We discuss the Dirac equation in a curved 5-dimensional spherically symmetric space-time. The angular part of the solutions is thoroughly studied, in a formulation suited for extending to rotating space-times with equal angular momenta. It has a symmetry $SU(2) \times U(1)$ and is implemented by the Wigner functions. The radial part forms a Dirac-Schrödinger type equation, and existence of the analytical solutions of the massless and the massive modes is confirmed. The solutions are described by the Jacobi polynomials. Also, the spinor of the both large and small components is obtained numerically. As a direct application of our formulation, we evaluate the spectrum of the Dirac fermion in Einstein-Gauss-Bonnet space-time and the space-time of a boson star.

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I. INTRODUCTION

We study the Dirac equation in the background of a curved 5-dimensional spherically symmetric space-time. The angular part of the solutions is thoroughly studied, in a formulation suited for extending to rotating space-times with equal angular momenta. It has a symmetry $SU(2) \times U(1)$ and is implemented by the Wigner functions. The radial part forms a Dirac-Schrödinger type equation, and existence of the analytical solutions of the massless and the massive modes is confirmed. The solutions are described by the Jacobi polynomials. Also, the spinor of the both large and small components is obtained numerically. As a direct application of our formulation, we evaluate the spectrum of the Dirac fermion in Einstein-Gauss-Bonnet space-time and the space-time of a boson star.

I. INTRODUCTION

We study the Dirac equation in the background of a curved 5-dimensional space-time where the angular part is expressed as a $S^1$ bundle over $CP^1$. This form arises naturally in the case of the rotating space-time with equal angular momenta, for instance for the Myers-Perry black hole [1].

While the Dirac equation in 5-dimensional Myers-Perry with single angular momentum background has been studied in the literature (see for instance [2–4]), we have not find a detailed discussion about the angular harmonic with a manifest $SU(2) \times U(1)$ symmetry. Note that in the case mentioned above, the angular part is expressed as spheroidal harmonics. In this paper, we provide a detailed discussion on the representation of the angular sector of the Dirac field with such a manifest symmetry. We will focus on the problems of the non-rotating, since the computations are much simpler and the angular basis has the same symmetries than the isometry group of the rotating case with equal angular momenta. This work will serve as a basis for treating the rotating case, which is currently under consideration.

In the case of equal angular momenta, the space-time is cohomogeneous one, contrary to the case of a single angular momentum, which in fact has more in common with the four dimensional Kerr space-time. The angular separation of bosonic fields in such a background relies on the enhancement of the symmetry of the case of the equal angular momenta and has been addressed in [5]. In this work it was shown that the angular harmonics are expressed in terms of Wigner functions (see also [6] for an application). In this paper, we extend this result to the Dirac case, again, starting with the simpler limit of no rotation.

We construct the spinorial harmonics in this background and derive a Schrödinger like equation in the static case. As illustrative applications, we construct
AdS/CFT potentially has many applications in wave basis in flat space-time. In Sec.VI, we construct the radial part of the Dirac-Schrödinger equation is given. Sec.V is devoted to the complete, orthonormal plane wave basis in flat space-time. We examine the spectrum of the fermion coupled with a boson star. Finally, Sec.IX contains some concluding remarks.

II. BACKGROUND SPACE-TIME

A. Metric and angular isometry group

Let us consider a 5-dimensional background space-time of the form

\[ ds^2 = -b(r)dt^2 + \frac{1}{f(r)}dr^2 + g(r)d\tilde{\theta}^2 + h(r)\sin^2\tilde{\theta}(d\varphi_1 - \omega(r)dt)^2 + h(r)\cos^2\tilde{\theta}(d\varphi_2 - \omega(r)dt)^2 + (g(r) - h(r))\sin^2\tilde{\theta}\cos^2\tilde{\theta}(d\varphi_1 - d\varphi_2)^2 \]  

where \( \tilde{\theta} \) runs from 0 to \( \pi/2 \) while \( \varphi_1, \varphi_2 \) have the range from 0 to 2\( \pi \). Introducing new coordinates

\[ \theta = 2\tilde{\theta}, \varphi = \varphi_2 - \varphi_1, \psi = \varphi_1 + \varphi_2, \]

the line element then reduces to

\[ ds^2 = -b(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{g(r)}{4}(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{h(r)}{4}(d\psi + \cos\theta d\varphi - 2\omega dt)^2. \]  

This form of the space-time is well suited to study the spectrum of the Dirac equation in several contexts such as those considered in [12]. In fact, in the present paper, we will first consider the static case, i.e., no rotations, but keeping this representation of the angular sector, where the symmetries in the case of rotation with equal angular momenta is more manifest. Note that this form of the metric also accommodates black holes space-time, such as the 5-dimensional Myers-Perry black hole metric in both asymptotically AdS and flat spaces.

In [5], the authors extensively studied separability of field equations in the degenerate Myers-Perry black hole metric (1). They successfully decompose scalar, vector and tensor fields on the metric and then we employ the method to a spinor fields on the static case.

The static form of the metric (i.e. with \( g = h, \omega = 0 \)) has symmetry of \( SO(4) \cong SU(2)_L \otimes SU(2)_R \) rotation group, which breaks to \( SU(2)_R \times U(1)_L \) when the rotation is present. We define two invariant one-forms \( \sigma^R_{aL}(a = 1, 2, 3) \) of \( SU(2) \) which satisfy \( d\sigma^R_a = 1/2\epsilon^{abc}\sigma^R_b \wedge \sigma^R_c \) and \( d\sigma^L_a = -1/2\epsilon^{abc}\sigma^L_b \wedge \sigma^L_c \). The explicit forms are

\[ \sigma^R_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\varphi \]
\[ \sigma^R_2 = \cos \psi d\theta + \sin \psi \sin \theta d\varphi \]
\[ \sigma^R_3 = d\psi + \cos \theta d\varphi \]

and

\[ \sigma^L_1 = \sin \varphi d\theta - \cos \varphi \sin \theta d\psi \]
\[ \sigma^L_2 = \cos \varphi d\theta + \sin \varphi \sin \theta d\psi \]
\[ \sigma^L_3 = d\varphi + \cos \theta d\psi. \]  

symmetric plane wave solutions in flat space-time. We further construct normal normalizable and regular modes in both massive and massless AdS cases. These normal modes are obtained algebraically. This was first considered in [7] for the massive case, and we do recover the same spectrum.
The metric (2) is given by
\[
    ds^2 = -f(r)a^2(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{g(r)}{4}(\sigma_1 R^L)^2 + (\sigma_2 R^L)^2 + (\sigma_3 R^L)^2
    - \frac{g(r)}{4}(\sigma_3 R^L)^2 + h(r)\left(\sigma_3 R - 2\omega dt\right)^2.
\] (5)

In the static case, the metric has the following Killing vector \(\xi_\alpha\) (\(\alpha = 1, 2, 3\)):
\[
    \xi_1^R = -\sin \psi \partial_\theta + \frac{\cos \psi}{\sin \theta} \partial_\phi - \cot \theta \cos \psi \partial_\psi
    \xi_2^R = \cos \psi \partial_\theta + \frac{\sin \psi}{\sin \theta} \partial_\phi - \cot \theta \sin \psi \partial_\psi
    \xi_3^R = \partial_\psi
\] (6)
and
\[
    \xi_1^L = \sin \varphi \partial_\theta - \frac{\cos \varphi}{\sin \theta} \partial_\phi + \cot \theta \cos \varphi \partial_\phi
    \xi_2^L = \cos \varphi \partial_\theta + \frac{\sin \varphi}{\sin \theta} \partial_\phi - \cot \theta \sin \varphi \partial_\phi
    \xi_3^L = \partial_\varphi
\] (7)
which are dual to the one-forms, i.e., \(\langle \xi^{R,L}, \sigma^{R,L} \rangle = \delta_{\alpha\beta}\).

The symmetry can be explicitly shown by using the relation \(\mathcal{L}_{\xi^R} \sigma^R = 0\) where the \(\mathcal{L}_{\xi^R}\) is a Lie derivative along the curve generated by the Killing vector field \(\xi^R\).

In the case where the rotation is present, the metric (5) has less symmetries, the \(SU(2)_R\) breaks to \(U(1)_R\) and the remaining Killing vectors are given by \(\xi_1^L, \xi_2^L, \xi_3^L\) and \(\xi_3^R\).

Let us define two kinds of angular momenta
\[
    L^L_{\alpha} = i \xi^L_{\alpha}, \quad L^R_{\alpha} = i \xi^R_{\alpha}
\] (8)
which satisfy the commutation relations
\[
    [L^L_{\alpha}, L^L_{\beta}] = i \epsilon_{\alpha\beta\gamma} L^L_{\gamma}, \quad [L^R_{\alpha}, L^R_{\beta}] = -i \epsilon_{\alpha\beta\gamma} L^R_{\gamma}, \quad [L^L_{\alpha}, L^R_{\beta}] = 0.
\] (9)
Note that \((L^L_{\alpha})^2 = (L^R_{\alpha})^2 = L^2\) and the operators \(L^L_{\alpha}, L^R_{\alpha}\) have the common eigenfunction called Wigner D function:
\[
    L^2 D^L_{K,M} = J(J + 1) D^L_{K,M}
    L^3 D^L_{K,M} = K D^L_{K,M}
    L^L_{K,M} = M D^L_{K,M}
\] (10-12)
where \(J, K, M\) are integers satisfying \(J \geq 0\) and \(K, M \leq |J|\) (we obey the convention at [13]). In [5], the authors demonstrated how scalar and vector fields expand via invariant basis \(\sigma^R\) times Wigner functions \(D^L_{K,M}\).

Let us stress that these common eigenfunctions are associated with the symmetry generators of the rotating case. In other words, \(G, K, M\) are ‘good’ quantum numbers for the rotating case, where \(G\) is related to \(J\) but takes into additional account the spin of the fermion.

This will be discussed in more details elsewhere. In this paper, we construct the fermionic angular basis which preserves these quantum numbers. For sake of simplicity we consider the static case which includes these symmetries without using arguments about the symmetry enhancement in this case. Our construction can then be transposed in the more complicated case of rotating space-time with equal angular momenta. From now on, we assume
\[
    \omega(r) = 0, g(r) = h(r) = r^2, b(r) = f(r)a^2(r)
\] (13)
reducing the line element to
\[
    ds^2 = -f(r)a^2(r)dt^2 + \frac{1}{f(r)}dr^2
    + \frac{r^2}{4}(\sigma_1 R^L)^2 + (\sigma_2 R^L)^2 + (\sigma_3 R^L)^2
\] (14)
where the coordinates have the range in \(0 \leq \theta < \pi, 0 \leq \varphi < 2\pi\) and \(0 \leq \psi < 4\pi\).

III. VIELBEIN AND GAMMA MATRICES

Let us first remind the relation between the Cartesian coordinates \((x_1, x_2, x_3, x_4) \equiv (x, y, z, w)\) and the polar coordinates \((r, \theta, \varphi, \psi)\):
\[
    x_1 = r \sin \theta \cos \frac{\psi - \varphi}{2}, \quad x_2 = r \sin \theta \sin \frac{\psi - \varphi}{2}
    x_3 = r \cos \theta \cos \frac{\psi + \varphi}{2}, \quad x_4 = r \cos \theta \sin \frac{\psi + \varphi}{2}.
\] (15)

Accordingly, we choose the following form for the vielbein
\[
    e^\alpha_i = \sqrt{fa}
    e^1_r = \frac{1}{\sqrt{f}} \sin \frac{\theta}{2} \cos \frac{\psi - \varphi}{2}, \quad e^2_r = \frac{1}{\sqrt{f}} \sin \frac{\theta}{2} \sin \frac{\psi - \varphi}{2},
    e^3_r = \frac{1}{\sqrt{f}} \cos \frac{\theta}{2} \cos \frac{\psi + \varphi}{2}, \quad e^4_r = \frac{1}{\sqrt{f}} \cos \frac{\theta}{2} \sin \frac{\psi + \varphi}{2},
    e^1_\theta = \frac{r}{2} \cos \frac{\theta}{2} \cos \frac{\psi - \varphi}{2}, \quad e^2_\theta = \frac{r}{2} \cos \frac{\theta}{2} \sin \frac{\psi - \varphi}{2},
    e^3_\theta = -\frac{r}{2} \sin \frac{\theta}{2} \cos \frac{\psi + \varphi}{2}, \quad e^4_\theta = -\frac{r}{2} \sin \frac{\theta}{2} \sin \frac{\psi + \varphi}{2},
    e^1_\varphi = \frac{r}{2} \sin \frac{\theta}{2} \cos \frac{\psi - \varphi}{2}, \quad e^2_\varphi = -\frac{r}{2} \sin \frac{\theta}{2} \sin \frac{\psi - \varphi}{2},
    e^3_\varphi = \frac{r}{2} \cos \frac{\theta}{2} \cos \frac{\psi + \varphi}{2}, \quad e^4_\varphi = \frac{r}{2} \cos \frac{\theta}{2} \sin \frac{\psi + \varphi}{2},
    e^1_\psi = -\frac{r}{2} \cos \frac{\theta}{2} \cos \frac{\psi - \varphi}{2}, \quad e^2_\psi = \frac{r}{2} \cos \frac{\theta}{2} \sin \frac{\psi - \varphi}{2},
    e^3_\psi = \frac{r}{2} \sin \frac{\theta}{2} \cos \frac{\psi + \varphi}{2}, \quad e^4_\psi = \frac{r}{2} \sin \frac{\theta}{2} \sin \frac{\psi + \varphi}{2}.
\] (16)

The vielbein satisfies the following relations:
\[
    g_{MN} = e^\alpha_M e_{\alpha N} = \eta_{\alpha\beta} e^\alpha_M e^\beta_N.
\] (17)
The form of the gamma matrices that we employ is

\[
\gamma^0 = i \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = i \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},
\]

where \( \tau^i (i = 1, 2, 3) \) are the standard Pauli matrices. The gamma matrices satisfy \( \{ \gamma^a, \gamma^b \} = 2\eta^{ab} \).

### IV. DIRAC–SCHRÖDINGER EQUATION

#### A. Dirac Hamiltonian

The Lagrangian of the fermion field is

\[
L_{\text{fermion}} = \bar{\Psi}(\Gamma^M D_M - m) \Psi
\]

where \( m \) is the mass of the fermion. The gamma matrices of the curved space-time \( \Gamma^M \) are defined with the help of the vielbein \( e^a_M \) and those of the flat space-time \( \gamma^a \), i.e., \( \Gamma^M = e^a_M \gamma^a \). The covariant derivative for the fermion is defined as

\[
D_M = \partial_M + \frac{1}{8} \omega_{Mab}[\gamma^a, \gamma^b]
\]

where \( \omega_{Mab} := \frac{1}{2} \Gamma^N e_N M e_{bN} \) is the spin connection. \( M, N = 0, \ldots, 4 \) are the 4+1 dimensional curved space-time indexes and \( \hat{a}, \hat{b} = 0, 1, \ldots, 4 \) correspond to the flat tangent 4+1 Minkowski space-time.

The Dirac equation corresponding to the lagrangian (19) is thus

\[
\left( e^\hat{c}_M \gamma^c \left( \partial_M + \frac{1}{8} \omega_{Mab}[\gamma^a, \gamma^b] \right) - m \right) \Psi = 0.
\]

The Eq. (21) can be written such as \( i\partial_t \Psi = \mathcal{H} \Psi \). Further, we assume that the spinor can be decomposed as \( \Psi(t, r, \theta, \varphi, \psi) = e^{iEt} (\chi_1(r, \theta, \varphi, \psi), \chi_2(r, \theta, \varphi, \psi))^T \). This leads to an equation of the form

\[
\mathcal{H} \Psi = E \Psi,
\]

where the Dirac Hamiltonian \( \mathcal{H} \) and the Dirac spinor \( \Psi \) are given by

\[
\mathcal{H} = \begin{pmatrix} \sqrt{\mathcal{T}} a m & -i\mathcal{T} \tilde{p}_\mu \\ i\mathcal{T} \tilde{p}_\mu & -\sqrt{\mathcal{T}} a m \end{pmatrix}, \quad \Psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},
\]

where \( \tilde{p}_\mu = (-i\tau^j, I_2) \) (\( \mu = 1, \ldots, 4 \)) is same as one known as the quaternion basis. The ‘momentum’ is defined in terms of the coordinates and the derivatives:

\[
p_1 = i\partial_1 + \frac{i}{4r} \sin \frac{\theta}{2} \cos \frac{\varphi}{2} \left( a(-6\sqrt{f} + 6f + rf') + 2rf \sigma^A \right)
\]

\[
i\partial_2 := i\partial_2 + \frac{i}{4r} \sin \frac{\theta}{2} \cos \frac{\varphi}{2} \left( a(-6\sqrt{f} + 6f + rf') + 2rf \sigma^A \right)
\]

\[
p_3 = i\partial_3 + \frac{i}{4r} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \left( a(-6\sqrt{f} + 6f + rf') + 2rf \sigma^A \right)
\]

\[
p_4 = i\partial_4 + \frac{i}{4r} \cos \frac{\theta}{2} \sin \frac{\varphi}{2} \left( a(-6\sqrt{f} + 6f + rf') + 2rf \sigma^A \right)
\]

The Dirac Hamiltonian does not commute with the space-time symmetry generators \( L_p^a, L_\alpha^R, L_\alpha^L \). The reason is that these operators are the generators of the angular momenta, instead Dirac fields carry a spin, which couple to the background angular momenta. Defining the total angular momenta:

\[
G_a^R = L_a^R - \frac{1}{2} \begin{pmatrix} \tau_a & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
G_a^L = L_a^L + \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \tau_a \end{pmatrix}
\]

which are commute with the Hamiltonian:

\[
[G_a^R, \mathcal{H}] = [G_a^L, \mathcal{H}] = 0.
\]

We mention here that in the case of rotation, the commutation relations reduce to

\[
[G_3^R, \mathcal{H}^{\text{rot}}] = [G_3^L, \mathcal{H}^{\text{rot}}] = 0,
\]

where \( \mathcal{H}^{\text{rot}} \) is the Hamiltonian constructed in the rotating background. This case will be consideration elsewhere.
As a consequence, the spinorial angular harmonics can be constructed from eigenvectors of the \((\mathcal{G}^L)^2, \mathcal{G}^L, \mathcal{G}^R\). This basis would then also be suitable for the rotating case.

### C. Reduction to Schrödinger form

By standard procedure, one can eliminate the component \(\chi_2\) from (22), (23); the result is

\[
(E^2 - f a^2 m^2)\chi_1 = \{\tilde{\tau}_\mu p_\mu \bar{\tau}_\nu p_\nu \chi_1\}
\]

\[+(E + \sqrt{f} am) \times \left(\tilde{\tau}_\mu i \tilde{\partial}_\mu \frac{1}{E + \sqrt{f} am}\right) \bar{\tau}_\nu p_\nu \chi_1. \tag{28}\]

Here the parentheses \(\{\}\) indicate the range of operation of differential in \(\tilde{\tau}_\mu p_\mu\). The first term in the right hand side of (28) can be computed as

\[
\tilde{\tau}_\mu p_\mu \bar{\tau}_\nu p_\nu \chi_1 = -a^2 f^2 \frac{\partial^2 \chi_1}{\partial r^2} - \frac{a f}{2r} \left(4rf a' + 3a(2f + rf')\right) \frac{\partial \chi_1}{\partial r} - \frac{4a^2 f}{r^2} \Theta^2 \chi_1
\]

\[+ \frac{a \sqrt{f}}{r^2} \left(2raf' + a(2\sqrt{f} - 2f + rf')\right) i \mathcal{D} \chi_1 - \Omega \chi_1. \tag{29}\]

where \(\Theta, i \mathcal{D}\) are angular differential operators and \(\Omega\) is a function of \(f, f' = df/dr\) defined by

\[
\Theta^2 = \csc^2 \theta (\partial^2_\theta - 2 \cos \theta \partial_\theta \partial_\phi + \partial^2_\phi + \sin^2 \theta \partial^2_\theta
\]

\[+ \sin \theta \cos \theta \partial_\theta)\]

\[i \mathcal{D} = i \tau_1 (\sin \psi \partial_\psi + \cos \psi \cot \theta \partial_\phi - \cos \psi \csc \theta \partial_\phi)
\]

\[+ i \tau_2 (-\cos \psi \partial_\psi + \sin \psi \cot \theta \partial_\phi - \sin \psi \csc \theta \partial_\phi)
\]

\[- i \tau_3 \partial_\phi)\]

\[\Omega = \frac{1}{16r^2} \left\{ 4r^2 f^2 a^2 + 8raf\left(-3\sqrt{f} a' + 2ra f' + f(6a' + ra'') + a^2 \left(24f^{3/2} + 12f^2 - 12r\sqrt{f} f' + r^2 f''\right) + 4f(-9 + 9rf' + r^2 f'') \right) \right\}. \tag{30}\]

The second term of the right hand side of (28) is given by

\[
(E + \sqrt{f} am) \times \left(\tilde{\tau}_\mu i \tilde{\partial}_\mu \frac{1}{E + \sqrt{f} am}\right) \bar{\tau}_\nu p_\nu \chi_1
\]

\[= 4Af \frac{\partial \chi_1}{\partial r} + A \left(a rf' - 6\sqrt{f} + 6f) + 2rf a'\right) \chi_1
\]

\[+ 8A \sqrt{f} i \mathcal{D} \chi_1 \tag{31}\]

where \(A\) is defined by

\[A = \frac{m a f' + 2fa'}{8r E + \sqrt{f} am}. \tag{32}\]

### D. Angular operator

The operator \(i \mathcal{D}\) can be written in terms of the matrix form

\[i \mathcal{D} := \begin{pmatrix} -L_R^R & \sqrt{2}L_R^L \\ -\sqrt{2}L_R^L & L_R^R \end{pmatrix} \tag{33}\]

where \(L_R^{\pm} = (i L_R^R \pm L_R^L)/\sqrt{2}\).

Both \(\Theta^2, i \mathcal{D}\) can be diagonalized by using some special combinations of the Wigner functions \(D^L_{K,M}(\theta, \psi, \phi)\), defined as

\[\Theta^2 D^L_{K,M} = -J(J+1)D^L_{K,M}\]

\[L_R^R D^L_{K,M} = MD^L_{K,M}\]

\[L_R^K D^L_{K,M} = \pm \sqrt{2}L_R^L D^L_{K,M\pm 1}. \tag{36}\]

Note that \(\Theta^2 = -(\vec{L})^2 = -(\vec{L}^R)^2\) and that the Wigner function further satisfies

\[L_R^R D^L_{K,M} = KD^L_{K,M}. \tag{37}\]

The Wigner functions are the higher dimensional generalizations of the standard spherical harmonics. The spinorial harmonics are then constructed as follows,

\[|0\rangle_r = \begin{pmatrix} \sqrt{2}G-M+1 \ && \sqrt{2}G-M+1/2 \\ 2G+1/2 \ && \sqrt{2}G-M-1/2 \end{pmatrix} \tag{38}\]

satisfying

\[i \mathcal{D}|0\rangle_r = \left(G - \frac{1}{2}\right)|0\rangle_r \tag{39}\]

\[\Theta^2|0\rangle_r = -\left(G - \frac{1}{2}\right)\left(G + \frac{1}{2}\right)|0\rangle_r. \tag{40}\]

Similarly, a second linearly independent harmonic is given by

\[|1\rangle_r = \begin{pmatrix} \sqrt{2}G-M+1 \ && \sqrt{2}G-M+1/2 \\ 2G+1/2 \ && \sqrt{2}G-M-1/2 \end{pmatrix} \tag{41}\]

which satisfies

\[i \mathcal{D}|1\rangle_r = -\left(G + \frac{3}{2}\right)|1\rangle_r \tag{42}\]

\[\Theta^2|1\rangle_r = -\left(G + \frac{1}{2}\right)\left(G + \frac{3}{2}\right)|1\rangle_r. \tag{43}\]

Finally we further introduce

\[|2\rangle_r = \begin{pmatrix} \sqrt{2}G-M+1/2 \ && \sqrt{2}G-M+1/2 \\ 2G+1/2 \ && \sqrt{2}G-M-1/2 \end{pmatrix} \tag{44}\]

\[|3\rangle_r = \begin{pmatrix} -\sqrt{2}G-M+1 \ && -\sqrt{2}G-M+1/2 \\ -2G+1/2 \ && -\sqrt{2}G-M-1/2 \end{pmatrix}. \tag{45}\]
satisfying
\[ iD|2\rangle_r = -(G + 1)|2\rangle_r \] (46)
\[ iD|3\rangle_r = G|3\rangle_r. \] (47)

This basis is a basis for the upper component of the Dirac spinor, satisfying
\[ G^L \left( \begin{array}{c} \langle i | \rangle_r \\ 0 \end{array} \right) = K \left( \begin{array}{c} \langle i | \rangle_r \\ 0 \end{array} \right), \]
\[ G^G \left( \begin{array}{c} \langle i | \rangle_r \\ 0 \end{array} \right) = M \left( \begin{array}{c} \langle i | \rangle_r \\ 0 \end{array} \right), \]
\[ (G^G)^2 \left( \begin{array}{c} \langle i | \rangle_r \\ 0 \end{array} \right) = -G(G + 1) \left( \begin{array}{c} \langle i | \rangle_r \\ 0 \end{array} \right). \] (48)

Similarly, one can define a left basis for the lower components of the Dirac spinor:
\[
|0\rangle_L = \left( \begin{array}{c} \sqrt{\frac{G + K}{2G}} D^{G-1/2} K^{-1/2, M} \\ \sqrt{\frac{G - K}{2G}} D^{-1/2} K^{1/2, M} \end{array} \right), \\
|1\rangle_L = \left( \begin{array}{c} \sqrt{\frac{-G + K + 1}{2G}} D^{G+1/2} K^{-1/2, M} \\ \sqrt{\frac{G + K + 1}{2G}} D^{G+1/2} K^{1/2, M} \end{array} \right), \\
|2\rangle_L = \left( \begin{array}{c} \sqrt{\frac{-G - K}{2G}} D^{G} K^{-1/2, M} \\ \sqrt{\frac{G + K}{2G}} D^{G} K^{1/2, M} \end{array} \right), \\
|3\rangle_L = \left( \begin{array}{c} \sqrt{\frac{G - K + 1}{2G}} D^{G} K^{-1/2, M} \\ \sqrt{\frac{-G + K}{2G}} D^{G} K^{1/2, M} \end{array} \right). \] (49)

E. Parity of the spinorial harmonics

Here we discuss the parity of the basis. In the four dimensional polar coordinates \((r, \theta, \varphi, \psi)\), the parity transformation corresponds to
\[(r, \theta, \varphi, \psi) \rightarrow (r, 2\pi - \theta, \varphi + \pi, \psi - \pi). \] (50)

The periodicity conditions for \(D^G_{K,M}(\varphi, \theta, \psi)\) are (page 80, Eq.(4) of [13])
\[ D^G_{K,M}(\varphi, -\theta, \psi) = (-1)^{-M} D^G_{K,M}(\varphi, \theta, \psi) \]
\[ D^G_{K,M}(\varphi, \theta \pm 2n\pi, \psi) = (-1)^{2nG} D^G_{K,M}(\varphi, \theta, \psi) \]
\[ D^G_{K,M}(\varphi \pm n\pi, \theta, \psi) = (-i)^{2nK} D^G_{K,M}(\varphi, \theta, \psi) \]
\[ D^G_{K,M}(\varphi, \theta, \psi \pm n\pi) = (-i)^{2nM} D^G_{K,M}(\varphi, \theta, \psi) \] (51)
where \(n\) is integer. By the parity transformation, Wigner \(D\) function becomes
\[ D^G_{K,M}(\varphi + \pi, 2\pi - \theta, \psi - \pi) = (-1)^{2G+2K-2M} D^G_{K,M}(\varphi, \theta, \psi). \] (52)

The parity of the angular basis is defined according to the parity property of the upper component. We have four possibilities, namely \(|0\rangle_r, |1\rangle_r, |2\rangle_r, |3\rangle_r\). The lower components are constructed from (22), (23) and are respectively proportional to \(|2\rangle_l, |3\rangle_l, |0\rangle_l, |1\rangle_l\). It can be seen by direct inspection that parity even spinors have \(|0\rangle_r, |1\rangle_r\) in the upper part, while parity odd spinors have \(|2\rangle_r, |3\rangle_r\).

In the next section, we present a detailed basis for the Dirac spinor in the case of the flat vacuum space-time.

V. RADIAL DIRAC-SCHRÖDINGER EQUATION

Now, the eigenfunction \(\Psi\) can be separated by the angular basis and the radial part such as
\[ \Psi^{(i)} = \left( \begin{array}{c} \mathcal{F}_i(r) \langle i | \rangle_r \\ G_i(r) \langle i' | \rangle_r \end{array} \right), \] (53)
in which we have four possible basis \(i = 0, \cdots, 3\). Note that for satisfying the full Dirac equation, special combinations \(\{i, i'\}\) are allowed. It is well-known that by eliminating the lower (“the smaller”) component in (53), one can obtain the Schrödinger like radial equations for the positive eigenvalues
\[
a^2 f^2 \mathcal{F}''_i + P_i \mathcal{F}_i \\
+ \left( P^0_i - P^0(G_i - 2) - \frac{a^2 f}{r^2} G_i(G_i - 2) \right) \mathcal{F}_i = 0 \] (54)
where \(G_0 = 2G + 1, G_1 = -2G - 1, G_2 = -2G, G_3 = 2G + 2\) and
\[ P_1 := \frac{af}{2r} (4rf + 3a(2f + rf')) - 4Ar a f^{3/2} \]
\[ P^0 := E^2 - f a^2 m^2 \\
+ \Omega - a \sqrt{f} (2rf + a(2f - 6 \sqrt{f} + 7f')) \] (55)
\[ P^0 := \frac{a \sqrt{f}}{2r^2} (2rf + a(2 \sqrt{f} - 2f + rf')) - 4Af; \]
\[ A = \frac{m}{8r} a f' + 2f a f'' \] (56)
for \(E > 0\). The lower component \(\chi_2\) can be computed by using
\[ G_{i'}\langle i' | \rangle_r = \frac{i \tau^2 p_r}{E + a \sqrt{f} m} \mathcal{F}_i \langle i | \rangle_r, \] (57)
In the similar way, for the negative eigenvalues, the equations are obtained by eliminating the smaller (in this case, upper) component and then
\[ a^2 f^2 G_{i'}'' + Q_1 G_{i'} \\
+ \left( Q^0_0 - Q^0(G_{i'} - 2) - \frac{a^2 f}{r^2} G_{i'}(G_{i'} - 2) \right) G_{i'} = 0 \] (58)
where \(G_0 = 2G + 1, G_1 = -2G - 1, G_2 = -2G, G_3 = \]
where \( \tilde{\{ \right. \} } \) for \( E < 0 \). Now, the upper component can be obtained by using

\[
\mathcal{F}_n | \ell \rangle = \frac{i \tau \mu p_\mu}{E + m} G_\ell | \ell \rangle .
\]

VI. PLANE WAVES IN FLAT SPACE-TIME

In the case of the flat space-time, i.e., \( f(r) = 1 \), both Eqs.(54),(58) reduce to the standard Bessel equation:

\[
r^2 \tilde{F}_0'' + r \tilde{F}_0' + \tilde{F} (r^2 (E^2 - m^2) - G_1^2) = 0,
\]

where \( \tilde{F} = r \tilde{F} \). The general solution is then given by

\[
\tilde{F} = A J_{|G_1|}(kr) + B Y_{|G_1|}(kr),
\]

where \( k \) is such that \( E^2 = k^2 + m^2 \) and \( J \) and \( Y \) are the Bessel function of first and second kind.

\[
\chi_1 \sim \frac{J_{2G+1}(kr)}{r} |0\rangle ,
\]

with some multiplicative constants. The lower component \( \chi_2 \) is estimated via

\[
\chi_2 = \frac{i \tau \mu p_\mu}{E + m} \chi_1 \sim \frac{J_{2G+1}(kr)}{r} |2\rangle .
\]

Here we summarize the final result of the plane wave basis set. The four component basis set \( \{ u_i \} \) which is parity even of the upper component is

\[
u^a = N_k \begin{pmatrix} \frac{i \omega_{E_k}^+}{E_k} J_{2G+1}(kr) |0\rangle_r \\ \frac{- \omega_{E_k}^-}{E_k} J_{2G+1}(kr) |2\rangle_r \end{pmatrix}
\]

\[
u^b = N_k \begin{pmatrix} \frac{i \omega_{E_k}^+}{E_k} J_{2G+2}(kr) |1\rangle_r \\ \frac{- \omega_{E_k}^-}{E_k} J_{2G+2}(kr) |3\rangle_r \end{pmatrix}
\]

There is another four set \( \{ v_i \} \) with the opposite parity which is

\[
u^a = N_k \begin{pmatrix} \frac{i \omega_{E_k}^+}{E_k} J_{2G+1}(kr) |2\rangle_r \\ \frac{- \omega_{E_k}^-}{E_k} J_{2G+1}(kr) |0\rangle_r \end{pmatrix}
\]

\[
u^b = N_k \begin{pmatrix} \frac{i \omega_{E_k}^+}{E_k} J_{2G+2}(kr) |3\rangle_r \\ \frac{- \omega_{E_k}^-}{E_k} J_{2G+2}(kr) |1\rangle_r \end{pmatrix}
\]

where \( \omega_{E_k}^+ \equiv \text{sgn}(E_k), \omega_{E_k}^- \equiv 0 \). Now, the upper component can be obtained by using

\[
\mathcal{F}_n | \ell \rangle = \frac{i \tau \mu p_\mu}{E + \sqrt{f} m} G_\ell | \ell \rangle .
\]

VII. NORMAL MODES IN AdS

A. Massless case

For AdS vacuum, we set \( a = 1, f = 1 + r^2/\ell^2 \). In the massless case \( m = 0 \), the asymptotic form of the solutions of (54) or (58) is

\[
\mathcal{F}_i(r) = \frac{c_1}{r^3} + \frac{c_2}{r^2}.
\]

Note that this result is different from the one with the setting the limit \( m \to 0 \) in the \( m > 0 \) equation which will be seen in the next section. This is due to the peculiar form of the term \( \mathcal{A} \). The normalizability of the solutions then requires \( c_2 = 0 \). The expansion of the solutions around the origin leads to

\[
\mathcal{F}_i(r) = a_1 r^{G_i - 2} + a_2 r^{-G_i}
\]

independently of \( m \).

Let us perform the following change of variables

\[
r = \frac{1}{\ell} \sqrt{1 - \frac{r^2}{\ell^2}}.
\]

The Schrödinger like equation then reduces to

\[
(1 - x^2) \mathcal{F}'' - \frac{4 \mathcal{F}}{x} + \epsilon^2 \mathcal{F} - V(x) \mathcal{F} = 0
\]
which has to be considered for \( x \in [0, 1] \) and where \( \epsilon = E\ell \), and prime denotes derivative with respect to \( x \). The potential is given by

\[
V(x) = -\frac{6}{x^2} + \frac{G_i(G_i - 2)}{2(1 - x)} + \frac{(G_i + 1)(G_i - 1)}{2(1 + x)} + \frac{1}{4} .
\]

(75)

This equation presents singular points of second orders at \( x = 0, \pm 1 \), but it can be reduced to well-known equations after an appropriate change of the function of the form

\[
\mathcal{F}_i(x) = x^a(1 - x)^b(1 + x)^c H(x) .
\]

(76)

We find eight possible sets of the powers \( a, b, c \) which reduce the order of the singular points. Namely, for \( a = 3 \) and \( a = 2 \); we find \( b = (G_i - 2)/2 \) or \( b = -G_i/2 \) and \( c = (G_i - 1)/2 \) or \( c = -(G_i + 1)/2 \). The normalizability condition of the wave function naturally suggests \( a = 3 \) but this leads to a Heun equation with three singular points at \( x = 0, \pm 1 \) which we find difficult to treat. The choice \( a = 2 \) leads to an hypergeometric equation for the factor \( H(x) \). Assuming \( G_i > 0 \), the natural choice ensuring the regularity of the wave function at the origin is \( b = (G_i - 2)/2 \); then further choosing \( c = -(G_i + 1)/2 \) leads to the equation

\[
(1 - x^2)H'' + (1 - 2G_i - x)H' + \epsilon^2 H = 0 .
\]

(77)

Note that the other choice of the parameter \( c \) leads to a supplementary factor \( (1 + x)^{G_i} \) which is regular on the domain of the interest of the variable \( x \).

Natural solutions of the equations are the Jacobi polynomials. They correspond to the integer values of the energy

\[
H(x) = P_n^{(G_i - 1, -G_i)}(x) , \quad E = n .
\]

(78)

However, these polynomials do not vanish for \( x = 0 \) and the corresponding wave function is non-normalizable.

In order to construct the normalizable solutions, we have to construct the solutions of Eq.(78) for arbitrary values of the energy parameter and fine-tune \( E \) in such a way to have \( H(0) = 0 \).

For generic values of \( E \), the solutions are given by hypergeometric functions

\[
H(x) = _2F_1 \left( E, -E, G_i, \frac{1 - x}{2} \right) .
\]

(79)

A numerical integration of the equation leads, for fixed \( G_i \), to a family of solutions which fulfill the condition \( H(0) = 0 \). As can be expected, the solutions are labeled by the number of zeros in \([0, 1]\).

Setting \( G_0 = 3 \), the profiles of the four first functions are presented in Fig. 1. The normalization \( H(1) = 1 \) is chosen. For \( G_0 = 3, 4 \), we find respectively for the four first energy levels

\[
\epsilon = 3.725, 5.65, 7.62, 9.6 \quad \text{and} \quad \epsilon = 4.76, 6.68, 8.64, 10.62 .
\]

(80)

Let us stress that the spectrum is symmetric for \( \epsilon \to -\epsilon \), as can already be seen in Eq.(74).

We believe there is a direct connection with the results of [10]. Indeed, we find a general solution in terms of the hypergeometric function of half variable. In the case of [10], the spectrum was continuous because the spinor lived on a compact space. Here the space is not compact and we need to impose a normalizability condition.

Note that we indirectly recover the results of [9] for deSitter space. Indeed, the condition that the solution is regular at its poles implies that the eigenvalue is proportional to an integer. Here, this case gives non-normalizable eigenstates.

### B. Massive case

The equations in \( AdS \) vacuum are invariant under the rescaling \( m' = mL \), \( E' = E\ell \) and \( r' = r/\ell \), we therefore set arbitrarily \( \ell = 1 \) without loss of generality.

The asymptotic solution to (54) is given by

\[
\mathcal{F} \approx \mathcal{F}_\infty = A_{\infty} r^{-2-m} + B_{\infty} r^{2+m} .
\]

(81)

The spinor is normalizable if it decays at least as \( r^{-3} \), implying that no normalizable mode exists for \( |m| < 1 \).

It is important to remark that as mentioned in the previous section, the asymptotic behavior of the massless case differs from the \( m = 0 \) limit of the massive case. Thanks to this fact, normalizable modes exist in the massless case.

Close to the origin, the function \( \mathcal{F} \) behaves like

\[
\mathcal{F} \approx A_0 r^{G_i - 2} + B_0 r^{-G_i} .
\]

(82)

In order to construct the spectrum of the equation for \( m > 0 \), it is useful to use the change of variable (73) and
to perform the following change of function on the radial function $F_i(r)$ appearing in Eq. (54):

$$F_i(x) = x^{m+2} (1-x)^{(G_i-2)/2} (1+x)^{(G_i-1)/2} H(x). \quad (83)$$

If there exists solutions $H(x)$ which are regular at $x = 0$ and $x = 1$, the factorization of the power of $x$ and of the power of $(1-x)$ ensure respectively the normalizability (i.e. for $r \to \infty$) and the regularity (i.e. for $r = 0$) of the corresponding wave function $F_i(r)$.

The equation for the new function $H(x)$ is then found to be

$$x(1-x^2)(Ex+m) \frac{d^2H}{dx^2} + P_3(x) \frac{dH}{dx} + P_2(x) H = 0, \quad (84)$$

where

$$P_3(x) = -E(2G_i + 2m + 1)x^3$$
$$\quad + (E - 2G_im - 2m^2 - 2m)x^2$$
$$\quad + m(2E + 1)x + m(2m + 1), \quad (85)$$

$$P_2(x) = -(G_i + m - E)(E + m + G_i)x^2$$
$$\quad + m(E + m + G_i + 1)x - m). \quad (86)$$

It turns out that Eq. (84) admits a family of solutions which are polynomials of degree $n$ in $x$, say $H(x) = H_n(x)$. The energy $E$ of the solution $H_n(x)$ is given by

$$E_n = (-1)^n(G_i + m + n). \quad (87)$$

Note stress that only even $n$ (i.e., $E_n > 0$) in (87) are allowed because the Eq.(54) is of the positive eigenstates. To get the solutions of the negative eigenstates, we should start with (58). We have the same form as (87), but at that case we should adopt only for the odd $n$.

The generic coefficients of the polynomials $H_n(x)$ can be constructed by solving some recurrence relations. The even and odd parts of the polynomials split naturally, leading to a form

$$H_n(x) = H_{n,e}(x) + H_{n,o}(x), \quad (88)$$
$$H_{n,e}(-x) = H_{n,e}(x), \quad H_{n,o}(-x) = -H_{n,o}(x).$$

Assuming first that the degree of the polynomial $H_n$ is even, i.e. $n = 2p$, then we have

$$H_{n,e}(x) = \sum_{j=0}^{p} (-1)^j C_j a_j^e x^{2j}, \quad (89)$$

$$H_{n,o}(x) = \sum_{j=0}^{p} (-1)^j C_j a_j^o x^{2j}, \quad (90)$$

FIG. 2: The ground state of the spinor for the case of mass parameter $m = 1, 4$ of $\mathcal{F}_0$ (the top two figures) and $\mathcal{G}_0$ (the bottom two figures). For larger value of $m$, the small components decrease.
the coefficients of the series defining our solutions turn combination of Jacobi polynomials, defined as

\[ H_{n,o}(x) = n \sum_{j=0}^{p-1} (-1)^j C_j^{p-1} a_j^{o} x^{2j+1} \quad (90) \]

where \( C_j^{p} \) denotes the combinatoric symbols. The ratios of double factorials are in fact finite products. In the above expressions, we have defined for compactness

\[
\begin{align*}
    a_j^{ee} &= (2m + 2p + 1)!!(2G_i + 2m + 2p + 2j - 1)!! \\
    (2m + 2j - 1)!!(2G_i + 2m + 4p - 1)!! \\
    a_j^{eo} &= (2m + 2p + 1)!!(2G_i + 2m + 2p + 2j - 1)!! \\
    (2m + 2j + 1)!!(2G_i + 2m + 4p - 1)!!.
\end{align*}
\]

For odd values of \( n \), i.e. \( n = 2p + 1 \) the polynomials have similar form:

\[ H_{n,e}(x) = \sum_{j=0}^{p} (-1)^j C_j^{p} a_j^{oe} x^{2j} , \quad (91) \]

\[ H_{n,o}(x) = -\sum_{j=0}^{p} (-1)^j C_j^{p} a_j^{oo} x^{2j+1} \quad (92) \]

where

\[
\begin{align*}
    a_j^{oe} &= (2m + 2p + 1)!!(2G_i + 2m + 2p + 2j + 1)!! \\
    (2m + 2j - 1)!!(2G_i + 2m + 4p - 1)!! \\
    a_j^{oo} &= (2m + 2p + 1)!!(2G_i + 2m + 2p + 2j + 1)!! \\
    (2m + 2j + 1)!!(2G_i + 2m + 4p + 1)!!.
\end{align*}
\]

It is possible to express these series in term of a combination of Jacobi polynomials, defined as

\[ P_{p}^{\alpha,\beta}(z) = p! \frac{\Gamma(\alpha + p + 1)}{\Gamma(\alpha + \beta + p + 1)} \times \sum_{j=0}^{p} C_j^{p} \frac{\Gamma(\alpha + \beta + p + j + 1)}{\Gamma(\alpha + j + 1)} (z - \frac{1}{2})^j. \quad (93) \]

Indeed, noting that for a constant \( N \),

\[ N!! = 2^{N/2} \Gamma \left( \frac{N}{2} + 1 \right) , \quad (94) \]

the coefficients of the series defining our solutions turn out to be precisely those of the Jacobi polynomials.

For even degree polynomials, we find

\[
H_{2p}(x) = \frac{1}{C_p^{m+G+p-\frac{1}{2}}} \times \left( P_{p-m-\frac{1}{2},G_i}^m (1 - 2x^2) + x P_{p-1-m-\frac{1}{2},G_i}^m (1 - 2x^2) \right) \quad (95)
\]

with the associates eigenvalue given by \( E_{2p} = G_i + m + 2p \).

Similarly, for odd degree polynomials, we find

\[
H_{2p+1}(x) = \frac{m + p + \frac{1}{2}}{(p + 1) C_p^{m+G+2p+\frac{1}{2}}} \times P_{p-m-\frac{1}{2},G_i+1}^m (1 - 2x^2) \]

\[ -\frac{x}{C_p^{m+G+2p+\frac{1}{2}}} P_{p-m-\frac{1}{2},G_i}^m (1 - 2x^2). \quad (96) \]

The lower component of the Dirac spinor formally can be reconstructed by using (57), but it is not straightforward and is tedious task to get the explicit form because the solution (95) is already quite complicated. The massive spectrum we build here agrees with the results of [7].

\[ \text{FIG. 3: The eigenvalues as a function of the mass parameter } m. \text{ The dots on } m = 0 \text{ denote the result of the massless case.} \]

\[ \text{C. Numerical study} \]

We have found analytically the normalizable solutions for the Schrödinger like equations (54), (58) for both the massless/massive cases. If one obtains the smaller component of the solutions, formally (57) and (61) should be used. It is, however, tedious task to compute analytically. In this section we numerically solve the equations and obtain both the larger and the smaller components.

It is convenient to use the rescaled coordinate defined as \( y := r/(1 + r) \) which runs from 0 to 1. The equations can be written by the new coordinate

\[
\begin{align*}
    \{ (1 - y)^2 + y^2 \}^2 &\frac{d^2F_i}{dy^2} \\
    + \left[ p_i(1 - y)^2 - 2 \left( \frac{(1 - y)^2 + y^2}{1 - y} \right)^2 \right] \frac{dF_i}{dy} \\
    + \left[ p_{0i}^2 - p_{0i}^2(G_i - 2) \right] \frac{F_i}{y^2} G_i(G_i - 2) \right] F_i = 0
\end{align*}
\]  

(97)
The process (i)-(iii) is repeated until the convergence is attained. If the analysis reaches the correct eigenfunction, it no longer has discontinuity and the computation is successfully terminated.

We present some typical results of the eigenvalues in Table I.

| $n$ | $F_i$ | $G_i$ |
|-----|-------|-------|
| 0   | 4.00000003 | 6.00000008 |
| 1   | 8.00000015 | -9.00000107 |
| 2   | -5.00000024 | -7.00000069 |

TABLE I: The eigenvalues for $l = 1, m = 1, G_i = 3$.

Now we numerically calculate the lower component of $F_i$ and also the upper component of $G_i$. In Fig. 2, we plot the $F_0, G_0$ and the corresponding lower component for the case of $m = 1$ and $m = 4$. Ratio of the lower component to the upper component is smaller for $m = 4$. Thus, we speculate that for $m \to \infty$, only upper component remains, corresponding to the “non-relativistic limit”. In Fig. 3, we show the eigenvalues for the case of the normalizable mode.

**VIII. GAUSS-BONNET GRAVITY AND BOSON STAR**

The above discussion can be extended in numerous directions. Namely: (i) the gravity sector can be extended by a Gauss-Bonnet term, (ii) various matter fields can be supplemented to the Einstein or Einstein-Gauss-Bonnet (EGB) action. To be more concrete, the Dirac equation can be studied in the background of a space-time constructed out of the model

$$S = \frac{1}{16\pi G} \int d^5 x (R - 2\Lambda + \alpha \nabla^M \Pi)$$

where $R$ is the Ricci scalar, $\Lambda = -6/\ell^2$ is the cosmological constant, $\Pi$ is a complex field, $\alpha$ denotes the Gauss-Bonnet coupling constant and $L_{GB}$ is the Gauss-Bonnet term, constructed out of the Riemann tensor in the standard way:

$$L_{GB} = R_{MNKL} R_{MNKL} - 4 R_{MN} R_{MN} + R^2,$$
with $M, N, K, L \in \{0, 1, 2, 3, 4\}$. In the case of EGB gravity, the usual $AdS$ space-time is modified by the Gauss-Bonnet interaction, the metric function $f(r)$ takes the form:

$$f(r) = 1 + \frac{r^2}{\ell^2}, \quad \frac{1}{f} = \frac{1}{\ell^2} \left[ 1 - \sqrt{1 - \frac{2\alpha}{\ell^2}} \right]^{\frac{1}{2}}$$  \hspace{1cm} (107)

In the presence of the cosmological constant, the range of the Gauss-Bonnet coupling constant is limited: $\alpha \in [0, \frac{\ell^2}{2}]$, the upper limit is called the Chern-Simons limit [14].

The model (105) is one of the simplest way to couple matter (minimally) to gravity. The coupled system admits regular, localized, stationary solutions called boson stars (see e.g. [15] for a review). For $d > 4$ boson stars were constructed namely [16–18]. Even in the absence of a self-interacting potential of the scalar field, the asymptotically $AdS$ space-time renders possible the existence boson stars.

The simplest boson star can be constructed by performing a spherically symmetric ansatz for the scalar field $\phi$: $\Pi(x) = \exp(-i\omega t)\phi(r)$, where the time-dependant harmonic factor contains the frequency parameter $\omega$ and $r$ is the radial variable. With this ansatz and the metric (1), the field equations reduces to a system of three differential equations for the functions $f, b, \phi$. These equations have to be solved numerically. However, in the limit where the scalar field decouples from gravity (the so called probe limit) the gravity part is determined by (107) while the the Klein-Gordon equation of the massless boson takes the form

$$\left( r^3 f \phi' \right)' - \frac{r^3 \omega^2}{f} \phi = 0$$  \hspace{1cm} (108)

which can be solved in terms of hypergeometric functions

$$\phi(r) = \frac{\ell_c^4}{(r^2 + \ell_c^2)^2} \, _2F_1 \left( \frac{4 - \omega \ell_c^2}{2}, \frac{4 + \omega \ell_c^2}{2}, 3, \frac{r^2}{r^2 + \ell_c^2} \right).$$  \hspace{1cm} (109)

The regularity of this solution on the full line requires $\omega \ell_c = 4 + 2k$ with an integer $k$. These corresponding solutions, called oscillons [19], are regular, localized in space and labelled by the integer $k$ which sets the number of nodes of the profile $\phi(r)$ and the frequency $\omega$.

Once coupled to gravity, the oscillons solutions get deformed by the geometry and exist on a finite range of the frequency $\omega$: from now on, we pose $\kappa = 16\pi G$. The three coupled differential equations for the functions $f, b, \phi$ have to be solved with appropriate boundary conditions. At the center of the boson star, the regularity of the solutions requires $f(0) = 1$, $b(0) = 0$, $\phi(0) = 0$. Expanding the fields about the origin leads to the following behaviour

$$f(r) = 1 + F_2 r^2 + O(r^4),$$

$$b(r) = B_0 + \frac{6B_0 F_2 - \kappa \omega^2 \Pi_0^2 - 12B_0/\ell^2}{6(\alpha F_2 - 1)} r^2 + O(r^4),$$

$$\phi(r) = \Pi_0 - \frac{\Pi_0 \omega^2}{8B_0} r^2 + O(r^4)$$  \hspace{1cm} (110)

where the parameters $B_0, \Pi_0$ are underdetermined while

$$F_2 = \frac{6B_0 - \sqrt{36B_0^2 + 6\alpha B_0 \kappa \omega^2 \Pi_0^2 - 72\alpha B_0^2/\ell^2}}{6\alpha B_0}. \hspace{1cm} (111)$$

Asymptotically, the scalar field is required to vanish while the metric (107) is approached. More precisely, for $\alpha < \ell^2/2$, the fields should obey

$$f(r) = 1 + \frac{r^2}{\ell_c^2} + \frac{\hat{M}}{r^2} + O(r^{-4}),$$

$$b(r) = 1 + \frac{r^2}{\ell_c^2} + \frac{\hat{M}}{r^2} + O(r^{-4}),$$

$$\phi(r) = \frac{\Pi_\infty}{r^4} + O(r^{-6}).$$  \hspace{1cm} (112)

(The limit $\alpha = \ell^2/2$ is special, see e.g. [20]). Technically, the equations are integrated numerically by a fine tuning the parameters $\hat{M}, B_0$ and $\Pi_0$ in such a way that the boundary conditions are obeyed. The Newton constant $\kappa$ can be rescaled in the scalar field and the Gauss-Bonnet parameter is fixed by hand. For simplicity, we address only the deformations of the fundamental oscillons characterized by $k = 0$. With a choice of $\alpha$, a branch of boson stars labelled by the frequency $\omega$ (or equivalently by the value $\phi(0) = \Pi_0$) can be constructed. The boson stars are namely characterized by mass:

$$M = \frac{V_3}{16\pi G} (\hat{M} - 4M)(1 - \frac{2\alpha}{\ell^2})^{1/2}, \quad V_3 = 2\pi^2$$  \hspace{1cm} (113)

where the parameters $\hat{M}, M$ appear in the asymptotic of the metric (112). We construct numerically a large number of solutions which allow an understanding of the pattern of the EGB boson stars. For all values parameter $\phi(0)$ that we have considered, the EGB boson star could be constructed up to the maximal value $\alpha = 1/2$: it was checked that the expression under the square root in (111) stays strictly positive. (Potentially this term can become negative for some values of $\alpha$, limiting the domain of existence of the solutions. For instance, this happens for spinning solitons [21]).

The dependence of the mass of the fundamental (i.e. with $k = 0$) boson stars on the frequency $\omega$ is reported on Fig. 4 for several values of $\alpha$ (the mass is in the unit $V_3/16\pi G$).

In the case of pure Einstein gravity, the curve is represented by the solid-black line. In the limit $\omega \to 4$, the scalar field tends uniformly to zero and the $AdS$ space-time is approached uniformly. Increasing gradually the central value $\phi(0)$ reveals that the mass reaches a maximum and that the boson star exists only on a finite interval of frequencies, $\omega \in [3.5, 4.0]$. In that limit $\phi(0) \to \infty$, the metric seems to approach a configuration with a singularity of the Ricci scalar at the origin. The mass remains finite and the $M - \omega$ line takes the form of a spiral.

When the Gauss-Bonnet parameter is non-zero, the solution is gradually deformed. In particular the curve...
$M - \omega$ changes smoothly, but the spirals have a tendency to disappear. This feature seems to be generic for EGB boson stars; it is first observed in [18] in the case of asymptotically flat space-time (and in the presence of a self-interacting potential of the scalar field) and more recently in [21, 22] for asymptotically AdS spinning boson and non-spinning solutions. When the parameter $\alpha$ becomes large enough, our numerical analysis strongly indicates that the interval of allowed frequencies is extended to $[0, 4/\ell_c]$ (in particular $\omega \in [0, 4\sqrt{2}]$ for the Chern-Simons limit $\alpha = 1/2$). A numerical evaluation of the critical value of $\alpha$ is quite involved and is not aimed for this paper.

One question which occurs naturally is the study of the evolution of the spectrum of the fermion in the EGB space-time and/or in the space-time of a boson star. In order to study the effect of the scalar field on the fermion spectrum, we introduce a Yukawa type coupling, by extending the fermionic Lagrangian (19) to

$$\mathcal{L}_{\text{fermion}} = \bar{\Psi}(i\Gamma^M D_M - m - \mu |\phi|) \Psi,$$  \hspace{1cm} (114)

where $\mu$ is the Yukawa coupling and $|\phi|$ is the norm of the scalar field. The Dirac and Dirac-Schrödinger equations are extended straightforwardly, according to the procedure described in Sec.IV. All the symmetry consideration of that section remain unaltered, in particular the angular sector is unaffected.

In the absence of matter field and for a massless fermion the fermionic levels obey the scaling rule $E(\alpha) = E(0)/\ell_c(\alpha)$. As a consequence, the energy levels increase with the increasing $\alpha$. In the presence of a fermion mass, the scaling rule is only approximative, but the increase of the fermion levels with $\alpha$ still holds.

We finally study the response of the fermion eigenmodes to boson stars in EGB space-time by considering the reduced Dirac equation in the background of an oscillon. The sketch of these results are summarized in Fig. 5 for the first two fermionic modes for Einstein and EGB oscillons. We check that the qualitative properties of the fermionic eigenvalues in the case of the boson stars (i.e. for scalar minimally coupled to gravity) are qualitatively the same. In the case of Einstein boson star, the fermionic levels, obtained numerically, increase slightly with the Yukawa parameter, indicating that the fermion becomes more strongly bounded to the space of the boson star. As expected, in the case of the boson stars in the EGB-gravity (and then also in the Chern-Simons limit) the fermion binding energy becomes much stronger. This result can be explained by the stronger ‘harmonic oscillator’ coupling played by the underlying space-time asymptotically.

**IX. CONCLUSION**

In this paper, we discuss in details the separation of the radial and angular variables for the Dirac equation in a 5-dimensional space-time. The full isometry group of the angular sector of the metric is $SU(2)_R \times SU(2)_L$ symmetry. However, only the subgroup $SU(2)_R \times U(1)_L$ appears as a manifest symmetry in the coordinates used here. This subgroup is precisely the isometry group of the 5-dimensional Myers-Perry space-time with equal angular momentum. The angular basis obtained here can be used for the case of rotating space-time with equal angular momenta. As a crosscheck of our equations we (re)derived the plane wave basis in flat vacuum space-time and the normal basis for the case of AdS vacuum in both massless and massive case.

The massless spinor in AdS is expressed in terms of the hypergeometric functions, and the spectrum is symmetric under change of sign of the eigenenergy.

The construction of the eigenvector in this case cannot
be done till the end algebraically, and the corresponding eigenvalue has to be determined numerically by imposing the suitable boundary condition.

In contrast, the massive case can be treated in terms of Jacobi polynomials and lead to a set of explicit eigenvalues [9]. Accordingly, the spectrum in the massless case doesn’t exhibit the same amount of regularity as for the massive case. Furthermore, they are not continuously connected. This constitutes one of our original result.

The massive eigenvalues can be labeled by an integer \( n \) denoting the degree of the polynomial \( H(x) \) appearing in the construction. They are alternatively positive and negative as \( n \) increases, and the associated eigenfunction is expressed as a peculiar linear combination of two Jacobi polynomials. Positive eigenvalues are associated to even degree polynomials, while the negative energy modes have odd degree.

We note that the solution we describe here are analytic on the whole \( AdS \) space-time since both normalizability and regularity at the origin are imposed.

Our results are relevant for many purposes. For instance, they be used as a basis of the Dirac spinor in asymptotically \( AdS \) and flat space-time, which can be used to construct the spectrum of the Dirac equation in more elaborate setup, e.g. fermionic modes around a holographic superconductor.

The simplest model discribing an such a physical system consists of a charged, complex scalar field minimally coupled to gravity. The relevant classical solution is determined numerically. In Sect VIII we made a step forward in this direction by studying the evolution of the Dirac equation in the background of the space-time of a boson star supported by Einstein and Einstein-Gauss-Bonnet gravity. Here the eigenvalues are determined numerically and our results show that the spectrum is smoothly deformed by the boson star.

Finally, let us stress again that the symmetry preserved in the construction is the isometry group of the rotating case (with equal angular momenta). This is the main result of this technical paper. This works settles the issue of the angular part for this case. Of course, more work is needed and is in progress, but will be presented elsewhere due to the various technical points involved in the full construction.

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[1] R. C. Myers and M. Perry, “Black Holes in Higher Dimensional Space-Times,” *Annals Phys.* **172** (1986) 304.

[2] D. Ida, K.-y. Oda, and S. C. Park, “Rotating black holes at future colliders: Greybody factors for brane fields,” *Phys.Rev.* **D67** (2003) 064025, arXiv:hep-th/0212108 [hep-th].

[3] M. Casals, S. Dolan, P. Kanti, and E. Winstanley, “Brane Decay of a (4+n)-Dimensional Rotating Black Hole. III. Spin-1/2 particles,” *JHEP* **0703** (2007) 019, arXiv:hep-th/0608193 [hep-th].

[4] A. Flachi, M. Sasaki, and T. Tanaka, “Spin polarization effects in micro black hole evaporation,” *JHEP* **0905** (2009) 031, arXiv:0809.1006 [hep-ph].

[5] K. Murata and J. Soda, “A Note on separability of field equations in Myers-Perry spacetimes,” *Class.Quant.Grav.* **25** (2008) 035006, arXiv:0710.0221 [hep-th].

[6] J. V. Rocha, R. Santarelli, and T. Delsate, “Collapsing rotating shells in Myers-Perry-AdS5, spacetime: a perturbative approach,” arXiv:1406.1461 [gr-qc].

[7] I. I. Cotaescu, “Dirac fermions in de Sitter and anti-de Sitter backgrounds,” *Rom.J.Phys.* **52** (2007) 895–940, arXiv:gr-qc/0701118 [gr-qc].

[8] S.-Q. Wu, “Separability of the massive Dirac’s equation in 5-dimensional Myers-Perry black hole geometry and its relation to a rank-three Killing-Yano tensor,” *Phys.Rev.* **D78** (2008) 064052, arXiv:0807.2114 [hep-th].

[9] A. Lopez-Ortega, “Dirac quasinormal modes of D-dimensional de Sitter spacetime,” *Gen.Rel.Grav.* **39** (2007) 1011–1029, arXiv:0704.2468 [gr-qc].

[10] R. Camporesi and A. Higuchi, “On the Eigen functions of the Dirac operator on spheres and real hyperbolic spaces,” *J.Geom.Phys.* **20** (1996) 1–18, arXiv:gr-qc/9505009 [gr-qc].

[11] Y. Brihaye, B. Hartmann, and S. Tojiev, “Stability of charged solitons and formation of boson stars in 5-dimensional Anti-de Sitter space-time,” *Class.Quant.Grav.* **30** (2013) 115009, arXiv:1301.2452 [hep-th].

[12] Y. Brihaye and B. Hartmann, “A Scalar field instability of rotating and charged black holes in (4+1)-dimensional Anti-de Sitter space-time,” *JHEP* **1203** (2012) 050, arXiv:1112.6315 [hep-th].

[13] D. Varshalovich, A. Moskalev, and V. Khersonsky, “Quantum Theory of Angular Momentum: Irreducible Tensor, Spherical Harmonics, Vector coupling coefficients, 3 NJ Symbols”,.

[14] A. H. Chamseddine *Phys. Lett.* **B233** 291 (1989) .

[15] E. W. Mielke and F. E. Schunck, “Boson stars: Early history and recent prospects,” gr-qc/9801063 .

[16] D. Astefanesei and E. Radu, “Boson stars with negative cosmological constant,” *Nucl. Phys.* **B665** 594 (2003) .

[17] A. Prikas, “Q stars in extra dimensions,” *Phys. Rev. D* **69** 125008 (2004) .
[18] B. Hartmann, J. Riedel, and R. Suciu, “Gauss-Bonnet boson stars,” *Phys. Lett. B* **726**, 906–912 (2013), arXiv:1308.3391 [gr-qc].

[19] V. Cardoso, O. J. Dias, J. P. Lemos, and S. Yoshida, “The Black hole bomb and superradiant instabilities,” *Phys. Rev. D* **70** (2004) 044039, arXiv:hep-th/0404096 [hep-th].

[20] Y. Brihaye and E. Radu, “Black hole solutions in $d = 5$ chern-simons gravity,” *JHEP* **1311**, 049 (2013).

[21] L. J. Henderson, R. B. Mann, and S. Stotyn, “Gauss-Bonnet Boson Stars with a Single Killing Vector,” arXiv:1403.1865 [gr-qc] (2014), arXiv:1403.1865 [gr-qc].

[22] Y. Brihaye and J. Riedel, “Rotating Boson Stars in Einstein-Gauss-Bonnet gravity,” *Phys. Rev. D* **89** (2014) 104060, arXiv:1310.7223 [gr-qc].