Quantum states in rotating electromagnetic fields

B. V. Gisin

IPO, Ha-Tunnaim St. 9, Tel-Aviv 69209, Israel. E-mail: borsi2011@bezeqint.net

(Dated: June 3, 2011)

We describe a new class of exact square integrable solutions of the Pauli and Dirac equation in rotating electromagnetic fields. Solutions obtained by putting equations in the stationary form with help of a coordinate transformation corresponding to the transition into a rotating frame. The transformation is assumed to be Galilean one however a non-Galilean transformation is of particular interest for such solutions. Obtained solutions, especially of Dirac’s equation, are valid for arbitrary values of parameters and may be tested experimentally.

PACS numbers: 03.65.Pm, 03.65.Ta, 31.30.jx, 06.20.Jr

INTRODUCTION

Exact solutions of field equations are of fundamental importance as in theoretical as experimental physics. Such solutions of the Pauli and Dirac equation are described in a variety of books (see, for example, [1]). However, as far as the author knows, there are no publications on two-dimensional (2D) square integrable solutions of these equations in rotating electromagnetic fields with a constant magnetic field along the rotation axis. In this paper we present such solutions for the Pauli equation in a rotating magnetic field and for the Dirac equation in the field of a traveling circularly polarized electromagnetic wave.

For finding the solution the translation of spinor into a rotating frame is necessary. This translation must be accompanied by the transformation of coordinates and time. These manipulations turn the non-stationary into stationary problem. However what must be the transformation? In the appendix we discuss some physical arguments for the choice of the reasonable transformation.

The Galilean or Lorentz transformation is the term pertaining to mechanics. We use here this term keeping in mind that the role of velocity is played by frequency.

THE PAULI EQUATION

We start from the Pauli equation

\[ i\hbar \frac{\partial}{\partial t} \Psi = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 \Psi - \mu(\sigma \mathbf{H}) \Psi \]  

(1)

in standard notations. \( \mathbf{A} \) is the potential: \( A_x = -H_z y/2, \quad A_y = H_x x/2, \quad A_z = H \left[ -x \sin(\Omega t) + y \cos(\Omega t) \right] \), \( \mathbf{H} \) is the constant component of the magnetic field along the \( z \)-axis, \( \hbar \) is the amplitude of the rotating transverse component, \( m, e, c, \mu \) and \( \sigma_k \) is the Plank constant, mass, charge, speed of light, magnetic moment and Pauli’s matrix respectively, \( \Omega \) is the frequency of rotation. \( \Omega \) may take as positive as negative values. Negative \( \Omega \) corresponds to the reverse rotation. We consider solutions with the definite momentum \( p \) along the \( z \)-axis. Solutions must be continuous and square integrable over all the cross-section transversely to this axis.

The search for solutions consists of the wave function translation into the rotating frame \( \tilde{\Psi} = \exp(i\sigma_3 \Omega t/2) \Psi \) and a transformation of coordinates

\[
\tilde{\varphi} = \varphi - \Omega t, \quad \tilde{r} = r, \quad \tilde{z} = z, \quad \tilde{t} = t,
\]

(2)

where \( r^2 = x^2 + y^2 \), \( \tan \varphi = y/x \). Above two operations reduce the Pauli equation to the stationary case. Further this equation is putted to the diagonal form \( \tilde{\Psi}_d = \exp(i\sigma_2 \gamma/2) \tilde{\Psi} \), where

\[
\tan \gamma = \frac{2mH}{\hbar^2 \Delta}, \quad \Delta = \frac{m \Omega}{\hbar} + \frac{2m}{\hbar^2} \mu H_z.
\]

(3)

After that we use new coordinates

\[
\tilde{x} = r \cos \tilde{\varphi}, \quad \tilde{y} = r \sin \tilde{\varphi},
\]

(4)

and investigate stationary states \( \tilde{\Psi}_d = \exp(-iE\tilde{t}/\hbar + ipz/\hbar)\tilde{\Psi} \), where \( E \) is the “energy” in the rotating frame. Finally denote

\[
g_1 = \frac{e^2}{\hbar^2 c^2} \frac{1}{4} H_z^2, \quad g_2 = \frac{e^2}{\hbar^2 c^2} \left( \frac{1}{4} H_z^2 + H^2 \right),
\]

(5)

\[
b = \frac{2pe}{\hbar^2 c}, \quad \epsilon = \sqrt{\frac{2m}{\hbar^2} E - \frac{p^2}{\hbar^2} + \sigma_3 \rho},
\]

(6)

\[
f = \frac{2m\Omega}{\hbar} + \frac{eH_z}{\hbar c}, \quad \rho = \sqrt{\frac{2m}{\hbar^2} \mu H^2 + \Delta^2}.
\]

(7)

Then the Pauli equation may be written in the following stationary form

\[
\psi_{\tilde{x},\tilde{z}} + \psi_{\tilde{y},\tilde{y}} - if \left( \tilde{x}\psi_{\tilde{y}} - \tilde{y}\psi_{\tilde{x}} \right) - (g_1 \tilde{x}^2 + g_2 \tilde{y}^2 - b\tilde{y} - \epsilon)\psi = 0.
\]

(8)

In this equation the comma means the differentiation. In Eq. (8) two components of the wave function are defined by independent equations. Exact solutions for
It may be shown with help these relations that if \( f \) in (12), (14). Parameters \( d \) 2D axially symmetric harmonic oscillator.

\[
D = \frac{1}{2} d_{11} \ddot{x}^2 + d_{12} \ddot{x} \ddot{y} + \frac{1}{2} d_{22} \ddot{y}^2 + d_1 \dot{x} + d_2 \dot{y}, \quad (9)
\]

\( \psi_{n}^{\pm} \) is a polynomial in \( \ddot{x} \) and \( \ddot{y} \), the quantum number \( n \geq 0 \) is defined as maximal number \( n = n_1 + n_2 \) in the product \( \ddot{x}^{n_1} \ddot{y}^{n_2}; n_1, n_2 \) are integers. The complete two-component wave function has to be defined as a combination of two solutions \( \psi_{n}^{\pm} \). In the initial frame the wave function is

\[
\Psi = \cos \frac{1}{2} \gamma \left( C^+ \exp \left( \frac{i}{\hbar} \Omega t - i E_n^+ t \right) \right) \exp \left( i \frac{p}{\hbar} z + D \right) + \sin \frac{1}{2} \gamma \left( -C^- \exp \left( \frac{i}{\hbar} \Omega t - i E_n^- t \right) \right) \exp \left( i \frac{p}{\hbar} z + D \right), \quad (10)
\]

where \( C^+, C^- \) are constants. The wave function describes a state with the spin rotation.

Every \( n \) is connected with \( 2(n + 1) \) energy levels \( E_n^\pm \equiv E_n^\sigma \) with condition \( E_n^+ - E_n^- = -\hbar^2 \rho / m \). The levels are determined from an algebraic equation of the power \( n + 1 \). In particular, first three energy levels are

\[
E_0^\sigma = \frac{p^2}{2m} - \sigma_3 \frac{\hbar^2}{2m} \rho + \epsilon_0, \quad (11)
\]

\[
E_1^{\sigma, \tau} = \frac{p^2}{2m} - \sigma_3 \frac{\hbar^2}{2m} \rho \mp \epsilon_1, \quad (12)
\]

\[
E_2^\sigma = \frac{p^2}{2m} - \sigma_3 \frac{\hbar^2}{2m} \rho + \epsilon_2, \quad (13)
\]

\[
E_2^{\sigma, \tau} = \frac{p^2}{2m} - \sigma_3 \frac{\hbar^2}{2m} \rho \pm \epsilon_2, \quad (14)
\]

\[
\epsilon_n = -\frac{\hbar^2}{2m} [(n + 1)(d_{11} + d_{22}) + d_1^2 + d_2^2], \quad (15)
\]

\[
\tau = \frac{\hbar^2}{2m} \sqrt{(d_{11} - d_{22})^2 + d_{12}^2 + f^2}. \quad (16)
\]

The subscript of \( E \) equals \( n \), energy levels are numbered by superscripts \( \sigma, \tau \). Two values of both superscripts \( \sigma \) and \( \tau \) corresponds two sign of \( \sigma_3 \) and two sign before \( \tau \) in (12), (14). Parameters \( d_{ki}, \) \( d_k \) are defined by equations

\[
d_{11}^2 + d_{12}^2 - i f d_{12} - g_1 = 0, \quad (17)
\]

\[
d_{22}^2 + d_{12}^2 + i f d_{12} - g_2 = 0, \quad (18)
\]

\[
2d_{11}d_{12} + 2d_{22}d_{12} - i f (d_{22} - d_{11}) = 0, \quad (19)
\]

\[
2d_{11}d_1 + 2d_{12}d_2 - if d_2 = 0, \quad (20)
\]

\[
2d_{22}d_2 + 2d_{12}d_1 + if d_1 + b = 0. \quad (21)
\]

It may be shown with help these relations that if \( f \to 0 \) then \( H \to 0 \). In this limit solutions turn out into that of 2D axially symmetric harmonic oscillator.

The wave function normalization

\[
\int \Psi^\ast \Psi dx dy = 1, \quad (22)
\]

where the integration is over all the cross-section, imposes one condition on two coefficients \( C^+, C^- \)

\[
C^+ C^- + C^+ C^- = \frac{d_{11}d_{22}}{\pi} \exp \left( \frac{d_2^2}{d_{22}} \right). \quad (23)
\]

The second one follows application scenarios. In particular, at the magnetic resonance we have following expression for the average value of spin

\[
s_3 = \frac{1}{2} \int \Psi^\ast \sigma_3 \Psi dx dy = \mp \frac{1}{2} \cos \left( \frac{2\mu H_z}{\hbar} t \right). \quad (24)
\]

The first condition of the magnetic resonance \( C^+ = \pm C^- \) is defined by making the constant part of \( s_3 \) equal zero. It is achieved by the initial polarization. The second condition \( \gamma = \pi / 2 \), is defined by making the amplitude of the variable part equal maximum. It is achieved by the adjustment of \( \Omega \) or \( H_z \). The same procedure for the magnetic resonance definition is used in the next section for solutions of Dirac’s equation.

Desirable solutions does not exist if \( d_{11} \geq 0, d_{22} \geq 0 \), since in this case the integral (22) tends to infinity and if

\[
4g_1 \leq f^2 \leq 4g_2, \quad (25)
\]

since in this interval energy has complex values. At boundaries of (25) energy is real but at the lower and upper boundary \( d_{11} = 0 \) and \( d_{22} = 0 \) respectively, i.e., solutions are not square integrable. With help of (25) we may evaluate the forbidden zone for values of the magnetic moment-or rather its \( g \) factor-using the condition of the magnetic resonance \( \gamma = \pi / 2 \) or

\[
h\Omega + \mu H_z = 0, \quad \mu = g e \frac{\hbar}{2mc} \quad (26)
\]

(for electron \( e = -|e| \) and \( \Omega, H_z \) has the same sign). From (20) and (17) we obtain \( f = (1 - g) e H_z / \hbar c \). Then from (25) we find the forbidden zone for the \( g \) factor

\[
1 - \sqrt{1 + \frac{4H^2}{H_z^2}} \leq g \leq 0, \text{ or } 2 \leq g \leq 1 + \sqrt{1 + \frac{4H^2}{H_z^2}}. \quad (27)
\]

Usually \( H \ll H_z \) and the zone represents a narrow vicinity of two points \( g \approx 0 \) and \( g \approx 2 \). The second inequality in Eq. (27) establishes an important result. Namely, the magnetic moment must be always anomalous.

Note that the Pauli principle in the given case may be reformulated as follows. Any state in a rotating magnetic field may have two electrons with the opposite sense of the spin rotation.

In conclusion of this Section note that similar exact solutions are not found for the Dirac equation. On the
other hand solutions in the field of a traveling circularly polarized electromagnetic wave are not found for the Pauli equation but they exist for the Dirac equation. It is connected with the unusual character of solutions. Solutions of the Dirac equation considered below do not consist of a small and large two-component spinor. Therefore they cannot be presented in the first approximation as solutions of the Pauli equation

THE DIRAC EQUATION

Consider exact solutions of the Dirac equation

\[ i\hbar \frac{\partial \Psi}{\partial t} + \alpha(c \mathbf{p} - e\mathbf{A})\Psi + \beta mc^2 \Psi = 0 \]  

(28)

in the field of a powerful traveling circularly polarized wave and a constant magnetic field directed along the axis of rotation (propagation). Such a field corresponds to the potential \( A_x = -\frac{1}{2}H_z y + \frac{1}{2} H \cos(\Omega t - kz) \), \( A_y = \frac{1}{2}H_z x + \frac{1}{2} H \sin(\Omega t - kz) \); \( k = \varepsilon/\Omega/c \) is the propagation constant, where values \( \varepsilon = +1 \) and \( \varepsilon = -1 \) correspond to propagation of fermions and wave in the same and opposite direction respectively. Make manipulations analogously to previous section. However we must take into account the coordinate \( z \) in the translation of the wave function \( \Psi = \exp[-\alpha_1 \alpha_2 (\Omega t - k z)/2] \bar{\Psi} \) as well as in the transformation of coordinates

\[ \bar{\varphi} = \varphi - \Omega t + k z, \quad \bar{r} = r, \quad \bar{z} = z, \quad \bar{t} = t. \]  

(29)

Further we consider the stationary case with the definite momentum \( p \) : \( \bar{\Psi} = \exp(-iE/\hbar + i p \bar{z}/\hbar)\bar{\Psi}_s \). Then in coordinates \( \bar{x} = r \cos \bar{\varphi}, \quad \bar{y} = r \sin \bar{\varphi} \) exact solutions, somewhat more symmetric than that of the Pauli equation, are \( \bar{\Psi}_s = \psi_n \exp D \), where

\[ D = -\frac{1}{2} i d(\bar{x}^2 + \bar{y}^2) + d_1 \bar{x} + d_2 \bar{y}, \]  

(30)

\( \psi_n \) is the spinor polynomial in \( \bar{x} \) and \( \bar{y} \). Such solutions exist as for \( \epsilon H_z < 0 \), as for \( \epsilon H_z > 0 \). In the first case \( d = -\epsilon H_z/2\hbar c \), the parameter \( d \) is always positive since solutions must decrease at infinity, \( d_1 = -id_2 \),

\[ d_2 = \frac{mch\epsilon_0}{2\bar{h}(\epsilon - \epsilon_0)}, \quad h = \frac{e}{kmc^2} H, \]  

(31)

\[ \bar{\epsilon} = \frac{E - c\epsilon h}{mc^2}, \quad \epsilon_0 = \frac{2\bar{h}d}{\Omega m}. \]  

(32)

As an example, we present here the normalized wave function of the ground state for the first case

\[ \psi_0 = N \sqrt{\frac{d}{2\pi}} \begin{pmatrix} \frac{-\epsilon \bar{h}\epsilon}{h\epsilon} \\ \frac{\bar{\epsilon}(\bar{\epsilon} - 1)(\bar{\epsilon} - \epsilon_0)}{h\epsilon} \end{pmatrix} \exp D_0, \]  

(33)

\[ N = \frac{1}{\sqrt{(\bar{\epsilon}^2 + 1)(\bar{\epsilon} - \epsilon_0)^2 + h^2\bar{\epsilon}^2}}, \]  

(34)

where \( D_0 = (D - d_2^2/2d) \).

A distinguish feature of the solution \([33]\) is that bispinor does not consist of a small and large two-component spinor. It means that this solution cannot be reduced to that of the Pauli equation. Solution in the form of an expansion in infinite series of terms like \([10]\) is possible. But such series evidently do not converge to a finite value.

Another distinguish feature is that energy of the ground state is defined by an algebraic equation of the third order

\[ \bar{\epsilon}^3 - (\bar{\epsilon}_0 - \nu)\bar{\epsilon}^2 - (1 + \epsilon_0 \nu + h^2)\bar{\epsilon} + \epsilon_0 = 0, \]  

(35)

where \( \nu = (2c\epsilon h/\Omega)/mc^2 \). For \( \nu, h \to 0 \) Eq. \([35]\) has two positive and one negative roots \( \bar{\epsilon} \to \pm 1, \bar{\epsilon} \to \epsilon_0 > 0 \). Therefore a combination of two solutions with positive roots may be used as a solution.

In the second case \( d = eH_z/4\hbar c > 0 \), \( d_1 = id_2 \),

\[ d_2 = \frac{mch\epsilon_0}{2\bar{h}(\epsilon + \epsilon_0)}. \]  

and above expressions \([34] [31]\) are still valid by substituting \( \psi_0 \to -\alpha_1 \alpha_3 \beta \psi_0 \).

In the initial frame the wave function is

\[ \psi = \exp[-iE\bar{t}/\hbar + i p \bar{z}/\hbar - i\alpha_1 \alpha_2 (\Omega t - k z)/2 + D] \psi_n. \]  

(36)

According to the procedure of the previous section consider the average value of spin

\[ s_3 = -i \int \Psi^* \alpha_1 \alpha_2 \Psi dxdy \]  

(37)

and its temporal evolution. The wave function consists of two solutions with two positive roots of Eq. \([35]\): \( \psi = C_1 \psi_1 + C_2 \psi_2 \). The normalization integral \([22]\) and the equality to zero of the constant part of \( s_3 \) results in following values of normalization constants \( C_1 = \cos \theta, C_2 = \sin \theta, \)

\[ \cos 2\theta = \frac{\hbar^2 \Pi \bar{\epsilon}^3}{(\Pi + 1)^2[(\bar{\epsilon}_0^2 - \Pi^2)\bar{\epsilon}^2 + 2\Pi(\Pi - 1)\bar{\epsilon}^0]}, \]  

(38)

where

\[ \Pi = \bar{\epsilon}_1 \bar{\epsilon}_2, \quad \bar{\epsilon}^+ = \bar{\epsilon}_1 + \bar{\epsilon}_2, \quad \bar{\epsilon}^- = \bar{\epsilon}_1 - \bar{\epsilon}_2. \]  

(39)

For the average spin we obtain

\[ s_3 = h^2 \Pi N_1 N_2 \exp \left[ -\frac{(d_2 - d_2')^2}{d} \right] \sin 2\theta \cos \Phi, \]  

(40)

where \( N_1, N_2 \) and \( d_2, d'_2 \) is the normalization parameter \([34]\) and the parameter \([31]\) for \( \bar{\epsilon}_1, \bar{\epsilon}_2 \) respectively, \( \Phi \) defines the frequency of the spin oscillation

\[ \omega = \frac{\Phi}{t} = \frac{\bar{\epsilon}^-}{\hbar}. \]  

(41)
It may be shown that for small $\eta, v, h$ in the first approximation the amplitude maximum of $s_3$ is realized at $E_0 = 1$. It is easily to see that this equality is the condition of the magnetic resonance (29) for $q = 2$. However next approximations lead to conflicting results. Once again emphasize the solution (10) is not an approximation of (33). The solution (33) represents a new class of solutions.

In the general case the amplitude of the average spin as well as the frequency of the spin oscillation is a complicated function of all parameters. However both variables can be calculated in accordance with (40), (41), adjusted and tested experimentally.

**APPENDIX. THE TRANSFORMATION FOR POINT ROTATION COORDINATE FRAMES**

The concept of the point rotation frame arises in electrooptics, when light propagates through electrooptical crystal with the rotating optical indicatrix (index ellipsoid) [3]. The optical indicatrix is an example of such a frame however we attach a more fundamental meaning to this concept. The concept is applicable to any rotating field, in particular to rotating magnetic field in quantum mechanics. A distinguish feature of the point rotation frame is the existence of the rotation axis at every point. All points in the plane perpendicular to the rotation axis are equivalent but time in frames rotating at different frequencies may be different. Centrifugal forces are absent in this frame. Coordinates of the frame are an angle and time; the frequency of rotation is a parameter [4]. Such non-rotating frames are used in quantum field theory for a long time.

Above it is shown that the first step in studying solutions of the Pauli and Dirac equation in the rotating electromagnetic field is the translation of the wave function into the rotating frame. This translation must accompanied by the transformation of coordinates and time. For finding of above solutions the Galilean transformation (2) or (29) is used. However there is possible a phase mismatch if the real transformation is non-Galilean and time in the initial and rotating frame is different. The problem in a sense is similar to that in mechanics where the consideration in the system of the mass center is preferable. The correct result in mechanics can be obtained only if the Lorentz transformation is used for the transition into this system. In the opposite case some corrections of the result are necessary. In this sense the frame with the resting magnetic field is of particular importance as an analog of the mass center in mechanics.

The general linear 2D transformation for the point rotating frames was considered in [5]. It was assumed that symmetry exists by the exchange angle $\leftrightarrow$ time and shown that in this condition a non-trivial transformation together with the Lorentz transformation is possible. This transformation possesses unusual properties and enables the existence of allowable frequency regions with lower and upper boundaries (see an example in [6]). From the general physical viewpoint it seems more preferable in comparison with the Galilean (with infinite frequencies) or Lorentz (with the sole limiting frequency) transformation. However the transformation cannot be uniquely determined with help of the assumption on the symmetry only. In [4] the 2D transformation was used for a phenomenological description of a circularly polarized light wave in the electrooptical single-sideband modulator [7], [8]. This description is similar to an elastic collision in mechanics, however such a consideration is not applicable to quantum mechanics.

The 3D Galilean transformation (29) is used for finding solutions of the Dirac equation in the field of traveling circularly polarized wave. Some aspects of the general 3D transformation is considered in [9]. This transformation also cannot be uniquely determined. In both cases there is lacking a principle which would allow determining the transformation explicitly.

**CONCLUSION**

We have presented the new class of exact solutions of the Pauli and Dirac equation in rotating electromagnetic fields. Their properties differ significantly from known presently exact solutions. That especially is related to the Dirac equation. In the standard approach the Pauli equation is not the first approximation of Dirac’s equation for such solutions. The obtained solutions may be a basis for experiments at large values of parameters, in particular, for relativistic electron.

We have discussed non-Galilean transformation. Main problem is the absence of a principle for determining the transformation. Main advantage of such a transformation is the possibility of different lower and upper frequency boundaries for different fields. This is important from the general physical viewpoint and calls for detail investigations of this issue.

[1] L. D. Landau and E. M. Lifshiz, *Quantum Mechanics*. (Pergamon, Oxford, 1965)
[2] J. Schwinger, Phys. Rev. 73, 416 (1948); S. S. Schweber, *QED and Men Who Made It: Dyson, Feyman Schwinger, and Tomonaga*, (Princeton University Press, Princeton, New Jersey, 1994).
[3] W. Pauli, Zs. f. Phys., 31, 765 (1925).
[4] B. V. Gisin, Journal of Physics A 40, 1341-1347 (2007).
[5] C. F. Buhrer, D. H. Baird, and E. M. Conwell, Appl. Phys. Lett., 1, 46 (1962).
[6] B. V. Gisin, arXiv: 0710.3007v3 [physics.optics], 2007; arXiv: 0908.3998v2 [physics.optics], 2009.
[7] D. H. Baird and C. F. Buhrer, *Single-Sideband Light Modulator*, (U.S. Patent 3204104, 1965).

[8] J. P. Campbell, and W. H. Steier, IEEE J. Quantum Electron. **QE-7**, 450(1971).