A universal divergence rate for symmetric Birkhoff Sums in infinite ergodic theory

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Abstract
We show that there exists a universal gap in the failure of the ergodic theorem for symmetric Birkhoff sums in infinite ergodic theory.

1 Introduction
For an ergodic infinite measure preserving system, the ergodic theorem fails in the sense that there does not exist a normalizing sequence for its Birkhoff sums. That is for every conservative, ergodic, measure preserving system $(X, B, m, T)$ with $m(X) = \infty$, $0 \leq f \in L^1(X, m)$ and $a_n \to \infty$, either

$$\liminf_{n \to \infty} \frac{S_n(f)}{a_n} = 0 \, \text{a.e.}$$

or

$$\limsup_{n \to \infty} \frac{S_n(f)}{a_n} = \infty \, \text{a.e.}$$

Here $S_n(f) := \sum_{k=0}^{n-1} f \circ T^k$ denotes the Birkhoff sum of $f$. For an invertible transformation one can consider Symmetric (two-sided) Birkhoff sums

$$\Sigma_n (1_A)(x) := \sum_{|k| \leq n} 1_A(T^k x),$$

where the summation is in a symmetric time interval. In [AKW, MS], examples of infinite measure preserving transformations for which there exists normalizing constants $a_n \to \infty$ such that for every $f \in L^1(X, m)_+$,

$$\lim_{n \to \infty} \frac{\Sigma_n(f)}{a_n} > 0 \text{ and } \lim_{n \to \infty} \frac{\Sigma_n(f)}{a_n} < \infty.$$  \hfill (1.1)

The examples of [AKW] include some natural transformations in infinite ergodic theory such as the class of rank one transformations with bounded cutting sequence and generalized recurrent events (some null recurrent Markov chains are in this class). This shows that symmetric Birkhoff sums can behave better than their one sided counterparts. However in the work with Jon Aaronson and Benjamin Weiss we proved that for an invertible infinite measure preserving transformation, there is no ergodic theorem for symmetric Birkhoff sums of infinite measure preserving transformations. That is for every normalizing sequence $a_n \to \infty$ and $f \in L^1(X, m)_+$,

$$\lim_{n \to \infty} \frac{\Sigma_n(f)}{a_n} < \lim_{n \to \infty} \frac{\Sigma_n(f)}{a_n}.$$  

The purpose of this note (which is largely taken from the authors Ph.D. thesis) is to prove a universal quantitative divergence rate for symmetric Birkhoff sums.
**Theorem 1.** There exists a universal constant $\epsilon > \frac{1}{10002}$ so that for every conservative, ergodic, measure preserving system $(X, \mathcal{B}, m, T)$ with $m(X) = \infty$, $0 \leq f \in L_1(X, m)$ and $a_n \to \infty$,

$$\lim_{n \to \infty} \frac{\sum_{n} a_n f}{\sum_{n} a_n} \leq 1 - \epsilon.$$  

After proving the theorem we give an application to the study of fluctuations of symmetric Birkhoff integrals of horocyclic flows on geometrically finite surfaces.

**Notation**

From now on we will write

$$S_n^{-}(f) := \sum_{k=1}^{n} f \circ T^{-k} = \Sigma_n(f) - S_n(f).$$

For eventually positive sequences $a_n, b_n$ we write:

- $a_n \sim b_n$ if $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$.
- $a_n \lesssim b_n$ if $\lim_{n \to \infty} \frac{a_n}{b_n} \leq 1$.
- For an infinite subset $K \subset \mathbb{N}$, $a_n \lesssim b_n$ if $\lim_{n \to \infty, n \in K} \frac{a_n}{b_n} \leq 1$.

**1.1 Preliminaries**

**Bounded Rational Ergodicity**

As in [Aar1], a conservative, ergodic, measure preserving transformation $(X, \mathcal{B}, m, T)$ is called **boundedly rationally ergodic** (BRE) if $\exists A \in \mathcal{B}$, $0 < m(A) < \infty$ so that

$$S_n^{-}(1_A)(x) \leq M a_n(A) \text{ a.e. on } A \forall n \geq 1 \quad (1.2)$$

where $a_n(A) = \sum_{k=0}^{n-1} \frac{m(A \cap T^{-k}A)}{m(A)^2}$. In this case [Aar1], $(X, \mathcal{B}, m, T)$ is **weakly rationally ergodic** (WRE), that is, writing $a_n(T) := a_n(A)$ (where $A$ as in (1.2)), there is a dense hereditary ring

$$\mathcal{R}(T) \subset \mathcal{F} := \{ F \in \mathcal{B} : m(F) < \infty \}$$

(including all sets satisfying (1.2)) so that

$$a_n(F) \sim a_n(T) \quad \forall F \in \mathcal{R}(T), \ m(F) > 0$$

and

$$\sum_{k=0}^{n-1} m(F \cap T^{-k}G) \sim m(F)m(G)a_n(T), \ \forall F, G \in \mathcal{R}(T).$$

For invertible transformations, the one sided properties (BRE and WRE) are equivalent to their two sided analogues: $(X, \mathcal{B}, m, T)$ is:

- two sided, boundedly rationally ergodic if $\exists A \in \mathcal{B}$, $0 < m(A) < \infty$ so that

$$\Sigma_n^{-}(1_A)(x) \leq M \overline{a}_n(A) \text{ a.e. on } A \forall n \geq 1 \quad (1.3)$$

where $\overline{a}_n(A) = \sum_{|k| \leq n} \frac{m(A \cap T^kA)}{m(A)^2} \sim 2a_n(A)$.  


two sided, weakly rationally ergodic if there is a dense hereditary ring
\( \mathcal{R}(T) \subset \mathcal{F} := \{ F \in \mathcal{B} : m(F) < \infty \} \)

(including all sets satisfying (1.3)) so that
\( \pi_n(F) \sim 2a_n(T) \forall F \in \mathcal{R}(T), m(F) > 0. \)

Again it follows that for \( F, G \in \bar{\mathcal{R}}(T) \),
\[
\hat{F} \Sigma_n(1G) \sim \frac{1}{2a_n(T)} \sum_{k=-n}^{n} m(F \cap T^{-k}G)
\] (1.4)
\[
\sim m(F)m(G) \text{ as } n \to \infty.
\]

We will make use of the following proposition from [AKW].

**Proposition 2.** [AKW, Prop. 1] Let \((X, \mathcal{B}, m, T)\) be an invertible, conservative, ergodic, measure preserving transformation.

(i) If \( T \) satisfies (1.1) w.r.t. to some normalizing constants, then \( T \) is bounded rationally ergodic and
\[ a_n \sim 2a_n(T). \]

(ii) If \( T \) is bounded rationally ergodic, then
\[ \alpha(T) = \alpha(T^{-1}), \]
whence
\[ \beta(T) \leq \alpha(T) \leq 2\beta(T) \text{ and } \]
\[
\beta(T) \leq \frac{\alpha(T)}{2}. \quad (1.5)
\]

1.1.1 A minimal gap between the limit inferior and the limit superior of two sided Birkhoff sums

**Theorem 3.** Let \((X, \mathcal{B}, m, T)\) be an infinite, invertible, conservative, ergodic, measure preserving transformation, then
\[
\beta(T) - \underline{\beta}(T) \geq \frac{1}{5000}.
\]

**Remark 4.** The constant \( \delta := \frac{1}{5000} \) was chosen so that
\[ (1 - 50\delta) \leq 0.99 \quad (1.7) \]
and
\[
\frac{1}{2} \left( \frac{100}{99} \right)^2 \frac{1 + \delta}{1 - \delta} \leq \frac{1}{\sqrt{3}}. \quad (1.8)
\]

We would like to point out that by a more careful bookkeeping one can obtain a better constant for \( \delta \). This will amount in more technical arguments which we chose not to follow. As for now, we don’t know of any examples with \( \beta - \underline{\beta} < \frac{1}{2} \), it is interesting to find out what is the minimal \( \delta \) so that there exists a conservative, ergodic infinite measure preserving transformation \( T \) with \( \delta = \beta(T) - \underline{\beta}(T) \).
Proof: Suppose otherwise that 
\[ \overline{\beta(T)} - \underline{\beta(T)} < \delta := \frac{1}{5000}. \]
Let \( a(n) := a_n(T) \). Then \( T \) satisfies \( (1.3) \), since otherwise either \( \overline{\beta} := \overline{\beta(T)} = \infty \) or \( \overline{\beta} \geq 1 \) & \( \underline{\beta} = 0 \).
Since \( \underline{\beta}(T) \leq 1 \) we can assume that for all \( A \in F_+ \)
\[ (1 - \delta) m(A) \lesssim \frac{1}{2a_n(T)} \Sigma_n(1_A) \lesssim (1 + \delta) m(A). \]
We claim that
\[ 2 - 2\delta < \alpha := \alpha(T) < 2 + 2\delta \] (1.9)
Indeed, by \( (1.6) \), \( \alpha \geq 2\beta = 2 - 2\delta \) and by \( (1.5) \) \( \alpha \leq 2\overline{\beta} \leq 2 + 2\delta \).

The rest of the proof is a quantitative version of the “single orbit” argument in [AKW], which we proceed to specify.

- Fix \( A \in F_+ \). By Egorov there exists \( B \in F_+ \cap A \), \( m(B) > \frac{3}{4} m(A) \) and \( N_0 \in \mathbb{N} \) so that for all \( n \geq N_0 \) and \( x \in B 
\]
\[ (2 - 2\delta) a(n) \leq \sup_{N \geq n} S_n(1_A)(x), \Sigma_n(1_A)(x) \leq (2 + 2\delta) a(n). \] (1.10)
- Call a point \( x \in B \) admissible if
\[ \frac{S_n(1_A)(x)}{S_n(1_B)(x)} \xrightarrow{n \to \infty} \frac{m(B)}{m(A)}, \] (A1)
\[ \frac{1}{2a(n)} \Sigma_n(1_A)(x) = (1 \pm \delta), \text{ for all } n \geq N_0; \] (A2)
\[ \sup_{N \geq n} \alpha a(N) S_n(1_B) \xrightarrow{n \to \infty} m(B), \] (A3)

and \( \exists K \subset \mathbb{N} \), an \( x \)-admissible subsequence in the sense that
\[ T^n x \in B, \ \forall n \in K \text{ and } \] (A4)
\[ \frac{1}{\alpha a(n)} S_n(1_B)(x) \xrightarrow{n \to \infty, n \in K} m(B). \] (A5)

An admissible pair is \( (x, K) \in B \times 2^\mathbb{N} \) where \( x \) is an admissible point and \( K \) is an \( x \)-admissible subsequence.

Note that if \( (x, K) \) is an admissible pair, then by (A1) and (A5)
\[ \frac{1}{\alpha a(n)} S_n(1_A)(x) \xrightarrow{n \to \infty, n \in K} m(A). \]

Lemma 5. Almost every \( x \in B \) is admissible.

Proof. By (1.9), (1.10) and the ratio theorem, almost every \( x \in B \) satisfies (A1), (A2) and (A3).

Also since \( \alpha = \alpha(T) < \infty \), for a.e. \( x \in B \), \( \exists K \subset \mathbb{N} \) satisfying (A5).
We claim that if \( K := \{k_n : n \geq 1\}, k_n \uparrow \), then \( K' := \{k'_n : n \geq 1\} \) where \( k'_n = \max \{j \leq k_n : T^j x \in B\} \)
is \( x \)-admissible. Evidently \( K' \) is infinite and satisfies (A4). To check (A5):
\[ \alpha a(k_n) m(B) \geq \alpha a(k'_n) m(B) \gtrsim S_{k'_n}(1_B)(x) = S_{k_n}(1_B)(x) \sim \alpha a(k_n) m(B). \] (A3) (A5)
Lemma 6. If \( x \in B, \ K \subset \mathbb{N} \) and \( \{J_n : n \in \mathbb{N}\} \) satisfy

\[
\frac{1}{\alpha a(n)} S_n (1_A) (x) \xrightarrow{n \to \infty, \ n \in K} m(A); \quad \quad n \geq J_n \xrightarrow{n \to \infty, \ n \in K} \infty; \quad \lim_{n \to \infty, n \in K} \frac{a(J_n)}{a(n)} =: \rho \in (0, 1/2),
\]

then

\[
\frac{1}{\alpha a(J_n)} S_{J_n} (1_A) (x) \gtrsim_{n \in K} m(A) \left( 1 - \frac{4\delta}{\rho} \right).
\]

Proof. Since \( x \in B \), for \( n \in K \) large

\[
S_n (1_A) (x) + S_n^- (1_A) (x) = \Sigma_n (1_A) (x) \lesssim 2 (1 + \delta) a(n)m(A).
\]

Consequently

\[
S_n^- (1_A) (x) \lesssim_{n \in K} [2 + 2\delta - \alpha] a(n)m(A) \leq 4\delta a(n)m(A),
\]

and

\[
\frac{1}{\alpha a(J_n)} S_{J_n}^- (1_A) (x) \leq \frac{1}{\rho \alpha a(J_n)} S_{J_n}^- (1_A) (x) \lesssim_{n \in K} \frac{4\delta a(n) m(A)}{\alpha a(J_n)} \lesssim_{n \in K} \frac{4\delta}{\alpha \rho} a(n)m(A).
\]

Therefore as \( x \in B \),

\[
\frac{1}{\alpha a(J_n)} S_{J_n} (1_A) (x) = \frac{1}{\alpha a(J_n)} \Sigma_J (1_A) (x) - \frac{1}{\alpha a(J_n)} S_{J_n}^- (1_A) (x) \gtrsim_{n \in K} \frac{(2 - 2\delta)}{\alpha} m(A) - \frac{1}{\alpha a(J_n)} S_{J_n}^- (1_A) (x) \gtrsim_{n \in K} \frac{1}{\alpha} \left[ (2 - 2\delta) - \frac{4\delta}{\rho} \right] m(A) \geq (1 - 4\delta/\rho) m(A).
\]

Here the last inequality follows from

\[
\frac{2 - 2\delta}{\alpha} \geq \frac{2 - 2\delta}{2 + 2\delta} \geq 1 - 2\delta,
\]

\( \rho < \frac{1}{2} \) and \( \alpha > 2 - 2\delta > 3/2 \).

\[
\square
\]

Lemma 7. Let \((x, K) \in B \times 2^\mathbb{N}\) be an admissible pair then

\[
\frac{2}{25} \leq \lim_{n \to \infty, n \in K} \frac{a(n)}{a(n)} \& \lim_{n \to \infty, n \in K} \frac{a(n)}{a(n)} \leq \frac{1}{3}.
\]

Proof. We show first that

\[
\lim_{n \to \infty, n \in K} \frac{a(n)}{a(n)} \geq \frac{2}{25}, \quad \text{(a)}
\]
Define
\[ J_l := \min \left\{ l \geq \frac{\ln 9}{9} : T^l x \in B \right\} \wedge \frac{(l + 1)n}{9}; \quad (0 \leq l \leq 8), \]
then
\[ \alpha m(B) a(n) \overset{(A5)}{\leq} S_n(1_B)(x) = \sum_{l=0}^{8} S_{\frac{n}{9}}(1_B) \left( T^{\frac{ln}{9}} x \right) \]
\[ = \sum_{l=0}^{8} S_{\frac{n}{9} - J_l}(1_B) \left( T^{J_l} x \right) \leq \sum_{l=0}^{8} S_{\frac{n}{9}}(1_A) \left( T^{J_l} x \right) \]
\[ \leq \sum_{l=0}^{8} \left\| S_{\frac{n}{9}}(1_A) \right\|_{L^\infty(B)} \overset{\Delta}{\lesssim} 9(2 + 2\delta) a \left( \frac{n}{9} \right) m(A), \]
Thus
\[ \lim_{n \to \infty, n \in K} \frac{a \left( \frac{n}{3} \right)}{a(n)} \geq \frac{\alpha m(B)}{18(1 + \delta)m(A)} > \frac{2}{25}. \quad \square (a) \]
Next we show
\[ \lim_{n \to \infty, n \in K} \frac{a \left( \frac{n}{3} \right)}{a(n)} \leq \frac{1}{\sqrt{3}}. \quad (b) \]
By (a), \( \{n/3 : n \in K\} \) satisfies the conditions of Lemma 3 with \( \rho = 2/25 \), hence
\[ S_{\frac{n}{3}}(1_A)(x) \overset{\Delta}{\gtrsim} a \left( \frac{n}{3} \right) (1 - 50\delta) m(A). \]
By (A1),
\[ S_{\frac{n}{3}}(1_B)(x) \overset{n \in K}{\gtrsim} a \left( \frac{n}{3} \right) (1 - 50\delta) m(B) \overset{11.2}{\geq} \frac{99\alpha}{100} a \left( \frac{n}{3} \right) m(B). \]
For \( n \in K \), let
\[ J_n := \max \{ j \leq n/3 : T^j x \in B \}. \]
We claim that \( a(J_n) \gtrsim 0.99a(n/3) \), since
\[ a(J_n) m(B) \gtrsim S_{J_n}(1_B)(x) = S_{\frac{n}{3}}(1_B)(x) \]
\[ \gtrsim \frac{99\alpha}{100} a \left( \frac{n}{3} \right) m(B). \]
Finally
\[ (2 + 2\delta) a(n) \overset{(\epsilon \in B)}{\gtrsim} \Sigma_{n}(1_A)(T^{J_n} x) = \sum_{-n+J_n}^{n+J_n} 1_A(T^k x) \]
\[ \gtrsim \Sigma_{J_n}(1_A)(T^{J_n} x) + \Sigma_{J_n}(1_A)(T^m x) \]
\[ \overset{(*)}{\gtrsim} 2(2 - 2\delta) a(J_n) \overset{\Delta}{\gtrsim} (4 - 4\delta) \left( 0.99a \left( \frac{n}{3} \right) \right). \]
In (*) we used the fact that \( T^{J_n} x, T^m x \in B \). Therefore
\[ \lim_{K \ni n \to \infty} \frac{a(n/3)}{a(n)} \leq \frac{100}{198} \frac{(1 + \delta)}{(1 - \delta)} \leq \frac{1}{\sqrt{3}}. \quad \square (b) \]
Next, we show that
\[ \lim_{n \to \infty, n \in K} \frac{a \left( \frac{n}{9} \right)}{a(n)} \leq \frac{1}{3}. \] (c)

Let
\[ L_n := \min \left\{ J \geq \frac{n}{3} : T^J x \in B \right\}. \]

For \( n \in K, T^n x \in B \), whence \( L_n \leq n \).

Since
\[ \frac{a \left( \frac{L_n}{9} \right)}{a \left( \frac{n}{3} \right)} \geq \frac{a \left( \frac{n}{9} \right)}{a(n)} \geq \frac{2}{25}, \]
it follows from Lemma 6 that
\[ a \left( \frac{n}{3} \right) \geq \frac{99}{100} a \left( \frac{L_n}{3} \right) m(A). \] (1.12)

Define
\[ j_n := \max \left\{ j \leq \frac{L_n}{3} : T^j x \in B \right\}. \]

By repeating the previous argument with \( L_n \) replaced by \( L_n/3 \) and \( n/3 \) replaced by \( j_n \), one proves that
\[ S_{L_n/3} \left( 1_A \right)(x) \geq \frac{99}{100} a \left( \frac{L_n}{3} \right) m(A). \]

and
\[ a \left( \frac{n}{3} \right) \geq \frac{99}{100} a \left( \frac{L_n}{3} \right) m(A). \]

Therefore
\[ a \left( \frac{L_n}{3} \right) \geq \frac{99}{100} \cdot \frac{99}{100} a \left( \frac{L_n}{3} \right) m(A). \]

The argument in the proof of (b) shows that
\[ 2 \left( 1 - 2\delta \right) a \left( j_n \right) m(A) \leq \sum_{j_n} \left( 1_A \right) (T^j x) + \sum_{j_n} \left( 1_A \right) (T^{L_n} x) \]
\[ \leq \sum_{L_n} \left( 1_A \right) (T^{j_n} x) \leq (2 + 2\delta) a \left( L_n \right) m(A). \]

Here we used in the first inequality the fact that \( T^{j_n} x, T^{L_n} x \in B \) and in the last inequality the fact \( T^{j_n} x \in B \).

Therefore
\[ a \left( j_n \right) \geq \frac{99}{100} a \left( \frac{L_n}{3} \right) m(A). \]

and
\[ \frac{a \left( \frac{n}{9} \right)}{a \left( \frac{n}{3} \right)} \leq \frac{a \left( \frac{L_n}{3} \right)}{a \left( \frac{n}{3} \right)} \leq \frac{100}{99} \cdot \frac{a \left( \frac{L_n}{3} \right)}{a \left( L_n \right)} \]
\[ \leq \frac{1}{\sqrt{3}}. \] (1.14)
Thus
\[
\frac{a(n/9)}{a(n)} = \frac{a(n/9)}{a(n/3)} \cdot \frac{a(n/3)}{a(n)} \lesssim \frac{1}{3}. \quad \Box \text{(c)}
\]

Lemma 8. If \((x, K)\) is an admissible pair then
\[
\lim_{n \to \infty, n \in K} \frac{a(n/9)}{a(n)} \leq 0.93.
\]

Proof. First we show that
\[
S^-_\frac{n}{9} (1_A) (T^m x) \gtrsim (2 - 52\delta) a \left(\frac{n}{9}\right) m(A) \gtrsim \frac{96\alpha}{100} a \left(\frac{n}{9}\right) m(A), \quad (1.15)
\]
here the last inequality follows from \(\alpha \leq 2 + 2\delta = \frac{10002}{5000}\) and \((2 - 52\delta) = \frac{9648}{5000} \geq \frac{96\alpha}{100}\).

Indeed, since
\[
S^-_n (1_A) (T^m x) = S_n (1_A) (x) \sim \alpha a(n) m(A),
\]
then
\[
S_n (1_A) (T^m x) = \sum_n (1_A) (T^m x) - S^-_n (1_A) (T^m x) \sim \sum_n (1_A) (T^m x) - \alpha a(n) m(A).
\]

In addition for every \(n \in K\), \(T^m x \in B\), it follows from (A2) that as \(K \ni n \to \infty\),
\[
\sum_n (1_A) (T^m x) \lesssim (2 + 2\delta) a(n) m(A).
\]

Therefore since \(\alpha > 2 - 2\delta\),
\[
S_n (1_A) (T^m x) \lesssim ((2 + 2\delta) - \alpha) a(n) m(A) \leq 4\delta a(n) m(A). \quad (1.16)
\]

Finally
\[
S^-_\frac{n}{9} (1_A) (T^m x) \gtrsim \sum_n (1_A) (T^m x) - S^-_n (1_A) (T^m x) \gtrsim (2 - 2\delta) a \left(\frac{n}{9}\right) - 4\delta a(n).
\]

Indeed,
\[
(2 - 2\delta) a \left(\frac{n}{9}\right) - 50\delta a \left(\frac{n}{9}\right) = (2 - 52\delta) a \left(\frac{n}{9}\right). \quad \Box \text{(1.15)}
\]

Finally since
\[
\frac{a(\frac{n}{9})}{a(n)} \gtrsim \frac{a(\frac{n}{9})}{a(n)} \gtrsim \frac{2}{25},
\]
then by Lemma (6)
\[
\alpha \left(1 - 50\delta\right) a \left(\frac{8n}{9}\right) \lesssim \sum_{n \in K} S^-_{\frac{8n}{9}} (1_A) (x) = S_{\frac{8n}{9}} (1_A) (x) - S^-_{\frac{8n}{9}} (1_A) (T^m x) \quad \Box \text{(1.15)}
\]

and \((A5)\)
\[
\alpha m(A) \left(a(n) - \frac{96}{100} a \left(\frac{n}{9}\right)\right) \lesssim \alpha m(A) a(n) \left[1 - \frac{96}{100} \cdot \frac{2}{25}\right] = \alpha m(A) a(n) \left[\frac{92}{100}\right].
\]

Whence
\[
\frac{a(\frac{8n}{9})}{a(n)} \lesssim \frac{92}{100 (1 - 50\delta)} \leq 0.93.
\]

\[\Box\]
Proof of Theorem 3. Fix an admissible pair \((x, K) \in B \times 2^\mathbb{N}\), then
\[
\lim_{n \to \infty, n \in K} \frac{a \left( \frac{n}{9} \right)}{a(n)} \leq \frac{1}{3} \quad \text{and} \quad \lim_{n \to \infty, n \in K} \frac{a \left( \frac{8n}{9} \right)}{a(n)} \leq 0.93.
\]
For \(n \in K\), let
\[
J_n = J_n(x) := \min \left\{ j \geq \frac{8n}{9} : T^j x \in B \right\}.
\]
We claim that \(J_n \leq \frac{8n}{9}\); else as \(n \to \infty, n \in K\):
\[
\alpha a(n) m(B) \sim S_n(1_B)(x) = S_{J_n}(1_B)(x) + S_{n-J_n,0}(1_B)(T^{J_n}x) = S_{\frac{8n}{9}}(1_B)(x) + S_{\frac{n}{9}}(1_B)(T^{J_n}x) \leq S_{\frac{8n}{9}}(1_B)(x) + S_{\frac{n}{9}}(1_B)(T^{J_n}x) \leq 2 \alpha a \left( \frac{n}{9} \right).
\]
The inequality of \((\diamond)\) is where we assume that \(J_n \geq \frac{8n}{9}\).

Thus
\[
\frac{1}{2} \leq \frac{a \left( \frac{n}{9} \right)}{a(n)} \leq \frac{1}{3}.
\]
This contradiction shows that \(J_n \leq \frac{8n}{9}\).

Finally since \(\frac{n}{9} \leq J_n \leq \frac{8n}{9}\):
\[
[0, n] \subset \left[ J_n - \frac{8n}{9}, J_n + \frac{8n}{9} \right],
\]
therefore as \(n \to \infty, n \in K\),
\[
(2 + 2 \delta) a \left( \frac{8n}{9} \right) m(A) \geq \sum_{n \in K} m(A) \geq \sum_{n \in K} S_{\frac{8n}{9}}(1_A)(T^{J_n}x) \geq S_n(1_A)(x) \sim \alpha a(n) m(A) \geq (2 - 2 \delta) a(n) m(A),
\]
whence
\[
\frac{1 - \delta}{1 + \delta} \leq \frac{a \left( \frac{8n}{9} \right)}{a(n)} \leq 0.93.
\]
This is a contradiction since \(\frac{1 - \delta}{1 + \delta} = 0.93 > 0.93\). This proves the theorem. \(\square\)

2 The main step to move from a return sequence to a universal bound

Lemma 9. Let \((X, B, m, T)\) be an infinite, invertible, conservative, bounded rationally ergodic, measure preserving transformation then for any sequence \(a_n \to \infty\),
\[
\lim_{n \to \infty} \frac{\sum a_n f}{a_n} \leq \sqrt{\frac{\beta(T)}{\beta(T)}}.
\]
Proof. Assume in the contra negative that for all \(0 \leq f \in L^1(X, m)\),
\[
\lim_{n \to \infty} \frac{\sum a_n f}{a_n} \sqrt{\frac{\beta(T)}{\beta(T)}} > 0.93.\]
Since this property is true, \((X, \mathcal{B}, m, T)\) is bounded rationally ergodic and thus \(\exists A \in \mathcal{R}(T)\) with 
\[ 0 < m(A) < \infty. \]
By multiplying \(a_n\) by constants we can assume that,
\[
\lim_{n \to \infty} \frac{\sum_n (1_A) (x)}{a_n} = m(A) \quad \text{and} \quad \lim_{n \to \infty} \frac{\sum_n (1_A)}{a_n} > \sqrt{\frac{\beta(T)}{\beta(T)}} m(A).
\]
As before, it follows from Egorov’s theorem that for all \(\gamma < \sqrt{\frac{\beta(T)}{\beta(T)}} < 1 < \lambda\), there exists \(B \subset A\) of positive measure so that for all \(n\) large,
\[
\gamma m(A) = \frac{\sum_n (1_A) (x)}{a_n} \leq \lambda m(A) \quad \text{uniformly in} \quad x \in B,
\]
and thus for large \(n\)
\[
\gamma m(A) m(B) \leq \int_B \frac{\sum_n (1_A) (x)}{a_n} dm \leq \lambda m(A) m(B)
\]
Since \(\mathcal{R}(T)\) is hereditary, \(B \in \mathcal{R}(T)\). It follows from (1.4),
\[
\int_B \frac{\sum_n (1_A) (x)}{a_n} dm = \frac{2 a_n(T)}{a_n} \int_B \frac{\sum_n (1_A) (x)}{2 a_n(T)} dm 
\sim \frac{2 a_n(T)}{a_n} m(A) m(B).
\]
This shows that
\[
\gamma a_n \precsim 2 a_n(T) \precsim \lambda a_n,
\]
Consequently for all \(0 \leq f \in L^1(X, m)\),
\[
\lim_{n \to \infty} \frac{\sum_n (f)}{a_n} \leq \frac{\lambda \lim_{n \to \infty} \frac{\sum_n (f)}{2 a_n(T)}}{\gamma \lim_{n \to \infty} \frac{\sum_n (f)}{2 a_n(T)}} 
= \frac{\lambda \beta(T)}{\gamma \beta(T)}
\]
Since \(\gamma\) is arbitrary close to \(\sqrt{\frac{\beta(T)}{\beta(T)}}\) and \(\lambda\) is arbitrarily close to 1,
\[
\lim_{n \to \infty} \frac{\sum_n (f)}{a_n} \leq \sqrt{\frac{\beta(T)}{\beta(T)}},
\]
a contradiction. \(\square\)

**Remark 10.** In [AKW] we considered two important subclasses of infinite measure preserving transformations. Namely the “Rank one transformations” and “transformations admitting a generalized recurrent event” (the latter includes the class of null recurrent Markov shifts). In those examples when (1.1) happens then
\[
\frac{\beta(T)}{\beta(T)} \leq \frac{1}{2}.
\]
This together with the previous Lemma shows that for those examples for all \(a_n \to \infty\),
\[
\lim_{n \to \infty} \frac{\sum_n (f)}{a_n} \leq \frac{1}{\sqrt{2}}.
\]
3 Proof of Theorem 3

Let \((X, \mathcal{B}, m, T)\) be a conservative, ergodic, measure preserving transformation with \(m(X) = \infty\) and \(a_n \to \infty\). By Lemma 9

\[
\lim_{n \to \infty} \frac{\sum f}{a_n} \leq \sqrt{\beta(T)}
\]

and by Theorem 3 one has

\[
\bar{\beta}(T) - \beta(T) \geq \frac{1}{5000}.
\]

The theorem follows from

\[
\frac{\beta(T)}{\beta(T)} \leq \max \left\{ \frac{y}{x} : y < 1 < x, \ |x - y| > \frac{1}{5000} \right\}
= \max \left\{ \frac{y}{y + 1/5000} : y < 1 \right\}
= \frac{5000}{5001}
\]

and

\[
\sqrt{\frac{5000}{5001}} \leq 1 - \frac{1}{10002}.
\]

4 Applications for horocyclic flows on geometrically finite hyperbolic spaces

In [MS], Maucourant and Schapira considered the horocycle flow on geometrically finite hyperbolic spaces and showed examples where the invariant measure is infinite yet one still has precise knowledge of the fluctuations of the symmetric Birkhoff integrals which we now proceed to specify.

In this setting, let \(\Gamma_0\) be a non elementary finitely generated discrete subgroup of \(G = SL(2, \mathbb{R})\) without Torsion elements other than \(-Id\). Equivalently the surface \(S = \Gamma_0 \setminus \mathbb{H}\) where \(\mathbb{H}\) is the hyperbolic plane, is a geometrically finite Hyperbolic surface. On the tangent bundle of \(S\) one can consider two measures. The first is the measure of maximal entropy for the geodesic flow, also called the Bowen-Margulis or Patterson Sullivan measure which we will denote by \(m^{ps}\). This measure is supported on \(\Omega\), the non wandering set of the geodesic flow. The non wandering set \(E\) of the horocyclic flow is the union of horocycles intersecting \(\Omega\). By [Bu, Ro], the horocyclic flow has a unique ergodic invariant probability measure of full support on \(E\). This measure, denoted by \(m\), is often called the Burger-Roblin measure. The critical exponent of \(\Gamma := \pi_1(S)\) is defined by

\[
\delta := \limsup_{T \to \infty} \log \frac{1}{T} \# \{ \gamma \in \Gamma_0 : d(o, \gamma o) \leq T \},
\]

for any fixed point \(o \in \mathbb{H}\). In words \(\gamma\) is the exponential growth rate of the orbits of \(\Gamma\) on \(\mathbb{H}\). The ergodic theorem of [MS] is the following (We took the liberty of rephrasing it in a way that will explain the connection with symmetric Birkhoff sums).

**Theorem.** [MS] (1) Let \(S\) be a non elementary geometrically finite hyperbolic surface. Let \(u \in E\) be a non periodic and non wandering vector for the horocyclic flow. If \(f : T^1 S \to \mathbb{R}\) is continuous with compact support, then

\[
\lim_{t \to \infty} \frac{1}{m^{-H(u)}(h^s u)_{|s| \leq t}} \int_{-t}^t f(h^s u) ds = \frac{1}{m^{ps}(T^1 S)} \int_{T^1 S} f dm.
\]

Here \(m^{-H(u)}\) is the conditional measure of the Patterson-Sullivan measure on the strong stable horocycle \(H^{-}(u) = (h^s u)_{s \in \mathbb{R}}\).

(2) Writing \(\tau(u) := m^{-H(u)}((h^s u)_{|s| \leq 1})\), then \(m^{-H(u)}((h^s u)_{|s| \leq t}) = t^{\delta} \tau(g^{\log t} u)\).
If $S$ is convex cocompact, the non wandering set $\Omega \subset E$ of the geodesic flow is compact, the map $\tau$ is bounded from above and below on $\Omega$. Thus there exists constants $c_S, C_S > 0$ such that

$$\frac{c_S t}{m^p (T^1 S)} \int_{T^1 S} f dm \lesssim \int_{-t}^t f (h^s u) ds \lesssim \frac{C_S t}{m^p (T^1 S)} \int_{T^1 S} f dm, \text{ as } t \to \infty$$

The question arises of how close to 1 can $\frac{c_S}{C_S}$ be? For example is it true that there exists a sequence of convex cocompact geometrically finite surfaces $S_n = \Gamma_n \backslash \mathbb{H}$ such that

$$\frac{c_{S_n}}{C_{S_n}} \longrightarrow 1?$$

By modifying our proof for flows one sees that the answer to the last question is negative. The proof carries on verbatim once one makes the following adjustments:

- Definition of bounded rational ergodicity for flows.
- Showing that if for a monotone increasing function $a : [0, \infty) \to [0, \infty)$ and a set $A \subset E$ of positive $m$-measure,

  $$0 < c \lesssim \frac{1}{a(t)} \int_0^t 1_A (h^s(u)) \, dt \lesssim C < \infty$$

  for $m$ a.e. $u \in E$, then the functions

  $$E \times [0, \infty) \ni (u, t) \mapsto F_t(u) := \frac{1}{a(t)} \int_0^t 1_A (h^s(u)) \, dt$$

  satisfy the conditions of the Egorov type theorem for continuous parameter flows. In fact this case is much simpler and can be verified by applying Egorov on a discretization of the time parameter (a discrete skeleton) and then using the equicontinuity in $t$ of the map $F_t$.

- By the previous step one can carry the proof verbatim by first showing that the flow is bounded rationally ergodic and then applying our argument on a single orbit with minor modifications (in the definition of the stopping times).

The concluding statement is as follows.

**Corollary 11.** There exists a universal $\epsilon > 0$ such that for any $S$ a convex cocompact geometrically finite hyperbolic surface

$$\frac{c_S}{C_S} > 1 - \epsilon$$

where $c_S, C_S$ are the constants defined by

$$c_S := \liminf_{t \to \infty} \frac{1}{m^p (T^1 S)} \int_{-t}^t f (h^s u) ds \quad m - \text{a.e. } u \in E$$

and

$$C_S := \limsup_{t \to \infty} \frac{1}{m^p (T^1 S)} \int_{-t}^t f (h^s u) ds \quad m - \text{a.e. } u \in E,$$

for any $f : T^1 S \to \mathbb{R}$ continuous with compact support. Equivalently

$$c_S := \text{ess-} \liminf_{T \to \infty} \tau \left( g^{\log T} u \right)$$

$$C_S := \text{ess-} \limsup_{T \to \infty} \tau \left( g^{\log T} u \right).$$

**Acknowledgements:** Theorem 3 is part of the Author’s PhD thesis in Tel Aviv University done under the supervision of Jon Aaronson. I would like to thank Benjamin Weiss and Jon Aaronson for introducing me to the question of divergence of symmetric sums and Eli Glasner for suggesting the problem of universality in the divergence theorem. This research was supported in part by the European Advanced Grant StochExtHomog (ERC AdG 320977).
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