Projective structures on a hyperbolic 3–orbifold

Joan Porti and Stephan Tillmann

Abstract We compute and analyse the moduli space of those real projective structures on a hyperbolic 3–orbifold that are modelled on a single ideal tetrahedron in projective space. Parameterisations are given in terms of classical invariants, traces, and geometric invariants, cross ratios.

AMS Classification 57M25, 57N10

Keywords 3–orbifold, projective geometry

1 Introduction

This note studies certain real projective structures on a 3-dimensional orbifold, \( O \), which is obtained by taking the one tetrahedron triangulation with two vertices of \( S^3 \), deleting the vertices and modelling the edge neighbourhoods on \( \mathbb{R}^3/\langle r \rangle \), where \( r \) is a rotation by 120° (see Figure 1). This orbifold supports a unique complete hyperbolic structure; this has two Euclidean (3, 3, 3)–pillow case cusps and is of finite volume.

In [3], the philosophy was put forward that strictly convex projective manifolds behave like hyperbolic manifolds sans Mostow rigidity. This paper computes a moduli space of projective structures on \( O \) that are modelled on an ideal tetrahedron. This moduli space, denoted \( \text{Mod}(O) \), turns out to be the union of two disjoint smooth, open 2–dimensional discs, \( \text{Mod}(O) = D_0 \cup D_1 \). We obtain two parameterisations of \( \text{Mod}(O) \): one in terms of algebraic invariants, traces, and one in terms of geometric invariants, cross-ratios.

The complete hyperbolic structure on \( O \) is singled out as the only structure on \( D_0 \) having standard cusps, whilst the remaining structures on \( D_0 \) all have generalised cusps. It is also characterised as the unique fixed point of a natural involution on \( D_0 \). The problem to decide which of the structures on \( D_0 \) are properly convex appears to be difficult by elementary means, but is completely solved by the theoretical results of Cooper, Long and Tillmann [3] [4]; the answer is all of them. This problem has also motivated some of Choi’s work [2]. To complete the discussion of the moduli space, we show that none of the structures carried by \( D_1 \) are properly or strictly convex, and we show that the unique fixed point of a natural involution on \( D_1 \) corresponds to an action of the alternating group Alt(5) on a tesselation of \( \mathbb{R}P^3 \) by fifteen 3–simplices. This lifts to an action of the binary icosahedral group on the 3–sphere.

Acknowledgements. Research of the first author is supported by FEDER-MEC (grant number PGC2018-095998-B-I00). Research of the second author is supported by an Australian Research Council Future Fellowship (project number FT170100316).
2 Projective structures modelled on triangulations

In $n$–dimensional real projective space, $n$–simplices are overly congruent. Given any two $n$–simplices, there is a projective transformation taking one to the other. Given an $n$–simplex, there is a $n$–dimensional family of projective transformations taking it to itself whilst fixing each of its vertices. The following notions can be defined in all dimensions, but we restrict to the case $n = 3$.

The space $\mathbb{R}P^3$ will be viewed as the set of 1–dimensional vector subspaces of $\mathbb{R}^4$ with the induced topology. The set of projective transformations, $\text{PGL}(4, \mathbb{R})$, then corresponds to the quotient of $\text{GL}(4, \mathbb{R})$ by its centre, the group of all non-zero multiples of the identity matrix. If $\Delta$ is the 3–simplex with vertices corresponding to the standard unit vectors $e_1, \ldots, e_4$ in $\mathbb{R}^4$ and containing $\sum e_i$ in its interior, then the family of projective transformations stabilising $\Delta$ and fixing its vertices corresponds to the set of diagonal matrices in $\text{GL}(4, \mathbb{R})$ having all entries positive or all entries negative. This gives a 3–dimensional family of projective transformations.

Let $M$ be an arbitrary, ideally triangulated 3–orbifold with the property that the ideal triangulation restricts to an ideal triangulation of the (possibly empty) singular locus. A real projective structure on $M$ is a pair $(\text{dev}, \rho)$, where $\text{dev}: \tilde{M} \to \mathbb{R}P^3$ is a locally injective map and $\rho: \pi_1(M) \to \text{PGL}(4, \mathbb{R})$ is a representation of the orbifold fundamental group which makes $\text{dev}$ equivariant. Since $\tilde{M}$ is non-compact, some of its ends may be homeomorphic to $\mathbb{R}^2 \times (0, 1)$. We therefore make some additional assumptions.

1. **Structure is modelled on projective simplices:** Let $\hat{M}$ be the end-compactification of $M$, and $\hat{\tilde{M}}$ be the end compactification of $\tilde{M}$. We may view $M \subset \hat{M}$ and $\tilde{M} \subset \hat{\tilde{M}}$, and the complements consist of discrete sets of points. Lift the ideal triangulation of $\tilde{M}$ to an ideal triangulation of $\hat{\tilde{M}}$, then each ideal 3–simplex in $\hat{\tilde{M}}$ corresponds to a 3–simplex in $\hat{M}$. The induced map $\hat{\text{dev}}: \hat{\tilde{M}} \to \mathbb{R}P^3$ is simplicial.

   **We assume that** $\text{dev}: \tilde{M} \to \mathbb{R}P^3$ **extends to a continuous, equivariant map** $\hat{\text{dev}}: \hat{\tilde{M}} \to \mathbb{R}P^3$.

   In this case, $\hat{\text{dev}}$ is equivariantly homotopic to a map $\hat{\text{dev}}_0$ with the property that $\hat{\text{dev}}_0(\Delta)$ is a projective simplex (of dimension 0, 1, 2 or 3) for every simplex $\Delta$ in $\hat{M}$. In particular, the map is possibly not locally injective.

2. **Structure is non-collapsed:** Let $\Delta$ a 3–simplex in $\hat{M}$.

   **We assume that the images of the vertices of $\Delta$ under $\hat{\text{dev}}$ are in general position.**

   In this case, the above homotopy can be assumed to fix $\hat{M}$, and $\hat{\text{dev}}_0$ maps any simplex to a simplex of the same dimension. Moreover, it may be assumed to do so by a linear map and in particular, it is locally injective at interior points of 3–simplices.

**Definition 1** Let $M$ be an ideally triangulated 3–orbifold with the property that the ideal triangulation restricts to an ideal triangulation of the singular locus. A real projective structure modelled on the triangulation is a real projective structure $(\text{dev}, \rho)$ on $M$ which is modelled on projective simplices and non-collapsed. If the triangulation consists of a single 3–simplex, we will also say that the structure is modelled on a 3–simplex.
3 The moduli space

Figure 1: To obtain $O$, first glue the faces meeting along one of the edges with cone angle $2\pi/3$ to obtain a spindle, and then identify the boundary discs of the spindle. The result is $S^3$ minus two points, with the labelled graph (minus its vertices) as the singular locus. The hyperbolic structure can be obtained by identifying the ideal 3–simplex with an ideal hyperbolic 3–simplex with shape parameter $\frac{1}{3} + \sqrt{-3}/6$. The fundamental group of $O$ admits, up to conjugation, exactly two irreducible representations into $\text{SL}(2, \mathbb{C})$. They are complex conjugates and correspond to holonomies for the hyperbolic structure.

**Theorem 2** The set of real projective structures on $O$ modelled on a 3–simplex is parameterised by the set, $X$, of all $(w, x, y, z) \in \mathbb{R}^4$ subject to the following two equations:

\begin{align}
  w + x + y + z &= 3 + wy, \\
  wy &= zx.
\end{align}

The structures corresponding to any two distinct points of $X$ are neither isotopic nor projectively equivalent. Moreover, $X$ is a disjoint union of two smooth open discs, $D_0$ and $D_1$. The involution $(w, x, y, z) \rightarrow (y, z, w, x)$ on $X$ has exactly two fixed points, $(3, 3, 3, 3) \in D_0$, and $(1, 1, 1, 1) \in D_1$. It hence restricts to an involution on each of the components. The fixed point $(3, 3, 3, 3)$ corresponds to the complete hyperbolic structure on $O$. Moreover, $D_0$ parameterises properly convex projective structures on $O$. The fixed point $(1, 1, 1, 1)$ corresponds to an action of the alternating group $\text{Alt}(5)$ on $\mathbb{R}P^3$. Moreover, no point on $D_1$ corresponds to a properly or strictly convex projective structure on $O$.

**Proof** Denote the ideal 3–simplex by $[v_1, v_2, v_3, v_4]$. The face pairings are $\alpha [v_1, v_2, v_3] = [v_1, v_2, v_4]$ and $\beta [v_2, v_3, v_4] = [v_1, v_3, v_4]$. The orbifold fundamental group is generated by the face pairings, and we have the following presentation:

\[ \pi_1^{\text{orb}}(O) = \langle \alpha, \beta : \alpha^3 = \beta^3 = (\alpha \beta \alpha^{-1} \beta^{-1})^3 = 1 \rangle. \]

(3.3)

To determine all projective structures of $O$ modelled on a 3–simplex up to projective equivalence, it suffices to fix a projective 3–simplex, $\Delta$, and to determine all representations $\rho$ of $\pi_1(\Delta)$ with $\rho(\alpha)$ and $\rho(\beta)$ as the corresponding face pairings. Choosing $\Delta = [e_1, e_2, e_3, e_4]$, the most general form of lifts of the face pairings is:

\[ A = \begin{pmatrix} s_1 & 0 & 0 & a_1 \\ 0 & s_2 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & s_4 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & t_1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & t_3 & 0 \\ b_4 & 0 & 0 & t_4 \end{pmatrix}, \]

(3.4)
subject to $s_1s_2s_4a_3 \neq 0$ and $t_1t_3t_4b_2 \neq 0$.

Since the above does not take division by the centre into account, the equation $A^3 = cI_4$ for $c \neq 0$ is projectively equivalent to $A^3 = I_4$, since the equation $c^3 = 1$ always has a non-zero real root. Similarly for $B$. This gives:

$$A = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & -a_3^{-1} & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -b_2^{-1} & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & 1 & 0 \\ b_4 & 0 & 0 & 1 \end{pmatrix}, \quad (3.5)$$

subject to $a_3b_2 \neq 0$. Note that both matrices are elements of $\text{SL}(4, \mathbb{R})$. Since we are interested in representations up to conjugacy, one may conjugate the above to give:

$$A = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ b_3 & 0 & 1 & 0 \\ b_4 & 0 & 0 & 1 \end{pmatrix} \quad (3.6)$$

It now remains to analyse $(ABA^{-1}B^{-1})^3 = cI_4$. With (3.6), one first notes that $c = 1$. In particular, any representation of $\pi_1(O)$ into $\text{PGL}(4, \mathbb{R})$ lifts to a representation into $\text{SL}(4, \mathbb{R})$. One obtains the following cases:

**Case 1:** $a_1 = a_2 = b_3 = b_4 = 0$. In this case there is a single representation which in fact satisfies $A^3 = B^3 = ABA^{-1}B^{-1} = I_4$. The corresponding developing map is not locally injective, and hence that there is no corresponding real projective structure.

**Case 2:** One obtains a single equation:

$$(a_1 + a_2)(b_3 + b_4) = 3 + a_1a_2b_3b_4. \quad (3.7)$$

Analysis of which representations are conjugate yields that pairs $A, B$ and $A', B'$ give conjugate representations in $\text{PGL}(4, \mathbb{R})$ if and only if they are conjugate by

$$M = \text{diag}(m, m, m^{-1}, m^{-1})$$

for some $m \neq 0$. The effect on the quadruples is:

$$(a'_1, a'_2, b'_3, b'_4) = (ma_1, ma_2, m^{-1}b_3, m^{-1}b_4). \quad (3.8)$$

Note that $M$ corresponds to a projective transformation stabilising any subsimplex of $\Delta$. Since no face pairing is a reflection, it remains to check local injectivity at the edges of the triangulation. It follows from inspection that local injectivity at the axis of $A$ is equivalent to not both $a_1$ and $a_2$ to be contained in $(-\infty, 0]$, and local injectivity at the axis of $B$ is equivalent to not both $b_1$ and $b_2$ to be contained in $(-\infty, 0]$. Thus, the corresponding pair of equivariant map and representation, $(\text{dev}, \rho)$, can be replaced by a projective structure given by $(M \circ \text{dev}, M \circ \rho \circ M^{-1})$ with $M = \text{diag}(m, m, m^{-1}, m^{-1})$ that is locally injective at both the axes of $A$ and $B$ unless $(a_1, a_2 \leq 0$ and $b_1, b_2 \geq 0)$ or $(a_1, a_2 \geq 0$ and $b_1, b_2 \leq 0)$. But equation (3.7) has no solutions of this form. Moreover, injectivity around the axis of the commutator $ABA^{-1}B^{-1}$ follows from this. Hence every representation found generates (up to conjugacy) a real projective structure on $O$.  


The coordinates given in the statement of the theorem can be expressed in terms of classical invariants of the chosen lift of \( \rho \):

\[
\begin{align*}
  w &= a_1 b_4 = 2 + \text{tr} \ AB, \\
  x &= a_1 b_3 = 2 + \text{tr} \ A^{-1} B, \\
  y &= a_2 b_3 = 2 + \text{tr} \ A^{-1} B^{-1}, \\
  z &= a_2 b_4 = 2 + \text{tr} \ AB^{-1},
\end{align*}
\]

and (3.7) can be expressed in terms of these, giving the first equation given in (3.1). The second arises from \( w y = a_1 a_2 b_3 b_4 = x z \). It can now be verified that \( X \) corresponds to the quotient of the action of \( \mathbb{R} \setminus \{0\} \) given in (3.8) on the set of all \( \langle a_1, a_2, b_3, b_4 \rangle \) subject to (3.7).

In order to apply the results of [4] on deformations of properly convex structures, we look at the peripheral subgroups of \( \pi^m_b(O) \). There are two conjugacy classes of peripheral subgroups, corresponding to the two ends of \( O \) and represented respectively by the stabilizer of \( v_1 \) and the stabilizer of \( v_4 \). The stabilizer of \( v_1 \) is the group generated by \( \alpha \) and \( \gamma \), and \( \gamma = [\alpha, \beta] \), it is the fundamental group of the link of the end of \( O \) corresponding to \( v_1 \), the 2-orbifold with underlying space a sphere and three cone points of order 3: \( \langle \alpha, \gamma \mid \alpha^3 = \gamma^3 = (\alpha^2 \gamma)^3 = 1 \rangle \). Similarly, for \( v_4 \) the peripheral group is generated by \( \beta \) and \( \gamma \). The maximal torsion-free subgroup of the peripheral group of \( v_1 \) has index 3, it is generated by \( \alpha \gamma \) and \( \gamma \alpha \) and it is isomorphic to \( \mathbb{Z}^2 \). Next we claim that all eigenvalues of \( AC \) are real, for each value of the parameters \( (w, x, y, z) \in X \). For that purpose, we compute the characteristic polynomial of \( AC \):

\[
(\lambda - 1)(\lambda^3 + (-yx - w + 2 x + 2 y - z) \lambda^2 + (zw - 2 w + x + y - 2 z) \lambda - 1)
\]

and we check that the discriminant of its degree three factor is nonnegative. To check that, we write \( w \) and \( z \) in terms of \( x \) and \( y \) from (3.1) and (3.2); this only parameterizes an open dense subset of \( X \), but it is sufficient to determine the non-negativity of the discriminant. With this parameterisation of a subset \( X \), the discriminant is

\[
\frac{(y^2 - 3 y + 3)^2 (x^2 - 3 x + 3)^2 (-y + x)^2 (x^2 y^2 - 3 x^2 y - 3 xy^2 + 3 x^2 + 3 xy + 3 y^2)^2}{(xy - x - y)^6},
\]

which is always nonnegative. As \( \alpha \gamma \) is conjugate to \( \gamma \alpha \), the holonomy of \( \langle \alpha \gamma, \gamma \alpha \rangle \cong \mathbb{Z}^2 \) preserves a flag in \( \mathbb{R}^4 \), for any structure in \( X \). We can argue similarly for \( v_4 \) and the peripheral subgroup generated by \( \beta \) and \( \gamma \). Hence, we may apply Theorem 0.2 of [4] to say that set of properly convex structures is open in \( X \). In addition, it is closed by Theorem 0.14 of [4], because the injectivity radius for the Hilbert metric of the barycenter of the simplex is uniformly bounded below away from zero (by the injectivity radius in the Hilbert metric of the simplex). Thus the set of properly convex structures on \( X \) is open and closed. As the projective structure induced by the hyperbolic metric corresponds to \( (w, x, y, z) = (3, 3, 3, 3) \), every structure in \( D_0 \) is properly convex.

So it remains to show that there is a projective structure in \( D_1 \) that is not properly convex, this is the structure for \( (w, x, y, z) = (1, 1, 1, 1) \) that we detail in the next paragraph.

When \( (w, x, y, z) = (1, 1, 1, 1) \), we take \( a_1 = a_2 = b_3 = b_4 = 1 \). Then both matrices \( A \) and \( B \) preserve the projectivization of the set of five points \( \{\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle, \langle e_4 \rangle, \langle e_1 + e_2 - e_3 - e_4 \rangle\} \) in \( \mathbb{R}P^3 \). As both \( A \) and \( B \) act as cycles of order three on this set, and their
fixed points are disjoint, this is the action of the alternating group Alt(5). It follows that
the group generated by $A$ and $B$ is finite, in particular the projective structure cannot be
properly convex.

\[ \square \]

**Remark 3** In the previous proof, when $(w, x, y, z) = (1, 1, 1, 1)$ we have used that \( \langle A, B \rangle \)
acts as Alt(5) by permuting five points, the vertices of $\Delta$ and a fifth point in the orbit.
It can be shown that those are the vertices of a tessellation of $\mathbb{RP}^3$ defined by the orbits
of $\Delta$. The stabilizer of $\Delta$ has 4 elements and the tessellation consists of 15 tetrahedra
(each tetrahedron shares its vertices with two other tetrahedra of the tessellation). The
quotient of $\mathbb{RP}^3$ by this action of $A_5$ is the spherical orbifold $J \times^*_J J$ in [5].

**Remark 4** The above proof gives an interpretation of the coordinates in terms of traces.
We wish to point out that there is an alternative interpretation in terms of cross ratios
as follows. We look for cross ratios of points in the projective line

\[ l = \langle e_3, e_4 \rangle \]

that contains the edge $e_3e_4$, namely the fixed point set of $B$. We consider the projective
plane

\[ \Pi = \langle e_1, e_2, e_3 \rangle, \]

so that both $\Pi$ and $A(\Pi)$ contain a face of the 3-simplex $\Delta$. The first three points of
$l$ that we consider are the respective intersection of $l$ with $\Pi$, $A(\Pi)$ and $A^2(\Pi)$. Their
homogeneous coordinates are

\[ p_3 = \Pi \cap l = [0 : 0 : 1 : 0], \quad p_4 = A(\Pi) \cap l = [0 : 0 : 0 : 1], \quad \text{and} \quad p_5 = A^2(\Pi) \cap l = [0 : 0 : 1 : 1]. \]

Each coordinate $x$, $y$, $z$ and $w$ appears as a cross ratio of $p_3$, $p_4$ and $p_5$ with one of the
following:

\[ p_x = AB^{-1}A(\Pi) \cap l = [0 : 0 : 1 : 1 - x] \]
\[ p_y = ABA(\Pi) \cap l = [0 : 0 : 1 : 1 - y], \]
\[ p_z = A^{-1}B(\Pi) \cap l = [0 : 0 : 1 - z : 1], \]
\[ p_w = A^{-1}B^{-1}(\Pi) \cap l = [0 : 0 : 1 - w : 1]. \]

Namely:

\[ x = (p_x, p_3; p_5, p_4), \]
\[ y = (p_y, p_3; p_5, p_4), \]
\[ z = (p_z, p_4; p_5, p_3), \]
\[ w = (p_w, p_4; p_5, p_3). \]

**Remark 5** Ballas and Casella [1] study the same example but instead of properly convex
projective structures they consider structures equipped with peripheral flags. Their
deformation space is just a point, in contrast with this setting.
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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain
School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia

Email: porti@mat.uab.cat stephan.tillmann@sydney.edu.au