Contrastive learning, multi-view redundancy, and linear models

Christopher Tosh\textsuperscript{*1}, Akshay Krishnamurthy\textsuperscript{†2}, and Daniel Hsu\textsuperscript{‡1}

\textsuperscript{1}Columbia University, New York, NY
\textsuperscript{2}Microsoft Research, New York, NY

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Abstract

Self-supervised learning is an empirically successful approach to unsupervised learning based on creating artificial supervised learning problems. A popular self-supervised approach to representation learning is contrastive learning, which leverages naturally occurring pairs of similar and dissimilar data points, or multiple views of the same data. This work provides a theoretical analysis of contrastive learning in the multi-view setting, where two views of each datum are available. The main result is that linear functions of the learned representations are nearly optimal on downstream prediction tasks whenever the two views provide redundant information about the label.

1 Introduction

Learning useful representations from unlabeled data is a standard problem in machine learning. While there are many approaches to tackle this problem, self-supervised learning is currently among the most popular. Broadly, self-supervised learning refers to techniques that take advantage of naturally occurring structure in unlabeled data to create artificial supervised learning problems and then solves these problems using machine learning methods such as deep learning. The hope is that in solving these problems, a learning algorithm will also create internal representations for data that are useful for other downstream learning tasks. Self-supervised techniques include de-noising autoencoders (Vincent et al., 2008), image inpainting (Pathak et al., 2016), and the focus of this work, contrastive learning (Hadsell et al., 2006; Oord et al., 2018; Logeswaran and Lee, 2018; Hjelm et al., 2018; Arora et al., 2019; Bachman et al., 2019; Tian et al., 2019; Tosh et al., 2020; Chen et al., 2020).

A common theme among many of these self-supervised representation learning works is the exploitation of naturally occurring similar points, or multiple views of the same data points. To train a de-noising autoencoder, one first creates an alternate “view” of data points by corrupting them with added noise and then trains the autoencoder to reconstruct the original. Image inpainting removes patches of images and trains models to reconstruct the original image. Contrastive learning trains models to distinguish naturally occurring similar pairs of points, such as neighboring sentences (Logeswaran and Lee, 2018) or randomly cropped and blurred versions of images (Chen et al., 2020), from random pairs of points.

Exploiting multiple views of data for representation learning is not a new technique. Canonical correlation analysis (CCA) (Hotelling, 1936) is a classical (unsupervised) technique that finds the linear transformation that aligns two views of data so that the resulting coordinates are uncorrelated. A fascinating line of work

\textsuperscript{*}c.tosh@columbia.edu
\textsuperscript{†}akshaykr@microsoft.com
\textsuperscript{‡}djhsu@cs.columbia.edu
(Ando and Zhang, 2005; Kakade and Foster, 2007; Ando and Zhang, 2007; Foster et al., 2009) investigated
the quality of representations produced by CCA (and related linear methods) for downstream regression
problems. Taken together, these works demonstrated that linear regression with the CCA representations
in a low-dimensional space will have optimal (or nearly optimal) performance relative to the best linear
function of the original representation in two settings. In the first of these settings, there is some hidden
(random) vector \( H \) such that: (a) the two views and the label are conditionally independent given \( H \), and
(b) the conditional expectation of each of these quantities given \( H \) is a linear transformation of \( H \). In the
second setting, no probabilistic assumption is made except that there is some redundancy among the two
views, i.e., that the best linear prediction of the label on each individual view is nearly as good as the best
linear prediction of the label when both views are used together.

In this work, we examine contrastive learning from the perspective of multi-view redundancy, analogously
to the CCA analysis of Kakade and Foster (2007) and Foster et al. (2009). We show that when there is
some redundancy between the two views on the label, contrastive learning leads to representations such
that linear functions of these representations are competitive with the (possibly non-linear) Bayes optimal
predictor of the label. Our analysis is rather general, and we are able to bound the dimensionality of the
representations that is sufficient to lead to good performance in the downstream prediction task. We consider
two specific representations based on contrastive learning. The first of these is a general-purpose construction
that uses the “landmark embedding” technique of Tosh et al. (2020). The second representation is formed by
solving a particular bivariate optimization problem. In both cases, we show that we can use low-dimensional
representations and still achieve near-optimal downstream performance with linear methods. We instantiate
our results in some simple latent variable models for illustration.

1.1 Overview of results

We are interested in the problem of prediction in the multi-view setting, in which data points may be
represented as triples of random variables \((X, Z, Y)\) where \(X\) and \(Z\) represent two views of the data and \(Y\)
is some label or regression value to be predicted. “Views” should be interpreted liberally here. For example,
they could correspond to the first and second halves of a document or to two different distortions of the same
image. However, the main property that we will require of our views is that they share redundant information
with respect to predicting \(Y\). That is, predicting \(Y\) from \(X\) or \(Z\) individually should be nearly as accurate as
predicting \(Y\) from \(X\) and \(Z\) together.

When \(X\) and \(Z\) do satisfy this redundancy property, we show that there is a surprisingly effective
prediction strategy: given \(X\), first try to infer \(Z\) and then predict \(Y\) based only on the inferred \(Z\). Specifically,
we prove the following lemma.

**Lemma 1** (Restated). If \(X, Z, Y\) are random variables, then

\[
\mathbb{E} \left[ \left( \mathbb{E} [Y | Z] - \mathbb{E} [Y | X, Z] \right)^2 \right] \leq \varepsilon_X + 2\sqrt{\varepsilon_X \varepsilon_Z} + \varepsilon_Z
\]

where \(\varepsilon_W = \mathbb{E} \left[ \left( \mathbb{E} [Y | W] - \mathbb{E} [Y | X, Z] \right)^2 \right] \) for each \(W \in \{X, Z\}\).

The strategy of Lemma 1 is reminiscent of the information bottleneck method (Tishby et al., 1999),
in which predicting \(Y\) from \(X\) is done by first compressing \(X\) to a “smaller” representation \(\hat{X}\) and then
predicting \(Y\) using \(\hat{X}\). In our case, the separate view \(Z\) acts as a natural intermediate target instead of \(\hat{X}\). As
it will turn out, Lemma 1 is the basis of all of the following results.

We are primarily concerned with the setting where we have lots of unlabeled data, i.e., \((X, Z)\) pairs,
and rather less labeled data \((X, Z, Y)\). In such situations, one natural strategy is to use the unlabeled data
to learn a representation of \(X\) (or \(Z\) or \((X, Z)\)), and then use the small collection of labeled data to learn a
simple function, like a linear predictor, on top of this new representation. In this work, we will look at the specific representation learning algorithm posed by Tosh et al. (2020), which is a type of contrastive learning algorithm.

Roughly speaking, the approach of Tosh et al. (2020) is to learn a function $f$ that distinguishes between true data points $(X, Z)$, and fabricated data points $(X, \tilde{Z})$, where $X$ and $\tilde{Z}$ come from independently sampled data points $(X, Z)$ and $(\tilde{X}, \tilde{Z})$. The idea is that such a function $f$ will learn enough about the relationship between $X$ and $Z$ to allow us to predict the label $Y$ from $X$ through $Z$ as in Lemma 1.

In Section 3, we show that one can extract an embedding of a view $X$ from the learned function $f$ such that linear functions on top of this embedding are competitive with the best predictor of $Y$ from $X$. Moreover, this embedding will be the same one proposed by Tosh et al. (2020) which uses landmark views $Z_1, \ldots, Z_m$ that are i.i.d. copies of $Z$ and embeds a point $x$ with the prediction values $f(x, Z_1), \ldots, f(x, Z_m)$.

**Theorem 3 (Restated).** Given a solution $f^* : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ to the contrastive learning problem and embedding points $Z_1, \ldots, Z_m$ sampled i.i.d. from the marginal distribution of $Z$, the landmark embedding $\varphi^* : \mathcal{X} \to \mathbb{R}$ based on $f^*$ and $Z_1, \ldots, Z_m$ defined in Section 3 satisfies, with high probability,

$$
\min_{w \in \mathbb{R}^m} \mathbb{E} \left[ \left( w^\top \varphi^*(X) - \mathbb{E}[Y \mid X, Z] \right)^2 \right] \leq \varepsilon_X + 2\sqrt{\varepsilon_X \varepsilon_Z} + \varepsilon_Z + O_m \left( \frac{1}{m} \right).
$$

To avoid clutter, we have used big-$O_m$ in the above statement to suppress all factors that do not depend on $m$, including logarithmic factors in the failure probability and other quantities that depend on the distribution of $X$ and $Z$. The full statement is provided in Section 3.

While the results of Section 3 demonstrate that contrastive learning can lead to useful representations in the presence of redundancy, the landmark embedding technique is not reflective of what is done in practice. In practice, self-supervised representation learning algorithms typically optimize embedding functions directly (Hadsell et al., 2006; Chen et al., 2020). To address this, in Section 4, we investigate the strategy of trying to learn the embedding functions directly. That is, we look at the bivariate architecture setting where we learn $\mathbb{R}^m$-valued functions $\eta, \psi$ such that $\eta(x)^\top \psi(z)$ distinguishes between the real and fake data points. As we will see, the benefit of this approach is that when there exist pairs of accurate functions, $\eta$ (and also $\psi$) allow us to do useful linear predictions. However, it is unclear a priori how large the output dimension of $\eta$ and $\psi$ needs to be to achieve this.

As a first step towards understanding this dimensionality question, we consider the setting where there is some hidden variable $H$ that renders the two views $X$ and $Z$ conditionally independent. Note that there is always a trivial hidden variable to achieve this, namely $H = (X, Z)$. However, we show that when the hidden variable obeys a nicer structure, the dimensionality of the embedding can be drastically improved. Specifically, we show the following.

(a) When the hidden variable $H$ takes values in a finite set, the cardinality of this set is an upper bound on the dimensionality needed for an exact embedding.

(b) In the general setting, there exist approximate embeddings where the approximation factor decreases at a rate of $O_m(1/m)$, where the big-$O_m$ notation suppresses dependence a particular variance quantity of the hidden variable structure.

Importantly, there is no assumption that the hidden variable structure is known. Rather, our results imply that solving the bivariate contrastive learning problem automatically recovers embeddings whose performance can be bounded by factors that depend on the underlying hidden variables.

Finally, in Section 5, we analyze how errors in optimizing the contrastive objectives propagate to the performance of these representations on downstream linear prediction tasks. We investigate this error
propagation for both the landmark and direct embeddings, and we show that the downstream prediction risk has a smooth relationship with the excess contrastive loss.

Along the way, we illustrate these results with running examples of a simplified topic modeling (Blei et al., 2003) as well as a simple Gaussian latent variable model. However, our results are applicable to other multi-view settings, including co-training, certain mixture models, hidden Markov models, and phylogenetic tree models (e.g., Blum and Mitchell, 1998; Dasgupta et al., 2002; Mossel and Roch, 2005; Chaudhuri et al., 2009; Allman et al., 2009; Anandkumar et al., 2012).

1.2 Related work
A number of recent works have sought to theoretically explain the success of contrastive learning specifically, and self-supervised learning more generally. Arora et al. (2019) presented a theoretical treatment of contrastive learning that considered the specific setting of trying to minimize the loss 

\[ L(\phi) = \mathbb{E}[\ell(\phi(X)^T(\phi(X_+) - \phi(X_-)))] \]

where \((X, X_+)\) is a random “positive” pair, \((X, X_-)\) is a random “negative” pair, and \(\ell\) is a binary classification loss such hinge or logistic loss. They showed that if there is an underlying collection of latent classes and positive examples are generated by draws from the same class, then minimizing the contrastive loss over embedding functions \(f\) yields good representations for distinguishing latent classes with linear models.

In work concurrent with the present paper, Lee et al. (2020) considered a self-supervised scheme in which two views \((X, Z)\) are available for each data point, and the representation learning objective is a reconstruction error of \(Z\) based on a function of \(X\):

\[ L(\phi) = \mathbb{E}\|Z - \phi(X)\|^2. \]

(Here, it is assumed that \(Z\) takes values in a suitable normed space.) They showed that if the two views are approximately independent conditioned on the label, then linear functions of the learned representation are capable of predicting the label. This approach bears resemblance to representation learning methods of Ando and Zhang (2005, 2007), as well as methods for learning predictive state representations (Littman and Sutton, 2002) and related approaches for learning dynamical systems (Hsu et al., 2009; Langford et al., 2009; Song et al., 2010). The self-supervised problem we study is instead a classification problem rather than a (possibly multidimensional output) regression problem.

Most relevant to the current work, Tosh et al. (2020) also considered the problem of contrastive learning under certain generative assumptions. Specifically, they showed that when the two views of the data point correspond to random partitions of a document, contrastive learning recovers information related to the underlying topics that generated the document. The contrastive learning problem they study is also a classification problem rather than a regression problem.

Also related is the application of self-supervised learning in the context of exploration in a model for reinforcement learning called Block MDPs (Du et al., 2019; Misra et al., 2020). In these settings, self-supervised learning is used to derive decoders of unobserved latent state from observations. The analyses in these works apply to cases where exact decoding of the state is possible. In particular, the method studied by Misra et al. (2020) uses a similar contrastive learning objective approach to the one that we analyze. In our work, we give an example that resembles the Block MDP, although our analyses applies more broadly to scenarios where latent variables cannot be perfectly decoded from the observations.

The contrastive estimation technique we study, now known as “Noise Contrastive Estimation” (NCE; Gutmann and Hyvärinen, 2010), was also theoretically analyzed in other contexts, including density level set estimation (Steinwart et al., 2005; Abe et al., 2006) and parametric estimation (Gutmann and Hyvärinen, 2010; Ma and Collins, 2018). Although the setups in these works do not consider the use of a learned
representation in a downstream task, NCE has inspired many empirical works that use the technique in this way. The primary motivation for NCE given in these works is the relationship between NCE and maximizing mutual information (e.g., Oord et al., 2018; Hjelm et al., 2018; Bachman et al., 2019; Tian et al., 2019), and the usefulness of the learned representation is attributed to this connection. Although this connection also makes an appearance in our work, it is subordinate to multi-view redundancy in our analysis. McAllester and Stratos (2020) highlight some limitations on measuring mutual information in these contexts, which raises some doubt that this mutual information perspective can solely explain the success of NCE. Other doubts about the mutual information perspective are raised by Tschannen et al. (2019).

2 Contrastive learning

In this section, we formalize contrastive learning in the multi-view setting, and introduce the redundancy assumption that is key to our analysis.

2.1 Multi-view data distribution and notation

We consider the multi-view setting, in which data points take the form \((x, z, y) \in \mathcal{X} \times \mathcal{Z} \times \mathbb{R}\), for some pair of data spaces \((\mathcal{X}, \mathcal{Z})\). Here \(x\) and \(z\) refer to the separate views of the data point, and \(y\) refers to its label or regression value. We will assume that there is some distribution over \((x, z, y)\) triples, and we will denote the corresponding random variables with capital letters \((X, Z, Y)\). We will also refer to the joint and marginal distributions of random variables with their corresponding letters. So, for example, the joint distribution over \((X, Z, Y)\) is denoted by \(p_{X,Z,Y}\), and the marginal distribution of just \(X\) is denoted by \(p_X\). For simplicity, we assume these random variables have either (joint) probability mass functions or probability density functions, and denote them by \(p_{X,Z,Y}\), \(p_X\), etc. In some examples, as well as in Section 4, we will introduce a hidden variable \(H\), and there we shall use \(p_{H|X}\) to denote the conditional distribution of \(H\) given \(X\). For instance, \(p_{H|X}(h \mid x) = \Pr(H = h \mid X = x)\) when \(H\) is discrete. Finally, we use the notation \(p \otimes q\) to refer to the product distribution with marginals \(p\) and \(q\).

Our main interest is in the semi-supervised learning setting, in which we have both unlabeled data from \(\mathcal{X} \times \mathcal{Z}\), typically modeled as i.i.d. copies of \((X, Z)\), as well as labeled data from \(\mathcal{X} \times \mathcal{Z} \times \mathbb{R}\), modeled as i.i.d. copies of \((X, Z, Y)\). In many cases, the unlabeled data are plentiful, whereas the labeled data are very few due to the cost of obtaining labels. In this setting, we will use contrastive learning on the unlabeled data to learn a representation that ultimately simplifies the downstream supervised learning task which uses the labeled data. (In particular, the downstream supervised learning will be accomplished using just linear predictors that, we prove, are competitive even with non-linear predictors.)

2.2 Contrastive distribution

Following Tosh et al. (2020), we define the contrastive distribution \(D_{\text{contrast}}\) via the following process:

- Let \((\tilde{X}, \tilde{Z})\) be an independent copy of \((X, Z)\), so each of \((X, Z)\) and \((\tilde{X}, \tilde{Z})\) has distribution \(p_{X,Z}\).
- Independently toss a fair coin; if heads, output \((X, Z, 1)\); otherwise, output \((X, \tilde{Z}, 0)\).

We let \((X_c, Z_c, Y_c) \sim D_{\text{contrast}}\). Note that \(Y_c\) has nothing to do with the random variable \(Y\); it is simply the outcome of the fair coin in the generative process above. Therefore, sampling from \(D_{\text{contrast}}\) can be accomplished using the process described above as long as one can sample from \(p_{X,Z}\)—the distribution of unlabeled data. In practice, this process provides a way to create a self-supervised data set of \((x_c, z_c, y_c)\) triples using only unlabeled data.
2.3 Contrastive learning problem

The goal of the contrastive learning problem is to find a predictor of $Y_c$ from $(X_c, Z_c)$; that is, predict whether the two views $(X_c, Z_c)$ are from the same data point (i.e., $(X_c, Z_c) \sim p_{X,Z}$), or from two independent data points (i.e., $(X_c, Z_c) \sim p_X \otimes p_Z$). This is a binary classification problem, and a standard approach for doing this is to minimize the expected cross-entropy loss between the label $Y_c$ and our prediction $f(X_c, Z_c)$. Formally, we solve the following optimization problem:

$$f^* \in \arg\min_{f: \mathcal{X} \times \mathcal{Z} \to \mathbb{R}} L_{lm}(f),$$

where

$$L_{lm}(f) := \mathbb{E}\left[Y_c \log \left(1 + \exp \left(-f(X_c, Z_c)\right)\right) + (1 - Y_c) \log \left(1 + \exp \left(f(X_c, Z_c)\right)\right)\right].$$

(The lm subscript is for “landmark”, discussed later.) Note that the optimal solution $f^*$ to (1) (over all functions from $\mathcal{X} \times \mathcal{Z}$ to $\mathbb{R}$) predicts the pointwise mutual information between two views:

$$f^*(x, z) = \log \frac{p_{X,Z}(x, z)}{p_X(x)p_Z(z)}.$$

Given $f^*$, it is easy to compute the density ratio of the joint distribution of $X$ and $Z$ and the product of their marginals:

$$g^*(x, z) := \exp \left(f^*(x, z)\right) = \frac{p_{X,Z}(x, z)}{p_X(x)p_Z(z)}.$$

This density ratio will be central to our analysis.

Of course, in practice, we typically cannot minimize the objective in (1) over all (measurable) functions directly. Instead, we may use an empirical approximation to the objective based on a finite sample, and (attempt to) minimize this approximation over a particular class of functions (e.g., neural networks). The issues that arise from such discrepancies are important, but a detailed study is beyond the scope of the present work. Our results in Section 5 provide some generic analysis of how errors introduced due to these concerns can affect performance in the downstream supervised learning task.

2.4 Redundancy

We will assume that for the task of predicting $Y$, there is a certain amount of redundancy between $X$ and $Z$. That is, the quantities

$$\varepsilon_X := \mathbb{E}\left[\left(\mathbb{E}[Y \mid X] - \mathbb{E}[Y \mid X, Z]\right)^2\right] \quad \text{and} \quad \varepsilon_Z := \mathbb{E}\left[\left(\mathbb{E}[Y \mid Z] - \mathbb{E}[Y \mid X, Z]\right)^2\right]$$

are both small. We opt not to formalize the redundancy assumption, since our theoretical results hold for any values of $\varepsilon_X$ and $\varepsilon_Z$. We stress that these predictors $(x \mapsto \mathbb{E}[Y \mid X = x], z \mapsto \mathbb{E}[Y \mid Z = z], (x, z) \mapsto \mathbb{E}[Y \mid (X, Z) = (x, z)])$ are not assumed to be linear.

Intuitively, when the redundancy assumption holds, one should be able to get a good prediction on $Y$ by first predicting $Z$ from $X$, and then using the resulting information to predict $Y$. This strategy is formalized by the following function:

$$\mu(x) := \mathbb{E}[\mathbb{E}[Y \mid Z] \mid X = x].$$

The following lemma tells us that this strategy does indeed work.

**Lemma 1.** $\mathbb{E}\left[(\mu(X) - \mathbb{E}[Y \mid X, Z])^2\right] \leq \varepsilon_X + 2\sqrt{\varepsilon_X\varepsilon_Z} + \varepsilon_Z =: \varepsilon_\mu.$

6
Proof. By the law of total expectation and Jensen’s inequality, we have
\[
\mathbb{E}[(\mu(X) - \mathbb{E}[Y \mid X])^2] = \mathbb{E}[(\mathbb{E}[\mathbb{E}[Y \mid Z] \mid X] - \mathbb{E}[Y \mid X])^2]
\]
\[
= \mathbb{E}[(\mathbb{E}[\mathbb{E}[Y \mid Z] - \mathbb{E}[Y \mid X, Z] \mid X])^2]
\]
\[
\leq \mathbb{E}[(\mathbb{E}[\mathbb{E}[Y \mid Z] - \mathbb{E}[Y \mid X, Z])^2 \mid X)] \quad \text{(Jensen’s inequality)}
\]
\[
= \mathbb{E}[(\mathbb{E}[Y \mid Z] - \mathbb{E}[Y \mid X, Z])^2]
\]
\[
= \varepsilon_Z.
\]
Using the AM/GM inequality, for any \(\lambda > 0\),
\[
\mathbb{E}[(\mu(X) - \mathbb{E}[Y \mid X, Z])^2] = \mathbb{E}[(\mathbb{E}[\mathbb{E}[Y \mid Z] - \mathbb{E}[Y \mid X, Z] + \mu(X) - \mathbb{E}[Y \mid X])^2]
\]
\[
\leq (1 + 1/\lambda)\mathbb{E}[(\mathbb{E}[\mathbb{E}[Y \mid Z] - \mathbb{E}[Y \mid X, Z])^2] + (1 + \lambda)\mathbb{E}[(\mu(X) - \mathbb{E}[Y \mid X])^2]
\]
\[
\leq (1 + 1/\lambda)\varepsilon_X + (1 + \lambda)\varepsilon_Z.
\]

Optimizing the bound with respect to \(\lambda\) gives
\[
\mathbb{E}[(\mu(X) - \mathbb{E}[Y \mid X, Z])^2] \leq \varepsilon_X + 2\sqrt{\varepsilon_X\varepsilon_Z} + \varepsilon_Z. \quad \square
\]

The function \(\mu\) is related to our contrastive function \(g^*\) in the following way:
\[
\mu(x) = \mathbb{E}[\mathbb{E}[Y \mid Z] \mid X = x] = \int \mathbb{E}[Y \mid Z = z]\frac{p_X(x, z)}{p_X(x)} \, dz = \int \mathbb{E}[Y \mid Z = z]g^*(x, z)p_Z(z) \, dz.
\]

Thus, \(g^*\) provides the change-of-measure from the marginal distribution of \(Z\) to the conditional distribution of \(Z\) given \(X = x\). Note that \(g^*\) depends only on (the distributions of) \(X\) and \(Z\); it does not depend on \(Y\) at all. Therefore \(g^*\) is useful for all prediction targets \(Y\) for which the redundancy assumption holds.

2.5 Landmark embeddings

How should one use \(g^*\) to produce a finite dimensional embedding of \(x \in \mathcal{X}\)? When \(g^*\) is implemented using a neural network, a common approach is construct a mapping defined by some internal hidden units (e.g., removing the top layer or two of the network). How well the resulting embedding fares in downstream tasks may depend on details of the implementation, such as the specific architecture and connection weights.

A different, and generic, approach to constructing an embedding from \(g^*\), proposed by Tosh et al. (2020), is to use the external behavior of \(g^*\) on a random sample \(Z_1, \ldots, Z_m\) of views, called landmarks, and embed according to
\[
\varphi^*(x) := (g^*(x, Z_1), \ldots, g^*(x, Z_m)).
\]
We assume \(Z_1, \ldots, Z_m\) are taken from i.i.d. copies \((X_1, Z_1, Y_1), \ldots, (X_m, Z_m, Y_m)\) of \((X, Z, Y)\). In practice, these landmarks can be taken from a random sample of \(m\) unlabeled data. (One can, of course, construct an embedding of \(z \in Z\) in a completely symmetric way.)

To gain intuition on why this approach is sound, note that if \(w \in \mathbb{R}^m\) satisfies \(w_i = \frac{1}{m}\mathbb{E}[Y_i \mid Z_i]\), then as \(m \to \infty\) we have
\[
w^T \varphi^*(x) = \frac{1}{m} \sum_{i=1}^m \mathbb{E}[Y_i \mid Z_i]g^*(x, Z_i) \to \int \mathbb{E}[Y \mid Z = z]g^*(x, z)p_Z(z) \, dz = \mu(x).
\]

Thus, in the limit, \(\varphi^*(x)\) provides a useful representation for downstream linear prediction under the redundancy assumption. In Section 3, we show that this holds for a finite number of landmarks as well.
2.6 Direct embeddings

An alternative to learning the bivariate function \( f^*(\cdot, \cdot) \) and using the landmark embedding technique is to directly search for embedding functions by solving the following optimization problem:

\[
(\eta^*, \psi^*) \in \arg\min_{\eta, \psi} L_{\text{direct}}((x, z) \mapsto \eta(x)^T \psi(z)),
\]

\[
L_{\text{direct}}(f) := \mathbb{E} \left[ Y \log \left( 1 + \frac{1}{f(X_c, Z_c)} \right) + (1 - Y) \log \left( 1 + f(X_c, Z_c) \right) \right]. \tag{2}
\]

Here, the minimization is over functions \( \eta: \mathcal{X} \to \mathbb{R}^m \) and \( \psi: \mathcal{Z} \to \mathbb{R}^m \) for some embedding dimension \( m \). Note that this is the same as minimizing the cross-entropy loss when our prediction on \((x, z) \in \mathcal{X} \times \mathcal{Z}\) is \( \log(\eta(x)^T \psi(z)) \). The loss in (2) bears a resemblance to the contrastive losses proposed by Hadsell et al. (2006), but differing in the fact that (a) the embedding functions are allowed to differ in the two views and (b) inner products are used as opposed to distances. Under similar conditions as the landmark setting, we show in Section 4 that solutions to (2) give rise to good linear predictors for the downstream task.

3 Landmark embedding representations

Recall the landmark embedding:

\[
\varphi^*(x) := (g^*(x, Z_1), \ldots, g^*(x, Z_m))
\]

for \( Z_1, \ldots, Z_m \) drawn i.i.d. from \( p_Z \). As illustrated in the previous section, linear functions of \( \varphi^* \) are capable of reproducing \( \mu \) in the limit as \( m \to \infty \). In this section, we analyze the approximation error that arises when we are restricted to a finite number of landmarks.

3.1 Landmark embedding error

The following lemma quantifies the error from using finite-dimensional landmark embeddings.

**Lemma 2.** Let \( (X_1, Z_1, Y_1), \ldots, (X_m, Z_m, Y_m), (X, Z, Y) \) be i.i.d., and suppose the landmarks used to define \( \varphi^* \) are \( Z_1, \ldots, Z_m \). With probability \( 1 - \delta \), there exists a weight vector \( w \in \mathbb{R}^m \) such that

\[
\mathbb{E}[|w^T \varphi^*(X) - \mu(X)|^2 \mid Z_1, \ldots, Z_m] \leq \varepsilon_{\text{lm}}
\]

where

\[
\varepsilon_{\text{lm}} := \frac{2}{m/\log_2(1/\delta)} \text{var}(\mathbb{E}[Y_1 \mid Z_1]g^*(X, Z_1)).
\]

**Proof.** We partition the \( m \) coordinates of the embedding into blocks of \( n := \lfloor m/\log_2(1/\delta) \rfloor \) coordinates per block. We first consider the part of the embedding corresponding to the first block, say, \( \varphi_{1:n}^* : \mathcal{X} \to \mathbb{R}^n \).

Define the weight vector \( v \in \mathbb{R}^n \) by

\[
v = v(Z_1, \ldots, Z_n) := \frac{1}{n} (\mathbb{E}[Y_1 \mid Z_1], \ldots, \mathbb{E}[Y_n \mid Z_n]).
\]

Define \( A_i(x) = \mathbb{E}[Y_i \mid Z_i]g^*(x, Z_i) - \mu(x) \) for all \( x \in \mathcal{X} \), so \( A_1(x), \ldots, A_n(x) \) are i.i.d. mean-zero random variables. This implies

\[
\mathbb{E}[(v^T \varphi_{1:n}^*(x) - \mu(x))^2] = \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^n A_i(x) \right)^2 \right] = \frac{\mathbb{E}[A_1(x)^2]}{n}.
\]
Now replacing $x$ with $X$ and taking expectations gives
\[E[(v^T \varphi^*_1(X) - \mu(X))^2] = \frac{E[A_1(X)^2]}{n} = \frac{\text{var}(E[Y_1 \mid Z_1]g^*(X, Z_1))}{n}.\]

By Markov’s inequality, the event
\[E[(v^T \varphi^*_1(X) - \mu(X))^2 \mid Z_1, \ldots, Z_m] \leq 2 \frac{\text{var}(E[Y_1 \mid Z_1]g^*(X, Z_1))}{n} \]
has probability at least $1/2$. We can analogously define such a “good” event for each block of coordinates. With probability at least $1 - \delta$, at least one of these good events occurs; in this event, we can pick any such “good” block, set the corresponding weights in $w$ according to the construction above, and set the remaining weights in $w$ to zero. This produces the desired guarantee.

The upshot of Lemma 2 is that to get an embedding that can linearly approximate $\mu$ within squared loss error of $\epsilon$, it suffices to embed using no more than $O\left(\frac{\text{var}(E[Y_1 \mid Z_1]g^*(X, Z_1))}{\epsilon}\right)$ landmarks.

Thus, we have the following immediate consequence of Lemma 1 and Lemma 2 (and the AM/GM inequality).

**Theorem 3.** Let $(X_1, Z_1, Y_1), \ldots, (X_m, Z_m, Y_m), (X, Z, Y)$ be i.i.d., and suppose the landmarks used to define $\varphi^*$ are $Z_1, \ldots, Z_m$. With probability $1 - \delta$, there exists a weight vector $w \in \mathbb{R}^m$ such that
\[E[(w^T \varphi^*(X) - \mathbb{E}[Y \mid X, Z])^2 \mid Z_1, \ldots, Z_m] \leq \epsilon_\mu + 2\sqrt{\epsilon_\mu \epsilon_{lm}} + \epsilon_{lm}\]
where $\epsilon_\mu$ is defined in Lemma 1 and $\epsilon_{lm}$ is defined in Lemma 2.

### 3.2 Topic model

We now turn to a simple topic modeling example to illustrate the error bound.

Let $P_1, \ldots, P_K$ be distributions over a finite vocabulary $V$. Each distribution corresponds to a topic, and we assume that their supports are disjoint; this is similar to the setting studied by Papadimitriou et al. (2000). For simplicity, suppose each document is exactly two tokens long, so that the two views $(X, Z)$ are individual tokens (and $X = Z = V$). We assume these tokens are drawn from a random mixture of $P_1, \ldots, P_K$, where the mixing weights are drawn from a symmetric Dirichlet distribution with parameter $\alpha > 0$ (following the LDA model of Blei et al., 2003). Thus the generative model for a single document is:

- Draw $\Theta = (\Theta_1, \ldots, \Theta_K) \sim \text{Dirichlet}(\alpha)$.

- Given $\Theta$, draw $X$ and $Z$ independently from the mixture distribution $\sum_{k=1}^{K} \Theta_k P_k$.

**Proposition 4.** Assume that $Y$ takes values in $[-1, 1]$. In the topic model setting,
\[\epsilon_{lm} \leq \begin{cases} O\left(\frac{\log(1/\delta)}{m}\right) & \text{if } \alpha = \Theta(1) \text{ as } K \to \infty; \\ O\left(\frac{K^2 \log(1/\delta)}{m}\right) & \text{if } \alpha \leq 1/K. \end{cases}\]

The proof is given in Appendix A.2. The $\alpha = \Theta(1)$ and $\alpha \leq 1/K$ correspond to the “non-sparse” and “sparse” regimes of LDA; here, sparsity is considered in an approximate sense (Telgarsky, 2013).
3.3 Gaussian model

As another example, we consider a simple multi-view Gaussian latent variable model:

\[ H \sim \mathcal{N}(0, \sigma^2), \]
\[ X \mid H \sim \mathcal{N}(H, 1), \]
\[ Z \mid H \sim \mathcal{N}(H, 1), \]

and we assume \( X \perp Z \mid H \).

**Proposition 5.** Assume that \( Y \) takes values in \([-1, 1]\). In the Gaussian model setting, for any \( \sigma^2 > 0 \),

\[ \varepsilon_{\text{im}} \leq \frac{2}{\lfloor m/\log_2(1/\delta) \rfloor} \cdot \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2} = O \left( \frac{\log(1/\delta)}{m} \right) \cdot \left( 1 + \frac{\sigma^4}{1 + 2\sigma^2} \right). \]

The proof is given in Appendix A.3. The analysis shows that the variance of \( H \) captures the difficulty of obtaining a low-dimensional representation in this model. We will revisit this example when we consider direct embeddings in Section 4.

4 Direct embeddings under hidden variable structure

The results of Section 3 demonstrate that contrastive learning, coupled with the landmark embedding method, produces representations that are useful for linear prediction. However, in practice, contrastive learning is not coupled with some landmark embedding. Rather, practitioners directly optimize \( \mathbb{R}^m \)-valued embedding functions and use these for downstream learning tasks.

Of crucial importance here is the size of the embedding dimension \( m \) needed to guarantee good performance on downstream linear predictions. In this section, we tackle this problem in the setting where there is some hidden variable \( H \) that renders the two views \( X \) and \( Z \) conditionally independent. Note that, in general, there is always such a random variable (by taking \( H = (X, Z) \)); however there may be a much more succinct hidden variable structure, and when this is the case, we show that relatively low-dimensional embeddings can achieve good predictive performance.

4.1 Discrete hidden variables

When \( H \) is a discrete random variable, taking values in some finite set \( S \), Tosh et al. (2020) observed that we may write

\[ g^\star(x, z) = \frac{p_{X,Z}(x, z)}{p_X(x)p_Z(z)} = \sum_{h \in S} \Pr(H = h)p_{X|H}(x \mid h)p_{Z|H}(z \mid h) \]
\[ = \sum_{h \in S} \Pr(H = h \mid X = x) \Pr(Z = z \mid H = h) = \eta^\star(x)^T \psi^\star(z) \]

where \( \eta^\star : \mathcal{X} \to \mathbb{R}^{|S|} \) and \( \psi^\star : \mathcal{Z} \to \mathbb{R}^{|S|} \) satisfy

\[ \eta^\star(x) = (\Pr(H = h \mid X = x))_{h \in S} \quad \text{and} \quad \psi^\star(z) = \frac{1}{p_Z(z)} (p_{Z|H}(z \mid h))_{h \in S}. \]

It is not too hard to show that there is a linear function of \( \eta^\star \) that reproduces \( \mu \). Namely, we may take \( w \in \mathbb{R}^{|S|} \) to be

\[ w = \mathbb{E}[\psi^\star(Z)\mathbb{E}[Y \mid Z]] = \int \psi^\star(z)\mathbb{E}[Y \mid Z = z]p_Z(z) \, dz. \]
For this choice of \( w \), we have
\[
    w^T \eta^*(x) = \int \eta^*(x)^T \psi^*(z) \mathbb{E}[Y \mid Z = z] p_Z(z) \, dz = \int \mathbb{E}[Y \mid Z = z] g^*(x, z) p_Z(z) \, dz = \mu(x).
\]

Models with discrete hidden variables are very common. They include multi-view mixture models from Chaudhuri et al. (2009) and Anandkumar et al. (2012), as well as models with richer hidden variable structure, such as hidden Markov models and phylogenetic trees (Mossel and Roch, 2005; Allman et al., 2009).

### 4.2 General hidden variables

In general, there may not be a discrete random variable that makes \( X \) and \( Z \) conditionally independent. However, there is always some random variable that does satisfy this. How much such a random variable buys us will naturally depend on its structure and relationship with the views \( X \) and \( Z \). We will look at two embedding constructions in the general random variable setting. The first is a natural, although possibly naïve, construction via discretizing the hidden variable state space. The second embedding construction is probabilistic and can result in a much lower embedding dimension.

However, before we proceed to the constructions, we first verify that approximate solutions to (2) are sufficient to get good predictions.

**Lemma 6.** For every \( \eta : \mathcal{X} \to \mathbb{R}^m \) and \( \psi : \mathcal{Z} \to \mathbb{R}^m \), there exists a \( w \in \mathbb{R}^m \) such that
\[
    \mathbb{E}[(w^T \eta(X) - \mu(X))^2] \leq \mathbb{E}[Y^2] \cdot \varepsilon_{\text{direct}}(\eta, \psi)
\]
where
\[
    \varepsilon_{\text{direct}}(\eta, \psi) := \mathbb{E} \left[ \left( \eta(X)^T \psi(\tilde{Z}) - g^*(X, \tilde{Z}) \right)^2 \right]
\]
and \((X, \tilde{Z}) \sim p_X \otimes p_Z\).

**Proof.** We take \( w \) to be
\[
    w := \mathbb{E} \left[ \mathbb{E}[Y \mid Z] \psi(Z) \right].
\]
Let \((\tilde{X}, \tilde{Z}, \tilde{Y})\) be an independent copy of \((Z, X, Y)\), and observe that
\[
    \mu(X) = \mathbb{E} \left[ \mathbb{E}[Y \mid Z] \mid X \right] = \mathbb{E} \left[ \mathbb{E}[\tilde{Y} \mid \tilde{Z}] g^*(X, \tilde{Z}) \mid X \right].
\]
Therefore
\[
    \mathbb{E}[(w^T \eta(X) - \mu(X))^2] = \mathbb{E} \left[ \left( \mathbb{E} \left[ \mathbb{E}[\tilde{Y} \mid \tilde{Z}] \psi(\tilde{Z}) \mid \eta(X) - \mu(X) \right] \right)^2 \right]
\]
\[
    = \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{E}[\tilde{Y} \mid \tilde{Z}] \psi(\tilde{Z})^T \eta(X) - \mathbb{E}[\tilde{Y} \mid \tilde{Z}] \eta(X) \mid X \right] \right] \mathbb{E} \left[ \left( \psi(\tilde{Z})^T \eta(X) - g^*(X, \tilde{Z}) \right)^2 \right]
\]
\[
    \leq \mathbb{E} \left[ \mathbb{E}[\tilde{Y}^2] \right] \mathbb{E} \left[ \left( \psi(\tilde{Z})^T \eta(X) - g^*(X, \tilde{Z}) \right)^2 \right]
\]
\[
    = \mathbb{E}[Y^2] \mathbb{E} \left[ \left( \psi(\tilde{Z})^T \eta(X) - g^*(X, \tilde{Z}) \right)^2 \right].
\]
where

Above, the first inequality follows from Cauchy-Schwarz, the subsequent equality uses the independence of \((\tilde{Z}, \tilde{Y})\) and \(X\), and the second the inequality follows from Jensen’s inequality and the law of total expectation.

Thus, it suffices to find embeddings whose inner product only approximates \(g^*\) with respect to \(p_X \otimes p_Z\).

### 4.2.1 A discretization construction

One generic approach to the general hidden variable case is to discretize the hidden variable state space. Specifically, if the hidden variables take values in \(\mathcal{H}\), find an \(\epsilon\)-covering \(h_1, \ldots, h_m\) of \(\mathcal{H}\) such that

\[
\sup_{h \in \mathcal{H}} \min_{i=1, \ldots, m} \mathbb{E} \left[ \left( \frac{p_{X|H}(X | h)p_{Z|H}(\tilde{Z} | h)}{p_X(X)p_Z(\tilde{Z})} - \frac{p_{X|H}(X | h_i)p_{Z|H}(\tilde{Z} | h_i)}{p_X(X)p_Z(\tilde{Z})} \right)^2 \right] \leq \epsilon,
\]

where \(\tilde{Z}\) is an independent copy of \(Z\). Let \(R_i \subseteq \mathcal{H}\) denote the set of \(h \in \mathcal{H}\) for which the min above is achieved by \(i\) (breaking ties arbitrarily). Given such a covering, we can define the embeddings

\[
\eta(x) = \frac{1}{p_X(x)} \left( \sqrt{p_H(R_i)p_X(x | h_i)} \right)^m, \quad \psi(z) = \frac{1}{p_Z(z)} \left( \sqrt{p_H(R_i)p_Z(z | h_i)} \right)^m
\]

where \(p_H(R_i) = \Pr(H \in R_i)\). Given this construction, it is not hard to show that \(\eta(x)^T\psi(z)\) approximates \(g^*(x, z)\) on average with respect to \(p_X \otimes p_Z\). By noting that \(R_1, \ldots, R_m\) partition the space \(\mathcal{H}\), we have

\[
\begin{align*}
\mathbb{E} \left[ (\eta(X)^T\psi(\tilde{Z}) - g^*(X, \tilde{Z}))^2 \right] & = \mathbb{E} \left[ \left( \sum_{i=1}^m p_H(R_i) \int_{R_i} \left( \frac{p_{X|H}(X | h_i)p_{Z|H}(\tilde{Z} | h_i)}{p_X(X)p_Z(\tilde{Z})} - \frac{p_{X|H}(X | h)p_{Z|H}(\tilde{Z} | h)}{p_X(X)p_Z(\tilde{Z})} \right) \frac{p_H(h)}{p_H(R_i)} \, dh \right)^2 \right] \\
& \leq \sum_{i=1}^m p_H(R_i) \int_{R_i} \mathbb{E} \left[ \left( \frac{p_{X|H}(X | h_i)p_{Z|H}(\tilde{Z} | h_i)}{p_X(X)p_Z(\tilde{Z})} - \frac{p_{X|H}(X | h)p_{Z|H}(\tilde{Z} | h)}{p_X(X)p_Z(\tilde{Z})} \right)^2 \right] \frac{p_H(h)}{p_H(R_i)} \, dh \\
& \leq \epsilon.
\end{align*}
\]

The first inequality follows from two applications of Jensen’s inequality and using linearity of expectation, while the second inequality comes from the construction of the \(\epsilon\)-covering as well as the fact that \(\sum_i p(R(h_i)) = 1\).

Thus, discretization gives us an \(\epsilon\)-approximate embedding, and therefore, by Lemma 6, gives us a useful linear representation.

**Topic model, revisited.** Let us return to our topic modeling example from Section 3.2. In this case, for any \(\theta, \tilde{\theta} \in \Delta^{K-1}\), it can be checked that

\[
\begin{align*}
\mathbb{E} \left[ \left( \frac{p_{X|\theta}(X | \theta)p_{Z|\theta}(\tilde{Z} | \theta)}{p_X(X)p_Z(\tilde{Z})} - \frac{p_{X|\tilde{\theta}}(X | \tilde{\theta})p_{Z|\tilde{\theta}}(\tilde{Z} | \tilde{\theta})}{p_X(X)p_Z(\tilde{Z})} \right)^2 \right] & = K^2 \sum_{k=1}^K \sum_{k'=1}^K (\theta_k \theta_{k'} - \tilde{\theta}_k \tilde{\theta}_{k'})^2.
\end{align*}
\]

Thus, the discretization argument above requires a covering of the space \(\{\theta\theta^T : \theta \in \Delta^{K-1}\}\) equipped with the Frobenius norm metric. As \(\Delta^{K-1}\) is a \((K - 1)\)-dimensional space, we can expect that such a covering will require size at least exponential in \(K\).
4.2.2 A probabilistic construction

Note that a direct embedding will be found by solving the optimization problem in (2). Thus, we do not need to give an explicit embedding construction. Rather, we only need to show the existence of embeddings \( \eta, \psi \) that are low-dimensional and are approximate solutions to (2). The following lemma does exactly that.

**Lemma 7.** Let \( H \) denote a random variable that renders \( X \) and \( Z \) conditionally independent. For any \( m > 0 \), there exist \( \eta: \mathcal{X} \to \mathbb{R}^m \) and \( \psi: \mathcal{Z} \to \mathbb{R}^m \) such that

\[
\mathbb{E} \left[ (\eta(X)^T \psi(\tilde{Z}) - g^*(X, \tilde{Z}))^2 \right] \leq \frac{1}{m} \text{var} \left( \frac{p_{X|H}(X | \hat{H})p_{Z|H}(\tilde{Z} | \hat{H})}{p_X(X)p_Z(\tilde{Z})} \right)
\]

where \((X, \tilde{Z}, \hat{H}) \sim p_X \otimes p_Z \otimes p_H\).

**Proof.** We will prove this using the probabilistic method, constructing a random embedding of dimension \( m \) that satisfies the lemma in expectation. This will suffice to show that there exists such an embedding.

Let \( H_1, \ldots, H_m \) be i.i.d. copies of \( H \). Define, for each \((x, z) \in \mathcal{X} \times \mathcal{Z}\),

\[
\eta(x) = \frac{1}{p_X(x)\sqrt{m}}(p_{X|H}(x | H_i))_{i=1}^m
\]

\[
\psi(z) = \frac{1}{p_Z(z)\sqrt{m}}(p_{Z|H}(z | H_i))_{i=1}^m
\]

\[
B_i(x, z) = \frac{p_{X|H}(x | H_i)p_{Z|H}(z | H_i)}{p_X(x)p_Z(z)} - g^*(x, z).
\]

Observe that \( B_1(x, z), \ldots, B_m(x, z) \) are i.i.d. mean-zero random variables, and

\[
\eta(x)^T \psi(z) - g^*(x, z) = \frac{1}{m} \sum_{i=1}^m B_i(x, z),
\]

and

\[
\mathbb{E}[(\eta(x)^T \psi(z) - g^*(x, z))^2] = \frac{\mathbb{E}[B_1(x, z)^2]}{m}.
\]

Now replacing \((x, z)\) with \((X, \tilde{Z}) \sim p_X \otimes p_Z\) and taking expectations gives

\[
\mathbb{E} \left[ (\eta(X)^T \psi(\tilde{Z}) - g^*(X, \tilde{Z}))^2 \right] = \frac{\mathbb{E}[B_1(X, \tilde{Z})^2]}{m} = \frac{1}{m} \text{var} \left( \frac{p_{X|H}(X | \hat{H})p_{Z|H}(\tilde{Z} | \hat{H})}{p_X(X)p_Z(\tilde{Z})} \right).
\]

4.2.3 Examples

We revisit the topic model and Gaussian model examples from Section 3.2 and Section 3.3.

**Proposition 8.** Assume that \( Y \) takes values in \([-1, 1]\). In the topic model setting, there exists \( \eta: \mathcal{X} \to \mathbb{R}^m \) and \( \psi: \mathcal{Z} \to \mathbb{R}^m \) such that

\[
\varepsilon_{\text{direct}}(\eta, \psi) \leq \begin{cases} O \left( \frac{1}{m} \right) & \text{if } \alpha = \Theta(1) \text{ as } K \to \infty; \\ O \left( \frac{K^2}{m} \right) & \text{if } \alpha \leq 1/K. \end{cases}
\]
The proof is given in Appendix A.1. This is essentially the same bound as what was obtained in Proposition 4 for the landmark embedding. We see that even though the hidden variable structure is not discrete, we still obtain bounds that are polynomial in the dimension of the hidden variable.

**Proposition 9.** Assume that $Y$ takes values in $[-1, 1]$. In the Gaussian model setting, if $\sigma^2 < \frac{1}{2}$, then there exist $\eta: \mathcal{X} \to \mathbb{R}^m$ and $\psi: \mathcal{Z} \to \mathbb{R}^m$ such that

$$
\varepsilon_{\text{direct}}(\eta, \psi) \leq \frac{1}{m} \cdot \frac{(1 + \sigma^2)^2}{\sqrt{1 - 4\sigma^2}}.
$$

The proof is given in Appendix A.4. Here, we note that the existence argument for $\eta$ and $\psi$ requires a stronger condition than what was required for the landmark embedding to work. This is reflected in the condition $\sigma^2 < \frac{1}{2}$.

5 Error analysis

We now turn to the problem of bounding the error in the representation incurred by lack of data, imprecise optimization, or restricted function classes. Specifically, we will be interested in the following measure of risk for an embedding function $\phi: \mathcal{X} \to \mathbb{R}^m$:

$$
R(\phi) := \inf_{w \in \mathbb{R}^m} \mathbb{E} \left[ (w^T \phi(X) - \mu(X))^2 \right].
$$

(3)

To obtain a guarantee on the mean squared error in approximating $\mathbb{E}[Y \mid X, Z]$, we can simply appeal to Lemma 1.

Our goal is to relate the risk of an embedding to the excess loss in terms of either $L_{\text{lm}}$ from (1) or $L_{\text{direct}}$ from (2). Note that these two loss functions operate on different scales: the minimizer of $L_{\text{lm}}$ is the log-odds ratio function $\log \circ g^*$, and the minimizer of $L_{\text{direct}}$ is the odds ratio function $g^*$.

5.1 Error analysis for landmark embeddings

We begin with the error analysis for the landmark embedding method. The analysis is a modification of the argument from Tosh et al. (2020).

**Theorem 10.** Assume $Y$ takes values in $[-1, 1]$, and let $g_{\text{max}} := \sup_{(x,z) \in \text{supp} p_X \otimes p_Z} g^*(x, z)$. Pick any $f: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $g := \exp \circ f$ satisfies $\sup_{(x,z) \in \text{supp} p_X \otimes p_Z} g(x, z) \leq g_{\text{max}}$. Let $\varphi: \mathcal{X} \to \mathbb{R}^m$ be the embedding function given by

$$
\varphi(x) = (g(x, Z_1), \ldots, g(x, Z_m))
$$

where $Z_1, \ldots, Z_m$ are i.i.d. copies of $Z$. With probability at least $1 - \delta$ (over the realization of $Z_1, \ldots, Z_m$),

$$
R(\varphi) \leq 2\varepsilon_{\text{lm}} + 4(1 + g_{\text{max}})^2 \sqrt{2\varepsilon_{\text{opt,lm}}\varepsilon_{\text{lm}}} + 16(1 + g_{\text{max}})^4 \varepsilon_{\text{opt,lm}}^2,
$$

where $\varepsilon_{\text{lm}}$ is defined in Lemma 2 and $\varepsilon_{\text{opt,lm}}$ is the excess $L_{\text{lm}}$-loss:

$$
\varepsilon_{\text{opt,lm}} := L_{\text{lm}}(f) - L_{\text{lm}}(\log \circ g^*).
$$

**Proof.** Let $(X_1, Z_1, Y_1), \ldots, (X_m, Z_m, Y_m), (X, Z, Y)$ be i.i.d., and let $\varphi^*(x)_i := g^*(x, Z_i)$ for all $i = 1, \ldots, m$. Let $n := \lceil m/\log_2(1/\delta) \rceil$. We shall adopt the same block repetition argument as in Lemma 2, where the $m$ coordinates are partitioned into groups of $n$ coordinates each.
We first analyze what happens in the first block of coordinates. From the arguments in Lemma 2, we know with probability \( \geq 3/4 \),
\[
\mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Y_i | Z_i] g^*(X, Z_i) - \mu(X) \right)^2 | Z_1, \ldots, Z_m \leq \frac{4 \text{var}(\mathbb{E}[Y_i | Z_i] g^*(X, Z_i))}{n} = 2\varepsilon_{\text{lm}}.
\]
(4)

We also claim that, with probability \( \geq 3/4 \),
\[
\mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} (g(X, Z_i) - g^*(X, Z_i))^2 | Z_1, \ldots, Z_m \right) \leq 4(1 + g_{\text{max}})\varepsilon_{\text{opt,lm}}.
\]
(5)

To see this, we make the following definitions
\[
p^*(x, z) := \frac{g^*(x, z)}{1 + g^*(x, z)}, \quad \text{and} \quad p(x, z) := \frac{g(x, z)}{1 + g(x, z)}.
\]

Recall that \((X_c, Z_c, Y_c) \sim \mathcal{D}_{\text{contrast}}\). Now we have
\[
\varepsilon_{\text{opt,lm}} = L_{\text{lm}}(f) - \inf_{f^* : \mathcal{X} \times Z \rightarrow \mathbb{R}} L_{\text{lm}}(\log \circ g^*)
\]
\[
= \mathbb{E} Y_c \log \left( \frac{p^*(X_c, Z_c)}{p(X_c, Z_c)} \right) + (1 - Y_c) \log \left( \frac{1 - p^*(X_c, Z_c)}{1 - p(X_c, Z_c)} \right)
\]
\[
= \mathbb{E} \left[ p^*(X_c, Z_c) \log \left( \frac{p^*(X_c, Z_c)}{p(X_c, Z_c)} \right) + (1 - p^*(X_c, Z_c)) \log \left( \frac{1 - p^*(X_c, Z_c)}{1 - p(X_c, Z_c)} \right) \right]
\]
where the second-to-last line follows from the fact that \( g^* \) is the odds ratio for the contrastive learning problem and \( \text{KL}(p, q) \) denotes the KL divergence between two Bernoulli random variables. Pinsker’s inequality tells us that, for any \((x, z) \in \mathcal{X} \times \mathcal{Z}\),
\[
\text{KL}(p^*(x, z), p(x, z)) \geq 2(p^*(x, z) - p(x, z))^2 \geq \frac{2}{(1 + g_{\text{max}})^4} (g^*(x, z) - g(x, z))^2.
\]

Since \((X_c, Z_c) \sim \frac{1}{2} p_{X,Z} + \frac{1}{2} p_X \otimes p_Z\),
\[
\mathbb{E} \left[ (g^*(X_c, Z_c) - g(X_c, Z_c))^2 \right] = \frac{1}{2} \mathbb{E} \left[ (g^*(X, Z) - g(X, Z))^2 \right] + \frac{1}{2} \mathbb{E} \left[ (g^*(X, Z_1) - g(X, Z_1))^2 \right]
\]
\[
\geq \frac{1}{2} \mathbb{E} \left[ (g^*(X, Z_1) - g(X, Z_1))^2 \right].
\]

Therefore, we conclude that
\[
\mathbb{E} \left[ (g^*(X, Z_1) - g(X, Z_1))^2 \right] \leq (1 + g_{\text{max}})^4 \varepsilon_{\text{opt,lm}},
\]
and hence also
\[
\mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} (g^*(X, Z_i) - g(X, Z_i))^2 \right) \leq (1 + g_{\text{max}})^4 \varepsilon_{\text{opt,lm}}.
\]
By Markov’s inequality, (5) holds with probability \(3/4\). A union bound grants that (4) and (5) hold simultaneously with probability \(\geq 1/2\). Call this the “good” event for this first block of landmarks.

Now considering all blocks, with probability \(1 - \delta\), the good event holds for at least one group of coordinates. As in the proof of Lemma 2, we will set \(w_i = \frac{1}{n}\mathbb{E}[Y_i | Z_i]\) for the coordinates in the good group and we set \(w_i = 0\) for all other coordinates. Thus, with probability \(1 - \delta\) we can conclude two facts. First, that \(\varphi^*\) satisfies

\[
\mathbb{E}\left[\left(\mathbf{w}^T \varphi^*(X) - \mu(X) \mid Z_1, \ldots, Z_m\right)^2\right] \leq 2\varepsilon_{\text{lm}}.
\]

Second, there is some block of \(n\) coordinates (which we take to be \(\{1, \ldots, n\}\)) without loss of generality such that

\[
\mathbb{E}\left[\left(\mathbf{w}^T \varphi(X) - \mathbf{w}^T \varphi^*(X)\right)^2 \mid Z_1, \ldots, Z_m\right] \leq \left\|\mathbf{w}\right\|^2 \cdot \mathbb{E}\left[\left\|\varphi(X) - \varphi^*(X)\right\|^2 \mid Z_1, \ldots, Z_m\right] \\
\leq \frac{1}{n} \mathbb{E}\left[\left\|\varphi(X) - \varphi^*(X)\right\|^2 \mid Z_1, \ldots, Z_m\right] \\
= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} (g(X, Z_i) - g^*(X, Z_i))^2 \mid Z_1, \ldots, Z_m\right] \\
\leq 4(1 + g_{\text{max}})\varepsilon_{\text{opt,lm}}
\]

where the first inequality follows from Cauchy-Schwarz, the second inequality comes from the boundedness of \(Y\), and the third inequality is (5). Putting it all together with the AM/GM inequality gives us the theorem statement.

\[\square\]

### 5.2 Error analysis for direct embeddings

We now turn to bounding the risk associated with direct embeddings.

**Theorem 11.** Let \(g_{\text{max}} := \sup_{(x,z) \in \text{supp}_{\mathcal{P}_X \otimes \mathcal{P}_Z}} g^*(x,z)\). Pick any embedding functions \(\eta: \mathcal{X} \rightarrow \mathbb{R}^m\) and \(\psi: \mathcal{Z} \rightarrow \mathbb{R}^m\) such that \(\sup_{(x,z) \in \text{supp}_{\mathcal{P}_X \otimes \mathcal{P}_Z}} \langle \eta(x) \psi(z) \rangle \leq g_{\text{max}}\). We have

\[
R(\eta) \leq \mathbb{E}[Y^2] \left(1 + g_{\text{max}}\right)^4 \varepsilon_{\text{opt,direct}}
\]

where \(\varepsilon_{\text{opt,direct}}\) is the excess \(L_{\text{direct}}\)-loss:

\[
\varepsilon_{\text{opt,direct}} := L_{\text{direct}}((x, z) \mapsto \eta(x)^T \psi(z)) - L_{\text{direct}}(g^*)
\]

We point out that \(\varepsilon_{\text{opt,direct}}\) is the excess loss relative to the odds ratio \(g^*\), and not necessarily the best \(m\)-dimensional representation. Thus, when there are no perfect \(m\)-dimensional embedding functions \(\eta^*, \psi^*\) satisfying \(g^*(x,z) = \eta^*(x)^T \psi^*(z)\), the quantity \(\varepsilon_{\text{opt}}\) accounts for both the error due to optimization and the error due to representational non-realizability. Ideally, one would like an error transformation result that teases apart the contributions of both of these factors, but this is a challenging problem in general that we leave to future work. Thus, this result is primarily relevant for the settings where finite dimensional embeddings can exactly represent \(g^*\), as discussed in Section 4, so that \(\varepsilon_{\text{opt}}\) actually does correspond to the excess loss from imprecise optimization.

**Proof.** From Lemma 6,

\[
R(\eta) = \inf_{\mathbf{w} \in \mathbb{R}^m} \mathbb{E}\left[\left(\mathbf{w}^T \eta(X) - \mu(X)\right)^2\right] \leq \mathbb{E}[Y^2] \mathbb{E}\left[\left(\eta(X)^T \psi(Z) - g^*(X, Z)\right)^2\right]
\]
Therefore, we focus on bounding the second factor on the right-hand side. For the most part, the proof uses similar arguments as in that of Theorem 10.

Using the definitions

\[ p^*(x, z) := \frac{g^*(x, z)}{1 + g^*(x, z)} \quad \text{and} \quad p(x, z) := \frac{\eta(x)^T \psi(z)}{1 + \eta(x)^T \psi(z)} \]

we have

\[ \varepsilon_{\text{opt, direct}} = L_{\text{direct}}((x, z) \mapsto \eta(x)^T \psi(z)) - L_{\text{direct}}(g^*) \]

\[ = \mathbb{E} \left[ Y_c \log \left( \frac{p^*(X_c, Z_c)}{p(X_c, Z_c)} \right) + (1 - Y_c) \log \left( \frac{1 - p^*(X_c, Z_c)}{1 - p(X_c, Z_c)} \right) \right] \]

\[ = \mathbb{E} \left[ p^*(X_c, Z_c) \log \left( \frac{p^*(X_c, Z_c)}{p(X_c, Z_c)} \right) + (1 - p^*(X_c, Z_c)) \log \left( \frac{1 - p^*(X_c, Z_c)}{1 - p(X_c, Z_c)} \right) \right] \]

\[ = \mathbb{E} \left[ \text{KL}(p^*(X_c, Z_c), p(X_c, Z_c)) \right]. \]

By Pinsker’s inequality, for any \((x, z) \in \mathcal{X} \times \mathcal{Z},\)

\[ \text{KL}(p^*(x, z), p(x, z)) \geq 2(p^*(x, z) - p(x, z))^2 \geq \frac{2}{(1 + g_{\text{max}})^4} (g^*(x, z) - \eta(x)^T \psi(z))^2. \]

Finally, since \((X_c, Z_c) \sim \frac{1}{2} p_{X,Z} + \frac{1}{2} p_X \otimes p_Z,\)

\[ \mathbb{E} \left[ (g^*(X_c, Z_c) - \eta(X_c)^T \psi(Z_c))^2 \right] \geq \frac{1}{2} \mathbb{E} \left[ (g^*(X, Z) - \eta(X)^T \psi(Z))^2 \right]. \]

Putting it all together gives us the theorem statement.

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A Omitted proofs

A.1 Proof of Proposition 8

By Lemma 7, it suffices to bound

$$\text{var} \left( \frac{p_{X|\Theta}(X|\Theta)p_{Z|\Theta}(\tilde{Z}|\Theta)}{p_X(X)p_Z(Z)} \right),$$

for \((X, \tilde{Z}, \Theta) \sim p_X \otimes p_Z \otimes p_\Theta\). This, in turn, is bounded above by

$$E \left[ \left( \frac{p_{X|\Theta}(X|\Theta)p_{Z|\Theta}(\tilde{Z}|\Theta)}{p_X(X)p_Z(Z)} \right)^2 \right].$$

So Proposition 8 follows immediately from the following Proposition.

Proposition 12.

$$E \left[ \left( \frac{p_{X|\Theta}(X|\Theta)p_{Z|\Theta}(\tilde{Z}|\Theta)}{p_X(X)p_Z(Z)} \right)^2 \right] = \begin{cases} \Theta(1) & \text{if } \alpha = \Theta(1) \text{ as } K \to \infty; \\ \Theta(K^2) & \text{if } \alpha \leq 1/K. \end{cases}$$

Proof. For any word \(v \in \mathcal{V}\), let \(k(v) \in [K]\) denote the unique topic for which \(P_{k}(v) > 0\). For any \(v \in \mathcal{V}\) and \(\theta \in \Delta^{K-1}\), we have

$$p_X(v) = \frac{1}{K} P_{k(v)}(v),$$

$$p_{X|\Theta}(v|\theta) = \theta_{k(v)} P_{k(v)}(v).$$

Therefore, we have for any \(\theta \in \Delta^{K-1}, x \in \mathcal{V}, \text{and } z \in \mathcal{V},\)

$$\left( \frac{p_{X|\Theta}(x|\theta)p_{Z|\Theta}(z|\theta)}{p_X(x)p_Z(z)} \right)^2 = K^4 \cdot \theta^2_{k(x)} \theta^2_{k(z)}.$$ 

Replacing \((x, z, \theta)\) with \((X, \tilde{Z}, \Theta)\) and taking expectations gives

$$E \left[ \left( \frac{p_{X|\Theta}(X|\Theta)p_{Z|\Theta}(\tilde{Z}|\Theta)}{p_X(X)p_Z(Z)} \right)^2 \right] = E \left[ K^4 \cdot \Theta^2_{k(X)} \Theta^2_{k(\tilde{Z})} \right]$$

$$= K^4 \cdot \sum_{k=1}^{K} \sum_{k'=1}^{K} \Pr(k(X) = k) \Pr(k(\tilde{Z}) = k') E \left[ \Theta^2_{k} \Theta^2_{k'} \right]$$

$$= K^2 \cdot \sum_{k=1}^{K} \sum_{k'=1}^{K} E \left[ \Theta^2_{k} \Theta^2_{k'} \right]$$

$$= K^3 \cdot \left( E[\Theta^4_1] + (K - 1)E[\Theta^2_1 \Theta^2_2] \right)$$

where the fourth and fifth steps follow by symmetry. The fourth-moments in the final expression are:

$$E[\Theta^2_1 \Theta^2_2] = \frac{\Gamma(K \alpha)}{\Gamma(K \alpha + 4)} \cdot \left( \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha)} \right)^2$$

$$E[\Theta^4_1] = \frac{\Gamma(K \alpha)}{\Gamma(K \alpha + 4)} \cdot \frac{\Gamma(\alpha + 4)}{\Gamma(\alpha)}.$$

Therefore, we have the following:
1. For $\alpha = \Theta(1)$ and $K \to \infty$,

$$
\mathbb{E} \left[ \left( \frac{p_{X|\Theta}(X \mid \hat{\Theta})p_{Z|\Theta}(Z \mid \hat{\Theta})}{p_X(X)p_Z(Z)} \right)^2 \right] = K^3 \left( \Theta \left( \frac{1}{K^4} \right) + \Theta \left( \frac{1}{K^3} \right) \right) = \Theta(1).
$$

2. For $\alpha \leq 1/K$,

$$
\mathbb{E} \left[ \left( \frac{p_{X|\Theta}(X \mid \hat{\Theta})p_{Z|\Theta}(Z \mid \hat{\Theta})}{p_X(X)p_Z(Z)} \right)^2 \right] = K^3 \left( \Theta \left( \frac{1}{K^2} \right) + \Theta(\alpha) \right) = \Theta(K^2).
$$

A.2 Proof of Proposition 4

We first make a simple observation about the relationship between terms in the definitions of $\epsilon_{\text{lm}}$ and $\epsilon_{\text{direct}}$.

Proposition 13. If $X \perp Z \mid H$, then for any $(x,z) \in \mathcal{X} \times \mathcal{Z}$,

$$
g^*(x,z)^2 \leq \mathbb{E} \left[ \left( \frac{p_{X|H}(x \mid H)p_{Z|H}(z \mid H)}{p_X(x)p_Z(z)} \right)^2 \right]
$$

where $H \sim p_H$.

Proof. Fix any $(x,z) \in \mathcal{X} \times \mathcal{Z}$. Then

$$
g^*(x,z)^2 = \left( \frac{p_{X,Z}(x,z)}{p_X(x)p_Z(z)} \right)^2
$$

$$
= \frac{1}{(p_X(x)p_Z(z))^2} \left( \int p_{X,Z,H}(x,z,h) \, dh \right)^2
$$

$$
= \frac{1}{(p_X(x)p_Z(z))^2} \left( \int p_{X|H}(x \mid h)p_{Z|H}(z \mid h)p_H(h) \, dh \right)^2
$$

$$
\leq \frac{1}{(p_X(x)p_Z(z))^2} \int \left( p_{X|H}(x \mid h)p_{Z|H}(z \mid h) \right)^2 p_H(h) \, dh
$$

$$
= \mathbb{E} \left[ \left( \frac{p_{X|H}(x \mid H)p_{Z|H}(z \mid H)}{p_X(x)p_Z(z)} \right)^2 \right],
$$

where the inequality follows from Jensen’s inequality.

Now we return to the proof of Proposition 4. Since $Y$ takes values in $[-1,1]$, it suffices to bound $\mathbb{E}[g^*(X, Z_1)^2]$. By Proposition 13, we have

$$
g^*(X, Z_1) \leq \mathbb{E} \left[ \left( \frac{p_{\Theta}(X \mid \hat{\Theta})p_{Z|\Theta}(Z \mid \hat{\Theta})}{p_X(X)p_Z(Z)} \right)^2 \mid X, Z_1 \right]
$$

After taking expectations on both sides, the result follows by Proposition 12.

We note that a direct analysis of $\mathbb{E}[g^*(X, Z_1)^2]$ is also straightforward, and would ultimately involve only second-moments of $\Theta$ (as opposed to fourth-moments, considered in Proposition 12). However, the final bound is the same.
A.3 Proof of Proposition 5

The marginal distribution of \((X, Z)\) is \(\mathcal{N}(0, \Sigma)\), where

\[
\Sigma := \begin{pmatrix} 1 + \sigma^2 & \sigma^2 \\ \sigma^2 & 1 + \sigma^2 \end{pmatrix}.
\]

Thus, letting \(v := (x, z)\), we have

\[
p_{X,Z}(v) = \frac{1}{2\pi\sqrt{1 + 2\sigma^2}} \exp \left( -\frac{1}{2} v^\top \Sigma^{-1} v \right)
\]

\[
p_X(x) = \frac{1}{\sqrt{2\pi(1 + \sigma^2)}} \exp \left( -\frac{x^2}{2(1 + \sigma^2)} \right)
\]

\[
p_Z(z) = \frac{1}{\sqrt{2\pi(1 + \sigma^2)}} \exp \left( -\frac{z^2}{2(1 + \sigma^2)} \right).
\]

Therefore, for any fixed \((x, z) \in \mathbb{R}^2\),

\[
g^\star(x, z)^2 = \frac{p_{X,Z}(x, z)^2}{p_X(x)^2 p_Z(z)^2} = \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2} \exp \left( -v^\top \Sigma^{-1} v + \frac{x^2 + z^2}{1 + \sigma^2} \right)
\]

\[
= \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2} \exp \left( v^\top A v \right)
\]

where

\[
A := -\frac{\sigma^2}{1 + 2\sigma^2} \begin{pmatrix} \frac{\sigma^2}{1 + \sigma^2} & -1 \\ -1 & \frac{\sigma^2}{1 + \sigma^2} \end{pmatrix} = \lambda_1 u_1 u_1^\top + \lambda_2 u_2 u_2^\top
\]

has eigenvalues \(\lambda_1 = \frac{\sigma^2}{(1 + \sigma^2)(1 + 2\sigma^2)}\) and \(\lambda_2 = -\frac{\sigma^2}{1 + 2\sigma^2}\) corresponding to some orthonormal eigenvectors \(u_1\) and \(u_2\). Now replacing \((x, z)\) with \(V := (X, \tilde{Z}) \sim p_X \otimes p_Z\) and taking expectation gives

\[
\mathbb{E}[g^\star(X, \tilde{Z})^2] = \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2} \mathbb{E} \left[ \exp \left( \lambda_1 (u_1^\top V)^2 + \lambda_2 (u_2^\top V)^2 \right) \right]
\]

\[
= \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2} \mathbb{E} \left[ \exp \left( \lambda_1 (u_1^\top V)^2 \right) \right] \mathbb{E} \left[ \exp \left( \lambda_2 (u_2^\top V)^2 \right) \right]
\]

\[
= \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2} \cdot \frac{1}{\sqrt{1 - 2\lambda_1 (1 + \sigma^2)}} \cdot \frac{1}{\sqrt{1 - 2\lambda_2 (1 + \sigma^2)}}
\]

\[
= \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2} \cdot \frac{1}{\sqrt{1 - \frac{2\sigma^2}{1 + 2\sigma^2}}} \cdot \frac{1}{\sqrt{1 + 2\sigma^2}}
\]

\[
= \frac{(1 + \sigma^2)^2}{1 + 2\sigma^2}.
\]

Above, we use the fact that \(u_1^\top V\) and \(u_2^\top V\) are independent \(\mathcal{N}(0, 1 + \sigma^2)\) random variables. \(\square\)
A.4 Proof of Proposition 9

The proof is similar to that of Proposition 5. Using similar computations, we obtain for any fixed \((x, z) \in \mathbb{R}^2\),

\[
E \left[ \left( \frac{p_{X|H}(x | H)p_{Z|H}(z | H)}{p_X(x)p_Z(z)} \right)^2 \right] = \frac{(1 + \sigma^2)^2}{\sqrt{1 + 4\sigma^2}} \exp \left( \frac{2\sigma^2(x + z)^2}{1 + 4\sigma^2} - \frac{\sigma^2(x^2 + z^2)}{1 + \sigma^2} \right)
\]

\[
= \frac{(1 + \sigma^2)^2}{\sqrt{4\sigma^2 + 1}} \exp (u^T A u)
\]

where

\[
A := \sigma^2 \left( \frac{1 - 2\sigma^2}{2} \frac{2}{1 + \sigma^2} \right) = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T
\]

has eigenvalues \(\lambda_1 = \frac{3\sigma^2}{(1 + \sigma^2)(1 + 4\sigma^2)}\) and \(\lambda_2 = -\frac{\sigma^2}{1 + \sigma^2}\) corresponding to some orthonormal eigenvectors \(u_1\) and \(u_2\). Now replacing \((x, z)\) with \(V := (X, \tilde{Z}) \sim p_X \otimes p_Z\) and taking expectation gives, for \((X, \tilde{Z}, \tilde{H}) \sim p_X \otimes p_Z \otimes p_H\),

\[
E \left[ \left( \frac{p_{X|H}(X | \tilde{H})p_{Z|H}(\tilde{Z} | \tilde{H})}{p_X(X)p_Z(\tilde{Z})} \right)^2 \right] = \frac{(1 + \sigma^2)^2}{\sqrt{4\sigma^2 + 1}} \mathbb{E} \left[ \exp \left( \frac{3\sigma^2(u_1^T V)^2}{(1 + \sigma^2)(1 + 4\sigma^2)} - \frac{\sigma^2(u_2^T V)^2}{1 + \sigma^2} \right) \right].
\]

Since \(u_1^T V\) and \(u_2^T V\) are independent \(N(0, 1 + \sigma^2)\) random variables, this expression simplifies to

\[
E \left[ \left( \frac{p_{X|H}(X | \tilde{H})p_{Z|H}(\tilde{Z} | \tilde{H})}{p_X(X)p_Z(\tilde{Z})} \right)^2 \right] = \frac{(1 + \sigma^2)^2}{\sqrt{4\sigma^2 + 1}} \cdot \frac{1}{\sqrt{1 - \frac{6\sigma^2}{1 + 4\sigma^2}}} \cdot \frac{1}{\sqrt{1 + 2\sigma^2}} = \frac{(1 + \sigma^2)^2}{\sqrt{1 - 4\sigma^4}}.
\]

The condition \(\sigma^2 < 1/2\) is used to ensure that the expectation in the last equation display is finite. \(\square\)