Maximal violation of Clauser-Horne-Shimony-Holt inequality for four-level systems

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Clauser-Horne-Shimony-Holt inequality for bipartite systems of 4-dimension is studied in detail by employing the unbiased eight-port beam splitters measurements. The uniform formulae for the maximum and minimum values of this inequality for such measurements are obtained. Based on these formulae, we show that an optimal non-maximally entangled state is about 6% more resistant to noise than the maximally entangled one. We also give the optimal state and the optimal angles which are important for experimental realization.

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I. INTRODUCTION

Recently, the inequalities 1 2 have been found that generalize the CHSH inequality to systems of high dimension, which give the analytic description of previous numerical results 3. In this brief report, we study the CHSH inequality for bipartite systems for 4-dimension by employing the unbiased eight-port beam splitters measurements. The uniform formulae of the maximal and minimal values of this inequality are obtained. Based on these formulae, we find an optimal non-maximally entangled state violates the inequality more strongly than the maximally entangled one, and then is about 6% more resistant to noise than the maximally entangled one. Similar to what we have pointed out for three dimensional systems 4, we also find that the left hand of the inequality can not be violated by the maximally entangled state. However, we find that the left hand of the inequality can be violated by some non-maximally entangled states, and the optimal non-maximally entangled state for
the left hand of the inequality is not the optimal one for the right hand.

II. THE INEQUALITY

In this section, let us recall the Bell inequality for four dimension obtained in Refs. [2, 5]. The scenario of the inequality involves two parties: Alice, can carry out two possible measurements, $A_1$ or $A_2$, on one of the particles, whereas the other party, Bob, can carry out two possible measurements, $B_1$ or $B_2$, on the other one. For the composed systems of 4-dimensional parties (or bipartite systems of spin $S = 3/2$ particles with the relation $d = 2S + 1$), each measurement may have 4 possible outcomes: $A_1, A_2, B_1, B_2 = 0, 1, 2, 3$. The joint probabilities are denoted by $P(A_i = m, B_j = n)$, which are required to satisfy the normalization condition: $\sum_{m,n=0}^{3} P(A_i = m, B_j = n) = 1$ [6]. We define the correlation functions $Q_{ij}$ as follows,

$$Q_{ij} \equiv \frac{1}{S} \sum_{m=0}^{3} \sum_{n=0}^{3} f^{ij}(m,n)P(A_i = m, B_j = n),$$

in which $S = 3/2$, and the function $f^{ij}(m,n)$ can be one of the following forms,

$$f^{ij}(m,n) = S - M(\varepsilon(i - j)(m \pm n), 4),$$

where $\varepsilon(x)$ is the sign function: $\varepsilon(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$, and $M(x, 4)$ is defined as follows: $M(x, 4) = (x \mod 4)$ and $0 \leq M(x, 4) \leq 3$. Then one can consider the following Bell expression:

$$I_4 = Q_{11} + Q_{12} - Q_{21} + Q_{22}.$$  \hspace{1cm} (3)

Especially, if taking $f^{ij}(m,n) = S - M(\varepsilon(i - j)(m - n), 4)$, one can prove that the above expression is just the Bell expression presented in Ref. [2].

The authors of [2, 5] proved that the maximum value of the above Bell expressions is 2 and minimum value of it is $-10/3$ for local variable theories. Then we get the following Bell inequality:

$$-\frac{10}{3} \leq I_4 \leq 2.$$ \hspace{1cm} (4)

However, the inequality will be violated for some entangled states by quantum predictions. For the maximally entangled state of two 4-dimensional systems $\psi = \frac{1}{2} \sum_{j=0}^{3} |l\rangle_A |l\rangle_B$, the authors of [2, 5] obtained the strongest violation of Bell expression $I_4$ for such a state,
\[ I_d(QM) = \frac{3}{4} \left( \sqrt{2} + \sqrt{10 - \sqrt{2}} \right) \approx 2.89624. \] In the presence of uncolored noise the quantum state
\[ \rho = (1 - F) |\psi\rangle \langle \psi| + F \frac{I}{16}. \] (5)
Following [3], we define the threshold noise admixture \( F_{thr} \) (the minimal noise admixture fraction for \( |\psi\rangle \)) \( F_{thr} = 1 - 2/I_d(QM) \), then we get \( F_{thr} \approx 0.30945 \). This result equals to the numerical results of [3].

III. THE MAXIMAL VIOLATION

Here, as in the previous works, we restricted our analysis to unbiased eight-port beam splitters [7, 8], more specifically to Bell multi-port beam splitters [3]. Unbiased Bell 2d-port beam splitters have the property that a photon entering into any single port (out of the \( d \)) has equal chances to exit from any output port. In additional, for Bell 2d-port beam splitters the elements of their unitary transition matrix, \( U^d \), are solely powers of the \( d \)-th root of unity \( \gamma_d = \exp(i2\pi/d) \), namely,
\[ U^d_{ij} = \frac{1}{\sqrt{d}} \gamma_d^{(i-1)(j-1)}. \] (7)

Let us now imagine spatially separated Alice and Bob who perform the Bell type experiment via the eight-port beam splitters on the state
\[ |\phi\rangle = \frac{1}{2} \sum_{i=0}^{3} a_i |i\rangle_A |i\rangle_B, \] (6)
with real coefficients \( a_i \), where, e.g., \( |i\rangle_A \) describes a photon in mode \( i \) propagating to Alice. One has \( \langle i|j\rangle_x = \delta_{ix} \), with \( x = A, B \). The overall unitary transformation performed by such a device is given by
\[ U^4_{ij} = \frac{1}{2} \gamma^4 e^{i\varphi_{ij}}, \quad i,j = 0, 1, 2, 3 \] (7)
where \( \gamma_4 = e^{i\pi/2} \) and \( j \) denotes an input beam to the device, and \( i \) an output one; \( \varphi_{ij} \) are the four phases that can be set by the local observer, denoted as \( \vec{\varphi} = (\varphi_0, \varphi_1, \varphi_2, \varphi_3) \) . The transformations at Alice’s side are denoted as \( \vec{\varphi}^A = (\varphi_0^A, \varphi_1^A, \varphi_2^A, \varphi_3^A) \), and \( \vec{\varphi}^B = (\varphi_0^B, \varphi_1^B, \varphi_2^B, \varphi_3^B) \) for Bob’s side. In this way the local observable is defined. The quantum prediction for the joint probability \( P^{QM}(A_i = m, B_j = n) \) to detect a photon at the \( m \)-th output of the multiport \( A \) and another one at the \( n \)-th output of the multiport \( B \) calculated for the state (6) is given by
\[ P^{QM}(A_i = m, B_j = n) = \sum_{k=0}^{3} \sum_{l=0}^{3} a_k a_l \gamma_4^{(k-l)(m+n)} e^{i(\varphi_k^A + \varphi_k^B - \varphi_i^A - \varphi_i^B)}, \] (8)
For convenience, we use the definition of the correlation functions \( Q_{ij} \) with the function \( f^{ij} = S - M(\varepsilon(i - j)(m + n), d) \). By substituting (8) into (3), after some elaborate, we can obtain

\[
I_4 = \sum_{k<l} a_k a_l T_{kl}, \quad (k = 0, 1, 2; l = 1, 2, 3),
\]

where \( T_{kl} \) are six continuous functions of sixteen angles \( \varphi^A_i \) and \( \varphi^B_j \) \((i, j = 1, 2)\). Let us define \( \varphi_{ij}^a = \varphi^{A_i} - \varphi^{A_j} + \varphi^{B_j} - \varphi^{B_i} \), then we can list these six functions as the follows:

\[
T_{01} = \frac{1}{6} \left\{ \left[ \cos(\varphi_{01}^{11}) - \cos(\varphi_{01}^{21}) - \cos(\varphi_{01}^{12}) + \cos(\varphi_{01}^{22}) \right] - \left[ \sin(\varphi_{01}^{11}) + \sin(\varphi_{01}^{21}) + \sin(\varphi_{01}^{12}) + \sin(\varphi_{01}^{22}) \right] \right\},
\]

\[
T_{02} = -\frac{1}{6} \left[ \cos(\varphi_{02}^{11}) - \cos(\varphi_{02}^{21}) + \cos(\varphi_{02}^{12}) + \cos(\varphi_{02}^{22}) \right],
\]

\[
T_{03} = \frac{1}{6} \left\{ \left[ \cos(\varphi_{03}^{11}) - \cos(\varphi_{03}^{21}) - \cos(\varphi_{03}^{12}) + \cos(\varphi_{03}^{22}) \right] + \left[ \sin(\varphi_{03}^{11}) + \sin(\varphi_{03}^{21}) + \sin(\varphi_{03}^{12}) + \sin(\varphi_{03}^{22}) \right] \right\},
\]

\[
T_{12} = \frac{1}{6} \left\{ \left[ \cos(\varphi_{12}^{11}) - \cos(\varphi_{12}^{21}) - \cos(\varphi_{12}^{12}) + \cos(\varphi_{12}^{22}) \right] - \left[ \sin(\varphi_{12}^{11}) - \sin(\varphi_{12}^{21}) + \sin(\varphi_{12}^{12}) + \sin(\varphi_{12}^{22}) \right] \right\},
\]

\[
T_{13} = -\frac{1}{6} \left[ \cos(\varphi_{13}^{11}) - \cos(\varphi_{13}^{21}) + \cos(\varphi_{13}^{12}) + \cos(\varphi_{13}^{22}) \right],
\]

and

\[
T_{23} = \frac{1}{6} \left\{ \left[ \cos(\varphi_{23}^{11}) - \cos(\varphi_{23}^{21}) - \cos(\varphi_{23}^{12}) + \cos(\varphi_{23}^{22}) \right] - \left[ \sin(\varphi_{23}^{11}) - \sin(\varphi_{23}^{21}) + \sin(\varphi_{23}^{12}) + \sin(\varphi_{23}^{22}) \right] \right\},
\]

We can know that \(|T_{kl}| \leq \frac{\sqrt{10-\sqrt{2}(2+3\sqrt{2})}}{21}\). However, they are strongly correlated, so \( T_{kl} \) can not reach their maximum value at the same time. As has been proven in Ref. [4], the maximum (minimum) values of \( I_4 \) can only be found on the vertices of the polyhedron formed by \( T_{kl} \). There are three sets of the vertices of the polyhedron. By denoting \( \Gamma_1 = \frac{\sqrt{10-\sqrt{2}(2+3\sqrt{2})}}{21} \approx 0.87104, \Gamma_2 = \frac{\sqrt{7}}{3} \approx 0.4714, \) and \( \Gamma_3 = \frac{\sqrt{10-\sqrt{2}(4-\sqrt{2})}}{21} \approx 0.3608 \), we list them in the following three tables.

In these tables, we have used the stipulations, e.g., \( T_{ab} = T_{ab} \) for \( a < b \) or \( T_{ab} = T_{ba} \) for \( b < a \). As has been proven in Ref. [4], for each vertices of the polyhedron one can
obtain an extreme value of the $I_4$, and the maximum (or minimum) value can be obtained by comparing the values among them.

Assuming \( \{A_i, \ (i = 0, 1, 2, 3)\} = \{|a_0|, |a_1|, |a_2|, |a_3|\} \), where \( \ast = \) means the equality of two sets, and \( A_i \) are in decreasing order, i.e. \( A_0 \geq A_1 \geq A_2 \geq A_3 \). For convenience, we denote \( \Gamma_1 = \frac{\sqrt{10 - \sqrt{2}}}{21}, \Gamma_2 = \frac{\sqrt{2}}{3}, \) and \( \Gamma_3 = \frac{\sqrt{10 - \sqrt{2}}}{21} \). Let us define

\[
B_1(\phi) = (A_0A_1)\Gamma_1 + (A_0A_2 + A_1A_3)\Gamma_2 + (A_0A_3 + A_1A_2 + A_2A_3)\Gamma_3,
\]

and

\[
B_2(\phi) = (A_0A_1)\Gamma_3 + (A_0A_2 + A_1A_3)\Gamma_2 + (A_0A_3 + A_1A_2 - A_2A_3)\Gamma_1.
\]

Then, the maximum value of \( I_4 \) for \( |\phi\rangle \) must be

\[
I_{4}^{\text{max}}(\phi) = \text{Max}(B_1(\phi), B_2(\phi)).
\]

From (18), one can immediately get that for the maximally entangled states, i.e., \( |a_i| = 1 \) \( (i = 0, 1, 2, 3) \), the maximum value of \( I_4 \) for such states are

\[
I_{4}^{\text{max}} = \Gamma_1 + 2\Gamma_2 + 3\Gamma_3 = \frac{2}{3} \left( \sqrt{2} + \sqrt{10 - \sqrt{2}} \right).
\]

This is just the result obtained in Ref. [3], and gives \( F_{\text{thr}} \approx 0.30945 \) which equals to the numerical results of [3].

Consider \( a_i \) as variables, we can obtain the maximal value of \( I_{4}^{\text{max}} \), denoted as \( \bar{I}_{\text{max}} \). The value of Eq. (18) should be maximum for \( A_0 = A_1 \) and \( A_2 = A_3 \). One can easily find that \( B_1(\phi) > B_2(\phi) \) for any state. For this case, By calculating the extreme value of \( B_1(\phi) \) with \( A_0 = A_1 \) and \( A_2 = A_3 \), after some elaboration, we get

\[
\bar{I}_{\text{max}} = \bar{A}_+^2\Gamma_1 + 2\bar{A}_+\bar{A}_-(\Gamma_2 + \Gamma_3) + \bar{A}_-^2\Gamma_3,
\]

where

\[
\bar{A}_\pm = \sqrt{1 \pm \frac{1}{791} \left( 357 + 7\sqrt{2} - 20\sqrt{10 - \sqrt{2}} - 58\sqrt{2}(10 - \sqrt{2}) \right)}
\]

with \( \bar{A}_+ = A_0 = A_1 \) and \( \bar{A}_- = A_2 = A_3 \). We then have \( \bar{S}_{\text{max}} \approx 2.9727 \), and the threshold amount of noise is about \( F_{\text{thr}} \approx 0.3272 \), which was also obtained in Ref. [9] by calculating the maximum eigenvalue of the Bell operator [10]. One can see this optimal non-maximally entangled state is about 6% more resistant to noise than the maximally entangled one. One can check the above results with the following optimal angles for \( a_i > 0 \) \( (i = 0, 1, 2, 3) \):

\[
\varphi^{A_1} = (0, \frac{\pi}{6}, -\pi, \frac{4\pi}{9}), \quad \varphi^{A_2} = (0, -\frac{5\pi}{9}, \frac{5\pi}{9}, -\frac{\pi}{3});
\]

(21)
and
\[ \vec{\phi}^{B_1} = \left(0, -\frac{\pi}{2}, \frac{13\pi}{18}, -\frac{11\pi}{18} \right), \quad \vec{\phi}^{B_2} = \left(0, \frac{7\pi}{36}, -\frac{27\pi}{36}, -\frac{7\pi}{18} \right). \] (22)

On the other hand, we can also calculated the minimum value of \( I_4 \). Let us define
\[ S_1(|\phi\rangle) = -(A_0A_1 + A_0A_3 + A_1A_2)\Gamma_1 - (A_0A_2 + A_1A_3)\Gamma_2 + (A_2A_3)\Gamma_3, \] (23)
and
\[ S_2(|\phi\rangle) = -2(A_0A_1 + A_0A_3 + A_1A_2 + A_2A_3)/3 - (A_0A_2 + A_1A_3)/3, \] (24)
The minimum value of \( I_4 \) for \(|\phi\rangle\) should be
\[ I_4^{\min}(|\phi\rangle) = \text{Min}(S_1(|\phi\rangle), S_2(|\phi\rangle)). \] (25)
Then, for the maximally entangled state, the minimum value of \( I_4 \) is
\[ I_4^{\min} = \frac{-10}{3}. \] (26)
One sees that the maximally entangled state does not violate the left hand of the inequality. However, for a non-maximally entangled state with \( A_0 = A_1 = K_+ \) and \( A_2 = A_3 = K_- \) where
\[ K_\pm = \sqrt{1 \pm \sqrt{\frac{1}{791} \left(357 - 7\sqrt{2} - 80\sqrt{10 - \sqrt{2}} + 6\sqrt{2(10 - \sqrt{2})}\right)}}, \] (27)
we find the minimum value of \( I_4^{\min} \), denoted as \( \bar{I}_{\min} \)
\[ \bar{I}_{\min} = -K_+^2\Gamma_1 - 2K_+K_- (\Gamma_1 + \Gamma_2) + K_-^2\Gamma_3 \approx -3.46424. \] (28)
Obviously, such states violate the left hand of the inequality.

IV. DISCUSSION

In summary of the present paper, we study the CHSH inequality for \( d = 4 \) in details on the Bell type experiment via the eight-port beam splitters which is realizable for nowadays technique. We give the analytic formulae of the maximum and minimum values of this inequality for such an experimental consideration. The maximal violations we obtained are the same as Refs. \( ^{[9]} \). We also give the optimal state and the optimal angles which are important for experimental realization.
It is well-known that for bipartite systems of 2-dimension, the CHSH inequality is symmetry. For any entangled state the inequality is violated symmetrically, and will be maximally violated by the maximally entangled states. However, for the higher dimensional systems, namely $d$-dimensional systems ($d > 2$), the inequality is asymmetry (see in Ref. [2, 5]),

$$\frac{-2(d + 1)}{(d - 1)} \leq I_d \leq 2.$$  \hfill (29)

The authors of Ref. [2, 5] studied the violation of the right hand of the above inequality for maximally entangled states and reproduced the results of previous numerical works [3]. For $d = 3$, one can immediately obtain that the left hand of the above inequality is $-4$, which can never be violated by any state [2, 5]. In Ref. [4], the authors shown that the minimal values of $I_3$ for maximally entangled states is just $-4$. They also found that an optimal non-maximally entangled state violates the inequality more strongly than the maximally entangled one. For $d = 4$, we also find that the left hand of the inequality can be violated by some non-maximally entangled states, and the optimal non-maximally entangled state for the left hand of the inequality is not the optimal one for the right hand.

In fact, the relations (29) and (3) define two inequalities, namely, the right ones and the left ones. The right inequalities are optimal and tight but the left ones are not tight. The asymmetry of the CHSH inequalities is due to the asymmetry of Hilbert space for higher dimensional systems [11]. For the systems of 2-dimension, the Hilbert space is symmetry, so the CHSH inequalities are symmetry as well. But for the higher dimensional systems, the Hilbert space is asymmetry, so the inequalities which are optimal for the right hand will be not optimal for the left. In other word, we can not find a inequality for higher dimensional systems which is optimal for the both sides.

V. ACKNOWLEDGMENT

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\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\{T_{ij}\} & T_{[ab]} & T_{[ac]} & T_{[ad]} & T_{[bc]} & T_{[bd]} & T_{[cd]} \\
\hline
\Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_3 & \Gamma_2 & \Gamma_3 \\
\hline
-\Gamma_1 & -\Gamma_2 & -\Gamma_3 & \Gamma_3 & \Gamma_2 & \Gamma_3 \\
\hline
-\Gamma_1 & \Gamma_2 & \Gamma_3 & -\Gamma_3 & -\Gamma_2 & \Gamma_3 \\
\hline
\Gamma_1 & -\Gamma_2 & \Gamma_3 & -\Gamma_3 & \Gamma_2 & -\Gamma_3 \\
\hline
\Gamma_1 & \Gamma_2 & -\Gamma_3 & \Gamma_3 & -\Gamma_2 & -\Gamma_3 \\
\hline
\Gamma_1 & -\Gamma_2 & -\Gamma_3 & -\Gamma_3 & -\Gamma_2 & \Gamma_3 \\
\hline
-\Gamma_1 & \Gamma_2 & -\Gamma_3 & -\Gamma_3 & \Gamma_2 & -\Gamma_3 \\
\hline
-\Gamma_1 & -\Gamma_2 & \Gamma_3 & \Gamma_3 & -\Gamma_2 & -\Gamma_3 \\
\hline
\end{tabular}
\caption{TABLE I: The first set of vertices, in which $i \neq j$.}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\{T_{ij}\} & T_{[ab]} & T_{[ac]} & T_{[ad]} & T_{[bc]} & T_{[bd]} & T_{[cd]} \\
\hline
-\Gamma_1 & -\Gamma_2 & -\Gamma_1 & -\Gamma_1 & -\Gamma_2 & \Gamma_3 \\
\hline
\Gamma_1 & \Gamma_2 & \Gamma_1 & -\Gamma_1 & -\Gamma_2 & \Gamma_3 \\
\hline
\Gamma_1 & -\Gamma_2 & \Gamma_1 & \Gamma_1 & \Gamma_2 & \Gamma_3 \\
\hline
-\Gamma_1 & \Gamma_2 & -\Gamma_1 & -\Gamma_1 & -\Gamma_2 & -\Gamma_3 \\
\hline
-\Gamma_1 & -\Gamma_2 & \Gamma_1 & -\Gamma_1 & \Gamma_2 & -\Gamma_3 \\
\hline
-\Gamma_1 & \Gamma_2 & \Gamma_1 & \Gamma_1 & -\Gamma_2 & -\Gamma_3 \\
\hline
\Gamma_1 & -\Gamma_2 & \Gamma_1 & \Gamma_1 & \Gamma_2 & \Gamma_3 \\
\hline
\Gamma_1 & \Gamma_2 & -\Gamma_1 & -\Gamma_1 & \Gamma_2 & -\Gamma_3 \\
\hline
\end{tabular}
\caption{TABLE II: The second set of vertices, in which $i \neq j$.}
\end{table}
| \(\{T_{ij}\}\) | \(T_{[ab]}\) | \(T_{[ac]}\) | \(T_{[ad]}\) | \(T_{[bc]}\) | \(T_{[bd]}\) | \(T_{[cd]}\) |
|---|---|---|---|---|---|---|
| \(-\frac{2}{3}\) | \(-\frac{1}{3}\) | \(-\frac{2}{3}\) | \(-\frac{2}{3}\) | \(-\frac{1}{3}\) | \(-\frac{2}{3}\) |
| \(\frac{2}{3}\) | \(\frac{1}{3}\) | \(\frac{2}{3}\) | \(-\frac{2}{3}\) | \(-\frac{1}{3}\) | \(-\frac{2}{3}\) |
| \(\frac{2}{3}\) | \(-\frac{1}{3}\) | \(-\frac{2}{3}\) | \(\frac{2}{3}\) | \(\frac{1}{3}\) | \(-\frac{2}{3}\) |
| \(-\frac{2}{3}\) | \(\frac{1}{3}\) | \(-\frac{2}{3}\) | \(\frac{2}{3}\) | \(-\frac{1}{3}\) | \(\frac{2}{3}\) |
| \(-\frac{2}{3}\) | \(-\frac{1}{3}\) | \(\frac{2}{3}\) | \(-\frac{2}{3}\) | \(\frac{1}{3}\) | \(\frac{2}{3}\) |
| \(-\frac{2}{3}\) | \(\frac{1}{3}\) | \(\frac{2}{3}\) | \(\frac{1}{3}\) | \(-\frac{2}{3}\) |
| \(\frac{2}{3}\) | \(-\frac{1}{3}\) | \(\frac{2}{3}\) | \(\frac{2}{3}\) | \(-\frac{1}{3}\) | \(\frac{2}{3}\) |
| \(\frac{2}{3}\) | \(\frac{1}{3}\) | \(-\frac{2}{3}\) | \(-\frac{2}{3}\) | \(\frac{1}{3}\) | \(\frac{2}{3}\) |

**TABLE III:** The third set of vertices, in which \(i \neq j\)