On \((1, 2)^*\)-fuzzy soft \(b\)-continuity in fuzzy soft bitopological spaces

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Abstract
The purpose of this paper is to introduce the concepts of \((1, 2)^*\)-fuzzy soft \(b\)-continuous maps, \((1, 2)^*\)-fuzzy soft \(b\)-irresolute maps and the relations with other weak forms of fuzzy soft continuous maps in fuzzy soft bitopological spaces. Also we investigate the basic properties of \((1, 2)^*\)-fuzzy soft \(b\)-irresolute map and \((1, 2)^*\)-fuzzy soft \(b\)-open (closed) maps.

Keywords \((1, 2)^*\)-fuzzy soft \(b\)-continuous \cdot \((1, 2)^*\)-fuzzy soft \(b\)-irresolute \cdot \((1, 2)^*\)-fuzzy soft \(b\)-open map \cdot \((1, 2)^*\)-fuzzy soft \(b\)-closed map

1 Introduction
In the year 1965, Zadeh (1965) introduced the concept of fuzzy set theory and its applications that are found in many branches of mathematics and engineering sciences including management science, computer science, artificial intelligence (see, e.g., Chou et al. 2012; Eksin et al. 2008). The Russian researcher Molodtsov (1999) initiated the concept of soft sets as a new mathematical tool to deal with uncertainties while modeling problems in engineering physics, computer science, economics, social sciences and medical sciences (see, e.g., Özgür and Taş 2015; Yüksel et al. 2013). Maji et al. (2003) studied the theory of soft sets initiated by Molodtsov and then defined equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set and absolute soft set with examples. Soft binary operations like AND, OR and also the operations of union and intersection were also defined. Chen (2005) presented a new definition of soft set parametrization reduction and a comparison of it with attribute reduction in rough set theory.

The concept of bitopological spaces was introduced and studied by Ittanagi (Ittanagi 2014), and other authors have contributed to its development (see, e.g., Kandil et al. 2016, 2017; Revathi and Bageerathi 2017, 2015). The notion of soft bitopological space was introduced using different soft topologies on an initial universe set. On the other hand, the mixed type of soft set theory was given using different soft topologies (see, e.g., Taş and Aıkgöz 2014; Açıklgöz and Taş 2014; Açıklgöz et al. 2016; Öztürk and Karademr 2017).

Mukherjee1 and Park (2015) extended this notion to fuzzy soft bitopological space, and they introduced the notions of \(\tau_1 \tau_2\)-fuzzy soft open (closed) sets, \(\tau_1 \tau_2\)-fuzzy soft interior (resp. closure and studied some of their basic properties. Also, Sayed (2017, 2018, 2020) extended this study by characterizing a new type of fuzzy soft sets and introducing some separation axioms in fuzzy soft bitopological spaces. The purpose of this paper is to introduce the concepts of \((1, 2)^*\)-fuzzy soft \(b\)-continuous maps, \((1, 2)^*\)-fuzzy soft \(b\)-irresolute maps and the relations with other weak forms of fuzzy soft continuous maps in fuzzy soft bitopological spaces. Also we introduce and investigate the basic properties of \((1, 2)^*\)-fuzzy soft \(b\)-irresolute map and \((1, 2)^*\)-fuzzy soft \(b\)-open (closed) maps.

2 Preliminaries
In this section, we are going to present the basic definitions and results of fuzzy soft set and fuzzy soft bitopological space which will play a central role in our paper.
Throughout our discussion, $X$ refers to an initial universe, $E$ denotes the set of all parameters for $X$, and $P(X)$ denotes the power set of $X$.

**Definition 2.1** (Zadeh 1965) A fuzzy set $A$ in a non-empty set $X$ is characterized by a membership function $\mu_A : X \to [0, 1]$ whose value $\mu_A(x)$ represents the “degree of membership” of $x$ in $A$ for every $x \in X$. Let $I^X$ denote the family of all fuzzy sets on $X$.

A member $A$ in $I^X$ is contained in a member $B$ of $I^X$ denoted $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$ (see Zadeh 1965).

Let $A, B \in I^X$, we have the following properties on fuzzy sets (see Zadeh 1965).

1. **Equality:** $A = B$ if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.
2. **Intersection:** $C = A \cap B \in I^X$ by $\mu_C(x) = \min(\mu_A(x), \mu_B(x))$ for all $x \in X$.
3. **Union:** $D = A \cup B \in I^X$ by $\mu_D(x) = \max(\mu_A(x), \mu_B(x))$ for all $x \in X$.
4. **Complement:** $E = E^c \in I^X$ by $\mu_E(x) = 1 - \mu_A(x)$ for all $x \in X$.

**Definition 2.2** (Zadeh 1965) The empty fuzzy set on $X$ denoted by $\emptyset$ is a function which maps each $x \in X$ to 0. That is, $\emptyset(x) = 0$ for all $x \in X$.

A universal fuzzy set denoted by $\bar{1}$ is a function which maps each $x \in X$ to 1. That is, $\bar{1}(x) = 1$ for all $x \in X$.

**Definition 2.3** (Molodtsov 1999) Let $A \subseteq E$. A pair $(F, A)$ is called a soft set over $X$ if $F$ is a mapping $F : A \to P(X)$.

**Definition 2.4** (Maji et al. 2001) Let $A \subseteq E$. A pair $(f, A)$, denoted by $f_A$, is called a fuzzy soft set over $X$, where $f$ is a mapping given by $f : A \to I^X$ defined by $f_A(e) = \mu_{f_A}(e)$ where

$$
\mu_{f_A}(e) = \begin{cases}
\bar{0}, & \text{if } e \not\in A; \\
\bar{1}, & \text{otherwise, if } e \in A.
\end{cases}
$$

$(X, E)$ denotes the family of all fuzzy soft sets over $(X, E)$.

**Definition 2.5** (Maji et al. 2003) A fuzzy soft set $f_A \in (X, E)$ is said to be:

(a) NULL fuzzy soft set, denoted by $\phi$, if for all $e \in A$, $f_A(e) = \bar{0}$.
(b) absolute fuzzy soft set, denoted by $\bar{E}$, if for all $e \in E$, $f_A(e) = \bar{1}$.

**Definition 2.6** (Roy and Samanta 2012) The complement of a fuzzy soft set $f_A$, denoted by $f_A^c$ where $f_A^c : E \to I^X$, is a mapping given by $\mu_{f_A^c} = \bar{1} - \mu_{f_A}$, for all $e \in E$ and where $\bar{1}(x) = 1$, for all $x \in X$. Clearly, $(f_A^c)^c = f_A$.

**Definition 2.7** (Roy and Samanta 2012) Let $f_A, g_B \in (X, E)$, $f_A$ is fuzzy soft subset of $g_B$, denoted by $f_A \subseteq g_B$, if $A \subseteq B$ and $\mu_{f_A} \leq \mu_{g_B}$ for all $e \in A$, i.e., $\mu_{f_A}(x) \leq \mu_{g_B}(x)$ for all $x \in X$ and for all $e \in A$.

**Definition 2.8** (Roy and Samanta 2012) Let $f_A, g_B \in (X, E)$. The union of $f_A$ and $g_B$ is also a fuzzy soft set $h_C$, where $C = A \cup B$, and for all $e \in C$, $h_C(e) = \mu_{h_C}(e) = \mu_{f_A}(e) \lor \mu_{g_B}(e)$.

Here we write $h_C = f_A \cup g_B$.

**Definition 2.9** (Roy and Samanta 2012) Let $f_A, g_B \in (X, E)$. The intersection of $f_A$ and $g_B$ is also a fuzzy soft set $d_C$, where $C = A \cap B$ and for all $e \in C$, $d_C(e) = \mu_{d_C}(e) = \mu_{f_A}(e) \land \mu_{g_B}(e)$.

Here we write $d_C = f_A \cap g_B$.

**Definition 2.10** (Mahanta and Das 2012) The fuzzy soft set $f_A \in (X, E)$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu_{f_A}(x) = \alpha(0 < \alpha \leq 1)$ and $\mu_{f_A}(y) = 0$ for each $y \in X - \{x\}$, and this fuzzy soft point is denoted by $x_e^f$ or $f_e$.

**Definition 2.11** (Mahanta and Das 2012) The fuzzy soft point $f_e$ is said to be belonging to the fuzzy soft set $(g, A)$, denoted by $f_e \in (g, A)$, if for the element $e \in A$, $\alpha \leq \mu_{g_A}(x)$, $(0 < \alpha \leq 1)$.

**Definition 2.12** (Beaulaa and Gunaseilib 2014) Let $f_A$ be fuzzy soft set over $X$. The two fuzzy soft points $f_{e_1}^c, f_{e_2}^c \in f_A$ are said to be equal if $\mu_{f_{e_1}}(x) = \mu_{f_{e_2}}(x)$ for all $x \in X$. Thus, $f_{e_1}^c \neq f_{e_2}^c$ if and only $\mu_{f_{e_1}}(x) \neq \mu_{f_{e_2}}(x)$ for all $x \in X$.

**Definition 2.13** (Roy and Samanta 2012) A fuzzy soft topology $\tau$ over $(X, E)$ is a family of fuzzy soft sets over $(X, E)$ satisfying the following properties:

(i) $\emptyset, \bar{1} \in \tau$,
(ii) if $f_A, g_B \in \tau$, then $f_A \cap g_B \in \tau$,
(iii) if $f_{A\alpha} \in \tau$ for all $\alpha \in A$ an index set, then $\bigcup_{\alpha \in A} f_{A\alpha} \in \tau$.

**Definition 2.14** (Mukherjee1 and Park 2015) If $\tau$ is a fuzzy soft topology on $(X, E)$, the triple $(X, E, \tau)$ is said to be a fuzzy soft topological space. Also each member of $\tau$ is called a fuzzy soft open set in $(X, E, \tau)$.

The complement of a fuzzy soft open set is a fuzzy soft closed set.

**Definition 2.15** (Mukherjee1 and Park 2015) Let $(X, E, \tau_1)$ and $(X, E, \tau_2)$ be the two different fuzzy soft topologies on $(X, E)$. Then, $(X, E, \tau_1, \tau_2)$ is called a fuzzy soft bitopological space on which no separation axioms are assumed unless explicitly stated.
The members of $\tau_i(i = 1, 2)$ are called $\tau_i(i = 1, 2)$-fuzzy soft open sets, and the complement of $\tau_i(i = 1, 2)$-fuzzy soft open sets is called $\tau_i(i = 1, 2)$-fuzzy soft closed sets.

**Definition 2.16** (Mukherjee1 and Park 2015) A fuzzy soft set $f_E \in (X, E)$ is called $\tau_1 \tau_2$-fuzzy soft open set if $f_E = g_E \cup h_E$ such that $g_E \in \tau_1$ and $h_E \in \tau_2$.

The complement of $\tau_1 \tau_2$-fuzzy soft open set is called $\tau_1 \tau_2$-fuzzy soft closed set.

The family of all $\tau_1 \tau_2$-fuzzy soft open (closed) sets in $(X, E, \tau_1, \tau_2)$ is denoted by $\tau_1 \tau_2 FSO(X, \tau_1, \tau_2)E (\tau_1 \tau_2 FSC(X, \tau_1, \tau_2)E)$, respectively.

**Remark 2.17** (Mukherjee1 and Park 2015) $\tau_i(i = 1, 2)$-fuzzy soft open is $\tau_1 \tau_2$-fuzzy soft open, but the converse is not true.

**Theorem 2.18** (Mukherjee1 and Park 2015) If $(X, E, \tau_1, \tau_2)$ is a fuzzy soft bitopological space, then $\tau = \tau_1 \tilde{\cap} \tau_2$ is a fuzzy soft topological space over $(X, E)$.

**Remark 2.19** (Mukherjee1 and Park 2015) If $(X, E, \tau_1, \tau_2)$ is a fuzzy soft bitopological space, then $\tau = \tau_1 \tilde{\cap} \tau_2$ is not a fuzzy soft topological space over $(X, E)$.

**Definition 2.20** (Mukherjee1 and Park 2015) Let $(X, E, \tau_1, \tau_2)$ be a fuzzy soft bitopological space and $f_E \in (X, E)$. Then, the $\tau_i(i = 1, 2)$-fuzzy soft closure of $f_E$, denoted by $\tau_i cl(f_E)$, is the intersection of all $\tau_i(i = 1, 2)$-fuzzy soft closed supersets of $f_E$.

Clearly, $\tau_i cl(f_E)$ is the smallest $\tau_i(i = 1, 2)$-fuzzy soft closed set over $(X, E)$ which contains $f_E$.

**Definition 2.21** (Mukherjee1 and Park 2015) Let $(X, E, \tau_1, \tau_2)$ be a fuzzy soft bitopological space and $f_E \in (X, E)$. Then, the $\tau_1 \tau_2$-fuzzy soft closure of $f_E$, denoted by $\tau_1 \tau_2 cl(f_E)$, is the intersection of all $\tau_1 \tau_2$-fuzzy soft closed supersets of $f_E$.

Clearly, $\tau_1 \tau_2 cl(f_E)$ is the smallest $\tau_1 \tau_2$-fuzzy soft closed set over $(X, E)$ which contains $f_E$.

**Remark 2.22** (Mukherjee1 and Park 2015) If $(X, E, \tau_1, \tau_2)$ is a fuzzy soft bitopological space and $f_E \in (X, E)$, then $\tau_1 \tau_2 cl(f_E) \subseteq \tau_i cl(f_E)(i = 1, 2)$.

**Definition 2.23** (Mukherjee1 and Park 2015) Let $(X, E, \tau_1, \tau_2)$ be a fuzzy soft bitopological space and $f_E \in (X, E)$. Then, the $\tau_i(i = 1, 2)$-fuzzy soft interior of $f_E$, denoted by $\tau_i int(f_E)$, is the union of all $\tau_i(i = 1, 2)$-fuzzy soft open subsets of $f_E$.

Clearly, $\tau_i int(f_E)$ is the largest $\tau_i(i = 1, 2)$-fuzzy soft open set over $(X, E)$ which contained in $f_E$.

**Definition 2.24** (Mukherjee1 and Park 2015) Let $(X, E, \tau_1, \tau_2)$ be a fuzzy soft bitopological space and $f_E \in (X, E)$. Then, the $\tau_1 \tau_2$-fuzzy soft interior of $f_E$, denoted by $\tau_1 \tau_2 int(f_E)$, is the union of all $\tau_1 \tau_2$-fuzzy soft open subsets of $f_E$.

Clearly, $\tau_1 \tau_2 cl(f_E)$ is the largest $\tau_1 \tau_2$-fuzzy soft open set over $(X, E)$ which contained in $f_E$.

**Remark 2.25** (Mukherjee1 and Park 2015) If $(X, E, \tau_1, \tau_2)$ is a fuzzy soft bitopological space and $f_E \in (X, E)$, then $\tau_1 int(f_E)(i = 1, 2) \subseteq \tau_1 \tau_2 int(f_E)$.

**Definition 2.26** (Sayed 2017) Let $(X, E, \tau_1, \tau_2)$ be a soft bitopological space. Then, the family of all $\tau_1 \tau_2$-fuzzy soft open sets is a supra fuzzy soft topology on $(X, E)$. This supra fuzzy soft topology will denoted by $\tau_2$, i.e., $\tau_2 = (X, \tau_1 \tau_2 FSO(X, \tau_1, \tau_2)E) = \{g_E = g_1 \cup g_2 : g_i \in \tau_i, i = 1, 2\}$ and the triple $(X, \tau_1 \tau_2)$ is the supra fuzzy soft topological space associated with the fuzzy soft bitopological space $(X, E, \tau_1, \tau_2)$.

**Definition 2.27** (Sayed 2020) A fuzzy soft set $f_E$ in a fuzzy soft bitopological space $(X, E, \tau_1, \tau_2)$ is called

- (i) $(1, 2)^*-$fuzzy soft preopen set if $f_E \subseteq \tau_1 \tau_2 int(\tau_1 \tau_2 cl(f_E))$ and $(1, 2)^*-$fuzzy soft preclosed set if $\tau_1 \tau_2 cl(\tau_1 \tau_2 int(f_E)) \subseteq f_E$.
- (ii) $(1, 2)^*-$fuzzy soft semiopen set if $f_E \subseteq \tau_1 \tau_2 cl(\tau_1 \tau_2 int(f_E))$ and $(1, 2)^*-$fuzzy soft semiclosed set if $\tau_1 \tau_2 int(\tau_1 \tau_2 cl(f_E)) \subseteq f_E$.
- (iii) $(1, 2)^*-$fuzzy soft $\beta$-open set if $f_E \subseteq \tau_1 \tau_2 cl(\tau_1 \tau_2 int(f_E))$ and $(1, 2)^*-$fuzzy soft $\beta$-closed set if $\tau_1 \tau_2 int(\tau_1 \tau_2 cl(f_E)) \subseteq f_E$.

**Definition 2.28** (Sayed 2020) Let $(X, E, \tau_1, \tau_2)$ be a fuzzy soft bitopological space and $f_E \in (X, E)$. Then, $f_E$ is called $(1, 2)^*-$fuzzy soft $b$-open set (briefly, $(1, 2)^*-$fsb $b$-open) if $f_E \subseteq \tau_1 \tau_2 int(\tau_1 \tau_2 cl(\tau_1 \tau_2 int(f_E))) \cup \tau_1 \tau_2 cl(\tau_1 \tau_2 int(f_E))$.

**Theorem 2.29** (Sayed 2020) Let $(X, E, \tau_1, \tau_2)$ be a fuzzy soft bitopological space. Then,

- (i) Every $(1, 2)^*-$fuzzy soft preopen set is $(1, 2)^*-$fuzzy soft $b$-open set.
- (ii) Every $(1, 2)^*-$fuzzy soft $b$-open set is $(1, 2)^*-$fuzzy soft $\beta$-open set.
- (iii) Every $(1, 2)^*-$fuzzy soft semiopen set is $(1, 2)^*-$fuzzy soft $b$-open set.

**Definition 2.30** (Sayed 2020) Let $(X, E, \tau_1, \tau_2)$ be a fuzzy soft bitopological space and $f_E \in (X, E)$. Then,

- (i) $(1, 2)^*-$fuzzy soft $b$-closure $((1, 2)^*-$fsbcl$(f_E))$ of a set $f_E$ in $(X, E, \tau_1, \tau_2)$ defined by $(1, 2)^*-$fsbcl$(f_E) = \cap\{g_E \supseteq f_E : g_E$ is a $(1, 2)^*-$fuzzy soft $b$-closed set in $(X, E, \tau_1, \tau_2)\}$. 

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(ii) $(1, 2)^{-}$-fuzzy soft $b$-interior $((1, 2)^{-}$-$fsbint(f_E))$ of a set $f_E$ in $(X, E, \tau_1, \tau_2)$ defined by $(1, 2)^{-}$-$fsbint(f_E) = \bigcup\{g_E \subseteq f_E : g_E$ is $(1, 2)^{-}$-fuzzy soft $b$-open in $(X, E, \tau_1, \tau_2)\}$.

$(1, 2)^{-}$-$fsbcl(f_E)$ is the smallest $(1, 2)^{-}$-fuzzy soft $b$-closed set in $(X, E, \tau_1, \tau_2)$ which contains $f_E$ and $(1, 2)^{-}$-$fsbcl(f_E)$ is the largest $(1, 2)^{-}$-fuzzy soft $b$-closed set in $(X, E, \tau_1, \tau_2)$ which is contained in $f_E$.

**Definition 2.31** (Aygünul and Aygün 2009) Let $(X, E)$ and $(Y, K)$ be the families of all fuzzy soft sets over $X$ and $Y$, respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two functions. Then, the pair $(\varphi, \psi)$ is called a fuzzy soft mapping from $X$ to $Y$ and denoted by $(\varphi, \psi) : (X, E) \to (Y, K)$.

In the diagram, $\tilde{f_A} = \tilde{f_A}(X, E, \psi)\tilde{g_B}(Y, K)$ and Let $\varphi : I^X \to I^Y$ is the forward powerset operator (see, e.g., Rodabaugh 1999), that is, $\tilde{\varphi^{-1}}(h) := \varphi(h)$ for all $h \in I^X$. Since componentwise composition of two fuzzy soft functions $(\varphi, \psi)$ from $E(X, E)$ to $(Y, K)$ and $(\varphi, \psi)$ from $(Y, K)$ to $(Z, T)$ is obviously a fuzzy soft function $(\varphi \circ \varphi, \psi \circ \psi)$ from $(X, E)$ to $(Z, T)$, where $\psi : E \to K$ and $\psi : K \to T$ and $\varphi : I^X \to I^T$ and $\psi : I^Y \to I^Z$ and the pair of identities $(id_X, id_E)$ from $(X, E)$ to $(X, E)$ is the identical morphism.

(1) Let $f_A \in \tilde{f_A}(X, E)$. Then, the image of $f_A$ under the fuzzy soft mapping $(\varphi, \psi)$ is the fuzzy soft set over $Y$ defined by $(\varphi, \psi)(f_A)$, where $\forall k \in \psi(E)$, $\forall y \in Y$,

$\varphi(f_A)(k)(y) = \begin{cases} \bigvee_{x \in \psi^{-1}(y)} \bigvee_{f_A(e)(x)} = y & \text{if } e \in \psi^{-1}(y) \\ 0_E & \text{otherwise} \end{cases}$

(2) Let $g_B \in \tilde{g_B}(Y, K)$. Then, the inverse image of $g_B$ under the fuzzy soft mapping $(\varphi, \psi)$ is the fuzzy soft set over $X$ defined by $\psi^{-1}(g_B)$, where $\forall e \in \psi^{-1}(K), \forall x \in X, f_A(e)(x) = g_B(\psi(e))(\varphi(x))$. If both $\varphi$ and $\psi$ are injective, then the fuzzy soft mapping $(\varphi, \psi)$ is said to be injective. If both $\varphi$ and $\psi$ are surjective, then the fuzzy soft mapping $(\varphi, \psi)$ is said to be surjective. $(\varphi, \psi)$ is said to be bijective if it is both injective and surjective.

**3 $(1, 2)^{-}$-fuzzy soft $b$-continuous maps**

In this section, we introduce the concept of $(1, 2)^{-}$-fuzzy soft $b$-continuous functions and some of their properties are discussed.

We begin by the following concepts which will be used in the sequel.

**Definition 3.1** A fuzzy soft mapping $(\varphi, \psi) : (X, E, \tau_1, \tau_2) \to (Y, K, \sigma_1, \sigma_2)$ is said to be

(i) $(1, 2)^{-}$-fuzzy soft-continuous (briefly $(1, 2)^{-}$-f-continuous) if the inverse image of every $\sigma_1\sigma_2$-fuzzy soft open set in $(Y, K, \sigma_1, \sigma_2)$ is a $\tau_1\tau_2$-fuzzy soft open set in $(X, E, \tau_1, \tau_2)$.

(ii) $(1, 2)^{-}$-fuzzy soft semicontinuous (briefly $(1, 2)^{-}$-f-continuous) if the inverse image of every $\sigma_1\sigma_2$-fuzzy soft open set in $(X, K, \sigma_1, \sigma_2)$ is a $(1, 2)^{-}$-fuzzy soft semiopen set in $(X, E, \tau_1, \tau_2)$.

(iii) $(1, 2)^{-}$-fuzzy soft precontinuous (briefly $(1, 2)^{-}$-f-continuous) if the inverse image of every $\sigma_1\sigma_2$-fuzzy soft open set in $(Y, K, \sigma_1, \sigma_2)$ is a $(1, 2)^{-}$-fuzzy soft preopen set in $(X, E, \tau_1, \tau_2)$.

(iv) $(1, 2)^{-}$-fuzzy soft $\beta$-continuous (briefly $(1, 2)^{-}$-f-$\beta$-continuous) if the inverse image of every $\sigma_1\sigma_2$-fuzzy soft open set in $(Y, K, \sigma_1, \sigma_2)$ is a $(1, 2)^{-}$-fuzzy soft $\beta$-open set in $(X, E, \tau_1, \tau_2)$.

**Definition 3.2** A fuzzy soft mapping $(\varphi, \psi) : (X, E, \tau_1, \tau_2) \to (Y, K, \sigma_1, \sigma_2)$ is said to be $(1, 2)^{-}$-fuzzy soft $b$-continuous (briefly $(1, 2)^{-}$-fsb-continuous) if the inverse image of every $\sigma_1\sigma_2$-fuzzy soft open set in $(Y, K, \sigma_1, \sigma_2)$ is a $(1, 2)^{-}$-fuzzy soft $b$-open set in $(X, E, \tau_1, \tau_2)$.

**Example 3.3** Let $X = \{x, y\}, E = K = \{e_1, e_2\}$. Let $\tau_1 = \{0_E, 1_E, f_{e_1}, f_{e_2}\}, \tau_2 = \{0_E, 1_E, f_{e_2}\}$, where $f_{e_1} = \{f_1(e_1) = \{x/0.5, y/0.0\}, f_1(e_2) = \{x/0.0, y/0.0\} = 0\}$, $f_{e_2} = \{f_2(e_1) = \{x/0.0, y/0.7\}, f_2(e_2) = \{x/0.0, y/0.0\} = 0\}$.

It is clear that $(X, E, \tau_1, \tau_2)$ is a fuzzy soft bitopological space.

The $\tau_1\tau_2$-fuzzy soft open sets are $\{0_E, 1_E, f_{e_1}, f_{e_2}, f_{e_3}, f_{e_4}, f_{e_5}, f_{e_6}, f_{e_7}, f_{e_8}, f_{e_9}, f_{e_10}, f_{e_11}, f_{e_12}\}$, where $f_{e_1} = \{f_1(e_1) = \{x/0.5, y/0.0\}, f_4(e_2) = \{x/0.3, y/0.6\}, f_{e_3} = \{x/0.5, y/0.7\}, f_6(e_2) = \{x/0.3, y/0.0\} = 0\}$, $f_{e_4} = \{f_4(e_1) = \{x/0.5, y/0.0\}, f_4(e_2) = \{x/0.3, y/0.0\} = 0\}$.
\( f_{10E} = \{ f_{10}(e_1) = \{x/0.5, y/0.7\}, f_{10}(e_2) = \{x/0.3, y/0.0\}\}, f_{11E} = \{ f_{11}(e_1) = \{x/0.5, y/0.7\}, f_{11}(e_2) = \{x/0.0, y/0.0\}\}. \)

Let \( \sigma_1 = \{0K, \tilde{1}K, g_{1E}, g_{2E}\}, \sigma_2 = \{\tilde{0}K, \tilde{1}K, g_{3E}\}, \) where \( g_{1E} = \{g_1(e_1) = \{x/0.5, y/0.0\}, g_1(e_2) = \{x/0.0, y/0.0\} = \emptyset\}, g_{2E} = \{g_2(e_1) = \{x/0.5, y/0.0\}, g_2(e_2) = \{x/0.3, y/0.0\}\}, g_{3E} = \{g_3(e_1) = \{x/0.5, y/0.7\}, g_3(e_2) = \{x/0.0, y/0.0\} = \emptyset\}. \) It is clear that \( (X, E, \tau_1, \tau_2) \) is a fuzzy soft bitopological space. The \( \sigma_1 \sigma_2 \)-fuzzy soft open sets are \( \{0K, \tilde{1}K, g_{1E}, g_{2E}, g_{3E}, g_{4E}\} \), where

\[
\begin{align*}
g_{2E} &= g_1 \cup g_{2E}, g_{3E} = g_{1E} \cup g_{3E}, \\
g_{4E} &= g_{2E} \cup g_{3E} = \{g_4(e_1) = \{x/0.5, y/0.7\}, g_4(e_2) = \{x/0.0, y/0.0\}\} = \emptyset.
\end{align*}
\]

It is clear that \( (Y, K, \sigma_1, \sigma_2) \) is a fuzzy soft bitopological space.

Now define \( \varphi : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( \psi : E \rightarrow K \) such that

\[
\varphi(x, \mu g(e_1)(x)) = (x, \mu g(e_2)(x)), \psi(e_i) = e_i, i = 1, 2.
\]

Then, the inverse image of every \( \sigma_1 \sigma_2 \)-fuzzy soft open sets is \( \tau_1 \tau_2 \)-fuzzy b-soft open sets are \( (1, 2)^* \)-fuzzy soft b-open sets in \( (X, E, \tau_1, \tau_2) \). Thus, \( (\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2) \) is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous.

**Theorem 3.4** Every \( (1, 2)^* \)-fuzzy soft continuous map is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous.

**Proof** Let \( (\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2) \) be a \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous map. Let \( f_A \) be \( \sigma_1 \sigma_2 \)-fuzzy soft open set in \( (Y, K, \sigma_1, \sigma_2) \). Since \( (\varphi, \psi) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous map, \( (\varphi, \psi)^{-1}(f_A) \) is \( \tau_1 \tau_2 \)-fuzzy soft open set in \( (X, E, \tau_1, \tau_2) \). And so \( (\varphi, \psi)^{-1}(f_A) \) is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous map in \( (X, E, \tau_1, \tau_2) \). Therefore, \( (\varphi, \psi) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous map.

The converse of the above theorem is not true as shown in the following example.

**Example 3.5** Consider Example 3.3, \( (\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2) \) is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous but not \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous.

**Theorem 3.6** Let \( (\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2) \) be a fuzzy soft mapping. Then, the following statements are equivalent:

(i) \( (\varphi, \psi) \) is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous.

(ii) The inverse image of each \( \sigma_1 \sigma_2 \)-fuzzy soft closed set in \( (Y, K, \sigma_1, \sigma_2) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-closed set in \( (X, E, \tau_1, \tau_2) \).

**Proof** (i) \( \Rightarrow \) (ii): Let \( g_K \) be a \( \sigma_1 \sigma_2 \)-fuzzy soft closed set in \( (Y, K, \sigma_1, \sigma_2) \). Then, \( g_K^c \) is a \( \sigma_1 \sigma_2 \)-fuzzy soft open set in \( (Y, K, \sigma_1, \sigma_2) \). Thus, \( (\varphi, \psi)^{-1}(g_K^c) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-open set in \( (X, E, \tau_1, \tau_2) \) and \( (\varphi, \psi)^{-1}(g_K^c) = \{(\varphi, \psi)^{-1}(g_K^c)\} \). Hence, \( (\varphi, \psi)^{-1}(g_K) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-closed set in \( (X, E, \tau_1, \tau_2) \).

(ii) \( \Rightarrow \) (i): Let \( g_K \) be a \( \sigma_1 \sigma_2 \)-fuzzy soft closed set in \( (Y, K, \sigma_1, \sigma_2) \). Then, \( g_K^c \) is a \( \sigma_1 \sigma_2 \)-fuzzy soft open set, and (ii), we have \( (\varphi, \psi)^{-1}(g_K^c) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-closed set in \( (X, E, \tau_1, \tau_2) \) and \( (\varphi, \psi)^{-1}(g_K^c) = \{(\varphi, \psi)^{-1}(g_K^c)\} \). Hence, \( (\varphi, \psi)^{-1}(g_K) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-open set in \( (X, E, \tau_1, \tau_2) \). Therefore, \( (\varphi, \psi) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous map.

**Theorem 3.7** Let \( (\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2) \) be a fuzzy soft map. Then,

(i) Every \( (1, 2)^* \)-fuzzy soft continuous map is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous.

(ii) Every \( (1, 2)^* \)-fuzzy soft precontinuous map is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous.

(iii) Every \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous map is \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous.

**Proof** (i) Assume that \( (\varphi, \psi) \) is a \( (1, 2)^* \)-fuzzy semi continuous map and \( g_K \) be a \( \sigma_1 \sigma_2 \)-fuzzy soft in \( (Y, K, \sigma_1, \sigma_2) \). Then, \( (\varphi, \psi)^{-1}(g_K) \) is a \( (1, 2)^* \)-fuzzy semiopen set in \( (X, E, \tau_1, \tau_2) \). Since every \( (1, 2)^* \)-fuzzy semiopen set is \( (1, 2)^* \)-fuzzy soft \( \beta \)-open set, \( (\varphi, \psi)^{-1}(g_K) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-open set in \( (X, E, \tau_1, \tau_2) \). Therefore, \( (\varphi, \psi) \) is a \( (1, 2)^* \)-fuzzy soft \( \beta \)-continuous map.

The proof of (ii) and (iii) is similar to (i).
\[f_{0E} = \{f_0(e_1) = \{x/0.0, y/0.0\} = \emptyset, f_0(e_2) = \{x/0.0, y/0.6\}\},\]
\[f_{1E} = \{f_1(e_1) = \{x/0.0, y/0.0\} = \emptyset, f_1(e_2) = \{x/0.3, y/0.6\}\},\]
\[f_{8E} = \{f_8(e_1) = \{x/0.5, y/0.0\}, f_8(e_2) = \{x/0.3, y/0.0\}\},\]
\[f_{9E} = \{f_9(e_1) = \{x/0.5, y/0.0\}, f_9(e_2) = \{x/0.0, y/0.6\}\},\]
\[f_{10E} = \{f_{10}(e_1) = \{x/0.5, y/0.0\}, f_{10}(e_2) = \{x/0.3, y/0.6\}\},\]
\[f_{11E} = \{f_{11}(e_1) = \{x/0.0, y/0.7\}, f_{11}(e_2) = \{x/0.3, y/0.0\}\},\]
\[f_{12E} = \{f_9(e_1) = \{x/0.0, y/0.7\}, f_9(e_2) = \{x/0.0, y/0.6\}\},\]
\[f_{13E} = \{f_{10}(e_1) = \{x/0.0, y/0.7\}, f_{10}(e_2) = \{x/0.3, y/0.0\}\},\]
\[f_{14E} = \{f_{11}(e_1) = \{x/0.0, y/0.7\}, f_{11}(e_2) = \{x/0.0, y/0.6\}\}.\]

and \((1, 2)^*\)-fuzzy soft semiopen sets are \([\emptyset_E, \emptyset_E, f_{1E}, f_{2E}]\). Let \(\sigma_1 = [0\_K, \mathbb{I}_E, g_{1E}, g_{2E}], \sigma_2 = [0\_K, \mathbb{I}_E, g_{3E}],\)

where \(g_{1E} = g_1(e_1) = \{x/0.0, y/0.7\}, g_1(e_2) = \{x/0.0, y/0.0\} = \emptyset,\)
\[g_{2E} = g_2(e_1) = \{x/0.0, y/0.7\}, g_2(e_2) = \{x/0.3, y/0.6\}.\]

It is clear that \((X, E, \tau_1, \tau_2)\) is a fuzzy soft bitopological space.

The \(\sigma_1\_\sigma_2\)-fuzzy soft open sets are \([0\_K, \mathbb{I}_E, g_{1E}, g_{2E}, g_{3E}]\).

\[f_{1K} = \{f_1(e_1) = \{x/0.0, y/0.0\}, f_1(e_2) = \{x/0.0, y/0.6\}\},\]
\[f_{2E} = \{f_2(e_1) = \{x/0.0, y/0.7\}, f_2(e_2) = \{x/0.0, y/0.0\}\}.\]

It is clear that \((X, E, \tau_1, \tau_2)\) is a fuzzy soft bitopological space.

The \(\tau_1\_\tau_2\)-fuzzy soft open sets are \([\emptyset_E, \emptyset_E, f_{1E}, f_{2E}]\).

Example 3.10 Consider Example 3.2, \((\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2)\) is \((1, 2)^*\)-fuzzy soft b-continuous but not \((1, 2)^*\)-fuzzy soft precontinuous. Since the \((1, 2)^*\)-fuzzy soft preopen set in \((X, E, \tau_1, \tau_2)\) is \([\emptyset_E, \emptyset_E, f_{1E}, f_{2E}, f_{10E}, f_{11E}]\) and \((\varphi, \psi)^{-1}(g_{2E}) = f_{2E}\), which is not \((1, 2)^*\)-fuzzy soft preopen set in \((X, E, \tau_1, \tau_2)\).
Theorem 3.13 Let \((\varphi, \psi): (X, E, \tau_1, \tau_2) \to (Y, K, \sigma_1, \sigma_2)\) is \((1, 2)^*\)-fuzzy soft b-continuous, then \((\varphi, \psi)((1, 2)^*\text{-}\text{fscl}(f_E)) \subseteq \tau_1 \tau_2 \text{fscl}((\varphi, \psi)(f_E))\) for every \(f_E \in (X, E, \tau_1, \tau_2)\).

Proof Assume that \((\varphi, \psi)\) is \((1, 2)^*\)-fuzzy soft b-continuous and \(f_E\) be any fuzzy soft subset in \((X, E, \tau_1, \tau_2)\). Then, \(\tau_1 \tau_2 \text{fscl}((\varphi, \psi)(f_E))\) is a \(\sigma_1 \sigma_2\)-fuzzy soft closed set in \((Y, K, \sigma_1, \sigma_2)\). Since \((\varphi, \psi)\) is \((1, 2)^*\)-fuzzy soft b-continuous, \((\varphi, \psi)^{-1}((\tau_1 \tau_2 \text{fscl}((\varphi, \psi)(f_E)))\) is \((1, 2)^*\)-fuzzy soft b-closed set in \((X, E, \tau_1, \tau_2)\) and it contains \(f_E\). But \((1, 2)^*\)-fscl\((f_E)\) is the intersection of all \((1, 2)^*\)-fuzzy soft b-closed sets containing \(f_E\). Therefore, \((1, 2)^*\)-fsbc\((f_E)\) is \((\varphi, \psi)^{-1}((\tau_1 \tau_2 \text{fscl}((\varphi, \psi)(f_E)))\) and so \((\varphi, \psi)((1, 2)^*\text{-}\text{fscl}(f_E)) \subseteq \tau_1 \tau_2 \text{fscl}((\varphi, \psi)(f_E))\) for every fuzzy soft subset \(f_E \in (X, E, \tau_1, \tau_2)\). \(\square\)

Remark 3.14 The composition of two \((1, 2)^*\)-fuzzy soft b-continuous maps need not be \((1, 2)^*\)-fuzzy soft b-continuous as is shown in the following example.

Example 3.15 Let \(X = Y = Z = \{x, y\}, E = K = T = \{e_1, e_2\} \). Let \(\tau_1 = \{\mathcal{O}_E, \mathcal{I}_E, f_{1E}, f_{2E}, f_{3E}\}\) and \(\tau_2 = \{\mathcal{O}_E, \mathcal{I}_E, f_{1E}, f_{2E}\}\) where \(f_{1E} = \{f_1(e_1) = \{x, 0.5, y, 0\}, f_1(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(f_{2E} = \{f_2(e_1) = \{x, 0, 0, y, 0\}, f_2(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\).

It is clear that \((X, E, \tau_1, \tau_2)\) is a fuzzy soft bitopological space. The \(\tau_1 \tau_2\)-fuzzy soft open sets are \(\{\mathcal{O}_E, \mathcal{I}_E, f_{1E}, f_{2E}, f_{3E}\}\) where \(f_{3E} = f_{1E} \mathcal{U} f_{2E} = \{f_3(e_1) = \{x, 0.5, y, 0\}, f_3(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \((1, 2)^*\)-fuzzy soft b-open sets are \(\{\mathcal{O}_E, \mathcal{I}_E, f_{1E}, f_{2E}, f_{3E}, f_{4E}, f_{5E}, f_{6E}, f_{7E}, f_{8E}, f_{9E}, f_{10E}, f_{11E}\}\), where \(f_{1E}, f_{2E}, f_{3E}\) are shown as above and \(f_{3E} = \{f_3(e_1) = \{x, 0.5, y, 0\}, f_3(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\).

It is clear that \((Z, T, \eta_1, \eta_2)\) is a fuzzy soft bitopological space. The \(\eta_1 \eta_2\)-fuzzy soft open sets are \(\{\mathcal{O}_T, \mathcal{I}_T, h_{1T}, h_{2T}, h_{3T}, h_{4T}, h_{5T}\}\), \((1, 2)^*\)-fuzzy soft b-open sets are \(\{\mathcal{O}_T, \mathcal{I}_T, h_{1T}, h_{2T}, h_{3T}, h_{4T}, h_{5T}\}\) where \(h_{1T} = \{h_1(e_1) = \{x, 0.5, y, 0\}, h_1(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(h_{2T} = \{h_2(e_1) = \{x, 0.5, y, 0\}, h_2(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(h_{3T} = \{h_3(e_1) = \{x, 0.5, y, 0\}, h_3(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(h_{4T} = \{h_4(e_1) = \{x, 0.5, y, 0\}, h_4(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\).

It is clear that \((Z, T, \eta_1, \eta_2)\) is a fuzzy soft bitopological space. The \(\eta_1 \eta_2\)-fuzzy soft open sets are \(\{\mathcal{O}_T, \mathcal{I}_T, h_{1T}, h_{2T}, h_{3T}, h_{4T}, h_{5T}\}\), \((1, 2)^*\)-fuzzy soft b-open sets are \(\{\mathcal{O}_T, \mathcal{I}_T, h_{1T}, h_{2T}, h_{3T}, h_{4T}, h_{5T}\}\) where \(h_{1T} = \{h_1(e_1) = \{x, 0.5, y, 0\}, h_1(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(h_{2T} = \{h_2(e_1) = \{x, 0.5, y, 0\}, h_2(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(h_{3T} = \{h_3(e_1) = \{x, 0.5, y, 0\}, h_3(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(h_{4T} = \{h_4(e_1) = \{x, 0.5, y, 0\}, h_4(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\), \(h_{5T} = \{h_5(e_1) = \{x, 0.5, y, 0\}, h_5(e_2) = \{x, 0, 0, y, 0\}\} \neq \{x, 0, 0, y, 0\}\).
b-continuous since \((\hat{\psi}, \hat{\psi}) \circ (\varphi, \psi))^{-1}(h_{\delta_{T}})\) which is not a 
(1,2)*-fuzzy soft b-open set in \((X, E, \tau_{1}, \tau_{2})\).

**Theorem 3.16** Let \((\varphi, \psi) : (X, E, \tau_{1}, \tau_{2}) \rightarrow (Y, K, \sigma_{1}, \sigma_{2})\) and \((\hat{\varphi}, \hat{\psi}) : (Y, K, \sigma_{1}, \sigma_{2}) \rightarrow (Z, T, \eta_{1}, \eta_{2})\) be two maps in fuzzy soft bitopological spaces. Then, \((\hat{\varphi}, \hat{\psi}) \circ (\varphi, \psi) : (X, E, \tau_{1}, \tau_{2}) \rightarrow (Z, T, \eta_{1}, \eta_{2})\) is (1,2)*-fuzzy soft b-continuous, if \((\varphi, \psi)\) is (1,2)*-fuzzy soft b-continuous and \((\hat{\psi}, \hat{\varphi})\) is (1,2)*-fuzzy soft b-continuous.

**Proof** Let \(h_{T}\) be a \(\eta_{1} \eta_{2}\)-fuzzy soft closed set of \((Z, T, \sigma_{1}, \sigma_{2})\). Since \((\hat{\varphi}, \hat{\psi}) : (Y, K, \sigma_{1}, \sigma_{2}) \rightarrow (Z, T, \eta_{1}, \eta_{2})\) is (1,2)*-fuzzy soft b-continuous, \((\hat{\varphi}, \hat{\psi})^{-1}(h_{T})\) is \(\sigma_{1} \sigma_{2}\)-fuzzy soft b-continuous set of \((X, E, \tau_{1}, \tau_{2})\). Now, \((\varphi, \psi) : (X, E, \tau_{1}, \tau_{2}) \rightarrow (Y, K, \sigma_{1}, \sigma_{2})\) is (1,2)*-fuzzy soft b-continuous, \((\varphi, \psi)^{-1}(h_{T})\) is \(\sigma_{1} \sigma_{2}\)-fuzzy soft b-continuous set in \((Y, K, \sigma_{1}, \sigma_{2})\), and \((\varphi, \psi)^{-1}(h_{T})\) is (1,2)*-fuzzy soft b-continuous set of \((X, E, \tau_{1}, \tau_{2})\) for every \((1,2)*\)-fuzzy soft b-closed set of \((X, E, \tau_{1}, \tau_{2})\) in \((Y, K, \sigma_{1}, \sigma_{2})\).

**Definition 4.1** A fuzzy soft mapping \((\varphi, \psi) : (X, E, \tau_{1}, \tau_{2}) \rightarrow (Y, K, \sigma_{1}, \sigma_{2})\) is said to be (1,2)*-fuzzy soft b-irresolute mapping (briefly, \((1,2)*\)-fsb-irresolute) if \((\varphi, \psi)^{-1}(g_{K})\) is a \(1,2)*\)-fuzzy soft b-closed set in \((X, E, \tau_{1}, \tau_{2})\).

**Theorem 4.2** A fuzzy soft mapping \((\varphi, \psi) : (X, E, \tau_{1}, \tau_{2}) \rightarrow (Y, K, \sigma_{1}, \sigma_{2})\) is (1,2)*-fuzzy soft b-irresolute if and only if the inverse image of every \((1,2)*\)-fuzzy soft b-open set in \((Y, K, \sigma_{1}, \sigma_{2})\) is (1,2)*-fuzzy soft b-open set in \((X, E, \tau_{1}, \tau_{2})\).

**Proof** \(\Rightarrow\): Let \(g_{K}\) be a (1,2)*-fuzzy soft b-open set in \((Y, K, \sigma_{1}, \sigma_{2})\). Then, \(g_{K}^{c}\) is a (1,2)*-fuzzy soft b-closed set in \((Y, K, \sigma_{1}, \sigma_{2})\). Hence, \((\varphi, \psi)^{-1}(g_{K}^{c})\) is a \(1,2)*\)-fuzzy soft b-closed set in \((X, E, \tau_{1}, \tau_{2})\). But \((\varphi, \psi)^{-1}(g_{K}) = (\varphi, \psi)^{-1}(g_{K}^{c})^{c}\), so that \((\varphi, \psi)^{-1}(g_{K})\) is a \(1,2)*\)-fuzzy soft b-open set in \((X, E, \tau_{1}, \tau_{2})\).

\(\Leftarrow\): Let \(g_{K}\) be a (1,2)*-fuzzy soft b-closed set in \((Y, K, \sigma_{1}, \sigma_{2})\). Then, \(g_{K}^{c}\) is a (1,2)*-fuzzy soft b-open set in \((Y, K, \sigma_{1}, \sigma_{2})\). Thus, \((\varphi, \psi)^{-1}(g_{K}^{c})\) is a (1,2)*-fuzzy soft b-open set in \((X, E, \tau_{1}, \tau_{2})\). Also, \((\varphi, \psi)^{-1}(g_{K}) = (\varphi, \psi)^{-1}(g_{K}^{c})^{c}\). Thus, \((\varphi, \psi)^{-1}(g_{K})\) is a (1,2)*-fuzzy soft b-closed set in \((X, E, \tau_{1}, \tau_{2})\). Therefore, \((\varphi, \psi) : (X, E, \tau_{1}, \tau_{2}) \rightarrow (Y, K, \sigma_{1}, \sigma_{2})\) is (1,2)*-fuzzy soft b-irresolute mapping.

**Definition 5.1** A fuzzy soft map \((\varphi, \psi) : (X, E, \tau_{1}, \tau_{2}) \rightarrow (Y, K, \sigma_{1}, \sigma_{2})\) is said to be (1,2)*-fuzzy soft b-open (closed) maps.

**Theorem 4.3** Every (1,2)*-fuzzy soft b-irresolute mapping is (1,2)*-fuzzy soft b-continuous.
(i) a $(1, 2)^*$-fuzzy soft $b$-open (briefly, $(1, 2)^*$-fsb-open) map if the image of every $(1, 2)^*$-fuzzy soft $b$-open set in $(X, E, \tau_1, \tau_2)$ is $(1, 2)^*$-fuzzy soft $b$-open set in $(Y, K, \sigma_1, \sigma_2)$.

(ii) a $(1, 2)^*$-fuzzy soft $b$-closed (briefly, $(1, 2)^*$-fsb-closed) map if the image of every $(1, 2)^*$-fuzzy soft $b$-closed set in $(X, E, \tau_1, \tau_2)$ is $(1, 2)^*$-fuzzy soft $b$-closed set in $(Y, K, \sigma_1, \sigma_2)$.

Theorem 5.2  If $(\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2)$ is $(1, 2)^*$-fuzzy soft $b$-closed map and $(\varphi, \psi) : (Y, K, \sigma_1, \sigma_2) \rightarrow (Z, T, \eta_1, \eta_2)$ is $(1, 2)^*$-fuzzy soft $b$-closed map. Then, $(\varphi, \psi) \circ (\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Z, T, \eta_1, \eta_2)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

Proof  Let $f_E$ be a $(1, 2)^*$-fuzzy soft closed set in $(X, E, \tau_1, \tau_2)$, $(\varphi, \psi)(f_E)$ be $(1, 2)^*$-fuzzy soft closed set in $(Y, K, \sigma_1, \sigma_2)$. Since $(\varphi, \psi) : (Y, K, \sigma_1, \sigma_2) \rightarrow (Z, T, \eta_1, \eta_2)$ is $(1, 2)^*$-fuzzy soft $b$-closed map, $(\varphi, \psi)((\varphi, \psi)(f_E))$ is $(1, 2)^*$-fuzzy soft closed set in $(Z, T, \eta_1, \eta_2)$. Therefore, $(\varphi, \psi) \circ (\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

Theorem 5.3  Let $(\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2)$ and $(\varphi, \psi) : (Y, K, \sigma_1, \sigma_2) \rightarrow (Z, T, \eta_1, \eta_2)$ be two fuzzy soft maps such that $(\varphi, \psi) \circ (\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

(i) If $(\varphi, \psi)$ is a $(1, 2)^*$-fuzzy soft continuous and surjective, then $(\varphi, \psi) \circ (\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

(ii) If $(\varphi, \psi) \circ (\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-irresolute and injective, then $(\varphi, \psi) \circ (\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

Proof  (i) Let $g_K$ be $(\sigma_1, \sigma_2)$-fuzzy soft closed set in $(Y, K, \sigma_1, \sigma_2)$. If $(\varphi, \psi)$ is a $(1, 2)^*$-fuzzy soft continuous and surjective, then $(\varphi, \psi)^{-1}(g_K)$ is a $(\tau_1, \tau_2)$-fuzzy soft closed set in $(X, E, \tau_1, \tau_2)$. Hence, $(\varphi, \psi) \circ (\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map, $(\varphi, \psi) \circ (\varphi, \psi)((\varphi, \psi)^{-1}(g_K)) = (\varphi, \psi)(g_K)$ is $(1, 2)^*$-fuzzy soft $b$-closed map in $(Z, T, \eta_1, \eta_2)$. Hence, $(\varphi, \psi) : (Y, K, \sigma_1, \sigma_2) \rightarrow (Z, T, \eta_1, \eta_2)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

(ii) Let $f_E$ be a $(\tau_1, \tau_2)$-fuzzy soft closed subset of $(X, E, \tau_1, \tau_2)$. If $(\varphi, \psi) \circ (\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Z, T, \eta_1, \eta_2)$ is $(1, 2)^*$-fuzzy soft $b$-closed map, then $(\varphi, \psi) \circ (\varphi, \psi)(f_E)$ is a $(1, 2)^*$-fuzzy soft $b$-closed set in $(Z, T, \eta_1, \eta_2)$, and so $(\varphi, \psi)^{-1}((\varphi, \psi) \circ (\varphi, \psi))(f_E) = (\varphi, \psi)(f_E)$ is $(1, 2)^*$-fuzzy soft $b$-closed set in $(Y, K, \sigma_1, \sigma_2)$. Therefore, $(\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-irresolute and injective, $(\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

Proposition 5.4  For any bijection fuzzy soft map $(\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2)$, the following statements are equivalent.

(i) The inverse fuzzy soft map $(\varphi, \psi)^{-1} : (Y, K, \sigma_1, \sigma_2) \rightarrow (X, E, \tau_1, \tau_2)$ is $(1, 2)^*$-fuzzy soft $b$-continuous.

(ii) $(\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-open map.

(iii) $(\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

Proof  (i) $\Rightarrow$ (ii): Given that $(\varphi, \psi)^{-1} : (Y, K, \sigma_1, \sigma_2) \rightarrow (X, E, \tau_1, \tau_2)$ is $(1, 2)^*$-fuzzy soft $b$-continuous. Let $f_E$ be a $(\tau_1, \tau_2)$-fuzzy soft open set in $(X, E, \tau_1, \tau_2)$. Then, $((\varphi, \psi)^{-1})(f_E) = (\varphi, \psi)(f_E)$ is $(1, 2)^*$-fuzzy soft $b$-open set in $(Y, K, \sigma_1, \sigma_2)$.

(ii) $\Rightarrow$ (iii): Let $(\varphi, \psi) : (X, E, \tau_1, \tau_2) \rightarrow (Y, K, \sigma_1, \sigma_2)$ be a $(1, 2)^*$-fuzzy soft $b$-open map. Let $f_E$ be a $(\tau_1, \tau_2)$-fuzzy soft open set in $(X, E, \tau_1, \tau_2)$. Then, $f_E^c$ be a $(\tau_1, \tau_2)$-fuzzy soft closed set in $(X, E, \tau_1, \tau_2)$. Thus, $(\varphi, \psi)(f_E^c)$ is $(1, 2)^*$-fuzzy soft $b$-closed set in $(Y, K, \sigma_1, \sigma_2)$. Therefore, $(\varphi, \psi)$ is $(1, 2)^*$-fuzzy soft $b$-closed map.

(iii) $\Rightarrow$ (i): Let $(\varphi, \psi)$ be a $(1, 2)^*$-fuzzy soft $b$-closed map. Let $f_E$ be a $(\tau_1, \tau_2)$-fuzzy soft closed set in $(X, E, \tau_1, \tau_2)$, then $((\varphi, \psi)^{-1})(f_E) = (\varphi, \psi)(f_E)$ is $(1, 2)^*$-fuzzy soft $b$-closed set in $(Y, K, \sigma_1, \sigma_2)$. Therefore, $(\varphi, \psi)^{-1} : (Y, K, \sigma_1, \sigma_2) \rightarrow (X, E, \tau_1, \tau_2)$ is $(1, 2)^*$-fuzzy soft $b$-continuous.
Aygünolu A, Aygün H (2009) Introduction to fuzzy soft groups. Comput Math Appl 58:1279–1286
Beaulaa T, Gunaseelib C (2014) On fuzzy soft metric spaces. Malaya J Mat 44(8–9):197–202
Chen D (2005) The parametrization reduction of soft sets and its applications. Comput Math Appl 49:757–763
Chou CC, Yih JM, Ding JF, Han TC, Lim YH, Liu LJ, Hsu WK (2012) Application of a fuzzy EOQ model to the stock management in the manufacture system. Key Eng Mater 499:361–365
Eksin C, Güzelkaya M, Yesil E, Eksin I (2008) Fuzzy logic approach to mimic decision making behaviour of humans in stock management game. In: Proceedings of the 2008 System Dynamics Conference
Ittanagi BM (2014) Soft bitopological spaces. Int J Comput Appl 107(7):1–4
Kandil A, Tantawy OAE, El-Sheikh SA, Hazza SA (2016) Pairwise open (closed) soft sets in soft bitopological spaces. Ann Fuzzy Math Inf 11(4):1–20
Kandil A, Tantawy OAE, El-Sheikh SA, Hazza SA (2017) Pairwise soft separation axioms in soft bitopological spaces. Ann Fuzzy Math Inf 13(5):563–577
Kelly JC (1963) Bitopological spaces. Proc. London Math Soc 13:71–81
Mahanta J, Das PK (2012) Results on fuzzy soft topological spaces. arXiv:1203.0634v1
Maji PK, Biswas R, Roy AR (2001) Fuzzy soft sets. J Fuzzy Math 9(3):589–602
Maji PK, Biswas R, Roy R (2003) Soft set theory. Comput Math Appl 45:555–562
Molodtsov DA (1999) Soft set theory-first results. Comput Math Appl 37:19–31
Özgür NY, Taş N (2015) A note on application of fuzzy soft sets to investment decision making problem. J New Theory 1(7):1–10
Öztürk TY, Karademir M (2017) Soft pair-wise b-continuity on soft bitopological spaces. Celal Bayar Univ J Sci 13(2):413–422
Mukherjee P, Park C (2015) On fuzzy soft bitopological spaces. Math Comput Sci J 10(7):1–8
Patty CW (1967) Bitopological spaces. Duke Math J 34:387–392
Reilly IL (1972) On bitopological separation properties. Nanta Math 29:14–25
Revathi N, Bageerathi K (2015) On soft B-open sets in soft bitopological space. Int J Appl Res 1(11):615–623
Revathi N, Bageerathi K (2017) (1, 2)-soft b-continuous and (1, 2)*-soft b-closed map. Asian J Math Comput Res 17(2):111–122
Rodabaugh SE (1999) Powerset operator foundations for poslat fuzzy set theories and topologies. In: Hhlc U, Rodabaugh SE (eds) Chapter 2 in mathematics of fuzzy sets: logic, topology and measure theory. Kluver Academic Publishers, New York, pp 91–116
Roy S, Samanta TK (2012) A note on fuzzy soft topological spaces. Ann Fuzzy Math Inf 3(2):305–311
Sayed AF (2017) On characterizations of some types of fuzzy soft sets in fuzzy soft bitopological spaces. J Adv Math Comput Sci 24(3):1–12
Sayed AF (2018) Some separation axioms in fuzzy soft bitopological spaces. J Math Comput Sci (JMCS) 8(1):28–45
Sayed AF (2020) On fuzzy soft b-open sets in fuzzy soft bitopological space. J Math Comput Sci 21:31–44
Taş NA, Aıkgöz A (2014) Some mixed soft operations and extremely soft disconnectedness via two soft topologies. Appl Math 5:490–500
Yüksel SY, Dizman T, Yildizdan G, Sert Ü (2013) Application of soft sets to diagnose the prostate cancer risk. J Inequal Appl 2013:229
Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353

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