\((L_p, L_q)\) estimates of potentials with oscillating kernel

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Abstract

\((L_p, L_q)\) estimates are obtained for oscillatory potentials

\[
(K^\alpha f)(x) = \int_{\mathbb{R}^n} \frac{\exp(i|y|)}{|y|^{n-\alpha}} f(x - y)dy, \quad 0 < \alpha < n, \quad n \geq 2,
\]

whose symbol has a singularity on the unit sphere. These potentials are natural modifications of the celebrated Bochner-Riesz operator and Helmholtz potential arising in Fourier analysis and PDE. For some values of \(\alpha\), determination of the corresponding pairs \((L_p, L_q)\) represents an open problem. The range of \(\alpha\) for which the problem is open is just the same as for the Bochner-Riesz means.

1 Introduction

We consider the potential operator with oscillating kernel

\[
(K^\alpha f)(x) = \int_{\mathbb{R}^n} \frac{\exp(i|y|)}{|y|^{n-\alpha}} f(x - y)dy, \quad 0 < \alpha < n, \quad n \geq 2.
\] (1.1)

Up to now, the \((L_p, L_q)\) estimates for oscillatory potentials on \(\mathbb{R}^n\) have been investigated for some special cases only. We mention the oscillation generated by the Bessel function (Bochner-Riesz means) and that generated by the Hankel function (Helmholtz potential). Such oscillatory integrals arise from the classical problems of Fourier analysis concerning the summability of series and inversion of the Fourier transform and from the Helmholtz equation with the Dirichlet
boundary condition. Respectively, more general potentials (L.1) are natural modification of those operators.

The multiplier problem for the Bochner-Riesz means has a long history. We refer to [3, 11, 12, 13, 14, 16, 28] for the background information. We also point out the papers [1, 2, 3, 27] and [18, 22] where one can find $(L_p, L_q)$ estimates for the Bochner-Riesz operators of negative order and the Helmholtz potentials, respectively.

For different $\alpha$, we establish the boundedness of $K^\alpha$ taking $L_p$ into $L_q$. The sets of pairs $(\frac{1}{p}, \frac{1}{q})$ for which the operator $K^\alpha$ is bounded from $L_p$ into $L_q$ are convex and of the special form on the $(\frac{1}{p}, \frac{1}{q})$-plane (see Figure). The $\mathcal{L}$-characteristic of the operator $K^\alpha$ is constructed for either $n = 2$, or $n > 2$, provided $0 < \alpha < \frac{n(n-1)}{2(n+1)}$ or $\frac{n}{2} \leq \alpha < n$. In other cases we have gaps between necessary and sufficient conditions of boundedness similar to those for the Bochner-Riesz operator. Note, that the principal difficulties in the oscillatory potentials theory are related to controlling the oscillation of kernels. The techniques used here to get through those difficulties goes back to Stein [28], Fefferman [13], Bak [1, 2] and Börjeson [3].

The paper is organized as follows. In Section 2 we formulate our main result (Theorem 2.1). Section 3 contains necessary preliminaries that will be used throughout this paper. Section 4 deals with some auxiliary statements. First, the symbol $m_\alpha(|\xi|)$ of the operator $K^\alpha$ is estimated. Since the symbol has a singularity on the unit sphere, the operator $K^\alpha$ shares properties of both the Bochner-Riesz means and Riesz potential. More precisely, if we split the integral in (L.1) into two over $|y| \geq 1$ and $|y| < 1$ and

$$K^\alpha f = S^\alpha f + N^\alpha f, \quad (1.2)$$

where

$$\left( S^\alpha f \right)(x) = \int_{|y| \geq 1} \frac{\exp(i|y|)}{|y|^{n-\alpha}} f(x - y)dy, \quad (1.3)$$

$$\left( N^\alpha f \right)(x) = \int_{|y| < 1} \frac{\exp(i|y|)}{|y|^{n-\alpha}} f(x - y)dy, \quad (1.4)$$

then $S^\alpha$ will be responsible for the Bochner-Riesz means properties while $N^\alpha$ for properties of the Riesz potential. Furthermore, we give estimates of an auxiliary operator, arising from Stein’s decomposition of the operator $S^\alpha$ (see (2.2)). We next establish $L_p$—boundedness of $K^\alpha$. Clearly, it can be reduced to the $L_p$—boundedness property of the operator $S^\alpha$. So the situation here is just the same as in the Bochner-Riesz case, and the problem of precise $p$ for
which $K^\alpha$ is bounded on $L_p$ is correspondingly for $n > 2$, $\frac{n(n-1)}{2(n+1)} \leq \alpha < \frac{n-1}{2}$ (the problem of Stein). In Subsection 4.4 we give $(L_p, L_q)$ estimates of $K^\alpha$ along a segment through $(\frac{1}{p'}, \frac{1}{p})$, where $p' = p/(p-1)$, and perpendicular to the line of duality $\frac{1}{p} + \frac{1}{q} = 1$ provided $\frac{n-1}{2} < \alpha < n$. We consider the cases $n = 2$ and $n > 2$ separately. In the first one the result obtained here is sharp. When $n > 2$, the result is sharp for $n/2 \leq \alpha < n$ only, as in the case of Bochner-Riesz operator with negative index. The proof of Theorem 2.1 is given in Section 3. For the sake of convenience, we give one technical estimate in Appendix.

2 The main result

To formulate our main result we label some points in $Q = [0; 1] \times [0; 1]$ (see Figure):

$L(\frac{1}{2}; \frac{1}{2}), E(1; 0), F(1; 1), A(1; 1 - \frac{\alpha}{n}), A'(\frac{\alpha}{n}; 0), H(1 - \frac{\alpha}{n}; 1 - \frac{\alpha}{n}), H'(\frac{\alpha}{n}; \frac{\alpha}{n}), C(\frac{3}{2} - \frac{2\alpha}{n-1}; \frac{3}{2} - \frac{2\alpha}{n-1}), C'(\frac{2\alpha}{n-1} - \frac{1}{2}; 2\frac{\alpha}{n-1} - \frac{1}{2}), D(\frac{2\alpha}{n-1}; \frac{2\alpha}{n-1}), B(1 - \frac{n-1(n-\alpha)}{n(n+1)}; 1 - \frac{\alpha}{n}), B'(\frac{n-1}{n+1} - \frac{n}{n(n+1)}; 1 - \frac{\alpha}{n}), G(\frac{n+3}{2n+2}; 1 - \frac{\alpha}{n}), G'(\frac{n-1}{n+1; \frac{2n-4n-1}{2n+2}), P(\frac{3\alpha-n+3}{2n+2}; \frac{1}{2}), P'(\frac{3\alpha-4n-1}{2n+2})$.

Let $(ABC\ldots P)$ be an open polygon in $Q$ with vertices $A, B, C, \ldots, P$, and let $[ABC\ldots P]$ be its closure. Let also $(AB)$ be an open interval in $Q$ with the ends $A$ and $B$, $[AB], [AB], (AB)$, be the closed and half open intervals, respectively. For linear operator $A$ defined on $L_p$ spaces we denote $\mathcal{L}(A) = \{(\frac{1}{p}, \frac{1}{q}) \in Q : \|A\|_{L_p \to L_q} < \infty\}$.

We are now in a position to formulate our results.

**Theorem 2.1.** I. The following embeddings are valid:

1) If $0 < \alpha < \frac{n(n-1)}{2(n+1)}$, $n > 2$ or $0 < \alpha < \frac{1}{2}$, $n = 2$ then $(A'H'H) \cup (A'A) \cup (H'H) \subset \mathcal{L}(K^\alpha)$;  
2) If $\frac{n(n-1)}{2(n+1)} \leq \alpha < \frac{n-1}{2}$, $n > 2$ then $(A'G'C'CGA) \cup (A'A) \cup (C'C) \subset \mathcal{L}(K^\alpha)$;  
3) If $\frac{1}{2} < \alpha < 2$, $n = 2$ or $\frac{\alpha}{2} \leq \alpha < n$, $n > 2$ then $(A'B'BA) \cup (A'A) \cup (BB') \subset \mathcal{L}(K^\alpha)$;  
4) If $\frac{n-1}{2} < \alpha < \frac{3}{2}$, $n > 2$ then $(A'G'P'PGA) \cup (A'A) \cup (P'P) \subset \mathcal{L}(K^\alpha)$;  
5) If $\alpha = \frac{1}{2}$, $n = 2$ then $(A'B'BA) \cup (A'A) \subset \mathcal{L}(K^\alpha)$;  
6) If $\alpha = \frac{n-1}{2}$, $n > 2$ then $(A'G'LGA) \cup (A'A) \subset \mathcal{L}(K^\alpha)$.

II. The operator $K^\alpha$ is unbounded from $L_p$ into $L_q$ whenever

1) $\left(\frac{1}{p}, \frac{1}{q}\right) \in [HAF] \cup [H'A'O]$;  
2) $\left(\frac{1}{p}, \frac{1}{q}\right) \in [A'AE] \setminus (AA')$;  
3) $\left(\frac{1}{p}, \frac{1}{q}\right) \in (BB'H'H')$ provided $\frac{n-1}{2} < \alpha < n$.

We wish here to formalize a part of the reasoning presented in the proof of the positive results. As it was mentioned above, the idea of splitting $K^\alpha$ into $\left[\frac{1}{2}; \frac{1}{2}\right]$ leads us to controlling the oscillation of the kernel of $S^\alpha$. For this goal, we apply a standard model to get control
of operators whose symbol has a singularity on the unit sphere. The method is based on the decomposition

\[ S^\alpha = \sum_{\ell=0}^{\infty} S_\ell^\alpha \]  

(2.1)

into the sum of operators with kernels supported on dyadic annuli. In fact, we use Stein’s decomposition ([28]) setting

\[ (S_\ell^\alpha f)(x) = 2^{(\alpha-n)\ell} \int_{|y| \geq 1} \exp(i|y|)|\psi(y/2^\ell)|f(x-y)dy, \]  

(2.2)

where \( \psi(y) = |y|^{\alpha-n}[\eta(y) - \eta(2y)] \) is a smooth function supported in \( \frac{1}{2} < |y| < 2 \), \( \eta(y) = 1 \) as \( |y| \leq 1 \) and \( \eta(y) = 0 \) as \( |y| \geq 2 \).

\[
\begin{array}{c}
\text{Figure}
\end{array}
\]

Then the study of the operator \( S^\alpha \) can be reduced to the study of the oscillatory operator \( G_\lambda \) (see (3.8)). \( L_p \)– boundedness property and \((L_p, L_q)\) estimates of the latter (Lemma 3.4 and 3.5) have been investigated in [28] and [27], respectively (see also [1], [17]). Such argument is used to prove \( L_p \)– boundedness of the operator \( S^\alpha \) and \((L_p, L_q)\) estimates on the open segment
(BB′) in the case \( n = 2 \). It is to be noted that the method of proof of Lemma 4.3 which involves the estimates for characteristic functions \( f = \chi_E \) implying the Lorentz space estimate is not new (see e.g. [1]).

To obtain \((L_p, L_q)\) estimates of \( K^\alpha \) along a segment through \((\frac{1}{p}, \frac{1}{p'})\) and perpendicular to the line of duality when \( n > 2 \), we apply the modification of an interpolation theorem for analytic families of operators (Theorem 3.1). A special case of Stein’s interpolation theorem ([29]) established in [2] need a slight modification for our case. Such approach developed in [2] and adopted here allows to obtain \((L_p, L_q)\)- boundedness of \( K^\alpha \) not only for the point \( D \), but for indices \((\frac{1}{p}, \frac{1}{q})\) off the line of duality. Altogether then, it is relatively simple matter to construct the \( L^- \) characteristic of \( K^\alpha \) in two dimension as well as in higher dimensions provided \( 0 < \alpha < \frac{n(n-1)}{2(n+1)} \) or \( \frac{n}{2} \leq \alpha < n \). In other cases we establish additional estimates for the operators \( S^\alpha_\ell \). The arguments that are required for proofs are based on properties of the symbol of \( S^\alpha_\ell \), restriction theorem for the Fourier transform and the Riesz-Thorin interpolation theorem.

3 Preliminaries

3.1 On the analyticity of an integral depending on a parameter

Lemma 3.1 (see e.g. [23]). Let \( f(x, z) \) be analytic in \( z \in D \subset \mathbb{C} \) for almost all \( x \in \mathbb{R}^n \). If there is a function \( F(x) \in L_1(\mathbb{R}^n) \) such that \(|f(x, z)| \leq F(x)\) for almost all \( x \in \mathbb{R}^n \) and for all \( z \in D \), then the integral \( \int_{\mathbb{R}^n} f(x, z) dx \) is an analytic function on \( D \).

3.2 Modified theorem for analytic families of operators

Let \( E \) be the set of simple functions, that is, the set of linear combinations of characteristic functions of sets in \( \mathbb{R}^n \) having finite measure. We recall (see e.g. [29]), that the family of operators \( \{T_z\}_{z \in S} \), \( S = \{ z \in \mathbb{C} : 0 \leq \text{Re} \ z \leq 1 \} \), taking the set of simple functions from \( L_1(\mathbb{R}^n) \) into the space of functions measurable on \( \mathbb{R}^n \), is said to be admissible growth on \( \mathbb{R}^n \) if for arbitrary \( f, g \in E \), the function \( F(z) = \int_{\mathbb{R}^n} (T_z f)(x) g(x) dx \) is integrable on \( \mathbb{R}^n \) and has the following properties:

i) \( F(z) \) is analytic in the interior of \( S \);

ii) \( F(z) \) is continuous on \( S \);

iii) \( \sup_{\gamma \in \mathbb{R}_1} \{ -a |\text{Im} \ z| \} \ln F(z) < \infty \) for some \( a < \pi \).
We need the following modification of Theorem 1’ ([4]) which is a special case of Stein’s interpolation theorem [29] (cf. [18]).

**Theorem 3.1.** Let \( \{T_z\}_{z \in S} \), \( S = \{z \in \mathbb{C} : 0 \leq \text{Re } z \leq 1\} \) be an admissible growth family of multiplier operators satisfying, for each \( f \in E \), the relations

\[
\|T_{\gamma}f\|_2 \leq M_1(\gamma)\|f\|_2, \quad \|T_{1+i\gamma}f\|_\infty \leq M_2(\gamma)\|f\|_1,
\]

where \( M_j(\gamma) \), \( j = 1, 2 \), are independent of \( f \), and \( \sup_{\gamma \in \mathbb{R}} \exp\{-a|\gamma|\} \ln M_j(\gamma) < \infty \) for some \( a < \pi \). Let also the family \( \{T_z\}_{z \in S} \) satisfy the additional assumption

\[
|\mu_z(\xi)|^2 \leq \mu_2 \text{Re } z(\xi) \quad \text{(pointwise)} \quad \text{if} \quad 0 < \text{Re } z < \mu \leq 1/2
\]

provided \( \hat{T_z}(\xi) = \mu_z(\xi)\hat{f}(\xi) \). Here \( C_z \) is a non-negative function such that \( \log C_z \leq K\exp(k|\text{Im } z|) \) for some \( K > 0 \) and \( k < \pi \). Then if for some \( 0 < t < 1 \) and \( 1/p_t - 1/q_t = t \) the operator \( T_z \) is continuous from \( L_{p_t} \) into some topological space \( X \) in which \( L_{q_t} \) is continuously embedded, then it is continuous from \( L_{p_t} \) into \( L_{q_t} \), provided

\[
\frac{1 + t}{2} \leq \frac{1}{p} \leq \frac{1 + 2t}{2}, \quad \text{for} \quad 0 < t < \mu,
\]

and

\[
\frac{1 + t}{2} \leq \frac{1}{p} \leq \frac{1 + t - 2t\mu}{2(1 - \mu)}, \quad \text{for} \quad \mu \leq t < 1.
\]

### 3.3 Uniform asymptotic expansion for the Bessel function

**Lemma 3.2.** Let \( z \in \Omega = \{z \in \mathbb{C} : |z| > \eta, \ |\text{arg } z| < \theta\} \), \( \eta > 0 \), \( \theta \in (0; \pi/2) \). Then

\[
J_\nu(z) = \left(\frac{\pi z}{2}\right)^{-1/2}[e^{-iz}\left(\sum_{m=0}^{M} C_{m,-}\nu z^{-m} + R_{M,-}(z)\right) + e^{iz}\left(\sum_{m=0}^{M} C_{m,+}\nu z^{-m} + R_{M,+}(z)\right)], \quad (3.1)
\]

where remainders \( R_{M,\pm}(z) \) are analytic in \( \Omega \) and

\[
|R_{M,\pm}(z)| \leq C|z|^{M-1}, \quad (3.2)
\]

with \( C \) independent of \( \nu \) and \( z \).
3.4 Estimates for the hypergeometric function

Lemma 3.3. Let $|z| < 1$, Re $c \geq -1/2$ and there is a constant $M$ such that $|\text{Re } a|, |\text{Re } b|, |\text{Re } c| \leq M$. Then

$$|\, _2F_1(a, b; c; z)\, | \leq C \left\{ \begin{array}{ll}
\exp(\pi |\text{Im } c|) + \frac{|\Gamma(c)|}{|\Gamma(a)\Gamma(b)|}, & \text{if } a, b \neq -m; \quad \text{Re } (c - a - b) > 0 \\
\exp(\pi |\text{Im } c|), & \text{if } a = -m \quad \text{or} \quad b = -m,
\end{array} \right. \tag{3.3}$$

where $m = 0, 1, 2, \ldots$; $C = C(M)$.

Proof. Let us fix $N > 0$ and write

$$\, _2F_1(a, b; c; z) = \left( \sum_{m=0}^{N} + \sum_{m=N+1}^{\infty} \right) \frac{(a)_m(b)_m z^m}{(c)_m m!}.$$

For $m = 0, 1, 2, \ldots, N$, in view of the estimate

$$|\Gamma(\beta + 1/2)|^{-1} \leq C \exp(\pi |\text{Im } \beta|), \quad \text{Re } \beta \geq -1, \tag{3.4}$$

with $C$ independent of $\beta$ (see [3], p. 48) we get

$$\left| \frac{(a)_m(b)_m}{(c)_m m!} \right| = \left| \frac{(a)_m(b)_m \Gamma(c)}{m! \Gamma(c + m)} \right| \leq C(M, N) \exp(\pi |\text{Im } c|). \tag{3.5}$$

For $m = N + 1, N + 2, \ldots$, the relation

$$\lim_{m \to \infty} e^{-a \ln m} \frac{\Gamma(a + m)}{\Gamma(m)} = 1$$

(see [3], p. 47) yields

$$\left| \frac{(a)_m(b)_m}{(c)_m m!} \right| \leq C(m) \frac{|\Gamma(c)|}{|\Gamma(a)\Gamma(b)|} m^{\text{Re } (a+b-c-1)}, \tag{3.6}$$

where $\lim_{m \to \infty} C(m) = 1$. Making use of (3.5) and (3.6), we arrive at (3.3), provided $a$ and $b$ both differ from $0, -1, -2, \ldots$. If $a = -m$ or $b = -m$, then the hypergeometric series is finite and (3.3) follows from (3.5).

Remark 3.1. A simple analysis of the proof shows that it is possible to omit the restriction Re $(c - a - b) > 0$, if $|z| \leq 1/2$. 

■
3.5 Restriction theorem for the Fourier transform

Let $S$ be the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$, and let $\hat{f}(\sigma)$ denote restriction of the Fourier transform of $f \in S$ to the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. We need the following $(L_p, L_2)$ restriction property.

**Theorem 3.2 ([28]).** Let $f \in S$. Then

$$\left( \int_{S^{n-1}} |\hat{f}(\sigma)|^2 d\sigma \right)^{1/2} \leq C_p \|f\|_p$$

(3.7)

if and only if $1 \leq p \leq p_0$, $p_0 = \frac{2n+2}{n+3}$.

**Remark 3.2.** Since $S$ is dense in $L_p$, we can define $\hat{f}$ on $S^{n-1}$ for each $f \in L_p$ whenever $1 \leq p \leq p_0$.

**Remark 3.3.** Observe that (3.7) holds uniformly for any sphere of radius $1/2 \leq r \leq 2$ on place of $S^{n-1}$.

3.6 $(L_p, L_q)$ estimates for certain oscillatory operator

Let $\psi$ be a fixed smooth function of compact support on $\mathbb{R}^n$ that vanishes in a neighborhood of the origin, and set

$$(G_\lambda f)(x) = \int_{\mathbb{R}^n} \exp(i\lambda|x-y|)\psi(x-y)f(y)dy.$$  (3.8)

**Lemma 3.4 ([28]).** Let $1 \leq p \leq \frac{2n+2}{n+3}$ for $n > 2$, and $1 \leq p < \frac{4}{3}$ for $n = 2$. Then

$$\|G_\lambda f\|_p \leq C\lambda^{-n/p'}\|f\|_p.$$  (3.9)

**Lemma 3.5 ([1], [27]).** Let $1 \leq p \leq 2$, $q = \frac{(n+1)p'}{n-1}$ for $n > 2$, and $1 \leq p < 4$, $q = 3p'$ for $n = 2$. Then

$$\|G_\lambda f\|_q \leq C\lambda^{-n/q}\|f\|_p.$$  (3.10)
4 Auxiliary statements

4.1 On the symbol of the operator $K^\alpha$

We first show that $K^\alpha$ is Fourier multiplier operator. Set

$$m_\alpha(|\xi|) = \begin{cases} 
\frac{2\pi^{n/2}\Gamma(n/2)}{\Gamma(n/2)} \ 2F_1\left(\frac{n}{2}, \frac{n+1}{2}, |\xi|^2\right), & \text{if } |\xi| < 1, \\
\frac{2\pi^{n/2}\Gamma(n/2)}{\Gamma(n-\alpha/2)} |\xi|^{-\alpha} \ 2F_1\left(\frac{\alpha+n+2}{2}, \frac{1}{2}, \frac{1}{|\xi|^2}\right) + \\
i \frac{2\pi^{n/2}\Gamma((n+1)/2)}{\Gamma(n-\alpha/2)} |\xi|^{-\alpha-1} \ 2F_1\left(\frac{n-\alpha-1}{2}, \frac{3}{2}, \frac{1}{|\xi|^2}\right), & \text{if } |\xi| > 1.
\end{cases}$$

(4.1)

Lemma 4.1. Let $f \in E$, $0 < \text{Re } \alpha < \frac{n}{2}$. Then

$$(\mathcal{F}K^\alpha f)(\xi) = m_\alpha(|\xi|)(\mathcal{F}f)(\xi)$$

(4.2)

where the Fourier transform is interpreted as

$$(\mathcal{F}K^\alpha f)(\xi) = \lim_{k \to \infty} \int_{|x| < k} (K^\alpha f)(x) \exp(ix \cdot \xi) dx.$$

Proof. Let $\chi(|y|)$ be the indicator of the unit ball and let $K^{(1)}_\alpha(y) = |y|^{\alpha-n} \exp(i|y|)\chi(|y|)$, $K^{(2)}_\alpha(y) = |y|^{\alpha-n} \exp(i|y|)(1-\chi(|y|))$. Evidently, $K^{(1)}_\alpha \in L_1(\mathbb{R}^n)$ for $\text{Re } \alpha > 0$, and $K^{(2)}_\alpha \in L_2(\mathbb{R}^n)$ for $\text{Re } \alpha < n/2$. We put

$$K^{(2)}_{\alpha,N}(y) = \begin{cases} 
K^{(2)}_\alpha(y), & \text{if } |y| < N \\
0, & \text{if } |y| \geq N.
\end{cases}$$

Then there is a subsequence $\mathcal{F}K^{(2)}_{\alpha,N_j}$ such that $\lim_{j \to \infty} (\mathcal{F}K^{(2)}_{\alpha,N_j})(\xi) = (\mathcal{F}K^{(2)}_\alpha)(\xi)$ exists almost everywhere. Therefore (4.2) is valid, where

$$m_\alpha(|\xi|) = \lim_{j \to \infty} \int_{|y| < N_j} (K^{(1)}_\alpha(y) + K^{(2)}_\alpha(y)) \exp(iy \cdot \xi) dy = \lim_{j \to \infty} \int_{|y| < N_j} |y|^{\alpha-n} \exp(i|y| + iy \cdot \xi) dy.$$

Making use of the formula

$$\int_{\mathbb{R}^n} \exp(ix \cdot y) \varphi(|y|) dy = \frac{(2\pi)^{n/2}}{|x|^{n/2}} \int_0^\infty \varphi(\rho) \rho^{n/2} J_{n/2-1}(\rho|x|) d\rho,$$

(4.3)
(see e.g. [30]), we obtain

\[ m_\alpha(|\xi|) = \left(\frac{2\pi}{|\xi|^{(n-2)/2}}\right) \int_0^\infty \rho^{\alpha-n/2} \exp(i\rho) J_{n-2}(\rho|\xi|) d\rho, \tag{4.4} \]

where the integral is understood as improper integral provided \( \Re \alpha \geq \frac{n-1}{2} \). Note, that \( m_\alpha(|\xi|) \) is defined when \( \Re \alpha < \frac{n+1}{2} \). Now the representation of the integral in (4.4) via hypergeometric functions (see e.g. 2.12.15.3, [20]) yields (4.1).

**Remark 4.1.** In what follows we shall need sometimes to understand (4.4) as Abel summable to \( m_\alpha(|\xi|) \) when \( \frac{n-1}{2} \leq \Re \alpha < \frac{n+1}{2} \), that is,

\[ m_\alpha(|\xi|) = \lim_{\varepsilon \to 0} m_{\alpha,\varepsilon}(|\xi|), \tag{4.5} \]

where

\[ m_{\alpha,\varepsilon}(|\xi|) = \left(\frac{2\pi}{|\xi|^{(n-2)/2}}\right) \int_0^\infty \rho^{\alpha-n/2} \exp((i\rho - \varepsilon \rho) J_{n-2}(\rho|\xi|) d\rho. \]

**Lemma 4.2.** The function \( m_\alpha(|\xi|) \) admits the following estimates:

1. If \( 0 < \Re \alpha \leq \delta_1 < \frac{n-1}{2} \), then

\[ |m_\alpha(|\xi|)| \leq C \exp\left(\frac{\pi}{2} |\Im \alpha|\right), \tag{4.6} \]

where \( C = C(n, \delta_1) \);

2. If \( \frac{n-1}{2} \leq \Re \alpha \leq \delta_2 < \frac{n+1}{2} \), \( \alpha \neq \frac{n-1}{2} \), then

\[ |m_\alpha(|\xi|)| \leq C \exp(\pi |\Im \alpha|) \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{2} \text{ or } |\xi| \geq 2, \\ |\Gamma(\alpha - \frac{n-1}{2})| (1 + |1 - |\xi||^{\frac{n-1}{2} - \Re \alpha}), & \text{if } \frac{1}{2} < |\xi| < 2, \text{ or } |\xi| \neq 1 \end{cases} \tag{4.7} \]

where \( C = C(n, \delta_2) \).

3. If \( \alpha = \frac{n-1}{2} \), then

\[ |m_\alpha(|\xi|)| \leq C \begin{cases} \exp\left(\frac{\pi}{2} |\Im \alpha|\right), & \text{if } |\xi| \leq \frac{1}{2} \text{ or } |\xi| \geq 2, \\ 1 + \ln |1 - |\xi||, & \text{if } \frac{1}{2} < |\xi| < 2, \text{ or } |\xi| \neq 1 \end{cases} \tag{4.8} \]

where \( C = C(n) \).

**Proof.** Estimates (4.6), provided \( |\xi| \neq 1 \) and (4.7), (4.8), provided \( |\xi| \leq \frac{1}{2} \) or \( |\xi| \geq 2 \) can be readily verified by means of (3.3). In the case \( \frac{n-1}{2} \leq \Re \alpha \leq \delta_2 < \frac{n+1}{2} \), \( \frac{1}{2} < |\xi| < 2 \), \( |\xi| \neq 1 \)
we will use the representation (4.3) of the symbol \( m_\alpha(|\xi|) \). Let \( \psi(\rho) \) be a smooth function such that \( \psi(\rho) = 0 \) if \( \rho < 1/2 \) and \( \psi(\rho) = 1 \) if \( \rho > 1 \). Then by means of (3.1) we obtain

\[
m_{\alpha,\varepsilon}(|\xi|) = \sum_{m=0}^{\infty} \frac{1}{|\xi|^{|\alpha+n|+m}} \left[ C_{m,+} I_{m,+}^{\alpha,\varepsilon}(|\xi|) + C_{m,-} I_{m,-}^{\alpha,\varepsilon}(|\xi|) \right] + \frac{(2\pi)^{\nu/2}}{|\xi|^{(\alpha+n)/2}} I_{\alpha,\varepsilon}(|\xi|) + \frac{1}{|\xi|^2} \left[ C_{M,+} I_{M,+}^{\alpha,\varepsilon}(|\xi|) + C_{M,-} I_{M,-}^{\alpha,\varepsilon}(|\xi|) \right],
\]

where

\[
I^{\alpha,\varepsilon}(|\xi|) = \int_0^{\infty} (1 - \psi(\rho))\rho^{\alpha-n/2} \exp(i\rho - \varepsilon\rho) J_{\nu/2}(\rho|\xi|) d\rho,
\]

\[
I_{m,\pm}^{\alpha,\varepsilon}(|\xi|) = \int_0^{\infty} \psi(\rho)\rho^{\alpha-n/2-m} \exp(i\rho(1 \pm |\xi|) - \varepsilon\rho) d\rho,
\]

\[
I_{M,\pm}^{\alpha,\varepsilon}(|\xi|) = \int_0^{\infty} \psi(\rho)\rho^{\alpha-n/2-1} \exp(i\rho(1 \pm |\xi|) - \varepsilon\rho) R_{M,\pm} d\rho.
\]

We need only to consider \( I_{0,\pm}^{\alpha,\varepsilon} \), since \( |\lim_{\varepsilon \to 0} I^{\alpha,\varepsilon}(|\xi|)| \leq C(n, \delta_2) \), \( |\lim_{\varepsilon \to 0} I_{M,\pm}^{\alpha,\varepsilon}(|\xi|)| \leq C(n, \delta_2) \), and the integrals \( I_{m,\pm}^{\alpha,\varepsilon} \), \( m = 1, 2, \ldots \), have better decay properties then \( I_{0,\pm}^{\alpha,\varepsilon} \). The calculation in the case \( \frac{n-1}{2} < \Re \alpha \leq \delta_2 \) is easy by virtue of (2.3.3.1), [19]:

\[
I_{0,\pm}^{\alpha,\varepsilon}(|\xi|) = \int_0^{\infty} (\psi(\rho) - 1)\rho^{\alpha-n+1} \exp(i\rho(1 \pm |\xi|) - \varepsilon\rho) d\rho + \Gamma(\alpha - \frac{n-1}{2})[i(1 \pm |\xi|) - \varepsilon]^{\frac{n-1}{2}-\alpha}.
\]

Thus,

\[
|\lim_{\varepsilon \to 0} m_{\alpha,\varepsilon}(|\xi|)| \leq C \exp(\pi|\Im \alpha|)\Gamma(\alpha - \frac{n-1}{2})(1 + |1 - |\xi||^{\frac{n-1}{2}-\Re \alpha}).
\]

Then, for \( \alpha = \frac{n-1}{2} + i\gamma \), \( \gamma \in \mathbb{R}^1 \) and integrating by parts yields

\[
I_{0,\pm}^{\alpha,\varepsilon}(|\xi|) = -\frac{i(1+|\xi|-\varepsilon)}{\gamma} \int_0^{\infty} (\psi(\rho) - 1)\rho^{\gamma} \exp(i\rho(1 \pm |\xi|) - \varepsilon\rho) d\rho
\]

\[
- \Gamma(i\gamma)[i(1 \pm |\xi|) - \varepsilon]^{-i\gamma} - \int_0^{\infty} \psi'(\rho)\rho^{\gamma} \exp(i\rho(1 \pm |\xi|) - \varepsilon\rho) d\rho,
\]

if \( \gamma \neq 0 \), and

\[
I_{0,\pm}^{\alpha,\varepsilon}(|\xi|) = c + \ln(\varepsilon - i(1 \pm |\xi|)) - \int_0^{\infty} (\psi(\rho) - 1) \ln \rho \exp(i\rho(1 \pm |\xi|) - \varepsilon\rho) d\rho
\]

\[
- \int_0^{\infty} \psi'(\rho) \ln \rho \exp(i\rho(1 \pm |\xi|) - \varepsilon\rho) d\rho,
\]

if \( \gamma = 0 \), in view of 4.331.1 [15], where \( c \) is Euler’s constant. The desired result then readily follows from this.
4.2 Auxiliary estimates

Let us consider the operator $S^\alpha$ defined by (1.3). Decompose $S^\alpha$ into (2.2) and denote by $K^\alpha_\ell$ the kernel of the operator $S^\alpha_\ell$.

Lemma 4.3. Let $\ell = 0, 1, 2, \ldots$. Then

$$|\hat{K}^\alpha_\ell(\xi)| \leq C 2^{-M\ell} \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{2}, \\ (1 + |\xi|)^{-M}, & \text{if } |\xi| \geq 2 \end{cases}$$

(4.10)

for any $M > 0$, and

$$|\hat{K}^\alpha_\ell(\xi)| \leq C 2^{(\alpha - \frac{n-1}{2})\ell}, \quad \text{if } \frac{1}{2} < |\xi| < 2.$$ (4.11)

Proof. Let $\ell = 1, 2, \ldots$. Making use of (1.3), we get

$$\hat{K}^\alpha_\ell(\xi) = \frac{2^{\ell(n/2+1)}(2\pi)^{n/2}}{|\xi|^{n/2}} \int_0^\infty \rho^{n/2} \exp(i2^\ell \rho) J_{n/2}(2^\ell \rho \xi) \psi(\rho) d\rho.$$ 

In the case $|\xi| \leq \frac{1}{2}$ we apply the integral representation for the Bessel function

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_{-1}^1 (1 - t^2)^{\nu-1/2} e^{itz} dt,$$

(see [4]) and obtain

$$\hat{K}^\alpha_\ell(\xi) = C(n) 2^{\alpha \ell} \int_{-1}^1 (1 - t^2)^{\frac{n-1}{2}-1} dt \int_0^\infty \rho^{n/2} \exp(i2^\ell \rho + i2^\ell \rho t) \psi(\rho) d\rho.$$ 

Then the estimate (4.10) for $|\xi| \leq \frac{1}{2}$ can be easily obtained from the relation

$$\int_0^\infty \rho^{n/2} \exp(i2^\ell \rho + i2^\ell \rho t) \psi(\rho) d\rho = \left[ \frac{i}{2^\ell(1 + \xi t)} \right]^k \int_0^\infty \frac{d}{d\rho} \left( \rho^{n/2} \psi(\rho) \right) \exp(i2^\ell \rho + i2^\ell \rho t) d\rho.$$ 

The latter obviously holds for any $k > 0$ by means of repeated integration by parts. Letting $|\xi| > \frac{1}{2}$, in view of (3.1) we have

$$\hat{K}^\alpha_\ell(\xi) = 2^{\ell(n-\frac{n-1}{2})} \left[ \sum_{m=0}^{M-1} \frac{1}{|\xi|^{n/2-m}} \left( C_{m,-} I_{m,-}(|\xi|) + C_{m,+} I_{m,+}(|\xi|) \right) + \frac{1}{|\xi|^{-\frac{n-1}{2}}} \left( C_{M,-} I_{M,-}(|\xi|) + C_{M,+} I_{M,+}(|\xi|) \right) \right],$$
where
\[ I_{m,\pm}(\xi) = \int_0^{\infty} \rho^{\frac{n-1}{2} - m} \exp(i2^\ell \rho(1 \pm |\xi|)) \psi(\rho) \, d\rho, \]
\[ I_{M,\pm}(\xi) = \int_0^{\infty} \rho^{\frac{n-1}{2}} \exp(i2^\ell \rho(1 \pm |\xi|)) \psi(\rho) R_{M,\pm}(2^\ell |\xi|) \, d\rho. \]

Evidently, \(|\hat{K}_\ell^\alpha(\xi)| \leq C 2^{\ell(\alpha - \frac{n-1}{2})}\), provided \(\frac{1}{2} < |\xi| < 2\). Repeated integration by parts yields
\[ \int_0^{\infty} \rho^{\frac{n-1}{2} - m} \exp(i2^\ell \rho(1 \pm |\xi|)) \psi(\rho) \, d\rho = \left[ \int_0^{1/2} \frac{i}{2^\ell (1 \pm \xi)} \right]^k \int_0^{\infty} \left( \frac{d}{d\rho} \right)^k (\rho^{\frac{n-1}{2} - m} \psi(\rho)) \exp(i2^\ell \rho(1 \pm |\xi|)) \, d\rho. \]

Taking into account (3.2), we obtain (4.10) for \(|\xi| \geq 2\). \[\Box\]

**Lemma 4.4.** Let \(\ell = 0, 1, 2, \ldots, f \in L_p, 1 \leq p \leq \frac{2(n+1)}{n+3}\). Then
\[ \|S_\ell^\alpha f\|_2 \leq C 2^{\ell(\alpha - n/2)} \|f\|_p \] (4.12)
with \(C\) independent of \(\ell\).

**Proof.** Since the kernel \(K_\ell^\alpha\) of the operator \(S_\ell^\alpha\) is supported in the annulus \(2^{\ell-1} < |y| < 2^{\ell+1}\), it suffices to proof (4.12) for functions supported in the ball of radius \(2^{\ell+1}\). For such \(f\) we obtain
\[ \|S_\ell^\alpha f\|_2^2 = \left( \int_{|\xi| \leq 1/2} + \int_{|\xi| \geq 2} + \int_{1/2 < |\xi| < 2} \right) |\hat{K}_\ell^\alpha(\xi)|^2 |\hat{f}(\xi)|^2 d\xi = I_1 + I_2 + I_3. \]

The integrals \(I_1\) and \(I_2\) are easily treated by (4.11). For \(I_3\) we use (3.7), the radiality of \(\hat{K}_\ell^\alpha\) and the estimate \(\|K_\ell^\alpha\|_2 \leq C 2^{\ell(\alpha - n/2)}\). Hence
\[ I_3 \leq \|f\|_p^2 \|K_\ell^\alpha\|_2^2 \leq C 2^{2\ell(\alpha - n/2)} \|f\|_p^2, \]
and (4.12) is proved. \[\Box\]

### 4.3 \(L_p\)– boundedness property of the operator \(S^\alpha\)

**Theorem 4.1.** The operator \(S^\alpha\) is bounded on \(L_p\) whenever
1. \(0 < \alpha < \frac{1}{2}, \frac{2}{2-\alpha} < p < \frac{2}{\alpha}\), if \(n = 2\);
2. \(0 < \alpha < \frac{n(n-1)}{2(n+1)}, \frac{n}{n-\alpha} < p < \frac{n}{\alpha}\), if \(n \geq 3\);
3. \(\frac{n(n-1)}{2(n+1)} \leq \alpha < \frac{n-1}{2}, \frac{2(n-1)}{3(n-1)-4\alpha} < p < \frac{2(n-1)}{4\alpha-n+1}\), if \(n \geq 3\).
Proof. Making use of (2.1), we have \( \| S_\alpha^\ell \| = 2^{\alpha \ell} \| G_{2^{\ell}} \| \), and
\[
\| S_\alpha^\ell f \|_p \leq C 2^{\ell (\alpha - \frac{n}{2p})} \| f \|_p,
\]
for \( 1 \leq p \leq \frac{2n+2}{n+3} \) when \( n > 2 \) and \( 1 \leq p < \frac{4}{3} \) when \( n = 2 \) in view of Lemma 3.4. Summation over \( \ell \) and duality give the desired conclusions in the first two cases.

To obtain the last desired result, we will interpolate between (4.13) and the estimate
\[
\| S_\alpha^\ell f \|_2 \leq C 2^{\ell (\alpha - \frac{n-1}{2})} \| f \|_2.
\]
By this we arrive at the inequality
\[
\| S_\alpha^\ell f \|_p \leq C 2^{\ell (\alpha + \frac{n-1}{2p} - \frac{3(n-1)}{4})} \| f \|_p,
\]
and we are done.

\[\boxed{\text{Remark 4.2. A necessary condition for } S^\alpha \text{ to be bounded on } L_p \text{ is } \frac{n}{n-a} < p < \frac{n}{a}.}\]

### 4.4 \((L_p, L_q)\) estimates along a segment through \((\frac{1}{p}, \frac{1}{p'})\) and perpendicular to the line of duality

Lemma 4.5. Let \( n = 2, \frac{1}{2} < \alpha < 2, f \in L_p \) and \((\frac{1}{p}, \frac{1}{q})\) is on the open segment \((B'B)\). Then there is \( C = C(\alpha) \) such that \( \| S^\alpha f \|_q \leq C \| f \|_p \).

Proof. Let us fix point \( P(\frac{1}{p}, \frac{1}{q}) \) on \((BB')\) such that \( \frac{1}{3} - \frac{\alpha}{6} < \frac{1}{q} < \min\{\frac{1}{4}, 1 - \frac{\alpha}{2}\} \). Making use of (3.10) and the assertion
\[
(S_\alpha^\ell f)(x) = 2^{\alpha \ell} G_{2^{\ell}}(f_{2^{\ell}})(2^{-\ell} x)
\]
where \( f_\lambda(x) = f(\lambda x) \), we have
\[
\| S_\alpha^\ell f \|_q \leq C 2^{(\alpha - 2 + 6q)/(q)} \| f \|_r, \quad q = 3r'.
\]
Note that by duality (3.10) is equivalent to the estimate
\[
\| G_\lambda f \|_b \leq C \lambda^{-2/3b} \| f \|_a,
\]
for \( b > 4/3 \) and \( 1/b = 3(1 - 1/a) \). It follows from (4.13) and (4.17) that
\[
\| S_\alpha^\ell f \|_q \leq C 2^{(\alpha - 2 + 2/q)/(q)} \| f \|_a,
\]
where \(1/a = 1 - 1/3q\). Estimates (4.16) and (4.18) applied to a characteristic function \(f = \chi_E\) give

\[
\|S^{\alpha}_\ell \chi_E\|_q \leq C \min\{2^{(\alpha-2+6/q)\ell} |E|^{1-3/q}, \ 2^{(\alpha-2+2/q)\ell} |E|^{1-1/3q}\}.
\]

Observe that the first term in the braces is smaller then the second precisely when \(2\ell < |E|^{2/3}\). Let \(v = |E|\) and for \(v > 0\) let \(N = N(v)\) be the integer such that \(2^N < v^{2/3} \leq 2^{N+1}\). Then

\[
\sum_{\ell=1}^{\infty} \|S^{\alpha}_\ell \chi_E\|_q \leq C \sum_{\ell=-\infty}^{N} 2^{(\alpha-2+6/q)\ell} v^{1-3/q} + C \sum_{\ell=N+1}^{\infty} 2^{(\alpha-2+2/q)\ell} v^{1-1/3q}.
\]

Since \(1/3 - \alpha/6 < \frac{1}{q} < 1 - \frac{\alpha}{2}\), the two geometric series are convergent, and we obtain

\[
\sum_{\ell=1}^{\infty} \|S^{\alpha}_\ell \chi_E\|_q \leq C 2^{N(\alpha-2+6/q)\ell} v^{1-3/q} + C 2^{(N+1)(\alpha-2+2/q)\ell} v^{1-1/3q} \leq
\]

\[
\leq C v^{(2/3)(\alpha-2+6/q)\ell} v^{1-3/q} + v^{(2/3)(\alpha-2+2/q)\ell} v^{1-1/3q} \leq C v^{1/q-1/3+2\alpha/3}.
\]

Hence we get

\[
\|S^{\alpha}_\ell \chi_E\|_q \leq C \|\chi_E\|_p, \quad (4.19)
\]

whenever \(\frac{1}{3} - \frac{\alpha}{6} < \frac{1}{q} < \min\{\frac{1}{q}; 1 - \frac{\alpha}{2}\}\) and \(\frac{1}{p} = \frac{1}{q} - \frac{1}{3} + \frac{2\alpha}{3}\). It is well known ([30]) that (4.19) is equivalent to the Lorentz space estimate

\[
\|S^{\alpha} f\|_q \leq C \|f\|_{L^{p,1}} \quad (4.20)
\]

for the same \((p,q)\). In the case \(\alpha \geq 3/2\) the estimate (4.20) implies

\[
\|S^{\alpha} f\|_q \leq C \|f\|_p \quad (4.21)
\]

for any given point \((1/p, 1/q)\) on \((BB')\). For \(\alpha < 3/2\) the estimate (4.21) for any given point \((\frac{1}{p}, \frac{1}{q})\) on \((BB')\) follows by duality and interpolation. \[\Box\]

**Lemma 4.6.** Let \(n > 2\), \(\frac{n-1}{2} < \alpha < n\), \(f \in L_p\) and \((\frac{1}{p}, \frac{1}{q})\) is either on the open segment \((PP')\), provided \(\alpha < \frac{n}{2}\) or on \((B'B)\), provided \(\frac{n}{2} \leq \alpha < n\). Then there is \(C = C(\alpha)\) such that

\[
\|K^{\alpha} f\|_q \leq C \|f\|_p.
\]
Proof. Let us consider the family of operators $T_z$, $z \in S = \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \}$, defined on simple functions $f$ as follows:

$$T_z f = \begin{cases} \frac{1}{\Gamma(\frac{n+1}{2})} K^{\frac{n+1}{2}} f, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases} \quad (4.22)$$

We prove that this family is admissible growth. Given $f, g \in E$, let us put $F(z) = \int (T_z f)(x) g(x) dx$. The condition i) obviously holds by virtue of Lemma 3.1. To prove ii) it remains to show that $F(z)$ is continuous in a strip $S_\delta = \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq \delta \}$ of arbitrarily small width $\delta$. Let us fix $\delta < \frac{1}{n+1}$. Since $T_z f \in L_2$ for $z \in S_\delta$ the application of Parseval’s formula yields

$$F(z) = \frac{(2\pi)^{-n}}{\Gamma(\frac{z(n+1)}{2})} \int_{\mathbb{R}^n} m_{z(n+1) + \frac{n+1}{2}}(\xi) (F f)(\xi) (F g)(-\xi) d\xi. \quad (4.23)$$

The integral on the right-hand side is continuous on $S_\delta$, provided $z \neq 0$, in view of (4.7). In the case $z = 0$ we have $\lim_{z \to 0} T_z f = 0$, hence $F(z)$ is continuous at the point $z = 0$. Assertion iii) easily follows from (4.23) and (4.4), if $0 \leq \text{Re} z \leq \delta, \delta < \frac{1}{n+1}$. For $\frac{1}{n+1} < \text{Re} z \leq 1$ we have

$$|F(z)| \leq \int_{\mathbb{R}^n} |g(x)|(|Q_z(x)| + |R_z(x)|) dx, \quad (4.24)$$

where

$$Q_z(x) = \frac{1}{\Gamma(\frac{n+1}{2})} \int_{|y| < 1} \frac{\exp(i|y|)}{|y|^\frac{n+1}{2}(1-z)} f(x-y) dy$$

and

$$R_z(x) = \frac{1}{\Gamma(\frac{n+1}{2})} \int_{|y| > 1} \frac{\exp(i|y|)}{|y|^\frac{n+1}{2}(1-z)} f(x-y) dy.$$
\[ \|T_{1+i\gamma}f\|_{\infty} \leq C \exp\left(\frac{\pi(n+1)}{2} |\gamma|\right)\|f\|_{1}. \]

Moreover,
\[ |\mu_z(|\xi|)|^2 \leq C \exp\left(\frac{\pi \text{Im} z(n+1)}{2} \Gamma(\text{Re} z(n+1))\right) \frac{\mu_{2\text{Re} z}(|\xi|)}{|\Gamma\left(\frac{z(n+1)}{2}\right)|^2}, \quad 0 \leq \text{Re} z < \frac{1}{n+1}, \]

where \( \mu_z(|\xi|) = \frac{1}{\Gamma\left(\frac{z(n+1)}{2}\right)} m_{z(n+1)+\frac{1}{2}}(|\xi|) \) (see Appendix). Next, we show that the operator \( T_t \), \( t \in [0,1] \), is continuous from \( L_{p_t} \) into \( X = L_{1,\beta} = \{f : \int \frac{|f(x)|}{|x|^{n+1}} dx < \infty, \quad \beta > n\} \), provided \( \frac{1}{p_t} > 1 - \frac{(n+1)(1-t)}{2n} \). Let \( f \in L_{p_t} \), then
\[ |(T_t f)(x)| \leq \frac{1}{|\Gamma\left(\frac{z(n+1)}{2}\right)|} \left( \int_{|y|<1} + \int_{|y|>1} \right) \frac{|f(x-y)|}{|y|^{rac{n+1}{2}(1-t)}} dy = \frac{1}{|\Gamma\left(\frac{z(n+1)}{2}\right)|} \|(T_{o,t} f)(x) + (T_{\infty,t} f)(x)\|. \]

The operator \( T_{o,t} \) is bounded on \( L_{p_t} \), while the operator \( T_{\infty,t} \) is bounded from \( L_{p_t} \) into \( L_{\infty} \).

Putting \( t = \frac{2n-n+1}{n+1}, \frac{1}{p_t} - \frac{1}{q_t} = t \) and interpolating, we obtain the desired result.

\section{5 Proof of the main result}

We now turn to the proof of Theorem 2.1

I. Let us decompose \( K^\alpha \) into \( (1.2) \). Since the operator \( N^\alpha \) is bounded from \( L_p \) into \( L_q \), provided \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{n}{\alpha} \) by Hardy-Littlewood-Sobolev theorem and the mapping \( f \rightarrow N^\alpha f \) is of "weak-type" \((1,1-\frac{n}{\alpha})\), we obtain \([OFAA] \backslash (\{A\} \cup \{A'\}) \subset \mathcal{L}(N^\alpha)\).

So it remains to investigate the operator \( S^\alpha \). We will consider the following situations separately.

1) The cases \( 0 < \alpha < \frac{n(n-1)}{2(n+1)}, n > 2 \) and \( 0 < \alpha < \frac{1}{2}, n = 2. \)

In view of Hölder’s inequality we have \( \|S^\alpha f\|_{\infty} \leq C\|f\|_{p}, \) provided \( \frac{1}{p} > \frac{n}{\alpha} \). It follows from theorem \( 1.1 \) and the Riesz-Thorin interpolation theorem that \( (A'H'HA) \cup (H'H) \cup (A'E) \cup [EA] \in \mathcal{L}(S^\alpha) \).

2) In the cases \( 1/2 < \alpha < 2, n = 2 \) and \( \frac{n}{2} \leq \alpha < n, n > 2 \) the conclusion \( (A'B'BAE) \cup (B'B) \cup (A'E) \cup [EA] \in \mathcal{L}(S^\alpha) \) is an easy consequence of Lemma \( 4.5 \) and Lemma \( 4.6 \).

3) The case \( \frac{n(n-1)}{2(n+1)} < \alpha < \frac{n}{2}, \alpha \neq \frac{n-1}{2}, n > 2. \)

Having decomposition \( 2.1 \) of the operator \( S^\alpha \) we shall now apply Riesz-Thorin interpolation theorem for the operators \( 2.2 \). Let us first interpolate between the estimates \( (4.13) \) and \( (4.12) \), taking both of them at the endpoint \( p_0 = \frac{2n+2}{n+3} \). Then we arrive at the inequality
\[ \|S^\alpha f\|_{q} \leq C 2^{\left(\frac{n}{q}+\alpha-n\right)} \|f\|_{p_0}, \]
which is valid for any \( p_0 \leq q \leq 2 \). We note, that in view of (5.1) the operator \( S^\alpha \) is bounded from \( L_{p_0} \) into \( L_q \) in the case \( \alpha \leq n/2 \) whenever \( \frac{1}{q} < 1 - \frac{\alpha}{n} \). Our second step will be interpolation between (5.1) and the estimate

\[
\|S_\ell f\|_\infty \leq C2^{\ell(\alpha-n)}\|f\|_1,
\]

which gives

\[
\|S_\ell f\|_q \leq C2^{\ell(\frac{\alpha}{q}+\alpha-n)}\|f\|_p,
\]

where \( 1 \leq p \leq p_0, q = \frac{(n-1)p_0q_1}{2(n+1)}, p_0 \leq q_1 \leq 2 \). The exponent of \( 2^\ell \) is negative if \( \frac{1}{q} < 1 - \frac{\alpha}{n} \). By virtue of the restriction \( \frac{1}{q} \leq \frac{1}{q_1} \) we have to require \( \frac{1}{q_1} \geq 1 - \frac{\alpha}{n} \). Then \( \frac{1}{p} > \frac{n+3}{2(n+1)} \) and (4.21) is valid for such \((p,q)\). It follows immediately from Lemma 4.6 and Theorem 4.1, that

\[
\begin{align*}
(A'G'P'PAE) \cup (A'E) \cup [EA] \cup (P'P) \subset \mathcal{L}(S^\alpha), & \quad \text{if } \frac{n-1}{2} < \alpha < \frac{n}{2}, \\
(A'G'CC'GAE) \cup (A'E) \cup [EA] \cup (CC') \subset \mathcal{L}(S^\alpha), & \quad \text{if } \frac{n(n-1)}{2(n+1)} \leq \alpha < \frac{n-1}{2}.
\end{align*}
\]

4). The case \( n = 2, \alpha = \frac{1}{2} \).

Interpolation between (4.13) and (5.2) yields

\[
\|S_\ell f\|_q \leq 2^{\ell(\frac{\alpha}{q}+\alpha-2)}\|f\|_p,
\]

where \( \frac{1}{p} = 1 - \frac{1}{q}(p_1 - 1) \), provided \( 1 \leq p_1 < \frac{3}{2} \). Thus, (4.21) holds whenever \( \frac{1}{p} > 1 - \frac{2-\alpha}{6} \), \( \frac{1}{q} < 1 - \frac{\alpha}{2} \). In particular, this is the case for \( \alpha = \frac{1}{2} \) and consequently \((A'H'H')AE) \cup (A'E) \cup [EA] \subset \mathcal{L}(S^\alpha)\).

5). The case \( n > 2, \alpha = \frac{n-1}{2} \). In a similar way, making use of the interpolation between (4.14) and (5.2), we obtain

\[
\|S_\ell f\|_{p'} \leq 2^{\ell(\frac{n+1}{p'}+\alpha-2)}\|f\|_p.
\]

It is clear that \((DE) \subset \mathcal{L}(S^\alpha)\) and therefore \((A'G'LGAE) \cup (A'E) \cup [EA] \subset \mathcal{L}(S^\alpha)\) for \( \alpha = \frac{n-1}{2} \) by means of (5.3).

II. One can show by an elementary argument that the operator \( K^\alpha \) is unbounded from \( L_p \) into \( L_q \) whenever either \((\frac{1}{p}, \frac{1}{q}) \in [HAF] \cup [H'A'O] \) or \((\frac{1}{p}, \frac{1}{q}) \in [A'AE] \setminus (AA')\). The proof of the negative result 3) is precisely that used in [6].

6 Appendix

To verify (1.23), we first obtain the auxiliary estimate

\[
|1 - |\xi|^2| - \Re z^{(n+1)} \leq Cz\mu_2\Re z(|\xi|), \quad \Re z < \frac{1}{n+1},
\]

(6.1)
where \( C_z = C \Gamma(\Re z(n+1)) \) (with \( C \) independent of \( z \)). In the case \(|z| < 1\) we have
\[
\mu_{2Re z}(|\xi|) = \frac{2\pi^{n/2} \Gamma(\Re z(n+1) + \frac{n-1}{2}) \exp\left[\left(\Re z(n+1) + \frac{n-1}{2}\pi i\right)\right]}{\Gamma\left(\frac{n}{2}\right)} (1 - |\xi|^2)^{-\Re z(n+1)}
\times \ _2F_1\left( \frac{n+1}{4} - \frac{\Re z(n+1)}{2}, \frac{n-1}{4} - \frac{\Re z(n+1)}{2}; \frac{n}{2}; |\xi|^2 \right).
\]

Making use of the integral representation for the hypergeometric function
\[
_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \quad 0 < \Re b < \Re c
\]
(see [21], p. 431), it is not difficult to verify that
\[
\mu_{2Re z}(|\xi|) \geq C \Gamma(\Re z(n+1) + \frac{n-1}{2})(1 - |\xi|^2)^{-\Re z(n+1)},
\]
and therefore (6.1) is proved provided \(|\xi| < 1\). In the case \(|\xi| > 1\) we use the representation (4.3) of the symbol \( m_{\Re z(n+1)+\frac{n-1}{2}}(|\xi|) \). Applying decomposition (4.3) and the arguments used in the proof of Lemma 4.2, we obtain (6.1). Since
\[
|\mu_z(|\xi|)|^2 = \frac{4\pi^n |\Gamma\left(\frac{z(n+1)}{2} + \frac{n-1}{2}\right)|^2 \exp\left[\left(z(n+1) + \frac{n-1}{2}\pi i\right)^2\right]}{|\Gamma\left(\frac{z(n+1)}{2}\right)||\Gamma\left(\frac{n}{2}\right)|^2} (1 - |\xi|^2)^{-\Re z(n+1)}
\times \ _2F_1\left( \frac{n+1}{4} - \frac{z(n+1)}{4}, \frac{n-1}{4} - \frac{z(n+1)}{4}; \frac{n}{2}; |\xi|^2 \right)^2, \quad |\xi| < 1,
\]
it follows from (6.1) and (3.3) that (4.25) is valid. We prove (4.25) for \(|\xi| > 1\) in just the same way as is proved for \(|\xi| < 1\).

Acknowledgments. I would like to express my gratitude to Prof. V. A. Nogin for suggesting this problem to me as well as for his patient help and criticisms. I am thankful to Prof. E. Liflyand for numerous discussions and valuable comments to the manuscript of the paper, and to Prof. B. Rubin for useful discussions of the results.

References

[1] J.-G. Bak, Sharp estimates for the Bochner–Riesz operator of negative order in \( \mathbb{R}^2 \). Proc. Amer. Math. Soc., 125 (1997), No. 7, 1977–1986.

[2] J.-G. Bak, D. McMichael, D. Oberlin, \( L_p - L_q \) estimates off the line of duality. J. Austral. Math. Soc., (Series A) 58 (1995), No. 7, 154-166.
[3] H. Bateman, A. Erdelyi, Higher transcendental functions, Vol I, McGraw-Hill Book Company, 1953.

[4] H. Bateman, A. Erdelyi, Higher transcendental functions, Vol II, McGraw-Hill Book Company, 1953.

[5] L. Börjeson, Estimates for the Bochner–Riesz operator with negative index. Indiana Univ. Math. J., 35 (1989), 225–233.

[6] J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis. Geom. Funct. Anal. 1 (1991), 147–187.

[7] L. Brandolini, L. Colzani, Bochner–Riesz means with negative index of radial functions in Sobolev spaces. Rend. Circ. Mat. Palermo (2) 42 (1993), No. 1, 117-128.

[8] A. Carbery, F. Soria, Almost-everywhere convergence of Fourier integrals for functions in Sobolev spaces, and an $L^2$-localisation principle. Rev. Mat. Iberoamericana 4 (1988), 319–337.

[9] L. Carleson, P. Sjölin, Oscillatory integrals and a multiplier problem for the disc, Studia Math. 44 (1972), 287–299.

[10] L. Colzani, G. Travaglini, M. Vignati, Bochner-Riesz means of functions in weak $L^p$. Rend. Circ. Mat. Palermo (2) 42 (1993), No. 1, 117–128.

[11] A. Cordoba, A note on Bochner–Riesz operators. Duke Math. J. 46 (1979), 505–511.

[12] K. M. Davis, Y. C. Chang, Lectures on Bochner-Riesz means, London Math. Society Lecture Notes, Series 114.

[13] C. Fefferman, A note on spherical summation multipliers, Israel J. Math. 15 (1973), 44–52.

[14] C. Fefferman, Inequalities for strongly singular convolution operators. Acta Math. 124 (1970), 9–36.

[15] I. S. Gradshteyn, I. M. Ryzhik, Table of integrals, series and products. Acad. Press, New York and London, 1965.

[16] C. S. Herz, On the mean inversion of the Fourier and Hankel transforms. Proc. Nat. Acad. Sci. 40 (1954), 996–999.
[17] L. Hörmander, Oscillatory integrals and multipliers on $F L^p$. Ark. Mat. 11 (1973), 1–11.

[18] V. A. Nogin, B. S. Rubin, Bounds for potentials with oscillating kernels related to the Helmholtz equation. Differential equations, 9 (1991), 1195–1200.

[19] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, Integrals and series. Vol. 1: Elementary functions. Gordon and Breach Sci. Pub., 1988.

[20] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, Integrals and series. Vol. 2: Special functions. Gordon and Breach Sci. Pub., 1988.

[21] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, Integrals and series. Vol. 3: More special functions. Gordon and Breach Sci. Pub., 1988.

[22] B. Rubin, $L_p$-properties of solutions of the singular Helmholtz equations in the half-space. Frac. Cal. Appl. An., 2 (1999), No. 1, 47–61.

[23] B. Rubin, Fractional integrals and potentials. Pitman Monographs and Surveys in Pure and Appl. Math., 82, Longman, Harlow, 1996.

[24] T. Tao, Weak-type endpoint bounds for Riesz means. Proc. Amer. Math. Soc. 124 (1996), No. 9, 2797–2805.

[25] T. Tao, The Bochner-Riesz conjecture implies the restriction conjecture. Duke Math. J. 96 (1999), no. 2, 363–375.

[26] T. Tomas, A restriction theorem for the Fourier transform. Bull. Amer. Math. Soc. 81 (1975), 477–478.

[27] C. D. Sogge, Oscillatory integrals and spherical harmonics. Duke Math. J., 53 (1986), No. 1, 43–65.

[28] E. Stein, Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals. Princeton Univ. Press, Princeton, 1993.

[29] E. Stein, Interpolation of linear operators. Trans. Amer. Math. Soc., 83 (1956), 482–492.

[30] E. Stein, G. Weiss, Introduction to Fourier analysis on Euclidean spaces. Princeton Univ. Press, Princeton, N.J., 1971.