Closing the convergence gap of SGD without replacement

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Abstract

Stochastic gradient descent without replacement sampling is widely used in practice for model training. However, the vast majority of SGD analyses assumes data sampled with replacement, and when the function minimized is strongly convex, an $O \left( \frac{1}{T} \right)$ rate can be established when SGD is run for $T$ iterations. A recent line of breakthrough work on SGD without replacement (SGDo) established an $O \left( \frac{n^3}{T^2} \right)$ convergence rate when the function minimized is strongly convex and is a sum of $n$ smooth functions, and an $O \left( \frac{1}{T^2} + \frac{n^3}{T^2} \right)$ rate for sums of quadratics. On the other hand, the tightest known lower bound postulates an $\Omega \left( \frac{n^2}{T^3} \right)$ rate, leaving open the possibility of better SGDo convergence rates in the general case. In this paper, we close this gap and show that SGD without replacement achieves a rate of $O \left( \frac{n^2}{T^2} \right)$ when the sum of the functions is a quadratic, and offer a new lower bound of $\Omega \left( \frac{n}{T^2} \right)$ for strongly convex functions that are sums of smooth functions.

1 Introduction

Stochastic gradient descent (SGD) is a widely used first order optimization technique used to approximately minimize a sum of functions

$$F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x).$$

In its most general form, SGD produces a series of iterates

$$x_{i+1} = x_i - \alpha \cdot g(x, \xi_i)$$

where $x_i$ is the $i$-th iterate, $g(x, \xi_i)$ is a stochastic gradient defined below, $\xi_i$ is a random variable that determines the choice of a single or a subset of sampled functions $f_i$, and $\alpha$ represents the stepsize. With-and without-replacement sampling of the individual component functions are regarded as some of the most popular variants of SGD. During SGD with replacement sampling, the stochastic gradient is equal to $g(x, \xi_i) = \nabla f_{\xi_i}(x)$ and $\xi_i$ is a uniform number in $\{1, \ldots, n\}$, i.e., a with-replacement sample from the set of gradients $\nabla f_1, \ldots, \nabla f_n$. In the case of without-replacement sampling, the stochastic gradient is equal to $g(x, \xi_i) = \nabla f_{\xi_i}(x)$ and $\xi_i$ is the $i$-th ordered element in a random permutation of the numbers in $\{1, \ldots, n\}$, i.e., a without-replacement sample.

In practice, SGD without replacement is much more widely used compared to its with-replacement counterpart, as it can empirically converge significantly faster [1, 2, 3]. However, in the land of theoretical guarantees, with-replacement SGD has been the focal point of convergence analyses. The reason for this is that analyzing stochastic gradients born with replacement is significantly more tractable for a simple reason: in expectation, the stochastic gradient is equal to the “true” gradient of $F$, i.e., $E_{\xi} \nabla f_{\xi_i}(x) = \nabla F(x)$. This makes SGD amenable to analyses very similar to that of vanilla gradient descent (GD), which has been extensively studied under a large variety of function classes and geometric assumptions, e.g., see [4].

Unfortunately, the same cannot be said for SGD without replacement, which has long resisted non-vacuous convergence guarantees. For example, although we have long knew that SGD with replacement can achieve a $O \left( \frac{1}{T} \right)$ rate for strongly convex functions $F$, the best bounds we had for SGD without replacement did not even match that rate, in contrast to empirical evidence. However, a recent series of breakthrough results have established convergence guarantees for SGD without replacement establishing, similar or better rates than SGD with replacement.

In [5], the authors establish for the first time that for sums of quadratics or smooth functions, there exist parameter regimes under which SGDo achieves an $O(n^2/T^2)$ rate compared to the $O(1/T)$ rate
A detailed comparison of current and proposed bounds can be found in Table 1.

The recent flurry of work on without replacement sampling in stochastic optimization extends to several variants of stochastic algorithms beyond SGD. In [9, 10], the authors provide convergence rates for random cyclic coordinate descent, establishing for the first time that it can provably converge faster than stochastic coordinate descent with replacement sampling. This work is complemented by a lower bound of with-replacement sampling SGD. In this case, if $n$ is considered a constant, then SGD becomes $T$ times faster than SGD with replacement. More recently, [6] showed that for functions that are sums of quadratics, or smooth functions under a Hessian smoothness assumption, one could obtain an even faster rate of $O\left(\frac{1}{T^2} + \frac{n^2}{T^3}\right)$. [7] show that for Lipschitz convex functions SGD is at least as fast as SGD with replacement, and for functions that are strongly convex and sum of $n$ smooth components one can achieve a rate of $O\left(\frac{n}{T^2}\right)$. This latter result was the first convergence rate that provably establishes the superiority of SGD without replacement even for the regime that $n$ is not a constant, as long as $T$ grows faster than $n$.

This new wave of upper bounds has also been followed by new lower bounds. [5] establishes that there exist sums of quadratics on which SGD cannot converge faster than $\Omega\left(\frac{1}{T^2} + \frac{n^2}{T^3}\right)$. This lower bound gave rise to a gap between achievable rates and information theoretic impossibility. On one hand, there exists this aforementioned lower bound for sums of $n$ quadratics. On the other hand the best upper bound for sums of quadratics is $O\left(\frac{1}{T^2} + \frac{n^2}{T^3}\right)$ and for the more general class of strongly convex functions that are sums of smooth functions the best rate is $O\left(\frac{n}{T^2}\right)$. This leaves open the question of whether the upper or lower bounds are loose. This is precisely the gap we close in this work.

Our Contributions: In this work, we establish tight bounds for SGD. We close the gap between lower and upper bounds on two of the function classes that prior works have focused on: strongly convex functions that are i) sums of quadratics and ii) sums of smooth functions. Specifically, for i), we offer tighter convergence rates, i.e., an upper bound that matches the lower bound of [5]; as a matter of fact our convergence rates apply to general quadratic functions that are strongly convex, which is a little more general of a function class. For ii), we provide a new lower bound that matches the upper bound by [7].

A detailed comparison of current and proposed bounds can be found in Table 1.

A few words on the techniques used are in place. For our convergence rate on quadratic functions, we heavily rely on and combine the approaches of [7] and [6]. The convergence rate analyses proposed by [6] can be tightened by a more careful analysis that employs iterate coupling similar to [7], combined with new bounds on the deviation of the stochastic, without-replacement gradient from the true gradient of $F$.

For our lower bound, we use a similar construction to the one in [5], with the difference that each of the individual function components is not a quadratic function, but rather a piece-wise quadratic. This particular function has the property we need: it is smooth, but not quadratic. By appropriately scaling the sharpness of the individual quadratics we construct a function that behaves in a way that SGD without replacement cannot converge faster than a rate of $n/T^2$, no matter what stepsize one chooses.

We note that although our methods have an optimal dependence on $n$ and $T$, we believe that the dependence on function parameters, e.g., strong convexity, Lipschitz, and smoothness, can potentially be improved.

| $F$ is strongly convex and a sum of $n$ quadratics | $F$ is strongly convex and a sum of $n$ smooth functions |
|--------------------------------------------------|--------------------------------------------------|
| Lower bound for quadratics [5] | $\Omega\left(\frac{1}{T^2} + \frac{n^2}{T^3}\right)$ |
| Upper bound in [6] | $O\left(\frac{1}{T^2} + \frac{n^2}{T^3}\right)$ |
| Our upper bound | $O\left(\frac{1}{T^2} + \frac{n^2}{T^3}\right)$ |
| Lower bound for quadratics [5] | $\Omega\left(\frac{1}{T^2} + \frac{n^2}{T^3}\right)$ |
| Upper bound in [7] | $O\left(\frac{n}{T^2}\right)$ |
| Our lower bound | $\Omega\left(\frac{n}{T^2}\right)$ |

Table 1: Comparison of our lower and upper bounds to current state-of-the-art results. With magenta we denote the state-of-the-art bounds we are comparing against, and with cyan the new bounds. Our matching bounds establish information theoretically optimal rates for SGD. NOTE: $\tilde{O}(\cdot)$ hides the logarithmic factors.
on the gap between the random and non-random permutation variant of coordinate descent [11]. Several other works have focused on the random permutation variant of coordinate descent, e.g., see [12, 13]. In [14], novel bounds are given for incremental Newton based methods. In [15], the authors present convergence bounds for with replacement sampling and distributed SGD. Finally, [16] present asymptotic bounds for SGD for strongly convex functions, and show that with a constant stepsize it approaches the global optimizer to within smaller error radius compared to SGD with replacement.

3 Preliminaries and Notation

We focus on using SGD to approximately find \( x^\ast \), the global minimizer of the following unconstrained minimization problem

\[
\min_{x \in \mathbb{R}^d} \left( F(x) := \frac{1}{n} \sum_i f_i(x) \right).
\]

In our convergence bounds, we denote by \( T \) the total number of iterations of SGD, and by \( K \) the total number of epochs, i.e., passes over the data. Hence, \( T = nK \). In our derivations, we denote by \( x_{j}^{i} \) the \( i \)-th iterate of the \( j \)-th epoch. Consequentlly, we have that \( x_{j}^{i+1} \equiv x_{j}^{i} \).

Our results in the following sections rely on the following assumptions.

Assumption 1. (Convexity of Components) \( f_i \) is convex for all \( i \in [n] \).

Assumption 2. (Strong Convexity) \( F \) is strongly convex with strong convexity parameter \( \mu \), that is \( \forall x, y : F(y) \geq F(x) + \langle \nabla F(x), y - x \rangle + \frac{\mu}{2} \| y - x \|^2 \)

Assumption 3. (Bounded Domain) \( \forall x : \| x - x^\ast \| \leq D \)

Assumption 4. (Bounded Gradients) \( \forall i, x : \| \nabla f_i(x) \| \leq G \)

Assumption 5. (Lipschitz Gradients) \( \forall i, x, y : \| \nabla f_i(x) - \nabla f_i(y) \| \leq L \| x - y \| \)

4 Optimal SGD rates for quadratics

In this section, we will focus on strongly convex functions that are quadratic. We will provide a tight convergence rate that improves upon the \( O \left( \frac{1}{T^2} + \frac{n^3}{T^3} \right) \) rate by [6] and matches the \( \Omega \left( \frac{1}{T^2} + \frac{n^2}{T^3} \right) \) lower bound of [5] up to logarithmic factors.

For strongly convex functions that are a sum of smooth functions, [7] offer a rate of \( O \left( \frac{n}{T^2} \right) \), whereas for strongly convex quadratics [6] give a convergence rate of \( O \left( \frac{n}{T^2} + \frac{n^3}{T^3} \right) \). A closer comparison of these two rates reveals that neither of them can be tight due to the following observation. Assume that \( n \ll K \).

Then, that implies that

\[
\left( \frac{1}{T^2} + \frac{n^3}{T^3} \right) < \frac{n}{T^2}.
\]

At the same time if we assume that the number of data points is significantly larger than the number of epochs that we run SGD for, i.e., \( n \gg K \) we have that

\[
\left( \frac{1}{T^2} + \frac{n^3}{T^3} \right) > \frac{n}{T^2}.
\]

In comparison, the known lower bound for quadratics given by [5] is \( \Omega \left( \frac{n}{T^2} + \frac{n^2}{T^3} \right) \). This makes one wonder what is the true convergence rate of SGD in this case. We settle the optimal rates for quadratics here by providing an upper bound which matches the known lower bound (up to logarithmic factors).

For the special case of one dimensional quadratics, [5] proved an upper bound matching the one we prove in this paper. Further, the paper conjectures that the proof can be extended to the generic multidimensional case. However, the authors claim that the main technical barrier for this extension is that it requires a special case of a matrix-valued arithmetic-geometric mean inequality which has only been conjectured to be true but not yet proven. The authors further conjecture that their proof can be extended to general smooth and strongly convex functions, which turns out to not be true, as we show in Theorem [5]. On the other hand, we believe that our proof can be extended to the more general family of
strongly convex functions, where the Hessian is Lipschitz, similar to the way [6] extend their proof to that case.

In addition to assumptions 1-5 above, here we also assume the following:

**Assumption 6.** \( F \) is a quadratic function

\[
F(x) = \frac{1}{2} x^T H x + b^T x + c
\]

where \( H \) is a positive semi-definite matrix.

Note that this assumption is a little more general than the assumption that \( F \) is a sum of quadratics. Also, note that this assumption, in combination with the assumptions on strong convexity and Lipschitz gradients implies bounds on the minimum and maximum eigenvalues of the Hessian of \( F \), that is,

\[
\mu I \preceq H \preceq LI,
\]

where \( I \) is the identity matrix and \( A \preceq B \) means that \( x^T (A - B) x \leq 0 \) for all \( x \).

**Theorem 1.** Under Assumptions 1-6, let the step size of SGDo be

\[
\alpha = \frac{8 \log T}{T \mu}
\]

and the total number of epochs we run it for be

\[
K \geq 128 \frac{L^2}{\mu^2} \log T.
\]

Then, after \( T \) iterations SGDo achieves the following rate

\[
\mathbb{E}[\|x_T - x^*\|^2] \leq \tilde{O} \left( \frac{1}{T^2} + \frac{n^2}{T^3} \right),
\]

where \( \tilde{O} \) hides logarithmic factors.

We note at this point that in our bounds we hide factors relating to problem parameters, like \( L, \mu, D \), etc, as we believe that a possibly better dependence is possible.

The proof for Theorem 1 uses ideas from [6] and [7]. In particular, one of the central ideas in these two papers is that they aim to quantify the amount of progress made by SGDo over a single epoch. Both analyses decompose the progress of the iterates in an epoch as \( n \) steps of full gradient descent plus some noise term.

Similar to [6], we use the fact that the Hessian \( H \) of \( F \) is constant, which helps us better estimate the value of gradients around the minimizer. In contrast to that work, we do not require all individual components \( f_i \) to be quadratic, but rather the entire \( F \) to be a quadratic function.

An important result proved by [7] is that during an epoch, the iterates do not steer off too far away from the starting point of the epoch. This allows one to obtain a reasonably good bound on the noise term, when one tries to approximate the stochastic gradient with the true gradient of \( F \). In our analysis, we prove a slightly different version of the same result using an iterate coupling argument similar to the one in [7].

The analysis of [7] relies on computing the Wasserstein distance between the unconditional distribution of iterates and the distribution of iterates given a function sampled during an iteration. In our analysis, we use the same coupling, but we bypass the Wasserstein framework that [7] suggests and directly obtain a bound on how far the coupled iterates move away from each other during the course of an epoch. This results, in our view, to a somewhat simpler and shorter proof.

### 4.1 Sketch of proof for Theorem 1

Now we give an overview of the proof. As mentioned before, similar to the previous works, the key idea is to perform a tight analysis of the progress made during an epoch. This is captured by the following Lemma.
Lemma 1. Let the step size be
\[ \alpha = \frac{8 \log T}{T \mu} \]
and the total number of epochs we run SGD be
\[ K \geq 128 \frac{L^2}{\mu^2} \log T. \]

Then there exist universal constants \( C_1 \) and \( C_2 \) such that for any epoch \( j \) we have
\[ \mathbb{E} \left[ \|x_n^j - x^*\|^2 \right] \leq \left( 1 - n \alpha \frac{L \mu}{2(L + \mu)} \right) \mathbb{E} \left[ \|x_0^j - x^*\|^2 \right] + n \alpha^3 C_1 + \alpha^4 n^3 C_2. \]

At this point, we would like to remark that the bound on epochs \( K \geq 128 \frac{L^2}{\mu^2} \log T \) may be a bit surprising as \( K \) and \( T \) are dependent. However, note that since \( T = nK \), we can show that the bound on \( K \) above is satisfied if we set the number of epochs to be greater than \( C \log n \) for some constant \( C \).

Furthermore, we note that the dependence of \( K \) on \( \frac{L}{\mu} \) (i.e., the condition number of \( F \)) is most probably not optimal. In particular both \([7]\) and \([8]\) have a better dependence on the condition number.

Given the result in Lemma 1, proving Theorem 1 is a simple exercise. To do so, we can simply unroll the recursion (1) across \( K \) consecutive epochs:
\[ \mathbb{E} \left[ \|x_n^K - x^*\|^2 \right] \leq \left( 1 - n \alpha \frac{L \mu}{2(L + \mu)} \right)^K \mathbb{E} \left[ \|x_0^K - x^*\|^2 \right] + \alpha^3 n^3 C_2 \sum_{j=1}^{K} \left( 1 - n \alpha \frac{L \mu}{2(L + \mu)} \right)^j. \]

We can now use the fact that \( (1 - x) \leq e^{-x} \) and consequently
\[ \left( 1 - n \alpha \frac{L \mu}{2(L + \mu)} \right) \leq 1. \]
This leads to the following bound
\[ \mathbb{E} \left[ \|x_n^K - x^*\|^2 \right] \leq e^{-n \alpha \frac{L \mu}{2(L + \mu)}} K \mathbb{E} \left[ \|x_0^K - x^*\|^2 \right] + (n \alpha^3 C_1 + \alpha^4 n^3 C_2) K. \]

We further know that for \( L \)-smooth and \( \mu \)-strongly convex functions, the strong convexity parameter cannot be larger than the smoothness parameter, that is \( L \geq \mu \). This simple inequality further implies that
\[ \frac{L}{L + \mu} \geq \frac{1}{2}. \]

By setting the stepsize to be \( \alpha = \frac{8 \log T}{T \mu} \) and noting that \( T = nK \), and \( \|x_n^0 - x^*\|^2 \leq D^2 \) we have the following bound
\[ \mathbb{E} \left[ \|x_n^K - x^*\|^2 \right] \leq e^{-n \alpha \frac{L \mu}{2(L + \mu)}} K \mathbb{E} \left[ \|x_n^K - x^*\|^2 \right] + (n \alpha^3 C_1 + \alpha^4 n^3 C_2) K \]
\[ \leq e^{-2 \log T} \mathbb{E} \left[ \|x_n^K - x^*\|^2 \right] + O \left( \frac{1}{T^2} + \frac{n^2}{T^3} \right) \]
\[ \leq \frac{D^2}{T^2} + O \left( \frac{1}{T^2} + \frac{n^2}{T^3} \right). \]
This establishes Theorem 1.
4.2 With- and without-replacement stochastic gradients are close

One of the key lemmas in [7] establishes that once SGDo iterates get closer to the global minimizer \( x^* \), then any iterate at any time during an epoch \( x_j^t \) stays close to the iterate at the beginning of that epoch. To be more precise, the lemma we refer to is the following.

**Lemma 2.** *(adapted from Lemma 5 in [7])*

\[
\mathbb{E}[\|x_j^t - x_0^t\|^2] \leq 5\alpha^2 G^2 + 2\alpha(F(x_0^t) - F(x^*))
\]

We would like to note that Lemma 2 is slightly different from the one in [7], which instead uses \( \mathbb{E}[F(x_j^t) - F(x^*)] \) rather than \( (F(x_j^t) - F(x^*)) \), but their proof can be adapted to obtain this version above.

Now, consider the case when the iterates are very close to the optimum and hence \( F(x_j^t) - F(x^*) \approx 0 \). Then, Lemma 2 implies that \( \mathbb{E}[\|x_j^t - x_0^t\|^2] \) does not grow quadratically in \( i \) which would generically happen for \( i \) gradient steps, but it rather grows linearly in \( i \). This is an important and useful fact for SGDo: it shows that all iterates within an epoch remain close to \( x_0^t \).

Hence, since the iterates of SGDo don’t move too much during an epoch, then the gradients computed throughout the epoch at points \( x_j^t \) should be well approximated by gradients computed on the \( x_0^t \) iterate. Roughly, this translates to the following observation: the \( n \) gradient steps taken through a single epoch are almost equal to \( n \) steps of full gradient descent computed at \( x_0^t \). This is in essence what allows SGDo to achieve better convergence than SGD - an epoch can more often than not be approximated by \( n \) steps of gradient descent, rather than by a single step.

Now, let \( \sigma^t \) represent the random permutation of the \( n \) functions \( f_i \) during the \( j \)-th epoch. Thus, \( \sigma^t(i) \) is the index of the function chosen at the \( i \)-th iteration of the \( j \)-th epoch. Proving Lemma 2 requires proving that the function value of \( f_{\sigma^t(i)}(x_j^t) \), in expectation, is almost equal to \( F(x_j^t) \). In particular, we prove the following lemma in our supplemental material.

**Lemma 3.** For any epoch \( j \) and \( i \)-th ordered iterate during that epoch, we have that

\[
\|\mathbb{E}[F(x_j^t)] - \mathbb{E}[f_{\sigma^t(i)}(x_j^t)]\| \leq 2\alpha G^2. \tag{2}
\]

This lemma establishes that SGDo behaves almost like SGD with replacement, for which the following is true

\[ \mathbb{E}[f_{\sigma^t(i)}(x_j^t)] = \mathbb{E}[F(x_j^t)]. \]

To prove this lemma, [7] consider the conditional distribution of iterates, given the current function index, that is \( x_j^t|\sigma(t(j)) \), and the unconditional distribution of the iterates \( x_j^t \). Then, they prove that the absolute difference \( \|\mathbb{E}[F(x_j^t)] - \mathbb{E}[f_{\sigma^t(i)}(x_j^t)]\| \) can be upper bounded by the Wasserstein distance between these two distributions. To further upper bound the Wasserstein distance, they propose a coupling between the two distributions. To prove our slightly different version of Lemma 2, we proved (2) without using this Wasserstein framework. Instead, we use the same coupling argument to directly get a bound on (2).

Below we explain the coupling and provide a short intuition.

Consider the conditional distribution of \( \sigma^t|\sigma(i) = s \). If we take the distribution of \( \sigma|\sigma(i) = 1 \), we can generate the support of \( \sigma^t|\sigma(i) = s \) by taking all permutations \( \sigma|\sigma(i) = 1 \) and by swapping 1 and \( s \) among them. This is essentially a coupling between these two distributions, proposed in [7]. Now, if we use this coupling to convert a permutation in \( \sigma|\sigma(i) = 1 \) to a permutation \( \sigma|\sigma(i) = s \), the corresponding \( x_i|\sigma(i) = 1 \) and \( x_i|\sigma(i) = s \) would be within a distance of \( 2\alpha G \). This distance bound is Lemma 2 from [7], and for completeness, the proof of this is provided inside the detailed proof of Lemma 3 in the appendix.

We can now use such distance bound, and let \( v \) denote a (random) vector whose norm is less than...
2αG. Then,

\[
\mathbb{E} \left[ f_{\sigma(i)} (x_i) \right] = \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ f_{\sigma(i)} (x_i) | \sigma(i) = s \right]
\]

\[
= \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ f_s (x_i) | \sigma(i) = s \right]
\]

\[
= \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ f_s (x_i + v) | \sigma(i) = 1 \right]
\]

\[
\leq \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ f_s (x_i) + (2\alpha G^2) | \sigma(i) = 1 \right]
\]

\[
= \mathbb{E} \left[ f(x_i) | \sigma(i) = 1 \right] + 2\alpha G^2.
\]

Similarly, for any \( s \in \{1, \ldots, n\} \):

\[
\mathbb{E} \left[ f_{\sigma(i)} (x_i) \right] \leq \mathbb{E} [F (x_i) | \sigma(i) = s] + 2\alpha G^2.
\]

Therefore,

\[
\mathbb{E} \left[ f_{\sigma(i)} (x_i) \right] \leq \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} [F (x_i) | \sigma(i) = s] + 2\alpha G^2
\]

\[
\leq \mathbb{E} [F (x_i)] + 2\alpha G^2.
\]

Similarly, we can prove that

\[
\mathbb{E} \left[ f_{\sigma(i)} (x_i) \right] \geq \mathbb{E} [F (x_i)] - 2\alpha G^2.
\]

Combining these two results we obtain (2).

The full proof of Theorem 1 requires some more nuanced bounding derivations, and the complete details can be found in the Appendix.

5 Lower bound for general case

In the previous section, we establish that for quadratic functions the \( \Omega \left( \frac{1}{T^2} + \frac{n^2}{T^2} \right) \) lower-bound by [5] is essentially tight. This still leaves open the possibility that a tighter lower bound may exist for strongly convex functions that are not quadratic. After all, the best convergence rate known for strongly convex functions that are sums of smooth functions is of the order of \( \frac{n}{T^2} \).

Indeed, in this section, we show that the convergence rate of \( O \left( \frac{n^2}{T^2} \right) \) established by [7] is tight. In particular, we have the following for different step sizes. For steplength \( \alpha \leq \frac{1}{2T} \), [5] shows that \( \mathbb{E}[\|x_T - x^*\|^2] = \Omega(1) \). For any constant \( C \), a steplength of \( \alpha \geq \frac{C}{n} \), the theorem of [5] can be adapted directly to get \( \mathbb{E}[\|x_T - x^*\|^2] = \Omega \left( \frac{1}{n} \right) \). Finally, for a certain constant \( C \) (see the proof of Theorem 2 in the Appendix for the value of \( C \)), we show the following theorem

**Theorem 2.** For \( \frac{1}{2T} \leq \alpha \leq \frac{C}{n} \), there exists a strongly convex function that is sum of \( n \) smooth convex functions such that

\[
\mathbb{E}[\|x_T - x^*\|^2] = \Omega \left( \frac{n}{T^2} \right)
\]

Thus overall, we get that for any fixed step size \( \mathbb{E}[\|x_T - x^*\|^2] = \Omega \left( \frac{n}{T^2} \right) \).

Next, we try to explain the function construction and proof technique behind Theorem 2. The construction of the lower bound is similar to that in [5]. The difference is that the prior work considers quadratic functions, while we consider a slightly modified “piece-wise” quadratic function.

We specifically construct the following function \( F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \):

\[
F(x) = \begin{cases} 
\frac{\lambda x^2}{2}, & \text{if } x \geq 0 \\
\frac{\lambda R x^2}{2}, & \text{if } x < 0.
\end{cases}
\]
Now, of the \( n \) component functions \( f_i \), \( n/2 \) are of the 1st kind

\[
f_i(x) = \begin{cases} \frac{\lambda x^2}{2} + \frac{Gx}{2}, & \text{if } x \geq 0, \ i \leq n/2 \\ \frac{\lambda R x^2}{2} + \frac{Gx}{2}, & \text{if } x < 0, \ i \leq n/2 \end{cases}
\]

and \( n/2 \) functions are of the 2nd kind

\[
f_i(x) = \begin{cases} \frac{\lambda x^2}{2} - \frac{Gx}{2}, & \text{if } x \geq 0, \ i > n/2 \\ \frac{\lambda R x^2}{2} - \frac{Gx}{2}, & \text{if } x < 0, \ i > n/2. \end{cases}
\]

For our construction, we let set \( R \) to be a big enough positive constant. See for example, Fig. 1.

![Figure 1: Lower bound construction. Note that \( f_1(x) \) represents functions of the first kind, and \( f_2(x) \) represents functions of the second kind, and \( F(x) \) represents the overall function.](image)

Next we ought to verify that this function abides to Assumptions 1-5.

Note that Assumption 1 is satisfied, as it can be seen that functions \( f_i \)'s are all continuous and convex.

Next, we need to show that Assumption 2 holds, that is \( F \) is strongly convex. We will show that this is true by proving

\[
\forall x, y : (\nabla F(x) - \nabla F(y))(x - y) \geq (x - y)^2
\]

It is easy to see that this holds in the case \( xy > 0 \). Therefore, wlog let us assume \( x \geq 0 \) and \( y \leq 0 \). Then,

\[
(\nabla F(x) - \nabla F(y))(x - y) = (x - R y)(x - y) \geq (x - y)(x - y) = (x - y)^2
\]

In the proof of Theorem 2, we initialize at the origin. In that case, in the proof we also prove that Assumptions 3 and 4 hold. In particular, we show that the iterates do not go outside of a bounded domain, and inside this domain, the gradient is bounded by \( G \). Finally, let us focus on Assumption 5. To prove that these functions have Lipschitz gradients, we need to show

\[
\forall x, y : |\nabla f_i(x) - \nabla f_i(y)| \leq R\lambda|x - y|.
\]

If \( xy \geq 0 \), that is \( x \) and \( y \) lie on the same side of the origin, then this is simple to see because they both lie on the same quadratic. Otherwise, assume \( x < 0 \) and \( y > 0 \). Also, assume that \( f_i \) is function of the
first kind, that is the linear term in $f_i(x)$ is $\frac{G \lambda x}{2}$. Then,

$$|\nabla f(x) - \nabla f(y)| = \left| R\lambda x + \frac{G}{2} - \lambda y - \frac{G}{2} \right|$$

$$= \lambda y - R\lambda x$$

$$\leq R\lambda y - R\lambda x$$

$$\leq R\lambda |y - x|.$$

Overall, the difficulty in the analysis comes from the fact that unlike the functions considered in [5], our functions are piecewise quadratics.

Let us initialize at $x_0 = 0$ (the minimizer). We will show that in expectation, at the end of $K$ epochs, the iterate would be at a certain distance (in expectation). Note that the progress made over an epoch is just the sum of gradients (multiplied by $-\alpha$) over the epoch:

$$x_n^j - x_0^j = -\alpha \sum_{i=1}^{n} \nabla f_{\sigma(i)}(x_i)$$

where $\sigma(i)$ represents the index of the $i$-th function chosen in the $j$-th epoch. Next, note that the gradients from the linear components $\pm \frac{G}{2} x$ are equal to $\pm \frac{G}{2}$, that is they are constant. Thus, they will cancel out over an epoch.

However the gradients from the quadratic components do not cancel out, and in fact that part of the gradient will not even be unbiased, in the sense that if $x_t \geq 0$, the gradient at $x_t$ from the quadratic component $\frac{\lambda x^2}{2}$ will be less in magnitude than the gradient from the quadratic component $\frac{R\lambda x^2}{2}$ at $-x_t$.

The idea is to now ensure that if an epoch starts off near the minimizer, then the iterates spend a certain amount of time in $x < 0$ region, so that they ‘accumulate’ a lot of gradients of the form $R\lambda x$, which makes the sum of the gradients at the end of the epoch biased away from the minimizer.

To ensure that the iterates spend some time in the $x < 0$ region, we need to analyze the contribution of the linear components during the epoch. This is because when the iterates are already near the minimizer $x \approx 0$, the gradient contribution of the quadratic terms would be small, and the dominating component would come from the linear terms. What this means is that during the epoch, it is the linear terms which contribute the most towards the ‘iterate movement’.

Then, to obtain a lower bound matching the upper bound of [7], observe that it is indeed this contribution of the linear terms that we require to get a tight bound on. This is because, the upper bound from [7] was in fact directly dependent on the “noisy” movement during such an epoch (Lemma 5 from [7]). We give below the informal version of the main lemma for the proof:

**Lemma 4.** Let $(\sigma_1, \ldots, \sigma_n)$ be a random permutation of $(+1, \ldots, \frac{n}{2} \text{ times}, \ldots, +1, -1, \ldots, \frac{n}{2} \text{ times}, \ldots, -1)$. Then for $i < n/2$ we have that

$$E \left[ \sum_{j=1}^{i} \sigma_j \right] \geq C \sqrt{i}.$$

For the purpose of intuition, ignore the contribution of gradients from the quadratic terms. Then, the lemma above says that during an epoch, the gradients from the linear terms would move the iterates approximately $\Omega(\alpha \sqrt{n/2})$ away from the minimizer (after we multiply by the stepsize $\alpha$).

This implies that in the middle of an epoch, with (almost) probability 1/2 the iterates would be near $x \approx -\Omega(\alpha \sqrt{n/2})$ and with (almost) probability 1/2 the iterates would be near $x \approx \Omega(\alpha \sqrt{n/2})$. Hence, over the epoch, the accumulated quadratic gradients multiplied by the stepsize would look like

$$\sum_{i=1}^{n} \frac{1}{2} \left( -\alpha R\lambda \Omega \left( -\alpha \sqrt{n} \frac{G}{2} \right) \right) + \frac{1}{2} \left( -\alpha \lambda \Omega \left( \alpha \sqrt{n} \frac{G}{2} \right) \right)$$

$$= \Omega(R\lambda \alpha^2 n \sqrt{n}).$$

If this happens for $K$ epochs, we get that the accumulated steps would be $\Omega(R\lambda \alpha^2 n \sqrt{n} K) = \Omega \left( \frac{1}{\sqrt{\alpha K}} \right)$ for $\alpha \in \left[ \frac{1}{2}, \frac{1}{n} \right]$. Since $E[\|x_T\|] \geq \frac{1}{\sqrt{n} K}$, we know that $E[\|x_T - 0\|^2] \geq \frac{1}{n} K^2 = n/\alpha^2$. Since 0 is the minimizer of our function in this setting, we have constructed a case where SGD0 achieves error

$$E[\|x_T - x^*\|^2] \geq n/T^2.$$

This completes the sketch of the proof and the complete proof of Theorem 2 is given in the Appendix.
5.1 Discussion on possible improvements

Theorem 2 hints that for faster convergence rates in the epoch based random shuffling SGD, we would not just require smooth and strongly convex functions, but also potentially require that the Hessians of such functions to be Lipschitz.

We conjecture that Hessian Lipschitzness is sufficient to get the convergence rate of Theorem 4. We think that this is interesting, because the optimal rates for both SGD with replacement and vanilla gradient descent only require strong convexity (and gradient smoothness for SGD). However, here we prove that an optimal rate for SGDo requires the function to be quadratic as well (or at the very least Hessian Lipschitz), and SGDo seems to converge slower if the Hessian is not Lipschitz.

6 Conclusions and Future Work

SGD without replacement has long puzzled researchers. From a practical point of view, it always seems to outperform SGD with replacement, and is the algorithm of choice for training modern machine learning models. From a theoretical point of view, SGDo has resisted tight convergence analysis that establish its performance benefits. A recent wave of work established that indeed SGDo can be faster than SGD with replacement sampling, however a gap still remained between the achievable rates and the best known lower bounds.

In this paper we settle the optimal performance of SGD without replacement for functions that are quadratics, and strongly convex functions that are sums of $n$ smooth functions. Our results indicate that a possible improvement in convergence rates may require a fundamentally different stepsize rule and significantly different function assumptions.

As future directions, we believe that it would be interesting to establish rates for variants of SGDo that do not re-permute the functions at every epoch. This is something that is common in practice, where a random permutation is only performed once every few epochs without a significant drop in performance. Current theoretical bounds are insufficient to explain this phenomenon, and a new theoretical breakthrough may be required to tackle it.

We however believe that one of the strongest new theoretical insights introduced by [7] and used in our analyses can be of significance in a potential attempt to analyze other variants of SGDo as the one above. This insight is that of iterate coupling. That is the property that SGDo iterates are only mildly perturbed after swapping only two elements of a permutation. Such a property is reminiscent to that of algorithmic stability, and a deeper connection between that and iterate coupling is left as a meaningful intellectual endeavor for future work.

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A Proof of Theorem 1

Remark: The proofs of all the theorems, lemmas, claims and corollaries are provided in this Appendix (except the ones cited from other papers).

Proof. The proof for upper bound uses the framework of [6], combined with some crucial ideas from [7]. In the block diagram below we connect the pieces needed to establish the proof. All lemmas and proofs follow.

Figure 2: A dependency graph for the proof of Theorem 1 giving short descriptions of the components required.

First, we use the following lemma which quantifies the expected progress made per epoch

**Lemma 1.** Let the step size be

\[ \alpha = \frac{8 \log T}{T \mu} \]

and the total number of epochs we run SGD be

\[ K \geq 128 \frac{L^2}{\mu^2} \log T. \]

Then there exist universal constants \( C_1 \) and \( C_2 \) such that for any epoch \( j \) we have

\[
\mathbb{E} \left[ \| x_j^0 - x^* \|^2 \right] \leq \left( 1 - \frac{\mu}{2L + \mu} \right) \| x_{j-1}^0 - x^* \|^2 + \frac{L \mu}{2(L + \mu)} \| x_{j-1}^0 - x^* \|^2 + \alpha^3 C_1 + \alpha^3 n^3 C_2
\]

(1)
Simply unrolling this equation for $K$ epochs, we get:

$$
E[\|x_T - x^*\|] = E[\|x_n^K - x^*\|] \\
\leq \left(1 - n\alpha \frac{L\mu}{2(L + \mu)}\right)^K \|x_1^0 - x^*\|^2 + \sum_{j=1}^K \left(1 - n\alpha \frac{L\mu}{2(L + \mu)}\right)^j (n\alpha^3C_1 + \alpha^4n^3C_2) \\
\leq \left(1 - n\alpha \frac{L\mu}{2(L + \mu)}\right)^K D_2^2 + \sum_{j=1}^\infty \left(1 - n\alpha \frac{L\mu}{2(L + \mu)}\right)^j (n\alpha^3C_1 + \alpha^4n^3C_2) \\
= \left(1 - n\alpha \frac{L\mu}{2(L + \mu)}\right)^K D_2^2 + \frac{2(L + \mu)}{n\alpha L\mu} (n\alpha^3C_1 + \alpha^4n^3C_2) \\
(\text{Using the formula for sum of Geometric Progression}) \\
\leq e^{-nK\alpha \frac{L\mu}{2(L + \mu)}} D_2^2 + \frac{2(L + \mu)}{L\mu} (\alpha^2C_1 + \alpha^3n^2C_2) \\
(\text{Since } (1 - x) \leq e^{-x}) \\
\leq e^{-4\log T \frac{\mu}{T\mu}} D_2^2 + \frac{2(L + \mu)}{L\mu} \left( \frac{\log^2 T}{T^2} C_1 + \frac{\log^3 T}{T^3} n^2 C_2 \right) \\
(\text{Substituting } \alpha = 8 \log T/T\mu) \\
\leq \frac{D_2^2}{T^2} + \frac{2(L + \mu)}{L\mu} \left( \frac{\log^2 T}{T^2} C_1 + \frac{\log^3 T}{T^3} n^2 C_2 \right)
$$

where in the last inequality we used the fact that $L \geq \mu$ and hence $\frac{L}{L + \mu} \geq \frac{1}{2}$. Therefore, we get that

$$
E[\|x_T - x^*\|^2] \leq \tilde{O} \left( \frac{1}{T^2} + \frac{n^2}{T^3} \right).
$$

\[\square\]

### A.1 Proof of Lemma 1

**Proof.** Throughout this proof we will be working inside an epoch, so we would be skipping the super script $j$ in $x^j$ which denotes the $j$-th epoch. Thus in this proof, $x_0$ refers to the iterate at beginning of that epoch. Let $\sigma$ denote the permutation of $[n]$ used in this epoch. Therefore at the $i$-th iteration of this epoch, we compute the gradient on $f_{\sigma(i)}$. Next we define the error term (this is the same error term as the one defined in [R])

$$
R := \sum_{i=1}^n (\nabla f_{\sigma(i)}(x_{i-1}) - \nabla F(x_0)) = \sum_{i=1}^n (\nabla f_{\sigma(i)}(x_{i-1}) - \nabla f_{\sigma(i)}(x_0)) .
$$

Then,

$$
\|x_n - x^*\|^2 = \|x_0 - x^*\|^2 - 2\alpha \left( x_0 - x^* \sum_{i=1}^n \nabla f_{\sigma(i)}(x_{i-1}) \right) + \frac{\alpha^2}{2} \left( \sum_{i=1}^n \nabla f_{\sigma(i)}(x_{i-1}) \right)^2 \\
(\text{Using the formula for gradient descent: } x_{i+1} = x_i - \alpha \nabla f_{\sigma(i)}(x_i)) \\
= \|x_0 - x^*\|^2 - 2\alpha \langle x_0 - x^*, \nabla F(x_0) \rangle - 2\alpha \langle x_0 - x^*, R \rangle + \alpha^2 \|n\nabla F(x_0) + R\|^2 \\
\leq \|x_0 - x^*\|^2 - 2\alpha \langle x_0 - x^*, \nabla F(x_0) \rangle + 2\alpha^2 n^2 \|\nabla F(x_0)\|^2 + 2\alpha^2 \|R\|^2 - 2\alpha \langle x_0 - x^*, R \rangle \\
(\text{Since } \|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2) \\
\leq \|x_0 - x^*\|^2 - 2\alpha \left( \frac{L\mu}{L + \mu} \|x_0 - x^*\|^2 + \frac{1}{L + \mu} \|\nabla F(x_0)\|^2 \right) \\
+ 2\alpha^2 n^2 \|\nabla F(x_0)\|^2 + 2\alpha^2 \|R\|^2 - 2\alpha \langle x_0 - x^*, R \rangle \\
(\text{Property of strong convexity (see Theorem 3 given below)}) \\
= \left(1 - 2\alpha \frac{L\mu}{L + \mu} \right) \|x_0 - x^*\|^2 - 2\alpha \left( \frac{1}{L + \mu} - \alpha n \right) \|\nabla F(x_0)\|^2 + 2\alpha^2 \|R\|^2 - 2\alpha \langle x_0 - x^*, R \rangle .
$$

(5)
Theorem 3. (Theorem 2.1.11 from [17]) For strongly convex function $F$,

$$\langle \nabla F(x) - \nabla F(y), x - y \rangle \geq \frac{\mu L}{\mu + L} \|x - y\|^2 + \frac{1}{\mu + L} \|\nabla F(x) - \nabla F(y)\|^2$$

Next, we need to control $\mathbb{E}[2\alpha^2 \|R\|^2]$ and $\mathbb{E}[-2\alpha \langle x_0 - x^*, R \rangle]$. To achieve that we introduce the following two lemmas.

**Lemma 5.**

$$\mathbb{E}[\|R\|^2] \leq L^2 n^2 \|x_0 - x^*\|^2 + 5n^3 \alpha^2 L^2 G^2.$$  \hfill (6)

**Lemma 6.**

$$\langle x_0 - x^*, \mathbb{E}[R] \rangle \geq -\frac{1}{2} \alpha n^2 \|\nabla F(x_0)\|^2 - \frac{1}{4} \left(n - 1\right) \mu + 4\alpha^2 n^3 \mu^{-1} L^4 \|x_0 - x^*\|^2$$

$$- 20\mu^{-1} \alpha L^4 G^2 n^4 - 16n^3 \alpha^2 L^2 \mu^{-1}.$$  \hfill (7)

Substituting these two inequalities in (6), we get that

$$\mathbb{E}[\|x_n - x^*\|^2] \leq \left(1 - 2n\alpha \frac{L\mu}{L + \mu}\right) \|x_0 - x^*\|^2 - 2n\alpha \left(\frac{1}{L + \mu} - \alpha\right) \|\nabla F(x_0)\|^2 + 2\alpha^2 (L^2 n^2 \|x_0 - x^*\|^2 + 5n^3 \alpha^2 L^2 G^2)$$

$$- 2\alpha \left(-\frac{1}{2} \alpha n^2 \|\nabla F(x_0)\|^2 - \frac{1}{4} \left(n - 1\right) \mu + 4\alpha^2 n^3 \mu^{-1} L^4 \|x_0 - x^*\|^2ight) - 20\mu^{-1} \alpha L^4 G^2 n^4 - 16n^3 \alpha^2 L^2 \mu^{-1})$$

$$\leq \left(1 - 2n\alpha \frac{L\mu}{L + \mu} + 2\alpha^2 L^2 n^2 + \frac{1}{2} \alpha \mu (n - 1) + 8\alpha^2 n^3 \mu^{-1} L^4 \alpha^3 \right) \|x_0 - x^*\|^2$$

$$- n\alpha \left(\frac{2}{L + \mu} - 3\alpha\right) \|\nabla F(x_0)\|^2 + 10n^3 \alpha^4 L^2 G^2 + 40\mu^{-1} \alpha^5 L^4 G^2 n^4 + 32n^3 \alpha^2 G^2 \mu^{-1}.$$  \hfill (8)

Since $L \geq \mu$, $K \geq 128 \frac{L^2}{\mu^2} \log T$ and $\alpha = \frac{8 \log T}{T \mu}$, we can see that

$$n\alpha \frac{L\mu}{L + \mu} - \frac{1}{2} \alpha \mu (n - 1) \geq 0,$$  \hfill (9)

$$n\alpha \frac{L\mu}{2(L + \mu)} - 2\alpha^2 L^2 n^2 - 8\mu^{-1} n^3 L^4 \alpha^3 \geq 0,$$  \hfill (10)

$$\frac{2}{L + \mu} - 3\alpha \geq 0,$$  \hfill (11)

$$10n^3 \alpha^4 L^2 G^2 - 40\mu^{-1} \alpha^5 L^4 G^2 n^4 \geq 0.$$  \hfill (12)

Finally, using (6), (10), (11) and (12) in (8), we get

$$\mathbb{E}[\|x_n - x^*\|^2] \leq \left(1 - n\alpha \frac{L\mu}{2(L + \mu)}\right) \|x_0 - x^*\|^2 + 20n^3 \alpha^4 L^2 G^2 + 32n^3 \alpha^2 G^2 L^2 \mu^{-1}.$$  

This completes the proof. The only thing left is to prove the following inequalities (a), (b), (c) and (d):

(a) : $n\alpha \frac{L\mu}{L + \mu} \geq (n - 1)\alpha \frac{L\mu}{L + \mu} \geq (n - 1)\alpha \frac{\mu}{L + \mu} \geq (n - 1)\alpha \frac{\mu}{1 + 1} \geq (n - 1)\alpha \frac{\mu}{2}$.

(b) : It can be shown that $2\alpha^2 L^2 n^2 \geq 8\mu^{-1} n^3 L^4 \alpha^3$ (See proof of (d) below). So, it is sufficient to show

$$n\alpha \frac{L\mu}{2(L + \mu)} \geq 4\alpha^2 L^2 n^2$$

which is equivalent to proving

$$\alpha \leq \frac{\mu}{8nL^2}.$$  

14
Simply substituting $K \geq 128 \frac{L^2}{\mu^2} \log T$ and $\alpha = \frac{8 \log T}{T \mu}$ proves this.

(c):

$$\alpha n = \frac{8 \log T}{K \mu} \leq \frac{\mu^3 \log T}{16 L^2 \mu \log T} = \frac{\mu}{16 L^2} \leq \frac{1}{16 L} \leq \frac{1}{8 (L + \mu)} \leq \frac{2}{3 (L + \mu)}.$$ (13)

(d):

$$\alpha = \frac{8 \log T}{T \mu} = \frac{8 \log T}{n K \mu} \leq \frac{8 \mu^2 \log T}{128 L^2 n \mu \log T} = \frac{\mu}{16 L^2 n} \leq \frac{\mu}{4 L^2 n} \Rightarrow 10 n^3 \alpha^4 L^2 G^2 \geq 40 \mu^{-1} \alpha^5 L^4 G^2 n^4.$$ (14)

A.2 Proof of Lemma 5

Proof. Throughout this proof we will be working inside an epoch, so we would be skipping the superscript $j$ in $x^j$ which denotes the $j$-th epoch. Thus in this proof, $x_0$ refers to the iterate at beginning of that epoch. Let $\sigma$ denote the permutation of $[n]$ used in this epoch. Therefore at the $i$-th iteration of this epoch, we compute the gradient on $f_{\sigma(i)}$. 

\[ E[\|R\|^2] = E \left[ \sum_{i=1}^{n} (\nabla f_{\sigma(i)}(x_{i-1}) - \nabla f_{\sigma(i)}(x_0)) \right]^2 \]
\[ \leq E \left[ \sum_{i=1}^{n} \|\nabla f_{\sigma(i)}(x_{i-1}) - \nabla f_{\sigma(i)}(x_0)\|^2 \right] \]  
(Triangle inequality)
\[ \leq L^2 E \left[ \sum_{i=1}^{n} \|x_{i-1} - x_0\|^2 \right] \]  
(Using gradient Lipschitzness)
\[ = L^2 E \sum_{i=1}^{n} \sum_{j=1}^{n} \|x_{i-1} - x_0\| \|x_{j-1} - x_0\| \]
\[ = L^2 \sum_{i=1}^{n} \sum_{j=1}^{n} E[\|x_{i-1} - x_0\| \|x_{j-1} - x_0\|] \]
\[ \leq L^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{E[\|x_{i-1} - x_0\|^2]} \sqrt{E[\|x_{j-1} - x_0\|^2]} \]  
(Cauchy-Schwartz inequality)
\[ \leq L^2 n^2 (2\alpha\|x_0 - x^*\|^2 + 5\alpha^2 G^2) \]  
(Using Lemma 7)
\[ \leq L^2 n^2 \|x_0 - x^*\|^2 + 5\alpha^2 G^2. \]  
(15)

In the last inequality, we used the fact that \( \alpha \leq 1/2nL \) (see (13)).

**Lemma 7.** (Lemma 5 from [7], but proved slightly differently)

\[ E[\|x_i^l - x_0^l\|^2] \leq E[\|x_i^l - x_0^l\|^2] \leq 5\alpha^2 G^2 + 2\alpha(F(x_0^l) - F(x^*)) \leq 5\alpha^2 G^2 + \alpha L\|x_0^l - x^*\|^2. \]

\[ \square \]

**A.3 Proof of Lemma 6**

**Proof.** Throughout this proof we will be working inside an epoch, so we would be skipping the super script in \( x_i \) which denotes the \( j \)-th epoch. Thus in this proof, \( x_0 \) refers to the iterate at beginning of that epoch. Let \( \sigma \) denote the permutation of \([n]\) used in this epoch. Therefore at the \( i \)-th iteration of this epoch, we compute the gradient on \( f_{\sigma(i)} \).

\[ R = \sum_{i=1}^{n} [\nabla f_{\sigma(i)}(x_{i-1}) - \nabla f_{\sigma(i)}(x_0)] \]
\[ = \sum_{i=1}^{n} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_{j-1}) \right) - \nabla f_{\sigma(i)}(x_0) \right] \]  
(Using the formula for gradient descent: \( x_{i+1} = x_i - \alpha \nabla f_{\sigma(i)}(x_i) \))
\[ = \sum_{i=1}^{n} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_{j-1}) \right) - \nabla f_{\sigma(i)}(x_0) \right] \]
\[ + \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) - \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) \]
\[ = \sum_{i=1}^{n} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) - \nabla f_{\sigma(i)}(x_0) \right] \]
\[ + \sum_{i=1}^{n} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_{j-1}) \right) - \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) \right] \]
\[ = A + B \]  
(16)
where

\[ A := \sum_{i=1}^{n} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) - \nabla f_{\sigma(i)}(x_0) \right], \]

\[ B := \sum_{i=1}^{n} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_{j-1}) \right) - \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) \right]. \]

Again, \( A \) and \( B \) are the same terms as the ones defined in [6]. The difference in our analysis and [6] is that we get tighter bounds on these terms.

In the following, we use \( u \) to denote a random vector with norm less than or equal to 1. Also, assume that \( H \) is the Hessian of \( F \).

**Claim 1.**

\[ \mathbb{E}[A] = (2n\alpha GL)u - \alpha \frac{n(n-1)}{2} H\nabla F(x_0). \quad (17) \]

**Claim 2.**

\[ \|\mathbb{E}[B]\|^2 \leq n^4 L^4 \alpha^2 \|x_0 - x^*\|^2 + 5n^5 L^4 \alpha^4 G^2. \quad (18) \]

Using (16) and Claim 1,

\[ \langle x_0 - x^*, \mathbb{E}[R] \rangle = \langle x_0 - x^*, \mathbb{E}[A] + \mathbb{E}[B] \rangle \\
= \langle x_0 - x^*, \mathbb{E}[A] \rangle + \langle x_0 - x^*, \mathbb{E}[B] \rangle \\
= -\alpha \frac{n(n-1)}{2} \langle x_0 - x^*, H\nabla F(x_0) \rangle + \langle (2n\alpha GL)u, x_0 - x^* \rangle + \langle x_0 - x^*, \mathbb{E}[B] \rangle \\
= -\alpha \frac{n(n-1)}{2} \|\nabla F(x_0)\|^2 + \langle (2n\alpha GL)u, x_0 - x^* \rangle + \langle x_0 - x^*, \mathbb{E}[B] \rangle. \quad (19) \]

For the middle term in (19), we use the bound

\[ \langle (2n\alpha GL)u, x_0 - x^* \rangle \geq -\|2n\alpha GL\| \|x_0 - x^*\| \geq - \left[ \lambda \frac{1}{2} \|x_0 - x^*\|^2 + \frac{1}{2\lambda} (2n\alpha GL)^2 \right]. \quad (20) \]

Similarly, for the last term in (19), we use the bound

\[ \langle x_0 - x^*, \mathbb{E}[B] \rangle \geq -\|x_0 - x^*\| \|\mathbb{E}[B]\| \geq - \left[ \lambda \frac{1}{2} \|x_0 - x^*\|^2 + \frac{1}{2\lambda} \|\mathbb{E}[B]\|^2 \right]. \quad (21) \]

Set \( \lambda = \frac{1}{4} \mu (n-1) \) in (20) and (21). Finally, substituting these into (19) and using Claim 2 results in the following

\[ \langle x_0 - x^*, \mathbb{E}[R] \rangle \geq - \frac{1}{2} \alpha n^2 \|\nabla F(x_0)\|^2 - \left( \frac{1}{4} (n-1) \mu + 4n^2 \alpha^2 L^4 \right) \|x_0 - x^*\|^2 \\
- 20\mu^{-1} \alpha^4 L^4 G^2 n^4 - 16n^2 G^2 L^2 \mu^{-1}. \]

\[ \square \]
A.3.1 Proof of Claim

Proof.

\[ \mathbb{E}[A] = \mathbb{E} \left[ \sum_{i=1}^{n} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) - \nabla f_{\sigma(i)}(x_0) \right] \right] \]

\[ = \sum_{i=1}^{n} \mathbb{E} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) - \nabla f_{\sigma(i)}(x_0) \right] \]

\[ \overset{(a)}{=} \sum_{i=1}^{n} \mathbb{E} \left[ \nabla F \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) + (2\alpha GL)u - \nabla F(x_0) \right] \]

\[ = (2n\alpha GL)u + \sum_{i=1}^{n} \mathbb{E} \left[ \nabla F \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) - \nabla F(x_0) \right] \]

\[ = (2n\alpha GL)u - \alpha \sum_{i=1}^{n} \mathbb{E} \left[ H \left( \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) \right] \quad \text{(For quadratics, } \nabla f(x) - \nabla f(y) = H(x - y) \text{)} \]

\[ = (2n\alpha GL)u - \alpha H \sum_{i=1}^{n} \left( \sum_{j=1}^{i-1} \mathbb{E} \left[ \nabla f_{\sigma(j)}(x_0) \right] \right) \]

\[ = (2n\alpha GL)u - \alpha \frac{n(n-1)}{2} H \nabla F(x_0) \]

where we used the following claim for (a):

Claim 3.

\[ \mathbb{E} \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) \right] = \mathbb{E} \left[ \nabla F \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)}(x_0) \right) \right] + (2\alpha GL)u \]

\[ \square \]
A.3.2 Proof of Claim 2

Proof.

\[ \|E[B]\|^2 \leq (E[|B|])^2 \] (Jensen’s inequality)

\[ \leq \left( E \left[ \sum_{i=1}^{n} \| \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)} (x_{j-1}) \right) \right] - \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)} (x_0) \right) \right)^2 \] (Triangle inequality and definition of B)

\[ \leq L^2 \alpha^2 \left( \sum_{i=1}^{n} \sum_{j=1}^{i-1} \| \nabla f_{\sigma(j)} (x_{j-1}) - \nabla f_{\sigma(j)} (x_0) \| \right)^2 \] (Gradient Lipschitzness)

\[ \leq L^2 \alpha^2 \left( E \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \| \nabla f_{\sigma(j)} (x_{j-1}) - \nabla f_{\sigma(j)} (x_0) \| \right)^2 \right) \] (The inner summation doesn’t depend on i)

\[ \leq n^2 L^4 \alpha^2 \left( E \left[ \sum_{j=1}^{n} \| x_{j-1} - x_0 \| \right)^2 \right) \] (Gradient Lipschitzness)

\[ \leq n^2 L^4 \alpha^2 E \left[ \sum_{j=1}^{n} \| x_{j-1} - x_0 \| \right)^2 \] (Jensen’s inequality)

\[ = n^2 L^4 \alpha^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \| x_{i-1} - x_0 \| \| x_{j-1} - x_0 \| \] (Using Lemma 7)

\[ \leq n^2 L^4 \alpha^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{E \left[ \| x_{i-1} - x_0 \|^2 \right]} \sqrt{E \left[ \| x_{j-1} - x_0 \|^2 \right]} \] (Cauchy-Schwartz inequality)

\[ \leq n^4 L^4 \alpha^2 \left( 2n \alpha L \| x_0 - x^* \|^2 + 5n \alpha^2 G^2 \right) \]

In the last inequality, we used the fact that \( \alpha \leq 1/2nL \) (see (13)).

\[ \square \]

A.3.3 Proof of Claim 3

Proof.

\[ E \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)} (x_0) \right) \right] = \frac{1}{n} \sum_{s=1}^{n} E \left[ \nabla f_{\sigma(i)} \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)} (x_0) \right) | \sigma(i) = s \right] \] (Since \( \mathbb{P}(\sigma(i) = j) = 1/n \))

\[ = \frac{1}{n} \sum_{s=1}^{n} E \left[ \nabla f_s \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla f_{\sigma(j)} (x_0) \right) | \sigma(i) = s \right]. \]
Note that the distribution of $\sigma|\sigma(i) = s$ can be created from the distribution of $\sigma|\sigma(i) = 1$ by taking all permutations from $\sigma|\sigma(i) = 1$ and swapping 1 and s in the permutations (this is essentially a coupling between the two distributions, same as the one in [17]). This means that when we convert a permutation from $\sigma|\sigma(i) = 1$ to $\sigma|\sigma(i) = s$ in this manner, the sum $\sum_{j=1}^{i-1} \nabla \sigma(i) (x_0)$ would have a change of at most one component before and after the swap. In the following, we use $u$ and $v$ to denote random vectors with norms less than or equal to:

\[
\mathbb{E} \left[ \nabla \sigma(i) \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) \right] = \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ \nabla \sigma(i) \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) | \sigma(i) = s \right] 
\]

\[
= \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ \nabla \sigma(i) \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) + (2\alpha G)v \right) | \sigma(i) = 1 \right] 
\]

\[
= \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ \nabla \sigma(i) \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) + (2\alpha G)u|\sigma(i) = 1 \right] 
\]

\[
= \mathbb{E} \left[ \nabla F \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) | \sigma(i) = 1 \right] + (2\alpha G)u. 
\]

Similarly, for any $s$:

\[
\mathbb{E} \left[ \nabla \sigma(j) \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) \right] = \mathbb{E} \left[ \nabla F \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) | \sigma(i) = s \right] + (2\alpha G)u. 
\]

Hence,

\[
\mathbb{E} \left[ \nabla \sigma(i) \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) \right] = \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} \left[ \nabla F \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) | \sigma(i) = s \right] + (2\alpha G)u 
\]

\[
= \mathbb{E} \left[ \nabla F \left( x_0 - \alpha \sum_{j=1}^{i-1} \nabla \sigma(j) (x_0) \right) \right] + (2\alpha G)u. 
\]

\[\square\]

A.4 Proof of Lemma [7]

We will prove $\mathbb{E}[\|x_i - x_0\|^2] \leq 5\alpha^2 + 2\alpha (F(x_0) - F(x^*))$, the rest follows from Jensen’s inequality and smoothness.

First we have the following claim (Lemma 4 from [7] proved in a slightly different way: we skip the Wasserstein framework but use the same coupling)

Claim 4.

\[
|\mathbb{E} \left[ F \left( x_i \right) - f_{\sigma(i)} \left( x_i \right) \left| x_0 \right] \right| \leq 2\alpha G^2. 
\]

The rest of the proof is identical to the proof in [7]. We will be working inside an epoch, so we will skip the super script in the notation,

\[
\|x_{i+1} - x_0\|^2 = \|x_i - x_0\|^2 - 2\alpha \langle \nabla f_{\sigma(i)}(x_i), x_i - x_0 \rangle + \alpha^2 \|\nabla f_{\sigma(i)}(x_i)\|^2 
\]

\[
\leq \|x_i - x_0\|^2 - 2\alpha \langle \nabla f_{\sigma(i)}(x_i), x_i - x_0 \rangle + \alpha^2 G^2 
\]

(Bounded gradients)

\[
\leq \|x_i - x_0\|^2 + 2\alpha (f_{\sigma(i)}(x_0) - f_{\sigma(i)}(x_i)) + \alpha^2 G^2 
\]

(Convexity of $f_{\sigma(i)}$)
Taking expectation both sides:
\[
\mathbb{E}[\|x_{i+1} - x_0\|^2 | x_0] \leq \mathbb{E}[\|x_i - x_0\|^2 | x_0] + 2\alpha \mathbb{E}[f_{\sigma(i)}(x_0) - f_{\sigma(i)}(x_j) | x_0] + \alpha^2 G^2
\]
\[
= \mathbb{E}[\|x_i - x_0\|^2 | x_0] + 2\alpha F(x_0) - 2\alpha \mathbb{E}[f_{\sigma(i)}(x_j) | x_0] + \alpha^2 G^2
\]
\[
= \mathbb{E}[\|x_i - x_0\|^2 | x_0] + 2\alpha F(x_0) + 2\alpha \mathbb{E}[-f_{\sigma(i)}(x_j) | x_0] + \alpha^2 G^2
\]
\[
= \mathbb{E}[\|x_i - x_0\|^2 | x_0] + 2\alpha F(x_0) + 2\alpha \mathbb{E}[F(x_j) - f_{\sigma(i)}(x_j) - F(x_j) | x_0] + \alpha^2 G^2
\]
\[
\leq \mathbb{E}[\|x_i - x_0\|^2 | x_0] + 2\alpha F(x_0) + 2\alpha \mathbb{E}[F(x_j) - f_{\sigma(i)}(x_j) - F(x^*) | x_0] + \alpha^2 G^2
\]
\[
(x^* is the minimizer of F)
\]
\[
= \mathbb{E}[\|x_i - x_0\|^2 | x_0] + 2\alpha F(x_0) - 2\alpha \mathbb{E}[F(x_j) - f_{\sigma(i)}(x_j) | x_0] + \alpha^2 G^2
\]
\[
\leq \mathbb{E}[\|x_i - x_0\|^2 | x_0] + 2\alpha F(x_0) - 2\alpha (F(x^*) + \alpha 2G^2) + \alpha^2 G^2
giving us the required result.

Unrolling this for \(i\) iterations gives us the required result.

### A.4.1 Proof of Claim 4

**Proof.** We will work inside an epoch, so we will skip the superscript in this proof.

\[
\mathbb{E} [f_{\sigma(i)} (x_i)] = \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} [f_{\sigma(i)} (x_i) | \sigma(i) = s]
\]
\[
= \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} [f_s (x_i) | \sigma(i) = s].
\]

Note that the distribution of \(\sigma | \sigma(i) = s\) can be created from the distribution of \(\sigma | \sigma(i) = 1\) by taking all permutations from \(\sigma | \sigma(i) = 1\) and swapping 1 and \(s\) in the permutations (this is essentially a coupling between the two distributions, same as the one in [7]). This means that when we convert a permutation from the distribution \(\sigma | \sigma(i) = 1\) to a permutation from the distribution \(\sigma | \sigma(i) = s\) in this manner, the corresponding \(x_i | \sigma(i) = 1\) and \(x_i | \sigma(i) = s\) would be within a distance of \(2\alpha G\). Here is why this is true: let \(x'\) be an iterate reached using a permutation \(\sigma'\) from the distribution \(\sigma | \sigma(i) = 1\). Now, create \(\sigma''\) by swapping 1 and \(s\) in \(\sigma'\) and let \(x''\) be an iterate reached using \(\sigma''\). Then we have the following Lemma 2 from [7] adapted to our setting.

**Lemma 8. **(Lemma 2 from [7]) Let \(\alpha \leq 2/L\). Then almost surely, \(\forall i \in [n],\)
\[
\|x' - x''\| \leq 2G\alpha.
\]

In the following, we use \(u\) to denote a random vector with norm less than or equal to 1, and \(w\) to denote a random scalar with absolute value less than or equal to 1.

\[
\mathbb{E} [f_{\sigma(i)} (x_i)] = \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} [f_s (x_i) | \sigma(i) = s] = \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} [f_s (x_i + (2\alpha G)u) | \sigma(i) = 1]
\]
\[
= \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} [f_s (x_i + (2\alpha G)w) | \sigma(i) = 1] = \mathbb{E} [F(x_i) | \sigma(i) = 1] + (2\alpha G^2)w.
\]

Similarly, for any \(s:\)
\[
\mathbb{E} [f_{\sigma(i)} (x_i)] = \mathbb{E} [F(x_i) | \sigma(i) = s] + (2\alpha G^2)w.
\]

Hence,
\[
\mathbb{E} [f_{\sigma(i)} (x_i)] = \frac{1}{n} \sum_{s=1}^{n} \mathbb{E} [F(x_i) | \sigma(i) = s] + (2\alpha G^2)w
\]
\[
= \mathbb{E} [F(x_i)] + (2\alpha G^2)w.
\]

This implies
\[
\left| \mathbb{E} \left[ F \left( x_i \right) - f_{\sigma(i)} \left( x_i^{\prime} \right) \mid x_0 \right] \right| \leq 2\alpha G^2.
\]

\(\square\)
B Proof for Theorem 2

As with Theorem 1, we first start with a block diagram of the components required to establish the lower bound.

The function $F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$ that we construct is of the form

$$F(x) = \frac{\lambda x^2}{2}, \text{ if } x \geq 0$$

$$F(x) = \frac{\lambda R x^2}{2}, \text{ if } x < 0$$

Now, of the $n$ functions $f_i$, $n/2$ are of the form (we call these as functions of 1st kind)

$$f_i(x) = \frac{\lambda x^2}{2} + \frac{G x}{2}, \text{ if } x \geq 0, i \leq n/2$$

$$f_i(x) = \frac{\lambda R x^2}{2} + \frac{G x}{2}, \text{ if } x < 0, i \leq n/2$$

and $n/2$ functions are of the form (we call these as functions of 2nd kind)

$$f_i(x) = \frac{\lambda x^2}{2} - \frac{G x}{2}, \text{ if } x \geq 0, i > n/2$$

$$f_i(x) = \frac{\lambda R x^2}{2} - \frac{G x}{2}, \text{ if } x < 0, i > n/2.$$}

As we will see during the proof, there are universal constants $C_R$ and $C_\alpha$, such that we need $R \geq C_R$ and step length needed is

$$\frac{1}{nK} \leq \alpha \leq \frac{C_\alpha}{nR \lambda}.$$
Figure 4: Lower bound construction. Note that \( f_1(x) \) represents functions of the first kind, and \( f_2(x) \) represents functions of the second kind, and \( F(x) \) represents the overall function.

Let us initialize at \( x_0 = 0 \). Next we prove an upper bound on gradient norm.
The minimizer of the functions of first kind is at \( x = -G/2R\lambda \), and the minimizer of the functions of the second kind is at \( x = G/2\lambda \). Between these two values, the norm of gradient of any of the two function kinds is always less than \( G \) (upper bound on gradient norm). We need to show that the iterates stay between the two minimizers.
The step size we have chosen is small enough (smaller than \( 1/R\lambda \)) to ensure that the iterates do not cross the two minimizers. To see this imagine that we’re doing gradient descent on \( g(x) = ax^2/2 \). Then if \( x > 0 \) and step length \( \alpha < 1/a \), we have that the next iterate \( x_{t+1} = x_t - \alpha(ax_t) = x_t(1-\alpha a) > 0 \), that is the iterates never ‘cross over’ the minimizer 0. A similar logic here implies that iterates in our case stay within \([-G/2R\lambda, G/2\lambda]\).
Thus gradient norm would never exceed \( G \). Then, we have the following key lemma of the proof:

**Lemma 9.** There are positive universal constants \( C_1, C_2, C_a \) and \( C_R \) such that if \( R \geq C_R \) and \( \alpha \leq C_a/nR\lambda \), then

\[
\mathbb{E} \left[ x_0^{j+1} \left| x_0^j \right| \right] \leq C_1 G\alpha \sqrt{n} = x_0^j + C_2 R\lambda G\alpha^2 n\sqrt{n}
\]

Intuitively, what Lemma 9 implies is that \( \mathbb{E} \left[ x_0^j \right] \) should keep increasing at rate \( \Omega(\alpha^2n\sqrt{n}) \) until \( |x_0^j| = \Omega(\alpha\sqrt{n}) \). The rest of the proof is making this intuition rigorous.
First we have the following helper lemma:

**Lemma 10.** There are positive universal constants \( C_3, C_4 \) such that if \( \alpha \leq 1/nR\lambda \), then

\[
\mathbb{E} \left[ \left| x_0^{j+1} - x_0^j \right| \right] \leq C_3 R\lambda |x_0^j|\alpha n + C_4 \alpha^2 R\lambda Gn \sqrt{n}
\]

Define \( C_5 = 2 \max \left\{ C_3, \frac{C_4}{C_1} \right\} \). Then, Lemma 10 implies the following corollary

**Corollary 1.**

\[
\mathbb{E} \left[ x_0^{j+1} \left| x_0^j \right| > C_1 G\alpha \sqrt{n} \right] \geq x_0^j - C_5 R\lambda |x_0^j|\alpha n
\]
\[ \mathbb{E} \left[ |x_0^{j+1}| \right] \geq |x_0^j|(1 - C_3 R\lambda n) - C_4 R\lambda G\alpha \, n \sqrt{n} \]

Consider two cases:

**Case 1:** If \( P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \) \( \mathbb{E} \left[ |x_0^j| \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] \leq \min \left\{ \frac{C_1}{2}, \frac{C_2}{4C_5} \right\} G\alpha \sqrt{n} \). Then,

\[ \mathbb{E} \left[ x_0^{j+1} \right] = P \left( |x_0^j| \leq C_1 G\alpha \sqrt{n} \right) \mathbb{E} \left[ x_0^{j+1} \bigg| |x_0^j| \leq C_1 G\alpha \sqrt{n} \right] + P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \mathbb{E} \left[ x_0^{j+1} \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] \]

(Decomposing expectation into sum of conditional expectations)

\[ \geq P \left( |x_0^j| \leq C_1 G\alpha \sqrt{n} \right) \left( \mathbb{E} \left[ x_0^j \bigg| |x_0^j| \leq C_1 G\alpha \sqrt{n} \right] + C_2 R\lambda G\alpha^2 n \sqrt{n} \right) \]

\[ + P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \left( \mathbb{E} \left[ x_0^j \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] - C_3 R\lambda \mathbb{E} \left[ x_0^j \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] \right) \]

(Using Lemma 9 and Corollary 1)

\[ = \mathbb{E} \left[ x_0^j \right] + P \left( |x_0^j| \leq C_1 G\alpha \sqrt{n} \right) C_2 R\lambda G\alpha^2 n \sqrt{n} \]

(Deciding sum of conditional expectations back into unconditional expectation)

Since \( P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \mathbb{E} \left[ x_0^j \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] \leq \frac{C_1}{2} G\alpha \sqrt{n} \), we know that \( P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \leq \frac{1}{2} \). This means \( P \left( |x_0^j| \leq C_1 G\alpha \sqrt{n} \right) \geq \frac{1}{2} \). Hence,

\[ \mathbb{E} \left[ x_0^{j+1} \right] \geq \mathbb{E} \left[ x_0^j \right] + \frac{1}{2} C_2 R\lambda G\alpha^2 n \sqrt{n} \]

\[ - C_3 R\lambda P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \mathbb{E} \left[ x_0^j \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] \]

\[ \geq \mathbb{E} \left[ x_0^j \right] + \frac{1}{2} C_2 R\lambda G\alpha^2 n \sqrt{n} - C_3 R\lambda \left( \frac{C_2}{4C_5} G\alpha \sqrt{n} \right) \]

\[ \left( \text{Since } P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \mathbb{E} \left[ x_0^j \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] \leq \frac{C_2}{4C_5} G\alpha \sqrt{n} \right) \]

\[ \geq \mathbb{E} \left[ x_0^j \right] + \frac{1}{4} C_2 R\lambda G\alpha^2 n \sqrt{n}. \] (22)

**Case 2:** Otherwise, if \( P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \mathbb{E} \left[ x_0^j \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] > \min \left\{ \frac{C_1}{2}, \frac{C_2}{4C_5} \right\} G\alpha \sqrt{n} \).

Then,

\[ \mathbb{E} \left[ |x_0^j| \right] \geq P \left( |x_0^j| > C_1 G\alpha \sqrt{n} \right) \mathbb{E} \left[ |x_0^j| \bigg| |x_0^j| > C_1 G\alpha \sqrt{n} \right] \]

\[ > \min \left\{ \frac{C_1}{2}, \frac{C_2}{4C_5} \right\} G\alpha \sqrt{n}. \]

Let \( C_6 = \min \left\{ \frac{C_1}{2}, \frac{C_2}{4C_5} \right\} \).

What we have shown is that if \( \mathbb{E} \left[ |x_0^j| \right] \leq C_6 G\alpha \sqrt{n} \), then \( \mathbb{E} \left[ x_0^{j+1} \right] \geq \mathbb{E} \left[ x_0^j \right] + \frac{1}{4} C_2 R\lambda G\alpha^2 n \sqrt{n} \).

We initialize \( x_0^1 = 0 \) and we run for \( K \) epochs. Then, because \( \mathbb{E} \left[ x_0^j \right] \geq \mathbb{E} \left[ x_0^j \right] \), we know that
What we have shown is that throughout this process, as a reminder, the function $\alpha$ which is the following:

\[
\forall \quad \text{we prove that once } \mathbb{E} \left[ x_0^j \right] \geq C_7 \mathbb{G} \alpha \sqrt{n}, \text{ we have that } \mathbb{E} \left[ x_0^j \right] \text{ remains } \geq C_8 \mathbb{G} \alpha \sqrt{n} \text{ for some universal constant } C_8. \text{ This would complete the proof.}
\]

Assume $\alpha \leq \frac{1}{nR\lambda} \min \left\{ \frac{1}{2C_3}, \frac{C_7}{4C_4} \right\}$.

Let $j$ be such that $\mathbb{E} \left[ x_0^j \right] \geq C_7 \mathbb{G} \alpha \sqrt{n}$ and $\mathbb{E} \left[ x_0^{j+1} \right] < C_7 \mathbb{G} \alpha \sqrt{n}$. Then,

\[
\mathbb{E} \left[ x_0^{j+1} \right] \geq \mathbb{E} \left[ x_0^j \right] (1 - C_3 R\lambda n) - C_4 R\lambda \mathbb{G} \alpha^2 n \sqrt{n} \tag{Using Corollary 1}
\]

\[
\geq \frac{1}{2} \mathbb{E} \left[ x_0^j \right] - C_4 R\lambda \mathbb{G} \alpha^2 n \sqrt{n} \quad \text{(Since } \alpha \leq \frac{1}{2C_3 nR\lambda})
\]

\[
\geq \frac{1}{2} C_7 \mathbb{G} \alpha \sqrt{n} - C_4 R\lambda \mathbb{G} \alpha^2 n \sqrt{n}
\]

\[
\geq \frac{1}{4} C_7 \mathbb{G} \alpha \sqrt{n}. \quad \text{(Since } \alpha \leq \frac{C_7}{4C_3 nR\lambda})
\]

Now by Corollary 1, after each epoch $l > j + 1$, $\mathbb{E}[x_0^l]$ increases by at least $\frac{1}{4} C_2 R\lambda \mathbb{G} \alpha^2 n \sqrt{n}$. Further by Corollary 1 $\mathbb{E}[x_0^l]$ can decrease by at most $C_4 R|x_0^{l-1}|n + C_4 R\lambda \mathbb{G} \alpha^2 n \sqrt{n} \leq (C_7 C_3 + C_4) R\lambda \mathbb{G} \alpha^2 n \sqrt{n}$. This combined with the facts that $\forall l : \mathbb{E}[x_0^l] \geq 0$ (Lemma 11) and $\mathbb{E}[x_0^l] \leq \mathbb{E}[x_0^l]$, gives that $\mathbb{E}[x_0^l]$ can decrease for at most $\frac{1}{4} C_2 R\lambda \mathbb{G} \alpha^2 n \sqrt{n} + (C_7 C_3 + C_4) R\lambda \mathbb{G} \alpha^2 n \sqrt{n}$ epochs. This means $\mathbb{E}[x_0^l]$ can decrease to at most

\[
\frac{1}{4} C_7 \mathbb{G} \alpha \sqrt{n} - \frac{1}{4} C_2 R\lambda \mathbb{G} \alpha^2 n \sqrt{n} + (C_7 C_3 + C_4) R\lambda \mathbb{G} \alpha^2 n \sqrt{n} = \frac{1}{4} C_7 \mathbb{G} \alpha \sqrt{n} \left( 1 - \frac{C_2}{C_2 + 4C_7 C_4 + 4C_4} \right) = \Omega(\alpha \sqrt{n}).
\]

**Lemma 11.** If $\alpha \leq 1/R\lambda$, then $\forall i, j : \mathbb{E}[x_i^j] \geq 0$.

After this $\mathbb{E}[|x_0^j|]$ will have to keep increasing till it reaches $C_7 \mathbb{G} \alpha \sqrt{n}$ again, and this cycle repeats. What we have shown is that throughout this process, $\mathbb{E}[x_0^j] = \Omega(\alpha \sqrt{n})$.

Finally, given $\alpha = \Omega \left( \frac{1}{nR} \right)$, we get that $\mathbb{E}[|x_0^K|] = \Omega \left( \frac{1}{\sqrt{n}R} \right)$. Applying Jensen’s inequality on this gives $\mathbb{E}[|x_0^K|^2] \geq (\mathbb{E}[|x_0^K|])^2 = \Omega \left( \frac{1}{nR^2} \right) = \Omega \left( \frac{n}{\lambda^2} \right)$.

The last remaining thing that we need to do is gather all the required bounds on $\alpha$ into one place, which is the following:

\[
\frac{1}{2nR} \leq \alpha \leq \frac{C_\alpha}{nR\lambda}
\]

**B.1 Proof of Lemma 9**

As a reminder, the function $F(x) = \sum_{i=1}^{n} f_i(x)$ that we construct is of the form

\[
F(x) = \frac{\lambda x^2}{2}, \text{ if } x \geq 0
\]

25
\[ F(x) = \frac{\lambda R x^2}{2}, \text{ if } x < 0 \]

where, of the \( n \) functions \( f_i \), \( n/2 \) are of the form (we call these as functions of 1st kind)

\[ f_i(x) = \frac{\lambda x^2}{2} + \frac{G x}{2}, \text{ if } x \geq 0, i \leq n/2 \]

\[ f_i(x) = \frac{\lambda R x^2}{2} + \frac{G x}{2}, \text{ if } x < 0, i \leq n/2 \]

and \( n/2 \) functions are of the form (we call these as functions of 2nd kind)

\[ f_i(x) = \frac{\lambda x^2}{2} - \frac{G x}{2}, \text{ if } x \geq 0, i > n/2 \]

\[ f_i(x) = \frac{\lambda R x^2}{2} - \frac{G x}{2}, \text{ if } x < 0, i > n/2. \]

Let \( \sigma^j \) be the permutation of \( f_i \)'s used in the \( j \)-th epoch. Note that \( \sigma^j \) can just be treated as a permutation of \((+1, \ldots, +\frac{n}{2} \text{ times} \ldots, +1, -1, \ldots, -\frac{n}{2} \text{ times} \ldots, -1)\): if \( \sigma^j_i = +1 \), we assume in the \( i \)-th iteration of the \( j \)-th epoch, a function of the 1st kind was picked. Similarly if \( \sigma^j_i = -1 \), we assume in the \( i \)-th iteration of the \( j \)-th epoch, a function of the 2nd kind was picked.

\[
\mathbb{E}[x_0^{j+1}] = x_0^j - \alpha \frac{G}{2} \sum_{i=1}^{n} \sigma_i^j - \alpha \sum_{i=0}^{n} \mathbb{E}\left[ \left( \mathbb{1}_{x_i^j \leq 0} R + \mathbb{1}_{x_i^j > 0} \right) \lambda x_i^j \right]
\]

\[
= x_0^j - \alpha \sum_{i=0}^{n} \mathbb{E}\left[ \left( \mathbb{1}_{x_i^j \leq 0} R + \mathbb{1}_{x_i^j > 0} \right) \lambda x_i^j \right]
\]

\[
= x_0^j - \alpha \sum_{i=n/4}^{n/2} \mathbb{E}\left[ \left( \mathbb{1}_{x_i^j \leq 0} R + \mathbb{1}_{x_i^j > 0} \right) \lambda x_i^j \right]
\]

\[
- \alpha \sum_{i \notin \left\{n/4, n/2\right\}} \mathbb{E}\left[ \left( \mathbb{1}_{x_i^j \leq 0} R + \mathbb{1}_{x_i^j > 0} \right) \lambda x_i^j \right]
\]

\[
= x_0^j - \alpha \lambda \sum_{i=n/4}^{n/2} \mathbb{P}\left( \left( \sum_{p=1}^{i} \sigma_p^j \right) > 0 \right) \mathbb{E}\left[ \left( \mathbb{1}_{x_i^j \leq 0} R + \mathbb{1}_{x_i^j > 0} \right) \sum_{p=1}^{i} \sigma_p^j \right] > 0 \right]
\]

\[
- \alpha \lambda \sum_{i=n/4}^{n/2} \mathbb{P}\left( \left( \sum_{p=1}^{i} \sigma_p^j \right) \leq 0 \right) \mathbb{E}\left[ \left( \mathbb{1}_{x_i^j \leq 0} R + \mathbb{1}_{x_i^j > 0} \right) \sum_{p=1}^{i} \sigma_p^j \right] \leq 0 \right]
\]

\[
- \alpha \lambda \sum_{i \notin \left\{n/4, n/2\right\}} \mathbb{E}\left[ \left( \mathbb{1}_{x_i^j \leq 0} R + \mathbb{1}_{x_i^j > 0} \right) \lambda x_i^j \right].
\]

(Decomposing expectation into sum of conditional expectations)

Now, let \( r \) be any random variable and \( \mathbb{E}[r] \leq \zeta \) and we know \( R \geq 1 \).

\[
\mathbb{E}[\mathbb{1}_{r \leq R} + \mathbb{1}_{r > R} \mathbb{1}_{r > 0} r] = R \mathbb{P}(r < 0) \mathbb{E}[r | r < 0] + \mathbb{P}(r \geq 0) R (r \geq 0) \mathbb{E}[r | r \geq 0]
\]

\[
= R \left[ \mathbb{P}(r < 0) \mathbb{E}[r | r < 0] + \mathbb{P}(r \geq 0) R (r \geq 0) \right] + (1 - R) \mathbb{P}(r \geq 0) \mathbb{E}[r | r \geq 0]
\]

\[
\leq R \zeta
\]

(24)

and

\[
\mathbb{E}[\mathbb{1}_{r \leq R} + \mathbb{1}_{r > R} \mathbb{1}_{r > 0} r] = R \mathbb{P}(r < 0) \mathbb{E}[r | r < 0] + \mathbb{P}(r \geq 0) R (r \geq 0) \mathbb{E}[r | r \geq 0]
\]

\[
= 1(\mathbb{P}(r < 0) \mathbb{E}[r | r < 0] + \mathbb{P}(r \geq 0) R (r \geq 0)) + (R - 1) \mathbb{P}(r < 0) \mathbb{E}[r | r < 0]
\]

\[
\leq 1 \zeta.
\]

(25)

Next, we have the following lemma:
Lemma 12. There are universal constants $C_{11} \geq 0, C_{12} \geq 0, C_{13} \geq 0$ such that if $|x_0^i| \leq C_{11}G\sqrt{n\alpha}$ and $\alpha \leq \frac{C_{12}}{nR\lambda}$, then for $n/4 \leq i \leq n/2$, we have

$$E \left[ |x_i^j - x_0^j| \left( \sum_{p=1}^i \sigma_p^j \right) \right] > 0 \leq -C_{13}G\sqrt{n\alpha}.$$

We have the following Lemma and Corollary which actually come up in the proof of Lemma 12, but have been copied here because we will use them here as well:

Lemma 13. Suppose we have a uniform at random permutation $\sigma$ of $\{+1, \ldots, \frac{n}{2} \text{ times} \ldots, +1, -1, \ldots, \frac{n}{2} \text{ times} \ldots, -1\}$ (where $n$ is an even number greater than 8). Let $\sigma_i$ be the $i$-th element of this permutation. Then for $i \leq n/2$, there are universal constants $C_9 > 0, C_{10} > 0$ such that

$$C_9\sqrt{i} \leq E \left[ \sum_{j \leq i} \sigma_j \right] \leq \sqrt{i} \quad \tag{26}$$

and for $i \in [n/4, n/2]$: $P\left( \left( \sum_{p=1}^i \sigma_p^j \right) < 0 \right) \geq C_{10} - P\left( \left( \sum_{p=1}^i \sigma_p^j \right) > 0 \right) \geq C_{10}.$

Corollary 2. There is a universal constant $C_9$ such that

$$E \left[ |x_i^j - x_0^j| \left( \sum_{p=1}^i \sigma_p^j \right) \right] \leq \frac{1}{C_{10}} \left( \sqrt{5iG\alpha} + |x_0^i|\sqrt{2i\alpha R\lambda} \right),$$

$$E \left[ |x_i^j - x_0^j| \left( \sum_{p=1}^i \sigma_p^j \right) \right] > 0 \leq \frac{1}{C_{10}} \left( \sqrt{5iG\alpha} + |x_0^i|\sqrt{2i\alpha R\lambda} \right).$$

Assume $|x_0^i| \leq \min \left\{ C_{11}, \frac{C_{13}}{4} \right\} G\sqrt{n\alpha}$ and $\alpha \leq 1/nR\lambda$.

- for $i \in [n/4, n/2], \left( \sum_{p=1}^i \sigma_p^j \right) > 0$:

$$E \left[ (1 \cdot_{x_i^j \leq 0} R + 1 \cdot_{x_i^j > 0}) x_i^j \left( \sum_{p=1}^i \sigma_p^j \right) > 0 \right] \leq RE \left[ x_i^j \left( \sum_{p=1}^i \sigma_p^j \right) > 0 \right] \quad \text{(Using (24))}$$

$$\leq RE \left[ x_i^j - x_0^j \left( \sum_{p=1}^i \sigma_p^j \right) > 0 \right] + Rx_0^j$$

$$\leq -RC_{13} \sqrt{n^4/4}G\alpha + R \frac{C_{13}}{4} G\sqrt{n\alpha} \quad \text{(Using Lemma 12)}$$

$$\leq -R \frac{C_{13}}{4} G\sqrt{n\alpha}. \quad \tag{27}$$

- for $i \in [n/4, n/2], \left( \sum_{p=1}^i \sigma_p^j \right) \leq 0$:

$$E \left[ (1 \cdot_{x_i^j \leq 0} R + 1 \cdot_{x_i^j > 0}) x_i^j \left( \sum_{p=1}^i \sigma_p^j \right) \leq 0 \right] \leq 1 E \left[ x_i^j \left( \sum_{p=1}^i \sigma_p^j \right) \leq 0 \right] \quad \text{(Using (25))}$$

$$\leq E \left[ x_i^j - x_0^j \left( \sum_{p=1}^i \sigma_p^j \right) \leq 0 \right] + |x_0^i|$$

$$\leq \frac{1}{C_{10}} \left( \sqrt{5iG\alpha} + |x_0^i|\sqrt{2i\alpha R\lambda} \right) + |x_0^i| \quad \text{(Using Corollary 2)}$$

$$\leq C_{14} G\sqrt{n\alpha}. \quad \tag{28}$$

($C_{14}$ is universal constant since it is dependent only on $C_{11}, C_{13}, C_{10}$. Also used $\alpha \leq 1/nR\lambda$)
• for \( i \notin [n/4, n/2] \):

\[
E \left[ \left( \mathbb{1}_{x_i' \leq 0} R + \mathbb{1}_{x_i' > 0} 1 \right) x_i' \right] \leq 1E \left[ x_i' \right] \quad \text{(Using (25))}
\]

\[
\leq E \left[ x_i' - x_0' \right] + |x_0'|
\]

\[
\leq \left( \sqrt{5} \alpha |x_0'| + 2\alpha R\lambda \right) + |x_0'|
\]

\[
\leq C_{14} G \sqrt{n} \alpha.
\]

(30)

Assume \( R \geq \max \left\{ 1, \frac{32C_{14}}{C_{10}C_{13}} \right\} \). Continuing on from (23) and substituting (27), (28) and (30):

\[
E \left[ x_{i+1}^{t+1} \right] \geq x_0' - \alpha \lambda \sum_{i = n/4}^{n/2} \mathbb{P} \left( \left( \sum_{p=1}^{i} \sigma_p^i \right) > 0 \right) E \left[ \left( \mathbb{1}_{x_i' \leq 0} R + \mathbb{1}_{x_i' > 0} 1 \right) x_i' \right] \left( \sum_{p=1}^{i} \sigma_p^i \right) > 0
\]

\[
- \alpha \lambda \sum_{i = n/4}^{n/2} \mathbb{P} \left( \left( \sum_{p=1}^{i} \sigma_p^i \right) \leq 0 \right) E \left[ \left( \mathbb{1}_{x_i' \leq 0} R + \mathbb{1}_{x_i' > 0} 1 \right) x_i' \right] \left( \sum_{p=1}^{i} \sigma_p^i \right) \leq 0
\]

\[
\geq x_0' - \alpha \lambda \sum_{i = n/4}^{n/2} \mathbb{P} \left( \left( \sum_{p=1}^{i} \sigma_p^i \right) > 0 \right) \left( -R \frac{C_{13}}{4} G \sqrt{n} \alpha \right)
\]

\[
- \alpha \lambda \sum_{i = n/4}^{n/2} \mathbb{P} \left( \left( \sum_{p=1}^{i} \sigma_p^i \right) \leq 0 \right) \left( C_{14} G \sqrt{n} \alpha \right) - \alpha \lambda \sum_{i \notin [n/4, n/2]} \left( C_{14} G \sqrt{n} \alpha \right)
\]

\[
\geq x_0' - \alpha \lambda \sum_{i = n/4}^{n/2} \mathbb{P} \left( \left( \sum_{p=1}^{i} \sigma_p^i \right) > 0 \right) \left( -R \frac{C_{13}}{4} G \sqrt{n} \alpha \right)
\]

\[
- \alpha \lambda \sum_{i = 1}^{n} \left( C_{14} G \sqrt{n} \alpha \right)
\]

\[
\geq x_0' + \alpha \lambda \frac{n}{4} C_{10} \left( R \frac{C_{13}}{4} G \sqrt{n} \alpha \right) - \alpha \lambda n \left( C_{14} G \sqrt{n} \alpha \right)
\]

\[
\geq x_0' + R \lambda \frac{C_{13}}{32} G \alpha^2 n \sqrt{n}.
\]

(Using Lemma 13)

(Since \( R \geq \frac{32C_{14}}{C_{10}C_{13}} \))
B.2 Proof of Lemma 10

We assume \( \alpha \leq 1/nR\lambda \).

\[
\begin{align*}
E[|x_0^{j+1} - x_0^j|] &= E \left[ -\alpha \lambda \sum_{i=1}^{n} \sigma_i^j - \alpha \sum_{i=0}^{n} (\mathbb{1}_{x_i^j \leq 0} + \mathbb{1}_{x_i^j > 0}) x_i^j \right] \\
&= E \left[ -\alpha \lambda \sum_{i=0}^{n} (\mathbb{1}_{x_i^j \leq 0} + \mathbb{1}_{x_i^j > 0}) x_i^j \right] \\
&\leq \alpha \lambda \sum_{i=0}^{n} (\mathbb{1}_{x_i^j \leq 0} + \mathbb{1}_{x_i^j > 0}) x_i^j \\
&\leq \alpha \lambda E \left[ \sum_{i=0}^{n} (\mathbb{1}_{x_i^j \leq 0} + \mathbb{1}_{x_i^j > 0}) x_i^j \right] \\
&\leq \alpha \lambda E \left[ \sum_{i=0}^{n} \sqrt{2\alpha R} \left( x_i^j - |x_i^j| \right) + \sqrt{5\alpha G} + |x_i^j| \right] \quad \text{(Using Lemma 7)} \\
&\leq \alpha \lambda \left( \sqrt{5\alpha G} + \sum_{i=0}^{n} \sqrt{5\alpha G} \right) \quad \text{(Since } \alpha \leq 1/nR\lambda) \\
&\leq \alpha \lambda n \left( \sqrt{5\alpha G} + 3 \right) |x_0^j| \\
&\leq \alpha \lambda n \left( \sqrt{5\alpha G} + 3 \right) |x_0^j|. 
\end{align*}
\]

(31)

B.3 Proof of Corollary 1

\[
E \left[ |x_0^{j+1} - x_0^j| \right] \leq C_3 R |x_0^j| |an + C_4 R G \alpha^2 n \sqrt{n} |
\]

Define \( C_5 = 2 \max \left\{ \frac{C_3}{C_1}, \frac{C_4}{C_1} \right\} \). Then,

\[
\begin{align*}
E \left[ |x_0^{j+1} - x_0^j| \right] &> C_1 G \alpha \sqrt{n} \\
&= E \left[ x_0^j + x_0^{j+1} - x_0^j \left| x_0^j \right| > C_1 G \alpha \sqrt{n} \right] \\
&\geq x_0^j - E \left[ |x_0^{j+1} - x_0^j| \left| x_0^j \right| > C_1 G \alpha \sqrt{n} \right] \\
&\geq x_0^j - (C_3 |x_0^j| |an + C_4 R G \alpha^2 n \sqrt{n}|) \quad \text{(Using Lemma 10)} \\
&\geq x_0^j - \left( C_3 |x_0^j| \frac{|an + C_4 R G \alpha^2 n \sqrt{n}|}{C_1} \right) \quad \text{(Since } |x_0^j| > C_1 G \alpha \sqrt{n}) \\
&\geq x_0^j - C_5 R |x_0^j| an.
\end{align*}
\]

This proves the first equation of the corollary. For the other equation,

\[
\begin{align*}
E \left[ |x_0^{j+1}| \right] &= E \left[ x_0^j + x_0^{j+1} - x_0^j \right] \\
&\geq E \left[ x_0^j \right] - E \left[ |x_0^{j+1} - x_0^j| \right] \\
&\geq |x_0^j| (1 - C_3 R an) - C_4 R G \alpha^2 n \sqrt{n}. \quad \text{(Using Lemma 10)}
\end{align*}
\]

B.4 Proof of Lemma 11

Let us denote the iterate at time \( t \) by \( x_{t,R} \), where the \( R \) in subscript is to show dependence on \( R \). Now, consider the problem setup when \( R = 1 \). In that case, the two kinds of functions are simply quadratics
instead of piecewise quadratics. The two kinds of functions in this case are \( f_1(x) = \frac{\lambda}{2}x^2 + \frac{G}{2}x \) and 
\( f_2(x) = \frac{\lambda}{2}x^2 - \frac{G}{2}x \). Then, due to symmetry of the function, the expected value of iterate at any time \( t \) is 0: \( \mathbb{E}[x_{t,1}] = 0 \).

Now, take the problem setup for the case \( R \geq 1 \), where we denote the iterates by \( x_{t,R} \). We couple the two iterates \( x_{t,1} \) and \( x_{t,R} \) such that both \( x_{t,R} \) and \( x_{t,1} \) are created using the exact same random permutations of the functions but \( x_{t,1} \) had \( R = 1 \) and \( x_{t,R} \) had \( R \geq 1 \). Then, we show that \( x_{t,R} \geq x_{t,1} \) and since \( \mathbb{E}[x_{t,1}] = 0 \), we get that \( \mathbb{E}[x_{t,R}] \geq 0 \).

We prove this by induction. Note that \( \alpha \leq \frac{1}{R\lambda} \).

- **Base case:** Both are initialized at 0, and thus \( x_{0,1} = x_{0,R} = 0 \).

- **Inductive case (assume \( x_{i,R} \geq x_{i,1} \)):**
  1. \( x_{i,1} \leq x_{i,R} \leq 0 \). Then note that regardless of the choice of the functions, the contribution of the linear gradients in the gradients would be the same for both \( x_{i+1,1} \) and \( x_{i+1,R} \) (since we use the exact same permutation of functions for both). Hence,

\[
x_{i+1,R} - x_{i+1,1} = (1 - \alpha R\lambda)x_{i,R} - (1 - \alpha \lambda)x_{i,1} \\
\geq (1 - \alpha R\lambda)x_{i,R} - (1 - R\alpha \lambda)x_{i,1} \\
\geq 0.
\]

  2. \( 0 \leq x_{i,1} \leq x_{i,R} \). Again, regardless of the choice of the functions, the contribution of the linear gradients in the gradients would be the same for both \( x_{i+1,1} \) and \( x_{i+1,R} \) (since we use the exact same permutation of functions for both). Hence,

\[
x_{i+1,R} - x_{i+1,1} = (1 - \alpha \lambda)x_{i,R} - (1 - \alpha \lambda)x_{i,1} \\
\geq 0.
\]

  3. \( x_{i,1} \leq 0 \leq x_{i,R} \). Again, regardless of the choice of the functions, the contribution of the linear gradients in the gradients would be the same for both \( x_{i+1,1} \) and \( x_{i+1,R} \) (since we use the exact same permutation of functions for both). Hence,

\[
x_{i+1,R} - x_{i+1,1} = (1 - \alpha \lambda)x_{i,R} - (1 - \alpha \lambda)x_{i,1} \\
\geq 0.
\]

**B.5 Proof of Lemma 12**

As a reminder, the function \( F(x) = \sum_{i=1}^{n} f_i(x) \) that we construct is of the form

\[
F(x) = \frac{\lambda x^2}{2}, \text{ if } x \geq 0
\]
\[
F(x) = \frac{\lambda Rx^2}{2}, \text{ if } x < 0
\]

where, of the \( n \) functions \( f_i, n/2 \) are of the form (we call these as functions of 1st kind)

\[
f_i(x) = \frac{\lambda x^2}{2} + \frac{Gx}{2}, \text{ if } x \geq 0, i \leq n/2
\]
\[
f_i(x) = \frac{\lambda Rx^2}{2} + \frac{Gx}{2}, \text{ if } x < 0, i \leq n/2
\]

and \( n/2 \) functions are of the form (we call these as functions of 2nd kind)

\[
f_i(x) = \frac{\lambda x^2}{2} - \frac{Gx}{2}, \text{ if } x \geq 0, i > n/2
\]
\[
f_i(x) = \frac{\lambda Rx^2}{2} - \frac{Gx}{2}, \text{ if } x < 0, i > n/2
\]
$$f_i(x) = \frac{\lambda R x^2}{2} - \frac{G x}{2}, \text{ if } x < 0, i > n/2$$

Let $\sigma^j$ be the permutation of $f_i$’s used in the $j$-th epoch. Note that $\sigma^j$ can just be treated as a permutation of $(+1, \ldots, \frac{n}{2} \text{ times}, +1, -1, \ldots, \frac{n}{2} \text{ times}, -1)$: if $\sigma^j_i = +1$, we assume in the $i$-th iteration of the $j$-th epoch, a function of the 1st kind was picked. Similarly if $\sigma^j_i = -1$, we assume in the $i$-th iteration of the $j$-th epoch, a function of the 2nd kind was picked. Hence,

$$E[|x_i - x_0| \left( \sum_{p=1}^{i} \sigma^j_p \right)] > 0 = E \left[ -\alpha \sum_{p=1}^{i} \nabla f_{\sigma^j_p} (x_{p-1}) \left( \sum_{p=1}^{i} \sigma^j_p \right) > 0 \right]$$

$$= -\alpha E \left[ \sum_{p=1}^{i} \left( \frac{G}{2} \sigma^j_p + \lambda x^j_{p-1} \left( \|x_{p-1}^j\|_0 + R \|x_{p-1}^j\|_0 < 0 \right) \right) \left( \sum_{p=1}^{i} \sigma^j_p \right) > 0 \right]$$

$$\leq -\alpha E \left[ \frac{G}{2} \sum_{p=1}^{i} \sigma^j_p - \lambda R \sum_{p=1}^{i} |x^j_{p-1}| \left( \sum_{p=1}^{i} \sigma^j_p \right) > 0 \right]$$

$$\leq -\alpha E \left[ \frac{G}{2} \sum_{p=1}^{i} \sigma^j_p - \lambda R \sum_{p=1}^{i} |x^j_{p-1}| + \sum_{p=1}^{i} \sigma^j_p \right] > 0 \right] . \ (32)$$

We have the following helpful lemma

**Lemma 13.** Suppose we have a uniform at random permutation $\sigma$ of $(+1, \ldots, \frac{n}{2} \text{ times}, +1, -1, \ldots, \frac{n}{2} \text{ times}, -1)$ (where $n$ is an even number greater than 8). Let $\sigma_i$ be the $i$-th element of this permutation. Then for $i \leq n/2$, there are universal constants $C_0 > 0, C_{10} > 0$ such that

$$C_0 \sqrt{i} \leq E \left[ \sum_{j<i} \sigma_j \right] \leq \sqrt{i} \ \ \ (26)$$

and for $i \in [n/4, n/2]$: $P \left( \sum_{p=1}^{i} \sigma^j_p < 0 \right) \geq C_{10}, \ \ P \left( \sum_{p=1}^{i} \sigma^j_p > 0 \right) \geq C_{10}$.

Using Lemma 13 and Lemma 7 (Lemma 5 from [7]), we have

**Corollary 2.** There is a universal constant $C_{10}$ such that

$$E \left[ |x^j_i - x^j_0| \left( \sum_{p=1}^{i} \sigma^j_p \right) \right] \leq \frac{1}{C_{10}} \left( \sqrt{5iG} + |x^j_0| \sqrt{2i\alpha R\lambda} \right),$$

$$E \left[ |x^j_i - x^j_0| \left( \sum_{p=1}^{i} \sigma^j_p \right) \right] \geq \frac{1}{C_{10}} \left( \sqrt{5iG} + |x^j_0| \sqrt{2i\alpha R\lambda} \right).$$

Continuing from (32), and using Corollary 2, we have that for $n/4 \leq i \leq n/2$

$$E \left[ |x^j_i - x^j_0| \left( \sum_{p=1}^{i} \sigma^j_p \right) \right] \leq -\frac{C_0}{2} \sqrt{i}G + \alpha \lambda R i \left( \frac{1}{C_{10}} \sqrt{5iG} + |x^j_0| \left( 1 + \frac{1}{C_{10}} \sqrt{2i\alpha R\lambda} \right) \right).$$

If $|x^j_0| \leq \frac{\sqrt{5iG}}{4C_{10}}$ and $\alpha \leq \frac{1}{R\lambda} \min \left\{ C_{10}, C_{10}C_0 \right\}$, then for $n/4 \leq i \leq n/2$, we have

$$E \left[ |x^j_i - x^j_0| \left( \sum_{p=1}^{i} \sigma^j_p \right) \right] \leq -\frac{1}{4} C_0 \sqrt{i}G.$$
B.6 Proof of Lemma 13

Proof. Let \( s_i = \sum_{j<i} \sigma_j \). First, we prove the upper bound. We have that

\[
\mathbb{E}[s_i] = \mathbb{E} \left[ \sum_{j<i} \sigma_j \right]
\leq \sqrt{\mathbb{E} \left[ \left( \sum_{j<i} \sigma_j \right)^2 \right]}
= \sqrt{\sum_{j<i} \mathbb{E}[\sigma_j^2] + 2 \sum_{k<j<i} \mathbb{E}[\sigma_k \sigma_j]}
= \sqrt{i + 2 \sum_{k<j<i} \mathbb{E}[\sigma_k \sigma_j]}.
\]

(Jensen’s inequality)

We can see that \( \mathbb{E}(\sigma_k \sigma_j) < 0 \) (they are negatively correlated). Hence,

\[\mathbb{E}[s_i] \leq \sqrt{i}.\]

For the lower bound,

\[\mathbb{E}[s_i] = \mathbb{P}(s_{i-1} = 0).\mathbb{E}[s_i | s_{i-1} = 0] + \mathbb{P}(s_{i-1} \neq 0). (\mathbb{E}[s_i | s_{i-1} = 0] + \mathbb{P}(s_{i-1} \neq 0). (\mathbb{E}[s_i | s_{i-1} = 0])\]

(Decomposing expectation into sum of conditional expectations)

\[= \mathbb{P}(s_{i-1} = 0).1 + \mathbb{P}(s_{i-1} > 0). (\mathbb{E}[s_i | s_{i-1} = 0] + \mathbb{P}(s_{i-1} \neq 0). (\mathbb{E}[s_i | s_{i-1} = 0] - 1)\]

(Got the expectation back from sum of conditional expectations)

\[= \mathbb{E}[s_{i-1}] + \mathbb{P}(s_{i-1} = 0).1 + \mathbb{P}(s_{i-1} > 0).1 + \mathbb{P}(s_{i-1} < 0).(-1)\]

\[= \mathbb{E}[s_{i-1}] + \mathbb{P}(s_{i-1} = 0).1 + \sum_{j=1}^{i-1} \mathbb{P}(s_{i-1} = j) ( \mathbb{P}(s_{i-1} = j) (n-i+j+1/2) . 1 + \sum_{j=1}^{i-1} \mathbb{P}(s_{i-1} = j) \left( \frac{n-i+j+1/2}{n-i+1} \right).(-1)\]

\[= \mathbb{E}[s_{i-1}] + \mathbb{P}(s_{i-1} = 0) - \sum_{j=1}^{i-1} \mathbb{P}(s_{i-1} = j) \left( \frac{j}{n-i+1} \right)\]

\[= \mathbb{E}[s_{i-1}] + \mathbb{P}(s_{i-1} = 0) - \mathbb{E}(s_{i-1}) \frac{1}{n-i+1}\]

\[= \mathbb{E}[s_{i-1}] (1 - 1/(n-i+1)) \quad \mathbb{P}(s_{i-1} = 0)\]

\[\geq \mathbb{E}[s_{i-1}] (1 - 2/n) + \mathbb{P}(s_{i-1} = 0)\]

(Since \( i \leq n/2 \))

\[= \mathbb{E}[s_{i-1}] (1 - 2/n) + 1 \quad (s_{i-1} = 0 \text{ only if } i - 1 \text{ is even})\]

To compute \( \mathbb{P}(s_{i-1} = 0) \), we first see that it is just the ratio of the number of ways of choosing \((i-1)/2\) positions for +1s in the first \(i-1\) positions and then choosing \((n-i+1)/2\) positions from the remaining \(n-i+1\) positions for the remaining +1s, to the total number of ways of choosing \(n/2\) positions for +1s.
from \( n \) positions. Using this,
\[
E[s_i] \geq E[s_{i-1}](1 - 2/n) + \mathbb{1}_{1-i} \text{ is even} \mathbb{P}(s_{i-1} = 0) \\
= E[s_{i-1}](1 - 2/n) + \mathbb{1}_{1-i} \text{ is even} \frac{(i-1/2)(n-i+1/2)}{(n/2)}
\]
\[
\geq E[s_{i-1}](1 - 2/n) + \mathbb{1}_{1-i} \text{ is even} \frac{8\pi^2}{e^5} \sqrt{\frac{n}{(i-1)(n-i+1)}}
\]
(Using the approximation of the factorial: \( \sqrt{2\pi n^{k+0.5}e^{-k}} \leq k! \leq ek^{k+0.5}e^{-k} \))
\[
\geq E[s_{i-1}](1 - 2/n) + \mathbb{1}_{1-i} \text{ is even} \frac{8\pi^2}{e^5} \sqrt{\frac{1}{i-1}}
\]
\[
= E[s_{i-2}](1 - 2/n)^2 + \mathbb{1}_{1-2} \text{ is even} \frac{8\pi^2}{e^5} \sqrt{\frac{1}{i-2}(1 - 2/n)} + \mathbb{1}_{1-i} \text{ is even} \frac{8\pi^2}{e^5} \sqrt{\frac{1}{i-1}}
\]
\[
\geq E[s_{i-2}](1 - 2/n)^2 + \frac{8\pi^2}{e^5} \sqrt{\frac{T}{i}}(1 - 2/n)
\]
\[
\geq \frac{8\pi^2}{e^5} \sqrt{\frac{T}{i}} \sum_{j=1}^{\lfloor i/2 \rfloor} (1 - 2/n)^{2j-1}
\]
\[
= \frac{8\pi^2}{e^5} \sqrt{\frac{T}{i}} (1 - 2/n) \frac{1 - (1 - 2/n)^{i/2}}{2/n}
\]
\[
\geq \frac{8\pi^2}{e^5} \sqrt{\frac{T}{i}} (1 - 2/n) \frac{1 - 1/2n}{2/n}
\]
\[
= \frac{8\pi^2}{e^5} \sqrt{\frac{T}{i}} \frac{n + 2}{2(n + 1)}
\]
\[
\geq \frac{\pi^2}{e^5} \sqrt{i}
\]

Next we prove that \( \mathbb{P}\left(\sum_{p=1}^{i} \sigma_p^i < 0\right) \geq C_{10}, \mathbb{P}\left(\sum_{p=1}^{i} \sigma_p^i > 0\right) \geq C_{10}, \) for some universal constant \( C_{10}. \) Firstly, we can see that \( \mathbb{P}\left(\sum_{p=1}^{i} \sigma_p^i < 0\right) = P\left(\sum_{p=1}^{i} \sigma_p^i > 0\right) \) by symmetry and hence we need to only an upper bound on \( \mathbb{P}\left(\sum_{p=1}^{i} \sigma_p^i = 0\right). \) Note that we want to prove this for \( i \in \lfloor n/4, n/2\rfloor. \)
\[
\mathbb{P}\left(\sum_{p=1}^{i} \sigma_p^i = 0\right) = \frac{\binom{i}{n/2} \binom{n-i}{(n-i)/2}}{\binom{n}{n/2}}
\]
\[
\leq \frac{e^4}{\pi^2} \sqrt{\frac{n}{i(n-i)}}
\]
(Using approximation of factorial: \( \sqrt{2\pi n^{k+0.5}e^{-k}} \leq k! \leq ek^{k+0.5}e^{-k} \))
\[
\leq \frac{C_n}{\sqrt{n}}
\]
(Since \( n/4 \leq i \leq n/2 \))

For some universal constant \( C_n. \) Therefore for \( n \geq (2C_n)^2, \) this is less than 1/2. Thus, we need to prove that \( \mathbb{P}\left(\sum_{p=1}^{i} \sigma_p^i = 0\right) \) is upper bounded by a constant less than 1 even for \( n < (2C_n)^2. \) Note that the ways of choosing \( n/2 \) positions for +1 out of \( n \) possible positions can be decomposed as the sum over \( j, \) of the number of ways of choosing \( j \) positions for +1 from the first \( i \) positions times the number of ways of choosing \( n-j \) positions for +1 from the remaining \( n-i \) positions. That is,
\[
\binom{n}{n/2} = \sum_{j=0}^{i} \binom{i}{j} \binom{n-i}{n/2-j}
\]
which implies
\[ \sum_{i=0}^{i} (\binom{i}{j}) \binom{n-i}{n/2} = 1 \]
and hence,
\[ \forall j \in [0, i] : \left( \binom{i}{j} \binom{n-i}{n/2} \right) < 1. \]
Therefore there is a universal constant \( C \) defined as
\[ C = \max_{n<(2Cn)^2, i \in [n/4, n/2]} \left\{ \binom{i}{i/2} \binom{n-i}{n/2} \right\} < 1. \]
Hence, \( \mathbb{P} \left( \sum_{i=1}^{i} \sigma^i_p = 0 \right) \) is upper bounded by a universal constant less than 1; specifically, \( \mathbb{P} \left( \sum_{i=1}^{i} \sigma^i_p = 0 \right) \leq \max\{1/2, C\}. \)

**B.7 Proof of Corollary 2**

Lemma 7 says that
\[ \mathbb{E}[|x^i_j - x^i_0|] \leq \sqrt{5iG\alpha} + |x^i_0|\sqrt{2\alpha R\lambda} \]
and Lemma 13 says that \( \mathbb{P} \left( \sum_{p=1}^{i} \sigma^i_p < 0 \right) \geq C_{10}. \) Hence,
\[ \mathbb{P} \left( \sum_{p=1}^{i} \sigma^i_p < 0 \right) \mathbb{E} \left[ |x^i_j - x^i_0| \left| \sum_{p=1}^{i} \sigma^i_p < 0 \right. \right] \leq \mathbb{E}[|x^i_j - x^i_0|] \]
\[ \Rightarrow \mathbb{E} \left[ |x^i_j - x^i_0| \left| \sum_{p=1}^{i} \sigma^i_p < 0 \right. \right] \leq \frac{\mathbb{E}[|x^i_j - x^i_0|]}{\mathbb{P}(\sum_{p=1}^{i} \sigma^i_p < 0)} \]
\[ \Rightarrow \mathbb{E} \left[ |x^i_j - x^i_0| \left| \sum_{p=1}^{i} \sigma^i_p < 0 \right. \right] \leq \frac{\mathbb{E}[|x^i_j - x^i_0|]}{C_{10}}. \]
The other inequality can be proved similarly.