DIVISION PROBLEM FOR SPATIALLY PERIODIC DISTRIBUTIONS

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Abstract. We give a sufficient condition for the surjectivity of partial differential operators with constant coefficients on a class of distributions on $\mathbb{R}^{n+1}$ (here we think of there being $n$ space directions and one time direction), that are periodic in the spatial directions and tempered in the time direction.

1. Introduction

An important milestone in the general theory of partial differential equations is the solution to the division problem: Let $D$ be a nonzero partial differential operator with constant coefficients and $T$ be a distribution; can one find a distribution $S$ such that $DS = T$? That this is always possible was established by Ehrenpreis [3]. See also [1], [2], [4], [5], [6], [7], [10] for the solution of various avatars of the division problem for different spaces of distributions.

In this article, we study the division problem in spaces of distributions on $\mathbb{R}^{n+1}$ (where we think of there being $n$ space directions and one time direction) that are periodic in the spatial directions. The study of such solution spaces arises naturally in control theory when one considers the so-called “spatially invariant systems”; see [5]. In the “behavioural approach” to control theory for such spatially invariant systems, a fundamental question is whether this class of distributions is an injective module over the ring of partial differential operators with constant coefficients; see for example [11]. In light of this, one can first ask what happens with the division problem. Thus besides being a purely mathematical question that fits in the classical theme mentioned in the previous paragraph, there is also a behavioural control theoretic motivation for studying the division problem for distributions that are periodic in the spatial directions. Upon taking Fourier transform with respect to the spatial variables, the problem amounts to the following.

Problem: For which $P(\tau, \xi) \in \mathbb{C}[\tau, \xi_1, \cdots, \xi_n]$ is $P\left(\frac{d}{dt}, i\xi\right) : X \to X$ surjective, where

\[
X = \{(T_\xi) = (T_\xi)_{\xi \in \mathbb{Z}^n} \in \mathcal{D}'(\mathbb{R})^{\mathbb{Z}^n} : \forall \varphi \in \mathcal{D}(\mathbb{R}), \exists k \in \mathbb{N} : \forall \xi \in \mathbb{Z}^n, |\langle \varphi, T_\xi \rangle| \leq k \cdot (1 + |\xi|^k) \}.
\]

($| \cdot |$ denotes the 1-norm in $\mathbb{R}^n$.)

An obvious necessary condition is that

for all $\xi \in \mathbb{Z}^n$, $P\left(\frac{d}{dt}, i\xi\right) \neq 0$.

However, that this condition is not sufficient is demonstrated by considering the following example.

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Example 1.1. Let \( c = \sum_{j=1}^{\infty} \frac{1}{2^j} \) (a “Liouville number”). With \( p_k := \sum_{j=1}^{k} \frac{1}{2^{j+1}} q_k := 2^k \),
\[
\left| c - \frac{p_k}{q_k} \right| = \sum_{j=k+1}^{\infty} \frac{1}{2^{j+1}} + \frac{1}{2^{k+1}} + \cdots < 2 \cdot \frac{1}{2^{k+1}} \leq \left( \frac{1}{2^{k+1}} \right)^k = \frac{1}{q_k^k}, \quad k \in \mathbb{N}.
\]
Now take \( P = \xi_1 + c \xi_2 \). Then (1) \( \xi \in \mathbb{Z}^2 \) does not belong to the range of \( P \) because \[
\frac{1}{\xi_1 + c \xi_2} \notin X.
\]
Indeed, otherwise there would exist an \( m \) such that
\[
\left| \frac{1}{\xi_1 + c \xi_2} \right| \leq m(1 + |\xi_1| + |\xi_2|)^m \text{ for all } \xi_1, \xi_2 \in \mathbb{Z},
\]
and in particular, with \( \xi_1 = -p_k \), and \( \xi_2 = q_k \), \( k \in \mathbb{N} \),
\[
q_k^{k-1} \leq \frac{1}{|\xi_1 + c \xi_2|} \leq m(1 + p_k + q_k)^m \leq m(kq_k + kq_k + kq_k)^m = m(3kq_k)^m,
\]
a contradiction.

We consider a simpler situation and set
\[
Y = \{(T\xi) \in S'({\mathbb Z}^n) : \forall \varphi \in S({\mathbb R}), \exists k \in \mathbb{N} : \forall \xi \in \mathbb{Z}^n, |\langle \varphi, T\xi \rangle| \leq k \cdot (1 + |\xi|)^k\}.
\]
Our main result is the following:

**Theorem 1.2.** Let \( P(\tau, i\xi) \in C[\tau, \mathbb{C}] = C[\xi][\tau] \), and for \( \xi \in \mathbb{Z}^n \),
\[
P(\tau, i\xi) = c_{\xi} \cdot \prod_{j=1}^{m_{\xi}} (\tau - \lambda_{j, \xi}),
\]
with \( m_{\xi} \in \mathbb{N}_0, c_{\xi} \in \mathbb{C} \setminus \{0\}, \lambda_{1, \xi}, \cdots, \lambda_{m_{\xi}, \xi} \in \mathbb{C}. \) (The roots \( \lambda_{j, \xi} \) are arbitrarily arranged.) Let
\[
d_{\xi} := \begin{cases} 
1 & \text{if for all } j = 1, \cdots, m_{\xi}, \text{ Re}(\lambda_{j, \xi}) = 0, \\
\min_{j: \text{Re}(\lambda_{j, \xi}) \neq 0} |\text{Re}(\lambda_{j, \xi})| & \text{otherwise}. 
\end{cases}
\]
If
1. \( (c_{\xi}^{-1}) \in s'({\mathbb Z}^n) \) and
2. \( (d_{\xi}^{-1}) \in s'({\mathbb Z}^n) \),

then \( P \left( \frac{d}{dt}, i\xi \right) : Y \to Y \) is surjective.

From Example 1.1, it follows that the first condition is not superfluous. Here is an example demonstrating that the second condition is not superfluous either.

**Example 1.3.** Take
\[
P \left( \frac{d}{dt}, i\xi \right) := \frac{d}{dt} + \xi_1 + c \xi_2
\]
with the same \( c \) as in Example 1.1 and \( T\xi := 1, \xi \in \mathbb{Z}^2 \). Then \( (S\xi) = \left( \frac{1}{\xi_1 + c \xi_2} \right) \notin Y. \)
2. Preliminaries

There holds that
\[ Y \simeq \mathcal{L}(S(\mathbb{R}), S'(\mathbb{R}^n)) \simeq S'(\mathbb{R}) \otimes S'(\mathbb{R}^n). \]
\[ (T_\xi) \quad \mapsto \quad (\varphi \mapsto \langle \varphi, T_\xi \rangle). \]
(For the first isomorphism we use the Closed Graph Theorem, while the second isomorphism follows from [9, Proposition 50.4].) Also, by [9, Theorem 51.3, p. 528, and Corollary to Theorem 51.6, p. 531],
\[ S'(\mathbb{R}) \otimes S'(\mathbb{R}^n) \simeq S'(\mathbb{R}) \otimes S'(\mathbb{R}/\mathbb{Z}^n) \simeq S'(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}^n)) = \hat{Y}, \]
that is, \( \hat{Y} \) is the dual of a Fréchet space and there holds that
for all \( \hat{T} \in \hat{Y} \) there exists \( k \in \mathbb{N} \) such that for all \( \psi \in S(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}^n)), \)
\[ |\langle \psi, \hat{T} \rangle| \leq k \cdot \sum_{|\alpha| \leq k} \| (1 + |t|)^{\alpha} \varphi \|_\infty. \]
Hence it follows that in \( Y \):
for all \( (T_\xi) \in Y \) there exists \( k \in \mathbb{N} \) such that for all \( \varphi \in S(\mathbb{R}) \) and all \( \xi \in \mathbb{Z}^n, \)
\[ |\langle \varphi, T_\xi \rangle| \leq k \cdot (1 + |\xi|)^k \cdot \sum_{j=0}^{k} \| (1 + |t|)^{j} \varphi^{(j)} \|_\infty. \]
In particular for \( \varphi \in S(\mathbb{R}) \) and \( t \in \mathbb{R}, \)
\[ |(\varphi \ast T_\xi)(t)| = |\langle \tau \mapsto \varphi(t + \tau), T_\xi \rangle| \leq k \cdot (1 + |\xi|)^k \cdot \sum_{j=0}^{k} \| (1 + |t - \tau|)^{j} \varphi^{(j)}(\tau) \|_\infty \]
\[ \leq C_{\varphi, k} \cdot (1 + |\xi|)^k \cdot (1 + |t|)^k. \]
We will also need the following lemma.

**Lemma 2.1.** Consider a monic polynomial
\[ P = \tau^d + c_{d-1, \xi} \tau^{d-1} + \cdots + c_1, \xi \tau + c_0, \xi \in S'(\mathbb{Z}^n)[\tau]. \]
For \( \xi \in \mathbb{Z}^n \), we factorize
\[ P(\tau, i\xi) = \prod_{j=1}^{d}(\tau - \lambda_j, \xi), \]
with \( \lambda_1, \xi, \ldots, \lambda_d, \xi \in \mathbb{C} \). (The roots \( \lambda_i, \xi \) are arbitrarily arranged.) Then \( (\lambda_j, \xi) \xi \in S'(\mathbb{Z}^n), \)
\( j = 1, \ldots, d. \)

**Proof.** Let \( (\tau + \alpha, \xi)(\tau^{d-1} + b_{d-2, \xi} \tau^{d-2} + \cdots + b_1, \xi \tau + b_0, \xi) = P \). Then by comparing coefficients of the powers of \( \tau \), we obtain
\[ \alpha, \xi \cdot b_0, \xi = c_0, \xi, \]
\[ \alpha, \xi \cdot b_1, \xi + b_0, \xi = c_1, \xi, \]
\[ \vdots \]
\[ \alpha, \xi \cdot b_{d-2, \xi} + b_{d-3, \xi} = c_{d-2, \xi}, \]
\[ \alpha, \xi + b_{d-2, \xi} = c_{d-1, \xi}. \]
We first show that \((\alpha_\xi) \in s'(\mathbb{Z}^n)\). Suppose this is not true. Then there exists a sequence \((\xi_k)_k\) such that
\[
\lim_{k \to \infty} |\xi_k| = \infty
\]
and \(|\alpha_{\xi_k}| > k(1 + |\xi_k|)^k\) for all \(k\). But then from the first equation in the above equation array, it follows that
\[
|b_{0,\xi_k}| = \frac{|c_{0,\xi_k}|}{|\alpha_{\xi_k}|} \xrightarrow{k \to \infty} 0.
\]
Then from the second equation in the above equation array, we also obtain that
\[
|b_{1,\xi_k}| = \frac{|c_{1,\xi_k} - b_{0,\xi_k}|}{|\alpha_{\xi_k}|} \xrightarrow{k \to \infty} 0.
\]
Proceeding in this manner, we get eventually that
\[
1 \leq \frac{|c_{d-1,\xi_k} - b_{d-2,\xi_k}|}{|\alpha_{\xi_k}|} \xrightarrow{k \to \infty} 0,
\]
a contradiction.

We will be done once we show that \((b_{d-2,\xi}), \ldots, (b_{1,\xi}), (b_{0,\xi})\) all belong to \(s'(\mathbb{Z}^n)\) by an inductive argument. Suppose that \(k\) is the least index such that \((b_{k,\xi}) \notin s'(\mathbb{Z}^n)\). But then by a similar argument as above, it follows from
\[
\alpha_\xi \cdot b_{k+1,\xi} + b_{k,\xi} = c_{k+1,\xi}
\]
that \((b_{k+1,\xi}) \notin s'(\mathbb{Z}^n)\) (since we have already established that \((\alpha_\xi) \in s'(\mathbb{Z}^n)\)). Proceeding in this manner, we eventually obtain that \(b_{d-2,\xi} \notin s'(\mathbb{Z}^n)\), which clearly contradicts the last equation in the above equation array, namely that \(\alpha_\xi + b_{d-2,\xi} = c_{d-1,\xi}\). \(\square\)

3. Proof of the Main Result

**Proof.** (a) The pointwise multiplication map
\[
c_\xi : s'(\mathbb{Z}^n) \to s'(\mathbb{Z}^n) : (\alpha_\xi) \mapsto (c_\xi \cdot \alpha_\xi)
\]
is a topological vector space isomorphism, thanks to the first assumption that \((c_\xi^{-1}) \in s'(\mathbb{Z}^n)\). So it is enough to prove the surjectivity of
\[
Q := \frac{1}{c_\xi} P \left( \frac{d}{dt}, i\xi \right) : Y \to Y,
\]
that is, of \(Q\) given by
\[
Q(T_\xi) = \prod_{j=1}^{m_\xi} \left( \frac{d}{dt} - \lambda_j, i \xi \right) T_\xi.
\]
By Lemma 2.1 since the coefficients of \(\tau \mapsto P(\tau, i\xi)\) are polynomials in \(\xi\) and because \((c_\xi^{-1}) \in s'(\mathbb{Z}^n)\), it follows that \((\lambda_j, i \xi) \in s'(\mathbb{Z}^n)\), that is, there exists a \(k \in \mathbb{N}\) such that for all \(\xi \in \mathbb{Z}^n\), and all \(j = 1, \ldots, m_\xi\), \(|\lambda_j, i \xi| \leq k(1 + |\xi|)^k\). Thus it is enough to show that for every \((T_\xi) \in Y\), there exists a \((S_\xi) \in Y\) such that for all \(\xi \in \mathbb{Z}^n\),
\[
\begin{cases}
S_\xi = T_\xi & \text{if } m_\xi = 0, \\
\left( \frac{d}{dt} - \lambda_1, i \xi \right) S_\xi = T_\xi & \text{if } m_\xi > 0.
\end{cases}
\]
Then this process can be inductively continued for the other \(j\)’s.
(b) For \( \lambda \in \mathbb{C} \), let
\[
E_+ = Y(t)e^{\lambda t}, \\
E_- = -Y(-t)e^{\lambda t},
\]
where \( Y(\cdot) \) denotes the Heaviside step function. Let \( \chi_+ \in C^\infty(\mathbb{R}) \) be such that
\[
\chi_+(t) = \begin{cases} 
1 & \text{if } t \geq 0, \\
0 & \text{if } t \leq -1,
\end{cases}
\]
and define \( \chi_- \in C^\infty(\mathbb{R}) \) by \( \chi_+ + \chi_- = 1 \) on \( \mathbb{R} \). For \( U \in S'(\mathbb{R}) \), define \( R_\lambda(U) \) by
\[
R_\lambda(U) = \begin{cases} 
U \ast E_+ & \text{if } \Re(\lambda) < 0, \\
U \ast E_- & \text{if } \Re(\lambda) > 0, \\
(\chi_+ \ast U) \ast E_+ + (\chi_- \ast U) \ast E_- & \text{if } \Re(\lambda) = 0.
\end{cases}
\]
(Note that for \( \pm \Re(\lambda) < 0 \), since \( E_\pm \in \mathcal{O}'(\mathbb{R}) \), the space of distributions rapidly decreasing at infinity, it follows that \( R_\lambda(U) \) is well-defined.) Then there holds that
\[
\left( \frac{d}{dt} - \lambda \right) R_\lambda(U) = U.
\]
For \( U \in S'(\mathbb{R}) \), \( \Re(\lambda) < 0 \) and \( \varphi \in S(\mathbb{R}) \),
\[
\langle \varphi, R_\lambda(U) \rangle = \langle \varphi, U \ast E_+ \rangle = \langle \varphi \ast \hat{U}, E_+ \rangle = \int_0^{\infty} \langle \varphi \ast \hat{U}(t) \rangle e^{\lambda t} dt.
\]
Thus \( \varphi \ast \hat{U} \in \mathcal{O}_M \) (smooth functions slowly increasing at infinity), and so there exists a \( k \in \mathbb{N} \) such that for all \( t \in \mathbb{R} \),
\[
|\langle \varphi \ast \hat{U}(t) \rangle| \leq k(1 + |t|)^k.
\]
Consequently, when \( \Re(\lambda) < 0 \), we have
\[
|\langle \varphi, R_\lambda(U) \rangle| \leq k \int_0^{\infty} (1 + t)^k |e^{\lambda t}| dt \leq C_k (1 + (\Re(\lambda))^{-k-1}).
\]
Now the second assumption that \((d_{-1}) \in S'(\mathbb{R}^n)\) will allow us to conclude that for \((T_\xi) \in Y\),
\[
S_\xi := \begin{cases} 
T_\xi & \text{if } m_\xi = 0, \\
R_{\lambda_1, \xi}(T_\xi) & \text{if } m_\xi > 0
\end{cases}
\]
belongs to \( Y \) as well. The details are as follows. For \((T_\xi) \in Y\), \( \varphi \in S(\mathbb{R}) \), there exist \( k \in \mathbb{N} \), \( C_{\varphi, k} \in \mathbb{R} \) such that for all \( t \in \mathbb{R} \), \(|\langle \varphi \ast T_\xi(t) \rangle| \leq C_{\varphi, k} \cdot (1 + |\xi|)^k (1 + |t|)^k\). So for \( \xi \) such that \( \Re(\lambda_1, \xi) \neq 0 \), we have
\[
|\langle \varphi, R_{\lambda_1, \xi}(T_\xi) \rangle| \leq C_{\varphi, k} \cdot (1 + |\xi|)^k \cdot \int_0^{\infty} (1 + t)^k e^{-|\Re(\lambda_1, \xi)|t} dt
\]
\[
\leq C_{\varphi, k} \cdot (1 + |\xi|)^k \cdot \bar{C}_k \cdot (1 + |\Re(\lambda_1, \xi)|^{-k-1})
\]
\[
\leq C_{\varphi, k} \cdot (1 + |\xi|)^k \cdot \bar{C}_k \cdot (1 + d_{-\xi}^{-k-1}). \tag{3.1}
\]
On the other hand, when \( \Re(\lambda_1, \xi) = 0 \) and \( \varphi \in S(\mathbb{R}) \), we have \( \langle \varphi \ast \hat{E}_+ \rangle \cdot \chi_+ \in S(\mathbb{R}) \) because for example using the fact that for each \( k \) there exists a constant \( C \) such that for all \( t, \tau \geq 0 \),
Since $B$ are periodic in the space variables, that is, distributions in $S(R)$ as well. Furthermore, the $S$-seminorms of $\psi = (\varphi \ast \tilde{E}_\pm) \cdot \chi_\pm$ grow at most polynomially in $|\lambda_{1, \xi}|$. This gives
\begin{align*}
|\langle \varphi, R_{\lambda_{1, \xi}}(T_\xi) \rangle| &\leq |\langle (\varphi \ast \tilde{E}_+) \cdot \chi_+, T_\xi \rangle| + |\langle (\varphi \ast \tilde{E}_-) \cdot \chi_-, T_\xi \rangle| \\
&\leq k \cdot (1 + |\xi|^k \cdot (1 + |\lambda_{1, \xi}|)^k,
\end{align*}
for $\text{Re}(\lambda_{1, \xi}) = 0$ and a suitable $k \in \mathbb{N}$ which depends on $\varphi$ and $(T_\xi)$. Using $(d_\xi^{-1}) \in S'(\mathbb{Z}^n)$ and the estimates (3.1) and (3.2), this completes the proof. \hfill \Box

Let us finally go back to the original problem. We consider tempered distributions which are periodic in the space variables, that is, distributions in
$$
\tilde{Y} = \{ T \in S'(\mathbb{R}^{n+1}_{t, x}) : \forall a \in AZ^n : T(t, x + a) = T(t, x) \}
$$
where $A \in \mathbb{R}^{n \times n}$ is a non-singular matrix. Upon using partial Fourier transform with respect to $x$, we obtain the isomorphism
$$
F : \tilde{Y} \xrightarrow{\sim} Y : T \mapsto (T_\xi)_{\xi \in \mathbb{Z}^n}, \quad \text{where } (F_x T)(\eta) = \sum_{\xi \in \mathbb{Z}^n} T_\xi(t) \delta(\eta - B\xi)
$$
with $B = 2\pi A^{-1}T$ and
$$
\langle \phi, F_x T \rangle = \left\langle \int_{\mathbb{R}^n} \phi(t, \eta) e^{-ix\eta} d\eta, T \right\rangle \text{ for } \phi \in S(\mathbb{R}^{n+1}).
$$
Since
$$
F(P(\partial) T) = \left( P\left( \frac{d}{dt} + iB\xi \right) T_\xi \right)_{\xi \in \mathbb{Z}^n}
$$
holds if $FT = (T_\xi)_{\xi \in \mathbb{Z}^n}$, we obtain the following corollary to Theorem 2.1.

**Corollary 3.1.** Let $P(\partial) \in \mathbb{C}[\partial_1, \partial_2, \ldots, \partial_n]$, $A \in \text{Gl}_n(\mathbb{R})$, $B = 2\pi A^{-1}T$ and $\tilde{Y}$ as above. We assume, furthermore, that
$$
P(\tau, iB\xi) = c_{\xi} \cdot \prod_{j=1}^{m_{\xi}} (\tau - \lambda_{j, \xi}),
$$
with $m_{\xi} \in \mathbb{N}_0$, $c_{\xi} \in \mathbb{C} \setminus \{0\}$, $\lambda_{1, \xi}, \cdots, \lambda_{m_{\xi}, \xi} \in \mathbb{C}$, and that the conditions (1) and (2) in Theorem 2.1 hold. Then $P(\partial) : \tilde{Y} \rightarrow \tilde{Y}$ is surjective.

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