Latest Results on Lattice Calculation Concerning $K \rightarrow \pi \ell^+ \ell^-$ Decays

P Boyle$^{1,2}$, A Jüttner$^3$, F Ó hÓgáin$^1$ and A Portelli$^1$.

$^1$ School of Physics, University of Edinburgh, Edinburgh EH9 3JZ, UK
$^2$ Physics Department, Brookhaven National Laboratory, Upton NY, USA
$^3$ School of Physics and Astronomy, University of Southampton Southampton, SO17 1BJ, UK
E-mail: fionn.o.hogain@ed.ac.uk

Abstract. The $K \rightarrow \pi \ell^+ \ell^-$ decay is a flavor changing neutral current process which is forbidden at tree level in the Standard Model. This suppression causes the decay to be sensitive to potential New Physics. The decay channels are dominated by long-distance contributions, which require non-perturbative methods of investigation. Previous lattice calculations by the RBC and UKQCD collaborations, at unphysical kaon/pion masses, have successfully extracted the matrix elements needed to describe the form factor of the decays. A new lattice calculation, on a gauge configuration with $m_\pi \approx 140$ MeV and $m_K \approx 500$ MeV, is underway and will be discussed here.

1. Introduction

The rare kaon decays $K \rightarrow \pi \nu \bar{\nu}$ and $K \rightarrow \pi \ell^+ \ell^-$ are flavor changing neutral current (FCNC) processes that occur in the Standard Model (SM) through WW and $\gamma/Z$ exchange diagrams. These second order electroweak interactions are suppressed in the SM and so these decays can be used to probe areas of potential New Physics, such as lepton flavor universality violation [1].

Experiments are underway at NA62 at CERN to measure the decay rate of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ [2][3], at KOTO at J-PARC to measure the decay rate of $K_L \rightarrow \pi^0 \nu \bar{\nu}$ [4], and there are prospective experiments planned at LHCb to study the $K_S \rightarrow \pi^0 \ell^+ \ell^-$ decay [5]. In order to aid comparisons of experimental and theoretical results the RBC and UKQCD collaborations have performed an exploratory calculation of the long distance contributions to the $\nu \bar{\nu}$ channel at unphysical pion mass [6] and are currently performing a calculation at close to physical physical pion mass [7]. An exploratory calculation of the $\ell^+ \ell^-$ channel has been performed at unphysical masses [8].

For $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ the relevant long-distance Minkowski amplitude that is to be computed is given by

$$A^j_\mu(q^2) = \int d^4x \left\langle \pi^j(p) | T [J_\mu(0) H_W(x)] | K^j(k) \right\rangle \quad (1)$$

where $q = k - p$ and $j = +, 0$. The electromagnetic current $J_\mu$ is the flavor-diagonal operator

$$J_\mu = \frac{1}{3} \left( 2V^u_\mu - V^d_\mu - V^s_\mu + 2V^c_\mu \right). \quad (2)$$
The effective weak Hamiltonian is given by [9]

\[ H_W(x) = \frac{G_F}{\sqrt{2}} V_{us} V_{ud} [C_1(Q^q_1 - Q^q_1) + C_2(Q^q_2 - Q^q_2)] , \tag{3} \]

where the \( C_i \) are the Wilson coefficients in a chosen renormalization scheme. \( Q^q_{1,2} \) are the following current-current local operators:

\[ Q^q_1 = (s_a \gamma^L_\mu d_a)(\bar{q}_b \gamma^L \mu q_b) \quad \text{and} \quad Q^q_2 = (s_a \gamma^L_\mu q_a)(\bar{q}_b \gamma^L \mu d_b) , \tag{4} \]

where \( a \) and \( b \) are summed color indices and \( \gamma^L_\mu = \gamma_\mu (1 - \gamma_5) \).

Using electromagnetic gauge invariance the nonlocal matrix element in equation 1 can be written in terms of an electromagnetic form factor, \( V(z) \), [10]

\[ A^j_q(z^2) = -i G_F V^j_q(z^2) \left[ q^2(k + p)_\mu - (M_K^2 - M^2_\pi)q_\mu \right] \tag{5} \]

where \( z = q^2/M^2_K \). This form factor can be parameterized as

\[ V_j(z) = a_j + b_j z + V_{j,\pi\pi}(z) . \tag{6} \]

\( V_{\pi\pi}(z) \) is introduced to account for \( \gamma^* \to \pi\pi \) effects and the coefficients \( a_j \) and \( b_j \) are found by fitting experimental spectra and it is our aim to perform ab initio calculations of these coefficients using Lattice QCD.

2. Lattice Implementation

In order to do so we must consider the time-integrated 4pt correlator,

\[ I^j_q(T_a,T_b) = \int_{T_a}^{T_b} dt_H \left\langle \pi^j(p)|T[J_{\mu}(0)H_W(t_H)]|K^j(k)\right\rangle , \tag{7} \]

in the limit \( T_a,T_b \to \infty \). The Euclidean spectral decomposition for this correlator is given by

\[ I^j_q(T_a,T_b) = -\int_{0}^{\infty} dE \frac{\rho(E)}{2E} \frac{\left< \pi^j(p) | J_{\mu} | E,k \right> \left< E,k | H_W | K^j(k) \right>}{E_K(k) - E} \left( 1 - e^{(E_K(k) - E)T_a} \right) \]

\[ + \int_{0}^{\infty} dE \frac{\rho_s(E)}{2E} \frac{\left< \pi^j(p) | H_W | E,p \right> \left< E,p | J_{\mu} | K^j(k) \right>}{E - E_s(p)} \left( 1 - e^{-(E - E_s(p))T_b} \right) . \tag{8} \]

Contributions from states with \( E < E_K(k) \) diverge exponentially as \( T_a \to \infty \) and must be subtracted. Two methods for removing the single pion exponentially growing term have been and have been seen to be successful [8]. The first option is to measure the \( \left< \pi^j(p) | J_{\mu} | \pi^j(k) \right> \) and \( \left< \pi^j(k) | H_W | K^j(k) \right> \) matrix elements explicitly such the the exponentially growing term can be calculated and subtracted. The second option is to shift the effective weak Hamiltonian by \( c_s \bar{s}d \) and tune the \( c_s \) parameter such that the \( K \to \pi \) matrix element vanishes:

\[ \left< \pi^j(k) | H'_W | K^j(k) \right> = \left< \pi^j(k) | H_W | K^j(k) \right> - c_s \left< \pi^j(k) | \bar{s}d | K^j(k) \right> . \tag{9} \]

The two- and three- pion states did not contribute in the exploratory simulation as \( m_\pi \approx 430 \) MeV and \( m_K \approx 600 \) MeV. For the physical point calculation the \( \pi\pi \) state is not expected to contribute to this decay due to parity conservation and the \( \pi\pi\pi \) state contribution is expected
Figure 1. The diagrams contributing to the Wick contractions for $K \rightarrow \pi H_W$ 3-pt functions.

...to be negligibly small for the target precision of the simulation.

Defining the subtracted integrated correlator as $\bar{I}_\mu$ we recover the rare kaon decay by taking the limit

$$A_\mu^i(q^2) = -i \lim_{T_{a,b} \rightarrow \infty} \bar{I}_\mu(q^2, T_a, T_b).$$

The four classes of diagrams that result from performing the Wick contractions in equation 1 are shown in figure 1. The currents are inserted on each quark line in each topology, as well as on an additional disconnected loop. This results in 20 total diagrams to compute. Logarithmic divergences arise in the Eye and Saucer diagrams as $J_{\mu}$ and $H_W$ approach each other. By simulating up and charm quarks for the loops the GIM mechanism [11] will be employed to cancel this divergence.

3. Previous Exploratory Calculation
The exploratory calculation was performed on a $24^3 \times 64$ lattice with an inverse lattice spacing of $a^{-1} = 1.78$ GeV, using Shamir Domain Wall Fermions (DWFs) with Iwasaki Gauge action [12]. The pion mass was $\sim 430$ MeV, the kaon mass $\sim 625$ MeV and an unphysical charm quark mass of $m_c^{MS}(2 \text{ GeV}) = 533$ MeV was used for the GIM subtraction. 128 configurations were used for the sample, separated by 20 molecular dynamics time units. The $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ decay was studied with the kaon at rest decaying into a pion with momentum $p = \frac{1}{L} (1, 0, 0)$, $p = \frac{1}{L} (1, 1, 0)$, $p = \frac{1}{L} (1, 1, 1)$, where $L$ is the spatial length of the lattice. The kaon and pion were separated in time by 28 lattice units with the current inserted halfway between them.

The exponentially growing term in equation 8 as $T_{a,b} \rightarrow \infty$ can be seen in figure 2 along with one method of subtraction of the term for $p = \frac{1}{L} (1, 0, 0)$, different methods for the subtraction are shown in figure 3 for the three momenta, and the $z$-dependence of the form factor from equation 6 is plotted in figure 4. The results obtained for the $V_\perp(z)$ parameters differ from phenomenological fits to experimental data however this is to be expected given the unphysical masses used in the calculation.

4. Current Physical Point Calculation
The physical point calculation is being performed on a $48^3 \times 96$ lattice with an inverse lattice spacing of $a^{-1} = 1.73$ GeV, using ZMöbius DWFs for the light (degenerate up/down) and charm quarks and Shamir DWFs for the strange quarks [13]. The pion mass is $\sim 140$ MeV, the kaon mass $\sim 500$ MeV. Three unphysical charm quark masses of $am_c = 0.20, 0.25, 0.30$ will be used to extrapolate to the physical charm quark mass. The need for this extrapolation is that the same quark action must be used for the up and the charm in order to perform the GIM subtraction but the ZMöbius action breaks down for heavy quark masses. 60 configurations are currently...
Figure 2. The integrated 4pt correlator for $\int_{t_J-A}^{t_J+A} \tilde{\Gamma}_0^{(4)} dt_H$, where $\tilde{\Gamma}_0^{(4)}$ is the normalized 4-point correlator and the subtraction is performed by measuring the matrix elements explicitly.

Figure 3. Amplitudes (in lattice units) obtained using different methods of subtracting the exponentially growing contribution.

Figure 4. Dependence of the form factor for the decay $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ on $z = q^2/M_K^2$.

available to use for the sampling, separated by 20 molecular dynamics time units, and the same kinematic set up as the exploratory calculation will be used with the kaon and pion separated in time by 32 lattice units and the current inserted halfway between them.

Currently the non-eye (Wing and Connected) diagrams have been computed on 15 of these configurations, $\Gamma_0^{(4)}$ and $I_0^{(4)}$ are plotted without normalization factors for the non-eye contributions in figures 5 and 6.
5. Future Outlook and Conclusions

Further noise reduction techniques for the loop must be investigated in order to compute the eye diagrams for the first set of kinematics and the signal for the logarithmic divergence on the loop must be carefully observed going forward. If the charm quark extrapolation is not sufficient a 3-flavor theory may be used along with non-perturbative renormalization. If these results are produced with a sufficiently clear signal then the first pion momentum investigated gives a kinematic result of \( z = 0.01 \), which will give a good indication of what the value of \( a_+ \) is. More results would then follow, with further configuration and momenta being run, such that a fit of the lattice data to a linear ansatz could be performed in order to obtain \( a_+ \) and \( b_+ \).

Acknowledgments

P.B. received support from the Royal Society Wolfson Research Merit award WM/60035. A.J. acknowledges funding from STFC consolidated grant ST/P000711/1. F.Ó.h. is funded by a scholarship from the Scottish Funding Council. A.P. is supported in part by UK STFC grants ST/P000630/1. A.P. also received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme under grant agreements No 757646 & 813942. This work used the DiRAC Extreme Scaling service at the University of Edinburgh, operated by the Edinburgh Parallel Computing Centre on behalf of the STFC DiRAC HPC Facility (www.dirac.ac.uk). The equipment was funded by BEIS capital funding via STFC capital grants ST/R00238X/1 and ST/S002537/1 and STFC DiRAC Operations grant ST/R001006/1. DiRAC is part of the National e-Infrastructure.

References

[1] Crivellin A et al. 2016 Phys. Rev. D 93 074038
[2] Gil E C 2019 Physics Letters B 791 156
[3] Gil E C 2019 Physics Letters B 797 134794
[4] Ahn J K et al. 2019 Phys. Rev. Lett. 122 021802
[5] Alves A A et al. 2019 J. High Energ. Phys. 2019
[6] Bai Z et al. 2017 Phys. Rev. Lett. 118 252001
[7] Christ N et al. 2019 Preprint arXiv:1910.10644
[8] Christ N et al. 2016 Phys. Rev. D 94 114516
[9] Isidori G et al. 2006 Physics Letters D 633 75
[10] D’Ambrosio G et al. 1998 Journal of High Energy Physics 08 004
[11] Glashow S et al. 1970 Phys. Rev. D 2 1285
[12] Aoki Y et al. 2011 Phys. Rev. D 83 074508
[13] Blum T et al. 2016 Phys. Rev. D 93 074505