Self-Binding Transition in Bose Condensates with Laser-Induced “Gravitation”

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In our recent publication (D. O’Dell, et al, Phys. Rev. Lett. 84, 5687 (2000)) we proposed a scheme for electromagnetically generating a self-bound Bose-Einstein condensate with 1/r attractive interactions: the analog of a Bose star. Here we focus upon the conditions necessary to observe the transition from external trapping to self-binding. This transition becomes manifest in a sharp reduction of the condensate radius and its dependence on the laser intensity rather than the trap potential.

I. INTRODUCTION

We have recently proposed a scheme for inducing a 1/r gravitational-like attractive interatomic potential in an atomic Bose-Einstein condensate (BEC) contained in the near-zone volume of intersecting triads of orthogonal laser beams. For sufficiently strong self-“gravitation” the BEC becomes self-bound. In this unique, novel regime the 1/r attraction balances the outward pressure due to the zero point kinetic energy and the short range s-wave scattering. Here we focus upon the transition from external trapping to self-binding. This transition becomes manifest in a sharp reduction of the condensate radius and its dependence on the laser intensity rather than the trap potential. We analyze the conditions for the observability of the self-binding transition: the threshold laser intensity (Sec. I), the bounds on the number of atoms imposed by the near-zone condition (Sec. II), as well as the loss rates (Sec. III). Sec. IV summarizes the findings.

II. SELF-BINDING THRESHOLD INTENSITY

A. Threshold condition

We need to find a situation where the mean-field self-“gravitation” energy associated with the near-zone laser-induced attractive 1/r potential can become (at least) comparable with the short-range s-wave scattering energy. To this end, we examine the mean-field solution for a condensate of atoms interacting via Thirumalachan

\[
U_{\text{iso}}(\tilde{r}) = -\frac{15\pi u}{11\lambda_L} \left( \frac{\sin(4\pi \tilde{r})}{(2\pi \tilde{r})^2} + 2\frac{\cos(4\pi \tilde{r})}{(2\pi \tilde{r})^3} - 5\frac{\sin(4\pi \tilde{r})}{(2\pi \tilde{r})^4} - 6\frac{\cos(4\pi \tilde{r})}{(2\pi \tilde{r})^5} + 3\frac{\sin(4\pi \tilde{r})}{(2\pi \tilde{r})^6} \right) \tag{1}
\]

where \( \tilde{r} = r/\lambda_L \), is normalized to the laser wavelength \( \lambda_L \), and

\[
u = (11\pi/15)(I\alpha^2 / c\epsilon_0^2 \lambda_L^2) , \tag{2}
\]

\( I \) being the sum of the intensities of all the lasers, and \( \alpha \) the atomic polarizability. The potential begins to oscillate (i.e. becomes alternately repulsive and attractive) at distances beyond \( \sim 0.36\lambda_L \). However, this potential can support a self-bound condensate with a larger radius as shown below.

We use the mean-field approximation (MFA), as embodied in the following generalized Gross-Pitaevskii equation, to calculate the ground-state order parameter \( \Psi(R) \) of a BEC subject to a laser-induced interatomic interaction

\[
\mu \Psi(R) = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(R) + V_{\text{sc}}(R) \Psi(R) \tag{3}
\]

where \( m \) is the atomic mass, \( V_{\text{ext}}(R) = m\omega_0^2 R^2 / 2 \) is an isotropic external trap potential (which will be considered negligible—see below), and \( V_{\text{sc}}(R) \) is the self-consistent potential

\[
V_{\text{sc}}(R) = g\rho(R) + \int d^3R' U_{\text{iso}}(R' - R) \rho(R') \tag{4}
\]

where \( \rho(R) = \Psi^2(R) \) is the density and \( g = 4\pi\alpha a^2 / m \), \( a \) being the s-wave scattering length.

In cold dilute atomic BECs with short-range s-wave scattering, the validity of the MFA (i.e. the Gross-Pitaevskii equation), is well established providing \( pa^3 \ll 1 \). However, the MFA is also valid for the long range repulsive coulomb potential, \( +u/r \), provided many atoms lie within an interaction sphere with a Bohr-type radius, \( a_\star = h^2 / mu \), so that \( pa_\star^3 \gg 1 \). This condition means that the potential must be weak. Remarkably, self-gravitating BECs simultaneously satisfy both of these MFA validity conditions as can be readily verified using the ensuing expressions.

The condensate radius can be studied using the variational wavefunction \( \Psi_w(R) = \sqrt{N} \exp(-R^2/2u^2\lambda_L^2) / (\pi u^2 \lambda_L^2)^{3/4} \), where \( w \) is a dimensionless variational parameter giving the width of the condensate. The variational solution in the limit of negligible kinetic energy (Thomas-Fermi limit) yields a self-bound condensate, i.e. finite \( w \).
(see Fig. 1 and Fig. 2 below), if the laser intensity exceeds
the following threshold value (in S.I. units)
\[
I_0 = \frac{48 \pi \hbar^2 c \alpha^2}{7 m a^2 \ell}.
\] (5)

Here \(I_0\) is the total intensity supplied by all the laser beams: for a trial each laser should have 1/3 of the
above value and for the 6 triad configuration \(\parallel 12\) of
the lasers should have 1/15, and the remaining 6 should
have 1/30, of the above value. The threshold \(I_0\) signifies
the equality of the gravitational-like potential and the
s-wave scattering potential.

With an intensity 1.5 times the threshold value (Eq.
\(\parallel 3\)) (arrow in Fig 2) the expectation value of the rms
condensate radius \(R_{\text{rms}} = \sqrt{\langle R^2 \rangle}\) is a fraction of the
laser wavelength \(\lambda_L (R_{\text{rms}} \approx 0.43 \times \lambda_L)\). The condensate
is less and less confined as one approaches the threshold \(\parallel 3\) — see Fig. 2, from above. Increasing the intensity \(I\) reduces the condensate radius which becomes, in the
asymptotic limit, proportional to \(1/\sqrt{I}\). Thus the depen-
dence \(R_{\text{rms}} \sim (I_0/I)^{1/2}\lambda_L\) is a distinct experimental
signature of self-binding.

\[
\begin{align*}
\text{FIG. 1.} & \quad (a) \text{ Variational mean field energies per particles} \\
& \quad \text{in the case of negligible kinetic energy (TF-G regime) and} \\
& \quad \lambda_L/Na \ll 1 \text{ plotted versus the trial size } w \text{ for different values} \\
& \quad \text{of } I/I_0. \quad \text{(b) Equilibrium value of } w \text{ versus } I/I_0 \text{ in the limit} \\
& \quad \text{of negligible kinetic energy (Thomas-Fermi limit). Only for} \\
& \quad I/I_0 > 1 \text{ are self-bound variational solutions (having minimum} \\
& \quad \text{at finite } w \text{) observed. Inset - Schematic phase portrait} \\
& \quad \text{of the transition from unbound to self-bound regime for neg-
ligible external trapping is plotted versus } \log_{10}(\lambda_L/Na) \text{ and} \\
& \quad \log_{10}(I/I_0).
\end{align*}
\]

At the threshold intensity an external harmonic trap
becomes negligible when \(\rho_0 \lambda_L a \gg 1\), where \(\rho_0^2 = \hbar/m\omega_0\)
and \(\rho\) is the density. As the laser intensity is in-
creased beyond this value the trap becomes increas-
ingly “irrelevant” — it is not necessary to turn it off to access
the TF-G regime, where \(r^{-1}\) and s-wave scattering domi-
nate.

The threshold \(I_0\) \(\parallel 3\) is evaluated neglecting the ki-
netic energy. The role of kinetic energy can be discussed
in terms of \(\lambda_L/Na\) (approximately the ratio between
the kinetic energy \(N\hbar^2/m\lambda_L^2\) and the scattering energy
\(N^2\hbar^2a/m\lambda_L^2\)), as shown schematically in the phase por-
trait in Fig. 1 (drawn for negligible external trapping)
which can modify the threshold for self-binding. The G
regime, representing the purely “gravitational” counter-
part of the TF-G regime, where only “self-gravitation”
and kinetic energy play a role \(\parallel 3\) (as in a Bose star \(\parallel 3\))
is accessed when
\[
\frac{\lambda_L}{Na} \approx \frac{I}{I_0} \lesssim \left(\frac{\lambda_L}{Na}\right)^2
\] (6)
that implies \(1 \lesssim \lambda_L/Na\).

At this point the variety of choices can be mainly di-
vided into two categories: i) to work with long laser wave-
lengths in order to contain many atoms within the near
zone, at the price of very high threshold power; ii) use
laser wavelengths moderately detuned from an atomic
resonance, so as to benefit from the increased polarizabil-
ity, at the price of considerably fewer self-bound atoms.

B. Long-wavelength (static polarizability) threshold

The threshold intensity (Eq. \(\parallel 3\)) is independent of the
laser wavelength \(\lambda_L\), as long as the dynamic polarizabil-
ity \(\alpha(q)\) is too. The \(I_0\) threshold takes the following zero-
frequency (static) values: \(I_0 = 5.65 \times 10^9 \text{ Watts/cm}^2\)
for sodium, \(I_0 = 8.19 \times 10^8 \text{ Watts/cm}^2\) for rubidium. It is
sufficient to use 20 W \(\times\) 3 beams of Nd:Yag lasers focused
down to 10 \(\mu\)m for rubidium to exceed the threshold. By
contrast, we require multi kW CO\(_2\) lasers focussed
down to 100 \(\mu\)m for the same purpose. A laser beam with a
gaussian profile focussed to 10 \(\lambda_L\) would exert a large in-
ward radial dipole force on each atom, so non-gaussian
optics giving a very flat intensity profile \(\parallel 3\) over the con-
densate region may be required in the long-wavelength
(static) case. There remains the problem of random noise
in the intensity profile, but fortunately this can only ex-
ist on scales larger than the wavelength and so may be
overcome.

An additional option is to reduce the scattering length
\(a\) (to which the threshold intensity \(\parallel 3\) is proportional).
This is possible in the vicinity of (but somewhat off) a
Feshbach resonance, as demonstrated experimentally \(\parallel 3\): 
reduction of \(a\), and correspondly \(I_0\), by one to two orders
of magnitude would eliminate the need for non-gaussian
optics in the static polarizability case.

C. Moderate-detuning threshold

Using a moderate detuning from an atomic resonance
one can increase the polarizability by many orders of
magnitude compared to its zero frequency value.

In a recent experiment on superradiance \(\parallel 3\) the laser
was red detuned by 1.7 GHz from the 3S1/2, \(F=1 \rightarrow
3P_{3/2}, F = 0,1,2,\) transition of sodium. With this de-
tuning, the polarizability in cgs units is \(\alpha = 3.534 \times\)
$10^{-18} \text{cm}^3$, which is $\approx 1.5 \times 10^5$ times the static value of the polarizability. The threshold intensity (1) is then reduced by a factor $\approx 2.3 \times 10^{10}$ compared to the static polarizability case, becoming $I_0 \approx 262$ mW/cm$^2$ for sodium, which is close to the values used in Ref. [1].

With this value of threshold intensity the gradient forces can be negligible if the focal spots of the lasers are much wider than $\lambda_L$.

### D. Moderate-detuning saturation and repulsion

The potential (1) is the result of a 4th order, two-atom, QED process [2], valid when the laser is far detuned from any atomic transitions. This means that the initial absorption of a laser photon and the subsequent intermediate steps are virtual processes (which are most significant in the near-zone), followed by photon emission back into the original laser mode. A different process can take place when the laser is on-resonance. Genuine absorption of a laser photon by a single atom (measured by the saturation), followed by spontaneous emission of this real photon is a process that radiates energy. If another atom absorbs this radiation then in the far-zone it feels a repulsive Coulomb-like force $F_{\text{repuls}} = K/r^2$ [3], which has been recently measured in rubidium molasses [4]. For moderate detuning, can this force counteract our attractive gravitation-like force $F_{\text{grav}} = -u/r^2$?

For detuning $\delta$ much larger than both the Rabi frequency $\Omega$ and the linewidth $\gamma$ of the resonance, the saturation parameter $s = I d^2/\langle \epsilon_0 \hbar^2 \delta^2 \rangle$ [3], where $d$ is the dipole matrix element, becomes independent of the detuning when calculated at the threshold intensity (2):

$$s(I = I_0) = \frac{4\pi a \epsilon_0 \hbar^2}{7 md^2}. \quad (7)$$

It is then found that $K \approx \sigma_0^3 I \Omega^2/(16 \epsilon_0 \delta^2)$, where $\sigma_0$ is the resonant absorption cross section and $I_0$ is the corresponding saturation intensity. On comparing this expression with $u$ (Eq. (2)), we find that, in terms of the saturation parameter $s$, $K \approx su$. [3]

For the sodium transition and 1.7 GHz detuning referred above, Eq. (3) yields very small value $s \approx 0.0003$. This implies that under the moderate-detuning conditions discussed above, the repulsive force has a negligible effect on self-binding.

### III. NUMBER OF SELF-BOUND ATOMS

A key experimental restriction on self-binding is that the atoms should be in the near-zone to feel the $1/r$ potential: a condensate smaller than the laser wavelength limits the number of atoms involved. Let us assume we have the maximum density of some $10^{15}$ atoms/cm$^3$. Using the gaussian wavefunction one can have of the order of $10^6$ or $10^3$ atoms in the condensate irradiated by a CO$_2$ laser or Nd:Yag laser, respectively (see Fig. 2).

The price of moderately detuned wavelengths ($\approx .589 \mu m$ for sodium) is the small number of atoms involved. With an intensity $I \approx 1.5 I_0$ the atom cloud contains $\approx 40$ atoms as the peak density ranges from $10^{15}$ to $10^{16}$ atoms/cm$^3$. Although this number is small, it is sufficient to demonstrate the self-binding effect.

For given values of $I$, $\Omega$, $\alpha$, and $m$, we are either in the G regime or the TF-G regime, depending on whether the number of atoms $N$ is smaller or larger than the number $N_{\text{border}} \approx \sqrt{3\pi \hbar^2/(2\mu a)}$ which corresponds to the line separating the two regions in the inset of Fig. 1.

It so happens that 40, the lower estimate of the number of self-bound sodium atoms obtainable in the moderate-detuning regime, is very close to $N_{\text{border}}$. This is an interesting region, because both the kinetic energy and the $s$-wave scattering are significant and together with the $r^{-1}$ attraction determine the condensate properties.

### IV. LOSS RATES

#### A. Spontaneous Rayleigh losses

The single-atom Rayleigh scattering rate $\Gamma_{\text{Ray}}$ leads to depletion of the condensate. The probability amplitude for inelastic scattering from the ground state $|0\rangle$ of the near-zone condensate to any excited state $|n\rangle$ due to an external field with wavevector $\mathbf{q}$ is proportional to

![Fig. 2. Range of numbers N of Na condensate atoms as a function of $\lambda_L$ that are compatible with a TF-G or G solution. The density is $10^{13} - 10^{16}$ atoms/cm$^3$ and the intensity is 1.5 times the threshold intensity (2). The region above $10^{16}$ cm$^{-3}$ corresponds to excessive density. The vertical long-dashed line corresponds to the moderate-detuning choice discussed for Na.](image)
\[ \sqrt{N} \sum_{n \neq 0} \langle n | (q \cdot r) | 0 \rangle. \] Hence, for sample sizes less than a wavelength we expect the spontaneous Rayleigh scattering rate to be reduced by a factor at least as small as \((q R_{\text{rms}})^2\), analogously to the Lamb-Dicke effect \([3]\). The lifetime of the condensate, when determined from spontaneous Rayleigh scattering alone, is estimated to be

\[ \tau_{\text{ray}} \geq \left( \Gamma_{\text{ray}} (q R_{\text{rms}})^2 \right)^{-1}. \] (9)

Since \( \Gamma_{\text{ray}} = I q^3 \alpha^2 / (3 \hbar c^2 c) \) \([3]\), it can be expressed in terms of the electromagnetically induced energy \( U(r) = -u/r \) of a single pair of atoms separated by a distance equal to the wavelength

\[ \Gamma_{\text{ray}} = \left( \frac{20\pi}{11} \right) \frac{u}{\hbar \lambda} \] (10)

where \( u \) is defined in Eq. (2). Using this relation, we can compare the upper bound on the condensate lifetime set by Rayleigh scattering with the time scale of the dynamics, the requirement being that the system exists long enough to equilibrate. In the TF-G and G (self-bound) regions a characteristic time scale for the dynamics is provided by the following “plasma” frequency

\[ \omega_p^2 = \frac{4 \pi u \rho_{\text{peak}}}{m} \] (11)

where \( \rho_{\text{peak}} \) is the peak density. We can express \( \omega_p \) in terms of the recoil energy \( E_R = \hbar^2 q^2 / 2m \) (\( q \) being the mean laser wavelength) and the Rayleigh scattering rate \( \Gamma_{\text{ray}} \) using Eq. (10)

\[ \omega_p \approx 0.25 \frac{\hbar \Gamma_{\text{ray}}^2}{E_R} N^2 f^{-3/2} \] (12)

where the factor

\[ f = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{N^2}{N_{\text{border}}^2}} \] (13)

is asymptotically equal to 1 in the G region and \( N/N_{\text{border}} \) in the TF-G region. It follows from (13) that the characteristic oscillation frequency \( \omega_p \) can be much bigger than \( \Gamma_{\text{ray}} \), by a factor proportional to \( N^2 \) in the G or TF-G region, respectively. Thus the lifetime can be considerably longer than the characteristic time scale of the dynamics.

Even for the small number of 40 sodium atoms in the self-bound moderate-detuning regime (\( I = 1.5 \times 10^4 \text{ s}^{-1} \), \( \delta = 1.7 \text{ GHz} \)) for which the recoil energy is \( E_R / \hbar = 1.57 \times 10^5 \text{ s}^{-1} \) and \( \Gamma_{\text{ray}} = 1.58 \times 10^4 \text{ s}^{-1} \), we find \( \omega_p \approx 20 \times \Gamma_{\text{ray}} \). This implies that several oscillation periods of the self-bound condensate can occur within the Rayleigh lifetime.

### B. Interference losses

We revisit the expressions for the loss rate \( \Gamma_{\text{interf}} \) due to multi-beam interference as obtained in \([1]\). We can express \( \Gamma_{\text{interf}} \) in terms of the recoil energy \( E_R \) and Rayleigh scattering rate \( \Gamma_{\text{ray}} \) as in Sec. IV A.

\[ \Gamma_{\text{interf}} \approx 0.05 \left( \frac{\hbar \Gamma_{\text{ray}} N}{E_R} \right)^4 \sqrt{\frac{m}{E_R}} \frac{\Gamma_{\text{ray}} f^{-3}}{E_R} \] (14)

where \( \Omega \) is the relative detuning of beams in the triad. In the example given in Sec. IV A above, \( \Gamma_{\text{interf}} \) turns out to be few times bigger than \( \Gamma_{\text{ray}} \), when \( \Omega \) is chosen to be of the order of \( \omega_p \).

### V. CONCLUSIONS

Our main conclusion is that at least the TF-G self-bound region is experimentally accessible, although such an experiment would be challenging. Moderate detuning is preferable to the longer wavelength case due to the huge enhancement in the polarizability, but it allows the self-binding of few (less than 100) atoms. If the scattering length were reduced via a Feshbach resonance then this would further facilitate the self-trapping of many more atoms using near-infrared lasers.

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