The covariant entropy bound and loop quantum cosmology

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We examine Bousso’s covariant entropy bound conjecture in the context of radiation filled, spatially flat, Friedmann-Robertson-Walker models. The bound is violated near the big bang. However, the hope has been that quantum gravity effects would intervene and protect it. Loop quantum cosmology provides a near ideal setting for investigating this issue. For, on the one hand, quantum geometry effects resolve the singularity and, on the other hand, the wave function is sharply peaked at a quantum corrected but smooth geometry which can supply the structure needed to test the bound. We find that the bound is respected. We suggest that the bound need not be an essential ingredient for a quantum gravity theory but may emerge from it under suitable circumstances.

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I. INTRODUCTION

Over 30 years ago, Bekenstein made the seminal suggestion that black holes carry an entropy equal to a quarter of their surface area in Planck units [1, 2]. The discovery of the first law of black hole thermodynamics by Bardeen, Carter and Hawking [3, 4], coupled with Hawking’s discovery of black hole radiance [5] made this suggestion compelling. The ensuing generalized second law of thermodynamics [6] in turn motivated the more recent holographic principle by ’t Hooft and Susskind [7, 8] as well as several entropy bound conjectures. It has been suggested that the holographic principle is a powerful hint and should be used as an essential building block for any quantum gravity theory, much as the principle of equivalence is fundamental to general relativity [7, 8, 9].

Perhaps the most promising of the entropy bound conjectures is due to Bousso [9, 10]. It is formulated in precise geometric terms and limits the entropy flux through a surface’s “light-sheet” by the area of that surface. More precisely, let $A$ be the area of an arbitrary spatial 2-surface $B$. A three-dimensional null hypersurface $L$ is a light-sheet of $B$ if it is generated by nonexpanding light rays that begin at $B$ and extend orthogonally away from $B$. Let $S$ denote the total entropy flux of the matter fields across any light-sheet $L$ of $B$. Then (in units where $c=k=1$)

$$S \leq \frac{A}{4G\hbar}. \quad (1.1)$$

In general, $S$ is difficult to compute unless an entropy current $s^a$ can be introduced, in which

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\[ S = \int_L s^a \epsilon_{abcd}, \]  
\[(1.2)\]

where \( \epsilon_{abcd} \) is the space-time volume 4-form. Although the existence of an entropy current is not necessary for the bound to be meaningful, it often is not clear how to compute \( S \) if \( s^a \) does not exist.

The conjecture has several curious features. First, for it to be easily tested, matter should admit an entropy current. This is a strong requirement since a well-defined expression of entropy current is generally not available unless the matter can be represented as a fluid. Therefore, standard space-times with “fundamental” matter fields —such as scalar, Maxwell, and gauge fields— that one often considers in general relativity cannot readily be tested. Thus, circumstances where the conjecture can be tested are rather limited. However, it is still a highly nontrivial question as to whether the bound holds for matter sources that do admit a well-defined entropy current. Remarkably, the answer is in the affirmative provided the entropy current and stress-energy tensor satisfy certain inequalities along \( L \) \[11, 12\]. Moreover, these inequalities can be motivated from statistical physics of standard bosons and fermions provided one remains away from the Planck scale. Finally, there is also a necessary and sufficient condition \[13\] for the validity of the bound, based on a plausible assumption on the time scale on which thermodynamic equilibrium can be reached.

A second feature of the conjecture is that its current proofs \[11, 12\] require that matter fields satisfy the dominant (or at least null) energy condition; otherwise it would be possible to add entropy across \( L \) without adding energy and appreciably changing the area of \( B \). This assumption is completely reasonable within classical general relativity and indeed constitutes a building block of many of the hard results in this theory. However, since the right side of \((1.1)\) becomes infinite in the limit \( \hbar \) goes to zero, the conjecture is nontrivial only in the realm of quantum physics. But in quantum field theory, expectation values of the matter stress-energy tensor fail to satisfy the dominant (or the null) energy condition. Thus there is a clear tension between the classical formulation of the covariant entropy bound and the quantum world it is trying to represent. What would happen in full quantum gravity? Would a suitable generalization of the conjecture survive? As remarked above, it has been suggested that an appropriate generalization should in fact be a building block of any quantum gravity theory.

Finally, as we will see in Sec. II, even in the k=0, radiation dominated Friedmann-Robertson-Walker (FRW) space-time —a classical solution of Einstein’s equation, which admits a perfect fluid as source— the conjecture is violated near the big bang.\(^1\) (In this regime the inequalities assumed in \[11, 12, 13\] fail.) It is natural to suppose that this violation is spurious, because quantum gravity effects would be dominant in this regime and modify the dynamics of general relativity. But one’s first expectation would be that in this Planck regime the space-time metric would fluctuate violently,\(^2\) making it impossible to speak of a spacelike surface \( B \) and especially its light-sheet \( L \). If so, the basic ingredients in the very statement of the conjecture would be unavailable. However, loop quantum cosmology (LQC) provides a rather unexpected situation \[18, 19, 20\]: Einstein’s equations do get modified, the classical singularity is resolved and replaced by a quantum bounce, but

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\(^1\) Such problems in the deep Planck regime very near the big bang were encountered also in the earlier versions of entropy bounds \[14\].

\(^2\) Explicit examples of such phenomena are provided by some symmetry reduced models. See, e.g., \[15, 16\].
all this physics is well captured by a quantum corrected metric, which is again a smooth
tensor field (although its expression contains \(\hbar\) corrections, which dominate the dynamics
near the bounce). Therefore, the structure required for Bousso’s formulation is available.
It is then natural to ask: Does the bound hold? We will show that the answer is in the
affirmative.

This result by itself is not a strong statement in favor of either the covariant bound or
LQC. Had it been violated, one could have argued that the fluid approximation is inap-
propriate near the bounce where the matter density is close to the Planck density, or that
LQC, with its focus on homogeneous and isotropic situations, is too restrictive. Nonetheless,
the fact that there is a coherence between ideas that were independently developed from
completely different motivations and perspectives is quite striking. There may well be a
deeper underlying reason behind this apparent unity. Our viewpoint, motivated by results
of loop quantum gravity (LQG), is the following. In the Planck regime, space-time geometry
is quantum mechanical and has a fundamental, in-built discreteness (see, e.g., [21, 22, 23]).
This discreteness is directly responsible for many of the novel features of the theory, including
the resolution of the big bang singularity through a quantum bounce [18, 19, 20]. Because of
this discreteness, the true degrees of freedom of nonperturbative quantum gravity are very
different from those of standard quantum field theories on background space-times. This fact
has important consequences which have a “holographic character” —e.g. the derivation of
the entropy of black holes and cosmological horizons [24, 25, 26]. Bousso’s covariant entropy
bound could thus emerge from LQG in suitable circumstances, including some in which the
assumptions made in [11, 12, 13] are violated. Thus, rather than being an essential building
block for quantum gravity, the bound could emerge from a background independent, non-
perturbative theory which appropriately incorporates the quantum nature of geometry in
the Planck regime.

The paper is organized as follows. In Sec. II we show that in the k=0, radiation filled
FRW cosmology, the covariant entropy bound is violated near the big bang singularity. In
Sec. III we first recall the basic results from LQC and then show that the bound is in fact re-
spected in the quantum corrected “effective” space-time geometry that descends from LQC.
In the classical as well the LQC analysis of the bound, we only consider “round” 2-spheres \(\mathcal{B}\).
As in other discussions —e.g., of marginally trapped surfaces— in general relativity, treat-
ment of general surfaces would increase the technical complexity very significantly. Section
IV summarizes the results and discusses related issues.

II. FRW UNIVERSE

The line element of a flat FRW universe is given by
\[
ds^2 = -d\tau^2 + a(\tau)^2 \left( dr^2 + r^2 d\Omega^2 \right),
\]
and the Friedmann equations describing the evolution of this universe are
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho,
\]
\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + P) = 0.
\]
We will restrict ourselves to a radiation-dominated universe where the required entropy current can be constructed using standard statistical mechanical considerations. The equation of state \( P = \frac{1}{3} \rho \) of the fluid gives the solution

\[
a(\tau) = \left( \frac{32\pi G K_0}{3} \tau \right)^{1/4} \quad \text{and} \quad \rho(\tau) = \frac{K_0}{a(\tau)^4},
\]

(2.4)

where \( K_0 \) is an integration constant without direct physical significance, which drops out of the expressions of measurable quantities.

We will take the radiation fluid to be a photon gas and assume that the universe is always in instantaneous equilibrium. Then, from standard statistical mechanics we have

\[
\rho = \frac{\pi^2}{15\hbar^3} T^4,
\]

(2.5)

where \( T \) is the temperature. From (2.4) we then conclude as usual that the temperature has the following time dependence:

\[
T = \left( \frac{45\hbar^3}{32\pi^3 G} \right)^{1/4} \frac{1}{\sqrt{\tau}}.
\]

(2.6)

The entropy density \( s \) of a photon gas is given by

\[
s = \rho + P T.
\]

(2.7)

Therefore, the entropy flux through a light-sheet \( L \) is given by

\[
S = \int_L s^a \epsilon_{abcd},
\]

(2.8)

where the entropy density 4-vector is \( s^a = su^a \) with \( u^a \), the unit vector normal to constant \( \tau \) slices and where, as before, \( \epsilon_{abcd} \) is the space-time volume 4-form. For our calculations, we will choose our spatial 2-surface \( B \) to be a metric 2-sphere at time \( \tau_f \). Since the FRW space-time does not admit a trapped surface, \( B \) must admit a past light-sheet. Furthermore, since the FRW space-times are conformally flat and \( B \) is a round 2-sphere, \( L \) would either terminate on the big bang singularity or be the future null cone of a point. Without loss of generality we can assume that \( L \) is generated by the null vector \( k^a = (-1, -\frac{1}{a}, 0, 0) \). Then, if \( L \) is the null cone of a point at \( \tau = \tau_i > 0 \), we find that

\[
S = \frac{16\pi}{9} \left( \frac{\pi^2 K_0^3}{15\hbar^3} \right)^{1/4} R_f^3,
\]

(2.9)

where

\[
R_f = \int_{\tau_i}^{\tau_f} \frac{d\tau'}{a(\tau')}
\]

(2.10)

is the radius of \( B \). \( R_f \), and hence \( S \), is well defined so long as \( \tau_i > 0 \). Since the area of \( B \) is simply

\[
A = 4\pi a(\tau_f)^2 R_f^2,
\]

(2.11)
we obtain
\[
S \frac{A}{\tau_f} = \frac{1}{6} \left( \frac{2}{45 \pi G^3 \hbar^3} \right)^{1/4} \frac{1}{\sqrt{\tau_f}} \left( 1 - \sqrt{\frac{\tau_i}{\tau_f}} \right)
\]
(2.12)
as the ratio of the entropy flux through L to the area of B. Clearly this ratio diverges as \(\tau_f\) approaches zero (keeping \(\tau_i < \tau_f\)). Therefore, one can violate the covariant entropy bound by an arbitrary amount by choosing B sufficiently close to the big bang.

However, by plugging in numbers it is easy to show that the bound does hold if \(\tau_f \gtrsim 0.06 t_{Pl}\) or, equivalently, the matter density satisfies \(\rho \lesssim 8.3 \rho_{Pl}\). Thus, the bound is violated only in the deep Planck regime where quantum gravity effects are expected to be dominant. Indeed, since the phenomenological fluid approximation is likely to fail for much larger values of \(\tau_f\), it is surprising that the bound has such a large domain of validity! Nonetheless, since it does fail, it is natural to ask whether the quantum gravity effects that resolve the big bang singularity can also restore the validity of the bound. In the next section, we will analyze this issue in the context of LQC and show that the answer is in the affirmative.

III. QUANTUM BOUNCE AND THE ENTROPY BOUND

This section is divided into two parts. In order to make the paper self-contained, in the first part we provide a concise summary of ideas that underlie LQC and results that have emerged from it, focusing on those features that are most relevant to our main result. Readers who are already familiar with LQC will find the streamlined summary of “effective” equations useful. Readers who are interested only in the entropy bound can skip this material and go directly to the next subsection where we analyze the covariant entropy bound in the quantum corrected, “effective” space-time that emerges from LQC.

A. Loop Quantum Cosmology

Loop quantum cosmology is an application of the techniques of LQG \[21, 22, 23\] to space-times that are homogenous and isotropic. Currently, symmetry reduction is carried out at the classical level; LQC is yet to be derived systematically from LQG. However, one quantizes the reduced system by mimicking full LQG as closely as possible. Therefore, the mathematical structure of LQC closely resembles that of full LQG, thus differing from the older Wheeler-DeWitt quantum cosmology already at the kinematical level.

In LQG the gravitational phase space is the same as that in the SU(2) Yang-Mills theory. Thus, the configuration variable is an SU(2) connection \(A_a^i\) and its conjugate momentum is a vector density \(E^a_i\) of weight 1 which also takes values in the Lie algebra of SU(2).\(^4\) However, \(A_a^i\) is now the gravitational connection used to parallel propagate chiral spinors. Similarly, the “electric field” \(E^a_i\) now has a direct geometrical interpretation: it represents a (density weighted) orthonormal triad that determines the spatial Riemannian geometry. The fundamental quantum algebra \(a\) is generated by holonomies \(h_e\) defined by the gravitational spin-connection \(A_a^i\) along (1-dimensional curves or) edges \(e\), and fluxes \(E_S\) of the

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\(^3\) For a relatively short review of LQC and its relation to LQG, see, e.g., \[17\].
\(^4\) Thus, indices \(a, b, c, \ldots\) refer to the tangent space of the “spatial” 3-manifold and \(i, j, k, \ldots\) to the Lie algebra of SU(2).
“electric fields” $E_i^a$ across 2-surfaces $S$. The key task in quantum kinematics is to find an appropriate representation of $a$ on a Hilbert space $\mathcal{H}_{\text{kin}}$. Somewhat surprisingly, it turns out that the requirement of background independence selects the representation uniquely \[27\]! This representation provides the arena to formulate quantum dynamics. Now, the classical dynamics of the gravitational and matter fields is generated by a set of first class constraints. These classical constraints have to be promoted to well-defined self-adjoint operators on $\mathcal{H}_{\text{kin}}$. Finally, the physical Hilbert space is built from suitable solutions to these operator constraints. As in any background independent system, these physical states already encode quantum dynamics which can be made explicit by choosing one of the degrees of freedom —such as a matter field or, in cosmology, the scale factor— as an internal clock with respect to which other degrees of freedom evolve.

When one symmetry reduces general relativity by imposing homogeneity and isotropy, it turns out simplest to gauge fix and solve the so-called Gauss and vector constraints, which respectively generate the internal SU(2) rotations of triads and spatial diffeomorphisms. One is then left just with the Hamiltonian constraint, which generates evolution (in proper time). In the $k=0$, FRW model, it is given by

$$\mathcal{H}_{\text{grav}} + \mathcal{H}_{\text{matt}} = 0,$$

where

$$\mathcal{H}_{\text{grav}} = -\gamma^{-2} \epsilon^{ij}_k \frac{E_i^a E_j^b}{\sqrt{E}} F_{ab}^k,$$

and

$$\mathcal{H}_{\text{matt}} = \rho \sqrt{E},$$

(3.1)

where $\gamma$ is the so-called Barbero-Immirzi parameter, $F_{ab}^k$ the field strength of the connection $A_i^a$, $\rho$, the matter density and $E$, the determinant of $E_i^a$ (or equivalently, of the spatial metric $q_{ab}$ determined by $E_i^a$). Let us focus on the gravitational part $\mathcal{H}_{\text{grav}}$ of this constraint. The term containing triad can be taken over to a quantum operator in a natural fashion, first spelled out by Thiemann \[23, 28\]. Therefore, the key problem in passage to the quantum theory lies in the construction of an operator $\hat{F}_{ab}^k$ representing field strength. For, while the unique representation of $a$ selected by the requirement of background independence admits an operator $\hat{h}_e$ representing holonomies, these fail to be continuous in the edge $e$, whence there is no operator corresponding to the connection $A_i^a$ itself. The strategy then is to use holonomy operators to obtain $\hat{F}_{ab}^k$.

Now, it is well known that in classical geometry, the field strength can be recovered as the limit of the ratio of the holonomy around an appropriate closed loop divided by the area of the loop, as one shrinks the loop thereby sending its area to zero. We can use the same idea for quantization of $F_{ab}^k$. However, now because of quantum geometry, eigenvalues of the area operator are discrete. Therefore, to define the action of $\hat{F}_{ab}^k$ on a given LQC state, we are led to shrink the loop only until the physical area it encloses reaches the minimum nonzero eigenvalue, say $\alpha_o \ell_2^2 (\text{Pl})$ (in the class of LQG states compatible with the LQC state under consideration). The resulting $\hat{F}_{ab}^k$ is a self-adjoint operator on the kinematical Hilbert space $\mathcal{H}_{\text{kin}}^{LQC}$ with a built-in nonlocality at a scale $\alpha_o$. The viewpoint in LQC is that this nonlocality is fundamental and the more familiar, local expression of the field strength arises only in the classical limit. (For details, see \[17, 19, 29\].) This nonlocality is important for quantum dynamics: it is the principal reason why the LQC Hamiltonian constraint is qualitatively different from the Wheeler-DeWitt differential equation.

Solutions to the LQC Hamiltonian constraints were first obtained numerically \[19\] and an analytical understanding was developed more recently \[30\]. If one begins with a suitable quantum state that is sharply peaked on a classical trajectory at late times and evolves it...
both forward and backward in time, the trajectory remains sharply peaked for all times; the theory admits “dynamical coherent states”. However the trajectory on which they remain peaked are classical solutions only till the matter density \( \rho \) reaches about 1% of the Planck density \( \rho_{Pl} \). Then, the quantum evolution departs from the classical trajectory. Rather than evolving into the singularity in the backwards evolution, the trajectory undergoes a quantum bounce. The density then starts decreasing again. Once it falls below 1% of \( \rho_{Pl} \), classical general relativity again becomes an excellent approximation. Thus, the effect of quantum geometry is to produce an effective repulsive force which is negligible till \( \rho \sim 0.01 \rho_{Pl} \) but rises extremely quickly once the density further increases, so much so that it is able to overwhelm the classical gravitational attraction, thereby triggering the bounce and avoiding the classical singularity. It is this short but critical interval that will be important for our entropy considerations in the next subsection.

The LQC quantum evolution is extremely well approximated by quantum corrected “effective” equations. These are obtained \[33\] using a geometrical formulation of quantum mechanics.\footnote{In this formulation, the space of quantum states is represented by an infinite dimensional phase space \( \Gamma_{\text{Quant}} \) on which the exact quantum dynamics defines a Hamiltonian flow (see, e.g., \[32\]). The quantum corrected equations are obtained by projecting this flow on a suitably chosen subspace which is isomorphic to the classical phase space \( \Gamma_{\text{Cl}} \). To distinguish them from effective equations obtained from other methods, these equations should really be referred to as “geometric quantum mechanics projected equations”. However, for brevity we will refer to them just as “effective” equations.} We will conclude this subsection by sketching the structure of these equations. Recall first that in the FRW models one generally introduces a fiducial frame \( e^a_i \) and the corresponding co-frame \( \omega_i^a \) and expresses the dynamical variables in terms of them. Homogeneity and isotropy imply that the connection \( A^a_i \) is proportional to \( \omega_i^a \) — \( A^a_i \sim c \omega_i^a \) — and the triad \( E^a_i \) to \( e^a_i \) — \( E^a_i \sim p e^a_i \) — where, for simplicity we have omitted some kinematical factors that are irrelevant for dynamics. \( c,p \) is then the basic canonically conjugate pair. The kinematical factors make \( c \) dimensionless, endow \( p \) with dimensions of \((\text{length})^2\), and provide the following Poisson bracket: \( \{c, p\} = 8\pi \gamma G/3 \). The geometrical meaning of these variables is the following: \( c \sim \gamma \dot{a} \) and \( |p| \sim a^2 \) where, as in (2.1), \( a \) is the scale factor, and \( \gamma \) is the Barbero-Immirzi parameter which disappears in this limit, as they must. Note that only the gravitational part of \( \mathcal{H}_{\text{eff}} \) has acquired quantum corrections; as one would expect, they arise from the fundamental nonlocality of \( \hat{F}_{ab} \) in LQC. However, to compare with the standard form of the classical constraint — the Friedmann equation — it is

\[
\mathcal{H}_{\text{eff}} = -\frac{3|p|^{3/2}}{8\pi G \gamma^2 \alpha_o \ell_{Pl}^2} \sin^2 \sqrt{\frac{\alpha_o \ell_{Pl}^2 c^2}{|p|}} + |p|^{3/2} \rho = 0.
\](3.2)

As in the classical theory, the canonical transformation generated by this constraint yields evolution equations (in proper time). However, \( \mathcal{H}_{\text{eff}} \) has a dependence on Planck’s constant through \( \ell_{Pl} = \sqrt{G \hbar} \). The classical Hamiltonian constraint results when we take the limit \( \hbar \to 0 \). Both \( \alpha_o \) and the Barbero-Immirzi parameter \( \gamma \) disappear in this limit, as they must.

In terms of these variables the effective Hamiltonian constraint turns out to be:

\[
\mathcal{H}_{\text{eff}} = -\frac{3|p|^{3/2}}{8\pi G \gamma^2 \alpha_o \ell_{Pl}^2} \sin^2 \sqrt{\frac{\alpha_o \ell_{Pl}^2 c^2}{|p|}} + |p|^{3/2} \rho = 0.
\](3.2)
Convenient to rewrite this equation in a slightly different form. Using the equation of motion

\[ \dot{p} := \{p, H_{\text{eff}}\} = \frac{2|p|}{\gamma \alpha_o \ell^2_{\text{Pl}}} \sin \sqrt{\frac{\alpha_o c^2 \ell^2_{\text{Pl}}}{|p|}} \cos \sqrt{\frac{\alpha_o c^2 \ell^2_{\text{Pl}}}{|p|}} \]

for \( p \), the fact that \( H^2 = (\dot{p})^2/4p^2 \), and (3.2) we obtain

\[ H^2 = \frac{8\pi G \rho}{3} \left( 1 - \frac{\rho}{\rho_{\text{crit}}} \right), \quad \text{with} \quad \rho_{\text{crit}} = \frac{3}{8\pi \gamma^2 \alpha_o G \ell^2_{\text{Pl}}}. \] (3.4)

Thus, using the equation of motion \( H_{\text{eff}} \) generates, we have moved the quantum correction from the “geometric” to the “matter” side. (This procedure is quite general and holds so long as \( H_{\text{matt}} \) does not depend on the gravitational connection \( c \).) As \( \hbar \) goes to zero, \( \rho_{\text{crit}} \) diverges and the second term in the parenthesis on the right-hand side disappears; we recover the classical Friedmann equation, Eq. (2.2). The quantum correction is negligible when \( \rho \ll \rho_{\text{crit}} \sim \rho_{\text{Pl}} \). But it dominates dynamics when \( \rho \approx \rho_{\text{crit}} \). In particular, when \( \rho = \rho_{\text{crit}} \) the right side vanishes, whence \( \dot{a} = 0 \) and we have a quantum bounce.

The expression of the critical density contains two dimensionless parameters, \( \gamma \) and \( \alpha_o \). In LQG, the value of the Barbero-Immirzi parameter \( \gamma \) is fixed through a black hole entropy calculation \(^{24, 25} \) and turns out to be \( \gamma \approx 0.2375 \). The second parameter is \( \alpha_o \). In the LQC literature to date it has been taken to be the minimum nonzero eigenvalue of the area operator, \( \alpha_o = 2\sqrt{3\pi \gamma} \), on gauge invariant states (see, e.g., \(^{19, 20, 30} \)). The corresponding area eigenstate has two edges: an edge \( e_1 \), which terminates transversally at a vertex \( v \) on the surface whose area is being computed, and meets there an edge \( e_2 \) that is tangential to the surface (each edge carrying a spin-label \( j = 1/2 \)). However, we recently realized that if one sets up a semiheuristic correspondence between LQG and LQC quantum states, such states would not occur in homogeneous models: at \( v \), \( e_1 \) would meet an edge \( e_2 \), which is also transversal to the surface. Therefore in LQC one should take the minimum area eigenvalue within this class. That minimum is \( \alpha_o = 4\sqrt{3\pi \gamma} \approx 5.166 \). Thus, in this paper we will set \( \gamma = 0.2375 \) and \( \alpha_o = 5.166 \).

### B. Entropy bound in the LQC-corrected FRW model

Since we have restricted ourselves to the \( k=0 \), isotropic homogeneous situation, the effective space-time metric in LQC again has the form

\[ ds^2 = -d\tau^2 + a(\tau)^2 \left( dr^2 + r^2 d\Omega^2 \right). \] (3.5)

Any symmetric, second rank tensor field \( T_{ab} \) that is invariant under these symmetries can be written as \( T_{ab} = \rho \nabla_a \tau \nabla_b \tau + P(g_{ab} + \nabla_a \tau \nabla_b \tau) \). Hence the stress energy tensor is necessarily that of a perfect fluid. Since we are working with a radiation filled universe, \( P = \frac{1}{3} \rho \). As usual we can model it using kinetic theory on a curved space-time. The resulting conservation

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\(^6\) These considerations become quite important in a systematic treatment of the Hamiltonian constraint in non-isotropic models.
equation \( \nabla^a T_{ab} = 0 \) again yields the continuity equation:

\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + P) = 0.
\]  

(3.6)

Thus the form of the metric and the continuity equations are the same as in the classical theory. However as we saw in Sec. IIIA the Friedmann equation is now replaced by:

\[
H^2 = \frac{8\pi G \rho}{3} \left(1 - \frac{\rho}{\rho_{\text{crit}}}\right), \quad \text{with} \quad \rho_{\text{crit}} = \frac{3}{8\pi \gamma^2 \alpha_o G \ell_{Pl}^2}.
\]  

(3.7)

We can repeat the procedure followed in Sec. II and use these three equations to solve for \( a(\tau) \). The result is:

\[
a(\tau) = \left(\frac{32\pi G K_o}{3} \tau^2 + \frac{K_o}{\rho_{\text{crit}}}\right)^{1/4}, \quad \text{and} \quad \rho(\tau) = \frac{K_o}{a(\tau)^4}.
\]  

(3.8)

where, as before, \( K_o \) is an integration constant which does not have a direct physical significance. This functional form of \( a(\tau) \) is very similar to Eq. (2.4) in the classical case, and reduces to it in the limit \( \hbar \to 0 \) as well as in the limit \( \tau \to \pm\infty \). The critical density \( \rho_{\text{crit}} \) at which the universe bounces is a measure of the strength of the effects of quantum gravity. The smaller \( \rho_{\text{crit}} \) is (or, the larger the area gap \( \alpha_o \) is), the earlier the onset of the quantum regime will be. Now, in the classical theory, the ratio \( S/A \) could be made arbitrarily large by choosing the surface \( B \) sufficiently close to the singularity. However this is precisely the regime in which LQC effects become dominant. Therefore we are now led to ask:

- Do these effects naturally bound the ratio \( S/A \) under consideration?
- If so, for the values of parameters \( \gamma \) and \( \alpha_o \) that are currently used in LQG and LQC, is the bound less than \( 0.25/\ell_{Pl}^2 \)?

These questions can now be answered by a straightforward calculation. Let us again choose a round 2-sphere \( B \) at an instant of time \( \tau_f \) and consider its past lightsheet. Since the space-time metric (3.5) is conformally flat and nonsingular, these null rays necessarily converge to a point \( p \), say at time \( \tau_i \), so that \( B \) is a cross section of the (future) light cone of \( p \). However as one moves along the light cone from \( B \) to \( p \), the expansion may become positive somewhere, in which case the light sheet of \( B \) would only be a portion of the (future) light cone of \( p \). If not, the light-sheet is the entire light cone from \( p \) to \( B \). Both these possibilities occur. But in either case, the area of \( B \) is just

\[
A = 4\pi [a(\tau_f)]^2 R_f^2 \quad \text{where} \quad R_f = \int_{\tau_i}^{\tau_f} \frac{d\tau'}{a(\tau')}.
\]  

(3.9)

\footnote{It is not necessary to consider, in addition, past light cones because the quantum corrected “effective” solution is symmetric about the bounce point at \( \tau = 0 \).}
FIG. 1: The \((\tau_i, \tau_f)\) region (in Planck units) where LQC corrections are significant. Since \(\tau_i < \tau_f\), the striped portion is excluded. For \((\tau_i, \tau_f)\) in the black region, surfaces \(B\) do not admit past light-sheets. In the grey region, the expansion is negative near \(B\) but then becomes positive so that the light-sheet \(L\) of \(B\) is incomplete. Points in the white region are the most interesting ones for testing the entropy bound. For \((\tau_i, \tau_f)\) in this region, light-sheets of surfaces \(B\) are complete.

Using the space-time metric it is easy to calculate the entropy current \(s^a = [(\rho + P)/T]u^a\) through the full light cone, where \(u^a\) is the unit normal to the \(\tau = \text{const}\) slices. We obtain:

\[
\frac{S}{A} = \frac{4}{9} \left( \frac{\pi^2 K_o}{15 \hbar^4} \right)^{1/4} \frac{R_f}{\sqrt{\frac{32 \pi G \rho_{\text{crit}}^7}{3} + \frac{1}{\rho_{\text{crit}}}}}.
\]

where \(R_f\) is determined by a hypergeometric function:

\[
R_f(\tau_i, \tau_f) = \left[ \tau \left( \frac{\rho_{\text{crit}}}{K_o} \right)^{1/4} \right]^{2F_1 \left( \frac{1}{2}, \frac{1}{4}; \frac{3}{2}, -\frac{32 \pi G \rho_{\text{crit}}^7}{3} \right)}_{\tau_f}. \tag{3.11}
\]

If the light-sheet of \(B\) is only a portion of the light cone, the \(S/A\) relevant for the covariant entropy bound will be smaller.

As noted in Sec. IIIA, the LQC solution is extremely close to the classical FRW solution once the matter density \(\rho\) falls to about 1\% of the Planck density \(\rho_{\text{Pl}}\). Since the covariant entropy bound is respected in the classical solution in this regime, it suffices to examine a neighborhood of the bounce in which \(\rho > 10^{-2} \rho_{\text{Pl}}\). In the conventions used above, the
FIG. 2: \( S/A \) is plotted in Planck units for points \((\tau_i, \tau_f)\) in the white region of Fig. 1. (a) Shows the full plot and (b) zooms in on the portion of the plot where \( S/A \) is near its maximum. The maximum value is \( S/A \approx 0.244/\ell_{Pl}^2 \).

bounce occurs at \( \tau = 0 \), and the matter density is approximately \( 1.86 \times 10^{-3} \rho_{pl} \) at \( \tau = \pm 4t_{Pl} \). Therefore, it is sufficient to examine the region \( \tau_i, \tau_f \in [-4t_{Pl}, 4t_{Pl}] \).

Because \( \tau_i \) is necessarily less than \( \tau_f \), the allowed range of these parameters excludes the striped region in Fig. 1. For \((\tau_i, \tau_f)\) in the allowed region, consider the (future) null cone of any point \( p \) at \( \tau = \tau_i \) till it intersects the surface \( \tau = \tau_f \) in a round 2-sphere \( B \). If the point \((\tau_i, \tau_f)\) lies in the black region, the expansion of the light rays is positive at \( B \), whence no portion of the null cone is a light-sheet of \( B \). (Thus, all \( B \) in the black region are future trapped surfaces.) Therefore the black portion is irrelevant for the conjecture. If a point \((\tau_i, \tau_f)\) lies in the grey area, then the expansion is negative near \( B \) but becomes positive as we move further back in time. So, in this case the light-sheet is only a portion of the light cone. An explicit calculation shows that, because of this truncation, the light-sheet \( L \) is too short for the entropy flux through it to violate the bound. Thus, to test the entropy bound, the nontrivial portion of the \((\tau_i, \tau_f)\) plane is just the white region. The corresponding surfaces \( B \) have complete past light-sheets.

Figure 2a shows the ratio \( S/A \) in Planck units for \((\tau_i, \tau_f)\) in this region, while Fig. 2b focuses on the region where the ratio reaches its maximum. The maximum attained is

\[
\frac{S}{A} \approx \frac{0.244}{\ell_{Pl}^2},
\]

for \( \tau_i \approx -2.035t_{Pl} \) and \( \tau_f \approx 0.0619t_{Pl} \). Thus the ratio does approach the \( 1/4\ell_{Pl}^2 \) specified in the covariant bound but does not quite reach it.

To conclude, let us return to the two questions raised earlier in this subsection. In the above calculations we used the specific values of \( \gamma = 0.2375 \) and \( \alpha_o = 5.166 \) for the parameters \( \gamma \) and \( \alpha_o \) that appear in the expression of the quantum corrected space-time
metric. The value of $\gamma$ is well motivated by black hole entropy considerations \cite{25} but an independent check is still lacking. The basis for the value of $\alpha_0$ is more tentative, because we do not have a systematic procedure to arrive at the Hamiltonian constraint of LQC from a specific, well-defined proposal in LQG. However, since the quantum corrected geometry is well defined for any value of these parameters, it follows that $S/A$ would remain bounded even if these values were to shift. Thus, the affirmative answer to the first question is robust. The second question, on the other hand, refers to specific values of the parameters. The fact that the bound is satisfied but almost saturated near the bounce brings out an unforeseen coherence, thereby providing independent circumstantial evidence that the current values of $\gamma$ and $\alpha_0$ may be correct to a good approximation.

IV. DISCUSSION

In Sec. II we found that in the classical, radiation filled FRW model, the covariant entropy bound is violated near the big bang singularity. However, the violation occurs in the region where space-time curvature and matter density are of Planck scale. Therefore one would expect quantum gravity effects to be dominant. The question is whether they can protect the bound. A priori this appears to be a very difficult question to address because quantum gravity effects would typically introduce large fluctuations of geometry in the Planck regime, making the notion of a light-sheet $L$ of a spacelike surface $B$ —and hence the very statement of the bound— ill defined.

LQC provides a near ideal setting to investigate this issue because of two features. First, the singularity is resolved: quantum states can be evolved “through” the putative singularity. The universe simply undergoes a quantum bounce and LQC equations provide a deterministic evolution from a pre-big-bang branch to our current post-big-bang branch. In FRW models, these results are robust: they hold irrespective of whether there is a cosmological constant \cite{31}, whether we have a $k=0$ or $k=1$ universe \cite{19, 20}, and the singularity resolution is valid for all states \cite{30}. Second, the LQC wave function remains sharply peaked on a smooth geometry even near the bounce point. Therefore, there is a quantum corrected, “effective” metric —obtained by taking expectation values— which reproduces the full quantum dynamics to an excellent degree of approximation. Although its coefficients involve $\hbar$, the metric is smooth and can be used to introduce spacelike surfaces $B$ and their light-sheets $L$. So, we can now ask: Is the covariant entropy bound respected in this quantum corrected geometry? We cannot simply use one of conditions \cite{11, 12, 13} that ensures the satisfaction of the bound because Einstein’s equations do not hold on the quantum corrected space-time. However, a direct calculation in Sec. III B showed that the answer is in the affirmative.

While this calculation provides an interesting convergence of very different ideas related to quantum gravity, it has some important limitations that stem from the current formulation of the bound. First, the statement of the entropy bound requires a well-defined notion of entropy current $s^a$. In practice this means that the matter be described in hydrodynamical terms. Matter described by “fundamental” classical and quantum fields does not readily admit such a description. Furthermore, even in the hydrodynamic context, entropy current is typically well-defined only in equilibrium. These restrictions led us to use radiation fluid as matter and assume that it is in instantaneous equilibrium throughout the history of the universe. These approximations are suspect especially in the Planck regime. The fact that the entropy bound persists in spite of these seemingly crude approximations is intriguing. Indeed even in the classical Friedmann universe, the bound is respected after $\tau \approx 0.06 \tau_{\text{Pl}}$, 

a time at which the matter density is about $8.3 \rho_{Pl}$. Why does the bound continue to hold in such circumstances which are clearly beyond the scope of approximations that are made? Such “unreasonable” successes have led to the suggestion that the bound may have truly fundamental significance and should be a cornerstone of any quantum gravity theory.

The most serious limitation of the current formulation of the bound is that it requires a smooth classical geometry and, even on such a geometry, one does not readily allow quantum matter fields. As discussed in Sec. II, since quantum matter violates the dominant (or null) energy condition, the available proofs break down. Can one somehow incorporate quantum violations of energy conditions? In the context of 2-dimensional, semiclassical space-times which include the back reaction of the Hawking radiation, there is an interesting proposal: modify the right-hand side of the bound by addition of a term corresponding to “entanglement entropy” which could protect the bound in spite of quantum violations [34]. Perhaps the arguments available in the classical theory [11, 12, 13] can be generalized to prove a version of an appropriately modified bound also in four dimensions. However the next step, dropping reference to a smooth classical geometry, would be significantly more difficult.

What is to become of the light-sheet $L$ or indeed of a spacelike surface $B$ in situations [15, 16] in which, unlike quantum cosmology, there are large fluctuations of geometry?

Our overall viewpoint on the bound can be summarized as follows. The bound is strongly motivated by the generalized second law (which also does not have a well-defined, definitive formulation). Now, already the standard second law of thermodynamics is a deep fact of Nature but it has a “fuzziness” which is not shared by other deep laws such as the conservation of energy-momentum and angular momentum. In particular, the second law requires a coarse graining in an essential way. It is not a statement about the evolution of micro-states; in a fundamental theory their dynamics is always time reversible (leaving aside, for simplicity, quantum measurements). Rather, it is a statement about how the number of micro-states compatible with a prespecified coarse graining changes in time. For instance if we have a gas that is confined to the left half of a box and we open the partition and wait till equilibrium is again reached, the entropy increases. Any one of the final micro-states (occupying the entire box) of these experiments will, under time-reversal, evolve back to a micro-state that is confined to the left half. Nonetheless, entropy increases because the total number of states compatible with the final coarse graining —most of which would not have resulted as end points of an actual evolution in our experiment— is enormously larger than that corresponding to the initial coarse graining. Thus, while an increase of entropy can be calculated using statistical mechanics, it has little relevance to the fundamental dynamics of micro-states. It is also not an input in the construction of statistical mechanics. In the same vein, we believe that the covariant entropy bound —and its appropriate generalizations that could encompass quantum field theory processes even on “quantum corrected” but smooth space-times— should emerge from a fundamental quantum gravity theory. These bounds would be valuable but are not essential ingredients in the construction of such a theory. The distinguishing feature about LQC, for example, is the underlying quantum geometry. The covariant entropy bound was never an input or even a motivation. Nonetheless, it emerged on the quantum corrected space-time as a consequence of LQC.

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