Geometry of Large Extra Dimensions versus Graviton Emission

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ABSTRACT

We study how the geometry of large extra dimensions may affect field theory results on a three-brane. More specifically, we compare cross sections for graviton emission from a brane when the internal space is a $N$-torus and a $N$-sphere for $N = 2$ to 6. The method we present can be used for other smooth compact geometries. We find that the ability of high energy colliders to determine the geometry of the extra dimensions is limited but there is an enhancement when both the quantum gravity scale and $N$ are large. Our field theory results are compared with the low energy corrections to the gravitational inverse square law due to large dimensions compactified on other spaces such as Calabi-Yau manifolds.

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I. INTRODUCTION

In recent years we have learned there could be more than meets the eye concerning gravity. While this is expected what is rather surprising is that we can appreciate this statement without entering the realm of M-theory. The idea that there might be more than the commonsensical four dimensions of our everyday world has been floating around for many decades now. This very fruitful idea was used in many contexts sometimes, it seems, according to taste. Its most fundamental incarnation is found in string/M-theory where the extra dimensions are introduced for consistency. Which vacuum this elegant unifying scheme ultimately chooses is an open problem. Consequently, there are many possibilities as to what the low energy theory of gravitation in our universe can be. For example, effective theories with a factorizable metric and a Planck scale of energy lower than the one associated with the four-dimensional gravitational coupling ($M_P \approx 1.30 \times 10^{19}$ GeV) are not excluded [1]. In fact, there is a vast literature on the idea of using extra dimensions and a lower quantum gravity scale to devise effective gravitational models. Inspired by the $D_p$-brane concept of string theory, the Standard Model (SM) fields are assumed to be localized on a three-brane (the classical version of a stack of $D3$-brane) while gravity propagates in the entire spacetime. An interesting feature of this scenario is the potentially large size of the extra dimensions. For example, a brane world model with $N = 2$ transverse dimensions and a quantum gravity scale, $M_D$, of the order of 1 TeV leads to dimensions that can be as large as one millimeter. Although this particular set of parameters seems to be ruled out by astrophysical bounds [2], it is still worth investigating the large dimensions scenario for other values of $N$ and $M_D$.

When the effects of large extra dimensions on SM processes are studied these are usually compactified on a $N$-dimensional torus [3-6]. In this work, we carefully study some of the consequences of having the extra dimensions compactified on a $N$-sphere. In sections [1] and [11] we describe the linear theory of gravity on which we build our work and comment on the Kaluza-Klein compactification scheme we use. In the following section we perform the mode decomposition of the graviton both for toric and spherical internal manifolds. In section [7] we devise a method for computing cross sections when large dimensions are compactified on smooth geometries. Explicit calculations are performed in order to compare models with spaces compactified on a torus and a sphere. We find that gravitons having a small momentum (transverse to the brane) with respect to the curvature of the extra dimensions are the most efficient for studying the geometry of the compact space. This leads to the observation that high energy colliders have limited power in determining the geometry of the extra dimensions since the relative importance of the low energy modes becomes less important for processes with large $\sqrt{s}$. We also compare our results with the low energy potential obtained from gravitational models with large extra dimensions compactified on tori, spheres and some Calabi-Yau manifolds. Finally we comment on possible extensions of our work leading to effects potentially detectable at the LHC.

II. LINEAR GRAVITY AND KALUZA-KLEIN COMPACTIFICATION

The degrees of freedom we consider are those of general relativity which we use as our effective theory. Of course, the physics at high energy will leave residual effects at low energy but these are highly suppressed non-renormalizable interactions. The quantized $(4+N)$-dimensional version of gravity we use follows the covariant approach (for example see Ref. [7]) in which the graviton field, $h_{AB}$, is a small perturbation,
\[ g_{AB} = \bar{g}_{AB} + h_{AB} \quad |h_{AB}| \ll 1, \]  

where \( A, B = 0, 1, \ldots (D - 1) \). The result is a perturbative theory on a background with metric \( \bar{g}_{AB} \). We are considering quantization around \( M^4 \times T^N \) and \( M^4 \times S^N \) where \( M^4 \) is four-dimensional Minkowski space. This work is therefore based on the linear version of the Einstein equations with the following metric ansatz

\[ g_{AB} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix} + h_{AB}, \]

where \( g_{ij} \) \((i = 1 \) to \( N \)) is the metric of the large transverse dimensions (manifold \( B^N \)) and \( \mu = 0, 1, 2, 3 \). We denote coordinates on \( M^4 \) (the three-brane) with \( x^\mu \) and those in the internal space with \( y^i \). It is understood that a model with non-trivial \( g_{ij} \) might not be a solution of the field equations without the addition of matter to the system. We address some such issues in Ref. [8]. In Sec. [II], we show what ingredients are needed to build a model with a flat brane and the extra dimensions compactified on a \( N \)-sphere in a manner which is consistent with the field equations.

Once quantized, the states of this theory are \( D \)-dimensional spin-two plane waves. The physical states are the gauge fixed ones. Picking the harmonic gauge condition takes out \((4 + N)\) degrees of freedom but there is a residual set of diffeomorphisms preserving this gauge choice \( i.e. \)

\[ x_A \rightarrow x_A + \xi_A, \]  

with \( \Box_d \xi_A = 0 \) \( (\Box_d \) is the \( D \)-dimensional d’Alembertian operator on \( M^4 \times B^N \)). Once all the gauge freedom is used up, we are left with physical \((4 + N)\)-dimensional gravitons having

\[ \frac{(N + 2)(N + 3)}{2} - 1 \]

polarization states.

We now explain how small perturbations around \( \bar{g}_{AB} \) manifest themselves from a four-dimensional point of view. Since we assume \( B^N \) to be compact the correct procedure is to perform a Kaluza-Klein decomposition of \( h_{AB} \) (see for example Refs. [I]). Symmetric spaces are characterized by Killing vectors. The set of such vectors for a manifold with Euclidean signature represents the family of one-parameter diffeomorphic transformations leaving the metric invariant,

\[ \mathcal{L}_{K^a} g_{ij} = \nabla_{(i} K^a_{j)} = 0, \]

where \( \mathcal{L}_{K^a} \) is the Lie derivative along \( K^a \). A maximally symmetric \( N \)-dimensional manifold has \( N(N + 1)/2 \) Killing vectors \( (e.g. T^N, S^N) \). The \( K^a \)'s obey the Lie algebra of a group \( G \) that depends on the isometries of \( B^N \),

\[ [K^a, K^b] = f_{c}^{ab} K^c. \]

For example, \( S^2 \) has three Killing vectors transforming under \( SO(3) \).

A special class of coordinate transformations on \( B^N \) is the isometry group

\[ y^i \rightarrow y^i + \epsilon^a(x) K^a(y), \]
\[ h_{AB} = f(V_N) \left( h_{\mu\nu} + \eta_{\mu\nu}\phi_{ij} \right) \begin{pmatrix} A^a_i K_i^a & -N\phi_{ij} \\ A^a_i & \end{pmatrix}, \] (7)

where \( \phi_{ij} \) is a matrix of scalar fields and \( f(V_N) \) is a function of the compact space volume. The diffeomorphic transformations (6) are equivalent to

\[ A^\alpha \rightarrow A^\alpha + D\mu^\alpha, \] (8)

where \( D\mu \) is the covariant derivative associated with the resulting fiber bundle on \( M^4 \). The transformation rule (8) could, for example, correspond to the gauge transformations of a non-abelian theory. By varying the topology of \( B^N \) we can obtain different gauge theories. So \( G \) is a subgroup of the \((4+N)\)-dimensional diffeomorphisms perceived as an internal symmetry from a four-dimensional point of view. The simplest case is when \( B^N = T^N \). Then the Kaluza-Klein ansatz for the metric perturbation takes the following form,

\[ h_{AB} = \frac{1}{\sqrt{V_{T^N}}} \left( h_{\mu\nu} + \eta_{\mu\nu}\phi_{ij} \right) \begin{pmatrix} A^\alpha_i \delta^\alpha_i & -N\phi_{ij} \\ A^\alpha_i & \end{pmatrix}, \] (9)

where \( V_{T^N} \) is the volume of a \( N \)-torus and \( A^\alpha_i \) are abelian gauge bosons.

Ultimately, we want to compare cross sections for processes taking place on \( M^4 \) when the topology of \( B^N \) is modified. For the kind of processes we are investigating only the spin-two part of the metric perturbation, \( h_{\mu\nu} \), is relevant.

**III. FIELD EQUATIONS OF THE SPIN-TWO PERTURBATION WHEN THE EXTRA DIMENSIONS ARE COMPACTIFIED ON A N-SPHERE**

Our aim is to compare cross sections for processes in models with extra dimensions compactified on spheres and tori. To accomplish this we use models containing flat three-branes. In a scenario with toric manifolds this is easily realized since the curvature in the internal space is zero. This is not the case when the geometry of the extra dimensions is spherical. Then, the curvature of the extra dimensions forces us to modify the model by, for example, introducing a bulk cosmological constant and an abelian gauge field trapped inside the compact manifold. Only then can we obtain a model with a flat brane that is consistent with the Einstein equations.

The field equations are then

\[ R_{AB} - \frac{1}{2}g_{AB}R = -8\pi G_D \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{AB}}, \] (10)

where \( S_M \) is the action for a bulk cosmological constant, \( \Lambda \), and a p-form gauge field, \( A_p \), with field strength \( F_{p+1} = dA_p \).

\[ S_M = \int d^4x d^N y\sqrt{-g} \left[ -2\Lambda - \frac{1}{2(p + 1)!} F_{A_1...A_{p+1}} F^{A_1...A_{p+1}} \right]. \] (11)
The resulting field equations are
\[
R_{AB} - \frac{1}{2} g_{AB} R = -\Lambda g_{AB} - \frac{4\pi G_D}{(p + 1)!} \left[ \frac{1}{2} g_{AB} F_{A_1 \ldots A_{p+1}} F^{A_1 \ldots A_{p+1}} - (p + 1) F_A^{A_2 \ldots A_{p+1}} F_{BA_2 \ldots A_{p+1}} \right].
\]
(12)

We now specialize to a model with a flat tangent space metric \(\eta_{\mu\nu}\), a transverse space with the metric \(g_{ij}\) of an \(N\)-sphere and a \((N - 1)\)-form (magnetic) field trapped in the compact space. The latter is accomplished by using the ansatz \(F_{A_1 \ldots A_N} = k \delta^{i_1}_{A_1} \ldots \delta^{i_{N}}_{A_{N}} \epsilon_{i_1 \ldots i_{p+1}}\) where \(k\) is a constant.

The Einstein equations then reduce to
\[
- \frac{1}{2} \eta_{\mu\nu} \tilde{R} = [-\Lambda - 2\pi G_D k^2] \eta_{\mu\nu},
\]
(13)

\[
R_{ij} - \frac{1}{2} g_{ij} \tilde{R} = [-\Lambda + 2\pi G_D k^2] g_{ij},
\]
(14)

where \(\tilde{R} = N(N - 1)/a^2\) is the Ricci scalar in the internal space and

\[
R_{ij} - \frac{1}{2} g_{ij} \tilde{R} = -\frac{1}{2a^2} (N - 2)(N - 1) g_{ij},
\]
(15)

with \(a\) the radius of the spheres. Using the expression for \(\tilde{R}\) and Eq. (13), Eqs. (13) and (14) reduce to

\[
\Lambda + 2\pi G_D k^2 = \frac{N(N - 1)}{2a^2},
\]
(16)

\[
\Lambda - 2\pi G_D k^2 = \frac{1}{2a^2} (N - 2)(N - 1).
\]
(17)

Inspection of the last two equations shows why it is necessary to introduce both a bulk cosmological constant and a gauge field for a model with a flat three-brane to satisfy the gravitational field equations when the extra dimensions are compactified on a \(N\)-sphere.

We now derive the linear free field equations that the spin-two perturbation, \(h_{\mu\nu}\), must satisfy when propagating in the background \(M^4 \times S^N\) described above. In order to achieve this we use the simplified Kaluza-Klein ansatz

\[
h_{AB}(x, y) = \frac{1}{\sqrt{V_S^N}} \begin{pmatrix} h_{\mu\nu}(x, y) & 0 \\ 0 & 0 \end{pmatrix}.
\]
(18)

In the linear limit, the free field equations with both the cosmological constant and the gauge field turned on are

\[
R^{(L_1)}_{\mu\nu} + R^{(L_2)}_{\mu\nu} = 0
\]
(19)

and

\[
R^{(L_1)}_{ij} = -\frac{1}{2} \partial_i \partial_j h = 0,
\]
(20)

where \(h\) is the trace of \(h_{\mu\nu}\) and
\[ R^{(L_1)}_{\mu\nu} = -\frac{1}{2} \partial^\lambda \partial_\lambda h_{\mu\nu} + \frac{1}{2} [\partial_\mu (\partial_\lambda h_{\nu}^\lambda - \frac{1}{2} \partial_\nu h) + \partial_\nu (\partial_\lambda h_{\mu}^\lambda - \frac{1}{2} \partial_\mu h)], \]  

(21)

\[ R^{(L_2)}_{\mu\nu} = -\frac{1}{2} \partial^i \partial_i h_{\mu\nu} - \frac{1}{2} [\partial_j h_{\mu\nu} \partial_j g^{ij} + \Gamma^i_{ij} g^{ik} \partial_k h_{\mu\nu}]. \]  

(22)

Eq. (21) is a constraint on the trace, \( h \), of the spin-two field which is present even when the internal space is flat. It constrains all Kaluza-Klein modes of the graviton to be traceless (see Sec. IV for more details on the Kaluza-Klein decomposition procedure). To simplify Eq. (21) we use the harmonic gauge condition \( g^{\mu\nu} \Gamma^C_{\mu\nu} = 0 \) which is equivalent to

\[ \partial_\mu h^\mu_\lambda - \frac{1}{2} \partial_\lambda h = 0. \]  

(23)

Using Eq. (23) the tensor \( R^{(L_1)}_{\mu\nu} \) simplifies to

\[ R^{(L_1)}_{\mu\nu} = -\frac{1}{2} \partial^\lambda \partial_\lambda h_{\mu\nu}. \]  

(24)

Now from \( \nabla_i g^{ij} = 0 \) we find

\[ \partial_i g^{ij} = -\Gamma^i_{ik} g^{kj} - \Gamma^j_{ik} g^{ik}, \]  

(25)

which allows us to write

\[ R^{(2)}_{\mu\nu} = -\frac{1}{2} \nabla^i \nabla_i h_{\mu\nu}, \]  

(26)

where use has also been made of the fact that, when applied on a scalar function, the following operator equality holds,

\[ g^{ij} \nabla_i \nabla_j = g^{ij} \partial_i \partial_j - g^{ij} \Gamma^k_{ij} \partial_k. \]  

(27)

Combining Eqs. (24) and (26) the linear free field equations for the spin-two field \( h_{\mu\nu} \) become

\[ \Box_D h_{\mu\nu}(x, y) = 0, \]  

(28)

where \( \Box_D \) is the d’Alembertian operator on \( M^4 \times S^N \). Similarly, the \((i, j)\) components of perturbation \((\s)\), when the \((\mu \nu)\) and \((\mu i)\) components are turned off, satisfy an equation similar to Eq. (28). Moreover, the off-diagonal \((\mu i)\) components satisfy Maxwell-like field equations when the \((\mu \nu)\) and \((i j)\) components are ignored. When all components of the perturbation are turned on, the physical fields correspond to mixings between the metric components. The full calculation is presented in Ref. [10] for the case of large extra dimensions compactified on a torus.

**IV. GRAVITON MODES IN THE INTERNAL SPACE**

A detailed analysis of the Kaluza-Klein reduction and gauge fixing procedure for compactification on a \( N \)-torus can be found in Ref. [10]. Using a similar procedure one expects to find the following \((4 + N)\)-dimensional equations of motion for the spin-two perturbation (which we call from now on the graviton):
\[ \Box_D h_{\mu\nu}(x, y) = 0, \] (29)

where \( \Box_D \) is now the d’Alembertian operator on \( M^4 \times B^N \) (although we have demonstrated this to be true explicitly only for \( B^N = S^N \)). Because of the compact nature of \( B^N \) the gravitational field can be recast as an infinite tower of Kaluza-Klein (KK) modes,

\[ h_{\mu\nu}(x, y) = \sum_{\{n\}} h^{[n]}_{\mu\nu}(x)\psi^{[n]}(y), \] (30)

where \( \{n\} \) is a set of quantum numbers related to the isometry group of the compact space and \( \psi^{[n]}(y) \) is a normalized wave function. As can be read from Eq. (29) the wave functions must be such that

\[ \left[ \nabla_i \nabla^i + m^2_{\{n\}} \right] \psi^{(n)}(y) = 0, \] (31)

which we solve both for the \( T^N \) and \( S^N \) geometries. We see that the KK modes are the eigenmodes of the appropriate Laplacian on the internal space. They depend completely on its geometry and topology. Compact hyperbolic spaces have been considered in Ref. [11].

\section*{A. Compactification on a N-Torus}

The simplest compact geometry for the extra dimensions is a \( N \)-dimensional torus with a unique radius \( a \). Cases with toric extra dimensions characterized by different length scales are studied in Ref. [12]. The wave function in transverse space is simply obtained by solving

\[ \left[ \partial_i \partial^i + m^2_n \right] \psi^n(y) = 0, \] (32)

which leads to

\[ \psi^n(y) = \frac{1}{(2\pi a)^{N/2}} e^{i\frac{\mathbf{n} \cdot \mathbf{y}}{a}}, \] (33)

where \( \mathbf{n} = \{n_1, n_2, ..., n_N\} \) with the \( n_i \)'s integers running from \(-\infty\) to \(+\infty\) and \( 0 < y_i \leq 2\pi a \) are the components of the vector \( y \). Based on Eq. (3) the \( \mathbf{n} = \mathbf{0} \) modes correspond to a massless graviton (2 degrees of freedom), a set of \( N \ U(1) \) massless gauge bosons (2\( N \) degrees of freedom) and moduli composed of \( N(N+1)/2 \) massless scalars for an expected total of \( (2 + N)(3 + N)/2 - 1 \) degrees of freedom. For \( \mathbf{n} \neq \mathbf{0} \) the analysis is somewhat different since then the momentum of the graviton in the transverse space is not zero. For an observer on the brane this transverse momentum is perceived as a four-dimensional mass \( (m^2 = \mathbf{n}^2/a^2) \). The spectrum then consists (for each level \( \mathbf{n} \neq \mathbf{0} \)) of one massive spin-two particle, \( (N-1) \) massive vector bosons and a set of \( N(N-1)/2 \) massive scalars. For later reference we write down the wave function at \( y = 0 \),

\[ \psi^n(0) = \frac{1}{(2\pi a)^{N/2}} = \frac{1}{\sqrt{V_{T^N}}}, \] (34)

where \( V_{T^N} \) is the volume of the torus.
B. Compactification on a \(N\)-Sphere

The derivation of the wave function for the graviton propagating on a \(N\)-sphere is more challenging. In that case we need to solve the following equation,

\[
\left( \nabla_{sN}^2 + m_{\{n\}}^2 \right) \psi^{\{n\}}(y) = 0,
\]

where \( \nabla_{sN}^2 \) is the Laplacian on a \(N\)-sphere of fixed radius \(a\),

\[
\nabla_{sN}^2 = \frac{1}{a^2 \sin^{N-1} \phi_1 \sin^{N-2} \phi_2 \cdots \sin \phi_{N-1}} \sum_{i=1}^{N} \frac{\partial}{\partial \phi_i} \left( \frac{\partial}{\partial \phi_i} \sin^{N-1} \phi_1 \sin^{N-2} \phi_2 \cdots \sin \phi_{N-1} \right) \frac{\partial}{\partial \phi_i}.
\]

The \(S^N\) geometry is characterized by \(N\) angles, \((N-1)\) of which run from 0 to \(\pi\) (\(\phi_1, \ldots, \phi_{N-1}\)). The azimuthal angle \(\phi_N\) varies from 0 to \(2\pi\). Introducing \(\nabla_{sN}^2 = \frac{1}{a^2} \nabla^2\), Eq. (35) becomes

\[
\left( \nabla^2 + m_{\{n\}}^2 a^2 \right) \psi^{\{n\}}(\{\phi_i\}) = 0,
\]

which we solve using the ansatz

\[
\psi^{\{n\}}(\{\phi_i\}) = \psi_1(\phi_1)\psi_2(\phi_2) \cdots \psi_N(\phi_N).
\]

Eq. (35) can then be recast in the following form,

\[
\left( \begin{array}{c}
\frac{1}{\psi_1} \frac{1}{\sin^{N-1} \phi_1} \frac{\partial}{\partial \phi_1} \psi_1 \\
\frac{1}{\sin^2 \phi_2} \frac{1}{\sin^{N-3} \phi_3 \sin \phi_3} \frac{\partial}{\partial \phi_3} \psi_3 + \cdots + \frac{1}{\sin^2 \phi_{N-1}} \frac{1}{\sin \phi_{N-1}} \frac{\partial^2}{\partial \phi_{N-1}^2} \psi_{N-1}
\end{array} \right) = 0.
\]

Using the change of variables \(x_i = \cos \phi_i\) and introducing the parameter \(2\alpha_i = N - i\), Eq. (36) can be written as a set of \(N\) differential equations:

\[
(1 - x_i^2) \frac{\partial^2 \psi_i}{\partial \phi_i^2} - (2\alpha_i + 1)x_i \frac{\partial \psi_i}{\partial \phi_i} + \left[ n_i(n_i + 2\alpha_i) - \frac{n_k(n_k - 1 + 2\alpha_k)}{1 - x_i^2} \right] \psi_i = 0,
\]

where the quantum numbers \(n_i\) are such that \(n_1 \geq n_2 \geq \cdots \geq n_{N-1}\) with \(n_i = 0, 1, \ldots, \infty\) and the range of the azimuthal quantum number is \(-n_{N-1} \leq n_N \leq n_{N-1}\). The mass spectrum is uniquely controlled by \(n_1\),

\[
m^2 = \frac{n_1(n_1 + 2\alpha_1)}{a^2} = \frac{n_1(n_1 + N - 1)}{a^2}.
\]

The solutions to Eqs. (40) can be expressed as a product of normalized associated Gegenbauer polynomials [13].
\[
\psi_{n_i}^{(\alpha_k)}(x_i) = N_{n_i}^{(\alpha_k)} n_k (1 - x_i^2)^{n_k/2} \frac{d^{n_k}}{dx_i^{n_k}} C_n^{(\alpha_k)}(x_i), \quad (43)
\]

where the \(N_{n_i}^{(\alpha_k)} n_k\)'s are normalization constants and

\[
C_n^{(\alpha)}(x) = \frac{(-1)^n}{2^n} \frac{\Gamma(2\alpha + n)}{\Gamma(2\alpha) \Gamma\left(\frac{2\alpha + 1}{2}\right)} (1 - x^2)^{1/2 - \alpha} \frac{d^n}{dx^n} \left[(1 - x^2)^{\alpha + n - 1/2}\right]. \quad (44)
\]

The Gegenbauer polynomials can also be obtained from the hyper-spherical generating functional,

\[
\frac{1}{(1 - 2xt + t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x) t^n. \quad (45)
\]

Using the measure on a \(N\)-sphere of radius \(a\),

\[
dV = a^N \sin^{N-1} \phi_1 \sin^{N-2} \phi_2 \ldots \sin \phi_{N-1}, \quad (46)
\]

the normalization constants are found to be

\[
N_{n_i}^{(\alpha_k)} n_k = \frac{\Gamma(\alpha_i + n_k)}{2^n \alpha_i (\alpha_i + 2) \ldots (\alpha_i + n_k - 1)} \left[\frac{(n_i - n_k)!(\alpha_i + n_i)}{\pi 2^{(1 - 2\alpha_i - 2n_k)} \Gamma(2\alpha_i + n_i + n_k)}\right]^{1/2}. \quad (47)
\]

For later reference we evaluate the graviton wave function at a given point on \(S^N\). For simplicity, we choose to evaluate it at \(y = 0\) which on a \(N\)-dimensional sphere can be taken to correspond to \(\phi_1 = 0\) with \(\phi_2 \ldots \phi_N\) being irrelevant variables. We use the following property of the associated Gegenbauer polynomials at \(x_i = \cos \phi_i = 1\),

\[
\psi_{n_i}^{(\alpha_k)} n_k(1) = \frac{(2\alpha_i + n_i - 1)!}{n_i!(2\alpha_i - 1)!} \delta_0^{n_k}, \quad (48)
\]

where \(\delta_0^{n_k}\) is the Kronecker delta setting \(n_k\) to zero. The wave function evaluated at a specific point can then be shown to be proportional to \(\delta_0^{n_1} \delta_0^{n_2} \ldots \delta_0^{n_N}\). Consequently, in a mode expansion of the graviton on a \(N\)-sphere only the \(n_1\) quantum number is seen to play a role. One might be led to think that this greatly reduces the density of states allowed to propagate on the sphere but this turns out not to be the case. In fact, the multiplicity at each quantum level \(n_1\), which is \(|\psi^{n_1}(y = 0)|^2 V_{S^N}\), reappears in the coupling terms of SM matter with the graviton modes through the normalization constant. For later use, we write down the graviton wave function at a specific point on the sphere for \(N = 2\) to \(6\) (from now on we use the convention \(n_1 = n\)),

\[
\psi^n(y = 0) = \left[\frac{2n + 1}{V_{S^2}}\right]^{1/2}, \quad N = 2 \quad (49)
\]

\[
\psi^n(y = 0) = \left[\frac{(n + 1)^2}{V_{S^3}}\right]^{1/2}, \quad N = 3 \quad (50)
\]

\[
\psi^n(y = 0) = \left[\frac{(2n + 3)(n + 2)(n + 1)}{6V_{S^4}}\right]^{1/2}, \quad N = 4 \quad (51)
\]
\[ \psi^n(y = 0) = \left( \frac{(n + 3)(n + 2)^2(n + 1)}{12V_{S^5}} \right)^{1/2}, \quad N = 5 \]  

\[ \psi^n(y = 0) = \left( \frac{(5 + 2n)(n + 4)(n + 3)(n + 2)(n + 1)}{120V_{S^6}} \right)^{1/2}, \quad N = 6 \]  

where \( V_{S^N} = \frac{2\pi^{(N+1)/2}a_N}{\Gamma((N+1)/2)} \) is the volume of a \( N \)-sphere. As will be made clear in Sec. \( V \), the wave function evaluated at a given point (directly related to the multiplicity at each KK level) is crucial for understanding how processes on the three-brane are affected by the geometry of the internal space.

V. PROBING THE EXTRA DIMENSIONS

We now consider in detail how the geometry of the internal space affects the couplings of SM fields to the gravitational sector. The linear coupling, which is universally determined by general covariance, is of the form (see Ref. [10])

\[ \frac{1}{M_D^{1/2}} \int d^Dx \, h^{\mu\nu}T_{\mu\nu}, \]  

where \( T_{\mu\nu} \) is the stress-energy tensor associated with SM fields on the three-brane. We are studying gravity on the product space \( M^4 \times B^N \) with SM fields localized on the \( M^4 \) submanifold. It is a reasonable approximation for our purposes to use the following form for the stress-energy tensor:

\[ T_{\mu\nu}(x, y) = \delta_\mu^\mu \delta_\nu^\nu T_{\mu\nu}(x) \delta^{(N)}(y). \]  

This expression is written in the so-called static gauge which consists in ascribing four bulk coordinates to the three-brane \( (A = 0, 1, 2, 3 \rightarrow \mu = 0, 1, 2, 3) \) and the remaining \( N \) coordinates to the internal space \( (A = 4, ..., D - 1 \rightarrow i = 1, ..., N) \). Using Eq. (55) is equivalent to considering an infinitely thin and tensionless brane. A consequence of this simplification is that tree level diagrams involving the exchange of off-shell gravitons are not finite which goes against intuition. In fact, we expect loop diagrams to diverge but not the tree level ones. Let us pause and consider this problem. The incoming and outgoing states in a typical process are SM fermions that are confined to the flat submanifold \( M^4 \). From the point of view of an observer on the brane, a graviton is emitted at one vertex, propagates into the bulk, and is reabsorbed at the second vertex. Using a stress-energy tensor of the form (55) to work out the expression for the tree level amplitude shows that momentum is conserved on the brane but not in the internal space. In other words, there is no constraint at the vertices on the transverse momentum of the graviton. This is similar to what happens in a loop diagram so one should

*When constructing the effective interactions due to virtual KK modes exchange, an infinite sum needs to be evaluated. It is in most cases a divergent quantity unless one introduces by hand an ultraviolet cut-off. This is not a very natural way to cure the problem as the cut-off remains in the final expression.
not be surprised that tree level amplitudes may diverge. This puzzle is resolved in Ref. [14] where the authors give the brane a finite tension and take into account its fluctuation modes in the transverse directions. This induces an exponential factor at the vertices which naturally cuts off the problematic ultraviolet modes responsible for the divergences. This phenomenon is of no concern to us since we are only considering finite interactions involving on-shell gravitons. Nevertheless, such a suppression factor for KK modes emitted from the three-brane should be included in a thorough analysis.

Using Eqs. (54) and (55) insures that everything coupling to the gravitational sector is located on the three-brane. Using the ansatz (18), the coupling term becomes

$$ S^L = \frac{1}{M_{D}^{1+\frac{N}{2}}} \int d^N y \, \delta^{(N)}(y) \int d^4 x \, T_{\mu\nu}(x) \delta^\mu_{\mu} \delta^\nu_{\nu} h^{AB}(x, y) $$

$$ = \frac{1}{M_{D}^{1+\frac{N}{2}}} \int d^4 x \, Tr \left( \sum h^{(n)}_{\mu\nu}(x)\psi^{(n)}(y = 0) \right) \left( T^{\mu\nu}(x) \right) \psi^{(n)}(y = 0). $$

Consequently, each KK mode is characterized by the coupling

$$ \frac{1}{M_{D}^{1+\frac{N}{2}}} \psi^{(n)}(y = 0) \int d^4 x \, h^{(n)}_{\mu\nu}(x) T^{\mu\nu}(x). \quad (56) $$

When the compact geometry is a torus this expression becomes

$$ \frac{1}{M_{P}} \int d^4 x \, h^{n}_{\mu\nu}(x) T^{\mu\nu}(x), \quad (57) $$

where use has been made of Eq. (31) and the fact that $M_{P} = V^{1/2} M_{D}^{1+N/2}$. When considering a spherical compact geometry, we obtain the same kind of expression:

$$ \frac{f_{n}(n)}{M_{P}} \int d^4 x \, h^{n}_{\mu\nu}(x) T^{\mu\nu}(x), \quad (58) $$

where $f_{n}(n)$ represents a family of polynomials in $n$ (see Eqs. (49)-(53)) related to the multiplicity of the states propagating on the sphere at each KK level.

Using Eq. (54) we can find the Feynman rules for processes involving gravitons coupled to SM fields on the three-brane [14,13]. We restrict ourselves to studying the potential relevance of the process $e^+e^- \rightarrow \gamma h$ for probing the geometry of the transverse space. From the four-dimensional point of view on submanifold $M^4$ this corresponds to the emission of a kinematically cut-off tower of massive graviton modes during a high energy collision. The differential cross section for the emission of a QED photon and a massive graviton (denoted $h_m$) following an $e^+e^-$ collision with CM energy $\sqrt{s}$ is [3,5]

$$ \frac{d\sigma_m}{dt}(e^+e^- \rightarrow \gamma h_m) = \frac{\alpha}{16 M^2_p} F(x, y), \quad (59) $$

where $\alpha$ is the electromagnetic fine-structure constant and

$$ F(x, y) = \frac{-4x(1+x)(1+2x+2x^2) + y(1+6x+18x^2+16x^3) - 6y^2x(1+2x) + y^3(1+4x)}{x(y-1-x)} \quad (60) $$
with $x = t/s$ and $y = m^2/s$. From our limited four-dimensional point of view we do not distinguish between gravitons of different transverse momenta (mass-squared). Thus, the actual cross section for graviton emission from the brane is obtained by summing Eq. (59) over all kinematically allowed values of $m^2$ i.e. up to $m^2 = s$. Note that when we use the variable $y = m^2/s$ the sum conveniently runs from $y = 0$ to 1.

From an experimental perspective, the $e^+e^- \rightarrow \gamma h$ process is competing with the Standard Model background $e^+e^- \rightarrow \gamma \nu\bar{\nu}$. Of course, when there are either no or extremely small extra dimensions the gravitational process, being supressed by a $M_P^{-2}$ factor, is completely undetectable. With large extra dimensions the relatively important number of KK modes enhances the graviton signal which leads to a potentially detectable departure from the SM signature. This corresponds to $\sigma(e^+e^- \rightarrow \gamma h)$ no longer being suppressed by a $M_P^{-2}$ factor but by a $M_D^{(2+N)}$ factor. Consequently, picking $M_D$ as small as possible leads to larger gravitational signals. There is a fundamental limitation in our freedom to do that though. In fact, using the Gauss law one finds the following low energy constraint [1]:

$$M_P^2 = M_D^{2+N}V_N,$$ (61)

where $V_N$ is the volume of the compact space. Requiring the effective low energy four-dimensional gravitational coupling to be the observed Newton constant $G_N$ is equivalent to imposing Eq. (61) which is a relationship between the size of the extra dimensions, their number, $N$, and the true quantum gravity scale, $M_D$. It is interesting to note that experiments have been performed probing gravity down to approximately one millimeter [15] without finding any discrepancies with the usual $1/r$ potential. Based on Eq. (61), this implies that for $N = 2$ the quantum gravity scale could be as low a 1 TeV. Although this particular set of parameters seems to be excluded by astrophysical constraints [2], it does not mean that other values of $N$ and $M_D$ leading to detectable signatures have to be rejected. It should be clear that the graviton signal is increasingly suppressed relative to the background process $e^+e^- \rightarrow \gamma \nu\bar{\nu}$ as $M_D$ and $N$ are increased.

### A. Phase Space Integrals on $T^N$

For a $T^N$ geometry the wave function for the transverse graviton modes at $y = 0$ is independent of $n$. Based on Eq. (57) this means that there is no restriction on the quantum numbers of the modes that are emitted at a given point on the torus (from the three-brane). Then the operator we use to sum over transverse momenta is [3]

$$\mathcal{O}_{T^N} = \sum_k \frac{1}{V_{|k|}} \int d^N k \frac{\Omega_N R_N}{2} \int m^{N-2} dm^2 = \frac{\Omega_N R_N s^{N/2}}{2} \int_0^1 dy \frac{y^{N-2}}{2},$$ (62)

where $V_{|k|}$ is the volume occupied by one state in $k$-space, $y = m^2/s$ and

$$\Omega_N = \frac{2 \pi^{N/2}}{\Gamma(N/2)}$$ (63)

is the volume of a unit sphere embedded in a $N$-dimensional space. Because the extra dimensions are assumed to be large (in TeV$^{-1}$ units) with respect to the inverse center of mass energy of the process, it is reasonable to take the continuum limit when performing the sum over momenta (see the Appendix). For future comparison with $S^N$ phase space integrals we explicitly write down Eq. (62) for $N = 2$ to 6:
FIG. 1. Cross sections (arbitrary units) for $e^+e^- \rightarrow \gamma h$, in models with the extra dimensions compactified on $T^N$, as a function of $s$ (in TeV$^2$) for $M_D = 1$ TeV. Starting from the top, curves are for $N = 2$ to 6.

\[
O_{T^2} \rightarrow V_{T^2} \frac{s^2}{4\pi^2} \int_0^1 dy
\]

(64)

\[
O_{T^3} \rightarrow V_{T^3} \frac{s^{3/2}}{4\pi^2} \int_0^1 dy \ y^{1/2}
\]

(65)

\[
O_{T^4} \rightarrow V_{T^4} \frac{s^2}{16\pi^2} \int_0^1 dy \ y 
\]

(66)

\[
O_{T^5} \rightarrow V_{T^5} \frac{s^{5/2}}{24\pi^3} \int_0^1 dy \ y^{3/2}
\]

(67)

\[
O_{T^6} \rightarrow V_{T^6} \frac{s^3}{128\pi^3} \int_0^1 dy \ y^2.
\]

(68)

To evaluate the differential cross section for the emission of a (4+$N$)-dimensional graviton from the three-brane we need only apply operator $O_{x^N}$ to Eq. (59). As an example Fig. 1 shows the total cross sections (after the integration over angles has been performed) as a function of $s$ for graviton emission when $M_D = 1$ TeV for $N = 2$ to 6.

**B. Phase Space Integrals on $S^N$**

As shown in Sec. [IV B] the graviton wave function evaluated at a given point on $S^N$ only depends on the quantum number $n$. Using Eqs. (64)-(68), it is straightforward to write down the operators analogous to Eqs. (64)-(68) when the compact geometry is $S^N$,

\[
O_{S^2} \rightarrow V_{S^2} \frac{s^2}{4\pi} \int_0^1 dy
\]

(69)
\[ O_{S^3} \rightarrow V_{S^3} \frac{s^{3/2}}{4 \pi^2} \int_0^1 dy \left( y + \frac{1}{a^2 s} \right)^{1/2} \]  \hspace{1cm} (70)

\[ O_{S^4} \rightarrow V_{S^4} \frac{s^2}{16 \pi^2} \int_0^1 dy \left( y + \frac{2}{a^2 s} \right) \]  \hspace{1cm} (71)

\[ O_{S^5} \rightarrow V_{S^5} \frac{s^{5/2}}{24 \pi^2} \int_0^1 dy \left( y + \frac{3}{a^2 s} \right) \left( y + \frac{4}{a^2 s} \right)^{1/2} \]  \hspace{1cm} (72)

\[ O_{S^6} \rightarrow V_{S^6} \frac{s^3}{128 \pi^2} \int_0^1 dy \left( y + \frac{4}{a^2 s} \right) \left( y + \frac{6}{a^2 s} \right) \]  \hspace{1cm} (73)

The parameter playing a role in distinguishing a spherical from a toric geometry depends on the size of the internal space. We label it

\[ d_a = \frac{1}{n_{\text{max}}^2}, \]  \hspace{1cm} (74)

where \( n_{\text{max}} = a \sqrt{s} \) is (if \( N \ll n_{\text{max}} \)) the maximum quantum number over which we integrate when performing the phase space sum on a sphere. If we integrate over an overwhelmingly large number of states (\( d_a \rightarrow 0 \)) the \( e^+e^- \rightarrow \gamma h \) cross section evaluated on \( T^N \) is expected to be very close to the one evaluated on \( S^N \). This corresponds to a sector of the theory for which the typical size of the extra dimensions (in TeV\(^{-1} \) units) is large with respect to the inverse center of mass energy of the process (\( a \gg 1/\sqrt{s} \)). In this case the spacing between KK levels is small compared with the CM energy. For example, it can be seen that \( a \approx 2 \times 10^5 \text{ TeV}^{-1} \) if \( N = 6 \), \( M_D = 1 \text{ TeV} \) and \( a \approx 200 \text{ TeV}^{-1} \) with \( M_D = 30 \text{ TeV} \). As we increase both \( M_D \) and \( N \) we expect the difference between cross sections on \( T^N \) and \( S^N \) to increase since this corresponds to taking larger values of \( d_a \) (a smaller number of KK modes are summed over). The numerical factors multiplying \( d_a \) in \( O_{S^N} \) are more important for large \( N \) which also contributes in enhancing the difference in cross sections between the two geometries.

When the internal space has a typical length scale which is extremely small (with respect to the inverse CM energy of the process), one expects processes taking place on a torus to be indistinguishable from processes on a sphere (or on any other smooth manifold for that matter). This is not reflected in our phase space integral procedure. In the limit when the extra dimensions are extremely small the procedure we are using is not valid anymore since it is highly probable that, for the range of CM energies considered, only the zero-mode of the graviton will be excited. The phase space integral procedure is useful only when numerous modes are excited \( i.e. \) when the extra dimensions are large. The \( N = 2 \) case is special since it is then impossible, for large extra dimensions, to distinguish between the \( T^2 \) and the \( S^2 \) geometries using a process such as \( e^+e^- \rightarrow \gamma h \) (\( O_{T^2} = O_{S^2} \)).

If \( d_a \) is not too small we expect differences in cross sections evaluated on \( T^N \) and \( S^N \) for \( N > 2 \). Comparing Eqs. (64)-(68) and Eqs. (69)-(73) shows that models with a spherical transverse space will lead to larger cross sections. Fig. 2 represents the behavior of cross sections for the two geometries studied when \( M_D = 30 \text{ TeV} \) and \( N = 6 \). We see that as \( s \) is augmented the difference between the cross sections slightly increases. This suggests that overall the number of KK modes excited on a \( N > 2 \) sphere is larger than on a torus. Based on Eq. (74) we see that \( s \) is related to the maximum quantum number (\( n_{\text{max}} \)) over which the
FIG. 2. Comparison of cross sections (in units of $[\alpha/\pi^3 M_D^8]$) for $e^+e^- \rightarrow \gamma h$ in models with compact manifold $T^6$ and $S^6$ when $M_D = 30$ TeV for $s$ varying from 0.01 to 0.1 TeV$^2$. The upper curve corresponds to an internal space with a spherical geometry and the lower curve to a toric geometric. The difference between the two curves slightly increases with $s$.

integration is performed. Consequently, the larger the CM energy is, the larger we expect the cross section differences to be (a large $s$ corresponds to integrating over more modes). Since highly energetic modes are not expected to differentiate between smooth geometries (their wavelength is assumed to be much smaller than the inverse curvature-squared of the internal space), there exists a CM energy beyond which the multiplicity at each level is the same both for the sphere and the torus$^\dagger$. Past this critical $s$-value, we expect the difference between cross sections to become constant. Although this is not obvious from Fig. 3, we have shown numerically that this is in fact what happens.

Approximating sums over graviton modes by integrals is valid for the range of $M_D$ we are studying because the spacing between quantum levels (in momentum space) is small. In fact, this is of the order of magnitude $1/a$ where, for example, $a \approx 2 \times 10^5$ TeV$^{-1}$ when $N = 6$ and $M_D = 1$ TeV. As we increase $M_D$ the typical size of the extra dimensions decreases therefore leading to a larger spacing. This has the potential of invalidating the sums by integrals approximation for sufficiently large values of the quantum gravity scale. For more details on potential errors induced by the approximation see the Appendix.

A parameter we can use to quantify the effect of the compact geometry on graviton emission from the three-brane is the ratio

$$D_N(s, M_D) = \frac{\sigma_{SN}(e^+e^- \rightarrow \gamma h) - \sigma_{TN}(e^+e^- \rightarrow \gamma h)}{\sigma_{SN}(e^+e^- \rightarrow \gamma h)},$$

which is a function of the size of the extra dimensions through $N$ and $M_D$. This quantity characterizes relative rather than absolute differences. As shown earlier, the ratio $D_N$ is zero

$^\dagger$We do not specify what this critical value of $s$ is as it depends both on $M_D$ and $N$. It represents a natural separation between the low and high energy modes propagating on the compact manifold. For example, when $M_D = 30$ TeV and $N = 6$, the critical value is around $s = 0.15$ TeV$^2$. 

FIG. 3. Comparison of cross sections (in units of $[\alpha/\pi^3M_D^3]$) for $e^+e^- \rightarrow \gamma h$ in models with compact manifold $T^6$ and $S^6$ when $M_D = 30$ TeV for $s$ varying from 0.2 to 0.3 TeV$^2$. The upper curve corresponds to an internal space with a spherical geometry and the lower curve to a toric geometry. Beyond a critical value of $s$ (around $s = 0.15$ TeV$^2$ in this case) the difference between the two curves becomes constant.

For $N = 2$. For $N = 3$ and $N = 4$, all considered values of $M_D$ and $\sqrt{s}$ lead to a function $D_N$ which is a negligible fraction of a percent. $D_5$ is also negligible for $M_D = 1$ TeV but reaches 0.002% for $M_D = 10$ TeV (for small values of $s$). When $M_D = 30$ TeV the ratio $D_5$ goes as high as 0.04% for $\sqrt{s} = 0.1$ TeV but goes down to 0.002% for $\sqrt{s} = 0.5$ TeV. The most noticeable effects occur for $N = 6$. Then, with $M_D = 30$ TeV, $D_6$ varies from 5% to 0.1% as $\sqrt{s}$ spans the 0.1 to 0.5 TeV range. Still for $N = 6$, when $M_D = 10$ TeV $D_6$ varies from 0.3% to 0.01% but is negligible for $M_D = 1$ TeV.

In conclusion, we find that the relative difference between cross sections (for a given $s$) in models with spherical and toric geometries takes larger values when the quantum gravity scale is large and the dimensions are numerous. While the absolute difference increases with $s$ (for a given $M_D$ and $N$) the relative difference, $D_N$, does just the opposite. This is expected as cross sections are rising functions of the CM energy.

VI. CONCLUDING REMARKS

The contribution of each KK mode to the effective four-dimensional gravitational potential is proportional to

$$e^{-\text{(const.)}r}/r,$$

where the constant is related to the mass of the mode. In the large $r$ limit the potential takes the following form:

$$V(r) \simeq -\frac{1}{r}(1 + \kappa e^{-m_{(1)} r}),$$

where $\kappa$ is the multiplicity of the first massive KK mode and $m_{(1)}$ its mass. It can be shown that $\kappa$ is somewhat larger for a torus (with all compactification radii assumed to be the same) than
it is for a sphere. This suggests that to an observer on the three-brane the force due to gravity is slightly stronger if the extra dimensions are toric. This is not in contradiction with our result that cross sections for graviton emission associated with a spherical manifold are larger than those evaluated with a toric internal space. While it is true that low energy gravity is stronger for the torus, it simply appears otherwise from a microscopic point of view when gravity is probed with high energies. It is argued in Ref. [16] that large extra dimensions compactified on some Calabi-Yau manifolds are such that $\kappa$ can be as large as 20 which is noticeably larger than the corresponding multiplicities on the 6-sphere and the 6-torus. It is hopeless to try and find a generic solution for multiplicities at all KK levels for Calabi-Yau manifolds as there exists an overwhelmingly large number of such spaces (vacuum degeneracy problem of string theory). Nevertheless, it is conceivable that models with large extra dimensions compactified on a Calabi-Yau might lead to more significant discrepancies for microscopic processes compared with models where the internal space is either toric or spherical.

Having graviton modes propagating on a $N$-torus is exactly the same as having them existing in a $N$-dimensional box. Such a geometry has no intrinsic curvature so whether the modes have low or high discretized momenta does not matter. By that we mean that all modes perceive the space as being $T^N$. If the compact geometry is $S^N$ the situation is different. The high energy modes, having a small wavelength in transverse space, do not behave differently than when they are propagating on a $N$-torus (the wavelength-squared is assumed to be much smaller that the inverse local curvature). It then makes sense to say that the physics resulting from these high energy modes cannot be used to distinguish between processes taking place on different compact geometries (unless the associated curvature is large). The graviton modes to which are associated small quantum numbers (low energy modes) are the ones that can be used to study the shape of the extra dimensions. In fact, their wavelength is presumably large enough to allow them to recognize a sphere from a torus say. The multiplicity of states (or density of states) at each quantum level on the compact geometry lattice grows as the norm of the momentum is increased. We have shown that overall this multiplicity is larger on the sphere. This explains why the cross section for a process like $e^+e^- \rightarrow \gamma h$ is larger when the compact geometry has a spherical symmetry. We have seen that past a certain transverse momenta the multiplicity on a sphere and a torus become equal. This means that beyond some critical value for the CM energy, the difference between the cross sections evaluated for different geometries stabilizes to a constant value. This is what we have found for the $N = 6$, $M_D = 30$ TeV case but this is true in general. As $M_D$ is augmented the geometry of the compact space plays an increasingly important role. While this is true, it is also worth noting that when $M_D$ is increased, deviations from the $e^+e^- \rightarrow \gamma \nu \bar{\nu}$ process progressively become negligible. In fact, the size of the extra dimensions then becomes small which allows only a limited number of modes to propagate in the extra dimensions.

In summary, high energy colliders are limited in their ability to determine the geometry of large extra dimensions. This is due to two factors:

- as $\sqrt{s}$ is increased the relative difference between cross sections on different geometries decreases because the contribution of the low energy modes becomes increasingly small;
- the impact of the geometry is more important for a large quantum gravity scale and numerous extra dimensions but this also corresponds to a smaller gravitational contribution to SM background processes.

In fact, the geometrical effect we find for the $e^+e^- \rightarrow \gamma h$ process is rather small. Nevertheless, it is still conceivable that it could be detected at the upcoming LHC, using the process $q \bar{q} \rightarrow \gamma h$, 

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for certain values of $N$ and $M_D$. This experimental study and the effects of the geometry on supernova constraints will be considered in upcoming work [18].

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**APPENDIX: ESTIMATING SUMS BY INTEGRALS**

We now verify that approximating sums over KK modes propagating in the extra dimensions by integrals is valid for the range of parameters ($s$, $N$, and $M_D$) considered in this paper. In order to do that we use the Euler-MacLaurin formula (see for example [17]) which we write down schematically:

$$
\sum_k f(k) = \frac{1}{w} \int_{k_{\text{min}}}^{k_{\text{max}}} dk \ f(k) + \frac{1}{2} [f(k_{\text{min}}) + f(k_{\text{max}})] \\
+ \sum_{s=1}^{m} \left[ \frac{B_{2s}}{2s!} w^{2s-1} \left( f^{(2s-1)}(k_{\text{min}}) - f^{(2s-1)}(k_{\text{max}}) \right) + R_s \right],
$$

where $w$ characterizes the spacing between levels in the Kaluza-Klein tower, the $B$'s are Bernoulli numbers ($B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30$, etc.) and

$$
R_s = w^{2s-1} \int_k dk \ f^{(2s+1)}(k) P_{2s+1}(k)
$$

where $P_{2s+1}$ is a polynomial made of an infinite sum over oscillating functions of $k$, $k_{\text{min}}$ and $w$, the details of which are not important for our purpose.

The mass of the KK modes propagating in the extra dimensions depends on integer-valued quantum numbers. In the case of a space compactified on a 6-sphere the following sum needs to be evaluated,

$$
\sigma_{s^6}(e^+e^- \rightarrow \gamma h) = \frac{\alpha}{16 M_D^2} \sum_n F \left( x, m^2 = \frac{n(n+5)}{a^2} \right).
$$

The function $F(x, y)$ where $y = m^2/s$ is introduced in Sec. 4. Using Eq. (78) it can be shown that the sum in Eq. (80) is equivalent to an integral plus corrections parametrized by $\Delta_{s^6}$. To a good approximation we can write

$$
\Delta_{s^6} = \frac{1}{2 s a^2} \left[ \frac{24}{s^2 R^4} F(x, 0) + \left( 1 + \frac{4}{s a^2} \right) \left( 1 + \frac{6}{s a^2} \right) F(x, 1) \right]
$$

where there remains to be performed an angular integration ($x$-variable). The other corrections include all order derivatives of $F(x, y)$ with respect to $y$ and are proportional to rising powers of $1/(\sqrt{s} a)$ the most significant contribution being $O(1/s^{3/2} a^3)$. Since $F(x, y)$ is a smooth slowly varying function of $y$ and because $1/s^{3/2} a^3$ is small (varies from $0.002$ to $2 \times 10^{-7}$ for $\sqrt{s} = 0.1$ TeV to $\sqrt{s} = 1$ TeV when $M_D = 30$ TeV and $N = 6$) these contributions are small compared with the one obtained using Eq. (81). The corresponding expression for a 6-torus is
\[
\Delta x^6 = \frac{1}{2s a^2} F(x, 1). \tag{82}
\]

We note that the corrections (81) and (82) are more significant when \(s\), \(M_D\) and \(N\) are large. We computed those explicitly for \(N = 6\), \(M_D = 30\) TeV and \(\sqrt{s} = 0.1\) to 1 TeV. The absolute value of the corrections are to a good approximation equal for the spherical and toroidal compactifications. Consequently, the corrections slightly rise up both curves in Fig. 2 and Fig. 3 by the same amount for each given \(s\). The relative importance of the corrections is approximately 1/5 the value of the relative differences we found between cross sections evaluated on a torus and on a sphere. Of course, as \(M_D\) is increased the approximation becomes less precise as an increasingly small number of modes are summed over.
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