REMARKS ON CONGRUENCE OF 3–MANIFOLDS

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Dedicated to Lou Kauffman on the occasion of his 60th birthday

Abstract. We give two proofs that the 3–torus is not weakly $d$–congruent to $\#^3 S^1 \times S^2$, if $d > 2$. We study how cohomology ring structure relates to weak congruence. We give an example of three 3–manifolds which are weakly 5–congruent but are not 5–congruent.

Let $d$ be an integer greater than one. In [G], we considered two equivalence relations generated by restricted surgeries on oriented closed 3–manifolds. Weak type–$d$ surgery is $q/ds$ Dehn surgery along a simple closed curve. Here $s$ and $q$ (which must be relatively prime to $d$ and $s$) may vary but $d$ is held fixed. The label $q/ds$ indicating which surgery is given with respect to some meridional and a longitudinal pair on the boundary of a solid torus neighborhood of the surgery curve. A meridian bounds a disk in the solid torus which meets the surgery curve transversely in one point, and a longitude meets the meridian transversely in one point in the boundary torus. The set of surgeries described as weak type–$d$ surgeries does not depend on the choice of meridional and a longitudinal pair. If $q \equiv \pm 1 \mod d$, we say the surgery is type–$d$ surgery. This concept is also independent of the choice of meridian and longitude.

The equivalence relation on the set of closed oriented 3–manifolds generated by weak type–$d$ surgery is called weak $d$–congruence. The equivalence relation generated by type–$d$ surgery is called $d$–congruence. The equivalence relation $d$–congruence is coarser [G] than an equivalence relation which was first considered by Lackenby [L] : congruence modulo $d$. It is not known that $d$–congruence is strictly coarser than congruence modulo $d$, but this seems likely.

The notion of $d$–congruence of 3–manifolds is closely related to the notion of $t_d$–move (now called $d$–move) equivalence of links [P1, remark before proof of Theorem p.639]: A $d$–move between links implies that there is $1/d$ Dehn surgery relating the double branched covers of $S^3$ along the links. Similarly, weak $d$–equivalence of 3–manifolds is closely related to rational move equivalence of links as analyzed in [P2, footnote 5], [P3, footnote 22] and [DP, DIP]. Completing this circle of ideas, we note that $d$–move equivalence of links is a special case of congruence modulo $(d,q)$ of links due to Fox [F1]. Lackenby’s study of congruence modulo $(d,q)$ of links lead him to define congruence modulo $d$ of 3–manifolds.

We will give two proofs of the following theorem. The first proof will use Burnside groups and second will use cohomology ring structure. We let $T^3$ denotes the 3–torus.

**Theorem 1.** $T^3$ is not weakly $d$–congruent to $\#^3 S^1 \times S^2$ for any $d > 2$. 

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We remark that we don’t know whether or not the 3–torus is weakly 2–congruent to \(\#^3 S^1 \times S^2\). It seems unlikely. If one could prove that the 3–torus is not weakly 2–congruent to \(\#^3 S^1 \times S^2\), it would provide a second proof of Fox’s result \[2\] that the 3–torus is not the double branched cover of a link. By the trick of Montesinos \[M\], the double branched cover of \(S^3\) 2–congruent to \# \(Z \) preserves the ring structure. Using Poincare duality, this simply means the trilinear group is preserved by weak \(c\)–congruence.\[t\]

First Proof of Theorem \[7\]. The \(d\)th Burnside group of a group \(G\) obtained by quotienting \(G\) by the subgroup normally generated by the \(d\)th powers of all elements. The \(d\)th Burnside group of a manifold \(M\) is the \(d\)th Burnside group of the fundamental group of the manifold. Slightly generalizing an observation of Dabkowski and Przytycki \[DP\] proof of Theorem (1.2), we noted in \[G\] that the \(d\)th Burnside group is preserved by weak \(d\)–congruence.

The \(d\)th Burnside group of \(T^3\) is abelian. In fact it is \(Z_3^d\). According to \[MKS\] Exercise 2.2.19, the \(d\)th Burnside group of a free group on \(r\) generators is nonabelian if \(d > 2\) and \(r > 1\). Of course the fundamental group of \(\#^3 S^1 \times S^2\) is free on three generators. \[□\]

In \[G\] Theorem (2.7), we observed that a weak \(d\)–congruence induces an isomorphism of \(Z_d\)–cohomology groups of \(M\). Moreover if \(d\) is odd, this isomorphism preserves the ring structure. Using Poincare duality, this simply means the trilinear pairing \(t_M\) on \(H_1(M, Z_d)\) with values in \(Z_d\) is preserved.

Let \(t_M\) denote the trilinear form on \(H_1(M, Z_d)\) with values in \(Z_d\) which sends \((\chi_1, \chi_2, \chi_3)\) to \((\chi_1 \cup \chi_2 \cup \chi_3) \cap [M]\). If \(d\) is even, let \(\rho: Z_d \rightarrow Z_{d/2}\) be reduction modulo \(d/2\).

**Theorem 2.** A weak \(d\)–congruence between \(M\) and \(M'\) induces an isomorphism \(c: H^1(M, Z_d) \rightarrow H^1(M', Z_d)\). If \(d\) is odd,

\[
t_M(\chi_1, \chi_2, \chi_3) = t_{M'}(c(\chi_1), c(\chi_2), c(\chi_3)).
\]

If \(d\) is even,

\[
\rho(t_M(\chi_1, \chi_2, \chi_3)) = \rho(t_{M'}(c(\chi_1), c(\chi_2), c(\chi_3))).
\]

**Proof.** The case \(d\) odd is \[G\] Theorem (2.7). The proof, in the case \(d\) even, proceeds in exactly the same way. At the end, we need to see that the triple intersection number \(\tau\) must satisfy \(\rho(\tau) = 0 \pmod{d/2}\). This follows from \(\tau = -\tau \pmod{d}\), which holds since the triple intersection number of surfaces is skew symmetric. \[□\]

**Proposition 3.** In the case, \(d\) is even,

\[
t_M(\chi_1, \chi_2, \chi_3) = t_{M'}(c(\chi_1), c(\chi_2), c(\chi_3))
\]

need not hold. The \(\tau\) that appears in the proof of Theorem \[2\] is congruent to \(d/2\) modulo \(d\).

**Proof.** One may pass from \(S^1 \times S^2\) to \(L(ds, q)\) by a weak type-d surgery. Let \(\psi\) denote a generator for \(H^1(L(ds, q), Z_d)\). One has that \(\psi \cup \psi\) is \(d/2\) times a generator for \(H^1(L(ds, q), Z_d)\) \[H\] Example 3.41. It follows that \(t_{L(ds, q)}(\psi, \psi, \psi) = d/2 \pmod{d}\). On the other hand, \(t_{S^1 \times S^2}\) is the zero trilinear form. We note that it follows that \(\tau = d/2 \pmod{d}\). \[□\]

Second Proof of Theorem \[7\]. We apply Theorem \[2\]. If \(d\) is odd, we note that \(t_{T^3}\) is non–trivial and \(t_{\#^3 S^1 \times S^2}\) is zero. If \(d\) is even, we observe that \(\rho \circ t_{T^3}\) is non–trivial and \(\rho \circ t_{\#^3 S^1 \times S^2}\) is zero. \[□\]
Let $P$ denote the Poincare homology sphere. $P$ can also be described as the Brieskorn manifold $\Sigma(2, 3, 5)$. Let $\Sigma$ denote the Brieskorn homology sphere $\Sigma(2, 3, 7)$.

**Proposition 4.** $P$, $\Sigma$ and $S^3$ are all weakly 5–congruent to each other. However no two of them are 5-congruent.

**Proof.** The last statement is contained in [G, Corollary 3.10]. By [Mi, Lemma (1.1)], $P$ and $\Sigma$ are double branched covers of $S^3$ along the respectively the $(3, 5)$ and $(3, 7)$ torus knots. Both of these knots are closures of 3–braids. According to [DIP, Theorem 2.2], the closure of any 3–braid is $(2, 2)$–move equivalent to a trivial link or one of four specified 3–component links. We have that $(2, 2)$ moves are covered in the double branched covers of links by $\pm 2/5$ surgeries [DP, DIP]. So $P$ and $\Sigma$ must each be weakly 5–congruent to $S^3$ (the double branched cover of the unknot) or the double branched covers of a link with more than one component. But the double branched cover of a $c$–component link will have first homology with $\mathbb{Z}_5$ coefficients $\mathbb{Z}_5^c - 1$. As this homology group is preserved by weak 5–congruence, and both $P$ and $S$ are homology spheres, $P$ and $S$ must be weakly 5–congruent to $S^3$. $\square$

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