Graphs on Surfaces and the Partition Function of String Theory

J. Manuel García-Islas*

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Abstract. Graphs on surfaces is an active topic of pure mathematics belonging to graph theory. It has also been applied to physics and relates discrete and continuous mathematics. In this paper we present a formal mathematical description of the relation between graph theory and the mathematical physics of discrete string theory. In this description we present problems of the combinatorial world of real importance for graph theorists.

The mathematical details of the paper are as follows: There is a combinatorial description of the partition function of bosonic string theory. In this combinatorial description the string world sheet is thought as simplicial and it is considered as a combinatorial graph. It can also be said that we have embeddings of graphs in closed surfaces.

The discrete partition function which results from this procedure gives a sum over triangulations of closed surfaces. This is known as the vacuum partition function.

The precise calculation of the partition function depends on combinatorial calculations involving counting all non-isomorphic triangulations and all spanning trees of a graph. The exact computation of the partition function turns out to be very complicated, however we show the exact expressions for its computation for the case of any closed orientable surface. We present a clear computation for the sphere and the way it is done for the torus, and for the non-orientable case of the projective plane.

1 Introduction

String theory is considered to be a quantum theory of all forces of nature including of course quantum gravity [12] [18]. When strings propagate over space-time they sweep a two dimensional surface known as world sheet. The bosonic action of this world sheet is proportional to its area, and when propagating in

*e-mail: jmgislas@gmail.com
Minkowski space-time it is given by

\[ S_P = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\hbar} \eta_{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \]  

(1)

which is known as the Polyakov action. The partition function of this theory for a fixed surface \( \Sigma \) is given by

\[ Z(\Sigma) = \int DX Dg e^{-S_P} \]  

(2)

where the integral is over all embeddings of the surface \( \Sigma \) in space-time (for example \( \mathbb{R}^n \)), and over all metrics on the surface.

The combinatorial description of the string partition function is as follows:

Consider the world sheet (two dimensional surface) \( \Sigma \), and triangulate \( T \) it. Denote the vertices of the triangulated surface by latin indices \( i, j \).

Following \[4\], the Polyakov action can be written in a discrete form as

\[ S(\Sigma T)(X) = \frac{1}{2} \sum_{i \sim j} (X_i - X_j)^2 + \mu F(T) \]  

(3)

where \( X_i, X_j \) denotes the image of vertices \( i, j \) under the embedding \( X \) of the triangulated surface in space-time.

\( i \sim j \) means the vertices \( i \) and \( j \) are joined by an edge, \( \mu \) is a parameter and \( F(T) \) denotes the number of triangles of the triangulation \( T \). As explained in \[4\], the analogous of all metrics in the world-sheet corresponds in the discrete theory to all non-isomorphic triangulations. Therefore the partition function for a fixed topology \( \Sigma \) is given by a sum over triangulations

\[ Z(\Sigma) = \sum_T \int \prod_{i \in V(T)} dX_i \ e^{-S(\Sigma T)(X)} \]  

(4)

In \[2\], \[3\] this partition function was studied. In this paper we now give a precise mathematical description which stresses the mathematical side related to graph theory, and more specifically to problems that graph theorist are interested in at present and which are important for a deep understanding of the partition function.

For example, in the above partition function sum it is evident that we have to know how to generate all non-isomorphic triangulations of an arbitrary two dimensional surface which is clearly understood in graph theory. The procedure is to start with the irreducible triangulations of a surface which number grows as the genus of the surface grows. Then an arbitrary triangulation is always obtained from the set of irreducible ones by certain moves known as vertex splittings. The problem is that we do not know how many irreducible triangulations there are for any surface. The most studied cases have been until now, the sphere, the torus, the torus of genus 2, the projective plane, the Klein
bottle which are only a few cases. It is certain that the problem becomes more difficult as the genus of the surface grows. Recent studies on this direction have been considered in [1]. There has been also studies of similar problem of finding embeddings of complete graphs on surfaces [5], [11], [14].

Let us mention a well known interesting combinatorial problem. When we have a fixed triangulation of a world-sheet surface $\Sigma$, the integral

$$Z(\Sigma_T) = \int \prod_{i \in V(T)} dX_i \, e^{-S(\Sigma_T)(X)}$$

is related to the well known Matrix-Tree theorem of combinatorics [10].

When summing over different triangulations for a fixed closed surface $\Sigma$, we will show how each summand of the partition function is calculated and see that the number of spanning trees of the triangulation is relevant. The matrix tree theorem tells us how to calculate this number for any graph. The problem is that if the graph has numerous vertices it is not very practical to use the matrix tree theorem but an estimate number of the number of spanning trees is needed for our calculations.

Besides for a fixed number of vertices there are many non-isomorphic number of triangulations of a surface.

We divide this paper as follows. In section 2 we introduce the discrete partition function of string theory and show how the partition function is calculated for any closed surface. We will see what each term of the sum is, even though this is not sufficient to know what the sum converges to. More will be needed for that as we will understand in this paper. In section 3 we introduce the mathematical concept of graphs on surfaces and the way to generate all triangulations of a surface from the irreducible set of triangulations as well as the number of non-isomorphic triangulations with a fixed number of vertices and the number of spanning trees. In section 4 we go to the main part of the paper which is to do explicit calculations on surfaces of the partition function which was our motivation. We consider the cases of the sphere which is the only one we can do more formally; we also show how it is done (less formally but still with rigor) for the torus and the projective plane; and finally in general the calculation for any surface.

2 The partition function

In this section we describe the discrete partition function. The nice thing about it is that it is completely combinatorial. Consider first a closed vacuum string world-sheet embedded in a space-time of dimension $D$. The sheet is a compact, connected two dimensional surface without boundaries. Let $T$ be a non-
degenerate triangulation of it. This means that $T$ itself can be seen as a graph, i.e. a finite collection of vertices and edges with the following properties: for any two different vertices it can exist one edge only which joins them; otherwise there is no edge between two different vertices. Moreover, a single vertex can not be joint to itself, i.e there are no loops. With these conditions we think of the non-degenerate triangulation of the world-sheet surface as a graph. Consider the discrete Polyakov action for a particular surface $\Sigma$ and triangulation $T$ which is given by equation (3).

Define the combinatorial Laplacian of a graph (which extends to our triangulation) as follows

$$\Delta = \begin{cases} 
    d & \text{if } i = j \\
    -1 & \text{if } i \sim j \\
    0 & \text{otherwise}
\end{cases}$$

where $d$ is the number of edges incident to a vertex which is known as its valance. With this combinatorial Laplacian it is not difficult to see that the discrete Polyakov action can be written as

$$S(\Sigma_T)(X) = \frac{1}{2} \sum_{i \sim j} X_i \Delta X_j + \mu F(T)$$

(6)

In this discrete theory as mentioned in the introduction, for a fixed surface, all non-isomorphic triangulations play the role of the metrics, and the maps which are defined on vertices, are just the different embeddings of the triangulated sheet in space-time. The partition function for a closed surface $\Sigma$ is given by

$$Z(\Sigma) = \sum_T \int \prod_{i \in V(T)} dX_i \ e^{-S(\Sigma_T)(X)}$$

(7)

where we sum over all non-isomorphic triangulations of the surface. The most general partition function is given by summing over different topologies

$$Z = \sum_{\Sigma} \sum_T \int \prod_{i \in V(T)} dX_i \ e^{-S(\Sigma_T)(X)}$$

(8)

Consider the integral (5) for a fixed triangulation $T$ of the surface $\Sigma$. Let $v$ be any vertex of this triangulation $T$ and consider the graph $T - v$ which is given by considering the complement of the vertex $v$ and of all the edges incident to it.

Let the image of vertex $v$, $X_v$ be fixed. The partition function reduces then to an integral of over all embeddings of the remaining vertices, with the

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1 We give a formal mathematical description of graphs on surfaces and in particular of triangulations in section 4
requirement that the image $X_v$ is fixed. Integral (5) up to a factor can be rewritten as

$$Z(\Sigma_T) = e^{-\mu F(T)} \int \prod_{i \in V(T)} dX_i \ e^{\frac{1}{2} \sum_{i \sim j} X_i \Delta_{(T-v)} X_j}$$

where $\Delta_{(T-v)}$ is the combinatorial Laplacian assigned to the graph $T - v$ which can be obtained from the Laplacian $\Delta$ associated to the triangulation $T$ by removing from its associated matrix the column and row labeled by the vertex $v$. The resulting amplitude is up to a factor given by

$$Z(\Sigma_T) = e^{-\mu F(T)} \left( \frac{(2\pi)^{N(V)-1}}{\text{Det}\Delta_{(T-v)}} \right)^{\frac{D}{2}}$$

where $N(V) - 1$ is the number of vertices of the graph $T - v$, which is just the number of vertices of the triangulation $T$ minus one and $D$ is the dimension of the space-time in which our triangulated surface lives.

The integral above in combinatorics is related to the Matrix-Tree theorem \cite{10} where it is described that the determinant $\text{Det}\Delta_{(T-v)}$ equals the number of spanning trees of the triangulation $T$. The Laplacian can be generalized to a vertex or edge weights description where the above integral is generalized, however we do not describe it here.

Observe that for a non-degenerate triangulation $T$, the number of spanning trees is clearly greater than one. Therefore as the determinant appears as a denominator in the evaluation of the partition function it is clear that the partition function evaluation is bounded from above as

$$Z(\Sigma_T) < \left( (2\pi)^{N(V)-1} \right)^{\frac{D}{2}}$$

which comes from a degenerate case in which it could exist only one tree.

The partition function is given by the sum over all triangulations

$$Z(\Sigma) = \left( \frac{1}{2\pi} \right)^{\frac{D}{2}} \sum_T e^{-\mu F(T)} \left( \frac{(2\pi)^{N(V)}}{\text{Det}\Delta_{(T-v)}} \right)^{\frac{D}{2}}$$

The question now is how do we perform the above sum over all triangulations for any arbitrary surface. This is what we do in section 4 by considering the general orientable and non-orientable case, giving details of the calculations for some examples. We also need to know how are all triangulations of a surface generated. This is what we describe in the following section.

\footnote{See appendix for a description of the Matrix-Tree Theorem}
3 Graphs on surfaces: Triangulations

We first give some definitions. A graph is a pair $G = (V(G), E(G))$ where $V(G) \neq \emptyset$ is called vertex set, and $E(G)$ is a set where each element $e \in E(G)$ consist of a pair of elements of $V(G)$. The elements of $E(G)$ are called edges. Two vertices are said to be adjacent, if there is an element of $E(G)$ which joins them. A graph with $n$ vertices is complete and denoted $K_n$ if any two vertices are adjacent.

A graph $G$ is said to be embedded in a surface $\Sigma$ if the vertices of $G$ are distinct points of $\Sigma$ and every edge of $G$ is a curve in $\Sigma$ connecting the corresponding points.

A triangulation of a surface $\Sigma$ will be defined as an embedded graph $T$ in the surface such that $\Sigma$ is divided into regions called faces, such that each face is bounded by exactly three vertices and three edges, and any two faces have either one common vertex or one common edge or no common elements of the graph.

Two triangulations $T_1$ and $T_2$ are said to be isomorphic if there is a one to one and onto mapping $\phi : V(T_1) \rightarrow V(T_2)$ such that $\phi(u)\phi(v) \in E(T_2)$ whenever $uv \in E(T_1)$.

Let $T$ be a triangulation of a surface $\Sigma$, and consider an edge $e$ and their two triangles which contain it. Contract the edge $e$ and replace the two double edges by single ones. This lead us to a new triangulation see fig[1]. The inverse move is called vertex splitting.

![Figure 1: Vertex splitting and edge contraction](image)

Given a triangulation $T$ of a surface $\Sigma$ we can perform a vertex splitting or an edge contraction in order to obtain a new triangulation. When in a triangulation we cannot perform none edge contractions which lead to a new triangulation again, we say that our triangulation is minimal.

The sphere has only one 3-connected minimal triangulation given by the embedded graph $K_4$ in the sphere [16], [19].

And it is also known that all triangulations of the sphere are obtained from the singular minimal triangulation $K_4$ [19] by vertex splittings.

Now it is known that there are two minimal triangulations of the projective plane [6] one given by the embedding of $K_6$ and the other given in figure[2]. All the triangulations of the projective plane are obtained from these two minimal triangulations by vertex splittings.
Finally, for the torus it was shown \[15\] that there are 21 minimal triangulations of it. For instance one is given by the embedding of $K_7$ in the torus, 15 triangulations with 8 vertices, 4 non-isomorphic ones with 9 vertices and 1 irreducible one with 10 vertices, all of them non-isomorphic. And from these 21 triangulations we can obtain all the triangulations of the torus by vertex splittings moves.

It is known that the set of minimal triangulations for every surface $\Sigma$ is finite \[7, 8\] and the number grows rapidly.

Given a graph $G$, a subgraph $H$ is given by $V(H) \subseteq V(G)$ and $E(F) \subseteq E(G)$. When it happens that $V(H) = V(G)$, $H$ is called a spanning subgraph.

A tree is a connected graph without cycles. Given a graph $G$ we say that $H$ is a spanning tree of $G$ if $H$ is a tree and a spanning graph.

Given a graph $G$(or triangulation of a surface $T$), it is also important to know the number of spanning trees of it, as will be used in the next section. There is way to calculate the number of different spanning trees of the graph by the matrix-tree theorem given in the appendix.

However, if we need to know how the number of spanning trees grows as the triangulation of a surface has more and more vertices(as is needed for our calculations in the next section) the matrix-tree theorem is not very useful for the purposes of computing the partition function.

We therefore need to know a new way to calculate it which does not require such a tedious calculation. Or we can try to give upper bounds for this number. This is what we do in the following section, we use an upper bound found in \[13\]. As a mathematical problem it will be interesting to have a better bound; or even better an exact way to describe it.

### 4 Computations of the partition function

In this section we compute our partition function following all the mathematical details we described in our previous section. Our description is mathematically formal which gives a precision rule for doing any calculation for any surface.

However it will be clear that even this combinatorial computations are far from being trivial and when the genus of the surface grows the computations become so difficult and completely unknown. This is because we do not know the number of irreducible triangulations of all closed surfaces, and some studies...
in this direction by finding upper bounds have been studied in [17] and recently approached by [1].

Our next step is to show the way to perform this computation. The thing is that we can only give an approximation of it, and give an lower bound explicitly for the sphere only. Part of the calculation can also be given for the case of the torus. The combinatorial problem is complicated since as the number of spanning trees grows when the triangulation has a larger number of vertices, the number of non-isomorphic triangulations with a certain number of vertices increases a lot as well. In fact this latter problem is a very complicated one in the field of combinatorics. We proceed now to our calculations. We denote a surface of genus \( g \) by \( S_g \).

4.1 The Sphere

In our notation we denote the sphere by \( S_0 \). Generally we saw that the partition function was given by equation (12). We are summing over triangulations, but it can be seen that such a sum can be translated into a series sum over integers, as we now show.

As we have mentioned before, all of the triangulations of the sphere can be obtained by refining a single simple triangulation which is a minimal one [19]. This minimal single triangulation of the sphere is given by the complete graph \( K_4 \), that is, the tetrahedron graph.

In the language of topological graph theory we say that the complete graph \( K_4 \) is embeddable in the sphere. This graph is our first summand of our partition function. From this single triangulation we start taking vertex splittings.

It is clear that the following summands are given when we take vertex splittings over and over; observe that the number of faces is always even, that is, \( N(F) = n = 2k \), and the number of vertices is giving by \( N(V) = k + 2 \), which can be seen to be \( k + \chi(S_0) \).

This lead us to rewrite the partition function sum (12) as follows

\[
Z(S_0) = \left( \frac{1}{2\pi} \right)^\frac{D}{2} \sum_{k=2}^\infty e^{-\mu^2k} C(T_{k+2}) \frac{(2\pi)^{k+2}}{\kappa(T_{k+2})} \frac{D}{2} \tag{13}
\]

where by \( C(T_{k+2}) \) we denote the number of non-isomorphic triangulations with \( N(V) = k + 2 \) vertices, and \( \kappa(T_{k+2}) \) denotes the number of spanning trees of a triangulation with \( N(V) = k + 2 \) vertices. For instance the first summand is given by only one single graph which is \( K_4 \) where \( C(T_4) = 1 \) and \( \kappa(T_4) = 16 \).

Each summand has contributions from the number of non-isomorphic triangulations with a fixed number of vertices and from the number of trees of this triangulations.

The number of non-isomorphic triangulations of the sphere with a fixed number of vertices has an asymptotic behavior [20]. This number is giving by
\[ C(T_{k+2}) \sim \frac{1}{16} \left( \frac{3}{2\pi} \right)^{\frac{3}{2}} (k + 2)^{-\frac{5}{2}} \left( \frac{256}{27} \right)^{k+3} \]  
(14)

The number of spanning trees for two non-isomorphic triangulations \( T_1 \) and \( T_2 \) with the same number of vertices \( N(V) = k + 2 \), are different since their Tutte polynomial invariant is different. For this reason, the calculation is harder than thought. The only thing we can do now is to use an upper bound for the number of trees on any triangulation with \( N(V) = k + 2 \) vertices. As proved in \cite{13} any triangulation with \( k + 2 \) vertices has an upper bound for the number of spanning trees given by

\[ \kappa(T_{k+2}) \leq \frac{1}{k + 2} \left( \frac{3(k + 2)}{k + 1} \right)^{k+1} \]  
(15)

We therefore have

\[ Z(S_0) \geq \left( \frac{1}{2\pi} \right)^{\frac{3}{2}} \frac{1}{16} \left( \frac{3}{2\pi} \right)^{\frac{3}{2}} \sum_{k=2}^{\infty} e^{-\mu 2k} (k+2)^{-\frac{5}{2}} \left( \frac{256}{27} \right)^{k+3} \left( \frac{(2\pi)^{k+2}(k+2)(k+1)^{k+1}}{(3(k+2))^{k+1}} \right)^{\frac{3}{2}} \]  
(16)

which tells us that we have a lower bound. It is now a task to study its convergence for values of the parameter \( \mu \) and of the dimension \( D \). It can be seen that the above partition function converges for any value of \( \mu \geq 2 \).

Computations can be done with the help of a computer; Fix for instance \( \mu = 2 \), for \( D = 1 \) we have \( Z(S_0) \geq 0.5115676 \); for \( D = 2 \) \( Z(S_0) \geq 2.2794931 \). It can be seen that for larger values of \( D \), the partition function goes to infinity, but also we have that for larger values of \( \mu \) the partition function converges more rapidly. We therefore have that for a fixed value of the dimension \( D \), the partition function always converges for \( \mu \geq 2 \).

4.2 The Torus and more surfaces

As in the case of the sphere, all the triangulations of the torus can be obtained by refining the minimal triangulations of it. In the case of the sphere we had only one minimal triangulation. For the case of the torus we have 21 non-isomorphic minimal triangulations \cite{15} from which we start in order to obtain all of the remaining ones.

For instance the torus \( S_1 \) has as its simpler triangulation one given by the embedding of the complete graph \( K_7 \). Therefore it has 7 vertices, 21 edges and 14 faces. We also have 15 non-isomorphic triangulations with 8 vertices, 4 non-isomorphic ones with 9 vertices and 1 irreducible one with 10 vertices.

\[ ^3\text{The number of spanning trees in a graph is a special case of the Tutte Polynomial(see appendix)} \]
In all of these triangulations we can see that the number of faces is always even, that is, \( N(F) = n = 2k \); we also have that \( N(V) = k = k + \chi(S_1) \). This leads to the following sum

\[
Z(S_1) = \left( \frac{1}{2\pi} \right)^2 \prod_{k=7}^{\infty} e^{-\mu 2k} C(T_k) \left( \frac{(2\pi)^k}{\kappa(T_k)} \right)^{\frac{D}{2}}
\]

(17)

where again \( C(T_k) \) denotes the number of non-isomorphic triangulations with \( N(V) = k \) vertices for the torus, and \( \kappa(T_k) \) is the number of spanning trees of a triangulation graph with \( k \) vertices. The upper bound number of spanning trees is the same we used before for the sphere since it is just a number which depends on the number of vertices of the graph. But now our problem is that the number of non-isomorphic triangulations \( C(T_k) \) of the torus is not known in any way. It is just as simple as noticing that we now have \( C(T_7) = 1, C(T_8) = 15, C(T_9) = 4, C(T_{10}) = 1 \). Then the sum above can be taken to the following expression

\[
Z(S_1) \sim \left( \frac{1}{2\pi} \right)^2 21 \sum_{k=10}^{\infty} e^{-\mu 2k} C(T_k) \left( \frac{(2\pi)^k}{\kappa(T_k)} \right)^{\frac{D}{2}}
\]

(18)

where the major contribution is obviously given by

\[
Z(S_1) \sim \left( \frac{1}{2\pi} \right)^2 21 \sum_{k=10}^{\infty} e^{-\mu 2k} C(T_k) \left( \frac{(2\pi)^k}{\kappa(T_k)} \right)^{\frac{D}{2}}
\]

(19)

The real thing is that if we do not know anything about the number \( C(T_k) \), except for the irreducible triangulations, we cannot compare the torus partition function to the sphere one.

However we would like to show only a partial comparison. This partial comparison will be done by considering that there is only one triangulation with a fixed number of vertices, for the sphere and for the torus.

Suppose then that there is only one triangulation for the sphere with \( k + 2 \) vertices, that is \( C(T_{k+2}) = 1 \). Then

\[
Z(S_0)_{\text{partial}} \sim \left( \frac{1}{2\pi} \right)^2 \sum_{k=2}^{\infty} e^{-\mu 2k} \left( \frac{(2\pi)^{k+2}}{\kappa(T_{k+2})} \right)^{\frac{D}{2}}
\]

(20)

For the torus we have that there are 21 irreducible triangulations from which we generate all triangulations. Suppose then that each of the 21 irreducible triangulations generate only one respective class of triangulations with a fixed number of vertices. We write
both sums are partial but they still contain a sum over a very large number of
triangulations. The thing is that if we take $\mu \geq 2D$ we have that
\[
Z(S_0)_{\text{partial}} \gg Z(S_1)_{\text{partial}}
\]  
(22)

The above inequality is a very strict one and it tells us that the partial contribution of the sphere is really much more bigger than the partial contribution of the torus. Of course this is not telling us that the original sums obey the same inequality, but the interesting thing is the following. The number of non-isomorphic triangulations with a fixed number of vertices for the torus, is bigger than the one for the sphere with the same number of fixed vertices. We have also mentioned that this number grows exponentially when the genus of the surface grows. Therefore it is expected that the inequality (22) changes when considering the complete calculation.

It can also be suggested that partial contributions from other topological surfaces are also dominated by the lowest genus surface.

Let us now give the partition function sum expression for any orientable closed surface.

Observe first the following, which we assume happens for all of the different topologies: The sums for the sphere and the partial sum of the torus show that in the summands $2\pi$ has exponent $k + \chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of the surface. We also have a factor which multiplies the summand given by the number of non-isomorphic irreducible triangulations of the surface, which for the sphere it was one and for the torus it was 21. The sum starts also from a higher number when the genus of the surface increases. For the torus it starts for $k = 10$ where 10 is the number of vertices of the irreducible triangulation with more vertices. The sum for any surfaces of any genus is given by
\[
Z(S_g) = \left(\frac{1}{2\pi}\right)^D 2^{10} \sum_{k=10}^{\infty} e^{-\mu^2 k} \left(\frac{(2\pi)^k}{\kappa(T_k)}\right)^{\frac{D}{2}}
\]  
(23)

where again we have that $C(T_{k + \chi(S_g)})$ denotes the number of non-isomorphic triangulations with $N(V) = k + \chi(S_g)$ vertices for the surface of genus $\chi(S_g)$, and $\kappa(T_{k + \chi(S_g)})$ is the number of spanning trees of a triangulation graph with $k + \chi(S_g)$ vertices.

4.3 The Projective Plane and non-orientable surfaces

With the calculation of the sphere and the way we explained how the sum for the torus and any surface is to be obtained we could easily know how the
calculations follows for any non-orientable surface. The only difference would be the appearance of the non-orientable euler characteristic.

For instance recall that the projective plane has two irreducible triangulations from which we can obtain all of its triangulations by the vertex splitting moves. One is given by the embedding of the complete graph $K_6$, with 6 vertices, 15 edges and 10 faces. Then each triangulation obtained from this irreducible one, by the splitting moves, will have an even number of faces $2k$ and $k + 1$ vertices where $k$ starts from 5. The second irreducible triangulation has 7 vertices, 18 edges and 12 faces, and all of the triangulations obtained from this irreducible one, will have also an even number of faces $2k$ and $k + 1$ vertices where $k$ starts from 6. Denote the projective plane by $N_0$. Therefore the partition function is given by

$$Z(N_0) = \left( \frac{1}{2\pi} \right)^{\frac{D}{2}} \exp\left[ -10\mu \left( \frac{(2\pi)^6}{\kappa(T_6)} \right)^{\frac{D}{2}} + 2 \sum_{k=6}^{\infty} e^{-\mu 2k} C(T_{k+1}) \left( \frac{(2\pi)^{k+1}}{\kappa(T_{k+1})} \right)^{\frac{D}{2}} \right]$$ (24)

We can easily guess and generalize the above sum to any non-orientable surface of genus $g$. Denote such surface by $N_g$. We therefore have the generalized partition function given by

$$Z(N_g) = \left( \frac{1}{2\pi} \right)^{\frac{D}{2}} \sum_{k=n}^{\infty} e^{-\mu 2k} C(T_{k+\chi(N_g)}) \left( \frac{(2\pi)^{k+\chi(N_g)}}{\kappa(T_{k+\chi(N_g)})} \right)^{\frac{D}{2}}$$ (25)

where as for the orientable case $C(T_{k+\chi(N_g)})$ denotes the number of non-isomorphic triangulations with $N(V) = k + \chi(N_g)$ vertices for the non-orientable surface of genus $\chi(S_g)$, and $\kappa(T_{k+\chi(N_g)})$ is the number of spanning trees of a triangulation graph with $k + \chi(N_g)$ vertices.

5 Conclusion

We have seen in this paper that there is a need to understand deeper a pure mathematics problem in order to have a complete calculation of the partition function of any two dimensional surface. In order to have a complete sum over all triangulations of a surface we learnt that we need to know first all the non-isomorphic irreducible triangulations of the surface.

The problem clearly would be to have an asymptotic expression for the number of non-isomorphic triangulations of any surface. Until now, we have this expression for the sphere only [20]. And it is even very hard to find at least the number of irreducible triangulations of a surface. There have been only upper bounds for the number of irreducible triangulations of a surface of genus $\chi(S_g)$ [11], [17]. Even finding non-isomorphic complete graph orientable or non-orientable embeddings of complete graphs on surfaces gives a huge number of families [5], [11], [14].
Therefore the problem of computing partition functions for any surface is incomplete. We therefore have that the discrete formulation which we presented here, is not an advantage over the continuous evaluations. It will be an advantage if we first solve the combinatorial problems we presented here.

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A The spanning trees of a triangulation

This appendix describes the matrix-tree theorem. This is in order to just understand how it is used in the paper. For a deeper description of it see [9].

Let $G$ denote a connected graph with vertex set $V(G)$ and edges set $E(G)$. The combinatorial Laplacian $\Delta_G$ for the graph $G$ is defined in section 2, and it is given by a square matrix indexed by their vertices. This square matrix is completely symmetric and has determinant zero. Given any vertex $v$ of $G$ consider the cofactor $\Delta_{G-v}$ of the matrix Laplacian $\Delta_G$ given by deleting from $\Delta_G$ the row and column indexed by the vertex $v$.

Matrix-Tree Theorem. The determinant $Det(\Delta_{G-v})$ is independent of the vertex $v$ and equals the number of spanning trees of $G$.

There is also a generalization of the matrix-tree theorem when considering graphs with edge weights. The number of spanning trees of a graph can be thought as an invariant of the graph. This is because this number is a particular case of a more general invariant associated to graphs via a polynomial discovered by Tutte [21].

The Tutte polynomial of a graph is a two variable one $T(G; x, y)$ which is defined by the contraction-deletion rule.

1.- If $G$ has no edges then $T(G; x, y) = 1$

2.- $T(G; x, y) = T(G - e; x, y) + T(G \setminus e; x, y)$ where $e$ is neither a loop nor a bridge and $G - e$ and $G \setminus e$ denote the result of deleting and contracting the edge $e$.

3.- $T(G; x, y) = yT(G - e; x, y)$ when $e$ is a loop

4.- $T(G; x, y) = xT(G/e; x, y)$ when $e$ is a bridge

This are the properties which define the Tutte Polynomial. It happens that when $x = 1, y = 1$, the Tutte polynomial of the graph $G$ gives the number of its spanning trees.
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