H\textsuperscript{2} BLOWUP RESULT FOR A SCHRÖDINGER EQUATION WITH NONLINEAR SOURCE TERM

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Abstract. In this paper, we consider the nonlinear Schrödinger equation on \( \mathbb{R}^N \), \( N \geq 1 \),
\[
\partial_t u = i\Delta u + \lambda|u|^{\alpha} u,
\]
with \( H^2 \)-subcritical nonlinearities: \( \alpha > 0 \), \( (N - 4) \alpha < 4 \) and \( \text{Re} \lambda > 0 \). For any given compact set \( K \subset \mathbb{R}^N \), we construct \( H^2 \) solutions that are defined on \( (0, T) \) for some \( T > 0 \), and blow up exactly on \( K \) at \( t = 0 \). We generalize the range of the power \( \alpha \) in the result of Cazenave, Han and Martel [5]. The proof is based on the energy estimates and compactness arguments.

1. Introduction. In this paper, we consider the nonlinear Schrödinger equation with the power nonlinearity
\[
\partial_t u = i\Delta u + \lambda|u|^{\alpha} u
\]
on \( \mathbb{R}^N \), where
\[
N \geq 1, \quad \alpha > 0, \quad (N - 4) \alpha < 4,
\]
and \( \lambda \in \mathbb{C} \) such that
\[
\text{Re} \lambda > \begin{cases} 
0, & \text{if } 1 \leq N \leq 3, \\
\frac{\alpha}{2} |\text{Im} \lambda|, & \text{if } N \geq 4.
\end{cases}
\]
Under the assumption (1.2), the equation (1.1) is \( H^2 \)-subcritical, so that the corresponding Cauchy problem is locally well posed in \( H^2(\mathbb{R}^N) \), see [12] and [21]. It is well-known that if \( \alpha < \frac{4}{N} \) and the equation (1.1) has a dissipative nonlinearity, i.e. \( \text{Re} \lambda < 0 \), then all \( H^1 \) solutions are global, see [2]. If \( \alpha < \frac{2}{N} \) and the nonlinearity is not dissipative, i.e. \( \text{Re} \lambda > 0 \), it is proved in [2] that the equation (1.1) has no global in time \( H^1 \) solution that remains bounded in \( H^1 \). The question of the finite-time blow-up is still open. With the restriction \( \alpha \geq 2 \), it is proved in [6] that under the assumption that \( (N - 2) \alpha \leq 4 \) and \( \text{Re} \lambda = 1 \), finite time blowup occurs. The construction is based on an appropriate ansatz. This result is extended in [13] to the case \( \alpha > 1 \) and \( (\alpha + 2) \text{Re} \lambda \geq \alpha|\lambda| \). Moreover, by refining the initial ansatz (2.7) inductively, the blow-up result is extended to the whole range of \( H^1 \) subcritical powers and arbitrary \( \text{Re} \lambda > 0 \) in [5]. There are some similarly results for the focusing energy subcritical nonlinear wave equation, see [7, 8].

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In this paper, we extend the previous blow-up result in [5] to the $H^2$-subcritical case under the additional technical assumptions (1.3).

**Theorem 1.1.** Under the conditions (1.2) and (1.3), for any nonempty compact subset $K \subset \mathbb{R}^N$, there exist $S \in (-1,0)$ and a solution $u \in C([S,0), H^2(\mathbb{R}^N)) \cap C^1([S,0), L^2(\mathbb{R}^N))$ of the equation (1.1) which blows up at time $0$ exactly on $K$ in the following sense.

1. If $x_0 \in K$ then for any $r > 0$,
   \[
   \lim_{t \uparrow 0} \|u(t)\|_{L^2(|x-x_0|<r)} = \infty. \tag{1.4}
   \]

2. If $K$ is a open subset of $\mathbb{R}^N$ such that $K \subset U$, then
   \[
   \lim_{t \uparrow 0} \|\nabla u(t)\|_{L^2(U)} = \infty, \text{ and } \lim_{t \uparrow 0} \|\Delta u(t)\|_{L^2(U)} = \infty. \tag{1.5}
   \]

3. If $\Omega$ is a open subset of $\mathbb{R}^N$ such that $\overline{\Omega} \cap K = \emptyset$, then
   \[
   \sup_{t \in (S,0]} \|u(t)\|_{H^2(\Omega)} < \infty. \tag{1.6}
   \]

**Remark 1.1.** Under the assumptions that $\alpha > 0$, $(N-2)\alpha \leq 4$ and Re$\lambda > 0$, Cazenave-Han-Martel [5] proved that given any nonempty compact subset $K$ of $\mathbb{R}^N$, there exists a $H^1$ solution of (1.1) which blows up exactly on $K$ when $t = 0$. We generalize the range of $\alpha$ to the $H^2$-subcritical case, following the technique developed in [6]. For technical reasons, we require that Re$\lambda > \frac{2}{N} |\text{Im}\lambda|$ when the dimension $N \geq 4$, which is used in the proof of the estimates of $\|\partial_t \varepsilon_n\|_{L^2}$, see (3.29)-(3.41).

**Remark 1.2.** It follows from (1.4) and (1.5) that both $\|u(t)\|_2$, $\|\nabla u(t)\|_2$ and $\|\Delta u(t)\|_2$ blow up when $t \uparrow 0$.

**Remark 1.3.** The estimate (1.4) can be refined. More precisely, it follows from (4.8) that
   \[
   (-t)^{-\frac{N}{2} + \frac{N}{2k}} \|u(t)\|_{L^2(|x-x_0|<r)} \lesssim (-t)^{-\frac{N}{2}}
   \]
   where $k > N\alpha$ is given by (2.2).

We prove Theorem 1.1 by the strategy of [1]. More precisely, we consider the sequence $\{u_n\}_{n \geq 1}$ of solutions of (1.1) with the initial datum $u_n(t) = U_f(-\frac{r}{n})$, where $U_f$ is a refined blowup profile defined in Lemma 2.3. It follows that $u_n$ is defined on $(s_n, -\frac{1}{n})$ for some $s_n < -\frac{1}{n}$. Letting $\varepsilon_n(t) = u_n(t) - U_0(t)$, following the ideas of [5, 15], we show that $\{\varepsilon_n\}_{n \geq 1}$ is uniformly bounded in $L^\infty((S,\tau), H^2) \cap W^{1,\infty}((S,\tau), L^2)$ ($S$ is given by Proposition 3.1) for any $\tau \in (S,0)$ by the energy arguments. Moreover, by the compactness argument, we find $\varepsilon \in L^\infty((S,0), H^2) \cap W^{1,\infty}((S,0), L^2)$ and a subsequence of $\{\varepsilon_n\}_{n \geq 1}$ weakly converges to $\varepsilon$. Therefore, setting $u(t) = U_f(t) + \varepsilon(t)$, we see that $u$ is a $H^2$ solution of (1.1). Finally, note that $\varepsilon$ is bounded in $H^2(\mathbb{R}^N)$ and $U_f$ blows up at time 0 exactly on $K$, we deduce that $u(t)$ also blows up at time 0 exactly on $K$.

The solution $u$ given by Theorem 1.1 blows up at $t = 0$ like the function $U_f$ defined in Lemma 2.3. Since the function $U_0$ defined by (2.7) satisfying $U_t = \lambda U^\alpha U$, and $U_f$ is a refinement of $U_0$, we see that the solution $u$ displays an ODE-type blowup. We recall that there are many ODE-type blowup results for several other nonlinear equations, refer to [10, 17, 20] for results in the parabolic context,
refer to [1, 18, 23] for the nonlinear wave equations. Recently, there are many well-posedness results for the nonlinear Schrödinger equation, see [9, 14, 24, 25] and references therein.

The rest of the paper is organized as follows. In Section 2, we introduce the blow-up ansatz and the corresponding estimates which are from [5], and recall some useful estimates. Section 3 is devoted to the construction of a sequence of solutions of (1.1) close to the blow-up ansatz and some a priori estimates of the approximate solutions. Finally, we complete the proof of Theorem 1.1 in Section 4 by passing to the limit in the approximate solutions.

2. The blow-up ansatz. In this section, we introduce the blow-up ansatz constructed in [5].

The first candidate $U_0$ is defined by (2.7) below, which is a solution of the ordinary differential equation $U_t = \lambda |U_0|^\alpha U_0$. Since the error term $i\Delta U_0$ is not integrable in time near the singularity when $\alpha$ is small, the method used in [1] does not applicable to the case $0 < \alpha \leq 1$. To treat any subcritical $\alpha$ and any $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$, Cazenave-Han-Martel [5] refine the blow-up ansatz inductively, using only ODE techniques, see (2.18)-(2.22) for more details. Throughout this section, we choose two integers

$$J = \left\lceil \frac{2}{\alpha} + 4\sigma \right\rceil + 1$$

and

$$k = \max\{2J + 4, \frac{16}{3\gamma \sigma}, N\alpha, \frac{1}{(1 - \frac{3}{2} \gamma)\sigma}\}$$

where

$$\gamma = \min\left\{\frac{1}{2}, \frac{\alpha}{\alpha + 2}, \frac{4}{N}\right\},$$

$$\sigma = \max\left\{\frac{4}{\gamma}, (2^{\alpha+1} + 4 + 4K_1)(\alpha + 1)|\lambda|\left|M(\alpha \Re \lambda)^{-1}\right|\right\},$$

where $M$ is given by Lemma 2.4 and $K_1 = |\Re \lambda - \frac{\alpha}{2} \Im \lambda|^{1-\alpha}(4(\alpha + 1)|\lambda|\lambda)^{\alpha-1}$. Let $K$ be any nonempty compact set of $\mathbb{R}^N$ included in the ball of center 0 and radius $R > 0$. It is well-known that there exists a smooth function $Z : \mathbb{R}^N \to [0, \infty)$ which vanishes exactly on $K$ (see Lemma 1.4 in [19]). Define the function $A : \mathbb{R}^N \to [0, \infty)$ by

$$A(x) = (Z(x)\chi(|x|) + (1 - \chi(|x|))|x|)^k$$

where

$$\chi \in C^\infty(\mathbb{R}, \mathbb{R}), \quad \chi(s) = \begin{cases} 1, & 0 \leq s \leq R, \\ 0, & s \geq 2R, \end{cases} \quad \chi'(s) \leq 0 \leq \chi(s) \leq 1, \quad s \geq 0.$$

It follows that the function $A \in C^{k-1}(\mathbb{R}^N, \mathbb{R})$, vanishes exactly on $K$, satisfies

$$\begin{cases} A \geq 0 \text{ and } |\partial_x^\beta A| \lesssim A^{1-\frac{|\beta|}{k}}, & \text{on } \mathbb{R}^N \text{ for } |\beta| \leq k - 1, \\ A(x) = |x|^{k}, & \text{for } x \in \mathbb{R}^N, |x| \geq 2R. \end{cases}$$

Set

$$U_0(t, x) = (\Re \lambda)^{-\frac{\alpha}{2}} (-\alpha t + A(x))^{-\frac{1}{2} - i \frac{\lambda}{\lambda \Re \lambda}}, \quad t < 0, x \in \mathbb{R}^N.$$

From (1.2), (2.2) and (2.6), we have

$$U_0 \text{ is } C^\infty \text{ in } t < 0 \text{ and } C^{k-1} \text{ in } x \in \mathbb{R}^N,$$

$$\partial_t U_0 = \lambda |U_0|^\alpha U_0, \quad t < 0, x \in \mathbb{R}^N,$$
\[ |U_0| = (\text{Re}\lambda)^{-\frac{1}{2}} (\alpha t + A(x))^{-\frac{1}{2}} \leq (\text{Re}\lambda)^{-\frac{1}{2}} (-t)^{-\frac{1}{2}}, \] (2.9)

and
\[ \partial_t |U_0| = \text{Re}\lambda |U_0|^{\alpha+1} \geq 0. \] (2.10)

Next we estimate the profile \( U_0 \) given by (2.7). We collect the estimates on \( U_0 \) which are from [5] and slight modifications.

**Lemma 2.1.** Under the conditions (1.2), (2.2) and (2.6), then we have
\[ \|U_0(t)\|_{L^p} \lesssim (-t)^{-\frac{1}{2}} \] (2.11)
for all \( p \geq 1 \) and \( -1 \leq t < 0 \). In addition, for every \( \rho \in \mathbb{R}, \ell \in \mathbb{N} \) and \( |\beta| \leq k - 1 \),
\[ |\partial_\ell^\beta \partial_2^\beta U_0| \lesssim |U_0|^{1+\ell+\frac{1}{2} |\beta|} \lesssim (-t)^{-\ell-\frac{|\beta|}{2}} |U_0|, \] (2.12)
\[ |\partial_\ell^\beta (|U_0|^{\rho})| \lesssim |U_0|^{\rho+\frac{1}{2} |\beta|} \lesssim (-t)^{-\frac{|\beta|}{2}} |U_0|^\rho, \] (2.13)
\[ |\partial_\ell^\beta (|U_0|^{\rho-1} U_0)| \lesssim |U_0|^{\rho+\frac{1}{2} |\beta|} \lesssim (-t)^{-\frac{|\beta|}{2}} |U_0|^\rho, \] (2.14)
\[ |\partial_\ell^\beta \partial_2^\beta |U_0|^\alpha U_0| \lesssim (-t)^{-1-\frac{|\beta|}{2}} |U_0|^{\alpha+1}, \] (2.15)
for all \( x \in \mathbb{R}^N, t < 0 \), and
\[ U_0 \in C^\infty((-\infty, 0), H^{k-1}(\mathbb{R}^N)). \] (2.16)

Furthermore, for any \( x_0 \in \mathbb{R}^N \) such that \( A(x_0) = 0 \), for any \( r > 0, -1 \leq t < 0 \) and \( 1 \leq p \leq \infty \),
\[ C_{r,p} (-t)^{-\frac{1}{2} + \frac{N}{2}} \lesssim \|U_0(t)\|_{L^p(|x-x_0|<r)}, \] (2.17)
where the constant \( C_{r,p} \) depends on \( r \) and \( p \).

**Proof.** Estimates (2.11)-(2.14) and the property (2.16) follows by the calculation in [5].

Note that \( |U_0| \) is positive for any time \( t < 0 \), we have
\[ \partial_t (|U_0|^\alpha U_0) = \frac{\alpha + 2}{2} |U_0|^\alpha \partial_t U_0 + \frac{\alpha}{2} |U_0|^{\alpha-2} U_0^2 \partial_2 U_0. \]

It follows from Leibnitz’s formula, (2.12)-(2.14) that
\[ |\partial_\ell^\beta \partial_2^\beta (|U_0|^\alpha U_0)| \lesssim \sum_{\beta_1 + \beta_2 = \beta} |\partial^\beta \partial_2^\beta (|U_0|^\alpha \partial_2^\beta \partial_2 U_0| + \sum_{\beta_1 + \beta_2 = \beta} |\partial^\beta (|U_0|^{\alpha-2} U_0^2 \partial_2^\beta \partial_2 U_0)|
\lesssim \sum_{\beta_1 + \beta_2 = \beta} (-t)^{-\frac{|\beta_1|}{2}} |U_0|^{\alpha} \cdot (-t)^{-1-\frac{|\beta_2|}{2}} |U_0| \lesssim (-t)^{-\frac{|\beta|}{2}} |U_0|^{\alpha+1}, \]
which proves (2.15).

To prove (2.17), we set \( x_0 \in \mathbb{R}^N \) such that \( A(x_0) = 0 \). For any fixed \( x \in \mathbb{R}^N \) satisfying \( |x-x_0| < r \), choosing \( x_1 \in \mathbb{R}^N \) satisfying \( |x_1-x_0| \leq |x-x_0| \) and
\[ |A(x_1)| = \max_{|y-x_0| \leq |x-x_0|} |A(y)|. \]
From (2.6), we have,
\[ |A(x_1)| = |A(x_1) - A(x_0)| = |\nabla A(\eta x_1 + (1-\eta)x_0) \cdot (x_1-x_0)| \leq C|A(\eta x_1 + (1-\eta)x_0)|^{1-\frac{1}{k}} |x_1-x_0| \leq C|A(x_1)|^{1-\frac{1}{k}} |x_1-x_0|, \]
for some \( \eta \in [0,1] \), and
\[ |A(x_1)| \leq C|x_1-x_0|^k. \]
Then, we have
\[ |A(x)| \leq |A(x_1)| \leq C|x_1 - x_0|^k \leq C|x - x_0|^k, \quad \forall \ |x - x_0| < r, \]
and
\[
\int_{|x-x_0| < r} |U_0|^p \, dx \gtrsim \int_{|x-x_0| < r} (-t + |x-x_0|^k)^{-\frac{p}{2}} \, dx
\]
\[ \gtrsim (-t)^{-\frac{p}{2} + \frac{K}{2}} \int_{|y| < r} (1 + |y|^k)^{-\frac{p}{2}} \, dy \geq C_{r,p}(-t)^{-\frac{p}{2} + \frac{K}{2}}. \]
This completes the proof of (2.17).

Next, we introduce a procedure to reduce the singularity of the error term at any order of \((-t)\) by refining the approximate solution. We consider the linearization of the equation (2.8),
\[
\partial_t w = \lambda \frac{\alpha + 2}{2} |U_0|^\alpha w + \lambda \frac{\alpha}{2} |U_0|^\alpha - 2 U_0^2 w
\]  
(2.18)
The equation (2.18) has two solutions \( w = iU_0 \) and \( w = \partial_t U_0 = \lambda |U_0|^\alpha U_0 \). By means of variation of constants, it is not hard to see that the corresponding nonhomogeneous equation
\[
\partial_t w = \lambda \left( \frac{\alpha + 2}{2} |U_0|^\alpha w + \frac{\alpha}{2} |U_0|^\alpha - 2 U_0^2 w \right) + G
\]  
(2.19)
has the solution \( w = \mathcal{P}(G) \), where
\[
\mathcal{P}(G) = \frac{\lambda}{\text{Re} \lambda} |U_0|^\alpha U_0 \int_0^t \left[ |U_0|^{-\alpha - 2} \text{Re}(\overline{U_0}G) \right](s) \, ds
\]
\[ + i \frac{1}{\text{Re} \lambda} U_0 \int_0^t \left[ |U_0|^{-2} \text{Im}(\overline{U_0}G) \right](s) \, ds \]  
(2.20)
We define \( U_j, w_j, \mathcal{E}_j \) by
\[
w_0 = iU_0, \quad \mathcal{E}_0 = -\partial_t U_0 + i\Delta U_0 + \lambda |U_0|^\alpha U_0 = i\Delta U_0
\]  
(2.21)
and then recursively
\[
w_j = \mathcal{P}(\mathcal{E}_{j-1}), \quad U_j = U_{j-1} + w_j \quad \mathcal{E}_j = -\partial_t U_j + i\Delta U_j + \lambda |U_j|^\alpha U_j
\]  
(2.22)
for \( j \geq 1 \), as long as they make sense. We will see that for \( j \leq \frac{k-4}{2} \), \( \mathcal{P}(\mathcal{E}_{j-1}) \) is well defined at each step, on a sufficiently small time interval. From similar arguments in Lemma 3.2 in [5], by Lemma 2.1 and Faà di Bruno’s formula (see Corollary 2.10 in [11]), we have the following estimates. For the convenience of the reader, we briefly sketch the proof.

**Lemma 2.2.** Under the conditions (1.2), (2.2) and (2.6), then there exists \(-1 < T < 0\) such that the following estimates hold for all \( 0 \leq j \leq \frac{k-4}{2} \).

1. If \( 0 \leq |\beta| \leq k - 1 - 2j \), then
\[
|\partial_x^\beta w_j| \lesssim (-t)^{(1-\frac{j}{2})-\frac{\beta}{2}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N,
\]  
(2.23)
\[
|\partial_x^\beta (U_j - U_0)| \lesssim (-t)^{1-\frac{\beta+2}{2}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N,
\]  
(2.24)
\[
|\partial_x^\beta \partial_t w_j| \lesssim (-t)^{-1+j(1-\frac{j}{2})-\frac{\beta}{2}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N.
\]  
(2.25)
2. If \( 0 \leq |\beta| \leq k - 3 - 2j \), then
\[
|\partial_x^\beta \mathcal{E}_j| \lesssim (-t)^{(1-\frac{j}{2})-\frac{\beta+2}{2}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N.
\]  
(2.26)
Moreover
\[
\frac{1}{2} |U_0| \leq |U_j| \leq 2 |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N, \quad (2.27)
\]
\[
U_j \in C^1 ((T, 0), H^{k-1-2j} (\mathbb{R}^N)), \quad (2.28)
\]
\[
|\partial_t U_j| \lesssim (-t)^{-1} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N, \quad (2.29)
\]
\[
|\partial_j \mathcal{E}_j| \lesssim (-t)^{-1+j(1-\frac{2}{d})-\frac{2}{d}} |U_0|, \quad T \leq t < 0, x \in \mathbb{R}^N. \quad (2.30)
\]

**Proof.** The proof is based on the induction on \(j\). From (2.12), we get that (2.23)-(2.30) hold with \(j = 0\).

Assume (2.23)-(2.30) hold with \(j \leq n\). Then, we only prove (2.25), (2.29) and (2.30) with \(j = n + 1\), and the other estimates with \(j = n + 1\), follows from Lemma 3.2 in [5].

In view of (2.20) and (2.22), we see that
\[
\partial_t w_{n+1} = \lambda \left( \frac{\alpha + 2}{2} |U_0|^\alpha w_{n+1} + \frac{\alpha}{2} |U_0|^{\alpha-2} U_0^2 \right) + \mathcal{E}_n. \quad (2.31)
\]
It follows from Leibnitz’s formula, (2.9), (2.13)-(2.14), (2.23) with \(j = n + 1\) and (2.26) with \(j = n\) that
\[
|\partial_t \partial_x^\beta w_{n+1}| \lesssim (-t)^{-1+(n+1)(1-\frac{2}{d})-\frac{2}{d}} |U_0|,
\]
which implies (2.25) with \(j = n + 1\).

Next by (2.22), we see that
\[
U_{n+1} = U_n + w_{n+1} = \cdots = w_{n+1} + w_n + \cdots + w_1 + U_0, \quad (2.32)
\]
so that \(|\partial_t U_{n+1}| \lesssim (-t)^{-1} |U_0|\) by (2.12) and (2.25) with \(j \leq n + 1\). Then (2.29) holds with \(j = n + 1\).

Finally, we prove (2.30) with \(j = n + 1\). Since \(U_{n+1} - U_n = w_{n+1}\), it follows from (2.19), (2.20) and (2.22) that
\[
\mathcal{E}_{n+1} - \mathcal{E}_n = -\partial_t w_{n+1} + i \Delta w_{n+1} + \lambda (|U_{n+1}|^\alpha U_{n+1} - |U_n|^\alpha U_n)
\]
\[
= -\mathcal{E}_n + i \Delta w_{n+1} + \lambda (|U_{n+1}|^\alpha U_{n+1} - |U_n|^\alpha U_n)
\]
\[
- \frac{\alpha + 2}{2} |U_0|^\alpha w_{n+1} - \frac{\alpha}{2} |U_0|^{\alpha-2} U_0^2 \frac{w_{n+1}}{|U_0|^\alpha}. \quad (2.32)
\]
Writing
\[
|U_{n+1}|^\alpha U_{n+1} - |U_n|^\alpha U_n = \int_0^1 \frac{d}{d\theta} |U_n + \theta w_{n+1}|^\alpha (U_n + \theta w_{n+1}) \, d\theta
\]
\[
= \int_0^1 \frac{\alpha + 2}{2} |U_n + \theta w_{n+1}|^\alpha w_{n+1} + \frac{\alpha}{2} |U_n + \theta w_{n+1}|^{\alpha-2} (U_n + \theta w_{n+1})^2 \frac{w_{n+1}}{|U_0|^\alpha} \, d\theta,
\]
we have
\[
\mathcal{E}_{n+1} = i \Delta w_{n+1} + \lambda \int_0^1 \frac{\alpha + 2}{2} A_{n+1}(t, \theta) w_{n+1} + \frac{\alpha}{2} B_{n+1}(t, \theta) \frac{w_{n+1}}{|U_0|^\alpha} \, d\theta, \quad (2.33)
\]
where
\[
A_{n+1}(t, \theta) = |U_n + \theta w_{n+1}|^\alpha - |U_0|^\alpha,
\]
\[
B_{n+1}(t, \theta) = |U_n + \theta w_{n+1}|^{\alpha-2} (U_n + \theta w_{n+1})^2 - |U_0|^{\alpha-2} U_0^2.
\]
By the directly computation, one can get
\[
A_{n+1}(t, \theta) = \int_0^1 \frac{d}{ds} |U_0 + sg_{n+1}(\theta)|^\alpha ds
= \int_0^1 \alpha \text{Re} \left[ |U_0 + sg_{n+1}(\theta)|^{\alpha-2} (U_0 + sg_{n+1}(\theta)) \frac{\partial g_{n+1}(\theta)}{\partial s} \right] ds
\]
where \( g_{n+1}(\theta) = U_n + \theta w_{n+1} - U_0 \). From (2.12), (2.23) with \( j = n+1 \), (2.24) with \( j = n \), (2.25) with \( j = n+1 \), (2.32), choosing \( T \) satisfying
\[
C_0 T^{1-\frac{\alpha}{2}} \leq \frac{1}{2},
\]
we obtain
\[
|g_{n+1}(\theta)| \leq C_0 (-t)^{-\frac{\alpha}{2}} |U_0| \leq \frac{1}{2} |U_0|,
\]
\[
|\partial_t g_{n+1}(\theta)| \lesssim (-t)^{-\frac{\alpha}{2}} |U_0|,
\]
\[
|\partial_t (U_0 + sg_{n+1}(\theta))| \lesssim (-t)^{-1} |U_0|.
\]
It follows from (2.34)-(2.38) and Leibnitz’s formula that
\[
|A_{n+1}(t, \theta)| \lesssim (-t)^{-\frac{\alpha}{2}}, \quad |\partial_t A_{n+1}(t, \theta)| \lesssim (-t)^{-1 - \frac{\alpha}{2}}.
\]
Similarly, using Leibnitz’s formula, we see that
\[
|B_{n+1}(t, \theta)| \lesssim (-t)^{-\frac{\alpha}{2}}, \quad |\partial_t B_{n+1}(t, \theta)| \lesssim (-t)^{-1 - \frac{\alpha}{2}}.
\]
Now it follows from (2.25) with \( j = n+1 \), (2.33), (2.39)-(2.40) and Leibnitz’s formula that
\[
|\partial_t \mathcal{E}_{n+1}| \lesssim |\partial_t \Delta w_{n+1}| + \int_0^1 (|A_{n+1}| + |B_{n+1}|) |\partial_t w_{n+1}| dt
+ \int_0^1 (|\partial_t A_{n+1}| + |\partial_t B_{n+1}|) |w_{n+1}| dt
\lesssim (-t)^{-1+(n+1)(1-\frac{\alpha}{2})-\frac{\alpha}{2}} |U_0|,
\]
which implies (2.30) with \( j = n+1 \). Thus (2.23)-(2.30) hold for all \( 0 \leq j \leq \frac{\alpha-2}{\alpha+2} \) by the induction.

Then, we get the following lemma immediately.

**Lemma 2.3.** Under the conditions in Lemma 2.2, we have
\[
|\partial^\beta_x (U_J - U_0)| \lesssim (-t)^{-1 - \frac{\alpha|\beta|}{2}} |U_0|, \quad 0 \leq |\beta| \leq k - 1 - 2J,
\]
\[
|\partial^\beta_x \mathcal{E}_J| \lesssim (-t)^{-1 - \frac{\alpha|\beta|}{2}} |U_0|, \quad 0 \leq |\beta| \leq k - 3 - 2J,
\]
\[
\frac{1}{2} |U_0| \leq |U_J| \leq 2 |U_0|,
\]
\[
U_J \in C^1 ((T, 0), H^{k-1-2J} (\mathbb{R}^N)) ,
\]
\[
|\partial_t U_J| \lesssim (-t)^{-1} |U_0|,
\]
\[
|\partial_t \mathcal{E}_J| \lesssim (-t)^{-1 - 1+J(1-\frac{\alpha}{2})-\frac{\alpha}{2}} |U_0|,
\]
\[
\mathcal{E}_J = -\partial_t U_J + i \Delta U_J + \lambda |U_J|^\alpha U_J,
\]
where \( T \leq t < 0, x \in \mathbb{R}^N, T \in (-1, 0) \).

Finally, we introduce some useful estimates, which will be used in Section 3.
Lemma 2.4. There exists a constant $M \geq 1$ such that
\begin{align*}
||u + v|| - |v|^\alpha &\leq M(||u||^\alpha + 1_{\alpha>1}|u||v|^\alpha), \\
||u + v||^\alpha - |v|^\alpha &\leq M(||u||^\alpha + 1_{\alpha>1}|u||v|^\alpha), \\
||u + v|| - |v| &\leq M(||u||^\alpha + |v|^\alpha)|u - v|,
\end{align*}
(2.48)
and if $0 < \alpha \leq 1$,
\begin{align*}
||u + v||^\alpha - |u|^\alpha &+ ||u + v||^\alpha - |v|^\alpha - 2 - |u|^\alpha - 2| \leq M |u|^\alpha - |v|,
\end{align*}
(2.51)
for all $u, v \in \mathbb{C}$, where
\[
1_{\alpha>1} = \begin{cases} 0, & \text{if } 0 < \alpha \leq 1, \\ 1, & \text{if } \alpha > 1. \end{cases}
\]

Proof. From (2.10) in [4], we can get (2.48) and (2.49), (also see formulas (2.26)-(2.27) in [3]). By the directly computation, one can get (2.50) easily, and omit the details. We prove (2.51) for completeness. Let $z \in \mathbb{C}, |z| \geq \frac{1}{2}$. From $|z|^\alpha \leq C|z|$, (2.48) and (2.49) we have
\begin{align*}
||1 + z|| - |1 + z|^\alpha - 1 + ||1 + z||^\alpha - 2(1 + z)^2 - 1 &\leq C|z|^\alpha \leq C|z|.
\end{align*}
(2.52)
For $|z| \leq \frac{1}{2}$, writing
\begin{align*}
||1 + z||^\alpha - 1 &+ ||1 + z||^\alpha - 2(1 + z)^2 - 1 \\
&= \int_0^1 \frac{d}{d\theta} \left[||1 + \theta z||^\alpha - 1 + ||1 + \theta z||^\alpha - 2(1 + \theta z)^2 - 1\right] d\theta,
\end{align*}
(2.53)
we get
\begin{align*}
\left|\frac{d}{d\theta} \left[||1 + \theta z||^\alpha - 1 + ||1 + \theta z||^\alpha - 2(1 + \theta z)^2 - 1\right]\right| &\leq C(\min_{0 \leq \theta \leq 1} |1 + \theta z|)^{\alpha-1}|z| \leq C|z|,
\end{align*}
(2.54)
which yields (2.52). Now let $u, v \in \mathbb{C}$ with $u \neq 0$, setting $z = v/u$ in (2.52), we obtain that the inequality (2.51) by choosing $M$ larger enough. \qed

Lemma 2.5. Assume that $\lambda \in \mathbb{C}, 0 < \alpha, (N-4)\alpha < 4$, $I \subset \mathbb{R}$ is a compact interval and $u \in C(I, H^2(\mathbb{R}^N)) \cap C^1(I, L^2)$ is a strong $H^2$ solution of the equation
\[
\partial_t u = i\Delta u + \lambda|u|^\alpha u,
\]
then we have
\[
\partial_t (|u|^\alpha u) \in \begin{cases} L^2(I, L^{\frac{2N}{N+2\alpha}}(\mathbb{R}^N)), & \text{if } 2 \leq (N-2)\alpha, \\ L^2(I, L^2(\mathbb{R}^N)), & \text{if } (N-2)\alpha < 2. \end{cases}
\]

Proof. Firstly we recall that $u$ is bounded in $W^{1,q}(I, L^r(\mathbb{R}^N)) \cap L^q(I, H^{2,r}(\mathbb{R}^N))$ for every admissible pair $(q, r) \in \Lambda$ where
\[
\Lambda = \{(q, r) : 2 \leq q, r, \infty, \frac{2}{q} + \frac{N}{r} = \frac{N}{2}, (q, r, N) \neq (2, \infty, 2)\},
\]
see [12, 21].

Then if $2 \leq (N-2)\alpha$, we choose two real numbers $r = \frac{2N(\alpha+1)}{(N-2)(\alpha+1)}$, $q = \frac{4(\alpha+1)}{(N-2)\alpha-2}$ such that $\frac{N+2}{2N} = \frac{1}{2} + \frac{\alpha}{2(\alpha+2)}$, and $(q, r) \in \Lambda$. By Hölder’s inequality and note that
2 ≤ r, (N − 2)r < 2N, q ≥ 2, \( H^2 \hookrightarrow L^{2\alpha+2} \), we deduce that
\[
\|\partial_t(|u|^\alpha u)\|_{L^2(I,L^{2\alpha+2}(\mathbb{R}^N))} \leq \|\partial_t u\|_{L^\infty(R^N)} \|u\|_{L^{2\alpha+2}(\mathbb{R}^N)} \|L^2(I)\|
\leq \|u\|_{L^\infty(I,H^2(\mathbb{R}^N))} \|\partial_t u\|_{L^2(I,L^\infty(\mathbb{R}^N))}
\leq C(I) \|\partial_t u\|_{L^{\infty}(I,L^\infty(\mathbb{R}^N))} \|L^2(I,L^\infty(\mathbb{R}^N))\) < +\infty.
\]

In the case \((N-2)\alpha < 2\), we may choose \( q = \frac{4(\alpha+1)}{N} > 2 \) such that \((q,2\alpha+2) \in \Lambda\). Thus, by Hölder’s inequality and \( H^2 \hookrightarrow L^{2\alpha+2} \), we deduce that
\[
\|\partial_t(|u|^\alpha u)\|_{L^2(I,L^2(\mathbb{R}^N))} \leq \|\partial_t u\|_{L^2(\mathbb{R}^N)} \|u\|_{L^{2\alpha+2}(\mathbb{R}^N)} \|L^2(I)\|
\leq C(I) \|\partial_t u\|_{L^2(I,L^{2\alpha+2}(\mathbb{R}^N))}
\leq C(I) \|\partial_t u\|_{L^{2\alpha}\cap L^\infty(I,L^{2\alpha+2}(\mathbb{R}^N))} < +\infty.
\]

\(\Box\)

3. Construction and estimates of approximate solutions. In this section, we construct a sequence of solutions \( u_n \) of (1.1), close to the ansatz \( U_J \) in Lemma 2.3, which will eventually converge to the blowing-up solution of Theorem 1.1. We will estimate \( \varepsilon_n = u_n - U_J \) by the energy method. More precisely, we estimate
\[
(-t)^{-\sigma}\|\varepsilon_n\|_2 \leq (t)^{(1-\frac{\gamma}{2})\sigma}\|\nabla\varepsilon_n\|_2 + (t)^{(1-\gamma)\sigma}\|\Delta\varepsilon_n\|_2 + (t)^{(1-\frac{\gamma}{2})\sigma}\|\partial_t \varepsilon_n\|_2
\]
for some appropriate parameters \( \gamma, \sigma \) given in (2.3) and (2.4).

Let the ansatz \( U_J \) and \( T < 0 \) be given in Lemma 2.3. From \( 2J \leq k-4 \) by (2.2), \( U_J \left( \frac{-1}{n} \right) \in H^2(\mathbb{R}^N) \) by (2.2) and (2.28), we obtain that there exist \( s_n \) and a unique solution \( u_n \in C^\infty\left( (s_n, -\frac{1}{n}) \cap C^1\left( (s_n, -\frac{1}{n}) \cap L^2(\mathbb{R}^N) \right) \right) \) of the following nonlinear Schrödinger equation
\[
\begin{cases}
\partial_t u_n = i\Delta u_n + \lambda |u_n|^\alpha u_n, \\
u_n \left( \frac{-1}{n} \right) = U_J \left( \frac{-1}{n} \right),
\end{cases}
\]
(3.1)
defined on the maximal interval \((s_n, -\frac{1}{n})\), with the blow-up alternative that if \( s_n > -\infty \), then
\[
\|u_n(t)\|_{H^2} \rightarrow \infty.
\]
(3.2)
see [12]. Letting \( \varepsilon_n \in C(I_n,H^2(\mathbb{R}^N)) \cap C^1(I_n,L^2(\mathbb{R}^N)) \) be defined by
\[
u_n = U_J + \varepsilon_n,
\]
(3.3)
with \( I_n = (\max\{s_n,T\}, -\frac{1}{n}) \), we have the following estimate.

**Proposition 3.1.** There exist \( T \leq S < 0 \) and \( n_0 > -\frac{1}{2} \) such that \( s_n \leq S \), for all \( n \geq n_0 \). Moreover,
\[
\|\varepsilon_n(t)\|_{L^2} \leq (-t)^{\sigma}, \quad \|\nabla\varepsilon_n(t)\|_{L^2} \leq (-t)^{(1-\frac{\gamma}{2})\sigma},
\]
(3.4)
\[
\|\Delta\varepsilon_n(t)\|_{L^2} \leq (-t)^{(1-\frac{\gamma}{2})\sigma}, \quad \|\partial_t \varepsilon_n(t)\|_{L^2} \leq (-t)^{(1-\frac{\gamma}{2})^\sigma},
\]
(3.5)
for all \( n \geq n_0 \) and \( t \in [S, -\frac{1}{n}] \).

**Proof.** Throughout the proof, we write \( \varepsilon \) instead of \( \varepsilon_n \). Moreover, \( C \) denotes a constant that may change from line to line, but is independent of \( n \) and \( t \). Unless otherwise specified, all integrals are over \( \mathbb{R}^N \). Using (2.22) and (3.3), we have
\[
\begin{cases}
\partial_t \varepsilon = i\Delta \varepsilon + \lambda |U_J + \varepsilon|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \mathcal{E}_J,
\varepsilon \left( \frac{-1}{n} \right) = 0.
\end{cases}
\]
(3.6)
Let
\[ \tau_n = \inf \left\{ t \in \left[ \max\{ T, s_n \}, -\frac{1}{n} \right] : \| \varepsilon(s) \|_{L^2} \leq (-s)^{\sigma} \right\}, \]
\[ \| \nabla \varepsilon(s) \|_{L^2} \leq (-s)^{(1-\frac{\alpha}{2})\sigma}, \quad \| \Delta \varepsilon(s) \|_{L^2} \leq (-s)^{(1-\gamma)\sigma}, \]
\[ \| \partial_t \varepsilon(s) \|_{L^2} \leq (-s)^{(1-\frac{\alpha}{2})\sigma}, \quad \text{for all } t < s \leq -\frac{1}{n}. \] (3.7)

Since \( \varepsilon(-\frac{1}{n}) = 0 \), we see that \( T \leq \tau_n < -\frac{1}{n} \). Moreover, it follows from the blow-up alternative (3.2) that \( s_n < \tau_n \).

We first estimate \( \| \varepsilon(t) \|_{L^2} \). Multiplying (3.6) by \( \varepsilon \) and taking the real part, we obtain
\[ \frac{1}{2} \frac{d}{dt} \| \varepsilon(t) \|_{L^2}^2 = \text{Re} \left( \lambda \int \| U_J + \varepsilon \|^2 (U_J + \varepsilon) \cdot |U_J|^\alpha U_J \varepsilon \right) + \text{Re} \int \mathcal{E}_J \varepsilon. \]
Using Lemma 2.4, we deduce that
\[ \frac{1}{2} \frac{d}{dt} \| \varepsilon(t) \|_{L^2}^2 \geq -|\lambda| M \int \| U_J \|^\alpha \| \varepsilon \|^2 - \| \mathcal{E}_J \|_{L^2} \| \varepsilon \|_{L^2}. \] (3.8)

By (2.9) and (2.43), we have
\[ \int \| U_J \|^\alpha \| \varepsilon \|^2 \leq 2^\alpha (\alpha \text{Re} \lambda)^{-1} (-t)^{-1} \| \varepsilon \|_{L^2}^2. \] (3.9)

In addition, by Gagliardo-Nirenberg’s inequality and (3.7), we get
\[ \int \| \varepsilon \|^{\alpha + 2} \leq C \| \varepsilon \|_{H^2}^{\alpha + 2} \| \Delta \varepsilon \|_{L^2}^{\frac{\alpha}{2}} \leq C \| \varepsilon \|_{H^2}^{\alpha + 2} \leq C(-t)^{(\alpha + 2)(1-\gamma)\sigma}. \] (3.10)

Next, by (2.42), we obtain
\[ \| \mathcal{E}_J \|_{L^2} \| \varepsilon \|_{L^2} \leq C(-t)^{J(1-\frac{\alpha}{2})-\frac{\sigma}{2}+\frac{\alpha}{2}+\sigma} = C(-t)^{-1+J+1(1-\frac{\alpha}{2})-\frac{\sigma}{2}}. \] (3.11)

By (2.1), (2.2) and (2.3), we have
\[ (J + 1) \left( 1 - \frac{2}{k} \right) - \frac{1}{\alpha} + \sigma \geq \frac{1}{2} (J + 1) - \frac{1}{\alpha} + \sigma \geq 3\sigma, \]
\[ (\alpha + 2)(1-\gamma)\sigma \geq 2\sigma, \] (3.12)
and
\[ |\lambda| M \int \| \varepsilon \|^{\alpha + 2} + \| \mathcal{E}_J \|_{L^2} \| \varepsilon \|_{L^2} \leq C(-t)^{2\sigma}, \] (3.13)
where \( T \in (-1,0) \) and \( \sigma > 1 \) by (2.4). It follows from (3.8), (3.9) and (3.13) that
\[ \frac{d}{dt} \| \varepsilon(t) \|_{L^2}^2 \geq -2^{\alpha+1} (\alpha \text{Re} \lambda)^{-1} |\lambda| M (-t)^{-1} \| \varepsilon \|_{L^2}^2 - C(-t)^{2\sigma} \]
and
\[ \frac{d}{dt} ((-t)^{-\sigma} \| \varepsilon(t) \|_{L^2}) = \sigma (-t)^{-\sigma-1} \| \varepsilon(t) \|_{L^2}^2 + (-t)^{-\sigma} \frac{d}{dt} \| \varepsilon(t) \|_{L^2}^2 \]
\[ \geq [\sigma - 2^{\alpha+1} (\alpha \text{Re} \lambda)^{-1} |\lambda| M] (-t)^{-\sigma-1} \| \varepsilon(t) \|_{L^2}^2 - C(-t)^{\sigma}. \]

Using (2.4), we obtain
\[ \frac{d}{dt} ((-t)^{-\sigma} \| \varepsilon(t) \|_{L^2}^2) \geq -C(-t)^{\sigma}. \]

Integrating on \( (t, -\frac{1}{n}) \) and using \( \varepsilon(-\frac{1}{n}) = 0 \), we deduce that
\[ \| \varepsilon(t) \|_{L^2} \leq C_1 (-t)^{\sigma + \frac{1}{2}} \] (3.14)
for all $t \in (\tau_n, \frac{1}{n})$.

Multiplying the equation (3.6) by $-\Delta \varepsilon$ and taking the real part, we obtain after integrating by parts

$$\frac{1}{2} \frac{d}{dt} \|\nabla \varepsilon\|^2_{L^2} = \text{Re} \lambda \int \nabla(|U_J + \varepsilon|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J - |\varepsilon|^\alpha \sigma) \cdot \nabla \varepsilon$$

$$+ \text{Re} \lambda \int \nabla(|\varepsilon|^\alpha \varepsilon) \cdot \nabla \varepsilon + \text{Re} \int \nabla \mathcal{E}_J \cdot \nabla \varepsilon := N_1 + N_2 + N_3.$$  

By Hölder’s and Gagliardo-Nirenberg’s inequality, and note that

$$\nabla(|\varepsilon|^\alpha \varepsilon) = \alpha + \frac{2}{2} |\varepsilon|^\alpha \nabla \varepsilon + \frac{\alpha}{2} |\varepsilon|^{\alpha - 2} \varepsilon^2 \nabla \varepsilon$$

we deduce that

$$|N_2| \leq C \int |\varepsilon|^{\alpha} |\nabla \varepsilon|^2 \leq C \left( \int |\varepsilon|^{2\alpha + 2} \right)^{\frac{\alpha}{\alpha + 2}} \left( \int |\nabla \varepsilon|^{\frac{4(\alpha + 1)}{\alpha + 2}} \right)^{\frac{\alpha + 2}{2\alpha + 2}}$$

$$\leq C \|\varepsilon\|_{H^2}^{\alpha + 2} \leq C(-t)^{(\alpha + 2)(1 - \gamma)\sigma} \leq C(-t)^{-1 + 2\sigma},$$

where $(N - 4)(2\alpha + 2) < 2N$ and $4(N - 2)(\alpha + 1)/(\alpha + 2) < 2N$ by (1.2), $(\alpha + 2)(1 - \gamma)\sigma \geq -1 + 2\sigma$ by (3.12). Next by (2.42) and (3.7), we see that

$$|N_3| \leq \|\nabla \mathcal{E}_J\|_{L^2} \|\nabla \varepsilon\|_{L^2} \leq C(-t)^{-1} \|U_0\|_{L^2} \|\nabla \varepsilon\|_{L^2}$$

$$\leq C(-t)^{-1} \|1 - \frac{\varepsilon}{\varepsilon^2} - \frac{1}{2} + (1 - 3\gamma)\sigma \leq C(-t)^{-1 + 2\sigma},$$

where

$$J \left(1 - \frac{2}{k}\right) - \frac{3}{k} - \frac{1}{\alpha} + (1 - \frac{3}{8}\gamma)\sigma = -1 + (J + 1) \left(1 - \frac{2}{k}\right) - \frac{1}{\alpha} + (1 - \frac{3}{8}\gamma)\sigma$$

$$> -1 + \frac{J + 1}{2} - \frac{1}{k} - \frac{1}{\alpha} + (1 - \frac{3}{8}\gamma)\sigma$$

$$> -1 + 2\sigma - \frac{1}{k} + (1 - \frac{3}{8}\gamma)\sigma \geq -1 + 2\sigma$$

by (2.1) and (2.2). We now estimate $N_1$. By the directly computation, we have

$$\nabla(|U_J + \varepsilon|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J - |\varepsilon|^\alpha \varepsilon)$$

$$= \alpha + \frac{2}{2}(|U_J + \varepsilon|^\alpha - |\varepsilon|^\alpha) \nabla \varepsilon + \frac{\alpha}{2}(|U_J + \varepsilon|^\alpha - 2(U_J + \varepsilon)^2 - |\varepsilon|^{\alpha - 2} \varepsilon^2) \nabla \varepsilon$$

$$+ \alpha + \frac{2}{2}(|U_J + \varepsilon|^\alpha - |U_J|^\alpha) \nabla U_J + \frac{\alpha}{2}(|U_J + \varepsilon|^\alpha - 2(U_J + \varepsilon)^2 - |U_J|^\alpha - 2U_J^2) \nabla \overline{U_J},$$

and

$$|N_1| \leq (\alpha + 1)\lambda \left( \int B_1 |\nabla \varepsilon|^2 + \int B_2 |\nabla U_J \nabla \varepsilon| \right)$$

with

$$B_1 = ||U_J + \varepsilon|^\alpha - |\varepsilon|^\alpha| + ||U_J + \varepsilon|^\alpha - 2(U_J + \varepsilon)^2 - |\varepsilon|^{\alpha - 2} \varepsilon^2|,$$

$$B_2 = ||U_J + \varepsilon|^\alpha - |U_J|^\alpha| + |U_J + \varepsilon|^\alpha - 2(U_J + \varepsilon)^2 - |U_J|^\alpha - 2U_J^2|.$$  

It follows from Lemma 2.4 and (2.43) that

$$B_1 \leq 2a M|U_0|^\alpha + 2M \lambda_{\alpha > 1} |\varepsilon|^{\alpha - 1} |U_0|.$$  

(3.20)

If $\alpha > 1$, then $|\varepsilon|^{\alpha - 1} |U_0| \leq |\varepsilon|^\alpha + |U_0|^\alpha$, so that

$$B_1 \leq (2a + 2) M |U_0|^\alpha + C|\varepsilon|^\alpha.$$  

(3.21)
Then, from (3.20)-(3.21), we obtain
\[
\int B_1|\nabla \varepsilon|^2 \leq (2^\alpha + 2)M(\alpha \Re \lambda)^{-1}(-t)^{\frac{1}{2}}\|\nabla \varepsilon\|_{L^2}^2 + C \int |\varepsilon|^{\alpha}|\nabla \varepsilon|^2 \\
\leq (2^\alpha + 2)M(\alpha \Re \lambda)^{-1}(-t)^{\frac{1}{2}}\|\nabla \varepsilon\|_{L^2}^2 + C(-t)^{-2\sigma + 2\sigma} \tag{3.22}
\]
by (2.9) and (3.16).
Next we estimate $B_2$, separately the cases $\alpha \leq 1$ and $\alpha > 1$. When $\alpha \leq 1$, using (2.9), (2.12), (2.41), (3.7) and Lemma 2.4, we deduce that
\[
\int B_2|\nabla U J \nabla \varepsilon| \leq C \int |U J|^{\alpha - 1}|\varepsilon||\nabla U J||\nabla \varepsilon| \\
\leq C(-t)^{-1 - \frac{\alpha}{2}}\|\varepsilon\|_{L^2}^\alpha \|\nabla \varepsilon\|_{L^2} \\
\leq C(-t)^{-1 - \frac{\alpha}{2} + (2 - \frac{3}{8})\gamma} \sigma, \tag{3.23}
\]
When $\alpha > 1$, we deduce from Lemma 2.4 and (2.41) that
\[
\int B_2|\nabla U J \nabla \varepsilon| \leq C \int |U J|^{\alpha - 1} + |\varepsilon|^{\alpha - 1})|\varepsilon||\nabla U J||\nabla \varepsilon| \\
\leq C\|U J\|_{L^\infty}^{\alpha - 1}||\nabla U J||_{\infty}||\varepsilon||_{L^2}||\nabla \varepsilon||_{L^2} + C\|U J\|_{L^\infty}||\varepsilon||_{L^2}^{\alpha} \|\nabla \varepsilon\|_{L^2} \\
\leq C(-t)^{-1 - \frac{\alpha}{2} + (\alpha - \alpha) \frac{3}{8} + (2 - \frac{3}{8})\gamma} \sigma + (1 - \frac{3}{4})\gamma)^\alpha \sigma, \tag{3.24}
\]
where $\|\varepsilon\|_{L^2} \leq C\|\varepsilon\|_{L^2}^{\alpha - 1} + |\varepsilon|^{\alpha - 1})|\varepsilon||\nabla U J||\nabla \varepsilon|$ by Gagliardo-Nirenberg’s inequality.
Note that
\[
\alpha - \alpha N \frac{1}{2} + \frac{1}{2\alpha} - 1 = -\frac{\alpha - 1}{1 - \frac{N}{4}} \geq 0
\]
by (2.3) and $\alpha > 1$, we deduce that $\int B_2|\nabla U J \nabla \varepsilon| \leq C(-t)^{-1 - \frac{\alpha}{2} + (2 - \frac{3}{8})\gamma} \sigma$. Moreover, we see that $-\frac{1}{k} + (2 - \frac{3}{8})\gamma \sigma \geq 2(1 - \frac{3}{8})\gamma \sigma + \frac{3\gamma}{16}$ by $k > \frac{16}{3\gamma}$ in (2.2), hence
\[
\int B_2|\nabla U J \nabla \varepsilon| \leq C(-t)^{-1 + 2(1 - \frac{3}{8})\gamma} \sigma + \frac{3\gamma}{16} . \tag{3.25}
\]
so that
\[
|N_1| \leq \frac{(\alpha + 1)(2^\alpha + 2)M|\lambda|}{\alpha \Re \lambda(-t)} \|\nabla \varepsilon\|_{L^2}^2 + C(-t)^{-1 + 2(1 - \frac{3}{8})\gamma} \sigma + \frac{3\gamma}{16} . \tag{3.26}
\]
Combining (3.15)-(3.17), (3.26) and $-1 + 2\sigma > -1 + 2(1 - \frac{3}{8})\gamma \sigma + \frac{3\gamma}{16}$, we obtain
\[
\frac{d}{dt}\|\nabla \varepsilon(t)\|_{L^2}^2 \geq -2(\alpha + 1)(2^\alpha + 2)M(\alpha \Re \lambda)^{-1}|\lambda|(-t)^{-1} \|\nabla \varepsilon\|_{L^2}^2 \\
- C(-t)^{-1 + 2(1 - \frac{3}{8})\gamma} \sigma + \frac{3\gamma}{16}.
\]
Using (2.4), we deduce that
\[
\frac{d}{dt} [(-t)^{-\sigma} \|\nabla \varepsilon(t)\|_{L^2}^2] = \sigma(-t)^{-\sigma - 1} \|\nabla \varepsilon(t)\|_{L^2}^2 + (-t)^{-\sigma} \frac{d}{dt} \|\nabla \varepsilon(t)\|_{L^2}^2 \\
\geq (\sigma - 2(\alpha + 1)(2^\alpha + 2)|\lambda|M(\alpha \Re \lambda)^{-1})(-t)^{-1 - \sigma} \|\nabla \varepsilon\|_{L^2}^2 \\
- C(-t)^{-1 + 1 + (1 - \frac{3}{8})\gamma} \sigma + \frac{3\gamma}{16} \\
\geq -C(-t)^{-1 + 1 + (1 - \frac{3}{8})\gamma} \sigma + \frac{3\gamma}{16}.
\]
Integrating on $(t, -\frac{1}{n})$, using $\varepsilon(-\frac{1}{n}) = 0$, and multiplying by $(-t)^{\sigma}$, we obtain
\[
\|\nabla \varepsilon(t)\|_{L^2} \leq C_2(-t)^{1 - \frac{3}{8})\gamma} \sigma + \frac{3\gamma}{16} . \tag{3.27}
\]
for all $\tau_n < t \leq \frac{1}{n}$. 

Thus, multiplying the equation (3.6) by $\Delta\varepsilon$ and taking the imaginary part, we obtain
\[
||\Delta\varepsilon||^2 \leq \lambda \int | \nabla (U_J + \varepsilon)^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J - |\varepsilon|^\alpha \varepsilon|| \nabla \varepsilon |
\]
\[
+ \lambda \int | \nabla (|\varepsilon|^\alpha \varepsilon)|| \nabla \varepsilon | + \int | \nabla \varepsilon \nabla \varepsilon | + \int | \partial_t \varepsilon \Delta \varepsilon |
\]
\[
\leq C(N_1 + N_2 + N_3) + ||\partial_t \varepsilon||_L^2 ||\Delta \varepsilon||_L^2
\]
\[
\leq C(-t)^{-1 + 2(1 - \frac{\gamma}{2})\sigma} + C(-t)^{(2 - \gamma - \frac{\gamma}{2})\sigma}
\]
\[
\leq C(-t)^{2(1 - \gamma)\sigma + \frac{\gamma}{2}}
\]
where $-1 + 2(1 - \frac{\gamma}{2})\sigma \geq 2(1 - \gamma)\sigma + \frac{\gamma}{2}$ by (2.4), and the (3.16), (3.17), (3.26) for the estimates of $N_1, N_2, N_3$. So we deduce that
\[
||\Delta \varepsilon||_2 \leq C_\delta (-t)^{(1 - \gamma)\sigma + \frac{\gamma}{2}}. \tag{3.28}
\]

Finally, we estimate $||\partial_t \varepsilon||_L^2$, which is similarly to $||\nabla \varepsilon||_L^2$ and slight modifications. We choose $\rho \in C_0^\infty(\mathbb{R}^N)$ with $\int \rho \, dx = 1$, and $\rho_s(x) = \rho(\frac{x}{s})\delta_{-N}(\delta > 0)$. Applying time derivative $\partial_t$ to the equation (3.6), taking convolution with $\rho_s$ and then multiplying it by $\partial_t \varepsilon \ast \rho_s$, taking the real part, we obtain after integrating by parts
\[
\frac{1}{2} \frac{d}{dt} ||\partial_t \varepsilon \ast \rho_s||_L^2 \tag{3.29}
\]
\[
= \operatorname{Re}[\lambda \int (\partial_t(U_J + \varepsilon)^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon) \ast (\partial_t \varepsilon \ast \rho_s)].
\]
Multiplying the equation (3.29) by $(-t)^{-\sigma}$, and then integrating it on the interval $(t, -\frac{1}{n})$, we obtain
\[
-\frac{1}{2} (-t)^{-\sigma} ||\partial_t \varepsilon \ast \rho_s||_L^2 = \int_{s_n}^{t} (-s)^{-\sigma} \left( \operatorname{Re}[\lambda \int (\partial_t(U_J + \varepsilon)^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon) \ast (\partial_t \varepsilon \ast \rho_s)] + \frac{\sigma}{2} (-s)^{-1} ||\partial_t \varepsilon \ast \rho_s||_L^2 \right) ds,
\]
where $s_n < t < -\frac{1}{n}$. Now by Lemma 2.5, (2.11), (2.41)-(2.42) and (2.45)-(2.46), we have that $\partial_t(U_J \ast |U_J|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon)$ is bounded in $L^2([T, -\frac{1}{n}], L^\infty(\mathbb{R}^N))$ if $2 \leq (N - 2)\alpha$ or bounded in $L^2([T, -\frac{1}{n}], L^2(\mathbb{R}^N))$ if $(N - 2)\alpha < 2$ for any $s_n < T < -\frac{1}{n}$. Note also that $\partial_t \varepsilon$ is bounded in $L^2([T, -\frac{1}{n}], L^\frac{2N}{N-2}(\mathbb{R}^N))(N \geq 3) \cap L^2([T, -\frac{1}{n}], L^2(\mathbb{R}^N))$ for any $s_n < T < -\frac{1}{n}$. Then, for a.e. $t \in (s_n, -\frac{1}{n})$, we deduce that
\[
\partial_t(U_J \ast |U_J|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon) \in L^\frac{2N}{N-2}(\mathbb{R}^N), \partial_t \varepsilon \in L^\frac{2N}{N-2}, \text{ if } 2 \leq (N - 2)\alpha,
\]
or
\[
\partial_t(U_J \ast |U_J|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon) \in L^2(\mathbb{R}^N), \partial_t \varepsilon \in L^2, \text{ if } (N - 2)\alpha < 2.
\]
By Young’s and Hölder’s inequality we deduce that for a.e. $t \in (s_n, -\frac{1}{n})$
\[
\operatorname{Re}[\lambda \int (\partial_t(U_J \ast |U_J|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon) \ast (\partial_t \varepsilon \ast \rho_s)] \rightarrow \operatorname{Re}[\lambda \int (\partial_t(U_J \ast |U_J|^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon)) \cdot \partial_t \varepsilon], \tag{3.31}
\]
and the left hand side of (3.31) is dominated by the integrable function

\[ |\lambda| \| \partial_t ((U_J + \varepsilon^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon)) \|_{L^2(R^N)} \| \partial_t \varepsilon \|_{L^2(R^N)}, \]

if \(2 \leq (N-2)\alpha\), or dominated by

\[ |\lambda| \| \partial_t ((U_J + \varepsilon^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J + \varepsilon)) \|_{L^2(R^N)} \| \partial_t \varepsilon \|_{L^2(R^N)}, \]

if \((N-2)\alpha < 2\). In both cases, the dominated function is integrable on interval \([T, -\frac{1}{n}]\) for any \(s_n < T < -\frac{1}{n}\). Thus, we can passing the limit \(\delta \to 0\) in (3.30) to get that

\[
-\frac{1}{2}(-t)^{-\sigma} \| \partial_t \varepsilon \|_{L^2}^2 \\
= \int_{-1}^{-\frac{1}{n}} (-s)^{-\sigma} \left( \text{Re}\lambda \int (\partial_t ((U_J + \varepsilon^\alpha (U_J + \varepsilon) - |U_J|^\alpha U_J - |\varepsilon|^\alpha \varepsilon)) + \partial_t (|\varepsilon|^\alpha \varepsilon) + \partial_t \varepsilon \cdot (\partial_t \varepsilon) + \frac{\sigma}{2} (-s)^{-1} \| \partial_t \varepsilon \|_{L^2}^2 \right) ds. \\
= \int_{-1}^{-\frac{1}{n}} (-s)^{-\sigma} (M_1 + M_2 + M_3 + \frac{\sigma}{2} (-s)^{-1} \| \partial_t \varepsilon \|_{L^2}^2) ds \tag{3.32}
\]

We first estimate \(M_2\). If \(N \geq 4\), then

\[
M_2 = \text{Re}\lambda \int \partial_t (|\varepsilon|^\alpha \varepsilon) \partial_t \varepsilon \\
= (\text{Re}\lambda) \text{Re} \int \partial_t (|\varepsilon|^\alpha \varepsilon) \partial_t \varepsilon - \text{Im}\lambda \cdot \text{Im} \int \partial_t (|\varepsilon|^\alpha \varepsilon) \partial_t \varepsilon \\
\geq \text{Re}\lambda \int |\varepsilon|^\alpha \partial_t \varepsilon|^2 - \text{Im}\lambda \frac{\alpha}{2} \text{Im} \int |\varepsilon|^\alpha - 2 \varepsilon^2 (\partial_t \varepsilon)^2 \\
\geq (\text{Re}\lambda - \frac{\alpha}{2} |\text{Im}\lambda|) \int |\varepsilon|^\alpha \partial_t \varepsilon|^2 = \mu \int |\varepsilon|^\alpha \partial_t \varepsilon|^2. \tag{3.33}
\]

where \(\mu = \text{Re}\lambda - \frac{\alpha}{2} |\text{Im}\lambda|\), and

\[
\text{Re} \int \partial_t (|\varepsilon|^\alpha \varepsilon) \partial_t \varepsilon = \frac{\alpha + 2}{2} \int |\varepsilon|^\alpha |\partial_t \varepsilon|^2 + \text{Re} \frac{\alpha}{2} \int |\varepsilon|^\alpha - 2 \varepsilon^2 (\partial_t \varepsilon)^2 \geq \int |\varepsilon|^\alpha |\partial_t \varepsilon|^2.
\]

When \(1 \leq N \leq 3\), we deduce that

\[
|M_2| \leq C \int |\varepsilon|^\alpha |\partial_t \varepsilon|^2 \leq C \| \varepsilon \|_{H^\alpha}^\alpha \| \partial_t \varepsilon \|_{H^2}^2 \leq C \| \varepsilon \|_{H^2}^\alpha \| \partial_t \varepsilon \|_{H^2}^2 \\
\leq C (-s)^{(2-\gamma)-(1-\gamma)\alpha} \leq C (-s)^{-1+2(1-\frac{n}{2})\sigma+\frac{\alpha}{4}}, \tag{3.34}
\]

by Sobolev’s embedding \(H^2(R^N) \hookrightarrow L^\infty(R^N)\) and (2.3), (3.7).

Next we estimate \(M_3\). By using (2.46), (3.7) and note that \(-1 + J(1 - \frac{n}{2}) - \frac{n}{2} + (1 - \frac{n}{2})\sigma - \frac{\alpha}{4} \geq -1 + 2(1 - \frac{n}{2})\sigma + \frac{\alpha}{4}\) by (2.1) and (2.2), we see that

\[
|M_3| \leq |\text{Re} \int \partial_t \varepsilon \cdot (\partial_t \varepsilon)| \leq C (-s)^{-1+J(1-\frac{n}{2})-\frac{n}{2}+(1-\frac{n}{2})\sigma-\frac{\alpha}{4}} \\
\leq C (-s)^{-1+2(1-\frac{n}{2})\sigma+\frac{\alpha}{4}}. \tag{3.35}
\]
We now estimate \( M_1 \). By the directly computation, we have
\[
\partial_t([U_j + \varepsilon]^{1/(\alpha)}[U_j + \varepsilon] - |U_j[1/\alpha]U_j - |\varepsilon|) \\
= \frac{\alpha + 2}{2}(|U_j + \varepsilon|^{1/\alpha} - |\varepsilon|)\partial_t\varepsilon + \frac{\alpha}{2}(|U_j + \varepsilon|^{1/\alpha - 2}(U_j + \varepsilon)^2 - |\varepsilon|^{1/\alpha - 2}\varepsilon^2)\partial_t\varepsilon \\
+ \frac{\alpha + 2}{2}(|U_j + \varepsilon|^{1/\alpha} - |U_j|^{1/\alpha})\partial_tU_j + \frac{\alpha}{2}(|U_j + \varepsilon|^{1/\alpha - 2}(U_j + \varepsilon)^2 - |U_j|^{1/\alpha - 2}U_j^2)\partial_tU_j,
\]
and
\[
|M_1| \leq (\alpha + 1)|\lambda|\left(\int B_1|\partial_t\varepsilon|^2 + \int B_2|\partial_tU_j\partial_t\varepsilon|\right). \tag{3.36}
\]
If \( \alpha > 1 \), then \(|\varepsilon|^{1-\alpha}|U_0| \leq \mu(4(\alpha + 1)|\lambda|M)^{-1}|\varepsilon|^{1/\alpha} + K_1|U_0|^{1/\alpha} \) by Young’s inequality, where \( K_1 = |\mu|^{1-\alpha}(4(\alpha + 1)|\lambda|M)^{\alpha - 1} \). By the inequality (2.48), we get
\[
\int B_1|\partial_t\varepsilon|^2 \leq \frac{(2^\alpha + 2K_1)M}{\alpha\text{Re}(\lambda(-s))}||\partial_t\varepsilon||_{L^2}^2 + \frac{\mu}{2(\alpha + 1)|\lambda|} \int |\varepsilon|^{1/\alpha}||\partial_t\varepsilon||_{L^2}^2. \tag{3.37}
\]
Moreover, if \( 1 \leq N \leq 3 \), we have by (3.34) that
\[
\int B_1|\partial_t\varepsilon|^2 \leq \frac{(2^\alpha + 2K_1)M}{\alpha\text{Re}(\lambda(-s))}||\partial_t\varepsilon||_{L^2}^2 + C(-s)^{-1+2(1-\frac{\sigma}{2})\gamma + \frac{\sigma}{2}}. \tag{3.38}
\]
Next we estimate \( B_2 \) term, separately the cases \( \alpha \leq 1 \) and \( \alpha > 1 \). When \( \alpha \leq 1 \), from Lemma 2.4, (2.9), (2.43), (2.45) and (3.7), we get that
\[
\int B_2|\partial_tU_j\partial_t\varepsilon| \leq C \int |U_j|^{1-\alpha}|\varepsilon||\partial_tU_j||\partial_t\varepsilon| \leq C(-s)^{-2+2(1-\frac{\sigma}{2})\gamma}. \tag{3.39}
\]
When \( \alpha > 1 \), we deduce from Lemma 2.4, (2.9), (2.43), (2.45) and (3.7), that
\[
\int B_2|\partial_tU_j\partial_t\varepsilon| \leq C \int ([U_j]^{1-\alpha} + |\varepsilon|^{1-\alpha})||\partial_tU_j||\partial_t\varepsilon| \\
\leq C(-s)^{-2}||\varepsilon||_{L^2}||\partial_t\varepsilon||_{L^2} + C(-s)^{-2}||\varepsilon||_{L^2}||\partial_t\varepsilon||_{L^2} \\
\leq C(-s)^{-2+2(1-\frac{\sigma}{2})\gamma} + C(-s)^{-2+1+2(1-\frac{\sigma}{2})\gamma} + C(-s)^{-2+2(1-\frac{\sigma}{2})\gamma} \\
\leq C(-s)^{-2+2(1-\frac{\sigma}{2})\gamma}, \tag{3.40}
\]
where \(-2 + (\alpha - \frac{N}{2}(\frac{1}{2} - \frac{1}{2\alpha}))\gamma + (1 - \frac{\gamma}{2})\sigma \geq -2 + 2(1 - \frac{\gamma}{2})\sigma \) by \( \alpha > 1 \) and (2.3). Combining (3.32)-(3.40), and note that \(-2 + (2 - \frac{\gamma}{2})\sigma \geq -1 + 2(1 - \frac{\gamma}{2})\sigma + \frac{2\gamma}{2} \) by (2.4), we obtain for all \( N \geq 1 \)
\[
-\frac{1}{2}(t)^{\gamma}||\partial_t\varepsilon||_{L^2}^2 \\
\geq \int_t^{-\frac{1}{2}} (-s)^{-\gamma} - \frac{1}{2} - (2^\gamma + 2K_1)(\alpha + 1)|\lambda|^{1/(\alpha\text{Re}(\lambda))} \cdot ||\partial_t\varepsilon||_{L^2}^2 \cdot ds \\
- C \int_t^{-\frac{1}{2}} (-s)^{-1+2(1-\gamma)\sigma + \frac{\gamma}{2}} \cdot ds \\
\geq -C \int_t^{-\frac{1}{2}} (-s)^{-1+2(1-\gamma)\sigma + \frac{\gamma}{2}} \cdot ds \geq -C(t)^{(1-\gamma)\sigma + \frac{\gamma}{2}},
\]
which implies that
\[
||\partial_t\varepsilon(t)||_{L^2} \leq C_4(-t)^{(1-\frac{\gamma}{2})\sigma + \frac{\gamma}{2}} \tag{3.41}
\]
for all \( \tau_n < t \leq -\frac{1}{2} \).
By (3.14), (3.27), (3.28), and (3.41), there exists \( S \in [T, 0) \) satisfying
\[
C_1(-S)^{\frac{\gamma}{2}} \leq 1, \quad C_2(-S)^{\frac{\gamma}{2}} \leq 1, \quad C_3(-S)^{\frac{\gamma}{2}} \leq 1, \quad C_4(-S)^{\frac{\gamma}{2}} \leq 1, \quad (3.42)
\]
such that for $n$ sufficiently large such that $S < -\frac{1}{n}$,

$$
\|\varepsilon\|_{L^2} \leq (-t)^\gamma, \|\nabla\varepsilon\|_{L^2} \leq (-t)^{(1-\frac{2}{r})\gamma}, \|\Delta\varepsilon\|_{L^2} \leq (-t)^{(1-\gamma)\gamma}, \|\partial_t\varepsilon\|_{L^2} \leq (-t)^{(1-\frac{2}{r})\gamma}
$$

for all $\tau_n < t < -\frac{1}{n}$ such that $t \geq S$. By the definition (3.7) of $\tau_n$, this implies that $\tau_n \leq S$. Using the blow-up alternative (3.2), we conclude that $s_n < S$, (3.4) and (3.5) hold.

4. Proof of Theorem 1.1. Using estimate (3.4) and (3.5), we deduce that $\{\varepsilon_n\}_{n \geq \frac{1}{2}}$ is bounded in $L^\infty([S, \Sigma], H^2(\mathbb{R}^N)) \cap W^{1,\infty}([S, \Sigma], L^2(\mathbb{R}^N))$ for any given $\Sigma \in (S, 0)$. Therefore, there exists $\varepsilon \in L^\infty([S, \Sigma], H^2(\mathbb{R}^N)) \cap W^{1,\infty}([S, \Sigma], L^2(\mathbb{R}^N))$ such that (after extracting a subsequence)

$$
\varepsilon_n \rightharpoonup \varepsilon, \quad \text{weak * in } L^\infty([S, \Sigma], H^2(\mathbb{R}^N)), \quad \partial_t \varepsilon_n \rightharpoonup \partial_t \varepsilon, \quad \text{weak * in } L^\infty([S, \Sigma], L^2(\mathbb{R}^N)). \tag{4.1}
$$

Moreover, note that for any bounded domain $\Omega \subset \mathbb{R}^N$, we have the embedding relation $H^2(\Omega) \hookrightarrow L^{2+\alpha}(\Omega) \hookrightarrow L^2(\Omega)$. Since $\{\varepsilon_n\}_{n \geq \frac{1}{2}}$ is uniformly bounded in $L^\infty([S, \Sigma], H^2(\Omega)) \cap W^{1,\infty}([S, \Sigma], L^2(\Omega))$, then we have (after extracting a subsequence),

$$
\varepsilon_n \rightharpoonup \varepsilon \text{ in } L^\infty([S, \Sigma], L^{2+\alpha}(\Omega)) \tag{4.2}
$$

by Aubin-Lions Theorem, see Simon [22]. Moreover, using $L^\infty(\Omega) \hookrightarrow L^{2+\alpha}(\Omega)$, we see that

$$
\varepsilon_n \rightharpoonup \varepsilon \text{ in } L^{2+\alpha}([S, \Sigma] \times \Omega) \tag{4.3}
$$

By the arbitrariness of $\Sigma$, a standard argument of diagonal extraction shows that there exists $\varepsilon \in L^\infty_{\text{loc}}([S, 0], H^2(\mathbb{R}^N)) \cap W^{1,\infty}_{\text{loc}}([S, 0], L^2(\mathbb{R}^N))$, such that (after extracting a subsequence) (4.1)-(4.3) hold for all $S < \tau < 0$, and

$$
\|\varepsilon(t)\|_{L^2} \leq (-t)^\gamma, \quad \|\nabla\varepsilon(t)\|_{L^2} \leq (-t)^{(1-\frac{2}{r})\gamma}, \quad \|\Delta\varepsilon(t)\|_{L^2} \leq (-t)^{(1-\gamma)\gamma}, \quad \|\partial_t\varepsilon(t)\|_{L^2} \leq (-t)^{(1-\frac{2}{r})\gamma}, \tag{4.4}
$$

for all $S < t < 0$. Moreover, it follows easily from (3.6) and the convergence properties (4.1)-(4.3) that

$$
\partial_t \varepsilon = i\Delta \varepsilon + \lambda(|U_J + \varepsilon|^\alpha(U_J + \varepsilon) - |U_J|^\alpha U_J) + \mathcal{E}_J, \quad \text{in } L^\infty_{\text{loc}}([S, 0], L^2(\mathbb{R}^N)). \tag{4.6}
$$

Therefore, setting

$$
u(t) = U_J(t) + \varepsilon(t), \quad S \leq t < 0, \tag{4.7}
$$

we see that $u \in L^\infty_{\text{loc}}([S, 0], H^2(\mathbb{R}^N)) \cap W^{1,\infty}_{\text{loc}}([S, 0], L^2(\mathbb{R}^N))$ and that

$$
\partial_t u = i\Delta u + \lambda|u|^\alpha u, \quad \text{in } L^\infty_{\text{loc}}([S, 0], L^2(\mathbb{R}^N)),
$$

by (2.47), (4.6) and (4.7). From the local existence in $H^2(\mathbb{R}^N)$ and the uniqueness in $L^\infty H^2_0$, we conclude that $u \in C([S, 0], H^2(\mathbb{R}^N)) \cap C^1([S, 0], L^2(\mathbb{R}^N))$.

We now prove (1.4)-(1.6) in Theorem 1.1. Let $\Omega$ be an open subset of $\mathbb{R}^N$ such that $\Omega \cap K = \emptyset$. It follows from (2.5) that $A > 0$ on $\Omega$ and $A(x) = |x|^k$ when $|x| > 2R$; and so there exists a constant $c > 0$, such that $A(x) \geq c(1 + |x|)^k$ on $\Omega$. Moreover using (2.6), (2.7) and (2.9), we deduce that

$$
|U_0| \leq C(1 + |x|)^{-\frac{k}{2}}, \quad |\nabla U_0| \leq C(1 + |x|)^{-\frac{k}{2} - 1} \quad \text{and} \quad |\Delta U_0| \leq C(1 + |x|)^{-\frac{k}{2} - 2}, \quad \text{on } \Omega.
$$

Since $(1 + |x|)^{-\frac{k}{2}} \in L^2(\mathbb{R}^N)$ by (2.2), applying (2.41) and (2.43), we conclude that

$$
\limsup_{t \to 0} \|U_J\|_{H^2(\Omega)} < \infty.
$$
Then the estimate (1.6) follows from (4.7) and the $L^\infty([S,0), H^2(\mathbb{R}^N))$ boundedness of $\varepsilon$ (4.4)-(4.5). Let now $x_0 \in K$ and $r > 0$, it follows from (2.11), (2.17) and (2.43) that

$$(-t)^{-\frac{1}{2} + \frac{N}{2N}} \leq \|U_J(t)\|_{L^2(|x-x_0| < r)} \lesssim (-t)^{-\frac{1}{2}}.$$ (4.8)

Using (4.7) and the embedding $H^2(|x-x_0| < r) \hookrightarrow L^2(|x-x_0| < r)$, we deduce that

$$\|u(t)\|_{L^2(|x-x_0| < r)} \geq \|U_J(t)\|_{L^2(|x-x_0| < r)} - \|\varepsilon(t)\|_{L^2(|x-x_0| < r)} \gtrsim (-t)^{-\frac{1}{2} + \frac{N}{2N}} - C\|\varepsilon(t)\|_{H^2(\mathbb{R}^N)},$$

which proves the estimate (1.4) in Theorem 1.1. Next, we prove the estimate (1.5) in Theorem 1.1. Since $k$ satisfies $(2 + \frac{4\alpha}{N})(N - 2) < 2N$ by (2.2), we fix a real number $p$ satisfying

$$p > 2 + \frac{4\alpha}{k} \text{ and } p(N - 2) < 2N.$$ (4.9)

We apply (2.11), (2.17), (2.43) and Gagliardo-Nirenberg’s inequality to obtain

$$(-t)^{-\frac{1}{2} + \frac{N}{2N}} \leq \|U_J\|_p \lesssim \|\Delta U_J\|_2^{\frac{N}{2}(\frac{1}{2} - \frac{k}{p})} \|U_J\|_2^{1 - \frac{N}{2}(\frac{1}{2} - \frac{k}{p})} \lesssim \|\Delta U_J\|_2^{\frac{N}{2}(\frac{1}{2} - \frac{k}{p})} (-t)^{-\frac{1}{2}(1 - \frac{N}{2}(\frac{1}{2} - \frac{k}{p}))}$$

and

$$(-t)^{-\frac{1}{2} + \frac{N}{2N}} \leq \|U_J\|_p \lesssim \|\nabla U_J\|_2^{N(\frac{1}{2} - \frac{k}{p})} \|U_J\|_2^{1 - N(\frac{1}{2} - \frac{k}{p})} \lesssim \|\nabla U_J\|_2^{N(\frac{1}{2} - \frac{k}{p})} (-t)^{-\frac{1}{2}(1 - N(\frac{1}{2} - \frac{k}{p}))},$$

which implies that

$$(-t)^{\frac{\alpha}{N}(\frac{1}{2} - \frac{k}{p} + \frac{1}{pk})} \lesssim \|\Delta U_J\|_2, \quad (-t)^{\frac{\alpha}{N}(\frac{1}{2} - \frac{k}{p} + \frac{1}{pk})} \lesssim \|\nabla U_J\|_2.$$ From (4.9), we have

$$\frac{1}{pk} - \frac{1}{2\alpha} + \frac{1}{p\alpha} < \frac{1}{pk} - \frac{1}{4\alpha} + \frac{1}{2p\alpha} < 0$$

and

$$\lim_{t \to 0} \|\nabla U_J\|_2 = \lim_{t \to 0} \|\Delta U_J\|_2 = \infty.$$ Combining (4.7) and (4.4)-(4.5), we have the estimate (1.5), and finish the proof of Theorem 1.1. □

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