Non-local Probes of Entanglement in the Scale Invariant Gravity

R. Pirmoradian and M. Reza Tanhayi

Department of Physics, Central Tehran Branch, Islamic Azad University (IAUCTB), P.O. Box 14676-86831, Tehran, Iran

E-mail: *mtanhayi@ipm.ir

Abstract

The most general action for the scale invariant gravity in four dimensions was introduced in Ref. [1,2]. In this paper, we first study this theory and then by making use the holographic methods, we compute the holographic entanglement entropy, the mutual and the tripartite information. We show that mutual information is subadditive while tripartite information becomes superadditive. This indeed recovers the monogamy property of mutual information in this context.

Contents

1 Introduction

2 General Scale Invariant Gravity: Entanglement Entropy
2.1 Entanglement entropy

3 Holographic mutual and tripartite Information

4 Concluding Remarks
1 Introduction

Since its original formulation more than two decades ago, the AdS/CFT correspondence \[3\] has taught us a lot about the quantum nature of gravity. In addition, with the discovery of the Ryu-Takayanagi (RT) formula \[4, 5\] and its generalizations \[6, 7\], holography and in particular the AdS/CFT correspondence has become an exceedingly important tool in calculating the entanglement entropy (and other entanglement measures) for holographic systems. In this paper, we will use a generalization of RT formula for higher derivative theories, due to Dong \[6\], in order to calculate the entanglement entropy, mutual information, and tripartite information in a scale invariant theory of gravity in four dimensions. In order to arrive at the action of this scale invariant theory, we note that using the procedure of holographic renormalization, the holographic Weyl anomaly in four dimensions is given by \[8\]

$$\langle T^\mu_\mu \rangle = c_4 \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right),$$

(1)

where $c_4$ is a numerical factor. Given the above expression for the Weyl anomaly, one would expect the action of the conformal gravity in four dimensions to have the following form \[9,10\]:

$$I_{CG} = -\frac{\kappa}{16\pi} \int d^4x \sqrt{-g} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) + \text{total derivative terms},$$

(2)

where $\kappa$ is a dimensionless constant. It is well known that the total derivatives do not contribute to the equations of motion, but in order to have a well defined variational principle, such terms play a crucial role. Also, the boundary terms contribute to the physical quantities such as the free energy, the entropy, entanglement entropy and the correlation function of the holographic stress tensor. Thus in this sense, fixing the boundary terms seems to be important task.

In four dimensional gravitational theory, there exists a natural surface term, the Gauss-Bonnet term, which gives the Euler character of the manifold and as a result is topological and does not change the equations of motion. The Gauss-Bonnet term in four dimensions is given by:

$$\text{GB}_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2,$$

(3)

After adding this boundary term the action reads as follows

$$I_{CG} = -\frac{\kappa}{32\pi} \int d^4x \sqrt{-g} \left( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \right),$$

(4)

which is, indeed, the Weyl squared action. On the other hand for the scale invariant gravity, one can say that the most general action of a scale invariant four dimensional gravity can be written as follows

$$I_{SI} = -\frac{\kappa}{32\pi} \int d^4x \sqrt{-g} \left( \sigma_0 C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} + R^2 - \sigma_1 \text{GB}_4 \right),$$

(5)

Where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor and $\sigma_0$, $\sigma_1$ and $\kappa$ are dimensionless constants. Note that, we rescale the action, so that, the coefficient of $R^2$ term is set to one. It is known that the Weyl squared term is regularized and does not give a divergent term when the action is evaluated on an asymptotically AdS geometry. Nonetheless, the on shell action could still be divergent because of $R^2$ term. We fix the coefficient of Gauss-Bonnet term $\sigma_1$ such that to remove the divergent parts of $R^2$ term. It was shown that in order to get a finite action one should set $\sigma_1 = -6$ \[2\]. As a result we will consider the action of 4D scale invariant gravity as follows

$$I_{SI} = -\frac{\kappa}{32\pi} \int d^4x \sqrt{-g} \left( \sigma_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + R^2 - 6 \text{GB}_4 \right),$$

(6)
which is guaranteed to get finite on shell action for asymptotically AdS geometries. More explicitly the action may be written in the following form

\[ I_{SI} = -\frac{\kappa}{32\pi} \int d^4x \sqrt{-g} \left[ (\sigma_0 - 6)R_{\mu\nu\rho\sigma}^2 - 2(\sigma_0 - 12)R_{\mu\nu}^2 + \frac{(\sigma_0 - 5)}{3}R^2 \right]. \tag{7} \]

Therefore, we note that the most general action of scale invariant (pure) gravity in four dimensions is a one family parameter action denoted by \( \sigma_0 \). Now the main aim of this paper is to study further this model, and precisely, we want to consider some non-local quantum measures such as entanglement entropy and mutual information. Note that in this model of scale invariant theory we recover conformal gravity at point \( \sigma_0 \to \infty \) while at \( \sigma_0 = 0 \) one gets \( R^2 \) gravity.

This paper is organized as follows: in the next section, we will study and review the model at a generic point \( \sigma_0 \neq 0, \infty \) and its solutions. Also, we will further study the model at \( \sigma_0 = 0 \) where we shall use holographic methods and study the holographic entanglement entropy as a probe. In section three, we will consider holographic mutual and tripartite information. Finally at the conclusion part, we will briefly discuss our results and also some possible future directions in the conclusion.

\section{General Scale Invariant Gravity: Entanglement Entropy}

As mentioned, the boundary Gauss-Bonnet term does not contribute to the equation of motion, and the equations of motion derived from the action (5) are as follows

\[ \left( \nabla^\sigma \nabla^\rho - \frac{1}{2} R^\sigma^\rho \right) C_{\mu\sigma\nu\rho} = \frac{1}{2\sigma_0} \left( RR_{\mu\nu} - g_{\mu\nu} \frac{R^2}{4} - \nabla_\mu \nabla_\nu R + g_{\mu\nu} \Box R \right), \tag{8} \]

where \( \Box = \nabla^\mu \nabla_\mu \). These equations of motion admit several black hole solutions. Indeed, restricting to Einstein solutions, the above equations of motion are solved by the following black hole solutions (see for example \( \Pi \))

\[ ds^2 = \frac{L^2}{r^2} \left( -F(r) dt^2 + \frac{dr^2}{F(r)} + d\Sigma_{2,k}^2 \right), \quad F(r) = \lambda + kr^2 + c_3 r^3, \tag{9} \]

where \( L \) is the radius of curvature and \( k = 1, -1, 0 \) corresponds to \( \Sigma_{2,k} = S^2, H_2, R^2 \), respectively. Note that being Einstein solutions they satisfy \( R_{\mu\nu} = \frac{3\lambda}{L^2} g_{\mu\nu} \) with \( \lambda = \pm 1, 0 \).

For \( \sigma_0 > 6 \) it has Lifshitz solution which is given by (see \( \Pi \) for Einstein-Weyl gravity)

\[ ds^2 = \frac{L^2}{r^2} \left( -\frac{dt^2}{z^2} + dr^2 + dx_1^2 + dx_2^2 \right), \quad z = \frac{\sigma_0 - 6 + \sqrt{(\sigma_0 - 6)(4\sigma_0 + 3)}}{\sigma_0 + 3}. \tag{10} \]

We note that for \( \sigma_0 = 6 \) the action (7) reads

\[ I_{SI} = -\frac{3\kappa}{32\pi} \int d^4x \sqrt{-g} \left[ 4R_{\mu\nu}^2 - R^2 \right], \tag{11} \]

which is the curvature squared terms.

\subsection{Entanglement entropy}

In order to further explore the role of the Gauss-Bonnet in the scale invariant gravity, we study holographic entanglement entropy \( \Pi \) for theories whose gravitational dual is provided by four dimensional scale invariant gravity on asymptotically AdS backgrounds. Since the model under consideration includes higher order terms, in order to compute the holographic entanglement entropy one should use the generalized Ryu-Takayanagi prescription \( \Pi \). In particular, for an action containing the most general curvature squared terms

\[ I = -\frac{\kappa}{32\pi} \int d^4x \sqrt{-g} \left( \lambda_1 R^2 + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right), \tag{12} \]
the holographic entanglement entropy can be obtained by minimizing the following entropy functional [13]

\[ S_A = \frac{\kappa}{8} \int d^2 \zeta \sqrt{\gamma} \left[ 2\lambda_1 R + \lambda_2 \left( R_{\mu\nu} n^\mu n^\nu - \frac{1}{2} K^i K_i \right) + 2\lambda_3 \left( R_{\mu\nu\rho\sigma} n^\mu n^\nu n^\rho n^\sigma - \mathcal{K}_{\mu\nu}^i \mathcal{K}_i^{\mu\nu} \right) \right], \]  

(13)

with \( i = 1, 2 \) we denote the two transverse directions to a co-dimension two hypersurface in the bulk, \( n_i^\mu \) are two mutually orthogonal unit vectors to the hypersurface and \( \mathcal{K}_{\mu\nu}^i \) are the traces of two extrinsic curvature tensors defined by

\[ \mathcal{K}_{\mu\nu}^i = \pi^\sigma_{\mu} \pi^\rho_{\nu} \nabla_\rho (n_i)_\sigma, \quad \text{with} \quad \pi^\sigma_{\mu} = \epsilon^\sigma_{\mu} + \xi \sum_{i=1,2} (n_i)_\sigma (n_i)_\mu, \]  

(14)

where \( \xi = -1 \) for space-like and \( \xi = 1 \) for time-like vectors. Moreover, \( \gamma \) is the induced metric on the hypersurface whose coordinates are denoted by \( \zeta \).

Let us now consider holographic entanglement entropy for a strip. To do so, it is useful to parametrize the AdS metric as follows

\[ ds^2 = \frac{L^2}{r^2} (-dt^2 + dr^2 + dx^2 + dy^2), \]  

(15)

by which the entangling region may be given by

\[ t = \text{constant}, \quad -\frac{\ell}{2} \leq y \leq \frac{\ell}{2}, \quad 0 \leq x \leq H. \]  

(16)

In this case the co-dimension two hypersurface in the bulk may be parametrized by \( y = f(r) \). Then, going through the procedure we have explored above, one arrives at [14]

\[ S = \kappa H \left[ (\sigma_0 - 6) \left( \frac{1}{\epsilon} - \frac{2\pi \Gamma \left( \frac{3}{4} \right)^2}{\Gamma \left( \frac{1}{4} \right)^2} \frac{1}{\ell} \right) + (\sigma_0 - 6) \left( -\frac{1}{\epsilon} \right) \right], \]  

(17)

where the first term is the contributions of the dynamical terms, though the last term is that of the Gauss-Bonnet term which obviously plays the role of regulator. It also interesting to compare the result with that of Einstein gravity which is

\[ S_{\text{Ein}} = \frac{L^2 H}{2G} \left( \frac{1}{\epsilon} - \frac{2\pi \Gamma \left( \frac{3}{4} \right)^2}{\Gamma \left( \frac{1}{4} \right)^2} \frac{1}{\ell} \right), \]  

(18)

which is again the same as that of dynamical part if one identifies \( \kappa = \frac{L^2}{2G(\sigma_0 - 6)\ell}. \)

To conclude this section, we have shown that holographic entanglement entropy for an Einstein solution of the action (7) obtained from the dynamical part of the action is the same as that of Einstein gravity if we identify \( \kappa = \frac{L^2}{2(\sigma_0 - 6)\ell G} \) which makes sense as long as \( \sigma_0 \neq 6 \).

In the following section we will consider other non-local measurements of entanglement, namely mutual information and tripartite information.

### 3 Holographic mutual and tripartite Information

In this section we are taking into account two and three subsystems and extend the results of the previous section for more subsystems. In this way mutual and tripartite information have been used in order to quantify the amount of quantum correlation between two or three subsystems, respectively. Namely, the mutual and the tripartite information are used as the proper measures to evaluate the amount of shared information, or the correlation, between the entangling regions [15]. For two systems \( A \) and \( B \), the mutual information is given by [16,17]

\[ I(A, B) = S(A) + S(B) - S(A \cup B), \]  

(19)

\[ S(A) = \int \rho(A) \log \rho(A), \]  

\[ S(B) = \int \rho(B) \log \rho(B), \]  

\[ S(A \cup B) = \int \rho(A \cup B) \log \rho(A \cup B). \]
where $S(A_i)$’s are the entanglement entropies. For three subregions, the tripartite information is defined by

$$I^{[3]}(A, B, C) = S(A) + S(B) + S(C) - S(A \cup B) - S(A \cup C) - S(B \cup C) + S(A \cup B \cup C).$$  \hspace{1cm} (20)

In the above relations, $S(A_i \cup A_j)$ is the entanglement entropy for the union of the entangling regions. It is worth to mention that in writing the tripartite information, the union parts plays an important role. To explore this point let us write the tripartite information in terms of the mutual information as follows

$$I^{[3]}(A, B, C) = I(A, B) + I(A, C) - I(A, B \cup C),$$  \hspace{1cm} (21)

which helps us to investigate the sign of mutual information. Now the aim is to compute the mutual information. For two strips with the same length of $\ell$ separated by distance $h$, after making use of the corresponding holographic entanglement entropy, the holographic mutual information is given by

$$I(A, B) = \frac{4\pi H}{G_N} \frac{(\frac{3}{4})^2}{(\frac{1}{4})^2} \left( \frac{2}{\ell} - \frac{1}{h} - \frac{1}{2\ell + h} \right).$$  \hspace{1cm} (22)

On the other hand for three entangling regions with the same length $\ell$ separated by distance $h$, the main point is finding the union part of entanglement entropy. According to the holographic principle, the minimal configuration in the bulk space is needed. In Fig.2 we have plotted all possible diagrams of the union of three regions and $S(A_i \cup A_j)$ and $S(A \cup B \cup C)$ are given by the minimum among the possible diagrams. For two and three strips as the entangling regions with the same length $\ell$ one can write

$$S(A_i \cup A_j) : \begin{cases} 
2S(\ell) \equiv S_1 \\
S(2\ell + h) + S(h) \equiv S_2 \\
S(3\ell + 2h) + S(\ell + 2h) \equiv S_3
\end{cases}$$
and for three entangling regions one has
\[
S(A \cup B \cup C) : \begin{cases}
3S(\ell) = S_4 \\
S(3\ell + 2h) + S(\ell + 2h) + S(\ell) = S_5 \\
S(2\ell + h) + S(\ell) + S(h) = S_6 \\
S(3\ell + 2h) + 2S(h) = S_7
\end{cases}
\]

From these possible configurations, and after making use of the minimum expression in each case, the holographic tripartite information can be written as follows
\[
I^{[3]}(A, B, C) = 3S(\ell) - 2 \min\{S_1, S_2\} - \min\{S_2, S_3\} + \min\{S_4, S_5, S_6, S_7\}.
\]

We mean by \(\min\{S_1, S_2\}\) the minimum configuration between \(S_1\) and \(S_2\).

In the most general scale invariant theory in four dimensions, in figure 3, we have plotted the mutual and tripartite informations, numerically for some certain value of \(\ell\) and \(h\). As it is clear, for our parameters the mutual information is positive on the other hand the tripartite information remains negative.

\section{Concluding Remarks}

In this paper, we have studied various features of holographic entanglement measures for the scale invariant gravity in four dimensions. The action of the scale invariant theory can indeed be parametrized by a one family of free parameter say as \(\sigma_0\). And for \(\sigma_0 \to \infty\), the theory reduces to the four dimensional conformal gravity, while at \(\sigma_0 = 0\) it reduces to pure \(R^2\) gravity. In this theory, we studied the holographic mutual and tripartite information. From the relation (21) one can say that the sign of tripartite information constraints the monogamy of mutual information such that for any three regions when \(I^{[3]}(A, B, C) < 0\) the \(I(A, B)\) becomes monogamous. Noting that in the context of quantum information for any measure of quantum entanglement, any inequalities in the form of \(F(A, B) + F(A, C) \leq F(A, BUC)\), are known as monogamy relations. This property is related to the security of quantum cryptography, as long as, the quantum entanglement is not a shareable resource. Unlike classical correlation, this is indeed comes from the quantum nature of information. In other words, entangled correlations between \(A\) and \(B\) cannot be shared with a third system \(C\) without spoiling the original entanglement [17]. For more related works see Ref.s [18,21].

In this study we investigated this issue in the scale invariant gravity in four dimensions and our numerical results indicated that mutual information is still monogamous in scale invariant theories of gravity.
As a future work, we will study the other solutions of the scale invariant gravity in four dimensions. For example, the theory that we have considered, for $\sigma_0 = 6$, exhibits a new solution. In this critical point all entanglement entropies we have computed are identically zero indicating that at this critical point the vacuum solution of the model is given by the logarithmic solution. It is worth mentioning that in the context of gauge/gravity duality adding the log term to an AdS solution may be holographically identified to a deformation of the dual CFT by an irrelevant operator. Therefore, adding this term would destroy the conformal symmetry at UV and it is not clear how to apply the AdS/CFT correspondence. Nevertheless, following [22], one may assume that the deformation is sufficiently small and this term may be treated perturbatively. We leave further investigation to other works.

Acknowledgments

The main idea of this paper belongs to M. Alishahiha. We would like to specially thank him for all his support and generosity and also for his useful comments and hints. R.P. would also like to thank A. Naseh and B. Taghavi for useful conversations and comments.

References

[1] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst and A. Riotto, Fortsch. Phys. 64, no.2-3, 176-189 (2016) doi:10.1002/prop.201500100 [arXiv:1505.07657 [hep-th]].

[2] M. Alishahiha, “On 4D Scale Invariant Gravity,” unpublished.

[3] J. M. Maldacena, Int. J. Theor. Phys. 38, 1113-1133 (1999) doi:10.1023/A:1026654312961 [arXiv:hep-th/9711200 [hep-th]].

[4] S. Ryu and T. Takayanagi, "Holographic Derivation of Entanglement Entropy from AdS/CFT,” Phys. Rev. Lett. 96 (2006) 181602 [hep-th/0603001].

[5] S. Ryu and T. Takayanagi, ”Aspects of Holographic Entanglement Entropy,” JHEP 0608 (2006) 045 [hep-th/0605073].

[6] X. Dong, “Holographic Entanglement Entropy for General Higher Derivative Gravity,” JHEP 1401, 044 (2014) arXiv:1310.5713 [hep-th], arXiv:1310.5713.

[7] J. Camps, “Generalized entropy and higher derivative Gravity,” JHEP 1403, 070 (2014) arXiv:1310.6659 [hep-th]].

[8] M. Henningson and K. Skenderis, JHEP 07, 023 (1998) doi:10.1088/1126-6708/1998/07/023 [arXiv:hep-th/9806087 [hep-th]].

[9] R. J. Riegert, “Birkhoff’s Theorem in Conformal Gravity,” Phys. Rev. Lett. 53, 315 (1984).

[10] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lust and A. Riotto, “Aspects of Quadratic Gravity,” arXiv:1505.07657 [hep-th].

[11] “Black hole solutions in $R^2$ gravity,” JHEP 1505, 143 (2015) arXiv:1502.04192 [hep-th]].

[12] H. Lu, Y. Pang, C. N. Pope and J. F. Vazquez-Poritz, “AdS and Lifshitz Black Holes in Conformal and Einstein-Weyl Gravities,” Phys. Rev. D 86, 044011 (2012) arXiv:1204.1062 [hep-th]].

[13] D. V. Fursaev, A. Patrushev and S. N. Solodukhin, “Distributional Geometry of Squashed Cones,” arXiv:1306.4000 [hep-th].
[14] M. Alishahiha, A. F. Astaneh and M. R. M. Mozaffar, “Holographic Entanglement Entropy for 4D Conformal Gravity,” JHEP 1402, 008 (2014) [arXiv:1311.4329 [hep-th]].

[15] A. Bernamonti, N. Copland, B. Craps and F. Galli, “Holographic thermalization of mutual and tripartite information in 2d CFTs,” PoS Corfu 2012, 120 (2013) [arXiv:1212.0848 [hep-th]].

[16] H. Casini and M. Huerta, “Remarks on the entanglement entropy for disconnected regions,” JHEP 0903, 048 (2009) doi:10.1088/1126-6708/2009/03/048 [arXiv:0812.1773 [hep-th]].

[17] P. Hayden, M. Headrick and A. Maloney, “Holographic Mutual Information is Monogamous,” Phys. Rev. D 87, no. 4, 046003 (2013) doi:10.1103/PhysRevD.87.046003 [arXiv:1107.2940 [hep-th]].

[18] M. Alishahiha, M. R. Mohammadi Mozaffar and M. R. Tanhayi, “On the Time Evolution of Holographic n-partite Information,” JHEP 1509, 165 (2015) doi:10.1007/JHEP09(2015)165 [arXiv:1406.7677 [hep-th]].

[19] S. Mirabi, M. R. Tanhayi and R. Vazirian, “On the Monogamy of Holographic n-partite Information,” Phys. Rev. D 93, no. 10, 104049 (2016) doi:10.1103/PhysRevD.93.104049 [arXiv:1603.00184 [hep-th]].

[20] M. R. Tanhayi, “Universal terms of holographic entanglement entropy in theories with hyperscaling violation,” Phys. Rev. D 97, no. 10, 106008 (2018) doi:10.1103/PhysRevD.97.106008 [arXiv:1711.10526 [hep-th]].

[21] H. Bagheri and M. R. Tanhayi, ‘Higher-curvature corrections to holographic mutual information,” Journal of Theoretical and Applied Physics, doi.org/10.1007/s40094-020-00367-4

[22] K. Skenderis, M. Taylor and B. C. van Rees, “Topologically Massive Gravity and the AdS/CFT Correspondence,” JHEP 0909, 045 (2009) [arXiv:0906.4926 [hep-th]].