Comments on a boundary-driven open XXZ chain: asymmetric driving and uniqueness of steady states

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Abstract

In this paper, we provide two extensions of recent explicit results on the matrix-product ansatz for the non-equilibrium steady state of a Markovian boundary-driven anisotropic Heisenberg XXZ spin-1/2 chain. We write a perturbative solution for the steady-state density matrix in the system–bath coupling for an arbitrary (asymmetric) set of four spin-flip rates at the two chain ends, generalizing the symmetric-driving ansatz of Prosen (2011 Phys. Rev. Lett. 106 217206). Furthermore, we generalize the exact (non-perturbative) form of the steady state for just two Lindblad channels (spin-up flipping on the left and spin-down flipping on the right) to an arbitrary (asymmetric) ratio of the spin-flipping rates (Prosen 2011 Phys. Rev. Lett. 107 137201). In addition, we also indicate a simple proof of the uniqueness of our steady states.

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1. Introduction

The anisotropic Heisenberg (XXZ) model [1] of $n$ coupled quantum spins 1/2 with the Hamiltonian

$$ H = \sum_{j=1}^{n-1} \left( 2\sigma_j^+ \sigma_{j+1}^- + 2\sigma_j^- \sigma_{j+1}^+ + \Delta \sigma_j^z \sigma_{j+1}^z \right) $$

(1)

can be considered as a prototype of a many-body quantum model of strong interactions. We write Pauli operators on a tensor product space $\mathcal{F}_n = (\mathbb{C}^2)^\otimes_n$ as $\sigma_j^+ = 1_{2j-1} \otimes \sigma^x \otimes 1_{2n-2j}$, with $1_d$ being a $d$-dimensional unit matrix, where $\sigma^\pm = \frac{1}{2} (\sigma^x \pm \sigma^y)$ and $\sigma^{x,y,z}$ are the standard $2 \times 2$ Pauli matrices.

The XXZ model exhibits a rich variety of equilibrium and non-equilibrium physical behaviors. In nature it provides an excellent description of the so-called spin-chain materials [2], and it is believed to provide the key for understanding various collective quantum phenomena in low-dimensional strongly interacting systems, such as magnetic or superconducting transitions in two dimensions. Although equilibrium (thermodynamic) properties of the XXZ chain are well understood in terms of Bethe ansatz [3], as the model represents a paradigmatic example of quantum integrable systems, its non-equilibrium properties at finite temperature are a subject of considerable debate [4].

We consider the non-equilibrium quantum transport model [5] based on a Markovian master equation for the XXZ chain in the Lindblad form [6, 7]

$$ \frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_k 2L_k \rho(t)L_k^\dagger - \{L_k^\dagger L_k, \rho(t)\} $$

(2)

with a set of four boundary-supported Lindblad operators

$$ L_1 = \sqrt{\alpha} \sigma^+_1, \quad L_2 = \sqrt{\beta} \sigma^+_n, \quad L_3 = \sqrt{\gamma} \sigma^-_n, \quad L_4 = \sqrt{\delta} \sigma^+_n, $$

(3)

where $\alpha$, $\beta$, $\gamma$ and $\delta$ represent, respectively, spin-down/spin-up incoherent transition rates at the left and the right boundary of the chain. These are assumed to be the only incoherent processes in the model, whereas its bulk dynamics is fully specified by the Hermitian many-body Hamiltonian $H$.

In particular, we are interested in the non-equilibrium steady state (NESS), with the density operator $\rho_\infty = \lim_{t \to \infty} \rho(t)$ satisfying the fixed point condition

$$ \hat{L} \rho_\infty = 0, $$

(4)
where \( \hat{L} \) is the Liouvillian, decomposed into the unitary and dissipative parts:
\[
\hat{L} := -i \text{ad} \, H + \hat{D},
\]
\[
\hat{D} := \alpha \hat{D}_{\alpha}^{-} + \beta \hat{D}_{\alpha}^{+} + \gamma \hat{D}_{\gamma}^{-} + \delta \hat{D}_{\gamma}^{+},
\]
\[
\hat{D}_{\rho}(\rho) := 2\sigma \rho \sigma^{+} - (\sigma^{+} \sigma, \rho).
\]

The boundary-driven open XXZ model (2) with (1) and (3) can be considered a hybrid classical/quantum Markovian lattice gas (see, e.g., [8] for related ideas), namely the boundary injection/absorption rates are the same as those in some classical Markovian stochastic many-body processes (i.e. simple exclusion processes [9, 10]), while the bulk dynamics is fully coherent. Note that the effect of incoherent processes on the boundary, which works against developing strong macroscopic entanglement in the course of time evolution, also enables the efficient application of Liouville space density-matrix-renormalization-group methods for computing the NESSs for generic (non-solvable) local spin chain Hamiltonians [11]. The applicability of incoherent boundary processes to model the (magnetic) baths can be indeed justified if there is a finite correlation (coherence) length in the microscopic model of the baths.

Below we make a few remarks concerning the recently developed exact solution to the NESS of the boundary-driven open XXZ model [14, 15]. In section 2, we show how the uniqueness of NESS simply follows from a theorem of Evans [12]. In section 3, we extend the zeroth and the first order of the NESS density matrix in the weak-coupling perturbative expansion [14] to the case of arbitrary boundary spin-flipping rates \( \alpha, \beta, \gamma \) and \( \delta \), whereas in section 4 in a similar way we extend the non-perturbative exact solution of [15].

2. Proof of the uniqueness of the steady state

Let us first show that under quite general conditions, the open-XXZ model (2) with (1) and (3) possesses a unique NESS (5), i.e. the fixed point \( \rho_{\infty} \) is independent of the initial state \( \rho(0) \).

We start by noting a theorem of Evans [12] (which is a generalization of [13]) that essentially states that NESS is unique iff the set of operators \( \{ H, L_{1}, L_{2}, \ldots \} \) generates, under multiplication and addition, the entire algebra of (bounded) operators, in our case the Pauli algebra \( \mathcal{B}(\mathcal{F}_{n}) \) of the spin-1/2 chain on \( n \) sites. Indeed, this is true even if we take only the Hamiltonian \( H \) and a single pair of one up-flip and one down-flip Lindblad operators out of four (3), say \( \sigma_{+}^{i} \) and \( \sigma_{-}^{j} \). Note that the scalar prefactors \( \sqrt{\alpha}, \sqrt{\beta} \) are not important for this discussion as we are interested only in the generators of the algebras and not in the operators themselves.

One then observes the following recursive operator identities:
\[
\sigma_{+}^{n} = \frac{1}{2} \sigma_{+}^{i} [\sigma_{+}^{i}, H, \sigma_{-}^{j}],
\]
\[
\sigma_{-}^{j} = -\sigma_{-}^{j-2} - \frac{1}{2} \sigma_{-}^{j-1} [\sigma_{-}^{j-1}, H, \sigma_{+}^{n-1}], \quad j = 3, 4, \ldots, n,
\]
which generate the entire set \( \{ \sigma_{+}^{j}; j = 1, \ldots, n \} \) starting from just \( H \) and \( \sigma_{+}^{1} \). Similarly, \( \{ \sigma_{-}^{j}; j = 1, \ldots, n \} \) are generated by Hermitian adjoints of (8) and (9).

3. Perturbative (weak coupling) solution

We start by considering the weak-coupling regime, where all the four rates are small,
\[
\alpha = \varepsilon a, \quad \beta = \varepsilon b, \quad \gamma = \varepsilon c, \quad \delta = \varepsilon d,
\]
and \( \varepsilon \) is considered to be a small parameter. Let us write the NESS density operator as a formal power series
\[
\rho_{\infty} = \sum_{p=0}^{\infty} (i\varepsilon)^{p} \rho^{(p)}.
\]

Plugging the ansatz (11) into the fixed point condition (4) results in an operator-valued recurrence relation for the sequence \( \{ \rho^{(p)}; p = 0, 1, 2, \ldots \} \)
\[
[H, \rho^{(p)}] = \begin{cases} 
0 & \text{if } p = 0, \\
-\hat{D}_{\rho} \rho^{(p-1)} & \text{if } p = 1, 2, \ldots,
\end{cases}
\]
where \( \hat{D}_{\rho} = a \hat{D}_{\alpha} + b \hat{D}_{\beta} + c \hat{D}_{\gamma} + d \hat{D}_{\delta} \), so that \( \hat{D} \equiv \varepsilon \hat{D}_{\rho} \).

In [14], we have shown that for a particular case of symmetric driving
\[
asym \equiv d_{sym} = \frac{1 - \mu}{2}, \quad b_{sym} \equiv c_{sym} = \frac{1 + \mu}{2},
\]
one can express the zeroth, the first and the second order of the perturbation series
\[
2^{n} \rho^{(0)}_{sym} = \mathds{1},
\]
\[
2^{n} \rho^{(1)}_{sym} = \mu (Z - Z^{\dagger}),
\]
\[
2^{n} \rho^{(2)}_{sym} = \frac{\mu^{2}}{2} (Z - Z^{\dagger})^{2} - \frac{\mu}{2} [Z, Z^{\dagger}],
\]
in terms of a non-Hermitian matrix product operator
\[
Z = \sum_{(s_{1}, \ldots, s_{n})(r_{1}, \ldots, r_{n})(\varepsilon_{1}, \ldots, \varepsilon_{n})} \langle L | A_{s_{1}}^{(1)} \cdots A_{s_{n}}^{(n)} | R \rangle \sigma^{s_{1}} \cdots \sigma^{s_{n}} \sigma^{r_{1}} \cdots \sigma^{r_{n}},
\]
where \( \sigma^{0}_{\infty} = \mathds{1} \). \( A_{s}, A_{s} \) is a triple of near-diagonal matrix operators acting on an auxiliary Hilbert space \( \mathcal{H} \) spanned by an orthonormal basis \( \{ | \lambda \rangle, | R \rangle, | 1 \rangle, | 2 \rangle, \ldots \}:
\[
A_{0} = | L \rangle \langle L | + | R \rangle \langle R | + \sum_{r=1}^{\infty} \cos (r \lambda) | r \rangle \langle r |,
\]
\[
A_{+} = | L \rangle \langle 1 | + c \sum_{r=1}^{\infty} \sin (2 \left( \frac{r+1}{2} \right) \lambda) | r \rangle \langle r+1 |,
\]
\[
A_{-} = | 1 \rangle \langle R | - c^{-1} \sum_{r=1}^{\infty} \sin (2 \left( \frac{r}{2} + 1 \right) \lambda) | r+1 \rangle \langle r |,
\]
where $\lambda = \arccos A \in \mathbb{R} \cup i\mathbb{R}$ and $[x]$ is the largest integer not larger than $x$. The constant $c \in \mathbb{C} - \{0\}$ is arbitrary, but it is perhaps suitable to choose $c = 1$ for $|\Delta| \leq 1$ ($\lambda \in \mathbb{R}$) and $c = i$ for $|\Delta| > 1$ ($\lambda \in i\mathbb{R}$) making the matrices (18) always real. The key property of the $Z$ operator that is proven and used extensively in [14] is the almost-commutation (or conservation law) property

$$[H, Z] = -\sigma^z_1 + \sigma^z_n. \quad (19)$$

We shall proceed to show now that this solution can also be used to express (perturbatively) the leading orders of the NESS density operator for arbitrary driving (arbitrary rates $a, b, c, d$). This we will do by writing quite general ansätze for the zeroth and the first order

$$2^n \rho^{(0)} = (\sigma^0 + v\sigma^z)^{\otimes n} =: R, \quad (20)$$

$$2^n \rho^{(1)} = \mu (Z - Z^\dagger) R, \quad (21)$$

where $v$ (related to an average magnetization in the zeroth order $v = (\sigma^z_1)_{-\infty}^0$) and $\mu$ are still undetermined parameters. We note that the operator $R$ can be expressed in terms of an exponentiated total spin projection,

$$R = \sqrt{1 - v^2} \exp[(\tanh v)M^z], \quad M^z = \sum_{j=1}^n \sigma^z_j, \quad (22)$$

hence, it can be shown to commute with the Hamiltonian (1) and the $Z$-operator (17)$^1$

$$[H, R] = 0, \quad [Z, R] = 0, \quad (23)$$

guaranteeing the zeroth order $p = 0$ condition (12) and making the ordering of the terms in the first order (21) not important.

Plugging the ansätze (20) and (21) into the equation $[H, \rho^{(1)}] = -D\rho^{(0)}$ and using equations (19) and (23), one gets an identity $\mu \sigma^z_1 - \mu \sigma^z_n = \frac{1}{2}v([-b - a - v(a + b))\sigma^z_1 + (d - c - v(c + d))\sigma^z_n - v(b - a - v(a + b) + d - c - v(c + d))\mathbb{I}],$ which immediately fixes the unknown parameters $v, \mu.$ Namely, the average magnetization reads

$$v = \frac{b + d - a - c}{a + b + c + d} \quad (24)$$

and the effective driving is

$$\mu = \frac{2(bc - ad)}{(1 - v^2)(a + b + c + d)}. \quad (25)$$

We note that the ansatz for the second order $\rho^{(2)}$ (16) cannot be extended to general asymmetric boundary conditions in a similar way as the zeroth and the first orders $\rho^{(p)} = R\rho^{(p)}_{\text{sym}}, \quad p \in \{0, 1\}$.

4. The non-perturbative (extreme driving) solution

Let us now focus on a non-perturbative (exact) solution for the case with just two Lindblad channels

$$L_1 = \sqrt{\gamma} \sigma^+_n, \quad L_2 = \sqrt{\gamma} \sigma^-_n. \quad (26)$$

For the symmetric situation $\beta = \gamma =: \varepsilon$, this corresponds to the extreme driving case $\mu = 1$ of (13) which has been solved exactly in [15], namely

$$\rho^{(2)}_{\text{sym}} = \frac{S_n S_n^\dagger}{\text{tr}(S_n S_n^\dagger)}, \quad (27)$$

and $S_n$ is a non-Hermitian matrix product operator

$$S_n = \sum_{(s_1, \ldots, s_n) \in \{0, 1\}^n} \langle 0 | \mathcal{A}^s_{n-1} \cdots \mathcal{A}^s_{2} | 0 \rangle \sigma^{s_1} \otimes \sigma^{s_2} \otimes \cdots \otimes \sigma^{s_n}, \quad (28)$$

where $\sigma^0 = 1_2, \mathcal{A}^s_{0, 1}$ is a triple of near-diagonal matrix operators acting on an infinite-dimensional auxiliary Hilbert space $\mathcal{H}$ spanned by an orthonormal basis $\{0\}, \{1\}, \{2\}, \ldots$:

$$\mathcal{A}^s_0 = |0\rangle \langle 0 | + \sum_{r=1}^{\infty} a^s_r |r\rangle \langle r |, \quad (29)$$

$$\mathcal{A}^s_r = i\varepsilon |1\rangle \langle 1 | + \sum_{r=1}^{\infty} a^s_{-r} |r+1\rangle \langle r |,$$

with matrix elements (writing again $\lambda = \arccos \Delta$)

$$a^s_0 = \cos (r \lambda) + i\frac{\sin (r \lambda)}{2 \sin \lambda},$$

$$a^s_{2k-1} = c \sin (2k \lambda) + i\varepsilon \frac{c \sin ((2k-1) \lambda) \sin (2k \lambda)}{2\cos ((2k-1) \lambda + \tau_{2k-1}) \sin \lambda},$$

$$a^s_{2k} = c \sin (2k \lambda) - i\varepsilon \frac{c \cos (2k \lambda) + \tau_{2k}}{2 \sin \lambda},$$

$$a^s_{2k-1} = -\frac{\sin ((2k-1) \lambda)}{c} + i\varepsilon \frac{\cos ((2k-1) \lambda) + \tau_{2k-1}}{2c \sin \lambda},$$

$$a^s_{2k} = -\frac{\sin (2k \lambda)}{c} - i\varepsilon \frac{\sin (2k \lambda) \sin (2k \lambda) + \tau_{2k}}{2c \cos (2k \lambda) + \tau_{2k}} \sin \lambda.$$

The constant $c \in \mathbb{C} - \{0\}$ and signs $\tau_r \in \{\pm 1\}$ are arbitrary, i.e. all choices of $c, \tau_r$, for $r = 1, 2, \ldots$, give identical operator $S_n$ (28).

The crucial ingredient in the proof of [15] was the following recursive identity satisfied by operators $S_n$:

$$[H, S_n] = -i\varepsilon (\sigma^z \otimes S_{n-1} - S_{n-1} \otimes \sigma^z), \quad (31)$$

which can be understood as a ‘non-perturbative’ analog of the commutator relation (19).

We shall now show how one can generalize the ansatz (27) and (28) in order to incorporate the asymmetric driving (26) for any rates $\beta > \gamma$. We start by recognizing the following non-unitary symmetry of the XXZ dynamics:
Lemma: Let $\nu \in (-1, 1)$ be a real parameter and $\hat{V}_1 : B(\mathcal{F}_1) \to B(\mathcal{F}_1)$ a non-unitary, but non-degenerate linear map of a set of $2 \times 2$ matrices onto itself, which is completely specified by its action on the Pauli basis $\hat{\sigma}^t = \hat{V}_1(\hat{\sigma}^t)$, $s \in \{0, +, -, z\}$:
\begin{align}
\hat{\sigma}^z &= \sigma^z, \\
\hat{\sigma}^0 &= \sigma^0 - \nu \sigma^z, \\
\hat{\sigma}^z &= \frac{1}{1 - \nu^2}(\sigma^z - \nu \sigma^0).
\end{align}

$\hat{V}_1$ induces a one-to-one linear map $\hat{V} = \hat{V}_1^{\otimes n}$ of $n$-spin Pauli algebra $B(\mathcal{F}_n)$, which is completely specified by $\hat{V}(\sigma^s \otimes \sigma^s \cdots \sigma^s) = \hat{\sigma}^s \otimes \hat{\sigma}^s \cdots \otimes \hat{\sigma}^s$. Then, $\hat{V}$ commutes with the Heisenberg dynamics of the XXZ chain (1), i.e. for any $x \in B(\mathcal{F}_n)$:
\begin{equation}
[H, \hat{V}(x)] = \hat{V}(H[x]),
\end{equation}

Proof. Writing the Hamiltonian (1) as a sum of two-body terms $H = \sum_{j=1}^n h_{j,j+1}$, a sufficient condition for (33) to hold is $[h_{j,j+1}, \hat{V}(x)] = \hat{V}(h_{j,j+1})$. Due to the linearity of this relation in $x$, the latter can be considered to be of the form $x = x_1 \otimes x_2 \otimes \cdots \otimes x_n$, where the first, or the last, factor is taken as trivial if $j = 1$, or $j = n-1$, respectively. Proving the lemma is then equivalent to showing (33) for $n = 2$, i.e.
\begin{equation}
[h, \hat{\sigma}^t \otimes \hat{\sigma}^t] = \hat{V}([h, \sigma^t \otimes \sigma^t]),
\end{equation}

where $h = 2\sigma^+ \otimes \sigma^+ + 2\sigma^- \otimes \sigma^- + \Delta \sigma^z \otimes \sigma^z$. This follows from observing
\begin{align}
[h, \sigma^t \otimes \sigma^0] &= 2\Delta \sigma^+ \otimes \sigma^+ - 2\sigma^+ \otimes \sigma^+, \\
[h, \sigma^0 \otimes \sigma^t] &= 2\Delta \sigma^+ \otimes \sigma^+ - 2\sigma^0 \otimes \sigma^+, \\
[h, \sigma^t \otimes \sigma^-] &= 2\Delta \sigma^+ \otimes \sigma^+ - 2\sigma^0 \otimes \sigma^+, \\
[h, \sigma^0 \otimes \sigma^-] &= 4\sigma^+ \otimes \sigma^- - 4\sigma^- \otimes \sigma^+,
\end{align}

for $s \in \{0, +, -, z\}$, together with related Hermitian conjugate identities and identities with swapped tensor factors, and checking exactly identical identities with $\sigma^t$ replaced by $\hat{\sigma}^t$. \hfill \square

We note that the non-unitary symmetry map $\hat{V}$ is not a canonical transformation, namely $\hat{\sigma}^t$ do not satisfy the same commutation relations as the Pauli matrices $\sigma^t$.

Now, we are in a position to state the main result:

**Theorem 1.** The unique NESS of the flow (2) with (1) and (26) can be written as
\begin{equation}
\rho_{\infty} = \frac{\hat{S}_n^{\dagger} R \hat{S}_n^{\dagger} R}{\mathrm{tr}(\hat{S}_n^{\dagger} R \hat{S}_n^{\dagger} R)},
\end{equation}

where $R = (\sigma^0 + \nu \sigma^z)^{\otimes n}$ is the (unnormalized) weak-coupling limit (20), and the operators $\hat{S}_n$ are given in terms of the same form of the MPO as in the symmetric case (28)
\begin{equation}
\hat{S}_n = \sum_{(i_1, \ldots, i_n) \in \{+, -, 0\}^n} (0| \hat{A}'_{i_1} \hat{A}'_{i_2} \cdots \hat{A}'_{i_n} |0) \hat{\sigma}^{i_1} \otimes \hat{\sigma}^{i_2} \cdots \otimes \hat{\sigma}^{i_n},
\end{equation}

but in a different basis (32), i.e. $\hat{S}_n = \hat{\mathcal{V}}(\hat{S}_n)$, with the magnetization parameter:
\begin{equation}
\nu = \frac{\beta - \gamma}{\beta + \gamma},
\end{equation}

and the effective coupling $\varepsilon$ (to be used in explicit expressions (29) and (30) for the triple of matrices $\hat{A}'_{i}, \hat{A}'_{-}, \hat{A}'_{0}$):
\begin{equation}
\varepsilon = \frac{2\beta \gamma}{\beta + \gamma}.
\end{equation}

Proof. The uniqueness of the fixed point $\rho_{\infty}$ has been shown also for the restricted case of deriving (26), in section 2.

We need to show that (36) satisfies equation (4), which, together with $[R, \hat{S}_n] = 0$ again following from the fact that all the Pauli terms of (37) contain the same number of $\sigma^+ \equiv \sigma^+$ and $\sigma^- \equiv \sigma^-$ tensor factors, is equivalent to the condition
\begin{equation}
i[H, \hat{S}_n^{\dagger} R] = \hat{D}(\hat{S}_n^{\dagger} R) R^{-1}.
\end{equation}

Note that, since $\hat{S}_n = \hat{\mathcal{V}}(\hat{S}_n)$, the lemma implies
\begin{equation}
[H, \hat{S}_n] = -i\delta \left((\sigma^z - \nu \sigma^0) \otimes \hat{S}_n^{-1} \otimes \hat{S}_n^{-1} \otimes (\sigma^z - \nu \sigma^0) \right),
\end{equation}

where $\delta = \varepsilon/(1 - \nu^2)$ and $\delta$ is still unspecified. Following the idea of the proof of [15], we can use the algebraic relations among $\hat{A}'_{i}$ to write $\hat{S}_n := \sigma^0 \otimes \hat{S}_n^{-1} + \hat{\sigma} \otimes \hat{P}_n^{-1} := \hat{S}_n^{-1} \otimes \sigma^0 + \hat{Q}_n^{-1} \otimes \sigma^0$, defining $\hat{Q}_n^{-1}$ and $\hat{P}_n^{-1}$ as some linear operators over $\mathcal{F}_{n-1}$. This merely expresses the fact that in $\hat{S}_n$ the first $\sigma^t$ always comes to the left of all $\sigma^-$, and the last $\sigma^-$ comes to the right of all $\sigma^+$. Straightforward calculation, employing (41) and definitions of $\hat{P}_n, \hat{Q}_n$, then expresses the left-hand side of (40) as
\begin{equation}
\hat{S}_n \left[2(\sigma^+ - \nu \sigma^0) \otimes \hat{S}_n^{-1} \hat{S}_n^{-1} \otimes (\nu + 1) \sigma^+ \otimes \hat{P}_n^{-1} \hat{S}_n^{-1} \right. \\
- (\nu + 1) \sigma^- \otimes \hat{S}_n^{-1} \hat{P}_n^{-1} \otimes (\nu - 1) \sigma^- \otimes \hat{P}_n^{-1} \hat{S}_n^{-1} \otimes (\sigma^z - \nu \sigma^0) \\
+ (\nu - 1) \hat{Q}_n^{-1} \hat{S}_n^{-1} \otimes \sigma^- \otimes (\nu - 1) \hat{S}_n^{-1} \hat{Q}_n^{-1} \otimes \sigma^+ \right].
\end{equation}

For the right-hand side of (40), we use the fact that the map
\begin{equation}
\rho \to \hat{D}(\rho R) R^{-1}
\end{equation}

non-trivially acts only locally, on sites $j = 1$ and $j = n$, yielding
\begin{align}
&\frac{2\beta}{1 + \nu} (\sigma^z - \nu \sigma^0) \otimes \hat{S}_n^{-1} \hat{S}_n^{-1} = -\beta \sigma^+ \otimes \hat{P}_n^{-1} \hat{S}_n^{-1} \\
- \beta \sigma^- \otimes \hat{S}_n^{-1} \hat{P}_n^{-1} = -\frac{2\gamma}{1 - \nu} \hat{S}_n^{-1} \hat{S}_n^{-1} \otimes (\sigma^z - \nu \sigma^0) \\
- \gamma \hat{Q}_n^{-1} \hat{S}_n^{-1} \otimes \sigma^- \otimes \hat{S}_n^{-1} \hat{Q}_n^{-1} \otimes \sigma^+.
\end{align}

Indeed, these two expressions, (42) and (44), are linear combinations of exactly the same terms. Comparing them term-wise results in conditions $\delta = \beta/(1 + \nu) = \gamma/(1 - \nu)$, yielding, with $\delta = (1 - \nu^2)\varepsilon$, exactly expressions (38) and (39). \hfill \square
5. Conclusion

Computation of expectation values of physical observables in NESS, based on our new results for generalized boundary conditions, can be facilitated in terms of the transfer matrices, which can be constructed exactly as in [14, 15]. The effect of the asymmetric boundary conditions is essentially an offset of average magnetization, which is a consequence of the non-unitary symmetry $\hat{V}$ described in the lemma of section 4.

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References

[1] Heisenberg W 1928 Z. Phys. 49 619
[2] Sologubenko A V, Lorentz T, Ott H R and Freimuth A 2007 J. Low Temp. Phys. 147 387
[3] Klümp A 1993 Z. Phys. B 91 507
[4] Takahashi M 1999 Thermodynamics of One-Dimensional Solvable Models (Cambridge: Cambridge University Press)
[5] Giamarchi T 2004 Quantum Physics in One Dimension (Oxford: Clarendon)
[6] Sirker J, Pereira R G and Affleck I 2009 Phys. Rev. Lett. 103 216602
[7] Sirker J, Pereira R G and Affleck I 2011 Phys. Rev. B 83 035115
[8] Langer S, Heidrich-Meisner F, Gemmer J, McCulloch I P and Schollwöck U 2009 Phys. Rev. B 79 214409
[9] Jesenko S and Žnidarič M 2011 Phys. Rev. B 84 174438
[10] Wichterich H, Heinrich M-J, Breuer H-P, Gemmer J and Michel M 2007 Phys. Rev. E 76 031115
[11] Lindblad G 1976 Commun. Math. Phys. 48 119
[12] Gorini V, Kossakowski A and Sudarshan E C G 1976 J. Math. Phys. 17 821
[13] BreuerH-P and Petruccione F 2002 The Theory of Open Quantum Systems (New York: Oxford University Press)
[14] Eisler V 2011 J. Stat. Mech. P06007
[15] Blythe R A and Evans M R 2007 J. Phys. A: Math. Theor. 40 R333
[16] Schütz G M 2001 Exactly solvable models for many-body systems far from equilibrium Phase Transitions and Critical Phenomena vol 19 ed C Domb and J L Lebowitz (London: Academic) pp 1–251
[17] Prosen T and Žnidarič M 2009 J. Stat. Mech. P02035
[18] Evans D E 1977 Commun. Math. Phys. 54 293
[19] Spohn H 1977 Lett. Math. Phys. 2 33
[20] Prosen T 2011 Phys. Rev. Lett. 106 217206
[21] Prosen T 2011 Phys. Rev. Lett. 107 137201