Gauge Invariant Bosonization of Quantum Hall Systems and Skyrmions: Kinematics

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Abstract

We develop a systematic semiclassical approximation scheme for quantum Hall skyrmions near filling factors $\nu = \frac{1}{2n+1}$, which is exact in the long wavelength limit. We construct a coherent state basis for the Hilbert space of Chern-Simons gauge fields and composite bosons with spin. These states are projected to the physical gauge invariant subspace and their wavefunctions explicitly evaluated. The lowest Landau level (LLL) condition is shown to be equivalent to an analyticity condition on the parameters.

The matrix elements of physical observables between these states are shown to be calculable in the limit of small amplitude long wavelength density fluctuations. The electric charge density is shown to be proportional to the topological charge density if and only if the LLL condition is satisfied.

We then show that these states themselves form a generalised coherent state basis, parameterised by the values of physical observables. The theory can therefore be written in terms of these gauge invariant bosonic fields in the long wavelength regime. The off diagonal matrix elements of observables in these coherent states are computed and shown to vanish in the long wavelength limit. Thus we are able to prove that the classical description of the skyrmion is exact in the limit of large skyrmions.
I. INTRODUCTION

The combination of a low value of effective mass and a low value of the Zeeman coupling in GaAs makes the spin degree of freedom relevant for quantum Hall systems in this material. Initial calculations by Chakraborty and Zhang [1] showed that the low energy quasiparticles in such systems were spin reversed. Kane and Lee [2] then argued that the quasiparticles could be extended spin textures that are described by topological solitons of the non-linear sigma model (NLSM): skyrmions. Sondhi et. al. [3] estimated the energy of skyrmions using an effective NLSM and showed that they are energetically favoured over single spin reversed quasiparticles. This was corroborated by Hartree-Fock calculations by Fertig et. al. [4].

Experimental evidence for the quasiparticles being extended spin textures initially came from Knight shift measurements of Barret et. al. [5], optical magneto-absorption experiments of Aifer et. al. [6] and tilted field transport measurements of Schrieffer et. al [7]. These experiments found the average value of the spin of the quasiparticle to be $\sim 2 - 7$ depending on the system parameters. Subsequently, experiments have been done in systems where the Landé g factor is reduced by high pressures [8] or by engineering the material [9] yielding evidence of quasiparticles with very large, ($\sim 20 - 30$), values of spin. A negative result has also been reported by Kukushkin et. al. [10], where optical measurements found the quasiparticles to have spin $\frac{1}{2}$, i.e. no spin textures. There has been no explanation of this result in the literature so far.

In the NLSM approach, the quasiholes (particles) are assumed to be described by the classical solutions of the model. The electrical charge density is assumed to be equal to the topological charge density, thus the quasiparticles are the classical solutions in the topological charge, $Q_{top} = 1$, sector. Near $\nu = 1$, the NLSM energy functional and the relation between the topological charge density and electrical charge density has been derived by many workers in the LLL, long wavelength approximation [3][11][12].

What is the regime of validity of this model and is there a limit where the classical approximation is exact? This is the main question we address and answer in this paper.
This question has previously been discussed by Girvin et. al. [13], where they numerically compare the skyrmion energies form the NLSM, Hartree-Fock and exact diagonalisation. The energies match in the limit of large sized skyrmions (with size $R \geq 15l_c$, $l_c$ being the magnetic length. This calculation suggests that the classical approximation may be exact in long wavelength limit. In this paper we start from the microscopic theory and analytically address the following questions:

1(a). When is the topological charge proportional to the electric charge ?

1(b). When are the corresponding densities proportional to each other ?

2. What is the limit in which the classical approximation is exact ?

These questions automatically throw up a third one:

3. How can the LLL condition be imposed in the classical theory ?

The composite boson formalism [14] is the obvious choice of the formalism to use to address the question of the classical limit. However, we would also like to impose the LLL condition, which is a condition on the quantum states. We therefore develop a coherent state formalism for composite bosons.

In section II, we construct a coherent state basis for the Hilbert space of the bosons and the Chern-Simons gauge fields. The anticommuting electron operators are then constructed in terms of the bosonic fields. This construction provides the explicit mapping between the electronic Hilbert space and operators and the gauge invariant sector of the composite boson theory.

Section III concentrates on the gauge invariant sector of the theory. The coherent states are projected on to the gauge invariant sector and their wave functions calculated. We then find that the wave functions thus obtained are exactly of the form written down previously by Ezawa [16]. The LLL condition is easily seen to be equivalent to an analyticity condition on the parameters labelling the coherent states. The coherent states are labelled by the values of the bosonic spinor field $\phi_\sigma(x)$ and the Chern-Simons gauge field $\alpha_i(x)$. The states projected to the gauge invariant sector depend only on gauge invariant combinations of these fields i.e the longitudinal part of $\alpha(x)$ gets related to the phase of $\phi(x)$. An inspection of
the wavefunctions reveals another local invariance that relates the transverse part of $\alpha(x)$ to the magnitude, $\phi^\dagger(x)\phi(x)$ of the bosonic fields. This transformation is not unitarily realised. Nevertheless it implies that the physical states can be labelled by a single bosonic spinor field which we denote by $W_\sigma(x)$. We then show that the matrix elements of the observables between these projected coherent states can be computed in the limit of small amplitude, longwavelength density fluctuations, which we refer to as the hydrodynamic limit. We then discuss the relation between the charge and the topological charge densities. Our approach is related to previous approaches of both that of Ezawa [16] and to that of Murthy and Shankar [15]. Our coherent state wavefunctions are of the same form as in reference [16] and our calculations are done in the same physical regime (the hydrodynamic limit) as in reference [17].

The issue of the classical limit is dealt with in section IV. We show (in the hydrodynamic limit), that the projected coherent states satisfy the properties of generalized coherent states [17]. We also show that the off diagonal matrix elements of the observables vanish in this limit. The theory thus admits a classical description in this limit. Further, it can be written in completely in terms of bosonic fields corresponding to gauge invariant physical observables.

The concluding section VI summarises and discusses our results.

II. COMPOSITE BOSON THEORY AND COHERENT STATES

In this section, we do the usual bosonization wherein the electronic theory is written in terms of bosons attached to fluxes of Chern-Simons fields. The electron operators and the procedure of flux attachment is very transparent in terms of coherent states. So we first define the Hilbert space of the composite bosons and construct the coherent state basis for it. We then construct the electron operators and obtain the explicit mapping between the electronic Hilbert space and observables and the gauge invariant states and observables in the bosonic theory.
A. The Composite Boson Hilbert Space

The bosonic degrees of freedom are described by the spinor field operators\[ \hat{\phi}_\sigma(x), \hat{\phi}^\dagger_\sigma(x), \sigma = 1, 2 \] and the Chern-Simons gauge fields, \( a_i(x), i = 1, 2 \). The \( \hat{\phi} \) operators act on the Hilbert space, \( \mathcal{H}_B \) and satisfy the canonical commutation relations,\[ \begin{align*}
[\hat{\phi}_\sigma(x), \hat{\phi}^\dagger_{\sigma'}(y)] & = \delta_{\sigma\sigma'}\delta^2(x - y) \\
[\hat{\phi}_\sigma(x), \hat{\phi}_{\sigma'}(y)] & = 0
\end{align*} \tag{1} \]

The Chern-Simons gauge fields act on the Hilbert space \( \mathcal{H}_{CS} \) and satisfy,\[ [a_i(x), a_j(y)] = \sqrt{\frac{\hbar c}{\kappa}} \epsilon_{ij} \delta^2(x - y) \tag{2} \]
where \( \kappa = \frac{e^2}{2\pi \hbar c(2n+1)} \). If we define the complex fields, \( a(x) \) and \( \bar{a}(x) \) as,
\[ \begin{align*}
a(x) & \equiv \frac{a_2(x) + ia_1(x)}{\sqrt{2}} \sqrt{\frac{\kappa}{\hbar c}} \\
\bar{a}(x) & \equiv \frac{a_2(x) - ia_1(x)}{\sqrt{2}} \sqrt{\frac{\kappa}{\hbar c}}
\end{align*} \tag{3} \]
then the commutation relation between them is given by,\[ [a(x), \bar{a}(y)] = \delta^2(x - y) \tag{4} \]

The full Hilbert space of the composite boson theory is the direct sum of the above two spaces and we denote it by,\[ \mathcal{H}_{CB} = \mathcal{H}_B \oplus \mathcal{H}_{CS} \tag{5} \]
We denote the gauge invariant sector of this space by \( \mathcal{H}_{phy} \subset \mathcal{H}_{CB} \). \( \mathcal{H}_{phy} \) consists of the states which respect the Chern-Simons Gauss law constraint,\[ \hat{G}(x)|\psi\rangle_{phy} = 0 \tag{6} \]
where \( \hat{G}(x) \) are the generators of gauge transformations given by,\[ \hat{G}(x) = \kappa \nabla \times \bar{a}(x) - e\hat{\phi}^\dagger(x)\hat{\phi}(x) \tag{7} \]
We will refer to the gauge invariant observables, the operators that commute with \( \hat{G}(x) \) as physical observables.
B. Coherent State Basis

In this section, we construct the coherent state basis for $\mathcal{H}_{CB}$. The displacement operators are defined to be,

$$D(\alpha) \equiv e^{\int_x [\alpha(x)\hat{a}(x) - \bar{\alpha}(x)a(x)\]}$$

$$U(\varphi) \equiv e^{\int_x [\varphi(x)\hat{\varphi}(x)\dagger - \bar{\varphi}(x)\hat{\varphi}(x)\]}$$

(8)

where, $\alpha(x) \equiv \frac{\alpha_2(x) + i\alpha_1(x)}{\sqrt{2}}\sqrt{\frac{\hbar c}{\kappa}}$. The coherent states $|\alpha, \varphi\rangle$, parameterised by the gauge field $\alpha(x)$ and the spinor field $\varphi(x)$ are then given by,

$$|\alpha, \varphi\rangle \equiv U(\varphi)D(\alpha)|0\rangle$$

(9)

where,

$$a(x)|0\rangle = \hat{\varphi}_\sigma(x)|0\rangle = 0$$

(10)

The states defined in equation(9) can be interpreted as gaussian wave packets peaked around the classical field configuration $(\alpha(x), \varphi(x))$. They satisfy the three standard properties of coherent states [17] namely,

1. Resolution of unity:

$$\int D[\alpha, \varphi] |\alpha, \varphi\rangle \langle \alpha, \varphi| = I$$

(11)

where $D[\alpha, \varphi] = \prod_{x,\sigma} \frac{d\alpha(x)d\bar{\alpha}(x) d\varphi(x)d\bar{\varphi}(x)}{2\pi i}$

2. Continuity of overlaps:

$$\langle \alpha_1, \varphi_1|\alpha_2, \varphi_2\rangle = e^{-\frac{i}{\hbar c} \int_x \bar{\alpha}_1(x)\times\bar{\alpha}_2(x)} e^{-\frac{1}{2\hbar c} \int_x (\bar{\alpha}_1(x) - \bar{\alpha}_2(x))^2}$$

$$e^{\frac{i}{2\hbar c} \int_x [\bar{\varphi}_1(x)\varphi_2(x) - \varphi_1(x)\bar{\varphi}_2(x)] e^{-\frac{i}{2\hbar c} \int_x |\varphi_1(x) - \varphi_2(x)|^2}}$$

(12)

3. Values of Observables:

$$\langle \alpha, \varphi \rangle : O(a, \bar{a}, \hat{\varphi}, \hat{\varphi}^\dagger) : |\alpha, \varphi\rangle = O(\alpha, \bar{\alpha}, \varphi, \varphi^\dagger)$$

(13)

The coherent states are not gauge invariant. Under gauge transformations,
\[ |\alpha, \varphi\rangle \rightarrow e^{ie\frac{\hat{\alpha}}{\hbar c}} \int G(x) \Omega(x) |\alpha, \varphi\rangle = e^{ie\frac{\hat{\alpha}}{\hbar c}} \int a(x) \times \nabla \Omega(x) |\alpha - \nabla \Omega, \varphi e^{-\frac{\hbar}{\alpha}} \Omega) \]  

(14)

We will now construct a projection operator that projects any state into the gauge invariant subspace, \( \mathcal{H}_{phy} \). Consider,

\[
P \equiv \frac{1}{V_G} \int \Omega e^{ie\int x \Omega(x) \hat{G}(x)}
\]

(15)

where \( \hat{G}(x) \) is the generator of gauge transformations, as given in equation(11) and \( V_G = f_{\Omega} \) is the volume of the gauge group. Shifing the integration variable \( \Omega \) by \( \beta \) in the projection operator \( P \),

\[
e^{ie\int x \hat{G}(x)} P = P
\]

(16)

Taking \( \beta \rightarrow 0 \),

\[
\hat{G}(x) P = 0 \Rightarrow \hat{G}(x) P |\psi\rangle = 0
\]

(17)

This proves that \( P \) is an operator that projects any state into \( \mathcal{H}_{phy} \).

The above three properties (11 - 13) and the projection operator defined in equation(15) can be used to derive the path integral representation of the gauge invariant evolution operator. This is done in the Appendix A to obtain,

\[
Z = \int \mathcal{D}[a_0(x, t)] \mathcal{D}[a_i(x, t)] \mathcal{D}[\varphi(x, t)] e^{i\frac{\hbar}{\alpha} \int dtd^2 x L(x, t)}
\]

(18)

where \( L(x, t) \) is the standard lagrangian of matter fields coupled to Chern-Simons gauge fields. This confirms the equivalence of our formalism to the standard lagrangian formalism.

**C. Bosonization**

We will now construct gauge invariant anticommuting operators that create and annihilate flux carrying bosons. These operators satisfy the fermionic canonical anticommutation relations and can hence be used to represent the electron creation and annihilation operators.
in $\mathcal{H}_{CB}$. We will thus be able to map the gauge invariant sector of the composite boson Hilbert space, $\mathcal{H}_{phy}$, to the Hilbert space of the electronic system, $\mathcal{H}_{el}$. The mapping is then used to map the observables of the electronic system to gauge invariant operators in $\mathcal{H}_{CB}$.

We define $c^\dagger_\sigma(x)$ as

$$c^\dagger_\sigma(x) \equiv D(x)\hat{\varphi}^\dagger_\sigma(x)K(x)$$  \hspace{1cm} (19)

We have used $D(x)$ as short notation for $D(\alpha^v_x)$. $\alpha^v_x$ is the classical configuration of a vortex with a delta function flux density at the point $x$.

$$\kappa \nabla \times \vec{\alpha}^v_x(z) = e\delta^2(z - x)$$  \hspace{1cm} (20)

$D(x)$ therefore creates a gaussian wave packet peaked around this classical vortex configuration. When $c^\dagger(x)$ acts on a state, $\hat{\varphi}^\dagger(x)$ creates a bosonic particle at $x$ and $D(x)$ attaches Chern-Simons flux to it. The operator $K(x)$ gives the Aharanov-Bohm phase corresponding to all the other particles already present in the state. It is defined as,

$$K(x) \equiv e^{i(2n+1)\int_x^z \theta(x-z)\hat{\varphi}^\dagger(z)\hat{\varphi}(z)}$$  \hspace{1cm} (21)

where $\theta(x)$ is the angle the vector, $x$, makes with the x-axis.

Using the commutation relations given in equations (1 and 2), it can be verified that the following canonical anti-commutation relations hold good.

$$\{c_\sigma(x), c^\dagger_{\sigma'}(y)\} = \delta_{\sigma\sigma'}\delta^2(x - y)$$  \hspace{1cm} (22)

$$\{c_\sigma(x), c_{\sigma'}(y)\} = \{c^\dagger_\sigma(x), c^\dagger_{\sigma'}(y)\} = 0$$  \hspace{1cm} (23)

Hence $c^\dagger_\sigma(x)$ and $c_\sigma(x)$ provide a representation of the electron creation and annihilation operators in $\mathcal{H}_{CB}$.

Under gauge transformations,

$$\hat{\varphi}_\sigma(x) \rightarrow e^{i\frac{e}{\hbar c}\Omega(x)}\hat{\varphi}_\sigma(x) \hspace{1cm}, \hspace{1cm} \hat{\varphi}^\dagger_\sigma(x) \rightarrow e^{-i\frac{e}{\hbar c}\Omega(x)}\hat{\varphi}^\dagger_\sigma(x)$$

$$a_i(x) \rightarrow a_i(x) + \partial_i\Omega(x) \hspace{1cm}, \hspace{1cm} D(x) \rightarrow e^{i\frac{e}{\hbar c}\Omega(x)}D(x)$$  \hspace{1cm} (24)
We see that \( c_\sigma(x) \) and \( c_\sigma^\dagger(x) \) are gauge invariant.

We are now in a position to map \( \mathcal{H}_{el} \) into \( \mathcal{H}_{phy} \). We map the state with 0 number of electrons, \( |0\rangle_{el} \), to the vacuum state of \( \mathcal{H}_{CB} \), defined in equation (10), projected to \( \mathcal{H}_{phy} \).

\[
|0\rangle_{el} \rightarrow P|0\rangle
\]  

(25)

Since \( c_\sigma(x) \) are gauge invariant, they commute with \( P \). Then from equation (10) it follows that,

\[
c_\sigma(x)P|0\rangle = 0
\]  

(26)

The state with \( N \) electrons at \((x_1, x_2, ..., x_N)\) with spins \((\sigma_1, \sigma_2, ..., \sigma_N)\), \(|\{x_n, \sigma_n\}_N\rangle\), is then mapped into,

\[
|\{x_n, \sigma_n\}_N\rangle \rightarrow \prod_{n=1}^{N} c_{\sigma_n}^\dagger(x_n)P|0\rangle
\]  

(27)

Since the states in the RHS of equations (25) and (27) form a basis for \( \mathcal{H}_{el} \), these equations specify the explicit mapping of \( \mathcal{H}_{el} \) into \( \mathcal{H}_{phy} \).

It is now easy to map the observables as well. The density is given by,

\[
\hat{\rho}(x) = c_\sigma^\dagger(x)c_\sigma(x) = \hat{\phi}_\sigma^\dagger(x)\hat{\phi}_\sigma(x)
\]  

(28)

The spin density is,

\[
\hat{S}^a(x) = \frac{1}{2} c_\sigma^\dagger(x)\tau_{a\sigma'}c_\sigma(x) = \frac{1}{2} \hat{\phi}_\sigma^\dagger(x)\tau_{a\sigma'}\hat{\phi}_\sigma(x)
\]  

(29)

The current density is,

\[
\hat{J}_i(x) = \frac{1}{2}(c_\sigma^\dagger(x)[-i\hbar \partial_i - \frac{e}{c} A_i(x)]c_\sigma(x) + h.c)
\]

\[
= \frac{1}{2}(\hat{\phi}_\sigma^\dagger(x)[-i\hbar \partial_i - \frac{e}{c} a_i(x) - \frac{e}{c} A_i(x)]\hat{\phi}_\sigma(x) -
\]

\[
\frac{1}{c}\hat{\phi}_\sigma^\dagger(x)\hat{\phi}_\sigma(x) \int_z \alpha_{\chi_1}^\nu(z)\hat{G}(z) + h.c)
\]

(30)

The last term acting on physical states is zero. Thus for matrix elements between physical states, we have,
\[ \hat{J}_i(x) = \frac{1}{2} \hat{\phi}^\dagger(\sigma^i(x))[-i\hbar \partial_i - \frac{e}{c} a_i(x) - \frac{e}{c} A_i(x)]\hat{\phi}(x) + h.c \]  

(31)

Similarly, the kinetic energy density, \( \hat{T}(x) \) is computed to be,

\[ \hat{T}(x) = \frac{1}{2m} \hat{\phi}^\dagger(\sigma^i(x))[-i\hbar \partial_i - \frac{e}{c} a_i(x) - \frac{e}{c} A_i(x)]^2 \hat{\phi}(x) \]  

(32)

### III. GAUGE INVARIANT BASIS STATES

In this section, we study the coherent states projected into \( \mathcal{H}_{phy} \). We show that these states form a basis of \( \mathcal{H}_{phy} \). Their wavefunctions and expectation values of observables are computed. The LLL condition can then be seen to be equivalent to an analyticity condition on the parameters. We then discuss the relation between the charge density and the topological charge density. Finally, we describe the parametrisation of the projected coherent states in terms of a single complex spinor field \( W_\sigma(x) \) discussed in the end of section II and derive expressions for the observables in terms of \( W_\sigma(x) \)

#### A. Projected coherent states

Consider the set of coherent states, projected to \( \mathcal{H}_{phy} \),

\[ |\alpha, \varphi\rangle_p \equiv P|\alpha, \varphi\rangle \]  

(33)

Using the fact that \( P^2 = P \) and equation(11), we have,

\[ \int \mathcal{D}[\alpha] \mathcal{D}[\varphi^\dagger] \mathcal{D}[\varphi] |\alpha, \varphi\rangle_p \langle \alpha, \varphi | = P \]  

(34)

\( P \) is the identity operator in \( \mathcal{H}_{phy} \) so the projected coherent states form a basis for it.

The coherent states are not eigenstates of the number operator. Thus they have a non-zero overlap with states containing any number of particles. The wavefunction in the \( N \) particle sector is the overlap with the states given in equation(27),

\[ \psi_N(\{x_i, \sigma_i\}) = \langle \{x_i, \sigma_i\}_N |\alpha, \varphi\rangle_p \]  

(35)
Using equations (9), (15), (19) and (27), the RHS can be written as,

\[ \psi_N(\{x_i, \sigma_i\}) = \prod_{i>j} e^{(2n+1)\theta(x_i-x_j)} \frac{1}{V_G} \int_\Omega \left( e^{\frac{i}{2h}\int \nu(x)(\nabla \times \vec{a})(x)} \prod_{i=1}^N \varphi_{\sigma_i}(x_i) e^{-\frac{\hbar c}{e} \sum_{i=1}^N \Omega(x_i)} e^{-\frac{\hbar c}{2e} \int_\nu |\varphi(x)|^2} \right) \]

(36)

The details of this calculation and what follows is given in Appendix B. The \( \Omega \) integral in the RHS above is gaussian and can be done exactly. After some algebra, we obtain,

\[ \psi_N(\{x_i, \sigma_i\}) = \text{const.} e^{-\frac{1}{4l^2c} \sum_{i=1}^N |z_i|^2} \]

(37)

where \( \psi_L(\{x_i\}) \) is the Laughlin wavefunction,

\[ \psi_L(\{x_i\}) = \prod_{i>j} (z_i - z_j)^{2n+1} e^{-\frac{1}{4l^2c} \sum_{i=1}^N |z_i|^2} \]

(38)

\( \alpha \) has been written as

\[ \alpha_i(x) = \epsilon_{ij} \partial_j \Omega_T(x) + \partial_i \Omega_L(x) \]

(39)

and \( \Omega_T(x) = -\frac{\hbar c |x|^2}{4l^2c} \). Note that when \( \varphi_{\sigma}(x) = \text{constant} \) and \( \Omega_T(x) = \Omega_T(x) \Rightarrow \nabla \times \vec{a} = B \), the wavefunction reduces to the Laughlin wavefunction. Thus the ”mean field” state is the Laughlin state. In this case the \( \Omega \) integral in equation (36) is equivalent to an \( N \) vertex operator correlation function in a \( c=1 \) conformal field theory. These wavefunctions are exactly of the form written down by Ezawa [16].

**B. Parameterisation and LLL condition**

Apart from an overall factor that only affects the norm, the wavefunction in equation (37) depends on the parameters \( \alpha \) and \( \varphi \) through a spinor field \( W_{\sigma}(x) \) defined as,

\[ W_{\sigma}(x) \equiv \varphi_{\sigma_i}(x_i) e^{\frac{\hbar c}{e} (\Omega_T(x_i) - \bar{\Omega}_T(x_i) - i\Omega_L(x_i))} \]

(40)

\( W_{\sigma}(x) \) and hence the wavefunction is gauge invariant (as it should be), since under gauge transformation,
\[ \Omega_T(x) \rightarrow \Omega_T(x) \]
\[ \Omega_L(x) \rightarrow \Omega_L(x) + \Omega(x) \]
\[ \varphi_{\sigma}(x) \rightarrow \varphi_{\sigma}(x)e^{\frac{i\hbar}{2}\chi(x)} \]  
\[ \Omega_L(x) \rightarrow \Omega_L(x) + \chi(x) \]
\[ \varphi_{\sigma}(x) \rightarrow \varphi_{\sigma}(x)e^{-\frac{i\hbar}{2}\chi(x)} \]  
(41)  

\( \alpha \) and \( \varphi \) have 6 real field components. The gauge invariance of the wavefunctions reduces the number of parameters to 5. There is another local invariance of \( W \), i.e.

\[ \Omega_T(x) \rightarrow \Omega_T(x) + \chi(x) \]
\[ \Omega_L(x) \rightarrow \Omega_L(x) \]
\[ \varphi_{\sigma}(x) \rightarrow \varphi_{\sigma}(x)e^{-\frac{i\hbar}{2}\chi(x)} \]  
(42)  

Only the norm of the state changes under this transformation and the physical state remains the same. Clearly this transformation is not unitarily implemented in \( \mathcal{H}_{CB} \). Nevertheless it reduces the number of independent real fields that parameterize the states to 4, the components of the spinor field \( W \). Thus we can define the normalised projected coherent states, that are parameterised by \( W \) as,

\[ |W\rangle = \frac{1}{\mathcal{N}}|\alpha,\varphi\rangle_p \]  
(43)  

where, \( \mathcal{N} = \langle \alpha, \varphi | \alpha, \varphi \rangle_p \), is the norm of \( |\alpha, \varphi\rangle_p \).

From equations (37) and (40) it is clear that the LLL condition is equivalent to the condition that \( W \) is analytic,

\[ \partial_{\bar{z}} W_{\sigma}(x) = 0 \]  
(44)  

Thus the LLL condition is easily implemented in this formalism as it is equivalent to an analyticity condition on the parameters.

C. Observables

We will now compute the expectation values of gauge invariant operators in the projected coherent states. This is given by,
\[ \langle \hat{O} \rangle = \langle W | \hat{O} | W \rangle \] (45)

where \( \hat{O} \) is a gauge invariant observable.

We do all our calculations the limit of \( W_\sigma(x) \) being a slowly varying function of \( x \) (over a length scale of \( l_c \)). As we will see, this is also the limit of small density fluctuations. We refer to this limit as the hydrodynamic limit. We note that this is also the limit in which the analytic calculations of Murthy and Shankar [15] are done.

Just as in the case of the Laughlin wavefunction, the computation of \( \mathcal{N} \) reduces to the computation of the partition function of a classical 2-d plasma problem. Except that here, the plasma density is coupled to an external field which is a function of \( W \). In the hydrodynamic limit, the partition function can be evaluated by the saddle point approximation. The details of the calculation are presented in Appendix C, where we evaluate the norm to be,

\[
\mathcal{N}[W, \Omega_T] = \text{const} \times e^{-\int_x W_\dagger(x) W(x) e^{-\frac{\bar{\rho}}{\hbar c} \{\Omega_T(x) - \bar{\Omega}_T(x)\}}} e^{-\frac{\hbar c}{4\pi} \int_x \nabla \Omega_T(x) \cdot \nabla \Omega_T(x)} e^{-\frac{1}{4\pi(2n+1)} \int_x [\ln(W_\dagger(x)W(x)) + 2\frac{\bar{\rho}}{\hbar c} \Omega_T(x)]} \nabla^2 [\ln(W_\dagger(x)W(x))] + 2\frac{\bar{\rho}}{\hbar c} \Omega_T(x)} \] (46)

Expectation values of observables similarly reduce to the computation of expectation values in the plasma problem. These are also computed in the saddle point approximation in Appendix C. We get the density to be,

\[
\rho(x) - \bar{\rho} \equiv \langle W | \hat{\rho}(x) - \bar{\rho} | W \rangle = -\frac{1}{4\pi(2n+1)} \nabla^2 \ln(W_\dagger(x)W(x)) \] (47)

where \( \bar{\rho} \) is the mean density.

Similarly, the spin density is computed to be,

\[
s^a(x) \equiv \langle W | \hat{s}^a(x) | W \rangle = \frac{\rho(x)}{2} Z_\dagger(x) \tau^a Z(x) \] (48)

where we have denoted the normalised spinor by \( Z \).
\[ Z_\sigma(x) \equiv \frac{W_\sigma(x)}{\sqrt{W^\dagger(x)W(x)}} \]  

(49)

and \( \tau^a \) are the Pauli spin matrices. The current density and the kinetic energy density are also computed to yield,

\[
J_i(x) \equiv \langle W | \hat{J}_i(x) | W \rangle = \rho(x)[\hbar \dot{L}^3_i(x) - \frac{e}{c}(A_i(x) - \alpha_i(x))] 
\]

(50)

where \( L^3_i \equiv \frac{1}{\hbar}(Z^\dagger \partial_i Z - h.c) \) and \( \kappa \nabla \times \vec{\alpha}(x) = e\rho(x) \),

\[
\mathcal{T}(x) \equiv \langle W | \hat{T}(x) | W \rangle = \hbar \omega_c \rho(x) \frac{\partial_i W_\sigma(x) \partial_j W_\sigma'(x)}{W_\sigma'(x) W_\sigma'(x)} 
\]

(51)

Note that the kinetic energy density is zero when \( W \) is analytic.

**D. Charge and Topological Charge Densities**

The topological charge density is given by

\[
q(x) = \frac{1}{8\pi} \epsilon_{ij} \hat{n}(x) \cdot \partial_i \hat{n}(x) \times \partial_j \hat{n}(x) 
\]

(52)

where \( \hat{n}(x) \) is the local direction of spin polarization, \( \vec{s}(x) = \frac{1}{2} \rho(x) \hat{n}(x) \). In terms of \( Z \), it is given by,

\[
q(x) = \frac{1}{2\pi i} \epsilon_{ij} \partial_i Z^\dagger(x) \partial_j Z(x) 
\]

(53)

As can be seen from equations (47) and (53), the topological charge density is not necessarily proportional to the electrical charge density. In fact, in general, they are independent of each other since \( W^\dagger(x)W(x) \) and \( Z_\sigma(x) \) are independent variables. However, if the LLL condition is satisfied, then the analyticity of \( W_\sigma(x) \) relates the modulus and the phase of each component. Then \( W^\dagger(x)W(x) \) and \( Z_\sigma(x) \) are no longer independent. In fact if we use the analyticity condition, \( \partial_i W_\sigma(x) = -i\epsilon_{ij} \partial_j W(x) \), in the RHS of equation (47), we get,
\[ \rho(x) - \bar{\rho} = -\frac{1}{2n+1} q(x) \]  

(54)

Thus the topological charge density is proportional to the electrical charge density if and only if the LLL condition is satisfied. The relation (54) will therefore not be true in presence of Landau level mixing.

When the densities are proportional, the total excess charge, \( Q \), will of course be proportional to the total topological charge, \( Q_{\text{top}} \). However, the total charges could be proportional without the densities being so. We will now investigate this possibility. Integrating equation (47) over all space, we have,

\[ Q = \frac{1}{4\pi(2n+1)} \oint dx^i \epsilon_{ij} \partial_j \ln(W^*(x)W(x)) \]  

(55)

where the contour is at infinity. If \( W \) is analytic at infinity, then the RHS of equation (55) can be written as,

\[ Q = -\frac{1}{2\pi(2n+1)} \oint dx^i \frac{1}{2i}(Z^*(x)\partial_i Z(x) - \partial_i Z^*(x)Z(x)) \]

\[ = -\frac{1}{2n+1} Q_{\text{top}} \]  

(56)

Thus if there is no Landau level mixing in the ground state, the total charge is always proportional to the topological charge. Note that \( Z \) and hence \( q(x) \) is well defined only if \( \rho(x) \) is non-zero everywhere. So all our considerations are true only in this case. They will not hold for polarized vortices where \( \rho(x) \) will vanish at some point.

**IV. GAUGE INVARIANT BOSONIZATION**

In the previous section, we saw that the projected coherent states are labelled by a spinor field \( W \), and that the expectation values of observables could be computed in the hydrodynamic limit in terms of \( W \). The states can therefore be labelled by the values of the physical observables, the density \( \rho(x) \) and the normalised spinor \( Z_\sigma(x) \). In this section, we will show that these states themselves satisfy the generalised coherent state properties [17].
in $\mathcal{H}_{phy}$. Namely, the resolution of unity and continuity of overlaps. This implies that, in the hydrodynamic limit, the original electronic theory can be expressed completely in terms of bosonic field operators corresponding to $\rho(x)$ and $Z_\sigma(x)$. Thus in this limit, the theory can be bosonized in a gauge invariant way with no redundant degrees of freedom.

**A. Resolution of Unity**

The fact that the identity operator in $\mathcal{H}_{phy}$ can be resolved in terms of the projected coherent states has already been shown in equation (34). Here we express this same equation in terms of the gauge invariant parameters. We perform the following change of variables in equation (34),

$$\alpha_i(x), \varphi_\sigma(x) \rightarrow \Omega_L(x), \Omega_T(x), W_\sigma(x) \quad (57)$$

further, using equations (43) and (46), we get

$$I = \int \mathcal{D}[\alpha, \varphi]|\alpha, \varphi\rangle_p \langle \alpha, \varphi| = \text{const} \int \mathcal{D}[\Omega_T(x)] \mathcal{D}[\Omega_L(x)] \mathcal{D}[W] e^{-\int x \frac{\hbar}{\sqrt{c}} \Omega_T(x) \mathcal{N}[W, \Omega_T]} \langle W | \langle W | \quad (58)$$

where the factor $e^{-\frac{\hbar}{\sqrt{c}} \Omega_T(x)}$ is the Jacobian due to the change of variables $\varphi \rightarrow W$, 

$$\mathcal{D}[W] = \prod_{x, \sigma} \frac{dW_\sigma(x) \bar{d}W_\sigma(x)}{2\pi i} \quad (59)$$

and

$$G[W] \equiv \int \mathcal{D}[\Omega_T] e^{-\int x \frac{\hbar}{\sqrt{c}} \Omega_T(x) \mathcal{N}[W, \Omega_T]}$$

(60)

The integral over $\Omega_T$ can be done in the hydrodynamic limit, the details are given in Appendix D, to get,

$$G[W] = \text{const} \prod_x \frac{1}{|W^+(x)W(x)|^2} \quad (61)$$
We now make another change of variables from $W_\sigma(x)$ to $\rho(x), Z_\sigma(x)$, defined by equations (47) and (49), to get,

$$I = const \int D[\rho] D[Z] \langle \rho, Z \rangle \langle \rho, Z \rangle$$

where,

$$D[\rho] = \prod_x d\rho(x) \quad D[Z] = \prod_x \sin^2 \theta(x) \sin \theta(x) d\theta(x) d\phi(x) d\psi(x)$$

$Z$ has been parameterised as

$$Z = \left( \cos \frac{\theta}{2} e^{i\frac{\psi + \phi}{2}}, \sin \frac{\theta}{2} e^{i\frac{\psi - \phi}{2}} \right)$$

B. Overlaps

The overlap of two gauge invariant coherent states $|W_1\rangle$ and $|W_2\rangle$ is computed in the hydrodynamic limit in Appendix E. The final answer is,

$$\langle W_1 | W_2 \rangle = e^{-F[\rho_1, Z_1, \rho_2, Z_2]}$$

$$F = -\frac{i}{4} \int_x (\rho_1(x) + \rho_2(x)) \Phi(\hat{n}_1, \hat{n}_2)$$

$$-\frac{1}{4} \int_x (\rho_1(x) + \rho_2(x)) \ln \left( 1 + \hat{n}_1 \cdot \hat{n}_2 \right)$$

$$-\frac{\pi}{2} (2n + 1) \int_{x,y} (\rho_1(x) - \rho_2(x)) (x|\frac{1}{\sqrt{2}}y\rangle (\rho_1(y) - \rho_2(y))$$

where $\Phi(\hat{n}_1, \hat{n}_2)$ is the solid angle subtended by the geodesic triangle with $\hat{n}_1, \hat{n}_2$ and some third point on the unit sphere as vertices.

Note that the overlap smoothly goes to 1 as $(\rho_1, Z_1) \to (\rho_2, Z_2)$. We will now evaluate the overlap for neighbouring states. We put $\rho_2(x) = \rho(x), \rho_1(x) = \rho(x) + \epsilon \partial_t \rho(x), Z_{2\sigma}(x) = Z_\sigma(x)$ and $Z_{1\sigma}(x) = Z_\sigma(x) + \epsilon \partial_t Z_\sigma(x).$ Keeping terms only to the order $O(\epsilon)$ we get,

$$\langle W + \epsilon \partial_t W | W \rangle = e^{-i\epsilon \int_x \rho(x) L_3^t(x) + O(\epsilon^2)}$$

where $L_\mu^3 \equiv \frac{1}{2i} (Z^\dagger \partial_\mu Z - h.c)$ for $\mu = t, 1$ and 2.
If we now impose the LLL condition, then the charge density fluctuations get tied up to the spin density fluctuations. i.e.,

\[ \rho(x) = \bar{\rho} - \frac{1}{2n+1} q(x) \]

\[ = \bar{\rho} - \frac{1}{2\pi(2n+1)} \epsilon_{ij} \partial_i L^3_j(x) \]  

(66)

The theory can then be expressed in terms of spin fluctuations alone. The expression for the overlap in equation (65) then gets written as,

\[ \langle W + \epsilon \partial_t W | W \rangle = e^{-i\bar{\rho} \int x \bar{L}^3_t(x) + \frac{1}{2} \int x \epsilon_{\mu\nu\lambda} L^\lambda_\mu(x) \partial_\nu L^3_\lambda(x) + O(\epsilon^2)} \]  

(67)

The second term in the exponent in the RHS of equation (67) is the Hopf term. Thus the theory, when restricted to the lowest Landau level is a NLSM with a Hopf term in the action.

V. THE CLASSICAL LIMIT

We will finally show that for large skyrmions, the theory becomes classical. Consider the set of states corresponding to configurations characterised by a size parameter, \( \lambda \). We parameterise them as,

\[ \rho^\lambda(x) = \bar{\rho} + \frac{1}{\lambda^2} \Delta \rho \left( \frac{x}{\lambda} \right) \]

\[ Z^\lambda(x) = Z \left( \frac{x}{\lambda} \right) \]  

(68)

Substituting \( \rho_1^\lambda(x), \ z_1^\lambda(x) \) and \( \rho_2^\lambda(x), \ z_2^\lambda(x) \) in equation (64) and changing the variable \( x \rightarrow \lambda x \) we get,

\[ \langle W_1 | W_2 \rangle = e^{\int x [\lambda^2 \bar{\rho} \ln(\frac{1+\hat{n}_1 \cdot \hat{n}_2}{2}) + 0(\lambda^0)]} \]  

(69)

\( \frac{1}{2} \ln(\frac{1+\hat{n}_1 \cdot \hat{n}_2}{2}) \) is zero \( \hat{n}_1 = \hat{n}_2 \) and negative otherwise. Thus for \( W_1 \neq W_2 \),

\[ \lim_{\lambda \rightarrow \infty} \langle W_1 | W_2 \rangle \rightarrow 0 \]  

(70)

The coherent states thus become orthogonal when \( \lambda \rightarrow 0 \). It can also be shown that the off-diagonal matrix elements of the observables in the coherent state basis, vanish in this limit. Hence the set of states corresponding to a system of skyrmions will behave classically in the limit of the skyrmion sizes tending to infinity.
VI. CONCLUSION

The motivation of this work was to examine the microscopic basis of the semiclassical NLSM for skyrmions at $\nu = 1/(2n+1)$. In the systems of interest, the energy scales are such that it is important to impose the LLL constraint. The coherent state basis is the ideal one to address questions of quantum states in a semiclassical approach. Therefore, we developed a coherent state formalism of the composite boson theory.

We showed that the coherent state basis of $\mathcal{H}_{CB}$, when projected to the physical subspace $\mathcal{H}_{phy}$, can be parameterized by a spinor field that we denoted by $W_\sigma(x)$. In the hydrodynamic limit we have shown that these states, $|W\rangle$ themselves satisfy the coherent state properties of the resolution of unity and continuity of overlaps. The LLL condition is equivalent to the condition that $W_\sigma(x)$ are analytic functions.

The charge density is determined by the modulus of $W$ i.e $W^\dagger(x)W(x)$ and the spin density by the normalised $CP_1$ spinor, $Z_\sigma(x)$. In general these are independent quantities and therefore the charge density is independent of the spin density. However if $W(x)$ is analytic, the modulus and phase of each of its components get tied up. We showed, that consequently, the excess charge density becomes proportional to the topological charge density which is determined by the spin density. Thus this proportionality will cease to hold in presence of Landau level mixing. We also showed that the condition for the total charge density to be proportional to the topological charge is weaker. It only requires $W(x)$ to be analytic at infinity, i.e. that the ground state does not have Landau level mixing.

Finally we showed that if we consider the set of states corresponding to classical configurations of characterised by a length scale $\lambda$, then they become orthogonal in the limit of $\lambda \to \infty$. This implies that a system of skyrmions will behave classically in the limit of their sizes going to infinity.
In the following appendix we derive the path integral representation of the partition function by splitting the time interval $t$ into $N$ segments of length $\epsilon$ and take the limit $\epsilon \to 0$, $N \to \infty$ such that $\epsilon N = t$. And at each intermediate step we insert the resolution of identity (34) of $H_{phy}$:

$$Z = \text{Tr} e^{\frac{i}{\hbar} H t} = \text{Tr} \left[ e^{\frac{i}{\hbar} e^{\frac{i}{\hbar} H t}} \right]^N$$

(A1)

$$Z = \prod_{n=0}^{N} \int_{\alpha_n,\varphi_n,\Omega_n} \prod_{n=0}^{N} \langle \alpha_{n+1},\varphi_{n+1} | P e^{\frac{i}{\hbar} e^{\frac{i}{\hbar} H t}} | \alpha_n,\varphi_n \rangle$$

(A2)

where $(\alpha_{N+1}, \varphi_{N+1}) \equiv (\alpha_0, \varphi_0)$

Since $H$ commutes with $P$ and $P^2 = P$, we get, after explicitly acting $P$ on $|\alpha_n \varphi_n\rangle$ and making use of gauge invariance of $|\alpha \varphi\rangle \langle \alpha \varphi|$, we get

$$Z = \prod_{n=0}^{N} \int_{\alpha_n,\varphi_n,\Omega_n} \prod_{n=0}^{N} t_n$$

(A3)

where,

$$t_n = e^{\frac{i}{\hbar} \int_x \bar{\alpha}_n(x) \times \nabla \Omega_n(x)} \langle \alpha_{n+1} - \nabla \beta_{n+1}, \varphi_{n+1} e^{-i \frac{e}{\hbar c} \beta_{n+1}} | e^{\frac{i}{\hbar} e^{\frac{i}{\hbar} H t}} | \alpha_n - \nabla \beta_n, \varphi_n e^{-i \frac{e}{\hbar c} (\Omega_n + \beta_n)} \rangle$$

(A4)

To the order $\epsilon$: $\alpha_{n+1} = \alpha_n + \epsilon \dot{\alpha}_n$, $\Omega_{n+1} = \Omega_n + \epsilon \dot{\Omega}_n$ and $\varphi_{n+1} = \varphi_n + \epsilon \dot{\varphi}_n$. And if we choose $\beta_{n+1} = \beta_n + \Omega_{n+1}$ then to the order $O(\epsilon)$ $t_n$ is:

$$t_n = e^{\frac{i}{\hbar} \int_x \bar{\alpha}_n(x) \times \nabla \Omega_n(x)} \langle \alpha_n - \nabla \beta_n - \nabla \Omega_n + \epsilon (\dot{\alpha}_n - \nabla \dot{\Omega}_n), \{ \varphi_n + \epsilon (\dot{\varphi}_n - i \frac{e}{\hbar c} \varphi_n \dot{\Omega}_n) \} e^{-i \frac{e}{\hbar c} (\beta_n + \Omega_n)} | e^{\frac{i}{\hbar} e^{\frac{i}{\hbar} H t}} | \alpha_n - \nabla \beta_n - \nabla \Omega_n, \varphi_n e^{-i \frac{e}{\hbar c} (\Omega_n + \beta_n)} \rangle$$

(A5)

Using the fact that $\epsilon \dot{\beta}_n = \beta_n - \beta_{n-1} + O(\epsilon^2)$ and

$$\langle \alpha + \delta \alpha, \varphi + \delta \varphi | e^{\frac{i}{\hbar} e^{\frac{i}{\hbar} H t}} | \alpha, \varphi \rangle = \langle \alpha + \delta \alpha, \varphi + \delta \varphi | \alpha, \varphi \rangle [1 - i \frac{e}{\hbar} \langle \alpha \varphi | H | \alpha \varphi \rangle] + O(\epsilon^2)$$

(A6)
where $\delta \alpha \sim O(\epsilon)$ and $\delta \varphi \sim O(\epsilon)$ the above expression for $t_n$, after defining $\alpha_{0n} \equiv \dot{\alpha}_n/c$, and making use of gauge invariance of $H$, we get

$$t_n = \exp\left[\frac{i\epsilon}{\hbar} \int_x \left\{ -\frac{\kappa}{2} \epsilon_{\mu\nu\lambda} \alpha_{\mu n}^\lambda(x) \partial^\nu \alpha_n^\lambda(x) + e\alpha_{0n}(x) \varphi_n(x) \varphi_n(x) - \frac{i\hbar}{2} [\varphi_n(x) \dot{\varphi}_n(x) - \dot{\varphi}_n(x) \varphi_n(x)] - H(\varphi_n, \alpha_n) \right\} \right]$$  \hspace{1cm} (A7)

If we now take the limit $\epsilon \to 0$ the partition function becomes (after calling $\alpha$ by $a$)

$$Z = \int \mathcal{D}[a_0(x, t)] \mathcal{D}[a_i(x, t)] \mathcal{D}[\varphi(x, t)] e^{\frac{i}{\hbar} \int dt dx L(x, t)}$$  \hspace{1cm} (A8)

where $L(x, t)$ is the standard Chern-Simons lagrangian.

**APPENDIX B: THE WAVEFUNCTIONS**

In this appendix we give the details of the calculation of equation (36). Using equations (15), (19) and (27) we get,

$$\psi_N(\{x_i, \sigma_i\}) = \frac{1}{V_G} \int_\Omega e^{\frac{i}{\hbar} \int_x x \cdot \nabla \Omega(x)} \langle 0 | \prod_{i=1}^N \left[ K(x_i) \dot{\varphi}_{\sigma_i}(x_i) D^\dagger(x_i) \right] | \alpha - \nabla \Omega, \varphi e^{-\frac{i\epsilon}{\hbar} \Omega} \rangle$$  \hspace{1cm} (B1)

Using the fact that,

$$\dot{\varphi}_{\sigma_i}(x_i) K(x_j) = e^{i(2n+1)\theta(x_i - x_j)} K(x_j) \dot{\varphi}_{\sigma_i}(x_i)$$  \hspace{1cm} (B2)

and $\langle 0 | K(x) = 0$, we can pull all the $K$'s to the left and rewrite equation (B1) as,

$$\psi_N(\{x_i, \sigma_i\}) = \prod_{i>j} e^{i(2n+1)\theta(x_i - x_j)} \frac{1}{V_G} \int_\Omega e^{\frac{i}{\hbar} \int_x \Omega(x) (\nabla \times \bar{a}(x))} \prod_{i=1}^N \varphi_{\sigma_i}(x_i) e^{-\frac{i\hbar}{\epsilon} \sum_{i=1}^N \Omega(x_i)} e^{-\frac{1}{2} \int_x |\varphi(x)|^2}$$

$$e^{-\frac{i}{\hbar} \int_x \left( \sum_{i=1}^N \bar{a}_{\sigma_i}(x) \times (\bar{a}(x) - \nabla \Omega(x)) \right) - \frac{i}{\hbar} \int_x (\bar{a}(x) - \nabla \Omega(x) - \sum_{i=1}^N \bar{a}_{\sigma_i}(x))^2}$$  \hspace{1cm} (B3)

This is equation (36). We now write,

$$\alpha^i_m(x) = \epsilon^{ij} \partial_j f_m(x)$$  \hspace{1cm} (B4)

where,

$$-\kappa \nabla^2 f_m(x) = e\delta(x - x_m)$$  \hspace{1cm} (B5)
for $m = 1 - N$.

The zero momentum mode of the $\Omega$ integral will make the wavefunction vanish unless the total number of flux quanta equals the total number of particles. i.e.

$$\kappa \int_x \nabla \times \vec{\alpha}(x) = eN$$

(B6)

The $\Omega$ integral is gaussian for the other modes and can be done exactly to give,

$$\text{const} \times \exp\left[-\frac{1}{4\hbar c} \int_x (\vec{\alpha}(x) - \sum_{i=1}^{N} \vec{\alpha}_i(x))^2\right]$$

(B7)

We then write the wavefunction as,

$$\psi_N(\{x_i, \sigma_i\}) = \text{const} \times \exp\left[-\frac{1}{2} \int_x |\varphi(x)|^2 \prod_{i>j} \left|e^{i\theta(x_i-x_j)}\right|^{2n+1} \prod_{i=1}^{N} \varphi_{\sigma_i}(x_i) \exp\left[-\frac{1}{2\hbar c} \int_x (\vec{\alpha}(x) - \sum_{i=1}^{N} \vec{\alpha}_i(x))^2\right]$$

(B8)

We also have,

$$\int_x \vec{\alpha}(x) \cdot \vec{\alpha}_m(x) = \int_x \nabla \Omega_T(x) \cdot \nabla f_m(x) = -\int_x \Omega_T(x) \nabla^2 f_m(x) = \frac{e}{\kappa} \Omega_T(x_m)$$

(B9)

Using the fact that the solution of equation (B5) is,

$$f_m(x) = -\frac{e}{\kappa} \frac{1}{2\pi} \ln |x - x_m|$$

(B10)

and proceeding as in equation (B9), we have,

$$\int_x \vec{\alpha}_m(x) \cdot \vec{\alpha}_n(x) = \frac{e}{\kappa} f_m(x_n) = -\frac{e^2}{\kappa^2} \frac{1}{2\pi} \ln |x_m - x_n|$$

(B11)

So we have the result,

$$\int_x (\vec{\alpha}(x) - \sum_{i=1}^{N} \vec{\alpha}_i(x))^2 = \int_x \nabla \Omega_T(x) \cdot \nabla \Omega_T(x) - \frac{e^2}{\kappa^2} \frac{1}{2\pi} \sum_{m=1}^{N} \sum_{n \neq m} \ln |x_m - x_n|$$

$$-2\frac{e}{\kappa} \sum_{m=1}^{N} \Omega_T(x_m) + \text{const}$$

(B12)
The (infinite) \( \text{const} \) comes from the \( m = n \) terms

Finally the wavefunction is written as,

\[
\psi_N(\{x_i, \sigma_i\}) = \text{const} \times e^{-\frac{1}{2} \int_x |\varphi(x)|^2} e^{-\frac{1}{2 \hbar c} \int_x \nabla \Omega_T(x) \cdot \nabla \Omega(x)} \times \prod_{i>j} (z_i - z_j)^{2n+1} \left[ \phi_{\sigma_i}(x_i) e^{i \Omega(x_i)} - \sum_{i} \ln \phi_{\sigma_i}(x_i) \right] \quad (B13)
\]

APPENDIX C: THE OBSERVABLES

We first evaluate the norm, \( \mathcal{N}[W, \Omega_T] \), in the hydrodynamic approximation, i.e., in the small amplitude and long wavelength limit. It is given by,

\[
\mathcal{N}[W, \Omega_T] = \prod_{i=1}^{N} \int_{x_i} \sum_{\sigma_i} |\psi_N(\{x_i, \sigma_i\})|^2 \\
= \text{const} \times e^{-\int_x W_0(x) W(x)} e^{-\frac{2\pi}{\hbar c} (\Omega_T(x) - \bar{\Omega}_T(x))} e^{-\frac{1}{2 \hbar c} \int_x \nabla \Omega_T(x) \cdot \nabla \Omega_T(x)} \times \prod_{i=1}^{N} \int_{x_i} e^{\sum_{i} \ln W_0(x_i) W(x_i)} + 2\pi \sum_{i} \bar{\Omega}_T(x_i) + (2n+1) \sum_{i,j} \ln |x_i - x_j| \quad (C1)
\]

As in the case of the Laughlin wave function, the norm has the form of a classical partition function of a 2D plasma. Here, there is also an "external potential" which is a function of \( W_0(x) W(x) \) and \( \Omega_T(x) \). In the hydrodynamic limit, we write this partition function as a functional integral over the density field and evaluate it using the saddle point approximation. So we change the variables from \( \{x_i\} \rightarrow \bar{\rho} \), where,

\[
\bar{\rho}(x) = \sum_{i=1}^{N} \delta(x - x_i) \quad (C2)
\]

then for any function \( F \) of \( \{x_i\} \), we have

\[
\frac{1}{N!} \prod_{i=1}^{N} \int dx_i F(\{x_i\}) = \int D[\bar{\rho}] J[\bar{\rho}] F[\bar{\rho}] \quad (C3)
\]

where the jacobian of the transformation is the entropy factor,

\[
J[\bar{\rho}] = e^{\int_x [\bar{\rho}(x) - \bar{\rho}(x) \ln \bar{\rho}(x)]} \quad (C4)
\]

Hence the norm can be written as,
\[ \mathcal{N}[W, \Omega_T] = const \times e^{-\int_x W_\sigma(x) W_\sigma(x) e^{\frac{\hbar c}{\hbar} (\Omega_T(x) - \Omega_T(x))} e^{-\int_x \nabla \Omega_T(x) \cdot \nabla \Omega_T(x)}} \times \int \mathcal{D}[\tilde{\rho}] e^{-S[\tilde{\rho}; W]} \] (C5)

where,

\[ S[\tilde{\rho}; W] = \int_x \left[ -\tilde{\rho}(x) + \rho(x) \ln \tilde{\rho}(x) - \tilde{\rho}(x) \ln \{W_\sigma(x) W_\sigma(x)\} - 2 \frac{e^{\frac{\hbar c}{\hbar}}}{\hbar} \tilde{\rho}(x) \Omega_T(x) \right. \\
\left. -2\pi(2n+1)\tilde{\rho}(x) \frac{1}{\nabla^2} \tilde{\rho}(x) \right] \] (C6)

We evaluate this functional integral in the saddle point limit. Dropping \( \tilde{\rho} \ln \tilde{\rho} \) term in comparison with \( \tilde{\rho} \frac{1}{\nabla^2} \tilde{\rho} \) and substituting the solution of the saddle point equation,

\[ \tilde{\rho}(x) = \bar{\rho} - \frac{1}{4\pi(2n+1)} \nabla^2 \ln \{W_\sigma(x) W_\sigma(x)\} \] (C7)

we get equation (H6).

We evaluate the values of density and spin density by proceeding along similar calculational steps employed in evaluating the norm. The density is given by,

\[ \rho(x) \equiv \langle W | \hat{\rho}(x) | W \rangle \\
= \frac{1}{\mathcal{N}(W, \Omega_T)} \prod_{i=1}^{N} \left( \int dx_i \sum_{\sigma_i} \bar{\psi}_N(\{x_i, \sigma_i\}) \sum_{i=1}^{N} \delta(x - x_i) \bar{\psi}_N(\{x, \sigma\}) \right) \\
= \frac{1}{\mathcal{Z}} \int \mathcal{D}[\tilde{\rho}] \tilde{\rho}(x) e^{-S[\tilde{\rho}; W]} \\
\equiv \langle \tilde{\rho}(x) \rangle \\
= \frac{-1}{4\pi(2n+1)} \nabla^2 \ln \{W_\sigma(x) W_\sigma(x)\} + \bar{\rho} \] (C8)

where \( \mathcal{Z} \equiv \int \mathcal{D}[\tilde{\rho}] e^{-S[\tilde{\rho}; W]} \)

The spin density is,

\[ s^a(x) \equiv \langle W | \hat{s}^a(x) | W \rangle \\
= \frac{1}{\mathcal{N}(W, \Omega_T)} \prod_{i=1}^{N} \left( \int dx_i \sum_{\sigma_i} \bar{\psi}_N(\{x_i, \sigma_i\}) \sum_{i=1}^{N} \frac{1}{2} \tau(i) \delta(x - x_i) \bar{\psi}_N(\{x, \sigma\}) \right) \\
= \frac{1}{2} \frac{\tilde{W}_\sigma(x) \tau_{\sigma \sigma'} W_{\sigma'}(x)}{W_{\sigma'}(x) W_{\sigma'}(x)} \langle \tilde{\rho}(x) \rangle \] (C9)

The current density is,
\[ J_j(x) \equiv \langle W|\hat{J}_j(x)|W \rangle \]

\[ = \frac{1}{\mathcal{N}(W,\Omega_T)} \prod_{i=1}^{N} \left( \int \text{d}x_i \sum_{\sigma_i} \bar{\psi}_N(\{x_i,\sigma_i\}) \sum_{i=1}^{N} \frac{1}{2} \delta(x - x_i) \left[-i\hbar \partial_{x_{ij}} - \frac{\epsilon}{c} A_j(x_i) \right] \psi_N(\{x_i,\sigma_i\}) \right) \]

\[ + \ h.c \]

\[ = \rho(x) \left[ \hbar L_j^3(x) - \frac{\epsilon}{c} (A_j(x) - \alpha_j(x)) \right] \tag{C10} \]

where \( i \) is the particle index and \( j \) is the coordinate index. \( \alpha \) is defined through the relation

\[ \kappa \nabla \times \vec{\alpha}(x) = e\rho(x) \text{ and } L_j^3 \equiv \frac{1}{2}(Z^i \partial_j Z - h.c) \text{ for } j = 1 \text{ and } 2. \]

The kinetic energy density is,

\[ \mathcal{T}(x) \equiv \langle W|\hat{T}(x)|W \rangle \]

\[ = \frac{1}{\mathcal{N}(W,\Omega_T)} \prod_{i=1}^{N} \left( \int \text{d}x_i \sum_{\sigma_i} \bar{\psi}_N(\{x_i,\sigma_i\}) \sum_{i=1}^{N} \frac{1}{2} \delta(x - x_i) \left[-i\hbar \partial_{x_{i}} - \frac{\epsilon}{c} A_j(x_i) \right] \psi_N(\{x_i,\sigma_i\}) \right)^2 \]

\[ = \hbar \omega_c \frac{\partial_x \bar{W}_{\sigma}(x) \partial_x W_{\sigma}(x)}{W_{\sigma'}(x) W_{\sigma'}(x)} \langle \tilde{\rho}(x) \rangle \tag{C11} \]

where \( D_i = \partial_{x_i} + \frac{1}{2} \bar{z}_i \) and \( z = \frac{x_1 + iz_2}{l_c \sqrt{2}}. \)

**APPENDIX D: THE \( \Omega_T \) INTEGRAL**

In this appendix we evaluate \( \mathcal{G}[W] \) by doing the \( \Omega_T \) integral in equation (60). The saddle-point approximation of the integral gives

\[ \mathcal{G}[W] = \text{const} \times e^{-\int_x \frac{\hbar}{2e} \tilde{\Omega}_T(x) \mathcal{N}[W,\tilde{\Omega}_T]} \tag{D1} \]

where \( \tilde{\Omega}_T \) is the solution to the saddle-point equation which, in long wavelength limit \( \nabla^2 \ln \bar{W} \ll \ln \bar{W} \) is,

\[ \tilde{\Omega}_T(x) = \Omega_T(x) + \frac{\hbar c}{2e} \ln\{\bar{W}_{\sigma}(x)W_{\sigma}(x)\} \tag{D2} \]

When this value for \( \tilde{\Omega}_T \) is substituted in the above equation we get equation (61).

**APPENDIX E: THE OVERLAPS**

The overlap of two gauge invariant coherent states \(|W_1\rangle\) and \(|W_2\rangle\), obtained by proceeding with steps similar to those involved in evaluating the norm, is
\begin{align*}
\langle W_1|W_2 \rangle &= \frac{1}{\sqrt{\mathcal{N}(W_1, \Omega_{T1})\mathcal{N}(W_2, \Omega_{T2})}} \prod_{i=1}^{N} \left( \int dx_i \sum_{\sigma_i} \bar{\psi}_1(\{x_i, \sigma_i\})\psi_2(\{x_i, \sigma_i\}) \right) \\
&= e^{-\frac{1}{8\pi(2n+1)} \int_x [f_{12}(x)\nabla^2 f_{12}(x) - \frac{1}{2}f_{11}(x)\nabla^2 f_{11}(x) - \frac{1}{2}f_{22}(x)\nabla^2 f_{22}(x)]}
\end{align*}

where

\[ f_{ab}(x) = \ln \{\tilde{W}_{a\sigma}(x)W_{b\sigma}(x)\} + 2\frac{e}{\hbar c} \bar{\Omega}(x) \]  

If we express \( W \) in terms of \( \rho \) and \( Z \) we get the overlap to be

\begin{align*}
\langle W_1|W_2 \rangle &= e^{-\frac{1}{8\pi(2n+1)} \int_x [\ln(Z_{1}(x)Z_{2}(x))\nabla^2 \ln(Z_{1}(x)Z_{2}(x))} \\
&\times e^{\frac{1}{2} \int_x (\rho_1(x) + \rho_2(x)) \ln(Z_{1}(x)Z_{2}(x))} \\
&\times e^{\frac{1}{2} \int_x (\rho_1(x) - \rho_2(x)) \frac{1}{\sqrt{2}}(\rho_1(x) - \rho_2(x))}
\end{align*}

The first term in the exponent of the RHS can be dropped with respect to the second one in the long wavelength limit. Making use of the relation, \( \bar{Z}_{1\sigma}Z_{2\sigma} = e^{\Phi(\vec{n}_1, \vec{n}_2)(1 + \frac{\vec{n}_1 \cdot \vec{n}_2}{2})^{\frac{1}{2}}} \) where \( \Phi(\vec{n}_1, \vec{n}_2) \) is the area of the spherical triangle with vertices at \( \vec{n}_1, \vec{n}_2 \) and a third point on the unit sphere, in the above equation we get the equation (64).
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