Application of Four-Point Newton-EGSOR iteration for the numerical solution of 2D Porous Medium Equations

J V L Chew¹ and J Sulaiman²
¹,²Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400, Kota Kinabalu, Sabah, Malaysia

E-mail: jackelchew93@gmail.com

Abstract. Partial differential equations that are used in describing the nonlinear heat and mass transfer phenomena are difficult to be solved. For the case where the exact solution is difficult to be obtained, it is necessary to use a numerical procedure such as the finite difference method to solve a particular partial differential equation. In term of numerical procedure, a particular method can be considered as an efficient method if the method can give an approximate solution within the specified error with the least computational complexity. Throughout this paper, the two-dimensional Porous Medium Equation (2D PME) is discretized by using the implicit finite difference scheme to construct the corresponding approximation equation. Then this approximation equation yields a large-sized and sparse nonlinear system. By using the Newton method to linearize the nonlinear system, this paper deals with the application of the Four-Point Newton-EGSOR (4NEGSOR) iterative method for solving the 2D PMEs. In addition to that, the efficiency of the 4NEGSOR iterative method is studied by solving three examples of the problems. Based on the comparative analysis, the Newton-Gauss-Seidel (NGS) and the Newton-SOR (NSOR) iterative methods are also considered. The numerical findings show that the 4NEGSOR method is superior to the NGS and the NSOR methods in terms of the number of iterations to get the converged solutions, the time of computation and the maximum absolute errors produced by the methods.

1. Introduction

Partial differential equations (PDEs) have made many researchers interested in studying complex natural phenomena. Porous medium equation, for instance, is known as one of the important nonlinear mathematical models that are used to describe heat and mass transfer in many areas such as analytical study of an imbibition phenomenon in two immiscible fluid flow through porous media [1], numerical investigation of an instability phenomenon arises during secondary oil recovery process [2], inverse heat conduction problem in determining the transient heat transfer coefficient from temperature measurements inside the casting [3] and so on. It is also used to describe the evolution of the scaled density of an ideal gas flowing isentropically in a homogeneous porous medium [4].

Due to the nonlinearity on the solution function, a porous medium equation can be difficult to be solved. In the case where the exact solution is almost impossible to be obtained, it is necessary to use a numerical procedure such as the finite differences method as an alternative solver. Then, in order to determine the efficiency of a particular numerical method, the capability of the method to give an approximate solution within the specified error with the least computational complexity becomes the main subject of study.
In the context of increasing the iteration convergence rate, there are several iterative methods have been proposed and one of the iterative methods is called the Explicit Group (EG) iterative method. This EG method, which was introduced by Evans [5], uses small fixed size groups of mesh point strategy in order to reduce the computational iteration in solving PDEs. The combination of EG scheme together with efficient iterative methods such as Successive Over Relaxation (SOR) and Modified Successive Over Relaxation (MSOR) namely EGSOR and EGMSOR respectively have shown promising results in solving a wide class of PDEs and since then, both methods were extensively investigated by several researchers [6-9].

This paper attempts to examine the efficiency of the Newton-EGSOR method by considering particularly the Four-Point Newtont-EGSOR (4NEGSOR) iteration for the numerical solution of 2D PMEs based on the use of the second-order implicit finite difference scheme. To do this, the efficiency of 4NEGSOR is investigated by studying the required number of iterations, the time of computation and the maximum absolute errors against the other two tested iterations namely, Newton-Gauss-Seidel (NGS) and Newton-SOR (NSOR) [10]. This paper considers the following general form of 2D PME [11]:

\[
\frac{\partial u}{\partial t} = \alpha \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + m u^{m-1} \left( \frac{\partial u}{\partial x} \right)^2 + m u^{m-1} \left( \frac{\partial u}{\partial y} \right)^2 \right]
\]

where \(m\) and \(\alpha\) can be any rational numbers. The solution function in the equation (1) considers the domain \(0 \leq x, y, t \leq 1\) and is denoted as \(u(x, y, t)\). Equation (1) is solved subject to the initial condition \(u(x, y, 0)\) and boundary conditions \(u(0, y, t), u(1, y, t), u(x, 0, t)\) and \(u(x, 1, t)\). As taking \(m > 1\), equation (1) will be uniformly parabolic in any region, where \(u\) is bounded away from zero and will degenerate in the neighborhood of any point where \(u = 0\). While \(m \leq 1\) equation (1) is called fast diffusion and the speed of propagation is infinite and it is well known for the classical equation of heat conduction when \(m = 1\) [4].

This paper is outlined as follows. In Section 2, the discretization of 2D PME is shown. Then, in Section 3, we derive the 4NEGSOR iteration. All numerical experiments are presented and discussed in Section 4. The final section presents the overall findings and conclusion.

2. Finite difference approximation equation

To formulate the finite difference approximation equation, let equation (1) be expanded into

\[
\frac{\partial u}{\partial t} = \alpha \left[ u^{m} \frac{\partial^2 u}{\partial x^2} + m u^{m-1} \left( \frac{\partial u}{\partial x} \right)^2 + u^{m} \frac{\partial^2 u}{\partial y^2} + m u^{m-1} \left( \frac{\partial u}{\partial y} \right)^2 \right]
\]

Before constructing the approximation equation of problem (2), let the solution function \(u(x, y, t)\) be discretized at the fixed distance on the spatial-temporal grid \(u(ih, jh, nk)\) with \(1 \leq i, j \leq M - 1, 0 < n < T\). Here, \(h\) and \(k\) are denoted as the spatial and temporal steps respectively. By replacing the derivative terms in problem (2) with the second order implicit finite difference operators, the finite difference approximation equation can be written as

\[
\begin{align*}
    f(u_{i,j,n+1}) &= u_{i,j,n+1} - A_1 u_{i,j,n+1}^m + A_2 u_{i,j,n+1}^{m-1} - A_3 u_{i,j,n+1}^{m+1} - A_4 u_{i,j,n+1}^{m+2} - u_{i,j,n+1} \\
    &= -2u_{i,j,n+1}^m + u_{i+1,j,n+1}^m + u_{i-1,j,n+1}^m + u_{i,j,n+1}^{m+1} + u_{i+1,j,n+1}^m + u_{i-1,j,n+1}^m + u_{i,j,n+1}^{m+2} - u_{i,j,n+1} \\
    &= A_1 \left( u_{i,j,n+1}^m - u_{i,j,n+1}^{m+1} \right) \left( u_{i+1,j,n+1}^m - u_{i-1,j,n+1}^m \right) - u_{i,j,n+1}
\end{align*}
\]

where \(A_1 = \alpha k / (h^2), A_2 = \alpha k / 4(h^2)\) with \(k = 1/T\) and \(h = 1/M\).
By considering the two-dimensional solution domain of problem (2), the iteration implementation of equation (3) can be done by imposing into each interior grid point over the solution domain, in which the grid points are linked to each other to both x- and y-directions. As a result, this implementation produces a large-sized and sparse system of nonlinear equations in the form of

\[ F\left(U_{n+1}\right) = 0 \]  

where \( F\left(U_{n+1}\right) = \left(f_{1,1}, f_{2,1}, \ldots, f_{1,2}, f_{2,2}, \ldots, f_{M-1,M-1}\right)' \) and

\[ U_{n+1} = \left(u_{1,1,n+1}, u_{2,1,n+1}, \ldots, u_{1,2,n+1}, u_{2,2,n+1}, \ldots, u_{M-1,M-1,n+1}\right). \]

### 3. Four-Point Newton-EGSOR iteration

There are various numerical methods that can be applied for an efficient solution to equation (4). One of them is the Newton-GS (NGS) iterative method which was introduced by Ortega and Rheinboldt [10]. Basically, this method linearizes a nonlinear system by using Newton method and then uses the GS iteration to solve the linearized system. They have also suggested that when an optimum relaxation parameter of \( \alpha < 2 \) is used, the rate of convergence in solving the linearized system can be accelerated. Furthermore, global convergence theorems of the NGS iteration and its related methods have been studied by Moré [13].

As mentioned previously, this paper proposes a 4NEGSOR iteration for the numerical solution of 2D PMEs. Firstly, Newton method is applied to equation (4) in order to transform it into the corresponding linear system as follows.

\[ J_F\left(U_{n+1}\right) \Delta H_{n+1} = \left(\frac{\partial F}{\partial U_{n+1}}\right) \Delta U_{n+1} = 0, \quad \ell = 1, 2, \ldots \]  

where \( J_F\left(U_{n+1}\right) \) is a \( (M-1)^2 \times (M-1)^2 \) penta-diagonal Jacobian matrix, while \( \Delta H_{n+1} \) and \( \Delta U_{n+1} \) are the column matrices with iterative index \( \ell \) and \( n+1 \) is the time level.

Now let a four-point group be selected from a sparse and large linear system as shown in equation (5). As taking \( M = p^2, \ p \geq 1 \), the solution domain will have several completed groups of four-points and ungroup points. According to Evans [5], the ungrouped points can be treated as a group of two points and/or point iteration schemes. Based on any completed group of four-points in the solution domain of problem (2), the general scheme of the Four-Point EG iterative method can be shown in the following equation.

\[
\begin{bmatrix}
\Delta h_{i,j} \\
\Delta h_{i+1,j} \\
\Delta h_{i,j+1} \\
\Delta h_{i+1,j+1}
\end{bmatrix}
^{(\ell+1)} =
\begin{bmatrix}
c_{i,j} & d_{i,j} & e_{i+1,j} \\
b_{i+1,j} & c_{i+1,j} & d_{i+2,j} \\
a_{i,j+1} & b_{i+1,j+1} & c_{i+1,j+1}
\end{bmatrix}
^{-1}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4
\end{bmatrix}
\]  

where

\[
S_1 = f_{i,j} - a_{i,j} \Delta h_{i,j} - b_{i,j} \Delta h_{i+1,j} - c_{i,j+1} \Delta h_{i,j+1}
\]

\[
S_2 = f_{i+1,j} - a_{i+1,j} \Delta h_{i+1,j} - b_{i+1,j} \Delta h_{i+2,j} - c_{i+1,j+1} \Delta h_{i+1,j+1}
\]

\[
S_3 = f_{i,j+1} - a_{i,j+1} \Delta h_{i,j+1} - b_{i,j+1} \Delta h_{i+1,j+1} - c_{i+1,j+1} \Delta h_{i+1,j+1}
\]

\[
S_4 = f_{i+1,j+1} - a_{i+1,j+1} \Delta h_{i+1,j+1} - b_{i+1,j+1} \Delta h_{i+2,j+1} - c_{i+1,j+2} \Delta h_{i+1,j+2}
\]
Then, by adding one relaxation parameter, $\omega$ over equation (6), the Four-Point EGSOR iteration can be derived as

$${\begin{bmatrix}
\Delta h_{i,j} \\
\Delta h_{i+1,j} \\
\Delta h_{i,j+1} \\
\Delta h_{i+1,j+1}
\end{bmatrix}}^{(t+1)} = (1-\omega) {\begin{bmatrix}
\Delta h_{i,j} \\
\Delta h_{i+1,j} \\
\Delta h_{i,j+1} \\
\Delta h_{i+1,j+1}
\end{bmatrix}}^{(t)} + \omega \begin{bmatrix}
c_{i,j} & d_{i,j} & e_{i+1,j} \\
b_{i+1,j} & c_{i+1,j} & d_{i,j+1} \\
a_{i,j+1} & b_{i+1,j+1} & c_{i+1,j+1}
\end{bmatrix}{\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4
\end{bmatrix}}^{(t)}$$

(7)

Based on equation (7), the iteration process can be summarized into Algorithm 1. Algorithm 1 is executed several times in order to find the optimum value of $\omega$. The optimum value of $\omega$ is determined when the least number of iterations is obtained. The convergence test in Algorithm 1 considered a tolerance error of $\varepsilon = 1 \times 10^{-10}$.

**Algorithm 1: 4NEGSOR iteration**

1. Calculate the values for $A_1, A_2, A_3,$ and $A_4$.
2. Calculate the values for $-F(U^{(t)})$ and $J_F(U^{(t)})$.
3. Iterate equation (7).
4. If $|\Delta H^{(t)} - \Delta H^{(t)}| < \varepsilon$, calculate the solution vectors $U^{(t)} = \frac{\Delta H^{(t)}}{\Delta H^{(t)}} + U^{(t)}$.
5. If $|U^{(t)} - U^{(t)}| < \varepsilon$, go to the next temporal step.
6. Display approximate solutions.

4. Numerical experiments

To examine the efficiency of 4NEGSOR iteration, three parameters such as the number of iterations ($t$), the time of computation that is measured in seconds ($s$) and the maximum absolute errors (error) will be considered. NSOR and NGS iteration are also used for the comparative analysis. Three 2D PME examples are used in the numerical experiments as follows:

**Example 1 [11]:**

$$\frac{\partial u}{\partial t} = 0.2 \bigg[ \frac{\partial}{\partial x} \bigg( u \frac{\partial u}{\partial x} \bigg) + \frac{\partial}{\partial y} \bigg( u \frac{\partial u}{\partial y} \bigg) \bigg]$$

(8)

The exact solution: $u(x, y, t) = x + y + (0.4)t$.

**Example 2 [11]:**

$$\frac{\partial u}{\partial t} = 0.2 \bigg[ \frac{\partial}{\partial x} \bigg( u^2 \frac{\partial u}{\partial x} \bigg) + \frac{\partial}{\partial y} \bigg( u^2 \frac{\partial u}{\partial y} \bigg) \bigg].$$

(9)

The exact solution: $u(x, y, t) = (5(x + y + t))^{1/2}$.

**Example 3 [14]:**

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \bigg( u^x \frac{\partial u}{\partial x} \bigg) + \frac{\partial}{\partial y} \bigg( u^y \frac{\partial u}{\partial y} \bigg).$$

(10)

The exact solution: $u(x, y, t) = (4(x + y + 2t)/5)^{1/4}$.
Throughout the numerical experiments, the different values of grid size, $M$ to be considered are 16, 32, 64, 128 and 256. The numerical results from the three tested iterations on the three 2D PME examples are tabulated in Table 1. Through these numerical results shown in Table 1, the efficiency between the 4NEGSOR and the NSOR iterations has been analyzed by calculating the reduction percentage in the number of iterations and the time of computation over the NGS iteration based on the following expressions.

$$\left( \Delta I \right)\% = \left| \frac{I_{NGS} - I_{propose}}{I_{NGS}} \right| \times 100\%,$$

$$\left( \Delta s \right)\% = \left| \frac{s_{NGS} - s_{propose}}{s_{NGS}} \right| \times 100\%$$

(11)

5. Conclusion
This paper applied the 4NEGSOR iteration for the numerical solution of 2D PMEs based on the implicit finite difference scheme. From the numerical results shown in Table 1 and the reduction percentages, the findings show that a decrement in number of iterations approximately 52.31%-97.56% and 23.08%-96.48% correspond to the 4NEGSOR and the NSOR against the NGS iteration. Similarly, the finding shows a decrement in the time of computation about 28.57%-96.33% and 20.00%-96.02% correspond to the 4NEGSOR and NSOR against the NGS iteration. In addition to these findings, the accuracy by the numerical solutions via the 4NEGSOR iteration is comparable to the NSOR and the NGS iterations. In future, this work will investigate the efficiency of the usage of two different relaxation parameters or known as the MSOR iterative method [15].

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Table 1. Comparison of the number of iterations ($\ell$), the time of computation in seconds ($s$) and the maximum absolute error (error) for three examples.

| M   | Method | Example 1          | Example 2          | Example 3          |
|-----|--------|---------------------|---------------------|---------------------|
| 16  | NGS    | $\omega$ 1.00 $\ell$ 136 $s$ 0.35 $\text{error}$ 8.87E-11 | $\omega$ 1.00 $\ell$ 130 $s$ 0.47 $\text{error}$ 7.57E-11 | $\omega$ 1.00 $\ell$ 739 $s$ 0.82 $\text{error}$ 1.10E-09 |
|     | NSOR   | $\omega$ 1.19 $\ell$ 100 $s$ 0.28 $\text{error}$ 3.93E-12 | $\omega$ 1.16 $\ell$ 100 $s$ 0.30 $\text{error}$ 2.58E-11 | $\omega$ 1.55 $\ell$ 291 $s$ 0.50 $\text{error}$ 2.84E-11 |
|     | 4NEGSOR | $\omega$ 1.12 $\ell$ 64 $s$ 0.25 $\text{error}$ 1.31E-12 | $\omega$ 1.15 $\ell$ 62 $s$ 0.28 $\text{error}$ 8.52E-13 | $\omega$ 1.50 $\ell$ 204 $s$ 0.50 $\text{error}$ 5.42E-12 |
| 32  | NGS    | $\omega$ 1.00 $\ell$ 438 $s$ 2.35 $\text{error}$ 2.97E-10 | $\omega$ 1.00 $\ell$ 400 $s$ 1.96 $\text{error}$ 2.31E-09 | $\omega$ 1.00 $\ell$ 2630 $s$ 8.51 $\text{error}$ 6.69E-09 |
|     | NSOR   | $\omega$ 1.43 $\ell$ 207 $s$ 1.43 $\text{error}$ 1.59E-11 | $\omega$ 1.37 $\ell$ 208 $s$ 1.48 $\text{error}$ 1.26E-10 | $\omega$ 1.74 $\ell$ 573 $s$ 2.92 $\text{error}$ 3.97E-11 |
|     | 4NEGSOR | $\omega$ 1.32 $\ell$ 131 $s$ 1.32 $\text{error}$ 7.28E-12 | $\omega$ 1.34 $\ell$ 124 $s$ 1.40 $\text{error}$ 2.11E-11 | $\omega$ 1.62 $\ell$ 395 $s$ 2.71 $\text{error}$ 2.89E-11 |
| 64  | NGS    | $\omega$ 1.00 $\ell$ 1526 $s$ 19.77 $\text{error}$ 1.88E-09 | $\omega$ 1.00 $\ell$ 1380 $s$ 19.26 $\text{error}$ 1.31E-08 | $\omega$ 1.00 $\ell$ 9478 $s$ 113.63 $\text{error}$ 3.56E-08 |
|     | NSOR   | $\omega$ 1.64 $\ell$ 412 $s$ 8.47 $\text{error}$ 5.83E-11 | $\omega$ 1.60 $\ell$ 420 $s$ 8.68 $\text{error}$ 2.97E-10 | $\omega$ 1.86 $\ell$ 1118 $s$ 18.86 $\text{error}$ 5.69E-11 |
|     | 4NEGSOR | $\omega$ 1.56 $\ell$ 262 $s$ 7.47 $\text{error}$ 1.69E-11 | $\omega$ 1.54 $\ell$ 255 $s$ 8.57 $\text{error}$ 7.12E-11 | $\omega$ 1.77 $\ell$ 780 $s$ 14.48 $\text{error}$ 4.70E-11 |
| 128 | NGS    | $\omega$ 1.00 $\ell$ 5459 $s$ 258.55 $\text{error}$ 8.96E-09 | $\omega$ 1.00 $\ell$ 4901 $s$ 248.82 $\text{error}$ 4.95E-08 | $\omega$ 1.00 $\ell$ 34098 $s$ 1653.85 $\text{error}$ 1.72E-07 |
|     | NSOR   | $\omega$ 1.80 $\ell$ 813 $s$ 55.72 $\text{error}$ 8.16E-11 | $\omega$ 1.77 $\ell$ 836 $s$ 56.86 $\text{error}$ 5.66E-10 | $\omega$ 1.93 $\ell$ 2178 $s$ 131.63 $\text{error}$ 6.65E-11 |
|     | 4NEGSOR | $\omega$ 1.74 $\ell$ 522 $s$ 48.45 $\text{error}$ 2.70E-11 | $\omega$ 1.72 $\ell$ 505 $s$ 51.68 $\text{error}$ 3.69E-10 | $\omega$ 1.88 $\ell$ 1525 $s$ 114.26 $\text{error}$ 4.06E-11 |
| 256 | NGS    | $\omega$ 1.00 $\ell$ 19388 $s$ 4275.66 $\text{error}$ 3.85E-08 | $\omega$ 1.00 $\ell$ 17458 $s$ 4243.79 $\text{error}$ 1.75E-07 | $\omega$ 1.00 $\ell$ 121649 $s$ 29234.80 $\text{error}$ 7.78E-07 |
|     | NSOR   | $\omega$ 1.89 $\ell$ 1608 $s$ 454.53 $\text{error}$ 1.44E-10 | $\omega$ 1.87 $\ell$ 1658 $s$ 478.09 $\text{error}$ 1.50E-09 | $\omega$ 1.96 $\ell$ 4276 $s$ 1163.89 $\text{error}$ 6.26E-11 |
|     | 4NEGSOR | $\omega$ 1.86 $\ell$ 1020 $s$ 416.83 $\text{error}$ 3.98E-11 | $\omega$ 1.84 $\ell$ 992 $s$ 455.23 $\text{error}$ 1.28E-09 | $\omega$ 1.94 $\ell$ 2971 $s$ 1073.27 $\text{error}$ 4.91E-11 |