}
\]

while, for the states at the beginning, it is growing to the right

\[
\rightarrow |1\rangle \otimes |1\rangle \otimes |\ast\rangle \otimes |\ast\rangle \otimes ... |\ast\rangle, \quad A_n = F_{n-1}. \tag{43}
\]
In generic case, when duplicated states appear at positions $k$ and $k+1$, where $k = 1, 2, ..., n - 1$, we have two Fibonacci trees, growing in both directions

$$|* \otimes * \otimes \ldots * \otimes |1 \otimes |1 \otimes * \otimes \ldots * \rangle_{k-1} \otimes |* \rangle_{k+1} \otimes |* \rangle_{n-k-1}$$ \hspace{1cm} (44)$$

and the number of allowed states is product

$$A_n = F_k \cdot F_{n-k}. \hspace{1cm} (45)$$

6.1 Probability for arbitrary position of duplicated states

Probability of states (44) with arbitrary position of duplicated states is

$$P_{n,k} = \frac{F_k \cdot F_{n-k}}{2^n} = \frac{L_n - (-1)^k L_{n-2k}}{5 \cdot 2^n}, \hspace{1cm} (46)$$

where we have used following identity with Lucas numbers $L_n$:

$$F_m \cdot F_n = \frac{L_{m+n} - (-1)^n L_{m-n}}{5}. \hspace{1cm} (47)$$

The number of states with pair $|1 \rangle |1 \rangle$ in all possible positions $k = 1, 2, ..., n-1$ is

$$\sum_{k=1}^{n-1} F_k \cdot F_{n-k} = \frac{nL_n - F_n}{5} \hspace{1cm} (48)$$

and probability of getting this pair anywhere, but only once is

$$\sum_{k=1}^{n-1} P_{n,k} = \sum_{k=1}^{n-1} \frac{F_k \cdot F_{n-k}}{2^n} = \frac{nL_n - F_n}{5 \cdot 2^n}. \hspace{1cm} (49)$$

7 Fibonacci Numbers for Separable States

Here we extend the above results to separable duplicated states.

7.1 Fibonacci numbers for bipartite states

For bipartite separable states

$$|\Psi\rangle_{n+m} = |**\ldots 11\rangle \otimes |**\ldots 11\rangle$$ \hspace{1cm} (50)$$
the number of allowed duplicated states is
\[ A = F_{n-1}F_{m-1} = \frac{1}{5}(L_{n+m-2} + (-1)^m L_{n-m}) \] (51)
and corresponding probability is equal
\[ p = \frac{F_{n-1}F_{m-1}}{2^{n+m}}. \] (52)

### 7.2 Fibonacci numbers for multi-partite states

For multi-partite states of following form
\[ |\Psi\rangle_n = |\ast\ldots|11\rangle \otimes |\ast\ldots|11\rangle \otimes \ldots |\ast\ldots|11\rangle, \] (53)
where partition \( n = n_1 + \ldots + n_N \), the number of allowed duplicated states is
\[ A = F_{n_1-1}F_{n_2-1}\ldots F_{n_N-1} \] (54)
and corresponding probability is equal
\[ p = \frac{F_{n_1-1}F_{n_2-1}\ldots F_{n_N-1}}{2^{n_1+n_2+\ldots+n_N}}. \] (55)

### 7.3 Arbitrary positions in bipartite states and Fibonacci garden

If we have bipartite \( n + m \) qubit state of the form
\[ |\Psi\rangle_{n+m} = |\ast\ldots|11\ast\ldots\rangle \otimes |\ast\ldots|11\ast\ldots\rangle, \] (56)
where duplicated states \( |1\rangle|1\rangle \) take place at positions \((k_1, k_1 + 1)\) in \( n \) qubit state and \((k_2, k_2 + 1)\) in \( m \) qubit state, then the number of allowed states is
\[ A = F_{k_1}F_{n-k_1}F_{k_2}F_{m-k_2} \] (57)
and corresponding probability equal to
\[ p = \frac{F_{k_1}F_{n-k_1}F_{k_2}F_{m-k_2}}{2^{n+m}}. \] (58)
7.4 Arbitrary positions in multipartite states

For separable $n$ qubit state with partition $n = n_1 + ... + n_N$, and arbitrary positions of duplicated states, $(k_s, k_{s+1})$ in corresponding $n_s$ qubit state ($s = 1, 2, ..., N$),

$$|\Psi\rangle_n = \underbrace{|**...*11*...*\rangle}_{n_1} \otimes \underbrace{|**...*11*...*\rangle}_{n_2} \otimes ... \otimes \underbrace{|**...*11*...*\rangle}_{n_N}$$  \hspace{1cm} (59)

the number of states is

$$A = F_{k_1} F_{n_1 - k_1} F_{k_2} F_{n_2 - k_2} ... F_{k_N} F_{n_N - k_N}$$  \hspace{1cm} (60)

and corresponding probability is equal

$$p = \frac{F_{k_1} F_{n_1 - 1} F_{k_2} F_{n_2 - 1} ... F_{k_N} F_{n_N - 1}}{2^{n_1 + n_2 + ... + n_N}}.$$  \hspace{1cm} (61)

8 Maximally Random $n$-qubit State as Qudit Coin

From computational $n$ qubit states $|k\rangle$, $k = 0, 1, 2, ..., 2^n - 1$, the Hadamard gate in $2^n$ dimensions

$$H = \frac{1}{\sqrt{2^n}} \sum_{k,l=0}^{2^n-1} q^{k\cdot l} |k\rangle \langle l|,$$  \hspace{1cm} (62)

where $q = e^{i\frac{2\pi}{2^n}}$ is primitive root of unity $q^{2^n} = 1$, can generate $2^n$ maximally random $n$-qubit states

$$|\phi_k\rangle = H|k\rangle, \hspace{0.5cm} k = 0, 1, ..., 2^n - 1.$$  \hspace{1cm} (63)

By using Silvester shift matrix $\Sigma_1$, computational states are expressible as

$$|k\rangle = \Sigma_1^k |0\rangle$$  \hspace{1cm} (64)

and as follows

$$|\phi_k\rangle = \frac{1}{\sqrt{2^n}} [2^n] q^{k \cdot \Sigma_1} |0\rangle = A_k^+ |0\rangle,$$  \hspace{1cm} (65)

where we have used following definition.
Definition 8.0.1 The matrix Q-number is defined by the sum
\[ I + \Sigma_1 + \Sigma_1^2 + \ldots + \Sigma_1^{2^n-1} \equiv [2^n]_{\Sigma_1}. \]

In Eq. (65), for every \( k \) we have this matrix for \( Q = \bar{q}^k \Sigma_1 \), \( k = 0, 1, 2, \ldots, 2^n - 1 \).

Every maximally random \( n \) qubit state \(|\phi_k\rangle\) represents a qudit quantum coin with number of states \( d = 2^n \) and the number of such coins is equal \( 2^n \).
The set of these quantum coin states is orthonormal and complete
\[ \langle \phi_k | \phi_l \rangle = \delta_{kl}, \quad \sum_{k=0}^{2^n-1} |\phi_k\rangle \langle \phi_k| = I. \quad (66) \]

8.1 \( n \)-qubit coin in M trials

For every quantum coin \(|\phi_k\rangle\) the result of measurement is one of the states \(|l\rangle\), \( l = 0, 1, \ldots, 2^n - 1 \), with equal probability
\[ p = \frac{1}{2^n} = |\langle l|\phi_k\rangle|^2. \quad (67) \]

This is why the state \(|\phi_k\rangle\) represents the qudit coin with \( d = 2^n \) states. The measurement of an arbitrary qudit coin in \( M \) trials, for duplicated states of the form
\[ \underbrace{|\ast\rangle_n \otimes \ast\rangle_n \otimes \ldots \ast\rangle_n}_M \otimes \underbrace{|1\rangle_n \otimes |1\rangle_n}_2 \equiv \underbrace{|\ast\rangle \ldots |\ast\rangle}_M \otimes |1\rangle |1\rangle \quad (68) \]
was described in [1]. Applying these results to our problem, we find that number of allowed states \( A_M = D_{M-1} \) is determined by generalized Fibonacci numbers \( D_M \), with recursion formula
\[ D_M = (2^n - 1)(D_{M-1} + D_{M-2}), \quad D_0 = 0, \quad D_1 = 1. \quad (69) \]

Corresponding probability for allowed states is equal
\[ P_M = \frac{D_{M-1}}{(2^n)^M} \quad (70) \]
and it satisfies the recursion relations
\[ P_{M+1} = \left(1 - 2^{-n}\right) \left(P_M + 2^{-n}P_{M-1}\right), \quad (71) \]
\[ P_2 = 2^{-2n}, \quad P_3 = 2^{-2n} - 2^{-3n}. \quad (72) \]
8.2 Arbitrary n-qubit state in M trials

The result can be generalized to generic $n$ qubit state

$$|\psi\rangle = \sum_{k=0}^{2^n-1} c_k |k\rangle_n$$

(73)

with probabilities to collapse

$$p_k = |c_k|^2 = |\langle k|\psi\rangle|^2, \quad \sum_{k=1}^{2^n-1} p_k = 1.$$  

(74)

For such state, considered as generic (not maximally random) $d = 2^n$ coin, probability of collapse for duplicated states $|1\rangle|1\rangle$ in M trials is determined by recursion relations

$$P_M = (1 - p_1)(P_{M-1} + p_1 P_{M-2}),$$  

(75)

$$P_2 = p_1^2, \quad P_3 = p_1^2(1 - p_1).$$  

(76)

References

[1] O. K. Pashaev, Quantum coin flipping, qubit measurement, and generalized Fibonacci numbers, Theoretical and Mathematical Physics 208 (2021) 1075 - 1092.

[2] O. K. Pashaev, S. Nalci, Golden quantum oscillator and Binet-Fibonacci calculus, Journal of Physics A: Mathematical and Theoretical 45 (2012) 015303.

[3] O. K. Pashaev, Quantum calculus of Fibonacci divisors and infinite hierarchy of Bosonic-Fermionic Golden quantum oscillators, International Journal of Geometric Methods in Modern Physics 18 (2021) 2150075.

[4] A. de Moivre, The Doctrine of Chances, (3rd edition), 1756, Chelsea Publ., New York (reprint), 1967.

[5] S. F. Kennedy, M. W. Stafford, Coin flipping, dynamical systems and the Fibonacci numbers, Mathematics Magazine 67 (1994) 380-382.
[6] T. Parlakgorur, O. K. Pashaev, Apollonius representation and complex geometry of entangled qubit states, Journal of Physics: Conf. Series 1194 (2019) 012086.

[7] D. Deutsch, M. W. Stafford, Uncertainty in quantum measurement, Physical Review Letters 50 (1983) 631-633.