RECONSTRUCTION OF HIGHER-DIMENSIONAL FUNCTION FIELDS

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ABSTRACT. We determine the function fields of varieties of dimension \( \geq 2 \) defined over the algebraic closure of \( \mathbb{F}_p \), modulo purely inseparable extensions, from the quotient by the second term in the lower central series of their pro-\( \ell \) Galois groups.

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INTRODUCTION

Fix two distinct primes \( p \) and \( \ell \). Let \( k = \overline{\mathbb{F}}_p \) be an algebraic closure of the finite field \( \mathbb{F}_p \). Let \( X \) be an algebraic variety defined over \( k \) and \( K = k(X) \) its function field. We will refer to \( X \) as a model of \( K \); we will generally assume that \( X \) is normal and projective. Let \( G_K^\ell \) be the abelianization of the pro-\( \ell \)-quotient \( G_K \) of the absolute Galois group of \( K \). Under our assumptions on \( k \), \( G_K^\ell \) is a torsion-free \( \mathbb{Z}_\ell \)-module. Let \( G_K^\circ \) be its canonical central extension - the second lower central series quotient of \( G_K \). It determines a set \( \Sigma_K \) of distinguished (primitive) finite-rank subgroups: a topologically noncyclic subgroup \( \sigma \in \Sigma_K \) iff

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• $\sigma$ lifts to an abelian subgroup of $G_K^c$;
• $\sigma$ is maximal: there are no abelian subgroups $\sigma' \subset G_K^a$ which lift to an abelian subgroup of $G_K^c$ and contain $\sigma$ as a proper subgroup.

Our main theorem is

**Theorem 1.** Let $K$ and $L$ be function fields over algebraic closures of finite fields $k$, resp. $l$, of characteristic $\neq \ell$. Assume that the transcendence degree of $K$ over $k$ is at least two and that there exists an isomorphism

$$\Psi = \Psi_{K,L} : G_K^a \sim \rightarrow G_L^a$$

(1.1)

of abelian pro-$\ell$-groups inducing a bijection of sets

$$\Sigma_K = \Sigma_L.$$

Then $k = l$ and there exists a constant $\epsilon \in \mathbb{Z}_\ell^*$ such that $\epsilon^{-1} \cdot \Psi$ is induced from a unique isomorphism of perfect closures

$$\bar{\Psi}^* : \bar{L} \sim \rightarrow \bar{K}.$$

In this paper we implement the program outlined in [1] and [2] describing the correspondence between higher-dimensional function fields and their abelianized Galois groups. We follow closely our paper [4], where we treated in detail the case of surfaces: The isomorphism (1.1) of abelianized Galois groups induces a canonical isomorphism

$$\Psi^* : \hat{L}^* \sim \rightarrow \hat{K}^*$$

between pro-$\ell$-completions of multiplicative groups. One of the steps in the proof is to show that under the assumptions of Theorem 1, $\Psi^*$ induces by restriction a canonical isomorphism

$$(1.2) \quad \Psi^* : L^*/l^* \otimes \mathbb{Z}_\ell(t) \sim \rightarrow (K^*/k^* \otimes \mathbb{Z}_\ell(t))^\epsilon, \quad \text{for some} \quad \epsilon \in \mathbb{Z}_\ell^*.$$ 

Here we proceed by induction on the transcendence degree, using [4] as the inductive assumption. We first recover abelianized inertia and decomposition subgroups of divisorial valuations using the theory of commuting pairs developed in [3]. Then we apply the inductive assumption (1.2) to residue fields of divisorial valuations. This allows to prove that for every normally closed one-dimensional subfield $F = l(f) \subset L$ there exists a one-dimensional subfield $E \subset K$ such that

$$\Psi^*(F^*/l^* \otimes \mathbb{Z}_\ell(t)) \subseteq (E^*/k^* \otimes \mathbb{Z}_\ell(t))^\epsilon,$$

for some constant $\epsilon \in \mathbb{Z}_\ell^*$, depending on $F$. The proof that $\epsilon$ is independent of $F$ and, finally, the proof of Theorem 1 are then identical to those in dimension two in [4].
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2. Basic Algebra and Geometry of Fields

Here we state some auxiliary facts used in the proof of our main theorem.

Lemma 2.1. Every function field over an algebraically closed ground field admits a normal model.

Lemma 2.2. For every one-dimensional subfield $E \subset K$ there is a canonical sequence of maps

$$X \xrightarrow{\pi_E} C' \xrightarrow{\mu_E} C,$$

where

- $\pi_E$ is dominant with irreducible generic fiber;
- $\mu_E$ is quasi-finite and dominant;
- $k(C') = E^K$ (the normal closure of $E$ in $K$) and $k(C) = E$.

Note that $C'$ and $C$ do not depend on the choice of a model $X$.

We call a divisor $p$-irreducible if it is a $p$-power of an irreducible divisor.

Lemma 2.3. Let $\pi : X \to C$ be a surjective map with irreducible generic fiber and $R \subset X$ an irreducible divisor surjecting onto $C$. Then the intersection $R \cdot \pi^{-1}(c)$ is $p$-irreducible for all but finitely many $c \in C$.

Proof. This is a positive-characteristic version of Bertini’s theorem.

Lemma 2.4. Let $\pi : T \to C$ be a separable map of degree $m$ with branch locus $\{c_1, \ldots, c_N\} \subset C$. Write

$$\pi^{-1}(c_j) = \sum_{r=1}^{m_j} e_{j,r} t_{j,r}, \quad t_{j,r} \in T, e_{j,r} \in \mathbb{N}, \quad \text{and} \sum_{r=1}^{m_j} e_{j,r} = m.$$

Let $e'_{j,r}$ be the maximal prime-to-$p$ divisor of $e_{j,r}$. Assume that

$$\sum_{r=1}^{m_j} (e'_{j,r} - 1) > m/2,$$

for all $j = 1, \ldots, N$. Then

$$g(T) > N - 3.$$
Proof. Hurwitz formula (for curves over a field of finite characteristics). □

Let $X \subset \mathbb{P}^N$ be a normal projective variety of dimension $n \geq 2$ over $k$. Consider the moduli space $\mathcal{M}(d)$ of complete intersection curves on $X$ of multidegree $d = (d_1, \ldots, d_{n-1})$. For $|d| \gg 0$ we have:

- for any codimension $\geq 2$ subvariety $Z \subset X$ there is a Zariski open subset of $\mathcal{M}(d)$ such that every curve $C$ parametrized by a point in this subset avoids $Z$ and intersects every irreducible divisor $D \subset X$.

Such families will be called **families of flexible curves**.

A Lefschetz pencil is a surjective map

$$\lambda : X \to \mathbb{P}^1$$

from a normal variety with irreducible fibers and normal generic fiber.

**Lemma 2.5.** Let $\lambda : X \to \mathbb{P}^1$ be a Lefschetz pencil on a normal projective variety. Then there exists an $m \in \mathbb{N}$ such that every irreducible normal fiber $D_t := \lambda^{-1}(t)$ contains a family of flexible curves of genus $\leq m$.

**Proof.** We can realize the fibers $D_t$ simultaneously as hyperplane sections in a fixed projective embedding and consider induced complete intersection curves. The degree calculation for $X$ yields a uniform genus estimate for corresponding flexible curves for all $D_t$. □

The following lemma is a consequence of the Merkuriev–Suslin theorem:

**Lemma 2.6.** [5, Lemma 25]. Let $K = k(X)$ be a function field of an algebraic variety of dimension $\geq 2$. Two functions $f_1, f_2 \in K^* / k^*$ are algebraically dependent if and only if the symbol in the (reduced) second Milnor $K$-group of $K$ vanishes:

$$(f_1, f_2) = 0 \in K^M_2(K) / \text{infinitely divisible elements}.$$  

3. Galois groups

Let $G^c_K$ the abelianization of the pro-$\ell$-quotient $G_K$ of the Galois group of a separable closure of $K = k(X)$,

$$G^c_K = G_K / [[G_K, G_K], G_K] \xrightarrow{\text{pr}} G^a_K$$

its canonical central extension and $\text{pr}$ the natural projection. By our assumptions, $G^a_K$ is a torsion-free $\mathbb{Z}_\ell$-module.
Definition 3.1. We say that $\gamma, \gamma' \in G^a_K$ form a commuting pair if for some (and therefore any) of their preimages $\tilde{\gamma} \in pr^{-1}(\gamma), \tilde{\gamma}' \in pr^{-1}(\gamma') \in G^c_K$, one has $[\tilde{\gamma}, \tilde{\gamma}'] = 0$. A subgroup $\mathcal{H}$ of $G^a_K$ is called liftable if any two elements in $\mathcal{H}$ form a commuting pair. A liftable subgroup is called maximal if it is not properly contained in any other liftable subgroup.

Definition 3.2. A fan $\Sigma^*_K = \{\sigma\}$ on $G^a_K$ is the set of all topologically noncyclic maximal liftable subgroups $\sigma \subset G^a_K$.

Notation 3.3. Let
\[ \mu_{\ell, n} := \{ \ell^n \mathbb{Z} \} \]
and
\[ \mathbb{Z}_\ell(1) = \lim_{n \to \infty} \mu_{\ell, n} \]
be the Tate twist of $\mathbb{Z}_\ell$. Write
\[ \hat{K}^* := \lim_{n \to \infty} K^*/(K^*)^{\ell^n} \]
for the $\ell$-adic completion of the multiplicative group $K^*$.

Theorem 3.4 (Kummer theory). For every $n \in \mathbb{N}$ we have a pairing
\[ [\cdot, \cdot]_n : G^a_K / \ell^n \times K^*/(K^*)^{\ell^n} \to \mu_{\ell, n} \]
\[ (\mu, f) \mapsto [\mu, f]_n := \mu(f)/f \]
which extends to a nondegenerate pairing
\[ [\cdot, \cdot] : G^a_K \times \hat{K}^* \to \mathbb{Z}_\ell(1). \]

Since $k$ is algebraically closed of characteristic $\neq \ell$ we can choose a non-canonical isomorphism
\[ \mathbb{Z}_\ell \simeq \mathbb{Z}_\ell(1). \]
From now on we will fix such a choice.

4. VALUATIONS

In this section we recall basic definitions and facts concerning valuations, and their inertia and decomposition subgroups of Galois groups (see [6]).

A (nonarchimedean) valuation $\nu = (\nu, \Gamma_\nu)$ on $K$ is a pair consisting of a totally ordered abelian group $\Gamma_\nu = (\Gamma_\nu, +)$ (the value group) and a map
\[ \nu : K \to \Gamma_{\nu, \infty} := \Gamma_\nu \cup \{\infty\} \]
such that
\begin{itemize}
  \item $\nu : K^* \to \Gamma_\nu$ is a surjective homomorphism;
\end{itemize}
• $\nu(\kappa + \kappa') \geq \min(\nu(\kappa), \nu(\kappa'))$ for all $\kappa, \kappa' \in K$;
• $\nu(0) = \infty$.

Every valuation of $K = k(X)$ restricts to a trivial valuation on $k = \mathbb{F}_p$.

Let $o_\nu$, $m_\nu$, and $K_\nu$ be the ring of $\nu$-integers in $K$, the maximal ideal of $o_\nu$ and the residue field $K_\nu := o_\nu/m_\nu$.

Basic invariants of valuations are: the $\mathbb{Q}$-rank $\text{rk}_\mathbb{Q}(\Gamma_\nu)$ of the value group $\Gamma_\nu$ and the transcendence degree $\text{tr} \deg_k(K_\nu)$ of the residue field. We have:

\begin{align}
\text{rk}_\mathbb{Q}(\Gamma_\nu) + \text{tr} \deg_k(K_\nu) \leq \text{tr} \deg_k(K).
\end{align}

A valuation on $K$ has an algebraic center $c_{\nu,X}$ on every model $X$ of $K$; there exists a model $X$ where the dimension of $c_{\nu,X}$ is maximal, and equal to $\text{tr} \deg_k(K_\nu)$. A valuation $\nu$ is called divisorial if

\begin{align}
\text{tr} \deg_k(K_\nu) = \dim(X) - 1;
\end{align}

it can be realized as the discrete rank-one valuation arising from a divisor on some normal model $X$ of $K$. We let $\mathcal{V}_K$ be the set of all nontrivial (nonarchimedean) valuations of $K$ and $\mathcal{D}\mathcal{V}_K$ the subset of its divisorial valuations.

It is useful to keep in mind the following exact sequences:

\begin{align}
1 \to o_\nu^* \to K^* \to \Gamma_\nu \to 1
\end{align}

and

\begin{align}
1 \to (1 + m_\nu)^* \to o_\nu^* \to K_\nu^* \to 1.
\end{align}

For every $\nu \in \mathcal{V}_K$ we have the diagram

\begin{align}
I_\nu^c \subseteq D_\nu^c \subseteq G_\nu^c \\
\downarrow \quad \downarrow \quad \downarrow
\end{align}

\begin{align}
I_\nu^a \subseteq D_\nu^a \subseteq G_\nu^a,
\end{align}

where $I_\nu^c, I_\nu^a, D_\nu^c, D_\nu^a$ are the images of the inertia and the decomposition group of the valuation $\nu$ in $G_\nu^c$, respectively, $G_\nu^a$; the left arrow is an isomorphism and the other arrows surjections. There are canonical isomorphisms

\begin{align}
D_\nu^c/I_\nu^c \simeq G_\nu^c \quad \text{and} \quad D_\nu^a/I_\nu^a \simeq G_\nu^a.
\end{align}

The group $D_\nu^c$ is the centralizer of $I_\nu^c = I_\nu$ in $G_\nu^c$, i.e., $I_\nu^a$ is the subgroup of elements forming a commuting pair with every element of $D_\nu^a$.

For divisorial valuations $\nu \in \mathcal{D}\mathcal{V}_K$, we have

\begin{align}
I_\nu^c = I_\nu^a \simeq \mathbb{Z}_\ell.
\end{align}
Kummer theory, combined with equations (4.2) and (4.3) yields

\[ I^a_{\nu} = \{ \gamma \in \text{Hom}(K^*, \mathbb{Z}_\ell) \mid \gamma \text{ trivial on } \sigma_{\nu}^* \} = \text{Hom}(\Gamma_{\nu}, \mathbb{Z}_\ell) \]

and

\[ D^a_{\nu} = \{ \gamma \in \text{Hom}(K^*, \mathbb{Z}_\ell) \mid \gamma \text{ trivial on } (1 + m_{\nu})^* \}. \]

In particular,

\[ \text{rk}_{\mathbb{Z}_\ell}(I^a_{\nu}) \leq \text{rk}_Q(\Gamma_{\nu}) \leq \text{tr deg}_k(K). \]

Two valuations \( \nu_1, \nu_2 \) are dependent if there exists a common coarsening valuation \( \nu \) (i.e., \( m_{\nu} \) is contained in both \( m_{\nu_1}, m_{\nu_2} \)), in which case

\[ D^a_{\nu_1}, D^a_{\nu_2} \subset D^a_{\nu}. \]

For independent valuations \( \nu_1, \nu_2 \) we have

\[ K^* = (1 + m_{\nu_1})^*(1 + m_{\nu_2})^*; \]

it follows that their decomposition groups have trivial intersection.

In [3, Proposition 6.4.1, Lemma 6.4.3 and Corollary 6.4.4] we proved:

**Proposition 4.1.** Every topologically noncyclic liftable subgroup \( \sigma \) of \( G_K^a \) contains a subgroup \( \sigma' \subseteq \sigma \) such that there exists a valuation \( \nu \in \mathcal{V}_K \) with

\[ \sigma' \subseteq I^a_{\nu}, \quad \sigma \subseteq D^a_{\nu}, \]

and \( \sigma / \sigma' \) topologically cyclic.

**Corollary 4.2.** For every \( \sigma \in \Sigma_K \) one has

\[ \text{rk}_{\mathbb{Z}_\ell}(\sigma) \leq \text{tr deg}_k(K). \]

**Proof.** By (4.7),

\[ \text{rk}_{\mathbb{Z}_\ell}(I^a_{\nu}) \leq \text{tr deg}_k(K). \]

We are done if \( \sigma = \sigma' \). Otherwise, \( D^a_{\nu} / I^a_{\nu} \) is nontrivial and \( \text{tr deg}_k(K_{\nu}) \geq 1 \). In this case, (4.7) and (4.1) yield that

\[ \text{rk}_{\mathbb{Z}_\ell}(\sigma') \leq \text{tr deg}_k(K) - 1, \]

and the claim follows. \( \square \)

**Corollary 4.3.** Assume that for \( \sigma_1, \sigma_2 \in \Sigma_K \) one has

\[ \sigma_1 \cap \sigma_2 \neq 0. \]

Then there exists a valuation \( \nu \in \mathcal{V}_K \) such that

\[ \sigma_1, \sigma_2 \subset D^a_{\nu}. \]
Proof. The valuations cannot be independent. Thus there exists a common coarsening.

This allows to recover the abelianized decomposition and inertia groups of valuations in terms of \( \Sigma_K \). Here is one possible description for divisorial valuations, a straightforward generalization of the two-dimensional case treated in [4, Proposition 8.3]:

**Lemma 4.4.** Let \( K = k(X) \) be the function field of an algebraic variety of dimension \( n \geq 2 \). Let \( \sigma_1, \sigma_2 \in \Sigma_K \) be liftable subgroups of rank \( n \) such that \( I := \sigma_1 \cap \sigma_2 \) is topologically cyclic. Then there exists a unique divisorial valuation \( \nu \) such that \( I = I^a_\nu \). The corresponding decomposition group \( D^a_\nu \subset G^a_K \) is the subgroup of elements forming a commuting pair with a topological generator of \( I^a_\nu \).

**Proof.** Let \( \nu_1, \nu_2 \in V_K \) be the valuations associated to \( \sigma_1, \sigma_2 \) in Proposition 4.1. By Corollary 4.3, there exists a valuation \( \nu \in V_K \) such that

\[
\sigma_j \subset D^a_{\nu_j} \subset D^a_\nu, \quad \text{for} \quad j = 1, 2.
\]

Let \( I^a_\nu \) be the corresponding inertia subgroup, the subgroup of elements commuting with all of \( D^a_\nu \). In particular, \( I^a_\nu \) commutes with all elements of \( \sigma_1 \) and \( \sigma_2 \). Since \( \sigma_1, \sigma_2 \) are maximal liftable subgroups of \( G^a_K \), we obtain that

\[
I^a_\nu \subseteq \sigma_1 \cap \sigma_2 = I \simeq \mathbb{Z}_\ell.
\]

Note that \( I^a_\nu \) cannot be trivial; otherwise, the residue field \( K_\nu \) would contain a liftable subgroup of rank \( n \), and have transcendence degree \( n \), by Corollary 4.2, which is impossible. It follows that \( \text{tr} \deg_k(K_\nu) = n - 1 \).

Now we apply Corollary 4.2 to

\[
\bar{\sigma}_j := \sigma_j/I^a_\nu \subset G^a_{K_\nu}, \quad \text{for} \quad j = 1, 2,
\]

liftable subgroups of rank \( n - 1 \). It follows that \( \text{tr} \deg_k(K_\nu) \geq n - 1 \), thus equal to \( n - 1 \), i.e., \( \nu \) is a divisorial valuation.

Conversely, an inertia subgroup \( I^a_\nu \) can be embedded into maximal liftable subgroups \( \sigma_1, \sigma_2 \) as above, e.g., by considering “flag” valuation with value group \( \mathbb{Z}^n \), with disjoint centers supported on the corresponding divisor \( D = D_\nu \subset X \).

The following is useful for the visualization of composite valuations:
Lemma 4.5. Let $\nu \in \mathcal{DV}_K$ be a divisorial valuation. There is a bijection between liftable subgroups $\sigma \in \Sigma_K$ with the property that

$$T^a_\nu \subset \sigma \subset \mathcal{D}^a_\nu$$

and liftable subgroups $\sigma_\nu \in \Sigma_{K_\nu}$.

**Proof.** We apply [4, Corollary 8.2]: let $\nu$ be a valuation of $K$ and $\iota_\nu \in T^a_\nu$. Let $\gamma \in G^a_K$ be such that $\iota_\nu$ and $\gamma$ form a commuting pair. Then $\gamma \in \mathcal{D}^a_\nu$. $\square$

In summary, under the assumptions of Theorem 1, we have obtained:

- a canonical isomorphism of completions $\Psi^* : \hat{K}^* \sim \hat{K}^*$ induced, by Kummer theory, from the isomorphism $\Psi : G^a_K \sim G^a_L$;
- a bijection on the set of inertia (and decomposition) subgroups of divisorial valuations

$$G^a_K \supset T^a_\nu \xrightarrow{\Psi} T^a_\nu \subset G^a_L.$$ 

Note that $K^*/k^* \subset \hat{K}^*$ determines a canonical topological generator $\delta_{\nu,K} \in T^a_\nu$, for all $\nu \in \mathcal{DV}_K$, by the condition that the restriction takes values in the integers

$$\delta_{\nu,K} : K^*/k^* \to \mathbb{Z} \subset \mathbb{Z}_\ell$$

i.e., that there exist elements $f \in K^*/k^*$ such that $\delta_{\nu,K}(f) = 1$. A topological generator of the procyclic group $T^a_\nu \simeq \mathbb{Z}_\ell$ is defined up to the action of $\mathbb{Z}^*_{\ell}$. We conclude that there exist constants

$$\varepsilon_\nu \in \mathbb{Z}^*_\ell, \quad \nu \in \mathcal{DV}_K = \mathcal{DV}_L$$

such that

$$\Psi(\delta_{\nu,K}) = \varepsilon_\nu \cdot \delta_{\nu,L}, \quad \forall \nu \in \mathcal{DV}_K.$$ 

The main difficulty is to show that there exists a conformally unique $\mathbb{Z}(\ell)$-lattice, i.e., a constant $\epsilon \in \mathbb{Z}^*_\ell$, unique modulo $\mathbb{Z}^*_\ell$, such that

$$\varepsilon_\nu = \epsilon, \quad \forall \nu \in \mathcal{DV}_K.$$ 

A proof of this fact will be carried out in Section 6.

Let $\nu$ be a divisorial valuation. Passing to $\ell$-adic completions in sequence (4.2) we obtain an exact sequence

$$1 \to \hat{\delta}^*_\nu \to \hat{K}^* \xrightarrow{\nu} \mathbb{Z}_\ell \to 0.$$
The sequence (4.3) gives rise to a surjective homomorphism
\[ \hat{\alpha}_\nu \to \hat{K}_\nu^*. \]
Combining these, we obtain a surjective homomorphism
\[ (4.9) \quad \text{res}_\nu : \text{Ker}(\nu) \to \hat{K}_\nu^*. \]
This homomorphism has a Galois-theoretic description, via duality arising from Kummer theory: We have
\[ I_\alpha^\nu \subset D_\alpha^\nu \subset G^a_K, \]
and
\[ \hat{K}_\nu^* = \text{Hom}(G^a_{K^\nu}, \mathbb{Z}_\ell) = \text{Hom}(D^a_\nu / I^a_\nu, \mathbb{Z}_\ell); \]
each \( \hat{f} \in \text{Ker}(\nu) \subset \hat{K}^* = \text{Hom}(G^a_K, \mathbb{Z}_\ell) \) gives rise to a well-defined element in \( \text{Hom}(D^a_\nu / I^a_\nu, \mathbb{Z}_\ell) \).

5. \( \ell \)-adic Analysis: Generalities

Here we recall the main issues arising in the analysis of \( \ell \)-adic completions of functions, divisors, and Picard groups of normal projective models of function fields \( K = k(X) \) (see [4, Section 11] for more details).

We have an exact sequence
\[ (5.1) \quad 0 \to K^*/k^* \xrightarrow{\text{div}_X} \text{Div}(X) \xrightarrow{\varphi} \text{Pic}(X) \to 0, \]
where \( \text{Div}(X) \) is the group of (locally principal) Weil divisors of \( X \) and \( \text{Pic}(X) \) is the Picard group. We will identify an element \( f \in K^*/k^* \) with its image under \( \text{div}_X \). Let \( \widehat{\text{Div}}(X) \) be the pro-\( \ell \)-completion of \( \text{Div}(X) \) and put
\[ \text{Div}(X)_\ell := \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \subset \widehat{\text{Div}}(X). \]
Every element \( \hat{f} \in \hat{K}^* \) has a representation
\[ \hat{f} = (f_n)_{n \in \mathbb{N}} \text{ or } f = f_0 f_1^\ell f_2^{\ell^2} \cdots, \]
with \( f_n \in K^* \). We have homomorphisms
\[ \begin{align*}
\text{div}_X : \hat{K}^* & \to \widehat{\text{Div}}(X), \\
\hat{f} & \mapsto \text{div}_X(\hat{f}) := \sum_{n \in \mathbb{N}} \ell^n \cdot \text{div}_X(f_n) = \sum_m \hat{a}_m D_m,
\end{align*} \]
where $D_m \subset X$ are irreducible divisors,
\[
\hat{a}_m = \sum_{n \in \mathbb{N}} a_{nm} \ell^n \in \mathbb{Z}_\ell, \quad a_{nm} \in \mathbb{Z}.
\]

Equation (5.1) gives rise to an exact sequence
\[
0 \to K^*/k^* \otimes \mathbb{Z}_\ell \xrightarrow{\text{div}_X} \text{Div}^0(X)_{\ell} \xrightarrow{\varphi_{\ell}} \text{Pic}^0(X)\{\ell\} \to 0,
\]
where
\[
\text{Div}^0(X)_{\ell} := \text{Div}(X)^0 \otimes \mathbb{Z}_\ell, \quad \text{and} \quad \text{Pic}^0(X)\{\ell\} = \text{Pic}^0(X) \otimes \mathbb{Z}_\ell
\]
is the $\ell$-primary component of the torsion group Pic$^0(X)$. The assignment
\[
\mathcal{T}_\ell(X) := \lim_{\leftarrow} \text{Tor}_1(\mathbb{Z}/\ell^n, \text{Pic}^0(X)\{\ell\}).
\]
is functorial:
\[
\mathcal{T}_\ell(Y) \to \mathcal{T}_\ell(X) \Rightarrow \mathcal{T}_\ell(Y) \to \mathcal{T}_\ell(X).
\]
We have $\mathcal{T}_\ell(X) \simeq \mathbb{Z}_{\ell}^{2g}$, where $g$ is the dimension of Pic$^0(X)$. Passing to pro-$\ell$-completions in (5.2) we obtain an exact sequence:
\[
0 \to \mathcal{T}_\ell(X) \to \hat{K}^* \xrightarrow{\text{div}_X} \hat{\text{Div}}^0(X) \to 0,
\]
\[
\text{since Pic}^0(X) \text{ is an } \ell\text{-divisible group. Note that all groups in this sequence are torsion-free. We have a diagram
\[
0 \to K^*/k^* \otimes \mathbb{Z}_\ell \xrightarrow{\text{div}_X} \text{Div}^0(X)_{\ell} \xrightarrow{\varphi_{\ell}} \text{Pic}^0(X)\{\ell\} \to 0
\]
\[
0 \to \mathcal{T}_\ell(X) \to \hat{K}^* \xrightarrow{\text{div}_X} \hat{\text{Div}}^0(X) \xrightarrow{\hat{\varphi}} 0.
\]
\]
Every $\nu \in \text{DV}_K$ gives rise to a homomorphism
\[
\nu : \hat{K}^* \to \mathbb{Z}_\ell.
\]
On a normal model $X$, where $\nu = \nu_D$ for some divisor $D \subset X$, $\nu(\hat{f})$ is the $\ell$-adic coefficient at $D$ of $\text{div}(\hat{f})$.

The following lemma generalizes [4, Lemmas 11.12 and 11.14] to normal varieties.

**Lemma 5.1.** Let $K$ be a function field over $k$ of transcendence degree $\geq 3$. Then there exists a normal projective model $X$ of $K$ such that for all birational maps $\hat{X} \to X$ from a normal variety $\hat{X}$ one has a canonical isomorphism
\[
\mathcal{T}_\ell(X) \to \mathcal{T}_\ell(\hat{X}).
\]
In particular, $T_{\ell}(X)$ is an invariant of $K$. Moreover, we have

\[(5.6)\quad T_{\ell}(X) = T_{\ell}(K) = \cap_{\nu \in DV_K} \text{Ker}(\nu) \subset \hat{K}^* .\]

Proof. For any projective $X$, its Albanese $\text{Alb}(X)$ is the maximal abelian variety such that

- there exists a morphism $X \to \text{Alb}(X)$ and
- $\text{Alb}(X)$ is generated, as an algebraic group, by the image of $X$.

This construction is functorial. Then there exists an abelian variety $\text{Alb}(K)$ which is maximal for all such models. Indeed, the dimension of $\text{Alb}(X)$ is bounded by the genus of a flexible curve on any birational model of $X$. Thus there exists a maximal $\text{Alb}(K)$ dominating all $\text{Alb}(X)$ and a class of normal models where $\text{Alb}(X) = \text{Alb}(K)$. It suffices to observe that $T_{\ell}(X) = T_{\ell}(\text{Alb}(K))$.

The second claim follows from the fact that every divisorial valuation can be realized as a divisor on a normal model $X$ of $K$.

\[\square\]

Lemma 5.2. Let $K = k(X)$ be the function field of a normal projective variety $X \subset \mathbb{P}^N$ of dimension $\geq 3$. For every divisorial valuation $\nu \in DV_K$ there is a canonical homomorphism:

$$\xi_{\nu,\ell} : T_{\ell}(K) \rightarrow T_{\ell}(K_{\nu}).$$

Assume that $\nu$ corresponds to an irreducible normal hyperplane section of $X$. Then $\xi_{\nu,\ell}$ is an isomorphism.

Proof. The map is induced from a canonical map of Albanese varieties (see [4, Lemma 11.12]). It suffices to apply Lefschetz’ theorem. \[\square\]

Lemma 5.3. Let $\lambda : X \rightarrow \mathbb{P}^1$ be a Lefschetz pencil on a normal variety of dimension $\geq 3$ and $D_t = \lambda^{-1}(t)$. Then:

1. For all but finitely many $t \in \mathbb{P}^1$,

$$\xi_{D_t,\ell} : T_{\ell}(X) \rightarrow T_{\ell}(D_t) ,$$

is an isomorphism.

2. For any $t \in \mathbb{P}^1$ and any surjection $D_t \rightarrow C_t$ onto a smooth projective curve we have $g(C_t) \leq \text{rk}_{K_t}(T_{\ell}(X))$.

Proof. Follows from standard facts for general hyperplane sections of normal varieties (see Lemma 5.2). \[\square\]

Lemma 5.4. Let $\pi : X \rightarrow C$ be a surjective map with irreducible fibers. Assume that $\hat{f} \in \text{Ker}(\nu)$ and that $\text{res}_\nu(\hat{f}) = 1 \in \hat{K}_\nu^*$, for infinitely many $\nu \in DV_K$ corresponding to fibers of $\pi$. Then $\hat{f}$ is induced from $\hat{k}(C)^*$. 
**Proof.** Assume that $\hat{f} \mod \ell^n$, for some $n \in \mathbb{N}$, contains a summand corresponding to a horizontal divisor $R$. By Lemma 2.3, $R$ intersects all but finitely many fibers $p^m$-transversally. In particular, $\text{div}_X(\hat{f})$ intersects infinitely many fibers nontrivially, contradiction to the assumption. Thus $\text{div}_X(\hat{f})$ is a sum of vertical divisors.

Hence $\hat{f} = \tau + \hat{g}$, where $\hat{g} \in \widehat{k(C)}^*$, and $\tau \in T_\ell(K)$. The triviality of $\tau$ on fibers $D_c = \pi^{-1}(c)$ implies that $\tau$ is induced from the image of $X$ in $\text{Alb}(X)/\text{Alb}(D_c)$. In particular, the triviality on infinitely many fibers implies that it is induced from the Jacobian $J(C)$ and hence $\hat{f} \in \widehat{k(C)}^*$. □

**Notation 5.5.** Let $X$ be a normal projective model of $K$. For $\hat{f} \in \hat{K}^*$ with
$$\text{div}_X(\hat{f}) = \sum_m \hat{a}_m D_m$$
we put
$$\text{supp}_K(\hat{f}) := \{ \nu \in \mathcal{DV}_K \mid \hat{f} \text{ nontrivial on } T_\nu \};$$
$$\text{supp}_X(\hat{f}) := \{ D_m \subset X \mid \hat{a}_m \neq 0 \};$$
$$\text{fibr}(\hat{f}) := \{ \nu \in \mathcal{DV}_K \mid \hat{f} \in \text{Ker}(\nu) \text{ and } \text{res}_\nu(\hat{f}) = 1 \in \hat{K}_\nu \},$$
where $\text{res}_\nu$ is the projection from Equation (4.9). Note that the finiteness of $\text{supp}_X(\hat{f})$ does not depend on the choice of the normal model $X$. Put
$$\text{supp}'_K(\hat{f}) := \text{fibr}(\hat{f}) \cup \text{supp}_K(\hat{f}).$$
If $X$ is a normal model of $K$ write
$$\text{supp}'_X(\hat{f}) \subset \text{supp}'_K(\hat{f})$$
for the subset of divisorial valuations realized by divisors on $X$. We have
$$\text{supp}'_K(\hat{f}) = \bigcup_X \text{supp}'_X(\hat{f}).$$

**Definition 5.6.** A $K$-divisor is a function
$$\mathcal{DV}_K \to \mathbb{Z}_\ell.$$ Each $\hat{f} \in \hat{K}^*$ defines a $K$-divisor by
$$\text{div}_K(\hat{f}) : \nu \mapsto [\delta_{\nu,K}, \hat{f}].$$

The different notions of support for elements in $\hat{K}^*$ introduced in Notation 5.5 extend naturally to $K$-divisors. The divisor of $\hat{f}$ on a normal model $X$ of $K$ coincides with the restriction of $\text{div}_K(\hat{f})$ to the set of divisorial valuations of $K$ which are realized by divisors on $X$. In particular, it has finite support on $X$ modulo $\ell^n$, for any $n \in \mathbb{N}$. (This fails for general $K$-divisors.)
Let $E \subset K$ be a one-dimensional subfield and $\pi_E : X \to C$ the corresponding surjective map with irreducible generic fiber. For all nontrivial $\hat{f}_1, \hat{f}_2 \in \hat{E}^*$, we have

$$\text{supp}'_{K}(\hat{f}_1) = \text{supp}'_{K}(\hat{f}_2).$$

This gives a well-defined invariant of $\hat{E}^*$. We have a decomposition

$$(5.7) \quad \text{supp}'_{K}(\hat{E}^*) = \bigsqcup_{c \in C} \text{supp}'_{K,c}(\hat{E}^*),$$

where $\text{supp}'_{K,c}(\hat{E}^*)$ are minimal nonempty subsets of the form

$$\text{supp}_{K}(\hat{f}_1) \cap \text{supp}_{K}(\hat{f}_2)$$

contained in $\text{supp}'_{K}(\hat{E})$; these correspond to sets of irreducible divisors supported in $\pi_{E}^{-1}(c)$, for $c \in C(k)$. Note that $\text{supp}'_{K}(\hat{E}^*)$ depends only on the normal closure of $E$ in $\hat{K}$. On the other hand, the decomposition (5.7) is preserved only under purely inseparable extensions of $E$. We formalize this discussion in the following definition.

**Definition 5.7.** A formal projection is a triple

$$\pi_{\hat{E}} = (C, \{R_c\}_{c \in C}, Q),$$

where $C$ is an infinite set, $\{R_c\}_{c \in C}$ is a set of $K$-divisors, and $Q \subset \hat{K}^*$ a subgroup of $\Z_{\ell}$-rank at least two satisfying the following properties:

1. for all $\hat{f}_1, \hat{f}_2 \in Q$ one has $\text{supp}'_{K}(\hat{f}_1) = \text{supp}'_{K}(\hat{f}_2)$;
2. $\text{supp}_{K}(R_{c_1}) \cap \text{supp}_{K}(R_{c_2}) = \emptyset$, for all pairs of distinct $c_1, c_2 \in C$;
3. for all nontrivial $\hat{f} \in Q$ one has

$$\text{div}_{K}(\hat{f}) = \sum_{c \in C} a_c R_c, \quad a_c \in \Z_{\ell},$$

and

$$\cup_{c \in C} \text{supp}_{K}(R_c) = \text{supp}'_{K}(\hat{f});$$

4. for all $c_1, c_2 \in C$ there exists an $m \in \N$ such that

$$m(R_{c_1} - R_{c_2}) = \text{div}_{K}(\hat{f}),$$

for some $\hat{f} \in Q$.

**Example 5.8.** A one-dimensional subfield $E = k(C) \subset K$ defines a formal projection $\pi_{\hat{E}} = (C, \{R_c\}_{c \in C}, Q)$, with $C$ the set of $k$-points of the image of $\pi_{E}$, $R_c$ the intrinsic $K$-divisors over $c \in C$, and $Q = \hat{E}^*$. 

Note that for normally closed subfields $E \subset K$, the corresponding subgroup $Q$ is maximal, for subgroups of $\hat{K}^*$ appearing in formal projections.

**Lemma 5.9.** The formal divisor $\text{div}_X(R_c)$ is finite mod $\ell^n$ for any model $X$.

**Proof.** The support of $\Psi^*(\hat{f})$ mod $\ell^n$ is finite for all $n \in \mathbb{N}$. Now observe that the $K$-divisors $R_c$ have disjoint support in $\text{supp}_K(\hat{Q})$, thus have no components in common. \qed

6. **ONE-DIMENSIONAL SUBFIELDS**

We recall the setup of Theorem 1:

$$\Psi : G^a_K \to G^a_L.$$ 

Our goal here is to show:

$$\hat{L}^* \xrightarrow{\Psi^*} \hat{K}^*$$

$$L^*/l^* \xrightarrow{} (K^*/k^*)^{\epsilon}$$

for some constant $\epsilon$. We know that $g \in K^*/k^* \otimes \mathbb{Z}_\ell$ have finite support supp$_X(g)$, on every normal model $X$ of $K$. In the second half of this section we will prove:

**Proposition 6.1** (Finiteness of support). For all $f \in L^*/l^*$ and all normal models $X$ of $K$ the support supp$_X(\Psi^*(f))$ is finite.

Assuming this, we will prove:

**Proposition 6.2** (Image of $\Psi^*$). For all $f \in L^*/l^*$ there exist a function $g \in K^*/k^*$ and constants $N \in \mathbb{N}$, $\alpha \in \mathbb{Z}_\ell$ such that

(6.1) $\Psi^*(f)^N = g^\alpha$.

Moreover, there exists a constant $\epsilon \in \mathbb{Z}_\ell^*$ such that

$$\Psi^*(l(f)^*/l^* \otimes \mathbb{Z}_\ell) \subseteq (k(g)^*/k^* \otimes \mathbb{Z}_\ell)^\epsilon.$$ 

Considerations in Section 4 imply that under the assumptions of Theorem 1 we have a canonical commutative diagram, for every $\nu \in \mathcal{DV}_K$:
Proof of Proposition 6.2. Let $X$ be a normal projective model of $K$ and put $\hat{f} := \Psi^*(f)$. By Proposition 6.1, we may assume that $\text{supp}_X(\hat{f})$ is finite, i.e.,

$$\text{div}(\hat{f}) = \sum_{j \in J} d_j D_j,$$

where $J$ is a finite set, $d_j \in \mathbb{Z}_\ell$ and $D_i$ are irreducible divisors on $X$. Then there exists an $N \in \mathbb{N}$ such that $\hat{f}^N \in \text{Div}^0(X)_\ell \subset \hat{\text{Div}}^0(X)$. By (5.5), we have

$$\hat{f}^N = t_{\hat{f}} \cdot \prod_{i \in I} g_i^{a_i},$$

with $I$ a finite set, $a_i \in \mathbb{Z}_\ell$ linearly independent over $\mathbb{Z}_\ell$, $g_i \in K^*/k^*$ multiplicatively independent, and $t_{\hat{f}} \in T_\ell(K)$.

The projective model $X$ contains a hyperplane section $D \subset X$ such that $T_\ell(K) = T_\ell(X) = T_\ell(D)$, under the natural restriction isomorphism $\xi_{D, \ell}$ from Lemma 5.3, and the restrictions of $g_i$ to $D$ are multiplicatively independent in $k(D)^*/k^* = K^*/k^*$, where $\nu = \nu_D$.

By the construction and the inductive assumption, we have $\text{res}_\nu(\hat{f}^N) = g_{\nu}^{b_{\nu}}$, where $b_{\nu} \in \mathbb{Z}_\ell$, $g_{\nu} \in K^*_\nu$:

$$\text{res}_\nu(\hat{f}^N) = \text{res}_\nu(t_{\hat{f}}) \cdot \prod_{i \in I} \text{res}_\nu(g_i)^{a_i} = g_{\nu}^{b_{\nu}}.$$

In particular, $\text{res}_\nu(t_{\hat{f}}) = 1$ and hence $t_{\hat{f}} = 1$. Since $\text{res}_\nu(g_i) \in K^*_\nu$ are independent, it follows that $\#I = 1$ and

$$\hat{f}^N = g^a, \quad g \in K^*/k^*, \quad a \in \mathbb{Z}_\ell.$$

This proves the first claim.
The function $g \in K^*/k^*$ defines a map $\pi : X \to C$ from some normal model of $K$ onto a curve, with generically irreducible fibers. For each $h \in l(f)^*/l^*$, consider $\text{div}_X(\Psi^*(h)) \subset \hat{\text{Div}}^0(X)$. Then divisors in $\text{div}_X(\Psi^*(h))$ are $\pi$-vertical. Indeed, the restriction of $g$ to a $\pi$-horizontal component $D$ would be defined and nontrivial. On the other hand, the restriction of $f$ to $D$ is either not defined or trivial, contradiction. By Lemma 5.4, $\Psi^*(h) \in \hat{k(C)^*} = \hat{k(g)^*}$.

Let $\nu = \nu_D$ be a divisorial valuation such that $f$ is defined and nontrivial on $D$. Then

$$f \in \hat{L}^*_\nu \supset l(f)^*/l^* \otimes \mathbb{Z}((\ell))$$

By the inductive assumption, this implies that there exists a constant $\epsilon \in \mathbb{Z}^*_\ell$ such that

$$\Psi^*(l(f)^*/l^* \otimes \mathbb{Z}((\ell))) \subseteq (k(g)^*/k^* \otimes \mathbb{Z}((\ell)))^\epsilon,$$

(see, e.g., [4, Proposition 13.1]).

We now prove Proposition 6.1. Fix a normal projective model $Y$ of $L$. The subfield $F = l(f)$ determines a surjective map $\pi_F : Y \to C$ with irreducible generic fibers. For each $c \in C$ we have an intrinsically defined formal sum

$$(6.2) \quad R_c = \sum_{\nu \in \mathcal{DV}_{L,c}} a_{c,\nu} R_{c,\nu}, \quad a_{c,\nu} \in \mathbb{N},$$

where $\mathcal{DV}_{L,c} \subset \mathcal{DV}_L = \mathcal{DV}_K$ is the subset of divisorial valuations supported in the fiber over $c$, $R_{c,\nu}$ is a divisor on some model $\tilde{Y} \to Y$ realizing $\nu$, and $a_{c,\nu}$ are local degrees. Note that $R_c$ do not depend on the model $Y$, and that $R_{c_1}$ and $R_{c_2}$ have no common components, for $c_1 \neq c_2$. Furthermore, the sets $\mathcal{DV}_{L,c}$ have an intrinsic Galois-theoretic characterization in terms of $\hat{F}^*$: these are minimal nonempty subsets of the form

$$\text{supp}^*_K(\hat{f}_1) \cap \text{supp}^*_K(\hat{f}_2), \quad f_1, f_2 \in \hat{F}^*,$$

contained in $\text{supp}^*_K(\hat{F}^*)$.

For each model $\tilde{Y} \to Y$ we have a map

$$R_c \mapsto R_{\tilde{Y},c} := \sum_{\nu : D_\nu \in \text{Div}(\tilde{Y})} a_{c,\nu} R_{c,\nu},$$
the fiber over $c$. The divisor of a function $f \in F^*/l^*$ on this model can be written as a finite sum
\[
\text{div}_Y(f) = \sum n_c R_{\tilde{Y},c}, \quad n_c \in \mathbb{N}.
\]

Given $\{\delta_{\nu,L}\}$, each $\hat{f} \in \hat{L}^*$ defines a $\mathbb{Z}_l$-valued function on $\mathcal{D}\mathcal{V}_L$ by the Kummer-pairing from Theorem 3.4
\[
\mathcal{D}\mathcal{V}_L \to \mathbb{Z}_l, \quad \nu \mapsto [\delta_{\nu,L}, \hat{f}].
\]
Similarly, each $R_c$ defines a function on $\mathcal{D}\mathcal{V}_L$ by setting
\[
\nu \mapsto \delta_{\nu,L} \cdot R_c = \delta_{\nu,L}(t),
\]
where $t$ is a local parameter along $c$ if $\nu$ is supported over $c$, and $\nu \mapsto 0$, otherwise.

For $\hat{f} \in \hat{F}^* \subset \hat{L}^*$ write
\[
\text{div}_C(\hat{f}) = \sum_{c \in C} b_{\hat{f},c} c, \quad b_{\hat{f},c} \in \mathbb{Z}_l,
\]
with decreasing coefficients $b_{\hat{f},c}$. Then (6.3) is given by
\[
\nu \mapsto b_{\hat{f},c} a_{\nu,c}.
\]

We face the following difficulty: we don’t know the image $\Psi^*(F^*/l^*)$ in $\hat{K}^*$, and in particular, we don’t know that $\Psi^*(R_c)$, resp. $\Psi^*(R_{\tilde{Y},c})$, as functionals on $\mathcal{D}\mathcal{V}_K$, correspond to fibers of any fibration on a model $X$ of $K$. However, we know the “action” of $\Psi^*$ on the coefficients in Equation (6.2):
\[
a_{c,\nu} \mapsto \varepsilon^{-1}_{\nu} a_{c,\nu}.
\]

**Lemma 6.3.** Either there is a nonconstant $f \in F^*/l^*$ such that $\text{supp}_X(\Psi^*(f))$ is finite or there is at most one $c \in C$, where $C$ corresponds to $F$, such that $\Psi^*(R_c)$ has finite support on every model $X$ of $K$.

**Proof.** Let $c_1, c_2 \in C$ be distinct points such that
\[
\text{supp}_X(\Psi^*(R_{c_1})) \cup \text{supp}_X(\Psi^*(R_{c_2}))
\]
is finite. Then there is a function $f$ with divisor supported in this set, thus finite $\text{supp}_X(\Psi^*(f))$. \hfill $\square$

**Proof of Proposition 6.1.** By contradiction. Assume that $\text{supp}_X(\Psi^*(f))$ is infinite. An argument as in the proof of Proposition 6.2 shows that the same holds for every $h \in l(f)^*/l^*$. 

Fix a Lefschetz pencil \( \lambda : X \rightarrow \mathbb{P}^1 \) such that for almost all fibers \( D_t \) of \( \lambda \) we have a well-defined
\[
\text{res}_\nu : l(f)^*/l^* \to L^*_\nu \xrightarrow{\Psi^*} \hat{K}^*_{\nu_t},
\]
where \( \nu_t \) is the divisorial valuation corresponding to \( D_t \). By the inductive assumption, there exist one-dimensional closed subfields \( E_t = k(C_t) \subset k(D_t) = K_{\nu_t} \) such that
\[
\Psi^*(\text{res}_{\nu_t}(l(f)^*/l^*) \otimes \mathbb{Z}(\ell)) \subseteq (E_t^* \otimes \mathbb{Z}(\ell))^{\ell_t}, \quad \ell_t \in \mathbb{Z}^*_t.
\]
We have an induced surjective map
\[
\pi_t : D_t \to C_t
\]
as in Lemma 2.2. Passing to a finite purely-inseparable cover of \( C_t \) we may assume that \( \pi_t \) is separable (this effects the constant \( \epsilon \) by multiplication by a power of \( p \) which is in \( \mathbb{Z}^*_t \)). We identify the sets \( C(k) \) and \( C_t(k) \), set-theoretically.

Fix a family of flexible curves \( \{T_t\} \) uniformly on all but finitely many \( D_t \) as in Lemma 2.5 and let \( m \) be the bound on the genus of these curves obtained in this Lemma. Put \( N := m + 4 \) and choose \( c_1, \ldots, c_N \in C_t(k) = C(k) \) such that supp \( X(R_{c_j}) \) is infinite for all \( j \), this is possible by Lemma 6.3.

For each \( c_j \) express the fiber over \( c_j \) as
\[
R_{c_j} := \sum_{e=0}^\infty \ell^e R_{c_j,e}, \quad R_{c_j,e} := \sum_{i \in I_{c,j}} \epsilon_{i,e,j} R_{i,e,j},
\]
where \( I_{c,j} \) are finite, and \( R_{i,e,j} \) irreducible divisors over \( c_j \), and \( \epsilon_{i,e,j} \in \mathbb{Z}^*_t \) (see Lemma 5.9). Let \( S_{c_j,e} = \cup R_{i,e,j} \) be the support of \( R_{c_j,e} \). Note that \( T_t \) intersect all \( S_{c_j,e} \) and write \( d_{j,e} := \deg(S_{c_j,e} \cdot T_t) \) for the degree of the intersection.

Choose \( M \) such that for all \( j = 1, \ldots, N \) one has
\[
d_{j,0} < \sum_{e=1}^M d_{j,e}, \quad (6.4)
\]
this is possible since the number of components over all \( c_j \) is infinite. Using Lemma 2.3 choose \( t \) so that the intersections
\[
R_{i,e,j,t} := D_t \cdot R_{i,e,j}
\]
are \( p \)-irreducible and pairwise distinct, this holds for all but finitely many \( t \). Choose a flexible curve \( T_t \subset D_t \) such that
- \( T_t \) does not pass through the points of indeterminacy of \( \pi_t : D_t \to C_t \);
- \( T_t \) is not contained in any of the \( R_{i,e,j,t} \).
• $T_t$ does not pass through pairwise intersections of these divisors.

Consider the restriction

$$
\pi_t : T_t \to C_t.
$$

By the choice of $T_t$, the number of nonramified points over each $c_j$ is at most $d_{j,0}$. On the other hand, the ramification index over $c_j$ is at least $\ell \sum_{e=1}^{m} d_{j,e}$.

By the choice (6.4), combined with Hurwitz formula in Lemma 2.4, we obtain that $g(T_t) > m$, contradicting the universal bound. □

**Proposition 6.4.** There exists a constant $\epsilon \in \mathbb{Z}_\ell^*$ such that

$$(6.5) \quad \Psi^*(L^* / l^* \otimes \mathbb{Z}(\ell)) = (K^* / k^* \otimes \mathbb{Z}(\ell))^{\epsilon}.$$  

**Proof.** By Proposition 6.2, for each one-dimensional subfield $F = l(f) \subset L$ there exists a one-dimensional subfield $E = k(g)$ and a constant $\epsilon_F \in \mathbb{Z}_\ell^*$ such that

$$\Psi^*(F^* / l^* \otimes \mathbb{Z}(\ell)) \subseteq (E^* / k^* \otimes \mathbb{Z}(\ell))^{\epsilon_F}.$$  

Moreover, for $f_1, f_2 \in L^*/l^*$ we have $\Psi^*(f_j)^{N_j} = g_j^{\epsilon_j}$, for some $N_j \in \mathbb{N}$ and $\epsilon_j \in \mathbb{Z}_\ell$. It follows that the symbol $(f_1, f_2)$ is infinitely divisible in $K^*_M(L)$ if and only if $(g_1, g_2)$ is infinitely divisible in $K^*_M(K)$. Thus $f_1, f_2$ are algebraically dependent if and only if $g_1, g_2$ are algebraically dependent, see Lemma 2.6. In particular, if $f_1, f_2$ are not powers of the same element in $L^*$ the same holds for $g_1, g_2$, i.e., the divisors of $g_1, g_2$, on any model $X$ of $K$, are not proportional. We have

$$\Psi^*(f_1)^{N_1} = g_1^{\alpha_1}, \quad \Psi^*(f_2)^{N_2} = g_2^{\alpha_2}, \quad \text{and} \quad \Psi^*(f_1 f_2)^{N_{12}} = g_{12}^{\alpha_{12}}.$$  

We need to show that $\alpha_1, \alpha_2, \alpha_{12}$ span a 1-dimensional lattice, modulo $\mathbb{Z}(\ell)$. We have

$$g_{12} = g_1^{\alpha_1 / \alpha_{12} N_1} g_2^{\alpha_2 / \alpha_{12} N_2}, \quad \text{for some } N_1, N_2 \in \mathbb{Z}(\ell).$$

In particular, for any divisorial valuation $\nu$ in the support of $g_{12}$ the integral coefficient $b_{12}(\nu) = [\delta_{\nu,K}, g_{12}] \in \mathbb{Z}$ equals

$$b_1(\nu) \alpha_1 / \alpha_{12} N_1 + b_2(\nu) \alpha_2 / \alpha_{12} N_2, \quad \text{for some } b_1(\nu), b_2(\nu) \in \mathbb{Z}.$$  

Since the divisors of $g_1, g_2$ are not proportional the rank of the corresponding system of equations, as we vary over $\nu$, is at least 2. Hence both $\alpha_1 / \alpha_{12} N_1, \alpha_2 / \alpha_{12} N_2$ are rational, as claimed. □
7. Proof

In this section we prove our main theorem.

Step 1. We have a nondegenerate pairing
\[ G^a_K \times \hat{K}^* \to \mathbb{Z}_\ell(1). \]
This implies a canonical isomorphism
\[ \Psi^* : \mathit{L}^* \to \hat{K}^*. \]

Step 2. By assumption, \( \Psi : G^a_K \to G^a_L \) is bijective on the set of liftable subgroups, in particular, it maps liftable subgroups \( \sigma \in \Sigma_K \) to a liftable subgroups of the same rank. In Section 4 we identify intrinsically the inertia and decomposition groups of divisorial valuations:
\[ \mathcal{I}^a_\nu \subset \mathcal{D}^a_\nu \subset G^a_K : \]
every liftable subgroup \( \sigma \in \Sigma_K \) contains an inertia element of a divisorial valuation (which is also contained in at least one other \( \sigma' \in \Sigma_K \)). The corresponding decomposition group is the “centralizer” of the (topologically) cyclic inertia group (the set of all elements which “commute” with inertia). This identifies \( \mathcal{D}V_K = \mathcal{D}V_L \).

By [4, Section 17, Step 7 and 8], an isomorphism
\[ \Psi^* : G^a_K \to G^a_L \]
of abelianized Galois group of function fields \( K = k(X) \) and \( L = l(Y) \) of surfaces over algebraic closures of finite fields of characteristic \( \neq \ell \) implies the existence of a constant \( \epsilon \in \mathbb{Z}_\ell^* \) and a canonical isomorphism
\[ L^*/l^* \otimes \mathbb{Z}_\ell \supset \bigcup_{n \in \mathbb{N}} (L^*/l^*)^{1/p^n} \simeq \bigcup_{n \in \mathbb{N}} (K^*/k^*)^\epsilon/p^n \subset K^*/k^* \otimes \mathbb{Z}_\ell. \]
By the induction hypothesis, we may assume that this holds in dimension \( \leq n - 1 \): Once we have identified decomposition and inertia subgroups of divisorial valuations, we have, for each \( \nu \in \mathcal{D}V_K \), an intrinsically defined sublattice
\[ \Psi^*(L^*_\nu/l^*) = (K^*_\nu/k^*)^\epsilon \subset \hat{K}^*_\nu \]
of elements of the form \( f^\epsilon \), with \( f \in K^*_\nu/k^* \) and \( \epsilon \in \mathbb{Z}_\ell^* \) in the completion of the residue field. Using Proposition 6.4, we prove that the same holds for the image of \( L^*/l^* \) in \( \hat{K}^* \).
Thus we obtained an isomorphism
\[ \epsilon^{-1} \cdot \Psi^* : L^*/l^* \otimes \mathbb{Z}_\ell \to K^*/k^* \otimes \mathbb{Z}_\ell \]
which maps multiplicative groups of one-dimensional subfields of \( L \) into multiplicative groups of one-dimensional subfields of \( K \). The same holds for multiplicative groups of subfields of transcendence degree two (see Proposition 6.4 and its proof; Lemma 2.6 allows to characterize algebraically independent elements). To conclude the proof it suffices to apply the inductive hypothesis, the case of surfaces: in [4] we showed that the additive structure is canonically encoded in these data.

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