Sharp estimates for the hypergeometric functions related to root systems of type $A$ and of rank 1
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ABSTRACT

In this article, we conjecture exact estimates for the Weyl-invariant Opdam-Cherednik hypergeometric functions. We prove the conjecture for the root system $A_n$ and for all rank 1 cases. We provide other evidence that the conjecture might be true in general.

1. Introduction and Conjecture

1.1. Basics on Opdam-Cherednik analysis. In Opdam-Cherednik analysis, the “curved” counterpart of Dunkl analysis for a root system $\Sigma$ on $\mathbb{R}^d$, a crucial role is played by the Opdam-Cherednik kernel $G_k(X,Y)$. Finding good estimates of the kernels $G_k$ is therefore important. In this paper we conjecture exact estimates of the $W$-radial Opdam-Cherednik kernel or hypergeometric functions related to root systems. We prove these estimates in the case of the root systems of type $A$ and for all rank one cases.

It is interesting to note that our “guess” for the behaviour of the hypergeometric functions related to the root system $A_n$ was completely informed by the rank one case $n = 1$. It is therefore encouraging for our conjecture that we were able to verify it for the other root system of rank 1, namely $BC_1$.

For a good introduction on Opdam-Cherednik theory, the reader should consider the paper [14] and the Lecture Notes [15] by Opdam (see also [1, 16]). We provide here some details and notations on Opdam-Cherednik analysis.

Let $\partial_\xi$ be the derivative in the direction of $\xi \in \mathbb{R}^d$. The Dunkl-Cherednik or Cherednik operators indexed by $\xi$ are then given by

$$D_\xi f(X) = \partial_\xi f(X) + \sum_{\alpha \in \Sigma^+} k_{\alpha} \alpha(\xi) \frac{f(X) - f(\sigma_\alpha X)}{1 - e^{-\alpha}} - \rho(k)(\xi) f(X),$$

where $\rho(k) = \sum_{\alpha \in \Sigma_+} k(\alpha)\alpha$. The $D_\xi$’s, $\xi \in \mathbb{R}^d$, form a commutative family.

For fixed $Y \in \mathbb{R}^d$, the kernel $G_k(\cdot, \cdot)$ is the only real-analytic solution to the system

$$D_\xi |_X G_k(X,Y) = \langle \xi, Y \rangle G_k(X,Y), \ \forall \xi \in \mathbb{R}^d$$

with $G_k(0,Y) = 1$. In fact, $G_k$ extends to a holomorphic function on $(\mathbb{R}^d + i U) \times \mathbb{C}^d$ where $U$ is a neighbourhood of 0 (refer to [13] Th. 3.15).

Its $W$-invariant version $G^W_k(X,\lambda)$ is called a hypergeometric function. One notes that in [1], the authors use the term “hypergeometric function” for the function $G_k(X,Y)$.

The (Weyl-invariant) hypergeometric functions related to root systems are the extension of the spherical functions $\phi_\lambda$ to arbitrary positive multiplicities. In this paper we use the latter terminology and notation. We have

$$\phi_\lambda(X) = G^W_k(X,\lambda) = \frac{1}{|W|} \sum_{w \in W} G_k(w \cdot X,\lambda)$$

and $\phi_\lambda(X)$ is the only real-analytic solution of the system

$$p(D_{e_1}, \ldots, D_{e_d})(k)|_X \phi_\lambda(X) = p(\lambda) \phi_\lambda(X), \ \forall \lambda \in \mathbb{R}^d$$

for every Weyl-invariant polynomial $p$ (here $e_1, \ldots, e_d$ represent the standard basis on $\mathbb{R}^d$).

Let $\omega_k(X) := \prod_{\alpha \in \Sigma^+} |\sinh(\alpha, X)|^{2k(\alpha)}$ be the Opdam-Cherednik weight function on $\mathbb{R}^d$. Recall that the Opdam-Cherednik transform of a $W$-invariant function $f$ on $\mathbb{R}^d$

$$\hat{f}(\lambda) := c_k^{-1} \int f(x) \phi_{-i\lambda}(X) \omega_k(X) dX, \ \lambda \in \mathbb{R}^d,$$

plays the role of the spherical Fourier transform in $W$-invariant Opdam-Cherednik analysis.
**1.2. Conjecture on sharp bound for the hypergeometric functions related to root systems.** The notation $f \asymp g$ in a domain $D$ means that there exists $C_1 > 0$ and $C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ with $C_1$ and $C_2$ independent of $x \in D$.

**Conjecture 1.1.** If $X, \lambda \in \overline{a^+}$, then we have for any root system

$$
\phi_{\lambda}(e^X) \asymp e^{(\lambda-\rho)(X)} \prod_{\alpha \in \Sigma^+} \frac{1 + \alpha(X)}{(1 + \alpha(\lambda) \alpha(X))^{k(\alpha) + k(2\alpha) - 1}} \left( \frac{1 + \alpha(\lambda) (1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k(\alpha) + k(2\alpha) - 1}
$$

where $\Sigma^+$ is the set of indivisible positive roots and $\Sigma^+_\lambda = \{ \alpha \in \Sigma^+: \alpha(\lambda) = 0 \}$.

**Remark 1.2.** According to Conjecture 1.1, we have

$$
\phi_{\lambda}(e^X) \asymp e^{(\lambda-\rho)(X)} \prod_{\alpha \in \Sigma^+} (1 + \alpha(X)) f_\alpha(\lambda, X)
$$

where the function

$$
f_\alpha(\lambda, X) = \frac{1}{(1 + \alpha(\lambda) \alpha(X))^{k(\alpha)}} \left( \frac{1 + \alpha(\lambda) (1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k-1}, \quad k = k(\alpha) + k(2\alpha),
$$

codifies in one $k$-rational function (i.e. a rational function with powers $k > 0$ non-necessarily an integer) four possible power function asymptotics:

$$
(1.1) \quad f_\alpha(\lambda, X) \asymp \begin{cases} 
1 & \text{if } \alpha_X \alpha_\lambda \leq 1, \\
1/(\alpha_X \alpha_\lambda)^k & \text{if } \alpha_X \alpha_\lambda \geq 1, \alpha_X \leq 1, \\
1/(\alpha_\lambda \alpha_X) & \text{if } \alpha_X \alpha_\lambda \geq 1, \alpha_X \geq 1, \alpha_\lambda \leq 1, \\
1/(\alpha_X^k \alpha_\lambda) & \text{if } \alpha_X \alpha_\lambda \geq 1, \alpha_X \geq 1, \alpha_\lambda \geq 1.
\end{cases}
$$

**Figure 1.** The four regions

The main result of this paper is the proof of Conjecture 1.1 in the $A_n$ case.

**Theorem 1.3.** If $X, \lambda \in \overline{a^+}$, then we have for the root systems of type $A$,

$$
\phi_{\lambda}(e^X) \asymp e^{(\lambda-\rho)(X)} \prod_{\alpha \in \Sigma^+} \frac{1 + \alpha(X)}{(1 + \alpha(\lambda) \alpha(X))^{k} \left( \frac{1 + \alpha(\lambda) (1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k-1}}.
$$

Naturally, this theorem is consistent with the results obtained in [7] for the complex case ($k = 1$).

**1.3. Outline of the paper.** In Section 2 we introduce notation and some results that will be useful to prove the various upper and lower estimates. The proof of Theorem 1.3 is found in Section 3. We conclude with Section 4 where we present other evidence for Conjecture 1.1 including the proof for the root system $BC_1$. In Section 4.1 we show that our conjecture is consistent with some known estimates [11].
2. Notation and technical results

**Notation 2.1.** We will write \( f(x) \lesssim g(x) \) \((f(x) \gtrsim g(x))\) for \( x \in D \) if there exists a constant \( C > 0 \) independent of \( x \) such that \( f(x) \leq C g(x) \) \((f(x) \geq C g(x))\) for all \( x \in D \).

In what follows, \( \Sigma_n^+ \) will be the set of positive roots of the root system \( A_n \).

We introduce here some technical results.

**Lemma 2.2.** Assume \( a \geq 0 \). Then for \( u \geq 0 \), the functions \( F_1, F_2, F_3, F_4 \) and \( F_5 \) defined by

\[
F_1(u) = \frac{u}{1 + a u}, \quad F_2(u) = \frac{u(1 + a (1 + u))^{-1}}{1 + a u}, \quad 0 < k \leq 1, \quad F_3(u) = \frac{u(1 + a (1 + u))}{1 + a u}
\]

are all increasing functions of \( u \).

**Proof.** It suffices to compute the derivatives \( F_1'(u), F_2'(u), F_3'(u), F_4'(u), F_5'(u) \) which are easily seen to be positive. \( \Box \)

**Lemma 2.3.** For \( k > 0 \) and \( x \geq 0 \), we have

\[
\int_0^x u^{k-1} e^{-u} du \asymp \left( \frac{x}{1 + x} \right)^k.
\]

**Proof.** The result is clearly true if \( 0 \leq x < 1 \) (use \( e^{-1} \leq e^{-x} \leq 1 \) and integrate). If \( x \geq 1 \) then

\[
\int_0^1 u^{k-1} e^{-u} du \leq \int_0^x u^{k-1} e^{-u} du < \int_0^\infty u^{k-1} e^{-u} du
\]

and the result follows. \( \Box \)

In a way, the next result contains the essence of the proof of Theorem [3.3] for the root system \( A_1 \).

**Lemma 2.4.** Suppose \( a \geq 0 \). For \( x \geq 0 \), we have

\[
\int_0^x e^{-a u} \left( \frac{u}{1 + u} \right)^{k-1} du \asymp \left( \frac{x}{1 + x} \right)^k \frac{1 + x}{1 + a} \frac{1 + a (1 + x)}{1 + a}.
\]

**Proof.** Let \( A \) represent the left hand side. If \( 0 \leq x \leq 2 \) then by Lemma [2.3] we have

\[
A \asymp \int_0^x e^{-a u} u^{k-1} du \asymp \left( \frac{x}{1 + a x} \right)^k
\]

which gives the result in that case.

If \( x \geq 2 \) then using Lemma [2.3] once more and the bound \( 1 - e^{-u} \asymp u/(1 + u) \) we have,

\[
A \asymp \int_0^1 e^{-a u} u^{k-1} du + \int_1^x e^{-a u} du \asymp \left( \frac{1}{1 + a} \right)^k + \frac{e^{-a} - e^{-a x}}{a} \asymp \left( \frac{1}{1 + a} \right)^k + e^{-a} \frac{x - 1}{1 + a (x - 1)} \asymp \left( \frac{1}{1 + a} \right)^k + e^{-a} \frac{x}{1 + a x}
\]

which gives the result in that case (consider separately \( 0 \leq a \leq 1 \) and \( a > 1 \)). \( \Box \)

3. Case \( A_n \). Proof of Theorem [1.3]

3.1. A recursive formula for spherical functions of type \( A_n \). The following result is an important tool of the proof of Theorem [1.3] (see for example [15]).

**Theorem 3.1.** For \( X \in a^+ \subset R^{n+1} \) and \( \lambda \in R^{n+1} \), we define \( \phi_\lambda(e^X) = e^{\lambda(X)} \) when \( n = 0 \) and, for \( n \geq 1 \),

\[
\phi_\lambda(e^X) = \frac{\Gamma(k (n + 1))}{\Gamma(k (n + 1))} e^{\lambda_{n+1} \sum_{j=1}^{n+1} x_j} \int_{E(X)} \phi_{\lambda_0}(e^Y) S^{(k)}(Y, X) d(Y)^{2 k} dY
\]
where \( E(X) = \{ Y = \text{diag}(y_1, \ldots, y_n) : x_{j+1} \leq y_j \leq x_j \} \), \( \lambda(X) = \sum_{j=1}^{n+1} \lambda_j x_j \), \( \lambda_0(Y) = \sum_{i=1}^{n} (\lambda_i - \lambda_{n+1}) y_i \), 
\( d(X) = \prod_{r<s} \sinh(x_r - x_s) \), \( d(Y) = \prod_{r<s} \sinh(y_r - y_s) \) and 
\[
S^{(k)}(Y, X) = d(X)^{1-2k} d(Y)^{1-2k} \left[ \prod_{r=1}^{n} \left( \prod_{s=r}^{r} \sinh(x_s - y_r) \prod_{s=r+1}^{n+1} \sinh(y_r - x_s) \right) \right]^{k-1}
\]

Then \( \phi_0 \) is the (Weyl-invariant) hypergeometric function for the root system \( A_n \).

### 3.2. An equivalent form of Theorem \[1.3\]

**Proposition 3.3.** Theorem \[1.3\] is equivalent to

\[
I(n) \cong \frac{\pi(X)^{2k-1}}{\prod_{\alpha \in \Sigma^+} (1 + \alpha(1 + \alpha(X)))^{k-1}} \prod_{\alpha \in \Sigma^+} (1 + \alpha(1 + \alpha(X)))^{k-1} \prod_{\alpha \in \Sigma^+} (1 + \alpha(1 + \alpha(X)))^{k-1}, \quad X \in \mathbb{R}^{n+1}, \quad \lambda, \alpha \in \mathbb{R}_+^n
\]

where, for \( \lambda, X \in \mathbb{R}^{n+1} \)

\[
I^{(n)}(\lambda, X) = \int_{x_{n+1}}^{x_1} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^{n} (\lambda_i - \lambda_{n+1}) (x_i - y_i)} P_n(\lambda_1, \ldots, n, Y) T_n(X, Y) dy_1 \cdots dy_n
\]

**Proof.** This follows using induction and from the fact that for \( x \geq 0 \), \( \sinh x \cong e^x x/(1+x) \) and some simplifications.

**Remark 3.4.** If we assume that \( \gamma = x_m - x_{m+1} \) is the largest positive root in \( X \) then we have either \( y_i - y_j \geq \gamma \) or \( y_i - y_j \leq \gamma \) for \( i < j \) (similarly for \( x_i - y_j, i \leq j \) and \( y_i - x_j, i < j \)). The proof of the estimate \[3.2\] will be done with the largest positive root \( \gamma \) fixed. Moreover, the following result will greatly simplify the proof of the estimate \[3.2\].

**Proposition 3.5.** Assume that \( \gamma = x_n - x_{n+1} \) is the largest positive root in \( X \) and let

\[
I_1 = \int_{M_n}^{x_n} \cdots \int_{x_2}^{x_1} e^{-\sum_{i=1}^{n} (\lambda_i - \lambda_{n+1}) (x_i - y_i)} P_n(\lambda_1, \ldots, n, Y) T_n(X, Y) dy_1 \cdots dy_n
\]

where \( M_n = (x_n + x_{n+1})/2 \). Then \( I_1 \cong I^{(n)} \), when \( X \in \mathbb{R}^{n+1} \).

**Proof.** Let \( I_2 = I^{(n)} - I_1 \). In \( I_1 \) and \( I_2 \), consider only the corresponding integral in \( y_n \), calling the resulting expressions \( I_1 \) and \( I_2 \). To prove the result, it suffices to show that \( I_2 \leq I_1 \).

Let \( Q_n = (3x_n + x_{n+1})/4 \). Observe that for \( y_n \in [M_n, Q_n] \), we have \( y_n - x_{n+1} \geq \gamma \), \( x_i - y_n \geq \gamma \), \( 1 \leq i \leq n \), and \( y_i - y_n \geq \gamma \), \( 1 \leq i \leq n - 1 \). Thus we have

\[
I_1 \geq \int_{M_n}^{Q_n} e^{-\sum_{i=1}^{n} (\lambda_i - \lambda_{n+1}) (x_i - y_i)} P_n(\lambda_1, \ldots, n, Y) T_n(X, Y) dy_1 \cdots dy_n
\]
Proof of Theorem 1.3.

Step 1:

Proof of Step 1. 

Note that for 

\[ \begin{align*}
\int M_n \left( \frac{y_n - x_{n+1}}{1 + y_n - x_{n+1}} \right)^{k-1} dy_n \leq \left( \frac{\gamma}{1 + \gamma} \right)^{k-1} \int x_{n+1} \left( y_n - x_{n+1} \right)^{k-1} dy_n \approx \left( \frac{1}{1 + \gamma} \right)^{k-1} \gamma^k.
\end{align*} \]

In both cases, we can conclude that \( \tilde{I}_2 \leq \tilde{I}_1 \). \( \square \)

We now prove that Theorem 1.3 holds.

Proof of Theorem 1.3. We will use Proposition 3.3 and use induction on \( n \) to show that the estimate of \( I^{(n)} \) given in (3.2) holds for the root system \( A_n \), \( n \geq 1 \) with root multiplicity \( k > 0 \).

We first prove the result for \( n = 1 \). Let \( \alpha = x_1 - x_2 \). Using Proposition 3.3 and Lemma 2.4, we have

\[ \begin{align*}
I^{(1)} &\approx \int_{M_1} \left( \frac{x_1 - y}{1 + x_1 - y} \right)^{k-1} dy \\
&\approx \left( \frac{\alpha}{1 + \alpha} \right)^{k-1} \int_{M_1} e^{-(\lambda_1 - \lambda_2)(x_1 - y)} \left( \frac{x_1 - y}{1 + x_1 - y} \right)^{k-1} dy \\
&\approx \left( \frac{\alpha}{1 + \alpha} \right)^{k-1} \int_0^{\alpha/2} e^{-(\lambda_1 - \lambda_2) u} \left( \frac{u}{1 + u} \right)^{k-1} du \\
&\approx \left( \frac{\alpha}{1 + \alpha} \right)^{2k-1} \left( \frac{1 + \alpha}{1 + (\lambda_1 - \lambda_2)(1 + \alpha)} \right)^{k-1} \lambda_1 - \lambda_2 \right) \alpha^k \right)^{k-1}.
\end{align*} \]

which is the desired result by Proposition 3.3.

Assume that the result holds for the root systems \( A_1, A_2, \ldots, A_{n-1} \).

Fix \( 1 \leq m < n \) and suppose that \( \gamma(X) = x_m - x_{m+1} \) is the largest simple positive root in \( X \). We will discuss the case \( m = n \) at the end. We divide the integral \( I^{(n)} \) in two parts \( I_1 \) and \( I_2 \) corresponding to integration in \( y_1 \) on the segment \( [M_m, x_m] \) and \( [x_{m+1}, M_m] \) (recall that \( M_m = (x_m + x_{m+1})/2 \) and \( Q_m = (x_m + x_{m+1})/4 \)), respectively. The proof consists in two steps:

Step 1: Show that \( I_1 \) has the asymptotics given in (3.2).

Step 2: Show that \( I_2 \lesssim I_1 \).

Proof of Step 1. Note that for \( y_m \in [M_m, x_m] \), we have \( x_i - y_j \Gamma, i \leq m, m < j \leq n, y_i - x_j \Gamma, i \leq m, j \geq m+2, y_i - y_j \Gamma, i \leq m, m < j \leq n \). It follows that

\[ \begin{align*}
I_1 &= \int_{x_{n+1}}^{x_m} \cdots \int_{x_{n+1}}^{x_m} \cdots \int_{x_2}^{x_2} e^{-\sum_{i=1}^{n} \left( \lambda_i - \lambda_{n+1} \right)(x_i - y_i)} T_n(X, Y) P_n(\lambda, Y) dy_1 \cdots dy_n \\
&= \int_{x_{n+1}}^{x_m} \cdots \int_{x_{n+1}}^{x_m} \cdots \int_{x_2}^{x_2} e^{-\sum_{i=1}^{n} \left( \lambda_i - \lambda_{n+1} \right)(x_i - y_i)} \\
&\quad \cdot T_m(X_1, \ldots, m+1, Y_1, \ldots) T_{n-m}(X_1, \ldots, n+1, Y_1, \ldots, m+1, \ldots) R_1(X, Y) \\
&\quad \cdot P_m(\lambda_1, \ldots, m+1, \ldots) P_{n-m}(\lambda_{m+1}, \ldots, n, \ldots) R_2(\lambda, Y) dy_1 \cdots dy_n,
\end{align*} \]
and the terms $R_1 = T_n/(T_n P_{n-m})$ and $R_2 = P_n/(P_n P_{n-m})$ have the estimates
\[
R_1(X,Y) \asymp \left( \frac{\gamma}{1 + \gamma} \right)^{(2m(n-m)(k-1))} =: r_1(X),
\]
\[
R_2(\lambda, Y) \asymp \prod_{i \leq m < j \leq n} \frac{\gamma((1 + (\lambda_i - \lambda_j)(1 + \gamma)k-1)}{1 + (\lambda_i - \lambda_j)\gamma}k := r_2(\lambda, X).
\]

After replacing the terms $R_1$ and $R_2$ by the estimates $r_1$ and $r_2$, the remaining integrand factorizes and by Fubini theorem and Proposition 3.5, we get
\[
I_1 \asymp r_1(X) r_2(\lambda, X) I^{(m)}(\lambda_{1,\ldots,m,n+1}, X_{1,\ldots,m+1}) I^{(n-m)}(\lambda_{m+1,\ldots,n+1}, X_{m+1,\ldots,n+1}).
\]

By the induction hypothesis on $A_m$ and on $A_{n-m}$, we finally obtain
\[
I_1 \asymp \left( \frac{\gamma}{1 + \gamma} \right)^{(2m(n-m)(k-1))} \prod_{i \leq m < j \leq n} \frac{\gamma((1 + (\lambda_i - \lambda_j)(1 + \gamma)k-1)}{1 + (\lambda_i - \lambda_j)\gamma}k
\]
\[
\prod_{i < j \leq m} (1 + (\lambda_i - \lambda_j)(1 + (x_i - x_j))k-1
\]
\[
\prod_{i < j \leq m} (1 + (\lambda_i - \lambda_j)(1 + x_i - x_j))k-1
\]
\[
\prod_{i = 1}^m (1 + (\lambda_i - \lambda_{n+1})(1 + x_i - x_{n+1}))k(1 + \lambda_i - \lambda_{n+1})k-1
\]
\[
\prod_{i = 1}^m \frac{\gamma(1 + (\lambda_i - \lambda_{n+1})(1 + \gamma))k}{1 + (\lambda_i - \lambda_{n+1})\gamma}k.
\]

Using the fact that $x_i - x_j \asymp \gamma$ when $i \leq m$ and $j \geq m + 1$, we see that the last expression has the desired asymptotics [4.2].

**Proof of Step 2.** We now show that $I_2 = I^{(m)} - I_1 \lesssim I_1$. As before, we show instead that $\hat{I}_2 \lesssim \hat{I}_1$ where $\hat{I}_1$ (resp. $\hat{I}_2$) represents the portion of $I_1$ (resp. $I_2$) where $y_m$ appears.

Note that for $y_m \in [M_m, Q_m]$, we have $x_i - y_m \asymp \gamma$, $i \leq m$ and $y_m - x_j \asymp \gamma$, $j > m$, and $|y_i - y_m| \asymp \gamma$, $i \neq m$.

It follows that
\[
\hat{I}_1 \lesssim \int_{M_m} e^{-(\lambda_m - \lambda_{n+1})(x_m - y_m)} T^{(m)}(X,Y) \prod_{i = m+1}^n \frac{(y_i - y_m)(1 + (\lambda_i - \lambda_m)(1 + y_i - y_m))k-1}{1 + (\lambda_i - \lambda_m)(y_i - y_m)k} dy_m
\]
\[
\gtrsim e^{-(\lambda_m - \lambda_{n+1})\gamma/2} \frac{\gamma}{4} \left( \frac{\gamma}{1 + \gamma} \right)^{(n+1)(k-1)} \prod_{i = m+1}^n \frac{\gamma(1 + (\lambda_i - \lambda_m)(1 + \gamma))k-1}{1 + (\lambda_i - \lambda_m)\gamma}k
\]
\[
\prod_{i = m+1}^n \frac{\gamma(1 + (\lambda_i - \lambda_m)(1 + \gamma))k}{1 + (\lambda_i - \lambda_m)\gamma}k.
\]

On the other hand, since $x_i - y_m \asymp \gamma$, $i \leq m$, and $y_i - y_m \asymp \gamma$, $i < m$, when $y_m \in [x_{m+1}, M_m]$, we have
\[
\hat{I}_2 \lesssim e^{-(\lambda_m - \lambda_{n+1})\gamma/2} \int_{x_{m+1}}^{M_m} T^{(m)}(X,Y) \prod_{i = m+1}^n \frac{(y_i - y_m)(1 + (\lambda_i - \lambda_m)(1 + y_i - y_m))k-1}{1 + (\lambda_i - \lambda_m)(y_i - y_m)k} dy_m
\]
\[
\gtrsim e^{-(\lambda_m - \lambda_{n+1})\gamma/2} \left( \frac{\gamma}{1 + \gamma} \right)^m \prod_{i = m+1}^n \frac{\gamma(1 + (\lambda_i - \lambda_m)(1 + \gamma))k-1}{1 + (\lambda_i - \lambda_m)\gamma}k
\]
\[
\geq e^{-(\lambda_m - \lambda_{n+1})\gamma/2} \left( \frac{\gamma}{1 + \gamma} \right)^m \prod_{i = m+1}^n \frac{\gamma(1 + (\lambda_i - \lambda_m)(1 + \gamma))k-1}{1 + (\lambda_i - \lambda_m)\gamma}k
\]
\[
(3.4) \gtrsim e^{-(\lambda_m - \lambda_{n+1})\gamma/2} \left( \frac{\gamma}{1 + \gamma} \right)^m \prod_{i = m+1}^n \frac{\gamma(1 + (\lambda_i - \lambda_m)(1 + \gamma))k-1}{1 + (\lambda_i - \lambda_m)\gamma}k
\]
If $k \leq 1$, referring to the function $F_2$ of Lemma 2.2 we have from (3.4)

$$
\tilde{I}_2 \lesssim e^{-(\lambda_m - \lambda_{n+1}) \gamma / 2} \left( \frac{\gamma}{1 + \gamma} \right)^{m(k-1)} \prod_{i=1}^{m-1} \frac{\gamma (1 + (\lambda_i - \lambda_m) (1 + \gamma))^{k-1}}{(1 + (\lambda_i - \lambda_m) \gamma)^k} \\
\int_{x_{m+1}}^{M_m} \prod_{i=m+1}^{n+1} \left( \frac{1}{1 + y_m - y_{i+1}} \right)^{k-1} \prod_{i=m+1}^{n} \frac{y_m - y_i}{1 + (\lambda_m - \lambda_i) (y_m - y_i)} \\
\prod_{i=m+1}^{n} \left( \frac{(y_m - y_i) (1 + (\lambda_m - \lambda_i) (1 + y_m - y_i))}{(1 + y_m - y_i) (1 + (\lambda_m - \lambda_i) (y_m - y_i))} \right)^{k-1} \\
\lesssim e^{-(\lambda_m - \lambda_{n+1}) \gamma / 2} \left( \frac{\gamma}{1 + \gamma} \right)^{m(k-1)} \prod_{i=1}^{m-1} \frac{\gamma (1 + (\lambda_i - \lambda_m) (1 + \gamma))^{k-1}}{(1 + (\lambda_i - \lambda_m) \gamma)^k} \\
\frac{\gamma}{2} \left( \frac{\gamma}{1 + \gamma} \right)^{k-1} \prod_{i=m+1}^{n} \frac{\gamma}{1 + (\lambda_m - \lambda_i) \gamma} \prod_{i=m+1}^{n} \frac{\gamma (1 + (\lambda_m - \lambda_i) (1 + \gamma))^{k-1}}{(1 + (\lambda_i - \lambda_m) \gamma)^k} \\
\frac{\gamma}{2} \prod_{i=m+1}^{n} \frac{\gamma (1 + (\lambda_m - \lambda_i) (1 + \gamma))^{k-1}}{(1 + (\lambda_m - \lambda_i) \gamma)^k} \lesssim \tilde{I}_1.
$$

If $k \geq 1$, using $y_m - x_i \leq y_m - y_i$, referring to the functions $F_1$ and $F_3$ of Lemma 2.2 and rewriting (3.4), we have

$$
\tilde{I}_2 \lesssim e^{-(\lambda_m - \lambda_{n+1}) \gamma / 2} \left( \frac{\gamma}{1 + \gamma} \right)^{m(k-1)} \prod_{i=1}^{m-1} \frac{\gamma (1 + (\lambda_i - \lambda_m) (1 + \gamma))^{k-1}}{(1 + (\lambda_i - \lambda_m) \gamma)^k} \\
\int_{x_{m+1}}^{M_m} \prod_{i=m+1}^{n} \left( \frac{y_m - x_{i+1}}{1 + y_m - x_{i+1}} \right)^{k-1} \prod_{i=m+1}^{n} \frac{y_m - y_i}{1 + (\lambda_m - \lambda_i) (y_m - y_i)} \\
\prod_{i=m+1}^{n} \left( \frac{(y_m - y_i) (1 + (\lambda_m - \lambda_i) (1 + y_m - y_i))}{(1 + y_m - y_i) (1 + (\lambda_m - \lambda_i) (y_m - y_i))} \right)^{k-1} \\
\lesssim e^{-(\lambda_m - \lambda_{n+1}) \gamma / 2} \left( \frac{\gamma}{1 + \gamma} \right)^{m(k-1)} \prod_{i=1}^{m-1} \frac{\gamma (1 + (\lambda_i - \lambda_m) (1 + \gamma))^{k-1}}{(1 + (\lambda_i - \lambda_m) \gamma)^k} \\
\frac{\gamma}{2} \left( \frac{\gamma}{1 + \gamma} \right)^{k-1} \prod_{i=m+1}^{n} \frac{\gamma}{1 + (\lambda_m - \lambda_i) \gamma} \prod_{i=m+1}^{n} \frac{\gamma (1 + (\lambda_m - \lambda_i) (1 + \gamma))^{k-1}}{(1 + (\lambda_i - \lambda_m) \gamma)^k} \\
\frac{\gamma}{2} \prod_{i=m+1}^{n} \frac{\gamma (1 + (\lambda_m - \lambda_i) (1 + \gamma))^{k-1}}{(1 + (\lambda_m - \lambda_i) \gamma)^k} \lesssim \tilde{I}_1.
$$

To conclude, we reason by symmetry, as explained below. By the structure of the root system $A_n$, the case $\alpha_n$ maximal is equivalent to the case $\alpha_1$ maximal. Indeed, in formula (3.1), one does not assume that $\lambda \in \pi^\ast$. We also know that $\phi_\lambda(e^X)$ is invariant under permutation of its $\lambda$ argument. Hence one can re-write (3.1) by exchanging $\lambda_1$ and $\lambda_{n+1}$,

$$
\phi_\lambda(e^X) = e^{\lambda(X)} \text{ if } n = 1 \text{ and } \\
\phi_\lambda(e^X) = \frac{\Gamma(k(n+1))}{\Gamma(k)^{n+1}} e^{\lambda_1 \sum_{j=1}^{n+1} x_j} \int_{E(X)} \phi_{\lambda_0}(e^Y) S^{(k)}(Y, X) d(Y)^2 k dY
$$
where \( \widetilde{\phi}_0(Y) = \sum_{r=2}^{n+1} (\lambda_r - \lambda_1) y_{r-1} \).

We used the fact that

\[
\phi_{|\lambda_n+1-\lambda_1,\lambda_2-\lambda_1,...,\lambda_n-\lambda_1|}(e^Y) = \phi_{|\lambda_2-\lambda_1,...,\lambda_n-\lambda_1,\lambda_{n+1}-\lambda_1|}(e^Y).
\]

Theorem \( \text{1.3} \) is then equivalent to

\[
J^{(n)}(\lambda, X) = \prod_{\alpha \in \Sigma_+^+} \prod_{\alpha \in \Sigma_+^+} (1 + \alpha(\lambda)(1 + \alpha(X)))^{k-1} \prod_{\alpha \in \Sigma_+^+} (1 + \alpha(X))^{2k-2} \prod_{i=2}^{n+1} (1 + \lambda_i - \lambda_i)^{k-1}
\]

where, for \( \lambda, X \in \mathbb{R}^{n+1} \),

\[
J^{(n)}(\lambda, X) = \int_{x_1}^{x_n} \cdots \int_{x_{n+1}}^{x_{n+1}} e^{-\sum_{i=1}^{n} (\lambda_i - \lambda_{i+1}) (y_{i-1} - x_{i+1})} P_n(\lambda, X, Y) \ d y_1 \cdots d y_n.
\]

The term \( J^{(n)} \) corresponds to a constant multiple of \( e^{-\lambda(X)} d(X)^{2k-1} \phi_{\lambda}(e^X) \) in which we have replaced \( \phi_{\lambda_0}(e^Y) \) by its asymptotic expression proposed in Theorem \( \text{1.3} \). One then proves the case \( \alpha_n \) maximal as one proves the case \( \alpha_1 \) maximal.

This concludes the proof of the Theorem \( \text{1.3} \) for \( X \in \alpha^+ \) (recall that the formula \( \text{3.1} \) holds for \( X \in \alpha^+ \)). The estimates that we find for \( \phi_{\lambda}(e^X) \) extend to \( X \in \alpha^+ \) by continuity. \( \square \)

4. OTHER EVIDENCE FOR CONJECTURE \( \text{1.1} \)

4.1. Comparison with known estimates. Recall the estimates from Narayanan and al in \( \text{11} \) (refer also to \( \text{10} \)):

\[
C_1(\lambda) e^{\lambda - \rho}(X) \prod_{\alpha \in \Sigma_+^+} (1 + \alpha(X)) \leq \phi_{\lambda}(X) \leq C_2(\lambda) e^{\lambda - \rho}(X) \prod_{\alpha \in \Sigma_+^+} (1 + \alpha(X))
\]

where, as before, \( \Sigma_+^+ = \{ \alpha \in \Sigma_+^+ : \alpha(\lambda) = 0 \} \).

We will show that our conjecture is consistent with the bound \( \text{4.1} \). We will need a technical Lemma.

**Lemma 4.1.** Assume \( u \geq 0 \) and \( a > 0 \). Then

\[
\frac{1}{1 + a} \leq \frac{1 + u}{(1 + a u)^k} (1 + (1 + u) a)^{-1} \leq \frac{(1 + a)^k}{a}.
\]

**Proof.** Refer to Lemma \( \text{2.2} \). We have

\[
f(u) = \frac{1 + u}{(1 + a u)^k} (1 + (1 + u) a)^{-1} = \frac{1 + u}{1 + (1 + u) a} \left( \frac{1 + u}{1 + a u} \right)^k = F_5(u)/F_4(u)^k.
\]

The function \( F_5(u) \) increases in \( u \) and therefore \( 1/(1 + a) = F_5(0) \leq F_5(u) \leq F_5(\infty) = 1/a \).

The function \( 1/F_4(u) \) decreases in \( u \) and therefore \( 1 + a = 1/F_4(0) \geq 1/F_4(u) \geq 1/F_4(\infty) = 1 \). \( \square \)

**Proposition 4.2.** Conjecture \( \text{1.4} \) is consistent with \( \text{4.1} \).

**Proof.** Assume the bound proposed in Conjecture \( \text{1.1} \):

\[
\phi_{\lambda}(e^X) \leq e^{(\lambda - \rho)(X)} \prod_{\alpha \in \Sigma_+^+} \frac{1 + \alpha(X)}{(1 + \alpha(\lambda) \alpha(X))^{k(\alpha) + k(2 \alpha) - 1}} \left( \frac{1 + \alpha(\lambda)(1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k(\alpha) + k(2 \alpha) - 1}
\]

\[
= e^{(\lambda - \rho)(X)} \prod_{\alpha \in \Sigma_+^+, \alpha(\lambda) = 0} \frac{1 + \alpha(X)}{(1 + \alpha(\lambda) \alpha(X))^{k(\alpha) + k(2 \alpha) - 1}} \left( \frac{1 + \alpha(\lambda)(1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k(\alpha) + k(2 \alpha) - 1}
\]

\[
= e^{(\lambda - \rho)(X)} \prod_{\alpha \in \Sigma_+^+, \alpha(\lambda) > 0} \frac{1 + \alpha(X)}{(1 + \alpha(\lambda) \alpha(X))^{k(\alpha) + k(2 \alpha) - 1}} \left( \frac{1 + \alpha(\lambda)(1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k(\alpha) + k(2 \alpha) - 1}
\]

\[
= e^{(\lambda - \rho)(X)} \prod_{\alpha \in \Sigma_+^+, \alpha(\lambda) = 0} \left( 1 + \alpha(\lambda) \right)
\]
\[ \prod_{\alpha \in \Sigma^+, \alpha(\lambda) > 0} \frac{1 + \alpha(X)}{(1 + \alpha(\lambda) \alpha(X))^{k(\alpha) + k(2 \alpha)}} \left( \frac{1 + \alpha(\lambda)(1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k(\alpha) + k(2 \alpha) - 1}. \]

In order to show that this is consistent with (4.1), we only have to show that each term

\[ \frac{1 + \alpha(X)}{(1 + \alpha(\lambda) \alpha(X))^{k(\alpha) + k(2 \alpha)}} \left( \frac{1 + \alpha(\lambda)(1 + \alpha(X))}{1 + \alpha(\lambda)} \right)^{k(\alpha) + k(2 \alpha) - 1} \]

is bounded below and above by expressions only depending on \( \lambda \) whenever \( \alpha(\lambda) > 0 \) for a positive root \( \alpha \). This follows from Lemma 4.1.

\[ \square \]

4.2. **BC$_1$ case.** Recall that the only rank 1 root systems are \( A_1 \) and \( BC_1 \). In this section we discuss Conjecture 1.1 for the system \( BC_1 \). Denote \( k_1 = k(\alpha) \) and \( k_2 = k(2\alpha) \). We have \( \rho = k_1 + 2k_2 \).

Conjecture 1.1 reads in this case

\[ \phi_{\lambda}(e^t) \asymp e^{(\lambda - \rho)(t)} \frac{1 + t}{(1 + \lambda t)^{k_1 + k_2}} \left( \frac{1 + \lambda(1 + t)}{1 + \lambda} \right)^{k_1 + k_2 - 1}. \]

We need to prove four following asymptotics, with the notation \( k = k_1 + k_2 \):

\[ e^{-(\lambda - \rho)(t)} \phi_{\lambda}(e^t) \asymp \begin{cases} 1 + t & \text{if } \lambda t \leq 1, \\ (\lambda t)^{-k} & \text{if } \lambda t \geq 1, t \leq 1, \\ \lambda^{-1} & \text{if } \lambda t \geq 1, t \geq 1, \lambda \leq 1, \\ \lambda^{-k} & \text{if } \lambda t \geq 1, t \geq 1, \lambda \geq 1. \end{cases} \]

**Lemma 4.3.** For \( 0 \leq r \leq 1 \) and \( 0 \leq t \leq 1 \), there exists \( C > 0 \) independent of \( t \) and \( r \) such that

\[ |\log(\cosh(t + r \sinh(t)) - r t)| \leq C t^2. \]

**Proof.** Let \( F(r) = \log(\cosh(t + r \sinh(t)) - r t) \). We find that \( F'(r) = 0 \) only if \( r = r_0 = (\sinh(t) - t \cosh(t)) / (t \sinh(t)) \).

The maximum and minimum of \( F(r) \) on \([0,1]\) can only occur if \( r = 0 \), \( r = 1 \) or \( r = r_0 \). This corresponds to the values of \( F(r) \) equal to \( \log(\cosh(t))/t^2 \), 0 or \( (\cosh(t) t + \log(\sinh(t)/t) \sinh(t) - \sinh(t))/t^2 \). The result follows.

\[ \square \]

**Lemma 4.4.** Assume \( x_1 \geq x_2 \). Let \( \phi^{(k)}(\lambda) \) denote the spherical function for the \( A_1 \) root system. Then

\[ \int_{x_2}^{x_1} e^{\mu y} ((e^{2x_1} - e^{2y})(e^{2y} - e^{2x_2}))^{k-1} \, dy = C_k e^{(k-1)(x_1 + x_2)} \sinh^{2k-1}(x_1 - x_2) \phi^{(k)}_{\mu + 2(k-1),0}(e^X). \]

**Proof.** Note that \((e^{2x_1} - e^{2y})(e^{2y} - e^{2x_2}) = 4e^{x_1 + x_2} e^{2y} \sinh(x_1 - y) \sinh(y - x_2)\).

\[ \square \]

**Proposition 4.5.** The spherical functions in the \( BC_1 \) case satisfy Conjecture 1.1.

**Proof.** The spherical functions for the \( BC_1 \) case are given by [10] (5.28) (where we corrected a small misprint):

\[ \phi_{\lambda}(a_t) = \frac{2 \Gamma(k_1 + k_2 + 1/2)}{\Gamma(1/2) \Gamma(k_1) \Gamma(k_2)} \int_0^1 \int_0^\pi |\cosh(t) + r e^{i \phi} \sinh(t)|^{-\rho} (1 - r^2)^{k_1 - 1} r^{2k_2 - 1} \sin^{2k_2 - 1} (\phi r dr d\phi) \]

with \( \rho = k_1 + 2k_2 \).

We divide the region \((\lambda, X) \in \mathbb{R}^+ \times \mathbb{R}^+\) in Regions I, I, II and IV based on [4.2] (with some technical variations).

**Figure I** in the Introduction illustrates these regions.

**Region I:** Suppose \( 0 \leq \lambda t \leq 1 \). Since, for \( 0 \leq r \leq 1 \),

\[ e^{-t} = \cosh(t) - \sinh(t) \leq |\cosh(t) + r e^{i \phi} \sinh(t)| \leq \cosh(t) + \sinh(t) = e^t, \]

\[ \phi_{\lambda}(a_t) \asymp \phi_{\lambda}(a_t) \asymp e^{-\rho t} (1 + t) \asymp e^{(\lambda - \rho) t} (1 + t) \]

(refer to [4.1] which proves the proposition in this case.

Now,

\[ |\cosh(t) + r e^{i \phi} \sinh(t)|^2 = |\cosh(t) + r e^{i \phi} \sinh(t)| |\cosh(t) + r e^{-i \phi} \sinh(t)| \]

\[ = \cosh^2 t + 2 r \cos \phi \sinh t \cosh t + r^2 \sinh^2 t \]
= \cosh^2 t + r \cos \phi \sinh(2t) + r^2 \sinh^2 t.

Hence, using $e^{2x} = \cosh^2 t + r \cos \phi \sinh(2t) + r^2 \sinh^2 t$ and noting that

$$\cos \phi = \frac{e^{2x} - \cosh^2 t - r^2 \sinh^2 t}{r \sinh(2t)},$$

$$\sin \phi = \sqrt{1 - \cos^2 \phi} = \frac{\left(\cosh t + r \sinh t\right)^2 - e^{2x}}{r \sinh(2t)} \left(e^{2x} - (\cosh t - r \sinh t)^2\right)^{1/2}, 0 \leq \phi \leq \pi,$$

we have (the constant $C$ may vary from line to line):

denoting $x_1(r,t) = \log(\cosh t + r \sinh t)$ and $x_2(r,t) = \log(\cosh t - r \sinh t)$,

$$\phi_\lambda(a_t) = C \int_0^1 \int_0^{\pi} \left[\cosh^2 t + r \cos \phi \sinh(2t) + r^2 \sinh^2 t\right]^{(\lambda-\rho)/2} (1 - r^2)^{k_1-1} r^{2k_2-1} \sin^{k_2-1} \phi \frac{e^{2x}}{2 \phi} d\phi dr$$

$$= \frac{C}{\sinh^2 k_2-1(2t)} \int_0^1 \int_{\log(\cosh t-r \sinh t)}^{\log(\cosh t+r \sinh t)} e^{(\lambda-\rho)x} \left(1 - r^2\right)^{k_1-1} r^{2k_2-1} \sin^{k_2-2} \phi \frac{e^{2x}}{2 \phi} dr dx$$

$$= \frac{C}{\sinh^2 k_2-1(2t)} \int_0^1 \left(1 - r^2\right)^{k_1-1} e^{(k_2-1)(x_1(r,t)+x_2(r,t))} \sinh^{k_2-1} \left(x_1(r,t) - x_2(r,t)\right) \frac{\gamma(k_2)}{\phi(\lambda-\rho+2k_2-1,0)}(x_1(r,t),x_2(r,t)) r dr$$

$$= \frac{C}{\sinh^2 k_2-1(2t)} \int_0^1 \left(1 - r^2\right)^{k_1-1} (\cosh^2 t - r^2 \sinh^2 t)^{k_2-1} \frac{r \sinh(2t)}{\cosh^2 t - r^2 \sinh^2 t} \frac{2k_2-1}{\phi(\lambda-\rho+2k_2,0)}(x_1(r,t),x_2(r,t)) r^{2k_2} dr$$

$$= C \int_0^1 \left(1 - r^2\right)^{k_1-1} (\cosh^2 t - r^2 \sinh^2 t)^{-k_1} \frac{\gamma(k_2)}{\phi(\lambda-k_1,0)}(x_1(r,t),x_2(r,t)) r^{2k_2} dr$$

since

$$\sinh(x_1(r,t) - x_2(r,t)) = \frac{r \sinh(2t)}{\cosh^2 t - r^2 \sinh^2 t}.$$

Remark that

$$-t = \log(\cosh t - \sinh t) \leq \log(\cosh t - r \sinh t) \leq \log(\cosh t + r \sinh t) \leq \log(\cosh t + \sinh t) = t.$$  

Region II: Suppose now that $\lambda \geq 1$ and $0 \leq t \leq T_0 = \min\{\log 2, 1/(2k_1), 1/(4C)\}$ where $C$ is as in Lemma [4.3].

In that case, $\lambda \geq 2k_1$,

$$1 = \cosh^2 t - \sinh^2 t \leq \cosh^2 t - r^2 \sinh^2 t \leq \cosh^2 t \leq \cosh^2(1/2),$$

$$rt \leq x_1(r,t) - x_2(r,t) \leq 2rt$$

(for the two last inequalities we studied variations of convenient functions and used $1 - \sinh^2 t \geq 0$ for $t \leq \log 2$).

Applying the estimates for $A_t$ with the multiplicity $k_2$ we get

$$\phi_\lambda(a_t) \asymp \int_0^1 \left(1 - r^2\right)^{k_1-1} (\cosh^2 t - r^2 \sinh^2 t)^{-k_1} e^{(\lambda-k_1)x_1(r,t)} e^{-k_2(x_1(r,t)-x_2(r,t))}$$

$$\frac{1 + x_1(r,t) - x_2(r,t)}{(1 + (\lambda - k_1)(x_1(r,t) - x_2(r,t)))^{k_2}}$$
whenever $0 \leq r \leq 1$.

Region III: Suppose that $\lambda t \geq 1$, $t \geq t_0$, $\lambda \leq \min\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. The constant $t_0$ will be defined in the proof.

According to [9] Ex. 8, p. 484 (see also [13] p. 325 and [17] p. 109),

\[(4.6) \quad \mu_\lambda(e^t) = 2F1\left( \frac{\rho}{2}, \frac{\lambda}{2}; \frac{\rho}{2} + k_1 + k_2 + \frac{1}{2}; -\sin^2 t \right)\]

We apply the formula [12] 15.8.2, with $z = -\sin^2 t$ and we get:

\[
\frac{-\sin(\pi \lambda)}{\pi \Gamma(k_1 + k_2 + \frac{1}{2})} \phi_\lambda(e^t) = \frac{(\sinh t)^{-(\rho + \lambda)}}{\Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{k_1 + 1 - \lambda}{2}\right)\Gamma(\lambda + 1)} \cdot 2F1\left( \frac{\rho + \lambda}{2}, \frac{\lambda - k_1 + 1}{2}, \frac{\lambda + 1}{2} \right)
\]
| (4.7) \[ \frac{(\sinh t)^{\lambda-\rho}}{\Gamma(\rho+\lambda)} \frac{\Gamma(k_1+1+\lambda)}{2} \Gamma(-\lambda+1) \frac{z}{2} \phi_F(\rho-\lambda, -\lambda-k_1+1, z) \] |

All the Gamma functions are bounded and bounded away from zero.

As \( \frac{1}{2} \) < 1, the hypergeometric functions on the right hand side of (4.7) are equal to the hypergeometric power series

\[ _2F_1(a, b; c; w) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} w^n \]

with \( w = \frac{1}{2} \). Observing that if \( |a| \leq a_0, |b| \leq b_0 \) and \( |c| \geq c_0 > 0 \) and \( w \leq 1/2 \) then by manipulating the defining power series, one finds that

\[ |_2F_1(a, b; c; w) - 1| \leq \frac{a_0 b_0}{c_0} \phi_F(a_0 + 1, b_0 + 1, c_0 + 1; 1/2) \]

This will allow us to show that the hypergeometric terms on the right hand side of (4.7) are bounded and bounded away from 0, if \( w = 1/2 \) is small enough. Indeed, let us take

\[ w_0 = \min \left\{ \frac{1}{2}, \frac{1}{2 (\rho/2 + 1/4) (3/2 + k_1)/1/2}, \frac{1}{2 \phi_F(\rho/2 + 5/4, k_1/2 + 7/4, 3/2; 1/2)} \right\} \]

In the assumptions of this case, we must suppose \( t \geq t_0 \) where \( t_0 \) is defined by \( t_0 = \text{arg sinh}(w_0^{-1/2}) \).

Finally, by (4.7)

\[ \phi_\lambda(e^t) \asymp \lambda^{-1}(-\sinh t)^{(\rho+\lambda)} + (\sinh t)^{\lambda-\rho} \asymp \lambda^{-1} e^{(\lambda-\rho)t} \]

what proves the conjecture in this case.

**Region IV:** Suppose that \( t \geq c_1 > 0, \lambda \geq c_2 > 0 \).

We use the formula [4.6]. The uniform asymptotic approximation [4], (3.76), p. 689], valid uniformly for large \( |z| \) when \( \lambda_0 \rightarrow \infty \), implies the following estimate when \( t \geq c_1 > 0, \lambda_0 \geq c_2 > 0 \), for \( c_1, c_2 \) large enough:

\[ _2F_1(a + \lambda_0, a - \lambda_0, c; -z) \asymp \lambda_0^{1/2-c} e^{-2a-1/2} z^{-a} \left( (z(1 + \zeta)^2)_{\lambda_0} + \frac{1}{(z(1 + \zeta)^2)_{\lambda_0}} \right), \]

where \( a = \rho/2, \lambda_0 = \lambda/2, c = k_1 + k_2 + 1/2, z = \sinh^2 t \) and \( \zeta = (1 + z^{-1})^{1/2} = \coth t. \) Consequently,

\[ z(1 + \zeta)^2 = (\sinh t + \cosh t)^2 = e^{2t}. \]

we get

\[ \phi_\lambda(e^t) \asymp \lambda^{-k} \coth^{k-\rho} t e^{-\rho t} (e^{\lambda t} + e^{-\lambda t}) \asymp \lambda^{-k} e^{(\lambda-\rho)t} \]

which proves the proposition in this case.

The closure of the complement of the four regions we have discussed above in the set \( \mathfrak{a}^+ \times \mathfrak{a}^+ \) is a compact set. Given that \( \phi_\lambda(e^{kx}) \) and the proposed bounds are both continuous in \( (\lambda, X) \) and nonzero (for \( \phi_\lambda \), this can be easily seen from (4.3)), the result follows.

**Remark 4.6.** Asymptotic expansions and approximations of the hypergeometric function \( _2F_1 \) for large values of parameters is an important and active research topic research see [3], [12] Chapter 15.12] and [4] as a survey of results.

In [15], Watson gave an asymptotic expansion of the function \( _2F_1(a + \lambda, b - \lambda, c; (1 - z)/2) \) for large \( |\lambda| \), resumed in [3] (17) p. 77. His result also yields the estimate \( \phi_\lambda(e^{kx}) \asymp \lambda^{-k} e^{(\lambda-\rho)t} \) for large \( \lambda \) and \( z \) although the uniformity on his bound in \( z \) is not clearly stated.

4.3. Proof of Conjecture [1.1] in a region for a symmetric space of noncompact type. We first recall a result from a previous paper ([8], Proposition 3.5)].

**Proposition 4.7.** Let \( \alpha_i \) be the simple roots and let \( A_{\alpha_i} \) be such that \( \langle X, A_{\alpha_i} \rangle = \alpha_i(X) \) for \( X \in \mathfrak{a} \). Suppose \( X \in \mathfrak{a}^+ \) and \( w \in W \setminus \{id\} \). Then we have

\[ (4.8) \quad Y - w Y = \sum_{i=1}^{r} \frac{a_i^w(Y)}{|\alpha_i|^2} A_{\alpha_i}, \]

where \( a_i^w \) is a linear combination of positive simple roots with non-negative integer coefficients for each \( i \).
Corollary 4.8. Suppose $X, \lambda \in \mathbb{R}^+$. Then there exists $M > 0$ depending only on the Lie algebra structure such that for $H \in C(X)$, the convex hull of $W \cdot X, \lambda(X) - M \max_{1 \leq i,j \leq n} \{\alpha_i(\lambda) \alpha_j(X)\} \leq \lambda(H) \leq \lambda(X)$.

Proof. Since $\lambda$ is a linear function, it attains its maximum at the extremal points of $C(X)$ namely on $W \cdot X$. The rest follows from the Proposition 4.7.

The next result shows, using the well known estimates for $\phi_0(e^X)$, that Conjecture 1.1 holds in a region of the variables $(\lambda, X)$ for a symmetric space of noncompact type.

Proposition 4.9. Let $X, \lambda \in \mathbb{R}^+$ and suppose $\alpha(\lambda) \alpha(X) \leq C$ for all $\alpha \in \Sigma^+$. Then there exists $M > 0$ depending only on the Lie algebra structure

$$e^{-MC} e^\lambda(X) \phi_0(e^X) \leq \phi_\lambda(e^X) \leq e^\lambda(X) \phi_0(e^X)$$

Proof. We consider every classical or exceptional Lie algebras. According to [2] Plates I–IX, every root system has a highest root of the form $\gamma = \sum_{i=1}^r n_i \alpha_i$ where $n_i \geq 1$ for each $i$. The condition $\gamma(\lambda) \gamma(X) \leq C$ then implies that $\max_{1 \leq i,j \leq n} \{\alpha_i(\lambda) \alpha_j(X)\} \leq C$.

Now,

$$\phi_\lambda(e^X) = \int_K e^{(\lambda - \rho)(H(e^X k))} dk = \int_K e^{\lambda(H(e^X k)) - \rho(H(e^X k))} dk.$$

Noting that $\{H(e^X k) : k \in K\} = C(X)$, the result follows from Corollary 4.8.$\Box$

5. ACKNOWLEDGEMENTS

We thank A. B. Olde Daalhuis and A. Nowak for advice on asymptotics of the hypergeometric function.

We are grateful to the grants IEA CNRS: Analyse liée aux racines et applications 2021–2022 and MIR Université d’Angers “Symétries” for their support of this research.

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