Low-lying excitations of a trapped rotating Bose-Einstein condensate

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We investigate the low-lying excitations of a weakly-interacting, harmonically-trapped Bose-Einstein condensate gas under rotation, in the limit where the angular momentum \( L \) of the system is much less than the number of the atoms \( N \) in the trap. We show that in the asymptotic limit \( N \rightarrow \infty \) the excitation energy, measured from the energy of the lowest state, is given by \( 27N_3(N_3 - 1)v_0/68 \), where \( N_3 \) is the number of octupole excitations and \( v_0 \) is the unit of the interaction energy.

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The behaviour of a trapped Bose-Einstein condensate under rotation has attracted much attention in the recent years. References [1,2] have dealt with this problem theoretically both in the Thomas-Fermi limit of strong interactions, as well as in the limit of weak interactions. Borrowed from the field of nuclear physics, the terminology “yrast state” refers to the state with the lowest excitation energy for a given angular momentum \( L \) that the system has. Of equal importance to the yrast state are the low-lying excitations, which determine the thermodynamic stability of the system.

In the present study we consider the hamiltonian

\[
H_0 = \sum_i \left[ \frac{\hbar^2}{2M} \nabla_i^2 + \frac{1}{2} M \omega^2 r_i^2 \right]
\]

includes the kinetic energy of the particles and the potential energy due to the trapping potential, where \( M \) is the atom mass, and \( \omega \) is the frequency of the harmonic confining potential, which we assume to be isotropic. Also the interaction between the particles \( V \) is assumed to be short-ranged,

\[
V = \frac{1}{2} U_0 \sum_{i \neq j} \delta(r_i - r_j),
\]

where \( U_0 = 4\pi\hbar^2 a/M \) is the matrix element for atom-atom collisions, with \( a \) being the scattering length. We consider the limit of weak interactions, where the typical interaction energy is much less than the typical oscillator quantum of energy,

\[
nU_0 \ll \hbar \omega,
\]

with \( n \) being the density of atoms. The above condition allows us to work in the subspace of single-particle states with no radial excitations,

\[
\Phi_m(r) = \frac{1}{(m! \pi a_0^3)^{1/2}} \left( \frac{\rho}{a_0} \right)^{|m|} e^{im\phi} e^{-(\rho^2 + z^2)/2a_0^2}.
\]

Here \( \rho, z, \) and \( \phi \) are cylindrical polar coordinates, and \( a_0 = (\hbar/M\omega)^{1/2} \) is the oscillator length. These states are degenerate in the absence of interactions (while states with radial excitations lie higher by an energy of order \( \hbar \omega \)). The whole problem thus reduces to incorporating the interactions between the atoms, which lift the degeneracy of the states.

One of us has determined in Ref. [6] the yrast line of a weakly-interacting Bose-Einstein condensate for an effective repulsion between the atoms. In the same reference the yrast line has also been determined for the case of an effective attraction between the atoms and for \( L \ll N \).

In Ref. [10] we have given a more detailed description of the yrast line within the mean-field Gross-Pitaevskii approximation for a wide range of values of the ratio \( L/N \).

If one studies the limit \( L \ll N \), as shown in Ref. [3], in a state with a \( 2\lambda \)-pole excitation,

\[
|\lambda\rangle = |(m = 0)^{N-1}, (m = \lambda)^1\rangle,
\]

where \( m \) is the state with angular momentum \( m\hbar \), the excitation (interaction) energy is

\[
e_\lambda = \langle \lambda | V | \lambda \rangle - \langle 0 | V | 0 \rangle = -Nv_0 \left( 1 - \frac{1}{2^{\lambda-1}} \right),
\]

where \( |0\rangle = |(m = 0)^N\rangle \) is the ground state, and \( v_0 = U_0/a_0^3 \). The above equation implies that the highest gain in the interaction energy per unit of angular momentum comes from the \( \lambda = 2 \) or 3 excitations, and therefore the quadrupole and octupole excitations are expected to carry the angular momentum for \( L \ll N \). As a result, in this limit there is a quasi-degeneracy between the low-lying states, as we discuss below in detail. By saying quasi-degenerate states, we mean that the energy separation between them is of order \( v_0 \), and not of order \( Nv_0 \).

Bertsch and Papenbrock [4] have examined the ground state of a weakly-interacting Bose-Einstein condensate under rotation numerically by diagonalizing the hamiltonian \( H \) in the subspace of degenerate states [4] for a given angular momentum. In Ref. [13] Nakajima and Ueda...
have performed similar numerical calculations in the limit where the angular momentum per particle \( L/N \) is much less than 1, and have found that the quasi-degenerate states which we mentioned in the previous paragraph lie above the yrast by an energy, which in the asymptotic limit \( N \to \infty \) is given by \( 1.59N_{v_0}(N_3 - 1)v_0/4 \), where \( N_3 \) is the number of octupole excitations.

In this study we give an analytical derivation of this result with use of a diagrammatic perturbation-theory approach. The starting point of our analysis is the fact that the quadrupole and octupole \( \lambda = 2, 3 \) excitations are dominant for \( L \ll N \), and we therefore assume that in this limit the angular momentum is carried by \( \lambda = 2 \) and \( \lambda = 3 \) excitations only. In addition, the condensate is dominated by atoms which do not have any angular momentum. Our approach therefore consists of considering a condensate with \( N_0 \) atoms in the state with \( m = 0 \), \( N_2 \) atoms in the state with \( m = 2 \), and \( N_3 \) in the state with \( m = 3 \), and then treating the other states perturbatively by keeping the appropriate diagrams and using perturbation theory to get the correction to the energy. The "bare" interaction energy with particles in the state \( \psi_0 \) in this limit is dominated by atoms which do not have any angular momentum. To conserve particle number and angular momentum, we have the following constraints

\[
N_0 + N_2 + N_3 = N \quad \text{and} \quad 2N_2 + 3N_3 = L. \tag{13}
\]

Table I gives the correction to the energy for the processes shown in Fig. 2. Adding these terms to \( \delta E^{(0)} \), we see that the corrected interaction energy \( \delta E^{(1)} \) is, in units of \( v_0 \) and to leading order in \( N^{-1} \),

\[
\delta E^{(1)} = \frac{1}{4}N(2N - 2) + \frac{27}{68}N_3(N_3 - 1). \tag{14}
\]

The number \( 27/68 \) coincides with the numerical result 1.59/4 reported in Ref. [13]. Here \( N_3 \) can take all the non-negative integer values that are consistent with the constraints of Eq. (13). The number of excited states described by Eqs. (13) and (14) is equal to the integer part of \( L/6 \). These statements are exact asymptotically, i.e., for \( N \to \infty \), since there are processes which couple the quadrupole \( m = 2 \) excitations with the octupole \( m = 3 \) excitations and they contribute terms of order \( 1/N \) to the excitation energy. Concerning the yrast state, a consequence of Eq. (13) is that its energy is given by the expression

\[
\delta E_0 = \frac{N}{4}(2N - L - 2), \tag{15}
\]

in agreement with Refs. [3,10,12]. Finally a result of Eq. (14) is that the yrast state for \( L \ll N \) is dominated by quadrupole excitations, i.e., to leading order,

\[
\frac{N_2}{N} = \frac{1}{2} \frac{L}{N}. \tag{16}
\]

In summary we have developed an effective theory which describes the ground state and the low-lying excited states of a weakly-interacting Bose-Einstein condensate under rotation, in the limit where the angular momentum is much less than the number of atoms. This study demonstrates that there are low-lying excited states which differ by an energy of order \( v_0 \) (and not \( Nv_0 \)), and we have found agreement with a previous numerical study [13] of the same problem. We should point
out that for all the values of $L/N$ we have examined numerically, except the case $L/N \ll 1$, we found that the low-lying excited states are separated from the yrast state by an energy of order $Nv_0$, and in that respect the limit $L \ll N$ seems to be unique.

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![Diagram 1](image1.png)

**FIG. 1.** The six diagrams contributing to the bare interaction energy $E^{(0)}$, Eq. (7). The straight lines denote atoms with angular momentum given by the numbers written above the lines; dashed lines denote the interaction.

![Diagram 2](image2.png)

**FIG. 2.** The five (additional) diagrams contributing to the interaction energy $E^{(1)}$, Eq. (14).

| Diagram | Energy |
|---------|--------|
| 1(a)    | $N_0(N_0 - 1)/2$ |
| 1(b)    | $3N_2(N_2 - 1)/16$ |
| 1(c)    | $5N_3(N_3 - 1)/32$ |
| 1(d)    | $N_0N_2/2$ |
| 1(e)    | $N_0N_3/4$ |
| 1(f)    | $5N_2N_3/8$ |
| 2(a)    | $-3N_2(N_2 - 1)/16$ |
| 2(b)    | $-5N_3(N_3 - 1)/544$ |
| 2(c)    | $-N_2N_3/8$ |
| 2(d)    | $-N_2/2$ |
| 2(e)    | $-3N_3(N_2 + 1)/4$ |

**TABLE I.** The contribution of the diagrams shown in Figs. 1 and 2 to the interaction energy (in units of $v_0$).

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