Dissipative superfluid dynamics from gravity

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Abstract:
Charged asymptotically $AdS_5$ black branes are sometimes unstable to the condensation of charged scalar fields. For fields of infinite charge and squared mass $-4$ Herzog was able to analytically determine the phase transition temperature and compute the endpoint of this instability in the neighborhood of the phase transition. We generalize Herzog’s construction by perturbing away from infinite charge in an expansion in inverse charge and use the solutions so obtained as input for the fluid gravity map. Our tube wise construction of patched up locally hairy black brane solutions yields a one to one map from the space of solutions of superfluid dynamics to the long wavelength solutions of the Einstein Maxwell system. We obtain explicit expressions for the metric, gauge field and scalar field dual to an arbitrary superfluid flow at first order in the derivative expansion. Our construction allows us to read off the leading dissipative corrections to the perfect superfluid stress tensor, current and Josephson equations. A general framework for dissipative superfluid dynamics was worked out by Landau and Lifshitz for zero superfluid velocity and generalized to nonzero fluid velocity by Clark and Putterman. Our gravitational results do not fit into the 13 parameter Clark-Putterman framework. Purely within fluid dynamics we present a consistent new generalization of Clark and Putterman’s equations to a set of superfluid equations parameterized by 14 dissipative parameters. The results of our gravitational calculation fit perfectly into this enlarged framework. In particular we compute all the dissipative constants for the gravitational superfluid.
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1. Introduction and Summary

It was pointed out by Gubser [1] that charged asymptotically $AdS_5$ black branes are sometimes unstable in the presence of charged scalar fields. The endpoint of this instability is a hairy black brane: a black brane immersed in a charged scalar condensate. The AdS/CFT correspondence maps the hairy black brane to a phase in which a global $U(1)$ symmetry is spontaneously broken by the vacuum expectation value of a charged scalar operator. In condensed matter physics a phase with a spontaneously broken global $U(1)$ symmetry is referred to as a superfluid.

In this paper we study aspects of the fluid dynamical description of superfluids using the AdS/CFT correspondence. Our work builds on a large body of earlier studies (see e.g. [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] ) but differs from these works in its emphasis on the study of dissipative terms in superfluid dynamics.

As we describe in detail in the next section, the variables of relativistic superfluid dynamics consist of two velocity fields; the normal fluid velocity $u\mu$ and a superfluid velocity field $u^\mu_s$; together with a
temperature and chemical potential field. The superfluid velocity is the unit vector in the direction of \(-\xi_\mu\) where \(\xi_\mu\) is the gradient of the phase of the scalar condensate. Conservation of the stress tensor and charge current together with the assertion that \(\xi_\mu\) is curl free constitute the equations of superfluid dynamics. These equations constitute a closed dynamical system once they are supplemented with constitutive relations that express the stress tensor, charge current and the component of \(\xi_\mu\) along the normal velocity as functions of the fluid dynamical variables. As superfluid dynamics is a long distance effective field theory, it is natural to specify the relevant constitutive relations in an expansion in derivatives.

Over fifty years ago Landau and Tiza [23, 24] presented a simple and elegant proposal for the structure of superfluid constitutive relations at leading (zero) order in the derivative expansion. Landau and Tiza (see §2 below) proposed a form for the constitutive relations that is entirely determined by a single thermodynamical ‘Free Energy’ \(P(T, \mu, \xi)\). An obvious first question in the study of holographic superfluids is the following: do the perfect fluid constitutive relations of holographic superfluids respect the Landau-Tiza ansatz? This question was largely answered in the affirmative in a beautiful recent paper by Sonner and Withers [7]. These authors used Einstein’s equations and the holographic dictionary to demonstrate that the stress tensor and charge current of a homogeneous stationary holographic superfluid flow takes the form predicted by Landau and Tiza, with the free energy or pressure interpreted as the value of Einstein’s action for the relevant bulk solutions. We re-derive and slightly extend the results of Sonner and Withers in §3 and Appendix B below. As we review in Appendix B, a beautiful feature of the gravitational derivation of the Landau-Tiza model is its abstract nature. The results of Sonner and Withers are derived on general grounds, and do not use the explicit form of the gravitational solution dual to homogeneous stationary superfluid flows. This is fortunate as no completely explicit analytic solutions are known for static hairy black branes (i.e. holographic superfluids at rest) much less for hairy black branes in motion.

There also exists a large literature on the subject of dissipative corrections to the equations of superfluid dynamics, accurate to first order in the derivative expansion (see e.g. [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 21]). In particular, a general first order theory of dissipative fluid dynamics was presented by Landau and Lifshitz [27] (under the assumption of small superfluid velocities) and was generalized to apply to flows with finite superfluid velocities in [25, 26]. It turns out that the most general one derivative corrections to the equations of perfect superfluid dynamics are parameterized by 36 dissipative corrections (assuming parity invariance). Like Landau and Lifshitz, in [25, 26] the authors assumed that the entropy current of superfluid dynamics takes a natural ‘canonical’ form at one derivative order. Once again, like Landau and Lifshitz, the authors in [25, 26] used the requirement of local positivity of entropy production, together with the ‘Onsager Reciprocity relations’ to cut the parameter space for physically permissible equations of superfluids down to a 13 parameter set (see Appendix VI in [26]). To our knowledge this Clark-Putterman formalism [25, 26] has not been tested by any first principles dynamical calculations within a quantum field theory (e.g. has not been derived from a field theoretically motivated set of Boltzmann equations). The AdS/CFT correspondence offers us the opportunity to test this abstract formulation of viscous superfluid dynamics.

In this paper we perform such a test in a very particular holographic super fluid. The system we study is a small perturbation of the ‘analytic superconductor’ studied by Herzog in [6]. In very broad terms we use the fluid gravity map [28, 30, 41, 42, 43, 44, 45] to derive the equations of superfluid dynamics from Einstein equations to first order in the derivative expansion. We then read off the constitutive relations of the stress tensor, charge current and Josephson equation at first order in the derivative expansion. In the rest of this introduction we describe our calculations, their results and their interpretation in more detail.
As we have mentioned we work with a generalization of the model studied by Herzog [6]. Herzog’s model consists of the Einstein Maxwell system interacting with a charged scalar field of $m^2 = -4$ in the so called probe limit of infinite charge $e$. Herzog demonstrated that this system undergoes a second order phase transition towards superfluidity whenever $|\mu| \geq 2$. The stable gravitational solution, for $|\mu|$, just larger that 2, has a background scalar vev. Let $\epsilon$ denote the value of this vev. In [6] Herzog analytically determined the relevant bulk solutions perturbatively in $\epsilon$ and separately in the difference between superfluid and normal velocities.

To start with we generalize Herzog’s infinite charge solutions beyond the strict probe approximation, to first nontrivial order in the $1/e^2$. This generalization is necessary in order to allow for the study of the response of the normal velocity and temperature fields to the dynamics of the superfluid velocity and chemical potential fields. As a check on our algebra we explicitly verify that our solutions obey all the predictions of the Landau Tiza model, to the order that we are able to compute, in accordance with the results of Sonner and Withers and of Appendix B. We then proceed to use these solutions as raw ingredients for the fluid gravity correspondence [38, 39, 40, 41, 42, 43, 44, 45].

Following the procedure of the fluid gravity correspondence, we search for solutions of the Einstein Maxwell scalar system that tube wise approximate the stationary solutions described in the previous paragraph. More explicitly, we study a perturbative expansion to the solutions of Einstein’s equations whose first term is given by the stationary solutions of the previous paragraph -written in ingoing Eddington Finklestein coordinates - with the eight parameters of equilibrium superfluid flows replaced by slowly varying functions of spacetime. The configuration described in this paragraph does not obey the bulk equations; however it may sometimes be systematically corrected, order by order in boundary derivatives, to yield a solution to these equations. This procedure works if and only if our eight fields are chosen such that $\xi^{\mu}(x)$ is curl free, and such that the energy momentum and charge current built out of these fields is conserved. The constitutive relations that allow us to express the stress tensor and charge current in terms of fluid dynamical fields is generated by the perturbative procedure itself. In other words the output of our perturbative procedure is a set of gravitational solutions that are in one to one correspondence with the solutions of superfluid dynamics, with superfluid constitutive relations that are determined by the bulk gravitational equations.

Note that the construction described in the previous paragraph is carried out in a triple expansion. We follow Herzog to expand our equilibrium solutions in a power series in the deviations from criticality (let us denote the relevant parameter by $\epsilon$), and further expand these solutions in a power series in $1/e^2$. We then go on to use the solutions as ingredients in a spacetime derivative expansion. We would like to emphasize that this procedure is sensible only if the derivatives times mean free path are assumed to be parametrically smaller than $\epsilon$. The physical reason for this is that precisely at $\epsilon = 0$ we have a new massless mode, corresponding to fluctuations of the vev of the charged scalar field. When $\epsilon$ is nonzero this mode is no longer massless, but it is light at small $\epsilon$. The fluid dynamical description ignores the dynamics of this light mode. This is justified only when derivatives are all much smaller than the mass of this mode, i.e. when derivatives are parametrically small compared to $\epsilon$. The fluid dynamical expansion breaks down if derivatives are held fixed as $\epsilon$ is taken to zero. This fact formally shows up in the blow up of several dissipative fluid coefficients at small $\epsilon$ in the equations presented in this paper.

We have implemented the perturbative procedure that determines the gravitational solution dual to superfluid dynamics to first order in the derivative expansion, and thereby determined the stress tensor, charge current and Josephson equation of our holographic superfluid to first order in the derivative expansion. Our results pass several consistency checks and also have several attractive features. To start with our results respect Weyl invariance. This fact, while necessarily true of the
results of any gravitational calculation in an asymptotically AdS space, is not algebraically automatic, and so yields consistency check on the rather involved algebraic manipulations of our paper.

More importantly, we are also able to compute the natural gravitational entropy current for our gravitational fluid flow. As explained in [39], a gravitational construction of fluid dynamics is always automatically accompanied a local entropy current that is guaranteed, by the area increase theorems of general relativity, to be of positive divergence. We demonstrate by explicit computation in our particular solution that the gravitational entropy current of [39] agrees precisely with the ‘canonical’ entropy current (see §2.3 for details) of Clark and Putterman (and Landau Lifshitz) [25, 26, 27]. We regard this agreement as a nontrivial check of one of the central physical assumptions of the Landau Lifshitz and Clark Putterman formulations of dissipative fluid dynamics. Despite this fact, however, the results we obtain for the dissipative coefficients for the gravitational superfluid described in this paper do not fit into Clark-Putterman’s 13 parameter framework for dissipative fluid dynamics.

Given the fact that the gravitational entropy current agrees with the current employed by Clark and Putterman [25, 26], it is puzzling that the gravitational results do not fit into Clark and Putterman’s framework. We believe that the resolution to this puzzle is that Clark and Putterman missed a parameter in their analysis. In §3 below we demonstrate that the most general equations of super fluid dynamics consistent with positivity of the canonical entropy current is parameterized by 21 dissipative coefficients rather than 20 as mistakenly asserted by Clark and Putterman. Imposition of the 7 Onsager relations then leaves us with a 14 (rather than 13) parameter set of consistent equations of first order dissipative super fluid dynamics. We present this generalized 14 parameter set of equations for dissipative fluid dynamics in §3 below. It turns out that our gravitational results fit perfectly into this enlarged 14 parameter framework.

Our paper is organized as follows. In §2, §3 and §4 we present a general framework for relativistic superfluid dynamics including dissipative effects at first order in the derivative expansion. These sections are purely fluid dynamical and makes no reference to the equations of gravity or the AdS/CFT correspondence. The chief new result of §3 is the presentation of the 14 parameter set of equations that generalize Clark and Putterman’s 13 parameter equations [25, 26]. We also demonstrate that the equations of Weyl invariant superfluid dynamics are parameterized by 10 dissipative parameters, and present the most general form for these equations. In §3 below we review and slightly generalize Sonner and Wither’s gravitational derivation of the Landau Tiza 2 fluid model from gravity. In §3 we perturb Herzog’s construction of hairy black branes away from the strict large charge or probe limit. In §3 we use the results of §3 as an input into the fluid gravity map to generate the gravitational solutions dual to superfluid flows. We compute the dissipative part of the stress tensor, charge current and entropy current dual to superfluid flows and also verify the Weyl invariance of our results.

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1We should, however, emphasize that we have checked this agreement only to leading nontrivial order in the $\epsilon$ expansion. It would certainly be useful to verify this agreement - and all the other results of our paper - at higher orders in the $\epsilon$ expansion, but we leave that for future work.

2In an earlier version of this draft we compared our results to those of Landau and Lifshitz but not to those of Clark and Putterman, as we were unaware of their work. We reported that our results did not agree with the predictions of Landau and Lifshitz; while this is true, the comparison itself is inappropriate, as Landau and Lifshitz apparently intended their analysis to apply only to the limit of zero superfluid velocity, while in this paper we work at arbitrary superfluid velocity. In this version of our paper we have instead compared our results to the predictions of Clark and Putterman who explicitly work at finite superfluid velocity. We once again find disagreement with their predictions, and have motivated us to construct a slight generalization of the Clark-Putterman formalism, with which our gravitational results now agree. We thank C. Herzog and A. Yarom for making us aware of Clark and Putterman’s work.

3Imposing conformal invariance (which is relevant for the gravitational calculation) sets 4 out of these 14 parameters to zero (see §3.4 for more details).
transform our gravitational stress tensor and charge current to a frame convenient for the study of fluid
dynamics, and demonstrate that our gravitational results are a special case of the 14 parameter set of
dissipative fluid dynamical equations derived in §. We also explicitly list the values of all dissipative
parameters of our gravitational superfluid. In appendix A we extend the discussion of the canonical
entropy current in §2.3 to demonstrate that it is independent of the choice of frame and also compute
its divergence. In appendix B we present an abstract geometrical derivation of the analogue of Gibbs
Duhem relations for superfluids in thermodynamic equilibrium directly from gravity. In appendix
C we show the existence of an upper bound for the superfluid velocity beyond which the superfluid
phase is unstable. Finally, in appendix D we complete the discussion of §7.5 by manifestly recovering
complete SO(3) rotational invariance in the zero superfluid velocity limit.

Note Added: Our paper has substantial overlap with [46], which was posted on the ArXiv simulta-
neously with the first version of this paper.

2. Review of relativistic superfluid dynamics

2.1 Relativistic superfluids in equilibrium

Consider a relativistic quantum field theory with a conserved U(1) charge. In the sequel we denote
the conserved U(1) current of this theory by $J^\mu$ and the conserved stress tensor by $T^{\mu\nu}$. Consider
this system at a finite temperature $T$ and finite chemical potential $\mu$. It is conceivable that a charged
operator $O$ (charged under the U(1) charge described above) develops a nonzero vev over a certa-
ing range of temperatures and chemical potentials. Whenever this happens our system is said to display
superfluidity. Superfluidity, then, is associated with the spontaneous breakdown of a global U(1)
symmetry. One of the most striking facts about a superfluid is that it admits more stationary homo-
genous solutions to the equations of motion, on $R^{3,1}$, that might naively have been supposed. Any
system with a U(1) charge admits at least a 5 parameter set of homogeneous solutions on $R^{3,1}$; these
solutions represent the system in equilibrium at temperature $T$ and chemical potential $\mu$, moving at
a uniform four velocity $u^\mu$. However superfluids actually admit an 8 parameter set of homogeneous
stationary solutions, as we will now explain.

Let $\psi$ denote the phase of the condensed operator $O$. $\psi$ is effectively a massless scalar field in
the superfluid phase. A massless scalar field admits solutions of the form $\psi(x) = e^{\xi_\mu x^\mu}$ for arbitrary
constant values of $\xi_\mu$ (here $e$ is the charge of the operator $O$). It turns out that a superfluid admits
homogeneous stationary solutions at every constant value of $T$, $w^\mu$ and $\xi_\mu$.

There is another way to think of the 8 parameter set of solutions listed above. We can ‘gauge’ the
global symmetry of the field theory, and so couple the theory to a non dynamical gauge field $A_\mu$. The
solutions described above may all, equivalently, be thought of as solutions with constant values of the
phase $\psi$, but with the non dynamical gauge fields

$$A_\mu = \xi_\mu$$

where $e$ is the charge of the field. The equivalence of these two ways of describing these solutions follows
from the fact that a phase $e^{\xi_\mu x^\mu}$ of a field of charge $e$ is gauge equivalent to the gauge field listed
above.

Finally some definitions. We define

$$\mu = \xi_\mu w^\mu.$$  

$\mu$ is referred to as the chemical potential of the system (or sometimes as the chemical potential of the
normal part of the system). This terminology is reasonable as $\mu$ is equal to the asymptotic value of
the time component of the gauge field in the frame in which the normal fluid is at rest (i.e. in the frame in which \( u^\mu = (1, 0, 0, 0) \)). Also let

\[ \xi = \sqrt{-\xi^\mu \xi_\mu}. \]

We define the normalized 4 vector

\[ u_s^\mu = -\frac{\xi^\mu}{\xi} \]

and will refer to \( u_s^\mu \) as the ‘superfluid velocity’ in the solution. Roughly speaking, \( u_s^\mu \) is the four velocity at which the stuff associated with the condensate, \( \langle O \rangle \) moves.

With all this terminology in place, we can reword the central assertion of the previous paragraph as follows: the stationary non dissipative solutions of a superfluid are labeled by the three components of \( u^\mu \), the ‘velocity’ of the normal component of the fluid, the three components of \( u_s^\mu \), the velocity of the superfluid (or condensate) of the fluid, the temperature \( T \) and the chemical potential \( \mu \). Most importantly we have exactly stationary solutions in which the ‘normal’ and ‘superfluid’ parts of the system move at arbitrary speeds with respect to one another.

We now turn to the quantitative characterization of the stationary homogeneous flows described above. We would first like quantitative expressions for the stress tensor and the charged current of the superfluid, as a function of the eight parameters characterizing flow. These eight parameters have three associated scalars, namely \( T, \mu, \xi \). Symmetry considerations immediately allow us to write expressions for the stress tensor and current in terms of a set of arbitrary functions of these scalar fields.

\[
\begin{align*}
T^{\mu \nu} &= (\rho_n + P)u^\mu u^\nu + P\eta^{\mu \nu} + \frac{\rho_s}{\xi^2} \xi^\mu \xi^\nu \\
&= (\rho_n + P)u^\mu u^\nu + P\eta^{\mu \nu} + \rho_s u_s^\mu u_s^\nu \\
J^\mu &= q_n u^\mu - q_s \frac{\xi^\mu}{\xi} \\
&= q_n u^\mu + q_s u_s^\mu \\
u^\mu \xi_\mu &= \mu
\end{align*}
\]

The third equation in (2.1) is simply a statement of definitions.

The reader may wonder why we have not allowed a \( u^\mu \xi^\nu \) cross term in the first of (2.1). The reason is as follows. Were such a cross term to in fact appear in the expansion, we could get rid of it by a redefinition of \( u^\mu \). The assertion that no such cross term appears in the expansion of the stress tensor constitutes our definition of the normal velocity \( u^\mu \).

The quantities \( \rho_n, P, \rho_s, q_n, q_s \) are all as yet arbitrary functions of the three scalar quantities \( (T, \mu, \xi) \). The claim of the Landau Tiza two fluid theory is that all these quantities may be derived from a single function, the pressure of these solutions, \( P = P(T, \mu, \xi) \) via the formulas

\[
\begin{align*}
\rho_n + P &= q_n \mu + Ts \\
\rho_s &= \mu_s q_s \\
\mu_s &= \xi = \xi^\mu u_s^\mu \\
dP &= s dT + q_s d\mu_s + q_n d\mu \\
&= s dT + q_s d\xi + q_n d\mu
\end{align*}
\]

\( - 7 - \)
2.2 Relativistic Superfluid Dynamics

At long distances (compared to an effective mean free path) a superfluid admits a fluid dynamical description. In this subsection we will present what appears to us to be the simplest and most logically satisfactory formulation of the theory of dissipative super fluid dynamics. Our presentation differs by a field redefinition from the more ‘traditional’ presentations of, for instance, Clark, Putterman and Landau Lifshitz [25, 26, 27]. We will later describe the field redefinitions that allow us to transform our formulation to the more traditional one.

In the relativistic context we choose the variables of superfluid dynamics simply to be $u^\mu(x)$, the four velocity of the ‘normal’ fluid, $T(x)$ the local temperature of the fluid, and $\xi^\mu(x)$, the local value of the gradient of the superfluid phase. More specifically, $\xi^\mu$ is given in terms of the superfluid phase $\psi(x)$ by

$$\xi^\mu(x) = -\partial^\mu \psi(x) \quad (2.3)$$

We have a total of eight fluid dynamical fields. We will also sometimes use the terminology

$$\mu(x) = u(x).\xi(x). \quad (2.4)$$

We emphasize, though, that within the formulation presented in this subsection, $\mu(x)$ is not an independent dynamical field, but merely terminology for the projection of $\xi(x)$ on the normal velocity field $u(x)$.

The equations of super fluid dynamics consist of the following relations

$$\partial_\mu T^{\mu\nu} = 0$$
$$\partial_\mu J^\mu = 0$$
$$\partial_\mu \xi_\nu - \partial_\nu \xi_\mu = 0 \quad (2.5)$$

The first two of these equations are simply the statement of the conservation of the stress tensor and the conservation of the charge current; these equations are true in any field theory. The last of these equations asserts that $\xi_\nu$ is the gradient of a scalar. In order to see that we have as many independent equations as variables, we note that (2.3) is a local solution of the last equation in (2.5). This leaves us with 5 variables $u^\mu(x)$, $T(x)$ and $\psi(x)$ subject to the 5 remaining equations in the first two lines of (2.5).

While $\xi_\mu(x)$ may be traded for $\psi(x)$ for counting purposes, the equations of fluid dynamics are more conveniently formulated directly in terms of the variables $\xi_\mu(x)$ rather than the phase field $\psi(x)$. The reason for this is that gradients of $\psi(x)$ are not necessarily small in the regime of validity of superfluid dynamics, while gradients of $\xi^\mu(x)$ necessarily are. The introduction of the variables $\xi_\mu(x)$ allows us to formulate the equations of superfluid dynamics in a systematic derivative expansion of all its participating fields.

The equations of superfluid dynamics constitute a closed dynamical system once they are combined with constitutive relations that determine the stress tensor and charge current in terms of the variables of fluid dynamics $u^\mu$, $T$, $\xi^\mu$. The constitutive relations take the form

$$T^{\mu\nu} = (\rho_n + P)u^\mu u^\nu + P\eta^{\mu\nu} + \frac{\rho_s}{\xi^2} \xi^\mu \xi^\nu + \pi^{\mu\nu}$$
$$J^\mu = q_n u^\mu - q_s \frac{\xi^\mu}{\xi} + J^\mu_{\text{diss}} \quad (2.6)$$

where $\pi^{\mu\nu}$ and $J^\mu_{\text{diss}}$ are respectively tensors and vectors that are first or higher order in an expansion in derivatives (of the fluid dynamical fields) and all other quantities were defined in the previous subsection.
2.2.1 A canonical fluid frame

The equations of superfluid dynamics change their detailed form under field redefinitions. The quantity \( \xi_{\mu}(x) \) has a microscopic definition. In the rest of this paper \( \xi_{\mu}(x) \) will always refer to this microscopically unambiguous value, and we will not permit arbitrary field redefinitions of \( \xi_{\mu}(x) \).

On the other hand the fluid variables \( T(x) \) and \( u^\mu(x) \) are only really defined in thermodynamical equilibrium, and so allow for possible field redefinitions at derivative order (as derivatives parameterize departures from thermodynamical equilibrium).

In order to completely specify the equations of superfluid dynamics we need to specify unambiguous definitions of the thermodynamical fields \( T(x) \), and \( u^\mu(x) \). We specify these definitions by prescribing certain conditions on the dissipative terms \( \pi^{\mu\nu} \) and \( J_{\text{diss}}^{\mu} \). We must choose these conditions so that they are not automatic, but can always be reached by an appropriate field redefinition, and completely fix field redefinition ambiguity. For instance we could work with the superfluid analogue of the ‘Landau Frame’ of ordinary fluid dynamics

\[
\pi_{\mu\nu} u^\nu = 0
\]

These 4 conditions serve to provide unambiguous definitions for the velocity, chemical potential \(^4\) and temperature fields. For reasons that will become clearer below, we will refer to this choice of fluid field variables as the \( \mu_{\text{diss}} = 0 \) frame.

The equation (2.7) may equivalently be written as

\[
T^{\mu\nu} u_\nu = -\rho_n u^\mu + (u.u_s)\rho_s u^\mu
\]

where it is important that the functions \( \rho_n \) and \( \rho_s \) on the RHS of (2.8) are not independent functions, but are related to each other by the thermodynamical relations \(2.2\) \(^5\).

2.3 ‘Fluid Frames’

While the frame presented in the previous subsection seems to us to be rather natural from several points of view, it turns out not to be the fluid frame most commonly employed in analysis of superfluid dynamics. In this subsection we will describe a generalized framework for superfluid dynamics. In the next subsection we will describe how to transform between fluid frames.

Conventional descriptions of superfluid dynamics are presented in terms of 9 fields subject to a single constraint rather than 8 independent fields. In this subsection we will take these 9 fields to be \( T(x) \), \( \mu(x) \), \( u^\mu(x) \) and \( \xi^\mu(x) \).

While in the the previous subsection the field \( \mu(x) \) was simply convenient notation for \( u^\mu(x)\xi_{\mu}(x) \), in this subsection \( \mu(x) \) is an independent field variable; the relation between \( \mu(x) \) and \( u(x)\xi(x) \) is taken to be given by the so called Josephson equation

\[
u(x)\xi(x) = \mu(x) + \mu_{\text{diss}}(x)
\]

where \( \mu_{\text{diss}}(x) \) is a function of derivatives of fluid variables (i.e. it vanishes when all fluid variables are constants in spacetime). The quantity \( \mu_{\text{diss}} \) will be chosen in order to ensure that another condition

\(^4\)Note that here \( \mu(x) \) is not an independent field and is given by (2.4).

\(^5\)In our presentation above we have used the temperature field as one of the independent fields of the superfluid dynamical description. Of course this choice is arbitrary; we could as well use any other thermodynamical field (for instance the energy density \( \rho_n \) instead of the temperature, where \( \rho_n \) is defined as the thermodynamical function of \( T, u, \xi \) and \( \xi \) in place of the temperature. As such thermodynamical reparameterizations are ultralocal (i.e. do not involve derivatives) they do not affect the split of the stress tensor and current into perfect fluid and dissipative parts, and so do not constitute a change of frame.
we specify below is satisfied. Comparing (2.4) with (2.9) explains why we referred to the frame of the previous section as the $\mu_{\text{diss}} = 0$ frame.

The stress tensor and charge current continue to be given by the form (2.6), with the understanding, however, that all thermodynamic functions in those formulas are to be regarded as functions of the fields $(T(x), \mu(x), \xi(x))$ rather than the fields $(T(x), u(x), \xi(x))$ (these two choices were identical in the previous subsection). As $\mu(x)$ is not equal to $u(x)$ for a general fluid flow, it follows that the perfect fluid part of the fluid stress tensor (at a given spacetime point) is, in general, not equal to stress tensor for any equilibrium flow (this is a complication that was absent in our formulation of the previous subsection).

In the current formulation we have 5 thermodynamical fields - $T(x)$, $u(\mu)(x)$ and $\mu(x)$ - that require precise definitions. This requires us to specify five equations (generalizing the four conditions in, e.g. (2.7)) to give meaning to these fields. One natural choice [29] for these equations is

$$\pi_{\mu\nu} u^\mu = 0$$
$$J_{\text{diss}}^\mu = 0.$$  \hfill (2.10)

We refer to the fluid frame defined by these equations as the Transverse Frame. These equations effectively determine the previously undetermined quantity $\mu_{\text{diss}}$.

Of course (2.10) is only one set of the possible set of five equations we must impose on our thermodynamical variables. Many other choices are possible. One other possible choice is

$$\pi_{\mu\nu} u^\mu = 0$$
$$\mu_{\text{diss}} = 0.$$  \hfill (2.11)

which defines the $\mu_{\text{diss}} = 0$ frame of the previous subsection. It follows that the formulation of superfluid dynamics, described in this subsection, is a generalization of the formulation of the previous subsection, and includes the later as a special case.

We end this subsection by emphasizing our terminology. We refer to the formulation of fluid dynamics, presented in this subsection, as the formulation in terms of fluid frames. The key feature of this description of fluid dynamics is that the full microscopically defined gradient of phase, $\xi^\mu$, is taken as one of the variables of description. The perfect fluid part of the stress tensor and charge current is written in terms of $\xi^\mu$ and thermodynamical functions of $\xi$. Superfluid dynamics in a fluid frame is to be contrasted with super fluid dynamics in a modified phase frames, introduced in §2.6.

### 2.4 Transforming between fluid frames

In this subsection we supply the equations that allow us to transform between fluid frames. Let us suppose we want to make a change of variables that will take us from a completely unspecified fluid frame labeled ‘our’ to another frame labeled ‘there’, where the frame ‘there’ is a well defined fluid frame (e.g. the transverse or the $\mu_{\text{diss}} = 0$ frame). We set

$$u^\mu_{\text{there}} = u^\mu_{\text{our}} + \delta u^\mu$$
$$T_{\text{there}} = T_{\text{our}} + \delta T$$
$$\mu_{\text{there}} = \mu_{\text{our}} + \delta \mu$$  \hfill (2.12)

The quantities $\delta u^\mu$, $\delta T$ and $\delta \mu$ are necessarily of first or higher order in derivatives. As we work only to first order in derivatives in this section, we will effectively work only to linear order in these variations.
Note that in order that $u^\mu_{\text{there}}$ and $u^\mu_{\text{our}}$ both be unit normalized, it is necessary that
\[ \delta u^\mu u^\mu = 0 \]
(at the order at which we work $\delta u^\mu u^\mu_{\text{our}} = \delta u^\mu u^\mu_{\text{there}}$ as the two differ at quadratic order in $\delta$. Whenever an equation is true both of - say - $u^\mu_{\text{our}}$ and $u^\mu_{\text{there}}$, we will simply omit the subscript in this subsection).

With infinitesimal variations restricted as above we find
\[
\begin{align*}
\delta \pi^\mu{}_{\nu} &= (u^\mu \delta u^\nu + u^\nu \delta u^\mu)(P + \rho_n) + u^\mu u^\nu d(P + \rho_n) + \frac{\xi \xi^\nu}{\xi^2} d(\rho_s) + \eta^\mu{}^\nu dP \\
\delta J^\mu_{\text{diss}} &= q_n \delta u^\mu + dq_n u^\mu - dq_s \frac{\xi^\mu}{\xi} \\
\delta \mu_{\text{diss}} &= -\delta u^\mu \xi^\mu + \delta \mu.
\end{align*}
\] (2.13)

where by
\[
\begin{align*}
\delta \pi^\mu{}^\nu &= \pi^\mu{}^\nu_{\text{our}} - \pi^\mu{}^\nu_{\text{there}} \\
\delta J^\mu_{\text{diss}} &= (J^\mu_{\text{diss}})_{\text{our}} - (J^\mu_{\text{diss}})_{\text{there}}
\end{align*}
\]

and in these equations the symbol $df(\mu, \xi, T)$ represents the change in the function $f$ under the first order variable change (2.12). These equations may then be used to obtain the 5 equations that determine the five unknowns $\delta \mu$, $\delta T$, and $\delta u^\mu$. For instance, if we are interested in transforming to the transverse frame (i.e. we want to set ‘there’ to be the transverse frame) we would require that $\pi^\mu{}^\nu_{\text{there}}$ and $J^\mu_{\text{there}}$ be orthogonal to $u^\mu_{\text{there}}$, giving the equations
\[
\begin{align*}
d\rho_n + \frac{\mu^2}{\xi^2} d\rho_s - \pi^\mu{}^\nu_{\text{our}} u^\mu u^\nu &= 0 \\
(P + \rho_n) \delta u^\mu &= \frac{\mu}{\xi^2} d\rho_s \xi^\mu - u^\mu d\rho_n - \pi^\mu{}^\nu_{\text{our}} u^\nu \\
dq_n + dq_s \frac{\mu}{\xi} + (J^\mu_{\text{diss}})_{\text{our}} u^\mu &= 0
\end{align*}
\] (2.14)

The 5 equations (2.14) determine $\delta u^\mu$, $\delta \mu$, $\delta T$, and so the change in the dissipative part of the stress tensor and current etc from (2.13). A similar procedure may be employed to transform into the $\mu_{\text{diss}} = 0$ frame or any other frame of interest.

2.5 A canonical Entropy Current

In this section we will define a ‘canonical’ entropy current in any fluid frame. Our definition is
\[
J^\mu_s = s u^\mu - \frac{\mu}{T} J^\mu_{\text{diss}} - \frac{u^\mu \pi^\mu{}^\nu}{T}
\] (2.15)

where $s$ is the thermodynamical entropy density of our fluid.

Although this is not obvious, we have shown in Appendix [A] that this current is frame invariant. This means, for instance, that (2.15) defines the same vector field in the transverse as well as the $\mu_{\text{diss}} = 0$ frames.

In the same Appendix we have also demonstrated that the divergence of this current is given, in any fluid frame, by
\[
\partial^\mu J^\mu_s = -\partial^\mu \left[ \frac{u^\mu}{T} \right] \pi^\mu{}^\nu - \partial^\mu \left[ \frac{\mu}{T} \right] J^\mu_{\text{diss}} + \frac{\mu_{\text{diss}}}{T} \partial^\mu \left( \frac{\rho_s \xi^\mu}{\xi^2} \right)
\] (2.16)
2.6 Modified Phase frames

As we have emphasized above, the theory of superfluid dynamics formulated in any fluid frame in which \( \mu_{\text{diss}} \neq 0 \) (like the transverse frame) has slightly hybrid features. The perfect fluid part of the charge current and stress tensor involves thermodynamical functions of \( \mu \) not \( u, \xi \), but is written (in index structure) in terms of the full field \( \xi^\mu \). This ensures that, for a general fluid flow, the prefect fluid stress tensor and charge current, at any given point, does not equal the stress tensor and charge current for any equilibrium flow.

Fluid dynamics formulated in a modified phase frame eliminates this slightly unpleasant feature by working with a modified gradient of phase field, \( \xi_0^\mu (x) \) defined as

\[
\xi_0^\mu = -\mu u^\mu + w^\mu \\
\xi_0 = \sqrt{\mu^2 - w^2}
\]

where \( w^\mu \) is defined as the part of \( \xi^\mu \) projected orthogonal to \( u^\mu \).

Note that \( \xi_0^\mu \) is not equal to the phase field \( \xi^\mu \) (because \( \mu \neq u, \xi^\mu \)). Instead the relationship between these two fields is given by

\[
\xi^\mu = \xi_0^\mu - \mu_{\text{diss}} u^\mu \\
\xi = \xi_0 + \mu_{\text{diss}} \frac{\mu}{\xi_0}
\]

Note also that, by construction, \( \xi_0, u = \mu \).

The fluid stress tensor, charge current, in a modified phase frame, are assumed to take the form

\[
T_{\mu\nu} = (\rho_n + P)u^\mu u^\nu + Pr_{\mu\nu} + \frac{\rho_n}{\xi_0} \xi_0^\mu \xi_0^\nu + \tilde{\pi}^{\mu\nu} \\
J^\mu = q_n u^\mu - \frac{\xi_0^\mu}{\xi_0} + \tilde{J}_{\text{diss}}^\mu
\]

where all thermodynamical functions are taken to be functions of \( (T, \mu, \xi_0) \).

As in the case of fluid frames, the precise definition of any given modified phase frame requires the specification of 5 additional conditions (to give precise meaning to the fields \( T(x), u^\mu(x) \) and \( \mu(x) \)). The ‘Landau-Lifshitz-Clark-Putterman’ frame is a modified phase frame in which the additional conditions are taken to be

\[
\tilde{J}_{\text{diss}}^\mu = u_\mu u_\nu \tilde{\pi}^{\mu\nu} = 0
\]

As the name suggests, this is the frame employed by Landau Lifshitz, Clark and Putterman \[25, 26, 27\] (in a non relativistic context) in their analysis of superfluid dynamics.

It is not difficult to generalize the analysis of \( \S 2.4 \) to describe the transformation from a modified phase frame to a fluid frame or vice versa. We will implement such a frame change in the next subsection.

2.7 The canonical entropy current in modified phase frames

The most general modified phase frame may be obtained starting from the most general fluid frame and then reexpressing all quantities in terms of \( \xi_0^\mu \) and \( \xi_0 \) rather than \( \xi^\mu \) and \( \xi \). Let the fluid frame we start with be characterized by \( \pi^{\mu\nu} \) and \( J_{\text{diss}}^\mu \) and \( \mu_{\text{diss}} \). The generalized phase frame, obtained from
this fluid frame by reexpressing $\xi^\mu$ as a function of $\xi^\mu_0$ has

$$\tilde{\pi}^{\mu\nu} = \pi^{\mu\nu} - d(\rho_n + P)u^\mu u^\nu - dP \eta^{\mu\nu} - df \xi^\mu \xi^\nu_0 - f \mu^{\text{diss}}(u^\mu \xi^\nu_0 + u^\nu \xi^\mu_0)$$

$$\tilde{J}^{\mu}_{\text{diss}} = J^{\mu}_{\text{diss}} - dq_n u^\mu + df \xi^\mu_0 + f \mu^{\text{diss}} u^\mu$$

$$\xi_0 u = \mu$$

$$\xi \cdot u = \mu + \mu^{\text{diss}}$$

where $f = \frac{q_n}{\xi_0} = \frac{\rho_n}{\xi_0}$. Further, any thermodynamical function $A$ in the fluid frame is related to the corresponding thermodynamical function $\tilde{A}$ in the modified phase frame by

$$dA = \tilde{A} - A = A(\xi_0) - A(\xi) = -\mu^{\text{diss}} \left( \frac{\mu}{\xi_0} \right) \frac{\partial A}{\partial \xi}$$

As we have explained above, the canonical entropy current in an arbitrary fluid frame is given by (2.15). Applying the transformation formulas described above, this entropy current may be expressed in terms of the modified phase thermodynamical and dissipative parameters as (see Appendix A.3.1)

$$\tilde{J}^{\mu}_{\text{s}} = s(\xi_0) u^\mu - \frac{\mu}{T} \tilde{J}^{\mu}_{\text{diss}} - \frac{u^\nu \tilde{\pi}^{\mu\nu}}{T} + f \frac{\mu^{\text{diss}}}{T} u^\mu$$

(2.21)

In obtaining (2.21) we have used the thermodynamical identity (2.2).

It is also possible to transform the equation for the divergence of the entropy current, (2.16), into the modified phase frame (or simply rederive this expression directly in the modified phase frame). We find (see Appendix A.3)

$$\partial^\mu \tilde{J}^{\mu} = - \partial^\mu \left[ \frac{u^\nu}{T} \right] \tilde{\pi}^{\mu\nu} - \partial^\mu \left[ \frac{\mu}{T} \right] \tilde{J}^{\mu}_{\text{diss}} + \mu^{\text{diss}} P^{\mu\nu} \partial^\nu \left( \frac{f w_{\mu}}{T} \right)$$

(2.22)

3. A theory of first order dissipative superfluid dynamics in fluid frames

In this subsection we will present a ‘theory’ of dissipative superfluid dynamics to first order in the derivative expansion. By this we mean that we will parameterize the allowed forms of $\pi^{\mu\nu}$ and $J^{\mu}_{\text{diss}}$ at first order in the derivative expansion. Our parameterization will be in terms of a certain number of undetermined functions of $T$ and $\mu$ and $\xi$. One of these functions is the viscosity of the normal part of the superfluid. Following standard (but slightly misleading) usage, we will refer to these functions as dissipative parameters of the superfluid.

3.1 Summary of arguments and results

As the analysis of this section will be rather lengthy, we first present a summary of our logic and our procedure. To start with we simply classify all onshell inequivalent one derivative contributions to $\pi^{\mu\nu}$, $J^{\mu}_{\text{diss}}$ and $\mu^{\text{diss}}$. It is not difficult to establish that, in any given frame (e.g. the transverse frame or $\mu^{\text{diss}} = 0$ frame or the Landau-Lifshitz-Clark-Putterman frame) there exists a 36 parameter space of inequivalent first derivative corrections to the equations of superfluid dynamics (assuming parity invariance).

In order to cut down the set of possibilities we then follow Landau Lifshitz, Clark and Putterman [25, 26, 27] to make the central assumption of this subsection. We assume that the entropy current takes the canonical form described in [25, 6]. A local form of the second law of thermodynamics then

---

6This assumption is not universally valid. In particular it seems certain to fail in situations in which the $U(1)$ symmetry in question has a $U(1)^3$ anomaly. It may also fail in other circumstances. We leave the investigation of the validity of this assumption to future work.
asserts that the equations of superfluid dynamics should be geared to ensure that the divergence of this entropy current is positive. Using the expression (2.16) we find that this requirement cuts down the 36 parameter space of possible one derivative corrections to the entropy current to a 21 parameter space of possibilities. The coefficients in this 21 parameter space are further constrained by a complicated set of inequalities that ensure positivity of the entropy production. (One of these inequalities, for instance, asserts the positivity of the normal viscosity). Finally, the Onsager reciprocity relations relate 7 of the remaining parameters to each other, leaving us with a 14 parameter space of dissipative coefficients. As mentioned above, these 14 parameters (each of which is a function of $T$, $\mu$ and $\xi$) are further constrained to obey a set of inequalities. As far as we are aware, there are no further restrictions on this 14 parameter space from general principles.

To end this summary we explain how the framework presented in this subsection relates to previous work. The programme outlined in the paragraph above was implemented by Landau and Lifshitz [27] for the special case of flows with zero (or negligibly small) superfluid velocities. Landau and Lifshitz found a set of equations with 5 first order dissipative parameters. Our 14 parameter set of equations indeed reduce to the Landau Lifshitz form upon setting the superfluid velocity to equal the normal velocity; consequently our framework agrees with that of Landau and Lifshitz within its domain of validity.

Clark and Putterman [25, 26] extended the Landau Lifshitz programme to the case of nonzero superfluid velocities. The end result of their analysis was a thirteen parameter set of equations. We believe that Clark and Putterman overlooked one allowed parameter. Reinstating that parameter yields our 14 parameter set of equations. Thus Clark and Putterman’s equations are a special case of our 14 parameter equations with one parameter set to zero.

As we have explained in the previous section, one of the complicating features of superfluid dynamics is that one can work in many different frames. In this section we work out the ‘theory’ of dissipative superfluid dynamics in the two natural fluid frames described in the previous subsection, namely the transverse frame and the $\mu_{diss} = 0$ frame. In the next section we present the equivalent analysis for in the Landau-Lifshitz-Clark-Putterman frame.

### 3.2 Counting of parameters

At a given spacetime point, a superfluid flow has two independent velocities; the normal velocity $u^\mu$ and the superfluid velocity $\xi_{s}^\mu$. These two velocities break the local Lorentz rotation group $SO(3,1)$ down to $SO(2)$, the group of spatial rotations in the plane orthogonal to both velocities.

In §7.2 we enumerate the onshell inequivalent first derivatives of all fluid dynamical fields at a point. We found it convenient to classify these derivatives as scalars (spin 0), vectors (spin \(\pm 1\)) and tensors (spin \(\pm 2\)) under the unbroken $SO(2)$ described in the previous paragraph. As explained in §7.2 it turns out that there are 6 onshell inequivalent parity even scalars, 5 onshell inequivalent parity even vector and 2 onshell inequivalent parity even tensor first derivatives of fluid dynamical fields.

In order to be specific we will assume in the rest of this subsection that we are working in the transverse frame. Very similar arguments can be made in a fluid or modified phase frame, and give identical results.

---

7Specifically, the traceless symmetric 3 index tensor listed in equation (A VI-9) of Putterman’s book is not unique. Another such tensor is given by

$$Y_{ijk} = w_i(w_jw_k - (1/3)w^2\delta_{jk}).$$

8In addition we have one additional parity odd scalar field. Further, every vector $V_\mu$ can be transformed to a pseudo vector $\tilde{V}_\mu$ according to the formula $\tilde{V}_\mu = \epsilon_{\mu\alpha\beta}V^\alpha n^\beta u^3$. 

---
In the transverse frame, $\pi^{\mu\nu}$ has two inequivalent scalar components $\xi^\mu \xi^\nu \pi^\mu_\xi$ and $\pi^\mu_\mu$, one vector component $\tilde{P}_\alpha^\mu \pi^{\alpha\beta} \xi^\nu$ and a single tensor component $\tilde{P}_\alpha^\mu \pi^{\alpha\beta} \xi^\nu$. Finally, $\mu_{\text{diss}}$ has one scalar component $\tilde{P}_\alpha^\mu J^{\alpha}_{\text{diss}}$. It follows that the total number of undetermined parameters in the arbitrary expansion of $\pi^{\mu\nu}$, $J^\mu_{\text{diss}}$ and $\mu_{\text{diss}}$ in terms of first derivatives of the fluid dynamical fields (assuming parity invariance) is given by

$$4 \times 6 + 2 \times 5 + 2 = 36$$

where the three terms above originate in the scalar, vector and tensor sector respectively.  

### 3.3 Constraints from positivity of entropy production and Onsager relations in the transverse frame

In this subsection we will explore the constraints on dissipative coefficients from the physical requirements of positivity of entropy production and the Onsager reciprocity relations. We will find these requirements cut down the 36 parameter set of possible dissipative coefficients (assuming parity invariance) to a 14 parameter set of coefficients that are further constrained by positivity requirements.

For concreteness we present our analysis in the transverse frame. We will record the results of similar analysis in other fluid frames in later subsections.

#### 3.3.1 Constraints from positivity of entropy production

The divergence of the ‘canonical’ entropy current, given by (2.16), involves only terms proportional to $\partial^\mu u_\nu \pi^{\mu\nu}$, $\partial^\nu (\mu/T) J^\nu_{\text{diss}}$ and $\mu_{\text{diss}} \partial^\mu (q_s \xi^\mu / \xi)$. Let us examine these terms one by one. In the transverse frame

$$\partial^\mu u_\nu \pi^{\mu\nu} = \sigma_{\mu\nu} \pi^{\mu\nu} + \left( \frac{\partial^\mu u^\nu}{3} \right) \pi^{\theta\theta}$$

where $\sigma_{\mu\nu}$ is the traceless symmetric part of $\partial^\mu u_\nu$, projected in the direction perpendicular to $u^\mu$.

$$\sigma_{\mu\nu} = P^\alpha_\mu P^\beta_\nu \left( \frac{\partial^\alpha u^\beta + \partial^\beta u^\alpha}{2} - \frac{\eta_{\alpha\beta}}{3} \right)$$

and

$$P_{\mu\nu} = \text{The projector} = \eta_{\mu\nu} + u_\mu u_\nu$$

Now the field $\sigma_{\mu\nu}$ has one scalar piece of data

$$S_w = n^\mu n^\nu \sigma_{\mu\nu}$$

one vector piece of data

$$[V_b]_\mu = \tilde{P}_\mu^\alpha n^\alpha \sigma_{\nu\alpha}$$

and a tensor piece of data

$$T_{\mu\nu} = \tilde{P}_\mu^\alpha \tilde{P}_\nu^\beta \sigma_{\alpha\beta}$$

---

9 Dropping the assumption of parity invariance we have $4 \times 7 + 2 \times 10 + 2 = 50$ independent dissipative coefficients.

10 In the terminology of §7.2 below, $S_w = \frac{2}{3} \tilde{P}_\mu^\alpha \tilde{P}_\nu^\beta \sigma_{\alpha\beta}$.

11 In the terminology of §7.2 below, $[V_b]_\mu = \frac{1}{2} [V_5]_\mu$.

12 In the terminology of §7.2 below, $T_{\mu\nu} = \frac{1}{2} [T_1]_{\mu\nu}$. 
The trace of $\pi^{\mu\nu}$ couples to another scalar piece of data $^{13}$

$$S_{\omega'} = \partial_\mu u^\mu.$$ 

In the expressions above $n^\mu$ is the unique normal vector in the plane spanned by $u^\mu$ and $\xi^\mu$ that is orthogonal to $u^\mu$ and is given by

$$n_\mu \equiv \frac{w_\mu}{w},$$

with $w$ being the norm of the $w_\mu$ vector.

Similarly, in the transverse gauge

$$\partial_\nu (\mu/T) J_{\nu}^{\text{diss}} = P_\alpha^\nu \partial_\nu (\mu/T) J_{\alpha}^{\text{diss}},$$

where $P_{\mu\nu}$ is the projection operator (defined in (3.1)) that projects orthogonal to $u_\mu$ only. The quantity $P_\alpha^\nu \partial_\nu (\mu/T)$ has one scalar piece of data

$$S_b = (n^\mu \partial_\mu) (\mu/T)$$

and one vector piece of data

$$[V_\alpha]_\mu = \tilde{P}_\mu^\sigma \partial_\sigma (\mu/T)$$

Finally

$$S_a = \frac{\partial_\mu (q_s \xi^\mu / \xi)}{T^3}$$

is itself a scalar piece of data.

In other words we conclude that the expression for the divergence of the entropy current, (2.16), depends explicitly (i.e. apart from the dependence of $\pi^{\mu\nu}$, $J_{\mu}^{\text{diss}}$ and $\mu_{\text{diss}}$ on these terms) only on 4 scalar expressions, 2 vector expressions and one tensor expression. Let us choose these 4 vectors scalars $S_a$, $S_b$, $S_w$ and $S_{w'}$, supplemented by 3 other arbitrarily chosen scalar expressions $S_m^j (m = 1 \ldots 3)$ as our 7 independent scalar expressions. Similarly we choose the 2 vectors $[V_\alpha]_\mu$ and $[V_0]_\mu$ supplemented by 3 other arbitrarily chosen expressions $[V_m^j]_\mu$ ($m = 1 \ldots 3$) as our four independent vector expressions. We also choose $T_{\mu\nu}$ as one of our two independent tensor expressions $^{14}$. We proceed to express $\pi^{\mu\nu}$, $J_{\mu}^{\text{diss}}$ and $\mu_{\text{diss}}$ as the most general linear combinations of all combinations of independent expressions allowed by symmetry.

$^{13}$In the terminology of §7.2 below, $S_{\omega'} = S_4 + S_6$.

$^{14}$We could now go ahead and use the perfect fluid equations to solve for for the dependent data in terms of independent data; however we will not need the explicit form of this solution in this subsection.
\[ \pi^{\mu \nu} = T^3 \left[ \left( P_a S_a + P_b S_b + P_w S_w + P_{w'} S_{w'} + \sum_{m=1}^{3} P^l m S^l m \right) \left( n_\mu n_\nu - \frac{P_{\mu \nu}}{3} \right) \right. \\
+ \left( T_a S_a + T_b S_b + T_w S_w + T_{w'} S_{w'} + \sum_{m=1}^{3} T^l m S^l m \right) P^{\mu \nu} \\\n+ E_a (V_a^{\mu} n^\nu + V_{a'}^{\mu} n^\nu) + E_b (V_b^{\mu} n^\nu + V_{b'}^{\mu} n^\nu) + \sum_{m=1}^{3} E^l m \left( [V^l m]^{\mu} n^\nu + [V^l m]^{\nu} n^\mu \right) \\\n+ \tau T^{\mu \nu} + \tau_2 T^{\mu \nu} \right] \]

\( J^\mu_{\text{diss}} = T^2 \left[ \left( R_a S_a + R_b S_b + R_w S_w + R_{w'} S_{w'} + \sum_{m=1}^{3} R^l m S^l m \right) n^\mu \right. \\
+ C_a V^\mu_a + C_b V^\mu_b + \sum_{m=1}^{3} C^l m [V^l m]^\mu \left. \right] \]

\( \mu_{\text{diss}} = - \left[ Q_a S_a + Q_b S_b + Q_w S_w + Q_{w'} S_{w'} + \sum_{m=1}^{3} Q^l m S^l m \right] \)

Plugging (3.18) into (2.16) we now obtain an explicit expression for the divergence of the entropy current as a quadratic form in first derivative independent data. We wish to enforce the condition that this quadratic form is positive definite. Now the quadratic form from (2.16) clearly has no terms proportional to \((S^l m)^2\). It does, however, have terms of the form (for instance) \(S_a S^l m\), and also terms proportional to \(S_2^a\). Now it follows from a moments consideration that no quadratic form of this general structure can be positive unless the coefficient of the \(S_a S^l m\) term vanishes. \(^{15}\) Using similar reasoning we can immediately conclude that the positive definiteness of (2.16) requires that

\[ P^l m = T^l m = E^l m = C^l m = R^l m = \tau_2 = 0. \]

(3.3) is the most important conclusion of this subsubsection. It tells us that a 21 parameter set of first derivative corrections to the constitutive relations are consistent with the positivity of the canonical entropy current.

Of course the remaining 21 parameters are not themselves arbitrary, but are constrained to obey inequalities in order to ensure positivity. In order to derive these conditions we plug (3.3) into (3.2) and use (2.16) so that the divergence of the entropy current is the linear sum of three different quadratic forms (involving the tensor terms, vector terms and scalar terms respectively)

\[ \partial_\mu J^\mu = T^2 (Q_s + Q_V + Q_T) \]

where

\[ Q_T = -\tau T^2 \]

\[ Q_V = -C_a V^2_a - (C_b + E_a) V_b V_a - E_b V^2_b \]

\[ = -C_a \left[ V_a + \left( \frac{C_b + E_a}{2 C_a} \right) V_b \right]^2 - \left[ E_b - \frac{(C_b + E_a)^2}{4 C_a} \right] V^2_b \]

\(^{15}\)For instance the quadratic form \(x^2 + cy \) (where \(c\) is a constant) can be made negative by taking \(\frac{c}{2}\) to either positive or negative infinity (depending on the sign of \(c\)) unless \(c = 0\).
In order that

This is simply the requirement that the normal component of our superfluid have a positive viscosity.

Note that this expression involves the RHS; the last inequality above is satisfied roughly, when

substitution. The coefficients that appear in these equations are further constrained by the Onsager reciprocity relations (see, for instance, the textbook [27], for a discussion). These relations assert, in the present context, that we should equate any two dissipative parameters that multiply the same terms in the formulas (3.5) and (3.6) for entropy production. This implies that

\[ Q_w = P_a, \quad Q_{w'} = T_a, \quad R_w = P_b \]

\[ R_{w'} = T_b, \quad R_a = Q_b, \quad T_w = P_{w'} \]

and

\[ C_b = E_a \]

In summary we are left with a 14 parameter set of equations of first order dissipative superfluid dynamics. The requirement of positivity constrains further these coefficients to obey the inequalities spelt out in the previous subsubsection.
3.4 Weyl Invariant Superfluid Dynamics in the transverse frame

Let us now specialize the results of §3.3.1 and §3.3.2 to the case of super fluid dynamics for a conformal superfluid. The analysis of §3.3.1 is simplified in this special case by the fact that the trace of the stress tensor vanishes in an arbitrary state (and so in the fluid limit) of a conformal field theory. This fact reduces the number of explicit scalars that appear in (3.10) from 4 to 3 (the scalar \( S_w' \) never makes an appearance). It follows that the requirement of Weyl invariance forces \( P_{w'} = T_{w'} = T_w = Q_{w'} = 0 \). Moreover the requirement that \( \pi^{\mu\nu} \) be traceless forces \( T_a = T_b = T_w = 0 \). It turns out that there are no further constraints from the requirement of Weyl invariance. The expansion of the dissipative part of the stress tensor and charge current for a conformal superfluid is given by

\[
\pi^{\mu\nu} = T^3 \left[ (P_a S_a + P_b S_b + P_w S_w) \left( n_{\mu} n_{\nu} - \frac{P_{\mu\nu}}{3} \right) + E_a (V_a^{\mu} n_{\nu} + V_a^{\nu} n_{\mu}) + E_b (V_b^{\mu} n_{\nu} + V_b^{\nu} n_{\mu}) + \tau T^{\mu\nu} \right]
\]

(3.10)

\[
J_{\text{diss}}^\mu = T^2 \left[ (R_a S_a + R_b S_b + R_w S_w) n^{\mu} + C_a V_a^{\mu} + C_b V_b^{\mu} \right]
\]

(3.11)

The entropy production is given by

\[
\partial_{\mu} J_s^\mu = T^2 (Q_s + Q_V + Q_T)
\]

where

\[
Q_T = -\tau T^2
\]

\[
Q_V = -C_a V_a^2 - (C_b + E_a) V_b V_a - E_b V_b^2
\]

\[
= -C_a \left[ V_a + \left( \frac{C_b + E_a}{2C_a} \right) V_b \right]^2 - \left[ E_b - \left( \frac{C_b + E_a}{4C_a} \right)^2 \right] V_b^2
\]

(3.12)

\[
Q_S = -P_w S_w^2 - Q_a S_a^2 - R_b S_b^2
\]

\[
+ (Q_w + P_a) S_w S_a - (R_a + Q_b) S_a S_b + (R_w + P_b) S_w S_b
\]

(3.13)

For the entropy current to be positive it is necessary and sufficient that \( \tau \leq 0 \) and that

\[
C_a \leq 0, \quad E_b \leq 0 \quad \text{and} \quad 4E_b C_a \geq (C_b + E_a)^2.
\]

(3.14)

and that the quadratic form

\[
Q_S = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + b_1 x_1 x_2 + b_2 x_2 x_3 + b_3 x_3 x_1
\]

\[
= a_1 \left[ x_1 + \left( \frac{b_1}{2a_1} \right) x_2 + \left( \frac{b_3}{2a_1} \right) x_3 \right]^2
\]

\[
+ \left( a_3 - \frac{b_1^2}{4a_1} \right) \left[ x_3 + \left( \frac{2a_1 b_2 - b_1 b_3}{4a_1 a_3 - b_1^2} \right) x_2 \right]^2
\]

\[
+ \left[ 4a_1 (a_1 - b_1) (4a_1 a_3 - b_1^2) - (2a_1 b_2 - b_1 b_3)^2 \right] \frac{x_2^2}{4a_1 (4a_1 a_3 - b_1^2)}
\]

(3.15)
is positive with \( x_1 = S_w, x_2 = S_a \) and \( x_3 = S_b \) and

\[
\begin{align*}
a_1 &= -P_w, \quad a_2 = -Q_a, \quad a_3 = -R_b, \quad b_1 = Q_w + P_a, \quad b_2 = -(Q_b + R_a), \quad b_3 = R_w + P_b
\end{align*}
\]

For the last quadratic form to be positive it is necessary and sufficient that

\[
\begin{align*}
a_1 &\geq 0 \\
4a_1a_2 &> b_1^2 \\
(4a_1a_2 - b_1^2)(4a_1a_3 - b_3^2) &> (2a_1b_2 - b_1b_3)^2
\end{align*}
\]

(3.16)

By rewriting (3.15) as a sum of squares in a cyclically permuted manner we can also derive the cyclical permutations of these equations.

In summary, the most general Weyl invariant fluid dynamics consistent with positivity on the entropy current is parameterized by a negative \( \tau_1 \), 4 parameters in the vector sector constrained by the inequalities (3.14) and 9 parameters in the scalar sector, subject to the inequalities (3.16). These 14 dissipative parameters are further constrained by the 4 Onsager relations

\[
\begin{align*}
Q_w &= P_a, \quad R_w = P_b, \quad R_a = Q_b, \quad C_b = E_a
\end{align*}
\]

(3.17)

leaving us with a 10 parameter set of final equations.

3.5 Weyl Invariant Super Fluid dynamics in the \( \mu_{diss} = 0 \) frame

In this section we present constitutive relations of super fluid dynamics in the frame that we consider the most natural, namely the \( \mu_{diss} = 0 \) fluid frame. We only present our final results, and specialize to the case of a Weyl invariant fluid for simplicity. We find

\[
\pi^{\mu\nu} = T^3 \left[ [PS + PbSb + PuSw] \left( n^\mu n^\nu - \frac{P^{\mu\nu}}{3} \right) + E_a (V_a^\mu n^\nu + V_a^\nu n^\mu) + E_b (V_b^\mu n^\nu + V_b^\nu n^\mu) + \tau T^{\mu\nu} \right]
\]

(3.18)

\[
J_{diss}^{\mu} = T^2 \left[ [QS + QbSb + QwSw] u^\mu + [RbSr + RaSr + RwSw] n^\mu + C_a V_a^\mu + C_b V_b^\mu \right]
\]

where \( S = (u, \partial) (\nabla) \)

The equation of entropy production is given by

\[
\partial_\mu J_{diss}^\mu = T^2 (Q_s + Q_V + Q_T)
\]

(3.19)

where

\[
Q_T = -\tau T^2
\]

\[
Q_V = -C_a V_a^2 - (C_b + E_a) V_a V_b - E_b V_b^2 = -C_a \left[ V_a + \left( \frac{C_b + E_a}{2C_a} \right) V_b \right]^2 - \left[ E_b - \frac{(C_b + E_a)^2}{4C_a} \right] V_b^2
\]

(3.20)
and
\[
Q_S = -P_w S_w^2 - Q^2 S^2 - R_b S_b^2 \\
+ (Q_w + P_S) S w S + (R_w + P_b) S w S_b - (R_S + Q_b) S S_b
\]  
(3.21)

The positivity of \( Q_T \) is equivalent to the requirement that \( \tau_1 \leq 0 \). The positivity of \( Q_V \) is equivalent to the condition
\[
C_a \leq 0, \quad E_b \leq 0 \quad \text{and} \quad 4 E_b C_a \geq (C_b + E_a)^2
\]

The implications of the requirement of the positivity of \( Q_S \) are exactly as in §3.4. The Onsagar relations imply that
\[
Q_w = P_S, \quad R_w = P_b, \quad R_S = Q_b, \quad C_b = E_a
\]

Once again we have a 10 parameter set of dissipative equations of super fluid dynamics.

### 4. Dissipative Superfluid dynamics in the Landau-Lifshitz-Clark-Putterman Frame

In the previous section we have presented a ‘theory’ of first order dissipative super fluid dynamics in the transverse fluid frame. In this subsection we present the equivalent ‘theory’ in the Landau-Lifshitz-Clark-Putterman modified phase frame. As the logic of this presentation mirrors that of the previous subsection in every way we will explicitly omit all derivations and present only our final results. After we have presented our results we will explicitly take the non relativistic limit and compare with the analogous expressions in Clark, Putterman and Landau and Lifshitz [25, 26, 27].

#### 4.1 Allowed forms for dissipative terms

As we have explained above, the Landau-Lifshitz-Clark-Putterman frame is a modified phase frame that specified by the relations
\[
u^{\mu} u^{\nu} \tilde{\pi}_{\mu \nu} = 0, \quad \tilde{J}^{\mu}_{\text{diss}} = 0
\]  
(4.1)

We take the stress tensor to be given by
\[
\tilde{\pi}_{\mu \nu} = (Q^{\mu} u^{\nu} + Q^{\nu} u^{\mu}) + \Pi^{\mu \nu} + \Pi_{t}^{\mu \nu}
\]
where
\[
u^{\mu} Q_{\mu} = 0, \quad u^{\mu} \Pi_{t}^{\mu \nu} = 0, \quad (\Pi_{t})^{\mu}_{\mu} = 0, \quad \tilde{\pi}_{\mu} = 3 \Pi
\]  
(4.2)

The divergence of the entropy current in this frame is easily obtained from (A.20) and takes the form
\[
\partial_{\mu} J_{S}^{\mu} = - \left( \frac{[\partial_{\mu} \Pi + \Pi_{t}^{\mu \nu} \sigma_{\mu \nu}]}{T} - \frac{Q^{\nu} [\partial_{\nu} T + T (u.\partial) u_{\nu}]}{T^2} + \mu_{\text{diss}} P^{\mu \nu} \left[ \partial_{\mu} \left( \frac{f w_{\nu}}{T} \right) \right] \right)
\]  
(4.3)

The first derivative expressions of fluid dynamical fields that appear explicitly in (4.3) may be written in terms of the following scalars
\[
\Sigma_{1} = n^{\nu} \left[ \partial_{\nu} T + T (u.\partial) u_{\nu} \right] / T, \quad \Sigma_{2} = n^{\nu} \sigma_{\mu \nu} n^{\nu}, \quad \Sigma_{3} = \partial_{\nu} u^{\mu}, \quad \Sigma_{4} = P^{\mu \nu} \left[ \partial_{\mu} \left( \frac{f w_{\nu}}{T^2} \right) \right]
\]

the following vectors
\[
\mathcal{W}^{\mu}_{1} = P^{\mu \nu} \left[ \partial_{\nu} T + T (u.\partial) u_{\nu} \right] / T, \quad \mathcal{W}^{\mu} = n^{\beta} \sigma_{\beta \alpha} \tilde{P}^{\alpha \mu}
\]
and the following tensors
\[ \mathcal{T}^{\mu\nu} = \tilde{\gamma}^{\mu\beta} \sigma_{\beta\alpha} \tilde{\gamma}^{\alpha\mu} \]

where
\[ n^\mu = \frac{w^\mu}{w}, \quad \tilde{\gamma}^{\mu\nu} = P^{\mu\nu} - n^\mu n^\nu \]

In terms of these expressions, positivity of the entropy current requires
\[ \mu_{\text{diss}} = -[\mu_1 \Sigma_1 + \mu_2 \Sigma_2 + \mu_3 \Sigma_3 + \mu_4 \Sigma_4] \]

\[ \Pi = T^3 \left( \pi_1 \Sigma_1 + \pi_2 \Sigma_2 + \pi_3 \Sigma_3 + \pi_4 \Sigma_4 \right) \]

\[ Q^\mu = T^3 \left[ Q_1^{(s)} \Sigma_1 + Q_2^{(s)} \Sigma_2 + Q_3^{(s)} \Sigma_3 + Q_4^{(s)} \Sigma_4 \right] n^{\mu} \]
\[ + T^3 Q_1^{(v)} W_1^\mu + T^3 Q_2^{(v)} W_2^\mu \]

(4.4)

\[ \Pi^{\mu\nu} = T^3 \left[ (P_1 \Sigma_1 + P_2 \Sigma_2 + P_3 \Sigma_3 + P_4 \Sigma_4) \left( n_\mu n_\nu - \frac{1}{3} P^{\mu\nu} \right) \right. \]
\[ + E_1 (W_1^\mu n_\nu + W_1^\nu n_\mu) + E_2 (W_2^\mu n_\nu + W_2^\nu n_\mu) \]
\[ + \tau T^{\mu\nu} \]

(4.5)

With this form for the dissipative parameters, the divergence of the entropy current is given by
\[ \partial_\mu J_\mu^2 = T^2 (Q_S + Q_v + Q_T) \]

where
\[ Q_S = -Q_1^{(s)} \Sigma_1^2 - P_2 \Sigma_2^2 - \pi_3 \Sigma_3^2 - \mu_4 \Sigma_4^2 \]
\[ - (Q_4^{(s)} + \mu_1) \Sigma_1 \Sigma_4 - (P_4 + \mu_2) \Sigma_2 \Sigma_4 - (\pi_4 + \mu_3) \Sigma_3 \Sigma_4 \]
\[ - (Q_2^{(s)} + P_1) \Sigma_1 \Sigma_2 - (Q_3^{(s)} + \pi_1) \Sigma_1 \Sigma_3 - (\pi_2 + P_3) \Sigma_3 \Sigma_2 \]

(4.6)

\[ Q_V = -Q_1^{(v)} W_1^2 - \left( Q_2^{(v)} + E_1 \right) W_1 W_2 - E_2 W_2^2 \]
\[ = -Q_1^{(v)} \left[ W_1 + \left( \frac{Q_2^{(v)} + E_1}{2 Q_1^{(v)}} \right) W_2 \right]^2 - \left[ E_2 - \frac{(Q_2^{(v)} + E_1)^2}{4 Q_1^{(v)}} \right] W_2^2 \]

\[ Q_T = -\tau T^{\mu\nu} T^{\mu\nu} \]

The positivity of entropy current requires that the each of the three quadratic forms \( Q_S, Q_v \) and \( Q_T \) are positive definite.

The Onsager relations take the form
\[ Q_4^{(s)} = \mu_1, \quad P_4 = \mu_2, \quad \pi_4 = \mu_3 \]
\[ Q_2^{(s)} = P_1, \quad Q_3^{(s)} = \pi_1, \quad P_3 = \pi_2 \]
\[ Q_2^{(v)} = E_1 \]

Once again we have 14 independent dissipative coefficients, constrained by inequalities that follow from the requirement of positivity of the entropy current.
4.2 The Limit of zero superfluid velocity

In this subsubsection we will examine how the description of the previous subsection must simplify in the limit \( w \to 0 \). In this limit the superfluid and normal velocity are coincident. In this limit the vector \( n^\mu \) no longer has any significance. The equations for the dissipative parts of the stress tensor, current and \( \mu_{diss} \) must all possess the full \( SO(3) \) invariance of rotations that leave \( u^\mu \) fixed. The expressions for all dissipative quantities must be analytic functions of the projected superfluid velocity vector field \( w^\mu = P_\nu^\mu \xi^\nu = v_c \zeta n^\mu \). This requirement imposes several constraints on the \( \zeta \to 0 \) behaviour of the coefficients in dissipative terms, including

\[
\begin{align*}
\lim_{w \to 0} Q_2^{(s)} &= \lim_{w \to 0} Q_3^{(s)} = \lim_{w \to 0} Q_4^{(s)} = \lim_{w \to 0} Q_2^{(v)} = 0 \\
\lim_{w \to 0} P_1 &= \lim_{w \to 0} P_2 = \lim_{w \to 0} P_3 = \lim_{w \to 0} P_4 = \lim_{w \to 0} E_2 = 0 \\
\lim_{w \to 0} \mu_1 &= \lim_{w \to 0} \mu_2 = \lim_{w \to 0} \pi_1 = \lim_{w \to 0} \pi_2 = 0 \\
\lim_{w \to 0} P_2 &= \lim_{w \to 0} E_1 = \lim_{w \to 0} \tau = \eta \\
\lim_{w \to 0} Q_1^{(s)} &= \lim_{w \to 0} Q_1^{(v)} = \kappa
\end{align*}
\]

(4.7)

It follows that, in this limit the 21 independent dissipative coefficients of the previous subsubsection (before imposing Onsager relations) reduce to only 6 non-zero coefficients

\[ \eta, \kappa, \mu_3, \mu_4, \pi_3, \pi_4. \]

The relations in (4.4) simplify to

\[
\begin{align*}
\mu_{diss} &= \mu_3 \Sigma_3 + \mu_4 \Sigma_4 \\
\Pi &= T^3 \left[ \pi_3 \Sigma_3 + \pi_4 \Sigma_4 \right] \\
Q^\mu &= T^3 \kappa \, P^\mu \left( \frac{\partial_\nu T}{T} + (u.\partial)u_\nu \right) \\
\Pi_t^{\mu\nu} &= T^3 \eta \, \sigma^{\mu\nu}
\end{align*}
\]

(4.8)

The Onsager relation set \( \mu_3 = \pi_4 \), finally leaving us with 5 dissipative coefficients, as predicted by Landau and Lifshitz [27].

4.3 The Non Relativistic Limit

In order to make contact with the results of Clark and Putterman [25, 26] (and with those of Landau and Lifshitz [27] in the limit of zero superfluid velocity) we will now explicitly take the non relativistic limit of the formulas presented in [4.1].

16To see where these conditions come from, consider, for example, a constitutive relationship for a quantity \( v_1 \) which we know is a full \( SO(3) \) vector when \( \zeta \) vanishes, but is simply the sum of an \( SO(2) \) vector and an \( SO(2) \) scalar at nonzero \( \zeta \). At arbitrary values of \( \zeta \), the scalar and vector parts of \( v_1 \) can be expanded as arbitrary linear combinations of all independent \( SO(2) \) scalars and vectors. However as \( \zeta \to 0 \), the scalar component can be expanded in only those scalars that arise from the decomposition of an \( SO(3) \) vector. As all \( SO(2) \) scalars are not of this form, this implies the vanishing of a number of coefficients in the expansion. Moreover the coefficient of these scalars has to equal the coefficient of the corresponding \( SO(2) \) vectors in the vector part of \( v_1 \). This requires that two otherwise independent coefficients are equal. Similar remarks hold for the expansion of vector fields.
The non relativistic limit is taken as follows. We set
\[ u_\mu = \{1, \vec{v}\}, \quad w_\mu = \{0, \vec{w}\}, \quad n_\mu = \{0, \vec{n}\} \]
(we ignore all terms of quadratic or higher order in \(\vec{v}\) and \(\vec{w}\)) and
\[ P_{\mu\nu} = \delta_{ij}, \quad \tilde{P}_{\mu\nu} = \tilde{P}_{ij} = \delta_{ij} - n_in_j \]
Note in particular that all vectors and tensor projected orthogonal to \(u^\mu\), in this limit, are purely spatial.

The various scalar, vector and the tensor terms listed in §4.1 reduce to the following expressions in the non relativistic limit:
\[ \Sigma_1 = \frac{(\vec{n}.\vec{\partial})T}{T}, \quad \Sigma_2 = n_in_j\sigma_{ij}, \quad \Sigma_3 = \vec{\partial}.\vec{v}, \quad \Sigma_4 = \frac{1}{T^2} \vec{\partial}\left(\frac{f\vec{w}}{T}\right) \]
\[ \left[ W_1\right]_i = \frac{\partial_i T}{T} - n_i\Sigma_1, \quad \left[ W_2\right]_i = n_j\sigma_{ji} - n_i\Sigma_2 \]
\[ T_{ij} = (\delta_{ik} - n_in_k)\sigma_{km}(\delta_{mj} - n_mn_j) \]
(note that all vectors and tensors are purely spatial, and we have only listed their spatial components).

It follows that
\[ \tilde{\pi}^{00} = 0 \]
\[ \tilde{\pi}^{0i} = Q^i \]
\[ \tilde{\pi}^{ij} = \Pi^{ij} + \Pi\delta^{ij} \]
(4.10)
where all the quantities in the equation above are given by plugging (4.9) into (4.8). The resultant equations are very close to equation AVI-11 - AVI-14 in Putterman’s book [26]. The main difference between our equations and those of Putterman is that our equations have 21 free parameters, in contrast with Putterman’s 20 free parameters. The equations of this subsection reduce to (AVI -11 to AVI-14) in [26] if we set \(Q_2^{(\sigma)} = 0\).

The Onsager relations of §4.1 carry over unchanged to the non relativistic limit. These relations are identical to the Onsager relations imposed by Putterman [26] and reduce the number of dissipative coefficients to 14 in our formulation, but to 13 according to Putterman [26].

The non relativistic version of the \(\vec{w} \rightarrow 0\) constitutive relations (4.8) is simply given by by plugging (4.3) into (4.3); the resulting 6 parameter set of constitutive (cut down to a 5 parameter set by the Onsager relations) agree with those presented by Landau and Lifshitz [27].

5. Equilibrium Superfluid Thermodynamics from Gravity

In this section we explain that the equilibrium superfluid thermodynamics, reviewed in [21], may be derived from gravity very simply. This section, which is abstract in nature, is a generalization and reworking of the beautiful paper of Sonner and Withers [28] on the same subject.
Consider a general gravitational system governed by the action

$$S = \frac{1}{16\pi G} \int \sqrt{g} \left( R + 12 - \frac{1}{e^2} (V_1(\phi^*) F_{ab} F^{ab} + V_2(\phi^*) D_a \phi D^a \phi^* + V_3(\phi^*)) \right)$$  \hspace{1cm} (5.1)$$

where $D_a = \nabla - iA_a$ and $\bar{D}_a = \nabla + iA_a$ and all the potentials $V_1$, $V_2$, $V_3$ are real. We assume that this system admits a homogeneous stationary asymptotically $AdS$ family of solutions - dual to homogeneous stationary superfluid flows - that take the form

\[
\text{Metric : } ds^2 = -2g\left(\frac{r}{r_c}\right)u_\mu dx^\mu dr - r_c^2 f\left(\frac{r}{r_c}\right)u_\mu u_\nu dx^\mu dx^\nu + r_c^2 k\left(\frac{r}{r_c}\right)n_\mu n_\nu dx^\mu dx^\nu \\
+ r_c^2 j\left(\frac{r}{r_c}\right)(n_\mu u_\nu + u_\mu n_\nu) dx^\mu dx^\nu + r^2 \bar{P}_{\mu\nu} dx^\mu dx^\nu, \\
\text{Gauge field : } r_c A = H\left(\frac{r}{r_c}\right) u^\mu \partial_\mu + L\left(\frac{r}{r_c}\right) n^\mu \partial_\mu \\
\text{Bulk scalar field = } \phi\left(\frac{r}{r_c}\right)
\]  \hspace{1cm} (5.2)

where

$$\bar{P}_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu. \hspace{1cm} (5.3)$$

Here $u_\mu$ and $n_\mu$ are two arbitrary constant vectors obeying

$$u^\mu u_\mu = 0; \quad u^\mu u_\mu = -1; \quad n^\mu n_\mu = 1. \hspace{1cm} (5.4)$$

We work in a gauge such that the scalar field is real $\phi^* = \phi$ (this implies $A^r = 0$), so that the boundary value of the gauge field gives the superfluid velocity. We choose the constant vector $u_\mu$ so as to ensure that the killing vector coincides with the generators of the event horizon of our solution. $n^\mu$ is then uniquely determined by $\mathcal{L}^\xi$ together with the requirement that $A_\mu$ at infinity (i.e. $\xi_\mu$) can be written as a linear combination of $u^\mu$ and $n^\mu$.

We now choose coordinates so that the killing vector $\partial_v$ points along the direction of the $u^\mu$ and the vector $\partial_x$ points in the direction of $n^\mu$. Our solution retains rotational invariance in the remaining two spatial directions. We also work in the rescaled variables $\frac{r}{r_c}$ and $r_c x^i$ in terms of which (5.2) reduces to

\[
\text{Metric : } ds^2 = 2g(r) dv dr - f(r) dv^2 - 2j(r) dv dx + k(r) dx^2 + r^2 \left( \sum \partial y_i^2 \right) \\
A^v = 0, \quad A^r = H(r), \quad A^x = L(r), \quad A^y = 0, \quad A^z = 0 \\
\text{Bulk scalar field = } \phi(r) \hspace{1cm} (5.5)
\]

The solutions above are translationally invariant in the field theory spacetime directions. They are characterized by an onshell action density (action $\mathcal{L}^\xi$ per unit volume), a stress tensor and a charge current. Merely from symmetry, and with an appropriate definition of $u^\mu$, the stress tensor and the charge current of our system necessarily take the form (2.1). In Appendix B below we demonstrate that the functions that appear in (2.1) obey all the thermodynamical relations of the Landau Tiza two fluid model described above.

While the details of our demonstration are presented in Appendix B, we very briefly review the main ideas here.

Our system has three inequivalent translationally invariant killing vectors: $u^\mu \partial_\mu = \partial_v$, $u^\mu \partial_\mu$ and $k^\mu \partial_\mu$ where $k^\mu$ is transverse to the normal as well as the superfluid velocity. As is standard in
derivations of black hole thermodynamics, we demonstrate using the equations of motion that $R^{\mu\nu} \theta_{\nu}$ (where $\theta^{\mu}$ is any of the vectors above) takes a particularly simple form. The resultant identities allow us to demonstrate the onshell vanishing of three total derivatives built out of the functions in the metric and gauge field (5.2). Integrating these expressions allows us to prove the Smarr-Gibbs-Duhem relations listed in the first two of equations (2.2). We are also able to show that the killing vector that reduces to the null generator of the horizon of our solution also defines the normal velocity of our fluid. Recall the later is defined to ensure the absence of $u^{\mu} u^{\nu}_s$ cross terms in the stress tensor.

Next we follow [7] to rewrite the action of an equilibrium configuration of the sort studied in this section as the integral of a total derivative. Performing the integral we find that the action evaluates to the difference of two terms, one at infinity and the second at the horizon. The term at the horizon vanishes while the term at infinity evaluates to the negative of the pressure of the solution.

Finally, the action density (hence the pressure) of this system may be regarded as a function of its temperature $T$, chemical potential $\mu$ and superfluid chemical potential $\mu_s$. It is well known in classical mechanics that the onshell action as a function of initial and final coordinates, $q_i^0$ and $q_i^f$ obeys the relation $dS = p_i^0 dq_i^0 - p_i^f dq_i^f$. Sonner and Withers [3] have demonstrated that the application of this result to our system (with $r$ being regarded as the time) gives

$$dP = q_n d\mu + q_s d\mu_s + s dT$$

(5.6)

where $s$ is the entropy density of the solution given by the Hawking-Beckenstein formula.

6. An analytically tractable limit of hairy black branes

As we have explained in the previous section, general gravitational methods are powerful enough to tightly constrain the gravitational solutions dual to fluid dynamics, in particular to demonstrate that the thermodynamics of these solutions is given by the Landau-Tizano two fluid model. In order to explicitly determine the thermodynamics of any particular gravitational system, however, we need to explicitly determine the solutions dual to uniform superfluid flows. Unfortunately, the ordinary differential equations that arise in this attempt have proved so complicated that it has not proved possible to analytically solve for hairy black branes (the gravitational duals to superfluids) in any reasonable gravitational system. The only analytic results that we are aware of, for hairy black brane solutions, are those of Herzog [3]. Herzog considered a very special model, the model of a charged scalar field of $m^2 = -4$ and infinite charge $e$ (i.e. a model in the so called probe limit). He demonstrated that this model displays a second order phase transition towards superfluidity whenever $|\frac{\mu}{T}| \geq 2$.

When $|\frac{\mu}{T}|$ is just larger that 2, the stable gravitational solutions develop a scalar vev. Let $\epsilon$ denote the value of this vev. Herzog was able to generate the relevant gravitational solutions perturbatively in $\epsilon$ and also perturbatively in the difference between superfluid and normal velocities.

In this paper we will be interested in probing the structure of viscous superfluid dynamics from gravity. In the infinite charge or probe limit of [3] scalar and gauge dynamics do not back react on spacetime. In order to probe the dynamics of the interaction between the stress tensor and the charge current we need to go beyond the infinite charge probe limit. In this section we generalize Herzog’s perturbative construction of gravitational solutions to go beyond the probe limit. In other words we generalize Herzog’s infinite $\epsilon$ solutions to retain the first nontrivial correction in a $\epsilon$ expansion.

In the next section we will use the results of this section as an input into the fluid gravity map, in order to generate gravitational solutions dual to viscous superfluid flows.

\[17\] Of course much attention has been focused on the numerical solutions of the relevant equations in several models.
6.1 The bulk system and the equations of motion

Following Herzog [6] we consider the system

\[ \mathcal{L} = \frac{1}{16\pi G} \int d^5 x \sqrt{-g} \left( \mathcal{R} + 12 + \frac{1}{e^2} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} |D_{\mu}\phi|^2 + 2|\phi|^2 \right) \right), \]  

(6.1)

Where \( D_{\mu} = \nabla_{\mu} - iA_{\mu} \), and \( \nabla_{\mu} \) is the gravitational covariant derivative. Note that this system is a special case of the system (5.1) (where we have chosen particular simple forms of the functions \( V_1, V_2 \) and \( V_3 \)).

The equation of motion for the scalar field and the gauge field that follows from (6.1) are respectively

\[ D_{\mu}D^{\mu}\phi + 4\phi = 0, \]  

(6.2)

and

\[ D_{\mu}F^{\mu\nu} = \frac{1}{2} J^\nu, \]  

(6.3)

where the current \( J_{\mu} = i(\phi^* D_{\mu}\phi - \phi(D_{\mu}\phi)^*) \). The Einstein Equation that follows from (6.1) is

\[ G_{\mu\nu} - 6g_{\mu\nu} = \frac{1}{e^2}((T_{\text{max}})_{\mu\nu} + (T_{\text{mat}})_{\mu\nu}), \]  

(6.4)

where

\[ (T_{\text{max}})_{\mu\nu} = -\frac{1}{2} \left( F_{\mu\beta} F^{\beta\nu} - \frac{1}{4} g_{\mu\nu} F_{\sigma\beta} F^{\beta\sigma} \right), \]  

(6.5)

\[ (T_{\text{mat}})_{\mu\nu} = \frac{1}{4} (D_{\mu}\phi D_{\nu}\phi^* + D_{\nu}\phi D_{\mu}\phi^*) - \frac{1}{4} g_{\mu\nu} (|D_{\beta}\phi|^2 - 4|\phi|^2). \]

6.2 Boundary Conditions and Solutions

We search for solutions of the form (5.5). The 4-vectors \( n^\mu \), defined in the previous section, may be computed as follows. Let

\[ r_c \zeta_\mu = (\eta_{\mu\nu} + u_{\mu}u_{\nu}) \xi^{\mu}. \]

It follows that \( n_{\mu} \) is given by

\[ n_{\mu} = \zeta_{\mu}/|\zeta|. \]

As explained in the previous section we choose gauge so that our scalar field is real \( \phi(r) = \phi^*(r) \) by a choice of gauge. This gauge choice, plus the static nature of our solution, forces \( A^\tau = 0 \) (as can be seen by comparing the equation of motion of \( \phi \) with that of \( \phi^* \)).

We search for solutions that obey the following large \( r \) boundary conditions

\[ k(r) = r^2 + \frac{k_2}{r^2}, \]

\[ f(r) = r^2 + \frac{f_2}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right), \]

\[ j(r) = \frac{j_2}{r^2} + \ldots \]  

(6.6)

\[ L(r) = \frac{L}{r^2} + \ldots \]

\[ \phi(r) = \frac{\phi}{r^2} + \ldots \]
It turns out that the conditions above, together with the equations of motion, automatically ensure
\[ \lim_{r \to \infty} g(r) = 1 \]
so that this condition, while true, does not have to be additionally imposed. Also, it turns out that the coefficient of \(1/r^2\) term in the asymptotic expansion of \(H(r)\) is fixed by equations of motion and the requirement that \(\phi\) be regular at the horizon.

Our functions are also constrained at \(r = 1\) as follows
\[ j(1) = f(1) = 0 \quad (6.7) \]
On the other hand the functions \(H(r), k(r), L(r)\) and \(\phi(r)\) are required only to be regular at \(r = 1\).

It is possible to argue that there exists an 8 parameter class of solutions of the form (5.2), to the system (6.1), subject to the boundary conditions listed above. One of these parameters is \(r_c\) in (5.2).

The three normal velocity parameters can be set to zero by a boost, and rotations can be used to point the superfluid velocity in the \(x\) direction, as in the previous section. This leaves us with a two parameter set of solutions, parameterized by \(\epsilon\) and \(\zeta\).

6.3 Perturbative Solutions

In this subsection we will generalize the work out in [6] to the hairy black branes of our system, as a function of \(\epsilon\) and \(\zeta\) at small values of those parameters. Our starting point is Herzog’s observation that, at \(\epsilon = \infty\), the linearized equations of motion about the Reissner Nordstrom black brane at \(|\#| = 2\) admit a regular static solution scalar solution proportional to \(\epsilon^{1+} r^2\). As was explained in [6] this solution can be taken to be the starting point for a perturbative expansion of hairy black brane solutions in a power series in \(\epsilon\). The solutions of [6] were further generalized to nonzero \(\zeta\).

In this subsection we generalize Herzog’s solutions away from the infinite charge limit, to first order in a power series expansion in \(O(\frac{1}{\epsilon^2})\), i.e to first order in deviations away from the probe approximation. This generalization will prove crucial for generating the equations of superfluid dynamics including effects of back reaction of the superfluid on the normal fluid.

The techniques for obtaining this perturbative expansion are standard. We do not pause to explain our computations in detail; in the rest of this section we simply present the results of our calculations. As a function of \(\epsilon\) and \(\zeta\) (with both taken to be small) we find that the scalar field is given by
\[
\phi(r) = \left\{ \epsilon \left[ \frac{1}{r^2 + 1} + \frac{\zeta^2 (2 \log(r) - \log (r^2 + 1))}{4r^2 + 4} + O(\zeta^4) \right] + \epsilon^3 \left[ -2 \frac{(r^2 + 1) \log(r) + (r^2 + 1) \log (r^2 + 1) - 2}{48 (r^2 + 1)^2} + O(\zeta^2) \right] + O(\epsilon^5) \right\} + O\left(\frac{1}{\epsilon^2}\right) \quad (6.8) \]

The functions in the gauge field in (5.2) are given by
\[
H(r) = (H_0(r) + H_1(r)\epsilon^2 + H_2(r)\epsilon^4 + O(\epsilon^6)) + O(1/\epsilon^2), \quad \text{and} \quad L(r) = (L_0(r) + L_1(r)\epsilon^2 + L_2(r)\epsilon^4 + O(\epsilon^6)) + O(1/\epsilon^2) \quad (6.9) \]
where

\[ H_0(r) = \frac{2}{r^2 + 1} + \frac{\zeta^2}{2(r^2 + 1)} - \frac{\zeta^4(1 - \log(2))}{4(r^2 + 1)} + O\left(\zeta^6\right), \]

\[ H_1(r) = \left(\frac{r^2 - 5}{48(r^2 + 1)^2} + \frac{\zeta^2}{288(r^2 - 1)(r^2 + 1)^2}\right)\left(10r^4 + 72r^4\log(r) - 27r^4\log(2) + 18r^2 + 18r^2\log(2) - 36r^4\log(r^2 + 1) - 28 + 45\log(2)\right) + O\left(\zeta^4\right), \]

\[ H_2(r) = -\frac{1}{55296}\frac{1}{(r^2 - 1)(r^2 + 1)^3}\left(253r^6 + 589r^4 - 589r^2 + 48(r^2 + 1)^2\log(64) - 336(r^2 - 1)(r^2 + 1)^2\log(2) + 576(r^2 + r^4)\log\left(\frac{r^2}{r^2 + 1}\right) - 253\right) + O\left(\zeta^2\right), \]

and

\[ L_0(r) = \frac{\zeta}{r^2}, \]

\[ L_1(r) = -\frac{\zeta}{8r^2(1 + r^2)} + O(\zeta^3), \]

\[ L_2(r) = \mathcal{O}(\zeta). \]

The functions in the metric in (6.2) are given by

\[ f(r) = \left(r^2 - \frac{1}{r^2}\right) + \frac{1}{e^r}(f_0(r) + f_1(r)e^2 + f_2(r)e^4 + O(e^6)) + O\left(\frac{1}{e^4}\right), \]

\[ g(r) = 1 + \frac{1}{e^r}(g_0(r) + g_1(r)e^2 + g_2(r)e^4 + O(e^6)) + O\left(\frac{1}{e^4}\right), \]

\[ j(r) = 0 + \frac{1}{e^r}(j_0(r) + j_1(r)e^2 + O(e^4)) + O\left(\frac{1}{e^4}\right), \]

\[ k(r) = r^2 + \frac{1}{e^r}(k_0(r) + k_1(r)e^2 + k_2(r)e^4 + O(e^6)) + O\left(\frac{1}{e^4}\right). \]

where

\[ f_0(r) = -4\frac{(r^2 - 1)}{3r^4} - 2\frac{(r^2 - 1)\zeta^2}{3r^4} + \frac{\zeta^4(3r^2 + r^4(-\log(16)) - 3 + \log(16))}{12r^4} + O\left(\zeta^6\right), \]

\[ f_1(r) = \frac{7r^4 + 12r^2 - 5}{36r^4(r^2 + 1)} + \frac{\zeta^2}{432r^2(r^2 + 1)}\left(\frac{54r^6 + r^4(54\log(2) - 23) - 36r^2(2 + \log(2)) + 41 - 90\log(2)}{r^2}\right) + 18\left(3r^6 + 3r^4 - 9r^2 - 1\right)\left(2\log(r) - \log(r^2 + 1)\right) + O\left(\zeta^4\right), \]

\[ f_2(r) = \frac{1}{48r^2}\left(-2\frac{(r^6 + r^4 - 2r^2)}{3(r^2 + 1)}\right) \left(2\log(r) - \log(r^2 + 1)\right) - \frac{1}{864r^2(2r^2 + 1)^2}\left(576r^{10} + 989r^8 + 624r^6\log(2) - 1538r^6 + 1248r^6\log(2) - 1044r^4 + 914r^2 - 1248r^2\log(2) + 103 - 624\log(2)\right)\right), \]
\[ g_0(r) = \mathcal{O}(\zeta^6), \]
\[ g_1(r) = -\frac{1}{6(r^2 + 1)^2} \frac{\zeta^2 \left( 54(r^2 + 1)^3 - 6(r^2 + 1)^2(-9r^4 - 18r^2 + 3)(2\log(r) - \log(r^2 + 1)) \right)}{864(r^2 + 1)^4} + \mathcal{O}(\zeta^4), \]
\[ g_2(r) = -\frac{6r^6 - 21r^4 - 6(r^2 + 1)^2(r^4 + 2r^2)(2\log(r) - \log(r^2 + 1)) + 4}{864(r^2 + 1)^4} + \mathcal{O}(\zeta^2), \]
\[ j_0(r) = \mathcal{O}(\zeta^6), \]
\[ j_1(r) = \frac{(r^2 - 1)\zeta}{8(r^2 + 1)} + \mathcal{O}(\zeta^3), \]
\[ k_0(r) = \mathcal{O}(\zeta^6), \]
\[ k_1(r) = \frac{r^2\zeta^2(-2(r^2 + 1)\log(r) + (r^2 + 1)\log(r^2 + 1) - 1)}{8(r^2 + 1)} + \mathcal{O}(\zeta^4), \]
\[ k_2(r) = \mathcal{O}(\zeta^2), \]

and

Upon setting \( \frac{1}{e^2} = 0 \), our result exactly matches with the equations 2.30, 2.31, 2.32 in [6], if we replace \( u = 1/r \) in those equations.

### 6.4 The first correction to the phase transition curve

As we have seen above, Herzog’s model undergoes a superfluidity nucleation phase transition at \( |\frac{\tilde{\phi}}{q}| = 2 \).

It is easy to work out how this phase transition curve is corrected at \( \mathcal{O}(\frac{1}{e^2}) \). The phase transition curve is determined by finding the black brane solution that admits a static regular and normalizable solution to the linearized scalar field equations about a Reissner Nordstrom black brane. Now the black brane solution, at fixed temperature and chemical potential, depends on the parameter \( e \). In the probe limit the metric of the charged black brane is simply independent of the chemical potential and equal to that of the uncharged black brane of the same temperature. This metric (and the accompanying gauge field) are slightly deformed at \( \mathcal{O}(\frac{1}{e^2}) \). The condition for the existence of a normalizable and regular zero mode to the scalar equation is also, consequently, deformed.

Let the restriction of the gauge field at infinity be given by

\[ \xi_\mu = r_c(-\nu u_\mu) \]

where \( r_c \) is the location of the horizon. It turns out that a normalizable and regular zero mode to the linearized scalar equation occurs at

\[ \nu_c = -2 + \frac{1}{e^2} \left( 4 - \frac{16\log(2)}{3} \right) + \mathcal{O}\left(\frac{1}{e^2}\right) \]

The temperature of this solution (taking the back reaction of the gauge field into account) is given by

\[ T = \frac{r_c}{\pi} \left[ 1 - \frac{2}{3e^2} + \mathcal{O}\left(\frac{1}{e^4}\right) \right] \]
It follows that \( \frac{\mu}{T} \) at the superfluidity phase transition is given by

\[
\left| \frac{\mu}{T} \right| = \frac{r_c V_c}{T} = \pi \left[ 2 + \frac{8}{3e^2} (2 \log(2) - 1) + \mathcal{O} \left( \frac{1}{e^2} \right) \right]
\]

### 6.5 Boundary Thermodynamics

Using the solution obtained in the previous section we evaluate the boundary stress tensor charge current. For this purpose we use the standard AdS/CFT formulas \([47, 48]\).

\[
\text{Boundary stress tensor} = T_{\mu \nu} = \frac{1}{16 \pi G} \lim_{r \to \infty} r^4 \left( 2 (\delta^\mu_{\nu} K_{\alpha \beta} \gamma^{\alpha \beta} - K_{\nu}^\mu) - 6 \delta^\mu_{\nu} + \frac{\phi^* \phi}{e^2} \delta^\mu_{\nu} \right)
\]

\[
\text{Boundary charge current} = j^\mu = \frac{1}{16 \pi G} \lim_{r \to \infty} r^3 F^{\mu \nu}
\]

\[
\text{Entropy density} = s = \sqrt{k(1)}
\]

\[
\text{Temperature} = T = \frac{f'(1)}{4 \pi g(1)}
\]

where \( \gamma_{\alpha \beta} \) and \( K_{\alpha \beta} \) are respectively the induced metric and extrinsic curvature of a constant \( r \) surface. The result for the stress tensor and current can be parameterized as the form

\[
T^{\mu \nu} = \frac{1}{16 \pi G} \left[ A u^\mu u^\nu + B n^\mu n^\nu + C (n^\mu u^\nu + u^\mu n^\nu) + \left( \frac{A - B}{2} \right) \tilde{p}^{\mu \nu} \right]
\]

\[
j^\mu = \frac{1}{16 \pi G} [Q_1 u^\mu + Q_2 n^\mu]
\]

\[\text{(6.18)}\]

\( A, B \) and \( C \) are given by the following expressions.

\[
A = 3r_c^4 + \frac{r_c^4}{e^2} \left\{ 4 + 2\zeta^2 + \zeta^4 \left( \log(2) - \frac{3}{4} \right) + \mathcal{O} (\zeta^6) \right\}
\]

\[
+ \epsilon^2 \left[ \frac{7}{12} + \zeta^2 \left( \frac{59}{144} - \frac{3 \log(2)}{8} \right) + \mathcal{O} (\zeta^4) \right]
\]

\[
+ \epsilon^4 \left[ \frac{624 \log(2) - 451}{13824} + \mathcal{O} (\zeta^2) + \mathcal{O} (\epsilon^6) \right] \right\} + \mathcal{O} \left( \frac{1}{e^4} \right)
\]

\[
B = r_c^4 + \frac{r_c^4}{e^2} \left\{ 4 + 2\zeta^2 + \zeta^4 \left( \log(2) - \frac{1}{4} \right) + \mathcal{O} (\zeta^6) \right\}
\]

\[
+ \epsilon^2 \left[ \frac{7}{36} + \zeta^2 \left( \frac{131}{432} - \frac{\log(2)}{8} \right) + \mathcal{O} (\zeta^4) \right]
\]

\[
+ \epsilon^4 \left[ \frac{624 \log(2) - 451}{41472} + \mathcal{O} (\zeta^2) + \mathcal{O} (\epsilon^6) \right] \right\} + \mathcal{O} \left( \frac{1}{e^4} \right)
\]

\[\text{(6.19)}\]

\( C = \frac{r_c^4}{e^2} \left\{ \zeta^2 \left[ \frac{1}{2} + \mathcal{O} (\zeta^3) \right] + \mathcal{O} (\epsilon^4) \right\} + \mathcal{O} \left( \frac{1}{e^4} \right) \]

While \( Q_1 \) and \( Q_2 \) are given by the following expressions
Further the chemical potential and \( \mu_s \) of our solution is given by

\[
\mu = u^s \xi_\mu = r_e \left\{ \left[ -2 - \frac{\zeta^2}{2} + \xi^4 \left( \frac{13}{64} + \log(2) \right) + O(\xi^6) \right] \right. \\
+ e^2 \left[ -\frac{1}{48} + \xi^4 \left( \frac{3 \log(2)}{32} - \frac{5}{144} \right) + O(\xi^4) \right] \\
+ e^4 \left[ \frac{253}{55296} - \frac{7 \log(2)}{1152} + O(\xi^2) \right] + O(\xi^6) \right\} + O\left( \frac{1}{e^4} \right)
\] (6.22)

Moreover we find

\[
s = \frac{r_e^3}{4G} \left\{ \left[ \frac{1}{32} \log(4) - \frac{1}{32} \zeta^2 + O(\xi^4) \right] + O(\xi^4) \right\} + O\left( \frac{1}{e^4} \right)
\] (6.23)
\[ T = \frac{r_c}{\pi} + \frac{r_c}{4\pi e^2} \left\{ \left[ -\frac{8}{3} - \frac{4\zeta^2}{3} + \zeta^4 \left( \frac{1}{2} - \frac{2\log(2)}{3} \right) + O(\zeta^6) \right] + \epsilon^2 \left[ \frac{1}{9} + \zeta^2 \left( \frac{\log(2)}{4} - \frac{23}{216} \right) + O(\zeta^4) \right] + \epsilon^4 \left[ \left( \frac{91}{20736} - \frac{\log(2)}{108} \right) + O(\epsilon^2) \right] + O(\epsilon^6) \right\} + O\left( \frac{1}{e^4} \right) \] (6.24)

Using these expressions and the quantities obtained in (6.21) we have verified all the relations (2.2) to the order to which we have evaluated our solution.

6.6 Stability

In Appendix C we have demonstrated that the solutions we have computed above are unstable whenever \( \frac{3}{4} \geq \frac{1}{4} \sqrt{\frac{5}{3}} \) (working at leading order in \( \frac{1}{e^2} \) and first nontrivial order in \( \zeta \) and \( \epsilon \) with both quantities taken to be of the same order).

It follows that superfluid flows with \( \zeta \gg \epsilon \) are unstable. For this reason in the rest of this paper we will specialize to the case that \( \zeta \leq O(\epsilon) \).

It is of some interest to characterize the superfluid phase on a phase diagram with axes labeled by \( \frac{\mu}{T} \) and \( \zeta \) (by conformal invariance the phase diagram can only depend on the ratio \( \frac{\mu}{T} \)). By restricting (6.22) to lowest order, and noting that \( \epsilon \geq 0 \), we conclude that the superfluid phase exists only when

\[ \left| \frac{\mu}{T} \right| - 2 \geq \frac{\zeta^2}{2} \] (6.25)

On the other hand this phase is unstable when

\[ \left| \frac{\mu}{T} \right| - 2 \geq \frac{9\epsilon^2}{14} \] (6.26)

As \( \frac{9}{14} > \frac{1}{2} \), it follows that as we increase \( \zeta \) at fixed \( |\frac{\mu}{T}| \), the superfluid phase first goes unstable and then stops existing. It would be interesting to investigate whether this qualitative behaviour persists at finite values of \( \zeta \).

7. Superfluid dynamics to first order in the derivative expansion

In the previous section we have determined the equilibrium solutions for hairy black branes, perturbatively in \( \epsilon \) and the superfluid velocity, and separately in an expansion in \( \frac{1}{e^2} \). In this section we use the results of the previous subsection as an input into the fluid gravity map.

The basic idea here is a simple generalization of the ideas spelt out in [38, 39, 40, 41, 42, 43, 44, 45]. We search for gravitational solutions that tube wise approximate the 8 parameter hairy black brane solutions described in the previous section, with values of the temperature, the chemical potential, \( \zeta^\mu \) and \( u^\mu \) varying in space and time. The tubes in question run along null ingoing geodesics, and foliate our spacetime. Technically, this programme is implemented by working in ingoing Eddington Finklestein coordinates (as we have been through this paper) but promoting the parameters of our solutions to fields that vary in spacetime.

The fluid gravity map generates the gravitational solutions dual to fluid flows perturbatively in a boundary derivative expansion. The zero order ansatz for such a solution is simply the solution (6.2)
The precise definitions of our field variables is given by the equations

\[ u_{\mu} \] and \( w^\mu \) promoted to arbitrary slowly varying functions of spacetime. This ansatz of course solves the equations of motion (1.2), (1.3) and (1.4), when all parameters are constant, but does not solve these equations when these parameter vary in spacetime. As in [22, 11, 12, 43] this ansatz may be corrected to obtain a true solution (systematically in a derivative expansion) provided the eight fluid fields that parameterize our ansatz obey certain constraint equations. These constraint equations are simply the fluid equations (2.5) with holographically generated constitutive relations for the stress tensor, the charge current, and a holographically generated correction to the Josephson equation.

In this section we implement this programme to first order in the derivative expansion.

### 7.1 The method

As we have explained above, we will search for gravity solutions that tube wise approximate the equilibrium solutions of the previous section. In principle our solutions could be labeled by a temperature and a chemical potential field in addition to the normal and superfluid velocity fields. However, for calculation purposes we will find it convenient to trade chemical potential for \( \rho \) and a chemical potential field in addition to the normal and superfluid velocity fields. However, for equilibrium solutions of the previous section. In principle our solutions could be labeled by a temperature.

As we have explained above, we will search for gravity solutions that tube wise approximate the equilibrium solutions of the previous section. In principle our solutions could be labeled by a temperature.

The fluid gravity map is generated by solving Einstein’s equations tube wise, point by point on the boundary. At any given boundary point we can always boost and rotate coordinates so that

\[ u_\mu = (-1, 0, 0, 0), \quad n_\mu = (0, 1, 0, 0) \]

In the neighborhood of our special point, however,

\[ u_\mu = \gamma_u (1, -1, \beta_1, \beta_2, \beta_3), \quad n_\mu = \gamma_n (-n_v, 1, n_2, n_3) \]

\[ \gamma_u = \frac{1}{\sqrt{1 - \beta_1^2 - \beta_2^2 - \beta_3^2}}, \quad \gamma_n = \frac{1}{\sqrt{n_v^2 - 1 - n_2^2 - n_3^2}} \]  

(7.2)

where \( \beta_i \) and \( n_i \) are of first or higher order in derivatives of fluid fields at the special point.

In this paper we will work only to first order in the derivative expansion. At this order we are sensitive only to first derivatives of \( \beta_1, \beta_2, \beta_3, n_2 \) and \( n_3 \) along with the first derivatives of \( \xi, r_c \) and \( \epsilon \).

The solution at our special point preserves an SO(2) symmetry (of rotations in a plane perpendicular to \( u_\mu \) and \( n_\mu \); the yz plane in our coordinates). This symmetry will help us organize our calculation. To start with it will prove useful to organize first derivative ‘fluid data’, i.e. all the first derivatives of the fluid fields at our special point, in terms of their SO(2) transformation properties.

We list our results
• First order derivative excitations with spin 0 (scalars):
  \[ S_1 = \frac{1}{\varepsilon} \partial_1 \zeta, \quad S_2 = \frac{1}{\varepsilon} \partial_1 \epsilon, \quad S_3 = \partial_0 \beta_1, \quad S_4 = \partial_1 \beta_1, \quad S_5 = \partial_0 n_i, \]
  \[ S_6 = \partial_i \beta_i, \quad S_7 = \varepsilon_i j \partial_i n_j, \]
  \[ S_8 = \partial_0 r_c, \quad S_9 = \partial_1 r_c, \quad S_{10} = \frac{1}{\varepsilon} \partial_0 \zeta, \quad S_{11} = \frac{1}{\varepsilon} \partial_0 \epsilon, \quad S_{12} = \varepsilon_i j \partial_i \beta_j \]

• First order derivative excitations with spin \pm 1 (vectors):
  \[ [V_1]_i = \frac{1}{\varepsilon} \partial_i \epsilon, \quad [V_2]_i = \frac{1}{\varepsilon} \partial_i \zeta, \quad [V_3]_i = \partial_i n_i, \quad [V_4]_i = \partial_0 \beta_i, \]
  \[ [V_5]_i = \partial_i \beta_1 + \partial_1 \beta_i \quad [V_6]_i = \partial_0 r_c, \quad [V_7]_i = \partial_0 n_i, \quad [V_8]_i = \partial_i \beta_1 - \partial_1 \beta_i \]

• First order derivative excitations with spin \pm 2 (traceless symmetric tensors):
  \[ [T_1]_{ij} = \partial_i \beta_j + \partial_j \beta_i - (\partial_k \beta_k) \delta_{ij}, \quad [T_2]_{ij} = \partial_i n_j + \partial_j n_i - (\partial_k n_k) \delta_{ij} \]

Here \( \{i, j\} = \{2, 3\} \).

Following the methods of \([38, 41, 42]\), in order to derive the metric dual to a fluid flow we need to solve the equations of motion, order by order, in the derivative expansion. That is we set the metric \( g \) of our solution to \( g_0 + \epsilon g_1 \ldots \) (and similarly for the gauge fields and the scalars) and solve the bulk equations of motion at first order in \( \epsilon \). As explained in \([38, 41, 42]\), the resulting equations are of two sorts. The Einstein and Maxwell constraint equations reduce simply to the equations of energy momentum and current conservation, and do not involve the unknown fields \( g_1 \) etc. These equations relate some of the independent derivatives listed above to others. On the other hand the dynamical Einstein and Maxwell equations allow you us compute the unknown fields \( g_1 \) etc in terms of the constrained derivative data listed above.

### 7.2 The constraint equations

We will now first describe the solution of the constraint equations, before turning to the dynamical equations.

In addition to the conservation equations described above, there is one additional source of constraints on the derivative data given in \([34]\). Our demand that our solution be asymptotically AdS requires, in particular that the boundary field strength vanishes, implying \( \partial_\mu \zeta_\mu - \partial_\mu \zeta_\mu \) vanishes. We must add this equation to the list of equations that constrain independent data.

It is convenient to decompose the constraint equations according to the its quantum numbers under the preserved \( SO(2) \). We now perform the relevant decompositions, and state which pieces of data we use these constraints to solve for.

- **Current conservation:** It is a spin-0 constraint. Using this we shall solve for \( S_{11} \).
- **Stress-tensor conservation:** It is effectively four equations. Among them two are spin-0 constraints and one spin-1 constraint. Using this we shall solve for \( S_8, S_9 \) and \( V_6 \).
- **Curl-free condition on \( \xi_\mu \):** This imposes a set of 6 equations.
  Two of them transform in spin-0 (\( [\partial_0 \xi_1 - \partial_1 \xi_0] \) and \( \epsilon_{ij} \partial_i \xi_j) \). Using these we solve for \( S_{10} \) and \( S_{12} \) respectively.
  Four of them transform in two separate spin-1 (\( [\partial_1 \xi_1 - \partial_1 \xi_1] \) and \( [\partial_0 \xi_0 - \partial_0 \xi_0] \)). Using these we solve for \( V_7 \) and \( V_8 \).
After solving for dependent data \(^{18}\), the remaining independent one derivative pieces of data are given as follows. We have seven spin-0 \((S_1, \cdots, S_7)\), five spin-1 \((V_1, \cdots, V_5)\) and two spin-2 \((T_1, T_2)\) boundary data.

For later use we will find it useful to list covariant expressions for the independent data. These expressions are most usefully written in terms of the projector normal to the velocity/superfluid velocity frame

\[
\tilde{P}_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu} - n_\mu n_\nu
\]

Using this projector one can write the following covariant expressions for our choices of independent boundary data as follows:

**Spin-0**

\[
\begin{align*}
S_1 &= \frac{1}{\epsilon} (n^\mu \partial_\mu) \zeta, \\
S_2 &= \frac{1}{\epsilon} (n^\mu \partial_\mu) \epsilon, \\
S_3 &= u^\mu n^\nu \partial_\mu u_\nu, \\
S_4 &= n^\mu n^\nu \partial_\mu u_\nu, \\
S_5 &= \tilde{P}^{\mu\nu} \partial_\mu n_\nu, \\
S_6 &= \tilde{P}^{\mu\nu} \partial_\mu u_\nu, \\
S_7 &= \epsilon^{\mu\nu\rho\sigma} n_\mu u_\mu \partial_\rho n_\sigma
\end{align*}
\]

\[\text{(7.3)}\]

**Spin-1**

\[
\begin{align*}
[V_1]_\mu &= \frac{1}{\epsilon} \tilde{P}^\sigma_\mu \partial_\sigma \epsilon, \\
[V_2]_\mu &= \frac{1}{\epsilon} \tilde{P}^\sigma_\mu \partial_\sigma \zeta, \\
[V_3]_\mu &= \tilde{P}^\nu_\mu n^\sigma \partial_\sigma n_\nu, \\
[V_4]_\mu &= \tilde{P}^{\nu\sigma}_\mu \partial_\sigma u_\nu, \\
[V_5]_\mu &= \tilde{P}^{\nu\sigma}_\mu (\partial_\nu u_\sigma + \partial_\sigma u_\nu)
\end{align*}
\]

\[\text{(7.4)}\]

**Spin-2**

\[
\begin{align*}
[T_1]_{\mu\nu} &= \tilde{P}^\sigma_\mu \tilde{P}^\rho_\nu [\partial_\sigma u_\rho + \partial_\rho u_\sigma] - S_5 \tilde{P}^{\mu\nu}, \\
[T_2]_{\mu\nu} &= \tilde{P}^\sigma_\mu \tilde{P}^\rho_\nu [\partial_\sigma n_\rho + \partial_\rho n_\sigma] - S_5 \tilde{P}^{\mu\nu}
\end{align*}
\]

\[\text{(7.5)}\]

### 7.3 The dynamical equations

Following earlier work on the fluid gravity correspondence \[^{38, 41, 42, 43}\], we work in the gravitational gauge \(g_{rr} = 0\) and \(g_{r\mu} = u_\mu\). For the \(U(1)\) field we continue to demand that the scalar field be real. With derivatives taken into account this requirement no longer sets \(A^r\) to zero, but allows us to determine \(A^r\) rather simply, by demanding the consistency of the equations for \(\phi\) and \(\phi^*\).

We will now solve for the first derivative corrections about the basic fluid gravity ansatz. As we have determined the equilibrium solutions, in the previous section, only to order \(1/\epsilon^2\), we can of course compute the metric dual to fluid flows only at the same order in \(1/\epsilon^2\).

We now describe in rough terms how we determine the deviations away from the zero order fluid ansatz. Let us start with the gauge field and scalars. At leading order in \(1/\epsilon^2\) we take derivative corrections to the gauge field and the scalar field to have the form \((\delta A^M\) is the derivative correction

\[^{18}\text{As we have indicated above, we solve for some first derivatives of fluid fields in terms of other derivatives. The relevant equations are linear and easy to solve; the solutions are explicit but lengthy and we do not present them here.}\]
for the gauge field)

\[
\delta A^r = \sum_{i=1}^{7} \delta A_i \left( \frac{r}{r_c} \right) S_i + \mathcal{O} \left( \frac{1}{r^2} \right)
\]

\[
\delta A^\mu = \frac{1}{r_c^2} \left[ u^\mu \sum_{i=1}^{7} \delta H_i \left( \frac{r}{r_c} \right) S_i + n^\mu \sum_{i=1}^{7} \delta L_i \left( \frac{r}{r_c} \right) S_i + \sum_{i=1}^{5} X_i \left( \frac{r}{r_c} \right) \left[ V_i \right]^\mu \right] + \mathcal{O} \left( \frac{1}{r^2} \right) \quad (7.6)
\]

\[
\delta \phi = \frac{1}{r_c} \left[ \sum_{i=1}^{7} \delta \phi_i \left( \frac{r}{r_c} \right) S_i \right] + \mathcal{O} \left( \frac{1}{r^2} \right)
\]

We now describe in structural terms how we have solved for these functions, emphasizing boundary conditions

1. It turns out that \( \delta A_i (r) \) obeys a first order differential equation in \( r \). The general solution of \( \delta A^r \) diverges linearly in \( r \) while expanded around \( r = \infty \). We fix the constant of integration (coefficient of the homogeneous solution in this equation) by setting the coefficient of the linear term in \( r \) to zero. This choice of boundary conditions is forced on us by the requirement that the bulk current goes to zero at the boundary so that the boundary current is really conserved.

2. \( \delta H_i (r) \) obeys a second order differential equation (arising from the \( r \) component of the Maxwell equation). The two integration constants for this equation are fixed as follows. One of the integration constant is determined from the requirement of regularity at the horizon. The other integration constant is obtained from the requirement that there exist a regular scalar field solution (see below).

3. \( \delta L_i (r) \) obeys a second order differential equation given by the \( x \) component of the Maxwell equation. Here one of the integration constant is determined imposing the regularity of the solution at the horizon. The other integration constant is fixed using the fact that according to equation (7.1) \( \zeta^\mu \) does not receive any derivative correction. A generic solution of \( \delta L_i (r) \) dies at infinity like \( \frac{1}{r^2} \); the coefficient of this \( \frac{1}{r^2} \) must be set to zero.

4. The equation for \( X_i (r) \) comes from the \( y \) or \( z \) component of the Maxwell equation. This is also a second order differential equation and its integration constants are determined in a similar way as in \( \delta L_i (r) \).

5. The equation for the scalar field determines \( \delta \phi_i (r) \). Normalizability and the definition of \( \epsilon (x) \) as given in equation (7.1) fixes the two integration constants here. More specifically, an expansion about infinity of a generic solution to the scalar field equation takes the form

\[
\delta \phi_i (r) = a_i \ln \left( \frac{r}{r_c} \right) + b_i \frac{1}{r^2} + \mathcal{O} \left( \frac{1}{r^2} \right)
\]

Our boundary conditions are that both \( a_i \) vanishes (from the requirement of normalizability) and that \( b_i \) vanishes (from our definition of \( \epsilon \)). These two requirements completely fix the scalar fluctuation. As described above, the further requirement that the scalar fluctuation be regular at the horizon yields a boundary condition on \( \delta H_i (r) \) (see above).
Let us now turn to the metric field. In the strict limit of \( r_c \to 0 \) the scalar and gauge field do not back react on the metric. The derivative expansion of the metric in this limit is thus that of uncharged fluid dynamics and was determined in [38] to be

\[
\left. ds^2 = -2u_\mu dx^\mu dr + r^2 \left[ -\left(1 - \frac{r_c^4}{r^4}\right) u_\mu u_\nu + + P_{\mu\nu} \right] dx^\mu dx^\nu \right.
\]
\[
- \left. dx^\nu dx^\mu \left[ 2ru_\mu \left( u,\partial \right) u_\nu - \frac{1}{3} (\partial, u) u_\nu \right] + r_c F \left( \frac{r}{r_c} \right) \sigma_{\mu\nu} \right]\]

(7.7)

Where

\[
F(r) = -\frac{r_c^2}{2} \left[ -\log (r^2 + 1) + 4 \log(r - 1) + 2 \tan^{-1}(r) - \pi \right]
\]

and

\[
\sigma_{\mu\nu} = P_{\mu}^\alpha P_{\nu}^\beta \left( \frac{\partial, u_\alpha + \partial, u_\beta}{2} - \frac{\partial, u_\alpha}{3} \eta_{\alpha\beta} \right)
\]

The new results of this paper are for the derivative correction to the metric at order \( O \left( \frac{1}{r^2} \right) \). We parameterize the corrections to the metric as

\[
\delta(ds^2) = -\frac{2}{e^2} \left[ \sum_{i=1}^{7} S_i \left( \frac{r}{r_c} \right) \right] u_\mu dx^\mu dr
\]
\[
+ \frac{r_c}{e^2} \left[ \sum_{i=1}^{7} S_i \left( \frac{r}{r_c} \right) \right] u_\mu u_\nu + \left[ \sum_{i=1}^{7} S_i \delta K_i \left( \frac{r}{r_c} \right) \right] n_\mu n_\nu
\]
\[
+ \left[ \sum_{i=1}^{7} S_i \delta J_i \left( \frac{r}{r_c} \right) \right] \left( n_\mu u_\nu + n_\nu u_\mu \right)
\]
\[
+ \frac{r_c}{e^2} \left[ \sum_{i=1}^{5} Y_i \left( \frac{r}{r_c} \right) \left( u_\mu [V_i]_\nu + u_\nu [V_i]_\mu \right) \right]
\]
\[
+ \left[ \sum_{i=1}^{2} Z_i \left( \frac{r}{r_c} \right) [T_i]_{\mu\nu} \right] + O \left( \frac{1}{e^4} \right)
\]

(7.8)

We now describe, very qualitatively, how we have solved for these functions.

1. \( \delta g_i(r) \), \( \delta f_i(r) \) and \( \delta K_i(r) \) are determined solving three coupled equations obtained from the \((rr)\), \((rv)\) and \((xx)\) component of the Einstein equations. Once decoupled using appropriate combination of these functions, two of the equations become first order and the third one is a second order differential equation. Two of the four integration constants are determined using the asymptotic AdS condition of the metric. 19 A third integration constant is fixed by demanding the regularity of the function \( \delta K_i(r) \) at \( r = r_c \). The last integration constant is fixed to ensure that the \( uv \) component of the boundary stress tensor receives no derivative corrections so that the equation (7.1) is satisfied.

19After this condition is imposed \( \frac{1}{r^4} \) expansion of the functions \( \delta g_i(r) \) and \( \delta K_i(r) \) take the form

\[
\lim_{r \to \infty} \delta g_i(r) = O \left( \frac{1}{r^4} \right), \quad \lim_{r \to \infty} \delta K_i(r) = O \left( \frac{1}{r^2} \right)
\]
2. The function $\delta J_i(r)$ is determined using the $(rx)$ or $(vx)$ component of the Einstein equation. This is a second order differential equation in $r$. The two integration constants are determined using the normalizability and the definition of boundary stress tensor (according to (7.1) the $vx$ component of the boundary stress tensor should not receive any derivative correction). The general solution for $\delta J_i(r)$ has the following expansion around $r = \infty$.

$$\lim_{r \to \infty} \delta J_i(r) = j_0 r^2 + \frac{j_1}{r^2} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

Our boundary condition is that $j_0$ and $j_1$ both vanish.

3. $Y_i(r)$ is determined from the $(vy)$ or $(vz)$ component of the Einstein equation. This is a second order differential equation in $r$. The two integration constants are determined using normalizability of the metric and the definition of boundary stress tensor (the $(vy)$ or $(vz)$ component of the stress tensor should not receive any derivative corrections). This condition is exactly same as that of $\delta J_i(r)$ in terms of the coefficients of $\frac{1}{r}$ expansion.

4. $W_i(r)$ is determined from the $(xy)$ or $(xz)$ component of the Einstein equation. This is a second order differential equation in $r$. The two integration constants are determined using normalizability and regularity of the metric respectively. A generic solution of $W_i$ behaves like $r^2$ at large $r$. Our boundary conditions are that the leading coefficient of this leading $r^2$ piece vanish.

5. $Z_i(r)$ is determined from the $(yz)$ component of the Einstein equation. This is a second order differential equation in $r$. The two integration constants are determined exactly the same way as for $W_i(r)$.

### 7.4 Results for Bulk Fields

Without further ado in this subsection we simply present our final results for all the fields defined in the previous subsection.

We have performed all our computations in this section using Mathematica. In several instances we have carried out calculations to higher order, in the Mathematica file, than we have presented below, mainly to avoid burdening the reader with very lengthy expressions.

The solutions presented in this subsection determine the full first order correction to the gauge field, scalar field and metric to the relevant order in an expansion in $\epsilon$ and $\frac{1}{\zeta}$. Now we choose to scale $\zeta$ like $\epsilon$. We present our results below in terms of the order one field

$$\chi = \frac{\zeta}{\epsilon}$$

Recall that, according to the results of §6.4, our fluid becomes unstable whenever $\chi$ exceeds a number of order unity. So while $\chi$ can be arbitrarily small, it is unphysical for $\chi$ to be made arbitrarily large.
Results for the gauge field and scalar field

\[ \delta A_1(r) = \epsilon \left[ \frac{r^2 (96\chi^2 - 5) + 48\chi^2 + 1}{14r^3} \right] + O(\epsilon^3) \]

\[ \delta A_2(r) = \epsilon \left[ \frac{(2 - 3r^2)\chi}{7r^3} \right] + O(\epsilon^3) \]

\[ \delta A_3(r) = -\epsilon \left[ \frac{(2r^2 + 1)\chi}{7r^3} \right] + O(\epsilon^3) \]

\[ \delta A_4(r) = \frac{16 (2r^2 + 1)\chi^2}{7r^3} - \frac{2r}{3 (r^2 + 1)} + O(\epsilon^2) \]  

(7.9)

\[ \delta A_5(r) = \epsilon \left[ \frac{(1 - 5r^2)\chi}{14r^3} \right] + O(\epsilon^3) \]

\[ \delta A_6(r) = -\frac{2r}{3 (r^2 + 1)} - \frac{8 (2r^2 + 1)\chi^2}{7r^3} + O(\epsilon^2) \]

\[ \delta H_1(r) = \epsilon \left[ \frac{r(r + 2)(96\chi^2 - 5) - 48\chi^2 - 1}{14r(r + 1)(r^2 + 1)} \right] + O(\epsilon^3) \]

\[ \delta H_2(r) = \epsilon \left[ \frac{3r^2 + 6r + 2)\chi}{7r^3 + r^2 + r + 1} \right] + O(\epsilon^3) \]

\[ \delta H_3(r) = \epsilon \left[ \frac{(5r^2 + 10r + 1)\chi}{7r (r^3 + r^2 + r + 1)} \right] + O(\epsilon^3) \]

(7.10)

\[ \delta H_4(r) = \frac{16 (2r^2 + 4r - 1)\chi^2}{ir (r^3 + r^2 + r + 1)} + O(\epsilon^2) \]

\[ \delta H_5(r) = \epsilon \left[ \frac{(5r^2 + 10r + 1)\chi}{14r (r^3 + r^2 + r + 1)} \right] + O(\epsilon^3) \]

\[ \delta H_6(r) = -\frac{8 [2r(r + 2) - 1] \chi^2}{7r(r + 1)(r^2 + 1)} + O(\epsilon^2) \]
\[ \delta L_1(r) = \epsilon^2 \left[ \frac{\chi \left( \log \left( r^2 + 1 \right) - 2 \log(r + 1) + 2 \tan^{-1}(r) - \pi \right)}{4r^2} \right] + \mathcal{O}(\epsilon^4) \]

\[ \delta L_2(r) = \epsilon^2 \left[ \frac{\log \left( r^2 + 1 \right) - 2 \log(r + 1) + 2 \tan^{-1}(r) - \pi}{96r^2} \right] + \mathcal{O}(\epsilon^4) \]

\[ \delta L_3(r) = \mathcal{O}(\epsilon^4) \]

\[ \delta L_4(r) = -\epsilon \left[ \frac{\chi \left( \log \left( r^2 + 1 \right) - 4 \log(r) + 2 \log(r + 1) - 2 \tan^{-1}(r) + \pi \right)}{3r^2} \right] + \mathcal{O}(\epsilon^3) \]

\[ \delta L_5(r) = \mathcal{O}(\epsilon^4) \]

\[ \delta L_6(r) = -\epsilon \left[ \frac{\chi \left( \log \left( r^2 + 1 \right) - 4 \log(r) + 2 \log(r + 1) - 2 \tan^{-1}(r) + \pi \right)}{6r^2} \right] + \mathcal{O}(\epsilon^3) \]

\[ X_1(r) = \epsilon^2 \left[ \frac{\log \left( r^2 + 1 \right) - 2 \log(r + 1) + 2 \tan^{-1}(r) - \pi}{96r^2} \right] + \mathcal{O}(\epsilon^4) \]

\[ X_2(r) = \epsilon^2 \left[ \frac{\chi \left( \log \left( \frac{r^2+1}{r+1} \right) + 2 \tan^{-1}(r) - \pi \right)}{4r^2} \right] + \mathcal{O}(\epsilon^4) \]

\[ X_3(r) = \mathcal{O}(\epsilon^4) \]

\[ X_4(r) = \mathcal{O}(\epsilon^4) \]

\[ X_5(r) = \epsilon \left[ -\frac{\chi \left( \log \left( r^2 + 1 \right) - 4 \log(r) + 2 \log(r + 1) - 2 \tan^{-1}(r) + \pi \right)}{4r^2} \right] + \mathcal{O}(\epsilon^3) \]

\[ (7.11) \]

\[ (7.12) \]
\[ \delta \phi_1 (r) = \epsilon ^2 \left[ \frac{3}{14} \left( 1 - 8 \chi^2 \right) \left( \tan^{-1} (r) - \log (1 + r) - \frac{\pi}{2} \right) 
\quad + \frac{2}{7} \left( 30 \chi^2 - 1 \right) \log (r) + \left( \frac{1}{4} - 6 \chi^2 \right) \log (r^2 + 1) \right] + O(\epsilon^4) \]
\[ \delta \phi_2 (r) = \epsilon ^2 \left[ - \frac{1}{28} \chi \left( -7 \log (r^2 + 1) + 4 \log (r) + 10 \log (r + 1) - 10 \tan^{-1} (r) + 5 \pi \right) \right] 
\quad + O(\epsilon^4) \]
\[ \delta \phi_3 (r) = \epsilon ^2 \left[ \frac{\chi \left( -7 \log (r^2 + 1) + 8 \log (r) + 6 \log (r + 1) - 6 \tan^{-1} (r) + 3 \pi \right)}{14 \left( r^2 + 1 \right)} \right] 
\quad + O(\epsilon^4) \]
\[ \delta \phi_4 (r) = \epsilon \left[ \frac{4 \chi^2 \left( -7 \log (r^2 + 1) + 12 \log (r) + 2 \log (r + 1) - 2 \tan^{-1} (r) + \pi \right)}{7 \left( r^2 + 1 \right)} \right] 
\quad + O(\epsilon^3) \]
\[ \delta \phi_5 (r) = \epsilon \left[ - \frac{\chi}{28} \left( -7 \log (r^2 + 1) + 8 \log (r) + 6 \log (r + 1) - 6 \tan^{-1} (r) + 3 \pi \right) \right] 
\quad + O(\epsilon^4) \]
\[ \delta \phi_6 (r) = \epsilon \left[ - \frac{2}{7} \chi^2 \left( -7 \log (r^2 + 1) + 12 \log (r) + 2 \log (r + 1) - 2 \tan^{-1} (r) + \pi \right) \right] 
\quad + O(\epsilon^3) \]

Results for the metric
\[ \delta f_1 (r) = \epsilon \left[ - \frac{2 \left( 80 \chi^2 - 3 \right)}{7r^4} \right] + O(\epsilon^3) \]
\[ \delta f_2 (r) = \epsilon \left[ \frac{16 \chi}{21r^4} \right] + O(\epsilon^3) \]
\[ \delta f_3 (r) = - \epsilon \left[ \frac{12 \chi}{7r^4} \right] + O(\epsilon^3) \]
\[ \delta f_4 (r) = \left[ - \frac{320 \chi^2}{21r^4} - \frac{\left( r^4 + 1 \right) \left( -5 \log (r^2 + 1) + 8 \log (r) + 2 \log (r + 1) + 4 \tan^{-1} (r) \right)}{9r^2} \right. 
\quad + \left. \frac{2 \left( r^4 + 1 \right) \left( r^2 \left( r^2 - r + \pi - 3 \right) - 2 \right)}{9 \left( r^6 + r^4 \right)} \right] + O(\epsilon^2) \] (7.14)
\[ \delta f_5 (r) = \epsilon \left[ \frac{6 \chi}{7r^4} \right] + O(\epsilon^3) \]
\[ \delta f_6 (r) = - \frac{1}{2} \left[ - \frac{320 \chi^2}{21r^4} - \frac{\left( r^4 + 1 \right) \left( -5 \log (r^2 + 1) + 8 \log (r) + 2 \log (r + 1) + 4 \tan^{-1} (r) \right)}{9r^2} \right. 
\quad + \left. \frac{2 \left( r^4 + 1 \right) \left( r^2 \left( r^2 - r + \pi - 3 \right) - 2 \right)}{9 \left( r^6 + r^4 \right)} \right] + O(\epsilon^2) \]
\[\delta g_1(r) = O(\varepsilon^3)\]
\[\delta g_2(r) = O(\varepsilon^3)\]
\[\delta g_3(r) = O(\varepsilon^3)\]
\[\delta g_4(r) = \left[ \frac{1}{18} (-5 \log (r^2 + 1) + 8 \log(r) + 2 \log(r + 1) + 4 \tan^{-1}(r) - 2\pi) \right.\]
\[+ \frac{r^6 + 4r^5 + 4r^4 + 6r^3 + r^2 - 2r - 2}{9(r + 1)(r^3 + r)^2} \left. \right] + O(\varepsilon^2) \quad (7.15)\]
\[\delta g_5(r) = O(\varepsilon^3)\]
\[\delta g_6(r) = -\frac{1}{2} \left[ \frac{1}{18} (-5 \log (r^2 + 1) + 8 \log(r) + 2 \log(r + 1) + 4 \tan^{-1}(r) - 2\pi) \right.\]
\[+ \frac{r^6 + 4r^5 + 4r^4 + 6r^3 + r^2 - 2r - 2}{9(r + 1)(r^3 + r)^2} \left. \right] + O(\varepsilon^2)\]
\[\delta K_1(r) = O(\varepsilon^3)\]
\[\delta K_2(r) = O(\varepsilon^3)\]
\[\delta K_3(r) = O(\varepsilon^4)\]
\[\delta K_4(r) = \left[ \frac{r^2}{3} (-5 \log (r^2 + 1) + 8 \log(r) + 2 \log(r + 1) + 4 \tan^{-1}(r)) \right.\]
\[+ \frac{4 - 2r^2 (\pi r^2 - r + \pi - 3)}{3(r^2 + 1)} \left. \right] + O(\varepsilon^2) \quad (7.16)\]
\[\delta K_5(r) = O(\varepsilon^3)\]
\[\delta K_6(r) = -\frac{1}{2} \left[ \frac{r^2}{3} (-5 \log (r^2 + 1) + 8 \log(r) + 2 \log(r + 1) + 4 \tan^{-1}(r)) \right.\]
\[+ \frac{4 - 2r^2 (\pi r^2 - r + \pi - 3)}{3(r^2 + 1)} \left. \right] + O(\varepsilon^2)\]
\[ \delta J_1(r) = -\epsilon^2 \left[ \frac{(r^4 - 1)}{4r^2} \left( \log (r^2 + 1) - 2 \log(r + 1) - 2 \tan^{-1}(r) + \pi \right) 
+ \frac{\chi}{6r^2} (2 - 3r) \right] + \mathcal{O}(\epsilon^4) \]

\[ \delta J_2(r) = -\epsilon^2 \left[ \frac{(r^4 - 1)}{96r^2} \left( - \log (r^2 + 1) + 2 \log(r + 1) - 16 \tan^{-1}(r) + 8\pi \right) 
+ \frac{27r^4 - 3r^3 + 20r^2 - 3r - 19}{144(r^2 + 1)} \right] + \mathcal{O}(\epsilon^4) \] (7.17)

\[ \delta J_3(r) = -\epsilon^2 \left[ \frac{r}{6(r^2 + 1)^2} \right] + \mathcal{O}(\epsilon^3) \]
\[ \delta J_4(r) = \mathcal{O}(\epsilon^3) \]
\[ \delta J_5(r) = \mathcal{O}(\epsilon^3) \]
\[ \delta J_6(r) = \mathcal{O}(\epsilon^3) \]

\[ Y_1(r) = -\epsilon^2 \left[ \frac{(r^4 - 1)}{96r^2} \left( - \log (r^2 + 1) + 2 \log(r + 1) - 16 \tan^{-1}(r) + 8\pi \right) 
+ \frac{27r^4 - 3r^3 + 20r^2 - 3r - 19}{144(r^2 + 1)} \right] + \mathcal{O}(\epsilon^4) \]

\[ Y_2(r) = -\epsilon^2 \left[ \frac{(r^4 - 1)}{4r^2} \left( \log (r^2 + 1) - 2 \log(r + 1) - 2 \tan^{-1}(r) + \pi \right) 
+ \frac{\chi}{6r} (2 - 3r) \right] + \mathcal{O}(\epsilon^4) \] (7.18)

\[ Y_3(r) = \mathcal{O}(\epsilon^4) \]
\[ Y_4(r) = \epsilon^2 \left[ \frac{r}{6(r^2 + 1)^2} \right] + \mathcal{O}(\epsilon^4) \]
\[ Y_5(r) = \mathcal{O}(\epsilon^3) \]

\[ W_1(r) = \epsilon^3 \left[ \frac{-3\pi (r^3 + r) + 6 (r^3 + r) \tan^{-1}(r) + 6r^2 + 4}{16(r^3 + r)} \right] + \mathcal{O}(\epsilon^5) \]

\[ W_2(r) = \epsilon^3 \left[ \frac{-3\pi (r^3 + r) + 6 (r^3 + r) \tan^{-1}(r) + 6r^2 + 4}{32(r^3 + r)} \right] + \mathcal{O}(\epsilon^5) \]

\[ W_3(r) = \epsilon^3 \left[ \frac{\chi [-3\pi (r^3 + r) + 6 (r^3 + r) \tan^{-1}(r) + 6r^2 + 4]}{32(r^3 + r)} \right] + \mathcal{O}(\epsilon^4) \] (7.19)

\[ W_4(r) = \epsilon^3 \left[ \frac{\chi [-3\pi (r^3 + r) + 6 (r^3 + r) \tan^{-1}(r) + 6r^2 + 4]}{32(r^3 + r)} \right] + \mathcal{O}(\epsilon^4) \]
\[ W_5(r) = \frac{-r^2}{6} \left[ - \frac{2(r + 1)}{r^2 + 1} - \frac{4}{r^2} + 5 \log (r^2 + 1) - 8 \log(r) - 2 \log(r + 1) 
- 4 \tan^{-1}(r) + 2\pi \right] + \mathcal{O}(\epsilon^2) \]
\[ Z_1(r) = \frac{r^2}{3} \left[ \frac{2(r + 1)}{r^2 + 1} + \frac{4}{r^2} - 5 \log(r^2 + 1) + 8 \log(r) \right] + 2 \log(r + 1) + 4 \tan^{-1}(r) - 2\pi + O(\epsilon^2) \]

\[ Z_2(r) = \epsilon^3 \chi \left[ \frac{-3\pi (r^3 + r) + 6(r^3 + r) \tan^{-1}(r) + 6r^2 + 4}{16(r^3 + r)} \right] + O(\epsilon^4) \]  

(7.20)

7.5 The \( \zeta \to 0 \) limit

The gravitational solutions presented above are complicated largely because they possess very little rotational symmetry. At any given spacetime point we have a normal fluid velocity and an independent superfluid velocity. These two velocities together break the local Lorentz group at a point down to the abelian group \( SO(2) \). While we have usefully organized the results of our gravitational calculation in representations of \( SO(2) \), as representations of \( SO(2) \) are all one dimensional, our solutions admit several different functions of \( r \).

In the special case that \( \zeta = 0 \), however, the residual symmetry group about a point is \( SO(3) \). \( SO(3) \) representation theory is considerably more constraining than \( SO(2) \) representation theory. This implies that the gravitational dual to superfluid dynamics should be considerably simpler in the special limit \( \zeta \to 0 \) than in the generic case.

Let us first present a brief ab initio analysis of the nature of the gravitational solution when \( \zeta = 0 \). All independent first derivative data may be organized into \( SO(3) \) scalars, vectors and tensors. These may be chosen as follows

**Scalar**

\[ \partial_\mu u^\mu \text{ and } P^{\mu\nu} \partial_\mu \zeta_\nu \]

**Vector**

\[ u^\mu \partial_\mu u^\nu, \ P^{\mu\nu} \partial_\nu \epsilon \text{ and } \epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda \zeta_\sigma \]

**Tensor**

\[ \sigma_{\mu\nu} \text{ and } \sigma^{(Q)}_{\mu\nu} = P_\mu^\alpha P_\nu^\beta \left( \frac{\partial_\alpha \zeta_\beta + \partial_\beta \zeta_\alpha}{2} - \frac{P_{\theta_1 \theta_2} \partial_{\theta_1} \zeta_{\theta_2}}{3} \right) \eta_{\alpha\beta} \]

Note of course that an \( SO(3) \) vector or an \( SO(3) \) may be decomposed into an \( SO(2) \) vector and a scalar, while an \( SO(3) \) tensor is composed of an \( SO(2) \) tensor, vector and scalar. In \( SO(2) \) terms, therefore, the data listed above totals to 7 scalars, 5 vectors and two tensor.

It follows from symmetry considerations (and the fact that our parity conserving gravitational system will never generate a parity violating vector term, so we can ignore the third vector above)
that it must be possible to write the metric and gauge field, in the $\zeta \to 0$ limit, in the form

$$ds^2 = -2g \left( \frac{r}{r_c} \right) u^\mu dx^\mu dr + \left[ - r_c^2 f \left( \frac{r}{r_c} \right) u_\mu u_\nu + r^2 P_{\mu\nu} \right] dx^\mu dx^\nu + r_c^2 F \left( \frac{r}{r_c} \right) \sigma_{\mu\nu} dx^\mu dx^\nu$$

$$+ \frac{1}{e^2} \left[ - 2 \left( \frac{r}{r_c} \right) \left( \partial_\mu u^\mu \right) + G_1 \left( \frac{r}{r_c} \right) \left( P_{\mu\nu} \partial_\mu \zeta_\nu \right) \right] u_\mu dx^\mu dr$$

$$+ \frac{1}{e^2} \left[ - 2 \left( \frac{r}{r_c} \right) \left( \partial_\mu u^\mu \right) + G_2 \left( \frac{r}{r_c} \right) \left( P_{\mu\nu} \partial_\mu \zeta_\nu \right) \right] u_\mu dx^\mu dr$$

$$+ \frac{1}{e^2} \left[ \partial_\mu u^\mu \right] + \frac{1}{e^2} \left[ \partial_\mu u^\mu \right]$$

(7.21)

$$A = \frac{1}{r_c} \left\{ H \left( \frac{r}{r_c} \right) u^\mu \partial_\mu + \frac{1}{r_c} \left\{ A_1 \left( \frac{r}{r_c} \right) \left( \partial_\mu u^\mu \right) + A_2 \left( \frac{r}{r_c} \right) \left( P_{\mu\nu} \partial_\mu \zeta_\nu \right) \right\} \partial_r$$

$$+ \frac{1}{r_c} \left\{ \partial_\mu u^\mu \right\} + \frac{1}{r_c} \left\{ \partial_\mu u^\mu \right\}$$

$$+ \frac{1}{r_c} \left\{ \partial_\mu u^\mu \right\} + \frac{1}{r_c} \left\{ \partial_\mu u^\mu \right\}$$

The results of the previous subsection must obey several relations in the limit $\zeta \to 0$ for them to agree with the form presented in (7.21). In Appendix D we have explicitly verified that each required relation is indeed obeyed. The results presented in the previous subsection are consistent with the form (7.21) once we make the identifications.

---

\[20\text{A direct comparison between these two forms is complicated by an irritating feature; the coordinate choice of the previous subsection differs from the one above (it breaks manifest SO(3) invariance) even in the limit } \zeta \to 0. \text{ In Appendix D we have explicitly performed the coordinate change that allows one to transform the results between coordinates.}\]
\[ V_1(r) = e^2 \left[ \frac{r}{3(r^2 + 1)^2} \right] + O(e^4) \]
\[ V_2(r) = O(e^4) \]
\[ T_1(r) = -\frac{r^2}{3} \left[ -\frac{2(r + 1)}{r^2 + 1} - \frac{4}{r^2} + 5 \log (r^2 + 1) - 8 \log(r) - 2 \log(r + 1) \right. \]
\[ \left. - 4 \tan^{-1}(r) + 2\pi \right] + O(e^2) \tag{7.22} \]
\[ T_2(r) = O(e^3) \]
\[ G_1(r) = 0, \quad \text{(Required by Weyl invariance)} \]
\[ F_1(r) = O(e^2) \]
\[ G_2(r) = O(e^3) \]
\[ F_2(r) = \frac{6e}{7r^4} + O(e^3) \]

and

\[ A_1(r) = -\frac{2r}{3(r^2 + 1)} + O(e^2) \]
\[ A_2(r) = \frac{e}{4r^3} + O(e^3) \]
\[ H_1(r) = O(e^2) \]
\[ H_2(r) = e \left[ -\frac{5r(r + 2)}{14r(r + 1)(r^2 + 1)} \right] + O(e^3) \tag{7.23} \]
\[ L_1(r) = O(e^4) \]
\[ L_2(r) = e^2 \left[ \frac{\log (r^2 + 1) - 2 \log(r + 1) + 2 \tan^{-1}(r) - \pi}{96r^2} \right] + O(e^4) \]

7.6 Stress Tensor, Charge current and the Josephson Equation

The results of the previous subsection may be used to read off the values of the boundary stress tensor, the boundary current and the correction to the Josephson equation at first order in the derivative expansion. Like all the calculations in this paper our results are obtained in a power series expansion in \( \epsilon \) and \( \frac{1}{r^2} \).

We parameterize our boundary stress tensor and current as

\[ T^{\mu \nu} = \frac{1}{16\pi G} \left[ A u^\mu u^\nu + B n^\mu n^\nu + C (n^\mu u^\nu + u^\mu n^\nu) + \left( \frac{A - B}{2} \right) \tilde{P}^{\mu \nu} \right] + \tilde{\Pi}^{\mu \nu} \]
\[ J^{\mu} = \frac{1}{16\pi G} [Q_1 u^\mu + Q_2 n^\mu] + \tilde{J}^{\mu}_{\text{diss}} \tag{7.24} \]

where \( A, B, C, Q_1 \) and \( Q_2 \) are functions of \( \epsilon(x), \zeta(x) \) and \( r_c(x) \) as given in equations (6.13) and (5.20). We further expand the corrections to the perfect fluid stress tensor and current as
\[ 16\pi G \tilde{\pi}_{\mu\nu} = -2r_c^3 \sigma_{\mu\nu} + \frac{1}{e^2} \left[ r_c^3 \sum_{i=1}^{7} S_i P_i \left( n_{\mu} n_{\nu} - \frac{1}{2} \tilde{\rho}_{\mu\nu} \right) \right] + \frac{r_c^3}{e^2} \sum_{i=1}^{5} v_i \left( n_{\mu} [V_i]_{\nu} + n_{\nu} [V_i]_{\mu} \right) + \frac{r_c^3}{e^2} \sum_{i=1}^{2} t_i \left[ T_i \right]_{\mu\nu} + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ 16\pi G \tilde{j}^\mu_{\text{diss}} = \frac{r_c^2}{e^2} \sum_{i=1}^{7} S_i \left( a_i u^\mu + b_i n^\mu \right) + \frac{r_c^2}{e^2} \sum_{i=1}^{5} c_i \left[ V_i \right]^{\mu} + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ \mu_{\text{diss}} = \sum_{i=1}^{7} \delta \mu_i S_i + \mathcal{O} \left( \frac{1}{e^2} \right) \]

Our results are given as follows.

**Results for stress tensor:**

\[ P_1 = \mathcal{O}(\epsilon^4), \quad P_2 = \mathcal{O}(\epsilon^4), \quad P_3 = \mathcal{O}(\epsilon^4), \quad P_4 = \mathcal{O}(\epsilon^4), \quad P_5 = \mathcal{O}(\epsilon^3), \quad P_6 = \mathcal{O}(\epsilon^3) \]

\[ v_1 = \mathcal{O}(\epsilon^5), \quad v_2 = \mathcal{O}(\epsilon^5), \quad v_3 = \mathcal{O}(\epsilon^5), \quad v_4 = \mathcal{O}(\epsilon^5), \quad v_5 = \mathcal{O}(\epsilon^4) \]

\[ t_1 = \mathcal{O}(\epsilon^4), \quad t_2 = \mathcal{O}(\epsilon^4) \]

**Results for current:**

\[ a_1 = \epsilon \left[ \frac{3}{7} (3 - 80 \chi^2) \right] + \mathcal{O}(\epsilon^3), \quad b_1 = \epsilon^2 \chi + \mathcal{O}(\epsilon^4) \]

\[ a_2 = \epsilon \left( \frac{8 \chi}{7} \right) + \mathcal{O}(\epsilon^3), \quad b_2 = -\epsilon^2 \left[ -\frac{1}{24} \right] + \mathcal{O}(\epsilon^4) \]

\[ a_3 = -\epsilon \left( \frac{18 \chi}{7} \right) + \mathcal{O}(\epsilon^3), \quad b_3 = \mathcal{O}(\epsilon^4) \]

\[ a_4 = -\left( \frac{160 \chi^2}{7} \right) + \mathcal{O}(\epsilon^3), \quad b_4 = \mathcal{O}(\epsilon^4) \]

\[ a_5 = \epsilon \left( \frac{9 \chi}{7} \right) + \mathcal{O}(\epsilon^3), \quad b_5 = \mathcal{O}(\epsilon^4) \]

\[ a_6 = \left( \frac{80 \chi^2}{7} \right) + \mathcal{O}(\epsilon^2), \quad b_6 = \mathcal{O}(\epsilon^4) \]

\[ c_1 = \epsilon^2 \frac{1}{24} + \mathcal{O}(\epsilon^4) \]

\[ c_2 = \epsilon^2 \chi + \mathcal{O}(\epsilon^4) \]

\[ c_3 = \mathcal{O}(\epsilon^4) \]

\[ c_4 = \mathcal{O}(\epsilon^4) \]

\[ c_5 = \epsilon^3 \chi \left( \frac{-1 + 2 \log(2)}{16} \right) + \mathcal{O}(\epsilon^4) \]
Results for the correction to the Josephson equation:

\[
\begin{align*}
\delta\mu_1 &= \epsilon \left[ \frac{1}{14} \left( 5 - 96 \chi^2 \right) \right] + \mathcal{O}(\epsilon^3) \\
\delta\mu_2 &= \epsilon \left( \frac{3 \chi}{7} \right) + \mathcal{O}(\epsilon^3) \\
\delta\mu_3 &= -\epsilon \left( \frac{5 \chi}{7} \right) + \mathcal{O}(\epsilon^3) \\
\delta\mu_4 &= -\left( \frac{32 \chi^2}{7} \right) + \mathcal{O}(\epsilon^2) \\
\delta\mu_5 &= \epsilon \left( \frac{5 \chi}{14} \right) + \mathcal{O}(\epsilon^3) \\
\delta\mu_6 &= \left( \frac{16 \chi^2}{7} \right) + \mathcal{O}(\epsilon^2)
\end{align*}
\]

(7.28)

In \( \zeta \to 0 \) limit this derivative corrections to stress tensor, charge current and the phase equation take the following form

\[
\lim_{\zeta \to 0} \tilde{\pi}^{\mu\nu} = -2r^3 c \sigma^{\mu\nu} + \mathcal{O}(\epsilon^3)
\]

\[
\lim_{\zeta \to 0} \tilde{J}^{\mu}_{\text{diss}} = r^2_c \left\{ \alpha_1 u^{\mu} \left[ P^{ab} \partial_a \zeta_b \right] + \alpha_2 P^{\mu\nu} \partial_{\nu} \epsilon \right\}
\]

\[
\lim_{\zeta \to 0} \tilde{\mu}^{\text{diss}} = \alpha_3 \left[ P^{ab} \partial_a \zeta_b \right]
\]

(7.29)

where

\[
\alpha_1 = \frac{9}{7} + \mathcal{O}(\epsilon^3), \quad \alpha_2 = \frac{\epsilon}{24} + \mathcal{O}(\epsilon^4), \quad \alpha_3 = \frac{5}{14} + \mathcal{O}(\epsilon^3)
\]

7.7 Weyl Covariance of our bulk fields and boundary currents

In this section we will demonstrate that our fluid dynamical solutions must, on general grounds, obey certain constraints that follow from the requirement of Weyl invariance. We then verify that our explicit solution does indeed obey these constraints, providing a nontrivial check on these solutions.

All computations reported in this paper have been performed for superfluid motion on a flat boundary metric. However our final results must be the restriction to a flat boundary of results that apply in a general weakly curved space. The (boundary) generally covariant version of our final bulk metric, stress tensor etc are all given simply by promoting all derivatives to covariant derivatives (ambiguities in this procedure and boundary curvature terms all show up only at second order in the derivative expansion).

Given these results in a general boundary spacetime, it follows on general grounds (see [40]) that our bulk metric, gauge field and scalar fields must enjoy invariance under the following spacetime dependent Weyl transformations and coordinate redefinitions.

\[
\tilde{\epsilon} = \epsilon, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} e^{-2\psi(v,x_i)}, \quad \tilde{u}^{\mu} = u^{\mu} e^{-\psi(v,x_i)}, \quad \tilde{n}_\mu = n_\mu e^{-\psi(v,x_i)}, \quad \tilde{\zeta} = \zeta, \quad \tilde{\epsilon} = \epsilon
\]

Note that the Weyl transformed metric \( \tilde{g}_{\mu\nu} \) is, in general, not flat even if the original metric is. Let us work in the special case that the original metric \( g_{\mu\nu} \) is taken to be flat. The boundary connection with respect to \( \tilde{g}_{\mu\nu} \) is non zero and is given by

\[
\tilde{\Gamma}^{\sigma}_{\mu\nu} = - \left( \delta^{\sigma}_{\mu} \partial_\nu \psi + \delta^{\sigma}_{\nu} \partial_\mu \psi - \eta_{\mu\nu} \partial^{\sigma} \psi \right)
\]
The new frame covariant derivatives of $u_\mu$ and $n_\mu$ are given by

$$\tilde{\nabla}_\mu \tilde{u}_\nu = e^{-\psi} [\partial_\mu u_\nu + u_\mu \partial_\nu \psi - \eta_{\mu\nu} (u.\partial) \psi]$$

$$\tilde{\nabla}_\mu \tilde{n}_\nu = e^{-\psi} [\partial_\mu n_\nu + n_\mu \partial_\nu \psi - \eta_{\mu\nu} (n.\partial) \psi]$$

Using these expressions one can deduce the transformation properties of the scalar, vector and the tensor forms appearing in the bulk solution

$$\tilde{S}_1 = e^\psi S_1, \quad \tilde{S}_2 = e^\psi S_2$$

$$\tilde{S}_3 = e^\psi [S_3 - (n.\partial) \psi], \quad \tilde{S}_4 = e^\psi [S_4 - (u.\partial) \psi],$$

$$\tilde{S}_5 = e^\psi [S_5 - 2(n.\partial) \psi], \quad \tilde{S}_6 = e^\psi [S_6 - 2(u.\partial) \psi],$$

$$\tilde{V}_1 = [V_1]_\mu, \quad \tilde{V}_2 = [V_2]_\mu,$$

$$\tilde{V}_3 = [V_3]_\mu + \tilde{P}^\mu_\nu \partial_\nu \psi, \quad \tilde{V}_4 = [V_4]_\mu - \tilde{P}^\mu_\nu \partial_\nu \psi, \quad \tilde{V}_5 = [V_5]_\mu$$

$$\tilde{T}_1 = e^{-\psi} [T_1]_{\mu\nu}, \quad \tilde{T}_2 = e^{-\psi} [T_2]_{\mu\nu}$$

If we transform the gauge field and the metric from the new Weyl frame (the frame with tilde variables) to the old Weyl frame (the frame where the variables are denoted without tilde), the equilibrium solution itself generates some new terms due to the $r$ coordinate redefinition. In the new frame the coordinates are $\tilde{r} = re^{\psi(v,x)}$ and $\tilde{x}^\mu = x^\mu$. This implies the following transformation rule for the differentials.

$$d\tilde{r} = e^{\psi(v,x)} (dr + rd\mu \partial_\mu \psi)$$

$$d\tilde{x}^\mu = dx^\mu$$

$$\tilde{\partial}_\mu = \frac{\partial r}{\partial \tilde{x}_\mu} \partial_\mu + \partial_\mu$$

This induces the following transformations on gauge field

$$\tilde{A} = \frac{1}{r_c} \left[H \left(\frac{\tilde{r}}{r_c}\right) (\tilde{u}.\tilde{\partial}) + L \left(\frac{\tilde{r}}{r_c}\right) (\tilde{n}.\tilde{\partial})\right]$$

$$= -\frac{e^{\psi}}{r_c} \left[H \left(\frac{r}{r_c}\right) (u.\partial \psi) + L \left(\frac{r}{r_c}\right) (n.\partial \psi)\right] \partial_\tau$$

$$+ \frac{1}{r_c} \left[H \left(\frac{r}{r_c}\right) (u.\partial) + L \left(\frac{r}{r_c}\right) (n.\partial)\right]$$

$$= -re^{\psi} \left[H (r) (u.\partial \psi) + L (r) (n.\partial \psi)\right] \partial_\tau + r \left[H (r) (u.\partial) + L (r) (n.\partial)\right]$$

In the last line we have used the scaling symmetry to set $r_c = 1$. 

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Similarly the equilibrium metric also transforms and the nontrivial transformation is generated due to the term \( dx^\mu dr \):

\[
-2g\left(\frac{r}{r_c}\right) \tilde{u}_\mu dx^\mu d\tilde{r} = -2g\left(\frac{r}{r_c}\right) u_\mu dx^\mu (dr + r\partial_\nu \psi dx^\nu) \\
= -2g\left(\frac{r}{r_c}\right) u_\mu dx^\mu dr - 2rg\left(\frac{r}{r_c}\right) u_\mu u_\nu (u.\partial)\psi dx^\mu dx^\nu \\
- rg\left(\frac{r}{r_c}\right) \left[ (u_\mu n_\nu + u_\nu n_\mu)(n.\partial)\psi + \left( u_\mu \tilde{P}_\nu^\sigma + u_\nu \tilde{P}_\mu^\sigma \right) \partial_\sigma \psi \right] dx^\mu dx^\nu
\]

(7.33)

Here also in the last step the scaling symmetry is used to set \( r_c = 1 \).

Combining these transformations we find the transformed metric, gauge field and scalar have the expected form (expected according to (7.7) ) together with some additional pieces that multiply a single derivative of \( \psi \). The coefficients of these unwanted pieces themselves have no derivatives, and must vanish in order that our result respect Weyl invariance. This requirement imposes the following simple algebraic conditions on the fields in the metric, scalar and gauge field:

\[
\delta A_3(r) + 2 \delta A_5(r) - rL(r) = 0 \\
\delta A_4(r) + 2 \delta A_6(r) - rH(r) = 0 \\
\delta H_3(r) + 2 \delta H_5(r) = 0, \quad \delta H_4(r) + 2 \delta H_6(r) = 0 \\
\delta L_3(r) + 2 \delta L_5(r) = 0, \quad \delta L_4(r) + 2 \delta L_6(r) = 0 \\
\delta \phi_3(r) + 2 \delta \phi_5(r) = 0, \quad \delta \phi_4(r) + 2 \delta \phi_6(r) = 0 \\
X_3(r) - X_4(r) = 0
\]

(7.34)

\[
\delta f_3(r) + 2 \delta f_5(r) = 0 \\
\delta f_4(r) + 2 \delta f_6(r) + 2rg(r) = 0 \\
\delta J_3(r) + 2 \delta J_5(r) + rg(r) = 0, \quad \delta J_4(r) + 2 \delta J_6(r) = 0 \\
\delta K_3(r) + 2 \delta K_5(r) = 0, \quad \delta K_4(r) + 2 \delta K_6(r) = 0 \\
Y_3(r) - Y_4(r) - rg(r) = 0, \quad W_3(r) - W_4(r) = 0
\]

(7.35)

We also require that the stress tensor, charge current and Josephson equation in our model are invariant under Weyl transformations. As these boundary quantities are all independent of \( r \), the redefinition of \( r \) is irrelevant to the study of Weyl transformations of these quantities. Using only the
equations (7.30) we find the following constraints on the coefficients in (7.25)

\[ P_3 + 2 P_5 = P_4 + 2 P_6 = 0 \]
\[ v_3 - v_4 = 0 \]
\[ a_3 + 2 a_5 = a_4 + 2 a_6 = 0 \quad (7.36) \]
\[ b_3 + 2 b_5 = b_4 + 2 b_6 = 0 \]
\[ c_3 - c_4 = 0 \]
\[ \delta \mu_3 + 2 \delta \mu_5 = \delta \mu_4 + 2 \delta \mu_6 = 0 \]

The equations (7.35), (7.34) and (7.36) must apply to any consistent asymptotically AdS solution of gravitational equations. In particular these equations must apply to the results of this paper, and constitute a nontrivial consistency check on our algebra. We have explicitly checked that the results of §7.4 and §7.6 obey these constraints, to the calculated order in \( \epsilon \) and \( \frac{1}{e^2} \).

### 7.8 Entropy Current from Gravity

Fluid flows obtained from the fluid gravity correspondence are automatically equipped with families of local entropy currents of positive divergence. A particularly natural choice for this entropy current was presented in equation 3.11 of [39]. Using this formula for our solution we have computed the entropy current dual to our fluid flow. This entropy current has a piece at \( \mathcal{O}(1) \) and a piece at \( \mathcal{O}(1/e^2) \), and takes the form

\[
4GJ^\mu_s = r_c^3 u^\mu + \frac{r_c^2}{e^2} \sum_{i=1}^{7} S_i \left( \kappa^{(u)}_i u_{\mu} + \kappa^{(n)}_i n_{\mu} \right) + \frac{r_c^2}{e^2} \sum_{i=1}^{5} \kappa^{(v)}_i [V_i]_{\mu} + \mathcal{O} \left( \frac{1}{e^2} \right) \quad (7.37)
\]

where

\[
k^{(u)}_1 = \frac{1}{2} \left[ \frac{3}{7} \left( 3 - 80 \chi^2 \right) \right] + \mathcal{O}(\epsilon^3), \quad k^{(n)}_1 = \frac{1}{2} \epsilon^2 \chi + \mathcal{O}(\epsilon^3)
\]
\[
k^{(u)}_2 = \frac{1}{2} \left( \frac{8 \chi}{7} \right) + \mathcal{O}(\epsilon^3), \quad k^{(n)}_2 = -\frac{1}{2} \epsilon^2 \left[ -\frac{1}{24} \right] + \mathcal{O}(\epsilon^3)
\]
\[
k^{(u)}_3 = -\frac{1}{2} \epsilon^2 \left[ -\frac{18 \chi}{7} \right] + \mathcal{O}(\epsilon^3), \quad k^{(n)}_3 = \mathcal{O}(\epsilon^3)
\]
\[
k^{(u)}_4 = -\frac{1}{2} \epsilon^2 \left[ -\frac{160 \chi^2}{7} \right] + \mathcal{O}(\epsilon^3), \quad k^{(n)}_4 = \mathcal{O}(\epsilon^3)
\]
\[
k^{(u)}_5 = \frac{1}{2} \epsilon^2 \left[ \frac{9 \chi}{7} \right] + \mathcal{O}(\epsilon^3), \quad k^{(n)}_5 = \mathcal{O}(\epsilon^3)
\]
\[
k^{(u)}_6 = \frac{1}{2} \epsilon^2 \left[ \frac{80 \chi^2}{7} \right] + \mathcal{O}(\epsilon^3), \quad k^{(n)}_6 = \mathcal{O}(\epsilon^3)
\]

\[
k^{(v)}_1 = \frac{e^2}{24} + \mathcal{O}(\epsilon^3)
\]
\[
k^{(v)}_2 = \frac{1}{2} \epsilon^2 \chi + \mathcal{O}(\epsilon^3)
\]
\[
k^{(v)}_3 = \mathcal{O}(\epsilon^3)
\]
\[
k^{(v)}_4 = \mathcal{O}(\epsilon^3)
\]
\[
k^{(v)}_5 = \mathcal{O}(\epsilon^3)
\]

(7.39)
Quite remarkably this gravitational entropy current agrees exactly with the simple fluid dynamical current described in (2.21) (to the order at which we have done the calculation).

Actually, all but the first two terms on the RHS of (2.21) are $O(\epsilon^3)$ or higher. It follows that to $O(\epsilon^2)$

$$J^\mu_S = su^\mu - \frac{\mu}{T} J^\mu_{diss} + O(\epsilon^3)$$

(7.40)

We have checked that the gravitational entropy current, presented in this subsection, exactly agrees with this form to this order.

8. Transformation to standard frames and identification of the dissipative parameters

The equations of gravitational super fluid dynamics, derived in the previous section are presented in a modified phase frame that is adapted to the expectation value of the operator $\epsilon(x)$, and is not particularly natural from a fluid dynamical point of view.

In this section we will transform our results to the $\mu_{diss} = 0$ fluid frame and the transverse fluid frames. We will then compare these results with the general ‘theory’ of dissipative dynamics presented in §3. We will find perfect agreement with the general structures predicted in §3, and so be able to read off the values of all 10 nonzero dissipative fluid parameters.

To begin this subsection we first recall the structure of the boundary equations that emerged from our gravitational computations of the previous section. In the previous section our gravitational solutions were parameterized by the eight fields $u(\mathbf{x}), \epsilon(\mathbf{x}), r_c(\mathbf{x})$ and $\zeta(\mathbf{x})$. In this section we will find it convenient to use the first of (6.22) to define a new field $\xi_0$ and to eliminate $\epsilon(\mathbf{x})$ in favour of the new field $\mu_0(x)$. We will always suppose that this has been done in what follows.

Our gravitational results for the stress tensor, current and superfluid phase are of the form

$$T^{\mu\nu} = \rho_n(r_c, \mu_0, \xi_0) \ u^\mu u^\nu + P(r_c, \mu_0, \xi_0) \eta^{\mu\nu} + f(\xi_0, \mu_0, r_c) \xi^\mu_0 \xi^\nu_0 + \tilde{\pi}^{\mu\nu}$$

$$J^\mu = q_n(r_c, \xi_0, \mu_0) u^\mu - f(r_c, \xi_0, \mu_0) \xi^\mu_0 + J^\mu_{diss}$$

$$\xi^\mu = -(\mu_0 + \mu_{diss}) u^\mu + r_c \zeta^\mu$$

(8.1)

where the functions $\rho_n, P$ etc are the thermodynamical functions derived in §6. Our results are presented in a modified phase frame. We remind the reader that this means that we have presented our answers in terms of a new auxiliary phase field $\xi^\mu_0$ and its modulus $\xi_0$ defined as

$$\xi^\mu_0 = -\mu_0 u^\mu + r_c \zeta^\mu$$

$$\xi_0 = \sqrt{\mu_0^2 - r_c^2 \zeta^2}$$

(8.2)

Note that $\xi^\mu_0$ is not equal to the phase field $\xi^\mu$ (because $\mu_0 \neq u\xi$). Instead the relationship between these two fields is given by

$$\xi^\mu = \xi^\mu_0 - \mu_{diss} u^\mu$$

$$\xi = \xi_0 - \mu_{diss} \frac{\mu_0}{\xi_0}$$

(8.3)

In this section we wish to transform our gravitational results into the $\mu_{diss} = 0$ fluid frame and the transverse fluid frame. As we have explained above, (8.1) is in a modified phase frame not a fluid
frame. This means that (8.1) differs from the form (2.6) in two ways. First thermodynamical functions, like the pressure, in (8.1), are functions of \((\mu_0, r_c, \xi_0)\). According to the form specified in (2.6) the choice and definition of two of the three thermodynamical variables—e.g. the chemical potential and temperature—is up to the user, and can reasonably chosen in a fluid frame to be \(\mu_c\) and \(r_c\). If we work in a fluid frame, however, the third variable has to be \(\xi\), the magnitude of the phase field. As we have seen from (8.3) \(\xi_0 \neq \xi\).

The second way in which (8.1) differs from the standard fluid frame form (2.6) is that the fourth term in the expression for \(T^\mu{}_{\nu}\) in (8.1) is proportional to \(\xi^\mu_0 \xi^\nu_0\). Similarly the second term in the expression for \(J^\mu_{\text{diss}}\) in (8.1) is proportional to \(\xi^\mu_0\). The corresponding terms in (2.6), however, are proportional to \(\xi^\mu \xi^\nu\) and \(\xi^\mu\) respectively.

These discrepancies are easily cured. In order to move to a fluid frame, all one needs to do is substitute the expressions for \(\xi_0\) and \(\xi^\mu_0\) as functions of \(\xi\) and \(\xi^\mu\) (see (8.2)) into (8.1). That is we must perform a prescribed field redefinition on \(\xi^\mu_0\) to take it to the field \(\xi^\mu\). In addition we are free to perform any additional field redefinitions

\[
\begin{align*}
\mu_0 &= \mu + \delta \mu \\
r_c &= \tilde{r}_c + \delta r_c \\
u^\mu &= \tilde{u}^\mu + \delta u^\mu
\end{align*}
\]  
(8.4)

in order to transform to any particular fluid dynamical frame. Of course \(\delta \mu\), \(\delta r_c\) and \(\delta u^\mu\) above are all necessarily of first or higher order in derivatives.

We will now describe how these general ideas can be used in practice to transform our gravitational results into the \(\mu_{\text{diss}} = 0\) frame and the transverse frames respectively.

### 8.1 Transformation to the \(\mu_{\text{diss}} = 0\) frame

In order to transform to the \(\mu_{\text{diss}} = 0\) frame we must take \(\delta \mu = -\mu_{\text{diss}}\) in (8.4). We can immediately work out the effect of this field redefinition, combined with the change of variables from \(\xi^\mu_0\) to \(\xi^\mu\), on the current and stress tensor. We find

\[
J^\mu = q_n(r_c, \xi_0, \mu_0)u^\mu - f(r_c, \xi_0, \mu_0)\xi^\mu_0 + \tilde{J}_{\text{diss}}^\mu
\]

\[
= q_n(r_c, \xi, \mu)u^\mu - f(r_c, \xi, \mu)\xi^\mu + df \cdot \xi^\mu + \tilde{J}_{\text{diss}}^\mu
\]

(8.5)

and

\[
T^{\mu\nu} = \left[\rho_n(r_c, \mu_0, \xi_0) + P(r_c, \mu_0, \xi_0)\right] u^\mu u^\nu + \left[\rho_n(r_c, \mu, \xi) + P(r_c, \mu, \xi)\right] u^\mu u^\nu + \left[\rho_n(r_c, \mu, \xi) + P(r_c, \mu, \xi)\right] u^\mu u^\nu + f(r_c, \xi_0, \mu_0)\xi^\mu \xi^\nu + \tilde{\pi}^{\mu\nu}
\]

\[
= \left[\rho_n(r_c, \mu, \xi) + P(r_c, \mu, \xi)\right] u^\mu u^\nu + \left[\rho_n(r_c, \mu, \xi) + P(r_c, \mu, \xi)\right] u^\mu u^\nu + \left[\rho_n(r_c, \mu, \xi) + P(r_c, \mu, \xi)\right] u^\mu u^\nu + f(r_c, \xi, \mu)\xi^\mu \xi^\nu
\]

\[
+ df \cdot \xi^\mu \xi^\nu + \delta \rho \delta u^\mu + \mu_{\text{diss}} f(r_c, \xi_0, \mu_0)(u^\mu \xi^\nu_0 + u^\nu \xi^\mu_0)
\]

\[
+ df \cdot \xi^\mu \xi^\nu + \delta \rho \delta u^\mu + \mu_{\text{diss}} f(r_c, \xi, \mu_0)(u^\mu \xi^\nu + u^\nu \xi^\mu)
\]

(8.6)

where \(f(r_c, \xi, \mu) = \frac{q_n(r_c, \xi, \mu)}{\xi}\) = \(\frac{\rho_n(r_c, \xi, \mu)}{\xi}\), and the operation ‘\(\delta\)’ acting on any function of \(r_c, \xi\) and \(\mu\) is given by

\[
dA = -\mu_{\text{diss}} \frac{\mu_0}{\xi_0} \left[\frac{\partial A(r_c, \xi, \mu)}{\partial \xi}\right] - \mu_{\text{diss}} \left[\frac{\partial A(r_c, \xi, \mu)}{\partial \mu}\right]
\]
The gravitational stress tensor and charge current have now been recast into the fluid dynamical form with
\[
\pi^{\mu\nu} = \left[ d\rho + \mu_0^2 \frac{d}{d\tau} \right] u^\rho u^\nu + \mu_{\text{diss}} f(r_c, \xi_0, \mu_0)(u^\rho \xi_0^\nu + u^\nu \xi_0^\rho) + d\frac{d}{d\tau} \xi_0^\rho \xi_0^\nu + dP \, \Gamma^\rho_\nu + \tilde{\pi}^{\mu\nu}
\]
\[
J_{\text{diss}}^\mu = (dq_n + \mu_0 \frac{d}{d\tau}) u^\rho + \mu_{\text{diss}} f(r_c, \xi_0, \mu_0) u^\rho - d\frac{d}{d\tau} \xi_0^\rho + J_{\text{diss}}^\mu
\]
\[
\mu_{\text{diss}} = 0
\]

While \( \tilde{\pi}^{\mu\nu} \) was orthogonal to the velocity field \( u^\rho \), the same is not true of \( \pi^{\mu\nu} \) in (8.7). In order to enforce the transversality of \( \pi^{\mu\nu} \) (a defining condition for the \( \mu_{\text{diss}} = 0 \) frame), we must now redefine \( r_c \) and \( u^\rho \). The determination of \( \delta r_c \) and \( \delta u^\rho \) is particularly simple at leading order in \( \frac{1}{c^4} \) (the order to which our gravitational results have been obtained). Recall that, to leading (unit) order in \( \frac{1}{c^4} \), \( \tilde{\pi}^{\mu\nu} \) is already transverse to \( u_\mu \). The piece of \( T^{\mu\nu} \) that is not transverse to \( u_\mu \) starts out at \( \mathcal{O}(\frac{1}{c^4}) \). It follows that the \( \delta r_c \) and \( \delta u^\rho \) must both be of order \( \mathcal{O}(\frac{1}{c^4}) \). It is important to implement this field redefinition only in the \( \mathcal{O}(1) \) part of the equilibrium or perfect fluid part of \( T^{\mu\nu} \) (as we are keeping track of the final answer only \( \mathcal{O}(\frac{1}{c^4}) \) and only to first order in fluid derivatives). This piece and its transformations are given by

\[
T^{\mu\nu}_{\text{equilibrium at } \mathcal{O}(\frac{1}{c^4})} = r_c^4(4u^\rho u^\nu + \eta^{\mu\nu})
\]
\[
= r_c^4(4u^\rho u^\nu + \eta^{\mu\nu}) + 4r_c^4 \left[ \frac{\delta r_c}{r_c} (3u^\rho u^\nu + P^{\mu\nu}) + (\delta u^\rho u^\nu + \delta u^\nu u^\rho) \right]
\]

Adding (8.8) to (8.4) gives the complete expression of \( \pi^{\mu\nu} \) after this variable redefinition. \( \pi^{\mu\nu} \) is transverse if we choose

\[
12r_c^3 \delta r_c = - (d\rho_n + \mu_0^2 \frac{d}{d\tau}) f \mu_{\text{diss}}
\]
\[
4r_c \delta u^\rho = (df - f \mu_{\text{diss}}) \xi_0^\rho
\]

(as can be verified by dotting \( \pi^{\mu\nu} \) with \( u^\rho u^\nu \) and with \( u^\rho u^\nu \)). With \( \delta r_c \) and \( \delta u^\rho \) chosen as above

\[
\pi^{\mu\nu} = df \, \xi_0^\rho \xi_0^\nu + (dP + 4r_c^3 \delta r_c) \, P^{\mu\nu} + \tilde{\pi}^{\mu\nu}
\]

(8.10)

(all expressions involving \( \delta u^\rho \) cancel).

Note that the corrections to \( J_{\text{diss}}^\mu \) that arise from the field redefinitions of \( r_c \) and \( u^\rho \) are all at \( \mathcal{O}(\frac{1}{c^4}) \). As we have not kept track of the current to this order in our gravitational computation, we will ignore all such terms; \( J_{\text{diss}}^\mu \) continues to be given by (8.7).

Note that the final expression (8.10) for \( \pi^{\mu\nu} \) depends on \( \delta r_c \) but not on \( \delta u^\rho \) (this is, of course, true only to first order in the derivative expansion). While \( \delta r_c \) is given by (8.4), it is equally well given by solving the equation

\[
df \, \xi_0^\rho + 3 \left( dP + 4r_c^3 \delta r_c \right) = 0
\]

(8.11)

That (8.11) must be true follows from the trace of (8.10) and the observation that \( \pi^{\mu\nu} \) and \( \tilde{\pi}^{\mu\nu} \) are both traceless. That the expressions for \( \delta r_c \) obtained from (8.11) and (8.4) agree follows from acting the operator ‘\( d^2 \)’ on the equation

\[
(\rho_n + 3P - f \xi_0^2) = 0
\]
which itself is simply an assertion of the tracelessness of the perfect fluid stress tensor 21.

Inserting the expression for $\delta r$ from (8.11) into the expression for $\pi^{\mu\nu}$ in (8.10) we conclude that the gravitational expressions for $\pi^{\mu\nu}$ and $J_{\text{diss}}^{\mu}$ in the $\mu_{\text{diss}} = 0$ frame are given by

$$J_{\text{diss}}^{\mu} = (dq_n + \mu_0 df) u^\mu + \mu_{\text{diss}, f}(\xi_0, \mu_0) u^\mu - df \, \zeta^\mu + \tilde{J}_{\text{diss}}^{\mu}$$

$$\pi^{\mu\nu} = \frac{df}{3} \left( 2n^\mu n^\nu - \tilde{P}^{\mu\nu} \right) + \tilde{\pi}^{\mu\nu}$$ \hfill (8.13)

In the rest of this subsection we will now compare this gravitational result to the ‘standard form’ for the dissipative corrections to the stress tensor and charge current predicted by our ‘theory’ of Weyl Invariant fluid dynamics in §3. This standard form was given in (3.18) and is reproduced here for the convenience of the reader

$$\pi^{\mu\nu} = T^3 \left[ [P_S S + P_b S_b + P_w S_w] \left( n^\mu n^\nu - \frac{P^{\mu\nu}}{3} \right) ight.$$

$$\left. + E_a (V_a^{\mu} n^\nu + V_a^{\nu} n^\mu) \right]$$

$$+ \tau T^{\mu\nu} \right]$$

$$J_{\text{diss}}^{\mu} = T^2 \left[ [Q_S S + Q_b S_b + Q_w S_w] u^\mu 

+ [R_S S + R_b S_b + R_w S_w] n^\mu 

+ C_a V_a^{\mu} + C_b V_b^{\mu} \right]$$ \hfill (8.14)

where

$$S = (u.\partial) \left( \frac{\mu}{T} \right), \quad S_b = (n.\partial) \left( \frac{\mu}{T} \right), \quad S_w = n^\mu n^\nu \sigma_{\mu\nu} = \frac{2 S_a - S_b}{3}$$

and

$$V_a^{\mu} = \tilde{P}^{\mu\nu} \partial_{\nu} \left( \frac{\mu}{T} \right), \quad V_b^{\mu} = \tilde{P}^{\mu\alpha} \sigma_{\alpha\beta} n^\beta$$

The standard form described above uses a basis of independent first derivative terms that differs from our choice of independent first derivative terms in our gravitational solution. In order to compare the two forms we simply reexpress $S_a, S_b$ and $V_a$ in terms of the independent first derivative forms

$$0 = d(-\rho_n + 3P - f\xi^2) = -d\rho_n + 3dP - \xi^2 df - 2\xi f d\xi$$

$$= -d\rho_n + 3dP - \xi^2 df + 2\xi f \left[ \frac{\mu}{3} \mu_{\text{diss}} \right] = \xi^2 df + 3dP - [d\rho_n + \mu^2 df - 2\mu f \mu_{\text{diss}}]$$ \hfill (8.12)

$$= \xi^2 df + 3dP + 12r^3 c dr_c$$

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employed in the gravity calculation, utilizing the equations of motion. We get

\[
S = \epsilon^2 \frac{2\chi^2}{7} (2S_4 - S_6) + \epsilon^3 \left( \frac{1 + 48\chi^2}{56} \right) S_1 + \epsilon^3 \frac{\chi}{14} S_2
- \epsilon^3 \frac{\chi}{56} (2S_3 - S_5) + \mathcal{O}(\epsilon^4)
\]

\[
S_b = -\epsilon^2 \chi S_1 - \frac{\epsilon^2}{24} S_2 + \mathcal{O}(\epsilon^4)
\]

\[
V_\mu^\alpha = -\epsilon^2 \chi V_2^\mu - \frac{\epsilon^2}{24} V_1^\mu + \mathcal{O}(\epsilon^4)
\]

(8.15)

We then compare the expressions so obtained with our gravitational results (8.13). We find that the two expressions match perfectly if we choose

\[
16\pi GQ_S = \frac{\pi}{\epsilon^2} \left[ -\frac{208}{\epsilon^2} + \mathcal{O}(\epsilon^0) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

\[
16\pi GQ_b = \frac{\pi}{\epsilon^2} \left[ -240 \frac{\chi}{\epsilon} + \mathcal{O}(\epsilon^0) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

\[
16\pi GQ_w = \frac{\pi^2}{\epsilon^2} \left[ 240\chi^2 + \mathcal{O}(\epsilon) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

(8.16)

\[
16\pi GR_S = \frac{\pi}{\epsilon^2} \left[ -240 \frac{\chi}{\epsilon} + \mathcal{O}(\epsilon^0) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

\[
16\pi GR_b = \frac{\pi}{\epsilon^2} \left[ -1 - 288\chi^2 + \mathcal{O}(\epsilon) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

\[
16\pi GR_w = \frac{\pi^2}{\epsilon^2} \left[ 288 \epsilon \chi^3 + \mathcal{O}(\epsilon^2) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

\[
16\pi GP_S = \frac{\pi^2}{\epsilon^2} \left[ 240\chi^2 + \mathcal{O}(\epsilon) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

\[
16\pi GP_b = \frac{\pi^2}{\epsilon^2} \left[ 288 \epsilon \chi^3 + \mathcal{O}(\epsilon^2) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]

\[
16\pi GP_w = -3\pi^3 + \frac{\pi^3}{\epsilon} \left[ -6 - \left( \frac{1}{4} - 3\chi^2 - 288\chi^4 \right) \epsilon^2 + \mathcal{O}(\epsilon^3) \right] + \mathcal{O} \left( \frac{1}{\epsilon^4} \right)
\]
\[16\pi GC_a = \frac{\pi}{e^2} \left[-1 + O(\epsilon)\right] + O\left(\frac{1}{e^4}\right)\]
\[16\pi GC_b = \frac{\pi^2}{e^2} \left[\epsilon^3 \chi \left(-1 + \log(\epsilon^4)\right) + O(\epsilon^4)\right] + O\left(\frac{1}{e^4}\right)\]

\[16\pi GE_a = O(\epsilon^3) + O\left(\frac{1}{e^4}\right)\]
\[16\pi GE_b = -2\pi^3 + \pi^3 \left[-4 + \left(\frac{1}{6} - 2\chi^2\right) \epsilon^2 + O(\epsilon^4)\right] + O\left(\frac{1}{e^4}\right)\]
\[16\pi G\tau = -2\pi^3 + \pi^3 \left[-4 + \left(\frac{1}{6} - 2\chi^2\right) \epsilon^2 + O(\epsilon^4)\right] + O\left(\frac{1}{e^4}\right)\]

Note that \(Q_w = P_a, \ R_w = P_b, \ R_a = Q_b\) and \(C_b = E_a\), so that our results obey the Onsager relations listed in §3.5. It may also be verified that these results obey all the positivity constraints required on general grounds in §3.5.

8.2 Transformation to the Transverse Frame

Following the discussion of the previous subsection, it is also possible to cast our gravitational results into the transverse fluid frame. As mentioned in §3.3, the basis of first derivative quantities with non zero coefficients most suitable for this frame are

\[S_a = \partial_{\mu} \left(\frac{q_2 \xi^\mu}{\xi}\right); \quad S_b = (n^\mu \partial_\nu) \left(\frac{\mu}{T}\right); \quad S_w = n^\mu \eta^\nu \sigma_{\mu\nu};\]
\[V_\mu^a = \tilde{P}^\mu \sigma_\nu \eta^\nu; \quad V_\mu^b = \tilde{P}^\mu \tilde{\sigma}_{\alpha\beta} n^\nu; \quad T_{\mu\nu} = \tilde{\sigma}_{\mu\nu}\]

These quantities may be expressed in term of the quantities (defined in (7.3)) used for the gravity calculation as follows

\[S_a = \frac{1}{16\pi G} \frac{\sigma^2}{e^2} \left[ 2S_1^2 - S_0 \right] + \epsilon^3 \left(\frac{5 + 96\chi^2}{28}\right) S_1 - \epsilon^3 \chi \left(\frac{5 + 96\chi^2}{28}\right) S_2 + O(\epsilon^4)\]
\[S_b = -\epsilon^2 \chi S_1 - \frac{\epsilon^2}{24} S_2 + O(\epsilon^4); \quad S_w = \frac{2S_1 - S_0}{3};\]
\[V_\mu^a = -\epsilon^2 \chi V_2^\mu - \frac{\epsilon^2}{24} V_1^\mu + O(\epsilon^4); \quad V_\mu^b = \frac{V_2^\mu}{2}; \quad T_{\mu\nu} = \frac{T_{\mu\nu}}{2}\]

Let us rewrite the first derivative corrections to charge current obtained from gravity (given in (7.25) and (7.27)) in the following schematic form

\[\tilde{j}_{diss}^\mu = \left(\frac{1}{16\pi G}\right) \frac{\sigma^2}{e^2} \left(\tilde{j}_{u\nu}^\mu + \tilde{j}_{n\nu}^\mu + \sum \xi c_i V_i^\mu\right).\]

In the gravity solution the stress tensor (\(\tilde{\sigma}_{\mu\nu}\)) is given as the following (see §7.26).

\[16\pi G \tilde{\sigma}_{\mu\nu} = -2\epsilon^3 \sigma_{\mu\nu} + O(\epsilon^3) + O\left(\frac{1}{e^4}\right)\]
Then using the procedure outlined in the previous section and in § (2.4) we can compute first
derivative corrections to stress tensor, charge current and chemical potential in the transverse frame
(which we denote by \( \pi^{(T)}_{\mu\nu}, (J_{\text{diss}}^{(T)})_{\mu} \) and \( \mu_{\text{diss}}^{(T)} \) respectively). We find

\[
16\pi G \left( \pi^{(T)}_{\mu\nu} \right) = \frac{6}{5} \left[ j_u - 4\mu_{\text{diss}} \right] \xi^2 \left( n^\mu n^\nu - \frac{P^\mu}{3} \right) + \tilde{\pi}^{\mu\nu},
\]

\[
16\pi G \left( J_{\text{diss}}^{(T)} \right)_{\mu} = \frac{\xi^2}{\epsilon^2} \left( \left[ j_n - \frac{6\xi}{5} \left[ j_u - 4\mu_{\text{diss}} \right] \right] n^\mu + \sum_i c_i [V_i]^\mu \right),
\]

\[
\mu_{\text{diss}}^{(T)} = \frac{1}{10} \left[ j_u - 14 \mu_{\text{diss}} \right],
\]

where \( \mu_{\text{diss}} \) is the first derivative correction to the chemical potential obtained from gravity given in
\((7.25)\) and in \((7.28)\). Here the formulas presented in \((8.21)\) are valid only at the leading order in \( \epsilon \) for
each independent data at one derivative order.

As in the previous subsection, we then consider the expected standard form fluid expression which
is given by

\[
\pi^{\mu\nu} = T^3 \left[ \left( P_a S_a + P_b S_b + P_w S_w \right) \left( n^\mu n^\nu - \frac{P^\mu}{3} \right) \right.
\]

\[
+ \left. E_a (V^\mu_a n^\nu + V^\nu_a n^\mu) + E_b (V^\mu_b n^\nu + V^\nu_b n^\mu) \right] + \tau T^{\mu\nu}
\]

\[
J_{\text{diss}}^\mu = T^2 \left[ \left( R_a S_a + R_b S_b + R_w S_w \right) n^\mu \right.
\]

\[
+ \left. C_a V^\mu_a + C_b V^\mu_b \right] \]

\[
\mu_{\text{diss}} = - \left[ Q_a S_a + Q_b S_b + Q_w S_w \right]
\]

\[(8.22)\]

Just as in the previous section the gravity result after the frame transformation \([8.21]\) perfectly

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fits into the above form provided we identify equation 8.22

\[ Q_a = 16\pi G \left( \frac{e^2}{\pi^4} \right) \left[ - \frac{52}{25e^2} + \mathcal{O}(e)^0 \right] + \mathcal{O} \left( \frac{1}{e^4} \right)^0 \]

\[ R_a = \frac{1}{\pi} \left[ - \frac{24\chi}{25e} + \mathcal{O}(e)^0 \right] + \mathcal{O} \left( \frac{1}{e^2} \right) \]

\[ P_a = \frac{24}{25\chi^2} + \mathcal{O}(e) + \mathcal{O} \left( \frac{1}{e^2} \right) \]

\[ Q_B = \frac{1}{\pi} \left[ - \frac{24\chi}{25e} + \mathcal{O}(e)^0 \right] + \mathcal{O} \left( \frac{1}{e^2} \right) \]

\[ R_B = \frac{\pi}{(16\pi G)e^2} \left[ - \frac{888}{25} \chi^2 + \mathcal{O}(e) \right] + \mathcal{O} \left( \frac{1}{e^2} \right) \]

\[ P_B = \frac{\pi^2}{(16\pi G)e^2} \left[ \frac{288}{25} e \chi^3 + \mathcal{O}(e)^2 \right] + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ Q_w = \frac{24}{25\chi^2} + \mathcal{O}(e) + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ R_w = \frac{\pi^2}{(16\pi G)e^2} \left[ \frac{288}{25} e \chi^3 + \mathcal{O}(e)^2 \right] + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ P_w = -\frac{3\pi^3}{16\pi G} + \frac{\pi^3}{(16\pi G)e^2} \left[ -6 - \left( \frac{1}{4} - 3\chi^2 + \frac{288}{25} \chi^4 \right) e^2 + \mathcal{O}(e)^3 \right] \]

\[ + \mathcal{O} \left( \frac{1}{e^4} \right) \]

and

\[ 16\pi G E_a = \frac{\pi^2}{e^2} \left[ \mathcal{O}(e)^3 \right] + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ 16\pi G C_a = \frac{\pi}{e^2} \left[ -1 + \mathcal{O}(e) \right] + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ 16\pi G E_b = -2\pi^3 + \frac{\pi^3}{e^2} \left[ -4 + \left( \frac{1}{6} - 2\chi^2 \right) e^2 + \mathcal{O}(e)^4 \right] + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ 16\pi G C_b = \frac{\pi^2}{e^2} \left[ e^3 \chi \left( \frac{-1 + \log(4)}{8} + \mathcal{O}(e)^4 \right) \right] + \mathcal{O} \left( \frac{1}{e^4} \right) \]

\[ 16\pi G \tau = -2\pi^3 + \frac{\pi^3}{e^2} \left[ -4 + \left( \frac{1}{6} - 2\chi^2 \right) e^2 + \mathcal{O}(e)^4 \right] + \mathcal{O} \left( \frac{1}{e^4} \right) \]

Note that in this transverse frame also we have \( Q_w = P_a, R_w = P_b, R_a = Q_b \) and \( C_b = E_a \), which constitutes the expected Onsager relations. All the positivity constraints given in [§8.3] are also obeyed in this frame.

In \( \zeta \to 0 \) limit derivative corrections to stress tensor, charge current and the phase equation in transverse frame take the following form

\[ \lim_{\zeta \to 0} \left[ \pi^{(T)} \right]^{\mu\nu} = T^{3\beta_1} \sigma^{\mu\nu} \]

\[ \lim_{\zeta \to 0} \left[ \mu_{\text{diss}}^{(T)} \right]^{\mu} = T^{2\beta_2} \varphi_{\text{diss}}^{\mu} \left( \frac{\mu}{T} \right) \]

\[ \lim_{\zeta \to 0} \mu_{\text{diss}}^{(T)} = \beta_3 \partial_{\mu} \left( \frac{\varphi_{T}}{\zeta} \right) \]

(8.25)
where

\[
\begin{align*}
\beta_1 &= -2\pi^3 + \frac{\pi^3}{e^2} \left[ -4 + \frac{e^2}{6} + O(\epsilon)^4 \right] + O \left( \frac{1}{e^4} \right) \\
\beta_2 &= \frac{1}{16\pi G} \left[ -\frac{\pi}{e^2} + O(\epsilon^3) \right] + O \left( \frac{1}{e^4} \right) \\
\beta_3 &= 16\pi G \left( \frac{e^2}{\pi^3 T^3} \right) \left[ -\frac{52}{25 e^2} + O(\epsilon^0) \right] + O \left( \frac{1}{e^4} \right) \tag{8.26}
\end{align*}
\]

8.3 Transformation to the Landau-Lifshitz-Clark-Putterman Frame

Our results may also be transformed to the Landau-Lifshitz-Clark-Putterman frame. We have not fully worked through the complicated algebra needed for this process. However, we have verified that this transformation yields

\[
16\pi G \, Q_2^{(s)} = \frac{\pi^3}{e^2} \left[ -\frac{6}{7} \epsilon^3 + O(\epsilon^4) \right] + O \left( \frac{1}{e^4} \right)
\]

Note in particular that \( Q_2^{(s)} \) is not zero, predicted by the formulas of Clark and Putterman.

9. Discussion

Our paper suggests many directions for future research. In this paper, we have studied the bulk dual to Einstein-Maxwell systems in the absence of a bulk Chern Simons term. A Chern Simon’s term has already been shown to lead to yield qualitatively new contributions to charged fluid dynamics even in the absence of charged scalar fields [49, 43, 41, 50]. It seems certain to contribute new and interesting terms to charged superfluid dynamics as well. We leave a study of the effect of this Chern Simons term on holographic superfluid flows to future work.

Relatedly, we should emphasize that the framework for dissipative fluid dynamics presented in this paper crucially assumes that the superfluid entropy current takes the canonical form discussed in §2.5. While this assumption turned out to be true of the gravitational system studied in this paper, we do not know of any physical reason for it always to be true. In particular, we think it very likely that the entropy current will be modified in superfluids dual to a gravitational system with a nonzero Chern Simons term as described in the previous paragraph. It would be very interesting to understand the general rules for the entropy current at first order in the derivative expansion, and thereby to construct the most general framework for first order dissipative superfluid dynamics.

It would of course be interesting to study dissipative corrections to holographic superfluid dynamics away from the rather strange limit (of very large \( \epsilon \) and perturbatively in the superfluid condensate) employed in this paper. In fact, it is interesting that a continuation of the model studied in this paper to a particular order one value of \( \epsilon \) and supplemented with a particular Chern Simons term and a more complicated scalar potential (however with squared mass \( = -4 \) as in this paper) is a consistent truncation of IIB supergravity on \( AdS_5 \times S^5 \) [51]. It would be fascinating to be able to work out the map from gravity to fluid dynamics in this system, though that might require numerical work. It would also be interesting to follow the investigation of [49] to study whether hairy black holes in global \( AdS_5 \) (see e.g. [52, 51, 53]) can be thought of as stationary superfluid flows.

The task described in the previous paragraphs is hampered by the lack of explicit analytic solutions for equilibrium configurations. One could imagine circumventing this lack in two different ways. First, it would be very interesting to attempt an abstract analysis of dissipative fluid flows, along the lines...
described in Appendix B (following Sonner and Withers) for equilibrium flows. The aim of such an abstract analysis could be to prove that the gravitational entropy current always agrees with the canonical form presented in §3 in the absence of bulk Chern Simons terms, and to prove that all Onsager relations are obeyed. This would constitute a proof that gravitational superfluid dynamics falls within the framework of dissipative superfluid dynamics described in §3 above (in the absence of bulk Chern Simons terms). A second possible approach to the same problem is numerical; a clever use of numerical techniques should permit the numerical generation of the gravitational solutions dual to arbitrary first order superfluid flows at finite values of $\epsilon$.

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A. More about the canonical entropy current

A.1 Frame invariance of the fluid frame entropy current

In this subsection we will demonstrate that the entropy current

$$J_\mu^\text{s} = su^\mu - \frac{\mu}{T} J_\text{diss}^\mu - \frac{u_\nu \pi^{\mu\nu}}{T}$$

is frame invariant.

Let us perform the following change of variables (we work accurately only to first order in the derivatives)

$$u^\mu = \bar{u}^\mu + \delta u^\mu$$
$$T = \bar{T} + \delta T$$
$$\mu = \bar{\mu} + \delta \mu$$

(A.1)

Under this redefinition $J_{\text{diss}}^\mu$ and $\pi^{\mu\nu}$ change according to the formulae given in (2.13).

$$\delta \pi^{\mu\nu} = (u^\mu \delta u^\nu + u^\nu \delta u^\mu)(P + \rho_n) + u^\mu u^\nu d(P + \rho_n) + \frac{\xi_\mu \xi_\nu}{\xi^2} d(\rho_s) + \eta^{\mu\nu} dP$$

$$\delta J_{\text{diss}}^\mu = q_n \delta u^\mu + dq_n u^\mu - dq_s \frac{\xi_\mu}{\xi}$$

(A.2)

In the new frame the entropy current is given by
\[ \hat{J}_s^\mu = \hat{s}(\hat{\mu}, \hat{T}) \hat{u}^\mu - \frac{\hat{\mu}}{\hat{T}} (J_{\text{diss}}^\mu + \delta J_{\text{diss}}^\mu) - \frac{u_\nu (\pi^{\mu\nu} + \delta \pi^{\mu\nu})}{\hat{T}} \]
\[ = J_s^\mu - ds \ u^\mu - \frac{\hat{\mu}}{\hat{T}} \left( q_n \delta u^\mu + dq_n u^\mu - dq_s \xi^\mu \right) \]
\[ + \frac{P + \rho_n}{\hat{T}} \delta u^\mu + \frac{u^\mu}{\hat{T}} d(P + \rho_n) - \frac{\mu d \rho_s \xi^\mu}{\hat{T} \xi} - \frac{u^\mu}{\hat{T}} dP \]
\[ = J_s^\mu + u^\mu \left( \frac{d \rho_n}{\hat{T}} - ds - \frac{\mu}{\hat{T}} dq_n \right) + \delta u^\mu \left( \frac{P + \rho_n}{\hat{T}} - s - \frac{\mu}{\hat{T}} q_n \right) \]
\[ - \frac{\xi^\mu \mu}{\hat{T} \xi} \left( \frac{d \rho_s}{\xi} - dq_s \right) \]

(A.3)

In the above expression each of the terms inside the bracket vanishes because of thermodynamic identities (recall also that \( \xi = \mu_s \)).

It follows that

\[ \hat{J}_s^\mu = J_s^\mu \]

A.2 The Divergence of the fluid frame entropy current

Using the relation that \( \mu_s = \xi = \sqrt{-\xi^\mu \xi_\mu} \) the stress tensor current and the phase as given in (2.6) can be rewritten as

\[ \begin{align*}
T^{\mu\nu} &= (\rho_n + P) u^\mu u^\nu + P \eta^{\mu\nu} + \frac{\rho_s}{\mu_s^2} \xi^\mu \xi^\nu + \pi^{\mu\nu} \\
J^\mu &= q_n u^\mu - \frac{q_s}{\mu_s} \xi^\mu + J_{\text{diss}}^\mu \tag{A.4} \\
u^\mu \xi_\mu &= \mu + \mu_{\text{diss}}
\end{align*} \]

We will find use, below, for the following thermodynamical relationships:

\[ s = \frac{\rho_n + P - \mu q_n}{\hat{T}} \]
\[ d \rho_n = \mu dq_n + T ds - q_s d \mu_s \tag{A.5} \]

Now divergence of \( su^\mu \) gives the following expression.

\[ \begin{align*}
\partial_\mu [su^\mu] &= u. \partial s + s(\partial . u) \\
&= \frac{1}{\hat{T}} [u. \partial \rho_n + (\rho_n + P)(\partial . u)] - \frac{\mu}{\hat{T}} [u. \partial q_n + q_n(\partial . u)] + \frac{q_s}{\hat{T}} u. \partial \mu_s
\end{align*} \]

(A.6)

Using conservation of stress tensor (the equation to be used is \( u_\nu \partial_\mu T^{\mu\nu} = 0 \)) one can evaluate the

\[ 22 \text{Recall that the symbol } d \text{ denotes the change of a quantity under a frame change field redefinition, and that the microscopically defined field } \xi^\mu(x) \text{ is taken to be the same in all frames so that, in particular, } d \xi = 0. \]
Similarly using the current conservation equation one can evaluate the expression \[ u \cdot \partial \rho_n + (\rho_n + P) (\partial \cdot u) \]

\[ u \cdot \partial \rho_n + (\rho_n + P) (\partial \cdot u) = \frac{\rho_s}{\mu_s} \left[ (u \cdot \xi)(\partial \cdot \xi) + u^v (\xi \cdot \partial \xi_v) \right] + (u \cdot \xi)(\xi \cdot \partial) \left( \frac{\rho_s}{\mu_s^2} \right) + u_\nu \partial_\mu \pi^{\mu \nu} \]

\[ = \frac{\rho_s}{\mu_s} \left[ \mu (\partial \cdot \xi) + u^v (\xi \cdot \partial \xi_v) + \mu (\xi \cdot \partial) \left( \frac{\rho_s}{\mu_s^2} \right) + \mu \partial \mu \partial_\nu \pi^{\mu \nu} \right] + u_\nu \partial_\mu \pi^{\mu \nu} \]  

(A.7)

In the last line we have used the following identity derived from the curl-free condition.

\[ u^\mu (\partial_\mu \xi_\nu - \partial_\nu \xi_\mu) \xi_\nu = 0 \]

\[ \Rightarrow u^\nu (\xi \cdot \partial) \xi_\nu = \frac{1}{2} (u \cdot \partial) (\xi_\mu \xi^\mu) = - \mu_s (u \cdot \partial) \mu_s \]  

(A.8)

Similarly using the current conservation equation one can evaluate the expression \[ u \cdot \partial q_n + q_n (\partial \cdot u) \]

\[ u \cdot \partial q_n + q_n (\partial \cdot u) = \frac{q_s}{\mu_s} (\partial \cdot \xi) + (\xi \cdot \partial) \left( \frac{q_s}{\mu_s} \right) - \partial \cdot J_{\text{diss}} \]

(A.9)

Adding all these equations one finds the following expression for the divergence of the entropy current

\[ \partial_\mu \left[ su^\mu \right] = \left( \frac{\rho_s}{\mu_s^2} - \frac{q_s}{\mu_s} \right) \left[ \partial_\mu \xi_\nu - \mu_s (\xi \cdot \partial) \mu_s \right] + \frac{\mu}{T} (\xi \cdot \partial) \left( \frac{\rho_s}{\mu_s^2} - \frac{q_s}{\mu_s} \right) + \frac{\mu}{T} \left[ \mu \partial \mu \partial_\nu \pi^{\mu \nu} \right] + \frac{\mu}{T} (\partial \cdot J_{\text{diss}}) \]

(A.10)

The first line is zero if \( \rho_s = \mu_s q_s \) and the second line can be written as

\[ \partial_\mu \left[ su^\mu - \frac{\mu}{T} J^\mu_{\text{diss}} - \frac{u_\nu \pi^{\mu \nu}}{T} \right] = - \partial_\mu \left[ \frac{u_\nu}{T} \right] \pi^{\mu \nu} - \partial_\mu \left[ \frac{\mu}{T} J^\mu_{\text{diss}} + \frac{\mu}{T} \partial_\mu \left( \frac{\rho_s}{\mu_s^2} \right) \right] \]

(A.11)

A.3 Direct Thermodynamical determination of the modified phase entropy current and its divergence

As we have explained in the main text, a modified phase fluid description works in terms of a gradient vector \( \xi_\mu^\nu \) defined by

\[ \xi_\mu^\nu = \xi_\mu^\nu - \mu_{\text{diss}} u^\mu \]

where \( \xi_\mu^\nu = - \mu u^\nu + \xi_\mu^\nu = - \mu u^\nu + \zeta n^\mu \)

In a modified phase frame, the stress tensor and charge current are taken to have the form

\[ T_{\mu \nu} = (\rho_s + P) u^\nu u^\mu + P \eta_{\mu \nu} + f \xi_\mu^\nu \xi_\nu^\nu + \tilde{\pi}_{\mu \nu} \]

\[ J^\mu = q_s u^\mu + f \xi_0^\mu + J^\mu_{\text{diss}} \]

(A.12)

where \( f = \frac{\rho_s}{\xi_0} = \frac{q_s}{\xi_0} \), and all thermodynamical quantities are functions of \( T, \mu, \xi_0 \).
We will find use, below, for the following thermodynamical relationships:

$$s = \frac{\rho_n + P - \mu q_n}{T}$$

$$d\rho_n = \mu dq_n + T ds - f\xi_0 d\xi_0$$  \hspace{1cm} (A.13)

We now compute the divergence of the vector \( su^\nu \)

$$\partial_\mu [su^\nu] = u.\partial s + s(\partial u) = \frac{1}{T} [u.\partial \rho_n + (\rho_n + P)(\partial u)] - \frac{\mu}{T} [u.\partial q_n + q_n(\partial u)] + \frac{q_s}{T} u.\partial s$$  \hspace{1cm} (A.14)

Using conservation of stress tensor (the equation to be used is \( u_\nu \partial_\mu T^{\mu\nu} = 0 \)) one can evaluate the expression \([u.\partial \rho_n + (\rho_n + P)(\partial u)]\) as

$$u.\partial \rho_n + (\rho_n + P)(\partial u) = \mu \partial_\mu (f \xi_0^\mu) + f u_\nu (\xi_0^\mu, \partial_\nu) \xi_0^\nu + u_\nu \partial_\mu \tilde{\pi}^{\mu\nu}$$  \hspace{1cm} (A.15)

Similarly using the current conservation equation one can evaluate the expression \([u.\partial q_n + q_n(\partial u)]\) as

$$u.\partial q_n + q_n(\partial u) = \partial_\mu (f \xi_0^\mu) - \partial_\mu \tilde{J}^\mu_{\text{diss}}$$  \hspace{1cm} (A.16)

The fact that the vector \( \xi_0 \) is curl free gives the following identity

$$\xi_0 (u, \partial) \xi_0 + u_\nu (\xi_0, \partial) \xi_0^\nu = - \mu_{\text{diss}} \xi_0^\nu (u, \partial) u_\nu + (\xi, \partial) \mu_{\text{diss}}$$  \hspace{1cm} (A.17)

Using all these equations one finds the following expression for the divergence of the entropy current

$$\partial_\mu [su^\nu] = \frac{\mu}{T} \partial_\mu \tilde{J}^\mu_{\text{diss}} + \frac{u_\nu \partial_\mu \tilde{\pi}^{\mu\nu}}{T} - \frac{f}{T} \mu_{\text{diss}} \xi_0^\nu (u, \partial) u_\nu - \frac{f}{T} (\xi, \partial) \mu_{\text{diss}}$$  \hspace{1cm} (A.18)

It follows that if we define an entropy current by

$$\tilde{J}^\mu_S = s(\xi_0) u^\mu - \frac{\mu}{T} \tilde{J}^\mu_{\text{diss}} - \frac{u_\nu \tilde{\pi}^{\mu\nu}}{T} + \frac{f}{T} \mu_{\text{diss}} \xi_0^\nu$$  \hspace{1cm} (A.19)

then its divergence is given by

$$\partial_\mu \tilde{J}^\mu_S = - \partial_\mu \left[ \frac{u_\nu}{T} \tilde{\pi}^{\mu\nu} \right] - \partial_\mu \left[ \frac{\mu}{T} \tilde{J}^\mu_{\text{diss}} + \mu_{\text{diss}} P^{\mu\nu} \partial_\mu \left( \frac{f \xi_0^\nu}{T} \right) \right]$$  \hspace{1cm} (A.20)

In deriving this expression we have used the following equations.

$$\frac{f}{T} \left[ \xi_0 (u, \partial) \xi_0 + u_\nu (\xi_0, \partial) \xi_0^\nu \right] = \frac{f}{T} \left[ \xi_0 (u, \partial) \xi_0 + u_\nu (\xi_0, \partial) \xi_0^\nu \right]$$

$$= - \partial_\mu \left( \frac{f}{T} \xi_0^\mu \mu_{\text{diss}} \right) + \mu_{\text{diss}} \left[ \partial_\mu \left( \frac{f}{T} \xi_0^\mu \right) - \frac{f}{T} \xi_0^\nu (u, \partial) u_\nu \right]$$  \hspace{1cm} (A.21)
A.3.1 Fluid frame entropy current transformed to the modified phase frame

In this subsection we will check that the modified frame entropy current \( (A.19) \) is precisely the current what we get by transforming the general fluid frame entropy current to modified phase variables. As we have explained in the main text, the relation between dissipative parameters in a general fluid frame and the corresponding modified phase frame is given by

\[
\pi_{\mu\nu} = \tilde{\pi}_{\mu\nu} + d\left( \rho_n + P \right) u^\mu u^\nu + dP u^\mu u^\nu + df \xi_0 \xi^\mu
\]

\[
+ f \mu_{diss}(u^\mu \xi^\nu + u^\nu \xi^\mu)
\]

\[
J^\mu_{diss} = \tilde{J}^\mu_{diss} + d\eta_n u^\mu - df \xi^\mu - f \mu_{diss} u^\mu
\]

where ‘d’ of any function denotes

\[
dA = A(\xi_0) - A(\xi) = -\mu_{diss} \left( \frac{\mu}{\xi_0} \right) \frac{\partial A}{\partial \xi}
\]

The transformation of the canonical fluid frame entropy current into the modified phase frame is given by

\[
J^\mu_S = s(\xi_0) u^\mu - \frac{u_\nu \pi^\mu_{\nu}}{T} - \frac{\mu}{T} J^\mu_{diss}
\]

\[
= s(\xi_0) u^\mu - \frac{u_\nu \tilde{\pi}^\mu_{\nu}}{T} - \frac{\mu}{T} J^\mu_{diss} - \left( \frac{\mu}{T} dq_n - \frac{d\rho_n}{T} + ds \right) u^\mu + \frac{\mu_{diss}}{T} \xi_0^\mu
\]

\[
= s(\xi_0) u^\mu - \frac{u_\nu \tilde{\pi}^\mu_{\nu}}{T} - \frac{\mu}{T} J^\mu_{diss} - \frac{f}{T} \xi_0 d\xi_0^\mu + \frac{\mu_{diss}}{T} \xi_0^\mu
\]

\[
= s(\xi_0) u^\mu - \frac{u_\nu \tilde{\pi}^\mu_{\nu}}{T} - \frac{\mu}{T} J^\mu_{diss} + \frac{f}{T} \mu_{diss} (\mu u^\mu + \xi_0^\mu)
\]

\[
= s(\xi_0) u^\mu - \frac{u_\nu \tilde{\pi}^\mu_{\nu}}{T} - \frac{\mu}{T} J^\mu_{diss} + \frac{f}{T} \mu_{diss} \xi^\mu
\]

In going from the second to third line we have used the following thermodynamic relation

\[
d\rho_n = \mu dq_n + T ds - f d\xi
\]

In going from the third line to fourth line we have used the fact that

\[
d\xi_0 = -\mu_{diss} \left( \frac{\mu}{\xi_0} \right)
\]

Using equation \( (A.23) \) and \( (A.19) \) it follows that

\[
\tilde{J}^\mu_S = J^\mu_S
\]

B. Details on Superfluid Thermodynamics from Gravity

In this section we will use the bulk equations of motion to demonstrate that our solution obeys two Gibbs Duhem type relations. We will also review the demonstration of [7] that the onshell action for the solution is the negative of its pressure, and that the equation for the infinitesimal variation of this pressure obeys a thermodynamical first law.
B.1 Bulk equations, symmetries, conventions etc

The Lagrangian we study is given by

\[
\mathcal{L} = \frac{\sqrt{g}}{16\pi G} \left( R + 12 \right.
- \frac{1}{e^2} \left[ V_1(\phi \phi^*) F_{ab} F^{ab} + V_2(\phi \phi^*) D_a\phi D^a\phi^* + V_3(\phi \phi^*) \right] \right),
\]

where \( V_3(x) \) is taken to vanish at its minimum, \( D_a = \nabla - i A_a \) and \( \bar{D}_a = \nabla + i A_a \) and all the potentials \( V_1, V_2, V_3 \) are real.

The bulk equations that follow by extremizing this Lagrangian are given by

\[
\begin{align*}
\mathcal{D}_a [V_2(\phi \phi^*) D^a \phi^*] &= \frac{\partial V_1}{\partial \phi} F_{ab} \frac{F^{ab}}{e^2} + \frac{\partial V_2}{\partial \phi} D_a \phi D^a \phi^* + \frac{\partial V_3}{\partial \phi^*} \\
\mathcal{D}_a [V_2(\phi \phi^*) D^a \phi] &= \frac{\partial V_1}{\partial \phi^*} F_{ab} \frac{F^{ab}}{e^2} + \frac{\partial V_2}{\partial \phi^*} D_a \phi D^a \phi^* + \frac{\partial V_3}{\partial \phi^*} \\
\nabla_a [V_1(\phi \phi^*) F^{ab}] &= -\frac{i V_2(\phi \phi^*)}{4} \left[ \phi \mathcal{D}^b \phi^* - \phi^* \mathcal{D}^b \phi \right] \\
R_{ab} - \frac{1}{2} R g_{ab} - 6 g_{ab} = \mathcal{T}^{em}_{ab} + \mathcal{T}^{rc}_{ab}
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{T}^{em}_{ab} &= \frac{2 V_1(\phi \phi^*)}{e^2} \left( F_{ac} F^c_b + \frac{1}{4} g_{ab} F_{c_1 c_2} F^{c_1 c_2} \right) \\
\mathcal{T}^{rc}_{ab} &= \frac{V_2(\phi \phi^*)}{e^2} \left( \frac{D_a \phi D_b \phi^* + \bar{D}_a \phi^* D_b \phi}{2} - \frac{1}{2} g_{ab} D_c \phi D^c \phi^* \right) - \left[ \frac{V_3(\phi \phi^*)}{2 e^2} \right] g_{ab}
\end{align*}
\]

Here small Latin letters denote the bulk coordinates (\( \{r, v, x, y, z\} \)) and Greek letters denote the boundary coordinates (\( \{v, x, y, z\} \)). Greek indices are lowered or raised using the metric \( \eta \).

In this section we study solutions that preserve translational invariance in the field theory directions. By scaling our solution we can always ensure that its horizon is located at \( r = 1 \); we make this choice in what follows. Our spacetime has a \( d \) parameter set of killing vectors that generate translations in the field theory directions. We now identify two special killing vectors among this set. Let \( k^a = \{0, u^\mu\} \), be the unique killing vector that is null on the horizon. Further let \( l^a = \{0, n^\mu\} \).

Where \( n^\mu \) is defined as

\[
n^\mu = \frac{P^{\mu\nu} \xi_\nu}{\sqrt{\xi_\mu P^{\mu\nu} \xi_\nu}}, \quad \xi_\nu = \lim_{r \to \infty} g_{\mu \alpha} A^\alpha
\]

where \( A = A^\alpha \partial_\alpha \) is the gauge field and \( P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu \).

Below we will identify the vector \( u^\mu \) with the normal fluid four velocity, while the vector \( n^\mu \) points in the spatial direction of the superfluid velocity, in a frame in which the normal fluid is at rest.

As we have explained, the most general ansatz for our solution is given by

\[
ds^2 = -2 g(r) \ u_\mu dx^\mu dr - f(r) \ u_\mu u_\nu dx^\mu dx^\nu + j(r) (u_\mu n_\nu + u_\nu n_\mu) \ dx^\mu dx^\nu
+ k(r) n_\mu u_\nu dx^\mu dx^\nu + r^2 \bar{P}_{\mu \nu} dx^\mu dx^\nu
\]

\[
A = A^\nu(r) \partial_\nu + H(r) \ u^\mu \partial_\mu + L(r) \ n^\mu \partial_\mu
\]

where

\[
\bar{P}_{\mu \nu} = \eta_{\mu \nu} + u_\mu u_\nu - n_\mu n_\nu
\]
As we have mentioned above, we work in a gauge in which the scalar field is set to be real. This will be consistent with the equation of motion if the difference between the first and the second equation in (B.2) vanishes when \( \phi^*(r) \) is set equal to \( \phi(r) \). One can check that when all potentials are real and there are no explicit dependence on the boundary coordinates the scalar field can be consistently chosen to be real if \( A^r(r) \) is set identically to zero. Consequently, on our solutions,

\[
A^r = 0 \quad \text{and} \quad \phi(r) = \phi^*(r)
\]

We now make the following convenient coordinate choices. Let \( v \) represent a flat normalized boundary coordinate in in the direction of constant vector \( u^\mu \) and let \( x \) represent the coordinate is in the direction of constant vector \( n^\mu \). In other words, in our new coordinate system,

\[
k^a = \{0,1,0,0,\cdots\} \quad \text{and} \quad l^a = \{0,0,1,0,\cdots\}
\]

where we list a vector by \((k_r,k_v,k_x,\cdots)\). In this coordinate system the metric and the gauge field take the form

\[
ds^2 = 2g(r) \ dv \ dr - f(r) dv^2 - 2j(r) \ dv \ dx + k(r) \ dx^2 + r^2 \left( \sum dy_i^2 \right) \tag{B.4}
\]

\[
A^r = 0, \quad A^v = H(r), \quad A^x = L(r)
\]

Recall that in our coordinates the horizon occurs at \( r_H = 1 \). Because the horizon is a null surface, the norm of the normal vector vanishes. As the oneform dual to the normal is simply \( dr \) we conclude that \( g_{rr}(r = 1) = 0 \). It follows then from (B.4) that \( j(r = 1) = 0 \). None of the other functions that appear in equation (B.4) are restricted at the horizon, except that they should be regular.

At infinity we will impose the condition that our space is asymptotically AdS which implies that

\[
\lim_{r \to \infty} g(r) = 1 + \frac{g_1}{r^4} + O(r^{-6})
\]

\[
\lim_{r \to \infty} f(r) = r^2 + \frac{f_2}{r^2} + O(r^{-4})
\]

\[
\lim_{r \to \infty} k(r) = r^2 + \frac{k_2}{r^2} + O(r^{-4})
\]

\[
\lim_{r \to \infty} j(r) = \frac{j_2}{r^2} + O(r^{-4})
\]

and that \(^{23}\)

\[
\lim_{r \to \infty} A_v = \xi_v = \text{finite}
\]

\[
\lim_{r \to \infty} A_x = \xi_x = \text{finite}
\]

\[
\lim_{r \to \infty} \left[ \sqrt{g} \ V_1 (\phi \phi^*) F^{vr} \right] = \text{finite}
\]

\[
\lim_{r \to \infty} \left[ \sqrt{g} \ V_1 (\phi \phi^*) F^{xr} \right] = \text{finite}
\]

\(^{23}\)The existence of such solution will impose some constraints on the potential such that the contribution of gauge field and matter field to the bulk stress tensor vanishes sufficiently rapidly as \( r \to \infty \).
The first two equations in (B.7) may be regarded as definitions of the constants \( \xi_v \) and \( \xi_x \). The last two equations assert the finiteness of the boundary charge density and the boundary current in the \( x \) direction.

**B.2 Derivation of the Smarr Gibbs Duhem Relations**

**B.2.1 Basic Strategy**

It is a geometrical fact (see e.g. section 5.3 of [54]) that if \( \zeta^a \) is a killing vector

\[
R_{ab}^c \zeta^b = \nabla_b \nabla_a \zeta^b.
\]

It follows that, in our background,

\[
R_{ab}^c \zeta^b = \nabla_b \nabla^a \zeta^b = -\nabla_b \nabla^a \zeta^a = -\frac{1}{\sqrt{g}} \partial_b \left[ \sqrt{g} g^{bc} g^{ac2} \nabla_c \zeta_{c2} \right] = -\frac{1}{\sqrt{g}} \partial_r \left[ \sqrt{g} g^{rc} g^{ac2} \nabla_c \zeta_{c2} \right]
\]

where we have used the Killing equation in the second line, the fact that \( \nabla_b \zeta^a \) is antisymmetric in the third line and the fact that all functions in our metric and gauge field depend only on \( r \) in the last line. Plugging in the explicit form of our metric and making different choices for the free indices we find

\[
\sqrt{g} R_v^v = -\partial_r \left[ \sqrt{g} g^{va} g^{vb} \nabla_a k_b \right] = -\partial_r \left( \frac{r^2 [k(r) f'(r) + j(r) j'(r)]}{2g(r) \sqrt{k(r)}} \right)
\]

\[
\sqrt{g} R_x^x = -\partial_r \left[ \sqrt{g} g^{xa} g^{xb} \nabla_a b_b \right] = -\partial_r \left( \frac{r^2 [f(r) k'(r) + j(r) j'(r)]}{2g(r) \sqrt{k(r)}} \right)
\]

\[
\sqrt{g} R_y^y = -\partial_r \left[ \sqrt{g} g^{ya} g^{yb} \nabla_a q_b \right] = -\partial_r \left( \frac{r^2 [k(r) f(r) + j^2(r)]}{g(r) \sqrt{k(r)}} \right)
\]

\[
\sqrt{g} R_v^x = -\partial_r \left[ \sqrt{g} g^{va} g^{vb} \nabla_a l_b \right] = -\partial_r \left( \frac{r^2 [j(r) f'(r) - f(r) j'(r)]}{2g(r) \sqrt{k(r)}} \right)
\]

where \( q^a = \{0, 0, 0, 1, 0, \cdots\} \).

The RHS of each of (B.9) are total derivatives. Our basic strategy is to find appropriate linear combinations of the four relations above so that the LHS of these equations also reduce to total derivatives onshell (i.e. upon using Einstein’s equations). We will find three such linear combinations and so deduce the vanishing of three different expressions, each of which is a total derivative. We will then integrate these three equations to obtain three relations between quantities at infinity (i.e. conserved charges, currents, and superfluid velocities) and quantities at the horizon (entropy and temperature). Using these relationships we will be able to identify the vector \( u^\mu \) (defined in terms of the generator of the horizon) with the normal velocity of the fluid, and also deduce two distinct Smarr Gibbs Duhem relations. In the next subsubsection we present the algebra involved in identifying the relevant total derivatives. In the subsequent subsection we will integrate these total derivatives to deduce physical conclusions.
B.2.2 Total derivatives that vanish onshell

We use Einstein equations to simplify the LHS of the equations above.

\[ R^a_b = T^a_b - \left( \frac{T^a_a}{3} \right) \delta^a_b - 4 \delta^a_b \]  

(B.10)

Here \( T_{ab} \) is the bulk stress tensor.

In order to do this we will now present and manipulate explicit expressions for the relevant bulk stress tensors.

The contribution of electromagnetic and matter fields to the stress tensor above is given by

\[ S^{em}_{ab} = T^{em}_{ab} - \left( \frac{T^{em}_{aa}}{3} \right) g_{ab} \]

\[ S^{sc}_{ab} = T^{sc}_{ab} - \left( \frac{T^{sc}_{aa}}{3} \right) g_{ab} \]  

(B.11)

where we have used the fact that the scalar field is real in the last equation.

Using the fact that all components of the gauge field, metric and scalar field are functions of \( r \) only, one can simplify those components of the Maxwell equations where the free index is the boundary direction.

\[ \nabla_a [V_1(\phi \phi^*) F^{a\mu}] = -\frac{i V_2(\phi \phi^*)}{4} \left[ \phi \tilde{D}^\mu \phi^* - \phi^* D^\mu \phi \right] \]

\[ \Rightarrow \partial_r [\sqrt{g} V_1(\phi) F^{r\mu}] = \frac{\sqrt{g}}{2} A^\mu V_2(\phi) \phi^2 \]  

(B.12)

We now find explicit expressions (in terms of the functions that appear in our ansatz for the metric and gauge field) for the components of \( S^{em}_{ab} \) that we will need in the sequel. Using the equation (B.12).

\[ \sqrt{g} [S^{em}]^v_v = -\frac{2}{e^2} \sqrt{g} V_1(\phi) \left[ \frac{2}{3} (\partial_r A_v) F^{v\mu} - \frac{1}{3} (\partial_r A_x) F^{x\mu} \right] \]

\[ = -\frac{2}{e^2} \partial_r \left[ \sqrt{g} V_1(\phi) \left( \frac{2}{3} A_v F^{v\mu} - \frac{1}{3} A_x F^{x\mu} \right) \right] \]

\[ + \frac{2}{e^2} \left[ \frac{2}{3} A_v \partial_r (\sqrt{g} V_1(\phi) F^{v\mu}) - \frac{2}{3} A_x \partial_r (\sqrt{g} V_1(\phi) F^{x\mu}) \right] \]

\[ = -\frac{2}{e^2} \partial_r \left[ \sqrt{g} V_1(\phi) \left( \frac{2}{3} A_v F^{v\mu} - \frac{1}{3} A_x F^{x\mu} \right) \right] \]

\[ - \frac{\sqrt{g} \phi^2 V_2(\phi)}{e^2} \left[ \frac{2}{3} A_v A^v - \frac{A_x A^x}{3} \right] \]

(B.13)
Similarly

\[
\sqrt{g} [S_{em}]^x = - \frac{2}{e^2} \sqrt{g} V_1(\phi) \left[ \frac{2}{3} (\partial_t A_x) F^{xx} - \frac{1}{3} (\partial_t A_v) F^{vv} \right] \\
= - \frac{2}{e^2} \partial_t \left[ \sqrt{g} V_1(\phi) \left( \frac{2}{3} A_x F^{xx} - \frac{1}{3} A_v F^{vv} \right) \right] \\
- \frac{\sqrt{g} \phi^2 V_2(\phi)}{e^2} \left[ \frac{2}{3} A_x A^x - A_v A^v \right] 
\]

(B.14)

\[
\sqrt{g} [S_{em}]^y = \frac{2}{3} e^2 \left[ \frac{\sqrt{g} V_1(\phi)}{e^2} \right] \left[ (\partial_t A_x) F^{xx} + (\partial_t A_v) F^{vv} \right] \\
= \frac{2}{(3)e^2} \partial_t \left[ \sqrt{g} V_1(\phi) (A_x F^{xx} + A_v F^{vv}) \right] \\
+ \frac{\sqrt{g} \phi^2 V_2(\phi)}{3e^2} \left[ A_x A^x + A_v A^v \right] 
\]

(B.15)

\[
\sqrt{g} [S_{em}]^z = \frac{2}{e^2} V_2(\phi) \left[ (\partial_t A_v) F^{xx} \right] \\
= \frac{2}{e^2} \partial_t \left[ \sqrt{g} V_1(\phi) A_v F^{xx} \right] - \frac{\sqrt{g} V_2(\phi) \phi^2}{e^2} A^x A_v 
\]

(B.16)

In a similar manner we now find explicit expressions for the same components of \( S^a_{ab} \)

\[
\sqrt{g} [S_{ec}]^x = \frac{\sqrt{g}}{e^2} \left( V_2(\phi) A^x A_v \phi^2 + \frac{V_3(\phi)}{3} \right) \\
\sqrt{g} [S_{ec}]^y = \frac{\sqrt{g}}{e^2} \left( V_2(\phi) A^x A_v \phi^2 + \frac{V_3(\phi)}{3} \right) \\
\sqrt{g} [S_{ec}]^z = \frac{\sqrt{g} \phi^2 V_2(\phi)}{(3)e^2} \left[ A^x A_v \right] 
\]

(B.17)

With these expressions in hand it is not difficult to determine three linear combinations of the equations appearing in \( B.10 \) so that the RHS of of these equations is a total derivative. We find
\[ \sqrt{g} \left( R^v_v - R^y_y \right) \]
\[ = \sqrt{g} \left( \left[ S^{\text{sem}} \right]_{v}^v + \left[ S^{\text{sc}} \right]_{v}^v - \left[ S^{\text{sem}} \right]_{y}^y - \left[ S^{\text{sc}} \right]_{y}^y \right) \]
\[ = - \frac{2}{e^2} \partial_r \left[ A_v \left( \sqrt{g} V_1(\phi) F^{v v} \right) \right] \]
\[ \sqrt{g} \left( R^x_x - R^y_y \right) \]
\[ = \sqrt{g} \left( \left[ S^{\text{sem}} \right]_{x}^x + \left[ S^{\text{sc}} \right]_{x}^x - \left[ S^{\text{sem}} \right]_{y}^y - \left[ S^{\text{sc}} \right]_{y}^y \right) \]
\[ = - \frac{2}{e^2} \partial_r \left( A_x \left( \sqrt{g} V_1(\phi) F^{x x} \right) \right) \]

\[ \sqrt{g} R^x_v \]
\[ = \sqrt{g} \left( \left[ S^{\text{sem}} \right]_{v}^x + \left[ S^{\text{sc}} \right]_{v}^x \right) \]
\[ = - \frac{2}{e^2} \partial_r \left( A_v \left( \sqrt{g} V_1(\phi) F^{v x} \right) \right) \]

\textbf{B.2.3 Integrating the Total Derivatives}

We now plug \((B.18)\) into \((B.9)\) to obtain three total derivatives that vanish onshell. We now integrate these expressions from the horizon to infinity to obtain

\[
\left[ \frac{r [ k(r) f(r) + j^2(r) ]}{g(r) \sqrt{k(r)}} - \frac{r^2 [ k(r) f'(r) + j(r) j'(r) ]}{2g(r) \sqrt{k(r)}} \right]_{r=1}^{r=\infty} = - \frac{2}{e^2} \left[ A_v \left( \sqrt{g} V_1(\phi) F^{v v} \right) \right]_{r=1}^{r=\infty} \tag{B.19}
\]

and

\[
\left[ \frac{r [ k(r) f(r) + j^2(r) ]}{g(r) \sqrt{k(r)}} - \frac{r^2 [ f(r) k'(r) + j(r) j'(r) ]}{2g(r) \sqrt{k(r)}} \right]_{r=1}^{r=\infty} = - \frac{2}{e^2} \left[ A_x \left( \sqrt{g} V_1(\phi) F^{x x} \right) \right]_{r=1}^{r=\infty} \tag{B.20}
\]

and

\[
\left[ \frac{r^2 [ j(r) f'(r) - f(r) j'(r) ]}{2g(r) \sqrt{k(r)}} \right]_{r=1}^{r=\infty} = \frac{2}{e^2} \left[ A_x \left( \sqrt{g} V_1(\phi) F^{x x} \right) \right]_{r=1}^{r=\infty} \tag{B.21}
\]

The asymptotic expansion \((B.40)\) and \((B.7)\) at infinity and the fact that

\[ f(r = 1) = j(r = 1) = 0 \]
at the horizon allow us to simplify the LHSs of (B.19), (B.20) and (B.21) as follows:

\[
\begin{align*}
\left[ r[k(r)f(r) + j^2(r)] - \frac{r^2[k(r)f'(r) + j(r)j'(r)]}{2g(r)\sqrt{k(r)}} \right]_{r=\infty} &= 2f_2 + \frac{\sqrt{k(1)f'(1)}}{2g(1)} \\
\left[ \frac{r[k(r)f(r) + j^2(r)]}{g(r)\sqrt{k(r)}} - \frac{r^2[j(r)f'(r) + j(r)j'(r)]}{2g(r)\sqrt{k(r)}} \right]_{r=\infty} &= 2k_2 \quad \text{(B.22)} \\
\left[ \frac{r^2[j(r)f'(r) - f(r)j'(r)]}{2g(r)\sqrt{k(r)}} \right]_{r=1} &= 2j_2
\end{align*}
\]

The RHSs of (B.19), (B.20) and (B.21) may also be simplified upon noting that according to the prescription of AdS/CFT the boundary current \( J^\mu \) is given by

\[
J^\mu = \frac{1}{16\pi G} \lim_{r \to \infty} \left( \frac{4}{\epsilon^2} \sqrt{g} V_1(\phi) F^{\mu r} \right), \quad \xi_\mu = \lim_{r \to \infty} g_{\mu a} A^a
\]

and also that

\[
A_v(r = 1) = - [f(r)H(r) + j(r)L(r)]_{r=1} = 0
\]

(because \( f \) and \( j \) vanish at the horizon) and that

\[
F^{x r}(r = 1) = [\nabla^r A^x]_{r=1} = [g^{x r} \nabla_r A^x]_{r=1} = 0
\]

because \( g^{x r}(r = 1) = 0 \). Using these relations the RHSs of those three equations may be simplified as follows:

\[
- \frac{2}{\epsilon^2} [A_v(\sqrt{g} V_1(\phi) F^{x r})]_{r=1} = - \frac{16\pi G}{2} \xi_v J^v = \frac{16\pi G}{2} \xi_v J_v
\]

\[
- \frac{2}{\epsilon^2} [A_x(\sqrt{g} V_1(\phi) F^{x r})]_{r=1} = - \frac{16\pi G}{2} \xi_x J^r = - \frac{16\pi G}{2} \xi_x J_x
\]

\[
- \frac{2}{\epsilon^2} [A_v(\sqrt{g} V_1(\phi) F^{x r})]_{r=1} = - \frac{16\pi G}{2} \xi_v J^r = - \frac{16\pi G}{2} \xi_v J_x
\]

Finally plugging the simplifications (B.22) and (B.23) into (B.19), (B.20) and (B.21) obtain the final result of this subsubsection

\[
\begin{align*}
f_2 &= - \frac{\sqrt{k(1)f'(1)}}{4g(1)} + \frac{16\pi G}{4} \xi_v J_v \\
k_2 &= - \frac{16\pi G}{4} \xi_x J_x \quad \text{(B.24)} \\
j_2 &= - \frac{16\pi G}{4} \xi_v J_x
\end{align*}
\]
B.2.4 Constraints on boundary stress tensor and current etc

Let us first use the identities in (B.24) of the previous subsubsection to demonstrate that the boundary stress tensor dual to our gravitational solution takes the form

\[ T_{\mu\nu} = (\rho n + P) \ u_\mu u_\nu + \rho_s (u_s)_\mu (u_s)_\nu + P \eta_{\mu\nu} \]  

(B.25)

In other words, \( u^\mu \), defined in this section as the killing vector that reduces to the black hole horizon, is also the normal fluid velocity, according to our definition of the normal fluid. In order to check the absence of a normal fluid super fluid cross term in (B.25) it is necessary and sufficient, in our coordinates, to check that

\[ \frac{T_{xx} - T_{yy}}{T_{vx}} = \frac{\xi_x}{\xi_v} \]  

(B.26)

Now using the usual definition of the stress tensor we find

\[ T_{vv} = \frac{1}{16\pi G} \left( -3f_2 + 6g_4 + 4k_2 + \frac{\phi^2}{e^2} \right) \]
\[ T_{vx} = \frac{1}{16\pi G} (-4j_2) \]
\[ T_{xx} = \frac{1}{16\pi G} \left( -f_2 - 6g_4 + \frac{\phi^2}{e^2} \right) \]
\[ T_{yy} = T_{zz} = \frac{1}{16\pi G} \left( -f_2 - 6g_4 - 4k_2 - \frac{\phi^2}{e^2} \right) \]  

(B.27)

Using these relations, the LHS of (B.26) reduces to \( \frac{k_2}{j_2} \). Using the last two equations in (B.24) then immediately yields (B.26).

Let us now demonstrate the first relation in (2.2). In order to do this we note that the horizon area of our solution is equal to \( \sqrt{k(1)} \) and so that the entropy density of our solution is given by \( \frac{\sqrt{k(1)}}{4G} \). The periodicity of the Euclidean time circle in our circle is equal to \( \beta = \frac{1}{T} = 4\pi \left( \frac{g(1)}{f(1)} \right) \). It follows that

\[ T_S = \frac{1}{16\pi G} \frac{\sqrt{k(1)} f'(1)}{g(1)} \]  

(B.28)

All other thermodynamical charges may be evaluated from the stress tensor as follows. In our coordinates

\[ \rho_s = T_{vx} \left( \frac{\xi^2}{\xi_v \xi_x} \right) \]
\[ \rho_n + P = T_{vv} + T_{yy} - T_{vx} \left( \frac{\xi_v}{\xi_x} \right) \]
\[ q_s = -J_x \left( \frac{\xi}{\xi_v} \right) \]
\[ q_n = - \left[ J_v - J_x \left( \frac{\xi_v}{\xi_x} \right) \right] \]  

(B.29)

\[ ^{24} \text{In order to obtain these relations we worked, for concreteness, with a scalar field is dual to an operator of dimension } 2, \text{ i.e. one whose normalizable solutions look like} \]
\[ \lim_{r \to \infty} \phi(r) = \frac{\phi^2}{r^2} + O(r^{-4}). \]

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Substituting (B.27) in the first two equations of (B.29) and then using (B.24) and the last two equations of (B.29) one finds the following relations
\[
16\pi G(P + \rho_n) = \frac{\sqrt{k(1)f'(1)}}{g(1)} + 16\pi G\xi vq_n = \frac{\sqrt{k(1)f'(1)}}{g(1)} + 16\pi G(\xi u^\mu)q_n \tag{B.30}
\]
\[
16\pi G\rho_s = 16\pi G\xi q_s
\]
Finally using (B.28) these two equations turn into
\[
P + \rho_n = T_s + \xi u^\mu q_n \tag{B.31}
\]
\[
\rho_s = \mu_s q_s
\]
the two Smarr-Gibbs-Duhem relations we set out to prove.

### B.3 The on-shell action

In this subsection we will demonstrate that the onshell bulk action of our solution is the negative of its pressure.

The bulk stress-tensor appearing in the Einstein equation that follows from the action in (5.1) is given by
\[
T_{ab} = T_{ab}^{em} + T_{ab}^{sc} + 6g_{ab}, \tag{B.32}
\]
where
\[
T_{ab}^{em} = -\frac{2V_1(\phi\phi^*)}{e^2} \left( F_{ac}F^c{}_b + \frac{1}{4}g_{ab}F^{c_1c_2}F_{c_1c_2} \right)
\]
\[
T_{ab}^{sc} = \frac{V_2(\phi\phi^*)}{e^2} \left( \frac{D_a\phi D_b\phi^* + D_a\phi^* D_b\phi}{2} - \frac{1}{2}g_{ab}D\phi D\phi^* \right) - \frac{V_3(\phi\phi^*)}{2e^2}g_{ab} \tag{B.33}
\]

We consider solution of the form (5.5). Then it follows that the $yy$-component of the stress-tensor is given by
\[
T_{yy} = \frac{1}{2} g_{yy} (\mathcal{L} - \mathcal{R}) \tag{B.34}
\]
In deriving this relation we have crucially used the fact that the term $D_y\phi D_y\phi$ in $T_{mat}$ and $F_{yc}F^c{}_y$ in $T_{max}$ is zero. This is because $\phi$ and the non-zero components of the gauge field is not a function of the coordinate $y$ and the $y$-component of the gauge field is zero in our chosen form of the solution. Also we recall the identity for the Einstein Tensor $E_{\mu\nu}$,
\[
E_{\mu}^{\mu} = -\frac{3}{2}\mathcal{R}.
\]
Using the above identity and the Einstein equation $T_{yy} = E_{yy}$, we have from (B.34)
\[
\mathcal{L} = 2\frac{E_{yy}}{g_{yy}} - \frac{2}{3}E_{\mu}^{\mu}. \tag{B.35}
\]
Now for the metric ansatz in (5.5) this expression reduces to
\[
\mathcal{L} = \frac{1}{\sqrt{-g}} \frac{d}{dr} \left( \frac{\sqrt{-g}}{r} g^{rr} \right). \tag{B.36}
\]
With the help of this relation our onshell action reduces to an integration of a total derivative

\[ S_{OS} = \frac{1}{16\pi G} \int d^5x \frac{d}{dr} \left( \frac{2}{r} \sqrt{-g} g^{rr} \right) \]  

(B.37)

Hence, after performing the above integration we get surface terms from the horizon and boundary. Since \( g^{rr} \) is zero at the horizon only the boundary term contributes and we have

\[ S_{OS} = \frac{\text{Vol}_4}{16\pi G} \left( \frac{2}{r} \sqrt{-g} g^{rr} \right) \bigg|_{\text{bdy}}. \]  

(B.38)

In order to avoid the singularities near the boundary we have to add the required counterterms to the on shell action

\[ S_{CT} = \frac{2}{16\pi G} \int_{\text{bdy}} \sqrt{-\gamma} \left( K - 3 + \lambda \phi^2 \right), \]  

(B.39)

where \( \gamma \) and \( K \) are respectively the induced metric and extrinsic curvature of a constant \( r \) surface and \( \nu \phi^2 \) is the mass counterterm for the scalar field. This mass counterterm is used to cancel any boundary divergence of the scalar field. In this subsection we keep it arbitrary as our argument here is independent of the specific value of \( \nu \); we assume the scalar field to be normalizable.

Let us now consider the following asymptotic expansion of the functions that appear in the metric and the scalar field

\[
\lim_{r \to \infty} g(r) = 1 + \frac{g_4}{r^4} + \mathcal{O}(r^{-6}) \\
\lim_{r \to \infty} f(r) = r^2 + \frac{f_2}{r^2} + \mathcal{O}(r^{-4}) \\
\lim_{r \to \infty} k(r) = r^2 + \frac{k_2}{r^2} + \mathcal{O}(r^{-4}) \\
\lim_{r \to \infty} j(r) = \frac{j_2}{r^2} + \mathcal{O}(r^{-4}) \\
\lim_{r \to \infty} \phi(r) = \frac{\phi_2}{r^2} + \mathcal{O}(r^{-3})
\]  

(B.40)

With these expansions we find that renormalized on shell action evaluates to

\[ S_{ROS} = S_{OS} - S_{CT} = \frac{1}{16\pi G} \left( f_2 + 6 \ g_4 + 4 \ k_2 - 2 \ \lambda \ \phi_2^2 \right) \]  

(B.41)

Now the boundary stress-tensor is given by

\[ T^{(\text{bdy})}_{\mu\nu} = -\frac{1}{8\pi G} \lim_{r \to \infty} r^4 \left( K_{\nu}^{\mu} - K_{\nu}^{\mu} + 3 \delta_{\nu}^{\mu} - \frac{\lambda \phi^2}{2} \delta_{\nu}^{\mu} \right) \]  

(B.42)

In the fluid dynamic limit the \( yy \)-component of this boundary stress tensor yields the pressure of the fluid. Plugging in the metric ansatz (5.5) into (B.42) we have

\[ T^{(\text{bdy})}_{yy} \equiv P = -\frac{1}{16\pi G} \left( f_2 + 6 \ g_4 + 4 \ k_2 - 2 \ \lambda \ \phi_2^2 \right) \]  

(B.43)

Hence, comparing (B.43) and (B.41) we have

\[ S_{ROS} = -(\text{Vol}_4) \ P. \]

which implies that the renormalized on-shell action density is given by \(-P\), as we set out to prove \(^{25}\).

\(^{25}\)Here \( \lambda \) is some constant fixed by the requirement that the limit in (B.42) is finite even when the non normalizable mode of the bulk scalar field is turned on. But when the scalar field is normalizable the limit exists for any constant \( \lambda \) and in such cases the identity, proved here, is true irrespective of the value of \( \lambda \).
C. Instabilities at large superfluid velocities

In this section we study the linearized equations of superfluid dynamics in Herzog’s model in a very simple context. First we restrict attention to the strict probe limit $\epsilon \to \infty$. In this limit the only dynamical variables are $\xi_\mu$ and $\epsilon$. Next we restrict attention to perfect fluids. Finally we work at leading order in the $\epsilon$ expansion.

We assume that $\partial_0 \phi = \mu(\epsilon)$ and $\partial_\nu \phi = \zeta$ where $\phi$ is the superfluid phase. It is consistent, to linear order, to truncate to this special form (where we retain only two of the variables of fluid dynamics) if we also assume that $\epsilon = \epsilon(x,t)$ and $\zeta = \zeta(x,t)$. More specifically we set

$$\epsilon = \epsilon_0 + \delta \epsilon e^{i\omega t + ikx}$$
$$\zeta = \zeta_0 + \delta \zeta e^{i\omega t + ikx} \quad (C.1)$$

In order to obtain the dispersion relation for our fluctuations we need to solve the equations

$$\partial_\mu J^\mu = 0, \quad \partial_\mu \xi_\nu - \partial_\nu \xi_\mu = 0 \quad (C.2)$$

where the second equation is nontrivial only when we choose $(\mu, \nu)$ to be $(0,1)$. In the first equation $J^\mu$ is given by (6.17) retaining only terms of quadratic order in smallness of $\epsilon$ and $\zeta$, assuming that the two quantities scale in the same way. In other words the terms in the current that we have neglected are of the order $O(\epsilon^3, \zeta^3, \epsilon^2 \zeta, \zeta^2 \epsilon)$.

The equations of motion have a solution of the form (C.1) only when the following ‘characteristic’ equation is obeyed

$$\begin{pmatrix}
\frac{2\omega k}{12} & \frac{k \xi_0 + \omega}{12} \\
\frac{6\epsilon_0 k \xi_0 + 7\epsilon_0 \omega}{24k} & 3\epsilon_0 \omega + 24\epsilon_0 \omega
\end{pmatrix}
\begin{pmatrix}
\delta \epsilon \\
\delta \zeta
\end{pmatrix} = 0. \quad (C.3)$$

This equation yields the dispersion relation

$$\omega = -\frac{6}{7} k \xi_0 \pm \frac{k}{28} \sqrt{144 \epsilon_0^2 - 96 \xi_0^2}$$

Notice that $\omega$ develops an imaginary piece when $\frac{\omega \epsilon_0}{\xi_0} \geq \sqrt{\frac{7}{48}} = \frac{1}{4} \sqrt{\frac{7}{2}}$ demonstrating that the system we study has an instability when this inequality is satisfied.

At the point of onset of instability the eigenvector that corresponds to the zero eigenvalue is given by

$$\begin{pmatrix}
\delta \epsilon \\
\delta \zeta
\end{pmatrix} = \begin{pmatrix}
1 \\
-\frac{1}{2} \sqrt{\frac{7}{3}}
\end{pmatrix} \quad (C.4)$$

D. Limit $\zeta \to 0$

As we have explained in the main text, the metric and gauge field dual to a super fluid flow must take the simplified form (7.21) in the limit $\zeta \to 0$. In this Appendix we will explicitly verify that the results for our metric and gauge field at nonzero $\zeta$ reduce to this rotationally invariant form when $\zeta$ is set to zero, and read off all the unknown functions of radius in (7.21).

Let us start with a metric and gauge field of the form (7.21) and rewrite it in the language of our metric and gauge field at nonzero $\zeta$. That is we set $\xi^\mu = \zeta n^\mu$ where $n^\mu$ is a unit vector orthogonal
to \( u^\mu \), and eventually set \( \zeta \) to zero. The scalar vector and the tensor quantities defined above (7.21) become

\[
\begin{align*}
\partial_\mu \zeta_\nu &= n_\nu \partial_\mu \zeta \\
P^{\mu \nu} \partial_\mu \zeta_\nu &= n^\nu \partial_\mu \zeta = S_1 \\
\sigma^{(\xi)}_{\mu \nu} &= \frac{2S_1}{3} \left( n_\mu n_\nu - \frac{\bar{P}_{\mu \nu}}{2} \right) + \frac{n_\nu [V_2]_\mu + n_\mu [V_2]_\nu}{2} \\
\partial_\mu u^\mu &= S_4 + S_6 \\
u \partial u_\mu &= S_3 n_\mu + [V_4]_\mu \\
\sigma_{\mu \nu} &= \frac{2S_4 - S_6}{3} \left( n_\mu n_\nu - \frac{\bar{P}_{\mu \nu}}{2} \right) + \frac{n_\nu [V_5]_\mu + n_\mu [V_5]_\nu + \left[ T_1 \right]_{\mu \nu}}{2}
\end{align*}
\]

Plugging these expressions into (7.21) we find

\[
d^2 s^2 = -2g \left( \frac{r}{r_c} \right) u^\mu dx^\mu dr + \left[ -r_c^2 \left( \frac{r}{r_c} \right) u_\mu u_\nu + r^2 P_{\mu \nu} \right] dx^\mu dx^\nu \\
+ r_c F \left( \frac{r}{r_c} \right) \sigma_{\mu \nu} dx^\mu dx^\nu \\
+ \frac{1}{e^2} \left\{ -\frac{2}{r_c} \left[ G_1 \left( \frac{r}{r_c} \right) (S_4 + S_6) + G_2 \left( \frac{r}{r_c} \right) S_1 \right] u_\mu dx^\mu dr \\
+ r_c \left[ F_1 \left( \frac{r}{r_c} \right) (S_4 + S_6) + F_2 \left( \frac{r}{r_c} \right) S_1 \right] u_\mu u_\nu dx^\mu dx^\nu \\
+ r_c \left[ V_1 \left( \frac{r}{r_c} \right) S_3 + V_2 \left( \frac{r}{r_c} \right) S_2 \right] u_\mu n_\nu dx^\mu dx^\nu \\
+ r_c \left[ \frac{2T_2 \left( \frac{r}{r_c} \right)}{3} S_1 + \frac{T_1 \left( \frac{r}{r_c} \right)}{3} (2S_4 - S_6) \right] \left[ n_\mu n_\nu - \frac{\bar{P}_{\mu \nu}}{2} \right] dx^\mu dx^\nu \\
+ r_c u_\mu \left[ V_1 \left( \frac{r}{r_c} \right) [V_4]_\nu + V_2 \left( \frac{r}{r_c} \right) [V_1]_\nu \right] dx^\mu dx^\nu \\
+ r_c n_\mu \left[ T_1 \left( \frac{r}{r_c} \right) [V_5]_\nu + T_2 \left( \frac{r}{r_c} \right) [V_2]_\nu \right] dx^\mu dx^\nu + \frac{r_c}{2} T_1 \left( \frac{r}{r_c} \right) \left[ T_1 \right]_{\mu \nu} dx^\mu dx^\nu \right\} + \mathcal{O} \left( \frac{1}{e^4} \right)
\]

\[
A = \frac{1}{r_c} H \left( \frac{r}{r_c} \right) u^\mu \partial_\mu \\
+ \left[ A_1 \left( \frac{r}{r_c} \right) (S_4 + S_6) + A_2 \left( \frac{r}{r_c} \right) S_1 \right] \partial_r \\
+ \frac{1}{r_c} \left[ H_1 \left( \frac{r}{r_c} \right) (S_4 + S_6) + H_2 \left( \frac{r}{r_c} \right) S_1 \right] u^\mu \partial_\mu \\
+ \frac{1}{r_c} \left[ L_1 \left( \frac{r}{r_c} \right) S_3 + L_2 \left( \frac{r}{r_c} \right) S_2 \right] u^\mu \partial_\mu + \frac{1}{r_c} \left[ L_1 \left( \frac{r}{r_c} \right) [V_4]_\mu + L_2 \left( \frac{r}{r_c} \right) [V_1]_\mu \right] \partial_\mu \\
+ \mathcal{O} \left( \frac{1}{e^2} \right)
\]
We would now like to compare these expressions with the $\zeta \to 0$ limit of the metric and gauge field at nonzero $\zeta$. There is, however, an immediate complication. The metric in (D.2) is not in the gauge employed in our computation at nonzero $\zeta$. In that more general calculation we have chosen a gauge such that the coefficient of $\tilde{P}_{\mu \nu} dx^\mu dx^\nu$ term in the metric is $r^2$. But in (D.2) the coefficient of $\tilde{P}_{\mu \nu} dx^\mu dx^\nu$ term is given by $r^2 + \frac{h(r)}{e^2}$ where

$$h(r) = -\frac{r_c}{2} \left[ 2T_2 \left( \frac{\tilde{r}}{r_c} \right) S_1 + \frac{T_1}{3} \left( \frac{\tilde{r}}{r_c} \right) (2S_4 - S_6) \right]$$

So in order to compare (D.2) with the results at nonzero $\zeta$ we must first perform a coordinate redefinition up to first order in derivative. Define

$$\tilde{r}^2 = r^2 + \frac{h(r)}{e^2}$$

such that

$$r \approx \tilde{r} - \frac{h(\tilde{r})}{2e^2 \tilde{r}}$$

$$dr \approx d\tilde{r} \left[ 1 - \left( \frac{h(\tilde{r})}{2e^2 \tilde{r}} \right) \right]$$

Since this coordinate redefinition has a factor of $\frac{1}{e^2}$ it does not affect the gauge field as the gauge field is computed only up to order $O \left( \frac{1}{e^2} \right)$. However the coordinate transformation does affect the metric, though only in the scalar sector (i.e. in the coefficients of scalar terms), basically because the coordinate transformation performed above is a scalar operation. Once this shift of $r$ coordinate is implemented new terms are generated from the expansion of the uncharged black-brane metric (the metric at order $O \left( \frac{1}{e^2} \right)$).

After performing the relevant coordinate transformation, the metric takes the following form

$$ds^2 = -2g \left( \frac{\tilde{r}}{r_c} \right) u^\mu dx^\mu dr + \left[ -r_c^2 f \left( \frac{\tilde{r}}{r_c} \right) u_\mu u_\nu + \tilde{r}^2 \tilde{P}_{\mu \nu} \right] dx^\mu dx^\nu$$

$$- \left[ 2ru_\mu \left( u, \partial_\mu \right) u_\nu - \frac{1}{3} \left( \partial_\mu u \right) u_\nu \right] + r_c F \left( \frac{\tilde{r}}{r_c} \right) \sigma_{\mu \nu} \right] dx^\mu dx^\nu$$

$$+ \frac{1}{e^2} \left\{ - \frac{2}{r_c} \left[ G_1 \left( \frac{\tilde{r}}{r_c} \right) (S_4 + S_6) + G_2 \left( \frac{\tilde{r}}{r_c} \right) S_1 - r_c \left( \frac{h(\tilde{r})}{2\tilde{r}} \right) \right] u_\mu u_\nu \right\}$$

$$+ \frac{1}{r_c} \left[ F_1 \left( \frac{\tilde{r}}{r_c} \right) (S_4 + S_6) + F_2 \left( \frac{\tilde{r}}{r_c} \right) S_1 + \frac{h(\tilde{r})}{r_c} \left( 1 + \frac{r_c^2}{\tilde{r}^2} \right) \right] u_\mu u_\nu dx^\mu dx^\nu$$

$$+ \frac{1}{r_c} \left[ V_1 \left( \frac{\tilde{r}}{r_c} \right) S_3 + V_2 \left( \frac{\tilde{r}}{r_c} \right) S_2 \right] u_\mu n_\nu dx^\mu dx^\nu$$

$$+ \frac{1}{r_c} \left[ \frac{2T_2}{3} \left( \frac{\tilde{r}}{r_c} \right) S_1 + \frac{T_1}{3} \left( \frac{\tilde{r}}{r_c} \right) (2S_4 - S_6) - \frac{h(\tilde{r})}{r_c} \right] n_\mu n_\nu dx^\mu dx^\nu$$

$$+ \frac{1}{r_c} \left[ V_1 \left( \frac{\tilde{r}}{r_c} \right) [V_1]_\mu + V_2 \left( \frac{\tilde{r}}{r_c} \right) [V_1]_\mu \right] dx^\mu dx^\nu$$

$$+ \frac{1}{r_c} \left[ T_1 \left( \frac{\tilde{r}}{r_c} \right) [V_2]_\mu + T_2 \left( \frac{\tilde{r}}{r_c} \right) [V_2]_\mu \right] dx^\mu dx^\nu + \frac{r_c}{2} \frac{T_1}{r_c} \left( \frac{\tilde{r}}{r_c} \right) [T_1]_{\mu \nu} dx^\mu dx^\nu + O \left( \frac{1}{e^4} \right)$$
Now this metric matches the $\zeta \to 0$ limit of the metric and gauge field in this paper provided, among other conditions, that

The constraints for the metric:

\[
\begin{align*}
\lim_{\chi \to 0} \delta g_2(r) &= \lim_{\chi \to 0} \delta g_3(r) = \lim_{\chi \to 0} \delta g_5(r) = 0 \\
\lim_{\chi \to 0} \delta f_2(r) &= \lim_{\chi \to 0} \delta f_3(r) = \lim_{\chi \to 0} \delta f_5(r) = 0 \\
\lim_{\chi \to 0} \delta K_2(r) &= \lim_{\chi \to 0} \delta K_3(r) = \lim_{\chi \to 0} \delta K_5(r) = 0 \\
\lim_{\chi \to 0} \delta J_1(r) &= \lim_{\chi \to 0} \delta J_4(r) = \lim_{\chi \to 0} \delta J_5(r) = \lim_{\chi \to 0} \delta J_6(r) = 0 \\
\lim_{\chi \to 0} \delta J_3(r) &= \lim_{\chi \to 0} Y_4(r) = \frac{1}{2} V_1(r) \\
\lim_{\chi \to 0} \delta J_2(r) &= \lim_{\chi \to 0} Y_1(r) = \frac{1}{2} V_2(r) \\
\lim_{\chi \to 0} \delta K_6(r) &= - \lim_{\chi \to 0} \frac{\delta K_4(r)}{2} = \lim_{\chi \to 0} \delta Z_1(r) = - \lim_{\chi \to 0} \delta W_5(r) = - \frac{1}{2} T_1(r) \\
\lim_{\chi \to 0} \delta K_1(r) &= 2 \lim_{\chi \to 0} \delta W_2(r) = T_2(r)
\end{align*}
\]  

(D.5)

The constraints for the gauge field

\[
\begin{align*}
\lim_{\chi \to 0} \delta A_4(r) &= \lim_{\chi \to 0} \delta A_6(r) = A_1(r) \\
\lim_{\chi \to 0} \delta H_4(r) &= \lim_{\chi \to 0} \delta H_6(r) = H_1(r) \\
\lim_{\chi \to 0} \delta L_3(r) &= \lim_{\chi \to 0} \delta X_4(r) = L_1(r) \\
\lim_{\chi \to 0} \delta L_2(r) &= \lim_{\chi \to 0} \delta X_1(r) = L_2(r)
\end{align*}
\]  

(D.6)

It turns out all these constraints (and all others that are required to ensure that the bulk metric is an analytic function of the vector field $w^\mu$ in the $\zeta \to 0$ limit) are true up to the order the metric is calculated and using them one can read off $V_1(r), V_2(r), T_1(r)$ and $T_2(r)$ and hence the gauge transformation parameter $h(r)$. Similarly the functions $G_1(r), G_2(r), F_1(r)$ and $F_2(r)$ appearing in $u_\mu dx^\mu dr$ and $u_\mu u_\nu dx^\mu dx^\nu$ terms are related to the corresponding functions appearing in the metric at nonzero $\zeta$. These relations also imply some constraint on the $\chi \to 0$ limit. But they are a bit complicated to write because they involve the function $h(r)$ and its derivative with respect to $r$. Once $V_1(r), V_2(r), T_1(r)$ and $T_2(r)$ are all fixed, using them it becomes easier read off $G_1(r), G_2(r), F_1(r)$ and $F_2(r)$.

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