Analysis of Cummer–Schurig acoustic cloaking

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Abstract. This paper presents an in-depth analysis of the scattering characteristics of the Cummer–Schurig acoustic cloaking design proposed by Cummer and Schurig (2007 \textit{New J. Phys.} 9 45). The analysis uses an analytical solution for orthotropic media in conjunction with approximating the radially varying properties by multiple layers of uniform shells. The analysis shows that the cloaking is effective for both planar incident waves and line sources, but not perfect for the entire computed spatial frequency range up to $ka = 10$, where $k$ is the wavenumber in the host medium and $a$ is the outer radius of the cloaking shell. Furthermore, the cloaking remains effective but less perfect when the cloaked region is not rigid. For a penetrable medium in the cloaked region, the cloaking is penetrated only at the resonant frequencies.

Contents

1. Introduction
2. Theory
3. Results and discussions
4. Conclusions
Acknowledgments
References

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1. Introduction

In the past few years, the concept of metamaterials has emerged as the result of active research on photonic and phononic band gap materials. Such materials exhibit novel properties in response to wave loading when the wavelength is much greater than the feature length of the material. These novel properties include negative permittivity and negative permeability for electromagnetic waves, and negative mass density and negative bulk modulus for mechanical waves. One of the applications of such materials that has attracted much attention from the mass media is the so-called cloaking device, owing to its Star Trek fame. Two cloaking designs for electromagnetic waves were proposed in 2006 [1, 2]. Later in the same year, cloaking of electromagnetic waves was demonstrated [3] using one of the designs.

One of the above-mentioned designs, by Leonhardt [2], with more detailed explanations given in [4], is based on conformal mapping and is also applicable to acoustic waves. Connections between cloaking, negative refraction, perfect lenses and event horizons have been discussed by Leonhardt and Philbin [5]. In fact, the idea of acoustic cloaking was first explored by Milton et al [6]. Their paper also described a long and interesting history of invisibility. Recently, the other design for electromagnetic waves has also been extended to acoustic cloaking by Cummer and Schurig [7], two members of the team that demonstrated the electromagnetic cloaking.

In this paper, the proposed Cummer–Schurig design for acoustic cloaking is analyzed. The design has three main ingredients: axisymmetric geometry, anisotropic mass density and radial gradient material properties. The mass density is a tensor having only non-zero entries in the diagonal entries when using a polar coordinate system. Such an anisotropy may be appropriately called circular orthotropy. This mass anisotropy gives rise to the following acoustic equations [7] in a cylindrical coordinate system (r and θ), for a steady-state problem at frequency ω:

\[
\begin{align*}
\rho_\theta v_\theta &= -\frac{1}{r} \frac{\partial p}{\partial \theta}, \\
\rho_r v_r &= -\frac{1}{r} \frac{\partial p}{\partial r}, \\
\rho_\theta v_\theta &= -\frac{1}{\lambda} \left( \frac{1}{r} \frac{\partial (rv_r)}{\partial r} - \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \right),
\end{align*}
\]

where \(\lambda\) is the bulk modulus, \(v_r\) and \(v_\theta\) are components of the particle velocity vector, \(p\) is the acoustic pressure, \(\rho_r\) and \(\rho_\theta\) are the components of the diagonal mass density tensor, and \(i = \sqrt{-1}\). The gradient material properties are

\[
\begin{align*}
\rho_r &= \frac{r}{r - r_1}, \\
\rho_\theta &= \frac{r - r_1}{r}, \\
\frac{\lambda_\theta}{\lambda_0} &= \left( \frac{r - r_1}{r - r_2} \right)^2 \frac{r}{r - r_1},
\end{align*}
\]

where \(r_1\) and \(r_2\) are the inner and outer radii of the cloaking shell, \(\lambda_\theta\) is the bulk modulus of the cloaking shell, and quantities with subscript 0 are those of the host medium. This design
exemplifies one of new challenges posed by the concept of metamaterials: many conventional assumptions about materials are no longer valid, and new analysis tools are needed.

In this paper, the Cummer–Schurig acoustic cloaking shell is approximated as a multi-layer scatterer and then analyzed, by utilizing two theoretical developments. Furthermore, letting the layers become infinitesimally thin, the solution would approach the exact solution. This procedure allows analyses of the scattering characteristics of the cloaking shell that are generally not possible with other numerical methods.

2. Theory

The first theoretical development is the general solution for acoustical scattering in a uniform circularly orthotropic material. Equations (1) through (3) can be combined, by eliminating the velocities $v_r$ and $v_\theta$, to obtain the following equation for acoustic pressure $p$:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2 \rho_\theta} \frac{\partial^2 p}{\partial \theta^2} + \frac{\omega^2 \rho_r}{\lambda} p = 0.$$

This equation can be solved via the classical separation of variables method, giving the following general solution:

$$p(r, \theta) = \sum_{n=0}^{\infty} \left[ \alpha_n J_{\nu_n}(kr) + \beta_n J_{-\nu_n}(kr) \right] e^{in\theta},$$

where $\alpha_n$ and $\beta_n$ are constants yet to be determined, $k$ is the wavenumber defined as,

$$k = \omega \sqrt{\frac{\rho_r}{\lambda}},$$

and $J_{\nu_n}(\cdot)$ is a Bessel function of the first kind of order $\nu_n$ with

$$\nu_n = n \sqrt{\frac{\rho_r}{\rho_\theta}}.$$

Note that, for non-integer orders, Bessel functions of the first kind $J_{\nu_n}(\cdot)$ and $J_{-\nu_n}(\cdot)$ are linearly independent [8]; and $J_{\nu_n}(kr)$ in the above solution can be replaced by other Bessel functions $Y_{\nu_n}(kr)$, $H^{(1)}_{\nu_n}(kr)$ or $H^{(2)}_{\nu_n}(kr)$.

This general solution is essentially the same as that for an isotropic medium, with the exceptions: (i) that the order $n$ of Bessel and Hankel functions is replaced by $\nu_n$, and (ii) that the wavenumber is determined by $\rho_r$. The material orthotropy does not affect the orthogonality of eigenfunction $e^{in\theta}$. Thus, all scattering solutions obtained for isotropic media remain the same, with the modifications just noted. For brevity, those solutions are not repeated here.

The second theoretical development is the solution procedure for acoustical scattering by a general multilayer scatterer. The term ‘general’ here refers to the fact that the number of layers is arbitrary. There are two basic approaches to analyzing a multi-layer cylindrical scatterer. One is directly listing all the continuity conditions at all the interfaces [9], resulting in a linear equation system which can be readily solved. Another approach is generally known as the transfer-matrix method, which was originally developed for a multilayer infinite medium [10], and was later extended to circular cylindrical geometries [11]. For a more detailed review of these basic approaches, the readers are referred to a paper by the present authors [13] that presents the novel
approach used in this paper. In order to accommodate the mass orthotropy of the layers, this novel approach is based on the multiple-scattering analysis within a single dual-layer scatterer.

A dual-layer scatterer consists of a core and an intermediate layer. The solution for a dual-layer scatterer was obtained by one of the authors [12] for the scattering of anti-plane elastic shear waves. The solution follows the physical process of multiple scattering in a single scatterer, leading to a solution form that is valid for all types of waves. Thus, the adaptation for acoustical scattering problems is straightforward.

This dual-layer solution is recursively used to obtain the solution for a general multilayer scatterer, as follows. Assume that the scatterer has \( N \) layers, which are numbered consecutively inward, with the host being layer 0 and the inner-most core being layer \( N \). In the first step, layers \( N \) and \( N - 1 \) are considered as a dual-layer scatterer embedded in a host made of the material of layer \( N - 2 \). In the next step, layers \( N \) and \( N - 1 \) are treated as a new composite core and layer \( N - 2 \) is the intermediate layer, embedded in a host made of the material of layer \( N - 3 \). This process is repeated until the host in the dual-layer problem is the actual host. The end result of this procedure is the \( T \)-matrix for the multilayer scatterer. Details of this solution procedure, as well as in-depth analyses of its computational characteristics, will be published elsewhere [13].

The Cummer–Schurig acoustic cloaking shell is then approximated as consisting of multiple uniform orthotropic layers of concentric thin shells. Such a shell system can be analyzed by a combined use of the solutions for an orthogonal medium and for a multilayer scatterer. Numerical difficulties are expected, due to the singularity in material properties: at the interface between the cloaked region and the cloaking shell. According to equations (4) and (5), at \( r = r_1 \), \( \rho_r \rightarrow \infty \) and \( \rho_\theta = 0 \). This singularity apparently leads to \( \nu_n \rightarrow \infty \). In computations, this limits the number of terms in the serial solution that can be included, and, in turn, the accuracy of the solution.

3. Results and discussions

In this analysis, it is assumed that the host is water, with a sound speed of 1450 m s\(^{-1}\) and a mass density of 1000 kg m\(^{-3}\). The radii of the cloaking shell are \( a \) and 0.5\( a \). The cloaked region is rigid. Both the host and the cloaked region are isotropic. To avoid the aforementioned singularity, equations (4) and (5) are slightly modified: \( r - r_1 \) is replaced by \( r - 0.99r_1 \).

Using 15 and 25 evenly divided layers, figure 1 shows the number of terms that can be truncated from the serial expressions before over- and under-flows occur, and the associated errors. The error is measured by the modulus of the last term in the resulting \( T \)-matrix, which is diagonal, for the concentric multilayer shell system. Using 10\(^{-2}\) as the error tolerance, figure 1 indicates that the maximum computable normalized spatial frequency is approximately \( ka = 21 \) and 12 for using 15 and 25 layers, respectively, where \( k \) is the wavenumber in the host. The computational results shown in the remainder of this paper use 25 layers.

The acoustic pressure field at a time instant, which is represented by the real part of the complex amplitude \( p \), is shown in figure 2 at spatial frequency \( ka = 9.5 \), in comparison with the result from [7], which was obtained for a case when the diameter of the cloaked region is 1.5 wavelengths, that is, \( ka = 3\pi \), using the finite difference method. Despite minor differences, the images show excellent agreement between the two computations, as well as a clear cloaking effect. The differences in these two computations include: a planar Gaussian beam was used as the incident wave in [7], whereas a planar incident wave is used in the present computation; the
Figure 1. Truncation terms and associated errors for using 15 and 25 layers to approximate the cloaking shell. The green horizontal line indicates the error tolerance of $10^{-2}$.

Figure 2. Pressure field comparison. Left: Cummer and Schurig [7], at $ka = 3\pi$ with a planar Gaussian beam as the incident wave. Right: present results, at $ka = 9.5$ with a planar incident wave. The color scale is for the present result only. Cummer and Schurig [7] do not provide a color scale.

difference in the wavenumber; and slight modifications for the mass gradients in order to avoid the singularity. This also confirms that the cloaking is rather robust, although not perfect.

Figure 3 shows the pressure distribution due to a line source of spatial frequency $ka = 9.5$ located at $(x, y) = (-3a, 0)$. The left image shows the snap shot of the pressure field (the real part of $p$), and the right image shows the amplitude of the pressure. All pressure fields are normalized by the pressure due to the line source in free space at a distance $a$ from the source. Since, theoretically, waves of different geometrical shapes of wavefronts can be produced by combining line sources of different strengths and phases in different arrangements, figure 3 implies that the cloaking works for any shapes of wavefronts.

Figure 4 shows the snap shots of pressure distributions when the Cummer–Schurig acoustic cloaking encloses a void (left) and a penetrable (right) region. In the latter case, the cloaked
Figure 3. Pressure distribution due to line source located at \((-3a, 0)\). Left: snapshot of the field. Right: total pressure amplitude. The color scale is for the right image; the color scale for the left image is the same as in figure 2.

Figure 4. Pressure distribution when the Cummer–Schurig acoustic cloaking shelters a void (left) and a penetrable (right) region. The color scale is the same as in figure 2.

region has the same material properties as the host (water). It is observed that the cloaking effect is noticeably weakened. However, it is interesting to observe that the two cases are almost indistinguishable based on the fields outside the cloaking shell. Since the void represents the other extreme end of the spectrum for possible material properties in the cloaked area, this figure suggests that the cloaking shell works for a wide range of possible materials in the cloaked region, although the best cloaking is in the original design; that is, a rigid core as the cloaked region.

In order to further analyze the last conclusion above, recall that the total acoustic pressure at any given point in the two-dimensional space can be written as

\[
p_{\text{total}}(r, \theta) = p_{\text{inc}}(r, \theta) + p_{\text{scr}}(r, \theta),
\]

where \(p_{\text{inc}}\) and \(p_{\text{scr}}\) are pressure contributions by the incident wave and the scattered wave, respectively. By using the asymptotic expression for Hankel functions for large arguments, the

New Journal of Physics 9 (2007) 450 (http://www.njp.org/)
pressure due to the scattered wave can be written as

\[
\lim_{r \to \infty} |p_{\text{sc}}| = \sqrt{\frac{2}{\pi kr}} f(\theta),
\]

(12)

where \( f(\theta) \) is called the scattering form factor when the incident wave is a planar wave of unit amplitude. The scattering form factor, also known as the scattering amplitude, is independent of radius \( r \) and thus is often used to describe the scattering pattern. Figure 5 shows the scattering form factor, normalized by that of the incident wave, for the Cummer–Schurig cloaking shell at \( ka = 9.5 \), in comparison with other three cases: the rigid core alone, a void as the cloaked region, and the cloaked region has the same properties as the host. In figure 5, the form factor is plotted in a linear scale. The form factor can be said to be normalized by the amplitude of the incident wave, since the incident wave is assumed to have a unit amplitude.

For the Cummer–Schurig cloaking, the scattering pattern has only one primary beam in the forward direction and it is almost quiescent in other directions. The other two cloaked cases show similar shapes but have higher pressure in the backward direction, indicating that the backward scattering is detectable but at a much lower level than the case of the rigid core alone. Figure 5 also shows that the form factors for the cases of void and water as the cloaked region are almost indistinguishable, consistent with an earlier observation from the pressure distribution at the same spatial frequency in the vicinity of the shell. However, this is not generally true, as will be noted later.

In order to observe the scattering characteristics of the Cummer–Schurig cloaking shell at different frequencies, a spectrum for the total scattering cross-section is computed for spatial frequencies up to \( ka = 10 \) and compared with the other three cases in figure 6. The total scattering cross-section, \( \sigma_t \), is defined as the total energy transmitted through a closed surface by the scattered wave in the far field when the incident wave is a planar wave of unit amplitude,
and has the following expression [14]:

$$
\sigma = \frac{2}{\pi k} \int_0^{2\pi} |f(\theta)|^2 \, d\theta. 
$$

This is a single numerical value that represents the overall scattering strength of a scatterer. It is observed that the total scattering cross-section for the Cummer–Schurig cloaking is significantly and consistently lower than the rigid core alone in the entire computed spatial frequency range, by an order of magnitude in a wide range of spatial frequency. At higher frequencies, the total scattering cross-section appears to be increasing at an increasing rate (positive second derivative) as the spatial frequency increases. This could be due to the errors from approximating continuously varying material gradients by a series of discrete layers. Such errors are generally more pronounced at higher frequencies. Unfortunately, the computational limitation as demonstrated in figure 1 prohibits further investigation of this hypothesis.

One of the interesting features in figure 6 is a few resonance peaks for the case when the cloaked region is water. There are two types of resonance peaks: one appears as narrow spikes such as $ka \approx 4.8$ and $7.7$; and the other appears as broader peaks such as $ka \approx 1.5$ and $8.0$. The resonance peak at $ka \approx 1.5$ appears to degenerate into a singularity at $ka = 0$ when the cloaked region becomes a void. Overall, figure 6 suggests that the resonance peaks are the only places where the cloaking shell is punctured. There is no resonance when the cloaked region is rigid; there is a resonance at $ka = 0$ when the cloaked region is void; and there are numerous resonance peaks when the cloaked region is penetrable.

At $ka = 1.5$, the first broad resonance peak for the case of water as the cloaked region, the form factors for the same four cases are shown in figure 7. At this resonant frequency, the scattering pattern for the case of water being the cloaked region is omni-directional, and significantly larger than the other cases. This resonance bears a close resemblance to the so-called Minnaert resonance [15], also known as the giant monopole resonance, which occurs at extremely low frequency for an air bubble in water when modeling fish schools. For the case when the cloaked region is a void, this resonance becomes a singularity at $ka = 0$. Mathematically the singularity is due to a peculiar property of the Bessel function of the first kind: $J_\nu(0) \to \infty$ when $\nu$ is non-integer; and $J_\nu(0) = 0$ when $\nu$ is a positive integer. However,
the physics behind this singularity is still unclear, as the mass-orthotropy and mass-singularity of a metamaterial are new concepts of metamaterials that are challenging many conventional notions about materials. This singularity is worthy of further investigation.

4. Conclusions

In summary, two theoretical developments enable an in-depth analysis of Cummer–Schurig acoustic cloaking. The analysis shows that the cloaking is effective but not perfect for the entire computed spatial frequency range up to $ka = 10$, where $a$ is the outer radius of the cloaking shell. The cloaking is effective for both planar incident waves and line sources. The latter suggests that the cloaking effect is independent of the geometrical shape of the wavefront. Furthermore, the cloaking remains effective but less perfect when the cloaked region is not rigid. For a penetrable medium in the cloaked region, the shell is penetrated at the resonant frequencies.

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References

[1] Pendry J B, Schurig D and Smith D R 2006 Science 312 1780
[2] Leonhardt U 2006 Science 312 1777
[3] Schurig D, Mock J J, Justice B J, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Science 314 977
[4] Leonhardt U 2006 New J. Phys. 8 118
[5] Leonhardt U and Philbin T G 2006 New J. Phys. 8 247
[6] Milton G W, Briane M and Willis J R 2006 New J. Phys. 8 248
[7] Cummer S A and Schurig D 2007 New J. Phys. 9 45
[8] Abramowitz M and Stegun I A 1965 Handbook of Mathematical Functions (New York: Dover)
[9] Rao T C K and Hamid M A K 1975 Int. J. Electron. 38 667–73
[10] Thompson W T 1950 J. Appl. Phys. 21 89–93
[11] Yeh C and Lindgren G 1977 Appl. Opt. 16 483–93
[12] Cai L-W 2004 J. Acoust. Soc. Am. 115 986–95
[13] Cai L-W and Sánchez-Dehesa J submitted
[14] Ochiai T and Sánchez-Dehesa J 2002 Phys. Rev. B 65 245111
[15] Gaunaud G, Scharnhorst K P and Überall H 1979 J. Acoust. Soc. Am. 65 573