RESONANCE NLS SOLITONS AS BLACK HOLES
IN MADELUNG FLUID

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Abstract

A new resonance version of NLS equation is found and embedded to the reaction-diffusion system, equivalent to the anti-de Sitter valued Heisenberg model, realizing a particular gauge fixing condition of the Jackiw-Teitelboim gravity. The space-time points where dispersion change the sign correspond to the event horizon, and the soliton solutions to the AdS black holes. The soliton with velocity bounded above describes evolution on the hyperboloid with nontrivial winding number and create under collisions the resonance states with a specific life time.

1. Resonance NLS equation.

The connection between black hole physics and the theory of supersonic acoustic flow was established by Unruh$^1$ and has been developed to investigate the Hawking radiation and other phenomena for understanding quantum gravity$^2$. Recently, the similar idea for simulation of quantum effects related to event horizons and ergoregions but in superfluids, which in contrast to the usual liquids allow nondissipative motion of the flow, was proposed$^3$. In this case a "superluminally" moving inhomogeneity of the order parameter like solitons, provides black holes like quasi-equilibrium states, exhibiting an event horizon. Although Jacobson and Volovik considered a simplified profile of soliton and mentioned unimportance of exact structure of the solution, an exactly soluble model of solitons related to black holes allows one to describe the scattering process of black holes and corresponding quantum phenomena. Very recently, some attempt was done to describe solitons of integrable models like the Liouville$^4$, the Sine-Gordon$^{5,15}$ and the Reaction-Diffusion system$^6,7$, as black holes of Jackiw-Teitelboim (JT) gravity. In the last paper the scattering of two identical soliton-like structures called dissipatons, relating to the black holes and creating the metastable state was considered. Continuing in this direction, in the present paper we present a new integrable version of the Nonlinear Schrödinger equation (NLS) in 1+1 dimensions, admitting solitons with a rich resonance scattering phenomenology and interpreted as black holes of JT gravity.

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The NLS, well known also as the Gross-Pitaevskiy equation, appears in the phenomenological description of superfluidity of an almost ideal Bose gas. In this case, the squared modulus of the wave function $\bar{\psi}\psi$ is interpreted as the particle number density in the condensate state, while the gradient of the phase is proportional to the superfluid velocity $v = \nabla \text{arg} \psi$. The similar hydrodynamical interpretation for the decomposition of quantum mechanical wave function was considered by Madelung long time ago. It can be extended to the NLS, representing a generic envelope equation and appearing in the wide range of phenomena, with many applications. Besides superfluidity, the most popular on is probably in the nonlinear optics, describing light propagation in optical fibers. The decomposition shows that quadratic dispersion term in the NLS results from contribution of two terms: the superflow (classical drift) dispersion and the so called ”quantum potential”

$$U_Q(x) = 2|\psi|_{xx}/|\psi|. \quad (1.1)$$

The potential was introduced by de Broglie and explored by D. Bohm to make a hidden-variable theory and is responsible for producing the quantum behavior, so that all quantum features are related to its special properties. In more recent context it is considered in the stochastic mechanics as producing non-classical diffusion. Relation of non-classical motion with the internal spin motion or the zitterbewegung was established in a series of papers. In the optical context the superflow term corresponds to the geometrical optics, while the quantum potential to the diffraction, which leads to the spontaneous pattern formation and allows the quantum effects even at room temperature and macroscopic scale. We emphasize that existence of the envelope solitons for nonlinear equations is possible only with this potential, since its dispersive contribution is exactly compensated by the nonlinearity. Changing the sign of quantum potential contribution to the NLS, modify the behavior of solitons drastically. The potential $U_Q$ is invariant under the complex constant rescaling transformations, $\psi(x,t) \rightarrow \lambda \psi(x,t)$, $\lambda \in \mathbb{C}$, and thus does not depend on the strength of the wave, associated with soliton, but depends only of the form of the wave. Therefore, its effect could be large even for the well separated and far enough solitons.

Below we consider the NLS soliton propagation in the quantum potential (1.1):

$$i\partial_t \psi + \partial_x^2 \psi + \frac{\Lambda}{4}|\psi|^2 \psi = 2 \frac{|\psi|_{xx}}{|\psi|} \psi, \quad (1.2)$$

which we call the resonance Nonlinear Schrödinger equation (RNLS). It could corresponds to the response of a hypothetical resonance medium to an action of a quasimonochromatic wave with complex amplitude $\psi(x,t)$, which is slowly varying function of the coordinate and the time. Eq.(1.2) can be considered as the third integrable version of NLS, intermediating between the defocusing and focusing cases (repulsive and attractive non-ideal Bose gas correspondingly). The additional term in the right hand side of NLS (1.2), depending on the form of the amplitude profile of the wave, can be interpreted also as a specific electrostriction pressure or the diffraction ”force”. Worth to note that the model (1.2) is integrable only with the chosen coefficient 2 on the r.h.s, but the resonance properties can be valid as well in a more general case.
2. Madelung fluid and the reaction-diffusion system.

By decomposing the wave function \( \psi \in \mathbb{C} \), \( \psi = \exp(R - iS) \), \( \bar{\psi} = \exp(R + iS) \), in terms of two real functions \( R, S \in \mathbb{R} \), the model (1.2) can be represented as the Madelung fluid\(^9\)

\[-\partial_t R + \partial_x^2 S + 2 \partial_x R \partial_x S = 0, \quad (2.1a)\]

\[-\partial_t S + \partial_x^2 R + (\partial_x R)^2 + (\partial_x S)^2 - \frac{\Lambda}{4} e^{2R} = 0. \quad (2.1b)\]

Introducing two new real functions \( e^+ = \exp(R + S) \), \( -e^- = \exp(R - S) \), or

\[-e^+ e^- = e^{2R} = |\psi|^2, \quad S = \frac{1}{2} \ln \frac{e^+}{-e^-} = \frac{1}{2i} \ln \frac{\bar{\psi}}{\psi}, \quad (2.2)\]

we have the system

\[-\partial_t e^+ + \partial_x^2 e^+ + \frac{\Lambda}{4} e^+ e^- e^+ = 0, \quad (2.3a)\]

\[+\partial_t e^- + \partial_x^2 e^- + \frac{\Lambda}{4} e^+ e^- e^- = 0, \quad (2.3b)\]

representing a particular form of a 2-component reaction-diffusion (RD) system. We note that unusual negative value for diffusion coefficient in the second equation is crucial for the existence of Hamiltonian structure and the integrability of the model. In this case the system (2.3) is time reversible \( t \to -t \) and invariant under the global \( SO(1,1) \) transformations \( e^\pm \to e^\pm \alpha e^\pm \). It admits solution with exponentially growing and decaying components,

\[e^\pm = \pm \left( \frac{8}{-\Lambda} \right)^{\frac{1}{2}} k e^{\pm \left( \frac{1}{4} v^2 + k^2 \right) t - \frac{1}{2} vx} \cosh^{-1} \left[ k(x - vt - x_0) \right], \quad (2.4)\]

but with perfect solitonic shape for the \( O(1,1) \) scalar product

\[-e^+ e^- = |\psi|^2 = \frac{8}{-\Lambda} k^2 \cosh^{-2} \left[ k(x - vt - x_0) \right]. \quad (2.5)\]

By analogy with the dissipative structures in the pattern formation we called these dissipative soliton solutions as \textit{dissipatons}\(^6\).

Using (2.2) we can find one-soliton solution of Eq.(1.2) corresponding to the one-dissipaton solution (2.4)

\[\psi = \left( \frac{8}{-\Lambda} \right)^{\frac{1}{2}} k e^{-i\left( \frac{1}{4} v^2 + k^2 \right) t - \frac{1}{2} vx} \cosh k(x - vt), \quad (2.6)\]

As we see the role of the "quantum potential" is to change the dispersion for the NLS. Indeed, when \( U_Q = 0 \), Eq.(1.2) is the so-called defocusing NLS, admitting the “dark” soliton solution with non-vanishing boundary values. While with \( U_Q \neq 0 \), we have the “bright” soliton (2.6), with vanishing boundaries, which usually appears in the focusing NLS.
3. The gravitational interpretation.

Dissipatons are related to the black hole solutions of the Jackiw-Teitelboim gravity and provide interesting tools to study nonperturbative sector of the General Relativity.

Defining the 2-dimensional metric tensor in terms of Einstein-Cartan zweibeins

\[ g_{\mu\nu} = e^a_{\mu} e^b_{\nu} \eta_{ab} = \frac{1}{2} (e^+_{\mu} e^-_{\nu} + e^+_{\nu} e^-_{\mu}), \quad (3.1) \]

where \( e^\pm_{\mu} = e^0_{\mu} \pm e^1_{\mu} = (e^\pm_0, e^\pm_1) \), \( \eta_{ab} = \text{diag}(-1, 1) \), and

\[ e^+_0 = \pm \frac{\partial}{\partial x} e^\pm, \quad e^+_1 \equiv e^\pm, \quad (3.2) \]

such that

\[ g_{00} = -\frac{\partial e^+}{\partial x} \frac{\partial e^-}{\partial x}, \quad g_{11} = e^+ e^- \quad g_{01} = \frac{1}{2} \frac{\partial e^+}{\partial x} e^- - e^+ \frac{\partial e^-}{\partial x}, \quad (3.3) \]

we find that for \( e^\pm \) satisfying (2.3) the metric describes two dimensional pseudo-Riemannian space-time with constant curvature \( \Lambda \):

\[ R = g^{\mu\nu} R_{\mu\nu} = \Lambda. \quad (3.4) \]

This, low dimensional (lineal) gravity model is known as the Jackiw-Teitelboim model. It can be used also to describe the S-wave sector of the extremal \( D = 4 \) supersymmetric black hole solutions of models with specific dilaton coupling. When the curvature vanishes, \( \Lambda = 0 \), the nonlinear term in Eqs. (1.2) and (2.3) disappears. Then the system (2.3) becomes the decoupled linear heat equation and the time-reversal one, while (1.2) reduces to the unusual, sign indefinite dispersive modification of the linear Schrödinger equation:

\[ i \partial_t \psi + \partial_x^2 \psi - 2 \frac{|\psi|_{xx}}{|\psi|} \psi = 0. \quad (3.5) \]

Worth to note that the classical theories corresponding to the usual quantum mechanical Schrödinger equation and to our modified model (3.5) would be equivalent, since the "quantum potential" is proportional to \( \hbar \), and in the \( \hbar \to 0 \) limit both models lead to the same Hamilton-Jacobi equations. It turns out that this model is relevant to the black hole solutions of the CGHS string-inspired gravitational theory.

The gauge fixing condition (3.2) is equivalent to the classical \( SO(2, 1)/SO(1, 1) \) Heisenberg model on the anti-de Sitter space (\( \Lambda < 0 \)),

\[ \partial_0 s = s \wedge \partial_2^2 s, \quad (3.6) \]

where \( e^\pm_{\mu} \) are local coordinates in the tangent plane \( \partial_{\mu} s = \left( \frac{-\Lambda}{s} \right)^{1/2} (e^+_{\mu} n_- - e^-_{\mu} n_+) \), so that the metric tensor is

\[ g_{\mu\nu} = \left( \frac{2}{-\Lambda} \right) (\partial_{\mu} s \partial_{\nu} s), \quad (3.7) \]
and \( s^2 = -(S^1)^2 + (S^2)^2 - (S^3)^3 = -1 \). Actually in Sec.7 we show that (3.6) is gauge equivalent to (2.3).

Formulas (3.3) allow us to establish a correspondence between the geometrical and physical characteristics of the model. Indeed, in terms of \( \psi \) variables the metric tensor is given by

\[
g_{00} = 2 \left( \partial |\psi| \partial x \right)^2 - \partial \bar{\psi} \partial \psi, \quad g_{11} = -|\psi|^2, \quad g_{01} = i \left( \partial \bar{\psi} \partial x - \bar{\psi} \partial \psi \partial x \right),
\]

so that \( g_{00} \) component is the dispersive part of energy density, while \( g_{11} \) and \( g_{01} \) the mass and momentum densities correspondingly. The mass, momentum and energy conserved quantities

\[
M = - \int_{-\infty}^{\infty} e^+ e^- dx = \int_{-\infty}^{\infty} |\psi|^2 dx,
\]

\[
P = - \int_{-\infty}^{\infty} (e^+ \partial_x e^- - \partial_x e^+ e^-) dx = i \int_{-\infty}^{\infty} (\partial_x \bar{\psi} \partial_x \psi - \bar{\psi} \partial_x \psi) dx,
\]

\[
E = 2 \int_{-\infty}^{\infty} [\partial_x e^+ \partial_x e^- - \frac{\Lambda}{8} (e^+ e^-)^2] dx = 2 \int_{-\infty}^{\infty} [\partial_x \bar{\psi} \partial_x \psi - 2 \partial_x |\psi| \partial_x |\psi| - \frac{\Lambda}{8} |\psi|^4] dx,
\]

for one-dissipaton solution (2.4) or the one-soliton solution (2.6) are

\[
M = \frac{16}{-\Lambda} |k|, \quad P = Mv, \quad E = \frac{Mv^2}{2} + \frac{\Lambda^2}{384} M^3.
\]

These expressions show that it can be interpreted as non-relativistic quasi-particle of non-negative mass \( M \) and momentum \( P \), with the positive rest energy \( E_0 = E(v = 0) = \frac{\Lambda^2}{384} M^3 \).

4. Resonance dispersion and black holes.

As we will see below an interaction of these particles leads to the creation and annihilation processes, forming resonance states. The existence of these states follows from the form of dispersion part of energy density (3.9c), written in terms of variables (2.2)

\[
\epsilon_0 \equiv 2 (\partial_x \bar{\psi} \partial_x \psi - 2 \partial_x |\psi| \partial_x |\psi|) = 2 [ (\partial_x S)^2 - (\partial_x R)^2 ] e^{2R}.
\]

For the NLS case, when \( U_Q = 0 \), we have the positive definite dispersion energy

\[
\epsilon_0 \equiv 2 \partial_x \bar{\psi} \partial_x \psi = 2 [ (\partial_x S)^2 + (\partial_x R)^2 ] e^{2R},
\]

while in contrast for \( U_Q \neq 0 \), from (4.1) it becomes indefinite. Function \( \epsilon_0 \) changes the sign at space-time points where

\[
(\partial_x S)^2 - (\partial_x R)^2 = (\partial_x S - \partial_x R)(\partial_x S + \partial_x R) = 0,
\]

or \( \partial_x S = \pm \partial_x R \). Now, if we compare (4.1) with (3.8), so that the dispersion energy density has geometrical meaning of the metric tensor component \( \epsilon_0 = -2g_{00} \), then conditions (4.3)
are equivalent to existence of the event horizon space-time points \((x_H, t_H)\), where \(g_{00}\) change the sign. In this way we relate the resonance dispersion of NLS with the "quantum potential" (1.2), to the event horizon in two dimensional space-time. The existence of event horizon indicates the nontrivial causal structure of the space-time and the black hole phenomena. In fact, if we calculate the metric (3.8) for one-soliton solution (2.6)

\[
ds^2 = \frac{8}{\Lambda}[ (k^2 \tanh^2 k(x - vt) - \frac{1}{4} v^2)(dt)^2 - (dx)^2 - vdxdt)|\psi|^2, \tag{4.4}
\]

it shows a horizon singularity at

\[
\tanh k(x - vt) = \pm \frac{v}{2k}, \text{ only if, } |v| \leq 2|k| \equiv |v_{\text{max}}|. \tag{4.5}
\]

Consequently, a black hole dissipaton cannot move faster than the maximal value of the velocity \(|v_{\text{max}}| = 2|k|\). In this case the metric can be transformed to the Schwarzschild type form and shows the causal structure in terms of Kruskal-Szekeres coordinates\(^7\).

5. Hydrodynamical interpretation.

The Madelung fluid representation gives simple hydrodynamical explanation of of the resonance states and the event horizon existence. In (2.2) representation we introduce the fluid density \(\rho \equiv e^{2\mathcal{R}} = |\psi|^2\) and the local velocity \(V \equiv -2\partial_x S\), describing the center of mass motion. Then, characterizing the internal motion referred to the center of mass frame, the stochastic diffusion or the zitterbewegung\(^13\) non-classical contribution from the "quantum potential", can be described by velocity \(V_Q \equiv \partial_x \rho\). For the normal Madelung fluid (the NLS, when \(U_Q = 0\) the dispersion energy density (4.2) is just the sum of kinetic energies of these two motions

\[
\epsilon_0 = \left(\frac{\rho V^2}{2} + \frac{\rho V_Q^2}{2}\right). \tag{5.1}
\]

In contrast, for the non-vanishing \(U_Q \neq 0\), the density (4.1) is the difference

\[
\epsilon_0 = \left(\frac{\rho V^2}{2} - \frac{\rho V_Q^2}{2}\right). \tag{5.2}
\]

In this hydrodynamical representation, the metric tensor (3.8) has the form

\[
g_{00} = \frac{1}{4}\rho(V_Q^2 - V^2), \quad g_{11} = -\rho, \quad g_{01} = \frac{1}{2}\rho V, \tag{5.3}
\]

which is similar to the ADM split of a (1+1)-dimensional Lorentzian spacetime corresponding to the so-called acoustic metric, derived by Unruh\(^1\) for the sound waves in a fluid\(^2\). Then, the existence of resonance states or the event horizon has the meaning of the equality of the center of mass and internal motion velocities \(V = \pm V_Q\). For the one-soliton solution (2.6) velocity of the classical center of mass motion is constant and coincides with the soliton propagation velocity \(V = v\), while the "quantum" velocity is \(V_Q = -2k \tanh k(x - vt)\).
The last one is bounden above by the constant value $|V_Q| \leq 2|k|$, why the velocities compensation is possible only in this region. Then, the compensation condition $V = \pm V_Q$ is equivalent to the event horizon Eq. (4.5).

To derive the black hole we first rewrite the metric (5.3) in the moving frame $(\xi, t) = (x - vt, t)$ with constant velocity $v$. Then it has convenient form in terms of the new “shifted” local velocity $W(\xi, t) = 2v - V(\xi, t)$:

$$\tilde{g}_{00} = \frac{1}{4}\rho(V_Q^2 - W^2), \quad \tilde{g}_{11} = -\rho, \quad \tilde{g}_{01} = -\frac{1}{2}\rho W.$$  \hspace{1cm} (5.4)

For a one-soliton solution (2.6) or dissipaton (2.4), moving with a constant velocity $v$, we have $W = v$ and the metric (5.3), where $V = v = \text{const.}$, but with stationary space-time geometry due to the time independence of $\rho = \rho(\xi), \quad V_Q = V_Q(\xi)$.

In the general case the metric contains the off-diagonal terms. The time synchronization of this space-time is possible if function $2W/(V_Q^2 - W^2)$ is integrable. Then we can define new time coordinate $d\tau = dt - 2W/(V_Q^2 - W^2)d\xi$ and get the metric

$$ds^2 = \rho\left[\frac{1}{4}(V_Q^2 - W^2)(d\tau)^2 - \frac{V_Q^2}{V_Q^2 - W^2}(d\xi)^2\right].$$  \hspace{1cm} (5.5)

Following the same arguments as for the black holes, the Hawking temperature can be derived from this metric. In particular case of one soliton solution everything can be done explicitly. Synchronization of the stationary metric is given by the above time transformation integrated as

$$\tau = f(\xi, t) = t + \frac{v}{2k^2(1 - \gamma^2)} \left[-k\xi + \frac{1}{2|\gamma|}\ln \left|\frac{|\gamma| + \tanh k\xi}{|\gamma| - \tanh k\xi}\right|\right],$$

where $|\gamma| \equiv |v/2k| < 1$. The Hawking temperature in this case is $T_H = \frac{1}{2\pi}k^2(1 - \gamma^2)$.

Comparing (5.2) with (3.9c) and (3.10) we can see that the kinetic part of the internal motion gives the negative sign contribution to the rest energy $E_0$, but due to the positive potential part, the resulting energy is positive. This is the reason why we can have the resonance behavior for our model, but not for the NLS. Indeed, decay of a soliton in the rest means creation of a pair with the positive energy, which is allowed only if the rest energy is positive. From conservation of the mass (3.9a) follows that the mass defect for soliton decay is always zero, $\Delta M = M - (M_1 + M_2) = 0$. Then, the rest energy satisfies the condition

$$E_0 = \frac{\Lambda^2}{384}M^3 = \frac{\Lambda^2}{384}(M_1 + M_2)^3 > \frac{\Lambda^2}{384}(M_1^3 + M_2^3) = E_{0(1)} + E_{0(2)},$$  \hspace{1cm} (5.6)

such that $\Delta E_0 = E_0 - (E_{0(1)} + E_{0(2)}) > 0$, which allows creating two solitons. In contrast, in the NLS case, the sign of the rest energy is negative and instead of inequality (5.6) one has $E_0 < E_{0(1)} + E_{0(2)}$, which forbids decay of the bright soliton.
6. Resonance interaction of solitons.

Rewriting RD system (2.3) in the bilinear form
\[(\pm D_t - D_x^2)(G^\pm \circ F) = 0, \quad D_x^2(F \circ F) = -2G^+G^- \tag{6.1}\]
where three new real functions are defined by \(e^\pm = (-8/\Lambda)^{\pm}G^\pm/F\), with the product \(e^+e^- = -\frac{8\Lambda}{\Lambda}(\log F)_{xx}\), we apply the Hirota bilinear approach to construct solutions of (2.3).

The one-dissipaton is given by following solution of the system (6.1)
\[G^\pm = \pm e^\eta^\pm, \quad F = 1 + e^{\eta^+_1 + \eta^-_1 + \phi_{1,1}}, \quad e^{\phi_{1,1}} = (k^+_1 + k^-_1)^{-2}, \tag{6.2}\]
where \(n^\pm_1 \equiv k^\pm_1 x \pm (k^\pm_1)^2t + \eta^\pm_1(0)\), and in terms of redefined parameters, \(k \equiv (k^+_1 + k^-_1)/2\), \(v \equiv -(k^+_1 - k^-_1)\) it has the form (2.4).

Now we compare the one-dissipaton boundary conditions with the horizon condition (4.5). In the space of parameters \((v, k)\) there exist the critical value \(v_{\text{crit}} = 2k\). For solution (6.2) when \(v < v_{\text{crit}}\), one has \(e^\pm \to 0\) at infinities. So the vanishing b.c. for dissipaton are equivalent to the black hole (BH) existence. The corresponding heavy particle we call the BH dissipaton. At the critical value the solution is a kink steady state in the moving frame \(e^\pm = \pm k e^{\pm k \xi_0} (1 \mp \tanh k \xi)\), with constant asymptotics \(e^\pm \to \pm 2ke^{\pm k \xi_0}\) for \(x \to \mp \infty\) and \(e^\pm \to \mp 0\) for \(x \to \mp \infty\). In this case we have the extremal black hole or the EBH dissipaton. In the over-critical case \(v > v_{\text{crit}}\), \(e^\pm \to \infty\) for \(x \to \mp \infty\) and \(e^\pm \to \pm 0\) for \(x \to \mp \infty\), no black hole exists and we have very fast and light particles called the LD.

For the two-dissipaton solution we have
\[G^\pm = \pm [e^{\eta^+_1} + e^{\eta^-_2} + (\frac{k^{\pm\pm}_{12}}{k^{\pm\pm}_{11}})^2 e^{\eta^+_1 + \eta^-_1 + \eta^-_2} + (\frac{k^{\pm\pm}_{12}}{k^{\pm\pm}_{22}})^2 e^{\eta^+_2 + \eta^-_2}]\tag{6.3a}\]
where \(k_{ij}^\pm \equiv k^a_i + k^b_j, \quad \tilde{k}_{ij}^{ab} \equiv k^a_i - k^b_j, \quad \eta^\pm_i \equiv k^\pm_i x \pm (k^\pm_i)^2t + \eta^\pm(0)\).

First we consider the degenerate case of (6.3), when \(k^+_1 = k^-_1 \equiv p_1, \quad k^+_2 = k^-_2 \equiv p_2\). Then the solution can be simplified and after some transformation be represented in the following form
\[e^\pm = \pm (\frac{8}{\Lambda}^{\pm})^{\frac{1}{2}} p_+ p_- \frac{p_1 \cosh \theta_2 e^{\pm p^2_2 t} + p_2 \cosh \theta_1 e^{\pm p^2_1 t}}{p_2^2 \cosh \theta_+ + p_1^2 \cosh \theta_- + 4p_1 p_2 \cosh(p_+ + p_- t)}, \tag{6.4}\]
where \(p_\pm \equiv p_1 \pm p_2, \quad \theta_\pm \equiv \theta_1 \pm \theta_2, \quad \theta_i \equiv p_i(x - x_{0i}), \quad (i = 1, 2)\). It describes collision of two identical dissipatons with amplitudes \(p_+/2\), moving with the equal velocities \(|v| = |p_-|\).
and creating the resonance state with the life time, $\Delta T \approx 2 p_d / p_+ p_-$, depending of the relative distance $d$, where $x_{01} = 0, x_{02} = d$. This particular solution was considered in Ref. 7.

In a more general case, when $k_i^\pm > 0$, ($i = 1, 2$), and $\tilde{k}_{12}^+ > 0$, $\tilde{k}_{12}^- > 0$, $\tilde{k}_{21}^+ > 0$, $\tilde{k}_{21}^- < 0$, (6.3) describes collision of two different dissipatons with amplitudes $k_{12}^+ / 2$ and $k_{21}^+ / 2$ and velocities $v_{12} = -\tilde{k}_{12}^+$ and $v_{21} = -\tilde{k}_{21}^+$ correspondingly. Depending of the positions shift the resonance states can be created.

As a simplest case we consider conditions for decay of a BH dissipaton at the rest ($v = 0$) on two dissipatons (2.4) with parameters $(k_1, v_1)$ and $(k_2, v_2)$. From the mass, momentum and energy conservation laws we obtain relations

$$v_1^2 = 4k_2^2, \quad v_2^2 = 4k_1^2.$$  

Two possibilities exist:

a) $|v_1| = |v_2|$. In this case $|k_1| = |k_2|$ and both dissipatons have the equal mass $M_1 = M_2 = M / 2$ and velocities, satisfying the extremal conditions $v_i^2 = 4k_i^2$, ($i = 1, 2$) and corresponding to two EBH illustrated on Fig.1 (for $t > 30$).

b) $|v_1| > |v_2|$ (without lose of generality). In this case $v_1^2 > 4k_1^2$ and $v_2^2 < 4k_2^2$, and we have decay on the one BH dissipaton and the one LD as illustrated in Fig.1 (for $t < 0$).

On Fig.2 the interaction of two BH dissipatons by exchange of LD is shown. A more complicated interaction (Feynman diagram) with 4 vertices is shown on Fig.3. Detailed description of various interactions simulated by MATHEMATICA would be published elsewhere.

7. Black holes as topological solitons.

The gauge equivalence between (2.3) and (3.6) allows us to construct exact solutions of (3.6) providing simple geometrical visualization of the event horizon position and to treat the black hole as topological soliton. Under collision they should have similar resonance properties as solutions of Eqs. (1.2) or (2.3). In the matrix representation for $SO(2, 1)$

$$S = i \left( \begin{array}{cc} S^3 & S^- \\ S^+ & -S^3 \end{array} \right) = (s, \tau) = g \tau g^{-1},$$  

where $S^\pm = S^1 \pm S^2$, $S^2 = -I$, det $S = 1$, Eq.(3.6) has the standard matrix form

$$\partial_t S = \frac{1}{2i} [S, \partial_x S],$$  

with the Lax pair

$$J_1^{HM} = \frac{i}{4} \lambda S, \quad J_0^{HM} = \frac{i}{8} \lambda^2 S + \frac{\lambda}{4} S \partial_x S.$$  

This model is gauge equivalent to the resonance NLS (1.2) and RD (2.3). Although the Lax pair for the first one can be derived\textsuperscript{17}, for the second one it has simple Zakharov-Shabat form

$$J_1^{RD} = \left( \begin{array}{cc} \frac{1}{4} \lambda & q^- \\ q^+ & -\frac{1}{4} \lambda \end{array} \right), \quad J_0^{RD} = \left( \begin{array}{cc} \frac{1}{8} \lambda^2 - q^+ q^- & -(\partial_x - \frac{1}{2} \lambda)q^- \\ (\partial_x + \frac{1}{2} \lambda)q^+ & -\frac{1}{8} \lambda^2 + q^+ q^- \end{array} \right).$$  

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where $q^\pm = (\frac{\pm 1}{\sqrt{8}})^2 e^\pm$. The gauge equivalence is implemented in the standard way by non-Abelian transformation $J^{\mu}_{HM} = g J^{RD}_{\mu} g^{-1} - \partial_{\mu} g g^{-1}$, $(\mu = 0, 1)$ where the matrix $g(x, t)$ is solution of the linear problem $\partial_{\mu} g = g J^{RD}_{\mu}(\lambda = 0)$. To construct the magnetic analog of dissipaton solution (2.4) we solve this system first. The result is

$$g(x, t) = \begin{pmatrix}
\tanh z - \gamma & \frac{1}{\cosh z} e^{\gamma z - k^2(1-\gamma^2)t} \\
-\frac{1}{\cosh z} e^{-\gamma z + k^2(1-\gamma^2)t} & \tanh z + \gamma
\end{pmatrix},$$

(7.5)

where $z \equiv k(x - vt), \gamma \equiv v/2k$, det $g = 1 - \gamma^2$. Thus, from (7.1) we find one magnetic dissipaton solution with components

$$S^3 = -1 + \frac{2}{1 - \gamma^2} \cosh^{-2} z,$$

(7.6a)

$$S^- = \frac{2}{1 - \gamma^2} \cosh^{-1} z (\tanh z - \gamma) e^{\gamma z - k^2(1-\gamma^2)t},$$

(7.6b)

$$S^+ = \frac{2}{1 - \gamma^2} \cosh^{-1} z (\tanh z + \gamma) e^{-\gamma z + k^2(1-\gamma^2)t}.$$ 

(7.6c)

It describes magnetic (curved) analog of dissipaton (2.4). Indeed, in the moving frame with velocity $v$ the $S^3$ component is time independent while the $S^-$ and $S^+$ components are decaying and growing in time. As well as for dissipatons, properties of solution (7.6) essentially depend on the velocity $v$. We have following cases.

a) $\gamma^2 < 1$, or $|v| < 2|k|$, then $-1 \leq S^3 \leq \frac{1 + \gamma^2}{1 - \gamma^2}$. At any fixed time $t$, $S^+$ and $S^-$ vanish when $z \to \pm \infty$, while $S^3 \to -1$. Due to $s = (0, 0, -1)$ when $z \to \pm \infty$, the real line $R$ is compactified. Since the hyperboloid $s^2 = -(S^1)^2 + (S^2)^2 - (S^3)^2 = -1$ has topology of cylinder $R \times S^1$, the solution (7.6) describes the mapping $S^1 \to S^1$ with degree one. Thus, we see that (7.6) is the topological soliton, travelling with constant velocity $v$. When $z = z_H$, so that $\tanh z = \pm \gamma$, one of the components $S^+$ or $S^-$ is zero. According to (3.7), in this case the metric component $g_{00} = 0$ and like dissipatons we have the event horizon at (4.5) (see Fig.4). Since any topological soliton configuration cross each of the lines $S^+ = 0$ and $S^- = 0$ at least once, the intersection points correspond to the event horizon. This fact relates the black holes solution with topological soliton on hyperboloid.

b) $\gamma^2 > 1$, or $|v| > 2|k|$. In this case $\frac{1 + \gamma^2}{1 - \gamma^2} \leq S^3 \leq -1$. At $z \to \pm \infty$ one of the components $S^+$ or $S^-$ is exponentially growing. Moreover, asymptotics at $+\infty$ and $-\infty$ are orthogonal. This is why the solution (7.6) has trivial winding. In fact it can be considered as the turning travelling wave, which never cross the asymptotic lines at finite distance. Thus, in this case there is no event horizon and black holes.

From the above analysis we can see that existence of black holes and event horizon intimately related to the topologically nontrivial solutions of the model (3.6).

8. Conclusions

The black hole picture with resonance interaction described above can be applied to physical models of slowly varying quasimonochromatic wave in nonlinear media with the
sign indefinite dispersion. The 1+1 dimensional case can be realized particularly in the 
nonlinear optics. The great variety of optical solitons is due to many different properties 
of the media involved, including nonlinearity, material and geometric dispersion, passive 
or active properties, etc. The crucial role in soliton properties play the group-velocity 
dispersion of the optical fibers, which depends not only on the property of glass material, 
but also on the waveguide property of the fiber. According to the sign of dispersion (positive 
or negative(anomalous)), two types of NLS are known - defocusing and focusing cases, 
which admit the ”dark” and the ”bright” solitons correspondingly. Since the envelope 
wave function is a complex quantity, the quadratic dispersion in general consists of two 
parts: the phase dispersion and the modulus dispersion. The first one corresponds to 
the geometrical optics, while the second one is responsible for the diffraction. But in 
both focusing and defocusing cases, contribution to dispersion from the phase and the 
modulus has the same sign (positive and negative correspondingly). In the present paper 
we considered the nonlinear media with the sign indefinite quadratic dispersion, which is 
the result of competition between contributing with the opposite signs of the phase and 
the modulus dispersions. Not going into microscopic details we confined ourself here only 
by phenomenological description of some hypothetical medium. Then, the response of the 
medium to an action of a quasimonochromatic wave with complex amplitude \( \psi(x,t) \), which 
is slowly varying function of the coordinate and the time, is described by our resonance 
version of NLS. If it can be realized experimentally, the resonance collisions of solitons 
would be an interesting tool in optical communication systems with the black hole physics 
flavor.

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**Figures**

**Fig. 1a.** 3D plot of two dissipatons resonance-type collision for \( k_1^+ = 0.1, k_1^- = 1, k_2^+ = 1, k_2^- = 0 \) and \( d = 30 \) in the \((x,t)\) plane.

**Fig. 1b.** Contour plot of two dissipatons collision with BH resonance in the \((x,t)\) plane.

**Fig. 2a.** 3D plot of two BH dissipatons exchange-type collision for \( k_1^+ = 2, k_1^- = 1, k_2^+ = -1.7, k_2^- = -1.9 \) and \( d = 50 \).

**Fig. 2b.** Contour plot of two BH dissipatons exchange-type collision in the \((x,t)\) plane.

**Fig. 3.** Contour plot of two-BH dissipatons 4 vertex-type collision for \( k_1^+ = 2, k_1^- = 1, k_2^+ = 1, k_2^- = 0.3 \) and \( d = 30 \) in the \((x,t)\) plane.

**Fig. 4.** Parametric plot of topological soliton projection on \((S^1, S^2)\) plane for \( k = 1, \gamma = 0.5 \). Positions of the BH event horizon correspond to intersection points with \( S^+ = 0 \) and \( S^- = 0 \) lines, while asymptotics on \( \pm \infty \) to the beginning of coordinates \((0, 0)\).
