Stability of the Weak Martingale Optimal Transport Problem

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Abstract

While many questions in (robust) finance can be posed in the martingale optimal transport (MOT) framework, others require to consider also non-linear cost functionals. Following the terminology of Gozlan, Roberto, Samson and Tetali [27] for classical optimal transport, this corresponds to weak martingale optimal transport (WMOT).

In this article we establish stability of WMOT which is important since financial data can give only imprecise information on the underlying marginals. As application, we deduce the stability of the superreplication bound for VIX futures as well as the stability of the stretched Brownian motion and we derive a monotonicity principle for WMOT.

Keywords: VIX futures, Robust finance, Weak optimal transport, Martingale Optimal Transport, Stability, Convex order, Martingale couplings.

1 Introduction

1.1 Overview

Gozlan, Roberto, Samson and Tetali [27] recently introduced a non-linear relaxation of classical optimal transport. On the one hand, this framework of weak optimal transport (WOT) still retains many characteristics of usual optimal transport, allowing for a compelling theory. On the other hand, this type of relaxation is suitable to cover a number of problems from geometric inequalities over optimal mechanism design to the Schrödinger problem that lie outside the scope of the classical theory, see [26, 25, 2, 8]. The same non-linear relaxation is also required in a number of challenges appearing in mathematical finance: this concerns the pricing of VIX futures, stretched Brownian motion, robust pricing for fixed income markets and the optimal Skorokhod embedding problem. Compared to other applications of WOT, the financial applications stick out through the additional martingale condition that results from the no arbitrage assumption prevalent in mathematical finance. This leads to a weak martingale optimal transport problem (WMOT) which is the main focus of this article.

The reason why a transport type problem pops up in finance is that the marginals are fixed based on the celebrated observation of Breeden–Litzenberger [17] that prices of traded vanilla options fix the marginals of the asset price process (S_t) at the respective maturity times.

This is only an approximation to reality as we will know only the prices of finitely many derivatives (up to a bid ask spread). To fit the challenges from finance into the WMOT framework, it is thus crucial to establish stability of WMOT w.r.t. to the initial data, which is the main aim of this paper. As particular applications we consider the robust pricing problem for VIX options and the stretched Brownian motion and we will derive a monotonicity principle for the WMOT problem. These results will be presented in some detail in the remainder of the introduction, while the corresponding proofs are postponed to later sections.
1.2 WMOT-framework and main result

The main contribution of this article is to establish stability for the weak and weak martingale optimal transport, extending and unifying previously known stability results in MOT and WOT. To state our results rigorously, we need to introduce some concepts.

Let $X$ and $Y$ be Polish spaces endowed with the compatible and complete metrics $d_X$ and $d_Y$. Let $\mu$ be in the set $\mathcal{P}(X)$ of probability measures on $X$, $\nu \in \mathcal{P}(Y)$ and $C : X \times \mathcal{P}(Y) \to \mathbb{R}_+$ be a nonnegative measurable function which is convex in the second argument. We denote by $\Pi(\mu, \nu)$ the set of couplings between $\mu$ and $\nu$, that is $\pi \in \Pi(\mu, \nu)$ if and only if $\pi \in \mathcal{P}(X \times Y)$ is such that for any measurable subsets $A \subset X$ and $B \subset Y$, $\pi(A \times Y) = \mu(A)$ and $\pi(X \times B) = \nu(B)$. Then the WOT problem consists in the minimisation of

$$V_C(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx), \quad \text{(WOT)}$$

where for all $\pi \in \Pi(\mu, \nu)$, $(\pi_x)_{x \in \mathbb{R}}$ denotes a disintegration of $\pi$ with respect to its first marginal, which we write $\pi(dx, dy) = \mu(dx) \pi_x(dy)$, or with a slight abuse of notation, $\pi = \mu \otimes \pi_x$ if the context is not ambiguous. Note that for a measurable map $c : X \times Y \to \mathbb{R}_+$, the WOT problem with the cost function $C : (x, p) \mapsto \int_Y c(x, y) p(dy)$ linear in the measure argument amounts to the classical Optimal Transport (OT) problem.

To recall some basic results on existence, duality and stability of the WOT problem, we introduce the appropriate topologies of the underlying spaces. We fix $r \geq 1$ and $x_0, y_0$ two arbitrary elements of $X$ and $Y$ respectively, their specific value having no impact on our study. Let $\mathcal{P}^r(X)$ denote the set of all probability measures on $X$ with finite $r$-th moment, i.e. $\mathcal{P}^r(X) = \{p \in \mathcal{P}(X) : \int_X d_X(x, x_0)^r p(dx) < +\infty\}$. Let $\mathcal{C}(X)$ denote the set of all real-valued continuous functions on $X$. The set $\mathcal{P}^r(X)$ is equipped with the weak topology induced by $\Phi^r(X) = \{f \in \mathcal{C}(X) : \forall x \in X, |f(x)| \leq \alpha(1 + d_X(x, x_0))\}$. A compatible metric is given by the $r$-Wasserstein distance $W_r$, see [4, 34, 36, 37] for more details.

In analogy to the classical transport problem it is natural to assume that the cost function $C : X \times \mathcal{P}^r(Y) \to \mathbb{R}_+$ is lower semicontinuous. As in the classical case, the WOT problem then admits a minimiser and a natural duality relation holds, see [5]. As in the case of classical optimal transport, stability is a more delicate question; in [9, Theorem 1.3], it is established for continuous cost functions satisfying an appropriate growth condition.

Motivated by the model-independent pricing problem, the martingale optimal transport (MOT) has received considerable attention, see [31, 23, 11, 21, 19]. We also refer to [33, 20, 24] for the multi-dimensional case and to [10, 14] for connections to Skorokhod problem. Stability of MOT, that is the continuous dependence of value and optimiser on the marginals, is a crucial property in light of its numerical resolution and the imperfect data coming available in financial applications and was recently established in [29, 9, 38].

In one time step, the MOT problem reads as

$$\inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} c(x, y) \pi(dx, dy), \quad \text{(MOT)}$$

where $\mu, \nu \in \mathcal{P}(\mathbb{R})$, $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is a measurable cost function, and

$$\Pi_M(\mu, \nu) := \left\{ \pi = \mu \otimes \pi_x \in \Pi(\mu, \nu) : \int_{\mathbb{R}} y \pi_x(dy) = x \mu \text{-a.s.} \right\}$$

The martingale constraint reflects the condition for a financial market with vanishing interest rates to be free of arbitrage. According to Strassen’s theorem, $\Pi_M(\mu, \nu)$ is not empty if and only if $\mu$ is smaller than $\nu$ in the convex order in the following sense:

$$\forall \varphi : \mathbb{R} \to \mathbb{R} \text{ convex }, \int_{\mathbb{R}} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}} \varphi(y) \nu(dy),$$

$^1$Convexity of $C$ in the second argument is a standing assumption in weak transport theory, in fact the theory for the general case can be reduced to the convex instance, see [1, Theorem 3.7].
which we also write \( \mu \leq_C \nu \).

Different problems in finance require to consider also the ‘weak’ version of this problem (WMOT) which permits costs \( C \) of the form \( C : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}_+ \). The WMOT problem consists in the minimisation of

\[
V^C_M(\mu, \nu) := \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R}} C(x, \pi_x) \mu(dx).
\]  

For a measurable map \( c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_+ \), the WMOT problem associated to the cost function \( C : (x, p) \mapsto \int_{\mathbb{R}} c(x, y) p(dy) \) linear in the measure argument amounts to an ordinary MOT problem.

Formally, WMOT is of course captured by WOT if one sets \( C(x, p) \) to be \( \infty \) if the barycenter of \( p \) does not equal \( x \). However this has previously not found much use as this cost functions does not fall in the realm of the typical regularity assumptions. Specifically, stability of WOT and MOT is established in two separate results in [9]. In the present article, we unify and extend these results, in particular we will prove the following:

**Theorem 1.1 (Stability).** Assume that \( C : \mathbb{R} \times \mathcal{P}^1(\mathbb{R}) \rightarrow \mathbb{R} \) is continuous, convex in the second argument and that there exists a constant \( K > 0 \) such that for \( (x, p) \in \mathbb{R} \times \mathcal{P}^1(\mathbb{R}) \)

\[
|C(x, p)| \leq K \left( |x| + \int_{\mathbb{R}} |y| p(dy) \right).
\]  

(1.1)

For \( k \in \mathbb{N} \), let \( \mu^k, \nu^k \in \mathcal{P}^1(\mathbb{R}) \), (resp. satisfying \( \mu^k \leq_C \nu^k \)) converge in \( \mathcal{P}^1(\mathbb{R}) \) to \( \mu \) and \( \nu \), respectively. Then

\[
\lim_{k \rightarrow \infty} V_C(\mu^k, \nu^k) \stackrel{\text{resp.}}{=} \lim_{k \rightarrow \infty} V^M_C(\mu^k, \nu^k) = V_C(\mu, \nu) \quad \text{(resp. } V^M_C(\mu, \nu) \text{).}
\]

There exist minimisers \( \pi^k \in \Pi(\mu^k, \nu^k) \) (resp. \( \pi^k \in \Pi_M(\mu^k, \nu^k) \)) of \( V_C(\mu^k, \nu^k) \) (resp. \( V^M_C(\mu^k, \nu^k) \)) and any accumulation point of \( (\pi^k)_{k \in \mathbb{N}} \) for the weak convergence topology is a minimiser of \( V_C(\mu, \nu) \) (resp. \( V^M_C(\mu, \nu) \)).

In Section 2.3 we will present several stronger and more general versions of Theorem 1.1. Growth- and continuity assumptions may be weakened and in the case of WOT, the results apply to abstract Polish spaces. In contrast to Theorem 1.1, according to an intriguing counter-example given by Brückerhoff and Juillet [18], stability fails for MOT and, as a consequence, for the more general WMOT problem in \( \mathbb{R}^d \) with \( d \geq 2 \). That is why all the statements concerning those problems only address the one-dimensional case. We note that in view of financial applications, this represents no restriction.

### 1.3 Robust pricing of VIX-futures

The domain of robust mathematical finance distinctly surpasses the one of MOT. As example consider the Volatility Index (VIX), often referred to as the Fear Index, which is a popular measure to determine market sentiment. When investors expect the market to move vigorously, they typically tend to purchase more options, which has an impact on implied volatility levels. The VIX is by definition the implied volatility calculated on a 30 days horizon on the S&P 500. The more the VIX increases, the more demand is expressed for options, which become more expensive. In that case the market is described as volatile. Conversely, a decreasing VIX often means less demand and therefore decreasing option prices, hence the market is perceived as calm.

We consider the S&P 500 \((S_t)_{t \in (T_1, T_2)}\), tradable at dates \( T_1 \) and \( T_2 = T_1 + 30 \) days. We suppose known the market price of call options for any strike \( K \geq 0 \), so that by the Breeden-Litzenberger formula [17] we get the respective probability distributions \( \mu \) and \( \nu \) of \( S_{T_1} \) and \( S_{T_2} \). We allow trading at time 0 in vanilla options with maturities \( T_1 \) and \( T_2 \), and trading at time \( T_1 \) in the S&P 500 and the forward-starting log-contract, that is the option with payoff \( \frac{-2}{T_2 - T_1} \ln \frac{S_{T_2}}{S_{T_1}} \) at \( T_2 \). In this setting, Guyon, Menegaux and Nutz [30] derive the model-independent arbitrage-free upper bound for the VIX future expiring at \( T_1 \), given by the smallest superreplication price at time 0

\[
P_{\text{super}}(\mu, \nu) = \inf\{\mu(u_1) + \nu(u_2)\},
\]  

(1.2)
where the infimum is taken over all \((u_1, u_2) \in L^1(\mu) \times L^1(\nu)\) and measurable maps \(\Delta^S, \Delta^L\) such that for all \((x, y, v) \in (0, +\infty)^3 \times [0, +\infty),\)

\[
u_1(x) + u_2(y) + \Delta^S(x, v)(y - x) + \Delta^L(x, v)\left(\frac{2}{T_2 - T_1} \ln \frac{y}{x} - v^2\right) - v \geq 0. \quad (1.3)
\]

Note that the primal problem \(P_{\sup} (\mu, \nu)\) given by (1.3) involves three variables \((x, y, v)\), which stand respectively for the S&P 500 at time \(T_1\), the S&P 500 at time \(T_2\), and the VIX at time \(T_1\). Naturally we would then expect to find three marginals in the dual formulation. Strikingly, in [30, Proposition 4.10] the dual side of the superreplication of the VIX is reformulated as a WMOT problem.

**Proposition 1.2** (Guyon, Menegaux, Nutz, 2017). Let \(0 < T_1 < T_2\). Let \(\mu, \nu\) be probability measures on \((0, +\infty), \) in the convex order, and which finitely integrate \(|\ln(x)| + |x|\). Then the dual problem \(D_{\sup}\) consists of

\[
D_{\sup}(\mu, \nu) = \sup_{\pi \in \Pi_M(\mu, \nu)} \int_{[0, +\infty)} \sqrt{\frac{2}{T_2 - T_1} \int_{[0, +\infty)} \ln \frac{x}{y} \pi_x(dy) \mu(dx)},
\]

(1.4)

The values of \(P_{\sup}(\mu, \nu)\) and \(D_{\sup}(\mu, \nu)\) coincide.

We contribute to the theory by establishing stability of (1.4):

**Theorem 1.3.** In the setting of Proposition 1.2, for \(k \in \mathbb{N}\) let each pair \((\mu^k, \nu^k), (\mu, \nu)\) of probability measures on \((0, +\infty)\) be in the convex order and finitely integrate \(f(x) := |\ln(x)| + |x|\). For \(k \to +\infty\), let \((\mu^k, \nu^k)\) converge weakly to \((\mu, \nu)\) and

\[
\mu^k(f) \to \mu(f), \quad \nu^k(f) \to \nu(f).
\]

Then

\[
\lim_{k \to +\infty} D_{\sup}(\mu^k, \nu^k) = D_{\sup}(\mu, \nu),
\]

for each \(k \in \mathbb{N}\), there exist in \(\Pi_M(\mu^k, \nu^k)\) maximisers of \(D_{\sup}(\mu^k, \nu^k)\), and any weak accumulation point of a sequence \((\pi^{k,*})_{k \in \mathbb{N}}\) of such maximizers maximizes \(D_{\sup}(\mu, \nu)\).

### 1.4 Stretched Brownian motion

In [7] a Benamou-Brenier type formulation for MOT is suggested. In dimension one, this problem consists in maximising for two probabilities \(\mu, \nu\) in the convex order

\[
MT(\mu, \nu) := \sup \mathbb{E} \left[ \int_0^1 \sigma_t \, dt \right]
\]

(MBB)

over all filtered probability spaces \((\Omega, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P}), \) real-valued \((\mathcal{F}_t)_{t \in [0,1]}\)-progressive process \((\sigma_t)_{t \in [0,1]}\) and real-valued \((\mathcal{F}_t)_{t \in [0,1]}\)-Brownian motions \((B_t)_{t \in [0,1]}\) such that the process

\[
(M_t)_{t \in [0,1]} = \left( M_0 + \int_0^t \sigma_s \, dB_s \right)_{t \in [0,1]}
\]

is a continuous martingale which satisfies \(M_0 \sim \mu\) and \(M_1 \sim \nu\). When the second moment of \(\nu\) is finite, then (MBB) has a unique maximiser \((M_0^*, M_1^*)_{t \in [0,1]}\) [7, Theorem 1.5] called the stretched Brownian motion from \(\mu\) to \(\nu\), since it is the martingale subject to the constraints \(M_0^* \sim \mu\) and \(M_1^* \sim \nu\) which correlates the most with the Brownian motion.

Let \(C_2 : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}\) be defined for all \((x, p) \in \mathbb{R} \times \mathcal{P}_2(\mathbb{R})\) by \(C_2(x, p) = \mathcal{W}_2^2(p, \mathcal{N}(0, 1))\), where \(\mathcal{N}(0, 1)\) denotes the unidimensional standard normal distribution. Let \(\mu, \nu \in \mathcal{P}_2(\mathbb{R})\) be in the convex order and \(V_{C_2}^M(\mu, \nu)\) be the value function given by (WMOT) for the cost function \(C_2\). Let \(\pi^* \in \Pi_M(\mu, \nu)\) be optimal for \(V_{C_2}^M(\mu, \nu)\) and \(M^*\) be the stretched Brownian motion from \(\mu\) to \(\nu\). Then Remark 2.1, Theorem 2.2 and Remark 2.3 from [7] imply that

4
(a) \(MT(\mu, \nu) = \frac{1}{2}(1 + \int_{\mathbb{R}} |y|^2 \nu(dy) - V_{C_2}^M(\mu, \nu))\);

(b) \(\pi^*\) is the joint probability distribution of \((M_0^*, M_1^*)\), and conversely

\[
\forall t \in [0, 1], \quad M_t^* = \mathbb{E}\left[F_{x_1}^{-1}(F_{N(0,1)}(B_1))|X, (B_s)_{0 \leq s \leq t}\right],
\]

where \(X \sim \mu\) is a random variable independent of the Brownian motion \((B_t)_{t \in [0,1]}\), and \(F_\eta(x) = \eta((-\infty, x])\), resp. \(F_\eta^{-1}(u) = \inf\{x \in \mathbb{R} : F_\eta(x) \geq u\}\) denotes the cumulative distribution function, resp. the quantile function of a probability distribution \(\eta \in \mathcal{P}(\mathbb{R})\).

As a consequence of Theorems 2.3 and 2.8, we obtain the following stability result for the stretched Brownian motion:

**Corollary 1.4** (Stability of stretched Brownian motion). *Let \(r \geq 2\) and \(\mu^k, \nu^k, \mu, \nu \in \mathcal{P}^r(\mathbb{R})\), \(k \in \mathbb{N}\) be such that for all \(k \in \mathbb{N}\), \(\mu^k \leq \nu^k\) and \(\mu^k\), resp. \(\nu^k\), converges to \(\mu\), resp. \(\nu\), in \(W_r\).

For \(k \in \mathbb{N}\), let \(M^k\) be the stretched Brownian motion from \(\mu^k\) to \(\nu^k\) and \(M^*\) be the stretched Brownian motion from \(\mu\) to \(\nu\). Equipping \(\mathcal{C}([0,1])\) with the supremum distance and denoting by \(\mathcal{L}(Z)\) the law of any random variable \(Z\), we have \(\lim_{k \to \infty} \mathcal{W}_r((M^k_{t \in [0,1]}), \mathcal{L}((M^*_{t \in [0,1]}))) = 0\).

### 1.5 Monotonicity principle

Recently the notion of martingale \(C\)-monotonicity in [9] was introduced for WMOT to show stability of MOT.

**Definition 1.5** (Martingale \(C\)-monotonicity). We say that a Borel set \(\Gamma \subseteq \mathbb{R} \times \mathcal{P}^1(\mathbb{R})\) is *martingale \(C\)-monotone* if for any \(N \in \mathbb{N}\), any collection \((x_1, p_1), \ldots, (x_N, p_N) \in \Gamma\), and \(q_1, \ldots, q_N \in \mathcal{P}^1(\mathbb{R})\) such that \(\sum_{i=1}^N p_i = \sum_{i=1}^N q_i\) and \(\int_{\mathbb{R}} y p_i(dy) = \int_{\mathbb{R}} y q_i(dy),\) we have

\[
\sum_{i=1}^N C(x_i, p_i) \leq \sum_{i=1}^N C(x_i, q_i).
\]

So far, it was known that martingale \(C\)-monotonicity is a necessary optimality criterion in the following sense, c.f. [9, Theorem 3.4]: let \(\pi^* \in \Pi_M(\mu, \nu)\) be a martingale coupling which minimises (WMOT), then there is a martingale \(C\)-monotone set \(\Gamma\) with

\[
(x, \pi_x) \in \Gamma \quad \text{for } \mu(dx)\text{-almost every } x.
\]

**Remark 1.6.** Conversely, if \(\pi \in \Pi_M(\mu, \nu)\) is a finitely supported coupling of the form \(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}(dx) p_i(dy)\) for \(x_1 < \cdots < x_n \in \mathbb{R}\) and \(p_1, \ldots, p_N \in \mathcal{P}^1(\mathbb{R})\) and satisfies (1.6) for some martingale \(C\)-monotone set \(\Gamma\), then it is optimal. Indeed, in that case \((x_1, p_1), \ldots, (x_N, p_N) \in \Gamma\) and any martingale coupling \(\pi' \in \Pi_M(\mu, \nu)\) is of the form \(\pi'(dx, dy) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}(dx) q_i(dy)\), where \(q_1, \ldots, q_N \in \mathcal{P}^1(\mathbb{R})\) are such that \(\sum_{i=1}^N p_i = \sum_{i=1}^N q_i\) and for each \(i \in \{1, \ldots, N\}, \int_{\mathbb{R}} y p_i(dy) = x_i = \int_{\mathbb{R}} y q_i(dy).\) By definition of martingale \(C\)-monotonicity, we get

\[
\int_{\mathbb{R} \times \mathbb{R}} C(x, \pi_x) \mu(dx) = \frac{1}{N} \sum_{i=1}^N C(x_i, p_i) \leq \frac{1}{N} \sum_{i=1}^N C(x_i, q_i) = \int_{\mathbb{R}^2} C(x, \pi'_x) \mu(dx),
\]

hence \(\pi\) is optimal.

However, the question remained open if any martingale coupling satisfying (1.6) is optimal. Our stability results allow us to confirm that this is indeed the case.

**Theorem 1.7** ( Sufficiency). *Let \(r \geq 1, \mu, \nu \in \mathcal{P}^r(\mathbb{R})\) be in convex order, and \(C: \mathbb{R} \times \mathcal{P}^r(\mathbb{R}) \to \mathbb{R}\) be a measurable cost function, continuous in the second argument and such that there exists a finite constant \(K\) which satisfies

\[
\forall (x, p) \in \mathbb{R} \times \mathcal{P}^r(\mathbb{R}), \quad C(x, p) \leq K \left(1 + |x|^r + \int_{\mathbb{R}} |y|^r p(dy)\right).
\]

\[
\forall (x, p) \in \mathbb{R} \times \mathcal{P}^r(\mathbb{R}), \quad C(x, p) \leq K \left(1 + |x|^r + \int_{\mathbb{R}} |y|^r p(dy)\right).
\]
Let $\Gamma$ be martingale $C$-monotone and $\pi \in \Pi_M(\mu, \nu)$ be such that we have (1.6). Then $\pi$ is optimal for (WMOT).

In turn Theorem 3.3 allows us to strengthen [13, Lemma A.2] and [28, Theorem 1.3] by relaxing the required continuity of the cost function to conclude that a martingale coupling is optimal if it is concentrated on a finitely optimal set. The notion of finite optimality was developed in the spirit of cyclical monotonicity for MOT in [13] and [28]. The notion of cyclical monotonicity (see for instance [37]) is a remarkable tool in the theory of OT, which allows to determine optimality of a coupling only by knowing its support.

Definition 1.8 (Competitor). Let $\alpha = \mu \otimes \alpha_x \in \mathcal{P}^1(\mathbb{R} \times \mathbb{R})$. We call $\alpha' = \mu' \otimes \alpha'_x \in \mathcal{P}^1(\mathbb{R} \times \mathbb{R})$ a competitor of $\alpha$, if

$$\mu = \mu', \quad \int_{x \in \mathbb{R}} \alpha_x(dy) \mu(dx) = \int_{x \in \mathbb{R}} \alpha'_x(dy) \mu'(dx) \quad \text{and} \quad \int_{x \in \mathbb{R}} y \alpha_x(dy) = \int_{x \in \mathbb{R}} y \alpha'_x(dy), \quad \mu(dx) \text{-a.e.}$$

Definition 1.9 (Finite optimality). Let $c: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a cost function. We say that a Borel set $\tilde{\Gamma} \subset \mathbb{R} \times \mathbb{R}$ is finitely optimal for $c$ if for every probability measure $\alpha \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ finitely supported on $\tilde{\Gamma}$, we have

$$\int_{x \in \mathbb{R}} \alpha(dy) \int_{y \in \mathbb{R}} c(x, y) \alpha(dx, dy) \leq \int_{x \in \mathbb{R}} c(x, y) \alpha'(dx, dy),$$

for every competitor $\alpha'$ of $\alpha$.

Corollary 1.10 (Monotonicity principle for MOT). Let $r \geq 1, \mu, \nu \in \mathcal{P}^r(\mathbb{R})$ be in convex order, $c: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be measurable and such that $y \mapsto c(x, y)$ is continuous for all $x \in \mathbb{R}$ and $\sup_{(x, y) \in \mathbb{R}^2} \frac{|c(x, y)|}{1 + |x|^r + |y|^r} < \infty$. Then $\pi \in \Pi_M(\mu, \nu)$ is optimal for (MOT) if and only if it is concentrated on a finitely optimal set.

1.6 Organization of the paper

In Section 2 we introduce a number of concepts that are needed later on and state our main results on the stability of the WOT and the WMOT problems in full strength. Section 3 is devoted to the proofs. Subsection 3.1 consists of the unified proof of the stability of the WOT and the WMOT problems. Subsections 3.2 and 3.3 respectively address the proofs of the stability of the VIX superreplication price and of the stretched Brownian motion. Subsection 3.4 consists in showing that martingale $C$-monotonicity is sufficient for optimality for the WMOT problem. Finally several auxiliary lemmas are collected in the Appendix A.

2 Notation and comprehensive stability result

2.1 Adapted weak topologies

The usual weak topology is sometimes not appropriate to handle settings where the time structure and flow of information play a distinct role. In particular in the context of mathematical finance, this topology is too weak to provide continuity of sequential decision making problems such as optimal stopping and utility maximization. Remarkably, in discrete time there exists a canonical extension of the weak topology which adequately captures the temporal structure of stochastic processes, the adapted weak topology [6].

In our setting, the adapted topology (of index $r$) is induced by the adapted Wasserstein distance $A\mathcal{W}_r$ of index $r$ defined for all probabilities $\pi, \pi' \in \mathcal{P}^r(X \times Y)$ with first marginals $\mu, \mu' \in \mathcal{P}^r(X)$ by

$$A\mathcal{W}_r(\pi, \pi') := \inf_{\chi \in \Pi(\mu, \mu')} \left( \int_{X \times X} (d_X(x, x') + \mathcal{W}_r^r(\pi_x, \pi'_x)) \chi(dx, dx') \right)^{\frac{1}{r}},$$

where $(\pi_x)_{x \in X}$ and $(\pi'_x)_{x \in X}$ denote disintegrations of $\pi$ and $\pi'$, respectively, w.r.t. the first coordinate. In the following we may occasionally write $\mu \otimes \pi_x$ in lieu of $\pi$, and use this notation to define probabilities on $X \times Y$. The adapted Wasserstein distance satisfies

$$A\mathcal{W}_r(\pi, \pi') = \mathcal{W}_r(J(\pi), J(\pi')),$$  \hspace{1cm} (2.1)
see for instance [35], where \( J \) is the embedding map from \( \mathcal{P}(X \times Y) \) to \( \mathcal{P}(X \times \mathcal{P}(Y)) \), namely
\[
J : \mathcal{P}(X \times Y) \ni \pi = \mu \otimes \pi_x \mapsto \mu(dx) \delta_{x_\pi}(dp) \in \mathcal{P}(X \times \mathcal{P}(Y)).
\] (2.2)

In the companion paper [12] we prove that any coupling whose marginals are approximated by probability measures can be approximated by couplings with respect to the adapted Wasserstein distance (see Proposition 2.5 below).

2.2 An extension of the weak and adapted topologies

For \( r \geq 1 \), the Wasserstein distance \( \mathcal{W}_r \) is widely used to measure the distance between two probability measures with finite \( r \)-th moment. In order to measure the distance between two couplings, one could also use the stronger adapted Wasserstein distance for reasons discussed above. Despite being very handy, those distances sometimes lack topological convenience. For example, the \( \mathcal{W}_r \)-balls \( \{ p \in \mathcal{P}^r(X) : \mathcal{W}_r(p, \delta_{x_0}) \leq R \} \), \( R > 0 \), are not compact for the \( \mathcal{W}_r \)-distance topology. Since the proof of sufficiency of martingale \( C \)-monotonicity (see Section 3.4 below) relies on a compactness argument, we choose in the present paper to work with a finer topology. We give the definition here as well as some insight to understand its basic properties. All proofs and technical details are deferred to Section A.1 below.

**Definition 2.1.** Let \( f : X \to [1, +\infty) \) be continuous. We consider the space
\[
\mathcal{P}_f(X) = \{ p \in \mathcal{P}(X) : p(f) < +\infty \}.
\]

We equip \( \mathcal{P}_f(X) \) with the topology induced by the following convergence: a sequence \( (p_k)_{k \in \mathbb{N}} \in \mathcal{P}_f(X)^\mathbb{N} \) converges in \( \mathcal{P}_f(X) \) to \( p \) if and only if one of the two following equivalent assertions is satisfied:

(i) \( p_k \xrightarrow[k \to +\infty]{} p \) in \( \mathcal{P}(X) \) endowed with the weak convergence topology and \( p_k(f) \xrightarrow[k \to +\infty]{} p(f) \).

(ii) \( p_k(h) \xrightarrow[k \to +\infty]{} p(h) \) for all \( h \in \mathcal{P}_f(X) := \{ h \in \mathcal{C}(X) : \exists \alpha > 0, \forall x \in X, |h(x)| \leq \alpha f(x) \} \) with \( \mathcal{C}(X) \) denoting the space of continuous functions from \( X \) to \( \mathbb{R} \).

Unless explicitly stated otherwise, \( \mathcal{P}(X) \) is endowed with the weak convergence topology; for \( r \geq 1 \), \( \mathcal{P}^r(X) \) is endowed with the \( \mathcal{W}_r \)-distance topology; for \( f : X \to [1, +\infty) \) continuous, \( \mathcal{P}_f(X) \) is endowed with the topology induced by the convergence introduced in Definition 2.1. When \( f \) is the map \( x \mapsto 1 + d_X(x, x_0) \), then \( \mathcal{P}_f(X) = \mathcal{P}^r(X) \) and the two topologies coincide. Hence the reader who is not willing to consider this extension may completely disregard it and consistently view \( \mathcal{P}_f(X) \) as the usual Wasserstein space \( \mathcal{P}^r(X) \).

We will mainly address convergences of probability measures from a topological point of view. However it will sometimes prove useful to consider the metric \( \overline{W}_f \) defined on \( \mathcal{P}_f(X) \) by
\[
\forall p, q \in \mathcal{P}_f(X), \quad \overline{W}_f(p, q) := \sup_{h : X \to [1, 1], h \text{ is } 1\text{-Lipschitz}} (p(fh) - q(fh)),
\] (2.3)
which is a complete metric compatible with the topology on \( \mathcal{P}_f(X) \).

A continuous function \( g : Y \to [1, +\infty) \) can naturally be lifted to a continuous function \( \hat{g} : \mathcal{P}_g(Y) \to [1, +\infty) \) by setting
\[
\forall p \in \mathcal{P}_g(Y), \quad \hat{g}(p) = p(g).
\] (2.4)

A convenient aspect of this topology is that the spaces \( \mathcal{P}_g(\mathcal{P}(Y)) \) and \( \mathcal{P}_g(\mathcal{P}_g(Y)) \) and their respective topologies, a priori different, coincide. If moreover \( \mathcal{P}_g(Y) \) is endowed with the metric \( \overline{W}_g \), then those topological spaces are also equal to \( \mathcal{P}^1(\mathcal{P}_g(Y)) \), with \( (1, \mathcal{P}_g(Y), \overline{W}_g) \) replacing \( (r, X, d_X) \). Therefore one can freely switch between the topological spaces \( \mathcal{P}_g(\mathcal{P}(Y)) \), \( \mathcal{P}_g(\mathcal{P}_g(Y)) \) and \( \mathcal{P}^1(\mathcal{P}_g(Y)) \).

It is also possible to extend the adapted weak topology in the spirit of (2.1). Recall the map \( J \) defined by (2.2) which embeds \( \mathcal{P}(X \times Y) \) into \( \mathcal{P}(X \times \mathcal{P}(Y)) \). For two real-valued functions \( f \) and \( g \) respectively defined on \( X \) and \( Y \), we denote by \( f \oplus g \) the map \( X \times Y \ni (x, y) \mapsto f(x) + g(y) \).
if and only if for any measurable subset

\[ AW \]

on the marginals is a crucial property: often, these problems are computationally solvable when the marginals

The continuous dependence of the WOT and WMOT problems, or of optimal transport problems in general,

2.3 Stability

Moreover, in practice marginals are only approximately known and usually derived from noisy data, which

Again a useful fact is that \( \mathcal{P}_{f \# g} \) and \( \mathcal{P}_{f \# g}(X \times \mathcal{P}(Y)) \) and their respective topologies are equal, hence we can rephrase (i) as

\[
J(\pi^k) \xrightarrow{k \to +\infty} J(\pi) \text{ in } \mathcal{P}_{f \# g}(X \times \mathcal{P}(Y)).
\]

When \( f(x) = 1 + d^c(x, x_0) \) and \( g(y) = 1 + d^c(y, y_0) \), then \((\pi^k)_{k \in \mathbb{N}}\) converges in \( AW_{f \# g} \) to \( \pi \) if and only if it converges in \( AW_r \). Once again, the reader may skip this extension and consider as she wishes that convergences in \( AW_{f \# g} \) mean convergences in \( AW_r \).

2.3 Stability

The continuous dependence of the WOT and WMOT problems, or of optimal transport problems in general, on the marginals is a crucial property: often, these problems are computationally solvable when the marginals are finite. It is therefore natural to discretise the marginals and solve discretised versions. This approach works only if we know that the discretised problem sufficiently converges to the original one. Moreover, in practice marginals are only approximately known and usually derived from noisy data, which again emphasizes the importance of stability. Therefore we are interested in the continuity of the maps \((\mu, \nu) \to V_C(\mu, \nu) \) and \((\mu, \nu) \to V_{C^d}(\mu, \nu)\). First we derive the lower semicontinuity of those maps.

Recall that sequence \((\mu^k)_{k \in \mathbb{N}}\) of probability measures on \( X \) is said to converge strongly to some \( \mu \in \mathcal{P}(X) \) if and only if for any measurable subset \( A \subset X \), \( \mu^k(A) \) converges to \( \mu(A) \) as \( k \) goes to \(+\infty\). If \( X \subset \mathbb{R} \), then the Borel \( \sigma \)-field \( \mathcal{B}(X) \) of \( X \) for the induced topology (the coarsest topology which makes the canonical injection \( X \ni x \mapsto \iota(x) = x \in \mathbb{R} \) continuous) contains \( \{A \cap X : A \in \mathcal{B}(\mathbb{R})\} \) by measurability of \( \iota \). Since \( \{A \cap X : A \in \mathcal{B}(\mathbb{R})\} \) is a \( \sigma \)-field which contains the open subsets of \( X \) for the induced topology and therefore \( \mathcal{B}(X) \), one has \( \mathcal{B}(X) = \{A \cap X : A \in \mathcal{B}(\mathbb{R})\} \). For \( \mu \in \mathcal{P}(X) \), one may define \( \bar{\mu} \in \mathcal{P}(\mathbb{R}) \) by

\[
\forall A \in \mathcal{B}(\mathbb{R}), \quad \bar{\mu}(A) = \mu(A \cap X) \tag{2.5}
\]

(\( \bar{\mu} \) is the image of \( \mu \) by \( \iota \)). By the characterization of the weak convergence through open sets in the Portmanteau theorem, a sequence \((\mu^k)_{k \in \mathbb{N}}\) converges weakly to \( \mu \) in \( \mathcal{P}(X) \) if and only if the sequence \((\mu^k)_{k \in \mathbb{N}}\) converges weakly to \( \bar{\mu} \) in \( \mathcal{P}(\mathbb{R}) \). When \( X \in \mathcal{B}(\mathbb{R}) \), then \( \mathcal{B}(X) \subset \mathcal{B}(\mathbb{R}) \) and if \( \nu \in \mathcal{P}(\mathbb{R}) \) is such that

\( \nu(X) = 1 \) then the restriction \( \nu|_{\mathcal{B}(X)} \) of \( \nu \) to \( \mathcal{B}(X) \) belongs to \( \mathcal{P}(X) \) and is such that \( \nu|_{\mathcal{B}(X)} = \nu \). The subset \( X \subset \mathbb{R} \) is called a Polish subspace of \( \mathbb{R} \) if it Polish for the induced topology. By Alexandrov’s theorem, \( X \) is a Polish subspace of \( \mathbb{R} \) if and only if it is a countable intersection of open subsets of the real line, which ensures that \( X \in \mathcal{B}(\mathbb{R}) \). We denote by \( \mathcal{F}^{\uparrow}(X) \) the set of continuous functions \( f : X \to [1, +\infty) \) such that

\( \forall x \in X, \quad f(x) \geq |x| \). When \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \) with \( X \) and \( Y \) Polish subspaces of \( \mathbb{R} \) and \( f \in \mathcal{F}^{\uparrow}(X) \), \( g \in \mathcal{F}^{\uparrow}(Y) \), then we say that \( \mu \leq_C \nu \) if the image \( \bar{\nu} \) of \( \nu \) by the injection from \( Y \) to \( \mathbb{R} \) dominates for the convex order the image \( \bar{\nu} \) of \( \nu \) by the injection from \( X \) to \( \mathbb{R} \). Then \( \Pi_M(\bar{\mu}, \bar{\nu}) \neq 0 \) by Strassen’s theorem and for any \( \bar{\pi} \in \Pi_M(\bar{\mu}, \bar{\nu}) \) the probability measure defined by \( \pi(A) = \bar{\pi}(A) \) for all Borel subset of \( X \times Y \) belongs to

\[
\Pi_M(\mu, \nu) := \left\{ \pi = \mu \otimes \pi_x \in \Pi(\mu, \nu) : \int_Y y \pi_x(dy) = x \mu\text{-a.s.} \right\}.
\]

For a measurable cost function on \( X \times \mathcal{P}(Y) \), we define

\[
V_C^M(\mu, \nu) := \inf_{\pi \in \Pi_M(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx). \tag{WMOT}
\]
and denote by $\tilde{C}$ the cost function on $X \times \mathcal{P}_g(Y)$ defined by
\[
\tilde{C}(x, p) = C(x, p) 1_{\{x = \int_Y y \, p(dy)} + \infty 1_{\{x \neq \int_Y y \, p(dy)}.
\] (2.6)

In the following statements which address both the WOT and the WMOT problems, namely Theorems 2.3, 2.7 and 2.8, the necessary changes in the martingale case are added in parentheses.

**Theorem 2.3.** Let $X$ and $Y$ be Polish spaces (resp. $X$ and $Y$ Polish subspaces of $\mathbb{R}$), $f : X \to [1, +\infty)$ and $g : Y \to [1, +\infty)$ be continuous (resp. $f \in \mathcal{F}^{||}(X)$ and $g \in \mathcal{F}^{||}(Y)$), $C : X \times \mathcal{P}_g(Y) \to \mathbb{R} \cup \{+\infty\}$ be measurable. We assume that $C$ (resp. $\tilde{C}$) is convex and lower semicontinuous in the second argument, and such that there exists a constant $K > 0$ which satisfies for all $(x, p) \in X \times \mathcal{P}_g(Y)$
\[
C(x, p) \geq -K \left( f(x) + \int_Y g(y) \, p(dy) \right) \quad \text{(resp.} \quad \tilde{C}(x, p) \geq -K \left( f(x) + \int_Y g(y) \, p(dy) \right) \). \tag{2.7}
\]

For $\mu \in \mathcal{P}_f(X)$ and $\nu \in \mathcal{P}_g(Y)$ (resp. with $\mu \leq_c \nu$), there exists $\pi^* \in \Pi(\mu, \nu)$ which minimises $V_C(\mu, \nu)$ (resp. $V^M_C(\mu, \nu)$) and if $C$ is strictly convex in the second argument and $V_C(\mu, \nu)$ (resp. $V^M_C(\mu, \nu)$) is finite, then $V_C(\mu, \nu)$ (resp. $V^M_C(\mu, \nu)$) admits a unique minimiser.

For $k \in \mathbb{N}$, let $\mu^k \in \mathcal{P}_f(X)$ and $\nu^k \in \mathcal{P}_g(Y)$ (resp. with $\mu^k \leq_c \nu^k$) converge in $\mathcal{P}_f(X)$ and $\mathcal{P}_g(Y)$ as $k \to +\infty$ to $\mu$ and $\nu$, respectively. Suppose moreover that one of the following holds true:

(A) $C$ (resp. $\tilde{C}$) is lower semicontinuous in both arguments.

(B) $\mu^k$ converges strongly to $\mu$ as $k \to +\infty$.

Then
\[
V_C(\mu, \nu) \leq \liminf_{k \to +\infty} V_C(\mu^k, \nu^k) \quad \text{(resp.} \quad V^M_C(\mu, \nu) \leq \liminf_{k \to +\infty} V^M_C(\mu^k, \nu^k) \). \tag{2.8}
\]

**Remark 2.4.** If $X$ and $Y$ are Polish subspaces of $\mathbb{R}$ and $C : X \times \mathcal{P}_g(Y) \to \mathbb{R} \cup \{+\infty\}$ with $g \in \mathcal{F}^{||}(Y)$ is convex in its second argument (resp. is lower semicontinuous in either its second argument or in both arguments, resp. satisfies (2.7)), then so does $\tilde{C}$. Indeed, as $Y \ni y \mapsto y \in \mathbb{R}$ belongs to $\Phi_g(Y)$, $\{(x, p) \in X \times \mathcal{P}_g(Y) : x = \int_Y y \, p(dy)\}$ is a closed subset of $X \times \mathcal{P}_g(Y)$ and for fixed $x \in X$, $\{p \in \mathcal{P}_g(Y) : x = \int_Y y \, p(dy)\}$ is convex.

To conclude with stability of the WOT problem, we derive upper semicontinuity. This relies on the following (trivial) extension of Proposition 2.3 in our companion paper [12]. This extension is then an easy consequence of the equivalent definitions stated in Definition 2.2.

**Proposition 2.5.** Let $f : X \to [1, +\infty)$ and $g : Y \to [1, +\infty)$ be continuous. Let $\mu^k \in \mathcal{P}_f(X)$, $\nu^k \in \mathcal{P}_g(Y)$, $k \in \mathbb{N}$, respectively converge to $\mu$ and $\nu$ in $\mathcal{P}_f(X)$ and $\mathcal{P}_g(Y)$ respectively. Then there is for any $\pi \in \Pi(\mu, \nu)$ a sequence of couplings $\pi^k \in \Pi(\mu^k, \nu^k)$, $k \in \mathbb{N}$ converging to $\pi$ in $\mathcal{AW}_{f,g}$.

In the martingale setting, we recall the main result of the companion paper [12, Theorem 2.6], namely that any martingale couplings whose marginals are approximated by probability measures in the convex order can be approximated by martingale couplings with respect to the adapted Wasserstein distance. We invoke here a trivial extension of the aforementioned result which is again a direct consequence of the equivalent definitions stated in Definition 2.2.

**Theorem 2.6.** Let $X$ and $Y$ be Polish subspaces of $\mathbb{R}$, and let $f \in \mathcal{F}^{||}(X)$ and $g \in \mathcal{F}^{||}(Y)$. Let $\mu^k \in \mathcal{P}_f(X)$ and $\nu^k \in \mathcal{P}_g(Y)$, $k \in \mathbb{N}$, be in the convex order and respectively converge to $\mu$ and $\nu$ in $\mathcal{P}_f(X)$ and $\mathcal{P}_g(Y)$. Let $\pi \in \Pi_M(\mu, \nu)$. Then there exists a sequence of martingale couplings $\pi^k \in \Pi_M(\mu^k, \nu^k)$, $k \in \mathbb{N}$ converging to $\pi$ in $\mathcal{AW}_{f,g}$.
Proof of Theorem 2.6. Let for $\eta \in \mathcal{P}(X)$ (resp. $\mathcal{P}(Y)$) $\tilde{\eta}$ denote the image of $\eta$ by the injection $\iota_X : X \ni x \mapsto x \in \mathbb{R}$ (resp. $\iota_Y : Y \ni y \mapsto y \in \mathbb{R}$). By continuity of the injections, the sequences $(\tilde{\mu}^k)_{k \in \mathbb{N}}$ and $(\tilde{\nu}^k)_{k \in \mathbb{N}}$ respectively converge weakly to $\tilde{\mu}$ and $\tilde{\nu}$. Since $\iota_X \in \Phi_f(X)$ and $\iota_Y \in \Phi_g(Y)$, $\int_{\mathbb{R}} |x| \tilde{\mu}^k(dx) = \int_{X} |X(x)| \mu^k(dx) \rightarrow \int_{X} |X(x)| \mu(dx) = \int_{\mathbb{R}} |x| \mu(dx)$ and $\int_{\mathbb{R}} |y| \tilde{\nu}^k(dy) \rightarrow \int_{\mathbb{R}} |y| \tilde{\nu}(dy)$ and the convergences hold in $\mathcal{P}_1(\mathbb{R})$. Let $\pi \in \Pi_M(\mu, \nu)$ and $\tilde{\pi}$ denote the image of $\pi$ by the injection from $X \times Y \rightarrow \mathbb{R}^2$. Then $\tilde{\pi} \in \Pi_M(\tilde{\mu}, \tilde{\nu})$ and, by [12], Theorem 2.6, there is a sequence $\tilde{\pi}^k \in \Pi_M(\tilde{\mu}^k, \tilde{\nu}^k)$, $k \in \mathbb{N}$ such that $J(\tilde{\pi}^k)$ converges to $J(\tilde{\pi})$ in $\mathcal{P}_1(\mathbb{R} \times \mathcal{P}_1(\mathbb{R}))$ as $k \rightarrow +\infty$.

The restrictions $\pi^k = \tilde{\pi}^k|_{B(X \times Y)}$ and $\pi = \tilde{\pi}|_{B(X \times Y)}$ respectively belong to $\Pi_M(\mu^k, \nu^k)$ and $\Pi_M(\mu, \nu)$ and are such that $J(\pi^k)$ is the image of $J(\pi)$ by the mapping $\iota_{X \times \mathcal{P}(Y)} : X \times \mathcal{P}(Y) \ni (x, p) \mapsto (x, \tilde{p}) \in \mathbb{R} \times \mathcal{P}(\mathbb{R})$. The open ball in $X \times \mathcal{P}(Y)$ centred at $(x, p)$ with radius $\varepsilon > 0$ for the sum of the usual real distance on $X$ and the Prokhorov metric with the usual real distance on $\mathcal{P}(Y)$ is the preimage by $\iota_{X \times \mathcal{P}(Y)}$ of the open ball centred at $(x, \tilde{p})$ with radius $\varepsilon$ for the sum of the usual real distance on $\mathbb{R}$ and the Prokhorov metric on $\mathcal{P}(\mathbb{R})$. Since the induced topologies on $X$ and $Y$ are metrizable by the usual distance on the real line, and, as $X$ is separable for the induced topology, the Prokhorov metric metrizes the weak convergence topology in $\mathcal{P}(X)$, we deduce that any open subset of $X \times \mathcal{P}(Y)$ is the preimage of an open subset of $\mathbb{R} \times \mathcal{P}(\mathbb{R})$ by $\iota_{X \times \mathcal{P}(Y)}$. By the open subset characterization of the weak convergence in Portmanteau’s theorem, we deduce that $J(\pi^k)$ converges weakly to $J(\pi)$ in $\mathcal{P}(X \times \mathcal{P}(Y))$. With the convergence of the marginals $\mu^k$ to $\mu$ in $\mathcal{P}_f(X)$ and $\nu^k$ to $\nu$ in $\mathcal{P}_g(Y)$, we conclude in view of Definition 2.2 that $\pi^k$ converges to $\pi$ in $\mathcal{A}W_{f \otimes g}$ as $k \rightarrow +\infty$. 

**Theorem 2.7** (Upper semicontinuity). Let $X$ and $Y$ be Polish spaces (resp. $X$ and $Y$ be Polish subspaces of $\mathbb{R}$). Let $f : X \rightarrow [1, +\infty)$ and $g : Y \rightarrow [1, +\infty)$ be continuous (resp. $f \in \mathcal{F}^{(\uparrow)}(X)$ and $g \in \mathcal{F}^{(\uparrow)}(Y)$), $C : X \times \mathcal{P}_g(Y) \rightarrow \{-\infty\} \cup \mathbb{R}$ be measurable, upper semicontinuous in the second argument, and such that there exists a constant $K > 0$ which satisfies for all $(x, p) \in X \times \mathcal{P}_g(Y)$

\[
C(x, p) \leq K \left( f(x) + \int_Y g(y) p(dy) \right). \tag{2.9}
\]

For $k \in \mathbb{N}$, let $\mu^k \in \mathcal{P}_f(X)$ and $\nu^k \in \mathcal{P}_g(Y)$ (resp. with $\mu^k \leq_c \nu^k$) converge in $\mathcal{P}_f(X)$ and $\mathcal{P}_g(Y)$ as $k \rightarrow +\infty$ to $\mu$ and $\nu$ respectively. Suppose moreover that one of the following holds true:

(A') $C$ is upper semicontinuous in both arguments.

(B') $\mu^k$ converges strongly to $\mu$ as $k \rightarrow +\infty$.

Then

\[
\limsup_{k \rightarrow +\infty} V_C(\mu^k, \nu^k) \leq V_C(\mu, \nu) \quad \left( \text{resp.} \quad \limsup_{k \rightarrow +\infty} V_C^M(\mu^k, \nu^k) \leq V_C^M(\mu, \nu) \right). \tag{2.10}
\]

**Theorem 2.8** (Stability). Let $X$ and $Y$ be Polish spaces (resp. $X$ and $Y$ be Polish subspaces of $\mathbb{R}$). Let $f : X \rightarrow [1, +\infty)$ and $g : Y \rightarrow [1, +\infty)$ be continuous (resp. $f \in \mathcal{F}^{(\uparrow)}(X)$ and $g \in \mathcal{F}^{(\uparrow)}(Y)$), $C : X \times \mathcal{P}_g(Y) \rightarrow \mathbb{R}$ be measurable, continuous in the second argument and such that there exists a constant $K > 0$ which satisfies for all $(x, p) \in X \times \mathcal{P}_g(Y)$

\[
|C(x, p)| \leq K \left( f(x) + \int_Y g(y) p(dy) \right). \tag{2.11}
\]

Also assume that $C$ (resp. $\tilde{C}$ defined in (2.6)) is convex in its second argument. For $k \in \mathbb{N}$, let $\mu^k \in \mathcal{P}_f(X)$ and $\nu^k \in \mathcal{P}_g(Y)$ (resp. with $\mu^k \leq_c \nu^k$) converge in $\mathcal{P}_f(X)$ and $\mathcal{P}_g(Y)$ as $k \rightarrow +\infty$ to $\mu$ and $\nu$ respectively. Suppose moreover that one of the following holds true:

(A') $C$ is continuous in both arguments.

(B') $\mu^k$ converges strongly to $\mu$ as $k \rightarrow +\infty$. 

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Then
\[ V_C(\mu^k,\nu^k) \to V_C(\mu,\nu) \quad \text{(resp. } V_C^M(\mu^k,\nu^k) \to V_C^M(\mu,\nu)\text{),} \quad (2.12) \]

For \( k \in \mathbb{N} \) let \( \pi^{k,*} \in \Pi(\mu^k,\nu^k) \) (resp. \( \pi^{k,*} \in \Pi^M(\mu^k,\nu^k) \)) be a minimiser of \( V_C(\mu^k,\nu^k) \) (resp. \( V_C^M(\mu^k,\nu^k) \)), the existence of which is granted by Theorem 2.3. Then any accumulation point of \( (\pi^{k,*})_{k \in \mathbb{N}} \) for the weak convergence topology is a minimiser of \( V_C(\mu,\nu) \) (resp. \( V_C^M(\mu,\nu) \)). If the latter has a unique minimiser \( \pi^* \), then
\[ \pi^{k,*} \to \pi^* \quad \text{in } \mathcal{P}_{f\otimes g}(X \times Y) \quad \text{(resp. in } \mathcal{P}_{f\otimes g}(X \times Y)) \]. \quad (2.13)

If \( C \) is moreover strictly convex in the second argument, then the convergence (2.13) holds in \( \mathcal{AW}_{f\otimes g} \).

3 Proofs

3.1 Proofs of the stability theorems

This section is devoted to the proof of Theorem 2.3, Theorem 2.7 and Theorem 2.8 about the stability of (WOT) and (WMOT).

**Proof of Theorem 2.3.** Since in the martingale case, for \( \mu \leq_c \nu \), \( V_C^M(\mu,\nu) = V_C(\mu,\nu) \), the statements concerning (WMOT) follow from those concerning (WOT) that we now prove.

Let \((\pi^n)_{n \in \mathbb{N}} \in \Pi(\mu,\nu)\) be such that \( f_X(C(x,\pi^n_x)) \mu(dx) \) converges to \( V_C(\mu,\nu) \) as \( n \to +\infty \). By tightness of \( \mu \) and \( \nu \) we deduce the existence of a subsequence \( (\pi^{n_i})_{i \in \mathbb{N}} \) of \((\pi^n)_{n \in \mathbb{N}}\) which converges to some \( \pi^* \in \Pi(\mu,\nu) \) with respect to the weak convergence topology and therefore the topology of \( \mathcal{P}_{f\otimes g}(X \times Y) \) since \( \pi^n(f \otimes g) = \mu(f) + \nu(f) = \pi^*(f \otimes g) \) for all \( l \in \mathbb{N} \). Since, for each \( l \in \mathbb{N} \), the first marginal of \( \pi^n \) is \( \mu \), combining Lemma A.10 and Proposition A.12 (d) below we then have
\[ V_C(\mu,\nu) \leq \int_X C(x,\pi^n_x) \mu(dx) \leq \liminf_{l \to +\infty} \int_X C(x,\pi^{n_i}_x) \mu(dx) = V_C(\mu,\nu), \]
which shows that \( \pi^* \) is a minimiser for \( V_C(\mu,\nu) \). The finiteness of \( V_C(\mu,\nu) \) and the strict convexity of \( C(x,\cdot) \) for all \( x \in X \) yield uniqueness of the minimisers. Indeed when \( \pi, \tilde{\pi} \in \Pi(\mu,\nu) \) then \( \frac{1}{2}(\pi + \tilde{\pi}) \in \Pi(\mu,\nu) \) when, moreover, \( \pi \neq \tilde{\pi} \), then \( \mu(\{x \in X : \pi_x \neq \tilde{\pi}_x\}) > 0 \) and since \( C(x,\frac{1}{2}(\pi_x + \tilde{\pi}_x)) \leq \frac{1}{2}(C(x,\pi_x) + C(x,\tilde{\pi}_x)) \) with strict inequality when \( \pi_x \neq \tilde{\pi}_x \),
\[ \int_X C \left( x, \frac{\pi_x + \tilde{\pi}_x}{2} \right) \mu(dx) < \frac{1}{2} \left( \int_X C(x,\pi_x) \mu(dx) + \int_X C(x,\tilde{\pi}_x) \mu(dx) \right). \quad (3.1) \]

We now show (2.8). Let \((V_C(\mu^{k_l},\nu^{k_l}))_{l \in \mathbb{N}}\) be a subsequence of \((V_C(\mu^k,\nu^k))_{k \in \mathbb{N}}\) such that
\[ \lim_{l \to +\infty} V_C(\mu^{k_l},\nu^{k_l}) = \liminf_{k \to +\infty} V_C(\mu^k,\nu^k), \]
and denote, for \( l \in \mathbb{N} \), an optimizer of \( V_C(\mu^{k_l},\nu^{k_l}) \) by \( \pi^{k_l,*} \). Let \( \tilde{\pi} \in \Pi(\mu,\nu) \) be an accumulation point of \((\pi^{k_l,*})_{l \in \mathbb{N}} \) in \( \mathcal{P}_{f\otimes g}(X \times Y) \), which exists by relative compactness of the marginals. Then, by Proposition A.12 (b) below under (A) and by Lemma A.10 and Proposition A.12 (d) below under (B), we find that
\[ \liminf_{k \to +\infty} V_C(\mu^k,\nu^k) = \lim_{l \to +\infty} \int_X C(x,\pi^{k_l,*}_x) \mu^{k_l}(dx) \geq \int_X C(x,\tilde{\pi}_x) \mu(dx) \geq V_C(\mu,\nu). \]

**Proof of Theorem 2.7.** Let \( \tilde{\Pi}(\mu,\nu) = \Pi(\mu,\nu) \) and \( \tilde{V}_C(\mu,\nu) = V_C(\mu,\nu) \) in the classic setting, and \( \tilde{\Pi}(\mu,\nu) = \Pi_M(\mu,\nu) \) and \( \tilde{V}_C(\mu,\nu) = V_C^M(\mu,\nu) \) in the martingale setting. Let \( \pi \in \tilde{\Pi}(\mu,\nu) \). By Proposition 2.5 in the
classic setting and Theorem 2.6 in the martingale setting, there exists a sequence $\pi^k \in \hat{\Pi}(\mu^k, \nu^k)$, $k \in \mathbb{N}$, which converges to $\pi$ in $\mathcal{AW}_{f \oplus g}$, which is equivalent to $J(\pi^k)$ converging to $J(\pi)$ in $\mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y))$.

We then have by Lemma A.11 (a) under Assumption (A') and by Lemma A.10 and Lemma A.11 (c) under (B') (Lemma A.11 is applied with $c$, $Y$ and $g$ replaced by $-C$, $\mathcal{P}_g(Y)$ and $\hat{g}$) that

$$\limsup_{k \to \infty} \int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\pi^k)(dx, dp) \leq \int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\pi)(dx, dp). \quad (3.2)$$

Since

$$\hat{V}_C(\mu^k, \nu^k) \leq \int_X C(x, \pi_x^k) \mu(dx) = \int_X C(x, p) J(\pi^k)(dx, dp),$$

we deduce that

$$\limsup_{k \to \infty} \hat{V}_C(\mu^k, \nu^k) \leq \int_X C(x, \pi_x) \mu(dx)$$

and conclude by taking the infimum of the right-hand side with respect to $\pi \in \hat{\Pi}(\mu, \nu)$.

**Proof of Theorem 2.8.** We adopt the same notation as in the proof of Theorem 2.7. Combining Theorems 2.3 and 2.7 and Remark 2.4 (to check the lower semicontinuity of $\hat{C}$ in the martingale case) yields the stability result (2.12). For $k \in \mathbb{N}$, let $\pi^{k, *}_\ell \in \hat{\Pi}(\mu^k, \nu^k)$ be a minimiser of $\hat{V}_C(\mu^k, \nu^k)$. For any subsequence $(\pi^{k, *}_\ell)_{\ell \in \mathbb{N}}$ of $(\pi^{k, *}_k)_{k \in \mathbb{N}}$ converging weakly to some $\bar{\pi}$, the convergence also holds in $\mathcal{P}_{f \oplus g}(X \times Y)$ and we have by Proposition A.12 (b) below under Assumption (A') and by Lemma A.10 and Proposition A.12 (d) below under (B') that

$$\hat{V}_C(\mu, \nu) = \lim_{\ell \to +\infty} \hat{V}_C(\mu^{k, \ell}, \nu^{k, \ell}) = \lim_{\ell \to +\infty} \int_X C(x, \pi_x^{k, *}) \mu(dx) \geq \int_X C(x, \bar{\pi}_x) \mu(dx) \geq \hat{V}_C(\mu, \nu),$$

so $\bar{\pi}$ is a minimiser of $\hat{V}_C(\mu, \nu)$. In particular if $\hat{V}_C(\mu, \nu)$ has a unique minimiser $\pi^*$, it is the unique accumulation point with respect to the weak convergence topology of the tight sequence $(\pi^{k, *}_k)_{k \in \mathbb{N}}$, which therefore converges to $\pi^*$ weakly and even in $\mathcal{P}_{f \oplus g}(X \times Y)$ since its marginals converge in $\mathcal{P}_f(X)$ and $\mathcal{P}_g(Y)$ respectively.

Let us now assume that $C$ is moreover strictly convex in its second argument. By Theorem 2.3, this implies uniqueness of the minimisers. Let $\pi^*$ be the only minimiser of $\hat{V}_C(\mu, \nu)$. To conclude the proof, it is enough to show that $J(\pi^{k, *}_k)$ converges to $J(\pi^*)$ in $\mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y))$ as $k$ goes to $+\infty$. Let $P^* \in \mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y))$ be an accumulation point of $(J(\pi^{k, *})_k)_{k \in \mathbb{N}}$, which exists by Lemma A.7.

To conclude, it suffices to show that $P^* = J(\pi^*)$, which is achieved in three steps. Let $\Lambda(\mu, \nu) = \Lambda(\mu, \nu)$ (see the definition (A.4) below) in the classic setting and $\Lambda(\mu, \nu) = \Lambda_M(\mu, \nu)$ (see the definition (A.6) below) in the martingale setting. First we show that

$$P^* \in \hat{\Lambda}(\mu, \nu). \quad (3.3)$$

Next, we show that $J(\pi^*)$ and $P^*$ both minimise

$$\hat{V}_C(\mu, \nu) := \inf_{P \in \hat{\Lambda}(\mu, \nu)} \int_{X \times \mathcal{P}_g(Y)} C(x, p) P(dx, dp).$$

Finally, we show the uniqueness of minimisers of $\hat{V}_C(\mu, \nu)$.

Let $(\pi^{k, *}_\ell)_{\ell \in \mathbb{N}}$ be a subsequence converging to $P^*$ in $\mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y))$. By Lemma A.8 below we have

$$\int_{(x, p) \in X \times \mathcal{P}_g(Y)} p(dy) J(\pi^{k, *}_\ell)(dx, dp) \quad \ell \to +\infty \quad \int_{(x, p) \in X \times \mathcal{P}_g(Y)} p(dy) P^*(dx, dp),$$

where the convergence holds in $\mathcal{P}_g(Y)$ as $\ell$ goes to $+\infty$. Since the left-hand side is $\nu^{k, *}$, which converges to $\nu$ in $\mathcal{P}_g(Y)$ and therefore in the weak topology, we deduce by uniqueness of the limit that the right-hand
side is $\nu$, hence $P^* \in \Lambda(\mu, \nu)$. In the martingale setting, since, as $f, g \in F^1(\mathbb{R})$, $X \times \mathcal{P}_g(Y) \ni (x, p) \mapsto |x - \int_Y y p(dy)| \in \Phi f \otimes g(X \times \mathcal{P}_g(Y))$, we have that

$$0 = \int_{X \times \mathcal{P}_g(Y)} \left| x - \int_Y y p(dy) \right| J(\pi_{k\varepsilon}^*) dx, dp \quad \lim_{\ell \to +\infty} \int_{X \times \mathcal{P}_g(Y)} \left| x - \int_Y y p(dy) \right| P^*(dx, dp),$$

hence $P^* \in \Lambda_M(\mu, \nu)$.

Let us show that $J(\pi^*)$ and $P^*$ both minimise $\hat{V}_C(\mu, \nu)$. Note that since $P^* \in \hat{\Lambda}(\mu, \nu)$, we have $P^*(X \times \mathcal{P}_g(Y)) = 1$. Since $(J(\pi_{k\varepsilon}^*))_{\varepsilon \in \mathbb{N}}$ converges to $P^*$ in $\mathcal{P}_f \otimes g(X \times \mathcal{P}(Y))$, we find by Lemma A.11

$$\int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\pi_{k\varepsilon}^*) dx, dp \quad \lim_{\varepsilon \to +\infty} \int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\pi_{k\varepsilon}^*) dx, dp$$

Then (3.4) and the definition of $\pi_{k\varepsilon}^*$ yield

$$\int_{X \times \mathcal{P}_g(Y)} C(x, p) P^*(dx, dp) = \int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\pi_{k\varepsilon}^*) dx, dp \quad \lim_{\varepsilon \to +\infty} \hat{V}_C(\mu \varepsilon, \nu \varepsilon)$$

$$= \hat{V}_C(\mu, \nu)$$

$$= \int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\pi^*) dx, dp.$$

Let $P(dx, dp) = \mu(dx) P_x(dp) \in \hat{\Lambda}(\mu, \nu)$. Then $\mu(dx) \int_{p \in \mathcal{P}_g(Y)} p(dy) P_x(dp) \in \hat{\Pi}(\mu, \nu)$, and by applying Proposition A.9 below in the last inequality, we find

$$\int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\pi^*) dx, dp = \int_X C(x, \pi_x^*) \mu(dx)$$

$$= \hat{V}_C(\mu, \nu)$$

$$\leq \int_X C \left( x, \int_{p \in \mathcal{P}_g(Y)} p(dy) P_x(dp) \right) \mu(dx)$$

$$\leq \int_X \int_{\mathcal{P}_g(Y)} C(x, p) P_x(dp) \mu(dx),$$

which proves $\hat{V}_C(\mu, \nu) = \hat{V}_C(\mu, \nu)$ with the latter attained by $J(\pi^*)$ and, according to (3.5), by $P^*$.

Let us finally prove that $J(\pi^*)$ is the only minimiser of $\hat{V}_C(\mu, \nu)$ by $J$. To conclude, it is enough to check that any minimiser $\hat{P} \in \hat{\Lambda}(\mu, \nu)$ actually belongs to this image. For $x \in X$, let $\hat{\pi}_x(dy) = \int_{p \in \mathcal{P}_g(Y)} p(dy) \hat{P}_x(dp)$ and $\hat{\pi}(dx, dy) = \mu(dx) \hat{\pi}_x(dy)$. Then $J(\hat{\pi}) \in \hat{\Lambda}(\mu, \nu)$ and Proposition A.9 below yields

$$\int_{X \times \mathcal{P}_g(Y)} C(x, p) J(\hat{\pi}) dx, dp = \int_X C(x, \hat{\pi}_x) \mu(dx) \leq \int_X \int_{\mathcal{P}_g(Y)} C(x, p) \hat{P}_x(dp) \mu(dx).$$

By optimality of $\hat{P}$, this inequality is an equality, hence for $\mu(dx)$-almost every $x \in X$ we have

$$C(x, \hat{\pi}_x) = \int_{\mathcal{P}_g(Y)} C(x, p) \hat{P}_x(dp),$$

and therefore $\hat{P}_x = \delta_{\hat{\pi}_x}$ by the equality case of Proposition A.9 below, or equivalently $\hat{P} = J(\hat{\pi})$. 

3.2 Proof of the stability of the dual value function of VIX-futures

Proof of Theorem 1.3. The open interval $(0, +\infty)$ is a Polish subspace of $\mathbb{R}$ and the function $f(x) = |\ln x| + |x|$ belongs to $\mathcal{F}^1((0, +\infty))$ (the fact that $f$ is bounded from below by 1 follows from the inequality $|\ln x| \geq 1 - x$ valid for $x \in (0, 1]$). The cost function

$$(0, +\infty) \times \mathcal{P}_f((0, +\infty)) \ni (x, p) \mapsto C(x, p) = -\sqrt{\frac{2}{T_2 - T_1} \int_{(0, +\infty)} \ln \left(\frac{x}{y}\right) p(dy)} \vee 0$$

is continuous and such that

$$|C(x, p)| \leq \frac{1}{2} \cdot \frac{1}{T_2 - T_1} \int_{(0, +\infty)} \ln \left(\frac{x}{y}\right) p(dy) \leq \left(\frac{1}{4} + \frac{1}{T_2 - T_1}\right) (f(x) + p(f)),$$

and that the modified cost function $\tilde{C}$ defined in (2.6) is convex in its second argument by concavity of the square root function. Since for each couple $(\mu, \nu) \in \mathcal{P}_f((0, +\infty))$ with $\mu \leq_c \nu$,

$$-D_{\text{super}}(\mu, \nu) = V^M_C(\mu, \nu),$$

the conclusion follows from Theorem 2.8 applied with $X = Y = (0, +\infty)$ and $g = f$. \hfill \Box

3.3 Proof of the stability of the stretched Brownian motion

The stability of the unidimensional stretched Brownian motion in $\mathcal{W}_r$-topology stated in Corollary 1.4 actually holds in the stronger $\mathcal{AW}_r$-topology.

Proposition 3.1 (Stability of the unidimensional stretched Brownian motion). Under the assumptions of Corollary 1.4,

$$\mathcal{AW}_r^r \left(\mathcal{L}(M^k_0, (M^k_t)_{t \in [0,1]}), \mathcal{L}(M^q_0, (M^q_t)_{t \in [0,1]})\right) \leq \left(\frac{r}{r - 1}\right)^r \mathcal{AW}_r^r \left(\mathcal{L}(M^k_0, M^k_t), \mathcal{L}(M^q_0, M^q_t)\right),$$

and the right-hand side vanishes as $k$ goes to $+\infty$.

The proof of the proposition relies on the following Lemma.

Lemma 3.2. Let $\rho > 1$, and $C_\rho : \mathbb{R} \times \mathcal{P}_\rho(\mathbb{R}) \to \mathbb{R}$ be defined for all $(x, p) \in \mathbb{R} \times \mathcal{P}_\rho(\mathbb{R})$ by $C_\rho(x, p) = W^\rho_\rho(p, \gamma)$, where $\gamma \in \mathcal{P}_\rho(\mathbb{R})$ does not weight points. Let $V^M_{C_\rho}$ be the value function given by (WMOT) for the cost function $C_\rho$.

Let $r \geq \rho$ and $\mu^k, \nu^k \in \mathcal{P}_\rho(\mathbb{R})$, $k \in \mathbb{N}$ be in convex order and converge respectively to $\mu$ and $\nu$ in $\mathcal{W}_r$. Then $\lim_{k \to +\infty} V^M_{C_\rho}((\mu^k, \nu^k)) = V^M_{C_\rho}(\mu, \nu)$ and the optimisers are converging in $\mathcal{AW}_r$.

Proof. By Theorem 2.8 applied with $X = Y = \mathbb{R}$, $f(x) = 1 + |x|^r$ and $g(y) = 1 + |y|^r$, it is sufficient to show that $p \mapsto W^\rho_\rho(\gamma, p)$ is strictly convex. Since $\gamma$ does not weight points, the unique $\mathcal{W}_\rho$-optimal coupling between $\gamma$ and $p \in \mathcal{P}_\rho(\mathbb{R})$ is the comonotonous coupling $\chi^p$ given by the map $x \mapsto F^{-1}_p(F_\gamma(x))$ i.e. the image of $\gamma$ by $x \mapsto (x, F^{-1}_p(F_\gamma(x)))$. For $q \in \mathcal{P}_\rho(\mathbb{R})$ and $\lambda \in (0, 1)$ the coupling $\chi = (1 - \lambda)\chi^p + \lambda \chi^q$ between $\gamma$ and $(1 - \lambda)p + \lambda q$ is not given by a map unless $F^{-1}_q(u) = F^{-1}_p(u)$ for all $u \in (0, 1)$ i.e. $p = q$. Therefore, when $p \neq q$,

$$(1 - \lambda)W^\rho_\rho(\gamma, p) + \lambda W^\rho_\rho(\gamma, q) = \int |x - y|^\rho \chi(dx, dy) > W^\rho_\rho(\gamma, (1 - \lambda)p + \lambda q).$$

\hfill \Box
Proof of Proposition 3.1. Let $\gamma \sim \mathcal{N}(0,1)$ be the unidimensional standard normal distribution and $C_2 : \mathbb{R} \times \mathcal{P}^2(\mathbb{R}) \to \mathbb{R}$ be defined for all $(x, p) \in \mathbb{R} \times \mathcal{P}^2(\mathbb{R})$ by $C_2(x, p) = \mathcal{W}_2^3(p, \gamma)$. Let $V_{C_2}^M$ be the value function given by (WMOT) for the cost function $C_2$.

In the setting of Corollary 1.4, let $\pi^* \in \Pi_M(\mu, \nu)$, resp. $\pi^k \in \Pi_M(\mu^k, \nu^k)$ be optimal for $V_{C_2}^M(\mu, \nu)$, resp. $V_{C_2}^M(\mu^k, \nu^k)$. For $(x, b) \in \mathbb{R} \times \mathbb{R}^{[0,1]}$, let $B = (B_t)_{t \in [0,1]}$ be a Brownian motion and

$$G^k(x, b) = \left( \mathbb{E} \left[ F_{\pi^k}^{-1}(F_{\gamma}(B_1 - B_t + b_t)) \right] \right)_{t \in [0,1]}$$

According to (1.5), $(M_0^k, (M_t^k)_{t \in [0,1]})$ and $(M_0^*, (M_t^*)_{t \in [0,1]})$ are respectively distributed according to

$$\eta^k(dx, df) := \mu^k(dx) (G^k(x, \cdot), W)(df) \quad \text{and} \quad \eta^*(dx, df) := \mu(dx) (G^*(x, \cdot), W)(df),$$

where $W$ denotes the Wiener measure on $C([0,1])$. Let $\chi^k \in \Pi(\mu^k, \mu)$ be optimal for $\mathcal{AW}_r(\pi^k, \pi)$. Then

$$\mathcal{AW}_r(\eta^k, \eta^*) \leq \int_{\mathbb{R} \times \mathbb{R}} (|x - x'|^r + \mathcal{W}_r^r(G^k(x, \cdot), W, G(x', \cdot), W)) \chi^k(dx, dx').$$

According to (1.5), for $\mu(dx)$-almost every $x \in \mathbb{R}$, $G^k(x, B)$ is the stretched Brownian motion from $\delta_x$ to $\pi^k_x$, hence it is a continuous $(F_t)_{t \in [0,1]}$-martingale, where $(F_t)_{t \in [0,1]}$ is the natural filtration associated to $B$. Similarly, for $\mu(dx)$-almost every $x \in \mathbb{R}$, $G^*(x', B)$ is a continuous $(F_t)_{t \in [0,1]}$-martingale. Therefore, for $\chi^k(dx, dx')$-almost every $(x, x') \in \mathbb{R} \times \mathbb{R}$, $G^k(x, B) - G^*(x', B)$ is a continuous $(F_t)_{t \in [0,1]}$-martingale. Using Doob’s martingale inequality for the second inequality, the fact that $F_1(B_1)$ is uniformly distributed on $(0,1)$ for the first equality and the fact that the convolutional coupling between $\pi^k_x$ and $\pi^*_x$ is optimal for $\mathcal{W}_r(\pi^k_x, \pi^*_x)$ for the second equality, we get for $\chi^k(dx, dx')$-almost every $(x, x') \in \mathbb{R} \times \mathbb{R}$

$$\mathcal{W}_r^r(G^k(x, \cdot), W, G(x', \cdot), W) \leq \mathbb{E} \left[ \sup_{t \in [0,1]} |G^k(x, B)_t - G^*(x', B)_t|^r \right]$$

$$\leq \left( \frac{r}{r-1} \right)^r \mathbb{E}[|G^k(x, B)_1 - G^*(x', B)_1|^r]$$

$$= \left( \frac{r}{r-1} \right)^r \mathbb{E}[|F_{\pi^k}^{-1}(F_{\gamma}(B_1)) - F_{\pi^*_x}^{-1}(F_{\gamma}(B_1))|^r]$$

$$= \left( \frac{r}{r-1} \right)^r \mathcal{W}_r^r(\pi^k_x, \pi^*_x).$$

We deduce that

$$\mathcal{AW}_r(\eta^k, \eta^*) \leq \left( \frac{r}{r-1} \right)^r \int_{\mathbb{R} \times \mathbb{R}} (|x - x'|^r + \mathcal{W}_r^r(\pi^k_x, \pi^*_x)) \chi^k(dx, dx') = \left( \frac{r}{r-1} \right)^r \mathcal{AW}_r^r(\pi^k, \pi^*),$$

where the right-hand side vanishes as $k$ goes to $+\infty$ by virtue of Lemma 3.2.

3.4 Proof of sufficiency of martingale C-monotonicity

In this section we prove the claim that martingale C-monotonicity is sufficient for optimality for (WMOT). Theorem 1.7 is a special case of the next statement for $f(x) = 1 + |x|^r$ and $g(y) = 1 + |y|^r$.

**Theorem 3.3 (Sufficiency).** Let $f, g \in \mathcal{F}^{\leq 1}(\mathbb{R})$, $\mu \in \mathcal{P}_f(\mathbb{R})$, $\nu \in \mathcal{P}_g(\mathbb{R})$ be in convex order, and $C : \mathbb{R} \times \mathcal{P}_g(\mathbb{R}) \to \mathbb{R}$ be a measurable cost function, continuous in the second argument and such that there exists a constant $K > 0$ which satisfies

$$\forall (x, p) \in \mathbb{R} \times \mathcal{P}_g(\mathbb{R}), \quad C(x, p) \leq K \left( f(x) + \int_{\mathbb{R}} g(y) p(dy) \right),$$

Let $\Gamma$ be martingale C-monotone and $\pi \in \Pi_M(\mu, \nu)$ be such that we have (1.6). Then $\pi$ is optimal for (WMOT).
For $g : X \rightarrow [1, +\infty)$ continuous, we denote
\[
\mathcal{F}_g(X) := \{ f : \mathbb{R} \rightarrow [1, +\infty) \text{ continuous : } \forall y \in X, \ f(y) \geq g(y) \},
\]
and
\[
\mathcal{F}_g^+(X) := \left\{ f \in \mathcal{F}_g(X) : \exists h : \mathbb{R}_+ \rightarrow [1, +\infty), \ \frac{h(t)}{t} \underset{t \rightarrow +\infty}{\rightarrow} +\infty \text{ and } f = h \circ g \right\}.
\]

Proof of Theorem 3.3. Let $h \in \mathcal{F}_g(\mathbb{R})$ be such that $\nu(h) < +\infty$, whose purpose will be revealed later in the proof. To demonstrate the main idea without further technical details, we assume for now that $\mu$ is concentrated on a Polish subspace $X$ of $\mathbb{R}$, and the restriction $C|_{X \times \mathcal{P}_h(\mathbb{R})}$ is continuous. Moreover, we denote by $\hat{f}$ the restriction of $f$ to $X$, thus $\hat{f} \in \mathcal{F}_g^+(X)$. Let $X_n : \Omega \rightarrow \mathbb{R}, n \in \mathbb{N}$ be independent random variables identically distributed according to $\mu$ and $\mathcal{G} \subset \Phi_{f \otimes \hat{g}}(X \times \mathcal{P}(\mathbb{R}))$ be a countable family which determines the convergence in $\mathcal{P}_{f \otimes \hat{g}}(X \times \mathcal{P}(\mathbb{R}))$ (see [22, Theorem 4.5.(b)]). By the law of large numbers, almost surely, for all $\psi \in \mathcal{G} \cup \{ C|_{X \otimes \mathcal{P}(\mathbb{R})} \}$,
\[
\frac{1}{n} \sum_{k=1}^{n} \int_{X \times \mathcal{P}(\mathbb{R})} \psi(x, p) \delta_{(X_k, \pi_{X_k})}(dx, dp) = \frac{1}{n} \sum_{k=1}^{n} \psi(X_k, \pi_{X_k}) \underset{n \rightarrow +\infty}{\rightarrow} \mathbb{E}[\psi(X_1, \pi_{X_1})] = \int_{X} \psi(x, \pi_{x}) \mu(dx) = \int_{X \times \mathcal{P}(\mathbb{R})} \psi(x, p) J(\pi)(dx, dp),
\]
Moreover, almost surely, for all $n \in \mathbb{N}$,
\[
(X_n, \pi_{X_n}) \in \Gamma \cap (X \times \mathcal{P}_g(\mathbb{R})),
\]
and by the law of large numbers again, we have almost surely
\[
\frac{1}{n} \sum_{k=1}^{n} \pi_{X_k}(h) \underset{n \rightarrow +\infty}{\rightarrow} \mathbb{E}[\pi_{X_1}(h)] = \int_{X} \pi_{x}(h) \mu(dx) = \nu(h).
\]
Let then $\omega \in \Omega$ be such that (3.9), (3.10) and (3.11) hold when evaluated at $\omega$ and set $x_n = X_n(\omega)$ and $\pi^n(dx, dy) = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}(dx) \pi_{x_k}(dy)$ for $n \in \mathbb{N}$. Then $\pi^n$ has first marginal $\mu^n = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}$ and second marginal $\nu^n = \int_{x \in X} \pi_x(dy) \mu^n(dx)$. We deduce that $\pi^n$ is a martingale coupling between $\mu^n$ and $\nu^n$ such that
\[
J(\pi^n) = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k, \pi_{x_k}} \underset{n \rightarrow +\infty}{\rightarrow} J(\pi) \text{ in } \mathcal{P}_{f \otimes \hat{g}}(X \times \mathcal{P}(\mathbb{R})).
\]
In particular we have convergence of the marginals in $\mathcal{P}_{f}(X)$ and $\mathcal{P}_g(\mathbb{R})$ respectively. The second marginals are even converging in $\mathcal{P}_h(\mathbb{R})$ since $\nu^n(h)$ converges to $\nu(h)$ as $n \rightarrow +\infty$ by (3.11) evaluated at $\omega$. Thus, the convergence of $(J(\pi^n))_{n \in \mathbb{N}}$ to $J(\pi)$ even holds in $\mathcal{P}_{f \otimes \hat{g}}(X \times \mathcal{P}(\mathbb{R}))$ (see Definition 2.2) and therefore $\mathcal{P}_{f \otimes \hat{g}}(X \times \mathcal{P}_h(\mathbb{R}))$ by Lemma A.2 (b) below.

Since $(x, \pi^n) \in \Gamma$ for $\mu^n(dx)$-almost every $x$, we have according to Remark 1.6 that
\[
V^M_C(\mu^n, \nu^n) = \int_{X} C(x, \pi^n_x) \mu^n(dx),
\]
where we recall that the value function $V^M_C$ is defined in (WMOT). In that context, $C|_{X \times \mathcal{P}_h(\mathbb{R})}$ is a continuous function on $X \times \mathcal{P}_h(\mathbb{R})$ that is dominated from above by a positive multiple of $f \otimes \hat{g}$. We apply Theorem 2.7 to get
\[
\int_{X} C(x, \pi_x) \mu(dx) = \int_{X \times \mathcal{P}_h(\mathbb{R})} C(x, p) J(\pi)(dx, dp)
\]
which is a compact subset of $\pi$. To synthesize the two cases, $\lim \inf_{\mu} \int_X C(x, \pi^\mu) \mu^n(dx)$
\[= \lim_{n \to +\infty} \int_X C(x, \pi^\mu) \mu^n(dx) \leq \lim_{n \to +\infty} V^M_C(\mu^n, \nu^n) \leq V^M_C(\mu, \nu),\]
where we used (3.9) for the second equality. Hence, $\pi$ is optimal for $V^M_C(\mu, \nu)$.

Next, we drop the additional joint-continuity assumption on $C$. Since $\nu(g) < +\infty$, there exists by the de La Vallée Poussin theorem $h \in \mathcal{F}_+^1(\mathbb{R})$ such that $\nu(h) < +\infty$. For $N \in \mathbb{N}^*$, let $B_N = \{p \in \mathcal{P}_g(\mathbb{R}) : p(h) \leq N\}$, which is a compact subset of $\mathcal{P}_g(\mathbb{R})$ by Lemma A.6 below, and $\mathcal{C}(B_N)$ be the set of continuous functions from $B_N$ to $\mathbb{R}$, endowed with the topology of uniform convergence. The map $\phi^N : \mathbb{R} \to \mathcal{C}(B_N)$ given by $\phi^N(x) = C(x, \cdot)|_{B_N}$ is Borel measurable due to [3, Theorem 4.55]. Let $\varepsilon \in (0, 1)$. By Lusin’s theorem there is for every $N \in \mathbb{N}^*$ a compact set $K^N \subset \mathbb{R}$ such that the restriction $\phi^N|_{K^N}$ is continuous and $\mu(K^N) \geq 1 - \frac{\varepsilon}{2N}$.

We have
\[\mu \left( \bigcap_{N \in \mathbb{N}^*} K^N \right) \geq 1 - \sum_{N \in \mathbb{N}^*} \mu \left( (K^N)^c \right) \geq 1 - \sum_{N \in \mathbb{N}^*} \frac{\varepsilon}{2N} = 1 - \varepsilon.\]

Let $K^c = \bigcap_{N \in \mathbb{N}^*} K^N$, then for all $N \in \mathbb{N}^*$ the restriction $\phi^N|_{K^c}$ is continuous. We claim that $C|_{K^c \times \mathcal{P}_h(\mathbb{R})}$ is continuous w.r.t. the product topology of $\mathbb{R} \times \mathcal{P}_h(\mathbb{R})$. To this end, take any sequence $(x_k, p_k)_{k \in \mathbb{N}} \in (K^c \times \mathcal{P}_h(\mathbb{R}))^\mathbb{N}$ with limit point $(x, p) \in K^c \times \mathcal{P}_h(\mathbb{R})$. Since $p_k \to p$ in $\mathcal{P}_h(\mathbb{R})$ as $k$ goes to $+\infty$, the sequence $(p_k(h))_{k \in \mathbb{N}}$ is convergent to $p(h)$ and therefore bounded so there exists $N \in \mathbb{N}$ such that $p, p_k \in B_N$ for all $k \in \mathbb{N}$. As $\phi^N(x_k)$ converges uniformly to $\phi^N(x)$, we have
\[C(x_k, p_k) = \phi^N(x_k)(p_k) \to_{k \to +\infty} \phi^N(x)(p) = C(x, p).\]

Therefore, $C|_{K^c \times \mathcal{P}_h(\mathbb{R})}$ is continuous.

Let $\mu^\varepsilon = \frac{1}{\nu(K^c)} \mu|_{K^c}$, $\pi^\varepsilon = \frac{\nu}{\nu(K^c)} \pi|_{K^c \times \mathbb{R}}$ and $\nu^\varepsilon$ be the second marginal of $\pi^\varepsilon$. Obviously $\mu^\varepsilon$ is concentrated on $K^c$. Since $\mu(K^c) \mu^\varepsilon \leq \mu$ and $\pi^\varepsilon = \pi|_{K^c \times \mathbb{R}}$, $\pi^\varepsilon$ is a martingale coupling and satisfies $(x, \pi_x^\varepsilon) \in \Gamma$ for $\mu^\varepsilon(dx)$-almost every $x$. Finally, $\mu(K^c) \nu^\varepsilon(h) = \int_{K^c} \pi_x^\varepsilon(h) \mu(dx) \leq \nu(h) < +\infty$, hence $\nu^\varepsilon \in \mathcal{P}_h(\mathbb{R})$. Therefore the first part applied with $(K^c, \mu^\varepsilon, \nu^\varepsilon, \pi^\varepsilon)$ replacing $(X, \mu, \nu, \pi)$ ensures that $\pi^\varepsilon$ is optimal for $V^M_C(\mu^\varepsilon, \nu^\varepsilon)$.

Since $C(x, p)$ is bounded from above by a positive multiple of $f(h) + h(p)$ and $\nu(f) + \nu(h) < +\infty$, either $\int_{K^c} C(x, \pi_x) \mu(dx) = -\infty$ or $\int_{K^c} |C(x, \pi_x)| \mu(dx) < +\infty$. In the latter case, by Lebesgue’s theorem, $\int_{K^c} C(x, \pi_x) \mu(dx)$ converges to $\int_{K^c} C(x, \pi_x) \mu(dx)$ as $\varepsilon \to 0$ and so does
\[\int_{K^c} C(x, \pi_x) \mu(dx) = \int_{K^c} C(x, \pi_x) \mu(dx) = V^M_C(\mu^\varepsilon, \nu^\varepsilon).\]

To synthesize the two cases, $\lim \inf_{\varepsilon \to 0} V^M_C(\mu^\varepsilon, \nu^\varepsilon) \geq \int_{K^c} C(x, \pi_x) \mu(dx)$. The marginals $(\mu^\varepsilon)_{\varepsilon > 0}$ converge to $\mu$ in $\mathcal{P}_f(\mathbb{R})$ and strongly, whereas the marginals $(\nu^\varepsilon)_{\varepsilon > 0}$ converge to $\nu$ in $\mathcal{P}_h(\mathbb{R})$ for $\varepsilon \searrow 0$. By Theorem 2.7, we conclude that
\[V^M_C(\mu, \nu) = \lim \sup_{\varepsilon \to 0} V^M_C(\mu^\varepsilon, \nu^\varepsilon) = \lim \sup_{\varepsilon \to 0} V^M_C(\mu^\varepsilon, \nu^\varepsilon) \geq \int_{\mathbb{R}} C(x, \pi_x) \mu(dx),\]
proving optimality of $\pi$.

In the same way, Corollary 1.10 is a special case of the next corollary for the choice $f(x) = 1 + |x|^r$ and $g(y) = 1 + |y|^r$.

**Corollary 3.4.** Let $f, g \in \mathcal{F}_+^1(\mathbb{R})$, $\mu \in \mathcal{P}_f(\mathbb{R})$ and $\nu \in \mathcal{P}_g(\mathbb{R})$ be in convex order, $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be measurable and such that $y \mapsto c(x, y)$ is continuous for all $x \in \mathbb{R}$ and $\sup_{(x,y) \in \mathbb{R}^2} \frac{|c(x,y)|}{f(x)+g(y)} < +\infty$. Then $\pi \in \Pi_M(\mu, \nu)$ is concentrated on a set finitely optimal for $c$ if and only if $\pi$ is optimal for (MOT).
Proof of Corollary 3.4. The sufficient condition is stated in Lemma 1.11 [13] under mere measurability of the cost function c. For the necessary condition let us suppose that \( \pi \) is concentrated on a set \( \tilde{\Gamma} \) finitely optimal for \( c \) and define \( \Gamma = \{(x, p) \in \mathbb{R} \times \mathcal{P}_d(\mathbb{R}) : \int_\mathbb{R} \mathcal{J}_c(x, y)p(dy) = 1 \} \) and \( \mathbb{R} \times \mathcal{P}_d(\mathbb{R}) \ni (x, p) \mapsto C(x, p) = \int_\mathbb{R} c(x, y)p(dy) \). The growth condition satisfied by \( c \) and the continuity of this function in its second variable ensure that \( C \) is continuous in its second argument and satisfies the growth assumption in Theorem 3.3. Since \( 1 = \pi(\tilde{\Gamma}) = \int_\mathbb{R} \int_\mathbb{R} \mathcal{J}_c(x, y)p(dy)dx = \int_\mathbb{R} \int_\mathbb{R} g(y)\pi_x(dy)\mu(dx) = \int_\mathbb{R} g(y)\nu(dy) < \infty \), \( \pi \) is finitely supported in \( \mathcal{P}_d(\mathbb{R}) \) and \( \int_\mathbb{R} g(y)\nu(dy) < \infty \). Let \( \lambda \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i \). For each \( i \in \{1, \ldots, N\} \), let \( g_i \) be optimal for \( \lambda \sum_{i=1}^{N} p_i \) (resp. \( \lambda \sum_{i=1}^{N} q_i \)) in \( \mathcal{P}_d(\mathbb{R}) \) (resp. in \( \mathcal{P}_d(\mathbb{R}) \)) as \( k \to \infty \), by Theorem 2.6 applied with \( X = Y = \mathbb{R} \), there exists a sequence \( (\delta_i^k)_{k \in \mathbb{N}} \) of elements of \( \mathcal{P}_d(\mathbb{R}) \) as \( k \to \infty \), by Theorem 2.6 applied with \( X = Y = \mathbb{R} \), there exists a sequence \( (\delta_i^k)_{k \in \mathbb{N}} \) of elements of \( \mathcal{P}_d(\mathbb{R}) \) as \( k \to \infty \), by Theorem 2.6 applied with \( X = Y = \mathbb{R} \), there exists a sequence \( (\delta_i^k)_{k \in \mathbb{N}} \) of elements of \( \mathcal{P}_d(\mathbb{R}) \) as \( k \to \infty \), by Theorem 2.6 applied with \( X = Y = \mathbb{R} \), there exists a sequence \( (\delta_i^k)_{k \in \mathbb{N}} \) of elements of \( \mathcal{P}_d(\mathbb{R}) \) as \( k \to \infty \), by Theorem 2.6 applied with \( X = Y = \mathbb{R} \), there exists a sequence \( (\delta_i^k)_{k \in \mathbb{N}} \) of elements of \( \mathcal{P}_d(\mathbb{R}) \) as \( k \to \infty \). Hence

\[
\sum_{i=1}^{N} C(x, p^k_i) = N \int_{\mathbb{R} \times \mathbb{R}} c(x, y)\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(dx)p^k_i(dy) \leq N \int_{\mathbb{R} \times \mathbb{R}} c(x, y)\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(dx)q^k_i(dy) = \sum_{i=1}^{N} C(x, q^k_i).
\]

Letting \( k \to \infty \), we deduce by continuity of \( C \) in the measure argument that \( \sum_{i=1}^{N} C(x_i, p_i) \leq \sum_{i=1}^{N} C(x_i, q_i) \). Therefore \( \Gamma \) is martingale \( C \)-monotone and by Theorem 3.3, \( \pi \) is optimal for \( \text{(WMOT)} \) and equivalently for \( \text{(MOT)} \) in view of the definition of \( C \).

Proof of Lemma 3.5. For \( U \) uniformly distributed on \((0, 1)\) and \( n \in \mathbb{N}^* \), let \( p^n_U := \frac{1}{n} \sum_{j=1}^{n} \delta_{F^{-1}_p(U/n)} \). For \( \varphi : \mathbb{R} \to \mathbb{R} \) measurable and bounded, one has

\[
\mathbb{E} \left[ \int_{\mathbb{R}} \varphi(y)p^n_U(dy) \right] = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \left[ \varphi \left( F^{-1}_p \left( \frac{j-U}{n} \right) \right) \right] = \frac{1}{n} \sum_{j=1}^{n} \int_{\mathbb{R}} \varphi \left( F^{-1}_p(v) \right) dv = \int_{0}^{1} \varphi \left( F^{-1}_p(v) \right) dv,
\]

with the last integral equal to \( \int_{\mathbb{R}} \varphi(y)p(dy) \) by the inverse transform sampling. Hence the intensity of the random probability \( p^n_U \) is equal to \( p \). In particular, \( \mathbb{E}[p^n_U(A)] = p(A) = 1 \) and

\[
\mathbb{E} \left[ \int_{\mathbb{R}} g(y)p^n_U(dy) \right] = \int_{\mathbb{R}} g(y)p(dy).
\]

Moreover, when \( \varphi \) is continuous and bounded,

\[
\int_{\mathbb{R}} \varphi(y)p^n_U(dy) = \int_{0}^{1} \varphi \left( F^{-1}_p \left( \frac{[nv] - U}{n} \right) \right) dv \to \int_{0}^{1} \varphi \left( F^{-1}_p(v) \right) dv = \int_{0}^{1} \varphi(y)p(dy) \ \text{as} \ n \to \infty,
\]
by Lebesgue’s theorem, since the set of discontinuities of $F_n^{-1}$ is at most countable. Hence $p_n^p$ converges weakly to $p$ as $n \to \infty$ from which we deduce that $\liminf_{n \to \infty} \int_{\mathbb{R}} g(y)p_n^p(dy) \geq \int_{\mathbb{R}} g(y)p(dy)$. With (3.12) and Fatou lemma, we deduce that a.s. $\liminf_{n \to \infty} \int_{\mathbb{R}} g(y)p_n^p(dy) = \int_{\mathbb{R}} g(y)p(dy)$ so that a.s. there is a subsequence of $(p_n^p)_n$ giving full weight to $A$ and converging to $p$ in $\mathcal{P}_g(\mathbb{R})$.

\section{Appendix}

We recall that the adapted weak topology can be defined as the initial topology under the embedding map $J$ from $\mathcal{P}(X \times Y)$ to $\mathcal{P}(X \times \mathcal{P}(Y))$, namely

$$J : \mathcal{P}(X \times Y) \ni \pi = \mu \otimes \pi_x \mapsto \mu(dx) \delta_{\pi_x}(dp) \in \mathcal{P}(X \times \mathcal{P}(Y)).$$

(A.1)

Conversely, we can associate to a probability measure $P \in \mathcal{P}(\mathcal{P}(Y))$ its intensity $I(P)$

$$I(P)(dy) := \int_{p \in \mathcal{P}(Y)} p(dy) P(dp) \in \mathcal{P}(Y).$$

(A.2)

For the extended space $\mathcal{P}(X \times \mathcal{P}(Y))$ we naturally define the extended intensity $\hat{I}$ by

$$\hat{I} : \mathcal{P}(X \times \mathcal{P}(Y)) \ni P \mapsto \int_{p \in \mathcal{P}(Y)} p(dy) P(dx, dp) \in \mathcal{P}(X \times Y),$$

(A.3)

which associates to each $P \in \mathcal{P}(X \times \mathcal{P}(Y))$ a coupling $\hat{I}(P) \in \mathcal{P}(X \times Y)$. We note that $\hat{I}$ is the left-inverse of $J$.

For $(\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}(Y)$, we define the set of extended couplings $\Lambda(\mu, \nu)$ between $\mu$ and $\nu$ as the set of probability measures on $\mathcal{P}(X \times \mathcal{P}(Y))$ whose extended intensity is a coupling between $\mu$ and $\nu$, that is

$$\Lambda(\mu, \nu) = \left\{ P = \mu \otimes P_x \in \mathcal{P}(X \times \mathcal{P}(Y)) : \hat{I}(P) \in \Pi(\mu, \nu) \right\}.$$

(A.4)

If $f : X \to \mathbb{R}^+$ and $g : Y \to \mathbb{R}^+$ are measurable functions, then any $P \in \Lambda(\mu, \nu)$ satisfies

$$\int_{X \times \mathcal{P}(Y)} f(x) P(dx, dp) = \mu(f), \quad \int_{X \times \mathcal{P}(Y)} \int_{Y} g(y) p(dy) P(dx, dp) = \nu(g).$$

(A.5)

For $\mu, \nu \in \mathcal{P}^1(\mathbb{R})$, the martingale counterpart $\Lambda_M(\mu, \nu)$ of $\Lambda(\mu, \nu)$ is given by the set of probabilities on $\mathcal{P}^1(\mathbb{R} \times \mathcal{P}^1(\mathbb{R}))$ satisfying

$$\Lambda_M(\mu, \nu) = \left\{ P \in \Lambda(\mu, \nu) : \int_{\mathbb{R}} y p(dy) = x, \ P(dx, dp)\text{-a.s.} \right\}.$$

(A.6)

\section{A.1 Extension from $\mathcal{P}^r$ to $\mathcal{P}_f$.}

We recall that unless explicitly stated otherwise, $\mathcal{P}(Y)$ is endowed with the weak convergence topology, and for any continuous map $f : Y \to [1, +\infty)$ we endow the space $\mathcal{P}_f(Y) = \{ p \in \mathcal{P}(Y) : p(f) < +\infty \}$ with the topology induced by the following convergence: a sequence $(p_k)_{k \in \mathbb{N}} \in \mathcal{P}_f(Y)^{\mathbb{N}}$ converges in $\mathcal{P}_f(Y)$ to $p$ if and only if $p_k$ converges weakly to $p$ and $p_k(f)$ converges to $p(f)$ as $k \to +\infty$.

As mentioned in Section 2, this extension emerged from the need to overcome the inconvenience of the non-compactness of the $W_1$-balls $\{ p \in \mathcal{P}^r(Y) : W_1(p, \delta_y) \leq R \}, \ R > 0$, for the $W_1$-topology. The following lemmas show that this extension enjoys flexibility as the usual Wasserstein topology and most importantly benefits of a helpful compactness result, see Lemma A.6 below.

**Remark A.1.** We continue with some remarks on the structure of $\mathcal{P}_f(Y)$:
(1) Convergence in $\mathcal{P}_f(Y)$ can be described differently: let $(p_k)_{k \in \mathbb{N}}$ converge to $p$ in $\mathcal{P}_f(Y)$, and let $g \in \mathcal{C}(Y)$ be such that $0 \leq g \leq f$. We have $p(g) = \liminf_{k \to +\infty} p_k(g)$ and $p(f) - p(g) = p(f) - \limsup_{k \to +\infty} p_k(g)$, hence $\limsup_{k \to +\infty} p_k(g) \leq p(g)$. We deduce that

$$p_k \xrightarrow{k \to +\infty} p \text{ in } \mathcal{P}_f(Y) \iff p_k(g) \xrightarrow{k \to +\infty} p(g), \quad \forall g \in \Phi_f(Y),$$

(A.7)

when $\Phi_f(Y) := \{g \in \mathcal{C}(Y) : g \text{ is absolutely dominated by a positive multiple of } f\}$.

It is immediate that for $r \geq 1$, this topology is finer than the one induced by $W_r$ on $\mathcal{P}_f(Y)$ if $f$ is bounded from below by $y \mapsto 1 + d_r^Y(y, y_0)$.

(2) The set $\mathcal{P}_f(Y)$ is naturally embedded into the set $\mathcal{M}_+(Y)$ of all bounded positive Borel measures on $Y$, endowed with the weak topology, via the following continuous injection

$$\nu: \mathcal{P}_f(Y) \to \mathcal{M}_+(Y), \quad \nu(p)(dy) = f(y) p(dy).$$

Clearly, the topology on $\mathcal{P}_f(Y)$ coincides with the initial topology under $\nu$. Even more, the set $\nu(\mathcal{P}_f(Y)) = \{m \in \mathcal{M}_+(Y) : m(\hat{1}) = 1\}$ is a closed subset of $\mathcal{M}_+(Y)$ since $\hat{1}$ is continuous and bounded. As such, we deduce that $\mathcal{P}_f(Y)$ is a Polish space.

(3) By [15, Theorem 8.3.2 and the preceding discussion], we have that the weak topology on $\mathcal{M}_+(Y)$ is induced by the norm

$$\|m_1 - m_2\|_0 := \sup_{g: Y \to [-1, 1], \text{g is 1-Lipschitz}} (m_1(g) - m_2(g)).$$

This permits us to define a metric on $\mathcal{P}_f(Y)$ via

$$\overline{W}_f(p, q) := \sup_{g: Y \to [-1, 1], \text{g is 1-Lipschitz}} (p(fg) - q(fg)) = \|\nu(p) - \nu(q)\|_0.$$ (A.8)

Thus, $\overline{W}_f$ is a complete metric compatible with the topology on $\mathcal{P}_f(Y)$.

From now on, we equip $\mathcal{P}_f(Y)$ with $\overline{W}_f$. A continuous function $f: Y \to [1, +\infty)$ can naturally be lifted to a continuous function $\hat{f}: \mathcal{P}_f(Y) \to [1, +\infty)$ by setting

$$\hat{f}(p) := p(f).$$ (A.9)

For any probability measure $P \in \mathcal{P}(\mathcal{P}(Y))$, we then have $P(\hat{f}) = I(\nu(P))(f)$ where the intensity $I(P)$ is defined in (A.2).

For two maps $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$, we denote $f \circ g : X \times Y \ni (x, y) \mapsto f(x) + g(y)$.

As we are solely interested in topological properties, the next lemma shows that we can freely switch between the spaces $\mathcal{P}_f(\mathcal{P}(Y))$, $\mathcal{P}_f(\mathcal{P}_f(Y))$, and $\mathcal{P}_f(\mathcal{P}_f(Y))$, where the latter is defined as $\mathcal{P}_f(X)$ with $(1, \mathcal{P}_f(Y), \overline{W}_f)$ replacing $(r, X, d_X)$.

**Lemma A.2.** (a) Let $f : Y \to [1, +\infty)$ be continuous. Then

$$\mathcal{P}_f(\mathcal{P}(Y)) = \mathcal{P}_f(\mathcal{P}_f(Y)),$$ (A.10)

and their topologies coincide. Moreover, if $\mathcal{P}_f(Y)$ is endowed with the metric $\overline{W}_f$, see (2.3), then

$$\mathcal{P}_f(\mathcal{P}(Y)) = \mathcal{P}_f(\mathcal{P}_f(Y)) = \mathcal{P}_f(\mathcal{P}_f(Y)),$$ (A.11)

and their topologies are equal.
(b) Let \( f : X \to [1, +\infty) \) and \( g : Y \to [1, +\infty) \) be continuous. Then

\[
\mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y)) = \mathcal{P}_{f \oplus g}(X \times \mathcal{P}_g(Y)),
\]

(A.12)

and their topologies coincide.

Remark A.3. The equalities (A.10), (A.11) and (A.12) are to be understood up to an identification: For two measurable sets \( Z' \subset Z \), we say a probability measure \( p \in \mathcal{P}(Z) \) also belongs to \( \mathcal{P}(Z') \) if \( p(Z') = 1 \). This is achieved by identifying \( p \in \mathcal{P}(Z) \) with \( p' \in \mathcal{P}(Z') \) where for any measurable subset \( A \subset Z' \), \( p'(A) = p(A \cap Z') \).

Proof. Let us prove (a). The inclusion \( \mathcal{P}_f(\mathcal{P}(Y)) \supset \mathcal{P}_f(\mathcal{P}_f(Y)) \) is straightforward. Conversely, let \( P \in \mathcal{P}_f(\mathcal{P}(Y)) \). Then by definition,

\[
P(\hat{f}) = \int_{\mathcal{P}(Y)} p(f) \, P(dp) < +\infty,
\]

which can only hold if \( p(f) \) is \( P(dp) \)-almost everywhere finite, or equivalently \( P(\mathcal{P}_f(Y)) = 1 \), hence \( \mathcal{P}_f(\mathcal{P}(Y)) \subset \mathcal{P}_f(\mathcal{P}_f(Y)) \) and therefore we have equality. To see that the two topologies match, let us show that

\[
P^k \xrightarrow{k \to +\infty} P \text{ in } \mathcal{P}_f(\mathcal{P}_f(Y)) \iff P^k \xrightarrow{k \to +\infty} P \text{ in } \mathcal{P}_f(\mathcal{P}(Y)).
\]

Since the topology on \( \mathcal{P}_f(\mathcal{P}(Y)) \) is finer than the weak topology on \( \mathcal{P}_f(Y) \), we have \( \mathcal{C}(\mathcal{P}(Y)) \subset \mathcal{C}(\mathcal{P}_f(Y)) \), so the direct implication is trivial. Conversely, suppose that \( P^k \) converges in \( \mathcal{P}_f(\mathcal{P}(Y)) \) to \( P \) as \( k \) goes to \( +\infty \). Let \( h \in \mathcal{C}(\mathcal{P}(Y)) \) be bounded. Then \( \hat{h} \in \mathcal{C}(\mathcal{P}(Y)) \) is bounded, and \( I(P^k)(\hat{h}) = P^k(\hat{h}) \) converges to \( P(\hat{h}) = I(P)(\hat{h}) \) as \( k \) goes to \( +\infty \). Moreover \( I(P^k)(f) = P^k(\hat{f}) \) converges to \( P(\hat{f}) = I(P)(f) \). This shows that \( \{I(P^k) : k \in \mathbb{N}\} \) converges in \( \mathcal{P}_f(Y) \) to \( I(P) \). Therefore \( \{I(P^k) : k \in \mathbb{N}\} \) is relatively compact in \( \mathcal{P}_f(Y) \).

We deduce by Lemma A.4 below that \( \{P^k : k \in \mathbb{N}\} \) is relatively compact in \( \mathcal{P}_f(\mathcal{P}_f(Y)) \). Let \( Q \) be an accumulation point of \( \{P^k : k \in \mathbb{N}\} \) in \( \mathcal{P}_f(\mathcal{P}(Y)) \). In particular \( Q \) is by the direct implication shown above an accumulation point of \( \{P^k : k \in \mathbb{N}\} \) in \( \mathcal{P}_f(\mathcal{P}(Y)) \), hence \( Q = P \) by uniqueness of the limit since the topology is metrisable and therefore Hausdorff.

Let us now prove the second part of (a). We endow \( \mathcal{P}_f(Y) \) with the metric \( \mathcal{W}_f \). To see that the sets \( \mathcal{P}_f(\mathcal{P}_f(Y)) \) and \( \mathcal{P}(\mathcal{P}(Y)) \) are the same, we find

\[
P(\hat{f}) < +\infty \iff \int_{\mathcal{P}(Y)} p(f) \, P(dp) < +\infty \iff \int_{\mathcal{P}(Y)} \mathcal{W}_f(p, \delta_{y_0}) \, P(dp) < +\infty,
\]

which is an easy consequence of

\[
\forall p \in \mathcal{P}_f(Y), \quad p(f) - f(y_0) \leq \mathcal{W}_f(p, \delta_{y_0}) \leq p(f) + f(y_0).
\]

Let us now prove (b). We derive the equality \( \mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y)) = \mathcal{P}_{f \oplus g}(X \times \mathcal{P}_g(Y)) \) as in (a) since

\[
P(f \oplus g) = \int_{X \times \mathcal{P}(Y)} (f(x) + p(g)) \, P(dx, dp) < +\infty,
\]

which can only hold if the second marginal of \( P \) is concentrated on \( \mathcal{P}_g(Y) \). To see that the topologies are equal, the only nontrivial part is, as in (a), to show that if \( \{P^k : k \in \mathbb{N}\} \) converges in \( \mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y)) \), then \( \{P^k : k \in \mathbb{N}\} \) is relatively compact in \( \mathcal{P}_{f \oplus g}(X \times \mathcal{P}_g(Y)) \). Let then \( \{P^k : k \in \mathbb{N}\} \) converge in \( \mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y)) \) to some \( P \). Recall moreover the definition of the extended intensity \( \hat{I} \) given by (A.3). Let \( h : X \times Y \to \mathbb{R} \) be continuous and bounded. Then the map \( H : X \times \mathcal{P}(Y) \ni (x, p) \mapsto \int_Y h(x, y) \, p(dy) = \int_{X \times Y} h(x, y) \delta_y(dx) \, p(dy) \) is continuous and bounded. We deduce that \( \hat{I}(P^k)(h) = P^k(H) \) converges to \( P(H) = \hat{I}(P)(h) \) as \( k \) goes to \( +\infty \). Hence \( \{\hat{I}(P^k) : k \in \mathbb{N}\} \) converges weakly to \( \hat{I}(P) \). Then by continuity of the projections the first marginal \( \mu^k \), resp. the second marginal \( \nu^k \) of \( \hat{I}(P^k) \) converges weakly to the first marginal \( \mu \), resp. the second marginal.
\[ \nu \text{ of } \hat{I}(P). \] Since the maps \( f \oplus \hat{g} : (x, p) \mapsto f(x) \) and \( 0 \oplus \hat{g} : (x, p) \mapsto \hat{g}(p) \) belong to \( \mathcal{L}(X \times \mathcal{P}(Y)) \) and are dominated by \( f \oplus \hat{g} \), we also have that
\[
\mu^k(f) = P^k(f \oplus 0) \xrightarrow{k \to +\infty} P(f \oplus 0) = \mu(f) \quad \text{and} \quad \nu^k(g) = P^k(0 \oplus g) \xrightarrow{k \to +\infty} P(0 \oplus g) = \nu(g),
\]
which shows that \((\mu^k, \nu^k)_{k \in \mathbb{N}}\) converges in \( \mathcal{P}(X) \times \mathcal{P}(Y) \) to \((\mu, \nu)\). Therefore \((\hat{I}(P^k))_{k \in \mathbb{N}}\) is tight in \( \mathcal{P}(X \times Y) \) and both projections \( \{\mu^k : k \in \mathbb{N}\} \) and \( \{\nu^k : k \in \mathbb{N}\} \) are relatively compact respectively in \( \mathcal{P}(X) \) and \( \mathcal{P}(Y) \), so by Lemma A.7 below \( \{P^k : k \in \mathbb{N}\} \) is relatively compact in \( \mathcal{P}_{f \oplus \hat{g}}(X \times \mathcal{P}(Y)) \), which proves the claim. \( \square \)

**Lemma A.4.** A set \( \mathcal{A} \subset \mathcal{P}_f(\mathcal{P}_f(Y)) \) is relatively compact if and only if the set of its intensities \( I(\mathcal{A}) \subset \mathcal{P}_f(Y) \) is relatively compact.

**Proof.** The first implication follows as in [5, Lemma 2.4] by continuity of \( I \), c.f. Lemma A.8 below. The reverse implication can be shown by pursuing the same idea as in [5, Lemma 2.4] with slight modification: instead of considering the map \( y \mapsto d_Y(y, y') \) we use \( y \mapsto f(y) \).

**Lemma A.5.** A set \( \mathcal{A} \subset \mathcal{P}_f(Y) \) is relatively compact if and only if it is tight and
\[
\forall \varepsilon > 0, \exists R > 0, \sup_{P \in \mathcal{A}} \int_{y \in Y : f(y) \geq R} f(y) \mu(dy) < \varepsilon.
\]

**Proof.** The proof of this lemma runs along the lines of [5, Lemma 2.5] when replacing \( y \mapsto d_Y(y, y') \) by \( y \mapsto f(y) \).

**Lemma A.6.** Let \( g : \mathbb{R} \to [1, +\infty) \) be continuous and such that \( \lim_{|x| \to \infty} g(x) = +\infty \). Then for all \( f \) in the set \( \mathcal{F}_g^+(\mathbb{R}) \) defined in (3.8), the set \( B_R := \{p \in \mathcal{P}(\mathbb{R}) : p(f) \leq R\} \) is a compact subset of \( \mathcal{P}_g(\mathbb{R}) \).

**Proof.** Let \( R \geq 0 \), \( (p_n)_{n \in \mathbb{N}} \) be a sequence in \( B_R^+ \) and \( \varepsilon > 0 \). There exists \( r > 0 \) such that for all \( x \in \mathbb{R} \), \( |x| \geq r \) implies \( f(x) \geq \frac{R}{\varepsilon} \). Let \( K = \{x \in \mathbb{R} : |x| \leq r\} \). For all \( n \in \mathbb{N} \), we have \( R \geq p_n(f) \geq p_n(\mathbb{R}\setminus K)^{\frac{R}{\varepsilon}} \), hence \( p_n(\mathbb{R}\setminus K) \leq \varepsilon \). So \( (p_n)_{n \in \mathbb{N}} \) is tight, and by Prokhorov's theorem there exists a subsequence, still denoted \( (p_n)_{n \in \mathbb{N}} \) for notational simplicity, which converges weakly to \( p \in \mathcal{P}(\mathbb{R}) \). Since \( f \) is continuous and nonnegative, we have
\[
p(f) \leq \liminf_{n \to +\infty} p_n(f) \leq R,
\]
so \( p(f) \in B_R \). It remains to show that this convergence also holds in \( \mathcal{P}_g(\mathbb{R}) \). By Skorokhod's representation theorem, there exists for all \( n \in \mathbb{N} \) a random variable \( Z_n \sim p_n \), such that \( (Z_n)_{n \in \mathbb{N}} \) converges almost surely to a random variable \( Z \sim p \). For all \( n \in \mathbb{N} \) we have
\[
p_n(g) = \mathbb{E}[g(Z_n)] \leq \mathbb{E}[f(Z_n)] = p_n(f) \leq R,
\]
so by the de La Vallée Poussin theorem, \( (g(Z_n))_{n \in \mathbb{N}} \) is uniformly integrable. We deduce that
\[
\lim_{n \to +\infty} p_n(g) = p(g)
\]
and \( (p_n)_{n \in \mathbb{N}} \) converges in \( \mathcal{P}_g(\mathbb{R}) \) to \( p \), so \( B_R \) is compact. \( \square \)

For a probability measure \( \pi \in \mathcal{P}(X \times Y) \), we denote by \( \text{proj}_X(\pi) \) and \( \text{proj}_Y(\pi) \) its \( X \)-marginal and \( Y \)-marginal, respectively. Recall moreover the definition of the extended intensity \( \hat{I} \) given by (A.3).

**Lemma A.7.** Let \( f : X \to [1, +\infty) \) and \( g : Y \to [1, +\infty) \) be continuous. The following are equivalent:

(a) A set \( \Pi \subset \mathcal{P}(X \times Y) \) is tight and both projections, \( \text{proj}_X(\Pi) \subset \mathcal{P}_f(X) \) and \( \text{proj}_Y(\Pi) \subset \mathcal{P}_g(Y) \), are relatively compact.

(b) \( J(\Pi) \) as a subset of \( \mathcal{P}_{f \oplus \hat{g}}(X \times \mathcal{P}(Y)) \) is relatively compact.
Conversely, the following are equivalent:

(a') \( \Lambda \subset \mathcal{P}_{f \oplus \hat{g}}(X \times \mathcal{P}_g(Y)) \) is relatively compact.

(b') \( \hat{I}(\Lambda) \subset \mathcal{P}(X \times Y) \) is tight, and both projections, \( \text{proj}_X(\hat{I}(\Lambda)) \subset \mathcal{P}_f(X) \) and \( \text{proj}_Y(\hat{I}(\Lambda)) \subset \mathcal{P}_g(Y) \), are relatively compact.

**Proof.** For this lemma works the same proof as in [5, Lemma 2.6] when using Lemma A.4, the characterisation of relative compactness given in Lemma A.5 and continuity of \( \hat{I} \), see Lemma A.8. \( \square \)

**Lemma A.8.** Let \( f : X \to [1, +\infty) \) and \( g : Y \to [1, +\infty) \) be continuous. The maps

\[
I : \mathcal{P}_g(\mathcal{P}(Y)) \to \mathcal{P}_g(Y), \quad I(P)(dy) := \int_{\mathcal{P}(Y)} p(dy) P(dp),
\]

\[
\hat{I} : \mathcal{P}_{f \oplus \hat{g}}(X \times \mathcal{P}(Y)) \to \mathcal{P}_{f \oplus \hat{g}}(X \times Y), \quad \hat{I}(P)(dx, dy) := \int_{\mathcal{P}(Y)} p(dy) P(dx, dp),
\]

are continuous.

**Proof.** Let \( (P^k)_{k \in \mathbb{N}} \) be a sequence in \( \mathcal{P}_g(\mathcal{P}(Y)) \) with limit point \( P \). Let \( h : Y \to \mathbb{R} \) be continuous and bounded, then \( \hat{h} : \mathcal{P}(Y) \to \mathbb{R} \) is continuous and bounded. Thus,

\[
\lim_{k \to +\infty} I(P^k)(h) = \lim_{k \to +\infty} P^k(\hat{h}) = P(\hat{h}) = I(P)(h),
\]

\[
\lim_{k \to +\infty} I(P^k)(g) = \lim_{k \to +\infty} P^k(\hat{g}) = P(\hat{g}) = I(P)(g),
\]

which shows continuity of \( I \).

Next, let \( (P^k)_{k \in \mathbb{N}} \) be a sequence in \( \mathcal{P}_{f \oplus \hat{g}}(X \times \mathcal{P}(Y)) \) converging to \( P \). Let \( h : X \times Y \to \mathbb{R} \) be continuous and bounded. Then the map \( H : X \times \mathcal{P}(Y) \ni (x, p) \mapsto \int_Y h(x, y)p(dy) = \int_{X \times Y} h(z, y)\delta_x(dz)p(dy) \) is continuous and bounded. Again, we find

\[
\lim_{k \to +\infty} \hat{I}(P^k)(h) = \lim_{k \to +\infty} P^k(H) = P(H) = \hat{I}(P)(h),
\]

\[
\lim_{k \to +\infty} \hat{I}(P^k)(f \oplus g) = \lim_{k \to +\infty} P^k(f \oplus \hat{g}) = P(f \oplus \hat{g}) = \hat{I}(P)(f \oplus g),
\]

whereby we derive continuity of \( \hat{I} \). \( \square \)

**Proposition A.9.** Let \( g : Y \to [1, +\infty) \) be continuous, \( C : \mathcal{P}_g(Y) \to \mathbb{R} \) be convex, lower semicontinuous and lower bounded by a negative multiple of \( \hat{g} \). Then for all \( Q \in \mathcal{P}_g(\mathcal{P}(Y)) \) holds

\[
C(I(Q)) \leq \int_{\mathcal{P}_g(Y)} C(p) Q(dp).
\]

If moreover \( C \) is strictly convex, then (A.15) is an equality if and only if \( Q = \delta_{I(Q)} \).

**Proof.** Let \( Q \in \mathcal{P}_g(\mathcal{P}(Y)) \), \( P_n : \Omega \to \mathcal{P}(Y) \), \( n \in \mathbb{N}^* \) be independent random variables identically distributed according to \( Q \) and \( \mathcal{G} \subset \Phi_g(\mathcal{P}(Y)) \) be a countable family which determines the convergence in \( \mathcal{P}_g(\mathcal{P}(Y)) \) (see [22, Theorem 4.5.(b)]). By the law of large numbers, almost surely, for all \( \psi \in \mathcal{G} \),

\[
\frac{1}{n} \sum_{k=1}^{n} \psi(P_k) \xrightarrow{n \to +\infty} \mathbb{E}[\psi(P_1)] = Q(\psi) \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} C(P_k) \xrightarrow{n \to +\infty} \mathbb{E}[C(P_1)] = Q(C).
\]

Let \( \omega \in \Omega \) be such that (A.16) holds when evaluated at \( \omega \) and set \( p_n = P_n(\omega) \) for \( n \in \mathbb{N}^* \). Then \( \left( \frac{1}{n} \sum_{k=1}^{n} \delta_{p_k} \right)_{n \in \mathbb{N}} \) converges in \( \mathcal{P}_g(\mathcal{P}(Y)) \) to \( Q \). By Lemma A.8, \( \frac{1}{n} \sum_{k=1}^{n} p_k \) converges to \( I(Q) \) in \( \mathcal{P}_g(Y) \) as
\( n \to +\infty \). By lower semicontinuity of \( C \) for the first inequality, convexity of \( C \) for the second one and (A.16) evaluated at \( \omega \) for the equality, we get
\[
C(I(Q)) \leq \liminf_{n \to +\infty} C \left( \frac{1}{n} \sum_{k=1}^{n} p_k \right) \leq \liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} C(p_k) = Q(C). \quad \text{(A.17)}
\]

If \( Q = \delta_{I(Q)} \) we have trivially equality in (A.15). So, assume that \( Q \) is not concentrated on a single point, and that \( C \) is strictly convex. There are \( h \in \Phi_g(Y) \) and \( b \in \mathbb{R} \) such that \( A = \{ p \in \mathcal{P}_g(Y) : p(h) \leq b \} \) satisfies
\[
Q(A) > 0 \text{ and } Q(A^c) > 0. \quad \text{(A.18)}
\]

Indeed, pick any points \( p_1, p_2 \in \mathcal{P}_g(Y), p_1 \neq p_2 \) in the support of \( Q \), then the Hahn-Banach separation theorem provides \( h \in \Phi_g(Y) \) and \( b \in \mathbb{R} \) such that \( p_1(h) < b < p_2(h) \). As both points lie in the support of \( Q \), and \( \{ p \in \mathcal{P}_g(Y) : p(h) < b \} \) and \( \{ p \in \mathcal{P}_g(Y) : p(h) > b \} \) are open subsets containing \( p_1 \) and \( p_2 \), respectively, we obtain (A.18). Write \( Q_1(dp) := \mathbb{1}_A \frac{Q(dp)}{Q(A)} \) and \( Q_2(dp) := \mathbb{1}_{A^c} \frac{Q(dp)}{Q(A^c)} \). By the definition of \( A \), we have that \( I(Q_1)(h) \leq b < I(Q_2)(h) \) and especially \( I(Q_1) \neq I(Q_2) \). By (A.15) we find
\[
\int_{\mathcal{P}_g(Y)} C(p) Q_1(dp) \geq C(I(Q_1)) \quad \text{and} \quad \int_{\mathcal{P}_g(Y)} C(p) Q_2(dp) \geq C(I(Q_2)).
\]

Hence, as \( Q = Q(A)Q_1 + (1 - Q(A))Q_2 \) we get
\[
\int_{\mathcal{P}_g(Y)} C(p) Q(dp) = \int_{\mathcal{P}_g(Y)} C(p)Q(A)Q_1(dp) + \int_{\mathcal{P}_g(Y)} C(p)Q(A^c)Q_2(dp) \\
\geq Q(A)C(I(Q_1)) + (1 - Q(A))C(I(Q_2)) \\
> C(Q(A)I(Q_1) + (1 - Q(A))I(Q_2)) = C(I(Q)),
\]
where we used \( I(Q_1) \neq I(Q_2) \) and strict convexity for the last inequality. \( \square \)

### A.2 A Pompantuel-like theorem for Carathéodory maps

Let \((\pi^k)_{k \in \mathbb{N}}\) be a sequence of probability measures defined on \( X \times Y \) converging in \( \mathcal{P}_{f \otimes g}(X \times Y) \) to \( \pi \), and \( c : X \times Y \to \mathbb{R} \) be a (lower) Carathéodory map, that is a measurable function which is (lower semi-) continuous in its second argument. The goal of the present section is to connect the asymptotic behaviour of \( \int_{X \times Y} c(x,y) \pi^k(dx,dy) \) and \( \int_{X \times Y} c(x,y) \pi(dx,dy) \) under relaxed assumption on the cost \( c \). We recall that \((\pi^k)_{k \in \mathbb{N}}\) is said to converge stably to \( \pi \) if and only if for every bounded measurable map \( g : X \to \mathbb{R} \) and bounded continuous map \( h : Y \to \mathbb{R} \)
\[
\int_{X \times Y} g(x) h(y) \pi^k(dx,dy) \quad \stackrel{k \to +\infty}{\longrightarrow} \quad \int_{X \times Y} g(x) h(y) \pi(dx,dy).
\quad \text{(A.19)}
\]

We say that a sequence \((\mu^k)_{k \in \mathbb{N}}\) of probability measures on \( \mathcal{P}(X) \) \( K \)-converges in total variation to \( \mu \) if and only if for every subsequence \((\mu^{k_i})_{i \in \mathbb{N}}\) we have
\[
\frac{1}{n} \sum_{i=1}^{n} \mu^{k_i} \quad \stackrel{n \to +\infty}{\longrightarrow} \quad \mu \quad \text{in total variation.}
\]

**Lemma A.10.** Let \( \pi, \pi^k \in \mathcal{P}(X \times Y) \), \( k \in \mathbb{N} \) be with respective first marginal \( \mu \), \( \mu^k \). All of the following statements are equivalent:

(a) \((\pi^k)_{k \in \mathbb{N}}\) converges to \( \pi \) stably.

(b) \((\pi^k)_{k \in \mathbb{N}}\) converges to \( \pi \) weakly and \((\mu^k)_{k \in \mathbb{N}}\) converges strongly to \( \mu \).
(c) \((\pi^k)_{k \in \mathbb{N}}\) converges to \(\pi\) weakly and every subsequence of \((\mu^k)_{k \in \mathbb{N}}\) has an in total variation \(K\)-convergent sub-subsequence with limit \(\mu\).

Proof. We prove \(\text{“(a) } \implies (b)\)”. The definition of stable convergence given by (A.19) is in the Polish set-up by [16, Theorem 8.10.65 (ii)] equivalent to

\[
\int_{X \times Y} c(x, y) \pi^k(dx, dy) \xrightarrow[k \to +\infty]{} \int_{X \times Y} c(x, y) \pi(dx, dy)
\]

for all \(c: X \times Y \to \mathbb{R}\) which are bounded and Carathéodory. Thus, stable convergence is stronger than weak convergence. For all measurable subsets \(A \subset X\), we find by setting \(g = \mathbb{I}_A\) and \(h = 1\) in (A.19) that

\[
\mu^k(A) \xrightarrow[k \to +\infty]{} \mu(A).
\]

Next we show \(\text{“(b) } \implies (c)\)”. Let \(\mu^k(dx) = \rho^k(x) \mu(dx) + \eta^k(dx)\) be the Lebesgue decomposition of \(\mu^k\) w.r.t. \(\mu\). Since \(\eta^k\) is singular to \(\mu\) there is \(N^k \in \mathcal{B}(X)\) such that \(\eta^k(N^k) = \eta^k(X)\) and \(\mu(N^k) = 0\). Define \(N = \bigcup_{k \in \mathbb{N}} N^k \in \mathcal{B}(X)\), then \(\eta^k(N) = \eta^k(X)\) for all \(k \in \mathbb{N}\) and \(\mu(N)\) vanishes as a countable union of null sets. Thus, \(\eta^k(X) = \mu^k(N) \to \mu(N) = 0\) as \(k \to +\infty\). Since \((\rho^k)_{k \in \mathbb{N}}\) is bounded in \(L^1(\mu)\) there is by Komlós theorem a \(K\)-convergent subsequence to some limiting function \(\rho \in L^1(\mu)\). We have

\[
\frac{1}{n} \sum_{i=1}^{n} \rho^{ki} \xrightarrow[n \to +\infty]{} \rho, \quad \mu\text{-a.s.}
\]

By [15, Corollary 4.5.7] the above convergence even holds in \(L^1(\mu)\). We find for any measurable subset \(A \subset X\),

\[
\int_A \frac{1}{n} \sum_{i=1}^{n} \rho^{ki}(x) \mu(dx) \xrightarrow[n \to +\infty]{} \int_A \rho(x) \mu(dx).
\]

On the other hand \(\int_A \frac{1}{n} \sum_{i=1}^{n} \rho^{ki}(x) \mu(dx) = \frac{1}{n} \sum_{i=1}^{n} \mu^{ki}(A)\) converges to \(\mu(A)\). Hence \(\int_A \rho(x) \mu(dx) = \mu(A)\) so that \(\rho(x) = 1, \mu(dx)\)-almost surely and

\[
TV \left( \frac{1}{n} \sum_{i=1}^{n} \mu^{ki}, \mu \right) = \eta^k(X) + \int_X \left| \frac{1}{n} \sum_{i=1}^{n} \rho^{ki}(x) - 1 \right| \mu(dx) \xrightarrow[n \to +\infty]{} 0.
\]

Finally we show \(\text{“(c) } \implies (a)\)”. If \((\pi^k)_{k \in \mathbb{N}}\) does not converge stably to \(\pi\), then there is a bounded Carathéodory function \(c: X \times Y \to \mathbb{R}\), such that

\[
\limsup_{k \to +\infty} \left| \int_{X \times Y} c(x, y) \pi^k(dx, dy) - \int_{X \times Y} c(x, y) \pi(dx, dy) \right| > 0.
\]

Hence, w.l.o.g. there is a subsequence \((\pi^{kj})_{j \in \mathbb{N}}\) such that \(\pi^{kj}(c) \geq \pi(c) + \delta\) for some \(\delta > 0\). Especially, we have for any sub-subsequence \((\pi^{kj_i})_{i \in \mathbb{N}}\) of \((\pi^{kj})_{j \in \mathbb{N}}\) that

\[
\frac{1}{n} \sum_{i=1}^{n} \pi^{kj_i}(c) \geq \pi(c) + \delta,
\]

whereby the Cesàro-means of the sub-subsequence are not stably convergent. By assumption there exists a subsequence \((\mu^{kj_i})_{i \in \mathbb{N}}\) of \((\mu^k)_{j \in \mathbb{N}}\) which \(K\)-converges in total variation to \(\mu\). For \(n \in \mathbb{N}^*\) define

\[
\hat{\mu}^n = \frac{1}{n} \sum_{i=1}^{n} \mu^{kj_i} \quad \text{and} \quad \hat{\pi}^n = \frac{1}{n} \sum_{i=1}^{n} \pi^{kj_i}.
\]
We will show that \((\tilde{\pi}^n)_{n \in \mathbb{N}^*}\) converges stably to \(\pi\), which will contradict (A.20) and end the proof. Let 
\[ \hat{\mu}^n(dx) = \tilde{\pi}^n(x) \mu(dx) + \tilde{\eta}^n(dx) \]
be the Lebesgue decomposition of \(\mu^n\) w.r.t. \(\mu\). Define the auxiliary sequence 
\[ \hat{\pi}^n(dx, dy) = (1 \wedge \tilde{\pi}^n(x)) \tilde{\pi}^n(dy) + (1 - \tilde{\pi}^n(x))^+ \pi_x(dy) \mu(dx). \]

Let \(c: X \times Y \to \mathbb{R}\) be Carathéodory and absolutely bounded by \(K\), then 
\[
\left| \int_{X \times Y} c(x, y) \hat{\pi}^n(dx, dy) - \int_{X \times Y} c(x, y) \tilde{\pi}^n(dx, dy) \right| \\
\leq K \left( \int_X |\tilde{\pi}^n(x) - 1 \wedge \tilde{\pi}^n(x)| \mu(dx) + \int_X (1 - \tilde{\pi}^n(x))^+ \mu(dx) + \tilde{\eta}^n(X) \right) \\
\leq K \left( \int_X |\tilde{\pi}^n(x) - 1| \mu(dx) + 2\tilde{\eta}^n(X) \right) \\
\leq 2K \text{TV}(\hat{\mu}^n, \mu) \to_{n \to +\infty} 0.
\]

Since \((\hat{\pi}^n)_{n \in \mathbb{N}^*}\) converges to \(\pi\) weakly, we deduce that so does \((\tilde{\pi}^n)_{n \in \mathbb{N}^*}\). Note that the first marginal \(\hat{\pi}^n\) is \(\mu\), and therefore [32, Lemma 2.1] or Lemma A.10 yield stable convergence of \(\tilde{\pi}^n\) to \(\pi\) as \(n \to +\infty\). By (A.21), we find that \((\hat{\pi}^n)_{n \in \mathbb{N}^*}\) also stably converges to \(\pi\).

**Lemma A.11.** Let \(f: X \to [1, +\infty)\) and \(g: Y \to [1, +\infty)\) be continuous, and let \((\pi^k)_{k \in \mathbb{N}}\) converge to \(\pi\) in \(\mathcal{P}_{fg}(X \times Y)\).

(a) If \(c: X \times Y \to \mathbb{R} \cup \{+\infty\}\) is lower semicontinuous and bounded from below by a negative multiple of \(f \oplus g\), then 
\[
\liminf_{k \to +\infty} \int_{X \times Y} c(x, y) \pi^k(dx, dy) \geq \int_{X \times Y} c(x, y) \pi(dx, dy).
\]

(b) If \(c: X \times Y \to \mathbb{R}\) is continuous and absolutely bounded by positive multiple of \(f \oplus g\), then 
\[
\lim_{k \to +\infty} \int_{X \times Y} c(x, y) \pi^k(dx, dy) = \int_{X \times Y} c(x, y) \pi(dx, dy).
\]

(c) If \(c: X \times Y \to \mathbb{R} \cup \{+\infty\}\) is lower Carathéodory and bounded from below by a negative multiple of \(f \oplus g\), and \(\pi^k\) converges to \(\pi\) stably, then 
\[
\liminf_{k \to +\infty} \int_{X \times Y} c(x, y) \pi^k(dx, dy) \geq \int_{X \times Y} c(x, y) \pi(dx, dy).
\]

(d) If \(c: X \times Y \to \mathbb{R}\) is Carathéodory and absolutely bounded by a positive multiple of \(f \oplus g\), and \(\pi^k\) converges to \(\pi\) stably, then 
\[
\lim_{k \to +\infty} \int_{X \times Y} c(x, y) \pi^k(dx, dy) = \int_{X \times Y} c(x, y) \pi(dx, dy).
\]

**Proof.** These results are well-known. Note that by [15, Theorem 8.10.65] we have for every bounded lower Carathéodory map \(c\) that \(\pi \to \pi(c)\) is lower semicontinuous w.r.t. the topology of stable convergence.

When \(f: X \to [1, +\infty)\) and \(g: Y \to [1, +\infty)\) are continuous functions, we say that a sequence \(\pi^k \in \mathcal{P}_{f \oplus g}(X \times Y)\) converges stably to \(\pi\) if and only if, for \(k \to +\infty\), \(\pi^k\) converges to \(\pi\) in the topology of stable convergence and \(\pi^k(f \oplus g)\) to \(\pi(f \oplus g)\).

**Proposition A.12.** Let \(f: X \to [1, +\infty)\) and \(g: Y \to [1, +\infty)\) be continuous functions, and \(C: X \times \mathcal{P}_g(Y) \to \mathbb{R} \cup \{+\infty\}\) be measurable and bounded from below by a negative multiple of \(f \oplus \hat{g}\). Then
(a) If \( C \) is lower semicontinuous, then
\[
\mathcal{P}_{f \oplus g}(X \times \mathcal{P}_g(Y)) \ni P \mapsto \int_{X \times \mathcal{P}_g(Y)} C(x, p) \, P(dx, dp)
\]
(A.22)

is lower semicontinuous.

(b) Suppose in addition that for all \( x \in X \), the map \( p \mapsto C(x, p) \) is convex. Then
\[
\mathcal{P}_{f \oplus g}(X \times Y) \ni \pi \mapsto \int_X C(x, \pi_x) \, \mu(dx),
\]
(A.23)

where \( \mu \) denotes the \( X \)-marginal of \( \pi \), is lower semicontinuous.

(c) On the other hand, if \( C \) is lower Carathéodory, then
\[
\mathcal{P}_{f \oplus g}(X \times \mathcal{P}_g(Y)) \ni P \mapsto \int_{X \times \mathcal{P}_g(Y)} C(x, p) \, P(dx, dp)
\]
(A.24)

is lower semicontinuous w.r.t. stable convergence on \( \mathcal{P}_{f \oplus g}(X \times \mathcal{P}_g(Y)) \).

(d) Suppose in addition that for all \( x \in X \), the map \( p \mapsto C(x, p) \) is convex. Then
\[
\mathcal{P}_{f \oplus g}(X \times Y) \ni \pi \mapsto \int_X C(x, \pi_x) \, \mu(dx),
\]
(A.25)

where \( \mu \) denotes the \( X \)-marginal of \( \pi \), is lower semicontinuous w.r.t. stable convergence on \( \mathcal{P}_{f \oplus g}(X \times Y) \).

**Proof.** Lower semicontinuity of (A.22) and (A.24) is a consequence of Lemma A.11 with \( Y \) and \( g \) replaced by \( \mathcal{P}_g(Y) \) and \( \hat{g} \). To see (A.23) (resp. (A.25)), let \((\pi_k)_{k \in \mathbb{N}} \in \mathcal{P}_{f \oplus g}(X \times Y)^\mathbb{N}\) converge (resp. stably) in \( \mathcal{P}_{f \oplus g}(X \times Y) \) to some \( \pi \). We find by the first part of Lemma A.7 an accumulation point \( P \in \mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y)) \) of \((J(\pi^k))_{k \in \mathbb{N}}\). By possibly passing to a subsequence we can assume that \( P^k := J(\pi^k) \) converges to \( P \) in \( \mathcal{P}_{f \oplus g}(X \times \mathcal{P}(Y)) \) as \( k \) goes to \(+\infty\). In the case that \((\pi^k)_{k \in \mathbb{N}}\) converge stably in \( \mathcal{P}_{f \oplus g}(X \times Y) \) to \( \pi \), we deduce by Lemma A.10 that \((P^k)_{k \in \mathbb{N}}\) converges stably in \( \mathcal{P}_{f \oplus g}(X \times \mathcal{P}_g(Y)) \) to \( P \). Write \( \mu^k, k \in \mathbb{N} \) and \( \mu \) for the \( X \)-marginal of \( \pi^k \) and \( \pi \), respectively. Due to (A.22) (resp. (A.24)), we obtain

\[
\liminf_{k \to +\infty} \int_X C(x, \pi^k_x) \, \mu^k(dx) = \liminf_{k \to +\infty} \int_{X \times \mathcal{P}_g(Y)} C(x, p) \, P^k(dx, dp)
\]

\[
\geq \int_{X \times \mathcal{P}_g(Y)} C(x, p) \, P(dx, dp)
\]

\[
\geq \int_X C(x, I(P_x)) \, \mu(dx)
\]

\[
= \int_X C(x, \hat{I}(P)_x) \, \mu(dx),
\]

where we used Proposition A.9 for the last inequality. Since \( \hat{I} \) is continuous by Lemma A.8, we find that \( \hat{I}(P^k) \to \hat{I}(P) \) and \( \hat{I}(P^k) = \pi^k \to \pi \) as \( k \to +\infty \). But the weak topology is Hausdorff and therefore \( \pi = \hat{I}(P) \) yielding

\[
\liminf_{k \to +\infty} \int_X C(x, \pi^k_x) \, \mu^k(dx) \geq \int_X C(x, \pi_x) \, \mu(dx),
\]

and thus (A.23) (resp. (A.25)). \(\square\)
References

[1] B. Acciaio, M. Beiglböck, and G. Pammer. Weak transport for non-convex costs and model-independence in a fixed-income market. *Math. Finance*, 31(4):1423–1453, 2021.

[2] J.-J. Alibert, G. Bouchitté, and T. Champion. A new class of costs for optimal transport planning. *European Journal of Applied Mathematics*, 30(6):1229–1263, 2019.

[3] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis: A hitchhiker’s guide*. Springer, 3rd edition, 2006.

[4] L. Ambrosio and N. Gigli. A user’s guide to optimal transport. In *Modelling and optimisation of flows on networks*, volume 2062 of *Lecture Notes in Math.*, pages 1–155. Springer, Heidelberg, 2013.

[5] J. Backhoff, M. Beiglböck, and G. Pammer. Existence, duality, and cyclical monotonicity for weak transport costs. *Calculus of Variations and Partial Differential Equations*, 58(6):1–28, 2019.

[6] J. Backhoff-Veraguas, D. Bartl, M. Beiglböck, and M. Eder. All adapted topologies are equal. *Probab. Theory Related Fields*, 178(3-4):1125–1172, 2020.

[7] J. Backhoff-Veraguas, M. Beiglböck, M. Huesmann, and S. Källblad. Martingale Benamou-Brenier: a probabilistic perspective. *Ann. Probab.*, 48(5):2258–2289, 2020.

[8] J. Backhoff-Veraguas and G. Pammer. Applications of weak transport theory. *Bernoulli*, 28(1):370–394, 2022.

[9] J. Backhoff-Veraguas and G. Pammer. Stability of martingale optimal transport and weak optimal transport. *Ann. Appl. Probab.*, 32(1):721–752, 2022.

[10] M. Beiglböck, A. Cox, and M. Huesmann. Optimal transport and Skorokhod embedding. *Invent. Math.*, 208(2):327–400, 2017.

[11] M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices: A mass transport approach. *Finance Stoch.*, 17(3):477–501, 2013.

[12] M. Beiglböck, B. Jourdain, W. Margheriti, and G. Pammer. Approximation of martingale couplings on the line in the adapted weak topology. *Probab. Theory Relat. Fields, to appear*, 2021.

[13] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.*, 44(1):42–106, 2016.

[14] M. Beiglböck, M. Nutz, and F. Stebegg. Fine Properties of the Optimal Skorokhod Embedding Problem. *J. Eur. Math. Soc. (JEMS), to appear*, Mar 2021.

[15] V. I. Bogachev. *Measure theory*, volume 1. Springer Science and Business Media, 2007.

[16] V. I. Bogachev. *Measure theory*, volume 2. Springer Science and Business Media, 2007.

[17] D. T. Breeden and R. H. Litzenberger. Prices of state-contingent claims implicit in option prices. *The Journal of Business*, 51(4):621–51, 1978.

[18] M. Brückerhoff and N. Juillet. Instability of martingale optimal transport in dimension $d \geq 2$. *arXiv preprint arXiv:2101.06964*, 2021.

[19] P. Cheridito, M. Kuiiski, D. J. Prömel, and H. M. Soner. Martingale optimal transport duality. *Math. Ann.*, 379(3-4):1685–1712, 2021.
[20] H. De March and N. Touzi. Irreducible convex paving for decomposition of multidimensional martingale transport plans. *Ann. Probab.*, 47(3):1726–1774, 2019.

[21] Y. Dolinsky and H. M. Soner. Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Relat. Fields*, 160(1-2):391–427, 2014.

[22] S. N. Ethier and T. G. Kurtz. *Markov processes: characterization and convergence*, volume 282. John Wiley & Sons, 2009.

[23] A. Galichon, P. Henry-Labordère, and N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.*, 24(1):312–336, 2014.

[24] N. Ghoussoub, Y.-H. Kim, and T. Lim. Structure of optimal martingale transport plans in general dimensions. *Ann. Probab.*, 47(1):109–164, 2019.

[25] N. Gozlan and N. Juillet. On a mixture of Brenier and Strassen theorems. *Proceedings of the London Mathematical Society*, 120(3):434–463, 2020.

[26] N. Gozlan, C. Roberto, P.-M. Samson, Y. Shu, and P. Tetali. Characterization of a class of weak transport-entropy inequalities on the line. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1667–1693, 2018.

[27] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali. Kantorovich duality for general transport costs and applications. *J. Funct. Anal.*, 273(11):3327–3405, 2017.

[28] C. Griessler. An extended footnote on finitely minimal martingale measures. *ArXiv e-prints*, June 2016.

[29] G. Guo and J. Obłój. Computational methods for martingale optimal transport problems. *Ann. Appl. Probab.*, 29(6):3311–3347, 2019.

[30] J. Guyon, R. Menegaux, and M. Nutz. Bounds for VIX futures given S&P 500 smiles. *Finance and Stochastics*, 21(3):593–630, 2017.

[31] D. Hobson and A. Neuberger. Robust bounds for forward start options. *Math. Finance*, 22(1):31–56, 2012.

[32] R. Lassalle. Causal transference plans and their Monge-Kantorovich problems. *Stochastic Analysis and Applications*, 36(3):452–484, 2018.

[33] J. Obłój and P. Siorpaes. Structure of martingale transports in finite dimensions. arXiv:1702.08433, 2017.

[34] F. Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[35] J. B. Veraguas, M. Beiglböck, M. Eder, and A. Pichler. Fundamental properties of process distances. *Stochastic Process. Appl.*, 130(9):5575–5591, 2020.

[36] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[37] C. Villani. *Optimal Transport. Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2009.

[38] J. Wiesel. Continuity of the martingale optimal transport problem on the real line. *ArXiv e-prints*, 2020.