POISSON LIMIT FOR ASSOCIATED RANDOM FIELDS

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Abstract. We prove that under an easily verifiable set of conditions a sequence of associated random fields converges under rescaling to the Poisson Point Process and give a couple of examples.

1. Introduction and Main result

In this note we prove a Poisson scaling limit for a sequence of associated random fields. Let us recall that a finite family (vector) \((X_1, \ldots, X_m)\) of random variables (r.v.’s) is called associated if for every pair of bounded and coordinatewise nondecreasing functions \(f, g : \mathbb{R}^m \to \mathbb{R}\),

\[ \text{cov}(f(X_1, \ldots, X_m), g(X_1, \ldots, X_m)) \geq 0. \]

An infinite family of r.v.’s is called associated if its every finite subfamily is associated.

The notion of association was introduced and studied in [4]. Inequalities (1) with their equivalents have been often referred to as FKG inequalities by the initials of authors of [5] who studied this type of positive correlation independently.

Associated r.v.’s arise frequently in various problems of statistical mechanics and many other areas, see numerous examples, a historic overview, theory and applications in a recent monograph [1].

Basic properties of associated random vectors: jointly independent r.v’s form an associated family; monotone transformations of associated random vectors are associated, too.

A number of limit theorems for sums of associated r.v.’s have been proved, see [1] and references therein. To the best of our knowledge, no theorem on convergence to a Poisson Point Process has appeared in the literature, although some results on Poisson approximations for systems satisfying FKG inequalities can be found in [2] and references therein.

We proceed to describe the setting. We fix a dimension \(d \in \mathbb{N}\), and for each \(n \in \mathbb{N}\), let \((X_j^{(n)})_{j \in \mathbb{Z}^d}\) be a weakly stationary (i.e. in the sense of first moment and covariance) associated random field. We assume that for all \(n, j\), r.v. \(X_j^{(n)}\) takes two values, 0 and 1, and there is a number \(\lambda > 0\) such that

\[ p_n = \frac{\lambda + o(1)}{n^d}, \]

where \(p_n = P\{X_0^{(n)} = 1\}\).
We also assume that
\[
\lim_{n \to \infty} n^d \sigma(n) = 0,
\]
where
\[
\sigma(n) = \sum_{j \neq 0} \text{cov}(X_0^{(n)}, X_j^{(n)}).
\]

For any \(n\) we define a random measure \(\mu_n\) on \(\mathbb{R}^d\) via
\[
\mu_n(A) = \sum_{j \in \mathbb{Z}^d \cap nA} X_j^{(n)},
\]
where \(nA = \{nx : x \in A\}\).

The vague topology on locally bounded Borel measures is defined by its base, the class of finite intersections of sets of the form \(\{\nu : s < \int_{\mathbb{R}^d} f \, d\nu < t\}\) with arbitrary nonnegative continuous function \(f\) with bounded support and \(s, t \in \mathbb{R}\), see \cite{8} Appendix 7.

**Theorem 1.** Under the conditions stated above, the sequence of measures \(\mu_n\) converges in distribution in the vague topology to the Poisson measure \(\mu\) with parameter \(\lambda\).

**Proof.** By \cite{8} Theorem 4.2, it is sufficient to check that for every continuous nonnegative function \(f\) with compact support,
\[
\int_{\mathbb{R}^d} f \, d\mu_n \xrightarrow{\text{Law}} \int_{\mathbb{R}^d} f \, d\mu, \quad \text{as } n \to \infty.
\]

Take a continuous function \(f\) with compact support and a number \(t \in \mathbb{R}\), and find
\[
\mathbb{E} e^{it \int f \, d\mu_n} = \mathbb{E} e^{it \sum_{j \in \mathbb{Z}^d} f(\frac{j}{n}) X_j^{(n)}}
\]
\[
= \prod_{j \in \mathbb{Z}^d} \mathbb{E} e^{it f(\frac{j}{n}) X_j^{(n)}} + \mathbb{E} e^{it \sum_{j \in \mathbb{Z}^d} f(\frac{j}{n}) X_j^{(n)}} - \prod_{j \in \mathbb{Z}^d} \mathbb{E} e^{it f(\frac{j}{n}) X_j^{(n)}}
\]
\[
= I_1(n) + I_2(n).
\]

Notice that, in fact, the product in \(I_1(n)\) involves finitely many factors, and
\[
I_1(n) = \prod_{j \in \mathbb{Z}^d} \left(1 + p_n(e^{it f(\frac{j}{n})} - 1)\right).
\]

Choosing the main branch of the natural logarithm \(\ln\), we can write
\[
I_1(n) = \exp \left\{ \sum_{j \in \mathbb{Z}^d} \ln(1 + p_n(e^{it f(\frac{j}{n})} - 1)) \right\}.
\]
Using the boundedness of $f$ and the Taylor expansion for the logarithm we derive that

$$I_1(n) = \exp \left\{ \frac{\lambda + o(1)}{n^d} \sum_{j \in \mathbb{Z}^d} (e^{itf(\frac{j}{n})} - 1) \right\} (1 + o(1)).$$

Obviously, the r.h.s converges to

$$\phi(t) = \exp \left\{ \lambda \int_{\mathbb{R}^d} (e^{itf(x)} - 1) dx \right\},$$

the characteristic function of $\int_{\mathbb{R}^d} f d\mu$, and the proof will be finished as soon as we show that

$$\lim_{n \to \infty} I_2(n) = 0.$$

To estimate $I_2(n)$ we need Newman’s inequality:

**Theorem 2** ([9]). If $(Y_1, \ldots, Y_m)$ is a family of associated r.v.’s with finite second moment then

$$\left| E e^{i \sum_{j=1}^m r_j Y_j} - \prod_{j=1}^m E e^{i r_j Y_j} \right| \leq \frac{1}{2} \sum_{j_1 \neq j_2} |r_{j_1} r_{j_2}| \text{cov}(Y_{j_1}, Y_{j_2}),$$

for any real numbers $r_1, \ldots, r_m$.

Applying this inequality to $I_2(n)$ we see that

$$I_2(n) \leq \frac{t^2 \|f\|^2_{L^\infty} K n^d}{2} \sum_{j_1, j_2 \in \mathbb{Z}^d \cap n \text{supp}(f)} \text{cov}(X_{j_1}^{(n)}, X_{j_2}^{(n)}),$$

where $\text{supp}(f)$ denotes the support of $f$, and $| \cdot |$ denotes the number of elements. Since $|\mathbb{Z}^d \cap n \text{supp}(f)| \leq Kn^d$ for some constant $K > 0$ and all $n > 0$, (3) follows from (2). 

**Remark 1.** The crucial step in the proof above is the application of Newman’s inequality for associated random variables. Covariance inequalities of this type can be obtained for a wide class of dependent r.v.’s. In particular the theorem is also applicable if one replaces association by quasi-association, see [2] and proof of Theorem 2 in [9].

2. Examples

Let $G$ be a finite subset of $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. Denote $m = |G|$ and for each $n$ consider an i.i.d. family $(Y_k^{(n)})_{k \in \mathbb{Z}^d}$ of Bernoulli random variables with

$$P\{Y_0^{(n)} = x\} = \begin{cases} q_n, & x = 1, \\ 1 - q_n, & x = 0, \end{cases}$$

for each $x = 0, 1$. 

where

\[ q_n = \frac{1}{n^{d/m}}. \]

For any finite subset \( H \) of \( \mathbb{Z}^d \) and every \( n \), we denote

\[ \chi^{(n)}_H = \prod_{j \in H} Y^{(n)}_j = 1_{\{Y^{(n)}_j = 1, j \in H\}}, \]

and define a random field \((X^{(n)}_k)_{k \in \mathbb{Z}^d}\) via

\[ X^{(n)}_k = \chi^{(n)}_{k+G}, \]

where \( k + G = \{k + j : j \in G\} \). Poisson approximations for a similar model with rectangular \( G \) was considered in [6].

Let us verify that \( X^{(n)} \) satisfies the conditions of Theorem 1. Random field \( Y^{(n)} \) is associated since it is composed of independent components. Therefore, \( X^{(n)} \) is associated being a monotone transform of the associated field \( Y^{(n)} \). It is also stationary due to stationarity of \( Y^{(n)} \).

For each \( n \), \( X^{(n)}_0 \) is a Bernoulli r.v. with

\[ P\{X^{(n)}_0 = 1\} = P\{Y^{(n)}_j = 1, j \in G\} = \left(\frac{1}{n^{d/m}}\right)^m = \frac{1}{n^d}. \]

Let us now estimate \( \sigma(n) \). Notice that \( \text{cov}(X^{(n)}_0, X^{(n)}_j) = 0 \) for sufficiently large values of \( |j| \), so that there is a number \( M \) such that for all \( n \),

\[ \sigma(n) \leq M \max_{j \neq 0} \text{cov}(X^{(n)}_0, X^{(n)}_j). \]

(4)

Notice that

\[ \text{cov}(X^{(n)}_0, X^{(n)}_j) = \text{E}X^{(n)}_{G \cup (j+G)} - \text{E}X^{(n)}_G \text{E}X^{(n)}_{j+G}. \]

Since a finite set cannot be invariant under a translation, \( |G \cup (j+G)| \geq m+1 \) for any \( j \). Therefore,

\[ \text{cov}(X^{(n)}_0, X^{(n)}_j) \leq \frac{1}{n^{d(m+1)/m}} = o(1/n^d), \]

which, together with (4), implies (2), so that all the conditions of Theorem 1 are satisfied.

For an associated random field \( X^{(n)} \), condition (2) means that \( X^{(n)}_0 \) is asymptotically independent of the rest of the random field. There is a variety of situations that can happen if this condition is replaced with weaker restrictions on dependence. The next example illustrates the convergence to a compound Poisson point process (with nonrandom mass 2 assigned to each atom), see [3] for the definition and properties of compound Poisson point processes.

Consider \( d = 1 \), and for every \( n \) and all \( k \in \mathbb{Z} \),

\[ X^{(n)}_k = Y^{(n)}_k \vee Y^{(n)}_{k+1} = Y^{(n)}_k + Y^{(n)}_{k+1} - Y^{(n)}_k Y^{(n)}_{k+1}, \]
where $Y^{(n)}$ is a sequence of i.i.d. Bernoulli r.v.’s with $P\{Y_0^{(n)} = 1\} = 1/n$. Then, as an easy computation shows, $\sigma(n) \sim 1/n$ so that (2) is violated. One can also show that the sequence of random measures $\mu_n$ converges in distribution to $2\mu$, where $\mu$ is the Poisson process with unit intensity, so that the conclusion of Theorem 1 is violated as well. Indeed, take a continuous function $f$ with compact support, and write

$$E e^{it \int_R f d\mu_n} = E e^{it \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) (Y_j^{(n)} + Y_{j+1}^{(n)}) - it \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) Y_j^{(n)} Y_{j+1}^{(n)}}.$$ 

Notice that

$$\sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) Y_j^{(n)} Y_{j+1}^{(n)} \overset{P}{\to} 0, \quad n \to \infty,$$

due to the Markov inequality, since the expectation of l.h.s. is $O(1/n)$. Therefore, we see that

$$\lim_{n \to \infty} E e^{it \int_R f d\mu_n} = \lim_{n \to \infty} E e^{it \sum_{j \in \mathbb{Z}} f\left(\frac{j}{n}\right) + f\left(\frac{j-1}{n}\right)} Y_j^{(n)}$$

$$= \exp \left\{ \int_R (e^{it2f(x)} - 1) dx \right\},$$

by the same argument we used to analyze $I_1(n)$.

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