On Useful Conformal Transformations In General Relativity

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Abstract. Local conformal transformations are known as a useful tool in various applications of the gravitational theory, especially in cosmology. We describe some new aspects of these transformations, in particular using them for derivation of Einstein equations for the cosmological and Schwarzschild metrics. Furthermore, the conformal transformation is applied for the dimensional reduction of the Gauss-Bonnet topological invariant in $d = 4$ to the spaces of lower dimensions.

1 Introduction

General Relativity (GR) is a successful relativistic theory of gravitation. According to the existing data, its predictions fit nicely to the most of the tests and the limits of its validity are likely to emerge only in the extreme situations when one has to account for, e.g. quantum effects [1, 2, 3, 4]. The quantum theory of matter fields in curved space-time has achieved many solid results (see e.g. [5, 6, 7], and may be viewed as a source of important applications in cosmology and black hole physics. Some of these achievements are related to the so-called conformal anomaly, which is nothing but the quantum violation of the local conformal symmetry which is imposed at the classical level. In order to illustrate this statement it is sufficient to mention that this anomaly is related to such relevant phenomenon as the black hole evaporation (Hawking effect). The anomaly is also related to the derivation of the low-energy effective action of (super)string - the main candidate to be the unified theory of quantum gravity and other fundamental interactions.

The importance of local conformal symmetry makes natural the idea to include it into the basic course of GR. However, the standard point of view is that such extension of the program is difficult because of the shortage of time. In this article we shall suggest a new possibility that including the conformal transformation into the program may be compatible with certain saving of time and efforts due to the relevant simplification in the derivation of Einstein equations for the metrics related to the most fundamental applications of GR such as cosmological and Schwarzschild solutions. Furthermore, we shall use the conformal transformation to investigate a problem which goes slightly beyond the standard course of GR, that is establish the relations between the topological invariants in four and two dimensional
spaces via the restricted dimensional reduction. It is supposed that this relation will be useful
for those willing to learn more advanced aspects of the gravitational theory.

The paper is organized as follows. In section 2 we shall present a brief review of the main
properties of the curvature tensor (see, e.g. [8, 2, 9] for the introduction). In particular, we
shall prove a simple factorization theorem describing important properties of curvature for
the metric of the special factorized form. In section 3 the list of formulas concerning the
local conformal transformation in the space of an arbitrary dimension is presented. For the
sake of generality we shall include the transformations not only for the curvature tensors but
also for the quadratic invariants built from this tensor and its derivatives. In section 4 the
derivation of the Einstein equations for the cosmological and spherically symmetric metrics is
presented. Section 5 is devoted to establishing the relation between the topological invariants
in dimensions 4, 2 and 3 via the factorization theorem from the section 2 and conformal
transformations. Finally, in the last section we draw our conclusions.

2 Brief review of curvature tensor

Let us start with the properties of the curvature tensor which will be used below. A com-
prehensive introduction can be found in the monographs on GR and Differential Geometry
[8, 2, 4, 3, 9].

The action of covariant derivative on the tensor $T_{\mu_1 \mu_2 ... \nu_1 \nu_2 ...}$ is defined by the relation

$$\nabla_\alpha T_{\mu_1 \mu_2 ... \nu_1 \nu_2 ...} = \partial_\alpha T_{\mu_1 \mu_2 ... \nu_1 \nu_2 ...} + \Gamma^\lambda_{\alpha \lambda} T_{\lambda \mu_2 ... \nu_1 \nu_2 ...} + \Gamma^\mu_{\lambda \lambda} T_{\mu_1 \lambda ... \nu_1 \nu_2 ...} + ...$$

providing tensor transformation rule under the general coordinate transformations. The co-
efficients $\Gamma^\lambda_{\alpha \beta}$ are defined according to

$$\Gamma^\lambda_{\mu \nu} = g^{\lambda \tau} (\partial_\mu g_{\tau \nu} + \partial_\nu g_{\tau \mu} - \partial_\tau g_{\mu \nu}),$$

such that the covariant derivative $\nabla_\alpha$ satisfies the metricity condition $\nabla_\alpha g_{\mu \nu} = 0$ and also
is torsionless $\Gamma^\lambda_{\mu \nu} = \Gamma^\lambda_{\nu \mu}$.

Consider the commutator of the two covariant derivatives over the vector $T^\alpha$. By direct
calculations we find

$$[\nabla_\mu, \nabla_\nu] T^\alpha = \nabla_\mu \nabla_\nu T^\alpha - \nabla_\nu \nabla_\mu T^\alpha = - T^\lambda \cdot R^\alpha_{\lambda \mu \nu},$$

where

$$R^\alpha_{\lambda \mu \nu} = \partial_\nu \Gamma^\alpha_{\lambda \mu} - \partial_\mu \Gamma^\alpha_{\nu \lambda} + \Gamma^\gamma_{\nu \mu} \Gamma^\alpha_{\gamma \lambda} - \Gamma^\gamma_{\nu \lambda} \Gamma^\alpha_{\gamma \mu}$$

is called (Riemann) curvature tensor.
The relevance of the Riemann tensor is due to its universality. One can easily show that the commutator of the two covariant derivatives acting on any tensor is a linear combination of the curvature tensors, e.g.

\[
[\nabla_\mu, \nabla_\nu] W^\alpha_\beta = R^\lambda_{\beta \nu \mu} W^\alpha_\lambda - R^\alpha_{\lambda \nu \mu} W^\lambda_\beta. \tag{5}
\]

The curvature tensor has following algebraic symmetries, which can be better seen on the completely covariant version \( R_{\alpha \beta \mu \nu} = g_{\alpha \gamma} R^\gamma_{\beta \mu \nu} \):

\[
R_{\alpha \beta \mu \nu} = - R_{\alpha \beta \nu \mu} = R_{\beta \alpha \nu \mu} = R_{\mu \nu \alpha \beta}, \tag{6}
\]

where symmetry between the pairs of the indices supplements the antisymmetry in the indices of each pair, and

\[
R_{\alpha \beta \mu \nu} + R_{\alpha \nu \beta \mu} + R_{\alpha \mu \nu \beta} = 0. \tag{7}
\]

It proves useful to define the Ricci tensor as the contraction of the Riemann curvature

\[
R_{\mu \alpha} = R^\beta_{\cdot \alpha \beta \mu} = g^\nu_\beta R_{\nu \alpha \beta \mu}. \tag{8}
\]

The Ricci tensor is symmetric \( R_{\mu \nu} = R_{\nu \mu} \). Further contraction produces the scalar curvature \( R = R^\alpha_\alpha \).

The algebraic properties of the Riemann, Ricci tensors and of the scalar curvature are the same in the Riemann or pseudo-Riemann spaces of any dimension. However, in the spaces of lower dimensions the curvature tensors are more restricted than in the general \( D \)-dimensional case. The extreme case is the \( D = 1 \) space, where all curvatures vanish identically.

In order to understand the possible role of \( D \) better, one has to calculate the number of independent components of the tensors \( R_{\mu \nu \alpha \beta} \) and \( R_{\mu \nu} \). Due to the algebraic symmetries (6), the curvature tensor with 3 equal indices vanish. Therefore, we have only three possible situations:

1). Two couples of equal indices, that is the construction like \( R_{0101} \). The combinatorial calculus tells us that the number of distinct combinations of this type in a \( D \)-dimensional space is \( n_1 = C_D^2 = \frac{D(D-1)}{2} \).

2). One couple of equal indices plus two distinct indices, that is the construction like \( R_{0102} \). The number of distinct combinations here is \( n_2 = D C_{D-1}^2 = \frac{D(D-1)(D-2)}{2} \).

3). Four distinct indices, that is the construction like \( R_{0123} \). In this case we have

\[
n_3 = \frac{2D(D-1)(D-2)(D-3)}{4!}.
\]

Summing up the three expressions, we obtain the number of independent components of the Riemann tensor in \( D \)-dimensional space.

\[
N_D = n_1 + n_2 + n_3 = \frac{D^2(D^2 - 1)}{12}. \tag{9}
\]
Now we are in a position to consider the particular cases of lower dimensions. Let us start from the $D = 2$ case, where only the option 1) takes place. $N_2 = 1$ and we see that the number of independent components of the Riemann tensor equals to the one of the scalar curvature. Since both of them are linear combinations of the partial derivatives of the metric, they must be proportional. Taking the algebraic symmetries into account, we obtain

$$R_{\mu\nu\alpha\beta} = t R (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) ,$$

where $t$ is an unknown coefficient. Making a contraction, we arrive at

$$R_{\mu\alpha} = t R g_{\nu\beta} \quad \text{and} \quad R = 2 t R .$$

Hence $t = 1/2$ and we obtain well-known relations

$$R_{\mu\nu\alpha\beta} = \frac{1}{2} R (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) , \quad R_{\mu\alpha} = \frac{1}{2} R g_{\nu\beta} .$$

(11)

For $D = 3$ space we have, according to (9), six independent components of both Riemann and Ricci tensors. The Riemann tensor is a linear combination of the scalar curvature and the Ricci tensor $R_{\mu\nu}$. Taking into account the algebraic symmetries and making contractions we obtain the relation which always holds for $D = 3$ but not necessary for $D \geq 3$

$$R_{\mu\nu\alpha\beta} = (R_{\mu\alpha} g_{\nu\beta} - R_{\mu\beta} g_{\nu\alpha} + R_{\nu\beta} g_{\mu\alpha} - R_{\nu\alpha} g_{\mu\beta}) - \frac{1}{2} R (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) .$$

(12)

Obviously, the relation (12) means that there is some part of the Riemann tensor which automatically vanish in $D = 3$ but may be non-zero for $D > 3$. The corresponding object is called Weyl tensor and is defined as follows

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - \frac{1}{D - 2} (R_{\mu\alpha} g_{\nu\beta} - R_{\mu\beta} g_{\nu\alpha} + R_{\nu\beta} g_{\mu\alpha} - R_{\nu\alpha} g_{\mu\beta})$$

$$+ \frac{1}{(D - 1)(D - 2)} R (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) .$$

(13)

The Weyl tensor has, by construction, the same algebraic symmetries as the Riemann tensor and also two additional properties

$$g^{\mu\alpha} C_{\mu\nu\alpha\beta}(D) = 0 , \quad C_{\mu\nu\alpha\beta}(D = 3) \equiv 0 .$$

(14)

Another important feature of the Weyl tensor will be discussed in the next section.

The last sort of curvature which we have to define is called Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} .$$

(15)

The Einstein tensor is constructed, exactly as Riemann, Ricci, Weyl tensors and scalar curvature, from the metric and its derivatives. All these tensors are homogeneous functions of the second order in the derivatives. The importance of the Einstein tensor is due to the
fact that the Einstein equations, which define the gravitational fields in General Relativity, have the form

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}, \]  

where \( G \) is the Newton constant and \( T_{\mu\nu} \) is a source term which is called Energy-Momentum Tensor (or stress tensor). In the next sections we shall develop the efficient method to calculate the Einstein tensor and, if necessary, also Riemann, Ricci and Weyl tensors, for the particular metrics of special physical interest.

The metric which provides \( R_{\mu\nu\alpha\beta} = 0 \) are necessary flat. However, there are situations when \( R_{\mu\nu\alpha\beta} \neq 0 \), but some of the reduced versions of curvature vanish. One can distinguish Ricci flat or Weyl flat metrics, which are not flat in the proper sense but provide the vanishing Ricci or Weyl tensors, correspondingly. As we already know, any \( D = 3 \) space is Weyl flat, for example.

In many problems of GR we do not need an arbitrary metric but instead just a particular form with a smaller number of independent components than the general one. Sometimes it is possible to reduce the derivation of the curvature tensors for these particular metrics to the one in a lower dimensional case. Let us now formulate the theorem about the factorized metrics which will prove very useful in what follows.

**Theorem.** Consider the \( D \)-dimensional Riemann or pseudo-Riemann manifold with the coordinates \( x^\mu = (x^a, x^i) \), where \( a, b, c, ... = 1, 2, ..., n \) and \( i, j, k, ... = n+1, n+2, ..., D \). Let us assume that the metric is factorized

\[ g_{\mu\nu} = \begin{pmatrix} g_{ab}(x^a) & 0 \\ 0 & g_{ij}(x^i) \end{pmatrix}. \]  

(17)

For this metric the elements with the mixed indices \( g_{ai} \) are zero, and only the blocks with \( g_{ab} \) and \( g_{ij} \) do not vanish. Moreover, the metric elements in each block depend only on their “own” coordinates \( g_{ab} = g_{ab}(x^a) \) and \( g_{ij} = g_{ij}(x^i) \). The statement is that, for the metric (17), the Riemann tensor is also factorized. This means that all its components with the mixed indices do vanish.

**Proof.** First of all, it clear that the inverse metric will also have the factorized block structure

\[ g^{\mu\nu} = \begin{pmatrix} g^{ab}(x^a) & 0 \\ 0 & g^{ij}(x^i) \end{pmatrix}. \]  

(18)

Consider the Christoffel symbol with the mixed indices, e.g.

\[ \Gamma^a_{ib} = g^{ac} (\partial_i g_{bc} + \partial_b g_{ic} - \partial_c g_{ib}) \]  

(19)

Taking into account the relations \( g_{ic} = 0 \) and \( \partial_i g_{bc} = 0 \), we obtain \( \Gamma^a_{ib} = 0 \). In the same way one can prove that other mixed components vanish

\[ \Gamma^i_{jb} = \Gamma^i_{ab} = \Gamma^a_{ij} = 0. \]  

(20)
Thus, only the “pure” components of the Christoffel symbol $\Gamma^a_{bc}(x^a)$ and $\Gamma^k_{ij}(x^i)$ can be non-zero.

Let us now consider the Riemann tensor with the mixed components. According to the formula (4) it is zero because $i)$ the “mixed” derivatives (e.g. $\partial_i \Gamma^a_{bc}$) equal zero; $ii)$ the $\Gamma$-type terms with the mixed indices can not be constructed from the $\Gamma$-symbols with the “pure” indices.

The last observation is that one can easily extend the factorization theorem for the case of Ricci tensor. The scalar curvature is given by the sum $R(g_{\mu\nu}) = R(g_{ab}) + R(g_{ij})$ and the form of factorization of the Weyl tensor can be easily obtained from its definition (13).

### 3 Local conformal transformations

Many properties of the physically interesting metrics become more apparent and easier to prove if we use the so called local conformal transformation

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) e^{2\sigma(x)}.$$ \hfill (21)

(21) is not the coordinate transformation, it can be seen as an alternative form of parametrizing the metric via the new variables $\bar{g}_{\mu\nu}(x)$ and $\sigma(x)$. Our purpose in this section will be to derive the relevant quantities like curvatures in this new parametrization. Since some of the formulas below are bulky, we shall use the condensed notations $(\nabla\sigma)^2 = g^{\mu\nu}g_{\mu\sigma}g_{\nu\sigma}$ and $(\nabla^2\sigma) = \bar{g}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\sigma$.

The transformation laws for the inverse metric and for the metric determinant have the form

$$g^{\mu\nu} = \bar{g}^{\mu\nu} e^{-2\sigma}, \quad g = \bar{g} e^{2D\sigma}.$$ \hfill (22)

For the Christoffel symbols we have

$$\Gamma^\lambda_{\alpha\beta} = \bar{\Gamma}^\lambda_{\alpha\beta} + \delta \Gamma^\lambda_{\alpha\beta}, \quad \delta \Gamma^\lambda_{\alpha\beta} = \delta^\lambda_\alpha \nabla_{\beta}\sigma + \delta^\lambda_\beta \nabla_{\alpha}\sigma - \bar{g}_{\alpha\beta} \nabla^\lambda\sigma,$$ \hfill (23)
and furthermore
\begin{equation}
R_{\mu\nu} = \bar{R}_{\mu\nu} - \bar{g}_{\mu\nu}(\bar{\nabla}^2\sigma) + (D - 2) \left[ (\bar{\nabla}_\mu\sigma)(\bar{\nabla}_\nu\sigma) - (\bar{\nabla}_\mu\bar{\nabla}_\nu\sigma) - \bar{g}_{\mu\nu}(\bar{\nabla}\sigma)^2 \right]
\end{equation}

and
\begin{equation}
R = e^{-2\sigma} \left[ \bar{R} - 2(D - 1)(\bar{\nabla}^2\sigma) - (D - 1)(D - 2)(\bar{\nabla}\sigma)^2 \right].
\end{equation}

It is easy to see, using (25), (26) and (27), that the Weyl tensor (13) transforms in a most simple possible way
\begin{equation}
C_{\alpha\beta\mu\nu} = \bar{C}_{\alpha\beta\mu\nu}, \quad \text{i.e.} \quad C_{\alpha\beta\mu\nu} = e^{2\sigma} \bar{C}_{\alpha\beta\mu\nu}.
\end{equation}

Consider the transformation of the quadratic contractions of the curvature tensors
\begin{equation}
\sqrt{-g} R^2_{\mu\nu\alpha\beta} = \sqrt{-g} e^{(D-4)\sigma} \left\{ \bar{R}^2_{\mu\nu\alpha\beta} - 2\bar{R}(\bar{\nabla}^2\sigma) + (3D - 4)(\bar{\nabla}^2\sigma)^2 + (D - 2)(\bar{\nabla}\sigma)^4 \right\}
\end{equation}

where we multiplied the expression by \(\sqrt{-g}\) for convenience.

\begin{equation}
\sqrt{-g} R^2_{\mu\nu} = \sqrt{-g} e^{(D-4)\sigma} \left\{ \bar{R}^2_{\mu\nu} - 2\bar{R}(\bar{\nabla}^2\sigma) + (3D - 4)(\bar{\nabla}^2\sigma)^2 + (D - 2)(\bar{\nabla}\sigma)^4 \right\}
\end{equation}

and
\begin{equation}
\sqrt{-g} R^2 = \sqrt{-g} e^{(D-4)\sigma} \left[ \bar{R} - 2(D - 1)(\bar{\nabla}^2\sigma) - (D - 1)(D - 2)(\bar{\nabla}\sigma)^2 \right]^2.
\end{equation}

For the square of the Weyl tensor the transformation is very simple
\begin{equation}
\sqrt{-g} C^2_{\mu\nu\alpha\beta} = \sqrt{-g} e^{(D-4)\sigma} \bar{C}^2_{\mu\nu\alpha\beta}.
\end{equation}

Another important combination of the quadratic in curvature invariants is
\begin{equation}
E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2.
\end{equation}

In \(D = 4\) it is the integrand of the Gauss-Bonnet topological term (or Euler characteristic, see, e.g. [9])
\begin{equation}
\int d^4x \sqrt{-g} E.
\end{equation}
But, even for the \( D \neq 4 \) the expression does not contribute to the propagator of gravitons (traceless and completely transverse modes of the metric perturbations on the flat background). For this reason this term plays very special role in string theory [10]. The conformal transformation of the Gauss-Bonnet integrand is

\[
\sqrt{-g} e^{(D-4)\sigma} \left[ \tilde{E} + 8(D - 3) \tilde{R}^{\mu\nu} (\tilde{\nabla}_\mu \tilde{\nabla}_\nu \sigma - \tilde{\nabla}_\mu \sigma \tilde{\nabla}_\nu \sigma) - 2(D - 3)(D - 4) \tilde{R}(\tilde{\nabla}^2 \sigma)^2 + 4(D - 2)(D - 3)^2 (\tilde{\nabla}^2 \sigma)(\tilde{\nabla}^2 \sigma)^2 - 4(D - 2)(D - 3)(\tilde{\nabla}^2 \sigma)^2 + 4(D - 2)(D - 3)(\tilde{\nabla}^4 \sigma) - 4(D - 3)R(\tilde{\nabla}^2 \sigma) \right].
\]

(35)

The last remaining invariant is the surface term, which has the following transformation rule:

\[
\sqrt{-\bar{g}} (\nabla^2 \bar{R}) = \sqrt{-\bar{g}} e^{(D-4)\sigma} \left[ \nabla^2 \bar{R} - 2(D - 4)R(\nabla^2 \sigma)^2 - 2R \nabla^2 \sigma - 2(D - 1)(D - 6)(\nabla^\mu \sigma)(\nabla_\mu \nabla^2 \sigma) + (D - 6)(\nabla^\mu \sigma)(\nabla_\mu \tilde{R}) + 2(D - 1)(D - 2)(D - 4)\nabla^4 \sigma - 2(D - 1)(D - 2)(D - 4)\nabla^2 \sigma \right].
\]

(36)

In deriving the last expression we used the transformation of the operator \( \nabla^2 \) acting on scalars

\[ \nabla^2 = e^{-2\sigma} [\tilde{\nabla}^2 + (D - 2)(\tilde{\nabla}^\mu \sigma)\tilde{\nabla}_\mu]. \]

It is remarkable that in \( D = 4 \) the following simple relation takes place

\[
\sqrt{-g} \left( E - \frac{2}{3} \nabla^2 \bar{R} \right) = \sqrt{-g} \left( \tilde{E} - \frac{2}{3} \tilde{\nabla}^2 \tilde{R} + 4\Delta_4 \sigma \right),
\]

(37)

where \( \Delta_4 \) is the fourth order Hermitian conformal invariant operator acting on conformally invariant scalar field [11, 12]

\[
\Delta_4 = \nabla^4 + 2 R^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\nu \nabla_\rho \nabla_\sigma - \frac{2}{3} R \nabla^2 + \frac{1}{3} (\nabla^\mu \nabla_\mu R). \]

(38)

In section 4 we shall see that the formula (37) is useful in understanding the relation between the topological terms in \( D = 4 \) and \( D = 2 \).

### 4 Practical use of conformal transformation

Let us consider the practical application of the \( D \)-dimensional conformal transformations derived in the previous section. We shall also use the factorization theorem of section 2.
4.1 Derivation of Einstein equations for the FRW metric

Consider the derivation of the Einstein tensor for the Friedmann-Robertson-Walker (FRW) metric in terms of the conformal time $\eta$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left( d\eta^2 - dl^2 \right),$$  \hspace{1cm} (39)

where (see, e.g. [2])

$$dl^2 = \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2,$$  \hspace{1cm} (40)

where $k = 0, \pm 1$. In order to fit with our notations in section 3 we denote

$$a(\eta) = e^{\sigma(\eta)}, \hspace{1cm} g_{\mu\nu} = \bar{g}_{\mu\nu} e^{2\sigma}.$$  \hspace{1cm} (41)

It is easy to see that the new metric

$$\bar{g}_{\mu\nu} = \text{diag} \left( 1, \frac{1}{1 - kr^2}, -r^2, -r^2 \sin^2 \theta \right)$$  \hspace{1cm} (42)

is factorized and therefore satisfies the conditions of the factorization theorem. One of the consequences is that the FRW metric is Weyl flat. This becomes clear if we remember the transformation rule for the Weyl tensor (28) and also that $C^\alpha{}_{\beta\mu\nu} = 0$ in $D = 3$ dimensions (see also [13]).

The relation between $R_{\mu\nu}$ and $\bar{R}_{\mu\nu}$ is given by Eq. (26) and the relation between $R$ and $\bar{R}$ by eq. (27). In both cases one has to set $D = 4$. Consider, as an example, $R_{00}$. Using (26) we arrive at

$$R_{00} = \bar{R}_{00} - 3a'',$$

where the prime indicates the derivative with respect to $\eta$. Furthermore $\bar{R}_{00} = 0$, because it is a Ricci tensor of a one-dimensional metric (remember factorization theorem). Therefore

$$R_{00} = -3a'' = -\frac{3}{a^2} \left( aa'' - a'^2 \right).$$  \hspace{1cm} (43)

Further calculations involve $\bar{R}_{ij}$, where $i, j = 1, 2, 3$. $\bar{R}_{i0} = 0$ automatically. So, in fact we reduced the $D = 4$ calculations to the simpler $D = 3$ ones. The components of $\bar{R}_{ij}$ can be easily calculated directly (this is technically simpler) or through another conformal transformation in $D = 3$

$$\bar{g}_{ij} = \text{diag} \left( -\frac{1}{1 - kr^2}, -r^2, -r^2 \sin^2 \theta \right) = \gamma_{ij} e^{2\rho(r)}, \hspace{1cm} \rho(r) = \ln r.$$  \hspace{1cm} (44)

It is easy to see that the new metric is again factorized and the calculation can be performed in a metric which is a product of the one-dimensional metric $\gamma_{11}$ and the two-dimensional
metric diag \((\gamma_{22}, \gamma_{33})\). Let us present only a final result \(\bar{R}_{ij} = -2k \bar{g}_{ij}\). After performing contraction, we obtain \(\bar{R} = -6k\). Now we are in a position to derive

\[
R_{i0} = 0 \quad \text{and} \quad R_{ij} = -\bar{g}_{ij} \left(2k + \sigma'' + 2\sigma'^2\right).
\]  

(45)

Then, after applying Eq. (27) with \(D = 4\) we arrive at

\[
R = -6e^{-2\sigma} \left(k + \sigma'' + \sigma'^2\right) = -6 \left(\frac{k}{a^2} + \frac{a''}{a^3}\right).
\]  

(46)

Finally, the result for the Einstein tensor has the form

\[
G_{\eta\eta} = R_{\eta\eta} - \frac{1}{2} R g_{\eta\eta} = 3k + 3 \left(\frac{a'}{a}\right)^2,
\]

\[
G_{\eta i} = 0,
\]

\[
G^i_j = \delta^i_j \left(\frac{k}{a^2} + \frac{2a''}{a^3} - \frac{a'^2}{a^4}\right).
\]  

(47)

The transition to the physical time \(t\) can be easily performed through the tensor transformation rule and the standard relation between the two time units \(dt = a(\eta)d\eta\)

\[
G^t_t = G_{tt} = \left(\frac{d\eta}{dt}\right)^2 G_{\eta\eta} = \frac{3(\dot{a}^2 + k)}{a^2}.
\]

\[
G^i_j = \delta^i_j \left(\frac{k}{a^2} + \frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right).
\]  

(48)

Using the expressions (48) one can easily write down the standard form of the Friedmann equations

\[
\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G \rho}{3}, \quad \frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G P,
\]  

(49)

where \(P\) and \(\rho\) are pressure and energy density of the ideal fluid filling the Universe. These two quantities are related by the equation of state specific for the given type of fluid (matter, radiation, vacuum energy etc). From our point of view, the method of deriving the Friedmann equations presented above is simpler than the standard one. Of course, this is true only if one has calculated the conformal transformations for the Ricci tensor and scalar curvature first, but these calculations are not involved.

### 4.2 Derivation for the Schwarzschild metric

The Schwarzschild metric is another important particular example of the metrics used in GR. It is characterized by spherical symmetry and therefore applies to the description of the gravitational field in the exterior of numerous objects like stars at the different stages of
their evolution. The extreme example is the point-like spherically symmetric mass distribution which corresponds to the black hole solution of GR. The formation and properties of the black holes is extremely interesting subject (see, e.g. the comprehensive monograph [15]). For this reason, and also due to its relative simplicity, the Schwarzschild solution is compulsory element of any course of GR. The consideration of this solution always starts with the derivation of Einstein equations for the corresponding metric

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2 d\Omega, \]

where \[ d\Omega = \sin^2 \theta d\varphi + d\theta^2. \]

The purpose of this section is the perform this calculation in a more economic way\(^1\) using the local conformal transformation described in section 3. But, due to the relative simplicity of this way of calculations, we will be able to consider a little bit more complicated metric

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - e^{2\Phi(r,t)} d\Omega, \]

where \( \Phi(r,t) \) is an arbitrary scalar function. Let us notice that the more general metric (51) is widely used in the black hole physics (see, e.g. [15]), in particular because it has some advantages in the theories with vacuum quantum corrections to the Einstein GR [16]. The particular case (50) may be always achieved by replacing \( \Phi(r,t) = \ln r \) into the corresponding expressions.

Our strategy in calculating the Einstein equations for the metric (51) will be the following. The first step is the conformal transformation

\[ g_{\mu\nu} = \bar{g}_{\mu\nu} e^{2\Phi}. \]

The new metric \( \bar{g}_{\mu\nu} \) is diagonal and factorized

\[ \bar{g}_{ab} = \text{diag} (\bar{g}_{00}, \bar{g}_{11}) , \quad \bar{g}_{ij} = \text{diag} (\bar{g}_{22}, \bar{g}_{33}) , \]

where

\[ \bar{g}_{00} = e^A , \quad \bar{g}_{11} = -e^B , \quad \bar{g}_{22} = -1 , \quad \bar{g}_{33} = -\sin^2 \theta , \]

and

\[ A = A(r,t) = \nu(r,t) - 2\Phi(r,t) , \quad B = B(r,t) = \lambda(r,t) - 2\Phi(r,t) . \]

The relations between the two Ricci tensors and the two scalar curvatures are given by the particular form of (26)

\[ R_{\mu\nu} = \bar{R}_{\mu\nu} - \bar{g}_{\mu\nu}(\nabla^2 \sigma) + 2(\nabla_{\mu}\sigma)(\nabla_\nu \sigma) - 2(\nabla_{\mu} \nabla_{\nu} \sigma) - 2\bar{g}_{\mu\nu}(\nabla \sigma)^2 , \]

\[ R = e^{-2\sigma} \left[ \bar{R} - 6(\nabla^2 \sigma) - 6(\nabla \sigma)^2 \right] . \]

\(^1\)The method equivalent to the one described below was used in the field theory textbook by W. Siegel [14].
According to the factorization theorem of section 2, all components of the curvature tensor with the mixed indices vanish and we need to calculate only the curvature tensors and other quantities which emerge in (55) for the two-dimensional metrics in (53). For the curvatures we have

$$K_{abcd} = \frac{1}{2} K (\bar{g}_{ac} \bar{g}_{bd} - \bar{g}_{ad} \bar{g}_{bc}) , \quad a, b, ... = t, r ,$$  \hspace{1cm} (56)

where $K_{abcd}$ is the Riemann tensor for the metric $\bar{g}_{ab}$ and

$$k_{ijkl} = \frac{1}{2} k (\bar{g}_{ik} \bar{g}_{jl} - \bar{g}_{il} \bar{g}_{jk}) , \quad i, j, ... = \theta, \varphi ,$$  \hspace{1cm} (57)

where $k$ is nothing but the scalar curvature of the $D = 2$ sphere $k = -2$, the negative sign appears due to the negative sign of the metric components in (53).

For the metric $\bar{g}_{ab}$ we obtain, by direct calculation

$$\bar{\Gamma}_{tt} = \frac{\dot{A}}{2} , \quad \bar{\Gamma}_{rt} = \frac{\dot{A}'}{2} e^{A-B} , \quad \bar{\Gamma}_{rt} = \frac{\dot{A}'}{2} ,$$

$$\bar{\Gamma}_{rr} = \frac{\dot{B}}{2} , \quad \bar{\Gamma}_{rr} = \frac{\dot{B}}{2} e^{B-A} , \quad \bar{\Gamma}_{rr} = \frac{\dot{B}'}{2} ,$$  \hspace{1cm} (58)

where the dot stands for $d/dt$ and the prime for the $d/dr$.

After a small algebra we obtain

$$K = \frac{1}{2} e^{-A} (\dot{A} \dot{B} - 2 \dot{B} - \ddot{B}) + \frac{1}{2} e^{-B} (2A'' - A'B' + A'^2) ,$$  \hspace{1cm} (59)

and check that $K_{tt} = K \frac{e^A}{2}$ and $K_{rr} = -K \frac{e^B}{2}$ according to (56). Furthermore

$$\bar{\nabla}_t \bar{\nabla}_t \Phi = \ddot{\Phi} - \frac{1}{2} \dot{A} \dot{\Phi} - \frac{1}{2} A' \Phi' e^{A-B} ,$$

$$\bar{\nabla}_r \bar{\nabla}_r \Phi = \ddot{\Phi}' - \frac{1}{2} B \dot{\Phi}' - \frac{1}{2} \dot{B} \dot{\Phi} e^{B-A} ,$$

$$\bar{\nabla}_t \bar{\nabla}_r \Phi = \ddot{\Phi}' - \frac{1}{2} A' \Phi' - \frac{1}{2} \dot{B} \dot{\Phi} .$$  \hspace{1cm} (60)

and

$$\left( \bar{\nabla} \Phi \right)^2 = e^{-A} \ddot{\Phi}^2 - e^{-B} \Phi'^2 ,$$

$$\bar{\nabla}^2 \Phi = e^{-A} \left( \ddot{\Phi} - \frac{1}{2} \dot{A} \dot{\Phi} + \frac{1}{2} \dot{B} \dot{\Phi} \right) + e^{-B} \left( \frac{1}{2} B \dot{\Phi}' - \frac{1}{2} A' \Phi' - \Phi'' \right) .$$  \hspace{1cm} (61)

Now we are in a position to calculate $R$ and $R_{\mu\nu}$. Using the second equation (55), relation $\bar{R} = K + 2$ and formulas (61), we arrive at

$$R = -2 e^{-2\Phi} + e^{-\nu} \left[ (\ddot{\nu} - \lambda) \left( 2 \dot{\Phi} + \frac{\dot{\lambda}}{2} \right) - \dot{\lambda} - 4 \ddot{\Phi} - 6 \dot{\Phi}^2 \right]$$

$$+ e^{-\lambda} \left[ \nu'' + (2 \Phi' + \frac{\nu'}{2}) (\nu' - \lambda') + 4 \Phi'' + 6 \Phi'^2 \right] .$$  \hspace{1cm} (62)
Similarly, using the first equation (55) we obtain the components of the Ricci tensor and finally of the Einstein tensor. For example,

\[ R_{\theta\theta} = -2e^{-2\Phi} + e^{-\nu} \left[ \Phi \left( \frac{\dot{\nu}}{2} - \frac{\dot{\lambda}}{2} - 2\dot{\Phi} \right) - \ddot{\Phi} \right] + e^{-\lambda} \left[ \Phi'' + \Phi' \left( \frac{\nu'}{2} - \frac{\lambda'}{2} + 2\Phi' \right) \right] \]  

leads to

\[ G^\theta_\theta = R^\theta_\theta - \frac{1}{2} R \delta^\theta_\theta = e^{-\nu} \left[ \Phi + \dot{\lambda} + \dot{\Phi}^2 + \dot{\Phi} \left( \frac{\lambda - \nu}{2} \right) + \frac{\lambda (\dot{\lambda} - \nu + 2\dot{\Phi})}{4} \right] 
+ e^{-\lambda} \left[ \frac{\Phi'}{2} (\lambda' - \nu') - \Phi'' - \Phi'^2 - \frac{\nu''}{2} + \frac{\nu' (\lambda' - \nu')}{4} \right]. \]  

In the special case \( \Phi = \ln r \) (64) boils down into

\[ G^\theta_\theta = e^{-\nu} \left( \frac{\dot{\lambda}^2}{4} - \frac{\dot{\nu} \dot{\lambda}}{4} + \frac{\ddot{\lambda}}{2} \right) + e^{-\lambda} \left( \frac{\nu' \Phi'}{4} - \frac{\nu'^2}{4} - \frac{\nu''}{2} + \frac{\lambda' - \nu'}{r} \right), \]  

in a perfect fit with the well-known result [1].

In a similar way, using (55) and (60) we obtain

\[ G^r_\theta = g^{rr} G_{tr} = g^{rr} R_{tr} - \frac{1}{2} R_{tr}, \]

\[ G^r_r = g^{rr} R_{rr} - \frac{1}{2} R_{rr}, \]

\[ G^t_t = g^{tt} R_{tt} - \frac{1}{2} R_{tt}. \]

In the special case \( \Phi = \ln r \) these expressions become the standard ones [1]

\[ G^r_r = e^{-2\Phi} - e^{-\lambda} \left( \nu' \Phi' + \Phi'^2 \right) + e^{-\nu} \left( 2\ddot{\Phi} + 3\dot{\Phi}^2 - \nu \ddot{\phi} \right), \]  

\[ G^t_t = e^{-2\Phi} + e^{-\nu} \left( \ddot{\Phi} + \dot{\Phi}^2 \right) + e^{-\lambda} \left[ -2\Phi'' + \lambda' \Phi' - 3\Phi'^2 \right]. \]  

Let us remark that if the derivation is performed directly for the usual metric (50), it is technically very simple and definitely less cumbersome than the standard one without conformal transformations.
5 Dimensional reduction of the topological invariant

The purpose of this section is to illustrate the calculational power of the conformal transformation for the theories beyond the GR. Let us use this transformation for deriving the form of the $D = 4$ Gauss-Bonnet topological invariant (34) in the particular metric

$$g_{\mu\nu} = \begin{pmatrix} g_{ab}(x^a) & 0 \\ 0 & e^{2\Phi(x^a)} h_{ij}(x^i) \end{pmatrix}. \quad (70)$$

Here $g_{ab}(x^a)$ and $h_{ij}(x^i)$ are both two-dimensional metrics, but $h_{ij}(x^i)$ has constant scalar curvature $k$ while $g_{ab}(x^a)$ is arbitrary. $\Phi(x^a)$ is a scalar field in the two-dimensional space. The result of this dimensional reduction is supposed to be the topological or surface term in the two-dimensional space with the metric $g_{ab}$, $a, b = 0, 1$. Taking into account the simplicity of the conformal transformation for the combination (37), we shall consider not the proper Gauss-Bonnet term, but instead its linear combination with the surface term

$$S_{top} = \int d^4 x \sqrt{-\bar{g}} \left( E - \frac{2}{3} \nabla^2 R \right). \quad (71)$$

After the conformal transformation $g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot \exp(2\Phi)$, according to (37), we obtain

$$S_{top} = \int d^4 x \sqrt{-\bar{g}} \left( \bar{E} - \frac{2}{3} \bar{\nabla}^2 \bar{R} + 4 \bar{\Delta} \Phi \right). \quad (72)$$

Since the metric $\bar{g}_{\mu\nu}$ is factorized

$$\bar{g}_{\mu\nu} = \begin{pmatrix} \bar{g}_{ab}(x^a) & 0 \\ 0 & h_{ij}(x^i) \end{pmatrix}, \quad (73)$$

we need to calculate all the relevant (and complicated) curvature-squared expressions only for the two-dimensional metrics $g_{ab}(x^a)$ and $h_{ij}$. The quantities based on the metric of the two-dimensional sphere $h_{ij}$ are very simple due to Eq. (11), and the whole problem can be resolved immediately. Let us denote the curvature tensor corresponding to the metric $g_{ab}$ by $\mathcal{K}_{abcd}$ and the curvature tensor corresponding to the metric $g_{\bar{ab}}$ by $\bar{\mathcal{K}}_{abcd}$. Indeed, as always in $D = 2$, we have (11), that is

$$\bar{\mathcal{K}}_{abcd} = \frac{1}{2} \bar{\mathcal{K}} (g_{ac} g_{bd} - g_{ad} g_{bc}) , \quad \bar{\mathcal{K}}_{ab} = \frac{1}{2} \bar{\mathcal{K}} \bar{g}_{ab}$$

$$\mathcal{K}_{abcd} = \frac{1}{2} \mathcal{K} (g_{ac} g_{bd} - g_{ad} g_{bc}) , \quad \mathcal{K}_{ab} = \frac{1}{2} \mathcal{K} g_{ab}. \quad (74)$$

Furthermore, in the $h_{ij}$ sector the Riemann and Ricci tensors are

$$k_{ijkl} = \frac{1}{2} k (h_{ik} h_{jl} - h_{il} h_{jk}) , \quad k_{ij} = \frac{1}{2} k h_{ij}.$$

Let us calculate the elements of the expression (72).

$$\bar{E} = \bar{\mathcal{K}}_{abcd} \bar{\mathcal{K}}_{abcd} - 4 \bar{\mathcal{K}}_{ab} \bar{\mathcal{K}}^{ab} + k_{ijkl} k^{ijkl} - 4 k_{ij} k^{ij} + (\bar{\mathcal{K}} + \bar{\kappa})^2 = 2 k \bar{\mathcal{K}} \quad (75)$$
The covariant derivatives corresponding to the metrics $g_{ab}$ and $\bar{g}_{ab}$ will be denoted by $\mathcal{D}_a$ and $\bar{\mathcal{D}}_a$ correspondingly. When acting on scalar (e.g. $\Psi$), the transformation rule for the d'Alembert operator looks like

$$\mathcal{D}^2\Psi = e^{-2\Phi}\bar{\mathcal{D}}^2\Psi.$$  (76)

For the last term in Eq. (72), it proves useful to rewrite the operator (38) in another form, that shows explicitly that it is a derivative operator,

$$\bar{\Delta}_4 = \bar{\nabla}^4 + 2\bar{\nabla}_\mu\bar{R}^{\mu\nu}\bar{\nabla}_\nu - \frac{2}{3}\bar{\nabla}^\mu\bar{R}\bar{\nabla}_\mu,$$  (77)

where we use the notations $\nabla A = A\nabla + (\nabla A)$.

Using (74), (27), (76) and (77), we obtain

$$\bar{\Delta}_4\Phi = \bar{\mathcal{D}}^4\Phi + \bar{\mathcal{D}}^a\bar{K}\bar{\mathcal{D}}_a\Phi - \frac{2}{3}\bar{\mathcal{D}}^a(k + \bar{K})\bar{\mathcal{D}}_a\Phi$$

$$= e^{2\Phi}\left[\mathcal{D}^2 e^{2\Phi} \mathcal{D}^2\Phi + \frac{1}{3}\mathcal{D}^a e^{2\Phi} \left(\mathcal{K} + 2\mathcal{D}^2\Phi\right) \mathcal{D}_a\Phi - \frac{8}{3}k\mathcal{D}^2\Phi\right]$$  (78)

and

$$\bar{\nabla}^2\bar{R} = e^{2\Phi} \mathcal{D}^2\left[e^{2\Phi} \left(\mathcal{K} + 2\mathcal{D}^2\Phi\right)\right].$$  (79)

Finally, for the modified topological term (72) we find, adding the overall factor of the volume $V_2$ of the space with the the metric $h_{ij}$,

$$S_{top} = V_2 \int d^2x \sqrt{-g(x^a)} \left\{ 2k\mathcal{K} + \frac{4}{3}k\mathcal{D}^2\Phi - \frac{2}{3}\mathcal{D}^2(e^{2\Phi}\mathcal{K}) + \frac{8}{3}\mathcal{D}^2(e^{2\Phi}\mathcal{D}\Phi) + \frac{4}{3}\mathcal{D}^a \left[e^{2\Phi}\left(\mathcal{K} + 2\mathcal{D}^2\Phi\right)\mathcal{D}_a\Phi\right]\right\}.$$  (80)

The first term here is the Einstein-Hilbert action (with the coefficient $2k$) which is indeed topological term in $D = 2$. Other terms are surface integrals which depend on the boundary conditions only. Therefore the expression (80) is a kind of a direct relation between the topological invariants in $D = 4$ and $D = 2$.

If we consider similar dimensional reduction to $D = 3$ space, the topological invariant does not emerge, and we meet only surface integrals. In order to see this one does not need to perform calculations. It is enough to look at (72) and remind that the potential source of the topological terms $\bar{E}$ vanish in $D = 3$, because it coincides with the square of the Weyl term $\bar{C}_{abcd}^2$.

6 Conclusions

We have pedagogically reviewed the local conformal transformations and demonstrated their utility in relatively easy derivation of various quantities related to the gravitational theory.
In particular, using these transformations and the factorization theorem of section 2 one can essentially reduce the amount of work needed to obtain the Einstein equations for the FRW and Schwarzschild metrics. From our point of view, this way of calculation provides certain advantages compared to the direct one, and can be used in the basic courses of GR.

Furthermore, we applied the same scheme of calculation to the fourth order terms in the gravitational action. These terms emerge frequently as quantum corrections to GR, in particular in the frameworks of semiclassical approach to gravity and in string theory. The application of conformal transformations made the calculations fairly easy, while the direct way of deriving the same quantities would require an enormous amount of work. As a result of this calculus we have obtained a direct relation between the topological invariants in $D = 4$ and $D = 2$ spaces. In $D = 3$ case the same procedure reduces the Gauss-Bonnet term to the integral of total derivative and there is no relation to the corresponding topological (Chern-Simons) action.

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