ON THE COMPATIBILITY BETWEEN BASE CHANGE AND HECKE ORBITS OF HILBERT NEWFORMS

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Let \( F/E \) be a Galois extension of totally real number fields, with Galois group \( \text{Gal}(F/E) \). Let \( \mathfrak{N} \) be an integral ideal which is \( \text{Gal}(F/E) \)-invariant, and \( k \geq 2 \) an integer. In this note, we study the action of \( \text{Gal}(F/E) \) on the Hecke orbits of Hilbert newforms of level \( \mathfrak{N} \) and weight \( k \). We also discuss the geometric counterpart to this action, which is closely related to the notion of abelian varieties potentially of \( \text{GL}_2 \)-type. The two actions have some consequences in relation with Langlands Functoriality.

We conclude with an example over the maximal totally real subfield \( F = \mathbb{Q}(\zeta_{32}) \) of the cyclotomic field of 32nd root of unity. Let \( D \) be the quaternion algebra over \( F \) ramified exactly at the unique prime above 2 and 7 real places, and \( X_D^0(1) \) the Shimura curve attached to \( D \). Among other things, our example shows that the field of 2-torsion of the Jacobian of the curve \( X_D^0(1) \) (and its Atkin-Lehner quotient) is the unique Galois extension \( \mathbb{N}/\mathbb{Q} \) unramified outside 2, with Galois group the Frobenius group \( F_{17} \). This completes Noam Elkies’ answer [Elk15] to a question posed by Jeremy Rouse on mathoverflow.net.

Introduction

Let \( F \) be a totally real number field of degree \( d \), with ring of integers \( \mathcal{O}_F \). Let \( \mathfrak{N} \) be an integral ideal of \( F \) and \( k \geq 2 \) an integer. Let \( f \) be a Hilbert newform of level \( \mathfrak{N} \), and (parallel) weight \( k \); and \( L_f \) the field of coefficients of \( f \). We recall that \( L_f = \mathbb{Q}(a_m(f) : m \subseteq \mathcal{O}_F) \), where \( a_m(f) \) is the Hecke eigenvalue of \( f \) at the Hecke operator \( T_m \) for the integral ideal \( m \). By [Shi78, Proposition 2.8], \( L_f \) is either totally real or CM.

Let \( \tau : L_f \to \overline{\mathbb{Q}} \) be a complex embedding. Then, by the Strong Multiplicity One Theorem [Miy71] and [Shi78, Proposition 2.6], there is a newform \( f^\tau \) of level \( \mathfrak{N} \) and weight \( k \) determined by its Hecke eigenvalues

\[ a_p(f^\tau) := \tau(a_p(f)), \text{ for all primes } p. \]

The Hecke orbit of the form \( f \) is defined as the finite set

\[ [f] := \{ f^\tau : \tau \in \text{Hom}(L_f, \overline{\mathbb{Q}}) \}. \]

Let us further assume that \( F \) is Galois over some subfield \( E \), and write \( G := \text{Gal}(F/E) \). Let us also assume that \( \mathfrak{N} = \mathfrak{N}\sigma \) for all \( \sigma \in G \). Then, similarly, for all \( \sigma \in G \), there exists a newform \( f^\sigma \) of level \( \mathfrak{N} \) and weight \( k \) determined by its Hecke eigenvalues

\[ a_p(f^\sigma) := a_{\sigma|p}(f), \text{ for all primes } p. \]

The form \( f^\tau \) is often called an exterior twist while \( f^\sigma \) is known as an inner twist.

In the literature, exterior twists and inner twists have been studied quite extensively, but separately, to the best of our knowledge. However, there is clearly an

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Let \( G \) be a totally real field and \( \sigma \in \text{Hom}(\mathbb{Q}(\mathbb{Z}_2), G) \). Then, \( L_\sigma = L_f \), and for \( p \) prime, we have
\[
\sigma_p(a(f)) = \sigma_\sigma(p)(f) = \tau(a_{\sigma(p)}(f)) = \tau(\sigma_p(f)) = \sigma_p(\langle f \rangle^\sigma).
\]
So that we have a well-defined action of \( G \) on the set of Hecke orbits of newforms of level \( \mathfrak{N} \) and weight \( k \) given by \( \sigma \cdot [f] = [f]^\sigma \). Therefore, understanding the action of \( G \) on those orbits seems a rather natural question.

In this short note, we describe the compatibility between base change and Hecke orbits of Hilbert newforms when the base field is Galois. Our results are essentially a generalisation of those in [CD17] to non-solvable extensions under the assumption that the Base Change Conjecture is true. Although the results are somewhat straightforward, and are probably known to most experts, they seem to have rather strong implications relating to Langlands Functoriality. That was illustrated in [CD17] where the functorial connection between the Eichler-Shimura conjecture and the Gross-Langlands conjecture on the modularity of abelian varieties was discussed. We conclude the note with an example pertaining to this connection. Let \( F \) be the maximal totally real subfield of \( \mathbb{Q}(\mathbb{Z}_2) \), and \( D \) the quaternion algebra over \( F \) ramified at the unique prime above \( 2 \) and \( 7 \) real places. Let \( X_0^0(1) \) be the Shimura curve attached to \( D \). Our example shows that the field of 2-torsion \( \mathbb{F}_2 \times (\mathbb{Z}/17\mathbb{Z})^\times \). This completes Noam Elkies' answer [Elk15] to a question posed by Jeremy Rouse on mathoverflow.net.

The outline of the paper is as follows. In Sections 1 and 2, we recall the necessary background on Hilbert modular forms, and base change. In Section 3, we discuss the compatibility of base change with Hecke orbits. In Section 4, we introduce the notion of abelian varieties potentially of \( \text{GL}_2 \) type. We then discuss their Galois descent properties, and its implications for Langlands Functoriality. Finally, in Section 5, we conclude with an example.

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1. Hilbert modular forms

In this section, we summarise the results we need on Hilbert modular forms, their associated automorphic representations and Galois representations. We refer to [Car86b, Hid88, Shi78, Tay89] for further details.

Let \( F \) be a totally real field, and \( J_F \) the set of real embeddings of \( F \). Let \( \mathcal{H}_F := \mathcal{H}^{J_F} \), where \( \mathcal{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \) is the Poincaré upper halfplane. Fix an integer \( k \geq 2 \).

1.1. Hilbert modular forms. We let \( \mathcal{G} = \text{Res}_F/Q(\text{GL}_2) \), this is the algebraic group obtained by restriction of scalars of \( \text{GL}_2 \) from \( F \) to \( Q \). By definition of \( G/Q \), we have
\[
G(\mathbb{A}_Q) = G_f \times G_\infty = \text{GL}_2(\mathbb{A}_F) = \text{GL}_2(\mathbb{A}_F^0) \times \text{GL}_2(\mathbb{R})^{J_F}.
\]
Let \( G^+ \) be the connected component of the identity element. Then, \( G^+ \) acts on \( \mathcal{H}_F \) component-wise by Mobius transforms. This action extends uniquely to \( G_\infty \) component-wise, such that on the \( v \)-th factor, the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) acts by \( z_v \mapsto -z_v \).
We denote the space of cusp forms by \( \mathcal{S}_k \). We say that \( f \) is a \( \mathcal{S}_k \) if
\[
(cz + d)^k := \prod_{v \in J_F} (c v z_v + d_v)^{k};
\]
\[
\det(\gamma)^{k-1} := \prod_{v \in J_F} \det(\gamma_v)^{k-1}.
\]

We let \( U = \prod_{p} U_p \) be a compact open subgroup of \( \Gamma(\mathbf{Z}) = \Gamma_0(\mathbf{Z}) \subset \Gamma_f \), and consider the space of functions \( f : \Gamma_f \backslash \mathbf{U} \to \mathbf{C} \). There is a (left) action of \( \Gamma(\mathbf{Q}) \) on this space, given by
\[
(f \gamma)(x, z) := \det(\gamma)^{k-1}(cz + d)^{-k} f(\gamma x, \gamma z), \quad x \in \Gamma_f, \ z \in \mathbf{U}_F.
\]
The space of Hilbert modular forms \( \mathcal{M}_k(U) \) of weight \( k \) and level \( U \) is the set of functions \( f : \Gamma_f \backslash \mathbf{U} \to \mathbf{C} \) such that
\[
\begin{align*}
(1) & \quad f|\gamma = f \quad \text{for all } \gamma \in \Gamma(\mathbf{Q}); \\
(2) & \quad \text{for all } x \in \Gamma_f, \text{ the map } (f_x : \mathbf{U}_F \to \mathbf{C}, \ z \mapsto f(x, z)) \text{ is holomorphic.}
\end{align*}
\]
By Condition (1) (and the Koehler principle \cite[sec. 2]{Shi78}), \( f_x \) admits a \( q \)-expansion of the form
\[
f_x(z) = a_0(f_x) + \sum_{\mu \in \mathfrak{U}_F^+} a_\mu e^{2\pi i \mu z},
\]
where
\[
\text{Tr}_{\mathbf{Q}/\mathbf{Q}}(\mu z) := \sum_{v \in J_F} \mu_v z_v.
\]
We say that \( f \) is a cusp form if, in addition to (1) and (2), we have
\[
(3) \quad f_x|\gamma \text{ has no constant term, i.e. } a_0(f_x|\gamma) = 0, \text{ for all } x \in \Gamma_f \text{ and } \gamma \in \Gamma(\mathbf{Q}).
\]
We denote the space of cusp forms by \( \mathcal{S}_k(U) \).

Let \( \mathfrak{N} \) be an integral ideal, we defined the compact opens:
\[
U_0(\mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathbf{Z}_F) : c = 0 \mod \mathfrak{N} \right\};
\]
\[
U_1(\mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathbf{Z}_F) : c = 0 \mod \mathfrak{N}, \ d = 1 \mod \mathfrak{N} \right\}.
\]
For \( U = U_0(\mathfrak{N}) \) or \( U_1(\mathfrak{N}) \), the strong approximation theorem implies that
\[
\Gamma(\mathbf{Q}) \backslash \Gamma_f / U \cong \mathbf{A}_F / \mathbf{F}_F \times \mathbf{R}^{\mathfrak{f}} \hat{\mathbf{O}}^\infty_F \simeq \text{Cl}^+(F),
\]
where \( \text{Cl}^+(F) \) is the narrow class group of \( F \). Let \( \mathfrak{d} \) be the different ideal of \( F \), and \( \mathfrak{c}_i, i = 1, \ldots, h^+ \), a complete set of representatives for the classes in \( \text{Cl}^+(F) \). For each \( i \), let \( x_i \in \Gamma_f \) be such that the idél \( \text{det}(x_i) \) generates the ideal \( \mathfrak{c}_i \), and define
\[
\Gamma_0(\mathfrak{c}_i, \mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left( \begin{array}{cc} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{c}_i \mathfrak{N} & \mathfrak{N} \end{array} \right) : ad - bc \in \mathcal{O}_F^\times \right\};
\]
\[
\Gamma_1(\mathfrak{c}_i, \mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{c}_i, \mathfrak{N}) : d = 1 \mod \mathfrak{N} \right\}.
\]
Let \( \Gamma_i = \Gamma_0(\mathfrak{c}_i, \mathfrak{N}) \) or \( \Gamma_1(\mathfrak{c}_i, \mathfrak{N}) \) according as to \( U = U_0(\mathfrak{N}) \) or \( U_1(\mathfrak{N}) \). Then, we have a bijection
\[
\mathcal{S}_k(U) \to \bigoplus_{i=1}^{h^+} \mathcal{S}_k(\Gamma_i),
\]
\[
f \mapsto (f_1, \ldots, f_{h^+}),
\]
where \( f_{i} = f_{x_{i}} \), and \( S_{k}(\Gamma_{i}) \) is the space of classical Hilbert cusp forms of level \( \Gamma_{i} \) and weight \( k \) (see [Shi78, sec. 2]). Let \( \mathfrak{m} \subseteq \mathcal{O}_{F} \) be an ideal; then there is a unique \( 1 \leq i \leq h^{+} \) and \( \mu \in F_{i}^{+} \) such that \( \mathfrak{m} = \mu c_{i}^{-1} \). We set

\[
 a_{\mathfrak{m}}(f) = a_{\mu}(f_{i}).
\]

Since \( f_{i} \) is invariant under \( \mathcal{O}_{F}^{+} \), which acts as \( z \mapsto \varepsilon z \), we see that

\[
 a_{\mu}(f_{i}) = a_{\mu}(f_{i}).
\]

So the quantity \( a_{\mathfrak{m}}(f) \) is well-defined. We call it the Fourier coefficient of \( f \) at \( \mathfrak{m} \).

### 1.2. Petersson inner product

We have the measure \( d\mu(z) = \prod_{v \in J_{p}} y_{v}^{-2} dx_{v} dy_{v} \).

We define the Petersson inner product on \( S_{k}(\Gamma) \) by

\[
\langle f, g \rangle_{\Gamma} = \frac{1}{\mu(\Gamma \setminus \mathcal{O}_{F})} \int_{\Gamma \setminus \mathcal{O}_{F}} \overline{f(z)g(z)} y^{k} d\mu(z).
\]

The Petersson inner product on \( S_{k}(U) \) is given by

\[
\langle f, g \rangle = \sum_{i=1}^{h^{+}} \langle f_{i}, g_{i} \rangle_{\Gamma_{i}}.
\]

### 1.3. Hecke operators

For \( x \in G_{f} \), we define the Hecke operator

\[
[U \times U] : S_{k}(U) \to S_{k}(U)
\]

\[
f \mapsto \sum_{i=1}^{h^{+}} f_{i} x_{i},
\]

where \([U \times U] = \bigcup_{i=1}^{h^{+}} x_{i} U\). For every prime \( p \) (resp. \( p \nmid \mathfrak{N} \)), we define the Hecke operators

\[
T_{p} := \begin{bmatrix} U & 0 \\ \varpi_{p} & 0 \\ 0 & 1 \end{bmatrix} U \quad \text{and} \quad S_{p} := \begin{bmatrix} U & 0 \\ \varpi_{p} & 0 \\ 0 & \varpi_{p} \end{bmatrix} U,
\]

where \( \varpi_{p} \) is a uniformizer at \( p \) (resp. \( p \nmid \mathfrak{N} \)). We define the Hecke algebra \( \mathbf{T}_{k}(U) \) to be the \( \mathbf{Z} \)-subalgebra of \( \text{End}_{C}(S_{k}(U)) \) generated by the operators \( T_{p} \) for all primes \( p \) and \( S_{p} \) for all primes \( p \nmid \mathfrak{N} \). We will simply write \( \mathbf{T} \) if there is no confusion.

### 1.4. Cusp forms with characters

Let \( \chi : F^{\times} \setminus \mathbf{A}_{F}^{\times} \to \mathbf{C}^{\times} \) be a Hecke character of modulus \( \mathfrak{N} \) and infinite type \((-n_{0}, \ldots, -n_{0})\). We define the space of Hilbert cusp forms of level \( \mathfrak{N} \), weight \( k \) and character \( \chi \) by

\[
S_{k}(\mathfrak{N}, \chi) := \left\{ f \in S_{k}(U_{1}(\mathfrak{N})) : S_{p} f = \chi(p) f \right\}.
\]

We will simply write \( S_{k}(\mathfrak{N}) \) when \( \chi := 1 \) is the trivial character. Then, we have the following decomposition (see [Shi78, sec. 2]), which is compatible with the Hecke action:

\[
S_{k}(U_{1}(\mathfrak{N})) = \bigoplus_{\chi} S_{k}(\mathfrak{N}, \chi),
\]

where \( \chi \) runs over all Hecke character of modulus \( \mathfrak{N} \) and infinite type \((-n_{0}, \ldots, -n_{0})\).

We will denote by \( \mathbf{T}_{k}(\mathfrak{N}, \chi) \) the Hecke algebra acting on \( S_{k}(\mathfrak{N}, \chi) \). Again, we will simply write \( \mathbf{T} \) if there is no confusion.

### 1.5. Eigenforms

We say that \( f \in S_{k}(\mathfrak{N}, \chi) \) is an eigenform if it is an eigenvector for all the operators \( T_{m}, S_{m} \in \mathbf{T}_{k}(\mathfrak{N}, \chi) \). In addition, we say that \( f \) is normalised if \( a_{(1)}(f) = 1 \). In this case, there is a ring homomorphism \( \lambda_{f} : \mathbf{T}_{k}(\mathfrak{N}, \chi) \to \mathbf{C} \) such that \( T_{m} f = \lambda_{f}(T_{m}) f \) for all \( \mathfrak{m} \subseteq \mathcal{O}_{F} \). By Shimura [Shi78, Proposition 2.8] we have \( \lambda_{f}(T_{m}) = a_{\mathfrak{m}}(f) \), and the field \( L_{f} = \mathbf{Q}(a_{m}(f) : m \subseteq \mathcal{O}_{F}) \) is a number field. It is either totally real or CM.
1.6. Newforms. Let \( \mathfrak{M} \mid \mathfrak{N} \) be divisible by the conductor of \( \chi \), and \( \mathfrak{Q} \mid \mathfrak{M}\mathfrak{M}^{-1} \). Then, by the Multiplicity One Theorem, there is a well-defined map \( \iota_\mathfrak{Q} : S_k(\mathfrak{M}, \chi) \to S_k(\mathfrak{N}, \chi) \)
\[
g \mapsto g|_\mathfrak{Q},
\]
determined by setting \( a_m(g|_\mathfrak{Q}) = a_m|_{\mathfrak{M}\mathfrak{N}^{-1}}(g) \). The old subspace of level \( \mathfrak{M} \), weight \( k \) and character \( \chi \) is defined by
\[
S_k(\mathfrak{M}, \chi)^{old} = \sum_{m \mid \mathfrak{M}|\mathfrak{M}^{-1}} \text{im}(\iota_\mathfrak{Q}).
\]
This is stable under the action of \( T_k(\mathfrak{M}, \chi) \). We define the new subspace \( S_k(\mathfrak{M}, \chi)^{new} \) of level \( \mathfrak{M} \), weight \( k \) and character \( \chi \) to be the orthogonal complement of \( S_k(\mathfrak{M}, \chi)^{old} \) under the Petersson inner product. So, we have
\[
S_k(\mathfrak{M}, \chi) = S_k(\mathfrak{M}, \chi)^{old} \oplus S_k(\mathfrak{M}, \chi)^{new}.
\]
We say that \( f \in S_k(\mathfrak{M}, \chi) \) is a newform if it is a normalised eigenform which belongs to \( S_k(\mathfrak{M}, \chi)^{new} \). By [Hi88, Theorem 5.2], we have a perfect pairing
\[
T_k(\mathfrak{M}, \chi) \times S_k(\mathfrak{M}, \chi) \to \mathbb{C}, \quad (T, f) \mapsto a(t,f)|_\mathfrak{M}.
\]
Let \( f \in S_k(\mathfrak{M}, \chi)^{new} \) be a newform. Then, we have a \( \mathbb{Q} \)-algebra homomorphism
\[
\lambda_f : T_k(\mathfrak{M}, \chi)^{new} \to \overline{\mathbb{Q}}
\]
\[
T_m \mapsto a_m(f).
\]
By considering the \( \overline{\mathbb{Q}} \)-structure \( S_k(\mathfrak{M}, \chi; \overline{\mathbb{Q}}) \) on \( S_k(\mathfrak{M}, \chi) \), we have an isomorphism of \( \overline{\mathbb{Q}} \)-vector spaces
\[
S_k(\mathfrak{M}, \chi; \overline{\mathbb{Q}})^{new} \to \text{Hom}_{\overline{\mathbb{Q}}}(T_k(\mathfrak{M}, \chi; \overline{\mathbb{Q}})^{new}, \overline{\mathbb{Q}})
\]
\[
f \mapsto \lambda_f.
\]
We also have a natural bijection
\[
\text{Hom}_{\overline{\mathbb{Q}}\text{-alg}}(T_k(\mathfrak{M}, \chi; \overline{\mathbb{Q}})^{new}, \overline{\mathbb{Q}}) \cong \text{Spec}(T_k(\mathfrak{M}, \chi; \overline{\mathbb{Q}})^{new})
\]
\[
\lambda \mapsto \ker(\lambda).
\]

1.7. Hecke orbits. Let \( f \in S_k(\mathfrak{M}, \chi)^{new} \) be a newform. Let \( \tau : L_f \hookrightarrow \overline{\mathbb{Q}} \) be a complex embedding. Then, by the Strong Multiplicity One Theorem [Miy71] and [Shi78, Proposition 2.6], there is a newform \( f^\tau \) of level \( \mathfrak{M} \), weight \( k \) and character \( \chi^\tau := \tau \circ \chi \) determined by its Hecke eigenvalues
\[
a_p(f^\tau) := \tau(a_p(f)), \quad \text{for all primes } p.
\]
The Hecke orbit of the form \( f \) is defined as the finite set
\[
[f] := \{f^\tau : \tau \in \text{Hom}(L_f, \overline{\mathbb{Q}})\}.
\]

1.8. Automorphic representations attached to Hilbert newforms. Let \( f \in S_k(\mathfrak{M}, \chi)^{new} \) be a newform. Then there is an automorphic representation \( \pi_f \) of \( \text{GL}_2(A_F) \) attached to \( f \), which by [Fla79] admits a factorisation as a restricted tensor product
\[
\pi_f = \bigotimes_v \pi_{f,v},
\]
where \( \pi_{f,v} \) is an admissible representation of \( \text{GL}_2(F_v) \) for all places \( v \) of \( F \).
1.9. Galois representations attached to Hilbert newforms. We let $G_F := \text{Gal}({\overline{Q}}/F)$ be the absolute Galois group of $F$. For each prime $p$, we also let $D_p$ and $W_p$ be the decomposition and the Weil group at $p$, respectively.

Let $f \in S_k(\mathfrak{N}, \chi)^\text{new}$ be a newform, and $\pi_f$ the associated automorphic representation. The following theorem is the result of the work of many people [Car86b, Tay89, BLGGT14].

**Theorem 1.1.** Let $\ell$ be a rational prime, and $\lambda$ a prime of $L_f$ above $\ell$. Then, there exists a Galois representation
\[ \rho_{f,\lambda} : \text{Gal}({\overline{Q}}/F) \to \text{GL}_2(T_{f,\lambda}), \]
such that

1. For all $p \nmid \mathfrak{N}$, the characteristic polynomial of Frobp is determined by
\[ \text{Tr}(\rho_{f,\lambda}(\text{Frob}_p)) = a_p(f) \text{ and } \det(\rho_{f,\lambda}(\text{Frob}_p)) = \chi(p) Np^{k-1}. \]
2. More generally, we have $\text{WD}(\rho_{f,\lambda}|_{D_p})^F \simeq \text{rec}_{F_p}(\pi_{f,p} \otimes | \det|^{-1/2}).$
3. For $p | \ell$, $\rho_{f,\lambda}|_{D_p}$ is de Rham with Hodge-Tate weights $\text{HT}_{\tau}((\rho_{f,\lambda})) = \{k, k-1\}$, for each embedding $\tau : F \hookrightarrow \mathbb{C}$ lying over the place at $p$. If $\pi_{f,p}$ is unramified then $\rho_{f,\lambda}|_{D_p}$ is crystalline.
4. If $c_\ell$ is a complex conjugation, then $\det(\rho_{f,\lambda}(c_\ell)) = -1.$

2. Base change

In this section, we recall the main conjecture regarding non-solvable base change for $\text{GL}_2$, and refer to [Lan80] for more details.

Let $F$ be as before and assume that $E$ is a subfield of $F$ such that $F/E$ is Galois. Assume that $\mathfrak{N}$ is stable under the action of $\text{Gal}(F/E)$.

Since $U$ is Galois stable, the group $\text{Gal}(E/F)$ acts on $G_f/U \times S_f$ by
\[ \sigma : G_f/U \times S_f \to G_f/U \times S_f \]
\[ (x, z) \mapsto (x^\sigma, z^\sigma). \]
This induces an action on the space $S_k(U)$. It also acts on $T_k(U)$ by sending $T_p$ to $T_{\sigma(p)}$, and $S_p$ to $S_{\sigma(p)}$. Via the surjection
\[ G_f \times G_\infty \to G_f/U \times S_f \]
\[ (g_f, g_\infty) \mapsto (g_f U, g_\infty(\sqrt{-1}, \ldots, \sqrt{-1})), \]
this induces an action of $\text{Gal}(F/E)$ on the set of automorphic representations of $\text{GL}_2(A_F)$ of level $U_1(\mathfrak{N})$ and weight $k$.

Let $f \in S_k(\mathfrak{N}, \chi)^\text{new}$ be a newform, and $\sigma \in G$. Then, there is an automorphic representation $\sigma \pi_f = \pi_f \circ \sigma$ of level $\mathfrak{N}$, weight $k$ and character $\sigma \chi := \chi \circ \sigma$. Let $\sigma f \in S_k(\mathfrak{N}, \sigma \chi)$ be the newform such that $\sigma f$ is a new vector in $\sigma \pi_f$. By the Strong Multiplicity One Theorem [Miy71], $\sigma f$ is uniquely determined by the relation
\[ a_p(\sigma f) := a_{\sigma(p)}(f), \]
for all primes $p$. We say that an automorphic representation $\pi$ is a base change if $\pi \circ \sigma \simeq \pi$ for all $\sigma \in G$. We say that $f$ is a base change if $\pi_f$ is a base change. This is equivalent to saying that $\sigma f = f$. By the Multiplicity One Theorem, this is also equivalent to saying that $a_{\sigma(p)}(f) = a_p(f)$, for all $\sigma \in G$ and almost all primes $p$. We note that, among other things, this will imply that $\chi \circ \sigma = \chi$ for all $\sigma \in G$. In other words, $\chi$ must factor through the norm map $N_{F/E} : F \to E$. 
Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\overline{\mathbb{Q}}) \) be an \( \ell \)-adic representation. For \( \sigma \in G \), let \( \tilde{\sigma} \in \text{Gal}(\overline{\mathbb{Q}}/E) \) be a lift, and set

\[
\rho^\sigma(\tilde{\sigma}) = \rho(\tilde{\sigma}\tilde{\sigma}^{-1}).
\]

Then, \( \rho^\sigma \) is well-defined and depends only on \( \sigma \). We call it the \( \sigma \)-conjugate of \( \rho \).

**Conjecture 2.1** (Base Change [Lan80]). Let \( f \) be a Hilbert newform of \( \text{Gal}(F/E) \)-invariant level \( \mathfrak{N} \), weight \( k \) and character \( \chi \). Then, the followings are equivalent:

(a) For all \( \sigma \in G \), we have \( \circ \pi f \cong \pi f \);

(b) For all \( \sigma \in G \), we have \( \circ f = f \);

(c) For all prime \( \lambda \) in \( L_f \) and \( \sigma \in G \), we have \( \rho_{f, \lambda}^\sigma = \rho_{f, \lambda} \);

(d) There exists a Hilbert newform \( \hat{f} \) of level \( U_1(\mathfrak{N}') \) and weight \( k \) over \( E \) such that

\[
\rho_{f, \lambda} \mid_{\text{Gal}(\overline{\mathbb{Q}}/F)} = \rho_{f, \lambda}
\]

for all primes \( \lambda' \) in \( L_f \) and \( \lambda \) in \( L_f \) above \( \lambda' \);

(e) There exists a Hilbert newform \( \hat{f} \) of level \( U_1(\mathfrak{N}') \) and weight \( k \) over \( E \) such that

\[
L(f, s) = \prod_{\eta \in \mathbb{C}} L(\hat{f} \otimes (\eta \circ \text{Art}_F), s),
\]

where \( \text{Art}_F : F^\times \setminus \mathbb{A}_F^\times \to \mathbb{G}_m^\times \) is the Global Artin reciprocity map.

(The level \( \mathfrak{N}' \) depends on the relative discriminant \( \mathfrak{D}_{F/E} \) and the level \( \mathfrak{N} \), see Remark 2.4.)

**Remark 2.2.** One can reformulate Statements (a) and (b) of Conjecture 2.1 by saying that \( \text{Spec}(T_k(\mathfrak{N}, \chi; \overline{\mathbb{Q}})_{\text{new}}) \) is a \( G \)-set whose fixed points correspond to newforms that are base change.

**Remark 2.3.** We recall that Conjecture 2.1 is true when \( F/E \) is a cyclic extension (see [Lan80]). In the non-solvable case, there have been some progress for \( F/\mathbb{Q} \) totally real thanks to Hida [Hid09] and Dieulefait [Die12].

**Remark 2.4.** The level \( U_1(\mathfrak{N}') \) of the form \( \hat{f} \) in Conjecture 2.1 can be determined explicitly using the local-global compatibility conditions in Theorem 1.1. We note that, for \( F/\mathbb{Q} \) cyclic of prime degree, there is earlier work of Saito [Sai79, Theorem 4.5] which gives \( U_1(\mathfrak{N}') \).

3. Base change and Hecke orbits

We keep the notation of Section 2. In particular, \( \mathfrak{N} \) is an integral ideal which is \( \text{Gal}(F/E) \)-invariant, and \( S_k(\mathfrak{N}) \) is the space of cusp forms of level \( \mathfrak{N} \), weight \( k \) and trivial character.

Let \( \mathcal{F} \) be the set of Hecke orbits of the newforms in \( S_k(\mathfrak{N}) \). Let \( f \in S_k(\mathfrak{N}) \) be a newform, \( \tau \in \text{Hom}(L_f, \overline{\mathbb{Q}}) \) and \( \sigma \in G \). Then, we have that \( L_{f, \tau} = L_f \), and that

\[
a_p(\circ f) = a_p(f^\tau) = \tau(a_p(f^\prime)) = \tau(a_p(\circ f)) = a_p((\circ f)^\tau).
\]

This means that, by setting \( \sigma \cdot [f] := [\circ f] \), we get a well-defined action of \( G \) on \( \mathcal{F} \).

We can write \( \mathcal{F} \) as a disjoint union

\[
\mathcal{F} = \bigsqcup_L \mathcal{F}_L,
\]

where \( L \) runs over all fields of coefficients and \( \mathcal{F}_L := \{ [f] : L_f = L \} \). Our goal is to understand the orbits of this action on this union. To this end, we start with the following lemma, which is somewhat straightforward.
Lemma 3.1. Assume that Conjecture 2.1 is true. Let \( f \in S_k(\mathcal{R}) \) be a newform with field of coefficients \( L \) such that the \( G \)-orbit of \([f]\) is a singleton.

(a) There is a group homomorphism

\[
\phi : G \longrightarrow \text{Aut}(L)
\]

\[
\sigma \longmapsto \tau
\]

where, for every \( \sigma \in G \), \( \phi(\sigma) := \tau \) is the unique element in \( \text{Aut}(L) \) such that \( \sigma f = f^\tau \).

(b) If \( f \) is not a base change from \( E \) then \( \phi \) is non-trivial.

(c) If \( f \) is not a base change from any intermediate field \( E'/E \), then the map \( \phi \) is injective.

Proof. (a) Let \( \sigma \in G \). Since \([f]\) is unique in its \( G \)-orbit, we have \([\sigma f] = [f]\). Hence, there exists \( \tau \in \text{Hom}(L, \overline{\mathbb{Q}}) \) such that \( \sigma f = f^\tau \). So, we have

\[
a_{\sigma(p)}(f) = \tau(a_p(f)),
\]

for all primes \( p \). By the Strong Multiplicity One Theorem [Miy71], the element \( \tau \) is well-defined and uniquely determined. Furthermore, since \( L_{\sigma f} = L_f = L \), and \( L \) is generated by the Hecke eigenvalues of \( f \), we have that \( \tau \in \text{Aut}(L) \). So, setting \( \phi(\sigma) := \tau \), we get a well-defined map \( \phi : G \rightarrow \text{Aut}(L) \).

To prove that \( \phi \) is a group homomorphism, let \( \phi(\sigma_1) = \tau_1 \) and \( \phi(\sigma_2) = \tau_2 \). Then, for all primes \( p \), we have

\[
a_{\sigma_1,\sigma_2}(p)(f) = a_{\sigma_1}(a_{\sigma_2}(p))(f) = \tau_1(\tau_2(a_p(f))) = (\tau_1\tau_2)(a_p(f)).
\]

So we have \( \sigma_1\sigma_2 f = f^{\tau_1\tau_2} \), and hence \( \phi(\sigma_1\sigma_2) = \phi(\sigma_1)\phi(\sigma_2) \) by the Multiplicity One Theorem.

(b) Since \( f \) is not a base change from \( E \), there exists \( \sigma \in G \) such that \( \sigma f \neq f \) and \([\sigma f] = [f]\). Therefore, \( \phi(\sigma) \neq 1 \), hence it must be non-trivial.

(c) Assume that there were \( \sigma \neq 1 \) such that \( \phi(\sigma) = 1 \). This would imply that \( \sigma f = f \). So \( f \) would be a base change from the subfield \( E' = F^{\sigma} \), which would be a contradiction. So \( \phi \) must be injective. \( \square \)

Remark 3.2. Letting \( K = L^\Delta \) in Lemma 3.1, where \( \Delta := \text{im}(\phi) \), we see that \( \text{Gal}(L/K) = \Delta \). So, in other words, Lemma 3.1 (c) implies that we have an injection \( \text{Gal}(F/E) \hookrightarrow \text{Gal}(L/K) \). As we will see later, in all our examples where \( \text{Gal}(F/E) \) is solvable, this injection is in fact an isomorphism. But there is no reason for this to persist forever as the degree of the field \( L \) becomes larger.

Theorem 3.3. Assume Conjecture 2.1 is true. Let \( f \in S_k(\mathcal{R}) \) be a newform with field of coefficients \( L \), \( G' := \text{Stab}_G([f]) \) and \( E' := F^{G'} \), so that \( \text{Gal}(F/E') = G' \).

(a) There exists a group homomorphism

\[
\phi : G' \longrightarrow \text{Aut}(L)
\]

\[
\sigma \longmapsto \tau
\]

where, for every \( \sigma \in G' \), \( \phi(\sigma) := \tau \) is the unique element in \( \text{Aut}(L) \) such that \( \sigma f = f^\tau \).

(b) If \( G' \) is non-trivial, and \( f \) is not a base change from \( E' \), then \( \phi \) is non-trivial.

(c) If \( G' \) is non-trivial, and \( f \) is not a base change from any intermediate field \( F'/E' \), then \( \phi \) is an injection.

Proof. The \( G' \)-orbit of \([f]\) is a singleton, so we apply Lemma 3.1 relative to the extension \( F/E' \). \( \square \)
Corollary 3.4. Let $F/E$ be a cyclic extension, and $f \in S_\kappa(\mathfrak{M})$ be a newform with field of coefficients $L$. Let $\text{Gal}(F/E) = \langle \sigma \rangle$, $\text{Stab}_G([f]) = \langle \sigma^* \rangle$ and $E' = F^{\langle \sigma^* \rangle}$.

(a) There exists a group homomorphism
\[
\text{Gal}(F/E') \longrightarrow \text{Aut}(L)
\]
\[
\sigma^* \longmapsto \tau
\]
where $\tau \in \text{Aut}(L)$ is the unique element such that $\sigma^* f = f^\tau$.

(b) If $G'$ is non-trivial, and $f$ is not a base change from $E'$, then $\phi$ is non-trivial.

(c) If $G'$ is non-trivial, and $f$ is not a base change from any intermediate field $F'/E'$, then $\phi$ is an injection.

Proof. Since $F/E$ is cyclic, Conjecture 2.1 is true in that case. \qed

There is a wide range of combinatorial results that one could derive from Theorem 3.3. We only state the following corollaries as an illustration.

Corollary 3.5. Assume Conjecture 2.1 is true. Let $f \in S_\kappa(\mathfrak{M})$ be a newform whose Hecke constituent is the unique constituent of dimension $d_f := [L_f : \mathbb{Q}]$. If $d_f < |G|$ then there exists an intermediate field $E'/E$ such that $f$ is a base change from $E'$.

Proof. By assumption, we have $\text{Stab}_G([f]) = G$. Assume that there is no intermediate field $E'$ such that $f$ is a base change from $E'$. Then, the map $\phi$ is injective by Theorem 3.3. This implies that $d_f \geq |G|$, which is a contradiction. \qed

Corollary 3.6. Assume Conjecture 2.1 is true. Let $f \in S_\kappa(\mathfrak{M})$ be a newform with field of coefficients $L_f$. Let $s_f := |\text{Stab}_G([f])|$ and $n_f$ be the number of Hecke constituents of dimension $d_f := [L_f : \mathbb{Q}]$. Then, we have $n_f \geq |G|/s_f$. In particular, when $\text{Stab}_G([f]) = \{1\}$, there are at least $|G|$ Hecke constituents of dimension $d_f$ (including the ones in the Hecke orbit of $f$).

Corollary 3.7. Assume Conjecture 2.1 is true. Let $f \in S_\kappa(\mathfrak{M})$ be a newform with field of coefficients $L_f$. Let $n_f$ be the number of Hecke constituents of dimension $d_f := [L_f : \mathbb{Q}]$. Assume that $d_f$ is coprime with $|G|$. Then, either $f$ is a base change from $E$ and $n_f = 1$, or $f$ is not a base change from any proper subfield of $E'/E$ and $n_f = |G|$.

Remark 3.8. In practice, it can be very hard to decide whether a newform $f$ is a base change or not. The thrust of Corollaries 3.5, 3.6 and 3.7 is that they allow us to do so by using purely combinatorial arguments sometimes.

Remark 3.9. An immediate consequence of Corollaries 3.6 and 3.7 is that a naive generalisation of the Maeda conjecture [HM97, Conjecture 1.2] to totally real number fields will not work. Any proper generalisation must be sensitive to the the action of $\text{Gal}(F/E)$ on Hecke orbits of newforms.

4. Abelian varieties potentially of $\text{GL}_2$-type

Theorem 3.3 and its corollaries have natural implications for the theory of descent of abelian varieties. Indeed, let $f$ be a newform of weight 2 and level $\mathfrak{M}$. Assume that $f$ is not a base change from $E$, and that the map $\phi : \text{Gal}(F/E) \to \text{Aut}(L)$ is an injection. Let $K := L^\Delta$ and $g := [L : K]$, where $\Delta = \text{im}(\phi)$. Further assume that there exists an abelian variety $A_f$ which satisfies the Eichler-Shimura construction for $f$ and recall that, we have
\[
a_{\sigma(p)}(f) = \tau(a_{p}(f)), \quad \text{for all primes} \ p \ \text{and} \ \sigma \in \text{Gal}(F/E),
\]
where $\tau = \phi(\sigma)$. Consider the family of $\lambda$-adic representations

$$\rho_{f,\lambda}: \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(L_{f,\lambda})$$

attached to $A_f$, where $\lambda$ runs over all primes in $L_f$. The identity above implies that $A_f$ is isogenous to all its Galois conjugates, and that the induced representations

$$\text{Ind}_{\mathbb{Q}}^{L} (\rho_{f,\lambda}) : \text{Gal}(\overline{\mathbb{Q}}/E) \to \text{GSp}_{2g}(L_{f,\lambda}),$$

are irreducible, and are in fact defined over $K$. This suggests that the isogeny class of the abelian variety $A_f$ descends to that of an abelian variety $B/E$ such that $\text{End}_E(B) \otimes \mathbb{Q} \cong K$. We note that $A_f$ cannot be an $E$-variety, in the terminology of [Rib04, Pyl04] or [Gui10], since $B$ is not of $\text{GL}_2$-type. So that motivates the following definition.

**Definition 4.1.** Let $A/E$ be an abelian variety. We say that $A$ is potentially of $\text{GL}_2$-type if there exists an extension $F/E$ such that $A \times_E F$ is of $\text{GL}_2$-type.

We see that an abelian variety that is of $\text{GL}_2$-type is clearly potentially of $\text{GL}_2$-type. We also see that being potentially of $\text{GL}_2$-type is slightly stronger than simply acquiring extra endomorphism after base change.

Let $A$ be an abelian variety defined over $F$, and $[A]$ its isogeny class. For any $\sigma \in G_E$, set

$$\sigma : [A] := [A^\sigma],$$

where $A^\sigma$ is the Galois conjugate of $A$ by $\sigma$. This defines an action of $G_E$ on isogeny classes of abelian varieties defined over $F$ since $A \sim A'$ implies that $A^\sigma \sim A'^\sigma$. We note that, since $A$ is defined over $F$, this action factor through $\overline{G} = \text{Gal}(F/E)$. If $A$ is of $\text{GL}_2$-type, then $A^\sigma$ is also of $\text{GL}_2$-type and $\text{End}_F(A) \otimes \mathbb{Q} = \text{End}_F(A^\sigma) \otimes \mathbb{Q}$. Let $\text{Stab}_G([A])$ be the stabiliser of the isogeny class of $A$. Then, $\text{Stab}_G([A])$ is trivial if and only if $A$ is not isogenous to any of its Galois conjugate. Similarly $\text{Stab}_G([A]) = G$ means that $A$ is isogenous to all its Galois conjugates.

We recall the following well-known lemma.

**Lemma 4.2.** Let $A/F$ be an abelian variety of $\text{GL}_2$-type, and $L = \text{End}_F(A) \otimes \mathbb{Q}$. Assume that $A$ is isogenous to all its Galois conjugates. Then, there exists a group homomorphism $\phi: G \to \text{Aut}(L)$.

**Proof.** Let $\{\mu_{\sigma} : A^\sigma \to A\}_{\sigma \in G_E}$ be a system of isogenies. For each $\sigma \in G_E$, we define

$$\tau_\sigma(\alpha) := \mu_\sigma \circ \alpha^\sigma \circ \mu_\sigma^{-1}, \quad \alpha \in L.$$

Let $\{\mu'_{\sigma} : A'^\sigma \to A\}_{\sigma \in G_E}$ be another system of isogenies. Then, we have

$$\mu'_\sigma \circ \alpha^\sigma \circ \mu'^{-1}_\sigma = (\mu'_\sigma \circ \mu^{-1}_\sigma) \circ (\mu_\sigma \circ \alpha^\sigma \circ \mu_\sigma^{-1}) \circ (\mu_\sigma \circ \mu'^{-1}_\sigma) = \mu_\sigma \circ \alpha^\sigma \circ \mu^{-1}_\sigma$$

since $L$ is commutative. Therefore, $\tau_\sigma$ is well-defined and independent of the choice of the system of isogenies. It is not hard to see that $(\phi : G_E \to \text{Aut}(L), \sigma \mapsto \tau_\sigma)$ defines a group action, hence is a homomorphism. Since $A$ is defined over $F$, which is Galois over $E$, the map $\phi$ factors through $G$. This concludes the lemma. \hfill \Box

We recall that an abelian variety $A/F$ of $\text{GL}_2$-type, which is isogenous to all its $\text{Gal}(F/E)$-conjugates, is called an $E$-variety if the homomorphism $\phi$ in Lemma 4.2 is trivial. In this case, Ribet [Rib94, Theorem 1.2] shows that there exists a 2-extension $\overline{F}$ of $F$, and a system of isogenies $\{\mu_{\sigma} : A^\sigma \to A\}_{\sigma \in G}$ defined over $\overline{F}$.

**Proposition 4.3.** Let $F/E$ be a Galois extension, and $A$ a non CM abelian variety of $\text{GL}_2$-type defined over $F$. Assume that $A$ is an $E$-variety with a system of isogenies $\{\mu_{\sigma} : A^\sigma \to A\}_{\sigma \in G}$ defined over $F$. Then, there is an abelian variety $B$ of $\text{GL}_2$-type defined over $E$ such that $A$ is a simple factor of $B \times_E F$. 

Proof. This is an adaptation of the proof of [Pyš04, Proposition 4.5] to arbitrary number fields (see also Ribet [Ribš04, Theorem 6.1]).

Let $A/F$ be of GL$_2$-type, and set $L = \operatorname{End}_F(A) \otimes \mathbb{Q}$. Let $\lambda$ be a prime in $L$ and consider the Galois representation on the $\lambda$-adic Tate module of $A$

$$\rho_{A,\lambda} : \operatorname{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(L_\lambda).$$

Let $p$ be a prime in $F$, and set

$$a_p = \text{Tr}(\rho_{A,\lambda}(\text{Frob}_p)) \in L.$$

Ribet [Ribš04, Proposition 3.3] shows that $L$ is generated by the $a_p$. The content of the following result is that Galois action on isogeny classes of abelian varieties of GL$_2$-type should mirror that on Hecke orbits of Hilbert newforms.

**Theorem 4.4.** Let $F/E$ be a Galois extension, and $A$ a non CM abelian variety of GL$_2$-type defined over $F$ with $\operatorname{End}_F(A) \otimes \mathbb{Q} = L$. Assume that $A$ is isogenous to all its Galois conjugates.

(a) There exists a group homomorphism $\phi : G \to \operatorname{Aut}(L)$ such that

$$a_{\sigma(p)} = \tau(a_p)$$

for all primes $p$ and all $\sigma \in G$, where $\tau := \phi(\sigma)$.

(b) If $A$ is not an $E$-variety, then $\phi$ is non-trivial.

(c) If $A$ is not an $E'$-variety for any proper subfield $E'/E$, then $\phi$ is an injection.

Proof. (a) By Lemma 4.2, there is a homomorphism $\phi : G \to \operatorname{Aut}(L)$. We only need to show that $\phi$ is compatible with the action of $\operatorname{Gal}(F/E)$. Let $\ell$ be a rational prime, and $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}_\ell$ the $\ell$-adic Tate module attached to $A$. This is a 2-dimensional $L \otimes \mathbb{Q}_\ell$ vector space, and we let

$$\rho_{A,\ell} : \operatorname{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}(V_\ell(A))$$

be the corresponding Galois representation. For $\sigma \in G_E$, the isogeny $\mu_\sigma$ induces a $\mathbb{Q}_\ell[\operatorname{Gal}(\overline{\mathbb{Q}}/F)]$-module isomorphism $\mu_\sigma : V_\ell(A)^\sigma \xrightarrow{\sim} V_\ell(A)$. For all $\alpha \in L$, and $x \in V_\ell(A^\sigma)$, we have

$$\mu_\sigma(\alpha^\sigma x) = (\mu_\sigma \circ \alpha^\sigma \circ \mu_\sigma^{-1})(\mu_\sigma(x)) = \tau(\alpha) \mu_\sigma(x),$$

where $\tau = \phi(\sigma)$. This means that $\mu_\sigma$ becomes an $L$-linear isomorphism by letting $L$ acts on $V_\ell(A^\sigma)$ (resp. $V_\ell(A)$) via $\sigma$ (resp. $\tau$). Similarly, by definition, there is an isomorphism $V_\ell(A) \xrightarrow{\sim} V_\ell(A^\sigma)$ induced by the map $A(\overline{\mathbb{F}}) \to A^\sigma(\overline{\mathbb{F}})$ sending $x$ to $x^\sigma$, which is $L$-linear if we let $L$ acts on $V_\ell(A^\sigma)$ via $\tau$. From this, we get the diagram below, which is compatible with the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/F)$ as indicated.

```
\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\operatorname{Frob}_p & \operatorname{Frob}_p & \operatorname{Frob}_p \\
V_\ell(A) & \xrightarrow{\alpha} & V_\ell(A^\sigma) \\
\downarrow & \downarrow & \downarrow \\
V_\ell(A) & \xrightarrow{\alpha^\sigma} & V_\ell(A) \\
\end{array}
```

This implies that

$$\text{Tr}(\rho_{A,\ell}(\text{Frob}_p)) = \tau(\text{Tr}(\rho_{A,\ell}(\text{Frob}_p))),$$

which is the stated identity.

(b) By definition $A$ is an $E$-variety if and only if $\phi$ is trivial.

(c) For every proper subfield $E'/E$, $A$ is not an $E'$-variety. So, $\phi|_{\operatorname{Gal}(F/E')}$ is non-trivial. Therefore, $\phi$ must be injection. \qed
The following result is a generalisation of [CD17, Theorem 4.2] for cyclic extensions $F/E$.

**Theorem 4.5.** Let $F/E$ be a Galois extension, and $A$ a non CM abelian variety of $GL_2$-type defined over $F$ such that $\text{End}_F(A) \otimes \mathbb{Q} = L$ is totally real. Assume that $A$ is isogenous to all its Galois conjugates, and that $A$ is not an $E'$-variety for any proper subfield $E'/E$ of $F$. Then, there exists an abelian variety $B$ defined over $E$, potentially of $GL_2$-type, such that $\text{End}_E(B) \otimes \mathbb{Q} = K$ and $A \sim B \times_E F$, where $K = L^\Delta$, $\Delta = \text{im}(\phi)$.

**Proof.** A careful inspection shows that the proof of [CD17, Theorem 4.2] only uses the fact that $L$ is totally real, and nothing about the solvability of the extension $F/E$. \qed 

**Proposition 4.6.** Assume Conjecture 2.1 is true. Let $f \in S_2(\mathfrak{N})$ be a newform with a totally real field of coefficients $L$ such that $\text{Stab}_G([f]) = G$, and the homomorphism $\phi : G \to \text{Aut}(L)$ in Theorem 3.3 is injective. Assume that the Eichler-Shimura conjecture for totally real fields is true for $f$, i.e., there exists an abelian variety $A/F$, with $\text{End}_F(A) \otimes \mathbb{Q} = L$, such that

$$L(A, s) = \prod_{f' \in [f]} L(f', s).$$

Then, there is an abelian variety $B/E$, potentially of $GL_2$-type, such that $\text{End}_E(B) \otimes \mathbb{Q} = K$ and $A \sim B \times_E F$, where $K = L^\Delta$, $\Delta = \text{im}(\phi)$.

**Proof.** By construction, the abelian variety $A$ satisfies the condition of Theorem 4.4. So, it descends to an abelian variety $B/E$ potentially of $GL_2$-type. \qed 

**Remark 4.7.** By the Gross-Langlands conjecture on the modularity of abelian varieties (see [Gro15] and also [CD17, Conjecture 5.2]), there exists a globally generic cuspidal automorphic representation $\pi$ on $GSp_{2g+1}(A_E)$ such that

$$L(B, s) = \prod_{\pi' \in [\pi]} L(\pi', s),$$

where $[\pi]$ is the Hecke orbit of $\pi$. Since $B$ is the Galois descent of $A$, Langlands Functoriality predicts that $\pi$ must be the automorphic descent from $GL_{2g}$ to $GSp_{2g+1}$ of the automorphic induction of $\pi_f$ from $F$ to $E$, whose existence also depends on Conjecture 2.1. In other words, Theorem 3.3 implies that there is a functorial connection between the Eichler-Shimura conjecture for totally real fields and the Gross-Langlands conjecture for abelian varieties [Gro15].

5. An example

The following beautiful example was first suggested by Benedict H. Gross in connection with his conjecture on the existence of non-solvable number fields ramified at one prime only, which we proved for $p = 2$ in [Dem09]. Unfortunately, all the residual Galois representations involved have solvable images. Recently, we realised that this example provides better evidence for the conjectures in [Gro15] (see also [CD17]). Our example also happens to be related to a mathoverflow.net question, which was partially answered by Elkies [Elk15].

5.1. The Shimura curve. Let $F = \mathbb{Q}(\zeta) = \mathbb{Q}(\zeta_{32} + \zeta_{32}^{-1})$ be the maximal totally real subfield of the cyclotomic field of the 32nd root of unity. This field is defined by the polynomial $x^3 - 8x^2 + 20x^3 - 16x^2 + 2$. Let $\mathcal{O}_F$ be the ring of integers of $F$. Let $v_1, \ldots, v_8$ be the real places of $F$. We consider the quaternion algebra $D/F$ ramified at $v_2, \ldots, v_8$ and the unique prime $q$ above 2. More concretely, we
Table 1. Eigenforms of level q and weight 2 on \( F = \mathbb{Q}(\zeta_{32})^+ \)

| Newform | Coefficient field \( L_f \) | Fixed field \( K_f = L_f^\mathbb{Q} \) | \( \text{Gal}(L_f/K_f) \) |
|---------|-----------------|-----------------|-----------------|
| \( f, f' \) | \( \mathbb{Q}(\zeta_{15})^+ \) | \( \mathbb{Q} \) | \( \mathbb{Z}/4\mathbb{Z} \) |
| \( g, g' \) | Quartic subfield of \( \mathbb{Q}(\zeta_{95})^+ \) | \( \mathbb{Q} \) | \( \mathbb{Z}/4\mathbb{Z} \) |
| \( h \) | Ray class field of modulus \( c = (\frac{1}{2}(c^2 - 16c + 25)) \) | \( r = x^3 + x^2 - 229x + 167 \) | \( \mathbb{Z}/8\mathbb{Z} \) |

have \( D = \left( \frac{w-1}{w} \right) \), where \( w = -\alpha^2 + \alpha \) has signature \((+, -, -, -)\). Let \( \mathcal{O}_D \) be a maximal order in \( D \), and \( X_0^D(1) \) the Shimura curve attached to \( \mathcal{O}_D \). Let \( w_D \) be the Atkin-Lehner involution at \( q \). We also let \( D'/F \) be the totally definite quaternion algebra ramified exactly at all the real places \( v_1, \ldots, v_8 \), and fix a maximal order \( \mathcal{O}_{D'} \) in \( D' \).

Let \( S_2(q)^{\text{new}} \) be the new subspace of cusp forms of level \( q \) and weight 2, this is a 40-dimensional space. Let \( S_2^D(1) \) (resp. \( S_2^{D'}(q)^{\text{new}} \)) be the space of cusp forms of level \( (1) \) on \( D \) (resp. new subspace of cusp forms of level \( q \) on \( D' \)). By the Jacquet-Langlands correspondence, we have isomorphisms of Hecke modules

\[
S_2(q)^{\text{new}} \simeq S_2^D(1) \simeq S_2^{D'}(q)^{\text{new}}.
\]

Moreover, we can canonical identify \( S_2^D(1) \) with the the space of 1-differential forms on \( X_0^D(1) \). The space \( S_2(q)^{\text{new}} \) decomposes into 5 Hecke constituents of dimensions 4, 4, 4, 4 and 24 respectively. (We note that all the computations have been performed using the Hilbert Modular Forms Package in \texttt{Magma} [BCP97], see also [DD08, DV13, GV11].) We let \( f, f', g, g' \) and \( h \) be newforms in those constituents. Then we have:

(i) The forms \( f \) and \( f' \) have the same coefficient field, which is the real quartic field \( \mathbb{Q}(\zeta_{15})^+ \) given by \( x^4 + x^3 - 4x^2 - 4x + 1 \);

(ii) The forms \( g \) and \( g' \) have the same coefficient field, which is the real quartic subfield of \( \mathbb{Q}(\zeta_{95})^+ \) given by \( x^4 + 19x^3 - 59x^2 + 19x + 1 \);

(iii) The coefficient field of the form \( h \) is a field \( L_h \) of degree 24, which is cyclic over the field \( K_h = \mathbb{Q}(c) \) defined by \( c^3 + c^2 - 229c + 167 = 0 \). More precisely, it is the ray class field of conductor \( c = (\frac{1}{2}(c^2 - 16c + 25)) \).

(We summarise that data in Table 1, and the relations among the forms in Table 2.) Let \( w \) and \( w_D \) be the Atkin-Lehner involutions acting on \( S_2(q)^{\text{new}} \) and \( S_2^D(1) \) respectively. The Atkin-Lehner involution \( w \) acts as follows:

\[
w f = -f, \quad w f' = -f', \quad w g = -g, \quad w g' = -g', \quad w h = h.
\]

We recall that \( w_D = -w \).

From the above discussion, it follows that \( X_0^D(1) \) is a curve of genus 40.

**Lemma 5.1.** The curve \( X_0^D(1) \) and the Atkin-Lehner involution \( w_D \) are both defined over \( \mathbb{Q} \).

**Proof.** Since \( \sigma(q) = q \) and the ray class group of modulus \( q v_2 \cdots v_8 \) is trivial, the curve \( X_0^D(1) \) is defined over \( F \) by [DN67, Corollary], and the field of moduli is \( \mathbb{Q} \). Furthermore, the field \( \mathbb{Q}(\zeta_{32}) \) is a splitting field for \( D \) whose class number is one. So, the CM point attached to the extension \( \mathbb{Q}(\zeta_{32})/F \) is defined over \( F \). Therefore, by [SV15, Corollary 1.11] the curve \( X_0^D(1) \) descends to \( \mathbb{Q} \).

Alternatively, by using the moduli interpretation in [Car86a], or the more recent work [TX16], one can show that both \( X_0^D(1) \) and \( w_D \) are defined over \( \mathbb{Q} \). \( \square \)

**Corollary 5.2.** The curve \( C := X_0^D(1)/(w_D) \) has genus 16, and descends to \( \mathbb{Q} \).
5.2. The Jacobian varieties $\text{Jac}(X_0^D(1))$ and $\text{Jac}(C)$. From the above discussion, we have the following decomposition for $\text{Jac}(X_0^D(1))$ over $F$:

$$\text{Jac}(X_0^D(1)) = A_f \times A_{f'} \times A_g \times A_{g'}.$$  

From (5.3), and the fact that $w_D = -w$, we see that $$\text{Jac}(C) = A_f \times A_{f'} \times A_g \times A_{g'}.$$ 

The fourfolds $A_f$ and $A_{f'}$ (resp. $A_g$ and $A_{g'}$) are Galois conjugate. We will see later that one of consequences of the compatibility between the base change action and Hecke orbits is that the decomposition (5.3) descends to subfields of $F$.

**Theorem 5.4.** The abelian variety $A_h$ descends to a 24-dimensional variety $B_h$ defined over $\mathbb{Q}$, with good reduction outside 2, such that $\text{End}_{\mathbb{Q}}(B_h) \otimes \mathbb{Q} = K_h$ and

$$L(B_h, s) = \prod_{\Pi^\prime \in [\Pi_0]} L(\Pi', s),$$

where $\Pi_h$ is the automorphic representation lifting $\pi_h$ to $\text{GSpin}_{17}(\mathbb{A}_Q)$ and $[\Pi_0]$ its Hecke orbit.

**Proof.** By Table 2, there exists a generator $\tau \in \text{Gal}(L_h/K_h)$ such that $h = h'$. So, by [CD17, Theorem 5.4], $\pi_h$ lifts to an automorphic representation $\Pi_h$ on a split form of $\text{GSpin}_{17}(\mathbb{A}_Q)$ with coefficients in the cubic field $K_h$. The Hecke orbit $[\Pi_0]$ of $\Pi_h$ has 3 elements, and by functoriality

$$L(B_h, s) = \prod_{\Pi^\prime \in [\Pi_0]} L(\Pi', s).$$

It follows that $\text{End}_{\mathbb{Q}}(B_h) \otimes \mathbb{Q} = K_h$. Since the level of the form $h$ is the unique prime 2 above 2, $B_h$ has good reduction outside 2. \qed

Now, we turn to the quotient $C := X_0^D(1)/\langle w_D \rangle$.

**Theorem 5.5.** The abelian varieties $A_f$ and $A_{f'}$ (resp. $A_g$ and $A_{g'}$) descend to pairwise conjugate fourfolds $B_f$ and $B_{f'}$ (resp. $B_g$ and $B_{g'}$) over $\mathbb{Q}(\sqrt{2})$ with trivial endomorphism rings such that

$$L(B_f, s) = L(\Pi_f, s) \text{ and } L(B_{f'}, s) = L(\Pi_{f'}, s),$$

$$L(B_g, s) = L(\Pi_g, s) \text{ and } L(B_{g'}, s) = L(\Pi_{g'}, s),$$

where $\pi_f, \pi_{f'}, \pi_g$ and $\pi_{g'}$ lift to the automorphic representations $\Pi_f, \Pi_{f'}, \Pi_g$ and $\Pi_{g'}$ on $\text{GSpin}_9/\mathbb{Q}(\sqrt{2})$. They have good reduction outside $(\sqrt{2})$.

**Proof.** The identity $\Pi$ in Table 2, combined with [CD17, Theorem 5.4], implies that $\pi_f, \pi_{f'}, \pi_g$ and $\pi_{g'}$ lift to automorphic representations $\Pi_f, \Pi_{f'}, \Pi_g$ and $\Pi_{g'}$ on $\text{GSpin}_9/\mathbb{Q}(\sqrt{2})$ (or $\text{SO}(9)$ after normalisation) with coefficients in $\mathbb{Q}$. Consequently, the fourfolds $A_f, A_{f'}$, $A_g$ and $A_{g'}$ descend to pairwise conjugate fourfolds $B_f$ and $B_{f'}$ (resp. $B_g$ and $B_{g'}$) such that

$$\text{End}_{\mathbb{Q}(\sqrt{2})}(B_f) = \text{End}_{\mathbb{Q}(\sqrt{2})}(B_{f'}) = \text{End}_{\mathbb{Q}(\sqrt{2})}(B_g) = \text{End}_{\mathbb{Q}(\sqrt{2})}(B_{g'}) = \mathbb{Z}.$$ 

The equalities of $L$-series follow by functoriality. For the same reason as above, the fourfolds have good reduction outside $(\sqrt{2})$. \qed

**Remark 5.6.** The decomposition (5.3) is only true $a\ a\ priori$ over $F$. However, Theorem 5.4 and Theorem 5.5 imply that it descends to $\mathbb{Q}(\sqrt{2})$. In fact, it will further descend to $\mathbb{Q}$ if we put the pairwise conjugates $A_f$ and $A_{f'}$ (resp. $A_g$ and $A_{g'}$) together.
Therefore \( T \) torsion for our varieties, we start with the following proposition. We recall the maximal ideals. Then, by the identities in Table \( \mathbf{L} \), \( \mathbf{F} \) corresponding mod 2 Hecke eigensystems. The residual Hecke algebras.

\[
\begin{array}{cccc}
\text{Newform} & \tau & I & II \\
\hline
f, f' & b \mapsto -b^3 + b^2 + 3b - 2 & \sigma f = f' & \sigma^2 f = f^r \\
g, g' & b \mapsto \prod_2 (-3b^3 - 58b^2 + 154b - 35) & \sigma g = g' & \sigma^2 g = g^r \\
h & \exists \tau & \sigma h = h^r & \\
\end{array}
\]

5.3. The residual Hecke algebras. Let \( \mathbf{T}^{\text{new}} \) be the \( \mathbb{Z} \)-subalgebra of \( \text{End}_{\mathbb{C}}(S_2(\mathbf{q})) \) acting on \( S_2(\mathbf{q})^{\text{new}} \); and \( \mathbf{T}_f, \mathbf{T}_{f'}, \mathbf{T}_g, \mathbf{T}_{g'} \) and \( \mathbf{T}_h \) the \( \mathbb{Z} \)-subalgebras acting on the constituents of \( f, f', g, g' \) and \( h \) respectively. From the above discussion, we have

\[
\mathbf{T}^{\text{new}} \otimes \mathbf{Q} = (\mathbf{T}_f \otimes \mathbf{Q}) \times (\mathbf{T}_{f'} \otimes \mathbf{Q}) \times (\mathbf{T}_g \otimes \mathbf{Q}) \times (\mathbf{T}_{g'} \otimes \mathbf{Q}) \times (\mathbf{T}_h \otimes \mathbf{Q}) = L_f \times L_{f'} \times L_g \times L_{g'} \times L_h.
\]

By direct calculations, we get the followings:

- \( [\mathcal{O}_{L_f} : \mathbf{T}_f] \) divides 3,
- \( [\mathcal{O}_{L_g} : \mathbf{T}_g] = [\mathcal{O}_{L_{g'}} : \mathbf{T}_{g'}] = 1, \)
- \( [\mathcal{O}_{L_h} : \mathbf{T}_h] \) divides 3 \( \cdot 5^6. \)

Therefore \( \mathbf{T}^{\text{new}} \otimes \mathbf{Z}_2 \) decomposes into \( \mathbf{Z}_2 \)-algebras as

\[
\mathbf{T}^{\text{new}} \otimes \mathbf{Z}_2 = (\mathbf{T}_f \otimes \mathbf{Z}_2) \times (\mathbf{T}_{f'} \otimes \mathbf{Z}_2) \times (\mathbf{T}_g \otimes \mathbf{Z}_2) \times (\mathbf{T}_{g'} \otimes \mathbf{Z}_2) \times (\mathbf{T}_h \otimes \mathbf{Z}_2).
\]

Since the prime 2 is inert in \( L_f = L_{f'}, \) and \( L_g = L_{g'}, \) the first four factors of this decomposition are local \( \mathbf{Z}_2 \)-algebras. Let \( m_f, m_{f'}, m_g \) and \( m_{g'} \) be the corresponding maximal ideals. Then, by the identities in Table 2, we have \( \sigma(m_f) = m_{f'} \) and \( \sigma^2(m_f) = \tau_f(m_f); \) and \( \sigma(m_g) = m_{g'} \) and \( \sigma^2(m_g) = \tau_g(m_g). \) We let \( \theta_f : \mathbf{T}_f \otimes \mathbf{Z}_2 \rightarrow \mathbf{F}_{16}, \theta_{f'} : \mathbf{T}_{f'} \otimes \mathbf{Z}_2 \rightarrow \mathbf{F}_{16}, \theta_g : \mathbf{T}_g \otimes \mathbf{Z}_2 \rightarrow \mathbf{F}_{16} \) and \( \theta_{g'} : \mathbf{T}_{g'} \otimes \mathbf{Z}_2 \rightarrow \mathbf{F}_{16} \) be the corresponding mod 2 Hecke eigensystems.

Next, we recall that \( L_h \) is the ray class field of conductor \( \mathcal{c} = (\frac{1}{2}(c^2 - 16c + 25)) \) over the field \( K_h = \mathbf{Q}(\mathcal{c}), \) with \( c^2 + c^2 - 229c + 167 = 0. \) The prime 2 is totally ramified in \( K_h. \) Letting \( p_2 \) be the unique prime above it, we get that \( p_2 = \mathfrak{P}_2 \mathfrak{P}' \), where \( \mathfrak{P} \) and \( \mathfrak{P}' \) are inert primes, and \( \tau(\mathfrak{P}) = \mathfrak{P}'. \) Therefore, there are two maximal primes \( m_2 \) and \( m_2' \) in \( \mathbf{T}_h \otimes \mathbf{Z}_2 \) such that \( \sigma(m_2) = m_2' \) and \( \sigma^2(m_2) = \tau_h(m_2). \) We let \( \theta_2 \times \theta_2' : \mathbf{T}_h \otimes \mathbf{Z}_2 \rightarrow \mathbf{F}_{16} \times \mathbf{F}_{16} \) be the resulting two mod 2 Hecke eigensystems.

By computing the socle of the underlying \( \mathbf{F}_2 \)-module to \( \mathbf{T}^{\text{new}} \otimes \mathbf{F}_2, \) we obtain that \( \theta_f \simeq \theta_g \simeq \theta_h, \) and \( \theta_{f'} \simeq \theta_{g'} \simeq \theta_{h'} \), up to rearranging. We will denote these two Hecke eigensystems by \( \theta \) and \( \theta' \) respectively.

5.4. The field of 2-torsion for Jac(\( \mathbf{X}_0^D(1) \)) and Jac(\( \mathbf{C} \)). To analyse the field of 2-torsion for our varieties, we start with the following proposition. We recall the following diagram

\[
\begin{array}{ccc}
\mathbf{Q}(\zeta_{64}) & \text{Q(}\zeta_{64}^{+}\text{)} & \mathbf{Q}(\zeta_{32}) \\
\downarrow & \downarrow & \downarrow \\
\mathbf{Q}(i(\zeta_{64} + \zeta_{64}^{-1})) & \mathbf{Q}(\zeta_{64}) & \mathbf{Q}(\zeta_{32}) \\
\end{array}
\]

The subfield \( K = \mathbf{Q}(\beta) = \mathbf{Q}(i(\zeta_{64} + \zeta_{64}^{-1})) \) is the unique CM extension of \( F \) with class number 17. For later, we observe that \( \beta^2 = -2 - \alpha, \) where \( \alpha = (2 + \alpha). \)
Proposition 5.7. Let $\bar{\rho}, \bar{\rho}' : \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\mathbb{F}_{16})$ be the mod 2 Galois representations attached to $\theta$ and $\theta'$ respectively. Then, there are characters $\chi, \chi' : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{F}_{2^5}$, with trivial conductor such that $\bar{\rho} = \text{Ind}_K^\overline{\mathbb{Q}} \chi$, and $\bar{\rho}' = \text{Ind}_K^\overline{\mathbb{Q}} \chi'$.

Proof. We already computed the Hecke constituents of the space $S_2(1)$ in [Dem09]. The mod 2 Hecke eigensystems in that case have coefficient fields $\mathbb{F}_2$ where $s = 1, 2, 8$. Therefore, since $\theta$ has coefficient field $\mathbb{F}_{16}$, it cannot arise from an eigenform of level 1. By the Serre conjecture for totally real fields (the totally ramified case) [GS11], it must appear on the quaternion algebra $D'$ with level (1) and non-trivial weight. The same is true for $\theta'$. In fact, the analysis conducted above shows that they are the only eigensystems that can appear at that weight. (We note that there are only two Serre weights in this case.)

Let $\chi : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{F}_2^\times$ be a character with trivial conductor such that $\chi^s \neq \chi$, where $\text{Gal}(K/F) = \langle s \rangle$. By class field theory, we can identify $\chi$ with its image under the Artin map. Since $\chi$ is unramified, it must factor as $\chi : K^\times \backslash \mathbb{A}_K^\times \to \text{Cl}_K \to \mathbb{F}_2^\times$. Furthermore, since $\text{Cl}_K \cong \mathbb{Z}/17\mathbb{Z}$, we must have $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{F}_2^\times$, and the representation $\bar{\rho}_\chi := \text{Ind}_K^\overline{\mathbb{Q}} \chi : \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\mathbb{F}_{16})$ has coefficients in $\mathbb{F}_{16}$. So, $\bar{\rho}_\chi$ has level (1) and non-trivial weight by the argument above. Therefore, it must be a Galois conjugate of $\bar{\rho}$. Up to relabelling, we can assume that $\bar{\rho} \simeq \bar{\rho}_\chi$. Since $\theta$ and $\theta'$ are $\text{Gal}(F/\mathbb{Q})$-conjugate, there is also a character $\chi' : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{F}_2^\times$ such that $\bar{\rho}' \simeq \bar{\rho}_{\chi'}$.

Alternatively, we can show that $\theta$ appears on $D'$ with the non-trivial weight without using the fact that it has coefficients in $\mathbb{F}_{16}$. Indeed, we have $\bar{\rho}_\chi|_{\mathfrak{p}_q} \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Let $K_\Omega$ be the completion of $K$ at $\Omega$, the unique prime above $\mathfrak{q}$. Since $K = F[\beta]$, and $\beta^2 = -2 - \alpha$ is a generator of $\mathfrak{q}$, then we have $K_\Omega = F_\mathfrak{q}[^2\sqrt{\alpha}]$, where $\alpha$ is a uniformiser of $F_\mathfrak{q}$. Therefore, $\bar{\rho}_\chi|_{D_\mathfrak{q}}$ doesn’t arise from a finite flat group scheme. Hence, $\bar{\rho}_\chi$ must have non-trivial weight.  

We are now ready to state the main theorem of this section.

Theorem 5.8. The Jacobians $\text{Jac}(X^D_0(1))$ and $\text{Jac}(C)$ of the curves $X^D_0(1)$ and $C$, both defined over $\mathbb{Q}$, have the same field of 2-torsion $N$. The field $N$ is the unique field unramified outside 2, with Galois group $\text{Gal}(N/\mathbb{Q}) = \mathbb{Z}/17\mathbb{Z} \times (\mathbb{Z}/17\mathbb{Z})^\times = F_{17}$.

Proof. Let $\bar{\rho}_{f,2}, \bar{\rho}_{f',2} : \text{Gal}(\overline{\mathbb{Q}}/F) \to \text{GL}_2(\mathbb{F}_{16})$ be the mod 2 Galois representations attached to $f$ and $f'$. By Proposition 5.7, $\bar{\rho}_{f,2}$ and $\bar{\rho}_{f',2}$ are dihedral and we have that $\text{im}(\bar{\rho}_{f,2}) = \text{im}(\bar{\rho}_{f',2}) = D_{17}$. Let $M_f, M_{f'}$ be the fields cut out by $\bar{\rho}_{f,2}$ and $\bar{\rho}_{f',2}$. Since $\sigma(m_f) = m_{f'}$, we have $M_{f'} = M_f^\sigma$. By construction $M_f$ and $M_{f'}^\sigma$ are unramified extension of $K$. So, by uniqueness of the Hilbert class field, we must have $M_f = M_{f'}^\sigma = M_fM_{f'}^\sigma = H_K$, where $M_fM_{f'}^\sigma$ is the compositum of $M_f$ and $M_{f'}^\sigma$; and $H_K$ is the Hilbert class field of $K$. Since $\sigma^2(m_f) = \tau_f(m_f)$, we have

$$\text{Gal}(N_f/\mathbb{Q}) = D_{17} \rtimes \mathbb{Z}/8\mathbb{Z} = F_{17}.$$  

Letting $N_g$ and $N_h$ be the normal closures of the fields cut out by the mod 2 representations attached to $g$ and $h$ respectively, the same argument shows that $\text{Gal}(N_g/\mathbb{Q}) = \text{Gal}(N_h/\mathbb{Q}) = F_{17}$.

By [Har94, Theorem 2.25], there is a unique field $N$ ramified at 2 and $\infty$, with Galois group $F_{17}$. Therefore, we must have $N = N_f = N_g = N_h$. By using
the decomposition in (5.3), we see that $N$ must be the field of 2-torsion for both $\Jac(X_D^0(1))$ and $\Jac(C)$. □

Remark 5.9. The field $N$ is the splitting field of the polynomial

$$H := x^{17} - 2x^{16} + 8x^{13} + 16x^{12} - 16x^{11} + 64x^9 - 32x^8 - 80x^7 + 32x^6 + 40x^5 + 80x^4 + 16x^3 - 128x^2 - 2x + 68.$$ 

This polynomial was computed by Noam Elkies. His computation, together with our Theorem 5.8, answer a question posed by Jeremy Rouse as to whether the field $N$ is the field of 2-torsion of some abelian variety. We thank David P. Roberts for bringing this matoverflow.net discussion [Elk15] to our attention.

One can show that the field $K$ splits the quaternion algebra $D$. Let $\mathcal{O}$ be the suborder of $\mathcal{O}_K$ of index $\Omega^2$, where $\Omega$ is the unique prime above $q$ in $K$. Let $\Pic(\mathcal{O})$ be the Picard group of $\mathcal{O}$. A quick Magma calculation show that $\# \Pic(\mathcal{O}) = 34 = 2 \cdot 17 + 2$. We conclude this note with the following claim.

Conjecture 5.10. The curve $C$ is hyperelliptic over $F$. Its Weierstrass points are the CM points arising from $\Pic(\mathcal{O})$.

Remark 5.11. We believe that the curve $C$ is in fact hyperelliptic over $\mathbb{Q}$.

Remark 5.12. We observe that $w_D$ is the unique Atkin-Lehner involution on $X_{D0}^0(1)$. Therefore, if Conjecture 5.10 is true, then the hyperelliptic involution on $C$ must be an exceptional one. We note that, for $F = \mathbb{Q}$, Michon [Mic81] provides a complete list of all Shimura curves with square-free level that are hyperelliptic. (This result was independently obtained by Ogg in an unpublished work.) But, in general, the question of finding those Shimura curves which admit a hyperelliptic quotient is still wide open even for $F = \mathbb{Q}$. In that regards, Conjecture 5.10 is rather striking.

Remark 5.13. One should be able to find an explicit equation for $C = X_{D0}^0(1)/\langle w_D \rangle$ using [VW14]. But, currently, the strategy for doing so is not fully implemented. It should also be possible to use a generalisation of the $p$-adic approach discussed in [FM14], which was inspired by [Kur94]. We hope to return to this problem in an upcoming paper.

References

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR 1484478

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, Potential automorphy and change of weight, Ann. of Math. (2) 179 (2014), no. 2, 501–609. MR 3152941

[Car86a] Henri Carayol, Sur la mauvaise réduction des courbes de Shimura, Compositio Math. 59 (1986), no. 2, 151–230. MR 860139

[Car86b] ______, Sur les représentations l-adiques associées aux formes modulaires de Hilbert, Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 3, 409–468. MR 870690

[CD17] Clifton Cunningham and Lassina Dembélé, Lifts of Hilbert modular forms and applications to modularity of abelian varieties, Preprint, 2017.

[DD08] Lassina Dembélé and Steve Donnelly, Computing Hilbert modular forms over fields with nontrivial class group, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 5011, Springer, Berlin, 2008, pp. 371–386. MR 2467859

[Dem09] Lassina Dembélé, A non-solvable Galois extension of $\mathbb{Q}$ ramified at 2 only, C. R. Math. Acad. Sci. Paris 347 (2009), no. 3-4, 111–116. MR 2538094

[Die12] Luis Dieulefait, Langlands base change for GL(2), Ann. of Math. (2) 176 (2012), no. 2, 1015–1038. MR 2950769

[DN67] Koji Doi and Hidehisa Naganuma, On the algebraic curves uniformized by arithmetic automorphic functions, Ann. of Math. (2) 86 (1967), 449–460. MR 0219537
Lassina Dembélé and John Voight, Explicit methods for Hilbert modular forms, Elliptic curves, Hilbert modular forms and Galois deformations, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Basel, 2013, pp. 135–198. MR 3184337

Noam Elkies, Degree 17 number fields ramified only at 2, unpublished, 2015.

G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, Invent. Math. 73 (1983), no. 3, 349–366. MR 718935

D. Flath, Decomposition of representations into tensor products, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 179–183. MR 546596

Cameron Franc and Marc Masdeu, Computing fundamental domains for the Bruhat-Tits tree for $GL_2(\mathbb{Q}_p)$, $p$-adic automorphic forms, and the canonical embedding of Shimura curves, LMS J. Comput. Math. 17 (2014), no. 1, 1–23. MR 3230854

Benedict Gross, On the Langlands correspondence for symplectic motives, preprint, 2015.

Toby Gee and David Savitt, Serre weights for mod $p$ Hilbert modular forms: the totally ramified case, J. Reine Angew. Math. 660 (2011), 1–26. MR 2855818

Matthew Greenberg and John Voight, Computing systems of Hecke eigenvalues associated to Hilbert modular forms, Math. Comp. 80 (2011), no. 274, 1071–1092. MR 2772112

David Harbater, Galois groups with prescribed ramification, Arithmetic geometry (Tempe, AZ, 1993), Contemp. Math., vol. 174, Amer. Math. Soc., Providence, RI, 1994, pp. 35–60. MR 1299733

Haruzo Hida, On $p$-adic Hecke algebras for $GL_2$ over totally real fields, Ann. of Math. (2) 128 (1988), no. 2, 295–384. MR 960949

Haruzo Hida and Yoshitaka Maeda, Non-abelian base change for $GL(2)$, Pure Appl. Math. Q. 5 (2009), no. 1, 81–125. MR 2520456

Haruzo Hida and Yoshitaka Maeda, Non-abelian base change for totally real fields, Pacific J. Math. (1997), no. Special Issue, 189–217, Olga Taussky-Todd: in memoriam. MR 1610859

Akira Kurihara, On $p$-adic Poincaré series and Shimura curves, Internat. J. Math. 5 (1994), no. 5, 747–763. MR 1297415

Robert P. Langlands, Base change for $GL(2)$, Annals of Mathematics Studies, vol. 96, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980. MR 574808

Jean-Francis Michon, Courbes de Shimura hyperelliptiques, Bull. Soc. Math. France 109 (1981), no. 2, 217–225. MR 623790

Toshitsune Miyake, On automorphic forms on $GL_2$ and Hecke operators, Ann. of Math. (2) 94 (1971), 174–189. MR 0299559

Elisabeth E. Pyle, Abelian varieties over $\mathbb{Q}$ with large endomorphism algebra and their simple components over $\mathbb{Q}$, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 189–239. MR 2058652

Kenneth A. Ribet, Fields of definition of abelian varieties with real multiplication, Arithmetic geometry (Tempe, AZ, 1993), Contemp. Math., vol. 174, Amer. Math. Soc., Providence, RI, 1994, pp. 107–118. MR 1299737

Kenneth A. Ribet, Abelian varieties over $\mathbb{Q}$ and modular forms, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 241–261. MR 2058653

Hiroshi Saito, Automorphic forms and algebraic extensions of number fields II, J. Math. Kyoto Univ. 19 (1979), no. 1, 105–123. MR 527398

Goro Shimura, The special values of the zeta functions associated with Hilbert modular forms, Duke Math. J. 45 (1978), no. 3, 637–679. MR 507462

Jeroen Sijling and John Voight, On explicit descent for marked curves and maps, preprint, 2015.

Richard Taylor, On Galois representations associated to Hilbert modular forms, Invent. Math. 98 (1989), no. 2, 265–280. MR 1016264

Yichao Tian and Liang Xiao, On Goren-Oort stratification for quaternionic Shimura varieties, Compos. Math. 152 (2016), no. 10, 2134–2220. MR 3570003

John Voight and John Willis, Computing power series expansions of modular forms, Computations with modular forms, Contrib. Math. Comput. Sci., vol. 6, Springer, Cham, 2014, pp. 331–361. MR 3381459
