NON-ABELIAN FINITE GROUPS WHOSE CHARACTER SUMS ARE INVARIANT BUT ARE NOT CAYLEY ISOMORPHISM

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Abstract. Let $G$ be a group and $S$ an inverse closed subset of $G \setminus \{1\}$. By a Cayley graph $Cay(G, S)$ we mean the graph whose vertex set is the set of elements of $G$ and two vertices $x$ and $y$ are adjacent if $x^{-1}y \in S$. A group $G$ is called a CI-group if $Cay(G, S) \cong Cay(G, T)$ for some inverse closed subsets $S$ and $T$ of $G \setminus \{1\}$, then $S^a = T$ for some automorphism $a$ of $G$. A finite group $G$ is called a BI-group if $Cay(G, S) \cong Cay(G, T)$ for some inverse closed subsets $S$ and $T$ of $G \setminus \{1\}$, then $\mu_{S^\nu} = \mu_{T^\nu}$ for all positive integers $\nu$, where $\mu_{S^\nu}$ denotes the set $\{ \sum_{s \in S} \chi(s) | \chi(1) = \nu, \chi$ is a complex irreducible character of $G \}$. It was asked by László Babai [J. Combin. Theory Ser. B, 27 (1979) 180-189] if every finite group is a BI-group; various examples of finite non BI-groups are presented in [Comm. Algebra, 43 (12) (2015) 5159-5167]. It is noted in the latter paper that every finite CI-group is a BI-group and all abelian finite groups are BI-groups. However it is known that there are finite abelian non CI-groups. Existence of a finite non-abelian BI-group which is not a CI-group is the main question which we study here. We find two non-abelian BI-groups of orders 20 and 42 which are not CI-groups. We also list all BI-groups of orders up to 30.

1. Introduction and Results

In this paper all graphs are finite and simple so graphs has a finite number of vertices and they are undirected without multiple edges and has no loop. Let $G$ be a group and $S$ an inverse closed subset of $G^* := G \setminus \{1\}$. By a Cayley graph $Cay(G, S)$ we mean the graph whose vertex set is the set of elements of $G$ and two vertices $x$ and $y$ are adjacent if $x^{-1}y \in S$.

There is a famous relation between complex irreducible characters of a finite group $G$ and spectra of Cayley graphs $Cay(G, S)$ (see [3], see Theorem 2.1 below). According to the latter result a character sum $\sum_{s \in S} \chi_i(s)$ on $S$ for any complex irreducible character $\chi_i$ of $G$ is equal to a sum of $\chi_i(1)$ eigenvalues of $Cay(G, S)$.

László Babai [3] has proposed whether the set of character sums of all complex irreducible characters of the same degree is an invariant of $Cay(G, S)$:

Problem 1.1. [3 Problem 3.3] Let $G$ be a finite group, let $\Gamma$ be a Cayley graph $Cay(G, S)$, $\nu$ a positive integer and

$$\mu_i = \sum_{s \in S} \chi_i(s), \quad (i = 1, \ldots, h),$$

where $\chi_1, \ldots, \chi_h$ are all irreducible characters of $G$. Is the set $M_{S^\nu} = \{ \mu_i | \chi_i(1) = \nu \}$ an invariant of $\Gamma$? (Thus, does $Cay(G, S) \cong Cay(G, S')$ imply $M_{S^\nu} = M_{S'^\nu}$?)

Remark 1.2. Babai in [3 Problem 3.3] mentions Problem 1.1 [for all Cayley digraphs and according to his definition of a Cayley digraph [3] p. 182] a Cayley graph can be considered as a Cayley digraph.

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As we mentioned above, here we only study the simple undirected Cayley graphs. In [1] various examples of finite groups not satisfying the property mentioned in Problem 1.1 are presented. So we need to name the class of groups satisfying the property.

**Definition 1.3.** A Cayley graph $\text{Cay}(G, S)$ is called a Babai Isomorphism graph or a BI-graph if $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ for some inverse closed subset $T$ of $G^*$ then $M^S = M^T$ for all complex irreducible characters of degree $\nu$ of $G$. A group $G$ is called a BI-group if all Cayley graphs of $G$ are BI-graphs.

A famous class of Cayley graphs is the class of Cayley isomorphism graphs which are not CI-graphs.

**Definition 1.4.** A Cayley graph $\text{Cay}(G, S)$ is called a Cayley Isomorphism graph or CI-graph if $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ for some inverse closed subset $T$ of $G^*$ then $S^\alpha = T$ for some automorphism $\alpha$ of $G$. A group $G$ is called a CI-group if $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ for some inverse closed subsets $S$ and $T$ of $G^*$, then $S^\alpha = T$ for some automorphism $\alpha$ of $G$.

Every Cayley isomorphism graph is a BI-graph [1, Proposition 2.5]. Every finite abelian group is a BI-group [1, Proposition 2.2] but not every abelian group is a CI-group [2, Theorem 2.11]. We give the first example of non-abelian finite BI-groups which are not CI-graphs.

**Theorem 1.5.** The group $G = \langle a, b | a^5 = b^4 = 1, a^b = a^3 \rangle$ is the smallest non-abelian BI-group which is not a CI-group.

**Theorem 1.6.** The group $H = \langle a, b | a^7 = b^6 = 1, a^b = a^3 \rangle$ is a non-abelian BI-group which is not a CI-group.

The question of which finite groups are BI-groups has been proposed in [1].

**Question 1.7.** [1] Question 3.9 Which finite groups are BI-groups?

Here we determine all finite BI-groups of order at most 30. We denote by $C_n$, $S_n$, $A_n$, $D_n$ and $Q_n$ the cyclic group of order $n$, the symmetric group of degree $n$, the alternating group of degree $n$, the dihedral group of order $n$ and the generalized quaternion group of order $n$, respectively. If $H$ and $K$ are two groups, we denote by $H \ltimes K$ the semidirect product of $K$ by $H$ and since in general using the notation $H \ltimes K$ for a group does not uniquely determine the group, we mention its GAP IdSmallGroup (see [12] for the usage of the latter code).

**Theorem 1.8.** Let $G$ be a finite non-abelian group of order at most 30. Then, $G$ is a BI-group if and only if $G$ is isomorphic to one of the following groups:

- $S_3$, $Q_8$, $D_{10}$,
- $C_4 \rtimes C_3 = \langle a, b | a^4 = b^3 = b^a b = 1 \rangle$ (GAP IdSmallGroup[12, 1]),
- $A_4$, $D_{14}$, $D_{18}$,
- $C_2 \ltimes (C_3 \times C_3) = \langle a, b, c | a^2 = b^3 = c^3 = [b, c] = b^a b = c^a c = 1 \rangle$ (GAP IdSmallGroup[18, 4]),
- $C_4 \rtimes C_5 = \langle a, b | a^4 = b^5 = b^a b = 1 \rangle$ (GAP IdSmallGroup[20, 1]),
- $C_4 \rtimes C_5 = \langle a, b | a^4 = b^5 = b^a b^2 = 1 \rangle$ (GAP IdSmallGroup[20, 3]),
- $C_2 \rtimes C_7 = \langle a, b | a^3 = b^7 = b^a b^3 = 1 \rangle$ (GAP IdSmallGroup[21, 1]),
- $D_{22}$, $C_4 \rtimes C_3 = \langle a, b | a^3 = b^3 = b^a b = 1 \rangle$ (GAP IdSmallGroup[24, 1]),
- $D_{26}$, $C_4 \rtimes C_7 = \langle a, b | a^4 = b^7 = b^a b = 1 \rangle$ (GAP IdSmallGroup[28, 1]),
- $C_5 \times S_3$, $C_3 \times D_{16}$, $D_{30}$.

2. Preliminaries

The following result of Babai [3] gives a useful relation between spectra of Cayley graphs and the irreducible characters of the underlying group.
Theorem 2.1. [3 Theorem 3.1] Let \( G \) be a finite group whose all complex irreducible characters are denoted by \( \chi_1, \ldots, \chi_h \) and the degree \( \chi_i(1) \) of \( \chi_i \) is denoted by \( n_i \) for each \( i = 1, \ldots, h \). Suppose \( S \) is an inverse closed subset of \( G^\ast \). Then, the spectrum of the Cayley graph \( \text{Cay}(G,S) \) can be arranged as

\[
\text{spec}(\text{Cay}(G,S)) = \{ \lambda_{ijk} | i = 1, \ldots, h; j, k = 1, \ldots, n_i \},
\]

where \( \lambda_{ij1} = \cdots = \lambda_{ijn_i} \), and this common value will be denoted by \( \lambda_{ij} \). Also:

\[
\chi_t^\ast = \sum_{\sigma_1, \ldots, \sigma_t \in S} \chi_1(s_1 \cdots s_t),
\]

for any positive integer \( t \).

It was shown in [1] that all CI-groups are BI-groups.

Theorem 2.2. [1 Propositions 2.6] Every finite CI-group is a BI-group.

For a finite group \( G \), \( \text{Irr}(G) \) denotes the set of all complex irreducible characters of \( G \); and \( 1_G \) denotes the principal character of \( G \).

Lemma 2.3. Let \( G \) be a finite group and \( S, T \) be inverse closed subsets of \( G^\ast \). If \( M^S_1 = M^T_1 \), then \( M^S_1 = M^T_1 \), where \( S^c = G^\ast \setminus S \).

Proof. We may write

\[
\sum_{s \in S^c} \chi(s) = \sum_{g \in G^\ast} \chi(g) - \sum_{s \in S} \chi(s)
\]

for any \( \chi \in \text{Irr}(G) \). By [11 Corollary 2.14]

\[
\sum_{g \in G^\ast} \chi(g) = \begin{cases} |G| - 1 & \chi = 1_G \\ -\chi(1) & \chi \neq 1_G \end{cases}
\]

Therefore

\[
\sum_{s \in S^c} \chi(s) = \begin{cases} |G| - 1 - |S| & \chi = 1_G \\ -\chi(1) - \sum_{s \in S} \chi(s) & \chi \neq 1_G \end{cases}
\]

It follows that

\[
\{ \sum_{s \in S^c} \chi(s) \mid \chi \in \text{Irr}(G), \chi(1) = \nu, \chi \neq 1_G \} = \{ \sum_{t \in T^c} \chi(t) \mid \chi \in \text{Irr}(G), \chi(1) = \nu, \chi \neq 1_G \}.
\]

Therefore \( M^S_1 = M^T_1 \) for all \( \nu > 1 \), since the degree of \( 1_G \) is 1.

Now assume that \( \nu = 1 \).

By Theorem 2.1 the set \( M^S_1 \) consists of some eigenvalues of \( \text{Cay}(G,S) \). Also \( |S| = \sum_{s \in S} \chi_1(s) \in M^S_1 \) is the largest eigenvalues of \( \text{Cay}(G,S) \) [7 Proposition 1.1.2]. Therefore \( |S| \) is the maximum of \( M^S_1 \) and so \( |S| = |T| \), since \( M^S_1 = M^T_1 \). It follows from the set equality (2.1) that \( M^S_1 = M^T_1 \). This completes the proof. \( \square \)

Corollary 2.4. Let \( G \) be a finite group and \( S \) be an inverse closed subset of \( G^\ast \). Then, \( \text{Cay}(G,S) \) is a BI-graph if and only if \( \text{Cay}(G,S^c) \) is a BI-graph.

Proposition 2.5. Let \( G \) be a finite group. Then, \( G \) is a BI-group if and only if \( \text{Cay}(G,S) \) is a BI-graph for all inverse closed subsets \( S \) of \( G^\ast \) such that \( G = \langle S \rangle \) and \( |G| - 1 \leq |S| \leq |G| - 1 \).

Proof. It follows from Corollary 2.4 and the fact that the complement of a disconnected graph is a connected graph, that a \( G \) is a BI-group if and only if \( \text{Cay}(G,S) \) is a BI-graph for all inverse closed subsets \( S \) of \( G^\ast \) such that \( G = \langle S \rangle \). Now if \( |S| < \frac{|G|}{2} - 1 \), then \( |S^c| > \frac{|G|}{2} \) and so \( G = S^cS^c \). In particular \( G = \langle S^c \rangle \). This completes the proof. \( \square \)
With the aid of the following codes written in GAP [18] one can check whether a given finite group is a BI-group or not.

The function $M$ determines whether for two given inverse closed subsets $S$ and $T$ without non-trivial element of a group $g$ and a given positive integer $v$, $M_S^v = M_T^v$ or not. The function $BIn$ determines for each character degree $v$ of a given finite group $g$ all pairs $(S,T)$ of inverse closed subsets of a given size $n$ of $g$ such that $Cay(g,S) \cong Cay(g,T)$ and $M_S^v \neq M_T^v$. The function $BI$ determines whether or not a given finite group $g$ is a BI-group by giving all pairs $(S,T)$ and positive integers $v$ such that $Cay(g,S) \cong Cay(g,T)$ and $M_S^v \neq M_T^v$.

LoadPackage("Grape");

\begin{verbatim}
M:=function(g,S,T,v)
local S1,T1,Xv,MSv,MTv,C,D;
D:=CharacterDegrees(g);
C:=ConjugacyClasses(g);
S1:=List(S,j->Filtered([1..Size(C)],i->(j in C[i])=true)[1]);
T1:=List(T,j->Filtered([1..Size(C)],i->(j in C[i])=true)[1]);
Xv:=Filtered(Irr(g),i->i[1]=v);
MSv:=Set(List(Xv,i->Sum(S1,j->i[j]));
MTv:=Set(List(Xv,i->Sum(T1,j->i[j]));
return MSv=MTv;
end;

BIn:=function(g,n)
local SS,TT,GG,NG,NN;
SS:=Filtered(Combinations(Difference(Elements(g),[One(g)]),n),i->
Set(i,k->k^-1)=i);
TT:=Set(Filtered(SS, i->GroupWithGenerators(i)=g));
GG:=Filtered(Combinations( TT , 2 ),i->IsIsomorphicGraph
(CayleyGraph(g,i[1]),CayleyGraph(g,i[2])));
NG:=List(Set( Irr( g ), DegreeOfCharacter ),i->
M(g,i[1],i[2],i)=false));
return NG;
end;

BI:=function(g)
return List([(Size(g)/2)-1..Size(g)-1],i->BIn(g,i));
end;
\end{verbatim}

3. Proof of Theorem 1.5

Definition 3.1. (1) Let $\Gamma$ be a graph. We denote by $\text{spec}(\Gamma) = (\mu_1^{l_1}, \mu_2^{l_2}, \ldots, \mu_k^{l_k})$ the spectrum of $\Gamma$ [2], where $\mu_i$’s are all distinct (real) eigenvalues of $\Gamma$ and $l_i$ is the multiplicity of $\mu_i$ for $i = 1, \ldots, k$.

(2) Let $G$ be a finite group. We denote by $S_k$ the set of all elements of order $k$ in $S \subset G$. So, if $\{l_1, \ldots, l_r\}$ be the set of orders of elements of $S$, then:

$$S = S_{l_1} \cup \cdots \cup S_{l_r}.$$

Theorem 3.2. Let $G = \langle a, b | a^5 = b^4 = 1, a^b = a^3 \rangle$, $S \subset G^*$, $S^{-1} = S$, $G = \langle S \rangle$ and $\Gamma = Cay(G,S)$. Then the spectrum of $\Gamma$ is either

$$\text{spec}(\Gamma) = (\mu_1^S, \mu_2^{4k+1}, \mu_3^{4k+2}, \mu_4^{4l_1}, \ldots, \mu_4^{4l_e})$$
and note that $S_4 = S_{4_1} \cup S_{4_2}$. The character table of $G$ is:

| Class | $G_1$ | $G_2$ | $G_{4_1}$ | $G_{4_2}$ | $G_5$ |
|-------|-------|-------|-----------|-----------|-------|
| Size  | 1     | 5     | 5         | 5         | 4     |
| $\chi_1$ | 1     | 1     | 1         | 1         | 1     |
| $\chi_2$ | 1     | 1     | $-1$     | $-1$     | 1     |
| $\chi_3$ | 1     | $-1$ | $i$       | $-i$      | 1     |
| $\chi_4$ | 1     | $-1$ | $-i$      | $i$       | 1     |
| $\psi$  | 4     | 0     | 0         | 0         | $-1$  |

TABLE 1. Character table of the group $G$

Now by Theorem 2.4 all eigenvalues of $\Gamma$ corresponding to linear characters $\chi_1, \ldots, \chi_4$ of $G$ are as follows:

$$
\lambda_1 = \sum_{s \in S} \chi_1(s) = |S| = |S_2| + |S_{4_1}| + |S_{4_2}| + |S_5| = |S_2| + |S_4| + |S_5|
$$
$$
\lambda_2 = \sum_{s \in S} \chi_2(s) = |S_2| - |S_{4_1}| - |S_{4_2}| + |S_5| = |S_2| - |S_4| + |S_5|
$$
$$
\lambda_3 = \sum_{s \in S} \chi_3(s) = -|S_2| + |S_5|
$$
$$
\lambda_4 = \sum_{s \in S} \chi_4(s) = -|S_2| + |S_5|
$$

So $\lambda_3 = \lambda_4$. Since $\Gamma$ is a connected $|S|$-regular graph, it follows from [7, Proposition 1.1.2] that $\mu_1 := \lambda_1 = |S|$ is the unique largest eigenvalue of $\Gamma$. Thus 1 occurs precisely one time in the multiplicity type of spectrum of $\Gamma$. By Theorem 2.4 there is a multiset $\Lambda := \{\lambda_5, \ldots, \lambda_{16}\}$ of 16 eigenvalues of $\Gamma$ corresponding to the character $\psi$ of degree 4 such that the multiplicity of each $\lambda \in \Lambda$ in $\Lambda$ is 4, 8, 12 or 16; note that the latter multiplicity is counted in $\Lambda$ and not in the spectrum of $\Gamma$. If $\lambda_2 \neq \lambda_3$, it follows from the above information that $\Gamma$ has the spectrum of the first type; otherwise the spectrum of $\Gamma$ is of the second type. This completes the proof.

Proof of Theorem 1.5. It follows from [13, Theorem 1.1] that the group $G$ is not a CI-group.

To prove that $G$ is a BI-group, by Proposition 2.3 it is enough to show that $M_\nu^S = M_\nu^T$ for every irreducible character degree $\nu$ of $G$ and for all inverse closed subsets $S$ and $T$ of $G^*$ such that $Cay(G, S) \cong Cay(G, T)$ and $G = \langle S \rangle = \langle T \rangle$.

First note that $\nu \in \{1, 4\}$ and

$$
M_1^S = \{|S|, |S_2| - |S_4| + |S_5|, |S_5| - |S_2|\},
$$
$$
M_4^S = \{-|S_5|\}
$$
and similarly
\[ M^T_i = \{ |T|, |T_2| - |T_4| + |T_5|, |T_5| - |T_2| \}, M^T = \{-|T_5|\}. \]

Since \( \text{Cay}(G, S) \cong \text{Cay}(G, T) \), \( \text{spec}(\text{Cay}(G, S)) = \text{spec}(\text{Cay}(G, T)) \). By Theorem 3.2, it follows that
\[
\begin{align*}
|S_2| + |S_4| + |S_5| &= |S| = |T| = |T_2| + |T_4| + |T_5| \\
|S_2| - |S_4| + |S_5| &= |T_2| - |T_4| + |T_5| \\
-|S_2| + |S_5| &= -|T_2| + |T_5|
\end{align*}
\]

Therefore,
\[ |S_2| = |T_2|, |S_4| = |T_4|, |S_5| = |T_5|, \]
and so \( M^S_i = M^T_i \) for all \( \nu \). This completes the proof. \( \square \)

4. Proof of Theorem 4.1

**Theorem 4.1.** Let \( H = \langle a, b | a^2 = b^3 = 1, a^b = a^2 \rangle \), \( S \subset H^* \), \( S^{-1} = S \), \( H = \langle S \rangle \) and \( \Gamma = \text{Cay}(H, S) \). Then the spectrum of \( \Gamma \) is one of the following four cases:

1. \( \text{spec}(\Gamma) = \{ \mu_1 = |S|, \mu_2 = |S|^{6k+1}, \mu_3 = |S|^{6k+2}, \mu_4 = |S|^{6k+3}, \mu_5 = |S|^{6k+4}, \mu_6 = |S|^{6k+5}, \mu_7 = |S|^{6k+6} \} \),
2. \( \text{spec}(\Gamma) = \{ \mu_1 = |S|, \mu_2 = |S|^{6k+1}, \mu_3 = |S|^{6k+2}, \mu_4 = |S|^{6k+3}, \mu_5 = |S|^{6k+4}, \mu_6 = |S|^{6k+5}, \mu_7 = |S|^{6k+6} \} \),
3. \( \text{spec}(\Gamma) = \{ \mu_1 = |S|, \mu_2 = |S|^{6k+1}, \mu_3 = |S|^{6k+2}, \mu_4 = |S|^{6k+3}, \mu_5 = |S|^{6k+4}, \mu_6 = |S|^{6k+5}, \mu_7 = |S|^{6k+6} \} \),
4. \( \text{spec}(\Gamma) = \{ \mu_1 = |S|, \mu_2 = |S|^{6k+1}, \mu_3 = |S|^{6k+2}, \mu_4 = |S|^{6k+3}, \mu_5 = |S|^{6k+4}, \mu_6 = |S|^{6k+5}, \mu_7 = |S|^{6k+6} \} \),
5. \( \text{spec}(\Gamma) = \{ \mu_1 = |S|, \mu_2 = |S|^{6k+1}, \mu_3 = |S|^{6k+2}, \mu_4 = |S|^{6k+3}, \mu_5 = |S|^{6k+4}, \mu_6 = |S|^{6k+5}, \mu_7 = |S|^{6k+6} \} \),

where \( k, k', k'', l_1, l_2, n \) are integers in \( \{0, \ldots, 6\} \).

**Proof.** The group \( H \) has 7 conjugacy classes: \( H_1, H_2 \) and \( H_7 \) are the conjugacy classes consisting of all elements of \( H \) of order 1, 2 and 7, respectively; the elements of order \( i \in \{3, 6\} \) of \( H \) are partitioned into 2 conjugacy classes denoted by \( H_1 \) and \( H_2 \), and if \( x \) is an element of order \( i \), then \( x \) and \( x^{-1} \) are not conjugate. Let \( S_i = S \cap H_i \) for \( i = 3, 6 \) and \( j = 1, 2 \), so \( S_i = S_{ik} \) for \( i = 3, 6 \). According to Definition 3.3,
\[
|S| = |S_2| + |S_{3a}| + |S_{3b}| + |S_{6a}| + |S_{6b}| + |S_7|,
\]
and note that \( S_3 = S_{3a} \cup S_{3b} \), \( |S_{3a}| = |S_{1a}| \), \( S_6 = S_{6a} \cup S_{6b} \) and \( |S_{6a}| = |S_{6b}| \). The character table of \( H \) is:
where \( \zeta_l = e^{2\pi i / l} \) (\( l = 3, 6 \)).

Now by Theorem 2.1 all eigenvalues of \( \Gamma \) corresponding to irreducible characters \( \chi_1, \ldots, \chi_6 \) of \( \mathcal{H} \) are as follows:

\[
\begin{align*}
\lambda_1 &= \sum_{s \in S} \chi_1(s) = |S| - |S_2| + |S_3| - |S_4| - |S_5| + |S_6| + |S_7| \\
\lambda_2 &= \sum_{s \in S} \chi_2(s) = -|S_2| - |S_3| - |S_4| + |S_5| - |S_6| + |S_7| \\
\lambda_3 &= \sum_{s \in S} \chi_3(s) = |S_2| - |S_3| - |S_4| + |S_5| - |S_6| + |S_7| \\
\lambda_4 &= \sum_{s \in S} \chi_4(s) = |S_2| - |S_3| + |S_5| + |S_7| \\
\lambda_5 &= \sum_{s \in S} \chi_5(s) = -|S_2| + |S_3| + |S_6| + |S_7| \\
\lambda_6 &= \sum_{s \in S} \chi_6(s) = -|S_2| - |S_3| + |S_6| + |S_7|
\end{align*}
\]

So \( \lambda_3 = \lambda_4 \) and \( \lambda_5 = \lambda_6 \). Since \( \Gamma \) is a connected \( |S| \)-regular graph, it follows from [7 Proposition 1.1.2] that \( \mu_1 := \lambda_1 = |S| \) is the unique largest eigenvalue of \( \Gamma \).

By Theorem 2.1 there exists a multiset \( \Lambda := \{ \lambda_2, \ldots, \lambda_6 \} \) of 36 eigenvalues of \( \Gamma \) corresponding to the character \( \psi \) of degree 6 such that the multiplicity of each \( \lambda \in \Lambda \) is a multiple of \( 6 \); note that the latter multiplicity is counted in \( \Lambda \) and not in the spectrum of \( \Gamma \).

Now, we must distinguish five cases to obtain the multiplicities of eigenvalues \( \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \):

**Case 1.** \( \lambda_2, \lambda_3 \) and \( \lambda_5 \) are pairwise distinct. Then \( \Gamma \) has the spectrum of the first type (1).

**Case 2.** \( \lambda_2 \neq \lambda_3 = \lambda_5 \). Then \( \Gamma \) has the spectrum of the second type (2).

**Case 3.** \( \lambda_2 = \lambda_3 \neq \lambda_5 \). Then \( \Gamma \) has the spectrum of the third type (3).

**Case 4.** \( \lambda_2 = \lambda_5 \neq \lambda_3 \). Then \( \Gamma \) has the spectrum of the third type (4).

**Case 5.** \( \lambda_2 = \lambda_3 = \lambda_5 \). Then \( \Gamma \) has the spectrum of the fourth type (5).

This completes the proof.

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**Proof of Theorem 1.6** It follows from [13 Theorem 1.1] that the group \( \mathcal{H} \) is not a CI-group.

To prove that \( \mathcal{H} \) is a BI-group, by Proposition 2.5 it is enough to show that \( M^S_{\nu} = M^T_{\nu} \) for every irreducible character degree \( \nu \) of \( \mathcal{H} \) and for all inverse closed subsets \( S \) and \( T \) of \( \mathcal{H}^* \) such that \( \Gamma := \text{Cay}(\mathcal{H}, S) \cong \text{Cay}(\mathcal{H}, T) =: \Gamma' \) and \( \mathcal{H} = \langle S \rangle = \langle T \rangle \).
First note that $\nu \in \{1,6\}$; and by the proof of Theorem 4.1

$$M_1^S = \{|S|, -|S_2| + |S_3| - |S_6| + |S_7|, S_2 - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7|, S_2 - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7|\}$$

and $M_2^S = \{-|S_7|\}$ and similarly

$$M_1^T = \{|T|, -|T_2| + |T_3| + |T_6| + |T_7|, T_2 - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7|, -|T_2| + \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7|\}$$

and $M_2^T = \{-|T_7|\}$. Since $\Gamma \cong \Gamma'$, $\text{spec}(\Gamma) = \text{spec}(\Gamma')$. By Theorem 4.1 the following cases may happen:

(i) $\Gamma$ and $\Gamma'$ have the same spectrum as type (1) in Theorem 4.1

(ii) $\Gamma$ has and $\Gamma'$ has the same spectrum as type (2);

(iii) $\Gamma$ and $\Gamma'$ have the same spectrum as type (3);

(iv) $\Gamma$ and $\Gamma'$ have the same spectrum as type (4);

(v) $\Gamma$ and $\Gamma'$ have the same spectrum as type (5);

(vi) $\text{spec}(\Gamma)$ is of type (3) and $\text{spec}(\Gamma')$ is of type (4);

By Theorem 4.1, $M_1^S$ is the set of eigenvalues of $\Gamma$ having the multiplicity $6k+\ell$ for some non-negative integer $k$ and positive integer $\ell \leq 5$. The similar statement is true for $M_1^T$ and so $M_1^S = M_1^T$ in all cases (i)-(vi).

Now it remains to prove that $M_2^S = M_2^T$ or equivalently $|S_7| = |T_7|$. With the notations of Theorem 4.1 let $\mu_1 := \mu_1(\Gamma)$ and $\mu'_1 := \mu_1(\Gamma')$.

According to the above cases (i)-(vi), we distinguish the following cases:

(i) $\Gamma$ and $\Gamma'$ have the same spectrum as type (1) in Theorem 4.1

$$\begin{cases}
\mu_1 = |S| = |S_2| + |S_3| + |S_6| + |S_7| = |T_2| + |T_3| + |T_6| + |T_7| = |T| = \mu'_1 \\
\mu_2 = -|S_2| + |S_3| - |S_6| + |S_7| = -|T_2| + |T_3| - |T_6| + |T_7| = \mu'_2 \\
\mu_3 = |S_2| - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7| = |T_2| - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7| = \mu'_3 \\
\mu_5 = -|S_2| - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7| = -|T_2| - \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7| = \mu'_5,
\end{cases}$$

or

$$\begin{cases}
\mu_1 = |S| = |S_2| + |S_3| + |S_6| + |S_7| = |T_2| + |T_3| + |T_6| + |T_7| = |T| = \mu'_1 \\
\mu_2 = -|S_2| + |S_3| - |S_6| + |S_7| = -|T_2| + |T_3| - |T_6| + |T_7| = \mu'_2 \\
\mu_3 = |S_2| - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7| = |T_2| - \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7| = \mu'_3 \\
\mu_5 = -|S_2| - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7| = -|T_2| - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7| = \mu'_5.
\end{cases}$$

If (I) happens, then as the matrix

$$\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1/2 & -1/2 & 1 \\
-1 & -1/2 & 1/2 & 1
\end{bmatrix}
$$

is invertible then $|S_7| = |T_7|$.

If (II) happens then

$$\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1/2 & -1/2 & 1 \\
-1 & -1/2 & 1/2 & 1
\end{bmatrix}
\begin{bmatrix}
|S_2| \\
|S_3| \\
|S_6| \\
|S_7|
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 \\
-1 & -1/2 & 1/2 & 1 \\
1 & -1/2 & -1/2 & 1
\end{bmatrix}
\begin{bmatrix}
|T_2| \\
|T_3| \\
|T_6| \\
|T_7|
\end{bmatrix}
$$

so

$$\begin{bmatrix}
|S_2| \\
|S_3| \\
|S_6| \\
|S_7|
\end{bmatrix}
= 
\begin{bmatrix}
-1/3 & 0 & 2/3 & 0 \\
0 & 1 & 0 & 0 \\
4/3 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
|T_2| \\
|T_3| \\
|T_6| \\
|T_7|
\end{bmatrix}.$$
(ii) \( \Gamma \) and \( \Gamma' \) have the same spectrum as type (2) in Theorem 4.1

\[
\begin{align*}
\mu_1 &= |S| = |S_2| + |S_3| + |S_6| + |S_7| = |T_2| + |T_3| + |T_6| + |T_7| = |T| = \mu_1' \\
\mu_2 &= -|S_2| + |S_3| - |S_6| + |S_7| = -|T_2| + |T_3| - |T_6| + |T_7| = \mu_2' \\
\mu_3 &= |S_2| - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7| = |T_2| - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7| = \mu_3' \\
\mu_5 &= -|S_2| - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7| = -|T_2| - \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7| = \mu_5'
\end{align*}
\]

\( \mu_3 = \mu_5 \)

(iii) \( \Gamma \) and \( \Gamma' \) have the same spectrum as type (3) in Theorem 4.1

\[
\begin{align*}
\mu_1 &= |S| = |S_2| + |S_3| + |S_6| + |S_7| = |T_2| + |T_3| + |T_6| + |T_7| = |T| = \mu_1' \\
\mu_2 &= -|S_2| + |S_3| - |S_6| + |S_7| = -|T_2| + |T_3| - |T_6| + |T_7| = \mu_2' \\
\mu_3 &= |S_2| - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7| = |T_2| - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7| = \mu_3' \\
\mu_5 &= -|S_2| - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7| = -|T_2| - \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7| = \mu_5'
\end{align*}
\]

\( \mu_2 = \mu_3 \)

(iv) \( \Gamma \) and \( \Gamma' \) have the same spectrum as type (4) in Theorem 4.1

\[
\begin{align*}
\mu_1 &= |S| = |S_2| + |S_3| + |S_6| + |S_7| = |T_2| + |T_3| + |T_6| + |T_7| = |T| = \mu_1' \\
\mu_2 &= -|S_2| + |S_3| - |S_6| + |S_7| = -|T_2| + |T_3| - |T_6| + |T_7| = \mu_2' \\
\mu_3 &= |S_2| - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7| = |T_2| - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7| = \mu_3' \\
\mu_5 &= -|S_2| - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7| = -|T_2| - \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7| = \mu_5'
\end{align*}
\]

\( \mu_2 = \mu_5 \)

(v) \( \Gamma \) and \( \Gamma' \) have the same spectrum as type (5) in Theorem 4.1

\[
\begin{align*}
\mu_1 &= |S| = |S_2| + |S_3| + |S_6| + |S_7| = |T_2| + |T_3| + |T_6| + |T_7| = |T| = \mu_1' \\
\mu_2 &= -|S_2| + |S_3| - |S_6| + |S_7| = -|T_2| + |T_3| - |T_6| + |T_7| = \mu_2' \\
\mu_3 &= |S_2| - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7| = |T_2| - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7| = \mu_3' \\
\mu_5 &= -|S_2| - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7| = -|T_2| - \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7| = \mu_5'
\end{align*}
\]

\( \mu_2 \equiv \mu_3 \equiv \mu_5 \)

(vi) \( \Gamma \) has the spectrum as type (3) and \( \Gamma' \) has the same spectrum but as type (4) in Theorem 4.1, conversely is true if \( \Gamma \) has the spectrum as type (4) and \( \Gamma' \) has the same spectrum but as type (3):

\[
\begin{align*}
\mu_1 &= |S| = |S_2| + |S_3| + |S_6| + |S_7| = |T_2| + |T_3| + |T_6| + |T_7| = |T| = \mu_1' \\
\mu_2 &= -|S_2| + |S_3| - |S_6| + |S_7| = -|T_2| + |T_3| - |T_6| + |T_7| = \mu_2' \\
\mu_3 &= |S_2| - \frac{1}{2}|S_3| - \frac{1}{2}|S_6| + |S_7| = |T_2| - \frac{1}{2}|T_3| - \frac{1}{2}|T_6| + |T_7| = \mu_3' \\
\mu_5 &= -|S_2| - \frac{1}{2}|S_3| + \frac{1}{2}|S_6| + |S_7| = -|T_2| - \frac{1}{2}|T_3| + \frac{1}{2}|T_6| + |T_7| = \mu_5'
\end{align*}
\]

\( \mu_2 \equiv \mu_3 \) or \( \mu_2 = \mu_5 \)
Then
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 \\
1 & -1/2 & -1/2 & 1 \\
-1 & -1/2 & 1/2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
|S_2| \\
|S_3| \\
|S_6| \\
|S_7| \\
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & 1 \\
-1 & -1/2 & 1/2 & 1 \\
1 & -1/2 & -1/2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
|T_2| \\
|T_3| \\
|T_6| \\
|T_7| \\
\end{bmatrix}
\]

so
\[
\begin{bmatrix}
|S_2| \\
|S_3| \\
|S_6| \\
|S_7| \\
\end{bmatrix} = 
\begin{bmatrix}
-1/3 & 0 & 2/3 & 0 \\
0 & 1 & 0 & 0 \\
4/3 & 0 & 1/3 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

Therefore, in all of cases one may conclude that \(|S_7| = |T_7|\); this completes the proof. \(\square\)

Remark 4.2. It can be proved that the cases (i)-II and (vi) will not happen in the proof of Theorem 1.6.

5. Proof of Theorem 1.8

We have used the following results to prove that some of groups of order at most 30 is not a BI-group.

Theorem 5.1. Let \(G\) be a non-abelian group. Suppose that there exist elements \(h \in G \setminus G'\) and \(k \in G'\) of the same order. Assume further that \(\lambda(h) = \lambda(h^{-1})\) for all linear characters \(\lambda\) of \(G\) (e.g. \(h\) is conjugate to \(h^{-1}\) or all linear characters of \(G\) are real on \(h\)). Then \(G\) is not a BI-group.

Proof. Let
\[
S = \{h, h^{-1}\} \text{ and } T = \{k, k^{-1}\}.
\]
Since \(o(h) = o(k), \text{Cay}(G, S) \cong \text{Cay}(G, T)\). Then
\[
M_T = \begin{cases}
\{1\} & o(k) = 2 \\
\{2\} & o(k) > 2
\end{cases},
\]
since \(T \subseteq G' = \bigcap\{\ker(\lambda)|\lambda \in \text{Irr}(G), \lambda(1) = 1\}\) by [11] Corollary 2.23. On the other hand \(S \not\subseteq G'\) implies that there exists a linear character \(\lambda \in \text{Irr}(G)\) such that \(\lambda(h) \neq 1\). It follows that \(M_T^S \neq M_T^T\). This completes the proof. \(\square\)

Theorem 5.2. Let \(G = H \ltimes K, \quad H\) is an abelian group and \(K\) is a non-abelian group. Assume that non-trivial elements \(h \in H\) and \(k \in K'\) have the same order and all linear characters of \(G\) has real value on \(h\). Then \(G\) is not a BI-group.

Proof. Note that \(K' \subset G'\) and \(G' \cap H = 1\). By the proof of Theorem 5.1 for \(S = \{(h, 1), (h, 1)^{-1}\}\) and \(T = \{(1, k), (1, k)^{-1}\}\) we have
\[
M_T^S = \begin{cases}
\{1\} & o(k) = 2 \\
\{2\} & o(k) > 2
\end{cases},
\]
and
\[
M_T^T = \begin{cases}
\{\lambda(1)|\lambda \in \text{Irr}(G), \lambda(1) = 1\} & o(h) = 2 \\
\{2\lambda(h), 1)|\lambda \in \text{Irr}(G), \lambda(1) = 1\} & o(h) > 2
\end{cases},
\]
since \(S \not\subseteq G'\) and \(T \subseteq G'\) and there is a linear character of \(G\) which is real on \(h\). So, \(M_T^S \neq M_T^T\). This completes the proof. \(\square\)

Theorem 5.3. Let \(G = H \times K, \quad H\) is an abelian group and \(K\) is a non-abelian group. Assume that non-trivial elements \(h \in H\) and \(k \in K'\) have the same order and all irreducible characters of \(H\) have real value on \(h\). Then \(G\) is not a BI-group.
TABLE 3. Non-abelian CI and BI groups of order at most 30. A reference is given for each item to establish if the group is CI or BI or not. Y. means Yes; N. means No. The reference [18] shows that we have used the above code written in GAP to determine the group is BI or not.

| ID  | Group                  | CI | BI |
|-----|------------------------|----|----|
| [6,1]| $S_3$                  | Y  | Y  |
| [8,4]| $Q_8$                  | Y  | Y  |
| [12,1]| $C_3 \times C_3$       | Y  | Y  |
| [12,4]| $D_{12}$              | N  | N  |

| ID  | Group                  | CI | BI |
|-----|------------------------|----|----|
| [10,1]| $D_{10}$              | Y  | Y  |
| [12,3]| $A_4$                 | Y  | Y  |
| [14,1]| $D_{14}$              | Y  | Y  |
| [18,3]| $C_3 \times S_3$      | Y  | Y  |
| [20,1]| $C_4 \times C_5$      | Y  | Y  |
| [20,4]| $D_{20}$              | N  | N  |
| [22,1]| $C_3 \times C_5$      | Y  | Y  |
| [23,3]| $SL(2, 3)$            | N  | N  |
| [24,5]| $C_4 \times S_3$      | N  | N  |
| [24,14]| $C_2 \times S_3$     | N  | N  |
| [24,12]| $S_4$                | Y  | Y  |
| [24,13]| $C_2 \times A_4$     | N  | N  |
| [27,3]| $C_3 \times (C_3 \times C_3)$ | N  | N  |
| [28,1]| $C_4 \times C_7$      | Y  | Y  |
| [30,1]| $C_5 \times S_3$      | Y  | Y  |
| [30,2]| $C_3 \times D_{10}$   | Y  | Y  |

Proof. Since the irreducible characters of $H$ on $h$ have the same values to corresponding irreducible characters of $G$ on $(h, 1)$, the proof is similar to the proof of Theorem 5.2.

Proposition 5.4. The group $SL(2, 3)$ is not a BI-group.

Proof. Let $S$ and $T$ as the following:

$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$

$T = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$

According to the character table of $G := SL(2, 3)$ it is easy to see $M_S^T = M_T^S$, while $Cay(G, S) \cong Cay(G, T)$, by using GAP [18]. So $SL(2, 3)$ is not a BI-group.

Theorem 5.5. Let $P$ be a Sylow $p$-subgroup of a CI-group. Then $P$ is either an elementary abelian group, or a cyclic group of order $p^l$, $(l, p \leq 3)$, or the quaternion group $Q_8$.

Theorem 5.6. Proposition 3.5] Dihedral group $D_{2n}$ of order $2n$ is not a BI-group for even $n$.

Proposition 5.7. All non-abelian groups of order 16 are not BI-groups.

Proof. It follows from to Theorems 5.3, 5.2 and 5.6.

Proof of Theorem 5.8 The proof follows from the results which are mentioned in Table 4.
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References

[1] A. Abdollahi and M. Zallaghi, Character sums for Cayley graphs, Comm. Algebra, 43 (12) (2015) 5159-5167.
[2] B. Alspach, Isomorphism and Cayley graphs on abelian groups, NATO Sci. Peace Secur. Ser. C Graph Symm., 497 (1997) 1-22.
[3] L. Babai, Spectra of Cayley graphs, J. Combin. Theory Ser. B, 27 (1979) 180-189.
[4] L. Babai, Isomorphism problem for a class of point-symmetric structures, Acta Math. Hungar., 29 (1977) 329-336.
[5] L. Babai and P. Frankl, Isomorphisms of Cayley graphs I, Colloq. Math. Soc. J. Bolyai, 18 (1976) 35-52.
[6] M. Conder and C. H. Li, On isomorphisms of finite Cayley graphs, European J. Combin., 19 (1998) 911-919.
[7] D. Cvetković, P. Rowlinson and S. Simić, An introduction to the theory of graph spectra, Cambridge, UK: Cambridge University Press, 2010.
[8] E. Dobson, Isomorphism problem for metacirculant graphs of order a product of distinct primes, Canad. J. Math., 50 (1998) 1176-1188.
[9] E. Dobson, On the Cayley isomorphism problem, Discrete Math., 247 (2002) 107-116.
[10] E. Dobson, J. Morris and P. Spiga, Further restrictions on the structure of finite DCI-groups: an addendum, J. Algebraic Combin., 42 (2015) 959-969.
[11] I. Martin Isaacs, Character theory of finite groups, Acad. Press, New York, 1976.
[12] C. H. Li, On isomorphisms of finite Cayley graphs—a survey, Discrete Math., 256 (2002) 301-334.
[13] C. H. Li, Z. P. Lu and P. P. Pálfy, Further restrictions on the structure of finite CI-groups, J. Algebraic Combin., 26 (2007) 161-181.
[14] L. Lovász, Spectra of graphs with transitive groups, Period. Math. Hungar., 6 (1975) 191-196.
[15] G. F. Royle, Constructive enumeration of graphs, Ph.D. thesis, University of Western Australia, 1987.
[16] G. Somlai, The Cayley isomorphism property for groups of order 8p, Ars Math. Contemp., 8 (2015) 433-444.
[17] P. Spiga, On the Cayley isomorphism problem for a digraph with 24 vertices, Ars Math. Contemp., 1 (2008) no. 1, 38-43.
[18] The GAP Group, GAP Groups, Algorithms, and Programming, Version 4.4.12 (2008), [http://www.gap-system.org].

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