Variability and skewness ordering of sample extremes from dependent random variables following the proportional odds model

Arindam Panja¹, Pradip Kundu²* and Biswabrata Pradhan¹
¹ SQC & OR Unit, ISI Kolkata, Kolkata-700108, India
² Decision Science and Operations Management, Birla Global University, Bhubaneswar, Odisha-751003, India

† Email Address: arindampnj@gmail.com (Arindam Panja), bis@isical.ac.in (Biswabrata Pradhan)

Abstract

The proportional odds (PO) model not only capable of generating new family of flexible distributions but also is a very important model in reliability theory and survival analysis. In this study, we investigate comparisons of minimums as well as maximums of samples from dependent random variables following the PO model and with Archimedean copulas, in terms of dispersive and star orders. Numerical examples are provided to illustrate the findings.

Keywords: Archimedean copula; Dispersive order; Proportional odds model; Star order.

1 Introduction

Let $X_{k:n}, k = 1, 2, \ldots, n$ denotes the $k$th order statistic corresponding to random variables (r.v.’s) $X_1, X_2, \ldots, X_n$. Order statistics play a crucial role in statistical inference, reliability theory, life-testing, operations research and economics. For example, in reliability theory, the smallest and the largest order statistics $X_{1:n}$ and $X_{n:n}$, respectively, represent the lifetimes of the series and the parallel systems, where the corresponding r.v.’s represent the lifetimes of $n$ components. Stochastic ordering has been widely used to compare the magnitude and variability of extreme order statistics. However, despite the importance and wide applications of the variability orders (e.g. dispersive order and star order), there are less research works in
this direction as compared to the magnitude orders (e.g., stochastic order, hazard rate order, reversed hazard rate order, and likelihood ratio order).

Dispersive order is one kind of variability order for comparing variability in probability distributions (Joen et al. [8], Kochar [12], Shaked and Shanthikumar [24]). Star order have been introduced in the literature to compare the relative skewness of probability distributions. The star order is also called more IFRA (increasing failure rate in average) order. Skewed distributions often serve as reasonable models for system lifetimes, auction theory, insurance claim amounts, financial returns etc. and thus it is of interest to compare skewness of probability distributions (Wu et al. [25]). Recently, some researchers have studied dispersive and star ordering of extreme order statistics from random samples come from different family of distributions (Ding et al. [5], Fang et al. [6, 7], Kochar and Xu [10, 11], Li and Fang [15], Nadeb et al. [20], Zhang et al. [27], Zhang et al. [28]). There are some research works on sample spacings also, like Xu and Balakrishnan [26] established dispersive and star ordering for sample spacing form heterogeneous exponential samples.

The proportional odds (PO) model (Bennett [2], Kirmani and Gupta [9]) is a very important model in reliability theory and survival analysis. Let \( X \) and \( Y \) be two r.v.’s with distribution functions \( F(\cdot) \), \( G(\cdot) \), and survival functions \( \bar{F}(\cdot) \), \( \bar{G}(\cdot) \) respectively. If the r.v. \( X \) denote a survival time, then the odds function \( \theta_X(t) \) defined by \( \theta_X(t) = \frac{\bar{F}(t)}{F(t)} \) represents the odds on surviving beyond time \( t \). The r.v.’s \( X \) and \( Y \) are said to satisfy PO model if \( \theta_Y(t) = \alpha \theta_X(t) \) for all admissible \( t \), where \( \alpha \) is a proportionality constant known as proportional odds ratio. Then the survival functions of \( X \) and \( Y \) are related as

\[
\bar{G}(t) = \frac{\alpha \bar{F}(t)}{1 - \alpha \bar{F}(t)},
\]

(1)

where \( \bar{\alpha} = 1 - \alpha \). From this relation, it can be observed that the ratio of hazard rate functions becomes \( 1/(1 - \bar{\alpha} \bar{F}(t)) \), so that the hazard ratio is increasing (decreasing) for \( \alpha > 1 \) (\( \alpha < 1 \)) and it converges to unity as \( t \) tends to \( \infty \). This is in contrast to the proportional hazard rate (PHR) model where this ratio remains constant with time. The convergence property of hazard functions makes the PO model reasonable in many practical applications as discussed by Bennett [2], Collett [3], Kirmani and Gupta [9], Lu and Zhang [17] and Rossini and Tsiatis [23]. Also, the model (1), with \( 0 < \alpha < \infty \), provides a method of generating more flexible new family of distributions known as Marshall-Olkin family of distributions or Marshall-Olkin extended distributions (Cordeiro et al. [4], Marshall and Olkin [19]), from an existing family of distributions. Extended Weibull distributions, extended linear failure-rate distributions and extended generalized exponential distributions are few examples those have been widely studied in the literature. Thus, model (1) has implications both in terms of the PO model and in extending any existing family of distributions to add flexibility in modeling. This makes the
Let $X_1, X_2, \ldots, X_n$ be a set of dependent r.v.'s with joint distribution function $F(\cdot)$ and joint survival function $\bar{F}(\cdot)$, marginal distribution functions $F_i(\cdot)$, and survival functions $\bar{F}_i(\cdot)$, $i = 1, 2, \ldots, n$. If there exist $C, \bar{C} : [0,1]^n \mapsto [0,1]$ such that $F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$ and $\bar{F}(x_1, \ldots, x_n) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n))$ for all $x_i, i \in I_n$, then $C$ and $\bar{C}$ are called the copula and survival copula respectively. If $\phi : [0, +\infty) \mapsto [0,1]$ with $\phi(0) = 1$ and $\lim_{t \to +\infty} \phi(t) = 0$, then $C(u_1, \ldots, u_n) = \phi(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_n)) = \varphi(\sum_{i=1}^n \phi(u_i))$ for all $u_i \in (0, 1)$, $i \in I_n$ is called an Archimedean copula with generator $\varphi$ provided $(-1)^k \varphi^{(k)}(t) \geq 0$, $k = 0, 1, \ldots, n - 2$ and $(-1)^{n-2} \varphi^{(n-2)}(t)$ is decreasing and convex for all $t \geq 0$. Here $\phi = \varphi^{-1}$ is the right continuous inverse of $\varphi$ so that $\phi(u) = \varphi^{-1}(u) = \sup\{t \in \mathbb{R} : \varphi(t) > u\}$. In case of dependent samples, Li and Fang [15] derived the dispersive order between maximums of two PHR samples having a common Archimedean copula. For samples following scale model, Li et al. [16] obtained the dispersive and the star order between minimums of one heterogeneous and one homogeneous samples sharing a common Archimedean copula. Fang et al. [6] investigated the dispersive order and the star order of extremes order statistics for the samples following PHR model with Archimedean survival copulas. Fang et al. [7] obtained the dispersive order between minimums of two scale proportional hazards samples with a common Archimedean survival copula. With resilience-scaled components, Zhang et al. [28] derived the dispersive and the star order between parallel systems, one consisting dependent heterogeneous components and another consisting homogeneous components sharing a common Archimedean survival copula.

In case of PO model, some authors, e.g. Kundu and Nanda [13], Kundu et al. [14], Nanda and Das [21] have investigated stochastic comparison of systems in the sense of magnitude orders. To the best of our knowledge, there is no related study on the variability of extreme order statistics arising from independent or dependent r.v.'s following the PO model. Motivated by this, in this paper we develop the dispersive and the star ordering for comparing the minimums and the maximums of dependent samples following the PO model. The organization of the rest of the paper is as follows. In Section 2, we briefly recalls definitions of variability orders. Section 3 investigates comparisons of minimum order statistics from dependent samples following the PO model, in terms of the dispersive order and the star order. Section 4 investigates the same in case of maximum order statistics. Section 5 presents some examples to illustrate the main results of the paper. In Section 6, we make concluding remarks.

## 2 Preliminaries

Assume that r.v.'s $X$ and $Y$ are absolutely continuous nonnegative r.v.'s with distribution functions $F(\cdot)$, $G(\cdot)$, respectively. Let $F^{-1}$ and $G^{-1}$ be the right continuous inverses of $F$ and $G$, respectively.
Definition 2.1 (Shaked and Shanthikumar [23]) Then X is said to be smaller than Y in the
(i) dispersive order (denoted as \( X \leq_{\text{disp}} Y \)) if if \( F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u) \) for all
\( 0 \leq u \leq v \leq 1 \). This is equivalently to \( G^{-1}F(x) - x \) is increasing in \( x \in \mathbb{R}_+ = [0, \infty) \).
When X and Y have densities \( f \) and \( g \), respectively, then \( X \leq_{\text{disp}} Y \) if, and only if,
\( g(G^{-1}(u)) \leq f(F^{-1}(u)) \) for all \( u \in (0,1) \);
(ii) star order (denoted by \( X \preceq_{*} Y \)) if \( G^{-1}F(x)/x \) is increasing in \( x \in \mathbb{R}_+ \).

Dispersive order is one kind of variability order for comparing variability in probability distributions. For more details and applications of dispersive ordering, see Joen et al. [8] and Kochar [12]. The star order is a partial orders to compare the skewness of two distributions. The star order is also called more IFRA (increasing failure rate in average) order. If one r.v. is smaller than another in terms of star order, then this can be interpreted as the former r.v. ages faster than the later in the sense of the star ordering. For more discussion and applications see Barlow and Proschan [1], Kochar [12] and Zhang et al. [27].

Definition 2.2 (Marshall and Olkin [18]) The distribution function \( F \) with hazard (failure) rate \( r(\cdot) \) and reversed hazard rate \( \tilde{r}(\cdot) \) is said to be of
(i) IFR/DFR (increasing/decreasing failure rate) if \( r(\cdot) \) is increasing/decreasing;
(ii) IRHR/DRHR (increasing/decreasing reversed hazard rate) if \( \tilde{r}(\cdot) \) is increasing/decreasing.

3 The dispersive ordering and the star ordering of minimums of dependent samples following the PO model

In this section we compare stochastically minimums of two dependent samples, one formed from heterogeneous r.v.’s and another from homogeneous r.v.’s.

Let \( X = (X_1, X_2, ..., X_n) \) is a set of dependent r.v.’s coupled with Archimedean survival copula with generator \( \varphi \) and following the PO model with baseline survival function \( \bar{F} \), denoted as \( X \sim PO(\bar{F}, \alpha, \varphi) \), where \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}_+^n \) is the proportional odds ratio vector. That is odds function of each r.v. \( X_i \) is proportional to an odds function (baseline odds) of a r.v. having distribution function \( F \), with proportionality constant \( \alpha_i \). We have the survival functions of \( X_{1:n} \) as
\[
\bar{F}_{X_{1:n}}(x) = \varphi \left( \sum_{i=1}^{n} \phi \left( \bar{F}_{X_i}(x) \right) \right),
\]
where \( \bar{F}_{X_i}(x) = \frac{\alpha_i F(x)}{1-A_i F(x)} \), \( \phi(u) = \varphi^{-1}(u) \), \( u \in (0,1] \).

The following theorem compares the minimum of two samples, one from \( n \) dependent heterogeneous r.v.’s following the PO model and another from \( n \) dependent homogeneous r.v.’s following the PO model, in terms of dispersive order.
Theorem 3.1 Suppose $X \sim PO(\bar{F}, \alpha, \phi)$ and $Y \sim PO(\bar{F}, \alpha, \varphi)$. Then for $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, $X_{1:n} \leq_{\text{disp}} Y_{1:n}$ if the baseline distribution is DFR, $\phi$ is log-convex, $\frac{\phi}{\varphi}$ is concave, and $0 \leq \alpha \leq 1$.

Proof: We have the distribution functions of $X_{1:n}$ and $Y_{1:n}$ as $F_1(x) = 1 - \varphi(\sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)))$, where $\bar{F}_{X_i}(x) = \frac{\alpha_i F(x)}{1 - \bar{\alpha}_i F(x)}$; and $G_1(x) = 1 - \varphi(n \phi(\bar{F}_{Y_1}(x)))$, where $\bar{F}_{Y_1}(x) = \frac{n \phi(x)}{1 - \bar{\alpha} F(x)}$, respectively. We get,

$$f_1(x) = \varphi^\prime \left( \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right) \sum_{i=1}^{n} \frac{\phi(\bar{F}_{X_i}(x))}{\varphi^\prime(\bar{F}_{X_i}(x))} \frac{r(x)}{1 - \bar{\alpha}_i F(x)},$$

(3)

$$G_1^{-1}(x) = \bar{F}^{-1} \left( \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)}{\alpha + \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)} \right),$$

$$g_1(x) = n \varphi^\prime \left( n \phi(\bar{F}_{Y_1}(x)) \right) \cdot \frac{r(x)}{1 - \bar{\alpha} F(x)} \cdot \frac{\varphi(\phi(\bar{F}_{Y_1}(x)))}{\varphi^\prime(\phi(\bar{F}_{Y_1}(x)))},$$

so that

$$G_1^{-1}(F_1(x)) = \bar{F}^{-1} \left( \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)}{\alpha + \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)} \right) = \bar{F}^{-1}(\gamma(x)),$$

(4)

where $\gamma(x) = \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)}{\alpha + \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)}$.

$$g_1(G_1^{-1}(F_1(x))) = n \varphi^\prime \left( n \phi \left( \frac{\alpha \gamma(x)}{1 - \bar{\alpha} \gamma(x)} \right) \right) \cdot \frac{r \left( \bar{F}^{-1}(\gamma(x)) \right)}{1 - \bar{\alpha} \gamma(x)} \cdot \frac{\varphi \left( \phi \left( \frac{n \gamma(x)}{1 - \gamma(x)} \right) \right)}{\varphi^\prime \left( \phi \left( \frac{n \gamma(x)}{1 - \gamma(x)} \right) \right)}$$

$$= n \varphi^\prime \left( \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right) \cdot \left( \alpha + \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right) \right) \cdot \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)}{\varphi^\prime \left( \frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)) \right)}.$$

(5)

Note that $\bar{F}_{X_i}(x)$ is increasing and concave in $\alpha_i$ and $1/(1 - \bar{\alpha}_i \bar{F}(x))$ is decreasing and convex in $\alpha_i$. Also it can be seen that $\phi(\bar{F}_{X_i}(x))$ is decreasing, and convex in $\alpha_i$ if $\varphi$ is log-convex. Now denote $\frac{1}{n} \sum_{i=1}^{n} \alpha_i = \alpha^{\text{avg}}$ and $\eta(\alpha_i) = \phi(\bar{F}_{X_i}(x))$. Then for $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i = \alpha^{\text{avg}}$, from the convexity and decreasing property of $\eta(\alpha_i) = \phi(\bar{F}_{X_i}(x))$ with respect to $\alpha_i$, we have
\[ \frac{1}{n} \sum_{i=1}^{n} \eta(\alpha_i) \geq \eta(\alpha_{\text{avg}}) \geq \eta(\alpha), \] which gives

\[ \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \geq \phi(F_{Y_i}(x)) \] (6)

implies 
\[ \frac{\alpha}{\hat{\alpha}} + \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right) \leq \frac{\alpha}{\hat{\alpha}} + \bar{F}_Y(x) \]

implies 
\[ 1 - \frac{\alpha}{\hat{\alpha}} + \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right) \leq 1 - \frac{\alpha}{\hat{\alpha}} + \bar{F}_Y(x) \]

implies 
\[ \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)}{\alpha + \alpha \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)} \leq \frac{\bar{F}_Y(x)}{\alpha + \alpha \bar{F}_Y(x)} \]

implies 
\[ \gamma(x) \leq \bar{F}(x). \]

Thus we have \( \bar{F}^{-1}(\gamma(x)) \geq x. \) Now if \( r(\cdot) \) is decreasing then

\[ r(\bar{F}^{-1}(\gamma(x))) \leq r(x). \] (7)

Now from (6), we have

\[ \alpha + \alpha \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right) \leq \alpha + \alpha \bar{F}_Y(x) \]

\[ = \frac{\alpha}{1 - \hat{\alpha} F(x)} \]

\[ \leq \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 - \alpha_i F(x)}. \] (8)

where the last inequality follows from the fact that \( \frac{1}{1 - \alpha_i F(x)} \) is decreasing and convex in \( \alpha_i. \)

If \( \varphi \) is concave, then we have

\[ - \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)}{\varphi' \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)} \leq - \frac{1}{n} \sum_{i=1}^{n} \varphi' \left( \phi(F_{X_i}(x)) \right). \] (9)

Thus we have

\[ \left( \alpha + \alpha \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right) \right) \left( - \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)}{\varphi' \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)} \right) \leq \alpha \frac{1}{n} \sum_{i=1}^{n} - \left( \frac{\varphi \left( \phi(F_{X_i}(x)) \right)}{\varphi' \left( \phi(F_{X_i}(x)) \right)} \right) \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 - \alpha_i F(x)} \] (10)

If \( \varphi \) is log-convex, then \( - \frac{\varphi(x)}{\varphi'(x)} \) is increasing in \( x, \) so that \( - \frac{\varphi(\phi(F_{X_i}(x)))}{\varphi'(\phi(F_{X_i}(x)))} \) is decreasing in \( \alpha_i. \) So
by Chebyshev’s inequality we have
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{\varphi(\bar{F}_{X_i}(x))}{\varphi'(\bar{F}_{X_i}(x))} \right) \cdot \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 - \alpha_i F(x)} \leq \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\varphi(\bar{F}_{X_i}(x))}{\varphi'(\bar{F}_{X_i}(x))} \right) \frac{1}{1 - \alpha_i F(x)} \tag{11}
\]

From (7), (10), (11) and the fact that the common factor \(\varphi'(\sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)))\) in (3) and (5) is negative, we have \(g_1(G_1^{-1}(F_1(x))) \leq f_1(x)\). Hence the theorem follows.

One may interested to know whether as in case of Theorem 3.1 we can establish dispersive ordering for \(\alpha \geq 1\) when the baseline distribution is IFR or DFR. The following counterexample shows that with these conditions, we cannot establish dispersive ordering even in case of samples from independent r.v.’s.

**Counterexample 3.1** Consider minimum of two samples, one having three independent and heterogeneous r.v.’s, and another having three independent and homogeneous r.v.’s with respective distribution functions \(F_1(x) = 1 - \prod_{i=1}^{3} \left( \frac{\alpha_i \bar{F}(x)}{1 - \alpha_i F(x)} \right)\) and \(G_1(x) = 1 - \left( \frac{\alpha \bar{F}(x)}{1 - \alpha F(x)} \right)^3\), where \(\alpha_1 = 7\), \(\alpha_2 = 25\), \(\alpha_3 = 100\), \(\alpha = (\alpha_1 + \alpha_2 + \alpha_3)/3 = 44\), and \(\bar{F}(x) = e^{-(9x)^{0.5}}\), so that the baseline distribution is DFR. We obtain

\[
g_1(G_1^{-1}(F_1(x))) = \frac{1}{\alpha} 3 \left( \prod_{i=1}^{3} \bar{F}_{X_i}(x) \right) \left( \alpha + \bar{\alpha} \left( \prod_{i=1}^{3} \bar{F}_{X_i}(x) \right)^{1/3} \right) r(\bar{F}^{-1}(\gamma(x))),
\]

where \(\gamma(x) = \frac{(\prod_{i=1}^{3} \bar{F}_{X_i}(x))^{1/3}}{\alpha + \bar{\alpha}(\prod_{i=1}^{3} \bar{F}_{X_i}(x))^{1/3}}\), and

\[
f_1(x) = \left( \prod_{i=1}^{3} \bar{F}_{X_i}(x) \right) r(x) \left( \sum_{i=1}^{3} \frac{1}{1 - \alpha_i F(x)} \right).
\]

We plot \(g_1(G_1^{-1}(F_1(x))) - f_1(x)\) by substituting \(x = t/(1 - t)\), so that for \(x \in [0, \infty)\), we have \(t \in [0, 1]\). From the Figure 4(a) we observe that \(g_1(G_1^{-1}(F_1(x))) \not\leq f_1(x)\) and also \(g_1(G_1^{-1}(F_1(x))) \not\geq f_1(x)\).

Next we take \(\alpha_1 = 0.78\), \(\alpha_2 = 0.97\), \(\alpha_3 = 67\), \(\alpha = (\alpha_1 + \alpha_2 + \alpha_3)/3 = 22.9167\), and \(\bar{F}(x) = e^{-x^3}\), so that the baseline distribution is IFR. We plot \(g_1(G_1^{-1}(F_1(x))) - f_1(x)\) by substituting \(x = t/(1 - t)\), so that for \(x \in [0, \infty)\), we have \(t \in [0, 1]\). From the Figure 4(b) we observe that \(g_1(G_1^{-1}(F_1(x))) - f_1(x) \not\leq 0\) and also \(g_1(G_1^{-1}(F_1(x))) - f_1(x) \not\geq 0\).

The following theorem compares the minimum of two samples, both from \(n\) dependent homogeneous r.v.’s following the PO model and with different Archimedean copulas.

**Theorem 3.2** Suppose \(X \sim PO(\bar{F}, \alpha f_1, \varphi_1)\) and \(Y \sim PO(\bar{F}, \alpha f_1, \varphi_2)\). Then \(X_{1:n} \leq_{disp} Y_{1:n}\) if baseline distribution is DFR, \(\varphi_2(\varphi_2(t)/n)/\varphi_1(\varphi_1(t)/n)\) is increasing in \(t\), and \(0 \leq \alpha \leq 1\).
Figure 1: Plot of $g_1(G_1^{-1}(F_1(x))) - f_1(x)$ for $x = t/(1-t)$, $t \in [0, 1]$ when baseline distribution is (a) DFR and (b) IFR.

**Proof:** The distribution functions of $X_{1:n}$ and $Y_{1:n}$ are given by $G_1(x) = 1 - \varphi_1(n\phi_1(\bar{F}_{X_1}(x)))$, and $G_2(x) = 1 - \varphi_2(n\phi_2(\bar{F}_{X_1}(x)))$, respectively, where $\bar{F}_{X_1}(x) = \frac{aF(x)}{1-aF(x)}$. We get,

$$g_1(x) = n\varphi'_1(n\phi_1(\bar{F}_{X_1}(x))) \frac{\varphi_1(\phi_1(\bar{F}_{X_1}(x)))}{\varphi'_1(\phi_1(F_{X_1}(x)))} \cdot \frac{r(x)}{1 - \alpha F(x)}, \quad \text{(12)}$$

$$g_2(x) = n\varphi'_2(n\phi_2(\bar{F}_{X_1}(x))) \frac{\varphi_2(\phi_2(\bar{F}_{X_1}(x)))}{\varphi'_2(\phi_2(F_{X_1}(x)))} \cdot \frac{r(x)}{1 - \alpha F(x)}$$

so that

$$G_2^{-1}(G_1(x)) = F^{-1} \left( \frac{\varphi_2(\frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))))}{\alpha + \bar{\alpha} \varphi_2(\frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))))} \right) = \tilde{F}^{-1}(\eta(x)) \quad \text{(say)}, \quad \text{(13)}$$

$$g_2(G_2^{-1}(G_1(x))) = n\varphi'_2(\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))) \frac{\varphi_2(\frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))))}{\varphi'_2(\frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))))} \cdot \left( \alpha + \bar{\alpha} \varphi_2 \left( \frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))) \right) \right). \quad \text{(14)}$$

From Lemma 3.9 of Fang et al. [6], for increasing $\varphi_2(\phi_2(t)/n)/\varphi_1(\phi_1(t)/n)$ we have $\varphi_2(n\phi_2(\bar{F}_{X_1}(x))) \geq \varphi_1(n\phi_1(\bar{F}_{X_1}(x)))$, which implies $\bar{F}_{X_1}(x) \geq \varphi_2(\frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x))))).$ Again from this we get $\bar{F}(x) \geq \eta(x)$ which implies $\tilde{F}^{-1}(\eta(x)) \geq x$. Thus if $r(\cdot)$ is decreasing then

$$r(\tilde{F}^{-1}(\eta(x))) \leq r(x). \quad \text{(15)}$$

Also for $\bar{\alpha} \geq 0$, $\varphi_2(\frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))) \leq \tilde{F}_{X_1}(x)$ implies

$$\alpha + \bar{\alpha} \varphi_2 \left( \frac{1}{n}\phi_2(\varphi_1(n\phi_1(\bar{F}_{X_1}(x)))) \right) \leq \frac{\alpha}{1 - \tilde{\alpha} F(x)}. \quad \text{(16)}$$
Again from Lemma 3.9 of Fang et al. [6] by substituting $t = \varphi_1 \left( n \varphi_1 \left( \bar{F}_{X_1}(x) \right) \right)$, we get

$$
\frac{\varphi'_2 \left( \varphi_1 \left( n \varphi_1 \left( \bar{F}_{X_1}(x) \right) \right) \right)}{\varphi'_2 \left( \frac{1}{n} \varphi_2 \left( \varphi_1 \left( n \varphi_1 \left( \bar{F}_{X_1}(x) \right) \right) \right) \right)} \leq \frac{\varphi'_1 \left( n \varphi_1 \left( \bar{F}_{X_1}(x) \right) \right) \varphi_1 \left( \bar{F}_{X_1}(x) \right)}{\varphi'_1 \left( \frac{1}{n} \varphi_1 \left( \bar{F}_{X_1}(x) \right) \right)}.
$$

(17)

Now using (15), (16) and (17), we have $g_2(G_2^{-1}(G_1(x))) \leq g_1(x)$. This completes the proof. \(\square\)

The following corollary follows from Theorems 3.1 and 3.2. This corollary compares the minimum of two samples, one from $n$ dependent heterogeneous r.v.’s following the PO model and another from $n$ dependent homogeneous r.v.’s following the PO model and with different Archimedean copulas.

**Corollary 3.1** Suppose $X \sim PO(\bar{F}, \alpha, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha, \varphi_2)$. Then for $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, $X_{1:n} \leq_{disp} Y_{1:n}$ if the baseline distribution is DFR, $\varphi_1$ is log-convex, $\frac{\bar{F}_{X_1}(x)}{\bar{F}_{Y_1}(x)}$ is concave, $\varphi_2(\phi(t)/n)/\varphi_1(\phi(t)/n)$ is increasing in $t$, and $0 \leq \alpha \leq 1$.

**Proof:** Let $Z \sim PO(\bar{F}, \alpha \mathbf{1}, \varphi_1)$. Then from Theorem 3.1 we have $X_{1:n} \leq_{disp} Z_{1:n}$. Again from Theorem 3.2 we have $Z_{1:n} \leq_{disp} Y_{1:n}$. \(\square\)

The following theorem compares the minimum of two samples, one from $n$ dependent heterogeneous r.v.’s following the PO model and another from $n$ dependent homogeneous r.v.’s following the PO model, in terms of star order.

**Theorem 3.3** Suppose $X \sim PO(\bar{F}, \alpha, \varphi)$ and $Y \sim PO(\bar{F}, \alpha, \varphi)$. Then for $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, $X_{1:n} \preceq_{\star} Y_{1:n}$ if $\varphi(x)$ is decreasing, $\varphi$ is log-convex, $\frac{\varphi}{\bar{F}}$ is concave, and $0 \leq \alpha \leq 1$.

**Proof:** Using equations (9), (11) and (13), we have

$$
x^2 \frac{d}{dx} \left( \frac{G_1^{-1}(F_1(x))}{x} \right)
= x \frac{d}{dx} \left( G_1^{-1}(F_1(x)) \right) - G_1^{-1}(F_1(x))
= x \frac{f_1(x)}{g_1 \left( G_1^{-1}(F_1(x)) \right)} - G_1^{-1}(F_1(x))
= \frac{\alpha x r(x) \frac{1}{n} \sum_{i=1}^{n} \frac{\varphi(F_{X_i}(x))}{\varphi(F(X_i(x)))} \frac{1}{1-\alpha F(x)}}{r \left( \bar{F}^{-1}(\gamma(x)) \right) \left( \alpha + \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right) \right) \frac{\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)}{\bar{F} \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_{X_i}(x)) \right)}} - \bar{F}^{-1}(\gamma(x)).
$$

(18)

In Theorem 3.1 for $0 \leq \alpha \leq 1$ we have already proved that

$$
\bar{F}^{-1}(\gamma(x)) \geq x.
$$

(19)
Now, if \( xr(x) \) is decreasing in \( x \), then we have \( xr(x) \geq \bar{F}^{-1}(\gamma(x))r(\bar{F}^{-1}(\gamma(x))) \), that is

\[
\frac{xr(x)}{r(\bar{F}^{-1}(\gamma(x)))} \geq \bar{F}^{-1}(\gamma(x)).
\] (20)

Now combining (10) and (11), we get

\[
\frac{\alpha}{n} \sum_{i=1}^{n} \left( \frac{\varphi(\phi(F_{X_i}(x)))}{\varphi'(\phi(F_{X_i}(x)))} \right) \frac{1}{1 - \alpha, F(x)} (\alpha + \bar{\alpha} \varphi (\frac{1}{n} \sum_{i=1}^{n} \phi(\bar{F}_{X_i}(x)))) \geq 1.
\] (21)

Using (20) and (21), from (18) we get

\[
x^2 \frac{d}{dx} \left( \frac{G^{-1}_1(F_1(x))}{x} \right) \geq 0.
\]

So, \( \frac{G^{-1}_1(F_1(x))}{x} \) is increasing in \( x \geq 0 \). Hence \( X_{n:n} \leq_s Y_{n:n} \).

We are interested to know whether as in case of Theorem 3.3 we can establish dispersive ordering for \( \alpha \geq 1 \) when when \( xr(x) \) is decreasing or increasing. The following counterexample shows that with these conditions, we cannot establish star ordering even in case of samples from independent r.v.'s.

**Counterexample 3.2** Consider maximums of two samples, one having four independent and heterogeneous r.v.'s, and another having four independent and homogeneous r.v.'s. Consider \( \alpha_1 = 0.75, \alpha_2 = 0.95, \alpha_3 = 23, \alpha_4 = 43, \alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 16.925, \) and \( \bar{F}(x) = (1 + \frac{x}{13})^{-0.9} \), so that \( xr(x) \) is increasing. We have

\[
G^{-1}_1(F_1(x)) = \bar{F}^{-1} \left( \frac{\prod_{i=1}^{4} \bar{F}_{X_i}(x)^{1/4}}{\alpha + \bar{\alpha} \left( \prod_{i=1}^{4} \bar{F}_{X_i}(x)^{1/4} \right)} \right).
\] (22)

We plot \( G^{-1}_1(F_1(x))/x \) by substituting \( x = t/(1 - t) \), so that for \( x \in [0, \infty) \), we have \( t \in [0, 1) \). From the Figure 2(a), we observe that \( G^{-1}_1(F_1(x))/x \) is neither increasing nor decreasing.

Next we take \( \alpha_1 = 2, \alpha_2 = 33, \alpha_3 = 63, \alpha_4 = 183, \alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 281/4, \) and \( \bar{F}(x) = \frac{1}{x^2} \), \( x \in [1, \infty) \) so that \( xr(x) \) is decreasing. We plot \( G^{-1}_1(F_1(x))/x \) by substituting \( x = 1/t \), so that for \( x \in [1, \infty) \), we have \( t \in [0, 1) \). From the Figure 2(b), we observe that \( G^{-1}_1(F_1(x))/x \) is neither increasing nor decreasing.

The following theorem compares the minimum of two samples, both from \( n \) dependent homogeneous r.v.'s following the PO model and with different Archimedean copulas. The proof can be done using the results of proof of Theorem 3.2 in the same line as of Theorem 3.3 and hence...
omitted.

**Theorem 3.4** Suppose \( X \sim \text{PO}(\bar{F}, \alpha_1, \varphi_1) \) and \( Y \sim \text{PO}(\bar{F}, \alpha_2, \varphi_2) \). Then \( X_{1:n} \leq_\ast Y_{1:n} \) if \( x_r(x) \) is decreasing, \( \varphi_2(\phi_2(t)/n)/\varphi_1(\phi_1(t)/n) \) is increasing in \( t \), and \( 0 \leq \alpha \leq 1 \).

The following corollary follows from Theorems \ref{thm:3.3} and \ref{thm:3.4}.

**Corollary 3.2** Suppose \( X \sim \text{PO}(\bar{F}, \alpha, \varphi_1) \) and \( Y \sim \text{PO}(\bar{F}, \alpha_1, \varphi_2) \). Then for \( \alpha \geq 1 \), 
\[
X_{1:n} \leq_\ast Y_{1:n} \quad \text{if} \quad x_r(x) \text{ is decreasing, } \varphi_1 \text{ is log-convex, } \frac{\varphi_2'}{\varphi_1'} \text{ is concave, } \varphi_2(\phi_2(t)/n)/\varphi_1(\phi_1(t)/n) \text{ is increasing in } t, \quad \text{and} \quad 0 \leq \alpha \leq 1.
\]

## 4 The dispersive ordering and the star ordering of maximum of dependent samples following the PO model

In this section we compare stochastically maximums of two dependent samples, one formed from heterogeneous r.v.’s and another from homogeneous r.v.’s.

The distribution function of \( X_i \) and \( Y_i \) are 
\[
F_{X_i}(x) = \frac{F(x)}{1-\alpha_i F(x)} \quad \text{and} \quad F_{Y_i}(x) = \frac{F(x)}{1-\alpha F(x)},
\]
respectively, where \( \alpha_i = 1 - \alpha \) for \( i = 1, 2, \ldots, n \), and \( \bar{\alpha} = 1 - \alpha \). The distribution functions of \( X_{n:n} \) and \( Y_{n:n} \) are given by
\[
F_{X_{n:n}}(x) = \varphi \left( \sum_{i=1}^{n} \phi (F_{X_i}(x)) \right),
\]
and
\[
F_{Y_{n:n}}(x) = \varphi (n\phi (F_{Y_1}(x))),
\]
where \( \phi(u) = \varphi^{-1}(u) \), \( u \in (0,1] \).

The following theorem compares the maximums of two samples, one from \( n \) dependent heterogeneous r.v.’s following the PO model and another from \( n \) dependent homogeneous r.v.’s following the PO model, in terms of dispersive order.
Theorem 4.1 Suppose $X \sim PO(F, \alpha, \varphi)$ and $Y \sim PO(F, \alpha 1, \varphi)$. Then for $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, $X_{n,n} \geq disp Y_{n,n}$ if the baseline distribution is IRHR, $\varphi$ is log-concave, and $\frac{\alpha}{n}$ is convex.

Proof: From equations (23) and (24), we have the distribution functions of $X_{n,n}$ and $Y_{n,n}$ $F_2(x) = \varphi \left( \sum_{i=1}^{n} \phi (F_X(x)) \right)$ and $G_2(x) = \varphi (n \phi (F_Y(x)))$, respectively, where $F_X(x) = \frac{F(x)}{\alpha_i + \tilde{\alpha} F(x)}$ and $F_Y(x) = \frac{F(x)}{\alpha + \tilde{\alpha} F(x)}$. We get,

$$f_2(x) = \varphi' \left( \sum_{i=1}^{n} \phi (F_X(x)) \right) \sum_{i=1}^{n} \varphi (\phi (F_X(x))) \frac{\alpha_i \tilde{r}(x)}{\alpha_i + \tilde{\alpha}_i F(x)},$$

$$G_2^{-1}(x) = F^{-1} \left( \frac{\alpha \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_X(x)) \right)}{1 - \tilde{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_X(x)) \right)} \right),$$

$$g_2(x) = n \varphi' \left( n \phi (F_Y(x)) \right) \cdot \frac{\alpha \tilde{r}(x)}{\alpha + \tilde{\alpha} F(x)} \cdot \frac{\varphi (\phi (F_Y(x)))}{\varphi' (\phi (F_Y(x)))},$$

so that

$$G_2^{-1}(F_2(x)) = F^{-1} \left( \frac{\alpha \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_X(x)) \right)}{1 - \tilde{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_X(x)) \right)} \right) = F^{-1}(\beta(x)),$$ (25)

where $\beta(x) = \frac{\alpha \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_X(x)) \right)}{1 - \tilde{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_X(x)) \right)}$.

$$g_2(G_2^{-1}(F_2(x))) = n \varphi' \left( \frac{\beta(x)}{\alpha + \tilde{\alpha} \beta(x)} \right) \cdot \frac{\alpha \tilde{r}(x)}{\alpha + \tilde{\alpha} \beta(x)} \cdot \frac{\varphi \left( \frac{\beta(x)}{\alpha + \tilde{\alpha} \beta(x)} \right)}{\varphi' \left( \frac{\beta(x)}{\alpha + \tilde{\alpha} \beta(x)} \right)} \cdot \tilde{r} \left( F^{-1}(\beta(x)) \right).$$

Note that $\alpha_i/(\alpha_i + \tilde{\alpha}_i F(x))$ is increasing and concave in $\alpha_i$. It can be seen that $\phi (F_X(x))$ is increasing and concave in $\alpha_i$ if $\varphi$ is log-concave. First we take $\alpha \leq 0$. For $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, from
For $\bar{\alpha} \geq 0$, from (28) we have

\[
1 - \frac{1}{1 - \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_{X_i}(x)) \right)} \geq 1 - \frac{1}{1 - \bar{\alpha} F_{Y_1}(x)} - 1
\]

implies

\[
\frac{\alpha \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_{X_i}(x)) \right)}{1 - \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_{X_i}(x)) \right)} \geq \frac{\alpha F_{Y_1}(x)}{1 - \bar{\alpha} F_{Y_1}(x)}
\]

implies $\beta(x) \geq F(x)$.

Thus we have $F^{-1}(\beta(x)) \geq x$. Now if $\tilde{r}(\cdot)$ is increasing then

\[
\tilde{r}(F^{-1}(\beta(x))) \geq \tilde{r}(x).
\]

Next we take $\bar{\alpha} \geq 0$. As $\varphi(x)$ is decreasing and convex, we have

\[
\varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_{X_i}(x)) \right) \leq \frac{1}{n} \sum_{i=1}^{n} \varphi (\phi (F_{X_i}(x)))
\]

implies $\bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_{X_i}(x)) \right) \leq \bar{\alpha} \frac{1}{n} \sum_{i=1}^{n} \frac{F(x)}{\alpha_i + \bar{\alpha}_i F(x)}$

\[
= \frac{1}{n} \sum_{i=1}^{n} \bar{\alpha}_i \cdot \frac{1}{n} \sum_{i=1}^{n} \frac{F(x)}{\alpha_i + \bar{\alpha}_i F(x)}
\]

\[
\leq \frac{1}{n} \sum_{i=1}^{n} \bar{\alpha}_i \frac{F(x)}{\alpha_i + \bar{\alpha}_i F(x)}
\]

implies

\[
1 - \bar{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi (F_{X_i}(x)) \right) \geq 1 - \frac{1}{n} \sum_{i=1}^{n} \frac{\bar{\alpha}_i F(x)}{\alpha_i + \bar{\alpha}_i F(x)}
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_i + \bar{\alpha}_i F(x)}
\]
Now for $\tilde{\alpha} \leq 0$, from (29), we have

$$1 - \tilde{\alpha} \phi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(X_i(x)) \right) \geq \frac{\alpha}{\alpha + \tilde{\alpha} F(x)} \geq \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_i + \tilde{\alpha}_i F(x)},$$

(32)

where the second inequality follows from the fact that $\frac{\alpha_i}{\alpha_i + \tilde{\alpha}_i F(x)}$ is increasing and concave in $\alpha_i$.

If $\frac{\phi}{\varphi'}$ is convex, then we have

$$-\frac{\phi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(X_i(x)) \right)}{\varphi' \left( \frac{1}{n} \sum_{i=1}^{n} \phi(X_i(x)) \right)} \geq -\frac{1}{n} \sum_{i=1}^{n} \frac{\phi(X_i(x))}{\varphi'(X_i(x))}.$$  

(33)

Thus we have

$$\left(1 - \tilde{\alpha}\phi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(X_i(x)) \right) \right) \left( -\frac{\phi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(X_i(x)) \right)}{\varphi' \left( \frac{1}{n} \sum_{i=1}^{n} \phi(X_i(x)) \right)} \right) \geq \frac{1}{n} \sum_{i=1}^{n} \left( -\frac{\phi(X_i(x))}{\varphi'(X_i(x))} \right) \frac{\alpha_i}{\alpha_i + \tilde{\alpha}_i F(x)}.$$  

(34)

If $\varphi$ is log-concave, then $-\frac{\phi(x)}{\varphi'(x)}$ is decreasing in $x$, so that $-\frac{\phi(X_i(x))}{\varphi'(X_i(x))}$ is decreasing in $\alpha_i$. So by Chebyshev's inequality we have

$$\frac{1}{n} \sum_{i=1}^{n} \left( -\frac{\phi(X_i(x))}{\varphi'(X_i(x))} \right) \leq \frac{1}{n} \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_i + \tilde{\alpha}_i F(x)} \geq \frac{1}{n} \sum_{i=1}^{n} \left( -\frac{\phi(X_i(x))}{\varphi'(X_i(x))} \right) \frac{\alpha_i}{\alpha_i + \tilde{\alpha}_i F(x)}.$$  

(35)

From (30), (34), (35) and the fact that the common factor $\varphi' \left( \sum_{i=1}^{n} \phi(X_i(x)) \right)$ in (25) and (27) is negative, we have $g_2(G_2^{-1}(F_2(x))) \geq f_2(x)$. Hence the theorem follows. \hfill \square

One may interested to know whether in case of Theorem 4.1 we can establish dispersive ordering when baseline distribution is DRHR. The following counterexample shows that with these conditions, we cannot establish dispersive ordering even in case of samples from independent r.v.’s.

**Counterexample 4.1** Consider maximums of two samples, one having four independent and heterogeneous r.v.’s, and another having four independent and homogeneous r.v.’s with respective distribution functions $F_2(x) = \prod_{i=1}^{4} \left( \frac{F(x)}{1 - F(x)} \right)$ and $G_2(x) = \left( \frac{F(x)}{1 - F(x)} \right)^4$, where $\alpha_1 = 0.9$, $\alpha_2 = 0.95$, $\alpha_3 = 27$, $\alpha_4 = 37$, $\alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 16.4625$, and $F(x) = 1 - e^{-5x^{0.5}}$, so that the baseline distribution is DRHR. We obtain

$$f_2(x) = \left( \sum_{i=1}^{4} \frac{\alpha_i \tilde{r}(x)}{\alpha_i + \tilde{\alpha}_i F(x)} \right) \prod_{i=1}^{4} \left( \frac{F(x)}{\alpha_i + \tilde{\alpha}_i F(x)} \right).$$
and
\[ g_2(G_2^{-1}(F_2(x))) = 4 \left( \prod_{i=1}^{4} F_{X_i}(x) \left( 1 - \tilde{\alpha} \prod_{i=1}^{4} (F_{X_i}(x))^{1/n} \right) \frac{\tilde{\alpha}}{\alpha + \tilde{\alpha}} F(x) \right) \]

where \( \beta(x) = \frac{\alpha \left( \prod_{i=1}^{n} F_{X_i}(x) \right)^{1/4}}{1 - \tilde{\alpha} \left( \prod_{i=1}^{n} F_{X_i}(x) \right)^{1/4}} \).

We plot \( g_2(G_2^{-1}(F_2(x))) - f_2(x) \) by substituting \( x = t/(1-t) \), so that for \( x \in [0, \infty) \), we have \( t \in [0, 1] \). From the Figure 3 we observe that \( g_2(G_2^{-1}(F_2(x))) - f_2(x) \not\leq 0 \) and also \( g_2(G_2^{-1}(F_2(x))) - f_2(x) \not\leq 0 \).

The following theorem compares the maximum of two samples, both from \( n \) dependent homogeneous r.v.’s following the PO model and with different Archimedean copulas.

**Theorem 4.2** Suppose \( X \sim PO(\bar{F}, \alpha 1, \varphi_1) \) and \( Y \sim PO(\bar{F}, \alpha 1, \varphi_2) \). Then \( X_{n:n} \geq_{disp} Y_{n:n} \) if baseline distribution is IPHR, \( \varphi_1(\phi_1(t)/n)/\varphi_2(\phi_2(t)/n) \) is increasing in \( t \), and \( \alpha \geq 1 \).

**Proof:** The distribution functions of \( X_{n:n} \) and \( Y_{n:n} \) are \( G_1(x) = \varphi_1 \left( n\varphi_1 \left( F_{X_1}(x) \right) \right) \) and \( G_2(x) = \varphi_2 \left( n\varphi_2 \left( F_{X_1}(x) \right) \right) \), respectively, where \( F_{X_1}(x) = \frac{F(x)}{\alpha + \tilde{\alpha} F(x)} \). We get,

\[
g_1(x) = n\varphi_1' \left( n\varphi_1 \left( F_{X_1}(x) \right) \right) \cdot \frac{\alpha \tilde{\alpha}}{\alpha + \tilde{\alpha}} \frac{\varphi_1 \left( F_{X_1}(x) \right)}{\varphi_1' \left( F_{X_1}(x) \right)} \]

(36)

\[
g_2(x) = n\varphi_2' \left( n\varphi_2 \left( F_{X_1}(x) \right) \right) \cdot \frac{\alpha \tilde{\alpha}}{\alpha + \tilde{\alpha}} \frac{\varphi_2 \left( F_{X_1}(x) \right)}{\varphi_2' \left( F_{X_1}(x) \right)} \]

so that

\[
G_2^{-1}(G_1(x)) = \frac{\alpha \varphi_2 \left( \frac{1}{n} \varphi_2 \left( \varphi_1 \left( n\varphi_1 \left( F_{X_1}(x) \right) \right) \right) \right)}{1 - \tilde{\alpha} \varphi_2 \left( \frac{1}{n} \varphi_2 \left( \varphi_1 \left( n\varphi_1 \left( F_{X_1}(x) \right) \right) \right) \right)} = F^{-1}(\zeta(x)) \text{ (say)},
\]

(37)
The following corollary follows from Theorems 4.1 and 4.2. This corollary compares the minimum of two samples, one from \( n \) dependent homogeneous r.v.’s following the PO model and another from \( n \) dependent heterogeneous r.v.’s following the PO model with different Archimedean copulas.

**Corollary 4.1** Suppose \( X \sim PO(\bar{F}, \alpha, \varphi_1) \) and \( Y \sim PO(\bar{F}, \alpha 1, \varphi_2) \). Then for \( \alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i \), \( X_{n:n} \geq_{disp} Y_{n:n} \) if the baseline distribution is IRHR, \( \varphi_1 \) is log-concave, \( \frac{\varphi_1}{\varphi_1(\varphi_1(1)/n)} \) is convex, \( \varphi_1(\varphi_1(t)/n)/\varphi_2(\varphi_2(t)/n) \) is increasing in \( t \), and \( \alpha \geq 1 \).

The following theorem compares the minimum of two samples, one from \( n \) dependent heterogeneous r.v.’s following the PO model and another from \( n \) dependent homogeneous r.v.’s following the PO model, in terms of star order.

**Theorem 4.3** Suppose \( X \sim PO(\bar{F}, \alpha, \varphi) \) and \( Y \sim PO(\bar{F}, \alpha 1, \varphi) \). Then for \( \alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i \), \( X_{n:n} \geq_{s} Y_{n:n} \) if \( x \tilde{\varphi}(x) \) is increasing in \( x \), \( \varphi \) is log-concave, \( \frac{\varphi}{\varphi} \) is convex.
Proof: Using equations (25), (26) and (27), we have

\[ x^2 \frac{d}{dx} \left( \frac{G_2^{-1}(F_2(x))}{x} \right) \]

\[ = x \frac{d}{dx} \left( G_2^{-1}(F_2(x)) - G_2^{-1}(F_2(x)) \right) \]

\[ = \frac{f_2(x)}{g_2(G_2^{-1}(F_2(x)))} - G_2^{-1}(F_2(x)) \]

\[ = \frac{x \tilde{r}(x) \frac{1}{n} \sum_{i=1}^{n} \frac{\phi(F_X(x))}{\phi(F_X(x))}}{\tilde{r} \left( F^{-1}(\beta(x)) \right) \frac{\frac{1}{n} \sum_{i=1}^{n} \phi(F_X(x))}{\phi(F_X(x))} \cdot (1 - \tilde{\alpha} \varphi \left( \frac{1}{n} \sum_{i=1}^{n} \phi(F_X(x)) \right)) - F^{-1}(\beta(x)). \] (42)

In Theorem 4.1, we have already proved that

\[ F^{-1}(\beta(x)) \geq x. \] (43)

Now, if \( x \tilde{r}(x) \) is increasing in \( x \), then we have from (43), \( x \tilde{r}(x) \leq F^{-1}(\beta(x)) \tilde{r}(F^{-1}(\beta(x))) \), that is

\[ \frac{x \tilde{r}(x)}{\tilde{r}(F^{-1}(\beta(x)))} \leq F^{-1}(\beta(x)). \] (44)

Now combining (34) and (35), we have

\[ \frac{1}{n} \sum_{i=1}^{n} \left( - \frac{\phi(F_X(x))}{\phi(F_X(x))} \right) \cdot \frac{\alpha_i}{\alpha_i + \tilde{\alpha} F(x)} \]

\[ \leq 1. \] (45)

Using (44) and (45), from (42) we get

\[ x^2 \frac{d}{dx} \left( \frac{G_2^{-1}(F_2(x))}{x} \right) \leq 0. \]

So, \( \frac{G_2^{-1}(F_2(x))}{x} \) is decreasing in \( x \geq 0 \). Hence \( X_{n:n} \geq_* Y_{n:n} \).

The following counterexample shows that we cannot establish star ordering as in case of Theorem 4.3 when \( x \tilde{r}(x) \) is decreasing or increasing even in case of samples from independent r.v.’s.

**Counterexample 4.2** Consider maximums of two samples, one having four independent and heterogeneous r.v.’s, and another having four independent and homogeneous r.v.’s. Consider \( \alpha_1 = 5 \), \( \alpha_2 = 15 \), \( \alpha_3 = 25 \), \( \alpha_4 = 45 \), \( \alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 45/2 \), and \( F(x) = 1 - (1 + x)^{-0.6} \), so that \( x \tilde{r}(x) \) is decreasing. We plot \( G_2^{-1}(F_2(x))/x \) by substituting \( x = t/(1 - t) \), so that for
Figure 4: Plot of $G^{-1}_2(F_2(x))/x$ for $x = t/(1-t), t \in [0, 1]$

$x \in [0, \infty)$, we have $t \in [0, 1)$. We obtain

$$G^{-1}_2(F_2(x)) = F^{-1}\left(\frac{\alpha \left(\prod_{i=1}^{4} F_{X_i}(x)\right)^{1/4}}{1 - \bar{\alpha} \left(\prod_{i=1}^{4} F_{X_i}(x)\right)^{1/4}}\right).$$

From the Figure 4, we observe that $G^{-1}_2(F_2(x))/x$ is neither increasing nor decreasing.

The following theorem compares the minimum of two samples, both from $n$ dependent homogeneous r.v.’s following the PO model and with different Archimedean copulas. The proof can be done using the results of proof of Theorem 4.2 in the same line as of Theorem 4.3, and hence omitted.

**Theorem 4.4** Suppose $X \sim PO(\bar{F}, \alpha_1, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha_2, \varphi_2)$. Then for $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, $X_{n:n} \geq_s Y_{n:n}$ if $x\bar{r}(x)$ is increasing in $x$, $\varphi_1(\phi_1(t)/n)/\varphi_2(\phi_2(t)/n)$ is increasing in $t$, and $\alpha \geq 1$.

The following corollary follows from Theorems 4.3 and 4.4.

**Corollary 4.2** Suppose $X \sim PO(\bar{F}, \alpha, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha, \varphi_2)$. Then for $\alpha \geq \frac{1}{n} \sum_{i=1}^{n} \alpha_i$, $X_{n:n} \geq_s Y_{n:n}$ if $x\bar{r}(x)$ is increasing in $x$, $\varphi_1$ is log-concave, $\frac{\varphi'}{\varphi'}$ is convex, $\varphi_1(\phi_1(t)/n)/\varphi_2(\phi_2(t)/n)$ is increasing in $t$, and $\alpha \geq 1$.

5 Examples

Here we demonstrate some of the proposed results numerically. The first example illustrates the result of Theorem 3.1.
Example 5.1 Consider the minimums of two samples, one from three dependent and heterogeneous r.v.’s, and another from three dependent and homogeneous r.v.’s, with respective distribution functions $F_1(x) = 1 - \varphi\left(\sum_{i=1}^{3} \phi\left(\frac{\alpha_i F(x)}{1-\alpha_i F(x)}\right)\right)$, and $G_1(x) = 1 - \varphi\left(3\phi\left(\frac{\alpha F(x)}{1-\alpha F(x)}\right)\right)$, where $\alpha_1 = 0.34$, $\alpha_2 = 0.65$, $\alpha_3 = 1.23$, $\alpha = 0.88 > 0.74 = (\alpha_1 + \alpha_2 + \alpha_3)/3$, and $\bar{F}(x) = e^{-x^{0.3}}$, so that the baseline distribution is DFR. We take $\varphi(x) = a/\log(x + e^a)$, $a \in (0, \infty)$ (4.2.19, Nelsen [22]) which satisfies all the conditions of Theorem 3.1. For this example we take $a = 5$. We plot $g_1(G_1^{-1}(F_1(x))) - f_1(x)$ by substituting $x = t/(1-t)$, so that for $x \in [0, \infty)$, we have $t \in [0, 1)$. From the Figure 5, we observe that $g_1(G_1^{-1}(F_1(x))) \leq f_1(x)$. Thus $X_{1:3} \leq \text{disp}, Y_{1:3}$.

Example 5.2 Consider the minimum of two samples, one from four dependent and heterogeneous r.v.’s, and another from four dependent and homogeneous r.v.’s. Consider $\alpha_1 = 0.24$, $\alpha_2 = 0.45$, $\alpha_3 = 0.57$, $\alpha_4 = 1.23$, $\alpha = 0.73 > (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 0.6225$, and $\bar{F}(x) = 1/\sqrt{x}$, $x \in [1, \infty)$ so that $x r(x)$ is constant, and so may be taken as decreasing. We take $\varphi(x) = a/\log(x + e^a)$, $a \in (0, \infty)$ which satisfies all the conditions of Theorem 3.3. For this example we take $a = 7$. We plot $(G_1^{-1}(F_1(x)))/x$ by substituting $x = 1/t$, so that for $x \in [1, \infty)$, we have $t \in (0, 1]$. From the Figure 6, we observe that $G_1^{-1}(F_1(x))/x$ is increasing. Thus $X_{1:4} \leq, Y_{1:4}$.

Example 5.3 Consider the maximums of two samples, one from four dependent and heterogeneous r.v.’s, and another from four dependent and homogeneous r.v.’s, with respective distribution functions $F_2(x) = \varphi\left(\sum_{i=1}^{3} \phi\left(\frac{F(x)}{\alpha_i + \alpha_i F(x)}\right)\right)$ and $G_2(x) = \varphi\left(3\phi\left(\frac{F(x)}{\alpha + \alpha F(x)}\right)\right)$, where
Figure 6: Plot of \((G_1^{-1}(F_1(x))/x)'\) for \(x = 1/t, t \in [0, 1]\) when \(x_r(x)\) is decreasing.

\(\alpha_1 = 0.95, \alpha_2 = 0.32, \alpha_3 = 1.54, \alpha_4 = 0.76, \alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 0.8925,\)

and \(\bar{F}(x) = e^{-(-3x)^{0.3}}, x \in (-\infty, 0],\) so that the baseline distribution is IRHR. We take
\(\varphi(x) = (1 - x)/(1 + (\lambda - 1)x), \lambda \in [1, \infty)\) \((4.2.8, Nelsen [22])\) which satisfies all the conditions of Theorem 4.1. For this example we take \(\lambda = 1.5.\) We plot \(g_2(G_2^{-1}(F_2(x)))/x\) by substituting \(x = t/(1 + t),\) so that for \(x \in (-\infty, 0],\) we have \(t \in (-1, 0].\) From the Figure 7 we observe that \(g_2(G_2^{-1}(F_2(x)))/x\) is decreasing. Thus \(X_{4:4} \geq_{\text{disp}} Y_{4:4}.\)

The following example illustrates the result of Theorem 4.3.

**Example 5.4** Consider the maximums of two samples, one from three dependent and heterogeneous r.v.’s, and another from three dependent and homogeneous r.v.’s. Consider \(\alpha_1 = 0.5, \alpha_2 = 0.8, \alpha_3 = 1.7, \alpha = 1.6 > (\alpha_1 + \alpha_2 + \alpha_3)/3 = 1,\) and \(F(x) = (e^x - 1)/(e - 1), x \in [0, 1]\) so that \(x_r(x)\) is decreasing. We take \(\varphi(x) = (1 - x)/(1 + (\lambda - 1)x), \lambda \in [1, \infty)\) \((4.2.8, Nelsen [22])\) which satisfies all the conditions of Theorem 4.3. For this example we take \(\lambda = 1.5.\) We plot \((G_2^{-1}(F_2(x)))/x\). From the Figure 8, we observe that \(G_2^{-1}(F_2(x))/x\) is decreasing. Thus \(X_{3:3} \geq_{\star} Y_{3:3}.\)

### 6 Conclusion

To the best of our knowledge, very few research works on dispersive order and star order of extreme order statistics from dependent samples can be found in literature, may be due to mathematical complexity. In those works, the corresponding r.v.’s follow either of scale model, PHR model, resilience-scale model and scale proportional hazards model. The PO model with its PO implications and its capability of extending any existing family of distributions to more flexible distributions, is served as very important model in various fields including reliability
Figure 7: Plot of $g_2(G_2^{-1}(F_2(x))) - f_2(x)$ for $x = t/(1 + t)$, $t \in [-1, 0]$ when baseline distribution function is IRHR.

Figure 8: Plot of $(G_2^{-1}(F_2(x))/x)'$, $x \in [0, 1]$
theory and survival analysis. Till now, there is no research work on dispersive and star ordering in case of PO model available in the literature. In this work, we derive the dispersive and the star order between both maximums and minimums of samples following the PO model and coupled with Archimedean copula. The results are illustrated with numerical examples.

It is expected that more research works in this direction will be done in future. For instance, comparing extreme order statistics by means of some other variability orders or skewness orders like the the excess wealth order, convex transform order and the Lorenz order.

References

[1] R.E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing, Silver Spring, Maryland, MD, USA, 1981.

[2] S. Bennett, Analysis of survival data by the proportional odds model, Statistics in Medicine 2 (1983) 273-277.

[3] D. Collett, Modelling survival data in medical research, 2nd edition, Chapman and Hall/CRC, 2004.

[4] G.M. Cordeiro, A.J. Lemonte, E.M.M. Ortega, The Marshall-Olkin family of distributions: mathematical properties and new models, Journal of Statistical Theory and Practice 8(2) (2014) 343-366.

[5] W. Ding, J. Yang, X. Ling, On the skewness of extreme order statistics from heterogenous samples, Communication in Statistics - Theory and Methods 46(5) (2017) 2315-2331.

[6] R. Fang, C. Li, X. Li, Stochastic comparisons on sample extremes of dependent and heterogenous observations, Statistics 50 (2016) 930-955.

[7] R. Fang, C. Li, X. Li, Ordering results on extremes of scaled random variables with dependence and proportional hazards, Statistics 52(2) (2018) 458-478.

[8] J. Jeon, S.C. Kochar, C.G. Park, Dispersive ordering-some applications and examples, Statistical Papers, 47(2) (2006) 227-247.

[9] S.N.U.A. Kirmani, R.C. Gupta, On the proportional odds model in survival analysis, Annals of the Institute of Statistical Mathematics 53(2) (2001) 203-216.

[10] S.C. Kochar, M. Xu, On the skewness of order statistics with applications, Annals of Operations Research 212(1) (2014) 127-138.

[11] S.C. Kochar, M. Xu, On the skewness of order statistics in multiple-outlier models, Journal of Applied Probability 48 (2011) 271-284.

[12] S. Kochar, Stochastic comparisons of order statistics and spacings: A Review, ISRN Probability and Statistics vol. 2012, Article ID 839473, 47 pages, 2012.

[13] P. Kundu, A.K. Nanda, Reliability study of proportional odds family of discrete distributions, Communications in Statistics - Theory and Methods 47(5) (2018) 1091-1103.
[14] P. Kundu, N.K. Hazra, A.K. Nanda, Reliability study of series and parallel systems of heterogeneous component lifetimes following proportional odds model, Statistics 54(2) (2020) 375-401.

[15] X. Li, R. Fang, Ordering properties of order statistics from random variables of Archimedean copulas with applications, Journal of Multivariate Analysis 133 (2015) 304-320.

[16] C. Li, R. Fang, X. Li, Stochastic comparisons of order statistics from scaled and interdependent random variables, Metrika 79 (2016) 553-578.

[17] W. Lu, H.H. Zhang, Variable selection for proportional odds model, Statistics in Medicine 26 (2007) 3771-3781.

[18] A.W. Marshall, I. Olkin, Life distributions, Springer, New York, 2007.

[19] A.W. Marshall, I. Olkin, A new method of adding a parameter to a family of distributions with applications to the exponential and Weibull families, Biometrika 84 (1997) 641-652.

[20] H. Nadeb, H. Torabi, A. Dolati, Stochastic comparisons between the extreme claim amounts from two heterogeneous portfolios in the case of Transmuted-G model, North American Actuarial Journal, 2020. DOI: 10.1080/10920277.2019.1671203.

[21] A.K. Nanda, S. Das, Stochastic orders of the Marshall-Olkin extended distribution, Statistics and Probability Letters 82 (2012) 295-302.

[22] R.B. Nelsen, An Introduction to Copulas, 2nd Ed., Springer, 2006.

[23] A.J. Rossini, A.A. Tsiatis, A Semiparametric proportional odds regression model for the analysis of current status data, Journal of the American Statistical Association 91(434) (1996) 713-721.

[24] M. Shaked, J.G. Shanthikumar, Stochastic Orders, Springer-Verlag, New York, 2007.

[25] J. Wu, M. Wang, X. Li, Convex transform order of the maximum of independent Weibull random variables, Statistics and Probability Letters 156 (2020) 108597.

[26] M. Xu, N. Balakrishnan, On the sample ranges from heterogeneous exponential variables, Journal of Multivariate Analysis 109 (2012) 1-9.

[27] Y. Zhang, W. Ding, P. Zhao, On variability of series and parallel systems with heterogeneous components, Probability in the Engineering and Informational Sciences, 2019. DOI: 10.1017/S0269964819000263.

[28] Y. Zhang, X. Cai, P. Zhao, H. Wang, Stochastic comparisons of parallel and series systems with heterogeneous resilience-scaled components, Statistics 53(1) (2019) 126-147.