NIEMEIER SELF-DUAL LATTICES
AND TOPOLOGICAL PHASE TRANSITIONS

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Abstract

A topological phase transition in two-dimensional nonlinear $\sigma$-models on tori, connected with self-dual (unimodular) 24-dimensional Niemeier lattices, is considered. It is shown that critical properties of these transitions are determined by corresponding Coxeter numbers of lattices. A case of general integer-valued lattices with minimal norm equal 1 and 2 and a possible application to string theory are discussed.

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Recently it was shown [1] (see also [2]), that two-dimensional nonlinear \( \sigma \)-models, defined on the maximal abelian Cartan tori \( T_G \) of simple compact Lie groups \( G \), have a topological phase transition with critical properties different from that of \( XY \)-model or of nonlinear \( \sigma \)-model on circle \( S^1 \) [3, 4]. There are the universality classes of critical behaviour, determined by the type of the corresponding dual root lattices \( L_v \), which can belong to the series \( A_n, D_n, E_n, Z_n \). More rigorously, all critical indices are determined by a set of minimal vectors and can be expressed through the Coxeter number \( h_G \) of these lattices. For example, a correlation length \( \xi \) as a function of temperature \( T \) (or equivalent coupling constant of the nonlinear \( \sigma \)-model) has an essential singularity

\[
\xi(T) \sim a \exp(A \tau^{-\nu_G}), \quad \tau = \frac{T - T_c}{T_c},
\]

(1)

\[
\nu_G = 2/(h_G + 2),
\]

(2)

where \( a \) is a small size cut-off parameter, \( A \) is some nonuniversal constant \( O(1) \) and \( T_c \) is a temperature of phase transition. To the initial \( XY \)-model, usual tori \( T^m = U(1)^n \) and \( Z^n \) lattices corresponds \( h_G = h_{A_1} = 2 \).

The Coxeter numbers of \( A, D, E \) root systems have the following values

\[
h_{A_n} = n + 1, \quad h_{D_n} = 2n - 2, \quad h_{E_6} = 12, \quad h_{E_7} = 18, \quad h_{E_8} = 30
\]

(3)

It means that in this relation tori \( T_G, G = A, D, E \), cannot be considered as a direct product of \( S^1 = U(1) \): \( T_G \neq U(1)^n \).

All tori \( T_G \) are determined by so called integer-valued lattices (in appropriate scale) of \( A, D, E, Z \) series. A series \( Z_n \) belongs to the odd self-dual (or unimodular) lattices and contains a minimal vectors with norm equal 1, while the series \( A_n, D_n, E_n \) belong to the even lattices with minimal norm equal 2. Among them only lattice \( E_8 \) is self-dual. There are a number of other integer-valued lattices, but the most interesting are self-dual ones. Besides possible application of these lattices in a construction of the anomaly-free string theory [5] one can show that for such lattices and connected with them tori and nonlinear \( \sigma \)-models all quantum numbers admit a topological interpretation: they can be represented as a topological charges of classical solutions [6].

From the point of view of topological phase transitions the most importance have the integer-valued lattices with minimal norm equal 2, since, in principle, only they can have new critical properties. But their number and
classification are not known at present (except some low-dimensional cases).
If for this reason we confine ourselves only by self-dual lattices, even then
their number is too large and grows rapidly with lattice dimension $d$. For
example, a number of such lattices in dimensions $d \leq 23$ is 106 [8]. Thus
we shall consider only even self-dual lattices. They can exist only in spaces
with dimension $d = 8n$, $n \in \mathbb{Z}^+$ [8]. In $d = 8$ there is one such lattice $E_8$,
in $d = 16$ there are two lattices: $E_8 \oplus E_8$ and $D_{16}^+$. Here a lattice $D_{16}^+$
is obtained from lattice $D_{16}$ by addition one gluing vector [8], but its set of
the minimal vectors coincides with that of lattice $D_{16}$, and for this reason
their critical properties also coincides.

In space with $d = 24$ there are 24 self-dual even lattices, enumerated and
constructed by Niemeier [8] and 23 from them have minimal norm equal 2
(note that a number of such odd lattices is much larger and equals 155 [8]).
The last lattice, the so called Leach lattice, has a minimal norm equal 4 [10].
In the next space with $d = 32$ there are no less than $8 \cdot 10^7$ even unimodular
lattices [8] and for this reason any analysis of such lattices in spaces with
d > 24 is now also impossible.

In this paper we consider critical properties of nonlinear $\sigma$-models, con-
nected with 23 even unimodular lattices with a minimal norm equal 2, which
we will call as a $N_n$ ($n = 1, \ldots, 23$) series. All lattices of this series are
constructed by a gluing method from lattices of $A, D, E$ series, such that
all components must have the same Coxeter number $h_n$ and a sum of their
dimensions must be equal to 24 [8, 9]. In analogy with lattices of $A, D, E$
type a set of minimal vectors with norm equal 2 are named the roots of these
lattices.

A geometry of the root systems $\{r\}_n$ of the Niemeier lattices, obtained
by using some theorems about dimension of the modular forms space, is the
following [8]:

1) a rank (or dimension $d$) of the root set is equal 24;
2) a number of roots in $N_n$ lattices are equal to $24h_n$;
3) an isotropy constant $B_n$ of the root systems $\{r\}_n$, defined by condition

$$B_n \delta_{ik} = \sum_{\{r^a\}} r^a_i r^a_k,$$

is equal

$$B_n = 2h_n$$

Compare this with corresponding data for root systems of $A, D, E$ latt-
tices one can show that the root systems of $N_n$ lattices break into direct

3
sum of the root systems of the lattices, composing a lattice $N_n$, and that the Coxeter numbers for $N_n$ lattices has the same property as the Coxeter numbers of the $A, D, E$ lattices, i.e. a number of all roots $\#_n$ in $N_n$ lattice is equal $\#_n = nh_n$. Now one can enumerate all Niemeier lattices, their number is a number of partitions of $d = 24$ into the dimensions of the root systems of $A, D, E$ types with condition, that all components must have the same Coxeter number.

Taking into account (3) one can show that there are only 23 combinations. They are listed together with their Coxeter numbers in a Table 1.

| $N_n$ | $A_{11}D_7E_6$ | $2A_{12}$ | $3D_8$ | $A_{15}D_9$ | $D_{16}2E_7$ |
|-------|----------------|-----------|--------|-------------|--------------|
| $h_{N_n}$ | 12            | 13        | 14     | 16          | 18           |

Table 1

| $N_n$ | $A_{17}D_7$ | $2D_{12}$ | $A_{24}$ | $3E_8$ | $D_{16}E_8$ | $D_{24}$ |
|-------|-------------|-----------|----------|--------|-------------|----------|
| $h_{N_n}$ | 18          | 22        | 25       | 30     | 30          | 46       |

Here a symbol $iA_rkDs1Et$ denotes that the corresponding $N_n$ lattice is composed from $iA_r$, $kD_s$ and $1E_t$ lattices. As was shown in [4] all coefficients of renorm-group equations, determining critical properties of the models, are expressed through Coxeter number. Since all components of each $N_n$ lattice have the same Coxeter numbers it means that the nonlinear $\sigma$-models, related with the Niemeier lattices, will have the same critical properties as the $\sigma$-models related with $A, D, E$ lattices with the same Coxeter numbers! Note also, that again, as in the case of $A, D, E$ lattices, there are some degenerations between different $N_n$ lattices on the Coxeter number (see Table 1) and consequently all critical properties of the corresponding models will coincide.

Thus we have shown that no any new universality classes appear in the case of 24-dimensional even self-dual lattices with minimal norm equal 2.
turns out that this fact takes place for all integer-valued lattices with minimal norm equal 1 or 2. This conclusion follows from the Witt theorem \[8\], proving that the set of their minimal vectors must be a direct sum of the root systems. But they can be only of \(A, D, E, Z\) types. Consequently, all integer-valued lattices with minimal norm equal 1 can have critical properties only of \(XY\)-model (or of \(Z^n\) lattice) type, while all integer-valued lattices with minimal norm equal 2 can have critical properties only of \(A, D, E\) lattice types. Now different components can have different Coxeter numbers and one can have a series of phase transitions in each component (discussion of some simple examples see below).

Due to existence of low-temperature massless phase a topological phase transition in nonlinear \(\sigma\) models on tori can be interpreted as a statistical decompactification transition \([11]\). It means that this transition does not really change a topology of the target space \(T^d\), but only effectively, i.e. it changes a behaviour of all correlations at large distances to those in free theory defined on covering target space \(R^d\). From this point of view such topological phase transition can have some interest for string theories. As we have shown a phase transition in \(\sigma\)-models on composite torus takes place simultaneously in the whole torus space, if all components have the same Coxeter number. For string theory a more interest would have a possibility of topological phase transition of above mentioned type from some compact space \(C\) into partially decompactified space of the form

\[\mathcal{M} = R^d \otimes C',\]

where \(C'\) is another compact space, which can be identified with internal (isotopic) space. For this one need to consider a \(\sigma\)-model on a target space \(C\) with homotopical group \(\pi_1 = L^d\), where \(L^d\) is some \(d\)-dimensional lattice of \(A, D, E, Z\) type. In general, \(C\) can contain some torus \(T^d\) as a submanifold and has corresponding relative homotopical group \([6, 11]\)

\[\pi_2(C, T^d) = L^d.\]

Below, for simplicity, we suppose that

\[C = T^d \otimes C'.\]

Here \(T^d\) is some \(d\)-dimensional torus, corresponding to the lattice \(L^d\). Let us first consider a case \(\pi_1(C') = 0\). Then topological phase transition will
effectively decompactified torus and target space becomes $\mathcal{M}$. Another possibility is when $\mathcal{C}$ is a composite torus

$$\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2.$$  

Then one can have two (or $n$, if a number of components is equal $n$) phase transitions at different temperatures $T_{ci}$ in $i$-th component of the composite torus, under this a compact composite torus $\mathcal{T}$ will behave itself at intermediate temperatures effectively as a partially noncompact space

$$\mathcal{M}' = R^{d_2} \otimes \mathcal{T}_1,$$

where $R^{d_2}$ corresponds to the decompactified component $\mathcal{T}_2$. This scenario is possible when components of the composite torus $\mathcal{T}$ have different Coxeter numbers $h_i$ and, for example, $h_1 > h_2$.

The corresponding schematic phase diagram is depicted in Fig.1. Here $g$ is a dimensionless amplitude of one vortex, $\delta$ is dimensionless parameter, connected with coupling constant and $\tau$. Lines 1,2 denote lines of phase transitions in torus $\mathcal{T}_1$ and torus $\mathcal{T}_2$ respectively, $X$ denotes an intermediate phase with partial decompactification: torus $\mathcal{T}_2$ is decompactified, torus $\mathcal{T}_1$ still not. Line 3 denotes a symmetric separatrix, which lies in massive phase, its incline is universal for all tori and on it there is an additional symmetry. For tori $T_G$ it is a symmetry of the corresponding continuous compact Lie group $G$ [12].

![Fig.1. Schematic phase diagram of $\sigma$-model on composite torus $\mathcal{T} = \mathcal{T}_1 \otimes \mathcal{T}_2$](image)
It follows from (3) that the Coxeter numbers $h_G$ increase with increase of dimension of lattices and for lattices with equal dimensions the next inequalities take place

$$h_{A_n} \leq h_{D_n} < h_{E_n}.$$  \hspace{1cm} (6)

Thus, if both tori belong to the same series, firstly a torus with smaller dimension must be decompactified. When one torus is of usual type $T_2 = R^{d_2}/L^d$ ($h = 2$ always corresponds to them) and other torus is of $T_G (G = A, D, E,)$ type with $d_1 > 1$, then the usual torus will decompactify first.

For obtaining 4-dimensional space $R^4$ (we confine here ourselves only by Euclidean case) one need take a torus $T_G$ of one of the simple compact Lie groups with rank equal 4. Among them there is $SU(5)$ group, which is used often as a group of Grand Unification Theories! Moreover, as was noted in [1], if we consider analogous decompactification transition in chiral model on $G = SU(5)$, we can obtain as an internal space a corresponding flag space $F_G = G/T_G$ with dimension $d = 20$ and with homotopical group $\pi_2(F_G) = L^G_v$ ($L^G_v$ is a lattice of dual roots of group $G$), which is needed for topological interpretation of the quantum numbers of group $G$ [6]. It is worth to note that quantum numbers of the fundamental (quark) representations of groups $SU(N)$ do not admit topological interpretation in terms of instantons on $F_G$ and maybe for this reason the quarks do not freely exist.

Spaces $\mathcal{M}'$ are also analogous to the partially compactified spaces of the string theory, but here they appear as an intermediate spaces. Under further cooling, in low $T$ phase, both torus become effectively decompactified. A physical picture of decompactification transition in chiral models on tori is very attractive. It can serve as a guide line for more realistic string models, which must describe a process of formation of the partially compactified space. In our opinion these processes must be similar to the cosmological processes of the ”birth of Universe” type and for this reason must take into account the cosmological aspects as well as the fact that string partially compactified spaces are the final (or even maybe the intermediate) spaces of the general process of the Universe development.

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