Tri-Scalar CFT and Holographic Bi-Fishchain Model

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Abstract

Bi-scalar CFT from $\gamma$ deformed $\mathcal{N}=4$ SYM describes the fishnet theory which is integrable in the planar limit. The holographic dual of the planar model is the fishchain model. The derivation of the weak-strong duality from the first principle was presented in a recent paper (”The Holographic Fishchain” arXiv:1903.10508). In this note we extend the investigation to the tri-scalar CFT which raises from the large twist limit of ABJM theory. We show that it becomes tri-scalar fishnet theory in planar limit and the dual theory is the holographic bi-fishchain model.

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1 Introduction

The AdS/CFT correspondence relates bulk AdS physics to a boundary conformal field theory. The archetypal example of the gauge-string correspondence is that of $N = 4$ super Yang-Mills theory, regarded as the boundary CFT, describes the type IIB superstrings in the $AdS_5 \times S^5$ bulk [1, 2, 3]. The correspondence has been applied to investigate several problems in large $N_c$ QCD such as the Wilson loop [4], the meson spectra [5], baryon dynamics [6] and so on.

Gauge-string correspondence, despite going through many successful checks and having been applied to study, again successfully, both strongly interacting QFTs and quantum gravity, still lacks a satisfactory microscopic derivation. One of the basic questions is how to reformulate CFT correlation functions and then, from which the emergence of gravitational dynamics becomes manifest. In other words, given a CFT and assuming a limit where the dual geometry is well defined, how to find an algorithm to determine this dual background directly from the correlators? In a recent paper [7, 8] Gromov and Sever showed a possible way to achieve the goal. The model theory they considered is a specially deformed SYM theory.

It is known that the $\mathcal{N}= 4$ SYM theory after $\gamma$ deformed in double scaling limit is conformal and integrable in the planar limit [9]. In this limit, the gauge fields decouple and the theory contains three complex scalars and fermions interacting through quartic and “chiral” Yukawa couplings. After turning off some couplings the fermions and scalar decouple and left the following bi-scalar Lagrangian [9]:

$$L_\phi = N_c \text{Tr} \left( \partial^\mu \phi^1_1 \partial_\mu \phi^1_1 + \partial^\mu \phi^2_2 \partial_\mu \phi^2_2 + 4(2\pi)^2 \xi^4 \phi^1_1 \phi^2_2 \phi^1_1 \phi^2_2 \right)$$

(1.1)

where $\phi^a_i = \phi^a_i T^a$ are complex scalar fields and $T^a$ are the generators of the SU($N_c$) gauge group. The 4-dimensional QFT is conformal in the large $N_c$ limit and integrable at any coupling in the 't Hooft limit, even in the absence of supersymmetry and gauge symmetry [9].

Since $\phi^a_i$ are complex matrix fields the interaction term in the Lagrangian $L_\phi$ is not real and the theory described by (1.1) is not unitary. Non-unitary CFTs appear in a condensed matter context, for example in [10] and references therein. The property of non-unitary CFTs is not clear as the bootstrap methods are not applicable for the non-unitary CFTs.

In the large $N_c$ limit the bi-scalar Lagrangian becomes planar model which is called as fishnet theory. Authors of [7] presented the first-principle derivation of a weak-strong duality between the fishnet theory in four dimensions and a discretized string-like model living in five dimensions, which is called as fishchain model.

It is known that the analogous fishnet diagrams also exist in 2, 3 and 6 dimensions Zamolodchikov [11] and it is interesting to consider the ABJM model which is a 3-dimensional theory [12]. In this note we extend the investigation of [7] to the tri-scalar CFT which raises from the large twist limit of ABJM theory. We will show, in step by step, that the dual theory is the holographic bi-fishchain model [1].

1 Authors of [7, 8] expected that their techniques can be generalized rather straightforwardly to this class of graphs. Our work simply confirms their expectations in a clear manner. For the convenience of readers, the derivation of each equation in this note is written in great detail.
In section 2 we describe the anomalous dimension of correlation function, Feynman diagram and the wheel-type Feynman graph. We plot several diagrams to illuminate how to use the graph-building element to build the Feynman graph and associated fishnet diagram. We first discuss the bi-scalar theory and then extend it to the tri-scalar theory. In section 3 we use the diagram to find the associated Lagrangian, which shows the strong-weak duality after using the gauge symmetry in the system. In section 4, from the property that the D-dimensionally conformal invariant quantity is a D+2 scalar on SO(D+1,1) we uplifte the action into a D+1 embedding space, which leads to the holographic fishchain model. These then explicitly derive holographic bi-fishchain model from tri-scalar CFT. A short conclusion is made in the last section.

The early literature on fishnet can be found in [13, 14, 15, 16, 17]. The associated exact states, scattering amplitudes, and correlators were calculated in [18, 19, 20, 21, 22, 23]. Fishnet from \( \gamma \) Deformed N=2, twist fishnet, and massive fishnets were studied in [24, 25, 26] respectively. The Regge properties of fishnet was analyzed detailly in [27, 28]. Some recent research can be found in [29, 30, 31, 32, 33, 34, 35] and references therein.

## 2 Anomalous Dimension and Graph-building Method

A family of scalar single-trace operators to be studied is [17]

\[
O_{J,n,\ell} = P_{2\ell} \text{tr} \left[ \phi_1^J \phi_2^n \phi_2^\dagger \right] + \ldots \tag{2.1}
\]

where \( P_{2\ell} \) denotes \( 2\ell \) derivatives acting on scalar fields inside the trace with all Lorentz indices contracted. The dots stand for similar operators with the scalar fields \( \phi_1, \phi_2 \) and \( \phi_2^\dagger \) exchanged inside the trace. The operators have scaling dimension \( \Delta = J + 2(n + \ell) \) for zero coupling \( \xi = 0 \).

We consider the simplest case of \( n = \ell = 0 \) and the operator takes the form

\[
O_J = \text{tr} [\phi_1^J] \tag{2.2}
\]

In \( \mathcal{N} = 4 \) SYM the similar operator is protected from quantum corrections dues to the supersymmetry therein and is known as the BMN vacuum operator. In the \( \gamma \)-deformed \( \mathcal{N} = 4 \) SYM, the operator is not protected and its scaling dimension will depend on the coupling strength \( \xi \).

The coupling dependent corrections to the scaling dimension \( \Delta \) is defined by

\[
D(x) = \langle O_J(x) \bar{O}_J(0) \rangle = \frac{d(\xi)}{(x^2)^{J+\gamma_J(\xi)}} \tag{2.3}
\]

where the normalization constant \( d(\xi) \) and the anomalous dimension \( \gamma_J(\xi) \) depend on the coupling constant \( \xi \). In the large \( N_c \) limit both only come from the wheel-type Feynman graphs. Each wheel contains \( J \) interaction vertices and the contribution to the scaling dimension of the wheel graph with \( M \) frames scales as \( (\xi^2)^{J-M} \), by the Feynman rule read from the Lagrangian of bi-scalar theory [1.1]. For example Fig. 1 is the \( J=M=3 \) wheel-type Feynman graph in which

\footnote{The time reparametrization invariant in (3.3) is a local symmetry and is called as gauge symmetry in here.}
three $\phi_1^\dagger(0)$ locate together at origin while three $\phi_1(x_1)$ locate separately at radius $r=x_1$. There are $3 \times 3$ vertices and is order of $(\xi^2)^9$ wheel-type Feynman graph which is used to calculate $\langle O_3(x)\bar{O}_3(0) \rangle$ in [2.3].

![Feynman graph](image)

Figure 1: $J=M=3$ wheel-type Feynman graph. There are $3 \times 3 = 9$ vertices.

The Feynman rules associated to the bi-scalar Lagrangian (1.1) are:

- **Propagator**: $\langle \phi_1(x_i)\phi_1^\dagger(y_i) \rangle = \frac{1}{(2\pi)^2} \frac{1}{(x_i-y_i)^2}$; $\langle \phi_2(y_i)\phi_2^\dagger(y_{i+1}) \rangle = \frac{1}{(2\pi)^2} \frac{1}{(y_i-y_{i+1})^2}$ (2.4)
- **Vertex**: $(4\pi)^2 \xi^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2$ (2.5)

Using above Feynman rules we can resume the wheel-type Feynman graphs from the “graph-building” operator $\hat{B}$ by defining its integral kernel

$$
\hat{B}(\{y_i\}_{i=1}^J, \{x_j\}_{j=1}^J) = \prod_{i=1}^{J} \frac{(4\pi)^2 \xi^2}{(2\pi)^2 (y_i-y_{i+1})^2 (x_i-y_i)^2} = \prod_{i=1}^{J} \frac{\xi^2/\pi^2}{(y_i-y_{i+1})^2 (x_i-y_i)^2}
$$

(2.6)

which is simply a product of several “graph-building element”. For example

$$
\hat{B}(y_1, y_2, y_3, x_1, x_2, x_3) = \prod_{i=1}^{3} \frac{\xi^2/\pi^2}{(y_i-y_{i+1})^2 (x_i-y_i)^2}
$$

(2.7)

is shown in figure 2.

![Graph-building method](image)

Figure 2: Graph-building method. Use “graph-building element” to build Feynman graph.

Note that we use periodic boundary condition $y_1 = y_{J+1}$. Each horizontal link is a free scalar propagator $(2\pi)^{-2} (y_j-y_{j+1})^{-2}$ while each vertical line has similar propagator connecting points.
$x_i$ and $y_i$, as described in (2.4). Applying this operator once, we add one wheel to the graph on above, thus the sum of all wheels inside the graph forms a geometric series:

$$\text{All wheels} = \frac{1}{1 - B}$$  \hspace{1cm} (2.8)

For example, by the definition (2.3)

$$\frac{1}{1 - B_3} = \langle O_3(x) O_3(0) \rangle = \langle \text{tr}[\phi_1(x_1) \phi_1(x_2) \phi_1(x_3)] \text{tr}[\phi_1(0)^3] \rangle = \frac{d(\xi)}{(x^2)^{3+\gamma_3(\xi)}}$$  \hspace{1cm} (2.9)

which is used to calculate the correlation function of J=M=3 wheel-type Feynman graph in Figure 1. The zeros of the denominator in (2.8) can be identified as the anomalous dimensions $\gamma(\xi)$ of the local operators $[19]$, which plays a special role in CFT.

An important observation is that the zeros of the denominator can be found by solving the eigenvalue equation

$$(-1 + B)\Psi = 0$$  \hspace{1cm} (2.10)

To proceed we can use the 4 dimensional Green function

$$\square \frac{1}{4\pi^2(\vec{x} - \vec{y})^2} = -\delta^4(\vec{x} - \vec{y})$$  \hspace{1cm} (2.11)

and definition of $\hat{B}$ in (2.6) to construct the bi-scalar Hamiltonian $H_{bi}$ form below relation

$$0 = \prod_i \square_i \left((-1 + \hat{B})\Psi\right) = \left(- \prod_i \square_i\right)\Psi + \left(\prod_i \hat{B}\right)\Psi$$

$$= H_{bi} \circ \Psi \{x_i\}, \quad H_{bi} = \sum_{i=1}^{J} \vec{p}_i^2 - \sum_{i=1}^{J} \frac{4\xi^2}{(\vec{x}_i - \vec{x}_{i+1})^2}$$  \hspace{1cm} (2.12)

To obtain above result we use the relation $\vec{p}_i = -i\vec{\partial}_{x_i}$ and property of delta function in Green function relation (2.11).

In this note we will extend the method to the large twist limit of ABJM theory. The associated Lagrangian of tri-scalar CFT is $[14]$.

$$L_{\phi} = N_c \text{Tr} \left( \partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + \partial^\mu \phi_3^\dagger \partial_\mu \phi_3 + 4(2\pi)^3 \xi^2 \phi_1^\dagger \phi_2^\dagger \phi_3^\dagger \phi_1 \phi_2 \phi_3 \right)$$  \hspace{1cm} (2.13)

The Lagrangian describes a pure 3D tri-scalar interacting theory and is integrable in the planar limit. We will first follow the previous method to find how the $H_{bi}$ in (2.12) will be modified.

It is easy to see that the “graph-building” operator $\hat{B}$ in (2.6) now becomes

$$\hat{B}([\vec{y}_i]_{i=1}^J, [\vec{x}_j]_{j=1}^J) = \prod_{i=1}^{J} \frac{4(2\pi)^3 \xi^2}{(2\pi)^2(\vec{y}_i - \vec{y}_{i+1})^2(2\pi)^2(\vec{x}_i - \vec{x}_{i+1})^2(2\pi)^2(\vec{y}_i - \vec{x}_{i+1})^2}$$  \hspace{1cm} (2.14)

which is simply a product of several “graph-building element”, as can be seen in Figure 3 for bi-scalar CFT and Figure 4 for tri-scalar CFT.
Figure 3: Use “graph-building element” to build Feynman graph of bi-scalar CFT.

Figure 4: Use “graph-building element” to build Feynman graph of tri-scalar CFT.

Note that using the Feynman graph in figure 4 we can construct the wheel-type Feynman graph for bi-scalar CFT and tri-scalar CFT, as shown in figure 5. By gluing the arrow of each color and putting green stars on the central of a wheel we can see that the left-hand side becomes wheel-type Feynman graph for bi-scalar CFT, as the Figure 1, and the right-hand side becomes wheel-type Feynman graph for tri-scalar CFT.

The paper [14] had also plotted the wheel-type Feynman graph for tri-scalar CFT (figure 11), which using the curved line. Our graph in figure 5 is plotted in the strength line and is easy to imagine for some readers.
Figure 5: Use “Feynman graph” to build “wheel-type Feynman graph”. Wheel-type Feynman graph is obtained by gluing the arrow of each color and putting green stars on central of a wheel.

To proceed we see that the relation \( \vec{z}_1 = \vec{x}_1 + \vec{y}_2 - \vec{y}_1 \) can simplify the propagator relation

\[
\frac{1}{(2\pi)^2(z_1 - y_1)^2} = \frac{1}{(2\pi)^2(x_1 + y_2 - 2y_1)^2} = \frac{1}{(2\pi)^2(x_1 + x_2 - 2x_1)^2} \tag{2.15}
\]

\[
= \frac{1}{(2\pi)^2(x_1 - x_2)^2} \tag{2.16}
\]

where the last relation in (2.15) is the result of delta function property in Green function relation (2.11). Therefore the eigenvalue equation associated to bi-scalar CFT in (2.12) now becomes

\[
0 = \mathcal{H}_{\text{tri}} \circ \Psi(\{x_i\}), \quad \mathcal{H}_{\text{tri}} = \left( \prod_{i=1}^{J} \vec{p}_i^2 - \prod_{i=1}^{J} \frac{4\xi^2}{(\vec{x}_i - \vec{x}_{i+1})^4} \right) \tag{2.17}
\]

in tri-scalar CFT. We see that just changes the factor \((\vec{x}_i - \vec{x}_{i+1})^{-2}\) in \(\mathcal{H}_{\text{bi}}\), i.e. eq.(2.12), to \((\vec{x}_i - \vec{x}_{i+1})^{-4}\) we then obtain \(\mathcal{H}_{\text{tri}}\), i.e. eq.(2.17).

In the following sections we will first finds the associated Lagrangian of tri-scalar CFT from \(\mathcal{H}_{\text{tri}}\) and proves that the corresponding dual Lagrangian describes a bi-fishchain model, which is plotted in figure 6. Below derivations totally follow the method in the original paper [7] while present them in step by step in mathematical relations and is more easy to read.

### 3 Fishnet Lagrangian and Strong-Weak Duality

Through the following standard calculations one can find the associated Lagrangian. First, using \(\mathcal{H}_{\text{tri}}\) in (2.17) we have a relation

\[
\dot{x}_k \equiv \frac{\partial \mathcal{H}_{\text{tri}}}{\partial \vec{p}_k} = \frac{2\prod_{i=1}^{J} \vec{p}_i^2}{\vec{p}_k} \tag{3.1}
\]

which leads to

\[
\prod_{i=1}^{J} \dot{x}_i = 2^J \left( \prod_{i=1}^{J} \vec{p}_i^2 \right)^{-\frac{1}{2}}; \quad \prod_{i=1}^{J} \vec{p}_i^2 = \frac{1}{2^J - 1} \left( \prod_{i=1}^{J} \dot{x}_i^2 \right)^{\frac{2}{J+1}} \tag{3.2}
\]

\[
\sum_{k=1}^{J} (\vec{p}_k \dot{x}_k) = \sum_{k=1}^{J} \left( 2 \prod_{i=1}^{J} \vec{p}_i^2 \right) = 2J \prod_{i=1}^{J} \vec{p}_i^2 \tag{3.3}
\]

The Lagrangian then becomes

\[
L(\gamma) = \sum_{k} \vec{p}_k \dot{x}_k - \mathcal{H}_{\text{tri}} = \frac{2J - 1}{2^{2J+1}} \left( \frac{1}{\gamma} \prod_{i=1}^{J} \dot{x}_i^2 \right)^{\frac{2}{J+1}} + \gamma \prod_{i=1}^{J} \frac{4\xi^2}{(\vec{x}_i - \vec{x}_{i+1})^4} \tag{3.4}
\]

where we add a gauge parameter \(\gamma\). If we consider the following gauge transformations (time reparameterization)

\[
t \rightarrow f(t); \quad \gamma \rightarrow f'(t) \tag{3.5}
\]
which imply
\[ dt \rightarrow f'(dt); \quad \dot{x}^2 = \frac{dx^2}{dt^2} \rightarrow \frac{dx^2}{f'^2} \dot{x}^2 \] (3.6)
the Lagrangian and action then transform as
\[
L(\gamma) = \left( \frac{2J - 1}{2^{2J-1}} \left( \frac{1}{\gamma} \prod_{i=1}^{J} \dot{x}_i^2 \right)^{J-1} + \gamma \prod_{i=1}^{J} \frac{4\xi^2}{(\dot{x}_i - \dot{x}_{i+1})^4} \right) \rightarrow \frac{1}{f'} L(\gamma) \] (3.7)
\[
S = \int dt L(\gamma) = \int dt f' \left( \frac{1}{f'} L(\gamma) \right) = \int dt L(\gamma) \equiv S \] (3.8)
We see that the action is invariant under the gauge transformations (3.5).

Now, we use the gauge freedom of \( \gamma \) to choose the its extremum by \( \partial_{\gamma_{max}} L(\gamma_{max}) = 0 \). The action and Lagrangian then become
\[
S(\gamma_{max}) = \xi \int dt L(\gamma_{max}) \] (3.9)
\[
L(\gamma_{max}) = 2J \left( \prod_{i=1}^{J} \frac{\dot{x}_i^2}{(\dot{x}_i - \dot{x}_{i+1})^4} \right)^{\frac{1}{2J}} \] (3.10)
Two interesting properties are shown in above action and Lagrangian:
1. In path integration the action factor \( e^{-\frac{S}{\hbar}} \) tells us that the overall factor of coupling \( \xi \) in action (3.9) plays the role of \( \frac{1}{\hbar} \). Since that small \( \hbar \) is classical the large \( \xi \) is classical too. Therefore, the strong coupling of Lagrangian (3.4) can be studied by classical model of (3.10) and strong-weak duality is shown up in here.
2. The action S in (3.9) is also invariant under global conformal transformations, which maybe relates to the property a CFT dual. We will, next, use the conformal symmetry to uplifte the action into a D+1 embedding space, which leads to the holographic fishchain model.

4 Holographic Bi-Fishchain Model

It is known that conformal algebra on D dimension is same as the algebra of SO(D+1,1) and D-dim conformal invariant quantity is a D+2 scalar on SO(D+1,1). Using this property we can express the Lagranginal
\[ L(\gamma_{max}) \] of (3.10) in higher dimension, which is called as Fishchain.

Let us denote \( X^A \) as D+2 dim coordinate
\[ X^A = (X^1, \ldots, X^D, X^{D+1}, X^{D+2}) \] (4.1)
In light cone coordinate
\[ X^\pm = X^5 \pm X^6, \quad ds^2 = -dX^+dX^- + \sum_{\mu} (dX^\mu)^2 \] (4.2)
To project it to 4 dimension we can choose it on null cone, since null is invariant. I.e. we choose
\[ X^A = (X^+, X^-, x^\mu) = (1, x^2, x^\mu) \] (4.3)
\[ \rightarrow X^2 = X^A X_A = -1 \cdot x^2 + x_\mu x^\mu = 0 \] (4.4)
Above choice leads to
\[
\dot{X}^A = (0, 2i\mu x_\mu, \dot{x}_\mu) \rightarrow \dot{X}^A \dot{X}_A = -0 \times 2i\mu x_\mu + \dot{x}_\mu \dot{x}_\mu = \dot{x}_\mu \dot{x}_\mu
\]
(4.5)
and Lagrangian \(L(\gamma_{\text{max}})\) of (3.10) can be expressed in D+1 coordinate \(X^A\) in (4.4) and then
\[
L_{D+1} = 2J \left( \frac{\dot{X}_i^2}{2^2 \alpha_i} \right) \prod_{i=1}^{j} \left( -2X_i \cdot X_{i+1} \right)^{\frac{1}{2}} \]
(4.6)
Having above D+1 Lagrangian the next work is to find a conventional Lagrangian, with both of kinetic term and interaction term, which describes above Lagrangian.

With the guiding in [7, 8], which investigate the bi-scalar CFT case, the associated Lagrangian in tri-scalar CFT case is
\[
\mathcal{L} = -\left( \sum_i \left[ \frac{\dot{X}_i^2}{2^2 \alpha_i} + \eta_i X_i^2 \right] \right) - J \left( \prod_k \alpha_k \right)^{\frac{1}{2}} \prod_k \left( -X_k \cdot X_{k+1} \right)^{-\frac{3}{2}}
\]
(4.7)
The \(\eta_i\) is a Lagrangian multiple used to constrain the theory on null plane \(X_i^2 = 0\). The Lagrangian has several symmetry: 1. Conformal symmetry. 2. Time reparameterization of \(t \rightarrow f(t), \alpha_i \rightarrow \alpha_i/f(t), \eta_i \rightarrow \eta_i/f(t)\). 3. Translation symmetry \(X_i \rightarrow X_i+1\).

To proceed we first extremeize (4.7) with respective to parametre \(\alpha_i\) we find
\[
0 = \partial_{\alpha_i} \mathcal{L} \rightarrow \frac{\dot{X}_i^2}{2^2 \alpha_i} = \left( \prod_k \alpha_k \right)^{\frac{1}{2}} \prod_k \left( -X_k \cdot X_{k+1} \right)^{-\frac{3}{2}}
\]
(4.8)
Note that the right-hand side in the last equation is independent of index "i". Above relation leads to constraint
\[
\prod_i \dot{X}_i^2 = 2^{2J} \left( \prod_k \alpha_k \right)^{2} \prod_k \left( -X_k \cdot X_{k+1} \right)^{-2}
\]
(4.9)
Using above last two relations and choosing the gauge of \(\eta_i = 1\) we find that (4.7) becomes
\[
\mathcal{L} = 2J \left( \prod_k \alpha_k \right)^{\frac{1}{2}} \prod_k \left( -X_k \cdot X_{k+1} \right)^{-\frac{3}{2}}
\]
(4.10)
\[
= 2J \left( \prod_i \dot{X}_i^2 \right) 2^{-2J} \prod_k \left( -X_k \cdot X_{k+1} \right)^{2} \prod_k \left( (-X_k \cdot X_{k+1})^2 \right)^{-\frac{3}{2}}
\]
\[
= 2J \prod_k \left( \frac{\dot{X}_k^2}{(-2X_k \cdot X_{k+1})^2} \right)^{\frac{1}{2J}}
\]
(4.11)
and \(\mathcal{L}\) goes back to \(L_{D+1}\). In the gauge \(\alpha_i = 1\) equation (4.10) can be written as
\[
S = \xi J \int dt \mathcal{L}, \quad \mathcal{L} = 4 \prod_k \left( -X_k \cdot X_{k+1} \right)^{-\frac{3}{2}}
\]
(4.12)
Therefore, the dual tri-scalar CFT is the the holographic bi-fishchain model on the lightcone of D+1 spacetime and it’s Lagrangian is \((4.12)\). Note that the Lagrangian of the holographic fishchain model related to bi-scalar CFT derives in \([7]\) is

\[
L = 2 \prod_k \left( -X_k \cdot X_{k+1} \right)^{-\frac{1}{2}}.
\]

We plot the model diagrams in figure 6 in which the black circle represents the lattice point \(X_k\). The strength of link between two lattice point \(X_k\) and \(X_{k+1}\) is \(-X_k \cdot X_{k+1} \). The holographic fishchain (upper diagram) has only one link, while holographic bi-fishchain (down diagram) has two links, between two nearest lattice points. The model chain has \(J\) sites and is closed.

Figure 6: Fishchain and bi-fishchain Models. Fishchain has one link. Bi-fishchain has two links. Each chain has \(J\) sites and is closed.

5 Conclusion

We derive the dual model of 3D tri-scalar CFT which raises from the large twist limit of ABJM theory. It is a weak to strong coupling duality between the single trace operators of the tri-fishnet theory and a quantum-mechanical system of particles which forms a bi-fishchain in light-cone limit of five dimensions. The gauge symmetry in the system plays important role to have the strong-weak duality between them.

Note that our derivations are totally following the original paper \([7]\) and thus the derivations of holographic single fishchain and bi-fishchain can be written in a unified form. One can begin with the eigenvalue equation associated to bi-scalar or tri-scalar CFT which is written as

\[
0 = \mathcal{H}_\kappa \circ \Psi(\{x_i\}), \quad \mathcal{H}_\kappa = \left( \prod_{i=1}^J p_i^2 - \prod_{i=1}^J \frac{4\xi^2}{(x_i - x_{i+1})^{2\kappa}} \right) (5.1)
\]

where \(\kappa = 1\) describes bi-scalar CFT in \((2.12)\) while \(\kappa = 2\) describes tri-scalar CFT in \((2.17)\). In this way, we can perform the similar derivations and find that the Lagrangian and constraint of the \(J\) closed bi-fishchain model become

\[
\dot{X}_k^2 = 2\kappa \prod_i \left( -X_i \cdot X_{i+1} \right)^{-\frac{1}{2}} = \mathcal{L}, \quad k = 1, \ldots, J (5.2)
\]

in which \(\kappa = 1\) describes fishchain in Gromov and Sever paper \([7]\) while \(\kappa = 2\) describes bi-fishchain in equations \((4.9)\) and \((4.12)\).

Finally, we make four comments to conclude this paper:

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1. Begin with the bi-fishchain Hamiltonian

$$H = \sum_i \frac{P_i^2}{2} - J \prod_i \left( -X_i \cdot X_{i+1} \right)^{\frac{\kappa}{2}}$$

(5.3)

where $\kappa = 2$ now. We can impose the primary constrains and secondary constrains, and then follow the method in [8] to quantize the bi-fishchain model. The derivation steps are the same as the fishchain model and result is the same: While the classical model was formulated on the lightcone of four dimensional flat spacetimed, described in (4.4), after the quantization the quantum bi-fishchain model will live on $AdS_4$ with radius $\sim \frac{1}{\xi}$, where $\xi$ is the ‘t Hooft coupling defined in tri-scalar Lagrangian (2.13).

2. Note that the conformal symmetry of the fishnet theory is nontrivial due to double trace coupling. While a solution to this aspect has been extensively studied, particularly for the Fishnet version of N = 4 SYM, there seems to be a lack of detailed investigations on conformality for the fishnet version of ABJM, apart from the study mentioned in [15]. This issue needs further clarification.

3. We can furthermore follow the method in [7] to check a consistent property of the bi-fishchain Lagrangian (5.2). Using global symmetries we can always go to the center of mass frame and set the last components zero: i.e. $X_{1,2} = \frac{r}{\sqrt{2}}(\cosh{s}, \sinh{s}, \pm \cos{\varphi}, \mp \sin{\varphi}, 0)$. (Note that $X_{1,2} = \frac{r}{\sqrt{2}}(\cosh{s}, \sinh{s}, \pm \cos{\varphi}, \mp \sin{\varphi}, 0, 0)$ for fishchain). The coordinates $s$ and $\varphi$ conjugate to conserved charges, $D = ir^2 \dot{s}$ and $S_1 = r^2 \dot{\varphi}$, and the constraint (5.2) gives

$$S_1 - D = 4\kappa$$

(5.4)

The case of fishnet model, $\kappa = 1$, above relation agrees with the exact spectrum derived in [19]. To see whether the bi-fishchain model, $\kappa = 2$, (5.2) has a similar consistent property we have to find the associated exact spectrum. The calculations are remained to further study.

4. It shall be mentioned that the fishnet limit itself assumes a limit where the coupling constant goes to zero. In this regard, the corresponding holographic gravity theory becomes a tensionless string theory. Also, Gromov and Sever [7, 8] obtained a dual theory by analyzing a segment of the string. Therefore, the confirmation of the duality is still far from established. It needs to do more tests.

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