Perturbative methods for the Painlevé test

R. Conte

Service de physique de l’état condensé, CEA Saclay,
F–91191 Gif-sur-Yvette Cedex, France

Abstract. There exist many situations where an ordinary differential equation admits a movable critical singularity which the test of Kowalevski and Gambier fails to detect. Some possible reasons are: existence of negative Fuchs indices, insufficient number of Fuchs indices, multiple family, absence of an algebraic leading order. Mainly giving examples, we present the methods which answer all these questions. They are all based on the theorem of perturbations of Poincaré and computerizable.
## Contents

1. Examples to be settled  
   2. Basic theorems  
   3. Meaning of the negative Fuchs indices  
   4. The Fuchsian perturbative method  
      4.1 The simplest constructive example  
      4.2 An example needing order seven to conclude  
   5. The non-Fuchsian perturbative method  
   6. Miscellaneous perturbations  
   7. Conclusion
The “usual Painlevé test” of Kowalevski and Gambier [9], also called method of pole-like expansions, establishes necessary conditions for the Painlevé property (PP) (defined as the absence of movable critical singularities in the general solution of an ordinary differential equation). It does so by checking the existence of all converging Laurent series with a finite principal part susceptible to represent either the general solution or a particular solution.

Whenever there exists a movable multivaluedness in the general solution and the test of Kowalevski and Gambier fails to detect it, there exists a plethora of other methods able to perform this detection. The purpose of these notes is to explain these algorithmic methods not in full theoretical detail but on examples selected for their simplicity.

More details can be found in the lecture notes of a Cargèse school [5].

1 Examples to be settled

The following three ODEs possess a general solution with movable logarithms undetected by the “usual Painlevé test” of Kowalevski and Gambier

\[ u'' + 4uu' + 2u^3 = 0, \tag{1} \]
\[ u''' + 3uu'' - 4u'^2 = 0, \tag{2} \]
\[ u'' + uu'' - 2u'^2 = 0. \tag{3} \]

The first equation (1) has a single family of movable singularities

\[ u \sim u_0 \chi^p, \chi = x - x_0 \] (one should avoid the term branch which induces a confusion with branching i.e. multivaluedness)

\[ p = -1, \quad E_0 = u_0(u_0 - 1)^2 = 0, \quad \text{indices } (-1, 0), \quad u = \chi^{-1}, \tag{4} \]

with the puzzling fact that \( u_0 \) should be at the same time equal to 1 according to the equation \( E_0 = 0 \), and arbitrary according to the Fuchs index 0. The Laurent series is here reduced to its first term and only depends on one arbitrary constant. The movable logarithms, initially found by the \( \alpha \)-method [12] §13, p. 221), are exhibited in section 4.

The second equation (2), isolated by Bureau [2] p. 79), possesses two families

\[ p = -2, \quad u_0 = -60, \quad \text{indices } (-3, -2, -1, 20), \quad u = -60\chi^{-2} + u_{20}\chi^{18} + \ldots \tag{5} \]
\[ p = -3, \quad u_0 \text{ arbitrary}, \quad \text{indices } (-1, 0), \quad u = u_0\chi^{-3} - 60\chi^{-2}. \tag{6} \]

The first family has enough indices but not enough positive ones, while the second one has not enough indices, therefore none of the two families can represent the general solution. The movable logarithms are found by two methods, in sections 4 and 5.

The third equation has no power-law leading behaviour. Chazy [3, 4] had to establish a special theorem, using divergent series, to exhibit the movable logarithms. The failure appears in section 6.

A feature common of the method of Kowalevski when applied to these three ODEs is the impossibility to represent the general solution by some Laurent series with a finite principal part. Accordingly, the possible presence of multivaluedness in the missing part of the general solution cannot be tested by that method.

The common principle to all the methods performing such a detection is to perturb a particular solution into the general solution. Let us first recall the relevant theorems.

2 Basic theorems

Boldface letters denote multicomponent quantities.
Theorem of perturbations (Poincaré, Mécanique céleste [3]). Consider an ODE of order \( N \), of degree one in the highest derivative, depending on a small complex parameter \( \varepsilon \), defined in the canonical form
\[
\frac{du}{dx} = K[x, u, \varepsilon], \quad x \in \mathcal{C}, \ u \in \mathcal{C}^N, \ \varepsilon \in \mathcal{C}.
\] (7)

Let \((x_0, u_0, 0)\) be a point in \( \mathcal{C} \times \mathcal{C}^N \times \mathcal{C} \) and \( D \) be a domain containing \((x_0, u_0, 0)\). If \( K \) is holomorphic in \( D \),

- there exists a solution \( u(x, \varepsilon) \) satisfying the initial condition \( u(x_0, 0) = u_0 \),
- it is unique,
- it is holomorphic in a domain containing \((x_0, u_0, 0)\).

**Proof.** See any textbook. Note that \( K \) may be independent of \( \varepsilon \), in which case this is just the existence theorem of Cauchy.

**Remark.** More practically, the canonical form can also be defined as
\[
\frac{d^N u}{dx^N} = K'[x, u, \ldots, u^{(N-1)}].
\] (8)

**Definition.** Given a differential equation (DE)
\[
E(x, u) = 0
\] (9)
and a point \( u_0 \), the linear DE
\[
E'(x, u^{(0)})v = \lim_{\lambda \to 0} \frac{E(x, u^{(0)} + \lambda v) - E(x, u^{(0)})}{\lambda} = 0
\] (10)
in the unknown \( v \) is called the *linearized equation* in the neighborhood of \( u^{(0)} \) associated to the equation \( E(x, u) = 0 \). This was introduced by Darboux [7] under the name “équation auxiliaire”. The derivative \( E' \) is known under various names: Gâteaux derivative, linearized map, tangent map, Jacobian matrix, and sometimes Fréchet derivative.

Let us define the formal Taylor expansions
\[
u = \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)}, \ K = \sum_{n=0}^{+\infty} \varepsilon^n K^{(n)}.
\] (11)
The single equation (7) is equivalent to the infinite sequence
\[
\begin{align*}
n = 0 : \quad & \frac{du^{(0)}}{dx} = K^{(0)} = K[x, u^{(0)}, 0] \\
n \geq 1 : \quad & \frac{du^{(n)}}{dx} = K^{(n)} = K'[x, u^{(0)}, 0]u^{(n)} + R^{(n)}(x, u^{(0)}, \ldots, u^{(n-1)}).\end{align*}
\] (12) (13)

At order zero, the equation is nonlinear.
At order one, the equation, in the particular important case when \( K \) is independent of \( \varepsilon \), is the linearized equation (without rhs, since \( R^{(1)} = 0 \)) canonically associated to the nonlinear equation.
At higher orders, this is the same linearized equation with different rhs \( R^{(n)} \) arising from the previously computed terms, and only a particular solution is needed to integrate.

**Theorem II** (Poincaré 1890, Painlevé 1900, Bureau 1939). Take the assumptions of previous theorem. If the general solution of (7) is single valued in \( D \) except maybe at \( \varepsilon = 0 \), then
• \( \varepsilon = 0 \) is no exception, i.e. the general solution is also single valued there,
• every \( u^{(n)} \) is single valued.

**Proof.** See [12] p. 208. The main difficulty is to prove the convergence of the series. This theorem remains valid if one replaces “single valued” (Painlevé version) by “periodic” (Poincaré version) or “free from movable critical points” (Bureau version).

This feature (one nonlinear equation \((12)\), one linear equation \((13)\) with different rhs) is a direct consequence of perturbation theory, it is common to all methods aimed at building necessary stability conditions (following Bureau, we call stable an ODE with the PP). The equations may be differential like \((12)\)–\((13)\), or simply algebraic. Moreover, all the methods which we are about to describe (except the one of Painlevé) will reduce the differential problems to algebraic problems keeping the same feature, and the overall difficulty will be to solve one nonlinear algebraic equation, then one linear algebraic equation with a countable number (practically, a finite number) of rhs.

All the methods of the Painlevé test are applications of the last theorem:

1. the method of pole-like expansions of Kowalevski and Gambier [9],
2. the \( \alpha \)-method of Painlevé [12],
3. the method of Bureau [1],
4. the Fuchsian perturbative method [6],
5. the non-Fuchsian perturbative method [11].

These methods establish necessary conditions for the Painlevé property by building one or two perturbed equations from the original unperturbed equation, then by applying the theorem II at a point \( x_0 \) which is movable. This movable point can be either regular (method of Painlevé) or singular noncritical (all the others), which will require its previous transformation to a regular point (by a transformation close to \( u \to u^{-1} \)) for theorem II to apply. One is thus led to the equations \((13)\), i.e. to one linear DE with a sequence of rhs. In order to avoid movable critical points in the original equation, one requires single valuedness in a neighborhood of \( x_0 \) for:
- the general solution of the linear homogeneous equation, a particular solution of each of the successive linear inhomogeneous equations.

### 3 Meaning of the negative Fuchs indices

Basically, they are just the consequence of a resummation of a series. In particular, they do not imply the existence of an essential singularity. This is easier to understand if one starts from a given general solution rather than from a given ODE whose general solution may not be known in closed form.

The ODE with a meromorphic general solution [1]

\[
E \equiv u'' + 3uu' + u^3 = 0, \quad u = \frac{1}{x - a} + \frac{1}{x - b}, \quad a \text{ and } b \text{ arbitrary},
\]

has two families,

(F1) \( p = -1, u_0 = 1 \), indices \((-1, 1)\), \( u = \chi^{-1} + \ldots \),

(F2) \( p = -1, u_0 = 2 \), indices \((-2, -1)\), \( u = 2\chi^{-1} \),
and the negative index $-2$ must coexist with the meromorphy. The representation of the general solution \((14)\) as a Laurent series of \(x - x_0\) is the sum of two copies of an expansion of \(1/(x - c)\), and there exist two expansions of \(1/(x - c)\)

\[
(x - c)^{-1} = \sum_{j=-\infty}^{1} (c - x_0)^{-1-j}(x - x_0)^j, \quad |c - x_0| < |x - x_0|
\]  

(15)

\[
= \sum_{j=0}^{+\infty} -(c - x_0)^{-1-j}(x - x_0)^j, \quad |x - x_0| < |c - x_0|.
\]  

(16)

The family \((F_1)\) corresponds to the sum (first expansion with \(c = a = x_0\)) plus (second expansion with \(c = b\)), while the family \((F_2)\) corresponds to the sum (first expansion with \(c = a\)) plus (second expansion with \(c = b\)).

The second family series terminates : \(u = 2/(x - x_0)\), and its algorithmic perturbation into the doubly infinite Laurent series is handled in Ref. \([6]\) by the method of section \(4\).

### 4 The Fuchsian perturbative method

It allows to extract the information contained in the negative indices \([8]\), thus building infinitely many necessary conditions for the absence of movable critical singularities of the logarithmic type \([8]\).

The perturbation which describes it is close to the identity

\[
x \text{ unchanged, } u = \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)} : E = \sum_{n=0}^{+\infty} \varepsilon^n E^{(n)} = 0,
\]  

(17)

where, like for the \(\alpha-\)method, the small parameter \(\varepsilon\) is not in the original equation.

Then, the single equation \((18)\) is equivalent to the infinite sequence

\[
n = 0 \quad E^{(0)} \equiv E(x, u^{(0)}) = 0
\]

\[
\forall n \geq 1 \quad E^{(n)} \equiv E'(x, u^{(0)}) u^{(n)} + R^{(n)}(x, u^{(0)}, \ldots, u^{(n-1)}) = 0,
\]  

(19)

with \(R^{(1)}\) identically zero. From Theorem II, necessary stability conditions are

- the general solution \(u^{(0)}\) of \((18)\) has no movable critical points,
- the general solution \(u^{(1)}\) of \((19)\) has no movable critical points,
- for every \(n \geq 2\) there exists a particular solution of \((19)\) without movable critical points.

Order zero is just the complete equation for the unknown \(u^{(0)}\), so, in order to get some additional information, one must apply Theorem II for a perturbation different from \((17)\). One then uses the method of pole-like expansions at this order zero, only to obtain the leading term \(u^{(0)} \sim u^{(0)} \chi^p\) of all the acceptable families of movable singularities.

The precise steps of the algorithm are detailed in Refs. \([3, 5]\). The algorithm is purely algebraic, i.e. one does not perform any integration. The expression of each Taylor coefficient \(u^{(n)}\) is found to be a Laurent series of \(\chi = x - x_0\),

\[
u^{(n)} = \sum_{j=n\rho}^{+\infty} u^{(n)}_{j} \chi^j + p,
\]  

(20)
in which $\rho$ is the smallest Fuchs index (therefore lower than or equal to $-1$), so that the full expression $u$ is a “full” Laurent series, i.e. one whose powers extend to both infinities.

The Fuchsian perturbative method (as well as the non-Fuchsian one which will be seen section 5) is useful if and only if the zeroth order $n = 0$ fails to describe the general solution. This may happen for two reasons. The most common one is a negative Fuchs index in addition to $-1$ counted once, the second, less common one is a multiplicity higher than one for some family, as in the single family of equation (1).

4.1 The simplest constructive example

The equation (1) is the simplest example to understand this method, because
1. there exists a movable logarithm, as shown by the $\alpha$-method ([12] §13, p. 221),
2. the method of pole-like expansions fails to find it,
3. the Fuchsian perturbative method finds it after a very short computation, easy to do by hand since all series terminate.

The single family, in the notation of this section, is $p = -1$, $E^{(0)}_0 = u^{(0)}_0 (u^{(0)}_0 - 1)^2 = 0$, indices $(-1, 0)$,

and the order $n = 0$ yields the one-parameter series

$$u^{(0)} = \chi^{-1} \text{ (the series terminates).}$$

At order $n = 1$, the derivative of $E$ at point $u = u^{(0)}$ is

$$E'(x, u^{(0)}) = \partial^2_x + 4\chi^{-1}\partial_x + 2\chi^{-2},$$

so that the linearized equation for $u^{(1)}$ is of Fuchsian type (by the way, this is why we call its indices “Fuchs indices” and not “resonances”, a word which refers to no resonance phenomenon).

The computation of the Laurent series (20) for $u^{(1)}$ (with $n = 1, p = -1, \rho = -1$) is made in one computer loop by increasing values of $j$ and exhibits no logarithms. The result is

$$u^{(1)} = u^{(1)}_{-1} \chi^{-2} + u^{(1)}_0 \chi^{-1}, \text{ } u^{(1)}_{-1} \text{ and } u^{(1)}_0 \text{ arbitrary.}$$

The sum $u^{(0)} + \epsilon u^{(1)}$ is the beginning of a series which now depends on two arbitrary parameters, namely $x_0$ and $\epsilon u^{(1)}_0$ (one can indeed set $u^{(1)}_{-1}$ to zero without loss of generality since it represents a perturbation of $x_0$). With the retained value

$$u^{(1)} = u^{(1)}_0 \chi^{-1}, \text{ } u^{(1)}_0 \text{ arbitrary,}$$

the order $n = 2$ is

$$E^{(2)} = E'(x, u^{(0)}) u^{(2)} + 6u^{(0)} u^{(1)}^2 + 4u^{(1)} u^{(1)'} = \chi^{-2}(\chi^2 u^{(2)})'' + 2u^{(1)}_0 \chi^{-3} = 0,$$

and the Laurent series (20) for the particular solution $u^{(2)}$ is found not to exist because of a movable logarithm

$$u^{(2)} = -2u^{(1)}_0 \chi^{-1}(\log \chi - 1).$$

The movable logarithmic branch point is therefore detected in a systematic way at order $n = 2$ and index $i = 0$.

The necessity to perform a perturbation arises from the multiple root of the equation for $u^{(0)}_0$, responsible for the insufficient number of arbitrary parameters in the zeroth order series $u^{(0)}$. 
4.2 An example needing order seven to conclude

In the equation (2), the first family provides, at zeroth order, only a two-parameter expansion and, when one checks the existence of the perturbed solution

\[ u = \sum_{n=0}^{+\infty} \varepsilon^n \left\{ \sum_{j=0}^{+\infty} u_j^{(n)} \chi^{j-2-3n} \right\}, \]

one finds that coefficients \( u_{20}^{(0)}, u_{-3}^{(1)}, u_{-2}^{(1)}, u_{-1}^{(1)} \) can be chosen arbitrarily, and, at order \( n = 7 \), one finds two violations \[ Q_{-1}^{(7)} \equiv u_{20}^{(0)} u_{-3}^{(1)} = 0, \]

implying the existence of a movable logarithmic branch point.

Remark [11]. The value \( n = 7 \) is the root of the linear equation \( n (i_{\min} - p) + (i_{\max} - p) = -1 \), with \( p = -2, i_{\min} = -5, i_{\max} = 18 \), linking the pole order \( p \) in the Fuchsian case \( c = 0 \), the smallest and the greatest Fuchs indices. It expresses the condition for the first occurrence of a power \( \chi^{-1} \), leading by integration to a logarithm, in the r.h.s. \( R^{(n)} \) of the linear inhomogeneous equation (19), r.h.s. created by the nonlinear terms \( 3uu'' - 4u^2 \).

As to the second family, it is useless for the Fuchsian perturbative method, because the two arbitrary coefficients corresponding to the two Fuchs indices \((-1, 0)\) are already present at zeroth order.

5 The nonFuchsian perturbative method

Every time the family under study has less Fuchs indices than the differential order \( N \), the Fuchsian perturbation method fails to build a representation of the general solution, thus possibly missing some stability conditions. Such an example is the second family of the equation (2). The missing solutions of the auxiliary equation (10) are then nonFuchsian solutions.

Although there is no difficulty to algorithmically compute expansions for the nonFuchsian solutions, these are of no immediate help, due to their generic divergence.

There is one situation where some stability conditions can be generated \( \text{algorithmically} \). It occurs when the two following conditions are met [11].

1. There exists a particular solution \( u = u^{(0)} \) which is known globally, is meromorphic and has at least one movable pole at a finite distance denoted \( x_0 \).

2. The only singular points of the linearized equation \( E^{(1)} = 0 \) are \( x = x_0 \), nonFuchsian, and \( x = \infty \), Fuchsian.

Then, the property that a fundamental set of solutions \( u^{(1)} \) be locally single valued near \( \chi = x - x_0 = 0 \) is equivalent to the same property near \( \chi = \infty \). This is the global nature of \( u^{(0)} \) which allows the study of the point \( \chi = \infty \), easy to perform with the Fuchsian perturbative method.

An important technical bonus is the lowering of the differential order \( N \) of equation \( E^{(1)} = 0 \) by the number \( M \) of arbitrary parameters \( c \) which appear in \( u^{(0)} \). Indeed, again since \( u^{(0)} \) is closed form, its partial derivatives \( \partial_c u^{(0)} \) are closed form and are particular solutions of \( E^{(1)} = 0 \), which allows this lowering of the order.

At each higher perturbation order \( n \geq 2 \), one similarly builds particular solutions \( u^{(n)} \) as expansions near \( \chi = \infty \) and one requires the same properties.
Let us just take one example. In section 4.2, the fourth order equation (2) has been proven to be unstable after a computation practically intractable without a computer. Let us now prove this result without computation at all [11]. For the global two-parameter solution

\[ u^{(0)} = c\chi^{-3} - 60\chi^{-2}, \quad (c, x_0) \text{ arbitrary}, \]  

(30)

the linearized equation

\[ E^{(1)} = E'(x, u^{(0)})u^{(1)} = [\partial_x^4 + 3u^{(0)}\partial_x^2 - 8u^{(0)'}\partial_x + 3u^{(0)''}]u^{(1)} = 0 \]  

(31)

has only two singular points \( \chi = 0 \) (nonFuchsian) and \( \chi = \infty \) (Fuchsian), it admits the two global single valued solutions \( \partial_{x_0}u^{(0)} \) and \( \partial_cu^{(0)} \), i.e. \( u^{(1)} = \chi^{-4}, \chi^{-3} \). The lowering by \( M = 2 \) units of the order of the linearized equation (31) is obtained with

\[ u^{(1)} = \chi^{-4}v : [\partial_x^2 - 16\chi^{-1}\partial_x + 3c\chi^{-3} - 60\chi^{-2}]v'' = 0, \]  

(32)

and the local study of \( \chi = \infty \) is unnecessary, since one recognizes the confluent hypergeometric equation. The two other solutions in global form are

\[ c \neq 0 : \quad v_1'' = \chi^{-3}F_1(24; -3c/\chi) = \chi^{17/2}J_{23}(\sqrt{12c/\chi}), \]  

\[ v_2'' = \chi^{17/2}N_{23}(\sqrt{12c/\chi}), \]  

(33)

(34)

where the hypergeometric function \( _0F_1(24; -3c/\chi) \) is single valued and possesses an isolated essential singularity at \( \chi = 0 \), while the function \( N_{23} \) of Neumann is multivalued because of a \( \log \chi \) term.

Remark. The local study of (31) near \( \chi = 0 \) provides the formal expansions for the two nonFuchsian solutions

\[ \chi \to 0, \ c \neq 0 : \quad u^{(1)} = e^{\pm\sqrt{-12c/\chi}}\chi^{31/4}(1 + O(\sqrt{\chi})), \]  

(35)

detecting the presence in (31) of an essential singularity at \( \chi = 0 \), but the generically null radius of convergence of the formal series forbids to conclude to the multivaluedness of \( u^{(1)} \). A nonobvious result is the existence, as seen above, of a linear combination of the two formal expansions (33) which is single valued.

An application to the Bianchi IX cosmological model can be found in Ref. [10].

6 Miscellaneous perturbations

The differential complexity of the \( \alpha \)-method explains why it usually succeeds in case of failure of all the other methods, which only have an algebraic complexity.

For equation (3), there exists no perturbation satisfying the assumptions of Theorem II page 4, there only exist singular perturbations, i.e. which discard the highest derivative. Since they however give the correct information, it would be desirable to extend Theorem II in that direction. Meanwhile, the reasoning given below cannot be considered as a proof, and one should refer to the proof of Chazy.

Equation (3) is handled by the singular perturbation

\[ u = \varepsilon^{-1} \sum_{n=0}^{+\infty} \varepsilon^n u^{(n)}, \quad E = \varepsilon^{-2} \sum_{n=0}^{+\infty} \varepsilon^n E^{(n)}, \]  

(36)
which excludes \( u''' \) from the simplified equation and for which all the series happen to terminate, which makes the computation quite easy

\[
E^{(0)} \equiv u^{(0)}u^{(0)''} - 2u^{(0)''} = 0 \tag{37}
\]

\[
u^{(0)} = c\chi^{-1}, \ \chi = x - x_0, \ (x_0, c) \text{ arbitrary}, \tag{38}
\]

\[
E^{(1)} \equiv c(\chi^{-3}(\chi^2 u^{(1)})'' - 6\chi^{-4}) = 0, \tag{39}
\]

\[
u^{(1)} = 6\chi^{-1}(\log \chi - 1). \tag{40}
\]

7 Conclusion

One should not be afraid of negative Fuchs indices. Indeed, at the level of the linearized equation i.e. of the term \( n = 1 \) of a perturbation process, the sign of these Fuchs indices does not matter at all. Their presence does not imply the existence of essential singularities, as shown on an elementary example, and there do exist algorithmic methods to build no-log conditions from them, in the same way as the widely known method of pole-like expansions builds no-log conditions from the positive indices.

Acknowledgments

The author would like to thank the Bharatidasan University for its warm hospitality and support, and the Ministère des affaires étrangères for travel support.

References

[1] F. J. Bureau, Sur la recherche des équations différentielles du second ordre dont l'intégrale générale est à points critiques fixes, Bulletin de la Classe des Sciences XXV (1939) 51–68.

[2] F. J. Bureau, Differential equations with fixed critical points, Annali di Matematica pura ed applicata LXVI (1964) 1–116.

[3] J. Chazy, Sur la limitation du degré des coefficients des équations différentielles algébriques à points critiques fixes, C. R. Acad. Sc. Paris 155 (1912) 132–135.

[4] J. Chazy, Sur la limitation du degré des coefficients des équations différentielles algébriques à points critiques fixes, Acta Math. 41 (1918) 29–69.

[5] R. Conte, The Painlevé approach to nonlinear ordinary differential equations, The Painlevé property, one century later, 112 pages, ed. R. Conte, CRM series in mathematical physics (Springer, Berlin, 1999) (Cargèse school, 3–22 June 1996). solv-int/9710020.

[6] R. Conte, A. P. Fordy and A. Pickering, A perturbative Painlevé approach to nonlinear differential equations, Physica D 69 (1993) 33–58.

[7] G. Darboux, Sur les équations aux dérivées partielles, C. R. Acad. Sc. Paris 96 (1883) 766–769.

[8] A. P. Fordy and A. Pickering, Analysing negative resonances in the Painlevé test, Phys. Lett. A 160 (1991) 347–354.
[9] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critiques fixes, Thèse, Paris (1909); Acta Math. 33 (1910) 1–55.

[10] A. Latifi, M. Musette and R. Conte, The Bianchi IX (mixmaster) cosmological model is not integrable, Phys. Letters A 194 (1994) 83–92; 197 (1995) 459–460.

[11] M. Musette et R. Conte, Non-Fuchsian extension to the Painlevé test, Phys. Lett. A 206 (1995) 340–346.

[12] P. Painlevé, Mémoire sur les équations différentielles dont l’intégrale générale est uniforme, Bull. Soc. Math. France 28 (1900) 201–261.

[13] H. Poincaré, Les méthodes nouvelles de la mécanique céleste, 3 volumes (Gauthier-Villars, Paris, 1892, 1893, 1899).