Zero-Hopf bifurcation in the general Van der Pol-Duffing equation

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Abstract

We investigate the dynamical behavior of the general Van der Pol-Duffing equation
\[
\dot{x} = -\nu (x^3 - \mu x - y), \quad \dot{y} = -hz + kx - \alpha y, \quad \dot{z} = \beta y.
\]
This model generalizes the behavior of a Van der Pol-Duffing circuit with parallel resistor. We prove the existence of periodic solutions that bifurcate from the origin of coordinates for different values of the parameters. For some of these values, we can prove the existence of up to three periodic solutions and also two invariant tori that may bifurcate from the origin of that system. These results are obtained using very recent results on averaging theory.

Keywords: Averaging theory, Periodic solutions, Zero–Hopf bifurcation, 2010 MSC: 34C29, 37C27.

1. General Van der Pol-Duffing Equation

Introduction and statements of the main results

The aim of this paper is to study the bifurcation of periodic solutions and invariant tori in the general Van der Pol-Duffing differential system

\[
\begin{align*}
\dot{x} &= -\nu (x^3 - \mu x - y), \\
\dot{y} &= -hz + kx - \alpha y, \\
\dot{z} &= \beta y,
\end{align*}
\]

where \(\alpha, h, \beta, k, \nu\) and \(\mu\) are real parameters. In 2008, Matouk and Agiza \textsuperscript{5} studied the Hopf bifurcation in system \textsuperscript{1} with \(\alpha = h = -1\). They also

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used numerical methods for detecting chaotic behavior in this system. It is well known that chaotic systems are very important in nonlinear dynamical systems and they have been intensively investigated by many authors. Roughly speaking a chaotic system is a nonlinear deterministic system that presents a complex and unpredictable behavior in the sense that it has a great sensitive dependence upon initial conditions.

The results of [5] show that for $\alpha = h = -1$ system (1) has periodic solutions and chaotic attractors bifurcating from certain equilibrium points.

The differential system (1) (with arbitrary $\alpha$ and $h$) was proposed by Zhao et. al. [9] as a generalization of the original Van der Pol-Duffing oscillator studied in [5]. Motivated by [5] Zhao et. al. detected the occurrence of hidden chaotic attractors as well as periodic orbits and chaotic attractors in system (1). The Hopf bifurcation analysis done in [9] is focused in the periodic orbits bifurcating from the pair of symmetric equilibria $P_{\pm} = \left( \pm \sqrt{\mu}, 0, \pm \sqrt{\mu k h} \right)$.

One of the purpose of this work is to prove that a rich bifurcating phenomena may also emerge from the origin of system (1) using the existence of Hopf and zero-Hopf bifurcations at this equilibrium point. Namely, we analytically provide sufficient conditions for the existence of periodic solutions emerging from the origin of system (1) and information about the stability of such solutions. Finally, we also provide sufficient conditions for a torus bifurcation in the periodic solutions emerging from the origin (when $P_+$ and $P_-$ coalesce). Consequently, our paper improves and complements the results presented in [5] and [9].

First of all, we shall study the classical Hopf bifurcation in the origin of coordinates of system (1). The technique used here is presented in Section 2 (see also [1, 6]).

Our first main result complements the bifurcation analysis provided in [9].

Theorem 1. Let $\bar{k} = -\frac{\alpha (a \mu \nu - \beta h - \mu^2 \nu^2)}{\nu (\alpha - \mu \nu)}$ and assume that $(\beta h \mu \nu)(\mu \nu - \alpha) > 0$. The following statements hold:

(a) The point $(0,0,0,\bar{k})$ is a Hopf point (see Section 3) of system (1) whose corresponding eigenvalues are $\pm i \sqrt{\frac{\beta h \mu \nu}{\nu (\alpha - \mu \nu)}}$ and $\alpha - \mu \nu$;

(b) Let $(\alpha, \beta, k, h, \nu) \in \mathbb{R}^5$ with $\alpha = \mu \nu \left( 1 - \frac{\beta h}{2\nu} \right)$ and $\omega > 0$. Then, for $k$ sufficiently close to the bifurcation value $\bar{k}$, the following statements hold:

(i) If $(\omega^2 - \beta h)(\beta h \mu^2 \nu^2 - \omega^4) > 0$, system (1) has a supercritical Hopf bifurcation at the origin of coordinates,

(ii) If $(\omega^2 - \beta h)(\beta h \mu^2 \nu^2 - \omega^4) < 0$, system (1) has a subcritical Hopf bifurcation at the origin of coordinates.

The proof of Theorem 1 is provided in Section 4.

Now we are going to study the zero-Hopf bifurcation at the origin of coordinates of system (1). We recall that a zero-Hopf equilibrium is an equilibrium of
the differential system where the Jacobian matrix has a zero eigenvalue and a pair of purely imaginary conjugate eigenvalues. By direct computations we can show that there exists five parameter families of the general Van der Pol Duffing differential system for which the origin of coordinates is a zero-Hopf equilibrium point. Namely, when the parameters of the system satisfy the following relations

\[ \alpha = 0, \quad \beta = \frac{\omega^2}{h}, \quad \nu = 0, \]

or

i) \[ h = 0, \quad k = \frac{-\mu^2 \nu^2 - \omega^2}{\nu} \quad \text{and} \quad \alpha = \mu \nu; \]

ii) \[ \beta = 0, \quad k = \frac{-\mu^2 \nu^2 - \omega^2}{\nu}, \quad \alpha = \mu \nu; \]

iii) \[ \alpha = 0, \quad k = \frac{\omega^2}{\nu}, \quad h = 0 \quad \text{and} \quad \mu = 0. \]

iv) \[ \alpha = 0, \quad \beta = \frac{\omega^2}{h} + \frac{\omega^2}{\nu}, \quad h = 0 \quad \text{and} \quad \mu = 0. \]

In the next theorems we will provide sufficient conditions for the existence of periodic orbits bifurcating from the origin of system (1) for parameters \( \epsilon \)-close to those provided in cases (i)–(iv). The first three theorems concern with the conditions (i)–(iii) and all of them will be proved in Section 4 using the averaging method (see Section 3) for detecting all periodic solutions.

**Theorem 2.** Consider system (1) with

\[ h = \epsilon h_1, \quad k = \frac{-\mu^2 \nu^2 - \omega^2}{\nu} + \epsilon k_1, \quad \alpha = \mu \nu + \epsilon \alpha_1, \]

where \( (h_1, k_1, \alpha_1) \in \mathbb{R}^3 \) with \( \epsilon > 0 \) being a parameter. If \((\alpha_1 \omega^2 + \beta_1 h_1 \nu) \nu < 0\), then for \( \epsilon \) sufficiently small, system (1) will have a periodic solution emerging from the origin of coordinates. Moreover, setting

\[ \lambda_1 = \beta_1 h_1 \nu \quad \text{and} \quad \lambda_2 = \alpha_1 \omega^2 + \beta_1 h_1 \nu, \]

the periodic solution will be an attractor if \( \lambda_1 < 0, \lambda_2 < 0 \), a repeller if \( \lambda_1 > 0, \lambda_2 > 0 \), and will have a two dimensional stable manifold and a two dimensional unstable manifold if \( \lambda_1 \lambda_2 < 0 \).

**Theorem 3.** Consider system (1) with

\[ \beta = \epsilon \beta_1, \quad k = \frac{-\mu^2 \nu^2 - \omega^2}{\nu} + \epsilon k_1 \quad \text{and} \quad \alpha = \mu \nu + \epsilon \alpha_1, \]

where \( (\beta_1, k_1, \alpha_1) \in \mathbb{R}^3 \) and \( \epsilon > 0 \) being a parameter. If \((\alpha_1 \omega^2 + \beta_1 h_1 \nu) \nu < 0\), then for \( \epsilon \) sufficiently small, system (1) will have a periodic solution emerging from the origin of coordinates. Moreover, setting

\[ \lambda_1 = \beta_1 h_1 \nu \quad \text{and} \quad \lambda_2 = \alpha_1 \omega^2 + \beta_1 h_1 \nu, \]

the periodic solution will be an attractor if \( \lambda_1 < 0, \lambda_2 < 0 \), a repeller if \( \lambda_1 > 0, \lambda_2 > 0 \), and will have a two dimensional stable manifold and a two dimensional unstable manifold if \( \lambda_1 \lambda_2 < 0 \).
Theorem 4. Consider system \([1]\) with
\[
\alpha = \varepsilon \mu_1 \nu + \varepsilon^2 \alpha_2, \quad k = -\frac{\omega^2}{\nu} + \varepsilon k_1 + \varepsilon^2 k_2, \quad \mu = \varepsilon \mu_1 + \varepsilon^2 \mu_2, \quad h = \varepsilon h_1 + \varepsilon^2 h_2,
\]
where \((\alpha_2, \mu_1, \mu_2, k_1, k_2, h_1, h_2) \in \mathbb{R}^7\) being \(\varepsilon > 0\) a parameter and assume that
\[
\nu (\omega^2 (\mu_2 \nu - \alpha_2) - \beta h_1 \mu_1 \nu) > 0.
\]
Then, for \(\varepsilon\) sufficiently small, system \([1]\) will have a periodic solution emerging from the origin of coordinates. Moreover, setting
\[
\lambda_1 = h_1 \beta \mu_1 \nu, \quad \lambda_2 = \omega^2 (\alpha_2 - \mu_2 \nu) + \beta h_1 \mu_1 \nu,
\]
the periodic solution will be an attractor if \(\lambda_1 < 0, \lambda_2 < 0\), a repeller if \(\lambda_1 > 0, \lambda_2 > 0\), and will have a two dimensional stable manifold and a two dimensional unstable manifold if \(\lambda_1 \lambda_2 < 0\).

In the next result we present sufficient conditions for the existence of periodic solutions and two invariant tori emerging from the origin of coordinates of system \([1]\) with the parameter vector \((\alpha, \beta, \mu)\) \(\varepsilon\)-close to \((0, (k \nu + \omega^2)/h, 0)\). It will be proved in Section 4 using averaging theory (see Theorems 8 and \([A]\)).

Theorem 5. Consider system \([1]\) with
\[
\alpha = \varepsilon \alpha_1, \quad \beta = \frac{k \nu + \omega^2}{h} + \varepsilon \beta_1 \quad \text{and} \quad \mu = \varepsilon \mu_1,
\]
where \((\alpha_1, \beta_1, \mu_1) \in \mathbb{R}^3_+\) being \(\varepsilon > 0\) is a parameter. Let \(\rho = \). Then, there exist \(\varepsilon_0 > 0\) sufficiently small such that for \(0 < \varepsilon < \varepsilon_0\) the following statements hold.

i) If \(\rho \neq 0\), system \([1]\) has a periodic solution \(\varphi_0(t, \varepsilon)\) emerging from the origin of coordinates. Moreover, setting
\[
\lambda_1 = \frac{(k \nu + \omega^2) (2 \alpha_1 \omega^2 + 5 \alpha_1 k \nu + k \mu_1 \nu^2)}{4 \omega^2 h^2}, \quad \lambda_2 = -\frac{8 \alpha_1 \omega^2 + 15 \alpha_1 k \nu + 2 k \mu_1 \nu^2}{4 \omega^2},
\]
the periodic solution \(\varphi_0(t, \varepsilon)\) will be an attractor if \(\lambda_1 < 0, \lambda_2 < 0\); a repeller if \(\lambda_1 > 0, \lambda_2 > 0\); and will have a two dimensional stable manifold and a two dimensional unstable manifold if \(\lambda_1 \lambda_2 < 0\);

ii) If \(\rho > 0\), system \([1]\) has two additional periodic solutions \(\varphi_{\pm}(t, \varepsilon)\) emerging from the origin of coordinates for \(\varepsilon\) sufficiently small. Moreover, setting
\[
\lambda_1 = A + \frac{\alpha_1}{8 \omega^2} \sqrt{B}, \quad \lambda_2 = A - \frac{\alpha_1}{8 \omega^2} \sqrt{B},
\]
with
\[
A = \frac{(k \nu + 2 \omega^2) (4 \alpha_1 \omega^2 + k \nu (5 \alpha_1 + 2 \mu_1 \nu))}{8 k \nu \omega^2},
\]
\[
B = \frac{1}{\alpha_1^2 k \nu^2} \left(64 \alpha_1^3 \omega^8 + k^4 \nu^4 (145 \alpha_1^2 + 276 \alpha_1 \mu_1 \nu + 36 \mu_1^2 \nu^2)
+ 4 k^3 \nu^3 \omega^2 (153 \alpha_1^2 + 136 \alpha_1 \mu_1 \nu + 12 \mu_1^2 \nu^2)
+ 4 k^2 \nu^2 \omega^4 (221 \alpha_1^2 + 84 \alpha_1 \mu_1 \nu + 4 \mu_1^2 \nu^2) + 32 \alpha_1 k \nu \omega^6 (15 \alpha_1 + 2 \mu_1 \nu)\right),
\]
the periodic solutions $\varphi_{\pm}(t, \varepsilon)$ will be an attractor if $\text{Re}(\lambda_1) < 0$, $\text{Re}(\lambda_2) < 0$; a repeller if $\text{Re}(\lambda_1) > 0$ and $\text{Re}(\lambda_2) > 0$; and will have a two dimensional stable manifold and a two dimensional unstable manifold if $\text{Re}(\lambda_1) \text{Re}(\lambda_2) < 0$.

iii) Let

$$\ell_{1,1} = -\frac{3\pi\nu^2 (5k^2\nu^2 + 16k\nu\omega^2 + 8\omega^4)}{8k\omega^5}.$$ 

If $\ell \neq 0$, then there exists a smooth curve $\mu_1(\varepsilon)$, satisfying $\mu_1(\varepsilon) = \mu_0 + O(\varepsilon)$, with $\mu_0 = -\left(\alpha_1(5k\nu + 4\omega^2)\right)/2k\nu^2$, such that a unique invariant torus bifurcates from each of the periodic orbits $\varphi_{\pm}(t; \varepsilon)$ whenever $\ell(\mu_1 - \mu_1(\varepsilon)) < 0$. Moreover, if $\ell > 0$ (resp. $\ell < 0$) these torus are unstable (resp. asymptotically stable), whereas the periodic orbits $\varphi_{\pm}(t; \varepsilon)$ are asymptotically stable (resp. unstable).

Theorem 5 states that up to three periodic solutions can bifurcate from the origin of (1) and that two invariant torus may also bifurcates from the periodic solutions $\varphi_{\pm}(t, \varepsilon)$ when it changes stability. Moreover, from Theorem 5 we can detect a rich bifurcation phenomena that takes place in the origin of system (1).

For certain parameter values system (1) has simultaneously 3 periodic orbits and two invariant tori in a neighborhood of the origin (see Figure 1). Namely, we take $\varepsilon = 1/70$, $\alpha_1 = -1$, $\mu_1 = 4.00133$, $h = 1$, $k = 4/3$, $\nu = 1$, $\omega = 1$ and $\beta_1 = 1/500$.

The stability of the periodic solutions are directly determined by Theorem 5 being $\varphi_{\pm}(t; \varepsilon)$ repeller and $\varphi_0(t; \varepsilon)$ having a two dimensional stable manifold and a two dimensional unstable manifold. In other hand, the invariant tori surrounding the periodic solutions $\varphi_{\pm}(t; \varepsilon)$ are stable (See Figure 2).

2. Hopf bifurcation and Lyapunov coefficients

In this section, we present some basic notions about Hopf bifurcations and the first Lyapunov coefficient. The theory of Lyapunov coefficients can be found in [3, Chapters 3 and 10]. We also refer the reader to [7] where the Lyapunov coefficients are calculated in great detail up to order 4.

Consider the differential equation

$$x' = f(x, \mu)$$

where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ are respectively vectors representing phase variables and control parameters.

A Hopf point $(x_0, \mu_0)$ is an equilibrium point of (2) where the Jacobian matrix $A = f_s(x_0, \mu_0)$ has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega$, $\omega > 0$ and admits no other eigenvalues with zero real part.

Denoting the variable $x - x_0$ also by $x$ we write

$$F(x) = f(x, \mu_0)$$
Figure 1: This figure shows two invariant tori surrounding the periodic orbits \( \varphi_\pm(t; \varepsilon) \) (represented by dashed lines). The periodic orbit \( \varphi_0(t; \varepsilon) \) is represented by a solid line.

as

\[
F(x) = Ax + \frac{1}{2} B(x, x) + \frac{1}{6} C(x, x, x) + \mathcal{O}(||x||^4)
\]

where \( A = f_x(0, \mu_0) \),

\[
B_i(x, y) = \sum_{j,k=1}^{n} \left. \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k} \right|_{\xi=0} x_j y_k
\]

and

\[
C_i(x, y, z) = \sum_{j,k,l=1}^{n} \left. \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\xi=0} x_j y_k z_l.
\]

Let \( p, q \in \mathbb{C}^n \) be vectors such that

\[
Aq = i\omega q, \quad A^T p = -i\omega p, \quad \bar{q} q = \bar{p} q = 1,
\]
where $A^\top$ is the transposed of the matrix $A$. We define the first Lyapunov coefficient as

$$l_1 = \frac{1}{2\omega} \text{Re} \left( \bar{p} C(q, q, \bar{q}) - 2\bar{p} B(q, A^{-1} B(q, \bar{q})) + \bar{p} B(\bar{q}, (2\omega i I_n - A)^{-1} B(q, q)) \right),$$  

where $I_n$ is the $n \times n$ identity matrix.

A Hopf point is called *transversal* if the parameter dependent complex eigenvalues cross the imaginary axis with non-zero derivative. In a neighborhood of a transversal Hopf equilibrium point with $l_1 \neq 0$ the dynamical behavior of the system (2), reduced to the family of parameter dependent continuations of the center manifold, is orbitally topologically equivalent to the following complex normal form

$$w' = (\eta + i\sigma)w + l_1 |w|^2.$$

Here $w \in \mathbb{C}$, $\eta$, $\sigma$ and $l_1$ are the real functions having derivatives of arbitrary high order, which are continuations of $0$, $\omega$ and the first Lyapunov coefficient at the transversal Hopf point. As $l_1 < 0$ ($l_1 > 0$) one family of stable (unstable) periodic orbits can be found on this family of manifolds, shrinking to this equilibrium point (cf. [7]). Thus we have the following lemma

**Lemma 6.** Consider the differential system (2) having the Hopf point $(x_0, \mu_0)$ and assume that $l_1 \neq 0$ and $\text{Re}(\lambda_{\pm}(\mu_0)) \neq 0$. Then following statements hold:

(i) If $l_1 > 0$, the differential system (2) has a supercritical Hopf bifurcation at $x_0$. 

Figure 2: The solutions of system (1) starting at $(0.267281, 0, 0.577784)$, $(0.12869, 0, 0.26998)$, $(0.226519, 0, 0.541664)$ and $(0.179934, 0, 0.455291)$ intersecting the cross section $y = 0$. The solutions are being attracted by the invariant tori.
(ii) If $l_1 < 0$, the differential system \((2)\) has a subcritical Hopf bifurcation at $x_0$.

3. Averaging theory for detecting periodic solutions and invariant tori

In this section we shall present the averaging method of first and second orders for detecting periodic solutions. Moreover, we also shall use averaging method for detecting invariant torus bifurcation. In the next result we introduce the classical version of the averaging theorem (Theorem 7). This theorem was used for detecting the periodic solutions in theorems 2, 3, 4 and 5.

The averaging theory provides sufficient conditions for the existence of periodic solutions of non-autonomous differential systems written in the following standard form:

$$
\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 \dot{F}(t, x, \varepsilon), \quad (t, x, \varepsilon) \in \mathbb{R} \times \Omega \times [-\varepsilon_0, \varepsilon_0],
$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^n$. We assume that $F_i$, for $i = 1, \ldots, 5$, and $\dot{F}$ are sufficiently smooth functions and $T$-periodic in the variable $t$.

For $i = 1, 2$, we define the averaged functions of order $i$ of system \((4)\) as

$$
g_i(z) = \frac{y_i(T, z)}{i!},
$$

where

$$
y_1(t, z) = \int_0^t F_1(\tau, z) d\tau,
$$

$$
y_2(t, z) = \int_0^t 2F_2(\tau, z) + 2\frac{\partial F_1}{\partial x}(\tau, x(\tau, x, 0))y_1(\tau, z) d\tau.
$$

**Theorem 7** ([4]). Assume that, for some $l \in \{1, 2\}$, $g_l$ is the first non-vanishing averaging function of system \((4)\). If there exists $z^* \in \Omega$ such that $g_l(z^*) = 0$ and $|Dg_l(z^*)| \neq 0$, then for $|\varepsilon| \neq 0$ sufficiently small there exists a $T$-periodic solution $\varphi(t, \varepsilon)$ of system \((4)\) such that $\varphi(0, 0) = z^*$.

**Theorem 8** ([8]). Consider the differential system \((4)\) and suppose that the conditions of Theorem 7 are satisfied. If all eigenvalues of $Dg_l(z^*)$ have negative real parts, then the corresponding periodic solution $\varphi(t, \varepsilon)$ of system \((4)\) is asymptotically stable for $\varepsilon > 0$ sufficiently small. Conversely, if one of the eigenvalues has positive real part, then $\varphi(t, \varepsilon)$ is unstable.

The next result will use averaging theory to detect torus bifurcation in two-parameter families of differential systems in the standard form \((4)\). Let

$$
g_l(x; \mu) = (g^1_l(x; \mu), g^2_l(x; \mu)) = \int_0^T F_1(t, x; \mu) dt.
$$
be the first order averaged function where \( \mu \in \mathbb{R} \) is a parameter. We define the *averaged system* corresponded with system (4) as
\[
\dot{x} = \varepsilon g_1(x; \mu).
\] (6)

Assume the following hypothesis

**A1.** There exists a continuous curve \( \mu \in J \mapsto x_0 \in \Omega \), defined on an interval \( J \ni \mu_0 \), such that \( g_1(x_0; \mu) = 0 \) for every \( \mu \in J \subset \mathbb{R} \), the pair of complex conjugated eigenvalues \( \alpha(\mu) \pm i\beta(\mu) \) of \( D_{x}g_1(x_0; \mu) \) satisfies \( \alpha(\mu_0) = 0 \) and \( \beta(\mu_0) = \omega_0 \), and \( D_{x}g_1(x_0; \mu_0) \) is in its real Jordan normal form.

**A2.** Let \( \alpha(\mu) \pm i\beta(\mu) \) be the pair of complex conjugated eigenvalues of \( D_{x}g_1(x_0; \mu) \) such that \( \alpha(\mu_0) = 0, \beta(\mu_0) = \omega_0 > 0 \). Assume that \( \alpha'(\mu_0) \neq 0 \).

Finally, define the number
\[
\ell_{1,1} = \frac{1}{8} \left( \frac{\partial^3 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^3} + \frac{\partial^3 g_1^2(x_{\mu_0}; \mu_0)}{\partial x \partial y^2} + \frac{\partial^3 g_1^2(x_{\mu_0}; \mu_0)}{\partial x^2 \partial y} + \frac{\partial^3 g_1^2(x_{\mu_0}; \mu_0)}{\partial y^3} \right)
\]
\[
+ \frac{1}{8 \omega_0} \left( \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x \partial y} + \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^2} + \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial y^2} \right) - \frac{\partial^2 g_1^1(x_{\mu_0}; \mu_0)}{\partial x^2} \frac{\partial^2 g_1^2(x_{\mu_0}; \mu_0)}{\partial y^2}.
\] (7)

The next result relates a *Hopf Bifurcation* in the averaged system (4) with a *Torus Bifurcation* in the differential equation (3).

**Theorem A ([2]).** In addition to hypotheses A1 and A2, assume that \( \ell_{1,1} \neq 0 \). Then, for each \( \varepsilon > 0 \) sufficiently small there exist a \( C^1 \) curve \( \mu(\varepsilon) \in J_0 \), with \( \mu(0) = \mu_0 \), and neighborhoods \( \mathbb{S}^1 \times \mathcal{U}_\varepsilon, \mathcal{U}_\varepsilon \subset \Omega \), of the periodic solution \( \varphi(t; \mu(\varepsilon), \varepsilon) \) and \( J_0 \subset J_0 \) of \( \mu(\varepsilon) \) for which the following statements hold.

(i) For \( \mu \in J_\varepsilon \) such that \( \ell_{1,1}(\mu - \mu(\varepsilon)) \geq 0 \), the periodic orbit \( \varphi(t; \mu(\varepsilon), \varepsilon) \) is unstable (resp. asymptotically stable), provided that \( \ell_{1,1} > 0 \) (resp. \( \ell_{1,1} < 0 \)), and the differential equation (4) does not admit any invariant tori in \( \mathcal{U}_\varepsilon \).

(ii) For \( \mu \in J_\varepsilon \) such that \( \ell_{1,1}(\mu - \mu(\varepsilon)) < 0 \), the differential equation (4) admits a unique invariant torus \( T_{\mu, \varepsilon} \) in \( \mathcal{U}_\varepsilon \) surrounding the periodic orbit \( \varphi(t; \mu, \varepsilon) \). Moreover, \( T_{\mu, \varepsilon} \) is unstable (resp. asymptotically stable), whereas the periodic orbit \( \varphi(t; \mu, \varepsilon) \) is asymptotically stable (resp. unstable), provided that \( \ell_{1,1} > 0 \) (resp. \( \ell_{1,1} < 0 \)).

(iii) \( T_{\mu, \varepsilon} \) is the unique invariant torus of the differential equation (4) bifurcating from the periodic orbit \( \varphi(t; \mu(\varepsilon), \varepsilon) \) in \( \mathcal{U}_\varepsilon \) as \( \mu \) pass through \( \mu(\varepsilon) \).
4. Proofs of Theorems 1, 2, 3, 4, and 5

In this section we will provide the proofs of Theorems 1, 2, 3, 4, and 5. We will do it separately.

Proof of Theorem 1. Statement (a) of Theorem 1 can be verified by direct computations.

The proof of statement (b) will be provided using Lemma 6. In order to simplify the computations we take
\[ \alpha = \mu \nu \left( 1 - \frac{\beta h}{\omega^2} \right) \] with \( \omega > 0 \). Then the bifurcation coefficient becomes
\[ \bar{k} = -\frac{\omega}{\beta h} \left( \mu \nu \right) \left( 1 - \beta h \right) \left( \mu^2 \nu^2 + \omega^2 \right) \], and the eigenvalues are
\[ \pm i\omega \quad \text{and} \quad \beta h \mu \nu \left( \mu^2 \nu^2 + \omega^2 \right) \].

The characteristic polynomial of the Jacobian matrix of system (1) at the origin is
\[ -\lambda^3 + \lambda^2 \frac{\beta h \mu \nu}{\omega^2} + \lambda \left( \nu (k + \mu^2 \nu) - \frac{\beta h (\mu^2 \nu^2 + \omega^2)}{\omega^2} \right) + \beta h \mu \nu = 0. \]
Assume that \( \lambda(k) \) is a root of the characteristic polynomial. It depends continuously on \( k \) and we can write
\[ \frac{d\lambda(k)}{dk} = \frac{\nu \omega^2 \lambda(k)}{\beta h (\mu^2 \nu^2 + \omega^2) - 2 \beta h \mu \nu \lambda(k) + 3 \omega^2 \lambda(k)^2 - \nu \omega^2 (k + \mu^2 \nu)}. \] (8)

Let \( \lambda(\bar{k}) = i\omega \) and taking \( k = \bar{k} \) in (8) we have that
\[ \text{Re} \left( \frac{d\lambda(k)}{dk}(\bar{k}) \right) = -\frac{\beta h \mu \nu^2 \omega^2}{2 (\beta^2 h^2 \mu^2 \nu^2 + \omega^4)} \neq 0. \]

Thus, in order to apply Lemma 6 accordingly with the arguments in Section 2, we calculate
\[ A = \begin{pmatrix} \mu h & 0 \\ \nu \omega^2 & \mu h \left( \frac{\beta h}{\omega^2} - 1 \right) \\ 0 & -h \end{pmatrix}, \]
and the multilinear functions \( B(x, y) = (0, 0, 0) \) and \( C(x, y, z) = (-6x_1 y_1 z_1, 0, 0) \).

We also obtain the eigenvectors
\[ p = \sigma_1 \begin{pmatrix} -i \beta \left( \beta h - \omega^2 \right) (\mu \nu - i \omega)^2 \\ -2i \omega (\beta h \mu \nu - i \omega^3) \\ 2 \beta h \mu \nu - 2i \omega^3 \end{pmatrix}, q = \sigma_2 \begin{pmatrix} i \omega \nu \omega \sgn(\beta)(\mu \nu - i \omega) \\ -i \omega \sgn(\beta)(\mu \nu - i \omega) \\ \beta \sgn(\beta)|\beta| \end{pmatrix}, \]
where
\[ \sigma_1 = \sqrt{\frac{\omega^2 (\nu^2 + \mu^2 \nu^2 + \omega^2)}{\beta^2 (\mu^2 \nu^2 + \omega^2)}}, \quad \sigma_2 = \sqrt{\frac{\mu^2 \nu^2 + \omega^2}{\beta^2 (\mu^2 \nu^2 + \omega^2) + (\mu^2 + 1) \nu^2 \omega^2 + \omega^4}}. \]
In this case the first Lyapunov coefficient (see (3)) is
\[ l_1 = \frac{1}{2\omega} \text{Re} \left( \bar{p}.C(q,q,\bar{q}) \right) \]
which can be written as
\[ l_1 = \frac{3\nu^3\omega (\omega^2 - \beta h) (\beta h\mu^2\nu^2 - \omega^4)}{2(\beta^2 (\mu^2\nu^2 + \omega^2) + (\mu^2 + 1)\nu^2\omega^2 + \omega^4) (\beta^2h^2\mu^2\nu^2 + \omega^4)}. \]
So, statement (b) of Theorem 1 follows directly from Lemma 6. \( \square \)

**Proof of Theorem 2.** Using the change of coordinates
\[ (x,y,z) = \varepsilon^{1/2} \left( X, -\frac{\mu
u X + Y\omega}{\nu} \beta X + \frac{\beta Y}{\omega} - Z \right) \]
the differential system can be written as
\[
\begin{align*}
\dot{X} &= -\omega Y - \varepsilon \nu - X^3, \\
\dot{Y} &= X\omega - \frac{\varepsilon}{\nu^2} (h_1(\beta \mu \nu Y - \beta X \omega + \nu \omega Z) + \omega \left( \nu X \left( k_1 + \mu \left( \alpha_1 - \nu X^2 \right) \right) \right. \\
&\quad \left. + \alpha_1 \omega Y \right)), \\
\dot{Z} &= \frac{\varepsilon}{\omega^3} \beta \left( h_1 \mu (\beta \mu \nu Y - \beta X \omega + \nu \omega Z) + \omega \left( \mu \nu X \left( k_1 + \mu \left( \alpha_1 - \nu X^2 \right) \right) \right) \\
&\quad - X^3 \omega^2 + \alpha_1 \mu \omega Y \right). 
\end{align*}
\]
Using the cylindrical change of variables \((X,Y,Z) = (r \cos \theta, \sin \theta, z)\) where \( r > 0 \), system (1) becomes
\[
\begin{align*}
\dot{r} &= \varepsilon \left( -\nu r^3 \cos^3 \theta - \frac{\sin \theta}{\omega^2} (h_1(\beta \mu \nu r \sin \theta - \beta r \omega \cos \theta + \nu \omega z) \\
&\quad + \omega \left( \nu r \cos \theta \left( \alpha_1 k_1 - \mu \nu r^2 \cos^2 \theta \right) + \alpha_1 r \omega \sin \theta \right) \right), \\
\dot{\theta} &= \omega + \frac{\varepsilon}{r} \left( \nu r^3 \sin \theta \cos^3 \theta - \frac{\cos \theta}{\omega^2} (h_1(\beta \mu \nu r \sin \theta - \beta r \omega \cos \theta + \nu \omega z) \\
&\quad + \omega \left( \nu r \cos \theta \left( \alpha_1 k_1 - \mu \nu r^2 \cos^2 \theta \right) + \alpha_1 r \omega \sin \theta \right) \right), \\
\dot{z} &= \frac{\varepsilon \beta}{\omega^3} \left( \mu r \omega \cos \theta \nu (\alpha_1 k_1 - \beta h_1) + \mu \left( r \sin \theta \left( \alpha_1 \omega^2 + \beta h_1 \mu \nu \right) \\
&\quad + h_1 \nu \omega z \right) - r^3 \omega^2 \cos^2 \theta \left( \mu^2 \nu^2 + \omega^2 \right) \right). 
\end{align*}
\]
System (9) can be written in the standard form for applying the averaging theory. Taking \( \theta \) as the new independent variable this system becomes
\[ (r', z') = \varepsilon F_1(r,z,\theta) + \varepsilon^2 \tilde{F}(t, x, \varepsilon), \]
where \( \dot{} = d/d\theta \) and

\[
\mathbf{F}_1(r, z, \theta) = \left( -\frac{\nu r^3 \cos^4 \theta}{\omega} - \frac{\sin \theta \cos \theta}{\omega^3} \left( r\omega \cos \theta \left( \nu(\alpha_1\mu + k_1) - \beta h_1 - \mu^2 r^2 \cos^2 \theta \right) \right) \\
+ r \sin \theta \left( \alpha_1 \omega^2 + \beta h_1 \mu \right) + h_1 \nu \omega z \right) \frac{\beta}{\omega^4} \left( \mu r \omega \cos \theta \left( \nu(\alpha_1\mu + k_1) - \beta h_1 \right) \right) \\
+ \mu r \sin \theta \left( \alpha_1 \omega^2 + \beta h_1 \mu \right) + h_1 \mu \nu \omega z - r^3 \omega \cos^3 \theta \left( \mu^2 \nu^2 + \omega^2 \right) \right),
\]

The first order averaging function (see (5)) of system (10) is

\[
g_1(r, z) = \left( -\frac{3\nu r^3}{8\omega} - \frac{r \left( \alpha_1 \omega^2 + \beta h_1 \mu \right)}{2\omega^3}, \frac{\beta h_1 \mu z}{\omega^3} \right).
\]

This function has the following simple zero

\[
(r_0, z_0) = \left( \frac{2}{\omega} \sqrt{-\alpha_1 \omega^2 - \beta h_1 \mu \nu \frac{3\nu}{3\nu}}, 0 \right)
\]

i.e. \( g_1(r_0, z_0) = 0 \). Moreover, the Jacobian matrix \( Dg_1(r_0, z_0) \) has the eigenvalues \( \lambda_1 = h_1 \beta \mu \nu / \omega^3 \) and \( \lambda_2 = (h_1 \beta \mu \nu + \alpha_1 \omega^2) / \omega^3 \). Thus the result follows from Theorem 7 and 8.

The proof of Theorems 3, 4, and 5 will follow the same main steps as the proof Theorem 2.

**Proof of Theorem 3** First change the coordinates of system (1) by setting

\[
(x, y, z) = \varepsilon^{1/2} \left( X + Z, \mu(-X - Z) - \frac{\omega Y}{\nu}, -\frac{\omega^2 Z}{h\nu} \right),
\]

obtaining the following system

\[
\dot{X} = -\omega Y - \varepsilon \frac{\beta h(\mu \nu X + Z + \omega Y) + \nu \omega^2 (X + Z)}{\omega^2},
\]

\[
\dot{Y} = X \omega + \varepsilon \frac{\mu^2 (X + Z)^3 - \nu (X + Z)(\alpha_1 \mu + k_1) - \alpha_1 \omega Y}{\omega},
\]

\[
\dot{Z} = \varepsilon \frac{\beta h(\mu \nu X + Z + \omega Y)}{\omega^2}.
\]

In order to write this system in normal form for applying the averaging theory, we use the cylindrical change of variables \((X, Y, Z) = (r \cos \theta, \sin \theta, z)\) where \( r > 0 \), and take \( \theta \) as the new independent variable. Then we obtain the following system

\[
(r', z') = \varepsilon \mathbf{F}_1(r, z, \theta) + \varepsilon^2 \mathbf{F}(t, x, \varepsilon),
\]

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where \( t = d/d\theta \) and

\[
\begin{align*}
\mathbf{F}_1(r, z, \theta) &= \left( \frac{1}{\omega^2} \left( \sin \theta \left( -\nu (\alpha_1 \mu + k_1) (r \cos \theta + z) - \alpha_1 r \omega \sin \theta \right. \right. \\
&\left. \left. + \mu \nu^2 (r \cos \theta + z)^3 \right) - \cos \theta \left( \beta_1 h (r \omega \sin \theta + \mu \nu (r \cos \theta + z)) \right. \right. \\
&\left. \left. + \nu \omega^2 (r \cos \theta + z)^3 \right) , \frac{\beta_1 h}{\omega^3} (r \omega \sin \theta + \mu \nu (r \cos \theta + z)) \right) .
\end{align*}
\]

Finally we have the following first order averaged function

\[
g_1(r, z) = \left( -\frac{3 \nu r^3}{8 \omega} - r \left( \frac{\alpha_1 \omega^2 + \beta_1 \mu \nu}{2 \omega^3} + \frac{3 \nu z^2}{2 \omega} \right) , \frac{\beta_1 h \mu \nu z}{3 \nu} \right) .
\]

We have that

\[
(r_0, z_0) = \left( \frac{2}{\omega} \sqrt{-\frac{\alpha_1 \omega^2 - \beta_1 h \mu \nu}{3 \nu}}, 0 \right)
\]

is a simple zero of the averaged function. Furthermore the Jacobian matrix \( Dg_1(r_0, z_0) \) has the eigenvalues \( \lambda_1 = h \beta_1 \mu \nu / \omega^3 \) and \( \lambda_2 = (h \beta_1 \mu \nu + \alpha_1 \omega^2) / \omega^3 \). Thus the result follows from Theorems 7 and 8.

**Proof of Theorem 4.** Using the change of coordinates

\[
(x, y, z) = \epsilon \left( X, \frac{\omega Y}{\nu}, \frac{\beta X}{\nu} + Z \right)
\]

in system (1) we obtain the following system

\[
\begin{align*}
\dot{X} &= -\omega Y - \epsilon \left( \frac{\beta_1 h (\mu \nu (X + Z) + \omega Y) + \nu \omega^2 (X + Z)^3}{\omega^2} \right), \\
\dot{Y} &= \omega X + \epsilon \left( \frac{\mu \nu^2 (X + Z)^3 - \nu (X + Z) (\alpha_1 \mu + k_1) - \alpha_1 \omega Y}{\omega} \right), \\
\dot{Z} &= \epsilon \frac{\beta_1 h (\mu \nu (X + Z) + \omega Y)}{\omega^2}.
\end{align*}
\]

The next step is to write it in the normal form for applying the averaging theory. Thus we use the cylindrical change of variables \((X, Y, Z) = (r \cos \theta, \sin \theta, z)\) where \( r > 0 \), and take \( \theta \) as the new independent variable. Writing the new systems up to order 2 in \( \epsilon \) we get the system

\[
(r', z') = \epsilon \mathbf{F}_1(r, z, \theta) + \epsilon^2 \tilde{\mathbf{F}}_2(r, z, \theta) + \epsilon^3 \tilde{\mathbf{F}}(t, x, \epsilon),
\]

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where $'=d/d\theta$ and

\[
F_1(r, z, \theta) = \left(\frac{\sin \theta (r \cos \theta (\beta h_1 - k_1 \nu) + h_1 \nu z) + \mu_1 \nu r \omega \cos (2\theta)}{\omega^2}, -\frac{\beta \mu_1 r \cos \theta}{\omega}\right).
\]

\[
F_2(r, z, \theta) = \left(\frac{1}{r^2 \omega^4} \left(\cos \theta (\cos \theta (k_1 \nu r - \beta h_1 r) - h_1 \nu z + 2 \mu_1 \nu r \omega \sin \theta) (\sin \theta
\right)
\]

\[
(r \cos \theta (\beta h_1 - k_1 \nu) + h_1 \nu z) + \mu_1 \nu r \omega \cos (2\theta)) + \frac{1}{\omega^2} \left(r \sin \theta \cos \theta (3h_1 - 2k_1 \nu) - \nu \omega^3 \cos \theta + \mu_1 \nu r \omega \cos^2 \theta\right),
\]

\[
\beta \cos \theta \left(r \cos^2 \theta (\beta h_1 - k_1 \nu + \mu_1 \nu r \omega \sin \theta - h_1 \nu z) + \mu_1 (-r)^2 \omega^2\right)\right).
\]

Using the expressions (5) we can verify that the first order averaged function is $g_1(r, z) = (0, 0)$. Thus computing the second order function we have

\[
g_2(r, z) = \left(\frac{\pi r \left(4h_1 \beta_1 \mu_1 \nu + \omega^2 \left(4\alpha_1 - 4\mu_1 \nu + 3r^2 \omega^2\right)\right)}{4\omega^3}, -\frac{2\pi h_1 \beta_1 \mu_1 \nu z}{\omega^3}\right).
\]

This nonlinear function has the following simple zero

\[
(r_0, z_0) = \left(\frac{2}{\omega} \sqrt{\frac{\omega^2 (\mu_1 \nu - \alpha_1) - \beta h_1 \mu_1 \nu}{3\nu}}, 0\right).
\]

The eigenvalues of the Jacobian matrix $Dg_2(r_0, z_0)$ are $\lambda_1 = 2\pi h_1 \beta_1 \mu_1 \nu / \omega^3$ and $\lambda_2 = 2\pi (h_1 \beta_1 \mu_1 \nu + (\alpha_1 - \mu_1 \nu) \omega^2) / \omega^3$. The result follows from Theorem 7 (taking $l = 2$) and Theorem 8.

Proof of Theorem 5: First we do the reescaling $(x, y, z) = \epsilon^{1/2}(\bar{x}, \bar{y}, \bar{z})$ obtaining the system

\[
\dot{x} = \bar{y} \nu + \epsilon \left(\mu_1 - \bar{x}^2\right),
\]

\[
\dot{y} = \bar{x} \nu - h \bar{z} - \epsilon \bar{y},
\]

\[
\dot{z} = \bar{y} \left(k \nu + \omega^2\right) + \epsilon \beta_1 \bar{y}.
\]

In order to write the linear part of this system in its Jordan normal form, we do the linear change of variables

\[
(x, y, z) = \left(Z - \frac{X \nu}{\omega}, Y, \frac{1}{h} \left(k \left(z - \frac{x \nu}{\omega}\right) - X \omega\right)\right).
\]

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Then the previous system writes
\[
\begin{align*}
\dot{X} &= -\omega Y + \frac{\varepsilon}{\omega^3} (k\nu(\nu X - \omega Z) (\nu^2 X^2 - 2\nu X \omega Z + \omega^2 (Z^2 - \mu_1)) - \beta_1 h\omega^3 Y), \\
\dot{Y} &= \omega X - \varepsilon\alpha_1 Y, \\
\dot{Z} &= \frac{\varepsilon}{\omega^3} (k\nu(X\nu - Z\omega) (X^2 \nu^2 - 2XZ\nu \omega + (Z^2 - \mu_1) \omega^2) + \omega^2 (-hY\beta_1 \omega \\
&\quad (X\nu - Z\omega) (X^2 \nu^2 - 2XZ\nu \omega + (Z^2 - \mu_1) \omega^2))).
\end{align*}
\]

Thus from item (i) of Theorem 7 and 8.

For analyzing the stability of these equilibria we study their eigenvalues. First, where here
\[
\rho = (2k\mu_1 \nu^2 - \alpha_1 \omega^2)/k.
\]

System (11) is \(2\pi\)-periodic in \(\theta\), and we can use the first order averaging method to write its averaged system
\[
\begin{align*}
\dot{r}' &= \frac{\varepsilon}{\omega^3} (-\beta_1 hr\omega^3 \sin \theta \cos \theta + k\nu \cos \theta (\nu r \cos \theta - \omega) \nu \cos \theta \\
&\quad (\nu r \cos \theta - 2\omega z) + \omega^2 (z^2 - \mu_1)) + \alpha_1 (-r) \omega^4 \sin^2 \theta) + O(\varepsilon^2) \\
&= F_{11}(\theta, r, z) + O(\varepsilon^2), \\
\dot{z}' &= \frac{\varepsilon}{\omega^6} (\nu (\nu + \omega^2) (\nu r \cos \theta - \omega z) (\nu r \cos \theta - \nu r \cos \theta - 2\omega z) \\
&\quad + \omega^2 (z^2 - \mu_1)) - \beta_1 hr\omega^3 \sin \theta) + O(\varepsilon^2) \\
&= F_{12}(\theta, r, z) + O(\varepsilon^2),
\end{align*}
\]

where here \(t' = d/d\theta\). System (11) is \(2\pi\)-periodic in \(\theta\), and we can use the first order averaging method to write its averaged system
\[
\begin{align*}
\dot{r}' &= r \left( \frac{-4\alpha_1 \omega^4 + 3k\nu^4 r^2 + 6k\nu^2 \omega^2 (3z^2 - \mu_1)}{8\omega^5} \right), \\
\dot{z}' &= \frac{\nu z}{2\omega^5} (k\nu + \omega^2) (3\nu^2 r^2 + 2\omega^2 (z^2 - \mu_1)).
\end{align*}
\]

The equilibrium point of the averaged system satisfying \(r > 0\) are
\[
S_{\pm} = \left( \frac{2\omega}{\nu} \sqrt{\frac{\rho}{15}}, \pm \sqrt{\frac{\rho}{5}} \right) \text{ and } S_0 = \left( \frac{2\omega}{\nu^2} \sqrt{\frac{\alpha_1 \omega^2 + k\mu_1 \nu^2}{3k}}, 0 \right),
\]

where \(\rho = (2k\mu_1 \nu^2 - \alpha_1 \omega^2)/k\). Consequently the following statements hold

(a) If \(\rho \leq 0\) the only equilibrium point of system (11) is \(S_0\).

(b) If \(\rho > 0\), system (11) has three equilibrium points \(S_{\pm}\) and \(S_0\).

For analyzing the stability of these equilibria we study their eigenvalues. First, the eigenvalues of the Jacobian matrix of system (11) at \(S_0\) are
\[
\lambda_1 = -\frac{(k\nu + \omega^2)(2\alpha_1 \omega^2 + k\mu_1 \nu^2)}{k\nu \omega^3} \quad \text{and} \quad \lambda_2 = \frac{\alpha_1 \omega^2 + k\mu_1 \nu^2}{\omega^3}.
\]

Thus from item (a) and (12), statement (i) of Theorem 5 is proved by applying
Theorem 7 and 8.
For analysing the stability of $S_\pm$ we assume $\rho > 0$. Then we write the characteristic polynomial of its Jacobian matrix $Dg_1(S_\pm)$ as

$$C(\lambda) = \lambda^2 + b\lambda + c$$

(13)

where

$$b = \frac{4\alpha_1\omega^2 + 5\alpha_1 k\nu + 2k\mu_1\nu^2}{5k\nu\omega} \quad \text{and} \quad c = \frac{2\rho(k\nu + \omega^2)(2\alpha_1\omega^2 + k\mu_1\nu^2)}{5\nu\omega^6}.$$

Consequently we have

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4c}}{2} \quad \text{and} \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

(14)

Then, from (b) and (14), statement (ii) of Theorem 5 is proved by applying Theorems 7 and 8.

For proving statement (iii) of Theorem 5 we are going to study the system (12) considering its dependence on $\mu_1$. The proof will be made for $S_+$ and follows analogously for $S_-$. First, we write the averaged system (12) as

$$(r', z') = g_1((r, z); \mu_1),$$

and its equilibrium point as

$$S_\pm(\mu_1) = \left(\frac{2\omega}{\nu} \sqrt{\frac{(2k\mu_1\nu^2 - \alpha_1\omega^2)}{15k}}, \sqrt{\frac{(2k\mu_1\nu^2 - \alpha_1\omega^2)}{5k}}\right).$$

Now we use the linear change of coordinates

$$(r, z) = \left(v + \frac{2\omega}{\nu} \sqrt{\frac{\rho}{15}}, \sqrt{\frac{\rho}{\omega}} \left(\frac{\sqrt{5}u - v}{6\alpha_1 k\nu} + \sqrt{\frac{\rho}{5}}\right)\right),$$

for having the equilibrium point $S_+(\mu_1)$ translated to the origin of coordinates and the linear part of system (12) in its Jordan normal form, obtaining the system

$$u' = \tilde{g}_1^1((u, v); \mu_1),$$
$$v' = \tilde{g}_1^2((u, v); \mu_1),$$

(15)
Thus taking \( \mu_1 = \mu_0 \) we get \( \lambda(\mu_0) = i\sqrt{5} \alpha_1 (kv + \omega^2)/\omega^3 \) and consequently
\[
\text{Re} \left( \frac{d\lambda(\mu_0)}{d\mu_1} \right) = -\frac{\nu}{5\omega}.
\]
Hence, system (15) satisfies hypothesis A2. Finally, we use the formula (7) for computing
\[
\ell_{1,1} = -\frac{3\pi \nu^2 (5k^2 \nu^2 + 16k^2 \omega^2 + 8\omega^4)}{8k\omega^5},
\]
Thus, the statement (iii) of Theorem 5 follows direct from Theorem A. This conclude the proof.
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