Radial Growth, Lipschitz and Dirichlet Spaces on Solutions to the Yukawa Equation

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Abstract. In this paper, we investigate some properties to solutions $f$ to the Yukawa PDE: $\Delta f = \lambda f$ in the unit ball $B^n$ of $\mathbb{C}^n$, where $\lambda$ is a nonnegative constant. First, we prove that the answer to an open problem of Girela and Peláez, concerning such solutions, is positive. Then we study relationships on such solutions between the bounded mean oscillation and Lipschitz-type spaces. At last, we discuss Dirichlet-type energy integrals on such solutions in the unit ball of $\mathbb{C}^n$ and give an application.

1. Introduction and Main results

Let $\mathbb{C}$ denote the complex plane. We write $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}$,

$$B^n(a, r) = \left\{ z \in \mathbb{C}^n : |z - a| = \left( \sum_{k=1}^{n} |z_k - a_k|^2 \right)^{1/2} < 1 \right\}$$

and $B^n = B^n(0, 1)$, the unit ball in $\mathbb{C}^n$. In particular, the unit disk of $\mathbb{C}$, i.e. $\mathbb{B}^2$, is denoted by $\mathbb{D}$. We use $d(z)$ to denote the Euclidean distance from $z$ to the boundary of $B^n$. Let $\lambda$ be a nonnegative constant and $f = u + iv$ be a complex-valued function of $B^n$ into $\mathbb{C}$, where $u$ and $v$ are real-valued and twice continuously differentiable functions of $B^n$ into $\mathbb{R}$. The following elliptic partial differential equation, or briefly PDE in the following,

$$(1.1) \quad \Delta f(z) = \lambda f(z)$$

in $\mathbb{B}^n$ is called the Yukawa PDE, where $\Delta$ represents the usual complex Laplacian operator

$$\Delta := \sum_{k=1}^{n} \left( \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2} \right) = 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z}_k}$$

for each $k \in \{1, \ldots, n\}$. Here $z_k = x_k + iy_k$.

Equation (1.1) arose out of an attempt by the Japanese physicist Hideki Yukawa to describe the nuclear potential of a point charge as $e^{-\sqrt{\lambda}r}/r$ (cf. [9, 24]). It is well known that each solution $f$ to (1.1) belongs to $C^\infty(\mathbb{B}^n)$, i.e., they are infinitely differentiable in $\mathbb{B}^n$. We refer to [1, 3, 22] for basic results on the theory of elliptic

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PDEs. Moreover, if \( \lambda \leq 0 \) in (1.1), then (1.1) is called *Helmholtz equation* (see [12]). Especially, if \( \lambda = 0 \) in (1.1), then \( f \) is a complex-valued *harmonic mapping* (cf. [2]). Moreover, if \( \lambda = 0 \) in (1.1) with \( n = 1 \), then \( f \) is a complex-valued planar harmonic mapping. It is known that every planar harmonic mapping \( f \) defined in \( \mathbb{D} \) admits a decomposition \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). We refer to [10] for basic results concerning planar harmonic mappings.

For a complex-valued and differentiable function \( f \) of \( \mathbb{B}^n \) into \( \mathbb{C} \), we introduce the following notations (cf. [4, 5]):

\[
f_z = (f_{z_1}, \ldots, f_{z_n}), \quad f_\overline{z} = (f_{\overline{z}_1}, \ldots, f_{\overline{z}_n}) \quad \text{and} \quad \nabla f = (f_z, f_\overline{z}).
\]

Let \( |\nabla f| \) be the *Hilbert-Schmidt norm* given by

\[
|\nabla f| = (|f_z|^2 + |f_\overline{z}|^2)^{1/2}.
\]

Let \( f = u + iv \) be a continuously differentiable mapping from \( \mathbb{B}^n \) into \( \mathbb{C} \), where \( u \) and \( v \) are real-valued functions. Then for \( z = (z_1, \cdots, z_n) = (x_1+iy_1, \cdots, x_n+iy_n) \in \mathbb{B}^n \),

\[
|f_z(z)| + |f_\overline{z}(z)| \leq |\nabla u(z)| + |\nabla v(z)|,
\]

where \( \nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial y_n} \right) \) and \( \nabla v = \left( \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial y_1}, \ldots, \frac{\partial v}{\partial x_n}, \frac{\partial v}{\partial y_n} \right) \). But the converse of (1.2) is not always true (see [7]).

For \( p \in (0, \infty) \), the Hardy space \( \mathcal{H}^p \) consists of those functions \( f : \mathbb{B}^n \to \mathbb{C} \) such that \( f \) is measurable, \( M_p(r, f) \) exists for all \( r \in (0, 1) \) and \( \|f\|_p < \infty \), where

\[
\|f\|_p = \begin{cases} 
\sup_{0<r<1} M_p(r, f), & \text{if } p \in (0, \infty), \\
\sup_{z \in \partial \mathbb{B}^n} |f(z)|, & \text{if } p = \infty,
\end{cases}
\]

and \( d\sigma \) denotes the normalized Lebesgue surface measure in \( \partial \mathbb{B}^n \).

A continuous increasing function \( \omega : [0, +\infty) \to [0, +\infty) \) with \( \omega(0) = 0 \) is called a *majorant* if \( \omega(t)/t \) is non-increasing for \( t > 0 \). Given a subset \( \Omega \) of \( \mathbb{C} \), a function \( f : \Omega \to \mathbb{C} \) is said to belong to the *Lipschitz space* \( \Lambda_{\omega}(\Omega) \) if there is a positive constant \( C \) such that

\[
|f(z) - f(w)| \leq C \omega(|z - w|) \quad \text{for all } z, w \in \Omega.
\]

For \( \delta_0 > 0 \), let

\[
\int_0^\delta \frac{\omega(t)}{t} \, dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0
\]

and

\[
\delta \int_{\delta}^{+\infty} \frac{\omega(t)}{t^2} \, dt \leq C \cdot \omega(\delta), \quad 0 < \delta < \delta_0,
\]

where \( \omega \) is a majorant and \( C \) is a positive constant. A majorant \( \omega \) is said to be *regular* if it satisfies the conditions (1.4) and (1.5) (see [11, 17]).

In [13], Girela and Peláez obtained the following result.
Theorem A. ([13, Theorem 1(a)]) Let $p \in (2, \infty)$. For $r \in (0, 1)$, if $f$ is an analytic function in $\mathbb{D}$ such that

$$M_p(r, f') = O\left(\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1,$$

then for all $\beta > 1/2$,

$$(1.6) \quad M_p(r, f) = O\left((\log \frac{1}{1-r})^{\beta}\right) \quad \text{as} \quad r \to 1.$$

In [13, P.464, Equation (26)], Girela and Peláez asked whether $\beta$ in (1.6) can be substituted by $1/2$. This problem was affirmatively settled by Girela, Pavlovic and Peláez in [14]. In [6], the authors proved further that the answer to this problem is affirmative for the setting of complex-valued harmonic mappings in $\mathbb{D}$. The first aim of this paper is to show that the answer to this problem is also affirmative for mappings $f$ satisfying (1.1) and $p \in [2, \infty)$. Our result is given as follows.

Theorem 1. Let $p \in [2, \infty)$, $\lambda \in [0, 4n/p)$ and $\omega$ be a majorant. For $r \in (0, 1)$, if $f$ is a solution to (1.1) such that

$$M_p(r, \tilde{\nabla}f) \leq C\omega\left(\frac{1}{1-r}\right),$$

then

$$M_p(r, f) \leq \left(\frac{4n}{4n-p\lambda}\right)^{1/2}\left(|f(0)|^2 + 2p(p-1)C^2\omega(1)T(r)\right)^{1/2},$$

where

$$T(r) = \int_0^1 \omega\left(\frac{1}{1-pr}\right) \, dp$$

and $C$ is a positive constant.

By taking $\omega(t) = t$ in Theorem 1, we obtain the following result.

Corollary 1.1. Let $p \in [2, \infty)$ and $\lambda \in [0, 4n/p)$. For $r \in (0, 1)$, if $f$ is a solution to (1.1) such that

$$M_p(r, \tilde{\nabla}f) = O\left(\left(\frac{1}{1-r}\right)\right) \quad \text{as} \quad r \to 1,$$

then

$$M_p(r, f) = O\left((\log \frac{1}{1-r})^{1/2}\right) \quad \text{as} \quad r \to 1.$$

Remark 1.1. Obviously, all analytic functions and complex-valued harmonic mappings defined in $\mathbb{B}^n$ are solutions to (1.1) with $\lambda = 0$, and there also are solutions which are neither analytic nor harmonic. For example, we can take $f(z) = e^{\sum_{k=1}^n (z_k + z_k/2)}$, where $z \in \mathbb{B}^n$. Hence Theorem 1 and Corollary 1.1 are generalizations of [14, Theorem 1.1], [6, Theorem 1(a)] and [20, Corollary 6]. But it is not clear for us that what the best upper bound of $\lambda$ in Theorem 1 is.
In [15], the author discussed the Lipschitz spaces on smooth functions. Dyakonov [11] discussed the relationship between the Lipschitz space and the bounded mean oscillation on analytic functions in \( D \), and obtained the following result.

**Theorem B.** [11, Theorem 1] Suppose that \( f \) is an analytic function in \( D \) which is continuous up to the boundary of \( D \). If \( \omega \) and \( \omega^2 \) are regular majorants, then

\[
 f \in L_\omega(D) \iff \left( \mathcal{P}_{|f|^2}(z) - |f(z)|^2 \right)^{1/2} \leq C \omega(d(z)),
\]

where \( \mathcal{P}_{|f|^2}(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} |f(e^{i\theta})|^2 d\theta \) and \( C \) is a positive constant.

For the solutions to (1.1), we also get the following theorem, which is similar to Theorem B.

**Theorem 2.** Let \( \omega \) be a majorant and \( f \) be a solution to (1.1). If \( f \) satisfies

\[
 |\bar{\nabla} f(z)| \leq C \omega \left( \frac{1}{d(z)} \right)
\]

in \( B^n \), then for all \( r \in (0, d(z)] \),

\[
 \frac{1}{|B^n(z, r)|} \int_{B^n(z, r)} |f(\zeta) - f(z)| dV(\zeta) \leq C r \omega \left( \frac{1}{r} \right),
\]

where \( C \) is a positive constant and \( dV \) denotes the Lebesgue volume measure in \( B^n \).

In particular, if \( f \) is a solution to (1.1) with \( \lambda = 0 \), then we have

**Theorem 3.** Let \( \omega \) be a majorant and \( f \) be a solution to (1.1) with \( \lambda = 0 \). Then \( f \) satisfies

\[
 |\bar{\nabla} f(z)| \leq C \omega \left( \frac{1}{d(z)} \right)
\]

in \( B^n \) if and only if for all \( r \in (0, d(z)] \),

\[
 \frac{1}{|B^n(z, r)|} \int_{B^n(z, r)} |f(\zeta) - f(z)| dV(\zeta) \leq C r \omega \left( \frac{1}{r} \right),
\]

where \( C \) is a positive constant.

**Definition 1.** Let \( f \) be a continuous function in \( B^n \). We say \( f \in BMO \) if

\[
 \|f\|_{BMO} = \sup_{B^n(z, r) \subset B^n} \frac{1}{|B^n(z, r)|} \int_{B^n(z, r)} \left| f(\zeta) - \frac{1}{|B^n(z, r)|} \int_{B^n(z, r)} f(\xi) dV(\xi) \right| dV(\zeta)
\]

is bounded, where \( r \in (0, d(z)] \).

In particular, by taking \( \omega(t) = t \) in Theorem 3, we get the following result.

**Corollary 1.2.** Let \( f \) be a solution to (1.1) with \( \lambda = 0 \). Then \( f \in BMO \) if and only if \( |\bar{\nabla} f(z)| \leq M \frac{1}{d(z)} \) holds in \( B^n \).
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For \( \nu, \gamma, t \in \mathbb{R} \),
\[
D_f(\nu, \gamma, t) = \int_{B^n} (1 - |z|)\nu |f(z)|\gamma |\tilde{\nabla}f(z)|^t dV_N(z)
\]
is called Dirichlet-type energy integral of \( f \) defined in \( B^n \), where \( dV_N \) denotes the normalized Lebesgue volume measure in \( B^n \) (cf. [12, 19, 20, 21]).

**Theorem 4.** Let \( f \) be a solution to (1.1). Then there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
\int_{B^n} (1 - |z|)^{\frac{1}{2}(n-1)} \Delta(|f(z)|^{\frac{2}{n}}) dV_N(z) \leq C_1 D_f(\beta - 1, 1, 1) + C_2,
\]
where \( \beta \in (0, 1] \).

As an application of Theorem 4, we get the following result.

**Corollary 1.3.** Let \( f \) be a solution to (1.1). If \( n = 1 \) and \( D_f(\beta - 1, 1, 1) < \infty \), then \( f \in \mathcal{H}^{\frac{2}{n}} \), where \( \beta \in (0, 1] \).

2. **Integral means and Lipschitz spaces**

We start this section by recalling the following result (cf. [18, 20, 23]).

**Theorem C. (Green’s Theorem)** Let \( g \) be a function of class \( C^2(B^n) \). If \( n \geq 2 \), then for \( r \in (0, 1) \),
\[
\int_{\partial B^n} g(re^{i\theta}) d\sigma(\zeta) = g(0) + \int_{B^n(0, r)} \Delta g(z) G_{2n}(z, r) dV_N(z),
\]
where \( G_{2n}(z, r) = \frac{|z|^{2(1-n)} - r^{2(1-n)}}{4n(n - 1)} \). Moreover, if \( n = 1 \), then for \( r \in (0, 1) \),
\[
\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) d\theta = g(0) + \frac{1}{2} \int_{B^n} \Delta g(z) \log \frac{r}{|z|} dA(z),
\]
where \( dA \) denotes the normalized area measure in \( B^n \).

Recall that a real-valued and continuous function \( u \) defined in \( B^n \) is subharmonic if for all \( z_0 \in B^n \), there is \( \varepsilon \in (0, 1 - |z_0|) \) such that
\[
u(z_0) \leq \int_{\partial B^n} u(z_0 + r\zeta) d\sigma(\zeta)
\]
holds for all \( r \in [0, \varepsilon) \). Moreover, if \( u \in C^2(B^n) \), then \( u \) is subharmonic if and only if \( \Delta u \geq 0 \) in \( B^n \) (cf. [8]).

**Lemma 1.** Suppose that \( f \) is a solution to (1.1). Then
(I) for \( p \in [2, \infty) \), \( M_p(r, f) \) is increasing in \( (0, 1) \) and \( |f|^p \) is subharmonic in \( B^n \);
(II) \( M_2^2(r, \tilde{\nabla}f) \) is increasing in \( (0, 1) \) and \( |\tilde{\nabla}f|^2 \) is subharmonic in \( B^n \).

Moreover, if \( f \) is a solution of (1.1) with \( \lambda = 0 \), then \( |f|^p \) is subharmonic in \( B^n \) for \( p \in [1, \infty) \).
Proof. We first prove (I). For this, we consider the case where \( p \in [2, 4) \) and the case where \( p \in [4, \infty) \), separately.

**Case 1. Suppose first \( p \in [2, 4) \).**

Let \( F_p^m = (|f|^2 + \frac{1}{m})^{p/2} \). By elementary calculations, we have

\[
\Delta(F_p^m) = 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z}_k} (F_p^m) = p(p - 2)(|f|^2 + \frac{1}{m})^{\frac{p}{2} - 2} \sum_{k=1}^{n} |\overline{f_k} f_k|^2 + 2p(1 + |f|^2)^{\frac{p}{2} - 1} |\nabla f|^2 + p \lambda |f|^2 (|f|^2 + 1)^{\frac{p}{2} - 1}.
\]

Let \( T_m = \Delta(F_p^m) \). Obviously, for \( r \in (0, 1) \), \( T_m \) is integrable in \( B^n(0, r) \) and \( T_m \leq F \), where

\[
F = p(p - 2)|f|^{p - 2} \sum_{k=1}^{n} (|f_k|^2 + |f_{\overline{k}}|^2)^2 + 2p(1 + |f|^2)^{\frac{p}{2} - 1} |\nabla f|^2 + p \lambda |f|^2 (|f|^2 + 1)^{\frac{p}{2} - 1}
\]

and \( F \) is integrable in \( B^n(0, r) \). By Theorem C and Lebesgue’s dominated convergence Theorem, we have

\[
\lim_{m \to \infty} r^{2n - 1} \frac{d}{dr} M_p^m(r, F_m) = \frac{1}{2n} \lim_{m \to \infty} \int_{B^n(0, r)} T_m dV_N(z) = \frac{1}{2n} \int_{B^n(0, r)} \lim_{m \to \infty} T_m dV_N(z) = \frac{1}{2n} \int_{B^n(0, r)} [p(p - 2)|f|^{p - 4} \sum_{k=1}^{n} |\overline{f_k} f_k|^2 + 2p|f|^{p - 2} |\nabla f|^2 + p \lambda |f|^2] dV_N(z) = r^{2n - 1} \frac{d}{dr} M_p^m(r, f) \geq 0,
\]

which implies that \( M_p^m(r, f) \) is increasing in \((0, 1)\) for \( p \in [2, 4) \).

**Case 2. Suppose then \( p \in [4, \infty) \).**

By computations, we get

\[
\Delta(|f|^p) = p(p - 2)|f|^{p - 4} \sum_{k=1}^{n} |\overline{f_k} f_k|^2 + 2p|f|^{p - 2} |\nabla f|^2 + p \lambda |f|^p \geq 0,
\]

which gives that \( M_p^m(r, f) \) is increasing in \((0, 1)\).

By Cases 1 and 2, we see that for \( p \in [2, \infty) \), \( M_p^m(r, f) \) is increasing in \((0, 1)\). This shows that for every point \( z_0 \in \mathbb{B}^n \),

\[
|f(z_0)|^p \leq \int_{\partial \mathbb{B}^n} |f(z_0 + r\zeta)|^p d\sigma(\zeta)
\]
for all \( r \in [0, 1 - |z_0|] \). Hence \(|f|^p\) is subharmonic in \( \mathbb{B}^n \). The proof of (I) is complete.

Now we come to prove (II).

\[
\Delta(|\nabla f|^2) = \Delta \left[ \sum_{k=1}^{n} (f_{z_k \bar{z}_k} + f_{\bar{z}_k z_k}) \right] 
\]

\[
= 4\lambda |\nabla f|^2 + 4 \sum_{k=1}^{n} \sum_{j=1}^{n} \left( |f_{z_k \bar{z}_j}|^2 + |f_{\bar{z}_k z_j}|^2 + |f_{z_k \bar{z}_j}|^2 + |f_{\bar{z}_k z_j}|^2 \right) 
\]

\[
\geq 0,
\]
which implies that \( M^2_2(r, |\nabla f|) \) is increasing in \((0, 1)\) and \(|\nabla f|^2\) is subharmonic in \( \mathbb{B}^n \).

In particular, if \( f \) is a solution to (1.1) with \( \lambda = 0 \), then \( f \) is a harmonic mapping. This implies that \(|f|^p\) is subharmonic in \( \mathbb{B}^n \) for \( p \in [1, \infty) \) (cf. [2]). The proof of this lemma is complete. \( \square \)

By using Theorem C and the similar argument as in the proof of Case 1 of Lemma 1, we obtain the following result.

**Lemma 2.** Let \( p \in [2, \infty), \ r \in (0, 1) \), and suppose that \( f \) is a solution to (1.1). Then

\[
M_p^p(r, f) = |f(0)|^p + \int_{\mathbb{B}^n(0, r)} \Delta(|f(z)|^p)G_{2n}(z, r)dV_N(z)
\]

and

\[
r^{2n-1} \frac{d}{dr} M_p^p(r, f) = \frac{1}{2n} \int_{\mathbb{B}^n(0, r)} \Delta(|f(z)|^p) dV_N(z),
\]

where \( G_{2n} \) is the function defined in Theorem C.

The following result is useful to the proof of Theorem 1.

**Lemma 3.** Let \( p \in [2, \infty), \ r \in (0, 1) \) and \( f \) be a solution to (1.1). Then

\[
\int_{\mathbb{B}^n(0, r)} |f(z)|^p G_{2n}(z, r) dV_N(z) \leq \frac{r^2}{4n} M_p^p(r, f).
\]

**Proof.** By Lemma 1, we see that \( M_p^p(\rho, f) \) is increasing on \( \rho \in (0, r] \). Let

\[
I(r) = \int_{\mathbb{B}^n(0, r)} |f(z)|^p G_{2n}(z, r) dV_N(z).
\]

Then

\[
I(r) = \frac{1}{2(n-1)} \int_0^r \left[ \int_{\partial \mathbb{B}^n} \left| f(\rho \zeta) \right|^p \left( \rho - \rho^{2n-1} r^{2(1-n)} \right) d\sigma(\zeta) \right] d\rho
\]

\[
= \frac{1}{2(n-1)} \int_0^r M_p^p(\rho, f) \left( \rho - \rho^{2n-1} r^{2(1-n)} \right) d\rho
\]

\[
\leq \frac{M_p^p(r, f)}{2(n-1)} \int_0^r (\rho - \rho^{2n-1} r^{2(1-n)}) d\rho
\]

\[
= \frac{r^2}{4n} M_p^p(r, f).
\]
The proof of this lemma is complete. □

Now we are ready to prove Theorems 1 and 2.

**Proof of Theorem 1.** Set

\[ \mathcal{A}(r, f) = \int_{\partial \mathbb{B}^n} |f(r \zeta)|^{p-2} |\nabla f(r \zeta)|^2 \, d\sigma(\zeta). \]

Then Hölder’s inequality yields

\[
\mathcal{A}(r, f) \leq \left( \int_{\partial \mathbb{B}^n} |\nabla f(r \zeta)|^p \, d\sigma(\zeta) \right)^{2/p} \left( \int_{\partial \mathbb{B}^n} |f(r \zeta)|^p \, d\sigma(\zeta) \right)^{(p-2)/p}
\]

\[ = M_p^2(r, \nabla f) \cdot M_p^{p-2}(r, f). \]

By using polar coordinates, we see from Lemmas 2 and 3 that

\[
M_p^2(r, f) = |f(0)|^p + \int_{\mathbb{B}^n(0, r)} \Delta(|f(z)|^p) G_{2n}(z, r) \, dV_N(z)
\]

\[
\leq |f(0)|^p + \int_{\mathbb{B}^n(0, r)} [2p(p-1)|f(z)|^{p-2} |\nabla f(z)|^2 + \lambda p|f(z)|^p] G_{2n}(z, r) \, dV_N(z)
\]

\[ = |f(0)|^p + \int_0^r \int_{\partial \mathbb{B}^n} 4np(p-1)\rho^{2n-1} |f(\rho \zeta)|^{p-2} |\nabla f(\rho \zeta)|^2 G_{2n}(\rho \zeta, r) d\sigma(\zeta) d\rho
\]

\[ + p\lambda \int_{\mathbb{B}^n(0, r)} |f(z)|^p G_{2n}(z, r) \, dV_N(z)
\]

\[ = |f(0)|^p + \int_0^r 4np(p-1)\rho^{2n-1} G_{2n}(\rho \zeta, r) \mathcal{A}(\rho, f) d\rho
\]

\[ + p\lambda \int_{\mathbb{B}^n(0, r)} |f(z)|^p G_{2n}(z, r) \, dV_N(z)
\]

\[ \leq |f(0)|^p + 4p(p-1) \int_0^r n\rho^{2n-1} G_{2n}(\rho \zeta, r) M_p^2(\rho, \nabla f) M_p^{p-2}(\rho, f) \, d\rho
\]

\[ + \frac{p\lambda r^2}{4n} M_p^p(r, f), \]

which, because \(M_p(r, f)\) is increasing on \(r\), implies

\[
\left(1 - \frac{p\lambda}{4n}\right) M_p^2(r, f) \leq \left(1 - \frac{p\lambda r^2}{4n}\right) M_p^2(r, f)
\]

\[ \leq |f(0)|^2 + 4p(p-1) \int_0^r n\rho^{2n-1} G_{2n}(\rho \zeta, r) M_p^2(\rho, \nabla f) d\rho
\]

\[ = |f(0)|^2 + 2p(p-1) \int_0^1 r^2 M_p^2(r, \nabla f) \cdot \rho(1 - \rho^{2n-2}) \, d\rho
\]
\[
\begin{align*}
\leq |f(0)|^2 + 2p(p-1)M^2_p(r, f) (1 - \rho) d\rho \\
\leq |f(0)|^2 + 2p(p-1)C^2 \int_0^1 \left[ \omega(\frac{1}{1-r\rho}) \right]^2 (1 - \rho) d\rho \\
\leq |f(0)|^2 + 2p(p-1)C^2 \int_0^1 \left[ \omega(\frac{1}{1-r\rho}) \right]^2 (1 - r\rho) (1 - r\rho) d\rho \\
\leq |f(0)|^2 + 2p(p-1)C^2 \omega(1) \int_0^1 \omega(\frac{1}{1-r\rho}) d\rho \\
= |f(0)|^2 + 2p(p-1)C^2 \omega(1) T(r),
\end{align*}
\]

where \( C \) is a positive constant. This observation gives the desired result:

\[
M_p(r, f) \leq \left( \frac{4n}{4n - p\lambda} \right)^{1/2} \left( |f(0)|^2 + 2p(p-1)C^2 \omega(1) T(r) \right)^{1/2}.
\]

The proof of this theorem is complete. \( \Box \)

**Proof of Theorem 2.** For \( z, w \in \mathbb{B}^n \) and \( t \in [0, 1] \), we have

\[
d(z + t(w - z)) = 1 - |z + t(w - z)| \geq d(z) - t|w - z|.
\]

Suppose that \( d(z) - t|w - z| > 0 \). Then

\[
|f(z) - f(w)| = \left| \int_0^1 \frac{df}{dt}(wt + (1-t)z) dt \right| \\
= \left| \sum_{k=1}^n (z_k - w_k) \int_0^1 \frac{df}{d\xi_k}(wt + (1-t)z) dt \right| \\
+ \left| \sum_{k=1}^n (\overline{z}_k - \overline{w}_k) \int_0^1 \frac{df}{d\xi_k}(wt + (1-t)z) dt \right| \\
\leq \sum_{k=1}^n |z_k - w_k| \cdot \left| \int_0^1 \frac{df}{d\xi_k}(wt + (1-t)z) dt \right| \\
+ \sum_{k=1}^n |\overline{z}_k - \overline{w}_k| \cdot \left| \int_0^1 \frac{df}{d\xi_k}(wt + (1-t)z) dt \right|
\]
\[ \leq \left( \sum_{k=1}^{n} |z_k - w_k|^2 \right)^{\frac{1}{2}} \left\{ \left[ \sum_{k=1}^{n} \left( \int_{0}^{1} \left| \frac{\partial f}{\partial \varsigma_k}(wt + (1-t)z) \right| dt \right)^2 \right]^{\frac{1}{2}} + \left[ \sum_{k=1}^{n} \left( \int_{0}^{1} \left| \frac{\partial f}{\partial \varsigma_k}(wt + (1-t)z) \right| dt \right)^2 \right]^{\frac{1}{2}} \right\} \]

\[ \leq \sqrt{n}|z - w| \left\{ \int_{0}^{1} |f_\varsigma(wt + (1-t)z)| dt 
+ \int_{0}^{1} |f_\tau(wt + (1-t)z)| dt \right\} \]

\[ \leq \sqrt{2n}|z - w| \int_{0}^{1} |\tilde{\nabla}f(wt + (1-t)z)| dt \]

\[ \leq C\sqrt{2n}|w - z| \int_{0}^{1} \omega \left( \frac{1}{d(z) - t|w - z|} \right) dt \]

\[ = C\sqrt{2n} \int_{0}^{r}|w - z| \omega \left( \frac{1}{d(z) - t} \right) dt. \]

This implies

\[ \frac{1}{|B^n(z, r)|} \int_{B^n(z, r)} |f(\zeta) - f(z)| dV(\zeta) \]

\[ \leq \frac{C\sqrt{2n}}{|B^n(0, r)|} \int_{B^n(0, r)} \left\{ \int_{0}^{|\zeta|} \omega \left( \frac{1}{d(z) - t} \right) dt \right\} dV(\zeta) \]

\[ = \frac{C2n\sqrt{2n}}{r^{2n}} \int_{0}^{r} \rho^{2n-1} \left\{ \int_{0}^{\rho} \omega \left( \frac{1}{d(z) - t} \right) dt \right\} d\rho \]

\[ \leq \frac{C\sqrt{2n}}{r^{2n}} \int_{0}^{r} \left\{ \int_{0}^{r} \rho^{2n-1} d\rho \right\} \omega \left( \frac{1}{r - t} \right) dt \]

\[ \leq \frac{C\sqrt{2n}}{r^{2n}} r^2 \omega \left( \frac{1}{r} \right) \int_{0}^{r} \left( r^{2n-1} + r^{2n-2} t + \cdots + t^{2n-1} \right) \omega \left( \frac{1}{r - t} \right) dt \]

\[ = C\sqrt{2n} \left( \sum_{j=1}^{2n} \frac{1}{j} \right) r^2 \omega \left( \frac{1}{r} \right), \]

where \( \varsigma = (\varsigma_1, \cdots, \varsigma_n) = wt + (1 - t)z \) and \( C \) is a positive constant. The proof of this theorem is complete. \( \square \)

In order to prove Theorem 3, we need the following lemma. Using the similar arguments as in the proof of [16, Lemma 2.5], we have
Lemma 4. Suppose that $f : \mathbb{B}^n(a, r) \to \mathbb{C}$ is a continuous function in $\mathbb{B}^n(a, r)$ and harmonic in $\mathbb{B}^n(a, r)$. Then

$$|\nabla f(a)| \leq \frac{4n\sqrt{n}}{r} \int_{\partial \mathbb{B}^n} |f(a + r\zeta) - f(a)|d\sigma(\zeta).$$

Proof. Let $f = u + iv$, where $u$ and $v$ are real harmonic functions in $\mathbb{B}^n(a, r)$. Without loss of generality, we may assume that $a = 0$ and $f(0) = 0$. Let

$$K(z, \zeta) = \frac{r^{2n-2}(r^2 - |z|^2)}{|z - r\zeta|^{2n}}.$$

Then

$$u(z) = \int_{\partial \mathbb{B}^n} K(z, \zeta)u(r\zeta)d\sigma(\zeta), \quad z \in \mathbb{B}^n(0, r).$$

By direct calculations, we have

$$\frac{\partial}{\partial x_j}K(z, \zeta) = r^{2n-2}\left[\frac{-2x_j}{|z - r\zeta|^{2n}} - \frac{2n(r^2 - |z|^2)(x_j - r\alpha_j)}{|z - r\zeta|^{2n+2}}\right]$$

and

$$\frac{\partial}{\partial y_j}K(z, \zeta) = r^{2n-2}\left[\frac{-2y_j}{|z - r\zeta|^{2n}} - \frac{2n(r^2 - |z|^2)(y_j - r\beta_j)}{|z - r\zeta|^{2n+2}}\right],$$

which gives

$$(2.1) \quad \frac{\partial}{\partial x_j}K(0, \zeta) = \frac{2n\alpha_j}{r} \quad \text{and} \quad \frac{\partial}{\partial y_j}K(0, \zeta) = \frac{2n\beta_j}{r},$$

where $z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$ and $\zeta = (\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n) \in \partial \mathbb{B}^n$. Then by (2.1), we have

$$|\nabla u(0)| = \left[\sum_{j=1}^n \left(\int_{\partial \mathbb{B}^n} \left|\frac{\partial}{\partial x_j}K(0, \zeta)u(r\zeta)d\sigma(\zeta)\right|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$\leq \sum_{j=1}^n \left(\int_{\partial \mathbb{B}^n} \left|\frac{\partial}{\partial y_j}K(0, \zeta)u(r\zeta)d\sigma(\zeta)\right| \right)^{\frac{1}{2}}$$

$$\leq \int_{\partial \mathbb{B}^n} |u(r\zeta)| \left[\sum_{j=1}^n \left(\left|\frac{\partial}{\partial x_j}K(0, \zeta)\right|^2 + \left|\frac{\partial}{\partial y_j}K(0, \zeta)\right|^2\right)\right]^{\frac{1}{2}} d\sigma(\zeta)$$

$$\leq \sqrt{2n} \int_{\partial \mathbb{B}^n} |u(r\zeta)| \left[\sum_{j=1}^n \left(\left|\frac{\partial}{\partial x_j}K(0, t)\right|^2 + \left|\frac{\partial}{\partial y_j}K(0, t)\right|^2\right)\right]^{\frac{1}{2}} d\sigma(\zeta)$$

$$= \frac{2n\sqrt{2\pi}}{r} \int_{\partial \mathbb{B}^n} |u(r\zeta)|d\sigma(\zeta).$$
Similarly, we have
\[ |\nabla v(0)| \leq \frac{2n\sqrt{2n}}{r} \int_{\partial B^n} |v(r\zeta)|d\sigma(\zeta). \]

Then by (1.2), we conclude that
\[ |\tilde{\nabla} f(0)| \leq |f_z(0)| + |f_\nu(0)| \leq |\nabla u(0)| + |\nabla v(0)| \leq \frac{2n\sqrt{2n}}{r} \int_{\partial B^n} |u(r\zeta)| + |v(r\zeta)|d\sigma(\zeta) \leq \frac{4n\sqrt{n}}{r} \int_{\partial B^n} |f(r\zeta)|d\sigma(\zeta). \]

The proof of the lemma is complete. □

**Proof of Theorem 3.** First, we show the “if” part. By Lemma 4, we have
\[ |\tilde{\nabla} f(z)| \leq \frac{4n\sqrt{n}}{\rho} \int_{\partial B^n} |f(z + \rho\zeta) - f(z)|d\sigma(\zeta), \]
where \( \rho \in (0, d(z)] \). Let \( r = d(z) \). Then we have
\[ \int_0^r |\tilde{\nabla} f(z)|\rho^{2n}d\rho \leq 2\sqrt{n} \int_0^r \left(2n\rho^{2n-1} \int_{\partial B^n} |f(z) - f(z + \rho\zeta)|d\sigma(\zeta)\right)d\rho, \]
which implies
\[ |\tilde{\nabla} f(z)| \leq \frac{2(2n + 1)\sqrt{n}}{r^{2n+1}} \int_0^r \left(2n\rho^{2n-1} \int_{\partial B^n} |f(z) - f(z + \rho\zeta)|d\sigma(\zeta)\right)d\rho \]
\[ = \frac{2(2n + 1)\sqrt{n}}{r|B^n(z, r)|} \int_{B^n(z, r)} |f(\xi) - f(z)|dV(\xi) \]
\[ \leq 2(2n + 1)\sqrt{n}C\omega\left(\frac{1}{r}\right) \]
\[ = 2(2n + 1)\sqrt{n}C\omega\left(\frac{1}{d(z)}\right). \]

The “only if” part easily follows from Theorem 2. The proof of the theorem is complete. □

3. **The finite Dirichlet energy integral and its application**

**Lemma 5.** Let \( f \) be a solution to (1.1). Then for \( p \in [2, \infty) \) and \( \beta \in (0, \infty) \),
\[ D_f(\beta, p - 2, 2) \leq \frac{\beta\sqrt{2}}{2} D_f(\beta - 1, p - 1, 1). \]
Proof. By Lemmas 1 and 2, we have
\[
 r^{2n-1} \frac{d}{dr} M_p^p(r, f) = \frac{1}{2n} \int_{\mathbb{B}^n(0, r)} \left[ p(p - 2)|f(z)|^{p-4} \sum_{k=1}^{n} |f(z)f_{z_k}(z) + f(z)f_{\bar{z}_k}(z)|^2 + 2p|f(z)|^{p-2} |\nabla f(z)|^2 + p\lambda |f(z)|^p \right] dV_N(z),
\]
(3.1)
which implies
\[
 \frac{d}{dr} \left( r^{2n-1} \frac{d}{dr} M_p^p(r, f) \right) = \int_{\partial\mathbb{B}^n} r^{2n-1} \left[ 2p|f(r\zeta)|^{p-2} |\nabla f(r\zeta)|^2 + p(p - 2)|f(r\zeta)|^{p-4} \sum_{k=1}^{n} |f(r\zeta)f_{z_k}(r\zeta) + f(r\zeta)f_{\bar{z}_k}(r\zeta)|^2 + p\lambda |f(r\zeta)|^p \right] d\sigma(\zeta),
\]
(3.2)
In addition, we see
\[
 \frac{d}{dr} M_p^p(r, f) = \int_{\partial\mathbb{B}^n} \frac{d}{dr} (|f(r\zeta)|^p) d\sigma(\zeta)
 = p \int_{\partial\mathbb{B}^n} |f(r\zeta)|^{p-2} \text{Re} \left[ \sum_{k=1}^{n} \left( f_{z_k}(r\zeta)f(r\zeta) + f(r\zeta)f_{\bar{z}_k}(r\zeta) \right) \zeta_k \right] d\sigma(\zeta)
\]
(3.3)
where $\zeta = (\zeta_1, \cdots, \zeta_n) \in \partial\mathbb{B}^n$.
It follows from (3.2) and (3.3) that
\[
 \beta p \sqrt{2} \int_{\mathbb{B}^n} (1 - |z|)^{\beta - 1} |f(z)|^{p-1} |\nabla f(z)| dV_N(z)
= \beta p \sqrt{2} \int_0^1 2nr^{2n-1} (1 - r)^{\beta - 1} \int_{\partial\mathbb{B}^n} |f(r\zeta)|^{p-1} |\nabla f(r\zeta)| d\sigma(\zeta) dr
\geq \beta \int_0^1 2nr^{2n-1} (1 - r)^{\beta - 1} \left( \frac{d}{dr} M_p^p(r, f) \right) dr
= \int_0^1 2n(1 - r)^{\beta} \frac{d}{dr} \left( r^{2n-1} \frac{d}{dr} M_p^p(r, f) \right) dr
\]
Suppose that Lemma 6. □ from which the proof follows.

By elementary computations, we easily see that

\[
1 - |z| \leq 2^{\max\{q-1,0\}}(a^q + b^q).
\]

**Proof of Theorem 4.** By Lemma 1, we know that \(|\nabla f|^2\) is subharmonic in \(\mathbb{B}^n\). Then for \(r \in [0, 1 - |z|]\), we have

\[
|\nabla f(z)|^2 \leq \int_{\partial \mathbb{B}^n} |\nabla f(z + r\zeta)|^2 d\sigma(\zeta).
\]

Integration and Lemma 5 yield

\[
\frac{(1 - |z|)^{2n} |\nabla f(z)|^2}{2^{2n}} \leq \int_{\partial \mathbb{B}^n} \int_0^{1 - |z|} 2nr^{2n-1} |\nabla f(z + r\zeta)|^2 dr d\sigma(\zeta)
\]

\[
= \int_{B^n(z, \frac{1 - |z|}{2})} |\nabla f(\xi)|^2 dV_N(\xi)
\]

\[
\leq 2^\beta (1 - |z|)^{-\beta} \int_{B^n(z, \frac{1 - |z|}{2})} (1 - |\xi|)^{\beta} |\nabla f(\xi)|^2 dV_N(\xi)
\]

\[
\leq 2^\beta D_f(\beta, 0, 2)(1 - |z|)^{-\beta}
\]

\[
\leq \beta 2^{\beta - \frac{3}{2}} D_f(\beta - 1, 1, 1)(1 - |z|)^{-\beta}
\]

which gives

\[
|\nabla f(z)| \leq \frac{C_3}{(1 - |z|)^{n + \frac{\beta}{2}}},
\]

where \(C_3 = \sqrt{\beta 2^{\beta - 1/2} 2^n D_f(\beta - 1, 1, 1)}\).
By (3.5), we have
\[ |f(z)| \leq |f(0)| + \left| \int_{[0,z]} df(\zeta) \right| \leq |f(0)| + \sqrt{2} \int_{[0,z]} |\nabla f(\zeta)||d\zeta| \leq |f(0)| + \frac{C_4}{(1 - |z|)^{\frac{\beta}{2} + n - 1}}, \]
where \( C_4 = \sqrt{2}C_3/(n - 1 + \beta/2) \) and \([0,z]\) denotes the segment from 0 to \( z \). Then by Lemma 6, we see that for \( z \in \mathbb{B}^n \),
\[ |f(z)|^\frac{\beta}{2} \leq \left[ |f(0)| + \frac{C_4}{(1 - |z|)^{\frac{\beta}{2} + n - 1}} \right]^\frac{\beta}{2} \leq 2^{\frac{\beta}{2} - 1} \left[ |f(0)|^\frac{\beta}{2} + \frac{C_4^\frac{\beta}{2}}{(1 - |z|)^{1 - \beta + (n - 1)(\frac{\beta}{2} - 2)}} \right] \]
and
\[ |f(z)|^{\frac{\beta}{2} - 2} \leq \left[ |f(0)| + \frac{C_4}{(1 - |z|)^{\beta/2 + n - 1}} \right]^{\frac{\beta}{2} - 2} \leq 2^{\frac{\beta}{2} - 2} \left[ |f(0)|^{\frac{\beta}{2} - 2} + \frac{C_4^{\frac{\beta}{2} - 2}}{(1 - |z|)^{1 - \beta + (n - 1)(\frac{\beta}{2} - 2)}} \right]. \]

Let \( p = 2/\beta \). We divide the rest of the proof into two cases.

**Case 3.** Let \( p \in [4, \infty) \).

By direct calculations, we get
\[ \Delta(|f|^p) = 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \overline{z}_k}(|f|^p) = p(p - 2)|f|^{p-2} \sum_{k=1}^{n} |f_{z_k} f + \overline{f_{z_k}} f|^2 + 2p|f|^{p-2}|\nabla f|^2 + p\lambda|f|^p \]
\[ \leq 2p(p - 1)|f|^{p-2}|\nabla f|^2 + p\lambda|f|^p. \]
Hence by (3.6), (3.7) and (3.8), we conclude that for \( z \in \mathbb{B}^n \),
\[ (1 - |z|)^{1+p(n-1)} \Delta(|f(z)|^p) \leq 2p(p - 1)(1 - |z|)^{1+p(n-1)}|f(z)|^{p-2}|\nabla f(z)|^2 \]
\[ + p\lambda(1 - |z|)^{1+p(n-1)}|f(z)|^p \leq 2p(p - 1)(1 - |z|)^\beta |\nabla f(z)|^2 (1 - |z|)^{1+p(n-1)-\beta} |f(z)|^{p-2} \]
\[ + p\lambda 2^{p-1} (C_5^p + |f(0)|^p) \]
\[ \leq C_5 + C_6(1 - |z|)^\beta |\nabla f(z)|^2, \]
where \( C_5 = p\lambda 2^{p-1}(C_4^p + |f(0)|^p) \) and \( C_6 = 2p(p - 1)2^{p-2}(|f(0)|^{p-2} + C_4^{p-2}) \). By Theorem 5, we know

\[
D_f(\beta, 0, 2) \leq \frac{\beta\sqrt{2}}{2}D_f(\beta - 1, 1, 1).
\]

Therefore, (3.9) and (3.10) imply that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
\int_{B_n}(1 - |z|)^{1+p(n-1)}\Delta(|f(z)|^p)dV_N(z) \leq C_1 D_f\left(\frac{2}{p} - 1, 1, 1\right) + C_2.
\]

\textbf{Case 4.} \( p \in [2, 4) \).

In this case, we let \( F_m = (|f|^2 + \frac{1}{m})^{p/2} \), and let \( T_m = \Delta(F_m) \). Obviously, for \( r \in (0, 1) \), \( T_m \) is integrable in \( B^n(0, r) \) and \( T_m \preceq F \), where

\[
F = p(p - 2)|f|^{p-2} \sum_{k=1}^{n}(|f_{z_k}| + |f_{\bar{z}_k}|)^2 + 2p(1 + |f|^2)^{\frac{p-1}{2}}|\nabla f|^2 + p\lambda |f|^2(|f|^2 + 1)^{\frac{p-1}{2}}
\]

and \( F \) is integrable in \( B^n(0, r) \).

Then, by Lebesgue’s Dominated Convergence Theorem together with (3.9), we have

\[
\lim_{n \to \infty} \int_{B^n(0, r)} (1 - |z|)^{1+p(n-1)}\Delta(F_m(z))dV_N(z)
= \int_{B^n(0, r)} (1 - |z|)^{1+p(n-1)} \lim_{n \to \infty} [\Delta(F_m(z))]dV_N(z)
= p \int_{B^n(0, r)} \left[(p - 2)|f(z)|^{p-4} \sum_{k=1}^{n} |f_{z_k}(z)f(z) + f_{\bar{z}_k}(z)f(z)|^2
+ 2|f(z)|^{p-2}|\nabla f(z)|^2 + \lambda |f(z)|^p\right](1 - |z|)^{1+p(n-1)}dV_N(z)
\leq \int_{B^n(0, r)} [C_5 + C_6(1 - |z|)^{\beta}|\nabla f(z)|^2]dV_N(z),
\]

and so we infer from (3.6), (3.7) and Theorem 5 that there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
\int_{B_n}(1 - |z|)^{1+p(n-1)}\Delta(|f(z)|^p)dV_N(z) \leq C_1 D_f\left(\frac{2}{p} - 1, 1, 1\right) + C_2.
\]

The proof of this theorem is complete. \( \square \)

\textbf{Proof of Corollary 1.3.} For a fixed \( r \in (0, 1) \), since

\[
\lim_{|z| \to r} \frac{\log r - \log |z|}{r - |z|} = \frac{1}{r},
\]

we see that there is \( r_0 \in (0, r) \) satisfying

\[
\log r - \log |z| \leq \frac{2}{r}(r - |z|)
\]
for \( r_0 \leq |z| < r \). Let \( p = 2/\beta \). The it follows from
\[
\lim_{\rho \to 0^+} \rho \log \frac{1}{\rho} = 0
\]
that
\[
(3.11) \quad \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} d\sigma(z) \leq \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{1}{|z|} d\sigma(z)
\]
\[
= \int_0^{2\pi} \int_0^{r_0} \Delta(|f(\rho e^{i\theta})|^p) \rho \log \frac{1}{\rho} d\rho d\theta
\]
\[
< \infty.
\]

Since \( D_f(\beta - 1, 1, 1) < \infty \), it follows from Theorem 4 that
\[
(3.12) \quad \int_{D \setminus D_{r_0}} \Delta(|f(z)|^p)(1 - |z|) d\sigma(z) < \infty.
\]

Hence by (3.11), (3.12) and Theorem C, we obtain that
\[
M_p^p(r, f) = |f(0)|^p + \frac{1}{2} \int_{D_r} \Delta(|f(z)|^p) \log \frac{r}{|z|} d\sigma(z)
\]
\[
= |f(0)|^p + \frac{1}{2} \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} d\sigma(z)
\]
\[
+ \frac{1}{2} \int_{D_r \setminus D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} d\sigma(z)
\]
\[
\leq |f(0)|^p + \frac{1}{2} \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} d\sigma(z)
\]
\[
+ \int_{D_r \setminus D_{r_0}} \Delta(|f(z)|^p) \frac{(r - |z|)}{r} d\sigma(z)
\]
\[
\leq |f(0)|^p + \frac{1}{2} \int_{D_{r_0}} \Delta(|f(z)|^p) \log \frac{r}{|z|} d\sigma(z)
\]
\[
+ \int_{D_r \setminus D_{r_0}} \Delta(|f(z)|^p)(1 - |z|) d\sigma(z)
\]
\[
< \infty.
\]

Since Lemma 1 shows that the function \( M_p^p(r, f) \) is increasing with respect to \( r \) in \((0, 1)\), we know that the limit
\[
\lim_{r \to 1^{-}} M_p(r, f)
\]
does exist, which implies \( f \in \mathcal{H}^p \). The proof of the corollary is complete. \( \square \)

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