Superconducting Quantum Critical Point

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We study the properties of a quantum critical point which develops in a BCS superconductor when pair-breaking suppresses the transition temperature to zero. The pair fluctuations are characterized by a dynamical critical exponent $z = 2$. Except for very low temperatures, anomalous contribution to the conductivity is proportional to $\sqrt{T}$ in three dimensions, but to $1/T$ in two dimensions. At lowest temperatures, the conductivity correction varies as $T^{1/4}$ in three dimensions, and as $\ln(1/T)$ in two.

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The possibility of quantum critical behavior in itinerant magnets has attracted great attention in recent years, in part, because quantum criticality affords the possibility of a controlled study of non-Fermi liquid behavior. At a quantum critical point, order parameter fluctuations develop an infinite correlation range in both space and time. The coupling between these fluctuations and conducting electron fluid is under certain circumstances, to eliminate the formation of well-defined quasi-particles in the electron liquid, giving rise to a new kind of metallic behavior.

In this paper we discuss the possibility of quantum critical behavior in superconductors. A quantum critical point implies a finite value of the electron interaction strengths. At first sight, this would appear to rule out the possibility of a superconducting quantum critical point, for conventional superconductivity develops for an arbitrarily small pair interaction. If the transition temperature is driven to zero in a pure BCS superconductor, the pairing interaction and the pair fluctuations are completely eliminated. Fortunately, this is not the case in the presence of pair-breaking, which cuts off the logarithmic singularity in the pair susceptibility, requiring that the pairing interaction reach a critical strength before superconductivity develops. In this paper we characterize the quantum critical behavior which develops at this special point. Our results can be tested experimentally on conventional superconductors, such as Ce-doped $La$. They may also provide a useful diagnostic tool for the understanding of unconventional, e.g. heavy fermion superconductors.

We begin with writing down the effective action of the pair field $\Delta$ in the vicinity of a quantum critical point:

$$S_{eff}[\Delta] = T \sum_{\nu,k} \Delta^\dagger \chi^{-1} \Delta + \frac{N(0)T^3}{T_0} \sum_{1,2,3} \Delta^\dagger_1 \Delta^\dagger_2 \Delta_3 \Delta_4,$$

(1)

$$\chi^{-1} = N(0) \left[ \frac{\delta F}{x_c} + \frac{T}{T_0} + \frac{|\nu|}{T_0} + \left[ \frac{v_F \nu}{T_0} \right]^2 \right],$$

(2)

which is valid for $T, vq, |\nu| \ll T_0$. Here $T$ is temperature, $N(0)$ is the density of states at the Fermi surface, $\nu$ is a Matsubara frequency, $q$ is momentum, $v_F$ is the Fermi velocity, and $\delta F$ is the deviation of the control parameter $x$ from its critical value $x_c$, where the transition temperature turns into zero.

We obtained this action by repeating the Abrikosov-Gorkov calculation for an $s$-wave BCS superconductor doped with magnetic impurities. Apart from the original calculation, the frequency and momentum dependence of the disorder-averaged pair propagator was kept and at the end both the pair field and the momentum were rescaled to give (1)-(2).

The quantity $T_0$ is the only characteristic energy scale of the effective action (1)-(2). In a BCS superconductor it is of the order of the pair-breaking rate, which at the critical point is of the order of the transition temperature in the clean system. Both quantities are much less than the Fermi energy $\epsilon_F$. However, in a strongly correlated material, $T_0$ may, in principle, be of the order of $\epsilon_F$.

These two limiting cases correspond to different physics. In the BCS case, the quartic term in (1) can be neglected at all experimentally relevant temperatures $T$ well below $T_0$. By contrast, if $T_0 \sim \epsilon_F$, the feedback of the quartic term dominates at $T \ll T_0$. In the latter case in the presence of pair-breaking, which cuts off the logarithmic singularity in the pair susceptibility, requiring that the pairing interaction reach a critical strength before superconductivity develops. In this paper we characterize the quantum critical behavior which develops at this special point. Our results can be tested experimentally on conventional superconductors, such as Ce-doped $La$. They may also provide a useful diagnostic tool for the understanding of unconventional, e.g. heavy fermion superconductors.

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TABLE I. Order of magnitude and leading temperature
dependences of the imaginary part of the electron self-energy
$\Sigma(\omega;p = p_F; T = 0)$ due to scattering by the fluctuations of
$\Delta$, the specific heat coefficient correction $\delta C(T)/T$ and the
correctivity correction $\delta \sigma(T)$ in the weak coupling (BCS)
limit ($T_0 << \epsilon_F$) and in the strong coupling limit ($T_0 \sim \epsilon_F$).
The value of $\sigma_0$ corresponds to the residual normal state con-
ductivity.

| $T_0$ | $d$ | $\text{Electron Specific Heat Conductivity} \quad \delta \sigma \quad \delta C \quad \delta \sigma$ |
|------|-----|-----------------------------------|-----------------|-----------------|
| $<< \epsilon_F$, $d = 3$ | $\frac{(T_0 \omega)^{3/2}}{\epsilon_F}$ | $\frac{T_0^{3/2}T^{-1/2}}{\epsilon_F}$ | $\sigma_0 \frac{T_0^{3/2}T^{-1/2}}{\epsilon_F}$ |
| $\sim \epsilon_F$, $d = 3$ | $\frac{\omega^{3/2}}{\epsilon_F^2}$ | $\frac{T^{1/2}}{\epsilon_F^2}$ | $\sigma_0 \frac{T}{\epsilon_F^{1/4}}$ |
| $<< \epsilon_F$, $d = 2$ | $\frac{\omega T_0}{\epsilon_F}$ | $\frac{T_0}{\epsilon_F} \ln \left( \frac{T_0}{T} \right)$ | $\sigma_0 \frac{T_0^2}{\epsilon_F T}$ |
| $\sim \epsilon_F$, $d = 2$ | $\omega$ | $\frac{1}{\epsilon_F} \ln \left( \frac{\epsilon_F}{T} \right)$ | $\sigma_0 \ln \left( \frac{\epsilon_F}{T} \right)$ |

In the case, one has to regard $T_0$ as a phenomenological par-
ter, resulting from a strong coupling, or non-BCS pairing
mechanism.

With (1)-(2) at hand, one can calculate various ther-
odynamic and transport properties of interest. Table I
presents the results for the zero-temperature quasipar-
ticle decay rate due to scattering by the $\Delta$ field, and for
the leading corrections to low-temperature thermodynamics
and transport at $T_c = 0$. In the BCS limit, corrections
to the specific heat coefficient come from Gaussian fluc-
tuations of the pair field. The correction for a strong
coupling case was found in using renormalization group
methods. The quasiparticle decay rate due to scatter-
ing off pair fluctuations is estimated by the diagram on
Fig.2 (a). It is essential for the calculation of conductivity
and of the quasiparticle decay rate, that the electron
vertex corrections are not singular, since the pair-breaking
makes the lifetime of a Cooper pair finite. The lead-
ing conductivity correction is given by the Aslamazov-
Larkin diagram shown on Fig.2(b), which can be re-
garded as conductivity of particles with the inverse prop-
gagator given by (2), at $\delta x = 0$.

The calculation was done by renormalization group
analysis of the expression for the Aslamazov-Larkin cor-
rection on Fig.2 (b). After transforming the Matsubara
sum into a contour integral and going to dimensionless
variables, it reads

$$\Delta \sigma = \int_1^1 d^d q \int_1^1 dz \frac{1}{T \sinh^2 \frac{z}{2T}} \left[ Im \chi(q + \omega) \right]^2,$$

where both the energy and the momentum cut-off have
been set to unity. When renormalizing, we will follow
first integrate out a thin outer shell in the momentum
space between 1 and $1 - 1/b$. Then rescale the momentum
$T \to q b^2$, mass term and the temperature $T \to Tb^2$ and,
finally, integrate out an energy shell to restore the
energy cut-off. At each step the quartic interaction
induces corrections to the mass term and to itself (see
for details of the renormalization group equations).

As a result, one arrives at the following transformation
law for the Aslamazov-Larkin correction:

$$\Delta \sigma(J) = b^{2-D} \Delta \sigma(J^*) + \ln b f[J],$$

where $J$ denotes the mass term, the temperature and the
quartic coefficient, $J^*$ denotes their renormalized values
and $D$ is the dimensionality of the sample. The precise
form of $f[J]$ can be easily obtained using the above reno-
malization procedure. However, only two features of $f[J]$ are
important: (a) as long as the running value of $T$ is
smaller than the cut-off, $f[J]$ has rather weak dependence
on its arguments; this corresponds to the quantum renor-
malization region; (b) in the classical renormalization
region, when $T$ exceeds the cut-off, $f[J]$ is proportional
to $T$. Thus $\Delta \sigma$ can be written as

$$\Delta \sigma = \int_0^{\ln b^*} dx f[J(e^x)] e^{(2-D)x},$$

where $b^*$ is the value of the rescaling factor at which
the mass term reaches the cut-off and the scaling process
stops. It is of the order of $T^{-3/4}$ in three dimensions
and of the order of $\sqrt{\ln(1/T)/T}$ in two. To evaluate
(4), one also needs the value of $b_1$ such that $T(b_1) \sim 1$,
which is $b_1 \sim T^{-1/2}$ independently of the dimensionality.
With these prerequisites, the answers in Table II follow
as soon as one neglects the dependence of $f[J(e^x)]$ on all
the couplings except temperature:

$$f[J(e^x)] = f[T e^{2x}].$$

In three dimensions, the same result can be obtained
just by replacing the “bare” $T^2$ mass term in (2) by its
renormalized value $T^{3/2}$, and then calculating the con-
ductivity correction as per (3).

Scaling analysis also allows to show that the vertex cor-
rections are negligible. Their inclusion reduces to putting
into the diagram of Fig.2 (b) extra bubbles such as one
on Fig.2 (c). Each bubble contributes two Green’s func-
tions plus an integral over the energy and momentum.
After an infinitesimal scaling transformation, this gives
an extra factor of $b^{2-D}$. Thus, in three dimensions, ver-
tex corrections are irrelevant. In two dimensions, they
appear marginal yet do not introduce any extra correc-
tions. This can be established by using the Ward identity
and writing the renormalization equation directly for the current vertex (see Fig.3), and then inserting the solution into the corresponding expression for the conductivity.

We would like to discuss theoretical and experimental implications of the results. As envisaged by Hertz, under quite general circumstances the superconducting quantum critical point falls into the $z = 2$ universality class. Although in principle different behavior cannot be ruled out, it appears to require additional tuning of parameters, as well as rather unusual features of the pairing phenomenon itself, such as the gap vanishing at the entire Fermi surface.

In a weakly interacting disordered metal, $T_0$ in (2) is of the order of the impurity scattering rate. In this case anomalous corrections due to quantum criticality in the Landau-Ginzburg regime are of the order of the weak localization corrections. Therefore, in two dimensions our results are fully consistent only as long as all corrections are small and additive. Moreover, in two dimensions, the electron-electron interaction is known to lead to the linear temperature dependence of the quasiparticle decay rate, regardless of closeness to the quantum critical point. Thus, for a weakly interacting two-dimensional system, the entire normal region corresponds to the two-dimensional disordered metallic regime (see Fig.1) with the quasiparticle decay rate proportional to the quasiparticle energy.

However, we expect that in a strongly correlated system with suppressed weak localization effects, superconducting quantum critical point still falls into the $z = 2$ universality class and Table I may describe the actual state of affairs for all dimensionalities. In this case, Fermi liquid turns marginal only at $T_c = 0$ and crosses over to the normal Fermi liquid behavior as the system is driven away from the quantum critical point into the metallic phase.

Finally, we should like to comment on the diagnostic opportunities, furnished by measurements at a superconducting quantum critical point. In light of the discussion in the beginning of the paper, one is led to conclude that in a clean time reversal invariant system, observation of singular behavior at the superconducting quantum critical point would mark a very peculiar phenomenon, as in a clean BCS superconductor suppression of $T_c$ completely eliminates pair fluctuations. Since any sample contains impurities, in reality the above conclusion refers to temperatures above the elastic scattering rate: $T > \tau^{-1}$. The latter can be extracted from the residual resistivity measurements of the sample.

To summarize, we studied the properties of a superconductor near a quantum critical point driven by pair-breaking disorder. Superconducting fluctuations are characterized by a dynamical critical exponent $z = 2$. In a BCS superconductor at experimentally accessible low temperatures, the singular contribution to the conductivity is proportional to $\sqrt{T}$ in three dimensions, but to $1/T$ in two dimensions. In a superconductor with the characteristic energy scale of the order of $\epsilon_F$ at the quantum critical point, the contribution to the conductivity varies as $T^{3/4}$ in three dimensions, and as $\ln(1/T)$ in two.

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1 see J. Phys. Cond. Mat. 8, (1996), for exhaustive references.
2 John A. Hertz, Phys. Rev. B 14, 1165 (1976).
3 A. J. Millis, Phys. Rev. B 48, 7183 (1993).
4 M. B. Maple et al., Phys. Rev. Lett. 23, 1375 (1969).
5 A. A. Abrikosov, L. P. Gor’kov, Sov. Phys. JETP 12, 1243 (1961).
6 see, e.g., A. M. Tsvelik, Quantum Field Theory in Conденsed Matter Physics, Cambridge University Press, 1995.
7 L. G. Aslamazov, A. I. Larkin, Sov. Phys. Solid State 10, 1104 (1968).
8 A. G. Aronov, S. Hikami, A. I. Larkin, Phys. Rev. B 51, 3880 (1995).
9 R. Ramazashvili, Phys. Rev. B 56, 5518 (1997), and references therein.
10 B. L. Altshuler, A. G. Aronov, in Electron-Electron Interactions in Disordered Systems, editors A. L. Efros and M. Pollak (Elsevier Science Publishers B. V., 1985).
11 E. Abrahams, P. W. Anderson, P. A. Lee, and T. V. Ramakrishnan, Phys. Rev. B 24, 6783 (1981);