On the spacetime connecting two aeons in conformal cyclic cosmology

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Abstract

It is shown that the contraction limit of a de Sitter spacetime for the cosmological term going to infinity satisfies a number of properties, including the Weyl curvature hypothesis, which qualify it as a candidate to represent the bridging spacetime connecting two aeons in Penrose’s conformal cyclic cosmology.

1 Introduction

The existence of an invariant length at the Planck scale indicates that the spacetime local kinematics should exhibit a departure from ordinary special relativity at that scale. One then has to look for a modified special relativity. An interesting attempt in this direction is the so-called “doubly special relativity” [1], obtained through the introduction of scale-suppressed terms of higher order in the momentum into the dispersion relation of special relativity, in such a way to allow the existence of a Lorentz-invariant length at the Planck scale. The importance of these terms is controlled by a parameter $\kappa$, which changes the kinematic group of special relativity from Poincaré to a $\kappa$-deformed Poincaré group. Far away from the Planck scale these terms are suppressed, and one obtains back ordinary special relativity.

A different solution to the same problem shows up from the simple observation that Lorentz transformations do not change the curvature of the homogeneous spacetime in which they are performed. Considering that the scalar curvature $R$ of any homogeneous spacetime is of the form

$$R \sim \pm l^{-2},$$

with $l$ the pseudo-radius, Lorentz transformations are found not to change the length parameter $l$. Although somewhat hidden in Minkowski space, because what is left invariant in this case is an infinite length — corresponding to a vanishing scalar curvature — in de Sitter and anti-de Sitter spacetimes, whose pseudo-radii are finite, this property becomes manifest. Contrary to the usual belief, therefore, Lorentz transformations do leave invariant a very particular length: that defining the scalar curvature of the homogeneous spacetime.

On account of this property, if the Planck length $l_P$ is to be invariant under Lorentz transformations, it is natural to assume that it represents the pseudo-radius of spacetime at the Planck scale, which will then be either a de Sitter or an anti-de Sitter space, with the scalar curvature given by

$$R \sim \pm l_P^{-2} \simeq \pm 10^{60} \text{cm}^{-2},$$

(1)

where the + (−) sign refers to the de Sitter (anti de Sitter) case. Such a huge $R$ — or cosmological term $\Lambda$ — is consistent with the idea that at the Planck scale spacetime must be strongly curved, but at the same time, since both de Sitter and anti de Sitter spacetimes have vanishing Weyl curvature, such curvature cannot be dynamic in the sense of general relativity.
From now on, taking into account recent astronomical observations indicating that the universe expansion is presently accelerating, we will restrict ourselves to the de Sitter case.

Now, if spacetime at the Planck scale is not Minkowski, but de Sitter, instead of ruled by the Poincaré group, spacetime kinematics will be ruled by the de Sitter group. This amounts to replace ordinary special relativity by a de Sitter-ruled special relativity \[2\,–\,4\]. Far away from the Planck scale, the de Sitter pseudo-radius will be large and the cosmological term will be small. Even at this low-energy scale, therefore, the spacetime kinematics will still be ruled by the de Sitter group — though the deviations from the Poincaré symmetry of ordinary special relativity will be small. Such property renders the theory universal in the sense that it holds from the Planck scale to the large scale of the universe. It is important to reinforce that, even though there is an invariant length-parameter related to the cosmological term, the Lorentz group remains part of the spacetime kinematics, which means that this symmetry is not broken at any energy scale. Taking into account the deep connection between Lorentz symmetry and causality \[5\], this theory predicts that causality is preserved even at the Planck scale. In the same way Einstein special relativity can be seen as a generalization of Galilei relativity for velocities comparable to the velocity of light, a de Sitter-ruled special relativity can be seen as a generalization of Einstein special relativity for energies comparable to the Planck energy.

Assuming that near the Planck scale spacetime is a de Sitter space with pseudo-radius of the order of the Planck length, and using the Inönü-Wigner process of group contractions \[6\], we will study the formal limit of the de Sitter spacetime for a vanishing pseudo-radius \(l\), which corresponds to an infinite cosmological term \(\Lambda\). Such limiting spacetime has already been shown to bear algebraic, geometric and thermodynamic properties that fit quite reasonably to what one would expect for an initial condition for the universe \[7\]. In this paper we show that it meets also the properties required for playing the role of the bridging spacetime connecting two aeons in Penrose’s conformal cyclic cosmology \[8\].

2 The de Sitter spacetime and group: possible limits

The maximally symmetric de Sitter spacetime, denoted \(dS\), can be seen as a hypersurface in the host pseudo-Euclidean space with metric \(\eta_{AB} = (+1, -1, -1, -1)\) \((A, B, ... = 0, ..., 4)\), whose points in Cartesian coordinates \(\chi^A\) satisfy the relation \[9\]
\[\eta_{AB} \chi^A \chi^B = -l^2, \tag{2}\]
or equivalently, in four-dimensional coordinates,
\[\eta_{\mu\nu} \chi^\mu \chi^\nu - (\chi^4)^2 = -l^2. \tag{3}\]
It has the de Sitter group \(SO(4, 1)\) as group of motions, and is homogeneous under the Lorentz group \(\mathcal{L} = SO(3, 1)\), that is \[10\]
\[dS = SO(4, 1)/\mathcal{L}. \tag{4}\]
In Cartesian coordinates \(\chi^A\), the generators of the infinitesimal de Sitter transformations are written in the form
\[L_{AB} = \eta_{AC} \chi^C \frac{\partial}{\partial \chi^B} - \eta_{BC} \chi^C \frac{\partial}{\partial \chi^A}. \tag{5}\]
They satisfy the commutation relations
\[[L_{AB}, L_{CD}] = \eta_{BC} L_{AD} + \eta_{AD} L_{BC} - \eta_{BD} L_{AC} - \eta_{AC} L_{BD}. \tag{6}\]
On account of the quotient character of the de Sitter spacetime, geometry and algebra turns out to be deeply connected. As a consequence, any deformation in the algebra and group will produce concomitant deformations in the geometry of the corresponding homogeneous spacetime. The use of the Inönü-Wigner contraction \[6\], therefore, constitutes a reliable method for studying geometrical limits of homogeneous spacetimes. In what follows we consider first, in the guise of completeness, the well-known contraction limit \( l \to \infty \). Subsequently we consider the contraction limit \( l \to 0 \), whose study constitutes the main part of the paper. Since these two limits require a different parameterisation, they must be performed separately.

2.1 Large pseudo-radius contraction

2.1.1 Parameterisation appropriate for large values of \( l \)

The four-dimensional stereographic coordinates \( \{ x^\mu \} \) are obtained through a stereographic projection from the de Sitter hypersurface into a target Minkowski spacetime. In the parameterisation appropriate to deal with large values of \( l \), they are defined by \[11\]

\[
\chi^\mu = \Omega x^\mu \tag{7}
\]

and

\[
\chi^4 = -\Omega \left( 1 + \sigma^2/4l^2 \right) \tag{8}
\]

where \( \Omega \equiv \Omega(x) \) is the function

\[
\Omega = (1 - \sigma^2/4l^2)^{-1} \tag{9}
\]

with \( \sigma^2 \) the Lorentz invariant quadratic form \( \sigma^2 = \eta_{\mu\nu} x^\mu x^\nu \). In these coordinates, the infinitesimal de Sitter quadratic interval

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{10}
\]

is written with the conformally flat metric

\[
g_{\alpha\beta} = \Omega^2 \eta_{\alpha\beta}. \tag{11}
\]

The corresponding Christoffel connection is \[12\]

\[
\Gamma^\lambda_{\mu\nu} = \frac{\Omega}{2l^2} \left( \delta^\lambda_\mu \eta_{\nu\alpha} x^\alpha + \delta^\lambda_\nu \eta_{\mu\alpha} x^\alpha - \eta_{\mu\nu} x^\lambda \right) \tag{12}
\]

with the Riemann tensor given by

\[
R^{\mu}_{\nu\rho\sigma} = \frac{\Omega^2}{l^2} \left( \delta^\mu_\rho \eta_{\nu\sigma} - \delta^\mu_\sigma \eta_{\nu\rho} \right). \tag{13}
\]

The Ricci and the scalar curvature are, consequently,

\[
R_{\nu\sigma} = \frac{3\Omega^2}{l^2} \eta_{\nu\sigma} \quad \text{and} \quad R = \frac{12}{l^2}. \tag{14}
\]

In terms of stereographic coordinates \( \{ x^\mu \} \), the de Sitter generators \[3\] are written in the form

\[
L_{\mu\nu} = \eta_{\mu\rho} x^\rho P_\nu - \eta_{\nu\rho} x^\rho P_\mu \tag{15}
\]
and
\[ L_{4\mu} = l P_\mu - \frac{1}{4l} K_\mu \]  
where
\[ P_\mu = \partial_\mu \quad \text{and} \quad K_\mu = (2\eta_{\mu\nu} x^\nu \sigma - \sigma^2 \delta_\mu^\nu) \partial_\rho \]  
are, respectively, the generators of translations and proper conformal transformations \[13\]. Generators \( L_{\mu\nu} \) refer to the Lorentz subgroup, whereas the elements \( L_{4\mu} \) define the transitivity on the homogeneous space. From Eq. (16) it follows that the de Sitter spacetime is transitive under a combination of translations and proper conformal transformations — usually called de Sitter “translations”. The relative importance between these two transformations is determined by the value of the pseudo-radius \( l \).

In order to study the limit of large values of \( l \), it is necessary to parameterise the generators (16) according to \[ 11 \]
\[ \Pi_\mu \equiv \frac{L_{4\mu}}{l} = P_\mu - \frac{1}{4l^2} K_\mu. \]  
In terms of these generators, the de Sitter algebra (6) assumes the form
\[ [L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\rho} L_{\nu\sigma}, \]  \[ [\Pi_\mu, L_{\rho\sigma}] = \eta_{\mu\rho} \Pi_\sigma - \eta_{\mu\sigma} \Pi_\rho, \]  \[ [\Pi_\mu, \Pi_\rho] = l^{-2} L_{\mu\rho}. \]  
The last commutator shows that the de Sitter “translation” generators are not really translations, but rotations.

### 2.1.2 The contraction limit \( l \to \infty \)

In the limit \( l \to \infty \), we see from Eq. (18) that the de Sitter generators \( \Pi_\mu \) reduce to generators of ordinary translations
\[ \Pi_\mu \to P_\mu. \]  
Concomitantly, the de Sitter algebra (19)-(21) contracts to
\[ [L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\rho} L_{\nu\sigma}, \]  \[ [P_\mu, L_{\rho\sigma}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \]  \[ [P_\mu, P_\rho] = 0 \]  
which is the Lie algebra of the Poincaré group \( \mathcal{P} = \mathcal{L} \circ \mathcal{T} \), the semi-direct product of the Lorentz (\( \mathcal{L} \)) and the translation (\( \mathcal{T} \)) groups. As a result of this algebra and group deformations, the de Sitter spacetime \( dS \) contracts to the flat Minkowski space \( M \):  
\[ dS \to M = \mathcal{P}/\mathcal{L}. \]  
In fact, as a simple inspection shows, the de Sitter metric (11) reduces to the Minkowski metric  
\[ g_{\mu\nu} \to \eta_{\mu\nu}, \]  
and the Riemann, Ricci and scalar curvatures vanish identically:  
\[ R^\mu_{\nu\rho\sigma} \to 0, \quad R_{\nu\sigma} \to 0, \quad R \to 0. \]  
From either (22) or (26) we see that Minkowski is transitive under ordinary translations.
2.2 Small pseudo-radius contraction

2.2.1 Parameterisation appropriate for small values of $l$

To deal with small values of $l$, it is convenient to define the ‘inverse’ host space coordinates

$$\bar{\chi}^A = \chi^A / 4l^2,$$

(29)

in terms of which relation (3) assumes the form

$$\eta_{\mu\nu} \bar{\chi}^\mu \bar{\chi}^\nu - (\bar{\chi}^4)^2 = - \frac{1}{16l^2}. \quad (30)$$

The stereographic projection is now defined by

$$\bar{\chi}^\mu = \Omega x^\mu \quad (31)$$

and

$$\bar{\chi}^4 = - l \bar{\Omega} \left( 1 + \sigma^2 / 4l^2 \right) \quad (32)$$

where

$$\bar{\Omega} \equiv \Omega / 4l^2 = \frac{1}{4l^2 - \sigma^2}. \quad (33)$$

In these coordinates, the infinitesimal de Sitter quadratic interval

$$ds^2 = \bar{g}_{\alpha\beta} dx^\alpha dx^\beta \quad (34)$$

is written with the metric

$$\bar{g}_{\alpha\beta} = \bar{\Omega}^2 \eta_{\alpha\beta}. \quad (35)$$

Considering that $\bar{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ differ by a constant, the corresponding Christoffel connections will coincide:

$$\bar{\Gamma}^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} = 2\bar{\Omega} \left( \delta^\lambda_{\mu} \eta_{\nu\alpha} x^\alpha + \delta^\lambda_{\nu} \eta_{\mu\alpha} x^\alpha - \eta_{\mu\nu} x^\lambda \right). \quad (36)$$

Of course, the same happens to the Riemann tensor

$$\bar{R}^\mu_{\nu\rho\sigma} \equiv R^\mu_{\nu\rho\sigma} = 16l^2 \bar{\Omega}^2 \left( \delta^\mu_{\rho} \eta_{\nu\sigma} - \delta^\mu_{\sigma} \eta_{\nu\rho} \right), \quad (37)$$

as well as to the Ricci tensor:

$$\bar{R}_{\nu\sigma} \equiv R_{\nu\sigma} = 16l^2 \bar{\Omega}^2 \eta_{\nu\sigma}. \quad (38)$$

The scalar curvature, however, due to a further contraction with the metric tensor, assumes a different form

$$\bar{R} \equiv 16l^4 R = 192l^2. \quad (39)$$

In this parameterisation, the sourceless Einstein equation reads

$$\bar{R}_{\nu\sigma} - \frac{1}{2} \bar{g}_{\nu\sigma} \bar{R} + 16l^4 \bar{g}_{\nu\sigma} \Lambda = 0, \quad (40)$$

whose trace yields

$$\Lambda = \frac{\bar{R}}{64l^4}. \quad (41)$$
Use of the relation (39) shows that the cosmological term keeps its original definition

\[ \Lambda = \frac{3}{l^2}. \]  

(42)

The de Sitter generators (15) and (16) are the same for both parameterisations. However, in order to study the limit of large values of \( l \), it is necessary to rewrite generators (16) in the form \[ \bar{\Pi}^\mu \equiv 4lL_{4\mu} = 4l^2P_\mu - K_\mu, \]  

(43)
in terms of which the de Sitter algebra (6) becomes

\[ \left[ L_{\mu\nu}, L_{\rho\sigma} \right] = \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\rho} L_{\nu\sigma}, \]  

(44)

\[ \left[ \bar{\Pi}_\mu, L_{\rho\sigma} \right] = \eta_{\mu\rho} \bar{\Pi}_\sigma - \eta_{\mu\sigma} \bar{\Pi}_\rho, \]  

(45)

\[ \left[ \bar{\Pi}_\mu, \bar{\Pi}_\rho \right] = 16l^2L_{\mu\rho}. \]  

(46)

This is the form appropriate to study small values of the de Sitter pseudo-radius \( l \).

### 2.2.2 The contraction limit \( l \to 0 \)

In the contraction limit \( l \to 0 \), the generators \( \bar{\Pi}_\mu \) reduce to (minus) the proper conformal generators:

\[ \bar{\Pi}_\mu \to -K_\mu. \]  

(47)

Accordingly, the de Sitter group \( SO(4,1) \) contracts to the conformal Poincaré group [13]

\[ \bar{P} = L \otimes \bar{T}, \]

the semi-direct product between the Lorentz \( L \) and the proper conformal group \( \bar{T} \), whose Lie algebra is

\[ \left[ L_{\mu\nu}, L_{\rho\sigma} \right] = \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\rho} L_{\nu\sigma}, \]  

(48)

\[ \left[ K_\mu, L_{\rho\sigma} \right] = \eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho, \]  

(49)

\[ \left[ K_\mu, K_\rho \right] = 0. \]  

(50)

Concomitant with the group contraction, on account of their quotient character, the de Sitter spacetime \( dS \) reduces to the homogeneous space \( \bar{M} \)

\[ dS \to \bar{M} = \bar{P}/L. \]  

(51)

The kinematic group \( \bar{P} \), like the Poincaré group, has the Lorentz group \( L \) as the subgroup accounting for the isotropy of the space. The homogeneity, however, is completely different: instead of ordinary translations, all points of \( \bar{M} \) are equivalent under special conformal transformations. In other words, the point-set of \( \bar{M} \) is that determined by special conformal transformations.

In the limit \( l \to 0 \), the de Sitter metric (35) assumes the form

\[ \bar{g}_{\mu\nu} \to \bar{\eta}_{\mu\nu} = \sigma^{-4} \eta_{\mu\nu}, \]  

(52)

which is the metric on \( \bar{M} \). The Christoffel connection, on the other hand, reduces to

\[ \bar{\Gamma}^\lambda_{\mu\nu} \to 0\bar{\Gamma}^\lambda_{\mu\nu} = -2\sigma^{-2} \left( \delta^\lambda_\mu \eta_{\nu\alpha} x^\alpha + \delta^\lambda_\nu \eta_{\mu\alpha} x^\alpha - \eta_{\mu\nu} x^\lambda \right). \]  

(53)
The corresponding Riemann, Ricci, and scalar curvatures vanish identically:

$$\bar{R}^\mu_{\nu\rho\sigma} \to 0, \quad \bar{R}^{\rho\sigma} \to 0, \quad \bar{R} \to 0.$$  (54)

The cosmological term, however, goes to infinity:

$$\Lambda \to \infty.$$  (55)

From these properties one can infer that $\bar{M}$ is a singular, four-dimensional cone spacetime (see Figure 1), transitive under proper conformal transformations [7]. It is important to remark that the contraction limit $l \to 0$ is not continuous — actually a general property of any group contraction. This can be seen by observing that, whereas the de Sitter group is semi-simple, the conformal Poincaré group is not, which means that it is not possible to continuously deform the former into the latter. The limit of a vanishing pseudo-radius $l$ of de Sitter space has sometimes been considered to be an infinitely curved spacetime [15]. This claim, however, is correct only if the limit is not fully accomplished, but halted at the Plank length $l = l_P$. In fact, for $l = l_P$, the resulting spacetime will have a huge scalar curvature, which up up this point is proportional to the cosmological term $\Lambda$. However, when the limit is fully accomplished to $l = 0$, its singular character produces a decoupling between the scalar curvature and the cosmological term, making the former going to zero and the latter to infinity.

3 Some attributes of the cone spacetime $\bar{M}$

As is well-known, in $(3 + 1)$ dimensions there are three homogeneous spaces: Minkowski, de Sitter and anti-de Sitter. The cone spacetime $\bar{M}$ is an additional four-dimensional maximally symmetric spacetime, with the conformal Poincaré group $\bar{P}$ as kinematic group, which is however singular. In this section we explore some of its properties.

3.1 Thermodynamic properties

Due to the common presence of a horizon, in the same way as in the Schwarzschild case, it is possible to attribute thermodynamic features to the de Sitter horizon [16]. For example, the
temperature associated to the de Sitter horizon has the form
\[ T = \frac{\hbar c}{2\pi k_B l}, \] (56)
where \( k_B \) is the Boltzmann constant. In a similar fashion, one can also associate the entropy
\[ S = \frac{k_B A_h}{4l_P^2} \equiv \frac{\pi c^3 k_B l^2}{G \hbar}, \] (57)
where \( A_h = 4\pi l^2 \) is the area of the horizon. The energy is then obtained from the relation
\[ dE = T \, dS. \] (58)
For a constant cosmological term, the energy, entropy and volume will be constant, and the above equation will be trivially satisfied. However, for a time varying cosmological term, the de Sitter pseudo radius \( l \) will change with time, and so will do \( E, S, \) and \( T. \) In this case, (58) can be integrated to give
\[ E = \frac{c^4 l}{2G}. \] (59)
The corresponding energy density is given by
\[ \varepsilon \equiv \frac{E}{V} = \frac{3c^4}{8\pi G l^2}. \] (60)

In the limit \( l \to 0, \) which corresponds to an infinite cosmological term \( \Lambda, \) the temperature, according to definition (56), becomes infinity: \( T \to \infty. \) The entropy, on the other hand, according to the expression (57), vanishes: \( S = 0. \) And finally, on account of the definition (59), the energy associated with the de Sitter horizon vanishes identically: \( E = 0. \) Although the energy vanishes, however, the energy density, according to the expression (60), becomes infinity: \( \varepsilon \to \infty. \) We can thus say that the cone spacetime \( \bar{M} \) is thermodynamically consistent with what one should expected for an initial condition for the universe.

### 3.2 Geometric relation between \( M \) and \( \bar{M} \)

Under the spacetime inversion
\[ x^\mu \to -\frac{x^\mu}{\sigma^2} \] (61)
the translation generators are led to the proper conformal transformations, and vice versa [13]:
\[ P_\mu \to K_\mu \quad \text{and} \quad K_\mu \to P_\mu. \] (62)
The Lorentz generators, on the other hand, remain unchanged:
\[ L_{\mu\nu} \to L_{\mu\nu}. \] (63)
This means that, under such inversion, the Poincaré group \( \mathcal{P} \) is led to the conformal Poincaré group \( \bar{\mathcal{P}}, \) and vice versa. Concomitantly, Minkowski \( M \) is transformed into the four-dimensional cone-spacetime \( \bar{M}, \) and vice versa. The corresponding spacetime metrics are also transformed into each other:
\[ \eta_{\mu\nu} \to \bar{\eta}_{\mu\nu} = \sigma^{-4} \eta_{\mu\nu} \quad \text{and} \quad \bar{\eta}_{\mu\nu} = \sigma^{-4} \eta_{\mu\nu} \to \eta_{\mu\nu}. \] (64)
Minkowski and the cone spacetimes can be considered a kind of dual to each other in the sense that their geometries are determined, respectively, by a vanishing and an infinite cosmological term. Observe that, in the limit \( l \to 0, \) the stereographic projection (31) reduces to a spacetime inversion.

8
3.3 The nature of the spacetime singularity

As shown in Section 2.2.2, the cone spacetime $\hat{M}$ has vanishing Riemann, Ricci and scalar curvature tensors. The cosmological term, on the other hand, is infinite, pointing to a singular spacetime. In fact, its metric tensor $\bar{\eta}_{\mu\nu}$, given by Eq. (52), is singular for points in which $\sigma^2 = 0$. However, if we perform a conformal re-scaling of the metric

$$\bar{\eta}_{\mu\nu} \rightarrow \tilde{\bar{\eta}}_{\mu\nu} = \omega^2(x) \bar{\eta}_{\mu\nu},$$

with the conformal factor given by

$$\omega^2(x) = \sigma^4 \alpha^2(x),$$

the resulting metric tensor

$$\tilde{\bar{\eta}}_{\mu\nu} = \alpha^2(x) \bar{\eta}_{\mu\nu}$$

is no longer singular. This kind of singularity, in which the metric is singular but the conformal equivalence class of the metric is not, is called a \textit{conformal gauge singularity}. This is an instrumental part of the Weyl curvature hypothesis, and consequently of Penrose’s conformal cyclic cosmology.

3.4 Transitivity and the notions of distance and time

Minkowski and de Sitter are both isotropic and homogeneous, but their homogeneity properties differ substantially: whereas Minkowski is transitive under translations, de Sitter is transitive under a combination of translations and proper conformal transformations. Transitivity is intimately related to the notions of space distance and time interval. Since any two points of Minkowski spacetime are connected by a spacetime translation, the corresponding notions of space distance and time interval will be essentially translational. On the other hand, any two points of de Sitter spacetime are connected by a combination of translation and proper conformal transformations — the so-called de Sitter “translations”. As a consequence, the notions of space distance and time interval will be described by a combination of translation and proper conformal transformations. The relative importance between the two notions is determined by the value of the de Sitter pseudo-radius $l$.

For large values of $l$, in which case the de Sitter transitivity generators are written in the parameterisation

$$\Pi_\mu = P_\mu - \frac{1}{4l^2} K_\mu,$$  \hspace{1cm} (68)

distance and time will be preponderantly determined by spacetime translations. In the formal limit $l \rightarrow \infty$, the de Sitter group contracts to the Poincaré group, and de Sitter spacetime reduces to Minkowski, where space and time are determined by ordinary translations only. In this case, the proper conformal degrees of freedom are turned off. On the other hand, for small values of $l$, in which case the de Sitter transitivity generators are written in the parameterisation

$$\tilde{\Pi}_\mu \equiv 4l L_{4\mu} = 4l^2 P_\mu - K_\mu,$$  \hspace{1cm} (69)

space and time will be preponderantly determined by proper conformal transformations. In the formal limit $l \rightarrow 0$, the de Sitter group contracts to the conformal Poincaré group, and de Sitter spacetime reduces to the singular, flat cone-spacetime $\hat{M}$. Such spacetime is transitive under proper conformal transformations, which implies that the translational degrees of freedom are
turned off — that is to say, the gravitational degrees of freedom are turned off. Notice that our conventional (translational) notions of space and time do not exist on $\mathcal{M}$, only the conformal notions exist \[18, 19\]. This means that local clocks cannot be defined, a result that is somehow in agreement with the general idea that (our conventional notion of) time should not exist at (or beyond) the Planck scale \[20\]. One should note finally that the cone spacetime metric $\tilde{\eta}_{\mu\nu}$ does not have the conventional meaning of a physical metric. In fact, it is not dimensionless, and the interval it defines,

$$ds^2 = \sigma^{-4} \eta_{\mu\nu} dx^\mu dx^\nu,$$

has to do with proper conformal distances, not with our usual notion of translational distances. Put together, the above properties match quite reasonably those normally attributed to the spacetime connecting two aeons in conformal cyclic cosmology \[21\].

## 4 Final remarks

In order to describe the initial state of the universe as a limit of the de Sitter spacetime, physics must be constructed, not on Minkowski, but on a de Sitter spacetime. This is possible because, as quotient spaces, Minkowski and de Sitter are both fundamental (non-gravitational) spacetimes in the sense that they are known \textit{a priori}, independently of Einstein equation. In particular, general relativity itself must be constructed on a de Sitter spacetime, which amounts to replace the usual Riemannian geometry by a Cartan geometry \[22\]. In this geometry, instead of reducing locally to Minkowski, spacetime reduces locally to de Sitter \[23\]. This, in turn, amounts to replace ordinary, Poincaré-ruled special relativity by a de Sitter-ruled special relativity \[2–4\]. Such construction does not change the dynamics of the gravitational field, which remains described by Einstein equation. The only change will be in the strong equivalence principle, which passes to state that \textit{in a locally inertial frame, where gravitation goes unnoticed, the laws of physics reduce to those of de Sitter-ruled special relativity.}

The contraction limit for an infinite cosmological term $\Lambda$ of a de Sitter spacetime yields a singular, flat, four-dimensional cone spacetime, which is transitive under proper conformal transformations. In addition to represent a possible initial singularity of a big bang universe \[15\], this spacetime complies also with the requirements to represent the bridging spacetime between two aeons in the conformal cyclic cosmology of Penrose \[8\]. It satisfies the Weyl curvature hypothesis \[24, 25\], according to which the initial spacetime singularity must have vanishing Weyl curvature. Thermodynamically, it is found to have infinite temperature, vanishing energy and entropy, and infinite energy density. Such properties allow this spacetime to be consistently interpreted as the cosmological initial singularity, a kind of “nothing” from where our universe could emerge.

It should be remarked that $l \to 0$ is just a formal limit in the sense that quantum effects preclude it to be fully performed. It is actually a contraction limit, on an equal footing with the classical contraction limit in which the speed of light goes to infinity $c \to \infty$, as well as with other contraction limits \[26–28\]. The cone-spacetime $\mathcal{M}$, which emerges as the output of this limit, should then be thought of as the \textit{frozen geometric structure} behind the spacetime quantum fluctuations taking place at the Planck scale. The quantum fluctuations from the cone spacetime with $l = 0$ to a de Sitter spacetime with $l = l_P$ give rise to a non-singular de Sitter universe with a huge cosmological term $\Lambda_P \simeq 10^{66}$ cm$^{-2}$, which would drive inflation. Once this transition occurs, the translational degrees of freedom are turned on, and our usual notions of time and space show up, as can be seen from the generators \[69\]. Considering
that these generators define the local transitivity of spacetime, the Noether conservation law corresponding to these transformations is of the form \[29\]
\[
\nabla_\mu \Pi^\mu = 0
\] (71)
where \(\Pi^\mu = 4l^2 T^\mu - K^\mu\), with \(T^\mu\) the energy-momentum tensor and \(K^\mu\) the proper conformal current \[30\]. Notice that in the cone spacetime \(\check{M}\), obtained in the limit \(l \to 0\), the translational degrees of freedom are turned off, and proper conformal current turns out to be conserved — a result consistent with the transitivity property of \(\check{M}\). The cone spacetime can thus be thought of as a universe in which all matter content is in the form of proper conformal current. The subsequent evolution of such universe is a problem yet to be explored. Its study amounts to reconsider general relativity in a Cartan geometry \[22, 23\], where the cosmological term would be allowed to evolve with the cosmological time \[31\].

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