On gauge theories for non-semisimple groups

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Abstract

We consider analogs of Yang-Mills theories for non-semisimple real Lie algebras which admit invariant non-degenerate metrics. These 4-dimensional theories have many similarities with corresponding WZW models in 2 dimensions and Chern-Simons theories in 3 dimensions. In particular, the quantum effective action contains only 1-loop term with the divergent part that can be eliminated by a field redefinition. The on-shell scattering amplitudes are thus finite (scale invariant). This is a consequence of the presence of a null direction in the field space metric: one of the field components is a Lagrange multiplier which 'freezes out' quantum fluctuations of the 'conjugate' field. The non-positivity of the metric implies that these theories are apparently non-unitary. However, the special structure of interaction terms (degenerate compared to non-compact YM theories) suggests that there may exist a unitary 'truncation'. We discuss in detail the simplest theory based on 4-dimensional algebra $E_6$. The quantum part of its effective action is expressed in terms of 1-loop effective action of $SU(2)$ gauge theory. The $E_6$ model can be also described as a special limit of $SU(2) \times U(1)$ YM theory with decoupled ghost-like $U(1)$ field.

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1. Introduction

In contrast to 2 dimensions the number of solvable or conformal models in 4 dimensions seems to be very limited. The only known examples are special Yang-Mills theories with (extended) supersymmetry. It may be of interest to study bosonic Yang-Mills-type $D = 4$ models which are analogs of certain 2-dimensional models and have simpler quantum properties than standard non-abelian gauge theories.

Many conformal field theories in two dimensions are described by (gauged) WZW models. It was realized in [1] that one can also construct WZW models for some non-semisimple groups (which admit a nondegenerate invariant bilinear form) and such models have much simpler structure than the standard semisimple models. One possible explanation for that is that they can be interpreted as formal limits of semisimple coset or WZW models where one makes simultaneous boost of coordinates and infinite rescaling of the level [2,4,9].

There is a certain analogy between semisimple WZW theory in 2 dimensions and Yang-Mills theory in 4 dimensions. Both are uniquely defined by specifying a Lie group and are classically conformally invariant. However, in contrast to WZW model, Yang-Mills theory is no longer conformal at the quantum level and has complicated dynamics. Below we shall construct Yang-Mills theories based on some non-semisimple groups and demonstrate that like their WZW counterparts they turn out to be very simple (though not exactly conformal) also at the quantum level.

The main difference compared to 2-dimensional WZW case is an apparent lack of unitarity of these non-semisimple 4-dimensional models. While in two dimensions the presence of one negative norm direction in the non-degenerate invariant bilinear form which replaces the Killing form may not be, in principle, a problem since there is an infinite dimensional conformal symmetry and hence (as in flat space case) we may be able to eliminate the ‘time-like’ field component by choosing the light-cone gauge (which indeed exists in these WZW models) this is no longer possible in four dimensions. However, the non-unitarity of the non-semisimple Yang-Mills models is much ‘milder’ than that of Yang-Mills theories based on non-compact simple groups. For example, the special ‘null’ structure of the interaction terms implies that one linear combination of the positive and negative norm fields is decoupled.

We shall start in Section 2 with a review of the structure on non-semisimple real Lie algebras with invariant non-degenerate bilinear forms following [6]. All explicitly discussed examples of such algebras are called indecomposable ‘depth 1’ algebras which are ‘double extensions’ of an abelian algebra by a simple or 1-dimensional one.

1 The issue of unitarity of WZW models based on non-compact semisimple groups where one is unable to choose a light-cone is nontrivial and remains under discussion (see [1] a recent suggestion of a unitary formulation of the model and also refs. there).
Given a Lie algebra with an invariant metric $\Omega_{ab}$ it is straightforward to define the non-semisimple generalizations of the WZW, Chern-Simons and Yang-Mills actions (Section 3). If $\Omega_{ab}$ is put into a diagonal form the actions have the same structure as in the case of a simple noncompact algebra but with ‘degenerate’ structure constants. Since it is the structure constants that determine the interactions, the quantum correction to $\Omega_{ab}$ in WZW and CS theories (represented exactly by the 1-loop term) is proportional to the degenerate Killing metric $g_{ab}$.

Similar renormalization of $\Omega_{ab}$ is found in non-semisimple Yang-Mills case (Section 4): $\Omega_{ab} \rightarrow \Omega_{ab} + b_1 g_{ab}$, where now $b_1$ is not finite but logarithmically divergent. The analogy with WZW and CS models goes beyond 1 loop: in contrast to semisimple Yang-Mills theories, in the non-semisimple case there is no two and higher loop renormalization. We demonstrate this for ‘depth 1’ algebras but this is likely to be true for generic indecomposable non-semisimple algebras with invariant metrics.

Moreover, as in the non-semisimple WZW and CS models, the full quantum effective action is given just by the 1-loop term. The divergent part of the effective action can actually be eliminated by a field redefinition (i.e. the infinite renormalization mentioned above is an off-shell artifact). This implies that the on-shell S-matrix is UV finite. The theory is very similar to that of a collection of quantized (abelian) positive norm vector fields interacting only with a background gauge field subject to the classical Yang-Mills equation. In particular, the conformal anomaly depends only on the background gauge field. This suggests a possibility of a (unitary, conformal) ‘free-field’ interpretation of this model.

The general discussion will be illustrated in Section 5 on the simplest nontrivial example of the 4-dimensional algebra $E_2^c$. We shall first review the structure of the $E_2^c$ WZW theory and then consider the $E_2^c$ Yang-Mills theory, making the analogies with the $E_2^c$ WZW and CS models explicit. We shall also describe the relation between the $E_2^c$ theory and the $SU(2) \times U(1)_{-}$ Yang-Mills theory (with decoupled $U(1)$ ghost field).

Our conclusions will be summarized in Section 6.

2. Non-semisimple Lie algebras with invariant metrics

For a real Lie algebra with generators $e_a \ (a = 1, \ldots, N)$ and structure constants $f^a_{bc}$ there may exist a symmetric, invariant and non-degenerate bilinear form $\Omega_{ab}$ which generalizes the Killing form $g_{ab}$

$$[e_a, e_b] = f^c_{\ a b} e_c , \quad g_{ab} \equiv -f^c_{\ a d} f^d_{\ b c} , \quad g_{c(d f^e_{(a)b})} = 0 , \quad (2.1)$$

Note that in contrast to, e.g., higher derivative or Pauli-Villars regularized theories which are also finite at the expense of being superficially non-unitary [13], here one has no extra scale-dependent parameters.
\[ \Omega_{ab} = \Omega_{ba} , \quad \Omega_{c(df)ab} = 0 , \quad \det \Omega_{ab} \neq 0 , \quad \Omega^{ab} \equiv (\Omega^{-1})^{ab} . \] (2.2)

In the semisimple case \( g_{ab} = c_2 \eta_{ab} \), \( \Omega_{ab} = \gamma g_{ab} = k \eta_{ab} \), where \( \gamma \) is a constant, \( k = c_2 \gamma \) and \( \eta_{ab} \) is diagonal matrix with \( \pm 1 \) entries. In general, we may always represent \( \Omega_{ab} \) in the form

\[ \Omega_{ab} = \gamma g_{ab} + \Omega^{(0)}_{ab} , \] (2.3)

where \( \Omega^{(0)}_{ab} \) also satisfies the invariance condition in (2.2) and can be considered a ‘complement’ of \( g_{ab} \) which makes \( \Omega_{ab} \) non-degenerate. Thus generic \( \Omega_{ab} \) contains at least one free parameter \( \gamma \). As we shall see below, it is this parameter that is renormalized by quantum corrections (and its renormalization is formally the same as in the semisimple case).

Examples of such algebras were considered in [12,1,2,4,7,10]. The general construction was presented in [6] and will be reviewed below. According to a theorem discussed in [6] the class of real Lie algebras with an invariant metric is the product of the class of semisimple algebras with the class obtained from the 1-dimensional algebra under the operations of taking direct sum and double extension by simple (or 1-dimensional) algebras. Indecomposable (i.e. not equal to a direct product) non-semisimple dimension \( > 1 \) Lie algebra \( \mathcal{L} \) with an invariant metric is always a double extension of another Lie algebra \( \mathcal{G} \) with an invariant metric \( \Omega^{(0)}_{ij} \) by a Lie algebra \( \mathcal{H} \) which is either simple or 1-dimensional. Equivalently, \( \mathcal{L} \) is a semidirect sum of \( \mathcal{H} \) with \( \mathcal{G} \oplus \mathcal{H}^* \), i.e. with a central extension of \( \mathcal{G} \) by the abelian algebra \( \mathcal{H}^* \) (\( \mathcal{H}^* \) is dual of \( \mathcal{H} \) and is an abelian ideal of \( \mathcal{L} \)). What that means is that \( \mathcal{L} = \mathcal{G} \oplus \mathcal{H} \oplus \mathcal{H}^* \) as a vector space, and the commutation relations in the basis \( e_a = \{ e_i, e_r, e^s \} \) \((i, j, k, l = 1, \ldots, \dim \mathcal{G}, \ r, s, t = 1, \ldots, \dim \mathcal{H} = \dim \mathcal{H}^*)\)

\[
\begin{align*}
[e_i, e_j] &= f^k_{ij} e_k + f_{ijr} e^r , \\
[e_r, e_i] &= f^j_{ri} e_j , \\
[e_r, e^s] &= f^s_{tr} e^t , \\
[e^r, e^s] &= 0 , \\
[e^r, e_i] &= 0 , \\
f_{[ijr]} &= \Omega^{(0)}_{ik} f^k_{jr} , \\
f_{(kj)r} &= 0 , \\
f_{[ijf^l_{r]}k} &= 0 .
\end{align*}
\] (2.4)

The corresponding Killing form is degenerate

\[
g_{ab} = f^c_{ad} f^d_{cb} = \begin{pmatrix}
g_{ij} & g_{is} & 0 \\
g_{rj} & g'_{rs} & 0 \\
0 & 0 & 0
\end{pmatrix} ,
\] (2.5)

\[
g_{ij} = f^k_{il} f^l_{kj} , \quad g_{ir} = g_{ri} = f^k_{il} f^l_{kr} , \quad g'_{rs} = f^i_{rj} f^j_{is} + 2g_{rs} , \quad g_{rs} = f^t_{rp} f^p_{ts} .
\]
but there exists a non-degenerate invariant form

\[
\Omega^{(0)}_{ab} = \begin{pmatrix}
\Omega^{(0)}_{ij} & 0 & 0 \\
0 & \Omega^{(0)}_{rs} & \delta^s_r \\
0 & \delta^r_s & 0 
\end{pmatrix},
\]

(2.6)

where \(\Omega^{(0)}_{rs}\) is an arbitrary (possibly degenerate) invariant bilinear form on \(H\). For a given \(L\) (2.4) the free parameters in \(\Omega^{(0)}_{ab}\) are at least the \(\Omega^{(0)}_{rs}\) and the scale of \(\Omega^{(0)}_{ij}\). Note that the signature of \(\Omega^{(0)}_{ab}\) is always indefinite. More general \(\Omega^{(0)}_{ab}\) in (2.3) is then given by

\[
\Omega_{ab} = \gamma g_{ab} + \Omega^{(0)}_{ab} = \begin{pmatrix}
\Omega_{ij} & g_{is} & 0 \\
g_{rj} & \Omega^{(0)}_{rs} & \delta^s_r \\
0 & \delta^r_s & 0 
\end{pmatrix},
\]

(2.7)

\[
\Omega_{ij} = \gamma g_{ij} + \Omega_{ij}^{(0)}, \quad \Omega_{rs} = \gamma g^{'}_{rs} + \Omega^{(0)}_{rs}.
\]

According to [6] \(G\) itself must be a double extension of some other algebra \(G_1\) with an invariant metric \(\Omega^{(0)}_1\) by some simple or 1-dimensional algebra \(H_1\), etc. After a number of double extension steps the final algebra \(G_n\) must be abelian. All previously explicitly discussed examples [1,2,4,7,10] correspond to a single double extension of some abelian Lie algebra \(G\), i.e. can be called ‘depth 1’ algebras [4]. In the general case of a ‘depth 1’ indecomposable non-semisimple \(L\) with simple or 1-dimensional \(H\) and abelian \(G\) (\(f''_{jk} = 0\)) the Killing metric (2.5) has \(g^{'}_{rs}\) as the only non-vanishing entry (\(g_{ij} = 0, g_{ir} = 0\)) and \(\Omega_{ab}\) (2.7) takes the same form as \(\Omega^{(0)}_{ab}\) (2.6).

\[
\Omega_{ab} = \begin{pmatrix}
\Omega_{ij}^{(0)} & 0 & 0 \\
0 & \Omega^{(0)}_{rs} & \delta^s_r \\
0 & \delta^r_s & 0 
\end{pmatrix}, \quad g_{ab} = \begin{pmatrix}
0 & 0 & 0 \\
0 & g^{'}_{rs} & 0 \\
0 & 0 & 0 
\end{pmatrix},
\]

(2.8)

\[
\Omega^{ab} = \begin{pmatrix}
\Omega_{ij}^{(0)} \delta^r_s & 0 & 0 \\
0 & \delta^r_s & -\Omega^{(0)}_{rs} \\
0 & 0 & \delta^s_r 
\end{pmatrix}, \quad \Omega_{rs} = \gamma g^{'}_{rs} + \Omega^{(0)}_{rs}.
\]

(2.9)

Note that since each double extension step adds an equal number (\(\dim H\)) of pluses and minuses to the signature of \(\Omega_{ab}\), to get the signature with just one minus (to ‘minimize’ the non-unitarity of a field theory constructed using \(\Omega_{ab}\)) one should consider only ‘depth 1’ algebras which have 1-dimensional \(H\). The only such algebras are the trivial \(N\)-dimensional generalization of \(E^c_2\) [1] or the Heisenberg algebra \(H_N\) [10] (see Section 5).

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4 Double extensions of semisimple \(G\) by 1-dimensional \(H\) is decomposable with simple part of \(G\) as a factor [3].
3. Wess-Zumino-Witten, Chern-Simons and Yang-Mills theories for non-semisimple groups

3.1. Classical actions

Given a Lie algebra \( \mathcal{L} \) with an invariant metric one may define the WZW model \[14\] by the following action \[1\]

\[
I = \frac{1}{16\pi} \Omega_{ab} \left( \int_{\partial M} d^2 \sigma \ A^a A^b + \frac{i}{3} \int_M d^3 x \ e^{\mu\nu\lambda} f^a_{\mu\nu\lambda} A^c A^d A^b \right),
\]

(3.1)

where 2-metric is flat and \( A^a_\mu \) is defined by

\[
A^a_\mu e_a = g^{-1} \partial_\mu g, \quad g = \exp(x^a e_a). \tag{3.2}
\]

Similarly, we can construct the ‘non-semisimple’ generalizations of the \( D = 3 \) Chern-Simons action \[15\]

\[
S_{CS} = -\frac{i}{8\pi} \Omega_{ab} \int d^3 x \ e^{\mu\nu\lambda} \left( A^a_\mu \partial_\nu A^b_\lambda + \frac{1}{3} f^a_{\mu\nu\lambda} A^c A^d A^b \right),
\]

(3.3)

and of the \( D = 4 \) Yang-Mills action

\[
S_{YM} = \frac{1}{4} \Omega_{ab} \int d^4 x \ F^a_{\mu\nu} F^b_{\mu\nu},
\]

(3.4)

\[
F^a_{\mu\nu} = 2 \partial_{[\mu} A^a_{\nu]} + f^a_{bc} A^b_{\mu} A^c_{\nu},
\]

(3.5)

which are invariant under

\[
\delta A^a_\mu = \partial_\mu \eta^a + f^a_{bc} A^b_\mu \eta^c, \quad \delta F^a_{\mu\nu} = f^a_{bc} F^b_{\mu\nu} \epsilon^c.
\]

(3.6)

On can also consider other actions, e.g., a combination of (3.3) with \( D = 3 \) analog of (3.4) (i.e. the topologically massive \( D = 3 \) gauge theory \[16\]) or the straightforward generalization of (3.4) including the topological term \( \theta \Omega_{ab} \int F^a_{\mu\nu} \tilde{F}^b_{\mu\nu} \). The theories (3.1), (3.3) and (3.4) are closely related as in the semisimple case \[17,18\] (the classical equation that follows from (3.3) is still \( F^a_{\mu\nu} = 0 \)). In (3.1), (3.3), (3.4) one may separate the overall loop counting parameter (‘Planck constant’) by setting \( \Omega_{ab} = k\Omega_{ab}, \ k = 1/g^2 \).

Since \( \Omega_{ab} \) is non-degenerate, it can be represented in the ‘diagonal’ form

\[
\Omega_{ab} = E^\alpha_a E^\beta_b \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \text{diag}(+1, ..., +1, -1, ..., -1), \quad \det E^\alpha_a \neq 0. \tag{3.7}
\]

One can then define the ‘rotated’ structure constants with ‘vierbein’ indices (\( \alpha, \beta, \gamma \)) which will encode the information about \( \Omega_{ab} \)

\[
f^\alpha_{\beta\gamma} \equiv E^\alpha_a E^\beta_b E^\gamma_c f^a_{bc}, \quad f_{\alpha\beta\gamma} \equiv \eta_{\alpha\delta} f^\delta_{\beta\gamma}, \tag{3.8}
\]
\[ f_{\alpha\beta\gamma} = 0 \quad \text{and} \quad g_{\alpha\beta} = -f^\gamma_{\alpha\delta} f^\delta_{\beta\gamma} = E^a_\alpha E^b_\beta g_{ab} . \]

Introducing the ‘rotated’ gauge field \( A^\alpha_\mu \equiv E^a_\alpha A^a_\mu \) one can put the actions (3.3) and (3.4) in the equivalent forms

\[
S_{CS} = -\frac{i}{8\pi} \eta_{\alpha\beta} \int d^3x \epsilon^{\mu\nu\lambda} \left( A^\alpha_\mu \partial_\nu A^\beta_\lambda + \frac{1}{3} f^\alpha_{\gamma\delta} A^\gamma_\mu A^\delta_\nu A^\beta_\lambda \right), \tag{3.9}
\]

\[
S_{YM} = \frac{1}{4} \eta_{\alpha\beta} \int d^4x \; F^{\alpha}_{\mu\nu} F^{\beta}_{\mu\nu}, \quad F^{\alpha}_{\mu\nu} = 2 \partial_{[\mu} A^\alpha_{\nu]} + f^{\alpha}_{\beta\gamma} A_\mu^\beta A_\nu^\gamma . \tag{3.10}
\]

These actions look the same as in the case of a semisimple non-compact group but with the structure constants corresponding to the non-semisimple algebra (2.1) in the ‘rotated’ basis \( e_\alpha = E^a_\alpha e_a \). Since \( \eta_{\alpha\beta} \) is not positive definite an apparently obvious conclusion is that the YM theory (3.10) is non-unitary. Indeed, the Hamiltonian or Euclidean action are not positive. However, this non-unitarity may be less serious than in the case of a non-compact simple group due to the ‘degenerate’ form of the structure constants which determine the interactions (for example, ghost-like gauge vectors may effectively decouple and one may be able to separate them from a unitary sector of the theory).

3.2. Quantum renormalization of couplings: WZW and CS theories

Let us now discuss the corresponding quantum theories. It is easy to show that WZW model (3.1) remains conformal for any \( \Omega_{ab} \) in (2.2). There is a finite quantum shift of \( \Omega_{ab} \) which can be determined, e.g., by generalizing the standard semisimple current algebra approach [19], or, equivalently, by solving the master equation [20]. Starting with the action (3.1), defining the currents \( J_a = \Omega_{ab} A^b_z \) and imposing the canonical commutation relations one finds that \( J_a \) generate the affine algebra with structure constants \( f^a_{bc} \) and the central term proportional to \( \Omega_{ab} \). The quantum analog of the classical stress tensor \( T_{zz} = \Omega_{ab} A^a_z A^b_z = \Omega_{ab} J_a J_b \) is \( \hat{T}_{zz} = \hat{\Omega}^{ab} : J_a J_b : \), where \( \hat{\Omega}^{ab} \) should satisfy [20]

\[
\hat{\Omega}^{ab} = \hat{\Omega}^{ac} \hat{\Omega}^{cd} \hat{\Omega}^{db} - \hat{\Omega}^{cd} \hat{\Omega}^{cf} f_{ce}^a f_{df}^b - 2 \hat{\Omega}^{cd} f_{ce}^a f_{df}^c f^{(a}_{df} \hat{\Omega}^{b)e} . \tag{3.11}
\]

Assuming that \( \hat{\Omega}^{ab} \) is non-degenerate and multiplying (3.11) by \( \hat{\Omega}^{aa'} \hat{\Omega}^{bb'} \), \( \hat{\Omega}_{ab} \equiv (\hat{\Omega}^{-1})_{ab} \) one finds (using (2.2)) the following solution

\[
\hat{\Omega}_{ab} = \Omega_{ab} + g_{ab} . \tag{3.12}
\]

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5 Our discussion (which is a generalization of that of [1]) is slightly different from that of [3] in that we start directly with the action (3.1) which fixes the form of the current algebra.

6 The relation to the notation in [20] is \( \Omega^{ab} = L^{ab} \), \( \Omega_{ab} = 2G_{ab} \to 2k_1 \eta_{ab} \), \( k = 2k_1 \).
The corresponding central charge of the Virasoro algebra is
\[ c = \Omega_{ab} \hat{\Omega}^{ab} = \text{Tr} \left( \frac{\Omega}{\Omega + g} \right) . \] (3.13)

As can be shown using (2.4)–(2.6) \( c \) is always integer for indecomposable non-semisimple \( \mathcal{L} \). These relations are the generalizations of the standard expressions in the semisimple case \( \hat{\Omega}_{ab} = (k + c_2)\eta_{ab} \), \( c = Nk/(k + c_2) \). The same results can be reproduced in the Lagrangian field theory approach, e.g., by generalizing the discussion in [21,22,23].

In particular, the local part of the quantum effective action will be given by (3.1) with \( \Omega_{ab} \) replaced by \( \hat{\Omega}_{ab} \).

The important observation that will be true also in CS and YM theories is that while the classical action depends on \( \Omega_{ab} \), the quantum correction contains only the contraction of the two structure constants, i.e., is proportional to the (degenerate) Killing metric. Thus the part of \( \Omega_{ab} \) which is not proportional to \( g_{ab} \) is not renormalized, i.e., the only effect of renormalization is to shift the constant \( \gamma \) in (2.3),
\[ \hat{\Omega}_{ab} = \hat{\gamma} g_{ab} + \Omega^{(0)}_{ab} , \quad \hat{\gamma} = \gamma + 1 . \] (3.14)

The reason underlying this conclusion is that the relevant quantum correction is exactly given by the 1-loop diagram with two propagators (containing \( \Omega^{-1} \)) and two vertices (proportional to \( \Omega_{ab} f^b_{cd} \)) so that the \( \Omega \)-factors cancel out because of the invariance property (2.2).

Similar quantum shift of \( k \) is known to occur in the semisimple Chern-Simons theory [17] (see also, e.g., [24,25]). It is straightforward to generalize, e.g., the perturbative background field method derivation of the induced P-odd (CS) term in the effective action [25] to the case of the action (3.3). Starting with (3.9) and splitting the gauge field into the background \( A^a_\mu \) and quantum \( B^a_\mu \) parts one can represent the term in the action which is bilinear in \( B \) and auxiliary gauge fixing field \( \phi \) in the form
\[ S^{(2)} = -\frac{i}{8\pi} \eta_{\alpha\beta} \int d^3 x (\epsilon^{\mu\nu\lambda} B^a_\mu D^\nu B^b_\lambda + 2 \phi^{\alpha} D^\mu B^a_\mu^\beta) , \] (3.15)
\[ (D_\mu)^\alpha_\gamma \equiv \delta^\alpha_\gamma \partial_\mu + f^\alpha_\beta A^b_\mu , \]
which can be then be represented as a fermionic action,
\[ -\frac{i}{8\pi} \eta_{\alpha\beta} \int d^3 x \psi^\alpha \Sigma^\mu D_\mu \psi^\beta , \] (3.16)
where \( \psi = (B_\mu, \phi) \) and \( \Sigma^\mu \) are 4×4 matrices with properties similar to those of Pauli matrices [27]. The resulting quantum contribution looks the same as in the semisimple case [21,17,24,25], i.e., is given by the classical action (3.3) with \( \Omega_{ab} \) replaced by the contraction of the two structure constants \( f^\alpha_\beta f^\beta_\alpha \), i.e., by the Killing metric \( g_{ab} \). As a consequence, the quantum effective action contains the same shifted \( \Omega_{ab} \) as in (3.12), (3.14).

7 The off-shell effective action in general contains other P-even terms but the shift of \( k \) that comes from the 1-loop determinant is the only genuine ‘on-shell’ effect [17,25]. This shift, of course, depends on the definition of the determinant, i.e., on a scheme choice.

8 The corresponding ghost term can be ignored since it does not contribute to the P-odd part of the effective action.
4. Non-semisimple Yang-Mills theory at the quantum level

4.1. One-loop renormalization and absence of higher loop divergences

Next, let us consider the renormalization of the coupling matrix $\Omega_{ab}$ in the YM theory (4.1). Again, one may expect that the 1-loop correction (which will now be infinite) will not depend on $\Omega_{ab}$ itself. Using the background field method \[27,28\] one can put the relevant ‘quantum’ part of the action (3.10) in the form (cf. (3.15))

$$S = \int d^4x \left( \frac{1}{2} \eta_{\alpha\beta} D_\mu B_\alpha^\mu D_\nu B_\beta^\nu + f_{\alpha\beta\gamma} F_{\mu\nu}^\alpha B_\beta^\mu B_\gamma^\nu + f_{\alpha\gamma\beta} D_\mu B_\alpha^\nu B_\beta^\mu B_\gamma^\nu \right)$$

where we have added the background gauge fixing term $\eta_{\alpha\beta} D_\mu B_\alpha^\mu D_\nu B_\beta^\nu$ and the corresponding ghost term. The 1-loop correction to the effective action $\Gamma(1\text{-loop})_{\text{\infty}}$ depends on $\Omega_{ab}$ through the structure constants $f_{\beta\gamma}^{\alpha}$ in (3.8). However, this dependence cancels out in the contraction of two structure constants which appears in the divergent part of $\Gamma(1\text{-loop})$. Indeed, repeating the computation in \[28\] (see also \[29\]) one finds that the divergent part of the 1-loop effective action is given by formally the same expression as in the semisimple case \[28,30\]

$$\Gamma(1\text{-loop})_{\text{\infty}} = \frac{1}{\epsilon} \beta_1 g_{ab} \int d^4x \ F_a^{\mu\nu} F_b^{\mu\nu}, \quad \beta_1 = -\frac{11}{6(4\pi)^2}, \quad \epsilon = 4 - D \to 0 . \quad (4.2)$$

For a simple Lie algebra $g_{ab} = c_2 \eta_{ab}$ while in the general case we learn that the 1-loop effective action contains the classical action term with the tree-level coupling matrix $\Omega_{ab}$ replaced by (cf. (3.12),(3.14))

$$\hat{\Omega}_{ab}^{(1\text{-loop})} = \Omega_{ab} + \frac{1}{\epsilon} \beta_1 g_{ab} , \quad \hat{\gamma}^{(1\text{-loop})} = \gamma + \frac{1}{\epsilon} \beta_1 . \quad (4.3)$$

As in WZW and CS models the non-degenerate part $\Omega_{ab}^{(0)}$ of the coupling matrix (2.3) which is ‘complementary’ to the Killing form $g_{ab}$ is not renormalized.\[8\]

In general, the divergent part of the $n$-loop term in the effective action in the theory (4.4) should have the form

$$\Gamma^{(n\text{-loop})}_{\text{\infty}} = \sum_{m=0}^{n} \frac{\beta_m}{\epsilon^m} C_{ab}^{(nm)} \int d^4x \ F_a^{\mu\nu} F_b^{\mu\nu} , \quad (4.4)$$

Note that had we set $\gamma = 0$ in the classical action (which could seem formally possible since $\Omega_{ab}^{(0)}$ can be chosen to be non-degenerate) the theory (3.4) would not be renormalizable, i.e. to get a renormalizable YM theory one needs to start with the most general $\Omega_{ab}$.\[9\]
where the matrices \( C_{ab}^{(nm)} \) are built out of \( f^a_{bc}, \Omega_{ab} \) and \( \Omega^{ab} \), or, equivalently, out of products of \( f_{\alpha \beta \gamma} \) and \( \eta_{\alpha \beta} \). Since \( f_{\alpha \beta \gamma} \) satisfies the same properties (total antisymmetry and Jacobi identity) as the structure constants of a semisimple algebra, higher loop contributions \( C_{ab}^{(nm)} \) will also have identically the same form as in the standard semisimple YM theory before one uses there the expressions for the products of the structure constants in terms of \( \eta_{\alpha \beta} \) which are true only in the semisimple case, \( f^\delta_{\alpha \gamma} f^\gamma_{\delta \beta} = \sigma_2 \eta_{\alpha \beta} \). Since the number of \( f \)'s in \( C_{ab}^{(nm)} \) is always even, and because of the properties of \( f_{\alpha \beta \gamma} \), the second rank tensors built out of them with the help of \( \eta_{\alpha \beta} \) can be represented in terms of products of their bilinear combination, i.e. the Killing metric \( g_{\alpha \beta} \),

\[
C_{\alpha \beta}^{(nm)} \sim g_{\alpha \alpha'} \eta^{\alpha' \gamma} ... g_{\sigma \sigma'} ... \eta^{\delta' \beta'} g_{\beta \beta'},
\]

where the number of \( g \) factors is \( n \) and the number of \( \eta^{\alpha \beta} \)-factors (equal to the power of the Planck constant) is \( n - 1 \). The known two-loop result \[31,29\] then implies that

\[
C_{\alpha \beta}^{(22)} = 0 , \quad C_{\alpha \beta}^{(21)} = \beta_2 g_{\alpha \alpha'} \eta^{\alpha' \beta'} g_{\beta \beta'} , \quad \beta_2 = \frac{17}{6(4\pi)^4} ,
\]

or, equivalently (cf. (4.3))

\[
\hat{\Omega}_{ab}^{(2\text{-loop})} = \Omega_{ab} + \frac{1}{\epsilon} \beta_1 g_{ab} + \frac{1}{\epsilon} \beta_2 g_{\alpha \alpha'} \Omega^{\alpha' \beta'} g_{\beta \beta'} .
\]

In principle, it could happen that in the general case of non-semisimple Lie algebra \( L \) the matrix \( g_{\alpha \alpha'} \Omega^{\alpha' \beta'} g_{\beta \beta'} \) would be non-vanishing and not proportional to \( \gamma_{ab} \) (cf. (2.5),(2.7)), i.e. that starting with the 2-loop level it would be necessary to renormalize also \( \Omega_{ab}^{(0)} \). This would be strange since then it would not be manifest that the resulting theory is renormalizable, while it must be such (on symmetry and dimensional consideration grounds) provided \( \Omega_{ab} \) was chosen in the most general possible form.

It is very likely, however, that the very special double extension structure of indecomposable algebras \( L \) actually implies that the conclusions for any of such \( L \) will always be the same as for the ‘depth 1’ special case \( L \) is a double extension of an abelian \( G \) by a simple or 1-dimensional \( H \). In this case (which includes all the explicit examples in \[1,2,4,7,10\]) we have according to (2.8),(2.9)

\[
g_{\alpha \alpha'} \Omega^{\alpha' \beta'} g_{\beta \beta'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{rs} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Omega_{ij}^{(0)} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_{rs} & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 ,
\]

\[
g_{\alpha \alpha'} \Omega^{\alpha' \beta'} g_{\beta \beta'} c ... g_{dd'} \Omega^{d' e'} g_{e' b} = 0 ,
\]

(4.8)
i.e., as in WZW and CS theories, there is no renormalization of $\Omega_{ab}$ beyond one loop\[^{10}\]
Moreover, as we shall show in the next section, in the case of the ‘depth 1’ algebras the full effective action is exactly given by the 1-loop term.

The non-semisimple YM theories thus occupy an ‘intermediate’ place between the abelian theories (where there is no renormalization at all) and the semisimple non-abelian YM theories (where there is a renormalization at each loop order). It is interesting to note that in the non-semisimple theory discussed above there are no higher loop divergences in spite of the fact that the subalgebra $\mathcal{H}$ of the full gauge algebra $\mathcal{L}$ may be non-abelian. This is due to a peculiar way in which $\mathcal{H}$ is embedded into $\mathcal{L}$. In fact, the non-abelian nature of $\mathcal{H}$ is only indirectly reflected even in the 1-loop renormalization (4.3): the $\mathcal{H}$ part of the 1-loop divergence (4.2) is proportional to $g'_rs$ (defined in (2.5)) which is different from the Killing metric $g_{rs}$ of $\mathcal{H}$ appearing in the 1-loop counterterm of $\mathcal{H}$ YM theory and is nonvanishing even if $\mathcal{H}$ is abelian (see Section 5).

4.2. On-shell finiteness and exact effective action

The special structure of non-semisimple theories leads to the conclusion that in spite of the presence of the (1-loop) divergence in the off-shell effective action, they are actually UV finite on-shell, i.e. have finite scattering amplitudes. To demonstrate this let us consider the structure of the divergence (4.2) in more detail. First, let us write down explicitly the YM action (3.4) in the case of generic indecomposable non-semisimple algebra (2.4). Introducing the $G, H, H^*$ components of the gauge potential and computing the field strengths

$$A^a_{\mu e_a} = A^e_{\mu i} + A^r_{\mu e_r} + A_{s\mu e^s} , \quad F^i_{\mu\nu} = 2\partial_{[\mu}A^i_{\nu]} + f^i_{jk}A^j_{\mu}A^k_{\nu} - 2f^i_{jr}A^j_{[\mu}A^r_{\nu]} , \quad (4.9)$$

$$F^r_{\mu\nu} = 2\partial_{[\mu}A^r_{\nu]} + f^r_{st}A^s_{\mu}A^t_{\nu} , \quad F_{s\mu\nu} = 2\partial_{[\mu}A^s_{\nu]} + 2f^t_{sr}A^r_{[\mu}A^t_{\nu]} + f_{ijs}A^i_{\mu}A^j_{\nu} ,$$

we find for the YM action (see (2.7))

$$S = \frac{1}{4} \int d^4x (\Omega_{ij}F^i_{\mu\nu}F^j_{\mu\nu} + 2g_{ir}F^i_{\mu\nu}F^r_{\mu\nu} + \Omega_{rs}F^r_{\mu\nu}F^s_{\mu\nu} + 2F^\nu_{\mu\nu}F_r^{\mu\nu}) \quad (4.10)$$

$$= \frac{1}{4} \int d^4x (\Omega_{ij}F^i_{\mu\nu}F^j_{\mu\nu} + 2g_{ir}F^i_{\mu\nu}F^r_{\mu\nu} + 2F^r_{\mu\nu}F_r^{\mu\nu} + 2F^\nu_{\mu\nu}F_r^{\mu\nu}) ,$$

where we have simplified the action by introducing the new variable

$$\hat{A}_{s\mu} = A_{s\mu} + \frac{1}{2}\Omega_{sr}A^r_{\mu} , \quad \hat{F}_{s\mu\nu} = F_{s\mu\nu} + \frac{1}{2}\Omega_{sr}F^r_{\mu\nu} . \quad (4.11)$$

\[^{10}\] In WZW and CS theories this conclusion holds (in a properly chosen scheme) not only in the non-semisimple but also in the semisimple algebra case.
This is possible, in particular, due to the fact that the field component $A_{s\mu}$ appears in the action only linearly. Thus it plays the role of a Lagrange multiplier which, when integrated out in the path integral, gives the factor $\delta(D_\mu F_{\mu\nu}^r)$, i.e. the constraint that $A_{r\mu}^r$ should satisfy the classical equations of motion of YM theory for the algebra $\mathcal{H}$. As a result, the only dynamical fields that propagate in quantum loops are $\mathcal{G}$-components $A_{i\mu}$. Since the structure of $\mathcal{G}$ is strongly constrained by the requirement that $\mathcal{L}$ is indecomposable, the resulting structure of the quantum effective action is also very special.

To see this let us further specify the discussion to the explicitly tractable case when the algebra $\mathcal{L}$ has ‘depth 1’, i.e. when $\mathcal{G}$ is abelian. Then $f_{ij}^k = 0$, the Killing form has only one ($g_{rs}^\prime$) non-vanishing entry (see (2.8)) and the action (4.10) takes the form

$$S = \frac{1}{4} \int d^4x \left[ \Omega_{ij}^{(0)} (2\partial_{[\mu} A_{\nu]}^i - 2f_{[i}^j A_{\mu]}^r A_{r\nu}^i) (2\partial_\mu A_{[i}^j - 2f_{k]}^j A_{r\mu}^r A_{r\nu}^i) \right]$$

(4.12)

$$+ 2(2\partial_{[\mu} A_{\nu]}^r + f_{st}^r A_{s\mu}^r A_{t\nu}^r) (2\partial_\mu \hat{A}_{[\mu}^i \hat{A}_{\nu]}^r + 2f_{r\mu}^i A_{\mu}^r A_{\nu}^i) .$$

The above field redefinition (4.11) eliminated the dependence of the action on $\gamma$, i.e. $\gamma$ (in fact, the whole $\Omega_{rs}$ in (2.8)) is not an ‘essential coupling’ [32] of the theory. Since we have seen that the divergence (4.2) can be absorbed into a shift of $\gamma$ (4.3), that means that the theory does not need a non-trivial renormalization at all, having finite $S$-matrix elements, i.e. being $UV$ finite on shell.

As it is clear from (1.12), the action is bilinear in $A_{i\mu}^i$. Combining this with the fact that the path integral over $\hat{A}_{r\mu}$ ‘kills’ the quantum fluctuations of $A_{r\mu}^r$ it is easy to show that the full effective action is exactly given by the $1$-loop term which is a combination of logarithms of determinants depending only on the background field $A_{r\mu}^r$,

$$\Gamma = S[A_{i\mu}^i, A_{r\mu}^r, \hat{A}_{r\mu}] + \Gamma^{(1-\text{loop})}[A_{r\mu}^r] .$$

(4.13)

The divergent part (4.2)

$$\Gamma^{(1-\text{loop})} = \frac{1}{\epsilon} \beta_1 g_{rs}^\prime \int d^4x F_{\mu\nu}^r F_{\mu\nu}^s ,$$

as expected, can be eliminated by the additional (infinite) field redefinition

$$\hat{A}_{s\mu}^{(1-\text{loop})} = \hat{A}_{s\mu} + \frac{2}{\epsilon} \beta_1 g_{sr}^\prime A_{r\mu}^r .$$

(4.14)

The theory has effectively reduced to that of abelian vector fields $A_{i\mu}^i$ coupled to the external (classical) gauge field $A_{r\mu}^r$ corresponding to $\mathcal{H}$. This interpretation makes it clear that though this model does not have non-trivial UV divergences (and therefore is scale invariant on shell) it is not conformally invariant in the usual sense. Assuming that the theory was first defined on a curved 4-space, (1.12) implies that the conformal factor of
the 4-metric does not decouple, i.e. the expectation value of the trace of the stress-energy tensor is
\[ < T^\mu_\mu > = \beta_1 g^\prime_{rs} F^r_\mu F^s_\mu . \]  
(4.15)
This is an ‘external’ conformal anomaly, which is not reflected in a scale dependence of on-shell correlation functions but present in the effective action. Analogous anomaly appears, for example, in the theory of quantized interacting Higgs field coupled to an external \( U(1) \) gauge field, where again it is not reflected in the Higgs field S-matrix since the gauge field is classical.

Since the action \( (4.12) \) of the ‘depth 1’ YM theories is linear in the field \( \hat{A}_{\rho\mu} \) which can thus be explicitly integrated out from the action, one may hope to be able to define a unitary subsector of theory in terms of the resulting ‘reduced’ action involving only positive norm fields \( A^i_\mu \). We shall return to the discussion of the unitarity issue at the end of Section 5.

5. Simplest examples: WZW, CS and YM theories for \( E^c_2 \) algebra

In this section we shall consider in detail the first non-trivial case of a non-semisimple algebra with a nondegenerate invariant form – the central extension of the euclidean algebra in 2 dimensions \( E^c_2 \) [12]. We will illustrate explicitly the general observations made above and make clear the analogy between non-semisimple WZW, CS and YM models.

The algebra \( E^c_2 \) is defined as
\[ [e_3, e_i] = \epsilon_{ij} e_j , \quad [e_i, e_j] = \epsilon_{ij} e_4 , \quad [e_4, e_i] = [e_4, e_3] = 0 , \quad i, j = 1, 2 , \]  
(5.1)
where \( e_i \) and \( e_3 \) generate the two translations and the rotation and \( e_4 \) belongs to the center. This is the simplest example of the general case \( (2.4) \) with \( e_i \) as the generators of \( \mathcal{G} \), \( e_3 = e_r \) as the generator of \( \mathcal{H} \) and \( e_4 = e^r \) as the generator of \( \mathcal{H}^* \). The algebra \( (5.1) \) has the straightforward \( N = 2M + 2 \) dimensional generalization [4] obtained by replacing the 2-dimensional \( \mathcal{G} \) by \( M \)-dimensional abelian one, i.e. by adding an extra index \( (I = 1, ..., M) \) to \( e_i \) and replacing \( [e_i, e_j] = \epsilon_{ij} e_4 \) by \( [e_i, e_j] = \delta_{IJ} \epsilon_{ij} e_4 \). This is a ‘depth 1’ non-semisimple algebra with a non-degenerate invariant form with only one minus sign in the signature (see Section 2). All of the discussion of \( E^c_2 \) theories below can be straightforwardly generalized to the case of arbitrary \( M > 1 \).

The algebra \( (5.1) \) may be compared to the algebra of \( SU(2) \times U(1) \)
\[ [\tilde{e}_3, \tilde{e}_i] = \epsilon_{ij} \tilde{e}_j , \quad [\tilde{e}_i, \tilde{e}_j] = \epsilon_{ij} \tilde{e}_3 , \quad [\tilde{e}_4, \tilde{e}_i] = [\tilde{e}_4, \tilde{e}_3] = 0 , \quad i, j = 1, 2 , \]  
(5.2)
where \( (\tilde{e}_i, \tilde{e}_3) \) are the generators of \( SU(2) \) and \( \tilde{e}_4 \) is the generator of \( U(1) \). The two algebras are related, e.g., by the following limit:
\[ \tilde{e}_i = \lambda^{-1} e_i , \quad \tilde{e}_3 = e_3 + \lambda^{-2} e_4 , \quad \tilde{e}_4 = e_4 , \quad \lambda \to 0 . \]  
(5.3)
The algebra (5.4) has the degenerate Killing form (cf. (2.3)) but admits a non-degenerate invariant bilinear form of signature $(+ + + -)$ (cf. (2.6),(2.7); $a, b = 1, \ldots, 4$)

\[
\Omega_{ab} = k \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 2\bar{\gamma} & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad g_{ab} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \bar{\gamma} \equiv k^{-1}\gamma.
\]

where $\gamma$ is an arbitrary constant. As we have already mentioned above, $\gamma$ is not be an essential parameter of the corresponding field theories constructed using (5.4) since the dependence on it can be eliminated from the actions by a field redefinition. However, since the $\gamma$-term is effectively generated at the 1-loop level in the off-shell effective action it is natural to keep $\gamma$ also in $\Omega_{ab}$ which defines the classical action, interpreting the quantum correction as its (off-shell) renormalization.

As for all ‘depth 1’ algebras, the value of the overall scale $k$ of $\Omega_{ab}$ in (3.1),(3.3),(3.4) will also not be essential – it will be possible to set it equal to 1 (or make it, e.g., arbitrarily small) by a rescaling of the fields. This is certainly different from what happens in semisimple theories which nontrivially depend on level $k$ or YM coupling.

5.1. $E_2^c$ WZW model

It is useful first to consider the $E_2^c$ WZW model \[ \] since there will be many parallels with subsequent discussion of CS and YM theories. Using the parametrization $g = \exp(x_i \epsilon_i) \exp(ue_3 + ve_4)$ the $E_2^c$ WZW action (3.1),(3.2) can be put into the form \[ \]

\[
I_{E_2^c} = \frac{k}{4\pi} \int d^2\sigma (\partial x_i \bar{\partial} x_i + \epsilon_{ij} \partial x_i \partial x_j + 2\bar{\gamma} \partial u \bar{\partial} u + 2\partial u \bar{\partial} v) \quad (5.5)
\]

\[
= \frac{k}{4\pi} \int d^2\sigma (\partial x_i \partial x_i + \epsilon_{ij} \partial x_j \partial x_i + 2\partial u \bar{\partial} v) , \quad \hat{v} = v + \bar{\gamma} u .
\]

Note that the value of $k$ can be changed by rescaling the coordinates since the action is invariant under

\[
x_i \rightarrow sx_i , \quad u \rightarrow u , \quad \hat{v} \rightarrow s^2 \hat{v} , \quad k \rightarrow s^{-2}k . \quad (5.6)
\]

That means that coordinate-invariant observables (e.g., the central charge) will not depend on $k$. Transforming the coordinates $x_1 \rightarrow x_1 + x_2 \cos u, \quad x_2 \rightarrow x_2 \cos u, \quad \hat{v} \rightarrow \hat{v} + \frac{1}{2} x_1 x_2 \cos u$, the action can be represented also in the ‘plane-wave’ form \[ \]

\[
I_{E_2^c} = \frac{k}{\pi} \int d^2\sigma (\partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2 + 2 \cos u \partial x_1 \bar{\partial} x_2 + 2\partial u \bar{\partial} v) . \quad (5.7)
\]

\[\]

\[11\] We ignore the total derivative term and use the following notation: $d^2\sigma = d\sigma_1 d\sigma_2$, $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$, $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. 

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is closely related \([4]\) to the action of the \(SU(2) \times U(1)_-\) WZW model (the index ‘−’ is used to indicate that the \(U(1)\) term appears in the action with the opposite sign). The latter is given by (in the \(SU(2)\) parametrization \(g = \exp(\frac{1}{2}i\sigma_1 \theta_L) \exp(\frac{1}{2}i\sigma_3 \phi) \exp(\frac{1}{2}i\sigma_1 \theta_R)\))

\[
I_{SU(2) \times U(1)_-} = \frac{\tilde{k}}{4\pi} \int d^2 \sigma \left( \partial \theta_L \bar{\partial} \theta_L + \partial \theta_R \bar{\partial} \theta_R + 2 \cos \phi \ \partial \theta_L \bar{\partial} \theta_R + \partial \phi \bar{\partial} \phi - \partial t \bar{\partial} t \right), \tag{5.8}
\]

where \(t\) is the coordinate corresponding to the \(U(1)\) factor.\(^{12}\) Ignoring the periodicity of coordinates in (5.8) we may set (cf. (5.3))

\[
\theta_L = \lambda x_1, \quad \theta_R = \lambda x_2, \quad \phi = u + \lambda^2 \hat{v}, \quad t = u, \quad \tilde{k} = \lambda^{-2} k, \quad \lambda \to 0, \tag{5.9}
\]

and then (5.8) goes into (5.7).

The special ‘null’ structure of (5.5) implies that there are no higher (than one) loop corrections: the path integral over \(\hat{v}\) constrains \(u\) to its classical values and what remains is a gaussian integral over \(x_i\) (equivalently, the only 1-PI Feynman diagrams that contribute to the effective action are the one-loop ones with internal \(x_i\)-propagators and \(u, v\)-lines as external ones \([1]\)). In addition, because of the chiral form of the interaction term in (5.5) (typical to WZW models) there are no 1-loop divergences, i.e. the model is conformal at the quantum level. This is also consistent with its relation through (5.9) to the \(SU(2) \times U(1)_-\) WZW model (note that since in the limit \(\tilde{k} \to \infty\) the central charge has free-theory value, \(c = 4\)). Furthermore, the full quantum effective action is given by the 1-loop term (\(x_i\)-determinant) depending only on \(u\)

\[
\Gamma_{E^{'2}} = I_{E^{'2}} + \frac{1}{2} \log \det \left( (\delta_{ij} \bar{\partial} + \epsilon_{ij} \bar{\partial} u) \partial \right). \tag{5.10}
\]

The ‘on-shell’ (\(\partial \bar{\partial} u = 0\)) part of this determinant can be computed explicitly. Introducing \(x = x_1 + ix_2, \ x^* = x_1 - ix_2\) one can set \(x = e^{iu} y, \ x^* = e^{-iu} y^*\). When \(\partial \bar{\partial} u = 0\) the resulting determinant becomes \(u\)-independent up to the anomaly contribution which (in the left-right symmetric scheme) is given by\(^{13}\)

\[
\frac{1}{2\pi} \int d^2 \sigma \partial u \bar{\partial} u.
\]

While there is no shift of the parameter \(k\), as expected, the constant \(\gamma\) gets the quantum correction

\[
\hat{\gamma} = \bar{\gamma} + \frac{1}{k}, \quad \hat{\gamma} = \gamma + 1. \tag{5.11}
\]

\(^{12}\) Note that this action does not have an invariance similar to (5.6), i.e. it nontrivially depends on \(\tilde{k}\).

\(^{13}\) This expression corresponds to the 2-sphere. It is easy also to compute the determinant on the 2-torus obtaining the partition function \([3]\) of this model \([3]\).
This agrees with the general result (3.12), (3.14) (the same one-loop shift was found in [1, 3]). The general ‘off-shell’ expression for the effective action for an arbitrary \( u \) contains also non-local terms which depend on \( \partial \bar{\partial} u \).

The simplicity of this theory is due to the special structure of the action which is linear in \( v \), quadratic in \( x^i \) and has chiral interaction term. As a consequence, can explicitly solve the corresponding field equations (and thus the resulting conformal field theory) in terms of free fields (the dependence on on-shell values of \( u \) can be effectively ‘eliminated’ by the ‘rotation’ of \( x^i \)). In spite of the existence of one time-like direction in the field space, the CFT corresponding to (5.5) is unitary since (in contrast to, e.g., \( SL(2, R) \) WZW model) here one can use the light-cone gauge [3, 33, 34].

5.2. \( E^c_2 \) Chern-Simons theory

The Chern-Simons action (3.3) for \( E^c_2 \) has the following explicit structure

\[
S_{E^c_2} = -\frac{ik}{8\pi} \int d^3x \epsilon^{\mu\nu\lambda} \left( A^i_\mu \partial_\nu A^i_\lambda + \epsilon_{ij} A^j_\mu A^i_\nu A^3_\lambda + 2\tilde{\gamma} A^3_\mu \partial_\nu A^3_\lambda + 2 A^3_\mu \partial_\nu A^4_\lambda \right) \tag{5.12}
\]

which is closely related to that of (5.5). As in WZW case, \( k \) can be changed by rescaling \( A^i_\mu, \tilde{A}^4_\mu \) and (5.12) can be also obtained as a limit of the semisimple \( SU(2) \times U(1) \) Chern-Simons theory (the derivation is similar to the one discussed below in the YM case). Since the integral over \( \tilde{A}^4_\lambda \) implies that \( A^3_\mu \) is subject to the classical equation, the quantum part of the effective action depends only on \( A^3_\mu \) and is exactly given by the one-loop term. The local ‘on-shell’ part of the determinant resulting from integration over \( A^i_\mu \) can be explicitly computed (see, e.g., [20, 23]) so that the final result is (cf. (3.13), (3.16), (5.11))

\[
\Gamma_{E^c_2} = S_{E^c_2} + \frac{1}{2} \log \det \left[ \epsilon^{\mu\nu\lambda} (\delta_{ij} \partial_\mu + \epsilon_{ij} A^3_\mu) \right] \tag{5.13}
\]

\[
= S_{E^c_2} - \frac{i}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A^3_\mu \partial_\nu A^3_\lambda + \ldots ,
\]

where dots stand for non-local \( P \)-even ‘off-shell’ terms which vanish when \( \partial_{[\mu} A^3_{\nu]} \rightarrow 0 \). Again, in contrast to the semisimple case [14, 24, 23], there is no quantum renormalization of \( k \) but only a shift of the coefficient \( \gamma \) which is the same as in \( E^c_2 \) WZW theory (5.11).
5.3. $E^\infty_2$ Yang-Mills theory

Starting with (5.1), (5.4) one finds from (3.4) (cf. (5.12))

$$S_{E^\infty_2} = \frac{1}{4g^2} \int d^4x \left( F^i_{\mu\nu} F^i_{\mu\nu} + 2\tilde{F}^3_{\mu\nu} F^3_{\mu\nu} + 2F^3_{\mu\nu} F^4_{\mu\nu} \right)$$

$$= \frac{1}{4g^2} \int d^4x \left( F^i_{\mu\nu} F^i_{\mu\nu} + 2\tilde{F}^3_{\mu\nu} \tilde{F}^3_{\mu\nu} \right),$$

$$F^i_{\mu\nu} = 2\partial_{[\mu} A^i_{\nu]} - 2\epsilon_{ij} A^3_{[\mu} A^j_{\nu]}, \quad i = 1, 2,$$

$$F^3_{\mu\nu} = 2\partial_{[\mu} A^3_{\nu]}, \quad F^4_{\mu\nu} = 2\partial_{[\mu} A^4_{\nu]} + \epsilon_{ij} A^i_{\mu} A^j_{\nu},$$

$$\tilde{A}^i_{\mu} = A^i_{\mu} + \tilde{\gamma} A^3_{\mu}, \quad g^2 = 1/k.$$  

This is a special case of the YM action for ‘depth 1’ non-semisimple algebras (4.12) (here $A^r_{\mu} = A^3_{\mu}, A_{r\mu} = A^4_{\mu}, \ f_{rst} = 0, \ f_{ijr} = \epsilon_{ij}$). It is invariant under (3.6),

$$\delta A^i_{\mu} = \partial_{\mu} \eta^i + \epsilon_{ij} A^j_{\mu} \eta^i, \quad \delta A^3_{\mu} = \partial_{\mu} \eta^3, \quad \delta A^4_{\mu} = \partial_{\mu} \eta^4 + \epsilon_{ij} A^i_{\mu} \eta^j.$$  

The correspondence between the YM and WZW actions (5.15) and (5.5) constructed using the same algebra and invariant form is established by

$$x_i \sim A^i_{\mu}, \quad u \sim A^3_{\mu}, \quad v \sim A^4_{\mu}, \quad k \sim g^{-2}.$$  

Like (5.5) the action (5.14) is quadratic in $A^i_{\mu}$ and the interaction term involves only $A^3_{\mu}$ and $A^i_{\mu}$. This action is also invariant under (cf. (5.6))

$$A^i_{\mu} \rightarrow s A^i_{\mu}, \quad A^3_{\mu} \rightarrow A^3_{\mu}, \quad \tilde{A}^4 \rightarrow s^2 \tilde{A}^4_{\mu}, \quad g \rightarrow sg,$$

implying that the theory can only have trivial dependence on $g$. In particular, as in the case of WZW and CS theories, the 1-loop approximation should be exact. The 1-loop term can only depend on $A^3_{\mu}$ (cf. (5.12)) which is not transformed under (5.18).

Like the WZW action (5.5), (5.7) the YM action (5.14) can be also obtained as a limit of the semisimple $SU(2) \times U(1)_-$ YM action

$$S_{SU(2) \times U(1)_-} = \frac{1}{4g^2} \int d^4x \left( \tilde{F}^i_{\mu\nu} \tilde{F}^i_{\mu\nu} + \tilde{F}^3_{\mu\nu} \tilde{F}^3_{\mu\nu} + \tilde{F}^4_{\mu\nu} \tilde{F}^4_{\mu\nu} \right),$$

$$\tilde{F}^i_{\mu\nu} = 2\partial_{[\mu} \tilde{A}^i_{\nu]} - 2\epsilon_{ij} \tilde{A}^3_{[\mu} \tilde{A}^j_{\nu]},$$

$$\tilde{F}^3_{\mu\nu} = 2\partial_{[\mu} \tilde{A}^3_{\nu]} + \epsilon_{ij} \tilde{A}^i_{\mu} \tilde{A}^j_{\nu}, \quad \tilde{F}^4_{\mu\nu} = 2\partial_{[\mu} \tilde{A}^4_{\nu]}.$$
where $\tilde{A}_\mu^i$, $\tilde{A}_\mu^3$, $\tilde{A}_\mu^4$ are the $SU(2)$ fields and $\tilde{A}_\mu^4$ is the ‘ghost’ $U(1)$ field (cf. (5.2), (5.1), (5.15)). Introducing the new variables and rescaling the coupling constant (cf. (5.3))

$$
\tilde{A}_\mu^i = \lambda A_\mu^i, \quad \tilde{A}_\mu^3 = A_\mu^3 + \lambda^2 \tilde{A}_\mu^4, \quad \tilde{A}_\mu^4 = A_\mu^4, \quad \tilde{g} = \lambda g,
$$

we can rewrite the $SU(2) \times U(1)_-$ YM action (5.19) as

$$
S_{SU(2) \times U(1)_-} = \frac{1}{4g^2} \int d^4x \left[\left(F_{\mu\nu}^i - 2\lambda^2 \epsilon_{ij} A_\mu^i A_\nu^j\right)^2 + (2F_{\mu\nu}^3 + \lambda^2 \tilde{F}_{\mu\nu}^4)\tilde{F}_{\mu\nu}^4\right]
$$

$$
= S_{E_2} + O(\lambda^2),
$$

where we have used $E^a_{\mu\nu}$ field strengths $F_{\mu\nu}^a$ given in (5.15). Thus $S_{E_2}$ is the $\lambda \to 0$ limit of $S_{SU(2) \times U(1)_-}$.

The quantum theory defined by the $E_2^a$ action (5.14) is much simpler than the $SU(2) \times U(1)_-$ one of which it is a limit. Describe the path integral corresponding to (5.14)

$$
Z = \int [dA_\mu^i][dA_\mu^3][d\tilde{A}_\mu^4] \exp \left( - \frac{1}{g^2} \int d^4 x \left[ (\partial_\mu A_\nu^i - \epsilon_{ij} A_\mu^j A_\nu^i)^2 + (2\partial_\mu A_\nu^3 + \epsilon_{ij} A_\mu^j A_\nu^i)\right\rangle \right).
$$

The integral over $(\tilde{A}_\mu^4)^\perp$ gives $\delta(\partial_{\mu} F_{\mu\nu}^3)$, i.e. fixes $A_\mu^3$ to be equal to its classical value. In the vacuum case ($A_\mu^i = 0$) what remains is the free integral over $A_\mu^i$. Since the 1-PI Feynman diagrams in this theory can only have $A_\mu^i$ (and their ghosts) propagating in the internal lines (with the external lines being represented only by the $A_\mu^3, \tilde{A}_\mu^4$ propagators), the possible loop diagrams are 1-loop ones only, i.e. the quantum effective action in this theory is exactly given by the one-loop term.

The latter can be found by treating $A_\mu^3$ as a background field and integrating over $A_\mu^i$,

$$
\exp(-\Gamma_{E_2}^{(quant)}[A_\mu^3])
$$

$$
= \int [dA_\mu^i] \exp \left( - \frac{1}{g^2} \int d^4 x \left[ (\partial_\mu A_\nu^i - \epsilon_{ij} A_\mu^j A_\nu^i)^2 \right\rangle \right).
$$

---

14 Note, in particular, that the norm of the $\tilde{A}_\mu^4$ field becomes null in the limit. Eq. (5.22) implies certain relations between the two quantum theories (the redefinition (5.21) is nondegenerate for any finite $\lambda$). For example, the vacuum partition function of $E_2^a$ YM theory $Z(g)$ equal to the $\lambda \to 0$ limit of $Z_{SU(2) \times U(1)_-}(\tilde{g})$ must be trivial since in this limit $\tilde{g} \to 0$.

15 One can use, e.g., the gauge $\partial_\mu A_\mu^a = 0$ in which the non-trivial part $\Delta_{gh} = \det(\partial_{\mu}(\delta_{ij}\partial_\mu - \epsilon_{ij} A_\mu^3))$ of the ghost determinant which follows from (5.16) depends only on $A_\mu^3$. 

16 Consider the path integral corresponding to (5.14).
It is interesting to note that \( \Gamma^{(quant)}_{E_2} [A^3_\mu] \) can be expressed in terms of the one-loop effective action in \( SU(2) \) YM theory. Suppose one starts with the \( SU(2) \) part of (5.19) and introduces a background only for the third component of the gauge field. Expanding the \( SU(2) \) action to the second order in the quantum fields we get (as in (4.1) here \( A^3_\mu \) is the background field and \( B^i_\mu, B^3_\mu \) are the quantum fields)

\[
S_{SU(2)} = \frac{1}{g^2} \int d^4x \left[ \left( \partial_\mu B^i_\nu - \epsilon_{ij} A^3_\mu B^j_\nu \right)^2 \right. \\
+ \left. \partial_\mu A^3_\nu \epsilon_{ij} B^i_\mu B^j_\nu + \left( \partial_\mu B^3_\nu \right)^2 + O((B^3_\mu)^3) \right].
\]

Observe that \( B^3_\mu \) is decoupled from \( A^3_\mu \) and comparing with (5.24) we conclude that the exact quantum part of the effective action in \( E_2^2 \) YM theory is equal to the one-loop effective action in \( SU(2) \) YM theory with only \( A^3_\mu \) as its non-vanishing argument,

\[
\Gamma_{E_2^2} [A^i_\mu, A^3_\mu, A^4_\mu] = \frac{1}{4g^2} \int d^4x \left( F^{i\mu}_{\nu} F^{i\nu}_{\mu} + 2F^{3\mu}_{\nu} \hat{F}^{3\nu}_{\mu} \right) + \Gamma^{(1-loop)}_{SU(2)} [A^3_\mu].
\]

If \( \Gamma^{(1-loop)}_{SU(2)} [A^3_\mu] \) is computed in the background field gauge \([27,28]\):

\[
D_\mu B^i_\mu \equiv \partial_\mu B^i_\mu - \epsilon_{ij} A^3_\mu B^j_\mu = \xi^i(x), \quad \partial_\mu B^3_\mu = \xi^3(x),
\]

then the effective action \( \Gamma_{E_2^2} \) is invariant under the classical gauge symmetry \((1.10)\), i.e., \( \delta A^3_\mu = \partial_\mu \eta^3 \). \( \Gamma^{(1-loop)}_{SU(2)} \) contains the well-known divergent term \((4.2)\), i.e.,

\[
(\Gamma^{(1-loop)}_{SU(2)} [A^3_\mu])_\infty = \frac{2}{\epsilon} \beta_1 \int d^4x F^{3\mu}_{\nu} \hat{F}^{3\nu}_{\mu}.
\]

As was already discussed above, this divergence can be absorbed into the renormalization of \( \gamma \) \((4.3)\) or can be eliminated by the field redefinition \((4.14)\) (cf. \((5.15)\))

\[
\hat{A}^{4\mu\text{(1-loop)}} = \hat{A}^4_\mu + \frac{4}{\epsilon} g^2 \beta_1 A^3_\mu.
\]

To determine the ‘ground state’ of the theory one is to solve the effective equations that follow from \( \Gamma_{E_2^2} \). The equation for \( A^3_\mu \) is still the classical Maxwell one, the equation for \( A^i_\mu \) is linear and does not receive quantum corrections\([16]\) and to find \( A^4_\mu \) one needs only to compute the derivative of \( \Gamma^{(1-loop)}_{SU(2)} [A^3_\mu] \) at the classical value of \( A^3_\mu \). The solution of the effective equations is thus parametrized by a solution \((A^3_\mu)_*\) of the Maxwell equation.

\[\text{[16] The equation for } A^i_\mu \text{ that follows from (5.14) has the same form as the linearized } SU(2) \text{ YM equation in the } A^i_\mu \text{ background (i.e. the equation for } B^i_\mu \text{ that follows from (5.25)). Introducing } W_\mu = A^1_\mu + iA^2_\mu, W'^* = A^1_\mu - iA^2_\mu, \text{ one finds: } -D_\mu D_\nu W_\nu - iF^3_{\mu\nu} W_\nu = 0, D_\mu = \partial_\mu + iA^3_\mu.\]
As we have already discussed in Section 4.2, the theory still has the ‘external’ conformal anomaly (4.15), i.e. $T_\mu^\mu = 2\beta_1 F_\mu^3 F_\nu^3$. The S-matrix (which is finite and scale invariant) has the following structure. The scattering amplitudes with external $A_\hat\mu^3$ points (ends of $< A_\hat\mu^3 A_\hat\mu^4 >$ propagator lines) vanish, the amplitudes with external $A_i^\mu$ lines are given by the tree diagrams only, and the loop correction (5.26) gives non-trivial scattering amplitudes with only $\hat A_\mu^4$ as external points (such amplitudes are absent at the tree level).

The fact that the Euclidean action and Hamiltonian of this theory are not positive definite implies the absence of unitarity. Still, the special structure of the interaction term in (5.14) (which does not involve $\hat A_\mu^4$) and of quantum scattering amplitudes (with no $A_\hat\mu^3$ fields appearing as external points) suggests that there may exist a re-interpretation of this model which is consistent with unitarity. The relation (5.22) to $SU(2) \times U(1)_-$ model (5.19) seems to make this plausible. While the $SU(2) \times U(1)_-$ action (5.13) is non-positive, the ‘non-unitarity’ of this theory caused by the negative sign of the $U(1)$ term is a trivial one since the $U(1)$ field is completely decoupled: the non-trivial part of the full S-matrix is just the unitary $SU(2)$ Yang-Mills S-matrix. Since the field redefinition (5.21) can not change this conclusion for any finite $\lambda$, one may hope that the theory obtained in the limit $\lambda \to 0$ is also unitary.

6. Conclusions

Our motivation for considering non-semisimple Yang-Mills theories was their close analogy with non-semisimple conformal WZW models in 2 dimensions (and CS models in 3 dimensions) which have particularly transparent structure and are explicitly solvable in terms of free fields. We have seen that these YM models are indeed much simpler than the standard compact non-abelian theories and come very close to be 4-dimensional ‘free-field’ analogs of the 2-dimensional models. They have UV finite yet non-trivial S-matrix controlled by the effective action containing only 1-loop term. The main issue is, of course, unitarity. It may, hopefully, be resolved by using the interpretation of these non-semisimple models as special limits of semisimple YM models which are superficially non-unitary but have ghosts that are decoupled from a unitary sector.

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17 The energy is non-positive since there are 6 negative norm components $A_0^\alpha, A_0^\dot\alpha, A_\alpha^-,$ $(\alpha = 1, 2, 3), \ A_\mu^\pm = A_\mu^3 \pm \hat A_\mu^4,$ and only 4 gauge symmetries.
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