Spaces of non-resultant systems of bounded multiplicity determined by a toric variety

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Abstract

The space of non-resultant systems of bounded multiplicity for a toric variety $X$ is a generalization of the space of rational curves on it. In our earlier work [24] we proved a homotopy stability theorem and determined explicitly the homotopy type of this space for the case $X = \mathbb{CP}^m$. In this paper we consider the case of a general non-singular toric variety and prove a homotopy stability theorem generalising the one for $\mathbb{CP}^m$.

1 Introduction

For a complex manifold $X$, let $\text{Map}^*(S^2, X) = \Omega^2 X$ (resp. $\text{Hol}^*(S^2, X)$) denote the space of all base point preserving continuous maps (resp. base point preserving holomorphic maps) from the Riemann sphere $S^2$ to $X$. The relationship between the topology of the space $\text{Hol}^*(S^2, X)$ and that of the space $\Omega^2 X$ has played a significant role in several different areas of geometry and mathematical physics (e.g. [2], [5]). In particular there arose the problem of whether the inclusion $\text{Hol}^*(S^2, X) \to \Omega^2 X$ is a homotopy equivalence (or homology equivalence) up to a certain dimension, which we will refer to as the stability dimension. Since G. Segal [32] studied this problem for the case $X = \mathbb{CP}^m$, a number of mathematicians have investigated various closely related ones. In particular, M. Guest [14] obtained a generalization of Segal’s
result to the case of compact non-singular toric varieties \( X \). More generally, J. Mostovoy and E. Munguia-Vilaneuva \cite{24} generalized the result of Guest to the case of spaces of holomorphic maps from \( \mathbb{C}P^m \) to a compact non-singular toric variety \( X \) for \( m \geq 1 \) and they also improved the homology stability dimension of Guest for the case \( m = 1 \). The authors \cite{25} also generalized the result of Mosotov-Vilaneuva for non-compact non-singular toric varieties \( X \) for \( m = 1 \) (see Theorem \ref{thm:1.10} in detail).

Similar stabilization results appeared in the work of Arnold \cite{3, 4}, and Vassiliev \cite{33, 34} in connection with singularity theory. They considered spaces of polynomials without roots of multiplicity greater than a certain natural number. These spaces are examples of “complement of discriminants” in Vassiliev’s terminology \cite{33}. In fact, a part of Segal’s proof in \cite{32} was based on an argument due to Arnold. The work of B. Farb and J. Wolfson \cite{11} was inspired by this argument, and they introduced a new family of spaces \( \text{Poly}^{d,m}_n(\mathbb{F}) \). They simultaneously generalized the ones studies by Segal, Arnold and Vassiliev, and they obtained algebro-geometric and arithmetic refinements of their topological results. Recall the definition of the space \( \text{Poly}^{d,m}_n(\mathbb{F}) \) as follows.

**Definition 1.1** (\cite{11}). Let \( \mathbb{N} \) be the set of all positive integers. For a field \( \mathbb{F} \) with its algebraic closure \( \overline{\mathbb{F}} \) and a pair \((m, n) \in \mathbb{N}^2\) with \((m, n) \neq (1, 1)\), let \( \text{Poly}^{d,m}_n(\mathbb{F}) \) denote the space of all \( m \)-tuples \((f_1(z), \cdots, f_m(z)) \in \mathbb{F}[z]^m\) of \( \mathbb{F} \)-coefficients monic polynomials of the same degree \( d \) such that polynomials \( f_1(z), \cdots, f_m(z) \) have no common root \( \alpha \in \overline{\mathbb{F}} \) of multiplicity \( \geq n \).

Note that the space \( \text{Poly}^{d,m}_n(\mathbb{F}) \) can be identified with \( \text{Hol}_d^*(S^2, \mathbb{C}P^{m-1}) \) for \((\mathbb{F}, n) = (\mathbb{C}, 1)\), where \( \text{Hol}_d^*(S^2, \mathbb{C}P^{m-1}) \) denotes the space of base point preserving holomorphic maps \( f : S^2 \to \mathbb{C}P^{m-1} \) of degree \( d \). Thus, the space \( \text{Poly}^{d,m}_n(\mathbb{C}) \) may be regarded as a generalizations of the space \( \text{Hol}_d^*(S^2, \mathbb{C}P^{m-1}) \).

For a monic polynomial \( f(z) \in \mathbb{F}[z] \) of degree \( d \), let \( F_n(f)(z) \) denote the \( n \)-tuple of monic polynomials in \( \mathbb{F}[z] \) of the same degree \( d \) defined by

\[
F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \cdots, f(z) + f^{(n-1)}(z)).
\]

In an earlier paper \cite{24} we determined the homotopy type of the space \( \text{Poly}^{d,m}_n(\mathbb{F}) \) explicitly for the case \( \mathbb{F} = \mathbb{C} \) and obtained the following homotopy stability result.

**Theorem 1.2** (\cite{24}). Let \( d, m, n \in \mathbb{N} \) be positive integers with \((m, n) \neq (1, 1)\), and let \( i_n^{d,m} : \text{Poly}^{d,m}_n(\mathbb{C}) \to \Omega^d_d \mathbb{C}P^{mn-1} \simeq \Omega S^{2mn-1} \) denote the natural map given by

\[
i_n^{d,m}(f)(\alpha) = \begin{cases} 
[F_n(f_1)(\alpha) : F_n(f_2)(\alpha) : \cdots : F_n(f_m)(\alpha)] & \text{if } \alpha \in \mathbb{C} \\
[1 : 1 : \cdots : 1] & \text{if } \alpha = \infty
\end{cases}
\]
for \((f, \alpha) = ((f_1(z), \cdots, f_m(z)), \alpha) \in \text{Poly}^{d,m}_n(\mathbb{C}) \times S^2\), where we identify \(S^2 = \mathbb{C} \cup \infty\).

Then the map \(i_{n,m}^d\) is a homotopy equivalence through dimension \(D(d; m, n)\) if \((m, n) \neq (1, 2)\) and it is a homology equivalence through dimension \(\lfloor \frac{d}{2} \rfloor\) if \((m, n) = (1, 2)\), where \(\lfloor x \rfloor\) denotes the integer part of a real number \(x\) and the positive integer \(D(d; m, n)\) is given by \(D(d; m, n) = (2mn - 3)(\lfloor \frac{d}{n} \rfloor + 1) - 1\).

\[\square\]

**Remark 1.3.** A map \(f : X \to Y\) is called a homotopy equivalence through dimension \(N\) (resp. a homology equivalence through dimension \(N\)) if the induced homomorphism \(f_* : \pi_k(X) \to \pi_k(Y)\) (resp. \(f_* : H_k(X; \mathbb{Z}) \to H_k(Y; \mathbb{Z})\)) is an isomorphism for any \(k \leq N\).

Our aim of this paper is to further generalize the above result to the case where the conditions on the roots are given in terms of the combinatorial information contained in a non-singular toric variety \(X_\Sigma\), where \(\Sigma\) denotes a fan in \(\mathbb{R}^n\) and let \(X_\Sigma\) be the toric variety associated to \(\Sigma\).

**Definition 1.4.** Let \(\mathbb{F}\) be a field with its algebraic closure \(\overline{\mathbb{F}}\), and let \(\Sigma\) be a fan in \(\mathbb{R}^n\) such that \(\Sigma(1) = \{\rho_1, \cdots, \rho_r\}\), where \(\Sigma(1)\) denotes the set of all one dimensional cones in \(\Sigma\).

For each \(r\)-tuple \(D = (d_1, \cdots, d_r) \in \mathbb{N}^r\), let \(\text{Poly}^{D,\Sigma}_n(\mathbb{F})\) denote the space of all \(r\)-tuples \((f_1(z), \cdots, f_r(z)) \in \mathbb{F}[z]^r\) of \(\mathbb{F}\)-coefficients monic polynomials satisfying the following two conditions:

1. \((1.2.1)\) \(f_i(z) \in \mathbb{F}[z]\) is an \(\mathbb{F}\)-coefficients monic polynomial of the degree \(d_i\) for each \(1 \leq i \leq r\).

2. \((1.2.2)\) For each \(\sigma = \{i_1, \cdots, i_s\} \in I(\mathcal{K}_\Sigma)\), polynomials \(f_{i_1}(z), \cdots, f_{i_s}(z)\) have no common root \(\alpha \in \overline{\mathbb{F}}\) of multiplicity \(\geq n\), where \(\mathcal{K}_\Sigma\) denotes the underlying simplicial complex of \(X_\Sigma\) on the index set \([r] = \{1, 2, \cdots, r\}\) defined by \((2.3)\) and \(I(\mathcal{K}_\Sigma)\) is the set \(I(\mathcal{K}_\Sigma) = \{\sigma \subset [r]: \sigma \notin \mathcal{K}_\Sigma\}\) defined by \((2.2)\).

\[\square\]

**Remark 1.5.** (i) By using the classical theory of resultants, one can show that \(\text{Poly}^{D,\Sigma}_n(\mathbb{F})\) is an affine variety over \(\mathbb{F}\) which is the complement of the set of solutions of a system of polynomial equations (called a generalised resultant) with integer coefficients. This is why we call it the space of non-resultant systems of bounded multiplicity of type \((\Sigma, n)\).

(ii) When \(X_\Sigma\) is a simply connected non-singular toric variety (over \(\mathbb{C}\)) satisfying the condition \((2.1)\), one can show that \(\text{Poly}^{D,\Sigma}_n(\mathbb{C}) = \text{Hol}^{D}_n(S^2, X_\Sigma)\) if \(n = 1\) (see Definition \((2.6)\) for the details). Moreover, note that \(\text{Poly}^{d,m}_n(\mathbb{F})\) = \(\{\text{Poly}^{D,\Sigma}_n(\mathbb{F})\}^d\).\(^{1}\)

\(^{1}\)The precise definition and notation concerning fans and toric varieties are explained in \(\square\).
Poly\textsubscript{\textit{n},\textit{D},\Sigma}(\mathbb{F}) when \(X_\Sigma = \mathbb{CP}^{m-1}\) and \(D = (d,d,\cdots,d)\). So we may regard the space Poly\textsubscript{\textit{n},\textit{D},\Sigma}(\mathbb{F}) as a generalization of the spaces Poly\textsubscript{\textit{n},\textit{D},\Sigma}(\mathbb{C}) and Hol\textsubscript{D}(S^2, X_\Sigma).

The principal motivation of this paper is to generalize the above result (Theorem 1.2) for the space Poly\textsubscript{\textit{n},\textit{D},\Sigma}(\mathbb{C}). From now on, we write

\[
(1.3) \quad \text{Poly}_{n}^{D,\Sigma} = \text{Poly}_{n}^{D,\Sigma}(\mathbb{C}) \quad \text{for} \quad \mathbb{F} = \mathbb{C}.
\]

Then the main result of this paper is the following

**Theorem 1.6 (Theorem 2.11).** Let \(D = (d_1,\cdots,d_r) \in \mathbb{N}^r, n \geq 2\), and let \(X_\Sigma\) be an \(m\) dimensional simply connected non-singular toric variety such that the condition (2.15.1) holds.

(i) If \(\sum_{k=1}^{r} d_k n_k = 0_m\), then the natural map (given by (2.20))

\[
i_D : \text{Poly}_{n}^{D,\Sigma} \to \Omega^2_D X_\Sigma(n) \simeq \Omega^2 Z_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})
\]

is a homotopy equivalence through dimension \(d(D; \Sigma, n)\), where \(d(D; \Sigma, n)\) denotes the positive integer defined in (2.22), and the spaces \(X_\Sigma(n)\) and \(Z_{\mathcal{K}}(X,A)\) are the orbit space and the polyhedral product of a pair \((X,A)\) given by (2.12) and Definition 2.1, respectively.

(ii) If \(\sum_{k=1}^{r} d_k n_k \neq 0_m\), there is a map

\[
j_D : \text{Poly}_{n}^{D,\Sigma} \to \Omega^2 Z_{\mathcal{K}_\Sigma}(D^{2n}, S^{2n-1})
\]

which is a homotopy equivalence through dimension \(d(D; \Sigma, n)\).

This paper is organized as follows. In §2 we recall several basic definitions and facts about toric varieties and holomorphic curves on toric varieties, which will be in the statements of the results of this paper. Precise statements of the main results are stated after these basic definitions and facts. In §3 we recall several basic facts related polyhedral products and toric varieties. In §4 we summarize the definitions of the non-degenerate simplicial resolution and the associated truncated simplicial resolution. Then we construct the Vassiliev spectral sequence converging to \(H_*(\text{Poly}_{n}^{D,\Sigma}; \mathbb{Z})\) by using them, and prove the homotopy stability result (Theorem 4.18, Corollary 4.19). In §5 we consider the configuration model for Poly\textsubscript{\textit{n},\textit{D},\Sigma} and recall the stabilized scanning map. Furthermore, we investigate the space \(E\Sigma_n(U, \partial U)\) and show that it is homotopy equivalent to the Davis-Januszkiewicz space \(DJ(K_\Sigma(n))\). In §6 we give the proof of stability result (Theorem 6.2) by using the stabilized scanning map, and finally in §7 we give the proofs of the main results (Theorem 2.11, Corollary 2.12).
2 Toric varieties and the main results

In this section we recall several basic definitions and facts related to toric varieties (convex rational polyhedral cones, toric varieties, a fan of toric variety, polyhedral products, homogeneous coordinate, rational curves on a toric variety etc) and give precise statements of the main results of this paper.

Fans and toric varieties

A convex rational polyhedral cone \( \sigma \) in \( \mathbb{R}^m \) is a subset of \( \mathbb{R}^m \) of the form

\[
\sigma = \text{Cone}(S) = \text{Cone}(m_1, \ldots, m_s) = \left\{ \sum_{k=1}^{s} \lambda_k m_k : \lambda_k \geq 0 \right\}
\]

for a finite set \( S = \{m_1, \ldots, m_s\} \subset \mathbb{Z}^m \). The dimension of \( \sigma \) is the dimension of the smallest subspace of \( \mathbb{R}^m \) which contains \( \sigma \). A convex rational polyhedral cone \( \sigma \) is called strongly convex if \( \sigma \cap (-\sigma) = \{0\} \), where we set \( 0_m = (0, 0, \ldots, 0) \in \mathbb{R}^m \).

A face \( \tau \) of a convex rational polyhedral cone \( \sigma \) is a subset \( \tau \subset \sigma \) of the form \( \tau = \sigma \cap \{x \in \mathbb{R}^m : L(x) = 0\} \) for some linear form \( L \) on \( \mathbb{R}^m \), such that \( \sigma \subset \{x \in \mathbb{R}^m : L(x) \geq 0\} \).

If we set \( \{k : 1 \leq k \leq s, L(m_k) = 0\} = \{i_1, \ldots, i_t\} \), we easily see that \( \tau = \text{Cone}(m_{i_1}, \ldots, m_{i_t}) \). Hence, if \( \sigma \) is a strongly convex rational polyhedral cone, so is any of its faces.

Let \( \Sigma \) be a finite collection of strongly convex rational polyhedral cones in \( \mathbb{R}^m \). Then it is called a fan (in \( \mathbb{R}^m \)) if the following two conditions (2.1.1) and (2.1.2) are satisfied:

(2.1.1) Every face \( \tau \) of \( \sigma \in \Sigma \) belongs to \( \Sigma \).

(2.1.2) If \( \sigma_1, \sigma_2 \in \Sigma \), \( \sigma_1 \cap \sigma_2 \) is a common face of each \( \sigma_k \) and \( \sigma_1 \cap \sigma_2 \in \Sigma \).

An \( m \) dimensional irreducible normal variety \( X \) (over \( \mathbb{C} \)) is called a toric variety if it has a Zariski open subset \( T^m_{\mathbb{C}} = (\mathbb{C}^*)^m \) and the action of \( T^m_{\mathbb{C}} \) on itself extends to an action of \( T^m_{\mathbb{C}} \) on \( X \). The most significant property of a toric variety is that it is characterized up to isomorphism entirely by its associated fan \( \Sigma \). We denote by \( X_\Sigma \) the toric variety associated to a fan \( \Sigma \) (see [9] for the details).

It is well known that there are no holomorphic maps \( \mathbb{CP}^1 = S^2 \to T^m_{\mathbb{C}} \) except the constant maps, and that the fan \( \Sigma \) of \( T^m_{\mathbb{C}} \) is \( \Sigma = \{0_m\} \). Hence, without loss of generality we always assume that \( X_\Sigma \neq T^m_{\mathbb{C}} \) and that any fan \( \Sigma \) in \( \mathbb{R}^m \) satisfies the condition \( \{0_m\} \subset \Sigma \).

\[\text{When} \ S \text{\ is the emptyset} \emptyset, \ \text{we set} \ \text{Cone}(\emptyset) = \{0_m\} \text{\ and we may also regard it as one of strongly convex rational polyhedral cones in} \ \mathbb{R}^m.\]
Polyhedral products  Now recall the basic definitions concerning polyhedral products and related spaces.

**Definition 2.1.** Let $K$ be a simplicial complex on the index set $[r] = \{1, 2, \cdots, r\}$ and let $(X, A)$ be a pairs of based spaces.

(i) Let $I(K)$ denote the some collection of subsets $\sigma \subset [r]$ defined by

$$I(K) = \{\sigma \subset [r] : \sigma \notin K\}.$$  

(ii) Define the polyhedral product $Z_K(X, A)$ with respect to $K$ by

$$Z_K(X, A) = \bigcup_{\sigma \in I(K)} (X, A)^\sigma,$$

where  

$$(X, A)^\sigma = \{(x_1, \cdots, x_r) \in X^r : x_k \in A \text{ if } k \notin \sigma \}.$$  

(iii) For each subset $\sigma = \{i_1, \cdots, i_s\} \subset [r]$, let $L_\sigma(C^n)$ denote the subspace of $C^{nr}$ defined by

$$L_\sigma(C^n) = \{(x_1, \cdots, x_r) \in C^{nr} : x_i \in C^n, x_{i_1} = \cdots = x_{i_s} = 0_n\}.$$  

and let $L_n(\Sigma)$ denote the subspace of $C^{nr}$ defined by

$$L_n(\Sigma) = \bigcup_{\sigma \subset I(K)} L_\sigma(C^n) = \bigcup_{\sigma \subset [r], \sigma \notin K} L_\sigma(C^n).$$  

Then it is easy to see that

$$Z_K(C^n, (C^n)^*) = C^{nr} \setminus L_n(\Sigma), \text{ where } (C^n)^* = C^n \setminus \{0_n\}. \square$$

**Homogenous coordinates of toric varieties**  Next we recall the basic facts about homogenous coordinates on toric varieties.

**Definition 2.2.** Let $\Sigma$ be a fan in $\mathbb{R}^m$ such that $\{0_m\} \not\subset \Sigma$, and let

$$\Sigma(1) = \{\rho_1, \cdots, \rho_r\}$$

denote the set of all one dimensional cones in $\Sigma$.

(i) For each $1 \leq k \leq r$, we denote by $n_k \in \mathbb{Z}^m$ the primitive generator of $\rho_k$, such that $\rho_k \cap \mathbb{Z}^m = \mathbb{Z}_{\geq 0} \cdot n_k$. Note that $\rho_k = \text{Cone}(n_k)$.

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3Let $K$ be some set of subsets of $[r]$. Then the set $K$ is called an abstract simplicial complex on the index set $[r]$ if the following condition holds: if $\tau \subset \sigma$ and $\sigma \in K$, then $\tau \in K$. In this paper by a simplicial complex $K$ we always mean an abstract simplicial complex, and we always assume that a simplicial complex $K$ contains the empty set $\emptyset$.  

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(ii) Let \( K_\Sigma \) denote the underlying simplicial complex of \( \Sigma \) defined by
\[
(2.8) \quad K_\Sigma = \left\{ \{i_1, \ldots, i_s\} \subseteq [r] : \text{\( n_i \) span a cone in } \Sigma \right\}.
\]
It is easy to see that \( K_\Sigma \) is a simplicial complex on the index set \([r]\).

(iii) Define the subgroup \( G_\Sigma \subseteq T_C = (\mathbb{C}^*)^r \) by
\[
(2.9) \quad G_\Sigma = \{ (\mu_1, \ldots, \mu_r) \in T_C : \prod_{k=1}^r (\mu_k)^{\langle n_k, m \rangle} = 1 \text{ for all } m \in \mathbb{Z}^m \},
\]
where \( \langle u, v \rangle = \sum_{k=1}^m u_kv_k \) for \( u = (u_1, \ldots, u_m) \) and \( v = (v_1, \ldots, v_m) \in \mathbb{R}^m \).

(iv) Now consider the natural \( G_\Sigma \)-action on \( \mathbb{Z}^{K_\Sigma(\mathbb{C}, (\mathbb{C}^*))} \) given by coordinate-wise multiplication, i.e.
\[
(2.10) \quad (\mu_1, \ldots, \mu_r) \cdot (x_1, \ldots, x_r) = (\mu_1x_1, \ldots, \mu_rx_r)
\]
for \( (\mu_1, \ldots, \mu_r), (x_1, \ldots, x_r) \in G_\Sigma \times \mathbb{Z}^{K_\Sigma(\mathbb{C}, (\mathbb{C}^*))} \), where we set
\[
(2.11) \quad \mu x = (\mu x_1, \ldots, \mu x_r). \quad \text{if } (\mu, x) = (\mu, (x_1, \ldots, x_r)) \in \mathbb{C} \times \mathbb{C}.
\]
Then define the space \( X_\Sigma(n) \) by the corresponding orbit space
\[
(2.12) \quad X_\Sigma(n) = \mathbb{Z}^{K_\Sigma(\mathbb{C}, (\mathbb{C}^*))}/G_\Sigma.
\]

**Remark 2.3.** (i) Let \( \Sigma \) be a fan in \( \mathbb{R}^m \) as in (2.7). Then the fan \( \Sigma \) is completely determined by the pair \( (K_\Sigma, \{n_k\}_{k=1}^r) \) (see [25, Remark 2.3] in detail).

(ii) Note that the group \( G_\Sigma \) acts on \( \mathbb{Z}^{K_\Sigma(\mathbb{C}, (\mathbb{C}^*))} \) freely (see Corollary 3.4). Moreover, one can show that \( X_\Sigma(n) \) is a toric variety (see Remark 5.11). \( \square \)

The following theorem plays a crucial role in the proof of the main result of this paper.

**Theorem 2.4** ([7], Theorem 2.1; [8], Theorem 3.1). Let \( \Sigma \) be a fan in \( \mathbb{R}^m \) as in Definition 2.2 and suppose that the set \( \{n_k\}_{k=1}^r \) of all primitive generators spans \( \mathbb{R}^m \) (i.e. \( \sum_{k=1}^r \mathbb{R} \cdot n_k = \{\sum_{k=1}^r \lambda_k n_k : \lambda_k \in \mathbb{R}\} = \mathbb{R}^m \)).

(i) Then there is a natural isomorphism
\[
(2.13) \quad X_\Sigma \cong \mathbb{Z}^{K_\Sigma(\mathbb{C}, \mathbb{C}^*)}/G_\Sigma = X_\Sigma(1).
\]

(ii) If \( f : \mathbb{CP}^s \rightarrow X_\Sigma \) is a holomorphic map, then there exists an \( r \)-tuple \( D = (d_1, \ldots, d_r) \in (\mathbb{Z}_{\geq 0})^r \) of non-negative integers satisfying the condition
∑_{k=1}^{r} d_k n_k = 0_m, and homogenous polynomials \( f_i \in \mathbb{C}[z_0, \ldots, z_s] \) of degree \( d_i \) (\( i = 1, 2, \ldots, r \)) such that the polynomials \( \{f_i\}_{i \in \sigma} \) have no common root except \( 0_{s+1} \in \mathbb{C}^{s+1} \) for each \( \sigma \in I(K_\Sigma) \) and that the diagram

\[
\begin{array}{ccc}
\mathbb{C}^{s+1} \setminus \{0_{s+1}\} & \xrightarrow{(f_1, \ldots, f_r)} & Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*) \\
\gamma_* \downarrow & & \downarrow q_* \\
\mathbb{CP}^s & \xrightarrow{f} & Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma
\end{array}
\]

is commutative, where we identify \( X_\Sigma = X_\Sigma(1) \) as in (2.13) and the two map \( \gamma_* : \mathbb{C}^{s+1} \setminus \{0_{s+1}\} \to \mathbb{CP}^s \) and \( q_* : Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*) \to X_\Sigma = X_\Sigma(1) \) denote the canonical Hopf fibering and the canonical projection induced from the identification (2.13), respectively. In this case, we call this holomorphic map \( f \) a holomorphic map of degree \( D = (d_1, \ldots, d_r) \) and we represent it as

\[
f = [f_1, \ldots, f_r].
\]

(iii) If \( g_i \in \mathbb{C}[z_0, \ldots, z_s] \) is a homogenous polynomial of degree \( d_i \) (\( 1 \leq i \leq r \)) such that \( f = [f_1, \ldots, f_r] = [g_1, \ldots, g_r] \), there exists some element \( (\mu_1, \ldots, \mu_r) \in G_\Sigma \) such that \( f_i = \mu_i \cdot g_i \) for each \( 1 \leq i \leq r \). Thus, the \( r \)-tuple \((f_1, \ldots, f_r)\) of homogenous polynomials representing a holomorphic map \( f \) is determined uniquely up to \( G_\Sigma \)-action.

Assumptions  From now on, let \( \Sigma \) be a fan in \( \mathbb{R}^m \) satisfying the condition \( (2.7) \) as in Definition 2.2 and assume that \( X_\Sigma \) is simply connected and non-singular. Moreover, we shall assume the following condition holds.

(2.15.1) There is an \( r \)-tuple \( D_* = (d'_1, \ldots, d'_r) \in \mathbb{N}^r \) such that \( \sum_{k=1}^{r} d'_k n_k = 0_m \).

Remark 2.5. It follows from [9, Theorem 12.1.10] that \( X_\Sigma \) is simply connected if and only if the fan \( \Sigma \) satisfies the following condition (*):

(*) The set \( \{n_k\}_{k=1}^{r} \) of all primitive generators spans \( \mathbb{Z}^m \) over \( \mathbb{Z} \), i.e.

\[
\sum_{k=1}^{r} \mathbb{Z} \cdot n_k = \mathbb{Z}^m.
\]

Thus, one can easily see that the set \( \{n_k\}_{k=1}^{r} \) of all primitive generators spans \( \mathbb{R}^m \) if \( X_\Sigma \) is simply connected. In particular, we can see that \( X_\Sigma \) is simply connected if \( X_\Sigma \) is a compact smooth toric variety (see Lemma 3.6).

Spaces of holomorphic maps  We let \( X_\Sigma \) be a smooth toric variety and make the identification \( X_\Sigma = Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma(1) \). Now consider a base point preserving holomorphic map \( f = [f_1, \ldots, f_r] : \mathbb{CP}^s \to X_\Sigma \) for the case
s = 1. In this case, we make the identification \( \mathbb{CP}^1 = S^2 = \mathbb{C} \cup \infty \) and choose the points \( \infty \) and \([1, 1, \cdots, 1]\) as the base points of \( \mathbb{CP}^1 \) and \( X_\Sigma \) respectively. Then, by setting \( z = \frac{w}{\bar{w}} \), for each \( 1 \leq k \leq r \), we can view \( f_k \) as a monic polynomial \( f_k(z) \in \mathbb{C}[z] \) of degree \( d_k \) in the complex variable \( z \). Now we can define the space of holomorphic maps as follows.

**Definition 2.6.** (i) Let \( P^d \) denote the space of all monic polynomials \( g(z) = z^d + a_1z^{d-1} + \cdots + a_{d-1}z + a_d \in \mathbb{C}[z] \) of degree \( d \), and we set

\[
P^D = P^{d_1} \times P^{d_2} \times \cdots \times P^{d_r} \quad \text{if} \quad D = (d_1, \cdots, d_r) \in \mathbb{N}^r.
\]

Note that there is a natural homeomorphism \( \phi : P^d \cong \mathbb{C}^d \) given by \( \phi(z^d + \sum_{k=1}^{d} a_kz^{d-k}) = (a_1, \cdots, a_d) \in \mathbb{C}^d \).

(ii) For any \( r \)-tuple \( D = (d_1, \cdots, d_r) \in \mathbb{N}^r \) satisfying the condition \( (2.15) \), let \( \text{Hol}_D^*(S^2, X_\Sigma) \) denote the space consisting of all \( r \)-tuples \( f = (f_1(z), \cdots, f_r(z)) \in P^D \) satisfying the following condition (\( 1_\Sigma \)):

\[
(\hat{1}_\Sigma) \text{ For any } \sigma = \{i_1, \cdots, i_s\} \in \mathcal{I}(K_\Sigma), \text{ the polynomials } f_{i_1}(z), \cdots, f_{i_s}(z) \text{ have no common root, i.e. } (f_{i_1}(\alpha), \cdots, f_{i_s}(\alpha)) \neq 0_s = (0, \cdots, 0) \text{ for any } \alpha \in \mathbb{C}.
\]

By identifying \( X_\Sigma = Z_{K_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma \) and \( \mathbb{CP}^1 = S^2 = \mathbb{C} \cup \infty \), one can define the natural inclusion map \( i_D : \text{Hol}_D^*(S^2, X_\Sigma) \to \text{Map}^*(S^2, X_\Sigma) = \Omega^2 X_\Sigma \) by

\[
i_D(f)(\alpha) = \begin{cases} [f_1(\alpha), f_2(\alpha), \cdots, f_r(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1, 1, \cdots, 1] & \text{if } \alpha = \infty \end{cases}
\]

for \( f = (f_1(z), \cdots, f_r(z)) \in \text{Hol}_D^*(S^2, X_\Sigma) \), where we choose the points \( \infty \) and \([1, 1, \cdots, 1]\) as the base points of \( S^2 \) and \( X_\Sigma \).

Since the representation of polynomials in \( P^D \) representing a base point preserving holomorphic map of degree \( D \) is uniquely determined, the space \( \text{Hol}_D^*(S^2, X_\Sigma) \) can be identified with the space of base point preserving holomorphic maps of degree \( D \). Moreover, since \( \text{Hol}_D^*(S^2, X_\Sigma) \) is path-connected, the image of \( i_D \) is contained in a certain path-component of \( \Omega^2 X_\Sigma \), which is denoted by \( \Omega_D^2 X_\Sigma \). Thus we have a natural inclusion

\[
i_D : \text{Hol}_D^*(S^2, X_\Sigma) \to \text{Map}_D^*(S^2, X_\Sigma) = \Omega_D^2 X_\Sigma. \quad \Box
\]

**Spaces of non-resultant systems of bounded multiplicity** Now consider the space \( \text{Poly}_{n,D}^{\Sigma, \Omega}(\mathbb{F}) \) for \( \mathbb{F} = \mathbb{C} \). For this purpose, we need the following notation.

\[\text{By [25, Remark 2.10]}, we see that } \text{Hol}_D^*(S^2, X_\Sigma) \text{ is path-connected.}\]
**Definition 2.7.** For a monic polynomial \( f(z) \in \mathbb{P}^d \) of degree \( d \), let \( F_n(f)(z) \) denote the \( n \)-tuple of monic polynomials of the same degree \( d \) given by
\[
F_n(f)(z) = (f(z), f(z) + f'(z), f(z) + f''(z), \ldots, f(z) + f^{(n-1)}(z))
\]
as in (1.1). Note that a monic polynomial \( f(z) \in \mathbb{P}^d \) has a root \( \alpha \in \mathbb{C} \) of multiplicity \( \geq n \) iff \( F_n(f)(\alpha) = 0_n \in \mathbb{C}^n \).

Then the space \( \text{Poly}_{n}^{D,\Sigma} = \text{Poly}_{n}^{D,\Sigma}(\mathbb{C}) \) can be redefined as follows.

**Definition 2.8.** (i) For each \( D = (d_1, \ldots, d_r) \in \mathbb{N}^r \), \( n \in \mathbb{N} \) and a fan \( \Sigma \) in \( \mathbb{R}^m \), let \( \text{Poly}_{n}^{D,\Sigma} \) denote the space of \( r \)-tuples \( (f_1(z), \ldots, f_r(z)) \in \mathbb{P}^D \) of \( \mathbb{C} \)-coefficients monic polynomials satisfying the following condition \((\dag_{\Sigma,n})\):
\[
(\dag_{\Sigma,n}) \quad \text{For any } \sigma = \{i_1, \ldots, i_s\} \in I(\mathcal{K}_\Sigma) \text{, polynomials } f_{i_1}(z), \ldots, f_{i_s}(z) \text{ have no common root of multiplicity } \geq n \text{ (but they may have common roots of multiplicity } < n).\]
Note that the condition \((\dag_{\Sigma})\) coincides with the condition \((\dag_{\Sigma,n})\) if \( n = 1 \).

(ii) When \( \sum_{k=1}^r d_k n_k = 0_n \), define the map \( i_D : \text{Poly}_{n}^{D,\Sigma} \rightarrow \Omega^2 X_{\Sigma}(n) \) by
\[
i_D(f)(\alpha) = \begin{cases} [F_n(f_1)(\alpha), F_n(f_2)(\alpha), \ldots, F_n(f_r)(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [e, e, \ldots, e] & \text{if } \alpha = \infty \end{cases}
\]
for \( f = (f_1(z), \ldots, f_r(z)) \in \text{Poly}_{n}^{D,\Sigma} \) and \( \alpha \in \mathbb{C} \cup \infty = S^2 \), where the space \( X_{\Sigma}(n) \) is the space defined as in (2.12) and we set \( e = (1,1,\ldots,1) \in \mathbb{C}^n \).

Since \( \text{Poly}_{n}^{D,\Sigma} \) is connected, the image of \( i_D \) is contained some path-component of \( \Omega^2 X_{\Sigma}(n) \), which is denoted by \( \Omega^2 D X_{\Sigma}(n) \). Thus we have the map
\[
i_D : \text{Poly}_{n}^{D,\Sigma} \rightarrow \Omega^2 D X_{\Sigma}(n). \n\]

The numbers \( r_{\text{min}}(\Sigma) \) and \( d(D;\Sigma,n) \) Before stating the main results of this paper, we need to define the positive integers \( r_{\text{min}}(\Sigma) \) and \( d(D;\Sigma,n) \).

**Definition 2.9.** We say that a set \( S = \{n_{i_1}, \ldots, n_{i_s}\} \) is a primitive if \( \text{Cone}(S) \notin \Sigma \) and \( \text{Cone}(T) \in \Sigma \) for any proper subset \( T \subseteq S \). Then we define \( d(D;\Sigma,n) \) to be the positive integer given by
\[
d(D;\Sigma,n) = (2nr_{\text{min}}(\Sigma) - 3) \left[ \frac{d_{\text{min}}}{n} \right] - 2,
\]
where \( r_{\text{min}}(\Sigma) \) and \( d_{\text{min}} = d_{\text{min}}(D) \) are the positive integers given by
\[
r_{\text{min}}(\Sigma) = \min \{ s \in \mathbb{N} : \{n_{i_1}, \ldots, n_{i_s}\} \text{ is primitive} \}, \quad d_{\text{min}} = d_{\text{min}}(D) = \min \{d_{1}, d_{2}, \ldots, d_{r} \}. \]

\(^{5}\)See Remark 2.4
For connected space $X$, let $\Omega^2_0 X$ denote the path-component of $\Omega^2 X$ which contains null-homotopic maps and recall the following result.

**Theorem 2.10** ([23]). Let $X_\Sigma$ be an $m$ dimensional simply connected non-singular toric variety such that the condition (2.13.1) holds. Then if $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ and $\sum_{k=1}^r d_k n_k = 0$, the inclusion map

$$i_D : \text{Hol}^*_D(S^2, X_\Sigma) \to \Omega^2_D X_\Sigma \simeq \Omega^2_0 X_\Sigma \simeq \Omega^2 Z_{K_\Sigma}$$

is a homotopy equivalence through dimension $d(D; \Sigma, 1) = (2r_{\min}(\Sigma) - 3)d_{\min} - 2$ if $r_{\min}(\Sigma) \geq 3$ and a homology equivalence through dimension $d(D; \Sigma, 1) = d_{\min} - 2$ if $r_{\min}(\Sigma) = 2$, where $Z_K$ denotes the moment-angle complex of a simplicial complex $K$.

The main results  The main result of this paper is a generalization of the above theorem (Theorem 2.10) to spaces of non-resultant systems of bounded multiplicity.

**Theorem 2.11.** Let $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, $n \geq 2$ and let $X_\Sigma$ be an $m$ dimensional simply connected non-singular toric variety such that the condition (2.13.1) holds.

(i) If $\sum_{k=1}^r d_k n_k = 0$, then the natural map

$$i_D : \text{Poly}^n_D X_\Sigma(n) \to \Omega^2_D X_\Sigma(n) \simeq \Omega^2_0 X_\Sigma(n) \simeq \Omega^2 Z_{K_\Sigma}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through dimension $d(D; \Sigma, n)$.

(ii) If $\sum_{k=1}^r d_k n_k \neq 0$, there is a map

$$j_D : \text{Poly}^n_D X_\Sigma(n) \to \Omega^2 Z_{K_\Sigma}(D^{2n}, S^{2n-1})$$

which is a homotopy equivalence through dimension $d(D; \Sigma, n)$.

**Corollary 2.12.** Let $n \geq 2$, $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, and let $X_\Sigma$ be an $m$ dimensional compact smooth toric variety over $\mathbb{C}$ such that the condition (2.13.1) holds. Let $\Sigma(1)$ denote the set of all one dimensional cones in $\Sigma$, and let $\Sigma_1$ be any fan in $\mathbb{R}^m$ such that $\Sigma(1) \subset \Sigma_1 \subset \Sigma$.

Then $X_{\Sigma_1}$ is a non-compact smooth toric subvariety of $X_\Sigma$ and the following two statements hold:

(i) If $\sum_{k=1}^r d_k n_k = 0$, the map

$$i_D : \text{Poly}^n_D \Sigma_1 \to \Omega^2_D X_{\Sigma_1}(n) \simeq \Omega^2 Z_{K_{\Sigma_1}}(D^{2n}, S^{2n-1})$$

is a homotopy equivalence through the dimension $d(D; \Sigma_1, n)$.

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6See Definition 3.1.
Example 2.14. Let \( R \) example the case where \( X \) given in Example 2.13. Then for an integer \( r \) defined by

\[
\sum_{k=1}^{r} d_{k} n_{k} \neq 0, \text{there is a map}
\]

\[
j_{D} : \text{Poly}_{n}^{D, \Sigma} \rightarrow \Omega^{2} \mathcal{Z}_{K_{\Sigma}}(D^{2n}, S^{2n-1})
\]

which is a homotopy equivalence through dimension \( d(D; \Sigma, n) \).

Since the case \( X_{\Sigma} = \mathbb{CP}^{n} \) was treated in [25] and [22], we will take as an example the case where \( X_{\Sigma} \) is the Hirzerbruch surface \( H(k) \).

**Definition 2.13.** For an integer \( k \in \mathbb{Z} \), let \( H(k) \) be the Hirzerbruch surface defined by

\[
H(k) = \{ ([x_{0} : x_{1} : x_{2}], [y_{1} : y_{2}) \in \mathbb{CP}^{2} \times \mathbb{CP}^{1} : x_{1}y_{1}^{k} = x_{2}y_{2}^{k} \} \subset \mathbb{CP}^{2} \times \mathbb{CP}^{1}.
\]

Since there are isomorphisms \( H(-k) \cong H(k) \) for \( k \neq 0 \) and \( H(0) \cong \mathbb{CP}^{1} \times \mathbb{CP}^{1} \), without loss of generality we can assume that \( k \geq 1 \). Let \( \Sigma_{k} \) denote the fan in \( \mathbb{R}^{2} \) given by

\[
\Sigma_{k} = \{ \text{Cone}(n_{1}, n_{i+1}) (1 \leq i \leq 3), \text{Cone}(n_{4}, n_{1}), \text{Cone}(n_{j}) (1 \leq j \leq 4), \{ 0 \} \},
\]

where we set \( n_{1} = (1, 0), n_{2} = (0, 1), n_{3} = (-1, k), n_{4} = (0, -1) \).

It is well-known that \( \Sigma_{k} \) is the fan of the toric variety \( H(k) \) and that the set of all one dimensional cones in \( \Sigma_{k} \) is \( \Sigma_{k}(1) = \{ \text{Cone}(n_{i}) : 1 \leq i \leq 4 \} \). Since \( \{ n_{1}, n_{3} \} \) and \( \{ n_{2}, n_{4} \} \) are the only primitive collections, \( r_{\text{min}}(\Sigma_{k}) = 2 \). Moreover, for a 4-tuple \( D = (d_{1}, d_{2}, d_{3}, d_{4}) \in \mathbb{N}^{4} \), the equality \( \sum_{k=1}^{4} d_{k} n_{k} = 0 \) holds if and only if \( (d_{1}, d_{2}, d_{3}, d_{4}) = (d_{1}, d_{2}, d_{1}, k d_{1} + d_{2}) \), and \( d_{\text{min}} = \min\{d_{1}, d_{2}, d_{3}, d_{4}\} = \min\{d_{1}, d_{2}\} \).

Hence, by Corollary 2.12 we have the following:

**Example 2.14.** Let \( k \geq 1 \) and \( n \geq 2 \) be positive integers. Let \( \Sigma \) be a fan in \( \mathbb{R}^{2} \) such that \( \Sigma_{k}(1) = \{ \text{Cone}(n_{i}) : 1 \leq i \leq 4 \} \subset \Sigma \not\subset \Sigma_{k} \), where \( \Sigma_{k} \) is the fan given in Example 2.13. Then \( X_{\Sigma} \) is a non-compact non-singular subvariety of \( H(k) \) and the following two statements hold.

(i) If \( D = (d_{1}, d_{2}, d_{1}, k d_{1} + d_{2}) \), the map

\[
i_{D} : \text{Poly}_{n}^{D, \Sigma} \rightarrow \Omega^{2} X_{\Sigma}(n) \simeq \Omega^{2} \mathcal{Z}_{K_{\Sigma}}(D^{2n}, S^{2n-1})
\]

is a homotopy equivalence through dimension \( (4n - 3)\left[ \frac{\min(d_{1}, d_{2})}{n} \right] - 2 \).

(ii) If \( D = (d_{1}, d_{2}, d_{3}, d_{4}) \in \mathbb{N}^{4} \), there is a map

\[
j_{D} : \text{Poly}_{n}^{D, \Sigma} \rightarrow \Omega^{2} \mathcal{Z}_{K_{\Sigma}}(D^{2n}, S^{2n-1})
\]

which is a homotopy equivalence through dimension \( (4n - 3)\left[ \frac{d_{\text{min}}}{n} \right] - 2 \), where we set \( d_{\text{min}} = \min\{d_{1}, d_{2}, d_{3}, d_{4}\} \).
3 Basic facts about polyhedral products

First, we recall some definitions and known results.

**Definition 3.1** ([6], Definition 6.27, Example 6.39). Let $K$ be a simplicial complex on the index set $[r]$. We denote by $\mathcal{Z}_K$ and $DJ(K)$ the moment-angle complex of $K$ and the Davis-Januszkiewicz space of $K$, respectively, which are defined by

\begin{equation}
\mathcal{Z}_K = \mathcal{Z}_K(D^2, S^1), \quad DJ(K) = \mathcal{Z}_K(\mathbb{C}P^\infty, *)).
\end{equation}

**Lemma 3.2** ([6]; Corollary 6.30, Theorem 6.33, Theorem 8.9). Let $K$ be a simplicial complex on the index set $[r]$.

(i) The space $\mathcal{Z}_K$ is 2-connected, and there is a fibration sequence

\begin{equation}
\mathcal{Z}_K \rightarrow DJ(K) \xrightarrow{\subset} \rightarrow (\mathbb{C}P^\infty)^r.
\end{equation}

(ii) There is an $(S^1)^r$-equivariant deformation retraction

\begin{equation}
\text{ret} : \mathcal{Z}_K(\mathbb{C}^n, (\mathbb{C}^n)^*) \xrightarrow{\sim} \mathcal{Z}_K(D^{2n}, S^{2n-1}).
\end{equation}

**Lemma 3.3** ([31]; (6.2) and Proposition 6.7). Let $\Sigma$ be a fan in $\mathbb{R}^m$ and let $X_\Sigma$ be a smooth toric variety such that the condition (2.15.1) holds. Then there is an isomorphism

\begin{equation}
G_\Sigma \cong T_{\mathbb{C}}^{r-m} = (\mathbb{C}^*)^{r-m},
\end{equation}

Moreover, the group $G_\Sigma$ acts on the space $\mathcal{Z}_{K_\Sigma}(\mathbb{C}, (\mathbb{C}^*)^*)$ freely by the coordinate-wise action and there is a principal $G_\Sigma$-bundle sequence

\begin{equation}
G_\Sigma \rightarrow \mathcal{Z}_{K_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \xrightarrow{q_\Sigma} X_\Sigma.
\end{equation}

**Corollary 3.4.** The group $G_\Sigma$ acts on the space $\mathcal{Z}_{K_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ freely and there is a principal $G_\Sigma$-bundle sequence

\begin{equation}
G_\Sigma \rightarrow \mathcal{Z}_{K_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \xrightarrow{q_\Sigma} X_\Sigma(n).
\end{equation}

Proof. If $n = 1$, the assertion follows from Lemma 3.3 and assume that $n \geq 2$. Since the action of $G_\Sigma$ on $\mathcal{Z}_{K_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)$ is the diagonal one of the case $n = 1$, this action is also free and we obtain the desired assertion.

**Lemma 3.5.** If the condition (2.14) is satisfied, the space $X_\Sigma$ is simply connected and $\pi_2(X_\Sigma) = \mathbb{Z}^{r-m}$.

Proof. This follows from [25] Lemma 3.4.

**Lemma 3.6** ([9]). Let $X_\Sigma$ be a toric variety determined by a fan $\Sigma$ in $\mathbb{R}^m$. Then $X_\Sigma$ is compact if and only if $\mathbb{R}^m = \bigcup_{\sigma \in \Sigma} \sigma$. 

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13
4 The Vassiliev spectral sequence

First, recall the definitions of the non-degenerate simplicial resolution and the associated truncated simplicial resolution ([28], [33]).

**Definition 4.1.** (i) For a finite set \( v = \{v_1, \ldots, v_l\} \subset \mathbb{R}^N \), let \( \sigma(v) \) denote the convex hull spanned by \( v \). Let \( h : X \to Y \) be a surjective map such that \( h^{-1}(y) \) is a finite set for any \( y \in Y \), and let \( i : X \to \mathbb{R}^N \) be an embedding. Let \( \mathcal{X}^\Delta \) and \( h^\Delta : \mathcal{X}^\Delta \to Y \) denote the space and the map defined by

\[
\mathcal{X}^\Delta = \{(y, u) \in Y \times \mathbb{R}^N : u \in \sigma(i(h^{-1}(y))) \} \subset Y \times \mathbb{R}^N, \quad h^\Delta(y, u) = y.
\]

The pair \((\mathcal{X}^\Delta, h^\Delta)\) is called the simplicial resolution of \((h, i)\). In particular, it is called a non-degenerate simplicial resolution if for each \( y \in Y \) any \( k \) points of \( i(h^{-1}(y)) \) span \((k - 1)\)-dimensional simplex of \( \mathbb{R}^N \).

(ii) For each \( k \geq 0 \), let \( \mathcal{X}^\Delta_k \subset \mathcal{X}^\Delta \) be the subspace of the union of the \((k - 1)\)-skeletons of the simplices over all the points \( y \) in \( Y \) given by

\[
\mathcal{X}^\Delta_k = \{(y, u) \in \mathcal{X}^\Delta : u \in \sigma(v), \quad v = \{v_1, \ldots, v_l\} \subset i(h^{-1}(y)), \quad l \leq k \}.
\]

We make the identification \( X = \mathcal{X}^\Delta_1 \) by identifying \( x \in X \) with the pair \((h(x), i(x)) \in \mathcal{X}^\Delta \), and we note that there is an increasing filtration

\[
\emptyset = \mathcal{X}^\Delta_0 \subset X = \mathcal{X}^\Delta_1 \subset \mathcal{X}^\Delta_2 \subset \cdots \subset \mathcal{X}^\Delta_k \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}^\Delta_k = \mathcal{X}^\Delta.
\]

Since the map \( h^\Delta : \mathcal{X}^\Delta \to Y \) is a proper map, it extends to the map \( h^\Delta_+ : \mathcal{X}^\Delta_+ \to Y_+ \) between the one-point compactifications, where \( X_+ \) denotes the one-point compactification of a locally compact space \( X \).

**Lemma 4.2** ([33], [34] (cf. Lemma 3.3 in [23])). Let \( h : X \to Y \) be a surjective map such that \( h^{-1}(y) \) is a finite set for any \( y \in Y \), and let \( i : X \to \mathbb{R}^N \) be an embedding. Then if \( X \) and \( Y \) are semi-algebraic spaces and the two maps \( h, i \) are semi-algebraic maps, then the map \( h^\Delta_+ : \mathcal{X}^\Delta_+ \xrightarrow{\sim} Y_+ \) is a homology equivalence.

**Proof.** The assertion follows from [33] Lemma 1 (page 90)].

**Remark 4.3.** Under the same assumption as Lemma 4.2, there exists always a non-degenerate simplicial resolution of the map \( h \). In fact, even for a surjective map \( h : X \to Y \) which is not finite to one, it is still possible to construct an associated non-degenerate simplicial resolution. See [25] Remark 6.4 in detail.

---

\( ^{7} \)It is known that \( h^\Delta_+ \) is actually a homotopy equivalence [34] page 156]. However, in this paper we do not need this stronger assertion.
Definition 4.4. Let $h : X \to Y$ be a surjective semi-algebraic map between semi-algebraic spaces, $j : X \to \mathbb{R}^N$ be a semi-algebraic embedding, and let $(\mathcal{X}^\Delta, h^\Delta : \mathcal{X}^\Delta \to Y)$ denote the associated non-degenerate simplicial resolution of $(h, j)$. Then for each positive integer $k \geq 1$, we denote by $h^\Delta_k : X^\Delta(k) \to Y$ the truncated (after the $k$-th term) simplicial resolution of $Y$ as in [29]. Note that there is a natural filtration $X^\Delta_0 \subset X^\Delta_1 \subset \cdots \subset X^\Delta_k \subset X^\Delta_{k+1} \subset \cdots \subset X^\Delta_l \subset X^\Delta_{l+1} = \cdots = X^\Delta(k)$, where $X^\Delta_0 = \emptyset$, $X^\Delta_l = X^\Delta_l$ if $l \leq k$ and $X^\Delta_l = X^\Delta(k)$ if $l > k$.

Lemma 4.5 ([29], cf. Remark 2.4 and Lemma 2.5 in [21]). Under the same assumptions as in Definition 4.4, the map $h^\Delta_k : X^\Delta(k) \to Y$ is a homotopy equivalence.

Next, we construct the Vassiliev spectral sequence. From now on, we always assume that $\Sigma$ is a fan in $\mathbb{R}^m$ such that $X_\Sigma$ is simply connected and that the condition (2.15) is satisfied. Moreover, $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$ will always denote a fixed $r$-tuple of positive integers.

Definition 4.6. (i) Let $\Sigma_D$ denote the discriminant of $\text{Poly}_{n,\Sigma}^D$ in $\mathbb{P}^D$ given by the complement

$$\Sigma_D = \mathbb{P}^D \setminus \text{Poly}_{n,\Sigma}^D$$

$$= \{(f_1(z), \cdots, f_r(z)) \in \mathbb{P}^D : (f_1(x), \cdots, f_r(x)) \in L_n(\Sigma) \text{ for some } x \in \mathbb{C}\},$$

where $L_n(\Sigma) = \bigcup_{\sigma \in I(\kappa_\Sigma)} L_\sigma(\mathbb{C}^n) = \bigcup_{\sigma \subset [r], \sigma \notin K_\Sigma} L_\sigma(\mathbb{C}^n)$ as in (2.5).

(ii) Let $Z_D \subset \Sigma_D \times \mathbb{C}$ denote the tautological normalization of $\Sigma_D$ consisting of all pairs $(G, x) = ((f_1(z), \cdots, f_r(z)), x) \in \Sigma_D \times \mathbb{C}$ satisfying the condition $F(x) = (F_n(f_1)(x), \cdots, F_n(f_r)(x)) \in L_n(\Sigma)$. Projection on the first factor gives a surjective map $\pi_D : Z_D \to \Sigma_D$.

Remark 4.7. Let $\sigma_k \in [r]$ for $k = 1, 2$. It is easy to see that $L_{\sigma_1}(\mathbb{C}^n) \subset L_{\sigma_2}(\mathbb{C}^n)$ if $\sigma_1 \subset \sigma_2$. Letting

$$Pr(\Sigma) = \{\sigma = \{i_1, \cdots, i_s\} \subset [r] : \{n_{i_1}, \cdots, n_{i_s}\} \text{ is a primitive collection}\},$$

we see that

$$L_n(\Sigma) = \bigcup_{\sigma \in Pr(\Sigma)} L_\sigma(\mathbb{C}^n)$$

and by using (2.23) we obtain the equality

$$\dim L_n(\Sigma) = 2n(r - r_{\min}(\Sigma)).$$
Our goal in this section is to construct, by means of the non-degenerate simplicial resolution of the discriminant, a spectral sequence converging to the homology of $\text{Poly}_D^\Delta$. 

**Definition 4.8.** (i) For an $r$-tuple $E = (e_1, \cdots, e_r) \in (\mathbb{Z}_{\geq 0})^r$ of non-negative integers, let $N(E)$ denote the non-negative integer given by

$$N(E) = \sum_{k=1}^{r} e_k.$$  

(ii) For each based space $X$, let $F(X, d)$ denote the ordered configuration space of distinct $d$ points in $X$ defined by

$$F(X, d) = \{(x_1, \cdots, x_d) \in X^d : x_i \neq x_j \text{ if } i \neq j\}.$$  

Since the symmetric group $S_d$ of $d$-letters acts on $F(X, d)$ freely by permuting coordinates and let $C_d(X)$ denote the unordered configuration space of $d$-distinct points in $X$ given by the orbit space

$$C_d(X) = F(X, d)/S_d.$$  

(iii) Let $L_k; \Sigma \subset (\mathbb{C} \times L_n(\Sigma))^k$ denote the subspace defined by

$$L_k; \Sigma = \{((x_1, s_1), \cdots, (x_k, s_k)) : x_j \in \mathbb{C}, s_j \in L_n(\Sigma), x_i \neq x_j \text{ if } i \neq j\}.$$  

The symmetric group $S_k$ on $k$ letters acts on $L_k; \Sigma$ by permuting coordinates. Let $C_k; \Sigma$ denote the orbit space

$$C_k; \Sigma = L_k; \Sigma/S_k.$$  

Note that $C_k; \Sigma$ is a cell-complex of dimension (by (4.5))

$$\text{dim } C_k; \Sigma = 2k (1 + nr - nr_{\text{min}}(\Sigma)).$$  

(iv) Let $(X^D, \pi_D : X^D \to \Sigma_D)$ be the non-degenerate simplicial resolution associated to the surjective map $\pi_D : Z_D \to \Sigma_D$ with the natural increasing filtration as in Definition 4.1.

$$\emptyset = X_0^D \subset X_1^D \subset X_2^D \subset \cdots \subset X^D = \bigcup_{k=0}^{\infty} X_k^D.$$  

By Lemma 4.2, the map $\pi_D^\Delta$ extends to a homology equivalence $\pi_D^\Delta : X^D_+ \to \Sigma_D$. Since $X_k^D/X_{k-1}^D \cong (X^D_k/X^D_{k-1})_+$, we have a spectral sequence

$$\{E_{t;D, s} : E_{t;D}^{k+s+1-t} \to H_{c}^{k+s}(\Sigma_D; \mathbb{Z})\}.$$
where $E_{k,s}^{1:D} = H_c^{k+s}(\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D; \mathbb{Z})$ and $H_c^k(X; \mathbb{Z})$ denotes the cohomology group with compact supports given by $H_c^k(X; \mathbb{Z}) = \hat{H}^k(X_+; \mathbb{Z})$.

Since there is a homeomorphism $P^n \cong \mathbb{C}^n$, by Alexander duality there is a natural isomorphism

$$(4.12) \quad \tilde{\tilde{H}}_k(\text{Poly}_n^{D,\Sigma}; \mathbb{Z}) \cong H_c^{2N(D) - k - 1}(\Sigma_D; \mathbb{Z}) \quad \text{for any } k.$$  

By reindexing we obtain a spectral sequence

$$(4.13) \quad \{ E_{k,s}^{1:D}, d^t : E_{k,s}^{t:D} \to E_{k+t,s+t-1}^{t:D} \} \Rightarrow H_{s-k}(\text{Poly}_n^{D,\Sigma}; \mathbb{Z}),$$

where $E_{k,s}^{1:D} = H_c^{2N(D) + k - s - 1}(\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D; \mathbb{Z})$.

**Lemma 4.9.** Let $d_{\min} = \min\{d_1, \ldots, d_r\}$ and suppose that $1 \leq k \leq \left\lfloor \frac{d_{\min}}{n} \right\rfloor$. Then the space $\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D$ is homeomorphic to the total space of a real affine bundle $\xi_{D,k,n}$ over $C_{k;\Sigma}$ with rank $l_{D,k,n} = 2N(D) - 2nrk + k - 1$.

**Proof.** Suppose that $1 \leq k \leq \left\lfloor \frac{d_{\min}}{n} \right\rfloor$. The argument is exactly analogous to the one in the proof of [1, Lemma 4.4]. Namely, an element of $\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D$ is represented by $(f, u) = ((f_1(z), \ldots, f_r(z)), u)$, where $f = (f_1(z), \ldots, f_r(z))$ is an $r$-tuple of monic polynomials in $\Sigma_D$ satisfying the condition

$$(4.14) \quad \mathbf{F}(x_j) = (F_n(f_1)(x_j), \ldots, F_n(f_r)(x_j)) \in L_n(\Sigma) \quad \text{for each } 1 \leq j \leq k$$

and $u$ is an element of the interior of the span of the images of some $k$ distinct points $\{x_1, \ldots, x_k\} \in C_k(\mathbb{C})$ under a suitable embedding. By using

[26, Lemma 2.5] one can show that the $k$ distinct points $\{x_j\}_{j=1}^k$ are uniquely determined by $u$. Thus there are projection maps $\pi_{k,D} : \mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D \to C_{k;\Sigma}$ defined by $((f_1(z), \ldots, f_r(z)), u) \mapsto \{(x_1, \mathbf{F}(x_1)), \ldots, (x_k, \mathbf{F}(x_k))\}$.

Let $c = \{(x_j, s_j)\}_{j=1}^k \subseteq C_{k;\Sigma}$, $(x_j \in \mathbb{C}, s_j \in L_n(\Sigma))$ be any fixed element and consider the fibre $\mathbf{F}(x_j)$. For this purpose, we write $s_j = (s_{1,j}, \ldots, s_{r,j})$ for each $1 \leq j \leq k$ with $s_{i,j} \in \mathbb{C}^n$ and consider the $r$-tuple $f = (f_1(z), \ldots, f_r(z)) \in \mathbb{P}^D$ of monic polynomials satisfying the condition

$$(4.15) \quad \mathbf{F}(x_j) = (F_n(f_1)(x_j), \ldots, F_n(f_r)(x_j)) = s_j \quad \text{for each } 1 \leq j \leq k \quad \Leftrightarrow \quad F_n(f_t)(x_j) = s_{t,j} \quad \text{for each } 1 \leq j \leq k \text{ and } 1 \leq t \leq r.$$  

If we set $s_{t,j} = (s_{t,j}^{(0)}, s_{t,j}^{(1)} + s_{t,j}^{(1)}, s_{t,j}^{(2)} + s_{t,j}^{(1)}, \ldots, s_{t,j}^{(0)} + s_{t,j}^{(n-1)}) \subseteq \mathbb{C}^n$ with $s_{t,j}^{(l)} \in \mathbb{C}$,

$$(4.16) \quad F_n(f_t)(x_j) = s_{t,j} \quad \Leftrightarrow \quad f_t^{(l)}(x_j) = s_{t,j}^{(l)} \quad \text{for each } 0 \leq l \leq n - 1.$$  

In general, the condition $f_t^{(l)}(x_j) = s_{t,j}^{(l)}$ gives one linear condition on the coefficients of $f_t$, and determines an affine hyperplanes in $\mathbb{P}^d(\mathbb{C})$. Indeed, if
we set \( f_i(z) = z^{d_i} + \sum_{i=0}^{d_i-1} a_i z^i \), then \( f_i(x_j) = s_{i,j}^{(0)} \) for any \( 1 \leq j \leq k \) if and only if \( A_1 \mathbf{x} = \mathbf{b}_1 \), where

\[
A_1 = \begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{d_1-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{d_2-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_k & x_k^2 & \cdots & x_k^{d_k-1}
\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d_1-1} \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} s_{1,1}^{(0)} - x_1^{d_1} \\ \vdots \\ s_{1,k}^{(0)} - x_k^{d_k} \end{bmatrix}
\]

Similarly, \( f_j''(x_j) = s_{i,j}^{(1)} \) for any \( 1 \leq j \leq k \) if and only if \( A_2 \mathbf{x} = \mathbf{b}_2 \), where

\[
A_2 = \begin{bmatrix}
0 & 1 & 2x_1 & 3x_1^2 & \cdots & (d_1-1)x_1^{d_1-2} \\
0 & 1 & 2x_2 & 3x_2^2 & \cdots & (d_2-1)x_2^{d_2-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 2x_k & 3x_k^2 & \cdots & (d_k-1)x_k^{d_k-2}
\end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} s_{1,1}^{(1)} - d_1x_1^{d_1-1} \\ \vdots \\ s_{1,k}^{(1)} - d_kx_k^{d_k-1} \end{bmatrix}
\]

and \( f_i'''(x_j) = s_{i,j}^{(2)} \) for any \( 1 \leq j \leq k \) if and only if \( A_3 \mathbf{x} = \mathbf{b}_3 \), where

\[
A_3 = \begin{bmatrix}
0 & 0 & 2 & 6x_1 & \cdots & (d_1-1)(d_1-2)x_1^{d_1-3} \\
0 & 0 & 2 & 6x_2 & \cdots & (d_2-1)(d_2-2)x_2^{d_2-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 2 & 6x_k & \cdots & (d_k-1)(d_k-2)x_k^{d_k-3}
\end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} s_{1,1}^{(2)} - d_1(d_1-1)x_1^{d_1-1} \\ \vdots \\ s_{1,k}^{(2)} - d_k(d_k-1)x_k^{d_k-1} \end{bmatrix}
\]

and so on. Since \( \{x_i\}_{i=1}^k \in C_k(\mathbb{C}) \), by Gaussian elimination of rows of matrices, the matrix \( A_1 \) reduces to the matrix \( B_1 \), where \( s_i(l) = \sum_{i_1+i_2+i_3+i_4=\cdots+i_i=i} x_1^{i_{1}} x_2^{i_{2}} \cdots x_l^{i_{i}} \)

and \( B_1 \) is the matrix given by

\[
\begin{bmatrix}
1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & \cdots & \cdots & x_1^{d_1-2} & x_1^{d_1-1} \\
0 & 1 & s_1(2) & s_2(2) & s_3(2) & s_4(2) & \cdots & \cdots & s_{d_1-3}(2) & s_{d_1-2}(2) \\
0 & 0 & 1 & s_1(3) & s_2(3) & s_3(3) & \cdots & \cdots & s_{d_1-4}(3) & s_{d_1-3}(3) \\
0 & 0 & 0 & 1 & s_1(4) & s_2(4) & \cdots & \cdots & s_{d_1-5}(4) & s_{d_1-4}(4) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & s_1(k) & s_2(k) & \cdots & s_{d_1-k}(k)
\end{bmatrix}
\]

Similarly, by easy Gaussian elimination of rows of matrices, the matrix \( A_2 \) reduces to the matrix \( B_2 \), where \( B_2 \) is the matrix of the following form

\[
\begin{bmatrix}
0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & \cdots & \cdots & (d_1-1)x_1^{d_1-2} \\
0 & 0 & 2 & 3s_1(2) & 4s_2(2) & 5s_3(2) & \cdots & \cdots & (d_1-1)s_{d_1-3}(2) \\
0 & 0 & 0 & 3 & 4s_1(3) & 5s_2(3) & \cdots & \cdots & (d_1-1)s_{d_1-4}(3) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & k-1 & ks_1(k) & \cdots & (d_1-1)s_{d_1-k-1}(k)
\end{bmatrix}
\]
Proof. Suppose that $1 \leq k \leq \left(\frac{d_{\min}}{n}\right)$. By Lemma 4.9, there is a homeomorphism $(X_k^D \setminus X_{k-1}^D)_+ \cong T(\xi_{D,k})$, where $T(\xi_{D,k,n})$ denotes the Thom space of $\xi_{D,k,n}$. Since $(2N(D) + k - s - 1) - l_{D,k,n} = 2nrk - s$, the Thom isomorphism gives a natural isomorphism $E^{1,d}_{k,s} \cong H_{c,2nrk-s}(C_{k};\Sigma; \pm \mathbb{Z})$.

**Lemma 4.10.** If $1 \leq k \leq \left(\frac{d_{\min}}{n}\right)$, there is a natural isomorphism

$$E^{1,D}_{k,s} \cong H_{c}^{2nrk-s}(C_{k};\Sigma; \pm \mathbb{Z}),$$

where the twisted coefficients system $\pm \mathbb{Z}$ comes from the Thom isomorphism.

**Proof.** Suppose that $1 \leq k \leq \left(\frac{d_{\min}}{n}\right)$. By Lemma 4.9, there is a homeomorphism $(X_k^D \setminus X_{k-1}^D)_+ \cong T(\xi_{D,k})$, where $T(\xi_{D,k,n})$ denotes the Thom space of $\xi_{D,k,n}$. Since $(2N(D) + k - s - 1) - l_{D,k,n} = 2nrk - s$, the Thom isomorphism gives a natural isomorphism $E^{1,d}_{k,s} \cong H_{c}^{2nrk-s}(C_{k};\Sigma; \pm \mathbb{Z})$.

**Definition 4.11.** For an $r$-tuple $D = (d_1, \cdots, d_r) \in \mathbb{N}^r$, let $U_D \subset \mathbb{C}$ denote the subspace defined by

$$(4.17) \quad U_D = \{ w \in \mathbb{C} : \text{Re}(w) < N(D) \},$$

and let $\varphi_D : \mathbb{C} \xrightarrow{\cong} U_D$ be any fixed homeomorphism. Moreover, we choose any mutually distinct $r$ points $x_1, \cdots, x_r \in \mathbb{C} \setminus U_D$ freely and fix them.

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(i) For each monic polynomial \( f(z) = \prod_{k=1}^{d}(z - \alpha_k) \in \mathbb{P}^d \) of degree \( d \), let \( \varphi_D(f) \) denote the monic polynomial of the same degree \( d \) given by

\[
(4.18) \quad \varphi_D(f) = \prod_{k=1}^{d}(z - \varphi_D(\alpha_k)) \in \mathbb{P}^d.
\]

(ii) For each \( r \)-tuple \( a = (a_1, \cdots, a_r) \in (\mathbb{Z}_{\geq 0})^r \) with \( a \neq 0_r \), define the stabilization map

\[
(4.19) \quad s_{D,D+a} : \text{Poly}_n^{D,\Sigma} \to \text{Poly}_{n+a}^{D,\Sigma} \quad \text{by}
\]

\[
(4.20) \quad s_{D,D+a}(f) = (\varphi_D(f_1)(z - x_1)^{a_1}, \cdots, \varphi_D(f_r)(z - x_r)^{a_r})
\]

for \( f = (f_1(z), \cdots, f_r(z)) \in \text{Poly}_n^{D,\Sigma}. \)

\[\square\]

**Remark 4.12.**

(i) Note that the definition of the map \( s_{D,D+a} \) depends on the choice of the homeomorphism \( \varphi_D \) and the points \( \{x_k : 1 \leq k \leq r\} \), but one shows that the homotopy type of it does not depend on these choices.

(ii) For each \( a, b \in (\mathbb{Z}_{\geq 0})^r \) be any two \( r \)-tuples such that \( a, b \neq 0_r \). Then it is easy to see that the equality

\[
(4.21) \quad (s_{D+a,D+a+b}) \circ (s_{D,D+a}) = s_{D,D+a+b} \quad \text{(up to homotopy)}
\]

holds. Thus we mainly only consider the stabilization map \( s_{D,D+e_i} \) for each \( 1 \leq i \leq r \), where \( e_1 = (1, 0, \cdots, 0) \), \( e_2 = (0, 1, 0, \cdots, 0) \), \( \cdots \), \( e_r = (0, 0, \cdots, 0, 1) \in \mathbb{R}^r \) denote the standard basis of \( \mathbb{R}^r \).

Now let \( 1 \leq i \leq r \) and consider the stabilization map

\[
(4.22) \quad s_{D,D+e_i} : \text{Poly}_n^{D,\Sigma} \to \text{Poly}_{n+e_i}^{D,\Sigma}.
\]

It is easy to see that it extends to an open embedding

\[
(4.23) \quad s_{D,i} : \mathbb{C} \times \text{Poly}_n^{D,\Sigma} \to \text{Poly}_{n+e_i}^{D,\Sigma}
\]

It also naturally extends to an open embedding \( \tilde{s}_{D,i} : P^D \to P^{D+e_i} \) and by restriction we obtain an open embedding

\[
(4.24) \quad \tilde{s}_{D,i} : \mathbb{C} \times \Sigma_D \to \Sigma_{D+e_i}.
\]

Since one-point compactification is contravariant for open embeddings, this map induces a map \( \tilde{s}_{D,i} : (\Sigma_{D+e_i})_+ \to (\mathbb{C} \times \Sigma_D)_+ = S^2 \wedge \Sigma_D^+ \) and we can easily see that the following diagram

\[
\begin{array}{ccc}
H_k(\text{Poly}_n^{D,\Sigma}; \mathbb{Z}) & \xrightarrow{s_{D,D+e_i}} & \tilde{H}_k(\text{Poly}_{n+e_i}^{D,\Sigma}; \mathbb{Z}) \\
\text{AD} \downarrow \cong & & \text{AD} \downarrow \cong \\
H_c^{2N(D)-k-1}(\Sigma_D; \mathbb{Z}) & \xrightarrow{\tilde{s}_{D,i}} & \tilde{H}_c^{2N(D)-k+1}(\Sigma_{D+e_i}; \mathbb{Z})
\end{array}
\]

(4.25)
is commutative, where $AD$ is the Alexander duality isomorphism and $\tilde{s}_{D,i+}$ denotes the composite of the suspension isomorphism with the homomorphism $(\tilde{s}_{D+})^*$,

$$H_c^M(\Sigma_D; \mathbb{Z}) \xrightarrow{\tilde{s}_{D,i+}} H_c^{M+2}(\mathbb{C} \times \Sigma_D; \mathbb{Z}) \xrightarrow{(\tilde{s}_{D+})^*} H_c^{M+2}(\Sigma_{D+e_i}; \mathbb{Z}),$$

where $M = 2N(D) - k - 1$. By the universality of the non-degenerate simplicial resolution \cite{[28]}, the map $\tilde{s}_{D,i}$ also naturally extends to a filtration preserving open embedding $\tilde{s}_{D,i} : \mathbb{C} \times \mathcal{X}^D \rightarrow \mathcal{X}^{D+e_i}$ between non-degenerate simplicial resolutions. This induces a filtration preserving map $(\tilde{s}_{D,i})_+ : \mathcal{X}^{D+e_i} \rightarrow (\mathbb{C} \times \mathcal{X}^D)_+ = S^2 \wedge \mathcal{X}^D$, and thus a homomorphism of spectral sequences

(4.26) \[
\{ \hat{\theta}^i_{k,s} : E^{i:D}_{k,s} \rightarrow E^{i:D+e_i}_{k,s} \}, \text{ where} \\
\begin{align*}
\{ E^{i:D}_{k,s}, \hat{\theta} : E^{i:D}_{k,s} \rightarrow E^{i:D}_{k+i,s+t-1} \} & \Rightarrow H_{s-k}(\text{Poly}^{D,2}_n; \mathbb{Z}), \\
\{ E^{i:D+e_i}_{k,s}, \hat{\theta} : E^{i:D+e_i}_{k,s} \rightarrow E^{i:D+e_i}_{k+i,s+t-1} \} & \Rightarrow H_{s-k}(\text{Poly}^{D+e_i}; \mathbb{Z}), \\
E^{i:D}_{k,s} & = H_c^{2N(D)+k-1-s}(\mathcal{X}^D \setminus \mathcal{X}^D_{k-1}; \mathbb{Z}), \\
E^{i:D+e_i}_{k,s} & = H_c^{2N(D)+k+1-s}(\mathcal{X}^D_{k+e_i} \setminus \mathcal{X}^D_{k-1}; \mathbb{Z}).
\end{align*}
\]

**Lemma 4.13.** If $1 \leq i \leq r$ and $0 \leq k \leq \left\lfloor \frac{d_{\text{min}}}{2} \right\rfloor$, $\hat{\theta}^1_{k,s} : E^{1:D}_{k,s} \rightarrow E^{1:D+e_i}_{k,s}$ is an isomorphism for any $s$.

**Proof.** Since the case $k = 0$ is clear, suppose that $1 \leq k \leq \left\lfloor \frac{d_{\text{min}}}{2} \right\rfloor$. It follows from the proof of Lemma \cite{[19]} that there is a homotopy commutative diagram of affine vector bundles

\[
\mathbb{C} \times (\mathcal{X}^D_{k} \setminus \mathcal{X}^D_{k-1}) \xrightarrow{\hat{s}_{D,i}} C_{k;\Sigma} \\
\downarrow \hat{s}_{D,i} \hspace{3cm} \downarrow || \\
\mathcal{X}^D_{k+e_i} \setminus \mathcal{X}^D_{k-1} \xrightarrow{} C_{k;\Sigma}
\]

Since one-point compactification is contravariant for open embeddings, the map $\hat{s}_{D,i+}$ induces the map

$$\hat{s}_{D,i+} : (\mathcal{X}^D_{k+e_i} \setminus \mathcal{X}^D_{k-1})_+ \rightarrow (\mathbb{C} \times (\mathcal{X}^D_{k} \setminus \mathcal{X}^D_{k-1}))_+ = S^2 \wedge (\mathcal{X}^D_{k} \setminus \mathcal{X}^D_{k-1})_+$$

between one-point compactifications. Recall from Lemma \cite{[19]} that $\xi_{D,k,n}$ (resp. $\xi_{D+e_i,k,n}$) is a real affine bundle over $C_{k;\Sigma}$ with rank $l_{D,k,n}$ (resp. $l_{D+e_i,k,n}$). Moreover, note that

$$(2N(D) + k - s + 1) - l_{D,k,n} - 2 = (2N(D) + k - s + 1) - l_{D+e_i,k,n} = 2nrk - s.$$
By the above commutative diagram and Alexander duality, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
E_{k,s}^{1:D} & \xrightarrow{\hat{\theta}_{k,s}^1} & E_{k,s}^{1:D+e_i} \\
H_c^{2N(D)+k-s-1}(\mathcal{X}_k^D \setminus \mathcal{X}_{k-1}^D; \mathbb{Z}) & \xrightarrow{\simeq} & H_c^{2N(D+e_i)+k-s-1}(\mathcal{X}_k^{D+e_i} \setminus \mathcal{X}_{k-1}^{D+e_i}; \mathbb{Z}) \\

\end{array}
\]

Hence, \( \hat{\theta}_{k,s}^1 \) is an isomorphism for any \( s \), and the assertion follows. \( \square \)

Now we consider the spectral sequences induced by truncated simplicial resolutions.

**Definition 4.14.** Let \( X^\Delta \) denote the truncated (after the \( \lfloor \frac{d_{\min}}{n} \rfloor \)-th term) simplicial resolution of \( \Sigma_D \) with the natural filtration

\[
\emptyset = X_0^\Delta \subset X_1^\Delta \subset \cdots \subset X_{\lfloor \frac{d_{\min}}{n} \rfloor}^\Delta \subset X_{\lfloor \frac{d_{\min}}{n} \rfloor+1}^\Delta = X_{\lfloor \frac{d_{\min}}{n} \rfloor+2} = \cdots = X^\Delta,
\]

where \( X_k^\Delta = X_k^D \) if \( k \leq \lfloor \frac{d_{\min}}{n} \rfloor \) and \( X_k^\Delta = X^\Delta \) if \( k \geq \lfloor \frac{d_{\min}}{n} \rfloor + 1 \).

Similarly, let \( Y^\Delta \) denote the truncated (after the \( \lfloor \frac{d_{\min}}{n} \rfloor \)-th term) simplicial resolution of \( \Sigma_{D+e_i} \) with the natural filtration

\[
\emptyset = Y_0^\Delta \subset Y_1^\Delta \subset \cdots \subset Y_{\lfloor \frac{d_{\min}}{n} \rfloor}^\Delta \subset Y_{\lfloor \frac{d_{\min}}{n} \rfloor+1}^\Delta = Y_{\lfloor \frac{d_{\min}}{n} \rfloor+2} = \cdots = Y^\Delta,
\]

where \( Y_k^\Delta = Y_k^{D+e_i} \) if \( k \leq \lfloor \frac{d_{\min}}{n} \rfloor \) and \( Y_k^\Delta = Y^\Delta \) if \( k \geq \lfloor \frac{d_{\min}}{n} \rfloor + 1 \). \( \square \)

By using Lemma 15 and the same method as in [29] §2 and §3 (cf. [21] Lemma 2.2), we obtain the following truncated spectral sequences

\[
\{ E_{k,s}^t; d^t : E_{k,s}^t \to E_{k+1,s+t-1}^t \} \Rightarrow H_{s-k}(\text{Poly}_n^{D,\Sigma}; \mathbb{Z}),
\]

\[
\{ 'E_{k,s}^t; d^t : 'E_{k,s}^t \to 'E_{k+1,s+t-1}^t \} \Rightarrow H_{s-k}(\text{Poly}^{D+e_i}_n; \mathbb{Z}),
\]

\[
E_{k,s}^1 = H_c^{2N(D)+k-1-s}(X_k^\Delta \setminus X_{k-1}^\Delta; \mathbb{Z}),
'\ E_{k,s}^1 = H_c^{2N(D)+k-1-s}(Y_k^\Delta \setminus Y_{k-1}^\Delta; \mathbb{Z}).
\]

By the naturality of truncated simplicial resolutions, the filtration preserving map \( \tilde{s}_{D,i} : C \times X^D \to X^{D+e_i} \) gives rise to a natural filtration preserving map \( \tilde{s}_{D,i}^D : C \times X^\Delta \to Y^\Delta \) which, in a way analogous to (4.26), induces a homomorphism of spectral sequences

\[
(4.27) \quad \{ \theta_{k,s}^t; E_{k,s}^t \to 'E_{k,s}^t \}.
\]
Lemma 4.15. (i) If \( k < 0 \) or \( k \geq \left\lfloor \frac{d_{\min}}{n} \right\rfloor + 2 \), \( E^1_{k,s} = E^1_{k,s} = 0 \) for any \( s \).

(ii) \( E^1_{0,0} = E^1_{0,0} = \mathbb{Z} \) and \( E^1_{0,s} = E^1_{0,s} = 0 \) if \( s \neq 0 \).

(iii) If \( 1 \leq k \leq \left\lfloor \frac{d_{\min}}{n} \right\rfloor \), there are isomorphisms

\[
E^1_{k,s} \cong E^1_{k,s} \cong H_c^{2nrk-s}(C_{k;\Sigma}; \pm \mathbb{Z}).
\]

(iv) If \( 1 \leq k \leq \left\lfloor \frac{d_{\min}}{n} \right\rfloor \), \( E^1_{k,s} = E^1_{k,s} = 0 \) for any \( s \leq (2nr_{\min}(\Sigma) - 2)k - 1 \).

(v) \( E^1_{\left [ \frac{d_{\min}}{n} \right ] + 1,s} = E^1_{\left [ \frac{d_{\min}}{n} \right ] + 1,s} = 0 \) for any \( s \leq (2nr_{\min}(\Sigma) - 2)\left\lfloor \frac{d_{\min}}{n} \right\rfloor - 1 \).

Proof. Let us write \( r_{\min} = r_{\min}(\Sigma) \) and \( d'_{\min} = \left\lfloor \frac{d_{\min}}{n} \right\rfloor \). Since the proofs of both cases are identical, it suffices to prove the assertions for \( E^1_{k,s} \).

(i), (ii), (iii) Since \( X_k^\Delta = X_k^D \) for any \( k \geq d'_{\min} + 2 \), the assertions (i) and (ii) are clearly true. Since \( X_k^\Delta = X_k^D \) for any \( k \leq d'_{\min} \), the assertion (iii) easily follows from Lemma 4.10.

(iv) Suppose that \( 1 \leq k \leq d'_{\min} \). Since \( \dim C_{k;\Sigma} = 2k(1 + nr - nr_{\min}) \) by \([4.10]\), \( 2nrk > \dim C_{k;\Sigma} \Leftrightarrow s \leq (2nr_{\min} - 2)k - 1 \). Thus, the assertion (iv) follows from (iii).

(v) It remains to prove (v). By Lemma \([29]\) Lemma 2.1, we see that

\[
\dim(X_{d'_{\min}+1}^\Delta \setminus X_{d'_{\min}}^\Delta) = \dim(X_{d'_{\min}}^D \setminus X_{d'_{\min}+1}^D) + 1 = l_{D,d'_{\min};n} + \dim C_{d'_{\min};\Sigma} + 1 = 2N(D) + 3d'_{\min} - 2nr_{\min}d'_{\min}.
\]

Since \( E^1_{d'_{\min}+1,s} = H_c^{2N(D)+d'_{\min}-s}(X_{d'_{\min}+1}^\Delta \setminus X_{d'_{\min}}^\Delta; \mathbb{Z}) \) and

\[
2N(D) + d'_{\min} - s > \dim(X_{d'_{\min}+1}^\Delta \setminus X_{d'_{\min}}^\Delta) = 2N(D) + 3d'_{\min} - 2nr_{\min}d'_{\min} \Leftrightarrow s \leq (2nr_{\min} - 2)d'_{\min} - 1,
\]

we see that \( E^1_{d'_{\min}+1,s} = 0 \) for any \( s \leq (2nr_{\min} - 2)d'_{\min} - 1 \) and the assertion (iv) follows.

Lemma 4.16. If \( n \geq 2 \), the space \( \text{Poly}_{n;\Sigma}^D \) is \((2nr_{\min}(\Sigma) - 5)\)-connected.

Proof. If \( d_{\min} < n \), \( \text{Poly}_{n;\Sigma}^D = \mathbb{P}^D \) is contractible and the assertion is clear and suppose that \( d_{\min} \geq n \). Consider the spectral sequence

\[
(4.28) \quad \{ E^t_{k,s}, d^t : E^t_{k,s} \to E^t_{k+t,s+t-1} \} \Rightarrow H_{s-k}(\text{Poly}_{n;\Sigma}^D; \mathbb{Z}).
\]

Then by using Lemma 4.15 we easily see that \( E^1_{k,s} = 0 \) if one of the following three conditions (a), (b) and (c) holds:
(a) $k < 0$, or $k > \lfloor \frac{d_{\text{min}}}{n} \rfloor + 2$, or $k = 0$ with $s \neq 0$.

(b) If $1 \leq k \leq \lfloor \frac{d_{\text{min}}}{n} \rfloor$, $s - k \leq (2nr_{\text{min}}(\Sigma) - 3)k - 1$.

(c) If $k = \lfloor \frac{d_{\text{min}}}{n} \rfloor + 1$, $s - (\lfloor \frac{d_{\text{min}}}{n} \rfloor + 1) \leq (2nr_{\text{min}}(\Sigma) - 3)\lfloor \frac{d_{\text{min}}}{n} \rfloor - 2$.

Hence, when $(k, s) \neq (0, 0)$, we see that $E_{k,s}^1 = 0$ for any $(k, s)$ if the condition $s - k \leq 2nr_{\text{min}}(\Sigma) - 2$ is satisfied. Thus, by the spectral sequence (4.28), we show that

\[ H_i(\text{Poly}_{\Sigma}^D, \Sigma; \mathbb{Z}) = 0 \quad \text{for any} \quad 1 \leq i \leq 2nr_{\text{min}}(\Sigma) - 5. \]

So it suffices to show that the space $\text{Poly}_{\Sigma}^D$ is simply connected. Note that an element of $\pi_1(\text{Poly}_{\Sigma}^D)$ can be represented by the $m$-tuple $(\eta_1, \cdots, \eta_m)$ of strings of $m$-different colors such that each $\eta_k$ $(1 \leq k \leq m)$ is a string with total multiplicity $d_k$ as in the case of strings of the classical braid group $\text{Br}_d = \pi_1(\mathbb{C}_d(\mathbb{C}))$ [17]. However, when all string of $m$-different colors moves continuously, the following case $(\ast)$ is not allowed to occur in this representation:

$(\ast)$ All strings of $m$-different colors with multiplicity $\geq n$ pass through a single point.

By using this string representation one can show that any strings can intersect, pass through one another (except the case $(\ast)$), and thus change the order as in [15, §Appendix]. Thus one can show that $a \cdot b = b \cdot a$ for any $a, b \in \pi_1(\text{Poly}_{\Sigma}^D)$. Hence, $\pi_1(\text{Poly}_{\Sigma}^D)$ is an abelian group.

On the other hand, since $n \geq 2$ and $r_{\text{min}}(\Sigma) \geq 2$, $2nr_{\text{min}}(\Sigma) - 5 \geq 8 - 5 = 3 > 1$. Hence, $H_1(\text{Poly}_{\Sigma}^D; \mathbb{Z}) = 0$ by (4.29). Thus by the Hurewicz theorem, we see that there is an isomorphism $\pi_1(\text{Poly}_{\Sigma}^D) \cong H_1(\text{Poly}_{\Sigma}^D; \mathbb{Z}) = 0$. \[\square\]

Now it is ready to prove the unstability result for $\text{Poly}_{\Sigma}^D$.

**Lemma 4.17.** If $0 \leq k \leq \lfloor \frac{d_{\text{min}}}{n} \rfloor$, $\theta^1_{k,s} : E_1^{k,s} \to E_1^{k,s}$ is an isomorphism for any $s$.

**Proof.** Since $(X_k^D, Y_k^D) = (X_k^{D+e_1}, Y_k^{D+e_1})$ for $k \leq \lfloor \frac{d_{\text{min}}}{n} \rfloor$, the assertion follows from Lemma 4.13. \[\square\]

**Theorem 4.18.** Let $n \geq 2$. Then for each $1 \leq i \leq r$, the stabilization map

\[ s_{D,D+e_i} : \text{Poly}_{\Sigma}^D \to \text{Poly}_{\Sigma}^{D+e_i} \]

is a homotopy equivalence through dimension $d(D; \Sigma, n)$, where $d(D; \Sigma, n)$ denotes the integer given by (2.22).
Proof. We write \( r_{\min} = r_{\min}(\Sigma) \) and \( d'_{\min} = \left\lfloor \frac{d_{\min}}{n} \right\rfloor \) as in the proof of Lemma 4.15. If \( n \geq 2 \), the spaces \( \text{Poly}^D_{n,\Sigma} \) and \( \text{Poly}^{D+e_i}_{n,\Sigma} \) are simply connected by Lemma 4.16. Thus it suffices to prove that the map \( s_{D,D+e_i} \) is a homology equivalence through dimension \( d(D;\Sigma,n) \).

Let us consider the homomorphism \( \theta^t_{k,s} : E^1_{k,s} \to E^t_{k,s} \) of truncated spectral sequences given in (4.27). By using the commutative diagram (4.25) and the comparison theorem for spectral sequences, it suffices to prove that the positive integer \( d(D;\Sigma,n) \) has the following property:

\[
\begin{align*}
\langle \rangle & \quad \theta^\infty_{k,s} \text{ is an isomorphism for all } (k,s) \text{ such that } s - k \leq d(D;\Sigma,n).
\end{align*}
\]

By Lemma 4.15, \( E^1_{k,s} = \delta^1_{k,s} = 0 \) if \( k < 0 \), or if \( k \geq d'_{\min} + 2 \), or if \( k = d'_{\min} + 1 \) with \( s \leq (2nr_{\min} - 2)d'_{\min} - 1 \). Since \( \{(2nr_{\min} - 2)d'_{\min} - 1\} - (d'_{\min} + 1) = (2nr_{\min} - 3)d'_{\min} - 2 = d(D;\Sigma,n) \), we see that:

\[
\begin{align*}
\langle \rangle_1 & \quad \text{if } k < 0 \text{ or } k \geq d'_{\min} + 1, \theta^\infty_{k,s} \text{ is an isomorphism for all } (k,s) \text{ such that } s - k \leq d(D;\Sigma,n).
\end{align*}
\]

Next, assume that \( 0 \leq k \leq d'_{\min} \), and investigate the condition that \( \theta^\infty_{k,s} \) is an isomorphism. Note that the groups \( E^1_{k_1,s_1} \) and \( \delta^1_{k_1,s_1} \) are not known for \( (u,v) \in S_1 = \{(d'_{\min} + 1, s) \in \mathbb{Z}^2 : s \geq (2nr_{\min} - 2)d'_{\min}\} \). By considering the differentials \( d^1 \)'s of \( E^1_{k,s} \) and \( \delta^1_{k,s} \), and applying Lemma 4.17, we see that \( \theta^2_{k,s} \) is an isomorphism if \( (k,s) \not\in S_1 \cup S_2 \), where

\[
\begin{align*}
S_2 = \{(u,v) \in \mathbb{Z}^2 : (u+1,v) \in S_1 \} = \{(d'_{\min}, v) \in \mathbb{Z}^2 : v \geq (2nr_{\min} - 2)d'_{\min}\}.
\end{align*}
\]

A similar argument shows that \( \theta^3_{k,s} \) is an isomorphism if \( (k,s) \not\in \bigcup_{t=1}^3 S_t \), where \( S_3 = \{(u,v) \in \mathbb{Z}^2 : (u+2,v+1) \in S_1 \cup S_2 \} \). Continuing in the same fashion, considering the differentials \( d^t \)'s on \( E^t_{k,s} \) and \( \delta^t_{k,s} \) and applying the inductive hypothesis, we see that \( \theta^\infty_{k,s} \) is an isomorphism if \( (k,s) \not\in S := \bigcup_{t \geq 1} S_t = \bigcup_{t \geq 1} A_t \), where \( A_t \) denotes the set

\[
\begin{align*}
A_t = \left\{ (u,v) \in \mathbb{Z}^2 \right. \left| \begin{array}{l}
\text{There are positive integers } l_1, \ldots, l_t \text{ such that, } \\
1 \leq l_1 < l_2 < \cdots < l_t, u + \sum_{j=1}^t l_j = d'_{\min} + 1, \\
v + \sum_{j=1}^t (l_j - 1) \geq (2nr_{\min} - 2)d'_{\min}
\end{array} \right. \}
\end{align*}
\]

Note that if this set was empty for every \( t \), then, of course, the conclusion of Theorem 4.18 would hold in all dimensions (this is known to be false in general). If \( A_t \neq \emptyset \), it is easy to see that

\[
\begin{align*}
a(t) = \min\{s - k : (k,s) \in A_t\} = (2nr_{\min} - 2)d'_{\min} - (d'_{\min} + 1) + t \\
= (2nr_{\min} - 3)d'_{\min} + t - 1 = d(D;\Sigma,n) + t + 1.
\end{align*}
\]
Hence, we obtain that \( \min \{ a(t) : t \geq 1, A_t \neq \emptyset \} = d(D; \Sigma, n) + 2 \). Since \( \theta_{k,s}^{\infty} \) is an isomorphism for any \( (k, s) \notin \bigcup_{t \geq 1} A_t \) for each \( 0 \leq k \leq d_{\text{min}}' \), we have the following:

\[
(\dagger)_2 \quad \text{If } 0 \leq k \leq d_{\text{min}}', \theta_{k,s}^{\infty} \text{ is an isomorphism for any } (k, s) \text{ such that } s - k \leq d(D; \Sigma, n) + 1.
\]

Then, by \((\dagger)_1\) and \((\dagger)_2\), we know that \( \theta_{k,s}^{\infty} : E_{k,s}^{\infty} \rightarrow E_{k,s}^{\infty} \) is an isomorphism for any \( (k, s) \) if \( s - k \leq d(D; \Sigma, n) \). Hence, by \((\dagger)\) we have the desired assertion and this completes the proof of Theorem 4.18.

**Corollary 4.19.** Let \( n \geq 2 \). Then for each \( a \in (\mathbb{Z}_{\geq 0})^{r} \) with \( a \neq 0_{r} \), the stabilization map

\[
s_{D,D+a} : \text{Poly}_{n}^{D,\Sigma} \rightarrow \text{Poly}_{n}^{D+a,\Sigma}
\]

is a homotopy equivalence through dimension \( d(D; \Sigma, n) \).

**Proof.** The assertion easily follows from \((4.21)\) and Theorem 4.18.

## 5 Scanning maps

In this section we consider configuration spaces and the scanning map.

**Definition 5.1.** For a positive integer \( d \geq 1 \) and a based space \( X \), let \( \text{SP}^{d}(X) \) denote the \( d \)-th symmetric product of \( X \) defined as the orbit space

\[
\text{SP}^{d}(X) = X^{d}/S_{d},
\]

where the symmetric group \( S_{d} \) of \( d \) letters acts on the \( d \)-fold product \( X^{d} \) in the natural manner.

**Remark 5.2.** (i) An element \( \eta \in \text{SP}^{d}(X) \) may be identified with a formal linear combination

\[
\eta = \sum_{k=1}^{s} n_{k} \alpha_{k},
\]

where \( \alpha_{1}, \ldots, \alpha_{s} \) are distinct points in \( X \) and \( n_{1}, \ldots, n_{s} \) are positive integers such that \( \sum_{k=1}^{s} n_{k} = d \). In this situation we shall refer to \( \eta \) as configuration (or divisor) of points, the points \( \alpha_{k} \in X \) having a multiplicity \( n_{k} \).

(ii) For example, when \( X = \mathbb{C} \), there is a natural homeomorphism

\[
f(z) = \prod_{k=1}^{s} (z - \alpha_{k})^{n_{k}} \quad \xrightarrow{\psi_{d}} \quad \eta = \sum_{k=1}^{s} n_{k} \alpha_{k}
\]

where \( n_{k} \in \mathbb{N} \) with \( \sum_{k=1}^{s} n_{k} = d \). 

\[ \square \]
Definition 5.3. (i) When $A \subset X$ is a closed subspace, define an equivalence relation “∼” on $SP^d(X)$ by

\[(5.4) \quad \eta_1 \sim \eta_2 \text{ if } \eta_1 \cap (X \setminus A) = \eta_2 \cap (X \setminus A) \text{ for } \eta_1, \eta_2 \in SP^d(X)\]

Define $SP^d(X, A)$ as the quotient space

\[(5.5) \quad SP^d(X, A) = SP^d(X) / \sim .\]

Note that the points in $A$ are ignored in $SP^d(X, A)$.

(ii) If $A \neq \emptyset$, we have a natural inclusion $SP^d(X, A) \subset SP^{d+1}(X, A)$ given by adding a point in $A$, and we can define $SP^\infty(X, A)$ as the union

\[(5.6) \quad SP^\infty(X, A) = \bigcup_{d=1}^{\infty} SP^d(X, A) .\]

(iii) For each $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 1})^r$, let $E^{D, \Sigma}_n(X)$ denote the subspace of $SP^d(X)^n$ given by

\[(5.7) \quad E^{D, \Sigma}_n(X) = \{(\xi_1, \cdots, \xi_r) \in \prod_{i=1}^r SP^d_i(X)^n : (5.7.1), (5.7.2)\},\]

where two conditions (5.7.1) and (5.7.2) are given by

\[(5.7.1) \text{ For each } 1 \leq i \leq r, \xi_i = (\xi_{i,1}, \cdots, \xi_{i,n}) \in SP^d_i(X)^n \text{ with } \xi_{i,j} \in SP^d_i(X).\]

\[(5.7.2) \text{ } \bigcap_{(i,j) \in \sigma \times [n]} \xi_{i,j} = \emptyset \text{ for any } \sigma \in I(K_\Sigma).\]

(iv) Let $A \subset X$ be a closed subspace and $A \neq \emptyset$. We define an equivalence relation “∼” on the space $E^{D, \Sigma}_n(X)$ by

\[(\xi_1, \cdots, \xi_r) \sim (\eta_1, \cdots, \eta_r) \text{ if } \xi_i \cap (X \setminus A) = \eta_i \cap (X \setminus A) \text{ for each } 1 \leq j \leq r.\]

Let $E^{D, \Sigma}_n(X, A)$ be the quotient space

\[(5.8) \quad E^{D, \Sigma}_n(X, A) = E^{D, \Sigma}_n(X) / \sim .\]

Adding points in $A$ gives a natural inclusion $E^{D, \Sigma}_n(X, A) \subset E^{D+e_i, \Sigma}_n(X, A)$ for each $1 \leq i \leq r$. So, one can define the space $E^{\Sigma}_n(X, A)$ as the union

\[(5.9) \quad E^{\Sigma}_n(X, A) = \bigcup_{D \in \mathbb{N}^r} E^{D, \Sigma}_n(X, A) .\]
Remark 5.4. (i) For each $D = (d_1, \ldots, d_r) \in \mathbb{N}^r$, there is a natural homeomorphism

\begin{equation}
\text{Poly}^D \xrightarrow{\Psi_d} E^D, (f_1(z), \ldots, f_r(z)) \mapsto (\Psi_d(f_1(z)), \ldots, \Psi_d(f_r(z)))
\end{equation}

where $\Psi_d(f(z)) \in \text{SP}^d(\mathbb{C})^n$ denotes the $n$-tuple of configuration given by

\begin{equation}
\Psi_d(f(z)) = (\psi_d(f(z)), \psi_d(f(z) + f'(z)), \ldots, \psi_d(f(z) + f^{(n-1)}(z)))
\end{equation}

for $f(z) \in \mathbb{P}^d$, where $\psi_d$ is the map defined in (5.3).

(ii) In general, $E^{D, \Sigma}(\mathbb{C})$ is path-connected. In fact, for any two points $\xi_0$ and $\xi_1$ in $E^{D, \Sigma}(\mathbb{C})$ one can construct a path $\omega : [0, 1] \to E^{D, \Sigma}(\mathbb{C})$ such that $\omega(i) = \xi_i$ for $i \in \{0, 1\}$ by the method explained in [15] §Appendix. Hence, the space $\text{Poly}^D$ is also path connected. \qed

Definition 5.5. Let $\varphi_D : \mathbb{C} \xrightarrow{\cong} U_D$ be any fixed homeomorphism, and we choose any mutually distinct fixed $r$ points $x_1, \ldots, x_r \in \mathbb{C} \setminus U_D$ as in Definition 4.11.

(i) Let $d$ be a positive integer and let $\eta = \sum_{k=1}^s n_k y_k \in \text{SP}^d(\mathbb{C})$ be any element such that $\{y_k\}_{k=1}^s \subset C_s(\mathbb{C})$ and $n_k \in \mathbb{Z}_{\geq 1}$ with $\sum_{k=1}^s n_k = d$. In this situation let $\tilde{\varphi}_D(\eta) \in \text{SP}^d(U_D)$ denote the configuration given by

\begin{equation}
\tilde{\varphi}_D(\eta) = \sum_{k=1}^s n_k \varphi_D(y_k).
\end{equation}

(ii) When $\eta = (\eta_1, \ldots, \eta_n) \in \text{SP}^d(\mathbb{C})^n$ with $\eta_k \in \text{SP}^d(\mathbb{C})$, let $\Phi_D(\eta) \in \text{SP}^d(\mathbb{C})^n$ denote the $n$-tuple of configurations given by

\begin{equation}
\Phi_D(\eta) = (\varphi_D(\eta_1), \ldots, \varphi_D(\eta_n)).
\end{equation}

(iii) For each $a = (a_1, \ldots, a_r) \neq 0_r \in (\mathbb{Z}_{\geq 0})^r$, define the stabilization map

\begin{equation}
\hat{s}_{D, D+a} : E^{D, \Sigma}(\mathbb{C}) \to E^{D+a, \Sigma}(\mathbb{C})
\end{equation}

by

\begin{equation}
\hat{s}_{D, D+a}(\xi_1, \ldots, \xi_r) = (\Phi_D(\xi_1) + a_1 \overline{x_1}, \ldots, \Phi_D(\xi_r) + a_r \overline{x_r})
\end{equation}

for $(\xi_1, \ldots, \xi_r) \in E^{D, \Sigma}(U_D)$ with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \text{SP}^d(\mathbb{C})^n$, where we set

\begin{equation}
\Phi_D(\xi_i) + a_i \overline{x_i} = (\tilde{\varphi}_D(\xi_{i,1}) + a_i x_i, \ldots, \tilde{\varphi}_D(\xi_{i,n}) + a_i x_i).
\end{equation}
It is easy to see that the diagram
\[
\begin{array}{c}
\text{Poly}^{D,\Sigma}_n \xrightarrow{s_{D,D+\alpha}} \text{Poly}^{D+\alpha}_n \\
\Psi_D \downarrow \cong \quad \Psi_{D+\alpha} \downarrow \cong \\
E_n^{D,\Sigma}(\mathbb{C}) \xrightarrow{\delta_{D,D+\alpha}} E_n^{D+\alpha}(\mathbb{C})
\end{array}
\]

is commutative.

Now we are ready to define the scanning map.

**Definition 5.6.** Let \( \epsilon_0 > 0 \) be any fixed sufficiently small number and let \( U = \{ w \in \mathbb{C} : |w| < 1 \} \). For each \( w \in \mathbb{C} \), let \( U_w = \{ x \in \mathbb{C} : |x - w| < \epsilon_0 \} \).
Then for an element \( \eta = (\eta_1, \ldots, \eta_r) \in E_n^{D,\Sigma}(\mathbb{C}) \), define a map \( sc_D(\eta) : \mathbb{C} \to E_n^{\Sigma}(\overline{U}, \partial\overline{U}) \) by
\[
w \mapsto \eta \cap \overline{U}_w = (\eta_1 \cap \overline{U}_w, \ldots, \eta_r \cap \overline{U}_w) \in E_n^{\Sigma}(\overline{U}_w, \partial\overline{U}_w) \cong E_n^{\Sigma}(\overline{U}, \partial\overline{U})
\]
for \( w \in \mathbb{C} \), where we identify \( (\overline{U}_w, \partial\overline{U}_w) \) with \( (\overline{U}, \partial\overline{U}) \) in the canonical way. Since \( \lim_{w \to \infty} sc(\eta)(w) = (\emptyset, \ldots, \emptyset) \), it naturally extends to a map
\[
sc(\eta) : S^2 = \mathbb{C} \cup \infty \to E_n^{\Sigma}(\overline{U}, \partial\overline{U})
\]
with \( sc(\eta)(\infty) = (\emptyset, \ldots, \emptyset) \). Now we choose the point \( \infty \) and the empty configuration \( (\emptyset, \ldots, \emptyset) \) as the base-points of \( S^2 = \mathbb{C} \cup \infty \) and \( E_n^{\Sigma}(\overline{U}, \partial\overline{U}) \), respectively. Then the map \( sc(\eta) \) is a base-point preserving map, and we obtain a map
\[
sc : E_n^{D,\Sigma}(\mathbb{C}) \to \Omega^2 E_n^{\Sigma}(\overline{U}, \partial\overline{U}).
\]
However, since \( E_n^{D,\Sigma}(\mathbb{C}) \) is connected, the image of the map \( sc \) is contained some path-component of \( \Omega^2 E_n^{\Sigma}(\overline{U}, \partial\overline{U}) \), which we denote by \( \Omega^2_D E_n^{\Sigma}(\overline{U}, \partial\overline{U}) \). Thus we have the map
\[
sc_D : E_n^{D,\Sigma}(\mathbb{C}) \to \Omega^2_D E_n^{\Sigma}(\overline{U}, \partial\overline{U}).
\]
Since we can identify \( \text{Poly}^{D,\Sigma}_n = E_n^{D,\Sigma}(\mathbb{C}) \) as in (5.10), we obtain the map
\[
sc_D : \text{Poly}^{D,\Sigma}_n \to \Omega^2_D E_n^{\Sigma}(\overline{U}, \partial\overline{U}).
\]

We refer to this map (and others defined by the same method) as “the scanning map”.

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Now let \(0_r \neq a \in (\mathbb{Z}_{\geq 0})^r\) be an \(r\)-tuple of integers. Then it is easy to see that there is a commutative diagram

\[
\begin{array}{ccc}
P\text{oly}_{n+\Sigma}^D & \xrightarrow{sc_{D}} & \Omega^2_0 E^\Sigma_n(U, \partial U) \\
\downarrow s_{D,D+a} & & \downarrow \simeq \\
P\text{oly}_{n+\Sigma}^D & \xrightarrow{sc_{D+a}} & \Omega^2_0 E^\Sigma_n(U, \partial U)
\end{array}
\]

(5.21)

Let \(P_{\Sigma}^n = \lim_{a \to \infty} P_{D+a:\Sigma}^n\) denote the colimit constructed from the stabilization maps \(\{s_{D,D+a}: a \in (\mathbb{Z}_{\geq 0})^r\}\), where the notation \(a = (a_1, \ldots, a_r) \to \infty\) means that \(\min\{a_k : 1 \leq k \leq r\} \to \infty\).

Then by using (5.21) we obtain the stabilized scanning map

\[
S : P_{\Sigma}^n \to \Omega^2_0 E^\Sigma_n(U, \partial U),
\]

(5.22)

where \(S = \lim_{a \to \infty} sc_{D+a} \) and \(\Omega^2_0 X\) denotes the path component of \(\Omega^2 X\) which contains the constant map.

**Theorem 5.7.** The stabilized scanning map

\[
S : P_{\Sigma}^n \xrightarrow{\simeq} \Omega^2_0 E^\Sigma_n(U, \partial U)
\]

is a homotopy equivalence.

**Proof.** The assertion can be proved by using Segal’s scanning method given in [14, Prop. 4.4] (cf. [13]) and [16].

Next we investigate about the homotopy type of the space \(E^\Sigma_n(U, \partial U)\).

**Definition 5.8.** Let \((X, \ast)\) be a based space, let \(I\) be a collection of some subsets of \([N] = \{1, 2, \ldots, N\}\), and let \(\Sigma\) be a fan in \(\mathbb{R}^m\).

(i) Let \(\vee^I X\) denote the subspace of \(X^N\) defined by

\[
\vee^I X = \{(x_1, \ldots, x_N) \in X^N : (\ddagger)_I\},
\]

(5.23) where \((\ddagger)_I\) For each \(\sigma \in I\), there is some \(j \in \sigma\) such that \(x_j = \ast\).

(ii) Recall the set

\[
[r] \times [n] = \{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq r, 1 \leq j \leq n\}
\]

of \(rn\) points and let \(I(\Sigma, n)\) denote the collection of subsets in \([r] \times [n]\) defined by

\[
I(\Sigma, n) = \{\sigma \times [n] : \sigma \in I(K_\Sigma)\}.
\]

(5.25)
(iii) Similarly let \( K_\Sigma(n) \) denote the simplicial complex on the index set \([r] \times [n]\) defined by
\[
(5.26) \quad K_\Sigma(n) = \{ \tau \subset [r] \times [n] : \sigma \times [n] \not\subset \tau \text{ for any } \sigma \in I(K_\Sigma) \}.
\]

Lemma 5.9 (cf. [22], Lemma 6.3). Let \( K \) be a simplicial complex on the index set \([N]\) and let \((X,*)\) be a based space.

(i) \( I(K) = \{ \sigma \subset [N] : \sigma \not\subset \tau \text{ for any } \tau \in K \} \).

(ii) \( Z_K(X,*) = \vee^{I(K)} X \).

Proof. The assertion (i) easily follows from the definition of simplicial complexes and the assertion (ii) follows from [23] Lemma 4.2.

Lemma 5.10. There is a homotopy equivalence
\[
r_\Sigma : E_\Sigma^\Sigma(U, \partial U) \xrightarrow{\sim} DJ(K_\Sigma(n)).
\]

Proof. For each \( \epsilon > 0 \), let \( U(\epsilon) = \{ w \in \mathbb{C} : |w| < \epsilon \} \subset \mathbb{C} \cup \infty = S^2 \). The proof is analogous to that of [32] Prop. 3.1, [18] Lemma 7.10 and [25] Lemma 4.3. Note that the space \( E_\Sigma^\Sigma(U, \partial U) \) is homeomorphic to the space
\[
(5.27) \quad E_\Sigma^\Sigma(S^2, \infty) = \{ (\eta_1, \ldots, \eta_r) \in (\text{SP}^\infty(S^2, \infty)^n)^r : (\ast)_n \}, \quad \text{where}
\]
\[
(\ast)_n \cap (i,j) \in \sigma \times [n] \eta_{i,j} = \emptyset \text{ for any } \sigma \in I(K_\Sigma), \quad \text{where } \eta_i = (\eta_{i,1}, \ldots, \eta_{i,n}) \in \text{SP}^\infty(S^2, \infty)^n \text{ with } \eta_{i,j} \in \text{SP}^\infty(S^2, \infty) \text{ for each } 1 \leq i \leq r.
\]

For each \( \epsilon > 0 \), let \( E_\epsilon^\Sigma \) denote the open subset of \( E_\Sigma^\Sigma(S^2, \infty) \) consisting of all \( r \)-tuples \((\eta_1, \ldots, \eta_r) \in E_\Sigma^\Sigma(S^2, \infty)\) such that, for any \( \sigma \in I(K_\Sigma) \) there exists some \((i,j) \in \sigma \times [n]\) satisfying the condition \( \eta_{i,j} \cap U(\epsilon) = \emptyset \).

Then the radial expansion defines a deformation retraction
\[
(5.28) \quad r_\epsilon : E_\epsilon^\Sigma \xrightarrow{\sim} \vee^{I(\Sigma,n)} \text{SP}^\infty(S^2, \infty)
\]
(in this case, if \( \eta_{i,j} \cap \overline{U(\epsilon)} = \emptyset \) and \((i,j) \in \sigma \times [n]\) (for any \( \sigma \in I(K) \), then the configuration \( \eta_{i,j} \) gets retracted to \( \infty \)). Since \( E_\epsilon^\Sigma(S^2, \infty) = \bigcup_{\epsilon>0} E_\epsilon^\Sigma \) and there is a homeomorphism \( \text{SP}^\infty(S^2, \infty) \cong \mathbb{C}\mathbb{P}^\infty \), there is a deformation retraction
\[
(5.29) \quad E_\epsilon^\Sigma(S^2, \infty) \xrightarrow{\sim} \vee^{I(\Sigma,n)} \text{SP}^\infty(S^2, \infty) \cong \vee^{I(\Sigma,n)} \mathbb{C}\mathbb{P}^\infty.
\]
Since \( I(\Sigma,n) = \{ \tau \subset [r] \times [n] : \tau \notin K_\Sigma(n) \} \), by Lemma 5.9 we can identify \( \vee^{I(\Sigma,n)} \mathbb{C}\mathbb{P}^\infty = Z_{K_\Sigma(n)}(\mathbb{C}\mathbb{P}^\infty,*) = DJ(K_\Sigma(n)) \). Thus we obtain the desired homotopy equivalence. \( \square \)
Remark 5.11. For each \((i, j) \in [r] \times [n]\), let \(n_{i,j} \in \mathbb{Z}^{mn}\) denote the lattice vector defined by
\[
(5.30) \quad n_{i,j} = (a_1, \cdots, a_n), \text{ where we set } a_k = \begin{cases} n_i & (k = j) \\ 0_m & (k \neq j) \end{cases}
\]
and define the fan \(F(\Sigma, n)\) in \(\mathbb{R}^{mn}\) by
\[
(5.31) \quad F(\Sigma, n) = \{ c_{\tau} : \tau \in K_\Sigma(n) \},
\]
where \(c_{\tau}\) denotes the cone in \(\mathbb{R}^{mn}\) generated by \(\{n_{i,j} : (i, j) \in \tau\}\).

Then one can show that there is a homeomorphism
\[
(5.32) \quad Z_{K_\Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \cong Z_{K_\Sigma(n)}(\mathbb{C}, \mathbb{C}^*),
\]
and that \(X_\Sigma(n)\) is a toric variety associated to the fan \(F(\Sigma, n)\). Since the proof is tedious and we do not need this fact, we omit the details. \(\square\)

6 The stable result

In this section we give the proof of the following stability result (Theorem 6.2) by using the stabilized scanning map and Theorem 5.7.

Definition 6.1. Let \(D = (d_1, \cdots, d_r) \in \mathbb{N}^r\) and \(a = (a_1, \cdots, a_r) \in \mathbb{N}^r\) be two \(r\)-tuples of positive integers such that
\[
(6.1) \quad \sum_{k=1}^{r} d_k n_k = \sum_{k=1}^{r} a_k n_k = 0_m,
\]
and consider the following homotopy commutative diagram
\[
\begin{align*}
\text{Poly}_{n}^{D, \Sigma} & \xrightarrow{i_D} \Omega_{D}^{2} X_{\Sigma}(n) \xrightarrow{\simeq} \Omega_{0}^{2} X_{\Sigma}(n) \\
\text{Poly}_{n}^{D+a, \Sigma} & \xrightarrow{i_{D+a}} \Omega_{D+a}^{2} X_{\Sigma}(n) \xrightarrow{\simeq} \Omega_{0}^{2} X_{\Sigma}(n)
\end{align*}
\]

Then by identifying \(\text{Poly}_{n}^{D+\infty, \Sigma} = \lim_{k \to \infty} \text{Poly}_{n}^{D+ka, \Sigma}\), we obtain the map
\[
(6.3) \quad i_{D+\infty} = \lim_{k \to \infty} i_{D+ka} : \text{Poly}_{n}^{D+\infty, \Sigma} = \lim_{k \to \infty} \text{Poly}_{n}^{D+ka, \Sigma} \rightarrow \Omega_{0}^{2} X_{\Sigma}(n).
\]

The purpose of this section is to prove the following result.
Theorem 6.2. The map $i_{D+\infty} : \text{Poly}_{n}^{D+\infty, \Sigma} \xrightarrow{\sim} \Omega^{2}_{0} X_{\Sigma}(n)$ is a homotopy equivalence.

To prove the above result (Theorem 6.2), we recall the following definitions and results.

Definition 6.3. (i) For an open set $X \subset \mathbb{C}$, let $\Sigma^{\Sigma}_{n}(X)$ denote the space of $r$-tuples $(f_{1}(z), \cdots, f_{r}(z)) \in \mathbb{C}[z]^{r}$ of (not necessarily monic) polynomials satisfying the following condition:

(6.3.1) For any $\sigma = \{i_{1}, \cdots, i_{s}\} \in I(\mathcal{K}_{\Sigma})$, the polynomials $f_{i_{1}}(z), \cdots, f_{i_{s}}(z)$ have no common root of multiplicity $\geq n$ in $X$.

Define the map $i^{\Sigma}_{n} : X \rightarrow Z_{\mathcal{K}_{\Sigma}}(\mathbb{C}^{n}, (\mathbb{C}^{n})^{\ast})$ by

(6.4) $i^{\Sigma}_{n}(f)(\alpha) = (F_{n}(f_{1})(\alpha), F_{n}(f_{2})(\alpha), \cdots, F_{n}(f_{r})(\alpha))$

for $(f, \alpha) = ((f_{1}(z), \cdots, f_{r}(z)), \alpha) \in F_{n}(X) \times X$.

(ii) Let $U = \{w \in \mathbb{C} : |w| < 1\}$ and let

(6.5) $ev_{0} : F_{n}^{\Sigma}(U) \rightarrow Z_{\mathcal{K}_{\Sigma}}(\mathbb{C}^{n}, (\mathbb{C}^{n})^{\ast})$

denote the map given by evaluation at 0, i.e.

(6.6) $ev_{0}(f) = (F_{n}(f_{1})(0), F_{n}(f_{2})(0), \cdots, F_{n}(f_{r})(0))$

for $f = (f_{1}(z), \cdots, f_{r}(z)) \in F_{n}(X)$.

(iii) Let $\tilde{F}_{n}^{\Sigma}(U) \subset F_{n}^{\Sigma}(U)$ denote the subspace of all $(f_{1}(z), \cdots, f_{r}(z)) \in F_{n}^{\Sigma}(X)$ such that no $f_{i}(z)$ is identically zero, and let

(6.7) be the map given by the restriction $ev = ev_{0}|\tilde{F}_{n}^{\Sigma}(U)$.

(iv) We denote by

(6.8) $u : \tilde{F}_{n}^{\Sigma}(U)/T_{\mathbb{C}} \rightarrow E_{n}^{\Sigma}(\overline{U}, \partial \overline{U})$

the natural map which assigns to an $r$-tuple $[f_{1}(z), \cdots, f_{r}(z)] \in \tilde{F}_{n}^{\Sigma}(U)/T_{\mathbb{C}}$ of polynomials the $r$-tuple of configurations in $\mathbb{C}^{n}$ represented by their roots of $F_{n}(f_{1})(z), \cdots, F_{n}(f_{r})(z)$ which lie in $U$.

Lemma 6.4. The map $ev : \tilde{F}_{n}^{\Sigma}(U) \xrightarrow{\sim} Z_{\mathcal{K}_{\Sigma}}(\mathbb{C}^{n}, (\mathbb{C}^{n})^{\ast})$ is a homotopy equivalence.
Proof. For each \( b = (b_0, b_1, \cdots, b_{n-1}) \in \mathbb{C}^n \), let \( f_b(z) \in \mathbb{C}[z] \) denote the polynomial defined by \( f_b(z) = b_0 + \sum_{k=1}^{n-1} \frac{(b_k - b_0)z^k}{k!} \), and define the map \( i_0 : Z_{K \Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \to F_n^{\Sigma}(U) \) by \( i_0(b_1, \cdots, b_r) = (f_{b_1}(z), \cdots, f_{b_r}(z)) \) for \( (b_1, \cdots, b_r) \in Z_{K \Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \). Since the degree of each polynomial \( f_{b_k}(z) \) is at most \( n - 1 \), it has no root of multiplicity \( \geq n \) and the map \( i_0 \) is well-defined. Clearly \( ev_0 \circ i_0 = id \).

On the other hand, let \( \Phi : F_n^{\Sigma}(U) \times [0,1] \to F_n^{\Sigma}(U) \) be the homotopy given by \( \Phi((f_1(z), \cdots, f_r(z)), t) = (f_1(tz), \cdots, f_r(tz)) \). This gives a homotopy between \( i_0 \circ ev_0 \) and the identity map, and this proves that \( ev_0 \) is a homotopy equivalence. Since \( F_n^{\Sigma}(U) \) is an infinite dimensional manifold and \( F_n^{\Sigma}(U) \) is a closed subspace of \( F_n^{\Sigma}(U) \) of infinite codimension, by using \cite[Theorem 2]{10}, one can show that the inclusion \( F_n^{\Sigma}(U) \to F_n^{\Sigma}(U) \) is a homotopy equivalence. Hence \( ev \) is also a homotopy equivalence. \( \blacksquare \)

Now it is ready to prove Theorem 6.2.

Proof of Theorem 6.2. Let \( U = \{ w \in \mathbb{C} : |w| < 1 \} \) as before and note that the group \( T_C \) acts freely on the space \( F_n^{\Sigma}(X) \) by coordinate multiplication for \( X = U \) or \( \mathbb{C} \). Let \( F_n^{\Sigma}(X) / T_C \) denote the corresponding orbit space. Note that \( u : F_n^{\Sigma}(U) / T_C \xrightarrow{\simeq} E_n^{\Sigma}(U, \partial U) \) is a homotopy equivalence. Indeed, this follows from \cite{27} (iii) of Lemma 8.4. Now let \( \text{scan} : F_n^{\Sigma}(U) \to \text{Map}(\mathbb{C}, \tilde{F}^{\Sigma}(U)) \) denote the map given by

\[
\text{scan}(f_1(z), \cdots, f_r(z))(w) = (f_1(z + w), \cdots, f_r(z + w))
\]

for \( w \in \mathbb{C} \), and consider the diagram

\[
\begin{array}{ccc}
F_n^{\Sigma}(U) & \xrightarrow{ev} & Z_{K \Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*) \\
p \downarrow & & \downarrow \\
\tilde{F}(U)^{\Sigma}/T_C & \xrightarrow{u} & E_n^{\Sigma}(U, \partial U)
\end{array}
\]

where \( p : F_n^{\Sigma}(U) \to \tilde{F}_n^{\Sigma}(U)/T_C \) denotes the natural projection map. Note that \( p \) is a \( T_C \)-principal bundle projection. Consider the diagram below

\[
\begin{array}{ccc}
\tilde{F}_n^{\Sigma}(U) & \xrightarrow{\text{Map}(\mathbb{C}, \tilde{F}_n^{\Sigma}(U))} & \text{Map}(\mathbb{C}, Z_{K \Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)) \\
\text{scan} \downarrow & \xrightarrow{ev \#} & \text{Map}(\mathbb{C}, Z_{K \Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*)) \\
p \downarrow & \xrightarrow{p \#} & \text{Map}(\mathbb{C}, Z_{K \Sigma}(\mathbb{C}^n, (\mathbb{C}^n)^*))
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{F}_n^{\Sigma}(U) & \xrightarrow{\text{Map}(\mathbb{C}, \tilde{F}_n^{\Sigma}(U))} & \text{Map}(\mathbb{C}, \tilde{F}_n^{\Sigma}(U)/T_C) \\
\text{scan} \downarrow & \xrightarrow{u \#} & \text{Map}(\mathbb{C}, \tilde{F}_n^{\Sigma}(U)/T_C) \\
p \downarrow & \xrightarrow{p \#} & \text{Map}(\mathbb{C}, \tilde{F}_n^{\Sigma}(U)/T_C)
\end{array}
\]

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induced from the above diagram. Observe that $\text{Map}(\mathbb{C}, \cdot)$ can be replaced by $\text{Map}^*(S^2, \cdot)$ by extending from $\mathbb{C}$ to $S^2 = \mathbb{C} \cup \infty$ (as base point preserving maps). Thus by setting

$$
\begin{align*}
\left\{ 
  j'_D & : \text{Poly}^D \xrightarrow{\cdot} \tilde{F}_n(\mathbb{C}) \xrightarrow{\text{scan}} \text{Map}^*(S^2, \tilde{F}_n(U)) = \Omega^2_D \tilde{F}_n(U) \\
  j''_D & : E^D(\mathbb{C}) \xrightarrow{\cdot} \tilde{F}_n(\mathbb{C})/\mathcal{T}_C \xrightarrow{\text{scan}} \text{Map}^*_D(S^2, \tilde{F}_n(U)/\mathcal{T}_C) = \Omega^2_D(\tilde{F}_n(U)/\mathcal{T}_C)
\end{align*}
$$

we obtain the following commutative diagram, where the suffix $D$ denotes the appropriate path component:

\begin{diagram}
\text{Poly}_n^D \xrightarrow{j'_D} \Omega^2_D \tilde{F}_n(U) \xrightarrow{\Omega^2_{\text{ev}}} \Omega^2_D Z_{X_{\Sigma}}(\mathbb{C}^n, (\mathbb{C}^n)^*) \xrightarrow{\Omega^2_{q_{\Sigma}}} \Omega^2_D X_{\Sigma}(n) \\
\cong \quad \quad \quad \quad \Omega^2_p \cong \\
E^D(\mathbb{C}) \xrightarrow{j''_D} \Omega^2_D \tilde{F}_n(U)/\mathcal{T}_C \xrightarrow{\Omega^2_u} \Omega^2_D E_{\Sigma}^n(\overline{U}, \partial \overline{U})
\end{diagram}

Note that the maps $\Omega^2_{q_{\Sigma}}, \text{ev}, \Omega^2_p$ and $u$ are homotopy equivalences. Moreover, from the definitions of the maps, one can see that the following two equalities hold (up to homotopy equivalence):

$$
(6.9) \quad \Omega^2_{q_{\Sigma}} \circ \Omega^2_{\text{ev}} \circ j'_D = i_D, \quad \Omega^2_p \circ j''_D = s_{CD}.
$$

Hence, the maps $i_D$ and $s_{CD}$ are homotopic up to homotopy equivalences. Thus, if we replace $D$ by $D + k a$ and let $k \to \infty$ then, by using Theorem 5.7, we see that the map $i_{D+\infty}$ is a homotopy equivalence.

\section{Proofs of the main results}

In this section we prove Theorem 2.11 and Corollary 2.12. For this purpose, from now on we always assume that $X_{\Sigma}$ is a simply connected smooth toric variety such that the condition (2.15.1) is satisfied. Now we can prove the main results.

\textbf{Proof of Theorem 2.11} The assertion (i) follows from Corollary 4.19 and Theorem 6.2. It remains to show (ii) and suppose that $\sum_{k=1}^r d_k n_k \neq 0_m$.

By the assumption (2.15.1), there is an $r$-tuple $D_\ast = (d_1^\ast, \ldots, d_r^\ast) \in \mathbb{N}^r$ such that $\sum_{k=1}^r d_k^\ast n_k = 0_m$. Then if we choose a sufficiently large positive integer $n_0$, the following equality holds:

$$
(7.1) \quad a = n_0 D_\ast - D = (n_0 d_1^\ast - d_1, \ldots, n_0 d_r^\ast - d_r) \in \mathbb{N}^r.
$$
Since the r-tuple $n_0D = D + a \in \mathbb{N}^r$ satisfies the condition $(2.15.1)$, the map $i_{D+a}$ is well-defined. Then one can define the map

$$(7.2) \quad j_D : \text{Poly}^{D,\Sigma}_n \to \Omega^2 \mathcal{Z}_{K_{\Sigma}}(D^{2n}, S^{2n-1})$$

by the composite $j_D = (i_{D+a}) \circ (s_{D,D+a})$,

$$\text{Poly}^{D,\Sigma}_n \xrightarrow{s_{D,D+a}} \text{Poly}^{D+a,\Sigma}_{D+a} \xrightarrow{i_{D+a}} \Omega^2_{D+a}X_{\Sigma}(n) \simeq \Omega^2_{\Sigma}X_{\Sigma}(n) \simeq \Omega^2 \mathcal{Z}_{K_{\Sigma}}(D^{2n}, S^{2n-1}) .$$

Note that the two maps $s_{D,D+a}$ and $i_{D+a}$ are homotopy equivalences through dimensions $d(D; \Sigma, n)$ and $d(D + a; \Sigma, n)$ (by Corollary 4.19 and Theorem 2.11). Since $d(D; \Sigma, n) \leq d(D + a; \Sigma, n)$, the map $j_D$ is a homotopy equivalence through dimension $d(D; \Sigma, n)$.

**Proof of Corollary 2.12.** Let $X_{\Sigma}$ be a compact smooth toric variety such that $\Sigma(1) = \{ \text{Cone}(n_k) : 1 \leq k \leq r \}$, where $\{n_k\}_{k=1}^r$ are primitive generators as in Definition 2.2. Since $X_{\Sigma}$ is a compact, by (ii) of Lemma 3.6 we easily see that the condition $(2.15.1)$ is satisfied for $X_{\Sigma}$. Since $\Sigma_1 \subset \Sigma$, by using Lemma 3.6 we see that $X_{\Sigma_1}$ is a non-compact smooth toric subvariety of $X_{\Sigma}$. Moreover, since $\Sigma(1) \subset \Sigma_1 \subset \Sigma$, we see that $\Sigma_1(1) = \Sigma(1)$. Hence, the condition $(2.15.1)$ holds for $X_{\Sigma_1}$, too. Thus, the assertion follows from Theorem 2.11. \qed

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