A NEW EFFICIENT ASYMMETRIC CRYPTOSYSTEM BASED
ON THE SQUARE ROOT PROBLEM

M.R.K. ARIFFIN, M.A.ASBOOLAH, AND N.A. ABU

ABSTRACT. The square root modulo problem is a known primitive in designing an asymmetric cryptosystem. It was first attempted by Rabin. Decryption failure of the Rabin cryptosystem caused by the 4-to-1 decryption output is overcome efficiently in this work. The proposed scheme (known as the AAβ-cryptosystem) has its encryption speed having a complexity order faster than the Diffie-Hellman Key Exchange, El-Gammal, RSA and ECC. It can also transmit a larger data set securely when compared to existing asymmetric schemes. It has a simple mathematical structure. Thus, it would have low computational requirements and would enable communication devices with low computing power to deploy secure communication procedures efficiently.

1. INTRODUCTION

The Rabin cryptosystem that utilizes the square root modulo problem, is said to be an optimal implementation of RSA with the encryption exponent $e = 2$ [9]. However, the situation of a 4-to-1 mapping during decryption has deterred it from being utilized. Mechanisms to ensure its possible implementation have been proposed, however the solutions either still have a possibility of decryption failure or the performance against the RSA is inadequate. As a consequence other underlying cryptographic primitives have taken centre stage. The discrete log problem (DLP) and the elliptic curve discrete log problem (ECDLP) has been the source of security for cryptographic schemes such as the Diffie-Hellman key exchange (DHKE) procedure, El-Gamal cryptosystem and elliptic curve cryptosystem (ECC) respectively [1], [7]. As for the world renowned RSA cryptosystem, the inability to find the $e$-th root of the ciphertext $C$ modulo $N$ from the congruence relation $C \equiv M^e \pmod{N}$ coupled with the inability to factor $N = pq$ for large primes $p$ and $q$ is its fundamental source of security [10]. It has been suggested that the ECC is able to produce the same level of security as the RSA with shorter key length. Thus, ECC should be the preferred asymmetric cryptosystem when compared to RSA [15]. Hence, the notion “cryptographic efficiency” is conjured. That is, to produce an asymmetric cryptographic scheme that could produce security equivalent to a certain key length of the traditional RSA but utilizing shorter keys. However, in certain situations where a large block needs to be encrypted, RSA is the better option than ECC because ECC would need more computational effort to undergo such a task [12]. Thus, adding another characteristic toward the notion of “cryptographic efficiency”
which is it must be less “computational intensive” and be able to transmit large blocks of data (when needed). In 1998 the cryptographic scheme known as NTRU was proposed with better “cryptographic efficiency” relative to RSA and ECC [4][5][6]. NTRU has a complexity order of $O(n^2)$ for both encryption and decryption as compared to DHKE, EL-Gammal, RSA and ECC (all have a complexity order of $O(n^3)$). As such, in order to design a state-of-the-art public key mechanism, the following are characteristics that must be “ideally” achieved (apart from other well known security issues):

1. Shorter key length. If possible shorter than ECC 160-bits.
2. Speed. To have speed of complexity order $O(n^2)$ for both encryption and decryption.
3. Able to increase data set to be transmitted asymmetrically. That is, not to be restricted in size because of the mathematical structure.
4. Simple mathematical structure for easy implementation.

In this paper, we attempt to efficiently enhance an asymmetric cryptosystem based on the square root problem as its cryptographic primitive. That is, we will efficiently redesign Rabin’s cryptosystem that has decryption failure due to a 4-to-1 mapping. We will show that in our design for encryption, it does not involve “expensive” mathematical operation. Only basic multiplication is required without division or modulo operation.

The layout of this paper is as follows. The Rabin cryptosystem will be discussed in Section 2. Previous designs to overcome the decryption failure of the Rabin cryptosystem will also be presented here. The mechanism of the $AA_B$-cryptosystem will be detailed in Section 3. In Section 4, the authors detail the decryption process and provide a proof of correctness. An example will also be presented. Continuing in Section 5, we will discuss a congruence attack, a Coppersmith type attack and a Euclidean division attack. An analysis of lattice based attack will be given in Section 6. Section 7 will be about the underlying security principles of the $AA_B$ scheme. A table of comparison between the $AA_B$ scheme against RSA,ECC and NTRU is given in Section 8. Finally, we shall conclude in Section 9.

2. The Rabin Cryptosystem

Let us begin by stating that the communication process is between A (Along) and B (Busu), where Busu is sending information to Along after encrypting the plaintext with Along’s public key.

- **Key Generation by Along**

  **INPUT:** Generate two random $n$-bit prime numbers $p$ and $q$.
  **OUTPUT:** The public key $N = pq$ and the private key pair $(p, q)$.

  Remark 2.1. To simplify computation one may choose $p \equiv q \equiv 3 \pmod{4}$.

- **Encryption by Busu**

  **INPUT:** The public key $N$ and the message $M$ where $0 \leq M \leq N - 1$.
  **OUTPUT:** The ciphertext $C = M^2 \pmod{N}$.
A NEW EFFICIENT ASYMMETRIC CRYPTO SYSTEM BASED ON THE SQUARE ROOT PROBLEM

• Decryption by Along

INPUT: The private key pair \((p, q)\) and the ciphertext \(C\).
OUTPUT: The plaintext \(M\).

Remark 2.2. Computing the square roots of \(C\) modulo \(N\) using the private keys \((p, q)\), would result in 4 square roots of \(C\) modulo \(N\). Thus, the “infamous” decryption failure scenario.

2.1. Redundancy Schemes for Unique Decryption. In order to overcome the decryption failure, it is necessary to have a scheme that could provide the plaintext upon decryption without having to guess. We provide here a brief description of 3 existing solution techniques.

   (1) **Redundancy in the message** [8]. This scheme has a probability decryption failure of approximately \(\frac{1}{2^l}\) where \(l\) is the least significant binary string of the message.

   (2) **Extra bits** [2]. One will send 2 extra bits of information to specify the square root. The encryption process requires the computation of the Jacobi symbol. This results in a computational overhead which is much more than just computing a single square modulo \(N\).

   (3) **Williams technique** [14]. The encryption process requires the encrypter to compute a Jacobi symbol. Hence, losing the performance advantage of Rabin over RSA (as in point no.2).

In the next section, we will present an efficient enhancement of Rabin’s cryptosystem that does not inherit the above properties.

3. THE AA\(_B\) PUBLIC KEY CRYPTO SYSTEM

• Key Generation by Along

INPUT: The size \(n\) of the prime numbers.
OUTPUT: A public key tuple \((n, e_{A1}, e_{A2})\) and a private key pair \((pq, d)\).

   (1) Generate two random and distinct \(n\)-bit strong primes \(p\) and \(q\) satisfying
   \[
   \begin{cases}
   p \equiv 3 \pmod{4}, & 2^n < p < 2^{n+1}, \\
   q \equiv 3 \pmod{4}, & 2^n < q < 2^{n+1}.
   \end{cases}
   \]

   (2) Choose random \(d\) such that \(d > (p^2q)^{\frac{1}{4}}\).

   (3) Choose random integer \(e\) such that \(ed \equiv 1 \pmod{pq}\) and add multiples of \(pq\) until \(2^{3n+4} < e < 2^{3n+6}\) (if necessary).

   (4) Set \(e_{A1} = p^2q\). We have \(2^{3n} < e_{A1} < 2^{3n+3}\).

   (5) Set \(e_{A2} = e\).

   (6) Return the public key tuple \((n, e_{A1}, e_{A2})\) and a private key pair \((pq, d)\).

We also have the fact that \(2^{2n} < pq < 2^{2n+2}\).
• Encryption by Busu

INPUT: The public key tuple \((n, e_A, e_B)\) and the message \(M\).
OUTPUT: The ciphertext \(C\).

(1) Represent the message \(M\) as a \(4n\)-bit integer \(m\) within the interval \((2^{4n-1}, 2^{4n})\) with \(m = m_1 \cdot 2^n + m_2\) where \(m_1\) is a \(3n + 1\)-bit integer within the interval \((2^{3n}, 2^{3n+1})\) and \(m_2\) is a \(n - 1\)-bit integer within the interval \((2^{n-2}, 2^{n-1})\).

(2) Choose a random \(n\)-bit integer \(k_1\) within the interval \((2^{n-2}, 2^n)\) and compute \(U = m_1 \cdot 2^n + k_1\). We have \(2^{4n} < U < 2^{4n+1}\).

(3) Choose a random \(n\)-bit integer \(k_2\) within the interval \((2^{n-1}, 2^n)\) and compute \(V = m_2 \cdot 2^n + k_2\). We have \(2^{2n-2} < V < 2^{2n-1}\).

(4) Compute \(C = U^{e_A} + V^{e_B}\).

(5) Send ciphertext \(C\) to Along.

4. Decryption

**Proposition 4.1.** Decryption by Along is conducted in the following steps:

INPUT: The private key \((pq, d)\) and the ciphertext \(C\).
OUTPUT: The plaintext \(M\).

(1) Compute \(W \equiv Cd \pmod{pq}\).
(2) Compute \(M_1 \equiv q^{-1} \pmod{p}\) and \(M_2 \equiv p^{-1} \pmod{q}\).
(3) Compute

\[
x_p \equiv W^{\frac{p+1}{p}} \pmod{p},
x_q \equiv W^{\frac{q+1}{q}} \pmod{q}.
\]

(4) Compute

\[
V_1 \equiv x_pM_1q + x_qM_2p \pmod{pq},
V_2 \equiv x_pM_1q - x_qM_2p \pmod{pq},
V_3 \equiv -x_pM_1q + x_qM_2p \pmod{pq},
V_4 \equiv -x_pM_1q - x_qM_2p \pmod{pq}.
\]

(5) For \(i = 1, 2, 3, 4\) compute \(U_i = \frac{C - V_i^{e_B}q}{e_A}\).
(6) Sort the pair \((U_j, V_j)\) for integer \(U_j\).
(7) Compute integral part \(m_1 = \lfloor U_1 \rfloor\).
(8) Compute integral part \(m_2 = \lfloor V_1 \rfloor\).
(9) Form the integer \(m = m_1 \cdot 2^n + m_2\).
(10) Transform the number \(m\) to the message \(M\).
(11) Return the message \(M\).

We now proceed to give a proof of correctness.

Along will begin by computing \(W \equiv Cd \equiv V^2 \pmod{pq}\). Along will then have to solve \(W \equiv V^2 \pmod{pq}\) using the Chinese Remainder Theorem.
Lemma 4.2. Let $p$ and $q$ be two different primes such that $p \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$. Define $x_p$ and $x_q$ by

$$x_p \equiv W^{\frac{p+1}{2}} \pmod{p}, \quad x_q \equiv W^{\frac{q+1}{2}} \pmod{q}.$$ 

Then the solutions of the equation $x^2 \equiv W \pmod{p}$ are $\pm x_p \pmod{p}$ and the solutions of the equation $x^2 \equiv W \pmod{q}$ are $\pm x_q \pmod{q}$.

Lemma 4.3. Let $p$ and $q$ be two different primes such that $p \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$. Define $x_p$ and $x_q$ by

$$x_p \equiv W^{\frac{p+1}{2}} \pmod{p}, \quad x_q \equiv W^{\frac{q+1}{2}} \pmod{q}.$$ 

Define $M_1 \equiv q^{-1}(mod \ p)$ and $M_2 \equiv p^{-1}(mod \ q)$. Then the solutions of the equation $V^2 \equiv W \pmod{pq}$ are

$$V_1 \equiv x_p M_1 q + x_q M_2 p \pmod{pq},$$
$$V_2 \equiv x_p M_1 q - x_q M_2 p \pmod{pq},$$
$$V_3 \equiv -x_p M_1 q + x_q M_2 p \pmod{pq},$$
$$V_4 \equiv -x_p M_1 q - x_q M_2 p \pmod{pq}.$$ 

To solve the equation $V^2 \equiv W \pmod{pq}$, we use the Chinese Remainder Theorem. Consider the equations $x_p^2 \equiv W \pmod{p}$ and $x_q^2 \equiv W \pmod{q}$. Then the solution of the equation $V^2 \equiv W \pmod{pq}$ are the four solutions of the four systems

$$\begin{align*}
V &\equiv \pm x_p (mod \ p) \\
V &\equiv \pm x_q (mod \ q)
\end{align*}$$

Define $M_1 \equiv q^{-1}(mod \ p)$ and $M_2 \equiv p^{-1}(mod \ q)$. We will get explicitly

$$V_1 \equiv x_p M_1 q + x_q M_2 p \pmod{pq},$$
$$V_2 \equiv x_p M_1 q - x_q M_2 p \pmod{pq},$$
$$V_3 \equiv -x_p M_1 q + x_q M_2 p \pmod{pq},$$
$$V_4 \equiv -x_p M_1 q - x_q M_2 p \pmod{pq}.$$ 

It can be seen that solving $V^2 \equiv W \pmod{pq}$, we will get four solutions $V_i$ for $i = 1, 2, 3, 4$.

We prove below that only one of them leads to the correct decryption and consequently, there is no decryption failure.

Lemma 4.4. Let $C$ be an integer representing a ciphertext encrypted by the AA$_\beta$ algorithm. The equation $C = U e_{A1} + V^2 e_{A2}$ has only one solution satisfying $V < 2^{2n-1}$.

Proof. Suppose for contradiction that there are two couples of solutions $(U_1, V_1)$ and $(U_2, V_2)$ of the equation $C = U e_{A1} + V^2 e_{A2}$ with $V_1 \neq V_2$ and $V_i < 2^{2n-1}$. Then $U_1 e_{A1} + V_1^2 e_{A2} = U_2 e_{A1} + V_2^2 e_{A2}$. Using $e_{A1} = p^2 q$, this leads to

$$(U_2 - U_1)p^2 q = (V_1 + V_2)(V_1 - V_2)e_{A2}.$$ 

Since $\gcd(p^2 q, e_{A2}) = 1$, then $p^2 q | (V_1 + V_2)(V_1 - V_2)$ and the prime numbers $p$ and $q$ satisfy one of the conditions

$$p^2 | (V_1 \pm V_2) \text{ or } \begin{cases} 
pq | (V_1 \pm V_2) \\
p | (V_1 \pm V_2)
\end{cases}$$
Observe that $p^2 > 2^{2n}$ and $pq > 2^{2n}$ while $|V_1 \pm V_2| < 2 \cdot 2^{2n-1} = 2^{2n}$. This implies that none of these conditions is possible. Hence the equation $C = Ue_{A1} + V^2e_{A2}$ has only one solution with the parameters of the scheme.

4.1. Example. Let $n = 16$. Along will choose the primes $p = 62683$ and $q = 62483$. The public keys will be

1. $e_{A1} = 245505609868187$
2. $e_{A2} = 4106878163802480$

The private keys will be

1. $pq = 3916621889$
2. $d = 2486483$

Busu’s message will contain the following parameters

1. $m_1 = 54464664056570$
2. $m_2 = 21777$

Busu will also generate the following ephemeral random session keys

1. $k_1 = 54433$
2. $k_2 = 33079$

Busu will then generate

1. $U = 35693832703611425953$
2. $V = 1427210551$ and consequently $V^2 = 2036929956885723601$

The ciphertext will be $C = 1712845932756226602243879187691$.

To decrypt Along will first compute $W = 3215349249$. Along will then obtain the following root values

$V_1 = 318887097$
$V_2 = 2489411338$
$V_3 = 1427210551$

and

$V_4 = 3597734792$.

Only $U_3 = \frac{C-V^2e_{A2}}{e_{A1}}$ will produce an integer value. That is $U_3 = 35693832703611425953$. Finally, $m_1$ and $m_2$ can be obtained.

5. Basic Attacks

5.1. Congruence attack. In this subsection we will observe the security of the ciphertext equation $C = Ue_{A1} + V^2e_{A2}$ when it is treated as a Diophantine equation. We will observe that solving the corresponding Diophantine equation parametric solution set for the unknown parameters $U$ and $V^2$ will result in exponentially many candidates to choose from.

From $C = Ue_{A1} + V^2e_{A2}$ and since $\gcd(e_{A1}, e_{A2}) = 1$ we have

$U \equiv Ce^{-1}_{A1} \equiv a \pmod{e_{A2}}$.

Hence $U = a + e_{A2}j$ for some $j \in \mathbb{Z}$. Replacing into $C$ we have

$C = Ue_{A1} + V^2e_{A2} = (a + e_{A2}j)e_{A1} + V^2e_{A2}$. 
A NEW EFFICIENT ASYMMETRIC CRYPTOSYSTEM BASED ON THE SQUARE ROOT PROBLEM

Then,
\[ V^2 = C - (a + e_{A2}j)e_{A1} = \frac{C - e_{A1}a}{e_{A2}} - e_{A1}j, \]
where \( \frac{C - e_{A1}a}{e_{A2}} = b \in \mathbb{Z} \). It follows that the equation \( C = U e_{A1} + V^2 e_{A2} \) has the parametric solutions
\[ U = a + e_{A2}j \text{ and } V^2 = b - e_{A1}j. \]

- Computing with \( U \)
To find \( U = a + e_{A2}j \), we should find an integer \( j \) such that \( 2^{4n} \cdot 2^{-4} < U < 2^{4n+1} \). This gives
\[ \frac{2^{4n} - a}{e_{A2}} < j < \frac{2^{4n+1} - a}{e_{A2}}. \]
We know that \( 2^{3n+4} < e_{A2} < 2^{3n+6} \). Then the difference between the upper and the lower bound is
\[ \frac{2^{4n+1} - a}{e_{A2}} - \frac{2^{4n} - a}{e_{A2}} = \frac{2^{4n}}{e_{A2}} > \frac{2^{4n}}{2^{3n+6}} = 2^{n-6}. \]
Hence the difference is very large and finding the correct \( j \) is infeasible.

- Computing with \( V^2 \)
To find \( V^2 = b - e_{A1}j \), we should find an integer \( j \) such that \( 2^{4n-4} \cdot 2^{-4} < V < 2^{4n-2} \). This gives
\[ \frac{2^{4n-4} - b}{-e_{A1}} > j > \frac{2^{4n-2} - b}{-e_{A1}}. \]
We know that \( 2^{3n} < e_{A1} < 2^{3n+3} \). Then the difference between the upper and the lower bound is
\[ \frac{2^{4n-4} - b}{-e_{A1}} - \frac{2^{4n-2} - b}{-e_{A1}} = \frac{3 \cdot 2^{4n-4}}{e_{A1}} = 3 \cdot 2^{n-7}. \]
Hence the difference is very large and finding the correct \( j \) is infeasible.

5.2. Coppersmith type attack.

**Theorem 5.1.** Let \( N \) be an integer of unknown factorization. Furthermore, let \( f_N(x) \) be an univariate, monic polynomial of degree \( \delta \). Then we can find all solutions \( x_0 \) for the equation \( f_N(x) \equiv 0(\mod N) \) with
\[ |x_0| < N^{\frac{1}{2^*}}, \]
in time polynomial in \( (\log N, \delta) \).

**Theorem 5.2.** Let \( N \) be an integer of unknown factorization, which has a divisor \( b > N^{\beta} \). Furthermore let \( f_b(x) \) be an univariate, monic polynomial of degree \( \delta \). Then we can find all solutions \( x_0 \) for the equation \( f_b(x) \equiv 0(\mod b) \) with
\[ |x_0| \leq \frac{1}{2} N^{\frac{\delta}{2^*} - \epsilon} \]
in polynomial time in \( (\log N, \delta, \frac{1}{2^*}) \).
• Attacking $V$

With reference to Theorem 1. Let $N = e_{A1} = p^2q$ and $d' ≡ e^{-1}(\text{mod } N)$. Compute $W = Cd' ≡ V^2(\text{mod } N)$. Let $f_N(x) ≡ x^2 - W ≡ 0(\text{mod } N)$. Hence, $δ = 2$. Thus the root $x_0 = V$ can be recovered if $V < N^{1.5} ≈ 2^{4n}$. But since $V ≈ 2^{2n}$, this attack is infeasible.

• Attacking $d$

With reference to Theorem 2. We begin by observing $f_b(x) = ex - 1 ≡ 0(\text{mod } pq)$ where $pq$ in an unknown factor of $N = e_{A1} = p^2q$. Since $pq > N^{1.5}$ we have $β = 2.3$. From $f_b(x)$ we also have $δ = 1$. By the Coppersmith theorem, the root $x_0 = d$ can be found if $|x_0| < N^{4.9}$. But since $d > N^{4.9}$, this attack is infeasible.

5.3. Euclidean division attack. From $C = Ue_{A1} + V^2e_{A2}$, the size of each public parameter within $C$ ensures that Euclidean division attacks does not occur. This can be easily deduced as follows:

(1) \[ \left\lfloor \frac{U}{e_{A1}} \right\rfloor \neq U \]
(2) \[ \left\lfloor \frac{V}{e_{A2}} \right\rfloor \neq V^2 \]

6. Analysis on lattice based attack

The square lattice attack has been an efficient and effective means of attack upon schemes that are designed based on Diophantine equations. The $AAβ$ scheme has gone through analysis regarding lattice attacks while it went through the design process. Let $C = Ue_{A1} + V^2e_{A2}$ be an $AAβ$ ciphertext. Consider the diophantine equation $e_{A1}x_1 + e_{A2}x_2 = C$. Introduce the unknown $x_3$ and consider the diophantine equation

$e_{A1}x_1 + e_{A2}x_2 - Cx_3 = 0$.

Then $(U, V^2, 1)$ is a solution of the equation. Next let $T$ be a number to fixed later. Consider the lattice $\mathcal{L}$ spanned by the matrix:

$$M_0 = \begin{pmatrix} 1 & 0 & e_{A1}T \\ 0 & 1 & e_{A2}T \\ 0 & 0 & CT \end{pmatrix}$$

Observe that

$$(x_1, x_2, x_3)M_0 = (x_1, x_2, T(e_{A1}x_1 + e_{A2}x_2 - Cx_3)).$$

This shows that the lattice $\mathcal{L}$ contains the vectors $(x_1, x_2, T(e_{A1}x_1 + e_{A2}x_2 - Cx_3))$ and more precisely the vector-solution $V_0 = (U, V^2, 0)$. Observe that the length of $V_0$ satisfies

$$\|V_0\| = \sqrt{U^2 + V^4} \approx 2^{4n}.$$ 

On the other hand, the determinant of the lattice is $\det(\mathcal{L}) = CT$ and the Gaussian heuristics for the lattice $\mathcal{L}$ asserts that the length of its shortest non-zero vector is usually approximately $\sigma(\mathcal{L})$ where

$$\sigma(\mathcal{L}) \approx \sqrt{3\det(\mathcal{L})^{\frac{1}{\dim(\mathcal{L})}}} = \sqrt{\frac{3}{2\pi e} (CT)^{\frac{1}{2}}}.$$
If we choose $T$ such that $\sigma(L) > \|V_0\|$, then $V_0$ can be among the short non-zero vectors of the lattice $L$. To this end, $T$ should satisfy

\[ T > \left( \frac{\pi e}{2} \right)^{\frac{3}{2}} \cdot \frac{2^{12n}}{C}. \]  

Next, if we apply the LLL algorithm to the lattice $L$, we will find a basis $(b_1, b_2, b_3)$ such that $\|b_1\| \leq \|b_2\| \leq \|b_3\|$ and

\[ b_i \leq 2^{n(n-1)} \det(L)^{1-\frac{1}{n}}, \text{ for } i = 1, ..., 4 \text{ and } n = 3. \]

For $i = 1$, we choose $T$ such that $\|V_0\| \leq \|b_1\| \leq 2^{\frac{3}{4}} (CT)^{\frac{3}{2}}$. Using the approximation $\|V_0\| \approx 2^{\frac{3}{4}} n$, this is satisfied if

\[ V > 2^{\frac{3}{4}} \cdot \frac{2^{12n}}{C}, \]

which follows from the lower bound of equation (3). We experimented this result to try to find $(U, V^2, 0)$. The LLL algorithm outputs a basis with a matrix in the form

\[ M_1 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & T \end{pmatrix} \]

If $(U, V^2, 0)$ is a short vector, then $(U, V^2, 0) = (x_1, x_2, x_3) M_1$ for some short vector $(x_1, x_2, x_3)$. We then deduce the system

\[ \begin{cases} a_{11} x_1 + a_{21} x_2 = U \\ a_{12} x_1 + a_{22} x_2 = V^2 \end{cases} \]

from which we can deduce that $x_3 = 0$. If we compute $(U e_{A_1} - V^2 e_{A_2}) / C$, we get $x_2 = 1$ for some $x_1$. It follows that

\[ \begin{cases} a_{11} x_1 + a_{21} = U \\ a_{12} x_1 + a_{22} = V^2 \end{cases} \]

This situation is similar to the congruence attack. We can also observe that this is a system of two equations with three unknowns (i.e. $x_1, U, V$).

**6.1. Example with lattice based attack.** We will use the parameters in the earlier example. Observe the lattice $L$ spanned by the matrix:

\[ M_0 = \begin{pmatrix} 1 & 0 & e_{A_1} T \\ 0 & 1 & e_{A_2} T \\ 0 & 0 & -CT \end{pmatrix} \]

the length of the vector $V = (U, V^2, 0)$ is approximately $\| V \| \approx 35751905917344588937$. We will use $T = 2^{20n}$ which would result in the length of the vector $V$ is shorter than the gaussian heuristic of the lattice $L$.

The LLL algorithm outputs the matrix $M_1$ given by:

\[
\begin{pmatrix}
-4106878163802480 & 245505609868187 & 0 \\
247367271832221073 & 415588887565804598 & 0 \\
-1118395942494397 & 66856738131713 & T
\end{pmatrix}
\]
7. Underlying security principles

7.1. The integer factorization problem. To find the unknown composite \( p \) and \( q \) such that \( e_{A1} = p^2q \).

7.2. The square root modulo problem. Since \( \gcd(e_{A1}, e_{A2}) = 1 \), one can obtain the relation \( V^2 \equiv \alpha (\mod e_{A1}) \). Since \( e_{A1} = p^2q \), then this is equivalent to calculating square roots modulo composite integers with unknown factorization which is infeasible.

7.3. The modular reduction problem. Since \( \gcd(e_{A1}, e_{A2}) = 1 \), one can obtain \( U \equiv \beta (\mod e_{A2}) \). Since \( U \gg e_{A2} \), to compute \( U \) prior to modular reduction by \( e_{A2} \) is infeasible.

7.4. Equivalence with integer factorization. From \( C = Ue_{A1} + V^2e_{A2} \) we have

\[
C \equiv V^2 (\mod e_{A1})
\]

where \( e_{A1} = p^2q \) is of unknown factorization. We show here that solving this congruence relation is equivalent to factoring \( e_{A1} \). If we know the factorization of \( e_{A1} \), then it is easy to solve the congruence relation. Conversely, suppose that we know all the solutions. By Lemma 2, the four solutions are

\[
\begin{align*}
V_1 &\equiv x_pM_1q + x_qM_2p \pmod{pq}, \\
V_2 &\equiv x_pM_1q - x_qM_2p \pmod{pq}, \\
V_3 &\equiv -x_pM_1q + x_qM_2p \pmod{pq}, \\
V_4 &\equiv -x_pM_1q - x_qM_2p \pmod{pq}.
\end{align*}
\]

and are such that \( V_i < pq \) for \( i = 1, 2, 3, 4 \). We will now have \( V_1 + V_3 = 2x_qM_2p + \alpha pq \) for some integer \( \alpha \). Then \( V_1 + V_3 \equiv 0 (\mod p) \). On the other hand, \( V_1 + V_3 < 2pq < p^2q \). Hence \( V_1 + V_3 \not\equiv 0 (\mod p^2q) \). Therefore

\[
p = \gcd(e_{A1}, V_1 + V_3) = \gcd(p^2q, V_1 + V_3).
\]

Hence \( q = \frac{p^2q}{p} \).

8. Table of Comparison

The following is a table of comparison between RSA, ECC, NTRU and AAβ.

| Algorithm | Encryption Speed | Decryption Speed | Ratio \( M : C \) | Ratio \( M : |E| \) |
|-----------|-----------------|-----------------|-----------------|-----------------|
| RSA       | \( O(n \log n^2) \) | \( O(n \log n^2) \) | 1 : 1           | 1 : 2           |
| ECC       | \( O(n^2) \)    | \( O(n^2) \)    | 1 : 2           | 1 : 2           |
| NTRU      | \( O(n^2) \)    | \( O(n^2) \)    | Varies [4]      | N/A             |
| AAβ       | \( O(n^2) \)    | \( O(n^2) \)    | 1 : 1.75        | 1 : 1.5         |

Table 1. Comparison table for input block of length \( n \)
A NEW EFFICIENT ASYMMETRIC CRYPTOSYSTEM BASED ON THE SQUARE ROOT PROBLEM

For the decryption process off the $AA_3^\beta$ scheme, the value of $k$ stated in its complexity is a “small” constant. Empirical evidence for length $n = 512$ (i.e. the length of the prime is 512 bits) which would result in total public key length to be $6n = 3072$ bits, when decrypting.

9. Conclusion

The asymmetric scheme presented in this paper provides a secure avenue for implementors who need to transmit up to $4n$-bits of data per transmission. With an expansion rate of $1 : 1.75$ the ciphertext to be transmitted is not much more larger than the ratio of the ECC. Eventhough its expansion rate is larger than RSA, this is only natural since it is transmitting a larger data set. This will give a significant contribution in a niche area for implementation of asymmetric type security in transmitting large data sets.

The scheme is also comparable to the Rabin cryptosystem with the advantage of having a unique decryption result. It has achieved an encryption and decryption speed with complexity order of $O(n^2)$ and it also has a simple mathematical structure for easy implementation.

Acknowledgments

The authors would like to thank Prof. Dr. Abderrahmane Nitaj of Département de Mathématiques, Université de Caen, France, Dr. Yanbin Pan of Key Laboratory of Mathematics Mechanization Academy of Mathematics and Systems Science, Chinese Academy of Sciences Beijing, China and Dr. Gu Chunsheng of School of Computer Engineering, Jiangsu Teachers University of Technology, Jiangsu Province, China for valuable comments and discussion on all prior $AA_3$ designs.

References

1. W. Diffie and M. E. Hellman, “New Directions in Cryptography,” Proc. IEEE Transactions on Information Theory, pp. 644-654, 1976.
2. S.D. Galbraith, Mathematics of Public Key Cryptography, Cambridge University Press, 2012.
3. J. Hoffstein, J. Pipher and J. H. Silverman, An Introduction to Mathematical Cryptography. New York: Springer, 2008.
4. J. Hoffstein, D. Lieman, J. Pipher and J. H. Silverman, NTRU: A Public Key Cryptosystem, NTRU Cryptosystems Inc.[Online]. Available: http://GROUPEI.IEEE.ORG/GROUPS/1363/LATTPK/SUBMISSIONS/NTRU.PDF 2008.
5. J. Hermans et al., “Speed Records for NTRU,” CT-RSA 2010, LNCS 5985, pp. 73–88, 2010.
6. J. Hoffstein, J. Pipher, and J. H. Silverman. “NTRU: A Ring Based Public Key Cryptosystem in Algorithmic Number Theory,” Lecture Notes in Computer Science 1423, pp. 267–288, 1998.
7. N. Koblitz, “Elliptic Curve Cryptosystems,” Math. Comp., pp. 203–209, 1987.
8. A.J. Menezes, P.C. van Oorschot, and S.A. Vanstone, Handbook of applied cryptography, CRC Press, 1996.
9. M.O. Rabin, Digitalized signatures and public-key functions as intractable as factorization, Tech. Report MIT/LCS/TR-212, MIT Laboratory for Computer Science, 1979.
10. R. L. Rivest, A. Shamir and L. Adleman, “A method for obtaining digital signatures and public key cryptosystems,” Commun. ACM, pp. 120–126, 1978.
11. B. Schneier, Key length in Applied Cryptography. New York: Wiley & Sons, 1996.
12. M. Scott, When RSA is better than ECC.[Online]. Available: http://WWW.DERKEILER.COM/NEWSGROUPS/SCI.CRYPT/2008-11/MSG00276.HTML 2008.
13. S. S. Wagstaff, Cryptanalysis of Number Theoretic Ciphers, Chapman & Hall, 2003.
14. H. C. Williams, A modification of the RSA public key encryption procedure, IEEE Trans. Inf. Theory 26 (1980), no. 6, 726 29.
15. S. Vanstone, *ECC holds key to next generation cryptography.* [Online]. Available: http://www.design-reuse.com/articles/7409/ecc-hold-key-to-next-gen-cryptography.html 2006.

**Al-Kindi Cryptography Research Laboratory, Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, MALAYSIA**

*Current address:* Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, MALAYSIA

*E-mail address:* rezal@putra.upm.edu.my

**Al-Kindi Cryptography Research Laboratory, Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, MALAYSIA**

*Current address:* Centre of Foundation Studies for Agricultural Science, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, MALAYSIA

*E-mail address:* ma_asyraf@putra.upm.edu.my

**Al-Kindi Cryptography Research Laboratory, Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, MALAYSIA**

*Current address:* Department of Computer Systems and Communication, Faculty of Information and Communication Technology, Universiti Teknikal Malaysia Melaka, 76109 Durian Tunggal, Melaka, MALAYSIA

*E-mail address:* nura@utem.edu.my