SOME INEQUALITIES FOR A CERTAIN SUBCLASS OF STARLIKE FUNCTIONS

R. KARGAR, H. MAHZOON AND N. KANZI

Abstract. In 2011, Sokół (Comput. Math. Appl. 62, 611–619) introduced and studied the class $SK(\alpha)$ as a certain subclass of starlike functions, consists of all functions $f$ ($f(0) = 0 = f'(0) - 1$) which satisfy in the following subordination relation:

$$zf'(z)/f(z) \prec 3 + (\alpha - 3)z - \alpha z^2 \quad |z| < 1,$$

where $-3 < \alpha \leq 1$. Also, he obtained some interesting results for the class $SK(\alpha)$. In this paper, some another properties of this class, including infimum of $\text{Re} f(z)/z$, order of strongly starlikeness, the sharp logarithmic coefficients inequality and the sharp Fekete-Szegő inequality are investigated.

1. Introduction

Let $A$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots,$$

which are analytic and normalized in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $A$ consisting of all univalent functions $f(z)$ in $\Delta$ is denoted by $S$. A function $f \in S$ is called starlike (with respect to 0), denoted by $f \in S^*$, if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0,1]$. The class $S^*(\gamma)$ of starlike functions of order $\gamma \leq 1$, is defined by

$$S^*(\gamma) := \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \gamma, \ z \in \Delta \right\}.$$  

Note that if $0 \leq \gamma < 1$, then $S^*(\gamma) \subset S$. Moreover, if $\gamma < 0$, then the function $f$ may fail to be univalent. A function $f \in S$ that maps $\Delta$ onto a convex domain, denoted by $f \in K$, is called a convex function. Also, the class $K(\gamma)$ of convex functions of order $\gamma \leq 1$, is defined by

$$K(\gamma) := \left\{ f \in A : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \gamma, \ z \in \Delta \right\}.$$  

In particular, we denote $S^*(0) \equiv S^*$ and $K(0) \equiv K$. The classes $S^*(\gamma)$ and $K(\gamma)$ introduced by Robertson (see [13]). Also, as usual, let

$$S^*_t(\gamma) := \left\{ f \in A : \left| \text{arg} \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\pi \gamma}{2}, \ z \in \Delta \right\},$$

be the class of strongly starlike functions of order $\gamma$ ($0 < \gamma \leq 1$) (see [13]). We note that $S^*_t(\gamma) \subset S^*$ for $0 < \gamma < 1$ and $S^*_t(1) \equiv S^*$. Define by $Q(\gamma)$, the class of all functions $f \in A$ so that satisfy the condition

$$\text{Re} \left( \frac{f'(z)}{z} \right) > \gamma \quad (0 \leq \gamma < 1).$$

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We denote by $B$ the class of analytic functions $w(z)$ in $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$, ($z \in \Delta$). If $f$ and $g$ are two of the functions in $A$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$, if there exists a $w \in B$ such that $f(z) = g(w(z))$, for all $z \in \Delta$.

Furthermore, if the function $g$ is univalent in $\Delta$, then we have the following equivalence:

$$f(z) \prec g(z) \iff (f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta)).$$

Also $|w(z)| \leq |z|$, by Schwarz’s lemma and therefore

$$\{f(z) : |z| < r \} \subset \{g(z) : |z| < r \} \quad (0 < r < 1).$$

It follows that

$$\max_{|z| \leq r} |f(z)| \leq \max_{|z| \leq r} |g(z)| \quad (0 < r < 1).$$

We now recall from [16], a one-parameter family of functions as follows:

$$p_b(z) := \frac{1}{1 - (1 + b)z + bz^2} \quad (z \in \Delta).$$

We note that if $|b| < 1$, then

$$\Re\{p_b(z)\} > \frac{1 - 3b}{2(1 - b)^2},$$

and if $b \in [-1/3, 1)$, then

$$\frac{1 - 3b}{2(1 - b)^2} < \Re\{p_b(e^{i\varphi})\} \leq \frac{1}{2(1 + b)} = p_b(-1) \quad (0 < \varphi < 2\pi).$$

Also, if $b \in [-1/3, 1]$, then the function $p_b$ defined in (1.3) is univalent in $\Delta$ and has no loops when $-1/3 \leq b < 1$.

By putting $b = -\alpha/3$ in the function (1.3), we have:

$$\tilde{q}_\alpha(z) := \frac{3}{3 + (\alpha - 3)z - \alpha z^2} \quad (z \in \Delta).$$

The function $\tilde{q}_\alpha(z)$ is univalent in $\Delta$ when $\alpha \in (-3, 1]$ (see Figure 1 for $\alpha = 1$).

![Figure 1. The graph of $\tilde{q}_\alpha(\Delta)$ for $\alpha = 1$](image)

Note that

$$\tilde{q}_\alpha(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$
where
\[
B_n = 3^{n+1} \left[ 1 + (-1)^n (\alpha/3)^{n+1} \right] \quad (n = 1, 2, \ldots).
\]

Over the years, the definition of a certain subclass of analytic functions by using the subordination relation has been investigated by many works including (for example) [5], [7], [8], [11], [12], [15] and [17]. We now recall from [16], the following definition which is used from subordination.

**Definition 1.1.** The function \( f \in A \) belongs to the class \( SK(\alpha) \), \( \alpha \in (-3, 1] \), if it satisfies the condition
\[
zf'(z) \prec \tilde{q}_\alpha(z) \quad (z \in \Delta),
\]
where \( \tilde{q}_\alpha \) is given by (1.4).

Since \( \Re \{ \tilde{q}_\alpha(z) \} > 9(1+\alpha)/2(3+\alpha)^2 \), therefore if \( f \in SK(\alpha) \), then
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \frac{9(1+\alpha)}{2(3+\alpha)^2} \quad (z \in \Delta).
\]

This means that if \( f \in SK(\alpha) \), then it is starlike of order \( \gamma \) where \( \gamma = 9(1+\alpha)/2(3+\alpha)^2 \). Also, \( SK(\alpha) \subset S^* \) when \(-1 \leq \alpha < 1\), \( SK(0) \equiv S^*(1/2) \), \( SK(1) \equiv S^*(9/16) \) and \( SK(-1) \equiv S^* \).

We denote by \( P \) the well-known class of analytic functions \( p(z) \) with \( p(0) = 1 \) and \( \Re(p(z)) > 0 \), \( z \in \Delta \).

For the proof of our results, we need the following Lemmas.

**Lemma 1.1.** [10, p.35] Let \( \Xi \) be a set in the complex plane \( \mathbb{C} \) and let \( b \) be a complex number such that \( \Re(b) > 0 \). Suppose that a function \( \psi : \mathbb{C}^2 \times \Delta \to \mathbb{C} \) satisfies the condition:

\[
\psi(\rho, \sigma; z) \notin \Xi,
\]
for all real \( \rho, \sigma \leq -|b - i\rho|^2/(2\Re b) \) and all \( z \in \Delta \). If the function \( p(z) \) defined by \( p(z) = b + b_1 z + b_2 z^2 + \cdots \) is analytic in \( \Delta \) and if

\[
\psi(p(z), zp'(z); z) \in \Xi,
\]
then \( \Re(p(z)) > 0 \) in \( \Delta \).

**Lemma 1.2.** [9] Let the function \( g(z) \) given by
\[
g(z) = 1 + c_1 z + c_2 z^2 + \cdots ,
\]
be in the class \( P \). Then, for any complex number \( \mu \)
\[
|c_2 - \mu c_1^2| \leq 2\max\{1, |2\mu - 1|\}.
\]
The result is sharp.

The structure of the paper is the following. In Section [2], at first, we obtain a lower bound for the \( \Re \{ zf'(z)/f(z) \} \) and by using it we get \( S^* \subset Q(1/2) \). In the sequel, we obtain the order of strongly starlikeness for the functions which belong to the class \( SK(\alpha) \). In Section [3] sharp coefficient logarithmic inequality and sharp Fekete-Szegö inequality are obtained.
2. Main results

The first result is the following. By using the Theorem 2.1 (below), we get the well-known result about the starlike univalent functions (Corollary 2.1).

**Theorem 2.1.** Let \( f \in A \) be in the class \( SK(\alpha) \) and \( 0 \leq \alpha \leq 1 \). Then

\[
\text{Re}\left(\frac{f(z)}{z}\right) > \gamma(\alpha) := \frac{2\alpha^2 + 3\alpha + 9}{3(\alpha^2 + 3\alpha + 6)} \quad (z \in \Delta).
\]

That is means that \( SK(\alpha) \subset Q(\gamma(\alpha)) \).

**Proof.** Put \( \gamma(\alpha) := \gamma \). Thus \( 0 < \gamma \leq 1/2 \) when \( 0 \leq \alpha \leq 1 \). Let \( p \) be defined by

\[
p(z) = \frac{1 - (1 - \gamma)p(z)}{1 - \gamma} \quad (z \in \Delta).
\]

Then \( p \) is analytic in \( \Delta \), \( p(0) = 1 \) and

\[
zf'(z) = 1 + \frac{(1 - \gamma)p'(z)}{(1 - \gamma)p(z) + \gamma} = \psi(p(z), \gamma).
\]

where

\[
\psi(a, b) := 1 + \frac{(1 - \gamma)a}{(1 - \gamma)a + \gamma}.
\]

By (1.8), we define \( \Omega_{\alpha'} \) as follows:

\[
\{\psi(p(z), \gamma) : z \in \Delta\} \subset \{w \in \mathbb{C} : \text{Re}\{w\} > \alpha'\} =: \Omega_{\alpha'},
\]

where \( \alpha' = 9(1 + \alpha)/2(3 + \alpha)^2 \). For all real \( \rho \) and \( \sigma \), which \( \sigma \leq -\frac{1}{2}(1 + \rho^2) \), we have

\[
\text{Re}\{\psi(i\rho, \sigma)\} = \text{Re}\left\{1 + \frac{(1 - \gamma)\sigma}{(1 - \gamma)i\rho + \gamma}\right\} = 1 + \frac{\gamma(1 - \gamma)\sigma}{(1 - \gamma)^2\rho^2 + \gamma^2}
\]

\[
\leq 1 - \frac{1}{2}\gamma(1 - \gamma) + \frac{1 + \rho^2}{(1 - \gamma)^2\rho^2 + \gamma^2}.
\]

Define

\[
h(\rho) = \frac{1 + \rho^2}{(1 - \gamma)^2\rho^2 + \gamma^2}.
\]

Then \( h'(\rho) = 0 \) occurs at only \( \rho = 0 \) and we get \( h(0) = 1/\gamma^2 \) and

\[
\lim_{\rho \to \infty} h(\rho) = \frac{1}{(1 - \gamma)^2}.
\]

Since \( 0 < \gamma \leq 1/2 \), we have

\[
\frac{1}{(1 - \gamma)^2} < h(\rho) \leq \frac{1}{\gamma^2}.
\]

Therefore

\[
\text{Re}\{\psi(i\rho, \sigma)\} \leq 1 - \frac{1}{2}\gamma(1 - \gamma) + \frac{1}{(1 - \gamma)^2} = \frac{2 - 3\gamma}{2(1 - \gamma)} =: \alpha'.
\]

This shows that \( \text{Re}\{\psi(i\rho, \sigma)\} \notin \Omega_{\alpha'} \). Applying Lemma 1.1, we get \( \text{Re}p(z) > 0 \) in \( \Delta \), and this shows that the inequality of (2.1) holds. This proves the theorem. \( \square \)

Setting \( \alpha = 0 \) in the Theorem 2.1 we get the following well known result:

**Corollary 2.1.** Let \( f \in A \) be defined by (1.1). Then \( S^* \subset Q(1/2) \), i.e.

\[
\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \Rightarrow \text{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2} \quad (z \in \Delta).
\]
We remark that in \[1 - 6\], the authors with a different method have shown that \( S^* \subset Q(1/2) \).

By putting \( z = re^{i\varphi} (r < 1) \), \( \varphi \in [0, 2\pi) \) and with a simple calculation, we have
\[
\tilde{q}_n(re^{i\varphi}) = \frac{3}{3 + (\alpha - 3)re^{i\varphi} - \alpha r^2e^{2i\varphi}} = \frac{9 + 3(\alpha - 3)r\cos\varphi - 3\alpha r^2\cos 2\varphi - 3i[(\alpha - 3)r\sin\varphi - \alpha r^2\sin 2\varphi]}{9 + (\alpha - 3)^2r^2 + \alpha^2r^4 + 2r(\alpha - 3)(3 - \alpha^2)\cos\varphi - 6\alpha r^2\cos 2\varphi}.
\]

Hence
\[
\begin{align*}
\left| \text{Im}\{\tilde{q}_n(re^{i\varphi})\} \right| & = \frac{3[(\alpha - 3)r\sin\varphi - \alpha r^2\sin 2\varphi]}{9 + 3(\alpha - 3)r\cos\varphi - 3\alpha r^2\cos 2\varphi} \\
\left| \text{Re}\{\tilde{q}_n(re^{i\varphi})\} \right| & < \frac{3(3 - \alpha)r + |\alpha|r^2}{9 - 3(3 - \alpha)r - 3|\alpha|r^2} =: \phi(r) \quad (r < 1, \ -3 < \alpha \leq 1).
\end{align*}
\]

For such \( r \) the curve \( \tilde{q}_n(re^{i\varphi}) \), \( \varphi \in [0, 2\pi) \), has no loops and \( \tilde{q}_n(re^{i\varphi}) \) is univalent in \( \Delta_r = \{ z : |z| < r \} \). Therefore
\[
\begin{align*}
\left| \frac{zf'(z)}{f(z)} \right| < \tilde{q}_n(z), \quad z \in \Delta_r \iff \left| \frac{zf'(z)}{f(z)} \right| < \tilde{q}_n(\Delta_r), \quad z \in \Delta_r.
\end{align*}
\]

The above relations give the following theorem.

**Theorem 2.2.** Let \(-1 < \alpha \leq 1\). If \( f \in SK(\alpha) \), then \( f \) is strongly starlike of order
\[
\frac{2}{\pi} \arctan \left\{ \frac{3(3 - \alpha) + |\alpha|}{9 - 3(3 - \alpha) - 3|\alpha|} \right\},
\]
in the unit disc \( \Delta \).

**Proof.** Since \( \text{Re}\{zf'(z)/f(z)\} > 0 \) in the unit disk, and from (2.5) and (2.6), we have
\[
\arg \left\{ \frac{zf'(z)}{f(z)} \right\} = \arctan \frac{\text{Im}\{zf'(z)/f(z)\}}{\text{Re}\{zf'(z)/f(z)\}} < \arctan \frac{\text{Im}\{\tilde{q}_n(re^{i\varphi})\}}{\text{Re}\{\tilde{q}_n(re^{i\varphi})\}} < \arctan \phi(r),
\]
where \( \phi(r) \) defined by (2.5). Now by letting \( r \to 1^- \) the proof of this theorem is completed. \( \square \)

3. ON COEFFICIENTS

The logarithmic coefficients \( \gamma_n \) of \( f(z) \) are defined by
\[
(3.1) \quad \log \frac{f(z)}{z} = \sum_{n=1}^{\infty} 2\gamma_n z^n \quad (z \in \Delta).
\]

These coefficients play an important role for various estimates in the theory of univalent functions. For functions in the class \( S^* \), it is easy to prove that \( |\gamma_n| \leq 1/n \) for \( n \geq 1 \) and equality holds for the Koebe function. Here, we get the sharp logarithmic coefficients inequality for the functions which belong to the class \( SK(\alpha) \). First, we present a subordination relation related with the class \( SK(\alpha) \). This relation is then used to obtain sharp inequality for their logarithmic coefficients.

**Theorem 3.1.** Let \( f \in A \) and \(-3 < \alpha \leq 1\). If \( f \in SK(\alpha) \), then there exists a function \( w(z) \in B \) such that
\[
(3.2) \quad \log \frac{f(z)}{z} = \int_0^z \frac{\tilde{q}_n(w(t)) - 1}{t} \, dt \quad (z \in \Delta).
\]
Proof. By Definition 1.1 if \( f \in SK(\alpha) \), then
\[
\frac{zf'(z)}{f(z)} \prec \tilde{q}_\alpha(z) \quad (z \in \Delta)
\]
or
\[
z \left\{ \log \frac{f(z)}{z} \right\}' \prec \tilde{q}_\alpha(z) - 1 \quad (z \in \Delta).
\]
From the definition of subordination, there exists a function \( w(z) \in \mathfrak{B} \) so that
\[
z \left\{ \log \frac{f(z)}{z} \right\}' = \tilde{q}_\alpha(w(z)) - 1 \quad (z \in \Delta).
\]
Now the assertion follows by integrating of the last equality. \( \square \)

Corollary 3.1. Let \( f \in \mathcal{A} \) and \(-3 < \alpha \leq 1\). If \( f \in SK(\alpha) \), then
\[
(3.3) \quad \log \frac{f(z)}{z} \prec \int_{0}^{z} \frac{\tilde{q}_\alpha(t) - 1}{t} \, dt \quad (z \in \Delta).
\]

The celebrated de Branges’ inequalities (the former Milin conjecture) for univalent functions \( f \) state that
\[
\sum_{n=1}^{k} (k-n+1)|\gamma_n|^2 \leq \sum_{n=1}^{k} \frac{k+1-n}{n} \quad (k = 1, 2, \ldots),
\]
with equality if and only if \( f(z) = e^{-i\theta}k(e^{i\theta}z) \) (see [1]). De Branges [1] used this inequality to prove the celebrated Bieberbach conjecture. Moreover, the de Branges’ inequalities have also been the source of many other interesting inequalities involving logarithmic coefficients of \( f \in S \) such as (see [3])
\[
\sum_{n=1}^{\infty} \left| \gamma_n \right|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]
Now, we have the following.

Theorem 3.2. Let \( f \in \mathcal{A} \) belongs to the class \( SK(\alpha) \) and \(-3 < \alpha \leq 1\). Then the logarithmic coefficients of \( f \) satisfy in the inequality
\[
(3.4) \quad \sum_{n=1}^{\infty} \left| \gamma_n \right|^2 \leq \frac{1}{4(3 + \alpha)^2} \left[ \frac{3\pi^2}{2} + 6\alpha Li_2(-\alpha/3) + \alpha^2 Li_2(\alpha^2/9) \right],
\]
where \( Li_2 \) is defined as following
\[
(3.5) \quad Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \int_{0}^{z} \frac{\ln(1-t)}{t} \, dt.
\]
The inequality is sharp.

Proof. Let \( f \in SK(\alpha) \). Then by Corollary 3.1 we have
\[
(3.6) \quad \log \frac{f(z)}{z} \prec \int_{0}^{z} \frac{\tilde{q}_\alpha(t) - 1}{t} \, dt \quad (z \in \Delta).
\]
Again, by using (3.1) and (3.6), the relation (3.6) implies that
\[
\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{B_n}{n} z^n \quad (z \in \Delta).
\]
Now by Rogosinski’s theorem [2, Sec. 6.2], we get
\[
4 \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} |B_n|^2
\]
\[
= \frac{9}{(3 + \alpha)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left|1 + (-1)^n (\alpha/3)^n + 1\right|^2
\]
\[
= \frac{9}{(3 + \alpha)^2} \left(\frac{\pi^2}{6} + \frac{2\alpha}{3} Li_2 \left(\frac{-\alpha}{3}\right) + \frac{\alpha^2}{9} Li_2 \left(\frac{-\alpha^2}{9}\right)\right),
\]
where $Li_2$ is given by (3.5). Therefore the desired inequality (3.4) follows. For the sharpness of (3.4), consider
\[
(3.7) \quad \phi_\alpha(z) = z \exp \int_0^z \tilde{q}_\alpha(t) - \frac{1}{t} \, dt.
\]
It is easy to see that $\phi_\alpha(z) \in SK(\alpha)$ and $\gamma_n(\phi_\alpha) = B_n/2n$, where $B_n$ is given by (1.6). Therefore, we have the equality in (3.4) and concluding the proof. □

**Theorem 3.3.** Let $f \in A$ be a member of $SK(\alpha)$. Then the logarithmic coefficients of $f$ satisfy
\[
|\gamma_n| \leq \frac{3 - \alpha}{6n} \quad (-3 < \alpha \leq 1, n \geq 1).
\]

**Proof.** Let $f \in SK(\alpha)$. Then by Definition 1.1 we have
\[
zf'(z) \prec \tilde{q}_\alpha(z) \quad (z \in \Delta)
\]
or
\[
(3.8) \quad z \left\{\log \frac{f(z)}{z}\right\}' \prec \tilde{q}_\alpha(z) - 1 \quad (z \in \Delta).
\]
Applying (1.5) and (3.1), the above subordination relation (3.8) implies that
\[
\sum_{n=1}^{\infty} 2n\gamma_n z^n < \sum_{n=1}^{\infty} B_n z^n.
\]
Applying the Rogosinski theorem [14], we get the inequality $2n|\gamma_n| \leq |B_1| = 1 - \alpha/3$. This completes the proof. □

The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the normalized analytic function class $A$ is known as the Fekete-Szegö problem. We recall here that, for a univalent function $f(z)$ of the form (1.1), the $k$th root transform is defined by
\[
F(z) = [f(z^k)]^{1/k} \triangleq z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \quad (z \in \Delta).
\]

Next we consider the problem of finding sharp upper bounds for the Fekete-Szegö coefficient functional associated with the $k$th root transform for functions in the class $SK(\alpha)$.

**Theorem 3.4.** Let that $f \in SK(\alpha)$, $-3 < \alpha \leq 1$ and $F$ is the $k$th root transform of $f$ defined by (3.9). Then, for any complex number $\mu$,
\[
|b_{2k+1} - \mu b_{2k+1}^2| \leq \frac{3 - \alpha}{6k} \max \left\{1, \frac{2\mu - 1}{k} \left(1 - \frac{\alpha}{3}\right) - \frac{\alpha^2 - 3\alpha + 9}{6(3 - \alpha)}\right\}.
\]
The result is sharp.
Proof. Since \( f \in SK(\alpha) \), from Definition 1.1 and definition of subordination, there exists \( w \in B \) such that
\[
zf'(z)/f(z) = \tilde{q}_\alpha(w(z)).
\]
We now define
\[
p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots.
\]
Since \( w \in B \), it follows that \( p \in P \). From (1.5) and (3.12) we have:
\[
\tilde{q}_\alpha(w(z)) = 1 + \frac{1}{2}B_1 p_1 z + \left( \frac{1}{4}B_2 p_1^2 + \frac{1}{2}B_1 \left( p_2 - \frac{1}{2}p_1^2 \right) \right) z^2 + \cdots.
\]
Equating the coefficients of \( z \) and \( z^2 \) on both sides of (3.11), we get
\[
a_2 = \frac{1}{2}B_1 p_1,
\]
and
\[
a_3 = \frac{1}{8} \left( B_2^2 + B_2 \right) p_1^2 + \frac{1}{4}B_1 \left( p_2 - \frac{1}{2}p_1^2 \right).
\]
A computation shows that, for \( f \) given by (1.1),
\[
F(z) = \left[ f(z^{1/k}) \right]^{1/k} = z + \frac{1}{k} a_2 z^{k+1} + \left( \frac{1}{k} a_3 - \frac{1}{2} \frac{k}{k^2} a_2^2 \right) z^{2k+1} + \cdots.
\]
From equations (3.9) and (3.16), we have
\[
b_{k+1} = \frac{1}{k} a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k} a_3 - \frac{1}{2} \frac{k}{k^2} a_2^2.
\]
Substituting from (3.14) and (3.15) into (3.17), we obtain
\[
b_{k+1} = \frac{1}{2k} B_1 p_1,
\]
and
\[
b_{2k+1} = \frac{1}{8k} \left( B_2 + B_1^2 \right) \rho_1 + \frac{1}{4k} B_1 \left( p_2 - \frac{1}{2} p_1^2 \right),
\]
so that
\[
b_{2k+1} - \mu \rho_{k+1}^2 = \frac{B_1}{4k} \left[ p_2 - \frac{1}{2} \left( \frac{2\mu - 1}{k} B_1 - \frac{B_2}{B_1} + 1 \right) \rho_1^2 \right].
\]
Letting
\[
\mu' = \frac{1}{2} \left( \frac{2\mu - 1}{k} B_1 - \frac{B_2}{B_1} + 1 \right),
\]
the inequality (3.10) now follows as an application of Lemma 1.2 and inserting \( B_1 = (3 - \alpha)/3, B_2 = (\alpha^2 - 3\alpha + 9)/18 \). It is easy to check that the result is sharp for the \( k \)th root transforms of the function
\[
f(z) = z \exp \left( \int_0^z \tilde{q}_\alpha(w(t)) \frac{dt}{t} \right).
\]
□

Putting \( k = 1 \) in Theorem 3.4, we have:

**Corollary 3.2.** (Fekete-Szegö inequality) Suppose that \( f \in SK(\alpha) \). Then, for any complex number \( \mu \),
\[
|a_3 - \mu a_2^2| \leq \frac{3 - \alpha}{6} \max \left\{ 1, (2\mu - 1)(1 - \alpha/3) - \frac{\alpha^2 - 3\alpha + 9}{6(3 - \alpha)} \right\}.
\]
The result is sharp.
If we take \( k = 1 \) and \( \alpha = -1 \) in Theorem 3.4, we get:

**Corollary 3.3.** Let \( f \) given by the form (1.1) be starlike function. Then

\[
|a_3 - \mu a_2^3| \leq \max \left\{ \frac{2}{3}, \frac{|8(2\mu - 1)|}{9 - 7/12} \right\}.
\]

The result is sharp.

Taking \( k = 1 \) and \( \alpha = 0 \) in Theorem 3.4, we have:

**Corollary 3.4.** Let \( f \) given by the form (1.1) be in the class \( S^{*}(1/2) \). Then

\[
|a_3 - \mu a_2^3| \leq \frac{1}{2} \max \{1, |2\mu - 3/2| \}.
\]

The result is sharp.

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), defined by

\[
f(f^{-1}(w)) = w \quad (|w| < r_0; \quad r_0 > 1/4),
\]

where

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.
\]

**Corollary 3.5.** Let the function \( f \), given by (1.1), be in the class \( S^K(\alpha) \). Also let the function \( f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \) be inverse of \( f \). Then

\[
|b_3| \leq \frac{3 - \alpha}{6} \max \left\{1, \frac{5a_2^2 - 33\alpha + 45}{6(3 - \alpha)} \right\}.
\]

Proof. The relation (3.23) gives

\[b_3 = 2a_2^2 - a_3\]

Thus, for estimate of \(|b_3|\), it suffices in Corollary 3.2 we put \( \mu = 2 \). Hence the proof of Corollary 3.5 is completed. \(\square\)

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Young Researchers and Elite Club, Urmia Branch, Islamic Azad University, Urmia, Iran

E-mail address: rkargar1983@gmail.com (Rahim Kargar)

Department of Mathematics, Islamic Azad University, Firoozkouh Branch, Firoozkouh, Iran.

E-mail address: mahzoon_hesam@yahoo.com (Hesam Mahzoon)

Department of Mathematics, Payame Noor University, Tehran, Iran

E-mail address: nad.kanzi@gmail.com (Nader Kanzi)