Global solutions to the three-dimensional full compressible Navier-Stokes equations with vacuum at infinity in some classes of large data

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Abstract

We consider the Cauchy problem for the full compressible Navier-Stokes equations with vanishing of density at infinity in \( \mathbb{R}^3 \). Our main purpose is to prove the existence (and uniqueness) of global strong and classical solutions and study the large-time behavior of the solutions as well as the decay rates in time. Our main results show that the strong solution exists globally in time if the initial mass is small for the fixed coefficients of viscosity and heat conduction, and can be large for the large coefficients of viscosity and heat conduction. Moreover, large-time behavior and a surprisingly exponential decay rate of the strong solution are obtained. Finally, we show that the global strong solution can become classical if the initial data is more regular. Note that the assumptions on the initial density do not exclude that the initial density may vanish in a subset of \( \mathbb{R}^3 \) and that it can be of a nontrivially compact support. To our knowledge, this paper contains the first result so far for the global existence of solutions to the full compressible Navier-Stokes equations when density vanishes at infinity (in space). In addition, the exponential decay rate of the strong solution is of independent interest.

Key Words: Full compressible Navier-Stokes equations, global classical and strong solutions, large-time behavior, vacuum.

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1 Introduction

The compressible Navier-Stokes equations, describing the motion of compressible fluids, can be written in the Eulerian coordinates in $\mathbb{R}^3$ as follows:

$$\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}(T), \\
(\rho E)_t + \text{div}(\rho Eu + Pu) &= \text{div}(Tu + \kappa \nabla \theta). 
\end{align*}$$

Here $T$ is the stress tensor given by

$$T = \mu (\nabla u + (\nabla u)^t) + \lambda \text{div} u I_3,$$

where $I_3$ is a $3 \times 3$ unit matrix; $\rho = \rho(x,t)$, $u = u(x,t) = (u_1, u_2, u_3)(x,t)$ and $\theta = \theta(x,t)$ are unknown functions denoting the density, velocity and absolute temperature, respectively; $P = P(\rho, \theta)$, $E$ and $\kappa$ denote pressure, total energy, and coefficient of heat conduction, respectively, where $E = e + \frac{|u|^2}{2}$ ($e$ is the internal energy); $\mu$ and $\lambda$ are coefficients of viscosity, satisfying the following physical restrictions

$$\mu > 0, \quad \lambda + \frac{2\mu}{3} \geq 0. \quad (1.2)$$

Assume that

$$P = R\rho \theta, \quad e = C_v \theta,$$

for some constants $R > 0$ and $C_v > 0$. Note that the temperature function can be written as $\theta = A \exp\{S/C_v\} \rho^{\gamma-1}$ where $S = S(x,t)$ is the entropy, $A > 0$ and $\gamma > 1$ are constants.

When the density has a positive lower bound (i.e., no vacuum at any point), the momentum equation and the energy equation are parabolic. In this case, there have been a lot of works so far on the well-posedness of solutions to the Cauchy problem and the initial-boundary-value problem for (1.1). Refer, for instance, to these elegant works [17, 18, 19, 25, 26, 28, 29, 32] for local and global existence of classical solutions from one dimension to high dimensions. In particular, Matsumura and Nishida in [28, 29] showed that the global classical solution in three dimensions exists provided that the initial data is small in $H^3$. In one dimension and high dimensions with spherically symmetric solutions, the global existence of classical solutions with large initial data has been obtained (see [26, 25, 18]). On the existence, asymptotic behavior of the weak solutions to the system (1.1), please refer, for instance, to [20, 21, 22, 14, 13, 10] from one dimension to high dimensions.

When vacuum is allowed, some new challenging difficulties arise, such as degeneracy of the equations. In spite of this, some important progress on global existence of weak solutions, local existence of strong solutions and global existence of classical solutions has been achieved by Feireisl, Bresch-Desjardins, Cho-Kim and Huang-Li [11, 11, 21, 15]. More precisely, Feireisl in his pioneering work [11] got the global existence of so-called variational solutions to (1.1).
with temperature-dependent coefficient of heat conduction in a bounded domain $\Omega \subseteq \mathbb{R}^N$ for $N \geq 2$. The temperature equation in [11] is satisfied only as an inequality in sense of distribution. This work is the very first attempt towards the existence of weak solutions to the full compressible Navier-Stokes equations in high dimensions. With viscosity coefficients which are only density-dependent, the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids in $\mathbb{T}^3$ or $\mathbb{R}^3$ was obtained by Bresch and Desjardins [1]. One of the important estimates in [11] is the Bresch-Desjardins entropy estimate which gives more regularity of the density function with the help of the density-dependent viscosity. As pointed out in [1], the assumption on the initial density that $\rho_0 - \rho_\infty \in L^1$ is necessary for some positive constant $\rho_\infty$. Huang and Li [15] obtained the existence and uniqueness of global classical solutions to the Cauchy problem for (1.1) in $\mathbb{R}^3$ provided that the initial energy is small. Vacuum is allowed interiorly but not at infinity in [15]. When the domain is bounded, with vacuum and large initial data, we got existence and uniqueness of global classical solutions to the initial-boundary-value problem of (1.1) in one dimension and in high dimensions with spherically and cylindrically symmetric initial data [33, 34]. Temperature-dependent coefficient of heat conduction plays a crucial role in the proof.

When the initial density and initial temperature vanish at infinity [1], there is no useful basic energy equality (or inequality). To our knowledge, there is only one result so far in this direction, please refer to [4] for the perfect gas with constant coefficients of viscosity and heat conduction, where the authors obtained the local existence and uniqueness of strong solutions to (1.1) with vacuum at infinity in $\mathbb{R}^3$. Two natural questions are: do there exist some global strong and more regular solutions to (1.1) when vacuum state is allowed at infinity? If do, then how do they behave when time goes to infinity? We will answer the questions in the paper.

We would like to introduce our main ideas in the paper. To prove the global existence of the strong solution, we establish a sharper blow-up criterion than that we have obtained in [35] for strong solution if the strong solution blows up in finite time. Then we get a crucial proposition (Proposition 4.1) which implies that the terms in the criterion will never blow up in finite time when the initial mass is small in some sense (refer to the proof of Corollary 4.2 for more details). This together with the contradiction arguments indicates that the strong solution exists globally in time. This is the main ingredient of the proof. Moreover, our result shows that the initial mass can be large if the coefficients of viscosity and heat conduction are taken to be large, which implies that large viscosity and heat conduction mean large solution. Furthermore, large-time behavior of the strong solution is considered, and a surprisingly exponential decay rate of the strong solution is obtained. Finally, we show that the global strong solution can become classical if the initial data is more regular.

The main challenges in studying the global well-posedness of solution are summarized as follows:

$(D_1)$: No useful basic energy equality (or inequality).

When $\rho \to \tilde{\rho} > 0$, $\theta \to \tilde{\theta} > 0$ as $|x| \to \infty$, the following classical basic energy equality holds:

$$C(t) + \int_0^t \int_{\mathbb{R}^3} \left( \frac{\lambda (\text{div} u)^2}{\theta} + \frac{\mu |\nabla u + (\nabla u)^\tau|^2}{\theta^2} + \frac{\kappa |\nabla \theta|^2}{\theta^2} \right) = C(0),$$

\footnote{In this case, initial vacuum state at infinity is allowed.}
where

\[ C(t) = \int \left( \rho(\frac{\theta}{\hat{\theta}} - \log \frac{\theta}{\hat{\theta}} - 1) + \frac{\rho|u|^2}{2\theta} + (\hat{\rho} - \rho + \rho \log \frac{\rho}{\hat{\rho}}) \right). \]

It is easy to verify that \( C(0) \geq 0 \). This equality plays an important role in the proof of the main theorems for instance in [15, 18, 22]. If \((\hat{\rho}, \hat{\theta}) = (0, 0)\), the energy equality (or inequality) is unavailable.

\( (D_2) \): Zlotnik inequality (see Appendix A in Section 7) which was used in [16] for isentropic flow to get the upper bound of the density does not work here. In [16], \( g(\rho) \) is defined as \( g(\rho) = -\frac{\rho P(\rho)}{\rho + \lambda} \) for the case that \( \hat{\rho} = 0 \), where \( P(\rho) = a\rho^\gamma \) for \( a > 0 \) and \( \gamma > 1 \). However, in the present paper, \( P = R\rho \theta \). Thus \( g(\infty) \neq -\infty \) due to the possible vanishing of \( \theta \).

Our strategies on handling \( (D_1) \) and \( (D_2) \) are as follows. Firstly, for \( (D_1) \), we define a new \( B(T) \) (see (4.3)) in Proposition 4.1 and do not need time-weighted terms with more regularity like those in [15]. Besides, we prove that the mass is conserved for all time with the regularity of the strong solution, i.e.,

\[ \int_{\mathbb{R}^3} \rho \, dx = \int_{\mathbb{R}^3} \rho_0 \, dx. \]

This gives that the mass is small for all time if we assume the initial mass is small. The “smallness” of the mass and the \textit{a priori} assumption of \( B(T) \) and density make us get the estimate of \( \nabla u \) in \( L^2_{\text{xt}} \) norm which is the very important starting point. Secondly, for \( (D_2) \), we use the idea of Lions ([27]) and Desjardins ([6]) to construct a “log \rho” equation. Then we define a different function \( g \) from the isentropic case, i.e., \( g(\rho) = -\frac{R\rho \theta}{\rho + \lambda} \). With the help of the “log \rho” equation, the “smallness” of mass and the \textit{a priori} assumptions of \( A(T) \) (see (4.2)), \( B(T) \) and \( \rho \), we have \( N_1 = 0 \) in Appendix A. This suggests that \( g \leq 0 \) is enough in stead of \( g(\infty) = -\infty \) in Appendix A. This is the main ingredient in the proof of the upper bound of the density. See Lemma 4.7 for more details.

Before we state our main results, we would like to give some notation which will be used throughout this paper.

\textbf{Notation:}

(i) \( \int_{\mathbb{R}^3} f = \int_{\mathbb{R}^3} f \, dx \).

(ii) For \( 1 \leq l \leq \infty \), denote the \( L^l \) spaces and the standard Sobolev spaces as follows:

\[ L^l = L^l(\Sigma), \quad D^{k,l} = \left\{ u \in L^1_{\text{loc}}(\Sigma) : \|\nabla^k u\|_{L^l} < \infty \right\}, \]

\[ W^{k,l} = L^1 \cap D^{k,l}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2}, \]

\[ D_0^k = \left\{ u \in L^6 : \|\nabla u\|_{L^2} < \infty \right\}, \]

\[ \|u\|_{D^k,l} = \|\nabla^k u\|_{L^l}. \]

(iii) \( G = (2\mu + \lambda)\text{div}u - P \) is the effective viscous flux.

(iv) \( \dot{h} = h_t + u \cdot \nabla h \) denotes the material derivative.
(v) \( m_0 = \int_{\mathbb{R}^3} \rho_0(x) \, dx \).

The rest of the paper is organized as follows. In Section 2, we present our main results. In Section 3 we establish a sharp blow-up criterion for strong solution. In Section 4 motivated by the blow-up criterion established in Section 3 we prove the global existence of strong solution provided that the initial mass is small in some sense. In Section 5 we study the large-time behavior of the solution and get the exponential decay estimate. In Section 6 based on the local well-posedness of the classical solution, we get the existence and uniqueness of global classical solution by establishing some higher-order \textit{a priori} estimates globally in time. In Section 7 we give the proof of the local well-posedness of classical solution.

2 Main results

Assume that \( \mu, \lambda \) and \( \kappa \) are constants. We assume \( R = C_\nu = 1 \) henceforth, since the constants \( R \) in the pressure function and \( C_\nu \) in the internal energy play no role in the analysis. In this case, if the solutions are regular enough (such as strong solutions and classical solutions), (1.1) is equivalent to the following system

\[
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla P(\rho, \theta) = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u, \\
\rho \theta_t + \rho u \cdot \nabla \theta + \rho \theta \text{div} u = \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\text{div} u)^2 + \kappa \Delta \theta,
\end{cases}
\]

in \( \mathbb{R}^3 \times (0, \infty) \).

System (2.1) is supplemented with initial conditions

\[
(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \quad x \in \mathbb{R}^3,
\]

with

\[
\rho(x, t) \to 0, \quad u(x, t) \to 0, \quad \theta(x, t) \to 0, \quad \text{as } |x| \to \infty, \quad \text{for } t \geq 0.
\]

We give the definition of the strong solution to (2.1)-(2.3) throughout this paper, which is similar to [4].

**Definition 2.1** (Strong solution) For \( T > 0 \), \( (\rho, u, \theta) \) is called a strong solution to the compressible Navier-Stokes equations (2.1)-(2.3) in \( \mathbb{R}^3 \times [0, T] \), if for some \( q \in (3, 6) \),

\[
0 \leq \rho \in C([0, T]; W^{1,q} \cap H^1), \quad \rho_t \in C([0, T]; L^2 \cap L^q),
\]

\[
(u, \theta) \in C([0, T]; D^2 \cap D^1_0) \cap L^2([0, T]; D^1 \cap D^2_0), \quad (u_t, \theta_t) \in L^2([0, T]; D^1_0),
\]

\[
(\sqrt{\rho} u_t, \sqrt{\rho} \theta_t) \in L^\infty([0, T]; L^2), \quad \theta \geq 0,
\]

and \( (\rho, u, \theta) \) satisfies (2.1) a.e. in \( \mathbb{R}^3 \times (0, T] \). In particular, the strong solution \( (\rho, u, \theta) \) of (2.1)-(2.3) is called global strong solution, if the strong solution satisfies (2.4) for any \( T > 0 \), and satisfies (2.1) a.e. in \( \mathbb{R}^3 \times (0, \infty) \).

2.1 A blow-up criterion

We state our main theorem in Section 2.1 which is on a blow-up criterion for strong solutions to (2.1)-(2.3), as follows:
Theorem 2.1.1 Assume $\rho_0 \geq 0, \rho_0 \in H^1 \cap W^{1,q} \cap L^1$, for some $q \in (3,6)$, $(u_0, \theta_0) \in D^2 \cap D^1_0$, and the following compatibility conditions are satisfied:

$$
\begin{aligned}
\mu \Delta u_0 + (\mu + \lambda) \nabla \text{div} u_0 - \nabla P(\rho_0, \theta_0) &= \sqrt{\rho_0} g_1, \\
\kappa \Delta \theta_0 + \frac{\mu}{\rho_0} |\nabla u_0 + (\nabla u_0)^T|^2 + \lambda (\text{div} u_0)^2 &= \sqrt{\rho_0} g_2, \quad x \in \mathbb{R}^3,
\end{aligned}
$$

(2.1.1)

for some $g_i \in L^2, i = 1, 2$. Let $(\rho, u, \theta)$ be a strong solution to (2.1)-(2.3) in $\mathbb{R}^3 \times [0, T]$. If $0 < T^* < +\infty$ is the maximal existence time of the strong solution, then

$$
\lim_{T \nearrow T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\rho\|_{L^4(0,T;L^{\frac{12}{5}})} \right) = \infty,
$$

(2.1.2)

provided that $3\mu > \lambda$.

Remark 2.1.2 Theorem 2.1.1 is an extension of our previous result in [32] where a blow-up criterion in terms of the upper bounds of $\rho$ and $\theta$ was established.

Remark 2.1.3 The additional restriction on the viscosity, i.e., $3\mu > \lambda$, is only used in Lemma 3.2. Thus, this additional restriction $3\mu > \lambda$ can be removed if (2.1.2) is replaced by

$$
\lim_{T \nearrow T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\rho\|_{L^4(0,T;L^{\frac{12}{5}})} + \|\rho^{\frac{1}{4}} \theta\|_{L^\infty(0,T;L^4)} + \|\|\nabla u\||_{L^2(0,T;L^2)} \right) = \infty.
$$

Remark 2.1.4 Theorem 2.1.1 gives a necessary condition that the strong solution blows up in finite time. Thus, to prove the global existence of strong solution to (2.1)-(2.3), it suffices to find some suitable initial data and explore some global a priori estimates to make the necessary condition fail. Please see Theorem 2.2.1 and its proof in Section 4 for more details.

Remark 2.1.5 Under the conditions of Theorem 2.1.1, the local existence and uniqueness of the strong solutions was obtained by Cho and Kim in [4]. Thus, the assumption $T^* > 0$ makes sense.

2.2 Global strong solution

The second result is concerned with global existence and uniqueness of strong solution.

Theorem 2.2.1 (Global strong solution) For any given $K_i > 0 (i = 1, 2)$ and $\bar{\rho} > 0$, assume that the initial data $(\rho_0, u_0, \theta_0)$ satisfies

$$
\rho_0 \geq 0, \quad \theta_0 \geq 0, \quad \text{in } \mathbb{R}^3, \quad \rho_0 \in H^1 \cap W^{1,q} \cap L^1, \quad (u_0, \theta_0) \in D^2 \cap D^1_0,
$$

(2.2.1)

for some $q \in (3,6)$, and

$$
\begin{aligned}
0 \leq \rho_0 \leq \bar{\rho}, \quad \text{in } \mathbb{R}^3, \\
\|\nabla u_0\|_{L^2} \leq \sqrt{K_i}, \quad \|\sqrt{\rho_0} \theta_0\|_{L^2} \leq \sqrt{K_2},
\end{aligned}
$$

(2.2.2)

and the compatibility conditions (2.1.1). Then there exists a unique global strong solution $(\rho, u, \theta)$ in $\mathbb{R}^3 \times [0,T]$ for any $T > 0$, provided that

$$
m_0 \leq \varepsilon_0 \triangleq \min \left\{ C_3, \frac{\tilde{C}(2\mu + \lambda)^6}{E^3}, \tilde{C}(2\mu + \lambda)^{\frac{36}{5}}, \frac{\tilde{C}\mu^{12}(2\mu + \lambda)^{12}}{E^{12}}, \tilde{C}(2\mu + \lambda)^{36} \right\},
$$

where $m_0 = \int_{\mathbb{R}^3} \rho_0(x) \, dx, \quad \tilde{E} = \frac{(14\mu + 9\lambda)}{2\mu} + \frac{6\mu K_2}{\mu(\mu + \lambda) K_1} + \frac{8\rho_0 K_2}{\mu(\mu + \lambda) K_1} + 1$, and

$$
C_3 = \min \left\{ \frac{\tilde{C}\kappa^6(\mu + \lambda)^6\mu^6}{(\kappa(\mu + \lambda) + 1)^6}, \frac{\tilde{C}\mu^3(2\mu + \lambda)^6}{E^2}, \frac{\tilde{C}\mu^2(2\mu + \lambda)^8}{(2\mu + \lambda)^8 E^6}, \frac{\tilde{C}\kappa^4 \mu^2}{E^4 \mu^2}, \frac{\tilde{C}\kappa^6}{E^3} \right\},
$$

(2.2.3)
for some constant $\tilde{C} > 0$ depending on $\tilde{\rho}, K_1, K_2$, and some other known constants but independent of $\mu, \lambda, \kappa$, and $t$.

**Remark 2.2.2** For the fixed viscosity and heat conduction, we need the “smallness” of initial mass. But the velocity and temperature can be large. Even the density can also be large in some small regions.

**Remark 2.2.3** In Theorem [2.2.1], for any fixed $\lambda$ satisfying (1.2), as $\mu$ and $\kappa$ are large enough, we have

$$
\tilde{E} \sim \frac{1}{\mu^2} + \frac{\kappa}{\mu^3} + 1,
$$

$$
C_3 \sim \min \left\{ \frac{\mu^6}{\kappa^2 + \mu^6}, \frac{\mu^9}{\kappa^6 + \mu^9}, \frac{\mu^{10}}{\kappa^6 \mu^2}, \frac{\kappa^6}{E^3} \right\}
$$

$$
\sim \min \left\{ \frac{\mu^{12}}{\kappa^2 + \mu^6}, \frac{\kappa^4 \mu^{12}}{\kappa^6 + \mu^{18}}, \frac{\kappa^4 \mu^{10}}{\kappa^3 + \mu^{12}}, \frac{\kappa^6 \mu^9}{\kappa^3 + \mu^9} \right\}.
$$

Thus,

$$
\varepsilon_0 \sim \min \left\{ \frac{\mu^{12}}{\kappa^2 + \mu^6}, \frac{\kappa^4 \mu^{12}}{\kappa^6 + \mu^{18}}, \frac{\kappa^4 \mu^{10}}{\kappa^3 + \mu^{12}}, \frac{\kappa^6 \mu^9}{\kappa^3 + \mu^9} \right\} \sim \mu^\alpha,
$$

for some $\alpha = \alpha(r_1) > 0$, provided that $\kappa \sim \mu^{r_1}$, for any $r_1 \in (\frac{3}{2}, 5)$. In this case, the initial mass (i.e., $C_0$) could be large if $\mu$ are sufficiently large. In fact, for isentropic flow (no temperature equation), there have been some works on the global large regular solutions with vacuum for the initial-boundary-value problem in one dimension and in high dimensions (symmetric initial data), Cauchy problem and periodic problem in two dimensions [7, 8, 23, 24]. For the full system, please refer to our previous works [33, 34] for the initial-boundary-value problem in one dimension and in high dimensions (symmetric initial data).

**Remark 2.2.4** For the coefficient of heat conduction $\kappa = 0$, two important works by Xin et al. [36, 37] indicate that there are no global smooth (classical) solutions to (2.1) with initial density of nontrivial compactly support or with initial density satisfying

$$
\begin{cases}
V \subset \bar{V} \subset U \subset \Omega, \\
\rho_0 = 0, \text{ in } U - V,
\end{cases}
$$

and $\rho_0$ is not identically equal to zero on $V$, where $U$ is a bounded and connected open set.

**Remark 2.2.5** For $\kappa > 0$, with small initial mass, one can not expect generally that the global solutions as in Theorems 2.2.1 and 2.4.2 are highly decreasing at infinity (in space) due to [30] even if they are initially, or that the entropy $S$ has better regularity due to [31].

**Remark 2.2.6** We would like to mention the Ref. [2] which shows that there is no global strong solution in $\mathbb{R}^3$ if the initial density is of nontrivial compactly support. While the authors in [2] essentially impose on the strong solution an assumption for the entropy function $S$ (i.e., $S = S(x, t) < \infty$ for all $(x, t) \in \mathbb{R}^3 \times [0, T]$) in order that the temperature function vanishes in the vacuum region. Thus here we find a global strong solution in a bigger space than that in [2]. As a byproduct, when the initial mass is small in some sense, our result shows that the entropy function $S$ of the strong solution is not always less than infinity in $\mathbb{R}^3 \times [0, T]$ even if it is initially.
2.3 Asymptotic behavior

The third result is concerned with the large-time behavior of the solution as well as its decay rate.

**Theorem 2.3.1** (Asymptotic behavior) Under the conditions of Theorem 2.2.1, it holds that
\[
\int_{\mathbb{R}^3} (\rho|\theta|^2 + |\nabla u|^2 + |\nabla \theta|^2) \to 0,
\]
(2.3.1)
as \(t \to \infty\), provided that
\[
m_0 \leq \varepsilon_0.
\]
Furthermore, the following decay rate
\[
\int_{\mathbb{R}^3} (\rho|\theta|^2 + |\nabla u|^2) \leq \bar{C} \exp\{-\bar{C}_1 t\},
\]
(2.3.2)
holds for any \(t \in [0, \infty)\), provided that
\[
m_0 \leq \min\{\varepsilon_0, \bar{\varepsilon}_0\},
\]
for some positive constants \(\bar{\varepsilon}_0\), \(\bar{C}\), and \(\bar{C}_1\) depending on \(\mu, \lambda, \kappa, K_1, K_2, \bar{\rho}\), and some other known constants but independent of \(t\).

**Remark 2.3.2** Some large-time behavior of the solutions to Cauchy problem for (1.1) with non-vacuum state at infinity have been studied before, see for instance [13, 15] and references therein. While, there seem few results on the decay rate. Here we get the exponential decay estimate (2.3.2) which seems surprising. The main ingredient is that here we have the integrability and the uniform upper bound of \(\rho\) in \(\mathbb{R}^3\) such that the inequality obtained from (5.17) (assume all the constant coefficients are 1 w.l.o.g.)
\[
\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla \theta|^2 + \rho|\dot{u}|^2) + \frac{d}{dt} \int_{\mathbb{R}^3} (\rho|u|^2 + \rho|\theta|^2 + |\text{curl} u|^2 + G^2) \leq 0
\]
is able to give the desired inequality (assume the constant coefficients are 1) which implies the exponential decay estimate, i.e.,
\[
\int_{\mathbb{R}^3} (\rho|u|^2 + \rho|\theta|^2 + |\text{curl} u|^2 + G^2) + \frac{d}{dt} \int_{\mathbb{R}^3} (\rho|u|^2 + \rho|\theta|^2 + |\text{curl} u|^2 + G^2) \leq 0.
\]
Refer to Section 5.2 for more details.

**Remark 2.3.3** From (2.13), Lemmas 5.2, 5.3, 5.5, Corollary 5.4, and the standard interpolation inequality, it is easy to verify that the large-time behavior (2.3.1) and the decay estimate (2.3.2) can be improved as follows:
\[
\int_{\mathbb{R}^3} (\rho|\theta|^{r_1} + |\nabla u|^r + |\nabla \theta|^r) \to 0
\]
for any \(r \in [2, 6]\) and any \(r_1 \in [2, \infty)\), as \(t \to \infty\), and
\[
\int_{\mathbb{R}^3} (\rho|\theta|^{r_1} + |\nabla u|^r) \leq \tilde{C} \exp\{-\tilde{C}_1 t\}
\]
for any \(t \in [0, \infty)\) and some positive constants \(\tilde{C}\) and \(\tilde{C}_1\) independent of \(t\).
2.4 Global classical solution

The global strong solution as in Theorem 2.2.1 can become classical, if there is more regularity of the initial density (c.f. [15, 24]). Before presenting the main result in this part, we would like to give a definition of classical solution in the paper.

**Definition 2.4.1** (classical solution) For \( T > 0 \), \((\rho, u, \theta)\) is called a classical solution to the compressible Navier-Stokes equations (2.1)-(2.3) in \( \mathbb{R}^3 \times [0, T] \), if for some \( q \in (3, 6) \),

\[
\begin{align*}
\rho &\in C([0, T]; H^2 \cap W^{2,q}), \quad \rho_t \in C([0, T]; H^1), \quad \rho \geq 0, \quad \theta \geq 0, \\
(u, \theta) &\in C([0, T]; D^2 \cap D^1_0) \cap L^2([0, T]; D^3), \quad (u_t, \theta_t) \in L^2([0, T]; D^1_0), \\
(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t) &\in L^\infty([0, T]; L^2), \quad \sqrt{t}\sqrt{\rho}u_{tt} \in L^2([0, T]; L^2), \quad t\sqrt{\rho}u_{tt} \in L^\infty([0, T]; L^2), \\
\sqrt{t}u &\in L^\infty([0, T]; D^3), \quad \sqrt{t}u_t \in L^\infty([0, T]; D^1_0) \cap L^2([0, T]; D^2), \\
tu &\in L^\infty([0, T]; D^{3,q}), \quad tu_t \in L^\infty([0, T]; D^2), \quad tu_{tt} \in L^2([0, T]; D^1_0), \\
t\theta &\in L^\infty([0, T]; D^3) \cap L^2([0, T]; D^4), \quad t\theta_t \in L^\infty([0, T]; D^1_0) \cap L^2([0, T]; D^2), \\
t^{\frac{3}{2}}\theta &\in L^\infty([0, T]; D^4), \quad t^{\frac{3}{2}}\theta_t \in L^\infty([0, T]; D^2), \quad t\sqrt{\rho}\theta_{tt} \in L^2([0, T]; L^2), \\
t^{\frac{3}{2}}\sqrt{\rho}\theta_{tt} &\in L^\infty([0, T]; L^2), \quad t^{\frac{3}{2}}\theta_{tt} \in L^2([0, T]; D^1_0),
\end{align*}
\]

and \((\rho, u, \theta)\) satisfies (2.1) in \( \mathbb{R}^3 \times (0, T) \). In particular, the classical solution \((\rho, u, \theta)\) of (2.1)-(2.3) is called global classical solution, if the classical solution satisfies (2.4.1) for any \( T > 0 \), and satisfies (2.1) in \( \mathbb{R}^3 \times (0, \infty) \).

Now we are in a position to state our main result in this part.

**Theorem 2.4.2** (Global classical solution) Under the conditions of Theorem 2.2.1, if in addition that

\[
\rho_0 \in H^2 \cap W^{2,q},
\]

for some \( q \in (3, 6) \), then there exists a unique global classical solution \((\rho, u, \theta)\) of (2.1)-(2.3).

**Remark 2.4.3** Though the initial data can be large if the coefficients of viscosity and heat conduction are large, it is still unknown whether the global classical solution exists when the initial data are large for the fixed coefficients of viscosity and heat conduction. It should be noted that a similar question of whether there exists a global smooth solution of the three-dimensional incompressible Navier-Stokes equations with smooth initial data is one of the most outstanding mathematical open problems ([27]).

3 A blow-up criterion

Let \( 0 < T^* < \infty \) be the maximal existence time of the strong solution \((\rho, u, \theta)\) to (2.1)-(2.3). Namely, \((\rho, u, \theta)\) is a strong solution to (2.1)-(2.3) in \( \mathbb{R}^3 \times [0, T] \) for any \( 0 < T < T^* \), but not a strong solution in \( \mathbb{R}^3 \times [0, T^*] \). We shall prove Theorem 2.1.1 by using a contradiction argument. Suppose that (2.1.2) is false, i.e.

\[
M := \lim\sup_{t \searrow T^*} (\|\rho(t)\|_{L^\infty} + \int_0^t \|\rho \theta(s)\|_{L^4}^4 \, ds) < \infty.
\]

Our aim is to show that under the assumption (3.1), there is a bound \( C > 0 \) depending only on \( M, \rho_0, u_0, \theta_0, \mu, \lambda, \kappa, \) and \( T^* \) such that

\[
\sup_{0 \leq t < T^*} \|\theta(t)\|_{L^\infty} \leq C.
\]
With (3.2) and (3.1), we showed in our previous paper [35] that \( T^* \) is not the maximal time, which is the desired contradiction.

Throughout the rest of the section, we denote by \( C \) a generic constant depending only on \( \rho_0, u_0, \theta_0, T^*, M, \lambda, \mu, \kappa \).

**Lemma 3.1** Under the conditions of Theorem 2.1, it holds that

\[
\int_{\mathbb{R}^3} \rho = \int_{\mathbb{R}^3} \rho_0 \triangleq m_0, \quad (3.3)
\]

for any \( t \in [0, T^*) \).

**Proof.** For any \( r > 1 \), let \( \phi_r \) be the classical cut-off function satisfying

\[ \phi_r \in C_0^\infty(\mathbb{R}^3), \text{ supp}\phi_r \subset B_r(0), \phi_r \equiv 1 \text{ in } B_{2r}(0), 0 \leq \phi_r \leq 1, \text{ and } |\nabla \phi_r| \leq \frac{C}{r} \text{ in } \mathbb{R}^3. \]

Multiplying (2.1) by \( \phi_r \), we have

\[ (\phi_r \rho)_t + \phi_r \nabla \cdot (\rho u) = 0. \quad (3.4) \]

Integrating (3.4) over \( \mathbb{R}^3 \times [0, t] \) for \( 0 \leq t < T^* \), and using integration by parts, we have

\[
\int_{\mathbb{R}^3} \phi_r \rho = \int_{\mathbb{R}^3} \phi_r \rho_0 + \int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \phi_r = \int_{\mathbb{R}^3} \phi_r \rho_0 + \int_0^t \int_{B_r(0)/B_{2r}(0)} \rho u \cdot \nabla \phi_r = I_1 + I_2. \quad (3.5)
\]

Since \( \rho_0 \in L^1 \), we have

\[ I_1 \to \int_{\mathbb{R}^3} \rho_0 \quad (3.6) \]

as \( r \to \infty \).

For \( I_2 \), we have

\[
|I_2| \leq C \int_0^t \int_{B_r(0)/B_{2r}(0)} \rho \frac{|u|}{r} \leq C \int_0^t \|u\|_{L^6(\mathbb{R}^3)} \|\rho\|_{L^2(B_r(0)/B_{2r}(0))} \|\frac{1}{r}\|_{L^3(B_r(0)/B_{2r}(0))}.
\]

Since \( \nabla u \in C([0, t]; L^2(\mathbb{R}^3)) \) and \( \rho \in C([0, t]; L^2(\mathbb{R}^3)) \) for \( t \in [0, T^*) \), let \( r \) go to \( \infty \), one has

\[ I_2 \to 0. \]

This together with (3.5) and (3.6) deduces

\[ \rho(\cdot, t) \in L^1, \]

and

\[ \int_{\mathbb{R}^3} \rho = \int_{\mathbb{R}^3} \rho_0 \]

for any \( t \in [0, T] \).
Lemma 3.2 Under the conditions of Theorem 2.1.1 and (3.1), if $3\mu > \lambda$, it holds that

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho|u|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} |u|^2 \nabla u^2 \, dx \, dt \leq C,$$

for any $T \in (0, T^*)$.

**Proof.** The detailed proof of Lemma 3.2 could be found in [35] (Lemma 4.2 therein), which might be slightly modified (only for the pressure term). \hfill \square

Lemma 3.3 Under the conditions of Theorem 2.1.1 and (3.1), it holds that for any $T \in [0, T^*)$

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} (\rho|\theta|^2 + |\nabla u|^2) \, dx + \int_0^T \int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho|u_t|^2) \, dx \, dt \leq C. \tag{3.8}$$

**Proof.** Multiplying (2.1) by $u_t$, and integrating by parts over $\mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \rho|u_t|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu|\nabla u|^2 + (\mu + \lambda)|\text{div}u|^2)$$

$$= \frac{d}{dt} \int_{\mathbb{R}^3} Pu - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P G - \int_{\mathbb{R}^3} \rho u \cdot \nabla u \cdot u_t \tag{3.9}$$

$$= \sum_{i=1}^4 (I)_i,$$

where $G = (2\mu + \lambda)\text{div}u - P$.

Recalling $P = \rho \theta$, we obtain from (2.1) and (2.1)3

$$P_t = -\text{div}(Pu) - \rho \theta \text{div}u + \mu (\nabla u + (\nabla u)' : \nabla u + \lambda \text{div}u) + \kappa \Delta \theta. \tag{3.10}$$

Substituting (3.10) into $(I)_3$, and using integration by parts and the Hölder inequality, we have

$$(I)_3 = -\frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} Pu \cdot \nabla G + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta \text{div}u G$$

$$+ \frac{\mu}{2\mu + \lambda} \int_{\mathbb{R}^3} (\nabla u + (\nabla u)' : (\nabla G \otimes u) + \frac{\lambda}{2\mu + \lambda} \int_{\mathbb{R}^3} \text{div}uu \cdot \nabla G$$

$$+ \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} (\mu \Delta u + (\mu + \lambda)\text{div}u) \cdot uG + \frac{\kappa}{2\mu + \lambda} \int_{\mathbb{R}^3} \nabla \theta \cdot \nabla G \tag{3.11}$$

$$\leq C \|\rho u\|_{L^2} \|\nabla G\|_{L^2} + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta \text{div}u G + C \|\nabla G\|_{L^2} \|u|\nabla u\|_{L^2}$$

$$+ C \|\nabla G\|_{L^2} \|\nabla \theta\|_{L^2} + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} (\mu \Delta u + (\mu + \lambda)\text{div}u) \cdot uG.$$

Substituting (2.1)2 into (3.11), and using the Sobolev inequality, (3.1) and integration by...
parts, we have
\[
(I)_3 \leq C \|
abla \rho u \|_{L^2}^2 + C \|
abla u \|_{L^2}^2 + C \|
abla \theta \|_{L^2}^2 + C \|
abla \|_{L^2}^2 + C \|
\nu \|_{L^2}^2 + C \|
\mu \|_{L^2}^2 + C \|
\nu \|_{L^2}^2
\]
\[
+ \int_{\mathbb{R}^3} \rho \nu \cdot \nabla \rho u \cdot u G - \frac{1}{2 \mu + \lambda} \int_{\mathbb{R}^3} P \rho \div G
\]
\[
\leq C \|
abla \rho u \|_{L^2}^2 (\|\rho \nu \|_{L^2}^2 + \|u \|_{L^2}^2 + \|\nabla \theta \|_{L^2}^2) + \frac{1}{6} \int_{\mathbb{R}^3} \rho |u|^2
\]
\[
+ C \int_{\mathbb{R}^3} \rho |u|^2 \|G\|^2 + C \|\nu \|_{L^2}^2
\]
\[
\leq C \|
abla \rho u \|_{L^2}^2 (\|\rho \nu \|_{L^2}^2 + \|u \|_{L^2}^2 + \|\nabla \theta \|_{L^2}^2) + \frac{1}{6} \int_{\mathbb{R}^3} \rho |u|^2
\]
\[
+ C \int_{\mathbb{R}^3} |u|^2 \|\nu \|^2 + C \int_{\mathbb{R}^3} \rho |u|^2 \|\rho \|^2.
\]
Taking \(\div\) on both side of (3.12), we get
\[
\Delta G = \div (\rho u + \rho u \cdot \nabla u).
\]

By (3.13) and the standard \(L^2\)-estimates together with (3.12), we get
\[
\|
\nabla \rho u \|_{L^2}^2 \leq C \|
\rho u \|_{L^2}^2 + C \|
\rho u \cdot \nabla u \|_{L^2} \leq C \|
\nabla \rho u \|_{L^2}^2 + C \|
\rho u \cdot \nabla u \|_{L^2}^2.
\]

Substituting (3.14) into (3.12), and using the Cauchy inequality, we have
\[
(I)_3 \leq C \|
\rho \nu \|_{L^2}^2 + C \|
\nu \|_{L^2}^2 + C \|
\nabla \theta \|_{L^2}^2 + \frac{1}{3} \int_{\mathbb{R}^3} \rho |u|^2.
\]

For the first term of the right hand side of (3.15), using the Hölder inequality, the Sobolev inequality and the Cauchy inequality, we have
\[
\|
\rho \nu \|_{L^2}^2 \leq \|
u \|^2 \|
\nu \|^2 \|
\rho \|^2 \|
\rho \|^2 \|
\nabla \nu \|^2 \|
\nabla \rho \|^2 \|
\nabla \rho \|^2 \|
\nabla \theta \|^2 \|
\nabla \theta \|^2 \|
\nabla \theta \|^2 \|
\nu \|^2 \|
\nu \|^2 \|
\nu \|^2
\]
\[
\leq C \|
u \|^2 \|
\nabla \nu \|^2 \|
\nu \|^2 + C \|
\rho \|^2 \|
\rho \|^2.
\]

Substituting (3.16) into (3.15), we have
\[
(I)_3 \leq C \|
\rho \|_{L^2}^2 + C \|
\nu \|^2 \|
\nabla \nu \|^2 \|
\nu \|^2 + C \|
\nabla \theta \|^2 \|
\nabla \theta \|^2 + \frac{1}{3} \int_{\mathbb{R}^3} \rho |u|^2.
\]

For (I)_4, using Cauchy inequality and (3.11), we have
\[
(I)_4 \leq \frac{1}{6} \int_{\mathbb{R}^3} \rho |u|^2 + C \int_{\mathbb{R}^3} \rho |u|^2 \|
\nabla \nu \|^2.
\]

Putting (3.17) and (3.18) into (3.9), and integrating it over \([0,t]\), for \(t<T^*\), we have
\[
\int_{0}^{t} \int_{\mathbb{R}^3} \rho |u|^2 + \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda)|\div u|^2)
\]
\[
\leq 2 \int_{\mathbb{R}^3} P \div u + C \int_{0}^{t} \|
\nabla \theta \|^2 \|
\nabla \theta \|^2 + C
\]
\[
\leq (\mu + \lambda) \int \div u + C \left( \int_{\mathbb{R}^3} \rho \theta^2 + \int_{0}^{t} \int_{\mathbb{R}^3} \|
\nabla \theta \|^2 \right) + C,
\]
where we have used Cauchy inequality, (3.1) and (3.7). Therefore,

\[
\int_0^t \int_{\mathbb{R}^3} \rho |u|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 \leq C \left( \int_{\mathbb{R}^3} \rho \theta^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \right) + C. \tag{3.19}
\]

Multiplying (2.1) by \(\theta\) and integrating by parts over \(\mathbb{R}^3\), we have

\[
\kappa \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho \theta^2 \\
= - \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u + \int_{\mathbb{R}^3} \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 \theta + \int_{\mathbb{R}^3} \lambda |\text{div} u|^2 \theta \\
= \sum_{i=1}^3 (II)_i. \tag{3.20}
\]

For \((II)_2\) and \((II)_3\), we have

\[
(II)_2 + (II)_3 = \int_{\mathbb{R}^3} \mu \left( \nabla u + (\nabla u)' \right) : \nabla u + \int_{\mathbb{R}^3} \lambda |\text{div} u|^2 \theta \\
= - \int_{\mathbb{R}^3} \mu (\Delta u + \nabla \text{div} u) \cdot u \theta - \int_{\mathbb{R}^3} \mu \left( \nabla u + (\nabla u)' \right) : (\nabla \theta \otimes u) \\
- \int_{\mathbb{R}^3} \lambda u \cdot \nabla \text{div} u \theta - \int_{\mathbb{R}^3} \lambda \text{div} uu \cdot \nabla \theta \\
= - \int_{\mathbb{R}^3} (\rho u_t + \rho \cdot \nabla u + \nabla P) \cdot u \theta - \int_{\mathbb{R}^3} \mu \left( \nabla u + (\nabla u)' \right) : (\nabla \theta \otimes u) \\
- \int_{\mathbb{R}^3} \lambda \text{div} uu \cdot \nabla \theta \\
= - \int_{\mathbb{R}^3} \rho u_t \cdot u \theta - \int_{\mathbb{R}^3} \rho (u \cdot \nabla) u \cdot u \theta + \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u + \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla \theta \\
- \int_{\mathbb{R}^3} \mu \left( \nabla u + (\nabla u)' \right) : (\nabla \theta \otimes u) - \int_{\mathbb{R}^3} \lambda \text{div} uu \cdot \nabla \theta,
\]

where we have used integration by parts and (2.1)2.

Using the H"older inequality, the Cauchy inequality, (3.1) and (3.7), we have

\[
(II)_2 + (II)_3 \\
\leq \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u\|_{L^4} \|\nabla \theta\|_{L^4} + \|u \nabla u\|_{L^2} \|\rho u\|_{L^3} \|\theta\|_{L^6} + \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u \\
+ \|\nabla \theta\|_{L^2} \|\sqrt{\rho} u\|_{L^4} \|\sqrt{\rho} \theta\|_{L^4} + C \|u \nabla u\|_{L^2} \|\nabla \theta\|_{L^2} \\
\leq C \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + \frac{\kappa}{2} \|\nabla \theta\|^2_{L^2} + C \|u \nabla u\|^2_{L^2} + \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u + C \|\sqrt{\rho} \theta\|^2_{L^4}. \tag{3.22}
\]

Substituting (3.22) into (3.20), we have

\[
\kappa \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \rho |\theta|^2 \leq C \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + C \|u \nabla u\|^2_{L^2} + C \|\sqrt{\rho} \theta\|^2_{L^4}. \tag{3.23}
\]

Integrating (3.23) over \([0,t]\) \((t < T^*)\), and using (3.7), we have

\[
\int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 + \int_{\mathbb{R}^3} \rho |\theta|^2 \leq C \int_0^t \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} \theta\|_{L^4} + C \int_0^t \|\sqrt{\rho} \theta\|^2_{L^4} + C. \tag{3.24}
\]
Multiplying (3.24) by 2C, and adding the resulting inequality into (3.19), we have
\[
\int_0^t \int_{\mathbb{R}^3} \rho |u|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 + \int_0^t \int_{\mathbb{R}^3} \rho |\theta|^2 \\
\leq C \int_0^t \|\sqrt{\rho u}\|_{L^2} \|\sqrt{\rho \theta}\|_{L^4} + C \int_0^t \|\sqrt{\rho \theta}\|_{L^2}^2 + C
\]
\[
\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \rho |u|^2 + C \int_0^t \left( \int_{\mathbb{R}^3} \sqrt{\rho \theta} |\sqrt{\rho \theta}|^3 \right)^{\frac{2}{3}} + C
\]
\[
\leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \rho |u|^2 + C \int_0^t \|\sqrt{\rho \theta}\|_{L^2}^2 \|\sqrt{\rho \theta}\|_{L^6}^2 + C
\]
where we have used the Young inequality, the H{"o}lder inequality, the Sobolev inequality and (3.1). This, together with the Gronwall inequality, gives (3.8).

**Lemma 3.4** Under the conditions of Theorem 2.1.1 and (3.1), it holds that for any \( t \in (0, T^*) \)
\[
\int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) \, dx + \int_0^t \int_{\mathbb{R}^3} (\rho |\theta|^2 + |\nabla \dot{u}|^2) \, ds \leq C. \tag{3.25}
\]

**Proof.** By the definition of \( \dot{u} \), (2.1) can be reformulated as follows:
\[
\rho \ddot{u} + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u. \tag{3.26}
\]
Denote
\[
f_\epsilon = f_\epsilon (x, \cdot) = \eta_\epsilon \ast f(x, \cdot), \quad f_\delta = f_\delta (\cdot, t) = \phi_\delta \ast f(\cdot, t), \quad f_{\epsilon, \delta} = \phi_\delta \ast (\eta_\epsilon \ast f),
\]
where
\[
\eta_\epsilon (\cdot) = \frac{1}{\epsilon} \eta (\frac{\cdot}{\epsilon}), \quad \phi_\delta (\cdot) = \frac{1}{\delta} \phi \left( \frac{\cdot}{\delta} \right).
\]
Here \( \eta \) and \( \phi \) are the standard mollifiers in one dimension and in three dimensions respectively. For any given \( \tau > 0 \), let \( \epsilon \in (0, \tau] \). Taking convolutions of both sides of (3.26) with \( \eta \) and \( \phi \), we have
\[
(\rho \ddot{u})_{\epsilon, \delta} + \nabla P_{\epsilon, \delta} = \mu \Delta u_{\epsilon, \delta} + (\mu + \lambda) \nabla \text{div} u_{\epsilon, \delta} \tag{3.27}
\]
in \( \mathbb{R}^3 \times (\tau, T - \tau) \) where \( T < T^* \).
Differentiating (3.27) with respect to \( t \), we have
\[
\frac{\partial}{\partial t} (\rho \ddot{u})_{\epsilon, \delta} + \nabla (P_t)_{\epsilon, \delta} = \mu \Delta (\dot{u})_{\epsilon, \delta} + (\mu + \lambda) \nabla \text{div} (\dot{u})_{\epsilon, \delta} - \mu \Delta [(u \cdot \nabla) u]_{\epsilon, \delta}
\]
\[
- (\mu + \lambda) \nabla \text{div} [(u \cdot \nabla) u]_{\epsilon, \delta}. \tag{3.28}
\]
Multiplying (3.28) by \( (\dot{u})_{\epsilon, \delta} \), and integrating by parts over \( \mathbb{R}^3 \), we have
\[
\int_{\mathbb{R}^3} \frac{\partial}{\partial t} (\rho \ddot{u})_{\epsilon, \delta} (\dot{u})_{\epsilon, \delta} + \int_{\mathbb{R}^3} \left[ \mu |\nabla (\dot{u})_{\epsilon, \delta}|^2 + (\mu + \lambda) |\text{div} (\dot{u})_{\epsilon, \delta}|^2 \right] \\
= \int_{\mathbb{R}^3} (P_t)_{\epsilon, \delta} \text{div} (\dot{u})_{\epsilon, \delta} + \mu \int_{\mathbb{R}^3} \nabla [(u \cdot \nabla) u]_{\epsilon, \delta} \cdot \nabla (\dot{u})_{\epsilon, \delta}
\]
\[
+ (\mu + \lambda) \int_{\mathbb{R}^3} \text{div} [(u \cdot \nabla) u]_{\epsilon, \delta} \text{div} (\dot{u})_{\epsilon, \delta}.
\]
Let $\delta \to 0^+$, we have
\[
\int_{\mathbb{R}^3} \frac{\partial}{\partial t} (p\dot{u})_\varepsilon(x,t) + \int_{\mathbb{R}^3} \left[ \mu |\nabla (\dot{u})_\varepsilon|^2 + (\mu + \lambda) |\text{div} (\dot{u})_\varepsilon|^2 \right] \\
= \int_{\mathbb{R}^3} (P)_\varepsilon \text{div} (\dot{u})_\varepsilon + \mu \int_{\mathbb{R}^3} \nabla [(u \cdot \nabla) u]_\varepsilon \cdot \nabla (\dot{u})_\varepsilon \\
+ (\mu + \lambda) \int_{\mathbb{R}^3} \text{div} [(u \cdot \nabla) u]_\varepsilon \text{div} (\dot{u})_\varepsilon.
\] (3.29)

Note that
\[
\frac{\partial}{\partial t} (p\dot{u})_\varepsilon = \frac{1}{\varepsilon^2} \int_0^T \eta'(\frac{t-s}{\varepsilon}) \rho(\cdot, s) \dot{u}(\cdot, s) \, ds \\
= \frac{1}{\varepsilon} \int_{-1}^1 \eta'(s) \rho(\cdot, t-\varepsilon s) \dot{u}(\cdot, t-\varepsilon s) \, ds \\
= \frac{1}{\varepsilon} \int_{-1}^1 \eta'(s) [\rho(\cdot, t-\varepsilon s) - \rho(\cdot, t)] \dot{u}(\cdot, t-\varepsilon s) \, ds + \rho(\cdot, t) \frac{1}{\varepsilon} \int_{-1}^1 \eta'(s) \dot{u}(\cdot, t-\varepsilon s) \, ds \\
= \frac{1}{\varepsilon} \int_{-1}^1 \eta'(s) [\rho(\cdot, t-\varepsilon s) - \rho(\cdot, t)] \dot{u}(\cdot, t-\varepsilon s) - \dot{u}(\cdot, t) \, ds \\
+ \frac{1}{\varepsilon} \dot{u}(\cdot, t) \int_{-1}^1 \eta'(s) \rho(\cdot, t-\varepsilon s) - \rho(\cdot, t) \, ds + \rho(\cdot, t) \frac{\partial}{\partial t} (\dot{u})_\varepsilon
\]

Thus,
\[
\int_{\mathbb{R}^3} \frac{\partial}{\partial t} (p\dot{u})_\varepsilon (\dot{u})_\varepsilon \, dx = \int_{\mathbb{R}^3} (\dot{u})_\varepsilon (x,t) \int_{-1}^1 \eta'(s) \rho(x, t-\varepsilon s) - \rho(x, t) \int_{-1}^1 \eta'(s) \dot{u}(x, t-\varepsilon s) - \dot{u}(x, t) \, ds \, dx \\
+ \int_{\mathbb{R}^3} (\dot{u})_\varepsilon (x,t) \dot{u}(x,t) \int_{-1}^1 \eta'(s) \rho(x, t-\varepsilon s) \, ds \, dx \\
+ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho(\dot{u})_\varepsilon^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \rho(\dot{u})_\varepsilon^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho(\dot{u})_\varepsilon^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \rho(\dot{u})_\varepsilon^2 \, dx,
\] (3.30)

where we have used the conclusions that
\[
\int_{-1}^1 \eta'(s) \, ds = 0, \quad \int_{-1}^1 \eta'(s) \rho(x, t-\varepsilon s) \, ds = (\rho_t)_\varepsilon.
\]

Here
\[
G_\varepsilon(t) = \int_{\mathbb{R}^3} (\dot{u})_\varepsilon (x,t) \int_{-1}^1 \eta'(s) \rho(x, t-\varepsilon s) - \rho(x, t) \int_{-1}^1 \eta'(s) \dot{u}(x, t-\varepsilon s) - \dot{u}(x, t) \, ds \, dx.
\]

By virtue of the mass equation, it is easy to check that $\rho_t \in L^\infty([0, T]; L^2_\mathcal{F})$ for $T < T^*$. This, combined with the H"older inequality, the Sobolev inequality and the fact that
\[
\lim_{\varepsilon \to 0^+} \| \nabla \dot{u}(\cdot, t-\varepsilon s) - \nabla \dot{u}(\cdot) \|_{L^2(\mathbb{R}^3 \times [\tau, T-\tau])} = 0,
\]
gives
\[
\lim_{\varepsilon \to 0^+} \| G_\varepsilon \|_{L^1([\tau, T-\tau])} = 0.
\] (3.31)
Putting (3.30) into (3.29), and integrating the result over \((\tau, t)\) for \(t \leq T - \tau\), we have

\[
\int_\tau^t G_\epsilon(s) \, ds + \int_\tau^t \int_{R^3} (\bar{u})_\epsilon (\rho_s)_\epsilon \, dx \, ds + \frac{1}{2} \int_\tau^t \int_{R^3} \rho(x, t) |(\bar{u})_\epsilon(x, t)|^2 \, dx \\
- \frac{1}{2} \int_\tau^t \int_{R^3} \rho_s |(\bar{u})_\epsilon|^2 \, dx \, ds + \int_\tau^t \int_{R^3} [\mu |\nabla (\bar{u})_\epsilon|^2 + (\mu + \lambda) |\text{div} (\bar{u})_\epsilon|^2] \, dx \, ds
\]

\[
= \frac{1}{2} \int_{R^3} \rho(x, \tau) |(\bar{u})_\epsilon(x, \tau)|^2 \, dx + \int_\tau^t \int_{R^3} (P_s) \epsilon \text{div} (\bar{u})_\epsilon \, dx \, ds
\]

\[
+ \mu \int_\tau^t \int_{R^3} \nabla [(u \cdot \nabla) u]_\epsilon \cdot \nabla (\bar{u})_\epsilon \, dx \, ds + (\mu + \lambda) \int_\tau^t \int_{R^3} \text{div} [(u \cdot \nabla) u]_\epsilon \text{div} (\bar{u})_\epsilon \, dx \, ds.
\]

Let \(\epsilon\) go to zero, and use (3.31) to conclude that

\[
\frac{1}{2} \int_\tau^t \int_{R^3} \rho_s |\bar{u}|^2 \, dx \, ds + \frac{1}{2} \int_\tau^t \int_{R^3} \rho(x, t) |\bar{u}(x, t)|^2 \, dx \\
+ \int_\tau^t \int_{R^3} (\mu |\nabla \bar{u}|^2 + (\mu + \lambda) |\text{div} \bar{u}|^2) \, dx \, ds
\]

\[
= \frac{1}{2} \int_{R^3} \rho(x, \tau) |\bar{u}(x, \tau)|^2 \, dx + \int_\tau^t \int_{R^3} P_s \text{div} \bar{u} \, dx \, ds + \mu \int_\tau^t \int_{R^3} \nabla [(u \cdot \nabla) u] \cdot \nabla \bar{u} \, dx \, ds
\]

\[
+ (\mu + \lambda) \int_\tau^t \int_{R^3} \text{div} [(u \cdot \nabla) u] \text{div} \bar{u} \, dx \, ds.
\]

For the first term on the left hand side of (3.33), we make use of the mass equation and integration by parts. Then we arrive at

\[
\frac{1}{2} \int_\tau^t \int_{R^3} \rho_s |\bar{u}|^2 \, dx \, ds = \int_\tau^t \int_{R^3} \rho \bar{u} \cdot [(u \cdot \nabla) \bar{u}] \, dx \, ds.
\]

(3.30) implies

\[
\rho \bar{u} = -\nabla P + \mu \Delta u + (\mu + \lambda) \nabla \text{div} u.
\]

Putting this into (3.34), we have

\[
\frac{1}{2} \int_\tau^t \int_{R^3} \rho_s |\bar{u}|^2 \, dx \, ds = -\int_\tau^t \int_{R^3} u \otimes \nabla P : \nabla \bar{u} \, dx \, ds + \mu \int_\tau^t \int_{R^3} u \otimes \Delta u : \nabla \bar{u} \, dx \, ds
\]

\[
+ (\mu + \lambda) \int_\tau^t \int_{R^3} u \otimes \nabla \text{div} u : \nabla \bar{u} \, dx \, ds.
\]

The combination of (3.33) and (3.35) gives

\[
\frac{1}{2} \int_{R^3} \rho(x, \tau) |\bar{u}(x, \tau)|^2 \, dx + \int_\tau^t \int_{R^3} (\mu |\nabla \bar{u}|^2 + (\mu + \lambda) |\text{div} \bar{u}|^2) \, dx \, ds
\]

\[
= \frac{1}{2} \int_{R^3} \rho(x, \tau) |\bar{u}(x, \tau)|^2 \, dx + \int_\tau^t \int_{R^3} (P_s) \text{div} \bar{u} + u \otimes \nabla P : \nabla \bar{u} \, dx \, ds
\]

\[
+ \mu \int_\tau^t \int_{R^3} [\nabla [(u \cdot \nabla) u] \cdot \nabla \bar{u} - u \otimes \Delta u : \nabla \bar{u}] \, dx \, ds
\]

\[
+ (\mu + \lambda) \int_\tau^t \int_{R^3} [\text{div} [(u \cdot \nabla) u] \text{div} \bar{u} - u \otimes \nabla \text{div} u : \nabla \bar{u}] \, dx \, ds.
\]
Let \( \tau \to 0^+ \) (take subsequence if necessary), we have

\[
\frac{1}{2} \int_{\mathbb{R}^3} \rho(x,t)|\dot{u}(x,t)|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} (\mu|\nabla \dot{u}|^2 + (\mu + \lambda)\| \text{div} \, \dot{u} \|^2) \, dx \, ds
\leq \int_0^t \int_{\mathbb{R}^3} (P_\delta \text{div} \, \dot{u} + u \otimes \nabla P : \nabla \dot{u}) \, dx \, ds
\]

\[
+ \mu \int_0^t \int_{\mathbb{R}^3} \left[ \nabla[(u \cdot \nabla)u] \cdot \nabla \dot{u} - u \otimes \Delta u : \nabla \dot{u} \right] \, dx \, ds
\]

\[
+ (\mu + \lambda) \int_0^t \int_{\mathbb{R}^3} \left[ \text{div} \, [(u \cdot \nabla)u] \cdot \nabla \dot{u} - u \otimes \nabla \text{div} \, u : \nabla \dot{u} \right] \, dx \, ds + C
\]

\[
= \sum_{i=1}^3 (III)_i + C.
\]

For \((III)_1\), using (2.1), integration by parts, (3.1) and Hölder inequality, we have

\[
(III)_1 = \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} [P_\delta \text{div} \, \dot{u} + u \otimes \nabla P : \nabla (\dot{u})_\delta]
\]

\[
= \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} \left[ (\rho \theta)_\delta \text{div} \, \dot{u} - \rho \theta (\nabla u)' : \nabla (\dot{u})_\delta - \rho \theta u \cdot \nabla \text{div} \, (\dot{u})_\delta \right]
\]

\[
= \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} \left[ (\rho \theta)_\delta \text{div} \, \dot{u} - \rho \theta (\nabla u)' : \nabla (\dot{u})_\delta + \text{div} \, (\rho \theta u) \text{div} \, (\dot{u})_\delta \right]
\]

\[
= \int_0^t \int_{\mathbb{R}^3} \left[ \rho \theta \text{div} \, \dot{u} - \rho \theta (\nabla u)' : \nabla \dot{u} \right]
\]

\[
\leq C \int_0^t \left( \| \sqrt{\rho \theta} \|_{L^2} \| \nabla \dot{u} \|_{L^2} + \| \sqrt{\rho \theta} \|_{L^4} \| \nabla u \|_{L^4} \| \nabla \dot{u} \|_{L^2} \right).
\]

For \((III)_2\), we have

\[
(III)_2 = \mu \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} \left[ \nabla [(u_\delta \cdot \nabla)u_\delta] \cdot \nabla (\dot{u})_\delta - u_\delta \otimes \Delta u_\delta : \nabla (\dot{u})_\delta \right]
\]

\[
= \mu \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} \left[ \text{div} \, (\Delta u_\delta \otimes u_\delta) - \Delta [(u_\delta \cdot \nabla)u_\delta] \right] \cdot (\dot{u})_\delta.
\]

It is not difficult to check that

\[
\text{div} \, (\Delta u_\delta \otimes u_\delta) - \Delta [(u_\delta \cdot \nabla)u_\delta] = \nabla_k (\text{div} \, u_\delta \nabla_k u_\delta) - \nabla_k (\nabla_k u_\delta^j \nabla_j u_\delta) - \nabla_j (\nabla_k u_\delta^j \nabla_k u_\delta),
\]

where \( \nabla_k = \frac{\partial}{\partial x_k} \). Thus we have

\[
(III)_2 = - \mu \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} (\text{div} \, u_\delta \nabla_k u_\delta) \cdot \nabla_k (\dot{u})_\delta + \mu \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} (\nabla_k u_\delta^j \nabla_j u_\delta) \cdot \nabla_k (\dot{u})_\delta
\]

\[
+ \mu \lim_{\delta \to 0^+} \int_0^t \int_{\mathbb{R}^3} (\nabla_k u_\delta^j \nabla_j u_\delta) \cdot \nabla_j (\dot{u})_\delta
\]

\[
= - \mu \int_0^t \int_{\mathbb{R}^3} (\text{div} \, u \nabla_k u) \cdot \nabla_k (\dot{u}) + \mu \int_0^t \int_{\mathbb{R}^3} (\nabla_k u^j \nabla_j u) \cdot \nabla_k (\dot{u})
\]

\[
+ \mu \int_0^t \int_{\mathbb{R}^3} (\nabla_k u^j \nabla_j u) \cdot \nabla_j (\dot{u})
\]

\[
\leq C \int_0^t \| \nabla \dot{u} \|_{L^2} \| \nabla u \|_{L^4}^2.
\]
where we have used integration by parts and the Hölder inequality. Note that

\[
\text{div} (\nabla \text{div} u_\delta \otimes u_\delta) - \nabla \text{div} \left((u_\delta \cdot \nabla) u_\delta\right) = \nabla (\nabla_j u_\delta \nabla_i u_\delta) - \nabla (\nabla_j u_\delta \nabla_i u_\delta) - \nabla (\nabla_j u_\delta \nabla_i u_\delta)
\]

Similar to the arguments for \((\text{III})_2\), we have

\[
(\text{III})_3 \leq C \int_0^t \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^4}^2.
\]  

(3.39)

Substituting (3.37), (3.38) and (3.39) into (3.36), and using the Cauchy inequality and (3.1), we have

\[
\frac{1}{2} \int_{\mathbb{R}^3} \rho|\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} \left(\mu|\nabla \dot{u}|^2 + (\mu + \lambda)|\text{div} \dot{u}|^2\right)
\]

\[
\leq \frac{\mu}{2} \int_0^t \|\nabla \dot{u}\|_{L^2}^2 + C \int_0^t \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \int_0^t \|\sqrt{\rho} \dot{\theta}\|_{L^4}^4 + C \int_0^t \|\nabla u\|_{L^4}^4 + C.
\]

The first term on the right hand side can be absorbed by the left. Thus we have

\[
\int_{\mathbb{R}^3} \rho|\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} \|\nabla \dot{u}\|_{L^2}^2 \leq C \int_0^t \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \int_0^t (\|\nabla u\|_{L^4}^4 + \|\nabla \theta\|_{L^2}^4) + C,
\]  

(3.40)

where we have used

\[
\int_0^t \|\sqrt{\rho} \dot{\theta}\|_{L^4}^4 = \int_0^t \int_{\mathbb{R}^3} \rho \dot{\theta}^4 \leq \int_0^t \|\rho\|_{L^3} \|\dot{\theta}\|_{L^6}^4
\]

\[
\leq C \int_0^t \|\nabla \theta\|_{L^2}^4.
\]

The next step is to get some estimates for \(\theta\). We rewrite (2.1) as follows:

\[
\rho \dot{\theta} + \rho \theta \text{div} u = \frac{\mu}{2} \left|\nabla u + (\nabla u)'\right|^2 + \lambda (\text{div} u)^2 + \kappa \Delta \theta.
\]

(3.41)

Multiplying (3.41) by \(\dot{\theta}\) and integrating by parts over \(\mathbb{R}^3\), we have

\[
\int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \frac{\kappa}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \theta|^2
\]

\[
= - \int_{\mathbb{R}^3} \rho \theta \text{div} u \dot{\theta} + \int_{\mathbb{R}^3} \left(\frac{\mu}{2} \left|\nabla u + (\nabla u)'\right|^2 + \lambda (\text{div} u)^2\right) \theta_t
\]

\[
+ \int_{\mathbb{R}^3} \left(\frac{\mu}{2} \left|\nabla u + (\nabla u)'\right|^2 + \lambda (\text{div} u)^2\right) u \cdot \nabla \theta + \kappa \int_{\mathbb{R}^3} \Delta \theta u \cdot \nabla 
\]

\[
= \sum_{i=1}^4 [(\text{IV})_i].
\]

For (IV)\(_1\), using Hölder inequality, (3.1), (3.3) and Cauchy inequality, we have

\[
(\text{IV})_1 \leq C \|\sqrt{\rho} \dot{\theta}\|_{L^2} \|\nabla u\|_{L^4} \|\sqrt{\rho} \dot{\theta}\|_{L^{12}} \|\theta\|_{L^5} \leq \frac{1}{8} \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2}^4.
\]  

(3.43)
Using integration by parts, we have

\[(IV)_2 = \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\text{div} u)^2 \right) \theta - \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u_t + (\nabla u_t)') \theta \\
- 2\lambda \int_{\mathbb{R}^3} \text{div} u \text{div} u_t \theta \\
= \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\text{div} u)^2 \right) \theta - \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u_t + (\nabla u_t)') \theta \\
+ \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u \cdot \nabla u + (\nabla u \cdot \nabla u)') \theta \\
+ \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') \cdot (u \cdot \nabla) (\nabla u + (\nabla u)') \theta - 2\lambda \int_{\mathbb{R}^3} \text{div} \text{div} u \theta \\
+ 2\lambda \int_{\mathbb{R}^3} \text{div}(\nabla u)' : \nabla u \theta + 2\lambda \int_{\mathbb{R}^3} u \cdot \text{div} \text{div} u \theta.
\]

Using integration by parts, we have

\[(IV)_2 = \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{\mu}{2} |\nabla u + (\nabla u)'|^2 + \lambda (\text{div} u)^2 \right) \theta - \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u_t + (\nabla u_t)') \theta \\
+ \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') : (\nabla u \cdot \nabla u + (\nabla u \cdot \nabla u)') \theta \\
- \mu \int_{\mathbb{R}^3} \frac{|\nabla u + (\nabla u)'|^2}{2} \text{div} u \theta \\
- \mu \int_{\mathbb{R}^3} (\nabla u + (\nabla u)') \cdot \nabla \theta - 2\lambda \int_{\mathbb{R}^3} \text{div} \text{div} u \theta + 2\lambda \int_{\mathbb{R}^3} \text{div}(\nabla u)' : \nabla u \theta \\
- \lambda \int_{\mathbb{R}^3} (\text{div} u)^3 \theta - \lambda \int_{\mathbb{R}^3} |\text{div} u|^2 u \cdot \nabla \theta
\]

\[= \sum_{i=1}^{9} (IV)_{2,i}.
\]

For \((IV)_{2,2}\) and \((IV)_{2,6}\), using the Hölder inequality, the Sobolev inequality, we have

\[(IV)_{2,2} + (IV)_{2,6} \leq C \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^3} \|\theta\|_{L^6} \leq C \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^3} \|\theta\|_{L^2}.
\]

Since \(\nabla u = \nabla \Delta^{-1} (\text{div} u - \nabla \times \text{curl} u)\), we apply Calderon-Zygmund inequality to get

\[\|\nabla u\|_{L^3} \leq C \|\text{curl} u\|_{L^3} + C \|\text{div} u\|_{L^3}.
\]

Taking curl on both sides of \((2.1)_2\), we have

\[\mu \Delta (\text{curl} u) = \text{curl}(\rho \dot{u}).
\]

By \((3.47)\), the \(L^2\)-estimates of the elliptic equations and \((3.1)\), we have

\[\|\nabla \text{curl} u\|_{L^2} \leq C \|\rho \dot{u}\|_{L^2} \leq C \|\sqrt{\rho \dot{u}}\|_{L^2}.
\]

By \((3.13)\), \((3.46)\) and \((3.48)\), together with the Sobolev inequality, we have

\[\|\nabla u\|_{L^3} \leq C \|\text{curl} u\|_{L^3} + C \|G\|_{L^3} + C \|ho \theta\|_{L^2} \leq C \|\text{curl} u\|_{H^1} + C \|G\|_{H^1} + C \|\rho\|_{L^6} \|\theta\|_{L^6} \leq C \|\text{curl} u\|_{L^2} + C \|\text{div} u\|_{L^2} + C \|\nabla (\text{curl} u)\|_{L^2} + C \|\nabla G\|_{L^2} + C \|\nabla \theta\|_{L^2} \leq C \|\nabla u\|_{L^2} + C \|\sqrt{\rho \dot{u}}\|_{L^2} + C \|\nabla \theta\|_{L^2},
\]

where we have used \((3.1)\), \((3.3)\) and \((3.8)\). Substituting \((3.49)\) into \((3.45)\), we obtain

\[(IV)_{2,2} + (IV)_{2,6} \leq C \|\nabla \dot{u}\|_{L^2} \left( \|\nabla u\|_{L^2} + \|\sqrt{\rho \dot{u}}\|_{L^2} + \|\nabla \theta\|_{L^2} \right) \|\nabla \theta\|_{L^2}.
\]
For \((IV)_{2.3}, \ (IV)_{2.4}, \ (IV)_{2.7}\) and \((IV)_{2.8}\), using the H"older inequality, the Sobolev inequality and the Calderon-Zygmund inequality, we have

\[
(IV)_{2.3} + (IV)_{2.4} + (IV)_{2.7} + (IV)_{2.8}
\leq C \int_{\mathbb{R}^3} |\nabla u|^3 |\theta| \leq C \|\nabla u\|_{L^{\frac{10}{3}}}^3 \|\theta\|_{L^6} \leq C \|\nabla u\|_{L^{\frac{10}{3}}}^3 \|\nabla \theta\|_{L^2} \tag{3.51}
\]

\[
\leq C \|\nabla \theta\|_{L^2} + C \|\nabla G\|_{L^2} \|\nabla \theta\|_{L^3} + C \|\nabla \theta\|_{L^2} + C \|\rho \theta\|_{L^{\frac{5}{2}}} \|\nabla \theta\|_{L^2} \tag{3.52}
\]

For \((IV)_{2.5}\) and \((IV)_{2.9}\), using the H"older inequality, the Cauchy inequality, the Sobolev inequality, the Gagliardo-Nirenberg inequality and \((3.53)\), we have

\[
(IV)_{2.5} + (IV)_{2.9} \leq C \int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta| \leq C \|\nabla u\|_{L^4}^2 \|u\|_{L^6} \|\nabla \theta\|_{L^3} \tag{3.53}
\]

\[
\leq C \|\nabla u\|_{L^4}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2} \leq C \|\nabla u\|_{L^4}^2 + C \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2} \tag{3.54}
\]

From the standard elliptic estimates and \((3.41)\), we have

\[
\|\nabla \theta\|_{L^2} \leq C \|\rho \theta\|_{L^2} + C \|\nabla \theta\|_{L^2} \|\nabla \theta\|_{L^2} \|\nabla \theta\|_{L^2} + C \|\nabla \theta\|_{L^2} \tag{3.55}
\]

Substituting \((3.51)\), \((3.52)\) and \((3.55)\) into \((3.44)\), and using the Cauchy inequality, we have

\[
(IV)_{2.5} + (IV)_{2.9} \leq \frac{1}{8} \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2}^4 + C \|\nabla \theta\|_{L^2}^2 \tag{3.56}
\]

For \((IV)_{3}\), using \((3.53)\) and \((3.55)\), we have

\[
(IV)_{3} \leq C \int_{\mathbb{R}^3} |\nabla u|^2 |u| |\nabla \theta| \leq \frac{1}{8} \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2}^4 + C \|\nabla \theta\|_{L^2}^2 \tag{3.57}
\]

For \((IV)_{4}\), using the H"older inequality, the Sobolev inequality, the Gagliardo-Nirenberg inequality, \((3.3)\), \((3.4)\) and the Young inequality, we have

\[
(IV)_{4} \leq C \|\Delta \theta\|_{L^2} \|u\|_{L^5} \|\nabla \theta\|_{L^3} \leq C \|\Delta \theta\|_{L^2} \|u\|_{L^2} \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2} \tag{3.58}
\]
Putting (3.43), (3.56), (3.57) and (3.58) into (3.42), integrating the resulting inequality over \([0, \varepsilon]\) for \(t \in (0, T^*)\), and using the Cauchy inequality, (3.1), (3.7) and (3.8), we have

\[
\int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \int_{\mathbb{R}^3} |\nabla \theta|^2 \leq C \int_0^t \|\nabla u\|_{L^4}^4 + C \int_{\mathbb{R}^3} |\nabla u|^2|\theta| + \varepsilon \int_0^t \|\nabla \dot{u}\|_{L^2}^2
\]
\[
+ C \varepsilon \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2\right) \|\nabla \theta\|_{L^2}^2 + C.
\]

(3.59)

For the second term of the right hand side of (3.59), we have

\[
C \int_{\mathbb{R}^3} |\nabla u|^2|\theta| \leq C \|\nabla u\|_{L^4}^2 \|\theta\|_{L^6} \leq C \|\nabla u\|_{L^4}^2 \|\nabla \theta\|_{L^2} + C \|\nabla u\|_{L^4}^2 \|\nabla \theta\|_{L^2}
\]
\[
\leq C \left(\|\nabla u\|_{L^4}^2 \|\theta\|_{L^2} + \|\nabla \theta\|_{L^2} \right) \|\nabla \theta\|_{L^2} + C \|\nabla \theta\|_{L^2}^2 \leq \frac{1}{2} \|\nabla \theta\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C,
\]

(3.60)

where we have used the Hölder inequality, the Calderon-Zygmund inequality, the Gagliardo-Nirenberg inequality, (3.1), (3.8), (3.13), (3.48), the Sobolev inequality and the Young inequality. Substituting (3.60) into (3.59), we have

\[
\int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + \int_{\mathbb{R}^3} |\nabla \theta|^2 \leq C \int_0^t \|\nabla u\|_{L^4}^4 + C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \varepsilon \int_0^t \|\nabla \dot{u}\|_{L^2}^2
\]
\[
+ C \varepsilon \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2\right) \|\nabla \theta\|_{L^2}^2 + C.
\]

(3.61)

Multiplying (3.61) by \(2C\) and adding the resulting inequality into (3.40), we have

\[
C \int_0^t \int_{\mathbb{R}^3} \rho |\dot{\theta}|^2 + 2C \int_{\mathbb{R}^3} |\nabla \theta|^2 + \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2
\]
\[
\leq 2C^2 \int_0^t \|\nabla u\|_{L^4}^4 + 2C^2 \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + 2\varepsilon C \int_0^t \|\nabla \dot{u}\|_{L^2}^2
\]
\[
+ 2CC \varepsilon \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2\right) \|\nabla \theta\|_{L^2}^2 + 2C^2.
\]

Taking \(\varepsilon\) sufficiently small, together with the Cauchy inequality, we have

\[
\int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) + \int_0^t (\rho |\dot{\theta}|^2 + |\nabla \dot{u}|^2)
\]
\[
\leq C \int_0^t \|\nabla u\|_{L^4}^4 + C \int_0^t \left(\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2\right) \|\nabla \theta\|_{L^2}^2 + C.
\]

(3.62)

For the first term of the right hand side of (3.62), similar to (3.49), we have

\[
\int_0^t \|\nabla u\|_{L^4}^4 \leq C \int_0^t \|\text{curl}u\|_{L^4}^4 + C \int_0^t \|G\|_{L^4}^4 + C \int_0^t \|\nabla \theta\|_{L^2}^4
\]
\[
\leq C \int_0^t \|\text{curl}u\|_{L^2}^2 \|\text{curl}u\|_{L^2}^2 + C \int_0^t \|G\|_{L^2}^4 \|\nabla G\|_{L^2}^4 + C \int_0^t \|\nabla \theta\|_{L^2}^4
\]
\[
\leq C \int_0^t \|\sqrt{\rho} \dot{u}\|_{L^2}^4 + C \int_0^t \|\nabla \theta\|_{L^2}^4.
\]

(3.63)
By (3.62), (3.63) and the Cauchy inequality, we have
\[
\int_{\mathbb{R}^3} (|\nabla \theta|^2 + \rho |\dot{u}|^2) + \int_0^t \int_{\mathbb{R}^3} (|\dot{\theta}|^2 + |\dot{\nabla} u|^2) \\
\leq C \int_0^t (\|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) \|\nabla \theta\|_{L^2}^2 + C \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) (\|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho} \dot{u}\|_{L^2}^2) + C.
\]
(3.64)

By (3.1), (3.7), (3.8), we have
\[
\int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) \leq C,
\]
for any \( t \in (0, T^*) \). This, together with (3.64) and the Gronwall inequality, deduces (3.25).

**Corollary 3.5** Under the conditions of Theorem 2.1.1 and (3.1), it holds that for any \( t \in (0, T^*) \)
\[
\|u\|_{L^\infty} + \|\nabla u\|_{L^r} + \int_0^t \int_{\mathbb{R}^3} (\rho |\dot{u}|^2 + |\nabla^2 \theta|^2) \ dx \, ds \leq C,
\]
(3.65)
for any \( r \in [2, 6] \), and any \( t \in [0, \infty) \).

**Proof.** Similar to (3.49), using (3.1), the Sobolev inequality and (3.25), we have
\[
\|\nabla u\|_{L^6} \leq C \|\nabla \rho u\|_{L^6} + C \|G\|_{L^6} + C \|\rho \theta\|_{L^6} \\
\leq C \|\nabla \rho u\|_{L^2} + C \|\nabla G\|_{L^2} + C \|\nabla \theta\|_{L^2} \\
\leq C \|\sqrt{\rho} \dot{u}\|_{L^2} + C \|\nabla \theta\|_{L^2} \leq C.
\]
(3.66)

By (3.8), we have
\[
\|\nabla u\|_{L^2} \leq C.
\]
This, together with (3.66) and the interpolation inequality and the Sobolev inequality, implies
\[
\|u\|_{L^\infty} + \|\nabla u\|_{L^r} \leq C,
\]
(3.67)
for any \( r \in [2, 6] \).

By (3.54), we have
\[
\|\nabla^2 \theta\|_{L^2}^2 \leq C \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 + C \|\nabla \theta\|_{L^2}^4 \\
\leq C \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^5}^3 + C \|\nabla \theta\|_{L^2}^2 \\
\leq C \|\sqrt{\rho} \dot{\theta}\|_{L^2}^2 + C \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2,
\]
(3.68)

where we have used the Gagliardo-Nirenberg inequality, (3.25), (3.66) and (3.67).

By (3.68), (3.25) and (3.3), we get
\[
\int_0^t \int_{\mathbb{R}^3} |\nabla^2 \theta|^2 \leq C.
\]
Recall $\dot{\theta} = \theta_t + u \cdot \nabla \theta$, we have
\[
\int_0^t \int_{\mathbb{R}^3} \rho|\theta_t|^2 \leq C \int_0^t \int_{\mathbb{R}^3} \rho|\dot{\theta}|^2 + C \int_0^t \int_{\mathbb{R}^3} \rho|u \cdot \nabla \theta|^2
\leq C \int_0^t \int_{\mathbb{R}^3} \rho|\dot{\theta}|^2 + C \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \leq C,
\]
where we have used (3.15), (3.8), (3.25), and (3.67).

Lemma 3.6 Under the conditions of Theorem 2.1.1 and (3.1), it holds that for any $t \in (0, T^*)$
\[
\int_{\mathbb{R}^3} \rho|\theta_t|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \, dx \, ds \leq C. \tag{3.69}
\]

Proof. To get the estimate (3.69), we use the arguments similar to the proof of Lemma 3.4, i.e., mollifying each term in (2.1), differentiating the result w.r.t. $t$, multiplying by $(\theta, \dot{\theta})_t$, integrating by parts, and passing to the limit. Then we arrive at
\[
\frac{1}{2} \int_0^t \int_{\mathbb{R}^3} \rho|\theta_t|^2 + \kappa \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \\
\leq - \int_0^t \int_{\mathbb{R}^3} \rho \left( \frac{\theta_t}{2} + u \cdot \nabla \theta + \theta \text{div} u \right) \theta_t - \int_0^t \int_{\mathbb{R}^3} \rho (u_t \cdot \nabla \theta + u \cdot \nabla \theta + \theta_t \text{div} u) \theta_t \\
- \int_0^t \int_{\mathbb{R}^3} \rho \theta \text{div} u \theta_t + \mu \int_0^t \int_{\mathbb{R}^3} \left( \nabla u + (\nabla u)^T \right) : (\nabla u_t + (\nabla u_t)^T) \theta_t \\
+ 2\lambda \int_0^t \int_{\mathbb{R}^3} \text{div} u \text{div} u \theta_t + C \\
= \sum_{i=1}^5 (V)_i + C,
\]
where
\[
(V)_1 = - \int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta \left( \frac{\theta_t}{2} + u \cdot \nabla \theta + \theta \text{div} u \right) - \int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta \theta_t \\
- \int_0^t \int_{\mathbb{R}^3} \rho u \cdot (\nabla u \cdot \nabla \theta + u \cdot \nabla \theta) \theta_t - \int_0^t \int_{\mathbb{R}^3} \rho u \cdot (\nabla \theta \text{div} u + \theta \text{div} u) \theta_t \tag{3.71}
\]
For $(V)_{1,1}$, we have
\[
(V)_{1,1} \leq \frac{\kappa}{24} \int_0^t \int_{\mathbb{R}^3} \|\nabla \theta_t\|^2 + C \int_0^t \|u\|_{L^\infty}^2 \|\sqrt{\rho \theta_t}\|_{L^2}^2 + C \int_0^t \|u\|_{L^\infty}^4 \|\nabla \theta\|^2_{L^2} \\
+ C \int_0^t \|u\|_{L^\infty}^4 \|\theta\|^2_{L^6} \|\nabla u\|^2_{L^3} \tag{3.72}
\]
\[
\leq \frac{\kappa}{24} \int_0^t \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_0^t \int_{\mathbb{R}^3} \rho|\theta_t|^2 + C \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2,
\]
where we have used the Cauchy inequality, (3.1) and (3.65).
For $(V)_{1,2}$ and $(V)_{1,3}$, using the Cauchy inequality, (3.1) and (3.65) again, we have
\[
(V)_{1,2} \leq \frac{\kappa}{24} \int_0^t \int_{\mathbb{R}^3} |\nabla \theta_t|^2 + C \int_0^t \int_{\mathbb{R}^3} \rho|\theta_t|^2, \tag{3.73}
\]
For \((V)_{1,4}\), integrating by parts, we have

\[
(V)_{1,4} = - \int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta \, d\nu \, d\xi - \frac{1}{2\mu + \lambda} \int_0^t \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla \theta \, d\xi \\
- \frac{1}{2\mu + \lambda} \int_0^t \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla (\rho \theta) \, d\xi \tag{3.75}
\]

Furthermore, we get

\[
(V)_{1,4} \leq C \int_0^t \int_{\mathbb{R}^3} \sqrt{\rho} \, |\nabla \theta|^2 \, d\xi + C \int_0^t \int_{\mathbb{R}^3} |\nabla G|^2 \, d\xi \tag{3.76}
\]

where we have used the H"older inequality, the Sobolev inequality, \((3.11), (3.13), (3.14), (3.25), (3.65)\) and the Cauchy inequality. Substituting \((3.72), (3.73), (3.74)\) and \((3.76)\) into \((3.71)\), we have

\[
(V)_{1} \leq \frac{\kappa}{24} \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \, d\xi + C \int_0^t \int_{\mathbb{R}^3} |\nabla \theta|^2 \, d\xi \tag{3.77}
\]

For \((V)_2\)

\[
(V)_2 = - \int_0^t \int_{\mathbb{R}^3} \rho \dot{u} \cdot \nabla \theta \, d\xi - \int_0^t \int_{\mathbb{R}^3} \rho (u \cdot \nabla) u \cdot \nabla \theta \, d\xi - \int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \theta \, d\xi \\
\leq C \int_0^t \int_{\mathbb{R}^3} \sqrt{\rho \dot{u}} \, |\nabla \theta| \, d\xi + C \int_0^t \int_{\mathbb{R}^3} \sqrt{\rho} \, |\nabla \theta|^2 \, d\xi \tag{3.78}
\]
where we have used the Hölder inequality, the Sobolev inequality, (3.1), (3.25) and (3.65). For \( (V)_3 \), integrating by parts, we have

\[
(V)_3 = - \int_0^t \int_{\mathbb{R}^3} \rho \theta \text{div} \theta_t + \int_0^t \int_{\mathbb{R}^3} \rho \theta \text{div}(u \cdot \nabla \theta_t)
= - \int_0^t \int_{\mathbb{R}^3} \rho \theta \text{div} \theta_t + \int_0^t \int_{\mathbb{R}^3} \rho \theta \nabla u : (\nabla u)^t \theta_t + \int_0^t \int_{\mathbb{R}^3} \rho \theta u \cdot \nabla \text{div} u \theta_t
= - \int_0^t \int_{\mathbb{R}^3} \rho \theta \text{div} \theta_t + \int_0^t \int_{\mathbb{R}^3} \rho \theta \nabla u : (\nabla u)^t \theta_t \tag{3.79}
\]

Furthermore, using the Hölder inequality, the Sobolev inequality, (3.1), (3.25) and (3.65), we have

\[
(V)_3 \leq C \int_0^t \| \nabla \tilde{u} \|_{L^2} \| \theta_t \|_{L^6} \| \theta \|_{L^6} \| \rho \|_{L^6} + C \int_0^t \| \rho \|_{L^\infty} \| \theta \|_{L^6} \| \nabla u \|_{L^2} \| \nabla \tilde{u} \|_{L^6} \| \theta_t \|_{L^6}
+ C \int_0^t \| \nabla u \|_{L^2} \| \theta \|_{L^6} \| \rho \|_{L^6} + C \int_0^t \| \nabla u \|_{L^2} \| \theta \|_{L^6} \| \theta_t \|_{L^6} \| \rho \|_{L^6}^2 \| \tilde{u} \|_{L^2}
+ C \int_0^t \| \tilde{u} \|_{L^6} \| \theta \|_{L^6} \| \rho \|_{L^6} \| \nabla \tilde{u} \|_{L^2} \tag{3.80}
\]

Similar to \( (V)_2 \) and \( (V)_3 \), for \( (V)_4 \) and \( (V)_5 \), we deduce

\[
(V)_4 + (V)_5 \leq C \int_0^t \| \nabla \tilde{u} \|_{L^2} \| \nabla \tilde{u} \|_{L^2} \| \theta_t \|_{L^6} \| \nabla \tilde{u} \|_{L^6} + C \int_0^t \| \nabla u \|_{L^3} \| \theta_t \|_{L^6}
+ C \int_0^t \| u \|_{L^6} \| \nabla \tilde{u} \|_{L^2} + C \int_0^t \| \nabla \tilde{u} \|_{L^2} + C \int_0^t \| \theta \|_{L^6}^2 \| \nabla \tilde{u} \|_{L^2} \tag{3.81}
\]

where we have used the Hölder inequality, integration by parts, the Cauchy inequality, (3.65), the interpolation inequality and the Sobolev inequality.

Putting (3.77), (3.78), (3.80) and (3.81) into (3.70), we have

\[
\frac{1}{2} \int_{\mathbb{R}^3} \rho | \theta_t |^2 + \kappa \int_{\mathbb{R}^3} \| \nabla \tilde{u} \|_{L^2}^2
\leq C \int_0^t \int_{\mathbb{R}^3} \rho | \theta_t |^2 + C \int_0^t \int_{\mathbb{R}^3} | \nabla \theta |^2 + C \int_0^t \| \nabla \tilde{u} \|_{L^2}^2 + C \int_0^t \| \nabla \tilde{u} \|_{L^2}^2 \tag{3.82}
\]

This combined with (3.3), (3.25) and (3.65) completes the proof of Lemma 3.6. \( \square \)
Corollary 3.7 Under the conditions of Theorem 2.1.1 and (3.1), it holds that for any $t \in (0, T^*)$

$$\int_{\mathbb{R}^3} |\nabla^2 \theta|^2 \, dx \leq C. \quad (3.83)$$

Proof. It follows from (2.1) 3, (3.1), (3.3), (3.25), (3.65), (3.69) and the interpolation inequality that

$$\|\nabla^2 \theta\|_{L^2} \leq C \|\rho \theta_t\|_{L^2} + C \|ho u \cdot \nabla \theta\|_{L^2} + C \|\rho \theta \text{div} u\|_{L^2} + C \|\nabla u\|_{L^4} \leq C.$$

By (3.25), (3.83) and the Sobolev inequality, we get the following corollary which is the desired one, i.e., (3.2).

Corollary 3.8 Under the conditions of Theorem 2.1.1 and (3.1), it holds that for any $t \in (0, T^*)$

$$\|\theta\|_{L^\infty(0,t;L^\infty)} \leq C. \quad (3.84)$$

4 Global strong solution

In this section, we shall prove the global existence and uniqueness of the strong solution. Since the local existence and uniqueness of the strong solution has been obtained in [4] under the conditions of Theorem 2.2.1, we assume that $T^* > 0$ is the maximal existence time of the strong solution. We shall prove $T^* = \infty$ by using contradiction arguments.

Remark 2.1.3 says that if $T^* < \infty$, then

$$\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|\rho \theta_t\|_{L^4(0,T;L^4)} + \|\rho^{\frac{4}{3}} u\|_{L^\infty(0,T;L^4)} + \|\rho \|
abla u\|_{L^2(0,T;L^2)} \right) = \infty \quad (4.1)$$

for all $\mu$ and $\lambda$ satisfying only the physical restriction (1.2).

If $T^* < \infty$, our aim is to prove that (4.1) is not true under the conditions of Theorem 2.2.1 which is the desired contradiction.

To do this, we define

$$A(T) = \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^T \int_{\mathbb{R}^3} \frac{\rho |\dot{u}|^2}{\mu}, \quad (4.2)$$

and

$$B(T) = \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} \rho \theta^2 + \int_0^T \int_{\mathbb{R}^3} \kappa |\nabla \theta|^2. \quad (4.3)$$

The following proposition plays a crucial role in the section.

Proposition 4.1 Assume that the initial data satisfies (2.2.1), (2.2.2), and (2.1.1). If the strong solution $(\rho, u, \theta)$ satisfies

$$A(T) \leq 2\bar{E} K_1, \ B(T) \leq 2K_2, \ 0 \leq \rho \leq 2\bar{\rho}, \ (x,t) \in \mathbb{R}^3 \times [0,T], \quad (4.4)$$
then
\[ A(T) \leq \frac{3}{2} \tilde{E}K_1, \quad B(T) \leq \frac{3}{2} K_2, \quad 0 \leq \rho \leq \frac{3}{2} \tilde{\rho}, \quad (x, t) \in \mathbb{R}^3 \times [0, T], \] (4.5)
provided that \( m_0 \leq \varepsilon_0. \) Here \( m_0 = \int_{\mathbb{R}^3} \rho_0(x) \, dx, \) \( \tilde{E} = \frac{(14\mu + 9\lambda)}{2\mu} + \frac{6\tilde{\rho}K_2}{\mu(\mu + \lambda)K_1} + \frac{8\tilde{\rho}K_2}{\mu(\mu + \lambda)^2K_1} + 1, \) and
\[ \varepsilon_0 = \min \left\{ C_3, \frac{\tilde{C}(2\mu + \lambda)^6}{E^3}, \frac{\tilde{C}(2\mu + \lambda)^6}{E^3}, \frac{\tilde{C}\mu^2(2\mu + \lambda)^8}{E^6}, \frac{\tilde{C}\kappa^4K_2}{(2\mu + \lambda)^6E^6}, \frac{\tilde{C}\kappa^6}{E^4}, \frac{\tilde{C}\kappa^6}{E^4} \right\}, \]
where
\[ C_3 = \min \left\{ \frac{\tilde{C}\kappa^6(\mu + \lambda)^6}{(\kappa(\mu + \lambda) + 1)^6}, \frac{\tilde{C}\mu^3(2\mu + \lambda)^6}{E^2}, \frac{\tilde{C}\mu^2(2\mu + \lambda)^8}{E^6}, \frac{\tilde{C}\kappa^4K_2}{(2\mu + \lambda)^6E^6}, \frac{\tilde{C}\kappa^6}{E^4}, \frac{\tilde{C}\kappa^6}{E^4} \right\}, \]
for some constant \( \tilde{C} > 0 \) depending on \( \tilde{\rho}, K_1, K_2, \) and some other known constants but independent of \( \mu, \lambda, \kappa, \) and \( t. \)

With Proposition 4.1 we shall get \( T^* = \infty. \) More precisely, we obtain the following corollary.

**Corollary 4.2** With Proposition 4.1 it holds that \( T^* = \infty \) with (4.4) valid for any \( 0 \leq T < \infty. \)

**Proof.** If \( T_1 > 0 \) is the maximal time such that (4.4) is valid, then \( T_1 = T^*. \) For otherwise, (4.3) implies that \( T_1 \) is not the maximal time.

With \( T_1 = T^*, \) (4.4) and the \( L^1 \)-bound of \( \rho \) (see Lemma 3.1), one can easily get
\[ \| \rho \|_{L^\infty(0, T; L^\infty)} + \| \rho \theta \|_{L^4(0, T; L^{12})} + \| \rho^\frac{1}{2}u \|_{L^\infty(0, T; L^3)} \leq \tilde{C}(1 + t^\frac{1}{4}) \] (4.6)
for all \( t \in [0, T^*], \) where \( \tilde{C} \) is a positive constant independent of \( t. \) Using the Hölder inequality, the inequality \( \| u \|_{L^6} \leq \tilde{C}\| \nabla u \|_{L^2} \) and (4.4), we have
\[ \| |u| \nabla u| \|_{L^2} \leq \| u \|_{L^6} \| \nabla u \|_{L^3} \leq \tilde{C}\| \nabla u \|_{L^3}. \]
This together with (4.4), the \( L^1 \)-bound of \( \rho \) and the estimate for \( \| \nabla u \|_{L^3} \) (see (4.15) for the detail) gives
\[ \| |u| \nabla u| \|_{L^2(0, T; L^2)} \leq \tilde{C}(1 + t^\frac{1}{4}). \] (4.7)

Therefore, if \( T^* < \infty, \) (4.6) and (4.7) will contradict with (4.1). Thus, \( T^* \) must be \( \infty. \) Then we get \( T_1 = T^* = \infty \) which implies that the strong solution exists globally in time and that (4.4) is valid for any \( T \in [0, \infty). \)

**Remark 4.3** Corollary 4.2 means that if Proposition 4.1 is valid, the global existence of strong solutions will be got. The uniqueness of the solutions can be referred to [2]. The proof of Theorem 2.2.1 is complete.

Let’s come back to prove Proposition 4.1. Throughout the rest of the paper, we denote generic constants by \( C \) depending on \( \tilde{\rho}, K_1, K_2, \) and some other known constants but independent of \( \mu, \lambda, \kappa, \) and \( t. \)

**Proof of Proposition 4.1**
Lemma 4.4 Under the conditions of Proposition 4.1, it holds that
\[
\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \leq m_0^{1/2},
\]
provided
\[
m_0 \leq \frac{\kappa^6(\mu + \lambda)^6 \mu^6}{C^6(\kappa(\mu + \lambda) + 1)^6} \triangleq C_1.
\]

Proof. Multiplying (2.1) by \( u \), integrating by parts over \( \mathbb{R}^3 \), and using the Cauchy inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^2 + \mu \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{1}{\mu + \lambda} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{4(\mu + \lambda)} \int_{\mathbb{R}^3} \rho \theta^2.
\]
This, together with the Hölder inequality, the Sobolev inequality, (4.4) and (3.3), deduces
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho |u|^2 + \mu \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{1}{\mu + \lambda} \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{Cm_0^{2/3}}{\mu + \lambda} \int_{\mathbb{R}^3} \rho \theta^2. \tag{4.9}
\]
Integrating (4.9) over \([0, T]\), and using (4.4) again, we have
\[
\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \leq \frac{C}{\mu} \int_0^T \rho_0 ||u||_L^2 + \frac{Cm_0^{2/3}}{\mu + \lambda} \int_0^T \|\theta\|_{L^2}^2 \leq \left(1 + \frac{1}{\kappa(\mu + \lambda)}\right) \frac{Cm_0^{2/3}}{\mu} \leq m_0^{1/2},
\]
provided that
\[
m_0 \leq \frac{\kappa^6(\mu + \lambda)^6 \mu^6}{C^6(\kappa(\mu + \lambda) + 1)^6} \triangleq C_1.
\]

Lemma 4.5 Under the conditions of Proposition 4.1, it holds that
\[
A(T) \leq \frac{3\tilde{E}K_1}{2}, \tag{4.10}
\]
provided that
\[
m_0 \leq \min \left\{ C_1, \frac{\mu^3 \kappa^3(2\mu + \lambda)}{216C^3}, \frac{\mu^6}{36C^2 \tilde{E}^2}, \frac{\mu^2(2\mu + \lambda)^8}{36C^2} \right\} \triangleq C_2,
\]
where
\[
\tilde{E} = \frac{(14\mu + 9\lambda)}{2\mu} + \frac{6\rho K_2}{\mu(\mu + \lambda) K_1} + \frac{8\rho_0 K_2}{\mu(\mu + \lambda)^2 K_1} + 1.
\]

Proof. Recall from (3.9)
\[
\int_{\mathbb{R}^3} \rho |u|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2)
= \frac{d}{dt} \int_{\mathbb{R}^3} P \text{div} u - \frac{1}{2(\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P^2 - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P_t G + \int_{\mathbb{R}^3} \rho (u \cdot \nabla) u \cdot \dot{u} \tag{4.11}
= \sum_{i=1}^4 \Pi_i,
\]
where $G = (2\mu + \lambda)\text{div} u - P$.

Substituting (3.10) into $II_3$, and using integration by parts, the H"older inequality and the Sobolev inequality, we have

$$II_3 \leq \frac{1}{2\mu + \lambda} \left\| \rho \theta \right\|_{L^3} \left\| \frac{\partial u}{\partial t} \right\|_{L^6} \left\| \nabla G \right\|_{L^2} + \frac{1}{2\mu + \lambda} \left\| \rho \theta \right\|_{L^3} \left\| \text{div} u \right\|_{L^2} \left\| G \right\|_{L^6} + C \left\| G \right\|_{L^6} \left\| \nabla u \right\|_{L^2} \left\| \nabla u \right\|_{L^3} + \frac{\kappa}{2\mu + \lambda} \left\| \nabla G \right\|_{L^2} \left\| \nabla \theta \right\|_{L^2}$$

$$\leq \frac{C}{2\mu + \lambda} \left\| \rho \right\|_{L^6} \left\| \theta \right\|_{L^5} \left\| \nabla u \right\|_{L^2} \left\| \nabla G \right\|_{L^2} + C \left\| \nabla G \right\|_{L^2} \left\| \nabla u \right\|_{L^2} \left\| \nabla u \right\|_{L^3} + \frac{\kappa}{2\mu + \lambda} \left\| \nabla G \right\|_{L^2} \left\| \nabla \theta \right\|_{L^2}.$$ (4.12)

By (3.13), (3.47), the standard $L^2$-estimates, and (1.4), we get

$$\left\| \nabla G \right\|_{L^2} \leq \left\| \rho \frac{\partial u}{\partial t} \right\|_{L^2} \leq \sqrt{2\rho} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2},$$ (4.13)

and

$$\left\| \nabla \text{curl} u \right\|_{L^2} \leq \frac{1}{\mu} \left\| \rho \frac{\partial u}{\partial t} \right\|_{L^2} \leq \frac{\sqrt{2\rho}}{\mu} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2}.$$ (4.14)

Similar to (3.49), using (4.12), (4.14), the Sobolev inequality, the H"older inequality, and the Gagliardo-Nirenberg inequality, we have

$$\left\| \nabla u \right\|_{L^3} \leq \left\| \text{curl} u \right\|_{L^3} + \frac{C}{2\mu + \lambda} \left\| G \right\|_{L^3} + C \left\| \rho \theta \right\|_{L^3}$$

$$\leq \left\| \text{curl} u \right\|_{L^2} \left\| \frac{\partial u}{\partial t} \right\|_{L^2} + \frac{1}{2\mu + \lambda} \left\| \rho \theta \right\|_{L^2} \left\| \nabla G \right\|_{L^2} + \frac{1}{2\mu + \lambda} \left\| \rho \right\|_{L^6} \left\| \theta \right\|_{L^6}.$$ (4.15)

Substituting (4.12) into (4.14), we have

$$II_3 \leq \frac{C}{2\mu + \lambda} \left\| \rho \right\|_{L^6} \left\| \theta \right\|_{L^5} \left\| \nabla u \right\|_{L^2} \left\| \nabla \theta \right\|_{L^2} + \sqrt{\rho} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2} \left\| \nabla u \right\|_{L^2} \left\| \nabla u \right\|_{L^3} + \frac{\kappa \sqrt{\rho}}{2\mu + \lambda} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2} \left\| \nabla \theta \right\|_{L^2}.$$ (4.16)

For $II_4$, using the H"older inequality, (4.3), and the Sobolev inequality, we have

$$II_4 \leq C \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2} \left\| u \right\|_{L^6} \left\| \nabla u \right\|_{L^3} \leq C \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2} \left\| \nabla u \right\|_{L^2} \left\| \nabla u \right\|_{L^3}.$$ (4.17)

Putting (4.16) and (4.17) together, and using (4.15), and the Young inequality, we have

$$II_3 + II_4 \leq \frac{C}{2\mu + \lambda} \left\| \rho \right\|_{L^6} \left\| \theta \right\|_{L^2} \left\| \nabla u \right\|_{L^2} \left\| \nabla \theta \right\|_{L^2} + \frac{C}{\sqrt{\mu}} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2} \left\| \nabla u \right\|_{L^2} + \frac{\kappa \sqrt{\rho}}{2\mu + \lambda} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2} \left\| \nabla \theta \right\|_{L^2}$$

$$\leq \frac{1}{2} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2}^2 + \frac{C}{(2\mu + \lambda)^2} \left\| \rho \theta \right\|_{L^2} \left\| \nabla u \right\|_{L^2}^2 + \frac{\kappa \sqrt{\rho}}{2\mu + \lambda} \left\| \sqrt{\rho} \frac{\partial u}{\partial t} \right\|_{L^2} \left\| \nabla \theta \right\|_{L^2}^2$$

$$+ \frac{C}{(2\mu + \lambda)^2} \left\| \nabla u \right\|_{L^2}^2 \left\| \rho \theta \right\|_{L^2}^2 + \frac{2\rho \kappa^2}{(2\mu + \lambda)^2} \left\| \nabla \theta \right\|_{L^2}^2.$$ (4.18)
Lemma 4.6 Under the conditions of Proposition 4.1, it holds that

\[ B(T) \leq \frac{3K_2}{2}, \]  

provided that

\[ m_0 \leq \min \left\{ C_1, \frac{\mu^3(2\mu + \lambda)^6}{216C^3}, \frac{\mu^6}{36C^2\tilde{E}^2}, \frac{\mu^2(2\mu + \lambda)^8}{36C^2} \right\} \equiv C_2, \]

where \( \tilde{E} = \frac{(14\mu + 9\lambda)}{2\mu} \frac{6\bar{\rho}_K}{\mu(\mu + \lambda)K_1} + \frac{8\bar{\rho}_K}{\mu(\mu + \lambda)^2}K_2 + 1. \)

By (4.20), we get (4.10). \( \square \)

Lemma 4.6 Under the conditions of Proposition 4.1, it holds that

\[ B(T) \leq \frac{3K_2}{2}, \]  

provided that

\[ m_0 \leq \min \left\{ C_2, \frac{\kappa^4\mu^2}{64C^4(2\mu + \lambda)^8\tilde{E}^6}, \frac{\kappa^4}{64C^4E^4\mu^2}, \frac{\kappa^6}{66C^6\tilde{E}^3} \right\} \equiv C_3, \]
Proof. Recall from (3.20)
\[
\kappa \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho \theta^2 = - \int_{\mathbb{R}^3} \rho \theta^2 \text{div} u \, + \, \int_{\mathbb{R}^3} \frac{\mu}{2} |\nabla u| \, + \, (\nabla u)' \theta \, + \, \int_{\mathbb{R}^3} \lambda (\text{div} u)^2 \theta
\]
(4.22)

\[
= \sum_{i=1}^{3} III_i.
\]

For $III_1$, using the Hölder inequality, and the Sobolev inequality, we have

\[
III_1 \leq \|\text{div} u\|_{L^2} \|\theta\|_{L^2}^2 \|\rho\|_{L^6} \leq C \|\text{div} u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \|\rho\|_{L^6}.
\]
(4.23)

For $III_2$ and $III_3$, using the Hölder inequality, and the Sobolev inequality again, together with (4.15), we have

\[
III_2 + III_3 \leq C(2\mu + \lambda) \|\nabla u\|_{L^2} \|\theta\|_{L^6} \leq C(2\mu + \lambda) \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \|\rho\|_{L^6}.
\]
(4.24)

Substituting (4.23) and (4.24) into (4.22), and using the Cauchy inequality, we have

\[
\kappa \int_{\mathbb{R}^3} |\nabla \theta|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho \theta^2 \leq C(2\mu + \lambda) \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \|\rho\|_{L^6}.
\]
(4.25)

Integrating (4.25) over $[0, t]$, and using (4.4), the Hölder inequality, (3.3), and (4.3), we have

\[
B(t) \leq \int_{\mathbb{R}^3} \rho_0 |\theta_0|^2 + \frac{C(2\mu + \lambda)^2}{\kappa \mu} \int_0^t \|\nabla u\|_{L^2}^3 \|\sqrt{\rho \hat{u}}\|_{L^2}
\]
\[
+ C \frac{\kappa}{\kappa} \int_0^t \|\nabla u\|_{L^2} \|\theta\|_{L^2} \|\sqrt{\rho \hat{u}}\|_{L^2} + C \int_0^t \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2} \|\rho\|_{L^6}
\]
\[
\leq K_2 + \frac{C(2\mu + \lambda)^2 A(T)^\frac{3}{2}}{\kappa \sqrt{\mu}} \|\nabla u\|_{L^2([0, t]; L^2)} + \frac{C \sqrt{K_2} A(T)}{\kappa} \|\nabla u\|_{L^2([0, t]; L^2)}
\]
\[
+ C \sqrt{A(T) m_0^\frac{3}{2}} \frac{K_2}{\kappa}
\]
\[
\leq K_2 + \frac{C(2\mu + \lambda)^2 \sqrt{\mu}}{\kappa \sqrt{\mu}} ((K_2 m_0^\frac{1}{2}) + \frac{C \sqrt{\mu}}{\kappa} (K_2 m_0^\frac{1}{2}) + C \sqrt{E m_0^\frac{1}{2}} K_2^\frac{1}{2}.
\]

Thus,

\[
B(t) \leq K_2 + \frac{K_2}{2} = \frac{3K_2}{2},
\]

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provided that
\[ m_0 \leq \min \left\{ C_2, \frac{\kappa^4 \mu^2}{6^4 C^4 (2\mu + \lambda)^8 E^6}, \frac{\kappa^4}{6^4 C^4 E^4 \mu^2}, \frac{\kappa^6}{6^6 C^6 E^3} \right\} \triangleq C_3. \]

\[ \Box \]

**Lemma 4.7** Under the conditions of Proposition [4.1] it holds that
\[ 0 \leq \rho \leq \frac{3\rho}{2}, \quad (4.26) \]
for any \((x, t) \in \mathbb{R}^3 \times [0, T] \), provided that
\[ m_0 \leq \min \left\{ C_3, \frac{(2\mu + \lambda)^6 (log \frac{3}{2})}{(4C)^6 E^3}, \frac{\mu^{12} (2\mu + \lambda)^{12} (log \frac{3}{2})^{12}}{(4C)^{12} E^{12}}, \frac{(2\mu + \lambda)^{36} \kappa^{12} (log \frac{3}{2})^{12}}{(4C)^{12}} \right\} \triangleq \varepsilon_0. \]

**Proof.** The first inequality of (4.26) is obvious. In fact, this has been obtained in [4] for any \((x, t) \in \mathbb{R}^3 \times [0, T] \subset \mathbb{R}^3 \times [0, T^\ast] \). We only need to prove the second inequality of (4.26).

Let us mention that the Zlotnik inequality (see Appendix A) used in [16] seems not working here. The main ingredient for handling such the difficulty is an equation obtained from (2.1) involving \( \log \rho \). It was introduced by P.L. Lions ([27]) to prove global existence of weak solutions of the compressible isentropic Navier-Stokes equations, and was later used by B. Desjardins ([6]) et al to study the regularity of weak solutions of the compressible isentropic Navier-Stokes equations for small time under periodic boundary conditions.

More precisely, for any given \((x, t) \in \mathbb{R}^3 \times [0, T] \). Denote
\[ \rho^\delta(y, s) = \rho(y, s) + \delta \exp\left\{ - \int_0^s \text{div} \left( X(\tau; x, t), \tau \right) d\tau \right\} > 0, \]
where \( X(s; x, t) \) is given by
\[ \left\{ \begin{array}{ll}
\frac{d}{ds} X(s; x, t) = u(X(s; x, t), s), & 0 \leq s < t, \\
X(t; x, t) = x. & 
\end{array} \right. \]
It is easy to verify that
\[ \frac{d}{ds} \rho^\delta \left( X(s; x, t), s \right) + \rho^\delta \left( X(s; x, t), s \right) \text{div} \left( X(s; x, t), s \right), \]
due to (2.1)\_1. This gives
\[ Y'(s) = g(s) + b'(s), \quad (4.27) \]
where
\[ Y(s) = \log \rho^\delta \left( X(s; x, t), s \right), \quad g(s) = - \frac{P(X(s; x, t), s)}{2\mu + \lambda}, \quad b(s) = - \frac{1}{2\mu + \lambda} \int_0^s G(X(\tau; x, t), \tau) d\tau, \]
and \( G = (2\mu + \lambda) \text{div} u - P = (2\mu + \lambda) \text{div} u - \rho \theta. \)
By (3.13) and (2.1), we have

\[ G(X(t;x,\tau),\tau) = -(-\Delta)^{-1} \text{div}[(\rho u)_\tau + \text{div}(\rho u \otimes u)] = \left[-[(-\Delta)^{-1} \text{div}(\rho u)]_\tau \right. \]

\[ -(-\Delta)^{-1} \text{divdiv}(\rho u \otimes u) \]

\[ = -[(-\Delta)^{-1} \text{div}(\rho u)]_\tau - u \cdot \nabla(-\Delta)^{-1} \text{div}(\rho u) + u \cdot \nabla(-\Delta)^{-1} \text{div}(\rho u) \]

\[ -(-\Delta)^{-1} \text{divdiv}(\rho u \otimes u) \]

\[ = \frac{d}{dt}[(-\Delta)^{-1} \text{div}(\rho u)] + u \cdot \nabla(-\Delta)^{-1} \text{div}(\rho u) - (-\Delta)^{-1} \text{divdiv}(\rho u \otimes u) \]

\[ = \frac{d}{dt}[(-\Delta)^{-1} \text{div}(\rho u)] + [u_i, R_{ij}](\rho u_j), \]

where \([u_i, R_{ij}] = u_i R_{ij} - R_{ij} u_i\) and \(R_{ij} = \partial_i(-\Delta)^{-1} \partial_j\). This deduces

\[ b(t) - b(0) = \frac{1}{2\mu + \lambda} \int_0^t \left[ \frac{d}{d\tau}[(-\Delta)^{-1} \text{div}(\rho u)] - [u_i, R_{ij}](\rho u_j) \right] d\tau \]

\[ = \frac{1}{2\mu + \lambda} (-\Delta)^{-1} \text{div}(\rho u) - \frac{1}{2\mu + \lambda} (-\Delta)^{-1} \text{div}(\rho_0 u_0) - \frac{1}{2\mu + \lambda} \int_0^t [u_i, R_{ij}](\rho u_j) d\tau \]

\[ \leq \frac{1}{2\mu + \lambda} \|(\rho_0 u_0)\|_{L^\infty} + \frac{1}{2\mu + \lambda} \|(-\Delta)^{-1} \text{div}(\rho_0 u_0)\|_{L^\infty} \]

\[ + \frac{1}{2\mu + \lambda} \int_0^t \|[u_i, R_{ij}](\rho u_j)\|_{L^\infty} d\tau = \sum_{i=1}^3 IV_i. \]

For \(IV_1\), using the Gagliardo-Nirenberg inequality, the Sobolev inequality, the Calderon-Zygmund inequality, the Hölder inequality, (4.4), and (3.3), we have

\[ IV_1 \leq \frac{C}{2\mu + \lambda} \|(\rho u)\|_{L^6} \|\nabla(-\Delta)^{-1} \text{div}(\rho u)\|_{L^3}^\frac{1}{6} \]

\[ \leq \frac{C}{2\mu + \lambda} \|\rho u\|_{L^2}^{\frac{1}{6}} \|\rho u\|_{L^4}^{\frac{5}{6}} \leq \frac{C}{2\mu + \lambda} \|\rho\|_{L^6}^{\frac{1}{6}} \|u\|_{L^6}^{\frac{5}{6}} \|\rho\|_{L^{12}} \|u\|_{L^6}^{\frac{5}{6}} \]

\[ \leq \frac{Cm_0^{\frac{1}{6}}}{2\mu + \lambda} \|\nabla u\|_{L^2} \leq \frac{Cm_0^{\frac{1}{6}} \sqrt{E}}{2\mu + \lambda}. \]

Similarly, for \(IV_2\), we have

\[ IV_2 \leq \frac{Cm_0^{\frac{1}{6}} \sqrt{E}}{2\mu + \lambda}. \]

Since \(u(\cdot, t) \in W^{1,6}(\mathbb{R}^3)\), \(pu(\cdot, t) \in L^{12}(\mathbb{R}^3)\) and \(\frac{1}{\alpha} = \frac{1}{6} + \frac{1}{12}\), in view of the conclusions by Desjardins ((33), [2]) or by Choe-Jin (Section 4. 35) and references therein, it holds that

\[ \|[u_i, R_{ij}](\rho u_j)\|_{W^{1,4}} \leq C\|u\|_{W^{1,6}}\|\rho u\|_{L^{12}}. \]

This, combined with (4.4), (3.3), the Sobolev inequality, the Calderon-Zygmund inequality similar to (3.46), gives

\[ \|[u_i, R_{ij}](\rho u_j)\|_{W^{1,4}} \leq Cm_0^{\frac{1}{6}} \|u\|_{W^{1,6}}\|u\|_{L^\infty} \leq Cm_0^{\frac{1}{6}} \|u\|_{W^{1,6}}^2 \]

\[ \leq Cm_0^{\frac{1}{6}} \left( \|\nabla u\|_{L^2}^2 + \|\text{curl} u\|_{L^6}^2 + \|\text{div} u\|_{L^6}^2 \right) \]

\[ \leq Cm_0^{\frac{1}{6}} \left( \|\nabla u\|_{L^2}^2 + \|\text{curl} u\|_{L^6}^2 + \frac{1}{(2\mu + \lambda)^2} \|G\|_{L^2}^2 + \frac{1}{(2\mu + \lambda)^6} \|\rho \theta\|_{L^6}^2 \right) \]

\[ \leq Cm_0^{\frac{1}{6}} \left( \|\nabla u\|_{L^2}^2 + \|\nabla \text{curl} u\|_{L^2}^2 + \frac{1}{(2\mu + \lambda)^2} \|\nabla G\|_{L^2}^2 + \frac{1}{(2\mu + \lambda)^6} \|\nabla \theta\|_{L^2}^2 \right). \]

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combined with the Sobolev inequality, (4.14) and (4.13) deduces
\[ \|u_i, R_{ij}(pu_j)\|_{L^\infty} \leq C m_0^{\frac{1}{2}} \left( \|\nabla u\|^2_{L^2} + \frac{1}{\mu^2} \|\sqrt{\rho}\dot{u}\|^2_{L^2} + \frac{1}{(2\mu + \lambda)^2} \|\nabla \theta\|^2_{L^2} \right). \]  
(4.31)

Substituting (4.31) into IV3, we have
\[ IV_3 \leq \frac{C m_0^{\frac{1}{2}}}{2\mu + \lambda} \int_0^t \left( \|\nabla u\|^2_{L^2} + \frac{1}{\mu^2} \|\sqrt{\rho}\dot{u}\|^2_{L^2} + \frac{1}{(2\mu + \lambda)^2} \|\nabla \theta\|^2_{L^2} \right) d\tau \]
\[ \leq \frac{C m_0^{\frac{1}{2}}}{2\mu + \lambda} \left( m_0^{\frac{1}{2}} + \frac{\tilde{E}}{\mu} + \frac{1}{(2\mu + \lambda)^2 \kappa} \right). \]
(4.32)

By (4.28), (4.29) and (4.32), we have
\[ b(t) - b(0) \leq \frac{C m_0^{\frac{1}{2}} \sqrt{\tilde{E}}}{2\mu + \lambda} + \frac{C m_0^{\frac{1}{2}}}{2\mu + \lambda} \left( m_0^{\frac{1}{2}} + \frac{\tilde{E}}{\mu} + \frac{1}{(2\mu + \lambda)^2 \kappa} \right) \]
\[ \leq \frac{C m_0^{\frac{1}{2}} \sqrt{\tilde{E}}}{2\mu + \lambda} + \frac{C m_0^{\frac{1}{2}}}{2\mu + \lambda} \frac{\tilde{E}}{\mu(2\mu + \lambda)} + \frac{C m_0^{\frac{1}{2}}}{2\mu + \lambda} \frac{\tilde{E}}{(2\mu + \lambda)^3 \kappa} \]
\[ \leq \log \frac{3}{2}, \]
(4.33)

provided that
\[ m_0 \leq \min \left\{ C_3, \frac{(2\mu + \lambda)^6 (\log \frac{\tilde{E}}{3})^6}{(4C)^6 E^3}, \frac{(2\mu + \lambda)^{12} (\log \frac{\tilde{E}}{3})^{12}}{(4C)^{12}}, \frac{\mu^{12} (2\mu + \lambda)^{12} (\log \frac{\tilde{E}}{3})^{12}}{(4C)^{12} E^{12}}, \frac{(2\mu + \lambda)^{36} \kappa^{12} (\log \frac{\tilde{E}}{3})^{12}}{(4C)^{12}} \right\} \]
\[ \triangleq \varepsilon_0. \]

Integrating (4.27) w.r.t. s over [0, t], we get
\[ \log \rho^\delta(x, t) = \log [\rho_0 (X(t; x, 0)) + \delta] + \int_0^t g(\tau) d\tau + b(t) - b(0) \]
\[ \leq \log (\bar{\rho} + \delta) + \log \frac{3}{2}, \]
provided that \( m_0 \leq \varepsilon_0 \). Let \( \delta \to 0^+ \), we have
\[ \rho \leq \frac{3\bar{\rho}}{2}. \]

\[ \square \]

5 Asymptotic behavior in time

In this section, we denote generic constants by \( \bar{C} \) depending on the initial data, coefficients of viscosity and heat conduction and some other known constants but independent of \( t \). Theorem 2.3.1 will be proved in Sections 5.1 and 5.2
5.1 Large-time behavior

The main result in Section 5.1 is stated as follows.

**Proposition 5.1** Under the conditions of Theorem 2.3.1, it holds that

$$\int_{\mathbb{R}^3} (\rho|\theta|^2 + |\nabla u|^2 + |\nabla \theta|^2) \to 0,$$

(5.1)

as $t \to \infty$.

To prove Proposition 5.1 we need some estimates uniform for $t$. In fact, the lower order estimates of the solutions have been made uniformly for $t$ in Section 4. More precisely,

**Lemma 5.2** Under the conditions of Theorem 2.3.1, it holds that

$$0 \leq \rho \leq \bar{C},$$

(5.2)

and

$$\int_{\mathbb{R}^3} (\rho + \rho|\theta|^2 + |\nabla u|^2) + \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla \theta|^2 + \rho|\dot{u}|^2) \leq \bar{C},$$

(5.3)

for any $(x, t) \in \mathbb{R}^3 \times [0, \infty)$.

With Lemma 5.2 one can follow the proofs of Lemma 3.4, Corollary 3.5 and Lemma 3.6 step by step, and easily get the following higher order estimates uniform for $t$, respectively.

**Lemma 5.3** Under the conditions of Theorem 2.3.1, it holds that

$$\int_{\mathbb{R}^3} (\rho|\dot{u}|^2 + |\nabla \theta|^2) + \int_0^t \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + \rho|\dot{\theta}|^2) \leq \bar{C},$$

(5.4)

for any $t \in [0, \infty)$.

**Corollary 5.4** Under the conditions of Theorem 2.3.1, it holds that

$$\|u\|_{L^\infty} + \|
abla u\|_{L^r} + \int_0^t \int_{\mathbb{R}^3} (\rho|\theta_t|^2 + |\nabla^2 \theta|^2) \leq \bar{C},$$

(5.5)

for any $r \in [2, 6]$, and any $t \in [0, \infty)$.

**Lemma 5.5** Under the conditions of Theorem 2.3.1, it holds that

$$\int_{\mathbb{R}^3} \rho|\theta_t|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \theta_t|^2 \leq \bar{C},$$

(5.6)

for any $t \in [0, \infty)$.

**Proof of Proposition 5.1**

Denote

$$F(t) = \int_{\mathbb{R}^3} \left(\mu|\text{curl} u|^2 + \frac{C^2}{2\mu + \lambda}\right).$$
By (5.2) and (5.3), we have
\[ F \in L^1(0, \infty). \] (5.7)
Moreover, by (4.11), (4.18), (5.2) and (5.3), we have
\[
\left| \frac{d}{dt} F(t) \right| \leq \bar{C} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \tilde{C} \rho \|\nabla\theta\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \bar{C} \|\nabla u\|_{L^2}^6
\]
\[
+ \bar{C} \|\nabla u\|_{L^2}^4 \|\rho \theta\|_{L^2}^2 + \bar{C} \|\nabla \theta\|_{L^2}^2
\]
\[
\leq \bar{C} \int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \tilde{C} \|\nabla \theta\|_{L^2}^2 + \bar{C} \|\nabla u\|_{L^2}^2,
\] (5.8)
where we have used \( \Delta u = \nabla \text{div} u - \nabla \times (\text{curl} u) \) such that
\[
\int_{\mathbb{R}^3} |\nabla u|^2 = \int_{\mathbb{R}^3} (|\text{div} u|^2 + |\text{curl} u|^2).
\] (5.9)

By (5.8), (5.3) and (5.7), we conclude that
\[ F \in W^{1,1}(0, \infty), \]
which deduces
\[
\int_{\mathbb{R}^3} \left( \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right) (t) = F(t) \to 0,
\] (5.10)
as \( t \to \infty \).

It follows from (5.3) and (5.6) that
\[ \|\nabla \theta\|_{L^2}^2 (\cdot) \in W^{1,1}(0, \infty), \]
which deduces
\[ \|\nabla \theta\|_{L^2}^2 (t) \to 0, \] (5.11)
as \( t \to \infty \).

By (5.10), (5.11), (5.9), (5.2) and (5.3), we get (5.1).

5.2 Decay estimates

Proposition 5.6 Under the conditions of Theorem 2.3.1, we get
\[
\int_{\mathbb{R}^3} (\rho \theta^2 + |\nabla u|^2) \leq \bar{C} \exp \{-\tilde{C}_1 t\},
\] (5.12)
for any \( t \in [0, \infty) \), provided that
\[ m_0 \leq \min \{\varepsilon_0, \tilde{\varepsilon}_0\}, \]
for some \( \tilde{\varepsilon}_0 > 0 \) depending on \( \mu, \lambda, \kappa, K_1, K_2, \bar{\rho} \), and some other known constants but independent of \( t \).

Remark 5.7 The decay rate of \( \|\nabla \theta\|_{L^2} \) is still unknown.
Proof. By (4.25), Corollary 4.2 and (4.3), we have

\[
\int_{\mathbb{R}^3} \rho |\dot{u}|^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \left( \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right)
\leq \frac{C}{(2\mu + \lambda)^2} \rho L^6_\theta \| \nabla \theta \|^2_{L^2} \| \nabla u \|^2_{L^2} + \frac{C}{\mu^2} \| \nabla u \|^6_{L^2} + \frac{C}{\mu^4} \| \nabla u \|^4_{L^2} \| \rho \theta \|^2_{L^2} \\
+ \frac{2\rho \kappa^2}{(2\mu + \lambda)^2} \| \nabla \theta \|^2_{L^2}
\leq \frac{C\tilde{E}m_0}{(2\mu + \lambda)^2} \| \nabla \theta \|^2_{L^2} + \frac{C\tilde{E}^2}{\mu^2} \| \nabla u \|^2_{L^2} + \frac{C\tilde{E}}{\mu^4} \| \nabla u \|^4_{L^2} + \frac{C\kappa^2}{(2\mu + \lambda)^2} \| \nabla \theta \|^2_{L^2}
= M_1 \| \nabla \theta \|^2_{L^2} + M_2 \| \nabla u \|^2_{L^2},
\]

where \( M_1 = \left( \frac{C\tilde{E}m_0}{(2\mu + \lambda)^2} + \frac{C\kappa}{(2\mu + \lambda)^4} \right) \), and \( M_2 = \left( \frac{C\tilde{E}^2}{\mu^2} + \frac{C\tilde{E}}{\mu^4} \right) \).

By (4.25), we have

\[
\kappa \int_{\mathbb{R}^3} \| \nabla \theta \|^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \rho \| \theta \|^2 
\leq \frac{C(2\mu + \lambda)^2}{\kappa \mu} \| \nabla u \|_{L^2} \| \sqrt{\rho \dot{u}} \|_{L^2} + \frac{C}{\kappa} \| \nabla u \|^2_{L^2} \| \rho \theta \|_{L^2} \| \sqrt{\rho \dot{u}} \|_{L^2}
+ \frac{C}{\kappa \mu} M_1 \sqrt{\tilde{E}} \| \nabla \theta \|^2_{L^2}
\leq \frac{1}{2} \| \sqrt{\rho \dot{u}} \|^2_{L^2} + \left( \frac{C(2\mu + \lambda)^4 \tilde{E}^2 M^2_1}{\kappa^4 \mu^2} + \frac{C M^2_1 \tilde{E}}{\kappa^4} \right) \| \nabla u \|^2_{L^2}
+ \frac{C m_0^4 M_1 \sqrt{\tilde{E}}}{\kappa} \| \nabla \theta \|^2_{L^2}.
\]

Multiplying (5.14) by \( \frac{2M_1}{\kappa} \), and using Corollary 4.2 and (4.3), and Cauchy inequality, we have

\[
2M_1 \int_{\mathbb{R}^3} \| \nabla \theta \|^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \rho \| \theta \|^2 
\leq \frac{C(2\mu + \lambda)^2 \tilde{E} M_1}{\kappa^2 \mu} \| \nabla u \|^2_{L^2} \| \sqrt{\rho \dot{u}} \|^2_{L^2} + \frac{C M_1 \sqrt{\tilde{E}}}{\kappa^2} \| \nabla u \|^2_{L^2} \| \sqrt{\rho \dot{u}} \|^2_{L^2}
+ \frac{C m_0^4 M_1 \sqrt{\tilde{E}}}{\kappa} \| \nabla \theta \|^2_{L^2}
\leq \frac{1}{2} \| \sqrt{\rho \dot{u}} \|^2_{L^2} + \left( \frac{C(2\mu + \lambda)^4 \tilde{E}^2 M^2_1}{\kappa^4 \mu^2} + \frac{C M^2_1 \tilde{E}}{\kappa^4} \right) \| \nabla u \|^2_{L^2}
+ \frac{C m_0^4 M_1 \sqrt{\tilde{E}}}{\kappa} \| \nabla \theta \|^2_{L^2}.
\]

Adding (5.15) into (5.13), we have

\[
M_1 \int_{\mathbb{R}^3} \| \nabla \theta \|^2 + \int_{\mathbb{R}^3} \rho \| \theta \|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \rho \| \dot{u} \|^2 + \frac{d}{dt} \int_{\mathbb{R}^3} \left( \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right)
\leq M_2 \| \nabla u \|^2_{L^2} + \left( \frac{C(2\mu + \lambda)^4 \tilde{E}^2 M^2_1}{\kappa^4 \mu^2} + \frac{C M^2_1 \tilde{E}}{\kappa^4} \right) \| \nabla u \|^2_{L^2} + \frac{C m_0^4 M_1 \sqrt{\tilde{E}}}{\kappa} \| \nabla \theta \|^2_{L^2}
\leq M_3 \| \nabla u \|^2_{L^2} + \frac{M_1}{2} \| \nabla \theta \|^2_{L^2},
\]

provided that

\[
m_0 \leq \min \left\{ \varepsilon_0, \frac{\kappa^6}{26 C^6 \tilde{E}^3} \right\}.
\]

Here \( M_3 = M_2 + \frac{C(2\mu + \lambda)^4 \tilde{E}^2 M^2_1}{\kappa^4 \mu^2} + \frac{C M^2_1 \tilde{E}}{\kappa^4} \).
Thus,
\[
\int_{\mathbb{R}^3} \left( \frac{M_1}{2} |\nabla \theta|^2 + \frac{1}{2} \rho |\dot{u}|^2 \right) + \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{2M_1}{\kappa} \rho |\theta|^2 + \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right) \leq M_3 \|\nabla u\|^2_{L^2}. \tag{5.16}
\]

Multiplying (4.9) by 2 \(\mu\), and adding the resulting inequality into (5.16), we have
\[
\int_{\mathbb{R}^3} \left( M_3 |\nabla u|^2 + \frac{M_1}{2} |\nabla \theta|^2 + \frac{1}{2} \rho |\dot{u}|^2 \right) + \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{M_3}{\mu} \rho |u|^2 + \frac{2M_1}{\kappa} \rho |\theta|^2 + \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right) \leq \frac{C m_0^2 M_3}{\mu(\mu + \lambda)} \|\nabla \theta\|^2_{L^2} \leq \frac{M_1}{4} \|\nabla \theta\|^2_{L^2}, \tag{5.17}
\]
provided that
\[m_0 \leq \min \{\varepsilon_0, \tilde{\varepsilon}_0\},\]
where
\[\tilde{\varepsilon}_0 = \min \left\{ \frac{\kappa^6}{2\eta \eta' E_3}, \frac{\mu^2 (\mu + \lambda)^2 M_1^2}{8C^2 M_3^2} \right\}.\]

By (5.17), together with the facts
\[
\int_{\mathbb{R}^3} \rho |u|^2 \leq \|\rho\|_{L^\frac{3}{2}} \|u\|^2_{L^6} \leq \tilde{C} \|\nabla u\|^2_{L^2},
\]
\[
\int_{\mathbb{R}^3} \rho |\theta|^2 \leq \tilde{C} \|\nabla \theta\|^2_{L^2},
\]
\[
\int_{\mathbb{R}^3} |\text{curl} u|^2 \leq \tilde{C} \|\nabla \text{curl} u\|^2_{L^\frac{6}{5}} \leq \tilde{C} \|\sqrt{\rho} \dot{u}\|^2_{L^\frac{6}{5}} \leq \tilde{C} \|\sqrt{\rho} u\|^2_{L^3} \leq \tilde{C} \|\text{ curl} u\|^2_{L^2},
\]
and
\[
\int_{\mathbb{R}^3} |G|^2 \leq \tilde{C} \|\sqrt{\rho} \dot{u}\|^2_{L^2},
\]
we get
\[
\tilde{C}_1 \int_{\mathbb{R}^3} \left( \frac{M_3}{\mu} \rho |u|^2 + \frac{2M_1}{\kappa} \rho |\theta|^2 + \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right) + \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{M_3}{\mu} \rho |u|^2 + \frac{2M_1}{\kappa} \rho |\theta|^2 + \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right) \leq 0, \tag{5.18}
\]
for some constant \(\tilde{C}_1 > 0\) depending on \(\mu, \lambda, \kappa, M_1, M_3\) and other known constants but independent of \(t\). (5.18) deduces
\[
\int_{\mathbb{R}^3} \left( \frac{M_3}{\mu} \rho |u|^2 + \frac{2M_1}{\kappa} \rho |\theta|^2 + \mu |\text{curl} u|^2 + \frac{G^2}{2\mu + \lambda} \right) \leq A \exp\{-\tilde{C}_1 t\}, \tag{5.19}
\]
where
\[A = \int_{\mathbb{R}^3} \left( \frac{M_3}{\mu} \rho_0 |u_0|^2 + \frac{2M_1}{\kappa} \rho_0 |\theta_0|^2 + \mu |\text{curl} u_0|^2 + \frac{G_0^2}{2\mu + \lambda} \right).\]

By (5.9) and (5.19), we get (5.12). \(\square\)
6 Global classical solution

The proof of local existence and uniqueness of the classical solution as in Theorem 2.4.2 can be found in Section 7 (see Appendix B below). Let \( T_1^* > 0 \) be the maximal existence time of the classical solution. Our aim is to prove \( T_1^* = \infty \). To do this, we use contradiction arguments similar to Section 4.

More precisely, we assume that \( 0 < T_1^* < \infty \). In this section, we denote generic constants by \( \tilde{C} \) depending on the initial data, \( \mu, \lambda, \kappa, T_1^* \) and some other known constants but independent of \( t \in [0, T_1^*) \). In this case, we shall prove

\[
\|\rho(\cdot, t)\|_{H^2 \cap W^{2, q}} \leq \tilde{C}, \tag{6.1}
\]

for any \( t \in [0, T_1^*) \), and

\[
\|\nabla u(\cdot, t)\|_{H^1} + \|\nabla \theta(\cdot, t)\|_{H^1} \leq \tilde{C}, \tag{6.2}
\]

for any \( t \in [0, T_1^*) \), and

\[
\|\sqrt{\rho}u_t(\cdot, t)\|_{L^2} + \|\sqrt{\rho} \theta_t(\cdot, t)\|_{L^2} \leq \tilde{C}, \tag{6.3}
\]

for a.e. \( t \in [0, T_1^*) \). With (6.1), (6.2) and (6.3), we can define a new initial data at \( T_1^* \)

\[
(\rho(\cdot, T_1^*), u(\cdot, T_1^*), \theta(\cdot, T_1^*)) = \lim_{t \to T_1^*} (\rho(\cdot, t), u(\cdot, t), \theta(\cdot, t)), \tag{6.4}
\]

which satisfies the conditions of Appendix B. This means that the life span of the classical solution beyond \( T_1^* \), which is the desired contradiction.

To get (6.1), (6.2) and (6.3), we begin with the following lemma which is essentially obtained in Section 4.

**Lemma 6.1** Under the conditions of Theorem 2.4.2, it holds that

\[
\|\rho(\cdot, t)\|_{L^\infty} + \|\rho(\cdot, t)\|_{H^1 \cap W^{1, q}} + \|\rho_t(\cdot, t)\|_{L^2 \cap L^q} \leq \tilde{C}, \tag{6.5}
\]

\[
\|u(\cdot, t)\|_{L^\infty} + \|\nabla u(\cdot, t)\|_{H^1} + \|\nabla \theta(\cdot, t)\|_{H^1} + \|\nabla^2 u\|_{L^2([0, t]; L^q)} + \|\nabla^2 \theta\|_{L^2([0, t]; L^q)} \leq \tilde{C}, \tag{6.6}
\]

\[
\|\nabla u_t\|_{L^2([0, t]; L^2)} + \|\nabla \theta_t\|_{L^2([0, t]; L^2)} + \|\sqrt{\rho} u_t(\cdot, t)\|_{L^2} + \|\sqrt{\rho} \theta_t(\cdot, t)\|_{L^2} \leq \tilde{C}, \tag{6.7}
\]

for a.e. \( t \in [0, T_1^*) \).

From Lemma 6.1, (6.2) and (6.3) have been obtained. What we need to do is to get (6.1).

**Lemma 6.2** Under the conditions of Theorem 2.4.2, it holds that

\[
\|\rho(\cdot, t)\|_{H^2} + \int_0^t \|\nabla u(\cdot, t)\|_{H^2}^2 dt \leq \tilde{C}, \tag{6.8}
\]

for any \( t \in [0, T_1^*) \).

**Proof.** Taking \( \nabla^2 \) on both sides of (2.1), we have

\[
\nabla^2 \rho_t + 2 \nabla u^j \otimes \nabla \nabla_j \rho + u^j \nabla^2 \nabla_j \rho + \nabla^2 u^j \nabla_j \rho + \nabla^2 \rho \text{div} u + 2 \nabla \rho \otimes \nabla \text{div} u + \rho \nabla^2 \text{div} u = 0. \tag{6.9}
\]
Multiplying \((6.9)\) by \(\nabla^2 \rho\), integrating by parts over \(\mathbb{R}^3\), and using the Cauchy inequality, the Hölder inequality, the Sobolev inequality and \((6.5)\), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 \rho|^2 \leq \tilde{C} \|\nabla u\|_{L^\infty} \int_{\mathbb{R}^3} |\nabla^2 \rho|^2 + \tilde{C} \int_{\mathbb{R}^3} |\nabla^2 \rho| |\nabla^3 u| + \tilde{C} \int_{\mathbb{R}^3} |\nabla^2 \rho| |\nabla^3 u|
\]

\[
\leq \tilde{C} \|\nabla u\|_{L^\infty} + 1 \int_{\mathbb{R}^3} |\nabla^2 \rho|^2 + \tilde{C} \|\nabla^2 u\|_{L^6}^2 \|\nabla \rho\|_{L^3}^2 + \tilde{C} \int_{\mathbb{R}^3} |\nabla^3 u|^2
\]

\[
\leq \tilde{C} \|\nabla u\|_{L^\infty} + 1 \int_{\mathbb{R}^3} |\nabla^2 \rho|^2 + \tilde{C} \int_{\mathbb{R}^3} |\nabla^3 u|^2.
\]

By \((2.1)_2\) and the standard elliptic estimates, we have

\[
\|\nabla^3 u\|_{L^2}^2 \leq \tilde{C} \|\nabla^2 \rho\|_{L^2}^2 \|\nabla u\|_{L^6}^2 + \tilde{C} \|\nabla u\|_{L^6}^2 \|\nabla \rho\|_{L^3}^2 + \tilde{C} \|\nabla^2 \rho\|_{L^2}^2 + \tilde{C} \|\nabla u\|_{L^4}^4
\]

\[
+ \tilde{C} \|\nabla^2 u\|_{L^2}^2 + \tilde{C} \|\nabla^2 \rho\|_{L^2}^2 + \tilde{C} \|\nabla \rho\|_{L^6}^2 \|\nabla \theta\|_{L^6}^2 + \tilde{C} \|\nabla^2 \theta\|_{L^2}^2
\]

\[
\leq \tilde{C} \|\nabla u\|_{L^2}^2 + \tilde{C} \|\nabla^2 \rho\|_{L^2}^2 + \tilde{C},
\]

where we have used the Hölder inequality, the Sobolev inequality, \((6.5)\) and \((6.6)\).

Substituting \((6.11)\) into \((6.10)\), and using \((6.5)\), \((6.6)\) and the Sobolev inequality again, together with \((6.7)\) and the Gronwall inequality, we have

\[
\|\rho(\cdot, t)\|_{H^2} \leq \tilde{C},
\]

for any \(t \in [0, T_1)\). By \((6.6)\), \((6.7)\), \((6.11)\), \((6.12)\), we get

\[
\int_0^t \|\nabla u(\cdot, t)\|_{H^2}^2 dt \leq \tilde{C},
\]

for any \(t \in [0, T_1)\). This completes the proof of Lemma \(6.2\).

\[\square\]

Corollary 6.3 Under the conditions of Theorem 2.4.2, it holds that

\[
\|\rho_t(\cdot, t)\|_{H^1} + \int_0^t \|\rho_{tt}\|_{L^2}^2 \leq \tilde{C},
\]

for a.e. \(t \in [0, T_1^\ast)\).

**Proof.** Taking \(\nabla\) on both sides of \((2.1)_1\), we have

\[
\nabla \rho_t = -\nabla (\rho \text{div} u + u \cdot \nabla \rho) = -\nabla \rho \text{div} u - \rho \nabla \text{div} u - \nabla u \cdot \nabla \rho - u \cdot \nabla \nabla \rho.
\]

This, together with \((6.5)\), the Hölder inequality, the Sobolev inequality, \((6.6)\) and \((6.8)\), deduces

\[
\|\rho_t(\cdot, t)\|_{H^1} = \|\rho_t(\cdot, t)\|_{L^2} + \|\nabla \rho_t(\cdot, t)\|_{L^2}
\]

\[
\leq \tilde{C} + \tilde{C} \|\nabla \rho(\cdot, t)\|_{L^2} \|\text{div} u(\cdot, t)\|_{L^6} + \tilde{C} \|\rho(\cdot, t)\|_{L^\infty} \|\nabla \text{div} u(\cdot, t)\|_{L^2}
\]

\[
+ \tilde{C} \|\nabla u(\cdot, t)\|_{L^6} \|\nabla \rho(\cdot, t)\|_{L^3} + \tilde{C} \|u(\cdot, t)\|_{L^\infty} \|\nabla^2 \rho(\cdot, t)\|_{L^2}
\]

\[
\leq \tilde{C}.
\]

Using \((2.1)_1\) again, similar to \((6.14)\), we have

\[
\|\rho_{tt}\|_{L^2} \leq \|((\rho \text{div} u + u \cdot \nabla \rho)_t\|_{L^2} = \|\rho_t \text{div} u + \rho \text{div} u_t + u_t \cdot \nabla \rho + u \cdot \nabla \rho_t\|_{L^2}
\]

\[
\leq \|\rho_t\|_{L^6} \|\text{div} u\|_{L^3} + \|\rho\|_{L^\infty} \|\text{div} u_t\|_{L^2} + \|u_t\|_{L^6} \|\nabla \rho\|_{L^3} + \|u\|_{L^\infty} \|\nabla \rho_t\|_{L^2}
\]

\[
\leq \tilde{C} \|\nabla u_t\|_{L^2} + \tilde{C}.
\]

By \((6.15)\) and \((6.7)\), we get

\[
\int_0^t \|\rho_t\|_{L^2}^2 dt \leq \tilde{C}.
\]

for a.e. \(t \in [0, T_1^\ast)\). The proof of Corollary 6.3 is complete. \[\square\]
Lemma 6.4 Under the conditions of Theorem 2.4.2 it holds that
\[ \int_{\mathbb{R}^3} t |\nabla u_t|^2 + \int_0^t \int_{\mathbb{R}^3} s \rho |u_{ss}|^2 \leq \tilde{C}, \]
(6.16)
for a.e. \( t \in [0, T_1) \).

Proof. Differentiating (2.1) w.r.t. \( t \), we have
\[ \rho u_{tt} + \rho_t u_t + \rho u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t + \nabla P_t = \mu \Delta u_t + (\mu + \lambda) \nabla \text{div} u_t. \]
(6.17)
Multiplying (6.17) by \( u_{tt} \), integrating by parts over \( \mathbb{R}^3 \), we have
\[
\begin{align*}
\int_{\mathbb{R}^3} \rho |u_{tt}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu |\nabla u_t|^2 + (\mu + \lambda) |\text{div} u_t|^2) \\
= - \int_{\mathbb{R}^3} (\rho u_t + \rho u \cdot \nabla u) \cdot u_{tt} - \int_{\mathbb{R}^3} (\rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t) \cdot u_{tt} - \int_{\mathbb{R}^3} \nabla P_t \cdot u_{tt} \\
= \sum_{i=1}^3 VI_i.
\end{align*}
\]
(6.18)
For \( VI_1 \), using (2.1), integration by parts, the Hölder inequality, the Sobolev inequality, (6.5), (6.6), (6.7) and (6.13), we have
\[
\begin{align*}
VI_1 = & -\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho_t |u_t|^2 + \rho_t (u \cdot \nabla) u \cdot u_t \right) + \frac{1}{2} \int_{\mathbb{R}^3} \rho u_t |u_t|^2 \\
& + \int_{\mathbb{R}^3} \left( \rho u_t \cdot \nabla u + \rho_t u_t \cdot \nabla u + \rho u \cdot \nabla u_t \right) \cdot u_t \\
\leq & -\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho_t |u_t|^2 + \rho_t (u \cdot \nabla) u \cdot u_t \right) + \int_{\mathbb{R}^3} (\rho_t u + \rho u_t) \cdot \nabla u_t \cdot u_t \\
& + \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|u_t\|_{L^6} \|u_t\|_{L^6} \|\rho_t\|_{L^2} \|u_t\|_{L^2} + C \|\rho u_t\|_{L^2}^2 \\
& + \|\sqrt{\rho} \|_{L^\infty} \|u_t\|_{L^2} \|\nabla u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} + \tilde{C} \|\rho u_t\|_{L^2}^2 \\
\leq & -\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho_t |u_t|^2 + \rho_t (u \cdot \nabla) u \cdot u_t \right) + C \|\nabla u_t\|_{L^2} \left( \|\nabla u_t\|_{L^2} + \|\nabla^2 u_t\|_{L^2} \right) \|\nabla u_t\|_{L^2} \\
& + \tilde{C} \|\rho u_t\|_{L^2}^2.
\end{align*}
\]
(6.19)
For \( VI_2 \), using the Hölder inequality, the Sobolev inequality, (6.5) and (6.6) again, we have
\[
\begin{align*}
VI_2 \leq & \frac{1}{4} \int_{\mathbb{R}^3} \rho |u_{tt}|^2 + \tilde{C} \|\rho\|_{L^\infty} \|u_{tt}\|_{L^6}^2 \|\nabla u\|_{L^3}^2 + \tilde{C} \|\rho\|_{L^\infty} \|u\|_{L^\infty}^2 \|\nabla u_t\|_{L^2}^2 \\
\leq & \frac{1}{4} \int_{\mathbb{R}^3} \rho |u_{tt}|^2 + \tilde{C} \|\nabla u_t\|_{L^2}^2.
\end{align*}
\]
(6.20)
For \( VI_3 \), we have
\[
\begin{align*}
VI_3 = & \frac{d}{dt} \int_{\mathbb{R}^3} P_t \text{div} u_t - \int_{\mathbb{R}^3} P_{tt} \text{div} u_t \\
= & \frac{d}{dt} \int_{\mathbb{R}^3} P_t \text{div} u_t - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P_{tt} G_t - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P_t^2.
\end{align*}
\]
(6.21)
By (2.1) and (2.3), we have

\[ P_t = - \div (\rho u_t) + (\rho_t \div u_t) + \mu (\nabla u + (\nabla u')^t) : (\nabla u_t + (\nabla u_t')^t) + 2\lambda (\div u) \div u_t + \kappa \Delta \theta_t. \]

Substituting this equality into the second term of the right side of (6.21), and using integration by parts, the Hölder inequality, the Sobolev inequality, (6.5), (6.6) and (6.7), we have

\[
- \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P_t G_t = - \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} (\rho u_t) \cdot \nabla G_t + \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} G_t (\rho \div u) \\
- \frac{\mu}{2\mu + \lambda} \int_{\mathbb{R}^3} G_t (\nabla u + (\nabla u')^t) : (\nabla u_t + (\nabla u_t')^t) \\
- \frac{2\lambda}{2\mu + \lambda} \int_{\mathbb{R}^3} G_t (\div u) \div u_t + \frac{\kappa}{2\mu + \lambda} \int_{\mathbb{R}^3} \nabla G_t \cdot \nabla \theta_t 
\leq \tilde{C} \| (\rho u_t) \|_{L^2} \| \nabla G_t \|_{L^2} + \tilde{C} \| G_t \|_{L^2} \| \rho u_t \|_{L^2} \\
+ \tilde{C} \| G_t \|_{L^6} \| \rho \div u \|_{L^{5/6}} + \tilde{C} \| G_t \|_{L^6} \| \rho \div u \|_{L^{5/6}} \\
+ \tilde{C} \| G_t \|_{L^6} \| \nabla u \|_{L^2} + \tilde{C} \| G_t \|_{L^2} \| \nabla \theta_t \|_{L^2} 
\leq \tilde{C} \| \nabla G_t \|_{L^2} (\| \nabla u_t \|_{L^2} + \| \nabla \theta_t \|_{L^2} + 1) + \tilde{C} \| \nabla u_t \|_{L^2} + \tilde{C}. \tag{6.22}
\]

It follows from (3.13) that

\[
\| \nabla G_t \|_{L^2} \leq \tilde{C} \| (\rho_t u_t + \rho u \cdot \nabla u_t) \|_{L^2} \\
\leq \tilde{C} \| \rho_t \|_{L^3} \| u_t \|_{L^6} + \tilde{C} \| \sqrt{\rho} \|_{L^\infty} \| \sqrt{\rho u_t} \|_{L^2} + \tilde{C} \| \rho_t \|_{L^3} \| u \|_{L^\infty} \| \nabla u \|_{L^6} \\
+ \tilde{C} \| \rho \|_{L^\infty} \| u_t \|_{L^6} \| \nabla u \|_{L^3} + \tilde{C} \| \rho \|_{L^\infty} \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} \\
\leq \tilde{C} \| \nabla u_t \|_{L^2} + \tilde{C} \| \sqrt{\rho u_t} \|_{L^2} + \tilde{C}. \tag{6.23}
\]

where we have used the Hölder inequality, the Sobolev inequality, (6.5) and (6.6).

Substituting (6.23) into (6.22), and using the Cauchy inequality, we have

\[
- \frac{1}{2\mu + \lambda} \int_{\mathbb{R}^3} P_t G_t \leq \frac{1}{4} \int_{\mathbb{R}^3} \rho |u_t|^2 + \tilde{C} \| \nabla u_t \|_{L^2}^2 + \tilde{C} \| \nabla \theta_t \|_{L^2}^2 + \tilde{C}. \tag{6.24}
\]

Putting (6.19), (6.20), (6.21) and (6.24) into (6.18), and using the Cauchy inequality, we have

\[
\frac{1}{2} \int_{\mathbb{R}^3} \rho |u_t|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (\mu |\nabla u_t|^2 + (\mu + \lambda) |\div u_t|^2) \\
\leq - \frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho_t |u_t|^2 + \rho_t (u \cdot \nabla) u \cdot u_t \right) + \frac{d}{dt} \int_{\mathbb{R}^3} P_t \div u_t - \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \int_{\mathbb{R}^3} P_t^2 \\
+ \epsilon \| \nabla^2 u_t \|_{L^2}^2 + \tilde{C} \| \nabla^2 u_t \|_{L^2}^2 + \tilde{C} \| \rho u_t \|_{L^2} + \tilde{C} \| \nabla \theta_t \|_{L^2}^2 + \tilde{C}, \tag{6.25}
\]

for \( \epsilon > 0 \) to be decided later.

By (6.17) and the elliptic estimates, together with the Hölder inequality, the Sobolev inequality, (6.5), (6.6) and (6.13), we have

\[
\| \nabla^2 u_t (\cdot, t) \|_{L^2} \leq \tilde{C} \| \rho u_t \|_{L^2} + \tilde{C} \| \rho u_t \|_{L^2} + \tilde{C} \| \rho u \cdot \nabla u \|_{L^2} + \tilde{C} \| \rho u_t \cdot \nabla u \|_{L^2} + \tilde{C} \| \rho u_t \cdot \nabla u \|_{L^2} \\
+ \tilde{C} \| \rho \|_{L^\infty} \| u_t \|_{L^6} \| \nabla u \|_{L^3} + \tilde{C} \| \rho \|_{L^\infty} \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} + \tilde{C} \| \rho u_t \|_{L^2} \| \theta \|_{L^\infty} \\
+ \tilde{C} \| \rho \|_{L^\infty} \| u_t \|_{L^6} \| \nabla u \|_{L^3} + \tilde{C} \| \rho \|_{L^\infty} \| u \|_{L^\infty} \| \nabla u_t \|_{L^2} + \tilde{C} \| \rho u_t \|_{L^2} \| \theta \|_{L^\infty} \\
\leq \tilde{C} \| \sqrt{\rho u_t} \|_{L^2} + \tilde{C} \| \nabla u_t \|_{L^2} + \tilde{C} \| \nabla \theta_t \|_{L^2} + \tilde{C}. \tag{6.26}
\]
Substituting (6.26) into (6.25), taking \( \epsilon \) sufficiently small, and then multiplying the result by \( t \), we have

\[
\frac{1}{4} \int_{\mathbb{R}^3} tp|u_t|^2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} t (\mu|\nabla u_t|^2 + (\mu + \lambda)|\text{div} u_t|^2)
\leq \frac{1}{2} \int_{\mathbb{R}^3} (\mu|\nabla u_t|^2 + (\mu + \lambda)|\text{div} u_t|^2) - \frac{d}{dt} \int_{\mathbb{R}^3} t \left( \frac{1}{2}\rho_t|u_t|^2 + \rho_t(u \cdot \nabla)u \cdot u_t \right)
+ \int_{\mathbb{R}^3} \left( \frac{1}{2}\rho_t|u_t|^2 + \rho_t(u \cdot \nabla)u \cdot u_t \right) + \frac{d}{dt} \int_{\mathbb{R}^3} tP_t \text{div} u_t - \int_{\mathbb{R}^3} P_t \text{div} u_t
- \frac{1}{2(2\mu + \lambda)} \int_{\mathbb{R}^3} (P_s + Ct\|\nabla u_t\|^2_{L^2} + Ct\|\mu_t\|^2_{L^2}) + Ct\|\nabla \theta_t\|^2_{L^2} + \tilde{C}.
\] (6.27)

Integrating (6.27) over \([0, t]\) for \( t \in [0, T^*_1]\), and using (2.11), integration by parts, (6.5), (6.6), (6.7), (6.13), the Cauchy inequality and the Hölder inequality, we have

\[
\frac{1}{4} \int_0^t \int_{\mathbb{R}^3} s|\nabla u_s|^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} t (\mu|\nabla u_t|^2 + (\mu + \lambda)|\text{div} u_t|^2)
\leq - \int_{\mathbb{R}^3} t (\rho \cdot \nabla) u_t \cdot u_t + \int_{\mathbb{R}^3} (\rho \cdot \nabla) u_s \cdot u_s + \frac{1}{2(2\mu + \lambda)} \int_0^t \int_{\mathbb{R}^3} P_s + C
\leq \frac{\mu t}{8} \int_{\mathbb{R}^3} |\nabla u_t|^2 + \tilde{C} t\|\rho_t\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^3} \|u_t\|_{L^6} + \tilde{C}
\leq \frac{\mu t}{4} \int_{\mathbb{R}^3} |\nabla u_t|^2 + \tilde{C}.
\]

This gives (6.13). \(\square\)

**Corollary 6.5** Under the conditions of Theorem 2.4.2, it holds that

\[
\|\nabla u(\cdot, t)\|^2_{H^2} + \int_0^t \int_{\mathbb{R}^3} s|\nabla^2 u_s|^2 \leq \tilde{C},
\] (6.28)

for a.e. \( t \in [0, T^*_1]\).

**Proof.** By (6.11), (6.12), (6.16) and (6.6), we have

\[
\|\nabla u(\cdot, t)\|^2_{H^2} \leq \tilde{C},
\] (6.29)

for a.e. \( t \in [0, T^*_1]\).

By (6.26), (6.7) and (6.16), we get

\[
\int_0^t \int_{\mathbb{R}^3} s|\nabla^2 u_s|^2 \leq \tilde{C},
\]

for a.e. \( t \in [0, T^*_1]\). \(\square\)

**Lemma 6.6** Under the conditions of Theorem 2.4.2, it holds that

\[
\|\nabla^2 \rho(\cdot, t)\|_{L^q} + \int_0^t \|\nabla \theta(\cdot, s)\|^2_{H^2} ds \leq \tilde{C},
\] (6.30)

for any \( t \in [0, T^*_1]\).
Proof. Multiplying (6.3.9) by $q|\nabla^2 \rho|^{q-2} \nabla^2 \rho$, integrating by parts over $\mathbb{R}^3$, and using the Hölder inequality, the Sobolev inequality and (6.5), we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 \rho|^q \leq \tilde{C} \||\nabla u\||_{L^\infty} \int_{\mathbb{R}^3} |\nabla^2 \rho|^q + \tilde{C} ||\nabla^2 \rho||^q_{L^q} ||\nabla^2 u||_{W^{1,q}} ||\nabla \rho||_{L^q} + C ||\nabla^2 \rho||^q_{L^q} ||\nabla^3 u||_{W^{1,q}} \]
\[
\leq \tilde{C} ||\nabla u||_{H^2} \int_{\mathbb{R}^3} |\nabla^2 \rho|^q + \tilde{C} ||\nabla^2 \rho||^q_{L^q} ||\nabla^2 u||_{W^{1,q}}.
\]

By (2.1), (2.1), and the elliptic estimates, together with the Sobolev inequality, (6.5), (6.6), (6.7), the Hölder inequality and the Gagliardo-Nirenberg inequality, we have

\[
||\nabla^2 u||_{W^{1,q}} \leq \tilde{C} ||\rho u||_{W^{1,q}} + \tilde{C} ||\rho u \cdot \nabla u||_{W^{1,q}} + \tilde{C} ||\nabla (\rho \theta)||_{W^{1,q}}
\]
\[
\leq \tilde{C} ||\nabla (\rho u)||_{L^2} + \tilde{C} ||\nabla (\rho u)||_{L^2} + \tilde{C} ||\nabla (\rho u \cdot \nabla u)||_{L^2} + \tilde{C} ||\nabla^2 (\rho \theta)||_{L^2} + \tilde{C}
\]
\[
\leq \tilde{C} ||\nabla u||_{L^2} + \tilde{C} ||\nabla u||_{L^2} + \tilde{C} ||\nabla u||_{L^2} + \tilde{C} ||\nabla^2 u||_{L^2} + \tilde{C} ||\nabla^3 u||_{L^2}
\]
\[
+ \tilde{C} ||\nabla^2 \rho||_{L^2} + \tilde{C} ||\nabla^3 \rho||_{L^2} + \tilde{C},
\]

and

\[
||\nabla^3 \theta||_{L^2} \leq \tilde{C} ||\rho \theta t + \rho \nabla \theta||_{L^2} + \tilde{C} ||\rho \nabla u \cdot \nabla \theta||_{L^2} + \tilde{C} ||\rho \nabla u \cdot \nabla \theta||_{L^2}
\]
\[
+ \tilde{C} ||\rho \nabla \theta ||_{L^2} + \tilde{C} ||\rho \nabla \theta ||_{L^2} + \tilde{C} ||\rho \nabla \theta ||_{L^2} + \tilde{C} ||\rho \nabla \theta ||_{L^2}
\]
\[
\leq \tilde{C} ||\nabla \theta||_{L^2} + \tilde{C} ||\nabla^3 u||_{L^2} + \tilde{C}.
\]

Substituting (6.32) and (6.33) into (6.31), and using the Young inequality, we have

\[
\frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 \rho|^q \leq \tilde{C} (||\nabla^3 u||_{L^2} + ||\nabla u||_{L^2} + ||\nabla \theta_t||_{L^2} + 1) \left( \int_{\mathbb{R}^3} |\nabla^2 \rho|^q + 1 \right)
\]
\[
+ \tilde{C} ||\nabla u||_{L^2} \frac{6-q}{2q} ||\nabla^2 u|| \frac{3q-6}{2q} \left( \int_{\mathbb{R}^3} |\nabla^2 \rho|^q + 1 \right).
\]

It is easy to see

\[
||\nabla u||_{L^2} \frac{6-q}{2q} ||\nabla^2 u|| \frac{3q-6}{2q} = (t ||\nabla u||_{L^2}^2) \frac{6-q}{4q} t^{-\frac{6-q}{4q}} (t ||\nabla^2 u||_{L^2}^2) \frac{3q-6}{4q} t^{-\frac{3q-6}{4q}}
\]
\[
\leq \tilde{C} t^{-\frac{q}{2q}} (t ||\nabla^2 u||_{L^2}^2) \frac{3q-6}{4q} t^{-\frac{3q-6}{4q}} (6.35)
\]
\[
\leq \tilde{C} t^{-\frac{q}{2q}} + \tilde{C} t ||\nabla^2 u||_{L^2}^2 \in L^1([0, T]),
\]

where we have used (6.16), (6.28), the Young inequality and $q < 6$.

By (6.7), (6.8), (6.35) and the Gronwall inequality, we have

\[
||\nabla^2 \rho(t)||_{L^q} \leq \tilde{C},
\]

for any $t \in [0, T^*_1)$. By (6.33), (6.7) and (6.8), we get

\[
\int_0^T ||\nabla \theta(t)||_{H^2}^2 \leq \tilde{C},
\]

for any $t \in [0, T^*_1)$.

□
7 Appendix

Appendix A (Zlotnik inequality)

Let the function \( y \) satisfy

\[
y'(t) = g(y) + b'(t) \quad \text{on} \quad [0, T], \quad y(0) = y^0,
\]

with \( g \in C(\mathbb{R}) \) and \( y, b \in W^{1,1}(0, T) \). If \( g(\infty) = -\infty \) and

\[
b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1)
\]

for all \( 0 \leq t_1 < t_2 \leq T \) with some \( N_0 \geq 0 \) and \( N_1 \geq 0 \), then

\[
y(t) \leq \max\{y^0, \bar{\zeta}\} + N_0 < \infty \quad \text{on} \quad [0, T],
\]

where \( \bar{\zeta} \) is a constant such that

\[
g(\zeta) \leq -N_1, \quad \text{for} \quad \zeta \geq \bar{\zeta}.
\]

Appendix B (Local classical solution) Assume that the initial data \((\rho_0, u_0, \theta_0)\) satisfies

\[
\rho_0 \geq 0, \quad \theta_0 \geq 0, \quad \text{in} \quad \mathbb{R}^3, \quad \rho_0 \in H^2 \cap W^{2,q}, \quad u_0 \in D^2 \cap D^1_0, \quad \theta_0 \in D^2 \cap D^1_0,
\]

(7.1)

for some \( q \in (3, 6) \), and the compatibility conditions

\[
\begin{cases}
\mu \Delta u_0 + (\mu + \lambda) \nabla \text{div} u_0 - \nabla P(\rho_0, \theta_0) = \sqrt{\rho_0} g_1, \\
\kappa \Delta \theta_0 + \frac{q}{2} |\nabla u_0 + (\nabla u_0)'|^2 + \lambda (\text{div} u_0)^2 = \sqrt{\rho_0} g_2, \quad x \in \mathbb{R}^3,
\end{cases}
\]

(7.2)

for some \( g_i \in L^2 \), \( i = 1, 2 \). Then there exist a positive constant \( T_0 > 0 \) and a unique classical solution \((\rho, u, \theta)\) in \( \mathbb{R}^3 \times [0, T_0] \) such that

\[
\rho \in C([0, T_0]; H^2 \cap W^{2,q}), \quad \rho_t \in C([0, T_0]; H^1), \quad \rho \geq 0, \quad \theta \geq 0 \quad \text{in} \quad \mathbb{R}^3 \times [0, T_0],
\]

(7.3)

\[
(u, \theta) \in C([0, T_0]; D^2 \cap D^1_0) \cap L^2([0, T_0]; D^3), \quad (u_t, \theta_t) \in L^2([0, T_0]; D^1_0),
\]

\[
(\sqrt{\rho u_t}, \sqrt{\rho \theta_t}) \in L^\infty([0, T_0]; L^2), \quad \sqrt{\rho} \sqrt{u_{tt}} \in L^2([0, T_0]; L^2), \quad u_{tt} \in L^\infty([0, T_0]; D^2), \quad \theta_{tt} \in L^2([0, T_0]; D^1_0),
\]

\[
tu \in L^\infty([0, T_0]; D^3), \quad t\theta_t \in L^\infty([0, T_0]; D^2), \quad t\theta_{tt} \in L^2([0, T_0]; D^1_0),
\]

Proof of Appendix B:

Using some arguments similar to [3, 4], we can construct a sequence of approximate classical solutions \((\rho^k, u^k, \theta^k)\) to \((7.1)-(7.3)\) satisfying

\[
\begin{cases}
\rho^k + \nabla \cdot (\rho^k u^{k-1}) = 0,
\rho^k u^k + \rho^k u^{k-1} \cdot \nabla u^k + \nabla P(\rho^k, \theta^k) = \mu \Delta u^k + (\mu + \lambda) \nabla \text{div} u^k,
\rho^k \theta^k + \rho^k u^{k-1} \cdot \nabla \theta^k + \rho^k \theta^{k-1} \text{div} u^k - \kappa \Delta \theta^k = \frac{\mu}{2} |\nabla u^{k-1} + (\nabla u^{k-1})'|^2 + \lambda (\text{div} u^{k-1})^2 + \kappa \Delta \theta^k,
\end{cases}
\]

(7.4)
with initial conditions

\[ (\rho^k, u^k, \theta^k)|_{t=0} = (\rho_0^\delta, u_0, \theta_0), \quad x \in \mathbb{R}^3, \quad (7.5) \]

and

\[ (\rho^k, u^k, \theta^k) \to (\delta, 0, 0) \text{ as } |x| \to \infty, \text{ for } t \geq 0, \quad (7.6) \]

where \( \rho_0^\delta = \rho_0 + \delta \) for \( \delta \in (0, 1) \), and \( k \geq 1 \). Here we take \((u^0, \theta^0) = (u_0, \theta_0)\).

From the compatibility condition \((7.2)\), we obtain

\[
\begin{aligned}
\mu \Delta u_0 + (\mu + \lambda) \nabla \Delta u_0 - \nabla P(\rho_0^\delta, \theta_0) &= \sqrt{\rho_0^\delta} g_1^\delta, \\
\kappa \Delta \theta_0 + \frac{\mu}{2} \left| \nabla u_0 + (\nabla u_0)' \right|^2 + \lambda (\nabla u_0)^2 &= \sqrt{\rho_0^\delta} g_2^\delta,
\end{aligned}
\]

where

\[ g_1^\delta = \left( \frac{\rho_0}{\rho_0^\delta} \right)^{\frac{1}{2}} g_1 - \frac{\delta}{\sqrt{\rho_0}} \nabla \theta_0, \quad \text{and} \quad g_2^\delta = \left( \frac{\rho_0}{\rho_0^\delta} \right)^{\frac{1}{2}} g_2. \]

It is easy to verify

\[ \| g_1^\delta \|_{L^2} \leq \| g_1 \|_{L^2} + \sqrt{\delta} \| \nabla \theta_0 \|_{L^2}, \quad \text{and} \quad \| g_2^\delta \|_{L^2} \leq \| g_2 \|_{L^2}. \quad (7.8) \]

**Step 1: Some estimates.**

From \([4]\) together with \((7.4)-(7.8)\), we get the following lemma.

**Lemma 7.1** Under the condition of \((7.1), (7.4)\) and \((7.8)\), there exists a constant \( T_0 \in (0,1) \) independent of \( k \) and \( \delta \), such that

\[ \rho^k > 0, \quad \| \rho^k (\cdot, t) \|_{L^\infty} + \| \rho^k (\cdot, t) - \delta \|_{H^1 \cap W^{1,4}} + \| \rho^k (\cdot, t) \|_{L^2 \cap L^4} \leq \tilde{C}, \quad (7.9) \]

\[ \| u^k (\cdot, t) \|_{L^\infty} + \| \nabla u^k (\cdot, t) \|_{H^1} + \| \nabla \theta^k (\cdot, t) \|_{H^1} + \| \nabla^2 u^k \|_{L^2([0,T];L^4)} + \| \nabla^2 \theta^k \|_{L^2([0,T];L^4)} \leq \tilde{C}, \quad (7.10) \]

\[ \| \nabla u^k \|_{L^2([0,T];L^2)} + \| \nabla^2 u^k \|_{L^2([0,T];L^2)} + \| \sqrt{\rho^k} u^k \|_{L^2} + \| \sqrt{\rho^k} \theta^k \|_{L^2} \leq \tilde{C}, \quad (7.11) \]

for any \( k \geq 1 \) and a.e. \( t \in [0,T_0] \), where \( \tilde{C} \) is independent of \( k, \delta \) and \( t \). Furthermore, \( \theta^k \geq \theta^0 \) a.e. \( t \in [0,T_0] \).

Based on **Lemma 7.1**, we derive the next lemma by using some arguments similar to Lemmas 6.2, 6.3 and Corollaries 5.3 and 6.5.

**Lemma 7.2** Under the condition of \((7.1), (7.4)\) and \((7.8)\), it holds that

\[
\left\{ \begin{array}{l}
\sqrt{T}\| \nabla u^k (\cdot, t) \|_{H^2} + \sqrt{T}\| \nabla u^k (\cdot, t) \|_{L^2} + \| \rho^k (\cdot, t) \|_{H^1} + \| \rho^k (\cdot, t) - \delta \|_{H^2 \cap W^{2,4}} \leq \tilde{C}, \\
\int_0^{T_0} \left( \| \rho^k \|_{L^2}^2 + \| \nabla u^k \|_{H^2}^2 + t \| \nabla^2 u^k \|_{L^2}^2 + t \| \nabla^2 \theta^k \|_{H^2}^2 \right) (\cdot, t) dt \leq \tilde{C},
\end{array} \right. \quad (7.12)\]

for any \( k \geq 1 \) and a.e. \( t \in [0,T_0] \).

---

\(^2\)This can be obtained by using \((7.4)\) and the maximal principle for the parabolic equation.

\(^3\)It is similar to **Lemma 6.1**.
We need some higher order estimates for \((\rho^k, u^k, \theta^k)\) which are included in the following lemmas.

**Lemma 7.3** Under the condition of (7.1), (7.7) and (7.8), it holds that
\[
\int_{\mathbb{R}^3} t^2 \left((\nabla \theta^k_t)^2 + \rho^k |u^k_t|^2\right) + \int_0^{T_0} \int_{\mathbb{R}^3} t^2 \left(\rho^k|\theta^k_t|^2 + |\nabla u^k_t|^2\right) \leq \bar{C},
\]
(7.13)
for any \(k \geq 1\) and a.e. \(t \in [0, T_0]\).

**Proof.** Differentiating (7.4) w.r.t. \(t\) two times, we have
\[
\rho^k u^k_{ttt} + 2\rho^k u^k_{tt} + \rho^k u^k_t + \rho_t^k u^k - \nabla u^k + 2\rho^k u^k_{k-1} \cdot \nabla u^k + 2\rho^k u^k_{k-1} \cdot \nabla u^k_t + \rho^k u^k_{k-1} \cdot \nabla u^k + \rho^k u^k_{k-1} \cdot \nabla u^k_t + \nabla P^k = \mu \Delta u^k_t + (\mu + \lambda) \nabla \text{div} u^k_t.
\]
(7.14)

Multiplying (7.14) by \(u^k_t\), integrating by parts over \(\mathbb{R}^3\), and using (7.3), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^k |u^k_t|^2 + \int_{\mathbb{R}^3} \left(\mu |\nabla u^k_t|^2 + (\mu + \lambda) |\text{div} u^k_t|^2\right)
\]
\[
= -2 \int_{\mathbb{R}^3} \rho_t^k u^k_{tt} - \int_{\mathbb{R}^3} \rho^k u^k_t \cdot u^k_t - \int_{\mathbb{R}^3} \rho_t^k u^k_{tt} u^k_t - 2 \int_{\mathbb{R}^3} \rho^k u^k_{k-1} \cdot \nabla u^k \cdot u^k_t
\]
\[
- 2 \int_{\mathbb{R}^3} \rho^k u^k_{k-1} \cdot \nabla u^k_t \cdot u^k_t - 2 \int_{\mathbb{R}^3} \rho^k u^k_{k-1} \cdot \nabla u^k_t \cdot u^k_t
\]
\[
+ \int_{\mathbb{R}^3} \rho^k |\nabla u^k_t|^2 = \sum_{i=1}^{8} VII_i.
\]
(7.15)

For \(VII_1\), using (7.4) and integration by parts again, together with the Cauchy inequality, (7.9) and (7.10), we have
\[
VII_1 = -4 \int_{\mathbb{R}^3} \rho^k u^k_{k-1} \cdot \nabla u^k_t \cdot u^k_t \leq \frac{\mu}{16} \int_{\mathbb{R}^3} |\nabla u^k_t|^2 + \bar{C} \int_{\mathbb{R}^3} \rho^k |u^k_t|^2.
\]
(7.16)

For \(VII_2\), we have
\[
VII_2 = - \int_{\mathbb{R}^3} \rho^k u^k_{k-1} \cdot \nabla (u^k_t \cdot u^k_t)
\]
\[
\leq \|\rho_t^k\|_{L^3} \|u^k - u^k_{k-1}\|_{L^\infty} \|\nabla u^k_t\|_{L^2} \|u^k_t\|_{L^6} + \|\rho^k\|_{L^3} \|u^k - u^k_{k-1}\|_{L^\infty} \|u^k_t\|_{L^6} \|\nabla u^k_t\|_{L^2}
\]
\[
+ \|\sqrt{\rho^k}\|_{L^\infty} \|\nabla u^k_t\|_{L^2} \|\nabla u^k_t\|_{L^2} + \|\rho^k u^k_t\|_{L^3} \|\rho^k u^k_{k-1} - \nabla u^k\|_{L^2}
\]
\[
\leq \frac{\mu}{16} \int_{\mathbb{R}^3} |\nabla u^k_t|^2 + \bar{C} \|\nabla u^k_t\|_{L^2} + \bar{C} \|\nabla u^k - u^k_{k-1}\|_{L^2} \|\nabla u^k_t\|_{L^2} \|\nabla u^k_t\|_{L^6}
\]
\[
+ \bar{C} \|\nabla u^k_t\|_{L^2}^2 + \bar{C} \|\nabla u^k_t\|_{L^2} \|\nabla u^k_t\|_{L^6} \|\nabla u^k - u^k_{k-1}\|_{L^2}
\]
\[
\leq \frac{\mu}{16} \int_{\mathbb{R}^3} |\nabla u^k_t|^2 + \bar{C} \|\nabla u^k_t\|_{L^2}^2 + \bar{C} \|\nabla u^k - u^k_{k-1}\|_{L^2}^2 + \bar{C} \|\nabla u^k_t\|_{L^2}^2 \|\nabla^2 u^k_t\|_{L^2}^2
\]
\[
+ \bar{C} \|\nabla u^k_t\|_{L^2}^2 + \bar{C},
\]
(7.17)
where we have used (7.3), integration by parts, the Hölder inequality, the Cauchy inequality, the Sobolev inequality, the Gagliardo-Nirenberg inequality, (7.9), (7.10) and (7.11).

Similarly, we have
\[
VII_3 + VII_4 + VII_5
\]
\[
\leq \bar{C} \|\rho_t^k\|_{L^2} \|u^k - u^k_{k-1}\|_{L^\infty} \|\nabla u^k_t\|_{L^2} \|u^k_t\|_{L^6} + \bar{C} \|\rho^k\|_{L^2} \|u^k - u^k_{k-1}\|_{L^6} \|\nabla u^k\|_{L^6} \|u^k_t\|_{L^6}
\]
\[
+ \bar{C} \|\rho^k\|_{L^2} \|u^k - u^k_{k-1}\|_{L^6} \|\nabla u^k_t\|_{L^6} \|u^k_t\|_{L^6}
\]
\[
\leq \frac{3\mu}{16} \int_{\mathbb{R}^3} |\nabla u^k_t|^2 + \bar{C} \|\rho^k\|_{L^2}^2 + \bar{C} \|\nabla u^k - u^k_{k-1}\|_{L^2}^2 + \bar{C} \|\nabla^2 u^k_t\|_{L^2},
\]
(7.18)
and

\[ \text{VII}_6 + \text{VII}_7 \leq \tilde{C} \| \rho^k u_t^k \|_{L^2} \| u_{tt}^k \|_{L^6} \| \nabla u^k \|_{L^3} + \tilde{C} \| \rho^k u_t^k \|_{L^2} \| u_{tt}^k \|_{L^6} \| \nabla u_t^k \|_{L^3} \]

\[ \leq \frac{\mu}{8} \int_{\mathbb{R}^3} |\nabla u_{tt}^k| \|^2 + \tilde{C} \sqrt{\rho \rho_t^k} \| u_{tt}^k \|^2 \leq \tilde{C} \| \nabla u_{tt}^k \|_{L^2} \| \nabla u_{tt}^k \|_{L^2} \| \nabla u_{tt}^k \|_{L^2} \]

\[ \leq \frac{\mu}{8} \int_{\mathbb{R}^3} |\nabla u_{tt}^k| \|^2 + \tilde{C} \sqrt{\rho \rho_t^k} \| u_{tt}^k \|^2 + \tilde{C} \| \nabla u_{tt}^k \|^4 \]

\[ \leq \frac{\mu}{8} \int_{\mathbb{R}^3} |\nabla u_{tt}^k| \|^2 + \tilde{C} \int_{\mathbb{R}^3} |\rho_t^k| \|^2 + \tilde{C} \int_{\mathbb{R}^3} |\nabla \theta_t^k| \|^2 + \tilde{C} \int_{\mathbb{R}^3} |\rho_t^k| \|^2. \]

Substituting (7.16), (7.17), (7.18), (7.19) and (7.20) into (7.15), and multiplying the result by \( t^2 \), we have

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} t^2 \rho^k \| u_{tt}^k \|_{L^2} \|^2 + \frac{5}{8} \int_{\mathbb{R}^3} t^2 \left( \mu |\nabla u_{tt}^k| \|^2 + (\mu + \lambda)|\text{div} u_{tt}^k| \|^2 \right) \]

\[ \leq \int_{\mathbb{R}^3} t^2 \rho^k \| u_{tt}^k \|_{L^2} \|^2 + \frac{\mu}{8} \int_{\mathbb{R}^3} t^2 |\nabla u_{tt}^k| \|^2 + \tilde{C} \int_{\mathbb{R}^3} t^2 \rho^k \| u_{tt}^k \|_{L^2} + \tilde{C} t^2 \| \nabla u_{tt}^k \|^4 \]

\[ \leq \tilde{C} t^2 \| \nabla u_{tt}^k \|^4 + \tilde{C} \| \rho_t^k \|^2_{L^2} + \tilde{C} \| \nabla u_{tt}^k \|^2 + \tilde{C} \| \nabla u_{tt}^k \|^2 \| \nabla u_{tt}^k \|^2 \]

\[ \leq \tilde{C} \int_{\mathbb{R}^3} t^2 |\nabla \theta_t^k| \|^2 + \tilde{C} \int_{\mathbb{R}^3} t^2 \rho^k |\theta_t^k| \|^2 + \tilde{C}. \]

Integrating (7.21) over \([0, t] \) for \( t \in [0, T_0] \), and using (7.11) and (7.12), for any given \( N \in \mathbb{Z}_+ \), we have

\[ \max_{1 \leq k \leq N} \int_{\mathbb{R}^3} t^2 \rho^k \| u_{tt}^k \|_{L^2} \|^2 + \max_{1 \leq k \leq N} \int_{\mathbb{R}^3} s^2 \| \nabla u_{ss}^k \|^2 \leq \tilde{C} \max_{1 \leq k \leq N} \int_{\mathbb{R}^3} t^2 \rho^k |\theta_{ss}^k| \|^2 + \tilde{C}. \]

Differentiating (7.4) w.r.t. \( t \), we have

\[ \rho^k \theta_{tt}^k + \rho_t^k \theta_{t}^k + \rho^k u_{tt}^k \cdot \nabla \theta^k + \rho^k u_{t}^k \cdot \nabla \theta^k + \rho^k u_{tt}^k \cdot \nabla \theta_{t}^k + \rho_t^k \theta_t^k \text{div} u_{tt}^k \]

\[ + \rho^k \theta_t^k \text{div} u_{tt}^k + \rho^k \theta_t^k \text{div} u_{tt}^k = \mu \left( \nabla u_{tt}^k + (\nabla u_{tt}^k)' \right) : \left( \nabla u_{tt}^k + (\nabla u_{tt}^k)' \right) \]

\[ + 2\lambda \text{div} u_{tt}^k \text{div} u_{tt}^k + \kappa \Delta \theta_t^k. \]

Multiplying (7.22) by \( \theta_t^k \), and integrating by parts over \( \mathbb{R}^3 \), we have

\[ \int_{\mathbb{R}^3} \rho^k |\theta_{tt}^k| \|^2 + \kappa \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla \theta_t^k| \|^2 \]

\[ = - \int_{\mathbb{R}^3} \rho^k \theta_t^k \theta_{tt}^k - \int_{\mathbb{R}^3} \rho^k u_{tt}^k \cdot \nabla \theta^k \theta_{tt}^k - \int_{\mathbb{R}^3} \rho^k u_{tt}^k \cdot \nabla \theta^k \theta_{tt}^k - \int_{\mathbb{R}^3} \rho^k u_{tt}^k \cdot \nabla \theta^k \theta_{tt}^k \]

\[ - \int_{\mathbb{R}^3} \rho_t^k \theta_t^k \text{div} u_{tt}^k \theta_t^k + \int_{\mathbb{R}^3} \rho^k \theta_t^k \text{div} u_{tt}^k \theta_t^k - \int_{\mathbb{R}^3} \rho_t^k \theta_t^k \text{div} u_{tt}^k \theta_t^k \]

\[ + \mu \int_{\mathbb{R}^3} \left( \nabla u_{tt}^k + (\nabla u_{tt}^k)' \right) : \left( \nabla u_{tt}^k + (\nabla u_{tt}^k)' \right) \theta_t^k + 2\lambda \int_{\mathbb{R}^3} \text{div} u_{tt}^k \text{div} u_{tt}^k \theta_t^k \]

\[ = \sum_{i=1}^{9} \text{VIII}_i. \]
For $VIII_1$ and $VIII_2$, similar to (6.19), we have

$$VIII_1 = -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} + \int_{\mathbb{R}^3} (\rho^k_{t} u^{k-1}_{t})_{t} \cdot \nabla \theta^k_{t}$$

$$\leq -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} + ||\rho^k_{t}||_{L^3} ||u^{k-1}_{t}||_{L^\infty} ||\nabla \theta^k_{t}||_{L^2} ||\theta^k_{t}||_{L^6}$$

$$+ ||\sqrt{\rho^k_{t}}||_{L^\infty} ||u^{k-1}_{t}||_{L^\infty} ||\nabla \theta^k_{t}||_{L^2} \sqrt{\rho^k_{t}} ||\theta^k_{t}||_{L^2}$$

$$\leq -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} + \tilde{C} ||\nabla \theta^k_{t}||_{L^2} + \tilde{C} ||\nabla u^{k-1}_{t}||_{L^2}^2 + \tilde{C} ||\nabla^2 u^{k-1}_{t}||_{L^2}^2,$$  \hspace{1cm} (7.25)

and

$$VIII_2 = -\frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} u^{k-1}_{t} \cdot \nabla \theta^k_{t} + \int_{\mathbb{R}^3} \rho^k_{t} u^{k-1}_{t} \cdot \nabla \theta^k_{t} + \int_{\mathbb{R}^3} \rho^k_{t} u^{k-1}_{t} \cdot \nabla \theta^k_{t}$$

$$+ \int_{\mathbb{R}^3} \rho^k_{t} u^{k-1}_{t} \cdot \nabla \theta^k_{t}$$

$$\leq -\frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} u^{k-1}_{t} \cdot \nabla \theta^k_{t} + ||\rho^k_{t}||_{L^2} ||u^{k-1}_{t}||_{L^\infty} ||\nabla \theta^k_{t}||_{L^3} ||\theta^k_{t}||_{L^6}$$

$$+ ||\sqrt{\rho^k_{t}}||_{L^\infty} ||u^{k-1}_{t}||_{L^\infty} ||\nabla \theta^k_{t}||_{L^2} \sqrt{\rho^k_{t}} ||\theta^k_{t}||_{L^2}$$

$$\leq -\frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} u^{k-1}_{t} \cdot \nabla \theta^k_{t} + \tilde{C} ||\nabla \theta^k_{t}||_{L^2} + \tilde{C} ||\nabla u^{k-1}_{t}||_{L^2}^2 + \tilde{C} ||\nabla^{2} u^{k-1}_{t}||_{L^2}^2.$$  \hspace{1cm} (7.26)

Similarly, for the rest terms of the right side of (7.24), we have

$$VIII_3 + VIII_4 + VIII_6 + VIII_7$$

$$\leq ||\sqrt{\rho^k_{t}}||_{L^\infty} ||\sqrt{\rho^k_{t}} \theta^k_{t}||_{L^2} \left( ||u^{k-1}_{t}||_{L^6} ||\nabla \theta^k_{t}||_{L^3} + ||u^{k-1}_{t}||_{L^\infty} ||\nabla \theta^k_{t}||_{L^2} \right)$$

$$+ ||\sqrt{\rho^k_{t}}||_{L^\infty} ||\sqrt{\rho^k_{t}} \theta^k_{t}||_{L^2} \left( ||\theta^k_{t}||_{L^6} ||\nabla \theta^k_{t}||_{L^3} + ||\theta^k_{t}||_{L^\infty} ||\nabla \theta^k_{t}||_{L^2} \right)$$

$$\leq \frac{1}{2} \frac{d}{dt} ||\sqrt{\rho^k_{t}} \theta^k_{t}||_{L^2}^2 + \tilde{C} ||\nabla u^{k-1}_{t}||_{L^2}^2 + \tilde{C} ||\nabla \theta^k_{t}||_{L^2}^2,$$  \hspace{1cm} (7.27)

and

$$VIII_5 = -\frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} \nabla u^{k-1}_{t} \cdot \theta^k_{t} + \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} \nabla u^{k-1}_{t} \cdot \theta^k_{t} + \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} \nabla u^{k-1}_{t} \cdot \theta^k_{t}$$

$$+ \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} \nabla u^{k-1}_{t} \cdot \theta^k_{t}$$

$$\leq -\frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} \nabla u^{k-1}_{t} \cdot \theta^k_{t} + ||\rho^k_{t}||_{L^2} ||\theta^k_{t}||_{L^\infty} ||\nabla u^{k-1}_{t}||_{L^3} ||\theta^k_{t}||_{L^6}$$

$$+ ||\rho^k_{t}||_{L^\infty} ||\theta^k_{t}||_{L^6}^2 ||\nabla u^{k-1}_{t}||_{L^6} + ||\rho^k_{t}||_{L^\infty} ||\theta^k_{t}||_{L\infty} ||\nabla u^{k-1}_{t}||_{L^2} ||\theta^k_{t}||_{L^6}$$

$$\leq -\frac{d}{dt} \int_{\mathbb{R}^3} \rho^k_{t} \theta^k_{t} \nabla u^{k-1}_{t} \cdot \theta^k_{t} + \tilde{C} ||\rho^k_{t}||_{L^2}^2 + \tilde{C} ||\nabla \theta^k_{t}||_{L^2}^2 + ||\nabla u^{k-1}_{t}||_{L^2}^2,$$  \hspace{1cm} (7.28)

and

$$VIII_8 \leq \frac{d}{dt} \int_{\mathbb{R}^3} \left( \nabla u^{k-1}_{t} + (\nabla u^{k-1})' \right) \cdot \left( \nabla u^{k-1}_{t} + (\nabla u^{k-1})' \right) \theta^k_{t}$$

$$+ \tilde{C} ||\nabla u^{k-1}_{t}||_{L^6} ||\nabla u^{k-1}_{t}||_{L^6} + \tilde{C} ||\nabla u^{k-1}_{t}||_{L^6} ||\nabla u^{k-1}_{t}||_{L^6} ||\theta^k_{t}||_{L^6}$$

$$\leq \frac{d}{dt} \int_{\mathbb{R}^3} \left( \nabla u^{k-1}_{t} + (\nabla u^{k-1})' \right) \cdot \left( \nabla u^{k-1}_{t} + (\nabla u^{k-1})' \right) \theta^k_{t}$$

$$+ \tilde{C} ||\nabla u^{k-1}_{t}||_{L^2} ||\nabla \theta^k_{t}||_{L^2} + \tilde{C} ||\nabla u^{k-1}_{t}||_{L^2} \left( ||\nabla u^{k-1}_{t}||_{L^2} + ||\nabla^2 u^{k-1}_{t}||_{L^2} \right) ||\nabla \theta^k_{t}||_{L^2}.$$  \hspace{1cm} (7.29)
and

\[
VIII_9 \leq 2\lambda \frac{d}{dt} \int_{\mathbb{R}^3} \nabla u_t^k \cdot \nabla u_t^k + C \| \nabla u_t^k \|_{L^2} \| \nabla \theta_t^k \|_{L^2} \\
+ C \| \nabla u_t^k \|_{L^2} \left( \| \nabla u_t^k \|_{L^2} + \| \nabla^2 u_t^k \|_{L^2} \right) \| \nabla \theta_t^k \|_{L^2}.
\] (7.30)

Substituting (7.25), (7.26), (7.27), (7.28), (7.29) and (7.30) into (7.24), and multiplying the result by \( t^2 \), we have

\[
\frac{1}{2} \int_{\mathbb{R}^3} t^2 \rho_t^k \theta_t^k \theta_t^k \|_{\mathbb{R}^3} + \frac{\kappa}{2} \frac{d}{dt} \int_{\mathbb{R}^3} t^2 \| \nabla \theta_t^k \|_{\mathbb{R}^3}^2 \\
\leq \kappa \int_{\mathbb{R}^3} t \| \nabla \theta_t^k \|_{\mathbb{R}^3} - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} t^2 \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 + \int_{\mathbb{R}^3} t \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 - \frac{d}{dt} \int_{\mathbb{R}^3} t^2 \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 \cdot \nabla \theta_t^k \|_{\mathbb{R}^3}^2 \\
+ 2 \int_{\mathbb{R}^3} t \rho_t^k u_t^k \cdot \nabla \theta_t^k \|_{\mathbb{R}^3} - \frac{d}{dt} \int_{\mathbb{R}^3} t^2 \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 \div \nabla u_t^k \|_{\mathbb{R}^3}^2 + 2 \int_{\mathbb{R}^3} t \rho_t^k \theta_t^k \div \nabla \theta_t^k \|_{\mathbb{R}^3}^2 \\
+ \mu \frac{d}{dt} \int_{\mathbb{R}^3} t^2 \left( \nabla u_t^k - \left( \nabla u_t^k \right) ' \right) \cdot \left( \nabla u_t^k - \left( \nabla u_t^k \right) ' \right) \theta_t^k \\
- 2 \mu \frac{d}{dt} \int_{\mathbb{R}^3} t \left( \nabla u_t^k - \left( \nabla u_t^k \right) ' \right) \cdot \left( \nabla u_t^k - \left( \nabla u_t^k \right) ' \right) \theta_t^k \\
+ 2 \lambda \frac{d}{dt} \int_{\mathbb{R}^3} t^2 \div \nabla u_t^k \|_{\mathbb{R}^3}^2 \div \nabla u_t^k \|_{\mathbb{R}^3}^2 - 4 \lambda \int_{\mathbb{R}^3} t \div \nabla u_t^k \|_{\mathbb{R}^3}^2 \\
+ C \| \nabla u_t^k \|_{L^2}^2 + C \| \nabla \theta_t^k \|_{L^2}^2 + C \| \nabla \theta_t^k \|_{L^2}^2 + C \| \nabla u_t^k \|_{L^2}^2 + C ,
\] (7.31)

where we have used (7.12), (7.13) and the Cauchy inequality.

Integrating (7.31) over \([0, t]\) for \( t \in [0, T_0] \), and using (7.11), integration by parts, the H"{o}lder inequality, (7.9), (7.10), (7.11), (7.12) and the Cauchy inequality, we have

\[
\frac{1}{2} \int_{\mathbb{R}^3} t^2 \| \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 + \frac{\kappa}{2} \int_{\mathbb{R}^3} t^2 \| \nabla \theta_t^k \|_{\mathbb{R}^3}^2 \\
\leq t^2 \| \rho_t^k \|_{L^\infty} \| \nabla \rho_t^k \|_{L^2} \| u_t^k \|_{L^2} \| \nabla \theta_t^k \|_{L^2}^2 + t^2 \| \rho_t^k \|_{L^2} \| u_t^k \|_{L^2} \| \nabla \theta_t^k \|_{L^2} \| \theta_t^k \|_{L^6} \\
+ t^2 \| \rho_t^k \|_{L^2} \| \theta_t^k \|_{L^\infty} \| \div u_t^k \|_{L^3} \| \theta_t^k \|_{L^6} + 4 \mu t^2 \| \div u_t^k \|_{L^3} \| \nabla u_t^k \|_{L^2} \| \theta_t^k \|_{L^6} \\
+ 2 \lambda t^2 \| \div u_t^k \|_{L^2} \| \nabla u_t^k \|_{L^2} \| \theta_t^k \|_{L^6} + C \int_0^t s \| \nabla u_t^k \|_{L^2} \| \nabla \theta_t^k \|_{L^2}^2 + C ,
\]

for \( \epsilon > 0 \) to be decided later. This gives

\[
\max_{1 \leq k \leq N} \int_0^t s \| \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 + \max_{1 \leq k \leq N} \frac{\kappa}{2} \int_{\mathbb{R}^3} t^2 \| \nabla \theta_t^k \|_{\mathbb{R}^3}^2 \leq \epsilon \max_{1 \leq k \leq N} \int_0^t s \| \nabla u_t^k \|_{L^2}^2 + C ,
\] (7.32)

for any given \( N \in \mathbb{Z}^+ \).

Multiplying (7.32) by \( 2 \tilde{C} \), putting the result into (7.22), and taking \( \epsilon > 0 \) sufficiently small, we get

\[
\int_0^t \int_{\mathbb{R}^3} \left( \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 + \nabla u_t^k \|_{\mathbb{R}^3}^2 \right) + \int_{\mathbb{R}^3} t^2 \left( \| \nabla \theta_t^k \|_{\mathbb{R}^3}^2 + \rho_t^k \theta_t^k \|_{\mathbb{R}^3}^2 \right) \leq \tilde{C},
\] (7.33)

for any \( k \in [1, N] \) and a.e. \( t \in [0, T_0] \). Since \( N \) is arbitrary, (7.33) implies (7.13). \( \square \)
Corollary 7.4 Under the condition of (7.1), (7.7) and (7.8), it holds that
\[ t\|\nabla^2 u_k^{t}(\cdot,t)\|_{L^2} + t\|\nabla^3 \theta^{k}(\cdot,t)\|_{L^2} + t\|\nabla^3 u^{k}(\cdot,t)\|_{L^2} \leq \hat{C}, \]
\[ \int_0^{T_0} t^2 \left(||\nabla^2 \theta^{k}(\cdot,t)^2 + ||\nabla^4 \theta^{k}||_{L^2}^2\right) \leq \hat{C}, \tag{7.34} \]
for any \( k \geq 1 \) and a.e. \( t \in [0, T_0]. \)

**Proof.** Similar to (6.26), (6.32), (6.33) and (6.35), together with (7.12) and (7.13), we have
\[ t\|\nabla^2 u_k^{t}(\cdot,t)\|_{L^2} \leq \hat{C} t \|\sqrt{\rho_1} u_k^{t}\|_{L^2} + \hat{C} t\|\nabla u_k^{t}\|_{L^2} + \hat{C} t\|\nabla u_k^{t-1}\|_{L^2} + \hat{C} t\|\nabla \theta_k^{t}\|_{L^2} + \hat{C} \]
\[ \leq \hat{C}, \tag{7.35} \]
and
\[ t\|\nabla^3 \theta^{k}(\cdot,t)\|_{L^2} \leq \hat{C} t\|\nabla \theta_k^{T}\|_{L^2} + \hat{C} t\|\nabla^3 u_k^{t-1}\|_{L^2} + \hat{C} \leq \hat{C}, \tag{7.36} \]
and
\[ t\|\nabla^3 u^{k}(\cdot,t)\|_{L^2} \leq \hat{C} t\|\nabla \theta_k^{T}\|_{L^2} + \hat{C} t\|\nabla^3 u_k^{t-1}\|_{L^2} + \hat{C} \leq \hat{C}, \tag{7.37} \]
By (7.23) and the \( H^2 \)-estimates for the elliptic equation, together with the Hölder inequality, the Sobolev inequality, (7.9), (7.10) and (7.11), we have
\[ \|\nabla^2 \theta^{k}(\cdot,t)\|_{L^2} \leq \hat{C} \|\rho \theta_k^{T}\|_{L^2} + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta^{k}\|_{L^6} + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta_k^{T}\|_{L^6} + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta_k^{T}\|_{L^6} \]
\[ + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta_k^{T}\|_{L^6} + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta_k^{T}\|_{L^6} \]
\[ \leq \hat{C} \|\sqrt{\rho} \theta_k^{T}\|_{L^2} + \hat{C} \|\nabla \theta_k^{T}\|_{L^2} + \hat{C} \|\nabla u_k^{T-1}\|_{L^2} + \hat{C} \|\nabla^2 u_k^{T-1}\|_{L^2} + \hat{C}, \tag{7.38} \]
which together with (7.12) and (7.13) deduces
\[ \int_0^{T_0} t^2 \|\nabla^2 \theta^{k}\|_{L^2}^2 \leq \hat{C}. \tag{7.39} \]

Using (7.33) and the elliptic estimates, together with the Hölder inequality, the Sobolev inequality, (7.9), (7.10), (7.11) and (7.12), we have
\[ \|\nabla^4 \theta^{k}\|_{L^2} \leq \hat{C} \|\nabla^2 \rho \theta^{k}\|_{L^2} + \hat{C} \|\nabla^2 \rho \theta^{k}\|_{L^2} + \hat{C} \|\rho \nabla^2 \theta^{k}\|_{L^2} + \hat{C} \|\nabla \rho \nabla \theta^{k}\|_{L^2} + \hat{C} \|\rho \nabla \theta^{k}\|_{L^2} \]
\[ + \hat{C} \|\rho \nabla \theta_k^{T}\|_{L^2} + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta^{k}\|_{L^6} + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta_k^{T}\|_{L^6} \]
\[ + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta_k^{T}\|_{L^6} + \hat{C} \|\rho \theta_k^{T}\|_{L^3} \|\nabla \theta_k^{T}\|_{L^6} \]
\[ \leq \hat{C} \|\nabla \theta_k^{T}\|_{L^2} + \hat{C} \|\nabla \theta_k^{T}\|_{L^2} + \hat{C} \|\nabla^3 u_k^{T-1}\|_{L^2} + \hat{C}, \tag{7.40} \]
which together with (7.12), (7.13), (7.36), (7.39) deduces
\[ \int_0^{T_0} t^2 \|\nabla^4 \theta^{k}\|_{L^2}^2 \leq \hat{C}. \]

The proof of Corollary 7.4 is complete. \( \blacksquare \)
Lemma 7.5 Under the condition of (7.16), (7.17) and (7.18), it holds that
\[
\int_{\mathbb{R}^3} t^3 \rho^k |\theta_{tt}^k|^2 + \int_{0}^{T_0} \int_{\mathbb{R}^3} t^3 |\nabla \theta_{tt}^k|^2 \leq \tilde{C},
\] (7.41)
for any \( k \geq 1 \) and a.e. \( t \in [0, T_0] \).

**Proof.** Differentiating (7.23) w.r.t. \( t \), multiplying the result by \( \theta_{tt}^k \), and integrating by parts over \( \mathbb{R}^3 \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^k |\theta_{tt}^k|^2 + \kappa \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2
\]
\[
= -2 \int_{\mathbb{R}^3} \rho^k |\theta_{tt}^k|^2 - \int_{\mathbb{R}^3} \rho^k \theta_{tt}^k \theta_{tt}^k - \int_{\mathbb{R}^3} \rho^k u_{tt}^{k-1} \cdot \nabla \theta_{tt}^k \theta_{tt}^k - 2 \int_{\mathbb{R}^3} \rho^k u_{tt}^{k-1} \cdot \nabla \theta_{tt}^k \theta_{tt}^k
\]
\[
- 2 \int_{\mathbb{R}^3} \rho^k \theta_{tt}^k \theta_{tt}^k - \int_{\mathbb{R}^3} \rho^k u_{tt}^{k-1} \cdot \nabla \theta_{tt}^k \theta_{tt}^k - 2 \int_{\mathbb{R}^3} \rho^k u_{tt}^{k-1} \cdot \nabla \theta_{tt}^k \theta_{tt}^k
\]
\[
- 2 \int_{\mathbb{R}^3} \rho^k \theta_{tt}^k \div u_{tt}^{k-1} \theta_{tt}^k - 2 \int_{\mathbb{R}^3} \rho^k \theta_{tt}^k \div u_{tt}^{k-1} \theta_{tt}^k - 2 \int_{\mathbb{R}^3} \rho^k \theta_{tt}^k \div u_{tt}^{k-1} \theta_{tt}^k
\]
\[
+ \mu \int_{\mathbb{R}^3} \left( \nabla u_{tt}^{k-1} + (\nabla u_{tt}^{k-1})' \right) \cdot \left( \nabla u_{tt}^{k-1} + (\nabla u_{tt}^{k-1})' \right) \theta_{tt}^k
\]
\[
+ \mu \int_{\mathbb{R}^3} \left( \nabla u_{tt}^{k-1} + (\nabla u_{tt}^{k-1})' \right)^2 \theta_{tt}^k + 2 \lambda \int_{\mathbb{R}^3} \div u_{tt}^{k-1} \div u_{tt}^{k-1} \theta_{tt}^k + 2 \mu \int_{\mathbb{R}^3} |\div u_{tt}^{k-1}|^2 \theta_{tt}^k
\]
\[
= \sum_{i=1}^{17} \text{IV}_i.
\]

From (7.16), (7.17), (7.18) and (7.19) with \( u^k \) replaced by \( \theta^k \), we obtain
\[
\text{IV}_1 \leq \frac{\kappa}{12} \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2 + \tilde{C} \int_{\mathbb{R}^3} \rho^k |\theta_{tt}^k|^2,
\] (7.43)
and
\[
\text{IV}_2 \leq \frac{\kappa}{12} \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2 + \tilde{C} \|\nabla \theta_{tt}^k\|^2_{L^2} + \tilde{C} \|\nabla u_{tt}^{k-1}\|^2_{L^2} \|\nabla \theta_{tt}^k\|^2_{L^2} \|\nabla^2 \theta_{tt}^k\|^2_{L^2}
\]
\[
+ \tilde{C} \sqrt{\rho^k} \|\theta_{tt}^k\|^2_{L^2} + \tilde{C} \|\nabla u_{tt}^{k-1}\|^4_{L^2},
\] (7.44)
and
\[
\text{IV}_3 + \text{IV}_4 + \text{IV}_5 \leq \frac{\kappa}{12} \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2 + \tilde{C} \|\rho_{tt}^k\|^2_{L^2} + \tilde{C} \|\nabla u_{tt}^{k-1}\|^2_{L^2} + \tilde{C} \|\nabla^2 \theta_{tt}^k\|^2_{L^2},
\] (7.45)
and
\[
\text{IV}_6 + \text{IV}_7 \leq \tilde{C} \int_{\mathbb{R}^3} |\nabla u_{tt}^{k-1}|^2 + \tilde{C} \sqrt{\rho} \|\theta_{tt}^k\|^2_{L^2} + \tilde{C} \|\nabla u_{tt}^{k-1}\|^2_{L^2} \|\nabla \theta_{tt}^k\|^2_{L^2} \|\nabla^2 \theta_{tt}^k\|^2_{L^2}.
\] (7.46)

Similarly, we have
\[
\text{IV}_8 + \text{IV}_9 + \text{IV}_{10}
\]
\[
\leq \|\rho_{tt}^k\|_{L^2} \|\theta_{tt}^k\|_{L^\infty} \|\div u_{tt}^{k-1}\|_{L^2} \|\theta_{tt}^k\|_{L^2} + 2 \|\rho_{tt}^k\|_{L^2} \|\theta_{tt}^k\|_{L^6} \|\div u_{tt}^{k-1}\|_{L^6} \|\theta_{tt}^k\|_{L^6}
\]
\[
+ 2 \|\rho_{tt}^k\|_{L^2} \|\theta_{tt}^k\|_{L^6} \|\div u_{tt}^{k-1}\|_{L^6} \|\theta_{tt}^k\|_{L^6}
\]
\[
\leq \frac{\kappa}{12} \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2 + \tilde{C} \|\rho_{tt}^k\|^2_{L^2} + \tilde{C} \|\nabla \theta_{tt}^k\|^2_{L^2} + \tilde{C} \|\nabla^2 u_{tt}^{k-1}\|^2_{L^2},
\] (7.47)
and
\[ VIV_{11} + VIV_{12} + VIV_{13} \]
\[ \leq \| \sqrt{\rho^k} \|_{L^\infty} \| \sqrt{\rho^k} \|_{L^2} \| \nabla u_{tt} \|_{L^3} \| \theta_{tt}^k \|_{L^6} + 2 \| \rho^k \|_{L^3} \| \nabla u_{tt} \|_{L^2} \| \theta_{tt}^k \|_{L^6} + \| \sqrt{\rho^k} \|_{L^\infty} \| \sqrt{\rho^k} \|_{L^2} \| \nabla u_{tt} \|_{L^2} \| \theta_{tt}^k \|_{L^2} \]
\[ \leq \tilde{C} \| \sqrt{\rho^k} \|_{L^2} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \sqrt{\rho^k} \|_{L^2} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \| \theta_{tt}^k \|_{L^2} \]
\[ \leq \frac{\kappa}{12} \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2 + \tilde{C} \| \sqrt{\rho^k} \|_{L^2} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \| \theta_{tt}^k \|_{L^2} . \tag{7.48} \]

and
\[ VIV_{14} + VIV_{15} + VIV_{16} + VIV_{17} \]
\[ \leq \tilde{C} \| \nabla u_{tt} \|_{L^3} \| \nabla u_{tt} \|_{L^2} \| \theta_{tt}^k \|_{L^6} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \| \theta_{tt}^k \|_{L^6} \]
\[ \leq \frac{\kappa}{12} \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2 + \tilde{C} \| \nabla u_{tt} \|_{L^2} \| \nabla u_{tt} \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \| \nabla \theta_{tt}^k \|_{L^2} \]
\[ + \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \sqrt{\rho^k} \|_{L^2} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \| \theta_{tt}^k \|_{L^2} . \tag{7.49} \]

Substituting (7.43), (7.44), (7.45), (7.46), (7.47), (7.48) and (7.49) into (7.42), and using the Cauchy inequality, we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho^k |\theta_{tt}^k|^2 + \frac{\kappa}{2} \int_{\mathbb{R}^3} |\nabla \theta_{tt}^k|^2 \]
\[ \leq \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \]
\[ + \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \sqrt{\rho^k} \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \]
\[ + \tilde{C} \| \nabla \theta_{tt}^k \|_{L^2} + \tilde{C} \| \nabla u_{tt} \|_{L^2} \] \[ \tag{7.50} \]

Multiplying (7.50) by \( t^3 \), integrating the result over \([0,t]\) for \( t \in [0,T_0] \), and using (7.42), (7.43), (7.44), we have
\[ \int_{\mathbb{R}^3} t^3 \rho^k |\theta_{tt}^k|^2 + \int_0^{T_0} \int_{\mathbb{R}^3} t^3 |\nabla \theta_{tt}^k|^2 \leq \tilde{C}. \]

The proof of Lemma 7.5 is complete. \( \square \)

**Corollary 7.6** Under the condition of (7.1), (7.7) and (7.8), it holds that
\[ t^3 \| \nabla^2 \theta^k (\cdot, t) \|_{L^2}^2 + t^3 \| \nabla^4 \theta^k (\cdot, t) \|_{L^2}^2 \leq \tilde{C}, \tag{7.51} \]
for any \( k \geq 1 \) and a.e. \( t \in [0, T_0] \).

**Proof.** It follows from (7.43), (7.44), (7.45), (7.46), (7.47), (7.48) and (7.49) that
\[ t^3 \| \nabla^2 \theta^k (\cdot, t) \|_{L^2} \leq \tilde{C} t^3 \| \sqrt{\rho^k} \|_{L^2} + \tilde{C} t^3 \| \nabla \theta^k (\cdot, t) \|_{L^2} \]
\[ + \tilde{C} t^3 \| \nabla \theta^k (\cdot, t) \|_{L^2} + \tilde{C} t^3 \| \nabla^2 \theta^k (\cdot, t) \|_{L^2} \]
\[ \leq \tilde{C}. \tag{7.52} \]

By (7.40), (7.41), (7.43), (7.44) and (7.45), we have
\[ t^3 \| \nabla^4 \theta^k (\cdot, t) \|_{L^2} \leq \tilde{C} t^3 \| \nabla \theta^k (\cdot, t) \|_{L^2} + \tilde{C} t^3 \| \nabla^2 \theta^k (\cdot, t) \|_{L^2} \]
\[ + \tilde{C} t^3 \| \nabla^2 \theta^k (\cdot, t) \|_{L^2} + \tilde{C} t^3 \| \nabla u_{tt} \|_{L^2} \]
\[ \leq \tilde{C}. \tag{7.53} \]

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Step 2: Completion of proof of Appendix B.

Using some arguments similar to [4], we obtain that the full sequence \((\rho^k, u^k, \theta^k)\) converges to a limit \((\rho^\delta, u^\delta, \theta^\delta)\) for any given \(\delta \in (0, 1)\) in the following strong sense:

\[
\begin{align*}
\rho^k & \to \rho^\delta \text{ in } L^\infty([0, T_0]; L^2), \text{ as } k \to \infty, \\
(u^k, \theta^k) & \to (u^\delta, \theta^\delta) \text{ in } L^2([0, T_0]; D^1_0), \text{ as } k \to \infty,
\end{align*}
\]

and \((\rho^\delta, u^\delta, \theta^\delta)\) is the unique solution to (2.1)-(2.2) with initial data replaced by \((\rho_0^\delta, u_0^\delta, \theta_0^\delta)\), where \(\rho^\delta > 0\) and \(\theta^\delta \geq 0\). With Lemmas 7.1, 7.2, 7.3 and 7.5, and Corollaries 7.4 and 7.6, and the lower semi-continuity of the norms, we have

\[
\begin{align*}
\|\rho^\delta(\cdot, t) - \delta\|_{H^2; W^{2, q}} & + \|\rho^\delta_t(\cdot, t)\|_{H^1} + \|\nabla u^\delta(\cdot, t)\|_{H^1} + \|\nabla \theta^\delta(\cdot, t)\|_{H^1} + \|\nabla^3 u^\delta\|_{L^2([0, T]; L^2)} + \|\nabla^3 \theta^\delta\|_{L^2([0, T]; L^2)} \leq \tilde{C}, \\
\|\sqrt{\rho^\delta} u^\delta_t(\cdot, t)\|_{L^2} & + \|\sqrt{\rho^\delta} \theta^\delta_t(\cdot, t)\|_{L^2} + \|\nabla u^\delta_t\|_{L^2([0, T]; L^2)} + \|\nabla \theta^\delta_t\|_{L^2([0, T]; L^2)} \leq \tilde{C},
\end{align*}
\] (7.54)

\[
\sqrt{t}\|\nabla u^\delta(\cdot, t)\|_{H^2} + \sqrt{t}\|\nabla u^\delta_t(\cdot, t)\|_{L^2} + \int_0^{T_0} \left( t\|\sqrt{\rho^\delta} u^\delta_t(\cdot, t)\|_{L^2}^2 + t\|\nabla^2 u^\delta(\cdot, t)\|_{L^2}^2 \right) dt \leq \tilde{C},
\] (7.56)

\[
t\|\nabla^2 u^\delta_t(\cdot, t)\|_{L^2} + t\|\nabla^3 \theta^\delta(\cdot, t)\|_{L^2} + t\|\nabla^3 u^\delta(\cdot, t)\|_{L^2} + t\|\nabla \theta^\delta_t(\cdot, t)\|_{L^2} + t\|\sqrt{\rho^\delta} u^\delta_t(\cdot, t)\|_{L^2} + t\|\sqrt{\rho^\delta} \theta^\delta_t(\cdot, t)\|_{L^2} + \int_0^{T_0} t^2 \left( \|\nabla^2 \theta^\delta_t\|_{L^2}^2 + \|\nabla^4 \theta^\delta(\cdot, t)\|_{L^2}^2 + \|\nabla u^\delta_t\|_{L^2}^2 \right) dt \leq \tilde{C},
\] (7.57)

and

\[
t^3\|\nabla^2 \theta^\delta_t(\cdot, t)\|_{L^2}^2 + t^3\|\nabla^4 \theta^\delta(\cdot, t)\|_{L^2}^2 + t^3\|\sqrt{\rho^\delta} \theta^\delta_t\|_{L^2}^2 + \int_0^{T_0} t^3 |\nabla \theta^\delta_t|^2 dx dt \leq \tilde{C}.
\] (7.58)

By (7.54), (7.55), (7.56), (7.57), (7.58), we pass \((\rho^\delta, u^\delta, \theta^\delta)\) to a limit \((\rho, u, \theta)\) (take subsequence if necessary) which is the unique solution to (2.1)-(2.3). By the lower semi-continuity of the norms and some arguments which are concerned with the time-continuity of the solutions as in [3, 4] and references therein, we get (7.3).

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