ON HOMOLOGY ROSES AND THE D(2)-PROBLEM

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Abstract. For a commutative ring $R$ with a unit, an $R$-homology rose is a
topological space whose homology groups with $R$-coefficients agree with those
of a bouquet of circles. In this paper, we study some special properties of
covering spaces and fundamental groups of $R$-homology roses, from which we
obtain some result supporting the Carlsson conjecture on free $(\mathbb{Z}_p)^r$ actions.
In addition, for a group $G$ and a field $\mathbb{F}$, we define an integer called the $\mathbb{F}$-gap
of $G$, which is an obstruction for $G$ to be realized as the fundamental group
of a 2-dimensional $\mathbb{F}$-homology rose. Furthermore, we discuss how to search
candidates of the counterexamples of Wall’s D(2)-problem among $\mathbb{F}$-homology
roses and $\mathbb{F}$-acyclic spaces.

1. Introduction

Let $R$ be a commutative ring with a unit. An $R$-homology rose $B$ with $m$-petals
($m \geq 1$) is a topological space with $H_*(B; R) \cong H_*(\bigvee_m S^1; R)$ where $\bigvee_m S^1$ is
a bouquet of $m$ circles. If $m = 1$, $B$ is called an $R$-homology circle. Our main
concern is when $R$ is $\mathbb{Z}$ or a field $\mathbb{F}$. When $R = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ ($p$ is a prime), $B$
is called a mod-$p$ homology rose (circle). In addition, an $R$-acyclic space is a
topological space $B$ with $H_*(B; R) = 0$.

Many important spaces that occur in geometry and topology are $R$-homology
roses. For example, for any knot $N$ in a 3-sphere $S^3$, its complement $S^3 \setminus N$ is a
$\mathbb{Z}$-homology circle.

Theorem 1.1. Suppose $(\mathbb{Z}_p)^r$ acts cellularly and freely on a finite CW-complex
$X$. Then the orbit space $X/(\mathbb{Z}_p)^r$ is a mod-$p$ homology rose if and only if $X$ is a
mod-$p$ homology rose.

Remark 1.2. Theorem [1.1] actually follows from [8, Proposition 1.2]. But the
proof of Theorem 2.2 given in [8] uses some results from Tate cohomology theory,

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ment) of Jiangsu higher education institutions.
which may not be so familiar to some people. So in the following, we give a proof of Theorem 1.1 via some elementary tools of algebraic topology.

A group $G$ acting cellularly on a CW-complex $X$ means that $G$ acts by homeomorphisms of $X$ that map each cell homeomorphically to another cell. If moreover $G$ acts freely, then the orbit space $X/G$ is also a CW-complex and so we can think of $X$ as a regular covering space over $X/G$ with deck transformation group $G$. So we have another way to state Theorem 1.1.

**Theorem 1.1'.** A regular $(\mathbb{Z}_p)^r$-covering $X$ of a finite CW-complex $K$ is a mod-$p$ homology rose if and only if $K$ is a mod-$p$ homology rose.

**Remark 1.3.** If we do not require the CW-complex $X$ in Theorem 1.1 to be finite, the conclusion of Theorem 1.1 may not be true. For example, $\mathbb{Z}_2$ can act freely $X = S^\infty \times S^1$ with orbit space $\mathbb{R}P^\infty \times S^1$.

For any topological space $X$ and any filed $\mathbb{F}$, let

$$b_i(X; \mathbb{F}) := \dim \mathbb{F}H_i(X; \mathbb{F}).$$

Similarly, for any group $\Gamma$, let

$$b_i(\Gamma; \mathbb{F}) = \dim \mathbb{F}H_i(\Gamma; \mathbb{F}), \quad \forall i \geq 0,$$

where $H_i(\Gamma; \mathbb{F}) = H_i(K(\Gamma, 1); \mathbb{F})$ is the group homology of $\Gamma$ with $\mathbb{F}$-coefficients.

For an $\mathbb{F}$-homology rose $K$ with $m$ petals, it is easy to see that

$$b_1(\pi_1(K); \mathbb{F}) = m, \quad b_2(\pi_1(K); \mathbb{F}) = 0. \quad (1)$$

For a regular covering space $\xi : X \to B$ with deck transformation group $G$, $\xi_* : \pi_1(X) \to \pi_1(B)$ maps $\pi_1(X)$ injectively into $\pi_1(B)$. And if we identify $\pi_1(X)$ with its image under $\xi_*$, we have $\pi_1(B)/\pi_1(X) \cong G$. If we assume that $B$ is a finite CW-complex and a mod-$p$ homology rose, Theorem 1.1 tells us that any regular $(\mathbb{Z}_p)^r$-covering $X$ over $B$ is also a mod-$p$ homology rose. So we have the following corollary.

**Corollary 1.4.** If a finite CW-complex $K$ is a mod-$p$ homology rose with $m$ petals, then there exists an infinitely long subnormal series of $\pi_1(K)$

$$\pi_1(K) = \Gamma_0 \triangleright \Gamma_1 \triangleright \cdots \triangleright \Gamma_k \triangleright \cdots \quad (2)$$

where $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}_p$ and $b_1(\Gamma_i; \mathbb{F}_p) = p^i(m - 1) + 1$, $b_2(\Gamma_i; \mathbb{F}_p) = 0$ for all $i \geq 0$.

In addition, there is an easy algebraic criterion to judge whether a group can be realized as the fundamental group of an $R$-homology rose.

**Proposition 1.5.** For any commutative ring $R$ with a unit, a group $\Gamma$ can be realized as the fundamental group of an $R$-homology rose with $m$ petals if and only if $H_1(\Gamma; R) = R^m$ and $H_2(\Gamma; R) = 0$. 

Indeed, we can show that for a finitely presentable group $\Gamma$ with $H_1(\Gamma; R) = R^n$ and $H_2(\Gamma; R) = 0$, there always exists a finite 3-complex $B$ with fundamental group $\Gamma$ and $B$ is an $R$-homology rose. But generally speaking, we can not choose $B$ to be 2-dimensional. Actually, for any field $F$, there is an obstruction (called the $F$-gap of $\Gamma$) to the existence of a 2-dimensional $F$-homology rose $B$ with $\pi_1(B) \cong \Gamma$. We will study the properties of the $F$-gap of a group and then use this notion to study Wall’s D(2)-problem among $F$-homology roses and $F$-acyclic spaces.

From Theorem 1.4 and Proposition 1.5, we obtain the following theorem.

**Theorem 1.6.** Let $B$ be a connected finite CW-complex with $H_2(\pi_1(B); \mathbb{F}_p) = 0$. Then for any regular $(\mathbb{Z}_p)^r$-covering $X$ over $B$, we have $b_1(X; \mathbb{F}_p) \geq 2^r - 1$.

**Corollary 1.7.** Let $B$ be a connected finite CW-complex with $H_2(B; \mathbb{F}_p) = 0$. Then for any regular $(\mathbb{Z}_p)^r$-covering $X$ over $B$, we have $\sum_{i=0}^{\infty} b_i(X; \mathbb{F}_p) \geq 2^r$.

The above result gives supporting evidences for the Carlsson conjecture which is an original motivation for our paper. Recall that the Carlsson conjecture claims that if $(\mathbb{Z}_p)^r$ can act freely on a finite CW-complex $X$, then $\sum_{i=0}^{\infty} b_i(X; \mathbb{F}_p) \geq 2^r$. In particular, if the free $(\mathbb{Z}_p)^r$-action on $X$ is cellular, the orbit space $X/(\mathbb{Z}_p)^r$ is also a finite CW-complex and $X$ is a regular $(\mathbb{Z}_p)^r$-covering over $X/(\mathbb{Z}_p)^r$. The Carlsson conjecture is also called the toral rank conjecture in some literature and remains open so far. The reader is referred to [1, 3, 16] for more information on the Carlsson conjecture.

In addition, from Proposition 1.5 we can classify all the finitely generated abelian groups that occur as the fundamental groups of mod-$p$ homology roses or mod-$p$ acyclic spaces (Proposition 3.6 and Proposition 3.10). Moreover, we show in Corollary 3.8 and Corollary 3.11 that if a finite 2-complex $K$ is a mod-$p$ homology rose (or mod-$p$ acyclic space) with abelian fundamental group, then $K$ is homotopy equivalent to $S^1$ (or a pseudo-projective plane).

The paper is organized as follows. In section 2, we give an elementary proof of Theorem 1.1. In section 3, we investigate the properties of the fundamental groups of $R$-homology roses and prove Proposition 1.5 and Theorem 1.6. In particular, we determine which finitely generated abelian groups can be realized as the fundamental groups of mod-$p$ homology roses and mod-$p$ acyclic spaces. In section 4, we introduce the notion of $F$-gap for a finitely presentable group and study some of its properties. This notion is a generalization of the efficiency of a group and is related to some other well known concepts. In section 5, we discuss how to search candidates for the counterexamples of Wall’s D(2)-problem among $F$-homology roses. Then in section 6 and section 7, we practice the search from the fundamental groups of closed 3-manifolds and from finite groups with trivial multiplicator and negative deficiency.
By abuse of terminology, we think of any 1-dimensional CW-complex in this paper as a 2-dimensional CW-complex without 2-cells. In addition, if no coefficients are specified, homology and cohomology groups are with $\mathbb{Z}$-coefficients.

2. $(\mathbb{Z}_p)^r$-COVERING SPACES OF MOD-$p$ HOMOLOGY ROSES

**Lemma 2.1.** If $K$ is a mod-$p$ homology rose with $m$ petals, then $H_1(K; \mathbb{Z}) \cong \mathbb{Z}^m \oplus T$ where $T$ is a finite abelian group without $p$-torsion.

**Proof.** By the universal coefficient theorem,

$$H_2(K; \mathbb{F}_p) \cong H^2(K; \mathbb{F}_p) = \text{Hom}(H_2(K); \mathbb{F}_p) \oplus \text{Ext}(H_1(K); \mathbb{F}_p)$$

Then $H_2(K; \mathbb{F}_p) = 0$ implies that $H_1(K; \mathbb{Z})$ has no $p$-torsion. In addition, since $H_1(K; \mathbb{F}_p) = H_1(K; \mathbb{Z}) \otimes \mathbb{F}_p \cong (\mathbb{F}_p)^m$, the free rank of $H_1(K; \mathbb{Z})$ must equal $m$. □

Before giving a proof of Theorem 1.1, we want to acknowledge that Theorem 1.1 is also a corollary of the following result in [8].

**Theorem 2.2** (Proposition 1.2 of [8]). Let $\tilde{B} \to B$ be a finite sheeted regular covering with deck transformation group $G$, where $B$ is a connected CW-complex and $\text{cd}(B) < \infty$. Let $M$ be a $\pi_1(B)$-module and $n$ an integer such that $H^i(\tilde{B}; M) = 0$ for $i > n$. Then $H^i(B; M) = 0$ for $i > n$, and $\tau: H^n(\tilde{B}; M)_G \to H^n(B; M)$ is an isomorphism.

Here $\text{cd}(B) := \sup\{i \in \mathbb{Z}; H^i(B; M) \neq 0 \text{ for some } \pi_1(B)\text{-module } M\}$, called the cohomological dimension of $B$. By the work of Wall ([35, 36]), $\text{cd}(B) < \infty$ implies that $B$ is homotopy equivalent to a CW-complex of finite dimension.

In addition, for any commutative ring $R$, we define

$$\text{cd}_R(B) := \sup\{i \in \mathbb{Z}; H^i(B; R) \neq 0\},$$

called the $R$-cohomological dimension of $B$. It is clear that $\text{cd}_R(B) \leq \text{cd}(B)$.

In the above theorem, let $M = \mathbb{F}_p$ (as a trivial $\pi_1(B)$-module). We obtain

$$\text{cd}_{\mathbb{F}_p}(\tilde{B}) \geq \text{cd}_{\mathbb{F}_p}(B).$$

Moreover, for any $\mathbb{F}_p[(\mathbb{Z}_p)^r]$-module $L$, the co-invariant $L(\mathbb{Z}_p)^r \neq \emptyset$ if and only if $L \neq 0$ (see [7, p.149]). So if $\tilde{B}$ is a regular $(\mathbb{Z}_p)^r$-covering of $B$, the last sentence of Theorem 2.2 implies that $\text{cd}_{\mathbb{F}_p}(\tilde{B}) = \text{cd}_{\mathbb{F}_p}(B)$. So we actually obtain

**Corollary 2.3.** If $\tilde{B} \to B$ is a regular $(\mathbb{Z}_p)^r$-covering space with $\text{cd}_{\mathbb{F}_p}(B) < \infty$, then $\text{cd}_{\mathbb{F}_p}(\tilde{B}) = \text{cd}_{\mathbb{F}_p}(B)$. 


Note that a mod-p homology rose is nothing but a path-connected space $K$ with $\text{cd}_{F_p}(K) = 1$. So Theorem 1.1 follows from the above corollary.

In the rest of this section, we give an alternative proof of Theorem 1.1 without referring to [8, Proposition 1.2]. Our proof involves a very simple spectral sequence and some elementary facts on the modular representations of $\mathbb{F}_p[\mathbb{Z}_p]$.

**Proof of Theorem 1.1.** Since $(\mathbb{Z}_p)^r$ acts freely and cellularly on $X$, its orbit space $K = X/(\mathbb{Z}_p)^r$ is also a finite CW-complex. It is easy to see that there is a sequence $X = X_r \rightarrow X_{r-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = K$ where $X_j \rightarrow X_{j-1}$ is a regular $\mathbb{Z}_p$-covering for each $1 \leq j \leq r$. So it is sufficient to prove the theorem for $r = 1$. In the rest of the proof, we assume that $\xi : X \rightarrow K$ is a regular $\mathbb{Z}_p$-covering.

(1) Assume that $K$ is a mod-$p$ homology rose with $m$ petals, we want to show that $X$ is also a mod-$p$ homology rose. By lemma 2.1, $H_1(K) \cong \mathbb{Z}^m \oplus T$ where $T$ is a finite abelian group without $p$-torsion. Let $\tilde{K}$ be the regular covering of $K$ with $\pi_1(\tilde{K})$ isomorphic to the kernel of the following group epimorphism

$$\tilde{\sigma} : \pi_1(K) \rightarrow H_1(K) = \mathbb{Z}^m \oplus T \rightarrow \mathbb{Z}^m.$$ 

In other words, $\tilde{K}$ is the maximal free abelian covering of $K$. Meanwhile, $\pi_1(X)$ is isomorphic to the kernel of a group epimorphism $\sigma_X : \pi_1(K) \rightarrow \mathbb{Z}_p$. Since $T$ has no $p$-torsion, $\sigma_X$ can factor through $\tilde{\sigma}$, i.e. there exists a group epimorphism $\eta_X : \mathbb{Z}^m \rightarrow \mathbb{Z}_p$ so that

$$\sigma_X = \eta_X \circ \tilde{\sigma} : \pi_1(K) \rightarrow \mathbb{Z}^m \xrightarrow{\eta_X} \mathbb{Z}_p.$$ 

So $\tilde{K}$ is also a regular covering of $X$. Let $D(\tilde{K})$ and $D(X)$ be the deck transformation group of $\tilde{K}$ and $X$ over $K$, respectively. It is clear that

$$D(\tilde{K}) \cong \mathbb{Z}^m, \quad D(X) \cong \mathbb{Z}_p.$$ 

Let $H$ be the normal subgroup of $D(\tilde{K})$ so that $D(\tilde{K})/H \cong D(X)$. So $H$ is a free abelian group of rank $m$ which can be identified with the kernel of $\eta_X$. Then we have a short exact sequence of abelian groups

$$0 \rightarrow H \cong \mathbb{Z}^m \xrightarrow{\Lambda_X} \mathbb{Z}^m \cong D(\tilde{K}) \xrightarrow{\eta_X} \mathbb{Z}_p \rightarrow 0.$$ 

The map $\Lambda_X$ can be represented by a diagonal matrix $\text{Diag}(1, \cdots, 1, p)$ with respect to a properly chosen basis of $D(\tilde{K})$, say $\{t_1, \cdots, t_{m-1}, t_m\}$. Then $H$ is generated by $\{t_1, \cdots, t_{m-1}, pt_m\}$. Let

$$H' := \langle t_1, \cdots, t_{m-1} \rangle \subset D(\tilde{K}), \quad H/H' \cong \mathbb{Z}.$$ 

Then $X' = \tilde{K}/H'$ is an regular $\mathbb{Z}$-covering of $K$ and, at the same time, $X'$ is a regular $\mathbb{Z}$-covering of $X$. Indeed, we have

$$X = \tilde{K}/H = (\tilde{K}/H')/(H'/H) = X'/(H'/H).$$
From the covering $X'$ over $K$, we obtain the following short exact sequence of chain complexes

$$0 \longrightarrow C_*(X'; \mathbb{F}_p) \xrightarrow{t_{m-1}} C_*(X'; \mathbb{F}_p) \longrightarrow C_*(K; \mathbb{F}_p) \longrightarrow 0. \quad (3)$$

By the homology long exact sequence of (3) and the assumption that $K$ is a mod-$p$ homology rose, we can conclude that $H_i(X'; \mathbb{F}_p) \xrightarrow{t_{m-1}} H_i(X'; \mathbb{F}_p)$ is an isomorphism for any $i \geq 2$ and an injection for $i = 1$.

From the covering $X'$ over $X$, we have the short exact sequence

$$0 \longrightarrow C_*(X'; \mathbb{F}_p) \xrightarrow{t_{m-1}} C_*(X'; \mathbb{F}_p) \longrightarrow C_*(X; \mathbb{F}_p) \longrightarrow 0. \quad (4)$$

Notice that, over $\mathbb{F}_p$, $(t_{m-1}^p - 1) = (t_m - 1)^p$. So from the homology long exact sequence of (1), we similarly obtain that $H_i(X'; \mathbb{F}_p) \xrightarrow{t_{m-1}} H_i(X'; \mathbb{F}_p)$ is an isomorphism for any $i \geq 2$ and an injection for $i = 1$. This implies that $H_i(X; \mathbb{F}_p) = 0$, $i \geq 2$, i.e. $X$ is a mod-$p$ homology rose.

Moreover, since $K$ is a finite CW-complex, the Euler characteristics of the fibration (5) converges to $H^1(X; \mathbb{F}_p) = H^1(K; \mathbb{F}_p)$ whose $E_2$-term is

$$E_2^{j,k} = H^j(B\mathbb{Z}_p; H^k(X; \mathbb{F}_p)) = H^j(\mathbb{Z}_p; H^k(X; \mathbb{F}_p))$$

where $H^k(X; \mathbb{F}_p)$ denotes $H^k(X; \mathbb{F}_p)$ as a $\mathbb{F}_p[\mathbb{Z}_p]$-module (see [32, Chapter 5]). Here $\mathbb{F}_p$-coefficient is implicitly assumed in all the cohomology groups.

Since $X$ is a mod-$p$ homology rose, $E_2^{j,k} = 0$ if $k \neq 0, 1$. For convenience, let

$$G_j := E_2^{j,1} = H^j(\mathbb{Z}_p; H^1(X; \mathbb{F}_p)).$$

The differential $d_2 : E_2^{j,k} \rightarrow E_2^{j+2,k-1}$ is as shown in Figure 1. To compute $G_j$, we need to know the $\mathbb{F}_p[\mathbb{Z}_p]$-module structure of $H^1(X; \mathbb{F}_p)$.

In the rest of the proof, we identify $\mathbb{F}_p[\mathbb{Z}_p]$ with $\mathbb{F}_p[x]/x^p$ where $x = t - 1$ and $t$ is a generator of $\mathbb{Z}_p$. Then by the theory of modular representation of finite groups (see [4]), any indecomposable $\mathbb{F}_p[\mathbb{Z}_p]$-module over $\mathbb{F}_p$ is isomorphic
to \( \mathcal{R}_k = \mathbb{F}_p[a]/a^k, 1 \leq k \leq p \) where \( x \in \mathbb{F}_p[x]/x^p \) acts on \( \mathcal{R}_k \) by multiplying each element of \( \mathcal{R}_k \) by \( a \). If we choose \( \{1, a, \cdots, a^{k-1}\} \) as a linear basis of \( \mathcal{R}_k \), then

\[
x \cdot (1, a, \cdots, a^{k-1}) = (1, a, \cdots, a^{k-1}) \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.
\]

(6)

Notice that \( \mathcal{R}_1 = \mathbb{F}_p \) is the trivial \( \mathbb{F}_p[\mathbb{Z}_p] \)-module. Let \( N_k \) denote the \( k \times k \) matrix on the right hand side of (6). It is clear that \( \text{rank}_{\mathbb{F}_p}(N_k) = k-1 \). Suppose

\[
\mathcal{H}^1(X; \mathbb{F}_p) \cong \mathcal{R}_1^{l_1} \oplus \cdots \oplus \mathcal{R}_p^{l_p}, \quad l_1, \cdots, l_p \geq 0.
\]

Then by the Lemma 2.4 below,

\[
G_j \cong \begin{cases} 
(\mathbb{F}_p)^{\sum_{i=1}^{p} l_i}, & j = 0; \\
(\mathbb{F}_p)^{\sum_{i=1}^{p-1} l_i}, & j \geq 1. 
\end{cases}
\]

(7)

Let us see how to determine \( l_1, \cdots, l_p \) from our conditions. Suppose \( X \) is mod-\( p \) homology rose with \( n \)-petals. Then

\[
\text{rank}_{\mathbb{F}_p}(\mathcal{H}^1(X; \mathbb{F}_p)) = n = l_1 + 2 \cdot l_2 + \cdots + p \cdot l_p
\]

(8)

In addition, the relation of Euler characteristics \( \chi(X) = p \cdot \chi(K) \) implies

\[
n = 1 - p \cdot \chi(K).
\]

(9)

Claim: \( l_1 = 1, l_2 = \cdots = l_{p-1} = 0, l_p = -\chi(K) \).

Indeed, the above spectral sequence converges to \( H^*(K; \mathbb{F}_p) \) which is nontrivial in only finite dimensions. Then since \( G_j, j \geq 1 \) are all isomorphic, we must have

\[
\text{rank}_{\mathbb{Z}_p}(G_j) = l_1 + \cdots + l_{p-1} = 1, \quad \forall j \geq 1.
\]

(10)
This is because if \( l_1 + \cdots + l_{p-1} = 0 \), every \( E_2^{j,0} \ (j \geq 3) \) will survive to \( E_\infty \)-term which contradicts the homological finiteness of \( K \) (see Figure 1). Similarly, if \( l_1 + \cdots + l_{p-1} > 1 \), \( G_j = E_2^{j,1} \ (j \geq 0) \) can not be killed by \( d_2 \), which again contradicts the homological finiteness of \( K \).

By (10), exactly one of \( l_1, \ldots, l_{p-1} \) is equal to one, others are all zero. If we assume \( l_k = 1 \) for some \( 1 \leq k \leq p-1 \), then \( n = k + pl_p \) by (8). By plugging this into (9), we get

\[
k + p \cdot l_p = 1 - p \cdot \chi(K) \implies p \cdot (l_p + \chi(K)) = 1 - k.
\]

Since \( 1 \leq k \leq p-1 \), the only possibility is \( k = 1 \) and \( l_p = -\chi(K) \). The claim is proved.

Therefore, \( H^j(X; \mathbb{F}_p) = \mathcal{R}_1 \oplus \mathcal{R}_p^{-\chi(K)} \) and so

\[
E_2^{1,1} = H^j(\mathbb{Z}_p; H^1(X; \mathbb{F}_p)) = H^j(\mathbb{Z}_p; \mathcal{R}_1 \oplus \mathcal{R}_p^{-\chi(K)}) = H^j(\mathbb{Z}_p; \mathcal{R}_1) \oplus H^j(\mathbb{Z}_p; \mathcal{R}_p^{-\chi(K)})
\]

Next, let us see what \( d_2 : E_2^{j,1} \to E_2^{j+2,0} \) should be when restricted to the \( H^j(\mathbb{Z}_p; \mathcal{R}_1) \cong \mathbb{F}_p \subseteq E_2^{j,1} \) (see Figure 2). In this case, since \( \mathcal{R}_1 = \mathbb{F}_p \) is the trivial \( \mathbb{F}_p[\mathbb{Z}_p] \)-module, the multiplicative structure of the spectral sequence implies that \( d_2|_{H^j(\mathbb{Z}_p; \mathcal{R}_1)} \) is determined by \( d_2|_{H^j(\mathbb{Z}_p; \mathcal{R}_1)} \). Then we claim that \( d_2 \) must map the \( H^0(\mathbb{Z}_p; \mathcal{R}_1) \subseteq E_2^{0,1} \) isomorphically onto \( E_2^{2,0} \cong \mathbb{F}_p \), because otherwise all the \( H^j(\mathbb{Z}_p; \mathcal{R}_1) \subseteq E_2^{j,1} \ (j \geq 0) \) will survive to \( E_\infty \), which contradicts the homological finiteness of \( K \).

Moreover, \( H^j(\mathbb{Z}_p; \mathcal{R}_p^{-\chi(K)}) = 0 \) for all \( j \geq 1 \), hence has no contribution to \( H^{\geq 2}(K; \mathbb{F}_p) \). So we obtain that \( H^i(K; \mathbb{F}_p) = 0 \) for all \( i \geq 2 \), i.e. \( K \) is mod-\( p \) homology rose. This finishes the proof of Theorem 1.1. \[\square\]
Lemma 2.4. For any $1 \leq k \leq p-1$, $H^j(\mathbb{Z}_p; R_k) = \mathbb{F}_p$ for all $j \geq 0$. For $k = p$, $H^j(\mathbb{Z}_p; R_p) = \begin{cases} \mathbb{F}_p, & j = 0, \\ 0, & j \neq 0. \end{cases}$

**Proof.** If we identity $\mathbb{F}_p[\mathbb{Z}_p]$ with $\mathbb{F}_p[x]/x^p$, a projective resolution of the trivial $\mathbb{F}_p[\mathbb{Z}_p]$-module $\mathbb{F}_p$ is:

$$
\cdots \mathbb{F}_p[\mathbb{Z}_p] \xrightarrow{x} \mathbb{F}_p[\mathbb{Z}_p] \xrightarrow{x^{p-1}} \mathbb{F}_p[\mathbb{Z}_p] \xrightarrow{x} \mathbb{F}_p[\mathbb{Z}_p] \xrightarrow{\varepsilon} \mathbb{F}_p
$$

where $\varepsilon$ is a ring homomorphism defined by $\varepsilon(x) = 0$ and $\varepsilon(1) = 1$. Then $H^j(\mathbb{Z}_p; R_k)$ is the $j$-th cohomology group of the following chain complex

$$
\cdots \text{Hom}_{\mathbb{F}_p[\mathbb{Z}_p]}(\mathbb{F}_p[\mathbb{Z}_p], R_k) \xrightarrow{\delta_2} \text{Hom}_{\mathbb{F}_p[\mathbb{Z}_p]}(\mathbb{F}_p[\mathbb{Z}_p], R_k) \xrightarrow{\delta_1} \text{Hom}_{\mathbb{F}_p[\mathbb{Z}_p]}(\mathbb{F}_p[\mathbb{Z}_p], R_k) \xrightarrow{\delta_0} \text{Hom}_{\mathbb{F}_p[\mathbb{Z}_p]}(\mathbb{F}_p[\mathbb{Z}_p], R_k)
$$

where $\delta_{\text{even}} = \text{Hom}_{\mathbb{F}_p[\mathbb{Z}_p]}(\cdot, x)$, $\delta_{\text{odd}} = \text{Hom}_{\mathbb{F}_p[\mathbb{Z}_p]}(\cdot, x^{p-1})$.

By identifying $\text{Hom}_{\mathbb{F}_p[\mathbb{Z}_p]}(\mathbb{F}_p[\mathbb{Z}_p], R_k)$ with $(\mathbb{F}_p)^k$ as a module over $\mathbb{F}_p$, we can write the above chain complex explicitly as

$$
\cdots (\mathbb{F}_p)^k \xrightarrow{N_k^t} (\mathbb{F}_p)^k \xrightarrow{(N_k^t)^{p-1}} (\mathbb{F}_p)^k \xrightarrow{N_k^t} (\mathbb{F}_p)^k \xrightarrow{\delta} (\mathbb{F}_p)^k
$$

where $N_k^t$ is the transpose of the matrix $N_k$. Then our lemma follows from the fact that $\text{rank}_{\mathbb{F}_p}(N_k) = k - 1$ and $\text{rank}_{\mathbb{F}_p}(N_k^{p-1}) = \begin{cases} 0, & 1 \leq k \leq p-1, \\ 1, & k = p. \end{cases}$

\[
\square
\]

3. **Fundamental Groups of Homology Roses**

First of all, let us prove Proposition 1.5.

**Proof of Proposition 1.5.** The necessity follows from Lemma 2.1. For the sufficiency, let us first choose a connected 2-complex $K$ with $\pi_1(K) = \Gamma$. By the following short exact sequence due to Hopf (see [7, Ch.II §5])

$$
\pi_2(K) \longrightarrow H_2(K) \longrightarrow H_2(\Gamma) \longrightarrow 0,
$$

we get a new short exact sequence

$$
\pi_2(K) \otimes R \longrightarrow H_2(K) \otimes R \longrightarrow H_2(\Gamma) \otimes R \longrightarrow 0.
$$

The assumption $H_2(\Gamma; R) = 0$ implies

$$
\text{Tor}(H_1(K); R) = \text{Tor}(H_1(\Gamma); R) = 0, \quad H_2(\Gamma) \otimes R = 0.
$$
So $H_2(K; R) = H_2(K) \otimes R$. Then from Equation (13), we obtain a surjective map $\pi_2(K) \otimes R \rightarrow H_2(K; R)$. This implies that any $\alpha \in H_2(K; R)$ can be represented by a continuous map $\varphi_\alpha : S^2 \rightarrow K$. Since $H_2(K)$ is clearly a free abelian group, $H_2(K; R)$ is a free $R$-module. Suppose $\{\alpha_i\}_{i \in I}$ is a a set of generators of $H_2(K; R)$ over $R$. Then for all $i \in I$, we glue a $3$-ball $D^3$ to $K$ via $\varphi_{\alpha_i}$, which gives us a $3$-complex $K'$. It is easy to check that $K'$ is an $R$-homology rose with $\pi_1(K') \cong \Gamma$. Note that if $\Gamma$ is finitely presentable, $K'$ can be taken to be a finite $3$-complex. 

Note that the same argument as the proof of Proposition 1.5 can be used to prove the following proposition.

**Proposition 3.1.** For any commutative ring $R$ with a unit, a group $\Gamma$ can be realized as the fundamental group of an $R$-acyclic space if and only if $H_1(\Gamma; R) = H_2(\Gamma; R) = 0$.

**Proof of Theorem 1.6.** Let $\Gamma = \pi_1(B)$ and let $b_1(\Gamma; \mathbb{F}_p) = b_1(B; \mathbb{F}_p) = m$. Then since $X$ is a regular $(\mathbb{Z}_p)^r$-covering over $B$, we must have $r \leq m$.

If $m = 0$, our claim clearly holds. If $m \geq 1$, Proposition 1.5 tells us that there exists a finite CW-complex $B'$ with $\pi_1(B') \cong \Gamma$ and $B'$ is a mod-$p$ homology rose with $m$ petals. In addition, the fundamental group $\pi_1(X) = N$ can be identified with a normal subgroup of $\Gamma$ so that $\Gamma/N \cong (\mathbb{Z}_p)^r$. Then we can also think of $N$ as a subgroup of $\pi_1(B')$, which determines a regular $(\mathbb{Z}_p)^r$-covering $X'$ over $B'$ with $\pi_1(X') \cong N$. By Theorem 1.1, $X'$ is also a mod-$p$ homology rose. Then from the fact that $b_1(B'; \mathbb{F}_p) = b_1(\Gamma; \mathbb{F}_p) = m$ and the Euler characteristic $\chi(X') = p^r \cdot \chi(B')$, we obtain that

$$b_1(X; \mathbb{F}_p) = b_1(N; \mathbb{F}_p) = b_1(X'; \mathbb{F}_p) = p^r(m - 1) + 1 \geq p^r(r - 1) + 1.$$ 

- When $r = 1$, $b_1(X; \mathbb{F}_p) \geq 1 = 2^r - 1$;

- When $r \geq 2$, $b_1(X; \mathbb{F}_p) \geq p^r + 1 > 2^r - 1$.

So the theorem is proved. 

**Proof of Corollary 1.7.** Note that $H_2(B; \mathbb{F}_p) = 0$ implies $H_2(B) \otimes \mathbb{F}_p = 0$ and $\text{Tor}(H_1(B); \mathbb{F}_p) = 0$. Then by (13), $H_2(\pi_1(B); \mathbb{F}_p) = 0$. So by Theorem 1.6, we obtain that $\sum_{i=0}^{\infty} b_i(X; \mathbb{F}_p) \geq b_0(X; \mathbb{F}_p) + b_1(X; \mathbb{F}_p) \geq 1 + (2^r - 1) = 2^r$. 

Suppose $G$ is a finitely presentable group. Let $\mathcal{P} = \langle a_1, \cdots, a_n | r_1, \cdots, r_m \rangle$ be a finite presentation of $G$. The integer $n - m$ is called the deficiency of $\mathcal{P}$, denoted by $\text{def}(\mathcal{P})$. The deficiency of $G$, denoted by $\text{def}(G)$, is the maximum over all its finite presentations, of the deficiency of each presentation. Note that if $G$ is not finitely presentable, we can still define the presentation complex of any presentation of $G$, but the notion of deficiency does not make sense anymore.
Any presentation $\mathcal{P}$ canonically determines a 2-dimensional CW-complex $K_{\mathcal{P}}$ called the \textit{presentation complex} of $\mathcal{P}$.

- $K_{\mathcal{P}}$ has a single vertex $q_0$, and one oriented 1-cell $\gamma_j$ attached to $q_0$ for each generator $a_j$ ($1 \leq j \leq n$). So the 1-skeleton of $K_{\mathcal{P}}$ is a bouquet of $n$ circles attached to $q_0$.
- $K_{\mathcal{P}}$ has one oriented 2-cell $\beta_i$ for each relator $r_i$ ($1 \leq i \leq m$), where $\beta_i$ is attached to the 1-skeleton of $K_{\mathcal{P}}$ via a map defined by $r_i$.

\textbf{Lemma 3.2.} \textit{For any finitely presentable group $G$ and any field $\mathbb{F}$,}
\[
def(G) \leq b_1(G; \mathbb{F}) - b_2(G; \mathbb{F}).
\]

\textit{Proof.} For any finite presentation $\mathcal{P} = \langle a_1 \cdots a_n \mid r_1, \cdots, r_m \rangle$ of $G$, there is an associated resolution of $\mathbb{Z}$ by projective $\mathbb{Z}G$-modules of the following form:
\[
0 \rightarrow \mathbb{Z}[G]^{\oplus m} \rightarrow \mathbb{Z}[G]^{\oplus n} \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{Z} \rightarrow 0
\]
Indeed, this comes from the cellular decomposition of the universal covering space of the $K(G,1)$ space built from the presentation complex $K_{\mathcal{P}}$ of $\mathcal{P}$. Then applying the functor $\otimes_{\mathbb{Z}[G]} \mathbb{F}$ to this projective resolution, we get a chain complex:
\[
0 \rightarrow \mathbb{Z}[G]^{\oplus m} \otimes_{\mathbb{Z}[G]} \mathbb{F} \rightarrow \mathbb{Z}[G]^{\oplus n} \otimes_{\mathbb{Z}[G]} \mathbb{F} \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} \mathbb{F} \rightarrow \mathbb{F} \rightarrow 0
\]
whose homology groups are just $H_*(G; \mathbb{F})$ (see [7]). Moreover, by the Morse inequality of this chain complex, we obtain that
\[
1 - n + m \geq b_0(G; \mathbb{F}) - b_1(G; \mathbb{F}) + b_2(G; \mathbb{F}).
\]
So $n - m \leq b_1(G; \mathbb{F}) - b_2(G; \mathbb{F})$. Since this argument works for arbitrary finite presentations of $G$, so we get $\def(G) \leq b_1(G; \mathbb{F}) - b_2(G; \mathbb{F})$. \hfill $\Box$

\textbf{Proposition 3.3.} \textit{If a finite 2-complex $K$ is an $\mathbb{F}$-homology rose with $m$ petals, then the deficiency of the fundamental group $\pi_1(K)$ of $K$ is equal to $m$. If $K$ is an $\mathbb{F}$-acyclic space, then the deficiency of $\pi_1(K)$ is equal to 0.}

\textit{Proof.} Without loss of generality, we can assume that $K$ has a single 0-cell and $n$ 1-cells. Then the number of 2-cells in $K$ is $n - m$. So $\def(\pi_1(K)) \geq m$. On the other hand, since $b_1(\pi_1(K); \mathbb{F}) = m$ and $b_2(\pi_1(K); \mathbb{F}) = 0$, Lemma 3.2 implies that $\def(\pi_1(K)) \leq m$. So we must have $\def(\pi_1(K)) = m$. The same argument clearly works for mod-$p$ acyclic spaces. \hfill $\Box$

It is clear that any group with positive deficiency must be infinite. So we obtain the following corollary.

\textbf{Corollary 3.4.} \textit{If a finite 2-complex $K$ is an $\mathbb{F}$-homology rose, $\pi_1(K)$ must be infinite.}
Next, we investigate which finitely generated abelian groups can be realized as the fundamental groups of mod-$p$ homology roses.

**Lemma 3.5.** For a finitely generated abelian group $A = \mathbb{Z}^r \oplus \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ where $d_1 > 1$ and $d_1|d_2|\cdots|d_k$, $\text{def}(A) = r - \binom{r+k}{2}$.

**Proof.** It is clear that $A$ can be presented by
\[ \mathcal{P}_A = \langle a_1, \ldots, a_r, b_1, \ldots, b_k \mid b_1^{d_1}, \ldots, b_k^{d_k}, [a_i, a_{i'}], [b_j, b_{j'}], [a_i, b_j], \rangle, \quad 1 \leq i < i' \leq r, 1 \leq j < j' \leq k. \]

So $\text{def}(A) \geq \text{def}(\mathcal{P}_A) = r - \binom{r+k}{2}$. On the other hand, for a finitely presented group $G$, it is well known that (see \cite[14.1.5]{[1]})
\[ \text{def}(G) \leq r(G_{ab}) - d(M(G)) \quad \text{(15)} \]
where $r(G_{ab})$ is the free rank of the abelianization $G_{ab}$ of $G$ (i.e. $r(G_{ab}) = b_1(G; \mathbb{Q})$), $M(G) \cong H_2(G; \mathbb{Z})$ is the Schur multiplicator of $G$ and $d(M(G))$ is the minimum number of elements that can generate $M(G)$. For our group $A$,
\[ M(A) = H_2(A; \mathbb{Z}) \cong \mathbb{Z}^{\binom{r}{2}} \oplus \mathbb{Z}_{d_1}^{r+k-1} \oplus \mathbb{Z}_{d_2}^{r+k-2} \oplus \cdots \oplus \mathbb{Z}_{d_k}^{r}. \quad \text{(16)} \]

The number of elements that can generate $M(A)$ is at least
\[ \binom{r}{2} + (r+k-1) + (r+k-2) + \cdots + r = \binom{r+k}{2}. \]

This is because if choose a prime $p|d_1$, then $M(A)/pM(A)$ will be a vector space over $\mathbb{F}_p$ of dimension $\binom{r+k}{2}$, which requires at least $\binom{r+k}{2}$ generators for $M(A)$. So
\[ d(M(A)) = \binom{r+k}{2}. \quad \text{(17)} \]

Then by \text{(16)}, we have $\text{def}(A) \leq r(A) - d(M(A)) = r - \binom{r+k}{2}$. The lemma is proved.

Suppose an abelian group $A = \mathbb{Z}^r \oplus \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ where $d_1 > 1$ and $d_1|d_2|\cdots|d_k$ is the fundamental group of a mod-$p$ homology rose $K$. Then $A \cong H_1(K)$ has no $p$-torsion and $H_2(A; \mathbb{F}_p) = 0$ by Proposition \text{(15)}. So by \text{(16)} and the fact that $A \cong H_1(K)$ is an infinite group (by Lemma \text{(2.1)}), we must have
\[ r = 1 \quad \text{and} \quad p \nmid d_i, \quad 1 \leq i \leq k. \quad \text{(18)} \]

Conversely, Proposition \text{(15)} implies that any finitely generated abelian group $A$ with free rank 1 and without $p$-torsion can be realized as the fundamental group of some mod-$p$ homology rose. Indeed, there is a more direct way to see this fact. Let $\mathbb{P}_m$ be the mapping cone of a map $f : S^1 \to S^1$ with degree $m$ (called a
pseudo-projective plane of order \( m \)). For any \( p \nmid m \), \( \mathbb{P}_m \) is a mod-\( p \) acyclic space. We can write \( A = \mathbb{Z} \oplus \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k} \) where \( p \nmid d_i \), \( 1 \leq i \leq k \). Define
\[
L_A = S^1 \times \mathbb{P}_{d_1} \times \mathbb{P}_{d_2} \times \cdots \times \mathbb{P}_{d_k}.
\]
It is clear that \( L_A \) is a mod-\( p \) homology circle whose fundamental group is \( A \). So we obtain the following result.

**Proposition 3.6.** A finitely generated abelian group \( A \) can be realized as the fundamental group of a mod-\( p \) homology rose if and only if \( A \) has free rank 1 and has no \( p \)-torsion.

**Corollary 3.7.** If the fundamental group of a mod-\( p \) homology rose \( K \) is a finitely generated abelian group, \( K \) must be a mod-\( p \) homology circle.

**Proof.** By Proposition 3.6, \( \pi_1(K) \) must be an abelian group with free rank 1 and has no \( p \)-torsion. Then \( b_1(\pi_1(K); \mathbb{F}_p) = 1 \), which implies that \( K \) is a mod-\( p \) homology circle. \( \square \)

**Corollary 3.8.** A 2-dimensional finite complex \( K \) is a mod-\( p \) homology rose with abelian fundamental group if and only if \( K \) is homotopy equivalent to \( S^1 \).

**Proof.** The sufficiency is trivial. For the necessity, let \( K \) be a mod-\( p \) homology rose with \( m \)-petals. By Proposition 3.6 we can assume that \( \pi_1(K) = H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k} \) where \( d_1 > 1 \), \( d_1|d_2|\cdots|d_k \) and \( p \nmid d_i \), \( 1 \leq i \leq k \). Then since \( K \) is 2-dimensional, Proposition 3.3 and Lemma 3.5 imply that
\[
1 - \left( \frac{k+1}{2} \right) = \text{def}(\pi_1(K)) = m.
\]
This forces \( m = 1 \) and \( k = 0 \), and hence \( \pi_1(K) \cong \mathbb{Z} \). Finally, Theorem 3.9 below implies that \( K \) must be homotopy equivalent to \( S^1 \). \( \square \)

**Theorem 3.9** (see [35] Proposition 3.3). Every compact, connected 2-complex with free fundamental group is homotopy equivalent to a finite bouquet of 1- and 2-dimensional spheres.

For mod-\( p \) acyclic spaces, we can easily obtain the following result from Proposition 3.1 and Lemma 3.5.

**Proposition 3.10.** A finitely generated abelian group \( A \) can be realized as the fundamental group of a mod-\( p \) acyclic space if and only if \( A \) is finite and has no \( p \)-torsion.

**Corollary 3.11.** A 2-dimensional finite complex \( K \) is a mod-\( p \) acyclic space with abelian fundamental group if and only if \( K \) is homotopy equivalent to a pseudo-projective plane \( \mathbb{P}_m \) with \( p \nmid m \).
Proof. The sufficiency is trivial. For the necessity, by Proposition 3.6 we can assume that $\pi_1(K) = H_1(K) = \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ where $d_1 > 1$, $d_1|d_2|\cdots|d_k$ and $p \nmid d_i$, $1 \leq i \leq k$. Then since $K$ is 2-dimensional, Proposition 3.3 and Lemma 3.5 imply that

$$\left(\frac{k}{2}\right) = \text{def}(\pi_1(K)) = 0.$$ 

So $k$ has to be 1, i.e. $\pi_1(K)$ is a finite cyclic group without $p$-torsion. Then the corollary follows from Theorem 3.12 below.

□

Theorem 3.12 (see [13]). Let $K$ be a connected 2-dimensional finite CW-complex with fundamental group $\mathbb{Z}_m$. Then $X$ has the homotopy type of the wedge sum $\mathbb{P}_m \vee S^2 \vee \cdots \vee S^2$ of the pseudo-projective plane $\mathbb{P}_m$ and rank $H_2(K)$-copies of the 2-sphere $S^2$.

4. $\mathbb{F}$-GAP OF A GROUP

For any finitely presentable group $G$ and a field $\mathbb{F}$, Lemma 3.2 tells us that

$$\text{def}(G) \leq b_1(G; \mathbb{F}) - b_2(G; \mathbb{F}).$$

Notice that both sides of the inequality are invariants of the group $G$. Let

$$\text{gap}(G; \mathbb{F}) := b_1(G; \mathbb{F}) - b_2(G; \mathbb{F}) - \text{def}(G) \in \mathbb{Z}_{\geq 0}.$$ 

We call $\text{gap}(G; \mathbb{F})$ the $\mathbb{F}$-gap of $G$ which is also an invariant of $G$.

Example 1. Let $A = \mathbb{Z}^r \oplus \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ where $d_1 > 1$ and $d_1|d_2|\cdots|d_k$. First of all, we compute $\text{gap}(A, \mathbb{Q})$ where $\mathbb{Q}$ is the rational numbers. By Lemma 3.5 and Equation (16), we obtain

$$\text{gap}(A, \mathbb{Q}) = r - \left(\frac{r}{2}\right) - \left(r - \left(\frac{r + k}{2}\right)\right) = \left(\frac{r + k}{2}\right) - \left(\frac{r}{2}\right).$$

So $\text{gap}(A, \mathbb{Q}) = 0$ if and only if $k = 0$ (i.e. $A$ is free abelian) or $r = 0$ and $k = 1$ (i.e. $A$ is a cyclic group).

Next, we compute $\text{gap}(A, \mathbb{F}_p)$ for any prime $p$.

- If $k = 0$, i.e. $A \cong \mathbb{Z}^r$, $\text{gap}(A; \mathbb{F}_p) = 0$.
- If $k \geq 1$, assume that $p|d_l$ but $p \nmid d_{l-1}$ for some $1 \leq l \leq k$, then

$$b_1(A; \mathbb{F}_p) = r + k - l + 1,$$

$$b_2(A; \mathbb{F}_p) = (k - l + 1) + \left(\frac{r}{2}\right) + \sum_{j=l}^{k}(r + k - j).$$
So by Lemma 3.5, we obtain
\[
\text{gap}(A; \mathbb{F}_p) = \left( \frac{r + k}{2} \right) - \left( \frac{r}{2} \right) - \sum_{j=l}^{k} (r + k - j) = \frac{1}{2} (2r + 2k - l)(l - 1).
\]
Then \( \text{gap}(A; \mathbb{F}_p) = 0 \) if and only if \( l = 1 \), i.e. the torsion of \( A \) is a \( p \)-group.

- If \( k \geq 1 \) and \( p \nmid d_i \) for all \( 1 \leq i \leq k \), we have \( \text{gap}(A; \mathbb{F}_p) = \left( \frac{r + k}{2} \right) - \left( \frac{r}{2} \right) \). In this case, \( \text{gap}(A; \mathbb{F}_p) = 0 \) if and only if \( r = 0 \) and \( k = 1 \), i.e. \( A \) is a finite cyclic group with no \( p \)-torsion.

By the above discussion, both \( \text{gap}(A; \mathbb{Q}) \) and \( \text{gap}(A, \mathbb{F}_p) \) can take arbitrarily large values among finitely generated abelian groups.

Next, let us interpret \( \mathbb{F} \)-gap of a group from some other viewpoints.

**Proposition 4.1.** For a finitely presentable group \( G \) and a field \( \mathbb{F} \),
\[
\text{gap}(G; \mathbb{F}) = \min \{ b_2(K_P; \mathbb{F}); P \text{ is a finite presentation of } G \} - b_2(G; \mathbb{F})
\]
\[
= \min \{ b_2(K; \mathbb{F}); K \text{ is a connected finite } 2\text{-complex with } \pi_1(K) \cong G \}
\]
\[
- b_2(G; \mathbb{F}).
\]

**Proof.** It is sufficient to prove the first equality of the proposition. Note that for any finite presentation \( P \) of \( G \), \( b_1(K_P; \mathbb{F}) = b_1(G; \mathbb{F}) \). So we have
\[
b_2(K_P; \mathbb{F}) - b_2(G; \mathbb{F}) = b_2(K_P; \mathbb{F}) - b_1(K_P; \mathbb{F}) - (b_2(G; \mathbb{F}) - b_1(G; \mathbb{F}))
\]
\[
= b_1(K_P; \mathbb{F}) - b_2(G; \mathbb{F}) - (b_1(K_P; \mathbb{F}) - b_2(K_P; \mathbb{F}))
\]
Observe that \( b_1(K_P; \mathbb{F}) - b_2(K_P; \mathbb{F}) \) coincides with \( \text{def}(P) \). Then
\[
\min \{- (b_1(K_P; \mathbb{F}) - b_2(K_P; \mathbb{F})); P \text{ is a finite presentation of } G \} = - \text{def}(G).
\]
This implies the first equality of the proposition. \( \square \)

For any group \( G \), the geometric dimension \( \text{gd}(G) \) of \( G \) is the minimal dimension of the \( K(G, 1) \)-complex. The cohomological dimension \( \text{cd}(G) \) of \( G \) is the minimal length of a projective resolution of (the trivial \( \mathbb{Z}[G] \)-module) \( \mathbb{Z} \). It is easy to see that \( \text{cd}(G) = \text{cd}(K(G, 1)) \) and \( \text{gd}(G) \geq \text{cd}(G) \).

The following corollary is immediate from Proposition 4.1.

**Corollary 4.2.** Let \( G \) be a finitely presentable group with \( \text{gap}(G; \mathbb{F}) > 0 \) for some field \( \mathbb{F} \). Then the geometric dimension of \( G \) is at least 3.

**Proposition 4.3.** For a finitely presentable group \( G \), the following statements are equivalent.
(i) \(\text{gap}(G; \mathbb{F}) = 0\).

(ii) There exists a \(K(G, 1)\) complex \(X\) so that \(X\) has only finitely many cells in each dimension and the cellular boundary map \(\partial_3 : C_3(X; \mathbb{F}) \to C_2(X; \mathbb{F})\) is trivial.

(iii) There exists a finite presentation \(\mathcal{P}\) of \(G\) so that the map \(h : \pi_2(K_{\mathcal{P}}) \to H_2(K_{\mathcal{P}}, \mathbb{F})\) induced from the Hurewicz map is the zero map.

**Proof.** First, let us assume \(\text{gap}(G; \mathbb{F}) = 0\). Then by Proposition 4.1, there exists a connected finite 2-complex \(K\) with \(b_2(K; \mathbb{F}) = b_2(G; \mathbb{F})\). Then by attaching finitely many 3-cells, we can kill \(\pi_2(K)\). Note that this process will not reduce \(b_2(K; \mathbb{F})\) because otherwise \(b_2(G; \mathbb{F})\) would be reduced as well, which is absurd. Furthermore, by adding some higher dimensional cells (finitely many in each dimension), we will obtain a \(K(G, 1)\) space that satisfies the requirements in (ii).

Conversely, suppose \(X\) is a \(K(G, 1)\) space with only finitely many cells in each dimension and \(\partial_3 : C_3(X; \mathbb{F}) \to C_2(X; \mathbb{F})\) is trivial. Then the 2-skeleton \(X(2)\) of \(X\) is a connected finite 2-complex with \(\pi_1(X(2)) \cong G\) and \(b_2(X(2); \mathbb{F}) = b_2(G; \mathbb{F})\). This implies that \(\text{gap}(G; \mathbb{F}) = 0\) by Proposition 4.1. So the equivalence of (i) and (ii) is proved. In addition, \(X(2)\) is clearly homotopy equivalent to the presentation complex of some finite presentation of \(G\). Then the equivalence of (ii) and (iii) follows easily from the construction of \(X\).

Remark 4.4. For a prime \(p\), a group \(G\) with \(\text{gap}(G; \mathbb{F}_p) = 0\) is called \(p\)-efficient (see [15]). A finite presentation \(\mathcal{P}\) of \(G\) is called \(p\)-Cockcroft (see [23]) if \(\mathcal{P}\) satisfies the condition in Proposition 4.3 (iii). So the \(\mathbb{F}\)-gap can be thought of as a generalization of \(p\)-efficiency and the existence of \(p\)-Cockcroft presentation of a group \(G\).

The \(\mathbb{Q}\)-gap of a group is related to two other known concepts. A finitely presentable group \(G\) is called efficient if there exists a connected finite 2-complex \(K\) with \(\pi_1(K) \cong G\) and \(b_2(K; \mathbb{Q}) = d(H_2(G))\) where \(d(-)\) is the minimal number of generators of a group. Indeed, we always have \(b_2(K; \mathbb{Q}) \geq d(H_2(G))\) since \(H_2(K)\) is free abelian and \(H_2(G)\) is a quotient of \(H_2(K)\) (see [12]). In addition, a connected 2-complex is said to have the Cockcroft property if the Hurewicz map \(h : \pi_2(K) \to H_3(K)\) is the zero map. The Cockcroft property was first studied by Cockcroft [10] in connection with the Whitehead conjecture (which states that a subcomplex of an aspherical 2-complex is itself aspherical).

**Proposition 4.5.** For any finitely presentable group \(G\), the following statements are equivalent.

(i) \(\text{gap}(G; \mathbb{Q}) = 0\).

(ii) \(G\) is efficient and \(H_2(G)\) has no torsion.
(iii) $G$ has a finite presentation $\mathcal{P}$ whose presentation complex $K_\mathcal{P}$ has the Cockcroft property.

Proof. These equivalences follow easily from the Hopf’s exact sequence (12) and Proposition 4.1. □

By the above proposition, the set of non-efficient finitely presentable groups is included in the set of groups with nonzero $\mathbb{Q}$-gaps. Moreover, this inclusion is strict. For example, any finitely generated abelian group $A$ is efficient. But our calculation in Example 1 shows that $\text{gap}(A; \mathbb{Q}) > 0$ if $A$ is not free abelian or cyclic (where $H_2(A)$ has torsion).

The notion of $\mathbb{Q}$-gap is also related to the Eilenberg-Ganea conjecture. The Eilenberg-Ganea conjecture claims that a group of cohomological dimension 2 is of geometric dimension 2. Let $G$ be a finitely presentable group with $\text{cd}(G) = 2$.

• If $\text{gap}(G; \mathbb{Q}) = 0$, then $G$ is efficient by Proposition 4.5. It is shown in [19] that when $\text{cd}(G) = 2$, $G$ is efficient if and only if there exists a finite 2-dimensional $K(G, 1)$-complex. So we obtain $\text{gd}(G) = 2$.

• If $\text{gap}(G; \mathbb{Q}) > 0$, $\text{gd}(G) \geq 3$ by Corollary 4.2. Then $G$ is a counterexample of the Eilenberg-Ganea conjecture.

So for a finitely presentable group $G$, the Eilenberg-Ganea conjecture is equivalent to say that $\text{cd}(G) = 2$ implies $\text{gap}(G; \mathbb{Q}) = 0$.

Example 2. If a finite 2-complex $K$ is an $F$-homology rose or $F$-acyclic space, then $\text{gap}(K; F) = 0$. In particular, for any knot $N$ in $S^3$, $\text{gap}(\pi_1(S^3 \setminus N); F) = 0$ for any field $F$ since $S^3 \setminus N$ is homotopy equivalent to a finite 2-complex.

Example 3. Let $M^3$ be a closed connected 3-manifold. If $M^3$ is orientable, there exists a finite cell decomposition of $M^3$ with only a single 3-cell. Let $K$ be the 2-skeleton of this cell decomposition. Since $M^3$ is orientable, the boundary of the 3-cell is 0. This implies that $H_2(K; F) = H_2(M^3; F)$ for any field $F$.

• If $M^3$ is aspherical, $b_2(\pi_1(M^3); F) = b_2(M^3; F)$. Then since

$$\text{gap}(\pi_1(M^3); F) \leq b_2(K; F) - b_2(\pi_1(M^3); F) = b_2(K; F) - b_2(M^3; F) = 0,$$

we obtain $\text{gap}(\pi_1(M^3); F) = 0$.

• If $M^3$ is a spherical manifold (i.e. the universal cover of $M^3$ is $S^3$), we also have $\text{gap}(\pi_1(M^3); F) = 0$. This is because $\pi_2(M^3) \cong \pi_2(S^3) = 0$, hence $M^3$ can serve as the 3-skeleton of a $K(\pi_1(M^3), 1)$ space.

If $M^3$ is non-orientable and aspherical, we can show that $\text{gap}(\pi_1(M^3); F_2) = 0$ by the same argument. For a general field $F$, we can only show $\text{gap}(\pi_1(M^3); F) \leq 1$. 
Next, we discuss the relation between the $\mathbb{Q}$-gap and an arbitrary $\mathbb{F}$-gap of a group. For a finitely presentable group $G$, let

$$H_i(G; \mathbb{Z}) \cong \mathbb{Z}^{b_i} \oplus T_i, \quad i = 1, 2,$$

where $b_i = b_i(G; \mathbb{Q})$ and $T_i$ stands for the torsion subgroup of $G$. So we have

$$H_1(G; \mathbb{F}) \cong H_1(G) \otimes \mathbb{F} \cong \mathbb{F}^{b_1} \oplus (T_1 \otimes \mathbb{F}),$$

$$H_2(G; \mathbb{F}) \cong H_2(G) \otimes \mathbb{F} \oplus \text{Tor}_2(H_1(G); \mathbb{F}) \cong \mathbb{F}^{b_2} \oplus (T_2 \otimes \mathbb{F}) \oplus \text{Tor}_2(T_1; \mathbb{F}).$$

(19)

Note that any finite abelian group $T$ can be defined by an exact sequence

$$0 \to \mathbb{Z}^n \to \mathbb{Z}^n \to T \to 0.$$

Then we have an exact sequence

$$0 \to \text{Tor}_2(T, \mathbb{F}) \to \mathbb{F}^n \to \mathbb{F}^n \to T \otimes \mathbb{F} \to 0.$$ This implies that $\dim F \text{Tor}_2(T, \mathbb{F}) = \dim F T \otimes \mathbb{F}$. So we obtain

$$\text{gap}(G; \mathbb{F}) = b_1(G; \mathbb{F}) - b_2(G; \mathbb{F}) - \text{def}(G) = \text{gap}(G; \mathbb{Q}) - \dim F (T_2 \otimes \mathbb{F}).$$

(21)

This relation implies the following.

**Proposition 4.6.** For a finitely presentable group $G$, $\text{gap}(G; \mathbb{Q}) = 0$ implies that $\text{gap}(G; \mathbb{F}) = 0$ for any field $\mathbb{F}$; and $\text{gap}(G; \mathbb{Q}) > 0$ implies that $\text{gap}(G; \mathbb{F}_p) > 0$ for almost all prime $p$.

Finally, let us discuss the property of $\mathbb{F}$-gap under the free product and direct product of groups. Let $G_1$ and $G_2$ be two finitely presentable groups.

- If $\text{gap}(G_1; \mathbb{F}) = \text{gap}(G_2; \mathbb{F}) = 0$, we must have $\text{gap}(G_1 \ast G_2; \mathbb{F}) = 0$ and $\text{def}(G_1 \ast G_2) = \text{def}(G_1) + \text{def}(G_2)$. The proof is parallel to the proof of [15, Lemma 1.6].
- That $\text{gap}(G_1; \mathbb{F}) = \text{gap}(G_2; \mathbb{F}) = 0$ does not imply $\text{gap}(G_1 \times G_2; \mathbb{F}) = 0$. For example, for $d_1, d_2 > 1$ with $p \nmid d_1$ and $p \nmid d_2$, $\text{gap}(\mathbb{Z}_{d_1}; \mathbb{F}_p) = 0$ and $\text{gap}(\mathbb{Z}_{d_1,d_2}; \mathbb{F}_p) = 0$. But by the calculation in Example [7] we have

$$\text{gap}((\mathbb{Z}_{d_1})^2; \mathbb{F}_p) = 1, \quad \text{gap}(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_1,d_2}; \mathbb{F}_p) = 1.$$

Conversely, we may ask the following question.

**Question:** If $\text{gap}(G_1 \times G_2; \mathbb{F}) = 0$, is it necessary that both $\text{gap}(G_1; \mathbb{F}) = 0$ and $\text{gap}(G_2; \mathbb{F}) = 0$.

The answer to this question is yes when $G_1$ and $G_2$ are both finitely generated abelian groups (see Example [7]). But the general answer is not clear to us.
5. Digression to the D(2)-problem

By Proposition 1.5 and Proposition 3.1, any finitely presentable group \( \Gamma \) with \( b_2(\Gamma; F) = 0 \) can be realized as the fundamental group of a finite 3-complex \( B \) which is an \( F \)-homology rose or an \( F \)-acyclic space. But generally speaking, we can not require \( B \) to be 2-dimensional. Indeed, Proposition 4.1 implies the following.

**Proposition 5.1.** For any field \( F \), a finitely presentable group \( \Gamma \) can be realized as the fundamental group of a connected finite 2-complex \( K \) with \( b_2(K; F) = 0 \) if and only if \( b_2(\Gamma; F) = 0 \) and \( \text{gap}(\Gamma; F) = 0 \).

So if \( B \) is a 3-complex with \( b_2(B; F) = 0 \) and \( \text{gap}(\pi_1(B); F) > 0 \), then \( B \) can not be homotopy equivalent to a 2-complex. More generally, we have:

**Lemma 5.2.** If \( B \) is a connected finite 3-complex with \( b_2(B; F) = b_2(\pi_1(B); F) \) and \( \text{gap}(\pi_1(B); F) > 0 \), then \( B \) is not homotopy equivalent to any finite 2-complex.

This simple fact motivates us to study Wall’s D(2)-problem (see [35]) among \( F \)-homology roses and \( F \)-acyclic spaces.

**D(2)-problem:** Let \( B \) be a finite 3-complex with \( \text{cd}(B) = 2 \), i.e. \( H^3(B; M) = 0 \) for all \( \pi_1(B) \)-module \( M \); is it true that \( B \) is homotopy equivalent to a finite 2-complex?

We say that the D(2)-property holds for a group \( \Gamma \) if any connected finite 3-complex \( B \) with \( \text{cd}(B) = 2 \) and \( \pi_1(B) \cong \Gamma \) is homotopy equivalent to a finite 2-complex. The D(2)-problem is equivalent to asking whether the D(2)-property holds for all finitely presentable groups. So far the D(2)-property is known to hold for

- free groups (Johnson [21]),
- finite abelian groups (Browning, Latiolais [9, 25]),
- \( \mathbb{Z} \times \mathbb{Z}_n \) and \( \mathbb{Z}^2 \) (Edwards [14]),
- dihedral groups of order 8 (Mannan [27]) and of order \( 4n+2 \) (Johnson [20]).

On the other hand, no counterexamples of the D(2)-problem are known so far, although people have tried many methods and proposed some candidates for such counterexamples (see [6, 14, 17, 21, 22, 23, 30]). Our discussion on \( F \)-gaps motivates us to search candidates for the counterexamples of the D(2)-problem among \( F \)-homology roses and \( F \)-acyclic spaces via the following strategy.

**Strategy:**

- Take a finitely presentable group \( \Gamma \) with \( H_2(\Gamma; F) = 0 \) and \( \text{gap}(\Gamma; F) > 0 \).
- By Proposition 1.5 and Proposition 3.1, there exists a connected finite 3-complex \( B \) with \( \pi_1(B) \cong \Gamma \) and \( b_2(B; F) = 0 \). If we can choose \( B \) with \( \text{cd}(B) = 2 \), then \( B \) is a counterexample of the D(2)-problem.
To actually put the above strategy into practice, we need to first construct enough examples of finite 3-complexes with cohomological dimension two. A general way to do so is to use Quillen’s plus construction (see [28]).

Let $K$ be a connected finite 2-complex. Assume that the fundamental group $\pi_1(K)$ of $K$ has a finitely closed perfect normal subgroup $N$. Here $N$ is \textit{finitely closed} means that it is the normal closure of a finitely generated subgroup of $\pi_1(K)$. And $N$ is \textit{perfect} means $H_1(N; \mathbb{Z}) = 0$ (i.e. $N = [N, N]$). Then Quillen’s plus construction on $K$ gives us a finite 3-complex $K^+$ with

\[ H_*(K^+; \mathbb{Z}) \cong H_*(K; \mathbb{Z}), \quad \pi_1(K^+) \cong \pi_1(K)/N, \quad \text{cd}(K^+) = \text{cd}(K) = 2. \tag{22} \]

It is shown in [28, Theorem 3.4] that any connected finite 3-complex with cohomological dimension 2 can be obtained from a finite 2-complex using the plus construction. So in particular, if a finite 3-complex $B$ with $b_2(B; \mathbb{F}) = 0$ has cohomological dimension 2, then there exists a finite 2-complex $K$ so that $B$ is homeomorphic to $K^+$ with respect to some finitely closed perfect normal subgroup of $\pi_1(K)$. This implies that $b_2(K; \mathbb{F}) = 0$, and so $b_2(\pi_1(K); \mathbb{F}) = 0$ and $\text{gap}(\pi_1(K); \mathbb{F}) = 0$ (by Proposition 5.1).

So if we assume that a finite 3-complex $B$ is a counterexample of the D(2)-problem and $B$ is an $\mathbb{F}$-homology rose or an $\mathbb{F}$-acyclic space, we necessarily have an exact sequence of the following form.

\[ 1 \rightarrow N \rightarrow G \rightarrow \Gamma \rightarrow 1 \quad \text{where} \tag{23} \]

- $N$ is a nontrivial finitely closed perfect normal subgroup of $G$.
- $G$ is a finitely presentable group with $b_2(G; \mathbb{F}) = 0$, $\text{gap}(G; \mathbb{F}) = 0$.
- $b_2(\Gamma; \mathbb{F}) = 0$, $\text{gap}(\Gamma; \mathbb{F}) > 0$, $H_1(\Gamma; \mathbb{Z}) = H_1(G; \mathbb{Z})$.

These conditions further imply that
- $\text{def}(G) = b_1(G; \mathbb{F}) = b_1(\Gamma; \mathbb{F}) \geq 0$ and $\text{def}(G) > \text{def}(\Gamma)$.

Conversely, Proposition 1.5 implies that any sequence as in (23) gives us a counterexample $B$ of the D(2)-problem where $B$ is an $\mathbb{F}$-homology rose or an $\mathbb{F}$-acyclic space with $\pi_1(B) \cong \Gamma$.

Note that in (23), $b_2(G; \mathbb{F}) = 0$ implies $b_2(G; \mathbb{Q}) = 0$ and $\text{tors}(H_2(G)) \otimes \mathbb{F} = 0$ (see [21]). Then since $\text{gap}(G; \mathbb{F}) = 0$, we get $\text{gap}(G; \mathbb{Q}) = 0$ (see [21]). Similarly, $b_2(\Gamma; \mathbb{F}) = 0$ implies $b_2(\Gamma; \mathbb{Q}) = 0$ and $\text{gap}(\Gamma; \mathbb{Q}) = \text{gap}(\Gamma; \mathbb{F}) > 0$. So if we want to search counterexamples of the D(2)-problem via the sequence (23), we only need to consider $\mathbb{Q}$-coefficients.

We can think of $\Gamma$ in the exact sequence (23) as the deck transformation group of a regular covering space $p : X \rightarrow K$ where $X$ and $K$ are connected finite 2-complexes with $\pi_1(X) \cong N$ and $\pi_1(K) \cong G$. From this viewpoint, we can try the following ways to construct possible counterexamples of the D(2)-problem.
(M1) Let $X$ be a connected 2-complex with $H_1(X;\mathbb{Z}) = 0$. Suppose $\Gamma$ is a group with gap($\Gamma;\mathbb{Q}$) > 0. If $\Gamma$ can act freely on $X$ so that the orbit space $K = X/\Gamma$ is a connected finite 2-complex with $b_2(K;\mathbb{Q}) = 0$, then the plus-construction $K^+$ with respect to $\pi_1(X)$ is a counterexample of the D(2)-problem.

(M2) Let $K$ be a connected finite 2-complex with $b_2(K;\mathbb{Q}) = 0$ and let $\Gamma$ be a group with gap($\Gamma;\mathbb{Q}$) > 0. If we can construct a regular $\Gamma$-covering space $\tilde{X}$ over $K$ with $H_1(\tilde{X};\mathbb{Z}) = 0$, the plus construction $K^+$ with respect to $\pi_1(X)$ is a counterexample of the D(2)-problem.

Moreover, the above way of searching candidates for the counterexamples of the D(2)-problem can be generalized as follows.

(M3) Let $G$ be a finitely presented group. If there exists a finitely closed perfect normal subgroup $N$ of $G$ so that the quotient group $\Gamma = G/N$ satisfies gap($\Gamma;\mathbb{Q}$) > 0 and $b_2(\Gamma;\mathbb{Q}) = b_2(K_\Gamma;\mathbb{Q})$, where $\mathcal{P}$ is a finite presentation of $G$, then the plus construction $K_\Gamma^+$ with respect to $N$ is a finite 3-complex with $b_2(K_\Gamma^+;\mathbb{Q}) = b_2(\Gamma;\mathbb{Q})$. So $K_\Gamma^+$ is a counterexample of the D(2)-problem.

Remark 5.3. The above way of searching counterexamples of the D(2)-problem is essentially equivalent to the one proposed in [28, Theorem 4.5].

The following discussion tells us that if we want to search counterexamples of the D(2)-problem among $\mathbb{F}$-homology roses, we need to assume, a priori, that the fundamental groups of the $\mathbb{F}$-homology roses are infinite, non-abelian and have trivial multiplicators.

Proposition 5.4. Let $B$ be a finite 3-complex.

(i) If $B$ is an $\mathbb{F}$-homology rose and $\pi_1(B)$ is finite, $\text{cd}(B) = 3$.

(ii) If $B$ is an $\mathbb{F}$-homology rose or an $\mathbb{F}$-acyclic space with $H_2(\pi_1(B)) \neq 0$, then $\text{cd}(B) = 3$.

Proof. (i) Consider the universal cover $\tilde{B} \to B$. Since $\chi(B) = 1 - b_1(B;\mathbb{F}) \leq 0$, $\chi(\tilde{B}) = |\pi_1(B)| \cdot \chi(B) \leq 0$. But since $\tilde{B}$ is simply connected, $\chi(\tilde{B}) = 1 + b_2(\tilde{B};\mathbb{Q}) - b_3(\tilde{B};\mathbb{Q})$.

So $H_3(\tilde{B}) \neq 0$. Then since $H^3(B;\mathbb{Z}[\pi_1(B)]) = H^3(\tilde{B}) \neq 0$, $\text{cd}(B) = 3$.

(ii) Since $H_2(B;\mathbb{F}) = 0$, we have $H_2(B;\mathbb{Q}) = 0$. So $H_2(B)$ is finite abelian. Then since $H_2(\pi_1(B)) \neq 0$, the sequence (12) implies that $H_2(B)$ is non-trivial. Hence $H_2(B)$ is a non-trivial finite abelian group. This implies $\text{cd}(B) = 3$. \qed

Corollary 5.5. Suppose a finite 3-complex $B$ with $\text{cd}(B) = 2$ is an $\mathbb{F}$-homology rose or an $\mathbb{F}$-acyclic space. Then the following statements are equivalent.
(i) $\pi_1(B)$ is abelian.

(ii) $\pi_1(B)$ is a cyclic group.

(iii) $B$ is homotopy equivalent to $S^1$ or $\mathbb{P}_m$, $m \geq 2$.

Proof. Since $\text{cd}(B) = 2$, we have $H_2(\pi_1(B)) = 0$ by Proposition 5.4. For a finitely generated abelian group $A$, $H_2(A) = 0$ if and only if $A$ is cyclic (see (16)). So we can derive (ii) from (i). Moreover, since the $D(2)$-property holds for all cyclic groups, (ii) implies that $B$ is homotopy equivalent to a finite 2-complex, which further implies (iii) according to Corollary 3.8 and Corollary 3.11. □

6. Search among 3-manifold groups

If we assume that the group $G$ in the exact sequence (23) is the fundamental group of a closed connected 3-manifold $M^3$, then [5, Theorem 4.2] implies

- $M^3$ is a rational homology 3-sphere, and so $b_1(G; \mathbb{Q}) = b_2(G; \mathbb{Q}) = 0$.
- $N$ is the fundamental group of an integral homology 3-sphere $\hat{M}^3$.
- $\Gamma$ is a finite group of periodic cohomology with period 1, 2 or 4, and $b_1(\Gamma; \mathbb{Q}) = b_2(\Gamma; \mathbb{Q}) = 0$ (since $\Gamma$ is the fundamental group of the plus-construction $(M^3)^+$).

Recall that a finite group $\Gamma$ is periodic and of period $m > 0$ if and only if $H^i(\Gamma; \mathbb{Z}) \cong H^{i+m}(\Gamma; \mathbb{Z})$ for all $i \geq 1$ where $\Gamma$ acts on $\mathbb{Z}$ trivially.

$M^3$ is the orbit space of a free orientation-preserving action of $\Gamma$ on $\hat{M}^3$. Note that the existence of Heegaard splittings for closed 3-manifolds implies that $\text{def}(G) = \text{def}(\pi_1(M^3)) \geq 0$. Then since $b_1(G; \mathbb{Q}) = 0$, the inequality (15) implies

$$\text{def}(G) = 0, \ H_2(G; \mathbb{Z}) = 0.$$ 

If $\hat{M}^3$ is the 3-sphere $S^3$, $\Gamma$ is the fundamental group of a spherical 3-manifold and hence $\text{gap}(\Gamma; \mathbb{Q}) = 0$ (see Example 3). So if we want to have $\text{gap}(\Gamma; \mathbb{Q}) > 0$, $\hat{M}^3$ has to be an integral homology sphere other than $S^3$.

Proposition 6.1. If a finite group $\Gamma$ with $\text{def}(\Gamma) < 0$ can act freely on an integral homology 3-sphere $\Sigma^3 \not\cong S^3$, then it gives a counterexample of the $D(2)$-problem.

Proof. Since $\Gamma$ is finite, the orbit space $\Sigma^3/\Gamma$ is a rational homology 3-sphere. Since $b_1(\Gamma; \mathbb{Q}) = b_2(\Gamma; \mathbb{Q}) = 0$, $\text{def}(\Gamma) < 0$ implies that $\text{gap}(\Gamma; \mathbb{Q}) > 0$. Let $K$ be the 2-skeleton of $\Sigma^3/\Gamma$. It is clear that $K$ is a 2-dimensional $\mathbb{Q}$-acyclic space. Then by our discussion, the plus construction $K^+$ with respect to the nontrivial group $\pi_1(\Sigma^3) < \pi_1(\Sigma^3/\Gamma) = \pi_1(K)$ is a counterexample of the $D(2)$-problem. □

By Smith’s theory, if a finite group $\Gamma$ can act freely on an integral homology 3-sphere, it is necessary that any elementary abelian $p$-subgroup of $\Gamma$ is cyclic for
all prime $p$. This is equivalent to say that any abelian subgroup of $\Gamma$ is cyclic (see [7, p.157]). Such finite groups are all periodic and have been classified by Suzuki and Zassenhaus (see [2, p.154] or [37, Chapter 6.3]).

But it is not true that any finite group whose abelian subgroups are all cyclic can act freely on an integral homology sphere. An additional restriction is the Milnor condition (see [29]) which says that if a group $\Gamma$ can act freely on an integral homology sphere, $\Gamma$ has at most one element of order two. This excludes, for example, dihedral groups. A list of the possible finite groups which may act freely on an integral homology 3-sphere is given in Milnor’s paper [29]. But the complete classification of such groups remains open (see [12] and [38] for more information). The difficulty lies in the understanding of a family of groups defined below. For relatively coprime positive integers $8n$, $k$ and $l$, let $Q(8n, k, l)$ denote the group with presentation

$$\langle x, y, z \mid x^2 = (xy)^2 = y^{2n}, z^k = 1, xzx^{-1} = z^r, yzy^{-1} = z^{-1} \rangle$$

where $r \equiv -1 \mod k$ and $r \equiv +1 \mod l$. The product of $Q(8n, k, l)$ with a cyclic group of coprime order are the only type of groups in Milnor’s list [29] which may possibly act freely on some integral homology 3-sphere other than $S^3$. In addition, a result in [26] further limits the possibility to: $n$ is odd and $n > k > l \geq 1$.

Therefore, if we assume the group $G$ in (23) to be the fundamental group of a closed 3-manifold, the groups $\mathbb{Z}_m \times Q(8n, k, l)$ with $n$ odd and $n > k > l \geq 1$ are the only candidates for $\Gamma$ for us to construct counterexamples of the D(2)-problem in this approach.

7. Search among finite groups with trivial multiplicator and negative deficiency

Let $\Gamma$ be a finite group with $H_2(\Gamma) = 0$ and def($\Gamma$) < 0. Then

$$b_1(\Gamma; \mathbb{Q}) = b_2(\Gamma; \mathbb{Q}) = 0, \quad \text{gap}(\Gamma; \mathbb{Q}) > 0.$$ 

Let $K$ be a finite 2-complex with $\pi_1(K) \cong \Gamma$. So by Hopf’s exact sequence

$$\pi_2(K) \rightarrow H_2(K) \rightarrow H_2(\Gamma) \rightarrow 0,$$

we can attach some 3-cells to $K$ to obtain a finite 3-complex $B$ so that $B$ is a $\mathbb{Q}$-acyclic space with $\pi_1(B) \cong \Gamma$ (see the proof of Proposition 1.5). Actually, we can require $H_2(B) = H_3(B) = 0$.

In addition, Proposition 5.1 (or Proposition 3.3) implies that $B$ can not be homotopy equivalent to a 2-complex. So $B$ would be a counterexample of the D(2)-problem if $\text{cd}(B) = 2$. 
Examples of finite groups with trivial multiplicator and negative deficiency can be found in Swan [33] and Kovács [24].

The following examples are taken from [33]. Let \( \mathbb{Z}_3 = \langle x \rangle \) acts on \((\mathbb{Z}_7)^k\) by 
\[ xyx^{-1} = y^2 \]
for all \( y \in (\mathbb{Z}_7)^k \). Let \( G_k \) be the corresponding semi-direct product of \( \mathbb{Z}_3 \) and \((\mathbb{Z}_7)^k\). It is shown in [33, p. 197] that

- \( H_2(G_k) = 0 \) for any \( k \).
- \( \text{def}(G_k) < 0 \) for any \( k \geq 3 \) and \( \text{def}(G_k) \to -\infty \) as \( k \to \infty \).
- \( G_k \) is not efficient for any \( k \geq 3 \).

So \( \text{gap}(G_k; \mathbb{Q}) > 0 \) for all \( k \geq 3 \). By the preceding discussion, there exists a finite 3-complex \( B_k \) with \( \pi_1(B_k) \cong G_k \) and \( \tilde{B}_k \) is a \( \mathbb{Q} \)-acyclic space. When \( k \geq 3 \), \( B_k \) is not homotopy equivalent to a 2-complex.

Let \( \tilde{B}_k \) be the universal covering of \( B_k \) whose cellular chain complex is:

\[
\cdots 0 \to C_3(\tilde{B}_k) \xrightarrow{\partial_3} C_2(\tilde{B}_k) \xrightarrow{\partial_2} C_1(\tilde{B}_k) \xrightarrow{\partial_1} C_0(\tilde{B}_k) \to 0.
\]

To prove \( \text{cd}(B_k) = 2 \), it is sufficient to show that \( \tilde{\partial}_3 \) splits, i.e. there exists a \( \mathbb{Z}[\pi_1(B_k)] \)-module homomorphism \( \beta : C_2(\tilde{B}_k) \to C_3(\tilde{B}_k) \) so that \( \beta \circ \tilde{\partial}_3 = \text{id}_{C_3(\tilde{B}_k)} \).

**Claim:** \( \text{cd}(B_k) = 2 \) if and only if \( H^3(\tilde{B}_k) = 0 \).

Since \( \tilde{B}_k \) is compact, we have

\[
H^3(B_k, \mathbb{Z}[G_k]) \cong H^3_c(\tilde{B}_k) = H^3(\tilde{B}_k) \; \text{(see [18, Proposition 3H.5])}.
\]

So if \( \text{cd}(B_k) = 2 \), we must have \( H^3(\tilde{B}_k) = 0 \). Conversely, assume \( H^3(\tilde{B}_k) = 0 \). We can think of \( \text{id}_{C_3(\tilde{B}_k)} \) as an element of \( H^3(B_k, C_3(\tilde{B}_k)) \). But \( H^3(B_k, C_3(\tilde{B}_k)) \) is zero because \( C_3(\tilde{B}_k) \) is a free \( \mathbb{Z}[G_k] \)-module and \( H^3(B_k, \mathbb{Z}[G_k]) = 0 \). Therefore, there exists a \( \mathbb{Z}[G_k] \)-module homomorphism \( \beta : C_2(\tilde{B}_k) \to C_3(\tilde{B}_k) \) so that \( \beta \circ \tilde{\partial}_3 = \text{id}_{C_3(\tilde{B}_k)} \). Then \( \tilde{\partial}_3 \) is a split injection. The claim is proved.

So to compute \( \text{cd}(B_k) \), it amounts to determine whether \( H^3(\tilde{B}_k, \mathbb{Z}) \) is zero. But the calculation is not clear to us.

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