Feedback Interconnected Density Estimation and Control

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Abstract—Swarm robotic systems have foreseeable applications in the near future. Recently, there has been an increasing amount of literature that employs partial differential equations (PDEs) to model the time-evolution of the probability density of swarm robotic systems and uses density feedback to design stable control laws that act on individuals such that their density converges to a target profile. However, it remains largely unexplored considering problems of how to estimate the real-time density, how the density estimation algorithms affect the control performance, and whether the estimation performance in turn depends on the control algorithms. In this work, we focus on studying the interplay of these algorithms. Specifically, we propose new density control laws which use the real-time density and its gradient as feedback, and prove that they are globally input-to-state stable (ISS) with respect to estimation errors. Then, we design filtering algorithms to obtain estimates of the density and its gradient, and prove that these estimates are convergent assuming the control laws are known. Finally, we show that the feedback interconnection of these estimation and control algorithms is still globally ISS, which is attributed to the bilinearity of the PDE system. An agent-based simulation is included to verify the stability of these algorithms and their feedback interconnection.

I. INTRODUCTION

Swarm robotic systems (such as drones) have foreseeable applications in the near future. Compared with small-scale robotic systems, the dramatic increase in the number of involved robots provides numerous advantages such as robustness, efficiency, and flexibility, but also poses significant challenges to their estimation and control problems.

Many methods have been proposed for controlling large-scale systems, such as graph theoretic design [1] and game theoretic formulation (especially potential games [2] and mean-field games [3]). Our work is inspired by the recent density-based modelling and control strategy. We note that unlike mean-field games/control [4] which use the density to approximate the collective effect of the swarm, we aim at the direct control of the density. Density-based models include Markov chains and PDEs. The first category partitions the spatial domain to obtain an abstracted Markov chain model and designs the transition probability to stabilize the density [5], [6], which usually suffers from the state explosion issue. In PDE-based models, individual robots are modelled by a family of stochastic differential equations and their density evolves according to a PDE. In this way, the density control problem of a robotic swarm is posed as a regulation problem of the PDE.

Considering the density control problem, early efforts usually adopt an optimal control formulation [7], [8]. While an optimal control formulation provides more flexibility for the control objective, the solution relies on numerically solving the optimality conditions, which are computationally expensive and essentially open-loop. Closed-loop optimal density control is studied in [9], [10] by relating density control problems with the so-called Schrödinger Bridge problems. However, numerically solving the associated Schrödinger Bridge problem is also known to suffer from the curse of dimensionality except for the linear case. In recent years, researchers have sought to design control laws that use the real-time density as feedback to form a closed-loop control framework [11]–[14]. These density feedback laws are able to guarantee closed-loop stability and can be efficiently computed on board. However, it remains largely unexplored considering problems of how to estimate the real-time density, how the density estimation algorithm affects the control performance, and whether the estimation performance in turn depends on the density control algorithms. These problems become more critical as it is observed that most of the density feedback laws more or less depend on the gradient of the density. Since the gradient operator is an unbounded operator, any density estimation algorithm that produces accurate density estimates may have arbitrarily large estimation error for the gradient. This brings significant concerns to the density estimation algorithms.

In this work, we aim to study the interplay of density feedback laws and estimation algorithms. In our previous work [14], we have proposed some density feedback laws and obtained preliminary results on their robustness to density estimation errors. In this work, we extend these feedback laws so that they are less restrictive and have more verifiable robustness properties in the presence of estimation errors. We have also presented a density filtering algorithm in [15] which is particularly for large-scale stochastic systems modelled by PDEs. In this work, we will extend this algorithm to directly estimate the gradient of the density, a quantity required by almost all existing density feedback control in the literature. Furthermore, we study the interconnection of these estimation and control algorithms and prove their closed-loop stability.

Our contribution includes three aspects. First, we propose new density feedback laws and show their robustness using the notion of ISS. Second, we design infinite-dimensional filters to estimate the gradient of the density and study their stability and optimality. Third, we prove that the feedback
interconnection of these estimation and control algorithms is still globally ISS.

The rest of the paper is organized as follows. Section II introduces some preliminaries. Problem formulation is given in Section III. Section IV is our main results in which we propose new density estimation and control laws, and study their interconnected stability. Section V presents an agent-based simulation to verify the effectiveness.

II. PRELIMINARIES

A. Notations

Let $E \subset \mathbb{R}^n$ be a measurable set. Consider $f : E \to \mathbb{R}$. For $p \in [1, \infty)$, denote $L^p(E) = \{ f \mid \| f \|_{L^p(E)} := \left( \int_E |f(x)|^p \, dx \right)^{1/p} < \infty \}$, endowed with the norm $\| f \|_{L^p(E)}$. Denote $L^\infty(E) = \{ f \mid \| f \|_{L^\infty(E)} := \sup_{x \in E} |f(x)| < \infty \}$, endowed with the norm $\| f \|_{L^\infty(E)}$. Given $g(x)$ with $\inf_{x \in E} g(x) > 0$ and $\sup_{x \in E} g(x) < \infty$, the weighted norm $\| f \|_{L^p(E; g)} := \left( \int_E |f(x)|^p g(x) \, dx \right)^{1/p}$ is equivalent to $\| f \|_{L^p(E)}$. Let $D^2f$ be the weak derivatives of $f$ for all multi-indices $\alpha$ of length $|\alpha|$. For $p \in [1, \infty)$, denote $W^{k,p}(E) = \{ f \mid \| f \|_{W^{k,p}(E)} := \sum_{|\alpha| \leq k} \| D^\alpha f \|_{L^p(E)} < \infty \}$, endowed with the norm $\cdot \|_{W^{k,p}(E)}$. Analogously, $W^{k,\infty}(E)$ is defined, equipped with the norm $\cdot \|_{W^{k,\infty}(E)}$. We also denote $H^k = W^{k,2}$. We will omit $E$ in the norms when it is clear. The gradient and Laplacian of a scalar function $f$ are denoted by $\nabla f$ and $\Delta f$, respectively. The divergence of a vector field $F$ is denoted by $\nabla \cdot F$.

The following lemma is a consequence of Proposition 4.2.7 and Theorem 4.6.3 in [16].

Lemma 1 (Poincaré inequality for density functions):

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex open set with a Lipschitz boundary. Let $g$ be a continuous density function on $\Omega$ such that $0 < c_1 \leq \inf g \leq \sup g \leq c_2$ for some constants $c_1$ and $c_2$. Then, $\exists \alpha > 0$ such that for all $f \in H^1(\Omega)$,

$$
\int_{\Omega} |\nabla f|^2 g \, dx \geq C \int_{\Omega} \left| f - \int_{\Omega} f \, g \, dx \right|^2 \, g \, dx.
$$

B. Input-to-state stability

We introduce ISS for infinite-dimensional systems [17]. Define the following classes of comparison functions:

$$
\mathcal{P} := \{ \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0, \text{ and } \gamma(r) > 0 \text{ for } r > 0 \},
$$

$$
\mathcal{K} := \{ \gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing} \},
$$

$$
\mathcal{K}_\infty := \{ \gamma \in \mathcal{K} \mid \gamma \text{ is unbounded} \},
$$

$$
\mathcal{L} := \{ \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t \to \infty} \gamma(t) = 0 \},
$$

$$
\mathcal{K} \cap \mathcal{L} := \{ \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta(\cdot, t) \in \mathcal{K}, \forall t \geq 0, \beta(\cdot, \cdot) \in \mathcal{L}, \forall r > 0 \}.
$$

Let $(X, \| \cdot \|_X)$ and $(U, \| \cdot \|_U)$ be the state and input space, endowed with norms $\| \cdot \|_X$ and $\| \cdot \|_U$, respectively. Denote $U_\gamma = PC(\mathbb{R}_+; U)$, the space of piecewise right-continuous functions from $\mathbb{R}_+$ to $U$, equipped with the supremum norm. Consider a control system $\Sigma = (X, U_\gamma, \phi)$ where $\phi : \mathbb{R}_+ \times X \times U \to X$ is a transition map. Let $x(t) = \phi(t, x_0, u)$.

**Definition 1:** $\Sigma$ is called locally input-to-state stable (LISS), if $\exists \rho_x, \rho_u > 0$, $\beta \in \mathcal{K} \cap \mathcal{L}$, and $\gamma \in \mathcal{K}$, such that

$$
\| x(t) \|_X \leq \beta(\| x_0 \|_X, t) + \gamma \left( \sup_{0 \leq s \leq t} \| u(s) \|_U \right), \quad t \geq 0
$$

$$
\forall x_0 \in X : \| x_0 \|_X \leq \rho_x, \forall u \in U_\gamma : \| u \|_U \leq \rho_u.
$$

It is called input-to-state stable (ISS), if $\rho_x = \infty$ and $\rho_u = \infty$.

**Definition 2:** A function $V : \mathbb{R}_+ \times D \to \mathbb{R}_+$, $D \in X$ is called an LISS-Lyapunov function for $\Sigma$, if $\exists \rho_x, \rho_u > 0$, $\psi_1, \psi_2 \in \mathcal{K} \cap \mathcal{L}$, and $W \in \mathcal{P}$, such that:

(i) $\psi_1(\| x \|_X) \leq V(t, x), \forall t \geq 0, \forall x \in X$

(ii) $\forall x \in X : \| x \|_X \leq \rho_x, u \in U_\gamma : \| u \|_U \leq \rho_u$ with $u(0) = 0$ it holds:

$$
\| x \|_X \leq \psi_2(\| x \|_X) \Rightarrow \dot{V}(t, x) \leq -W(\| x \|_X).
$$

If $D = X$, $\rho_x = \infty$ and $\rho_u = \infty$, then the function $V$ is called an ISS-Lyapunov function.

**Theorem 1:** If $\Sigma$ possesses an (L)ISS-Lyapunov function, then it is (L)ISS.

ISS is a convenient tool for studying the stability of cascade systems. Consider two systems $\Sigma_i = (X_i, U_{\gamma,i}, \phi_i), i = 1, 2$, where $U_{\gamma,i} = PC(\mathbb{R}_+; U_i)$ and $X_i \subset U_2$. We say they form a cascade connection if $u_2(t) = \phi_1(t, t_0, \phi_{01}, u_1)$.

**Theorem 2:** [18] The cascade connection of two ISS systems is ISS. If one of them is LISS, then the cascade connection is LISS.

C. Infinite-dimensional Kalman filters

We introduce the infinite-dimensional Kalman filters presented in [19]. Let $\mathcal{H}, \mathcal{K}$ be real Hilbert spaces. Consider the following infinite-dimensional linear system:

$$
du(t) = A(t)u(t) dt + B(t)dw(t), \quad u(0) = u_0
$$

$$
dz(t) = C(t)u(t) dt + F(t)du(t), \quad z(0) = 0,
$$

where $A(t)$ is a linear closed operator on $\mathcal{H}$, $B(\cdot) \in L^\infty([0, T]; \mathcal{L}(\mathcal{H}))$, $C(\cdot) \in L^\infty([0, T]; \mathcal{L}(\mathcal{K}))$, and $F(\cdot) (F(\cdot))^{-1} \in L^\infty([0, T]; \mathcal{L}(\mathcal{K}))$. $w(t)$ and $v(t)$ are independent Wiener processes on $\mathcal{H}$ and $\mathcal{K}$ with covariance operators $\mathcal{W}$ and $\mathcal{V}$, respectively. The infinite-dimensional Kalman filter is given by:

$$
d\hat{u}(t) = A(t)\hat{u}(t) dt + K(t)(dz(t) - C(t)u(t) dt), \quad \hat{u}(0) = 0
$$

where $K(t) = P(t)C^*(t)(F(t)VF^*(t))^{-1}$ is the Kalman gain, and $P(t)$ is the solution of the operator Riccati equation:

$$
dP(t) dt = A(t)P(t) + P(t)A^*(t) + B(t)WB^*(t)
$$

$$
- P(t)C^*(t)(F(t)VF^*(t))^{-1}C(t)P(t) + P(0) = P_0.
$$

III. PROBLEM FORMULATION

This work studies the density control problem of robotic swarms. Specifically, we want to design velocity commands for individual robots such that their density evolves to a target
profile. Consider $N$ robots in a bounded convex domain $\Omega \subset \mathbb{R}^n$. The robots are assumed to be homogeneous and satisfy:

$$dX_i^t = v(X_i^t, t)dt + \sqrt{2\sigma(t)}dB_i^t, \quad i = 1, \ldots, N,$$  
(2)

where $X_i^t \in \Omega$ is a stochastic process representing the position of the $i$-th robot, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is the velocity field acting on the robots, $\{B_i^t\}$ are standard Wiener processes assumed to be independent across the robots, and $\sqrt{2\sigma(t)} \in \mathbb{R}$ is the standard deviation. Let $p(x, t)$ be the probability density of the robots. The evolution of $p$ satisfies the Fokker-Planck equation and is given by:

$$\partial_t p = -\nabla \cdot (vp) + \Delta (\sigma p) \quad \text{in} \quad \Omega \times (0, T),$$

$$p = p_0 \quad \text{on} \quad \Omega \times \{0\},$$

$$n \cdot (\nabla (\sigma p) - vp) = 0 \quad \text{on} \quad \partial \Omega \times (0, T),$$

where $n$ is the unit normal inner to the boundary $\partial \Omega$, and $p_0(x)$ is the initial density. The last equation is the reflecting boundary condition to confine the robots within $\Omega$.

**Remark 1:** The relation of (2) and (3) holds regardless of the number of robots. However, if $N$ is small, using the density to represent a swarm doesn’t make much sense. Hence, we usually assume $N$ is large. Note that (2) and (3) share the same coefficients, which means that the velocity field we design for (3) can be easily implemented in (2).

This work focuses on the interconnected stability of density estimation and control. For clarity, the robots are assumed to be first-order integrators in (2). Density control for heterogeneous higher-order nonlinear systems is studied in a separate work [20]. The interconnected stability results to be presented later can be generalized to these more general systems by combining the stability results in [20].

The problems studied in this work are stated as follow.

**Problem 1 (Density control):** Given a target density $p_*(x)$, we want to design the velocity field $v$ such that the solution of (3) converges to $p_*(x)$.

**Problem 2 (Density estimation):** Given the density dynamics (3) and the collection of robots’ states $\{X_i^t\}_{i=1}^N$, we want to estimate the density $p$ and its gradient.

**Problem 3 (Feedback interconnection):** Given a target density $p_*(x)$, let $v$ in (3) be designed as a feedback function of certain density estimates that are in turn computed based on (3). We want to show that the feedback interconnected system still converges to $p_*(x)$.

### IV. MAIN RESULTS

#### A. Modified density feedback laws

Given a continuous target density $p_*(x)$, bounded from above and below by positive constants, we propose the following density feedback law:

$$v(x, t) = -\alpha(x, t)\nabla \frac{p(x, t)}{p_*(x)} + \frac{\sigma(t)\nabla p_*(x)}{p_*(x)}$$

(4)

where $\alpha \geq 0$ is a design parameter for individuals to adjust their velocity magnitude.

**Remark 2:** Compared with the density feedback laws proposed in [11], [13], [14], the most remarkable difference of (4) is that the feedback density $p$ does not appear in any denominator, which provides several advantages. First, it relaxes the requirement for $p$ to be strictly positive and avoids the phenomenon of producing large velocity when $p$ is close to 0. Second, it will enable us to obtain ISS results with respect to estimation errors in $L^2$ norm. (Such results are difficult to obtain for most of the existing density feedback laws.) The significance of this property will become apparent when we study the interconnected stability of control and estimation algorithms.

In this section, we focus on the stability and robustness of (4). We define $\Phi = p - p_*$ as the convergence error.

**Theorem 3:** Consider system (3). Let $v$ be given by (4). Then $\|\Phi\|_{L^2}$ converges to 0 exponentially.

**Proof:** Substituting (4) into (3), we obtain the closed-loop system:

$$\partial_t p = \nabla \cdot \left( \alpha p \nabla \frac{p}{p_*} \right) - \nabla \cdot \left( \frac{\sigma p}{p_*) \nabla p_* \right) + \Delta (\sigma p)$$

$$= \nabla \cdot \left( \alpha p \nabla \frac{p}{p_*} \right) + \nabla \cdot \left( \frac{\sigma p}{p_*) \nabla p_* - \nabla (\sigma p) \right)$$

$$= \nabla \cdot \left( \alpha p \nabla \frac{p}{p_*} \right) + \nabla \cdot \left( \sigma p \nabla \frac{p}{p_*} \right).$$

Consider a Lyapunov function $V(\Phi) = \|\Phi\|_{L^2}^2$. By the divergence theorem and the boundary condition, we have

$$\frac{dV}{dt} = \int_\Omega \frac{p - p_*}{p_*} \partial_t p dx$$

$$= \int_\Omega \frac{p - p_*}{p_*} \left[ \nabla \cdot \left( \alpha p \nabla \frac{p}{p_*} \right) + \nabla \cdot \left( \frac{\sigma p}{p_*} \nabla p_* \right) \right] dx$$

$$= \int_\Omega -\alpha p \left( \nabla \frac{p}{p_*} \right)^2 - \sigma p \frac{\nabla \frac{p}{p_*}}{p_*^2} \sigma p_* \nabla \frac{p}{p_*}^2 dx$$

$$\leq - (\alpha_1(t) + \alpha_2(t)) \left\| \nabla \frac{p}{p_*} \right\|_{L^2}^2$$

$$= - (\alpha_1(t) + \alpha_2(t)) \left\| \frac{p - p_*}{p_*} \right\|_{L^2}^2.$$

By the Poincâré inequality (1) (where we set $f = \frac{p - p_*}{p_*}$ and $g = p_*$) and the fact that $\int_\Omega (p - p_*) dx = 0$, we have

$$\frac{dV}{dt} \leq - (\alpha_1(t) + \alpha_2(t)) C^2 \left\| \frac{p - p_*}{p_*} \right\|_{L^2}^2$$

$$= - (\alpha_1(t) + \alpha_2(t)) C^2 \|\Phi\|_{L^2(1/p_*)}^2$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are scalar functions of $t$ satisfying $0 \leq \min_x \alpha p \leq \alpha_1(t) \leq \max_x \alpha p$ and $0 < \min_x \sigma p_* \leq \alpha_2(t) \leq \max_x \sigma p_*$, and $C > 0$ is the constant in the Poincâré inequality. By the strong maximum principle [21], there exist $t_1, a > 0$ such that $p(x, t) \geq a$ for $t \geq t_1$. Hence, $\alpha_1 = 0$ if and only if $\alpha = 0$ for $t \geq t_1$.

**Remark 3:** Notice that we allow $\alpha = 0$ in (4). In this case, (4) is reduced to $v = \frac{\sigma \nabla p}{p_*}$, which is a well-known strategy to drive stochastic particles towards a target distribution in physics [22]. However, the convergence speed will be extremely slow since $\sigma$ is small in general. The term $-\alpha \nabla \frac{p}{p_*}$ is added to provide extra and locally adjustable speed for the
convergence. The acceleration effect can be seen from the term $\alpha_1$ in the proof.

The density $p$ in (4) is a probability density which cannot be measured directly and needs to be estimated using some estimation algorithms. We will introduce how to obtain the estimates in the next section. We first establish some robustness results regardless of what estimation algorithm to use. It is useful to rewrite (4) as:

$$v = -\alpha \frac{p_p \nabla p - p_p \nabla p_*}{p_*} + \frac{\sigma(t) \nabla p_*}{p_*}. \quad (5)$$

We will design algorithms to estimate $p$ and $\nabla p$ separately. The reason for estimating $\nabla p$ separately is that the gradient operator $\nabla$ is an unbounded operator, i.e., any algorithm that produces accurate estimates of $p$ may have arbitrarily large estimation errors for $\nabla p$. Thus, we need to design additional algorithms to estimate $\nabla p$.

Let $\hat{p}$ and $\hat{\nabla} p$ be estimates of $p$ and $\nabla p$, respectively. According to (5), the density feedback law using estimates is given by:

$$v = -\alpha \frac{p_p \nabla \hat{p} - \hat{p} \nabla p_*}{p_*} + \frac{\sigma(t) \nabla p_*}{p_*}. \quad (6)$$

Define $\epsilon = \hat{p} - p$ and $\epsilon_g = \hat{\nabla} p - \nabla p$ as the estimation errors. Substituting (6) into (3), we obtain the closed-loop system:

$$\partial_t \Phi = \partial_t \hat{p} = \nabla \cdot \left( \alpha \frac{p_p \nabla \hat{p} - \hat{p} \nabla p_*}{p_*} + \sigma_p \nabla \hat{p} \right). \quad (7)$$

We have the following robustness result with respect to these estimation errors.

**Theorem 4 (ISS of density feedback):** If $\alpha > 0$, then $\|\Phi\|_{L^2}$ is ISS with respect to $\|\epsilon\|_{L^2}$ and $\|\epsilon_g\|_{L^2}$.

**Proof:** Consider a Lyapunov function $V(\Phi) = \|\Phi\|_{L^2(1/p_*^2)}$. By the divergence theorem and the boundary condition, we have

$$\frac{dV}{dt} = \int_{\Omega} \frac{p_p - p_*}{p_*} \left[ \nabla \cdot \left( \alpha \frac{p_p \nabla \hat{p} - \hat{p} \nabla p_*}{p_*} + \sigma_p \nabla \hat{p} \right) \right] dx$$

$$= \int_{\Omega} \alpha \frac{p_p \nabla \hat{p} - \hat{p} \nabla p_*}{p_*} \nabla \cdot \left( \frac{p_p \nabla \hat{p} - \hat{p} \nabla p_*}{p_*} + \sigma_p \nabla \hat{p} \right) dx$$

$$- \sigma_p \nabla \phi_{\nabla \hat{p}}^2 dx$$

$$= \int_{\Omega} \alpha \frac{p_p \nabla \hat{p}}{p_*^2} \cdot \left( \frac{p_p}{p_*^2} \nabla p_* + \frac{1}{p_*} \epsilon_g - \nabla \phi_{\nabla \hat{p}} \epsilon \right)$$

$$- \sigma_p \nabla \phi_{\nabla \hat{p}}^2 dx$$

$$\leq - (\alpha_1(t) + \alpha_2(t)) \left\| \frac{\nabla \hat{p}}{p_*} \right\|^2_{L^2}$$

$$+ \alpha_3(t) \left\| \frac{\nabla \hat{p}}{p_*} \right\|_{L^2} \left\| \frac{1}{p_*} \epsilon_g - \nabla \phi_{\nabla \hat{p}} \epsilon \right\|_{L^2}$$

where $\alpha_1$ and $\alpha_2$ are defined in the same way as in the proof of Theorem 3 and $\alpha_3(t)$ is a scalar function of $t$ satisfying $0 \leq \min \epsilon \alpha_p \leq \alpha_3(t) \leq \max \epsilon \alpha_p$. Using a constant $\theta \in (0, 1)$ to split the first term and applying the Poincaré inequality [1], we have

$$\frac{dV}{dt} \leq - (\alpha_1(t) + \alpha_2(t)) (1 - \theta) C^2 \left\| \frac{p - p_*}{p_*} \right\|_{L^2}$$

$$- (\alpha_1(t) + \alpha_2(t)) \theta C \left\| \frac{\nabla \hat{p}}{p_*} \right\|_{L^2} \left\| \frac{p - p_*}{p_*} \right\|_{L^2}$$

$$+ \alpha_3(t) \left( \|\epsilon_g\|_{L^2(1/p_*^2)} + \|\nabla p_*\|_{L^\infty} \|\epsilon\|_{L^2(1/p_*^2)} \right) \left\| \frac{\nabla \hat{p}}{p_*} \right\|_{L^2}$$

$$\leq - (\alpha_1(t) + \alpha_2(t)) (1 - \theta) C^2 \left( \|\epsilon_g\|_{L^2(1/p_*^2)} + \|\nabla p_*\|_{L^\infty} \|\epsilon\|_{L^2(1/p_*^2)} \right) \left\| \frac{\nabla \hat{p}}{p_*} \right\|_{L^2}$$

The ISS property follows from Theorem 1.
First, we need to derive an evolution equation for $\nabla p$. By applying the gradient operator on both sides of $\frac{\partial f}{\partial p}$, we have
\[
\frac{\partial}{\partial t}(\nabla p) = \nabla(\partial_t p) = \nabla[-\nabla \cdot (v_p) + \Delta(\sigma p)]
= \nabla[-\nabla \cdot (v \mathcal{F}(\nabla p)) + \Delta(\sigma \mathcal{F}(\nabla p))]
\]
where $\mathcal{F}$ is an integration operator defined as follows. For a given vector field $F : \Omega \to \mathbb{R}^n$, define $\mathcal{F}(F) = f$, where $f : \Omega \to \mathbb{R}$ is uniquely determined by the following relations:
\[
\nabla f = F \text{ and } n \cdot (\nabla (\sigma f) - v f) = 0 \text{ on } \partial \Omega.
\]
For simplicity, denote $q = \nabla p$, i.e., $q : \Omega \times (0, T) \to \mathbb{R}^n$. We obtain a partial-integro-differential equation for $q$:
\[
\partial_t q = \nabla[-\nabla \cdot (v \mathcal{F}(q)) + \Delta(\sigma \mathcal{F}(q))]
\]
which is a linear equation. We will rewrite (3) and (9) as evolution equations in $H_1 = L^2(\Omega)$ and $H_2 = L^2(\Omega)^n$, respectively, and use KDE to construct noisy measurements for their states. Specifically, define the following linear operators:
\[
\mathcal{A}(t)p := -\nabla \cdot [v(t)p] + \Delta[\sigma(t)p],
\mathcal{A}_g(t)q := \nabla[-\nabla \cdot (v(t)\mathcal{F}(q)) + \Delta[\sigma(t)\mathcal{F}(q)].
\]
For any $t$, we use KDE and the samples $\{X_i\}_{i=1}^N$ to construct a priori estimate $p_{KDE}(x,t)$, and treat $p_{KDE}(x,t)$ as a noisy measurement of $p(x,t)$. (See Appendices for how to construct $p_{KDE}$.). We also treat $\nabla p_{KDE}(x,t)$, the gradient of $p_{KDE}(x,t)$, as a noisy measurement of the density gradient $q(x,t)$. Define $w(x,t) = p_{KDE}(x,t) - p(x,t)$ and $w_g(x,t) = \nabla p_{KDE}(x,t) - q(x,t)$. Then, $w(x,t)$ and $w_g(x,t)$ are approximately infinite-dimensional Gaussian noises with diagonal covariance operators $R(t) = k \text{ diag}(p(t))$ and $\mathcal{R}_g(t) = k_g \text{ diag}(q(t))$, respectively, where $k, k_g > 0$ are constants depending on the kernels and $N$. (See Appendices for why $w$ and $w_g$ are approximately Gaussian.)

We obtain the following linear evolution equations:
\[
\dot{p} = \mathcal{A}(t)p
y = p_{KDE}(x,t) - p(x,t)
\]
and
\[
\dot{q} = \mathcal{A}_g(t)q
w_g = \nabla p_{KDE}(x,t) - q + w_g(x,t).
\]
We can design an optimal density filter for (10) and an optimal gradient filter for (11) according to infinite-dimensional Kalman filters. However, we notice that $\mathcal{R}$ and $\mathcal{R}_g$ are unknown because they depend on $p$ and $q$, the states that we want to estimate. Hence, we need to approximate $\mathcal{R}$ and $\mathcal{R}_g$ with some available quantities $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}_g$, respectively. A reasonable choice is to let $\mathcal{R}(t) = k \text{ diag}(\hat{p}(t))$ and $\mathcal{R}_g(t) = k_g \text{ diag}(\nabla \hat{p}(t))$. In this way, the "suboptimal" density filter is given by:
\[
\dot{\hat{p}} = \mathcal{A}(t)\hat{p} + \mathcal{P}(t)\mathcal{R}^{-1}(t)(y(t) - \hat{p}),
\]
with $\hat{p}(0) = p_{KDE}(0)$ and $\mathcal{P}(0) = \mathcal{P}_0$, and the "suboptimal" gradient filter is given by:
\[
\dot{\hat{q}} = \mathcal{A}_g(t)\hat{q} + \mathcal{Q}(t)\mathcal{R}_g^{-1}(t)(y_g(t) - \hat{q}),
\]
\[
\dot{\tilde{\mathcal{Q}}} = \mathcal{A}(t)\tilde{\mathcal{Q}} + \mathcal{Q}_g(t)\mathcal{R}_g^{-1}(t)\tilde{\mathcal{Q}},
\]
with $\hat{q}(0) = \nabla p_{KDE}(0)$ and $\mathcal{Q}(0) = \mathcal{Q}_0$, where $\hat{p}$ and $\hat{q}$ are our estimates for $p$ and $q$ (i.e., $\nabla p$).

Now we study the stability and optimality of these two filters. Let $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ be the corresponding solutions of the operator Riccati equations (13) and (15) when $\mathcal{R}$ and $\mathcal{R}_g$ are respectively replaced by $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}_g$, their true but unknown values. Then, $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ represent the optimal flows of estimation error covariance. We denote $\mathcal{L} = \mathcal{P}\mathcal{R}^{-1}$ and $\mathcal{L}_g = \mathcal{Q}_g\mathcal{R}_g^{-1}$ to represent the suboptimal Kalman gains, and denote $\tilde{\mathcal{L}} = \tilde{\mathcal{P}}\tilde{\mathcal{R}}^{-1}$ and $\tilde{\mathcal{L}}_g = \tilde{\mathcal{Q}}_g\tilde{\mathcal{R}}_g^{-1}$ to represent the optimal Kalman gains. Define $\epsilon = \hat{p} - p$ and $\epsilon_g = \hat{q} - q$ as the estimation errors of the filters. It is easy to see that $\epsilon$ and $\epsilon_g$ satisfy the following equations:
\[
\dot{\epsilon} = (\mathcal{A}(t) - \mathcal{P}(t)\mathcal{R}^{-1}(t))\epsilon + \mathcal{P}(t)\mathcal{R}^{-1}(t)w
\]
\[
\dot{\epsilon}_g = (\mathcal{A}_g(t) - \mathcal{Q}(t)\mathcal{R}_g^{-1}(t))\epsilon_g + \mathcal{Q}(t)\mathcal{R}_g^{-1}(t)w_g.
\]
We can show that the suboptimal filters are stable and remain close to the optimal ones. The following theorem is for the density filter (12) and (13), whose proof is given in [15].

**Theorem 5:** [15] Assume that $\|\mathcal{P}(t)\|$ and $\|\mathcal{Q}(t)\|$ are uniformly bounded, and $3c_3, c_4 > 0$ such that for $t \geq 0$,
\[
c_1I \preceq \mathcal{R}^{-1}(t), \tilde{\mathcal{R}}^{-1}(t), \mathcal{P}^{-1}(t), \tilde{\mathcal{Q}}^{-1}(t) \preceq c_2I.
\]

Then we have:
(i) $\|\epsilon\|_{L^2}$ is ISS with respect to (w.r.t.) $\|w\|_{L^2}$ (and is uniformly exponentially stable if $w = 0$);
(ii) $\|\hat{p} - p\|$ is LISS w.r.t. $\|\mathcal{R}^{-1} - \tilde{\mathcal{R}}^{-1}\|$;
(iii) $\|\mathcal{L} - \tilde{\mathcal{L}}\|$ is LISS w.r.t. $\|\mathcal{R}^{-1} - \tilde{\mathcal{R}}^{-1}\|$.

Property (i) means that the suboptimal filter (12) is stable even if $\mathcal{R}$ is only an approximation for $\mathcal{R}$. Property (ii) means that the solution of the suboptimal operator Riccati equation (13) remains close to the solution of the optimal one (when $\mathcal{R}$ is replaced by $\tilde{\mathcal{R}}$). Property (iii) means that the suboptimal Kalman gain remains close to the optimal Kalman gain. We have similar results for the gradient filter (14) and (15). The proof is similar to the proof of Theorem 5 and thus omitted.

**Theorem 6:** Assume that $\|\mathcal{Q}(t)\|$ and $\|\hat{\mathcal{Q}}(t)\|$ are uniformly bounded, and $3c_3, c_4 > 0$ such that for $t \geq 0$,
\[
c_3I \preceq \mathcal{R}_g^{-1}(t), \tilde{\mathcal{R}}_g^{-1}(t), \mathcal{Q}^{-1}(t), \tilde{\mathcal{Q}}^{-1}(t) \preceq c_4I.
\]

Then we have:
(i) $\|\epsilon_g\|_{L^2}$ is ISS w.r.t. $\|w_g\|_{L^2}$ (and is uniformly exponentially stable if $w_g = 0$);
(ii) $\|\hat{q} - q\|$ is LISS w.r.t. $\|\mathcal{R}_g^{-1} - \tilde{\mathcal{R}}_g^{-1}\|$;
(iii) $\|\mathcal{L}_g - \tilde{\mathcal{L}}_g\|$ is LISS w.r.t. $\|\mathcal{R}_g^{-1} - \tilde{\mathcal{R}}_g^{-1}\|$.

**C. Stability of feedback interconnection**

Now we discuss the stability of the feedback interconnection of density estimation and control. We collect equations
in the following for clarity:
\[
\partial_t \Phi = \nabla \cdot \left( \alpha_p p_* (\epsilon_g + \nabla p) - (\epsilon + p) \nabla p_* + \sigma_p \nabla p \right),
\]
\[
\dot{\epsilon} = (A(t; \Phi, \epsilon, \epsilon_g) - \mathcal{P}(t) \mathcal{R}^{-1}(t)) \dot{\epsilon} + \mathcal{P}(t) \mathcal{R}^{-1}(t) \dot{v},
\]
\[
\dot{\epsilon}_g = (A_g(t; \Phi, \epsilon, \epsilon_g) - \mathcal{Q}(t) \mathcal{R}_g^{-1}(t)) \dot{\epsilon} + \mathcal{Q}(t) \mathcal{R}_g^{-1}(t) \dot{w}_g,
\]
(20)

where we write \( A(t; \Phi, \epsilon, \epsilon_g) \) and \( A_g(t; \Phi, \epsilon, \epsilon_g) \) to emphasize their dependence on \( \Phi, \epsilon \) and \( \epsilon_g \) through \( v \).

A critical observation is that the ISS results we have established for \( \epsilon \) and \( \epsilon_g \) are valid in spite of this dependence. This is because when we design the estimation algorithms, \( A \) and \( A_g \) (and \( v \)) can be treated as known system coefficients. In this way, the bilinear control system (3) becomes a linear system. The dependence on \( \Phi, \epsilon \) and \( \epsilon_g \) can be seen as part of the time-varying nature of \( A \) and \( A_g \). Since the theory of Kalman filters applies to linear time-varying systems, the stability results for our density and gradient filters will not be affected. In this regard, we can treat (20) as a cascade system. By Theorem 2 we have the following stability result for (20).

Theorem 7 (Interconnected stability): Under the assumptions in Theorems 5 and 6 \( \| \Phi \|_{L^2}, \| \epsilon \|_{L^2} \) and \( \| \epsilon_g \|_{L^2} \) are all ISS w.r.t. \( \| w \|_{L^2} \) and \( \| w_g \|_{L^2} \).

This theorem has two implications. First, the stability results for the filters are independent of the density control laws. Hence, they can be used for any control design when density feedback is required. Second, the interconnected system is always ISS as long as the density feedback laws are designed in a way such that the closed-loop system is ISS w.r.t. the \( L^2 \) norm of estimation errors. Considering that norms of infinite-dimensional vectors are not equivalent in general, obtaining an ISS result w.r.t. \( L^2 \) norms is very critical. This in turn highlights the advantage of the modified feedback law (4), because it is difficult to obtain such an ISS result for most of the existing density feedback laws [11, 13, 14].

Remark 4: The density control strategy in this work is essentially centralized because of the requirement of knowing the real-time density (and its gradient). This constraint can be relaxed if each robot is able to estimate the density in a distributed way. In [23], we have presented a distributed density filter for this purpose, where each robot estimates the global density using only local observation and communication. The interconnected stability of distributed density estimation and control is left as our future work.

V. SIMULATION STUDIES

An agent-based simulation using 1024 robots is performed on Matlab to verify the proposed control law. We set \( \Omega = (0, 1)^2, \sqrt{2} \alpha = 0.01 \) and \( \alpha = 0.003 \). Each robot is simulated according to (2) where \( v \) is given by (6). The robots’ initial positions are drawn from a uniform distribution on \([0.15, 0.85]^2\). The desired density \( p_*(x) \) is illustrated in Fig. 1a. The implementation of the filters and the feedback controller is based on the finite difference method. We discretize \( \Omega \) into a \( 30 \times 30 \) grid, and the time difference is 0.01s. We use KDE (in which we set \( h = 0.04 \)) to obtain \( p_{KDE} \) and \( \nabla p_{KDE} \). The densities and the operators are approximated by finite-dimensional vectors and matrices.

Simulation results are given in Fig. 2. It is seen that the swarm is able to evolve towards the desired density. The convergence error \( \| \dot{\rho} - \dot{p}^* \|_{L^2}(\omega) \) is given in Fig. 1b which shows that the error converges exponentially to a small neighbourhood around 0 and remains bounded, which verifies the ISS property of the proposed algorithm.

VI. CONCLUSION

This paper studied the interplay of density estimation and control algorithms. We proposed new density feedback laws for robust density control of swarm robotic systems and filtering algorithms for estimating the real-time density and its gradient. We also proved that the interconnection of these algorithms is globally ISS. In implementation, we need a communication center to perform the estimation algorithms and broadcast the estimates to the robots. This is suitable for surveillance applications such as UAV-based environment surveillance. Our future work is to incorporate the density control algorithm for higher-order nonlinear systems in [20] and the distributed density estimation algorithm in [23], and study their interconnected stability.

APPENDICES: KERNEL DENSITY ESTIMATION

KDE is a non-parametric way to estimate an unknown probability density and its derivatives. Let \( X_1, \ldots, X_N \in \mathbb{R}^n \) be independent identically distributed random variables having a common probability density \( f \). The kernel density estimators for \( f \) and its gradient \( \nabla f \) are given by (24), (25)

\[
f_N(x) = \frac{1}{Nh^n} \sum_{i=1}^N K\left( \frac{x - X_i}{h} \right),
\]
and

\[
\nabla f_N(x) = \frac{1}{Nh^{n+1}} \sum_{i=1}^N \nabla K\left( \frac{x - X_i}{h} \right),
\]
(22)

respectively, where \( K(x) \) is a kernel function [26] and \( h \) is the bandwidth, usually chosen as a function of \( n \) such that \( \lim_{N \to \infty} h = 0 \) and \( \lim_{N \to \infty} Nh = \infty \). The Gaussian kernel is frequently used due to its infinite order of smoothness, given by

\[
K(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}x^T x\right).
\]
It is known that the \( f_N(x) \) is asymptotically normal and that \( f_N(x_i) \) and \( f_N(x_j) \) are asymptotically uncorrelated for \( x_i \neq x_j \), as \( N \to \infty \), which is summarized as follow.

**Lemma 2 (Asymptotic normality [27]):** Under conditions \( \lim_{N \to \infty} h = 0 \) and \( \lim_{N \to \infty} Nh = \infty \), if the bandwidth \( h \) tends to zero faster than the optimal rate, i.e., \( h^* = o \left( \frac{1}{N} \right)^{1/(n+4)} \), then as \( N \to \infty \), we have

\[
\sqrt{Nh^2} (f_N(x) - f(x)) \to N \{0, f(x) \int [K(u)]^2 du \}.
\]

**Lemma 3 (Asymptotic uncorrelatedness [27]):** Let \( x_i, x_j \) be two distinct continuity points of \( f \). Under \( \lim_{N \to \infty} Nh = 0 \), the covariance of \( f_N(x_i) \) and \( f_N(x_j) \) satisfies

\[
Nh \text{Cov}(f_N(x_i), f_N(x_j)) \to 0, \text{ as } N \to \infty.
\]

These two lemmas together imply that when \( N \) is large, \( f_N - f \) is approximately Gaussian with zero mean and diagonal covariance. Similar results exist for \( \nabla f \) which are omitted here due to space limit.

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