Heuristic approach to the Schwarzschild geometry

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Abstract. In this article I present a simple Newtonian heuristic for deriving a weak-field approximation for the spacetime geometry of a point particle. The heuristic is based on Newtonian gravity, the notion of local inertial frames [the Einstein equivalence principle], plus the use of Galilean coordinate transformations to connect the freely falling local inertial frames back to the “fixed stars”. Because of the heuristic and quasi-Newtonian manner in which the spacetime geometry is obtained, we are at best justified in expecting it to be a weak-field approximation to the true spacetime geometry. However, in the case of a spherically symmetric point mass the result is coincidentally an exact solution of the full vacuum Einstein field equations — it is the Schwarzschild geometry in Painlevé–Gullstrand coordinates.

This result is much stronger than the well-known result of Michell and Laplace whereby a Newtonian argument correctly estimates the value of the Schwarzschild radius — using the heuristic presented in this article one obtains the entire Schwarzschild geometry. The heuristic also gives sensible results — a Riemann flat geometry — when applied to a constant gravitational field. Furthermore, a subtle extension of the heuristic correctly reproduces the Reissner–Nordström geometry and even the de Sitter geometry. Unfortunately the heuristic construction is not truly generic. For instance, it is incapable of generating the Kerr geometry or anti-de Sitter space.

Despite this limitation, the heuristic does have useful pedagogical value in that it provides a simple and direct plausibility argument for the Schwarzschild geometry — suitable for classroom use in situations where the full power and technical machinery of general relativity might be inappropriate. The extended heuristic provides more challenging problems — suitable for use at the graduate level.

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1. Introduction

The heuristic construction presented in this article arose from combining three quite different trains of thought:

- For an undergraduate course, I wanted to develop a reasonably clean motivation for looking at the Schwarzschild geometry suitable for students who had **not** seen any formal differential geometry. These students had however been exposed to Taylor and Wheeler’s “Spacetime Physics” [1], so they had seen a considerable amount of Special Relativity, including the Minkowski space invariant interval. They had also already been exposed to the notion of local inertial frames [local “free-float” frames], which notion is equivalent to introduction of the Einstein equivalence principle. But there is no justification in the framework of [1] for introducing the Schwarzschild geometry.

- Additionally, I was of course aware of the Newtonian idea of a “dark star”; this notion going back to the Reverend John Michell (1783) [2, 3], and popularized by Pierre Simon Marquis de Laplace (1799) [4]. Michell noted that in Newtonian physics the escape velocity from the surface of a star can exceed the speed of light when

\[
\frac{1}{2} v_{\text{escape}}^2 = \frac{GM}{R} > \frac{1}{2} c^2.
\]

That is, in Newtonian physics, (adopting the “corpuscular model” [5]), light cannot escape from the surface of a star once

\[
R < R_{\text{escape}} = \frac{2GM}{c^2},
\]

and this critical radius is (in suitable coordinates) exactly the same as the Schwarzschild radius of general relativity.

- Finally, from exposure to the “analogue models” of general relativity [6, 7, 8, 9], I was aware of the large number of different ways in which effective Lorentzian spacetime geometries can arise in quite different physical systems. In particular, Bondi accretion [10] (spherically symmetric accretion onto a gravitating point mass) leads to an “acoustic geometry” qualitatively similar to the Schwarzschild geometry.

By combining these ideas I found it was possible to develop a good heuristic for the weak-field metric, which can be presented at a level appropriate for undergraduate students. (Though some of the technical comments made below are definitely not appropriate at this level.) The remarkable feature of this heuristic is that for the Schwarzschild geometry it happens to be **exact**. This appears to be a coincidence, but is a good way of introducing students who may not intend to specialize in general relativity to the notion of a black hole.

Now there is a long history of attempts using semi-Newtonian plausibility arguments to “derive” approximate forms of the Schwarzschild metric. One of the earliest was that of Lenz (as reported in [11] and [12]). Related but distinct plausibility arguments
have been developed by Schiff [13] and Harwit [14]; though the physical basis of those attempts have been questioned [15, 16, 17]. Specifically, Rindler develops a “reductio ad absurdum” argument by applying Schiff’s construction to a constant gravitational field [15], while Gruber et al develop an “impossibility proof” based on the distinction between space curvature and gravitational potential [16].

In contrast to the Lenz and Schiff plausibility arguments, the heuristic developed in this article gives correct results for a constant gravitational field, thus avoiding the criticism of Rindler, and also side-steps the “impossibility proof” of Gruber et al [16]. It does so by evading some of the basic choices made in setting up those analyses. The key technical difference is the use of off-diagonal components in the metric, and the fact that “space” (though not “spacetime”) is exactly flat. Though this is “merely” a coordinate change, it severely modifies the presentation, analysis and conclusions of [15, 16].

2. The heuristic

2.1. Free float frames:

Start with a mass $M$ which has Newtonian gravitational potential

$$\Phi = -\frac{GM}{r}. \tag{3}$$

Take a collection of local inertial frames [local free-float frames] that are stationary out at infinity, and drop them. In the Newtonian approximation these local free-float frames pick up a speed

$$\vec{v} = -\sqrt{2\frac{GM}{r}} \hat{r}. \tag{4}$$

In the local free-float frames, physics looks simple, and the invariant interval is simply given by the standard special relativity result

$$ds^2_{FF} = -c^2 dt_{FF}^2 + dx_{FF}^2 + dy_{FF}^2 + dz_{FF}^2, \tag{5}$$

where I want to emphasise that these are locally defined free-fall coordinates.

2.2. Rigid frame:

Let us now try to relate these freely falling local inertial coordinates to a rigidly defined surveyor’s system of coordinates that is tied down at spatial infinity — that is, we want a coordinate system connected to the “fixed stars”. Call these coordinates $t_{\text{rigid}}, x_{\text{rigid}}, y_{\text{rigid}},$ and $z_{\text{rigid}}$. Since we know the speed of the freely falling system with respect to the rigid system, and we assume velocities are small, we can write an approximate Galilean transformation: \footnote{Note that the weak-field approximation implies the velocities of the local “free fall” frames is low, which is what permits us to use the Galilean approximation.}

$$dt_{FF} = dt_{\text{rigid}}; \tag{6}$$
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\[ \mathbf{\Delta} x_{FF} = \mathbf{\Delta} x_{\text{rigid}} - \vec{v} \, dt_{\text{rigid}}. \] (7)

\textbf{Warning:} Most relativists will quite justifiably be concerned by the suggestion that there is a rigid background to refer things to. The only reason we have for even hoping to get away with this is because all of the discussion is at this stage in the weak-field approximation. For students with a special relativity background who have not been exposed to the mathematics of differential geometry the existence of these rigid coordinates is “obvious” and it is only the mathematically sophisticated students that have problems here. □

2.3. Approximate “metric”:

Substituting, we find that in terms of the rigid coordinates the spacetime interval takes the form

\[ ds_{\text{rigid}}^2 = -c^2 dt_{\text{rigid}}^2 + \|d \vec{x}_{\text{rigid}} - \vec{v} \, dt_{\text{rigid}}\|^2. \] (8)

Expanding

\[ ds_{\text{rigid}}^2 = -[c^2 - v^2] dt_{\text{rigid}}^2 - 2 \vec{v} \cdot d \vec{x} \, dt_{\text{rigid}} + \|d \vec{x}_{\text{rigid}}\|^2. \] (9)

That is

\[ ds_{\text{rigid}}^2 = - \left[ c^2 - \frac{2GM}{r} \right] dt_{\text{rigid}}^2 + 2 \sqrt{\frac{2GM}{r}} \, dr_{\text{rigid}} \, dt_{\text{rigid}} + \|d \vec{x}_{\text{rigid}}\|^2. \] (10)

Now this is only an approximation — we have used Newton’s gravity, Galileo’s relativity, and the notion of local inertial frames. There is no fundamental reason to believe this spacetime metric once $GM/r$ becomes large.

2.4. The miracle:

Dropping the subscript “rigid”, the invariant interval

\[ ds^2 = - \left[ c^2 - \frac{2GM}{r} \right] dt^2 + 2 \sqrt{\frac{2GM}{r}} \, dr \, dt + \|d \vec{x}\|^2 \] (11)

is an exact solution of Einstein’s equations of general relativity, $R_{ab} = 0$. It is the Schwarzschild solution in disguise. In spherical polar coordinates we have

\[ ds^2 = - \left[ c^2 - \frac{2GM}{r} \right] dt^2 + 2 \sqrt{\frac{2GM}{r}} \, dr \, dt + dr^2 + r^2 \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right]. \] (12)

This is one representation of the space-time geometry of a Schwarzschild black hole, in a particular coordinate system (the Painlevé–Gullstrand coordinates) [18, 19, 20]. There are many other coordinate systems you could use.

\textbf{Warning:} I emphasise that this is a heuristic that happens to give the exact result. I do not view this as a rigorous derivation of the Schwarzschild geometry from Newtonian physics, and on this issue disagree with reference [21]. The heuristic does however
provide a good motivation for being interested in this specific geometry, even if for pedagogical reasons you do not yet have the full vacuum Einstein equations available. □

Exercise: More advanced students could at this stage be asked to find the coordinate transformation required to bring the above into standard curvature coordinate form (see for instance [8]):

\[ t_{\text{Schwarzschild}} = t - 2r \sqrt{\frac{2GM}{rc^2}} - \frac{4GM}{c^3} \arctanh \left( \sqrt{\frac{2GM}{rc^2}} \right) \]  
(13)

or equivalently

\[ dt_{\text{Schwarzschild}} = dt - \frac{\sqrt{2GM/r}}{c^2 - 2GM/r} \, dr. \]  
(14)

Furthermore, advanced students could also be asked to use a symbolic algebra package to verify that the Ricci tensor is zero. □

The Painlevé–Gullstrand coordinates are in fact equivalent to the “rain frame” introduced in Project B of the book “Exploring black holes” [22]. [See in particular equation (15) on page B–13.] The underlying logic is rather different there however, as those authors are presupposing that one has somehow found the exact Schwarzschild solution, and are then trying to interpret it. In contrast, the presentation of the current article could be used to motivate interest in looking at the Schwarzschild solution.

2.5. Schwarzschild radius:

You can now easily see that something interesting happens at

\[ \frac{2GM}{r_S} = c^2; \quad r_S = \frac{2GM}{c^2}; \]  
(15)

where we essentially recover the observations of Reverend John Michell (1783) and Pierre Simon Marquis de Laplace (1799). In Einstein’s gravity the coefficient of $dt_{\text{rigid}}^2$ goes to zero at the Schwarzschild radius; in Newton’s gravity the escape velocity

\[ v_{\text{escape}} = \sqrt{\frac{2GM}{R}} \]  
(16)

reaches the speed of light once $R = r_S$.

Warning: This is a good point at which to introduce the students to the difference between “coordinate velocity” and “physical velocity”. For ingoing and outgoing null rays we have

\[ \frac{dr}{dt} = -\sqrt{\frac{2GM}{r}} \mp c. \]  
(17)

At the horizon the coordinate velocity of the infalling local inertial frames (relative to the “fixed” coordinates) exceeds the speed of light — but this is perfectly acceptable in general relativity as it is only a statement about coordinate systems, not a statement
about physical objects. The coordinate velocity of the outgoing light ray goes to zero. In addition, all physical velocities are limited by the speed of light and must lie in or on the light cone defined by the spacetime metric. □

Warning: Some students will at this stage take the notion of a “gravitational aether” a little too seriously. This is the major drawback of this heuristic, which can best be ameliorated by pointing out that this heuristic is not fundamental physics. The heuristic does not work well for the Reissner–Nordström black hole, and needs a subtle patch. More significantly, the heuristic fails utterly for the Kerr black hole. □

3. Constant gravitational field

Let us now consider a constant gravitational field in the $z$ direction, described by downward gravitational acceleration $g$. If we now take a collection if inertial frames and drop them from any point on the plane $z = 0$ with initial velocity zero, then they will in the Newtonian approximation pick up a speed

$$\vec{v} = -\sqrt{-2g z} \hat{z} \quad (z \leq 0).$$

(18)

Then switching from free fall coordinates [in which the metric is simple] to “rigid” coordinates by using the Galilean transformation of equations (6)–(7) we find [using equation (8)]

$$ds^2 = -[c^2 + 2g z] dt_{\text{rigid}}^2 + 2\sqrt{-2g z} \, dz_{\text{rigid}} \, dt_{\text{rigid}} + ||d\vec{x}_{\text{rigid}}||^2.$$  

(19)

Again, I emphasise that this is only an approximation — we have used the Newtonian idea of a constant gravitational field, Galileo’s relativity, and the notion of local inertial frames. There is no fundamental reason to believe this spacetime metric once $2g z/c^2$ becomes large. Again, we encounter a “miracle”. This metric is again an exact solution of the full Einstein equations — it is in fact Riemann flat, as it must be in order to be compatible with the full Einstein equivalence principle.

Exercise: More advanced students could at this stage be asked to verify that this metric is Riemann flat. (That this metric is indeed Riemann flat may even surprise many researchers.) □

Note that the metric above is in some sense the Painlevé–Gullstrand equivalent of the Rindler wedge. Though everywhere Riemann flat, the metric is real only for $z \leq 0$, and so it covers only part of the Minkowski spacetime (similar to the situation for the Rindler wedge).

Exercise: More advanced students could at this stage be asked to find the series of coordinate transformations required to bring the above into standard Rindler form. Initially, take

$$t_{\text{Rindler}} = t_{\text{rigid}} + \left\{ -\frac{\sqrt{-2g z}}{g} + \frac{c}{g} \arctanh \left( \frac{\sqrt{-2g z}}{c} \right) \right\},$$

(20)
or equivalently

\[ \text{d}t_{\text{Rindler}} = \text{d}t_{\text{rigid}} - \frac{\sqrt{-2 \, g \, z}}{c^2 + 2 \, g \, z} \, \text{d}z, \]  

(21)

to obtain §

\[ \text{d}s^2 = -[c^2 + 2 \, g \, z] \, \text{d}t^2 + \text{d}x^2 + \text{d}y^2 + \frac{c^2 \, \text{d}z^2}{c^2 + 2 \, g \, z} \]  

(22)

Furthermore, advanced students could also be asked to use a symbolic algebra package to verify that the Riemann tensor for this metric is also zero.

Exercise: Show that in the metric (22) the integral curves of the \( t \) coordinate have position dependent 4-acceleration

\[ a = \frac{g}{1 + 2 \, g \, z/c^2}. \]  

(23)

In contrast the red-shifted “local gravity”

\[ \kappa = \frac{\sqrt{|g_{tt}|}}{c} \, a = g \]  

(24)

is position independent. (Exactly the same phenomenon occurs in the Rindler wedge.) It is in this sense that we are dealing with constant gravitational acceleration. The metric (22) has a horizon at \( z = -c^2/(2g) \), where \( a \) diverges, but \( \kappa \) is well behaved. Indeed \( \kappa_H = g \) is the surface gravity of that horizon.

Exercise: To complete the transformation from (22) to Rindler space, now perform two additional coordinate transformations

\[ z \rightarrow z - \frac{c^2}{2g}, \]  

(25)

to obtain

\[ \text{d}s^2 = -2 \, g \, z \, \text{d}t^2 + \text{d}x^2 + \text{d}y^2 + \frac{c^2 \, \text{d}z^2}{2 \, g \, z}, \]  

(26)

followed by

\[ z \rightarrow \frac{g}{2 \, c^2} \, z^2, \]  

(27)

to obtain

\[ \text{d}s^2 = -\frac{g^2 \, z^2}{c^2} \, \text{d}t^2 + \text{d}x^2 + \text{d}y^2 + \text{d}z^2. \]  

(28)

This finally is the usual representation of the Rindler wedge.

§ While not being useful for the purposes of the heuristic developed in the current article, it is perhaps interesting to note that Rindler’s criticism of the Lenz and Schiff plausibility arguments [15] is misleading in that the Lenz and Schiff construction applied to a constant gravitational field should lead to the metric (22), not the standard Rindler wedge. It is only after several additional coordinate transformations that one recovers the Rindler wedge in the form (28).
4. Discussion

Overall, I feel that the benefits of this heuristic outweigh the risks — once the specific spacetime geometry has been motivated in this way, students not intending to specialize in general relativity can simply be told that this is the Schwarzschild solution, and the properties of this spacetime investigated in the usual manner [22, 23, 24]. Two key points are:

- This sort of argument cannot be fully general, even for weak fields.
- That it is exact for Schwarzschild seems to be an accident. ||

I expand on these points below. Some of the issues raised below are very definitely nontrivial and not suitable for an undergraduate audience. Suitably modified, some points may be of interest for mathematically sophisticated students who do not have a significant physics background.

4.1. Spherical symmetry:

That the heuristic presented above, or some variant thereof, has some chance of working for general time independent [stationary] spherically symmetric geometries, can be seen by appropriately choosing the coordinates. Stationarity plus spherical symmetry is enough to yield [24]

\[
ds^2 = -A(r) \, dt^2 + 2B(r) \, dr \, dt + E(r) \, dr^2 + F(r, t) \, d\Omega^2. \tag{29}
\]

The usual procedure at this point is to use the coordinate freedom in the \( r-t \) plane to eliminate the off-diagonal term, and also to normalize the \( d\Omega^2 \) coefficient, to locally obtain the manifestly static form

\[
ds^2 = -A(r) \, dt^2 + E(r) \, dr^2 + r^2 \, d\Omega^2. \tag{30}
\]

**Warning:** Coordinate arguments will only tell you that you can do this in suitably defined local coordinate patches. That global coordinate systems of this type exist for stars is a deep result that requires some assumptions about the the regularity of the centre, the nature of matter, and dynamical information from the Einstein equations. Specifically, if the null energy condition holds then there are no “wormhole throats” and the coordinate \( r \) is continuously increasing as one moves away from the center [27, 28]. □

In contrast, starting from (30) one could define a new time coordinate by

\[
dt_{\text{new}} = dt_{\text{old}} \pm \sqrt{\frac{E(r) - 1}{A(r)}} \, dr, \tag{31}
\]

|| However, it should be noted that Laszlo Gergely [25], adapting earlier work of Xanthopolous [26] has shown that there are certain situations in which a solution of the linearized weak-field Einstein equations can be bootstrapped into a solution of the full nonlinear Einstein equations. In the current heuristic the details are different, but the flavour of the result seems similar.
and so obtain [29, 30, 31, 32]
\[
\text{ds}^2 = -A(r) \, dt^2 \pm 2 \sqrt{\frac{E(r) - 1}{A(r)}} \, dr \, dt + dr^2 + r^2 \, d\Omega^2. \tag{32}
\]
There is a technical restriction here, that the \(E(r, t)\) occurring in (30) above be greater than unity. Otherwise one will encounter imaginary metric components in (32). This issue is relevant deep inside a Reissner–Nordström black hole, or more prosaically, in anti-de Sitter space.

One then defines functions \(N(r, t)\) and \(\beta(r, t)\) so that
\[
\text{ds}^2 = -\left[N(r, t)^2 - \beta(r, t)^2\right] \, dt^2 - 2 \beta(r, t) \, dr \, dt + dr^2 + r^2 \, d\Omega^2, \tag{33}
\]
implying
\[
\text{ds}^2 = -N(r, t)^2 \, dt^2 + (dr - \beta(r, t) \, dt)^2 + r^2 \, d\Omega^2, \tag{34}
\]
The interpretation is that in spherical symmetry one can almost always [patch-wise] choose coordinates to make the spatial slices [though not spacetime] flat. In the language of the ADM decomposition (see for instance [28, 33]), you bury all of the spacetime curvature in the lapse and shift functions, \(N(r, t)\) and \(\beta(r, t)\). The heuristic presented above consists of setting \(N(r, t) = c^2\) and \(\beta(r, t) = -\sqrt{2GM/r}\), but we now see that by instead choosing suitable ansatze for the lapse and shift we would be able to fit a wide class of spherically symmetric spacetimes.

Coordinates of this type are known as Painlevé–Gullstrand coordinates [18, 19, 20] and have many pedagogically and computationally useful properties [29, 30, 31, 32]. A particularly nice feature is that infalling observers cross the horizon in finite coordinate time, so that one does not have to confront the pseudo-paradox encountered in standard coordinates where one has to wait an infinite amount of coordinate time (but finite proper time) in order for a test particle to reach the horizon.

Historically the Painlevé–Gullstrand coordinates were developed in an attempt to show there was something wrong with the Schwarzschild coordinates [18, 19]. (More recently, see also [21].) However, as emphasised by Lemaître [20], these are just a specific choice of coordinates [albeit somewhat unusual ones] and their adoption or rejection cannot affect the underlying physics or mathematics.

The heuristic applied to a generic spherically symmetric field yields
\[
\text{ds}^2 = - \left[c^2 + 2\Phi(r)\right] \, dt^2 + 2 \sqrt{-2\Phi(r)} \, dr \, dt + dr^2 + r^2 \, d\Omega^2. \tag{35}
\]
If there is a well defined surface beyond which the object is vacuum, then in that region Newtonian physics gives \(\Phi(r) = -GM/r\) and so our heuristic reproduces the Birkhoff theorem [33]. But in general, in Newtonian gravity the gravitational acceleration in a situation with spherical symmetry is
\[
\vec{g} = -\frac{G m(r)}{r^2} \, \hat{r}. \tag{36}
\]
Integrating, this now implies
\[
\Phi(r) = \int g \, dr = -\frac{G m(r)}{r} + G \int \rho(r) \, r \, dr \tag{37}
\]
As long as the density falls off sufficiently rapidly at spatial infinity, \( \rho(r) \to C/r^{3+\epsilon} \), the second term is sub-dominant near spatial infinity, \( \int \rho(r) r \, dr \to C/r^{1+\epsilon} \), and we can (in the weak field limit) write
\[
\text{d}s^2 = - \left[ c^2 - \frac{2Gm(r)}{r} \right] dt^2 + 2 \sqrt{\frac{2Gm(r)}{r}} \, dr \, dt + dr^2 + r^2 \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right].
\] (38)

This geometry, while reasonably general, is not the most general weak-field metric possible in general relativity. For this reason our heuristic will not be able to exactly reproduce all spherically symmetric geometries. [You could also come to a similar conclusion, but without some of the interesting intermediate results, by noting that the general spherically symmetric geometry is specified by two arbitrary functions \( N(r, t) \) and \( \beta(r, t) \) whereas the heuristic depends on only one arbitrary function \( \Phi(r, t) \).]

### 4.2. Reissner–Nordström geometry:

The exact Reissner–Nordström geometry \([33]\) corresponds to the choice \( N(r, t) = c^2 \) and \( \beta(r, t) = -\sqrt{2GM/r - Q^2/r^2} \)

so that
\[
\text{d}s^2 = - \left[ c^2 - \frac{2GM}{r} + \frac{Q^2}{r^2} \right] dt^2 + 2 \sqrt{\frac{2GM}{r} - \frac{Q^2}{r^2}} \, dr \, dt + dr^2 + r^2 \text{d}\Omega^2.
\] (40)

Unfortunately, while we can put the Reissner–Nordström geometry into the Painlevé–Gullstrand form appropriate for our heuristic analysis, the precise details do not quite mesh with the most naive form of the heuristic. For a charged particle surrounded by an electric field we could argue that the equivalence of mass and energy requires
\[
\rho = M \delta^3(\vec{x}) + \frac{1}{8\pi} E^2 = M \delta^3(\vec{x}) + \frac{1}{8\pi} \frac{Q^2}{r^4},
\] (41)

so that
\[
m(r) = M - \frac{1}{2} \frac{Q^2}{r}.
\] (42)

Unfortunately this now implies
\[
\Phi = - \int_{r}^{\infty} g(\tilde{r}) \, d\tilde{r} = -G \int_{r}^{\infty} \left[ \frac{M}{\tilde{r}^2} - \frac{1}{2} \frac{Q^2}{\tilde{r}^3} \right] \, d\tilde{r} = -G \left[ \frac{M}{r} - \frac{Q^2}{4r^2} \right],
\] (43)

and the coefficient of the \( Q^2 \) term does not match the exact Reissner–Nordström geometry, with a missing factor of 2. Though the naive heuristic does not exactly reproduce the Reissner–Nordström geometry, it does get remarkably close.

There is a slightly less naive version of the heuristic that does the job: If we linearize the full Einstein equations around flat space then for a static situation the linearized equations imply
\[
\nabla^2 \Phi = G \left\{ \rho + \sum_{i} \frac{p_i}{c^2} \right\},
\] (44)
where the $p_i$ are the principal pressures. (Of course, deriving this result properly requires exactly the sort of technical analysis that I had hoped to avoid by adopting the heuristic. For non-technical students, one could simply assert that pressures contribute to the gravitational field in the same way that density does.) Spherical symmetry, plus the tracelessness of the electromagnetic stress-energy tensor now implies that in the radial and transverse directions

$$\rho = -\frac{p_r}{c^2} = \frac{p_t}{c^2},$$

so that (away from the central delta function)

$$\rho + \sum_i p_i c^2 \to 2\rho.$$ (46)

This factor of 2 now compensates the “missing” factor of 2 above, and with this extension to the heuristic we exactly reproduce the Reissner–Nordström geometry.

**Warning:** This extension of the heuristic brings in ideas that are considerably more subtle and advanced than those needed for the simple Schwarzschild solution. (At a minimum, you would need to motivate the idea that pressures and tensions also generate gravitational fields, and would need to be able to rely upon the student’s prior exposure to the Maxwell electromagnetic stress tensor.)

**Warning:** In addition the Reissner–Nordström geometry in Painlevé–Gullstrand coordinates suffers from the unpleasant feature that the shift vector becomes imaginary for $r < Q^2/2GM$. Fortunately this occurs inside the inner horizon [the Cauchy horizon] where we should not be trusting the geometry in any case [because the Cauchy horizon is unstable to any infalling stress-energy]. This type of coordinate singularity can be avoided by generalizing the form of the metric, see for instance [34], but this moves us outside the framework of the Newtonian heuristic, and is another reason for viewing the Newtonian heuristic as non-fundamental.

4.3. de Sitter geometry:

Once you have adopted the extended heuristic appropriate for the Reissner–Nordström geometry, a nice feature is that it automatically works for positive cosmological constant as well. The stress-energy equivalent to any cosmological constant satisfies

$$\rho = -\frac{p_i}{c^2},$$ (47)

so that

$$\rho + \sum_i p_i c^2 \to -2\rho.$$ (48)

This factor of $-2$ now implies (if $\rho$ is positive) a repulsive force, away from the origin. A local free float frame, *initially dropped at the origin* with velocity zero, will accelerate outwards at a rate

$$g = \frac{2Gm(r)}{r^2} = \frac{8\pi}{3} \rho r,$$ (49)
and pick up a speed
\[ \frac{1}{2} v^2 = \frac{4\pi}{3} G \rho r^2, \] (50)
whence we obtain
\[ ds^2 = -\left[ c^2 - \frac{8\pi}{3} G \rho r^2 \right] dt^2 + 2 \sqrt{\frac{8\pi}{3} G \rho r} dr dt + dr^2 + r^2 d\Omega^2. \] (51)

Though the extended heuristic has now reproduced the de Sitter solution, note that the free float frames are now dropped from the origin — not spatial infinity. This is a symptom of the fact that the heuristic is not well adapted to dealing with geometries that are not asymptotically flat.

**Warning:** If one now attempts to apply the heuristic to the Schwarzschild–de Sitter [Kottler] geometry, one has to face the very basic question of where the free float frames should be dropped from. There really is no good answer to this. Worse, if one attempts to deal with anti-de Sitter space, then the speed of the free float frames is everywhere imaginary — again the heuristic breaks down and we have another reason for viewing the Newtonian heuristic as non-fundamental.

4.4. Kerr geometry:

The heuristic approach definitely fails for the Kerr geometry — most fundamentally because the Kerr geometry is not spherically symmetric. More technically, the Painlevé–Gullstrand coordinates require the existence of flat spatial slices, and the Kerr geometry does not possess such a slicing. In fact the Kerr geometry does not even possess a conformally flat spatial slicing [35, 36, 37]. The closest that one seems to be able to get to Painlevé–Gullstrand coordinates seems to be Doran’s form of the metric [38], for which a brief computation shows that \( N(r, \theta) = c^2 \); the lapse function is a constant independent of position. Unfortunately the spatial slices in Doran’s coordinates are very definitely not flat. More critically I have not been able to find any useful set of coordinates that would make the Kerr geometry amenable to treatment along the lines of the heuristic approach considered above. For this reason, among others, the heuristic approach should not be thought of as fundamental physics.

4.5. Bondi acoustic geometry:

In contrast, a particularly nice feature of the heuristic analysis is the clean relationship with the acoustic geometry occurring in Bondi accretion [10]. Consider a fluid with a linear equation of state
\[ \rho(p) = \rho_0 + \frac{p}{c_s^2}; \quad p = (\rho - \rho_0) c_s^2; \] (52)
undergoing spherically symmetric accretion onto a compact object [10]. Here \( c_s \) is the speed of sound, assumed constant. Then as long as back-pressure can be neglected, the
infalling matter satisfies \( v = -\sqrt{2GM/r} \dot{r} \). Sound waves travelling on the background of this infalling matter will then travel at speed

\[
\left\| -\sqrt{2GM/r} \dot{r} + c_s \hat{n} \right\|
\]

with respect to the fixed stars. This situation is tailor-made for application of the acoustic geometry formalism [8, 9], and as long as the back-pressure is negligible the effective acoustic geometry is exactly the Schwarzschild geometry with the speed of light replaced by the speed of sound; that is, with the substitution \( c \rightarrow c_s \).

4.6. Spatially flat geometries:

Recently Nurowski, Schücking, and Trautman have used metrics with flat spatial slices (which include as a subset the spherically symmetric geometries in Painlevé–Gullstrand coordinates) to investigate general relativistic spacetimes with close Newtonian analogues [39]. That approach, since it starts from the full Einstein equations, is in some sense the converse of the heuristic developed here. Metrics with flat spatial slices also occur ubiquitously in the various “analogue model” geometries, not just the spherically symmetric ones. A necessarily incomplete set of references includes [6, 7, 8, 9, 40, 41, 42]. The class of spatially flat geometries appears to be of interest in its own right, even if it is not general enough to contain the Kerr geometry.

4.7. Summary:

The basic heuristic discussed in the first few pages of this article can easily be explained to undergraduate students who have no intention of specializing in general relativity, and can be used to motivate interest the Schwarzschild geometry and black hole physics. The remarkable feature of the heuristic is that it leads directly to what is certainly the physically most important exact solution of the full Einstein equations — the Schwarzschild geometry (albeit in Painlevé–Gullstrand coordinates). As we have seen in the commentary, this leads naturally to a number of rather more technical issues and questions (suitable for graduate student problems) hiding in this rather innocent looking heuristic. Though the heuristic should in no way be thought of as fundamental physics, it does have considerable pedagogical value.

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[1] E.F. Taylor and J.A. Wheeler, *Spacetime Physics*, (Freeman, New York, 1992).
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[2] Reverend John Michell, FRS, “On the Means of discovering the Distance, Magnitude, etc. of the Fixed Stars, in consequence of the Diminution of the Velocity of their Light, in case such a Diminution should be found to take place in any of them, and such other Data should be procured from Observations, as would be farther necessary for that Purpose”, Philosophical Transactions of the Royal Society of London 74 (1784) 35–57. (Warning: The correct spelling is Michell, not Mitchell, though usage is somewhat inconsistent.) The original reference is difficult to obtain and I provide a brief quotation:

If the semi-diameter of a sphere of the same density as the Sun in the proportion of five hundred to one, and by supposing light to be attracted by the same force in proportion to its [mass] with other bodies, all light emitted from such a body would be made to return towards it, by its own proper gravity.

It is also interesting to note that Reverend John Michell, though most commonly thought of as a geologist, had another major influence on the field of gravitation as the inventor of the torsion balance — subsequently used by Cavendish in his experimental determination of Newton’s constant.

[3] For recent comments on the historical connections between Michell, Cavendish, and Laplace, see: D. Lynden-Bell, “Why Do Disks Form Jets?”, arXiv:astro-ph/0203480; Published in The central kilo-parsec of starbursts and AGN: the La Palma Connection, ASP conference series, 249 (2001). Edited by J.H. Knapen, J.E. Beckman, I. Shlosman, and T.J. Mahoney.

[4] Pierre Simon Marquis de Laplace, Exposition du Systeme du Monde, 1796. The intimate connection between the work of Michell and Laplace can clearly be seen from the following quotation:

A luminous star, of the same density as the earth, and whose diameter should be two hundred and fifty times larger than that of the Sun, would not, in consequence of its attraction, allow any of its [light] rays to arrive at us; it is therefore possible that the largest luminous bodies in the universe may, through this cause, be invisible.

An easy-to-find English translation of an essay giving the technical justification for this statement is available as Appendix A of S.W. Hawking and G.F.R. Ellis, The large scale structure of spacetime, (Cambridge, England, 1972).

[5] Isaac Newton, Optics, 1704:

Query 1: And do not Bodies act upon Light at a distance and, by their action, bend its Rays, and is not this action strongest at the least distance?

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