On Moment Condition and Center Condition for Abel Equation

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Abstract. In this paper we consider Abel equation \( x' = g(t)x^2 + f(t)x^3 \), where \( f \) and \( g \) are analytical functions. We proved that if the equation has a center at \( x = 0 \), then the Moment Conditions, i.e., \( m_k = \int_{-1}^{1} f(t)(G(t))^k dt = 0 \), \( k = 0, 1, 2 \), is satisfied where \( G(t) = \int_{-1}^{t} g(s)ds \). Besides, we give partial a positive answer to a conjecture proposed by Y. Lijun and T. Yun in 2001.

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1 Introduction

1.1 Historical Aspect

Let the planar system

\[
\begin{align*}
\dot{x} &= -y + P(x, y) \\
\dot{y} &= x + Q(x, y),
\end{align*}
\]

where \(P(x, y)\) and \(Q(x, y)\) are polynomials, without constant term, of maximum degree \(n\). The singular point \((0, 0)\) is a center, if surrounded by closed trajectories; or a focus, if surrounded by spirals. The classical center-focus problem consists in distinguishing when a singular point is either a center or a focus. The problem started with Poincaré [29] and Dulac [20], and, in the present days, many questions remain unanswered. The basic results were obtained by A. M. Lyapunov [26]. He proved that if \(P(x, y)\) and \(Q(x, y)\) satisfy an infinite sequence of recursive conditions, then (1) has a center to the origin. He also presented conditions for the origin of the system (1) to be a focus.

If we write \(P(x, y) = \sum_{i=1}^{l} P_{m_i}(x, y)\) and \(Q(x, y) = \sum_{i=1}^{l} Q_{m_i}(x, y)\), where \(P_{m_i}(x, y)\) and \(Q_{m_i}(x, y)\) are homogeneous polynomials of degree \(m_i \geq 1\), then, from Hilbert’s theorem on the finiteness of basis of polynomial ideals ([23], Theorem 87, p. 58), it follows that, in the mentioned infinite sequence of recursive conditions, only a finite number of conditions for center are essential. The others result from them.

Let us consider a particular case of (1). Namely,

\[
\begin{align*}
\dot{x} &= -y + P_n(x, y) \\
\dot{y} &= x + Q_n(x, y),
\end{align*}
\]

where \(P_n(x, y)\) and \(Q_n(x, y)\) are homogeneous polynomials of degree \(n\).

When \(n = 2\), systems (2) are quadratic polynomial differential systems (or simply quadratic systems in what follows). Quadratic systems have been intensively studied over the last 30 years, and more than a thousand papers on this issue have been published (see, for example, the bibliographical survey of Reyn [30]).

A method for investigating if (2) has a center at the origin consists in transforming the planar system into an Abel equation. In polar coordinates \((r, \theta)\) defined by \(x = r \cos \theta, y = r \sin \theta\), the system (2) becomes

\[
\begin{align*}
\dot{r} &= A(\theta)r^n \\
\dot{\theta} &= 1 + B(\theta)r^{n-1},
\end{align*}
\]

where

\[
\begin{align*}
A(\theta) &= \cos \theta P_n(\cos \theta, \sin \theta) + \sin \theta Q_n(\cos \theta, \sin \theta), \\
B(\theta) &= \cos \theta Q_n(\cos \theta, \sin \theta) - \sin \theta P_n(\cos \theta, \sin \theta).
\end{align*}
\]

We remark that \(A\) and \(B\) are homogeneous polynomials of degree \(n + 1\) in the variables \(\cos \theta\) and \(\sin \theta\). In the region

\[R = \{(r, \theta) : 1 + B(\theta)r^{n-1} > 0\},\]
the differential system (3) is equivalent to the differential equation

\[ \frac{dr}{d\theta} = \frac{A(\theta)r^n}{1 + B(\theta)r^{n-1}}. \]

It is known that the periodic orbits surrounding the origin of the system (3) do not intersect the curve \( \theta = 0 \) (see the Appendix of [15]). Therefore, these periodic orbits are contained in the region \( R \). Consequently, they are also periodic orbits of equation (5).

The transformation \( (r, \theta) \rightarrow (\gamma, \theta) \) with

\[ \gamma = \frac{r^{n-1}}{1 + B(\theta)r^{n-1}} \]

is a diffeomorphism from the region \( R \) into its image. As far as we know, Cherkas was the first to use this transformation (see [16]). If we write equation (5) in the variable \( \gamma \), we obtain

\[ \frac{d\gamma}{d\theta} = -(n-1)A(\theta)B(\theta)\gamma^3 + [(n-1)A(\theta) - B'(\theta)]\gamma^2, \]

which is a particular case of an Abel differential equation. We notice that \( f(\theta) = -(n-1)A(\theta)B(\theta) \) and \( g(\theta) = (n-1)A(\theta) - B'(\theta) \) are homogeneous trigonometric polynomials of degree \( 2(n+1) \) and \( n+1 \), respectively.

Now the Center-Focus problem of equation (1) has a translation in equation (7). That is, given \( \gamma_0 \) small enough, we look for necessary and sufficient conditions on \( f(\theta) \) and \( g(\theta) \) in order to assure that the solution of equation (7) with the initial condition \( \gamma(0) = \gamma_0 \) has the property that \( \gamma(0) = \gamma(2\pi) \). We observe that this condition implies the periodicity of this solution.

In the present paper, we consider a certain variant of the Center-Focus problem related to the original one - the Center-Focus problem for the Abel differential equation

\[ \frac{dx}{dt} = g(t)x^2 + f(t)x^3 \]

This problem is to provide necessary and sufficient conditions on \( f \) and \( g \) and on \( a, b \in \mathbb{R} \) for all the solutions \( x(t) \) of (8) to satisfy \( x(a) = x(b) \). When \( f \) and \( g \) are polynomial functions, the equation (8) is called polynomial Abel equation. Notice that, now, such condition does not imply the periodicity of \( x(t) \). In recent years, the Center-Focus problem for the Polynomial Abel equation has advanced substantially, as observed in [8], [9], [12], [13], [14], [18], [32].

1.2 Composition conjecture

**Definition 1.1** An Abel differential equation (8) is said to have a center at \( a \leq t \leq b \) if \( x(a) = x(b) \) for any solution \( x(t) \) (with the initial value \( x(a) \) small enough), or equivalently, the equation (8) has a center at \( x = 0 \) if all the solutions close to \( x = 0 \) are closed.

The Center-Focus problem is to give necessary and sufficient conditions on \( f, g \) for the Abel equation above to have a center. The only, known to us, sufficient condition for the Center is the following "Polynomial Composition Condition" (PCC):
**Definition 1.2** The polynomials \( f = F' \), \( g = G' \) are said to satisfy the Polynomial Composition Condition (PCC) on \([a, b]\), if there exist polynomials \( \tilde{F}, \tilde{G} \) and \( W \), such that
\[
F(t) = \tilde{F}(W(t)), \quad G(t) = \tilde{G}(W(t)), W(a) = W(b).
\]

(PCC) is also known to be necessary for the Center for small degrees of \( f, g \) and in some other very special situations. Notice that (PCC) is described by a finite number of algebraic equations on the coefficients of \( f, g \).

**Proposition 1.3** If \( f, g \) satisfy the Polynomial Composition Condition (PCC) on \( A = [a, b] \), then the Abel equation (8) has a center on \( A \).

**Proof:** Indeed, after a change of variables \( w = W(t) \), we obtain a new polynomial Abel equation (9)
\[
\frac{d\tilde{y}}{dw} = \tilde{f}(w)\tilde{y}^3 + \tilde{g}(w)\tilde{y}^2
\]
with \( \tilde{f} = \tilde{F}', \tilde{g} = \tilde{G}' \). All the solutions of (8) are obtained from the solutions of (9) by the same substitution \( y(t) = \tilde{y}(W(t)) \). Fulfilling \( W(a) = W(b) \), one has \( y(a) = y(b) \).

All existing results in the literature so far supports the following conjecture:

**COMPOSITION CONJECTURE.** The polynomial Abel equation (8) has a center on the set of points \( A = [a, b] \) if and only if the Composition Condition (PCC) holds for \( f \) and \( g \) on \( A \).

This conjecture has been verified for small degrees of \( f \) and \( g \) and in many special cases in [4], [5], [6], [7], [8], [9], [10], [11], [18], [31], [32].

The equation (8) was studied in [1], where necessary and sufficient conditions were obtained for this equation to have a center at the origin, where \( f(t) \) and \( g(t) \) are particular continuous functions. More results that ensure the existence of a center at the origin for some subclasses of Abel equations were obtained in [24, 25].

When \( f \) and \( g \) are odd polynomials, then \( f(t) = t^k f(t^2) \) and \( g(t) = t^k g(t^2) \). Results on sufficient conditions presented below were obtained by Alwash and Lloyd [2].

**Proposition 1.4 ([2])** Assume \( f, g \in C([a, b]) \) to be expressed by
\[
f(t) = \hat{f}(\sigma(t))\sigma'(t), \quad g(t) = \hat{g}(\sigma(t))\sigma'(t)
\]
for some continuous functions \( \hat{f}, \hat{g} \) and a continuously differentiable function \( \sigma \), which is closed, i.e., \( \sigma(a) = \sigma(b) \). Then, the Abel equation
\[
x'(t) = f(t)x^3(t) + g(t)x^2(t), \quad t \in [a, b]
\]
has a center in \([a, b]\).

If true, the Composition Conjecture tells us a lot about the nature of the return map for Abel equations and its relationship with the coefficients of the system. However, it also highlights a significant difference between the polynomial and trigonometric cases. In the latter case, it is known that the class corresponding to the Composition Conjecture (that is, those systems with \( P \) and \( Q \) polynomials of a trigonometric polynomial), although a significant class, does not exhaust all possible center conditions, see [3].
1.3 Moment conditions and the Parametric composition conjecture

In this section, we consider the polynomial Abel differential equation

\[ x'(t) = g(t)x^2(t) + \epsilon f(t)x^3(t), \quad t \in [a,b], \]

where \( x \) is real, \( \epsilon \in \mathbb{R} \) and \( g(t) \) and \( f(t) \) are real polynomials. Let us assume that \( \int_a^b g(s)ds = 0 \).

One of the issues that can be tackled is characterizing when (11) has a center in \([a,b]\) for all \( \epsilon \) with \( |\epsilon| \) small enough. This type of centers are called infinitesimal centers or persistent centers, see [3], [17]. In [3], it was proved that a necessary and sufficient conditions for (11) to have a center in \([a,b]\) are

\[ \int_a^b f(t)(G(t))^k dt = 0, \]

for all natural numbers \( k \in \mathbb{N} \cup \{0\} \). Conditions (12) are called the moment conditions. The composition conjecture for moments is that the moment conditions imply the composition condition. Moreover, in [12] it is proved that “at infinity” the center conditions are reduced to the moment conditions.

A counterexample to the composition conjecture for moments in the polynomial case was given in [28], see too [21].

In the trigonometric case, that is, if one considers a trigonometric Abel differential equation of the form

\[ x'(\theta) = g(\theta)x^2(\theta) + \epsilon f(\theta)x^3(\theta), \quad \theta \in [0,2\pi], \]

where \( x \) is real, and \( \epsilon \) is a real value close to 0. One can define the composition conjecture for moments analogously to the polynomial case. The moment conditions in this case are written as

\[ \int_0^{2\pi} f(\theta)(G(\theta))^k d\theta = 0, \]

for all natural numbers \( k \in \mathbb{N} \cup \{0\} \). It is also possible to construct a counterexample of the composition conjecture for moments, see [21].

In Lijun and Yun [27, Theorem 5.5], the authors proved the following result.

**Proposition 1.5** Let \( f(\cdot) \) be a polynomial. Then, for \( \epsilon \) small enough the Abel equation

\[ x' = 2tx^2 + \epsilon f(t)x^3, \quad t \in [-1,1] \]

has a center \( x = 0 \) if and only if \( f(t) \) is an odd polynomial.

Also, the authors conjecture that the conclusion of Theorem 1.5 holds without \( \epsilon \).
1.4 Main results

This paper aims to study the following Abel’s equation

\[ x'(t) = f(t)x^3 + g(t)x^2, \; t \in [-1, 1], \]

where \( f \) and \( g \) are analytic functions. Now, we state our main results.

**Theorem 1.6** Consider the Abel’s equation

\[ x'(t) = f(t)x^3 + g(t)x^2, \; t \in [-1, 1], \]

where \( f \) and \( g \) are analytic functions. If this equation has a center at \( x = 0 \), then

\[ m_k = \int_{-1}^{1} f(t)(G(t))^k \, dt = 0, \quad k = 0, 1, 2, \]

where \( G(t) = \int_{t}^{-1} g(s) \, ds \).

**Remark 1.7** The following equation,

\[ \frac{dx}{d\theta} = (\sin \theta - \sin 2\theta + \sin 3\theta)x^3 + (\cos \theta + 2 \cos 2\theta)x^2, \; \theta \in [0, 2\pi], \]

studied by Giné, Grau and Santallusia [22], has a center at \( x = 0 \). Some computations show that \( m_0 = 0, m_1 = 0, m_2 = 0 \) and

\[ m_3 = \int_{0}^{2\pi} (\sin \theta - \sin 2\theta + \sin 3\theta) \left( \int_{0}^{\theta} (\cos s + 2 \cos 2s) \, ds \right)^3 \, d\theta = \frac{\pi}{2} \neq 0, \]

which demonstrates that the result of Theorem 1.6 is the best possible.

Now, considering \( f \) and \( g \) real polynomial functions in Abel’s equation (15), we obtain the following result.

**Theorem 1.8** Consider the Abel’s equation

\[ x'(t) = f(t)x^3 + g(t)x^2, \; t \in [-1, 1], \]

where \( f(t) = \sum_{j=0}^{d} a_j t^j = p(t^2) + tq(t^2) \) is such that \( p(t^2) \) changes sign at most two times in \([-1, 1]\) and \( g(t) = t^{n-1} \), where \( n \) is a positive even integer. If the equation (16) has a center at \( x = 0 \), then

\[ m_k = \int_{-1}^{1} f(t)(G(t))^k \, dt = 0, \quad k = 0, 1, 2, 3, 4 \ldots, \]

where \( G(t) = f(t) g(s) \, ds = \frac{1}{n}(t^n - 1) \).
If \( g(t) = t^n - 1 \) is as in Theorem 1.8, we have the following result.

**Corollary 1.9** Let \( f(\cdot) \) be a polynomial of degree \( d \leq 5 \). Then, the Abel equation

\[
x' = t^{n-1}x^2 + f(t)x^3, \quad t \in [-1, 1],
\]

has a center \( x = 0 \) if and only if \( f(t) \) is an odd polynomial.

**Remark 1.10** The results of Theorem 1.8 and Corollary 1.9 hold if \( g(t) = nt^n - 1 \), where \( n \) is a positive even integer. Hence, \( G(t) = \int_{-1}^{t} g(s)ds = t^n - 1 \).

According to Corollary 1.9 and Remark 1.10, we give a positive answer to the conjecture proposed by Lijun and Yun [27, Remark 5.6] when \( f(\cdot) \) is a polynomial of degree \( d \leq 5 \) and \( n = 2 \).

The following diagram gives us some relations and implications about the *Center problem*, *Moment conditions* and the *Composition Conjecture*.

\[\begin{align*}
\text{Composition Condition} \\
\text{Moment Condition} \\
y' = f(t)y^3 + g(t)y^2 \\
\text{has a center at origin}
\end{align*}\]

Implication (1) was proved in [4, p. 13].
Implication (2) was proved in [12, p. 442].
Implication (3) is not generally true. For example, see [28] and [21].
Implication (4) is generally an open problem, namely *Composition Conjecture*. Several particular cases were proved. See, for example, [7] and the reference therein. We proved some particular cases, see Corollaries 1.9.
Implication (5) is generally an open problem.
Implication (6) holds in several particular cases. One of these cases is the main result of this paper, see Theorem 1.8.

**Remark 1.11** Notice that, if (5) is true, then the composition conjecture (see (4)) is not true.
2 Preliminaries results

Following Yang Lijun and Tang Yun [27], we write the expression below, for a solution \( x(t, \rho) \) of the Abel equation (15) satisfying \( x(-1, \rho) = \rho \).

\[
x(t, \rho) = \rho + \sum_{k=2}^{\infty} r_k(t) \rho^k.
\]

(18)

To prove Theorems 1.6 and 1.8, we apply the following result of Yang Lijun and Tang Yun [27, Lemma 5.2, p 108].

Lemma 2.1 The origin \( x = 0 \) is a center of the Abel equation (15) if and only if

\[
\int_{-1}^{1} g(t) dt = 0 \quad \text{and} \quad \int_{-1}^{1} f(t) r_k(t) dt = 0, \quad k \geq 0
\]

or, equivalently, if and only if

\[
\int_{-1}^{1} g(t) dt = 0 \quad \text{and} \quad \int_{-1}^{1} f(t) x(t, \rho) dt = 0, \quad |\rho| < \rho_0
\]

for \( \rho_0 \) small enough, where

\[
x(t, \rho) = \frac{\rho}{1 - \rho \int_{-1}^{t} (f(s)x(s, \rho) + g(s)) ds}.
\]

(19)

Now, suppose that \( x = 0 \) is a center of the equation (15). Notice that a solution of (15) is equivalent to a solution of the integral equation

\[
x(t, \rho) = \frac{\rho}{1 - \rho \int_{-1}^{t} (f(s)x(s, \rho) + g(s)) ds}, \quad t \in [-1, 1]
\]

(21)

where \( x(-1) = \rho \), for \( \rho \) small enough such that \( \rho \int_{-1}^{t} (f(s)x(s, \rho) + g(s)) ds < 1 \) for all \( t \in [-1, 1] \). With the integral equation (21) of Abel equation we define the following nonlinear operator

\[
T_{\rho} : C[-1, 1] \rightarrow C[-1, 1]
\]

\[
T_{\rho}(x)(t) = \frac{\rho}{1 - \rho \int_{-1}^{t} (f(s)x(s, \rho) + g(s)) ds}
\]

(22)

(23)

where \( f, g \in C[-1, 1] \) and \( \rho \in \mathbb{R} \). Of course, \( T_{\rho} \) is well defined on arbitrary bounded set of \( C[-1, 1] \) if \( \rho \) is small enough. In [27], Lijun and Yun proved that \( T_{\rho} \) is a contraction. According to the well known Banach contraction theorem, \( T_{\rho} \) has a unique fixed point in \( B_1 = \{ f \in C[-1, 1]; \| f \| \leq 1 \} \), where \( \| f \| = \sup_{t \in [-1, 1]} |f(t)| \). In addition, Lijun and Yun proved that, for \( \rho \leq \rho_0 = (\sqrt{\| g \| + \| f \| + 1})^{-1} \), this fixed point is the solution \( x(t, \rho) \) of the Abel equation (15), with \( x(-1, \rho) = \rho \) and \( \| x \| \leq 1 \).
According to the Abel equation (15), 
\[ f(t, x) = f(t)x^3 + g(t)x^2, \]
with \( t \in [-1, 1] \) and \( x \in \mathbb{R} \), is analytic. Thus, the solution \( x(t, \rho) \) is also analytic in \((t, \rho)\) (see in [19, Th 8.2, p. 35]).

Set

\[ H(t, \rho) = \int_{-1}^{t} (f(s)x(s, \rho) + g(s))ds. \]  

Since \( f, g \) and \( x \) are analytic functions, we have that \( H \) is analytic. Furthermore, as \( ||x(t, \rho)|| \leq 1 \) and \( f \) and \( g \) are limited on the interval \([-1, 1]\), for small enough \( \rho \), we have
\[ -1 < \rho \int_{-1}^{t} (f(s)x(s, \rho) + g(s))ds < 1. \]
Thus, the following identities are well defined

\[ x(t, \rho) = \rho \frac{1}{1 - \rho \int_{-1}^{t} (f(s)x(s, \rho) + g(s))ds} = \rho \frac{1}{1 - \rho H(t, \rho)} = \rho (1 + H(t, \rho)\rho + H^2(t, \rho)\rho^2 + H^3(t, \rho)\rho^3 + \ldots) = \rho + H(t, \rho)\rho^2 + H^2(t, \rho)\rho^3 + H^3(t, \rho)\rho^4 + \ldots = \rho \sum_{k=0}^{\infty} H^k(t, \rho)\rho^k. \]

By Lemma 2.1, we obtain

\[ \int_{-1}^{1} f(t)x(t, \rho)dt = 0, \quad |\rho| < \rho_0 \]

The last identities lead us to conclude that

\[ \int_{-1}^{1} f(t)x(t, \rho)dt = \rho \sum_{k=0}^{\infty} \int_{-1}^{1} f(t)H^k(t, \rho)dt\rho^k = 0, \quad |\rho| < \rho_0. \]

In the proof of Theorem 1.6, we need to show that \( \int_{-1}^{1} f(t)H^k(t, \rho)dt = 0 \) in \( \rho = 0 \) and \( k = 0, 1, 2 \). For such, we need some lemmas, which are presented below.

**Lemma 2.2**

\[ \frac{\partial^j}{\partial \rho^j} (H(1, \rho)) \bigg|_{\rho=0} = j! \int_{-1}^{1} f(t)G^{j-1}(t)dt \quad \text{for each} \quad j = 1, 2 \]

and

\[ \frac{\partial^3}{\partial \rho^3} (H(1, \rho)) \bigg|_{\rho=0} = 3! \int_{-1}^{1} f(t)G^2(t)dt + 3! \int_{-1}^{1} f(t)F(t)dt, \]

where \( G(t) = \int_{-1}^{t} g(s)ds \) and \( F(t) = \int_{-1}^{t} f(s)ds \).
\textbf{Proof:} Since

\begin{equation}
\label{eq:proof1}
x(t, \rho) = \rho + H(t, \rho)\rho^2 + H^2(t, \rho)\rho^3 + H^3(t, \rho)\rho^4 + \ldots
\end{equation}

we obtain

\begin{equation}
\label{eq:proof2}
\frac{\partial}{\partial \rho} x(t, \rho) = 1 + \frac{\partial}{\partial \rho} (H(t, \rho)\rho^2) + \frac{\partial}{\partial \rho} (H^2(t, \rho)\rho^3) + \ldots,
\end{equation}

\begin{equation}
\label{eq:proof3}
\frac{\partial^2}{\partial \rho^2} x(t, \rho) = \frac{\partial^2}{\partial \rho^2} (H(t, \rho)\rho^2) + \frac{\partial^2}{\partial \rho^2} (H^2(t, \rho)\rho^3) + \ldots
= 2H(t, \rho) + R_1(t, \rho),
\end{equation}

where \( R_1(t, \rho)|_{\rho=0} = 0 \) and

\begin{equation}
\label{eq:proof4}
\frac{\partial^3}{\partial \rho^3} x(t, \rho) = \frac{\partial^3}{\partial \rho^3} (H(t, \rho)\rho^2) + \frac{\partial^3}{\partial \rho^3} (H^2(t, \rho)\rho^3) + \ldots
= 3!H^2(t, \rho) + 3! \frac{\partial}{\partial \rho} H(t, \rho) + R_2(t, \rho),
\end{equation}

where \( R_2(t, \rho)|_{\rho=0} = 0. \)

By definition of \( H \) and (26), we obtain

\begin{equation}
\frac{\partial}{\partial \rho} H(t, \rho) = \int_{-1}^{t} f(s) \frac{\partial}{\partial \rho} x(s, \rho) ds
\end{equation}

and

\begin{equation}
\frac{\partial}{\partial \rho} H(t, \rho)|_{\rho=0} = \int_{-1}^{t} f(s) ds = F(t).
\end{equation}

Hence,

\begin{equation}
\frac{\partial}{\partial \rho} H(1, \rho)|_{\rho=0} = \int_{-1}^{1} f(t) dt.
\end{equation}

By definition of \( H \) and (27), we obtain

\begin{equation}
\frac{\partial^2}{\partial \rho^2} H(t, \rho) = 2 \int_{-1}^{t} f(s)H(s, \rho) ds + \int_{-1}^{t} R_1(s, \rho) ds
\end{equation}

and

\begin{equation}
\frac{\partial^2}{\partial \rho^2} H(t, \rho)|_{\rho=0} = 2 \int_{-1}^{t} f(s)H(s, 0) ds = 2 \int_{-1}^{t} f(s)G(s) ds.
\end{equation}
Hence,

\[
\frac{\partial^2}{\partial \rho^2} H(1, \rho) |_{\rho=0} = 2 \int_{-1}^{1} f(t)G(t) dt.
\]

By definition of \( H \), (28) and (29), we obtain

\[
\frac{\partial^3}{\partial \rho^3} H(t, \rho) = 3! \int_{-1}^{t} f(s)H(s, \rho)ds + 3! \int_{-1}^{t} f(s) \frac{\partial}{\partial \rho} H(s, \rho)ds + \int_{-1}^{t} R_2(s, \rho)ds
\]

and

\[
\frac{\partial^3}{\partial \rho^3} H(t, \rho) |_{\rho=0} = 3! \int_{-1}^{t} f(s)H^2(s, 0)ds + 3! \int_{-1}^{t} f(s) \frac{\partial}{\partial \rho} H(s, 0)ds
\]

(33)

\[
= 3! \int_{-1}^{t} f(s)G^2(s)ds + 3! \int_{-1}^{t} f(s)F(s)ds.
\]

Hence,

\[
\frac{\partial^3}{\partial \rho^3} H(1, \rho) |_{\rho=0} = 3! \int_{-1}^{1} f(t)G^2(t)dt + 3! \int_{-1}^{1} f(t)F(t)dt.
\]

**Lemma 2.3** Let \( H \) be defined by (24), then

\[
\frac{\partial^j}{\partial \rho^j} (H(1, \rho)) |_{\rho=0} = j! \int_{-1}^{1} f(t)r_j(t)dt \text{ for each } j = 1, 2, 3.
\]

**Proof:** If we replace the expression (18) by the formula defining \( H(t, \rho) \), we obtain

\[
H(t, \rho) = \int_{-1}^{t} [f(s)x(s, \rho) + g(s)]ds
\]

\[
= \left( \int_{-1}^{t} f(s)ds \right) \rho + \sum_{k=2}^{\infty} \left( \int_{-1}^{t} f(s)r_k(s)ds \right) \rho^k + \int_{-1}^{t} g(s)ds
\]

\[
= G(t) + F(t)\rho + \sum_{k=2}^{\infty} \left( \int_{-1}^{t} f(s)r_k(s)ds \right) \rho^k.
\]

Hence,

\[
\frac{\partial^j}{\partial \rho^j} (H(t, \rho)) |_{\rho=0} = j! \int_{-1}^{t} f(s)r_j(s)ds \text{ for each } j = 1, 2, 3.
\]

Therefore,

\[
\frac{\partial^j}{\partial \rho^j} (H(1, \rho)) |_{\rho=0} = j! \int_{-1}^{1} f(t)r_j(t)dt \text{ for each } j = 1, 2, 3.
\]
3 Proof of Theorem 1.6

By Lemma 2.1 we have
\[ \int_{-1}^{1} f(t)r_k(t)dt = 0, \quad k \geq 0. \]

By Lemmas 2.2 and 2.3 we obtain
\[ \frac{\partial^j}{\partial \rho^j} (H(1, \rho)) \Big|_{\rho=0} = j! \int_{-1}^{1} f(t)G^j(t)dt = j! \int_{-1}^{1} f(t)r_j(t)dt = 0, \quad \text{for each } j = 1, 2 \]
and
\[ \frac{\partial^3}{\partial \rho^3} (H(1, \rho)) \Big|_{\rho=0} = 3! \int_{-1}^{1} f(t)G^2(t)dt + 3! \int_{-1}^{1} f(t)F(t)dt = 3! \int_{-1}^{1} f(t)r_3(t)dt = 0. \]

Since by Lemma 2.1 we obtain \( F(1) = \int_{-1}^{1} f(t)dt = 0 \), we conclude that
\[ \int_{-1}^{1} f(t)F(t)dt = \frac{1}{2} \int_{-1}^{1} \frac{d}{dt}F^2(t)dt = \frac{1}{2}[F^2(1) - F^2(-1)] = 0. \]

Therefore,
\[ \int_{-1}^{1} f(t)G^2(t)dt = \int_{-1}^{1} f(t)r_3(t)dt = 0. \]

Hence, we obtain
\[ m_k = \int_{-1}^{1} f(t)(G(t))^k dt = 0, \quad k = 0, 1, 2. \]

4 Proof of Theorem 1.8

In the proof of Theorem 1.6 we will use the result due to Briskin, Francoise and Yomdin [8 Theorem 4.1]. By [8], it is sufficient to analyze the sign changes of the function
\[ \psi(u) = \sum_{i=1}^{m} \frac{\text{sgn}(g(t_i(u)))f(t_i(u))}{g(t_i(u))} \]
where \( t_1(u), t_2(u), \ldots, t_m(u) \) are the solutions in \([-1, 1]\) of the equation \( G(t) = u \) where \( u \in [u_0, u_1] \), with \( u_0 = \min_{[-1, 1]} G(t) \) and \( u_1 = \max_{[-1, 1]} G(t) \).

Since \( g(t) = t^{n-1} \), we obtain \( G(t) = \frac{1}{n}(t^n - 1) \). Hence, \( u_0 = -\frac{1}{n} \) and \( u_1 = 0 \). In particular, if \( u \in [-\frac{1}{n}, 0] \), we obtain \( 0 \leq 1 + nu \leq 1 \). The solutions in \([-1, 1]\) of the equation
\[ G(t) = \frac{1}{n}(t^n - 1) = u \]
are \( t_1(u) = (1 + nu)^{\frac{1}{n}} \) and \( t_2(u) = -(1 + nu)^{\frac{1}{n}} \). Therefore, since \( n \) is even,

\[
\psi(u) = \frac{f(t_1(u))}{g(t_1(u))} + \frac{f(t_2(u))}{g(t_2(u))}
\]

Now, we define the polynomial \( h(t) = f(t) + f(-t) = p(t^2) \). Since according assumptions, the polynomial \( h(t) \) changes sign at most two times in \([-1, 1]\), we conclude that \( \psi(u) \) changes sign at most two times in \([-1, 0]\).

5 Proof of Corollary 1.9

We can write the polynomial \( f \) as

\[
f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5.
\]

By Theorem 1.6

\[
m_k = \int_{-1}^{1} f(t)(G(t))^k dt = 0, \quad k = 0, 1, 2
\]

where \( G(t) = \int_{-1}^{t} g(s) ds = \frac{1}{n} (t^n - 1) \).
Since $t^n - 1$ is an even function, we obtain
\[
\int_{-1}^{1} [a_0 + a_2 t^2 + a_4 t^4] dt = 0,
\]
and
\[
\frac{1}{n} \int_{-1}^{1} [a_0 + a_2 t^2 + a_4 t^4](t^n - 1) dt = 0
\]
and
\[
\frac{1}{n^2} \int_{-1}^{1} [a_0 + a_2 t^2 + a_4 t^4](t^n - 1)^2 dt = 0.
\]
By solving the integrals, we obtain the following system
\[
\begin{align*}
2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 &= 0 \\
-\frac{2n}{1+n}a_0 - \frac{2n}{3(n+1)}a_2 - \frac{2n}{5(n+1)}a_4 &= 0 \\
\frac{4n^2}{(1+2n)(1+n)}a_0 + \frac{4n^2}{3(3+2n)(3+n)}a_2 + \frac{4n^2}{5(5+2n)(5+n)}a_4 &= 0,
\end{align*}
\]
or equivalently
\[
\begin{pmatrix}
1 & \frac{1}{3} & \frac{1}{5} \\
\frac{1}{1+n} & \frac{1}{3(n+1)} & \frac{1}{5(n+1)} \\
\frac{1}{(1+2n)(1+n)} & \frac{1}{3(3+2n)(3+n)} & \frac{1}{5(5+2n)(5+n)}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_2 \\
a_4
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
Notice that the matrix associate
\[
M = \begin{pmatrix}
1 & \frac{1}{3} & \frac{1}{5} \\
\frac{1}{1+n} & \frac{1}{3(n+1)} & \frac{1}{5(n+1)} \\
\frac{1}{(1+2n)(1+n)} & \frac{1}{3(3+2n)(3+n)} & \frac{1}{5(5+2n)(5+n)}
\end{pmatrix}
\]
is nonsingular, with $\text{det}(M) = -\frac{16}{15(n+1)(n+3)(n+5)(1+2n)(3+2n)(5+2n)}$. Then, the system (35) has only the trivial solution, namely, $a_0 = a_2 = a_4 = 0$. Hence, $f(\cdot)$ has only odd powers of $t$ and this finishes the proof of Corollary 1.9.

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