LOCAL NORMAL FORMS OF SINGULAR LEVI-FLAT HYPERSURFACES

ARTURO FERNÁNDEZ-PÉREZ AND GUSTAVO MARRA

Abstract. We study normal forms of germs of singular real-analytic Levi-flat hypersurfaces. We prove the existence of rigid normal forms for singular Levi-flat hypersurfaces which are defined by the vanishing of the real part of complex quasihomogeneous polynomials with isolated singularity. This result generalizes previous results of Burns-Gong [6] and Fernández-Pérez [14]. Furthermore, we prove the existence of two new rigid normal forms for singular real-analytic Levi-flat hypersurfaces which are preserved by a change of isochore coordinates, that is, a change of coordinates that preserves volume.

1. Introduction

In this paper we study normal forms of germs of singular real-analytic Levi-flat hypersurfaces. Our first result is the following.

Theorem 1. Let \( M = \{ F = 0 \} \) be a germ of an irreducible singular real-analytic Levi-flat hypersurface at \( 0 \in \mathbb{C}^2 \) such that

(a) \( F(z) = \text{Re}(Q(z)) + H(z, \bar{z}) \);

(b) \( Q \) is a complex quasihomogeneous polynomial of quasihomogeneous degree \( d \) with isolated singularity at \( 0 \in \mathbb{C}^2 \).

(c) \( H \) is a germ of real-analytic function at \( 0 \in \mathbb{C}^2 \) of order strictly greater than \( d \) and \( H(z, \bar{z}) = \overline{H(z, \bar{z})} \).

Then there exists a germ of biholomorphism \( \phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) such that

\[
\phi(M) = \left\{ \text{Re} \left( Q(z) + \sum_{j=1}^{s} c_j e_j(z) \right) = 0 \right\},
\]

\( e_1, \ldots, e_s \) are the elements of the monomial basis of the local algebra of \( Q \) of quasihomogeneous degree strictly greater than \( d \) and \( c_j \in \mathbb{C} \).

When \( M \) is a germ of a singular real-analytic Levi-flat hypersurface at \( 0 \in \mathbb{C}^n, n \geq 3 \), the same result was proved by Fernández-Pérez in [13]. Therefore, the above theorem completes the study of normal forms of real-analytic Levi-flat hypersurfaces which are defined by the vanishing of real part of complex quasihomogeneous polynomials with isolated singularity. We also note

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that Theorem 1 generalizes the main results of [6] and [14], because the authors considered the same theorem for a generic Morse singularity and Arnold singularities of type $A_k$, $D_k$, $E_6$, $E_7$ and $E_8$, which are given by complex quasihomogeneous polynomials with inner modality zero (see for instance [2]).

Topics about singular real-analytic Levi-flat hypersurfaces have been previously studied by several authors, see for instance [5], [7], [15], [16], [22], and normal forms of CR singular codimension two Levi-flat submanifolds was studied in [20]. On the other hand, the study of normal forms of real-analytic hypersurfaces with Levi-form non-degenerate is given by the theory of Cartan [9] and Chern-Moser [10].

The second part of this paper is devoted to prove the existence of normal forms of singular real-analytic Levi-flat hypersurfaces which are preserved by a change of isochore coordinates, that is, a change of coordinates that preserve volume. Our main motivation are the Morse-type results for singularities of holomorphic functions given by J. Vey [25] and J-P Françoise [17]. More precisely, Vey proved an isochore version of Lemma of Morse for germs of holomorphic functions at $0 \in \mathbb{C}^n$, $n \geq 2$, and Françoise gave a new proof of the same result. A much more general statement was given by Garay [19]. In this same spirit, we propose here an analogous version of Vey’s theorem for singular real-analytic Levi-flat hypersurfaces which are defined by the vanishing of the real part of a generic Morse function. We state the following result.

**Theorem 2.** Let $M = \{F = 0\}$ be a germ of an irreducible singular real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, such that

$$F(z) = \text{Re}(z_1^2 + \ldots + z_n^2) + H(z, \bar{z}),$$

where $H(z, \bar{z}) = O(|z|^3)$ and $H(z, \bar{z}) = H(\bar{z}, z)$. Then, there exists a germ of a volume-preserving biholomorphism $\phi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ and a germ of an automorphism $\psi: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that

$$\phi(M) = \{\text{Re}(\psi(z_1^2 + \ldots + z_n^2)) = 0\}.$$

The above theorem can be viewed as an isochore version of Burns-Gong’s theorem [6]. On the other hand, in order to establish our next result we consider some definitions and notations that will be explained in the section 2: for a germ of a singular real-analytic Levi-flat hypersurface $M$ with Levi foliation $\mathcal{L}$ and singular set $\text{Sing}(M)$, we will define the complexification $\mathcal{M}_C$ of $M$, which will be a germ of complex analytic subvariety contained $M$. The singular set of $\mathcal{M}_C$ will be denoted by $\text{Sing}(\mathcal{M}_C)$. We will see that $\mathcal{M}_C$ is equipped with a germ of a singular codimension-one holomorphic foliation $\mathcal{L}_C$, which will be the complexification of the Levi foliation $\mathcal{L}$. The singular set of $\mathcal{L}_C$ will be denoted by $\text{Sing}(\mathcal{L}_C)$.

Recently in [24], A. Szawlowski presented a volume-preserving normal form for germs of holomorphic functions that are right-equivalent to the product of all coordinates. Motivated by [24], we will prove an analogous version for singular real-analytic Levi-flat hypersurfaces.

**Theorem 3.** Let $M = \{F = 0\}$ be a germ of an irreducible singular real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, such that $F(z) = \text{Re}(z_1 \cdots z_n) + H(z, \bar{z})$, where $H(z, \bar{z}) =$
$O(|z|^{n+1})$ and $H(z,\bar{z}) = \overline{H(z,\bar{z})}$. Suppose that

$$\text{Sing}(M_C) = \bigcup_{1 \leq i < j \leq n, 1 \leq k < \ell \leq n} V_{ijkl},$$

where $V_{ijkl} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n : z_i = z_j = w_k = w_\ell = 0\}$ and $\text{Sing}(M_C) \subset \text{Sing}(L_C)$. Then, there exists a germ of codimension-one holomorphic foliation $F_M$ tangent to $M$, with a non-constant holomorphic first integral $f(z) = z_1 \cdots z_n + O(|z|^{n+1})$ such that

$$M = \{\text{Re}(f(z)) = 0\}.$$ 

As a consequence of above theorem and the main result of Szawlowski [24] we have the following corollary.

**Corollary 1.** Let $M$ be a germ of an irreducible singular real-analytic Levi-flat hypersurface as in Theorem 3. If $f$ is right equivalent to the product of all coordinates, $f \sim_R z_1 \cdots z_n$. Then there exists a germ of a volume-preserving biholomorphism $\Phi : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ and a germ of an automorphism $\Psi : (\mathbb{C},0) \to (\mathbb{C},0)$ such that

$$\Phi(M) = \{\text{Re}(\Psi(z_1 \cdots z_n)) = 0\},$$

where $\Psi$ is uniquely determined by $f$ up to a sign.

Let us recall that two germs of holomorphic functions $f$ and $g$ are right equivalent $f \sim_R g$, if there exist a germ of biholomorphism $\phi$ around the origin such that $f \circ \phi^{-1} = g$. We remark that the normal forms of Theorem 3 and Corollary 1 are germs of real-analytic Levi-flat hypersurfaces whose singular set are of positive dimension. In general, the problem of finding normal forms of germs of real-analytic Levi-flat hypersurfaces with non-isolated singularities is very difficult and there are few results about it, see for instance [15].

To prove theorems 1, 2 and 3 we use the techniques of holomorphic foliations developed by D. Cerveau and A. Lins Neto in [8] and the first author in [12]. These are fundamental in order to find normal forms of Levi-flat hypersurfaces. Specifically, we apply a result of Cerveau-Lins Neto that gives sufficient conditions for a real-analytic Levi-flat hypersurface to be defined by the zeros of the real part of a holomorphic function and a key Lemma that will be stated in section 5.

This paper is organized as follows: in section 2, we recall some properties and known results about singular Levi-flat hypersurfaces. In section 3, we state some results about normal forms for a complex quasihomogeneous polynomial. Section 4 is devoted to recall the notions of weighted projective space and weighted blow-ups. In section 5 we prove Theorem 1 and give an application of Theorem 1. The section 6 is dedicated to establish the isochore normal forms for holomorphic functions given by Vey and Szawlowski. In section 7, we proved Theorem 2 and finally in section 8, we proved Theorem 3 and Corollary 1.

2. Singular Levi-flat hypersurfaces and holomorphic foliations

The following notation will be used in this paper:
Let $X$ be a compact connected complex manifold of complex dimension $n \geq 2$. A codimension-one singular holomorphic foliation $\mathcal{F}$ on $X$ is given by a covering of $X$ by open subsets $\{U_j\}_{j \in J}$ and a collection of integrable holomorphic 1-forms $\omega_j$ on $U_j$, $\omega_j \wedge d\omega_j = 0$, having zero set of complex codimension at least two such that, on each non-empty intersection $U_j \cap U_k$, we have

\begin{equation}
\omega_j = g_{jk} \omega_k, \quad \text{with} \quad g_{jk} \in \mathcal{O}^*(U_j \cap U_k).
\end{equation}

Let $\text{Sing}(\omega_j) = \{p \in U_j : \omega_j(p) = 0\}$. Condition (1) implies that $\text{Sing}(\mathcal{F}) := \bigcup_{j \in J} \text{Sing}(\omega_j)$ is a complex subvariety of complex codimension at least two in $X$.

Let $M$ be a germ of a real codimension-one irreducible real-analytic subvariety at $0 \in \mathbb{C}^n$, $n \geq 2$. Without loss of generality we may assume that $M = \{F(z) = 0\}$, where $F$ is a germ of irreducible real-analytic function at $0 \in \mathbb{C}^n$. We define the singular set of $M$ as

$$
\text{Sing}(M) = \{F(z) = 0\} \cap \{dF(z) = 0\}
$$

and its regular part is defined as $M^* = M \setminus \text{Sing}(M)$. Consider the distribution of complex hyperplanes $L$ on $M^*$ given by

$$
L_p := \ker(\partial F(p)) \subset T_p M^* = \ker(dF(p)), \quad p \in M^*.
$$

This distribution is called Levi distribution. When $L$ is integrable, in the sense of Frobenius, then we say that $M$ is Levi-flat. Since $M^*$ admits an integrable complex distribution, it is foliated locally by a real-analytic codimension-one foliation $L$ on $M^*$, the Levi foliation. Each leaf of $L$ is a codimension-one holomorphic submanifold immersed in $M^*$.

The distribution $L$ can be defined by the real-analytic 1-form $\eta = i(\partial F - \overline{\partial} F)$, the Levi form of $F$. The integrability condition is equivalent to

$$(\partial F - \overline{\partial} F) \wedge \partial \overline{\partial} F|_{M^*} = 0$$

which using the fact that $\partial F + \overline{\partial} F = dF$, is equivalent to

$$\partial F(p) \wedge \overline{\partial} F(p) \wedge \partial \overline{\partial} F(p) = 0 \quad \forall \ p \in M^*.$$

We refer to the book [3] for the basic language and background about Levi-flat hypersurfaces.

Suppose that $M$ is Levi-flat as above. If $\text{Sing}(M) = \emptyset$, then we say that $M$ is smooth. In this case, according to Cartan [3], around the origin of $\mathbb{C}^n$ one may find suitable coordinates $(z_1, ..., z_n)$ of $\mathbb{C}^n$ such that the germ of $M$ at $0 \in \mathbb{C}^n$ is given by

$$\{\Re(z_n) = 0\}.$$
This is called the *local normal form* for a smooth real-analytic Levi-flat hypersurface $M$ at $0 \in \mathbb{C}^n$.

In order to build singular real-analytic Levi-flat hypersurfaces which are irreducible, we consider the following lemma from [8].

**Lemma 2.1.** Let $f \in \mathcal{O}_n$, $f \neq 0$, $f(0) = 0$ which is not a power in $\mathcal{O}_n$. Then $\text{Im}(f)$ and $\text{Re}(f)$ are irreducible in $\mathbb{A}_{n \mathbb{R}}$.

Before proving our results, let us describe some known results and examples.

**Example 2.1.** Let $f \in \mathcal{O}_n$ be a germ of non-constant holomorphic function with $f(0) = 0$. Then the set $M = \{\text{Re}(f) = 0\}$ is Levi-flat and its singular set is given by $\text{crit}(f) \cap M$, where $\text{crit}(f)$ is the set of critical points of $f$. The leaves of the Levi foliation $\mathcal{L}$ on $M$ are the imaginary levels of $f$.

**Example 2.2.** In $\mathbb{C}^n$, $n \geq 2$, let $M$ be given as the set of zeros of

$$F(z_1, z_2, \ldots, z_n) = z_1z_1 - z_2z_2.$$ 

Then $M$ is Levi-flat and its singular set biholomorphic to $\mathbb{C}^{n-2}$. This real-analytic hypersurface is called *quadratic complex cone*. The leaves of the Levi foliation $\mathcal{L}$ on $M$ are the hyperplanes

$$L_c = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_1 - c \cdot z_2 = 0\} \quad \text{where} \quad c \in \mathbb{R}.$$

**Example 2.3.** Let $M$ be a germ of real-analytic hypersurface at $0 \in \mathbb{C}^n$ given by $\{F = 0\}$, where

$$F(z_1, \ldots, z_n) = \text{Re}(z_1^2 + \ldots + z_n^2) + H(z, \bar{z}), \quad \text{and} \quad H(z, \bar{z}) = O(|z|^3).$$

If $M$ is Levi-flat, then, according to [9], there exists a holomorphic coordinate system such that $M = \{\text{Re}(x_1^2 + \ldots + x_n^2) = 0\}$. We remark that this result was generalized in [13], where the first author considered the real part of a complex homogeneous polynomial of degree $k \geq 2$ with an isolated singularity.

**Example 2.4.** We consider the famous $A_k, D_k, E_k$ singularities or simple singularities of Arnold [1, 2]:

| Type | Normal form | Conditions |
|------|-------------|------------|
| $A_k$ | $z_1^2 + z_2^{k+1} + \ldots + z_n^2$ | $k \geq 1$ |
| $D_k$ | $z_1^2z_2 + z_2^{k-1} + z_3^2 + \ldots + z_n^2$ | $k \geq 4$ |
| $E_6$ | $z_1^3 + z_2^3 + z_3^2 + \ldots + z_n^2$ | |
| $E_7$ | $z_1^3z_2 + z_2^3 + z_3^3 + \ldots + z_n^2$ | |
| $E_8$ | $z_1^4 + z_2^4 + z_3^4 + \ldots + z_n^4$ | |

Let $M$ be a germ of singular real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^2$ defined by $\{F = 0\}$, where

$$F(z) = \text{Re}(Q(z)) + H(z, \bar{z}),$$
where \( Q \) is a complex quasihomogeneous polynomial of \( A_k, D_k, \) or \( E_k \) type of quasihomogeneous degree \( d \). Then in [14] it has been proved that there exists a holomorphic coordinate system such that
\[
M = \{ \Re e(Q(z)) = 0 \}.
\]
We remark that, in this case, the elements \( e_1, \ldots, e_s \) of the monomial basis of the local algebra of \( Q \) of quasihomogeneous degree strictly greater than \( d \) are zero, because the inner modality of the \( A_k, D_k, E_k \) singularities are zero.

2.1. Complexification of singular Levi-flat hypersurfaces. Let \( M \) be a germ of a singular real-analytic Levi-flat hypersurface at \( 0 \in \mathbb{C}^n \) defined by the set of zeros of \( F \in \mathbb{A}_{n,\mathbb{R}} \). Let \( \text{Sing}(M) \), \( M^* \) and \( L \) be the singular set, the regular part and the Levi foliation on \( M^* \) respectively.

We write the Taylor series of \( F \) around \( 0 \in \mathbb{C}^n \) as
\[
F(z) = \sum_{\mu, \nu} F_{\mu \nu} z^\mu \bar{z}^\nu,
\]
where \( F_{\mu \nu} \in \mathbb{C}, \mu = (\mu_1, \ldots, \mu_n), \nu = (\nu_1, \ldots, \nu_n), z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n} \) and \( \bar{z}^\nu = \bar{z}_1^{\nu_1} \cdots \bar{z}_n^{\nu_n} \). Since \( F \in \mathbb{A}_{n,\mathbb{R}} \), the coefficients verify \( F_{\mu \nu} = F_{\nu \mu} \). We define the complexification \( F_C \in \mathcal{O}_{2n} \) of \( F \) as the function defined by the power series
\[
F_C(z, w) = \sum_{\mu, \nu} F_{\mu \nu} z^\mu w^\nu.
\]
If the power series for \( F \) converges in a polydisc \( D^n_r = \{ z \in \mathbb{C}^n : |z_j| \leq r \} \) then the power series of the complexification \( F_C \) of \( F \) is convergent in the polydisc \( D^{2n}_r \) and therefore is holomorphic at \( 0 \in \mathbb{C}^{2n} \). Moreover,
\[
F(z) = F_C(z, \bar{z}) \quad \forall z \in D^n_r.
\]
This complexification does not depend on choice of coordinate system, see for instance [8].

As seen before, the Levi 1-form is given by \( \eta = i(\partial F - \bar{\partial} F) \). Its complexification is the germ of holomorphic 1-form
\[
\eta_C = i \sum_{j=1}^n \left( \frac{\partial F_C}{\partial z_j} dz_j - \frac{\partial F_C}{\partial w_j} dw_j \right) = i \sum_{\mu, \nu} (F_{\mu \nu} w^\nu d(z^\mu) - F_{\mu \nu} z^\mu d(w^\nu)).
\]
The complexification of \( M \) is defined as \( M_C = \{ F_C = 0 \} \). As before, \( M_C \) does not depend on choice of coordinate system. The regular part of \( M_C \) is
\[
M^*_C = M_C \setminus \{ dF_C = 0 \}
\]
and the singular part of \( M_C \) is
\[
\text{Sing}(M_C) = M_C \cap \{ dF_C = 0 \}.
\]
Since \( \eta \) is integrable on \( M^* \), then also \( \eta_C|_{M^*_C} \) is integrable and defines a codimension-one holomorphic foliation on \( M^*_C \), which will be denoted by \( L_C \). Such foliation is called complexification of \( L \).
Remark 2.1. We can write $\eta_C = i(\alpha - \beta)$, where
\[
\alpha := \sum_{j=1}^{n} \frac{\partial F}{\partial z_j} dz_j \quad \text{and} \quad \beta := \sum_{j=1}^{n} \frac{\partial F}{\partial w_j} dw_j.
\]
Note that $dF = \alpha + \beta$, then
\[
\eta_C\big|_{M^*} = (\eta_C + idF_C)|_{M^*} = 2i\alpha|_{M^*}.
\]
Analogously
\[
\eta_C\big|_{M^*} = (\eta_C - idF_C)|_{M^*} = -2i\beta|_{M^*}.
\]
In particular, $\alpha|_{M^*}$ and $\beta|_{M^*}$ define $L_C$ on $M^*$ and $\text{Sing}(L_C) = \text{Sing}(\eta_C|_{M^*})$.

Definition 2.1. Let $M = \{F = 0\}$ be a germ at $0 \in \mathbb{C}^n$ of a real-analytic Levi-flat hypersurface and $M_C$ its complexification. We define the algebraic dimension of $\text{Sing}(M)$ as the complex dimension of $\text{Sing}(M_C)$.

Let $W = M_C \setminus \text{Sing}(\eta_C|_{M^*})$ and let $L_p$ be the leaf of $L_C$ through $p \in W$. We have the following lemma from [8].

Lemma 2.2. For any $p \in W$, the leaf $L_p$ is closed (with the induced topology) in $M^*_C$.

The following theorem, due to D. Cerveau and A. Lins Neto [8] is the key ingredient for finding normal forms of singular Levi-flat hypersurfaces.

Theorem 2.3. Let $M = \{F = 0\}$ be a germ of an irreducible real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, with Levi 1-form $\eta$. Assume that the algebraic dimension of $\text{Sing}(M)$ is at most $2n - 4$. Then there exists a unique germ at $0 \in \mathbb{C}^n$ of codimension-one holomorphic foliation $F_M$ tangent to $M$, if one of the following conditions is fulfilled:

(a) $n \geq 3$ and $\text{cod}_{M^*_C}(\text{Sing}(\eta_C|_{M^*_C})) \geq 3$.

(b) $n \geq 2$, $\text{cod}_{M^*_C}(\text{Sing}(\eta_C|_{M^*_C})) \geq 2$ and $L_C$ has a non-constant holomorphic first integral. Moreover, in both cases the foliation $F_M$ has a non-constant holomorphic first integral $f$ such that $M = \{\Re(f) = 0\}$.

We recall that germ of holomorphic function $h$ is called a holomorphic first integral for a germ of codimension-one holomorphic foliation $F$ if its zeros set is contained in $\text{Sing}(F)$ and its level hypersurfaces contain the leaves of $F$.

3. Normal forms for a quasihomogeneous polynomial

The local algebra of $f \in \mathcal{O}_n$ is defined as
\[
A_f = \frac{\mathcal{O}_n}{\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \rangle}.
\]

The number $\mu(f, 0) = \text{dim}_\mathbb{C}(A_f)$ is the Milnor number of $f$ at $0 \in \mathbb{C}^n$. This number is finite if and only if $f$ has an isolated singularity at the origin. With these definitions, Morse lemma may be stated as follows: if $0 \in \mathbb{C}^n$ is an isolated singularity of $f \in \mathcal{O}_n$ with $\mu(f, 0) = 1$, then
$f$ is right equivalent to its second jet $j^2_0(f)$. The Morse lemma has the following generalization, and the proof can be found in [3].

**Theorem 3.1.** If $f \in \mathcal{M}_n$ has an isolated singularity at $0 \in \mathbb{C}^n$ with Milnor number $\mu$, then $f$ is right equivalent to $j^{\mu+1}_0(f)$.

**Definition 3.1.** A germ of function $f \in \mathcal{O}_n$ is quasihomogeneous with weights $w_1, \ldots, w_n \in \mathbb{Z}^*$ if, for each $\lambda \in \mathbb{C}^*$,

$$f(\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_1, \ldots, z_n).$$

The number $d$ is the quasihomogeneous degree of it.

The previous definition is equivalent to the following: $f(z)$ is quasihomogeneous of type $(w_1, \ldots, w_n)$ if it can be expressed as a linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which the equality

$$i_1 w_1 + \cdots + i_n w_n = d$$

holds. The number $d$ is the quasihomogeneous degree defined above.

**Definition 3.2.** The Newton support of germ $f = \sum a_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$ is defined as

$$\text{supp}(f) = \{(i_1, \ldots, i_n) : a_{i_1 \ldots i_n} \neq 0\}.$$

In the above situation, if $f = \sum a_I x^I$, $I = (i_1, \ldots, i_n)$, $x^I = x_1^{i_1} \cdots x_n^{i_n}$, then

$$\text{supp}(f) \subset \Gamma = \{I : w_1 i_1 + \cdots + w_n i_n = d\}.$$

The set $\Gamma$ is called the diagonal. One can define the quasihomogeneous filtration of the ring $\mathcal{O}_n$. It consists of the decreasing family of ideals $\mathcal{A}_d \subset \mathcal{O}_n$, $\mathcal{A}_{d'} \subset \mathcal{A}_d$ for $d < d'$. Here $\mathcal{A}_d = \{Q : \text{degrees of monomials from } \text{supp}(Q) \text{ are } \deg(Q) \geq d\}$; (the degree is quasihomogeneous). When $i_1 = \cdots = i_n = 1$, this filtration coincides with the usual filtration by the usual degree.

**Definition 3.3.** A function $f$ is semiquasihomogeneous if $f = Q + F'$, where $Q$ is quasihomogeneous of quasihomogeneous degree $d$ and $\mu(Q, 0) < \infty$, and $F' \in \mathcal{A}_{d'}$, $d' > d$.

From [2] we have the following result of V.I. Arnold.

**Theorem 3.2.** Let $f = Q + F'$ be a semiquasihomogeneous function. Then $f$ is right-equivalent to a function $Q(z) + \sum_j c_j e_j(z)$ where $e_1, \ldots, e_j$ are elements of the monomial basis of the local algebra $\mathcal{A}_Q$ of quasihomogeneous degree strictly greater than $d$ and $c_j \in \mathbb{C}$.

**Example 3.1.** Let $f = Q + F'$, where $Q(x, y) = x^2 y + y^k$, then $f$ is right equivalent to $Q$. Indeed, the basis of the local algebra

$$\mathcal{A}_Q = \mathcal{O}_2/(xy, x^2 + ky^{k-1})$$

is $1, x, y, y^2, \ldots, y^{k-1}$. Here $\mu(Q, 0) = k + 1$.

In the proof of Theorem 1 we will used the following Lemma of Saito [23].
Lemma 3.3. If $f \in \mathcal{M}_2$ is a complex quasihomogeneous polynomial, then $f$ factors uniquely as

$$f(z_1, z_2) = \mu z_1^{m_1} z_2^{m_2} \prod_{\ell=1}^{k} (z_2^p - \lambda_\ell z_1^q),$$

where $m, n, p, q \in \mathbb{Z}_+, \mu, \lambda_\ell \in \mathbb{C}^*$ for each $\ell = 1, \ldots, k$, and $\gcd(p, q) = 1$.

4. Weighted projective varieties and weighted blow-ups

In this section we present an overview of weighted projective spaces and weighted blow-ups. We refer to [11] and [21] for a more extensive presentation of the subject.

Let $\sigma := (a_0, \ldots, a_n)$ be positive integers. The group $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}\{0\}$ by

$$\lambda \cdot (x_0, \ldots, x_n) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n).$$

The quotient space under this action is the weighted projective space of type $\sigma$, $\mathbb{P}(a_0, \ldots, a_n) := \mathbb{P}_\sigma$. In case $a_i > 1$ for some $i$, $\mathbb{P}_\sigma$ is a compact algebraic variety with cyclic quotient singularities.

Let $[x_0 : \ldots : x_n]$ be the homogeneous coordinates on $\mathbb{P}(a_0, \ldots, a_n)$. The affine piece $x_i \neq 0$ is isomorphic to $\mathbb{C}^n / \mathbb{Z}_{a_i}$, here $\mathbb{Z}_{a_i}$ denote the quotient group modulo $a_i$. Let $\epsilon$ be an $a_i$th-primitive root of unity. The group acts by

$$z_j \mapsto \epsilon^{a_i} z_j$$

for all $j \neq i$, on the coordinates $(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ of $\mathbb{C}^n$; here $z_j$ is thought of as $x_j / x_i^{1/a_i}$. Compare this to the case of $\mathbb{P}^n$ where the affine coordinates on $x_i \neq 0$ are $z_j = x_j / x_i$.

Definition 4.1. $\mathbb{P}(a_0, \ldots, a_n)$ is well-formed if for each $i$

$$\gcd(a_0, \ldots, a_i, \ldots, a_n) = 1.$$

We have a natural orbifold map $\phi_\sigma : \mathbb{P}^n \to \mathbb{P}_\sigma$ defined by

$$[x_0 : \ldots : x_n] \mapsto [x_0^{a_0} : \ldots : x_n^{a_n}]_\sigma$$

(2)

Definition 4.2. Let $X$ be a closed subvariety of a weighted projective space $\mathbb{P}_\sigma$, and let $\rho : \mathbb{C}^{n+1}\{0\} \to \mathbb{P}_\sigma$ be the canonical projection. The punctured affine cone $C^*_X$ over $X$ is given by $C^*_X = \rho^{-1}(X)$, and the affine cone $C_X$ over $X$ is the completion of $C^*_X$ in $\mathbb{C}^{n+1}$.

Observe that $\mathbb{C}^*$ acts on $C^*_X$ giving $X = C^*_X / \mathbb{C}^*$. Note that we have the following fact.

Lemma 4.1. $C^*_X$ has no isolated singularities.

Definition 4.3. We say that $X$ in $\mathbb{P}_\sigma$ is quasi-smooth of dimension $m$ if its affine cone $C_X$ is smooth of dimension $m + 1$ outside its vertex $0 \in \mathbb{C}^{n+1}$.

When $X \subset \mathbb{P}_\sigma$ is quasi-smooth the singularities of $X$ are given by the $\mathbb{C}^*$-action and hence are cyclic quotient singularities. Notice that this definition is not equivalent to the smoothness of the inverse image $\phi_\sigma^{-1}(X)$ under the quotient map given in (2).

Another important fact (cf. [11], Theorem 3.1.6) is that a quasi-smooth subvariety $X$ of $\mathbb{P}_\sigma$ is a $V$-variety, that is, a complex space which is locally isomorphic to the quotient of a complex manifold by a finite group of holomorphic automorphisms.
Now, let \( X = \mathbb{C}^n / \mathbb{Z}_m(a_1, \ldots, a_n) \) be a cyclic quotient singularity. That is, \( X \) is the quotient variety \( \mathbb{C}^n / \tau \), where \( \tau \) is given by

\[
x_i \mapsto e^{\epsilon_i} x_i
\]

for all \( i \), where \( \epsilon \) is a \( m \)-th primitive root of unity.

4.1. **Weighted blow-ups.** In this part we will construct the blow-up of \( X \). First, we describe \( X \) using the theory of toric varieties (cf. [18]). Let

\[
e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1) \text{ and } e = \frac{1}{m}(a_1, \ldots, a_n).
\]

Then \( X = \mathbb{C}^n / \mathbb{Z}_m(a_1, \ldots, a_n) \) is the toric variety corresponding to the lattice \( N = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n + \mathbb{Z}e \) and the cone \( C = \mathbb{R}_{\geq 0} e_1 + \cdots + \mathbb{R}_{\geq 0} e_n \). Denote by \( \triangle \) the fan associated to \( X \) consisting of all the faces of \( C \).

Take \( \nu = \frac{1}{m}(a_1, \ldots, a_n) \in N \) with \( a_1, \ldots, a_n > 0 \) and assume that \( e_1, \ldots, e_n \) and \( \nu \) generate the lattice \( N \). Such \( \nu \in N \) will be called a weight. We can construct the weighted blow-up

\[
E : \tilde{X} \to X = \mathbb{C}^n / \mathbb{Z}_m(a_1, \ldots, a_n)
\]

with weight \( \nu \) as follows: we divide the cone \( C \) by adding the 1-dimensional cone \( \mathbb{R}_{\geq 0} \nu \), that is, we divide \( C \) into \( n \) cones

\[
C_i = \mathbb{R}_{\geq 0} e_1 + \cdots + \mathbb{R}_{\geq 0} \nu + \cdots + \mathbb{R}_{\geq 0} e_n \quad (i = 1, \ldots, n).
\]

Let \( \triangle' \) be the fan consisting of all the faces of \( C_1, \ldots, C_n \). Then \( \tilde{X} \) is the toric variety corresponding to \( N \) and \( \triangle' \), while \( E \) is the morphism induced from the natural map of fans \( (N, \triangle') \to (N, \triangle) \).

The variety \( \tilde{X} \) is covered by \( n \) affine open sets \( \tilde{U}_1, \ldots, \tilde{U}_n \) which correspond to the cones \( C_1, \ldots, C_n \) respectively. These affine open sets and \( E \) are described as follows:

\[
(3) \quad \tilde{U}_i = \mathbb{C}^n / \mathbb{Z}_m(-a_1, \ldots, \frac{1}{m}, \ldots, -a_n)
\]

\[
(4) \quad E|_{\tilde{U}_i} : \tilde{U}_i \ni (y_1, \ldots, y_n) \mapsto \left( y_1 y_i^{a_1/m}, \ldots, y_i^{a_i/m}, \ldots, y_n y_i^{a_n/m} \right) \in X.
\]

The exceptional divisor \( D \) of \( E \) is isomorphic to the weighted projective space \( \mathbb{P}(a_1, \ldots, a_n) \) and \( D \cap \tilde{U}_i = \{ y_i = 0 \} / \mathbb{Z}_m \).

5. FIRST INTEGRAL FOR THE LEVI FOLIATION AND THE PROOF OF THEOREM [1]

In this section, we give sufficient conditions (dynamical criteria) to find a non-constant holomorphic first integral for the complexification of the Levi foliation \( \mathcal{L}_C \) on \( M_C \) and then we prove Theorem [1].

Let \( \pi \) be a weighted blow-up on \( M_C \) with exceptional divisor \( E \). Denote by \( \tilde{M}_C \) the strict transform of \( M_C \) by \( \pi \) and by \( \tilde{F} = \pi^*(\mathcal{L}_C) \) the induced foliation on \( \tilde{M}_C \). Suppose that \( \tilde{M}_C \) is a
smooth variety and consider $\tilde{C} = \tilde{M}_C \cap E$. Assume that $\tilde{C}$ is invariant by $\tilde{F}$; i.e., it is a union of leaves and singularities of $\tilde{F}$.

Let $S := \tilde{C} \setminus \text{Sing}(\tilde{F})$. Then $S$ is a smooth leaf of $\tilde{F}$. Take a point $p_0$ in $S$ and a transverse section $\Sigma$ passing through $p_0$. Let $G \subset \text{Diff}(\Sigma, p_0)$ be the holonomy group of the leaf $S$; since $\dim(\Sigma) = 1$, we assume that $G \subset \text{Diff}(\Sigma, 0)$. In this context, we have the following result of Fernández-Pérez [14].

**Lemma 5.1.** Assume the following:

(a) For any $p \in S \setminus \text{Sing}(\tilde{F})$, the leaf $L_p$ of $\tilde{F}$ through $p$ is closed in $S$.

(b) $g'(0)$ is a primitive root of unity, for all $g \in G$, $g \neq id$.

Then $L_C$ has a non-constant holomorphic first integral.

To continue, we use the above lemma to prove the following proposition.

**Proposition 5.2.** Let $M$ be a germ of an irreducible singular real-analytic Levi-flat hypersurface at $0 \in \mathbb{C}^2$ satisfying the hypotheses of Theorem 1. Then we have the following:

(a) the algebraic dimension of $\text{Sing}(M)$ is 0;

(b) $\text{cod}_{M_C}(\text{Sing}(L_C)) = 2$;

(c) $L_C$ has a non-constant holomorphic first integral.

**Proof.** Let $M$ be as in Theorem 1. Then $M$ is given by $M = \{F = 0\}$, where

$$F(z) = \Re(Q(z)) + H(z, \bar{z}),$$

$Q$ is a complex quasihomogeneous polynomial of quasihomogeneous degree $d$ of type $(a, b)$ with an isolated singularity at $0 \in \mathbb{C}^2$ and $H$ is a germ of real-analytic function at $0 \in \mathbb{C}^2$ of order strictly greater than $d$. It follows from Lemma 3.3 that $Q$ can be written as

$$Q(x, y) = \mu x^m y^n \prod_{\ell=1}^{k} (y^p - \lambda_\ell x^q),$$

where $m, n, p, q \in \mathbb{Z}_+^*$, $\mu, \lambda_\ell \in \mathbb{C}^*$ for each $\ell = 1, \ldots, k$, and $\gcd(p, q) = 1$. Since $Q$ has an isolated singularity at $0 \in \mathbb{C}^2$, then we necessarily that both $m$ and $n$ are either 0 or 1.

On the other hand, since $Q$ has weights $(a, b)$ with $\gcd(a, b) = 1$ we have each polynomial $(y^p - \lambda_\ell x^q)$ has also weights $(a, b)$, which implies that $aq = bp$. Since $p, q$ are relatively prime, we get $a = p$ and $b = q$.

For simplicity, using (5), we write

$$Q(x, y) = \mu x^m y^n \prod_{\ell=1}^{k} Q_\ell(x, y),$$

where $Q_\ell(x, y) = (y^p - \lambda_\ell x^q)$. Without loss of generality, we can assume that $Q$ has real coefficients. Then the complexification $F_C$ of $F$ is given by

$$F_C(x, y, z, w) = \frac{1}{2} Q(x, y) + \frac{1}{2} Q(z, w) + H_C(x, y, z, w).$$
Since \( Q \) has an isolated singularity at \( 0 \in \mathbb{C}^2 \), we get \( M_C = \{ F_C = 0 \} \subset (\mathbb{C}^4, 0) \) has an isolated singularity at \( 0 \in \mathbb{C}^4 \) and so the algebraic dimension of \( \text{Sing}(M) \) is zero. Hence item (a) is proved. Consider the algebraic subvariety contained in \( \mathbb{P}(a, b, a, b) \)

\[
V_{M_C} = \{ Q(Z_0, Z_1) + Q(Z_2, Z_3) = 0 \},
\]

where \( [Z_0 : Z_1 : Z_2 : Z_3] \in \mathbb{P}(a, b, a, b) \). It is not difficult to see that \( \text{Sing}(M_C) = \text{Sing}(V_{M_C}) \). Note that \( V_{M_C} \) can be considered as \( V \)-variety

\[
V_{M_C} \subset Z \simeq \mathbb{C}^4 / \mathbb{Z}(a, b, a, b).
\]

Now we consider the weighted blow-up \( E : \tilde{Z} \to Z \), with weight \( \delta = (a, b, a, b) \). Let \( \tilde{M}_C \)
be the strict transform of \( M_C \) by \( E \) and \( D \simeq \mathbb{P}_\delta \) the exceptional divisor, with coordinates \( (Z_0, Z_1, Z_2, Z_3) \in \mathbb{C}^4 \setminus \{ 0 \} \). The intersection of \( \tilde{M}_C \) with \( \mathbb{P}_\delta \) is

\[
\tilde{C} := \tilde{M}_C \cap \mathbb{P}_\delta = \{ Q(Z_0, Z_1) + Q(Z_2, Z_3) = 0 \}.
\]

It follows from Remark [2.1] that \( \mathcal{L}_C \) can be defined by \( \alpha|_{M_C} = 0 \), where

\[
(6) \quad \alpha = Q(x, y) \left[ \left( \frac{m}{x} - qx^{q-1} \sum_{i=1}^{k} \frac{\lambda_i}{Q_i(x, y)} \right) dx + \left( \frac{n}{y} + py^{p-1} \sum_{i=1}^{k} \frac{1}{Q_i(x, y)} \right) dy \right] + \theta
\]

and \( \theta = 2 \left( \frac{\partial \alpha}{\partial x} dx + \frac{\partial \alpha}{\partial y} dy \right) \) is a holomorphic 1-form with order strictly greater than \( d \). It follows from (6) that \( \text{Sing}(\mathcal{L}_C) \) has codimension two proving item (b). The rest of the proof is devoted to the proof of item (c). Note that the leaves of \( \mathcal{L}_C \) are closed in \( M_C \setminus \text{Sing}(\mathcal{L}_C) \) by Lemma [2.2]. To apply Lemma [5.1] we need calculate the holonomy group associated to \( \mathcal{L}_C \).

For each \( i = 1, 2, 3, 4 \), we have the affine open sets

\[
\tilde{U}_i = \mathbb{C}^4 / \mathbb{Z}_a, (-a, ..., \frac{1}{1-ih}, ..., -b).
\]

We work in \( \tilde{U}_3 \) with coordinates \( (x_1, y_1, z_1, w_1) \). In this open subset, the blow-up \( E \) has the following expression

\[
E(x_1, y_1, z_1, w_1) = (x_1^q, y_1^q, z_1^q, w_1z_1^q),
\]

with \( D \cap \tilde{U}_3 = \{ z_1 = 0 \} / \mathbb{Z}_a \). In this chart, the pull-back of \( \alpha \) by \( E \) is given by

\[
E^* \alpha = z_1^{pm+qn+kpq-1} \alpha_1,
\]

where

\[
(7) \quad \alpha_1 = Q(x_1, y_1) \left[ \left( \frac{mz_1}{x_1} - qx_1^{q-1} z_1 \sum_{i=1}^{k} \frac{\lambda_i}{Q_i(x_1, y_1)} \right) dx_1 + \left( \frac{nz_1}{y_1} + py_1^{p-1} z_1 \sum_{i=1}^{k} \frac{1}{Q_i(x_1, y_1)} \right) dy_1 + \left( pm + qn - pqx_1^{q-1} \sum_{i=1}^{k} \frac{\lambda_i}{Q_i(x_1, y_1)} \right) dz_1 \right] + z_1 \theta_1
\]
and \( \theta_1 = E^*\theta/\omega_1^{pm+qn+kpq} \). The pull-back foliation \( \tilde{\mathcal{C}} \) is defined by \( \alpha_1|_{\tilde{\mathcal{C}}} = 0 \). The intersection of \( \tilde{\mathcal{C}} \) with the open subset \( \tilde{U}_3 \) is

\[
\tilde{C} \cap \tilde{U}_3 = \{ z_1 = Q(x_1, y_1) + Q(1, w_1) = 0 \}/\mathbb{Z}_a,
\]
which implies that \( \tilde{C} \) is invariant by \( \tilde{\mathcal{C}} \) by (7), and

\[
\text{Sing}(\tilde{\mathcal{C}}) \cap \tilde{U}_3 = \{ z_1 = Q(x_1, y_1) = Q(1, w_1) = 0 \}/\mathbb{Z}_a.
\]

In the chart \( \tilde{U}_4 \), with coordinates \( (x_2, y_2, z_2, w_2) \), the blow-up is

\[
E(x_2, y_2, z_2, w_2) = (x_2 w_2^2, y_2 w_2^k, z_2 w_2^2, w_2^3)
\]
and \( D \cap \tilde{U}_4 = \{ w_2 = 0 \}/\mathbb{Z}_a \). In this chart, the pull-back of \( \alpha \) is

\[
E^*\alpha = w_2^{pm+qn+kpq-1}\alpha_2,
\]
where

\[
\alpha_2 = Q(x_2, y_2) \left[ \left( \frac{mw_2}{x_2} - qx_2^{-1}w_2 \sum_{\ell=1}^{k} \frac{\lambda_{\ell}}{Q_{\ell}(x_2, y_2)} \right) dx_2 + \left( \frac{nw_2}{y_2} + py_2^{2}w_2 \sum_{\ell=1}^{k} \frac{1}{Q_{\ell}(x_2, y_2)} \right) dy_2 + \left( pm + qn - pqx_2^{2} \sum_{\ell=1}^{k} \frac{\lambda_{\ell}}{Q_{\ell}(x_2, y_2)} + pqy_2^{2} \sum_{\ell=1}^{k} \frac{1}{Q_{\ell}(x_2, y_2)} \right) dw_2 \right] + w_2 \theta_2
\]
and \( \theta_2 = E^*\theta/\omega_2^{pm+qn+kpq} \). The pull-back foliation \( \tilde{\mathcal{C}} \) is given by \( \alpha_2|_{\tilde{\mathcal{C}}} = 0 \). Similarly as before, the intersection of \( \tilde{C} \) with the open subset \( \tilde{U}_4 \) is

\[
\tilde{C} \cap \tilde{U}_4 = \{ w_2 = Q(x_2, y_2) + Q(z_2, 1) = 0 \}/\mathbb{Z}_a,
\]
which is invariant by \( \tilde{\mathcal{C}} \) by (8), and

\[
\text{Sing}(\tilde{\mathcal{C}}) \cap \tilde{U}_4 = \{ w_2 = Q(x_2, y_2) = Q(z_2, 1) = 0 \}/\mathbb{Z}_a.
\]
Now, we focus in the chart \( \tilde{U}_3 \). In this open subset, the action of the group is given by

\[
\begin{align*}
x_1 &\mapsto x_1, \\
y_1 &\mapsto e^\frac{2\pi i}{m} y_1, \\
w_1 &\mapsto e^\frac{2\pi i}{n} w_1.
\end{align*}
\]

The exceptional divisor in this chart is given by

\[
\text{Sing}(D) \cap \tilde{U}_3 = \{ y_1 = z_1 = w_1 = 0 \}/\mathbb{Z}_a
\]
and therefore the intersection of the singular set of \( \tilde{\mathcal{C}} \) with the singular set of the exceptional divisor is

\[
\text{Sing}(D) \cap \text{Sing}(\tilde{\mathcal{C}}) \cap \tilde{U}_3 = \{ y_1 = z_1 = w_1 = Q(1, 0) = 0 \}/\mathbb{Z}_a.
\]
Due to the factorization of \( Q \) given in (5), we investigated four cases.
\begin{itemize}
  \item $m = n = 1$. In this case, $Q(1,0) = 0$ and, since $Q(1,w_1)$ is a complex polynomial in $w_1$, there exists another complex polynomial $\tilde{Q}$ such that $Q(1,w_1) = w_1\tilde{Q}(w_1)$ such that $\tilde{Q}(0) \neq 0$. Note that the power for $w_1$ may not be higher than one, because this would conflict with the fact that $n = 1$ in the factorization of $Q$. Now, if $r$ is a root of $\tilde{Q}$, then $r \neq 0$ and therefore $(0,0,0,0) \in \text{Sing}(\tilde{L}_C) \cap \tilde{U}_3$ and $(0,0,0,0) \notin \text{Sing}(D) \cap \tilde{U}_3$. Hence, we get $\text{Sing}(D) \cap \tilde{U}_3 \subseteq \text{Sing}(\tilde{L}_C) \cap \tilde{U}_3$.
  \item $m = 0$, $n = 1$. The same argument as the previous one holds in this case and therefore we have $\text{Sing}(D) \cap \tilde{U}_3 \subseteq \text{Sing}(\tilde{L}_C) \cap \tilde{U}_3$.
  \item $m = 1$, $n = 0$. In this case, $Q(1,0) \neq 0$ and therefore $\text{Sing}(D) \cap \text{Sing}(\tilde{L}_C) \cap \tilde{U}_3 = \emptyset$.
  \item $m = 0$. Same as before, we conclude that $\text{Sing}(D) \cap \text{Sing}(\tilde{L}_C) \cap \tilde{U}_3 = \emptyset$.
\end{itemize}

We arrive to the same conclusions working in the chart $\tilde{U}_4$. In both cases we have shown that, either $\text{Sing}(D) \cap \text{Sing}(\tilde{L}_C) = \emptyset$ or that $\text{Sing}(D) \subseteq \text{Sing}(\tilde{L}_C)$.

Consider the set $S := \tilde{C} \setminus \text{Sing}(\tilde{L}_C)$. This set is a leaf of $\tilde{L}_C$. Let $q_0$ be a point in $S \setminus \text{Sing}(D)$ and a section $\Sigma$ transverse to $S$ passing through $q_0$. Working on the chart $\tilde{U}_3$, we may assume without loss of generality that $q_0 = (1,0,0,0)$ and $\Sigma = \{(1,0,t,0) : t \in \mathbb{C}\}$. Let $G$ be the holonomy group of the leaf $S$ of $\tilde{L}_C$ in $\Sigma$. Recall that

$$\text{Sing}(\tilde{L}_C) \cap \tilde{U}_3 = \{z_1 = Q(x_1,y_1) = Q(1,w_1)\} / \mathbb{Z}_a.$$ 

This set splits into several connected components, separated in the following cases:

\begin{itemize}
  \item $m = 1$, $n = 1$. In this case,
    $$Q(x,y) = xy \prod_{\ell=1}^k Q_{\ell}(x,y),$$

  where $Q_{\ell}(x,y) = (y^p - \lambda_\ell x^q)$, $\gcd(p,q) = 1$ and $aq = pb = d$. The set $\text{Sing}(\tilde{L}_C) \cap \tilde{U}_3$ splits as the union of the following connected components:

  $$C^{e}_{\ell \, rs} = \{z_1 = Q_{\ell}(x_1,y_1) = w_1 - \varepsilon^{(r)}_{\ell}(\lambda_s) = 0\} / \mathbb{Z}_a,$$

  $$C^{e_{rs}}_{\ell} = \{z_1 = x_1 = w_1 - \varepsilon^{(r)}_{\ell}(\lambda_s) = 0\} / \mathbb{Z}_a,$$

  $$C^{g_{rs}}_{\ell} = \{z_1 = y_1 = w_1 - \varepsilon^{(r)}_{\ell}(\lambda_s) = 0\} / \mathbb{Z}_a,$$

  where $s, \ell \in \{1,\ldots,k\}$ and $r \in \{1,\ldots,p\}$ and for each $r$, $\varepsilon^{(r)}_{\ell}(\lambda_s)$ is an $p$-th root of $\lambda_s$. According to [29], the fundamental group $\pi_1(S,q_0)$ may be written in terms of generators and its relations as

  $$\pi_1(S,q_0) = \langle \gamma_{\ell \, rs}, \delta_{\ell \, rs}, \xi_{\ell \, rs}, \tau_{\ell \, rs} : \gamma^p_{\ell \, rs} = \delta^r_{\ell \, rs} \rangle, \quad \ell, s = 1,\ldots,k; \quad r = 1,\ldots,p,$$

  where, for each $\ell, r, s$, the elements $\gamma_{\ell \, rs}$ and $\delta_{\ell \, rs}$ are loops around the connected component $C^{e}_{\ell \, rs}$ of $\text{Sing}(\tilde{L}_C) \cap \tilde{U}_3$, $\xi_{\ell \, rs}$ are loops around $C^{e_{rs}}_{\ell}$ and $\tau_{\ell \, rs}$ a loop around $C^{g_{rs}}_{\ell}$. If $G$ is the holonomy group of the leaf $S$ of $\tilde{L}_C$ in the section $\Sigma$, then

  $$G = \langle f_{\ell \, rs}, g_{\ell \, rs}, h_{\ell \, rs}, k_{\ell \, rs} : \gamma^p_{\ell \, rs} = \delta^r_{\ell \, rs} \rangle, \quad \ell, s = 1,\ldots,k; \quad r = 1,\ldots,p.$$
where \( f_{\ell rs}, g_{\ell rs}, h_{\ell rs} \) and \( k_{\ell rs} \) correspond to the equivalence classes of the loops \( \gamma_{\ell rs}, \delta_{\ell rs}, \xi_{\ell rs}, \tau_{\ell rs} \) in \( \pi_1(S,q_0) \), respectively. Each one of these loops lifts up to \( \Gamma_{\ell rs}(t), \Delta_{\ell rs}(t), \Xi_{\ell rs}(t), \Upsilon_{\ell rs}(t) \), respectively, under the condition that each one of these belong on the leaves of \( \tilde{L}_C \) and that this foliation is defined by \( \alpha_1|\lambda_{rs}^t = 0 \) (see for instance \( 7 \)). We have the coefficients of the linear terms of the holonomy maps are given by

\[
\begin{align*}
    f'_{\ell rs}(0) &= e^{-\frac{2(p+q)}{3} \pi i}, \\
    g'_{\ell rs}(0) &= e^{-\frac{2}{3} \frac{(p+q)^2}{p+q} \pi i}, \\
    h'_{\ell rs}(0) &= 1, \\
    k'_{\ell rs}(0) &= e^{-2 \left( \frac{1+p}{p+q} \right) \pi i}.
\end{align*}
\]

According to Lemma \ref{lemma:holonomy}, the foliation \( \tilde{L}_C \) has a holomorphic non-constant first integral and the proof in this case is finished.

- \( m = 0, n = 1 \). In this case,

\[
Q(x, y) = y^{\prod_{\ell=1}^k} Q_{\ell}(x, t),
\]

where \( Q_{\ell} = (y^p - \lambda_\ell x^q) \), gcd\( (p, q) = 1 \) and \( aq = pb = d \). The set \( \text{Sing}(\tilde{L}_C) \cap \tilde{U}_3 \) splits as the union of the following connected components:

\[
C_{\ell rs}^k = \{ z_1 = Q_\ell(x_1, y_1) = w_1 - \varepsilon^{(r)}_p(\lambda_s) = 0 \}/\mathbb{Z}_q,
\]

\[
C_{\ell rs}^y = \{ z_1 = y_1 = w_1 - \varepsilon^{(r)}_p(\lambda_s) = 0 \}/\mathbb{Z}_q,
\]

where \( s, \ell \in \{1, \ldots, k\}, r \in \{1, \ldots, p\} \) and, for each \( r, \varepsilon^{(r)}_p(\lambda_s) \) is a \( p \)-th root of \( \lambda_s \). The group \( \pi_1(S,q_0) \) is written in terms of generators and its relations as

\[
\pi_1(S,q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs}, \tau_{rs} : \gamma_{\ell rs}^p = \delta_{\ell rs}^p \rangle_{s=1,\ldots,k}^{r=1,\ldots,p}
\]

where, for each \( \ell, r, s, \gamma_{\ell rs} \) and \( \delta_{\ell rs} \) are loops around \( C_{\ell rs}^k \) and \( \tau_{rs} \) a loop around \( C_{\ell rs}^y \). If \( G \) is the holonomy group of the leaf \( S \) of \( \tilde{L}_C \) in the section \( \Sigma \) then

\[
G = \langle f_{\ell rs}, g_{\ell rs}, k_{\ell rs} \rangle_{s=1,\ldots,k}^{r=1,\ldots,p}
\]

where \( f_{\ell rs}, g_{\ell rs} \) and \( k_{\ell rs} \) correspond to the equivalence classes of the loops \( \gamma_{\ell rs}, \delta_{\ell rs}, \tau_{rs} \) in \( \pi_1(S,q_0) \), respectively. Each one of these loops lifts up to \( \Gamma_{\ell rs}(t), \Delta_{\ell rs}(t), \Upsilon_{\ell rs}(t) \), respectively, under the condition that each one of these belong on the leaves of \( \tilde{L}_C \) and that this foliation is defined by \( \alpha_1|\lambda_{rs}^t = 0 \) (see for instance \( 7 \)), we have the coefficients of the linear terms of the holonomy maps are given by

\[
\begin{align*}
    f'_{\ell rs}(0) &= e^{-\frac{2p}{3} \pi i}, \\
    g'_{\ell rs}(0) &= e^{-\frac{2}{3} \frac{p^2}{p+q} \pi i}, \\
    k'_{\ell rs}(0) &= 1.
\end{align*}
\]

Using Lemma \ref{lemma:holonomy} the proof in this case is finished.
• \( m = 1, n = 0 \). In this case

\[
Q(x, y) = x \prod_{\ell=1}^{k} Q_\ell(x, t),
\]

where \( Q_\ell = (y^p - \lambda_\ell x^q) \), \( \gcd(p, q) = 1 \) and \( aq = pb = d \). The set \( \text{Sing}(\mathcal{C}) \cap \tilde{U}_3 \) splits as the union of the following connected components:

\[
C_{rs}^\ell = \{ z_1 = Q_\ell(x_1, y_1) = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0 \}/\mathbb{Z}_a,
\]

\[
C_{rs}^{\eta} = \{ z_1 = x_1 = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0 \}/\mathbb{Z}_a,
\]

where \( s, \ell \in \{1, ..., k\} \), \( r \in \{1, ..., p\} \) and, for each \( r \), \( \varepsilon_p^{(r)}(\lambda_s) \) is a \( p \)-th root of \( \lambda_s \). The group \( \pi_1(S, q_0) \) is written in terms of generators and its relations as

\[
\pi_1(S, q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs} : \gamma_{\ell rs}^p = \delta_{\ell rs}^{\eta} \rangle \quad \ell, s = 1, ..., k \quad r = 1, ..., p
\]

where, for each \( \ell, r, s \), \( \gamma_{\ell rs} \) and \( \delta_{\ell rs} \) are loops around \( C_{rs}^\ell \) and \( \xi_{rs} \) a loop around \( C_{rs}^{\eta} \). If \( G \) is the holonomy group of the leaf \( S \) of \( \mathcal{C} \) in the section \( \Sigma \) then

\[
G = \langle f_{\ell rs}, g_{\ell rs}, h_{\ell rs} \rangle \quad \ell, s = 1, ..., k \quad r = 1, ..., p
\]

where \( f_{\ell rs}, g_{\ell rs} \) and \( h_{\ell rs} \) correspond to the equivalence classes of the loops \( \gamma_{\ell rs}, \delta_{\ell rs}, \xi_{rs} \) in \( \pi_1(S, q_0) \), respectively. Each one of these loops lifts up to \( \Gamma_{\ell rs}(t), \Delta_{\ell rs}(t), \Xi_{rs}(t) \), respectively, under the condition that each one of these belong on the leaves of \( \mathcal{C} \) and that this foliation is defined by \( \alpha_1|_{M_{\ell}} = 0 \) (see for instance [7]), we have the coefficients of the linear terms of the holonomy maps are given by

\[
\begin{align*}
\int_{\ell rs}(0) & = e^{-\frac{2\pi i}{p}}, \\
g_{\ell rs}(0) & = e^{-\frac{2\pi i}{p}}, \\
k_{\ell rs}(0) & = 1,
\end{align*}
\]

Again by Lemma 5.1, the proof in this case is finished.

• \( m = 0, n = 0 \). In this case, \( Q(x, y) = \prod_{\ell=1}^{k} Q_\ell(x, t) \), where \( Q_\ell = (y^p - \lambda_\ell x^q) \), \( \gcd(p, q) = 1 \) and \( aq = pb = d \). The set \( \text{Sing}(\mathcal{C}) \cap \tilde{U}_3 \) splits as the union of the following connected components:

\[
C_{rs}^\ell = \{ z_1 = Q_\ell(x_1, y_1) = w_1 - \varepsilon_p^{(r)}(\lambda_s) = 0 \}/\mathbb{Z}_a,
\]

where \( s, \ell \in \{1, ..., k\} \), \( r \in \{1, ..., p\} \) and, for each \( r \), \( \varepsilon_p^{(r)}(\lambda_s) \) is a \( p \)-th root of \( \lambda_s \). The group \( \pi_1(S, q_0) \) is written in terms of generators and its relations as

\[
\pi_1(S, q_0) = \langle \gamma_{\ell rs}, \delta_{\ell rs} : \gamma_{\ell rs}^p = \delta_{\ell rs}^{\eta} \rangle \quad \ell, s = 1, ..., k \quad r = 1, ..., p
\]
where, for each \( \ell, r, s \), \( \gamma_{\ell rs} \) and \( \delta_{\ell rs} \) are loops around \( C_{\ell rs}^\ell \). If \( G \) is the holonomy of the leaf \( S \) of \( \tilde{L}_C \) in the section \( \Sigma \) then

\[
G = \langle f_{\ell rs}, g_{\ell rs} \rangle_{\ell, r, s = 1, \ldots, k, \ell, r = 1, \ldots, p}
\]

where \( f_{\ell rs}, g_{\ell rs} \) correspond to the equivalence classes of the loops \( \gamma_{\ell rs}, \delta_{\ell rs} \) in \( \pi_1(S, q_0) \), respectively. Each one of these loops lifts up to \( \Gamma_{\ell rs}(t), \Delta_{\ell rs}(t) \), respectively, under the condition that each one of these belong on the leaves of \( \tilde{L}_C \) and that this foliation is defined by \( \alpha_1|_{\mathcal{L}_C} = 0 \) (see for instance [1]), we have the coefficients of the linear terms of the holonomy maps are given by

\[
f'_{\ell rs}(0) = e^{-\frac{2\pi i q}{\alpha_1}}, \quad g'_{\ell rs}(0) = e^{-\frac{2\pi i p}{\alpha_1}}.
\]

Finally Lemma 5.1 implies that \( \tilde{L}_C \) has a holomorphic non-constant first integral. \( \Box \)

5.1. **Proof of Theorem 1.** Note that Proposition 5.2 implies that the hypotheses of Theorem 2.3, part (b) are verified. Then there exists a germ of holomorphic foliation \( F_M \) with a non-constant holomorphic first integral \( f \in \mathcal{O}_2 \) such that \( M = \{ \Re(f) = 0 \} \). Without loss of generality, we can assume that \( f \) is not a power in \( \mathcal{O}_2 \) and therefore so \( \Re(f) \) is irreducible by Lemma 2.1. This implies

\[
\Re(f) = U \cdot F,
\]

where \( U \in \mathcal{A}_{nR} \) and \( U(0) \neq 0 \). Since \( F(z) = \Re(Q(z)) + H(z, \bar{z}) \) and \( Q \) is a quasihomogeneous polynomial of quasihomogeneous degree \( d \) with weights \( (a, b) \), we can write \( f \) as the decomposition

\[
f = \sum_{\ell \geq d} f_\ell,
\]

where each \( f_\ell \) is a quasihomogeneous polynomial of quasihomogeneous degree \( \ell \) with weights \( (a, b) \) (see [2, p. 193]). If the power series of \( U \) at \( 0 \in \mathbb{C}^2 \) is

\[
U(z) = U(0) + \tilde{U}(z) = U(0) + \sum_{\mu_1, \mu_2 \geq 1, \nu_1, \nu_2 \geq 1} c_{\mu_1, \mu_2, \nu_1, \nu_2} z^{\mu_1} z^2 \bar{z}^{\nu_1} \bar{z}^{\nu_2}
\]

then

\[
\Re(f) = \Re(U(0) + \tilde{U})(\Re(Q) + H) = U(0)\Re(Q) + \tilde{U}\Re(Q) + U(0)H + \tilde{U}H.
\]

We need to investigate what terms on the previous equality have quasihomogeneous degree \( d \) with weights \( (a, b) \), the sum of these terms will be equal to \( \Re(f_d) \). Set \( \tilde{H} = U(0)H + \tilde{U}H \), note that the quasihomogeneous terms of \( \tilde{H} \) has order strictly greater than \( d \). Writing

\[
\Re(f) = \Re(f_d) + \sum_{\ell > d} \Re(f_\ell),
\]
we have, for all $\lambda \in \mathbb{C}^*$

\[
\Re(f(\lambda^a z_1, \lambda^b z_2)) = U(0)\Re(Q(\lambda^a z_1, \lambda^b z_2)) + \tilde{U}(\lambda^a z_1, \lambda^b z_2) \]

\[
+ \tilde{H}(\lambda^a z_1, \lambda^b z_2) = U(0)\Re(Q(z_1, z_2)) + \tilde{H}(\lambda^a z_1, \lambda^b z_2)
\]

\[
= U(0)\left(\frac{\lambda^a Q(z) + \lambda^b Q(z)}{2}\right) + \left(c_{1000}\lambda^a z_1 + c_{0100}\lambda^b z_2 + c_{0010}\lambda^a z_2 + \cdots + \tilde{H}(\lambda^a z_1, \lambda^b z_2)\right)
\]

\[
\text{generalized degree is } d
\]

\[
\text{generalized degree is greater than } d
\]

which means that $f_0(z) = U(0)Q(z)$, hence $f(z) = U(0)Q(z) + \sum_{\ell > d} f_\ell$. Without any loss of generality we may assume that $U(0) = 1$. In particular, $\mu(f, 0) = \mu(Q, 0)$, since $Q$ has an isolated singularity at the origin. According to Theorem 3.2, there exists a germ of biholomorphism $\phi: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

\[
f \circ \phi^{-1}(z) = Q(z) + \sum_j c_j e_j(z),
\]

where $c_j \in \mathbb{C}$ and $e_j$ are elements of the monomial basis of $A_Q$ with $\deg(e_j) > d$. Hence

\[
\phi(M) = \left\{ \Re \left( Q(z) + \sum_j c_j e_j(z) \right) = 0 \right\}
\]

and this finishes the proof of Theorem 1.

**Example 5.1.** Now we give an application of Theorem 1. Consider the complex quasihomogeneous polynomial

\[
Q(x, y) = x^a + \lambda x^2 y^2 + y^b
\]

where $a \geq 4, \ b \geq 5, \ \lambda \neq 0$.

We have $Q$ has isolated singularity at $0 \in \mathbb{C}^2$ with $\mu(Q, 0) = a + b + 1$. According to [2, p. 33], every semiquasihomogeneous function $f$ with principal part $Q(x, y)$ is right equivalent to $Q(x, y)$. Consequently, if we consider $F(x, y) = \Re(Q(x, y)) + H(x, y)$ as a germ of real-analytic function at $0 \in \mathbb{C}^2$ such that $M = \{F = 0\}$ is Levi-flat then Theorem 1 implies that
$M$ is biholomorphic to germ at $0 \in \mathbb{C}^2$ of real-analytic Levi-flat hypersurface defined by
$$M' = \{\text{Re}(x^a + \lambda x^2 y^2 + y^b) = 0\}$$ for $a \geq 4$, $b \geq 5$, $\lambda \neq 0$.

6. Isochore normal forms for holomorphic functions

Let $f \in O_n$ be a germ of holomorphic function with an isolated singularity at $0 \in \mathbb{C}^n$ such that its Hessian form
$$h := \sum_{1 \leq i,j \leq n} \frac{\partial^2 f(0)}{\partial z_i \partial z_j} z_i z_j$$
is non-degenerate. The classical Morse’s lemma asserts that $f$ is right equivalent to $h$.

Let $\omega = a(z)dz_1 \wedge \ldots \wedge dz_n$, $a(0) \neq 0$ be a holomorphic volume form on a coordinate system $(z_1, \ldots, z_n)$ on an open set around $0 \in \mathbb{C}^n$. A coordinate system $(x_1, \ldots, x_n)$ is isochore or volume-preserve, if $\omega$ can be written as $dx_1 \wedge \ldots \wedge dx_n$ on these coordinates. Then, we say that a biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is isochore or volume-preserving if the coordinate system induced by it is isochore.

In 1977, J. Vey [25] has posed the following question: It is possible to find a coordinate system isochore such that $f$ is right equivalent to $h$? Vey answered negatively to question and proved the following result.

**Lemma 6.1 (Vey [25])**. Let $f \in O_n$, $n \geq 2$, with isolated singularity at $0 \in \mathbb{C}^n$ such that its Hessian form $h$ is non-degenerate. Then there exists a germ of a volume-preserving biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a germ of an automorphism $\psi \in O_1$, with $\psi(0) = 0$, such that
$$f \circ \phi^{-1} = \psi \circ h, \quad \psi(t) = t + c_2 t^2 + c_3 t^3 + \ldots$$
The function $\psi$ is uniquely determined by $f$ up to a sign.

This result was also proved by J-P Franoise [17]. The approach used by Franoise was later generalized by A. Szawlowski [24] to study of complex quasihomogeneous polynomials and to the germ of a holomorphic function that is right equivalent to the product of coordinates $z_1 \cdot \ldots \cdot z_n$, as stated by the following theorem.

**Theorem 6.2 (Szawlowski [24])**. Let $f \in O_n$, $n \geq 2$ be a germ of holomorphic function that is right equivalent to the product of all coordinates: $f \sim_R z_1 \cdot \ldots \cdot z_n$. Then there exists a germ of a volume-preserving biholomorphism $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a germ of an automorphism $\Psi \in O_1$, with $\Psi(0) = 0$, such that
$$f \circ \Phi = \Psi(z_1 \cdot \ldots \cdot z_n).$$
The function $\Psi$ is uniquely determined by $f$ up to a sign.

Note that the above normal form for $f$ is a germ of holomorphic function whose singular set is of positive dimension (non-isolated singularity). In general, normal forms of germs of functions with non-isolated singularities are very difficult of find, even for a change of coordinates non-isochore.
7. Theorem \[1\]

To prove Theorem 2, we use the following result proved in [13], although it is not stated as a separate theorem. We restate it here for completeness.

**Theorem 7.1** (Fernández-Pérez [13]). Let \( M = \{ F = 0 \} \) be a germ of an irreducible singular real-analytic Levi-flat hypersurface at \( 0 \in \mathbb{C}^n \), \( n \geq 2 \), such that

1. \( F(z) = \Re(P(z)) + H(z, \bar{z}) \),
2. \( P \) is a complex homogeneous polynomial of degree \( k \) with an isolated singularity at \( 0 \in \mathbb{C}^n \),
3. \( j_k^0(H) = 0 \) and \( H(z, \bar{z}) = H(\bar{z}, z) \).

Then there exists a germ at \( 0 \in \mathbb{C}^n \) of holomorphic codimension-one foliation \( F_M \) tangent to \( M \). Moreover, the foliation \( F_M \) has a non-constant holomorphic first integral \( f(z) = P(z) + O(|z|^{k+1}) \), and \( M = \{ \Re(f) = 0 \} \).

**7.1. Proof of Theorem 2.** Let \( M = \{ F = 0 \} \) be a germ at \( 0 \in \mathbb{C}^n \), \( n \geq 2 \), of an irreducible real-analytic Levi-flat hypersurface such that \( F(z) = \Re(z_1^2 + \ldots + z_n^2) + H(z, \bar{z}) \), where \( j_k^0(H) = 0 \), \( H(z, \bar{z}) = H(\bar{z}, z) \). Since \( P(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2 \) is a complex homogeneous polynomial of degree 2, we can apply Theorem 7.1 so that there exists a \( f \in \mathcal{O}_n \) such that \( f(z) = z_1^2 + \ldots + z_n^2 + O(|z|^3) \) and \( M = \{ \Re(f) = 0 \} \). On the other hand, applying Lemma 6.1 to \( f \), there exists a volume-preserving \( \phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) and an automorphism \( \psi_1 \in \mathcal{O}_1 \) with \( \psi_1(0) = 0 \), such that

\[
\psi_1(t) = t + c_2t^2 + c_3t^3 + \ldots
\]

Taking \( \psi := \psi_1(t/2) \in \mathcal{O}_1 \), we have \( f \circ \phi^{-1} = \psi \circ P \). Finally, \( \phi(M) = \{ \Re(\psi(z_1^2 + \ldots + z_n^2)) = 0 \} \) and the proof of Theorem 2 ends.

8. Proof of Theorem 3 and Corollary 1

Here we will use the same idea of the proof of Theorem 1. First of all, note that, in dimension two, under the change of variables \( z_1 = y + ix, \ z_2 = y - ix \), and we have \( z_1z_2 = x^2 + y^2 \) and then Theorem 3 follows from Theorem 2 because the singular set of \( M_{\mathbb{C}} \) is the origin of \( \mathbb{C}^4 \). Therefore, we only consider the case \( n \geq 3 \).

**Proposition 8.1.** Let \( M \) be a germ of a singular real-analytic Levi-flat hypersurface at \( 0 \in \mathbb{C}^n \), \( n \geq 3 \), satisfying the hypotheses of Theorem 3. Then \( L_{\mathbb{C}} \) has a non-constant holomorphic first integral.

**Proof.** Let \( M \) be as in Theorem 3. Then, \( M \) is given by \( \{ F = 0 \} \) where

\[
F(z) = \Re(z_1 \cdots z_n) + H(z_1, \ldots, z_n),
\]
and \( j_0^\pi(H) = 0 \). Its complexification is

\[
F_\mathcal{C}(z, w) = \frac{1}{2}(z_1 \cdots z_n) + \frac{1}{2}(w_1 \cdots w_n) + H_\mathcal{C}(z, w),
\]

and therefore \( M_\mathcal{C} = \{ F_\mathcal{C} = 0 \} \subset (\mathbb{C}^2)^n \). By hypotheses, \( \text{Sing}(M_\mathcal{C}) \) is the union of the sets

\[
V_{ijk\ell} = \{ z_i = z_j = w_k = w_\ell = 0 \}, \quad 1 \leq i < j \leq n, \quad 1 \leq k < \ell \leq n.
\]

Since \( V_{ijk\ell} \) has complex dimension \( 2n - 4 \), then the algebraic dimension of \( \text{Sing}(M) \) is \( 2n - 4 \).

On the other hand, it follows from Remark 2.1 that \( \mathcal{L}_\mathcal{C} \) is given by \( \alpha |_{M_\mathcal{C}} = 0 \), where

\[
\alpha = \sum_{i=1}^n \frac{\partial F_\mathcal{C}}{\partial z_i} dz_i.
\]

Using (9) we can write \( \alpha \) in coordinates \( (r_1, \ldots, r_n) \in \mathbb{C}^n \) as

\[
\alpha = \frac{1}{2} \sum_{i=1}^n \left( r_1 \cdots \hat{r}_i \cdots r_n + \frac{\partial R}{\partial r_i} \right) dr_i,
\]

where \( \frac{\partial R}{\partial r_i} = 2 \frac{\partial H_\mathcal{C}}{\partial r_i} \) for all \( i = 1, \ldots, n \). Then we can consider that \( \mathcal{L}_\mathcal{C} \) is defined by \( \tilde{\alpha} |_{M_\mathcal{C}} = 0 \), where

\[
\tilde{\alpha} = \sum_{i=1}^n \left( r_1 \cdots \hat{r}_i \cdots r_n + \frac{\partial R}{\partial r_i} \right) dr_i.
\]

Let us prove that \( \mathcal{L}_\mathcal{C} \) has a non-constant holomorphic first integral. We start with the blow-up \( \pi_1 \) at \( 0 \in \mathbb{C}^n \) with exceptional divisor \( D_1 \cong \mathbb{P}^{2n-1} \). Let \( [Z : Y] = [Z_1 : \ldots : Z_n : Y_1 : \ldots : Y_n] \) be the homogeneous coordinates of \( D_1 \). The intersection of \( M_\mathcal{C} = \pi_1^{-1}(M_\mathcal{C}) \) with the divisor \( D_1 \) is the algebraic hypersurface

\[
Q_1 := \tilde{M}_\mathcal{C} \cap D_1 = \{ [Z : Y] \in \mathbb{P}^{2n-1} : Z_1 \cdots Z_n + Y_1 \cdots Y_n = 0 \}.
\]

In the chart \( (W; (r, \ell) = (r_1, \ldots, r_n, \ell_1, \ldots, \ell_n)) \) of \( \tilde{\mathbb{C}}^{2n} \) where

\[
\pi_1(r, \ell) = (\ell_1 r_1, \ldots, \ell_1 r_2, \ldots, \ell_1 r_n, \ell_2, \ldots, \ell_n),
\]

Then

\[
\tilde{F}_\mathcal{C}(r, \ell) = F_\mathcal{C} \circ \pi_1(r, \ell) = \ell_1^n r_1 \cdots r_n + \ell_2 \cdots \ell_n + R(\pi_1(r, \ell))
\]

\[
= \ell_1^n (r_1 \cdots r_n + \ell_2 \cdots \ell_n + \ell_1 R_1(r, \ell)),
\]

where \( R_1(r, \ell) = R(\pi_1(r, \ell))/\ell_1^{n+1} \). Therefore

\[
\tilde{M}_\mathcal{C} \cap W = \{ r_1 \cdots r_n + \ell_2 \cdots \ell_n + R_1(r, \ell) = 0 \},
\]

and

\[
Q_1 \cap W = \{ \ell_1 = r_1 \cdots r_n + \ell_2 \cdots \ell_n = 0 \}.
\]
On the other hand, the pull-back of $\tilde{\alpha}$ by $\pi_1$ is

$$\pi_1^*(\tilde{\alpha}) = \sum_{i=1}^n \ell_1^{-1} (\ell_1 r_1 \cdot \cdots \cdot r_n) d(\ell_1 r_1) + \theta$$

$$= \ell_1^{-1} \left( \sum_{i=1}^n \ell_1 r_1 \cdot \cdots \cdot r_n dr_i + nr_1 \cdot \cdots \cdot r_n d\ell_1 + \ell_1 \theta \right),$$

where $\theta_1 = \theta/\ell_1^3$. In the chart $W$, the exceptional divisor is written as $D_1 = \{ \ell_1 = 0 \}$ and $\tilde{L}_C$ is given by $\alpha_1|_{\tilde{M}_C} = 0$, where

$$\alpha_1 = \sum_{i=1}^n \ell_1 r_1 \cdot \cdots \cdot r_n dr_i + nr_1 \cdot \cdots \cdot r_n d\ell_1 + \ell_1 \theta.$$ 

Note that $\tilde{M}_C \cap D_1$ is invariant by $\tilde{L}_C$ and moreover

$$\text{Sing}(\tilde{M}_C) \cap W = \bigcup_{i,j,k,s} W_{i,j,k,s},$$

where

$$W_{i,j,k,s} := \{ r_i = r_j = \ell_k = \ell_s = 0 \} \text{ where } i \neq j, k \neq s \text{ and } k \neq 1, s \neq 1.$$ 

Consider the irreducible component $W_{1,2,2,3}$ of $\text{Sing}(\tilde{M}_C) \cap W$. We make a blow-up along this component; the process of desingularization around the other components of $\text{Sing}(\tilde{M}_C) \cap W$ are similarly obtained by exchanging coordinates. Let $E$ be the exceptional divisor of $\pi_\ell : \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$. Let $\tilde{M}_C$ be the strict transform of $M_C$ and $\tilde{L}_C$ be the pull-back of $L_C$ by $\pi_\ell$ respectively. Let $U$ be an open subset with coordinates $(x_1, \ldots, x_{2n})$ where the blow-up is

$$\pi_\ell(x_1, \ldots, x_{2n}) = (x_1 x_n + x_2 x_n + x_3, \ldots, x_1 x_n + x_2 x_n + x_n+1, x_n+2 x_n+3, x_n+3, x_n+4, \ldots, x_{2n}),$$

we have

$$\tilde{F}_C = \tilde{F}_C \circ \pi_\ell = x_n^2 \cdot x_n^3 \cdot \cdots \cdot x_n + x_n+1 x_n+2 x_n+3 \cdots x_{2n} + x_n+1 x_n+3 R_2,$$

where $R_2 = R_1(\pi_\ell(x_1, \ldots, x_{2n}))/x_n^3$. Therefore

$$\tilde{M}_C \cap U = \{ x_1 \cdot \cdots \cdot x_n + x_{n+2} x_{n+4} \cdots x_{2n} + x_{n+1} x_{n+2} R_2 = 0 \}$$

hence

$$\tilde{M}_C \cap E \cap U = \{ x_{n+1} = x_{n+3} = x_1 \cdot \cdots \cdot x_n + x_{n+2} x_{n+4} \cdots x_{2n} = 0 \}.$$ 

The pull-back of $\alpha_1$ by $\pi_\ell$ is

$$\pi_\ell^*(\alpha_1) = \frac{x_{n+3}}{x_1} x_2 \cdots x_n x_{n+1} x_{n+3} dx_1 + x_1 x_3 \cdots x_n x_{n+1} x_{n+3} dx_2 + \sum_{i=3}^n \frac{x_{n+1} x_{n+3} dx_i}{x_1} + n x_1 x_n x_{n+3} dx_{n+1} + 2 x_1 x_2 \cdots x_n x_{n+1} dx_{n+3} + x_{n+1} x_{n+3} \theta_2,$$
where \( \theta_2 = \theta_1/x_{n+3}^2 \). In the chart \( U \), the exceptional divisor is written as
\[
D = D_1 \cup D_2 = \{ x_{n+1} = 0 \} \cup \{ x_{n+3} = 0 \}
\]
and \( \tilde{L}_C \) is given by \( \alpha_2|_{\tilde{L}_C} = 0 \), where
\[
\alpha_2 = x_2 \cdots x_n x_{n+1} x_{n+3} dx_1 + x_1 x_3 \cdots x_n x_{n+1} x_{n+3} dx_2 + \sum_{i=3}^{n} \frac{x_1 \cdots x_i x_{n+1} x_{n+3} dx_i}{x_i} + nx_1 \cdots x_n x_{n+3} dx_{n+1} + 2x_1 x_2 \cdots x_n x_{n+1} dx_{n+3} + x_{n+1} x_{n+3} \theta_2, \tag{10}
\]
which allows us to conclude that \( \tilde{M}_C \cap D \) is invariant by \( \tilde{L}_C \). The singularities of the foliation \( \tilde{L}_C \) on the exceptional divisor in this chart are given by
\[
\operatorname{Sing}(\tilde{L}_C) \cap D \cap U = \{ x_{n+1} = x_{n+3} = x_1 \cdots x_n = x_{n+2} x_{n+4} \cdots x_{2n} = 0 \}.
\]
If we define \( C_{i,n+j} = \{ x_{n+1} = x_{n+3} = x_i = x_{n+j} = 0 \} \cong \mathbb{C}^{2(n-2)} \), then we can write
\[
\operatorname{Sing}(\tilde{L}_C) \cap D \cap U = \bigcup_{1 \leq i,j \leq n, j \neq 1,3} C_{i,n+j}.
\]
Since \( \tilde{M}_C \cap D \) is invariant by \( \tilde{L}_C \), then
\[
S := (\tilde{M}_C \cap D) \setminus \operatorname{Sing}(\tilde{L}_C)
\]
is a leaf of \( \tilde{L}_C \). Let \( G \) be its holonomy group, and \( p_0 \in S \) given by
\[
p_0 = (x_1, \ldots, x_n, x_{n+1}, x_{n+3}, x_{n+4}, \ldots, x_{2n}) = (1, \ldots, 1, 0, -1, 0, 1, \ldots, 1).
\]
Take \( \Sigma \) the transversal section through \( p_0 \) given by
\[
\Sigma = \{(1, \ldots, 1, \lambda, -1, \lambda, 1, \ldots, 1) : \lambda \in \mathbb{C} \}.
\]
Let \( \delta_{i,j}(\theta) \) be a loop around \( C_{i,n+j} \), for \( 1 \leq i \leq n \) and \( 4 \leq j \leq n \), and \( \delta_{i,2}(\theta) \) a loop around \( C_{i,n+2} \), \( 1 \leq i \leq n \) with \( \theta \in [0, 1] \). Each one of these loops lifts up to \( \Gamma_{i,j}(\lambda, \theta) \) and \( \Gamma_{i,2}(\lambda, \theta) \), respectively, such that \( \Gamma_{i,j}(0, \theta) = 0 \), \( \Gamma_{i,j}(\lambda, 0) = \lambda \) and \( \Gamma_{i,j}(\lambda, \theta) = \sum_{k=1}^{\infty} \delta_{i,j}^k(\theta) \lambda^k \), for \( i = 1, \ldots, n \) and \( j = 2, 4, 5, \ldots, n \). The holonomy map with respect to these loops are
\[
h_{\delta_{i,j}}(\lambda) = \Gamma_{i,j}(\lambda, 1).
\]
Using the expression of \( \alpha_2 \) given in \( \text{[10]} \), we get
\[
h_{\delta_{i,j}}'(0) = e^{-\frac{\pi i}{4}} \lambda, \quad \text{for } i = 1, \ldots, n \text{ and } j = 2, 4, 5, \ldots, n.
\]
It follows from Lemma \( \text{[5.1]} \) that \( L_C \) has a non-constant holomorphic first integral. \( \Box \)
8.1. **Proof of Theorem 3**. Note that Proposition 8.1 implies that the hypotheses of Theorem 2.3, part (b) are verified. Then we get $f \in O_n$ such that the foliation $F$ given by $df = 0$ is tangent to $M$ and $M = \{\text{Re}(f) = 0\}$. Without loss of generality we may assume that $f$ is not a power in $O_n$ and therefore $\text{Re}(f)$ is irreducible in $A_{nR}$. We must have that $\text{Re}(f) = U \cdot F$ where $U \in A_{nR}$, $U(0) \neq 0$. If the Taylor expansion of $f$ at $0 \in \mathbb{C}^n$ is

$$f = \sum_{j \geq n} f_j,$$

where $f_j$ is a homogeneous polynomial of degree $j$, then

$$\text{Re}(f_n) = j_0^n(\text{Re}(f)) = j_0^n(U \cdot F) = U(0)\text{Re}(z_1 \cdots z_n),$$

which means $f_n(z) = U(0)z_1 \cdots z_n$. We can assume that $U(0) = 1$ and therefore

$$f(z) = z_1 \cdots z_n + O(|z|^{n+1}).$$

This finishes the proof of Theorem 3.

8.2. **Proof of Corollary 1**. If we assume that $f(z) \sim_R z_1 \cdots z_n$, it follows from Theorem 6.2 that there exists a germ of a volume-preserving biholomorphism $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a germ of an automorphism $\Psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, such that

$$f \circ \Phi^{-1}(z) = \Psi(z_1 \cdots z_n).$$

Hence

$$\Phi(M) = \{\text{Re}(\Psi(z_1 \cdots z_n)) = 0\}.$$

This finishes the proof of Corollary 1.

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(A. Fernández-Pérez) DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE MINAS GERAIS, UFMG Current address: Av. Antônio Carlos 6627, 31270-901, Belo Horizonte-MG, Brazil. E-mail address: fernandez@ufmg.br

(Gustavo Marra) DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE ITAJUBÁ Current address: Rua Irmã Ivone Drumond 200, 35903-087, Itabira-MG, Brazil. E-mail address: marra@unifei.edu.br