A Level Set Approach to Online Sensing and Trajectory Optimization with Time Delays

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Abstract: Presented is a method to compute certain classes of Hamilton–Jacobi equations that result from optimal control and trajectory generation problems with time delays. Many robotic control and trajectory problems have limited information of the operating environment a priori and must continually perform online trajectory optimization in real time after collecting measurements. The sensing and optimization can induce a significant time delay, and must be accounted for when computing the trajectory. This paper utilizes the generalized Hopf formula, which avoids the use of grids and numerical gradients that is typical of other methods for computing solutions to the Hamilton–Jacobi equation, which suffer exponential dimensional scaling. We present as an example a robot that incrementally predicts a communication channel from measurements as it travels. As part of this example, we introduce a seemingly new generalization of a non-parametric formulation of robotic communication channel estimation. New communication measurements are used to improve the channel estimate and online trajectory optimization with time-delay compensation is performed.

Keywords: Time Delay Systems, Hamilton–Jacobi Equation, Generalized Hopf Formula, Viscosity Solution, Optimal Control, Communication Seeking Robotics

1. INTRODUCTION

Time delays are common presence in real-world instantiations of dynamic systems. This causes issues with stability and robustness when attempting to design automatic or real-time control for these systems, and the challenges of design and analysis of systems with time delays have been well studied; see Richard (2003). Historically, there have been many attempts to compensate for time delays in control theory, such as the well-known Padé approximation in classical linear control theory (Franklin et al., 2006, Sec. 5.7.3), which approximates a pure delay as a rational transfer function or state transformations such as presented in Kwon and Pearson (1980). In a modern setting, real-time optimal control (RTOC) [Ross et al. (2006)] and model predictive control (MPC) [Camacho and Alba (2013)] have gained large scale acceptance in control and trajectory optimization problems. These seek to find a control sequence and the resulting trajectory that minimizes a pre-defined cost functional. RTOC optimizes the cost functional directly, while MPC computes on a smaller finite time horizon and is frequently referred to as receding horizon control. Because of this, MPC is sub-optimal and necessitates online re-computation of the current state frequently, while RTOC only needs to be re-computed online if the system is perturbed from the computed optimal trajectory.

Fast online computation is critical for these approaches, and delay from computation becomes more apparent as the dimensionality and model complexity of the optimization increases. In addition to time delays induced from computation, many robotics problems have limited information of the operating environment a priori. The robot must perform sensing in real-time and as a result, alters the cost functional for the MPC or RTOC problem, and necessitates online re-computation of the optimal trajectory with this new information. As methods for sensing in robotics have become more sophisticated, the time delay induced becomes larger. An illustrative example appeared in Usman et al. (2016), where a complicated non-parametric model was used to estimate the quality of the previous unknown communication signal from measurements of a transmitter with a known location. In that work, a RTOC scheme was developed to minimize combined motion and communication energy of the signal. The RTOC was re-computed every 10 seconds and it was noted that the combined sensing and trajectory optimization was around 2 seconds, or 20% of the entire compute interval. A delay this large can have drastic consequences if not accounted for. This was not addressed in Usman et al. (2016).

As noted in Lu (2008), attempts to directly account for time delays in MPC formulations are limited, with the most notable proposed in Kwon et al. (2004) for linear systems. However this work is restricted to a specific LQR cost functional and does not account for control saturation.
We present in this paper a more general approach based on Hamilton–Jacobi theory, which provides a natural way to deal with pure time delay in the control.

Generally, solutions to optimal control and trajectory problems can be found by solving a Hamilton–Jacobi (HJ) partial differential equation (PDE) as it establishes a sufficient condition for optimality [Osmolovskii (1998)]. Traditionally, numerical solutions to HJ equations require a dense, discrete grid of the solution space as in Osher and Fedkiw (2006); Mitchell (2008); Mitchell et al. (2005). Computing the elements of this grid scales poorly with dimension and has limited use for problems with dimension greater than four. The exponential dimensional scaling in optimization is sometimes referred to as the “curse of dimensionality” [Bellman (2015, 1957)].

A new result in Darbon and Osher (2016) discovered numerical solutions based on the Hopf formula [Hopf (1965)] that do not require a grid and can be used to efficiently compute solutions of a certain class of Hamilton–Jacobi PDEs. However, that only applied to systems with time-independent Hamiltonians of the form \( \dot{x} = f(u(t)) \), and has limited use for general linear control problems. Recently, the classes of systems was expanded upon and generalizations of the Hopf formula were used to solve optimal linear control problems in high-dimensions in Kimche et al. (2018a) and differential games as applied to a multi-vehicle, collaborative pursuit-evasion problem in Kirchner et al. (2018b).

Before proceeding we introduce some notation and assumptions. We consider linear dynamics

\[
\frac{d}{ds}x(s) = A x(s) + B a(s), \quad s \in [0,t],
\]

where \( x \in \mathbb{R}^n \) is the system state and \( a(s) \in \mathcal{A} \subset \mathbb{R}^m \) is the control input, constrained to the convex admissible control set \( \mathcal{A} \). We let \( \gamma(s; x, a(\cdot)) \in \mathbb{R}^n \) denote a state trajectory that evolves in time, \( s \in [0,t] \), with control input sequence \( a(\cdot) \in \mathcal{A} \) according to (1) starting from initial state \( x = 0 \).

The trajectory \( \gamma \) is a solution of (1) in that it satisfies (1) almost everywhere:

\[
\frac{d}{ds} \gamma(s; x, a(\cdot)) = A \gamma(s; x, a(\cdot)) + B a(\cdot),
\]

\( \gamma(0; x, a(\cdot)) = x. \)

We construct a cost functional for \( \gamma(s; x, a(\cdot)) \), given terminal time \( t \) as

\[
R(t, x, a(\cdot)) = \int_0^t C(s, x, a(s)) ds + J(\gamma(t; x, a(\cdot))),
\]

where the function \( C : (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \} \) is the running cost and represents the rate that cost is accrued over time. The value function \( v : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R} \) is defined as the minimum cost, \( R \), among all admissible controls for a given state \( x \) with

\[
v(x, t) = \inf_{a(\cdot) \in \mathcal{A}} R(t, x, a(\cdot)).
\]

The value function in (3) satisfies the dynamic programming principle [Bryson and Ho (1975); Evans (2010)] and also satisfies the following initial value Hamilton–Jacobi (HJ) equation by defining the function \( \varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) as \( \varphi(x, s) = v(x, t - s) \), with \( \varphi \) being the viscosity solution of

\[
\begin{aligned}
\frac{\partial \varphi(x, s)}{\partial s} (x, s) + H(s, x, \nabla_x \varphi(x, s)) &= 0, \\
\varphi(x, 0) &= J(x),
\end{aligned}
\]

where the Hamiltonian \( H : (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{ +\infty \} \) is defined by

\[
H(s, x, p) = \sup_{a(\cdot) \in \mathbb{R}^m} \{ -f(s, x, a(\cdot), p) - C(s, x, a(\cdot)) \}.
\]

The variable \( p \) in (5) denotes the costate, which in the HJ equation (4) is associated with the gradient of the value function. We denote by \( \lambda(s; x, a(\cdot)) \) the costate trajectory that satisfies almost everywhere:

\[
\begin{aligned}
\frac{d}{ds} \lambda(s; x, a(\cdot)) &= \nabla_x f(g(s; x, a(\cdot)), s) \lambda(s; x, a(\cdot)), \\
\lambda(t; x, a(\cdot)) &= \nabla_x J(g(t; x, a(\cdot), s)),
\end{aligned}
\]

for all \( s \in [0,t] \) with initial costate denoted by \( \lambda(0; x, a(\cdot)) = p \). With a slight abuse of notation, we will hereafter use \( \lambda(s) \) to denote \( \lambda(s; x, a(\cdot)) \), since the initial state and control sequence can be inferred through context with the corresponding state trajectory, \( \gamma(s; x, a(\cdot)) \).

### 2.1 Viscosity Solutions with the Hopf Formula

Consider simplified system dynamics represented as

\[
\frac{d}{ds} x(s) = f(a(s)).
\]

The associated HJ equation no longer depends on state and is given as

\[
\begin{aligned}
\frac{\partial \varphi(x, s)}{\partial s} + H(\nabla_x \varphi(x, s)) &= 0, \\
\varphi(x, 0) &= J(x).
\end{aligned}
\]

When \( J(x) \) is convex and continuous in \( x \), and \( H(p) \) is continuous in \( p \), it was shown in Evans (2010) that an
exact, point-wise viscosity solution to (7) can be found using the Hopf formula \cite{Hopf1965}:

\[ \varphi(x,t) = -\min_{p \in \mathbb{R}^n} \{ J^*(p) + tH(p) - \langle x, p \rangle \}, \tag{8} \]

with the Fenchel–Legendre transform, denoted \( J^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), defined for a convex, proper, lower semicontinuous function \( J : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) \cite{Hiriart-Urruty2012} as

\[ J^*(p) = \sup_{x \in \mathbb{R}^n} \{ \langle p, x \rangle - J(x) \}. \tag{9} \]

The transform defined in (9) is also referred to in literature as the convex conjugate.

Proceeding similar to Kirchner et al. (2018b), we can generalize the Hopf formula to (1) by making a change of variables

\[ z(s) = e^{-sA}x(s), \tag{10} \]

which results in the following system

\[ \frac{d}{ds}z(s) = f(s, \alpha(s)) = e^{-sA}B\alpha(s). \tag{11} \]

The terminal cost function is now defined in \( z \) with

\[ \varphi(z,0) = J(e^{tA}z). \tag{12} \]

For clarity in the sections to follow, we use the notation \( \hat{H} \) to refer to the Hamiltonian for systems defined by (11) and \( H \) for systems defined by (17). Notice that the system (11) does not depend on state but is now time-varying. It was shown in \cite[(1984, Section 5.3.2, p. 215)]{Kurzhanski2014} that the Hopf formula can be generalized for a time-dependent Hamiltonian to solve for the value function of the system in (11) with

\[ \varphi(x,t) = -\min_{p \in \mathbb{R}^n} \left\{ J^*(e^{-tA^\top}p) + \int_0^t \hat{H}(s,p) ds - \langle x, p \rangle \right\}, \tag{13} \]

with Hamiltonian defined by

\[ \hat{H}(s,p) = \sup_{\alpha \in \mathbb{R}^m} \{ \langle -e^{-sA}B\alpha, p \rangle - C(s,\alpha) \}. \tag{14} \]

For the remainder of the paper, we consider a time-optimal formulation in which \( \hat{H} \) is defined as

\[ \hat{H}(s,p) = \sup_{\alpha \in \mathbb{R}^m} \{ \langle -e^{-sA}B\alpha, p \rangle - I_\mathcal{A}(\alpha) \}, \tag{15} \]

where \( I_\mathcal{A} : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \) is the indicator function for the set \( \mathcal{A} \) and is defined by

\[ I_\mathcal{A}(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \mathcal{A} \\ +\infty & \text{otherwise}. \end{cases} \]

Suppose \( \mathcal{A} \) is a closed convex set such that \( 0 \in \text{int} \mathcal{A} \), where \( \text{int} \mathcal{A} \) denotes the interior of the set \( \mathcal{A} \). Then \( (I_\mathcal{A})^* \) defines a norm, which we denote with \( \|\cdot\|_{\mathcal{A}^*} \), which is the dual norm \cite{Hiriart-Urruty2012} to \( \|\cdot\|_{\mathcal{A}} \).

From this we can write (14) in general as

\[ \hat{H}(s,p) = \| -B^\top e^{-sA^\top}p \|_{\mathcal{A}^*}. \tag{16} \]

2.2 Time-Optimal Control to a Goal Set

Consider a goal set \( \Omega \subset \mathbb{R}^n \) and a task to determine the control that drives the system into \( \Omega \) in minimal time. We represent the set \( \Omega \) as an implicit surface with cost function \( J : \mathbb{R}^n \to \mathbb{R} \) such that

\[ \begin{cases} J(x) < 0 & \text{for any } x \in \text{int} \Omega, \\ J(x) > 0 & \text{for any } x \in (\mathbb{R}^n \setminus \Omega), \\ J(x) = 0 & \text{for any } x \in (\Omega \setminus \text{int} \Omega), \end{cases} \]

where \( \text{int} \Omega \) denotes the interior of \( \Omega \). Note that if the goal is a point in \( \mathbb{R}^n \), then we can represent this by choosing \( \Omega \) as a ball with arbitrarily small radius. As noted in Kirchner et al. (2018a) we solve for the minimum time to reach the set \( \Omega \) by constructing a newton iteration, starting from an initial guess, \( t_0 \), with

\[ t_{i+1} = t_i - \frac{\varphi(x,t_i)}{\frac{\partial \varphi}{\partial t}(x,t_i)}. \tag{17} \]

where \( \varphi(x,t_i) \) is the solution to (13) at time \( t_i \). Notice the value function must satisfy the HJ equation

\[ \frac{\partial \varphi}{\partial t}(x,t_i) = -H(\nabla \varphi(x,t_i), x), \]

where \( \nabla \varphi(x,t_i) \) is the argument of the minimizer in (13), which we will denote as \( p^* \). No change of variable is needed for the Newton update and we have

\[ H(p^*, x) = -x^\top A^\top p^* + \| -B^\top p^* \|_\ast. \]

We iterate (16) until convergence at the optimal time to reach, which we denote as \( t^* \). The optimal control can be found directly from the necessary conditions of optimality established by Pontryagin’s principal \cite{Pontryagin2018} by noting the optimal trajectory, denoted as \( \gamma^* \), must satisfy

\[ \frac{d}{ds} \gamma^*(s;x,\alpha^*(\cdot)) = -\nabla_p H(\lambda^*(s)) = A\gamma(s,x,\alpha^*(\cdot)) + B\nabla_p \| -B^\top \lambda^*(s) \|_\ast, \]

where \( \lambda^* \) is the optimal costate trajectory and is given by

\[ \lambda^*(s) = e^{-sA^\top}p^*. \]

This implies our optimal control is

\[ \alpha^*(s) = \nabla_p \| -B^\top e^{-sA^\top}p^* \|_\ast, \]

for all time \( s \in [0,t^*] \), provided the gradient exists.

Note that in the above formulation, we compute a viscosity solution to (7) without constructing a discrete grid and this method can provide a numerical solution that is efficient to compute even when the state space is high-dimensional. Additionally, no derivative approximations are needed with Hopf formula-based methods, and this eliminates the numeric dissipation introduced with the Lax–Friedrichs scheme that is necessary to maintain stability in grid-based methods.

3. ONLINE TRAJECTORY OPTIMIZATION WITH DELAY

We’ll be considering a RTOC framework for the rest of this work with a variable time horizon \( s \in [0,k^t] \), where each \( t^k \) is the optimal time-to-go as computed by Section 2.2. This can easily be applied to MPC problems by using a fixed, finite time horizon \( s \in [0,t] \). The superscript \( k \in \mathbb{N} \) is used to denote the computational update of each relevant quantity, with \( k = 0 \) being the first optimization. With this notation, \( x^k \) denotes our initial state when we initiate the
Fig. 1. Level set evolution of a double integrator system without time delay is shown in black. The level set computed of the double integrator system with time delay is shown in red. Both were computed at 10 equally spaced times on the interval from zero to two seconds.

computation, and likewise we denote by \( \gamma^*(s; x^k, \alpha^k(\cdot)) \) as the \( k \)-th computed optimal trajectory. We re-compute the control online after \( \delta^k \) seconds of traveling on the trajectory \( \gamma^*(s; x^k, \alpha^k(\cdot)) \), which gives the new initial state for the next optimization as

\[
x^{k+1} = \gamma^*(\delta^k; x^k, \alpha^k(\cdot)).
\]

Using \( x^{k+1} \) as the new initial condition, we proceed to use the methods of Section 2.2 to compute \( \alpha^{k+1}(s) \). Note that the time between updates, \( \delta^k \) need not be uniform. Now consider the following linear state space model with time delay

\[
d\frac{dx}{ds}(s) = Ax(s) + Bu(s - \tau^k),
\]

(17)

with delayed control input \( u^k \in \mathcal{A} \subset \mathbb{R}^m \), and \( \tau^k \) is the step-specific time delay. We can represent the delay dynamics in the same form as (1) by defining \( \alpha^k(\cdot) \), for \( k \neq 0 \), as

\[
\alpha^k(s) = \begin{cases} 
\alpha^k(s - \tau^k) & s \geq \tau \\
\alpha^{k-1}(s + \delta^{k-1}) & s < \tau.
\end{cases}
\]

We only consider causal systems and therefore assume each \( \tau^k \geq 0 \) and

\[
\alpha^0(s) = \begin{cases} 
u^0(s - \tau^0) & s \geq \tau \\
0 & s < \tau.
\end{cases}
\]

The Hamiltonian becomes

\[
\tilde{H}(s, p) = \begin{cases} 
-B^T e^{-sA^*} p & s \geq \tau \\
-p^T e^{-sA} B_\alpha k^{-1} \delta^{k-1} & s < \tau.
\end{cases}
\]

(18)

for solving the Hopf formula in (13), with \( p^k \) denoting the optimal initial costate for the update. Likewise we solve for the optimal time-to-reach of the delayed system using the methods of Section (2.2). The Hamiltonian for the Newton iteration becomes

\[
H(p^k, x) = \begin{cases} 
x^T A^* p^k + \| -B^T p^k \|_A^* & s \geq \tau \\
x^T A^* p^k - (p^k)^T B_\alpha k^{-1} (s + \delta^{k-1}) & s < \tau.
\end{cases}
\]

(19)

This implies our control becomes

\[
\alpha^k(s) = \begin{cases} 
\nabla_p \| -B^T e^{-sA^*} p^k \| & s \geq \tau \\
\alpha^{k-1}(s + \delta^{k-1}) & s < \tau.
\end{cases}
\]

Figure 1 shows the zero level set of the value function at different times for a double integrator system with and without delay, illustrating the extreme effect time delays can have on control optimization problems.

4. AN EXAMPLE OF ONLINE SENSING AND TRAJECTORY OPTIMIZATION

We present as an example a robotic vehicle that uses a radio transmitter to communicate data to a remote base station. The goal is to plan a trajectory to deliver the robot to a location with the best possible communication performance, in minimum time. The location of base station is unknown a priori and hence the communication link performance needs to be spatially estimated. Initially, the robot will have only a sparse number of samples of the communication link, or perhaps none at all, and needs to determine the channel-to-noise ratio (CNR). As the robot moves, more samples are collected and the estimate is improved. Each time the estimate is improved, a new trajectory needs to be generated, since the original trajectory is no longer optimal under the previous estimate.

This is similar to the problem addressed in Usman et al. (2016), but that work attempted to minimize total power in a non-time-optimal fashion and did not account for time delays in the sensing and computation. We choose this example to demonstrate the method presented since communication link performance needs to be sensed and computed while in motion and the computation time for the channel estimate is non-trivial. In the case of Usman et al. (2016), 2 seconds of computation was required for every 10 second computation cycle.

4.1 Channel Estimation with Unknown Emitter Location

It was proposed in Malmirchegini and Mostofi (2012) to model the CNR (in dB) at a particular spatial location \( q \in \mathbb{R}^3 \) from \( \ell \) observed measurements as
determine in the sum of the respective covariance functions. To along shown path. (a) Channel estimate with 5 measurements along shown path.

The channel estimates from measurements as a vehicle traverses a known path. The vehicle path is shown in red and the ground-truth transmitter location shown as a black diamond. Best viewed in color.

\[
\Upsilon_{dB}(q) = \Gamma(q; q_b) + \Delta(q),
\]

where \(\Gamma(q; q_b)\) is a parametric model of path loss, relating CNR to spatial distance to the (known) transmitter location, \(q_b\), and is given as

\[
\Gamma(q; q_b) = c_{PL} - 10n_{PL} \log (\|q - q_b\|),
\]

where \(c_{PL}\) and \(n_{PL}\) are constant parameters associated properties of the transmitter. The quantity \(\Delta(q)\) represents the deviation of the true CNR from \(\Gamma\) and is modeled nonparametrically as a Gaussian process Rasmussen (2004) with

\[
\Delta(q_i) \sim \mathcal{GP}(0, k_\Delta(q_i, q_i)),
\]

for each \(q_i\), where \(k_\Delta\) is referred to as the covariance function for modeling communication channels was suggested in Malmirchegini and Mostofi (2012) to be

\[
k_\Delta(q_i, q_j) = \xi^2 e^{-\frac{\|q_i - q_j\|^2}{\ell^2}} + \sigma^2_p,
\]

for any \(i, j \in \{1, \ldots, \ell\}\) channel measurements. The idea of constructing a model as a composition of a known parametric model and a non-parametric deviation is not uncommon and is generally formulated as Gaussian process regression with explicit basis functions (Rasmussen, 2004, Chapter 2.7). We present the model while omitting a detailed derivation as this is outside the scope of this work. For more information, the reader is encouraged to review Rasmussen (2004) for a thorough and complete review of Gaussian process regression and Malmirchegini and Mostofi (2012) for the derivation of the proposed covariance functions for communication models.

The model (20) assumes a priori knowledge of the location of the transmitter, which for many robotic path planning problems may not be available at run-time. We propose to generalize (20) by considering the case where the transmitting location is an unknown random variable distributed with probability density function \(g(q_0)\) and model \(\Upsilon\) with a single Gaussian process with

\[
\Upsilon_{dB}(q) \sim \mathcal{GP}(0, k(q_i, q_j)),
\]

where

\[
k(q_i, q_j) = k_\Delta(q_i, q_j) + k_\Gamma(q_i, q_j),
\]

with \(k_\Gamma(q_i, q_j)\) denoting the covariance function associated to the path loss of the signal. The last equation follows from the fact that the sum of two kernels, i.e. (20), results in the sum of the respective covariance functions. To determine \(k_\Gamma(q_i, q_j)\), we follow Neal (1996) to get

\[
k_\Gamma(q_i, q_j) = E_{q_b} [\Gamma(q_i; q_b) \cdot \Gamma(q_j; q_b)] = \int_{\mathbb{R}^2} g(q_0) \Gamma(q_i; q_0) \Gamma(q_j; q_0) dq_0.
\]

Letting \(K_q = [k(q_i, q_1), k(q_i, q_2), \ldots, k(q_i, q_6)]^T\) and \(K\) a matrix with entries defined by \(K_{ij} = k(q_i, q_j)\), the posterior estimate of the channel at an unknown location \(q\) can be found with mean

\[
\hat{\Upsilon}_{dB}(q) = K_q^T (K + \sigma^2_P I)^{-1} y,
\]

with \(y = [y_1, y_2, \ldots, y_6]^T\) a vector of \(6\) measurements, and \(I\) the identity matrix of the appropriate size. The variance of the channel estimate is given by

\[
\Sigma(q) = k(q, q) - K_q^T (K + \sigma^2_P I)^{-1} K_q.
\]

5. RESULTS

The methods presented above was implemented in MATLAB R2018b on a laptop equipped with an Intel Core i7-7500 CPU running at 2.70 GHz. We used as values for the parameters for communication channels what was presented in Usman et al. (2016) with \(\sigma_p = 1.64, \xi = 3.20\) \(c_{PL} = -41.34, \eta = 3.09,\) and \(n_{PL} = 3.86\). While the time to compute could vary as many measurements are gathered, we force a fixed value of time delay for consistency. We also follow Usman et al. (2016) for these values with \(\tau_k = 2\) seconds and \(\delta_k = 10\) seconds for all \(k\).

5.1 Planar Motion Example

We choose for dynamics (17) with state \(x \in [q, \dot{q}]^T\), where \(q \in \mathbb{R}^2\) is spatial position of a robot and \(\dot{q} \in \mathbb{R}^2\) is the velocity and

\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

The control \(u \in \mathbb{R}^2\) is constrained to lie in the set \(\|u\| \leq 1\). Since the 2-norm is self-dual, the Hamiltonian (18) for this example is

\[
\mathcal{H}(s, p) = \begin{cases} \| -B^T e^{-sA^T} p \|_2 & s \geq \tau \\ -p^T e^{-sA} B A^T p & s < \tau \end{cases},
\]

and the Hamiltonian (19) for the Newton update becomes

\[
H(p^k, x) = -x^T A^T p^k + Q(p^k),
\]
iteration. The matrix that defines the shape of the neighborhood. For this example, we use with only a single sample of channel at the vehicle’s initial trajectory for 3 optimization cycles. The vehicle starts located at $b = \begin{bmatrix} 25 \\ 50 \end{bmatrix}$ meters, the scale length of $k_2$ kernel function. The vehicle guides to a goal set that is an ellipsoidal neighborhood around the peak of the estimated CNR by setting

$$\Omega^k = \{ x : \langle (x - \tilde{x}^k), W^{-1} (x - \tilde{x}^k) \rangle \leq 1 \},$$

where

$$\tilde{x}^k = \begin{bmatrix} \tilde{x}^k_1 \\ \tilde{x}^k_2 \end{bmatrix}$$

and $\tilde{x}^k_2$ is the peak of the estimated CNR at the $k$-ith iteration. The matrix $W$ is positive definite and defines the shape of the neighborhood. For this example, we use

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \max V \end{bmatrix},$$

with $\max V$ being the maximal allowable velocity at the goal. We consider a vehicle that comes to close to rest at the goal, so set $\max V = 0.1$. This implies the follow initial value function

$$J^k (x) = \langle (x - \tilde{x}^k), W^{-1} (x - \tilde{x}^k) \rangle - 1.$$
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