Algebraically Special, Real Alpha-Geometries

Abstract
We exploit the spinor description of four-dimensional Walker geometry to describe the local geometry of four-dimensional neutral geometries with algebraically degenerate self-dual Weyl curvature and an integrable distribution of $\alpha$-planes (algebraically special real $\alpha$-geometry). In particular, we provide a derivation of the hyperheavenly equation from conformal rescaling formulae.

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1. Introduction

Throughout this paper, \((M, h)\) denotes a four-dimensional manifold \(M\) equipped with a metric \(h\) of neutral signature and will be referred to as a neutral geometry. Law (2006) contains the algebraic classification of the Weyl curvature spinors of a neutral geometry. Our aim in this paper is to shed light on those neutral geometries for which at least one of the two Weyl curvature spinors is algebraically special, i.e., admits a Weyl principal spinor (WPS), see Law (2006), of multiplicity greater than one.

An important class of examples is provided by Walker geometry, i.e., a neutral geometry \((M, g)\) which admits a parallel distribution \(D\) of totally null planes, see Walker (1950) and Law & Matsushita (2008). As shown in Law & Matsushita (2008), with its canonical choice of orientation a Walker geometry can be characterized (at least locally) by a (real) projective primed spinor field \([\pi^A]\). The projective spinor field defines, at each point \(p \in M\), an \(\alpha\)-plane

\[
Z_{[\pi]} := \{\eta^A \pi^{A'} : \eta^A \in S_p\},
\]

where \(S_p\) is the fibre at \(p\) of the bundle (defined at least locally) of unprimed spinors and \(\pi^{A'}\) is a scaled representative of \([\pi^A]\). The collection of these \(\alpha\)-planes constitutes the distribution \(D\). The condition for this distribution of \(\alpha\)-planes (hereafter, \(\alpha\)-distribution) to be parallel is

\[
S_a := \pi_{B'} \nabla_a \pi^{B'} = 0,
\]

where \(\pi^{A'}\) is any local scaled representative (LSR) of \([\pi^A]\), i.e., any local primed spinor field whose projective class coincides with \([\pi^A]\) (at each point). The condition for the \(\alpha\)-distribution to be merely integrable is

\[
\pi_{B'} \pi^{A'} \nabla_{A\alpha} \pi^{B'} = 0.
\]

Clearly (1.2) entails (1.3). Equation (1.2) is studied in Law & Matsushita (2008), where it is shown that a solution \([\pi^A]\) of (1.2) is a WPS of multiplicity at least two, and preliminary results for equation (1.3) presented in Law (2008). We will denote a Walker geometry by \((M, g, [\pi^A])\). For simplicity, we will phrase our discussion as if \((M, h)\) admits projective spinor bundles, which presumes that \((M, h)\) is \(\text{SO}^+\)-orientable; otherwise the results are valid locally.

The Generalized Goldberg-Sachs Theorem (GGST) for neutral geometry (see Law 2008 (6.2.17)) indicates an intimate connexion between solutions of (1.3) and multiple WPSs (real or complex; note that while the spinor spaces for neutral signature are real, solutions of spinorial equations may be complex valued, complex WPSs are just a particular instance). We denote by \((M, h, [\pi^A])\) a neutral geometry for which \([\pi^A]\) is a solution of (1.3) and refer to such as an \(\alpha\)-geometry. Every solution of (1.3) is automatically a WPS (Law 2008 (6.2.9)). We call \((M, h, [\pi^A])\) an algebraically special (AS) \(\alpha\)-geometry when \([\pi^A]\) is a multiple WPS (of the SD Weyl curvature spinor \(\tilde{\Psi}_{A'BC'D'}\)). We leave untouched the possibility of neutral geometries \((M, h)\) with a WPS \([\pi^A]\), of multiplicity \(p > 1\), which is not a solution of (1.3). By GGST, for such a neutral geometry:

\[
\nabla_{\bar{\pi}^A \cdots \pi^{C'}} D_D \tilde{\Psi}_{A'BC'D'} \neq 0.
\]

Note that if \([\pi^A]\) is complex, then \(p = 2\).

The cases of real and complex WPSs will be treated separately as the geometric interpretation of (1.3) differs in the two situations; we refer to neutral geometries admitting real (complex) solutions of (1.3) as, respectively, real (complex) \(\alpha\)-geometries. In this paper, we treat real \(\alpha\)-geometries and real multiple WPSs, and we may omit the qualifier ‘real’ when it is not needed for emphasis.

Consider a Walker geometry \((M, g, [\pi^A])\) and its integrable \(\alpha\)-distribution \(Z = Z_{[\pi]}\), defined as in (1.1). An \(\alpha\)-distribution is clearly a conformally invariant notion and integrability is a differential-topological property. Hence, in the neutral geometry \((M, h)\), where \(h := \bar{\Omega}^2 g\) for some smooth \(\mathbb{R}^+\)-valued function \(\Omega\), the distribution \(Z\) retains its character as the integrable \(\alpha\)-distribution \(Z_{[\pi]}\). Of course, \([\pi^A]\) is also a multiple WPS for \((M, h)\) since the Weyl curvature spinors are invariant under conformal rescalings of
the metric. Thus, from Walker geometries, conformal rescalings generate further examples of real AS-$\alpha$-geometries, which raises the question: is every real AS-$\alpha$-geometry $(M, h, [\pi^A])$ conformally Walker, i.e., is $h$ a conformal rescaling of a metric $g$, with $(M, g, [\pi^A])$ Walker? In §2, we review the geometry of conformal rescalings in the four-dimensional, neutral-signature context. In §3, we show that every real AS-$\alpha$-geometry is locally conformal to a Walker geometry and thus obtain a local characterization of all real AS-$\alpha$-geometries. This characterization includes a neutral-signature version of the hyperheavenly equation in complex general relativity obtained by Plebański and co-workers, see Plebański & Robinson (1976, 1977), Finley & Plebański (1976), and Boyer et al. (1980). In §4, we touch on some related null geometry.

As in Law & Matsushita (2008) and Law (2008), we employ the abstract index formalism and notation of Penrose & Rindler (1984) for indices (italic indices are ‘abstract’ indices while bold upright indices are ‘concrete’). See Law (2006), Law & Matsushita (2008) Appendix Two, and Law (2008) for brief accounts of the adaptation of the two-component spinor formalism of Penrose & Rindler (1984) to the context of neutral signature. $S$ will denote the bundle of unprimed spinors over $M$ (or over an open set $U$ if $S$ only exists locally) while $S'$ denotes the bundle of primed spinors.

2. Conformal Rescalings of the Metric

Penrose & Rindler (1984), §5.6, describe how, in the context of Lorentzian four-dimensional geometry, a spinor structure is related to a conformal class of metrics. This correspondence is easily adapted to the case of neutral signature. Let $(M, h)$ be a neutral geometry with, at least locally, a spinor structure. Let $\hat{h}_{ab} := \Omega^2 h_{ab}$, where $\Omega : M \rightarrow \mathbb{R}^+$ is smooth. The corresponding spinor structure for $(M, \hat{h})$ consists of the same spinor bundles but with skew scalar products given by

$$\hat{\epsilon}_{AB} := \theta \epsilon_{AB}$$

$$\hat{\epsilon}_{A'B'} := \tilde{\theta} \epsilon_{A'B'},$$

(2.1)

where $\theta$ and $\tilde{\theta}$ are smooth functions $M \rightarrow \mathbb{R}^+$ and

$$\theta\tilde{\theta} = \Omega^2.$$  

(2.2)

In the usual manner, the Levi-Civita connection induces unique connections (denoted simply by $\hat{\nabla}_{AA'}$) on the unprimed and primed spinor bundles with respect to which $\hat{\epsilon}_{AB}$ and $\hat{\epsilon}_{A'B'}$, respectively, are parallel. By the neutral-signature analogue of Penrose & Rindler (1984) (4.4.23),

$$\hat{\nabla}_{AA'} \xi^C = \nabla_{AA'} \xi^C + \Theta_{AA'B'} C^B \xi^B$$

$$\hat{\nabla}_{AA'} \xi'^C = \nabla_{AA'} \xi'^C + \tilde{\Theta}_{AA'B'} C^B \xi'^B,$$

(2.3)

where $\Theta_{aB'} C'$ and $\tilde{\Theta}_{aB'} C'$ are, in principle, independent quantities. Since, however, both $\nabla_{AA'}$ and $\hat{\nabla}_{AA'}$ are torsion free, one deduces, following Penrose & Rindler (1984), pp. 216–217, that $\Theta_{aB'} C'$ and $\tilde{\Theta}_{aB'} C'$ take the form

$$\Theta_{aB'} C' = \Lambda_a \epsilon_{B'} C' + \Upsilon_{A'B'} \epsilon_{A} C'$$

$$\tilde{\Theta}_{aB'} C' = \tilde{\Lambda}_a \epsilon_{B'} C' + \tilde{\Upsilon}_{A'B'} \epsilon_{A} C',$$

but with

$$\Lambda_a + \tilde{\Lambda}_a = 0$$

$$\Upsilon_a = \tilde{\Upsilon}_a,$$

in place of Penrose & Rindler (1984) (4.4.44 & 46). Hence

$$\Theta_{aB'} C' = \Lambda_a \epsilon_{B'} C' + \Upsilon_{A'B'} \epsilon_{A} C'$$

$$\tilde{\Theta}_{aB'} C' = -\Lambda_a \epsilon_{B'} C' + \Upsilon_{A'B'} \epsilon_{A} C'.$$

(2.4)

Now,

$$0 = \hat{\nabla}_a \epsilon_{BC}$$

$$= \hat{\nabla}_a (\theta \epsilon_{BC})$$

$$= \nabla_a (\theta \epsilon_{BC}) - \Theta_{aB} D \epsilon_{DC} \theta - \Theta_{aC} D \epsilon_{BD} \theta$$

$$= \epsilon_{BC} \nabla_a (\theta) - \epsilon_{BC} \Theta_{aD} D \theta$$

$$= \epsilon_{BC} \nabla_a (\theta) - \epsilon_{BC} \Theta_{aD} D \theta.$$
\[ \nabla_a (\ln \theta) = \Theta_a D^D = 2 \Lambda_a + \Upsilon_a \quad \text{and similarly} \quad \nabla_a (\ln \tilde{\theta}) = \tilde{\Theta}_a D^D = -2 \Lambda_a + \tilde{\Upsilon}_a. \] (2.5)

With \( \mu : M \to \mathbb{R}^+ \) smooth, write \( \theta = \mu \Omega \), whence \( \tilde{\theta} = \Omega/\mu \). Then (2.5) is equivalent to

\[ \Upsilon_a = \nabla_a (\ln \Omega) \quad \Lambda_a = \nabla_a (\ln \sqrt{\mu}). \] (2.6)

In the case of Lorentzian signature, the fact that the primed spin space is the complex conjugate of the unprimed spin space forces \( \tilde{\theta} \) to be the complex conjugate of \( \theta \), whence \( \mu \bar{\Omega} = 1 \). Provided \( \mu \) is required to be real valued, then \( \mu = \pm 1 \). The choice of negative sign is rejected as being discontinuous with the identity scaling, and \( \mu \equiv 1 \) results in the formulae of Penrose & Rindler (1984) §5.6. The independence of the primed and unprimed spin spaces for neutral signature is less restrictive and does allow nontrivial choices of \( \mu \). Clearly, \( \Lambda_a = 0 \) is equivalent to choosing \( \mu \) constant. Whether nontrivial choices of \( \mu \) are of interest in neutral geometry will not be pursued here; hereafter, we suppose \( \mu \equiv 1 \), whence unprimed and primed spinors are treated alike under conformal rescaling and \( \nabla_a \) is determined in terms of \( \nabla_a \) and \( \Upsilon_a \) exactly as in Penrose & Rindler (1984) 

\[ \hat{\nabla}_{AA'}^P \cdots S'_{\cdots} = \hat{\nabla}_{AA'}^P \cdots S'_{\cdots} - \Upsilon_A A' B C D E F \cdots + \cdots - \Upsilon_A A' B C D E F \cdots + \cdots + \chi A B C D E F \cdots + \cdots \] (2.7)

Our conventions for curvature are detailed in Law & Matsumoto (2008), Appendix One; they agree with those of Penrose & Rindler (1984), except that our Ricci tensor (whence Ricci scalar curvature) is the negative of theirs. But, we take the Ricci spinor and the scalar \( \Lambda \) to be unaffected by this different convention (one simply inserts an additional negative sign in Penrose & Rindler (1984) (4.6.20–23)). The curvature spinors of the metric \( \hat{h}_{ab} \) are then related to the curvature spinors of the metric \( h_{ab} \) as in Penrose & Rindler (1984):

\[ \hat{\Psi}_{A'B'C'D'} = \Psi_{A'B'C'D'} \quad \hat{\Phi}_{ABA'B'} = \Phi_{ABA'B'} + \Upsilon_A (A', B') B - \nabla_A (A', B') B \] (2.8)

\[ \hat{\Lambda} = \Omega^{-2} \left[ \Lambda + \frac{1}{4} \nabla^a \Upsilon_a + \frac{1}{4} \nabla_a \Upsilon_a \right] = \Omega^{-2} \left[ \frac{1}{4} \Omega^{-1} \Box \Omega \right]. \]

It will prove useful to define

\[ \omega := \ln \Omega. \] (2.9)

Notation for spin coefficients for neutral geometry (effectively a suitable definition of the priming operation to replace Penrose & Rindler (1984) (4.5.17)) was introduced in Law (2008). The notation of Law (2008) combined with (2.9) yields, for the components of \( \Upsilon_a \) with respect to spin bases \( \{o^A, \iota^A\} \) and \( \{o^A', \iota^A'\} \),

\[ \Upsilon_{00'} = D \omega \quad \Upsilon_{01'} = \delta \omega \quad \Upsilon_{10'} = \Delta \omega \quad \Upsilon_{11'} = D' \omega, \] (2.10)

which equations in effect define the operators \( D, \delta, \Delta \) and \( D' \). If the spin basis \( \{o^A, \iota^A\} \) is rescaled according to

\[ \hat{o}^A := \Omega^{o_0} o^A \quad \hat{\iota}^A := \Omega^{\iota_1} \iota^A, \] (2.11)

whence

\[ \hat{o}_A = \Omega^{o_0 + 1} o_A \quad \hat{\iota}_A = \Omega^{\iota_1 + 1} \iota_A, \] (2.12)

with

\[ \hat{\chi} := \iota^A \hat{o}_A = \Omega^{o_0 + 1} \iota^A \chi, \] (2.13)

where \( \chi := \iota^A o_A \). The spin basis \( \{o^A', \iota^A'\} \) may be independently rescaled as

\[ \hat{o}^A' := \Omega^{o_0'} o^A' \quad \hat{\iota}^A' := \Omega^{\iota_1'} \iota^A', \] (2.14)
whence
\[ \hat{\sigma}_A' = \Omega^{w_0 + 1} \sigma_A' \quad \text{and} \quad \hat{i}_A' = \Omega^{v_1 + 1} i_A', \]
with
\[ \hat{\chi}' := \hat{\epsilon}' \hat{\sigma}_A' = \Omega^{w_0 + w_1 + 1} \hat{\chi}, \]
where \( \hat{\chi} := \epsilon A' \sigma_A' \). With these choices, as \( \nabla_a = \nabla_a \) on functions, then on functions
\[ \hat{D} = \Omega^{w_0 + w_0} D \quad \hat{\delta} = \Omega^{w_0 + w_1} \delta \quad \hat{\Delta} = \Omega^{v_1 + w_0} \Delta \quad \hat{D}' = \Omega^{v_1 + w_1} D'. \]

The spin coefficients of \( \nabla_a \) with respect to the rescaled bases may be derived from Law (2008) (2.7–8), which are the neutral analogues of Penrose & Rindler (1984) (4.5.21). One obtains the following neutral analogue of Penrose & Rindler (1984) (5.6.25):

\[
\begin{array}{cccc}
\hat{\epsilon}' & \hat{\kappa}' & \hat{\tau}' & \hat{\lambda}' \\
\alpha & v_0 \Delta \omega & \kappa \Sigma & \tau' - \Delta \omega & \hat{\gamma}' + v_1 D \omega \Sigma \\
\hat{\alpha} & \hat{\beta} & \hat{\delta}' & \hat{\beta}' & \alpha' - v_1 \delta \\
\hat{\beta} & \hat{\beta}' & \hat{\gamma}' & \hat{\gamma}' & \beta + (v_0 + 1) \delta \omega & \sigma \Sigma & \beta' \delta \omega & \hat{\beta}' (v_0 + 1) \delta \omega \\
\hat{\gamma} & \hat{\beta}' & \hat{\kappa}' & \hat{\kappa}' & \gamma + v_0 D \omega \Sigma & \kappa \Sigma & \tau + \delta \omega & \kappa \Sigma - 2 & \hat{\gamma}' + (v_0 + 1) D \omega \Sigma \\
\end{array}
\]

\[
\begin{array}{cccc}
\hat{\epsilon}' & \hat{\kappa}' & \hat{\tau}' & \hat{\lambda}' \\
\alpha & v_0 \Delta \omega & \kappa \Sigma & \tau' - \Delta \omega & \hat{\gamma}' + v_1 D \omega \Sigma \\
\hat{\alpha} & \hat{\beta} & \hat{\delta}' & \hat{\beta}' & \alpha' - w_1 \Delta \omega \\
\hat{\beta} & \hat{\beta}' & \hat{\gamma}' & \hat{\gamma}' & \beta + (v_0 + 1) \Delta \omega & \sigma \Sigma & \beta' \delta \omega & \hat{\beta}' (w_0 + 1) \Delta \omega \\
\hat{\gamma} & \hat{\beta}' & \hat{\kappa}' & \hat{\kappa}' & \gamma + w_0 D \omega \Sigma & \kappa \Sigma & \tau + \Delta \omega & \kappa \Sigma - 2 & \hat{\gamma}' + (w_0 + 1) D \omega \Sigma \\
\end{array}
\]

where
\[ \Sigma := \Omega^{v_0 - w_1} \quad \hat{\Sigma} := \Omega^{w_0 - w_1}, \]
and the factors of \( \Omega \) to the right of the equality signs in (2.18–19) multiply each entry in the corresponding row.

### 3. Algebraically Special Real Alpha-Geometries

Let \((M, g, [\pi^A])\) be a Walker geometry. As discussed in §2, the \(\alpha\)-distribution \(Z_{[\pi]}\) of the Walker geometry retains its character as an integrable (though not necessarily parallel) \(\alpha\)-distribution with respect to the conformal class of metrics \([g]\) and may unambiguously be denoted by \(Z_{[\pi]}\) with respect to \([g]\).

Consider the neutral geometry \((M, \hat{g}), \hat{g} := \Omega^2 g\) with \(\Omega : M \to \mathbb{R}^+\) smooth. First note that as long as LSRs of \([\pi^A]\) are treated as conformal densities/invariants, then their status as LSRs is valid for the conformal class of metrics to which \(g\) belongs. With \(\hat{\pi}^A := \Omega^p \pi^A\), \((p = 0\) allowed), a simple calculation yields:

\[ \hat{\pi}^A \hat{\pi}^{B'} \hat{\nabla}_{b A'} \hat{\pi}^A = \Omega^{3p + 1} \pi^A \pi^{B'} \hat{\nabla}_{b A'} = \Omega^{3p + 1} \pi^A \pi^{B'} \left[ 2 \pi^A + \epsilon B' A' \gamma_{X A'} \right] = \Omega^{3p + 1} \pi^A \pi^{B'} \nabla_{b A'}, \]
i.e., as expected \(\hat{\pi}^A\) satisfies (1.3) for \((M, \Omega^2 g)\) iff \(\pi^A\) solves (1.3) for \((M, g)\).
Since \([\pi^A]\) is WPS of multiplicity at least two (Law & Matsushita 2008, 2.5) for \((M, g)\), by (2.8) it is so for \((M, \Omega^2 g)\), whence, as asserted in the Introduction, \((M, \Omega^2 g)\) is indeed a real ASo-geometry. It is natural to ask when the \(\alpha\)-distribution \(Z\) is parallel in \((M, \hat{g})\), i.e., when is \((M, \Omega^2 g, [\pi^A])\) itself Walker? A simple computation like the last yields the following result.

### 3.1 Lemma

If \((M, g, [\pi^A])\) is Walker, then \((M, \Omega^2 g, [\pi^A])\) is Walker iff \(\Upsilon_{B\mathcal{X}} \pi^X = 0\) for some, whence any, LSR of \([\pi^A]\), i.e., iff \(\Omega\) is constant on \(\alpha\)-surfaces. When \((M, \Omega^2 g, [\pi^A])\) is Walker, for any LSR, \(\nabla_b \pi^A = \nabla_b \pi^A\), whence \((M, g, [\pi^A])\) admits a parallel LSR iff \((M, \Omega^2 g, [\pi^A])\) does (see Law & Matsushita 2008 §3 for parallel LSRs in Walker geometry).

**Proof.** If \(\hat{\pi}^A := \Omega^p \pi^A\), then

\[
\hat{\pi}_A \nabla_b \hat{\pi}^A = \Omega^{2p+1} \pi_A \left[\nabla_b \pi^A + \epsilon_B \pi^A \Upsilon_{B\mathcal{X}} \pi^X\right] = \Omega^{2p+1} \pi_B \Upsilon_{B\mathcal{X}} \pi^X,
\]

whence \(\hat{\pi}^A\) solves (1.2) iff \(\Upsilon_{B\mathcal{X}} \pi^X = 0\). Thus, \((M, \Omega^2 g, [\pi^A])\) is Walker iff \(\Upsilon^a \in Z[\pi]\). When this condition holds, then clearly \(\nabla_b \pi^A = \nabla_b \pi^A\).

Suppose \((u, v, x, y)\) are Walker coordinates for \((M, g, [\pi^A])\) on some domain, i.e., they are Frobenius coordinates for the \(\alpha\)-distribution and the metric components with respect to these coordinates take the form

\[
(g_{ab}) = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & W \end{pmatrix}, \quad W = \begin{pmatrix} a & c \\ c & b \end{pmatrix} =: (W^{AB}),
\]

where \(a, b, \) and \(c\) are functions of \((u, v, x, y)\), see Walker (1950), Law & Matsushita (2008) 2.3. Whether \((M, \hat{g}, [\pi^A])\) is Walker or not, the coordinates \((u, v, x, y)\) are Frobenius coordinates for the \(\alpha\)-distribution irrespective of the metric and still provide useful coordinates for \((M, \hat{g}, [\pi^A])\), the metric \(\hat{g}_{ab}\) having components

\[
(\hat{g}_{ab}) = \Omega^2 \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & W \end{pmatrix}.
\]

We will call such coordinates *conformal Walker coordinates* for \((M, \hat{g}, [\pi^A])\).

There are at least two simple and natural choices of null tetrads related to conformal Walker coordinates. First, since the coordinates \(x\) and \(y\) label distinct \(\alpha\)-surfaces in the chart domain, for any LSR \(\pi^A\), one can write

\[
dx = \mu_A \pi_A, \quad dy = \nu_A \pi_A, \quad \mu_A \nu_A \neq 0.
\]

As in Law & Matsushita (2008) 2.4, given (conformal) Walker coordinates \((u, v, x, y)\), one can choose a unique (up to sign) LSR \(\pi^A\) so that (after possibly interchanging \(u\) and \(v\) and \(x\) and \(y\)) Law & Matsushita (2008) (2.8) holds:

\[
dx = \alpha_A \pi_A, \quad dy = \beta_A \pi_A, \quad \partial_u = \alpha_A \pi^A, \quad \partial_v = \beta_A \pi^A,
\]

where \(\{\alpha^A, \beta^A\}\) is a spin frame \((\beta^A \alpha_A = 1)\). We will refer to such (conformal) Walker coordinates as *oriented* because one can choose an atlas of such iff the \(\alpha\)-distribution is orientable, \(M\) itself being naturally oriented by the atlas of all Walker coordinates (see Law & Matsushita 2008 §1). The Walker null tetrad for \((M, g, [\pi^A])\) introduced in Law & Matsushita (2008) (2.11) is

\[
(\ell_a = dx = \alpha_A \pi_A, \quad \hat{m}_a = dy = \beta_A \pi_A, \quad n_a = du + \frac{a}{2} dx + \frac{c}{2} dy = \beta_A \xi_A, \quad m_a = -(dv + \frac{c}{2} dx + \frac{b}{2} dy) = \alpha_A \xi_A
\]

\[
\ell^a = \partial_u = \alpha^A \pi^A, \quad \hat{m}^a = \partial_v = \beta^A \pi^A, \quad n^a = -\frac{a}{2} \partial_u - \frac{c}{2} \partial_v + \partial_x = \beta^A \xi^A, \quad m^a = \frac{c}{2} \partial_u + \frac{b}{2} \partial_v - \partial_y = \alpha^A \xi^A
\]
where \( \{\pi^A, \xi^A\} \) is indeed a spin frame for the Walker geometry (these two spin frames corresponding to the Walker null tetrad are called Walker spin frames). If one takes \( L_a := \ell_a = dx \) and \( \tilde{M}_a = \tilde{m}_a = dy \), observe that \( dx \wedge dy = \epsilon_{AB}\pi^A \cdot \pi_B = \Omega^{-1/2}\tilde{\epsilon}_{AB}\pi^A \cdot \pi_B \), which suggests taking \( \hat{\pi}^A := \Omega^{-1/2}\pi^A \) as an LSR so that

\[
dx \wedge dy = 2\ell_a \tilde{m}_b = \epsilon_{AB}\pi^A \cdot \pi_B = \hat{\epsilon}_{AB}\hat{\pi}^A \cdot \hat{\pi}^B = 2L_a \tilde{M}_b.
\]

(3.7)

To maintain \( L_a = \ell_a \) and \( \tilde{M}_a = \tilde{m}_a \), write

\[
\ell_a = \alpha_A\pi^A = \hat{\alpha}_A\hat{\pi}^A = L_a \quad \text{and} \quad \tilde{m}_A = \beta_A\pi^A = \hat{\beta}_A\hat{\pi}^A = \tilde{M}_a.
\]

(3.8)

If one further defines \( N_a := \Omega^2 n_a, M_a := \Omega^2 m_a, \) and \( \hat{\xi}^A := \Omega^{3/2}\xi^A \), then

\[
\hat{\pi}^A = \Omega^{-1/2}\pi^A \quad \hat{\xi}^A = \Omega^{1/2}\xi^A \quad \hat{\alpha}_A = \Omega^{1/2}\alpha_A \quad \hat{\beta}_A = \Omega^{1/2}\beta_A
\]

(3.9)

where, of course, hatted objects have indices raised and lowered with respect to the geometry of \( (M, \hat{g}) \). Furthermore,

\[
L_a = \ell_a \quad \tilde{M}_a = \tilde{m}_a \quad N_a = \Omega^2 n_a \quad M_a = \Omega^2 m_a
\]

(3.10)

and \( \{L^a, N^a, M^a, \tilde{M}^a\} \) is a null tetrad for the geometry \( (M, \hat{g}) \) \( (\hat{g}_{ab} = 2L(a)N_b - 2M(a)\tilde{M}_b) \). Moreover, \( \{\hat{\alpha}^A, \hat{\beta}^A\} \) and \( \{\hat{\pi}^A, \hat{\xi}^A\} \) are spin frames in the geometry of \( (M, \hat{g}) \) and, indeed, the spin frames associated to the null tetrad \( \{L^a, N^a, M^a, \tilde{M}^a\} \), so that

\[
L^a = \hat{\alpha}^A\hat{\pi}^A \quad N^a = \hat{\beta}^A\hat{\xi}^A \quad M^a = \hat{\alpha}^{\frac{A}{\beta}}\hat{\xi}^A \quad \tilde{M}^a = \hat{\beta}^{\frac{A}{\beta}}\hat{\pi}^A
\]

(3.11)

This choice of spin frames for \( (M, \hat{g}) \) is of the form (2.11–15), with

\[
v_0 = v_1 = -\frac{1}{2} \quad w_0 = -\frac{3}{2} \quad w_1 = \frac{1}{2},
\]

(3.12)

whence the spin coefficients for \( (M, \hat{g}) \) with respect to these spin frames can be obtained directly from (2.18–19) using the formulae for the spin coefficients for the Walker geometry \( (M, g) \) with respect to the Walker spin frames given in Law (2008) (5.6). The spin frames \( \{\hat{\alpha}^A, \hat{\beta}^A\} \) and \( \{\hat{\pi}^A, \hat{\xi}^A\} \) are also convenient for taking components of the curvature spinors in (2.8) as the right hand side can be evaluated in terms of the components of the curvature spinors for \( (M, g) \) with respect to the Walker spin frames, see Law & Matsushita (2008) §2, and components of \( \Upsilon_a \). In particular,
where formulae for the nonzero spin coefficients of \((M, g)\) with respect to the Walker spin frames in terms of \(a, b, \) and \(c\) can be substituted from Law (2008) (5.6). Alternatively, one may instead require \(L^a = \ell^a\) and \(\tilde{M}^a = \tilde{m}^a\) so as to maintain this local frame for \(Z[\pi]\). By similar reasoning as above, since now
\[
2L^a\tilde{M}^b = 2\ell^a\tilde{m}^b = c^{AB}\pi^A\pi^B = \Omega c^{AB}\pi^A\pi^B,
\]
one is led to define
\[
\hat{\pi}^A := \Omega^{1/2}\pi^A \quad \hat{\xi}^A := \Omega^{-3/2}\xi^A \quad \hat{\sigma}^A := \Omega^{-1/2}\sigma^A \quad \hat{\beta}^A := \Omega^{-1/2}\beta^A,
\]
whence
\[
\hat{\pi}^A = \Omega^{3/2}\pi^A \quad \hat{\xi}^A = \Omega^{-1/2}\xi_A \quad \hat{\sigma} = \Omega^{1/2}\sigma \quad \hat{\beta} = \Omega^{1/2}\beta.
\]
Now
\[
L^a := \hat{\sigma}^A \hat{\pi}^A = \ell^a \quad N^a := \hat{\beta}^A \hat{\xi}^A = \Omega^{-2}n^a \quad M^a := \hat{\alpha}^A \hat{\hat{\alpha}}^A = \Omega^{-2}m^a \quad \tilde{M}^a := \hat{\beta}^A \hat{\pi}^A = \tilde{m}^a,
\]
and
\[
L_a = \Omega^2\ell_a \quad N_a = n_a \quad M_a = m_a \quad \tilde{M}_a = \Omega^2\tilde{m}_a,
\]
so
\[
\hat{g}_{ab} = 2L_aN_b - 2M_a\tilde{M}_b = \Omega^2(2\ell_a\tilde{m}_b - 2m_a\tilde{m}_b) = \Omega^2g_{ab}.
\]
Of course, \(\{\hat{\sigma}^A, \hat{\beta}^A\}\) and \(\{\hat{\pi}^A, \hat{\xi}^A\}\) are the spin frames associated to the null tetrad \(\{L^a, N^a, M^a, \tilde{M}^a\}\) for \((M, \hat{g})\). This alternative choice of spin frames is again of the form (2.11–15), but with
\[
v_0 = v_1 = -\frac{1}{2} \quad w_0 = \frac{1}{2} \quad w_1 = -\frac{3}{2}
\]
whence
\[
\begin{array}{c|c|c|c|c|c}
\hat{\ell} & \hat{\kappa} & \hat{\hat{\ell}} & \hat{\hat{\kappa}} & \Omega^0. & \frac{1}{2}D\omega \\
\hat{\alpha} & \hat{\beta} & \hat{\hat{\alpha}} & \hat{\hat{\beta}} & \Omega^0. & -\frac{1}{2}\Delta\omega \\
\hat{\hat{\beta}} & \hat{\hat{\alpha}} & \hat{\hat{\hat{\alpha}}} & \hat{\hat{\hat{\beta}}} & \Omega^0. & 0 \\
\hat{\gamma} & \hat{\hat{\gamma}} & \hat{\hat{\gamma}} & \hat{\hat{\hat{\gamma}}} & \Omega^0. & -\frac{1}{2}D\omega \\
\end{array}
\]
whence
\[
\begin{array}{c|c|c|c|c|c}
\hat{\ell} & \hat{\kappa} & \hat{\hat{\ell}} & \hat{\hat{\kappa}} & \Omega^0. & \frac{1}{2}\Omega^2D\omega \\
\hat{\alpha} & \hat{\beta} & \hat{\hat{\alpha}} & \hat{\hat{\beta}} & \Omega^0. & 0 \\
\hat{\hat{\beta}} & \hat{\hat{\alpha}} & \hat{\hat{\hat{\alpha}}} & \hat{\hat{\hat{\beta}}} & \Omega^0. & -\delta\omega \\
\hat{\gamma} & \hat{\hat{\gamma}} & \hat{\hat{\gamma}} & \hat{\hat{\hat{\gamma}}} & \Omega^0. & -\frac{1}{2}\Omega^2D\omega \\
\end{array}
\]
where again the nonzero spin coefficients on the right-hand sides are those for \((M, g)\) with respect to the Walker spin frames and are given in Law (2008) (5.6) in terms of \(a, b, \) and \(c\).

In particular, both (3.14) and (3.22) confirm that \(\hat{\kappa} = \hat{\sigma} = 0\), which are the conditions for the \(\alpha\)-distribution to be integrable (see Law 2008 (6.2.4)) and that \((M, \hat{g}, [\pi^A])\) is Walker iff \(\hat{\rho} = \hat{\tau} = 0\) (see Law 2008 (6.2.40)), i.e., iff \(D\omega = \Delta\omega = 0\) as required by (3.1).
When \((M, \hat{g}, [\pi^A])\) is itself Walker, (3.13-14) and (3.21–22) simplify considerably. One also readily observes, for example, that, employing either set of spin frames (3.9) or (3.15–18), by (2.8)

\[
\Phi_{\hat{A}B\hat{A}'B'} \hat{\pi}^A \hat{\pi}^{B'} = \Omega^k \left[ \Phi_{\hat{A}B\hat{A}'B'} \pi^A \pi^{B'} + \Upsilon_{AA'} \pi^A \Upsilon_{BB'} \pi^{B'} - \pi^A \pi^{B'} \nabla_{AA'} \Upsilon_{BB'} \right],
\]

for an appropriate value of \(k\). The first term on the right-hand side vanishes by Law & Matsushita (2008) 2.5; the second term directly from (3.1); and, upon writing \(\Upsilon_{BB'} = \kappa_B \pi^{B'}\) by virtue of (3.1), the third term is seen to vanish by (1.3). Thus, \(\Phi_{\hat{A}B\hat{A}'B'} \hat{\pi}^A \hat{\pi}^{B'}\) vanishes, confirming that \([\pi^A]\) is a Ricci principal spinor (RPS), as it must be in a Walker geometry (Law & Matsushita 2008, 2.5).

Clearly, even when \((M, \hat{g}, [\pi^A])\) is itself Walker, the metric components of \(\hat{g}_{ab} = \Omega^2 g_{ab}\) with respect to \((u, v, x, y)\) are not of the form \((3.2), i.e., (u, v, x, y)\) are not Walker coordinates for \((M, \hat{g}, [\pi^A])\), and the null tetrads and spin frames of (3.9–10) and (3.15–18) are not Walker null tetrads nor Walker spin frames. But it is easy to construct Walker coordinates for \((M, \hat{g}, [\pi^A]\) from \((u, v, x, y)\) if desired. Since the coordinates \(x\) and \(y\) label distinct \(\alpha\)-surfaces in the chart domain, for any LSR \(\pi^A\), as usual one can write

\[
dx = \mu_A \pi_A, \quad dy = \nu_A \pi_A, \quad \nu^A \mu_A \neq 0. \tag{3.23}
\]

Now define

\[
\hat{U}^a := \hat{g}^{ab} \mu_B \pi_B = \Omega^{-2} \hat{g}^{ab} \mu_B \pi_B =: \Omega^{-2} U^a, \quad \hat{V}^a := \hat{g}^{ab} \nu_B \pi_B = \Omega^{-2} \hat{g}^{ab} \nu_B \pi_B =: \Omega^{-2} V^a. \tag{3.24}
\]

Following Law & Matsushita (2008) 2.3, one can choose solutions \(\hat{u}\) and \(\hat{v}\) of the system of equations

\[
\hat{U} \hat{u} = 1 = \hat{V} \hat{v}, \quad \hat{U} \hat{v} = 0 = \hat{V} \hat{u},
\]

i.e.,

\[
\hat{U} \hat{u} = \Omega^2 = \hat{V} \hat{v}, \quad \hat{U} \hat{v} = 0 = \hat{V} \hat{u}.
\]

The coordinates \(u\) and \(v\) are constructed in Law & Matsushita (2008) 2.3 so that \(U = \partial_u\) and \(V = \partial_v\). By (3.1), \(\Omega\) is constant on \(\alpha\)-surfaces, i.e., a function of \(x\) and \(y\) only, when expressed in terms of the Walker coordinates \((u, v, x, y)\). Hence, one can choose

\[
\hat{u} := \Omega^2 u, \quad \hat{v} := \Omega^2 v, \quad \hat{x} := x, \quad \hat{y} := y, \tag{3.25}
\]

and \((\hat{u}, \hat{v}, \hat{x}, \hat{y})\) are Walker coordinates for \((M, \hat{g}, [\pi^A]\). Indeed,

\[
(\hat{\partial}_k) := (\partial_u \partial_v \partial_x \partial_y) = (\partial_u \partial_v \partial_x \partial_y), \quad \left(\begin{array}{cccc}
\Omega^{-2} & 0 & -2\Omega^{-3}\Omega_y \hat{u} & -2\Omega^{-3}\Omega_y \hat{u} \\
0 & \Omega^{-2} & -2\Omega^{-3}\Omega_x \hat{v} & -2\Omega^{-3}\Omega_y \hat{v} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) =: (\hat{\partial}_k)(J^k).
\]

whence the components of \(\hat{g}_{ab}\) with respect to \((\hat{u}, \hat{v}, \hat{x}, \hat{y})\) are

\[
\tau_{J} \Omega^2 \begin{pmatrix} 0 & 0 & 12 \\ 0 & 12 & W \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 12 \\ 12 & W \end{pmatrix}
\]

where

\[
\hat{W} := \Omega^2 \left( \begin{array}{cc}
a - 4\Omega^{-3}\Omega_x \hat{u} & c - 2\Omega^{-3}(\Omega_x \hat{v} + \Omega_y \hat{u}) \\
c - 2\Omega^{-3}(\Omega_x \hat{v} + \Omega_y \hat{u}) & b - 4\Omega^{-3}\Omega_y \hat{v}
\end{array}\right) =: \begin{pmatrix} \hat{a} & \hat{c} \\ \hat{c} & \hat{b} \end{pmatrix}.
\]

If \((u, v, x, y)\) are oriented Walker coordinates for \((M, g, [\pi^A]\) and the LSR \(\pi^A\) is chosen as in (3.5), then

\[
dx \wedge dy = \epsilon_{AB} \pi_A \pi_B = \epsilon_{AB} \hat{\pi}_A \hat{\pi}_B
\]
if one puts \( \hat{\pi}_{A'} := \Omega^{-1/2} \pi_{A'} \); equivalently,
\[
\partial_\alpha \wedge \partial_\beta = \Omega^{-4} \partial_\alpha \wedge \partial_\beta = \Omega^{-4} \epsilon^{\alpha \beta \pi_{A'} \pi_{B'}} = \hat{\epsilon}^{\alpha \beta \pi_{A'} \pi_{B'}},
\]
where \( \hat{\pi}_{A'} = \Omega^{-3/2} \pi_{A'} \). Thus, \( (\hat{u}, \hat{v}, \hat{x}, \hat{y}) \) are oriented Walker coordinates for \( (M, \hat{g}, [\pi^A]) \) and \( \hat{\pi}_{A'} \) is an LSR for these coordinates as in (3.5). If \( \{ L^a, N^a, M^a, M^a \} \) denotes the Walker null tetrad for \( (M, \hat{g}, [\pi^A]) \) and the coordinates \( (\hat{u}, \hat{v}, \hat{x}, \hat{y}) \), then \( L_a = \hat{d} \hat{x} = dx = \ell_a = \alpha_A \pi_{A'} = \hat{\alpha}_A \hat{\pi}_{A'} \) if \( \hat{\alpha}_A := \Omega^{1/2} \alpha_A \). Similarly, \( \hat{M}_a = \hat{d} \hat{y} = dy = \hat{m}_a \) suggests putting \( \hat{\beta}_A := \Omega^{1/2} \beta_A \). From (3.6),
\[
\begin{align*}
N_a &= \hat{d} \hat{u} + \frac{\hat{c}}{2} \hat{d} \hat{x} + \frac{\hat{d}}{2} \hat{d} \hat{y} = \Omega^2 n_a + (\Omega y u - \Omega z v) \hat{m}_a = \hat{\beta}_A [\Omega^{3/2} \xi_A + \Omega^{1/2} (\Omega y u - \Omega z v) \pi_{A'}]; \\
\hat{M}_a &= - \left( \frac{\hat{c} d \hat{v} + \hat{b} \hat{d} \hat{x} + \hat{b} \hat{d} \hat{y}}{2} \right) = \Omega^2 m_a + (\Omega y u - \Omega z v) \ell_a = \hat{\alpha}_A [\Omega^{3/2} \xi_A + \Omega^{1/2} (\Omega y u - \Omega z v) \pi_{A'}].
\end{align*}
\]
One can now identify the Walker spin frames \( \{ \hat{\alpha}^A, \hat{\beta}^A \} \) and \( \{ \hat{\pi}^A', \hat{\xi}^A' \} \) for \( (M, \hat{g}, [\pi^A]) \) and the coordinates \( (\hat{u}, \hat{v}, \hat{x}, \hat{y}) \):
\[
\hat{\alpha}^A := \Omega^{-1/2} \alpha^A \quad \hat{\beta}^A := \Omega^{-1/2} \beta^A \quad \hat{\pi}^A' := \Omega^{-3/2} \pi^A' \quad \hat{\xi}^A' := \Omega^{-1/2} \pi^A' + \Omega^{-1/2} (\Omega y u - \Omega z v) \pi_{A'}. \tag{3.26}
\]
The primed Walker spin frames for \( (M, \hat{g}, [\pi^A]) \) and \( (M, g, [\pi^A]) \) are not related as in (2.14) so (2.19) is inapplicable. Of course, the spin coefficients for the Walker spin frames for \( (M, \hat{g}, [\pi^A]) \) may be obtained directly from Law (2008) (5.6).

After these preliminaries on Walker geometry, we turn to the consideration of arbitrary real ASO-geometries.

### 3.27 Proposition

A real \( \alpha \)-geometry \( (M, h, [\pi^A]) \) is algebraically special iff locally conformally Walker, i.e., iff for each \( p \in M \), there exists a neighbourhood \( U_p \) and a metric \( g \) such that \( (U_p, g, [\pi^A]) \) is Walker and \( h = \Omega^2 g \) on \( U_p \) for some \( \Omega : U_p \to \mathbb{R}^+ \).

Proof. We seek a neighbourhood \( U_p \) and function \( \chi : U_p \to \mathbb{R}^+ \) such that the real ASO-geometry \( (U_p, \chi^2 h, [\pi^A]) \) is Walker, i.e., such that for any, whence every, LSR \( \pi^A', (1.2) \) holds with respect to the metric \( \hat{h} := \chi^2 h \). Hence, one requires a \( \chi \) such that
\[
0 = \pi^A' \hat{\nabla}_b \pi^A' = \pi^A' [\nabla_b \pi^A' + \epsilon_{B'R} \Upsilon_{BX'} \pi^{X'}] = \hat{S}_b + \pi_{B'} \Upsilon_{B'X'} \pi^{X'},
\]
where \( \Upsilon_b = \nabla_b \ln \chi \). Now, (1.3) is equivalent to \( \hat{S}_b = \pi^A' \nabla_b \pi^A' = : \pi_B \pi_{B'} \), for some spinor \( \omega_B \) (see Law 2008, (6.2.6) et seq., for the significance of \( \hat{S}_b \) and \( \omega_B \)). Thus, one seeks a function \( f : U_p \to \mathbb{R} \) satisfying
\[
f := \ln \chi^{-1}. \tag{3.27.1}
\]
(Notice that (3.27.1) does reduce to (3.1) when \( (M, h, [\pi^A]) \) is in fact Walker.) This equation has necessary and sufficient integrability condition for local solvability:
\[
\pi^A' \pi_{B'} \nabla_{B'} \omega_A = \omega_A \pi_{B'} \nabla_{B'} \pi^A' \tag{3.27.2}
\]
(see Law 2008, (6.2.16); Penrose & Rindler 1986, (7.3.20)). Equation (1.3) is also equivalent to \( \pi_{B'} \nabla_{B'} \pi^A' = \eta_B \pi_{A'} \), for some spinor \( \eta_B \) (again, see Law 2008, (6.2.6) et seq.), whence (3.27.2) becomes
\[
\pi^A' \pi_{B'} \nabla_{B'} \omega_A = \eta_B \omega_B \pi_{A'}. \tag{3.27.3}
\]
Now, on the one hand
\[
\pi_{B'} \nabla_{B'} S_a = \pi_{B'} \nabla_{B'} (\omega_A \pi_{A'}) = \pi_{A'} \pi_{B'} \nabla_{B'} \omega_A + \omega_A \pi_{B'} \nabla_{B'} \pi_{A'} = \pi_{A'} \pi_{B'} \nabla_{B'} \omega_A + \eta_A \omega_A \pi_{A'},
\]
while on the other,
\[ \pi^B_i \nabla^A_{B} S^a = \pi^B_i \nabla^A_{B} (\pi^C \nabla^a \pi^{C'}) \]
\[ = (\pi^B_i \nabla^A_{B} \pi^C) (\nabla^a \pi^{C'}) + \pi^C \pi^B_i \nabla^A_{B} \nabla^a \pi^{C'} \]
\[ = \eta^A \pi^C \nabla^a \pi^{C'} + \pi^C \pi^B_i \nabla^A_{B} \nabla^a \pi^{C'}, \]
whence
\[ \pi_A^i \pi^B_i \nabla^A_{B} \omega_A = \pi_C \pi^B \nabla^A_{B} \nabla^a \pi^{C'}. \] (3.27.4)

By Law (2008), (6.2.13)(c),
\[ \pi_C \pi^B_i \nabla^A_{B} \nabla^a \pi^{C'} = \eta^B \omega_B \pi^{A'} + 2 \tilde{\Psi}_{A'B'C'D'} \pi^B_i \pi^{C'} \pi^{D'}. \]

Hence, (3.27.4) becomes
\[ \pi_A^i \pi^B_i \nabla^A_{B} \omega_A = \eta^B \omega_B \pi^{A'} + 2 \tilde{\Psi}_{A'B'C'D'} \pi^B_i \pi^{C'} \pi^{D'}, \]
i.e., the integrability condition (3.27.3) is satisfied iff \( \tilde{\Psi}_{A'B'C'D'} \pi^B_i \pi^{C'} \pi^{D'} = 0 \). Thus, iff \( [\pi^A_i] \) is a multiple WPS can one solve (3.27.1) for \( f \) on some neighbourhood \( U_p \) of \( p \). Taking \( \Omega = \chi^{-1} = \exp(f) \), then \( h = \Omega^2 g \) on \( U_p \), with \( (U_p, g, [\pi^A_i]) \) Walker.

3.28 Remark
As a corollary of the computations in (3.27), one deduces that for any ASo-geometry (real or complex as the relevant computations are valid in both cases), and any LSR \( \pi^A_i \) of \( [\pi^A_i] \),
\[ \pi_A^i \pi^C \square \pi^{C'} = 2 \left[ \eta^B \omega_B \pi^{A'} + \tilde{\Psi}_{A'B'C'D'} \pi^B_i \pi^{C'} \pi^{D'} \right]. \]

In a Walker geometry, each of the terms in this equation vanishes.

Proof. One has, using standard results on curvature (see, for example, Law & Matsushita 2008, Appendix 2, or Law 2008, Appendix)
\[ \pi_C^i \pi^B_i \nabla^A_{B} \nabla^a \pi^{C'} = \pi^C_i \pi^B_i \nabla^a \pi^{C'} \]
\[ = \pi^C_i \pi^B_i \left[ \square_B \pi^{A'} + \frac{1}{2} \pi^B \square \pi^{C'} \right] \]
\[ = \tilde{\Psi}_{A'B'C'D'} \pi^B_i \pi^{C'} \pi^{D'} + \frac{1}{2} \pi_A^i \pi^{C'} \square \pi^{C'}. \]

Equating with the alternate expression given in the proof of (3.27) yields the assertion.

Thus, the local geometry of any real ASo-geometry \( (M, h, [\pi^A_i]) \) can be described in a suitable open set \( U \subseteq M \), as \( (U, \Omega^2 g, [\pi^A_i]) \), for some Walker geometry \( (U, g, [\pi^A_i]) \). It is straightforward to combine the results of Law & Matsushita (2008) and Law (2008) on Walker geometry with the conformal rescaling formulae of §2 to obtain this description. We will therefore denote the connection and curvature quantities of \( (M, h) \) by hatted symbols; unhatted symbols will refer to a (locally) conformally related Walker geometry.

The Weyl curvature spinors \( \hat{\Psi}_{A'B'C'D'} \) and \( \hat{\Psi}_{A'B'C'D'} \) of \( (U, h, [\pi^A_i]) \) of course coincide with those, \( \Psi_{A'B'C'D'} \) and \( \Psi_{A'B'C'D'} \) respectively, of \( (U, g, [\pi^A_i]) \). The Weyl curvature endomorphisms of \( (U, h, [\pi^A_i]) \) are simply scalar multiples of those for \( (U, g, [\pi^A_i]) \), \( \hat{\Psi}_{A'B'C'D'} = \Omega^{-2} \Psi_{A'B'C'D'} \) and \( \hat{\Psi}_{A'B'C'D'} = \Omega^{-2} \Psi_{A'B'C'D'} \), so, for example, the analysis of Law & Matsushita (2008) between (2.21) and (2.22) applies equally well to \( \hat{\Psi}_{A'B'C'D'} \) with exactly the same results for the algebraic classification as for \( \hat{\Psi}_{A'B'C'D'} \) recorded in Law & Matsushita
(2008) 2.6 (the eigenvalues of the Weyl curvature endomorphisms for \((U, h, [\pi^A])\) of course being \(\Omega^{-2}\) times those of the corresponding Weyl curvature endomorphism for \((U, h, [\pi^A])\)).

Given conformal oriented Walker coordinates, when convenient to employ a null tetrad \(\{L^a, N^a, M^a, \tilde{M}^a\}\) and spin frames \(\{\delta^A, \beta^A\}\) and \(\{\hat{\delta}^A, \hat{\beta}^A\}\) for \((U, h, [\pi^A])\), we will exploit those of (3.15–19), in which context \(\pi^A\) is always the LSR (unique up to sign) which satisfies (3.5) and which is the first element of the associated Walker primed spin frame for \((U, g, [\pi^A])\). The spin coefficients for \((U, h, [\pi^A])\) with respect to the chosen spin frames (3.15–16) are thus given by (3.21–22). As previously noted, one can substitute for \((u, v, x, y)\) within \((U, h, [\pi^A])\), \((u, v, x, y)\), and \((\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y})\) \(\alpha^A \subseteq \{\hat{\delta}^A, \hat{\beta}^A\}\), \((u, v, x, y)\), \((u, v, x, y)\), and \((\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y})\) \(\alpha^A \subseteq \{\hat{\delta}^A, \hat{\beta}^A\}\), \((u, v, x, y)\), \((u, v, x, y)\), and \((\tilde{u}, \tilde{v}, \tilde{x}, \tilde{y})\)

\[\delta_A := \pi^A \nabla_{AA'} = \alpha_A \delta_b \nabla_b - \beta_A \delta_b \nabla_b = \alpha_A \Delta - \beta_A D \]
\[= \alpha_A \partial_c - \beta_A \partial_a, \quad \text{when acting on functions.} \] (3.29)

This spinor operator in effect represents the (flat) induced connection, with respect to the Walker geometry, within \(\alpha\)-surfaces; see Law (2008) (6.2.54) for properties of \(\delta_A\).

Turning to the Ricci curvature, a natural geometric condition on \((M, h, [\pi^A])\) is that \([\pi^A]\) be a RPS, i.e., \(\hat{\Phi}_{ABAB'}B' \eta^A \eta^{B'} = 0\), for any LSR \(\eta^A\) of \([\pi^A]\). (Note that, by Law 2008 (6.2.18), any solution \([\pi^A]\) of (1.3) which is also a RPS is automatically a multiple WPS.) Working in \((U, h)\) with conformal oriented Walker coordinates, since the \(\tilde{\pi}^A\) of (3.15) is an LSR of \([\pi^A]\), the condition is \(\hat{\Phi}_{00} = \Phi_{10} = \Phi_{20} = 0\). From (2.8), and noting that \([\pi^A]\) is a PS of \(\Phi_{ABAB'}\) (Law & Matsushita 2008, 2.5),

\[\Phi_{ABAB'}\tilde{\pi}^A \tilde{\pi}^{B'} = \Omega \left( \Upsilon_{AA'} \pi^A \Upsilon_{BB'} \pi^{B'} - \pi^{B'} \pi^A \nabla_{AA'} \Upsilon_{BB'} \right) \]
\[= \Omega \left[ (\delta_A \omega)(\delta_B \omega) - \pi^{B'} \pi^A \nabla_{AA'} \Upsilon_{BB'} \right] \]
\[= \Omega \left[ (\delta_A \omega)(\delta_B \omega) - \delta_A \delta_B \omega \right], \] (3.30)

since, by Law (2008) (5.8), the Walker spin frames are parallel on \(\alpha\)-surfaces (in particular \(\delta_A \pi^{B'} = 0\)).

3.31 Lemma

For a real AS\(\alpha\)-geometry \((M, h, [\pi^A])\), \([\pi^A]\) is a RPS iff, with respect to any conformal Walker coordinates \((u, v, x, y)\) on \(U \subseteq M\) for \((M, h, [\pi^A])\) (i.e., \((U, h) = (U, \Omega^2 g)\) and \((u, v, x, y)\) are Walker coordinates on \(U\) for the Walker geometry \((U, g, [\pi^A])\)), \(\Omega^{-1}\) is affine as a function of \(u\) and \(v\), i.e.,

\[\Omega(u, v, x, y) = \left[ M(x, y)u + N(x, y)v + K(x, y) \right]^{-1}, \]

for some functions \(M, N,\) and \(K\) of \((x, y)\).

Proof. Without loss of generality, one can suppose the conformal Walker coordinates are oriented. From (3.30), \([\pi^A]\) is a PS of \(\Phi_{ABAB'}\) iff

\[\delta_A \delta_B \omega = (\delta_A \omega)(\delta_B \omega). \] (3.31.1)

But (3.31.1) is equivalent to

\[\Omega \delta_A \delta_B \Omega = 2(\delta_A \Omega)(\delta_B \Omega), \] (3.31.2)

which is equivalent to

\[\delta_A \delta_B \left( \Omega^{-1} \right) = 0. \] (3.31.3)

By Law (2008) (6.2.54), (3.31.3) is equivalent to the assertion of the lemma. One can also obtain the desired result from the spin coefficient field equations Law (2008) (3.4). For example, Law (2008) (3.4a), in conjunction with (3.21–22), yields

\[-\hat{D} \hat{\rho} = \hat{\rho}^2 - \hat{\rho}(\hat{\xi} + \hat{\tilde{\xi}}) + \hat{\Phi}_{00}. \]
Substituting for these nonzero spin coefficients the expressions in (3.21–22), and noting $\hat{D} = D$ on functions for (3.15–19), results in $-D^2\omega = (D\omega)^2 - (D\omega)(2D\omega) + \Phi_{00}$, i.e.,

$$\Phi_{00} = -D^2\omega + (D\omega)^2,$$

which is the result obtained upon transvecting (3.30) by $\hat{A}^A \hat{A}^B$. Similarly, the equations resulting from transvecting (3.30) by $\hat{A}^A \hat{B}^B$ (or $\hat{B}^A \hat{A}^B$) and $\hat{B}^A \hat{B}^B$ also result, respectively, from Law (2008) (3.1) (c) (or (d)) and (e).

Note that one can always introduce Frobenius coordinates $(p, q, x, y)$ for the integrable $\alpha$-distribution $Z_{[\pi]}$ in $(\cal M, h, [\pi^A])$, with $x$ and $y$ constant on $\alpha$-surfaces. With respect to such Frobenius coordinates, the metric $h$ takes the form

$$(h_{ab}) = \begin{pmatrix} O & V \\ \not{V} & W \end{pmatrix}. $$

for some $V, W \in \mathbb{R}(2)$, with $W$ symmetric. Since $dx$ and $dy$ vanish on $\alpha$-surfaces, for any LSR $\pi^A$ one can write

$$dx = \mu_A \pi^A, \quad dy = \nu_A \pi^A, \quad \nu_{A\mu} \neq 0.$$

The vector fields $U^a := \mu^A \pi^A$ and $V^a := \nu^A \pi^A$ span $Z_{[\pi]}$. Repeating the computation in Law & Matsushita (2008) 2.3 yields $U^b \nabla_b V^a = (\nu \cdot \mu) S^a = -V^b \nabla_b U^a$, whence $[U, V]^a = 2(\nu \cdot \mu) S^a$, explicitly revealing how $S^a$ blocks the construction of Walker coordinates. If desired, as in Law & Matsushita (2008) 2.4, there is a unique, up to sign, LSR $\pi^A$ such that $\nu \cdot \mu = \pm 1$.

If one wishes to retain the freedom to choose any LSR, however, one can choose spin frames so that $\alpha^A = \pi^A, \alpha^A \propto \mu^A$ and $\nu^A \propto \nu^A$ (assuming the coordinates $x$ and $y$ are oriented so that $\nu \leftarrow \mu > 0$), whence $\ell_a \propto dx, \hat{m}_a \propto dy$. Doing so, write

$$\ell_a = Ldx, \quad \hat{m}_a = Mdy.$$ 

Now, completing $\ell_a$ and $\hat{m}_a$ to a null tetrad $\{\ell_a, n_a, m_a, \hat{m}_a\}$,

$$m_a = A_1 dp + B_1 dq + C_1 dx + D_1 dy, \quad n_a = A_2 dp + B_2 dq + C_2 dx + D_2 dy,$$

where the $A_i, B_i, C_i$ and $D_i, i = 1, 2$, are functions of $(p, q, x, y)$. With $x$ and $y$ held constant, each of $\phi_1 := A_1 dp + B_1 dq$ and $\phi_2 := A_2 dp + B_2 dq$ are one-forms on the $\alpha$-surface in question. They are linearly dependent iff $A_1 B_2 - A_2 B_1 = 0$, in which case $B_2 m_a - B_1 n_a \in (dx, dy)_\mathbb{R} = \langle \ell_a, \hat{m}_a \rangle_\mathbb{R}$. Linear independence of the null tetrad one-forms then entails $B_1 = B_2 = 0$. Similarly, $A_1 = A_2 = 0$. But then $m_a$ and $n_a$ would be linear combinations of $\ell_a$ and $\hat{m}_a$. Hence, $\phi_1$ and $\phi_2$ are linearly independent. Now any one-dimensional distribution is integrable. In a two-dimensional surface, such a distribution is also of co-dimension one. By the differential form version of Frobenius' theorem, any one-form on a two-surface is therefore proportional to a gradient. Hence, one can write

$$\phi_1 = A_1 dp + B_1 dq =: \zeta dv, \quad \phi_2 = A_2 dp + B_2 dq =: \xi du,$$

where $\zeta, \xi, u$ and $v$ are functions of $(p, q, x, y)$. Now, on any $\alpha$-surface,

$$du = u_p dp + u_q dq = \frac{A_2}{\zeta} dp + \frac{B_2}{\xi} dq, \quad dv = v_p dp + v_q dq = \frac{A_1}{\xi} dp + \frac{B_1}{\zeta} dq,$$

whence the Jacobian of $(u, v, x, y)$ as a function of $(p, q, x, y)$ is nonsingular. Hence, one can use $(u, v, x, y)$ as Frobenius coordinates. Without restriction to a given $\alpha$-surface, one has

$$\zeta dv = A_1 dp + B_1 dq + C_2 dx + D_3 dy, \quad \xi du = A_2 dp + B_2 dq + C_4 dx + D_4 dy.$$
whence, for some functions \( \hat{C}_1, \hat{D}_1, \hat{D}_2, A, B, C, D, \)

\[
m_a = \zeta dv + \hat{C}_1 dx + \hat{D}_1 dy = \zeta(dx + Cdx + Ddy) \quad n_a = \zeta du + \hat{C}_2 dx + \hat{D}_2 dy = \zeta(dx + Adx + Bdy)
\]

and

\[
h_{ab} = 2[\ell_{(a}n_{b)} - m_{(a}\tilde{m}_{b)}]
\]

\[
= 2[L\zeta dx - M\zeta dy + L\zeta(Adx + Bdy) - M\zeta dy(Cdx + Ddy)]
\]

\[
= 2[\phi dx + \psi dy + \phi dx(Adx + Bdy) + \psi dy(Cdx + Ddy)]
\]

where \( \phi := L\zeta \) and \( \psi := M\zeta \), i.e.,

\[
h_{ab} = \begin{pmatrix}
0 & 0 & \phi & 0 \\
0 & 0 & \psi & 0 \\
\phi & \psi & 2\phi A + \psi C & \phi B + \psi C \\
0 & \psi & \phi B + \psi C & 2\psi D
\end{pmatrix}.
\] (3.32)

(3.32) is a generalization of Walker’s form of the metric when the integrable \( \alpha \)-distribution is not assumed to be parallel. Plebański and Robinson (1976), (1977) derived this coordinate form in the context of complex general relativity to study algebraically special Weyl curvature spinors. Their approach was developed into the hyperheavenly formalism for such complex space-times, in which the Einstein condition is reduced to a single PDE, the hyperheavenly equation, see Finley et al. (1976), Boyer et al. (1980). The hyperheavenly formalism is obviously equally applicable to real (four-dimensional) neutral geometry. The derivation of the hyperheavenly equation involves making further judicious changes of coordinates (in particular, the assumption of algebraic degeneracy of the relevant Weyl curvature spinor permits a change of coordinates which results in \( \phi = \psi \)) and other choices which simplify the expression of the Einstein condition. Two separate cases arise: the expanding and nonexpanding cases. In the real neutral context, the nonexpanding case corresponds to Walker geometry; an independent derivation of the hyperheavenly equation in this context, utilizing the spinor approach of Law & Matsushita (2008), was presented in Law (2008), (6.2.45–63). Real AS\(\alpha\)-geometries which are not Walker correspond to the expanding case in the hyperheavenly formalism. The approach to characterizing the local geometry of real AS\(\alpha\)-nonWalker geometries developed in this paper, based on the local conformally Walker structure, takes full advantage of Walker geometry. Though our approach has been independent of the hyperheavenly formalism, further development closely parallels the derivation of the hyperheavenly equation.

We next show it is possible to choose conformal (oriented) Walker coordinates which simplify the result of (3.31).

**3.33 Lemma**

For a real AS\(\alpha\)-geometry \((M, h, [\pi^A])\) for which \([\pi^A]\) is a RPS, every \( p \in M \) has a neighbourhood \( U \) admitting conformal (oriented) Walker coordinates \((U, V, X, Y)\) (i.e., \((U, h) = (U, \Omega^2 g)\)) for which \((U, V, X, Y)\) are (oriented) Walker coordinates for the Walker geometry \((U, g, [\pi^A])\) such that

\[
\Omega(U, V, X, Y) = (MU + NV)^{-1}
\]

for constants \( M \) and \( N \).

Proof. By (3.31), one can choose conformal (oriented) Walker coordinates \((u, v, x, y)\) for \((M, h, [\pi^A])\) on a neighbourhood of \( p \) so that \( \Omega(u, v, x, y) = M(x, y)u + N(x, y)v + K(x, y) \). We seek new coordinates \((U, V, X, Y)\), on a possibly smaller neighbourhood, which are still conformal (oriented) Walker coordinates but for which the desired result holds. Following Plebański & Robinson (1976), consider the following coordinate transformation:

\[
U = [uY_y - vY_x + P(x, y)]H(x, y) \quad X = X(x, y)
\]

\[
V = [vX_x - uX_y + Q(x, y)]H(x, y) \quad Y = Y(x, y),
\]
where \( X, Y, P, Q, H \) are yet-to-be-determined functions. The Jacobian for this transformation is

\[
J := \frac{\partial(U,V,X,Y)}{\partial(u,v,x,y)} = \begin{pmatrix} D & G \\ 0_2 & F \end{pmatrix},
\]

where

\[
D = H \begin{pmatrix} Y_y - Y_x \\ -X_y \\ X_x \end{pmatrix}, \quad F = \begin{pmatrix} X_x \\ Y_x \\ Y_y \end{pmatrix}.
\]

As \((u, v, x, y)\) are Walker coordinates for \((U, g, [\pi^A])\), so also are \((U, V, X, Y)\) iff \( F = \tau D^{-1} \), see Law & Matsushita (2008) (1.2), which is the case iff \( H(x, y) = (X_x Y_y - Y_x X_y)^{-1} \). This condition therefore determines \( H \) in terms of \( X \) and \( Y \).

For constants \( K_1 \) and \( K_2 \),

\[
K_1 U + K_2 V = (K_1 Y_y - K_2 X_y) H u + (K_2 X_x - K_1 Y_x) H v + (K_1 P + K_2 Q) H.
\]

To prove the lemma, one must therefore show that one can choose \( X, Y, P \) and \( Q \) so that \( (X_x Y_y - Y_x X_y) \neq 0 \) and

\[
\begin{pmatrix} M(x, y) \\ N(x, y) \end{pmatrix} = H \begin{pmatrix} Y_y - Y_x \\ -X_y \\ X_x \end{pmatrix} \cdot \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \quad \text{for } H(K_1 P + K_2 Q) = K.
\]

We may construe this problem as that of finding a change of coordinates \((X(x, y), Y(x, y))\) on some neighbourhood of the origin in \( \mathbb{R}^2 \) so that the given vector field \( V(x, y) := M(x, y) \partial_x + N(x, y) \partial_y \) transforms to the constant vector field \( Z(x, y) := K_1 \partial_X + K_2 \partial_Y \) and so that \( K_1 P + K_2 Q = KH^{-1} \). Given that \( V(x, y) \) is nonzero at the origin, it is always possible to find such a change of coordinates; once \( X \) and \( Y \) are chosen and \( H \) thereby determined, the final condition has many solutions for \( P \) and \( Q \).

The value of (3.33) is that such conformal (oriented) Walker coordinates simplify subsequent analysis of the Ricci curvature. This choice corresponds to a similar manoeuvre in the hyperheavenly formalism, e.g., in Finley et al. (1976), between (3.1) and (3.3), and in Boyer et al. (1980) at (5.21). In the remainder of this section, we will therefore employ such conformal oriented Walker coordinates \((U, V, X, Y)\) and will write

\[
\Omega = (MU + NV)^{-1} \quad \text{and} \quad W = \begin{pmatrix} A & C \\ C & B \end{pmatrix}, \tag{3.34}
\]

where \( M \) and \( N \) are constants and \( W \) is the analogue of \( W \) in (3.2–3) for \((U, V, X, Y)\). It is convenient to record:

\[
\frac{\partial^{p+q} \Omega}{\partial U^p \partial V^q} = (-1)^{p+q}(p+q)!M^p N^q \Omega^{p+q+1}; \quad \delta_A \Omega = -\Omega^2 (\alpha_A N - \beta_A M); \quad \delta_A \delta_B \Omega = 2\Omega^2 (\delta_A \Omega)(\delta_B \Omega). \tag{3.35}
\]

As an example of the utility of (3.33), consider the Ricci scalar curvature \( S = -24 \Lambda \). From (2.8), one needs to evaluate \( \Box \Omega \) in the Walker geometry \((U, g, [\pi^A])\). From Law & Matsushita (2008) (3.9), for any function \( F \), with respect to Walker coordinates \((U, V, X, Y)\) and with notation as in (3.34),

\[
\Box F = -AF_{UU} - 2CF_{UV} - BF_{VV} + 2F_{UX} + 2F_{VY} - (AV + CV) F_U - (BV + CU) F_V.
\]

In particular,

\[
\Box \Omega = -2(AM^2 + 2CMN + BN^2)\Omega^3 + (AV + CV)M\Omega^2 + (BV + CU)N\Omega^2;
\]

whence, noting from Law & Matsushita (2008) A1.6 that \(-24 \Lambda = S = A_{UU} + B_{VV} + 2C_{UV}\),

\[
\Lambda = \Omega^{-5} \left[ \Lambda - \frac{1}{2}(AM^2 + 2CMN + BN^2)\Omega^2 + \frac{1}{4}((AV + CV)M + (BV + CU)N)\Omega \right]
\]

\[
= \frac{\Omega^{-5}}{24} \left[ (A_{UU} \Omega^3 - 6A_{U}M\Omega^4 + 12AM^2\Omega^5) + (B_{VV} \Omega^3 - 6B_{V}N\Omega^4 + 12BN^2\Omega^5) \right]
\]

\[
= \frac{\Omega^{-5}}{24} \left[ (A\Omega^3)_{UU} + (B\Omega^3)_{VV} + 2(C\Omega^3)_{UV} \right]
\]

\[
= -\frac{\Omega^{-5}}{24} \delta_A \delta_B H^{AB} \quad \text{where} \quad (H^{AB}) := \Omega^3 W. \tag{3.36}
\]
As a result, it is convenient to label $W$ with concrete superscript ‘spinor’ indices.

Granted that $[\pi^A]$ is a PS of $\Phi_{ABA'B'}$, it is natural to investigate the condition for $[\pi^A]$ to be a multiple RPS; namely $\Phi_{ABA'B'}\eta_{B'} = 0$, for any LSR $\eta^A$ of $[\pi^A]$. Once again, work on an open set $U$ where $(U, h) = (U, U^2g)$, $(U, g, [\pi^A])$ is Walker, and on which $(u, v, x, y)$ are oriented Walker coordinates for $(U, g, [\pi^A])$. We continue to employ for spin frames $(3.15–19)$ constructed from the Walker spin frames $\{\alpha^A, \beta^A\}$ and $\{\pi^A, \xi^A\}$ associated to the oriented Walker coordinates $(u, v, x, y)$ for $(U, g, [\pi^A])$. Hence, the condition for $[\pi^A]$ to be a multiple PS of $\Phi_{ABA'B'}$ may be expressed as $\Phi_{ABA'B'}\hat{\pi}^{B'} = 0$, i.e., in addition to $\Phi_{00} = \Phi_{10} = \Phi_{20} = 0$, one also requires $\Phi_{01} = \Phi_{11} = \Phi_{21} = 0$. These quantities may be computed from Law (2008) (3.4) & (5.6) together with (3.21–22); but it is more efficient to exploit (2.8).

\section*{3.37 Lemma}

With assumptions and notation as in the previous paragraph, with $B_{AB} := -2\Phi_{ABA'B'}\xi^A\pi^{B'}$, see Law & Matsushita (2008) (2.32–33), and $\varsigma_A$ as defined in (A6), then

$$
\Phi_{AB1\nu'} := \Phi_{ABA'B'}\dot{\xi}^A\pi^{B'} = \Omega^{-1} \left[ -\frac{1}{2} B_{AB} - \Omega \delta(A) [\Omega^{-2}, \varsigma_B] \Omega \right].
$$

(3.37.1)

Proof. Recalling, from Law (2008) (5.8), that the Walker spin frames are parallel with respect to $\delta_A$, one computes from (2.8)

$$
\Phi_{AB1\nu'} = \Omega^{-1} \left[ \Phi_{ABA'B'} + Y_{A(A'} Y_{B')} B - \nabla_{A(A'} Y_{B')} B \right] \xi^{A'} \pi^{B'}
$$

$$
= \Omega^{-1} \left[ -B_{AB} + 2\varsigma(A]) [\delta_B] - \nabla_{A(A'} Y_{B')} - \delta_A \varsigma_B \right]
$$

$$
= \Omega^{-1} \left[ \frac{1}{2} B_{AB} + 2\varsigma(A]) [\delta_B] - \Omega^{-1} \left( \pi^{B'} \varsigma_{A} \nabla_{B'B'} + \delta_A \varsigma_B \right) \right]
$$

$$
= \Omega^{-1} \left[ -\frac{1}{2} B_{AB} + 2\varsigma(A]) [\delta_B] - \Omega^{-1} \left( \varsigma_{A} \nabla_{B'B'} + \delta_A \varsigma_B \right) \right]
$$

The term in round brackets does not appear to be symmetric in $A$ and $B$ but in fact it is (as can be seen by computing the unsymmetrized version of (A7)), as of course it must be. Symmetrizing over $A$ and $B$ in the previous equation and using (A7) yields

$$
\Phi_{AB1\nu'} = \Omega^{-1} \left[ -\frac{1}{2} B_{AB} + 2\varsigma(A]) [\delta_B] - \Omega^{-1} \delta(A) \varsigma_B \right],
$$

which proves the assertion.

Note that as the Walker spin frames are parallel with respect to $\delta_A$, the components of $\Phi_{AB1\nu'}$, with respect to the spin frames $(3.15–19)$ are just

$$
\Phi_{AB1\nu'} = \Omega^{-2} \left[ -\frac{1}{2} B_{AB} - \Omega \delta(A) \varsigma_B \right],
$$

(3.38)

where the components on the right-hand side of (3.38) refer to the Walker spin frames.

From Law & Matsushita (2008) (2.32–33) & A.1.8, one can write $B_{AB}$ in the form

$$
B_{AB} = \frac{1}{2} \delta(A) [W_{BC} \epsilon_{B}] D \epsilon_{BD} = -\frac{1}{2} \delta(A) [W_{BC} \epsilon_{B}] D \epsilon_{BD}.
$$

(3.39)

\section*{3.40 Lemma}

With assumptions as in (3.37), but utilizing the conformal oriented Walker coordinates $(U, V, X, Y)$ of (3.33) rather than arbitrary conformal oriented Walker coordinates, \[ \Phi_{AB1\nu'} = \Omega^{-3} [\Omega^2 \delta(A) [W_{BC} \epsilon_{B}] D] \epsilon_{KD}. \]
\( \zeta_A = \alpha_A \left[ \partial_X - \frac{1}{2} (AD + C\Delta) \right] - \beta_A \left[ \frac{1}{2} (CD + B\Delta) - \partial_Y \right] = \alpha_A \partial_X + \beta_A \partial_Y + \frac{1}{2} \epsilon_A^F \epsilon_F^{CD} \epsilon_D^D. \) (3.40.1)

With respect to \((U, V, X, Y)\), \(\Omega\) is as in (3.34), whence independent of \(X\) and \(Y\), so

\[ \zeta_B \Omega = \frac{1}{2} \epsilon_F^{CD} \epsilon_D^D \epsilon_B^F. \]

Hence,

\[
\delta_A (\Omega^{-2} \zeta_B \Omega) = -\frac{1}{2} \left[ \delta_A (\Omega^{-2} \epsilon^{CD} \epsilon_D^D \Omega) \right] \epsilon_F^{CF} \epsilon_B^F \epsilon_D^D
\]

\[ = -\frac{1}{2} \left[ -2 \Omega^{-3} \epsilon^{CD} (\delta_A \Omega)(\delta_A \Omega) + \Omega^{-2} \epsilon^{CD} \delta_A \delta_D \Omega + \Omega^{-2} (\delta_A \epsilon^{CD})(\delta_D \Omega) \right] \epsilon_F^{CF} \epsilon_B^F \epsilon_D^D
\]

\[ = -\frac{1}{2} \left[ \Omega^{-2} (\delta_A \epsilon^{CD})(\delta_D \Omega) \right] \epsilon_F^{CF} \epsilon_B^F \epsilon_D^D, \quad \text{upon using (3.35).}
\]

Substituting this last expression and (3.39) into (3.37.1) yields,

\[
\hat{\Phi}_{AB1\nu'} = \frac{\Omega^{-1}}{4} \left[ \delta_A (\delta_C \epsilon^{AC} \epsilon_B^D) \epsilon_B^D \right] + 2 \Omega^{-1} \left[ \delta_A \epsilon^{CD} \epsilon_B^D \epsilon_D^D \right] \epsilon_F^{CF} \epsilon_B^F \epsilon_D^D
\]

\[ = \frac{\Omega^{-3}}{4} \left[ \Omega^2 \delta_A \epsilon^{CD} \epsilon_B^D \epsilon_B^D \epsilon_D^D \right] + 2 \Omega \left[ \delta_A \epsilon^{CD} \epsilon_B^D \epsilon_D^D \right] \epsilon_F^{CF} \epsilon_B^F \epsilon_D^D
\]

\[ = \frac{\Omega^{-3}}{4} \delta_C \left[ \Omega^2 \delta_A \epsilon^{CD} \epsilon_B^D \epsilon_B^D \epsilon_D^D \right] \epsilon_F^{CF} \epsilon_B^F \epsilon_D^D
\]

Hence, with respect to conformal oriented Walker coordinates \((U, V, X, Y)\) as in (3.33), (3.38) specializes to

\[
\hat{\Phi}_{AB1\nu'} = -\frac{\Omega^{-4}}{4} \epsilon_F^{CF} \epsilon_B^F \epsilon_D^D \epsilon_B^D \epsilon_D^D \epsilon_C^C.
\] (3.41)

Specifically:

\[
\hat{\Phi}_{001\nu'} = -\frac{\Omega^{-2}}{4} \left[ C_{UU} + B_{UV} - 2 \Omega (M_{CU} + N_{BU}) \right] \quad \hat{\Phi}_{111\nu'} = \frac{\Omega^{-2}}{4} \left[ A_{UV} + C_{VV} - 2 \Omega (M_{AV} + N_{CV}) \right]
\]

\[
\hat{\Phi}_{011\nu'} = -\frac{\Omega^{-2}}{8} \left[ B_{VV} - A_{UU} + 2 \Omega ((A_{U} - C_{V}) M + (C_{U} - B_{V}) N) \right].
\]

### 3.42 Lemma

With assumptions as in (3.40) and exploiting the notation of (A10), if the Ricci scalar curvature \(\hat{S}\) of the real ASO-geometry \((M, h, [\pi^A])\) is constant then, on a suitable neighbourhood,

\[
\epsilon^{AB} = \Omega^{-3} \delta^{(A} \epsilon^{F} \epsilon^{B)} + \frac{\hat{S}}{12 \tau^2} K^A K^B,
\] (3.42.1)

for some pair of functions \(F^B\), and

\[
\hat{\Phi}_{AB1\nu'} = \frac{\Omega^{-3}}{8} \delta_A \delta_B \left( \delta_C (\Omega^{-2} \epsilon^{CD}) \right).
\] (3.42.2)

Proof. The proof is adapted from Finley & Plebański (1976). Notice that (3.36) reads \(\delta_A \delta_B H^{AB} = \hat{S} \Omega^5\), where \(H^{AB} = \Omega^3 \epsilon^{AB}\). With \(\hat{S}\) constant, the linear system of PDEs \(\delta_A E^A = \hat{S} \Omega^5\) has general solution
\(E^A = \delta^A H - (\mathcal{S}/4\tau)\Omega^4 K^A\), for arbitrary functions \(H\). Thus, \(\delta_B H^{AB} = \delta^A H - (\mathcal{S}/4\tau)\Omega^4 K^A\), for some specific \(H\). Write \(H =: (1/2)\delta_B Z^B\) for some (nonunique) quantities \(Z^B\). With

\[
F^{AB} := H^{AB} - \delta(A Z^B) - \frac{\mathcal{S} \Omega^3}{12\tau^2} K^A K^B,
\]

\(\delta_B F^{AB} = 0\). By the Poincaré lemma, \(F^{AB} = \delta^B G^A\), for some functions \(G^A\). But, as \(F^{AB} = F^{BA}\), then \(\delta^A G^B = \delta^B G^A\), i.e., \(-\partial_U(G^B) = \partial_V(G^A)\), whence, by the Poincaré lemma again, there exists a function \(K\) such that \(G^A = \delta^A K\). So, \(F^{AB} = \delta^A \delta^B K\), for some function \(K\). Hence,

\[
\mathcal{W}^{AB} = \Omega^{-3} H^{AB} = \Omega^{-3} (\delta^A \delta^B K + \delta(A Z^B)) + \frac{\tilde{S}}{12\tau^2} K^A K^B = \Omega^{-3} \delta(A F^B) + \frac{\tilde{S}}{12\tau^2} K^A K^B,
\]

where \(F^B = \delta^B K + Z^B\).

In the hyperheavily formalism, Plebański and co-workers showed that it is possible to choose coordinates in (3.32) so that \(\phi = \psi\), and then, assuming that \([\pi^A]\) is a RPS, so that \(\phi\) is independent of the \(x\) and \(y\) coordinates and linear in the \(u\) and \(v\) coordinates. In effect, one can take \(\phi\) to have the same functional form as \(\Omega^{-1}\) has as a function of the conformal oriented Walker coordinates \((U, V, X, Y)\). Formally identifying their \(\phi\) with \(\Omega^{-1}\), one sees that the expression given for their quantity \(C_{12\hat{A}\hat{B}}\) (see Finley and Plebański 1976, p. 2213, and their Appendix B), which is effectively equivalent to \(\hat{\Phi}_{AB1\hat{B}}\), is of the same form, up to a multiplicative factor, as (3.41). Hence, as the second term in (3.42.1) is independent of \(U\) and \(V\), by the same argument as given in Finley and Plebański (1976), p. 2213, substitution of (3.42.1) into (3.41) yields (3.42.2) (which result we have also confirmed directly).

### 3.43 Corollary

For a real ASO-geometry \((M, h, [\pi^A])\) for which \([\pi^A]\) is a RPS and the Ricci scalar curvature is constant, \([\pi^A]\) is a multiple RPS iff, with respect to conformal oriented Walker coordinates \((U, V, X, Y)\) satisfying (3.33), there is a function \(\vartheta(U, V, X, Y)\) such that

\[
\mathcal{W}^{AB} = \Omega^{-3} \delta(A \Omega^2 \vartheta^B) + \frac{2S \Omega^{-2} + \tilde{S}}{12\tau^2} K^A K^B,
\]

where \(S\) is the Ricci scalar curvature of the Walker geometry \((U, g, [\pi^A])\), with \(h = \Omega^2 g\) on the neighbourhood \(U\), and \(S \Omega\) is a function of \((X, Y)\) only.

Proof. Continuing to follow Finley & Plebański (1976), (3.42.2) is zero iff \(\delta_A \delta_B (\Omega \delta_C (\Omega^{-2} F^C)) = 0\), i.e., iff

\[
\Omega \delta_C (\Omega^{-2} F^C) = P(X, Y) U + Q(X, Y) V + R(X, Y),
\]

for some functions \(P, Q, R\) of \((X, Y)\). Put

\[
(H_A) := \frac{1}{2} \begin{pmatrix} P \\ Q \end{pmatrix} \quad (T^A) := \begin{pmatrix} U \\ V \end{pmatrix} \quad \text{whence} \quad \delta^A T^B = \epsilon^{AB}.
\]

With \(\beta^A (X, Y)\) functions such that \(\beta^A J_A = -R/2\), define

\[
L^A := F^A - \frac{1}{2\tau} K^B H_B T^A + \beta^A.
\]

Since \(\delta_A \beta^B = 0\) and \(\delta^A T^B = 0\), then \(\delta^A L^B = \delta^A F^B\), whence one can replace \(F^B\) by \(L^A\) in (3.42.1). In place of (3.42.1), one obtains

\[
\Omega \delta_C (\Omega^{-2} L^C) = \Omega \delta_C \left[ \Omega^{-2} (F^C - \frac{1}{2\tau} K^B H_B T^C + \beta^C) \right]
\]

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\[
(2T^B H_B + R) - \Omega \left[ \frac{K^B H_B}{2\tau} \delta_C (\Omega^{-2}T^C - \delta_C (\Omega^{-2} \tau^C) \right] \\
= 2T^B H_B + R - \Omega \left[ -2\Omega^{-3} \left( \frac{K^B H_B}{2\tau} T^C - \beta^C \right) \delta_C \Omega + \Omega^{-2} \frac{K^B H_B}{\tau} \right] \\
( \text{as } \delta_C \beta^C = 0, \quad \delta_C T^C = \epsilon_C^C = 2) \\
= 2T^B H_B + R - \frac{K^B H_B}{\tau} T^C J_C + 2\beta^C J_C - \frac{K^B H_B T^C J_C}{\tau} \\
(\text{noting that } \Omega^{-1} = T^C J_C) \\
= 2T^B H_A \left( \delta^A_B - \frac{K^A J_B}{\tau} \right) \\
= \frac{2T^B K_B J^A H_A}{\tau}, \quad \text{using (A10),} \\
= \frac{2\mu T^B K_B}{\tau}, \quad \text{where } \mu(X, Y) := -J^A H_A, \\
(3.43.5)
\]

which is simpler than (3.43.2). Thus, replacing \( F^C \) by \( L^C \) leaves (3.42.1) valid but now \( \Phi_{A'B'} \) vanishes iff \( \Omega \delta_C (\Omega^{-2} L^C) = 2(\mu/\tau) T^B K_B \). As this equation is again of the form \( \delta_C \phi^C = \psi \), one need only find a particular solution to write down the general solution, which is

\[
L^C = \Omega^2 \delta^C \vartheta + \frac{\mu}{\tau^2} T^B K_B K^C, \\
(3.43.6)
\]

where \( \vartheta \) is an arbitrary function of \( (U, V, X, Y) \). Noting that \( K^B \delta^A) T^D K_D = K^B (\epsilon^A) D K_D = K^A K^B \), substituting (3.43.6) for \( F^C \) in (3.42.1) yields

\[
\mathcal{W}^{AB} = \Omega^{-3} \delta^{(A} [\Omega^2 \delta^{B)} \vartheta] + \frac{\mu \Omega^{-3}}{\tau^2} K^A K^B + \frac{\hat{S}}{12\tau^2} K^A K^B \\
= \Omega^{-3} \delta^{(A} [\Omega^2 \delta^{B)} \vartheta] + \frac{12\mu \Omega^{-3} + \hat{S}}{12\tau^2} K^A K^B. \\
(3.43.7)
\]

At this point, \( \mu = -J^A H_A \) is determined by \( \Omega \) and the unknown functions \( P(X, Y) \) and \( Q(X, Y) \); however, one can obtain a simple determination of \( \mu \) in terms of known quantities as follows. By Law & Matsushita (2008) A1.6, the Ricci scalar curvature of the Walker geometry \((U, g, [\tau^A])\) is \( S = A_{UU} + B_{VV} + 2C_{UV} = \delta_A \delta_B \mathcal{W}^{AB} \). The first term on the right-hand side of (3.43.7) expands as

\[
\Omega^{-3} \delta^{(A} [\Omega^2 \delta^{B)} \vartheta] = -2j^{(A} \delta^{B)} \vartheta + \Omega^{-1} \delta^{A} \delta^{B} \vartheta, \\
(3.43.8)
\]

whence \( \delta_B (\Omega^{-3} \delta^{(A} [\Omega^2 \delta^{B)} \vartheta]) = J_B \delta^{A} \delta^{B} \vartheta \). Hence, from (3.43.7) and using (A.10),

\[
S = \delta_A \delta_B \mathcal{W}^{AB} = \delta_A \delta_B \left( \frac{12\mu \Omega^{-3} + \hat{S}}{12\tau^2} K^A K^B \right) \\
= \frac{\mu}{\tau^2} K^A K^B \delta_A \delta_B \Omega^{-3} \\
= 6\mu \Omega^{-1}. \\
(3.43.9)
\]

Substituting (3.43.9) into (3.43.7) yields (3.43.1).

The Einstein condition \( \Phi_{A'B' = 0} \) for ASO-geometries is of special interest as then, by the GGST, the two conditions defining ASO-geometries are equivalent to each other. Since \( \Phi_{A'B' = 0} \) entails, by the Bianchi identity, that \( S \) is constant, the Einstein condition for real ASO-geometries is characterized by the conditions derived above together with \( \Phi_{A'B' = 0} \). In the hyperheavenly formalism, this final condition leads to a single PDE of the form \( \delta_A \delta_B L = 0 \), where \( L \) is an expression in \( \vartheta, \Omega, \) and \( \mu \) (equivalently \( S \)). Thus,

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\( L \) is affine in \( U \) and \( V \), with coefficients functions of \( X \) and \( Y \). This constraint is called the hyperheavenly equation for \( \vartheta \). We obtain this result as follows.

From (2.8), (3.15) and Law & Matsushita (2008) (2.32),

\[
\Phi_{AB1'1'} = \Phi_{AB1'B} \xi^{A'} \xi^{B'}
\]

\[
= \Omega^{-3} [\Phi_{AB1'B} (\xi^{A'} \xi^{B'}) + (\xi^{A'} \xi^{B'} \xi^{AA'}) (\xi^{A'} \xi^{B'}) - \xi^{B'} \xi^{A'} \xi^{AA'} \xi_{BB'}]
\]

\[
= \Omega^{-3} [A_{AB} + (\varsigma_A \omega) (\varsigma_B \omega) - \xi^{B'} \varsigma_A \xi_{BB'} \varsigma_A]
\]

\[
= \Omega^{-3} [A_{AB} + 2 \Omega^{-2} (\varsigma_A \omega) (\varsigma_B \omega) - \Omega^{-1} \xi^{B'} \varsigma_A \xi_{BB'} \varsigma_A].
\] (3.44)

In terms of the Walker spin frames, \( \nabla_b \Omega = - (\delta_B \Omega) \xi_{BB'} + (\varsigma_B \Omega) \pi_{BB'} \), so the third term on the right-hand side of (3.44) may be expressed in terms of spin coefficients using Law (2008), (2.9), to obtain

\[
\xi^{B'} \varsigma_A \xi_{BB'} \varsigma_A = \varsigma_A \varsigma_B \Omega + (\kappa' \alpha_A + \sigma' \beta_A) \delta_B \Omega + (\gamma \alpha_A - \hat{\alpha} \beta_A) \varsigma_B \Omega.
\] (3.45)

We must now evaluate

\[
\Phi_{AB1'1'} = \Phi_{AB1'B} \xi^{A'} \xi^{B'} = \Omega^{-3} \Phi_{AB1'B} \xi^{A'} \xi^{B'},
\] (3.46)

and do so by evaluating the terms in (3.44–45). From Law (2008), (5.11), one observes that for oriented Walker coordinates

\[
A_{AB} = \delta(AZ_B),
\] (3.47)

is defined with respect to the corresponding (fixed) Walker spin frames (i.e., \( Z_A \) is not well-defined under transformations between different sets of Walker spin frames). We now restrict to conformal oriented Walker coordinates \((U, V, X, Y)\) for \((M, h, [\pi^A])\) so that (3.33–34), (A.10), and (3.40.1) all hold.

Using (3.40.1), one finds that, with

\[
\partial_B := (- \partial_Y, \partial_X),
\] (3.48)

\[
\varsigma_A \varsigma_B \Omega = - \frac{\Omega^2}{2} [-4 \Omega^{-3} \varsigma_A \Omega (\varsigma_B \Omega) + \varsigma_A (\epsilon_B \epsilon_F) \xi_{FC} \xi_{CD} J_D + \epsilon_B \epsilon_F \epsilon_C \epsilon_A \xi_{CD} \xi_{KD} + \frac{1}{2} \epsilon_{KP} \xi_{PQ} \varsigma_A \xi_{CD} \xi_{JD}].
\] (3.49)

Writing \((\kappa' \alpha_A + \sigma' \beta_A) \delta_B \Omega = -Z_A \delta_B \Omega = \Omega^2 Z_A J_B\), then upon substituting this expression, (3.45), (3.47) and (3.49) into (3.44/46) and symmetrizing over \( A \) and \( B \), one obtains

\[
\Phi_{AB1'1'} = \Omega^{-4} [\delta(AZ_B) - \Omega Z(A1_B) - \frac{\Omega}{2} \epsilon(C \delta A) \xi_{CD} J_D
\]

\[
+ \frac{\Omega}{2} (\epsilon(A \xi_B \xi_B) \varsigma_A (\epsilon_B \epsilon_F) \xi_{FC} \xi_{CD} J_D - \frac{1}{2} \epsilon(C \delta A B) \xi_{PQ} \xi_{PQ} \varsigma_A \xi_{CD} \xi_{JD} + (\gamma \alpha_A - \hat{\alpha} \beta_A) \epsilon_{BC} \xi_{CD} \xi_{JD})].
\] (3.50)

One computes:

\[
- \epsilon(C \delta A \xi_B) \xi_{CD} J_D - J^C \partial_C W_{AB} = - \tilde{Z}_A J_B,
\] (3.51)

where

\[
\tilde{Z}_B := \begin{pmatrix} C_Y - B_X \\ C_X - A_Y \end{pmatrix}_B = - \partial_C W_{CB}.
\] (3.52)

Writing \( Z_A = (1/2) \tilde{Z}_A + Y_A \), then

\[
4Y_B = \begin{pmatrix} AB_Y + C(B_Y - C_Y) - BC_V \\ -AC_U + C(A_U - C_V) + BA_V \end{pmatrix}_B = W_{CD} \delta_D W_{BC}.
\] (3.53)

Upon substituting (3.51), the first three terms inside the square brackets on the right-hand side of (3.50) may be rewritten in terms of \( \tilde{Z}_B \) and \( Y_B \) as follows:

\[
\Omega^{-1} \delta(A \Omega Y_B) + \frac{1}{2} \Omega^{-2} \delta(A \Omega^2 \tilde{Z}_B) + \frac{\Omega}{2} J^C \partial_C W_{AB}.
\]
Substituting for $W_{AB}$ using (3.42.1), but with the $L^B$ of (3.43.6) replacing $F^B$, and noting that the second summand of (3.42.1) is independent of $X$ and $Y$, one further obtains

$$
\Omega^{-1}\delta(A\Omega Y_B) + \Omega^{-2}\frac{1}{2}[\delta(A\Omega^2\hat{Z}_B) + J^C\partial_C\delta(A L_B)].
$$

Substituting now from (3.52) for $\hat{Z}_B$, and noting that $\partial_B$ and $\delta_B$ are just partial derivatives and so commute, yields

$$
\Omega^{-1}\delta(A\Omega Y_B) - \Omega^{-2}\frac{1}{2}\delta(A[\Omega^2\partial^C W_{B,C}] - J^C\partial_{[C|L_B]}).
$$

Hence, substituting from (3.53), one finds

$$
\text{side of (3.55) the second line together with the last summand of the first line together take the form}
$$

Thus, (3.50) may be written

$$
\hat{\Phi}_{AB1'} = \Omega^{-4}\left[ -\Omega^{-2}\frac{1}{2}\delta(A[\Omega^2\partial^C W_{B,C}] - J^C\partial_{[C|L_B]} + \Omega^{-1}\delta(A\Omega Y_B) + \frac{1}{2}\epsilon(A^B)\epsilon_{BC} \epsilon_{FC} W^{CD} J_D - \frac{1}{2}\epsilon(C(B^A)_P) W^{PQ} \delta_Q W^{CD} J_D + (\hat{\alpha}(A - \hat{\beta}A)\epsilon_{BC} W^{CD} J_D)\right].
$$

The first and third summands in the second line of (3.55) can be evaluated using Law (2008), (2.9) and (5.6). One finds

$$
\epsilon(A^B)\epsilon_{BC} \epsilon_{FC} W^{CD} J_D + (\hat{\alpha}(A - \hat{\beta}A)\epsilon_{BC} W^{CD} J_D) = -\frac{1}{2}J^C W^{CD} \delta_D W_{AB}. \tag{3.56}
$$

For the remaining summand in the second line of (3.55), observe that

$$
-\frac{1}{2}\epsilon_{CB} \epsilon_{AP} W^{PQ} \delta_Q W^{CD} J_D = \frac{1}{2}\epsilon_{AP} W^{PQ} \delta_Q W^{BD} J^D
$$

$$
= \frac{1}{2}(W^{PQ} \delta_Q W_{BAJ_P} - W^{PQ} \delta_Q W_{BPJ_A}).
$$

Hence, upon substituting from (3.53), one finds

$$
-\frac{1}{2}\epsilon_{CB} \epsilon_{AP} W^{PQ} \delta_Q W^{CD} J_D = \frac{1}{2}(J_P W^{PQ} \delta_Q W_{AB} - 4J(A Y_B)). \tag{3.57}
$$

Hence, (3.56) cancels the first term on the right-hand side of (3.57), which entails that on the right-hand side of (3.55) the second line together with the last summand of the first line together take the form

$$
\Omega^{-1}\delta(A\Omega Y_B) + \Omega^{-2}\frac{1}{2}[-2J(A Y_B)] = \delta(A Y_B) - 2\Omega J(A Y_B)
$$

$$
= \Omega^{-2}\delta(A\Omega^2 Y_B)
$$

$$
= \Omega^{-4}\frac{1}{4}\delta(A\Omega^2 W^{CD} \delta_D W_{B,C}). \tag{3.58}
$$

upon substituting back with (3.53). Hence, substituting (3.58) into (3.55) yields

$$
\hat{\Phi}_{AB1'} = \Omega^{-6}\frac{1}{4}\delta(A X_B), \tag{3.59}
$$

where

$$
X_B := \Omega^2 W^{CD} \delta_D W_{BC} - 2(\Omega^2 \partial^C W_{BC} - J^C \partial_C L_B). \tag{3.60}
$$

Equations (3.59–60) are of the same form, though with slight differences, as Finley and Plebański (1976) (A4–5) when one formally identifies their $\phi$ with our $\Omega^{-1}$. We may therefore follow their argument from this
where the last line follows from (3.43.5). If one now chooses altering (3.59), with the aim of choosing \( \lambda \) point. First note that by (3.43.3) one can add a term \( \lambda T \), then
\[
\begin{align*}
&\text{and observes that} \\
&\text{Using (3.43.7), one computes}
&\text{Expansion of the following first and third expressions confirms the identities:}
\end{align*}
\]

where the last line follows from (3.43.5). If one now chooses
\[
\lambda := -\frac{K^D \partial D \mu}{2\tau},
\]
and observes that
\[
\begin{align*}
\delta_B \left( \frac{(K^D T_D)(T^C \partial C \mu)}{2\tau} \right) &= \frac{K^D \epsilon_B D (T^C \partial C \mu) + (K^D T_D)\epsilon_B D C \partial C \mu}{2\tau} \\
&\quad \text{by (3.43.3)} \\
&= \frac{K^D (\partial B \mu T_D - \partial D \mu T_B) + K^D T_D \partial B \mu}{2\tau} \\
&= \frac{2K^D T_D \partial B \mu - K^D \partial D \mu T_B}{2\tau},
\end{align*}
\]
then
\[
-(\Omega^2 \partial C W_{BC} + J^C \partial C L_B) + \lambda T_B = \delta_B \left[ -\Omega^{-1} \partial C L_C + \frac{K^D T_D T^C \partial C \mu}{2\tau} \right].
\]

To treat the first summand on the right-hand side of (3.60), observe that
\[
\Omega^2 W_{BC} W^C_D = \epsilon_B D W, \quad \text{where } W := \frac{1}{2} \Omega^2 W_{BC} W^{BC}.
\]

Hence,
\[
\delta_B W = -\delta^D \epsilon_B D W = \delta_D (\Omega^2 W_{BC} W^{CD})
\]
and
\[
\Omega^2 W^{CD} \delta_D W_{BC} = \delta_B W - W_{BC} \delta_D (\Omega^2 W^{CD}).
\]

Using (3.43.7), one computes
\[
\delta_D (\Omega^2 W^{CD}) = 2\Omega^3 J^C J_D \delta^D \vartheta + \frac{\mu K^C}{\tau} - \frac{\delta \Omega^3 K^C}{6\tau}.
\]

Expansion of the following first and third expressions confirms the identities:
\[
\delta(A)[\Omega^2 \delta_B \vartheta] = \Omega^2 \delta_A \delta_B \vartheta - 2\Omega^2 J_A \delta_B \vartheta = \Omega \delta_A \delta_B (\Omega \vartheta) - 2\Omega^2 J_A J_B \vartheta.
\]
Substituting into (3.43.7) yields

$$W_{AB} = \Omega^{-2} \delta_A \delta_B (\Omega \vartheta) - 2 \Omega J_A J_B \vartheta + \frac{12 \mu \Omega^{-3}}{12 \tau^2} K_A K_B.$$  \hspace{1cm} (3.66)

The second summand in (3.64) can therefore be evaluated by multiplying together (3.65–66). Upon doing so, there is a single term not involving $\mu$ or $\hat{S}$:

$$2 J^C J_D \delta^D \vartheta (\Omega \delta_B \delta_C (\Omega \vartheta)) = - \delta_B \left[ (J^C \delta_C (\Omega \vartheta))^2 \right].$$  \hspace{1cm} (3.67)

The terms involving $\mu$ are

$$\frac{2\mu}{\tau} (J^D \delta_D \vartheta) K_B - 2 \mu \Omega \delta J_B + \frac{\mu \Omega^{-2}}{\tau} \delta_B (K^C \delta_C (\Omega \vartheta))$$  $$= \frac{2\mu}{\tau} (J^D \delta_D \vartheta) K_B - \frac{\mu}{\tau} (K^C \delta_C (\Omega \vartheta)) J_B - \mu \delta_B \vartheta + \frac{\mu \Omega^{-1}}{\tau} \delta_B (K^C \delta_C \vartheta),$$

upon expanding out the last summand of the left-hand side and rearranging terms. The right-hand side can be written

$$- 2 \frac{\mu}{\tau} (J^D J_B - J^D K_B) \delta_D \vartheta + \frac{\mu}{\tau} J_B K^D \delta_D \vartheta - \mu \delta_B \vartheta + \frac{\mu \Omega^{-1}}{\tau} \delta_B (K^D \delta_D \vartheta),$$

which, upon utilizing (A10), becomes

$$- 3 \mu \delta_B \vartheta + \frac{\mu}{\tau} J_B K^D \delta_D \vartheta + \frac{\mu \Omega^{-1}}{\tau} \delta_B (K^C \delta_C \vartheta) = \frac{\mu}{\tau} \delta_B \left( \Omega^{-4} K^D \delta_D (\Omega^3 \vartheta) \right),$$  \hspace{1cm} (3.68)

as is easily checked by expanding out the right-hand side of (3.68).

Finally, the terms involving $\hat{S}$ are:

$$\frac{\dot{\hat{S}}}{6} \left[ \frac{\Omega^3}{\tau} (J^D \delta_D \vartheta) K_B - \frac{\Omega}{\tau} \delta_B (K^C \delta_C (\Omega \vartheta)) + 2 \Omega J_B \vartheta \right]$$

which, expanding out the middle term, yields

$$\frac{\dot{\hat{S}}}{6} \left[ \frac{\Omega^3}{\tau} (J^D K_B + J_B K^D) \delta_D \vartheta - \frac{\Omega^2}{\tau} \delta_B (K^D \delta_D \vartheta) + \Omega^3 \delta_B \vartheta \right],$$

which in turn can be written

$$\frac{\dot{\hat{S}}}{6} \left[ \frac{\Omega^3}{\tau} (J^D K_B - K^D J_B) \delta_D \vartheta - \frac{2 \Omega^3}{\tau} J_B K^D \delta_D \vartheta - \frac{\Omega^2}{\tau} \delta_B (K^D \delta_D \vartheta) + \Omega^3 \delta_B \vartheta \right].$$

Upon using (A10), one finds the last expression simplifies to

$$- \frac{\dot{\hat{S}}}{6 \tau} \delta_B (\Omega^2 K^D \delta_D \vartheta).$$  \hspace{1cm} (3.69)

Thus, (3.59) remains valid with (3.60) replaced by

$$X_B := \Omega^2 \mathcal{W}^{CD} \delta_D \mathcal{W}_{BC} + 2 (\Omega^2 \delta^C \mathcal{W}_{BC} + J^C \partial_C L_B - \frac{K^D \partial_D \mu}{2 \tau} T_B) = \delta_B \mathcal{L}$$  \hspace{1cm} (3.70)

where, by (3.62), (3.64) and (3.67–69),

$$\mathcal{L} = - 2 \Omega^{-1} \partial^P L_D + \left( \frac{K^D T_D}{\tau} (T^C \partial_C \mu) \right) + W - \left[ - (J^C \delta_C (\Omega \vartheta))^2 + \frac{\mu}{\tau} \Omega^{-4} K^D \delta_D (\Omega^3 \vartheta) - \frac{\dot{\hat{S}}}{6 \tau} \Omega^2 K^D \delta_D \vartheta \right].$$
Defining
\[ w := \Omega^{-1} = MU + NV, \quad \text{then} \quad K^D\delta_D = -\frac{d}{dw} \] (3.71)
and substituting from (3.43.6) for \( L_C \) in favour of \( \theta \), one can write \( \mathcal{L} \) as
\[ \mathcal{L} = \mathcal{W} + (J^D\delta_D(\Omega\theta))^2 + 2\Omega\partial_D\delta_D\theta - \frac{\mu\Omega^{-4} d(\Omega^3\theta)}{dw} + \frac{T^D K_D}{\tau^2} [2\Omega^{-1}K^C - \tau T^C] \partial_C\mu + \frac{\hat{S}\Omega^2}{6\tau} d\theta. \] (3.72)

Hence, the Einstein condition, in terms of the conformal oriented Walker coordinates \((U, V, X, Y)\) for which \( \Omega^{-1} = MU + NV =: w\), \((M \text{ and } N \text{ constant})\) is the single condition
\[ \delta_A\delta_B\mathcal{L} = 0, \quad \text{i.e., } \mathcal{L} \text{ is affine in } U \text{ and } V: \quad \mathcal{L} = T^D\eta_D + K, \] (3.73)
for constants \( \eta_C \) and \( K \), which is our version of the hyperheavenly equation, cf. Finley & Plebański (1976), (3.14).

The consequences of various conditions imposed on the Ricci curvature of a real AS\(\alpha\)-geometry \((M, h, [\pi^A])\) derived in this section may be summarized as follows:

1) \((M, h, [\pi^A])\) may be characterized locally by four arbitrary functions \(a, b, c, \text{ and } \Omega \) of local coordinates \((u, v, x, y)\) with the metric given with respect to these coordinates by (3.3), with \( W \) as given in (3.2);
2) \((M, h, [\pi^A])\), \([\pi^A]\) a RPS, may be characterized locally as in (1) but with \( \Omega \) an affine function of \( u \) and \( v \); one may specialize this local characterization to local coordinates \((U, V, X, Y)\) for which \( \Omega = MU + NV \), with \( M \) and \( N \) constant;
3) \((M, h, [\pi^A])\), with constant scalar curvature and with \([\pi^A]\) a multiple RPS, may be characterized locally by three functions \( \vartheta, \Omega, \text{ and } S \) of local coordinates \((U, V, X, Y)\) such that \( \vartheta \) is arbitrary, \( \Omega = MU + NV \), \( M \) and \( N \) constant, \( S\Omega \) is a function of \((X, Y)\) only, and the metric with respect to \((U, V, X, Y)\) is given by (3.3) with \( W \) given by (3.43.1); the condition on \( S\Omega \) ensures that the (locally defined) Walker metric \( g := \Omega^{-2} h \) has scalar curvature \( S \);
4) \((M, h, [\pi^A])\) Einstein may be locally characterized as in (3), with \( \theta \) subject to a single constraint equation, the hyperheavenly equation (3.72–73).

Subsequent to our work, we learnt that the local conformally Walker nature of real AS\(\alpha\)-geometries (transcribed into the context of complex general relativity and without reference to Walkers’ work) was not just implicit in the work of Plebański and co-workers; e.g., a version of (3.27) is, in essence, given by Plebański & Różańska (1984) (their Lemma IV). Nevertheless, our approach, with its emphasis on conformal rescaling and Walker geometry, differs from that of Plebański and co-workers. But the end result naturally converges on that of the hyperheavenly formalism. One can therefore adapt applications of the hyperheavenly equation to the study of four-dimensional neutral geometry. In particular, various solutions of the hyperheavenly equation, e.g., Plebański & Torres del Castillo (1982), will be of interest. More recently, Chudecki & Przanowski (2008) obtained an explicit neutral metric for a real AS\(\alpha\)-geometry as a solution of the hyperheavenly equation.

4. Null Geometry

\((M, h, [\pi^A])\) again denotes a real AS\(\alpha\)-geometry and we shall continue to denote its curvature quantities etc., by hatted symbols. When convenient, one may suppose that any point \( p \in M \) has a neighbourhood \( U \) such that \( h = \Omega^2 g \) on \( U \), for some positive function \( \Omega \), with \((U, g, [\pi^A])\) a Walker geometry.

Suppose \((M, h, [\pi^A])\) is not Walker. As already noted, (1.3) entails that \( S^b = \pi_A^\flat \nabla^b \pi^A = \omega^B \pi^B \neq 0 \), for some nonzero spinor \( \omega^B \), where \( \pi^A \) is an LSR of \([\pi^A]\). While the geometry only determines \( S^b \) up to scale, the null distribution \( D := (S^b)_R \) is well defined. Indeed, the geometry defines the nested null distributions \( D \subseteq Z_{[\pi]} \subseteq \mathcal{K} \), where \( \mathcal{K} = D^+ \), (these distributions are null of types I, II, and III, respectively, in the sense of Law 2008). The geometry of these nested distributions was considered in Law (2008) for \( \alpha \)-geometries. By Law (2008) (6.2.37), the condition for \( D \) to be auto-parallel, i.e., for any local section \( q^a \) of \( D \), \( q^a \nabla_b q^a \propto q^a \), is \( \Phi_{AB^1A^2B^2} \omega^A \pi^A \pi^\flat = 0 \). If one supposes that \([\pi^A]\) is a RPS, then, by Law (2008)
(6.2.18), an $\alpha$-geometry is in fact an AS$\omega$-geometry. Furthermore, by the proof of (6.3.14) in Law (2008), in an AS$\omega$-geometry, $\Phi_{A'B'B'}\omega^B\pi^A\pi^B = 0$ is the condition for $\mathcal{H}$ to be integrable.

Hence, an (AS)$\omega$-geometry $(M, h, [\pi^A])$ for which $[\pi^A]$ is a RPS, which (consistent with Law 2008) we call a Ricci-aligned (AS)$\omega$-geometry, has nested integrable null distributions $\mathcal{D} \leq Z[\pi] \leq \mathcal{H}$, with $\mathcal{D}$ auto-parallel. The integral curves of $\mathcal{D}$, suitably parametrized, are the null geodesic generators of the null hypersurfaces which are the integral hypersurfaces of $\mathcal{H}$. These null hypersurfaces are also foliated by the $\alpha$-surfaces of $Z[\pi]$. Frobenius coordinates for these nested distributions were described in Law (2008) (6.3.16–18) and provide an alternative to conformal Walker coordinates.

Note that as distributions, each of $\mathcal{D}$, $Z[\pi]$, and $\mathcal{H}$ is of course defined and integrable in $(U, g)$. Moreover each retains its null character as these are conformally invariant. As $\mathcal{D}$ is null, the condition of being auto-parallel is conformally invariant too, so $\mathcal{D}$ is auto-parallel with respect to $g$.

By Law (2008) (6.2.29), for any spin frames $\{o^A, t^A\}$ and $\{o'^A, t'^A\}$ for which $o'^A$ is an LSR of $[\pi^A]$, $\omega_A = \hat{\tau}_A - \hat{\rho}_A$. Hence, with respect to the spin frames $\{\hat{A}^A, \hat{B}^A\}$ and $\{\hat{A}'^A, \hat{B}'^A\}$ of (3.15) for $(M, h, [\pi^A])$ associated to conformal oriented Walker coordinates $(u, v, x, y)$, by (3.21–22)

$$
\omega_A = \hat{\tau}_A - \hat{\rho}_A = \Omega^{1/2}(\Delta\omega)\rho_A - \Omega^{1/2}(D\omega)B_A
$$

The local conformal geometry also defines $\nabla_a\Omega = \nabla_a\xi_A \Omega$, i.e., the distribution $(\nabla^a)^R$. In terms of the conformally associated Walker spin frames, $\nabla_a\Omega = - (\delta_A\xi_A + (\xi_A\Omega)\pi_A)$. Note that $(M, h, [\pi^A])$ is Walker iff $\omega_A$ vanishes, i.e., iff $\delta_A\Omega$ vanishes, i.e., in accordance with (3.1), iff $\pi^A\nabla_{A'\Omega} \Omega$ vanishes, in which case $\nabla_a\Omega = (\xi_A\Omega)\pi_A$ and is null. More generally, for a nyRicci-aligned $\alpha$-geometry,

$$(\nabla_a\Omega)(\nabla^a\Omega) = -2(\delta_A\Omega)(\xi_A\Omega) = -W^{CD}(\delta_C\Omega)(\delta_D\Omega), \quad \text{by (3.40.1)}.$$

One does not expect, generically, $\nabla_a\Omega$ to be null. But, for any AS$\omega$-geometry

$$(\nabla_a\Omega)S^a \propto -(\delta_A\Omega)\xi_A + (\xi_A\Omega)\pi_A)\delta_A\Omega = 0.$$

Thus, if $(M, h, [\pi^A])$ is not Walker, $S^a$ does not vanish, $\mathcal{H}$ is well defined, and $\nabla^a\Omega \in \mathcal{H}$, which thereby relates this ingredient of the local conformal geometry to the geometry of the nested null distributions $\mathcal{D} \leq Z[\pi] \leq \mathcal{H}$. By (4.1), the null distribution $\mathcal{D}$ is aligned with the local conformal Walker geometry, i.e., $\omega_A$ is proportional to $\alpha_A$ or $\beta_A$, iff $D\Omega = 0$ or $\Delta\Omega = 0$, respectively, i.e., $\Omega$ is independent of $U$ or $V$, respectively.

If $U$ is a domain of $M$ on which there is an LSR $\pi^A$ and $\Omega$ and $\chi$ are local conformal factors whose domains intersect in $U$ then, by (4.1), $\delta_A(\Omega^{1/2} = \delta_A(\chi^{1/2})$, i.e., $\Omega^{1/2}$ and $\chi^{1/2}$ differ only by a constant on a given $\alpha$-surface within $U$.

Appendix

Let $(u, v, x, y)$ and $(p, q, r, s)$ be two sets of overlapping oriented Walker coordinates for a Walker geometry $(M, g)$. Suppose the metric takes components with respect to $(u, v, x, y)$ as in (3.2), and with respect to $(p, q, r, s)$ of the same form but with $\tilde{W}$ in place of $W$ and $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ in place of $a$, $b$, and $c$ respectively. For each set of coordinates one can construct the corresponding Walker null tetrads and spin frames as in (3.6). The notation employed in (3.6) will denote the Walker null tetrad and spin frames for $(u, v, x, y)$. The Walker null tetrad and spin frames for $(p, q, r, s)$ will be distinguished by the use of the ‘check’ mark over the relevant symbol. For notational convenience, the pair of Walker spin frames will also be denoted
here by $\epsilon^{A'}_A$ and $\epsilon^{A'}, A'$. Let $\{\partial_1, \ldots, \partial_4\}$ denote the coordinate basis for $(u, v, x, y)$ and $\{\upsilon_1, \ldots, \upsilon_4\}$ that for $(p, q, r, s)$. Then

$$b_j = \sum_{i=1}^{4} \partial_i J^i, \quad \text{where} \quad (J^i) := \frac{\partial(u, v, x, y)}{\partial(p, q, r, s)} = \left( D_{02} E D^{-1} \right).$$  

(A1)

with $E, D \in \mathbb{R}(2)$ and $\det(D) > 0$, see Law & Matsushita (2008) (1.2). Our aim in this Appendix is to record the relationships between the Walker spin frames for the two sets of oriented Walker coordinates and some simple observations. From (A1) and (3.6), one can express the two Walker null tetrads in terms of each other and then deduce the relationship between the Walker spin frames, obtaining:

$$\hat{\epsilon}^A_B = \epsilon^{A'}_A \Lambda^A_B, \quad \Lambda := \left( \Lambda^A_B \right) = \chi^{-1} D \in \text{SL}(2; \mathbb{R})$$

$$\hat{\epsilon}^{A'}_{B'} = \epsilon^{A'}_A \tilde{\Lambda}^{A'}_{B'}$$

$$\tilde{\Lambda} := \left( \tilde{\Lambda}^{A'}_{B'} \right) = \left( \begin{array}{cc} \chi & \chi^{-1} \mu \\ 0 & \chi^{-1} \end{array} \right) \in \text{SL}(2; \mathbb{R})$$

(A2)

where

$$\mu := \frac{(D^1_1 D^2_2 + D^2_1 D^1_2)c - D^1_1 D^2_1 b - D^2_2 D^1_2 a}{2 \chi^2} - \frac{\chi^2 \hat{\epsilon}}{2} + D^1_1 E^2 - D^2_1 E^1$$

$$= \frac{(D^1_1 D^2_2 + D^2_1 D^1_2)c - D^1_1 D^2_1 b - D^2_2 D^1_2 a}{2 \chi^2} + D^1_1 E^2 - D^2_1 E^1.$$  

(A3)

The ambiguity in sign for $\chi$ corresponds to the ambiguity in the sign of the LSR of $\pi^A$ satisfying (3.5) (Law & Matsushita 2008, (2.8)) and the ambiguity in overall sign for the Walker spin frames. Note that, by assumption,

$$\left( \begin{array}{cc} 0_2 & 1_2 \\ 1_2 & \tilde{W} \end{array} \right) = \hat{\gamma} J \left( \begin{array}{cc} 0_2 & 1_2 \\ 1_2 & \tilde{W} \end{array} \right) J.$$  

(A4)

The nontrivial condition in (A4) is

$$\tilde{W} = \hat{\gamma} E. D^{-1} + D^{-1} E + D^{-1} W. \hat{\gamma} D^{-1}.$$  

(A5)

The equality in (A3) is a consequence of the equation in the off-diagonal terms of (A5).

The quantity $\hat{\delta}_A := \pi^A \nabla \pi^{A'}$, where $\pi^A$ is any LSR for $[\pi^A]$ is clearly determined, up to scale, by the (Walker) geometry. Fixing the LSR $\pi^A$ to be that in (3.5), i.e., to be the element $\epsilon^{A'}_A$ of the Walker spin frames, fixes $\hat{\delta}_A$ up to sign. Moreover, from (A1–2), one confirms that $\hat{\delta}^A = \pm \chi \hat{\delta}_A$, as expected. On the other hand, defining, with respect to a given pair of Walker spin frames,

$$\zeta_A := \xi^{A'} \nabla \pi^{A'} = \alpha_A D' - \beta_A \delta, \quad \text{then} \quad \zeta_A = \chi^{-1} (\zeta_A + \mu \delta_A).$$  

(A6)

Using Law (2008), (5.8) & (5.10), one can show that, acting on functions,

$$\delta(A \pi B) = \zeta_A \delta_B - \left[ \zeta_A \pi B' \nabla B \right].$$  

(A7)

The hyperheavenly formalism, e.g., of Finley and Plebański (1976) and Boyer et al. (1980), might tempt one to define, for given Walker spin frames, $W^{AB} := W^{AB} \epsilon^{A'}_A \epsilon^{B'}_B$. From (A2–3) and (A5), one computes

$$\tilde{W}^{AB} = \chi^{-2} \left( (D. E + E. D) ^{CD} \epsilon^C_A \epsilon^D_B + W^{AB} \right).$$  

(A8)

Hence, for $W^{AB}$ to be a meaningful spinorial object, at least with respect to Walker spin frames, one would require

$$D \in \text{SL}(2; \mathbb{R}) \quad \text{and} \quad D. E \text{ skew.}$$  

(A9)
When utilizing oriented Walker coordinates \((U, V, X, Y)\) as in (3.33–34), and their associated Walker spin frames, it will be convenient to employ the following notation. Noting (3.34–35), define
\[
J_A := \delta_A \Omega^{-1} = -\Omega^{-2} \delta_A \Omega; \quad K^A := -(N^A + M_A^\beta); \quad \tau := K^A J_A = -2MN
\]
whence
\[
2K^{[A} J^{B]} = -\tau \epsilon^{AB}.
\] (A10)

If \((P, Q, R, S)\) are another set of oriented Walker coordinates satisfying (3.33–34), then
\[
\Omega = MU + NV = FP + GQ,
\] (A11)
say. Consequently, \(MdU + NdV = FdP + GdQ\), which is equivalent to, with \(J := J_A \epsilon_A^A\),
\[
^D J = \hat{J} \quad \quad ^E J = 0,
\] (A12)
the first equation of which is consistent with \(\hat{\delta} \Omega = \chi \delta \Omega\). \(K^A\), however, does not have form-preserving transformation and is best thought of as defined with respect to a fixed set of such oriented Walker coordinates \((U, V, X, Y)\), and the associated Walker spin frames.

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