Number of points on a family of curves over a finite field

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Abstract

In this paper we study a family of curves obtained by fibre products of hyperelliptic curves. We then exploit this family to construct examples of curves of given genus \( g \) over a finite field \( \mathbb{F}_q \) with many rational points. The results obtained improve the known bounds for a few pairs \((q, g)\).

Key words. Curves over finite fields, jacobian varieties.

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1 Introduction

Let \( C \) be a (projective, non-singular and geometrically irreducible) curve of genus \( g \) defined over the finite field \( \mathbb{F}_q \) with \( q \) elements. We denote by \( N_q(g) \) the maximum number of rational points on a curve of genus \( g \) over \( \mathbb{F}_q \), namely

\[ N_q(g) = \max \{|C(\mathbb{F}_q)| : C \text{ is a curve over } \mathbb{F}_q \text{ of genus } g\}. \]

In the last years, due mainly to applications in Coding Theory and Cryptography (see e.g.[8]), there has been considerable interest in computing \( N_q(g) \). It is a classical result that \( N_q(0) = q + 1 \). Deuring and Waterhouse[10], and Serre [6] computed \( N_q(1) \) and \( N_q(2) \) respectively. Serre also computed \( N_q(3) \) for \( q < 25 \) and Top [7] extended these computations to \( q < 100 \).

For \( g \geq 3 \) no such general formula is known. However, Serre [6] building on earlier results by Hasse and Weil proved the following upper-bound:

\[ N_q(g) \leq q + 1 + g[2\sqrt{q}], \]

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where \([x]\) denotes the floor part of a real number \(x\). This bound is now called Hasse-Weil-Serre bound.

Some partial results for some specific pairs \((q, g)\) are recorded on the website http://www.manypoints.org in the form \(N_q(g) \in [c_1, c_2]\) or \([\ldots, c_2]\), where

- \(c_2\) is the bound given by Hasse-Weil-Serre or by "explicit formulas", or by more intricate arguments.
- \(c_1\) is the bound given by the existence of a curve \(C\) with \(|C(\mathbb{F}_q)| = c_1\),
- \(\ldots\) when \(c_1 \leq c_2/\sqrt{2}\).

Our work consists in studying geometric families of curves and studying among them those which have many points over \(\mathbb{F}_q\). The curves we are considering are certain fibre products of hyperelliptic curves, we prove a formula for the genus and a formula for the number of rational points of such curves that depend are the polynomial \(f_i\)'s and the hyperelliptic they define. In particular by building curves of given genus explicitly, we have the following result:

Let \(k\) be an integer \(\geq 1\) and \(q \geq 2\) be a prime power. Let \(I\) be a non-empty subset \(\{1, 2, \ldots, k\}\) and denote by \(\mathcal{I}\) the set of all non-empty subsets of \(I\).

Let \(f_1, f_2, \ldots, f_k\) be polynomials of respective degrees \(d_i\)'s, with coefficients in the finite fields \(\mathbb{F}_q\) such that the polynomial \(f_1 \times f_2 \times \ldots \times f_k\) is separable.

We define the polynomial

\[ f_I(x) = \prod_{i \in I} f_i(x) \]

and denote by \(C_I\) the hyperelliptic curve of equation \(y^2 = f_I(x)\). Let \(A_I = q + 1 - |C_I(\mathbb{F}_q)|\). Consider now the fibre product \(C\) of which an open affine subset is given by

\[
\begin{cases} 
    y_1^2 = f_1(x) \\
    \vdots \\
    y_k^2 = f_k(x)
\end{cases}
\]

Let \(N = |C(\mathbb{F}_q)|\) where \(|C(\mathbb{F}_q)|\) is the number of \(\mathbb{F}_q\)-rational points on \(C\).

**Theorem 1.1.** In this notation:

- The genus \(g\) of \(C\) is given by:
  \[ g = 2^{k-2}(d_1 + \ldots + d_k - 4) + 1 + \delta_k, \]
Many points

with $\delta_k = 2^{k-2}$ (resp. $\delta_k = 0$) if one of the $d_i$’s is odd (resp. all of the $d_i$’s are even). There are $2^{k-1}$ (resp. $2^k$) points at infinity if one of the $d_i$’s is odd (resp. all of the $d_i$’s are even).

- The number of $\mathbb{F}_q$-rational points of $C$ is given by:

$$N = q + 1 - \sum_{I \in \mathcal{I}} A_I.$$ 

The proof of the above theorem follows from the lemma 2.1. and the lemma 3.3. for the computation of $g$ and $|C(\mathbb{F}_q)|$ respectively. In a second part of the paper, we use Theorem1.1 on specific examples in order to improve a current lower bounds for $N_q(g)$ for some values of $(q, g)$. The results we obtain are listed in Table 1.

| $g$ | $q$ | New enter | Old         | $g$ | $q$ | New enter | Old         |
|-----|-----|-----------|-------------|-----|-----|-----------|-------------|
| 5   | 17  | 48        | [...] 53    | 5   | 73  | 148       | [...] 156   |
| 19  | 52  | [...] 60  |             | 79  | 156 | [...] 165 |             |
| 23  | 62  | [...] 67  |             | 83  | 162 | [...] 172 |             |
| 29  | 72  | [...] 80  |             | 89  | 168 | [...] 180 |             |
| 31  | 76  | [...] 84  |             | 97  | 180 | [...] 193 |             |
| 37  | 88  | [...] 96  |             | $5^2$ | 64 | [...] 72  |             |
| 41  | 94  | [...] 102 |             | $13^2$ | 295 | [232, 300] |             |
| 43  | 100 | [...] 106 |             | $17^2$ | 454 | [376, 460] |             |
| 47  | 102 | [...] 113 |             | 23  | 66  | [60, 78]  |             |
| 53  | 120 | [...] 124 |             | 31  | 84  | [80, 92]  |             |
| 59  | 124 | [...] 133 |             | 41  | 104 | [102, 114] |             |
| 61  | 126 | [...] 137 |             | 59  | 134 | [132, 150] |             |
| 67  | 136 | [...] 148 |             | 7   | 29  | 80        | [72, 100]   |
| 71  | 144 | [...] 152 |             | 8   | 11  | 46        | [42, 55]    |

Table 1: The specified interval (Old) is that given by the website http://www.manypoints.org as of Avril 2016

2 Geometry of curves

Let $k$ be a positive integer. Consider now the fibre product $C = C_{f_1...f_k}$ of which an open affine subset is given by

$$\begin{align*}
y_1^2 &= f_1(x) \\
\vdots \\
y_k^2 &= f_k(x)
\end{align*}$$
where the $f_i$'s are coprime polynomials over $\mathbb{F}_q$ of respective degrees $d_i$'s.

**Lemma 2.1.** Suppose the polynomial $f(x) := \prod_i f_i(x)$ is separable, then the affine curve is smooth. The genus of the complete curve is

$$g = 2^{k-2}(d_1 + \ldots + d_k - 4) + 1 + \delta_k,$$

with $\delta_k = 2^{k-2}$ (resp. $\delta_k = 0$) if one of the $d_i$'s is odd (resp. all of the $d_i$'s are even). There are $2^{k-1}$ (resp. $2^k$) points at infinity if one of the $d_i$'s is odd (resp. all of the $d_i$'s are even).

**Proof.** Smoothness follows from the jacobian criterion applied to the matrix

$$
\begin{pmatrix}
    f'_1(x) & 2y_1 & 0 \\
    f'_2(x) & 0 & 2y_2 & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    f'_k(x) & 0 & \ldots & 2y_k
\end{pmatrix}
$$

If none of the $y_i$'s is zero, there exists a minor equal to

$$2^{k-1}y_1 \ldots y_{i-1}f'_i(x)y_{i+1} \ldots y_k \neq 0;$$

If for some $i$, $y_i = 0$, then $f_i(x) = 0$ and therefore $f'_i(x) \neq 0$ and, for $j \neq i$, we have $f_j(x) \neq 0$ and there exists a minor equal to

$$2^{k-1}y_1 \ldots y_{i-1}f'_i(x)y_{i+1} \ldots y_k \neq 0.$$

The group $G = \{\pm 1\}^k \cong (\mathbb{Z}/2\mathbb{Z})^k$ acts in an obvious way on $C$ by

$$[\varepsilon](x, y_1, \ldots, y_k) = (x, \varepsilon y_1, \ldots, \varepsilon y_k)$$

and is the Galois group of the covering

$$\phi : C \to \mathbb{P}^1$$

given by

$$(x, y_1, \ldots, y_k) \mapsto x.$$

The group $G$ acts transitively on the set $C_\infty$, of points at infinity; the inertia group is cyclic therefore is trivial or reduced to $\mathbb{Z}/2\mathbb{Z}$.

If one of the $d_i$'s is odd (resp. all $d_i$'s are even), there is ramification above $\infty \in \mathbb{P}^1$ hence we obtain $|C_\infty| = 2^{k-1}$ (resp. $2^k$). The Riemann-Hurwitz formula applied to the morphism $\phi$ gives the genus, observing that $\phi$ is branched at every point $x = \alpha_i$ with $f_i(\alpha_i) = 0$, i.e

$$(x, y_1, \ldots, y_k) = (\alpha_i, \pm \sqrt{f_1(\alpha_i)}, \ldots, 0, \ldots, \pm \sqrt{f_k(\alpha_i)}),$$

and possibly above $\infty$. 
3 Computation the number of points

We keep the same notation as before.

**Definition 3.1.** Let $I$ be a non-empty subset of $\{1, \ldots, k\}$, we define:

1. the polynomial $f_I(x) = \prod_{i \in I} f_i(x)$ and we denote by $d_I$ its degree;
2. the smooth projective curve $C_I$ whose affine model is given by $v^2 = f_I(u)$;
3. we denote by $g_I$ the genus of $C_I$;
4. the morphism $\phi_I : C \to C_I$ given by $\phi_I(x, y_1, \ldots, y_k) = (x, y_I)$ (where $y_I = \prod_{i \in I} y_i$).

We denote by $\mathcal{I}$ the set of non-empty subsets of $\{1, \ldots, k\}$.

**Lemma 3.2.** The morphism

$$\Psi : \prod_{I \in \mathcal{I}} \text{Jac}(C_I) \longrightarrow \text{Jac}(C) \quad (C_I)_{I \in \mathcal{I}} \longmapsto \sum_{I \in \mathcal{I}} \phi_I^*(C_I).$$

is a separable isogeny.

**Proof.** We first verify that $\sum_{I \in \mathcal{I}} g_I = g$; indeed, if we put $d_I = \sum_{i \in I} d_i = \text{deg} f_I$ then $g_I = \left\lfloor \frac{d_I - 1}{2} \right\rfloor$, which we may write $\frac{d_I - 1 - \varepsilon_I}{2}$ with $\varepsilon_I = 0$ or $1$.

So

$$\sum_{I \in \mathcal{I}} g_I = \sum_{I \in \mathcal{I}} \frac{d_I - 1 - \varepsilon_I}{2} = \frac{1}{2} \left( \sum_{i=1}^{k} d_i N_i - 2^k + 1 - \sum_{I} \varepsilon_I \right)$$

where $N_i$ is the number of $I$'s containing $i$, i.e. $N_i = 2^{k-1}$.

If all $d_i$ are even, all the $\varepsilon_I = 1$ and $\frac{1}{2} \sum_I (1 + \varepsilon_I) = 2^k - 1$ and the formula follows. If at least one of $d_i$ is odd, denote by $M$ the number of $I$ with $d_I$ odd, then $\frac{1}{2} \sum_I (1 + \varepsilon_I) = 2^k - 1 - \frac{M}{2}$; we conclude by using $M = 2^{k-1}$.

The abelian varieties $\prod_{I \in \mathcal{I}} \text{Jac}(C_I)$ and $\text{Jac}(C)$ have the same dimension.

Furthermore, if $\eta_j = u^{j-1} du/v$ is a regular differential form on $C_I$ (for $1 \leq j \leq g_I$) then $\omega_{I,j} := \phi_I^*(\eta_j) = x^{j-1} dx/y_I$ is regular on $C$ and these forms are linearly independent. Indeed an equality of type:

$$\sum_I \sum_{j=1}^{g_I} \lambda_{I,j} \omega_{I,j} = 0,$$

implies

$$\sum_I P_I(x) y_I = 0,$$
where $I^c := [1, k] \setminus I$; which implies the $P_I$’s are zero. The differential of $Ψ$ is an isomorphism, which proves that $Ψ$ is a separable isogeny.

**Remark.** We can deduce from the previous calculation, when $k \geq 2$, that the curve $C$ is not hyperelliptic. In fact the canonical morphism $P \mapsto (ω_{1,j}(P))$ may be written $P \mapsto (1, x, x^2, \ldots, y_1, \ldots, y_k, \ldots)$ thus is generically of degree 1, therefore an isomorphism (if $C$ were hyperelliptic, it would be a morphism of degree 2).

**Lemma 3.3.** Let $f_1, \ldots, f_k$ be polynomials with coefficients in the finite field $\mathbb{F}_q$ such that the product $f_1 \ldots f_k$ is separable, and let $C$ be the associated curve. Let $A_I = q + 1 - |C_I(\mathbb{F}_q)|$ then,

$$|C(\mathbb{F}_q)| = q + 1 - \sum_{I \in I} A_I.$$

**Proof.** The abelian varieties $\text{Jac}(C)$ and $\prod_I \text{Jac}(C_I)$ are $\mathbb{F}_q$-isogenous therefore have the same number of points on $\mathbb{F}_q^m$. If

$$|C(\mathbb{F}_q^m)| = q^m + 1 - (\beta_1^m + \ldots + \beta_{2g}^m)$$

then

$$|\text{Jac}(C)(\mathbb{F}_q^m)| = \prod_{1 \leq i \leq 2g} (1 - \beta_i^m)$$

and if

$$|C_I(\mathbb{F}_q^m)| = q^m + 1 - ((\alpha_1^{(I)})^m + \ldots + (\alpha_{2g}^{(I)})^m)$$

then

$$|\text{Jac}(C_I)(\mathbb{F}_q^m)| = \prod_{1 \leq i \leq 2g} (1 - (\alpha_i^{(I)})^m)$$

and thus therefore

$$|\text{Jac}(C)(\mathbb{F}_q^m)| = \prod_I |\text{Jac}(C_I)(\mathbb{F}_q^m)| = \prod_I \prod_j (1 - (\alpha_j^{(I)})^m),$$

$$|C_I(\mathbb{F}_q^m)| = q^m + 1 - ((\alpha_1^{(I)})^m + \ldots + (\alpha_{2g}^{(I)})^m) = q^m + 1 - \sum_{I} \sum_{1 \leq j \leq 2g} (\alpha_j^{(I)})^m,$$

and, in particular , for $m = 1$, we get

$$|C(\mathbb{F}_q)| = q + 1 - \sum_{I} \sum_{1 \leq j \leq 2g} \alpha_j^{(I)} = q + 1 - \sum_{I \in I} A_I.$$
4 Numerical examples

Examples of genus-\(g\) curves \(y_1^2 = f_1\), \(y_2^2 = f_2\) over \(\mathbb{F}_q\) having many points. The meaning of the quantities \(A_1, A_2, A_3\), and \(N\) is explained in the text. The following examples improve some results listed on the website http://www.manypoints.org and were computed using the computer algebra package Magma.

| \(q\) | \(f_1\) | \(A_1\) | \(N\) | \(f_2\) | \(A_2\) | \(A_3\) |
|-----|--------|--------|------|--------|--------|--------|
| 17  | \(x^4 + x^3 + 16x^2 + 15x + 1\) | \(-8\) | \(48\) \in \([\ldots, 53]\) | \(x^4 + 13x^3 + 16x^2 + 15\) | \(-6\) | \(-16\) |
| 19  | \(x^4 + x^3 + 18x^2 + 13x + 14\) | \(-8\) | \(52\) \in \([\ldots, 60]\) | \(x^4 + 4x^3 + 18x^2 + 7x + 12\) | \(-8\) | \(-16\) |
| 23  | \(x^4 + 19x^3 + 7\) | \(-9\) | \(60\) \in \([\ldots, 67]\) | \(x^3 + x + 11\) | \(-9\) | \(-18\) |
| 29  | \(x^4 + x^3 + 28x^2 + 28x + 18\) | \(-10\) | \(72\) \in \([\ldots, 80]\) | \(x^4 + 27x^3 + 27x^2 + 28x + 15\) | \(-10\) | \(-22\) |
| 31  | \(x^4 + x^3 + 30x^2 + 30x + 10\) | \(-11\) | \(76\) \in \([\ldots, 84]\) | \(x^4 + 30x^3 + 5x^2 + 19x + 14\) | \(-10\) | \(-23\) |
| 37  | \(x^4 + x^3 + 35x^2 + 32x + 1\) | \(-12\) | \(88\) \in \([\ldots, 96]\) | \(x^4 + 11x^3 + 29x^2 + 13x + 29\) | \(-12\) | \(-26\) |
| 41  | \(x^4 + x^3 + 40x^2 + 40x + 36\) | \(-12\) | \(94\) \in \([\ldots, 102]\) | \(x^4 + 23x^3 + 10x^2 + 20x + 36\) | \(-11\) | \(-29\) |
| 43  | \(x^4 + x^3 + 41x^2 + 34x + 1\) | \(-13\) | \(100\) \in \([\ldots, 106]\) | \(x^4 + 12x^3 + 17x^2 + 41x + 38\) | \(-13\) | \(-30\) |
| 47  | \(x^4 + 25x^3 + x^2 + 2x + 31\) | \(-12\) | \(102\) \in \([\ldots, 113]\) | \(x^3 + x + 38\) | \(-13\) | \(-29\) |
| 53  | \(f_1(x) = x^4 + x^3 + 52x^2 + 47x + 1\) | \(-14\) | \(120\) \in \([\ldots, 124]\) | \(f_2(x) = x^4 + 16x^3 + 36x^2 + 18x + 46\) | \(-14\) | \(-38\) |
| $q$ | $f_1(x)$ | $f_2(x)$ | $A_1$ | $A_2$ | $A_3$ | $N = 124 \in \ldots, 133$ |
|-----|----------|----------|-------|-------|-------|--------------------------|
| 59  | $x^4 + x^3 + 54x^2 + 6x + 1$ | $x^4 + 13x^3 + 22x^2 + 3x + 9$ | $-15$ | $-15$ | $-34$ |                       |
| 61  | $x^4 + x^3 + 58x^2 + 18x + 1$ | $x^4 + 27x^3 + 5x^2 + 47x + 28$ | $-15$ | $-14$ | $-35$ | $N = 126 \in \ldots, 137$ |
| 67  | $x^4 + x^3 + 66x^2 + 57x + 1$ | $x^4 + 2x^3 + 49x^2 + 24x + 1$ | $-16$ | $-16$ | $-36$ | $N = 136 \in \ldots, 148$ |
| 71  | $x^4 + x^3 + x^2 + 44x + 1$ | $x^4 + 9x^3 + 8x^2 + 21x + 64$ | $-16$ | $-16$ | $-40$ | $N = 144 \in \ldots, 152$ |
| 73  | $x^4 + x^3 + 66x^2 + 57x + 1$ | $x^4 + 2x^3 + 49x^2 + 24x + 1$ | $-17$ | $-17$ | $-40$ | $N = 148 \in \ldots, 156$ |
| 79  | $x^4 + x^3 + 3x^2 + 7x + 1$ | $x^4 + 4x^3 + 5x^2 + 24x + 68$ | $-16$ | $-16$ | $-36$ | $N = 156 \in \ldots, 165$ |
| 83  | $x^4 + x^3 + x^2 + 5x + 1$ | $x^4 + 72x^3 + 54x^2 + 29x + 36$ | $-18$ | $-16$ | $-44$ | $N = 162 \in \ldots, 172$ |
| 89  | $x^4 + x^3 + 3x^2 + 7x + 1$ | $x^4 + 4x^3 + 5x^2 + 24x + 68$ | $-16$ | $-16$ | $-36$ | $N = 168 \in [136, 180]$ |
| 97  | $x^4 + 8x^3 + 3x^2 + 23x + 1$ | $x^4 + 9x^3 + 63x^2 + 28x + 91$ | $-19$ | $-19$ | $-44$ | $N = 180 \in \ldots, 193$ |

| $q = 5^2$ | $f_1(x) = x^4 + x^2 + rx$ | $f_2(x) = x^4 + 2rx^2 + rx + 2$ | $r^2 + r + 1 = 0$ | $-9$ | $-9$ | $-24$ | $N = 68 \in \ldots, 72$ |
| $q = 13^2$ | $f_1(x) = (x-2)(x^2-2)$ | $f_2(x) = x^4 - 4x^2 + 1$ | $-26$ | $-21$ | $-78$ | $N = 295 \in [232, 300]$ |
| $q = 17^2$ | $f_1(x) = (x+2)(x+10)(x^2+6)$ | $f_2(x) = (x^2+10)(x^2+6x+3)$ | $-33$ | $-29$ | $-102$ | $N = 454 \in [376, 460]$ |

Table 2: Examples of genus-5.
Many points

\[
q = 23 \quad f_1(x) = x^5 + 12x^4 + 19x^3 + x + 2 \\
   f_2(x) = x^3 + 12x^2 + 18x + 4 \\
A_1 = -13 \quad N = 66 \in [60, 78]
\]

\[
q = 31 \quad f_1(x) = x^5 + 6x^4 + 4x^3 + 4x^2 + 17x + 16 \\
   f_2(x) = (x)^3 + 13x^2 + 15x + 13 \\
A_1 = -19 \quad N = 84 \in [80, 92]
\]

\[
q = 41 \quad f_1(x) = x^5 + 31x^4 + 11x^3 + 14x^2 + 35x + 40 \\
   f_2(x) = x^3 + 23x^2 + 5x + 17 \\
A_1 = -23 \quad N = 104 \in [102, 114]
\]

\[
q = 59 \quad f_1(x) = x^5 + 57x^4 + 17x^3 + 48 \\
   f_2(x) = x^3 + 2x + 22 \\
A_1 = -25 \quad N = 134 \in [132, 150]
\]

Table 3: Examples of genus-6.

\[
q = 29 \quad f_1(x) = x^6 + 9x^5 + 3x^4 + 1 \\
   f_2(x) = x^4 + 13x^3 + 28x^2 + 12x + 1 \\
A_1 = -46 \quad N = 80 \in [72, 100]
\]

\[
A_2 = -40 \\
A_3 = -54
\]

Table 4: Examples of genus-7.

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