Random Matrix Theory for Transition Strength Densities in Finite Quantum Systems: Results from Embedded Unitary Ensembles

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Abstract

Embedded random matrix ensembles are generic models for describing statistical properties of finite isolated interacting quantum many-particle systems. For the simplest spinless fermion (or boson) systems, with say $m$ fermions (or bosons) in $N$ single particle states and interacting via $k$-body interactions, we have EGUE($k$) [embedded GUE of $k$-body interactions] with GUE embedding and the embedding algebra is $U(N)$. A finite quantum system, induced by a transition operator, makes transitions from its states to the states of the same system or to those of another system. Examples are electromagnetic transitions (then the initial and final systems are same), nuclear beta and double beta decay (then the initial and final systems are different), particle addition to or removal from a given system and so on. Towards developing a complete statistical theory for transition strength densities (transition strengths multiplied by the density of states at the initial and final energies), we have derived formulas for the lower order bivariate moments of the strength densities generated by a variety of transition operators. Firstly, for a spinless fermion system, using EGUE($k$) representation for a Hamiltonian that is $k$-body and an independent EGUE($t$) representation for a transition operator that is $t$-body and employing the embedding $U(N)$ algebra, finite-$N$ formulas for moments up to order four are derived, for the first time, for the transition strength densities. Secondly, formulas for

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the moments up to order four are also derived for systems with two types of spinless fermions and a transition operator similar to beta decay and neutrinoless beta decay operators. In addition, moments formulas are also derived for a transition operator that removes $k_0$ number of particles from a system of $m$ spinless fermions. In the dilute limit, these formulas are shown to reduce to those for the EGOE version derived using the asymptotic limit theory of Mon and French [Ann. Phys. (N.Y.) 95 (1975) 90]. Numerical results obtained using the exact formulas for two-body ($k = 2$) Hamiltonians (in some examples for $k = 3$ and 4) and the asymptotic formulas clearly establish that in general the smoothed (with respect to energy) form of the bivariate transition strength densities take bivariate Gaussian form for isolated finite quantum systems. Extensions of these results to bosonic systems and EGUE ensembles with further symmetries are discussed.

**Keywords:** Finite many-particle quantum systems, Embedded ensembles, Transition strengths, Bivariate moments, $U(N)$ Wigner-Racah algebra, Asymptotics, Bivariate Gaussian

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1. **Introduction**

Wigner introduced random matrix theory (RMT) in physics in 1955 primarily to understand statistical properties of neutron resonances in heavy nuclei [1, 2]. Depending on the global symmetry properties of the Hamiltonian of a quantum system, namely rotational symmetry and time-reversal symmetry, we have Dyson’s tripartite classification of random matrices giving the classical random matrix ensembles, the Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE) ensembles. In the last three decades, RMT has found applications not only in all branches of quantum physics but also in many other disciplines such as econophysics, wireless communication, information theory, multivariate statistics, number theory, neural and biological networks and so on [3–7]. However, in the context of isolated finite many-particle quantum systems, classical random matrix ensembles are too unspecific to account for important features of the physical system at hand. One refinement which retains the basic stochastic approach but allows for such features consists in the use of embedded random matrix ensembles [8–13].

Finite quantum systems such as nuclei, atoms, quantum dots, small
metallic grains, interacting spin systems modeling quantum computing core and ultracold atoms, share one common property—their constituents predominantly interact via two-particle interactions. Therefore, it is more appropriate to represent an isolated finite interacting quantum system, say with \( m \) particles (fermions or bosons) in \( N \) single particle (sp) states by random matrix models generated by random \( k \)-body (note that \( k < m \) and most often we have \( k = 2 \)) interactions and propagate the information in the interactions to many \( (m) \) particle spaces. Then we have random matrix ensembles in \( m \)-particle spaces—these ensembles are defined by representing the \( k \)-particle Hamiltonian \((H)\) by GOE/GUE/GSE and then the \( m \) particle \( H \) matrix is generated by the \( m \)-particle Hilbert space geometry. The key element here is the recognition that there is a Lie algebra that transports the information in the two-particle spaces to the many-particle spaces. As a GOE/GUE/GSE random matrix ensemble in two-particle spaces is embedded in the \( m \)-particle \( H \) matrix, these ensembles are more generically called embedded ensembles (EE).

Embedded ensembles have proved to be rich in their content and wide in their scope. A book giving detailed discussion of the various properties and applications of a wide variety of embedded matrix ensembles is now available [13]. Significantly, the study of embedded random matrix ensembles is still developing. Partly this is due to the fact that deriving generic properties of these ensembles is not mathematically tractable and this is the topic of the present paper. A general formulation for deriving exact analytical results is to use the Wigner-Racah algebra of the embedding Lie algebra [13]. Till now, this has has been applied only in the study of one- and two-point functions in eigenvalues. The focus in the present paper is on transition strengths which on one hand probe the structure of the eigenfunctions of a quantum many-body system and on the other are important ingredients in many applications (for example, beta decay transition strengths are essential for nucleosynthesis studies). Also, as emphasized in [2, 8], there are many open questions in the random matrix theory for transition strengths in finite interacting quantum many-particle systems.

For finite quantum many-particle systems, induced by a transition operator, a given system makes transitions from its states to the states of the same system or to the states of another system. Examples are electromagnetic transitions (then the initial and final systems are same), nuclear beta and double beta decay (then the initial and final systems are different), particle addition to or removal from a given system and so on. Given a transition
operator $\mathcal{O}$ acting on the $m$-particle eigenstates $|E\rangle$ of $H$ will give transition matrix elements $|\langle E_f | \mathcal{O} | E_i \rangle|^2$. Fig. 1 shows a schematic picture of transition strengths. In the statistical theories, it is more useful to deal with the corresponding transition strength density (this will take into account degeneracies in the eigenvalues) defined by

$$I_{\mathcal{O}}(E_i, E_f) = I(E_f) \langle \langle E_f | \mathcal{O} | E_i \rangle \rangle^2 I(E_i).$$

In Eq. (1), $I(E)$ are state densities normalized to the dimension of the $m$-particle spaces. Note that $E_i$ and $E_f$ belong to the same $m$-particle system or different systems depending on the nature of the transition operator $\mathcal{O}$. In the discussion ahead, we will consider both situations. Random matrix theory has been used in the past to derive the form of the smoothed $I(E)$. In particular, exact (finite $N$) formulas for lower order moments $\langle H^p \rangle$, $p = 2$ and 4 of $I(E)$ are derived both for EGOE and EGUE ensembles using group theoretical methods directly [14] or indirectly [15]. Let us add that some results valid only in the asymptotic limit (essentially $N \to \infty$, $m \to \infty$, $m/N \to 0$) are also available in literature for the state densities [16, 17]. Going beyond the eigenvalue densities, here we will apply group theoretical methods and derive for the first time some exact formulas for the ensemble averaged lower order moments of the transition strength densities $I_{\mathcal{O}}(E_i, E_f)$. These then will give the general form of the smoothed (with respect to both $E_i$ and $E_f$) transition strength densities. We will focus on fermionic systems and at the end, discuss the extension to bosonic systems. It is important to mention that in the present paper fluctuations in transition strengths are not considered and they will be addressed in a future publication. For some discussion on strength fluctuations see [2, 8, 9]. More importantly, using smoothed transition strengths statistical spectroscopy analysis of transition strengths (for example for particle transfer, electromagnetic transitions, beta decay and even double beta decay matrix elements) in nuclei and also in atoms is possible [18, 19]. In addition, smoothed form of the strength densities can be used to calculate transition strengths that are needed for example in astrophysical applications [20, 21] and for neutrinoless double beta decay [19, 22, 23]. These formed the main motivation for the present study. Let us add that in the past, besides deriving the dilute limit formulas for the bivariate moments of the transition strength densities in some situations [24, 25], there are suggestions of using a polynomial expansion theory [26] (later in [24] it was shown that the polynomial expansion starts with the GOE result and hence in general inappropriate) for transition strengths, a specialized
theory for one-body transition operators \cite{18, 27–29} and using the bivariate t-distribution form for transition strength densities \cite{30}.

In general, the Hamiltonian may have many symmetries with the fermions (or bosons) carrying additional degrees of freedom such as spin, orbital angular momentum, isospin and so on. Also, the system may comprise of different types of fermions (or bosons); for example, in atomic nuclei we have protons and neutrons. In addition, a transition operator may preserve particle number and other quantum numbers or it may change them. Of all these various situations, here we will consider three different systems in detail. (i) Firstly, we will consider a system of spinless fermions and a transition operator that preserves particle number. (ii) Secondly, we will consider a system with two types of spinless fermions with the transition operator changing \( k_0 \) number of particles of one type to \( k_0 \) number of particles of other type as in nuclear beta decay and neutrinoless double beta decay. (iii) Third system considered is transition operators that remove (add) say \( k_0 \) number of particles from (to) a system of \( m \) spinless fermions. In all these we will restrict to EGUE. Before giving a short preview, let us mention that some of the results in the present paper are reported in three conferences earlier \cite{31–33} and in a thesis \cite{25}.

Section 2 gives some basic results for EGUE(\( k \)) for spinless fermion systems as derived in \cite{14} and some of their extensions. Using these results, formulas for the lower order bivariate moments of the transition strength densities for the situation (i) above are derived and they are presented in detail in Section 3. The corresponding asymptotic limit formulas are presented in Section 4. Similarly, results for the situation (ii) above are given in detail in Section 5 and their asymptotics are given in Section 6. In addition, in Section 7, results for the situation (iii) are given along with the corresponding asymptotic formulas. Extensions to bosonic systems is discussed briefly in Section 8. Finally, Section 9 gives conclusions and future outlook.

2. Basic EGUE(\( k \)) results for a spinless fermion system

Let us consider \( m \) spinless fermions in \( N \) degenerate sp states with the Hamiltonian \( \hat{H} \) a \( k \)-body operator,

\[
\hat{H} = \sum_{i,j} V_{ij}(k) A_i^\dagger(k) A_j(k) , \quad V_{ij}(k) = \left\langle k, i | \hat{H} | k, j \right\rangle .
\] (2)
Here $A_i^\dagger(k)$ is a $k$ particle (normalized) creation operator and $A_i(k)$ is the corresponding annihilation operator (a hermitian conjugate). Also, $i$ and $j$ are $k$-particle indices. Note that the $k$ and $m$ particle space dimensions are $\binom{N}{k}$ and $\binom{N}{m}$ respectively. We will consider $\hat{H}$ to be EGUE($k$) in $m$-particle spaces. Then $V_{ij}$ form a GUE with $V$ matrix being Hermitian. The real and imaginary parts of $V_{ij}$ are independent zero centered Gaussian random variables with variance satisfying,

$$V_{ab}(k) V_{cd}(k) = V_H^2 \delta_{ad} \delta_{bc}.$$  

Here the ‘overline’ indicates ensemble average. From now on we will drop the hat over $H$ and denote, when needed, $H$ by $H(k)$. Let us add that in physical systems, $k = 2$ is of great interest and in some systems such as atomic nuclei it is possible to have $k = 3$ and even $k = 4$ [34, 35].

The $U(N)$ algebra that generates the embedding, as shown in [14], gives formulas for the lower order moments of the one-point function, the eigenvalue density $I(E) = \langle \langle \delta(H - E) \rangle \rangle$ and also for the two-point function in the eigenvalues. In particular, explicit formulas are given in [13, 14] for $\langle H^P \rangle^m$, $P = 2, 4$ and $\langle H^P \rangle^m \langle H^Q \rangle^m$, $P + Q = 2, 4$. Used here is the $U(N)$ tensorial decomposition of the $H(k)$ operator giving $\nu = 0, \ldots, k$ irreducible parts $B^\nu(k)$ and then,

$$H(k) = \sum_{\nu=0}^{k} W_{\nu,\omega^\nu}(k) B^\nu(k).$$  

With the GUE($k$) representation for the $H(k)$ operator, the expansion coefficients $W$'s will be independent zero centered Gaussian random variables with, by an extension of Eq. (3),

$$W_{\nu_1,\omega^{\nu_1}}(k) W_{\nu_2,\omega^{\nu_2}}(k) = V_H^2 \delta_{\nu_1,\nu_2} \delta_{\omega^{\nu_1},\omega^{\nu_2}}.$$  

For deriving formulas for the various moments, the first step is to apply the Wigner-Eckart theorem for the matrix elements of $B^\nu(k)$. Given the $m$-fermion states $|f_m v_i\rangle$, we have with respect to the $U(N)$ algebra, $f_m = \{1^m\}$, the antisymmetric irreducible representation (irrep) in Young tableaux notation and $v_i$ are additional labels. Note that $\nu$ introduced above corresponds to the Young tableaux $\{2^\nu 1^{N-2\nu}\}$ and $\omega^\nu$ are additional labels. Now, Wigner-Eckart theorem gives

$$\langle f_m v_f | B^{\nu,\omega^{\nu}}(k) | f_m v_i \rangle = \langle f_m || B^{\nu}(k) || f_m \rangle C^{\nu,\omega^{\nu}}_{f_m v_f, f_m v_i}.$$  

6
Here, \( \langle \ldots || \ldots || \ldots \rangle \) is the reduced matrix element and \( C_{\ldots \ldots} \) is a \( U(N) \) Clebsch-Gordan (C-G) coefficient [note that we are not making a distinction between \( U(N) \) and \( SU(N) \)]. Also, if \( |f_m v_i\rangle \) represents a \( m \)-particle state, then \( \overline{|f_m v_i\rangle} \) represents a \( m \)-hole state (see [14] for details). In Young tableaux notation \( \overline{f_m} = \{1^{N-m}\} \). It is important to mention that \( \nu \omega \nu ') = |\nu \omega \nu ' \rangle \). Fig. 2 shows some typical Young tableaux and also the Young tableaux \( \overline{f} \) that corresponds to a given \( \{f\} \). Definition of \( B^{\nu \omega \nu '} (k) \) and the \( U(N) \) Wigner-Racah algebra will give,

\[
\langle f_m || B^{\nu} (k) || f_m \rangle^2 = \Lambda^{\nu} (N, m, m - k),
\]

\[
\Lambda^{\nu} (N', m', r) = \left( \begin{array}{c} m' - \mu' \\ r \end{array} \right) \left( \begin{array}{c} N' - m' + r - \mu \\ r \end{array} \right).
\]

Note that \( \Lambda^{\nu} (N, m, k) \) is nothing but, apart from a \( N \) and \( m \) dependent factor, a \( U(N) \) Racah coefficient [14]. This and the various properties of the \( U(N) \) Wigner and Racah coefficients give two formulas for the ensemble average of a product of any two \( m \)-particle matrix elements of \( H \),

\[
\langle f_m v_1 | H (k) | f_m v_2 \rangle \langle f_m v_3 | H (k) | f_m v_4 \rangle
\]

\[
= V_H^2 \sum_{\nu = 0}^{\nu'} \Lambda^{\nu} (N, m, m - k) C^{\nu \omega \nu'}_{f_m v_1, f_m v_2} C^{\nu' \omega \nu}_{f_m v_3, f_m v_4},
\]

and also

\[
\langle f_m v_1 | H (k) | f_m v_2 \rangle \langle f_m v_3 | H (k) | f_m v_4 \rangle
\]

\[
= V_H^2 \sum_{\nu = 0}^{\nu'} \Lambda^{\nu} (N, m, k) C^{\nu \omega \nu'}_{f_m v_1, f_m v_2} C^{\nu' \omega \nu}_{f_m v_3, f_m v_4}.
\]

Eq. (9) follows by applying a Racah transform to the product of the two C-G coefficients appearing in Eq. (8). Let us mention some properties of the \( U(N) \) C-G coefficients that are quite useful in deriving the formulas given in Sections 3 and 5,

\[
\sum_{v_i} C^{\nu \omega \nu \nu'}_{f_m v_i, f_m v_i} = \left( \begin{array}{c} N \\ m \end{array} \right)^{1/2} \delta_{\nu,0}, \quad C^{0 \nu \omega \nu'}_{f_m v_i, f_m v_i} = \left( \begin{array}{c} N \\ m \end{array} \right)^{-1/2} \delta_{v_i, v_j},
\]

\[
C^{F_{ab} v_{ab}}_{f_a v_a, f_b v_b} = (-1)^{\phi(f_a, f_b, f_{ab})} C^{F_{ab} v_{ab}}_{f_b v_b, f_a v_a},
\]

\[
\sum_{v_i, v_j} C^{\nu \omega \nu \nu'}_{f_m v_i, f_m v_j} C^{\nu' \omega \nu'}_{f_m v_j, f_m v_i} = \delta_{\nu \nu'} \delta_{\omega \omega'} \delta_{\nu \nu'} \delta_{\omega \omega'} .
\]
Here, \((-1)^{\phi}\) is a phase factor depending on the irreps \(f_a, f_b\) and \(f_{ab}\). See [14, 36–38] for further details on the phase relations for \(U(N)\) C-G coefficients. From now on we will use the symbol \(f_m\) only in the C-G coefficients, Racah coefficients and reduced matrix elements. However, for the matrix elements of an operator we will use \(m\) implying totally antisymmetric state for fermions (symmetric state for bosons).

Starting with Eq. (4) and using Eqs. (5), (9) and (10) will immediately give the formula,

\[
\langle H^2(k) \rangle^m = \left( \frac{N}{m} \right)^{-1} \sum_{v_i} \langle mv_i | H^2(k) | mv_i \rangle = V_H^2 \Lambda^0(N, m, k). \tag{11}
\]

Similarly, for \(\langle H^4(k) \rangle^m\), the ensemble average is decomposed into three terms as

\[
\langle H^4(k) \rangle^m = \left( \frac{N}{m} \right)^{-1} \sum_{v_i} \langle mv_i | H^4(k) | mv_i \rangle = \sum_{v_i, v_j, v_p, v_l} \langle mv_i | H(k) | mv_j \rangle \langle mv_j | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_l \rangle \langle mv_l | H(k) | mv_i \rangle
\]

\[
= \sum_{v_i, v_j, v_p, v_l} \left[ \langle mv_i | H(k) | mv_j \rangle \langle mv_j | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_l \rangle \langle mv_l | H(k) | mv_i \rangle + \langle mv_i | H(k) | mv_j \rangle \langle mv_j | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_l \rangle \langle mv_l | H(k) | mv_i \rangle + \langle mv_i | H(k) | mv_j \rangle \langle mv_j | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_l \rangle \langle mv_l | H(k) | mv_i \rangle \right]. \tag{12}
\]

It is easy to see that the first two terms simplify to give \(2 \left[ \langle H^2(k) \rangle^m \right]^2\) and the third term is simplified by applying Eq. (8) to the first ensemble average and Eq. (9) to the second ensemble average. Then, the final result is

\[
\langle H^4(k) \rangle^m = 2 \left[ \langle H^2(k) \rangle^m \right]^2 + V_H^4 \left( \frac{N}{m} \right)^{-1} \sum_{i=0}^{\min(k, m - k)} \Lambda^\nu(N, m, k) \Lambda^\nu(N, m, m - k) d(\nu : \nu), \tag{13}
\]

where

\[
d(\nu : \nu) = \left( \frac{N}{\nu} \right)^2 - \left( \frac{N}{\nu - 1} \right)^2. \tag{14}
\]
Finally, a by-product of Eqs. (9) and (10) is

\[
\sum_{v_j} \langle mv_i \mid H(k) \mid mv_j \rangle \langle mv_j \mid H(k) \mid mv_k \rangle = \langle H^2(k) \rangle^m \delta_{v_i,v_k},
\]

and we will use this in Section 3. Now we will discuss results for the moments of the transition strength densities generated by a transition operator \( \mathcal{O} \).

3. Lower-order moments of transition strength densities: \( H \) EGUE\((k)\) and \( \mathcal{O} \) an independent EGUE\((t)\)

For a spinless fermion system, similar to the \( k \)-body \( H \) operator, we will consider a \( t \)-body transition operator \( \mathcal{O} \) represented by EGUE\((t)\) in the \( m \)-particle spaces. Then, the matrix of \( \mathcal{O} \) in \( t \)-particle space will be a GUE with the matrix elements \( \mathcal{O}_{ab}(t) \) being zero centered independent Gaussian variables with the variance satisfying,

\[
\mathcal{O}_{ab}(t) \mathcal{O}_{cd}(t) = V^2 \delta_{ad} \delta_{bc}.
\]

Further, we will assume that the GUE representing \( H \) in \( k \)-particle spaces and the GUE representing \( \mathcal{O} \) in \( t \) particle spaces are independent (this is equivalent to the statement that \( \mathcal{O} \) does not generate diagonal elements \( \langle E_i \mid \mathcal{O} \mid E_i \rangle \); see [24]). It is important to mention that in the past there were attempts to numerically study, in some nuclear \((2s1d)\) shell examples, transition strengths using the eigenvectors generated by a EGOE (with symmetries) representation for \( H \) but taking \( \mathcal{O} \) to be a realistic transition operator. The results are found to be in variance with those obtained using a \( H \) defined by a realistic two-body interaction and \( \mathcal{O} \) a realistic transition operator [39]. As described in the present paper, a proper random matrix theory for transition strengths has to employ ensemble representation for both the Hamiltonian and the transition operator.

With EGUE representation, \( \mathcal{O} \) is Hermitian and hence, \( \mathcal{O}^\dagger = \mathcal{O} \). Moments of the transition strength densities \( I_{\mathcal{O}}(E_i,E_f) \) are defined by

\[
M_{PQ}(m) = \langle \mathcal{O}^{\dagger}(t)H^Q(k)\mathcal{O}(t)H^P(k) \rangle^m = \langle \mathcal{O}(t)H^Q(k)\mathcal{O}(t)H^P(k) \rangle^m.
\]

Here the ensemble average is w.r.t. both EGUE\((k)\) and EGUE\((t)\). Now we will derive formulas for \( M_{PQ} \) with \( P+Q = 2 \) and 4; the moments with odd value of \( P+Q \) will vanish by definition.
Firstly, the unitary decomposition of $\mathcal{O}(t)$ gives,

$$\mathcal{O}(t) = \sum_{\nu=0}^{t} U_{\nu \omega_{\nu}}(t) B_{\nu \omega_{\nu}}(t).$$

(18)

The $U$'s satisfy a relation similar to Eq. (5). Now, for $P = Q = 0$, using Eq. (11), we have

$$\langle \mathcal{O}(t) \mathcal{O}(t) \rangle^m = V_0^2 \Lambda^0(N, m, t).$$

(19)

Moreover, we have the relations

$$\langle \mathcal{O}(t)\mathcal{O}(t)H^P(k) \rangle^m = \langle \mathcal{O}(t)\mathcal{O}(t) \rangle^m \langle H^P(k) \rangle^m = \langle \mathcal{O}(t)H^P(k)\mathcal{O}(t) \rangle^m$$

(20)

and their proof is as follows. Let us consider $\langle \mathcal{O}(t)\mathcal{O}(t)H^P(k) \rangle^m$. Then,

$$\langle \mathcal{O}(t)\mathcal{O}(t)H^P(k) \rangle^m = \left(\frac{N}{m}\right)^{-1} \sum_{v_i,v_j} \langle mv_i | \mathcal{O}(t)\mathcal{O}(t) | mv_j \rangle \langle mv_j | H^P(k) | mv_i \rangle.$$ 

(21)

Now applying Eq. (15) gives $\langle mv_i | \mathcal{O}(t)\mathcal{O}(t) | mv_j \rangle = \langle \mathcal{O}(t)\mathcal{O}(t) \rangle^m \delta_{v_i,v_j}$. Substituting this into Eq. (21) will give the first equality in Eq. (20). The second equality $\langle \mathcal{O}(t)\mathcal{O}(t)H^P(k) \rangle^m = \langle \mathcal{O}(t)H^P(k)\mathcal{O}(t) \rangle^m$ follows from the cyclic invariance of the $m$-particle average. Equation (20) gives the moments,

$$M_{20}(m) = M_{02}(m) = \langle \mathcal{O}(t)\mathcal{O}(t) \rangle^m \langle H^2(k) \rangle^m,$$

$$M_{40}(m) = M_{04}(m) = \langle \mathcal{O}(t)\mathcal{O}(t) \rangle^m \langle H^4(k) \rangle^m.$$ 

(22)

Formulas for $\langle \mathcal{O}(t)\mathcal{O}(t) \rangle^m$, $\langle H^2(k) \rangle^m$ and $\langle H^4(k) \rangle^m$ follow from Eqs. (19), (11) and (13). Thus, the non-trivial moments $M_{PQ}$ for $P + Q \leq 4$ are $M_{11}$, $M_{13} = M_{31}$ and $M_{22}$.

It is easy to recognize that the bivariate moment $M_{11}$ has same structure as the third term in Eq. (12) for $\langle H^4(k) \rangle^m$ as the $\mathcal{O}$ and $H$ ensembles are independent. Then, the formula for $M_{11}$ follows directly from the second term in Eq. (13). This gives,

$$M_{11}(m) = \langle \mathcal{O}(t)H(k)\mathcal{O}(t)H(k) \rangle^m$$

$$= V_0^2 V_H^2 \left(\frac{N}{m}\right)^{-1} \sum_{\nu=0}^{\min(k,m-t)} \Lambda^\nu(N,m,t) \Lambda^\nu(N,m,m-k) d(N : \nu).$$

(23)
This equation has the correct $t \leftrightarrow k$ symmetry. Using Eqs. (23) and (11), we have formula for the bivariate correlation coefficient $\xi$,

$$
\xi(m) = \frac{M_{11}(m)}{\sqrt{M_{20}(m) M_{02}(m)}} = \frac{\sum_{\nu=0}^{\min(t,m-k)} \Lambda^\nu(N,m,k) \Lambda^\nu(N,m,m-t) d(N : \nu)}{\binom{N}{m} \Lambda^0(N,m,t) \Lambda^0(N,m,k)}.
$$

Turning to $M_{PQ}$ with $P + Q = 4$, the first trivial moment is $M_{13} = M_{31}$. For $M_{31}$ we have,

$$
M_{31}(m) = \langle \mathcal{O}(t) H(k) \mathcal{O}(t) H^3(k) \rangle^m
= \binom{N}{m}^{-1} \sum_{v_i^t, v_j^t, v_k^t} \langle mv_i | \mathcal{O}(t) | mv_j \rangle \langle mv_j | H(k) | mv_k \rangle \langle mv_k | \mathcal{O}(t) | mv_l \rangle \langle mv_l | H^3(k) | mv_j \rangle.
$$

In Eq. (25), the last equality follows from the fact that the EGUE’s representing $H$ and $\mathcal{O}$ are independent. The ensemble average of the product of two $\mathcal{O}$ matrix elements follows easily from Eq. (9) giving,

$$
\langle mv_i | \mathcal{O}(t) | mv_j \rangle \langle mv_k | \mathcal{O}(t) | mv_l \rangle = V_\mathcal{O}^2 \sum_{\nu=0}^{m-t} \Lambda^\nu(N,m,t) \sum_{f_{m_{vl}}, f_{m_{ij}}} C^{\nu}_{f_{m_{vl}}, f_{m_{ij}}} C^{\nu}_{f_{m_{ij}}, f_{m_{vl}}}.
$$

The ensemble average of the product of a $H$ matrix element and $H^3$ matrix
The first two terms in Eq. (25) simplify to give
\[
\langle mv_j | H(k) | mv_k \rangle \langle mv_l | H^3(k) | mv_i \rangle
\]
\[
= \sum_{v_p,v_q} \langle mv_j | H(k) | mv_k \rangle \langle mv_l | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_q \rangle \langle mv_q | H(k) | mv_i \rangle
\]
\[
= \sum_{v_p,v_q} \left[ \langle mv_j | H(k) | mv_k \rangle \langle mv_l | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_q \rangle \langle mv_q | H(k) | mv_i \rangle + \langle mv_j | H(k) | mv_k \rangle \langle mv_q | H(k) | mv_l \rangle \langle mv_l | H(k) | mv_p \rangle \langle mv_p | H(k) | mv_q \rangle + \langle mv_j | H(k) | mv_k \rangle \langle mv_p | H(k) | mv_q \rangle \langle mv_q | H(k) | mv_l \rangle \langle mv_l | H(k) | mv_i \rangle \right].
\]
(27)

The first two terms in Eq. (27) simplify to give \(2 \langle H^2(k) \rangle \) \(M_{11}(m)\), using Eq. (26). Simplifying the third term using Eqs. (8) and (9), we get
\[
\langle mv_j | H(k) | mv_k \rangle \langle mv_p | H(k) | mv_l \rangle = V_{hl}^4 \sum_{m-k} A^1(N, m, m - k) A^2(N, m, k) C_{\nu_1, \nu_2}^{\nu_1, \nu_2} C_{\nu_1, \nu_3}^{\nu_1, \nu_3} C_{\nu_2, \nu_4}^{\nu_2, \nu_4} C_{\nu_3, \nu_4}^{\nu_3, \nu_4}.
\]
(28)

Combining this with Eq. (26) and applying the orthonormal properties of the C-G coefficients will give the final formula for \(M_{31}\),
\[
M_{31}(m) = \langle O(t) H(k) O(t) H^3(k) \rangle^m
\]
\[
= 2 \langle |H^2(k)| \rangle^m M_{11}(m) + \sum_{m-t} V_{hl}^2 \left( \begin{array}{c} N \\ m \end{array} \right) A^1(N, m, m - t) A^2(N, m, t) A^3(N, m - k, m - k) d(N : \nu).
\]
(29)

Thus, \(M_{31}(m)\) involves only the \(\Lambda\) functions.

To derive the formula for \(M_{22}\), we will make use of the decompositions similar to those in Eqs. (25) and (27). Then it is easy to see that \(M_{22}\) will have three terms and let us say they are \(S_1\), \(S_2\) and \(S_3\). The first term is
\[ S_1 = \langle \mathcal{O}(t) \mathcal{O}(t) \rangle^m \left[ \langle H^2(k) \rangle^m \right]^2 \] and the second term \( S_2 \) is

\[ S_2 = \left( \frac{N}{m} \right)^{-1} \sum_{v_i, v_j, v_r, v_l, v_P, v_Q} S_{21}(v_i, v_j, v_r, v_l) S_{22}(v_i, v_j, v_l, v_P, v_Q) ; \]

\[ S_{21}(v_i, v_j, v_r, v_l) = \langle m, v_i | \mathcal{O}(t) | m, v_j \rangle \langle m, v_r | \mathcal{O}(t) | m, v_l \rangle, \]

\[ S_{22}(v_i, v_j, v_l, v_P, v_Q) = \langle m, v_j | H(k) | m, v_P \rangle \langle m, v_Q | H(k) | m, v_l \rangle \]

\[ \times \langle m, v_P | H(k) | m, v_r \rangle \langle m, v_Q | H(k) | m, v_i \rangle. \]

The sum involving \( S_{22} \) is simplified by applying Eq. (9) to the two ensemble averages in \( S_{22} \) and using the orthonormal properties of the C.G coefficients. Now, applying Eq. (8) to the ensemble average in \( S_{21} \) and then multiplying with \( S_{22} \) will (after using again the orthonormal properties of the C-G coefficients) lead to the final result for \( S_2 \),

\[ S_2 = \left( \frac{N}{m} \right)^{-1} \sum_{\nu=0}^{\min(t, m-k)} \Lambda^\nu(N, m, m-t) \left[ \Lambda^\nu(N, m, k) \right]^2 d(N : \nu). \] (31)

Similarly, the term \( S_3 \) is

\[ S_3 = \left( \frac{N}{m} \right)^{-1} \sum_{v_i, v_j, v_r, v_l, v_P, v_Q} S_{31}(v_i, v_j, v_r, v_l, v_P, v_Q) S_{32}(v_i, v_r, v_P, v_Q) ; \]

\[ S_{31}(v_i, v_j, v_r, v_l, v_P, v_Q) = \langle m, v_i | \mathcal{O}(t) | m, v_j \rangle \langle m, v_r | \mathcal{O}(t) | m, v_l \rangle \]

\[ \times \langle m, v_j | H(k) | m, v_P \rangle \langle m, v_l | H(k) | m, v_Q \rangle, \]

\[ S_{32}(v_i, v_r, v_P, v_Q) = \langle m, v_P | H(k) | m, v_r \rangle \langle m, v_Q | H(k) | m, v_i \rangle. \] (32)

The term \( S_{31} \) is simplified by first using Eqs. (4) and (18) and this gives,
after carrying out the ensemble averages,

\[ S_{31} = V_{\Omega}^2 V_H^2 \sum_{v_i, v_j, \nu_1, \nu_2 = 0}^t \sum_{\nu_1 = 0}^k \langle m, v_i | B^{\nu_1 \omega_1} (t) | m, v_j \rangle \]

\[ \times \langle m, v_r | B^{\nu_2 \omega_2} (k) | m, v_p \rangle \langle m, v_l | B^{\nu_2 \omega_2} (k) | m, v_Q \rangle \]

\[ = V_{\Omega}^2 V_H^2 \sum_{v_i, v_j, \nu_1 = 0}^t \sum_{\nu_1 = 0}^k \langle m, v_i | B^{\nu_1 \omega_1} (t) B^{\nu_2 \omega_2} (k) | m, v_p \rangle \]

\[ \times \langle m, v_r | B^{\nu_1 \omega_1} (t) B^{\nu_2 \omega_2} (k) | m, v_Q \rangle . \]  

(33)

Now, coupling the \( B \)'s in each matrix element in the last equality in Eq. (33) applying the Wigner-Eckart theorem for the matrix elements of the coupled tensor operator \( [B^{\nu_1} (t) B^{\nu_2} (k)]^{\nu_1 \omega_1} \) along with the application of Eq. (9) to the \( S_{32} \) term will give the formula for the \( S_3 \) term. With this, the \( M_{22} \) is given by,

\[ M_{22} (m) = \langle \mathcal{O}(t) \mathcal{O}(t) \rangle^m \left[ \langle H(k) H(k) \rangle^m \right]^2 \]

\[ + V_{\Omega}^2 V_H^4 \left( \frac{N}{m} \right)^{-1} \left\{ \sum_{\nu = 0}^{\min(t, m - k)} \Lambda^\nu (N, m, m - t) \left[ \Lambda^\nu (N, m, k) \right]^2 d(N : \nu) \right. \]

\[ + \left. \sum_{\nu = 0}^{\min(t + m - k)} \Lambda^\nu (N, m, k) d(N : \nu) \sum_{\nu_1 = 0}^t \sum_{\nu_2 = 0}^k \sum_{\rho} \langle f_m || \left[ B^{\nu_1} (t) B^{\nu_2} (k) \right]^{\nu_1 \rho} || f_m \rangle^2 \right\} . \]  

(34)

Here \( \rho \) labels multiplicity of the irrep \( \nu \) in the Kronecker product \( \nu_1 \times \nu_2 \rightarrow \nu \). We will see in Section 4 that Eq. (34) is useful in deriving asymptotic results.

In order to evaluate \( M_{22} \), we need a formula for the reduced matrix element in Eq. (34). This is obtained by considering the corresponding matrix element, decoupling the coupled operator, using complete set of states between the two operators, applying Wigner-Eckart theorem to the two matrix elements.
and then simplifying the sums over all the C-G coefficients. This gives,

\[
\langle f_m | [B^\nu_1(t) B^\nu_2(k)]^\nu, \rho | f_m \rangle = \sqrt{d(N : \nu_1) d(N : \nu_2)} \frac{d(N : \nu)}{N \binom{N}{m}}
\]

(35)

\[
\times \langle f_m | B^\nu_1(t) | f_m \rangle \langle f_m | B^\nu_2(k) | f_m \rangle U(f_m \nu_1 f_m \nu_2 ; f_m \nu_\rho).
\]

Eq. (35) gives the formula for \(M_{22}\) in terms of \(U(N)\) Racah coefficients,

\[
M_{22}(m) = \langle \mathcal{O}(t) H^2(k) \mathcal{O}(t) H^2(k) \rangle^m = V_\mathcal{O}^2 V_H^4 \left\{ \left[ \Lambda^0(N, m, k) \right]^2 \Lambda^0(N, m, t)
\right. \\
+ \binom{N}{m}^{-1} \min(t, m-k) \sum_{\nu=0} \Lambda^\nu(N, m, m-t) \left[ \Lambda^\nu(N, m, k) \right]^2 d(N : \nu) \\
+ \binom{N}{m}^{-2} \min(k+t, m-k) \sum_{\nu_1=0}^t \sum_{\nu_2=0}^k \Lambda^\nu(N, m, k) \Lambda^\nu_1(N, m, m-t) \\
\times \Lambda^\nu_2(N, m, m-k) d(N : \nu_1) d(N : \nu_2) \sum_\rho \left[ U(f_m \nu_1 f_m \nu_2 ; f_m \nu_\rho) \right]^2 \right\}.
\]

(36)

The \(U\) or Racah coefficient in Eq. (36) is with respect to \(U(N)\) and a formula for this is not available in closed form in the literature. Deriving formulas for this \(U\)-coefficient is an important open problem. We will show ahead that it is possible to derive a formula valid in the asymptotic \((N \to \infty, m \to \infty, m/N \to 0\) and \(k, t\) fixed) limit.

4. Asymptotic results for the bivariate moments for \(H\) \(\text{EGUE}(k)\) and \(\mathcal{O}\) an independent \(\text{EGUE}(t)\)

Lowest order (sufficient for most purposes) shape parameters of the bivariate strength density are the bivariate reduced cumulants of order four, i.e. \(k_{rs}(m), r + s = 4\). The \(k_{rs}(m)\) can be written in terms of the normalized central moments \(\hat{M}_{PQ}(m)\) where \(\hat{M}_{PQ}(m) = M_{PQ}(m) / M_{00}(m)\). Then, the scaled moments \(\mu_{PQ}(m)\) are

\[
\mu_{PQ}(m) = \frac{\hat{M}_{PQ}(m)}{\left[ \hat{M}_{20}(m) \right]^{P/2} \left[ \hat{M}_{02}(m) \right]^{Q/2}}, \quad P + Q \geq 2.
\]

(37)
Note that \( \mu_{20}(m) = \mu_{02}(m) = 1 \) and \( \mu_{11}(m) = \xi(m) \). Now the fourth order cumulants are,

\[
\begin{align*}
\mu_{40}(m) &= \mu_{04}(m) - 3, \\
\mu_{04}(m) &= \mu_{04}(m) - 3, \\
\mu_{31}(m) &= \mu_{31}(m) - 3 \xi(m), \\
\mu_{13}(m) &= \mu_{13}(m) - 3 \xi(m), \\
\mu_{22}(m) &= \mu_{22}(m) - 2 \xi^2(m) - 1.
\end{align*}
\]

The \( |k_{rs}(m)| \sim 0 \) for \( r + s \geq 3 \) implies that the transition strength density is close to a bivariate Gaussian (note that in our EGUE applications, \( k_{rs}(m) = 0 \) for \( r + s \) odd by definition). Numerical results for \( k_{rs}(m), r + s = 4 \) and also for \( \xi(m) \) for some typical values of \( N, m, k \) and \( t \) are shown in Table 1. It is seen that for sufficiently large values of \( N \), \( m \), \( k \) and \( t \) relatively small, in general the magnitude of the fourth order cumulants is very small \(< 0.3\) implying that for EGUE, transition strength densities approach a bivariate Gaussian. However, with increasing \( k + t \) value it is seen that \( \xi(m) \to 0 \) and this is the GOE result. Also, in this limit the marginal densities approach semi-circle form giving \( k_{40}(m) \to -1 \). For a better understanding of these results, it is useful to derive expressions for \( \mu_{PQ}(m) \) and thereby for \( k_{PQ}(m) \), using Eq. (38), in the asymptotic (asymp) limit defined by \( N \to \infty, m \to \infty, m/N \to 0 \) and fixed \( k \) and \( t \) with \( t < k \). First we will consider \( N \to \infty \) and \( m \) fixed with \( k, t << m \). Two relations we use are,

\[
\begin{align*}
\binom{N-p}{r} \xrightarrow{p/N \to 0} \frac{N^r}{r!},
\quad d(N : \nu) \xrightarrow{\nu/N \to 0} \frac{N^{2\nu}}{\nu!^2}.
\end{align*}
\]

Note that \( d(N : \nu) \) is defined in Eq. (14). Let us start with the formula for \( \xi \) given by Eq. (24). Applying Eq. (39), it is easily seen that only the term with \( \nu = t \) in the sum in Eq. (24) will contribute in the asymptotic limit. Using this and applying Eq. (39) to the formula for \( \Lambda' \) given by Eq. (7) will lead to,

\[
\begin{align*}
\xi(m) &\xrightarrow{m! N^{2t} \binom{m-t}{k} \binom{N-m+k-t}{k} \binom{N-2t}{m-t} \text{asymp} \binom{m}{k}^{-1} \binom{m-t}{k} \binom{m}{k}^{-1} \binom{m-k}{k}. \\
\mu_{40}(m) &\xrightarrow{2 + m! N^{2k} \binom{m-k}{k} \binom{N-m}{k} \binom{N-2k}{m-k} \text{asymp} \binom{m}{k}^{-1} \binom{m-k}{k}}.
\end{align*}
\]

Similarly, \( \mu_{40}(m) \) will be, using Eqs. (22), (13) and (11),

\[
\begin{align*}
\mu_{40}(m) &\xrightarrow{2 + m! N^{2k} \binom{m-k}{k} \binom{N-m}{k} \binom{N-2k}{m-k} \text{asymp} \binom{m}{k}^{-1} \binom{m-k}{k}}.
\end{align*}
\]
Turning to \( \mu_{31}(m) \), it is easy to see from Eq. (29) that \( M_{31}(m) \) has two terms. The first term is \( 2 \xi(m) \langle \mathcal{O}(t) \mathcal{O}(t) \rangle^m \left[ [H(k)H(k)]^m \right]^2 \) and in the second term, only \( \nu = k \) will survive in the asymptotic limit. These then will give,

\[
\mu_{31}(m) = 2 \xi(m) + \left( \begin{array}{c} N \\ m \end{array} \right) \frac{-1 \Lambda^k(N,m,t) \Lambda^k(N,m,k) \Lambda^k(N,m,m-k) d(N:k)}{\Lambda^0(N,m,t) [\Lambda^0(N,m,k)]^2} \]

\[
\text{asymp} \rightarrow \xi(m) \left[ 2 + \left( \begin{array}{c} m \\ k \end{array} \right)^{-1} \left( \begin{array}{c} m-k \\ k \end{array} \right) \right] = \xi(m) \mu_{40}(m). 
\]

(42)

Here, Eq. (39) is used in the final simplifications. Finally, let us consider \( \mu_{22}(m) \). Firstly, \( M_{22}(m) \) has three terms as seen from Eqs. (36) and (34) and let us call them \( X_1, X_2 \) and \( X_3 \). Then, there will be corresponding three terms in \( \mu_{22}(m) \) and we call them \( T_1, T_2 \) and \( T_3 \). It is seen that \( T_1 = 1 \) and \( T_2 \) is (in the corresponding \( X_2 \) sum, only \( \nu = t \) term will contribute in the asymptotic limit),

\[
T_2 \rightarrow \left( \begin{array}{c} N \\ m \end{array} \right) \frac{-1 \Lambda^t(N,m,m-t) \Lambda^t(N,m,k) \Lambda^t(N,m,m-k) d(N:t)}{\Lambda^0(N,m,t) [\Lambda^0(N,m,k)]^2} \rightarrow \left( \begin{array}{c} m \\ k \end{array} \right)^{-2} \left( \begin{array}{c} m-t \\ k \end{array} \right)^2. 
\]

(43)

As a formula for the \( U \)-coefficient appearing in Eq. (36) is not available, we use Eq. (34) for simplifying \( T_3 \). Then, in the \( X_3 \) sum only \( \nu = t + k \) term will survive in the asymptotic limit giving

\[
T_3 \rightarrow \left( \begin{array}{c} N \\ m \end{array} \right) \frac{-1 \Lambda^{t+k}(N,m,k) d(N:t+k)}{\Lambda^0(N,m,t) [\Lambda^0(N,m,k)]^2} \left| \langle f_m \mid [B^{t+k}(t)B^{k}(k)]^{t+k} \mid f_m \rangle \right|^2. 
\]

(44)

For further simplification of \( T_3 \), we will use the relation

\[
\left( \begin{array}{c} N \\ m \end{array} \right) \frac{[O(t)H(k)O(t)H(k)]^m}{\Lambda^0(N,m,t) [\Lambda^0(N,m,k)]^2} = V_0^2 V_H^2 \sum_{\nu_1=0}^{t} \sum_{\nu_2=0}^{k} \sum_{\nu=0}^{k} \left| \langle f_m \mid [B^{\nu_1}(t)B^{\nu_2}(k)]^{\nu_1 \nu_2} \mid f_m \rangle \right|^2 d(N: \nu). 
\]

(45)

Then, \( T_3 \) in the asymptotic limit will be

\[
T_3 \rightarrow \frac{\Lambda^{t+k}(N,m,k)}{\Lambda^0(N,m,k)} [\xi(N \rightarrow \infty)] \rightarrow \left( \begin{array}{c} m \\ k \end{array} \right)^{-2} \left( \begin{array}{c} m-t-k \\ k \end{array} \right) \left( \begin{array}{c} m-t \\ k \end{array} \right). 
\]

(46)
Notice that, here we have used Eq. (40) for $\xi (m)$ in the $N \to \infty$ limit. Now combining $T_1$, $T_2$ and $T_3$ we have

$$
\mu_{22}(m) \xrightarrow{\text{asymp}} 1 + \left( \frac{m}{k} \right)^{-2} \left( \frac{m-t}{k} \right)^{2} \left( \frac{m-t-k}{k} \right) \left( \frac{m-t}{k} \right).
$$

(47)

Comparing Eq. (46) with the formula for $T_3$ as given by Eq. (36), it is easy to see that in the asymptotic limit

$$
[U(f_m t f_m k ; f_m t + k)]^2 \xrightarrow{\text{asymp}} \left( \frac{m}{k} \right)^{-1} \left( \frac{m-t}{k} \right).
$$

(48)

The asymptotic formulas derived from the exact results and given by Eqs. (40), (41), (42) and (47) are same as those given before in [13, 24] where the theory given by Mon and French [16], that gives directly the asymptotic results, was used. This is a good test of the derivations in Section 3. It is important to add that in the derivations given in Section 3 we have considered only the binary correlated terms in the various sums [the binary correlations come in the ensemble average of the $W$’s and $U$’s defined by Eqs. (4) and (18)]. Within this constraint, the results in Section 3 are exact. On the other hand, in the Mon and French theory, the binary correlated terms are evaluated using formulas that are valid only in the asymptotic limit.

Asymptotic formulas for the bivariate moments will give the asymptotic formulas for the fourth order bivariate cumulants $k_{40}(m)$, $k_{31}(m)$ and $k_{22}(m)$ by applying Eqs. (37) and (38). Then we have,

$$
k_{40}(m) = k_{04}(m) = \left( \frac{m-k}{k} \right) \left( \frac{m}{k} \right)^{-1} - 1,
$$

$$
k_{31}(m) = k_{13}(m) = \xi (m) k_{40}(m) ,
$$

$$
k_{22}(m) = \xi^2(m) \left\{ \left( \frac{m-k-t}{k} \right) \left( \frac{m-t}{k} \right)^{-1} - 1 \right\} .
$$

(49)

Remember that $\xi (m)$ is given by Eq. (40). Now, the $1/m$ expansion of $k_{rs}(m)$ shows that $k_{rs}(m) = -k^2/m + O(1/m^2)$. This clearly demonstrates that we have for the transition strength densities bivariate Gaussian form in general in the dilute limit defined by $N \to \infty$, $m \to \infty$, $m/N \to 0$ with fixed $k$ and $t$. However, $\xi (m) \to 1$ in the dilute limit and this gives a singular bivariate
Gaussian that is unphysical. Therefore, in practice the dilute limit condition will not be realized and there will be departures from the bivariate Gaussian form. Thus, the corrections due to $k_{rs}(m)$, $r + s = 4$ should be included and out of the many ways to add the corrections, the Edgeworth expansion is considered to be the best [19, 40, 41]. The Edgeworth corrected bivariate Gaussian form including $k_{rs}(m)$ with $r + s = 4$ is given in Appendix A.

5. EGUE results for moments of transition strength densities: two types of spinless fermions with beta and double beta decay type transition operators

Here we will consider a system with two types of spinless fermions with $m_1$ fermions of type #1 in $N_1$ sp states and $m_2$ fermions of type #2 in $N_2$ sp states with the system $H$ operator preserving $(m_1, m_2)$. This is similar to protons ($p$) and neutrons ($n$) in an atomic nucleus. Thus, we have two orbits with the first one having $N_1$ sp states and the second $N_2$ sp states. Then the $H$ operator, assumed to be $k$-body, is given by,

$$H(k) = \sum_{i+j=k} \sum_{\alpha, \beta} \sum_{a, b} V_{\alpha a; \beta b}(i, j) A_\alpha^\dagger(i) A_\beta(i) A_a^\dagger(j) A_b(j) ;$$

(50)

$$V_{\alpha a; \beta b}(i, j) = \langle i, \alpha : j, a | H | i, \beta : j, b \rangle .$$

Here, we are using Greek labels $\alpha, \beta, \ldots$ to denote the many particle states generated by fermions occupying the first orbit and the Roman labels $a, b, \ldots$ for the many particle states generated by the fermions occupying the second orbit. Note that $A_\alpha^\dagger(i)$ creates the state $|i, \alpha\rangle$ with $i$ number of fermions in the first orbit and $A_a^\dagger(j)$ creates the state $|j, a\rangle$ with $j$ number of fermions in the second orbit. Similar is the action of the annihilation operators $A_\beta(i)$ and $A_b(j)$. For example for a two-body Hamiltonian, $(i, j) = (2, 0), (1, 1)$ and $(0, 2)$. This corresponds to $H = H_{pp} + H_{pn} + H_{nn}$ for atomic nuclei; with ‘p’ denoting protons and ‘n’ denoting neutrons. For each $(i, j)$ pair with $i + j = k$, we have a matrix $V(i, j)$ in the $k$-particle space and $H$ matrix is a direct sum of these matrices in the $k$ particle space. Their dimensions being $\left(\binom{N_1}{2}\right)$, $N_1 N_2$ and $\left(\binom{N_2}{2}\right)$ respectively. Action of the $H$ operator on the $|m_1, v_\alpha : m_2, v_a\rangle$ states of a $(m_1, m_2)$ system generates $(m_1, m_2)$ particle $H$ matrix; $v_\alpha$ and $v_a$ are respective additional labels. The $H$ matrix dimension is $d(m_1, m_2) = \left(\binom{N_1}{m_1}\right) \left(\binom{N_2}{m_2}\right)$. To proceed further, the $V(i, j)$ matrices are
represented by independent GUEs with matrix elements being zero centered Gaussian variables with variance,

\[ V_{\alpha \beta}(i, j) V_{\alpha' \beta'}(i', j') = V^2_H(i, j) \delta_{ii'} \delta_{jj'} \delta_{\alpha \beta} \delta_{\alpha' \beta'} . \] (51)

It is important to note that the embedding algebra for the EGUE generated by the action of the \( V \) ensemble on \(|m_1, v_\alpha : m_2, v_a\rangle\) states is the direct sum algebra \( U(N_1) \oplus U(N_2) \). Thus we have EGUE(\(k\))-\([U(N_1) \oplus U(N_2)]\) ensemble.

Going beyond the simple operators considered in Sections 2 and 3, here we will consider the following transition operator,

\[ \mathcal{O}(k_0) = \sum_{\alpha, a} O_{\alpha a} A^\dagger_{\alpha}(k_0) A_a(k_0) ; \quad O_{\alpha a} = \langle k_0, \alpha | O | k_0, a \rangle . \] (52)

It is easy to see from Eq. (52) that the operator \( \mathcal{O}(k_0) \) changes \( k_0 \) number of fermions in the second orbit to \( k_0 \) number of fermions in the first orbit. This is similar to the action of \( \beta \) decay operator (then \( k_0 = 1 \)) and neutrinoless double beta decay operator (\( k_0 = 2 \)). Eq. (52) gives the matrix elements of \( \mathcal{O} \) for the matrix representation of \( \mathcal{O} \) in the defining space and this is in general a rectangular \( D_A \times D_B \) matrix where \( D_A = (\binom{N_2}{k_0}) \) and \( D_B = (\binom{N_1}{k_0}) \). Just as in Section 3, we will assume GUE representation for the \( \mathcal{O} \) matrix in the defining space and then,

\[ O^\dagger_{\alpha a} O_{\beta b} = V^2_O \delta_{\alpha \beta} \delta_{ab} . \] (53)

Note that the \( O \) matrix in the defining space is a GUE in the sense that the real and imaginary parts of the \( O_{\alpha a} \) are zero centered independent Gaussian variables with variance given by Eq. (53). With the GUE representation for the \( O \) matrix in the defining space, we have EGUE for \( O \) in many particle spaces. This matrix will be again a rectangular matrix with matrix elements connecting \((m_1, m_2)\) states with \((m_1 + k_0, m_2 - k_0)\) states by the \( \mathcal{O} \) operator.

Fig. 3 shows an example for the \( H \) and \( O \) matrices in the defining and many-particle spaces. With independent GUE representations for \( H \) and \( O \) matrices in the defining spaces (then \( V_{\alpha a; \beta b}(i, j) \) are independent of \( O_{\alpha' a'} \)), we will derive formulas for the bivariate moments of the transition strength density

\[ I^{(m_1, m_2)}_{\mathcal{O}}(E_i, E_f) = I^{(m_1 + k_0, m_2 - k_0)}_{\mathcal{O}}(E_f) \langle (m_1 + k_0, m_2 - k_0, E_f | O | (m_1, m_2, E_i) \rangle^2 I^{(m_1, m_2)}(E_i) . \] (54)

20
Then, the ensemble averaged bivariate moments are

\[ M_{PQ}(m_1, m_2) = \langle \mathcal{O}^{\dagger}(k_0) H^Q(k) \mathcal{O}(k_0) H^P(k) \rangle^{(m_1, m_2)}. \]  

(55)

Note that \( \mathcal{O} \) takes \((m_1, m_2)\) to \((m_1 + k_0, m_2 - k_0)\) uniquely and therefore the later is not specified explicitly in Eqs. (55) and (54). Also, it is important to note that \( \mathcal{O}^{\dagger} \neq \mathcal{O} \).

Given one type of spinless fermions, we easily have

\[ \langle m, v_1 | \sum_\gamma A^\dagger_\gamma(k_0) A_\gamma(k_0) | m, v_2 \rangle^m_{m_1, m_2} = \binom{m}{k_0} \delta_{v_1, v_2}, \]  

(56)

\[ \langle m, v_1 | \sum_\gamma A_\gamma(k_0) A^\dagger_\gamma(k_0) | m, v_2 \rangle^m_{m_1, m_2} = \binom{N - m}{k_0} \delta_{v_1, v_2}. \]

Now, substituting complete set of state between the \( A^\dagger \) and \( A \) operators and applying the Wigner-Eckart theorem [note that \( A^\dagger_\alpha(k) \) and \( A^\dagger_\beta(k) \) transform as the \( U(N) \) tensors \( f_k = \{1^k\} \) and \( \{\bar{f}_k\} = \{1^{N-k^0}\} \)] will give,

\[ \langle m || A^\dagger(k_0) || m - k_0 \rangle \langle m - k_0 || A(k_0) || m \rangle = \binom{N - k_0}{m - k_0}, \]  

(57)

\[ \langle m || A(k_0) || m + k_0 \rangle \langle m + k_0 || A^\dagger(k_0) || m \rangle = \binom{N - k_0}{m}. \]

For the derivations given ahead it is important to recognize that \( A^\dagger_\alpha(k_0) \) and \( A^\dagger_\beta(k_0) \) transform as the \( U(N_1) \) tensors \( f_{k_0} = \{1^{k_0}\} \) and \( \{\bar{f}_{k_0}\} = \{1^{N_1-k_0}\} \) and similarly \( A^\dagger_\alpha(k_0) \) and \( A^\dagger_\beta(k_0) \) with respect to \( U(N_2) \). Moreover, using Eq. (56), we have

\[ \langle \mathcal{O}^{\dagger}(k_0) \mathcal{O}(k_0) \rangle^{(m_1, m_2)} = V_2^2 \binom{N_1 - m_1}{k_0} \binom{m_2}{k_0}, \]  

(58)

\[ \langle \mathcal{O}(k_0) \mathcal{O}^{\dagger}(k_0) \rangle^{(m_1, m_2)} = V_2^2 \binom{N_2 - m_2}{k_0} \binom{m_1}{k_0}. \]

Before turning to the bivariate moments \( M_{PQ}(m_1, m_2) \), let us consider the second and the fourth moment of the state densities \( I^{m_1, m_2}(E) \). For
deriving formulas for these moments, unitary decomposition of $H(k)$ needs to be carried out with respect to $U(N_1) \oplus U(N_2)$ algebra and this gives the $B^\nu\omega(k')$ tensors in $U(N_1)$ and $U(N_2)$ spaces. In order to distinguish the tensors, we call them $B$ and $C$ in the two spaces respectively. Denoting these tensorial ranks by $\nu_1$ and $\nu_2$ respectively, we have

$$H(k) = \sum_{i+j=k} \sum_{\nu_1=0}^{i} \sum_{\nu_2=0}^{j} W_{ij}(\nu_1, \omega_{\nu_1}; \nu_2, \omega_{\nu_2}) B^{\nu_1\omega_{\nu_1}}(i) C^{\nu_2\omega_{\nu_2}}(j).$$

(59)

Just as in Section 2, here also it can be proved that the $W$’s will be independent zero centered Gaussian variable with variances satisfying,

$$W_{ij}(\nu_1, \omega_{\nu_1}; \nu_2, \omega_{\nu_2}) W_{i'i'}(\nu_1', \omega_{\nu_1'}; \nu_2', \omega_{\nu_2'}) = V_H^2(i, j) \delta_{i, i'} \delta_{j, j'} \delta_{\nu_1, \nu_1'} \delta_{\nu_2, \nu_2'}.$$

(60)

Using Eqs. (59) and (60) and the results given in Section 2, it is straightforward to derive the formulas for $\langle H^P(k) \rangle^{(m_1, m_2)}$ for $P = 2$ and 4. We have for $P = 2$,

$$\langle H^2(k) \rangle^{(m_1, m_2)} = \sum_{i+j=k} V_H^2(i, j) \Lambda^0(N_1, m_1, i) \Lambda^0(N_2, m_2, j).$$

(61)

Similarly for $P = 4$,

$$\langle H^4(k) \rangle^{(m_1, m_2)} = 2 \left[ \langle H^2(k) \rangle^{(m_1, m_2)} \right]^2 + \sum_{i+j=k} \sum_{i'+j'=k} V_H^2(i, j) V_H^2(i', j') X(N_1, m_1, i, i') Y(N_2, m_2, j, j').$$

$$X(N_1, m_1, i, i') = \left[ \begin{array}{c} N_1 \\ m_1 \end{array} \right]^{-1} \min_{\nu_1=0} \sum_{\nu_1} \Lambda^{\nu_1}(N_1, m_1, i - \nu_1) \Lambda^{\nu_1}(N_1, m_1, i') d(N_1 : \nu_1),$$

$$Y(N_2, m_2, j, j') = \left[ \begin{array}{c} N_2 \\ m_2 \end{array} \right]^{-1} \min_{\nu_2=0} \sum_{\nu_2} \Lambda^{\nu_2}(N_2, m_2, j - \nu_2) \Lambda^{\nu_2}(N_2, m_2, j') d(N_2 : \nu_2).$$

(62)

It is also easy to show using Eqs. (56) and (53) that,

$$\langle \mathcal{O}^\dagger(k_0) \mathcal{O}(k_0) H^P(k) \rangle^{(m_1, m_2)} = \langle \mathcal{O}^\dagger(k_0) \mathcal{O}(k_0) \rangle^{(m_1, m_2)} \langle H^P(k) \rangle^{(m_1, m_2)},$$

$$\langle \mathcal{O}^\dagger(k_0) H^P(k) \mathcal{O}(k_0) \rangle^{(m_1, m_2)} = \langle \mathcal{O}^\dagger(k_0) \mathcal{O}(k_0) \rangle^{(m_1, m_2)} \langle H^P(k) \rangle^{(m_1+K_0, m_2-K_0)}.$$
The bivariate moments $M_{00}$, $M_{20}$, $M_{02}$, $M_{40}$ and $M_{04}$ follow directly by appropriately using Eqs. (58), (61)-(63) in Eq. (55).

Clearly, the first nontrivial bivariate moment is $M_{11}(m_1, m_2)$ and it is given by,

$$M_{11}(m_1, m_2) = \langle O \dagger(k_0) H(k) O(k_0) H(k) \rangle^{m_1, m_2} = \left\{ \binom{N_1}{m_1} \binom{N_2}{m_2} \right\}^{-1} \times \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4; a_1, a_2, a_3, a_4} \text{Eavg} \left\{ \langle m_1, \alpha_1; m_2, a_1 | O \dagger(k_0) | m_1 + k_0, \alpha_2; m_2 - k_0, a_2 \rangle \right. \left. \times \langle m_1 + k_0, \alpha_2; m_2 - k_0, a_2 | H(k) | m_1 + k_0, \alpha_3; m_2 - k_0, a_3 \rangle \right. \left. \times \langle m_1 + k_0, \alpha_3; m_2 - k_0, a_3 | O(k_0) | m_1, \alpha_4; m_2, a_4 \rangle \right. \left. \times \langle m_1, \alpha_4; m_2, a_4 | H(k) | m_1, \alpha_1; m_2, a_1 \rangle \right\}$$
\[
\left\{ \binom{N_1}{m_1} \binom{N_2}{m_2} \right\}^{-1} \times \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, a_1, a_2, a_3, a_4} \mathbb{E}_{\text{avg}} \left\{ \langle m_1, \alpha_1; m_2, a_1 \mid \mathcal{O}^\dagger(k_0) \mid m_1 + k_0, \alpha_2; m_2 - k_0, a_2 \rangle \right\} \\
\times \langle m_1 + k_0, \alpha_3; m_2 - k_0, a_3 \mid \mathcal{O}(k_0) \mid m_1, \alpha_4; m_2, a_4 \rangle \right\}
\times \mathbb{E}_{\text{avg}} \left\{ \langle m_1 + k_0, \alpha_2; m_2 - k_0, a_2 \mid H(k) \mid m_1 + k_0, \alpha_3; m_2 - k_0, a_3 \rangle \right\}
\times \langle m_1, \alpha_4; m_2, a_4 \mid H(k) \mid m_1, \alpha_1; m_2, a_1 \rangle \right\}
= V_3^2 \left\{ \binom{N_1}{m_1} \binom{N_2}{m_2} \right\}^{-1} \times \sum_{i+j=k} \sum_{\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'} \sum_{\nu, \omega, \nu', \omega'} \sum_{\nu, \omega, \nu', \omega'} V_H^2(i, j)
\times \langle m_1, \alpha_1; m_2, a_1 \mid A_\alpha^\dagger(k_0)A_\alpha(k_0) \mid m_1 + k_0, \alpha_2; m_2 - k_0, a_2 \rangle \right\}
\times \langle m_1 + k_0, \alpha_3; m_2 - k_0, a_3 \mid A_\alpha^\dagger(k_0)A_\alpha(k_0) \mid m_1, \alpha_4; m_2, a_4 \rangle \right\}
\times \langle m_1 + k_0, \alpha_2; m_2 - k_0, a_2 \mid B^{\nu, \omega}(i)C^{\nu', \omega'}(j) \mid m_1 + k_0, \alpha_3; m_2 - k_0, a_3 \rangle \right\}
\times \langle m_1, \alpha_4; m_2, a_4 \mid B^{\nu, \omega}(i)C^{\nu', \omega'}(j) \mid m_1, \alpha_1; m_2, a_1 \rangle \right\}
(64)
\]

Here we have used ‘\(\mathbb{E}_{\text{avg}}\)’ to denote ensemble average instead of using the ‘overline’. We will use this notation at a few other places in the remainder of this paper. In the second step above we have introduced complete set of states between the operators to write the ensemble average of \(\langle \mathcal{O}^\dagger(k_0)H(k)\mathcal{O}(k_0)H(k) \rangle\) as the ensemble average of four matrix elements. In the next step, this ensemble average is written as the product of the ensemble average of \(\mathcal{O}\) matrix elements and the ensemble average of \(H\) matrix elements using the independence of the \(\mathcal{O}\) and \(H\) ensembles. Finally we have used Eq. (59) to get the final form in Eq. (64). Now, it is easy to see that \(M_{11}(m_1, m_2)\) factorizes into product of terms in \(m_1\) and \(m_2\) spaces giving, \(M_{11}(m_1, m_2)\) to be of the form \(V_3^2 \left[ \binom{N_1}{m_1} \binom{N_2}{m_2} \right]^{-1} \sum_{i+j=k} V_H^2(i, j)X(N_1, m_1, i, k_0)Y(N_2, m_2, j, k_0)\).
where for example \( X(N_1, m_1, i, k_0) \) is

\[
X(N_1, m_1, i, k_0) = 
\sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4; \omega_1, \omega_2} \langle m_1, \alpha_1 | A_\alpha(k_0) | m_1 + k_0, \alpha_2 \rangle \langle m_1 + k_0, \alpha_3 | A^\dagger_\alpha(k_0) | m_1, \alpha_4 \rangle 
\times \langle m_1 + k_0, \alpha_2 | B^{\nu \omega \nu}(i) | m_1 + k_0, \alpha_3 \rangle \langle m_1, \alpha_4 | B^{\nu \omega \nu}(i) | m_1, \alpha_1 \rangle.
\]

(65)

Similarly \( Y(N_2, m_2, j, k_0) \) can be written. Now, applying the Wigner-Eckart theorem along with Eqs. (7) and (57), we will be left with four C-G coefficients. These can be simplified to give a \( U(N_1) \) \( U^- \) or Racah coefficient. Similarly, for the \( Y \) function we will get a \( U(N_2) \) Racah coefficient. Putting these together will give the final formula for \( M_{11} \) and for later usage we will write it in the following form,

\[
M_{11}(m_1, m_2) = V_2^2 \left\{ \left( \frac{N_1}{m_1} \right) \left( \frac{N_2}{m_2} \right) \right\}^{-1} \sum_{i+j=k} V_H^2(i, j) \left( \frac{N_1 - k_0}{m_1} \right) \left( \frac{N_2 - k_0}{m_2 - k_0} \right)
\times \left[ \sum_{\nu_1=0}^i X_{11}(N_1, m_1, k_0, i, \nu_1) \right] \left[ \sum_{\nu_2=0}^j Y_{11}(N_2, m_2, k_0, j, \nu_2) \right] ;
\]

\[
X_{11}(N_1, m_1, k_0, i, \nu) = \left[ \left( \frac{N_1}{k_0} \right) d(N_1 : \nu) \right]^{1/2}
\times [\Lambda^\nu(N_1, m_1, m_1 - i) \Lambda^\nu(N_1, m_1 + k_0, m_1 + k_0 - i)]^{1/2}
\times (-1)^{\phi(f_{m_1 + k_0, m_1, f_{k_0}}) + \phi(f_{m_1, f_{m_1, \nu}})} U(f_{m_1 + k_0, f_{m_1}, f_{m_1 + k_0, f_{m_1}}}; f_{k_0, \nu}) ;
\]

\[
Y_{11}(N_2, m_2, k_0, j, \nu) = X_{11}(N_2, m_2 - k_0, k_0, j, \nu)
= \left[ \left( \frac{N_2}{k_0} \right) d(N_2 : \nu) \right]^{1/2} \left[ \Lambda^\nu(N_2, m_2, m_2 - j) \Lambda^\nu(N_2, m_2 - k_0, m_2 - k_0 - j) \right]^{1/2}
\times (-1)^{\phi(f_{m_2 - k_0, f_{m_2, f_{k_0}}}) + \phi(f_{m_2 - k_0, f_{m_2 - k_0, \nu}})} U(f_{m_2 - k_0, f_{m_2}, f_{m_2 - k_0, f_{m_2}}}; f_{k_0, \nu}) .
\]

(66)

Note that \( f_r = \{1^r\} \) and \( \nu = \{2^\nu, 1^{N-2^\nu}\} \) with \( N = N_1 \) or \( N_2 \) as appropriate. Also, the \( U \)-coefficient in \( X_{11} \) is with respect to \( U(N_1) \) while the one in \( Y_{11} \) is
with respect to $U(N_2)$. Formula for the $U$-coefficient appearing in Eq. (66) is available in [36] and it is given by

$$U(\{n\} \{N-p\} \{n\} \{p\}; \{n-p\} \{2^\nu 1^{N-2\nu}\}) = (-1)^\phi(N,n,p,\nu)$$

$$\times \left[ \frac{(p!)^2 (n-\nu)! (N+1)! [(N-n)!]^2 (N-\nu-p)! (N-2\nu+1)}{(\nu!)^2 (p-\nu)! (n-p)! [(N+1-\nu)!]^2 (N-n+p)! (N-n-\nu)!} \right]^{1/2}$$

The phase factor $(-1)^\phi$ in Eq. (67) depends on the phase convention [36] and we will fix this phase later. Eqs. (64)-(66) show that in evaluating the bivariate moment $\langle \hat{O}^\dagger(k_0)H^Q(k)\hat{O}(k_0)H^P(k) \rangle_{m_1,m_2}$ we can use $H(k)$ as $\sum_{i+j=k} H_1(i)H_2(j)$ and then the moment will be a sum of terms where each term is a product of two functions with one in the $m_1$ space [generated by $H_1(i)$ with body rank $i$] and other in the $m_2$ space [generated by $H_2(j)$ with body rank $j$]. Also, these functions follow from the results in Sections 2 and 3 by appropriate application. This will be seen in the formulas for the fourth order moments that are discussed below.

Turning to the fourth order moments, we need $M_{13}, M_{31}$ and $M_{22}$ (we have already discussed $M_{40}$ and $M_{04}$). As $\hat{O}^\dagger \neq \hat{O}$, here $M_{13} \neq M_{31}$ and similarly $M_{40} \neq M_{04}$ (also $M_{20} \neq M_{02}$). Following the procedure used for deriving the formula for $M_{11}(m_1,m_2)$ and the results [Eqs. (25) and (27)] given in Section 3 for $M_{31}(m)$, we have for $M_{31}(m_1,m_2)$,

$$M_{31}(m_1,m_2) = \langle \hat{O}^\dagger(k_0)H(k)\hat{O}(k_0)H^3(k) \rangle_{m_1,m_2}^{m_1,m_2}$$

$$= 2 \langle H^2(k) \rangle_{m_1,m_2}^{m_1,m_2} M_{11}(m_1,m_2) + V_3^2 \left\{ \binom{N_1}{m_1} \binom{N_2}{m_2} \right\}^{-1}$$

$$\times \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1,j_1)V_H^2(i_2,j_2) X_{31}(N_1,m_1,i_1,i_2,k_0) Y_{31}(N_2,m_2,j_1,j_2,k_0) ;$$
\[ X_{31}(N_1, m_1, i_1, i_2, k_0) = \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_1, \omega_{1}, \nu_1, \omega_{2}, \nu_2} \langle m_1, \alpha_1 | A_o(k_0) | m_1 + k_0, \alpha_2 \rangle \times \langle m_1 + k_0, \alpha_3 | A_o^\dagger(k_0) | m_1, \alpha_4 \rangle \langle m_1 + k_0, \alpha_2 | B^{\nu_1, \omega_{1}}(i_1) | m_1 + k_0, \alpha_3 \rangle \times \langle m_1, \alpha_5 | B^{\nu_2, \omega_{2}}(i_2) | m_1, \alpha_6 \rangle \langle m_1, \alpha_4 | B^{\nu_2, \omega_{2}}(i_2) | m_1, \alpha_5 \rangle \times \langle m_1, \alpha_6 | B^{\nu_2, \omega_{2}}(i_2) | m_1, \alpha_1 \rangle , \]

\[ Y_{31}(N_2, m_2, j_1, j_2, k_0) = \sum_{b_1, b_2, b_3, b_4, b_5, b_6, \alpha_1, \omega_{1}, \nu_1, \omega_{2}, \nu_2} \langle m_2, b_1 | A_o^\dagger(k_0) | m_2 - k_0, b_2 \rangle \times \langle m_2 - k_0, b_3 | A_o(k_0) | m_2, b_4 \rangle \langle m_2 - k_0, b_2 | C^{\nu_3, \omega_{3}}(j_1) | m_2 - k_0, b_3 \rangle \times \langle m_2, b_5 | C^{\nu_4, \omega_{4}}(j_1) | m_2, b_6 \rangle \langle m_2, b_4 | C^{\nu_4, \omega_{4}}(j_2) | m_2, b_5 \rangle \times \langle m_2, b_6 | C^{\nu_4, \omega_{4}}(j_2) | m_2, b_1 \rangle . \] (68)

The term \( X_{31} \) (similarly \( Y_{31} \)) is simplified using Eqs. (6), (7), (9) and (57) giving,

\[ X_{31}(N_1, m_1, i_1, i_2, k_0) = \left( \frac{N_1 - k_0}{m_1} \right)^{i_1} \sum_{i_2=0}^{m_1 - i_2} \Lambda^{\nu_2}(N_1, m_1, i_2) \times [\Lambda^{\nu_1}(N_1, m_1 - i_1) \Lambda^{\nu_1}(N_1, m_1 + k_0, m_1 + k_0 - i_1)]^{1/2} \times \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_1} C^{\nu_1, \omega_{1}}_{f_{m_1+k_0} \alpha_2, \overline{f_{m_1+k_0} \alpha_2}, \overline{f_{m_1+k_0} \alpha_2}} C^{\nu_1, \omega_{1}}_{f_{m_1 \alpha_5}, \overline{f_{m_1 \alpha_5}, \overline{f_{m_1 \alpha_5}}}} C^{\nu_2, \omega_{2}}_{f_{m_1 \alpha_4}, \overline{f_{m_1 \alpha_4}, \overline{f_{m_1 \alpha_4}}}} \times \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_1} C^{\nu_2, \omega_{2}}_{f_{m_1 \alpha_6}, \overline{f_{m_1 \alpha_6}, \overline{f_{m_1 \alpha_6}}}} . \] (69)

The sum over \( \alpha_5 \) and \( \alpha_6 \) of \( C^{\nu_1, \omega_{1}}_{f_{m_1 \alpha_5}, \overline{f_{m_1 \alpha_5}}, \overline{f_{m_1 \alpha_5}}} C^{\nu_2, \omega_{2}}_{f_{m_1 \alpha_6}, \overline{f_{m_1 \alpha_6}}, \overline{f_{m_1 \alpha_6}}} \) will give \( \delta_{\nu_1, \nu_2} \delta_{\omega_{1}, \omega_{2}} \).

Now, the remaining four C-G coefficients sum up to give a Racah coefficient. Carrying out a similar simplification of the C-G coefficients in \( Y_{31} \), we finally
obtain,

\[ M_{31}(m_1, m_2) = \langle \mathcal{O}(k_0) H(k) \mathcal{O}(k_0) H^3(k) \rangle_{m_1, m_2}^{m_1, m_2} = 2 \langle H^3(k) \rangle_{m_1, m_2} M_{11}(m_1, m_2) \]

\[ + V_3^2 \left\{ \begin{pmatrix} N_1 \\ m_1 \end{pmatrix} \begin{pmatrix} N_2 \\ m_2 \end{pmatrix} \right\}^{-1} \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1, j_1) V_H^2(i_2, j_2) \]

\[ \times \left[ \begin{pmatrix} N_1 - k_0 \\ m_1 \end{pmatrix} \right]^{\min(i_1, m_1-i_2)} \sum_{\nu_1=0} \Lambda^{\nu_1} (N_1, m_1, i_2) X_{11}(N_1, m_1, k_0, i_1, \nu_1) \]

\[ \times \left[ \begin{pmatrix} N_2 - k_0 \\ m_2 - k_0 \end{pmatrix} \right]^{\min(j_1, m_2-j_2)} \sum_{\nu_2=0} \Lambda^{\nu_2} (N_2, m_2, j_2) Y_{11}(N_2, m_2, k_0, j_1, \nu_2) \right]. \]

(70)

The functions \( X_{11} \) and \( Y_{11} \) are defined in Eq. (66). Following the same procedure as above, the formula for \( M_{13} \) is,

\[ M_{13}(m_1, m_2) = \langle \mathcal{O}(k_0) H^3(k) \mathcal{O}(k_0) H(k) \rangle_{m_1, m_2}^{m_1, m_2} = 2 \langle H^3(k) \rangle_{m_1+k_0, m_2-k_0} M_{11}(m_1, m_2) \]

\[ + V_3^2 \left\{ \begin{pmatrix} N_1 \\ m_1 \end{pmatrix} \begin{pmatrix} N_2 \\ m_2 \end{pmatrix} \right\}^{-1} \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1, j_1) V_H^2(i_2, j_2) \]

\[ \times \left[ \begin{pmatrix} N_1 - k_0 \\ m_1 \end{pmatrix} \right]^{\min(i_2, m_1+k_0-i_1)} \sum_{\nu_1=0} \Lambda^{\nu_1} (N_1, m_1+k_0, i_1) X_{11}(N_1, m_1, k_0, i_2, \nu_1) \]

\[ \times \left[ \begin{pmatrix} N_2 - k_0 \\ m_2 - k_0 \end{pmatrix} \right]^{\min(j_2, m_2-k_0-j_1)} \sum_{\nu_2=0} \Lambda^{\nu_2} (N_2, m_2-k_0, j_1) Y_{11}(N_2, m_2, k_0, j_2, \nu_2) \right]. \]

(71)

Formula for \( M_{22}(m_1, m_2) \) is more complicated and we will turn to this now. Using the \( H \) decomposition \( H(k) = \sum_{i+j=k} H_1(i) H_2(j) \) mentioned just
after the $M_{11}(m_1, m_2)$ formula, the $M_{22}(m_1, m_2)$ can be written as,

$$M_{22}(m_1, m_2) = \left( \langle O^\dagger(k_0) H^2(k) O(k_0) H^2(k) \rangle \right)^{m_1,m_2}$$

$$= \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} \sum_{i_3+j_3=k} \sum_{i_4+j_4=k} \text{Eavg} \left\{ \langle O^\dagger(k_0) H_1(i_1) H_2(j_1) H_1(i_2) \right.$$  

$$\times H_2(j_2) O(k_0) H_1(i_3) H_2(j_3) H_1(i_4) H_2(j_4) \rangle ^{m_1,m_2} \}$$

$$= \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} \left[ \text{Eavg} \left\{ \langle O^\dagger(k_0) H_1(i_1) H_2(j_1) H_1(i_2) H_2(j_2) O(k_0) H_1(i_3) H_2(j_3) H_1(i_4) H_2(j_4) \rangle ^{m_1,m_2} \right\} \right.$$  

$$+ \text{Eavg} \left\{ \langle O^\dagger(k_0) H_1(i_1) H_2(j_1) H_1(i_2) H_2(j_2) O(k_0) H_1(i_1) H_2(j_2) H_1(i_3) H_2(j_3) \rangle ^{m_1,m_2} \right\}$$

$$+ \text{Eavg} \left\{ \langle O^\dagger(k_0) H_1(i_1) H_2(j_1) H_1(i_2) H_2(j_2) O(k_0) H_1(i_1) H_2(j_2) H_1(i_3) H_2(j_3) \rangle ^{m_1,m_2} \right\} \right.$$  

Thus, the ensemble averaged $M_{22}$ decomposes into three terms. The first term is simple and the next two terms can be decomposed into averages in $m_1$ and $m_2$ spaces. Then we have,

$$M_{22}(m_1, m_2) = \left( \langle O^\dagger(k_0) O(k_0) \rangle \right)^{m_1,m_2} \left( \langle H^2(k) \rangle \right)^{m_1,m_2} \left( \langle H^2(k) \rangle \right)^{m_1+k_0,m_2-k_0}$$

$$+ V_2^2 \left\{ \left( \begin{array}{c} N_1 \\ m_1 \end{array} \right) \left( \begin{array}{c} N_2 \\ m_2 \end{array} \right) \right\}^{-1} \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1, j_1) V_H^2(i_2, j_2)$$

$$\times X_{22:a}(N_1, m_1, i_1, i_2, k_0) Y_{22:a}(N_2, m_2, j_1, j_2, k_0)$$

$$+ V_2^2 \left\{ \left( \begin{array}{c} N_1 \\ m_1 \end{array} \right) \left( \begin{array}{c} N_2 \\ m_2 \end{array} \right) \right\}^{-1} \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1, j_1) V_H^2(i_2, j_2)$$

$$\times X_{22:b}(N_1, m_1, i_1, i_2, k_0) Y_{22:b}(N_2, m_2, j_1, j_2, k_0).$$

(73)

Note that the three terms in Eq. (72) correspond directly to the three terms in Eq. (73). The second term involves $X_{22:a}$ and $Y_{22:a}$ functions. The $X_{22:a}$
function is
\[ X_{22:a}(N_1, m_1, i_1, i_2, k_0) = \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \nu_1, \nu_2, \omega_1, \omega_2} (m_1, \alpha_1 \mid A_a(k_0) \mid m_1 + k_0, \alpha_2) \]
\times \langle m_1 + k_0, \alpha_4 \mid A_a^\dagger(k_0) \mid m_1, \alpha_5 \rangle \langle m_1 + k_0, \alpha_2 \mid B^{\nu_1, \omega_1}(i_1) \mid m_1 + k_0, \alpha_3 \rangle
\times \langle m_1, \alpha_6 \mid B^{\nu_2, \omega_2}(i_2) \mid m_1, \alpha_1 \rangle \langle m_1 + k_0, \alpha_3 \mid B^{\nu_2, \omega_2}(i_2) \mid m_1 + k_0, \alpha_4 \rangle
\times \langle m_1, \alpha_5 \mid B^{\nu_2, \omega_2}(i_2) \mid m_1, \alpha_6 \rangle
\]
\[ = \binom{N_1 - k_0}{m_1} \sum_{\nu_1 = 0}^{i_1} \sum_{\nu_2 = 0}^{i_2} \left[ \Lambda^{\nu_1}(N_1, m_1 + k_0, m_1 + k_0 - i_1) \Lambda^{\nu_2}(N_1, m_1, m_1 - i_1) \right]^{1/2}
\times \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha} C^{\nu_1, \omega_1}_{f_{m_1 + k_0, \alpha_2}, f_{m_1 + k_0, \alpha_3}} C^{\nu_2, \omega_2}_{f_{m_1 + k_0, \alpha_6}, f_{m_1 + k_0, \alpha_5}}
\times C^{\nu_2, \omega_2}_{f_{m_1, \alpha_5}, f_{m_1, \alpha_6}} C^{\nu_1, \omega_1}_{f_{m_1, \alpha_1}, f_{m_1, \alpha_2}} C^{\nu_0, \alpha}_{f_{m_1 + k_0, \alpha_4}, f_{m_1 + k_0, \alpha}} \right] .
\]

We have applied Eq. (7) and the Wigner-Eckart theorem to get the second form in Eq. (74). Simplification of the six C-G coefficients will finally give a compact formula in terms of the \( X_{11} \) functions introduced earlier,

\[ X_{22:a}(N_1, m_1, i_1, i_2, k_0) = \binom{N_1}{k_0}^{-1} \binom{N_1 - k_0}{m_1}
\times \sum_{\nu_1 = 0}^{i_1} X_{11}(N_1, m_1, k_0, i_1, \nu_1) \sum_{\nu_2 = 0}^{i_2} X_{11}(N_1, m_1, k_0, i_2, \nu_2) .
\]

Similarly, we have

\[ Y_{22:a}(N_2, m_2, j_1, j_2, k_0) = \binom{N_2}{k_0}^{-1} \binom{N_2 - k_0}{m_2}
\times \sum_{\nu_1 = 0}^{j_1} Y_{11}(N_2, m_2, k_0, j_1, \nu_1) \sum_{\nu_2 = 0}^{j_2} Y_{11}(N_2, m_2, k_0, j_2, \nu_2) .
\]

(74)
Finally, the third term in $M_{22}(m_1, m_2)$ involves the functions $X_{22,b}$ and $Y_{22,b}$. The expression for function $X_{22,b}$ is,

$$X_{22,b}(N_1, m_1, i_1, i_2, k_0) = \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta, \nu_1, \omega_1, \rho_2} \langle m_1, \alpha_1 | A_{\alpha}(k_0) | m_1 + k_0, \alpha_2 \rangle \times \langle m_1 + k_0, \alpha_4 | A_{\alpha}^b(k_0) | m_1, \alpha_5 \rangle \langle m_1 + k_0, \alpha_2 | B^\nu_1,\omega_2^1(i_1) \rangle \langle m_1 + k_0, \alpha_4 \rangle \times \langle m_1, \alpha_5 | B^\nu_1,\omega_2^1(i_1) \rangle \langle m_1, \alpha_1 \rangle \ . \tag{77}$$

Now, simplifying Eq. (77) and similarly, $Y_{22,b}$, finally we get

$$X_{22,b}(N_1, m_1, i_1, i_2, k_0) = \left( \frac{N_1 - k_0}{m_1} \right) \left( \begin{array}{c} N_1 \\ k_0 \end{array} \right) d(N_1 : \nu) \right)^{1/2} \times \sum_{\nu_1=0}^{i_1} \sum_{\nu_2=0}^{i_2} \sum_{\nu=0}^{i_1+i_2} \langle m_1 + k_0 || B^\nu(i_1)B^{\nu_2}(i_2) \rangle \langle m_1 \rangle \times \langle m_1 || B^\nu(i_1)B^{\nu_2}(i_2) \rangle \langle m_1 \rangle \times (-1)^{\phi(f_{m_1+k_0,f_{m_1},k_0})+\phi(f_{m_1,f_{m_1},\nu})} \times U(f_{m_1+k_0,f_{m_1},f_{m_1+k_0},f_{k_0,\nu}}) ,$$

$$Y_{22,b}(N_2, m_2, j_1, j_2, k_0) = \left( \frac{N_2 - k_0}{m_2} \right) \left( \begin{array}{c} N_2 \\ k_0 \end{array} \right) d(N_2 : \nu) \right)^{1/2} \times \sum_{\nu_1=0}^{j_1} \sum_{\nu_2=0}^{j_2} \sum_{\nu=0}^{j_1+j_2} \langle m_2 || B^\nu(j_1)B^{\nu_2}(j_2) \rangle \langle m_2 \rangle \times \langle m_2 - k_0 || B^\nu(j_1)B^{\nu_2}(j_2) \rangle \langle m_2 - k_0 \rangle \times (-1)^{\phi(f_{m_2,f_{m_2-k_0},f_{k_0}})+\phi(f_{m_2-k_0,f_{m_2-k_0},f_{k_0}})} U(f_{m_2,f_{m_2-k_0},f_{m_2-k_0},f_{k_0,\nu}}) \ . \tag{78}$$

Combining Eqs. (78), (75), and (76) with (73) will give the formula for $M_{22}(m_1, m_2)$. Note that the reduced matrix elements in Eq. (78) are given by Eq. (35) along with Eq. (7).
6. Asymptotic results for two types of spinless fermions with beta and double beta decay type transition operators

Employing the formulas derived in Section 5, for some typical values, appropriate for atomic nuclei, for \( N_1, N_2, m_1 \) and \( m_2 \) with \( k = 2 \) and \( k_0 \) taking values 1 and 2, numerical results for \( \xi(m_1, m_2) \) and \( k_{rs}(m_1, m_2) \) with \( r + s = 4 \) are shown in Tables 2 and 3. In the examples shown in the Table, \( N_1 \) and \( N_2 \) are sufficiently large but not \( m_1 \) and \( m_2 \) for all the examples. For the situation with \( m_1 \) and \( m_2 \) sufficiently large (typically larger than say 6), the \(|k_{rs}(m_1, m_2)| \) are \( \leq 0.3 \) pointing that the bivariate transition strength density is close to a bivariate Gaussian. Also, typically \( \xi \approx 0.6 \) shows that for the systems considered, EGUE is essential (as shown in [24], \( \xi = 0 \) for a GUE). For further understanding the smallness of \( k_{rs}(m_1, m_2) \), we will derive asymptotic formulas first for \( M_{rs}(m_1, m_2) \) and using them, for \( k_{rs}(m_1, m_2) \) with \( r + s = 4 \) and also for \( \xi(m_1, m_2) \).

Let us begin with the asymptotic limit defined by \( N_1 \to \infty \), \( N_2 \to \infty \), \( m_1 \), \( m_2 \) fixed with \( k \) and \( k_0 \) much smaller than \( m_1 \) and \( m_2 \). Note that in the dilute limit (or true asymptotic limit) we also have \( m_1 \to \infty \), \( m_2 \to \infty \), \( m_1/N_1 \to 0 \) and \( m_2/N_2 \to 0 \) and we will consider this in the later part of this section. First, we consider the following functions,

\[
A_1(N, m, i, t) = \left[ \binom{N}{t} d(N : i) \Lambda^i(N, m, m - i) \Lambda^i(N, m + t, m + t - i) \right]^{1/2} \\
\times \binom{N}{m}^{-1} \binom{N - t}{m} \left| U(f_{m+t}, \overline{f}_m, f_{m+t}, \overline{f}_m; f_t, i) \right|,
\]

\[
A_2(N, m, i, t) = \left[ \binom{N}{t} d(N : i) \Lambda^i(N, m, m - i) \Lambda^i(N, m - t, m - t - i) \right]^{1/2} \\
\times \binom{N}{m}^{-1} \binom{N - t}{m - t} \left| U(f_m, \overline{f}_{m-t}, f_m, \overline{f}_{m-t}; f_t, i) \right|,
\]

\[
T(N, m, i) = \Lambda^0(N, m, i),
\]

\[
F(N, m, i, j) = \binom{N}{m}^{-1} \Lambda^i(N, m, m - i) \Lambda^i(N, m, j) d(N : i).
\]

(79)
Using Eq. (7) for $\Lambda$, Eq. (14) for $d(N : i)$ and Eq. (67) for the $U$-coefficient, $A_1$ and $A_2$ in the asymptotic limit simplify to

$$A_1(N, m, i, t) = (\binom{m}{i}) (\binom{N}{i}) (\binom{N - m - i}{t}) (\binom{N - i + 1}{i})^{-1}$$

and

$$A_2(N, m, i, t) = (\binom{m}{i}) (\binom{N + 1}{i}) (\binom{N - m}{i}) (\binom{m - i}{t}) (\binom{N - i + 1}{i})^{-1}$$

Similarly $T$ and $F$ are given by,

$$T(N, m, i) \xrightarrow{\text{asymp}} (\binom{m}{i}) (\binom{N}{i}) ,$$

$$F(N, m, i, j) \xrightarrow{\text{asymp}} (\binom{m}{i}) (\binom{m - i}{j}) (\binom{N}{i}) (\binom{N}{j}) .$$

Using Eqs. (61) and (82),

$$\langle H^2(k) \rangle_{m_1, m_2} \xrightarrow{\text{asymp}} \sum_{i+j=k} V^2_H(i, j) T(N_1, m_1, i) T(N_2, m_2, j) .$$

Similarly for $M_{11}(m_1, m_2)$ given by Eq. (66), in the asymptotic limit only the terms with $\nu_1 = i$ in $X_{11}$ and $\nu_2 = j$ in $Y_{11}$ will contribute. Then, simplifying using Eqs. (80) and (81) we have

$$M_{11}(m_1, m_2) \xrightarrow{\text{asymp}} V^3 \sum_{i+j=k} V^2_H(i, j) \binom{N_1 - m_1 - i}{k_0} \binom{m_2 - j}{k_0} T(N_1, m_1, i) T(N_2, m_2, j) .$$

Turning to fourth order moments, firstly for $\langle H^4(k) \rangle_{m_1, m_2}$, using Eqs. (62)
and (82), we get

\[
\langle H^4(k) \rangle_{m_1,m_2} \overset{\text{asym}}{\to} 2 \left( \langle H^2(k) \rangle_{m_1,m_2} \right)^2 + \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1,j_1) V_H^2(i_2,j_2) 
\times F(N_1, m_1, i_1, i_2) F(N_2, m_2, j_1, j_2)
\]

(85)

This formula follows from the fact that in Eq. (62), in the asymptotic limit only terms with \( \nu_1 = i \) in \( X \) and \( \nu_2 = j \) in \( Y \) will contribute. Equation (85) also gives \( M_{40}(m_1, m_2) \) and \( M_{04}(m_1, m_2) \) via Eq. (63).

The first non-trivial fourth order moment is \( M_{31}(m_1, m_2) \) and it is given by Eq. (70). As only terms with \( \nu_1 = i_1 \) in \( X_{11} \) and \( \nu_2 = j_1 \) in \( Y_{11} \) in Eq. (70) will contribute in the asymptotic limit, we have

\[
M_{31}(m_1, m_2) = 2 \langle H^2(k) \rangle_{m_1,m_2} M_{11}(m_1, m_2) 
+ V_3^2 \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1,j_1) V_H^2(i_2,j_2) \left\{ \Lambda^{i_1}(N_1, m_1, i_2) A_1(N_1, m_1, i_1, k_0) 
+ \Lambda^{j_1}(N_2, m_2, j_2) A_2(N_2, m_2, j_1, k_0) \right\}
\]

(86)

Substituting Eqs. (80) and (81) for \( A_1 \) and \( A_2 \) respectively and Eq. (7) for \( \Lambda \) will give, \( \Lambda \)

\[
M_{31}(m_1, m_2) \overset{\text{asym}}{\to} 2 \langle H^2(k) \rangle_{m_1,m_2} M_{11}(m_1, m_2) 
+ V_3^2 \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1,j_1) V_H^2(i_2,j_2) \left( N_1 - m_1 - i_1 \right) \left( m_2 - j_1 \right) \]

(87)

\times F(N_1, m_1, i_1, i_2) F(N_2, m_2, j_1, j_2).

Moment \( M_{13}(m_1, m_2) \) is given by Eq. (71) and in the asymptotic limit only
the terms with \( \nu_1 = i_2 \) in \( X_{11} \) and \( \nu_2 = j_2 \) in \( Y_{11} \) will survive. Then,

\[
M_{13}(m_1, m_2) = 2 \langle H^2(k) \rangle^{m_1+k_0,m_2-k_0} M_{11}(m_1, m_2)
\]

\[
+ V^2_\delta \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V^2_H(i_1, j_1) V^2_H(i_2, j_2) \Lambda_2(N_1, m_1+i_1,k_0) A_1(N_1, m_1, i_2,k_0) A_1(N_1, m_1, i_2,k_0)
\]

\[
\times \Lambda_2(N_2, m_2-k_0, j_1) A_2(N_2, m_2, j_2,k_0).
\]

Substituting Eqs. (80) and (81) for \( A_1 \) and \( A_2 \) respectively and Eq. (7) for \( \Lambda \) will give,

\[
M_{13}(m_1, m_2) \xrightarrow{\text{asymp}} 2 \langle H^2(k) \rangle^{m_1+k_0,m_2-k_0} M_{11}(m_1, m_2)
\]

\[
+ V^2_\delta \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V^2_H(i_1, j_1) V^2_H(i_2, j_2) \left( \frac{m_1+k_0-i_2}{i_1} \right)
\]

\[
\times \left( \frac{N_1-m_1-k_0+i_1-i_2}{i_1} \right) \left( \frac{m_1}{i_2} \right) \left( \frac{N_1-m_1-i_2}{k_0} \right)
\]

\[
\times \left( \frac{m_2-k_0-j_2}{j_1} \right) \left( \frac{N_2-m_2+k_0+j_1-j_2}{j_1} \right) \left( \frac{m_2}{j_2} \right) \left( \frac{N_2}{j_2} \right) \left( \frac{m_2-j_2}{k_0} \right).
\]

Finally, the most complicated moment \( M_{22}(m_1, m_2) \) is given by Eq. (73) and it has three terms. The first term \( M^{(1)}_{22}(m_1, m_2) \) is simple,

\[
M^{(1)}_{22}(m_1, m_2) = \langle O^t(k_0) O(k_0) \rangle^{m_1,m_2} \langle H^2(k) \rangle^{m_1,m_2} (H^2(k))^{m_1+k_0,m_2-k_0}.
\]

The second term \( M^{(2)}_{22}(m_1, m_2) \) is given by Eq. (73) with functions \( X_{22,a} \) and \( Y_{22,a} \) given by Eqs. (75) and Eq. (76) respectively. In the asymptotic limit, only the terms with \( \nu_1 = i_1 \) and \( \nu_2 = i_2 \) will survive in Eq. (75). Similarly, only the terms with \( \nu_1 = j_1 \) and \( \nu_2 = j_2 \) will survive in Eq. (76). These will
give

\[
M_{22}^{(2)}(m_1, m_2) = V_2^D \sum_{i_1 + j_1 = k} \sum_{i_2 + j_2 = k} V_H^2(i_1, j_1) V_H^2(i_2, j_2) \times \left( \frac{N_1}{k_0} \right)^{-1} \left( \frac{N_1}{m_1} \right)^{-1} A_1(N_1, m_1, i_1, k_0) A_1(N_1, m_1, i_2, k_0)
\]

\[
\times \left( \frac{N_2}{k_0} \right)^{-1} \left( \frac{N_2}{m_2} \right)^{-1} A_2(N_2, m_2, j_1, k_0) A_2(N_2, m_2, j_2, k_0).
\]

Substituting Eqs. (80) and (81) for \( A_1 \) and \( A_2 \) respectively and further simplifications using the assumption that \( m_1 \gg k, k_0 \) and \( m_2 \gg k, k_0 \) will give,

\[
M_{22}^{(2)}(m_1, m_2) \xrightarrow{\text{asymp}} V_2^D \sum_{i_1 + j_1 = k} \sum_{i_2 + j_2 = k} V_H^2(i_1, j_1) V_H^2(i_2, j_2) \left( m_2 - j_2 \right)_0
\]

\[
\times \left( \frac{N_1 - m_1 - i_1 - i_2}{k_0} \right) T(N_1, m_1, i_1) T(N_1, m_1, i_2) T(N_2, m_2, j_1) T(N_2, m_2, j_2).
\]

Lastly, the third term \( M_{22}^{(3)}(m_1, m_2) \) of \( M_{22}(m_1, m_2) \) is given by Eq. (73) along with Eq. (78). In the asymptotic limit only the terms with \( \nu_1 = i_1, \nu_2 = i_2 \) and \( \nu = i_1 + i_2 \) will survive in \( X_{22, b} \) given by Eq. (78). Similarly, only terms with \( \nu_1 = j_1, \nu_2 = j_2 \) and \( \nu = j_1 + j_2 \) will survive in \( Y_{22, b} \). Then,

\[
M_{22}^{(3)}(m_1, m_2) = V_2^D \left( \frac{N_1}{m_1} \right)^{-1} \left( \frac{N_2}{m_2} \right)^{-1} \left( \frac{N_1}{k_0} \right)^{1/2} \left( \frac{N_2}{k_0} \right)^{1/2} \left( \frac{N_1 - k_0}{m_1} \right) \left( \frac{N_2 - k_0}{m_2} \right)
\]

\[
\times \sum_{i_1 + j_1 = k} \sum_{i_2 + j_2 = k} V_H^2(i_1, j_1) V_H^2(i_2, j_2) \{ d(N_1 : i_1 + i_2) d(N_2 : j_1 + j_2) \}^{1/2}
\]

\[
\times \left\langle m_1 + k_0 || [B^{i_1}(i_1)B^{i_2}(i_2)]^{i_1+i_2} || m_1 + k_0 \right\rangle \left\langle m_1 || [B^{i_1}(i_1)B^{i_2}(i_2)]^{i_1+i_2} || m_1 \right\rangle
\]

\[
\times \left\langle m_2 || [B^{j_1}(j_1)B^{j_2}(j_2)]^{j_1+j_2} || m_2 \right\rangle \left\langle m_2 || [B^{j_1}(j_1)B^{j_2}(j_2)]^{j_1+j_2} || m_2 \right\rangle
\]

\[
\times \left\langle m_2 - k_0 || [B^{j_1}(j_1)B^{j_2}(j_2)]^{j_1+j_2} || m_2 - k_0 \right\rangle \left\langle m_2 - k_0 || [B^{j_1}(j_1)B^{j_2}(j_2)]^{j_1+j_2} || m_2 - k_0 \right\rangle.
\]

\[
(93)
\]
Now, simplifying the reduced matrix elements as in Section 4 along with the $U$-coefficients using Eqs. (80) and (81) will give the final result,

$$M_{22}^{(3)}(m_1, m_2) \xrightarrow{\text{asympt}} V_{\delta}^2 \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1, j_1)V_H^2(i_2, j_2) \times \left( \frac{N_1 - m_1 - i_1 - i_2}{k_0} \right) \left( \frac{m_2 - j_1 - j_2}{k_0} \right) F(N_1, m_1, i_1, i_2) F(N_2, m_2, j_1, j_2) .$$

Combining Eqs. (90), (92) and (94) we have,

$$M_{22}(m_1, m_2) \xrightarrow{\text{asympt}} \langle \mathcal{O}(k_0) \rangle^m_{m_1, m_2} \langle H^2(k) \rangle^{m_1} \langle H^2(k) \rangle^{m_1 + k_0, m_2 - k_0} + V_{\delta}^2 \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} V_H^2(i_1, j_1)V_H^2(i_2, j_2) \left\{ \left( \frac{N_1 - m_1 - i_1 - i_2}{k_0} \right) \left( \frac{m_2 - j_1 - j_2}{k_0} \right) T(N_1, m_1, i_1) T(N_1, m_1, i_2) T(N_2, m_2, j_1) T(N_2, m_2, j_2) \right\} .$$

The asymptotic formulas given by Eqs. (83), (84), (85), (87), (89) and (95) are identical to those obtained using the asymptotic theory of Mon and French as derived in detail in [25] for EGOE. This agreement gives a good check of the exact formulas derived in Section 5. They also show (as the asymptotic results should be valid for any $k$ and $k_0$) that the term $(-1)^{\phi(...) + \phi(...)} U(...) \langle \mathcal{O}(k_0) \rangle^m_{m_1, m_2} \langle H^2(k) \rangle^{m_1} \langle H^2(k) \rangle^{m_1 + k_0, m_2 - k_0}$ in Eqs. (66) and (78) will be $|U(...) |$, i.e. the phase of the $U$-coefficient [see Eq. (67)] cancels with the phase factor $(-1)^{\phi(...) + \phi(...)}$ in these equations. Rewriting Eq. (67) in a form that extends easily to boson systems, we have for $U^2$,

$$[U(f_m, \overline{f}_p; f_m, f_p; f_{m-p}^\nu)]^2 = \frac{(N+1)^2(p-p^\nu)(N-p^\nu)}{(N-p^\nu)} \frac{(N-2\nu+1)}{(N-2\nu)} \frac{(N^\nu)}{(N)} \frac{(N-\nu)}{(N)} \frac{(N-\nu)}{(N)} \frac{(N+1)}{(N+1)} \frac{(N+1)}{(N+1)}. \quad (96)$$

We will discuss extension of Eq. (96) to boson systems in Section 8.

All the formulas given above simplify further in the dilute limit (or strict asymptotic limit) defined by $N_1 \to \infty$, $N_2 \to \infty$, $m_1 \to \infty$, $m_2 \to \infty$, $\nu \to \infty$.}

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\( m_1/N_1 \to 0 \) and \( m_2/N_2 \to 0 \) and \( k \) and \( k_0 \) fixed. Also, assuming that 
\( V^2_H(i,j) = V^2_H \) independent of \( i \) and \( j \), the reduced moments and cumulants will be independent of \( V^2_O \) and \( V^2_H \). With these, the dilute limit formulas for \( \xi \) and \( k_{rs} \) with \( r + s = 4 \) are obtained using Eqs. (37), (38), (58), (83), (84), (85), (87), (89) and (95). The results are as follows. Firstly, it is easy to see that 
\[ \hat{M}_{P0} = M_{P0}/M_{00} = \langle H^P(k) \rangle^{m_1,m_2} \] 
and 
\[ \hat{M}_{0P} = M_{0P}/M_{00} = \langle H^P(k) \rangle^{m_1+k_0,m_2-k_0} \]; \( P = 2, 4 \). Using Eqs. (83), (84), (85) and (58) we have
asymptotic formulas for bivariate cumulants $\xi$, $k_{40}$ and $k_{04}$,

$$\xi(m_1, m_2) = \frac{M_{11}(m_1, m_2)}{M_{00}(m_1, m_2) \left[ \tilde{M}_{20}(m_1, m_2) \tilde{M}_{02}(m_1, m_2) \right]^{1/2}}$$

$$\text{asym} \left[ \left. \left( \frac{m_2}{k_0} \right) \left\{ \sum_{i_1+j_1=k} T(N_1, m_1, i_1) T(N_2, m_2, j_1) \right\}^{1/2} \right]^{-1}$$

$$\times \sum_{i_2+j_2=k} T(N_1, m_1 + k_0, i_2) T(N_2, m_2 - k_0, j_2) \right)$$

$$\text{asym} \left[ \sum_{i+j=k} \left( \frac{m_2 - j}{k_0} \right) T(N_1, m_1, i) T(N_2, m_2, j) \right]^{-1}$$

$$k_{40}(m_1, m_2) = \frac{\tilde{M}_{40}(m_1, m_2)}{\left[ \tilde{M}_{20}(m_1, m_2) \right]^2} - 3$$

$$\text{asym} \left[ \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} F(N_1, m_1, i_1, i_2) F(N_2, m_2, j_1, j_2) \right]^{2} \left[ \sum_{i+j=k} T(N_1, m_1, i) T(N_2, m_2, j) \right]^{-1}$$

$$k_{04}(m_1, m_2) = \frac{\tilde{M}_{04}(m_1, m_2)}{\left[ \tilde{M}_{02}(m_1, m_2) \right]^2} - 3$$

$$\text{asym} \left[ \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} F(N_1, m_1 + k_0, i_1, i_2) F(N_2, m_2 - k_0, j_1, j_2) \right]^{2} \left[ \sum_{i+j=k} T(N_1, m_1 + k_0, i) T(N_2, m_2 - k_0, j) \right]^{-1}$$

Note that the functions $T$ and $F$ are given by Eq. (82). Similarly, Eqs. (87)
and (89) will give the formulas for $k_{31}$ and $k_{13}$ respectively,

$$k_{31}(m_1, m_2) = \frac{\tilde{M}_{31}(m_1, m_2)}{[\tilde{M}_{20}(m_1, m_2)]^{3/2}[\tilde{M}_{02}(m_1, m_2)]^{1/2}} - 3\xi(m_1, m_2)$$

$$\text{asymp} \quad \xrightarrow{\text{asymptotic}} \quad -\xi(m_1, m_2) + \left\{ \frac{m_2}{k_0} \left[ \sum_{i_1+j_1=k} T(N_1, m_1, i_1) T(N_2, m_2, j_1) \right] \right\}^{3/2}$$

$$\times \left[ \sum_{i_1+j_1=k} T(N_1, m_1 + k_0, i_2) T(N_2, m_2 - k_0, j_2) \right]^{-1}$$

$$\times \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} \left( \frac{m_2-j_1}{k_0} \right) F(N_1, m_1, i_1, i_2) F(N_2, m_2, j_1, j_2) ,$$

$$k_{13}(m_1, m_2) = \frac{\tilde{M}_{13}(m_1, m_2)}{[\tilde{M}_{20}(m_1, m_2)]^{1/2}[\tilde{M}_{02}(m_1, m_2)]^{3/2}} - 3\xi(m_1, m_2)$$

$$\text{asymp} \quad \xrightarrow{\text{asymptotic}} \quad -\xi(m_1, m_2) + \left\{ \frac{m_2}{k_0} \left[ \sum_{i_1+j_1=k} T(N_1, m_1, i_1) T(N_2, m_2, j_1) \right] \right\}^{1/2}$$

$$\times \left[ \sum_{i_1+j_1=k} T(N_1, m_1 + k_0, i_2) T(N_2, m_2 - k_0, j_2) \right]^{-1}$$

$$\times \sum_{i_1+j_1=k} \sum_{i_2+j_2=k} \left( \frac{m_2-j_2}{k_0} \right) T(N_1, m_1, i_2) T(N_2, m_2, j_2)$$

$$\times \left( \begin{array}{c} N_1 \\ i_1 \end{array} \right) \left( \begin{array}{c} m_1 + k_0 - i_2 \\ i_1 \end{array} \right) \left( \begin{array}{c} N_2 \\ j_1 \end{array} \right) \left( \begin{array}{c} m_2 - k_0 - j_2 \\ j_1 \end{array} \right).$$

(98)
Finally, using (95) we have,

\[
k_{22}(m_1, m_2) = \frac{\tilde{M}_{22}(m_1, m_2)}{\tilde{M}_{20}(m_1, m_2) \tilde{M}_{02}(m_1, m_2)} - 2 \xi^2(m_1, m_2) - 1
\]

\[
\text{asymp} - 2 \xi^2(m_1, m_2) + \left\{ \left( \frac{m_2}{k_0} \right) \sum_{i_1 + j_1 = k} T(N_1, m_1, i_1) T(N_2, m_2, j_1) \right\}^{-1}
\]

\[
\times \sum_{i_2 + j_2 = k} T(N_1, m_1 + k_0, i_2) T(N_2, m_2 - k_0, j_2)
\]

\[
\times \sum_{i_1 + j_1 = k} \sum_{i_2 + j_2 = k} \left( \frac{m_2 - j_1 - j_2}{k_0} \right) \left\{ T(N_1, m_1, i_1) T(N_1, m_1, i_2) \right\}
\]

\[
\times T(N_2, m_2, j_1) T(N_2, m_2, j_2) + F(N_1, m_1, i_1, i_2) F(N_2, m_2, j_1, j_2) \}
\]

(99)

Numerical results obtained using Eqs. (97) - (99) are shown in Tables 2 for \(k_0 = 2\) and \(k = 2\) and similarly in Table 3 for \(k_0 = 1\) and \(k = 2\). It is seen that in general \(|k_{PQ}(m_1, m_2)| \leq 0.3\) implying that the strength densities are close to a bivariate Gaussian. However, as seen from Tables 2 and 3, it is necessary to add the corrections due to \(k_{PQ}\). Also, expanding \(k_{PQ}\) in powers of \(1/m_1\) and \(1/m_2\) using Mathematica, it is seen that all the \(k_{PQ}\) with \(P + Q = 4\) behave as, for \(k = 2\) and \(k_0 = 2\),

\[
k_{PQ} = -\frac{4}{m_1} + O \left( \frac{1}{m_1^2} \right) + O \left( \frac{m_2^3}{m_1^3} \right) + \ldots .
\]

(100)

Therefore, for \(m_1 >> 1\) and \(m_2 << m_1^{3/2}\), the transition strength density approaches bivariate Gaussian form in general. It is important to recall that the strong dependence on \(m_1\) in Eq. (100) is due to the nature of the operator \(\mathcal{O}\) i.e., \(\mathcal{O}(k_0) |m_1, m_2\rangle = |m_1 + k_0, m_2 - k_0\rangle\). Thus, we conclude that bivariate Gaussian form with corrections (see Appendix A for the form with Edgeworth corrections) due to \(k_{PQ}\), \(P + Q = 4\) will form a good approximation for transition strength densities generated by beta and double beta decay type operators.
7. Lower-order moments of transition strength densities: results for particle removal operators

Particle removal (or addition) operators are of great interest in nuclear physics. For example one particle (proton or neutron) removal from a target nucleus gives information about the single particle levels in the target and similarly, two-particle removal gives information about pairing force [42, 43]. Let us begin with \( m \) spinless fermions in \( N \) sp states and a particle removal operator \( \mathcal{O} \) that removes \( k_0 \) number of particles when acting on a \( m \) fermion state. Then the general form of \( \mathcal{O} \) is,

\[
\mathcal{O} = \sum_{\alpha_0} V_{\alpha_0} A_{\alpha_0}(k_0) .
\]  

(101)

Here, \( A_{\alpha_0}(k_0) \) is a \( k_0 \) particle annihilation operator and \( \alpha_0 \) are indices for a \( k_0 \) particle state. Note that \( A_{\alpha_0}(k_0) \) transforms as \( \{ f_{k_0} \} = \{ 1^{N-k_0} \} \) with respect to \( U(N) \) and \( A_{\alpha_0}^\dagger(k_0) \) transforms as \( \{ f_{k_0} \} \). It important to recognize that the \( \mathcal{O} \) matrices will be rectangular matrices connecting \( m \) particle states to \( m - k_0 \) particle states. In the defining space, the matrix will be a \( 1 \times d_0 \) matrix with matrix elements given by \( V_{\alpha_0} \). Note that \( \alpha_0 \) takes \( d_0 \) values and \( d_0 = \binom{N}{k_0} \). We will represent the \( \mathcal{O} \) matrix in the defining space by GUE implying that the defining space matrix elements \( V_{\alpha_0} \) are zero centered independent Gaussian random variables. Also, the \( V_{\alpha_0} \) are assumed to be independent of the \( V_{ij}(k) \) variables in Eq. (2) and therefore also independent of the \( W \) variables in Eq. (4)] with variance satisfying

\[
\overline{V_\alpha V_\beta^\dagger} = V_\mathcal{O}^2 \delta_{\alpha\beta} .
\]  

(102)

In many particle spaces the \( \mathcal{O} \) matrices will be \( d_1 \times d_2 \) matrices connecting \( d_1 = \binom{N}{m} \) number of \( m \)-particle states to \( d_2 = \binom{N}{m-k_0} \) number of \( (m-k_0) \)-particle states. Fig. 4 gives an example for the \( \mathcal{H} \) and \( \mathcal{O} \) matrices in the defining space and in the \( m \) particle spaces. Using Eqs. (101) and (102), we have

\[
\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m = V_\mathcal{O}^2 \binom{m}{k_0} , \quad \langle \mathcal{O} \mathcal{O}^\dagger \rangle^m = V_\mathcal{O}^2 \binom{N-m}{k_0} .
\]  

(103)

Similarly, Eq. (15) gives the relations,

\[
\langle \mathcal{O}^\dagger \mathcal{O} \mathcal{H}^p \rangle^m = \langle \mathcal{O}^\dagger \mathcal{O} \rangle^m \langle \mathcal{H}^p \rangle^m , \quad \langle \mathcal{O}^\dagger \mathcal{H}^p \mathcal{O} \rangle^m = \langle \mathcal{O}^\dagger \mathcal{O} \rangle^m \langle \mathcal{H}^p \rangle^{m-k_0} .
\]  

(104)
We will also make use of Eq. (57) given before. Following the procedure used in Section 2, it is possible to derive formulas for the lower order bivariate moments of the transition strength densities generated by $O$ defined by Eq. (101). Just as in Sections 3 and 5, we will consider the bivariate moments

$$M_{PQ} = \langle O^\dagger H^P O H^Q \rangle^m$$

(105)

with $P + Q = 2$ and $4$ (the $P + Q = 3$ moments are zero as we are using independent EGUE representations for $O$ and $H$ matrices in many particle spaces).

7.1. Exact formulas for the bivariate moments

Firstly, Eqs. (104) gives,

$$M_{20} = \langle O^\dagger O \rangle^m \langle H^2 \rangle^m, \quad M_{02} = \langle O^\dagger O \rangle^m \langle H^2 \rangle^m - k_0,$$

$$M_{40} = \langle O^\dagger O \rangle^m \langle H^4 \rangle^m, \quad M_{04} = \langle O^\dagger O \rangle^m \langle H^4 \rangle^m - k_0.$$

(106)

Now, Eq. (103) along with Eqs. (11) and (13) will give the formulas for $M_{20}$, $M_{02}$, $M_{40}$ and $M_{04}$. Formula for the first non-trivial moment $M_{11}$ is derived by introducing complete set of states between $O^\dagger$ and $H$, $H$ and $O$ and $O$ and $H$ in the trace giving,

$$M_{11}(m) = \langle O^\dagger H O H \rangle^m =$$

$$\left( \begin{array}{c} N \\ m \end{array} \right)^{-1} \sum_{v_1,v_2,v_3,v_4} \langle m,v_1 \mid O^\dagger \mid m - k_0, v_2 \rangle \langle m - k_0, v_3 \mid O \mid m, v_4 \rangle$$

$$\times \langle m - k_0, v_2 \mid H \mid m - k_0, v_3 \rangle \langle m, v_4 \mid H \mid m, v_1 \rangle.$$  \hspace{1cm} (107)

Using Eq. (101) and applying Eq. (102) along with Eqs. (4) - (7) and the Wigner-Eckart theorem will give,

$$M_{11}(m) = V_3^2 V_H^2 \left( \begin{array}{c} N \\ m \end{array} \right)^{-1} \left( \begin{array}{c} N - k_0 \\ m - k_0 \end{array} \right)$$

$$\sum_{\nu=0}^k \left[ \Lambda^\nu(N,m-k_0,m-k_0-k) \Lambda^\nu(N,m,m-k) \right]^{1/2}$$

$$\times \sum_{v_1,v_2,v_3,v_4;\alpha;\omega} C_{f_{m_1}v_1,f_{m-k_0}v_2}^{J_{k_0}J_{k_0}} C_{f_{m-k_0}v_3,f_{m-k_0}v_4}^{J_{k_0}J_{k_0}} C_{f_{m-k_0}v_2,f_{m-k_0}v_3}^{\nu\nu} C_{f_{m}v_4,f_{m}v_1}^{\nu\nu}.$$  \hspace{1cm} (108)
Simplifying the four C-G coefficients will give finally,

\[ M_{11}(m) = V_2^2 V_H^2 \left( \frac{N}{m} \right)^{-1} (N - k_0) \sum_{\nu = 0}^{k} Z_{11}(N, m, k_0, k, \nu) ; \]

\[ Z_{11}(N, m, k_0, k, \nu) = \left( \begin{array}{c} N \\ k_0 \end{array} \right) d(N : \nu) \Lambda^\nu(N, m - k) \Lambda^\nu(N, m - k_0, m - k_0 - k)^{1/2} \times |U(f_m f_{m - k_0} f_m f_{m - k_0} : f_{k_0} \nu)|. \]

(109)

Here, \(|U|\) appears as discussed just after Eq. (95) and Eq. (96) gives the formula for \(U^2\).

Turning to the fourth order moments, we need \(M_{13}, M_{31}\) and \(M_{22}\). As \(O^\dagger \neq O\), here \(M_{13} \neq M_{31}\) [similarly \(M_{40} \neq M_{04}\) and \(M_{20} \neq M_{02}\) as seen from Eq. (106)]. Following the procedure used for deriving the formula for \(M_{11}(m)\), we have for \(M_{31}(m)\)

\[ M_{31}(m) = (O^\dagger H O H^3)^m = 2 \langle H^2 \rangle^m M_{11}(m) \]

\[ + V_2^2 V_H^2 \left( \frac{N}{m} \right)^{-1} \sum_{v_1, v_2, v_3, v_4, v_5, v_6 : \nu_1, \nu_2, \nu_3, \nu_4} \langle m, v_1 | A_{\nu_1}^\dagger(k_0) | m - k_0, v_2 \rangle \langle m - k_0, v_3 | A_{\nu_2}(k_0) | m, v_4 \rangle \times \langle m - k_0, v_2 | B^{v_3, \nu_3}(k) | m - k_0, v_3 \rangle \langle m, v_5 | B^{v_4, \nu_4}(k) | m, v_6 \rangle \times \langle m, v_4 | B^{v_5, \nu_5}(k) | m, v_5 \rangle \langle m, v_6 | B^{v_6, \nu_6}(k) | m, v_1 \rangle . \]

(110)

Now, applying the Wigner-Eckart theorem, using the results in Section 2 and simplifying the resulting C-G coefficients will give,

\[ M_{31}(m) = (O^\dagger H O H^3)^m = 2 \langle H^2 \rangle^m M_{11}(m) \]

\[ + V_2^2 V_H^2 \left( \frac{N}{m} \right)^{-1} (N - k_0) \sum_{\nu = 0}^{\min(k, m - k)} \Lambda^\nu(N, m, k) Z_{11}(N, m, k_0, k, \nu) . \]

(111)

The function \(Z_{11}\) is defined in Eq. (109). Following the same procedure as
above, the formula for $M_{13}$ is,

\[
M_{13}(m) = \langle \mathcal{O}^3 \mathcal{O}^3 \mathcal{O} \rangle^m = 2 \langle H^2 \rangle^{m-k_0} M_{11}(m)
\]

\[
+ V_0^2 V_p^2 \left( \frac{N}{m} \right)^{-1} \left( \frac{N-k_0}{m-k_0} \right) \sum_{\nu=0}^{\min(k,m-k_0-k)} \Lambda^\nu(N, m - k_0, k) Z_{11}(N, m, k_0, k, \nu).
\]

(112)

Formula for $M_{22}$ follows from the formula given in Section 5 for $M_{22}(m_1, m_2)$ by using the $m_2$ part appropriately. The final result (with $\rho$ a multiplicity label) is

\[
M_{22}(m) = \langle \mathcal{O}^2 \mathcal{O}^2 \mathcal{O}^2 \rangle^m = \langle \mathcal{O} \rangle^m \langle H^2 \rangle^m \langle H^2 \rangle^{m-k_0}
\]

\[
+ V_0^2 V_p^2 \left( \frac{N}{m} \right)^{-1} \left( \frac{N-k_0}{m-k_0} \right) \sum_{\nu_1=0}^{k} \sum_{\nu_2=0}^{k} \sum_{\nu=0}^{2k} \sqrt{\frac{N}{k_0}} d(N : \nu)
\]

\[
\times \sum_{\rho} \langle m || [B^{\nu_1}(k) B^{\nu_2}(k)]^{\nu_\rho} || m \rangle \langle m - k_0 || [B^{\nu_1}(k) B^{\nu_2}(k)]^{\nu_\rho} || m - k_0 \rangle
\]

\[
\times (-1)^{\phi(f_m, f_{m-k_0}, f_{m-k_0-k}) + \phi(f_{m-k_0}, f_{m-k_0-k}, \nu)} U(f_m f_{m-k_0} f_{m-k_0-k} f_{m-k_0} \nu).
\]

(113)

The moments $M_{PQ}$ can be converted into reduced (scale free) cumulants $k_{PQ}$ that gives information about the shape of the bivariate transition strength density. For our purpose the first non-trivial cumulants are the fourth order cumulants and they are defined by Eq. (38). The $k_{PQ}, P + Q = 4$ follow from Eqs. (11), (13), (103), (106), (109), (111), (112) and (113). Numerical results for some typical values of $(N, m, k, k_0)$ are shown in Table 4. These results show that in general $|k_{PQ}| \lesssim 0.3$ indicating that the bivariate strength density will be close to a bivariate Gaussian. For further confirming this result, we will derive asymptotic results for $k_{PQ}$.
7.2. Asymptotic formulas for bivariate moments and approach to bivariate Gaussian form

Here we will consider the asymptotic limit defined by $N \to \infty$ with $m$, $k$ and $k_0$ fixed and $k, k_0 << m$. Note that in the dilute limit (or true asymptotic limit) we also have $m \to \infty$ and $m/N \to 0$ with $k$ and $k_0$ fixed. Firstly, from Section 6 it is easy to see that in the asymptotic limit:

(i) $(N \choose m)^{-1} (N-k_0 \choose m-k_0) Z_{11}(N, m, k, k) \to (m \choose k) (N \choose k) (m-k_0 \choose k_0)$;

(ii) $\Lambda^0(N, m, k) \to (m \choose k)^2$;

(iii) $(N \choose m)^{-1} \Lambda^k(N, m, m-k) \Lambda^k(N, m, k) d(N : k) \to (m \choose k) (m-k_0 \choose k_0)^2$.

Starting with $\xi$, it should be clear that in the asymptotic limit only the term with $\nu = k$ in Eq. (109) will survive. Then, applying (i) and (ii) above we have

$$
\xi(m) = \frac{M_{11}(m)}{M_{00}(m) \left[ \tilde{M}_{20}(m) \tilde{M}_{02}(m) \right]^{1/2}} \xrightarrow{\text{asymp}} \left( \frac{m-k}{k} \right)^{1/2} \left( \frac{m}{k_0} \right)^{1/2} \left( \frac{m-k_0}{k} \right)^{1/2} \left( \frac{m}{k_0} \right)^{1/2} \left( \frac{m-k_0}{k} \right)^{1/2} . 
$$

Similarly, for $k_{40}$ and $k_{04}$ only the terms with $\nu = k$ in Eq. (13) will survive and then applying (ii) and (iii) above will give,

$$
k_{40}(m) = \frac{\tilde{M}_{40}(m)}{\left[ \tilde{M}_{20}(m) \right]^2} - 3 \xrightarrow{\text{asymp}} \left( \frac{m-k}{k} \right)^{1/2} \frac{m-k}{k} - 1 ,
$$

$$
k_{04}(m) = \frac{\tilde{M}_{04}(m)}{\left[ \tilde{M}_{02}(m) \right]^2} - 3 \xrightarrow{\text{asymp}} \left( \frac{m-k_0-k}{k} \right)^{3/2} \frac{m-k_0-k}{k} - 1 .
$$

For $M_{31}$, the first term in Eq. (111) is trivial and in the sum in the second term only the $\nu = k$ term will survive in the asymptotic limit. Now, applying (i)-(iii) above will give the result for $k_{31}(m),

$$
k_{31}(m) \xrightarrow{\text{asymp}} \left( \frac{m-k}{k} \right)^{1/2} \left( \frac{m-k_0}{k_0} \right)^{-1} - \xi(m) = \xi(m) k_{40}(m) .
$$

Similarly $k_{13}(m)$ is given by,

$$
k_{13}(m) \xrightarrow{\text{asymp}} \left( \frac{m-k_0-k}{k} \right)^{3/2} \frac{m-k_0-k}{k} - \xi(m) = \xi(m) k_{04}(m) .
$$
Finally, in $M_{22}$ only the third term in Eq. (113) is complicated. This is simplified using its relation, valid in the asymptotic limit, to $\xi(m)$ as described in Section 4. Following this we have for $k_{22}$,

$$k_{22}(m) \xrightarrow{\text{asymp}} -2[\xi(m)]^2 + \frac{(m_k)(m-k_k)^2}{(m-k_0)(m_k)} + \frac{(m-2k_k)(m-k_k)}{(m-k_0)(m-k_0)}$$

$$\approx -2[\xi(m)]^2 + \frac{(m-2k_k)}{(m_k)(m-k_0)} \left[ \binom{m}{k} + \binom{m-k}{k} \right].$$

(118)

In the dilute limit with $m \to \infty$ and $m/N \to 0$ and expanding the binomials in Eqs. (114) to (118), it is seen that to order $1/m$ the cumulants $k_{rs}$, $r + s = 4$ will be $-k^2/m$ (independent of $k_0$) and the correlation coefficient $\xi(m) \to 1 - (kk_0)/2m$. Thus, the cumulants will tend to zero giving bivariate Gaussian form. However, as $\xi \to 1$ as $m \to \infty$, in practice it is necessary to add the $k_{rs}$, $r + s = 4$ corrections to the bivariate Gaussian.

Let us add that the results given in this Section extend easily to particle addition operators $O^\dagger = \sum_\alpha V_\alpha A_\alpha^\dagger(k_0)$, acting on a $m$-particle state generating $m+k_0$ particle states, by using the results in Section 5 for the $m_1$ part appropriately.

In addition to the three fermionic systems considered so far, for example in nuclear physics applications in particular it is also important to consider explicitly the parity symmetry. For this situation, we need to consider two orbits one with $+$ve and other with $-$ve parity (for two types of fermions, we will have four orbits). By combining the formulation given in [44] for embedded ensembles with parity with the formulation in Sections 3 and 5, it is possible to derive formulas for the bivariate moments over spaces with fixed-$m$ [or fixed-$(m_1, m_2)$] and parity. Results for all these extensions will be discussed elsewhere.

8. Lower-order moments of transition strength densities: Results for boson systems

For $m$ spinless bosons in $N$ sp states with a general $k$-body Hamiltonian, we have $\text{BEGUE}(k)$ ['B' here stands for bosonic]. The embedding algebra for this system is again $U(N)$. As $m$ boson states should be symmetric under interchange of any two bosons, the irrep $f_m = \{m\}$; the totally symmetric irrep of $U(N)$. Similarly, $f_m = \{m^{N-1}\}$. For $H$ a $\text{BEGUE}(k)$ and
the transition operator an independent BEGUE\((k_0)\), the results of Section 3 translate to those of BEGUE\((k)\) by applying the well known \(N \to -N\) symmetry, i.e. in the fermion results replace \(N\) by \(-N\) and then take the absolute value of the final result; see [13, 14] for discussion on this property and [45] for explicit derivation of the formulas for boson systems for the moments of the one and two-point functions in eigenvalues without using the \(N \to -N\) symmetry. Firstly, it is easy to see that the \(m\) boson space dimension \(D_B(N, m) = {N + m - 1 \choose m}\) follows from the \(m\) fermion space dimension \(D_F(N, m) = {N \choose m}\) by replacing \(N\) by \(-N\) in \({N \choose m}\) and taking the absolute value giving in general,

\[
\left( \begin{array}{c} N + r \\ s \end{array} \right) \xrightarrow{N \to -N} \left( \begin{array}{c} N - r + s - 1 \\ s \end{array} \right).
\]

(119)

More strikingly, as pointed out in [14],

\[
\Lambda^\nu(N, m, k) \xrightarrow{bosons} \Lambda_B^\nu(N, m, k) = \left| \left( \begin{array}{c} m - \nu \\ k \end{array} \right) \left( \begin{array}{c} -N - m + k - \nu \\ k \end{array} \right) \right|
\]

\[
= \left( \begin{array}{c} m - \nu \\ k \end{array} \right) \left( \begin{array}{c} N + m + \nu - 1 \\ k \end{array} \right).
\]

(120)

Here we have used the formula given by Eq. (7) for fermion systems. Moreover, for bosons the irreps \(\nu\) for a \(k\)-body operator take the values \(\nu = 0, 1, \ldots, k\) as it is for fermions but for the fact that they correspond to the Young tableaux \(\{2\nu, \nu^{N-2}\}\). Also, the \(N \to -N\) symmetry and Eq. (14) will give

\[
d_B(N : \nu) = {N + \nu - 1 \choose \nu}^2 - {N + \nu - 2 \choose \nu - 1}^2.
\]

(121)

Using Eqs. (119), (120) and (121), it is possible to write the formulas for \(M_{rs}(m)\), \(r + s = 2, 4\), that correspond to the Eqs. (11), (13), (23), (29) and (36). As an example, the bivariate correlation coefficient for the system considered in Section 3 but for bosons is given by,

\[
\xi(m) = \frac{M_{11}(m)}{\sqrt{M_{20}(m)M_{02}(m)}} = \sum_{\nu=0}^{\min(m-k)} \Lambda_B^\nu(N, m, k) \Lambda_B^\nu(N, m, m - t) d_B(N : \nu)
\]

\[
= \left( \begin{array}{c} N + m - 1 \\ m \end{array} \right) \left( \begin{array}{c} N+m-1 \\ m \end{array} \right) \Lambda_B^\nu(N, m, t) \Lambda_B^\nu(N, m, k).
\]

(122)
Again, it can be verified that Eq. (122) has correctly the $k \leftrightarrow t$ symmetry. Let us add that for boson systems, asymptotic results correspond to the dense limit defined by $m \to \infty$, $N \to \infty$, $m/N \to \infty$ and $k, k_0$ fixed.

Two species boson systems are also important in quantum physics [13, 46] and for these systems a situation similar to the one we have considered in Section 5 is possible. Again, it is possible to apply the $N \to -N$ symmetry but now for both $N_1$ and $N_2$ using Eq. (119). Then, $\binom{N_1+r}{s}$ will change to $\binom{N_1-r+s-1}{s}$ and $\binom{N_2+t}{u}$ changes to $\binom{N_2-t+u-1}{u}$. Similarly $\Lambda^{\nu}(N_1, m_1, k)$, $d(N_1 : \nu)$, $\Lambda^{\nu}(N_2, m_2, k)$ and $d(N_2 : \nu)$ will change to the bosonic $\Lambda_B$ and $d_B$ via Eqs. (120) and (121). Also, for example for $k$-particle boson creation and annihilation operators $A^{\dagger}_\alpha(k)$ and $A_{\alpha}(k)$, it is easy to prove that

$$\langle m \mid A^{\dagger}(k) \mid m - k \rangle \langle m - k \mid A(k) \mid m \rangle = \binom{N + m - 1}{m - k}.$$  \hspace{1cm} (123)

This follows from the relation $\langle \sum_{\alpha} A^{\dagger}_\alpha(k)A_{\alpha}(k) \rangle^m = \binom{m}{k}$, the Wigner-Eckart theorem and the sum rules for C-G coefficients. More importantly, Eq. (123) follows also from the fermionic Eq. (57) by applying $N \to -N$ symmetry.

Going further, as seen from Sections 3, 5 and 7, the bivariate moments contain not only the functions $\Lambda^{\nu}(N, m, k)$ and $d(N : \nu)$ but also $U(N)$ Racah (or $U(-)$) coefficients [[see Eqs. (36) and (66)]. In principle the $N \to -N$ law applies also to the Racah coefficients by translating the irreps appropriately. For example, the fermionic $U$-coefficient given by Eq. (96) can be translated, using the $N \to -N$ law, for boson systems and the result is

$$U_B^2(f_m, f_p, f_m, f_p; f_{m-p}, \nu) = \frac{\binom{N + \nu - 2}{\nu}^2 \binom{m - \nu}{p - \nu} \binom{N + m + \nu - 1}{m - p}}{\left(\binom{N + m - 1}{p}^2 \binom{N + m - p - 1}{m - p}\right) (N - 1)} (N + 2\nu - 1).$$ \hspace{1cm} (124)

Note that in Eq. (124), $f_r = \{r\}$, $\bar{s} = \{s^{N-1}\}$ and $\nu = \{2\nu, \nu^{N-2}\}$. It is easy to verify Eq. (124) for $\nu = 0$. Using Eq. (124), it is possible to deal with boson systems that are similar to the fermionic systems considered in Sections 5 and 7. For example, Eqs. (124) and (109) will give the formula for the bivariate correlation coefficient for particle removal operator for bosons. Full details of the formulas for the bivariate moments for boson systems and their asymptotic structure will be presented elsewhere.
9. Conclusions and future outlook

Embedded random matrix ensembles were used with success in the past to understand the form of the eigenvalue density of finite quantum systems [16] and now there are good applications of the Gaussian form found for these systems [19, 47, 48]. Similarly, these ensembles are shown to provide the basis for the theory for expectation values of operators [19, 49]. In the present paper the focus is turned to transition strengths.

Employing embedded Gaussian unitary ensembles of random matrices, for the first time we have derived exact group theoretical formulas for the second and fourth order bivariate moments of the transition strength densities. Explicit results for a spinless many fermion system with $k$-body Hamiltonian and the transition operator a $k_0$-body operator are presented in Sections 3. Similarly, results for a system with two types of spinless fermions and beta decay (and double beta decay) type operators are presented in Section 5. In addition, results for a particle removal operator are presented in Section 7.1. The corresponding asymptotic results presented in Sections 4, 6 and 7.2 respectively and the numerical results from the exact formulas for the bivariate fourth order cumulants as presented in Tables 1-4 clearly show that the smoothed transition densities can be very well approximated by Edgeworth corrected bivariate Gaussian. Also, values of the bivariate correlation coefficient ($\xi$) shown in Tables 1-4 clearly confirm that the strength distribution will be a narrow distribution (unlike for a GOE). Briefly discussed in Section 8 are some results for boson systems. The formulation and results given in this paper hold for a general $k$-body Hamiltonian (similarly for the body rank of the transition operators considered). However, it should be emphasized that two-body Hamiltonians, i.e. EGUE(2)s, are in general more relevant for systems such as nuclei and atoms. Clearly, the work presented in the paper represents major progress in random matrix theory for smoothed transition strengths after the paper by French et al in 1988 [24].

One gap in the present results is that there is not yet a formula (or a good procedure) available for the $U(N)$ Racah coefficients of the type,

$$U(f_m, \nu_1, f_m, \nu_2 : f_m, \nu)$$

where $f_m = \{1^m\}$ for fermions (and $\{m\}$ for bosons) and $\mu = \{2^\mu, 1^{N-2\mu}\}$ (for bosons $\mu = \{2^\mu, \mu^{N-2}\}$). These coefficients are needed for the $M_{22}$ moment. In future, it is also important to consider fermions systems with spin as these are of direct interest in mesoscopic systems [50].
algebra for the ensembles with spin for the fermions is $U(\Omega) \otimes SU(2)$ with $SU(2)$ generating spin and $\Omega = N/2$ [51]. It is also possible to consider ensembles with more general $U(\Omega) \otimes SU(r)$ embedding [52]; the ensembles with $r = 1, 2$ and $4$ are important for fermionic systems and $r = 1, 2$ and $3$ for bosonic systems (for example $r = 3$ is appropriate for spinor BEC). The ensembles with $U(\Omega) \otimes SU(r)$ embedding [13, 51, 52] are studied so far only for one and two-point functions in eigenvalues. Even for these, the group theoretical results are incomplete [51, 52] as the Racah coefficients needed, for example for the fourth moment of the eigenvalue density, are not yet available. Another important extension is to consider ensembles with a mean-field one-body term which is more realistic for systems such as atomic nuclei and atoms.

In summary, in this paper a detailed analytical study of transition strengths has been carried out, going beyond the results in [24], using embedded random matrix ensembles and established clearly that the form of the transition strength densities for isolated finite interacting fermion systems will be generically a bivariate Gaussian with fourth order cumulant corrections. Fig. 5 shows a bivariate Gaussian with and without Edgeworth corrections. We expect that in the near future the bivariate Gaussian form will be used in practical calculations of transition strengths (see [21] for an attempt in the past) just as it is being done at present in a systematic manner by the Michigan group [47, 48] for nuclear level densities with good success using the Gaussian form given by the embedded ensembles.

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**APPENDIX A**

Given a bivariate distributions $\rho(x,y)$, its integral over $y[x]$ the marginal density $\rho_2(y)[\rho_1(x)]$. The centroids and variances of these are the marginal centroids and variances and say that they are $(\epsilon_1, \sigma_1^2)$ for $\rho_1(x)$ and similarly $(\epsilon_2, \sigma_2^2)$ for $\rho_2(y)$. Note that the $\widetilde{M}_{20}$ and $\widetilde{M}_{02}$ in Section 4 are $\sigma_1^2$ and $\sigma_2^2$ respectively. Bivariate Gaussian in terms of the standardized
variables $\hat{x} = (x - \epsilon_1)/\sigma_1$ and $\hat{y} = (y - \epsilon_2)/\sigma_2$ is given by,
\[
\eta_G(\hat{x}, \hat{y}) = \frac{1}{2\pi \sqrt{1 - \xi^2}} \exp \left\{ -\frac{\hat{x}^2 - 2\xi \hat{x} \hat{y} + \hat{y}^2}{2(1 - \xi^2)} \right\} .
\] (A-1)

Here $\xi$ is the correlation coefficient. Thus, a bivariate Gaussian is defined by the five variables $(\epsilon_1, \epsilon_2, \sigma_1, \sigma_2, \zeta)$. Now, the Edgeworth (ED) corrected bivariate Gaussian including $k_{rs}$ up to $r + s = 4$ is given by
\[
\eta_{biv-ED}(\hat{x}, \hat{y}) = \left\{ 1 + \left( \frac{k_{30}}{6} H e_{30}(\hat{x}, \hat{y}) + \frac{k_{21}}{2} H e_{21}(\hat{x}, \hat{y}) \right) \right. \\
+ \left. \frac{k_{12}}{2} H e_{12}(\hat{x}, \hat{y}) + \frac{k_{03}}{6} H e_{03}(\hat{x}, \hat{y}) \right) \\
+ \left\{ \frac{k_{10}}{24} H e_{40}(\hat{x}, \hat{y}) + \frac{k_{31}}{6} H e_{31}(\hat{x}, \hat{y}) \right) \\
+ \left. \frac{k_{22}}{4} H e_{22}(\hat{x}, \hat{y}) + \frac{k_{13}}{6} H e_{13}(\hat{x}, \hat{y}) + \frac{k_{04}}{24} H e_{04}(\hat{x}, \hat{y}) \right\} \\
+ \left\{ \frac{k_{30}^2}{72} H e_{60}(\hat{x}, \hat{y}) + \frac{k_{30} k_{21}}{12} H e_{51}(\hat{x}, \hat{y}) \right)$
\[
+ \left[ \frac{k_{21}^2}{8} + \frac{k_{30} k_{12}}{12} \right] H e_{42}(\hat{x}, \hat{y}) \\
+ \left[ \frac{k_{30} k_{03}}{36} + \frac{k_{12} k_{21}}{4} \right] H e_{33}(\hat{x}, \hat{y}) \\
+ \left[ \frac{k_{21}^2}{12} + \frac{k_{21} k_{03}}{12} \right] H e_{24}(\hat{x}, \hat{y}) + \frac{k_{12} k_{03}}{12} H e_{15}(\hat{x}, \hat{y}) \\
+ \left. \frac{k_{03}^2}{72} H e_{06}(\hat{x}, \hat{y}) \right\} \right\} \eta_G(\hat{x}, \hat{y}) .
\] (A-2)

Note that for EGUE and BEGUE, by definition the $k_{rs} = 0$ for $r + s = 3$. However in practical applications, these will be non-zero though expected to be small in magnitude. The bivariate Hermite polynomials $H e_{m_1 m_2}(\hat{x}, \hat{y})$ in Eq. (A-2) satisfy the recursion relation,
\[
(1 - \xi^2) H e_{m_1+1, m_2}(\hat{x}, \hat{y}) = (\hat{x} - \xi \hat{y}) H e_{m_1, m_2}(\hat{x}, \hat{y}) \\
- m_1 H e_{m_1-1, m_2}(\hat{x}, \hat{y}) + m_2 \xi H e_{m_1, m_2-1}(\hat{x}, \hat{y}) .
\] (A-3)
This can be solved using $He_{00}(\hat{x}, \hat{y}) = 1$, $He_{10}(\hat{x}, \hat{y}) = (\hat{x} - \xi \hat{y}) / (1 - \xi^2)$ and $He_{01}(\hat{x}, \hat{y}) = (\hat{y} - \xi \hat{x}) / (1 - \xi^2)$.

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Table 1: Bivariate correlation coefficient ($\xi$) and fourth order bivariate cumulants $k_{rs}$ with $r \geq s$ and $r + s = 4$ for the system considered in Section 5 for double beta decay type transition operators with $k_0 = 2$. Results are given for various values of number of spin state $N_1$ and $N_2$ and number of fermions $m_1$ and $m_2$ in these spin states respectively. The Hamiltonian body rank ($k$) is chosen to be $k = 2$. Results are obtained using the formulas given in Section 5. Note that for the $M_{22}$ that is needed for $k_{22}$ we have used Eq. (73) with the third term replaced by the corresponding asymptotic formula given by Eq. (94) as the $U$-coefficient needed for the finite-$(N, m)$ formula is not available. The finite $N$ and $m$ results are compared with the asymptotic limit results (these are given in the brackets). See text for further details.

| $N_1$ | $m_1$ | $N_2$ | $m_2$ | $\xi$     | $k_{40}$ | $k_{04}$ | $k_{31}$ | $k_{13}$ | $k_{22}$ |
|------|------|------|------|----------|----------|----------|----------|----------|----------|
|  20  |   8  |  20  |   8  |  0.66(0.76) | -0.34(-0.24) | -0.35(-0.24) | -0.22(-0.18) | -0.23(-0.19) | -0.01(-0.18) |
|  20  |  10  |  20  |  10  |  0.68(0.81) | -0.31(-0.19) | -0.32(-0.20) | -0.21(-0.16) | -0.22(-0.16) |  0.05(-0.16) |
|  32  |  10  |  32  |  10  |  0.74(0.81) | -0.26(-0.19) | -0.26(-0.20) | -0.19(-0.16) | -0.19(-0.16) | -0.04(-0.16) |
|  32  |  12  |  32  |  12  |  0.77(0.84) | -0.23(-0.16) | -0.24(-0.16) | -0.18(-0.14) | -0.18(-0.14) | -0.01(-0.14) |
|  32  |  16  |  32  |  16  |  0.78(0.88) | -0.21(-0.12) | -0.22(-0.12) | -0.17(-0.11) | -0.17(-0.11) | -0.06(-0.11) |
|  32  |  10  |  44  |   8  |  0.73(0.78) | -0.27(-0.22) | -0.29(-0.23) | -0.20(-0.17) | -0.21(-0.18) | -0.07(-0.18) |
|  32  |  10  |  44  |  15  |  0.79(0.85) | -0.21(-0.16) | -0.22(-0.16) | -0.17(-0.13) | -0.17(-0.13) | -0.03(-0.13) |
|  32  |  10  |  44  |  20  |  0.80(0.87) | -0.20(-0.13) | -0.20(-0.13) | -0.16(-0.11) | -0.16(-0.11) | -0.01(-0.11) |
|  32  |  12  |  44  |  20  |  0.81(0.88) | -0.19(-0.12) | -0.19(-0.12) | -0.16(-0.11) | -0.16(-0.11) |  0.02(-0.11) |
|  32  |  16  |  44  |  20  |  0.81(0.89) | -0.19(-0.11) | -0.19(-0.11) | -0.15(-0.10) | -0.15(-0.10) |  0.04(-0.10) |
|  44  |  10  |  58  |  20  |  0.83(0.87) | -0.18(-0.13) | -0.17(-0.13) | -0.15(-0.11) | -0.14(-0.11) | -0.03(-0.11) |
|  44  |  15  |  58  |  20  |  0.84(0.89) | -0.16(-0.11) | -0.16(-0.11) | -0.13(-0.10) | -0.14(-0.10) | -0.01(-0.10) |
|  44  |  20  |  58  |  20  |  0.85(0.90) | -0.15(-0.10) | -0.16(-0.10) | -0.13(-0.09) | -0.13(-0.09) |  0.01(-0.09) |
|  44  |  10  |  58  |  24  |  0.83(0.88) | -0.17(-0.12) | -0.16(-0.12) | -0.14(-0.10) | -0.14(-0.10) | -0.01(-0.10) |
|  44  |  10  |  58  |  28  |  0.84(0.90) | -0.16(-0.10) | -0.16(-0.10) | -0.14(-0.09) | -0.13(-0.09) |  0.01(-0.09) |
Table 2: Bivariate correlation coefficient ($\xi$) and fourth order bivariate cumulants $k_{rs}$ with $r \geq s$ and $r + s = 4$ for the system considered in Section 5 for beta decay type transition operators with $k_0 = 1$. Results are given for various values of number of sp state $N_1$ and $N_2$ and number of fermions $m_1$ and $m_2$ in these sp states respectively. The Hamiltonian body rank ($k$) is chosen to be $k = 2$. Results are obtained using the formulas given in Section 5. Note that for the $M_{22}$ that is needed for $k_{22}$ we have used Eq. (73) with the third term replaced by the corresponding asymptotic formula given by Eq. (94) as the $U$-coefficient needed for the finite-$\left(N,m\right)$ formula is not available. The finite $N$ and $m$ results are compared with the asymptotic limit results (these are given in the brackets). See text for further details.

| $N_1$ | $m_1$ | $N_2$ | $m_2$ | $\xi$  | $k_{40}$  | $k_{04}$  | $k_{31}$  | $k_{13}$  | $k_{22}$  |
|-------|-------|-------|-------|--------|----------|----------|----------|----------|----------|
| 20    | 8     | 30    | 8     | 0.83(0.88) | -0.32(-0.24) | -0.33(-0.25) | -0.26(-0.21) | -0.27(-0.22) | -0.12(-0.22) |
| 20    | 8     | 30    | 10    | 0.84(0.89) | -0.30(-0.22) | -0.30(-0.22) | -0.25(-0.19) | -0.25(-0.20) | -0.10(-0.20) |
| 20    | 10    | 30    | 10    | 0.84(0.90) | -0.29(-0.20) | -0.30(-0.20) | -0.24(-0.18) | -0.25(-0.18) | -0.07(-0.18) |
| 20    | 10    | 30    | 15    | 0.86(0.92) | -0.26(-0.16) | -0.26(-0.16) | -0.22(-0.14) | -0.23(-0.15) | -0.03(-0.15) |
| 36    | 8     | 36    | 8     | 0.85(0.88) | -0.29(-0.24) | -0.29(-0.24) | -0.24(-0.21) | -0.25(-0.21) | -0.15(-0.21) |
| 36    | 12    | 36    | 12    | 0.88(0.92) | -0.22(-0.16) | -0.22(-0.16) | -0.20(-0.15) | -0.20(-0.15) | -0.08(-0.15) |
| 36    | 16    | 36    | 16    | 0.90(0.94) | -0.20(-0.12) | -0.20(-0.12) | -0.18(-0.12) | -0.18(-0.12) | -0.03(-0.12) |
| 36    | 18    | 36    | 18    | 0.90(0.94) | -0.19(-0.11) | -0.19(-0.11) | -0.17(-0.10) | -0.17(-0.10) | -0.01(-0.10) |
| 36    | 8     | 36    | 10    | 0.86(0.89) | -0.27(-0.22) | -0.27(-0.21) | -0.23(-0.19) | -0.23(-0.19) | -0.13(-0.19) |
| 36    | 8     | 36    | 12    | 0.87(0.90) | -0.25(-0.20) | -0.25(-0.19) | -0.22(-0.18) | -0.22(-0.17) | -0.11(-0.17) |
| 36    | 8     | 36    | 16    | 0.88(0.92) | -0.23(-0.16) | -0.23(-0.16) | -0.21(-0.15) | -0.20(-0.15) | -0.08(-0.15) |
| 44    | 8     | 44    | 8     | 0.85(0.88) | -0.28(-0.24) | -0.28(-0.24) | -0.24(-0.21) | -0.24(-0.21) | -0.16(-0.21) |
| 44    | 12    | 44    | 12    | 0.89(0.92) | -0.21(-0.16) | -0.21(-0.16) | -0.19(-0.15) | -0.19(-0.15) | -0.09(-0.15) |
| 44    | 20    | 44    | 20    | 0.91(0.95) | -0.16(-0.10) | -0.16(-0.10) | -0.15(-0.09) | -0.15(-0.09) | -0.02(-0.09) |
Table 3: Bivariate correlation coefficient ($\xi$) and fourth order bivariate cumulants $k_{rs}$ ($= k_{sr}$) with $r \geq s$ and $r + s = 4$ for various values of number of sp state ($N$), number of fermions ($m$), Hamiltonian body rank ($k$) and the body rank ($k_0$) of the transition operator. Results are obtained using the formulas given in Section 3. Note that for the $M_{22}$ that is needed for $k_{22}$, we have used Eq. (36) with the third term replaced by the corresponding asymptotic formula given by Eq. (48) as the $U$-coefficient needed for the finite-($N,m$) formula is not available. The finite $N$ and $m$ results are compared with the asymptotic limit results (these are given in the brackets). See text for further details.

| $N$ | $m$ | $k$ | $k_0$ | $\xi$ | $k_{40}$ | $k_{31}$ | $k_{22}$ |
|-----|-----|-----|------|-----|--------|--------|-------|
| 20  | 10  | 2   | 1    | 0.68(0.8) | -0.54(-0.38) | -0.36(-0.3) | -0.09(-0.27) |
| 30  |     |     |      | 0.73(0.8) | -0.48(-0.38) | -0.35(-0.3) | -0.16(-0.27) |
| 50  |     |     |      | 0.76(0.8) | -0.43(-0.38) | -0.33(-0.3) | -0.21(-0.27) |
| 80  |     |     |      | 0.78(0.8) | -0.41(-0.38) | -0.32(-0.3) | -0.23(-0.27) |
| 40  | 8   | 2   | 1    | 0.71(0.75) | -0.52(-0.46) | -0.37(-0.35) | -0.23(-0.3) |
| 12  |     |     |      | 0.78(0.83) | -0.4(-0.32) | -0.31(-0.27) | -0.15(-0.24) |
| 15  |     |     |      | 0.81(0.87) | -0.36(-0.26) | -0.29(-0.22) | -0.1(-0.21) |
| 20  |     |     |      | 0.82(0.9) | -0.32(-0.2) | -0.27(-0.18) | -0.03(-0.17) |
| 60  | 8   | 2   | 1    | 0.72(0.75) | -0.5(-0.46) | -0.36(-0.35) | -0.26(-0.3) |
| 15  |     |     |      | 0.83(0.87) | -0.32(-0.26) | -0.26(-0.22) | -0.15(-0.21) |
| 20  |     |     |      | 0.86(0.9) | -0.27(-0.2) | -0.23(-0.18) | -0.09(-0.17) |
| 30  |     |     |      | 0.88(0.93) | -0.23(-0.13) | -0.2(-0.12) | -0.02(-0.12) |
| 24  | 8   | 2   | 1    | 0.67(0.75) | -0.56(-0.46) | -0.38(-0.35) | -0.18(-0.3) |
| 2   |     |     |      | 0.44(0.54) | -0.56(-0.46) | -0.24(-0.25) | -0.07(-0.17) |
| 3   |     | 1   |      | 0.54(0.63) | -0.88(-0.82) | -0.47(-0.51) | -0.24(-0.35) |
| 3   |     | 2   |      | 0.27(0.36) | -0.88(-0.82) | -0.23(-0.29) | -0.06(-0.12) |
| 4   |     | 1   |      | 0.41(0.5) | -0.99(-0.99) | -0.41(-0.49) | -0.16(-0.25) |
| 40  | 12  | 1   | 1    | 0.89(0.92) | -0.11(-0.08) | -0.1(-0.08) | -0.02(-0.08) |
| 2   |     |     |      | 0.78(0.83) | -0.4(-0.32) | -0.31(-0.27) | -0.15(-0.24) |
| 3   |     | 1   |      | 0.68(0.75) | -0.71(-0.62) | -0.48(-0.46) | -0.27(-0.37) |
| 3   |     | 2   |      | 0.45(0.55) | -0.71(-0.62) | -0.32(-0.34) | -0.12(-0.21) |
| 4   |     | 1   |      | 0.59(0.67) | -0.91(-0.86) | -0.53(-0.57) | -0.3(-0.4) |
| 4   |     | 2   |      | 0.33(0.42) | -0.91(-0.86) | -0.3(-0.36) | -0.1(-0.17) |
| 5   |     | 1   |      | 0.5(0.58) | -0.98(-0.97) | -0.49(-0.57) | -0.24(-0.34) |
| 5   |     | 2   |      | 0.24(0.32) | -0.98(-0.97) | -0.23(-0.31) | -0.05(-0.1) |
Table 4: Bivariate correlation coefficient ($\xi$) and fourth order bivariate cumulants $k_{rs}$ with $r+s=4$ for various values of number of sp states ($N$), number of fermions ($m$), Hamiltonian body rank ($k$) and the rank ($k_0$) of the particle removal transition operator. Results are obtained using the formulas given in Section 7.1. Note that for the $M_{22}$ that is needed for $k_{22}$, we have used Eq. (113) with the third term replaced by the corresponding asymptotic formula given by Eq. (118) as a formula for the reduced matrix elements in Eq. (113) is not available.

| $N$  | $m$  | $k$ | $k_0$ | $\xi$  | $k_{40}$ | $k_{04}$ | $k_{31}$ | $k_{13}$ | $k_{22}$ |
|------|------|-----|-------|--------|----------|----------|----------|----------|----------|
| 20   | 10   | 2   | 1     | 0.82   | -0.54    | -0.55    | -0.44    | -0.45    | -0.21    |
| 30   | 10   | 2   | 1     | 0.85   | -0.48    | -0.50    | -0.41    | -0.43    | -0.26    |
| 60   | 10   | 2   | 1     | 0.88   | -0.42    | -0.46    | -0.37    | -0.40    | -0.30    |
| 80   | 10   | 2   | 1     | 0.88   | -0.41    | -0.45    | -0.36    | -0.39    | -0.31    |
| 50   | 12   | 2   | 1     | 0.89   | -0.38    | -0.40    | -0.034   | -0.36    | -0.25    |
| 15   | 2    | 1   | 0.91  | -0.33  | -0.35    | -0.30    | -0.31    | -0.19    |          |
| 20   | 2    | 1   | 0.92  | -0.29  | -0.29    | -0.26    | -0.27    | -0.13    |          |
| 25   | 2    | 1   | 0.92  | -0.27  | -0.27    | -0.25    | -0.25    | -0.08    |          |
| 24   | 8    | 2   | 1     | 0.82   | -0.56    | -0.61    | -0.46    | -0.49    | -0.31    |
|      | 2    | 2   | 0.66  | -0.56  | -0.67    | -0.37    | -0.43    | -0.22    |          |
| 40   | 15   | 2   | 1     | 0.90   | -0.36    | -0.37    | -0.32    | -0.33    | -0.18    |
|      | 2    | 2   | 0.80  | -0.36  | -0.38    | -0.29    | -0.31    | -0.12    |          |
| 60   | 20   | 2   | 1     | 0.93   | -0.27    | -0.27    | -0.25    | -0.25    | -0.14    |
|      | 3    | 1   | 0.89  | -0.51  | -0.53    | -0.46    | -0.47    | -0.30    |          |
|      | 3    | 2   | 0.79  | -0.51  | -0.54    | -0.40    | -0.43    | -0.22    |          |
Figure 1: Schematic figure showing transition strengths. (a) Transition strengths for transitions induced by an operator $O_1$, from levels with energies (eigenvalues of the Hamiltonian) $E_i$ of a system ‘a’ to levels with energies $E_f$ of the same system ‘a’. The strengths $|\langle a, E_f | O_1 | a, E_i \rangle|^2$ are proportional to the widths of the lines in the figure. (b) Transition strengths for transitions induced by an operator $O_2$, from one particular level with energy (eigenvalue of the Hamiltonian) $E_i$ of a system ‘a’ to levels with energies $E_f$ of another system ‘b’. The strengths $|\langle b, E_f | O_1 | a, E_i \rangle|^2$ are proportional to the widths of the lines in the figure. In general, transitions from several levels of the system ‘a’ to the levels of system ‘b’ are possible as shown in the inset figure.
Young Tableaux $\{f\}$, $f_1 \geq f_2 \geq \ldots \geq f_\Omega \geq 0$

(a) general irrep

$\{f\} = \{6, 4, 2^3, 1\}$

(b) symmetric irrep

$\{f\} = \{1^7\}$

(c) anti-symmetric irrep

$\{f\} = \{8\}$

(d) conjugate irrep $\{\tilde{f}\}$, given the irrep $\{f\}$

$\{f\} = \{5, 3^2, 2, 1^2\}$

$\{\tilde{f}\} = \{6, 4, 3, 1^2\}$

conjugation

(e) definition of $\{\tilde{f}\}$ irrep, given the irrep $\{f\}$

$\{f\} = \{5, 3^2, 2, 1^2\}$

$\{\tilde{f}\} = \{5^{\Omega-6}, 4^2, 3, 2^2\}$

w.r.t. $U(\Omega)$

if $\{f\}$ is for a creation operator, then $\{\tilde{f}\}$ is for the corresponding annihilation operator

Figure 2: Young tableaux representation of the irreps of $U(\Omega)$. Shown are examples of: (a) a general irrep $\{f\}$; (b) symmetric irrep $\{m\}$; (c) antisymmetric irrep $\{1^m\}$; (d) conjugate irrep $\{\tilde{f}\}$ that corresponds to a given $\{f\}$; (e) irrep $\{\tilde{f}\}$ that corresponds to a given $\{f\}$. Note the importance of $\Omega$ in defining $\{\tilde{f}\}$. 

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Note that the bivariate moments \( M \) and \( N \) space. Note that \( N = 20, N_2 = 16, k = k_0 = 2 \).

(a) \( H \) matrix in the defining space

(b) \( \mathcal{O} \) matrix in the defining space

(c) \( A \) matrix connecting \((m_1, m_2) = (2, 0)\) to \((m_1, m_2) = (0, 2)\).

(d) \( \mathcal{O} \) matrix connecting \((6, 8)\) to \((8, 6)\)

(e) \( H \) matrix in the \((m_1, m_2) = (8, 6)\) space

(f) \( \mathcal{O}^\dagger \) matrix connecting \((8, 6)\) to \((6, 8)\)

Figure 3: Schematic diagram showing the matrix representations of the Hamiltonian \( H \) and the transition operator \( \mathcal{O} \) in the respective defining spaces and in the many particle spaces for the system considered in Section 5 with \( N_1 = 20, N_2 = 16, k = 2 \) and \( k_0 = 2 \). (a) \( H \) matrix in the defining space. Note that the matrix is in block diagonal form with 3 blocks and they correspond to \((m_1, m_2) = (2, 0), (1, 1), (0, 2)\) as shown in the figure. The matrix dimensions of each block are also shown in the figure. (b) \( \mathcal{O} \) matrix in the defining space. Note that \( \mathcal{O} \) is a rectangular matrix connecting \((m_1, m_2)_i = (0, 2)\) to \((m_1, m_2)_f = (2, 0)\). The number of rows and columns in the matrix are shown in the figure. (c) \( H \) matrix \( H_i \) in the initial \((m_1, m_2) = (6, 8)\) space with matrix dimension \( d_A = 498841200 \). (d) \( \mathcal{O} \) matrix connecting states in \((m_1, m_2)_i = (6, 8)\) space with the states in \((m_1, m_2)_f = (8, 6)\) space with matrix elements \((m_1, m_2)_f = (8, 6) | \mathcal{O} | (m_1, m_2)_i = (6, 8), \alpha \). Note that the matrix is a \( d_B \times d_A \) rectangular matrix with \( d_A \) given in (c) and \( d_B = 1008767760 \). (d) \( H \) matrix \( H_f \) in the final \((8, 6)\) space. (e) same as (d) but for the \( \mathcal{O}^\dagger \) matrix. It is useful to note that the bivariate moments \( M_{PQ} \) of the transition strength density are given by \((d_A)^{-1} \text{Tr} [\mathcal{O}^\dagger (H_f)^Q \mathcal{O}(H_i)^P] \) where \( \text{Tr} \) stands for the matrix trace.
$N = 20, k = 2, k_0 = 1$

Figure 4: Schematic diagram showing the matrix representations of the Hamiltonian ($H$) and the transition operator ($O$) in the respective defining spaces and in the many particle spaces for the system considered in Section 7 with $N = 20$, $k = 2$ and $k_0 = 1$. (a) $H$ matrix in the defining space. (b) $O$ matrix in the defining space. (c) $H$ matrix in the $m = 10$ space. (d) $O$ matrix connecting states in $m_i = 10$ space with the states in $m_f = 9$ space with matrix elements $\langle m = 9, \beta \mid O \mid m = 10, \alpha \rangle$. Note that the matrix is a $d_B \times d_A$ rectangular matrix with $d_A$ given in (c) and $d_B = 167960$. (e) $H$ matrix $H_f$ in the final $m = 9$ space. (f) same as (d) but for the $O^T$ matrix. See Fig. 3 for further details.
Figure 5: Schematic diagram showing (a) normalized bivariate Gaussian and (b) normalized bivariate Gaussian with Edgeworth (ED) corrections (see Appendix A). The fourth order cumulants used are $\xi = 0.83$, $k_{40} = -0.18$, $k_{04} = -0.17$, $k_{31} = -0.15$, $k_{13} = -0.14$, and $k_{22} = -0.03$ corresponding to $(N_1, m_1) = (44, 10)$ and $(N_2, m_2) = (58, 20)$ in Table 2.