Research article

Initial boundary value problem for fractional $p$-Laplacian Kirchhoff type equations with logarithmic nonlinearity

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Abstract: In this paper, we study the initial boundary value problem for a class of fractional $p$-Laplacian Kirchhoff type diffusion equations with logarithmic nonlinearity. Under suitable assumptions, we obtain the extinction property and accurate decay estimates of solutions by virtue of the logarithmic Sobolev inequality. Moreover, we discuss the blow-up property and global boundedness of solutions.

Keywords: Kirchhoff type; $p$-Laplacian; fractional; extinction; decay estimate

1. Introduction

In this paper, we study the extinction and the blow-up for the following fractional $p$-Laplacian Kirchhoff type equations with logarithmic nonlinearity.

\[
\begin{cases}
  u_t + M(\|u\|^{r})(-\Delta)^{s}_{p}u = \lambda |u|^{r-2}u \ln |u| - \beta |u|^{q-2}u, & \text{in } \Omega \times (0, T), \\
  u(0) = u_0, & \text{in } \Omega, \\
  u = 0, & \text{on } \partial \Omega \times (0, T),
\end{cases}
\]

where

\[
\|u\| = \left( \int_{Q} \frac{|u(x) - u(y)|^{p}}{|x - y|^{N + sp}} dx dy \right)^{\frac{1}{p}},
\]

\(Q = \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega), \ C\Omega = \mathbb{R}^{N} \setminus \Omega, \ \Omega \subset \mathbb{R}^{N} \ (N > 2s)\) is a bounded domain with Lipschitz boundary, \(s \in (0, 1), \ 1 < p < 2, \ 1 < q \leq 2, \ r > 1, \ \lambda, \ \beta > 0, \ (-\Delta)^{s}_{p}\) is the fractional $p$-Laplacian operator and satisfies

\[
(-\Delta)^{s}_{p}u(x) = 2 \lim_{\gamma \to 0^{+}} \int_{\mathbb{R}^{N} \setminus B_{\gamma}(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N + sp}}dy,
\]
where \( u(x) \in C^\infty \) and \( u(x) \) has compact support in \( \Omega \), \( B_\gamma(x) \subset \mathbb{R}^N \) is the ball with center \( x \) and radius \( \gamma \). \( u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \) is a nonzero non-negative function, where \( L^\infty(\Omega) \) and \( W_0^{1,p}(\Omega) \) are Lebesgue space and fractional Sobolev space respectively, which will be given in section 2. \( M(\cdot) \) is a Kirchhoff function with the following assumptions

\[
(M_1) \quad 0 < s < 1, M(\tau) := a + b\thetar^{\theta-1} \text{ for } \tau \in \mathbb{R}^*_+ := [0, +\infty) \quad (a > 0, b \geq 0 \text{ are two constants}), \theta \geq 1;
\]

\[
(M_2) \quad M : \mathbb{R}^*_+ \to \mathbb{R}^*_+ \setminus \{0\} \text{ is continuous and there exists } m_0 \geq 0 \text{ such } M(\tau) \geq m_0 \text{ for all } \tau \geq 0.
\]

It is worth pointing out that the interest in studying problems like (1.1) relies not only on mathematical purposes, but also on their significance in real models. For example, in the study of biological populations, we can use \( u(x, t) \) to represent the density of the population at \( x \) at time \( t \), the term \((-\Delta)^s u\) represents the diffusion of density, \( \mu|u|^{q-2}u \) represents the internal source and \( \lambda|u|^{-2}u \ln |u| \) denotes external influencing factors. For more practical applications of problems like (1.1), please refer to the studies [1–3].

Compared with integer-order equations, it is very difficult to study the problem (1.1), which contains both non-local terms (including fractional \( p \)-Laplacian operators and Kirchhoff functions) and logarithmic nonlinearity. For the fractional order theory, we refer the readers to the studies [4–6]. In [7, 8], the authors use Sobolev space and Nehari manifold to study the existence of solutions for fractional equations. In [9, 10], the solutions for fractional equations are discussed by virtue of Nehari manifold and fibrillation diagram. By using different methods from above, the properties of the solutions for such partial differential equations are considered by the method of variational principle and topological theory in the theoretical literature [11–13]. Moreover, the authors prefer to use potential well theory, Galerkin approximation and Nehari manifold method to prove the existence of solutions, decay estimation and blow-up, we refer the reader to the literature [14–16].

Existence, extinction and blow-up of solutions are three important topics which regard parabolic problems; in particular, the study of extinction properties has made great progress in recent years. In [17], Liu considered the following initial boundary value problem for the fractional \( p \)-Laplacian equation

\[
u_t - \text{div}(|\nabla u|^{p-2}\nabla u) + \beta u^q = \lambda u, \quad x \in \Omega, \quad t > 0,
\]

where \( 1 < p < 2, q \leq 1 \) and \( r, \lambda, \beta > 0 \). By employing the differential inequality and comparison principle, they obtained the extinction and the non-extinction of weak solutions. In [18], Sarra Toualbia et al. considered the following initial boundary value problem of a nonlocal heat equations with logarithmic nonlinearity

\[
u_t - \text{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p-2}u \log |u| - \int_{\Omega} |u|^{p-2}u \log |u|dx, \quad x \in \Omega, \quad t > 0
\]

where \( p \in (2, +\infty) \). By using the logarithmic Sobolev inequality and potential well method, they obtained decay, blow-up and non-extinction of solutions. In [19], Xiang and Yang studied the first initial boundary value problem of the following fractional \( p \)-Kirchhoff type

\[
u_t + M([u]_{s,p})^{s-2}(\nabla)^s u = \lambda|u|^{r-2}u - \mu|u|^{q-2}u, \quad (x, t) \in \Omega \times (0, \infty),
\]

where \( M : [0, \infty) \to (0, \infty) \) is a continuous function, \( 0 < s < 1 < p < 2, 1 < q \leq 2, r > 1, \lambda, \mu > 0 \). Under suitable assumptions, they proved the extinction and non-extinction of solutions and perfected the Gagliardo-Nirenberg inequality. For more information on the extinction properties of the solution, please refer to the studies [20–23].
Inspired by the above works, we overcome the research difficulties of logarithmic nonlinearity, \( p \)-Laplace operator and Kirchhoff coefficients in problem (1.1) based on the potential well theory, Nehari manifold and differential inequality methods, we give the extinction and the blow-up properties of solutions. In addition, we give the global boundedness of the solution by appropriate assumptions. To the best of our knowledge, it is the first result in the literature to investigate the extinction and blow-up of solutions for fractional \( p \)-Laplacian Kirchhoff type with logarithmic nonlinearity.

In order to introduce our main results, we first give some related definitions and sets.

**Definition 1.1 (Weak solution).** A function \( u(x, t) \) is said to be a weak solution of problem (1.1), if \((x, t) \in \Omega \times [0, T)\), \( u \in L^p(0, T; W_0^{s,p}(\Omega)) \cap C(0, T; L^2(\Omega))\), \( u_t \in L^2(0, T; L^2(\Omega)), \ u(x, 0) = u_0(x) \in W_0^{s,p}(\Omega)\), for all \( v \in W_0^{s,p}(\Omega) \), \( t \in (0, T) \), the following equation holds

\[
\int_{\Omega} u_t v dx + M(||u||^p)(u, v) = \lambda \int_{\Omega} v|u|^{r-2}u dx - \beta \int_{\Omega} v|u|^{q-2}u dx,
\]

where

\[
\langle u, v \rangle = \int_{\Omega} \frac{|u(x) - u(y)||^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dxdy.
\]

**Definition 1.2 (Extinction of solutions).** Let \( u(t) \) be a weak solution of problem (1.1). We call \( u(t) \) an extinction if there exists \( T > 0 \) such that \( u(x, t) > 0 \) for all \( t \in (t, T) \) and \( u(x, t) \equiv 0 \) for all \( t \in [T, +\infty) \).

Define the following two functionals on \( W_0^{s,p}(\Omega) \)

\[
E(u) = \frac{1}{p} a||u||^p + \frac{1}{p} b||u||^{q_p} - \lambda \frac{1}{r} \int_{\Omega} |u|^r \ln |u| dx + \frac{\lambda}{1} \frac{1}{r^2} \int_{\Omega} |u|^r dx + \frac{\beta}{q} \frac{1}{p} \int_{\Omega} |u|^q dx,
\]

\[
I(u) = a||u||^p + \theta b||u||^{q_p} - \lambda \int_{\Omega} |u|^r \ln |u| dx + \beta||u||^q.
\]

Let

\[
Z = \{ u \in L^p(0, T; W_0^{s,p}(\Omega)) \cap C(0, T; L^2(\Omega)), u_t \in L^2(0, T; L^2(\Omega)) \}.
\]

**Remark 1** Since \( u \in Z \), \( 1 < p < 2 \), \( M(\cdot) \) is a continuous function and

\[
\int_{\Omega} |u|^r \ln |u| dx \leq \frac{1}{\sigma} ||u||^{r+\sigma}_{r+\sigma} \leq \frac{1}{\sigma} C_{r+\sigma} ||u||^{r+\sigma},
\]

where \( 0 < \sigma < p^*_r - r \), then we can claim that \( E(u) \) and \( I(u) \) are well-defined in \( W_0^{s,p}(\Omega) \). Further, by arguing essentially as in [24], one can prove the that

\[
u \mapsto \int_{\Omega} |u|^r \ln |u| dx
\]

is continuous from \( W_0^{s,p}(\Omega) \) to \( \mathbb{R} \). It follows that \( E(u) \) and \( I(u) \) are continuous.

Define some sets as follows

\[
W := \{ u \in W_0^{s,p}(\Omega) \mid I(u) > 0, \ E(u) < h \} \cup \{0\}, \quad (1.7)
\]

\[
V := \{ u \in W_0^{s,p}(\Omega) \mid I(u) < 0, \ E(u) < h \}, \quad (1.8)
\]
Theorem 1

Moreover, we define

\[ N := \{ u \in W^{s,p}_0(\Omega) \setminus \{0\} | I(u) = 0 \}. \]  

(1.10)

Let \( \lambda_1 \) be the first eigenvalue of the problem

\[-\Delta u = \lambda |u|^{p-2}u \text{ in } \Omega, \quad u_{|\partial \Omega} = 0,\]  

(1.13)

and \( \phi(x) > 0 \) a.e. in \( \Omega \) be the eigenfunction corresponding to the eigenvalue \( \lambda_1 > 0 \), \( \phi(x) \in L^\infty(\Omega) \) and \( ||\phi||_{L^\infty(\Omega)} \leq 1 \).

First, we give some results satisfying \( I(u_0) > 0 \) and \( q = 2 \).

**Theorem 1.1** Assume that \( I(u_0) > 0 \), \( r = p \) and \( q = 2 \). Let \( m_0 \) be as in assumption (M2), and let

\[ l := \frac{2N - (s + N)p}{sp}, \quad P_1 := \frac{m_0}{\gamma_0 L(p, \Omega) + \ln(R)}, \quad P_2 := \frac{\lambda_1 m_0 p^{p-1}}{[\lambda_1 L(p, \Omega) + \ln(R)](p + l - 1)^{p-1}}, \]

where \( L(p, \Omega) \) and \( R \) are given in Lemma 2.1 and Lemma 2.5. Then, there exist positive constants \( C_1, \ C_2, \ T_1 \) and \( T_2 \) such that

(i) If \( \lambda < \lambda_1 P_1 \), then the weak solution of (1.1) vanishes in the sense of \( || \cdot ||_2 \) as \( t \to +\infty \).

(ii) If \( 2N/(N + 2s) < p < 2 \) and \( \lambda < \lambda_1 P_1 \) or \( 1 < p \leq 2N/(N + 2s) \) and \( \lambda < P_2 \), then the nonnegative solutions of (1.1) vanish in finite time, and

\[
\begin{cases}
||u||_2 \leq \left( ||u_0||_{L^2}^{2-p} + \frac{C_1}{\beta} \right)^{\frac{p-2}{p}} - \frac{C_1}{\beta}, & t \in [0, T), \\
||u||_2 \equiv 0, & t \in [T, \infty),
\end{cases}
\]

for \( 2N/(N + 2s) < p < 2 \), and

\[
\begin{cases}
||u||_{l+1} \leq \left( ||u_0||_{L^2}^{2-p} + \frac{C_2}{\beta} \right)^{\frac{p-2}{p}} - \frac{C_2}{\beta}, & t \in [0, T), \\
||u||_{l+1} \equiv 0, & t \in [T, \infty),
\end{cases}
\]

for \( 1 < p < 2N/(N + 2s) \).

**Theorem 1.2** Assume that \( I(u_0) > 0 \), \( 0 < \sigma \leq p^*_s - p, \ r > p \) and \( q = 2 \). Let \( m_0 \) be as in assumption (M2), and let

\[ P_3 := \max \left\{ \frac{m_0}{L(r, \Omega) R^r - p + \sigma \Phi}, \frac{\beta}{\Phi E^{r/(r-1)(s-p)}} \right\}, \quad P_4 := \max \left\{ \frac{m_0 p^p}{E^r_1(p + l - 1)^p}, \frac{\beta E^{r(1-s)/2}}{E^{r/(r-1)(s-p)}} \right\}.\]
where
\[ \phi = \ln(R)\Omega^{\frac{1}{2p - 1}} C_{p_1}^r(1 - \theta_1), \]
\[ p_s^* = \frac{Np}{N - s}, \quad s_1 = 2\theta_1 + p_s^*(1 - \theta_1), \quad \theta_1 = \frac{2(r - p)}{r(2 - p)}, \quad r_2 = \frac{p(l + r + \sigma - 1)}{l + p - 1}, \]

\( C_{p_1}^r \) is the best constant of embedding from \( W^{s,p}_0(\Omega) \) to \( L^{r, p}(\Omega) \), \( L(p, \Omega) \) and \( R \) are given in Lemma 2.1 and Lemma 2.5, and \( \varepsilon, \varepsilon_1 > 0 \) are two constants. Then, there exist positive constants \( C_4, C_5, C_6, C_7, T_3 \) and \( T_4 \) such that the non-negative weak solution of problem (1.1) vanishes in finite time and
\[ \|u\| \leq \left( \left\| u_0 \right\|_{L^{2,p}} + \frac{C_4}{C_5} \right)e^{(p - 2)C_4t} - \frac{C_4}{C_5}, \quad t \in [0, T_3), \]
\[ \|u\| = 0, \quad t \in [T_3, \infty), \]
for \( 2N/(N + 2s) \leq p < 2, \lambda < P_3, \) and
\[ \|u\| \leq \left( \left\| u_0 \right\|_{L^{2,p}} + \frac{C_4}{C_5} \right)e^{(p - 2)C_4t} - \frac{C_4}{C_5}, \quad t \in [0, T_4), \]
\[ \|u\| = 0, \quad t \in [T_4, \infty), \]
for \( 1 < p < 2N/(N + 2s), \lambda < P_4. \)

Secondly, we give some results satisfying \( I(u_0) > 0 \) and \( q < 2. \)

**Theorem 1.3** Assume that \( I(u_0) > 0, p = r, l_1 > l \geq 1 \) and \( 1 < q < 2. \) Let \( m_0 \) be as in assumption (\( M_2 \)), and let
\[ p_5 := \frac{\lambda_1 m_0 l_1 p^{p - 1}}{\lambda_1 L(p, \Omega) + \ln(R)(p + l_1 - 1)p^{p - 1}}, \]
where \( L(p, \Omega) \) and \( R \) are given in Lemma 2.1 and Lemma 2.5. If \( 2N/(N + 2s) < p < 2 \) with \( \lambda < \lambda_1 P_1 \) or \( 1 < p \leq 2N/(N + 2s) \) with \( \lambda < P_5 \), then the non-negative weak solution of problem (1.1) vanishes in finite time for any non-negative initial data.

**Theorem 1.4** Assume that \( I(u_0) > 0, l_1 > l \geq 1, r \leq 2 \) and \( 1 < q < 2. \) Let \( m_0 \) be as in assumption (\( M_2 \)), and let
\[ p_6 := \max\left\{ \frac{m_0}{L(r, \Omega)^{1 - \theta}}, \frac{\beta \varepsilon_2 \varepsilon_3}{\ln(R)^{1 - \theta}}, \frac{C_{p_1}^{(1 - \theta_1)p_1}(1 - \theta_1)r_4 R^{n - r}}{L(r, \Omega)^{1 - \theta}}, \frac{C_{p_1}^{(1 - \theta_1)p_1}(1 - \theta_1)r_4 R^{n - r}}{L(r, \Omega)^{1 - \theta}} \right\}, \]
\[ p_7 := \frac{1}{C\beta \varepsilon_2 \varepsilon_3^{(1 - \theta_1)p_1}(1 - \theta_1)r_4 R^{n - r}}, \]
\[ \theta_3 = \frac{[(q + l_1 - 1)p_5 - (l_1 + 1)p](p + l_1 - 1)}{[(q + l_1 - 1)p_5 - (p + l_1 - 1)p](l_1 + 1)}, \quad \theta_4 = \frac{(p_5 - 2)p}{(p_5 - q)2}, \quad C = \frac{1}{c\sigma \varepsilon_2 \varepsilon_3^{(1 - \theta_1)p_1}(1 - \theta_1)r_4 R^{n - r}}, \]
\[ r_3 = \frac{(q + l_1 - 1)(p + l_1 - 1)}{(q + l_1 - 1)(p + l_1 - 1)}, \quad r_4 = \frac{q}{q(1 - \theta_4) + \theta_4}, \quad r_5 = l_1 + r + \sigma - 1, \]
and \( \varepsilon_2, \varepsilon_3 > 0 \) are two constants. If \( 2N/(N + 2s) < p < 2 \) with \( \lambda < P_6 \) or \( 1 < p \leq 2N/(N + 2s) \) with \( \lambda < P_7 \), then the non-negative weak solution of problem (1.1) vanishes in finite time for any non-negative initial data.

Finally, we discuss the global boundedness and blow up of weak solutions.

**Theorem 1.5** Let \( u(x, t) \) be the weak solution of problem (1.1).
(i) If \( E(u_0) < 0, r > p > q \) and \( \theta = 1 \), then the weak solution \( u(x, t) \) blows up at \( +\infty \);
(ii) If \( 0 < E(u_0) \leq h \) and \( I(u_0) \geq 0 \), then the weak solution \( u(x, t) \) is globally bounded.

The rest of the paper is organized as follows. In Section 2, we give some related spaces and lemmas. In Section 3, we give the proof process for the main results of problem (1.1).
2. Preliminaries

In order to facilitate the proof of the main results, we start this section by introducing some symbols and Lemmas that will be used throughout the paper.

In this section, we assume that \(0 < s < 1 < p < 2\) and \(\Omega \in \mathbb{R}^N (N > 2s)\) is a bounded domain with Lipschitz boundary. We denote by \(\|u\|_i (i \geq 1)\) the norm of Lebesgue space \(L^i(\Omega)\). Let \(W^{s,p}(\Omega)\) be the linear space of Lebesgue measurable functions \(u\) from \(\mathbb{R}^N\) to \(\mathbb{R}\) such that the restriction to \(\Omega\) of any function \(u\) in \(W^{s,p}(\Omega)\) belongs to \(L^p(\Omega)\) and

\[
\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy < \infty,
\]

where \(Q = \mathbb{R}^{2N} \setminus (C\Omega \times C\Omega), C\Omega = \mathbb{R}^N \setminus \Omega\). The space \(W^{s,p}(\Omega)\) is equipped with the norm

\[
\|u\|_{W^{s,p}(\Omega)} = \left( \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{\frac{1}{p}}.
\]

We further give a closed linear subspace

\[ W^{s,p}_0(\Omega) = \{ u \in W^{s,p}(\Omega) | u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}. \]

As shown in [19], it can be concluded that

\[
\|u\| = \left( \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy \right)^{\frac{1}{p}}
\]

is an equivalent norm of \(W^{s,p}_0(\Omega)\).

Next we give the necessary Lemmas.

**Lemma 2.1 ([25])** Let \(u \in W^{s,p}_0(\Omega) \setminus \{0\}\). Then

\[
\int_{\Omega} |u|^p \ln |u| dx \leq L(p,\Omega) \|u\|_p^p + \ln(\|u\|) \int_{\Omega} |u|^p dx,
\]

where \(L(p,\Omega) := \frac{|\Omega|}{e^{\frac{1}{p}}} + \frac{1}{e^{(p-1)^{1/2}}} C_{p'}\) is the best constant of embedding from \(W^{s,p}_0(\Omega)\) to \(L^{p'}(\Omega)\).

**Lemma 2.2 ([26])** Let \(y(t)\) be a non-negative absolutely continuous function on \([T_0, +\infty)\) satisfying

\[
\frac{dy}{dt} + \alpha y^k + \beta y \leq 0, \quad t \geq 0, \quad y(0) \geq 0,
\]

where \(\alpha, \beta > 0\) are constants and \(k \leq 0, 1\). Then

\[
\begin{cases}
   y(t) \leq \left[ (y^{1-k}(T_0) + \frac{\alpha}{\beta}) e^{k(T-t)} - \frac{\alpha}{\beta} \right]^{\frac{1}{1-k}}, & t \in [T_0, T_*), \\
   y(t) \equiv 0, & t \in [T_*, +\infty),
\end{cases}
\]

where \(T_* = \frac{1}{1-k} \ln \left(1 + \frac{\alpha}{\beta} e^{1-k(T_0)}\right)\).

**Lemma 2.3 ([27])** Suppose that \(\beta^* \geq 0, N > sp \geq 1, 1 \leq r \leq q \leq (\beta^* + 1)p'_s\), then for \(u\) such that \(|u|^{p_s} u \in W^{s,p}_0(\Omega)\), we have

\[
\|u\|_q \leq C_{p'_s}^{\frac{1}{1-q}} \|u\|_p^p \|u|^{p'_s} u\|^{\frac{1}{p'_s}}.
\]

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with \( \theta = \frac{(p' + 1)r - q}{(p' + 1)r - 1} \), where \( C_{r'} \) is the embedding constant for \( W_0^{r,p}(\Omega) \hookrightarrow L^{r'}(\Omega) \).

**Lemma 2.4 ([19])** Let \( 1 < p < \infty \) and \( g : \mathbb{R} \to \mathbb{R} \) be an increasing function. Define

\[
G(t) = \int_0^t g'(\tau) \, d\tau, \quad t \in \mathbb{R}
\]

then

\[
|a - b|^{p-2}(a - b)(g(a) - g(b)) \geq |G(a) - G(b)|^p
\]

for all \( a, b \in \mathbb{R} \).

**Lemma 2.5** Assume that (\( M_1 \)) holds. Let \( u \in W_0^{r,p}(\Omega) \setminus \{0\}, 0 < \sigma \leq p_s^* - r \). We have

(i) if \( 0 < ||u|| \leq R \), then \( I(u) > 0 \);

(ii) if \( I(u) \leq 0 \), then \( ||u|| > R \).

where

\[
R = \left( \frac{a^\sigma}{\lambda C_{r+\sigma}} \right)^{\frac{1}{1-\sigma}},
\]

\( C_{r,\sigma} \) is the embedding constant for \( W_0^{r,p}(\Omega) \hookrightarrow L^{r+\sigma}(\Omega) \).

**Proof.** Since \( u \in W_0^{r,p}(\Omega) \setminus \{0\} \), and

\[
\sigma \ln |u(x)| < |u(x)|^\sigma \quad \text{for a.e.} x \in \Omega.
\]

Then by the definition of \( I(u) \), we obtain

\[
I(u) = a||u||^p + \theta b||u||^{p^*} + \beta ||u||_q^q - \lambda \int_\Omega |u|^r \ln |u| \, dx
\]

\[
> a||u||^p + \theta b||u||^{p^*} + \beta ||u||_q^q - \lambda \frac{1}{\sigma} ||u||_{r+\sigma}^{r+\sigma}
\]

\[
\geq a||u||^p - \lambda \frac{1}{\sigma} ||u||_{r+\sigma}^{r+\sigma}
\]

\[
\geq (a - \lambda \frac{1}{\sigma} C_{r+\sigma} ||u||^{r+\sigma} - p) ||u||^p.
\]

where \( C_{r+\sigma} \) is the embedding constant for \( W_0^{r,p}(\Omega) \hookrightarrow L^{r+\sigma}(\Omega) \).

We can get

\[
I(u) > (a - \lambda \frac{1}{\sigma} C_{r+\sigma} ||u||^{r+\sigma} - p) ||u||^p. \quad (2.1)
\]

If \( 0 < ||u|| \leq R \), then it follows from the definition of \( R \) that

\[
a - \lambda \frac{1}{\sigma} C_{r+\sigma} ||u||^{r+\sigma} - p \geq 0,
\]

thus (i) holds.

If \( I(u) \leq 0 \), by (2.1), we have

\[
a - \lambda \frac{1}{\sigma} C_{r+\sigma} ||u||^{r+\sigma} - p < 0,
\]

thus (ii) holds. \( \square \)

Lemmas 2.6 is similar to [28, Lemmas 9], so we ignore its proof.

**Lemma 2.6 ([28])** Assume that \( E(u_0) \leq h \), then the sets \( \mathcal{N}_+ \) and \( \mathcal{N}_- \) are both invariant for \( u(t) \), i.e., if \( u_0 \in \mathcal{N}_- \) (resp. \( u_0 \in \mathcal{N}_+ \)), then \( u(t) \in \mathcal{N}_- \) (resp. \( u(t) \in \mathcal{N}_+ \)) for all \( t \in [0, T] \).

**Lemma 2.7 ([29])** Let \( \alpha \) be positive. Then

\[
t^\theta \ln(t) \leq \frac{1}{e\alpha} t^{p+\alpha}, \quad \text{for all} \quad p, t > 0.
\]
3. Proof of main results

In this section, we prove that the main results of problem (1.1).

**Proof of Theorem 1.1**

(1) Taking \( v = u \) in Definition 1.1, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + M(||u||^p)||u||^p = \lambda \int_{\Omega} |u|^p \ln |u| \, dx - \beta \int_{\Omega} u^2 \, dx.
\]  

(3.1)

By Lemma 2.1, Lemma 2.5 and Lemma 2.6, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \left( m_0 - \lambda L(p, \Omega) - \frac{\lambda}{\lambda_1} \ln(R) \right)||u||^p + \beta \int_{\Omega} u^2 \, dx \leq 0.
\]  

(3.2)

Since \( \lambda < \lambda_1 \), we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \beta \int_{\Omega} u^2 \, dx \leq 0.
\]  

thus

\[
||u(\cdot, t)||_2^2 \leq ||u_0||_2^2 e^{-2\beta t}.
\]

which implies that \( ||u(\cdot, t)||_2 \to 0 \) as \( t \to +\infty \).

(2) We consider first the case \( 2N/(N+2s) < p < 2 \) with \( \lambda < \lambda_1 \). By (3.1) and Lemma 2.1, Lemma 2.5 and Lemma 2.6, we have

\[
\frac{1}{2} \frac{d}{dt} ||u||_2^2 + \left( m_0 - \frac{\lambda}{\lambda_1} (\lambda_1 L(p, \Omega) + \ln(R)) \right)||u||^p + \beta ||u||_2^2 \leq 0.
\]  

(3.3)

Using Hölder’s inequality and the fractional Sobolev embedding theorem, we have

\[
||u||_2 \leq ||\Omega||^{\frac{1}{2} - \frac{N-p}{N}} ||u||_{\frac{N}{N-p}} \leq C_{p'} ||\Omega||^{\frac{1}{2} - \frac{N-p}{N}} ||u||,
\]

(3.4)

where \( C_{p'} > 0 \) is the embedding constant. By (3.2), (3.3) and \( \lambda < \lambda_1 \), we obtain

\[
\frac{1}{2} \frac{d}{dt} ||u||_2^2 + C_1 ||u||^p_2 + \beta ||u||_2^2 \leq 0,
\]  

where

\[
C_1 = C_{p'} ||\Omega||^{\frac{N-p}{N}} \left( m_0 - \frac{\lambda}{\lambda_1} (\lambda_1 L(p, \Omega) + \ln(R)) \right) > 0.
\]  

(3.5)

Setting \( y(t) = ||u(\cdot, t)||_2^2, y(0) = ||u_0(\cdot)||_2^2, \) by Lemma 2.2, we obtain

\[
\begin{cases}
||u||_2 \leq \left( \left( ||u_0||_2^{2-p} + \frac{C_1}{\beta} \right) e^{(p-2)\beta t} - \frac{C_1}{\beta} \right)^{\frac{1}{p-2}}, & \text{for } t \in [0, T_1), \\
||u||_2 = 0, & \text{for } t \in [T_1, \infty),
\end{cases}
\]  

where

\[
T_1 = \frac{1}{(2-p)\beta} \ln \left( 1 + \frac{\beta}{C_1} ||u_0||_2^{2-p} \right).
\]  

(3.6)
Next, we consider the case $1 < p \leq 2N/(N + 2s)$ and $\lambda < \lambda_1P_2$. Taking $\nu = u'$ in Definition 1.1, where $l = \frac{2N-(s-N)p}{sp} \geq 1$, by Lemma 2.1, Lemma 2.4, Lemma 2.5 and Lemma 2.6, we obtain

\[
\frac{1}{l+1} \frac{d}{dt} ||u||_{l+1}^{l+1} + G||u||_{l+1}^{\frac{p+1}{p}} + \beta||u||_{l+1}^{l+1} \leq 0,
\]

(3.7)

where $G = \left( \frac{m_0lp}{(p+l-1)^p} - \frac{\lambda pL(p,\Omega)}{p+l-1} - \frac{\lambda p \ln(R)}{\lambda_1(p+l-1)} \right)$. By the very choice of $l$ and the fractional Sobolev embedding theorem, we have

\[
||u||_{l+1}^{\frac{p+1}{p}} = \left( \int_\Omega u^{\frac{p+1}{p}} \cdot \frac{Np}{N-p} dx \right)^{\frac{N-p}{np}} \leq C_{p^*} ||u||_{l+1}^{l+1}.
\]

(3.8)

Hence,

\[
\frac{1}{l+1} \frac{d}{dt} ||u||_{l+1}^{l+1} + C_2 ||u||_{l+1}^{l+1} \leq 0,
\]

(3.9)

where

\[
C_2 = C_{p^*} \left( \frac{m_0lp}{(p+l-1)^p} - \frac{\lambda pL(p,\Omega)}{p+l-1} - \frac{\lambda p \ln(R)}{\lambda_1(p+l-1)} \right),
\]

(3.10)

since $\lambda < P_2$, then $C_2 > 0$. Setting $y(t) = ||u(t)||_{l+1}$, $y(0) = ||u_0||_{l+1}$, by Lemma 2.2, we obtain

\[
\begin{cases} 
||u||_{l+1} \leq \left[ (||u_0||_{l+1}^{\frac{2p}{p}} + \frac{C_2}{p}) e^{p-2\beta t} - \frac{C_2}{p} \right]^{\frac{1}{p}}, & t \in [0, T_2), \\
||u||_{l+1} \equiv 0, & t \in [T_2, \infty),
\end{cases}
\]

where

\[
T_2 = \frac{1}{(2-p)\beta} \ln \left( 1 + \frac{\beta}{C_2} ||u_0||_{l+1}^{2-p} \right).
\]

(3.11)

The proof is completed. \hfill \Box

**Proof of Theorem 1.2**

We consider first the case $p < r < 2$ and $2N/(N + 2s) < p < 2$. Taking $\nu = u$ in Definition 1.1, we have

\[
\frac{1}{2} \frac{d}{dt} ||u||_2^2 + M(||u||^p) ||u|| = \int_\Omega |u|^r \ln |u| dx - \beta ||u||_2^2.
\]

(3.12)

By Lemma 2.1, Lemma 2.5 and Lemma 2.6, we obtain

\[
\frac{1}{2} \frac{d}{dt} ||u||_2^2 + M(||u||^p) ||u|| \leq \lambda L(r,\Omega) ||u||^r + \lambda \ln(R) ||u||^r - \beta ||u||_2^2.
\]

(3.13)

Using Hölder’s inequality and the interpolation inequality, the fractional Sobolev embedding theorem and Young inequality, we can easily obtain (see [19])

\[
||u||_2^r \leq |\Omega|^{\frac{s-r}{s}} ||u||_s^{\frac{r}{s}} \leq |\Omega|^{\frac{s-r}{s}} C_{p^*}^{\frac{1-r}{s}} e^{\frac{\theta_1}{r(s-r)} ||u||_p^p + \frac{\theta_1}{r(s-r)} ||u||_2^2},
\]

(3.14)

where $s > r$, $\theta_1 \in (0, 1)$, $\epsilon > 0$ and

\[
s_1 = 2\theta_1 + p^*_1 (1 - \theta_1), \quad \theta_1 = \frac{2(r-p)}{r(2-p)}.
\]
By (3.13) and (3.14), we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + C_3 \|u\|^p + C_4 \|u\|_2^2 \leq 0,
\end{equation}
where
\begin{equation}
C_3 = m_0 - \lambda L(r, \Omega) R^{-p} - \lambda \epsilon \ln(R) \|u\|_2^{1-p} C_{p'_1}^{(1-\theta_1)},
\end{equation}
\begin{equation}
C_4 = \beta - \lambda \ln(R) \|u\|_2^{1-p} C_{p'_1}^{(1-\theta_1)} e^{\frac{\mu (\theta_1 - 1)}{p'(1-\theta_1)}}.
\end{equation}
Since $2N/(N + 2s) < p < 2$ and $\lambda < P_3$, by the fractional embedding theorem and (3.3), we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + C_3 \|u\|^p + C_4 \|u\|_2^2 \leq 0,
\end{equation}
and $C_3 > 0$, $C_4 > 0$, where
\begin{equation}
C_5 = C_3 C_{p'_1}^{(1-\theta_1)} e^{\frac{\mu (\theta_1 - 1)}{p'(1-\theta_1)}}.
\end{equation}
Similar to the Theorem 1.1, one can prove the that
\begin{equation}
\begin{cases}
\|u\|_2 \leq \left(\|u_0\|_2^{1-p} + \frac{C_3}{C_5}\right) e^{(p-2)C_4 t} - \frac{C_3}{C_5}, & t \in [0, T_3), \\
\|u\|_2 = 0, & t \in [T_3, \infty),
\end{cases}
\end{equation}
where
\begin{equation}
T_3 = \frac{1}{(2 - p)C_4} \ln \left(1 + \frac{C_4}{C_5} \|u_0\|_2^{1-p}\right).
\end{equation}
When $1 < p \leq 2N/(N + 2s)$, $p < r \leq 2$ and $\lambda < P_4$. Taking $\nu = u'$ ($l = \frac{2N - sp - Np}{sp} \geq 1$) in Definition 1.1, by Lemma 2.4 and Lemma 2.7, we obtain
\begin{equation}
\frac{1}{l + 1} \frac{d}{dt} \|u\|_{l+1}^{p+1} + \frac{m_0 l^p}{(p + l - 1)^p} \|u\|_l^{(l+1)-p} \leq \lambda \frac{1}{e\sigma} \|u\|_l^{l+\nu+1} - \beta \|u\|_{l+1}^{p+1}
\end{equation}
further, we have
\begin{equation}
\frac{1}{l + 1} \frac{d}{dt} \|u\|_{l+1}^{p+1} + \frac{m_0 l^p}{(p + l - 1)^p} \|u\|_l^{(l+1)-p} \leq \lambda \frac{1}{e\sigma} \|u\|_l^{l+\nu+1} - \beta \|u\|_{l+1}^{p+1}
\end{equation}
where $r_1 = \frac{p(l+1)}{lp-1}$, $r_2 = \frac{p(l+\nu+1)}{lp-1}$. Note that, since $r_1 < p_3^*$, by the H"{o}lder's inequality and the fractional Sobolev embedding theorem, we have
\begin{equation}
\|u\|_{l+\nu+1}^{p+1} \leq \|u\|_l^{(l+1)-p} C_{p'_3}^{l+\nu+1} \|u\|_l^{(l+1)-p} \leq \|u\|_l^{(l+1)-p} C_{p'_3}^{l+\nu+1} \|u\|_l^{(l+1)-p} \|u\|_l^{(l+1)-p}.
\end{equation}
Using the same discussion as above, one can conclude that
\begin{equation}
\|u\|_{l+\nu+1}^{p+1} \leq \|u\|_l^{(l+1)-p} C_{p'_3}^{l+\nu+1} (\|u\|_l^{(l+1)-p} + \|u\|_l^{(l+1)-p} \|u\|_l^{(l+1)-p}),
\end{equation}
where $s_2 > r_2$, $\theta_2 \in (0, 1)$, $\epsilon_1 > 0$ and
\begin{equation}
s_2 = r_1 \theta_2 + p_3^*(1 - \theta_2), \quad \sigma_2 = \frac{r_1 (r_2 - p)}{r_2 (r_1 - p)}.
\end{equation}
Combining (3.22)-(3.24) and \( \lambda < P_4 \), we obtain
\[
\frac{1}{l+1} \frac{d}{dt} \|u\|_l^{\nu p} + C_6 \|u\|_l^{\nu p} + C_7 \|u\|_l^{\nu p} \leq 0,
\]
and \( C_6 > 0, C_7 > 0 \), where
\[
C_6 = \left( \frac{m_0 l^p}{l+1} - \frac{1}{e \sigma} \frac{1}{\rho} \left( \int_0^\infty \left( \int_0^\infty \frac{d\tau}{\tau} \right)^{\nu p} \right)^{\frac{1}{\nu p}} \right)
\]
\[
C_7 = \beta - \frac{1}{e \sigma} \left( \int_0^\infty \left( \int_0^\infty \frac{d\tau}{\tau} \right)^{\nu p} \right)^{\frac{1}{\nu p}}
\]
Using Lemma 2.2 and a direct calculation, we have
\[
\|u\|_{l+1} \leq \left( \left( \|u_0\|_{l+1}^2 + \frac{C_6}{C_7} \right) e^{(p-2)C_7 t} - \frac{C_6}{C_7} \right)^{-\frac{1}{p-2}}, \quad t \in [0, T_4),
\]
\[
\|u\|_{l+1} \equiv 0, \quad t \in [T_4, \infty),
\]
where
\[
T_4 = \frac{1}{(2-p)C_7} \ln\left( \frac{C_7}{C_6} \|u_0\|_{l+1}^{2-p} + 1 \right).
\]
The proof is completed. \( \square \)

**Proof of Theorem 1.3**

We consider first the case \( 2N/(N + 2s) < p < 2 \) with \( \lambda < \lambda_1 P_1 \). Taking \( v = u \) in Definition 1.1, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + M(\|u\|_p) \|u\|_p = \lambda \int_\Omega |u|^p \ln |u| dx - \beta \int_\Omega |u|^q dx.
\]
Note that \( \|u\|_p \geq \lambda_1 ||u||_p \) and by Lemma 2.1, Lemma 2.5 and Lemma 2.6, we have
\[
\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + (m_0 - \frac{\lambda}{A_1} (\lambda_1 L(p, \Omega) + \ln(R))) \|u\|_p + \beta \|u\|_q \leq 0.
\]
By (3.3) and \( \beta > 0 \) and \( \lambda < \lambda_1 P_1 \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + C_8 \|u\|_2^p \leq 0,
\]
where
\[
C_8 = C_p^{-p} |\Omega| \frac{2N-2p-Np}{2p} \left( m_0 - \frac{\lambda}{A_1} (\lambda_1 L(p, \Omega) + \ln(R)) \right) > 0.
\]
By a direct calculation, we obtain
\[
\begin{cases}
\|u\|_2 \leq \left( \left( \|u_0\|_2^{2-p} + C_8 (p-2) t \right)^{\frac{1}{p-2}}, \quad t \in [0, T_5),
\|u\|_2 \equiv 0, \quad t \in [T_5, \infty),
\end{cases}
\]
where
\[
T_5 = \frac{1}{(2-p)C_8} \|u_0\|_2^{2-p}.
\]
Next, we consider the case $1 < p \leq 2N(N + 2s)$ with $\lambda < P_2$. Multiplying (1.1) by $u^i$ ($l_1 > l \geq 1$) and integrating, by Lemma 2.1, Lemma 2.4, Lemma 2.5 and Lemma 2.6, we obtain

$$
\frac{1}{l_1 + 1} \frac{d}{dt} ||u||_{l_1+1}^i + G_1 ||u||_{l_1+1}^i \leq 0,
$$

where $G_1 = (\frac{m_0 l_1 p^p}{(p+1-i)^p} - \frac{\alpha p L(p, \Omega)}{l_1(p+1-i)})$. By Lemma 2.3, we obtain

$$
||u||_{l_1+1} \leq C \frac{\lambda^p}{\lambda^p + 1} ||u||_{l_1+1}^i \leq C \frac{\lambda^p}{\lambda^p + 1} ||u||_{q+l_1-i}^p,
$$

where

$$
\phi_3 = \frac{[(q + l_1 - 1)p^p - (l_1 + 1)l(p + l_1 - 1)}{[(q + l_1 - 1)p^p - (p + l_1 - 1)1]}.
$$

By the choice of $l_1$, we have $0 < \phi_3 < 1$. Hence, using the Young inequality, for every $r_3 > 0$ and $\varepsilon_2 > 0$, we obtain

$$
||u||_{l_1+1} \leq C \frac{\lambda^p}{\lambda^p + 1} \left(\varepsilon_2 ||u||_{l_1+1}^i + C \frac{\lambda^p}{\lambda^p + 1} ||u||_{q+l_1-i}^p \right).
$$

We now choose

$$
r_3 = \frac{(q + l_1 - 1)(p + l_1 - 1)}{\phi_3(p + l_1 - 1) + (q + l_1 - 1)(1 - \phi_3)}
$$

and we notice that

$$
\frac{\partial_3 r_3(p+l_1-i)}{(p+1-i) - (1-\phi_3)_r_3} = q + l_1 - 1.
$$

That means

$$
||u||_{l_1+1} \leq C \frac{\lambda^p}{\lambda^p + 1} \left(\varepsilon_2 ||u||_{l_1+1}^i + C \frac{\lambda^p}{\lambda^p + 1} ||u||_{q+l_1-i}^p \right).
$$

We choose

$$
\varepsilon_2 = \left[\frac{1}{\beta} \left(\frac{m_0 l_1 p^p}{p + l_1 - 1} - \frac{\lambda p L(p, \Omega)}{p + l_1 - 1} - \frac{\lambda p L(p, \Omega)}{p + l_1 - 1} \right) \right]^{\frac{\lambda^p}{\lambda^p + 1}}.
$$

By (3.34) and (3.37) and $\lambda < P_5$, we obtain

$$
\frac{1}{l_1 + 1} \frac{d}{dt} ||u||_{l_1+1}^i + C_9 ||u||_{l_1+1}^i \leq 0,
$$

where

$$
C_9 = C \frac{\lambda^p}{\lambda^p + 1} \left[\frac{1}{\beta} \left(\frac{m_0 l_1 p^p}{p + l_1 - 1} - \frac{\lambda p L(p, \Omega)}{p + l_1 - 1} - \frac{\lambda p L(p, \Omega)}{p + l_1 - 1} \right) \right]^{\frac{\lambda^p}{\lambda^p + 1}}.
$$

and $C_9 > 0$, which implies that

$$
\begin{aligned}
||u||_{l_1+1} &\leq \left||u_0||_{l_1+1}^{l_1+1-r_3} + (r_3 - l_1 - 1)C_9 t \right||_{l_1+1}^{l_1+1-r_3}, & t \in [0, T_6), \\
||u||_{l_1+1} &\equiv 0, & t \in [T_6, \infty),
\end{aligned}
$$

where

$$
T_6 = \frac{1}{(l_1 + 1 - r_3)C_9} ||u_0||_{l_1+1}^{l_1+1-r_3}.
$$

The proof is completed.

**Proof of Theorem 1.4**
We consider first the case $r < 2$. When $2N/(N + 2) < p < 2$, multiplying (1.1) by $u$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + M(\|u\|^p)\|u\|^p = \lambda \int_\Omega |u|^{p-1} \ln |u| dx - \beta \int_\Omega u^q dx.$$  \hfill (3.38)

Similar to the Theorem 1.2, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + (m_0 - \lambda L(r, \Omega) R^{p-r}) \|u\|^p \leq \lambda \ln(\Omega) \|\Omega\|^{\frac{2(p-1)}{p}} \|u\|_2^2 - \beta \|u\|_q^2.$$  \hfill (3.39)

Taking $\beta^* = 0$ in Lemma 2.3, we have

$$\|u\|_2 \leq \|u\|_p^{(1-\theta_4)}\|u\|_{q}^{\theta_4} \leq C_{p_5}^{(1-\theta_4)}\|u\|_p^{(1-\theta_4)}\|u\|_{q}^{\theta_4},$$  \hfill (3.40)

where $\theta_4 = \frac{(p_0')^p}{(p_0'-q)^2}$. Then, taking into account that $\theta_4 \in (0, 1)$, we can apply Young’s inequality: for every $r_4 > 0$ and $\varepsilon_3 > 0$, we have

$$\|u\|_2^r \leq C_{r_4}^{\varepsilon_3}(\|u\|_p^{\varepsilon_3} + \|u\|_q^{\varepsilon_3}).$$  \hfill (3.41)

Taking

$$r_4 = \frac{qp}{q(1-\theta_4) + \theta_4 p},$$

then $\frac{\theta_4 p}{p-\frac{1}{(1-\theta_4)}} = q$. By (3.39), (3.41), Lemma 2.5 and $\lambda < P_0$, and let $\varepsilon_3 = \left(\frac{m_0 - L(r, \Omega) R^{p-r}}{\beta} \right)^{\frac{1}{q(1-\theta_4)}}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_2^2 + C_{10} \|u\|_2^2 \leq 0,$$  \hfill (3.42)

where

$$C_{10} = \beta \varepsilon_3^{\frac{1}{q(1-\theta_4)}} = \frac{\beta}{C_{r_4}^{(1-\theta_4)r_4}} R^{p-r} - \lambda \ln(\Omega) \|\Omega\|^{\frac{2(p-1)}{p}} > 0,$$

which implies that our result holds.

When $1 < p \leq 2N/(N + 2)$, we multiply (1.1) by $u^l$ ($l_1 > l \geq 1$), by Lemma 2.4 and Lemma 2.7, we obtain

$$\frac{1}{l_1 + 1} \frac{d}{dt} \|u\|_{l_1+1}^2 + \frac{m_0 l_1 p^p}{(p + l_1 - 1)^p} \|u\|_{l_1+1}^{l_1+p-1} \|u\|_{l_1+1}^p \leq \lambda \frac{1}{e\sigma} \|u\|_{l_1+1}^r - \beta \|u\|_{q+l_1-1}^{q+l_1-1},$$  \hfill (3.43)

where $r_5 = l_1 + r + \sigma - 1$. Using the Holder inequality and (3.37), and we choose

$$\varepsilon_2 = \left[\frac{m_0 l_1 p^p}{\beta(p + l_1 - 1)^p} \right]^{\frac{1}{p(\frac{1}{p}\frac{l_1+1}{l_1+1} - 1)}},$$

then, we have

$$\frac{1}{l_1 + 1} \frac{d}{dt} \|u\|_{l_1+1}^2 + \beta \varepsilon_2^{\frac{1}{q(1-\theta_4)}} \|u\|_{l_1+1} \leq \lambda \frac{1}{e\sigma} \|u\|_{l_1+1}^{r_5} - \beta \|u\|_{q+l_1-1}^{r_5},$$  \hfill (3.44)

for $0 < \sigma < 2 - r$. After performing some simple calculations, we finally obtain

$$\frac{1}{l_1 + 1} \frac{d}{dt} \|u\|_{l_1+1}^2 + C_{11} \|u\|_{l_1+1}^{r_5} \leq 0,$$  \hfill (3.45)
where
\[ C_{11} = \beta e_2^{-(\phi_3-1)\nu_3} C_{10}^{(\phi_3-1)\nu_3} p \nu_3 - \lambda C, \quad C = \frac{1}{e^{C_1}} \left| \Omega \right|^{\frac{1}{p' + 1}} C_{10}^{\nu_3} R^{(q-1)\nu_3} \].

Note that \( \lambda < P_3 \), then \( C_{11} > 0 \), which implies that
\[
\begin{align*}
\frac{\|u\|_{l_{i+1}}}{\|u\|_{l_{i+1}} - 1} & \leq \left[ \|u_0\|_{l_{i+1}} + (r_3 - l_1 - 1)C_{11} t \right]^{\frac{1}{1 + (r_3 - l_1 - 1)}}, & t \in [0, T_7), \\
\|u\|_{l_{i+1}} & \equiv 0, & t \in [T_7, \infty),
\end{align*}
\]
where
\[
T_7 = \frac{1}{(l_1 + 1 - r_3)C_{11}} \|u_0\|_{l_{i+1}}^{l_1 + 1 - r_3}.
\]

The proof is completed. \( \square \)

**Proof of Theorem 1.5**

(i) By the definition of \( E(u) \) and \( I(u) \), we obtain
\[
E(u) = \frac{1}{p} I(u) + \frac{1}{p} b(1 - \theta)\|u\|^p + \beta \|u\|^q + \lambda \frac{1}{p^2} \|u\|^p. \tag{3.46}
\]
Choosing \( \nu = u_i \) in Definition 1.1, we have
\[
\int_\Omega u_i u_i dx = -M(\|u\|^p)(u, u_i) + \lambda \int_\Omega u_i |u|^{r-2}u \ln |u| dx - \beta \int_\Omega |u|^{q-2}u_i dx. \tag{3.47}
\]
Note that
\[
\frac{d}{dt} E(u) = \frac{1}{p} \frac{d}{dt} (a\|u\|^p + b\|u\|^q) - \lambda \int_\Omega |u|^{r-2}u_i \ln |u| dx + \beta \int_\Omega |u|^{q-2}u_i dx. \tag{3.48}
\]
By (3.47) and (3.48), we obtain
\[
\frac{d}{dt} E(u) + \int_\Omega u_i u_i dx = 0,
\]
which implies that
\[
E(u) = E(u_0) - \int_0^t \|u_\tau\|^2 d\tau. \tag{3.49}
\]
Setting \( \Gamma(t) = \frac{1}{2} \int_\Omega |u(x, t)|^2 dx \), then we have
\[
\Gamma'(t) = \int_\Omega u_i u_i dx = -I(u). \tag{3.50}
\]
By (3.46) and (3.49) and (3.50), we obtain
\[
\Gamma'(t) = -pE(u) + b(1 - \theta)\|u\|^p + \frac{1}{q} \beta (p - q)\|u\|^q + \lambda \frac{1}{p^2} \|u\|^p + \lambda \frac{1}{p} \|u\|^p + p \int_0^t \|u_\tau\|^2 d\tau.
\]
Since \( p > q \), then \( \frac{1}{q}u(p-q)||u||^p_q > 0 \), we obtain

\[
\Gamma'(t) \geq -pE(u_0) > 0.
\]

By a simple calculation, we get

\[
||u||^2_2 \geq -2pE(u_0)t + 2||u_0||^2_2, \text{ for all } t > 0,
\]

which implies that our result holds.

(ii) Here, we only prove the case of \( E(u_0) < h \), and the proof of \( E(u_0) = h \) is similar. Choosing \( \nu = u \) in Definition 1.1, we have

\[
\frac{1}{2} \frac{d}{dt}||u||^2_2 + M(||u||^p)||u||^p = \int_\Omega |u|\ln |u| dx - \beta||u||^q_q,
\]

namely

\[
\frac{1}{2} \frac{d}{dt}||u||^2_2 + I(u) = 0.
\]

Next, taking \( \nu = u_t \) in Definition 1.1 and integrating with respect to time from 0 to \( t \), we have

\[
\int_0^t ||u_t||^2_2 dt + E(u(t)) = E(u_0) < h, \text{ for } t > 0.
\]

We claim that \( u(x,t) \in W \) for any \( t > 0 \). If it is false, there exists a \( t_0 \in \mathbb{R}^+_0 \{0\} \) such that \( u(t_0) \in \partial W \), which implies

\[
I(u(x,t_0)) = 0 \text{ or } E(u(x,t_0)) = h.
\]

From (3.53), \( E(u(t_0)) = h \) is not true. So \( u(t_0) \in \mathcal{N} \), then by the definition of \( h \) in (1.9), we have \( E(u(t_0)) \geq h \), which also contradicts with (3.53). Hence, \( u(t_0) \in W \). By (3.52) and \( u(t) \in W \) for all \( t > 0 \), we obtain

\[
||u||^2_2 \leq ||u_0||^2_2.
\]

\[\square\]

Remark 2 Compared with problem (1.4), we not only discuss the extinction of weak solutions of problem (1.1) with logarithmic nonlinearity, but also prove that the weak solutions are globally bounded and blow up at infinity.

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Conflict of interest

The authors declare no potential conflict of interests.

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