The von Neumann entropy and information rate for ideal quantum Gibbs ensembles

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Abstract: A model of a quantum information source is proposed, based on the Gibbs ensemble of ideal (free) particles (bosons or fermions). We identify the (thermodynamic) von Neumann entropy as the information rate and establish the classical Lempel–Ziv universal coding algorithm in Grassberger’s form for such a source. This generalises the Schumacher theorem to the case of non-IID qubits.

1. Introduction and basic facts

In classical information theory, the fundamental unit is a ‘bit’, and the model behind it is a random variable taking values 0 and 1 with probability 1/2. We often refer to a sequence of random variables as a source – note that the physics of the way in which the random variables are generated is irrelevant, and that results on data-compression rely only on the statistics of long ‘strings’.

In the newer quantum information theory, the fundamental unit is a ‘qubit’, which is associated with a two-dimensional complex Hilbert space. Here the structure is much richer, since states can be not only $|0\rangle$ or $|1\rangle$ but any complex linear combination in between. However, the definition of a general quantum source producing a sequence of qubits remains open.

So far, the theory of quantum data compression has confined itself to the case of qubits emitted by an IID (independent identically distributed) source. Here, a qubit is a general $2 \times 2$ density matrix $\sigma$, and the assumption of independence is that the state of $n$ qubits is described by the tensor power $\sigma^{\otimes n}$. IID qubits can be implemented as photon pulses emitted by a laser. However, this model does not allow natural entanglement, and hence lacks interesting physical properties. Even the most enthusiastic proponents of modern quantum information theory consider the IID assumption as “an unfortunate restriction” (see Nielsen and Chuang [N-C], p. 554). It was noted that attempts to reliably produce a
qubit string by using various “random processes, such as the preparation and detection of photon pairs ... or atoms in thermal beam... suffer from inescapable signal degradation, ... as the probability of randomly generating the appropriate conditions decreases exponentially” ([S], p. 256).

On the other hand, in practice, qubits can be modelled by using physical particles or spins – electrons or atoms. Recent experimental results in quantum entanglement (see Sackett et al. [S]) indicate that perhaps the most reliable way to prepare a string of quantum qubits is to couple quantum particles in a coherent way. In the experiment reported in [S], these were ions of $^{9}$Be$^{+}$ interacting, approximately, via a Dicke–Lamb type potential and arranged in a one-dimensional lattice. A similar approach was put forward in [J-K-P]. Most recent experiments with physical implementation of Shor’s quantum factorisation algorithm also use quantum particle systems as a material base of a computational device [L]. For a mathematician, this stimulates interest in rigorous analysis of information coding methods for sources represented by ensembles of quantum particles.

The first step in this direction would be to consider the eigenvector distribution of a Gibbs density matrix of a large system of quantum particles or ‘spins’. An eigenvector $\phi$ of the density matrix can in principle be identified as a result of a quantum ‘measurement’ and the probability that in the grand canonical Gibbs ensemble the system chooses a pure eigenstate $\langle \phi | \phi \rangle$ is proportional to $\exp (-\beta \mu n - \beta \lambda)$. Here $n$ is the number of particles in state $\langle \phi | \phi \rangle$, $\mu$ represents the chemical potential (and $z = e^{\beta \mu}$ the ‘fugacity’), $\lambda$ is the corresponding eigenvalue and $\beta = 1/\kappa T$ where the $T$ is the absolute temperature and $\kappa$ the Boltzmann constant. The idea of our approach is that the corresponding eigenvector may usually be represented as a long sequence of numbers (‘digits’). If the quantum ensemble carries ‘enough randomness’, such a sequence can be treated as a sample of a random process or field. It seems interesting to analyse such a process or field from the point of view of (classical) information theory.

A natural (and simplest) example to consider is a system of free quantum particles in a volume $\Lambda \subset \mathbb{R}^d$ (an open bounded domain with piecewise smooth boundary $\partial \Lambda$). The interaction here is manifested through the chosen statistics (Bose or Fermi). The grand canonical Gibbs ensemble in $\Lambda$ is described by a quasi-free bosonic or fermionic density matrix $\rho^\Lambda$ in the Fock Hilbert space $\mathcal{F}^\Lambda$ associated with volume $\Lambda$ (index ± indicates the Bose or Fermi statistics). Such a state is generated by the one-particle Hamiltonian $H = H^\Lambda_1$, a self-adjoint operator in the one-particle complex Hilbert space $\mathcal{H} = \mathcal{H}^\Lambda_1$, given values of the thermodynamical parameters $\beta$ and $\mu$.

A typical model is where $\mathcal{H}^\Lambda_1 = L^2(\Lambda)$ and operator $H$ is minus one-half of the Laplacian with a ‘classical’ boundary condition on $\partial \Lambda$, see for example [B-R] Sections 5.2.4 and 5.2.5. In this case we assume that a) $\beta > 0$ and b) $\mu > 0$ for bosons and $-\infty < \mu < \infty$ for fermions. A lattice version of such a model is where $\mathcal{H}_1^A$ is the Hilbert space whose (complex) dimension equals $\# (\Lambda \cap \mathbb{Z}^d)$, the number of points $l = (l_1, \ldots, l_d) \in \mathbb{Z}^d$ with integer components $l_j$ within $\Lambda$. Here, $H$ may be minus one-half of the discrete Laplacian, again with a ‘classical’ boundary condition on $\partial (\Lambda \cap \mathbb{Z}^d)$.

Suppose that $H^A$ has a pure discrete spectrum and the eigenvalues of $H^A_1$ (counted with their multiplicities) are $\gamma^A_{\Lambda n}$ with $\min_{n \in N} \gamma^A_{n} = 0$. Here $n$ runs over a finite or denumerable set $N (= N^\Lambda)$ and $\sum_{n \in N} \exp (-\beta \gamma^A_{n}) < \infty$ for all $\beta > 0$. For instance, if $\Lambda \subset \mathbb{R}^d$ is a cube $(-L/2, L/2)^d$ and $H = -1/2 \Delta$ with
periodic boundary conditions, \( \mathcal{N} \) coincides with the integer cubic lattice \( \mathbb{Z}^d \) and
\[
\gamma_n^A = 4\pi^2 |n|^2 / L^2 \quad \text{where} \quad |n| = (n_1^2 + \ldots + n_d^2)^{1/2}, \quad n \in \mathbb{Z}^d.
\]

An eigenvector \( \phi \) of the quasi-free density matrix \( \rho^A \) is associated with a sequence of occupation numbers \( k = \{ k_n, n \in \mathcal{N} \} \) (we will also write \( \phi = \phi_k^A \)). More precisely, \( k_n \) is a non-negative integer equal to the number of particles in the eigenstate of \( H \) with the eigenvalue \( \gamma_n^A \); in the fermion case, \( k_n = 0 \) or 1. It is convenient to set \( \mathbb{K}_+ = \mathbb{Z}_+ := \{ 0, 1, 2, \ldots \} \) for the boson and \( \mathbb{K}_- = \{ 0, 1 \} \) for the fermion case. In both cases, the number of non-zero entries \( k_n \) in a given \( k \) is finite, with the sum \( \sum_{n \in \mathcal{N}} k_n \) representing the number of particles. The corresponding eigenvalue is
\[
\lambda = \lambda_k^A = \exp \left( -\beta \sum_{n \in \mathcal{N}} k_n (\mu + \gamma_n^A) \right).
\]

Thus the probability that the system will be found in pure state \( |\phi_k^A \rangle \langle \phi_k^A | \) is proportional to
\[
\exp \left( -\beta \sum_{n \in \mathcal{N}} k_n (\mu + \gamma_n^A) \right) = \prod_{n \in \mathcal{N}} \exp \left( -\beta (\gamma_n^A + \mu) k_n \right).
\]

In other words, a free quantum ensemble produces an ‘array’ \( K = \{ K_n, n \in \mathcal{N} \} \) of random variables \( K_n \) with probability determined by Equation (1). Throughout we use the convention that upper case letters refer to random variables, and lower case letters to the values that they take. This product form means that random variables \( K_n, n \in \mathcal{N}, \) are independent (but not identically distributed). The marginal distribution of \( K_n \) is geometric for bosons and two-point for fermions. Let \( \mathcal{P} = \mathcal{P}_A^A \) denote the induced probability distribution on \( \mathcal{K}_+ = \mathbb{K}_+ \mathcal{N}, \) supported by the set \( \mathcal{K}_+^A \) of arrays with finitely many non-zero components; it is convenient to think that \( \mathcal{P} \) is determined by the quadruple \( (\mathcal{H}_A, \mathcal{H}_A^A, \beta, \mu). \)

Now assume that \( \{ A \} \) is an increasing sequence of volumes in \( \mathbb{R}^d \) eventually covering the whole of \( \mathbb{R}^d, \) writing \( A \supsetneq \mathbb{R}^d. \) In is convenient to think that \( A \) is the result of the homothetic dilation of a fixed open bounded domain \( A^0 \subset \mathbb{R}^d \) containing the origin and with a piece-wise smooth boundary \( \partial A^0 \) consisting of finitely many smooth parts. In the above example, we can think of \( A^0 \) as a unit cube \( (-1/2, 1/2)^d \) and \( A = (-L/2, L/2)^d \) as its dilation by the linear factor \( L. \)

A question arises then: what are the properties of the ‘source’ \( (K, \mathcal{P}_A^A) \)? To what extent can classical coding theory be applied to such a source (or rather a sequence of sources, as \( A \supsetneq \mathbb{R}^d)? \) Some classical results are easily extended to the the case of \( (K, \mathcal{P}_A^A) \) (after all, \( \mathcal{P}_A^A \) is a product-distribution, albeit not stationary). For example, an asymptotic equipartition property (AEP) for \( (K, \mathcal{P}_A^A) \) is fairly straightforward (see Proposition [3]). [This property can be considered as an analog of the famous Shannon–McMillan–Breiman Theorem in the situation under consideration.] The corresponding information rate coincides with the von Neumann entropy per unit ‘volume’ of the limiting quantum free ensemble.

However, results beyond the AEP, such as the classical Lempel–Ziv universal encoding algorithm, are more tricky to establish. The Lempel-Ziv algorithm, in its various forms, is perhaps the most popular encoding method in modern information transmission. The idea of the algorithm, in the form of ‘parsing'
originally proposed by Ziv and Lempel \([Z-L]\) is very simple. Suppose we have a sequence \(x_0, x_1, x_2, \ldots\) of ‘letters’ from an ‘alphabet’ (say, \(x_i \in \{0,1\}\) (the binary alphabet)). We put a marker sign (say, a semi-colon ;) after \(x_0\). If \(x_1 \neq x_0\), we put the marker sign after \(x_1\), otherwise (i.e., if \(x_0 = x_1\)), we put it after \(x_2\). Continuing this procedure, given that the last marker sign was after \(x_j\), we put the next marker sign after letter \(x_{j'}\), \(j' > j\), if for all \(s = 1, \ldots, j' - j - 1\), the ‘word’ \((x_{j+1}, \ldots, x_{j+s})\) is among the ‘blocks’ formed between the subsequent marker signs already in place, but the ‘word’ \((x_{j+1}, \ldots, x_{j'})\) has not been seen before.

This gives rise to the following encoding method: each new parsed word has a ‘header’ (the word less the last letter) which has been seen before. Thus, to ‘encode’ this bit of sequence \(x_0, x_1, x_2, \ldots\) we need only to indicate the place where the header was seen in the past and in addition encode the last letter of the new block.

The popularity of this algorithm is due to its universal character (no knowledge of the properties of the source is required to implement it), and to the fact that asymptotically it achieves the data compression limit. However, this convergence is slow, leading to adaptations of the algorithm, including the so-called Grassberger \([G]\) form of the algorithm which also suits the multi-dimensional situation \((d > 1)\).

In Sections 2 and 3 we state our main results (see Theorem 1), that the Lempel–Ziv algorithm is valid (again with the von Neumann entropy as the information rate). In the higher-dimensional case, we establish this result in Grassberger’s form, and in the one-dimensional case, we also prove it in terms of the classical Lempel–Ziv algorithm. The proofs are given in Sections 4–8.

Our assumption on quadruple \((\mathcal{H}, H, \mu, \beta)\) follow the basic model outlined above where \(\Lambda = (-L/2, L/2)^d\), \(H^A = L_2(\Lambda)\) and \(H^A = -1/2\Delta\) with periodic boundary conditions. [The lattice version of this model can also be easily incorporated]. We consider fixed \(\beta > 0\) and \(\mu > 0\) for bosons and \(-\infty < \mu < \infty\) for fermions. Although this formally excludes the Bose–Einstein condensation, the fact is that the condensation is largely irrelevant to our results. We intend to discuss this issue in a separate paper. Furthermore, many of the properties obtained in this paper can be in turn extended to systems with interaction. The corresponding results are now in preparation.

We would like to point out an essential non-uniqueness of the definition of the quantum entropy (or entropies), see \([C-N-T]\). From this point of view, it would be interesting to clarify the relation of various concepts of quantum entropy with quantum information theory.

2. Preliminary results

Our main assumption is that a) the set \(\mathcal{N}\) coincides with \(\mathbb{Z}^d\), the cubic lattice, and so the collection of ‘arrays’ \(\mathcal{K}_+^d \subset \mathbb{R}_{\geq 0}^d\) consists of functions on \(\mathbb{Z}^d\) with finite supports and with values in \(\mathbb{K}_+ = \mathbb{Z}_+\) for bosons and \(\mathbb{K}_- = \{0,1\}\) for fermions, b) the eigenvalues \(\gamma_{n}^A, n \in \mathbb{Z}_+^d\), of the one-particle Hamiltonian \(H^A\) are of the form \(\theta(||n||/L)\), where c) \(L = L(\Lambda)\) is a parameter increasing to \(\infty\) as sequence of volumes \(\Lambda \supset \mathbb{Z}^d\) (it will be convenient to assume that \(L\) simply runs over the set of natural numbers), and d) \(\theta: [0, \infty) \to [0, \infty)\) is a given continuous function,
such that $\theta(x) > 0$ for $x > 0$, and such that the following integral is finite:

$$\int_{\mathbb{R}^d} \left( \mp \log \left( 1 \mp e^{-\beta(\|y\|+\mu)} \right) + \beta \left( \theta(\|y\|) + \mu \right) \left( e^{\beta(\theta(\|y\|)+\mu)}/\left( 1 \mp e^{-\beta(\|y\|)+\mu} \right) \right) dy, \right.$$  

(3)

where we take the first choice of all the $\mp$ for bosons, for all $\beta, \mu > 0$, and the second choice for for fermions, for all $\beta > 0$, $-\infty < \mu < \infty$.

Parameter $L$ can be thought of as a ‘linear size’ of $\Lambda$ and henceforth is used instead of $\Lambda$. In other words, we fix a sequence of positive numbers $L \to \infty$ replacing $\Lambda \nearrow \mathbb{R}^d$, say $L = 1, 2, \ldots$. As was suggested, it is convenient to think that $\Lambda$ is the cube $(-L/2, L/2)^d$.

In the model where $H^\Lambda = -\Delta/2$ with periodic boundary conditions, $\theta(t) = 4\pi^2 t^2$.

**Definition 1.** Integrals $\int_{\mathbb{R}^d}$ are called the von Neumann entropy per unit volume in the free boson/fermion limiting Gibbs ensemble and denoted by $h_{\pm}$. The restriction of the integral to a domain $\Gamma \subset \mathbb{R}^d$ is denoted by $h_{\Gamma}^\pm$.

**Remark 1.** The reasoning behind this definition is as follows. The probability measure $\mathcal{P}^L$ has been specified by Equation (1) as the product $\times_{n \in \mathbb{Z}^d} \pi_n$ where $\pi_n$ is the geometric distribution with parameter $e^{-\beta(\|n\|+\mu)}$ for bosons and the two-point distribution, with $\pi_n(\{0\}) = \frac{1}{1+e^{-\beta(\|n\|+\mu)}}$, $\pi_n(\{1\}) = \frac{e^{-\beta(\|n\|+\mu)}}{1+e^{-\beta(\|n\|+\mu)}}$, for fermions. The entropy of $\mathcal{P}^L$ divided by $L^d$ (the volume of $\Lambda$) is simply a Riemann sum for the integral $h_{\pm}$ and converges to $h_{\pm}$ as $L \to \infty$. On the other hand, the entropy of $\mathcal{P}^L$ is equal to the von Neumann entropy $\text{tr}_{\xi^L} \rho^L \log \rho^L$ of the density matrix $\rho^L$ corresponding to the Gibbs ensemble of free quantum particles in $\Lambda$, for given $\beta$ and $\mu$.

It is easy to check the following law of large numbers.

**Proposition 1.** Consider the random variable $\xi^L_{\pm} : k \to (1/L^d) \log \lambda^L(k)$, $k \in K^0_{\pm}$, where $\lambda^L(k)$ is the eigenvalue of Gibbs ensemble density matrix $\rho^L$ determined by function $k : \mathbb{Z} \to \mathbb{K}_{\pm}$. Then, for all $\varepsilon > 0$, $\lim_{L \to \infty} \mathcal{P}^L(\|\xi^L - h_{\pm}\| \geq \varepsilon) = 0$. Also, $\lim_{L \to \infty} \mathcal{P}^L(\|k^L\}) = h_{\pm}$ almost surely (a.s.) with respect to the product measure $\mathcal{P}^L \times k^L$ on the Cartesian power $K \times L$, with the sequence $(k^L)$ of $\mathcal{P}^L$-random elements.

A straightforward consequence of Proposition 1 is

**Corollary 1.** List the eigenvalues $\lambda^L(k)$, $k \in K^0_{\pm}$, in decreasing order: $\lambda_{(0)} \geq \lambda_{(1)} \geq \ldots$. Given $\epsilon \in (0, 1)$, select the eigenvalues in their order until the sum of the selected $\lambda$’s becomes greater than or equal to the value $1 - \epsilon$ for the first time. Let $M^L_{\pm}$ denote the number of selected eigenvalues. Then $\lim_{L \to \infty} \frac{1}{L^d} \log M^L_{\pm} = h_{\pm}$.

**Definition 2.** Remark 1. Proposition 1 and Corollary 1 specify an asymptotic equipartition property for probability measures $\mathcal{P}^L$, and $h_{\pm}$ can be considered as an analog of the information rate for $(K^0, \mathcal{P}^L)$. 

3. Main result

For the rest of the paper, \( k \in K_\pm^0 \) is a function \( \mathbb{Z}^d \to \mathbb{K}_\pm \) with compact support; we identify it with the collection of values \( k_n, n \in \mathbb{Z}^d \). Given a probability measure \( \mathcal{P}_L \) on \( K_\pm = \mathbb{K}_\pm^d \), \( K \) stands for an array of random variables \( \{ K_n \} \) representing the random element of \( K_\pm \). When considering the product-measure \( \mathcal{P}_L \times \mathcal{L} \) on the Cartesian product \( K \times L \), we denote by \( K^L \) the \( \mathcal{P}_L \)-random element of \( K_\pm \).

**Definition 2.** Given \( k \in K, \ u = (u_1, \ldots, u_d) \in \mathbb{Z}^d \) and \( s \geq 1 \), define the cubic box \( B_u(s) \) to have bottom corner \( u \) and side \( s \):

\[
B_u(s) = \{ n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : u_i \leq n_i \leq u_i + s - 1, \text{ for all } i = 1, \ldots d \}.
\]

Write \( k_u(s) = \{ k_n : n \in B_u(s) \} \) for the set of values of \( k \) confined to this box.

Now define \( r^L_u(k) \) to be the size of the smallest box with bottom corner at position \( u \) with values different to all the others with bottom corner in \( B_1(L) \):

\[
r^L_u(k) = \inf \{ s \geq 1 : k_u(s) \neq k_v(s) \text{ for all } v \neq u \in B_1(L) \}.
\]

For \( K \) a random array, we define the random variable \( R^L_u(K) \) in the same way.

These \( R^L_u \) have been studied by authors such as Grassberger [G], Kontoyiannis and Suhov [K-S], Quas [Q] and Shields [Sh1], [Sh2] first in the one-dimensional case and later for higher dimensions, partly because they serve as good entropy estimators for an ergodic process with a suitable degree of mixing. For example, Theorem 1 of [Q] shows:

If the array \( K \) is generated by a \( \mathbb{Z}^d \)-invariant ergodic probability measure on \( K_\pm \) with entropy \( h \), under a Doeblin condition,

\[
\lim_{L \to \infty} \sum_{u \in B_1(L)} \frac{R^L_u(K)}{L^d \log L} = \frac{1}{h}, \quad \lim_{L \to \infty} \sum_{u \in B_1(L)} \frac{\log L}{L^d R^L_u(K)} = h, \text{ a.s.}
\]

Our Theorem 1 below shows how a similar result looks for sequences \( (K_\pm^L, \mathcal{P}_L) \):

**Theorem 1.** For all fixed \( \zeta > 0 \), on \( K_\pm^L \),

\[
\lim_{L \to \infty} \sum_{u \in B_1(\zeta L)} \frac{\log L}{(\zeta L R^L_u(K))^d} = h^L_{B_0(\zeta)}, \mathcal{P}_L-\text{a.s.}
\]

Here \( h^L_{B_0(\zeta)} \) is the ‘truncated’ von Neumann entropy (cf Definition 3), where \( B_0(\zeta) \) is the cube \([0, \zeta]^n\).

We can deal with the case of \( \zeta \) increasing with \( L \), under extra assumptions on the behaviour of \( \theta \).

**Assumption 2** For all \( \eta \), there exist \( C, \delta \) such that uniformly in \( x > \eta \) for \( y < \delta \):

\[
\theta(x + y)/\theta(x) \leq C.
\]

**Assumption 3** Our \( \zeta \to \infty \), slowly enough that \( \zeta/\log L \to 0 \).
Theorem 4. If $\theta$ satisfies Assumption 3 and $\zeta$ satisfies Assumption 4 then on $K_{\pm}$,

$$
\lim_{L \to \infty} \sum_{u \in B_1(\zeta L)} \frac{\log L}{(\zeta L R_u^L(K))^d} = h_{\pm}, \mathcal{P} \times L \text{-a.s.}
$$

Remark 2. Alternatively, in the spirit of previous analysis, we can average the $R_u^L$ themselves. However, Theorem 1 in our view gives a more useful result for von Neumann entropy estimation.

For the sake of clarity, we focus on the case $\zeta = 1$ (though we indicate in due course how the case of $\zeta$ varying with $L$ can naturally be dealt with) and first prove the one-dimensional ($d = 1$) version of the result for geometric variables (bosons), in Sections 4 and 5. In Section 6, we indicate the adaptations needed in the case of two-valued variables (fermions), and in Section 7, we show how the method adapts to the case of higher dimensions. We split the proof of the result into 3 parts, corresponding to the Lemmas 6, 7 and 8 used in [Q]. In each case, writing $E_u^L$ for the entropy of $X_u$ under $\mathcal{P}$, we will show that $\log L/(R_u^L)^d$ is close to $E_u^L$.

Lemma 1. For any $\epsilon > 0$, then $\mathcal{P} \times L$-a.s.:

$$
\lim_{L \to \infty} \frac{1}{L^d} \# \left\{ u : R_u^L(K^L) \leq \left( \frac{\log L}{E_u^L} \right)^{1/d} \right\} = 0.
$$

Lemma 2. For any $\epsilon > 0$, then $\mathcal{P} \times L$-a.s.:

$$
\lim_{L \to \infty} \frac{1}{L^d} \# \left\{ u : R_u^L(K^L) \geq \left( \frac{\log L}{E_u^L} \right)^{1/d} \right\} = 0.
$$

Lemma 3. There exists a constant $c = c(\theta^*)$ such that $\mathcal{P} \times L$-a.s.:

$$
\limsup_{L \to \infty} \left( \max_{i \in B_1(\theta)} \frac{R_u^L(K^L)}{(\log L)^{1/d}} \right) \leq c.
$$

4. Proof of lower bound

Recall, in the next two sections, we concentrate on the one-dimensional geometric case and consider $\zeta = 1$. So, an array $k \in K$ is now a ‘string’ $\{k_i, i \in \mathbb{Z}\}$, where $k_i$ is a non-negative integer. Write $k_i(s)$ for a finite piece $(k_i, \ldots, k_{i+s-1})$ of string $k$ of length $s$ starting at position $i$ where $i, s \in \mathbb{Z}, s \geq 1$. Then $r^L_u(k)$ is the length of the shortest piece starting at position $i$ with values different to all the others pieces starting in $\{1, \ldots, L\}$: $r^L_u(k) = \inf\{s > 0 : k_i(s) \neq k_j(s) \text{ for all } j \neq i \in \{1, \ldots, L\}\}$. Accordingly, $r^L_u(k)$ is often called the match length. Here $\theta$ is a continuous function: $[0, \infty) \to [0, \infty)$ and $E_u^L$ is the entropy of the geometric distribution with parameter $e^{-\beta(\theta(|i|/L)+\mu)}$. Write $\theta^*$ for $\sup_{x \in [0,1]} \theta(x)$. 

We use the idea of a ‘typical set’, familiar from Ergodic and Information Theory. The aim is to show that usually we belong in this typical set $S$, which provides extra conditions so that the match length $R_i^L$ cannot be too low too often.

**Definition 3.** For a string $k = \{k_i, \ i \in \mathbb{Z}\}$, we define the centred log-likelihood:

$$ y_i^L(k) = -\log \mathcal{P}^L(K_i = k_i) - E_u^L = \beta(\mu + \theta(i/L))(k_i - \mathbb{E}^L K_i^L), $$

and for the random string $K$, we define $Y_i^L$ for the random variable $y_i^L(K)$. Here and below, $\mathbb{E}^L$ stands for the expectation relative to $\mathcal{P}^L$. Define the typical set by

$$ S_{j,M}^L = \left\{ k : \left| \sum_{i=j}^{j+M-1} y_i^L(k) \right| \leq M \theta(j/L)e' \right\}, $$

where $e' = \beta(\log (1 - e^{-\theta}))/2\theta^2$.

**Proof of Lemma 3.** Now for any sequence $M(i)$ and any $\eta > 0$, we deal with the first $\eta L$ variables separately:

$$ \left\{ k : \frac{1}{L} \# \left\{ i : R_i^L(k) \leq \frac{\log L(1 - \epsilon)}{E_i^L} \right\} > 3\eta \right\} \subseteq \left\{ k : \# \left\{ i : \frac{\log L(1 - \epsilon)}{E_i^L}, k \in S_{i,M(i)}^L \right\} > \eta L \right\} \cup \left\{ k : \# \left\{ i : \eta L : k \notin S_{i,M(i)}^L \right\} > \eta L \right\}. $$

We bound the size of the first set in Lemma 3 and the size of the second in Lemma 4.

**Lemma 4.** Given $\eta, \epsilon > 0$, we can find a sequence $M(i)$ and constant $C_1(\eta, \epsilon)$ such that for any $L \geq C_1$ and for any $k \in \mathcal{K}^0$

$$ \# \left\{ i : \eta L : R_i^L(k) \leq \frac{\log L(1 - \epsilon)}{E_i^L}, k \in S_{i,M(i)}^L \right\} \leq \eta L. $$

**Proof.** We can find intervals $J_i$ in which our variables have their means close together. Note that $f(x) = 1/(e^{\beta(\mu + x)} - 1)$ has derivative bounded below on $x > \epsilon > 0$. Hence, given $\theta$ and $\epsilon$, we can calculate $N = N(\epsilon)$ and $u_1, \ldots, u_N$ with $u_1 = \eta$, $u_N = 1$ such that

$$ \frac{1}{e^{\beta(\mu + \theta(u_i))} - 1} - \frac{1}{e^{\beta(\mu + \theta(u_{i+1}))} - 1} + \epsilon', \text{ for } i = 1, \ldots, N - 1, $$

where $\epsilon'$ is from Definition 3. Defining $J_i = \{m : m/L \in (u_i, u_{i+1})\}$, for each $j \in J_i$, define $M(j) = \log L(1 - \epsilon)/u_{i+1}$.

We compare $S_{j,M}^L$ with $D_{\gamma,M}$, a set which we can count and control more easily. For each $\gamma, M$, define $E_{\gamma}$ for the entropy of a geometric distribution with parameter $e^{-\gamma}$ and

$$ D_{\gamma,M} = \left\{ \mathbf{x}(M) = (x_1, \ldots, x_M) : \sum_{i=1}^{M} x_i \leq M \left( \frac{1}{e^{\gamma} - 1} + \frac{\epsilon E_{\gamma}}{\gamma} \right) \right\}. $$
For $x_1(M) \in D_{\gamma,M}$, writing $\mathbb{P}_\gamma$ for product measure for independent geometric random variables with parameter $e^{-\gamma}$:

$$
\mathbb{P}_\gamma(x_1(M)) = \exp\left(M \log(1 - e^{-\gamma}) - \gamma \sum_{i=1}^{M} x_i\right) \geq \exp(-ME_{\gamma}(1+\epsilon)).
$$

If $k \in S^L_{j,M}$, where $j \in J_i$, taking $\gamma = \sup_{x \in J_i} \beta(\mu + \theta(x))$:

$$
\sum_{i=j}^{j+M-1} k_i \leq M \left(\frac{1}{e^{\beta(\mu + \theta(j)/L)} - 1} + \epsilon'\right) \leq M \left(\frac{1}{e^{\gamma} - 1} + \epsilon E_{\gamma}\right),
$$

so $k_j(M) = (k_j, \ldots, k_{j+M-1}) \in D_{\gamma,M}$. We therefore know that if $k \in S^L_{j-j+M(j)-1}$ and $r^L_j = r^L_j(k) \leq \log L(1 - \epsilon)/E_j^L \leq M(j)$ then

$$
\mathbb{P}_\gamma\left(k_j, \ldots, k_{j+M(j)-1}\right) \geq \mathbb{P}_\gamma\left(k_j, \ldots, k_{j+M(j)-1}\right) \geq \exp(-M(j)E_{\gamma}(1+\epsilon)) = L^{-1+\epsilon^2}.
$$

Since these finite strings are distinct, the number of strings in $J_i$ such that these two conditions hold is less than $L^{1-\epsilon^2}$. Summing over intervals $J_i$, the total number of such strings is less than $L(L^{-\epsilon^2}N)$. Hence if $L \geq C_1(\eta, \epsilon) = (N(\epsilon)/\eta)^{1/\epsilon^2}$ then $L^{-\epsilon^2}N \leq \eta$ and the assertion holds.

If $\zeta/\log L \to 0$, then since $N$ grows linearly with $\zeta$, we know that $L^{-\epsilon^2}N$ still tends to zero as required. □

Note that the precise definition of match length $r^L_j(k)$ doesn’t matter, and that this analysis will go through for a variety of related definitions. For example, the original Lempel–Ziv parsing algorithm (see Introduction), or one-sided definitions of match lengths can be analysed in the same way. The key observation is that $k_j(r^L_j)$ are distinct strings. We deal with these issues in Section 8.

Next, we show that most of the time, we are in the typical set $S^L_{i,M(i)}$, using a series of applications of Chebyshev’s inequality.

**Lemma 5.** Suppose $\zeta$ is fixed, or that $\theta$ satisfies Assumption 3 and $\zeta$ satisfies Assumption 3. For any $\eta > 0$, $\mathbb{P}_{\times \in L}$-a.s.: 

$$
\frac{1}{L} \sum_{i=\eta L+1}^{\zeta L} I(K^L \notin S^L_{i,M(i)}) \geq \eta,
$$

for only finitely many values of $L$.

**Proof.** We require

$$
\frac{1}{\zeta L} \left(\sum_{i=\eta L+1}^{\zeta L} v_i^L\right) \leq \max_{i} v_i^L \to 0,
$$

where $v_i^L := \mathbb{P}(K^L \notin S^L_{i,M(i)})$. The key is a uniform bound on the 4th moment $E(Y_j^L)^4$ of $Y_j^L = \theta(j/L)(K_i - E^{L}K_i)$. Note that for $X$ a geometric variable
with parameter \( q \): \( \mathbb{E}(X - \mathbb{E}X)^4 = q(q^2 + 7q + 1)/(1 - q)^4 \leq 9/(1 - q)^4 \), so that writing \( \bar{\theta} \) for \( \inf_{x > q} \theta(x) \):

\[
\mathbb{E}^L \left( Y_j^L \right)^4 \leq 9 \left( \frac{\beta(\mu + \theta(j/L))}{1 - e^{-\beta(\mu + \theta(j/L))}} \right)^4 \leq C (\mu + \theta(j/L))^4.
\]

Hence for any set \( S \):

\[
\mathbb{E}^L \left( \sum_{j \in S} |Y_j^L| \right)^4 \leq \sum_{j \in S} \mathbb{E}^L \left( Y_j^L \right)^4 + 3 \sum_{j, k \in S, j \neq k} \mathbb{E}^L \left( Y_j^L \right)^2 \leq 3C \operatorname{max}_{j \in S} (\mu + \theta(j/L))^4 (\#S)^2.
\]

By Chebyshev, for any \( i \),

\[
\nu_i^L \leq \mathbb{E} \left( \frac{\sum_{j=i}^{i+M(i)-1} Y_j^L}{M(i) \theta(i/L) e^\theta L} \right)^4 \leq \frac{3C \operatorname{max}_{j \in (i, i+M(i)-1)} (\mu + \theta(j/L))}{\epsilon^4 \theta(i/L)} \left( \frac{\mu + \theta(j/L)}{\theta(i/L)} \right)^4 \frac{1}{M(i)^2},
\]

which is less than \( K/M(i)^2 \), under Assumption \( 1 \). So \( \max_i \nu_i^L \leq K/(\min_i M(i))^2 \), and since \( \min_i M(i) = \log L(1 - \epsilon)/\zeta \), if \( \zeta/\log L \to 0 \), then Equation \( 4 \) holds.

Now \( Z_i^L = I(K^L \notin S_{i,M(i)}^L) - \nu_i \) is a variable with mean 0, variance \( \nu_i \), and \( Z_i^L \) and \( Z_j^L \) are independent, if \( |i - j| \geq M(i) \). Hence \( \operatorname{Var} \left( \sum_{i = \eta L + 1}^{\zeta L} Z_i^L \right) \leq \sum_{i = \eta L + 1}^{\zeta L} \nu_i^L M(i) \leq K\zeta L/(\min_i M(i)) = K\zeta^2 L/(1 - \epsilon) \log L).

Overall, then, we deduce that for large enough \( L \):

\[
\mathcal{P}^L \left( \frac{1}{\zeta L} \sum_{i = \eta L + 1}^{\zeta L} I(K^L \notin S_{i,M(i)}^L) \geq \eta \right) \leq \frac{\operatorname{Var} \left( \sum_{i = \eta L + 1}^{\zeta L} Z_i^L \right)}{(\zeta L)^2 (\eta - \sum_i \nu_i/\zeta L)^2} \leq \frac{4K}{\eta^2 L \log L},
\]

which is summable in \( L \). \( \square \)

The proof of Lemma \( 3 \) is now complete.

5. Proof of upper bounds

We establish the upper bound in Lemma \( 3 \) by proving a related result about return times.

**Definition 4.** The return time \( T_{n,i}(k) \) is how long you have to wait until the substring \( k_i(n) \) is repeated in \( k \): \( T_{n,i}(k) = \inf\{j \geq 1 : k_{i+j}(n) = k_i(n)\} \), and \( T_{n,i}^{rev}(k) \) is a time-reversed version: \( T_{n,i}^{rev}(k) = \inf\{j \geq 1 : k_{i-j}(n) = k_i(n)\} \).

Theorem 1 of \( 2-W \) shows that

For a stationary ergodic probability measure on \( K \) with entropy \( h \), for any \( i \):

\[
\lim_{n \to \infty} \frac{\log T_{n,i}(K)}{n} = h \text{ a.s.}
\]
We need a version of this result for distributions $\mathcal{P}^L$. In the limit we are close to the IID case, so we lose little in comparison with that case.

Wyner and Ziv [W-Z] were the first to exploit the dual relationship between waiting times $T_{n,i}$ and match lengths $R_{n,i}^L$. We shall follow Shields’ approach, modified subsequently in [Q] and [Sh2] (to remove a confusion in [Sh1] in the way in which return times are defined – whether ‘overlapping matches’, when $T_{n,i} \leq n$, are counted).

A useful element introduced in [Q] is a truncation argument needed to cover the case of geometric random variables (the analysis in [Sh1] only holds for finite alphabet processes). [Q] introduces a truncation operation $\tau_m$ where $\tau_m(x) = \min(x, m)$ and $\tau_m(x) = \tau_m(x_i), i \in \mathbb{Z}$). Denote the match lengths and entropies of the truncated process by $\tau_m^L(k)$, $R_{n,i}^L(K)$ and $E_{n,i}^L$. First note that $r_m^L(k) \leq \tau_m^L(k) = \tau_m^L(k)$. Secondly, since $E_{n,i}^L/E_{n,i}^L = 1 - \exp(-\beta(\mu + \theta(i/L)m)$, we can ensure that $E_{n,i}^L < E_{n,i}^L(1 - \epsilon/2)/(1 + \epsilon)$ for all $i$. Hence, we need only prove that:

**Lemma 6.** For fixed $\theta, \eta, \epsilon$, then for each string $k$ defining:

$$U^L(k) = \left\{ i \mid \eta \leq \tau_m^L(k) > 1 + \log L \frac{E_{n,i}^L}{E_{n,i}^L(1 - \epsilon/2)} \right\}$$

then $\limsup_{L \rightarrow \infty} \#U^L(K)/L \leq \eta$, $\mathcal{P} \times L$-a.s.

**Proof.** We mirror the duality argument (cf Lemma 3 of [Sh1] and Appendix of [Sh2]). Define for $N = 1, 2, \ldots$ the forward count:

$$F_{n,i}^L(k) = \left\{ i : \frac{\log T_{n,i}^L(k)}{n} < E_{n,i}^L(1 - \epsilon/2) \text{ for some } n \geq N \right\},$$

and backwards count:

$$B_{n,i}^L(k) = \left\{ i : \frac{\log T_{n,i}^{rev}(k)}{n} < E_{n,i}^L(1 - \epsilon/2) \text{ for some } n \geq N \right\},$$

Now, if $i \in U^L(k)$, then there exists $j \neq i$ such that $k_i(s) = k_j(s)$, where $s = \tau_i^L(k) - 1$, so either

1. If $i < j$, then $T_{n,i}^L(k) \leq L$, so that $\log T_{n,i}^L(k)/n \leq \log L/n < E_{n,i}^L(1 - \epsilon/2)$
2. If $j < i$, then $T_{n,i}^{rev}(k) \leq L$, so that $\log T_{n,i}^{rev}(k)/n \leq \log L/n < E_{n,i}^L(1 - \epsilon/2)$

Hence if $i \in U^L(k)$, then $i$ is in $F_{n,i}^L(k)$ or $B_{n,i}^L(k)$ for some $N$. So, using the finiteness of the alphabet, if we can show that $F_{n,i}^L(K)$ and $B_{n,i}^L(K)$ are of low density (that is $|F_{n,i}^L(K)|/L$ and $|B_{n,i}^L(K)|/L$ are $\leq \eta$ eventually, $\mathcal{P} \times L$-a.s.) then so must $U^L(K)$ be, and the result follows.

First, we show that the number of overlapping matches is small. We mirror [Q] and define $A = \{ k : k_1(s) = k_{s+1}(s) \text{ for infinitely many } s \geq 1 \}$. We will show that this set has measure 0, by defining $B_m = \{ k : k_1(m) = k_{m+1}(m) \}$, so that $A = \cap_i \bigcup_{m \geq 1} B_m$. Following [Q], for each $m$ and $w \in \mathbb{Z}^+$, write $W(w)$ for the set of strings which begin with word (i.e., have $k_1(m) = w$), $W(ww)$ for the strings which begin with $w$ repeated twice. Now for $\delta > 0$ consider a
\( \delta \)-representative set \( \mathcal{V} = \{ w : -\log P^L(w) \geq \sum_{j=1}^{m} E_j^L - \delta \} \). On set \( \mathcal{V}_\delta \), since the entropy is bounded below:

\[
P^L(B_m \cap \mathcal{V}_\delta) = \sum_w P^L(\mathcal{W}(ww)) \leq \left( \sum_w P^L(\mathcal{W}(w)) \right) \exp \left( - \sum_{j=m+1}^{2m} E_j^L + \delta \right) = \exp \left( - \sum_{j=m+1}^{2m} E_j^L + \delta \right),
\]

which is summable in \( m \). So a Borel-Cantelli argument establishes the result.

Next, to bound \( F^L_K(K) \), we consider a word \( x = (x_1, \ldots, x_n) \) which lies in a \( \delta \)-representative set of the \( k_i(n) \)'s, that is for some \( \delta > 0 \):

\[
\sum_{i=1}^{n} |x_i - E^L K_{i-1+1}| < \delta / \beta. \tag{5}
\]

By direct calculation, we can bound from above the probability that \( x \) turns up later, that is for \( j > i \):

\[
P^L(K_j(n) = x) \leq P^L(K_i(n) = x) \exp(\theta^* \delta).
\]

Hence for any integer \( t \), if Equation (5) holds:

\[
P^L(n+1 \leq T_{n,i}(K) \leq t | K_i(n) = x) = \sum_{m=i+n+1}^{i+t} P^L(K_m(n) = x | K_i(n) = x) \leq t P^L(K_i(n) = x) \exp(\theta^* \delta) \leq t \exp(-n E_i^L + 2 \theta^* \delta).
\]

Then with \( t = \exp(n(E_i^L - \epsilon)) \), we need to pick \( \delta \) growing slowly enough that \( \theta^* \delta / n \) tends to zero – say \( \delta = n^{-7/8} \) (if \( \zeta_L \) is growing more slowly than \( \log L \), we can still choose appropriate \( \delta \)). Consider overlapping and non-overlapping matches separately:

\[
P^L \left( \frac{\log T_{n,i}(K_i)}{n} \leq E_i^L - \epsilon | K_i(n) = x \right) \leq P^L \left( \bigcup_{m \geq n/2} B_m \right) + \exp \left( -\frac{ne}{2} \right),
\]

for \( n \) sufficiently large. As \( n \to \infty \), the probability that Equation (5) holds tends to 1. We can bound the backward set \( B^L_K(K) \) similarly. \( \square \)

We can now prove the uniform upper bound in a more straightforward fashion

**Proof of Lemma 3.** Since for any \( j \), \( \max_i P^L(K_j = i) = P^L(K_j = 0) = 1 - \exp(-\beta(\mu + \theta(j/L))) \leq 1 - \exp(-\beta(\mu + \theta^*)) \), then for any \( N \):

\[
P^L(R_i^L(K) \geq N) = P^L(K_j(N) = K_i(N), \text{ for some } j \in \{1, \ldots, L\}, j \neq i \) \leq \sum_{j=1 \neq i}^{L} P^L(K_j(N) = K_i(N)) \leq L(1 - \exp(-\beta(\mu + \theta^*))\).
von Neumann entropy and information rate

So \( P^L(\max_i R_i^L \geq N) \leq L^2 (1 - \exp(-\beta(\mu + \theta^*)))^N \). Taking \( c > -3/\log(1 - \exp(-\beta(\mu + \theta^*))) \), and \( N = c \log L \), the result follows.

Again, if \( \zeta_L / \log L \to 0 \), the same bounds will work: since we need to make more comparisons, replace \( L^2 \) by \((L\zeta_L)^2\), and the logarithmic term is dominated by the polynomial.

\[ \square \]

6. Fermions

We can use the same techniques to consider the alternative model of two-point random variables (still in one dimension). We make the following observations, which ensure that the above proofs will carry through.

1. We adapt the proof of Lemma 4, introducing, for \( 0 < p < 1 \):

\[ D_{p,M} = \left\{ x_1(M) : \sum_{i=1}^M x_i \leq M \left( p + \frac{\epsilon E_p}{\log(1/p - 1)} \right) \right\} . \]

Here \( E_p \) stands for the entropy \(-p \log p - (1 - p) \log (1 - p)\). Again, it is true that for \( x_1(M) \in D_{p,M}, P_p(u) \geq \exp(-ME_p(1 + \epsilon)) \), where \( P_p \) is the Bernoulli measure on \( K_{p,Z} = \mathbb{K}^p_{\mathbb{Z}} \), with \( P_p(K_i = 0) = 1 - p, P_p(K_i = 1) = p \).

The assertion of Lemma 4 then follows in the same way as before.

2. For a string \( k = (k_i, i \in \mathbb{Z}) \in K_{\mathbb{Z}} \), we define

\[ y_i^L(k_i) = -\log P^L(K_i = k_i) - E_i^L = (k_i - E^L K_i) \log (1/P^L(K_i = 1) - 1) , \]

and for the random string \( K \), define \( Y_i^L(K_i) \) in the same fashion.

3. For random variable \( K \) taking values 0 with probability \( 1 - p \) and 1 with probability \( p \), if \( Y(k) = -\log P(K = k) - E_p \) then

\[ \mathbb{E}(Y(k))^4 = p(1 - 4p + 6p^2 - 3p^3) \log (1/p - 1)^4. \]

Since for \( p \in [0,1] \): \( 1 - 4p + 6p^2 - 3p^3 \leq 1 \), and making the substitution \( y = \log(1/p - 1) \), for \( p \leq 1/2 \) implies \( p \log(1/p - 1)^4 = y^4/(1 + e^y) \leq 24 \). By symmetry, the same result holds for \( p > 1/2 \). Hence the proof of Lemma 5 goes through.

4. Since we now deal with finite alphabets only, the proof of Lemma 6 simplifies. We don’t need the truncation argument previously described, and our observations about representative sets will go through as before.

5. The upper bound in Lemma 8 is proved in the same way, since a uniform bound \( \max_{i,j} P(X_j^L = i) \leq \max(1/(1 + e^{-\beta(\mu + \theta^*)}), 1/(1 + e^{\beta(\mu)}) \) holds.

7. Adapts to the higher-dimensional case

As in [Q], the generalization to higher dimensions goes through in a rather straightforward fashion.
1. The proof of Lemma 4 carries through; we still divide the larger region into sets \( J_i = \{ u = (u_1, \ldots, u_d) \in \mathbb{Z}^d : \|u/L\| \in (u_i, u_{i+1}) \} \) on which the variables are nearly IID. In general we need to replace \( M \) by \( M^d \), so for example:

\[
D_{\gamma,M} = \left\{ x_1(M) : \sum_{i \in B_1(M)} x_i \leq M^d \left( \frac{1}{e\gamma - 1} + \frac{E_\gamma}{\gamma} \right) \right\}.
\]

We introduce \( M(i) = (d \log L(1 - \epsilon)/E_{u_{i+1}})^{1/d} \).

2. The proof of Lemma 5 goes through as before, since the uniform bound on the 4th moment of \( Y_L \) still holds.

3. We can extend the definition of waiting time required in Section 5. Writing \( v = (v_1, \ldots, v_d) \geq 0 \) to mean that \( v_l \geq 0, 1 \leq l \leq d \), and with \( |v| = \max v_l \):

\[
T_{n,u}(k) = \inf\{|v|^+ : v \geq 0, k_{u+v}(n) = k_u(n)\}.
\]

4. The upper bound in Lemma 3 is proved in the same way, since a uniform bound on \( \max \{ P_L(K_u = j), u \in \mathbb{Z}^d, j \geq 1 \} \) holds.

8. Lempel-Ziv parsing

Now we establish the Lempel-Ziv parsing algorithm for one-dimensional free quantum systems. We use the notation from Sections 4–6. Recall the algorithm takes a string (or a ‘message’) \( k_1(L) \) and parses it into words; at each stage, we add a marker, ‘;’, so that the parsed block is the shortest word not already seen.

**Definition 5 (Lempel-Ziv parsing).** We parse the string \( k_1(L) = (k_1, \ldots, k_L) \) into words:

\[
k_1(L) = \{ k_{t(1)}(l(1)); k_{t(2)}(l(2)); \ldots; k_{t(c)}(l(c)); k_{t(c)+1}(r) \},
\]

according to the rule: \( t(1) = 1, t(i + 1) = t(i) + l(i) \),

\[
l(i + 1) = \min \{ m \geq 1 : k_{t(i)}(m) \notin \{ k_{t(1)}(l(1)), \ldots, k_{t(i)}(l(i)) \} \},
\]

where \( k_{t(c)+1}(r) \) is the remaining word, \( r = L - t(c) - 1 \) and \( c + 1 = c(k, L + 1) \) the total number of parsed words.

As was noted in the Introduction, this parsing rule is associated with a data-compression algorithm which is asymptotically efficient (achieves the upper bound provided by entropy) for ergodic processes. The algorithm relies on the fact that for each word \( k_{t(i)}(l(i)) \), we can describe it by first giving the point in the string between 1 and \( t(i) \leq L \) where block \( k_{t(i)}(l(i) - 1) \) previously occurs, and then by giving the extra symbol which is different. Thus we require \( \log L + 1 \) symbols to specify each parsed word in \( k_1(L) \) and the total length of the compressed message will be: \( c(k, L)(\log L + 1) \), cf Shields [Sh3], Chapter 11.
Theorem 5. For the one-dimensional quantum free ensemble, for all \( \zeta > 0 \),
\[
\lim_{L \to \infty} \frac{c(K_1(\zeta L)) \log L}{L} = h^{[0,\zeta]}_\pm, \ P_\pm^x - a.s.,
\]
(6)

Under Assumptions 2 and 3:
\[
\lim_{L \to \infty} \frac{c(K_1(\zeta L)) \log L}{L} = h^{\pm}, P^x - a.s.
\]

Proof. We know that the RHS is \( \lim_{L \to \infty} \sum_{i=1}^L E_i^L / L \) which represents the data compression limit. That is, Shannon’s Noiseless Coding Theorem (see for example Theorem 5.3.1 of [C-T]) states that the expected length of any decipherable code for a random variable \( X \) is greater than or equal to the entropy of \( X \).

Therefore, to prove Equation (6), it remains to establish the upper bound \( \limsup_{L \to \infty} \frac{c(K_1(\zeta L)) \log L}{L} \leq h^{[0,\zeta]}_\pm - a.s. \). We prove this using analysis similar to that of Section 4. As before, the proof goes in the same way for all values of \( \zeta \), so we fix \( \zeta = 1 \). Once again, we split the interval \([0,1]\) into subintervals \( J_i = (u_i, u_{i+1}] \), and for each \( i \), write \( k_i \) for \( (k_{Lu_i}, \ldots, k_{Lu_{i+1}-1}) \) and set \( G_i = \{t(j) : Lu_i \leq t(j) \leq Lu_{i+1}\} \) (the start-points of words which lie within the sub-interval). We also put \( N_i = \{r \in G_i : \sum_{s=r}^{r+1} E_s^L \leq \log L (1-\epsilon)\} \).

In the spirit of Lemma 4, we first observe that the cardinality \( |N_i| \leq L^{1-\epsilon^2} \), since again, these parsed words are short distinct strings, in the typical set. Then, considering the entropy present in these parsed words, we deduce that:
\[
\sum_{j \in J_i} E_j^L = \sum_{r \in G_i} \left( \sum_{s=r}^{r+1} E_s^L \right) \geq \log L (1-\epsilon)|G_i| - N_i.
\]

On rearranging we deduce that
\[
\limsup_{L \to \infty} \frac{|G_i| \log L}{L} \leq \frac{\sum_{j \in J_i} E_j^L}{L} + \epsilon.
\]

The theorem follows by summing over intervals \( J_i \), since \( \sum |G_i| = c(k, L) \).

We can deal with the case of \( \zeta / \log L \to 0 \) as before. \( \square \)

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