Universal properties of population dynamics with fluctuating resources *

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Abstract

Starting from the well-known field theory for directed percolation, we describe an evolving population, near extinction, in an environment with its own nontrivial spatio-temporal dynamics. Here, we consider the special case where the environment follows a simple relaxational (Model A) dynamics. Two new operators emerge, with upper critical dimension of four, which couple the two theories in a nontrivial way. While the Wilson-Fisher fixed point remains completely unaffected, a mismatch of time scales destabilizes the usual DP fixed point, suggesting a crossover to a first order transition from the active (surviving) to the inactive (extinct) state.

Key words: Renormalization group, population dynamics, directed percolation, ecology
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Introduction. The understanding of ecological catastrophes, where one or several species become extinct, is a major challenge in various areas of science. From a physics perspective, extinction events are frequently associated with a continuous phase transition from an active to an inactive (absorbing) state from which the population cannot recover. As the transition is approached, the dynamics is characterized by large fluctuations, whose characteristic length scale diverges. By virtue of this diverging scale, the large-distance long-time behavior of systems near continuous phase transitions is universal, i.e., independent of microscopic detail, and can be captured successfully by minimal

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models. Renormalization group (RG) approaches have been extremely successful in exploring the physics of such models near criticality. In this brief note, we explore the effects of a fluctuating environment, reflecting, e.g., a food supply with its own nontrivial dynamics, on the universal properties of this active-inactive state transition. We first summarize the field theory for this transition in a uniform environment. Next, we introduce the dynamics of the environment and discuss its effects on the evolving population. We conclude with a brief summary and some open questions.

The model. The minimal model describing the extinction of a single species in a uniform environment is well known [1,2,3]. It finds its applications in a broad range of problems, ranging from population dynamics to catalysis, forest fires, and directed percolation (DP). It is most easily expressed in the language of “chemical” reactions, describing the birth ($A \rightarrow 2A$), death ($A \rightarrow \emptyset$), and overcrowding ($A + A \rightarrow A$) of a population, with rates $\sigma$, $\mu$, and $g/2$, respectively. The individuals are free to diffuse in $d$ dimensions. Once the last individual has died, the species cannot recover. Denoting the space- and time-dependent density of the population by $s(x,t)$, the associated field theory is well known [1,2] and can be written in terms of a dynamic functional, with a response field $\tilde{s}(x,t)$:

$$J_{DP}\{\tilde{s}, s\} = \int d^d x \int dt \lambda \tilde{s} \left\{ \lambda^{-1} \partial_t + \left( \tau - \nabla^2 \right) + \frac{1}{2} \left( gs - \bar{g} \tilde{s} \right) \right\} s$$  \hspace{1cm} (1)

Here, $\tau \equiv \mu - \sigma$ is the net death rate which controls the distance from the (mean-field) critical point, and $\lambda$ sets the time scale. The stochastic character of the dynamics, along with the absorbing state condition, is reflected in the noise vertex, $\bar{g} \tilde{s}^2 s$. Invariance under rapidity reversal implies $g = \bar{g}$. For now, we retain different nonlinearities here, to allow for a breaking of this symmetry in the full model. The effects of the environment are encoded in the constant rates $\tau$, $g$, and the time scale $\lambda$. It is natural to study the consequences of fluctuating rates, reflecting an environment that has its own nontrivial spatiotemporal structure. Previously, such effects as a quenched random $\tau(x)$ [4] or a coupling to a diffusive mode [5,6,7] were investigated. Here, we study an environment with non-conserved relaxational dynamics, described by Model A [8,9]. Denoting the Model A local order parameter as $\varphi$, modelling, e.g., a nutrient supply for the population, the corresponding field theory is given by

$$J_{A}\{\tilde{\varphi}, \varphi\} = \int d^d x \int dt \gamma \left\{ \tilde{\varphi} \left[ \gamma^{-1} \partial_t \varphi + \left( r - \nabla^2 \right) \varphi + \frac{u}{3!} \varphi^3 \right] - \tilde{\varphi}^2 \right\}$$  \hspace{1cm} (2)

where $\tilde{\varphi}$ is, again, the corresponding response field, and $r$ is the critical parameter. Noting that both field theories have an upper critical dimension $d_c = 4$, we seek novel couplings, involving both Model A and DP fields, which might destabilize the familiar DP [2] or Wilson-Fisher [10] fixed points. If $\mu$ is an
external momentum scale, naive dimensional analysis results in $r \sim \tau \sim \mu^2$. Assuming that the net death rate depends on the environment (e.g., via the availability of nutrients) and that the presence of the population affects its environment, it is natural to write an expansion $\tau = \tau_o + \kappa \varphi + w \varphi^2 + \ldots$ and $r = r_o + vs + \ldots$. Dimensional analysis shows that $\kappa \sim \mu^{(6-d)/2}$, $v \sim \mu^{(4-d)/2}$, and $w \sim \mu^{4-d}$. The coupling $\kappa$ is relevant in $d = 4$ and must be tuned to zero to access the multi-critical point $r = \tau = \kappa = 0$. Alternately, if we demand that the up-down symmetry of the Ising model remain valid, $\kappa$ can be set to zero for physical reasons. Once zero, it is not generated under the RG. Collecting, our model is given by

$$J\{\tilde{\phi}, \phi, \tilde{s}, s\} = J_{DP}\{\tilde{s}, s\} + J_A\{\tilde{\phi}, \phi\} + J_{int}\{\tilde{\phi}, \phi, \tilde{s}, s\}$$

where

$$J_{int}\{\tilde{\phi}, \phi, \tilde{s}, s\} \equiv \int d^d x \int dt \left\{ \gamma v \tilde{\phi} \tilde{s} + \frac{1}{2} \lambda w \tilde{s} s \phi^2 \right\}$$

(3)

Expectation values of the four fields are given as functional integrals with weight $\exp(-J)$.

**RG results.** All five nonlinear couplings - $u$, $g$, $\bar{g}$, $v$, and $w$ - are marginal in the upper critical dimension $d_c = 4$. The new vertex $v$ violates the invariance under rapidity reversal but respects the absorbing state condition. We use renormalized perturbation theory, in $\epsilon = d_c - d$, combined with minimal subtraction. Due to the absence of a bare correlator in the DP field theory, no corrections to the Model A correlation and response functions are generated. As a result, the Model A fixed point remains at the Wilson-Fisher value, and all Model A exponents retain their familiar values. There are, however, corrections to the DP couplings, and nontrivial flow equations for the new mixed couplings. In one-loop order, we find numerous infrared unstable fixed points, and one nontrivial stable one, characterized by the renormalized values $u = 2\epsilon/3 + O(\epsilon^2)$, $g\bar{g} = 4\epsilon/3 + O(\epsilon^2)$, $v\bar{g} = 0 + O(\epsilon^2)$, and $w = 13(\lambda + \gamma)/(12\lambda)$. A geometric factor $G_\epsilon \equiv 2\Gamma(1 + \epsilon/2)(4\pi)^{-d/2}$ occurs in each loop integral and has been absorbed in the definition of the couplings. At first glance, this looks encouraging; in particular, $v\bar{g} = 0$ restores the rapidity reversal symmetry. However, at this fixed point, the ratio $\rho \equiv \gamma/\lambda$ of the two time scales flows towards $\rho = 0$. This implies that the dynamics of Model A freezes on the time scale of the DP fields. The limit $\rho \to 0$ turns out to be singular, so that the theory needs to be reanalyzed. At the most naive level, this involves adding the time-delocalized vertex, $\bar{w} \int d^d x \int dt \int dt' \tilde{s}(x, t) s(x, t') \tilde{s}(x, t') s(x, t')$, to the DP field theory. Once included, however, all fixed points become unstable [3,4], indicating that the active-inactive state transition might turn first order. At a more rigorous level, one should integrate out the Model A fields and then take the static limit. The resulting field theory will be analyzed elsewhere [11].

**Conclusions.** In summary, we have investigated the effect of a fluctuating environment on the evolution of a population near the brink of extinction.
These fluctuations may be due to a food source, for example, which has its own nontrivial spatio-temporal dynamics. Here, we consider the interactions of the well-known DP field theory with an auxiliary field obeying simple relaxational (Model A) dynamics. Investigating the combined field theory to one-loop order in $\epsilon = 4 - d$, we find that the RG flow implies a freezing of the Model A dynamics on the times scales of the DP fields. Reanalyzing the theory in this (singular) limit, all fixed points are now infrared unstable, indicating a possible first order transition. Two open questions remain, namely, first, to quantify the crossover to the time-delocalized theory, and second, to compare these field-theoretic findings near $d_c = 4$ to Monte Carlo simulations in two and three dimensions.

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