Homodyne measure of nonclassicality for light

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We propose a legitimate and easily computable measure of nonclassicality for the states of electromagnetic field based on the standard deviation in the measurement of the homodyne rotated quadrature operator. The proposed measure is the nonclassical area projected by the optical tomogram of the quantum state of light on the optical tomographic plane. If the nonclassical area projected by the optical tomogram of a quantum state is greater than zero, the state is strictly nonclassical, and the area is zero for the classical state. It is also noted that the nonclassical area of a quantum state increases with an increase in the strength of nonclassicality inducing operations on the state such as the squeezing, photon addition, etc. We have tested the validity of the nonclassical area measure by calculating the same for certain well-known nonclassical states and found that essential features of the nonclassicality shown by the states are captured in the nonclassical area measure. We have also shown that the nonclassical area measure is robust against environment-induced decoherence of the states. Nonclassical area projected by the optical tomogram of a quantum state of light is experimentally tractable using the balanced homodyne detection of the quadrature operator of the field, avoiding the reconstruction of the density matrix or the quasiprobability distribution of the state.

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I. INTRODUCTION

Properties of the coherent state of the radiation field can have analogous descriptions in the classical electrodynamics and are considered to be the most classical state of the field [1]. In contrast, the quantum states of the field which possesses certain exotic features that cannot have classical correspondence are referred to as the nonclassical states of light [2]. Precisely, an arbitrary quantum state of light is said to be nonclassical if the Glauber-Sudarshan $P$-function of the state is highly singular or takes negative values somewhere in the phase space [1,2]. Nonclassical states have received great theoretical attention as well as experiment interest over the past century primarily because of their use in technological applications ranging from gravitational wave detectors to quantum information protocols. Numerous varieties of nonclassical states have been experimentally prepared and were characterized by the optical homodyne tomography [3]. Significant theoretical efforts have been taken to study the various features shown by the nonclassical states of light [2]. One can say that the degree of nonclassicality of a state is directly related to the nonclassical features. Therefore, the state’s amount of nonclassicality or quantumness is a primary parameter to be known before using a nonclassical state for technological applications.

There are several theoretical probes to identify a nonclassical state, and various measures have been proposed to quantify the degree of nonclassicality associated with an arbitrary quantum state of light. Mandel’s $q$-parameter was used to characterize the deviation of photon-number statistics of the nonclassical states from the Poissonian photon-number statistics shown by the coherent state of the electromagnetic field [4]. The first attempt to quantify the degree of nonclassicality of a quantum state of light was proposed in terms of the nonclassical distance, which is the trace norm distance between the state and the set of all classical states [4]. Hereafter a variety of distance-based measures have been proposed based on the different types of metrics defining the distance [7,12]. Since the evaluation of these measures requires optimization over the set of all classical states, distance-based measures are not easily computable.

The entanglement potential of the state is another quantifier of the nonclassicality of state, which measures the entanglement created by a beam splitter in terms of the logarithmic negativity [13,14]. A quantifier of nonclassicality and entanglement related to the number of quantum superpositions of classical states was also introduced [15]. Quasiprobability distributions and the characteristic function of the field contain information about the degree of nonclassicality of the state. A nonclassicality indicator based on the volume of the negative part of the Wigner function in phase space was reported [16]. Since the squeezed states of light have a positive definite Wigner function, the nonclassicality of such states cannot be characterized by the negativity of the Wigner function. Convolution transformation between Glauber-Sudarshan $P$-function and Husimi $Q$ function was used to define a universal nonclassicality measure called nonclassical depth [17]. It can be viewed as the amount of thermal noise required to destroy whatever the nonclassical effects present in the quantum state. Recently, the characteristic function of the state has been used to quantify the nonclassicality of the state of light [18]. An opera-
tional measure of nonclassicality based on the negativity of an observable whose classical counterpart is positive semidefinite was introduced \[\ref{19}\]. Apart from the previously mentioned quantifiers, Schmidt-number witnesses have also been used to express the nonclassicality of the states \[\ref{21,22}\].

Nonclassicality measures mentioned so far are not directly related to the experiment because the experimental estimation of these measures requires the knowledge of the density matrix or the quasiprobability distribution of the state, which is not a directly measurable quantity. One way to experimentally determine the density matrix or the quasiprobability distribution is using the optical homodyne tomography \[\ref{4}\]. In this technique, a series of homodyne measurements of the rotated quadrature operator on an ensemble of identically prepared states generates an optical tomogram of the state \[\ref{23,24}\]. The optical tomogram of the state is a primary object characterizing the state of light. It contains all the information about the state, including the amount of nonclassicality contained in the field. This optical tomogram is used further to reconstruct the density matrix or the quasiprobability distributions of the quantum state by numerical methods. The systematic and statistical errors associated with measurement may propagate during the reconstruction process and lead to the loss of information contained in the state. It was shown that the properties of a quantum state of light could be inferred directly from the optical tomogram of the state \[\ref{24}\].

In this work, we have used the optical tomogram of the state to quantify the nonclassicality of the quantum states of an electromagnetic field. This paper aims to introduce an easily computable nonclassicality measure that is directly measurable using homodyne optical tomography. The proposed nonclassicality measure is based on the standard deviation in measuring the homodyne rotated quadrature operator in a quantum state. It can be considered as an effective area spanned by the optical tomogram of single-mode nonclassical states and the effect of decoherence on the nonclassical area measure. The definition of nonclassicality area measure for a two-mode field and the explicit calculations of the same for various two-mode states are given in Sec. \[\ref{V}\]. The effect of environment-induced decoherence of the two-mode state on the nonclassical area is also discussed. Section \[\ref{VI}\] generalizes the results obtained for single-mode and two-mode states to the case of a generic \(p\)-mode state of the electromagnetic field. Section \[\ref{VI}\] summarizes the main results of the paper.

II. OPTICAL TOMOGRAPHIC REPRESENTATION

The homodyne rotated quadrature operator for a single-mode electromagnetic field is given by

\[
\mathfrak{X}_\theta = \frac{1}{\sqrt{2}} \left( a e^{-i\theta} + a^\dagger e^{i\theta} \right),
\]

(1)

where \(\theta\) represents the phase of the local oscillator in homodyne detection arrangement \((\theta \in [0,2\pi])\), and \(a\) and \(a^\dagger\) are the ladder operators of the single-mode field, respectively. The optical tomogram of an arbitrary state \(|\psi\rangle\) is defined as the probability distribution of the rotated quadrature operator \(\mathfrak{X}_\theta\) in the state \(|\psi\rangle\). If \(|X_\theta,\theta\rangle\) is the eigenvector of the Hermitian operator \(\mathfrak{X}_\theta\) with eigenvalue \(X_\theta\) \[\ref{27}\], the optical tomogram a quantum state with density matrix \(\rho\) is given by \[\ref{24}\]:

\[
\omega(X_\theta,\theta) = \langle X_\theta,\theta | \rho | X_\theta,\theta \rangle.
\]

(2)

For a pure state with wave vector \(|\psi\rangle\), the Eq. (2) reduces to

\[
\omega(X_\theta,\theta) = |\langle X_\theta,\theta | \psi \rangle|^2,
\]

(3)

where \(\langle X_\theta,\theta | \psi \rangle\) is the quadrature representation of the state \(|\psi\rangle\). The normalization condition of the optical tomogram \(\omega(X_\theta,\theta)\) is given by

\[
\int_{-\infty}^{\infty} dX_\theta \omega(X_\theta,\theta) = 1.
\]

(4)

The optical tomogram is having the following symmetry property:

\[
\omega(X_\theta,\theta + \pi) = \omega(-X_\theta,\theta).
\]

(5)
The $n^{\text{th}}$ moments of the operator $X_\theta$ in an arbitrary quantum state can be calculated using the optical tomogram $\omega (X_\theta, \theta)$ of the corresponding state as

$$\langle X^n_\theta \rangle = \int_{-\infty}^{\infty} dX_\theta X^n_\theta \omega (X_\theta, \theta).$$  \hfill (6)

A plane with $X_p$- and $\theta$-axes defined is used to visualize the nonclassical features associated with the quantum state of the light in terms of the optical tomogram of the degree of nonclassicality of an arbitrary state. The standard deviation $\Delta X$ in the measurement of the rotated quadrature operator $X_\theta = X_\theta + iY_\theta$ can be written as

$$\Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2}.$$  \hfill (7)

where $\langle X \rangle$ is the eigenvalue of the $p$-mode rotated quadrature operator $X_{\theta_1, \theta_2, ..., \theta_p}$ with eigenvalue $X_{\theta_1, \theta_2, ..., \theta_p}$. The $n^{\text{th}}$ moment of the operator $X_{\theta_1, \theta_2, ..., \theta_p}$ for a generic $p$-mode state now becomes

$$\langle X^n_{\theta_1, \theta_2, ..., \theta_p} \rangle = \int_{-\infty}^{\infty} dX_{\theta_1} dX_{\theta_2} \cdots dX_{\theta_p} X^n_{\theta_1, \theta_2, ..., \theta_p} \times \omega (X_{\theta_1}, \theta_1; X_{\theta_2}, \theta_2; \ldots; X_{\theta_p}, \theta_p).$$  \hfill (8)

Optical tomograms of variety of nonclassical states have been theoretically investigated. Recently, it has been shown that the signatures of nonclassical effects such as quantum wave packet revivals, quadrature squeezing, and entanglement can be captured in the optical tomogram of the state. Quantitative estimation of the degree of nonclassicality of an arbitrary quantum state of the light in terms of the optical tomogram of the state has not been reported so far.

### III. NONCLASSICAL AREA FOR A SINGLE-MODE FIELD

Consider an arbitrary pure single-mode quantum state $|\psi\rangle$ whose optical tomogram is given by $\omega (X_\theta, \theta)$. We use the amount of quadrature fluctuation in the measurement of the rotated quadrature operator to quantify the nonclassicality associated with the state $|\psi\rangle$. Specifically, the quantity of interest is the standard deviation in the measurement of the rotated quadrature operator $X_\theta$ in the state $|\psi\rangle$, defined as

$$\Delta X_\theta = \sqrt{\langle X^2_\theta \rangle - \langle X_\theta \rangle^2}.$$  \hfill (9)

Deviation in the measurement of $X_\theta$ reflects the amount of spread of optical tomogram of the state $|\psi\rangle$ on the optical tomographic plane for a particular $\theta$.

Measurements of two rotated quadrature operators with $\theta'$ values differ by $\pi/2$ are limited by the Heisenberg uncertainty relation

$$\Delta X_{\theta} \Delta X_{\theta'+\pi/2} \geq \frac{1}{2},$$  \hfill (10)

as they are conjugate variables. In the expression and the rest of this manuscript, we have taken $\hbar = 1$. A schematic diagram of the distribution of standard deviation $\Delta X_\theta$ with respect to $\theta$ (shaded region in the optical tomographic plane) is given in Fig. [1] The uncertainty relation (given in Eq. (11)) equalizes for all $\theta$ values only for the classical states of the electromagnetic field. Hence, it turns out that the area projected the distribution of standard deviation $\Delta X_\theta$ on the optical tomographic plane (the shaded region) will have a lower bound for the classical states of the field.

Let us evaluate the lower bound of the area projected by the distribution of standard deviation $\Delta X_\theta$ on the optical tomographic plane for a single-mode classical state. We divide the optical tomographic plane into horizontal strips having minimal width $\Delta \theta$ so that the value of $\Delta X_\theta$ is constant throughout the strip. The symmetry property of the optical tomogram given in Eq. (11) allows us to confirm that the area spanned by the standard deviation distribution on the optical tomographic plane in the region between $\theta = 0$ to $\theta = 2\pi$ is two times the area of spanned by the same in the region between $\theta = 0$ to $\theta = \pi$. Again, we divide the region between $\theta = 0$ to $\theta = \pi$ into two halves. One between $\theta = 0$ to $\theta = \pi/2$ and other between $\theta = \pi/2$ to $\theta = \pi$. For each horizontal strip in the first half, let say with $\theta = \theta'$, there will be a corresponding strip in the second half with $\theta = \theta' + \pi/2$. The sum of the shaded area spanned by these strips are
The quantity \( \langle \psi | \theta \rangle \) subjected to the constraint given in Eq. (11) is always greater than \( \sqrt{2} \pi \). Therefore, the difference \( \sigma (|\psi\rangle) - \sqrt{2} \pi \) can be taken as a measure of nonclassicality for the single-mode states of the electromagnetic field. If the projection of the optical tomogram of a particular single-mode quantum state occupies an area more than \( \sqrt{2} \pi \) on the optical tomographic plane, the state is strictly a nonclassical state. In other words, a single-mode quantum state of light \( |\psi\rangle \) is said to be a nonclassical state if the nonclassical area

\[
\int_0^{2\pi} d\theta \Delta X_{\theta} = 2 \times \int_0^{\pi/2} d\theta' \left( \Delta X_{\theta'} + \Delta X_{\theta'+\pi/2} \right) .
\]  

(12)

The minimum value of the quantity inside bracket in Eq. (12) subjected to the constraint given in Eq. (11) is a constant for the classical states:

\[
(\Delta X_{\theta} + \Delta X_{\theta+\pi/2})_{\text{min}} = \sqrt{2}.
\]  

(13)

Therefore, the lower bound of the above integral is evaluated by plugging Eq. (13) in Eq. (12), and is calculated to be \( \sqrt{2} \pi \). This lower bound is a constant for all the pure single-mode classical states of the field. Since the value of the quantity inside bracket in Eq. (12) is always greater than \( \sqrt{2} \pi \) for the nonclassical states, the area projected by the standard deviation distribution will always be greater than \( \sqrt{2} \pi \) for all single-mode nonclassical states of the field. Hence, we can use the Eq. (12) to define a quantity

\[
\sigma (|\psi\rangle) = \int_0^{2\pi} d\theta \Delta X_{\theta},
\]  

(14)

in order to characterize the nonclassicality or quantumness of an arbitrary pure single-mode quantum state \( |\psi\rangle \). The quantity \( \sigma (|\psi\rangle) \) is the effective area projected by the optical tomogram of the single-mode quantum state \( |\psi\rangle \) on the optical tomographic plane. As mentioned earlier, the effective area \( \sigma (|\psi\rangle) \) has a lower bound for the classical states, which is \( \sqrt{2} \pi \). The projection of the optical tomogram of a particular single-mode quantum state occupies an area more than \( \sqrt{2} \pi \) on the optical tomographic plane, the state is strictly a nonclassical state. In other words, a single-mode quantum state of light \( |\psi\rangle \) is said to be a nonclassical state if the nonclassical area \( \sigma (|\psi\rangle) - \sqrt{2} \pi \) is a constant for the vacuum state \( |\psi\rangle \).

Let us analyze the effective area \( \sigma (|\psi\rangle) \) for the pure single-mode quantum states of the electromagnetic field. First, consider the case of a coherent state \( |\alpha\rangle \) (The eigenstate of the annihilation operator, \( a |\alpha\rangle = \alpha |\alpha\rangle \), where \( \alpha = |\alpha| e^{i\eta} \)). The optical tomogram of the coherent state \( |\alpha\rangle \) is given by

\[
\omega_{|\alpha\rangle} (X_{\theta}, \theta) = \frac{1}{\sqrt{\pi}} \exp \left[ - \left( \frac{X_{\theta} - \sqrt{2} |\alpha| \cos (\eta - \theta)}{\sqrt{\pi}} \right)^2 \right].
\]  

(15)

The projection of optical tomogram \( \omega_{|\alpha\rangle} (X_{\theta}, \theta) \) on the optical tomographic plane displays a structure with single sinusoidal strand \( [29] \). The standard deviation in the measurement of the homodyne rotated quadrature operator in the coherent state \( |\alpha\rangle \) is calculated to be \( \Delta X_{\theta} = 1/\sqrt{2} \). It is straightforward to calculate the effective area spanned by the optical tomogram of coherent state \( |\alpha\rangle \) as \( \sigma (|\alpha\rangle) = \sqrt{2} \pi \), which is independent of the value of \( \alpha \). As \( \alpha \to 0 \), the coherent state \( |\alpha\rangle \) reduces to the vacuum state \( |0\rangle \). The optical tomogram of the vacuum state \( |0\rangle \) is obtained as \( \omega_{|0\rangle} (X_{\theta}, \theta) = \exp (-X_{\theta}^2)/\sqrt{\pi} \), which is a structure with single straight strand in the optical tomographic plane \( [28] \). The standard deviation of the rotated quadrature and effective area for the vacuum state \( |0\rangle \) is same as that for the coherent state \( |\alpha\rangle \). That is, \( \Delta X_{\theta} = 1/\sqrt{2} \) and \( \sigma (|0\rangle) = \sqrt{2} \pi \). Therefore, the optical tomograms of the single-mode classical state of electromagnetic field span a constant effective area of \( \sqrt{2} \pi \) on the optical tomographic plane. As per the derivation of the nonclassical area measure discussed above, a nonzero value of the nonclassical area for a given single-mode quantum state is necessary to say that the state is nonclassical.

A nonclassicality-inducing operation on the coherent state or the vacuum state yields a nonclassical state for which the nonclassical area spanned by its optical tomogram must be greater than zero. Furthermore, the nonclassical area of a particular nonclassical state must increase with an increase in the strength of the nonclassicality-inducing operations such as the squeezing, photon addition, etc. By definition, the amount of quantumness associated with a state is independent of the displacement operations and rotations in the phase space. Therefore, the nonclassical area projected by the optical tomogram of the displaced and rotated quantum states must be the same as that of the original state. In the following, we check these key aspects and hence the validity of the nonclassical area measure by calculating...
the nonclassical area projected by the optical tomograms of different classes of nonclassical states of light.

A. Fock state

A simplest state of the field to start with would be an \( n \)-photon Fock state \( |n\rangle \), where \( n = 0, 1, 2, \ldots, \infty \). The optical tomogram of the Fock state \( |n\rangle \) is given by

\[
\omega_{|n\rangle} (X_\theta, \theta) = Q_{|n\rangle} (X_\theta, \theta) Q^*_{|n\rangle} (X_\theta, \theta),
\]

where \( Q_{|n\rangle} (X_\theta, \theta) \) is the quadrature representation of the Fock state \( |n\rangle \) [21]:

\[
Q_{|n\rangle} (X_\theta, \theta) = \langle X_\theta, \theta | n \rangle = e^{-X_\theta^2/2} H_n (X_\theta) e^{-i n \theta} \frac{1}{\sqrt{\pi} 2^{n/2} \sqrt{n!}}.
\]

Here \( H_n (\cdot) \) is the Hermite polynomial of order \( n \). The standard deviation in the measurement of the homodyne quadrature operator in the \( n \)-photon Fock state is \( \sqrt{\bar{n} + 1/2} \). Therefore, the optical tomogram given in Eq. (16) span an effective area

\[
\sigma (|n\rangle) = \sqrt{2(2n + 1)/\bar{n}},
\]

on the \( X_\theta - \theta \) plane. The case \( n = 0 \) corresponds to the vacuum state \( |0\rangle \), and the Eq. (18) once again shows that the nonclassical area \( \sigma (|0\rangle) = \sqrt{2\pi} \) spanned by the optical tomogram of the vacuum state \( |0\rangle \) is zero. But for the Fock state \( |n\rangle \) with \( n \neq 0 \), Eq. (16) tells that the degree of nonclassicality associated with the state \( |n\rangle \) is always greater than zero, that is \( \sigma (|n\rangle) > \sqrt{2\pi} > 0 \). It also demonstrates that the amount of nonclassicality contained in the state \( |n\rangle \) increases monotonically with the photon number \( n \) (see Fig. 2). Nonclassicality indicator based on the volume of the negative part of the Wigner function of the Fock state \( |n\rangle \) showed that the amount of nonclassicality of the state could be approximated as \( \sqrt{2\pi} \) for a large value of \( n \) [19]. The divergence of the amount of nonclassicality of the Fock state \( |n\rangle \) with the increase in the photon number \( n \) has also been demonstrated using the nonclassicality measures such as entanglement potential [19], the degree of nonclassicality based on characteristic function [18], quantifier based on the quantum superposition principle [18], etc.

B. Squeezed state

The action of the squeezing operator \( S (\xi) = \exp \left( (\xi^* a^2 - \xi a^2) / 2 \right) \) on the vacuum state \( |0\rangle \) generates a squeezed vacuum state which is defined as

\[
|\xi\rangle = S (\xi) |0\rangle,
\]

where \( \xi = re^{i\phi} \) Fock state representation of the state \( |\xi\rangle \) is given by [22]

\[
|\xi\rangle = \sum_{n=0}^{\infty} P_{2n} (\xi) |2n\rangle,
\]

where

\[
P_{2n} (\xi) = (-1)^n \sqrt{(2n)!} e^{in\phi} (\tanh r)^n / 2^{n/2} n!.
\]

Substituting Eq. (20) in Eq. (3), the optical tomogram of the squeezed state on the \( X_\theta - \theta \) plane is obtained as

\[
\omega_{|\xi\rangle} (X_\theta, \theta) = \sum_{n,n'=0}^{\infty} P_{2n} (\xi) P^*_{2n'} (\xi) Q_{|2n\rangle} (X_\theta, \theta) \times Q^*_{|2n'\rangle} (X_\theta, \theta).
\]

Standard deviation in the measurement of the homodyne quadrature operator in the squeezed vacuum state \( |\xi\rangle \) is calculated as

\[
\Delta X_\theta = \sqrt{(\cosh (2r) - \sinh (2r) \cos (\delta - 2\theta)) / 2}.
\]

Using Eqs. (20) and (16), the effective area spanned by the optical tomogram of the squeezed state on the \( X_\theta - \theta \) plane is evaluated to be

\[
\sigma (|\xi\rangle) = e^{-r} \sqrt{2} [E (2\pi - \delta/2 | k^2) + E (\delta/2 | k^2)],
\]

where \( k^2 = (1 - e^{4r}) \) and

\[
E (\phi | k^2) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \lambda} d\lambda
\]

is the incomplete elliptical integral of the second kind. The solid line in Fig. 2 shows the nonclassical area \( \sigma (|\xi\rangle) = \sqrt{2\pi} \) spanned by the optical tomogram of squeezed vacuum state on the optical tomographic plane as a function of the parameter \( r \). The degree of nonclassicality of the state \( |\xi\rangle \) increases monotonically as a function of \( r \) and is found to be independent of the value of \( \delta \). In the limit \( r \to 0 \), the nonclassical area of the state...
The degree of nonclassicality associated with the states $|\xi\rangle$ increases monotonically with the value of $r$, the argument of the squeezing parameter. This is evident from the area of the state, which becomes zero which is the value corresponding to the vacuum state $|0\rangle$.

When squeezing operator $S(\xi)$ acts on the $n$-photon Fock state, it generates the squeezed Fock state $|n,\xi\rangle$:

$$|n,\xi\rangle = S(\xi) |n\rangle = \sum_{j=0}^{\infty} C_j |j\rangle .$$  \hfill (26)

Here the Fock state expansion coefficients are given by [36]

$$C_j = \left( \frac{\nu}{2\mu} \right)^{j/2} \frac{\mu^j \sqrt{j!}}{n^j} \prod_{i=0}^{\min(n,j)} \left( \frac{n}{i} \right) \left( \frac{2\mu}{j-i+1} \right)^{\nu^2/2} \left( \frac{1}{j!} \right) \times \left( \frac{-\nu^2}{2\mu} \right)^{(n-j)/2} H_{j-i}(0) H_{n-i}(0)$$  \hfill (27)

where $\mu = \cosh(r)$, and $\nu = e^{i\delta} \sinh(r)$. For $n=0$, the state given in Eq. (26) retrieves the squeezed vacuum state $|\xi\rangle$. Using Eq. (26), the optical tomogram of the squeezed Fock state $|n,\xi\rangle$ is calculated as

$$\omega_{|n,\xi\rangle} = \sum_{j,j'=0}^{\infty} C_j C_{j'} Q_{|j\rangle} (X_\theta, \theta) Q^*_{|j\rangle} (X_\theta, \theta) .$$  \hfill (28)

For the state $|n,\xi\rangle$, the standard deviation in the measurement of the homodyne quadrature operator is obtained as

$$\Delta X_\theta = \frac{\sqrt{(n+1/2)}}{\sqrt{\cosh(2r) - \sinh(2r) \cos(\delta - 2\theta)}} .$$  \hfill (29)

Substituting Eq. (29) in Eq. (14), the effective area spanned by the optical tomogram of the squeezed Fock state $|n,\xi\rangle$ is found to be

$$\sigma (|n,\xi\rangle) = \sqrt{(n+1/2)} e^{-r} \left[ E(2\pi - \delta/2 | k^2 \right] + E \left( \delta/2 | k^2 \right]$$  \hfill (30)

It is the product of the effective area spanned by the optical tomogram of the Fock state $|n\rangle$ and that corresponds to a squeezed state $|\xi\rangle$, scaled down by a factor of $\sqrt{2\pi}$. That is,

$$\sigma (|n,\xi\rangle) = [\sigma (|n\rangle) \times \sigma (|\xi\rangle)] / \sqrt{2\pi} .$$  \hfill (31)

The degree of nonclassicality of the state $|n,\xi\rangle$ is then become $\sigma (|n,\xi\rangle) - \sqrt{2\pi}$. Variation of the nonclassical area spanned by the optical tomogram of the squeezed Fock states $|n,\xi\rangle$ with $n = 1$ (dashed line), 5 (dotted line), 10 (dash-dot line), are shown in Fig. 3. In general, the nonclassical area of the squeezed Fock state $|n,\xi\rangle$ increases monotonically with the parameter $r$, for all values of $n$. Figure 3 also shows that, in the limit $r \to 0$, the nonclassical area of the state $|n,\xi\rangle$ gives the value corresponding to the Fock state $|n\rangle$. It is worth noting that the state’s nonclassicality is independent of the value of $\delta$.

The combined action of the displacement operator $D(\alpha) = \exp \left( (\alpha a - a^\dagger \alpha^*) / 2 \right)$ followed by $S(\xi)$ on the vacuum state generates the squeezed coherent state $|\alpha,\xi\rangle$, for which the amount of nonclassicality of the state is same as that for the squeezed state $|\xi\rangle$. This feature is essentially captured in the nonclassical area measure as the effective area spanned by the optical tomogram of the state $|\alpha,\xi\rangle$ on the optical tomographic plane is calculated to be $\sigma (|\alpha,\xi\rangle) = \sigma (|\xi\rangle)$. It supports that the displacement operation does not induce any additional nonclassicality in the squeezed state $|\xi\rangle$. Furthermore, one can show that, $\sigma (|\alpha, n, \xi\rangle) = \sigma (|n,\xi\rangle)$. Hence the degree of nonclassicality of the displaced squeezed Fock state $|\alpha, n, \xi\rangle = D(\alpha) S(\xi) |n\rangle$, is same as that of the squeezed Fock state $|n,\xi\rangle$. Therefore, phase space rotations and displacements of the state will not increase the value of nonclassical area measure.

C. Photon-added coherent states

The addition of photons to a coherent state $|\alpha\rangle$ induces nonclassicality in the resultant states. The nonclassical state generated by the addition of $m$ photons to the coherent field $|\alpha\rangle$ is called an $m$-photon-added coherent state $|\alpha, m\rangle$, defined as [37, 38]

$$|\alpha, m\rangle = N_{\alpha, m} |\alpha\rangle |m\rangle ,$$  \hfill (32)

where $N_{\alpha, m}$ is the normalization constant and $|\alpha|^2$ is the mean number of photons in the coherent state $|\alpha\rangle$. The optical tomogram of the $m$-photon-added coherent states $|\alpha, m\rangle$ obtained by Substituting the Fock state representation of $|\alpha, m\rangle$ in Eq. (28)

$$\omega_{|\alpha, m\rangle} (X_\theta, \theta) = \frac{e^{-|\alpha|^2}}{m! L_m (-|\alpha|^2)}$$  \hfill (33)
Here $L_m(x)$ represents the Laguerre polynomial of order $m$. Nonclassical area $\sigma(\alpha, m) - \sqrt{2}\pi$ projected by the optical tomogram of $m$-photon-added coherent states on the optical tomographic plane was numerically evaluated using Eq. (14). Figure 4 shows the variation of the nonclassical area corresponding to the photon-added coherent state as a function of the number of photons $m$ added to the coherent state $|\alpha\rangle$. It clearly shows that the nonclassical area or the nonclassicality of the state $|\alpha, m\rangle$ increases with the increase in the number of photons added to the coherent field $|\alpha\rangle$.

Variation of the nonclassical area projected by the optical tomogram of $m$-photon-added coherent states $|\alpha, m\rangle$ with $|\alpha|^2 = 5$ on the optical tomographic plane. The nonclassicality of the state $|\alpha, m\rangle$ increases with the increase in the number of photons added to the coherent field $|\alpha\rangle$.

**FIG. 4.** Nonclassical area $\sigma - \sqrt{2}\pi$ projected by the optical tomogram of $m$-photon-added coherent states $|\alpha, m\rangle$ with $|\alpha|^2 = 5$ on the optical tomographic plane. The nonclassicality of the state $|\alpha, m\rangle$ increases with the increase in the number of photons added to the coherent field $|\alpha\rangle$.

Nonclassical states can also be obtained by superimposing two or more classical states. Consider the nonclassical states generated by the superposition of two coherent states with $\pi$ phase difference between them. Specifically, we consider the states defined by:

$$|\alpha\rangle_h = N_h \left[ |\alpha\rangle + e^{i\pi h} |\alpha\rangle \right],$$

where $N_h$ is the appropriate normalization constant, and the state with $h = 0$ corresponds to the even coherent state, and $h = 1$ corresponds to the odd coherent state. A suitable transformation in the symplectic tomogram of the even and odd coherent states [39] gives the corresponding optical tomogram of the states. The optical tomogram corresponds to the state $|\alpha\rangle_h$ is given by [28]

$$\omega_{|\alpha\rangle_h}(X_\theta, \theta) = N_h^2 \left[ \sum_{r=0}^{\infty} e^{i\pi hr} Q_{|\alpha e^{i\pi r}\rangle}(X_\theta, \theta) \right]^2,$$

where

$$Q_{|\alpha e^{i\pi r}\rangle}(X_\theta, \theta) = \frac{1}{\pi^{1/4}} \exp \left[ -\frac{X_\theta^2}{2} - \frac{(\alpha e^{i\pi r})^2 e^{-i2\theta}}{2} \right] \left[ -\frac{|\alpha|^2}{2} + \sqrt{2} \alpha e^{i\pi r} X_\theta e^{-i2\theta} \right]$$

is the quadrature representations of the coherent states $|\alpha e^{i\pi r}\rangle$ [27]. We have numerically computed the nonclassical area spanned by the optical tomogram given in Eq. (35) on the $X_\theta-\theta$ plane as a function of $|\alpha|^2$ (see Fig. 5). It is noted that the nonclassical area (degree of nonclassicality) of the state $|\alpha\rangle_h$ is independent of the value of the argument of $\alpha$. For large $|\alpha|^2$ values, the nonclassical area associated with the even and odd coherent states is the same, consistent with the results in [28] [40]. It has been shown that for large $|\alpha|^2$ values, the coherent states appearing in the superposition can be taken.
as orthonormal basis for representing the two-mode entangled states generated by the action of a 50/50 beam splitter on the state $|\alpha\rangle_h$ with vacuum state taken in the other input arm [41]. Here the amount entanglement of the states generated at the output of the beam splitter is also a measure of the amount of nonclassicality of the input state. For large $|\alpha|^2$ values, the entangled states generated by the even and odd coherent state have the same amount of entanglement [29]. The nonclassical area measure essentially captures this feature—note that the two curves in Fig. 6 merge for $|\alpha|^2$ values greater than 3.

In this section, we have defined nonclassical area measure for a single-mode field and described the calculations of the amount of nonclassicality of a pure single-mode field using the nonclassical area measure. Next, we investigate the effect of decoherence of the single-mode state of the field on the nonclassical area measure.

**E. Effect of decoherence on the nonclassical area**

Here we check the robustness of the nonclassical area measure by investigating the effect of environment-induced decoherence of the single-mode field on the nonclassical area measure. We consider an amplitude decay model of decoherence of the state due to the interaction of the field mode with the external environment. We model the external environment as a collection of infinite harmonic oscillators maintained at zero temperature. Within Born-Markov approximation, the zero temperature master equation in the interaction picture can be written as

$$\frac{\partial \rho}{\partial t} = \gamma \left( 2 \alpha a^\dagger \rho - a^\dagger a \rho - \rho a^\dagger a \right), \quad (37)$$

where $\gamma$ is the interaction strength of the field mode with the external environment. The above master equation can be solved using the Laplace transform method and the density matrix at time $t$ can be calculated in the Fock basis as

$$\rho(t) = \sum_{n, n'} \rho_{nn'} |n\rangle \langle n'|,$$

where the Fock state expansion coefficients are given by

$$\rho_{n, n'} = e^{-\gamma t (n + n')} \sum_{r=0}^{\infty} \sqrt{(n+r)C_r} (n'+r)C_r \times \left(1 - e^{-2\gamma t}\right)^r \rho_{n+r, n'+r} (t = 0) . \quad (39)$$

We have numerically computed the optical tomogram using Eq. (38), and hence the nonclassical area spanned by the optical tomogram of state given in Eq. (38) for various initial states. Figure 7 shows the exponential decay of nonclassical area for various initial states due the interaction of the system with the external environment. The smooth variation of the nonclassical area with respect to the interaction time indicates that the nonclassical area measure is robust against environment-induced decoherence.

**IV. NONCLASSICAL AREA FOR A TWO-MODE FIELD**

In this section, we extend our analysis of the nonclassical area measure given in Sec. [III] and introduce the
measure for quantifying the nonclassicality of a two-mode state of electromagnetic field. The optical tomogram of several two-mode states was theoretically calculated using Eq. [35, 29, 42]. Since the homodyne rotated quadrature operator \( X_{\theta_1, \theta_2} \) [see Eq. (7)] is in a separable form, and it obeys the uncertainty relation, it is straightforward to extend our analysis using the uncertainty relation for the case of a two-mode field and arrive at the following expression for the effective area projected by the optical tomogram of a generic two-mode state \( |\Psi\rangle \):

\[
\sigma (|\Psi\rangle) = \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \Delta X_{\theta_1, \theta_2}, \tag{40}
\]

where \( \Delta X_{\theta_1, \theta_2} \) is the standard deviation in the measurement of the operator \( X_{\theta_1, \theta_2} \). As mentioned earlier, the effective area \( \sigma (|\Psi\rangle) \) has a lower bound for the classical states such as a two-mode coherent state \( |\alpha_1 \rangle \otimes |\alpha_2 \rangle \) \((\alpha_1, \alpha_2 \in \mathbb{C})\) or vacuum state \( |0 \rangle \otimes |0 \rangle \). The effective area spanned by the optical tomogram of both of these states are calculated to be \( 2\sqrt{2} \pi^2 \). All the two-mode nonclassical states will be having an effective area greater than \( 2\sqrt{2} \pi^2 \). Therefore, the deviation

\[
\sigma (|\Psi\rangle) = 2\sqrt{2} \pi^2, \tag{41}
\]

can be taken as the nonclassical area which quantifies the nonclassicality associated with a two-mode field. A non-zero value of the nonclassical area \( \sigma (|\Psi\rangle) = 2\sqrt{2} \pi^2 \) is a necessary condition for an arbitrary two-mode state of the electromagnetic field to be nonclassical. The amount of nonclassicality associated with a state is proportional to the spanned area by the optical tomographic plane of the corresponding state on the optical tomographic plane. Next, we calculate the nonclassical area for specific well-known two-mode fields such as a two-mode squeezed vacuum state and two-mode even and odd coherent states and see whether the nonclassicality of these states is reflected in it. We also check the robustness of the nonclassical area measure by investigating the effect of decoherence of the two-mode state on the nonclassical area measure.

### A. Two-mode squeezed vacuum state

The action of a two-mode squeeze operator \( S_2 (\xi) = \exp \left( \xi^* a_1 a_2 - \xi a_1^* a_2^* \right) \), where \( a_1 \) and \( a_2 \) are the operators for the two modes and \( \xi = r e^{i\delta} \), on the two-mode vacuum state \( |0\rangle \otimes |0\rangle \) generates a two-mode squeezed vacuum state \( |\xi\rangle_2 \). In the Fock basis representation the two-mode squeezed vacuum state can written as [33]:

\[
|\xi\rangle_2 = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (-1)^n e^{i n \delta} (\tanh r)^n |n, n\rangle. \tag{42}
\]

It is known that the state \( |\xi\rangle_2 \) is highly nonclassical state exhibiting quadrature squeezing and entanglement [33].

Using Eq. (8), the optical tomogram of the state \( |\xi\rangle_2 \) is calculated to be

\[
\omega_{|\xi\rangle_2} (X_{\theta_1, \theta_2}; X_{\phi_1, \phi_2}) = \frac{1}{\cosh^2 r} \sum_{n, n'=0}^{\infty} (-1)^{n+n'} e^{i(n-n')\delta} 
\times (\tanh r)^{n+n'} Q_{\{\alpha\}} (X_{\phi_1}, \theta_1) Q_{\{\alpha\}} (X_{\phi_2}, \theta_2).
\]

(43)

The variance of the homodyne rotated quadrature operator \( X_{\theta_1, \theta_2} \) in the state \( |\xi\rangle_2 \) is calculated as

\[
(\Delta X_{\theta_1, \theta_2})^2 = \frac{1}{2} \left[ 1 + 2 \sinh^2 (r) - \sinh(2r) \cos (\delta - \theta_1 - \theta_2) \right]. \tag{44}
\]

We have numerically computed the nonclassical area of the state \( |\xi\rangle_2 \) using Eqs. (41) and (40) and is plotted as a function of the squeezing parameter \( r \) in Fig. 8. As we know from the other nonclassicality measures, the quantumness of the state \( |\xi\rangle_2 \) increases with increases in the squeezing parameter \( r \) and this trend is well reflected in the nonclassical area measure.

### B. Two-mode even and odd coherent state

Next, we look at the two-mode version of the even and odd coherent states given in Eq. (34). A beam splitter arrangement can be used to generate the two-mode even and odd coherent state by taking a single-mode even and odd coherent state at one input port and the vacuum at the other input port [24]. The state at the output of the beam splitter is the two-mode even and odd coherent state given by

\[
|\Psi\rangle^{(2)}_h = N_h \left[ |\alpha\rangle |\alpha\rangle + (-1)^h |-\alpha\rangle |-\alpha\rangle \right],
\]

(45)

where \( N_h \) is the normalization constant [same as in Eq. (44)], \( \alpha = |\alpha| e^{i\eta} \) and \( h = 0 \) (even state) or 1 (odd
state). The two-mode even and odd coherent state is a nonclassical state having entanglement property. In terms of the quadrature representation of a single-mode coherent given in Eq. \( \ref{eq:10} \), the optical tomogram of the state \( |\Psi \rangle^{(2)}_h \) is given by \( \ref{eq:20} \):

\[
\omega_{|\Psi \rangle^{(2)}_h} (X_{\theta_1}, \theta_1; X_{\theta_2}, \theta_2) = N_h^2 \sum_{r=0}^{\infty} Q_{|x=r\pi\rangle} (X_{\theta_1}, \theta_1) \times Q_{|x=r\pi\rangle} (X_{\theta_2}, \theta_2) \tag{46}
\]

The variance of the two-mode rotated quadrature \( X_{\theta_1, \theta_2} \) in the state \( |\Psi \rangle^{(2)}_h \) is found to be

\[
(\Delta X_{\theta_1, \theta_2})^2 = \frac{1}{2} + |\alpha|^2 \left\{ \cos^2(\eta - \theta_1) + \cos^2(\eta - \theta_2) - 1 \right\} + \tanh(2 |\alpha|^2)^{-(1)} \left[ 1 + \cos(\theta_1 - \theta_2) \right] + \cos(2 \eta - \theta_1 - \theta_2) \tag{47}
\]

The nonclassical area spanned by the optical tomogram

\[
\sigma = 2 \sqrt{2} \pi \tag{48}
\]

![Figure 9](image_url)

Variation of the nonclassical area spanned by the optical tomogram of two-mode even \( |\Psi \rangle^{(2)}_0 \) and odd \( |\Psi \rangle^{(2)}_1 \) coherent states as a function of the field strength \( |\alpha|^2 \).

C. Effect of decoherence

In this section, we study the effect of environment-induced decoherence of the state on the nonclassical area measure of nonclassicality for a two-mode field. We consider the two-mode extension of the zero-temperature master equation given in Eq. \( \ref{eq:37} \) to model the amplitude decay of a two-mode field. The zero-temperature master equation for a two-mode field with density matrix \( \rho^{(2)} \) is given by

\[
\frac{\partial \rho^{(2)}}{\partial t} = \sum_{k=1}^{2} \gamma_k \left( 2 a_k \rho^{(2)} a_k^\dagger - a_k^\dagger a_k \rho^{(2)} - \rho^{(2)} a_k^\dagger a_k \right) \tag{48}
\]

where \( \gamma_k \)'s are the interaction strengths of the field mode \( a_k \) with the external environment. One can analytically solve the master equation \( \ref{eq:48} \) using the disentangling theorem for SU(1,1) in thermofield dynamics notation \( \ref{eq:44} \). The density matrix elements of the two-mode field \( \rho^{(2)} \) at any instant \( t \) can be written in the Fock basis as

\[
\langle m_1, m_2 | \rho^{(2)} (t) | n_1, n_2 \rangle = \sum_{r_1, r_2=0}^{\infty} \Gamma_1 \Gamma_2 \times \langle m_1 + r_1, m_2 + r_2 | \rho^{(2)} (0) | n_1 + r_1, n_2 + r_2 \rangle \tag{49}
\]

where

\[
\Gamma_j = \sqrt{m_j + r_j} C_{m_j + r_j, n_1 + r_1} C_{m_j, n_1} \left[ 1 - \exp(-2 \gamma_j t) \right] + \exp(-2 \gamma_j t (m_j + n_j)) \tag{50}
\]

and \( \rho^{(2)} (0) \) is the initial density matrix of the two-mode field. Using the above equation and Eq. \( \ref{eq:36} \), we can evaluate the optical tomogram of the state \( \rho^{(2)} (t) \), which is undergoing amplitude decay due to interaction with the external environment. We have numerically calculated the standard deviation in the measurement of rotated quadrature operator \( X_{\theta_1, \theta_2} \) and hence the nonclassical area of the state \( \rho^{(2)} (t) \), which is initially prepared in a two-mode squeezed vacuum state as well as two-mode even and odd coherent states, as a function of time (see Fig. \( \ref{fig:10} \)). The figure shows that at the initial time, the nonclassical area of the state \( \rho^{(2)} (t) \) for initial \( |\Psi \rangle^{(2)}_h \) and \( |\xi \rangle_2 \) gives the value corresponding to the pure states \( |\Psi \rangle^{(2)}_h \) and \( |\xi \rangle_2 \), respectively. As the time progress the nonclassical area decay smoothly and final becomes zero for both the initial states. In the long time limit, the two-mode state reduces to the two-mode vacuum state \( |0, 0 \rangle \) for both initial states for which the nonclassical area is zero.

V. NONCLASSICAL AREA OF P-MODE STATE

Here we generalize the results described in the previous sections to the case of a generic \( p \)-mode state \( |\Psi \rangle^{(p)} \) of the electromagnetic field. Since the \( p \)-mode rotated quadrature operator given in Eq. \( \ref{eq:41} \) is in a separable form, our analysis using uncertainty relation given in Sec. \( \ref{sec:III} \) can
be repeated in a straightforward manner to arrive at an expression for the effective area projected by the optical tomogram of a p-mode field [given in Eq. (8)] as

$$
\sigma (|\Psi|^p) = \int_0^{2\pi} d\theta_1 d\theta_2 d\theta_3 \cdots d\theta_n \Delta X_{\theta_1, \theta_2, \theta_3, \ldots, \theta_n},
$$

where $\Delta X_{\theta_1, \theta_2, \theta_3, \ldots, \theta_n}$ is the standard deviation in the measurement of $p$-mode rotated quadrature operator $X_{\theta_1, \theta_2, \ldots, \theta_p}$ in the state $|\Psi|^p$. The effective area $\sigma (|\Psi|^p)$ has a lower bound for the $p$-mode classical state of the field which is $2^{p-1/2}\pi^p$. All the $p$-mode nonclassical states will have an effective area which greater than $2^{p-1/2}\pi^p$. Therefore, the nonclassical area which quantifies the nonclassicality of a $p$-mode state can be written as:

$$
\sigma (|\Psi|^p) - 2^{p-1/2}\pi^p.
$$

A nonzero value of nonclassical area $\sigma (|\Psi|^p) - 2^{p-1/2}\pi^p$ for a $p$-mode field is a necessary condition to say that state is nonclassical. For a $p$-mode coherent state or a $p$-mode vacuum state, which are considered to be the classical states of the field, the nonclassical area is zero. One limitation of nonclassical area measure of nonclassicality is to deal with the quantum states which are coherent mixtures. If the state is a coherent mixture, it contains classical information that will contribute to the standard deviation of the rotated quadrature operator and the actual quantumness of the state. Therefore, the nonclassical area of such states will be nonzero even though the state of coherent mixtures is considered to be a classical state. If one can remove the classical contribution from the standard deviation of the rotated quadrature operator, then the nonclassical area measure is good to use for the state of coherent mixture. One can also perform the Monte-Carlo method to simulate the data from the homodyne detection measurements and obtain the results presented in this paper.

VI. CONCLUSION

We have introduced a simple and easily computable measure of the degree of nonclassicality for the quantum states of light. The proposed measure is based on the standard deviation in the homodyne rotated quadrature operator’s measurement and is termed a nonclassical area. The nonclassical area can be viewed as an area projected by the optical tomogram of the state on the optical tomographic plane. If the nonclassical area spanned by the optical tomogram of a quantum state of light is nonzero, then the state is strictly nonclassical. The nonclassical area is zero for the classical states. The nonzero value of the nonclassical area for a given quantum state can be considered as a necessary condition for the state to be nonclassical. We have analyzed the nonclassical area spanned by the optical tomogram of several nonclassical states and observed that the essential features of nonclassicality shown by the states are consistently reflected in the nonclassical area measure. We have found that the nonclassical area associated with an arbitrary quantum state of light increases with the strength of any nonclassicality-inducing operations acting on the state, such as squeezing, photon addition, and superimposing two states. By investigating the effect of environment-induced decoherence on the nonclassical area, we have shown that the nonclassical area measure is robust against decoherence. The advantage of the nonclassical area measure is that the experimentalist can use the Eq. (52) to calculate the degree of nonclassicality associated with an arbitrary quantum state of light directly from the homodyne tomography data without using the density matrix or the quasiprobability distributions reconstructed from the optical tomogram. Thus, this method avoids the errors which may arise during the numerical method for reconstructing the density matrix or the quasiprobability distribution of the state.

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