We present a generic Markovian master equation inducing the gradual classicalization of a bosonic quantum field. It leads to the decoherence of quantum superpositions of field configurations, while leaving the Ehrenfest equations for both the field and the mode-variables invariant. We characterize the classicalization analytically and numerically, and show that the semiclassical field dynamics is described by a linear Boltzmann equation in the functional phase space of field configurations.

I. INTRODUCTION

Quantum systems with a large number of interacting constituents can often be effectively described in terms of quantum fields. Examples include degenerate quantum gases and fluids [1, 2], collective degrees of freedom in strongly correlated solid-state systems [3, 4], acoustic vibrations in superfluid Helium [5, 6], micromechanical oscillators [8, 9], and closely-spaced chains of harmonic oscillators realized, for example, in ion traps [10] and superconducting circuits [11]. For such many-body systems, a minimal but generic field-theoretic model that appropriately accounts for the decoherence dynamics towards a corresponding classical field theory is desirable.

The loss of quantum coherence and the emergence of classical behavior in systems with a finite number of degrees of freedom have been extensively and successfully studied using the framework of open quantum systems [12, 13]. Understanding these phenomena is of paramount importance for the development of quantum technologies, since their performance is ultimately limited by the coupling to the surrounding environment. Decoherence also plays a central role in probing the physics at the quantum-classical border [14] as superpositions of increasingly macroscopic and complex objects become experimentally accessible [15].

For systems with a large number of interacting constituents the combined decoherence dynamics become quickly intractable on an atomistic level. This calls for a field-theoretic description in terms of collective modes. The open quantum dynamics of fields have so far been formulated assuming a linear coupling with an environment [1, 16–20]. While such schemes are adequate for small fluctuations of the field amplitude, they cannot appropriately describe the decoherence of macroscopic superpositions, since the obtained rates become unrealistically large, growing above all bounds [14].

In this article we introduce a generic Lindblad master equation for non-relativistic bosonic fields that describes their gradual classicalization. That is, quantum superpositions of different field configurations quickly decay into a mixture while classical superpositions remain practically unaffected. We show that the semiclassical field dynamics is described by a linear Boltzmann equation in the functional phase space of field configurations, which in the linear-coupling approximation reduces to a linear Fokker–Planck equation. The noise term in the exact classicalizing dynamics is minimal, in the sense that it leaves the first moments of the field’s mode-quadratures unaffected while slowly increasing the field energy with a state-independent rate.

II. CLASSICALIZING MASTER EQUATION

We consider a bosonic scalar field confined to a one-dimensional region of length \( L \), subject to periodic boundary conditions (the generalization to higher dimensions is straightforward). In the Schrödinger picture, its quantum dynamics is described by the master equation

\[
\dot{\rho}_t = -\frac{i}{\hbar}[H, \rho_t] + \gamma \int_0^L \frac{dx}{L} \int d^2 \xi \ g(\xi) [U_x(\xi) \rho_t U_x^\dagger(\xi) - \rho_t].
\]

The first term describes the unitary dynamics of the field, as determined by the Hamiltonian \( H \), while the second...
term gives rise to the classicalization of the field. The latter involves unitary phase-space translation operators $U_x(\xi)$ acting on the field amplitude and its canonical conjugate momentum in the vicinity of position $x$. Combining the field variables into the complex field $\Psi(x)$, so that $[\Psi(x), \Psi^\dagger(y)] = \delta(x-y)$, these operators can be written as

$$
U_x(\xi) = \exp \left( \int_0^L dy f \left( \frac{y-x}{L} \right) \xi \Psi^\dagger(y) - \xi^* \Psi(y) \right). 
$$

Here $f$ is a real, square-integrable, $L$-periodic spread function with maximum $f(0) = 1$ and width $\sigma_x / L$. The argument $\xi$ of $U_x$ is a complex random number whose associated probability distribution $g(\xi)$ is assumed to be an even function.

From Eq. (1) it follows that the pure density $\rho = \text{Tr}(\rho^2)$ of any quantum state of the field decreases monotonically, with a decay rate given by (see Appendix)

$$
\dot{\rho}_t = -\gamma \int_0^L dx \int d^2\xi g(\xi) \|[\rho_t, U_x(\xi)]\|^2 < 0, 
$$

where $\|A\|^2 = \text{Tr}(A^\dagger A)$. Therefore, under this open-system dynamics any quantum superposition of the field will decay into a mixture. This loss of coherence will be assessed quantitatively below.

The master equation can be interpreted as describing a compound Poisson process with rate $\gamma$, in which the unitary evolution of the field is interrupted by generalized measurements of the canonical field variables whose outcomes are discarded \[21, 22\]. Whenever a measurement occurs around a position $x \in [0, L]$ it affects an entire neighborhood of width $\sigma_x$, as illustrated in Fig. [1]. The field degrees of freedom located at any point $y$ within this region experience a random phase-space kick of strength $\xi f(y/L - x/L)$, in accordance with the Heisenberg principle.

The operators in the master equation can be represented both in position space, in terms of the canonical field amplitude and its conjugate momentum, and in the eigenbasis of the field Hamiltonian, given by its mode variables. We will see below that the first representation is advantageous for semiclassical analysis, while the second enables the analytic treatment of the decoherence dynamics.

For definiteness, we take the bosonic field Hamiltonian as

$$
H = \frac{1}{2} \int_0^L dx \left[ \frac{1}{\mu} \Pi^2(x) + \mu \omega^2 \Phi^2(x) + \mu v^2 (\partial_x \Phi(x))^2 \right]. 
$$

The field amplitude $\Phi$ and its canonically conjugate momentum $\Pi$ are related to the complex field through

$$
\Psi(x) = \sqrt{\frac{\mu \omega}{2\hbar}} \Phi(x) + \frac{i}{\sqrt{2\hbar \mu \omega}} \Pi(x). 
$$

Here $\mu$ is the mass density of the field, $\omega$ is a frequency and $v$ is the phase velocity.

The Hamiltonian \[4\] can be written in diagonal form as $H = \sum_{k \in K} \hbar \omega_k c_k^\dagger c_k$, with $\omega_k^2 = \omega^2 + v^2 k^2$ and $K = \{2\pi/jL \mid j \in \mathbb{Z}\}$, thus $k$ runs over all integer multiples of $2\pi / L$. The bosonic mode operators

$$
c_k = \sqrt{\frac{\mu \omega}{2\hbar}} \Phi_k + \frac{i}{\sqrt{2\hbar \mu \omega}} \Pi_k, 
$$

satisfying the canonical commutation relations $[c_k, c_{k'}^\dagger] = \delta_{kk'}$, are defined in terms of the (non-Hermitian) normal coordinates that diagonalize the Hamiltonian, $\Phi_k = \int_0^L dx e^{-ikx} \Phi(x)/\sqrt{L}$ and $\Pi_k = \int_0^L dx e^{-ikx} \Pi(x)/\sqrt{L}$. In the basis \[6\], the field operators are expressed as

$$
\Psi(x) = \frac{1}{\sqrt{L}} \sum_{k \in K} \left[ e^{ikx} \Omega_k^+ c_k + e^{-ikx} \Omega_k^{-} c_k^\dagger \right], 
$$

with $\Omega_k^\pm = (\omega/\omega_k)^{1/2} \pm (\omega_k/\omega)^{1/2} / 2$.

It can be readily shown that the mode operators satisfy

$$
\{c_k, c_{k'}^\dagger\} = -i \omega \delta_{kk'}, 
$$

$$
\{\Pi_k, c_{k'}^\dagger\} = -ie^{ik \eta} \Omega_k^+ \Omega_k^{-}\delta_{kk'}, 
$$

where $f_k = \int_0^L dx e^{ikx} f(x)/\sqrt{L}$ are the Fourier coefficients of $f$. Since $g(\xi)$ is an even function, it follows that the Ehrenfest equation for the mode operators $\partial_t (c_k \eta) = -i \omega (c_k \eta)$ is unaffected by the incoherent part of the master equation \[1\]. The expectation values of the field amplitude $\langle \Phi(x) \rangle$ and its canonical momentum $\langle \Pi(x) \rangle$ therefore satisfy the field equations associated with the corresponding classical Hamiltonian.

The master equation \[1\] is most easily solved in the basis of the bosonic field Hamiltonian $H = \sum_{k \in K} \hbar \omega_k c_k^\dagger c_k$, which is denoted with a tilde, the equation of motion for $\tilde{\chi}_t(\{\eta_k\})$ is given by (see Appendix)

$$
\partial_t \tilde{\chi}_t(\{\eta_k\}) = -i \Gamma_t(\{\eta_k\}) \tilde{\chi}_t(\{\eta_k\}). 
$$

For an isotropic $g(\xi) = g_r(\xi) / 2\pi$, the decoherence rate

$$
\Gamma_t(\{\eta_k\}) = \gamma - \frac{\gamma}{L} \int_0^L dx \dot{g}_r(s_t(\{\eta_k\}, x)) 
$$

can be written in terms of the Hankel transform of $g$, $g_r(s) = \int_0^\infty dr r g_r(r) J_0(sr)$, evaluated at

$$
s_t(\{\eta_k\}, x) = \left[ \sum_{k \in K} \Omega_k^+ f_k e^{ikx} e^{i \omega_k t} \eta_k + \Omega_k^- f_k^* e^{-ikx} e^{-i \omega_k t} \eta_k^* \right]. 
$$

Equation \[9\] can be readily solved up to a quadrature. It will be used below to analyze the dynamics of the purity decay. Moreover, it serves as the starting point to derive the semiclassical dynamics of the field. In order to do that, we first reformulate the above results in the language of functional calculus.
III. EQUATION OF MOTION FOR THE WIGNER FUNCTIONAL

The semiclassical field dynamics induced by Eq. (1) is best described in the phase space of the canonical field variables $\Phi$ and $\Pi$. In this representation, and using the complex field (5), the Weyl operators take the form $D[\eta] = \exp \left( \int_0^L dx \left[ \eta(x) \psi^\dagger(x) - \psi^*(x) \psi(x) \right] \right)$. They are operator-valued functions describing a phase-space displacement of the canonical field variables at each point $x$ by the complex wave amplitude $\eta(x)$. (Note that $\eta(x)$ has dimension of reciprocal square root of length, the same as $\Psi(x)$ and $\xi$.)

In this representation, the equation of motion for the characteristic functional takes a form analogous to (9).

$$\partial_t \tilde{\chi}_t[\eta] = -\Gamma_t[\eta] \tilde{\chi}_t[\eta].$$  \hfill (12)

Here, the decoherence rate is a functional of the complex wave amplitude $\eta(x)$.

$$\Gamma_t[\eta] = \gamma - \frac{\gamma}{L} \int_0^L dx \int d^2 \xi \, g(\xi) e^{-\int_0^L dy [\eta^* y^* y + c.c.]}.$$  \hfill (13)

It involves the interaction-picture phase-space displacements $\Lambda_t(\xi, x; y) = \xi a_t(x; y) - \xi^* b_t(x; y)$, with

$$a_t(x; y) = \frac{1}{\sqrt{L}} \sum_{k \in K} \left[ \cos \omega_k t - \frac{i}{2} \left( \frac{\omega}{\omega_k} + \frac{\omega_k}{\omega} \right) \sin \omega_k t \right] \times f_k e^{ik(x-y)},$$  \hfill (14a)

$$b_t(x; y) = \frac{i}{2\sqrt{L}} \sum_{k \in K} \left( \frac{\omega}{\omega_k} - \frac{\omega_k}{\omega} \right) \sin(\omega_k t) f_k e^{ik(x-y)}.$$  \hfill (14b)

The Wigner functional of the field state is defined as the functional Fourier transform of $\tilde{\chi}_t[\eta]$ (see Appendix),

$$\tilde{W}_t[\lambda] = \int D^2[\eta] \chi_t[\eta] \exp \left( \int_0^L dy \left[ \lambda(y) \eta^* (y) - c.c. \right] \right).$$  \hfill (15)

Its equation of motion follows from the Fourier transform of (12) as

$$\partial_t \tilde{W}_t[\lambda] = -\gamma \tilde{W}_t[\lambda] + \frac{\gamma}{L} \int_0^L dx \int d^2 \xi \, g(\xi) \tilde{W}_t[\lambda - \Lambda_t(\xi, x)].$$  \hfill (16)

This equation describes the time evolution of a quasiprobability distribution on a functional phase space. Each point therein is described by a complex function $\lambda(y)$ corresponding to a linear combination of the canonical field variables. The latter are subject to random kicks whose strength is given by the function $\Lambda_t(\xi, x; y)$, playing the role of the random variable $\xi f(y/L - x/L)$ in the interaction picture. It thus follows that Eq. (16) can be considered the field-theoretic generalization of a quantum linear Boltzmann equation [23].

We note that in the diffusion limit of small and frequent kicks Eq. (16) can be approximated by the Fokker–Planck equation (see Appendix)

$$\partial_t \tilde{W}_t[\lambda] = \int dx_1 dx_2 \left[ Q_t^{\lambda^*}(x_2 - x_1) \frac{\delta}{\delta \lambda(x_1)} \frac{\delta}{\delta \lambda^*(x_2)} \tilde{W}_t[\lambda] \right] \tilde{W}_t[\lambda'] \tilde{W}_t[\lambda''] \tilde{W}_t[\lambda'''] \tilde{W}_t[\lambda'''] \tilde{W}_t[\lambda''''],$$  \hfill (17)

with the coefficients

$$Q_t^{\lambda^*}(x_2 - x_1) = -\gamma g_2^2 \int_0^L \frac{dx}{L} b_t(x; x_1) a_t^*(x; x_2),$$

$$Q_t^{\lambda^*}(x_2 - x_1) = \gamma g_2^2 \int_0^L \frac{dx}{L} b_t(x; x_1) b_t^*(x; x_2),$$

$$Q_t^{\lambda^*}(x_2 - x_1) = \gamma g_2^2 \int_0^L \frac{dx}{L} a_t(x; x_1) b_t^*(x; x_2),$$

$$Q_t^{\lambda^*}(x_2 - x_1) = -\gamma g_2^2 \int_0^L \frac{dx}{L} a_t(x; x_1) b_t^*(x; x_2),$$  \hfill (18)

where $g_2^2$ is the second moment of $g$. Since the Wigner functional (15) is analytic in the complex wave amplitude $\lambda$, the functional derivatives treat $\lambda$ and $\lambda^*$ as independent variables, like in the Wirtinger calculus [24]. This field-theoretic Fokker–Planck equation can be solved using the techniques introduced in [16].

The Wigner representation is also convenient for describing the gradual loss of quantumness induced by the exact Eq. (1). This is because quantum superpositions of macroscopically distinct field states are characterized by a quasiprobability distribution displaying strong oscillations between positive and negative values in a local phase-space region of volume $h$. The decoherence of a superposition due to random phase-space kicks results in the blurring of these fine structures. Once the Wigner functional is nonnegative, it can be viewed as a probability distribution in a functional phase space. The corresponding field state is then indistinguishable from a (mixed) classical field configuration. This loss of coherence will now be quantified through the decay of the purity of the state.

IV. DECOHERENCE DYNAMICS

Physically, one would expect that the purity decay rate of a quantum superposition of field states depends on their initial separation as compared to the characteristic width of $g$. Moreover, as we showed in Eq. (3), the purity of a superposition decays monotonically with time. Its
and choose the spread function \( a \), a generalized Faure sequence \([25–27]\).

The field is modeled as a harmonic chain of 32 local oscillators.

For definiteness, we take the kick distribution to be given by

\[
\psi(\tau) \propto (|\alpha\rangle_0 + |\beta\rangle_0) \{|0\}_{k \neq 0}
\]

where \( \gamma \) is the characteristic functional of the initial state. For definiteness, we take the kick distribution to be given by

\[
g(\xi) = \exp(-|\xi|^2/2\sigma_g^2)/2\pi\sigma_g^2,
\]

and choose the spread function \( f(s) \) so that

\[
f_k = \sqrt{L} \exp(-\sigma_g^2 k^2/2)/\sqrt{2\pi(0, exp(-2\pi^2 \sigma_g^2/L^2))},
\]

where \( \sigma_g \) is the Jacobi theta function. In the following we consider a broad spread function with \( \sigma_g/L \approx 1 \). In this case the approximations \( f_k \approx \sqrt{L} \delta_{k,0} \) and \( \Gamma(\eta_k) \approx \gamma - \gamma \exp(-2L\sigma_g^2|\eta_k|^2) \) can be used.

For the numerical calculation of Eq. (19) the quantum field is modeled as a harmonic chain of 32 local oscillators. The corresponding phase space is discretized using a generalized Faure sequence \([25–27]\).

Figure 2. (Color online) Purity decay of a superposition of single-mode coherent states \( |\psi\rangle \propto (|\alpha\rangle_0 + |\beta\rangle_0) \{|0\}_{k \neq 0} \), with \( \alpha = 2 + 2i \), \( v/L\omega = 0.01 \), and \( \sigma_g/L = 1 \). The dots correspond to the exact numerical calculation of Eq. (19). The top panel shows the case of a narrow kick distribution with \( \sigma_g^2 L = 0.32 \) and for \( \gamma/\omega = 1 \). The solid line is the analytic approximation of Eq. (19) for \( \sigma_g^2 L \ll 1 \) and \( \sigma_g/L \approx 1 \) (see Appendix). The opposite case of a broad kick distribution with \( \sigma_g^2 L = 32 \) and for \( \gamma/\omega = 0.2 \) is illustrated in the bottom panel. The solid line gives the long-time behavior obtained using the Laplace approximation (see Appendix), \( p_t \approx 1/4\gamma t\sigma_g^2 \), while the dashed line shows the short-time behavior \( p_t \approx \exp(-2\gamma t) \).

The purity of the time-evolved quantum field in the ground mode, \(|\psi\rangle \propto (|\alpha\rangle_0 + |\beta\rangle_0) \{|0\}_{k \neq 0} \)

is the Jacobi theta function. In the following we determine by the state of the field. In the following

\[
\frac{\text{p}}{\gamma} = \frac{\exp(-|\xi|^2/2\sigma_g^2)}{2\pi\sigma_g^2},
\]

and choose the spread function \( f(s) \) so that

\[
f_k = \sqrt{L} \exp(-\sigma_g^2 k^2/2)/\sqrt{2\pi(0, exp(-2\pi^2 \sigma_g^2/L^2))},
\]

where \( \sigma_g \) is the Jacobi theta function. In the following we consider a broad spread function with \( \sigma_g/L \approx 1 \). In this case the approximations \( f_k \approx \sqrt{L} \delta_{k,0} \) and \( \Gamma(\eta_k) \approx \gamma - \gamma \exp(-2L\sigma_g^2|\eta_k|^2) \) can be used.

For the numerical calculation of Eq. (19) the quantum field is modeled as a harmonic chain of 32 local oscillators. The corresponding phase space is discretized using a generalized Faure sequence \([25–27]\).

Figure 3. (Color online) Initial purity decay rate, \( R_0/\gamma \), of a superposition of single-mode coherent states \( |\psi\rangle \propto (|\alpha\rangle_0 + |\beta\rangle_0) \{|0\}_{k \neq 0} \) as a function of the separation \( |\alpha - \beta| \), for the case \( \sigma_g/L = 1 \), with \( v/L\omega = 0.01 \) and \( \gamma/\omega = 1 \). The curves show the behavior of \( R_0/\gamma \) for a narrow kick distribution with \( \sigma_g^2 L = 0.08 \). They correspond to superpositions such that \( \alpha + \beta = 8, 4, 0 \) (from left to right) with \( \alpha - \beta = i|\alpha - \beta| \). The circles give the exact numerical values of Eq. (22), and the solid lines show the analytic approximation of this equation for \( \sigma_g/L \approx 1 \) and \( \sigma_g^2 L \ll 1 \) (see Appendix).

Figure 2 shows the purity decay for two limiting values of the width of the kick distribution. The initial state is given by \( |\psi\rangle \propto (|\alpha\rangle_0 + |\beta\rangle_0) \{|0\}_{k \neq 0} \). In the limit of a narrow distribution \( g \), such that \( \sigma_g^2 L \ll 1 \), the purity can be calculated analytically from Eq. (19) (see Appendix). The resulting expression yields the solid line in the top panel of Fig. 2 which is in excellent agreement with the exact numerical calculation of Eq. (19). For the case of a broad distribution, \( \sigma_g^2 L \gg 1 \), the purity cannot be calculated analytically for all times. Its short-time behavior is given by \( p_t \approx \exp(-2\gamma t) \), as indicated by the dashed line in the bottom panel, and its long-time evolution can be determined using Laplace’s method \([28]\), yielding \( p_t \approx 1/4\gamma t\sigma_g^2 \) as \( \gamma t \to \infty \) (see Appendix). This asymptotic behavior is indicated by the solid curve in the bottom panel of Fig. 2.

In order to investigate how the purity decay depends on the separation \( |\alpha - \beta| \) between the superposed coherent states, we calculate the initial purity decay rate \( R_0 = -\rho_0 \) from Eq. (19),

\[
R_0 = 2 \int D^2(\eta_k) |\chi_0(\eta_k)|^2 \Gamma(\eta_k),
\]

for different initial states.

Figure 3 shows that \( R_0 \) in general does not increase monotonically with the separation \( |\alpha - \beta| \). For a narrow kick distribution, such that \( \sigma_g^2 L \ll 1 \), it exhibits oscillations for \( |\alpha - \beta| < 1.5 \). For large separations the classicalization rate approaches the maximum value (see Appendix)

\[
\frac{R_{\max}}{\gamma} = \frac{2 - 1}{1 + 2\sigma_g^2 L},
\]
The solid lines correspond to the approximation of Eq. (22) in the limit \( \sigma^2 L \ll 1 \) (see Appendix). They are in excellent agreement with the exact numerical calculation (circles). For a broad kick distribution \( R_0 \) does not vary appreciably with the separation and approaches the value \( R_0/\gamma = 2 \) in the limit of arbitrarily large separations (not shown).

V. MEAN ENERGY INCREASE

In addition to inducing decoherence, the presence of a classicalizing non-unitary dynamics will in general also affect the dynamics of otherwise conserved quantities such as energy [29]. Experimental bounds on the observed conservation of energy will therefore constrain the parameters entering the classicalizing master equation.

From Eq. (1) it follows that the field energy increases with a constant rate that is independent of the quantum state of the field (see Appendix),

\[
\partial_t \langle H \rangle_t = \gamma \sum_{k \in K} \hbar \omega_k |f_k|^2 \int d^2 \xi g(\xi) |\xi \Omega_k^+ - \xi^* \Omega_k^-|^2.
\]

(24)

In the limit of large \( L \) this heating rate reduces to (see Appendix)

\[
\partial_t \langle H \rangle_t = 2 \sqrt{\pi} \gamma \hbar \omega \sigma_x \sigma_g \left( 1 + \frac{v^2}{(2 \sigma_x \omega)^2} \right),
\]

(25)

where we used the distribution (20) and the spread function (21).

VI. CONCLUSIONS

We introduced a generic Lindblad master equation that serves to classicize a non-relativistic bosonic field. The Ehrenfest equations for the canonical field variables remain identical with the corresponding classical field equations, while quantum superpositions of distinct field configurations are rapidly turned into mixtures. In fact, the master equation induces a monotonic decay of the purity.

We showed that the Wigner functional is an appropriate representation to capture the gradual quantum-to-classical transition of the field. To the extent to which the functional turns positive, its dynamics can be regarded as being governed by a linear Boltzmann equation, which in the diffusion limit reduces to a Fokker–Planck equation. Using the characteristic functional, the decoherence rate of a quantum superposition of two effectively classical field configurations was shown to depend non-trivially on their phase-space separation, and to saturate for large separations.

The effect of the master equation on the field may be viewed as arising from a stochastic process of non-projective simultaneous measurements of the field amplitude and its canonical momentum, whose outcomes are discarded. These fictitious measurements have finite spatial resolution, as characterized by the spread function \( f \). It is important to remark that only due to the finite width \( \sigma_x > 0 \) of \( f \) in position space is the continuous field dynamics physically consistent and divergence-free. Moreover, the limited resolution \( \sigma_g > 0 \) associated with the generalized measurement of the phase-space coordinates ensures a finite back-action on all canonical field variables.

Notwithstanding the generality of the master equation and its complex classicalization dynamics, several analytical expressions were obtained for functionals of the field state. In particular, we obtained the functional dependence of the purity in the most important limiting cases, and we calculated the exact energy increase. The analytical results are in remarkable agreement with numerical calculations. This shows that for a superposition of field coherent states expectation values can be accurately calculated using quasi-Monte Carlo integration based on generalized Faure sequences, despite the high-dimensional character of the phase space.

We note that in Ref. [30] the stability of the quantum state of a macroscopic number of degrees of freedom against perturbation by a quantum or a classical noise was analyzed based on general considerations. Since the present work introduces a concrete classicalization model, it should encourage further investigations of macroscopic quantum systems described by quantum fields.

The methods discussed in this work can be straightforwardly generalized to non-relativistic bosonic tensor fields. In principle, a similar treatment can be developed for fermionic fields, even though their classical analogue is less evident. Finally, a relativistic generalization of the model presented here would enable the study of the quantum-to-classical transition of quantum electrodynamics.

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Appendix A: Classicalizing master equation

1. Monotonic decrease of the purity

Starting from the master equation
\[ \dot{\rho}_t = \mathcal{L}(\rho_t) = -\frac{i}{\hbar}[H, \rho_t] + \gamma \int_0^L \frac{dx}{L} \int d^2 \xi g(\xi) \left[ U_x(\xi) \rho_t U_x^\dagger(\xi) - \rho_t \right], \] (A1)
the rate of change of the purity is given by
\[ \dot{\rho}_t = 2 \text{Tr} (\rho_t \dot{\rho}_t) = 2 \text{Tr} (\rho_t \mathcal{L}(\rho_t)). \] (A2)

Now
\[ \text{Tr}(\rho_t \mathcal{L}(\rho_t)) = \text{Tr}(\rho_t[H, \rho_t]) + \gamma \int_0^L \frac{dx}{L} \int d^2 \xi g(\xi) \left( \text{Tr} (\rho_t U_x(\xi) \rho_t U_x^\dagger(\xi)) - \rho_t \right), \] (A3)
where \( \text{Tr}(\rho_t[H, \rho_t]) = \text{Tr}(\rho_t H \rho_t - \rho_t^2 H) = 0 \) and
\[ \text{Tr} (\rho_t U_x(\xi) \rho_t U_x^\dagger(\xi)) - \rho_t = -\frac{1}{2} \text{Tr} \left( (\rho_t U_x(\xi) - U_x(\xi) \rho_t) (\rho_t U_x(\xi) - U_x(\xi) \rho_t) \right) = -\frac{1}{2} \| [\rho_t, U_x(\xi)] \|^2, \] (A4)
with \( \| A \|^2 = \text{Tr}(A^\dagger A) \). Therefore,
\[ \dot{\rho}_t = -\gamma \int_0^L \frac{dx}{L} \int d^2 \xi g(\xi) \| [\rho_t, U_x(\xi)] \|^2 < 0. \] (A5)

2. Dynamics of the characteristic functional

In the mode basis the unitary phase-space displacement operators acting on the canonical variables of the field in the vicinity of position \( x \) are
\[ U_x(\xi) = \exp \left( \sum_{\lambda \in K} \left[ \frac{1}{\sqrt{L}} \int_0^L dy f \left( \frac{y-x}{L} \right) e^{-iky(\xi \Omega^+_k - \xi^* \Omega^-_k)} c_k - \frac{1}{\sqrt{L}} \int_0^L dy f \left( \frac{y-x}{L} \right) e^{iky(\xi^* \Omega^+_k - \xi \Omega^-_k)} c_k^\dagger \right] \right), \] (A6)
where \( K = \{ 2\pi j/L, j \in \mathbb{Z} \} \). That is, \( k \) runs over all integer multiples of \( 2\pi/L \). Using the change of variables \( z = y - x \),
\[ \frac{1}{\sqrt{L}} \int_0^L dy f \left( \frac{y-x}{L} \right) e^{iky} = e^{ikx} \frac{1}{\sqrt{L}} \int_{-x}^{L-x} dz f \left( \frac{z}{L} \right) e^{ikz}. \] (A7)
The last integral can be written as
\[ \int_{-x}^{L-x} dz f \left( \frac{z}{L} \right) e^{ikz} = \int_{-x}^0 dz f \left( \frac{z}{L} \right) e^{ikz} + \int_0^{L-x} dz f \left( \frac{z}{L} \right) e^{ikz}, \] (A8)
where, due to the periodicity,
\[ \int_{-x}^0 dz f \left( \frac{z}{L} \right) e^{ikz} = \int_{L-x}^L dz f \left( \frac{z}{L} \right) e^{ikz}. \] (A9)
Therefore, the above operator can be written as
\[ U_x(\xi) = \exp \left( \sum_k \left[ f_k e^{-ikx(\xi \Omega^+_k - \xi^* \Omega^-_k)} c_k + f_k e^{ikx(\xi^* \Omega^+_k - \xi \Omega^-_k)} c_k^\dagger \right] \right), \] (A10)
in terms of the Fourier coefficients of \( f \)
\[ f_k = \frac{1}{\sqrt{L}} \int_0^L dz f \left( \frac{z}{L} \right) e^{ikz}. \] (A11)
In the interaction picture with respect to the Hamiltonian
\[ H = \sum_{k} \hbar \omega_k c_k^\dagger c_k, \]  
(A12)
the unitary displacement operators are
\[ \hat{U}_x(\xi, t) = \exp \left( \sum_{k} \left[ f_k^* e^{-i(kx + \omega_k t)} (\xi \Omega_k^+ - \xi^* \Omega_k^-) c_k^\dagger - f_k e^{i(kx + \omega_k t)} (\xi^* \Omega_k^+ - \xi \Omega_k^-) c_k \right] \right). \]  
(A13)

The dynamics of the characteristic functional \( \tilde{\chi}_t(\eta_k) \) follows from (A1) using the cyclic property of the trace and the identity
\[ D^\dagger(\{\alpha_k\})D(\{\zeta_k\})D(\{\alpha_k\}) = D(\{\zeta_k\}) \exp \left[ \sum_{k} \alpha_k^* \zeta_k - \alpha_k \zeta_k^* \right]. \]  
(A14)

The equation of motion is
\[ \partial_t \tilde{\chi}_t(\eta_k) = -\Gamma_t(\{\eta_k\}) \tilde{\chi}_t(\eta_k), \]  
(A15)
with
\[ \Gamma_t(\{\eta_k\}) = \gamma - \gamma \int_{0}^{L} \frac{dx}{T} \int_{0}^{\infty} d^2 \xi \, g(\xi) \exp \left( \sum_{k} \left[ f_k e^{i(kx + \omega_k t)} (\xi^* \Omega_k^+ - \xi \Omega_k^-) \eta_k - \text{c.c.} \right] \right). \]  
(A16)

The \( \xi \)-integral can be calculated writing \( \xi = re^{i\theta} \) and \( d^2 \xi = r dr d\theta \). In order to calculate the \( \theta \)-integral, it is convenient to rewrite the argument of the exponential in (A16) in the form
\[ \xi^* z_t[\eta, x] - \xi z_t^*[\eta, x] = r|z_t[\eta, x]| \left[ e^{-i(\theta - \varphi_x)} - e^{i(\theta - \varphi_x)} \right], \]  
(A17)
with
\[ z_t[\eta, x] = |z_t[\eta, x]| e^{i\varphi_x} = \sum_{k} \Omega_k^+ f_k e^{ikx} e^{i\omega_k t} \eta_k + \Omega_k^- f_k^* e^{-ikx} e^{-i\omega_k t} \eta_k^*. \]  
(A18)

Recalling the Jacobi–Anger formula
\[ e^{ix \cos \theta} = J_0(x) + \sum_{n=1}^{\infty} i^n J_n(x) \cos(n\theta), \]  
(A19)
the following integral is invariant under translations of \( \theta \),
\[ \frac{1}{2\pi} \int_{0}^{2\pi} d\theta e^{-ix \cos \theta} = J_0(x). \]  
(A20)

Finally, the \( \theta \)-integral yields
\[ \Gamma_t(\{\eta_k\}) = \gamma - \gamma \int_{0}^{L} \frac{dx}{T} \int_{0}^{\infty} dr g_r(\tau) r J_0 \left( 2r \left| \sum_{k} \Omega_k^+ f_k e^{ikx} e^{i\omega_k t} \eta_k + \Omega_k^- f_k^* e^{-ikx} e^{-i\omega_k t} \eta_k^* \right| \right). \]  
(A21)
Since the \( r \)-integral corresponds to the Hankel transform of \( g_r \),
\[ \hat{g}_r(s) = \int_{0}^{\infty} dr g_r(\tau) J_0(rs) r, \]  
(A22)
Eq. (A21) can be compactly expressed as
\[ \Gamma_t(\{\eta_k\}, \{\eta_k^*\}) = \gamma - \gamma \int_{0}^{L} \frac{dx}{T} \hat{g}_r \left( 2 \left| \sum_{k} \Omega_k^+ f_k e^{ikx} e^{i\omega_k t} \eta_k + \Omega_k^- f_k^* e^{-ikx} e^{-i\omega_k t} \eta_k^* \right| \right). \]  
(A23)
Appendix B: Equation of motion for the Wigner functional

In the position basis and in the interaction picture with respect to the Hamiltonian (A1), the unitary displacement operators in the master equation (A1) are

\[ \hat{U}_x(\xi, t) = \exp \left( \int_0^t dy \Lambda_t(x, \xi; y) \Psi^\dagger(y) - \Lambda_t^\dagger(x, \xi; y) \Psi(y) \right), \]

(B1)

where

\[ \Lambda_t(x, \xi; y) = \xi a_t(x; y) - \xi^* b_t(x; y), \]

(B2)

with

\[ a_t(x; y) = \frac{1}{\sqrt{L}} \sum_{k \in K} f_k e^{i k (x - y)} \left[ \cos \omega_k t - \frac{i}{2} \left( \frac{\omega}{\omega_k} + \frac{\omega_k}{\omega} \right) \sin \omega_k t \right], \]

\[ b_t(x; y) = \frac{i}{2 \sqrt{L}} \sum_{k \in K} f_k e^{i k (x - y)} \left( \frac{\omega}{\omega_k} - \frac{\omega_k}{\omega} \right) \sin \omega_k t. \]

(B3)

The characteristic functional is defined as \( \chi_t[\eta] = \text{Tr}(\hat{\rho}_t D[\eta]) \) with

\[ D[\eta] = \exp \left( \int dy [\eta(y) \Psi^\dagger(y) - \eta^*(y) \Psi(y)] \right). \]

(B4)

Its equation of motion follows from the master equation (A1) as

\[ \partial_t \chi_t[\eta] = -\gamma + \gamma \frac{dx}{L} \int \frac{d^2 \xi g(\xi)}{\L} \int \frac{d \eta^* \chi_t[\eta]}{d \eta} \exp \left( \int_0^t dy [\eta(y) \Lambda_t^\dagger(x, \xi; y) - \eta^*(y) \Lambda_t(x, \xi; y)] \right) \chi_t[\eta]. \]

(B5)

The Wigner functional is defined as the functional Fourier transform of \( \chi_t[\eta] \) [1]

\[ \tilde{W}_t[\lambda] = \int D^2 \eta \chi_t[\eta] \exp \left( \int dy [\lambda(y) \eta^*(y) - \lambda^*(y) \eta(y)] \right), \]

(B6)

where the functional integral is defined as

\[ \int D^2 \eta F[\eta] = \lim_{n \to \infty} \int d^2 \eta_1 \ldots d^2 \eta_n F(\eta_1, \ldots, \eta_n). \]

(B7)

The functional Fourier transform of \( \chi_t[\eta] \) yields

\[ \partial_t \tilde{W}_t[\lambda] = \gamma \frac{dx}{L} \int d^2 \xi g(\xi) \int D^2 \eta \chi_t[\eta] \exp \left( \int dy [\lambda(y) - \Lambda_t(x, \xi; y)] \eta^*(y) - \text{c.c.} \right) - \gamma \tilde{W}_t[\lambda]. \]

(B8)

Therefore, we obtain a linear Boltzmann equation describing the dynamics of the Wigner functional

\[ \partial_t \tilde{W}_t[\lambda] = \gamma \frac{dx}{L} \int d^2 \xi g(\xi) \left( \tilde{W}_t[\lambda - \Lambda_t(x, \xi)] - \tilde{W}_t[\lambda] \right). \]

(B9)

1. Diffusion limit

First we rewrite Eq. (B5) as

\[ \partial_t \chi_t[\eta] = -\gamma \left[ 1 - \frac{dx}{L} \int d^2 \xi g(\xi) \exp(\xi^* z_t[\eta; x] - \xi z_t^*[\eta; x]) \right] \chi_t[\eta] \]

(B10)

with

\[ z_t[\eta; x] = \int_0^L dy [\eta(y) a_t(x; y) + \eta^*(y) b_t(x; y)]. \]

(B11)
To second order in $\xi$,

$$\exp(\xi^* z_1[\eta;x] - \xi z_1^*[\eta;x]) \simeq 1 + \xi^* z_1[\eta;x] - \xi z_1^*[\eta;x] + \frac{1}{2} (\xi^* z_1[\eta;x] - \xi z_1^*[\eta;x])^2.$$  \hspace{1cm} (B12)

Writing $\xi = |\xi|e^{i\theta}$ and given that $g(\xi)$ is an even function, the only term that contributes to the $\xi$-integral is $|\xi|^2 z_1[\eta;x]$. The dynamics of the characteristic functional is thus given by

$$\partial_t \bar{\chi}_t[\eta] = -\frac{\gamma \sigma_g^2}{L} \int_0^L dx \left\{ \int dx_1 dx_2 \eta(x_1)\eta(x_2) a_t(x_1) a_t^*(x_2) + \int dx_1 dx_2 \eta(x_1)\eta^*(x_2) a_t(x_1) a_t^*(x_2) + \int dx_1 dx_2 \eta(x_1)\eta^*(x_2)b_t(x_1) b_t^*(x_2) \right\} \bar{\chi}_t[\eta].$$  \hspace{1cm} (B13)

where the second moment of the distribution $g$ is

$$\int d^2 \xi g(\xi)|\xi|^2 = \sigma_g^2.$$  \hspace{1cm} (B14)

The functional Fourier transform of this equation is given by

$$\partial_t \tilde{W}_t[\lambda] = -\frac{\gamma \sigma_g^2}{L} \int dx \int dx_1 dx_2 \left\{ a_t(x_1) b_t^*(x_2) \int D^2 \eta \eta(x_1)\eta(x_2) e^{f dz [\lambda(z)\eta^*(z) - \lambda^*(z)\eta(z)]} \tilde{\chi}_t[\eta] + a_t(x_1) a_t^*(x_2) \int D^2 \eta \eta(x_1)\eta^*(x_2) e^{f dz [\lambda(z)\eta^*(z) - \lambda^*(z)\eta(z)]} \tilde{\chi}_t[\eta] + b_t(x_1) b_t^*(x_2) \int D^2 \eta \eta^*(x_1)\eta(x_2) e^{f dz [\lambda(z)\eta^*(z) - \lambda^*(z)\eta(z)]} \tilde{\chi}_t[\eta] + b_t(x_1) a_t^*(x_2) \int D^2 \eta \eta^*(x_1)\eta^*(x_2) e^{f dz [\lambda(z)\eta^*(z) - \lambda^*(z)\eta(z)]} \tilde{\chi}_t[\eta] \right\}. $$  \hspace{1cm} (B15)

Using

$$\frac{\delta \lambda(z)}{\delta \lambda(y)} = \delta(z - y),$$  \hspace{1cm} (B16)

we have

$$\frac{\delta}{\delta \lambda(x)} \tilde{W}_t[\lambda] = \int D^2 \eta \eta^*(x) \tilde{\chi}_t[\eta] \exp \left( \int dz [\lambda(z)\eta^*(z) - \lambda^*(z)\eta(z)] \right),$$  \hspace{1cm} (B17)

and

$$\frac{\delta}{\delta \lambda^*(x)} \tilde{W}_t[\lambda] = -\int D^2 \eta \eta^*(x) \tilde{\chi}_t[\eta] \exp \left( \int dz [\lambda(z)\eta^*(z) - \lambda^*(z)\eta(z)] \right).$$  \hspace{1cm} (B18)

From the above, we finally arrive at the Fokker–Planck equation

$$\partial_t \tilde{W}_t[\lambda] = \int dx dx_1 dx_2 \left\{ Q_{t \lambda}(x_2 - x_1) \lambda \frac{\delta}{\delta \lambda(x_1)} + Q_{t \lambda^*}(x_2 - x_1) \lambda^* \frac{\delta}{\delta \lambda^*(x_1)} \right\} \tilde{W}_t[\lambda],$$  \hspace{1cm} (B19)

with the coefficients

$$Q_{t \lambda}(x_2 - x_1) = -\gamma \sigma_g^2 \int_0^L dx \frac{d}{dx} b_t(x; x_1) a_t^*(x; x_2),$$

$$Q_{t \lambda^*}(x_2 - x_1) = \gamma \sigma_g^2 \int_0^L dx \frac{d}{dx} b_t(x; x_1) b_t^*(x; x_2),$$

$$Q_{t \lambda^*}(x_2 - x_1) = \gamma \sigma_g^2 \int_0^L dx \frac{d}{dx} a_t(x; x_1) a_t^*(x; x_2),$$

$$Q_{t \lambda^*}(x_2 - x_1) = -\gamma \sigma_g^2 \int_0^L dx \frac{d}{dx} a_t(x; x_1) b_t^*(x; x_2).$$  \hspace{1cm} (B20)
Appendix C: Decoherence dynamics

1. Purity in the mode representation.

In the mode basis the purity of the state of the field is given by

\[ p_t = \int \mathcal{D}^2[\{\eta_k\}] |\tilde{\chi}_t[\{\eta_k\}]|^2, \]  \hspace{1cm} (C1)

which following Eq. (A15) is equal to

\[ p_t = \int \mathcal{D}^2[\{\eta_k\}] |\chi_0[\{\eta_k\}]|^2 \exp \left( -2 \int_0^t dt \Gamma_t[\{\eta_k\}] \right), \]  \hspace{1cm} (C2)

with

\[ \Gamma_t[\{\eta_k\}] = -\gamma + \gamma \int_0^L dx \hat{g}_r(s_t[\{\eta_k\}, x]), \]  \hspace{1cm} (C3)

where

\[ s_t[\{\eta_k\}, x] = 2 \sum_{k \in K} \Omega_k^* f_k e^{i k x} e^{-i \omega_k t} \eta_k + \Omega_k f_k^* e^{-i k x} e^{-i \omega_k t} \eta_k^*. \]  \hspace{1cm} (C4)

In the following we will assume an isotropic kick distribution with

\[ \hat{g}_r(s) = \exp \left( -\frac{s^2 \sigma_g^2}{2} \right) \]  \hspace{1cm} (C5)

and a spread function such that

\[ f_k = \frac{\sqrt{L} \exp(-\sigma_k^2 k^2/2)}{\vartheta_3(0, \exp(-2\pi^2 \sigma_k^2/L^2))}, \]  \hspace{1cm} (C6)

where

\[ \vartheta_3(0, \exp(-2\pi^2 \sigma_k^2/L^2)) = \sum_{n \in \mathbb{Z}} \exp(-2\pi^2 \sigma_k^2 n^2/L^2). \]  \hspace{1cm} (C7)

Moreover, we will consider that the initial state is a superposition of coherent states in the mode \( k = 0 \), i.e. \( |\psi\rangle = \mathcal{N}^{1/2} (|\alpha_0\rangle + |\beta_0\rangle) \{0\}_{k \neq 0} \). The squared modulus of the corresponding characteristic functional is

\[ \frac{|\chi_0[\{\eta_k\}]|^2}{\mathcal{N}^2} = e^{-\Sigma_{k \neq 0} |\eta_k|^2} \left[ e^{-2|\Delta+\eta_0|^2} + e^{-2|\Delta-\eta_0|^2} \right] 
\]
\[ + 2 e^{-\Sigma_{k \neq 0} |\eta_k|^2} e^{-i|\eta_0|} \left[ 2 \cos^2 (2 \Im(\Delta \eta_0^*)) + \cos (4 \Im(\Delta C^*) e^{-|\Delta|^2}) \right] 
\]
\[ + 4 e^{-|\Delta|^2} \cos (2 \Im(\Delta C^*)) e^{-\Sigma_{k \neq 0} |\eta_k|^2} \cos (2 \Im(\Delta \eta_0^*)) \left[ e^{-|\Delta+\eta_0|^2} + e^{-|\Delta-\eta_0|^2} \right], \] \hspace{1cm} (C8)

where \( \mathcal{N} = \left[2 + 2 e^{-2|\Delta|^2} \cos (2 \Im(C^*\Delta)) \right]^{-1}, \Delta = (\alpha - \beta)/2 \) and \( C = (\alpha + \beta)/2 \).

For the case \( \sigma_x/L \simeq 1 \), the Fourier coefficients of the spread function are given by

\[ f_k \simeq \sqrt{L} \delta_{0k}, \]  \hspace{1cm} (C9)

and the purity is

\[ p_t \simeq \int \mathcal{D}^2[\{\eta_k\}] |\chi_0[\{\eta_k\}]|^2 \exp(-2\gamma t[1 - \exp(-2L \sigma_g^2 |\eta_0|^2)]). \]  \hspace{1cm} (C10)

In the limit of a narrow kick distribution, such that \( \sigma_g^2 L \ll 1 \), the above exponential can be approximated by

\[ \exp(-2L \sigma_g^2 |\eta_0|^2) \simeq 1 - 2L \sigma_g^2 |\eta_0|^2. \]  \hspace{1cm} (C11)
In order to calculate $p_t$, we must evaluate the integrals
\[
\int \frac{d^2\eta_0}{\pi} e^{-|\eta_0|^2(1+\mu^2)/2} \left[1 + \cos \left(4\text{Im} \Delta \eta_0^*\right)\right] = \frac{1}{1 + \mu^2} \frac{1}{2} \left(1 + e^{-\frac{4|\Delta|^2}{1+\mu^2}}\right),
\] (C12)
\[
\int \frac{d^2\eta_0}{\pi} e^{-|\Delta \pm \eta_0|^2 - |\eta_0|^2} \cos \left(2\text{Im} \Delta \eta_0^*\right) = \frac{1}{1 + \mu^2} e^{-|\Delta|^2},
\] (C13)
\[
\int \frac{d^2\eta_0}{\pi} e^{-|2\Delta \pm \eta_0|^2 - |\eta_0|^2} \mu^2 = \frac{1}{1 + \mu^2} e^{-\frac{4|\Delta|^2}{1+\mu^2}},
\] (C14)
where $\mu^2 = 4\gamma \sigma_g^2 L$. The final result is
\[
p_t \approx \int \mathcal{D}^2[\{\eta_k\}] |\chi_0[\{\eta_k\}]|^2 \exp(-4\gamma t L \sigma_g^2 |\eta_0|^2)
= \frac{2N^2}{1 + \mu^2} \left[1 + e^{-\frac{4|\Delta|^2}{1+\mu^2}} + e^{-\frac{4|\Delta|^2}{1+\mu^2}} + 4e^{-2|\Delta|^2} \cos \left(2\text{Im} \Delta \eta_0^*\right) + e^{-4|\Delta|^2} \cos \left(4\text{Im} \Delta \eta_0^*\right)\right].
\] (C15)

In the limit of a broad kick distribution, such that $\sigma_g^2 L \gg 1$, the purity decay can be characterized analytically for short and long times. To first order in time, the logarithm of the purity is given by
\[
\log p_t \approx \log p_0 + (\log p_t)|_{t=0} t + \cdots
\] (C16)
where
\[
(\log p_t)|_{t=0} = -2\gamma \int \mathcal{D}^2[\{\eta_k\}] |\chi_0[\{\eta_k\}]|^2 \left[1 - \exp(-2L \sigma_g^2 |\eta_0|^2)\right].
\] (C17)
Since $\sigma_g^2 L \gg 1$, then we obtain for the short-time behavior of the purity
\[
p_t \approx e^{-2\gamma t}. \tag{C18}
\]

The integral in Eq. (C10) can be written as
\[
p_t = \frac{1}{\pi} \int d^2\eta_0 g(\eta_0) \exp(-2\gamma t \eta(\eta_0)), \tag{C19}
\]
with
\[
g(\eta_0) = \int \mathcal{D}^2[\{\eta_k \neq \eta_0\}] |\chi_0[\{\eta_k\}]|^2 \tag{C20}
\]
and
\[
h(\eta_0) = 1 - \exp(-2L \sigma_g^2 |\eta_0|^2). \tag{C21}
\]
For $t \to \infty$, Eq. (C19) can be approximated using Laplace’s method:
\[
\int_{\mathbb{R}^d} dx g(x) e^{-t h(x)} \approx g(x_0) \left(\frac{2\pi}{t}\right)^{d/2} e^{-t h(x_0)} \left|H_h(x_0)\right|^{1/2} \text{ as } t \to \infty, \tag{C22}
\]
where $|\cdot|$ is the determinant of $H_h$, the Hessian matrix of $h$, and $x_0$ is a minimum of $h$, i.e., $\nabla h(x_0) = 0$ and $H_h(x_0) > 0$. Since $h$ has a minimum at $\eta_0 = 0$, the asymptotic approximation to the purity for $t \to \infty$ is given by
\[
p_t \approx \frac{g(0)}{4\gamma L \sigma_g^2} \frac{1}{t} \approx \frac{1}{4\gamma L \sigma_g^2} \frac{1}{t}, \tag{C23}
\]
valid for $1/4\gamma L \sigma_g^2 \gg 0$.

### 2. Initial purity decay rate

From Eq. (C2) and Eq. (C15), it follows that the initial decay rate of the purity, $R_0 = \frac{\partial}{\partial t} p_0$, is given by
\[
R_0 = 2\gamma - \frac{4\gamma N^2}{1 + \mu^2} \left[1 + e^{-\frac{4|\Delta|^2}{1+\mu^2}} + e^{-\frac{4|\Delta|^2}{1+\mu^2}} + 4e^{-2|\Delta|^2} \cos \left(2\text{Im} \Delta \eta_0^*\right) + e^{-4|\Delta|^2} \cos \left(4\text{Im} \Delta \eta_0^*\right)\right], \tag{C24}
\]
where $\mu^2 = 2\sigma_g^2 L$. We note that for arbitrarily large separations, i.e., for $\Delta \to \infty$, $N \simeq 1/2$ and
\[
\frac{R_0}{\gamma} \simeq 2 - \frac{1}{1 + 2\sigma_g^2 L}. \tag{C25}
\]
Appendix D: Mean energy increase

The time evolution of the expectation value of the Hamiltonian \( [A12] \) follows from the master equation \( [A1] \)

\[
\partial_t \langle H \rangle_t = -\frac{i}{\hbar} \text{Tr}(\{H, \rho_t\}H) + \gamma \int_0^L \frac{dx}{L} \int d^2 \xi \, g(\xi) \text{Tr} \left( U_\xi(\xi) \rho_t U_\xi^\dagger(\xi) H \right) - \gamma \langle H \rangle_t
\]

\[
= \gamma \int_0^L \frac{dx}{L} \int d^2 \xi \, g(\xi) \text{Tr} \left( \rho_t U_\xi^\dagger(\xi) H U_\xi(\xi) \right) - \gamma \langle H \rangle_t. \tag{D1}
\]

The unitary transformation of \( H \) is

\[
U_\xi^\dagger(\xi) H U_\xi(\xi) = \frac{1}{L} \sum_{k' \in K} \hbar \omega_{k'} \exp \left( \sum_{k \in K} \beta_k^\prime(\xi, \xi) c_k - \beta_k(\xi, \xi) c_k^\dagger \right) c_{k'} c_k \exp \left( \sum_{k \in K} \beta_k(\xi, \xi) c_k^\dagger - \beta_k^\prime(\xi, \xi) c_k \right), \tag{D2}
\]

with

\[
\beta_k(\xi, \xi) = e^{-ikx} f_k(\xi \Omega_k^+ \xi^* \Omega_k^-).
\]

We define the operators

\[
X = \sum_{k \in K} \beta_k(x, \xi) c_k^\dagger - \beta_k^\prime(\xi, \xi) c_k,
\]

and \( Y = c_{k'}^\dagger c_k \). Using the formula

\[
e^{-X} Y e^X = Y - [X, Y] + \frac{1}{2} [X, [X, Y]] + \cdots \tag{D5}
\]

and the commutator identity \([A, BC] = B [A, C] + [A, B] C\), it follows that

\[
[X, Y] = \sum_{k \in K} \beta_k(x, \xi) \left[ c_k^\dagger, c_{k'}^\dagger, c_k, c_{k'} \right] - \sum_{k \in K} \beta_k^\prime(\xi, \xi) \left[ c_k, c_{k'}^\dagger, c_k^\dagger, c_{k'} \right]
\]

\[
= - \sum_{k \in K} \beta_k(x, \xi) c_{k'}^\dagger \delta_{kk'} - \sum_{k \in K} \beta_k^\prime(\xi, \xi) c_k \delta_{kk'}
\]

\[
= - \beta_k(x, \xi) c_{k'}^\dagger - \beta_k^\prime(\xi, \xi) c_k,
\]

and

\[
[X, [X, Y]] = \sum_{k \in K} \left[ \beta_k(x, \xi) c_k^\dagger - \beta_k(x, \xi) c_k, -\beta_k^\prime(x, \xi) c_{k'}^\dagger - \beta_k^\prime(x, \xi) c_{k'} \right]
\]

\[
= \sum_{k \in K} \beta_k(x, \xi) \beta_k^\prime(x, \xi) \delta_{kk'} + \sum_{k \in K} \beta_k^\prime(x, \xi) \beta_k^\prime(x, \xi) \delta_{kk'}
\]

\[
= 2|\beta_k(x, \xi)|^2. \tag{D6}
\]

and

\[
\partial_t \langle H \rangle_t = \gamma \int_0^L \frac{dx}{L} \int d^2 \xi \, g(\xi) \text{Tr} \left( \rho_t \sum_{k \in K} \hbar \omega_k \left[ c_k^\dagger c_k + \beta_k(x, \xi) c_k^\dagger + \beta_k^\prime(x, \xi) c_k + |\beta_k(x, \xi)|^2 \right] \right) - \gamma \langle H \rangle_t. \tag{D8}
\]

Writing \( \xi = re^{i\theta} \), it follows that

\[
\int d^2 \xi g(\xi) \beta_k(x, \xi) = \int_0^r dr g_r(r) \int_0^{2\pi} d\theta \beta_k(x, r) = \int_0^r dr g_r(r) r^2 \int_0^{2\pi} d\theta \frac{1}{2\pi} (e^{i\theta} \Omega_k^+ - e^{-i\theta} \Omega_k^-)
\]

\[
\times \frac{1}{\sqrt{L}} \int_0^L dy f \left( \frac{y-x}{r} \right) e^{-iky} = 0. \tag{D9}
\]
Therefore, the space-averaged field energy density increases with the constant rate

$$\partial_t \langle H \rangle_t = \gamma \int d^2 \xi g(\xi) \sum_{k \in K} \hbar \omega_k |\beta_k(0, \xi)|^2.$$  \hspace{1cm} (D10)

This expression can be further simplified writing

$$|\beta_k(0, \xi)|^2 = r^2 |f_k|^2 (\cos^2 \theta (\Omega_k^+ - \Omega_k^-)^2 + \sin^2 \theta (\Omega_k^+ + \Omega_k^-)^2),$$  \hspace{1cm} (D11)

calculating the $\theta$-integral

$$\int d^2 \xi g(\xi) |\beta_k(0, \xi)|^2 = \int_0^\infty dr g_r(r) r^3 |f_k|^2 [(\Omega_k^+)^2 + (\Omega_k^-)^2].$$  \hspace{1cm} (D12)

and using the identity

$$(\Omega_k^+)^2 + (\Omega_k^-)^2 = \frac{1}{2} \left( \frac{\omega}{\omega_k} + \frac{\omega_k}{\omega} \right) = \frac{1}{2} \frac{\omega^2 + \omega_k^2}{\omega_k \omega}.$$  \hspace{1cm} (D13)

The result is

$$\partial_t \langle H \rangle_t = \gamma \hbar \int_0^\infty dr g_r(r) r^3 \sum_{k \in K} |f_k|^2 \left( \omega + \frac{v^2 k^2}{2 \omega} \right).$$  \hspace{1cm} (D14)

Given $f_k = \sqrt{L} \exp(-\sigma_f^2 L^2 k^2 / 2) / \partial_3 (0, \exp(-2\pi^2 \sigma_f^2))$, its squared modulus is

$$|f_k|^2 = L \exp(-\sigma_f^2 L^2 k^2 / 2) / \partial_3 (0, \exp(-2\pi^2 \sigma_f^2)).$$  \hspace{1cm} (D15)

In the limit of large $L$,

$$\frac{1}{L} \sum_{k \in K} \rightarrow \int_{-\infty}^{\infty} \frac{dk}{2\pi},$$  \hspace{1cm} (D16)

and therefore

$$\partial_3 (0, \exp(-2\pi^2 \sigma_f^2)) = \sum_k \exp(-\sigma_f^2 L^2 k^2 / 2)$$  \hspace{1cm} (D17)

becomes $L / \sqrt{2\pi \sigma_f^2}$. Moreover,

$$\int \frac{dk}{2\pi} |f_k|^2 \left( \omega + \frac{v^2 k^2}{2 \omega} \right) = 2\pi \sigma_f^2 \left( \frac{\omega}{\sqrt{4\pi \sigma_f^2}} + \frac{v^2}{2\omega \sqrt{16\pi \sigma_f^6}} \right).$$  \hspace{1cm} (D18)

Using $g_r(r) = \exp(-r^2 / 2\sigma_g^2) / \sigma_g^2$ the average energy increase is finally given by

$$\partial_t \langle H \rangle_t = 2\sqrt{\pi} \gamma \hbar \omega \sigma_x \sigma_y^2 \left( 1 + \frac{v^2}{(2\sigma_x \omega)^2} \right).$$  \hspace{1cm} (D19)

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