PBW PROPERTY FOR UNIVERSAL ENVELOPING ALGEBRAS OVER AN OPERAD

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Abstract. Given a symmetric operad \( P \) and a \( P \)-algebra \( V \), the universal enveloping algebra \( U_P \) is an associative algebra whose category of modules is isomorphic to the abelian category of \( V \)-modules. We study the notion of PBW property for universal enveloping algebras over an operad. In case \( P \) is Koszul a criterion for PBW property is found. Necessary condition on Hilbert series for \( P \) is found. Moreover, given any symmetric operad \( P \) together with a Gröbner basis \( G \), a condition is given on the structure of the underlying trees associated with leading monomials of \( G \) sufficient for the PBW property to hold. Examples are provided.

0. Introduction

Let \( P \) be a symmetric operad and \( V \) be a \( P \)-algebra. The natural notion of a module \( M \) over a \( P \)-algebra \( V \) was suggested in the beginning of the epoch of operads (see e.g. [11] §1.6). At the same time a definition of a universal enveloping algebra \( U_P(V) \) in terms of operadic trees was suggested (e.g. [11] Definition 1.6.4,[13]). On the other hand the associative algebra \( U_P(V) \) is defined by universal property that the abelian category of \( V \) isomorphic to the category of left modules over \( U_P(V) \). For example, one can easily verify that for a Lie algebra \( g \) (viewed as a \( P \)-algebra for the symmetric operad \( P = \text{Lie} \)) the operadic universal enveloping algebra \( U_{\text{Lie}}(g) \) coincides with the ordinary universal enveloping algebra \( U(g) \) (see also Example [14] below). Every textbook on Lie algebras includes the statement known as Poincaré-Birkhoff-Witt Theorem:

The universal enveloping algebra \( U(g) \) and the symmetric algebra \( S(g) \)
are isomorphic as vector spaces.

Let \( g_0 \) be the trivial Lie algebra structure on the vector space \( g \) (i.e. all commutators on \( g_0 \) are set to be zero). The symmetric algebra \( S(g) \) coincides with the universal enveloping algebra \( U(g_0) \) and the PBW property can be interpreted as stating that \( U(g) \) and \( U(g_0) \) are isomorphic. Restricting to Koszul operads \( P \), we shall prove a criterion to select those operads whose universal enveloping functor

\[ U_P : \mathcal{P}\text{-algebras} \to \text{Associative algebras} \]

satisfies the PBW property. In other words, we are interested in necessary and sufficient conditions on the Koszul operad \( P \) that implies an isomorphism of vector spaces \( U_P(V) \) and \( U_P(V_0) \). Here \( V \) is a (nontrivial) \( P \)-algebra and \( V_0 \) is the corresponding trivial \( P \)-algebra; all nontrivial elements of \( P \) act on \( V_0 \) by multiplication by zero.

We rephrase the notion of the universal enveloping functor and the PBW property in terms of colored operads whose first color corresponds to the algebra structure and the second color corresponds to the module. In particular, the numbers of inputs and outputs of the second color for the corresponding colored operads have to be equal. The colored suboperad spanned by elements with an output of the second color consists of a collection of \( S_n \)-representations \( Q(n) := \text{Res}_{S_m}^{S_{m+1}} P(n + 1) \). The operadic composition through the second color defines a monoidal product that is well known for symmetric collections:

\[ (Q \circ Q')(n) := \bigoplus_{k+m=n} \text{Ind}_{S_k \times S_m}^{S_n} Q(k) \otimes Q'(m). \]

The corresponding product of characters is given by multiplication of symmetric functions:

\[ \chi_{S_{m+1}}(Q(m) \circ Q'(n)) = \chi_{S_m}(Q(m)) \cdot \chi_{S_n}(Q'(n)) \in \mathbb{Z}[x_1, \ldots, x_{m+n}]^{S_{m+1}}. \]

Monoids in the category of symmetric collections with respect to monoidal product \( \circ \) are known under the name permutad (after [17]). The category of permutads is a nice category that has a lot of common properties with the category of associative algebras. In particular, one can easily define a notion of a quadratic and Koszul permutad. Our first main result relates PBW with the Koszul property of certain operads and permutads:
Theorem (Theorem 4.1). The universal enveloping functor $U_P$ associated with a symmetric Koszul operad $P$ satisfies PBW if and only if the permutad associated with the corresponding Koszul dual operad $P^!$ is Koszul.

Moreover, we explain that the PBW property of $U_P$ for a Koszul operad $P$ implies that the corresponding universal enveloping algebra $U_P(V)$ is a nonhomogeneous Koszul algebra for any $P$-algebra $V$ (Corollary 4.2). This observation crucially simplifies the homology theory and homological complexes associated with the abelian category of $V$-modules. In particular, the corresponding derived functors for modules over $U_P(V)$ coincide in some cases with the homological theories suggested in [1, 10, 12].

We also find a relationship on the generating series of $S$-characters of $P$ and of $U_P$ whenever $U_P$ satisfies PBW (Theorem 4.7). Finally, we work out necessary definitions of Gröbner bases for colored operads under consideration and prove the main sufficient condition for PBW of $U_P$:

Theorem (Theorem 4.16). If a symmetric operad $P$ admits a Gröbner basis (in the sense of [8]) whose leading monomials are trees with no branches growing to the right from none of the vertices then the universal enveloping functor $U_P$ satisfies PBW.

We illustrate our theory on the following list of examples that satisfy PBW:
- operad $\text{Lie}$ of Lie algebras – Examples 1.9, 2.7
- operad $\text{Comm}$ of commutative algebras – Example 1.10
- operad $\text{As}$ of associative algebras – Examples 1.11, 2.7
- operad $\text{Lie}_2$ of pairs of compatible Lie brackets – Section 5.1
- Koszul dual operad to the operad $O_A$ – Example 4.6
- operad $\text{PreLie}$ of PreLie algebras – Section 5.2
- operad $\text{Zinb}$ of Zinbiel algebras – Section 5.6

and the following list of examples that does not satisfy PBW:
- operad $\text{Pois}$ of Poisson algebras – Example 4.10, Section 5.4
- operad $\text{Perm} = (\text{PreLie})^!$ of permutative algebras – Section 5.3
- operad $\text{Leib}$ of Leibniz algebras – Section 5.5

We hope that the methods we suggest below are enough to decide for almost all Koszul operads mentioned in [23] whether the corresponding universal enveloping functor satisfies PBW.

0.1. Structure of the paper. Section 4 provides several equivalent definitions of the universal enveloping algebra and the universal enveloping functor. In particular, we recall the pictorial definition suggested in [11], definition via adjoint functors and give the handmade description. Section 2 is devoted to different equivalent definitions of PBW property. Section 3 is the core section containing the homological algebra around the universal enveloping functor. In particular, we end up with the bar-cobar resolutions of this functor in Section 3.2. Section 4 contains corollaries of the preceding Section 3 that leads do different necessary and sufficient conditions of PBW property that are of the most interest for applications. Certain interesting and relatively simple examples are outlined in Section 5.

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Notation

For simplicity, we deal with vector spaces (algebras, operads e.t.c) over a given field $k$ of zero characteristic. However, most of the results remains to be valid for fields of positive characteristic.

We use a lot the free object $F$, the bar-construction $B$, the cobar-construction $\Omega$, the contravariant $P \mapsto P^!$ and covariant $P \mapsto P^\dagger$ Koszul duality functors for different categories: for operads, colored operads, permutads and algebras. We use the standard notations for the aforementioned functors that
coincide with the textbook on the operads \cite{18}. We hope that it is clear from the notation which category is considered. However, in order to highlight the underlying category we use the additional lower index.

In particular,
- $\Omega\Omega_{\mathcal{P}}(\mathcal{P})$ stands for the cobar construction of a symmetric operad $\mathcal{P}$;
- $\mathcal{B}_{1,20}\mathcal{P} = \mathcal{B}_{1,2}(\mathcal{P})$ denotes the bar construction of a colored cooperad $\mathcal{P}$ on two colors, such that the number of inputs and outputs of the second color in $\mathcal{P}$ coincide;
- $\mathcal{P}_{\text{perm}}$ stands for the Koszul dual co-permutad of a (quadratic) permutad $\mathcal{P}$;
- $A_{\text{Alg}} \mathcal{P}$ denotes the Koszul dual algebra associated with associative algebra $A$;
- $\mathcal{F}_{\text{Sh-Alg}}(A)$ denotes the free shuffle algebra generated by $A$.

1. Universal enveloping of an algebra over an operad

1.1. Classical pictorial definition by Ginzburg and Kapranov. While describing the notion of a module over an algebra over an operad $V$, Ginzburg and M. Kapranov introduced a notion of a universal enveloping algebra $U_{\mathcal{P}}(V)$ for an algebra $V$ over an operad $\mathcal{P}$.

**Definition 1.1.** (III Definition 1.6.4 on page 225) The universal enveloping algebra $U_{\mathcal{P}}(V)$ is spanned by elements:

$$\gamma \in \mathcal{P}(n+1), \quad a_i \in A$$

subject to relations valid for all $\sigma \in S_m \subset S_{m+1}, a_i, b_j \in A, \gamma, \delta \in \mathcal{P}$:

$$a_1 a_2 \ldots a_m = \gamma a_{(1)} a_{(2)} \ldots a_{(m)}$$

with the product defined by

$$a_1 a_2 \ldots a_m b_1 b_2 \ldots b_m = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_m$$

1.2. Definition via colored 1-2-operads. Let us rephrase Definition 1.1 of the universal enveloping algebra using the language of colored operads.

**Definition 1.2.** With each symmetric algebraic operad $\mathcal{P} = \cup_{n \geq 1} \mathcal{P}(n)$ one can associate a colored operad $\mathcal{P}^+$ on two colors $\{1, 2\}$ such that the space $\mathcal{P}^+(m, n)^c$ of operations with $m$ inputs of the first color $\{1\}$ and $n$ inputs of the second color $\{2\}$ and with the output of the color $c \in \{1, 2\}$ is defined in the following way

$$\mathcal{P}^+(m, n)^c = \begin{cases} \mathcal{P}(m) & \text{if } n = 0, c = 1; \\ \mathcal{P}(m + 1) & \text{if } n = 1, c = 2; \\ 0 & \text{otherwise.} \end{cases}$$

The composition rules are completely defined by the compositions in the symmetric operad $\mathcal{P}$.

Roughly speaking, the operations in $\mathcal{P}^+$ are just the operations of $\mathcal{P}$ with at most one of the inputs colored by the second color and in the latter case the output should be also of the second color. We will use the dotted arrows for the second color:

$$\mathcal{P}(n) \simeq \left\{ \begin{array}{ccc} \sigma(1) \sigma(2) \ldots \sigma(n) \\ \sigma(1) \sigma(2) \ldots \sigma(n-1) \end{array} \right\} \simeq \mathcal{P}^+(n-1, 1)^2 \simeq \left\{ \begin{array}{ccc} \sigma(1) \sigma(2) \ldots \sigma(n-1) \end{array} \right\}$$

**Definition 1.3.** The colored operads with two colors whose operations differs from zero only if the number of inputs and outputs of the second color coincide will be called **symmetric 1-2-operad**.
In particular, the colored operad $\mathcal{P}_+$ and the colored suboperad $\mathcal{P}^2_+$ with the output of the second color are the examples of the 1-2-operads.

Remark 1.4. An algebra over $\mathcal{P}_+$ is a pair $(A, M)$ with $A$ an algebra over $\mathcal{P}$ and $M$ a module over $A$. This is a rephrasing of the Definition suggested by Ginzburg and Kapranov in section 1.6 of [11].

The operad $\mathcal{P}$ can be considered as a 2-colored operad with an empty set of nonidentical operation involving the second color. Consider the natural embedding $\iota: \mathcal{P} \rightarrow \mathcal{P}_+$ as an embedding of 2-colored operad. The corresponding forgetful functor $F$ between the categories of algebras over these operads admits a left adjoint called $U_{\mathcal{P}}:

\begin{align*}
U_{\mathcal{P}} : & \mathcal{P}\text{-algebras} \times \text{Vector spaces} \rightarrow \mathcal{P}_+\text{-algebras} \\
& (V, M) \mapsto (V, P^2_+ \circ (V, M))
\end{align*}

where $P^2_+ \circ (V, M)$ is the coequalizer of the diagram

$P^2_+ \circ \mathcal{P} \circ (V, M) \rightarrow (P^2_+ \circ \mathcal{P}) \circ (V, M) \rightarrow P^2_+(A, M)$

The main object of this paper was introduced in [11] (Definition 1.6.4) and is outlined in the following

Definition 1.5. In particular, the functor $U_{\mathcal{P}}$ maps a pair $(V, k)$ to the pair $(V, U_{\mathcal{P}}(V))$ where $U_{\mathcal{P}}(V)$ is called the universal enveloping algebra of a $\mathcal{P}$-algebra $V$.

Statement 1.6. The composition through the second color defines an associative multiplication on $U_{\mathcal{P}}(V)$ and there is a canonical equivalence of abelian categories:

left $U_{\mathcal{P}}(V)$-modules $\simeq$ modules over the $\mathcal{P}$-algebra $V$

We refer to [19] where all definitions are given in a more systematic way.

1.3. Handmade description of the universal enveloping. Let us first mention the following simple fact

Proposition 1.7. The functor $+: \mathcal{P} \rightarrow \mathcal{P}_+$ is exact.

Proof. This functor is exact on the level of underlying vector spaces of $n$-ary operations since they are just doubled. □

Note that the functor $+: \mathcal{P} \rightarrow \mathcal{P}_+$ maps free operads to free operads. Therefore, starting from presentation of the operad $\mathcal{P}$ in terms of generators and relations one easily derives the presentation of the 1-2-operad $\mathcal{P}_+$. This leads to the handmade description of the universal enveloping (associative) algebra. For simplicity, we will consider the case of operads generated by binary operations subject to binary relations, but we hope the reader can easily generalize Proposition 1.8 to the arbitrary arity of generators and relations.

Proposition 1.8. Suppose that $\mathcal{P}$ is an operad generated by binary operations $\gamma_1, \ldots, \gamma_k$ subject to the following set of quadratic relations numbered by the upper index $s \in S$:

$\sum_{i,j=1}^{k} \begin{pmatrix}
a^t_{ij} & 1 & 2 & 3 & 1 & 3 & 2 & 3 \\
1 & 2 & 3 & 1 & 3 & 2 & 3 & 1
\end{pmatrix} = 0$

Then the universal enveloping algebra $U_{\mathcal{P}}(V)$ of a $\mathcal{P}$-algebra $V$ is generated by the $k$ copies of $V$ where the $i$-th embedding of $V$ into the set of generators is denoted by $v \mapsto (v)_i$ subject to the following set of (quadratic-linear) relations numbered by pairs of elements $v, w \in V$:

$\sum_{i,j=1}^{k} \left(a^t_{ij}(v)(w)_j + b^t_{ij}(v)(w)_j + c^t_{ij}(\gamma_j(v, w))_i \right) = 0$
Proof. The \( i \)-th copy of the set of generators is spanned by \( \gamma \). Note that we suppose that \( \gamma_1, \ldots, \gamma_k \) is a basis of \( \mathcal{P}(2) \) and in particular we have an \( S_2 \) action on it, which leads to the \( S_2 \) action on the set of generators:

\[
(12)(v)_i = (12) \begin{pmatrix} \gamma_i \\ v \end{pmatrix} = (12) \gamma_i v.
\]

The linear span of all relations form an \( S_3 \)-module. Therefore, following the idea of Proposition 1.8, while working with algebra over the operad \( \mathcal{P}_+ \) one just have to color the path between the first leaf and the root by the second color and substitute \( v \) and \( w \) instead of leaves 2 and 3 respectively:

\[
\sum_{i,j=1}^{k} \left( a_{ij}^s \gamma_i \gamma_j v + b_{ij}^s \gamma_i \gamma_j w + c_{ij}^s \gamma_i \gamma_j wv \right) = 0.
\]

\[\square\]

**Example 1.9.** The operad \( \text{Lie} \) is generated by a unique anti-symmetric generator called the Lie bracket subject to the following relation:

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} +
\begin{array}{c}
2 \\
3 \\
1
\end{array} +
\begin{array}{c}
3 \\
1 \\
2
\end{array} = \begin{array}{c}
1 \\
2 \\
3
\end{array} - \begin{array}{c}
2 \\
1 \\
3
\end{array} - \begin{array}{c}
3 \\
2 \\
1
\end{array} = 0
\]

The classical universal enveloping algebra of a Lie algebra \( g \) coincides with the operadic universal enveloping:

\[
U_{\text{Lie}}(g) = k \langle g \langle g \otimes h - h \otimes g - [g, h] \rangle
\]

**Example 1.10.** The operad \( \text{Comm} \) of commutative algebras (without the unit) is generated by a symmetric operation called multiplication, subject to the following relations:

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} = \begin{array}{c}
2 \\
1 \\
3
\end{array} = \begin{array}{c}
3 \\
2 \\
1
\end{array}
\]

The universal enveloping \( U_{\text{Comm}}(V) \simeq k \otimes V \).

**Example 1.11.** The symmetric operad \( \text{As} \) of associative algebras is generated by two binary operations. That is a multiplication \( \oplus(a, b) := ab \) and its opposite \( \ominus(a, b) := ba \). The associativity relation \( \ominus(\ominus(ab)c) = ab \ominus\ominus(ab) \) is considered as 6 linearly independent quadratic relations (each expression below is a pair of relation that differ by reversing all signs in operations):

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} = \begin{array}{c}
2 \\
1 \\
3
\end{array},
\begin{array}{c}
2 \\
3 \\
1
\end{array} = \begin{array}{c}
3 \\
2 \\
1
\end{array},
\begin{array}{c}
1 \\
3 \\
2
\end{array} = \begin{array}{c}
1 \\
2 \\
3
\end{array}.
\]

Thus the corresponding universal enveloping of an associative algebra \( V \) has the following presentation:

\[
U_{\text{As}}(V) := k \left( V \oplus V \right) \begin{array}{c}
(v)_{11} w_1 = (v'w)_1,
(v)_{12} w_2 = (w'_2) v_1,
(v)_{22} w_2 = (w_2 v)_2
\end{array} \simeq (k \otimes V) \otimes (k \otimes V^\text{op})
\]

Many more nontrivial examples can be find in Section 5 below.
Remark 1.13. As one can see from the Example 1.11 that in all examples we are dealing with algebras over operads with a unit. On the level of operads this means that we consider operads without 0-ary operations. The functor \( + : \mathcal{P} \to \mathcal{P}_+ \) can be easily generalized to the case of the operads with nontrivial 0-ary operations. However, it is worth mentioning that the universal enveloping of an algebra over an operad \( \mathcal{P} \) with and without 0-ary operation are different as one can see in the example of the operad \( \text{As} \) and the operad \( \text{Pois} \) considered below in Example 4.10 without 0-ary operations and in [22] with a unit element.

2. PBW-property

Let \( \mathcal{P} \) be a symmetric operad. Let \( J(\mathcal{P}) \subset \mathcal{P} \) be the augmentation ideal. I.e. \( \mathcal{P}/(J(\mathcal{P})) \) is isomorphic to the trivial one-dimensional operad spanned by identity element \( \mathbb{1} \in \mathcal{P}(1) \). For an operad given by generators and relations the ideal \( J(\mathcal{P}) \) is the ideal generated by the generators of \( \mathcal{P} \). For each vector space \( V \) we can define a trivial \( \mathcal{P} \)-structure on \( V \) by setting \( J(\mathcal{P}) \)-action to be zero. In other words, it is enough to say that for each generator \( \gamma \in J(\mathcal{P}(n)) \subset \mathcal{P} \) and each collection \( v_1, \ldots, v_n \in V \) we pose \( \gamma(v_1, \ldots, v_n) := 0 \). We will denote the corresponding \( \mathcal{P} \)-algebra by \( V_0 \).

Definition 2.1. We say that the universal enveloping functor \( U_\mathcal{P} \) satisfies a PBW-property iff for all \( \mathcal{P} \)-algebras \( V \) the universal enveloping algebras \( U_\mathcal{P}(V) \) and \( U_\mathcal{P}(V_0) \) are isomorphic as vector spaces.

For any \( \mathcal{P} \)-algebra \( V \) one can define a PBW-filtration on \( U_\mathcal{P}(V) \) by the powers of \( V \).

Proposition 2.2. The universal enveloping functor \( U_\mathcal{P} \) satisfies PBW if and only if for any \( \mathcal{P} \)-algebra \( V \) there is an isomorphism of associative algebras:

\[
\text{gr}^{PBW} U_\mathcal{P}(V) \simeq U_\mathcal{P}(V_0)
\]

Let as rephrase the PBW property in the language of deformation theory. Suppose that the operad \( \mathcal{P} \) is generated by elements of arity greater or equal than 2. Consider an automorphism of \( \mathcal{P} \) given by dilation \( h : \mathcal{P}(n) \to h^{-1} \mathcal{P}(n) \). For each \( \mathcal{P} \)-algebra \( V \) let us denote by \( V_h \) the corresponding twisted \( \mathcal{P} \)-algebra structure on \( V \).

Corollary 2.3. The universal enveloping functor \( U_\mathcal{P} \) satisfies PBW-property if and only if the algebras \( U_\mathcal{P}(V_h) \) defines a flat deformation over \( k[h] \) of the algebra \( U_\mathcal{P}(V_0) \).

Proof. The algebra \( U_\mathcal{P}(V_h)|_{h=q} \) is isomorphic to \( U_\mathcal{P}(V) \) for \( q \neq 0 \) and is isomorphic to \( U_\mathcal{P}(V_0) \) for \( h = 0 \).

Theorem 2.4. The universal enveloping functor \( U_\mathcal{P} \) has a PBW-property if and only if \( \mathcal{P}_+ \) is a free right \( \mathcal{P} \)-module.

Proof. This easy statement is a particular case of the general PBW property for the embedding of monads \( \phi : \mathcal{M} \to \mathcal{N} \) nicely presented in [3]. See, in particular, the main Theorem 2 of [3].

Corollary 2.5. The universal enveloping functor \( U_\mathcal{P} \) has a PBW-property if and only if and only if the right \( \mathcal{P} \)-module \( \mathcal{P}_+ \) is free and there is an isomorphism of \( S \)-collections:

\[
\mathcal{P}_+ \simeq U^0_\mathcal{P} \circ \mathcal{P}
\]

where \( U^0_\mathcal{P} \) is the \( S \)-collection assigned to the Schur functor \( V \mapsto U_\mathcal{P}(V_0) \).

The abbreviation PBW comes after the following famous Poincaré-Birkhoff-Witt theorem stated for Lie algebras

Example 2.6. The associated graded to the PBW filtration of the universal enveloping algebra \( U(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \) is isomorphic to the symmetric algebra \( S(\mathfrak{g}) \). In other words, \( U_{\text{Lie}} \) admits a PBW property.

Example 2.7. The functors \( U_{\text{As}} : V \mapsto \mathbb{k} \oplus V \otimes (\mathbb{k} \oplus V^\text{op}) \) and \( U_{\text{Comm}} : V \mapsto \mathbb{k} \oplus V \) satisfy the PBW property.

The preceding Lemma 2.8 explains that there exists a lot of operads that do not satisfy PBW and, in particular, koszulness of an operad \( \mathcal{P} \) does not imply the PBW property of \( U_\mathcal{P} \).

Lemma 2.8. The universal enveloping functor \( U_\mathcal{P} \) of a finite-dimensional operad \( \mathcal{P} \) does not satisfy the PBW property.

Proof. Let \( n \) be the maximal integer such that \( \mathcal{P}(n) \neq 0 \). Then \( \mathcal{P}_+(n, 1)^2 \simeq \mathcal{P}(n+1) = 0 \). Consequently, \( \mathcal{P}_+(n, 1)^2 \) may not be isomorphic to a free right \( \mathcal{P} \)-module \( Q \circ \mathcal{P} \).
3. Permuts and 1-2-operads $P_+$

3.1. Recollection on Permuts. Recall that a collection of (possible empty) representations of symmetric groups $\cup_{n \geq 1} Q(n)$ is called an symmetric collection (or an $S$-collection). The most standard tensor product on the $S$ collections that reflects the multiplication of symmetric functions is given by inductions of representations for symmetric groups:

$$(Q \otimes Q')(n) := \oplus_{k+m=n} Ind_{S_k \times S_m}^{S_n} Q(k) \otimes Q'(m)$$

**Definition 3.1.** ([17]) A permutad is a monoid in the category of symmetric collections with respect to the tensor product $\circ$. In other words, a permutad is a symmetric collection $Q$ together with an associative product $\mu : Q \circ Q \to Q$.

While choosing a particular representatives by shuffle permutations in the equivalence classes $S_{m+n}/(S_m \times S_n)$ V. Dotsenko and the author introduced the algebraic object named a *Shuffle algebra* in [7] which is equivalent to the aforementioned notion of a *Permutad* introduced by J-L. Loday and M. Ronco in [17]. We do not want to recall these notions in full details and refer the reader to the papers [7, 17] since we hope that the definition is clear from the perspective of the following:

**Statement 3.2.** To each permutad $Q$ one can assign a 1-2-operad (two-colored operad) $\tilde{Q}$ such that

$$\tilde{Q}(m, n)^c := \begin{cases} Q(m), & \text{if } n = 1, c = 2; \\ 0, & \text{otherwise} \end{cases}$$

the unique nontrivial composition in the 1-2-operad $\tilde{Q}(k, 1)^2 \circ \tilde{Q}(m, 1)^2 \to \tilde{Q}(k + m, 1)^2$ is defined by the multiplicity $\circ$ in the permutad and, moreover, this assignment defines a fully faithful embedding.

In particular, the permutad is a particular case of a colored operad. Consequently, the notion of a free permutad as well as the homological algebra of permutads is covered by the same knowledge for colored operads. However, the restriction on the number of inputs and outputs in the colored operad $\tilde{Q}$ dramatically simplifies the underlying algebraic structure as we will see later on from the examples. We give a special name to the *permudadic* (co)bar construction $B_{\text{perm}}(Q)$ of a permutad $Q$. Recall, that $B_{\text{perm}}(Q)$ is a free co-permutad generated by a homologically shifted collection $s Q(n) \otimes S_{gn}$, tensored with a sign representation similar to the case of operads ([11], [18]). Respectively, the cobar construction of a co-permutad $Q^*$ will be denoted by $\Omega_{\text{perm}}(Q^*)$.

We do not repeat in full details here and refer for Bar-cobar duality and Koszul duality for shuffle algebras to [7] and for permuts to [17].

**Example 3.3.** If $P$ is a symmetric operad then the collection $P_+^2 := \cup_{n \geq 0} P_+(n, 1)^2$ with the composition with respect to the second color assembles a permutad. The $S_n$ action on $P_+(n, 1)^2$ comes from the permuting the inputs of the first color and is isomorphic to the restriction representation $Res_{S_n}^{S_n+1} P(n + 1)$. We will denote the corresponding permutad by $P_{\text{perm}}^2$.

**Example 3.4.** If $P$ is a symmetric operad such that the universal enveloping functor $U_P$ satisfies PBW, then the $S$-collection $U_P^2 := P_+^2 \circ_P k$ is a permutad.

Note that the free permutad is much smaller than the free operad, thus the set of generators of the permutad $P_+^2$ is typically much bigger than the set of generators of a given operad $P$.

**Proposition 3.5.** Let $U \circ_P \to P_+^2$ be a resolution of the permutad $P_+^2$ in the category of right $P$-modules. Then for any $P$-algebra $V$ there exists a structure of associative dg-algebra on $U(V)$ and, moreover, one has a quasi-isomorphism of associative algebras $U(V) \to U_P(V)$.

3.2. Bar-cobar resolution of $P_+^2$ as a right $P$-module.

**Proposition 3.6.** The functor $+: P \to P_+$ commutes with Koszul duality:

$$(3.7) \quad \Omega_P(P^*)_+ \simeq \Omega_{1, 20}(P^*)$$

Here $\Omega_Q(-)$ is the cobar construction of a symmetric operad and $\Omega_{1, 20}(P)(-)$ denotes the cobar construction of an operad with two colors.

**Proof.** The operations with the first color as an output are the same for the left hand side and the right hand side of (3.7) and coincide with the ordinary bar-construction $\Omega_P(P^*)$. Operations with the second color involved looks similar. On the left hand side of (3.7) one has to consider an operadic tree generated by $P$ and then make a path from the first leaf to the output dotted. Respectively, for the right
Theorem 4.1. Let $\varphi : Q \rightarrow P$ be a quasiisomorphism of symmetric operads. Then for a (dg) $P$-algebra $A$ one has an isomorphism of dg associative algebras:

$$
\varphi : \mathcal{U}_Q(A) \rightarrow \mathcal{U}_P(A).
$$

In particular, $\mathcal{U}_{P_\infty}(A)$ is quasi-isomorphic to $\mathcal{U}_P(A)$, where $P_\infty$ is generated by elements of arity greater than 1 then the universal enveloping functor $\mathcal{U}_{P_\infty}$ satisfies PBW property.

Corollary 3.8 may be considered as a nice definition of a homotopy universal enveloping algebra. Unfortunately the corresponding associative algebra will be infinitely generated and one might be interested to have a bit better description of a universal enveloping Thanks to Proposition 3.5 we are interested in a certain resolution of $P_+^2$.

Theorem 3.9. For each symmetric operad $P$ there exist a dg-permutad that is a free bar-coobar resolution of $P_+^2$ as a right $P$-module whose generators is the bar construction of a permutad $B(P)_+^2$ assigned to the operadic bar construction of $P$:

$$
\Omega_{\text{perm}}((B_{\text{Op}}(P))_+^2) \circ P \xrightarrow{\text{quis}} P_+^2.
$$

Proof. The well known bar-coobar construction predicts the following quasiisomorphism for any symmetric operad:

$$
\Omega_{\text{Op}}(B_{\text{Op}}(P)) \rightarrow P
$$

Thanks to Proposition 3.5 we have the following isomorphism $\Omega_{1-2\text{op}} ((B_{\text{Op}}(P))_+^2) \rightarrow P_+$. The subcooperad of $B_{\text{Op}}(P))_+$ spanned by operations with no inputs/outputs of the second color is isomorphic to $B_{\text{Op}}(P)$. The complement is spanned by operations that has a unique input and output of the second color. Thus, the subspace of the bar-coobar construction of operations with the output of the second color leads to the following quasiisomorphic surjection:

$$
\Omega_{1-2\text{op}} ((B_{\text{Op}}(P))_+^2) \rightarrow P_+^2
$$

As mentioned in Statement 3.2 the two-colored (co)bar construction of the 1-2-operad $P_+^2$ coincides with the (co)bar construction of the corresponding permutad: $\Omega_{\text{perm}} ((B_{\text{Op}}(P))_+^2)$. \hfill \Box

Corollary 3.11. If the symmetric operad $P$ is Koszul then there exists a smaller resolution of $P_+^2$ as a right $P$-module:

$$
\Omega_{\text{perm}}((P')_+^2) \circ P \xrightarrow{\text{quis}} P_+^2.
$$

Proof. The koszulness of the symmetric operad $P$ implies (and even is equivalent) to the following statement: "The surjection $B_{\text{Op}}(P) \rightarrow P'$ is a quasi-isomorphism." Consequently, the bar-coobar resolution (3.10) is equivalent to the resolution (3.12). \hfill \Box

4. Necessary and sufficient conditions on PBW of $\mathcal{U}_P$

4.1. Main criterion for PBW of $\mathcal{U}_P$ for a Koszul operad $P$. Thanks to corollary 3.11 we know that if the Koszul operad $P$ is an ordinary operad (meaning that there is no additional homological grading and a differential) then $\mathcal{U}_P$ satisfies PBW if and only if the homology of the complex $\Omega_{\text{perm}}((P')_+^2)$ differs from zero only in 0-homological degree.

Theorem 4.1. The universal enveloping functor $\mathcal{U}_P$ of a symmetric Koszul operad $P$ satisfies PBW if and only if the permutad $(P')_+^2$ assigned to a Koszul dual operad $P'$ has the same set of generators as the operad $P$ (resp. $P'$) and is a Koszul permutad.
Proof. Let $\mathcal{T}$ be the symmetric collection that generates the operad $\mathcal{P}$. Any basis of $\mathcal{T}$ will give the set of generators of $\mathcal{P}$. Let $\mathcal{T}'$ be the corresponding dual $S$-collection generating $\mathcal{P}'$. Denote by $\mathcal{T}_+$ the corresponding subspace of $(\mathcal{P}')^+_1$ that form an $S$-collection with respect to the first color. We have to show that $U_\mathcal{P}$ satisfies PBW if and only if the permutad $(\mathcal{P}')^+_1$ is generated by $\mathcal{T}_+$ and is quadratic and Koszul. The implication in one direction is easy. Namely, if the permutad $(\mathcal{P}')^+_1$ is Koszul then we have a quasi-isomorphism projection $\Omega_{\text{perm}}((\mathcal{P}'_{\text{op}})^+_1) \circ \mathcal{P} \rightarrow ((\mathcal{P}'_{\text{op}})^+_1)_{\text{perm}} \circ \mathcal{P}$ and thanks to Corollary 3.11 the latter is isomorphic to the permutad $\mathcal{P}'^+_1$.

In order to prove the if statement we have to observe several facts. First, since the operad $\mathcal{P}$ is quadratic the universal enveloping algebra $U_\mathcal{P}(V)$ is an associative algebra generated by $\mathcal{T}_+(V)$ subject to quadratic-linear relations (Proposition 1.3). Therefore if $U_\mathcal{P}$ satisfies PBW then the permutad $U_\mathcal{P}^0$ is quadratic and is generated by $\mathcal{T}_+$. Second, thanks to Corollary 2.5 we know that $\mathcal{P}_+$ is a free right $\mathcal{P}$-module. Third, since $\mathcal{P}$ is Koszul Corollary 3.11 predicts that the set of generators of $U_\mathcal{P}$ and the permutad $(\mathcal{P}'_{\text{op}})^+_1$ are bar-dual to each other. Since the permutad $(\mathcal{P}'_{\text{op}})^+_1$ is a real permutad without higher homotopy operations this may happen only if the permutad $U_\mathcal{P}^0$ is Koszul and $(\mathcal{P}'_{\text{op}})^+_1$ is the Koszul dual and consequently quadratic permutad.

**Corollary 4.2.** If $\mathcal{P}$ is a Koszul operad such that $U_\mathcal{P}$ satisfies PBW, then

1. There is an isomorphism of permutads:

$$U_\mathcal{P}^0 \cong (\mathcal{P}'_{\text{op}})^+_1)_{\text{perm}}$$

2. For any $\mathcal{P}$-algebra $V$ its universal enveloping $U_\mathcal{P}(V)$ is a quadratic-linear Koszul algebra, whose associated graded Koszul algebra is isomorphic to $U_\mathcal{P}(V_0)$, the corresponding Koszul dual coalgebra is isomorphic to

$$s(\mathcal{P}'_{\text{op}})^+_1(V_0) := \oplus_{n \geq 1} \mathcal{P}'_{\text{op}}(n) \otimes S_n V_0^{\otimes n}[n] \otimes \text{sgn}_n;$$

3. In particular, the Koszul complex

$$(4.3) \quad (U_\mathcal{P}(V) \otimes (s(\mathcal{P}'_{\text{op}})^+_1(V_0)), d) \rightarrow \mathbb{k}$$

defines a free resolution of $\mathbb{k}$ in the category of left $U_\mathcal{P}(V)$-modules.

**Proof.** Thanks to PBW property and the theory of nonhomogeneous Koszul duality (see [21] chapter 5 for details) it is enough to prove the second and the third items for the case of trivial $\mathcal{P}$-algebra $V_0$. We notice that for associated graded to PBW filtration the statement of the theorem becomes a reformulation of the theorem becomes a reformulation of the Koszul property for permutads.

Indeed, suppose that the permutad $\mathcal{Q}$ is Koszul. This is equivalent to the acyclicity of the Koszul complex:

$$(4.4) \quad (\mathcal{Q} \circ s\mathcal{Q}_{\text{perm}}, d) \rightarrow \mathbb{k}$$

Each graded component of the Koszul complex [4,3] defines a (polynomial) Schur functor of appropriate degree. To each given vector space $V$ we assign the associative algebra $\mathcal{Q}(V) := \oplus_{n \geq 0} \mathcal{Q}(n) \otimes S_n V_0^{\otimes n}$ and coassociative coalgebra $(s\mathcal{Q}_{\text{perm}})(V) \cong (\mathcal{Q}_{\text{perm}} \otimes \text{sgn})(V[1])$. The Schur-Weyl duality implies the acyclicity of the Koszul complex assigned to $\mathcal{Q}(V)$:

$$(\mathcal{Q}(V) \otimes s\mathcal{Q}_{\text{perm}}(V), d).$$

Therefore, the algebra $\mathcal{Q}(V)$ is Koszul and its Koszul dual coalgebra is isomorphic to $(s\mathcal{Q}_{\text{perm}})(V)$. □

**Example 4.5.** The operads $\text{Comm}$ and $\text{Lie}$ of commutative and Lie algebras are known to be Koszul dual to each other. The koszulness of the permutad $\text{Comm}^2_+$ was shown in [7] where we found a quadratic Gröbner basis for the corresponding shuffle algebra.

The permutad $\text{Lie}^2_+$ is the free (and hence Koszul) permutad generated by a single one-dimensional $S_1$-representation, since

$$ \text{Res}_{S_n}^{S_{n+1}} \text{Lie}(n+1) \cong \text{Res}_{S_n}^{S_{n+1}} \text{Ind}_{S_n}^{S_{n+1}} 1_{n+1} \otimes \mathbb{I} \cong \mathbb{k}[S_n]$$

Thus, Theorem 4.4.2 reproves the PBW property for $U_{\text{Lie}}$ and $U_{\text{Comm}}$, and Corollary 4.3.2 reproduce the well-known description of the corresponding universal enveloping algebras:

$$\text{gr}^{PBW} U_{\text{Lie}}(g) \cong S(g), \quad U_{\text{Comm}}(A) = \mathbb{k} \oplus A.$$
and rediscover the two famous resolutions:

\[(U(g) \otimes \Lambda^*(g[1]), d) \to k, \quad (A \otimes T(A[1]), d) \to k\]

**Example 4.6.** Recall that to each commutative associative graded algebra \(A := \oplus_{n \geq 0} A_n\) one can assign a symmetric operad \(O_A\) whose space of \(n\)-ary operations \(O_A(n)\) is isomorphic to \(A_{n-1}\) with the trivial \(S_n\) action and the composition rules are given by multiplication:

\[\circ : O_A(m) \otimes O_A(n) = A_{m-1} \otimes A_{n-1} \to A_{m+n-2} = O_A(m + n - 1)\]

Moreover, it was proved in [9] that if algebra \(A\) is Koszul then the corresponding operad \(O_A\) is also Koszul. The straightforward generalization to the case of colored operads (and in particular, for permutads) of the proof suggested in [9] shows that koszulness of \(A\) also implies the koszulness of the permutad \(O_A^2\).

Consequently, thanks to Theorem 4.1 the universal enveloping functor \(U_{(O_A)_\text{op}}\) satisfies the PBW property, and for any \(O\)-algebra \(V\) the associative algebra \(U_{(O_A)_\text{op}}(V)\) is a nonhomogeneous Koszul algebra whose Koszul-dual is isomorphic to the Hadamard product of graded quadratic Koszul algebras \(A\) and \(\Lambda^*(V)\):

\[\left(\text{gr}^{PBW} U_{(O_A)_\text{op}}(V)\right)_\text{Alg} \cong \oplus_{n \geq 0} A_n \otimes \Lambda^n V =: A \boxtimes \Lambda^*(V)\]

**4.2. Hilbert series and necessary condition for PBW of \(U_P\).** The necessary condition on the operad \(P\) to have a PBW property is formulated in terms of Hilbert series of dimensions (characters) of this operad. Recall that to each \(S\)-collection \(P\) (collection of possibly empty \(S_n\) representations \(P(n)\) for all \(n \geq 1\)) one can assign the two following generating series:

\[f_P(t) := \sum_{n \geq 1} \frac{\dim P(n)}{n!} t^n; \quad \chi_P(p_1, p_2, \ldots) = \sum_{n \geq 1} \chi_{S_n}(P(n))\]

Here \(\chi_{S_n}(V) := \sum_{\rho \in \mathfrak{S}_n} \frac{\rho}{\rho_p} \text{Tr}_V(\rho)\) is a symmetric function of degree \(n\) associated with the corresponding \(S_n\)-character of symmetric group given in the basis of Newton’s sums \(p_k := \sum x_i^n\).

**Theorem 4.7.** If the universal enveloping functor \(U_P\) of a symmetric operad \(P\) satisfies the PBW property then the generating series of the corresponding Schur functor \(U^P\) has the following description:

\[f_{U^P}(t) = -\left(\frac{\partial f_P(-t)}{\partial t}\right)^{-1}, \quad \text{with } f_P(-t) \circ f_P(-t) = f_P(-t) \circ f_P(-t) = t, \quad \chi_{U^P} = -\left(\frac{\partial}{\partial p_1} \chi_{\text{P}}(-p_1, -p_2, \ldots)\right)^{-1}\]

where \(\chi_{\text{P}}\) is the \(S\)-character of the Koszul dual operad \(\text{P'} := H^*(\Omega(s\text{P}'))\).

*Proof.* Note that \(f_Q(t)\) is a substitution of \(\chi_Q\) with \(p_1 = t\) and \(p_k = 0\) for \(k > 1\). Thus, it is enough to prove the relation for symmetric functions.

The PBW property gives the following relation:

\[(4.8) \quad \chi_{U^P} \circ \chi_{\text{P}} = \chi_{\text{P}} \cdot \frac{\partial}{\partial p_1} \chi_{\text{P}}\]

Recall that there is a standard automorphism \(\epsilon : \Lambda \to \Lambda\) of the ring of symmetric functions \(\Lambda := \mathbb{Z}[x_1, x_2, \ldots]^{S}\) that sends \(p_i \to -p_i\). On the level of \(S\)-representations \(\epsilon\) changes the parity and tensor it with a sign representation. The simplest relationship between \(S\)-characters of Koszul operad \(P\) and its Koszul dual \(\text{P}'\) has the following form:

\[\epsilon(\chi_{\text{P}}) \circ \epsilon(\chi_{\text{P}'}) = \chi_{\text{P}}(-p_1, -p_2, \ldots) \circ \chi_{\text{P}'}(-p_1, -p_2, \ldots) = p_1\]

Consequently, the Equation (4.13) is equivalent to the following

\[\chi_{U^P} = \chi_{U^P} \circ \epsilon(\chi_{\text{P}}) \circ \epsilon(\chi_{\text{P}'}) = \epsilon(\chi_{U^P} \circ \chi_{\text{P}}) \circ \epsilon(\chi_{\text{P}'}) = \epsilon\left(\frac{\partial}{\partial p_1} \chi_{\text{P}}\right) \circ \epsilon(\chi_{\text{P}'}) = -\left(\frac{\partial}{\partial p_1} \epsilon(\chi_{\text{P}'})\right) \circ \epsilon(\chi_{\text{P}'}) = -\frac{\partial}{\partial p_1} \left(\frac{\partial}{\partial p_1} \epsilon(\chi_{\text{P}'})\right) = -\left(\frac{\partial}{\partial p_1} \epsilon(\chi_{\text{P}'})\right)^{-1}\]
Corollary 4.9. The generating series \(-\left(\frac{\partial f_{\Omega}\left(-t, q\right)}{\partial t}\right)^{-1}\) of a symmetric (Koszul) operad \(P\) whose universal enveloping functor yields the PBW property have only nonnegative coefficients.

Example 4.10. The universal enveloping functor of a Poisson operad \(\text{Pois}\) does not satisfy PBW.

Proof. The Poisson operad is graded by the number of Lie brackets. Let us consider the corresponding generating series that depends on an additional parameter \(z\) that counts the number of Lie brackets:

\[
  f_{\text{Pois}}(t, q) = f_{\text{Comm}}(t) \circ f_{\text{Lie}}(qt) = (e^t - 1) \circ \left( -\frac{\ln(1 - qt)}{q} \right).
\]

The Poisson operad is known to be Koszul self-dual and consequently we have:

\[
  -\left(\frac{\partial f_{\text{Pois}}\left(-t, q\right)}{\partial t}\right)^{-1} = (1 + qt)^{-z-1} = 1 + (1 + q)t + \frac{(1 + q)^2}{2}t^2 + \frac{q^2 - 1}{6}t^3 + \ldots
\]

whose coefficient near \(t^3\) is already negative and thus \(U_{\text{Pois}}\) does not satisfy PBW. \(\square\)

Corollary 4.11. If the universal enveloping functor \(U_P\) associated with a symmetric (nongraded) operad \(P\) satisfies PBW then the symmetric function \(-\left(\frac{\partial \epsilon(\chi_P)}{\partial p}\right)^{-1}\) is Schur-positive.

4.3. Gröbner bases and sufficient condition for PBW of \(U_P\). Theorem 4.1 defines a nice sufficient condition for PBW property in terms of Koszul property for appropriate operads and permutads. At the moment the most effective way to check the Koszul property is to find a quadratic Gröbner bases. In this section we state a useful sufficient condition for PBW property in terms of Gröbner bases for operads and show how it works for particular examples in the subsequent Section §5. Let us briefly recall the main definition of [8] motivated by PBW theory suggested in [14].

Definition 4.12. A shuffle operad \(P\) in the category of vector spaces consist of a collection of vector spaces \(P(n)\) and composition rules \(\circ_\sigma: P(k) \otimes P(n_1) \otimes \ldots \otimes P(n_k) \rightarrow P(n_1 + \ldots + n_k)\) numbered by surjections \(\sigma : [1(n_1 + \ldots + n_k)] \rightarrow [k]\) that satisfy the shuffle condition:

\[
  1 \leq i < j \leq k \Rightarrow \min(\sigma^{-1}(i)) < \min(\sigma^{-1}(j))
\]

Yielding the appropriate associativity condition that can be roughly written in the following form:

\[
  P \circ_\sigma (P \circ_\tau P) = (P \circ_\sigma P) \circ_\tau P.
\]

With any symmetric operad \(P\) one can assign a shuffle operad \(\text{Sh}(P)\) by forgetting the action of symmetric group and part of the compositions. The forgetful functor \(\text{Sh}\) does not change the underlying collection of vector spaces and maps free symmetric operads to free shuffle operads.

The main advantage of shuffle operads is that there is a notion of monomials in the free shuffle operad. Each monomial can be uniquely presented as a planar tree whose vertices are labeled by generators and leaves are labeled such that for each inner vertex \(v\) the ordering of the minims of the leaves of subtrees of \(v\) respects the planar ordering. (See example (4.14) below of a shuffle monomial of arity 9 in the free shuffle operad on 4 generators of arities 2 and 3.)

\[
  (4.14)
\]

Any orderings of monomials compatible with compositions leads to the theory of Gröbner bases. We refer for all details of Gröbner bases and shuffle operads to [8].

Definition 4.15 ([17] Section 7.2). A shuffle monomial \(m\) in the free shuffle operad \(F(\gamma_1, \ldots, \gamma_n)\) is called a left comb if it is presented by a shuffle tree with no nontrivial subtrees growing to the right. For
example, if $\gamma_i$ are all binary operations then $m$ has to be of the following form:

$$
\gamma_1(\gamma_2(\ldots \gamma_k(x_1, x_{1+\sigma(1)}), \ldots , x_{1+\sigma(k-1)}, x_{1+\sigma(k)}) ) =
\begin{cases}
1 & \text{if } \sigma = 1 \\
1+\sigma(1) & \text{if } \sigma > 1
\end{cases}
$$

Theorem 4.16. If the operad $\mathcal{P}$ admits a Gröbner basis $G$ such that the set of leading monomials of $G$ are left comb shuffle monomials. Then the universal enveloping functor $U_\mathcal{P}$ satisfies the PBW property. If, in addition, the Gröbner basis $G$ is quadratic, then the conditions of Theorem 4.1 are satisfied: the operad $\mathcal{P}$ and the permutad $(\mathcal{P}_{\text{op}}^1)^2$ are Koszul.

In order to explain the evidence of this theorem we will extend the theory of shuffle operads for $1$-$2$-operads.

4.3.1. Shuffle 1-$2$-operads and applications. The notion of a shuffle operad and the theory of Gröbner bases can be generalized in a very straightforward way to the case of colored operads. In order to save the readers time and patience we will not present this theory here because the two-colored operads we consider are much easier to work out separately. Recall that we defined (Definition 1.3) 1-$2$-operads to be the operads on two colors such that the number of inputs and the outputs of the second color has to be the same.

Definition 4.17. A shuffle 1-$2$-operad $\mathcal{P}$ consist of two collections of vector spaces $\mathcal{P}(n)^1$ and $\mathcal{P}(n)^2$ together with compositions

$$
o_1^1 : \mathcal{P}(k)^1 \otimes \mathcal{P}(n_1)^1 \otimes \ldots \otimes \mathcal{P}(n_k)^1 \to \mathcal{P}(n_1 + \ldots + n_k)^1
$$

$$
o_1^2 : \mathcal{P}(k)^2 \otimes \mathcal{P}(n_1)^2 \otimes \mathcal{P}(n_2)^1 \otimes \ldots \otimes \mathcal{P}(n_k)^1 \to \mathcal{P}(n_1 + \ldots + n_k)^2
$$

numbered by surjections $\sigma : [1n_1 + \ldots + n_k] \to [1k]$ that satisfy the shuffle condition \[4.13\]. The composition rules satisfy operadic associativity.

Pictorially, shuffle 1-$2$-operad consists of operations of the first color that does not differ with the ordinary operad and of operations whose first input and the output is colored by the second color. Respectively, the second color in a shuffle monomial of the free shuffle 1-$2$-operad is a path from the leaf labelled by 1 and the root. Here is an example of a 1-$2$-shuffle monomial:

```
1
\begin{tikzpicture}
\node (a) at (0,0) {1};
\node (b) at (1,0) {2};
\node (c) at (2,0) {7};
\node (d) at (3,0) {6};
\node (e) at (4,0) {9};
\node (f) at (1,-1) {3};
\node (g) at (2,-1) {8};
\node (h) at (3,-1) {4};
\node (i) at (0,-2) {5};
\node (j) at (1,-2) {3};
\node (k) at (2,-2) {8};
\node (l) at (3,-2) {4};
\node (m) at (0,-3) {5};
\node (n) at (1,-3) {3};
\node (o) at (2,-3) {8};
\node (p) at (3,-3) {4};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\draw (d) -- (e);
\draw (f) -- (i);
\draw (i) -- (j);
\draw (j) -- (k);
\draw (k) -- (l);
\draw (l) -- (m);
\draw (m) -- (n);
\draw (n) -- (o);
\draw (o) -- (p);
\end{tikzpicture}
```

In other words, the second color in a 1-$2$-shuffle monomial is uniquely defined and the only thing we have to know is the underlying shuffle monomial with coloring omitted.

Proposition 4.18. There exists a linear ordering of 1-$2$-shuffle monomials in the free shuffle 1-$2$-operad that extends a given ordering of the generators and is compatible with the compositions.

Proof. Let $\mathcal{F}$ be the free shuffle 1-$2$-operad generated by $S := \{\alpha_1, \ldots, \alpha_s\} \subset \mathcal{F}^1$ with the output of the first color and $T := \{\beta_1, \ldots, \beta_t\} \subset \mathcal{F}^2$ with the output of the second color. Let $\mathcal{F}$ be the corresponding free shuffle operad with the set of generators $S \cup T$ where we just forget the colorings. Note that each 1-$2$-shuffle monomial in $\mathcal{F}$ can be considered as an ordinary shuffle monomial of $\mathcal{F}$. Thus we have an embedding of monomial basis of the free shuffle 1-$2$-operad $\mathcal{F}$ to the monomial basis of the free shuffle operad $\mathcal{F}$. Let us fix a linear ordering of the generators and let $\prec$ be a compatible ordering of the shuffle monomials in $\mathcal{F}$ which exists as shown in \[3\]. The restriction of the linear ordering $\prec$ to the set of 1-$2$-shuffle monomials defines a compatible ordering.

Corollary 4.19. There exists a theory of Gröbner bases for shuffle 1-$2$-operads.
We have two natural (exact) functors with shuffle 1-2-operads as a target category:

First, with each symmetric 1-2-operad \( Q \) we can associate a shuffle 1-2-operad \( \text{Sh}(Q) \) while forgetting the action of symmetric group and part of the compositions.

Second, with each shuffle operad \( P \) we can assign a shuffle 1-2-operad \( P_+ \) whose spaces of \( n \)-ary operations \( P_+(n) = P(n, 0)^1, P_+(n)^2 = P(n-1, 1)^2 \) are both isomorphic to \( P(n) \) and 1-2-compositions are restrictions of the underlying compositions in the shuffle operad \( P \).

**Proposition 4.20.** The functors \( \text{Sh} \) and \( + \) commutes and maps free objects to free objects:

\[
\begin{array}{ccc}
\text{Symmetric Operads} & \xrightarrow{\text{Sh}} & \text{Symmetric 1-2-Operads} \\
\downarrow & & \downarrow \\
\text{Shuffle Operads} & \xrightarrow{+} & \text{Shuffle 1-2-Operads}
\end{array}
\]

Moreover, the functors \( \text{Sh} \) keeps the number of generators of a free operad to be the same and the functor \( + \) doubles the set of generators: one copy of the first color and one copy has the first input and the output of the second color.

**Proof.** Direct observation. \( \square \)

**Corollary 4.21.** Let \( G \) be a Gröbner basis of the given shuffle operad \( P = \mathcal{F}(A|R) \). Then \( G_+ \) is a Gröbner basis of the shuffle 1-2-operad \( P_+ \) for an appropriate ordering of monomials in the free operad \( \mathcal{F}(A)_+ = \mathcal{F}(A_+) \).

**Proof.** We say that 1-2-shuffle monomial \( m \) is less or equal than \( n \) if the underlying shuffle (uncolored) monomials has the same comparability \( m \prec n \). This defines a linear ordering on monomials of the first color \( \mathcal{F}(A)_+ \) and on monomials of the second color \( \mathcal{F}(A)^2_+ \). Thus, in order to derive the theory of Gröbner basis we do not need to compare the monomials from these components. The functor \( + \) is compatible with filtrations given by the ordering of monomials by definition. \( \square \)

### 4.3.2. Proof of Theorem 4.16

Consider the filtration on the free shuffle operad and the induced filtration on \( P = \mathcal{F}(A|G) \) such that the associated graded operad \( \text{gr}P \) is an operad with monomial relations given by the leading terms of the Gröbner basis \( G \). Let \( A^1_+ \) and \( A^2_+ \) be the set of generators of \( P_+ \) with the output of the first and the second color correspondingly. Let \( \hat{G}^1_+ \) (resp. \( \hat{G}^2_+ \)) be the set of leading monomials of the Gröbner basis for \( P_+ \) with the output of the first (resp. second) color. Both sets \( \hat{G}^1_+ \) and \( \hat{G}^2_+ \) consists of left combs. Therefore, \( \hat{G}^1_+ \) belongs to the shuffle 1-2-suboperad generated by \( A^1_+ \). In particular, \( \forall g_1 \in \hat{G}^1_+, \forall g_2 \in \hat{G}^2_+ \) the greatest common divisor of \( g_1 \) and \( g_2 \) is 1. (We refer to \[8\]§3.3 for the notion of divisibility.) Consequently, the associated graded \( \text{gr}P^+_2 \) to the subspace of operations with the second color as an output is isomorphic to

\[
\left( \mathcal{F}_{\text{Op}}(A|\hat{G}) \right)^2_+ \simeq \left( \mathcal{F}_{1-2\text{op}}(A^1_+ \oplus A^2_+ | \hat{G}^1_+ \oplus \hat{G}^2_+) \right)^2 \simeq \mathcal{F}_{1-2}(A^2_+ | \hat{G}^2_+) \circ \mathcal{F}_{\text{Op}}(A^1_+ | \hat{G}^1_+) \simeq \mathcal{F}_{1-2}(A^2_+ | \hat{G}^2_+) \circ \text{gr}P
\]

Therefore, \( \text{gr}P^+_2 \) is the free right \( \text{gr}P \)-module. The freeness for associated graded implies the freeness for the initial operad \( P \) and, hence, the PBW property of the universal enveloping functor \( U_P \).

Note that the shuffle 1-2-suboperad \( \mathcal{F}_{1-2}(A^2_+ | \hat{G}^2_+) \) of \( \text{gr}P \) is a shuffle algebra in the sense of \([7]\), that is the permuted with forgotten action of symmetric group. Thus, we get even more. The shuffle algebra associated with the permuted \( U_P \) admits a filtration such that the associated graded is isomorphic to the shuffle algebra with monomial relations \( \mathcal{F}_{1-2}(A^2_+ | \hat{G}^2_+) \). Therefore, if the Gröbner basis \( G \) is quadratic then the operad \( P \) is Koszul by the results of \[5\] and the permuted \( U_P \) is Koszul thanks to the results of \[7\]. \( \square \)

### 5. Examples

We consider a short list of examples of operads generated by binary operations and we refer to \[23\] for the references on more detailed descriptions of the corresponding operads.
5.1. Pair of compatible Lie brackets. Let Lie₂ be an operad of pair of compatible Lie brackets (introduced in [6]). An algebra \( g \) over Lie₂ has two Lie brackets \([-,-]_1, [-,-]_2 : A^2 g \rightarrow g \) such that each linear combination \( \lambda [-,-]_1 + \mu [-,-]_2 \) defines a Lie bracket (satisfies the Jacobi identity).

Lemma 5.1. The universal enveloping of a Lie₂ algebra \( g \) has the following presentation:

\[
U_{\text{Lie}_2}(g) \simeq T^\otimes(g \otimes k^2) / (\Lambda^2 g \otimes S^2(k^2)) \simeq k \left\langle g_1 \oplus g_2 \right| \begin{array}{l}
g_1 \otimes h_1 - h_1 \otimes g_1 = [g,h]_1, \\
g_1 \otimes h_2 + g_2 \otimes h_1 - h_1 \otimes g_2 - h_2 \otimes g_2 = [g,h]_2, \\
g_2 \otimes h_2 - h_2 \otimes g_2 = [g,h]_2,
\end{array}
\]

Corollary 5.2. The universal enveloping functor with respect to the operad of compatible Lie brackets \( \text{Lie}_2 \) satisfies PBW. Moreover,

\[
\chi_{U_{\text{Lie}_2}} = \exp \left( \frac{\sum_{k \geq 1} h_k}{1 - \left( \sum_{k \geq 1} p_k \right) h_k} \right)
\]

Proof. The operad Koszul dual to \( \text{Lie}_2 \) is a particular example of the operads considered in Example [6]. Indeed, we have an isomorphism of operads \( \mathcal{O}_{\text{Lie},[x,y]} \simeq \text{Lie}_2 \). Thus, thanks to Corollary 4.1 the functor \( U_{\text{Lie}_2} \) satisfies PBW.

For the computation of the character \( \chi_{U_{\text{Lie}_2}} \) it is enough to recall that the space of \( n \)-ary operations in the Koszul dual operad to the operad \( \text{Lie}_2 \) is a trivial \( S_n \)-representation of dimension \( n \). Thus its generating series is very simple ([6]):

\[
\chi_{\text{Lie}_2}^n = \left( \sum_{k \geq 1} p_k \right)^{\sum_{k \geq 1} p_k / k} = \exp \left( \sum_{k \geq 1} \frac{p_k}{k} \right)
\]

The computer experiments shows that the expression \( (1 - (\sum_{k \geq 1} p_k))^{-1} \) is Schur positive. Let us denote the corresponding Schur functor by \( FL() \) and thus, we expect an isomorphism of Schur functors:

\[
U_{\text{Lie}_2}^\otimes(V) \simeq S(V) \otimes FL(V)
\]

5.2. PreLie algebras. The PreLie operad of Pre-Lie algebras introduced in [3] is also one of the frequently used Koszul operads. It is generated by one nonsymmetric operation \( x \triangleright y \) yielding the following condition:

\[
(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (x \triangleright z) \triangleright y - x \triangleright (z \triangleright y)
\]

It might be easier to define a universal enveloping algebra directly from relation (5.3). The universal enveloping algebra \( U_{\text{PreLie}}(g) \) of a PreLie algebra \( g \) is generated by two copies of \( g \) (whose elements are denoted by \( r_x \) and \( l_x \) respectively) subject to the following list of relations for all pairs of \( x, y \in g \):

\[
\begin{align*}
(r_x r_y - r_y r_x) & = (r_y r_x - r_x r_y), \\
l_x r_y - r_y l_x & = (l_y l_x - l_x l_y).
\end{align*}
\]

Theorem 5.4. The functor \( U_{\text{PreLie}} \) satisfies PBW. Moreover, on top of PBW-filtration there exists a filtration on \( U_{\text{PreLie}}(g) \) such that the associated graded algebra is isomorphic to \( S(g) \otimes T(g) \).

Proof. This theorem can be derived from the description of the universal enveloping in terms of generators and relations. However, we want to illustrate our methods and deal with relations in the operad. Note, that the shuffle operad \( \text{PreLie} \) is generated by two operations \( x_1 \triangleright x_2 \) and \( x_1 \triangleright x_2 = x_2 \triangleright x_1 \) and we know (Example 11 in [8]) that there exists a compatible ordering of monomials such that the quadratic Gröbner
basis for \textit{PreLie} consists of 3 relations (where we underline the leading monomial in each relation)

\[
\begin{align*}
\overset{\underset{\text{1 2}}{\text{3}}}{} & \overset{\underset{\text{1}}{\text{2}}}{} \overset{\underset{\text{3}}{\text{1}}}{} = \overset{\underset{\text{2}}{\text{1}}}{} \overset{\underset{\text{1}}{\text{2}}}{} \overset{\underset{\text{3}}{\text{1}}}{} ; \\
\overset{\underset{\text{1 2}}{\text{3}}}{} & \overset{\underset{\text{1}}{\text{3}}}{} \overset{\underset{\text{2}}{\text{3}}}{} = \overset{\underset{\text{2}}{\text{3}}}{} \overset{\underset{\text{1}}{\text{2}}}{} \overset{\underset{\text{1}}{\text{3}}}{} ; \\
\overset{\underset{\text{1 2}}{\text{3}}}{} & \overset{\underset{\text{1}}{\text{3}}}{} \overset{\underset{\text{2}}{\text{1}}}{} = \overset{\underset{\text{2}}{\text{1}}}{} \overset{\underset{\text{3}}{\text{2}}}{} \overset{\underset{\text{1}}{\text{3}}}{} .
\end{align*}
\]

Moreover, it is not hard to find out the filtration on a shuffle operad \textit{PreLie} such that the associated graded is still a quadratic operad subject to the following relations that form a Gröbner basis:

\[
\begin{align*}
\overset{\underset{\text{1 2}}{\text{3}}}{} & \overset{\underset{\text{1}}{\text{3}}}{} \overset{\underset{\text{2}}{\text{1}}}{} = \overset{\underset{\text{2}}{\text{1}}}{} \overset{\underset{\text{3}}{\text{2}}}{} \overset{\underset{\text{1}}{\text{3}}}{} = 0
\end{align*}
\]

The corresponding filtration descends to the filtration on \( U_{\text{PreLie}}(g) \) and its associated graded is isomorphic to \( S(g) \otimes T(g) \). \( \square \)

As predicted by Theorem 4.1 the PBW property of \( U_{\text{PreLie}} \) implies the simplification of the cohomology theory of the category of modules over a given \textit{PreLie}-algebra. Certain cohomological complexes were presented in [10] in 1999. We hope that our proof and description is simple enough to recover all corresponding cohomology theories.

5.3. \textbf{Operad} \textit{Perm} = \textit{PreLie}$.^1$

\textbf{Lemma 5.5.} The permutad \( \text{PreLie}^2_+ \) and the operad \textit{PreLie} has different sets of generators.

\textit{Proof.} Note that the space of \( n \)-ary operations in the free permutad \( F(V) \) generated by the vector space (\( S_1 \)-representation) \( V \subset F(1) \) is isomorphic to \( k[\mathcal{S}_n] \otimes V^\otimes n \). In particular, the dimension of the space of 2-ary operations of the free permutad on 2 binary generators is \( 2! \cdot 2^2 = 8 \).

On the other hand, there are two binary generators of the operad \textit{PreLie} and therefore there exists two generators in the permutad \( \text{PreLie}^2_+ \) that belongs to the subspace \( \text{PreLie}^2_+(1) = \text{PreLie}_+(1,1)^2 \). However,

\[
\dim(\text{PreLie}_+(2,1)^2) = \dim(\text{PreLie}(3)) = 9 > 8.
\]

\( \square \)

The Koszul-dual operad to the operad of pre-Lie algebras is called \( \text{Perm} \) (223). Recall that it is generated by two operations for which we will use the same notations as for \textit{PreLie}:

\[
x \triangleright y = y \triangleleft x
\]

subject to the following quadratic relations:

\[
(x \triangleright y) \triangleright z = x \triangleright (y \triangleright z) = x \triangleright (z \triangleright y)
\]

\textbf{Corollary 5.6.} The universal enveloping functor \( U_{\text{Perm}} \) does not satisfy PBW.

5.4. \textbf{Poisson algebras}. We have seen in Example 4.10 that the universal enveloping functor of the Poisson operad \textit{Pois} does not satisfy PBW. However, we insist that the homological description of this functor is of particular interest and we will discuss it in details elsewhere. The description of the permutad \( \text{Pois}^2_+ \) is the first step in this direction that illustrates the complexity of the theory of permutads.

\textbf{Proposition 5.7.} The permutad \( \text{Pois}^2_+ \) is generated by two elements \( \scriptsize{\begin{array}{c}1 \end{array}} \) (representing the image of commutative product) and \( \scriptsize{\begin{array}{c}1 \end{array}} \) (representing the image of Lie bracket) subject to the following
quadratic relations:

\[\begin{align*}
1 & \cdot 2 = \cdot 1 \cdot 2 \\
1 & \cdot 2 - 2 & \cdot 1 = -2 & \cdot 1 + 1 & \cdot 2
\end{align*}\]

but is not a Koszul permutad.

Proof. Note that the colored operad $\text{As}_+$ admits a filtration by the number of commutators such that the associated graded is isomorphic to $\text{Pois}$. Therefore, the same filtration exists on the permutad $\text{As}_+^2$. We checked already, that $U_{\text{As}}$ satisfies PBW (Example 1.11). Consequently, thanks to Theorem 4.1 the permutad $\text{As}_+^2$ is generated by $\text{As}_+ (1,1)^2$ subject to quadratic relations and is Koszul. We can rewrite these relations with respect to the basis:

\[\begin{align*}
1 & := 1 & 1 - 1 & 1, \\
1 & := 1 & 1 + 1 & 1
\end{align*}\]

and get the following:

\[\begin{align*}
& \cdot 1 2 - \cdot 2 1 = \cdot 1 2 \\
& \cdot 1 2 - \cdot 2 1 = -\cdot 2 1 + \cdot 1 2
\end{align*}\]

We see that the relations (5.8), (5.4) are the corresponding associated graded relations.

The permutad $\text{Pois}^2_+$ is not Koszul, because the Koszul operad $\text{Pois}$ is selfdual and the functor $U_{(\text{Pois})_b} = U_{\text{Pois}}$ does not satisfy PBW as we showed in Example 4.10.

5.5. Leibniz algebras. The operad of Leibniz algebras was introduced by J.-L. Loday ([15]) as an interesting generalization of Lie algebras. Let us recall, that a vector space $V$, equipped with binary (nonsymmetric) operation $[x,y]$ (called bracket) yielding the identity:

\[\forall x, y, z \in V \quad [x, [y, z]] - [[x, y], z] + [x, [z, y]] = 0\]

Which imply, in particular, the following identity:

\[\forall x, y, z \in V \quad [x, [y, z]] + [x[z, y]] = 0\]

Proposition 5.10. The corresponding symmetric operad $\text{Leib}$ is generated by two binary operations $x_1 \triangleleft x_2 := [x_1, x_2]$ and $x_1 \triangleright x_2 := [x_2, x_1]$ and the following list of quadratic relations:

\[\begin{align*}
& \begin{array}{l}
2 & 3 & 1 & 2 \ \triangleleft \ 3 & 1 & 2 + 2 & 3 & 1 & 2 = 2 & 3 & 1 & 2 + 2 & 3 & 1 & 2 = 0;
\end{array} \\
& \begin{array}{l}
2 & 3 & 1 & 2 \ \triangleright \ 3 & 1 & 2 + 2 & 3 & 1 & 2 = 2 & 3 & 1 & 2 + 2 & 3 & 1 & 2 = 0;
\end{array} \\
& \begin{array}{l}
1 & 2 & 3 \ \triangleleft \ 2 & 3 + 1 & 2 & 3 = 1 & 2 & 3 + 1 & 2 & 3 = 0;
\end{array} \\
& \begin{array}{l}
1 & 2 & 3 \ \triangleright \ 2 & 3 + 1 & 2 & 3 = 1 & 2 & 3 + 1 & 2 & 3 = 0;
\end{array}
\end{align*}\]

form a Gröbner basis with respect to convention $\triangleright > \triangleleft$ and the path opposite-degree lexicographical ordering of shuffle monomials. I.e. in order to compare two shuffle monomials $v$ and $w$ we first consider the paths (words in generators $\triangleleft$ and $\triangleright$) that goes from the root to the leaf 1 and say that
a monomial $v > w$ if the corresponding word assigned to $v$ is less than the corresponding word assigned to $w$ in degree-lexicographical ordering of words in two letters.

In particular, the operad $\text{Leib}$ is Koszul.

In [16] J.-L. Loday and T. Pirashvili discussed the structure of a universal enveloping algebra of a Leibniz algebra.

**Proposition 5.11.** (16) The universal enveloping algebra $U_{\text{Leib}}(g)$ of a Leibniz algebra $g$ is the algebra generated by two copies of $g$ (whose elements are denoted by $r_x$ and $l_x$ respectively) subject two the following list of quadratic-linear identities for all pairs $x, y \in g$:

\begin{align}
(5.12) & \quad r_{[x,y]} = -r_{[y,x]} = r_x r_y - r_y r_x; \\
(5.13) & \quad l_{[x,y]} = l_x r_y - r_y l_x; \\
(5.14) & \quad (r_y + l_y) l_x = 0
\end{align}

**Lemma 5.15.** The functor $U_{\text{Leib}}$ does not satisfy PBW property. However, the algebra $U_{\text{Leib}}(g)$ admits a PBW basis and, therefore, is a nonhomogeneous Koszul algebra generated by $L(g) := g$ spanned by $l_x, x \in g$ and by $R(g) := g/(\{[x,x]\})$ spanned by nontrivial right multiplications. Moreover, there exists a filtration such that associated graded algebra $\text{gr}U_{\text{Leib}}(g)$ is isomorphic to $S(L(g)) \otimes R(g)$.

**Proof.** The existence of filtration on $U_{\text{Leib}}(g)$ such that associated graded is isomorphic to $S(L(g)) \otimes R(g)$ was shown in [16]. The latter algebra is obviously Koszul. Therefore the algebra $U_{\text{Leib}}(g)$ is a nonhomogeneous Koszul algebra.

However, suppose that $g$ is a Leibniz algebra that has a nontrivial supercommutative part. In other words, there exists $x, y \in g$ such $[x,y] \neq -[y,x]$. Denote by $g_0$ the Leibniz algebra isomorphic to $g$ as a vector space but with all brackets to be zero. Then the universal enveloping algebras $U_{\text{Leib}}(g)$ and $U_{\text{Leib}}(g_0)$ are generated by different spaces and, consequently, has different size, because $R(g_0) \simeq g \neq R(g)$. Therefore, the functor $U_{\text{Leib}}$ does not satisfy PBW in the sense of Definition 2.11.

5.6. Zinbiel algebras. The operad of Zinbiel algebras is an operad Koszul-dual to the operad $\text{Leib}$ is generated by a binary nonsymmetric operation denoted by $(x \cdot y)$ subject to the following relation:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) + x \cdot (z \cdot y).$$

**Proposition 5.17.** The universal enveloping functor $U_{\text{Zinb}}$ satisfies PBW and the corresponding Schur functor $V \mapsto U_{\text{Zinb}}(V_0)$ is isomorphic to $k \oplus V \oplus V \oplus V \oplus V$.

**Proof.** Thanks to Proposition 5.11 we know that the operad $\text{Zinb}$ admits a quadratic Gröbner basis such that the set of leading monomials is the dual set to the leading monomials of Gröbner basis in $\text{Leib}$. This set consists of left comb monomials and thanks to Theorem 1.10 we get the PBW property of $U_{\text{Zinb}}$.

The $S_n$-representation $\text{Zinb}(n)$ is isomorphic to the regular representation $k[S_n]$. In particular, the symmetric collection $\cup_n \text{Zinb}(n)$ is isomorphic to symmetric collection $\cup_n \text{As}(n)$. Therefore, thanks to Theorem 4.7 the corresponding Schur functors assigned to universal enveloping functor should be the same for $\text{Zinb}$ and for $\text{As}$. The $S$-character of $U_{\text{As}}$ was already discussed in Proposition 1.12.

**Remark 5.18.** The universal enveloping functor of a Zinbiel algebra was discussed in details in [1].

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\[2r_x \text{ is multiplication by } x \text{ from the right given by } [-,x], \text{ respectively } l_x := [x,-] \]
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