Measures for a transdimensional multiverse

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Received April 28, 2010
Accepted May 29, 2010
Published June 24, 2010

Abstract. The multiverse/landscape paradigm that has emerged from eternal inflation and string theory, describes a large-scale multiverse populated by “pocket universes” which come in a huge variety of different types, including different dimensionalities. In order to make predictions in the multiverse, we need a probability measure. In $(3+1)d$ landscapes, the scale factor cutoff measure has been previously shown to have a number of attractive properties. Here we consider possible generalizations of this measure to a transdimensional multiverse. We find that a straightforward extension of scale factor cutoff to the transdimensional case gives a measure that strongly disfavors large amounts of slow-roll inflation and predicts low values for the density parameter $\Omega$, in conflict with observations. A suitable generalization, which retains all the good properties of the original measure, is the “volume factor” cutoff, which regularizes the infinite spacetime volume using cutoff surfaces of constant volume expansion factor.

Keywords: cosmology with extra dimensions, inflation, physics of the early universe, cosmological applications of theories with extra dimensions

ArXiv ePrint: 1004.4567
1 Introduction

Eternal inflation and string theory both suggest that we live in a multiverse which is home to many different types of vacua [1–5]. Many of these vacua are high-energy metastable states that can decay to lower-energy states through bubbles which nucleate and expand in the high-energy vacuum background [6]. Inverse transitions are also possible: bubbles of high-energy vacuum can nucleate in low-energy ones [7]. Once the process of eternal inflation begins, an unbounded number of bubbles will be created, one within the other, of every possible type. In addition each bubble has infinite spacelike slices when described in the maximally symmetric FRW coordinate system, and thus infinite spatial volume. The multiverse spacetime therefore has an infinite number of observers of every possible type.

In order to make predictions, we need to be able to compare the numbers of different types of events that can occur in the multiverse. This is non-trivial because of the aforementioned infinities, and has become known as “the measure problem” [9]. A number of measure proposals have been developed [10–28], and some have already been ruled out, as they either lead to paradoxes or yield predictions in conflict with observations. All these measures have been defined within the context of regular “equidimensional” (3 + 1)d eternal inflation: all parent and daughter vacua have the same dimensionality. However, in general we expect the string theory landscape to allow parent vacua to nucleate daughter vacua with different

\footnote{This is the so-called “recycling” process. In both cases, the radius of the bubbles asymptotically approaches the comoving horizon size in the parent vacuum at the moment of nucleation [8].}
effective dimensionality [29]. The purpose of the present work is to introduce a measure for a “transdimensional” multiverse.

One of the most attractive \((3 + 1)d\) measure proposals is the scale factor cutoff measure or, for brevity, scale factor measure \([10–14]\). It does not suffer from any obvious problems and gives prediction for the cosmological constant in a good agreement with the data \([26–28]\). In this measure the infinite spacetime volume of the multiverse is regularized with a cutoff at surfaces of constant scale factor. More precisely, one chooses some initial spacelike hypersurface \(\Sigma_0\), where the scale factor is set to be equal to one, \(a_0 = 1\), and follows the congruence of timelike geodesics orthogonal to that surface until the scale factor reaches some specified value \(a_{\text{max}}\). In the \((3 + 1)\)-dimensional case, the scale factor is defined as

\[
a = \mathcal{V}^{1/3},
\]

where \(\mathcal{V}\) is the volume expansion factor along the geodesics. Relative probabilities are calculated by counting the number of times a given event occurs in the finite spacetime volume spanned by the congruence, between the initial \((\Sigma_0)\) and final \((\Sigma_{\text{max}} : a = a_{\text{max}})\) hypersurfaces, in the limit that the scale factor cutoff \(a_{\text{max}} \to \infty\).

In this paper we consider possible extensions of the scale factor measure to a transdimensional landscape. Eq. (1.1) defining the scale factor can be generalized as

\[
a = \mathcal{V}^{1/D},
\]

where \(D\) is the number of expanding (non-compact) dimensions. As a straightforward extension of the \((3 + 1)d\) scale factor measure, one can still use constant-\(a\) surfaces as a cutoff. We will find, however, that this prescription leads to a problem: it strongly disfavors large amounts of slow-roll inflation inside bubbles and predicts low values of the density parameter \(\Omega\), in conflict with observations.

An alternative prescription is to use constant volume factor surfaces, \(\mathcal{V} = \text{const}\), for a cutoff. We shall call this the volume factor measure. In an equidimensional multiverse, a surface of constant scale factor is also a surface of constant volume factor. However, for transdimensional universes, different vacua generally have different values of \(D\), so constant scale factor and volume factor surfaces are not the same. We will find that the volume factor cutoff results in an expression for comparing relative probabilities that is analogous to that of the equidimensional scale factor measure and provides a natural framework in which to extend the successful features of the scale factor measure to transdimensional landscapes.

The rest of the paper is organized as follows: In section 2 we review the scale factor measure and outline how it can be implemented using the rate (master) equation formalism in four spacetime dimensions. In section III we review the spacetime structure of a transdimensional multiverse using the 6-dimensional Einstein-Maxwell theory as an example and generalize the master equation to the transdimensional case. This formalism is then applied in section IV to investigate the properties of the transdimensional volume factor and scale factor measures. Our conclusions are briefly summarized and discussed in section V.

2 The scale factor measure in \((3 + 1)d\)

In this section we shall review the key features of the scale factor measure as found in refs. \([26–28]\).
The scale factor time $\eta$ is defined in terms of the expansion factor of the geodesics $a(\tau)$ as

$$\eta = \ln a$$

and is related to the proper time $\tau$ via

$$d\eta = H(\tau) d\tau.$$  \hspace{1cm} (2.2)

Here, $H(\tau)$ is the local expansion rate of the geodesic congruence, which can be defined as the divergence of the four-velocity field $u^\mu(x)$ of the congruence,

$$H = (1/3) u^\mu_{;\mu}.$$  \hspace{1cm} (2.3)

We start with a congruence of geodesics orthogonal to the initial spacelike hypersurface $\Sigma_0$ and allow them to expand until the local scale factor reaches a critical value $a_{\text{max}}$ corresponding to $\eta_{\text{max}}$. This defines the cutoff hypersurface $\Sigma_{\text{max}}$. The relative probability to make an observation of type $i$ or $j$ is given by the ratio of the number of times that each type of observation ($N_i$ or $N_j$) occurs in the spacetime volume spanned by the congruence, in the limit that $\eta_{\text{max}} \rightarrow \infty$,

$$P(O_i) \equiv \lim_{\eta_{\text{max}} \rightarrow \infty} \frac{N_i}{N_j}.$$  \hspace{1cm} (2.4)

An intuitive picture of the scale factor cutoff emerges when we think of the geodesic congruence as a sprinkling of test “dust” particles. Imagine that this dust has a uniform density $\rho_0$ on our initial hypersurface $\Sigma_0$. As the universe expands, the dust density (in the dust rest frame) dilutes. The scale factor cutoff is then initiated when the density drops below a critical value. The scale factor time can be expressed in terms of the dust density as

$$\eta = -1/3 \ln \frac{\rho}{\rho_0}.$$  \hspace{1cm} (2.5)

We can also introduce the volume expansion factor $V = \rho_0/\rho$ and the corresponding “volume time”

$$T = \ln V.$$  \hspace{1cm} (2.6)

In a $(3 + 1)d$ multiverse this time is trivially related to the scale factor time,

$$T = 3\eta.$$  \hspace{1cm} (2.7)

The definition of the scale factor time $\eta$ in terms of a geodesic congruence encounters ambiguities in regions of structure formation, where geodesics decouple from the Hubble flow and eventually cross. In particular, eq. (2.5) suggests that scale factor time comes to a halt in bound structures, like galaxies, while it continues to flow in the intergalactic space. It is not clear whether one ought to use this local definition of $\eta$ for the scale factor cutoff, or use some nonlocal time variable that is more closely related to the FRW scale factor. Possible ways of introducing such a variable have been discussed in refs. [27, 33]. Here we shall assume that some such nonlocal procedure has been adopted, so that structure formation has little effect on the cutoff hypersurfaces.

The numbers of observations $N_{i,j}$ in eq. (2.4) are proportional to appropriately regularized volumes occupied by the corresponding vacua. We shall now review how these volumes can be calculated using the master equation of eternal inflation.
2.1 The rate equation

The comoving volume fractions $f_i(\eta)$ occupied by vacua of type $i$ on a constant scale factor time slice $\eta$ obeys the rate equation (also called the master equation) [8, 19]

$$\frac{df_i}{d\eta} = \sum_j (-\kappa_{ji} f_i + \kappa_{ij} f_j).$$  \hspace{1cm} (2.8)

Here, the first term on the right-hand side accounts for loss of comoving volume due to bubbles of type $j$ nucleating within those of type $i$, and the second term reflects the increase of comoving volume due to nucleation of type-$i$ bubbles within type-$j$ bubbles.

The transition rate $\kappa_{ij}$ is defined as the probability per unit scale factor time for a geodesic "observer" who is currently in vacuum $j$ to find itself in vacuum $i$,

$$\kappa_{ij} = \Gamma_{ij} \frac{4\pi}{3} H_j^{-4},$$  \hspace{1cm} (2.9)

where $\Gamma_{ij}$ is the bubble nucleation rate per unit physical spacetime volume,

$$H_j = (\Lambda_j/3)^{1/2}$$  \hspace{1cm} (2.10)

is the expansion rate and $\Lambda_j$ is the energy density in vacuum $j$. The total transition rate out of vacuum $i$ is given by the sum

$$\kappa_i = \sum_j \kappa_{ji}.$$  \hspace{1cm} (2.11)

We distinguish between the recyclable, non-terminal vacua with $\Lambda_j > 0$, which are sites of eternal inflation and further bubble nucleation, and the non-recyclable, “terminal vacua”, for which $\Lambda_j \leq 0$. Then, by definition, $\Gamma_{ij} = 0$ when the index $j$ corresponds to a terminal vacuum.

Eq. (2.8) can be written in a vector form,

$$\frac{df}{d\eta} = Mf,$$  \hspace{1cm} (2.12)

where $f(\eta) \equiv \{f_j(\eta)\}$ and

$$M_{ij} = \kappa_{ij} - \delta_{ij} \sum_r \kappa_{ri}.$$  \hspace{1cm} (2.13)

The asymptotic solution of (2.12) at large $\eta$ has the form

$$f(\eta) = f^{(0)} + se^{-q\eta} + ...$$  \hspace{1cm} (2.14)

Here, $f^{(0)}$ is a constant vector which has nonzero components only in terminal vacua. Any such vector is an eigenvector of the matrix $M$ with zero eigenvalue,

$$Mf_0 = 0.$$  \hspace{1cm} (2.15)

As shown in [19], all other eigenvalues of $M$ have a negative real part, so the solution approaches a constant at late times. Moreover, the eigenvalue with the smallest (by magnitude) real part is purely real. We have denoted this eigenvalue by $-q$ and the corresponding eigenvector by $s$. 

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The asymptotic values of the terminal components \( f_j^{(0)} \) depend on the choice of initial conditions. For any physical choice, we should have \( f_j^{(0)} \geq 0 \). It can also be shown [19] that \( s_j \leq 0 \) for terminal vacua and \( s_j \geq 0 \) for recycling vacua. In the case of recycling vacua, it follows from eq. (2.14) that

\[
    f_j \approx s_j e^{-q\eta}. \tag{2.16}
\]

The physical properties of bubble interiors, such as the spectrum of density perturbations, the density parameter, etc., depend not only on the vacuum inside the bubble, but also on the parent vacuum that it tunneled from (or, more precisely, on the properties of the bubble implementing the transition between the two vacua). Bubbles should therefore be labeled by two indices, \( ij \), referring to the daughter and parent vacua respectively. The fraction of comoving volume in type-\( ij \) bubbles, \( f_{ij} \), satisfies a slightly modified rate equation [30]

\[
    \frac{df_{ij}}{d\eta} = \sum_k ( -\kappa_{ki} f_{ij} + \kappa_{ij} f_{jk} ) = -\kappa_i f_{ij} + \kappa_{ij} f_{j}. \tag{2.17}
\]

Here and below we do not assume summation over repeated indices, unless the summation is indicated explicitly. The fractions \( f_{ij} \) and \( f_i \) are related by

\[
    f_i = \sum_j f_{ij}, \tag{2.18}
\]

and the original rate equation (2.8) can be recovered from (2.17) by summation over \( j \). Solutions of eq. (2.17) can also be expressed in the form (2.14), and for non-terminal bubbles we have

\[
    f_{ij} \approx s_{ij} e^{-q\eta}. \tag{2.19}
\]

Note that it follows from (2.18) that the eigenvalue \( q \) for the modified equation (2.17) should be the same as for the original equation.

The comoving volume fractions found on equal scale factor time hypersurfaces are directly related to the physical volume fractions. Thus we find that asymptotically the physical volumes in nonterminal bubbles are given by

\[
    V_{ij}(\eta) = V_0 s_{ij} e^{(3-q)\eta}, \tag{2.20}
\]

where \( V_0 \) is the volume of the initial hypersurface \( \Sigma_0 \) and \( e^{3\eta} \) is the volume expansion factor. Note that the growth of volume is given by \( e^{\gamma\eta} \), where

\[
    \gamma = 3 - q \tag{2.21}
\]

is the fractal dimension of the inflating region [31]. This is slightly less than 3, because volume is lost to terminal vacua.

Assuming that upward transitions in the landscape are strongly suppressed, the eigenvalue \( q \) can be approximated as [32] \( q \approx \kappa_{\min} \), where \( \kappa_{\min} \) is the decay rate of the slowest-decaying vacuum. In general, it can be shown [33] that

\[
    q \leq \kappa_{\min}. \tag{2.22}
\]

In the string theory landscape, \( \kappa_{\min} \) is expected to be extremely small, and thus \( \gamma \) is very close to 3.
The volume fractions can also be expressed in terms of the volume time \( T \), which in our case is simply \( T = 3\eta \),

\[
V_{ij}(T) = V_0 s_{ij} \tilde{\gamma}^T,
\]

where we have defined

\[
\tilde{\gamma} = 1 - \tilde{q}
\]

and

\[
\tilde{q} = q/3.
\]

### 2.2 The number of observers

Bubble interiors can be described by open FRW coordinates \((\tau, \xi, \theta, \phi)\),

\[
ds^2 = -d\tau^2 + a_{ij}(\tau)(d\xi^2 + \sinh^2 \xi d\Omega^2),
\]

where the functional form of \( a_{ij}(\tau) \) is generally different in different bubbles. We shall assume for simplicity that all observers in bubbles of type \( ij \) are formed at a fixed FRW time \( \tau_{ij}^{\text{obs}} \) with some average density \( n_{ij}^{\text{obs}} \). This defines the “observation hypersurface” within each habitable bubble. The infinite three-volume of this hypersurface is regularized by the scale factor cutoff at some \( \eta = \eta_{\text{max}} \). The regulated number of type-\( ij \) observers is

\[
N_{ij}^{\text{obs}} = V_{ij}^{\text{obs}} n_{ij}^{\text{obs}},
\]

where \( V_{ij}^{\text{obs}} \) is the total volume of observation hypersurfaces in all type-\( ij \) bubbles.

For our purposes in this paper, it will not be necessary to distinguish between the number of observers and the number of observations, so we shall use these terms interchangeably. If one is interested in some particular kind of observation, then \( n_{ij}^{\text{obs}} \) in eq. (2.27) should be understood as the density of observers who perform that kind of observation.

The calculation of the volume \( V_{ij}^{\text{obs}} \) has been discussed in refs. [27, 28, 34, 35]. Here we give a simplified derivation, referring the reader to the original literature for a more careful analysis.

As the geodesics cross from the parent vacuum into the bubble, they become comoving with respect to the bubble FRW coordinates (2.26) within a few Hubble times. This implies that a fixed amount of FRW time \( \tau \) roughly corresponds to a fixed amount of scale factor time \( \eta \) along the geodesics. (This will be increasingly accurate at large \( \tau \gg H^{-1} \).) We shall denote \( \Delta\eta_{ij}^{\text{obs}} \) to be the amount of scale factor time between crossing into a bubble and reaching the observation hypersurface \( \tau = \tau_{ij}^{\text{obs}} \). If the expansion rate in the daughter bubble right after nucleation is not much different from the expansion rate \( H_j \) in the parent vacuum, this can be approximated as

\[
\Delta\eta_{ij}^{\text{obs}} \approx \ln[H_j a_{ij}(\tau_{ij}^{\text{obs}})].
\]

From eq. (2.17), the comoving volume fraction that crosses into \( ij \) bubbles per time interval \( d\eta \) is

\[
d^+ f_{ij} = \kappa_{ij} f_j(\eta)d\eta.
\]

The corresponding physical volume is

\[
d^+ V_{ij} = V_0 e^{3\eta} d^+ f_{ij} = V_0 \kappa_{ij} s_j e^{(3-q)\eta} d\eta.
\]
where we have used eq. (2.16) for the comoving volume fraction in the recycling parent vacuum \( j \). The total regulated volume of observation hypersurfaces in type-\( ij \) bubbles can now be expressed as

\[
V_{ij}^{\text{obs}} = V_0 e^{3 \Delta \eta_{ij}^{\text{obs}}} e^{-\lambda_{ij} \Delta \eta_{ij}^{\text{obs}}} \int_0^{\eta_{\text{max}}} e^{(3-q)\eta} d\eta.
\]  

(2.31)

The factor \( e^{3 \Delta \eta_{ij}^{\text{obs}}} \) in front of the integral accounts for volume expansion in bubble interiors, and the factor \( e^{-\lambda_{ij} \Delta \eta_{ij}^{\text{obs}}} \) accounts for the volume loss due to bubble nucleation in the course of slow-roll inflation and subsequent non-vacuum evolution inside the bubbles. The corresponding nucleation rate is generally different from that in the vacuum; that is why we used a different notation. The integration in (2.31) is up to \( \eta_{\text{max}} - \Delta \eta_{ij}^{\text{obs}} \): regions added at later times are cut off before they evolve any observers. Performing the integration, we find

\[
V_{ij}^{\text{obs}} = \frac{1}{3-q} V_0 \kappa_{ij} s_j e^{(3-q)\eta_{\text{max}}} e^{(q-\lambda_{ij})\Delta \eta_{ij}^{\text{obs}}}.
\]  

(2.32)

The constant (\( ij \)-independent) factors in eq. (2.32) do not affect the relative numbers of observers in different vacua. Dropping these irrelevant factors and using eq. (2.27), we obtain

\[
N_{ij}^{\text{obs}} \propto \kappa_{ij} s_j e^{(q-\lambda_{ij})\Delta \eta_{ij}^{\text{obs}}}.
\]  

(2.33)

The decay rate \( \lambda_{ij} \) is typically exponentially suppressed, and according to eq. (2.22), the eigenvalue \( q \) should also be extremely small. One can expect therefore that, to a good approximation, the last factor in (2.33) can be dropped.

3 A transdimensional multiverse

3.1 Einstein-Maxwell landscape

We now wish to generalize the above formalism to a transdimensional multiverse. We will try to keep the discussion general, but it will be helpful to have a specific model in mind. So we start by reviewing the 6d Einstein-Maxwell model, which has been used as a toy model for flux compactification in string theory [36] and has been recently studied in the multiverse context in refs. [37–39].

The model includes Einstein’s gravity in (5 + 1) dimensions, a Maxwell field, and a positive cosmological constant. It admits solutions in which some of the spatial dimensions are compactified on a sphere. These extra dimensions are stabilized by either magnetic or electric flux. The landscape of this theory has turned out to be unexpectedly rich. It includes a 6-dimensional de Sitter vacuum (\( dS_6 \)), a variety of metastable 4-dimensional de Sitter and anti-de Sitter vacua (\( dS_4 \) and \( AdS_4 \)) with two extra dimensions compactified on a 2-sphere (\( S_2 \)), and a number of \( AdS_2 \) vacua with four extra dimensions compactified on \( S_4 \). The model also includes some perturbatively unstable \( dS_4 \times S_2 \) and \( dS_2 \times S_4 \) vacua (see also [40]).

Quantum tunneling transitions between all these vacua have been discussed in refs. [37–39]. Transitions between different \( dS_4 \times S_2 \) vacua (which were called “flux tunneling transitions” in [37]) are similar to bubble nucleation in 4d. They can be studied using the effective 4d theory, where they are described by the usual Coleman-De Luccia formalism. From a higher-dimensional viewpoint, the compactified \( S_2 \) in the new vacuum inside the bubble has a different radius and a different magnetic flux. The role of the bubble walls is played by magnetically charged black 2-branes.
Another kind of transition occurs when the extra dimensions are destabilized in the new vacuum, so they start growing inside the bubble. The geometry resulting from such a decompactification transition, $dS_4 \times S_2 \rightarrow dS_6$, is illustrated in figure 1. Observers in the parent vacuum see the nucleation and subsequent expansion of a bubble bounded by a spherical black 2-brane. The bubble interior is initially highly anisotropic, but as the compact dimensions expand, it approaches local isotropy, with the metric approaching that of a 6d de Sitter space.

Transdimensional compactification transitions in which the parent vacuum has higher dimensionality than the daughter vacuum lead to a rather different spacetime structure [38, 39]. Transitions from $dS_6 \rightarrow dS_4 \times S_2$ occur through spontaneous nucleation of horizon-size spherical black branes, with the new compactified regions contained in the brane interiors. In $dS_6 \rightarrow AdS_2 \times S_4$ transitions, the new regions are located inside spontaneously nucleated pairs of electrically charged black holes.

Transitions from $dS_4 \times S_2$ to $AdS_2 \times S_4$ may also occur, but the corresponding instantons have not yet been identified. An additional vacuum decay channel, which can exist in more complicated models, is the nucleation of the so-called “bubbles of nothing” [41–43] — that is, holes in spacetime bounded by higher-dimensional topological defects. In the case of flux compactification in 6d, such bubbles can be formed if the Maxwell field is embedded into a gauge theory with a spontaneously broken non-Abelian symmetry [43]. The boundary defect then has codimension 3 and its cross-section has the form of a non-Abelian magnetic monopole.

### 3.2 Scale factor and volume factor times

The scale factor and volume factor times can be defined as before, in terms of a congruence of geodesics orthogonal to some initial hypersurface $\Sigma_0$. The geodesics describe a dust of test particles which are initially uniformly spread over $\Sigma_0$. (If $\Sigma_0$ has some compact dimensions, the particles are assumed to be uniformly spread both in large and compact dimensions.) The scale factor time is defined as

$$d\eta = H(\tau) d\tau,$$

where $\tau$ is the proper time along geodesics, $H$ is given by

$$H = \frac{1}{D} u^{\mu}_{\mu},$$

and $D$ is the effective number of spatial dimensions (that is, the number of non-compact dimensions) in the corresponding vacuum.
The volume factor time $T$ is related to $\eta$ as

$$dT = D d\eta. \quad (3.3)$$

As the geodesics flow into bubbles where some dimensions get decompactified or into black branes which have compactified regions in their interiors, the value of $D$ changes and with it the relation between $T$ and $\eta$. Hence, in a transdimensional multiverse, equal scale factor hypersurfaces are not the same as equal volume factor hypersurfaces.

### 3.3 The rate equation

#### 3.3.1 Scale factor time

Let us first consider the comoving volume rate equation using the scale factor time $\eta$ as our time variable. The form of the rate equation is the same as in the equidimensional case,

$$\frac{df_{ij}}{d\eta} = -\kappa_{ij} f_{ij} + \kappa_{ij} f_j. \quad (3.4)$$

Here, $f_{ij}$ is the comoving volume fraction in regions of type $ij$ (which is proportional to the number of the geodesic “dust” particles in such regions) and $\kappa_{ij}$ is the probability per unit scale factor time for a geodesic which is currently in vacuum $j$ to cross into vacuum $i$. The effect of bubble or black brane nucleation is to remove some comoving volume from the parent vacuum and add it to the daughter vacuum (except for the bubbles of nothing, where the excised comoving volume is lost). As in the 4d case, the asymptotic radius of black branes and bubbles is equal to the comoving horizon size. This asymptotic value is approached very quickly, essentially in one Hubble time, and since the rate equation should be understood in the sense of coarse-graining over a Hubble scale, we can say it is reached instantly. The transition rates $\kappa_{ij}$ are given by

$$\kappa_{ij} = v_j H_j^{-1} \Gamma_{ij}, \quad (3.5)$$

where $D_j$ is the effective number of dimensions in the parent vacuum and $v_j$ is a geometric factor. For flux tunneling bubbles, decompactification bubbles, and bubbles of nothing, the entire horizon region is excised, and $v_j$ is the volume inside a unit sphere in $D_j$ dimensions,

$$v_j = \frac{2\pi^{(D_j+1)/2}}{\Gamma \left( \frac{D_j+3}{2} \right)}. \quad (3.6)$$

We note that $\Gamma_{ij}$ is the bubble (or black brane) nucleation rate in the large $(D_j + 1)$ dimensions, so that $\kappa_{ij}$ is dimensionless.

In the case of compactification transitions, the geometric factor is different. Consider, for example, transitions $dS_6 \to AdS_2 \times S_4$ in our 6d Einstein-Maxwell model. These transitions occur through nucleation of electrically charged black holes, which are described by 6-dimensional Reissner-Nordstrom-de Sitter solutions,

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 d\Omega_4^2, \quad (3.7)$$

2A similar transdimensional rate equation, but with somewhat different transition coefficients was presented by Carroll et.al in ref. [38]. In particular, they made no distinction between the geometric factors for compactification and decompactification transitions.
where
\[ f(r) = 1 - \frac{2M}{r^3} + \frac{Q^2}{r^6} - \frac{H_6^2 r^2}{r^6}. \] (3.8)

Here, \( H_6 \) is the expansion rate of \( dS_6 \) and the parameters \( M \) and \( Q \) are proportional to the black hole mass and charge, respectively. The requirement that the instanton describing black hole nucleation should be non-singular imposes a relation between \( M, Q \) and \( H_6 \). In the limit when the black hole is much smaller than the cosmological horizon,
\[ r_{bh} \ll H_6^{-1}, \] (3.9)

it can be shown \([39]\) that the black holes are nearly extreme, \( M \approx Q \), with a Schwarzschild radius \( r_{bh} \approx Q^{1/3} \).

Dust particles initially at rest in the metric (3.7) exhibit two types of behavior. For sufficiently small values of \( r < r_* \), particles are captured by the black hole, while for \( r > r_* \) they are driven away from the black hole by the cosmological expansion. The critical value \( r_* \) can be found from
\[ f'(r_*) = 0. \] (3.10)

In the limit (3.9), we have
\[ r_* \approx (3Q/H_6^2)^{1/5}. \] (3.11)

With \( D_j = 5 \), the geometric factor is
\[ v_j \approx \frac{8}{15} \pi^2 (H_j r_*)^6. \] (3.12)

This expression is not exact, since comoving geodesics in the parent vacuum will not be at rest in the coordinates (3.7), but it will give a good approximation when (3.9) is satisfied.

In the case of \( dS_6 \rightarrow dS_4 \times S_2 \) transitions, the metric of nucleating black branes is
\[ ds^2 = B^2(r)[-dt^2 + \cosh^2 t \, d\Omega_2^2] + dr^2 + \rho^2(r)d\Omega_2^2. \] (3.13)

The functions \( B(r) \) and \( \rho(r) \) are not known analytically, but have been found numerically in ref. \([39]\). Once again, the critical radius \( r_* \) below which the geodesics are captured by the brane can be found from the condition
\[ B'(r_*) = 0, \] (3.14)

and in the limit of \( r_*, H_j \ll 1 \) we have
\[ v_j \approx (4\pi)(4\pi r_*^2 H_j^3/3). \] (3.15)

Here, the first factor comes from the area of the horizon-size 2-brane, and the second factor is the fraction of the horizon volume occupied by the captured particles in the three dimensions transverse to the brane.\(^3\)

One complication of the transdimensional case is that the volumes \( V_{ij} \) occupied by different regions are no longer simply related to the comoving volume fraction \( f_{ij} \). The

\(^3\) We note that it follows from the analysis in ref.'s \([38, 39]\) that in order to have a positive-energy vacuum inside the nucleated black brane, the magnetic charge of the brane should be of the order \( Q_m \sim H_6^{-1} \). This implies that the black brane and cosmological horizons are comparable to one another, and therefore \( r_* \sim H_6^{-1} \). Hence, in this case the geometric factor is \( v_j \sim 1 \).
reason is that a given geodesic generally crosses regions with different values of $D_j$ and is thus exposed to different expansion rates at different times in its history. The rate equation for the physical volumes $V_{ij}$ on the constant-$\eta$ surfaces can be written as

$$\frac{dV_{ij}}{d\eta} = -\kappa_i V_{ij} + \kappa_{ij} V_j + D_i V_{ij}, \quad (3.16)$$

where we remind the reader that $\kappa_i = \sum_k \kappa_{ki}$ is the total decay rate out of vacuum $i$, $V_j = \sum_k V_{jk}$ is the physical volume fraction of type $j$ vacua, reached from all possible type $k$ parent vacua, and $D_i$ is simply the number of large inflating directions in vacuum $i$. Here, the volumes $V_{ij}$ should be understood in the higher-dimensional sense, that is, including both large and compact dimensions. With the ansatz

$$V_{ij} = V_0 s_{ij} e^{\gamma \eta}, \quad (3.17)$$

this can be rewritten as

$$(\gamma - D_i + \kappa_i) s_{ij} = \kappa_{ij} s_j, \quad (3.18)$$

where

$$s_j = \sum_k s_{jk}. \quad (3.19)$$

The right-hand side of (3.18) must be positive, since $s_j > 0$ for inflating vacua. If $i$ is also an inflating vacuum, we should have $s_{ij} > 0$ and

$$D_i - \kappa_i < \gamma. \quad (3.20)$$

Now, if we define

$$\gamma \equiv D_{\text{max}} - q, \quad (3.21)$$

where $D_{\text{max}}$ is the largest dimension of an inflating vacuum in the landscape, then

$$q < D_{\text{max}} - D_i + \kappa_i. \quad (3.22)$$

The bound in eq. (3.22) must be satisfied for all type $i$ vacua. The tightest constraint occurs when $D_i = D_{\text{max}}$, from which it follows that

$$q < \kappa_{\text{min}}(D_{\text{max}}), \quad (3.23)$$

where $\kappa_{\text{min}}(D_{\text{max}})$ is the smallest decay rate among the vacua of effective dimension $D_{\text{max}}$. As before, we expect $\kappa_{\text{min}}(D_{\text{max}})$ to be very small, and thus

$$\gamma \approx D_{\text{max}}. \quad (3.24)$$

\textsuperscript{4}Eq. (3.16) assumes that transitions between different vacua occur with conservation of volume, so the total volume is changed only due to cosmological expansion and due to volume loss to terminal bubbles. This would be so for a continuous evolution of the geodesic congruence, but is not obviously true for quantum transitions between the vacua. This issue requires further study. Here we note that accounting for a possible discontinuous volume change in the course of transitions could introduce an additional factor multiplying $\kappa_{ij}$ in (3.16). This would result in a renormalization of $\kappa_{ij}$ and would not change our general conclusions.

\textsuperscript{5}This derivation of eq. (3.23) follows closely the derivation of eq. (2.22) in ref. [33].
3.3.2 Volume factor time

The volume-factor-time rate equation can be written as

$$\frac{df_{ij}}{dT} = -\tilde{\kappa}_i f_{ij} + \tilde{\kappa}_{ij} f_j,$$

where the transition rates per unit time $T$ are given by

$$\tilde{\kappa}_{ij} = \frac{\kappa_{ij}}{D_j}.$$  \hspace{1cm} (3.26)

The asymptotic solution of this equation for non-terminal bubbles $ij$ is

$$f_{ij} \approx \tilde{s}_{ij} e^{-\tilde{q}T},$$  \hspace{1cm} (3.27)

where $-\tilde{q}$ is the smallest (by magnitude) eigenvalue of the matrix

$$\tilde{M}_{ij} = \tilde{\kappa}_{ij} - \delta_{ij} \tilde{\kappa}_i.$$  \hspace{1cm} (3.28)

Substitution of (3.27) into (3.25) gives the relation

$$(\tilde{\kappa}_i - \tilde{q}) \tilde{s}_{ij} = \tilde{\kappa}_{ij} \tilde{s}_j,$$

and it follows, since the right hand side of (3.29) must be positive, that $\tilde{q}$ is smaller than the decay rate per unit volume factor time $T$ of any inflating vacuum,

$$\tilde{q} < \tilde{\kappa}_{\text{min}}.$$  \hspace{1cm} (3.30)

Unlike the scale factor case, the volumes occupied by different vacua on the constant-$T$ surfaces are simply related to the comoving volume fractions,

$$V_{ij}(T) = V_0 f_{ij}(T)e^T = V_0 \tilde{s}_{ij} e^{\tilde{\gamma}T},$$

where

$$\tilde{\gamma} = 1 - \tilde{q} \approx 1.$$  \hspace{1cm} (3.32)

3.4 The number of observers

3.4.1 Scale factor cutoff

The number of observers can now be evaluated along the same lines as in section IIB. For the scale factor cutoff, the rate of volume increase in type-$ij$ bubbles due to new bubble formation is

$$d^+ V_{ij} = V_0 \kappa_{ij} s_j e^{\gamma \eta} d\eta,$$

and the total volume of “observation hypersurfaces” in $ij$ bubbles is given by

$$V^\text{obs}_{ij} = V_0 e^{D_i \Delta \eta^\text{obs}_{ij}} e^{-\lambda_{ij} \Delta \eta^\text{obs}_{ij}} \int_0^{\eta^\text{max} - \Delta \eta^\text{obs}_{ij}} \kappa_{ij} s_j e^{\gamma \eta} d\eta$$

$$= \frac{1}{\gamma} V_0 \kappa_{ij} s_j e^{\gamma \eta^\text{max} - (D_i - \gamma - \lambda_{ij}) \Delta \eta^\text{obs}_{ij}}.$$  \hspace{1cm} (3.34)

Dropping irrelevant constant factors, multiplying by the density of observers $n^\text{obs}_{ij}$ and using eq. (3.21), we obtain for the total number of observers

$$N^\text{obs}_{ij} \propto \kappa_{ij} s_j n^\text{obs}_{ij} e^{(D_i - D^\text{max} + q - \lambda_{ij}) \Delta \eta^\text{obs}_{ij}}.$$  \hspace{1cm} (3.36)

Note that here the density $n^\text{obs}_{ij}$ should be understood in a higher-dimensional sense, so that $n^\text{obs}_{ij} \propto 1/V^\text{comp}_i$, where $V^\text{comp}_i$ is the volume of compact dimensions in vacuum $i$. 

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3.4.2 Volume factor cutoff

Similarly, for the volume factor cutoff we find

\[ d^+ V_{ij} = V_0 \tilde{\kappa}_{ij} \tilde{s}_j e^{\tilde{\gamma} T} dT, \]  
\[ V_{ij}^{\text{obs}} = V_0 e^{\Delta T_{ij}^{\text{obs}}} e^{-\tilde{\lambda}_{ij} \Delta T_{ij}^{\text{obs}}} \int_0^{T_{\max} - \Delta T_{ij}^{\text{obs}}} \tilde{\kappa}_{ij} \tilde{s}_j e^{\tilde{\gamma} T} dT, \]  
\[ = \frac{1}{\tilde{\gamma}} V_0 \tilde{\kappa}_{ij} \tilde{s}_j e^{\tilde{\gamma} T_{\max}} e^{(\tilde{q} - \tilde{\lambda}_{ij}) \Delta T_{ij}^{\text{obs}}}, \]

and finally

\[ N_{ij}^{\text{obs}} \propto \tilde{\kappa}_{ij} \tilde{s}_j e^{(\tilde{q} - \tilde{\lambda}_{ij}) \Delta T_{ij}^{\text{obs}}}. \]

Here,

\[ \tilde{\lambda}_{ij} = \lambda_{ij} / D_i \]  
\[ \Delta T_{ij}^{\text{obs}} \approx D_i \Delta \eta_{ij}^{\text{obs}} \]

is the volume factor time it takes for observers to evolve in type-ij bubbles.

4 Properties of scale factor and volume factor measures

4.1 Number of e-folds of inflation and the density parameter $\Omega$

Suppose we have a landscape which includes bubbles (or black branes) with comparable nucleation rates and similar post-inflationary evolution, but different inflaton potentials. The bubbles then have different numbers $I_{ij}$ of e-folds of slow-roll inflation and different values for the density parameter $\Omega$ (measured at the same value of the CMB temperature, $T_{CMB} = 2.7 K$).\(^6\) We want to investigate the effect of these differences on the number of observers, $N$, regulated with scale factor and volume factor cutoffs.

The scale factor time $\Delta \eta_{ij}^{\text{obs}}$, which appears in eq. (3.36) for the number of observers, can be expressed as

\[ \Delta \eta_{ij}^{\text{obs}} \approx I_{ij} + \Delta \eta_{ij}^{\prime}, \]  
\[ N_{ij}^{\text{obs}} \propto e^{(D_i - D_{\max} + q - \lambda_{ij}) I_{ij}}. \]

For an equidimensional landscape with $D_i = D_{\max} = 3$, eq. (4.2) gives

\[ N_{ij}^{\text{obs}} \propto e^{(q - \lambda_{ij}) I_{ij}} \approx 1. \]

Since $q$ and $\lambda_{ij}$ are both extremely small, there is no selection effect favoring small or large values of $I_{ij}$ [26, 44]. What has happened here is that there has been an almost precise cancellation of two effects: the volume growth inside bubbles, and the decrease in the number of nucleated bubbles as we go back in time.

\[^6\]The primordial density contrast $Q$ will also generally vary. To simplify the analysis, here we consider only the section of the landscape where $Q$ is fixed to its observed value in our part of the universe. We have verified, using some simple models, that inclusion of variation of $Q$ does not significantly alter our conclusions.
Apart from the explicit dependence on $I_{ij}$ shown in eq. (4.2), $N_{ij}^{\text{obs}}$ also depends on $I_{ij}$ through the anthropic factor $n_{ij}^{\text{obs}}$ and through the “prior” distribution

$$P_{\text{prior}} \propto \kappa_{ij} s_j.$$  \hspace{1cm} (4.4)

Since a long slow-roll inflation requires fine-tuning, one can expect that $P_{\text{prior}}(I)$ is a decreasing function of $I$. Analysis of a simple landscape model in ref. [45] suggests a power-law dependence,

$$dP_{\text{prior}} \propto I^{-\alpha} dI$$ \hspace{1cm} (4.5)

with $\alpha = 4$.

The dependence of the anthropic factor $n_{ij}^{\text{obs}}$ on $I$ has been discussed in refs. [17, 44, 45]. This dependence is more conveniently expressed in terms of the density parameter $\Omega$, which is related to $I$ as

$$1 - \Omega = e^{2(C - I)}$$ \hspace{1cm} (4.6)

The value of the constant $C$ depends on the duration of the inflaton-dominated period, which may occur between the end of inflation and the onset of thermalization, and on the thermalization temperature $T_{th}$. Assuming instant thermalization at a GUT-scale temperature $T_{th} \sim 10^{15}$ GeV, we have $C \approx 61.5$. In the opposite extreme of low thermalization temperature, $T_{th} \sim 10^3$ GeV, $C \approx 34$.

The anthropic significance of $\Omega$ is due to its effect on structure formation. For $\Omega$ very close to 1, $(1 - \Omega) \ll 1$, this effect is negligible. For $\Omega$ significantly less than 1, the growth of density perturbations terminates at redshift $(1 + z_{\Omega}) \approx \Omega^{-1}$. For $z_{\Omega} \gg 1$ this strongly interferes with structure formation. The density of galaxies, and the density of observers, decreases somewhat with $\Omega$ in the range $0.9 \gtrsim \Omega \gtrsim 0.4$ and drops rapidly towards zero at $\Omega \lesssim \Omega_c \approx 0.3$ [44]. This behavior can be roughly accounted for by imposing a sharp cutoff at $\Omega_c \approx 0.3$. Using eq. (4.6), this translates into a cutoff on $I$ at $I \approx C - \frac{1}{2} \ln(0.7) \approx C$. The density of observers then effectively becomes a step function,

$$n_{ij}^{\text{obs}} \propto \theta(I - C).$$ \hspace{1cm} (4.7)

With the prior from eq. (4.5), the full distribution for $I$ is given by

$$dP(I) = (\alpha - 1)C^{\alpha - 1} \theta(I - C) I^{-\alpha} dI,$$ \hspace{1cm} (4.8)

where the prefactor is chosen so that the distribution is normalized to 1.

Observations indicate that in our part of the universe $|1 - \Omega| \lesssim 0.01$, which corresponds to

$$I > I_0 = C + \ln 10$$ \hspace{1cm} (4.9)

According to (4.8), the probability of this observation is

$$P(I > I_0) = (C/I_0)^{\alpha - 1}.$$ \hspace{1cm} (4.10)

Since $I_0$ in eq. (4.9) is only slightly greater than $C$, this probability is close to 1, as long as $\alpha$ is not very large (which is certainly the case for $\alpha = 4$). Hence, the observed value of $\Omega$ is fully consistent with the equidimensional scale factor measure [44].

The situation is very different in a transdimensional landscape. Omitting the exponentially small terms in the exponent of eq. (4.2), we have

$$N_{ij}^{\text{obs}} \propto e^{-(D_{\text{max}} - D_i)I_{ij}}.$$ \hspace{1cm} (4.11)
This shows that large amounts of slow-roll inflation are strongly disfavored in all bubbles with $D_i < D_{\text{max}}$. For example, in the case of $(3 + 1)$-dimensional bubbles or branes ($D_i = 3$) in our 6d Einstein-Maxwell theory ($D_{\text{max}} = 5$), $N_{ij}^{\text{obs}} \propto e^{-2I_{ij}}$. In string theory landscape, we could have $D_{\text{max}} = 9$ and an even stronger suppression, $N_{ij}^{\text{obs}} \propto e^{-6I_{ij}}$.

The reason is as follows: bubbles which undergo less e-folds of inflation before being able to produce observers get rewarded by being more plentiful, but undergo less expansion than other bubbles which have a longer “gestation” period. These are the same two competing effects we discussed above for regular $(3 + 1)d$ inflation. The difference here, in the transdimensional case, is that the internal volume reward (which grows like $e^{D_i I_{ij}}$) is not enough to compensate for the lower number of bubbles which nucleate at earlier times ($\propto e^{-D_{\text{max}}I_{ij}}$).

Using eq. (4.7) for $n^{\text{obs}}$ and disregarding the power-law prior compared to the steep dependence on $I$ in (4.11), we obtain

$$dP = (D_{\text{max}} - D)\theta(I - C)e^{-(D_{\text{max}} - D)(I - C)}dI.$$  \hspace{1cm} (4.12)

The probability for the observed bound on $\Omega$ is then

$$P(I > I_0) = e^{-(D_{\text{max}} - 3)(I_0 - C)} = 10^{-(D_{\text{max}} - 3)},$$  \hspace{1cm} (4.13)

where we have set $D = 3$ and used eq. (4.9) in the last step. For $D_{\text{max}} > 5$, this probability is smaller than $10^{-3}$, so the transdimensional scale factor measure can be ruled out at a high confidence level.

Turning now to the volume factor measure, we have from eqs. (4.1), (3.42)

$$\Delta T_{ij}^{\text{obs}} \approx D_i(I_{ij} + \Delta \eta'_{ij}),$$  \hspace{1cm} (4.14)

and eq. (3.40) for the number of observers gives

$$N_{ij}^{\text{obs}} \propto e^{(\tilde{q} - \tilde{\lambda}_{ij})D_i I_{ij}} \approx 1.$$  \hspace{1cm} (4.15)

This is essentially the same as eq. (4.3). There is no strong selection for large or small values of $I$, so the same analysis as we gave above for the equidimensional case should apply here. Thus, we conclude that the transdimensional volume factor measure is consistent with observations of $\Omega$.

### 4.2 Youngness bias

Another pitfall that a candidate measure needs to avoid is the “youngness paradox” \cite{46,47}. Consider two kinds of observers who live in the same type $ij$ of bubbles, but take different time $\Delta \eta'_{ij}$ to evolve after the end of inflation. Observers who take less time to evolve can live in bubbles that nucleate closer to the cutoff and thus have more volume available to them. With proper time slicing, this growth of volume is so fast that observers evolving even a tiny bit faster are rewarded by a huge volume factor, resulting in some bizarre predictions \cite{35,46–48}. On the other hand, the scale factor cutoff in $(3 + 1)d$ gives only a mild youngness bias \cite{26},

$$N_{ij}^{\text{obs}} \propto e^{-(3-q+\lambda_{ij})\Delta \eta'_{ij}} \approx e^{-3\Delta \eta'_{ij}},$$  \hspace{1cm} (4.16)

so observers evolving faster by a scale factor time $\delta \eta$ are rewarded by a bias factor

$$B \approx e^{3\delta \eta}.$$  \hspace{1cm} (4.17)
The generalization of eqs. (4.16) and (4.17) to a transdimensional landscape can be read off from eq. (3.36):

\[ N_{ij}^{\text{obs}} \propto e^{-(D_{\text{max}} - q + \lambda_{ij}) \Delta \eta_{ij}' \approx e^{-D_{\text{max}} \Delta \eta_{ij}'}, \]  

(4.18)

and

\[ B \approx e^{D_{\text{max}} \delta \eta}. \]  

(4.19)

Note that here, and in eq. (4.16) above, we omitted the factor \( \exp(D_{i} \Delta \eta_{ij}') \) which accounts for the growth of volume inside bubbles in the time interval \( \Delta \eta_{ij}' \). The reason is that post-inflationary expansion only dilutes matter (e.g., galaxies) and does not by itself result in any increase in the number of observers.\(^7\) In string theory landscape we could have \( D_{\text{max}} = 9 \), so the youngness bias (4.19) is stronger than in the equidimensional \((3 + 1)d\) case, but it is still mild enough to avoid any paradoxes.

Similarly, for the transdimensional volume factor cutoff we have

\[ N_{ij}^{\text{obs}} \propto e^{-(1 - \tilde{q} + \tilde{\lambda}_{ij}) D_{i} \Delta \eta_{ij}' \approx e^{-D_{i} \Delta \eta_{ij}'}, \]  

(4.20)

and

\[ B \approx e^{D_{i} \delta \eta}. \]  

(4.21)

For \((3 + 1)\)-dimensional regions this gives \( B \approx \exp(3 \delta \eta) \), the same as in the equidimensional case.

4.3 Boltzmann brains

Yet another potential problem for the measure is an excessive production of Boltzmann brains (BBs) — freak observers spontaneously nucleating as de Sitter vacuum fluctuations [49–54]. Measures predicting that BBs greatly outnumber ordinary observers should be ruled out.

The total number of BBs nucleated in type-ij bubbles can be estimated as

\[ N_{ij}^{\text{BB}} = \Gamma_{i}^{\text{BB}} \Omega_{ij}, \]  

(4.22)

where \( \Omega_{ij} \) is the total spacetime volume in such bubbles and \( \Gamma_{i}^{\text{BB}} \) is the BB nucleation rate. Here we disregard possible time variation of the nucleation rate during the early evolution in bubble interiors. The bulk of BB production tends to occur at late stages, after vacuum domination. The nucleation rate is then constant and depends only on the vacuum inside the bubble, not on the parent vacuum.

The spacetime volume \( \Omega_{ij} \) can be found by integrating the spatial volume over the proper time \( \tau \). With a scale factor cutoff, we find

\[ \Omega_{ij} = \int V_{ij} d\tau = H_{i}^{-1} \int_{\eta_{\text{max}}}^{\eta} V_{ij}(\eta)d\eta = \frac{V_{0}}{\gamma H_{i}} s_{ij} e^{\gamma \eta_{\text{max}}}, \]  

(4.23)

with \( \gamma \) from eq. (3.21). Combining this with eqs. (4.22) and (3.36), we obtain for the relative number of BBs and normal observers

\[ \frac{N_{ij}^{\text{BB}}}{N_{ij}^{\text{obs}}} \approx \frac{\Gamma_{i}^{\text{BB}}}{H_{i}} \frac{s_{ij}}{\kappa_{ij} s_{ij} r_{ij}^{\text{obs}}} e^{(D_{\text{max}} - D_{i}) \Delta \eta_{ij}'^{\text{obs}}} \]  

\[ \approx \frac{\kappa_{i}^{\text{BB}} H_{i}^{2} e^{(D_{\text{max}} - D_{i}) \Delta \eta_{ij}'^{\text{obs}}} \approx (D_{\text{max}} - D_{i} - q + \kappa_{i}) n_{ij}^{\text{obs}}, \]  

(4.24)

\[ \approx (D_{\text{max}} - D_{i} - q + \kappa_{i}) n_{ij}^{\text{obs}}. \]  

(4.25)

\(^7\)This means that the density of observers that we used in the preceding sections should be \( n_{ij}^{\text{obs}} \propto \exp(-D_{i} \Delta \eta_{ij}'). \) This relation accounts only for dilution of galaxies with cosmic time, but in fact some new galaxies will also be formed, so the dependence of \( n_{ij}^{\text{obs}} \) on \( \Delta \eta_{ij}' \) will be more complicated. This effect is unimportant in the context of the youngness paradox, so we disregard it here.
where

$$\kappa_i^{BB} = \Gamma_i^{BB} \frac{4\pi}{3} H_i^{-4}$$

(4.26)
is the dimensionless BB nucleation rate and we have used eq. (3.18) in the last step.

In an equidimensional landscape with $D_i = D_{\text{max}}$, eq. (4.25) gives

$$\frac{N_{ij}^{BB}}{N_{ij}^{\text{obs}}} \sim \frac{\kappa_i^{BB}}{\kappa_i} \frac{H_i^3}{n_{ij}^{\text{obs}}}$$

(4.27)
The rate of BB nucleation $\kappa_i^{BB}$ is extremely small, with estimates ranging from $\exp(-10^{16})$ to $\exp(-10^{120})$ [27, 33]. Compared to this, most quantities which can be large or small by ordinary standards, can be regarded as $O(1)$. In particular, this applies to the number of observers per horizon volume, $n_{ij}^{\text{obs}} H_i^{-3}$ [27, 33]. Hence,

$$\frac{N_{ij}^{BB}}{N_{ij}^{\text{obs}}} \sim \frac{\kappa_i^{BB}}{\kappa_i},$$

(4.28)
so BB domination in vacuum $i$ is avoided only if the vacuum decay rate is greater than the BB nucleation rate,

$$\kappa_i \gtrsim \kappa_i^{BB}.$$  

(4.29)
In particular, this condition should be satisfied by our vacuum [15].

Even if ordinary observers can evolve only in our vacuum, BBs might be able to nucleate in other vacua as well. For example, vacua having the same low-energy physics and differ from ours only by the value of the cosmological constant $\Lambda$ can nucleate human-like observers, as long as $\Lambda$ is small enough, so that the observers are not torn apart by the gravitational tidal forces. General conditions for the avoidance of BB domination have been discussed in refs. [27, 33]. The main condition is that eq. (4.29) should be satisfied for any vacuum $i$ that can support Boltzmann brains. This is a rather non-trivial condition. Vacuum decay rates can be extremely low, especially for vacua with a low supersymmetry breaking scale. Hence, the equidimensional scale factor measure may or may not be consistent with observations, depending on the details of the landscape.

In the transdimensional case, for vacua with $D_i < D_{\text{max}}$, eq. (4.25) gives

$$\frac{N_{ij}^{BB}}{N_{ij}^{\text{obs}}} \sim \kappa_i^{BB} \ll 1,$$

(4.30)
where, following [27, 33], we have set $e^{(D_{\text{max}}-D_i)\Delta n_{ij}^{\text{obs}}} \sim 1$. Thus, the Boltzmann brain problem of the scale factor measure disappears in the transdimensional case, even if it existed in the equidimensional landscape.

For the volume factor cutoff, eqs. (4.23) and (4.25) are replaced by

$$\Omega_{ij} = \frac{1}{D_i H_i} \int_{T_{\text{max}}}^{T_{ij}(T)} V_{ij}(T)dT = \frac{V_0}{\gamma D_i H_i} \bar{s}_{ij} e^{\gamma T_{\text{max}}},$$

(4.31)
and

$$\frac{N_{ij}^{BB}}{N_{ij}^{\text{obs}}} \approx \frac{\kappa_i^{BB} H_i^3 e^{(\bar{s}_{ij}-\bar{q})\Delta T_{ij}^{\text{obs}}}}{D_i n_{ij}^{\text{obs}} \kappa_i} \sim \kappa_i^{BB}.$$  

(4.32)
This is essentially the same as eq. (4.28) for the equidimensional scale factor measure. Hence, we expect the conditions for BB avoidance in a transdimensional landscape with a volume factor cutoff to be the same as derived in [27, 33] for the scale factor measure in $(3 + 1)d$.  

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5 Conclusions

In this paper we investigated possible generalizations of the scale factor measure to a trans-dimensional multiverse. We considered two such generalizations: a straightforward extension of the scale factor cutoff and the volume factor cutoff. Here we briefly summarize and discuss our results.

We found that the transdimensional scale factor measure is not subject to the youngness or Boltzmann brain paradoxes. One of the key features of this measure is that it exponentially disfavors large amounts of slow-roll inflation inside the bubbles. This results in preference for low values for the density parameter $\Omega$, while observations indicate that $\Omega$ is actually very close to $1$, 

$$|1 - \Omega_0| \lesssim 0.01. \quad (5.1)$$

The severity of the problem depends on the highest dimension $D_{\text{max}}$ of an inflating vacuum in the landscape. The probability for the bound (5.1) to hold is 

$$\mathcal{P}_0 \sim 10^{D_{\text{max}} - 3}. \quad (5.2)$$

For $D_{\text{max}} = 4$ this gives a reasonably large probability $\mathcal{P}_0 \sim 0.1$. If indeed we live in a multiverse with $D_{\text{max}} = 4$, then this measure makes a strong prediction that a negative curvature will be detected when the measurement accuracy is improved by another order of magnitude. On the other hand, if $D_{\text{max}} > 5$, as can be expected in string theory landscape, we have $\mathcal{P}_0 \lesssim 10^{-3}$, and the scale factor measure is observationally ruled out at a high confidence level.

One potential pitfall for the measure that we have not discussed in the body of the paper is the runaway problem, also known as $Q$ or $G$ catastrophe. This problem arises in measures that exponentially favor large amounts of inflationary expansion. The primordial density contrast $Q$ and the gravitational constant $G$ are strongly correlated with the number of e-folds of inflation $I$, and an exponential probability distribution for $I$ indicates that the distributions for $Q$ and $G$ should also be very steep, driving their expected magnitudes to extreme values [55–57]. These problems are often referred to as $Q$ and $G$ “catastrophes”. In our case, large inflation is exponentially disfavored, and the situation is different. The runaway toward small values of $I$ is stopped by the anthropic bound on $\Omega$. The resulting prediction for $\Omega$ may be too low, as described above. But if the distribution for $\Omega$ is acceptable, as it could be for $D_{\text{max}} = 4$, then $Q$ and $G$ catastrophes may also be avoided. This would happen if the distribution is cut off by the anthropic bound on $\Omega$ before it gets into the dangerous range of $Q$ or $G$. (Note that $\Omega$ is much more sensitive to the value of $I$ than $Q$ or $G$.)

The properties of the volume factor measure are essentially the same as those of the scale factor measure in $(3+1)$ dimensions. It is free from the $Q$ and $G$ catastrophes and from the youngness paradox. It may or may not have a Boltzmann brain (BB) problem, depending on the properties of the landscape. The conditions for BB avoidance for this measure are the same as found in refs. [27, 33] in the $(3 + 1)d$ case, the main condition being that the rate of BB nucleation in any vacuum should be smaller than the decay rate of that vacuum. Whether or not this condition is satisfied in the string theory landscape is presently an open question. The answer depends both on the unknown properties of the landscape and on our understanding of what exactly constitutes a Boltzmann brain. In any case, at our present state of knowledge, the volume factor measure is certainly not ruled out by the BB problem.

Our goal in this paper was to analyze and compare the properties of the scale factor and volume factor measures in a transdimensional multiverse. Our results suggest that the
volume factor cutoff is a more promising measure candidate. It avoids all potential measure problems, with the judgement still pending on the BB paradox. On the other hand, we found that the scale factor measure suffers from a serious problem, predicting low values for the density parameter $\Omega$. This problem becomes acceptably mild only if the highest dimension of the inflating vacua in the landscape is $D_{\text{max}} \leq 4$.

We note finally the recent work [58–60] which used holographic ideas for motivating the scale factor measure in $(3 + 1)d$. The proposal is that the dynamics of an eternally inflating universe has a dual representation in the form of Euclidean field theory defined at future infinity. An ultraviolet cutoff in this boundary theory then corresponds to a scale factor cutoff in the bulk. It would be interesting to see whether or not these ideas can be extended to a transdimensional multiverse.

Acknowledgments

We are grateful to Jaume Garriga for discussions and some crucial suggestions. We also benefited from very helpful discussions with Jose Blanco-Pillado and Michael Salem. This work was supported in part by grant PHY-0855447 from the National Science Foundation and by grant RFP2-08-30 from the Foundational Questions Institute (fqxi.org).

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