Monotonically controlled integrals

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Abstract

The monotonically controlled integral defined by Bendová and Malý, which is equivalent to the Denjoy-Perron integral, admits a natural parameter $\alpha > 0$ thereby leading to the whole scale of integrals called $\alpha$-monotonically controlled integrals. While the power of these integrals is easily seen to increase with increasing $\alpha$, our main results show that their exact dependence on $\alpha$ is rather curious. For $\alpha < 1$ they do not even contain the Lebesgue integral, for $1 \leq \alpha \leq 2$ they coincide with the Denjoy-Perron integral, and for $\alpha > 2$ they are mutually different and not even contained in the Denjoy-Khintchine integral.

1 Introduction

The monotonically controlled integral, or (MC) integral, defined by Hana Bendová and Jan Malý in [1], is an interesting variant of nowadays rather abundant equivalent definitions of the Denjoy-Perron integral. It is defined as follows.

Definition 1. Let $I \subset \mathbb{R}$ be an open interval and $f, F : I \to \mathbb{R}$ be functions. We say that $f$ is an (MC) derivative (monotonically controlled derivative) of $F$ on $I$, or that $F$ is an indefinite (MC) integral of $f$ on $I$, if there exists a strictly increasing function $\varphi : I \to \mathbb{R}$ (which is called control function for the pair $(F, f)$) such that for each $x \in I$,

$$
\lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(y) - \varphi(x)} = 0.
$$

As Jan Malý pointed out, this definition invites introduction of a natural parameter thereby leading to the whole scale of $\alpha$-monotonically controlled integrals, or (MC$_\alpha$) integrals, where $\alpha > 0$ is a parameter.

Definition 2. Let $I \subset \mathbb{R}$ be an open interval, $f, F : I \to \mathbb{R}$ be functions and $\alpha > 0$. We say that $f$ is an (MC$_\alpha$) derivative of $F$ on $I$, or that $F$ is an indefinite (MC$_\alpha$)
integral of $f$ on $I$, if there exists a strictly increasing function $\varphi : I \to \mathbb{R}$ (which we call the $\alpha$-control function for the pair $(F,f)$) such that for each $x \in I$,

$$\lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(x + \alpha(y - x)) - \varphi(x)} = 0.$$ 

The original (MC) integral is obtained for $\alpha = 1$ and, since (MC$_{\alpha}$) integrability implies (MC$_{\beta}$) integrability for $\beta > \alpha$, one of a number of natural questions, asked by Malý, is whether for $\alpha > 1$ the (MC$_{\alpha}$) integral is still equivalent to the Denjoy-Perron integral. Here we show that this is not the case, but that the situation is rather interesting: while for $\alpha \in [1,2]$ the (MC$_{\alpha}$) integral is equivalent to the Denjoy-Perron integral, for $\alpha > 2$ the (MC$_{\alpha}$) integrals are mutually different and even are not contained in the Denjoy-Khintchine integral. Although the case $\alpha < 1$ is perhaps less interesting, we at least show that these integrals do not contain the Lebesgue integral. More precisely, our main results imply the following

**Theorem 3.** On any open interval $I \subset \mathbb{R}$,

(i) There is a Lebesgue integrable function that is not (MC$_{\alpha}$) integrable for any $0 < \alpha < 1$.

(ii) For $1 \leq \alpha \leq 2$, the (MC$_{\alpha}$) integral coincides with the Denjoy-Perron integral.

(iii) For any $\alpha \geq 2$ there is a function which is not (MC$_{\alpha}$) integrable but is (MC$_{\beta}$) integrable for every $\beta > \alpha$.

(iv) There is a function that is (MC$_{\alpha}$) integrable for every $\alpha > 2$, but is not Denjoy-Khintchine integrable.

In connection with (i) and (iv), standard examples show that there is a function that is (MC$_{\alpha}$) integrable for any $\alpha < 1$ but not Lebesgue integrable, and a function that is (MC$_{\alpha}$) integrable for any $\alpha > 2$ but not Denjoy-Khintchine integrable.

Theorem 3, sometimes in a stronger or more precise form, will be proved in the last part of this paper: (i) in Theorem 14, (ii) in Theorem 18, (iii) in Theorem 20, and (iv) in Theorem 22.

Before coming to the above results, we show in Section 2 some basic properties of these integrals, especially that the notion is reasonable, namely that two indefinite integrals of the same function differ by a constant. However we do not develop more advanced theory of these integrals.

The main results are stated and proved in Section 3. We first remind ourself with the notions used to give equivalent definitions of the Lebesgue, Denjoy-Perron and Denjoy-Khintchine integrals that we use to compare them with the (MC$_{\alpha}$) integral. In addition to results stated in Theorem 3 we also show that bounded measurable functions are (MC$_{\alpha}$) integrable for every $\alpha > 0$ and that every function that is (MC$_{\alpha}$) integrable for some $\alpha > 0$ is Lebesgue integrable on some subinterval. In connection with the statement (ii) of Theorem 3 we reprove the result of Bendová and Malý that the (MC) integral is equivalent to the Denjoy-Perron integral by showing directly...
its equivalence with Perron’s original definition \[6\]. The argument in \[1\] is based on the definition of the Kurzweil-Henstock integral \[5, 4\], which is known to be equivalent to the Denjoy-Perron integral. However, we should point out that our approach is close to Kurzweil’s proof of equivalence of his integral (which later became the Kurzweil-Henstock integral) to the Perron integral. The closeness of the definition of the (MC) integral to Perron’s definition is perhaps surprising, since \[1\] quotes as the intermediate step to their definition the variational integral of \[3\] which arose from Henstock’s approach \[4\].

In this area, it is no wonder that most of our references are to the still best text on much of classical real analysis, *Theory of the Integral* by Stanislaw Saks. As this book numbers paragraphs and results in each chapter separately, we will refer, for example, to VII(§5) for paragraph 5 in chapter 7, or to Theorem VI(7.2) for Theorem (7.2) in chapter 6.

As in \[1\], although this makes little difference, we are interested in indefinite integrals rather than in definite ones. In particular, by saying that a function is integrable we mean that it has an indefinite integral. Finally, we mention that we will use the terms *positive* for $\geq 0$, *strictly positive* for $> 0$, and similarly *increasing* and *strictly increasing*.

### 2 Basic properties of (MC\(_\alpha\)) integrals

We begin by remarking that it is immediate to see that if $\varphi$ is an $\alpha$-control function for $(F, f)$, $c > 0$ and $\psi$ is increasing, then $c \varphi + \psi$ is also an $\alpha$-control function for $(F, f)$. It follows that, if $\varphi$ is an $\alpha$-control function for $(F, f)$, $\psi$ is an $\alpha$-control function for $(G, g)$ and $a, b \in \mathbb{R}$, then $\varphi + \psi$ is a control function for $(aF + bG, af + bg)$. In other words, the (MC\(_\alpha\)) integral is linear: If $F$ and $G$ are indefinite (MC\(_\alpha\)) integrals of $f$ and $g$, respectively, and $a, b \in \mathbb{R}$, then $aF + bG$ is an indefinite (MC\(_\alpha\)) integral of $af + bg$.

Some simple basic properties of indefinite (MC\(_\alpha\)) integrals are collected in the following statement.

**Proposition 4.** Suppose $\alpha > 0$ and $F$ is an indefinite (MC\(_\alpha\)) integral of $f$ on $(a, b)$. Then

(i) $F$ is continuous on $(a, b)$;

(ii) $F$ is an indefinite (MC\(_\beta\)) integral of $f$ on $(a, b)$ for every $\alpha > \beta$;

(iii) $F'(x) = f(x)$ for almost every $x \in (a, b)$.

**Proof.** Let $\varphi$ be an $\alpha$-control function for the pair $(F, f)$. Since $\varphi$ is bounded on a neighbourhood of $x$, taking limit as $y \to 0$ in

$$F(y) = F(x) + f(x)(y - x) + \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(x + \alpha(y - x)) - \varphi(x)}(\varphi(x + \alpha(y - x)) - \varphi(x))$$
gives
\[ \lim_{y \to x} F(y) = F(x). \]

The second statement follows immediately from
\[ \varphi(x + \alpha(y - x)) - \varphi(x) \leq \varphi(x + \beta(y - x)) - \varphi(x). \]

For the third statement we observe that for every \( x \) at which \( \varphi \) is differentiable,
\[
\lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{y - x} = \alpha \lim_{y \to x} \frac{\varphi(x + \alpha(y - x)) - \varphi(x)}{\alpha(y - x)} = 0.
\]
Since \( \varphi \) is differentiable almost everywhere, (iii) follows.

**Proposition 5.** Suppose \( F : (a, b) \to \mathbb{R} \) satisfies

(i) \( \limsup_{y \to x} F(y) \leq F(x) \leq \limsup_{y \to x} F(y) \) for every \( x \in (a, b) \);

(ii) there are a strictly increasing \( \varphi : (a, b) \to \mathbb{R} \) and \( \alpha > 0 \) such that for every \( x \in (a, b) \) except at most countably many,

\[ \liminf_{h \to 0} F(x + h) - F(x) \frac{\varphi(x + \alpha h) - \varphi(x)}{\alpha h} \geq 0. \] (2)

Then \( F \) is increasing on \( (a, b) \).

**Proof.** Fix \( \tau > 0 \) and notice that, since \( \varphi \) is increasing, \( F_\tau := F + \tau \varphi \) also satisfies (i).

We show that for every \( x \in (a, b) \) at which (2) holds
\[
\limsup_{h \to 0} \frac{F_\tau(x + h) - F_\tau(x)}{h} \geq 0. \] (3)

Then [7, Theorem VI(7.2)] says that \( F_\tau \) is increasing and the statement will follow by taking the limit as \( \tau \to 0 \).

Fix \( x \in (a, b) \) at which (2) holds. Suppose first that there is a sequence \( h_k \searrow 0 \) such that
\[ L := \lim_{k \to \infty} \frac{\varphi(x + \alpha h_k) - \varphi(x)}{h_k} \]
exists and is finite. Since the limit defining \( L \) consists of positive terms,
\[
\limsup_{h \searrow 0} \frac{F_\tau(x + h) - F_\tau(x)}{h} \geq L \liminf_{k \to \infty} \frac{F(x + h_k) - F(x)}{\varphi(x + \alpha h_k) - \varphi(x)} \geq 0.
\]

Suppose next that there is no \( L \) as above or, in other words, that \( \varphi \) has infinite right derivative at \( x \). Let \( u_k := \sup \{ h \in [0, \delta] : \varphi(x + h) \geq \varphi(x) + kh \} \). Using that \( \varphi \) has infinite right derivative at \( x \) and is bounded by \( \varphi(x + \delta) \), we see that
0 < u_k ≤ (ϕ(x + δ) − ϕ(x))/k. Since ϕ is increasing, ϕ(x + u_k) ≥ ϕ(x) + ku_k. For sufficiently large k we have αu_k ∈ [0, δ] and αu_k > u_k, hence ϕ(x + αu_k) − ϕ(x) ≤ kαu_k ≤ (ϕ(x + u_k) − ϕ(x))/α for large enough k. Passing to a subsequence of u_k, which we will denote h_k, we have h_k ց 0 and

\[ L := \lim_{k \to \infty} \frac{ϕ(x + αh_k) − ϕ(x)}{ϕ(x + h_k) − ϕ(x)} \]

exists and is finite. Hence

\[ \limsup_{h \searrow 0} \frac{F_\tau(x + h) − F_\tau(x)}{ϕ(x + h) − ϕ(x)} ≥ \tau + L \liminf_{k \to \infty} \frac{F(x + h_k) − F(x)}{ϕ(x + αh_k) − ϕ(x)} \geq \tau > 0. \]

Consequently, there are arbitrarily small h > 0 such that \( F_\tau(x + h) − F_\tau(x) > 0 \), and so (3) holds as well. Hence it holds in both cases, and the proof is finished.

In order to see the connection with the monotonicity result of [Theorem VI(7.2)] on which its proof based, Proposition 5 is stated in considerably greater generality than we need to prove the following Theorem. However, we did not attempt to find its strongest version. For example, our proof would allow the ϕ to depend on x, and, rather obviously, one may replace the lim inf in (ii) by lim sup for \( α ≤ 1 \).

**Theorem 6.** Suppose F is an indefinite \((\text{MC}_α)\) integral of \( f ≥ 0 \) on \((a, b)\). Then F is increasing on \((a, b)\).

**Proof.** This is immediate from Proposition 5 since its assumption (i) holds by continuity of F and (ii) holds with the \( α \)-control function for the pair \((F, f)\) by the definition of the \((\text{MC}_α)\) integral.

**Corollary 7.** For any \( α > 0 \), an indefinite \((\text{MC}_α)\) integral of a function on an interval is unique up to an additive constant.

**Proof.** If F, G are two indefinite \((\text{MC}_α)\) integrals of F, by linearity both \( F − G \) and \( G − F \) are indefinite \((\text{MC}_α)\) integrals of zero. Hence \( F − G \) and \( G − F \) are both increasing by Theorem 6 showing that they differ by a constant.

### 3 Relations between the \((\text{MC}_α)\), Lebesgue, Denjoy-Perron and Denjoy-Khintchine integrals

We begin by recalling notions related to the definitions or properties of the Lebesgue, Denjoy-Perron and Denjoy-Khintchine integrals that we use in our arguments. They all come from [7], where much more material on these integrals and notions may be found.

**Definition 8.** A real-valued function F defined on a set \( E ⊂ \mathbb{R} \) is said to be
• of bounded variation (VB) on $E$ if there is $v \in [0, \infty)$ such that $\sum_i |F(b_i) - F(a_i)| \leq v$ for every sequence of non-overlapping intervals whose end-points belong to $E$,

• absolutely continuous (AC) on $E$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every sequence of non-overlapping intervals whose end-points belong to $E$, the inequality $\sum_i (b_i - a_i) < \delta$ implies $\sum_i |F(b_i) - F(a_i)| < \varepsilon$,

• generalized absolutely continuous (ACG) on $E$ if it is continuous on $E$ and $E$ is the union of countably many sets on which $F$ is absolutely continuous,

For the next definition we recall that the oscillation of a function $F$ on a set is $\text{osc}(F, E) := \sup_{x, y \in E} |F(x) - F(y)|$.

**Definition 9.** Assuming in Definition 8 that $F$ is defined on an interval containing $E$, and replacing $\sum_i |F(b_i) - F(a_i)|$ by $\sum_i \text{osc}(F, [a_i, b_i])$ we obtain the notion of functions

• of bounded variation in the restricted sense (VB$^\star$) on $E$,

• absolutely continuous in the restricted sense (AC$^\star$) on $E$,

• generalized absolutely continuous in the restricted sense (ACG$^\star$) on $E$,

respectively.

Although we do not need it, we point out the important results that ACG$^\star$ functions are differentiable almost everywhere and ACG functions are approximately differentiable almost everywhere. For the definition of approximate derivative see [7, VII(§3)]; here we just need to know that if an ordinary derivative $F'(x)$ exists, then so does the approximate derivative $F'_{ap}(x)$ and $F'_{ap}(x) = F'(x)$.

Finally, we collect so called descriptive definitions of the three integrals that we use. Again, [7] is a reference for (often deep) proofs of equivalence to more usual definitions.

**Fact 10.** Suppose $F$ and $f$ are real-valued functions on an interval $(a, b) \subset \mathbb{R}$.

• $F$ is an indefinite Lebesgue integral of $f$ on $(a, b)$ if it is AC (or equivalently AC$^\star$) on $(a, b)$ and $F'(x) = f(x)$ for almost all $x \in (a, b)$,

• $F$ is an indefinite Denjoy-Perron integral of $f$ on $(a, b)$ if it is ACG$^\star$ on $(a, b)$ and $F'(x) = f(x)$ for almost all $x \in (a, b)$,

• $F$ is an indefinite Denjoy-Khintchine integral of $f$ on $(a, b)$ if it is ACG on $(a, b)$ and $F'_{ap}(x) = f(x)$ for almost all $x \in (a, b)$.
3.1 Relations between Lebesgue and \((\text{MC}_\alpha)\) integrals

From Proposition \(\text{Proposition 11}\) and the coincidence of the (MC) and Denjoy-Perron integral we know that \((\text{MC}_\alpha)\) and Lebesgue integrals agree provided that they both exist. Here we show that in general the “both” cannot be replaced by existence of just one of them. In our first result we observe that the functions providing an example cannot be bounded.

**Proposition 11.** Every measurable function that is locally bounded on \((a, b)\) has an indefinite \((\text{MC}_\alpha)\) integral on \((a, b)\) for every \(\alpha > 0\).

**Proof.** Let \(F\) be an indefinite Lebesgue integral of a bounded measurable function \(f\) on \((a, b)\). Let \(M := \{x : F'(x) = f(x)\}\) and \(N = (a, b) \setminus M\). Since \(N\) has Lebesgue measure zero, there are open sets \((a, b) \supset G_k \supset N\) such that \(|G_k| < 4^{-k}\). Define \(\varphi_k(x) := 2^k|(a, x) \cap G_k|\) and \(\varphi(x) := x + \sum_{k=1}^\infty \varphi_k(x)\). Clearly, \(\varphi\) is an increasing function on \((a, b)\). We fix \(\alpha > 0\) and show that \(\varphi\) is an \(\alpha\)-control function for the pair \((F, f)\).

If \(x \in M\), (1) follows immediately from \(F'(x) = f(x)\), since then

\[
\limsup_{y \to x} \frac{|F(y) - F(x) - f(x)(y - x)|}{|\varphi(x + \alpha(y - x)) - \varphi(x)|} \leq \lim_{y \to x} \frac{|F(y) - F(x) - f(x)(y - x)|}{\alpha |y - x|} = 0.
\]

If \(x \in N\), choose \(c \in (0, \infty)\) such that \(|f| \leq c\) on a neighbourhood of \(x\). Then \(|F(y) - F(x) - f(x)(y - x)| \leq 2c|y - x|\) and, given any \(k \in \mathbb{N}\), we see that

\[
\varphi(x + \alpha(y - x)) - \varphi(x) = 2^k\alpha(y - x)
\]

for \(y\) close enough to \(x\). Hence

\[
\limsup_{y \to x} \frac{|F(y) - F(x) - f(x)(y - x)|}{|\varphi(y) - \varphi(x)|} \leq \lim_{y \to x} \frac{2c|y - x|}{2^k |y - x|} = 2^{-k-1} c.
\]

Since \(k\) can be arbitrarily large, (1) follows. \(\square\)

The first part of the above proof also shows that, if \(F'(x) = f(x)\) at every point \(x \in (a, b)\), \(F\) is an indefinite \((\text{MC}_\alpha)\) integral of \(f\). Together with well known examples of non-absolutely integrable derivatives we get examples of functions that are \((\text{MC}_\alpha)\) integrable for every \(\alpha > 0\) but not Lebesgue integrable. For completeness, we record this in the following observation.

**Observation 12.** If \(F\) is everywhere differentiable on \((a, b)\), then it is an indefinite \((\text{MC}_\alpha)\) integral of \(F'\) for every \(\alpha > 0\). Hence there are functions that are \((\text{MC}_\alpha)\) integrable for every \(\alpha > 0\) but not Lebesgue integrable.

The following Lemma provides the main building block in definition of a function that is Lebesgue but not \((\text{MC}_\alpha)\) integrable for any \(0 < \alpha < 1\).

**Lemma 13.** For any interval \(J \subset \mathbb{R}\) and \(0 < \varepsilon, \tau < 1\) there is a measurable function \(f : \mathbb{R} \to [0, \infty)\) such that
(i) \( \int_{-\infty}^{\infty} f(x) \, dx \leq \varepsilon; \)

(ii) there is a finite collection of intervals \([a_i, b_i] \subset J\) such that \( \sum_{i} \int_{a_i}^{b_i} f(x) \, dx > 1/\varepsilon \) and the intervals \([a_i, a_i + \tau(b_i - a_i)]\) are mutually disjoint.

Proof. We choose \( \sigma \in (\tau, 1) \), an interval \([a, b] \subset J\) and \( m \in \mathbb{N} \) such that \( m > 1/\varepsilon^2 \).

For \( i = 0, 1, \ldots, m \) we recursively define \( a_0 := a \) and \( a_i := a_{i-1} + \sigma(b - a_i) \) and notice that \( a = a_0 < a_1 < \cdots < a_m < b \).

Let

\[
 f(x) := \begin{cases} 
 \varepsilon/(b - a_m) & x \in [a_m, b] \\
 0 & x \notin [a_m, b] 
\end{cases}
\]

Then (i) holds since \( \int_{-\infty}^{\infty} f(x) \, dx = \varepsilon \) and to satisfy (ii) we may take the intervals 
[\( [a_i, b_i] := [a_i, b] \), for \( i = 0, \ldots, m - 1 \)]. Indeed, with this choice \( \sum_{i} \int_{a_i}^{b_i} f(x) \, dx = m\varepsilon > 1/\varepsilon \) and \( a_i + \tau(b_i - a_i) < a_i + \sigma(b - a_i) = a_{i+1} \), and so the intervals \([a_i, a_i + \tau(b_i - a_i)]\) are mutually disjoint.

Theorem 14. There is a Lebesgue integrable function \( f : \mathbb{R} \to [0, \infty) \) that is, for any \( 0 < \alpha < 1 \), not (MC\( \alpha \)) integrable on any interval \( I \subset \mathbb{R} \).

Proof. Let \( r_k \) be an enumeration of all rational numbers, \( \varepsilon_k := 2^{-k} \) and \( \tau_k := 1 - 2^{-k} \). Obtain \( f_k \) from Lemma \( \ref{lem:existence} \) used with these \( \varepsilon_k, \tau_k \) and \( J_k \). Since the function \( \sum_{k=1}^{\infty} f_k \) is Lebesgue integrable, it is finite almost everywhere. Hence the set \( N := \{ x \in \mathbb{R} : \sum_{k=1}^{\infty} f_k(x) = \infty \} \cup \{ r_k : k = 1, 2, \ldots \} \) is Lebesgue null. Choose a \( G_{\delta} \) Lebesgue null set \( G \) containing \( N \) and define

\[
 f(x) := \begin{cases} 
 \sum_{k=1}^{\infty} f_k(x) & x \notin G, \\
 0 & x \in G. 
\end{cases}
\]

Then \( f : \mathbb{R} \to [0, \infty) \) is Lebesgue integrable; let \( F \) be its indefinite Lebesgue integral.

Suppose \( 0 < \alpha < 1 \) and \( f \) is (MC\( \alpha \)) integrable on some interval \( I \). By Proposition \( \ref{prop:integrability} \) \( F \) is an (MC\( \alpha \)) derivative of \( F \). Let \( \varphi \) be an \( \alpha \)-control function for \( (F, f) \). For each \( x \in N \) find \( \delta_x > 0 \) such that

\[
 |F(x + h) - F(x)| \leq \varphi(x + \alpha h) - \varphi(x)
\]

whenever \( 0 < h < \delta_x \).

Let \( N_I := \{ x \in G : \delta_x > 2^{-l} \} \) and infer from the Baire Category Theorem that there are \( l \in \mathbb{N} \) and an interval \( J = (u, v) \subset I \) such that \( N_I \cap J \) is dense in \( J \). Since \( \varepsilon_k \to 0 \) and \( \tau_k \to 1 \), the inequalities \( \alpha < \tau_k \) and \( \varphi(u) - \varphi(v) < 1/\varepsilon_k \) hold for all sufficiently large \( k \). Using also that \( J \) contains \( [r_k - \varepsilon_k, r_k + \varepsilon_k] \) for infinitely many \( k \), we find \( k > l \) such that \( \alpha < \tau_k, \varphi(u) - \varphi(v) < 1/\varepsilon_k = 2^k \) and \( [a, b] := [r_k - \varepsilon_k, r_k + \varepsilon_k] \subset J \).

Recalling that \( f_k \) has been found by using Lemma \( \ref{lem:existence} \) we infer from its statement (ii) that there is a finite collection of intervals \([a_i, b_i] \subset (a, b)\) such that the intervals \([a_i, a_i + \tau_k(b_i - a_i)]\) are mutually disjoint and \( \sum_{a_i}^{b_i} f_k(x) \, dx > 1/\varepsilon_k = 2^k \). Since
$N_i \cap (a, b)$ is dense in $(a, b)$ and $\alpha < \tau_k$, there are $x_i \in (a, b)$, $x_i \leq a_i$ close enough to $a_i$ such that the intervals $[x_i, x_i + \alpha(b_i - x_i)]$ are also mutually disjoint. Recalling that $k > l$, we see that $b_i - x_i \leq 2\varepsilon_k \leq 2^{-l}$, so $\delta_{x_i} > 2^{-l} \geq b_i - x_i$ and the definition of $\delta_{x_i}$ implies

$$\varphi(x_i + \alpha(b_i - a_i)) - \varphi(x_i) \geq F(b_i) - F(x_i).$$

Since $\varphi$ is increasing and $[x_i, x_i + \alpha(b_i - x_i)]$ are mutually disjoint subintervals of $(u, v)$, we get

$$2^k > \varphi(v) - \varphi(u) \geq \sum_i (\varphi(x_i + \alpha(b_i - a_i)) - \varphi(x_i)) \geq \sum_i (F(b_i) - F(x_i)) \geq \sum_i \int_{x_i}^{b_i} f_k(x) \, dx > \sum_i \int_{a_i}^{b_i} f_k(x) \, dx \geq 2^k,$$

which is the desired contradiction. \(\square\)

The final result of this section may seem to be a good start for showing the equivalence of Denjoy-Perron and $(MC_\alpha)$ integrals for any $\alpha \geq 1$ by the methods stemming from Denjoy's original definition \([2]\) of his integral. These methods have been used, for example, to show equivalence of the Denjoy and Perron integrals (see the proof of the Theorem of Hake-Alexandroff-Looman at \([7, \text{VIII(3)}]\)). However, the use of this idea to show equivalence of the $(MC_\alpha)$ and Denjoy-Perron integrals would need a similar statement with $I$ replaced by any closed subset of $I$. This turned out to be possible when $1 \leq \alpha \leq 2$, and will lead us to the proof of Theorem \([13]\) but for $\alpha > 2$ we actually used this information to guess how the counterexamples required in Theorem \([5, \text{iii}]) and \([iv]\) may look like.

**Proposition 15.** If, for some $\alpha > 0$, $F$ is an indefinite $(MC_\alpha)$ integral of $f$ on an interval $I$, there is a dense open subset $G$ of $I$ such that $F$ is absolutely continuous on every component of $G$. Consequently, $F$ is an indefinite Lebesgue integral of $f$ on every component of $G$.

**Proof.** It suffices to show that there is an interval $J \subset I$ on which $F$ has bounded variation. Indeed, since $f = F'$ a.e., we infer from \([7, \text{Lemma IV(7.4)}]\) that $f$ is Lebesgue integrable on $(a, b)$ and from Theorem \([14]\) that $F$ is an indefinite Lebesgue integral of $f$ on $J$. The required set $G$ is then obtained by a repeated use of this for suitable subintervals of $I$.

Let $\varphi$ be an $\alpha$-control function for $(F, f)$ and for every $x \in I$ choose $\delta_x > 0$ such that $|F(y) - F(x) - f(x)(y - x)| \leq |\varphi(x + \alpha(y - x)) - \varphi(x)|$ when $|y - x| \leq \delta_x$. Let $Q_k := \{x \in I : |f(x)| \leq k, \delta_x > 1/k\}$. Then $Q = \bigcup_{k=1}^{\infty} Q_k$, and hence by the Baire Category Theorem there is an open interval $(a, b) \subset I$ such that $Q_k \cap (a, b)$ is dense in $(a, b)$. We diminish $(a, b)$, if necessary, so that $b - a < 1/k$ and show

$$|F(y) - F(x)| \leq |\gamma(y) - \gamma(x)| \quad \text{for } x, y \in (a, b),$$

where $\gamma(x) = 2n(\varphi(x) + kx)$ and $n \in \mathbb{N}$ is the least integer greater than $3\alpha$.  

9
To prove (4), suppose \(x, y \in (a, b), x < y\) and \(\varepsilon > 0\), and find \(0 < \delta < (y - x)/6(1 + \alpha)\) such that \(|F(u) - F(v)| < \varepsilon/2\) when \(v = x, y\) and \(|u - v| < \delta\). Use that \(Q_k\) is dense in \((x, y)\) to find \(x < x_0 < x_1 < \cdots < x_{2n} < y\) such that \(x_i \in Q_k\) and \(|x_i - (x + i(y - x)/2n)| < \delta\). If \(1 \leq i \leq n\), we have \(x_{i-1} + \alpha(x_i - x_{i-1}) < x + (i-1)(y - x)/2n + \alpha(y - x)/2n + \delta < x + (y - x)/2 + (y - x)/6 + (y - x)/6 + (y - x)/6 = y\).

Since \(x_{i-1} \in Q_k\) and \(x_i - x_{i-1} < 1/k < \delta_x\), this and monotonicity of \(\varphi\) imply

\[
|F(x_i) - F(x_{i-1})| \leq \varphi(x_i - 1 + \alpha(x_i - x_{i-1})) - \varphi(x_{i-1}) + k(x_i - x_{i-1}) \\
\leq \varphi(y) - \varphi(x) + k(y - x) \leq (\gamma(y) - \gamma(x))/2n.
\]

If \(n < i \leq 2n\), we similarly have \(x_i - \alpha(x_i - x_{i-1}) \geq x\), and hence

\[
|F(x_i) - F(x_{i-1})| \leq |\varphi(x_i - \alpha(x_i - x_{i-1})) - \varphi(x_i)| + k(x_i - x_{i-1}) \leq (\gamma(y) - \gamma(x))/2n.
\]

Summing these inequalities gives \(|F(x_{2n}) - F(x_0)| \leq \gamma(y) - \gamma(x)|\), from which we infer

\[
|F(y) - F(x)| \leq |F(x_{2n}) - F(x_0)| + |F(x_{2n}) - F(y)| + |F(x) - F(x_0)| \leq \gamma(y) - \gamma(x) + 2\varepsilon
\]

and, since \(\varepsilon > 0\) is arbitrary, \(|F(y) - F(x)| \leq |\gamma(y) - \gamma(x)|\).

Having thus finished the proof of (4), we use it together with the monotonicity of \(\varphi\) to infer that

\[
\sum_i |F(u_i) - F(v_i)| \leq \sum_i (\gamma(v_i) - \gamma(u_i) \leq \gamma(b) - \gamma(a)
\]

for any mutually disjoint intervals \((u_i, v_i)\) with end-points in \((a, b)\). Consequently, \(F\) has bounded variation on \((a, b)\), which, as explained at the beginning of this proof, implies the statement of the Proposition.

\[\square\]

### 3.2 Coincidence of (MC) and Denjoy-Perron integrals

The coincidence the (MC) and Denjoy-Perron integrals, and so of the \((MC_\alpha)\) and Denjoy-Perron integrals when \(\alpha = 1\) was proved in [3] by Bendová and Malý. In their proof they used modern definitions of the Denjoy-Perron integral. Here we point out that it can be proved directly by using Perron’s original definition of the Denjoy-Perron integral. For this, we first briefly recapitulate the definition of the indefinite Perron integral. Full information may be found in [7] VI(§6) and VIII(§3)]

**Definition 16.** Let \(f, F\) be functions defined on an open interval \(I\). Then \(F\) is said to be an indefinite Perron integral of \(F\) on \(I\) if for every \(\varepsilon > 0\) and a compact interval \(J \subset I\) there are functions \(U, V : I \to \mathbb{R}\) such that \(D\left\{U(x) \geq f(x) \geq D\left\{V(x) \right.\right.\) for every \(x \in I\) and \(|U(x) - F(x)| + |V(x) - F(x)| < \varepsilon\) for every \(x \in J\). Here

\[
\overline{D}U(x) := \liminf_{y \to x} \frac{F(y) - F(x)}{y - x} \quad \text{and} \quad \underline{D}U(x) := \limsup_{y \to x} \frac{F(y) - F(x)}{y - x}.
\]

**Theorem 17** (Bendová and Malý). \(F\) is an indefinite (MC) integral of \(f\) on \(I\) if and only if it is its indefinite Perron integral on \(I\).
Proof. Write \( I \) as a union of an increasing sequence of compact intervals \( J_k = [a_k, b_k] \).

Let \( F \) be an indefinite (MC) integral of \( f \) on \( I \) and \( \varphi \) be a control function for \((F,f)\). Given \( \varepsilon > 0 \), we let \( U_\varepsilon := F + \varepsilon \varphi \) and \( V_\varepsilon := F - \varepsilon \varphi \). For every \( x \in I \) there is \( \delta > 0 \) such that \( |y - x| < \delta \) implies \( |F(y) - F(x) - f(y - x)| \leq \varepsilon |\varphi(y) - \varphi(x)| \). Rearranging this inequality gives \( (U_\varepsilon(y) - U_\varepsilon(x))/(y-x) \geq f(x) \); hence \( UV_\varepsilon(x) \geq f(x) \). A symmetric argument shows that \( DV_\varepsilon(x) \leq f(x) \). Finally, on each \( J_k \) we have \( |U(x) - F(x)| + |V(x) - F(x)| \leq 2\varepsilon \max(\|\varphi(a_k)\|,\|\varphi(b_k)\|) \). Hence \( F \) is an indefinite Perron integral of \( f \) on \( I \).

Assuming \( F \) is an indefinite Perron integral of \( f \) on \( I \), for every \( k \in \mathbb{N} \) there are functions \( U_k, V_k \) such that \( D(U_k) \geq f \geq D(V_k) \) on \( I \) and \( |U_k - V_k| \leq 2^{-k} \) on \( J_k \). Since \( D(U_k - V_j) \geq 0 \), \( U_k - V_j \) are increasing (see, for example, [7, Theorem VI(3.2)]), and so are also \( U_k - F \) and \( F - V_j \). Letting \( \varphi(x) := x + \sum_{k=1}^{\infty} k(U_k(x) - V_k(x)) \), which is well-defined since each \( x \in I \) belongs to all but finitely many \( J_k \), we show that \( \varphi \) is the required control function. Clearly, it is strictly increasing. For every \( x \in I \) and \( k \in \mathbb{N} \) there is \( \delta > 0 \) such that \( 0 < |y - x| < \delta \) implies

\[
\frac{U_k(y) - U_k(x)}{y - x} \geq f(x) - 1/k \quad \text{and} \quad \frac{V_k(y) - V_k(x)}{y - x} \leq f(x) + 1/k.
\]

Hence

\[
\frac{F(y) - F(x)}{y - x} - f(x) \leq \frac{U_k(y) - U_k(x)}{y - x} - \frac{V_k(y) - V_k(x)}{y - x} + \frac{1}{k} \leq \frac{\varphi(y) - \varphi(x)}{k(y - x)}
\]

and a symmetric argument gives

\[
\frac{F(y) - F(x)}{y - x} - f(x) \geq \frac{V_k(y) - V_k(x)}{y - x} - \frac{U_k(y) - U_k(x)}{y - x} - \frac{1}{k} \geq -\frac{\varphi(y) - \varphi(x)}{k(y - x)},
\]

which shows that \( \varphi \) is a control function for \((F,f)\).

\[\square\]

3.3 Relation between \((MC_\alpha)\) and Denjoy-Perron integrals

Our main argument here shows that every \((MC_2)\)-integrable function is Denjoy-Perron integrable, which together with the result of Bendorövá and Malý and Proposition 4 immediately implies that for \( 1 \leq \alpha \leq 2 \) the \((MC_\alpha)\) and Denjoy-Perron integrals coincide.

Theorem 18. For any \( 1 \leq \alpha \leq 2 \), the \((MC_\alpha)\) integral on any interval \( I \) coincides with the Denjoy-Perron integral.

Proof. By Theorem 17 and Proposition 4 it suffices to show that every \((MC_2)\) integrable function is Denjoy-Perron integrable. So suppose \( F \) is an indefinite \((MC_2)\) integral of \( f \) on \( I \). Denote by \( G \) the union of those open subintervals of \( I \) on which \( F \) is \( ACG_* \). By the Lindelöf property of the real line, \( G \) is the union of a countable family of such subintervals, and so \( F \) is \( ACG_* \) on \( G \). Since \( F' = f \) almost everywhere, \( F \) is an indefinite Denjoy-Perron integral of \( f \) on each component of \( G \). This means
that if \( G = (a, b) \), we are done; so assume \( Q := (a, b) \setminus G \neq \emptyset \). Notice that \( Q \) has no isolated points by [7] Lemma VIII(3.1). We show that there is an interval \([a, b] \subset I\) such that \( Q \cap (a, b) \neq \emptyset \) and

(a) \( f \) is Lebesgue integrable on \( Q \);

(b) the series of the oscillations of \( F \) on the components of \((a, b) \setminus Q\) converges.

By Lemma 3.4 in [7] Chapter 7 this will imply that \( F \) is an indefinite Denjoy-Perron integral of \( f \) on \((a, b)\). But then \((a, b) \subset G\), contradicting \( Q \cap (a, b) \neq \emptyset \).

To find the interval \((a, b)\), let \( \varphi \) be a 2-control function for \((F, f)\) and for every \( x \in I \) choose \( \delta_x > 0 \) such that \( |F(y) - F(x) - f(x)(y - x)| \leq |\varphi(x + 2(y - x)) - \varphi(x)| \) when \( |y - x| \leq \delta_x \). Let \( Q_k := \{x \in Q : |f(x)| \leq k, \delta_x > 1/k\} \). Then \( Q = \bigcup_{k=1}^\infty Q_k \), and hence by the Baire Category Theorem there is an open interval \((a, b)\) such that \( M := Q \cap (a, b) \neq \emptyset \) and \( Q_k \cap M \) is dense in \( M \). Since \( Q \) has no isolated points, we may diminish \((a, b)\), if necessary, to guarantee \( a, b \in M \) and \(|b - a| < 1/k\). We show that

\[
|F(y) - F(x)| \leq 2|\varphi(y) - \varphi(x)| + 2k|y - x| \quad \text{for} \quad x, y \in M. \tag{5}
\]

To prove this we use that \(|y - x| < \delta_x\) to infer that

\[
|F((x + y)/2) - F(x)| \leq |\varphi(y) - \varphi(x)| + |f(x)||y - x| \leq |\varphi(y) - \varphi(x)| + k|y - x|.
\]

Similarly we have \( |F((x + y)/2) - F(y)| \leq |\varphi(y) - \varphi(x)| + k|y - x| \). Hence

\[
|F(y) - F(x)| \leq |F((x + y)/2) - F(x)| + |F((x + y)/2) - F(y)| \leq 2|\varphi(y) - \varphi(x)| + 2k|y - x|,
\]

as claimed.

Clearly, monotonicity of \( \varphi \) and (5) show that

\[
\sum_i |F(u_i) - F(v_i)| \leq \sum_i (2|\varphi(v_i) - \varphi(u_i)| + 2k|v_i - u_i|) \leq 2|\varphi(b) - \varphi(a)| + 2k(b - a)
\]

for any mutually disjoint intervals \((u_i, v_i)\) with end-points in \( M \). Consequently, \( F \) has bounded variation on \( M \) and, since it is continuous, also on \( Q \). (See [7] VII(34)).

Since \( f = F' \) a.e., we infer (a) from [7] Lemma VIII(2.1) and [7] Lemma IV(7.4).

For (b) consider any component \((u, v)\) of \((a, b) \setminus Q\). If \( x \in [u, (u + v)/2] \), then

\[
|F(x) - F(u)| \leq |\varphi(u + 2(x - u)) - \varphi(u)| \leq \varphi(v) - \varphi(u),
\]

and if \( x \in [(u + v)/2, v] \), then

\[
|F(x) - F(v)| \leq |\varphi(v + 2(x - v)) - \varphi(v)| \leq \varphi(v) - \varphi(u).
\]

Using also (4), we get

\[
|F(x) - F(u)| \leq 3(\varphi(v) - \varphi(u)) + 2k(y - x),
\]

and hence the oscillation of \( F \) on \((u, v)\) is at most \( 6(\varphi(v) - \varphi(u)) + 4k(y - x) \). The sum of these oscillations over the components of \((a, b) \setminus Q\) is therefore at most

\[
6(\varphi(b) - \varphi(a)) + 4k(b - a),
\]

showing that (b) holds and so proving the Theorem.
3.4 Preliminaries to constructions for $\alpha > 2$

As we already said in connection with Proposition 13, the examples we have to construct to show the cases of Theorem 3 when $\alpha > 1$ need a closed, necessarily nowhere dense, set on which the function we construct is not ACG, respectively ACG. The (ternary) Cantor set is a good candidate, especially when we recall that the corresponding Cantor function has a very quick increase when passing through the Cantor set, thereby providing a good choice for a control function. We will actually use the ternary Cantor set to prove Theorem 3(iii), but for the proof of Theorem 3(iv) we need a bit more room that we gain by using one of the Cantor type sets with base 5. We therefore fix notation for the construction of these sets with any odd base $q \geq 3$, although we will use it only for $q = 3$ and $q = 5$.

The Cantor type sets we use are defined in a standard way. Let $q = 2^m + 1$ be an odd integer. We recursively define collections of closed intervals $C_k$, $k = 0, 1, 2, \ldots$ and collections of open intervals $R_k$, $k = 1, 2, \ldots$ as follows.

We let $C_0 := \{[0, 1]\}$ and, whenever $C_{k-1}$ has been defined, and $[u, v] \in C_{k-1}$, we put into $C_k$ the intervals $[u + j(v-u)/q, u + (j+1)(v-u)/q]$ where $0 \leq j \leq q - 1$ is even, and into $R_k$ the intervals $(u + j(v-u)/q, u + (j+1)(v-u)/q)$ where $0 \leq j \leq q - 1$ is odd.

The (base $q$) Cantor set is then defined as $C := \bigcap_{k=0}^{\infty} C_k$ where $C_k$ is the union of intervals from $C$. We will use the following straightforward facts.

- $C_0 \supset C_1 \supset \ldots$
- $C_k$ is the set of (connected) components of $C_k$.
- $R_k$ is the set of components of $C_k \setminus C_{k-1}$.
- The set of components of $[0, 1] \setminus C$ is $R := \bigcup_{k=1}^{\infty} R_k$, where the union is disjoint.
- For every $(u, v) \in R_k$, $u, v \in C$ and both $u, v$ are end-points of intervals from $C_k$.
- $C_k$ consists of $(m + 1)^k$ mutually disjoint closed intervals of length $q^{-k}$.
- $R_k$ consists of $m(m+1)^{k-1}$ mutually disjoint open intervals of length $q^{-k}$.
- If $k > p$, an interval from $C_p$ contains $m(m+1)^{k-p-1}$ intervals from $R_k$.

We will also use the corresponding Cantor function $\psi : \mathbb{R} \to [0, 1]$, which is characterized by being continuous, increasing, constant on each component of $\mathbb{R} \setminus C$, and, for each $k$, mapping each interval from $C_k$ onto an interval of length $(m + 1)^{-k}$. To define it, we may, for example, let $\psi_k(x) := (m + 1)^{-k}q^k|(-\infty, x) \cap C_k|$, observe that $|\psi_k - \psi_{k-1}| \leq (m + 1)^{-k}$ and define $\psi := \lim_{k \to \infty} \psi_k$.

3.5 (MC$_{\alpha}$) integrabilities differ for $\alpha \geq 2$

In this proof we will use the ternary Cantor set $C$, the collections of intervals $R_k$ and $R$ and the Cantor function $\psi$, as described in Section 3.4 when we take $q = 3$. 

13
Lemma 19. There are $Q_J > 0$, $J \in \mathcal{R}$, such that

(i) $\lim_{k \to \infty} \max\{Q_J : J \in \mathcal{R}_k\} = 0$;

(ii) $\sum_{J \in \mathcal{R}, J \subset I} Q_J = \infty$ whenever $I \subset (0,1)$ is an open interval meeting $C$.

(iii) for every $\eta > 0$,

$$Q_J \leq \eta \min(\psi(b + \eta(b - a)) - \psi(b), \psi(a) - \psi(a - \eta(b - a)))$$

(6)

holds for all but finitely many $J = (a, b) \in \mathcal{R}$;

Proof. We let $k_i := 4^i$, define $Q_J := 2^{-k - 2l}$ when $J \in \mathcal{R}_k$ and $k_{l-1} \leq k < k_l$ and notice that (1) holds.

For any open interval $J$ meeting $C$ there is $p \in \mathbb{N}$ such that $J$ contains an interval from $\mathcal{C}_p$. It follows that for $k > p$, $J$ contains at least $2^{k-p-1}$ intervals from $\mathcal{R}_k$. Choose $m \in \mathbb{N}$ such that $4^{m-1} > p$. Then, if $l \geq m$ and $k_{l-1} \leq k < k_l$, the sum of $Q_J$ over those $J \in \mathcal{R}_k$ that are contained in $J$ is at least $2^{-p-2l-1}$. Hence

$$\sum_{J \in \mathcal{R}, J \subset I} Q_J \geq \sum_{l=m}^{\infty} \sum_{k=k_{l-1}}^{k_{l-1}} 2^{-p-2l-1} = \sum_{l=m}^{\infty} 2^{-p-2l-1}(4^l - 4^{l-1}) \geq \sum_{l=m}^{\infty} 2^{-p-3} = \infty.$$ 

To prove (iii) suppose $\eta > 0$ and choose $n \in \mathbb{N}$ such that $2^{-n} < \eta$. Whenever $l \geq n$, $k_{l-1} \leq k < k_l$ and $J = (a, b) \in \mathcal{R}_k$, we notice that $b + \eta(b - a) \geq b + 2^{-n}3^{-k} \geq b + 3^{-k-n}$, and so that $b$ is the left-end point of an interval from $\mathcal{C}_{k+n}$ of length $3^{-k-n}$ that is mapped by $\psi$ onto an interval of length $2^{-k-n}$. Hence

$$\eta(\psi(b + \eta(b - a)) - \psi(b)) \geq 2^{-k-2n} \geq Q_J.$$ 

A symmetric argument shows $\eta(\psi(a) - \psi(a - \eta(b - a))) \geq Q_J$. Hence (6) holds for all $J \in \mathcal{R}_k$ when $k \geq n$, hence for all $J \in \mathcal{R}$ except finitely many. 

\[\square\]

Theorem 20. For any interval $I$ and $\alpha \geq 2$ there are functions $f, F : \mathbb{R} \to \mathbb{R}$ such that

(i) $F$ is the Denjoy-Khintchine indefinite integral of $f$ on $\mathbb{R}$;

(ii) $F$ is the $(\text{MC}_\beta)$ indefinite integral of $f$ on $\mathbb{R}$ for every $\beta > \alpha$;

(iii) $f$ is not $(\text{MC}_\alpha)$ integrable on $I$.

Proof. Without loss of generality we assume that $I$ is an open interval containing $[0, 1]$. Let $Q_J$ be as in Lemma 19. Choose a continuously differentiable function $\xi : \mathbb{R} \to [0, 1]$ with support in $[-1, 1]$ such that $\xi(x) = 1$ for $-1/2 \leq x \leq 1/2$. Let $\sigma := 1/\alpha$ and $\sigma_J := 3^{-k}\sigma$ for $J \in \mathcal{R}_k$. For $J = (a, b) \in \mathcal{R}$ let $u_J := a + \sigma(b - a)$ and define

$$\xi_J(x) := Q_J(\xi((x - u_J)/(\sigma_J(b - a)))$$

and $F(x) := \sum_{J \in \mathcal{R}} \xi_J(x)$.
Notice that the support of $\xi_J$ is contained in $[u_J - \sigma_J(b - a), u_J + \sigma_J(b - a)]$ which, since $\sigma_J < \sigma < 1/2$ is contained in $J$. Since the intervals from $\mathcal{R}$ are mutually disjoint, the functions $F_k(x) := \sum_{j=1}^{k} \sum_{J \in \mathcal{R}} \xi_J(x)$ satisfy $|F - F_k| \leq \max_{l>k} \max \{Q_J : J \in J_k\}$. Hence Lemma 19(ii) implies that $F_k$ converge to $F$ uniformly, and since $F_k$ are continuous, so is $F$. Moreover, $F = 0$ outside $[0, 1]$ and $F = \xi_J$ on $J \in \mathcal{R}$; hence $F$ is continuously differentiable on $\mathbb{R} \setminus C$ and we may define

$$f(x) := \begin{cases} F'(x) & \text{when } x \in \mathbb{R} \setminus C \\ 0 & \text{when } x \in C. \end{cases}$$

The first statement, that $F$ is the Denjoy-Khintchine indefinite integral of $f$ is straightforward. Since $F = 0$ on $C$, it is absolutely continuous on $C$ and since it is continuously differentiable outside $C$, it is ACG also on $\mathbb{R} \setminus C$. So $F$ is ACG on $\mathbb{R}$, continuous and $F' = f$ almost everywhere, which implies that indeed $F$ is the Denjoy-Khintchine indefinite integral of $f$.

To prove the second statement, we let $\varphi(x) := x + \psi(x)$, observe that $\varphi$ is strictly increasing and hence it suffices to show that for every $\beta > \alpha$, $x \in \mathbb{R}$ and $\varepsilon > 0$ there is $\delta > 0$ such that

$$|F(y) - F(x) - f(x)(y - x)| \leq \varepsilon|\varphi(x + \beta(y - x)) - \varphi(x)|$$  \hspace{1cm} (7)

whenever $y \in (x - \delta, x + \delta)$. So we fix $\beta > \alpha$, $x \in \mathbb{R}$ and $\varepsilon > 0$, and find such a $\delta$. This is easy when $x \notin C$, since then $f(x) = F'(x)$ and so for small enough $\delta > 0$,

$$|F(y) - F(x) - f(x)(y - x)| \leq \varepsilon|y - x| \leq \varepsilon|\varphi(x + \beta(y - x)) - \varphi(x)|$$

for every $y \in (x - \delta, x + \delta)$.

So assume $x \in C$. Choose $\eta > 0$ such that $0 < \eta < \varepsilon$ and, recalling that $\sigma = 1/\alpha$ and $\beta > \alpha$, that also $\beta \sigma > (1 + \eta)$. By Lemma 19(iii) we find $0 < \kappa < \sigma - (1 + \eta)/\beta$ such that for every $J = (a, b) \in \mathcal{R}$ with $b - a < \kappa$,

$$Q_J \leq \eta \min\{\psi(b + \eta(b - a)) - \psi(b), \psi(a) - \psi(a - \eta(b - a))\}. \hspace{1cm} (8)$$

Since $\sigma_J \leq b - a \leq \kappa$, we also have

$$\beta(\sigma - \sigma_J) \geq 1 + \eta. \hspace{1cm} (9)$$

Let $\delta := \kappa \sigma / 2$ and consider any $y \in (x - \delta, x + \delta)$. Since $F \geq 0$ and $f(x) = F'(x) = 0$, (7) reduces to showing that

$$F(y) \leq \varepsilon|\varphi(x + \beta(y - x)) - \varphi(x)|. \hspace{1cm} (10)$$

As this is obvious when $F(y) = 0$, we assume that $F(y) \neq 0$. Then there is $J = (a, b) \in \mathcal{R}$ such that $0 < F(y) \leq Q_J$ and $y \in [a + (\sigma - \sigma_J)(b - a), a + (\sigma + \sigma_J)(b - a)]$. In particular we have $|x - y| \geq \sigma(b - a)/2$ since $x \notin J$, $\sigma - \sigma_J \geq \sigma/2$ and $1 - (\sigma + \sigma_J) \geq 1/2 - \sigma_J \geq \sigma - \sigma_J \geq \sigma/2$. Hence $\kappa \sigma / 2 = \delta > |y - x| \geq \sigma(b - a)/2$, which gives $b - a < \kappa$ and hence (8) and (9) hold.

When $\alpha > 2$ the situations when $y > x$ and $y < x$ are not completely symmetric, so we continue by distinguishing these two cases.
Case $y > x$. Then $x \leq a < y$ and (10) gives
\[x + \beta(y - x) = x + \beta(a - x) + \beta(y - a) \geq x + (a - x) + \beta(y - a) \geq a + \beta(\sigma - \sigma_J)(b - a) \geq b + \eta(b - a).\]
Hence
\[F(y) \leq Q_J \leq \eta(\psi(b + \eta(b - a)) - \psi(b)) \leq \eta(\psi(x + \beta(y - x)) - \psi(x)) \leq \varepsilon(\varphi(x + \beta(y - x)) - \varphi(x)).\]

Case $y < x$. Then $x \geq b$ and, using $\sigma \leq 1/2$ and (10) to infer
\[\beta(1 - \sigma - \sigma_J) \geq \beta(\sigma - \sigma_J) \geq \eta,\]
we get
\[x + \beta(y - x) = x - \beta(x - b) - \beta(b - y) \leq x - (x - b) - \beta(b - y) \leq b - \beta(1 - \sigma - \sigma_J)(b - a) \leq a - \eta(b - a).\]
Hence
\[F(y) \leq Q_J \leq \eta(\psi(a) - \psi(a - \eta(b - a))) \leq \eta(\psi(x) - \psi(x + \beta(y - x))) \leq \varepsilon(\varphi(x + \beta(y - x)) - \varphi(x)).\]

In both cases we have proved that (10) holds for $y \in (x - \delta, x + \delta)$ as required, and we conclude that $f$ is (MC$_\beta$) integrable.

It remains to show that $f$ is not (MC$_\alpha$) integrable on $(0, 1)$. Arguing by contradiction and using Proposition 4, we assume that $f$ is an indefinite (MC$_\alpha$) integral of $f$ on $(0, 1)$. Then there is a strictly increasing function $\gamma : (0, 1) \to \mathbb{R}$ such that for every $x \in (0, 1)$ there is $\delta_x > 0$ such that for every $y \in (x, x + \delta_x)$,

\[|F(y) - F(x) - f(x)(y - x)| \leq \gamma(x + \alpha(y - x)) - \gamma(x).\]

Let $\Delta_k := \{x \in C \cap (0, 1) : \delta_x > 1/k\}$. Since $C = \bigcup_{k=1}^\infty \Delta_k$, the Baire Category Theorem implies that there are $k \in \mathbb{N}$ and an open interval $J \subset (0, 1)$ such that $J \cap C \neq \emptyset$ and $\Delta_k \cap J$ is dense in $C \cap J$. We diminish $J$ if necessary to achieve $|J| < 1/k$ and choose an interval $[a, b] \subset J$ such that $(a, b) \cap C \neq \emptyset$.

By Lemma 14 we find $n \in \mathbb{N}$ and intervals $J_i = (a_i, b_i) \in \mathcal{R}$, $i = 1, \ldots, n$ such that $(a_i, b_i) \subset (a, b)$ and $\sum_{i=1}^n Q_{J_i} \geq \gamma(b) - \gamma(a)$. Using that $a_i + \sigma(b_i - a_i) = u_{J_i}$, $C$ has no isolated points and $a_i \in C \cap J$, we find $x_i \in C \cap J \cap \Delta_k$ so close to $a_i$ that the intervals $(x_i, b_i)$, $i = 1, \ldots, n$ are mutually disjoint and
\[y_i := x_i + \sigma(b_i - x_i) \in [u_{J_i} - \sigma_{J_i}(b_i - a_i)/2, u_{J_i} + \sigma_{J_i}(b_i - a_i)/2].\]
Hence $F(y_i) = Q_{J_i}$ and since $F(x_i) = f(x_i) = 0$,
\[Q_{J_i} = F(y_i) - F(x_i) - f(x_i)(y_i - x_i) \leq \gamma(x_i + \alpha(y_i - x_i)) - \gamma(x_i) = \gamma(b_i) - \gamma(x_i),\]
Finally, we use that $\gamma$ is increasing to get
\[
\gamma(b) - \gamma(a) \geq \sum_{i=1}^{n} (\gamma(b_i) - \gamma(x_i)) \geq \sum_{i=1}^{n} Q_{j_i} > \gamma(b) - \gamma(a),
\]
which is the desired contradiction. \(\square\)

### 3.6 Relation between \((MC_\alpha)\) and Denjoy-Khintchine integrals

Since indefinite \((MC_\alpha)\) integrals are differentiable almost everywhere by Proposition 4(ii) but indefinite Denjoy-Khintchine integrals need not be differentiable almost everywhere (as an example one may take a continuous function that is everywhere approximately differentiable but not differentiable almost everywhere), we have

**Observation 21.** On any interval \(I\) there is a Denjoy-Khintchine integrable function that is not \((MC_\alpha)\) integrable for any \(\alpha > 0\).

For the opposite direction we need more work. To construct the required example, we will use the Cantor-type set \(C\) with base \(q = 5\), its approximating sets \(C_k\), the sets of intervals \(C_k, R_k\), and \(R\), and the corresponding Cantor-type function \(\psi\) described in Section 3.4.

**Theorem 22.** There is a function \(f : \mathbb{R} \to \mathbb{R}\) that is \((MC_\alpha)\) integrable on \(\mathbb{R}\) for every \(\alpha > 2\) but is not Denjoy–Khintchine integrable on \([0, 1]\).

**Proof.** For any interval \(I = [a, b]\) and \(0 < \tau < 1/2\) choose a continuously differentiable function \(g_{I, \tau} : \mathbb{R} \to \mathbb{R}\) which is increasing on \((-\infty, (a + b)/2]\), decreasing on \([(a + b)/2, \infty)\) and satisfies

\[
g_{I, \tau}(x) = \begin{cases} 0 & x \leq a + (1 + \tau)(b - a)/5; \\ 0 & x \geq b - (1 + \tau)(b - a)/5; \\ 1 & x \in [a + (2 - \tau)(b - a)/5, b - (2 - \tau)(b - a)/5]. \\
\end{cases}
\]

We let \(\sigma_k := 1/(k + 1)\), \(\tau_k := (k + 1)/(2(k + 2))\) and

\[
F(x) := \sum_{k=1}^{\infty} \sigma_k 3^{-k} \sum_{l \in \mathcal{C}_{k-1}} g_{I, \tau_k}(x).
\]

Since for each \(k\), \(\sum_{l \in \mathcal{C}_{k-1}} g_{I, \tau_k}(x)\) is continuous and bounded by one, \(F\) is the sum of a uniformly convergent series of continuous functions, and so it is continuous. We list the following easy but crucial properties of \(F\).

(a) \(|F(y) - F(x)| \leq \sigma_k 3^{-k+1}\) whenever \(x, y\) belong to the same interval from \(\mathcal{C}_{k-1}\);

(b) \(|F(b) - F(a)| = \sigma_k 3^{-k}\) whenever \((a, b)\) is an interval from \(\mathcal{R}_k\);

(c) if \((a, b)\) \(\in \mathcal{R}_k\), then \(F\) is constant on \([a, a + \tau_k (b - a)]\) as well as on \([b - \tau_k (b - a), b]\).
(d) On each \(J \in \mathcal{A}_k\) the sum defining \(F\) is finite, and hence \(F\) is continuously differentiable on \(\mathbb{R} \setminus C\).

The property \([d]\) allows us to define

\[
 f(x) := \begin{cases} 
 F'(x) & x \notin C; \\
 0 & x \in C.
\end{cases}
\]

Fix for a while \(\alpha > 2, x \in C\) and \(\varepsilon > 0\). We show that there is \(\delta > 0\) such that

\[
 |F(y) - F(x)| \leq \varepsilon(|\psi(x + \alpha(y - x)) - \psi(x)|)
\]  

whenever \(0 < y - x < \delta\). Since \(F = 0\) on \([1, \infty)\), any \(\delta > 0\) will do for \(x = 1\). So we assume \(x < 1\). To define \(\delta\), start by using that \(\alpha > 2\) to find \(l \in \mathbb{N}\) such that \(\alpha > 2(1 + 5^{l+1})\). Since \(\tau_j \to 1/2\) and \(\sigma_j \to 0\), there is \(m \in \mathbb{N}\) such that \(\alpha \tau_j > 1 + 5^{-l}\) and \(\sigma_j 3^{l+2} < \varepsilon\) for \(j \geq m\). Having done this, we let \(\delta = \min(1 - x, 5^{-m-1})\).

We are now ready to prove that \((11)\) indeed holds for \(y \in (x, x + \delta)\). Given such \(y\), find the least \(k \geq 1\) for which there is an interval \((u, v) \in \mathcal{A}_k\) that is contained in \((x, y)\). Clearly, \(k > m\). To finish the argument, we distinguish three cases.

**Case** \(y \in C_k\). Then \(x, y\) belong to the same interval from \(\mathcal{C}_{k-1}\) since otherwise \(k > 1\) and \([x, y]\) contains an interval from \(\mathcal{A}_{k-1}\). Moreover, \(x + \alpha(y - x) = x + \alpha(u - x) + \alpha(y - u) \geq u + \alpha(v - u)\). Since \(\alpha > 2\) and \(v - u < 5^{-k}\), we infer that \([x + \alpha(y - x), x] \supset [u + \alpha(v - u), u] \supset [v, v + 5^{-k}]\). Since \([v, v + 5^{-k}]\) belongs to \(\mathcal{C}_k\), we conclude from \([a]\) that

\[
 |F(y) - F(x)| \leq \sigma_k 3^{-k+1} = 3\sigma_k(\psi(v + 5^{-k}) - \psi(v)) \leq \frac{1}{2}\varepsilon(|\psi(x + \alpha(y - x)) - \psi(x)|).
\]

In the remaining cases \(y \notin C_k\). Since \(y \in (0, 1)\), there is \(1 \leq j \leq k\) such that \(y \in (w, z)\) for some \((w, z) \in \mathcal{A}_j\). Since \((x, y)\) contains an interval from \(\mathcal{A}_k\) but does not contain any interval from any \(\mathcal{A}_i\) with \(i < k\), we have \(w \in C_k\), \(x < w\) and \((x, w)\) contains an interval from \(\mathcal{A}_k\). By the previous case,

\[
 |F(w) - F(x)| \leq \frac{1}{2}\varepsilon(|\psi(x + \alpha(w - x)) - \psi(x)|).
\]  

**Case** \(y \notin C_k\) and \(y - w < \tau_j 5^{-j}\). Then \([c]\) implies that \(F\) is constant on \([w, y]\) and so \((12)\) gives

\[
 |F(y) - F(x)| = |F(w) - F(x)| \leq \frac{1}{2}\varepsilon(|\psi(x + \alpha(y - x)) - \psi(x)|).
\]

**Case** \(y \notin C_k\) and \(y - w > \tau_j 5^{-j}\). We use \(\delta > y - x \geq y - w \geq \tau_j 5^{-j} \geq 5^{j-1}\) to infer that \(j > m\). Hence \([x, x + \alpha(y - x)] \supset [w, w + \alpha(y - w)] \supset [z, z + 5^{-j-1}]\). Since \([z, z + 5^{-j-1}] \in \mathcal{C}_{j+1}\) and \([w, y]\) is contained in an interval from \(\mathcal{C}_{j-1}\), \([a]\) implies

\[
 |F(y) - F(w)| \leq \sigma_{j+1}(\psi(z + 5^{-j}) - \psi(z)) \leq \frac{1}{2}\varepsilon(|\psi(x + \alpha(y - x)) - \psi(x)|).
\]

This and \((12)\) give

\[
 |F(y) - F(x)| \leq |F(y) - F(w)| + |F(w) - F(x)| \leq \varepsilon(|\psi(x + \alpha(y - x)) - \psi(x)|).
\]
Having thus verified \( \text{(11)} \), we use it and \( f(x) = 0 \) (since \( x \in C \)) to conclude that

\[
\lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(y) - \varphi(x)} = 0.
\]

A symmetric argument shows

\[
\lim_{y \to x} \frac{F(y) - F(x) - f(x)(y - x)}{\varphi(y) - \varphi(x)} = 0.
\]

Finally, if \( x \notin C \), we infer from \( |\varphi(y) - \varphi(x)| \geq |y - x| \) and \( F'(x) = f(x) \) that

\[
\limsup_{y \to x} \frac{|F(y) - F(x) - f(x)(y - x)|}{|\varphi(y) - \varphi(x)|} \leq \lim_{y \to x} \frac{|F(y) - F(x) - f(x)(y - x)|}{|y - x|} = 0.
\]

Hence \( F \) is an indefinite (MC\( \alpha \)) integral of \( f \), as wanted.

It remains to show that \( f \) is not Denjoy-Khintchine integrable. Suppose the opposite and let \( H \) be its indefinite Denjoy-Khintchine integral. Notice that, in principle, \( H \) may be different from \( F \). However, on each interval \((a, b)\) from \( \mathcal{R} \), both \( H \) and \( F \) are Lebesgue indefinite integrals of \( f \) and hence \( H - F \) is constant on \([a, b]\). It follows that \( H(b) - H(a) = F(b) - F(a) \), and so \( \text{(10)} \) holds with \( F \) replaced by \( H \). This information will suffice for our arguments.

Since \( H \) is continuous, \( C \) is a union of closed sets on which it is AC. By the Baire Category Theorem, there is an open interval \( I \) meeting \( C \) such that \( H \) is AC on \( I \cap C \).

Find a component \([u, v]\) of some \( C_m \) contained in \( I \). For \( k \geq m \) the set \([u, v] \cap C_k \) has \( 3^{k-m} \) components and so \([u, v] \cap (C_{k+1} \setminus C_k) \) has \( 2 \cdot 3^{m-k} \) components. For each such component, say \((u, v)\), we have \( |H(v) - H(u)| = \sigma_{k+1} 3^{k+1} \) by validity of \( \text{(10)} \) for \( H \).

Let \((a_j, b_j)\) be an enumeration of the components of \([u, v] \setminus C \). Summing first over those \((a_j, b_j)\) that are components of \( C_{k+1} \setminus C_k \) and then over \( k > m \), we get

\[
\sum_{j=1}^{\infty} |H(b_j) - H(a_j)| = \sum_{k=m}^{\infty} 2 \sigma_{k+1} 3^{k+1} 3^{m-k} = \sum_{k=m}^{\infty} 2 \sigma_{k+1} 3^{m+1} = \infty.
\]

This shows that \( H \) is not AC on \( I \cap C \), and this contradiction shows that \( f \) is not Denjoy-Khintchine integrable.

\[\square\]

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