A special case of the existential version of the Non-commutative Khintchine inequality
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1 Introduction

The following theorem is known as the Non-commutative Khintchine inequality.

Theorem 1.1. Let $A_1, \ldots, A_n$ be $d \times d$ symmetric matrices. Let $e_1, \ldots, e_n$ be random variable taking values 1 or $-1$ with equal probability. Then there exists a constant $K$ such that

$$
\mathbb{E}[||\sum_i e_i A_i||] \leq K \sqrt{\log d} \sqrt{||\sum_i A_i^2||}.
$$

For further reading on this inequality we refer to the works of [Tro12], [BVH16], [LvHY18]. Given this inequality there is a question that arises out of similar inequalities in the vein of Spencer’s six sigma theorem [Spe85]. While there are logarithmic factors in the suprema of the random signing, is there a specific signing for which the logarithm is removed. In particular we ask the following question.

Does there exist a constant, $K$ independent of $d$, such that given $A_1, \ldots, A_n$ symmetric $d \times d$ matrices, is it true that there exists a signing, as in a sequence of 1 and $-1$, $e_1, \ldots, e_n$, such that

$$
\mathbb{E}[||\sum_i e_i A_i||] \leq K \sqrt{||\sum_i A_i^2||}?
$$

Here we prove a special case of the above.

Theorem 1.2. Let $A = \{a_{ij}\}_{i,j \in \mathbb{N}}$ be a bounded operator. Then there exists a signing of $A$ such that

$$
||A \circ S||_2 < 2||A||_{l_\infty(l_2)},
$$

where $A \circ S$ denotes the matrix generated by the entry-wise product of $A$ and $S$.

A similar result was proved in 1997 by Françoise Lust-Piquard [LP97].

Theorem 1.3. For every matrix $A = (a_{ij})$ such that $A$ and $A^*$ are bounded in $l^\infty(l^2)$ norm, there exists a matrix $B = (b_{ij})$ defining a bounded operator: $l^2 \to l^2$ such that

(i) $||B||_{2 \to 2} \leq K\text{max}\{||A||_{l_\infty(l_2)}, ||A^*||_{l_\infty(l_2)}\}$

(ii) $\forall i, j \in \mathbb{N}$, $|b_{ij}| \geq |a_{ij}|$,

where $K$ is an absolute constant and $|A|_{l_\infty(l_2)} \coloneqq \max_j \sqrt{\sum_i a_{ij}^2}$.
Our theorem is an improvement of this result in two ways. Firstly we show that there exists a signing of the matrix $A$ which satisfies the above theorem. (A signing is a matrix $B$ such that $|b_{ij}| = |a_{ij}|$). Secondly we get that the constant $K$ as $\sqrt{2}$ suffices. In fact our constant is tight in the case of signings. Results in this vein but for different norms have been proved by Pisier [Pis77]. In particular they prove that given a matrix $A$, there exists a signing, $B$ such that

$$||B||_\infty \rightarrow 1 \leq K||A||_{l_1(l_2)}.$$  

2 Notation and definitions

Given a $n \times n$ matrix $A = \{a_{ij}\}$, denote by

$$|A|_2 = \max_{x \in \mathbb{R}^n} ||Ax||_2 / ||x||_2.$$  

And denote

$$|A|_{l_\infty(l_2)} = \max_j \sqrt{\sum_i a_{ij}^2}.$$  

Definition 2.1. Given two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, define their Schur product, $A \circ B$ to be the matrix whose $(i, j)$’th entry is $a_{ij}b_{ij}$.

Definition 2.2. (Signings) A sign matrix is a $n \times n$ matrix, $S$ all of whose entries are $1$ or $-1$. A symmetric sign matrix is, as the name suggests, a symmetric matrix which is also a sign matrix. Let $S$ be the collection of all symmetric sign matrices of size $n$.

Given any matrix $A$ and a sign matrix $S$, a signing of $A$ by $S$ is simply the matrix $A \circ S$.

Definition 2.3. A dimer arrangement $D$ of size $d$ on the set $\{1, 2, ..., n\}$ is a set of tuples $\{(i_1, j_1), ..., (i_d, j_d)\}$ such that all the the $i$’s and $j$’s are distinct from one another. The size of the dimer arrangement $D$, denoted by $|D|$ is the number of tuples, $d$. Let $D$ be the set of all dimer arrangements of size $d$.

The canonical weight of a dimer arrangement on a matrix $A$ is defined to be $W_A(D) = \Pi_{(i,j)\in D} a_{ij}$.

Again given a matrix $A$, define the dimer partition function as

$$Z_d(A) = \sum_{|D|=d} W_A(D),$$

where the sum runs over all possible dimer arrangements of size $d$.

Definition 2.4. Finally given a $n \times n$ matrix $A$, the matching polynomial of $A$ is then defined to be

$$\mu_A(x) = \sum_{i=0}^{n/2} (-1)^d Z_d(A)x^{n-2d}.$$
3 Preliminaries

The following is a trivial modification of theorem 3.6 in [MSS13].

**Theorem 3.1.** Let $A$ be a symmetric matrix. As previously defined let $S$ be the set of all symmetric signing matrices. Let $S$ be a random signing chosen uniformly from $S$. Then

$$
\mathbb{E}_S[\det(xI - A \circ S)] = \mu_{A \circ A}(x).
$$

**Proof.** Let $\text{Sym}(T)$ denote the set of permutations of a set $T$. Let $|\sigma|$ denote the entropy or the number of inversions of a permutation $\sigma$. Then

$$
\mathbb{E}_S[\det(xI - A \circ S)] = \mathbb{E}_S \left[ \sum_{\sigma \in \text{Sym}([n])} (-1)^{|\sigma|} \prod_{i=1}^{n} (xI - A \circ S)_{i,\sigma(i)} \right]
$$

$$
= \mathbb{E}_S \left[ \sum_{k=0}^{n} x^{n-k} \sum_{T \subset [n];|T|=k} \sum_{\sigma \in \text{Sym}(T)} (-1)^{|\sigma|} \prod_{i=1}^{k} (A \circ S)_{i,\sigma(i)} \right]
$$

$$
= \sum_{k=0}^{n} x^{n-k} \sum_{T \subset [n];|T|=k} \sum_{\sigma \in \text{Sym}(T)} (-1)^{|\sigma|} \mathbb{E}_S \left[ \prod_{i=1}^{k} -a_{i,\sigma(i)} s_{i,\sigma(i)} \right].
$$

But the $s_{i,j}$ are all independent excepting $s_{i,j} = s_{j,i}$, with expectation, $\mathbb{E}(s_{i,j}) = 0$. Thus only even powers of $s_{i,j}$ survive the expectation. So we may only consider permutations which only have orbits of size 2. These are just the perfect matchings on $S$ or alternatively exactly all the dimer arrangements of size $|S|$. There are no such matchings when $|S|$ is odd. Otherwise its entropy is $|S|/2$. And since

$$
\mathbb{E}[(-a_{i,j} s_{i,j})^2] = a_{i,j}^2,
$$

we get

$$
\mathbb{E}_S[\det(xI - A \circ S)] = \sum_{k=0}^{n/2} x^{n-2k} \sum_{|D|=k; D \subset D} (-1)^k \prod_{(i,j) \in D} a_{i,j}^2 = \mu_{A \circ A}(x).
$$

\[ \square \]

The next theorem is the famous Heilman-Leib theorem which proves that the matching polynomial is real rooted and gives a bound for the maximum root of the matching polynomial of a matrix. It can be found in [HL72] as theorem 4.2 and 4.3.

**Theorem 3.2.** Let $A$ be a symmetric matrix with real positive entries. Let $b$ be the maximum row sum of $A$ i.e. $b = \max_{i \in [n]} \{\sum_j a_{i,j}\}$. Then $\mu_A(x)$ is real rooted and any root $\lambda$ satisfies, $\lambda < 2\sqrt{b}$.

An immediate corollary of the above theorem is the following.

**Corollary 3.3.** Let $A$ be any symmetric matrix. Let $r_1, ..., r_n$ be the rows of $A$. Let $\|r_i\|_2$ be the L2 norm of the vector $r_i$. Let $|A|_{\infty}(r_2) = \max_{i \in [n]} \|r_i\|_2 = \max_{\|x\|_\infty} \|Ax\|_2$.

Then every root $\lambda$ of $\mu_{A \circ A}(x)$ satisfies $|\lambda| < 2|A|_{\infty}(r_2)$. 

3
The final piece of the puzzle is the theory of interlacing families found in both [HL72] and [MSS13].

**Definition 3.4.** We say a polynomial \( g(x) = \prod_{i \in [n-1]} (x - \alpha_i) \) interlaces \( f(x) = \prod_{i \in [n]} (x - \beta_i) \) iff \( \beta_1 \leq \alpha_1 \leq \beta_2 \ldots \leq \alpha_{n-1} \leq \beta_n \).

We say that the polynomials \( f_1, \ldots, f_k \) have a common interlacing if there is a polynomial \( g \) such that \( g \) interlaces \( f_i \) for each \( i \).

It turns out that when all the polynomials are monic and of same degree, they have a common interlacing if and only if every convex combination is real-rooted.

**Definition 3.5.** Let \( S_1, \ldots, S_m \) be finite sets and for every assignment \( s_1, \ldots, s_m \in S_1 \times \ldots \times S_m \), let \( f_{s_1, \ldots, s_m} \) be a real rooted \( n \) degree polynomial with positive leading coefficient. For a partial assignment \( s_1, \ldots, s_k \in S_1 \times \ldots \times S_k \), with \( k < m \) define

\[
 f_{s_1, \ldots, s_k} = \sum_{s_{k+1} \in S_{k+1}, \ldots, s_m \in S_m} f_{s_1, \ldots, s_m},
\]

as well as

\[
 f_{\emptyset} = \sum_{s_1 \in S_1, \ldots, s_m \in S_m} f_{s_1, \ldots, s_m}.
\]

We say that \( \{f_{s_1, \ldots, s_m}\}_{s_1, \ldots, s_m} \) form an interlacing family, iff for all \( k < m \), and for all \( s_1, \ldots, s_k \in S_1 \times \ldots \times S_k \) the set of polynomials \( \{f_{s_1, \ldots, s_k, t}\}_{t \in S_{k+1}} \) have a common interlacing.

Then we have the following theorems from [MSS13] (thm 4.4)

**Theorem 3.6.** Let \( S_1, \ldots, S_m \) be finite sets and let \( \{f_{s_1, \ldots, s_m}\} \) be an interlacing family of polynomials. Then there exists some \( s_1, \ldots, s_m \in S_1 \times \ldots \times S_m \) such that the largest root of \( f_{s_1, \ldots, s_m} \) is less than or equal to the largest root of \( f_{\emptyset} \).

Finally let \( S_i = \{1, -1\} \) for \( 1 \leq i \leq m = n(n+1)/2 \). Then we note that each element of \( S_1 \times \ldots \times S_m \) corresponds to a symmetric signed matrix \( S_{s_1, \ldots, s_m} \). Thus we can define the polynomial \( f_{s_1, \ldots, s_m}(x) = \det(xI - A \circ S_{s_1, \ldots, s_m}) \). Again by [MSS13] (Theorem 5.2), we get that

**Theorem 3.7.** \( f_{s_1, \ldots, s_m}(x) \) forms an interlacing family.

While in the referred paper the authors use this theorem only on adjacency matrices of graphs (whose entries are only 0 or 1), its proof is valid over any symmetric matrix. The key idea behind the proof is that there is this class of functions on matrices called determinant-like, which remain determinant-like (and real-rooted) under a rank-one update.

4 Statement and Proof of main Theorem

Now we proceed to proving our main theorem.

**Theorem 4.1.** Let \( A \) be any \( n \times n \) matrix. Then there exists a signing matrix not necessarily symmetric such that

\[
 ||A \circ S||_2 \leq 2||A||_{f_2(t_2)}.
\]
Let $\mu$ be the largest root of

$$D = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix},$$

where $A^T$ denotes the transpose of $A$.

Note that $A_D$ is a symmetric matrix. Let $S$ be the set of all $2n \times 2n$ sign matrices. Let $\mathcal{S}$ be a sign matrix chosen uniformly from $\mathcal{S}$.

Then by Theorem 3.1,

$$\mathbb{E}_S[\det(x - A_D \circ S)] = \mu_{A_D \circ A_D}(x).$$

But by Theorem 3.7, the polynomials in the left hand side of the above equation form an interlacing family. Therefore by Theorem 3.6, there exists some signing matrix $S'$ such that, the largest root of $\det(x - A_D \circ S')$ is less than or equal to the largest root of $\mu_{A_D \circ A_D}(x)$.

But using Corollary 3.3, every root of $\mu_{A_D \circ A_D}(x)$ is in modulus smaller than $2|A_D|_{l_\infty(l_2)}$.

Combining all this we have a $2n \times 2n$ sign matrix $S'$ such that the largest eigen-value of $A_D \circ S'$ is less $2|A_D|_{l_\infty(l_2)}$. Let $S' = \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_1 \end{bmatrix}$ Then using Schur complements,

$$\det(xI - A_D \circ S) = x^n \det(x - x^{-1}(A \circ S_2)(A^T \circ S_2^T)) = \det(x^2 - (A \circ S_2)(A \circ S_2)^T).$$

Thus the largest eigenvalue of $A_D \circ S'$ is simply the largest singular value or the L2 norm of $A \circ S_2$.

So we have a signing matrix $S_2$, with

$$||A \circ S_2||_2 < 2||A \circ S_2||_{l_\infty(l_2)} = ||A||_{l_\infty(l_2)}.$$

\begin{proof}

Given $A$, define the dilation $A_D$ to be the $2n \times 2n$ matrix,

$$A_D = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix},$$

where $A^T$ denotes the transpose of $A$.

Note that $A_D$ is a symmetric matrix. Let $\mathcal{S}$ be the set of all $2n \times 2n$ sign matrices. Let $S$ be a sign matrix chosen uniformly from $\mathcal{S}$.

Then by Theorem 3.1,

$$\mathbb{E}_S[\det(x - A_D \circ S)] = \mu_{A_D \circ A_D}(x).$$

But by Theorem 3.7, the polynomials in the left hand side of the above equation form an interlacing family. Therefore by Theorem 3.6, there exists some signing matrix $S'$ such that, the largest root of $\det(x - A_D \circ S')$ is less than or equal to the largest root of $\mu_{A_D \circ A_D}(x)$.

But using Corollary 3.3, every root of $\mu_{A_D \circ A_D}(x)$ is in modulus smaller than $2|A_D|_{l_\infty(l_2)}$.

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Thus the largest eigenvalue of $A_D \circ S'$ is simply the largest singular value or the L2 norm of $A \circ S_2$.

So we have a signing matrix $S_2$, with

$$||A \circ S_2||_2 < 2||A \circ S_2||_{l_\infty(l_2)} = ||A||_{l_\infty(l_2)}.$$

\end{proof}

**Theorem 4.2.** (Extension to infinite dimensions). Let $A = \{a_{ij}\}_{i,j \in \mathbb{N}}$ be a bounded infinite dimensional operator. Then there exists a signing of $A$ such that

$$||A \circ S||_2 < 2||A \circ S||_{l_\infty(l_2)}.$$

\begin{proof}

For any integer $n$, let $A_n$ be the operator constructed from $A$ by taking the upper $n \times n$ part of $A$ and filling everything else with 0. Then by our previous result, there exists a signing $S_n$ such that

$$||A_n \circ S_n||_2 < 2||A_n \circ S_n||_{l_\infty(l_2)} = 2||A_n||_{l_\infty(l_2)} \leq 2||A||_{l_\infty(l_2)}.$$

Thus as the sequence $\{A_n \circ S_n\}$ is uniformly bounded, by using sequential Banach Alaoglu, there is a subsequence $k_n$ such that $A_{k_n} \circ S_{k_n}$ converges weakly to some matrix $B$. Note that $k_n$ approaches infinity, thus eventually every $i,j$ position of this subsequence is either $a_{ij}$ or $-a_{ij}$. Thus the weak limit is also a signing of $A$. Denote $B_n = A_{k_n} \circ S_{k_n}$.

Thus $B_n^*B_n$ also converges weakly to $B^*B$. Then for any $x$, we have that $\langle x, B_n^*B_nx \rangle$ converges to $\langle x, B^*Bx \rangle = ||Bx||_2$. Thus we have that for any $x$, such that $||x||_2 = 1$,

$$||Bx||_2 < 2||A||_{2,\infty}. $$

\end{proof}
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