A Continuation Method for Tensor Complementarity Problems

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Abstract
We introduce a Kojima–Megiddo–Mizuno type continuation method for solving tensor complementarity problems. We show that there exists a bounded continuation trajectory when the tensor is strictly semi-positive and any limit point tracing the trajectory gives a solution of the tensor complementarity problem. Moreover, when the tensor is strong strictly semi-positive, tracing the trajectory will converge to the unique solution. Some numerical results are given to illustrate the effectiveness of the method.

Keywords Tensor complementarity problems · Continuation method · Strictly semi-positive tensors · Strong strictly semi-positive tensors

Mathematics Subject Classification 90C33 · 15A69 · 65H20

1 Introduction

As a generalization of the linear complementarity problem [1], the tensor complementarity problem (TCP) was first introduced by Song and Qi [2]. Since then the tensor complementarity problems have received considerable attention [3–16]. They have found applications in several areas, including game theory and nonlinear compressed sensing [8,10]. Chapter 4 of the recently published book [17] summarizes many state-of-the-art results on the TCPs.

Being a special type of nonlinear complementarity problems, the TCPs inherit all the theory and algorithms developed for the nonlinear complementarity problems (see, for example, [18]). However, due to the multilinearity and homogeneity of the mappings, the TCPs have their own special and interesting properties that are not
covered by the theory of general nonlinear complementarity problems. For example, inspired by the fundamental role played by the structured matrices in the properties of solutions of the linear complementarity problems, researchers have investigated the existence, uniqueness, and boundedness of solutions of TCPs via structured tensors. Fruitful results have been obtained along this line [17]. Furthermore, as we shall show in Sect. 3, the proofs of boundedness of the trajectory of the continuation method for a general nonlinear complementarity problem in the literature do not apply to the TCP case when the tensor is strictly semi-positive, which is a useful structure that guarantees the existence and boundedness of the solutions of the TCP.

Various structured tensors have been proposed and investigated in the literature, such as P-tensors, strong P-tensors, B-tensors, Q-tensors, R-tensors, S-tensors, semi-positive tensors and strictly semi-positive tensors, strong semi-positive tensors and strong strictly semi-positive tensors. Of particular interest in this paper are the strictly semi-positive tensors and strong strictly semi-positive tensors. Song et al. [12–14] proved the existence and boundedness of solutions of TCPs with strictly semi-positive tensors. Liu et al. [9] proved that a TCP possesses the global uniqueness and solvability property if the tensor is strong strictly semi-positive, extending a similar result of Bai et al. [3] regarding strong P-tensors.

Some algorithms for solving tensor complementarity problems have been proposed recently. Luo et al. [10] proposed a method for finding the sparsest solution to a TCP with a Z-tensor by reformulating the TCP as an equivalent polynomial programming problem. Xie et al. [19] proposed an iterative method for finding the least solution to a TCP. Liu et al. [9] proposed a modulus-based nonsmooth Newton’s method for solving TCPs. Huang and Qi [8] proposed a smoothing type algorithm.

Continuation methods form an important class of methods for solving linear or nonlinear complementarity problems (see [20–24]). Recently homotopy continuation methods have been successfully developed for solving tensor eigenvalue problems and multilinear systems (see [25,26]). Motivated by this, we consider using a homotopy continuation method for solving TCPs in this paper. The convergence analyses of the continuation methods for general complementarity problems in the literature impose certain conditions on the mapping to ensure the existence and boundedness of a continuation trajectory [20–24]. Unfortunately, those analyses do not cover the TCP case when the tensor is strictly semi-positive, a condition that guarantees the existence and boundedness of solutions of the TCP. It is therefore desirable that a continuation method for TCPs produces a bounded continuation trajectory when the tensor is strictly semi-positive. We will prove that this is indeed true in this paper. We will also obtain some stronger results when the tensor is strong strictly semi-positive. We will implement the continuation method for TCPs with a strong strictly semi-positive tensor using an Euler–Newton predictor–corrector approach and provide some numerical results.

This paper is organized as follows. In Sect. 2, we give definitions of some structured tensors and summarize some known results. In Sect. 3, we introduce a Kojima–Megiddo–Mizuno type continuation method for solving TCPs and prove the existence of a bounded continuation path when the tensor is strictly semi-positive. Stronger results are proved when the tensor is strong strictly semi-positive. In Sect. 4, we present
an implementation of the continuation method when the tensor is strong strictly semi-positive and some numerical results. Some final remarks are given in Sect. 5.

2 Preliminaries

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}^{n+} = \{x \in \mathbb{R}^n : x > 0\}$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ mapping, i.e., $f$ is continuously differentiable. We consider the complementarity problem (CP) with the mapping $f$: Find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad f(x) \geq 0, \quad \langle x, f(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$.

Let $\mathbb{R}^{[m,n]}$ denote the set of all $m$th-order, $n$-dimensional real tensors. When $f(x) = Ax^{m-1} + q$, where $A \in \mathbb{R}^{[m,n]}$, $q \in \mathbb{R}^n$, and $Ax^{m-1}$ denotes the column vector whose $i$th entry is

$$(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n A_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \quad i = 1, 2, \ldots, n,$$

the CP (1) becomes the so-called tensor complementarity problem (TCP), denoted by TCP$(A, q)$: Find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Ax^{m-1} + q \geq 0, \quad \langle x, Ax^{m-1} + q \rangle = 0.$$

When $m = 2$, the TCP reduces to the well-studied linear complementarity problem (see, for example, [1]). When $m \geq 3$, the TCPs form a nontrivial class of nonlinear complementarity problems [17].

We will be concerned with the strictly semi-positive tensors and strong strictly semi-positive tensors, which are defined as follows.

**Definition 2.1** [9,11] Let $A \in \mathbb{R}^{[m,n]}$. Then $A$ is called

(a) A strictly semi-positive tensor if for every $x \neq 0$ in $\mathbb{R}^n_+$, there is an index $i : 1 \leq 1 \leq n$ such that $x_i > 0$ and $(Ax^{m-1})_i > 0$;

(b) A strong strictly semi-positive tensor if $f(x) = Ax^{m-1} + q$ is a $P$ function in $\mathbb{R}^n_+$ for any $q \in \mathbb{R}^n$, i.e., if for any distinct $x \in \mathbb{R}^n_+$ and $y \in \mathbb{R}^n_+$,

$$\max_{1 \leq i \leq n} (x_i - y_i)(f_i(x) - f_i(y)) > 0.$$

It is easy to see that strong strict semi-positiveness implies strict semi-positiveness but the converse is not true [9]. These two types of structured tensors have played an important role in the theoretical studies of TCPs. In particular, for TCPs with such tensors, we have the following results.
Theorem 2.1 [9,12–14] Let \( A \in \mathbb{R}^{m,n} \). Then for any \( q \in \mathbb{R}^n \),
(a) If \( A \) is strictly semi-positive, the TCP\((A,q)\) has a nonempty compact solution set;
(b) If \( A \) is strong strictly semi-positive, the TCP\((A,q)\) has a unique solution.

3 A Continuation Method

To solve a CP, typically a continuation method first reformulates it as an equivalent problem. Here we use one of the most frequently used equivalent reformulations in the literature for the TCP (2): Find a solution \((x,y) \in \mathbb{R}^{2n}\) such that
\[
\begin{bmatrix}
Xy \\
y - (Ax^{m-1} + q)
\end{bmatrix} = 0, \quad (x,y) \geq 0, \quad (3)
\]
where \( X = \text{diag}(x) \) is the diagonal matrix formed by the components of \( x \). We then choose vectors \( a \geq 0 \) and \( b > 0 \) from \( \mathbb{R}^n \) and define a Kojima–Megiddo–Mizuno [20] type homotopy mapping \( H : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}^{2n} \) by
\[
H(x, y, t) = \begin{bmatrix}
Xy - ta \\
y - (1 - t)(Ax^{m-1} + q) - tb
\end{bmatrix}. \quad (4)
\]

Starting with \( t = 1 \) and \((x^0, y^0) = (B^{-1}a, b)\), where \( B = \text{diag}(b) \), the continuation method follows a path from \( t = 1 \) to \( t = 0 \) by solving the system
\[
H(x, y, t) = 0, \quad (x, y) \geq 0. \quad (5)
\]

Let \( z = (x, y) \). Denote \( A = \text{diag}(a), B = \text{diag}(b), Y = \text{diag}(y) \). The partial derivatives matrices \( D_zH(x, y, t) \) and \( D_tH(x, y, t) \) of the homotopy \( H(x, y, t) \) play an important role in solving the system (5). To compute \( D_zH(x, y, t) \), we need the so-called semi-symmetric tensor \( \hat{A} = (\hat{A}_{i_1,i_2,...,i_m}) \) [27] defined by
\[
\hat{A}_{i_1i_2...i_m} = \frac{1}{p} \sum_{k=1}^{p} A_{i_1i_2^{(k)}...i_m^{(k)}}, \quad (6)
\]
where the sum is over all the \( p \) different permutations \( i_2^{(k)},...,i_m^{(k)} \) of \( i_2,...,i_m \). The partial derivatives matrix of \( Ax^{m-1} \) with respect to \( x \) is
\[
D_xAx^{m-1} = (m - 1)\hat{A}x^{m-2}. \quad (7)
\]

Therefore, the partial derivatives of \( H \) with respect to \( z \) and \( t \) are:
\[
D_zH(x, y, t) = \begin{bmatrix}
Y \\
- (1 - t)(m - 1)\hat{A}x^{m-2} \\
X \\
I_n
\end{bmatrix}. \quad (8)
\]
and
\[
D_t H(x, y, t) = \begin{bmatrix}
-a A x^{m-1} + q - b
\end{bmatrix},
\]
respectively, where \( I_n \in \mathbb{R}^{n \times n} \) is the identity matrix.

When \( t = 1 \), the matrix
\[
D z H(x^0, y^0, 1) = \begin{bmatrix}
B & B^{-1} A \\
0 & I
\end{bmatrix}
\]
is nonsingular. By the Implicit Function Theorem, there is \( \delta_1 \in (0, 1] \) such that the system (5) has a unique solution \( (x(t), y(t)) \) for each \( t \in [1 - \delta_1, 1] \) such that \( x(1) = x^0, y(1) = y^0 \), and \( x(t) \) and \( y(t) \) are smooth functions of \( t \). Thus, the homotopy (5) has a unique smooth trajectory emanated from \((x^0, y^0, 1)\) for \( t \in [1 - \delta_1, 1] \), where we choose \( \delta \in (0, \delta_1) \). Denote this trajectory by
\[
T_\delta = \{(x(t), y(t), t) : t \in [1 - \delta, 1]\}.
\]

Under rather mild conditions, we can show the existence of a continuation path that contains \( T_\delta \). We summarize the result in the following theorem, whose proof can be found in [20] for a general CP.

**Theorem 3.1** [20] Let \( a \in \mathbb{R}^n_+ \) be fixed. Then, for almost every \( b \in \mathbb{R}^n_{++} \), starting from \((x^0, y^0, 1)\), the homotopy system (5) yields a trajectory
\[
T = \{(u(s), v(s), t(s)) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times [0, 1] : 0 < s \leq 1\},
\]
which contains \( T_\delta \). Here \( u : [0, 1] \to \mathbb{R}^n_+ \), \( v : [0, 1] \to \mathbb{R}^n_+ \), and \( t : [0, 1] \to [0, 1] \) are piecewise \( C^1 \) mappings, and \( T \) is a one-dimensional manifold that is homeomorphic to \([0, 1]\).

If \( T \) is bounded, then \( \lim_{s \to 0} t(s) = 0 \).

If \( a \in \mathbb{R}^n_{++} \), then \( u, v, \) and \( t \) are \( C^1 \) mappings.

**Remark 3.1** When \( a \in \mathbb{R}^n_{++} \), the trajectory \( T \) in Theorem 3.1 is a one-dimensional smooth manifold. It is known that a one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval (see, for example, [28]). Since the matrix \( D z H(x^0, y^0, 1) \) is nonsingular, \( T \) is not diffeomorphic to a unit circle. Hence, \( T \) is diffeomorphic to \([0, 1]\).

To ensure the boundedness of a trajectory in a continuation method for a general CP (1), certain conditions need to be imposed on the mapping \( f \). Among several such conditions proposed in the literature, the following two conditions have been frequently used in theoretical studies and practice.

**Condition 1.** [20,24] (a) \( f \) is monotone on \( \mathbb{R}^n_+ \), i.e., \((x - y, f(x) - f(y)) \geq 0 \) for any \( x \) and \( y \) in \( \mathbb{R}^n_+ \).
(b) There exists a strictly feasible point \((\bar{x}, \bar{y})\) such that \( \bar{x} > 0 \) and \( \bar{y} = f(\bar{x}) > 0 \).
Condition 2. [21,22,24] (a) \( f \) is a \( P_0 \) function in \( \mathbb{R}_+^n \), i.e., for any distinct \( x \) and \( y \) in \( \mathbb{R}_+^n \), \( \max_{i \neq j} (x_i - y_i)(f_i(x) - f_j(y)) \geq 0 \).
(b) There exists a strictly feasible point \((\bar{x}, \bar{y})\).
(c) The set \( f^{-1}(D) = \{z = (x, y) \in \mathbb{R}_+^{2n} : f(z) \in D\} \) is bounded for every compact subset of \( D \) of \( \mathbb{R}_+^n \times B_++(f) \), where \( B_++(f) = \{u = y - f(x) \text{ for all } (x, y) > 0\} \).

The condition imposed in [23] does not require that \( f \) is a \( P_0 \) mapping or a monotone mapping. However, they require the following:
Condition 3. [23] (a) \( f \) is three times continuously differentiable.
(b) For any \( \{x^k\} \subset \mathbb{R}_+^n \), as \( k \to \infty \) and \( \|x^k\| \to \infty \), \( f(x^k) > 0 \) when \( k > K_0 \) for some \( K_0 > 0 \).
(c) The set \( S_+ = \{(x, y) \geq 0 : y = f(x)\} \) is nonempty.

Unfortunately, there exist strictly semi-positive tensors such that none of these conditions hold. As an example, we take Example 3.27 in [9] with the following \( A \in \mathbb{R}^{[3,2]} \):

\[
\begin{align*}
a_{111} &= 1, \ a_{121} = 2, \ a_{122} = 1, \ a_{222} = 1, \ a_{211} = -1, \ a_{221} = -1, \\
\text{and} \\
a_{ijk} &= 0, \text{ for other } i, j, k.
\end{align*}
\]

It is shown in [9] that this tensor is strictly positive semi-positive. However, its corresponding mapping \( f(x) = Ax^{m-1} + q \) is not a \( P_0 \) function or a monotone function for any \( q \in \mathbb{R}^n \). Moreover, as

\[
A x^2 = \begin{bmatrix} (x_1 + x_2)^2 \\ x_2^2 - x_1 x_2 - x_1^2 \end{bmatrix},
\]

clearly \( f(x) = A x^2 + q \) does not satisfy Condition (3.b).

Thus, the proofs of boundedness of the trajectory for a general CP in the literature do not apply to the TCP case when \( A \) is strictly semi-positive. According to Theorem 2.1, \( TCP(A, q) \) has a nonempty compact solution set if \( A \) is strictly semi-positive. Therefore, it is desirable to prove that the trajectory (11) is bounded if the tensor \( A \) is strictly semi-positive. As we will show in the following theorem, this is indeed true.

**Theorem 3.2** Under the conditions of Theorem 3.1, if \( A \in \mathbb{R}^{[m,n]} \) is a strictly semi-positive tensor, then the trajectory (11) is bounded. As \( s \to 0 \), any limit point of \((u(s), v(s))\) tracing this trajectory is a solution of the problem (3).

**Proof** Clearly, for \( t(s) \in [1 - \delta, 1] \), \( T = T_\delta \) is bounded. Thus, we only need to prove that the trajectory

\[
T = \{(u(s), v(s), t(s)) : 0 < s \leq 1\}
\]

is bounded when \( t(s) \in ]0, 1 - \delta] \).
Since $A$ is strictly semi-positive, as in [14], we define the following quantity

\[
\beta(A) = \min_{x \geq 0, \|x\|_\infty = 1} \max_{1 \leq i \leq n} x_i (Ax^{m-1})_i.
\]

Then $\beta(A) > 0$.

Note that

\[
v(s)_i = (1 - t(s))(Au(s)^{m-1}q)_i + t(s)b_i,
\]

and

\[
u(s)_i = t(s)a_i.
\]

Thus,

\[
t(s)a_i = (1 - t(s))u(s)_i (Au(s)^{m-1})_i + u(s)_i ((1 - t(s))q_i + t(s)b_i).
\]

By the definition of $\beta(A)$, we have

\[
(1 - t(s))\|u(s)\|^{m-1}_\infty \beta(A) \leq (1 - t(s)) \max_i u(s)_i (Au(s)^{m-1})_i
\]
\[
\leq \max_i t(s) a_i + \max_i u(s)_i ((1 - t(s))q_i - t(s)b_i)
\]
\[
\leq \max_i a_i + \|u(s)\|_\infty - (1 - t(s))\|q\| - t(s)\|b\|_\infty
\]
\[
\leq \max_i a_i + \|u(s)\|_\infty (\|q\| + \|b\|_\infty)
\]

Therefore,

\[
\|u(s)\|^{m-1}_\infty \leq \max \left\{ 1, \frac{\max_i a_i + \|q\| + \|b\|_\infty}{(1 - t(s))\beta(A)} \right\}.
\]

This implies that $u(s)$ is bounded when $t(s) \in ]0, 1 - \delta]$. Thus, it is bounded for $s \in ]0, 1]$. The boundedness of $v(s)$ follows from the boundedness of $u(s)$.

The boundedness of $T$ implies that $\lim_{s \to 0} t(s) = 0$ by Theorem 3.1. Moreover, $(u(s), v(s))$ has limit points when $s \to 0$. Each limit point is a solution of the problem (3).

We now study the behavior of the trajectory $T$ in (11) when the tensor $A$ is strong strictly semi-positive. By Remark 3.1 and Theorem 3.2, $T$ is a bounded smooth one-dimensional manifold that is diffeomorphic to $]0, 1]$. On this trajectory, we have

\[
H(u(s), v(s), t(s)) = 0,
\]
for $s \in [0, 1)$. Let $z = (u, v)$. Since $u$, $v$, and $t$ are smooth in $s$, differentiating this system gives

$$D_zH \cdot \frac{dz}{ds} + D_tH \cdot \frac{dt}{ds} = 0. \quad (12)$$

Following (8), the partial derivatives matrix $D_zH$ is of the form

$$D_zH(u(s), v(s), t(s)) = \begin{bmatrix} V & U \\ -M & I_n \end{bmatrix}, \quad (13)$$

where $V = \text{diag}(v(s))$, $U = \text{diag}(u(s))$, and $M = (1 - t(s))(m - 1)\hat{A}u(s)^{m-2}$. Since $A$ is strong strictly semi-positive, $Ax^{m-1}$ is a $P$ function in $R^n$. This implies that its Jacobian matrix $D_xAx^{m-1}$ defined in (7) is a $P_0$ matrix for $x \in R^n_+$ (see, [29]). Now as $u(s) \in R^n_+, v(s) \in R^n_+, t(s) \in [0, 1)$, the product matrix $VU^{-1}$ is a diagonal matrix with positive diagonal entries and the matrix $M$ is a $P_0$ matrix. It follows that the partial derivatives matrix $D_zH$ in (12) along the trajectory is nonsingular, as its determinant

$$\det(D_zH(u(s), v(s), t(s))) = \det(V + MU) = \det(VU^{-1} + M)\det(U) \neq 0.$$ 

This means that $dr/ds \neq 0$ for any $s \in [0, 1]$, because otherwise the nonsingularity of $D_zH$ would imply $dz/ds = 0$ for some $s$, contradicting that $T$ is diffeomorphic to $[0, 1]$. Therefore, the trajectory $T$ can be parametrized by the variable $t$:

$$T = \{(x(t), y(t), t) : 0 < t \leq 1\}, \subset R^n_+ \times R^n_+ \times [0, 1], \quad (14)$$

and we can trace $T$ by solving the following initial value problem

$$D_zH(z, t) \frac{dz}{dt} = -D_tH(z, t), \quad z(1) = (x^0, y^0). \quad (15)$$

where $z = (x, y)$.

We summarize the convergence results of the continuation method when $A$ is strong strictly semi-positive in the following theorem.

**Theorem 3.3** Let $a \in R_{++}^n$. Suppose that $A \in R^{[m,n]}$ is a strong strictly semi-positive tensor and $q \in R^n$. Then for almost every $b \in R^n_+$, solving the system (5) yields a smooth and bounded trajectory (14). Moreover, tracing this trajectory by solving the initial value problem (15) converges to the unique solution $(x_*, y_*)$ of the problem (3) as $t \to 0$, in which $x_*$ is the unique solution of the TCP (2).

**Remark 3.2** The analysis given before Theorem 3.3 is valid when the tensor $A$ is strictly semi-positive and the function $f(x) = Ax^{m-1} + q$ is a $P_0$ function in $R^n_+$, i.e., if for any distinct $x \in R^n_+$ and $y \in R^n_+$,

$$\max_{1 \leq i \leq n} (x_i - y_i)(f_i(x) - f_i(y)) \geq 0.$$
In this case, the solution set of the TCP (2) (as well as the solution set of (3)) is nonempty and compact, but the solutions are not necessarily unique. Therefore, any limit point of the trajectory (14) as \( t \to 0 \) is a solution of (3).

4 Numerical Results

We have implemented the continuation method described in the previous section when the tensor \( \mathcal{A} \) is strong strictly semi-positive. An Euler–Newton predictor–corrector method with adaptive step sizes (see, for example, [30]) is used to solve the initial value problem (15). We summarize our implementation in the following algorithm.

Algorithm HOM4TCP

Step 0. (Initialization) Choose positive vectors \( a, b \in \mathbb{R}_+^n \). Choose initial step size \( \Delta t_0 > 0 \), tolerances \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \). Let \( t_0 = 1, x_0 = B^{-1}a, y_0 = b \). Let \( z = (x, y) \) and \( z_0 = (x_0, y_0) \). Set \( k = 0 \).

Step 1. (Find new \( t_{k+1} \): Set \( t_{k+1} = t_k - \Delta t_k \). If \( t_k > 0 \) and \( t_{k+1} \leq 0 \) for some \( k \), then set \( N = k \) and reset \( t_{N+1} = 0 \) and \( \Delta t_N = t_N \).

Step 2. (Find a predictor using Euler’s method) Compute the tangent vector \( g \) to \( H(z_k, t) = 0 \) at \( t_k \) by solving the linear system

\[
D_zH(z_k, t_k)g = -D_tH(z_k, t_k)
\]

for \( g \). Then compute the approximation \( \tilde{z} \) to \( z_{k+1} \) by

\[
\tilde{z} = z_k + \Delta t_k g.
\]

Step 3. (Find a corrector using Newton’s method) Initialize \( w_0 = \tilde{z} \). For \( i = 0, 1, 2, \ldots \), compute

\[
w_{i+1} = w_i - [D_zH(w_i, t_{k+1})]^\dagger H(w_i, t_{k+1})
\]

until \( \| H(w_j, t_{k+1}) \| \leq \epsilon_1 \) if \( k < N \) or \( \| H(w_j, t_{k+1}) \| \leq \epsilon_2 \) if \( k = N \), where \( \dagger \) denotes the pseudo-inverse. Then let \( z_{k+1} = w_j \). If \( k = N \), we set \( (x_*, y_*) = z_{N+1} \) as the computed solution of problem (3) and stop.

Step 4. (Adaptively update the step size \( \Delta t_k \)) If more than five steps of Newton iterations were required to converge within the desired accuracy, then \( \Delta t_{k+1} = 0.25 \Delta t_k \). If \( \Delta t_{k+1} \leq 10^{-6} \), set \( \Delta t_{k+1} = 10^{-6} \). If two consecutive steps were not cut, then \( \Delta t_{k+1} = 2 \Delta t_k \). If \( \Delta t_{k+1} \geq 0.5 \), set \( \Delta t_{k+1} = 0.5 \). Otherwise, \( \Delta t_{k+1} = \Delta t_k \). Set \( k = k + 1 \). Go to Step 1.

Remark 4.1 Algorithm HOM4TCP is designed for TCPs with strong strictly semi-positive tensors. It relies on the uniqueness of the solution curve and the nonsingularity of the Jacobian matrix (13). As in the general nonlinear complementarity problem case, when the TCP has more than one solutions, one can implement a continuation method by tracking a parametrized curve \( (x(s), y(s), t(s)) \), where \( s \) is a parameter. The most commonly used parameter is the arc length of the curve. We refer to [30] for more details.
We have coded Algorithm HOM4TCP in MATLAB and done some numerical experiments to test its effectiveness. We first tested it on Examples 4.1–4.4. The tensors $\mathcal{A}$ in the first three examples are strong strictly semi-positive. In the fourth example, $\mathcal{A}$ is only strictly semi-positive, but the corresponding TCP has a unique solution for each $q$.

**Example 4.1** Let $\mathcal{A} \in \mathbb{R}^{[3,2]}$ be defined by:

\[
a_{111} = 1, a_{121} = 1, a_{122} = -1, a_{222} = 1, a_{211} = -1, a_{221} = 1,
\]

and $a_{i_1i_2i_3} = 0$ otherwise. This tensor is given in [9, Example 3.30]. It is strong strictly semi-positive. Different vectors $q \in \mathbb{R}^2$ are used in our experiments.

**Example 4.2** Consider the tensor $\mathcal{A} \in \mathbb{R}^{[4,2]}$ defined by:

\[
a_{1111} = 1, a_{1222} = -1, a_{1122} = 1, a_{2222} = 1, a_{2111} = -1, a_{2211} = 1,
\]

and $a_{i_1i_2i_3i_4} = 0$ otherwise. This tensor is first given in [3, Example 4.2]. It is a $P$ tensor, but not a strong $P$ tensor. However, it is strong strictly semi-positive [9]. Different vectors $q \in \mathbb{R}^2$ are used in our experiments.

**Example 4.3** Let $\mathcal{A} \in \mathbb{R}^{[5,3]}$ be defined by $a_{kkkk} = k$, for $k = 1, 2, 3$, and $a_{i_1i_2i_3i_4i_5} = 0$ otherwise. Clearly, this tensor is strong strictly semi-positive. Different vectors $q \in \mathbb{R}^3$ are used in our experiments.

**Example 4.4** Consider the tensor $\mathcal{A} \in \mathbb{R}^{[3,2]}$ defined by:

\[
a_{111} = 1, a_{121} = 2, a_{122} = 1, a_{222} = 1, a_{211} = -1, a_{221} = -1,
\]

and $a_{i_1i_2i_3} = 0$ otherwise. This tensor is given in [9, Example 3.27]. It is strictly semi-positive, but not strong strictly semi-positive. Nonetheless, the TCP (2) with this tensor has a unique solution for any $q \in \mathbb{R}^2$. Different vectors $q \in \mathbb{R}^2$ are used in our experiments.

Our experiments were done using MATLAB 2014b on a laptop computer with Intel Core i7-4600U at 2.10 GHz and 8 GB memory running Microsoft Windows 7. The tensor toolbox of [31] was used to compute tensor-vector products and to compute the semi-symmetric tensor $\hat{\mathcal{A}}$. We used $a = b = [1, 1, \ldots, 1]^T$, $\Delta t_0 = 0.1$, $\epsilon_1 = 10^{-5}$, and $\epsilon_2 = 10^{-12}$ in Algorithm HOM4TCP.

We now report the numerical results in Tables 1, 2, 3 and 4. In these tables, $\text{itr}$ and $\text{nwtitr}$ denote the number of prediction steps and the number of Newton iterations were used, respectively, solution denotes the solution of TCP (2) found by Algorithm HOM4TCP, and residue denotes the residue

\[
\left\| X_\ast y_\ast \begin{bmatrix} y_\ast \quad - (\mathcal{A} x_\ast^{m-1} + q) \end{bmatrix} \right\|_2
\]

at termination, where $X_\ast = \text{diag}(x_\ast)$. 

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Table 1  Numerical results for Example 4.1

| q         | itr | nwtitr | Solution          | Residue     |
|-----------|-----|--------|-------------------|-------------|
| $[-5, -3]^T$ | 5   | 12     | $[2.1286, 1.8791]^T$ | 8.8805e−15 |
| $[-5, 3]^T$  | 5   | 14     | $[2.0582, 0.4859]^T$ | 2.6746e−23 |
| $[5, 3]^T$   | 5   | 12     | $[0, 0]^T$        | 4.2730e−13 |
| $[0, 3]^T$   | 5   | 36     | $[0, 0]^T$        | 6.8709e−13 |
| $[2, -3]^T$  | 5   | 13     | $[0.3103, 1.6113]^T$ | 8.9179e−16 |
| $[0, -5]^T$  | 5   | 12     | $[1.2430, 2.0112]^T$ | 2.1817e−15 |

Table 2  Numerical results for Example 4.2

| q         | itr | nwtitr | Solution          | Residue     |
|-----------|-----|--------|-------------------|-------------|
| $[-5, -3]^T$ | 5   | 13     | $[1.6678, 1.5096]^T$ | 1.1586e−14 |
| $[-5, 3]^T$  | 5   | 14     | $[1.6714, 0.5409]^T$ | 1.9860e−15 |
| $[5, 3]^T$   | 5   | 12     | $[0, 0]^T$        | 1.0731e−18 |
| $[0, 3]^T$   | 5   | 33     | $[0, 0]^T$        | 5.2577e−13 |
| $[2, -3]^T$  | 5   | 13     | $[0.3906, 1.4167]^T$ | 6.2804e−16 |
| $[0, -5]^T$  | 5   | 13     | $[1.1143, 1.6331]^T$ | 3.8998e−15 |

Table 3  Numerical results for Example 4.3

| q         | itr | nwtitr | Solution          | Residue     |
|-----------|-----|--------|-------------------|-------------|
| $[1, 2, 3]^T$ | 5   | 12     | $[0, 0, 0]^T$    | 4.0969e−21 |
| $[1, -2, 3]^T$ | 5   | 12     | $[0, 1, 0]^T$    | 7.6027e−23 |
| $[-3, -2, -3]^T$ | 5   | 11     | $[1.3161, 1, 1]^T$ | 3.6186e−15 |
| $[3, 3, 3]^T$  | 5   | 12     | $[0, 0, 0]^T$    | 4.9693e−23 |
| $[-3, -1, -2]^T$ | 5   | 11     | $[1.3161, 0.8409, 0.9036]^T$ | 3.6748e−15 |
| $[0, -1, -2]^T$  | 5   | 31     | $[0, 0.8409, 0.9036]^T$ | 7.1789e−13 |

Table 4  Numerical results for Example 4.4

| q         | itr | nwtitr | Solution          | Residue     |
|-----------|-----|--------|-------------------|-------------|
| $[-5, -3]^T$ | 5   | 14     | $[0.3127, 1.9233]^T$ | 1.2942e−15 |
| $[-5, 3]^T$  | 5   | 11     | $[1.5513, 0.6847]^T$ | 1.7402e−14 |
| $[5, 3]^T$   | 5   | 13     | $[0, 0]^T$        | 6.0454e−26 |
| $[0, 3]^T$   | 5   | 36     | $[0, 0]^T$        | 6.3603e−13 |
| $[2, -3]^T$  | 5   | 13     | $[0, 1.7321]^T$   | 9.9301e−16 |
| $[0, -5]^T$  | 5   | 15     | $[0, 2.2361]^T$   | 1.9860e−15 |
From these tables, we observe that Algorithm HOM4TCP effectively computes the unique solution for each TCP in Examples 4.1–4.4. The algorithm is also efficient in terms of the number of prediction steps $\text{itr}$ and the number of Newton iterations $\text{nwtitr}$. We remark that the relatively large $\text{nwtitr}$ in the cases when $\mathbf{q} = [0, 3]^T$ in Examples 4.1, 4.2, and 4.4, and $\mathbf{q} = [0, -1, -2]^T$ in Example 4.3 is because more Newton iterations were used in the last step due to the singularity of the Jacobian matrix $D_zH(\mathbf{x}_*, \mathbf{y}_*, 0)$ at the solution $(\mathbf{x}_*, \mathbf{y}_*)$ of problem (3). Using a deflation method such as the one given in [32] can improve the performance of Newton’s method in such cases.

We then tested Algorithm HOM4TCP on some TCPs generated in the next two examples. The tensors in these examples are strictly semi-positive but not strong strictly semi-positive. For comparison, we also tested the modulus-based nonsmooth Newton’s (MNN) method for TCPs proposed in [9] on these problems. It is shown in [9] that the MNN method is an efficient method for solving TCPs with strong strictly semi-positive tensors.

**Example 4.5** Choose a randomly generated tensor $\mathbf{B} \in \mathbb{R}^{[m,n]}$

$$\mathbf{B} = \text{rand}(n, n, \ldots, n),$$

that is, each entry of $\mathbf{B}$ is randomly generated from $[0, 1]$ based on the uniform distribution. Set $\mathcal{A} = \mathbf{B} + 0.1\mathbf{I}$, where $\mathbf{I}$ is the identity tensor in $\mathbb{R}^{[m,n]}$. The vector $\mathbf{q} \in \mathbb{R}^n$ is chosen as $q_k = (-1)^k k$ for $k = 1, 2, \ldots, n$.

**Example 4.6** Choose the tensor $\mathbf{B} \in \mathbb{R}^{[m,n]}$ as

$$b_{i_1i_2\ldots i_m} = |\sin(i_1 + i_2 + \cdots + i_m)|,$$

and set $\mathcal{A} = \mathbf{B} + 0.1\mathbf{I}$. The vector $\mathbf{q}$ is the same as in the previous example.

Note that the tensors $\mathcal{A}$ in Examples 4.5 and 4.6 are tensors of positive entries. We set $\mathcal{A} = \mathbf{B} + 0.1\mathbf{I}$ in order to avoid some diagonal entries of $\mathcal{A}$ getting too close to zero. According to Theorem 4.22 and Corollary 4.3 in [17], a nonnegative tensor $\mathcal{A}$ is strictly semi-positive if and only if its diagonal entries are positive. Therefore, each tensor $\mathcal{A}$ in these examples is strictly semi-positive. However, $\mathcal{A}$ is not strong strictly semi-positive and the solutions of the corresponding TCP with the chosen $\mathbf{q}$ are not necessarily unique.

We used the initial guess $[1, 1, \ldots, 1]^T$ and set the maximum allowed iterations as 5000 when running the MATLAB code for the MNN method in [9]. The other parameters were the same as the ones used in the code in [9]. Let $\mathbf{x}_k$ be found by Algorithm HOM4TCP or the MNN method at termination, we record the following value

$$\text{RES} = \left\| \min \left\{ A\mathbf{x}_k^{m-1} + \mathbf{q}, \mathbf{x}_k \right\} \right\|_2 + \left\| (A\mathbf{x}_k^{m-1} + \mathbf{q})^T \mathbf{x}_k \right\|.$$

Note that $\text{RES} = 0$ if $\mathbf{x}_k$ is an exact solution of the TCP (2). Therefore, this value indicates how good the approximation $\mathbf{x}_k$ is. We tested Algorithm HOM4TCP and

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the MNN method on various TCPs generated in Examples 4.5 and 4.6 using different choices of \((m, n)\). The numerical results are reported in Tables 5 and 6, respectively. In these tables, \(\text{itr} / \text{nwtitr}\) for Algorithm HOM4TCP have the same meaning as in the previous tables, and \(\text{NIT}\) denotes the number of Newton iterations used by the MNN method.

From Tables 5 and 6, we see that Algorithm HOM4TCP successfully solves all the TCPs we tested but one: The \(m = 4, n = 20\) case in Example 4.6. This seems to
indicate that the continuation method is a promising method for solving TCPs with strictly semi-positive tensors. On the other hand, the performance of the MNN method on these TCPs is mixed: It performs well when \( n = 2, 4 \), but either fails or performs less satisfactorily when \( n \) is larger. This is perhaps due to that the MNN method is designed as an efficient method for solving TCPs with strong strictly semi-positive tensors, while the tensors in Examples 4.5 and 4.6 are strictly semi-positive only.

5 Conclusions

We have introduced a continuation method for solving TCPs. Under the assumption that the tensor is strictly semi-positive, we have proved the existence of a bounded continuation trajectory. This result is not covered by the theoretical results proved in the literature for general nonlinear complementarity problems. We have also proved that when the tensor is strong strictly semi-positive, tracing the trajectory will converge to the unique solution of the TCP. We have implemented the method for TCPs with strong strictly semi-positive tensors. Numerical results show the continuation method is promising for solving TCPs.

Various structured tensors have been introduced recently and they play an important role in studying theoretical properties of TCPs. An interesting direction for future research is to investigate how to use a continuation method to solve TCPs with other types of structured tensors.

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