GENERALIZED DRINFELD-SOKOLOV HIERARCHIES AND W-ALGEBRAS

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ABSTRACT

We review the construction of Drinfeld-Sokolov type hierarchies and classical W-algebras in a Hamiltonian symmetry reduction framework. We describe the list of graded regular elements in the Heisenberg subalgebras of the nontwisted loop algebra $\ell(gl_n)$ and deal with the associated hierarchies. We exhibit an $sl_2$ embedding for each reduction of a Kac-Moody Poisson bracket algebra to a W-algebra of gauge invariant differential polynomials.

1. Review of the Drinfeld-Sokolov Construction

In this talk I wish to describe some recent results on the construction of KdV type hierarchies and classical W-algebras. (Proofs and further details can be found in [1], [2].) First I review the relevant aspects of the Drinfeld-Sokolov (DS) construction of KdV type hierarchies [3] and the corresponding W-algebras concentrating on the simplest case. I shall raise some questions concerning the possible generalizations, which will be (partially) answered later in the talk.

As explained in detail in [1], the DS construction can be naturally understood in the framework of the Hamiltonian Adler-Kostant-Symes approach to integrable systems (e.g. [4]). The hierarchy results from a local symmetry reduction of the commuting family of Hamiltonian systems generated by the $ad^*$-invariant Hamiltonians on the dual $A^*$ of a Lie algebra $A$ of the form

$$A = \ell(G) := G \otimes \mathbb{C}[^1_{\lambda, \lambda^{-1}}],$$

(1.1)

where $G$ itself is a centrally extended loop algebra. The space $A^*$ carries the family of compatible R Lie-Poisson brackets induced by the classical r-matrices $R_k \in \text{End}(A)$ given by $R_k := (P_+ - P_-) \circ \lambda^k$, where $P_\pm \in \text{End}(A)$ project onto the subalgebras $A_\pm$ containing positive and negative powers of the spectral parameter $\lambda$, respectively, (see [5]).

For simplicity, let us concentrate on the case when $G = gl_n^\wedge$, the central extension of the algebra of smooth loops in $gl_n$, i.e., $G = \{ (X, a) | X : S^1 \to gl_n, a \in \mathbb{C} \}$ with the Lie bracket

$$[(X, a), (Y, b)] = XY - YX, \int_0^{2\pi} dx \text{tr} X'(x)Y(x)$$

(1.2)

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The periodic space variable parametrizing $S^1$ is denoted by $x \in [0, 2\pi]$ and tilde signifies “loops in $x$”. For any space $V$, we set $\ell(V) := V \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. In the usual way, the dual space $\mathcal{A}^\ast$ (or a dense subspace thereof) is represented as the space of first order matrix differential operators $\mathcal{L}$ of the form

$$\mathcal{L} = (e\partial_x + \mu(x)), \quad (1.3)$$

where $\mu(x) = \sum \mu_i(x)\lambda^i$ is a mapping from $S^1$ into $\ell(gl_n) := gl_n \otimes \mathbb{C}[\lambda, \lambda^{-1}]$, and $e = \sum e_i\lambda^i$ is an element of $\mathbb{C}[\lambda, \lambda^{-1}]$. The ad$^\ast$-invariant functions are generated by the invariants (eigenvalues) of the monodromy matrix $T(\lambda)$ of $\mathcal{L}$.

A crucial rôle in the construction is played by the “DS matrix” $\Lambda_n$ given by

$$\Lambda_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 1 \\ \lambda & 0 & \cdots & \cdots & 0 \end{bmatrix}. \quad (1.4)$$

This is a regular element of $\ell(gl_n)$, that is, we have

$$\ell(gl_n) = \text{Ker}(\text{ad}\Lambda) + \text{Im}(\text{ad}\Lambda), \quad \text{Ker}(\text{ad}\Lambda) : \text{abelian subalgebra}, \quad (1.5)$$

for $\Lambda = \Lambda_n$. In fact, $\text{Ker}(\text{ad}\Lambda_n)$ is the principal Heisenberg subalgebra of $\ell(gl_n)$ (it acquires the central extension in $\ell(gl_n)^\wedge$). Further, $\Lambda_n$ has grade 1 in the principal grading of $\ell(gl_n)$. A grading of $\ell(gl_n)$ can be defined by the eigenspaces of a derivation $d_{N,H} : \ell(gl_n) \to \ell(gl_n)$ of the form

$$d_{N,H} := N\lambda \frac{d}{d\lambda} + \text{ad}H, \quad (1.6)$$

where $N$ is an integer and $H \in gl_n$ is diagonalizable. The principal grading is obtained by taking $N := n$ and $H := H_n$, where

$$H_n := \frac{1}{2}\text{diag} [(n - 1), (n - 3), \ldots, -(n - 3), -(n - 1)]. \quad (1.7)$$

We shall need the decomposition

$$gl_n = gl_n^- + gl_n^0 + gl_n^+ \quad (1.8)$$

induced by the eigenvalues of $\text{ad}H_n$, where the summands are the subalgebras of (strictly) lower triangular, diagonal and upper triangular matrices, and also the constant matrices $C_0$ and $C_1$ defined by writing $\Lambda_n$ in the form

$$\Lambda_n := C_0 + \lambda C_1. \quad (1.9)$$
The DS construction starts by restricting to the subspace \( \mathcal{M} \subset \ell(\widetilde{gl}_n^\wedge)^\ast \) consisting of operators of the form

\[
\mathcal{L} = \partial + J + \lambda C_1, \quad J : S^1 \to gl_n .
\]

This is a Poisson subspace with respect to two out of the infinitely many \( R \) Lie-Poisson brackets on \( \ell(\widetilde{gl}_n^\wedge)^\ast \). The corresponding compatible Poisson brackets on \( \mathcal{M} \) are given by

\[
\{ \varphi, \psi \}_1(J) = - \int_{S^1} \text{tr} C_1 [ \frac{\delta \varphi}{\delta J}, \frac{\delta \psi}{\delta J} ] ,
\]

\[
\{ \varphi, \psi \}_2(J) = \int_{S^1} \text{tr} \left( J \frac{\delta \varphi}{\delta J} \frac{\delta \psi}{\delta J} + \left( \frac{\delta \varphi}{\delta J} \right)' \frac{\delta \psi}{\delta J} \right) ,
\]

where \( \frac{\delta \varphi}{\delta J} \) (resp. \( \frac{\delta \psi}{\delta J} \)) is the functional derivative of the function \( \varphi \) (resp. \( \psi \)) on \( \mathcal{M} \). The monodromy invariants of \( \mathcal{L} \) provide Hamiltonians on \( \mathcal{M} \) that form a commutative family with respect to both Poisson brackets.

It follows from

\[
[g_{l_n}^{-}, C_1] = 0
\]

that the group \( \mathcal{N} \) of transformations

\[
e^f : \mathcal{L} \mapsto e^f L e^{-f}, \quad \text{with} \quad f : S^1 \to g_{l_n}^{-},
\]

is a symmetry group of the commuting family of bihamiltonian systems carried by \( \mathcal{M} \). Indeed, these transformations preserve the two Poisson structures and the monodromy invariants. The KdV type hierarchy results from a symmetry reduction defined by using \( \mathcal{N} \) in such a way as to ensure the locality of the reduced system. Concretely, one considers the following two step reduction process. First, one restricts the system to the “constrained manifold” \( \mathcal{M}_c \subset \mathcal{M} \), defined as the set of \( \mathcal{L} \)’s of the following special form:

\[
\mathcal{L} = \partial + (j + C_0) + \lambda C_1 = \partial + j + \Lambda_n , \quad j : S^1 \to (g_{l_n}^{-} + g_{l_n}^0) .
\]

Second, one factorizes this constrained manifold by the symmetry group \( \mathcal{N} \), defining the reduced phase space

\[
\mathcal{M}_{\text{red}} = \mathcal{M}_c / \mathcal{N} .
\]

In other words, one factorizes out the “gauge transformations” generated by \( \mathcal{N} \) by declaring that only the \( \mathcal{N} \)-invariant functions of \( \mathcal{L} \) are physical. This reduction has the following nice features:

i) The eigenvalues of the monodromy matrix of \( \mathcal{L} \in \mathcal{M}_c \) can be computed by a recursive, algebraic procedure and thus they give commuting local Hamiltonians.

ii) The compatible Poisson brackets on \( \mathcal{M} \) induce compatible Poisson brackets on \( \mathcal{M}_{\text{red}} \).
iii) The gauge orbits in \( \mathcal{M}_c \) allow for global, differential polynomial gauge sections, which give rise to free generating sets for the gauge invariant differential polynomials in \( j \).

iv) The gauge invariant differential polynomials form a classical \( \mathcal{W} \)-algebra under the second Poisson bracket.

The monodromy invariants of a first order matrix differential operator are in general nonlocal objects. Statement i) is a consequence of the facts that \( \Lambda_n \in \ell(gl_n) \) is a graded regular element of nonzero grade, and the grades in \( j \) (1.15) are smaller than the grade of \( \Lambda_n \). The point is that by substituting the ansatz

\[
\Psi(x) = (I + Z(x))e^{F(x)}(I + Z(0))^{-1}\Psi(0),
\]

with \( F(x) \in \text{Ker}(\text{ad}\Lambda_n) \), \( Z(x) \in \text{Im}(\text{ad}\Lambda_n) \), into the linear problem \( \mathcal{L}\Psi = 0 \), one can determine both \( Z \) and \( F \) by quadrature by using the grading together with the decomposition (1.5). One can also easily diagonalize the monodromy matrix \( T = \Psi(2\pi)\Psi(0)^{-1} \).

The resulting \( \text{ad}^* \)-invariant Hamiltonians are local; i.e., are given by integrals of local densities formed from the components of \( j \) and their derivatives.

To construct new local hierarchies, it would be important to explore the possible constraints on the form of a first order matrix differential operator under which the monodromy invariants are local. It is known that one can associate such constraints to any positive graded regular element of any affine Lie algebra (see [6]). However, the list of the inequivalent graded regular elements of the affine Lie algebras seems to be unknown.

Statement ii) means that the compatible Poisson brackets carried by \( \mathcal{M} \) can be consistently restricted to the gauge invariant functions on \( \mathcal{M}_c \). This follows from the Dirac theory of reduction by constraints. By choosing some basis \( \{\gamma_i\} \) of \( gl_n^- \), the constraints defining \( \mathcal{M}_c \subset \mathcal{M} \) read

\[
\phi_i(x) = 0 \quad \text{where} \quad \phi_i(x) := \text{tr}\gamma_i(J(x) - C_0).
\]

It is easy to verify that they are first class with respect to any of the compatible Poisson brackets on \( \mathcal{M} \). The \( \phi_i(x) \) are in fact the generating densities of the \( \mathcal{N} \) symmetry transformations with respect to the second Poisson bracket (1.12). Therefore the Dirac theory tells us that we should factorize the constrained manifold by these transformations. The second Poisson bracket closes on the gauge invariant functions, which can be identified with the functions on \( \mathcal{M}_{\text{red}} \). Thus we obtain an induced Poisson bracket on the factor space (the Dirac bracket). On the other hand, the \( \phi_i \) do not generate any transformations on \( \mathcal{M} \) under the first Poisson bracket (1.11); i.e., they are “Casimir functions”. Therefore the first Poisson bracket can in principle already be restricted to \( \mathcal{M}_c \) without any factorization by \( \mathcal{N} \). Then \( \mathcal{N} \) becomes a group of Poisson (canonical) transformations with respect to the restricted bracket, which can further be reduced to a Poisson bracket on the invariant functions. In this way, we naturally obtain two induced Poisson brackets on \( \mathcal{M}_{\text{red}} \) from those on \( \mathcal{M} \), and the induced Poisson brackets are compatible because the original brackets (1.11,12) were compatible.
The gauges appearing in statement iii) are defined as follows [3]. Consider a direct sum decomposition
\[(gl_n^- + gl_n^0) = [C_0, gl_n^-] + V,\] (1.19a)
where the linear space \(V\) is graded by eigenvalues of \(\text{ad} H_n\) (i.e., \([H_n, V] \subset V\)). Then the subspace of \(\mathcal{M}_c\) consisting of operators of the form
\[L_V = \partial + j_V + \Lambda_n, \quad j_V : S^1 \to V,\] (1.19b)
defines a global gauge section. As proven in [3], a general element \(\mathcal{L} \in \mathcal{M}_c\), given by (1.15), can be brought to this gauge by a unique gauge transformation \(e^f \in \mathcal{N}\) and \(f\) is a differential polynomial in \(j\). It follows that the components of \(j_V\), when considered as functions on \(\mathcal{M}_c\), give a basis for the gauge invariant differential polynomials in \(j\), which thus form a freely generated differential ring. In [2] we gave a fairly general sufficient condition for the existence of this type of gauges (which we call “DS gauges”) in reductions by first class constraints. It should perhaps be noted that DS gauges are not available for the vast majority of reductions.

Let us now deal with statement iv). Note first that the differential polynomial
\[L_{H_n} := \frac{1}{2} \text{tr}(J^2) + \text{tr}(H_n J')\] (1.20)
satisfies the Virasoro algebra under the second Poisson bracket, and its restriction to \(\mathcal{M}_c\) is gauge invariant. Since it contains this Virasoro density, the second Poisson bracket algebra of the gauge invariant differential polynomials is an extended conformal algebra. Set \(M_0 := H_n, M_+ := C_0\) and choose \(M_- \in gl_n\) so that the \(sl_2\) relations
\[[M_0, M_\pm] = \pm M_\pm, \quad [M_+, M_-] = 2M_0\] (1.21)
hold. Consider the particular DS gauge belonging to
\[V := \text{Ker} (\text{ad}_{M_-}).\] (1.22)
A graded basis of this \(V\) is given by the matrices \((M_-)^k\) with \(k = 0, \ldots, (n - 1)\). It can be shown [7] that, with the exception of the \(M_-\) component, the gauge invariant differential polynomials corresponding to the components of the gauge fixed current \(j_V\) are in this case all primary fields (conformal tensors) with respect to the conformal action generated by \(L_{H_n}\). These primary fields and \(L_{H_n}\) together form a basis (free generating set) for the gauge invariant differential polynomials. This means that the extended conformal algebra of the gauge invariant differential polynomials is indeed a classical \(W\)-algebra.

The gauge transformations are generated by the constraints through the second Poisson bracket (1.12), which can be recognized as a “Kac-Moody Poisson bracket” (namely, the Lie-Poisson bracket corresponding to the affine Lie algebra \(\tilde{gl}_n^\wedge\)).
property that the gauge invariant differential polynomials form a $W$-algebra concerns only the reduction of the Kac-Moody (KM) Poisson bracket algebra, and is largely independent from other features of the hierarchy. We shall return to the generalizations of this KM $\rightarrow W$ reduction at the end of the talk.

2. Graded Regular Elements in Heisenberg Subalgebras of $\ell(gl_n)$

Drinfeld and Sokolov [3] associated integrable hierarchies to the grade 1 generators of the principal Heisenberg subalgebras of the loop algebras, given by $\Lambda_n$ (1.4) in the case of $\ell(gl_n)$. The fact that these are graded regular elements is crucial for obtaining the local monodromy invariants giving the Hamiltonians of the hierarchies. Recently, it has been proposed by De Groot et al [6] to construct new integrable hierarchies (and new $W$-algebras) by using any positive, graded regular element of any Heisenberg subalgebra of a loop algebra in a “generalized DS construction”. Roughly speaking, a set of constraints and a gauge group was associated to each graded regular element. Clearly, the actual content of this proposal depends on the supply of graded regular elements, which has not been investigated in [6]. The inequivalent graded Heisenberg subalgebras of the affine Lie algebras were classified by Kac and Peterson [8] and an explicit description of them was worked out by ten Kroode and van de Leur in Refs. [9], [10]. By using this explicit description, it is not hard to obtain the list of the graded regular elements by inspection.

The graded Heisenberg subalgebras of $\ell(gl_n)$ are classified by the partitions of $n$ in the following way [9]. Let a partition of $n$ be given by

$$n = n_1 + n_2 + \cdots + n_k , \quad \text{where} \quad n_1 \geq n_2 \geq \cdots \geq n_k \geq 1 . \quad (2.1)$$

The corresponding Heisenberg subalgebra consists of the $n \times n$ “block-diagonal” matrices $\Lambda$ of the form

$$\Lambda = \begin{bmatrix} y_1 \Lambda_{n_1}^{l_1} & & \\
 & y_2 \Lambda_{n_2}^{l_2} & \\
 & & \ddots & \\
 & & & y_k \Lambda_{n_k}^{l_k} \end{bmatrix} , \quad (2.2)$$

where the $l_i$ ($i = 1, 2, \ldots, k$) are arbitrary integers, the $y_i$ are arbitrary numbers, and $\Lambda_{n_i}$ is the $n_i \times n_i$ DS matrix, cf. (1.4). This maximal abelian subalgebra of $\ell(gl_n)$ is invariant under a grading operator $d_{N,H}$ of the form (1.6) with $N$ and $H$ determined by the partition [9]. An element $\Lambda$ in (2.2) is regular – gives rise to a decomposition of type (1.5) – if $\text{Ker}(\text{ad} \Lambda) \subset \ell(gl_n)$ is the Heisenberg subalgebra (and not a larger space).

The simplest case is that of the homogeneous Heisenberg subalgebra, belonging to the partition $n = 1+1+\cdots+1$. In this case the grading operator is $\Lambda_n^H$ and the graded regular elements are of the form $\Lambda = \lambda^k \text{diag}[y_1, y_2, \ldots, y_n]$, where $y_i \neq y_j$ for $i \neq j$ and $k$ is arbitrary integer. The other extreme case is that of the principal Heisenberg subalgebra, when $n$ is “not partitioned at all”. On account of $\Lambda_n^{l+mn} = \lambda^m \Lambda_n^l$, the generator $\Lambda_n^{l+mn}$ (of grade $(l + mn)$) is regular if and only if $\Lambda_n^l$ is regular. The DS
matrix $\Lambda_n$ itself is regular since its eigenvalues are the $n$ distinct $n$th-roots of $\lambda$. From this one verifies, by inspecting the eigenvalues of $\Lambda_l^l$, that for $1 \leq l \leq (n-1)$ $\Lambda_l^l$ is regular if and only if $n$ and $l$ are relatively prime. For the general case, we have the following result [1].

**Theorem 1.** Graded regular elements exist only in those Heisenberg subalgebras of $\ell(gl_n)$ which belong to the partitions of type

$$n = pr = r + \cdots + r, \quad \text{or} \quad n = pr + 1 = r + \cdots + r + 1. \quad (2.3)$$

In the equal block case $n = pr$ with $r > 1$, the graded regular elements are of the form

$$\Lambda = \lambda^m \begin{bmatrix} y_1 \Lambda_r^l & y_2 \Lambda_r^l & \cdots & y_p \Lambda_r^l \end{bmatrix}, \quad (2.4)$$

where

$$1 \leq l \leq (r-1), \quad y_i \neq 0, \quad y_i^r \neq y_j^r \quad i, j = 1, \ldots, p, \quad i \neq j,$$

with $l$ relatively prime to $r$ and $m$ any integer. The element $\Lambda$ is of grade $(l + mr)$, where the grading operator $d_{N,H}$ is given by (1.6) with $N = r$ and

$$H = \text{diag}[H_r, H_r, \ldots, H_r]. \quad (2.5)$$

In the equal-blocks-plus-singlet case $n = pr + 1$, the graded regular elements are those $n \times n$ matrices which contain an $(n-1) \times (n-1)$ block of the form given by (2.4) in the “top-left corner” and an arbitrary entry in the “lower-right corner”. The relevant grading operator is given by (1.6) with $N = r$,

$$H = \text{diag}[H_r, H_r, \ldots, H_r, 0] \quad \text{if } r \text{ is odd}; \quad (2.6a)$$

and with $N = 2r$,

$$H = \text{diag}[2H_r, 2H_r, \ldots, 2H_r, 0] \quad \text{if } r \text{ is even}. \quad (2.6b)$$

It would be interesting to know the list of graded regular elements and associated integrable systems for all loop algebras based on the simple Lie algebras. The above result, which is of course also valid in the case of $\ell(sl_n)$, makes it clear that graded regular
elements exist only in some “exceptional” Heisenberg subalgebras in general. Some
knew integrable systems can presumably be obtained by applying the DS construction
to each graded regular element, or in some cases one will recover known systems and
gain a better understanding of them in this way.

3. Matrix Gelfand-Dickey Hierarchy from DS Reduction

In [1] we gave a detailed analysis of the DS reduction based on a grade 1 regular
element of the Heisenberg subalgebra of $\ell(gl_n)$ defined by a partition of type $n = pr$
with $r > 1$, generalizing the $r = n$ case described in [3]. After a reordering of the basis,
our grade 1 regular element, $\Lambda_{r,p}$, can be written as

$$\Lambda_{r,p} = \Lambda_r \otimes D = \begin{bmatrix}
0 & D & 0 & \cdots & 0 \\
\vdots & 0 & D & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & D \\
\lambda D & 0 & \cdots & \cdots & 0
\end{bmatrix}, \quad (3.1)$$

where the $p \times p$ matrix $D := \text{diag} (y_1, y_2, \ldots, y_p)$ is such that $D^r$ has distinct, non-zero
eigenvalues (cf. (2.4)). In this basis, the grading operator is given by $d_{N,H} = r\lambda \frac{d}{dx} + \text{ad} H$
with

$$H = H_r \otimes 1_p = \text{diag} \left[ j1_p, (j-1)1_p, \ldots, -(j-1)1_p, -j1_p \right], \quad j = \frac{(r-1)}{2}. \quad (3.2)$$

In particular, this $H$ naturally gives every $n \times n$ matrix a block structure, with $p \times p$
blocks. The DS reduction is set up quite similarly as in the $p = 1$ case. We introduce
the matrices $C_0$ and $C_1$ through the equality $\Lambda_{r,p} := C_0 + \lambda C_1$, and define the spaces $\mathcal{M}$,
$\mathcal{M}_c$, and the gauge group $\mathcal{N}$ simply by substituting “block-triangular” for “triangular”
everywhere in the original definitions. The following statements identify the reduced
system as the $p \times p$ matrix version of the well-known (e.g. [11]) Gelfand-Dickey $r$-KdV
hierarchy.

First, the reduced space, $\mathcal{M}_{\text{red}} = \mathcal{M}_c/\mathcal{N}$, is the space of “matrix Lax operators”
of the form

$$L = (-D)^{-r} \partial^r + u_1 \partial^{r-1} + \ldots + u_{r-1} \partial + u_r, \quad (3.3)$$

where the $u_i$ are smooth, $p \times p$ matrix valued functions on $S^1$. Second, the Poisson
brackets on $\mathcal{M}_{\text{red}}$ induced by the reduction are the two compatible matrix Gelfand-
Dickey Poisson brackets, given by the well-known formulae

$$\{\varphi, \psi\}^{(1)}(L) = \int_{S^1} \text{tr res} \left( L[Y_-, X_-] \right), \quad (3.4)$$

$$\{\varphi, \psi\}^{(2)}(L) = \int_{S^1} \text{tr res} \left( YL(XL)_+ - LY(LX)_+ \right), \quad (3.5)$$
where $X := \nabla_L \varphi, Y := \nabla_L \psi$ are the gradients of the functions $\varphi, \psi$ on $\mathcal{M}_{\text{red}}$. (Similarly as in the scalar case, these gradients are pseudo-differential operators and the subindex $\pm$ refers to the splitting of the space of – now $p \times p$ matrix – pseudo-differential operators into the sum of the subspaces of pure differential and integral operators, containing positive and negative powers of $\partial$.) The second Poisson bracket algebra qualifies as a classical $\mathcal{W}$-algebra. Third, the Hamiltonians of the hierarchy, resulting from the monodromy invariants of $L \in \mathcal{M}_c$, allow for the following description in terms of $L$: Diagonalize $L$ by a recursive procedure; i.e., determine a $p \times p$ diagonal pseudo-differential operator $\hat{L} = (\Delta)^r \partial^r + \sum_{i=1}^{\infty} a_i \partial^{r-i}$ such that $L = \hat{L} g^{-1}$, where $g$ is of the form $g = I + \sum_{i=1}^{\infty} g_i \partial^{-i}$ with $g_i(x + 2\pi) = g_i(x)$. A natural generating set for the Hamiltonians of the hierarchy is obtained by integrating the componentwise residues of the fractional (including integral) powers $\hat{L}$. More precisely, the list of Hamiltonians reads

$$
\mathcal{H}_{0,i} = (-1)^r \int_{S^1} (D^r u_1)_{ii}, \quad \mathcal{H}_{k,i} = \frac{r}{k} \int_{S^1} \text{res} \left( \hat{L}^{k/r} \right)_{ii},
$$

(3.6)

where $i = 1, \ldots, p$ and $k = 1, 2, \ldots$ is arbitrary. These Hamiltonians satisfy “bihamiltonian ladder” relations, $\{ L, \mathcal{H}_{k,i}\}^{(2)} = \{ L, \mathcal{H}_{k+r,i}\}^{(1)}$. The number of independent bihamiltonian ladders is $n - 1$ since one has $\sum_{i=1}^{p} \mathcal{H}_{m+i,r,i} = 0$ for any $m = 1, 2, \ldots$, which is simply a consequence of the fact that $L$ is a purely differential operator.

The above description of the reduced system generalizes the result proven by Drinfeld and Sokolov in [3] for the scalar case $p = 1$. The proofs given in [1] use their methods, but at the same time introduce some conceptual simplifications (at least to our taste). The simplifications arise from the fact that we work entirely within the Hamiltonian Adler-Kostant-Symes approach. In this framework the existence of the compatible Poisson structures and commuting Hamiltonians is clear from the very beginning of the construction and the only problem is to describe them in terms of reduced variables as explicitly and nicely as possible.

The main difference between the $p \times p$ matrix and the $p = 1$ scalar case is that computing the Hamiltonians in the former case requires the diagonalization of $L$. The analogues of those Hamiltonians which are obtained from the integral powers of the diagonalized Lax operator $\hat{L}$ do not exist in the scalar case. The other Hamiltonians can also be expressed as integrals of trace-residues of independent fractional powers of $L$, without diagonalization.

The KdV type hierarchies based on matrix Lax operators of the type (3.3) have been investigated before in refs. [12-14], where the additional assumption was made that the diagonal part of $u_1$ vanishes. We verified that setting $[u_1]_{\text{diag}} = 0$ is consistent with the equations of the hierarchy resulting from the DS reduction and in fact corresponds to an additional Hamiltonian symmetry reduction.

Let us further comment on the relationship between the hierarchies and $\mathcal{W}$-algebras. It is known ([15], [2]) that one can associate a classical $\mathcal{W}$-algebra to every $sl_2$ subalgebra of $gl_n$. The $\mathcal{W}$-algebra arising in the above corresponds to the $sl_2$ subalgebra under which the defining representation of $gl_n$ decomposes into $p$ copies of the $r$-dimensional $sl_2$ irreducible representation. The other case in which a graded regular
element exists in the Heisenberg subalgebra is the case of the partition \( n = pr + 1 \). By taking an arbitrary regular element of minimal positive grade it may be verified that the generalized DS reduction proposed in [6] leads to a \( \mathcal{W} \)-algebra which is again equal to one of those studied in [15], [2]. (We note in passing that it is not clear to us whether the reductions belonging to regular elements of higher grade are related to \( \mathcal{W} \)-algebras or not.) Both the \( sl_2 \) subalgebras of \( gl_n \) and the Heisenberg subalgebras of \( \ell(gl_n) \) are classified by the partitions of \( n \). It is unclear whether there is a general relationship between all \( \mathcal{W} \)-algebras associated to \( sl_2 \) embeddings and KdV type hierarchies or not, since there is a \( \mathcal{W} \)-algebra for any partition, but graded regular elements exist only in exceptional cases. It is also worth noting that, in all cases, it is easy to construct families of “first Poisson structures” compatible with the “second one” giving the \( \mathcal{W} \)-algebra. This fact however does not automatically imply the existence of a corresponding local hierarchy.

4. Are \( sl_2 \) Embeddings Necessary for \( \mathcal{W} \)-Algebras?

Consider a finite dimensional Lie algebra \( G \) with an ad-invariant, nondegenerate scalar product \( \langle \cdot, \cdot \rangle \). Denote by \( K \) the space of \( G \)-valued smooth, periodic functions, \( K := \{ J : S^1 \to G \} \), and let \( K \) carry the “KM Poisson bracket algebra”:

\[
\{\langle u, J(x) \rangle, \langle v, J(y) \rangle \} = \langle [u, v], J(x) \rangle \delta(x - y) - \langle u, v \rangle \delta'(x - y),
\]

where \( u \) and \( v \) are arbitrary generators of \( G \). Choose a subalgebra \( \Gamma \subset G \) (with basis \( \{\gamma_i\} \)) and an element \( C_0 \in G \) in such a way that the following constraints

\[
\phi_i(x) = 0 \quad \text{where} \quad \phi_i(x) := \langle \gamma_i, J(x) - C_0 \rangle,
\]

are first class. The corresponding constrained manifold \( K_c \subset K \) consists of “currents” of the form

\[
J = C_0 + j \quad \text{with} \quad j : S^1 \to \Gamma^\perp.
\]

The first class constraints \( \phi_i \) generate gauge transformations on \( K_c \) and we are interested in the gauge invariant differential polynomials in \( j \). We would like to describe and classify the constraints for which the gauge invariant differential polynomials form a classical \( \mathcal{W} \)-algebra “similarly as in the standard DS case”. It has been recently recognized [15, 2] that one can find at least one such reduction for every \( sl_2 \) subalgebra of \( G \), by generalizing the standard case in a rather straightforward way. Thus it is natural to ask whether the presence of an \( sl_2 \) embedding is necessary in all “nice” cases. We do not have a complete classification of the nice cases yet, but, under the assumptions given below, we can answer this latter question in the positive.

Let \( \mathcal{R} \) be the set of gauge invariant differential polynomials in \( j \). This set is obviously closed with respect to linear combination, ordinary multiplication and application of \( \partial \). We express this by saying that \( \mathcal{R} \) is a differential ring. First of all, we assume that \( \mathcal{R} \) is freely generated on \( m := (\dim G - 2\dim \Gamma) \) gauge invariant differential polynomials. In other words, there exist generators \( W^a \in \mathcal{R} \ (a = 1, \ldots, m) \) such that any element
$W \in \mathcal{R}$ can be expressed in a unique way as a differential polynomial in the $W^a$'s. It follows that, upon imposing the first class constraints, the KM Poisson bracket algebra induces a Poisson bracket algebra of the form

$$\{W^b(x), W^c(y)\}^* = \sum_k p^{b,c}_k(x) \delta^{(k)}(x - y),$$

(4.4)

where the $p^{b,c}_k$ entering the finite sum on the right hand side are uniquely determined differential polynomials in the basis $W^a$ $(a = 1, \ldots, m)$. We assume that this induced Poisson bracket gives $\mathcal{R}$ the structure of a classical $\mathcal{W}$-algebra. By definition, this means that it is possible to choose a primary field basis in $\mathcal{R}$, that is, a basis such that $W^1$ is a Virasoro density and $W^a$ is a conformal primary field for $a = 2, \ldots, m$.

We now make a further assumption, which is more technical than the above, though we consider it still rather natural. Namely, we assume that the primary field basis is such that $W^1$ is equal to

$$L_H = \frac{1}{2} \langle J, J \rangle + \langle H, J' \rangle,$$

(4.5)

for some digonalizable element $H \in G$. We can translate the fact that $L_H$ is gauge invariant into the relations

$$[H, \Gamma] \subset \Gamma, \quad H \in \Gamma^\perp, \quad [H, C_0] = C_0.$$  

(4.6)

Furthermore, we assume that the DS type gauges are available by using this $H$ as grading operator. The meaning of the latter assumption is the following (cf. Eq. (1.19)). Take any graded linear space $V$ defining a direct sum decomposition

$$\Gamma^\perp = [C_0, \Gamma] + V,$$

(4.7)

and consider the subspace $\mathcal{C}_V \subset \mathcal{K}_c$ given by

$$\mathcal{C}_V := \{ J \mid J = C_0 + j_V, \ j_V : S^1 \to V \}.$$  

(4.8)

The assumption is that $\mathcal{C}_V$ defines a global gauge fixing in such a way that the components of the gauge fixed current $j_V$, when considered as functions on $\mathcal{K}_c$, provide a free generating set for $\mathcal{R}$. All in all, the above assumptions say that the main features of the standard case are valid for the reduction. Then the following result may be proven.

**Theorem 2.** Under the assumptions described above, there exists an element $M_- \in \Gamma$ which together with $H$ and $C_0$ generates an $\mathfrak{sl}_2$ subalgebra of $G$. More precisely, there exists an element $M_- \in \Gamma$ such that Eq. (1.21) holds with $M_0 := H$ and $M_+ := C_0$.

The proof of this result, and further results, implying for example that most KM reductions through conformally invariant first class constraints do not result in a (freely generated) $\mathcal{W}$-algebra, can be found in Refs. [2], [16].
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