Modular Bootstrap, Elliptic Points, and Quantum Gravity

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The modular bootstrap program for two-dimensional conformal field theories could be seen as a systematic exploration of the physical consequences of consistency conditions at the elliptic points and at the cusp of their torus partition function. The study at \( \tau = i \), the elliptic point stabilized by the modular inversion \( S \), was initiated by Hellerman, who found a general upper bound for the most relevant scaling dimension \( \Delta \). Likewise, analyticity at \( \tau = i \infty \), the cusp stabilized by the modular translation \( T \), yields an upper bound on the twist gap. Here we study consistency conditions at \( \tau = \exp[2\pi i/3] \), the elliptic point stabilized by \( ST \). We find a much stronger upper bound in the large-\( c \) limit, namely \( \Delta < \frac{c}{4} + 0.092 \), which is very close to the minimal mass threshold of the BTZ black holes in the gravity dual of \( AdS_3/CFT_2 \) correspondence.

INTRODUCTION

It is still an open question as to whether three-dimensional pure gravity exists as a quantum theory. In the case of negative cosmological constant, according to holographic duality\(^1\), solving pure quantum gravity means finding the two-dimensional conformal field theory (CFT) defined on the boundary of the asymptotically anti de Sitter (AdS) spacetime. At the classical level, pure 3d gravity is “trivial” in the sense that there are no gravitational waves; its degrees of freedom correspond to multitrace composites of the stress-tensor which map to the Virasoro module of the identity of the CFT.

As a consequence, if the degrees of freedom were only those of the identity module, pure gravity would not admit a quantum completion. The reason is very simple: the partition function of the CFT on the toroidal boundary is modular invariant, while the character of the identity module, as well as any other single Virasoro character, is not. Thus modular invariance of the boundary theory implies additional degrees of freedom in the bulk.

What is the meaning of modular invariance on the gravity side? In a quantum approach to gravity one expects to sum over different topologies of spacetime with fixed asymptotic boundary conditions. It is widely believed that modular invariance arises from the sum over saddle points of the gravitational path integral\(^2,4\). One such geometry is thermal AdS with periodic Euclidean time. Others correspond to the Euclidean black holes discovered by Bañados, Teitelboim and Zanelli (BTZ) in any three-dimensional gravity with negative cosmological constant\(^5\). Thus BTZ black holes are necessary degrees of freedom for a quantum description of pure gravity in asymptotic \( AdS_3 \) spacetimes. Are these enough? BTZ black holes can exist only above some mass threshold that in the large-\( c \) limit is holographically dual to a CFT primary of scaling dimension \( \Delta_{BTZ} = \frac{c}{12} + O(1) \), where \( c \) is the central charge.

Primary operators with \( \Delta < \Delta_{BTZ} \) can not be interpreted as black holes; they correspond to a new kind of matter\(^20\). Therefore, proving that a primary with \( \Delta < \Delta_{BTZ} \) is necessary for a consistent CFT at the boundary would be sufficient to argue that pure quantum gravity does not exist.

Modular invariance of the partition function constrains the possible spectra of 2d CFTs\(^7\). In particular, as first pointed out by Hellerman\(^8\), for general unitary 2d CFTs with \( c > 1 \) it is possible to find an upper bound for the allowed scaling dimension, \( \Delta_0 \), of the first non-trivial primary. Hellerman rigorously established the inequality \( \Delta_0 < \frac{c}{8} + 0.4737 \). This bound has since been improved numerically as well as analytically in various ways\(^9–15\), in particular using the linear programming method introduced in\(^10\). So far, in the large-\( c \) limit the best analytic bound is\(^13\) \( \Delta_0 \leq c/8.503 \), while the numerical upper bound, obtained by extrapolating large-\( c \) data, is\(^12\) \( \Delta_0 \leq c/9.08 \). Other bounds can be found by assuming complete factorization of the partition function into a holomorphic and an antiholomorphic part\(^14\). In this case, the CFT includes an infinite set of conserved higher spin currents, while it is widely believed that pure gravity is dual to a CFT in which the only conserved currents are those generated by the stress tensor.

\( PSL(2,\mathbb{Z}) \), the modular group of the torus, is the group of linear fractional transformations acting on the modular parameter \( \tau \in H_+ \) (\( H_+ \) is the upper half plane) and generated by the inversion \( S \) and the translation \( T \), satisfying \( S^2 = (ST)^3 = 1 \). One of the reasons why the modular invariance of the partition function \( Z \) is so constraining is that \( PSL(2,\mathbb{Z}) \) does not act freely on \( H_+ \); there are points of \( H_+ \) which are left invariant under the action of a non-trivial subgroup of \( PSL(2,\mathbb{Z}) \). The partition function is a smooth function of \( \tau \) only if it fulfills certain consistency conditions at these special points. \( PSL(2,\mathbb{Z}) \) admits three points of this kind. So far, only two of them have been explored in modular bootstrap studies. In this Letter we fill the gap by writing down the infinite set of linear equations associated with the third point. These give rise to a much stronger upper
bound, namely,

$$\Delta_0 < \frac{c - 1}{12} + \frac{1}{2\sqrt{3}\pi},$$  \hspace{1cm} (1)

valid in the large-$c$ limit, under the same assumptions of [8], with the further specification that the first non-trivial primary must have odd spin.

**CONSTRAINTS FROM MODULAR INVARIANCE**

Modular invariance of the partition function is guaranteed to be a universal property of any physically meaningful theory formulated on a torus, since modular transformations correspond to changes of basis on $\mathbb{Z} + i\mathbb{Z}$, its period lattice, while the physics can not depend on the choice of this basis. It reads

$$Z(\tau, \bar{\tau}) = Z \left( \frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \right) \in PSL(2, \mathbb{Z}),$$  \hspace{1cm} (2)

$\tau$ and $\bar{\tau}$ may be considered as two independent complex variables with $\tau \in H_+$ and $\bar{\tau} \in H_-$, where $H_+$ and $H_-$ are the upper and the lower half planes. $Z(\tau, \bar{\tau})$ becomes the partition function on a torus of modular parameter $\tau$ when $\bar{\tau}$ is chosen to be the complex conjugate of $\tau$, but equation (2) is more general. If the theory on the torus is conformally invariant, we can expand $Z$ as a sum over all contributing states

$$Z(\tau, \bar{\tau}) = \sum_{h,h} q^h \bar{q}^{\bar{h}} e^{2\pi i \tau},$$  \hspace{1cm} (3)

where $\Delta = h + \bar{h}$ are the scaling dimensions of the states and $j = h - \bar{h}$ their spins. For the sake of simplicity we have taken the same central charge $c$ for the left and the right Virasoro algebras. There is a unique vacuum with $h = \bar{h} = 0$. If one further assumes the theory unitary with a discrete spectrum, then equation (3) implies that $Z$ is a holomorphic function on $H_+ \times H_-.

The partition function of any physical theory formulated on a torus is assumed to be a smooth function on the fundamental domain $H_+ / PSL(2, \mathbb{Z})$ (or $H_- / PSL(2, \mathbb{Z})$). We can then apply a property which is at the core of modular bootstrap [2, 8]: a smooth function on the fundamental region lifts to a smooth function on its covering space $H_+$ (or $H_-$) if and only if it satisfies certain consistency conditions on its derivatives at the elliptic points and at the cusp, i.e. the special points of the fundamental region which are invariant under the action of non-trivial subgroups of $PSL(2, \mathbb{Z})$.

As already mentioned in the Introduction, the torus modular group admits three points of this kind. The cusp at $\tau = i\infty$ is stabilized by the subgroup generated by $T : \tau \rightarrow \tau + 1$. The elliptic point at $\tau = i$ is stabilized by $S : \tau \rightarrow -1/\tau$. The elliptic point at $\tau = e^{2\pi i/3}$ is stabilized by $ST : \tau \rightarrow -1/\tau$, the generator of a $\mathbb{Z}_3$ subgroup of $PSL(2, \mathbb{Z})$.

Invariance of the partition function under $T$ implies integer spins, i.e. $j = h - \bar{h} \in \mathbb{Z}$, and analyticity at the cusp $\tau = i\infty$ yields, for $c > 1$, the upper bound $\Delta - |j| \leq c/24$ on the twist gap [11]. Consistency conditions at $\tau = i$ demands [8]

$$(\tau \partial_{\tau})^m (\bar{\tau} \partial_{\bar{\tau}})^n Z(\tau, \bar{\tau})|_{\tau = -i, \bar{\tau} = i} = 0 \text{ for } m + n \text{ odd}. \hspace{1cm} (4)$$

This infinite set of homogeneous linear equations, dubbed modular bootstrap, yields the Hellerman upper bound on the scaling dimensions of the lightest primary operator and its refinements described in the Introduction.

What are the further constraints dictated by the consistency conditions at the point stabilized by $ST$? It suffices to take arbitrary derivatives of the identity

$$Z(\tau, \bar{\tau}) = Z \left( \frac{-1}{\tau + 1}, \frac{-1}{\bar{\tau} + 1} \right), \hspace{1cm} (5)$$

and evaluate them at $\tau = e^{2\pi i/3}$. In order to keep $Z$ real it is convenient to parameterize $\tau$ and $\bar{\tau}$ as

$$\tau = -\frac{1}{2} + \frac{i\beta}{2\pi}, \bar{\tau} = -\frac{1}{2} - \frac{i\bar{\beta}}{2\pi}, \hspace{1cm} (6)$$

where $\beta$ and $\bar{\beta}$ are two independent real variables, so we can set $q = e^{-\beta}$ and $\bar{q} = e^{-\bar{\beta}}$ in (3), thus obtaining a real partition function. Terms with even $j = h - \bar{h}$ are positive while those with odd $j$ are negative. The elliptic point corresponds to $\beta = \bar{\beta} = \sqrt{3}\pi \equiv \beta_c$.

Since $ST$ generates the cyclic group $\mathbb{Z}_3$, the identity [12] tells us that a derivative of arbitrary order in $\beta$ or $\bar{\beta}$ can be expressed at $\tau = e^{2\pi i/3}$ as a linear combination of derivatives of lower order in multiples of 3

$$\left( \partial_{\beta}^m \partial_{\bar{\beta}}^n - \sum_{3j \leq m, 3k \leq n} c_k^e e_j^m \partial_{\beta}^j \partial_{\bar{\beta}}^k \right) Z|_{\beta = \bar{\beta} = \beta_c} = 0, \hspace{1cm} (7)$$

where $c_k^e$ are suitable numerical coefficients. These are the equations to be added to (4) to complete the modular bootstrap program. The first few are, more explicitly,

$$\partial_{\beta}^m Z|_{\beta = \bar{\beta} = \beta_c} = 0, \hspace{1cm} \left( \partial_{\beta}^3 + \frac{2\sqrt{3}}{\pi} \partial_{\beta}^2 \right) Z|_{\beta = \bar{\beta} = \beta_c} = 0, \hspace{1cm} (8)$$

and identical equations for $\bar{\beta}$. One can check these identities by applying them for instance to the modular invariant

$$\sqrt{\tau - \bar{\tau}} \eta(\tau)\eta(-\bar{\tau}) = Z_b^{-1} \hspace{1cm} (9)$$

where $Z_b$ is the partition function of a free boson and $\eta$ is the Dedekind eta function.
In order to obtain useful information on the Virasoro primary spectrum, we have to separate primaries from descendants in the sum of states \( \chi \). \( Z(\tau, \bar{\tau}) \) can be expanded in Virasoro characters. If \( c > 1 \) and the theory is unitary, the modules of the Virasoro algebra are the identity degenerate module \( \chi_0(q) \) and a continuous family of non-degenerate modules \( \chi_A(q) \) labeled by a positive conformal weight \( h_A \)

\[
\chi_0(q) = \frac{q^{-\frac{c-1}{24}}}{\eta(\tau)}(1 - q), \quad \chi_A(q) = \frac{q^{h_A - \frac{c-1}{24}}}{\eta(\tau)}.
\]  

(10)

Assuming discreteness of the spectrum and no conserved currents beyond those of the Virasoro algebra yields the expansion \( Z = [0]_0 + \sum A N_A |\Delta_{jA}|, \) or more explicitly

\[
Z(\tau, \bar{\tau}) = \chi_0(q)\chi_0(\bar{q}) + \sum A N_A \chi_A(q)\chi_A(\bar{q}),
\]

(11)

where the multiplicity \( N_A \) is a non-negative integer.

In order to obtain the promised upper bound it suffices to apply to such an expansion the first few identities \( i \). We get rid of the \( \eta \) function and its derivatives by considering the modular-invariant combination \( Z(\tau, \bar{\tau})/Z_b = Z_{\text{vac}} + \sum A N_A Z_A \), with

\[
Z_{\text{vac}} = \sqrt{\beta + \bar{\beta} e^{\beta z + \bar{z}}} e^{\frac{\beta \tau + \bar{\beta} \bar{\tau}}{24}}, \\
Z_A = \sqrt{\beta + \bar{\beta} e^{\beta z + \bar{z}}} e^{\frac{\beta \tau + \bar{\beta} \bar{\tau}}{24}} e^{-\beta h_A} e^{-\bar{\beta} h_A},
\]

(12)

When applying \( i \) it turns out that the scaling dimensions \( \Delta_A = h_A + \bar{h}_A \) always appear in the combination \( \Delta_A - \Delta_+ \) with

\[
\Delta_+ = \frac{c - 1}{12} + \frac{1}{2\sqrt{3} \pi},
\]

(13)

and the resulting identities can be written as surprisingly simple sum rules; the first few are

\[
\sum_A (-1)^j A N_A e^{-\beta_\pm \Delta_A} (\Delta_A - \Delta_+) = v^2 \Delta_+ - uv,
\]

\[
\sum_A (-1)^j A N_A e^{-\beta_\pm j_A} j_A^2 = -u,
\]

\[
\sum_A (-1)^j A N_A e^{-\beta_\pm \Delta_A} \left( (\Delta_A - \Delta_+)^2 - \frac{1}{6\pi^2} \right) = -v^2 \Delta_+^2 + 2uv \Delta_+ - u(1 + u) + \frac{v^2}{6\pi^2},
\]

(14)

with \( u = 2e^{-\beta_\pm}, v = 1 + e^{-\beta_\pm} \). The LHS of these sum rules measures the difference in contributions of even and odd spins. We expect that at large \( j \) the two kinds of contributions cancel, as the density of the states is the same in this limit \( i \).

We can check the above equations in some specific model. An instructive example is the partition function \( Z_f \) of 8 free fermions with diagonal GSO projection, which saturates the unitarity bound at \( c = 4 \). The full partition function reads

\[
Z_f(\tau) = \sum_{i=1}^{4} \frac{\theta_i(\tau)^4 \theta_i(\bar{\tau})^4}{\eta(\tau)^{4} \eta(\bar{\tau})^{4}},
\]

(15)

where \( \theta_i \) denote the Jacobi theta functions. \( Z_f \) vanishes for \( \tau \to e^{2\pi i/3} \), as it becomes proportional to the Eisenstein series \( E_4(\tau) \), which is known to have a simple zero at this point. In our notations we have

\[
\sum_A (-1)^j A N_A e^{-\beta_\pm \Delta_A} = -v^2,
\]

(16)

so in this case the cancellation is complete \( i \).

We can easily expand \( Z_f \) in Virasoro characters. The first few terms are

\[
Z_f = [0]_0 + 28[1]_1 + 192[1_0] + 105[2]_2 + 1344[2_1] + 784[2]_3 + \ldots
\]

(17)

This expansion includes conserved currents, i.e. primaries of the form \( |j_1 \rangle \) that we had excluded in deriving \( i \). The contribution of a conserved current to the partition function is \( \chi_j(q)\chi_0(\bar{q}) + \chi_0(q)\chi_j(\bar{q}) \). One can repeat the calculation that led to \( i \) and check that the consequent modification is numerically negligible. As a matter of fact, the two-level decomposition \( i \) is enough to give a good numerical check of the first equation in \( i \), which involves only first derivatives, while the others require much more terms.

Equations \( i \) and their analogues with higher derivatives become particularly interesting in the large-\( c \) limit, where they simply read, for integer \( n \),

\[
\sum_A (-1)^j A N_A e^{-\beta_\pm \Delta_A} \left( \frac{\Delta_A - \Delta_+}{\Delta_+} \right)^n = v^2 (-1)^{n+1} + O(1/c).
\]

(18)

Owing to the higher spin cancellation, only a finite number of terms contribute to the sum; for even \( n \) all the terms of even spin give a positive contribution and all those of odd spin give a negative contribution, while when \( n \) is odd the sign of the terms is \textit{a priori} unknown. However, the alternating sign of the RHS requires at least one term of odd spin in which \( \Delta_A - \Delta_+ < 0 \), since only in this way we can have a positive value on the RHS when \( n \) is odd and a negative value when \( n \) is even. This is tantamount to saying that \( \Delta_+ \) is an upper bound of the largest maximal dimension of the first non-trivial primary in the large-\( c \) limit, and that such a primary has odd spin.

**DISCUSSION AND OUTLOOK**

In this Letter we pointed out that the partition function of a general CFT on a torus should obey a larger class of equations than those explored so far. The new equations allow us to derive a much stronger upper bound
for the maximal gap of the first non-trivial primary in the large-c limit. The latter is very close to the mass threshold of the corresponding BTZ black holes in $AdS_3/CFT_2$ correspondence.

For the sake of completeness, let us recall how such a threshold emerges in the holographic approach. A BTZ black hole of mass $M$ and spin $j$ in the bulk corresponds to a $(h, \bar{h})$ primary on the boundary with

$$h - \frac{c}{24} = \frac{1}{2} (M\ell + j), \quad \bar{h} - \frac{c}{24} = \frac{1}{2} (M\ell - j),$$

where $\ell$ is the radius of $AdS_3$, related to the central charge by $[18] \ c = \frac{24}{\beta}$, and $G$ is the Newton constant. Black holes have smooth horizons only if they fulfill the cosmic censorship condition $M\ell \geq |j|$, prohibiting naked singularities in spacetime. Therefore, according to (19), a CFT primary corresponds to a black hole in the bulk only if $h, \bar{h} \geq \frac{c}{24}$. One-loop corrections replace $c$ with $c - 1$ [19]. Thus primaries with $\Delta < \frac{c-1}{12}$ should correspond to a new kind of matter in the bulk. The threshold at $\Delta = \frac{c-1}{12}$ has a special meaning even on the CFT side [15, 19]. In particular, in a unitary CFT with boundary, a lower bound is derived for the boundary entropy when assuming $\Delta_0 > \frac{c-1}{12}$ [19], however lower values are possible. Actually some indication that a lower upper bound is necessary for saving unitarity has been recently found [14], since in the double limit $j \rightarrow \infty, h - \frac{c-1}{24} \rightarrow 0$ the density of states of the vacuum character contribution $\chi_0(q)\chi_0(\bar{q})$ becomes negative. A possibility to avoid this violation of unitarity is assuming a twist gap no larger than $\frac{c-1}{10}$.

Within this perspective, it would be interesting to try to improve our upper bound [1]. Note that to derive it only some simple properties of the new set of equations [7] were used. Just applying the linear programming technique to them would generate important information on the spectrum of primary states in a general CFT.

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