Testing cycle-freeness: Finding a certificate

C. Seshadhri
IBM Almaden Research Center

Abstract

We deal with the problem of designing one-sided error property testers for cycle-freeness in bounded degree graphs. Such a property tester always accepts forests. Furthermore, when it rejects an input, it provides a short cycle as a certificate. The problem of testing cycle-freeness in this model was first considered by Goldreich and Ron [13]. They give a constant time tester with two-sided error (it does not provide certificates for rejection) and prove a $\Omega(\sqrt{n})$ lower bound for testers with one-sided error. We design a property tester with one-sided error whose running time matches this lower bound (upto polylogarithmic factors). Interestingly, this has connections to a recent conjecture of Benjamini, Schramm, and Shapira [3]. The property of cycle-freeness is closed under the operation of taking minors. This is the first example of such a property that has an almost optimal $\tilde{O}(\sqrt{n})$-time one-sided error tester, but has a constant time two-sided error tester. It was conjectured in [3] that this happens for a vast class of minor-closed properties, and this result can seen as the first indication towards that.

1 Introduction

Given a massive data set, one may wish to learn some properties of this data set without reading the whole input. The standard decision problem is to accept an input if it satisfies some given property and reject it otherwise. In property testing [10,18], we deal with a relaxed version of this decision problem: accept the input if it satisfies the property, and reject if the input is far from satisfying the property. There are algorithms that can perform this relaxed version, for a vast array of properties, in sublinear (or even independent of the input size) time (see surveys [4,8,9,17]).

Goldreich, Goldwasser, and Ron first introduced property testing for dense graphs in [10], and gave testers for many different properties. Deep theorems about the class of properties that can tested in this model have been given in [1,2].

In this paper, we will deal with property testing in bounded degree graphs, as introduced by Goldreich and Ron [13]. The input undirected graph $G$ has degree bound $d$ (considered as a constant) and $n$ vertices. It is represented by an adjacency list, and therefore has a representation size of $nd$. We will assume that $G$ is simple without self-loops. Henceforth, we will only consider graphs of this form. For every vertex $v$, we have access to the at most $d$ neighbors of $v$. This allows a testing algorithm to perform walks in $G$, something that cannot be done with adjacency matrices. Given a property $\mathcal{P}$ and distance parameter $0 < \varepsilon < 1$, the graph $G$ is $\varepsilon$-far from $\mathcal{P}$ if $G$ must be changed at more than $\varepsilon nd$ edges to have property $\mathcal{P}$. A property tester accepts $G$ if it has property $\mathcal{P}$ and rejects if $G$ is $\varepsilon$-far from $\mathcal{P}$ (both hold with high probability). The running time of the property tester should be $\text{poly}(d\varepsilon^{-1}) o(n)$ (if there is no dependence on $n$, then we say that the running time is constant). A stronger tester, called a tester with one-sided error (for ease of notation, we will call this a one-sided tester) accepts with probability 1 if $G$ has $\mathcal{P}$. Also, if it rejects $G$, it provided a sublinear-sized certificate that $G$ does not have $\mathcal{P}$. This makes one-sided testers far more interesting, and harder to design. For one-sided testers, we have to prove that if $G$ is $\varepsilon$-far from having $\mathcal{P}$, then a sublinear sized certificate that $G$ is not in $\mathcal{P}$ can be found in sublinear time.
Goldreich and Ron [13] showed that many connectivity properties are testable in time independent of \( n \). Progress towards finding general classes of testable properties for bounded degree graphs was first made by Czumaj, Sohler, and Shapira [5, 7]. Benjamini, Schramm, and Shapira [3] made a breakthrough towards these characterizations, and showed that any minor-closed property is testable. These algorithms run in time independent of \( n \) but are not one-sided testers. Many interesting properties do not fall into this framework. Indeed, Goldreich and Ron [13] gave a lower bound of \( \Omega(\sqrt{n}) \) for property testing bipartiteness in the bounded degree model.

There has been a surprising dearth of knowledge about properties that require more than constant time for testing. Goldreich and Ron, in a very involved result, give a tester for bipartiteness [11] whose running time almost matches their lower bound of \( \sqrt{n} \). One of the very nice features of this work was the first use of random walks for property testing. The property of expansion received a lot of attention recently, and also has almost optimal property testers running in time \( \tilde{O}(\sqrt{n}) \) [6, 12, 14, 16].

1.1 Cycle-freeness

It is very intriguing to look at the natural and basic property of cycle-freeness, where the tester accepts if the input is a forest, and rejects if it is \( \varepsilon \)-far from being one. Goldreich and Ron in their initial paper [13] give a constant time tester for this property. Marko and Ron extend this to estimate the distance to cycle-freeness in constant time [15]. The tester of [13] is not one-sided, and fundamentally rests on the fact that in a tree, the number of edges is less than the number of vertices. They also prove the somewhat surprising lower bound that one-sided cycle-freeness testing cannot be done in \( o(\sqrt{n}) \) time. This is the problem that we shall attack. We resolve the problem of testing cycle-freeness by giving a one-sided property tester whose running time is almost optimal.

**Theorem 1.1** There exists a one-sided property tester for cycle-freeness whose running time is \( \sqrt{n} \ poly(d\varepsilon^{-1} \log n) \). In other words, this property tester always accepts the input \( G \), if \( G \) is a forest. If \( G \) is \( \varepsilon \)-far from being a forest, it rejects \( G \) with probability at least \( 2/3 \). Moreover, when the tester rejects \( G \), it provides a certificate in the form of a short cycle (of length \( poly(\varepsilon^{-1} \log n) \)).

Why is this challenging? The difficulty comes from the fact that the property tester needs to find a cycle in a graph that \( \varepsilon \)-far from being cycle-free. Most property testers for bounded degree graphs essentially sample some random vertices and looks at small neighborhoods around them. This cannot work for a one-sided cycle-freeness tester. The only testers that use superconstant time (actually \( O(\sqrt{n}) \) time) are the testers for bipartiteness and expansion. The main tool they use is that of random walks. It is very interesting to have another property that can be tested through a non-trivial use of random walks. Combinatorially, a one-sided cycle-freeness tester is more interesting because when it rejects the input \( G \), it provides a certificate in the form of a short cycle.

We use many ideas from the tester of Goldreich and Ron for bipartiteness [11]. One of their major contributions is to provide a framework to analyse short random walks in bounded degree graphs, and argue how it suffices to deal with “well-connected” graphs. Much of their result dealt with proving theorems about this framework. With this in place, they only need to test bipartiteness within these special subgraphs. It turns out that testing cycle-freeness within such subgraphs is much harder, and requires new ideas.

We develop combinatorial tools to study the behavior of random walks in graphs that are almost forests. We show how random walks in such graphs can be used to detect cycles. A major challenge

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1upto polylogarithmic factors

2This tester can reject forests and does not provide a certificate on rejection.
is to show that this can be done in $\tilde{O}(\sqrt{n})$ time, to match the lower bound of Goldreich and Ron. We are able to give a characterization of some special edges in terms of random walk behavior, such that these special edges form a forest. We then argue that the presence of many non-special edges allows our property tester to find a cycle.

Benjamini, Schramm, and Shapira [3] also observe the gap between property testers and their one-sided cousins. We first explain graph minors. A minor of a graph $G$ is obtained by removing vertices, edges, and contracting edges. A graph $G$ is $H$-minor free, if $H$ is not a minor of $G$. A property is minor-closed if the property is preserved on taking minors. They show that every minor-closed property is testable (with two-sided error) with a constant number of queries and conjecture that all such properties can be tested with one-sided error in $\tilde{O}(\sqrt{n})$ time\footnote{Based on lower bounds in [13], they conclude that for a large class of minor-closed properties, there is a lower bound of $\Omega(\sqrt{n})$ for one-sided testing.}. We provide the first such property with exactly this behavior, thereby giving some (very small) evidence for this conjecture.

The property of being cycle-free is certainly minor-closed. There is a constant time tester for cycle-freeness (with two-sided error) and we give an almost optimal $\tilde{O}(\sqrt{n})$-time one-sided tester. It is possible that the tools developed here could yield some insight into the conjecture of [3].

In the language of minors, we can think of cycle-free graphs as $K_3$-minor free graphs, where $K_3$ is the triangle. Our tester finds in sublinear time a forbidden minor in a graph that is far from being $K_3$-minor free. Our special edges form a $K_3$-minor free graph, and the remaining edges boost the probability of finding a $K_3$-minor in sublinear time. It would be very interesting to generalize our technique to other minors, making progress towards the conjecture of [3]. Specifically, if we could do the same for $K_5$ and $K_{3,3}$ minors, then we would have a sublinear time procedure to find forbidden minors in graphs far from being planar.

2 Preliminaries and intuition

The basic idea is to perform $O(\sqrt{n})$ random walks of length $O(\log n)$ from a random subset of $O(\varepsilon^{-1})$ vertices of $G$. We use a standard lazy random walk: each edge of $G$ is assigned a probability of exactly $1/2d$. For a vertex $v$, suppose that there are $d' \leq d$ edges incident to $v$. There is a self-loop at $v$ with probability $1 - d/2d' \geq 1/2$. We hope if $G$ is $\varepsilon$-far from being cycle-free, then we can find two different paths between two vertices. This provides us with a cycle that can be output as a witness. Note that it is not necessarily the case that there are an extremely large number of cycles in such a $G$. Consider the case where $G$ consists of $\varepsilon n$ disjoint cycles of size $1/\varepsilon$. The number of cycles is relatively small, but on the other hand, they are not hard to detect.

A technical lemma of [11] shows (roughly speaking) that we can partition the vertices of $G$ into “well-connected components”. Such a component $S$ has the following property: there is a special vertex $s \in S$ such that short (polylogarithmic) random walks from $s$ reach all vertices of $S$ with high probability. Furthermore, there are few edges connecting $S$ to the remaining vertices. These well-connected components do not necessarily have to look like expanders. Such a component could even be a very small (polylogarithmic) path. It is merely the case that random walks of a specified length from $s$ spread out to all of $S$. One of the nice features of this lemma is that we do not have to explicitly compute this partition. Note that the random walks are performed in $G$, not in the induced subgraph of $S$. It is some kind of a thought experiment that allows us to imagine that we are performing walks inside well-connected components. The testing algorithm is completely oblivious to this partition, and simply performs walks in $G$.

We can therefore focus on any one such well-connected component $S$ with $s$ as the special vertex. For simplicity, assume that all edges and vertices that we mention below are in $G_S$, the induced
subgraph on $S$. Suppose we perform a large number of random walks from $s$ and notice that there is an edge $e$ that always lies on a path from $s$ to $v$. The edge $e$ is a sort of bottleneck. Roughly speaking, we will show that there cannot be too many bottleneck edges since they all form a forest. Now, if $G_S$ was far being a forest, then there must be many edges in $G_S$ that are not bottlenecks. We then show that the presence of many non-bottleneck edges will boost the probability of detecting a cycle.

Now, we need to prove that a relatively small number of walks can find a cycle. This requires a delicate argument. We are able to argue that the probability of two random walks witnessing a cycle is around $1/n$. So, using a birthday paradox like argument, (as done in [11]), we hope that it suffices to perform $O(\sqrt{n})$ to detect a cycle. Let the random variable $X$ be an indicator for two random walks witnessing a cycle. It boils down to seeing the behavior of $X$. We showed that the expectation of $X$ behaves like $1/n$, but the variance can be too large. We look into when this variance is too large. This happens because there a relatively small number of paths which are part of most cycles. It turns out that we can handle this case separately (without dealing with bottleneck edges) in exactly $O(\sqrt{n})$ time.

3 Within a well-connected subgraph

We first perform an analysis in a well-connected component of $G$. This component will be referred to as $S$, and the induced subgraph on it as $G_S$. We fix some parameters: the length of the random walk $\ell = \text{poly}(\varepsilon^{-1} \log n)$, a probability $\frac{\varepsilon}{\sqrt{|S| \log n}} < \alpha < 1$, and $m = c\varepsilon^{-3}\sqrt{n\ell \log^2 n}$ ($c$ is a sufficiently large constant). Let $q_v$ be the probability that a random walk from $s$ reaches $v$. This means that the walk from $s$ encounters $v$ during its course.

Let $s \in S$ be a special vertex in $S$ if $q_v > \alpha$ for all $v \in S$. The analysis in this section will only deal with the vertices in $S$, although the random walk is being performed in $G$. From now on, the term random walk will refer only to walks from $s$ of length $\ell$.

Suppose that $G_S$ is far from being cycle-free. Our aim is to show that when many random walks are performed from $s$, a cycle is detected with high probability. A walk is a sequence of edges with possible backtracking and self-loops. The path induced by this walk is the sequence of edges from $s$ to $v$ obtained by removing self-loops and retraced paths. If we can find two different paths from $s$ to $v$, then this is a cycle in $G$. Note that two different walks can induce the same path, so finding two different walks from $s$ to $v$ yields no information. This is a major problem when trying to find a cycle. Just because there is a high probability of reaching $v$ from $s$ does not guarantee that different walks from $s$ to $v$ form a cycle. It could be possible that most walks from $s$ to $v$ induce the same path.

When we deal with a walk from $s$ that reaches $v$, we will not deal with the portion of the walk after $v$. In other words, we will say that two walks reach $v$ by the same path if the walks induce the same path upto $v$ (they can diverge after that).

We now describe a process CYCLE FINDER that attempts to find a cycle given such a vertex. This is analogous to the algorithm of [13] for bipartiteness. We remind the reader that a graph $G$ is $\varepsilon$-far from being cycle-free, if at least an $\varepsilon$-fraction of edges need to be removed from $G$ to eliminate all cycles. The main lemma of this section is:

Lemma 3.1 Let $s \in S$ be a special vertex for $S$ and $G_S$ be $\varepsilon$-far from being cycle-free. Then CYCLE FINDER with $s$ as input will detect a cycle with probability at least $2/3$.

\footnote{The exact polynomial factor will be decided later.}
Cycle Finder

Input: Vertex $s \in S$

1. Perform $m$ random walks of length $\ell$ from $s$.

2. Look at the subgraph $G'$ induced by all the vertices reached by the random walks. ($G'$ has at most $d m \ell$ edges and is therefore of $O(\sqrt{n})$ size.) If $G'$ is cycle-free, ACCEPT. Otherwise REJECT and output a cycle.

Before we study the behavior of this procedure, we will need a few definitions. We will look at every edge $(u, v)$ as a pair of directed edges $\langle u, v \rangle$ and $\langle v, u \rangle$. For the sake of clarity, angular braces represent a directed edge and parentheses denote an undirected edge.

**Definition 3.2** A vertex $v \in G_S$ is isolated if more than half of the walks that reach $v$ induce the same path (called the dominant path). Furthermore, the probability of a walk reaching $v$ and inducing a different path is $< \alpha/2$.

**Definition 3.3** A directed edge $\langle u, v \rangle$ is in $G_S$ is dominant if the following conditions hold:

1. The vertex $v$ is isolated.

2. The probability that a walk reaches $v$ without passing through the edge $(u, v)$ is $< \alpha/2$.

An undirected edge $(u, v)$ is recessive, if neither $\langle u, v \rangle$ or $\langle v, u \rangle$ are dominant.

The dominant edges are the bottlenecks that we mentioned in the previous section. We now prove some small claims about these edges that will allow us to bound their number.

**Claim 3.4** Given a dominant edge $\langle u, v \rangle$, the dominant path to $v$ passes through $(u, v)$.

**Proof:** Assume by contradiction that the dominant path does not contain the edge $(u, v)$. By the definition of the dominant path, the probability that a walk reaches $v$ by a path passing through $(u, v)$ is $< \alpha/2$. By the properties of $S$, we know that the probability of reaching $v$ is $\geq \alpha$. This implies that the probability of reaching $v$ by a path that does not contain $(u, v)$ is $> \alpha/2$. This violates the second part of Definition 3.3. □

**Claim 3.5** Both $\langle u, v \rangle$ and $\langle v, u \rangle$ cannot be dominant.

**Proof:** Since $\langle u, v \rangle$ is dominant, the probability that a walk reaches $v$ without going through $(u, v)$ is $< \alpha/2$. The probability of reaching $v$ is $\geq \alpha$. Therefore, the probability of a walk reaching $v$ after taking $(u, v)$ is $> \alpha/2$. Note that the portion of these walk from $s$ to $v$ cannot contain any other occurrence of $v$. Therefore, these walks reach $u$ without passing through $(u, v)$, contradicting that $\langle v, u \rangle$ is dominant. □

With these claims in hand, we are ready to connect dominant edges to cycle-freeness.

**Lemma 3.6** The set of dominant edges form a directed forest.
Proof: Consider the subgraph (directed) $U$ created by the dominant edges. First we show that no vertex in $U$ can have indegree $\geq 2$. Suppose there are edges $\langle u, v \rangle$ and $\langle w, v \rangle$. With probability $> \alpha/2$, the path reaching $v$ goes through edge $\langle u, v \rangle$ (or $\langle w, v \rangle$). Note that the probability of reaching $v$ through a dominant path is $> \alpha/2$. Since this must both pass through $\langle u, v \rangle$ and $\langle w, v \rangle$, we reach a contradiction.

Suppose the graph $U$ has an undirected cycle. By what was proven above, this must also be a directed cycle. Consider the directed cycle $\langle u_0, u_1 \rangle, \langle u_1, u_2 \rangle, \ldots, \langle u_{k-1}, u_k \rangle, \langle u_k, u_0 \rangle$. Let us look at all walks that reach $u_k$ through the dominant path (which passes through $u_{k-1}$). These walks have a cumulative probability of $> \alpha/2$. Suppose this path reaches the set of vertices $u_0, \ldots, u_k$ first at $u_i (i \neq 0)$. Since this path does not pass through $u_{i-1}$, it must be the case the with probability $> \alpha/2$, a walk reaches $u_i$ without passing through the edge $\langle u_{i-1}, u_i \rangle$. This contradicts the fact that $\langle u_{i-1}, u_i \rangle$ is dominant.

With probability $> \alpha/2$, a walk reaches $u_k$ inducing a path (which is dominant) that passes through $u_0$. Since $\langle u_k, u_0 \rangle$ is dominant, with probability $> \alpha/2$, a walk reaching $u_0$ induces a fixed path passing through $\langle u_k, u_0 \rangle$. Since $\langle u_{k-1}, u_k \rangle$ is dominant, this fixed path must follow the dominant path to $u_k$. But we showed that the dominant path to $u_k$ passes through $u_0$. This is absurd since it contradicts the fact that the dominant path to $u_0$ passes through $u_k$. This proves that $U$ is a directed forest.

\[ \square \]

3.1 Analysis

Assume that $G_S$ is $\varepsilon$-far from being cycle free. By Lemma 3.6, there must be at least $\varepsilon \text{nd}$ recessive edges. We will show that the presence of many recessive edges will imply that Cycle Tester will find cycles with high probability. Let $X$ be the indicator random variable for two independent random walks performed by Cycle Tester forming a cycle. The approach is to show that the expectation of $X$ is large enough (around $1/n$) because of the recessive edges. To ensure that a sublinear number of walks suffice to find a cycle, we need the variance of $X$ to be small. It turns out that the variance of $X$ can be quite large, and this is because of certain paths that we call heavy. Fortunately for us, cycles involving heavy paths can actually be detected very quickly.

We split our analysis into two parts. If there are many heavy paths, then we can argue directly that there is a high probability of finding a cycle. On the other hand, if there are not too many heavy paths, we can contain the variance of $X$. In either case, we only need $O(\sqrt{n})$ time to find cycles.

Definition 3.7 For a random walk $W$, let $\cyc_W$ be the probability that a random walk of length $\ell$ from $s$ creates a cycle with $W$.

If $\cyc_W > 1/\sqrt{n}$, then $P$ is heavy. Otherwise, it is light.

We first deal with heavy paths. We split the probability mass of walks $q_v$ into two disjoint parts: $q_v^L$ ($q_v^H$) is the probability that $v$ is reached by a light path (resp. heavy path). Obviously, $q_v = q_v^L + q_v^H$. Call vertex $v$ blue if $q_v^H > \alpha/4$.

Claim 3.8 If there are more than $\varepsilon|S|/2$ blue vertices, then Cycle Tester finds a cycle with probability $> 2/3$.

Proof: Consider a random walk $W$ of length $\ell$ from $s$. Let $X_W$ be the indicator random variable for the event that the path $P$ induced by $W$ is heavy. Let indicator random variable $X_v$ be 1 iff $P$ is heavy and passes through $v$. Since $W$ passes through at most $\ell$ vertices, $X_W \geq \sum_v X_v/\ell$. By
linearity of expectation, $E[X_W] \geq \sum_v E[X_v] / \ell = \sum_v q_v^H / \ell$. By the claim assumption, there are at least $\epsilon |S|/2$ vertices $v$ such that $q_v^H > \alpha/4$. That implies that $E[X_W] \geq \epsilon \alpha |S| / 8 \ell = \Omega(\epsilon^d \sqrt{\frac{|S|}{2n/\log n}})$. The expectation $E[X_W]$ is the probability that the random walk $W$ induces a heavy path. Consider the first $m/2$ walks performed by Cycle Tester. Since $m > c^{-1} \sqrt{n \ell \log n}$, with probability at least $5/6$, one of the first $m/2$ walks will induce a heavy path (abusing notation, call this $W$).

We know that $cyc_W > 1/\sqrt{n}$. Since $m > c^{1/2} |S|/n$, with probability at least $5/6$, the latter $m/2$ walks will contain a walk that creates a cycle with $W$. Therefore, by a union bound, with probability at least $2/3$, Cycle Tester finds a cycle. 

We will henceforth assume that there are at most $\epsilon |S|/2$ blue vertices. Call an edge non-blue if neither of its endpoints are blue. Since the degree bound is $d$, we have at least $\epsilon |S|/d/2$ non-blue recessive edges. Consider two independent random walks performed from $s$. Let $\mathcal{E}_v$ be the event that the two random walks are light and find a cycle in one of the following ways:

1. The two walks reach $v$, and the corresponding paths induces a cycle (which may not contain $v$; it is even possible that one walk by itself creates a cycle).
2. One walk reaches $v$, and the other walk reaches a neighbor $u \in S$. The two walks and the edge $(u, v)$ induce a cycle.

When the event $\mathcal{E} = \bigcup_v \mathcal{E}_v$ happens, then two walks performed by Cycle Tester detect a cycle. Goldreich and Ron [1] use similar events in their bipartiteness tester.

**Lemma 3.9** If there are less than $\epsilon |S|/2$ blue vertices, $\Pr(\mathcal{E}) \geq \epsilon |S|\alpha^2 / 16 \ell$.

**Proof:** We focus on two walks. There are at least $\epsilon |S|/d/2$ non-blue recessive edges. Consider such an edge $(u, v)$. Both $q_u^L, q_v^L$ are at least $3\alpha/4$. Neither $(u, v)$ or $(v, u)$ are dominant. We will split up into cases depending of which part of Definition 3.3 fails.

**Case 1:** For both $(u, v)$ and $(v, u)$, part 2 of Definition 3.3 fails. Therefore, the probability that a walk reaches $u$ (and $v$) without passing through $(u, v)$ is at least $\alpha/2$. By a union bound, the probability that such a walk is light is at least $\alpha/4$. Therefore, the probability of $\mathcal{E}_u$ (and $\mathcal{E}_v$) is at least $\alpha^2 / 16$.

**Case 2:** Wlog, let $v$ not be isolated. Now, suppose further that there is no dominant path to $v$. The probability that the first walk is light and reaches $v$ is at least $\alpha/2$, by the properties of $S$. The probability that the second walk reaches $v$ by a different path is at least $\alpha/2$, since no dominant path exists. The probability that this walk is light is at least $\alpha/4$. Therefore, the probability of $\mathcal{E}_v$ is at least $\alpha^2 / 16$. Now, suppose a dominant path (it must be light) to $v$ exists. Then the probability that a light walk reaches $v$ but induces a different path (from the dominant one) is at least $\alpha/4$. We can see that the probability of $\mathcal{E}_v$ is at least $\alpha^2 / 8$.

We can conclude that if $(u, v)$ is a non-blue recessive edge, then either $\Pr(\mathcal{E}_u)$ or $\Pr(\mathcal{E}_v)$ is at least $\alpha^2 / 16$. Therefore, there are at least $\epsilon |S|$ vertices $v$ such that $\Pr(\mathcal{E}_v) \geq \alpha^2 / 4$. Fix a pair of walks. How many different $\mathcal{E}_v$’s can this pair belong to? This number is at most $2\ell$, since that is the total number of vertices reached by these walks. Therefore,

$$\Pr(\mathcal{E}) \geq \sum_v \Pr(\mathcal{E}_v) / 2\ell \geq \epsilon |S| \alpha^2 / 16 \ell$$

We now look at the probability that Cycle Tester finds a cycle. Since Cycle Tester performs $m$ walks, we hope to boost the probability of rejection.
Lemma 3.10 If there are less than $\varepsilon |S|/2$ blue vertices, then Cycle Tester finds a cycle with probability at least 2/3.

Proof: There are $m$ walks performed. Let $F_{ij}$ be the event that for the $i$th and $j$th walks, the event $E$ happens (a cycle is detected in the way described above). Let $X_{ij}$ be the indicator random variable of this event. Obviously, the number of cycles detected is at least $X = \sum_{i,j} X_{ij}$. We have that $\mathbb{E}[X_{ij}] = 2\mathbb{P}(E)$ (denoted by $\mu$) and $\mathbb{E}[X] = (\binom{m}{2})\mu$. It will be convenient to work with random variables $\overline{X}_{ij} = X_{ij} - \mu$. We can see that $\mathbb{E}[\overline{X}_{ij}] = 0$ and

$$\mathbb{E}[\overline{X}_{ij}^2] = \mathbb{E}[X_{ij}^2] - 2\mu \mathbb{E}[X_{ij}] + \mu^2 = \mu - \mu^2 \leq \mu$$

. We now show that the variance of $X$ is not too large.

$$\text{var}(X) = \mathbb{E}[(X - \mu M)^2] = \mathbb{E}[(\sum_{i,j} X_{ij})^2] \leq \sum_{i,j} \mathbb{E}[X_{ij}^2] + \sum_{i\neq i', j\neq j'} \mathbb{E}[X_{ij}X_{i'j'}] + 6 \sum_{i<j<k} \mathbb{E}[X_{ij}X_{ik}] \leq \mu m^2 + 6 \sum_{i<j<k} \mathbb{E}[X_{ij}X_{ik}]$$

The second term is 0 because for pairs $(i, j), (i', j')$ where $i \neq i', j \neq j'$, $\mathbb{E}[X_{ij}]$ and $\mathbb{E}[X_{i'j'}]$ are independent. Dealing with the last term is more difficult. This is where the properties of light walks will help us bound this term. Fixing some $i, j, k$, we will deal with $\mathbb{E}[X_{ij}X_{ik}]$. When is $X_{ij}X_{ik}$ equal to 1? This happens when both the $j$th and $k$th walks form cycles with the $i$th walk (in the way prescribed by $E$). Given a light walk $W$, let $r_W$ denote the probability that a random walk forms a cycle with $W$ as described by $E$. In other words, given a random walk $W'$, $r_W$ is the probability that pair of walks $W$ and $W'$ are in $E$. Note that $r_W$ is at most $\text{cyc}_W$, which is bounded by $1/\sqrt{n}$.

$$\mathbb{E}[X_{ij}X_{ik}] = \sum_{W: W \text{ light}} \mathbb{P}(W)r_W^2 \leq (1/\sqrt{n}) \sum_{W: W \text{ light}} \mathbb{P}(W)r_W = (\mu/\sqrt{n})$$

We can now bound the variance of $X$, noting that $m \geq \sqrt{n}$.

$$\text{var}(X) \leq \mu m^2 + 6\mu m^3/\sqrt{n} \leq \frac{7\mu m^3}{\sqrt{n}}$$

By Chebyshev’s inequality,

$$\mathbb{P}(|X - \mu|^2 > k\sqrt{\frac{7\mu m^3 n^{-1/2}}{2}}) < \frac{1}{k^2}$$

By Lemma 3.9 and the choice of parameters:

$$\mu m\sqrt{n} = 2\mathbb{P}(E) \times \frac{cn\ell\log^2 n}{\varepsilon^3} \geq 2\varepsilon|S| \times \frac{\varepsilon^2}{16|S|n\ell\log^2 n} \times \frac{cn\ell\log^2 n}{\varepsilon^3} > 28$$

This implies that $\mu m^2 > 2\sqrt{\frac{7\mu m^3 n^{-1/2}}{2}}$, and that $X$ is zero with probability at most $1/4$. □

The proofs of Claim 3.8 and Lemma 3.10 prove Lemma 3.1.

\footnote{The factor of 2 comes about because in $X_{ij}$, we have labelled walks.}
4 The general case

We now use the lemmas proved in the previous section to give a property tester for the general case. We essentially use a lemma of Goldreich and Ron [11] to show how a tester for the a well-connected subgraph can be used for a general graph. The proof given here parallels that of [11] for bipartiteness. We slightly paraphrase for convenience:

**Lemma 4.1 [11]** Let \( H \) be subgraph of \( G \) with at least \( \varepsilon n/4 \) vertices. Let \( \ell = (\log(n/\varepsilon))^{6\varepsilon^{-8}} \). For at least half of the vertices \( s \) in \( H \), the following is true: there is a subset of vertices \( T_s \) in \( H \) such that,

1. The number of edges from \( T_s \) to the rest of \( H \) is at most \( \varepsilon d|T_s|/2 \).
2. For every \( v \in T_s \), \( q_v \geq \frac{\varepsilon}{\sqrt{|S||H|\log(n/\varepsilon)}} > \frac{\varepsilon}{\sqrt{|S||n\log n}|} \).

**Cycle-freeness Tester**

**Input:** Graph \( G \), and \( 0 < \varepsilon < 1 \)

1. Choose a random subset \( R \) of \( c/\varepsilon \) vertices (\( c \) is sufficiently large constant).
2. For every \( s \in R \), run **Cycle Finder** with \( s \) as input. Output ACCEPT iff all these runs output ACCEPT.

The property tester, **Cycle-freeness Tester** simply calls **Cycle Finder** from a sufficiently large random sample of vertices. We show that it rejects graphs far from being cycle-free with high probability.

**Theorem 4.2** If \( G \) is \( \varepsilon \)-far from being cycle-free, then the procedure **Cycle-freeness tester** rejects with probability > 2/3.

**Proof:** As in most property testing proofs, we will actually prove the contrapositive: if **Cycle-freeness tester** rejects \( G \) with probability less than 2/3, then \( G \) is \( \varepsilon \)-close to being cycle-free. This will be shown by removing \( \varepsilon n d \) edges and getting a forest. The behavior of **Cycle Finder** on the random subset \( R \) will guide us to the set of edges that need to be removed. Technically, we will run **Cycle-freeness tester** with input \( \varepsilon/4 \), instead of \( \varepsilon \).

Let vertex \( s \) be called **strong** if the probability that **Cycle Finder** (with \( s \) as input) detects a cycle is at least 2/3. We observe that at most \( \varepsilon n/10 \) vertices can be strong. If not, then with probability > 5/6, the random subset \( S \) contains at least \( c/20 \) strong vertices. The probability that at least one of these strong vertices is rejected by **Cycle Finder** is > 5/6. This shows that **Cycle-freeness Tester** rejects \( G \) with probability at least 2/3. This contradicts our initial assumption.

We will construct a special partition of the vertex set of \( G \) into \( U_1, U_2, \cdots, U_r \). For \( 1 \leq i < r \), the induced subgraph \( G_{U_i} \) is \( \varepsilon/4 \)-close to begin cycle-free. This is not true for \( U_r \), but the size \( |U_r| \) is at most \( \varepsilon n/4 \). Furthermore, the number of edges from \( U_i \) to \( \bigcup_{j>i} U_j \) is at most \( \varepsilon |U_i|d/2 \). If such a partition exists, we can remove \( \varepsilon n d \) edges to make \( G \) cycle-free. First, we remove all edges incident to \( U_r \) and any edge that crosses this partition. This is a total of \( \varepsilon n d/4 + \varepsilon n d/2 \) edges. This leaves us with \( G_{U_i} \), for \( i < r \). In each of these, we can remove at most \( \varepsilon |U_i|/4 \) edges to get a forest. The total number of edges removed is at most \( \varepsilon n d \) edges.

We construct this partition inductively. At any intermediate stage, we have a partition \( U_1, U_2, \cdots, U_i \). Each \( G_{U_j} \), \( j < i \) is \( \varepsilon/4 \)-close to being cycle-free, and the number of edges going out of each
$U_i$ is small (as given by the condition in previous paragraph). Initially, $U_1$ is simply the complete vertex set and previous condition is vacuously true. If $|U_i| \leq \varepsilon n/4$, then we are done. Suppose that is not the case, so $|U_i| > \varepsilon n/4$. By our initial assumption, there are at most $\varepsilon n/10$ strong vertices. These conditions together with Lemma 4.1 imply that there is at least one weak vertex $s$ that satisfies that conditions of Lemma 4.1. Therefore, $s$ is a special vertex for $T_s$ such that CYCLE FINDER with $s$ as input detects a cycle with probability less than $2/3$. By Lemma 3.1, the induced subgraph $G_{T_s}$ must be $\varepsilon/4$-close (since we provided an input of $\varepsilon/4$ to CYCLE-FREENESS TESTER). We set $U_i$ to be $T_s$, and $U_{i+1}$ to be the complement of $\bigcup_{j\leq i} U_j$. Repeating this procedure till it ends, we construct the desired partition. This proves that $G$ is $\varepsilon$-close to being cycle-free. \hfill \Box

It is easy to see that CYCLE-FREENESS TESTER always accepts forests and provides a certificate of size $\text{poly}(\varepsilon^{-1} \log n)$ when rejecting. The running time is $O(\sqrt{n} \text{poly}(d \varepsilon^{-1} \log n))$. This completes the proof of Theorem 1.1.

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