Algebra of conserved quantities in curved spacetime

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Abstract. In General Relativity (GR) finding out geodesics of a given spacetime manifold is an important task because it determines what are the classical processes which are dynamically forbidden. Conserved quantities play an important role in solving geodesic equations of a general spacetime manifold. Furthermore, knowing all possible conserved quantities of a system tells about the hidden symmetries of a system which is not apparent, since conserved quantities are deeply connected with symmetry properties of the system, which are very important in their own right. This connection is more transparent due to Noether’s theorem with the development of the Lagrangian and Hamiltonian formulations of Newtonian mechanics. Conserved quantities are also useful to capture certain features of spacetime manifold for an asymptotic observer. In this report, we have found out the possible methods of finding conserved quantities and their algebra in the dynamics of an object in curved spacetime both in presence and absence of spin of that object which will help in analyzing dynamics of objects in curved spacetime in future run of advanced LIGO with the help of gravitational memory effect and gravitational lensing effect using principles of electromagnetism and GR.
1 Introduction

For a system any physical quantity \( Q \) has, for each kinematically possible motion, a definite numerical value (which could be vectorial or tensorial in nature) at each instant of time. (Usually \( Q \) will be defined as some function of the position \( x \) and the momentum \( p \) (phase space variable), as well as possibly having an explicit dependence on the time \( t \)) If for every dynamically allowed motion of that system it happens that \( \frac{dQ}{dt} = 0 \), we then say that \( Q \) is conserved (or that \( Q \) is a constant of motion) for that particular physical system.

It is often found out that the same quantity \( Q \) (or a similar quantity) is a constant of motion, not only for one or two different systems and dynamical laws, rather for some broad and interesting class of systems. We then assert a conservation theorem which is in a way related to symmetries, characterizing the situations in which \( Q \) is conserved.

Conservation theorems are important for various reasons. First of all, conservation theorems are generic statements about the types of motions that a dynamical law (or a class of dynamical laws) allows. In particular, they give important information about certain types of motion which are forbidden. Conservation theorems also give partial information about the nature of a particular motion, even if the equations are too complicated to solve analytically [1–4] (e.g. the conservation of energy and/or angular momentum can often be used to find turning points, maximum height reached, etc., even when it’s hard to find explicit solution of equations of motion). A conserved quantity provides a “first integral” of the equations.
of motion (which is second order differential equation) and sometimes this is sufficient to essentially solve the problem; other times it can be used to decouple a set of coupled differential equations.

In GR which is an extension of Special Relativity in which Einstein unifies space and time, dynamics of a system is often talked w.r.t affine parameter $\lambda$ which in certain cases can be chosen as co-ordinate time $t$. In an arbitrary spacetime manifold one does not expect existence of any conserved quantities however existence of Killing symmetries in special class of spacetime manifolds makes it possible to define certain conserved quantities easily which we will discuss. Furthermore, if the metric of chosen spacetime manifold is asymptotically flat then it is straightforward to assign physical meaning to such quantities (for example angular momentum, energy of a particle etc.) Another context, in which existence of conserved quantities plays an important role is to find out integrability condition of a system which basically says a $2n$-dimensional Hamiltonian system of ordinary differential equations (ODE) is integrable [5–7] if it has $n$ (functionally) independent constants of the motion that are "in involution" where functionally independent means none of them can be written as a function of the others and "in involution" means that their Poisson Brackets all vanish.

The $T^{00}$-component of stress-energy tensor which coupled to metric-tensor in the weak-field limit of Einstein-Hilbert action, acts as Hamiltonian of the source from which Gravitational wave signals are measured. Hence, naturally all the dynamical information about that astrophysical object is captured in the dynamics or propagation of Gravitation waves emitted from that astrophysical source itself. These information can be extracted using Gravitational memory effect which has been suggested already specifically displacement memory [8, 9] and spin memory [10]. We believe combining all these information one would be able to find all the conserved quantities and their algebra would provide more conserved quantities which can then be directly checked from memory effect itself. Further, conserved quantities can also be measured from the Gravitational lensing effect [11–13] using principles of electrodynamics and GR, since this technique is mainly used to map the background dynamical geometry surrounding any astrophysical object. This way it would be possible to generate a closed algebra between all the conserved quantities. We have provided the existence of such conserved quantities and their algebra in this article in different context.

2 Diffeomorphism invariance

Given a spacetime manifold a general diffeomorphism is characterized vector field $\xi$ defined over spacetime manifold such that under following

$$x' = x + \xi \implies x'\mu = x^\mu + \xi^\mu(x)$$

(2.1)

co-ordinate transformation, variation of a geometric quantity is another geometric quantity which is characterized by

$$\delta\omega_\alpha = \mathcal{L}(\xi)\omega_\alpha$$

(2.2)

If this is true then for metric it would become

$$\delta g_{\alpha\beta}(x) = \mathcal{L}(\xi)g_{\alpha\beta} = \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha$$

$$= -\xi^\rho\partial_\rho g_{\alpha\beta} + \partial_\alpha\xi^\rho g_{\rho\beta} + \partial_\beta\xi^\rho g_{\alpha\rho}$$

(2.3)

which follows from co-ordinate transformation rule of metric and definition of metric compatible covariant derivative. Among these we are particularly interested in a class of diffeomorphisms under which metric remains invariant, which are known as isometries. The generators
of isometries are Killing vector fields and they satisfy following condition

\[ \mathcal{L} (\xi) g_{\alpha \beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \]  

(2.4)

Denoting \( H_{\mu \nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \) and starting from following identity

\[
0 = \nabla_\mu H_{\nu \rho} + \nabla_\rho H_{\mu \nu} - \nabla_\nu H_{\rho \mu} \\
= \nabla_\mu \nabla_\nu \xi_\rho + \nabla_\rho \nabla_\mu \xi_\nu + \nabla_\rho \nabla_\nu \xi_\mu - \nabla_\nu \nabla_\rho \xi_\mu \\
- \nabla_\nu \nabla_\rho \xi_\mu - \nabla_\rho \nabla_\mu \xi_\nu \\
= [\nabla_\mu, \nabla_\nu] \xi_\rho + [\nabla_\nu, \nabla_\mu] \xi_\rho + \{ \nabla_\mu, \nabla_\nu \} \xi_\rho \\
= R_{\rho \mu \nu \lambda} \xi_\lambda + R_{\rho \mu \nu \lambda} \xi_\lambda + 2 \nabla_\mu \nabla_\rho \xi_\nu + R_{\nu \rho \mu \lambda} \xi_\lambda \\
\implies \nabla_\mu \nabla_\rho \xi_\nu = - \frac{1}{2} R_{\nu \rho \mu \lambda} \xi_\lambda
\]

we obtain above relation.

This relation will help us in showing that all the isometries of a given spacetime form a Lie algebra. We will show that by showing that commutator of two isometries is another isometry through brute force method

\[
\nabla_\alpha [\xi_a, \xi_b] + \nabla_\beta [\xi_a, \xi_b] = \nabla_\alpha (\xi^a \nabla_\nu \xi^b - \xi^b \nabla_\nu \xi^a) \\
+ \nabla_\beta (\xi^b \nabla_\lambda \xi^a - \xi^a \nabla_\lambda \xi^b) \\
= \xi^a \nabla_\alpha \nabla_\mu \xi^b - \xi^b \nabla_\alpha \nabla_\mu \xi^a + \xi^b \nabla_\beta \nabla_\mu \xi^a - \xi^a \nabla_\beta \nabla_\mu \xi^b \\
+ \nabla_\alpha \xi^a \nabla_\mu \xi^b - \nabla_\alpha \xi^b \nabla_\mu \xi^a + \nabla_\beta \xi^b \nabla_\lambda \xi^a - \nabla_\beta \xi^a \nabla_\lambda \xi^b \\
= - \xi^a \nabla_\alpha \nabla_\beta \xi^b + \xi^b \nabla_\alpha \nabla_\beta \xi^a \\
= \frac{1}{2} \xi^a \xi^b (R_{\mu \beta \alpha \lambda} - R_{\lambda \beta \alpha \mu} + R_{\mu \alpha \beta \lambda} - R_{\lambda \alpha \beta \mu}) = 0
\]

(2.6)

Hence, this indicates that the mentioned commutator is a linear combination of other Killing vector with constant coefficients, known as structure constants of the isometry algebra. In general, if we consider \( \{ \xi_a, a = 1, \ldots, r \} \) represents a basis of linear space of Killing fields of a spacetime manifold, the following mathematical statement is true

\[
[\xi_a, \xi_b] = C^c_{ab} \xi_c 
\]

(2.7)

with \( C^c_{ab} = -C^c_{ba} \).

The isometry group is simply transitive if all Killing fields are linearly independent, otherwise the group is multiply transitive.

Now note that having a spacetime which exhibits at least one Killing vector field, contains a matter with covariant conserved stress-energy tensor \( T^{\mu \nu} \) (which satisfies \( \nabla_\mu T^{\mu \nu} \)) also possesses a conserved 4-current defined by

\[ J^\mu [\xi] = \xi^a T^{a \mu} \]

(2.8)

which is covariantly conserved

\[ \nabla_\mu J^\mu [\xi] = 0 \]

(2.9)

which follows from symmetry property of stress-energy tensor in two indices and its covariant conservation.

This section is important for further studies about method of finding conserved quantities for a single particle Hamiltonian mechanics in curved spacetime.
3 Canonical formulation

3.1 Hamiltonian mechanics

Most often to study dynamics of a particle on a curved spacetime one would start with following action

\[ S = - \int \sqrt{-g} dx^\mu dx^\nu = - \int \sqrt{-g(x)} \dot{x}^\mu \dot{x}^\nu d\lambda \] (3.1)

where \( \lambda \) is an affine parameter and \( \dot{\cdot} \) represents derivative w.r.t \( \lambda \). And extremizing this action which is equivalent to extremizing line-element leads to geodesic equations of that manifold.

There is another alternative choice of action which also leads to geodesic equations but characterestically of different form

\[ S = \frac{1}{2} \int g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu d\lambda \] (3.2)

but both the action is equivalent from 'einstein' formalism point of view. Therefore, we start with second action whose correspondence Hamiltonian is

\[ H = \frac{1}{2} g_{\mu\nu} p^\mu p^\nu \] (3.3)

which can be obtained through Legendre transformation and where \( p_\mu(\lambda) \) is the canonical conjugate momentum variable defined by

\[ p_\mu(\lambda) = g_{\mu\nu}(x(\lambda)) \dot{x}^\nu \] (3.4)

which translates geodesic equations in following form

\[
\begin{align*}
\frac{dp_\mu}{d\lambda} &= \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \frac{dx^\sigma}{d\lambda} \frac{dx^\nu}{d\lambda} + g_{\mu\nu}(x) \frac{d^2 x^\nu}{d\lambda^2} \\
&= \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \frac{dx^\sigma}{d\lambda} \frac{dx^\nu}{d\lambda} - g_{\mu\nu}(x) \Gamma^\rho_{\nu\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} \\
&= \frac{dx^\rho}{d\lambda} \left( \frac{\partial g_{\mu\sigma}}{\partial x^\rho} - \frac{1}{2} \delta_\sigma^{\rho} g_{\mu\sigma} - \frac{1}{2} \delta_\sigma^{\rho} g_{\mu\rho} + \frac{1}{2} \delta_\rho^{\sigma} g_{\mu\rho} \right) \\
&= \Gamma^\sigma_{\mu\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} = \Gamma^\sigma_{\mu\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} \\
&= \Gamma^\sigma_{\mu\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda}
\end{align*}
\] (3.5)

The equations (3.4) and (3.5) constitute a pair of first order ODEs which is equivalent to second order geodesic equations. Hamilton’s equations are the set of powerful results which shows how to derive above two equations from Hamiltonian \( H(\lambda) \) itself

\[ \frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\lambda} = - \frac{\partial H}{\partial x^\mu} \] (3.6)

where \( H(\lambda) = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \).
3.2 Poisson bracket

With the definitions (3.6), one can show that any function $F(x^\mu, p_\nu)$ defined over phase space spanned by canonical conjugate variables, changes along geodesic according to

$$\frac{dF}{d\lambda} = \frac{dx^\mu}{d\lambda} \frac{\partial F}{\partial x^\mu} + \frac{dp_\mu}{d\lambda} \frac{\partial F}{\partial p_\mu}$$
$$= \frac{\partial H}{\partial p_\mu} \frac{\partial F}{\partial x^\mu} - \frac{\partial H}{\partial x^\mu} \frac{\partial F}{\partial p_\mu}$$
$$= \{F, H\}$$

(3.7)

where Poisson bracket of two functions of phase space variable is defined as follows

$$\{F, G\} = -\{G, F\} = \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial G}{\partial x^\mu} \frac{\partial F}{\partial p_\mu}$$

(3.8)

Anti-symmetry of Poisson bracket automatically ensures that time-independent Hamiltonian (which does not have explicit time dependence) is a conserved quantity since

$$\{H, H\} = 0$$

(3.9)

Poisson bracket also satifies linearity in both of its arguments

$$\{F, \alpha_1 G_1 + \alpha_2 G_2\} = \alpha_1 \{F, G_1\} + \alpha_2 \{F, G_2\}, \forall \alpha, \beta \in \mathbb{C}$$

(3.10)

Another important property that it satisfies is the Jacobi identity

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0$$

(3.11)

This property also shows Poisson bracket of any two conserved quantity of geodesic flow is also a conserved quantity of geodesic flow. If $F, G$ satisfies $\{F, H\} = 0 = \{G, H\}$ then

$$\{\{F, G\}, H\} = \{F, \{G, H\}\} - \{G, \{F, H\}\} = 0$$

(3.12)

As $H$ is itself a constant of motion on geodesics the above result are enough to establish that constants of motion form a Lie-algebra, with Poisson bracket as Lie bracket. (We will alternatively use the terms constant of motion and conserved quantity which are equivalent.)

Note that definition of Poisson bracket mentioned above is nor manifestly covariant, which at first sight, seems to destroy the in buit covariance nature of GR. However, if one recalls that connetion used in GR is symmetric [14] in its lower two indices which helps us to extend the definition of Poisson bracket so that it captures general covariance in followig manner

$$\{F, G\}_* \equiv D_\mu F \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} D_\mu G$$

(3.13)

where for functions over phase-space variables (which are scalar by nature) we define covariant derivative as

$$D_\mu F = \partial_\mu F + \Gamma_\mu^{\lambda} p_\lambda \frac{\partial F}{\partial p_\nu}$$

(3.14)
Now note that for any two arbitrary phase-space functions

\[ \{ F, G \} = \partial_\mu F \frac{\partial G}{\partial p_\mu} - \partial_\mu F \frac{\partial G}{\partial p_\mu} + \Gamma^{\lambda}_{\mu\nu} p_\lambda \left( \frac{\partial F}{\partial p_\nu} \frac{\partial G}{\partial p_\mu} - \frac{\partial G}{\partial p_\nu} \frac{\partial F}{\partial p_\mu} \right) \]

\[ = \partial_\mu F \frac{\partial G}{\partial p_\mu} - \partial_\mu G \frac{\partial F}{\partial p_\mu} = \{ F, G \} \]  

(3.15)

whose shows both the definition of Poisson bracket numerically gives same value hence, for sake of convenience we will use the first definition from now onwards.

The second definition also preserves covariance manifestly for scalar functions which are of the form \( J(x, p) = J^\mu(x)p_\mu \):

\[ \mathcal{D}_\mu J = p_\nu \partial_\mu J^\nu + \Gamma^\lambda_{\mu\nu} p_\lambda J^\nu \]

\[ = (\partial_\mu J^\nu + \Gamma^\nu_{\mu\lambda})p_\nu \]  

\[ = (\nabla_\mu J^\mu)p_\nu \]  

(3.16)

The manifestly covariant nature of Poisson bracket can also be extended to completely symmetric tensors of rank-n by contracting all indices with \( p_\mu \)'s which yields

\[ T(x, p) = \frac{1}{n!} T^{\mu_1\ldots\mu_n} p_{\mu_1} \ldots p_{\mu_n} \]  

(3.17)

A similar result can be obtained for antisymmetric tensors using Grassmann algebra.

### 3.3 Conserved charges

Symmetry of geodesic structure generated by diffeomorphisms associated with Killing vector fields implies conservation of certain quantity under geodesic flow. That quantity is obtained through contraction of Killing vector field with momentum

\[ J[\xi] = \xi^\mu p_\mu \]  

(3.18)

which is also a generator of symmetry transformations.

Note that this is a scalar quantity and hence it is invariant under coordinate transformations. It generates a coordinate transformation to which we are familiar with

\[ \delta x^\mu = \{ J[\xi], x^\mu \} = -\xi^\mu(x) \]  

(3.19)

which is nothing but the infinitesimal diffeomorphism. Similarly we can also find out the variation of momentum under this generator which is

\[ \delta p_\mu = \{ J[\xi], p_\mu \} = \partial_\mu \xi^\nu p_\nu \]  

(3.20)

which is also a coordinate dependent transformation. One can explicitly check that under these two transformation Hamiltonian in eqn. (3.3) remains invariant which means

\[ \frac{dJ[\xi]}{d\lambda} = \{ J[\xi], H \} = 0 \]  

(3.21)
This result was expected since Hamiltonian depends on the coordinates only through metric which is invariant under diffeomorphism [15] associated with Killing vector fields.

As we have seen that Killing equations are linear in nature and hence, it forms a linear vector space. Let choose dimension of the vector space to be \( r \), then any arbitrary Killing vector field on this vector space can be written in terms of linear combination of basis vectors \( \{ e_i(x), \, i = 1, \ldots, r \} \)

\[
\xi(\alpha_i) = \alpha_1 e_1(x) + \ldots + \alpha_r e_r(x) \tag{3.22}
\]

Hence, each such diffeomorphism under which metric remains invariant can be written as a linear combination of \( r \) numbers of conserved generators/charges whose coefficients will also be \( \{ \alpha_i \} \)s

\[
J[\xi] = \alpha_1 J_1 + \ldots + \alpha_r J_r \tag{3.23}
\]

where \( J_i(x, p) = e_i^\mu(x)p_\mu \).

We will now show that these generators form a Lie-algebra under Poisson bracket:

\[
\{ J_i, J_j \} = \{ J[e_i], J[e_j] \} = \{ e_i^\mu p_\mu, e_j^\nu p_\nu \} = \{ e_i^\mu p_\mu, e_j^\nu p_\nu \} + \{ e_i^\mu, e_j^\nu p_\nu \} p_\mu \tag{3.24}
\]

\[
\begin{align*}
&= - e_i^\mu \partial_\mu e_j^\nu p_\nu + e_j^\nu \partial_\mu e_i^\mu p_\mu \\
&= (e_j^\nu \partial_\mu e_i^\mu - e_i^\nu \partial_\mu e_j^\mu)p_\mu
\end{align*}
\]

And as we have shown earlier in (2.7) that Killing vectors form a Lie-algebra, therefore,

\[
[e_i, e_j] = - (e_j^\nu \nabla_\nu e_i^\mu - e_i^\nu \nabla_\nu e_j^\mu) = -(e_j^\nu \partial_\nu e_i^\mu - e_i^\nu \partial_\nu e_j^\mu + e_j^\nu \epsilon_k^\mu \Gamma^\nu_{\nu \lambda} - e_i^\nu \epsilon_k^\mu \Gamma^\nu_{\nu \lambda}) = -(e_j^\nu \partial_\nu e_i^\mu - e_i^\nu \partial_\nu e_j^\mu) = C_{ij}^k e_k \tag{3.25}
\]

which shows

\[
\{ J_i, J_j \} = -C_{ij}^k e_k p_\mu = f_{ij}^k J_k \tag{3.26}
\]

where we denote \( f_{ij}^k = -C_{ij}^k \).

As we have shown that \( \{ J_i \} \) form a linear vector space with a closed algebra under a Lie bracket (which is Poisson bracket in this case) which is anti-symmetric and bilinear in its arguments hence, it is proven that \( \{ J_i \} \) form a Lie-algebra with structure constant \( f_{ij}^k \).

### 3.4 Conservation laws and algebra of conserved charges

It was established in previous section that associated with each continuous symmetries generated by Killing vector fields, one can define conservation laws or conserved quantities (denoted by generators \( J \)) along geodesics. But the reverse statement is not true which basically says that for each conserved quantity along geodesic flow there might not be a Killing symmetry, and one such example is Hamiltonian of the system. This raises the next question can we generate conserved quantities which are higher order in momenta.

Let define a general phase-space function \( J(x, p) \) on geodesic which is non-singular in momentum variables that means it is possible to express the quantity in following manner

\[
J(x, p) = \sum_{k=0}^{\infty} \frac{1}{k!} J^{[k]} p_{\mu_1} \ldots p_{\mu_k} \tag{3.27}
\]
where the expansion coefficients $J^{(k)\mu_1,\ldots,\mu_k}$ are completely symmetric in the interchange of indices. Now we ask under what circumstances

$$\frac{dJ}{d\lambda} = \{J, H\} = 0$$

(3.28)

In the first order (or leading order), one can check that it implies

$$\frac{\partial J^{(0)}}{\partial x^\mu} = 0$$

(3.29)

which shows $J^{(0)}$ to a constant which can be redefined and make it zero.

At first order one would find back the Killing equation

$$\nabla_\mu J^{(1)}_{\nu} + \nabla_\nu J^{(1)}_{\mu} = 0$$

(3.30)

because the term is linear in momentum. For the $k$th order following condition needs to be satisfied

$$0 = \frac{1}{2} [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k}, g^{\mu\nu} p_{\mu} p_{\nu}]$$

$$\implies 0 = \frac{1}{2} [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k}, g^{\mu\nu} p_{\mu} p_{\nu}]$$

$$= [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k}, g^{\mu\nu}] p_{\mu} p_{\nu}$$

$$+ [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k}, p_{\mu}] g^{\mu\nu} p_{\nu}$$

$$+ [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k}, p_{\nu}] g^{\mu\nu} p_{\mu}$$

$$= -\sum_{i=1}^{k} J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_i-1} p_{\mu_i+1} \cdots p_{\mu_k} \partial_{\mu} g^{\mu\nu} p_{\mu} p_{\nu}$$

$$+ \partial_{\mu} J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k} g^{\mu\nu} p_{\nu} + \partial_{\nu} J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k} g^{\mu\nu} p_{\mu}$$

$$= -\frac{1}{2} \sum_{i=1}^{k} [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_{i-1}} p_{\mu_{i+1}} \cdots p_{\mu_k} \partial_{\mu} g^{\mu\nu} p_{\mu} p_{\nu}]$$

$$+ \partial_{\mu} J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k} g^{\mu\nu} p_{\nu}$$

$$= \sum_{i=1}^{k} [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_{i-1}} p_{\mu_{i+1}} \cdots p_{\mu_k} \Gamma_{\alpha\beta\mu} g^{\alpha\nu} g^{\beta\nu} p_{\mu} p_{\nu}]$$

$$+ \partial_{\mu} J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k} g^{\mu\nu} p_{\nu}$$

which can be written as follows

$$0 = \sum_{i=1}^{k} [J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_{i-1}} p_{\mu_{i+1}} \cdots p_{\mu_k} \Gamma_{\mu_1,\ldots,\mu_k} g^{\mu\nu} p_{\mu} p_{\nu}]$$

$$+ \partial_{\mu} J^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k} g^{\mu\nu} p_{\nu}$$

$$= p_{\mu_1} \cdots p_{\mu_k} \left( \partial_{\mu} J^{(k)\mu_1,\ldots,\mu_k} + \sum_{i=1}^{k} \Gamma_{\mu_1,\ldots,\mu_k} J^{(k)\mu_1,\ldots,\mu_k} \right)$$

$$= p_{\mu_1} \cdots p_{\mu_k} \nabla_{\mu} J^{(k)\mu_1,\ldots,\mu_k}$$

$$\implies 0 = \nabla_{\mu} J^{(k)\mu_1,\ldots,\mu_k}$$

- 8 -
So, these are nothing but generalized Killing equations and solutions of these equations are called Killing tensors.

Just like Killing vectors, Killing tensors also act as a generator of transformations but on phase space variables. Under action of this $x^\mu$ changes by

$$\delta x^\mu = \{ J(x, p), x^\mu \} = -J^{(1)} \mu - \sum_{k=2}^{\infty} \frac{1}{(k-1)!} J^{(k)} \mu \mu_2 \ldots \mu_k p_{\mu_2} \ldots p_{\mu_k}$$  \hspace{1cm} (3.33)

Note that these transformations are in general velocity dependent (specifically momentum dependent) for higher order generators/charges.

Let’s now look at the algebra between charges/generators (which are constants along geodesic flow) of different order which would be closed, follows from Jacobi identity of Poisson bracket

$$\{ J^{(k)} \mu_1 \ldots \mu_k, J^{(l)} \nu_1 \ldots \nu_l \} = -\sum_{i=1}^{k} J^{(k)} \mu_1 \ldots \mu_{i-1} \mu_{i+1} \ldots \mu_k \partial_J^{(l)} \nu_1 \ldots \nu_{i-1} \nu_{i+1} \ldots \nu_l \partial_x \mu_i$$

$$+ \sum_{j=1}^{l} p_{\mu_1} \ldots p_{\mu_k} J^{(l)} \nu_1 \ldots \nu_{j-1} \nu_{j+1} \ldots \nu_l \partial_J^{(k)} \mu_1 \ldots \mu_{j-1} \mu_{j+1} \ldots \mu_k \partial_x \nu_j$$

$$\equiv J^{(k+l-1)} \sigma_1 \ldots \sigma_{k+l-1} p_{\mu_1} \ldots p_{\sigma_{k+l-1}}$$

$$\Rightarrow \{ J^{(k)}, J^{(l)} \} \sim J^{(k+l-1)}$$  \hspace{1cm} (3.34)

Note that for $l = 1$ above equation tell us

$$\{ J^{(k)}, J^{(1)} \} \sim J^{(k)}$$  \hspace{1cm} (3.35)

which means algebra of charge/generator of any order with charge/generator of order one (linear in momentum) is closed under Poisson bracket but it is not true for arbitrary values of $k$ and $l$.

Note also that the algebra shown in eqn. (3.34) is similar to Virasoro algebra [16, 17] which is the algebra of infinite dimensional Killing vector space for 2-dimensional conformal field theories [18–20].

### 4 For Hamiltonians linear in momentum

#### 4.1 Introduction

In this section, a motivation behind considering hamiltonians liner momentum is provided. As we are familiar with that a Hamiltonian quadratic in momentum naturally follows from Lagrangian which is quadratic in coordinate velocities through Legendre transformations. But in certain cases one can approximate a Hamiltonian quadratic in momentum such a way that only the linear momentum piece contained in it survives ultimately, one such example is shown here.

Consider a charged particle of charge $q$ on 2-dimensional space in a magnetic field in $z$-direction $\vec{B} = B_0 \hat{z}$ and for sake of simplicity we consider non-relativistic system. In that case, Hamiltonian is given by

$$H = \frac{1}{2m} \left( \vec{p} + \frac{q}{c} \vec{A} \right)^2 = \frac{\vec{p}^2}{2m} + \frac{q}{mc} \vec{A} \cdot \vec{p} + \frac{q^2}{2mc^2} \vec{A} \cdot \vec{A}$$  \hspace{1cm} (4.1)
Now we choose circular gauge in which $\vec{A} = B_0 \frac{r^2}{2} (-y, x)$ which yields

$$H = \frac{p^2}{2m} + qB_0 (xp_y - yp_x) + \frac{q^2 B_0^2}{8mc^2} (x^2 + y^2)$$  \hspace{1cm} (4.2)$$

If we consider the quantity $2B_0$ which is inverse of some time scale, is high in magnitude then we can drop the first term (which is the kinetic term) in the Hamiltonian which is valid for heavily charged astrophysical object moving around Neutron stars or in accretion disk then we obtain an effective Hamiltonian which is of following form

$$H_{eff} = \omega_c (xp_y - yp_x) + \frac{m\omega_c^2}{2} (x^2 + y^2), \ \omega_c = \frac{qB_0}{2mc}$$  \hspace{1cm} (4.3)$$

Note that Hamilton’s equation for coordinate velocity becomes

$$\dot{x} = -\frac{\omega_c}{m} y, \ \dot{y} = \omega_c x$$  \hspace{1cm} (4.4)$$

which leads to two simple harmonic oscillator equations

$$\ddot{x} = -\omega_c^2 x, \ \ddot{y} = -\omega_c^2 y$$  \hspace{1cm} (4.5)$$

which shows that particle is trapped in closed orbit [21–23].

If following condition happens to be true

$$\frac{p_i}{m\omega_c x^i} \gg 1$$  \hspace{1cm} (4.6)$$

then we can write effective Hamiltonian as follows

$$H_{eff} = \omega_c (xp_y - yp_x)$$  \hspace{1cm} (4.7)$$

With this motivation in mind in next section we will start with previous set of calculations mentioned in earlier section.

### 4.2 Conserved charges and their algebra

In this section we start with a general Hamiltonian linear in momentum which is of following form

$$H = \zeta^\mu p_\mu$$  \hspace{1cm} (4.8)$$

where $\zeta^\mu$ is a vector field defined over spacetime manifold and it’s a dimensionful quantity.

As earlier let define a general phase-space function $K(x, p)$ on geodesic which is non-singular in momentum variables that means it is possible to express the quantity in following manner

$$K(x, p) = \sum_{k=0}^{\infty} \frac{1}{k!} K^{(k)} p_{\mu_1} \cdots p_{\mu_k}$$  \hspace{1cm} (4.9)$$

where the expansion coefficients $K^{(k)} p_{\mu_1} \cdots p_{\mu_k}$ are completely symmetric in the interchange of indices. Now we ask under what circumstances

$$\frac{dK}{d\lambda} = \{K, H\} = 0$$  \hspace{1cm} (4.10)$$
Hence, existence of a conserved charge of rank-$k$ implies

\[
0 = \{ \mathcal{K}^{(k)}, H \} = \{ \mathcal{K}^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k}, \zeta^\nu p_\nu \}
\]

\[
= - \sum_{i=1}^{k} \mathcal{K}^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_{i-1}} \frac{\partial \zeta^\nu}{\partial x^\mu_i} p_\nu p_{\mu_{i+1}} \cdots p_{\mu_k} 
\]

\[
+ \frac{\partial \mathcal{K}^{(k)\mu_1,\ldots,\mu_k}}{\partial x^\nu} \zeta^\nu p_{\mu_1} \cdots p_{\mu_k}
\]

\[
= - \sum_{i=1}^{k} \mathcal{K}^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_{i-1}} \frac{\partial \zeta^\mu_i}{\partial x^\nu} p_\nu p_{\mu_{i+1}} \cdots p_{\mu_k}
\]

\[
= p_{\mu_1} \cdots p_{\mu_k} \left( \zeta^\nu \frac{\partial \mathcal{K}^{(k)\mu_1,\ldots,\mu_k}}{\partial x^\nu} - \sum_{i=1}^{k} \mathcal{K}^{(k)\mu_1,\ldots,\mu_k} \frac{\partial \zeta^\mu_i}{\partial x^\nu} \right)
\]

\[
= \mathcal{L}_\zeta \mathcal{K}^{(k)\mu_1,\ldots,\mu_k} p_{\mu_1} \cdots p_{\mu_k}
\]

\[
\Rightarrow \mathcal{L}_\zeta \mathcal{K}^{(k)\mu_1,\ldots,\mu_k} = 0
\]

which says mathematically that in order to be a conserved charge/generator Lie-derivative of $\mathcal{K}^{(k)\mu_1,\ldots,\mu_k}$ must be zero along the vector field $\zeta^\mu$. These set of conditions are completely different from the earlier set of conditions in eqn. (3.32). Hence, depending on the existence of solutions of above set of tensorial equation one can generate conserved charges of different rank. One of the simplest quantities to look at are the vector fields $\{\xi^{(i)}\}$ for which the Lie-bracket $[\zeta, \xi^{(i)}] = 0$ which gives set of conserved quantities $\{\xi^{(i)\mu} p_\mu\}$s.

Note that the algebra of charges of different rank remains same as we have shown earlier, which is

\[
\{ \mathcal{K}^{(k)}, \mathcal{K}^{(l)} \} \sim \mathcal{K}^{(k+l-1)}
\]

5 For Spinning objects

5.1 Introduction

The dynamics of angular momentum and spin of gravitating compact bodies has been a subject of great interest since the early days of relativity theory [24–30] and recent discovery of gravitational waves [31–35] attract more attract towards measurement of these properties in curved spacetime. Here, we start by mentioning Poisson algebra between phase-space variables for point-like objects [36], which is an idealization of a compact body since it neglects details of the internal structure by assigning the point-like object with overall fixed position, momentum and spin. This approach is known as the spinning-particle approximation, used for the semi-classical description of elementary particles as well. A large variety of models for spinning particles is found in the literature [37–39].

5.2 Covariant phase-space structure

To specify a Hamiltonian dynamical system three sets of ingredients are required which are the phase space, identifying the dynamical degrees of freedom, the Poisson brackets between these dynamical degrees of freedom defining a symplectic structure [40–42], and the Hamiltonian generating the evolution of the system with given initial conditions by specifying a curve in
the phase space passing through the initial point. The parametrization of phase-space is not unique, since changes in the parametrization can be compensated by redefining the Poisson brackets and the Hamiltonian.

We start by defining spin-degrees of freedom, described by an anti-symmetric tensor $\Sigma_{\mu \nu}$

$$S^\mu = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu \nu \lambda \kappa} u_\nu \Sigma_{\kappa \lambda}, \quad Z^\mu = \Sigma_{\mu \nu} u_\nu$$  \hspace{1cm} (5.1)

where $u^\mu$ is a time-like unit vector satisfies $u_\mu u^\mu = 1$. By construction above two quantities satisfy following two conditions

$$Z^\mu u_\mu = 0, \quad S^\mu u_\mu = 0$$ (5.2)

which means they are space-like in nature.

The full set of phase-space co-ordinates of a spinning particle can be constructed using position co-ordinate $x^\mu$, the covariant momentum $p_\mu$ and the spin tensor $\Sigma_{\mu \nu}$, with antisymmetric Poisson brackets

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \quad \{p_\mu, p_\nu\} = \frac{1}{2} \Sigma^\nu_\lambda \mathcal{R}_{\lambda \kappa \mu \nu}$$

$$\{\Sigma_{\mu \nu}, p_\lambda\} = \Gamma^\nu_{\lambda \kappa} \Sigma_{\kappa \mu} - \Gamma^\nu_{\lambda \kappa} \Sigma_{\mu \kappa}

\{\Sigma_{\mu \nu}, \Sigma^{\kappa \lambda}\} = g^{\mu \kappa} \Sigma_{\nu \lambda} - g^{\mu \lambda} \Sigma_{\nu \kappa} - g^{\nu \kappa} \Sigma_{\mu \lambda} + g^{\nu \lambda} \Sigma_{\mu \kappa}$$ (5.3)

Note that second Poisson bracket becomes trivial in a limit in which spin of the system vanishes which is consistent.

It is quite simple task to check that these brackets are indeed closed in the sense that they satisfy the Jacobi identities, hence a consistent symplectic structure is defined on the phase space. To have a well-defined dynamical system we need to complete the phase-space structure with a Hamiltonian that generates proper-time evolution of the system. Here, we choose free-particle Hamiltonian described in eqn. (3.3).

It can be shown explicitly with simple algebra that chosen Hamiltonian generates the following set of proper-time evolution equations

$$\dot{x}^\mu = \{x^\mu, H\} \implies p_\mu = g_{\mu \nu} \dot{x}^\nu$$

$$\dot{p}_\mu = \{p_\mu, H\} = \Gamma^\nu_{\lambda \mu} p_\nu \dot{x}^\lambda + \frac{1}{2} \Sigma^{\kappa \lambda} \mathcal{R}_{\kappa \lambda \mu \nu}$$ (5.4)

$$\dot{\Sigma}^\mu_\nu = \{\Sigma^\mu_\nu, H\} \implies \dot{\Sigma}^\mu_\nu + \Gamma^\mu_{\nu \lambda} \dot{x}^\nu \Sigma^\lambda_\mu + \Gamma^\mu_{\nu \lambda} \dot{x}^\lambda \Sigma^\nu_\mu = 0$$

Substituting first equation into second equation one would obtain

$$\ddot{x}^\mu + \Gamma^\mu_{\nu \lambda} \dot{x}^\nu \dot{x}^\lambda = \frac{1}{2} \Sigma^{\kappa \lambda} \mathcal{R}_{\kappa \lambda \mu \nu}$$ (5.5)

which reduces to familiar geodesic equation in $\Sigma = 0$ limit.

### 5.3 Conserved charges and their algebra

Since in this case phase-space variables include anti-symmetric tensor $\Sigma^\mu_\nu$ which makes it difficult to construct conserved charges. The old construction of conserved charges does not hold here since presence of $\Sigma^\mu_\nu$ tensor makes $\{p_\mu, p_\nu\} \neq 0$. Here we start by looking at conditions in order to construct conserved charges which are of following form

$$J = \sum_{n=0}^{\infty} J^{(2n)}_{\mu_1 \nu_1 \mu_2 \nu_2 \ldots \mu_n \nu_n} (x) \Sigma_{\mu_1 \nu_1} \ldots \Sigma_{\mu_n \nu_n}$$ (5.6)
Then to construct conserved charge of rank-2\(k\), following condition needs to be hold

\[
0 = \{ J^{(2k)\mu_1\nu_1\mu_2\nu_2\ldots\mu_k\nu_k} \, \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k}, \, \partial_{\mu} p_\nu \}
\]

\[
= \{ J^{(2k)\mu_1\nu_1\mu_2\nu_2\ldots\mu_k\nu_k} \, \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k}, \, g^{\mu\nu} p_\nu \}
\]

\[
= \frac{\partial J^{(2k)\mu_1\nu_1\mu_2\nu_2\ldots\mu_k\nu_k}}{\partial x^\mu} \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k} g^{\mu\nu} p_\nu
\]

\[
+ \sum_{i=1}^{k} J^{(2k)\mu_1\nu_1\ldots\mu_k\nu_k} \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k} g^{\mu\nu} p_\nu
\]

\[
= \frac{\partial J^{(2k)\mu_1\nu_1\ldots\mu_k\nu_k}}{\partial x^\mu} \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k} g^{\mu\nu} p_\nu
\]

\[
- \sum_{i=1}^{k} J^{(2k)\mu_1\nu_1\ldots\mu_i-1\nu_i\ldots\mu_k\nu_k} \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k} g^{\mu\nu} p_\nu
\]

\[
- \sum_{i=1}^{k} J^{(2k)\mu_1\nu_1\ldots\mu_i-1\nu_i\ldots\mu_k\nu_k} \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k} g^{\mu\nu} p_\nu
\]

\[
= \left[ \frac{\partial J^{(2k)\mu_1\nu_1\ldots\mu_k\nu_k}}{\partial x^\mu} - \sum_{i=1}^{k} \left( J^{(2k)\mu_1\nu_1\ldots\mu_i-1\nu_i\ldots\mu_k\nu_k} \Gamma^\lambda_{\mu_i} \right) \Gamma^\kappa_{\nu_i} + J^{(2k)\mu_1\nu_1\ldots\mu_i-1\nu_i\ldots\mu_k\nu_k} \Gamma^\lambda_{\mu_i} \Gamma^\kappa_{\nu_i} \right]
\]

\[
\times \sum_{i=1}^{\mu_1} \ldots \sum_{i=1}^{\nu_1} \ldots \sum_{i=1}^{\mu_k} \ldots \sum_{i=1}^{\nu_k} g^{\mu\nu} p_\nu
\]

\[
\Rightarrow \nabla_\mu J^{(2k)\mu_1\nu_1\ldots\mu_k\nu_k} = 0 \quad (5.7)
\]

where in the last line it is cleared that the transpositions of pairs (\(\mu_i, \nu_i\)) are symmetric for \(J^{(2k)}\) and it is anti-symmetric in permutation between \(\mu_i \leftrightarrow \nu_i\), \(\forall i\).

And also note that like previous case here also

\[
\{ J^{(2k)}, J^{(2l)} \} \sim J^{(2k+2l-2)} \quad (5.8)
\]

which follows from the algebra

\[
\{ \Sigma^{\mu\nu}, \Sigma^{\nu\lambda} \} = g^{\mu\kappa} \Sigma^{\nu\lambda} - g^{\mu\lambda} \Sigma^{\nu\kappa} - g^{\nu\kappa} \Sigma^{\mu\lambda} + g^{\nu\lambda} \Sigma^{\mu\kappa} \quad (5.9)
\]

Like previous case here also putting \(l = 1\) makes the algebra closed but not for any other arbitrary \(l\) value which require inclusion of infinite number higher-rank conserved charges

\[
\{ J^{(2k)}, J^{(2l)} \} \sim J^{(2k)} \quad (5.10)
\]

Next, we will look for conserved charges which are of the form \(Q^{(k)\mu_1\ldots\mu_k} p_{\mu_1} \ldots p_{\mu_k}\) which is
rank-\(k\) and satisfy

\[
0 = \left\{ Q^{(k)\mu_1\ldots\mu_k} p_{\mu_1} \ldots p_{\mu_k}, g^{\mu \nu} p_{\mu} \right\}
\]

\[
= \frac{1}{(n + 1)!} \nabla_{(\mu} Q^{(k)\mu_1\ldots\mu_k) p_{\rho_1} \ldots p_{\rho_k} p^{\rho_1} p^{\rho_k} p^\mu
\]

\[
+ \sum_{i=1}^k Q^{(k)\mu_1\ldots\mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \{ p_{\mu_i}, p_{i+1} \} p_{\mu_{i+1}} \ldots p_{\mu_k} g^{\mu \nu} p_{\nu}
\]

\[
= \frac{1}{(n + 1)!} \nabla_{(\mu} Q^{(k)\mu_1\ldots\mu_k) p_{\rho_1} \ldots p_{\rho_k} p^{\rho_1} p^{\rho_k} p^\mu
\]

\[
+ \frac{1}{2} \sum_{i=1}^k Q^{(k)\mu_1\ldots\mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \Sigma_{\kappa \lambda}^\mu \mathcal{R}_{\kappa \lambda \mu_1 \mu_k} p_{\mu_{i+1}} \ldots p_{\mu_k} g^{\mu \nu} p_{\nu}
\]

\[
= \frac{1}{(n + 1)!} \nabla_{(\mu} Q^{(k)\mu_1\ldots\mu_k) p_{\rho_1} \ldots p_{\rho_k} p^{\rho_1} p^{\rho_k} p^\mu
\]

\[
+ \frac{1}{2} \sum_{i=1}^k Q^{(k)\mu_1\ldots\mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \Sigma_{\kappa \lambda}^\mu \mathcal{R}_{\kappa \lambda \mu_1 \mu_k} p_{\mu_{i+1}} \ldots p_{\mu_k} g^{\mu \nu} p_{\nu}
\]

\[
= \frac{1}{(n + 1)!} \nabla_{(\mu} Q^{(k)\mu_1\ldots\mu_k) p_{\rho_1} \ldots p_{\rho_k} p^{\rho_1} p^{\rho_k} p^\mu
\]

\[
+ \frac{1}{2} \sum_{i=1}^k Q^{(k)\mu_1\ldots\mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \Sigma_{\kappa \lambda}^\mu \mathcal{R}_{\kappa \lambda \mu_1 \mu_k} p_{\mu_{i+1}} \ldots p_{\mu_k}
\]

\[
\Rightarrow \nabla_{(\mu} Q^{(k)\mu_1\ldots\mu_k)} = 0, \quad \sum_{i=1}^k Q^{(k)\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_k} \mathcal{R}_{\kappa \lambda}^{\mu_i} p_{\mu_{i+1}} \ldots p_{\mu_k} = 0
\]

where the second condition in the last line implies

\[
\sum_{i=1}^k Q^{(k)\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_k} \mathcal{R}_{\kappa \lambda}^{\mu_i} p_{\mu_{i+1}} \ldots p_{\mu_k} = [\nabla_\kappa, \nabla_\lambda] Q^{(k)\mu_1\ldots\mu_k} = 0
\]

(5.12)

As we can see that the inclusion of non-zero spin to the system in curved spacetime adds further condition on \(Q_{(k)}\) in order to make it conserved quantity.

This is an important point to notice unlike previous case here

\[
\{ Q^{(k)}, Q^{(l)} \} \sim Q^{(k+l-1)}
\]

(5.13)

since in presence of non-zero spin

\[
\{ p_\mu, p_\nu \} = \frac{1}{2} \Sigma_{\kappa \lambda}^\mu \mathcal{R}_{\kappa \lambda \mu \nu}
\]

(5.14)

which makes the Poisson-bracket between these conserved quantities not closed.

Now we are looking conserved quantities which are mixed both in \(\Sigma^{\mu \nu}\)s and \(p_\lambda\)s in following form

\[
C^{(2k,l)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k \lambda_1 \ldots \lambda_l} \Sigma_{\mu_1 \nu_1} \ldots \Sigma_{\mu_k \nu_k} p_{\lambda_1} \ldots p_{\lambda_l}
\]

(5.15)

which is of rank-(2k,l).
Conservation above quantities put following conditions on these quantities:

\[
0 = \{ C(2k,l) \lambda_1 \ldots \lambda_l \lambda_{i_1} \ldots \lambda_{i_l}, \Sigma_{\mu_1 \nu_1} \ldots \Sigma_{\mu_k \nu_k} p_{\lambda_1} \ldots p_{\lambda_l} g_{\mu \nu} p \}_{\mu \nu}
\]

\[
\Rightarrow \nabla_{\mu} C(2k,l) \lambda_1 \ldots \lambda_l \lambda_i = 0
\]

In the above set of condition, first equation refers to covariant derivative acting on \( \{ \lambda_i \} \) indices and third equation refers to covariant derivative acts on lower indices pairwise \((\mu_i, \nu_i)\).

Note that this suggests

\[
\{ J(2k), J(2l) \} \sim J(2k+2l-2)
\]

\[
\{ Q^{(k)}, Q^{(l)} \} \sim Q^{(k+l-1)} + \mathcal{C}^{(2k+l-2)}
\]

\[
\{ \mathcal{C}(2k,l), \mathcal{C}(2m,n) \} \sim \mathcal{C}^{(2k+2m, l+n-1)} + \mathcal{C}^{(2k+2m-2, l+n)} + \mathcal{C}^{(2k+2m+2, l+n-2)}
\]

which form a closed algebra under Poisson bracket.

Whole procedure can be repeated again in principle for analyzing the case for Hamiltonians linear in momentum.

Importance of finding such constants of motion or conserved quantities is that they are helpful in the analysis of spinning particle dynamics. An obvious such conserved quantity is the total spin

\[
I = \frac{1}{2} g_{\mu \nu} \Sigma_{\mu \nu} \Sigma_{\lambda \kappa} = S_{\mu} S^{\mu} + Z_{\mu} Z^{\mu}
\]

These quantities can be estimated from the operation of advanced LIGO detector [43–45] in near future using the spin memory effects of Gravitational waves.

6 Probing dynamical features of objects in curved spacetime

In this section, we would like to point out the usefulness of finding conserved quantities in order to probe dynamical features of objects in curved spacetime. As an example we consider a Hamiltonian of a moving object whose spin is dynamically coupled to curvature of spacetime in following way

\[
H = H_0 + H_{\Sigma}
\]

\[
H_0 = \frac{1}{2} g_{\mu \nu} p_{\mu} p_{\nu}
\]

\[
H_{\Sigma} = \kappa \mathcal{R}_{\mu \nu \lambda \kappa} \Sigma_{\mu \nu} \Sigma_{\lambda \kappa}
\]

where \( \kappa \) is a dimensionful quantity and strength of it is comparably small which also measures geodesic deviations of the object in the curved spacetime.

In this case, conditions that quantities need to be satisfied in order to become conserved charges becomes different because additional term of the Hamiltonian which captures spin-curvature coupling.

Now let’s start finding out the conditions for conserved quantities of different form in this case. First, start with following kind of quantities

\[
0 = \{ J^{(2k)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k}, \Sigma_{\mu_1 \nu_1} \ldots \Sigma_{\mu_k \nu_k}, H_0 + H_{\Sigma} \}
\]

\[
= p^\mu \nabla_{\mu} J^{(2k)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \Sigma_{\mu_1 \nu_1} \ldots \Sigma_{\mu_k \nu_k}
\]

\[
+ \{ J^{(2k)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k}, \Sigma_{\mu_1 \nu_1} \ldots \Sigma_{\mu_k \nu_k}, H_{\Sigma} \}
\]

(6.2)
Now we will figure out the second term in the above equation, since first term is already derived earlier.

\[
\{ \mathcal{J}^{(2k)}_{\mu_1 \ldots \mu_k, \nu_1 \ldots \nu_k}, H_\Sigma \} \\
= \frac{k}{2} \{ \mathcal{J}^{(2k)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_k \nu_k}, \Sigma^{\mu \nu} \} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\kappa \lambda} \\
= \frac{k}{2} \sum_{i=1}^{k} \mathcal{J}^{(2k)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_{i-1} \nu_{i-1}} \{ \Sigma^{\mu_i \nu_i}, \Sigma^{\mu_i \nu_i} \} \Sigma^{\mu_i+1 \nu_{i+1}} \ldots \Sigma^{\mu_k \nu_k} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\kappa \lambda} \\
= \frac{k}{2} \sum_{i=1}^{k} \mathcal{J}^{(2k)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_{i-1} \nu_{i-1}} (g^{\mu_i \nu_i} \nu_i \mu_i - g^{\mu_i \nu_i} \nu_i \mu_i + g^{\mu_i \nu_i} \nu_i \mu_i) \\
\times \Sigma^{\mu_{i+1} \nu_{i+1}} \ldots \Sigma^{\mu_k \nu_k} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\kappa \lambda} \\
= \frac{k}{2} \sum_{i=1}^{k} \left[ \mathcal{J}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\mu_i \nu_i} + \mathcal{J}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\mu_i \nu_i} - \mathcal{J}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\mu_i \nu_i} \right] \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_{i-1} \nu_{i-1}} \Sigma^{\mu_{i+1} \nu_{i+1}} \ldots \Sigma^{\mu_k \nu_k} \Sigma^{\kappa \lambda} \\
= \mathcal{J}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_k \nu_k} \Sigma^{\kappa \lambda} \\
(6.3)
\]

Hence, in a nutshell the condition put in (6.2) becomes

\[
p^{\mu} \nabla_\mu \mathcal{J}^{(2k)}_{\mu_1 \ldots \mu_k} - \kappa \Sigma^{\kappa \lambda} \left[ \nabla_\kappa \nabla_\lambda \mathcal{J}^{(2k)}_{\mu_1 \ldots \mu_k} \right] = 0, \quad \forall p_\mu, \Sigma^{\kappa \lambda}
\]

\[
\Rightarrow \nabla_\mu \mathcal{J}^{(2k)}_{\mu_1 \ldots \mu_k} = 0 = \left[ \nabla_\kappa \nabla_\lambda \mathcal{J}^{(2k)}_{\mu_1 \ldots \mu_k} \right] (6.4)
\]

Therefore, above two conditions need to be satisfied at same time in order \( \mathcal{J}^{(2k)}_{\mu_1 \nu_1 \ldots \mu_k \nu_k} \Sigma^{\mu_1 \nu_1} \ldots \Sigma^{\mu_k \nu_k} \) to be conserved charge.

For one more class of quantities we will show the conditions need to be imposed in order to make it conserved charge which is following

\[
0 = \{ Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_k}, H_0 + H_\Sigma \} \\
= \frac{1}{(n+1)!} p^{\mu} \nabla_\mu Q^{(k)}_{\mu_1 \ldots \mu_k} p^{\mu_1} \ldots p^{\mu_k} + \frac{1}{2} \Sigma^{\kappa \lambda} \left[ \nabla_\kappa \nabla_\lambda Q^{(k)}_{\mu_1 \ldots \mu_k} \right] p^{\mu_1} \ldots p^{\mu_k} \\
+ \{ Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_k}, H_\Sigma \} \\
(6.5)
\]
So similarly we need to evaluate the following quantity

\[
\{ Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_k}, H_\Sigma \}
\]

\[
= \frac{\kappa}{2} \{ Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_k}, \Sigma_{\mu \nu} \} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\kappa \lambda}
\]

\[
= \frac{\kappa}{2} \sum_{i=1}^{k} Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \{ p_{\mu_i}, \Sigma_{\mu \nu} \} p_{\mu_{i+1}} \ldots p_{\mu_k} \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\kappa \lambda}
\]

\[
- \frac{\kappa}{2} \sum_{i=1}^{k} Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \frac{\partial \mathcal{R}_{\mu \nu \kappa \lambda}}{\partial x^{\mu_i}} p_{\mu_{i+1}} \ldots p_{\mu_k} \Sigma^{\mu \nu} \Sigma^{\kappa \lambda}
\]

\[
= -\frac{\kappa}{2} \sum_{i=1}^{k} Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \left( \Sigma_{\mu \nu} \frac{\partial \mathcal{R}_{\mu \nu \kappa \lambda}}{\partial x^{\mu_i}} + \Gamma^\mu_{\mu \rho} \Sigma^{\mu \rho} \mathcal{R}_{\mu \nu \kappa \lambda} - \Sigma_{\mu \nu} \Sigma^{\mu \rho} \mathcal{R}_{\mu \nu \kappa \lambda} \right) p_{\mu_{i+1}} \ldots p_{\mu_k} \Sigma^{\kappa \lambda}
\]

\[
- \frac{\kappa}{2} \sum_{i=1}^{k} Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \left( \frac{\partial (\Sigma_{\mu \nu} R_{\mu \nu \kappa \lambda})}{\partial x^{\mu_i}} - \Gamma^\mu_{\mu \rho} \Sigma^{\mu \rho} R_{\mu \nu \kappa \lambda} - \Gamma^\nu_{\mu \rho} \Sigma^{\mu \rho} R_{\mu \nu \kappa \lambda} \right) p_{\mu_{i+1}} \ldots p_{\mu_k} \Sigma^{\kappa \lambda}
\]

\[
\times p_{\mu_{i+1}} \ldots p_{\mu_k} \Sigma^{\kappa \lambda}
\]

(6.6)

Let’s define an anti-symmetric rank-2 tensor \( \tilde{\Sigma}_{\kappa \lambda} \equiv \mathcal{R}_{\mu \nu \kappa \lambda} \Sigma^{\mu \nu} \), then

\[
\Rightarrow \{ Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_k}, H_\Sigma \}
\]

\[
= -\frac{\kappa}{2} \sum_{i=1}^{k} Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \left( \Sigma_{\mu \nu} \frac{\partial \tilde{\Sigma}_{\kappa \lambda}}{\partial x^{\mu_i}} - \Gamma^\mu_{\mu \rho} \Sigma^{\mu \rho} \tilde{\Sigma}_{\mu \nu} - \Gamma^\nu_{\mu \rho} \Sigma^{\mu \rho} \tilde{\Sigma}_{\mu \nu} \right) p_{\mu_{i+1}} \ldots p_{\mu_k}
\]

\[
= -\frac{\kappa}{2} \sum_{i=1}^{k} Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \left( \frac{\partial \tilde{\Sigma}_{\mu \nu}}{\partial x^{\mu_i}} - \Gamma^\mu_{\mu \rho} \tilde{\Sigma}_{\mu \rho} - \Gamma^\nu_{\mu \rho} \tilde{\Sigma}_{\mu \rho} \right) p_{\mu_{i+1}} \ldots p_{\mu_k}
\]

\[
= -\frac{\kappa}{2} \sum_{i=1}^{k} Q^{(k)}_{\mu_1 \ldots \mu_k} p_{\mu_1} \ldots p_{\mu_{i-1}} \left( \Sigma^{\mu \nu} \nabla_{\mu} \tilde{\Sigma}_{\nu \kappa} \right) p_{\mu_{i+1}} \ldots p_{\mu_k}
\]

\[
= -\frac{\kappa}{2} p_{\mu_1} \ldots p_{\mu_{k-1}} \left( Q^{(k)}_{\lambda_1 \ldots \lambda_{k-1}} \Sigma^{\mu \nu} \nabla_{\lambda} \tilde{\Sigma}_{\mu \nu} + \ldots + Q^{(k)}_{\mu_1 \ldots \mu_k-1} \Sigma^{\mu \nu} \nabla_{\lambda} \tilde{\Sigma}_{\mu \nu} \right)
\]

\[
= -\frac{\kappa}{2} p_{\mu_1} \ldots p_{\mu_{k-1}} \left( Q^{(k)}_{\lambda_1 \ldots \lambda_{k-1}} \Sigma^{\mu \nu} \nabla_{\lambda} \tilde{\Sigma}_{\mu \nu} \right)
\]

(6.7)

Therefore, altogether we have 3 conditions, need to be imposed

\[
\nabla_{\kappa} Q^{(k)}_{\mu_1 \ldots \mu_k} = 0, \ [\nabla_{\kappa}, \nabla_{\lambda}] Q^{(k)}_{\mu_1 \ldots \mu_k} = 0
\]

\[
Q^{(k)}_{\lambda_1 \ldots \lambda_{k-1}} \nabla_{\lambda} \tilde{\Sigma}_{\mu \nu} \Sigma^{\mu \nu} = 0
\]

(6.8)
in order to $Q^{(k)\mu_1...\mu_k}p_{\mu_1}...p_{\mu_k}$ be called conserved charge.

There is one more type of quantities one which conditions need to be imposed in order to make them conserved charge which are mixed in momentum and $\Sigma^{\mu\nu}$ but from the above mathematical procedure one can obtain those, hence we leave it as a task to check.

Note further that algebra between these charges remains as described in previous section since their construction is same.

Once these charge quantities are identified then one can talk about the nature of these kind of non-minimal coupled interactions from the data of LIGO, hence nature of Hamiltonian and its corresponding dynamics can be estimated. One can also matches the result consistently with measure geodesic deviations due to these kinds of coupling term in the Hamiltonian.

7 Conclusion

The gravitational two-body problem attracts a great amount of attention and interest long before the origin of General Relativity. In Newtonian mechanics, an isolated system of two point particles interacting under gravity is exactly solvable and the resulting motion is periodic. The energy and angular momentum are represented by two conserved integrals of motion which is a celebrated result in physics community. The dynamics of binaries are non-linear in nature in General Relativity due to its inherent construction and hence, the orbits are never periodic as the system emits gravitational waves it continuously loses energy and angular momentum which is not the case for Newtonian Gravity. That’s why looking at the conserved quantities which remains invariant under gravitational radiation is important in order to predict dynamics or extract information about dynamics in curved spacetime. Further, we have shown the conserved quantities in a generic curved spacetime with no Killing vectors at all and most importantly all these are derived based on Poisson bracket defined over phase space for dynamical objects in curved spacetime. Conserved quantities also play a huge role in numerical relativity in terms of generating new solutions for example, by adding a specified amount of angular momentum or spin to a solution of the vacuum Einstein equations, producing a new solution with specified angular momentum or spin but with only slightly perturbed energy-momentum vector for asymptotic flat spacetimes. Then, by taking a family of initial data set near the given one, and by doing the construction continuously, we obtain a perturbation with arbitrarily specified angular momentum while leaving the energy-momentum vector unchanged.

In this report, a wide class of conditions have been derived in most general form in order conserved charge to exist in a dynamical system which are made out of phase space variables and if it exists what are the algebra between them under Poisson bracket both in free-particle case and also with non-minimal spin-curvature coupling. These charges can be identified with symmetries inherited in spacetime, which acts as generators which is important in finding out quantities which are invariant under that symmetry. The conserved charges are useful in identifying hidden dynamical degrees of freedom, which are not apparent, like spin which we have discussed.

Further, it is also discussed that Gravitational memory and lensing effect could be used to extract such conserved quantities. This would help ultimately in determining the complete dynamics of any astrophysical objects and henceforth, correct the correct Hamiltonian that governs their dynamics. Knowing correct Hamiltonian also leads to the understanding of how these astrophysical objects (in terms of their momentum, spin etc.) coupled to spacetime in
terms of metric, Riemann tensors, Ricci-tensors etc. These information would help a lot in understanding complete dynamics of astrophysical objects under extreme conditions.

In order to find those conserved quantities it is important to invest a great amount of time in solving those mentioned partial differential equations at least numerically with suitable approach, if analytical treatment is not possible at all.

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