PLURIPOTENTIAL KÄHLER-RICCI FLOWS

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ABSTRACT. We develop a parabolic pluripotential theory on compact Kähler manifolds, defining and studying weak solutions to degenerate parabolic complex Monge-Ampère equations. We provide a parabolic analogue of the celebrated Bedford-Taylor theory and apply it to the study of the Kähler-Ricci flow on varieties with log terminal singularities.

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Introduction

The Ricci flow, first introduced by Hamilton [Ham82] is the equation

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij},$$

evolving a Riemannian metric by its Ricci curvature. If the Ricci flow starts from a Kähler metric -the underlying Riemannian manifold being complex Kähler-, the evolving metrics remain Kähler and the resulting PDE is called the Kähler-Ricci flow.

After the spectacular use of the Ricci flow by Perelman to settle the Poincaré and Geometrization conjectures, it is expected that the Kähler-Ricci flow can be used similarly to give a geometric classification of complex algebraic and Kähler manifolds, and produce canonical metrics at the same time.

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Understanding the existence of canonical Kähler metrics on compact Kähler manifolds has been a central question in the last fourty years, following Yau’s solution to the Calabi conjecture [Yau78]. The Kähler-Ricci flow provides a canonical deformation process towards such metrics, as shown by the works of many authors (see e.g. [Cao85, PSSW08, PSSW09, ST12, SW13, SSW13, CT15, TZ15, CS16, BBEGZ] and the references therein).

Writing locally $g_{ij} = \psi_{ij} = \partial_i \overline{\partial_j} \psi$, it is classical that the Kähler-Ricci flow can be reduced to a nonlinear parabolic scalar equation in $\psi$, of the form

$$\det(\psi_{ij}) = e^{\psi_t + H(t, x)} + \lambda \psi_t,$$

where $H$ is a smooth density, and $\lambda \in \mathbb{R}$ depends on $c_1(X)$.

The classification of complex algebraic manifolds requires to work on singular varieties, as advocated by the Minimal Model Program. Defining the Kähler-Ricci flow on midly singular projective varieties was undertaken by Song-Tian [ST17] and requires a theory of weak solutions for degenerate parabolic complex Monge-Ampère equations, where $\psi$ is no longer smooth and $H$ can blow up.

A parabolic viscosity approach has been developed in [EGZ16]. It applies to the Kähler context, but requires the densities to be continuous. This enabled one to study the behavior of the Kähler-Ricci flow on minimal models with positive Kodaira dimension and canonical singularities [EGZ18].

While both the approach of Song-Tian and the viscosity one permit a good understanding of the first singular situations encountered in the Minimal Model Program, one needs to extend these theories in order to treat the fundamental case of Kähler pairs with Kawamata log terminal (klt) singularities. This is the main objective of the present work.

From an analytic point of view klt singularities lead one to deal with densities that may blow up, though belonging to $L^p$ for some exponent $p > 1$ whose size is related to the algebraic nature of the singularities.

We develop in this article a parabolic pluripotential approach to the complex Monge-Ampère flows

$$(\omega_t + i \partial \overline{\partial} \varphi_t)^n = e^{\varphi_t + F(t, x, \varphi)} g(x) dV(x),$$

in $X_T := [0, T] \times X$, where $T \in [0, +\infty]$ and

- $X$ is a compact Kähler $n$-dimensional manifold.
- $t \mapsto \omega(t, x)$ is a $C^2$-family of closed semi-positive $(1, 1)$-forms such that $\theta(x) \leq \omega_t(x)$, where $\theta$ is a closed semi-positive big form with
  $$-A \omega_t \leq \dot{\omega}_t \leq \frac{A}{T} \omega_t \quad \text{and} \quad \ddot{\omega}_t \leq A \omega_t$$
  for some fixed constant $A > 0$;
- $(t, x, r) \mapsto F(t, x, r)$ is continuous in $[0, T] \times X \times \mathbb{R}$, quasi-increasing in $r$, locally uniformly Lipschitz and semi-convex in $(t, r)$;
- $g \in L^p(X, dV)$, $p > 1$, with $g \geq 0$ almost everywhere;
- $\varphi : [0, T] \times X \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t, \cdot)$. 


Here \(dV\) is a fixed normalized volume form on \(X\).

We introduce a notion of pluripotential solutions to such equations, a parabolic analogue of the theory developed by Bedford and Taylor in their celebrated articles \([BT76, BT82]\).

We interpret the above parabolic equation on \(X\) as a second order PDE on the \((2n+1)\)-dimensional manifold \(X_T\):

- the LHS becomes a positive Radon measure \(\omega_t + dd^c \varphi_t^n \wedge dt\), which is well defined for paths \(t \mapsto \varphi_t\) of bounded \(\omega_t\)-psh functions \([BT82]\),
- the RHS \(e^{\dot{\varphi}_t + F(t,x,\varphi_t)} g(x) dV(x) \wedge dt\) is a well-defined Radon measure if \(t \mapsto \varphi_t(x)\) is (locally) uniformly Lipschitz.

It is useful in practice to allow the Lipschitz constant to blow up as \(t\) approaches zero, so we introduce the corresponding class \(\mathcal{P}(X_T, \omega)\) of parabolic potentials (see Definition 1.1).

We develop the local side of this theory in \([GLZ1]\) by a direct approach, taking advantage of the euclidean structure of \(\mathbb{C}^n\). We approximate here \((CMAF)\) by smooth complex Monge-Ampère flows and establish various a priori estimates to prove our first main result:

**Theorem A.** Let \(\varphi_0\) be a bounded \(\omega_0\)-psh function. There exists a parabolic potential \(\varphi \in \mathcal{P}(X_T, \omega)\) such that

- \((t,x) \mapsto \varphi(t,x)\) is locally bounded in \([0,T[ \times X]\
- \((t,x) \mapsto \varphi(t,x)\) is continuous in \([0,T[ \times \text{Amp(}\theta]\)
- \(t \mapsto \varphi_t\) is locally uniformly semi-concave in \([0,T[ \times X]
- \varphi\) is a pluripotential solution to \((CMAF)\)
- \(\varphi_t \to \varphi_0\) as \(t \to 0^+\) in \(L^1(X)\) and pointwise.

Here \(\text{Amp(}\theta]\) denotes the ample locus of \(\theta\), i.e. the largest Zariski open subset of \(X\) where the cohomology class of \(\theta\) behaves like a Kähler class.

It turns out that \(t \mapsto \varphi_t(x) - n(t \log t - t) + Ct\) is increasing for some fixed \(C > 0\). The convergence at time zero is therefore rather strong (it is e.g. uniform in case \(\varphi_0\) is continuous).

The semi concavity information of the solution \(\varphi\) constructed in Theorem A is a crucial tool for approximation purpose (see Theorem 1.15). We show that it is the unique pluripotential solution with such time-regularity, by establishing the following comparison principle:

**Theorem B.** If \(\varphi \in \mathcal{P}(X_T, \omega)\) is a bounded pluripotential subsolution to \((CMAF)\) and \(\psi \in \mathcal{P}(X_T, \omega)\) is a bounded pluripotential supersolution which is locally uniformly semi-concave in \(t\) then

\[\varphi_0 \leq \psi_0 \implies \varphi \leq \psi.\]

In particular there is a unique bounded pluripotential solution \(\Phi(g,F,\omega_t, \varphi_0)\) to \((CMAF)\) which is locally uniformly semi-concave in \(t\).

This comparison principle also allows us to establish the following stability result which generalizes \([GLZ18, \text{Theorem B}]\) :
Theorem C. Assume
- \( (g_j) \) are densities which converge to \( g \) in \( L^p \),
- \( F_j \) converges to \( F \) with uniform constants;
- \( \omega_{t,j} \) are smooth semi-positive forms smoothly converging to \( \omega_t \),
- \( \varphi_{0,j} \) are bounded \( \omega_{0,j} \)-psh functions converging in \( L^1(X,dV) \) to \( \varphi_0 \).

Then \( \Phi(g_j, F_j, \omega_{t,j}, \varphi_{0,j}) \) locally uniformly converges to \( \Phi(g, F, \omega_t, \varphi_0) \).

It is delicate to compare pluripotential and viscosity concepts in general. We refer the interested reader to [GLZ3] where we prove, when \( g \) is continuous, that the viscosity solution constructed in [EGZ16] coincides with the pluripotential solution \( \Phi(g, F, \omega_t, \varphi_0) \).

The present pluripotential approach allows us to deal with non continuous data. We can, in particular, define a good notion of weak Kähler-Ricci flow on varieties with terminal singularities (and more generally on k.l.t. pairs), as we explain in section 5, where we prove the following:

Theorem D. Let \( (Y, \omega_0) \) be a compact \( n \)-dimensional Kähler variety with log terminal singularities and trivial first Chern class (\( Q \)-Calabi-Yau variety).

Fix \( S_0 \) a positive closed current with bounded potentials, whose cohomology class is Kähler. The Kähler-Ricci flow

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t)
\]

exists for all times \( t > 0 \), and deforms \( S_0 \) towards the unique Ricci flat Kähler-Einstein current \( \omega_{KE} \) cohomologous to \( S_0 \), as \( t \to +\infty \).

This extends previous results of [Cao85, Tsu88, TZ06], avoiding any projectivity assumption on \( X \) [ST17], nor any restriction on the type of singularities [EGZ16, EGZ18]. We refer the reader to section 5 for much more general and precise results.

Assumptions on the data and notations.

Assumptions on the manifold. In the whole article we let \( X \) be a compact Kähler \( n \)-dimensional manifold. We fix \( T \in ]0, +\infty[ \). Except for section 5 we are mainly concerned with finite time intervals, i.e. \( T < +\infty \), and we implicitly assume that our data are possibly defined in a slightly larger time interval, i.e. on \( (0, T + \varepsilon) \) for some \( \varepsilon > 0 \).

We let \( X_T \) denote the \((2n+1)\)-dimensional manifold \( X_T = ]0, T[ \times X \) with parabolic boundary

\[
\partial X_T := \{0\} \times X.
\]

We fix \( \theta \) a smooth closed semipositive \((1,1)\)-form whose cohomology class is big, i.e. contains a (singular) positive closed current of bidegree \((1,1)\) which dominates a Kähler form. We let \( \Omega \) denote the ample locus of \( \theta \),

\[
\Omega := \text{Amp}(\theta),
\]

which is a non empty Zariski open subset of \( X \).
Assumptions on the forms. We assume throughout the article that \((\omega_t)_{t\in[0,T]}\) is a \(C^2\)-smooth family of closed semipositive \((1,1)\)-forms on \(X\) satisfying
\[
\theta \leq \omega_t
\]
for all \(t \in [0, T]\). For finite times we can also assume without loss of generality that \(\omega_t \leq \Theta\) for some Kähler form \(\Theta\).

By the end of Section 2 we need to assume that \(t \mapsto \omega_t\) moreover satisfies
\[
\ddot{\omega}_t \leq A\omega_t,
\]
and
\[
(0.1) - A\omega_t \leq \dot{\omega}_t \leq A\omega_t,
\]
for some constant \(A > 0\). The lower bound in (0.1) is equivalent to the fact that \(t \mapsto e^{+At}\omega_t\) is increasing. In particular
\[
\omega_{t+s} \geq e^{-As}\omega_t \geq (1 - As)\omega_t, \quad s > 0.
\]
The latter will be used on several occasions in the sequel.

Assumptions on the densities. We assume throughout the article that
\begin{itemize}
  \item \(dV\) is a fixed volume form on \(X\);
  \item \(0 \leq g \in L^p(X, dV)\) for some \(p > 1\), and \(\text{Vol}\{\{g = 0\}\} = 0\),
  \item \((t, x, r) \mapsto F(t, x, r)\) is a continuous function on \([0, T] \times X \times \mathbb{R}\);
  \item \(r \mapsto F(\cdot, \cdot, r)\) is uniformly quasi-increasing, i.e. there exists a constant \(\lambda_F \geq 0\) such that for every \((t, x) \in [0, T] \times X\), the function
    \[
    (0.2) r \mapsto F(t, x, r) + \lambda_F r \text{ is increasing in } \mathbb{R}.
    \]
  \item \((t, r) \mapsto F(t, \cdot, r)\) is locally uniformly Lipschitz, i.e. for all \(J \Subset [0, T] \times \mathbb{R}\) there exists \(\kappa_J > 0\) such that for every \(x \in X\), \((t, r), (t', r') \in J\),
    \[
    (0.3) |F(t, x, r) - F(t', x, r')| \leq \kappa_J(|t - t'| + |r - r'|);
    \]
  \item \((t, r) \mapsto F(t, x, r)\) is locally uniformly semi-convex, i.e. for every compact \(J \Subset [0, T] \times \mathbb{R}\) there exists \(C_J > 0\) such that for every \(x \in X\),
    \[
    (0.4) (t, r) \mapsto F(t, x, r) + C_J(t^2 + r^2) \text{ is convex in } J.
    \]
\end{itemize}

Note that if \(F\) is \(C^2\)-smooth then the local conditions (0.3) and (0.4) are automatically satisfied, while (0.2) is a global assumption.

Invariance properties of the set of assumptions. We check in section 5.1.2 that the above conditions are satisfied for the parabolic equations that describe the evolution of the normalized (as well as the non-normalized) Kähler-Ricci flow on a mildly singular Kähler variety.

The family of parabolic complex Monge-Ampère equations we consider enjoy several useful invariance properties. We refer the reader to section 3.3 for more details.
Organization of the paper. We describe the class of potentials we are using in Section 1.1 and define parabolic complex Monge-Ampère operators in Section 1.2. We establish fundamental a priori estimates in Section 2, which are then used to prove Theorem A in Section 3. We study uniqueness and stability of pluripotential solutions in Section 4, establishing Theorem B and Theorem C. In Section 5 we use these tools to study the long term behavior of the normalized Kähler-Ricci flow on varieties with log terminal singularities and non-negative Kodaira dimension, proving Theorem D and several other convergence results.

Acknowledgements. This work is a natural continuation of [EGZ16, EGZ18]. We thank Philippe Eyssidieux for many useful discussions.

1. Parabolic potentials and Monge-Ampère operators

1.1. Families of quasi-plurisubharmonic functions.

1.1.1. Compactness properties. Recall that a function $u : X \to [-\infty, +\infty]$ is $\omega_t$-plurisubharmonic ($\omega_t$-psh for short), if it is locally given as the sum of a smooth and a plurisubharmonic function and the current $\omega_t + dd^c u \geq 0$ is positive on $X$. Here $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$ are both real operators.

Definition 1.1. The set of parabolic potentials $P(X_T, \omega)$ is the set of functions $\varphi : ]0, T[ \times X \to [-\infty, +\infty]$ such that

- $x \mapsto \varphi(t, x)$ is $\omega_t$-plurisubharmonic on $X$, for all $t \in ]0, T[$,
- $\varphi$ is locally uniformly Lipschitz in $]0, T[$.

We say that a family $\Phi \subset P(X_T, \omega)$ is locally uniformly Lipschitz in $]0, T[$ if the inequality (1.1) is satisfied for all $\varphi \in \Phi$ with a uniform constant $\kappa = \kappa(J, \Phi) > 0$ which only depends on $J$ and $\Phi$.

The last condition means that for any compact subset $J \subset ]0, T[$ there exists $\kappa = \kappa(J) > 0$ such that

(1.1) $\varphi(t, x) \leq \varphi(s, x) + \kappa|t - s|$, for all $s, t \in J$ and $x \in X$.

We say that a family $\Phi \subset P(X_T, \omega)$ is locally uniformly Lipschitz in $]0, T[$ if the inequality (1.1) is satisfied for all $\varphi \in \Phi$ with a uniform constant $\kappa = \kappa(J, \Phi) > 0$ which only depends on $J$ and $\Phi$.

A parabolic potential $\varphi \in P(X_T, \omega)$ can be extended as an upper semi-continuous function on $[0, T[ \times X$ with $\omega_t$-psh slices.

Proposition 1.2. Assume $\varphi_0$ is $\omega_0$-psh and $\varphi \in P(X_T, \omega)$ satisfies $\varphi_t \to \varphi_0$ in $L^1$, as $t \to 0$. Then the extension $\varphi : [0, T[ \times X \to [-\infty, +\infty]$ is upper semi-continuous.

Proof. It is classical that for all $x \in X$, $\varphi_0(x) = \limsup_{y \to x} \limsup_{t \to 0} \varphi_t(y)$. It therefore suffices to prove the following more general result: assume that $u \in P(X_T, \omega)$ is bounded from above near $t = 0$ and define

$$u_0(x) := \limsup_{y \to x} \left( \limsup_{t \to 0} u_t(y) \right).$$
Then the extension \( u : [0, T] \times X \to [-\infty, +\infty] \) is upper semicontinuous.

The upper semi-continuity inside \( X_T \) follows from the semi-continuity in space and Lipschitz regularity in time. Assume that \((t_j, x_j)\) is a sequence in \( X_T \) converging to \((0, x_0)\) with \( x_0 \in X \). We want to prove that

\[
\limsup_j u(t_j, x_j) \leq u_0(x_0).
\]

Since the problem is local we can assume that the functions \( u_{t_j} \) are psh and negative in a neighborhood \( B \subset \mathbb{C}^n \) of \( x_0 \).

Since \( u_0 \) is psh in \( B \) there exists \( r > 0 \) such that \( B(x_0, 2r) \subset B \) and

\[
(1.2) \quad \frac{1}{\text{Vol}(B(x_0, r))} \int_{B(x_0, r)} u_0(z) dV(x) \leq u_0(x_0) + \varepsilon.
\]

Fix \( \delta \in ]0, r[ \). For \( j \) large enough, \( x_j \in B(x_0, \delta) \) hence \( B(x_0, r) \subset B(x_j, r + \delta) \) and since \( u_{t_j} \leq 0 \) in \( B \) we have,

\[
\begin{align*}
  u(t_j, x_j) & \leq \frac{1}{\text{Vol}(B(x_j, r + \delta))} \int_{B(x_j, r + \delta)} u(t_j, x) dV(x) \\
  & \leq \frac{1}{\text{Vol}(B(x_j, r + \delta))} \int_{B(x_0, r)} u(t_j, x) dV(x) \\
  & = \frac{\text{Vol}(B(x_0, r))}{\text{Vol}(B(x_j, r + \delta)) \text{Vol}(B(x_0, r))} \int_{B(x_0, r)} u(t_j, x) dV(x).
\end{align*}
\]

Since \( \limsup_j u_{t_j}(x) \leq u_0(x) \), for all \( x \in X \), letting \( j \to +\infty \) and using (1.2) we obtain

\[
\limsup_j u(t_j, x_j) \leq \frac{\text{Vol}(B(z_0, r))}{\text{Vol}(B(z, r + \delta))}(u_0(x_0) + \varepsilon).
\]

Now, we first let \( \delta \to 0 \) and then \( \varepsilon \to 0 \) to obtain the result. \( \square \)

We next prove a compactness result for this class of functions.

**Theorem 1.3.** Let \((\varphi_j) \subset \mathcal{P}(X_T, \omega)\) be a sequence which

- is locally uniformly bounded from above in \( X_T \);
- is locally uniformly Lipschitz in \([0, T] \);
- does not converge locally uniformly to \( -\infty \) in \( X_T \).

Then \((\varphi_j)\) is bounded in \( L^1_{\text{loc}}(X_T) \) and there exists a subsequence which converges to some function \( \varphi \in \mathcal{P}(X_T) \) in the \( L^1_{\text{loc}}(X_T) \)-topology.

If \((\varphi_j)\) converges weakly (in the sense of distributions) to \( \varphi \) in \( X_T \), then it converges in \( L^p_{\text{loc}}(X_T) \) for all \( p \geq 1 \).

The classes \( L^p \) are here defined with respect to the \((2n+1)\)-dimensional Lebesgue measure associated to a fixed volume form \( dt \wedge dV \). For convenience we normalize \( dV \) so that \( \int_X dV = 1 \).

**Proof.** The proof of this result is local in nature and follows closely the classical proof of the analogous result for quasi-plurisubharmonic functions, once we have a substitute for the sub-mean value inequality.
We can thus assume here that $X = \Omega \subset \mathbb{C}^n$ is a bounded strictly pseudo-convex domain. The Poincaré lemma insures that $\omega_t = dd^c \rho_t$ for a family of plurisubharmonic functions $\rho_t$ which is Lipschitz in $t$. Changing $\varphi_t$ in $\varphi_t + \rho_t$, we reduce further to the case when $\omega_t = 0$. The corresponding compactness and convergence properties have then been obtained in [GLZ1]. □

**Corollary 1.4.** The class $\mathcal{P}(X_T, \omega)$ is a subset of $L^p_{\text{loc}}(X_T)$ for all $1 \leq p$, and the inclusions $\mathcal{P}(X_T, \omega) \hookrightarrow L^p_{\text{loc}}(X_T)$ are continuous.

The topologies induced by the classes $L^p$ are thus all equivalent when restricted to the class $\mathcal{P}(X_T, \omega)$.

1.1.2. **Slices and time-derivatives.** We now estimate the $L^1$-norm on slices.

**Lemma 1.5.** Fix $u, v \in \mathcal{P}(X_T, \omega)$ and $0 < T_0 < T_1 < T$. Then

$$
\|u(t, \cdot) - v(t, \cdot)\|_{L^1(X)} \leq 2M \max \left\{ \|u - v\|^{1/2}_{L^1(X_T)}, \|u - v\|_{L^1(X_T)} \right\},
$$

for all $T_0 \leq t \leq T_1$, where $M := \max \{ \sqrt{\kappa}, (T - T_1)^{-1} \}$, and $\kappa$ is the uniform Lipschitz constant of $u - v$ in $[T_0, T]$.

This lemma expresses in a quantitative way the following fact: for functions in $\mathcal{P}(X_T, \omega)$, the convergence in $L^1(X_T)$ implies the local uniform convergence of their slices in $L^1(X)$: if $(u_j) \subset \mathcal{P}(X_T, \omega)$ converges to $u$ in $L^1(X_T)$ and is locally uniformly Lipschitz in $[0, T]$, then $u_j(t, \cdot)$ converges to $u(t, \cdot)$ in $L^1(X)$ for each slice $t$.

**Proof.** The proof is identical to the corresponding one in the local context, we refer the reader to [GLZ1]. □

Fix $\mu$ a (finite) Borel measure on $X$, and let $\ell$ denote the Lebesgue measure on $\mathbb{R}^+$.  

**Lemma 1.6.** Fix $\varphi \in \mathcal{P}(X_T, \omega)$. Then $\partial_t \varphi(t, x)$ exists for all $(t, x) \notin E$, where $E \subset X_T$ is $\ell \otimes \mu$-negligible.

In particular $\partial_t \varphi \in L^\infty_{\text{loc}}(X_T)$ and for any continuous function $h \in C^0(\mathbb{R}, \mathbb{R})$, $h(\partial_t \varphi) \ell \otimes \mu$ is a well defined Borel measure on $X_T$.

**Proof.** The proof is identical to the corresponding one in the local context, we refer the reader to [GLZ1]. □

When $\varphi$ is semi-convex or semi-concave in $t$, we can improve this result.

**Definition 1.7.** We say that $\varphi : X_T \rightarrow \mathbb{R}$ is uniformly semi-concave in $[0, T]$ if for any compact $J \subseteq [0, T]$, there exists $\kappa = \kappa(J, \varphi) > 0$ such that for all $x \in X$, the function $t \mapsto \varphi(t, x) - \kappa t^2$ is concave in $J$.

The definition of uniformly semi-convex functions is analogous. Note that such functions are automatically locally uniformly Lipschitz.
Lemma 1.8. Let $\varphi : X_T \to \mathbb{R}$ be a continuous function which is uniformly semi-convex in $]0, T[$. Then

$$\partial^+_t \varphi(t, x) = \lim_{s \to 0^+} \frac{\varphi(t + s, x) - \varphi(t, x)}{s}$$

is upper semi-continuous in $X_T$, while

$$\partial^-_t \varphi(t, x) := \lim_{s \to 0^-} \frac{\varphi(t + s, x) - \varphi(t, x)}{s}$$

is lower semi-continuous in $X_T$. In particular, $\partial^+_t \varphi$ and $\partial^-_t \varphi$ coincide and are continuous $\ell \otimes \mu$-almost everywhere in $X_T$.

Proof. The proof is identical to the corresponding one in the local context, we refer the reader to [GLZ1]. □

1.1.3. Topology on $\mathcal{P}(X_T, \omega)$. We introduce a natural complete metrizable topology on the convex set $\mathcal{P}(X_T, \omega)$.

We first consider a partial Sobolev space $W^{(1,0), \infty}_{\loc}(X_T)$: this is the set of functions $u \in L^1_{\loc}(X_T)$ whose partial time derivative (in the sense of distribution) satisfies $\dot{u} = \partial_t u \in L^\infty_{\loc}(X_T)$. It follows from Lemma 1.6 that

$$\mathcal{P}(X_T, \omega) \subset W^{(1,0), \infty}_{\loc}(X_T).$$

The local uniform Lipschitz constant of $\varphi \in \mathcal{P}(X_T, \omega)$ on a compact subset $J \subset ]0, T[$ is given by

$$\sup_{t,s \in J, s \neq t} \sup_{x \in X} \frac{|\varphi(s, x) - \varphi(t, x)|}{|s - t|} = \|\dot{\varphi}\|_{L^\infty(J \times X)},$$

where $\sup^*$ is the essential sup with respect to a volume form $dV$ on $X$.

We can therefore consider the following semi-norms on $W^{(1,0), \infty}_{\loc}(X_T)$: given a compact subset $J \subset ]0, T[$ and $u \in W^{(1,0), \infty}_{\loc}(X_T)$, we set

$$\rho_J(u) := \|\dot{\varphi}\|_{L^\infty(J \times X)} + \int_J \int_X |u(t, x)| dV(x) dt.$$

Proposition 1.9. The space $W^{(1,0), \infty}_{\loc}(X_T)$ endowed with the semi-norms $(\rho_J)$ is a complete metrizable space and $\mathcal{P}(X_T, \omega)$ is a closed subset.

1.2. Parabolic complex Monge-Ampère operators. As explained in the introduction, we assume in this section (without loss of generality) that $\theta \leq \omega_t \leq \Theta$, where $\theta$ is a semi-positive and big $(1,1)$-form and $\Theta$ is a Kähler form.

1.2.1. Parabolic Chern-Levine-Nirenberg inequalities. We assume here that $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\loc}(X_T)$. For all $t \in ]0, T[$, the function

$$X \ni x \mapsto \varphi_t(x) = \varphi(t, x) \in \mathbb{R}$$

is $\omega_t$-psh and bounded, hence $(\omega_t + dd^c \varphi_t)^n$ is well defined as a positive Borel measure on $X$ as follows from the works of Bedford and Taylor [BT76, BT82].
Since $0 \leq \omega_t \leq \Theta$ for $0 \leq t \leq T$, the positive Borel measures $(\omega_t + dd^c \varphi_t)^n$ have uniformly bounded masses on $X$:

$$\int_X (\omega_t + dd^c \varphi_t)^n \leq \int_X (\Theta + dd^c \varphi_t)^n \leq \int_X \Theta^n.$$ 

These can be considered, alternatively, as a family of currents of degree $2n$ on the real $(2n+1)$-dimensional manifold $X_T = [0,T] \times X$. It follows from Bedford-Taylor’s convergence theorem [BT76, BT82] that $t \mapsto (\omega_t + dd^c \varphi_t)^n$ is continuous as a map from $[0,T]$ to the space $\mathcal{M}(X)$ of positive Radon measures on $X$ endowed with the weak*-topology. More generally we have

**Lemma 1.10.** Fix $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{loc}(X_T)$ and $\chi$ a continuous test function in $X_T$. The function $t \mapsto \int_X \chi(t, \cdot)(\omega_t + dd^c \varphi_t)^n$, is continuous in $[0,T]$ and bounded, with

$$\sup_{0 < t < T} \left| \int_X \chi(t, \cdot)(\omega_t + dd^c \varphi_t)^n \right| \leq (\max_{X_T} \left| \chi \right|) \int_X \Theta^n.$$ 

More generally if $\chi$ is upper semi-continuous (resp. lower semi-continuous, resp. Borel) on $X_T$, then so is the function $t \mapsto \int_X \chi(t, \cdot)(\omega_t + dd^c \varphi_t)^n$.

**Proof.** Fix a continuous test function $\chi$ on $X_T$ and fix a compact interval $J \Subset [0,T]$ such that $J \times X$ contains the support of $\chi$.

Fix $t_0 \in [0,T]$. The Lipschitz property of $\varphi$ ensures that $\varphi_t$ uniformly converges on $X$ to $\varphi_{t_0}$ as $t \to t_0$. The continuity of $t \mapsto \omega_t$ and Bedford-Taylor’s convergence theorem then ensure that $(\omega_t + dd^c \varphi_t)^n$ converges to $(\omega_{t_0} + dd^c \varphi_{t_0})^n$ as $t \to t_0$. Since $\chi_t$ uniformly converges on $X$ to $\chi_{t_0}$, the first statement follows. The second statement follows from the fact that $\int_X (\omega_t + dd^c \varphi_t)^n \leq \int_X \Theta^n$, for all $t \in [0,T]$.

**Definition 1.11.** Let $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{loc}(X_T)$. The map

$$\chi \mapsto \int_{X_T} \chi dt \wedge (\omega_t + dd^c \varphi_t)^n := \int_0^T dt \left( \int_X \chi(t, \cdot)(\omega_t + dd^c \varphi_t)^n \right).$$

defines a $(2n+1)$-current on $X_T$ denoted by $dt \wedge (\omega_t + dd^c \varphi_t)^n$, which can be identified with a positive Radon measure on $X_T$.

That (1.3) is well defined for continuous test (or Borel) functions $\chi$ follows from Lemma 1.10. The operator can also be defined by approximation in the spirit of Bedford and Taylor convergence results [BT76, BT82] :

**Proposition 1.12.** Fix $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{loc}(X_T)$ and let $\varphi_j$ be a monotone sequence of functions $(\varphi_j)$ in $\mathcal{P}(X_T, \omega) \cap L^\infty_{loc}(X_T)$ converging to $\varphi$ almost everywhere in $X_T$. Then

$$dt \wedge (\omega_t + dd^c \varphi_j^n) \to dt \wedge (\omega_t + dd^c \varphi_t)^n,$$

in the sense of measures on $X_T$. 

Proof. Let χ be a continuous test function in $X_T$. By definition, for all $j$, we have
\[
\int_{X_T} \chi dt \wedge \text{MA}(\varphi^j) := \int_0^T dt \left( \int_X \chi(t, \cdot) \text{MA}(\varphi^j_t) \right).
\]
We can apply Bedford and Taylor convergence theorems [BT82] to conclude that, for all $t \in ]0, T[$,
\[
\int_X \chi(t, \cdot)(\omega_t + dd^c \varphi^j_t) \to \int_X \chi(t, \cdot)(\omega_t + dd^c \varphi_t)^n.
\]
Since $\int_X \chi(t, \cdot)(\omega_t + dd^c \varphi^j_t)^n$ is a uniformly bounded (Lemma 1.10), the conclusion follows from Lebesgue convergence theorem. \hfill \square

It is classical that one can then define similarly mixed parabolic Monge-Ampère operators
\[
dt \wedge (\omega_t + dd^c \varphi^j_t) \wedge \cdots \wedge (\omega_t + dd^c \varphi^n_t)
\]
whenever $\varphi^1, \ldots, \varphi^n \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\text{loc}}(X_T)$. We note, for later use, the following stronger version of Chern-Levine-Nirenberg inequalities:

**Proposition 1.13.** Assume $\varphi^1, \ldots, \varphi^n \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\text{loc}}(X_T)$ and $\psi \in \mathcal{P}(X_T, \omega)$. Then, for all $J \subset ]0, T[$,
\[
\int_{J \times X} |\psi| dt \wedge (\omega_t + dd^c \varphi^1_t) \wedge \cdots \wedge (\omega_t + dd^c \varphi^n_t) \leq 
\]
\[
\leq \text{Vol}(\Theta) \int_J \left( \sup_X |\psi_t| + \sum_{j=1}^n \text{osc}(\varphi^j_t) \right) dt + \int_{J \times X} |\psi| dt \wedge \Theta^n.
\]
In particular, $\psi \in L^1_{\text{loc}}(X_T, dt \wedge (\omega_t + dd^c \varphi^1_t) \wedge \cdots \wedge (\omega_t + dd^c \varphi^n_t))$.

Proof. Fix $\psi \in \mathcal{P}(X_T, \omega)$ and $J \subset ]0, T[$. Setting $\psi_t = \tilde{\psi}_t + \sup_X \psi_t$ and using the triangle inequality we can write
\[
\int_{J \times X} |\psi| dt \wedge (\omega_t + dd^c \varphi^1_t) \wedge \cdots \wedge (\omega_t + dd^c \varphi^n_t) \leq 
\]
\[
\leq \int_{J \times X} |\tilde{\psi}| dt \wedge (\omega_t + dd^c \varphi^1_t) \wedge \cdots \wedge (\omega_t + dd^c \varphi^n_t) + \text{Vol}(\Theta) \int_J \sup_X |\psi_t| dt.
\]
We can thus assume that $\sup_X \psi_t = 0$ for all $t \in J$. A series of integration by parts as in [GZ05, Corollary 3.3] yields
\[
\int_X |\psi_t| (\omega_t + dd^c \varphi^1_t) \wedge \cdots \wedge (\omega_t + dd^c \varphi^n_t) \leq \sum_{j=1}^n \text{osc}_X(\varphi^j_t) \text{Vol}(\omega_t) + \int_X |\psi_t| \omega^n_t 
\]
\[
\leq \text{Vol}(\Theta) \sum_{j=1}^n \text{osc}_X(\varphi^j_t) + \int_X |\psi_t| \Theta^n.
\]
Integrating on $J$ yields the desired estimate. \hfill \square
1.2.2. Convergence results.

Definition 1.14. A family $\Phi \subset \mathcal{P}(X_T, \omega)$ is uniformly semi-concave in $]0, T[$, if for any compact subset $J \subset ]0, T[$, there exists a constant $\kappa = \kappa(J, \Phi) > 0$ such that any $\varphi \in \Phi$ is uniformly $\kappa$-concave in $J$.

Fix $\mu$ a Borel measure on $X$ and let $\ell$ denote the Lebesgue measure on $\mathbb{R}$.

Theorem 1.15. Let $(f_j)$ be a sequence of positive functions which converge to $f$ in $L^1(X_T, \ell \otimes \mu)$. Let $(\varphi^j)$ be a sequence of functions in $\mathcal{P}(X_T, \omega)$ which

- converge $\ell \otimes \mu$-almost everywhere in $X_T$ to a function $\varphi \in \mathcal{P}(X_T, \omega)$;
- is uniformly semi-concave in $]0, T[$.

Then $\lim_{j \to +\infty} \dot{\varphi}^j(t, x) = \dot{\varphi}(t, x)$ for $\ell \otimes \mu$-almost any $(t, x) \in X_T$, and

$h(\dot{\varphi}^j) f_j \ell \otimes \mu \to h(\dot{\varphi}) f \ell \otimes \mu,$

in the weak sense of Radon measures on $X_T$, for all $h \in C^0(\mathbb{R}, \mathbb{R})$.

Proof. The proof is identical to the corresponding one in the local context, we refer the reader to [GLZ1]. □

2. A priori estimates

In this section we assume that $\varphi(t, x) = \varphi_t(x)$ is a smooth $\omega_t$-psh solution to (CMAF), where $t \mapsto \omega_t$ is a smooth family of Kähler forms, and $F, g$ are smooth with $g$ positive. Our aim is to establish various a priori estimates that will allow us to construct weak solutions to the corresponding degenerate equations.

For convenience we will also assume that $\theta$ is Kähler, $\varphi_0$ is smooth and strictly $\omega_0$-psh. It follows however from [EGZ09, GZ17, To17] that all the a priori bounds below remain valid when, $\theta$ is semipositive and big, $\varphi_0$ is merely $\omega_0$-psh and bounded.

We will make various extra assumptions, depending on the a priori estimates that we are interested in.

2.1. Controlling the oscillation of $\varphi_t$. Recall that $\theta(x) \leq \omega_t(x) \leq \Theta(x)$, where $\theta, \Theta$ are Kähler forms. We let $V_1$ (resp. $V_2$) denote the volume of $\{\theta\}$ (resp. $\{\Theta\}$),

$$V_1 = \int_X \theta^n \text{ and } V_2 = \int_X \Theta^n.$$ 

We fix $c_1, c_2 \in \mathbb{R}$ normalizing constants such that $V_i = e^{c_i} \mu(X)$, where $\mu = gdV$. It follows from [Ko98, EGZ09] that there exists $\rho_1$ a bounded $\theta$-psh function (respectively $\rho_2$ a bounded $\Theta$-psh function) such that

$$(\theta + dd^c \rho_1)^n = e^{c_1} \mu \text{ and } (\Theta + dd^c \rho_2)^n = e^{c_2} \mu.$$ 

The functions $\rho_1, \rho_2$ are moreover unique once normalized by

$$\sup_X \rho_1 = \inf_X \rho_2 = 0.$$
Note that in proving existence of solutions the form \( \theta \) will no longer be Kähler but merely semipositive and big. But the \( L^\infty \) bound on \( \rho_1 \) remains uniform thanks to [EGZ09, Proposition 2.6].

**Proposition 2.1.** The following uniform a priori bound on \( \varphi_t \) holds:

\[
|\varphi_t(x)| \leq C_0 := C \left( e^{\lambda_F t} + \frac{e^{\lambda_F t} - 1}{\lambda_F} \right),
\]

where \( C \) is the following uniform constant

\[
C = \sup_{X_T} |F(t, x, 0)| + (\lambda_F + 1) \sup_X (|\rho_1| + |\rho_2|) + \sup_X |\varphi_0| + \max(-c_1, c_2).
\]

Recall that \( \lambda_F \geq 0 \) is a constant such that, for all \( (t, x) \in X_T \), the function \( r \mapsto F(t, x, r) + \lambda_F r \) is increasing on \( \mathbb{R} \).

**Proof.** Set, for \( t \in \mathbb{R} \),

\[
\gamma(t) := \sup_X |\varphi_0| e^{\lambda_F t} + \frac{C(e^{\lambda_F t} - 1)}{\lambda_F},
\]

where \( C \) is as in the statement of the proposition. A direct computation shows that \( \gamma(0) = \sup_X |\varphi_0| \) and \( \gamma'(t) - \lambda_F \gamma(t) = C \).

Set, for \( (t, x) \in X_T \), \( u(t, x) := \rho_1(x) - \gamma(t) \). Observe that \( u_t \) is \( \theta \)-psh hence \( \omega_t \)-psh, that \( u_0 \leq \varphi_0 \) and

\[
(\omega_t + \ddbar u_t)^n \geq (\theta + \ddbar \rho_1)^n = e^{c_1 \mu} \geq e^{\ddbar t + F(t, x, u_t)} \mu.
\]

The last inequality follows from our choice of \( C \): since \( r \mapsto F(r, \cdot, r) + \lambda_F r \) is increasing and \( \rho_1 \leq 0, u_t \leq 0 \), we obtain

\[
F(t, x, u_t(x)) + \ddbar t = F(t, x, u_t(x)) - \gamma'(t) \\
\leq F(t, x, 0) - \lambda_F(\rho_1 - \gamma(t)) - \gamma'(t) \\
\leq F(t, x, 0) + \lambda_F|\rho_1| - C \\
\leq c_1.
\]

It thus follows from the maximum principle that \( \varphi \geq u \) on \( X_T \).

Set now \( v(t, x) = \rho_2(x) + \gamma(t), (t, x) \in X_T \). We let the reader check similarly that \( v_t \) is \( \Theta \)-psh, it satisfies

\[
(\Theta + \ddbar v_t)^n \leq e^{\partial v(t, x) + F(t, x, v_t)} g dV,
\]

and \( v_0 \geq \varphi_0 \). Now \( \varphi \) is a subsolution to this new parabolic equation since \( \omega_t \leq \Theta \). It follows therefore from the maximum principle that \( \varphi \leq v \) on \( X_T \), and the desired estimates follow. \( \square \)

The following construction of subbarrier will be useful in showing that the pluripotential solution to (CMAF) has the right value at \( t = 0 \).

**Proposition 2.2.** For all \( 0 \leq t \leq 1 \),

\[
\varphi_t \geq (1 - t)e^{-\lambda_F t} \varphi_0 + t \rho_1 + n(t \log t - t) - C \frac{e^{\lambda_F t} - 1}{\lambda_F},
\]
where $C$ is the following uniform constant,
\[
C := \sup_{X_T} F(t, x, 0) + (A + \lambda_F + 1) \left( \sup_X |\varphi_0| + \sup_X |\rho_1| + n \right) - c_1.
\]

**Proof.** Recall that $A$ denotes a positive constant such that $\dot{\omega}_t \geq -A\omega_t$ for all $t \in [0, T]$. In particular $\omega_t \geq e^{-At}\omega_0$ and $\omega_t \geq \theta$. Recall also that $\lambda_F \geq 0$ is a constant so that $r \mapsto F(t, x, r) + \lambda_F r$ is increasing in $\mathbb{R}$ for all $(t, x) \in X_T$.

Consider the function
\[
\tau_t(x) := (1 - t) e^{-At}\varphi_0 + t\rho_1 + n(t \log t - t) - Ce^{\lambda_F t} - 1,\]

where $C$ is the uniform constant defined in the proposition.

Using that $\omega_t \geq e^{-At}\omega_0$ and $\omega_t \geq \theta$, we have
\[
\tau_t(x) \geq t^n(\omega_t + dd^c\rho_1)^n
\]
\[
\geq t^n e^{n\theta} gdV.
\]

Since $u_t \leq 0$ and $r \mapsto F(t, x, r) + \lambda_F r$ is increasing, a direct computation yields
\[
\dot{u}_t + F(t, x, u_t) = n \log t + \rho_1 + e^{-At}(A(1 - t) + 1)(-\varphi_0) - Ce^{\lambda_F t}
\]
\[
+ F(t, x, u_t) + \lambda_F u_t - \lambda_F u_t
\]
\[
\leq n \log t + (A + 1) \sup_X |\varphi_0| + \sup_{X_T} F(t, x, 0)
\]
\[
+ \lambda_F \left( \sup_X |\varphi_0| + \sup_X |\rho_1| + n \right) - C
\]
\[
\leq n \log t + c_1.
\]

It thus follows that $u_t$ is a subsolution to (CMAF) with $u_0 \leq \varphi_0$. The desired estimate follows from the classical maximum principle. \hfill $\square$

### 2.2. Controlling the average.

We establish the following control on the average of $\varphi_t$, which will be useful in proving convergence at zero.

**Proposition 2.3.** Set $\mu = gdV$. The following bound holds
\[
\int_X \varphi_t \, d\mu \leq \int_X \varphi_0 \, d\mu + Ct,
\]
where $C_3$ is the following uniform constant
\[
C := -\mu(X) \log(\mu(X)/V_2) - \inf_{X_T \times [-C_0, C_0]} F(t, x, r) \mu(X),
\]
and $C_0$ is the uniform constant defined in Proposition 2.1.

**Proof.** Set $-C' := \inf_{X_T \times [-C_0, C_0]} F(t, x, r) > -\infty$. It follows from the flow equation that
\[
\int_X e^{\bar{\varphi}_t - C'} \, d\mu \leq \int_X \omega_t^n \leq V_2.
\]
On the other hand it follows from Jensen’s inequality that
\[ \int_X \varphi_t \frac{d\mu}{\mu(X)} \geq \exp \left( \int_X \varphi_t \frac{d\mu}{\mu(X)} \right). \]
Combining these two estimates we arrive at
\[ \int_X \varphi_t g d\mu \leq C := C' \mu(X) + \mu(X) \log V_2 - \mu(X) \log \mu(X). \]

The function \( t \mapsto \int_X \varphi_t d\mu - Ct \) is therefore non-increasing, hence
\[ \int_X \varphi_t d\mu \leq \int_X \varphi_0 d\mu + Ct. \]

2.3. **Lipschitz control in time.** We now establish an a priori bound which will allow us to show that the solutions \( \varphi_t \) to degenerate complex Monge-Ampère flows are locally uniformly Lipschitz in time, away from zero.

For the convenience of the reader we first state and prove our theorem in the simpler case when \( t \mapsto \omega_t \) is affine and \( r \mapsto F(t, x, r) \) is increasing. A more technical statement follows, together with its proof.

2.3.1. **Affine dependence on time.**

**Theorem 2.4.** Assume \( t \mapsto \omega_t = \omega_0 + t\chi \) is affine and \( r \mapsto F(\cdot, \cdot, r) \) is increasing. Then for all \( (t, x) \in X_T \),
\[ n \log t - C \leq \dot{\varphi}_t(x) \leq \frac{C}{t}, \]
where \( C \) depends explicitly on \( T, \|\partial F/\partial r\|_{L^\infty}, \|\partial F/\partial t\|_{L^\infty}, \|g\|_p \), and \( C_0 \).

Here and below \( C_0 \) denotes the constant from Proposition 2.1, and the Lipschitz constants \( \|\partial F/\partial r\|_{L^\infty}, \|\partial F/\partial t\|_{L^\infty} \) are computed on \( X_T \times [-C_0, C_0] \).

**Proof.** For notational convenience we set \( \mu = g dV \). We first establish the bound from above. Consider
\[ H(t, x) = t\dot{\varphi}_t(x) - (\varphi_t - \varphi_0) - Bt, \]
where
\[ B = n + 1 - \inf_{X_T \times [-C_0, C_0]} \left[ t \frac{\partial F}{\partial t}(t, x, r) \right]. \]
Set \( S_t := \omega_t + dd^c \varphi_t \) and observe that
\[ \frac{\partial H}{\partial t} = t\ddot{\varphi}_t - B, \]
with
\[ \ddot{\varphi}_t = \log (S_t^n/\mu) - F(t, x, \varphi_t) \]
hence
\[ \ddot{\varphi}_t = \Delta S_t(\dot{\varphi}_t) + tr_{S_t}(\omega_t) - \frac{\partial F}{\partial t}(x, t, \varphi_t) - \dot{\varphi}_t \frac{\partial F}{\partial r}(x, t, \varphi_t), \]
where
\[ \Delta_{S_t} f := n \frac{dd^c f \wedge S_t^{n-1}}{S_t^n} \quad \text{and} \quad \text{tr}_{S_t}(\eta) := n \frac{\eta \wedge S_t^{n-1}}{S_t^n}. \]

On the other hand
\[ \Delta_{S_t}(H) = t \Delta_{S_t}(\dot{\varphi}_t) - n + \text{tr}_{S_t}(\omega_t + dd^c \varphi_0), \]
therefore
\[ \left( \frac{\partial}{\partial t} - \Delta_{S_t} \right)(H) = \left\{ -t \frac{\partial F}{\partial t} - t \dot{\varphi}_t \frac{\partial F}{\partial r} + n - B \right\} - \text{tr}_{S_t}(S_0) + \text{tr}_{S_t}(\omega_0 + t\omega_t - \omega_t). \]

The assumption that \( t \mapsto \omega_t \) is affine insures \( \text{tr}_{S_t}(\omega_0 + t\omega_t - \omega_t) = 0 \), while our choice of \( B \) yields
\[ \left( \frac{\partial}{\partial t} - \Delta_{S_t} \right)(H) \leq -1 - t \dot{\varphi}_t \frac{\partial F}{\partial r}(t, x, \varphi_t). \]

If \( H \) realizes its maximum \( H_{\text{max}} \) along \( (t = 0) \), we obtain
\[ H(t, x) \leq H_{\text{max}} = \sup_{x \in X} H(0, x) = 0 \]
which yields the desired upper-bound for \( \dot{\varphi}_t \).

If \( H \) realizes its maximum \( H_{\text{max}} \) at some point \( (t_0, x_0) \) with \( t_0 > 0 \), then
\[ 0 \leq \left( \frac{\partial}{\partial t} - \Delta_{S_t} \right)(H)(t_0, x_0) \]
hence
\[ t_0 \dot{\varphi}_{t_0}(x_0) \frac{\partial F}{\partial r}(t_0, x_0, \varphi_{t_0}(x_0)) \leq -1 < 0. \]
Since \( \frac{\partial F}{\partial r} \geq 0 \) and \( t_0 > 0 \), we infer \( \dot{\varphi}_{t_0}(x_0) < 0 \). hence
\[ H_{\text{max}} \leq -(\varphi_{t_0} - \varphi_0)(x_0) \leq C, \]
where the last inequality follows from Proposition 2.1. This yields again the desired upper-bound.

We now take care of the lower bound. We first deal with the particular case when
\[ \mu(x) = h(x)\theta^n(x), \]
where \( h \geq 0 \) is a bounded density. Fix \( D \gg 1 \) so large that all our quantities are well defined and under control on \([0, T + 1/D] \times X\). Observe that
\[ \chi + D\omega_t = D\omega_{t+1/D} \geq D\theta. \]

We set
\[ G(t, x) = \dot{\varphi}_t(x) + D\varphi_t(x) - n \log t, \]
and compute
\[
\left( \frac{\partial}{\partial t} - \Delta S_t \right) (G) = \text{tr}_t (\chi + D\omega_t) - Dn - \frac{\partial F}{\partial t} + \left[ D - \frac{\partial F}{\partial r} \right] \dot{\varphi}_t - \frac{n}{t}
\]
\[
\geq \frac{f_t^{-1/n}}{C_1} - C_2 - \frac{n}{t},
\]
where \( f_t = e^{\dot{\varphi}_t} \). We have used here
\[
\text{tr}_t (\chi + D\omega_t) \geq D \text{tr}_t (\theta) \geq nD \left( \frac{\theta^n}{S^n_t} \right)^{1/n} \geq f_t^{-1/n} \frac{C''}{C_1},
\]
(the last inequality uses our assumption that \( \mu = h\theta^n \) with \( h \) bounded), the fact that \( \frac{\partial F}{\partial t} \leq c, \frac{\partial F}{\partial r} \leq c' \), and the inequality
\[
\dot{\varphi}_t = \log f_t \geq -n\varepsilon f_t^{-1/n} - nC_e,
\]
valid for \( \varepsilon > 0 \) arbitrarily small (since \( \varepsilon x > \log x - C_e \) for \( x > 0 \)).

The function \( G \) attains its minimum on \([0,T] \times X\) at a point \((t_0, x_0)\) with \( t_0 > 0 \). At this point we therefore have a control on the density \( f_t \), namely
\[
f^{-1/n}_{t_0}(x_0) \leq C_1 C_2 + \frac{nC_2}{t_0}
\]

hence
\[
\dot{\varphi}_{t_0}(x_0) = \log f_{t_0}(x_0) \geq -n \log \left[ C_1 C_2 + \frac{nC_2}{t_0} \right].
\]

We infer
\[
G(t_0, x_0) \geq D\varphi_{t_0}(x_0) - n \log [C_1 C_2 t_0 + nC_2] \geq -C_3,
\]
using Proposition 2.1. The desired lower bound follows.

We now get rid of the extra assumption made on \( \mu \). We fix as earlier \( \rho_1 \) a smooth \( \theta \)-psh function such that
\[
(\theta + d\bar{\partial} \rho_1)^n = e^{c_1} \mu
\]
and \( \sup_X \rho_1 = 0 \). We set
\[
\tilde{\omega}_t = \omega_t + d\bar{\partial} \rho_1, \quad \tilde{\varphi}_t = \varphi_t - \rho_1 \in \text{PSH}(X, \tilde{\omega}_t)
\]
and
\[
\tilde{F}(t, x, r) = F(t, x, r + \rho_1(x)).
\]

Observe that \( \tilde{\varphi}_t = \varphi_t, \frac{\partial}{\partial t} \tilde{\omega}_t = \frac{\partial}{\partial t} \omega_t \) and
\[
\tilde{\omega}_t \geq \tilde{\theta} = \theta + d\bar{\partial} \rho_1.
\]

Moreover \( \tilde{F} \) has the same Lipschitz constant (in \( t \) and \( r \)) as that of \( F \) and
\[
(\tilde{\omega}_t + d\bar{\partial} \tilde{\varphi}_t)^n = e^{\tilde{\varphi}_t + \tilde{F}(t, x, \tilde{\varphi}_t)} \mu(x)
\]
with $\tilde{h}^n = e^{c_1} \mu$, hence $\mu = \tilde{h}\tilde{h}^n$ with bounded density $\tilde{h} = e^{-c_1}$. We can thus use the same reasoning as above to conclude.

2.3.2. Refining the hypotheses. We now establish similar uniform bounds on $\dot{\varphi}_t$ under less restrictive assumptions on $t \mapsto \omega_t$ and $r \mapsto F(\cdot, \cdot, r)$. Recall that $r \mapsto F(t, x, r) + \lambda_F r$ is increasing on $\mathbb{R}$.

**Theorem 2.5.** Assume $\dot{\omega}_t \geq -A\omega_t$. Then for all $(t, x) \in X_T$,

$$n \log t - C \leq \dot{\varphi}_t(x).$$

If there exists $A \geq 0$ with $\dot{\omega}_t \leq A\omega_t$, then for all $(t, x) \in X_T$,

$$\dot{\varphi}_t(x) \leq \frac{C}{t}.$$

Here again $C$ depends explicitly on $T, ||\partial F/\partial r||_{L^\infty}, ||\partial F/\partial t||_{L^\infty}, ||g||_{L^p}$ and $C_0$ (defined in Proposition 2.1). The norms $||\partial F/\partial r||_{L^\infty}, ||\partial F/\partial t||_{L^\infty}$ are computed on $[0, T] \times X \times [-C_0, C_0]$.

**Proof.** Consider

$$H(t, x) = t\dot{\varphi}_t(x) - C\varphi_t,$$

where $C := (A + \lambda_F)T + 2$. We let the reader check that

$$\left(\frac{\partial}{\partial t} - \Delta_S\right) (H) \leq C' - [t\partial_t F + C - 1] \dot{\varphi}_t.$$

The upper bound then follows just as in the proof of Theorem 2.4.

We now establish the lower bound. Consider, for $(t, x) \in [0, T] \times X$,

$$G(t, x) := \dot{\varphi}_t + A(2\varphi_t - \rho_1) - n \log t,$$

where $\rho_1 \in \text{PSH}(X, \theta)$ is the unique normalized solution to

$$(\theta + dd^c \rho_1)^n = e^{c_1} gdV.$$

Using the same notations as in the proof of Theorem 2.4 we obtain

$$\Delta_S G = \Delta_S \dot{\varphi}_t + A(\text{Tr}_S(2\omega_t + 2dd^c \varphi_t) - \omega_t - (\omega_t + dd^c \rho_1))$$

$$\leq \Delta_S \dot{\varphi}_t + 2nA - A\text{Tr}_S(\omega_t) - \text{Tr}_S(\theta + dd^c \rho_1)$$

$$\leq \Delta_S \dot{\varphi}_t + 2nA + \text{Tr}_S(\omega_t) - n e^{(c_1 - \dot{\varphi}_t - F(\omega_t, \cdot))}$$

$$\leq \text{Tr}_S(\omega_t + dd^c \dot{\varphi}_t) + 2nA - \frac{f_{t}^{-1/n}}{C_1}.$$

It thus follows that

$$\left(\frac{\partial}{\partial t} - \Delta_S\right) (G) = \ddot{\varphi}_t - 2A\dot{\varphi}_t - \frac{n}{t} - \Delta_S G$$

$$\geq -\frac{\partial F}{\partial t} - \left(2A + \frac{\partial F}{\partial r}\right) \dot{\varphi}_t + \frac{f_{t}^{-1/n}}{C_1} - \frac{n}{t} - 2nA.$$

We can then conclude as in the proof of Theorem 2.4. $\square$
Remark 2.6. The lower bound for $\dot{\varphi}_t$ ensures that
\[ \varphi_t \geq \varphi_0 + n(t \log t - t) - Ct, \]
which is a similar lower bound than the one provided by Proposition 2.2.

2.4. Semi-concavity in time. Our goal in this section is to establish that $\omega_t$-psh solutions to (CMAF) are $\kappa$-concave in time away from zero, with a uniform a priori constant $\kappa$.

2.4.1. A particular case.

Theorem 2.7. Assume that $t \mapsto \omega_t$ is affine and $r \mapsto F(\cdot, \cdot, r)$ is convex, increasing. Let $\varphi_t$ be a smooth solution to (CMAF). Then there exists $C > 0$ such that
\[ \ddot{\varphi}_t(x) \leq C/t \quad \text{for all } (t, x) \in X_T, \]
where $C$ depends explicitly on $T$, $\|\partial F/\partial r\|_{L^\infty}$, $\|\partial F/\partial t\|_{L^\infty}$, $\|\partial^2 F/\partial r \partial t\|_{L^\infty}$, $\|\partial^2 F/\partial t^2\|_{L^\infty}$, $\|g\|_p$ and $C_0$.

Recall that $C_0$ is an upper bound for $|\varphi_t|$ established in Proposition 2.1, and the norms on the partial derivatives of $F$ are computed on $X_T \times [-C_0, C_0]$.

We will establish a similar (though less precise) control under less restrictive assumptions on $t \mapsto \omega_t$ and $F$. We postpone this to the next subsection, as the a priori estimates are already quite involved.

Proof. Set $\omega_t = \omega + t\chi$ so that $\dot{\omega}_t = \chi$. Writing
\[ \dot{\varphi}_t = \log \left[ (\omega_t + dd^c \varphi_t)^n / g(x) dV(x) \right] - F(t, x, \varphi), \]
we differentiate in time to obtain
\[ \ddot{\varphi}_t = \Delta_{S_t}(\dot{\varphi}_t) + \text{tr}_{S_t}(\dot{\omega}_t) - \frac{\partial F}{\partial t}(t, x, \varphi_t) - \varphi_t \frac{\partial F}{\partial r}(t, x, \varphi_t), \]
where $S_t = \omega_t + dd^c \varphi_t$,
\[ \Delta_{S_t} f := n \frac{dd^c f \wedge S_t^{n-1}}{S_t^n} \quad \text{and} \quad \text{tr}_{S_t}(\eta) := n \frac{\eta \wedge S_t^{n-1}}{S_t^n}. \]

It follows from the Lipschitz a priori estimate (Theorem 2.4) that
\[ -C \leq t \dot{\varphi}_t \frac{\partial F}{\partial r}(t, x, \varphi_t) \leq C \]
is uniformly bounded on $X_T$, hence
\[ t \ddot{\varphi}_t = t \text{tr}_{S_t}(\chi + dd^c \dot{\varphi}_t) + O(1). \]

Differentiating again yields
\[ \dddot{\varphi}_t = \Delta_{S_t}(\dot{\varphi}_t) - n^2 \left( \frac{(\chi + dd^c \varphi_t) \wedge S_t^{n-1}}{S_t^n} \right)^2 + n(n-1) \left( \frac{(\chi + dd^c \varphi_t)^2 \wedge S_t^{n-2}}{S_t^n} \right) \]
\[ - \frac{\partial^2 F}{\partial t^2}(x, t, \varphi_t) - 2 \dot{\varphi}_t \frac{\partial^2 F}{\partial r \partial t}(x, t, \varphi_t) - \dot{\varphi}_t \frac{\partial F}{\partial r}(x, t, \varphi_t) - (\dot{\varphi}_t)^2 \frac{\partial^2 F}{\partial r^2}(x, t, \varphi_t). \]
Set $H(t,x) = t\dot{\varphi}_t - Bt$ where $B > 0$. It follows from the Lipschitz control
$t|\dot{\varphi}_t| \leq C$ and Lemma 2.8 below that
\[
\left( \frac{\partial}{\partial t} - \Delta S_t \right) H \leq \left[ 1 - t \frac{\partial F}{\partial r} \right] \dot{\varphi}_t - nt \left( \frac{(\chi + dd^c\varphi) \land S_t^{n-1}}{S_t^n} \right)^2
\]
if we choose $B > 0$ so large that
\[
t \frac{\partial^2 F}{\partial t \partial r} (t,x,\varphi_t) + 2t \dot{\varphi}_t \frac{\partial^2 F}{\partial r \partial t} (t,x,\varphi_t) + B \geq 0.
\]
We use here the simplifying assumption that $r \mapsto F(\cdot,\cdot,r)$ is convex so that $- (\dot{\varphi}_t)^2 \frac{\partial^2 F}{\partial r^2} (t,x,\varphi_t) \leq 0$. We will remove this assumption in the next subsection.

Let $(t_0,x_0) \in X_T$ be a point at which $H$ realizes its maximum. If $t_0 = 0$ then $H \leq 0$ hence $\dot{\varphi}_t \leq B$ and we are done. If $t_0 > 0$, then
\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta S_t \right) H \text{ at the point } (t_0,x_0) \text{ thus for } (t,x) = (t_0,x_0),
\]
\[
\frac{1}{n} (t \text{ tr}_{S_t}(\chi + dd^c\varphi_t))^2 \leq \left[ 1 - t \frac{\partial F}{\partial r} \right] t\ddot{\varphi}_t
\]
with
\[
t\ddot{\varphi}_t = t \text{ tr}_{S_t}(\chi + dd^c\varphi_t) - t \frac{\partial F}{\partial t} (t,x,\varphi_t) - t \dot{\varphi}_t \frac{\partial F}{\partial r} (t,x,\varphi_t) = t \text{ tr}_{S_t}(\chi + dd^c\varphi_t) + O(1).
\]
It follows that $t_0\ddot{\varphi}_{t_0}(x_0)$ is uniformly bounded from above, hence so is $H \leq C$. Thus $t\ddot{\varphi}_t \leq Bt + C \leq C'$ on $X_T$. \hfill \Box

We have used the following differential inequality which is probably well known. We include a proof for the reader’s convenience.

**Lemma 2.8.** Assume $n \geq 2$. Let $\omega$ be a Kähler form and let $\eta$ be a closed $(1,1)$-differential form. Then
\[
\frac{\eta^2 \land \omega^{n-2}}{\omega^n} \leq \left( \frac{\eta \land \omega^{n-1}}{\omega^n} \right)^2.
\]

**Proof.** This is a pointwise inequality, hence it reduces to linear algebra. Since $\omega$ is a Kähler form, we can assume that $\omega(x)$ is the euclidean Kähler metric. Perturbing $\eta(x)$ if necessary, we can also make a change of local coordinates so that $\eta(x)$ is given by a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. We infer
\[
\frac{\eta^2 \land \omega^{n-2}}{\omega^n} (x) = \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta
\]
while
\[
\left( \frac{\eta \land \omega^{n-1}}{\omega^n} \right)^2 (x) = \left( \frac{1}{n} \sum_\alpha \lambda_\alpha \right)^2.
\]
Theorem 2.9. Assume that
\[\begin{align*}
&\sum_{\alpha=1}^n \lambda_\alpha - \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta = \frac{1}{n^2(n-1)} \sum_{\alpha < \beta} (\lambda_\alpha - \lambda_\beta)^2 \geq 0.
\end{align*}\]

\[\square\]

2.4.2. More general bounds. We assume in this subsection that there exists a constant \(A > 0\) such that \(\forall t \in [0, T]\)
\[\begin{align*}
A\omega_t &\leq \dot{\omega}_t \leq +A\omega_t \quad \text{and} \quad \ddot{\omega}_t \leq A\omega_t.
\end{align*}\]

We also assume that \((t, r) \mapsto F(t, x, r)\) is uniformly semi-convex, i.e. there exists a constant \(C_F > 0\) such that for every \(x \in X\), the function
\[\begin{align*}
(t, r) \mapsto F(t, x, r) + C_F(t^2 + r^2)
\end{align*}\]
is convex in \([0, T] \times [-C_0, C_0]\).

Theorem 2.9. Assume that \(\omega_t\) and \(F(\cdot, \cdot, r)\) are as above. Let \(\varphi_t\) be a solution of the above parabolic Monge-Ampère equation. Then there exists \(C > 0\) such that
\[\varphi_t(x) \leq \frac{C}{t^2} \quad \text{for all} \quad (t, x) \in X_T,
\]
where \(C\) depends explicitly on \(A, T, C_F, \lambda_F, \|\partial F/\partial r\|_{L^\infty}, \|\partial F/\partial t\|_{L^\infty}, \|g\|_p, \text{ and } C_0\).

Here \(C_0\) is the constant given in Proposition 2.1 and the norms \(\|\partial F/\partial r\|_{L^\infty}, \quad \|\partial F/\partial t\|_{L^\infty}\) are computed on \(X_T \times [-C_0, C_0]\).

Proof. In the proof below we use \(C\) to denote various constants under control. Set \(\alpha_t := \dot{\omega}_t + dd^c\varphi_t, S_t := \omega_t + dd^c\varphi_t, \text{ and for } h \in C^\infty(X, \mathbb{R})\),
\[\Delta_t h := \text{Tr}_t(dd^c h) := \text{Tr}_t S_t(2dd^c h) = n \frac{dd^c h \wedge S_t^{n-1}}{S_t^n}.
\]
Writing
\[\varphi_t = \log \left(\frac{(\omega_t + dd^c\varphi_t)^n}{g(x) dV(x)}\right) - F(t, x, \varphi),\]
we differentiate twice in time to obtain, as in the proof of Theorem 2.7, that
\[\begin{align*}
t \dddot{\varphi}_t = t \text{Tr}_t \alpha_t - t \partial_t F - t \partial_t \varphi_t \partial_t F = t \text{Tr}_t \alpha_t + O(1),
\end{align*}\]
where we use the uniform bound \(t|\dot{\varphi}_t| \leq C\) (thanks to Theorem 2.5), and
\[\begin{align*}
\dddot{\varphi}_t &\leq \text{Tr}_t(\dot{\alpha}_t) + n(n-1) \frac{\alpha_t^2 \wedge S_t^{n-2}}{S_t^n} - n^2 \left(\frac{\alpha_t \wedge S_t^{n-1}}{S_t^n}\right)^2 - \varphi_t \frac{\partial F}{\partial r}(t, x, \varphi_t)
\end{align*}\]
\[\begin{align*}
&- \frac{\partial^2 F}{\partial r^2}(t, x, \varphi_t) - 2 \varphi_t \frac{\partial^2 F}{\partial r \partial t}(t, x, \varphi_t) - (\dot{\varphi}_t)^2 \frac{\partial^2 F}{\partial r^2}(t, x, \varphi_t)
\end{align*}\]
\[\begin{align*}
&\leq \text{Tr}_t(\dot{\alpha}_t) - \frac{1}{n} (\text{Tr}_t \alpha_t)^2 - \frac{\varphi_t^2}{\partial r}(t, x, \varphi_t) + C_F((\dot{\varphi}_t)^2 + 1),
\end{align*}\]
using the convexity condition (2.2) and Lemma 2.8. The Lipschitz control $t|\dot{\varphi}_t| \leq C$ (provided by Theorem 2.5) yields

\begin{equation}
(2.4) \quad t^2 \ddot{\varphi}_t \leq t^2 \text{Tr}_t \dot{\alpha}_t - t^2 n^{-1} [\text{Tr}_t(\alpha_t)]^2 - t^2 \varphi_t \frac{\partial F}{\partial r}(t, x, \varphi_t) + C.
\end{equation}

Set $H(t, x) = t^2 \varphi_t - ATt \varphi_t$. It follows from (2.1) and a direct computation that

\begin{equation}
(2.5) \quad \Delta_t H \geq t^2 \text{Tr}_t \dot{\alpha}_t + (ATt - A t^2) \text{Tr}_t(\omega_t) - AnTt \geq t^2 \text{Tr}_t \dot{\alpha}_t - C.
\end{equation}

It follows therefore from (2.4) and (2.5) that

\[
\left( \frac{\partial}{\partial t} - \Delta_t \right) H \leq t \ddot{\varphi}_t (2 - \theta_\alpha F) - t^2 n^{-1} (\text{Tr}_t \alpha_t)^2 + C.
\]

Let $(t_0, x_0) \in X_T$ be a point at which the function $H$ realizes its maximum. If $t_0 = 0$ then $H \leq 0$ hence $t^2 \varphi_t \leq C$ and we are done. If $t_0 > 0$, then

\[
t(\Delta_t H) = t \ddot{\varphi}_t (2 - \theta_\alpha F) - t^2 n^{-1} (\text{Tr}_t \alpha_t)^2 
\]

Using (2.3) we conclude that $t^2 \varphi_t \leq C$ on $X_T$, finishing the proof. \hfill \Box

2.5. Conclusion.

2.5.1. The estimates. We summarize here the a priori estimates we have obtained so far. We assume that the forms and densities are smooth and satisfy the uniform bounds listed in the introduction, involving the constants $A, p, \lambda_F, C_F$.

**Theorem 2.10.** There exists $C_0, C_1, C_2 > 0$ such that for all $(t, x) \in X_T$,

1. $-C_0 \leq \varphi_t(x) \leq C_0$;
2. $n \log t - C_1 \leq \varphi_t(x) \leq C_1/t$;
3. $\ddot{\varphi}_t(x) \leq C_2/t^2$;

where the $C_j$'s depend on $A, p, C_F, \lambda_F$ and

- $C_0$ explicitly depends on $T, \theta, \Theta, \inf_X \varphi_0, \sup_X \varphi_0$ and $\sup_{X_T} |F(t, x, 0)|$;
- $C_1$ explicitly depends on $C_0, T, ||\partial F/\partial r||_{L^\infty}, ||\partial F/\partial t||_{L^\infty}$ and $||g||_{L^p}$;
- $C_2$ explicitly depends on $C_0, C_1, T$.

The norms $||\partial F/\partial r||_{L^\infty}, ||\partial F/\partial t||_{L^\infty}$ are computed on $X_T \times [-C_0, C_0]$.

2.5.2. Convergence of semi-concave functions. It is useful to know when a sequence of $\omega_t$-psh functions is uniformly semi-concave. It allows one to obtain the convergence of the associated parabolic Monge-Ampère operators as the following result shows:

**Theorem 2.11.** Let $g_j(t, x)$ be a family of $L^1(X_T)$-densities such that $g_j \to g$ in $L^1(X_T)$. Let $F_j(t, x, r)$ be continuous densities which uniformly converge towards $F$. Let $\varphi_j(t, x)$ be a family of $\omega_t$-psh functions such that

- $(\varphi_j)$ is uniformly bounded;
- $\ddot{\varphi}_j \leq C/t^2$ for some uniform constant $C > 0$. 


Then there exists a bounded function $\varphi \in \mathcal{P}(X_T, \omega)$ such that, up to extracting and relabelling, $\varphi_j \to \varphi$ in $L^1_{\text{loc}}(X_T)$ and
\[
e^\varphi_j + F_j(t,x,\varphi_j(t,x)) g_j(t,x) \, dt \wedge dV(x) \longrightarrow e^{\varphi+F(t,x,\varphi(t,x))} g(t,x) \, dt \wedge dV(x),
\]
in the weak sense of Radon measures on $X_T$.

Proof. Since $(\varphi_j)$ is bounded in $L^2(X_T)$, it is weakly compact. Extracting and relabelling, we assume that $(\varphi_j)$ weakly converges to $\varphi \in L^2(X_T)$.

Fix a compact sub-interval $J \subset [0,T]$. There exists a constant $C = C_J > 0$ such that the functions $t \mapsto \varphi_j(t,x) - Ct^2$ are concave in $J$ for all $x \in X$ fixed. The same property holds for the limiting function $\varphi(t,x)$ by letting $j \to +\infty$. For $t$ fixed, the functions $x \mapsto \varphi_j(t,x)$ are $\omega_t$-psh and uniformly bounded, hence $x \mapsto \varphi(t,x)$ is $\omega_t$-psh and uniformly bounded in $X_T$.

It follows from Theorem 1.3 that $\varphi_j \to \varphi$ in $L^1_{\text{loc}}(X_T)$ and $\varphi_j(t,x) \to \varphi(t,x)$ almost everywhere in $X_T$ with respect to the Lebesgue measure. The conclusion follows by applying Theorem 1.15. \qed

3. Existence and properties of sub/super/solutions

From now on we assume that $t \mapsto \omega_t$ and the densities $g,F$ satisfy the conditions listed in the introduction.

For bounded parabolic potentials $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$, the equation (CMAF) should be understood in the sense of measures on $]0,T[ \times X$:

(CMAF) \[
(\omega_t + dd^c \varphi_t)^n \wedge dt = e^{\varphi_t + F(t,x,\varphi_t)} g(x) dV(x) \wedge dt.
\]

It follows from Definition 1.11 that the left hand side is a well defined Radon measure, while Lemma 1.6 ensures that so is the right hand side.

3.1. Stability estimates. We establish in this section uniform $L^\infty-L^1$ stability estimates needed in the proof of the existence theorem.

Proposition 3.1. Fix $0 \leq g_1, g_2 \in L^p(X)$ with $p > 1$, and $0 < T_0 < T_1 < T$. Assume $\varphi^1, \varphi^2 \in \mathcal{P}(X_T, \omega) \cap C^\infty(X_T)$ both satisfy
\[
dt \wedge (\omega_t + dd^c \varphi_t)^n = e^{\varphi^1(t,x) + F_1(t,x,\varphi^1)} g_1 dt \wedge dV.
\]
Then for all $(t,x) \in [T_0, T_1] \times X$,
\[
|\varphi^1(t,x) - \varphi^2(t,x)| \leq B \|\varphi^1 - \varphi^2\|_{L^1(X_T)},
\]
where $0 < \alpha = \alpha(n,p)$ while $0 < B$ depends on $T_0, T_1, \theta, \Theta, \omega_t$, and upper bounds for $\|g^1\|_{L^p(X)}$, $\|F_1\|_{L^\infty(X_T)}$, and $\|\partial_t F_1\|_{L^\infty(X_T)}$, $\|\partial_x F_1\|_{L^\infty(X_T)}$.

Proof. We are going to use the stability results of [EGZ09, GZ12]. These rely on important estimates which we recall for the convenience of the reader. The uniform bounds $\theta \leq \omega_t \leq \Theta$ and [EGZ08, Lemma 2.2] show that there exists a uniform constant $A_1 > 0$ such that
\[
\text{Vol}(K) \leq A_1 \text{Cap}_{\omega_t}(K)^2,
\]
for all \( t \in [0, T] \) and all compact sets \( K \subset X \), where
\[
\text{Cap}_{\omega_t}(K) := \sup \left\{ \int_K (\omega_t + dd^c u)^n ; u \in \text{PSH}(X, \omega_t) \text{ with } 0 \leq u \leq 1 \right\}
\]
is the Monge-Ampère capacity associated to the form \( \omega_t \).

Fix \( T_0 < T_1 < T \) and consider the densities
\[
f^i_t := e^{\psi^i(t, \cdot) + F_i(t, x, \psi^i)} g_i(x), \quad i = 1, 2.
\]
It follows from Theorem 2.5 that \( t \psi^1(t, x) \) is uniformly bounded by \( C_1 \), while Proposition 2.1 ensures that the \( \psi^i \)'s are uniformly bounded. The \( L^p \) norms of the densities \( f^i_t \) are thus uniformly bounded from above by
\[
A_2 := e^{C_1/T_0 + C_2 (\|g_0\|_{L^p(X)} + \|g_1\|_{L^p(X)})}
\]
when \( t \in [T_0, T_1] \). It follows therefore from [EGZ09, Proposition 3.3] that
\[
\max_X \|\psi^1(t, \cdot) - \psi^2(t, \cdot)\| \leq C \|\psi^1(t, \cdot) - \psi^2(t, \cdot)\|_{L^1(X)},
\]
for all \( t \in [T_0, T_1] \), where \( \gamma \in [0, 1[ \) only depends on \( p, n \). Lemma 1.5 yields
\[
\|\psi^1(t, \cdot) - \psi^2(t, \cdot)\|_{L^1(X)} \leq A \max \{\|\psi^1 - \psi^2\|_{L^1(X_T)}; \|\psi^1 - \psi^2\|_{L^1(X_T)}^{1/2}\},
\]
where \( A := 2 \max \{\sqrt{n}, (T - T_1)^{-1}\} \).

The proof is completed by combining the last two inequalities. \( \square \)

In practice, this proposition yields the following useful information:

**Corollary 3.2.** Assume \( \|g_j\|_{L^p}, \|F_j\|_{L^\infty}, \|\partial_t F_j\|_{L^\infty} \) and \( \|\partial_x F_j\|_{L^\infty} \) are uniformly bounded. If a sequence \( (\varphi^j) \) of solutions to \( (\text{CMAF})_{F_j, g_j} \) converges in \( L^1(X_T) \) to \( \varphi \), then it uniformly converges on compact subsets of \( ]0, T[ \times X \).

3.2. **The Cauchy problem.** We are now in position to prove Theorem A of the introduction.

**Definition 3.3.** A parabolic potential \( \varphi \in \mathcal{P}(X_T, \omega) \) is a pluripotential solution (sub/super solution respectively) to \( (\text{CMAF}) \) with initial values \( \varphi_0 \in \text{PSH}(X, \omega_0) \cap L^\infty(X) \) if \( \varphi \) satisfies \( (\text{CMAF}) \) (or the inequality \( \geq \leq \) respectively) in the sense of measures on \( X_T \) and \( \varphi_t \to \varphi_0 \) in \( L^1(X) \) as \( t \to 0^+ \).

**Theorem 3.4.** Assume that \( \varphi_0 \) a bounded \( \omega_0 \)-psh function in \( X \), and \( (\omega, F, g) \) is as in the introduction. There exists \( \varphi \in \mathcal{P}(X_T, \omega) \) solving \( (\text{CMAF}) \) such that for all \( 0 < T' < T \),
- \( (t, x) \mapsto \varphi(t, x) \) is uniformly bounded in \( ]0, T'[ \times X \);
- \( (t, x) \mapsto \varphi(t, x) \) is continuous in \( ]0, T[ \times \Omega \);
- \( t \mapsto \varphi(t, x) - n(t \log t - t) + C_1 t \) is increasing on \( ]0, T'[^{\text{for some}} \ C_1 > 0; \)
- \( t \mapsto \varphi(t, x) + C_2 \log t \) is concave on \( ]0, T'[ \) for some \( C_2 > 0; \)
- \( \varphi_t \to \varphi_0 \) as \( t \to 0^+ \) in \( L^1(X) \) and pointwise.
Recall that $\Omega$ is the ample locus of $\theta$. The solution we provide is in particular locally uniformly semi-concave in $t \in ]0,T[$. We will study the uniqueness issue in the next section.

**Proof.** Fix $0 < T' < T$. We prove the existence of a solution on $]0,T'[ \times X$. The uniqueness result (Corollary 4.5) then ensures that a solution exists in $]0,T[ \times X$.

We approximate

- $g$ by smooth densities $g_j > 0$ in $L^p(X)$;
- $F$ by smooth densities $F_j$ with uniform constants $\kappa_{F_j}, C_{F_j}, \lambda_{F_j}$;
- $\varphi_0$ from above on $X$ by smooth $(\omega_0 + 2^{-j}\Omega)$-psh functions $\varphi_{0,j}$.

It is well-known (see e.g. [Tos18]) that there exists a unique smooth solution $\varphi^j \in \mathcal{P}(X,T,\omega)$ to (CMAF)$_{F_j,g_j}$, i.e.

$$
(3.2) \quad dt \wedge (\omega_t + dd^c \varphi^j)^n = e^{\varphi^j(t,x) + F_j(t,x,\varphi^j(t,x))} g_j dt \wedge dV(x).
$$

It follows from Theorem 2.10 that the $\varphi^j$’s are uniformly bounded and the derivatives $\dot{\varphi}^j$ are locally uniformly bounded from above in $X_T$. Extracting and relabelling Theorem 2.11 ensures that there exists $\varphi \in \mathcal{P}(X_T) \cap L^\infty_{\text{loc}}(X_T)$ such that $\varphi^j \to \varphi$ in $L^1(X_T)$ and

$$
\begin{align*}
\lim_{n \to \infty} e^{\varphi^j(t,x) + F_j(t,x,\varphi^j(t,x))} g_j(t,x) dt \wedge dV(x) &\to e^{\varphi(t,x) + F(t,x,\varphi(t,x))} g(t,x) dt \wedge dV(x),
\end{align*}
$$

in the sense of currents on $X_T$.

We claim that $\varphi^j \to \varphi$ locally uniformly in $X_T$. This follows indeed from the stability estimates established in Proposition 3.1 above. Fix $0 < T_0 < T_1 < T$. Since the densities $g_j$ have uniform $L^p$ norms, Theorem 2.5 ensures that the sequence $(\dot{\varphi}^j)$ is uniformly bounded in $[T_0,T] \times X$. By (3.1), for all $j,k$ large enough, $t \in [T_0,T_1], x \in X$, we have

$$
|\dot{\varphi}^j(t,x) - \dot{\varphi}^k(t,x)| \leq C\|\dot{\varphi}^j - \dot{\varphi}^k\|^p_{L^1(X_T)},
$$

where $C > 0$ and $0 < \alpha < 1$ are uniform constants which do not depend on $j,k$, and $t \in [T_0,T_1]$. This proves our claim.

Therefore $dt \wedge (\omega_t + dd^c \varphi^j)^n \to dt \wedge (\omega_t + dd^c \varphi)^n$ in the sense of measures on $X_T$, hence $\varphi$ solves (CMAF).

One shows similarly that $\varphi$ is uniformly semi-concave in $]0,T[$: the densities $g_j$ in (3.2) are uniformly bounded in $L^p(X)$, hence Theorem 2.9 insures the existence of a uniform constant $C > 0$ such that

$$
\dot{\varphi}^j(t,x) \leq C/t^2
$$

for all $j \in \mathbb{N}, (t,x) \in X_T$. Thus, for each compact subinterval $J \subseteq ]0,T[$ there exists a constant $C_J > 0$ such that the functions $t \mapsto \dot{\varphi}^j(t) - C_J t^2$ are concave in $J$, and the same property holds for $\varphi$ by letting $j \to \infty$.

The continuity of $\varphi$ on $]0,T[ \times \Omega$ follows from the elliptic theory as will be shown in Proposition 3.12 below. The lower bound $\dot{\varphi}_t \geq n \log t - C_1$, provided by Theorem 2.5, ensures that $t \mapsto \varphi_t - n(t \log t - t) + C_1 t$ is increasing, hence any cluster point (in $L^1$-topology) of $\varphi_t$ (as $t \to 0^+$) is
greater than \( \varphi_0 \). On the other hand, it follows from Proposition 2.3 and Lemma 1.5 that
\[
\int_X \varphi_t g dV \leq \int_X \varphi_0 g dV + Ct,
\]
for a uniform constant \( C > 0 \). Let \( u_0 \) be any cluster point of \( (\varphi_t) \) as \( t \to 0^+ \). Then as explained above, \( u_0 \geq \varphi_0 \). On the other hand, the average control above ensures that
\[
\int_X u_0 g dV \leq \int_X \varphi_0 g dV.
\]
Since the set \( \{ g = 0 \} \) has Lebesgue measure zero we infer \( u_0 = \varphi_0 \) almost everywhere, hence everywhere. \( \square \)

**Remark 3.5.** Proposition 1.2 ensures that the pluripotential solution constructed above is upper semi-continuous on \([0, T] \times X \). The functions \( \varphi_t \) quasi-decrease to \( \varphi_0 \) as \( t \to 0 \). The convergence at time zero is thus quite strong: if \( \varphi_0 \) is continuous, it follows for instance that the convergence is uniform (for non continuous initial \( \varphi_0 \), there is convergence in capacity).

**Remark 3.6.** The way the density is allowed to vanish is crucial. Theorem A does not hold for an arbitrary density \( g \geq 0 \): if \( g \) vanishes in a non empty open set \( D \subset X \) then \( (CMAF) \) has no solution with initial value \( \varphi_0 \) unless \( \varphi_0 \) is a maximal \( \omega_0 \)-psh function in \( D \). Indeed the complex Monge-Ampère operator is continuous for the convergence in capacity, so \( (\omega_0 + dd^c \varphi_0)^n = 0 \) would converge to \( (\omega_0 + dd^c \varphi_0)^n = 0 \) in \( D \).

### 3.3. Invariance properties of the set of assumptions.

The family of parabolic complex Monge-Ampère equations we consider
\[
(\omega_t + dd^c \varphi_t)^n = e^{\varphi_t} + F(t, x, \varphi_t) g(x) dV(x),
\]
has several invariance properties, as we now briefly explain.

#### 3.3.1. Translations.

We can replace \( \varphi_t(x) \) by \( \psi_t(x) = \varphi_t(x) + C(t) \) without changing the Monge-Ampère term, while the density \( F \) is modified into
\[
\tilde{F}(t, x, r) = F(t, x, r - C(t)) - C'(t).
\]
We let the reader check that \( \tilde{F} \) satisfies the same set of assumptions as \( F \).

More generally we can replace \( \omega_t \) by \( \eta_t = \omega_t - dd^c \rho_t \), changing \( \varphi_t(x) \) in \( \varphi_t(x) + \rho_t(x) \). The density \( g \) remains unchanged while the new density \( F \) is
\[
\tilde{F}(t, x, r) = F(t, x, r - \rho(t, x)) - \partial_t \rho(t, x).
\]

#### 3.3.2. Scaling.

A more involved transformation consists in scaling in space and renormalizing in time, so that the equation keeps the same shape. Namely we replace \( \omega_t \) by \( \gamma(s) \omega_{t(s)} \) as well as \( \varphi_t(x) \) by \( \psi_s(x) = \gamma(s) \varphi_{t(s)}(x) \), where \( s \mapsto \gamma(s) > 0 \) is smooth and positive, \( t(0) = 0 \) and \( t'(s) = 1/\gamma(s) \), so that
\[
\partial_s \psi_s = \frac{\gamma'(s)}{\gamma(s)} \psi_s + \partial_t \varphi_{t(s)}.
\]
The density $g$ remains unchanged while the density $F$ is transformed into

$$F(s, x, R) = F\left(t(s), x, r(s, R)\right) + n \log \left(\gamma(s) - \frac{\gamma'(s)}{\gamma(s)} R\right),$$

where $r(s, R) = \frac{R}{\gamma(s)}$.

A classical example of such a transformation is when $\gamma(s) = e^s$ and $t(s) = 1 - e^{-s}$, allowing one to pass from the Kähler-Ricci flow to the normalized Kähler-Ricci flow.

We let the reader check that $\tilde{F}$ remains quasi-increasing in $R$ and locally uniformly Lipschitz in $(s, R)$. It is slightly more involved to keep track of the semi-convexity property :

**Lemma 3.7.** The function $(s, R) \mapsto \tilde{F}(s, x, R)$ is locally uniformly semi-convex in $(s, R)$.

**Proof.** Fix $0 < S_0 < S$, $T_0 = t(S_0)$, and a compact interval $J \subseteq \mathbb{R}$. We want to prove that $(s, R) \mapsto \tilde{F}(s, R)$ is semi-convex in $[0, S_0] \times J$. We omit in the sequel the dependence on $x$ as it is not affected by the transformation.

We can assume that $F$ is smooth and proceed by approximation. The goal is to prove that the Hessian matrix $H(s, R)$ of $(s, R) \mapsto \tilde{F}(s, R)$ satisfies

$$H(s, R) + CI_2 \geq 0,$$

where $I_2$ is the identity matrix in $M_2(\mathbb{R})$, and the constant $C$ is under control. Increasing $C$ we can also assume that $F$ is convex in $[0, S_0] \times J$.

Recall that $F$ is Lipschitz on $[0, T_0] \times J$ and $s \mapsto \gamma(s) > 0$ is smooth. Using this we can write

$$\frac{\partial^2 \tilde{F}}{\partial s^2} = \frac{\partial^2 F}{\partial t^2} \left(\frac{\partial t}{\partial s}\right)^2 + 2 \frac{\partial^2 F}{\partial t \partial r} \frac{\partial t}{\partial s} \frac{\partial r}{\partial s} + \frac{\partial^2 F}{\partial r^2} \left(\frac{\partial r}{\partial s}\right)^2 + O(1),$$

$$\frac{\partial^2 \tilde{F}}{\partial R^2} = \frac{\partial^2 F}{\partial r^2} \left(\frac{\partial r}{\partial R}\right)^2 + O(1),$$

$$\frac{\partial^2 \tilde{F}}{\partial R \partial s} = \frac{\partial^2 F}{\partial r^2} \frac{\partial r}{\partial R} \frac{\partial r}{\partial s} + \frac{\partial^2 F}{\partial r \partial t} \frac{\partial r}{\partial R} \frac{\partial t}{\partial s} + O(1).$$

It remains to check that

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \geq 0,$$

where

$$a = \frac{\partial^2 F}{\partial t^2} \left(\frac{\partial t}{\partial s}\right)^2 + 2 \frac{\partial^2 F}{\partial t \partial r} \frac{\partial t}{\partial s} \frac{\partial r}{\partial s} + \frac{\partial^2 F}{\partial r^2} \left(\frac{\partial r}{\partial s}\right)^2,$$

$$c = \frac{\partial^2 F}{\partial r^2} \left(\frac{\partial r}{\partial R}\right)^2,$$

$$b = \frac{\partial^2 F}{\partial r^2} \frac{\partial r}{\partial R} \frac{\partial r}{\partial s} + \frac{\partial^2 F}{\partial r \partial t} \frac{\partial r}{\partial R} \frac{\partial t}{\partial s}.$$
The convexity of $F$ and a direct computation ensure that $a, c \geq 0$ and $ac - b^2 \geq 0$.  

We note, for later use, that such a transformation allows one to reduce to the case when $r \mapsto F(\cdot, \cdot, r)$ is increasing:

**Lemma 3.8.** Assume that $r \mapsto F(\cdot, r)$ is quasi-increasing with $\lambda_F > 0$. Consider $\gamma : s \in [0, S] \mapsto 1 - \lambda_F s \in [0, T]$, where $S < \lambda_F^{-1}$ is defined by $\int_0^S (1 - \lambda_F r)^{-1} \, dr = T$. The function $R \mapsto \tilde{F}(s, R)$ is increasing, $\forall s \in [0, S]$.

**Proof.** The function $\tilde{F}$ is given, for $(s, R) \in [0, S] \times \mathbb{R}$, by

$$\tilde{F}(s, R) = F(t(s), R/\gamma(s)) + n \log \gamma(s) + \frac{\lambda R}{\gamma(s)}.$$

Using that $r \mapsto F(t, x, r) + \lambda r$ is increasing, it is straightforward to check that $\tilde{F}$ is increasing in $R$.  

3.4. **Pluripotential sub/supersolutions.**

3.4.1. **Definitions.** Our plan is to establish a pluripotential parabolic comparison principle. The latter is easier to obtain under an extra regularity assumption in the time variable, so we introduce the following terminology for convenience:

**Definition 3.9.** A parabolic potential $u \in \mathcal{P}(X_T, \omega)$ is called of class $C^{1/0}$ if for every $t \in ]0, T[$ fixed, $\partial_t u(t, x)$ exists and is continuous in $\Omega$.

**Definition 3.10.** A parabolic potential $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ is called a pluripotential subsolution of (CMAF) if

$$(\omega_t + dd^c \varphi_t)^n \wedge dt \geq e^{\varphi_t + F(t, x, \varphi)} g(x) \, dV(x) \wedge dt$$

holds in the sense of measures in $]0, T[ \times X$.

Similarly a parabolic potential $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ is called a pluripotential supersolution of (CMAF) if

$$(\omega_t + dd^c \varphi_t)^n \wedge dt \leq e^{\varphi_t + F(t, x, \varphi)} g(x) \, dV(x) \wedge dt$$

holds in the sense of measures in $]0, T[ \times X$.

In many cases one can interpret these notions by considering a family of inequalities on slices:

**Lemma 3.11.** Fix $u \in \mathcal{P}(X_T) \cap L^\infty(X_T)$.

1) If $u$ is a pluripotential subsolution of (CMAF) such that $\partial^+_t u$ exists and is lower semi-continuous in $t \in ]0, T[$, then for all $t \in ]0, T[$,

$$(\omega_t + dd^c u_t)^n \geq e^{\partial^+_t u + F} \, g dV$$

in the sense of measures in $X$.

2) If $u$ is a pluripotential supersolution of (CMAF) such that $\partial^-_t u$ exists and is upper semi-continuous in $t \in ]0, T[$, then for all $t \in ]0, T[$,

$$(\omega_t + dd^c u_t)^n \leq e^{\partial^-_t u + F} \, g dV$$

in the sense of measures in $X$. 

Proof. We will prove the result for subsolutions. The corresponding result for supersolutions follows similarly. Assume that the right-derivative \( \partial_t^+ u \) exists for all \( (t, x) \in X_T \) and is lower semi-continuous in \( t \) for \( x \) fixed. It follows from [GLZ1, Proposition 3.2] that for almost every \( t \in [0, T] \),
\[
(\omega_t + dd^c u_t)^n \geq e^{\partial_t^+ u + F} gdV,
\]
in the sense of measures on \( X \). Any \( t \in ]0, T[ \) can be approximated by a sequence \( (t_j)_{j \in \mathbb{N}} \) for which the inequality above holds. The limiting inequality follows from the lower semi-continuity of \( \partial_t^+ u(t, x) \) in \( t \) and Fatou’s lemma.

3.4.2. Properties of supersolutions. We use properties of solutions to complex Monge-Ampère equations to show that parabolic supersolutions automatically have continuity properties.

**Proposition 3.12.** Assume that \( \psi \in \mathcal{P}(X_T) \cap L^\infty(X_T) \) is a supersolution to (CMAF). Then \( \psi \) is continuous in \([0, T[ \times \Omega \).

**Proof.** Fix \( 0 < T_0 < T_1 < T \). For almost every \( t \in [0, T[ \) we have
\[
(\omega_t + dd^c \psi_t)^n \leq e^{\psi(t, \cdot) + F(t, \cdot, \psi_t)} gdV
\]
in the weak sense on \( X \).

Since \( \psi \) is locally uniformly Lipschitz in \( t \) and \( F \) is bounded, there exists \( M > 0 \) such that \( \dot{\psi}(t, \cdot) + F(t, \cdot, \psi_t) \leq M \) for almost any \( t \in [T_0, T_1] \). Thus
\[
(\omega_t + dd^c \psi_t)^n \leq e^{M} gdV,
\]
for almost every \( t \in [T_0, T_1] \). By (weak) continuity (in \( t \)) of the LHS, it follows that this inequality actually holds for any \( t \in [T_0, T_1] \).

The elliptic theory (see e.g. [GZ, Theorem 12.23]) implies that \( \psi_t \) is continuous in \( \Omega \) for any \( t \in [T_0, T_1] \). Since \( \psi \) is uniformly Lipschitz in \([T_0, T_1]\) it follows that \( \psi \) is continuous in \([T_0, T_1] \times \Omega \). Indeed let \( \kappa \) be the uniform Lipshitz constant of \( \psi \) on \([T_0, T_1]\). Then for any \( s, t \in [T_0, T_1] \) and \( x, y \in \Omega \) we have
\[
|\psi(s, x) - \psi(t, y)| \leq |\psi(s, x) - \psi(t, x)| + |\psi(t, x) - \psi(t, y)| \\
\leq \kappa |s - t| + |\psi(t, x) - \psi(t, y)|,
\]
which implies the continuity of \( \psi \) in \([T_0, T_1] \times \Omega \). \( \square \)

Supersolutions admit uniform bound from below:

**Proposition 3.13.** Assume that \( \psi \in \mathcal{P}(X_T) \cap L^\infty_{loc}(X_T) \) is a pluripotential supersolution to (CMAF) which is locally uniformly semi-concave in \( t \). There exists \( C > 0, t_0 > 0 \) such that for all \( (t, x) \in ]0, t_0[ \times X \),
\[
\psi(t, x) \geq (1 - t)e^{-At}\psi_0(x) + C(t \log t - t).
\]

Here \( A \) is a positive constant such that
\[
-A \omega_t \leq \dot{\omega}_t \leq A \omega_t, \quad \text{for all } t \in ]0, T[.
\]
This implies in particular that \( \omega_{t+s} \geq e^{-At}\omega_s \), for all \( t, s > 0 \) with \( t + s < T \).
Proof. Set $M := M_{\psi} := \sup_X |\psi|$. The Lipschitz condition on $F$ ensures that there exists a constant $\kappa = \kappa_F$ such that, for all $t, t' \in [0, T/2]$, $x \in X$, $r \in [-M, M]$,

$$|F(t, x, r) - F(t', x, r)| \leq \kappa|t - t'|.$$ 

Set $t_0 := \min(1, T/4, 1/2\lambda)$. As observed above, $\omega_{t+s} \geq e^{-At}\omega_s$ for all $s \in [0, t_0/2]$ and $t \in [0, t_0]$. Fix $s \in [0, t_0]$ and consider, for $t \in [0, t_0]$,

$$u_s(t, x) := (1 - t)e^{-At}\psi_s(x) + t\rho + C(t \log t - t),$$

$$v_s(t, x) := \psi(t + s, x) + 2\kappa ts,$$

where $\rho$ is a $\theta$-psh function on $X$, normalized by $\sup_X \rho = 0$, which solves $(\theta + dd^c \rho)^n = e^{c_1} gdV$ with a normalization constant $c_1$, and $C$ is a positive constant to be specified later.

The existence and boundedness of $\rho$ follows from [EGZ09]. Observe that $u_s$ is of class $C^{1/0}$ in $t$ and for each $t \in [0, t_0]$ fixed, $u_s(t, \cdot)$ is continuous in $\Omega$ (see Proposition 3.12).

A direct computation shows that, for $t \in [0, t_0]$,

$$(\omega_{t+s} + dd^c u_s)^n \geq (1 - t)\omega_{t+s} + t\omega_{t+s} + dd^c((1 - t)e^{-At}\psi_s) + tdd^c \rho)^n$$

$$\geq (1 - t)e^{-At}(\omega_s + dd^c \psi_s) + t(\theta + dd^c \rho))$$

$$\geq t^n gdV.$$

In the second line above we have used $\omega_{t+s} \geq e^{-At}\omega_s$ while in the last line we have used $\omega_s + dd^c \psi_s \geq 0$. Thus, since $\psi_s$ is uniformly bounded, by choosing $C > 0$ large enough (depending on $M_F$) we obtain

$$(\omega_{t+s} + dd^c u_s)^n \geq e^{\partial_t u_s(t, \cdot) + F(t, \cdot, u_s(t, \cdot))} gdV.$$ 

It is also clear from the definition that $u_s(t, \cdot)$ converge in $L^1(X, dV)$ to $u_s(0, \cdot) = \psi_s$ as $t \to 0^+$.

On the other hand, since $\psi$ is a supersolution to (CMAF), by Lemma 3.11 we have

$$(\omega_{t+s} + dd^c v_s)^n \leq e^{\partial_t \psi(t+s, \cdot) + F(t+s, \cdot, \psi(t+s, \cdot))} gdV$$

$$\leq e^{\partial_t v_s(t, \cdot) - 2\kappa s + F(t+s, \cdot, \psi(t+s, \cdot))} gdV.$$ 

The Lipschitz condition on $F$ ensures that, for all $t, s \in [0, t_0], x \in X$,

$$F(t + s, x, \psi(t + s, x)) \leq F(t, x, \psi(t + s, x)) + \kappa s.$$ 

The quasi-increasing property of $F$ ensures that

$$F(t, x, \psi(t + s, x)) = F(t, x, v_s(t, x) - 2\kappa ts)$$

$$\leq F(t, x, v_s(t, x)) + 2\kappa \lambda ts.$$ 

Thus for $t \leq t_0 \leq 1/2\lambda$,

$$(\omega_{t+s} + dd^c v_s)^n \leq e^{\partial_t v_s(t, \cdot) + F(t, \cdot, v_s(t, \cdot))} gdV.$$
It follows from Proposition 3.12 that \( v_s \) is continuous on \([0, t_0] \times \Omega\) and \( v_s(0, \cdot) = \psi_s \). We can thus apply Proposition 4.2 below and obtain \( u_s \leq v_s \) on \([0, t_0] \times X\). Letting \( s \to 0 \) we obtain that for all \((t, x) \in [0, t_0] \times X\),

\[
(1 - t)e^{-At}\psi_0(x) + t\rho + C(t \log t - t) \leq \psi(t, x).
\]

The result follows since \( \rho \) is bounded. \( \square \)

### 3.4.3. Regularization of subsolutions

We introduce a regularization process for subsolutions. Fix \( 0 < T' < T \) and \( \varepsilon_0 > 0 \) such that \((1 + \varepsilon_0)T' < T\). It follows from (0.1) that there exists \( A_1 > 0 \) such that for all \( t \in [0, T'] \) and \( s \in [1 - \varepsilon_0, 1 + \varepsilon_0] \),

\[
(3.3) \quad \omega_t \geq (1 - A_1|s - 1|)\omega_{ts},
\]

where \( \varepsilon_0 > 0 \) is a fixed small constant. For \( |s - 1| < \varepsilon_0 \) we set

\[
\lambda_s := \frac{|1 - s|}{s}, \quad \alpha_s := s(1 - \lambda_s)(1 - A_1|s - 1|) \in [0, 1[.
\]

Up to shrinking \( \varepsilon_0 \) we can also assume that for all \( |s - 1| \leq \varepsilon_0 \),

\[
\gamma_s := \frac{\lambda_s}{1 - \alpha_s} \geq \varepsilon_1,
\]

where \( \varepsilon_1 = (5 + A_1)^{-1} > 0 \).

We let \( \rho \in \text{PSH}(X, \theta) \), \( \sup_X \rho = 0 \), be the unique bounded solution to

\[
(\varepsilon_1 \theta + dd^c \rho)^n = e^{c_1} g dV,
\]

for some normalization constant \( c_1 \in \mathbb{R} \) (see [EGZ09]).

**Lemma 3.14.** Assume that \( u \in \mathcal{P}(X_T) \) is a bounded pluripotential sub-

solution of (CMAF). Then there exists a uniform constant \( C > 0 \), depending on \( M_u := \sup_X |u| \) and the data, such that for every \( s \in [1 - \varepsilon_0, 1 + \varepsilon_0] \),

\[
(t, z) \mapsto v_s(t, z) := \frac{\alpha_s}{s} u(ts, x) + (1 - \alpha_s)\rho(x) - C|s - 1|t
\]

is a pluripotential sub-solution of (CMAF) in \( X_T \).

**Proof.** For notational convenience we set

\[
\beta_s := \frac{1 - \lambda_s}{\alpha_s} = 1 + O(|s - 1|).
\]

Since \( u \) is a pluripotential sub-solution of (CMAF), using (3.3) we can write

\[
(\beta_s \omega_t + s^{-1}dd^c u(st, \cdot))^n \geq s^{-n}(\omega_{ts} + dd^c u(st, \cdot))^n
\]

\[
\geq e^{-n \log s + \partial_t u(st, \cdot) + F(ts, u(st, x))} g(x) dV.
\]

By the choice of \( \rho \) we also have

\[
(\gamma_s \omega_t + dd^c \rho)^n \geq (\varepsilon_1 \theta + dd^c \rho)^n = e^{c_1} g dV.
\]
Combining these with Lemma 3.15 below we arrive at
\[(\omega_t + dd^c v_s(t, \cdot))^n = [(1 - \lambda_s)\omega_t + \alpha_s dd^c u(st, \cdot) + \lambda_s \omega_t + (1 - \alpha_s)dd^c \rho]^n\]
\[= [\alpha_s(\beta_s \omega_t + s^{-1}dd^c u(st, \cdot)) + (1 - \alpha_s)(\gamma_s \omega_t + dd^c \rho)]^n\]
\[\geq e^{\alpha_s \partial_t u(st,x) + \alpha_s F(t,x,u(st,x)) + (1 - \alpha_s)C \log s} g(x) dV.\]

Since \(F(t, x, r)\) is uniformly bounded on \([0, T] \times X \times [-M_u, M_u]\) and \(\alpha_s - 1 = O(|s - 1|)\), up to enlarging \(C\) we infer
\[(\omega_t + dd^c v_s(t, \cdot))^n \geq e^{\beta_t v_s(t,x) + F(t,x,v_s(t,x))} g(x) dV.\]

This concludes the proof. \(\square\)

We have used the following mixed inequalities:

**Lemma 3.15.** Let \(\theta_1, \theta_2\) be two closed smooth semi-positive \((1,1)\)-forms on \(X\). Let \(u_1 \in \text{PSH}(X, \theta_1)\), \(u_2 \in \text{PSH}(X, \theta_2)\) be bounded and such that
\[(\theta_1 + dd^c u_1)^n \geq e^{f_1} \mu, \quad \text{and} \quad (\theta_2 + dd^c u_2)^n \geq e^{f_2} \mu,\]
where \(f_1, f_2\) are bounded measurable functions and \(\mu = hdV \geq 0\). Then, for every \(\alpha \in ]0, 1[,\)
\[(\alpha (\theta_1 + dd^c u_1) + (1 - \alpha) (\theta_2 + dd^c u_2))^n \geq e^{\alpha f_1 + (1 - \alpha) f_2} \mu.\]

**Proof.** The proof is identical to that of [GLZ1, Lemma 5.9] using the convexity of the exponential together with the mixed Monge-Ampère inequalities due to S. Kołodziej [Kol03] (see also [Diw09]). \(\square\)

Let \(\chi : \mathbb{R} \to [0, +\infty[\) be a smooth function with compact support in \([-1, 1]\) such that \(\int_{\mathbb{R}} \chi(s) ds = 1\). For \(\varepsilon > 0\) we set \(\chi_\varepsilon(s) := \varepsilon^{-1} \chi(s/\varepsilon)\).

**Proposition 3.16.** Assume that \(u \in \mathcal{P}(X_T)\) is a bounded pluripotential subsolution of \((\text{CMAF})\). Let \(v_s\) be defined as in Lemma 3.14.

If \(r \mapsto F(\cdot, \cdot, r)\) is convex then there exists a uniform constant \(B > 0\) such that, for \(\varepsilon > 0\) small enough, the function
\[u^\varepsilon(t, x) := \int_{\mathbb{R}} v_s(t, x) \chi_\varepsilon(s - 1) ds - B \varepsilon(t + 1)\]
is a pluripotential subsolution of \((\text{CMAF})\) which is \(C^{1/0}\) in \(t\) and such that
\[\sup_X [u^\varepsilon(0, x) - u_0(x)] \xrightarrow{\varepsilon \to 0} 0.\]

When \(r \mapsto F(\cdot, \cdot, r)\) is merely semi-convex, the same conclusion holds if we further assume that
\[|\partial_t u(t, x)| \leq C/t, \quad \forall (t, x) \in ]0, T'] \times X.\]

**Proof.** We first assume that \(r \mapsto F(\cdot, \cdot, r)\) is uniformly convex and increasing. In this case we do not need the assumption (3.4). Fix \(\varepsilon_0\) as in Lemma 3.14. For each \(|s| \leq \varepsilon_0\) the function \(v_s(t, z)\) is a pluripotential subsolution to \((\text{CMAF})\).
As in [GLZ1, Theorem 5.7, Step 3], we use [GLZ17, Main Theorem] and Jensen’s inequality to show that, for any \( t \in [0, T] \),

\[
(\omega_t + dd^c u^\varepsilon)^n \geq \exp \left( \partial_t u^\varepsilon(t, x) + B\varepsilon + \int_{\mathbb{R}} F(t, x, v_s(t, x))\chi(s-1)ds \right) \, gdV,
\]

in the weak sense on \( X \).

If \( F(t, x, \cdot) \) is convex for any \((t, x)\), then

\[
\int_{\mathbb{R}} F(t, x, v_s(t, x))\chi(s-1)ds \geq F(t, x, u^\varepsilon + B\varepsilon) \geq F(t, x, u^\varepsilon),
\]

since \( F \) is non decreasing in \( r \). Plugging this inequality in (3.5) we conclude that for any \( B \geq 0 \), \( u^\varepsilon \) is a subsolution to the equation (5.1).

If \( F \) is merely semi-convex, the function \( r \mapsto F(t, x, r) + \lambda r^2 \) is convex for any \((t, x) \in X_T \), for some constant \( \lambda > 0 \). Thus

\[
\int_{\mathbb{R}} F(t, x, v_s(t, x))\chi(s-1)ds \geq F(t, x, u^\varepsilon(t, x))\chi(s-1)ds + \lambda Q_\varepsilon(t, x),
\]

where

\[
Q_\varepsilon(t, x) := \int_{\mathbb{R}} v_s(t, x)^2\chi(s-1)ds - \left( \int_{\mathbb{R}} v_s(t, x)\chi(s-1)ds \right)^2.
\]

We claim that there is \( C_1 > 0 \) such that \(|Q_\varepsilon(t, x)| \leq C_1 \varepsilon \) for all \((t, x) \in [0, T] \times X \). Indeed the family of function \( s \mapsto v_s \) is uniformly bounded in \([0, T] \times X \) by a constant \( M > 0 \). Hence for any \((t, x) \in [0, T] \times X \) and \( \varepsilon > 0 \) small enough, we have

\[
|Q_\varepsilon(t, x)| \leq 2M \int_{\mathbb{R}} |v_s(t, x) - v_s^\varepsilon(t, x)|\chi(s-1)ds,
\]

where \( v_s^\varepsilon(t, x) := \int_{\mathbb{R}} v_s(t, x)\chi(s-1)ds \).

Recall that \( v_s(t, x) := \frac{\alpha_s}{s}u^s(t, x) + (1 - \alpha_s)p(x) - C|s-1|t \). The condition (5.3) ensures that the function \( \partial_s v_s \) is uniformly bounded in \( s \in [1-\varepsilon_0, 1+\varepsilon_0] \) and \((t, x) \in X_T \). Thus the family \( s \mapsto v_s \) is uniformly \( L \)-Lipschitz in \( s \in [1-\varepsilon_0, 1+\varepsilon_0] \), which proves our claim with \( C_1 := 2ML \).

By (3.6) this implies that for any \((t, x) \in X_T \),

\[
\int_{\mathbb{R}} F(t, x, v_s(t, x))\chi(s-1)ds \geq F(t, x, u^\varepsilon(t, x))\chi(s-1)ds - C_1 \varepsilon.
\]

Plugging this inequality in (3.5) and taking \( B \geq C_1 \) we see that \( u^\varepsilon \) is a subsolution to the equation (5.1). Taking \( B \) large enough we obtain furthermore that \( u^\varepsilon(0, x) \leq u_0(x) \) for all \( x \in X \).

We let the reader adapt these arguments to the situation when \( r \mapsto F(\cdot, \cdot, r) \) is merely quasi-increasing.
4. Uniqueness

We have shown in the previous section that the Cauchy problem for (CMAF) with bounded initial data \( \varphi_0 \in \text{PSH}(X, \omega_0) \) admits a pluripotential solution which is locally uniformly semi-concave in \( t \). We now prove that there is only one such solution.

4.1. Comparison principle 1. Our goal in this section is to establish the following comparison principle:

**Theorem 4.1.** Fix \( \varphi, \psi \in \mathcal{P}(X_T, \omega) \cap L^\infty(\Omega) \) and assume that

- (a) \( \varphi \) is a pluripotential subsolution to (CMAF);
- (b) \( \psi \) is a pluripotential supersolution to (CMAF);
- (c) \( x \to \varphi(\cdot, x) \) is continuous in \( \Omega \) and \( |\partial_t \varphi(t, x)| \leq C/t, \ \forall (t, x) \in X_T \);
- (d) \( \psi \) is locally uniformly semi-concave in \( t \in [0, T] \);
- (e) \( \varphi_t \to \varphi_0 \) and \( \psi_t \to \psi_0 \) in \( L^1 \) as \( t \to 0 \).

If \( \varphi_0 \leq \psi_0 \) then \( \varphi \leq \psi \).

We first establish this result under extra assumptions:

**Proposition 4.2.** Fix \( \varphi, \psi \in \mathcal{P}(X_T, \omega) \cap L^\infty(\Omega) \). Assume that \( \varphi \) (resp. \( \psi \)) is a pluripotential subsolution (resp. supersolution) to (CMAF) such that

- (a) \( \varphi \) is \( C^{1/0} \) in \( t \) and for any \( t > 0 \), \( \varphi(t, \cdot) \) is continuous on \( \Omega \);
- (b) \( \psi \) is locally uniformly semi-concave in \( t \);
- (c) \( \varphi_t \to \varphi_0 \) and \( \psi_t \to \psi_0 \) in \( L^1 \) as \( t \to 0 \);
- (d) the function \( (t, x) \to \psi(t, x) \) is continuous on \( [0, T] \times \Omega \).

Then

\[
\varphi_0 \leq \psi_0 \implies \varphi \leq \psi \text{ on } X_T.
\]

A particular case of this result was established in \cite[Theorem 3.1]{GLZ18}.

**Proof.** We fix \( T' \in [0, T] \) and we prove that \( \varphi \leq \psi \) on \( [0, T'] \times X \). The result then follows by letting \( T' \to T \). Using the invariance properties of the family of equations (see Section 3.3), we can assume without loss of generality that \( r \mapsto F(\cdot, \cdot, r) \) is increasing. We proceed in several steps.

**Construction of auxiliary functions.** We first introduce two auxiliary functions. Let \( \phi_1 \in \text{PSH}(X, \theta/2) \) be a \( \theta/2 \)-psh function with analytic singularities (in particular \( \phi_1 \) is smooth in \( \Omega \)) such that \( \phi_1 = -\infty \) on \( \partial \Omega \). We need to use this function in order to apply the classical maximum principle in \( \Omega \). The standard strategy is to replace \( \varphi \) by \( (1 - \delta)\varphi_t + \delta \phi_1 \). However, the time derivative \( \varphi_t \) may blow up as \( t \to 0 \) so we need to use a second auxiliary function. Let \( \phi_2 \in \text{PSH}(X, \theta/2) \) be the unique solution to

\[
(\theta/2 + dd^c \phi_2)^n = e^{c_1} g dV,
\]

normalized by \( \sup_X \phi_2 = 0 \), where \( c_1 \in \mathbb{R} \) is a normalization constant. Then \( \phi := \phi_1 + \phi_2 \) is a \( \theta \)-psh function which is continuous in \( \Omega \) and tends to \( -\infty \) on \( \partial \Omega \).
Adaptation of the arguments in [GLZ18]. Fix \( \varepsilon, \delta > 0 \) small enough and set

\[ w(t, x) := (1 - \delta) \varphi(t, x) + \delta \phi(x) - \psi(t, x) - 3 \varepsilon t, \quad t \in [0, T'], x \in \Omega. \]

This function is upper semi-continuous on \([0, T'] \times \Omega\) (see Proposition 1.2) and tends to \(-\infty\) on \(\partial \Omega\), hence attains a maximum at some \((t_0, x_0) \in [0, T'] \times \Omega\).

We claim that \(w(t_0, x_0) \leq 0\). Assume by contradiction that \(w(t_0, x_0) > 0\). Then \(t_0 > 0\) and we set

\[ K := \{ x \in \Omega ; w(t_0, x) = w(t_0, x_0) \}. \]

Since \(w\) tends to \(-\infty\) on \(\partial \Omega\) it follows from upper semi-continuity of \(\varphi\) that \(K\) is a compact subset of \(\Omega\). Since \(\varphi(\cdot, x_0)\) is differentiable in \([0, T]\], the classical maximum principle insures that for all \(x \in K\),

\[ (1 - \delta) \partial_t \varphi(t_0, x) \geq \partial_t^c \psi(t_0, x) + 3 \varepsilon. \]

By assumption the partial derivative \(\partial_t \varphi(t, x)\) is continuous on \(\Omega\). Moreover, Proposition 3.12 ensures that \(\psi_t\) is continuous on \(\Omega\), for all \(t \in [0, T]\). By local semi-concavity of \(t \mapsto \psi_t\) it then follows that, for \(t \in [0, T]\) fixed, \(\partial_t^c \psi(t, x)\) is upper semi-continuous in \(\Omega\). We thus can find \(\eta > 0\) small enough such that, by introducing the open set containing \(K\),

\[ D := \{ x \in \Omega ; w(t_0, x) > w(t_0, x_0) - \eta \} \Subset \Omega, \]

we have

\[ (1 - \delta) \partial_t \varphi(t_0, x) > \partial_t^c \psi(t_0, x) + 2 \varepsilon, \quad \forall x \in D. \]

Set \(u := (1 - \delta) \varphi(t_0, \cdot) + \delta \phi\) and \(v := \psi(t_0, \cdot)\). Since \(\varphi\) is a subsolution to (CMAF), using Lemma 3.15 we infer

\[ (\omega_{t_0} + dd^c u)^n \geq [(1 - \delta) (\omega_{t_0} + dd^c \varphi_{t_0}) + \delta (\theta/2 + dd^c \phi_2)]^n \geq e^{(1 - \delta) (\partial_t \varphi(t_0, \cdot) + F(t_0, x, \varphi(t_0, \cdot))) + \delta c} \int_{\Omega} i \cdot \theta \cdot dV. \]

Since \(F\) is bounded on \([0, T'] \times X \times [-M, M]\) for each \(M > 0\) and \(\varphi\) is bounded on \([0, T'] \times X\) there exists a constant \(C > 0\) under control such that

\[ (\omega_{t_0} + dd^c u)^n \geq e^{(1 - \delta) \partial_t \varphi(t_0, \cdot) + F(t_0, x, \varphi(t_0, \cdot)) - \delta C} \int_{\Omega} i \cdot \theta \cdot dV, \]

in the weak sense of measures in \(\Omega\). Using (4.1) and choosing \(\delta < C^{-1} \varepsilon\) we then have

\[ (\omega_{t_0} + dd^c u)^n \geq e^{\partial_t^c \psi(t_0, \cdot) + F(t_0, x, \psi(t_0, \cdot)) + \varepsilon} \int_{\Omega} i \cdot \theta \cdot dV, \]

in the weak sense of measures in \(D\). Since \(\psi\) is a pluripotential supersolution, Lemma 3.11 ensures

\[ (\omega_{t_0} + dd^c \psi_{t_0})^n \leq e^{\partial_t^c \psi(t_0, \cdot) + F(t_0, x, \psi(t_0, \cdot))} \int_{\Omega} i \cdot \theta \cdot dV, \]

in the weak sense of measures in \(D\). The last two estimates yield

\[ (\omega_{t_0} + dd^c u)^n \geq e^{F(t_0, u_{t_0} \cdot) - F(t_0, v(t_0) \cdot) + \varepsilon (\omega_{t_0} + dd^c v)^n}. \]

Recall that \(u(x) > v(x) + \varepsilon t_0\) for any \(x \in K\). Shrinking \(D\) if necessary, we can thus assume that \(u(x) > v(x)\) for all \(x \in D\).
Since $r \mapsto F(t, x, r)$ is increasing we thus get
\[(\omega_{t_0} + \dd c u)^n \geq e^\varepsilon (\omega_{t_0} + \dd c v)^n,\]
in the sense of positive measures in $D$.
Consider now $\tilde{u} := u + \min_{\partial D} (v - u)$. Since $v \geq \tilde{u}$ on $\partial D$, the comparison principle Proposition 4.3 below yields
\[
\int_{\{v < \tilde{u}\} \cap D} e^\varepsilon (\omega_{t_0} + \dd c v)^n \leq \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + \dd c u)^n \\
\leq \int_{\{v < \tilde{u}\} \cap D} (\omega_{t_0} + \dd c v)^n.
\]
It then follows that $\tilde{u} \leq v$ almost everywhere in $D$ with respect to the measure $(\omega_{t_0} + \dd c v)^n$, hence everywhere in $D$ by the domination principle (see Proposition 4.3). In particular,
\[
(4.2) \quad u(x_0) - v(x_0) + \min_{\partial D} (v - u) = \tilde{u}(x) - v(x) \leq 0,
\]
Since $K \cap \partial D = \emptyset$, we infer $w(t_0, x) < w(t_0, x_0)$, for all $x \in \partial D$, hence
\[
u(x) - v(x) < u(x_0) - v(x_0)
\]
for all $x \in \partial D$, contradicting (4.2). Altogether this shows that $t_0 = 0$, thus
\[
(1 - \delta) \varphi + \delta \psi - 3\varepsilon t \leq \delta \sup_X |\varphi_0|
\]
in $[0, T'] \times \Omega$. Letting $\delta \to 0$ and then $\varepsilon \to 0$ we arrive at $\varphi \leq \psi$ in $[0, T'] \times \Omega$ hence in $[0, T'] \times X$. \hfill \box

We have used the following semi-local version of the domination principle:

**Proposition 4.3.** Fix a non-empty open subset $D \subset X$ and let $u, v$ be bounded $\theta$-psh functions on $X$ such that
\[
\limsup_{D \ni z \to \partial D} (u - v)(z) \geq 0.
\]
Then
\[
\int_{\{u < v\} \cap D} \theta_v^n \leq \int_{\{u < v\} \cap D} \theta_u^n.
\]
Moreover, if $\text{MA}_\theta (u)(\{u < v\} \cap D) = 0$ then $u \geq v$ in $D$.

We now remove assumption $(d)$ in Proposition 4.2:

**Proposition 4.4.** Fix $\varphi, \psi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ such that $\varphi$ (respectively $\psi$) is a pluripotential subsolution (respectively supersolution) to (CMAF). If the assumptions $(a), (b), (c)$ of Proposition 4.2 are satisfied then
\[
\varphi_0 \leq \psi_0 \implies \varphi \leq \psi \text{ on } X_T.
\]
Proof. We assume without loss of generality that \( r \mapsto F(\cdot, \cdot, r) \) is increasing and use an argument similar to that of Lemma 3.14. Fix \( s > 0 \) small enough and consider
\[
v_s(t, x) := \psi(t + s, x) + Cst + Cs - Cs \log s,
\]
and
\[
u_s(t, x) := \alpha_s \varphi(t, x) + (1 - \alpha_s) \rho - Cst - Cs,
\]
where \( \alpha_s := (1 - s)(1 - As) \in [0, 1], A > 0 \) is defined in (0.1), \( \rho \) is a quasi-psh function and \( C \) is a positive constant to be specified later. The goal is to show that for \( C > 0 \) large enough (under control) \( v_s \) is a supersolution while \( u_s \) is a subsolution to a parabolic equation and \( u_s(0, \cdot) \leq v_s(0, \cdot) \). We can then invoke Proposition 4.2 and let \( s \to 0 \) to obtain the result.

By considering \( s \) small enough we can assume that
\[
\beta_s := \frac{(1 - As)s}{(1 - \alpha_s)} \geq \varepsilon_1 > 0
\]
for some uniform constant \( \varepsilon_1 \). We let \( \rho \) be the unique \( \varepsilon_1 \theta \)-psh function on \( X \) such that \( \sup_X \rho = 0 \) and \((\varepsilon_1 \theta + dd^c \rho)^n \) is \( C \theta \)-increasing. By definition one has that \( \alpha_s + (1 - \alpha_s) \beta_s = 1 - As \). Then one can show that
\[
(\omega_{t+s} + dd^c u_s)^n \geq [(1 - As)\omega_t + \alpha_s dd^c \varphi_t + (1 - \alpha_s)dd^c \rho]^n = [\alpha_s (\omega_t + dd^c \varphi_t) + (1 - \alpha_s)(\beta_s \omega_t + dd^c \rho)]^n \geq e^{\alpha_s (\partial_t \varphi_t + F(t, \cdot, \varphi(t, \cdot))) + (1 - \alpha_s)\varepsilon_1} gdV.
\]
where in the last line we use Lemma 3.15. Since \( \alpha_s = 1 + O(s) \) and \( F \) is bounded, by choosing \( C > 0 \) large enough (depending on \( M_F, \varepsilon_1 \)) we have
\[
(\omega_{t+s} + dd^c u_s)^n \geq e^{\partial_t u_s + F(t, u_s(t, \cdot))} gdV.
\]
On the other hand for \( C > 0 \) large enough (which depends on \( \kappa_F \)) we have
\[
(\omega_{t+s} + dd^c v_s)^n \leq e^{\partial_t v_s - Cs + F(t+s, v_s(t, \cdot))} gdV \leq e^{\partial_t v_s + F(t, v_s(t, \cdot))} gdV.
\]
Up to increasing \( C \) it follows from Proposition 3.13 that \( v_s(0, x) \geq \psi_0(x) \). Since \( u_s(0, x) \leq v_s(0, x) \) it then follows from Proposition 4.2 that
\[
u_s(t, x) \leq v_s(t, x), \forall (t, x) \in X_T.
\]
Letting \( s \to 0 \) we arrive at the conclusion, finishing the proof.

Proof of Theorem 4.1. Fix \( T' < T \). We regularize the subsolution \( \varphi \) by applying Proposition 3.16. The family of subsolutions obtained this way is denoted by \( u_{\varepsilon} \). Then \( u_{\varepsilon} \) is a pluripotential subsolution to (CMAF) and, up to enlarging the constant \( B > 0 \) in Proposition 3.16 we can also assume that \( u_{\varepsilon}(t, x) \) converges to \( \varphi_0 \) in \( L^1(X, dV) \) as \( t \to 0^+ \).

It follows moreover from Proposition 3.12 that for \( t \in [0, T] \) fixed, \( \psi_t \) is continuous in \( \Omega \). We can thus apply Proposition 4.2 and obtain \( u_{\varepsilon} \leq \psi \) on \([0, T'] \times X \). Letting \( \varepsilon \to 0 \) and then \( T' \to T \) we arrive at the conclusion.
Corollary 4.5. Fix $\varphi_0$ a bounded $\omega_0$-psh function. There exists a unique function $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ such that
- for each $t \in ]0, T[$, $x \mapsto \varphi(t, x)$ is continuous on $\Omega$;
- $|\partial_t \varphi(t, x)| \leq C/t$ for all $(t, x) \in ]0, T[ \times X$;
- $t \mapsto \varphi(t, \cdot)$ is locally uniformly semi-concave in $]0, T[$;
- $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0$;
- $\varphi$ solves (CMAF) in $X_T$.

In particular any smooth approximants converge towards this solution, hence the latter is independent of the approximants.

Definition 4.6. Given a data $(F, g, \omega, \varphi_0)$ we let $\Phi(F, g, \omega, \varphi_0)$ denotes the unique pluripotential solution to (CMAF) as in Corollary 4.5.

4.2. Uniqueness and stability. We now establish a more general uniqueness result. The proof relies on a delicate comparison principle and yields as well stability results.

4.2.1. Stability 1.

Proposition 4.7. Assume that $(g, F, \omega_t)$ and $(g_j, F_j, \omega_{t,j})$ satisfy the assumptions in the introduction with uniform constants independent of $j$, and
- $g_j, F_j$ are smooth
- $0 < g_j$ converge in $L^p(X)$ to $g \in L^p(X)$;
- $\varphi_{0,j}$ are uniformly bounded $\omega_0$-psh functions which converge in $L^1(X)$ towards $\varphi_0 \in L^\infty \cap \text{PSH}(X, \omega_0)$;
- $F_j$ uniformly converge to $F$ on $]0, T[ \times X \times J$, for each $J \subseteq \mathbb{R}$;
- $\omega_{t,j}$ uniformly converges to $\omega_t$.

Let $\varphi_j$ be the unique smooth solutions to (CMAF) with data $(g_j, F_j, \omega_{t,j}, \varphi_{0,j})$. Then $(\varphi_j)$ converges in $L^1_{\text{loc}}(X_T)$ to $\varphi$, where $\varphi$ is the unique solution of (CMAF) with data $(g, F, \omega_t, \varphi_0)$ provided by Corollary 4.5.

Proof. The sequence $(\varphi_j)$ satisfies the conditions of Theorem 2.10. Hence, Theorem 2.11 ensures that a subsequence of $(\varphi_j)$, still denoted by $(\varphi_j)$, converges in $L^1_{\text{loc}}(X_T)$ to a function $\varphi \in \mathcal{P}(X_T, \omega)$ which
- is a pluripotential solution to (CMAF);
- is locally uniformly semi-concave in $t \in ]0, T[$;
- satisfies $|\partial_t \varphi| \leq C/t$ on $]0, T[ \times X$.

It follows from Proposition 3.13 that there exists uniform constants $C > 0, t_0 > 0$ such that for all $(t, x) \in ]0, t_0[ \times X$,

$$\varphi_j(t, x) \geq (1 - t)e^{-Ct} \varphi_{0,j}(x) + C(t \log t - t),$$

Letting $j \to +\infty$ we obtain, for all $(t, x) \in ]0, t_0[ \times X$,

$$\varphi(t, x) \geq (1 - t)e^{-Ct} \varphi_0(x) + C(t \log t - t).$$

This lower bound and Proposition 2.3 ensure that $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$. It finally follows from Corollary 4.5 that $\varphi \in \mathcal{P}(X_T, \omega)$ is uniquely determined. ☐
4.2.2. Comparison principle 2. We now extend the comparison principle (Theorem 4.1), avoiding the space continuity assumption on the subsolution $\varphi$, as well as the Lipschitz type control at the origin:

**Theorem 4.8.** Assume that $\varphi, \psi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ are such that

- $\varphi$ is a pluripotential subsolution to (CMAF);
- $\psi$ is a pluripotential supersolution to (CMAF);
- $\psi$ is locally uniformly semi-concave in $t$;
- $\varphi_t \to \varphi_0$ and $\psi_t \to \psi_0$ in $L^1$ as $t \to 0$.

If $\varphi_0 \leq \psi_0$ then $\varphi \leq \psi$.

**Proof.** We fix $T' < T$ and prove that $\varphi \leq \psi$ on $[0, T'] \times X$. The result follows then by letting $T' \to T$. Using the invariance properties of our family of equations, we can assume that $r \mapsto F(\cdot, \cdot, r)$ is increasing.

If $\tilde{\psi}$ is the unique pluripotential solution constructed by approximation (see Corollary 4.5), Theorem 4.1 ensures that $\tilde{\psi} \leq \psi$ on $X_{T'}$. We can thus assume without loss of generality that $\psi = \tilde{\psi}$. We proceed in several steps.

**Step 1.** Assume that the data $(g, F, \psi_0)$ is smooth, $g > 0$, $\theta$ is Kähler, and the time derivative $\partial_t \varphi(t, x)$ is uniformly bounded in $[0, T'] \times X$.

Then $\psi$ is smooth since there exists a unique smooth solution (as follows from [GZ17, DNL17, To17] and Corollary 4.5).

If $x \mapsto \varphi(\cdot, x)$ were known to be continuous, we could invoke Theorem 4.1 to conclude. In absence of this extra assumption, we take a little detour inspired by viscosity techniques.

**Step 1.1.** Assume the following conditions

1. for all $x \in X$, $\dot{\varphi}_t(x)$ exists and $t \mapsto \dot{\varphi}_t(x)$ is continuous in $t \in [0, T']$.
2. $\varphi_t, \dot{\varphi}_t$ are uniformly quasi-continuous on $X$. This means, for any $\varepsilon > 0$ there exists an open subset $U$ with $\text{Vol}(U) < \varepsilon$ such that on the compact set $X \setminus U$ the functions $x \mapsto \varphi_t(x), \dot{\varphi}_t(x)$ are continuous for all $t \in [0, T']$.

The first condition ensures that the inequality

$$(\omega_t + df\varphi_t)^n \geq e^{\dot{\varphi}_t + F(t, \cdot, \varphi_t(\cdot))} gdV,$$

holds in the pluripotential sense on $X$ for all $t \in [0, T']$. As will be shown later, the regularizing family $\varphi_\varepsilon(t, x)$ as constructed in Theorem 3.16 satisfies these two conditions.

We introduce the following constants: $M := 2 + M_1 + M_2$, where

$M_1 := \sup_{[0, T'] \times X} |\varphi(t, x)|$;

and

$M_2 := \sup_{[0, T'] \times X} (|\dot{\varphi}_t(x)| + |F(t, x, \varphi_t(x))| + |F(t, x, \psi_t(x))|)$.

Fix $\varepsilon > 0$ small enough. By uniform quasi-continuity of $\varphi_t$ and $\dot{\varphi}_t$ there exists an open set $U$ such that
• Vol(U) < $e^{-4M/\varepsilon}$;
• for all $t \in [0, T']$, the functions $\varphi_t$ and $\dot{\varphi}_t$ are continuous on $X \setminus U$.

Let $\rho_\varepsilon$ be the unique bounded $\theta$-psh function on $X$ such that

$$(\theta + dd^c \rho_\varepsilon)^n = e^{2M/\varepsilon} 1_U g dV + a_\varepsilon g dV, \quad \sup_X \rho_\varepsilon = 0.$$ 

Here $0 \leq a_\varepsilon$ is a normalization constant. The boundedness of $g$ yields

$$\int_U e^{2M/\varepsilon} g dV \leq e^{-2M/\varepsilon} \sup_X g,$$

hence for $\varepsilon > 0$ small enough, $a_\varepsilon \geq 1/2$. Also, the $L^2$-norm of the density of $(\theta + dd^c \rho_\varepsilon)^n$ is uniformly bounded hence by [EGZ09, Proposition 2.6], $\rho_\varepsilon$ is uniformly bounded.

Set, for $(t, x) \in X_{t'}$,

$$u_\varepsilon(t, x) := (1 - \varepsilon) \varphi_t + \varepsilon \rho_\varepsilon.$$ 

We prove that $u_\varepsilon - \psi - C \varepsilon t \leq 0$ in $[0, T'] \times X$, where $C := 2M + M_1 \kappa_F$. By contradiction assume that it were not the case. Since the function is upper semicontinuous on the compact set $[0, T'] \times X$, its maximum is attained at some $(t_0, x_0) \in [0, T'] \times X$. We then have

$$u_\varepsilon(t_0, x_0) - \psi(t_0, x_0) \geq C \varepsilon t_0 > 0,$$

hence

$$(4.3) \quad \varphi(t_0, x_0) \geq \psi(t_0, x_0) - \varepsilon M_1.$$ 

By the classical maximum principle we have

$$(4.4) \quad (1 - \varepsilon) \dot{\varphi}_{t_0}(x_0) \geq C \varepsilon + \dot{\psi}_{t_0}(x_0).$$

By assumption (1), $t \mapsto \dot{\varphi}_t(x)$ is continuous on $[0, T']$ for all $x \in X$. Since $\varphi$ is a subsolution to (CMAF), Lemma 3.11 then ensures that

$$(\omega_{t_0} + dd^c \varphi_{t_0})^n \geq e^{\psi_{t_0} + F(t_0, \cdot, \varphi_{t_0}(\cdot))} g dV$$

holds in the sense of measures on $X$. By construction of $\rho_\varepsilon$ we also have

$$(\omega_{t_0} + dd^c \rho_\varepsilon)^n \geq (\theta + dd^c \rho_\varepsilon)^n \geq e^{f_\varepsilon} g dV,$$

where

$$f_\varepsilon := \begin{cases} 
\frac{2M}{\varepsilon}, & x \in U, \\
- \log 2, & x \in X \setminus U.
\end{cases}$$

It then follows from Lemma 3.15 that

$$(4.5) \quad (\omega_{t_0} + dd^c u_{\varepsilon}(t_0, \cdot))^n \geq e^{h_\varepsilon} g dV,$$

where

$$h_\varepsilon := \begin{cases} 
M, & x \in U, \\
(1 - \varepsilon) \dot{\varphi}_{t_0} + (1 - \varepsilon) F(t_0, x, \varphi_{t_0}(x)) - \varepsilon \log 2, & x \in X \setminus U.
\end{cases}$$
The assumption (2) ensures that $h_\varepsilon$ is lower semicontinuous on $X \setminus U$. The choice of $M$ then shows that $h_\varepsilon$ is lower semicontinuous on $X$ and

\begin{equation}
(4.6) \quad h_\varepsilon(x) \geq (1 - \varepsilon)\varphi_{t_0}(x) + (1 - \varepsilon)F(t_0, x, \varphi_{t_0}(x)) - \varepsilon \log 2, \quad \forall x \in X.
\end{equation}

It then follows from Lemma 4.9 that (4.5) holds in the viscosity sense. The function $x \mapsto \psi_{t_0}(x) - \psi_{t_0}(x_0) + u_\varepsilon(t_0, x_0)$ is a smooth upper test for $u_\varepsilon(t_0, \cdot)$ at $x_0$, hence

\begin{equation}
(\omega_{t_0} + dd^c \psi_{t_0})^n \geq e^{h_\varepsilon} g dV
\end{equation}

holds in the classical sense at $x_0$. Now from (4.4) and (4.6) we have

\begin{equation}
(4.7) \quad (\omega_{t_0} + dd^c \psi_{t_0})^n(x_0) \geq e^{\psi_{t_0}(x_0) + (C - \log 2)\varepsilon + F(t_0, x_0, \varphi_{t_0}(x_0))} g(x_0) dV.
\end{equation}

It follows from (4.3) and the monotonicity of $r \mapsto F(t, x, r)$ that

\begin{align*}
F(t_0, x_0, \varphi_{t_0}(x_0)) & \geq F(t_0, x_0, \psi_{t_0}(x_0) - M_1 \varepsilon) \\
& \geq F(t_0, x_0, \psi_{t_0}(x_0)) - \kappa_F M_1 \varepsilon.
\end{align*}

Hence

\begin{align*}
(1 - \varepsilon)F(t_0, x_0, \varphi_{t_0}(x_0)) & \geq F(t_0, x_0, \psi_{t_0}(x_0)) - \kappa_F M_1 \varepsilon - M \varepsilon \\
& \geq F(t_0, x_0, \psi_{t_0}(x_0)) - (C - M) \varepsilon.
\end{align*}

This together with (4.7) gives a contradiction since $\psi$ is a solution to (CMAF).

We thus have that $u_\varepsilon \leq \psi + C \varepsilon t$ on $[0, T'] \times X$. Letting $\varepsilon \to 0$ we arrive at the conclusion.

**Step 1.2.** We next remove the assumptions on $\varphi$ in Step 1.1.

Using Proposition 3.16 we can find $\varphi^\varepsilon$ which are pluripotential subsolutions to (CMAF) and which satisfy the assumptions in Step 1.1. Indeed, it suffices to check that $\varphi_t$ is uniformly quasi-continuous on $X$. But this holds by quasi-continuity of $\varphi_t$ and by the Lipschitz condition. More precisely, for fix $\varepsilon > 0$, there exists an open subset $U$ with $\text{Vol}(U) < \varepsilon$ such that $\varphi_t$ is continuous on $X \setminus U$ for all $t \in [0, T'] \cap \mathbb{Q}$. The continuity of $\varphi_t$ on $X \setminus U$ for irrational points $t$ follows from the Lipschitz property of the family $\varphi_t$.

Thus the previous step applies and yields $\varphi^\varepsilon \leq \psi + O(\varepsilon)$. Letting $\varepsilon \to 0$ we arrive at $\varphi \leq \psi$.

**Step 2.** We finally remove the smoothness assumption on the data and the Lipschitz condition on $\varphi$ by using the stability result above together with an argument from [GLZ18].

Let $(g_j, F_j)$ be smooth approximants of $(g, F)$. Fix $\varepsilon > 0$ small enough and consider

\begin{equation}
\varphi_{\varepsilon, j}(t, x) := (1 - \delta_j)\varphi(t + \varepsilon, x) + \delta_j \rho_j + n \log(1 - \delta_j) - (B\delta_j + C\varepsilon + \eta_j)t,
\end{equation}

where $\delta_j \in [0, 1/2]$ will be specified later, and $\rho_j \in \text{PSH}(X, \theta)$ is the unique solution to

\begin{equation}
(\theta + dd^c \rho_j)^n = \left(a_j + \frac{|g_j - g|}{\|g_j - g\|_p}\right) dV,
\end{equation}

normalized by $\sup_X \rho_j = 0$ for a normalization constant $a_j \geq 0$. 
We are going to prove that, for suitable choices of $B, M_j$, the function $\varphi_{\varepsilon,j}$ is a pluripotential subsolution to (CMAF) with data $(\omega_{\varepsilon,j}, g_j, F_j)$, where

- $\omega_{\varepsilon,j}(t) := \omega(t + \varepsilon) + 2^{-j}\Theta$;
- $0 < g_j$ is smooth and $\|g_j - g\|_p \to 0$;
- $F_j$ is smooth in $]0, T[ \times X \times \mathbb{R}$ with the same Lipschitz and semi-convexity constants as $F$, and $F_j$ locally uniformly converge to $F$.

Let $\psi_{\varepsilon,j}$ be the unique smooth solution to (CMAF) with the above data $(\omega_{\varepsilon,j}, g_j, F_j)$ such that $\psi_{\varepsilon,j}(0, \cdot) \geq (1 - \delta_j)\varphi(\varepsilon, \cdot)$ and

$$\int_X \psi_{\varepsilon,j}(0, x) dV(x) \leq \int_X (1 - \delta_j)\varphi(\varepsilon, x) dV(x) + 2^{-j}.$$  

It follows from Proposition 4.7 that $\psi_{\varepsilon,j}$ converge, as $j \to +\infty$ and $\varepsilon \to 0$, to some $\tilde{\psi} \in P(X_T \wedge)$ which is a pluripotential solution to (CMAF) with data $(\omega, g, F)$ and initial value $\varphi_0$. Moreover, $\tilde{\psi}, \psi$ satisfy the assumptions of Theorem 4.1, hence $\tilde{\psi} \leq \psi$.

We now prove that $\varphi \leq \tilde{\psi}$ by showing that $\varphi_{\varepsilon,j}$ is a subsolution to an approximate (CMAF). Since $\varphi$ is locally uniformly Lipschitz, there exists a constant $C_1$ (depending also on $\varepsilon$) such that

$$\sup_{[\varepsilon, T] \times X} |\psi| \leq C_1,$$

hence

$$\partial_t \varphi(t + \varepsilon, x) \geq (1 - \delta_j)\partial_t \varphi(t + \varepsilon, x) - C_1 \delta_j$$

$$= \partial_t \varphi_{\varepsilon,j}(t, x) + (B - C_1)\delta_j + C\varepsilon + \eta_j,$$

A direct computation yields

$$(\omega_{\varepsilon,j} + dd^c \varphi_{\varepsilon,j})^n \geq e^{n \log(1 - \delta_j) + \partial_t \varphi(t + \varepsilon, x) + F(t + \varepsilon, x, \varphi(t + \varepsilon, x) - C_1)\delta_j + C\varepsilon + \eta_j} dV + \delta_j \left| \frac{g_j - g}{\|g_j - g\|_p} \right| dV.$$

Set $M_\varphi := \sup_{X_T} |\varphi|$ and $J := [-M_\varphi, M_\varphi]$ and

$$\eta_j := \sup \{|F(t, x, r) - F_j(t, x, r)| ; (t, x, r) \in [0, T'] \times X \times J\}.$$

Then $\eta_j \to 0$ as $j \to +\infty$. Setting

$$\delta_j := e^{C_1 + M_F} \|g_j - g\|_p,$$

where

$$M_F := \sup \{|F(t, x, r)| ; (t, x, r) \in [0, T] \times J\},$$

and considering $j$ large enough (so that $\delta_j \leq 1/2$), we obtain

$$(\omega_{\varepsilon,j} + dd^c \varphi_{\varepsilon,j})^n \geq e^{n \log(1 - \delta_j) + \partial_t \varphi(t + \varepsilon, x) + F(t + \varepsilon, x, \varphi(t + \varepsilon, x))} g_j dV.$$

The Lipschitz condition on $F$ ensures that

$$F(t + \varepsilon, x, \varphi(t + \varepsilon, x)) - F(t, x, (1 - \delta_j)\varphi(t + \varepsilon, x)) \geq -C_2(\delta_j + \varepsilon),$$

resulting in

$$\psi_{\varepsilon,j}(0, x) \geq \psi_{\varepsilon,j}(0, x) \left(1 - \delta_j + C_2\varepsilon + \eta_j \right),$$

which is what we were after.
for a uniform constant $C_2 > 0$. Choosing $B, C > 0$ large enough and using
$\log(1 - \delta_j) \geq -\delta_j$, we conclude that

$$
(\omega_{\varepsilon,j}(t, x) + dd^c \varphi_{\varepsilon,j}(t, x))^n \geq e^{\delta \varphi_{\varepsilon,j} + F(t, x, \varphi_{\varepsilon,j}) + \eta} g_j dV

\geq e^{\delta \varphi_{\varepsilon,j} + F_j(t, x, \varphi_{\varepsilon,j})} g_j dV.
$$

Thus $\varphi_{\varepsilon,j}$ is a subsolution to (CMAF) for the data $(g_j, F_j, \omega_{\varepsilon,j})$.

We can now apply Step 1 to obtain $\varphi_{\varepsilon,j} \leq \psi_{\varepsilon,j}$. Letting $j \to +\infty$ and then $\varepsilon \to 0$ eventually shows that $\varphi \leq \psi$. □

We have used the following straightforward extension of [EGZ11, Proposition 1.5]:

**Lemma 4.9.** Assume that $u$ is a psh function in an open set $U \subset \mathbb{C}^n$ and
$(dd^c u)^n \geq e^f dV$ in the pluripotential sense, where $f$ is lower semicontinuous
in $U$. Then the inequality holds in the viscosity sense.

**Corollary 4.10.** There exists a unique solution to the Cauchy problem for
(CMAF) which is locally uniformly semi-concave in $t$. It is the envelope of
pluripotential subsolutions.

4.2.3. **Stability 2.** We are now ready to prove Theorem C of the introduction. We assume here that

- $F, G : X_T := [0, T] \times X \times \mathbb{R} \to \mathbb{R}$ are continuous;
- $F, G$ are increasing in the last variable;
- $F, G$ are uniformly Lipschitz in $r$ with Lipschitz constants $L_F, L_G$.
- $0 \leq f, g \in L^p(X)$ with $p > 1$.

The Lipschitz assumption on $F$ means that for all $(t, x) \in X_T$,

$$
|F(t, x, r) - F(t, x, r')| \leq L_F |r - r'|.
$$

**Theorem 4.11.** Assume that $\varphi \in \mathcal{P}(X_T, \omega)$ is a solution to the parabolic equation (CMAF) with admissible data $(F, f)$ and $\psi \in \mathcal{P}(X_T, \omega)$ is
a bounded solution to (CMAF) with admissible data $(G, g)$.

There exists $\alpha \in ]0, 1[$ and for any $\varepsilon > 0$ there exists $A(\varepsilon) > 0$ such that

$$
\sup_{[\varepsilon, T] \times X} |\varphi - \psi| \leq A(\varepsilon) \| \varphi - \psi \|^n_{L^1(X)} + T \sup_{X_T} |F - G| + A(\varepsilon) \| g - f \|_p^{1/n}.
$$

In particular if

- $(g_j)$ are densities which converge to $g$ in $L^p(X)$,
- $F_j$ converges to $F$ locally uniformly,
- $\varphi_{0,j}$ are bounded $\omega_0$-psh functions converging in $L^1(X)$ to $\varphi_0$,
then $\Phi(F_j, g_j, \varphi_{0,j})$ locally uniformly converges to $\Phi(F, g, \varphi_0)$.

We denote here by $\Phi(F, g, \varphi_0)$ the solution to the Cauchy problem for the admissible data $(F, g, \varphi_0)$.

**Proof.** Set $\Phi^j = \Phi(F_j, g_j, \varphi_{0,j})$ and $\Phi = \Phi(F, g, \varphi_0)$. The quantitative estimate is a simple consequence of Proposition 4.12 below. The norm
$|\Phi^j - \hat{\Phi}_\varepsilon|_{L^1(X)}$ is controlled by $|\Phi^j - \hat{\Phi}|_{L^1([\varepsilon,T] \times X)}$ as follows from Lemma 1.5. By Proposition 3.1, $\Phi^j$ converges in $L^1_{\text{loc}}(X_T)$ to $\Phi$, hence the last statement of the theorem follows. \hfill \Box

The stability result is a consequence of the following quantitative version of the comparison principle:

**Proposition 4.12.** Assume $\varphi \in \mathcal{P}(X_T, \omega)$ is a subsolution to (CMAF) with data $(F, f)$, $\psi \in \mathcal{P}(X_T, \omega)$ is a supersolution to (CMAF) with data $(G, g)$.

Fix $\varepsilon > 0$. There exists $\alpha, A, B > 0$ such that for all $(t, x) \in [\varepsilon, T] \times X$,

$$
\varphi(t, x) - \psi(t, x) \leq B|\varphi_\varepsilon - \psi_\varepsilon|_\alpha + \|G - F\|_{\text{sup}_{X_T}} + A\|g - f\|_{L_p}^{1/n};
$$

where $A, B > 0$ depend on $X, \theta, n, p$ and a uniform bound on $\hat{\varphi}, \hat{\psi}, \varphi, \psi$ on the set $[\varepsilon, T] \times X$, $\sup_{X_T} G(t, x, \sup_{X_T} \varphi)$ and $L_G$.

**Proof.** We use a perturbation argument as in [GLZ18] which goes back to the work of Kołodziej [Kol96]. For convenience we normalize $\theta$ so that $\int_X dV = \int_X \theta^n = 1$. Set

$$m_0 := \inf_{X_T} \varphi, \quad m_1(\varepsilon) := \inf_{[\varepsilon, T]} \hat{\varphi} \quad \text{and} \quad M := \sup_{X_T}(G - F)_+.$$

We first assume that $\|G - F\|_p > 0$. It follows from [EGZ09] (see also [Kol98] in the Kähler case) that there exists $\rho \in \text{PSH}(X, \theta) \cap L^\infty(X)$, normalized by $\max_X \rho = 0$, such that

$$\langle \theta + dd^c \rho \rangle^n = \left(1 + \frac{(g - f)_+}{\|G - F\|_p} \right) dV,$$

where $a \geq 0$ is a normalizing constant given by

$$a := 1 - \frac{\|G - F\|_p}{\|G - F\|_p} \in [0, 1].$$

We moreover have a uniform bound on $\rho$ which only depends on the $L^p$ norm of the density of $(\theta + dd^c \rho)^n$ which is here bounded from above by 2,

$$\|\rho\|_\infty \leq C_0(a + 1) \leq 2C_0,$$

where $C_0 > 0$ is a uniform constant depending only on $(X, \theta, p)$.

For $0 < \delta < 1$ and $(t, x) \in X_T$ we set

$$\varphi_\delta(t, x) := (1 - \delta)\varphi(t, x) + \delta \rho + n \log(1 - \delta) - B\delta t - M_t.$$

The plan is to choose $B > 0$ in such a way that $\varphi_\delta$ is a subsolution to (CMAF) on $[\varepsilon, T]$ with data $(G, g)$. The conclusion will then follow from the comparison principle (Theorem 4.8).

Observe that for almost all $t \in [\varepsilon, T]$ fixed, $\varphi_\delta(t, \cdot)$ is $\omega_t$-plurisubharmonic on $X$ and

$$\langle \omega_t + dd^c \varphi_\delta(t, \cdot) \rangle^n \geq (1 - \delta)^n(\omega_t + dd^c \varphi(t)) + \delta^n(\theta + dd^c \rho)^n.$$
Using that \( \varphi \) is a subsolution to (CMAF) with data \((F,f)\), we infer

\[
(4.10) \quad (\omega_t + dd^c \varphi(t, \cdot))^n \geq e^{\varphi + F(t, \varphi(t)) + n \log(1 - \delta)} f dV + \delta n \frac{(g - f)_+}{\| (g - f)_+ \|_p} dV.
\]

Noting that \( \varphi \geq \varphi_\delta + \delta \varphi \) and recalling that \( G \) is increasing in the last variable, we obtain

\[
\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n \log(1 - \delta)
\geq \dot{\varphi}_\delta(t, x) + \frac{\delta \varphi(t, x) + G(t, x, \varphi(t, x)) - M + n \log(1 - \delta) + B\delta}{\delta m_1(\varepsilon) + n \log(1 - \delta) + B\delta}.
\]

The Lipschitz condition on \( G \) yields, writing \( L = L_G \),

\[
\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n \log(1 - \delta)
\geq \dot{\varphi}_\delta(t, x) + G(t, x, \varphi_\delta(t, x) + B\delta - L\delta m_0 | + \delta m_1(\varepsilon) + n \log(1 - \delta).
\]

Using the elementary inequality \( \log(1 - \delta) \geq -2 \log(2) \delta \) for \( 0 < \delta \leq 1/2 \), it follows that for \( 0 < \delta \leq 1/2 \),

\[
B\delta - L\delta m_0 | + \delta m_1(\varepsilon) + n \log(1 - \delta) \geq (B - L m_0 | + m_1(\varepsilon) - 2n \log(2) \delta).
\]

We now choose \( B := L m_0 | + 2n \log(2) - m_1(\varepsilon) \) so that

\[
\dot{\varphi}(t, x) + F(t, x, \varphi(t, x)) + n \log(1 - \delta) \geq \dot{\varphi}_\delta(t, x) + G(t, x, \varphi(t, x)),
\]

which, together with (4.10), yields

\[
(4.11) \quad (\omega_t + dd^c \varphi_\delta(t, \cdot))^n \geq e^{\dot{\varphi}(t, \cdot) + G(t, \varphi(t, \cdot))} f dV + \delta n \frac{(g - f)_+}{\| (g - f)_+ \|_p}.
\]

On the other hand, if we set

\[
M_1(\varepsilon) := \sup_{[\varepsilon, T] \times X} \dot{\varphi}, \quad M_0 := \sup_X \varphi \quad \text{and} \quad M_2 := \sup_X G(t, x, M_0),
\]

then the Lipschitz property of \( G \) ensures, for \((t, x) \in [\varepsilon, T] \times X, \)

\[
\dot{\varphi}_\delta(t, x) + G(t, x, \varphi(t, t)) \leq (1 - \delta) \sup_{X_T} \dot{\varphi} + \sup_{X_T} G(t, x, (1 - \delta) \varphi(t, x))
\leq (1 - \delta) M_1(\varepsilon) + \sup_{X_T} G(t, x, (1 - \delta) M_0)
\leq (1 - \delta) M_1(\varepsilon) + M_0 L\delta + M_2
\leq M_2 + \max\{ L M_0, M_1(\varepsilon) \}.
\]

Using (4.11) we conclude that for \( 0 < \delta < 1/2, \varepsilon, x \in X \), and almost all \( t \in [\varepsilon, T]\),

\[
(4.12) \quad (\omega_t + dd^c \varphi_\delta(t, \cdot))^n \geq e^{\dot{\varphi}(t, \cdot) + G(t, \varphi(t, \cdot))} \left( f + \delta n e^{- M_3(\varepsilon)} \frac{(g - f)_+}{\| (g - f)_+ \|_p} \right) dV,
\]

where \( M_3(\varepsilon) := M_2 + \max\{ L M_0, M_1(\varepsilon) \} \).
To conclude that \( \varphi_\delta \) is a subsolution, we finally set
\[
\delta := \|(g - f)_+\|_p^{1/n} e^{M_\delta(\varepsilon)/n}.
\]
Assume first that \( \|(g - f)_+\|_p \leq 2^{-n} e^{-M_\delta(\varepsilon)} \) so that \( \delta \leq 1/2 \). It follows from (4.12) that, for almost all \( t \in [\varepsilon, T[ \),
\[
(\omega_t + dd^c \varphi_\delta(t, \cdot))_n \geq e^{\varphi_\delta(\varepsilon, \cdot) + G(t, \cdot; \varphi_\delta(\varepsilon, \cdot))}(f + (g - f)_+)dV,
\]
hence \( \varphi_\delta \) is a subsolution to (CMAF) for the data \((G, g)\) on \([\varepsilon, T[\). The comparison principle ensures that for all \((t, x) \in [\varepsilon, T[ \times X\),
\[
\varphi_\delta(t, x) - \psi(t, x) \leq \max_X(\varphi_\delta(\varepsilon, \cdot) - \psi(\varepsilon, \cdot))_+.
\]
Together with (4.9) and (4.13) we obtain, for \((t, x) \in [\varepsilon, T[ \times X\),
\[
\varphi(t, x) - \psi(t, x) \leq \max_X(\varphi(\varepsilon, \cdot) - \psi(\varepsilon, \cdot))_+ + TM + A_1(\varepsilon)\|(g - f)_+\|_p^{1/n},
\]
where
\[
A_1(\varepsilon) := (M_0 + 2n \log 2 + BT) e^{M_\delta(\varepsilon)/n}.
\]
When \( \|(g - f)_+\|_p > 2^{-n} e^{-M_\delta(\varepsilon)} \), we choose \( A_2(\varepsilon) > 0 \) so that
\[
\sup_{X_T}(\varphi(t, x) - \psi(t, x)) \leq \max_X(\varphi_\varepsilon - \psi_\varepsilon)_+ + A_2(\varepsilon)2^{-n} e^{-M_\delta(\varepsilon)}.
\]
We eventually set \( A(\varepsilon) = \max\{A_1(\varepsilon), A_2(\varepsilon)\} \).

Assume finally that \( \|(g - f)_+\|_p = 0 \) which means that \( g \leq f \) almost everywhere in \( X \). In this case, the function \((t, x) \mapsto \varphi(t, x) - Mt \) is a subsolution to (CMAF) with data \((G, g)\) and the conclusion follows from the comparison principle.

Now observe that \( \psi_\varepsilon \) is a supersolution to the degenerate elliptic equation
\[
(\omega_\varepsilon + dd^c \psi_\varepsilon)_n \leq e^{D(\varepsilon)} gdV,
\]
where \( D(\varepsilon) \) is an upper bound of \( \varphi(t, x) + G(t, x, \psi(t, x)) \) on \([\varepsilon, T[ \times X\).

By the \( L^\infty-L^1 \) semi-stability theorem of [EGZ09, Proposition 3.3], it follows that there exits \( \alpha \in [0, 1[ \) and a constant \( C(\varepsilon) > 0 \) depending on \( D(\varepsilon) \), \( p \), \( \theta \), and \( \|g\|_p \) such that
\[
\max_X(\varphi_\varepsilon - \psi_\varepsilon)_+ \leq C(\varepsilon)\|(\varphi_\varepsilon - \psi_\varepsilon)_+\|_{L^1(X)},
\]
concluding the proof. \( \square \)

5. Geometric applications

In this section we show that our hypotheses are satisfied when studying the Kähler-Ricci flow on a compact Kähler variety with log terminal singularities. We prove the existence and study the long term behaviour of the normalized Kähler-Ricci flow (NKRF for short) on such varieties starting from an arbitrary closed positive current with bounded potential.

The definition and study of the Kähler-Ricci flow on mildly singular projective varieties has been undertaken by Song and Tian in [ST12, ST17]. A
different viscosity approach has been developed by Eyssidieux-Guedj-Zeriahi in [EGZ16, EGZ18].

Our approach allows one to avoid any projectivity assumption on the varieties, to deal with more general singularities, to avoid any continuity assumption on the data, and also provide more general uniqueness and stability results. The whole discussion extends to the case of Kawamata log terminal pairs but we leave this discussion for later works.

5.1. Analytic approach to the Minimal Model Program.

5.1.1. Log terminal singularities. Let $Y$ be an irreducible compact Kähler normal complex analytic space with only terminal singularities. Let $\pi : X \to Y$ be a log-resolution, i.e.: $X$ is a compact Kähler manifold, $\pi$ is a bimeromorphic projective morphism and $\text{Exc}(\pi)$ is a divisor with simple normal crossings. Denote by $\{E\}_{E \in \mathcal{E}}$ the family of the irreducible components of $\text{Exc}(\pi)$. With this notation, one has furthermore

$$K_X \equiv \pi^* K_Y + \sum_{E} a_E E$$

where $-1 < a_E \in \mathbb{Q}$, $K_Y$ denote the first Chern class in Bott-Chern cohomology of the $\mathbb{Q}$-line bundle $O_Y(K_Y)$ on $Y$, whose restriction to the smooth locus is the line bundle whose sections are holomorphic top dimensional forms (canonical forms), $K_X$ the canonical class of $X$ and $E$ also denotes with a slight abuse of language the cohomology class of $E$ (we refer to [KM] for more details).

The log terminal condition $a_E > -1$ means that for every non-vanishing locally defined multivalued canonical form $\eta$ defined over $Y$, the holomorphic multivalued canonical form $\pi^* \eta$ on $X$ has poles or zeroes of order $a_E$ along $E$, so that the corresponding volume form decomposes as

$$\pi^*(c_n \eta \wedge \bar{\eta}) = e^{w^+ - w^-} dV(x),$$

where $w^+ = \sum_{a_E > 0} a_E \log |s_E|_{h_E}$ and $w^- = \sum_{0 > a_E > -1} a_E \log |s_E|_{h_E}$ are quasi-plurisubharmonic with $e^{w^+}$ continuous and $e^{-w^-} \in L^p$ for some $p > 1$ whose precise value depends on $\min_E(a_E + 1)$.

5.1.2. The (normalized) Kähler-Ricci flow. The Kähler-Ricci flow is the following evolution equation of Kähler forms on $Y$

$$\frac{\partial \theta_t}{\partial t} = -\text{Ric}(\theta_t),$$

starting from an initial Kähler form $\theta_0$. These can be written as

$$\theta_t = \chi_t + dd^c \phi_t, \text{ with } \chi_t = \theta_0 + t\chi,$$

where $\chi \in c_1(K_Y)$.

One can pull-back these forms via a log-resolution of singularities and consider the corresponding forms $\omega_t = \pi^* \chi_t$ which are big and semi-positive (they vanish along $\text{Exc}(\pi)$). The latter satisfy our main assumptions:
Lemma 5.1. Assume that \( \pi : X \to Y \) is a proper holomorphic map onto a compact normal Kähler space \( Y \), with \( \omega_t = \pi^* \chi_t \) and \( \theta = \pi^* \omega_Y \), where \( \omega_Y \) is a Kähler form on \( Y \). Then there exists \( A > 0 \) such that
\[
\frac{\theta}{A} \leq \omega_t, \quad -A\omega_t \leq \dot{\omega}_t \leq A\omega_t, \quad \text{and} \quad \ddot{\omega}_t \leq A\omega_t.
\]

Proof. The corresponding inequalities are valid on \( Y \) since \( \omega_Y, \theta_0 \) are Kähler and \( t \mapsto \chi_t \) is smooth, as long as we work on a finite interval of time (which is implicit). One can then transpose the inequalities from \( Y \) to \( X \) by the holomorphic mapping \( \pi \). \( \Box \)

One can similarly consider the normalized Kähler-Ricci flow on \( Y \),
\[
\frac{\partial \theta_t}{\partial t} = -\mathrm{Ric}(\theta_t) - \lambda \theta_t,
\]
starting from an initial data \( \theta_0 = \chi_0 + dd^c \phi_0 \) with \( \phi_0 \) being a bounded potential which is plurisubharmonic with respect to the given Kähler form \( \chi_0 \) on \( Y \), and where \( \lambda \in \mathbb{R} \).

By rescaling one can reduce to the cases \( \lambda = 1, 0, -1 \). To simplify the discussion we restrict to the case \( \lambda = 1 \). At the cohomological level, this yields a first order ODE showing that the cohomology class of \( \theta_t \) evolves as
\[
\{\theta_t\} = e^{-t}\{\theta_0\} + (1 - e^{-t})K_Y.
\]

We thus define by
\[
T_{\text{max}} := \sup\{t > 0 : e^{-t}\{\theta_0\} + (1 - e^{-t})K_Y \in \mathcal{K}(Y)\}
\]
the maximal (cohomological) time of existence of the flow.

Denote by \( \mathcal{K}(Y) \subset H^1(Y, \mathcal{PH}_Y) \) the open convex cone of Kähler classes and let \( \chi_0 \) be a smooth Kähler representative of the Kähler class \( \{\theta_0\} \). Assume \( h \) is a smooth hermitian metric on the holomorphic \( \mathbb{Q} \)-line bundle underlying \( \mathcal{O}_Y(K_Y) \). Then \( \chi := -dd^c \log h \) is a smooth representative of \( K_Y \in H^1(Y, \mathcal{PH}_Y) \) and we set
\[
\chi_t = e^{-t}\chi_0 + (1 - e^{-t})\chi.
\]

The solution to the normalized Kähler-Ricci flow can be written as \( \theta_t = \chi_t + dd^c \phi_t \), with \( \pi^* \phi \in \mathcal{P}(X_T) \). We now define
\[
\mu_{\text{NKRF}} = c_n \frac{\pi^* \eta \wedge \overline{\pi^* \eta}}{\pi^* \| \eta \|^2_h} \in C^0(X, \Omega_X^{n,n})
\]
which we view as a continuous element of \( C^0(X_T, \Omega_X^{n,n}) \) and \( c_n \) is the unique complex number of modulus 1 such that the expression is positive. As the notation suggests, \( \mu_{\text{NKRF}} \) is independent of the auxiliary multivalued holomorphic form \( \eta \) but depends on \( h \).

In local coordinates \( \mu_{\text{NKRF}} \) has density of the form
\[
v_{\text{NKRF}} = \prod_E |f_E|^{2\alpha_E} v
\]
where \( v > 0 \) is smooth and \( f_E \) is an equation of \( E \) in these local coordinates.
Theorem 5.2. The Cauchy problem with initial data $S_0 := \chi_0 + dd^c\phi_0$ for the normalized Kähler-Ricci flow on $Y$ admits a unique pluripotential solution defined on $[0, T_{\text{max}}] \times Y$.

Proof. Fix $T < T_{\text{max}}$. Since for any $t \in [0, T]$, $e^{-t}\{\omega_t\} + (1 - e^{-t})K_Y \in \mathcal{K}(Y)$, there exists a smooth family of Kähler forms $(\chi_t)_{0 \leq t \leq T} \in \mathcal{K}(Y)$ such that for any $t \in [0, T]$, $(\chi_t) = \{\omega_t\}$.

We can write $\theta_t = \chi_t + dd^c\phi_t$, where $\phi$ is a solution to the corresponding Monge-Ampère flow at the level of potentials,

$$(\chi_t + dd^c\phi_t)^n = e^{\partial_t \phi + \phi_t}v_Y,$$

on $Y_T$ for some admissible volume form $v_Y$ on $Y$, or equivalently

$$(\omega_t + dd^c\varphi_t)^n = e^{\partial_t \varphi + \varphi_t} \mu_{\text{NKRF}} = e^{\partial_t \varphi + F(t, x, \varphi t)} gdV_X,$$

on a log resolution $\pi : X \to Y$, where $\mu_{\text{NKRF}}$ is a volume form on $X$ which can be locally written

$$\mu_{\text{NKRF}} = \Pi_E |f_E|^{2a_E} dV_X = gdV_X,$$

where $g = \Pi_E |f_E|^{2a_E} \in L^p$ for some $p > 1$, since $-1 < a_E$ for all $E$, and $g > 0$ almost everywhere.

We write here $\varphi := \pi^* \phi$ and $\omega_t := \pi^* \chi_t$. Since $(\chi_t)_{0 \leq t \leq T}$ is a smooth family of Kähler forms on $Y$, it follows that the family of semi-positive forms $[0, T[ t \mapsto \theta_t$ satisfies all our requirements.

Theorem 3.4 can thus be applied (with $F(t, x, r) \equiv 0$) and guarantees the existence of a unique pluripotential solution to the Monge-Ampère flow on $X_T$ for any fixed $T < T_{\text{max}}$ starting at $\varphi_0$. By uniqueness all these solutions glue into a unique solution of the Monge-Ampère flow on $[0, T_{\text{max}}] \times X$ starting at $\varphi_0$. Pushing this solution down to $Y$ we obtain a solution to the NKRF starting at $S_0$. \qed

5.1.3. Song-Tian program. A natural and difficult problem is to understand the asymptotic behavior of $\omega_t$ as $t \to T_{\text{max}}$. Song and Tian have proposed in [ST17] an ambitious program, combining the Minimal Model Program and Hamilton-Perelman approach to the Poincaré conjecture.

We focus here on the case when $X$ has non-negative Kodaira dimension. One would ideally like to proceed as follows:

Step 1. Show that $(Y, \omega_t)$ converges to a midly singular Kähler variety $(Y_1, S_1)$ equipped with a singular Kähler current $S_1$, as $t \to T_{\text{max}}$;

Step 2. Restart the NKRF on $Y_1$ with initial data $S_1$;

Step 3. Repeat finitely many times to reach a minimal model $Y_\tau$ ($K_Y$ is nef);

Step 4. Study the long term behavior of the NKRF and show that $(Y_\tau, \omega_t)$ converges to a canonical model $(Y_{\text{can}}, \omega_{\text{can}})$, as $t \to +\infty$.

This program is more or less complete in dimension $\leq 2$ (see the lecture notes by Song-Weinkove in [SW13] or Tosatti [Tos18] and references therein). It is largely open in dimension $\geq 3$, but for Step 2 which has been completed in [ST17, GZ17, DNL17, EGZ16, To17].
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In the sequel we focus on the final Step 4, i.e. we assume that $T_{\max} = +\infty$, so that $Y$ is a minimal model with log terminal singularities. The normalized Kähler-Ricci flow is then well defined for all times $t > 0$, and our goal is to understand its asymptotic behavior, as $t \to +\infty$.

5.2. Convergence of the NKRF.

5.2.1. Convergence of the NKRF on l.t. varieties of general type. Let $Y$ be a compact Kähler variety with terminal singularities and assume $K_Y$ is big and nef. It has been shown in [EGZ09] that there exists a unique positive closed current $\omega_{KE}$ on $Y$ such that

- $\omega_{KE} \in C^1_1(K_Y)$ and it has bounded potentials;
- $\omega_{KE}$ is smooth in $\text{Amp}(K_Y)$ where it satisfies $\text{Ric}(\omega_{KE}) = -\omega_{KE}$.

The current $\omega_{KE}$ is called the singular Kähler-Einstein current.

Theorem 5.3. Fix $S_0$ a positive closed current with bounded potentials, whose cohomology class is Kähler. The normalized Kähler-Ricci flow continuously deforms $S_0$ towards $\omega_{KE}$, as $t \to +\infty$, at an exponential speed.

Proof. It is classical that the problem boils down to solving and studying the longtime behavior of the parabolic scalar equation

$$(\chi_t + dd^c \varphi_t)^n = e^{\varphi_t}v_Y,$$

with initial data $\varphi_0$, where $T_0 = \chi_0 + dd^c \varphi_0$, and $\chi_t = e^{-t}\chi_0 + (1 - e^{-t})\chi$.

Here $\chi$ is a Kähler form representing $c_1(K_Y)$.

The existence of the unique maximal solution $\varphi_t$ has been explained in Theorem 5.2, so the problem is to show that $\varphi_t \to \varphi_{KE}$, as $t \to +\infty$, where $\omega_{KE} = \chi + dd^c \varphi_{KE}$.

We let the reader check that $u(t, x) = e^{-t}\varphi_0 + (1 - e^{-t})\varphi_{KE} + h(t)e^{-t}$ is a subsolution to (5.1), where

$$h(t) = n(e^t - 1)\log(e^t - 1) - ne^t\log e^t = O(t).$$

The computations are the same as that of [EGZ16, Theorem 4.3, Step 1].

The comparison principle (Theorem 4.8) yields

$$\varphi_{KE}(x) - C(t + 1)e^{-t} \leq u(t, x) \leq \varphi(t, x),$$

for some uniform constant $C > 0$.

The proof for the upper bound is similar. Since $\chi$ is Kähler, we can fix $B > 0$ such that $\omega_0 \leq (1 + B)\chi$, thus $\chi_t \leq (1 + Be^{-t})\chi$ for all $t$. We set

$$v_t(x) := (1 + Be^{-t})\varphi_{KE} + Ce^{-t},$$

where $C$ is chosen so that $v_0 \geq \varphi_0$. The function $v$ is a supersolution to the Cauchy problem for the parabolic equation

$$(1 + Be^{-t})\chi + dd^c v_t)^n = e^{\partial_t v_t + v_t + n\log(1 + Be^{-t})}v_Y \leq e^{\partial_t v_t + v_t + nBe^{-t}}v_Y.$$
with initial data \( \varphi_0 \), while \( w(t, x) = \varphi(t, x) - nB e^{-t} \) is a subsolution to this equation since

\[
([1 + B e^{-t}] \chi + dd^c w)^n \geq (\chi_t + dd^c \varphi_t)^n = e^{\partial_t \varphi_t + \varphi_t v_Y} = e^{\partial_t w_t + w_t + nB e^{-t} v_Y}.
\]

The comparison principle thus yields

\[
\varphi(t, x) \leq \varphi_{KE}(x) + C'(t + 1)e^{-t}.
\]

The conclusion follows.

5.2.2. Convergence of the KRF on l.t. \( \mathbb{Q} \)-Calabi-Yau varieties. In this section we study the Kähler-Ricci flow on a \( \mathbb{Q} \)-Calabi-Yau variety \( Y \) (i.e. a Gorenstein Kähler space of finite index with trivial first Chern class and log-terminal singularities), and prove Theorem D of the introduction.

**Theorem 5.4.** Fix \( S_0 \) a positive closed current with bounded potentials, whose cohomology class is Kähler. The weak Kähler-Ricci flow

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t)
\]

exists for all times \( t > 0 \), and deforms \( S_0 \) towards the unique Ricci flat Kähler-Einstein current \( \omega_{KE} \) cohomologous to \( S_0 \), as \( t \to +\infty \).

The existence of the singular Ricci flat Kähler-Einstein current \( \omega_{KE} \) has been shown in [EGZ09], generalizing Yau’s celebrated solution to the Calabi conjecture [Yau78].

**Proof.** It is classical that the problem boils down to solving and studying the longterm behavior of the parabolic scalar equation

\[(\theta_0 + dd^c \varphi_t)^n = e^{\partial_t \varphi_t + \varphi_t v_Y},\]

with initial data \( \varphi_0 \), where \( S_0 = \theta_0 + dd^c \varphi_0 \).

The existence of the unique semi-concave solution \( \varphi_t \) has been explained in Theorem 5.2. We are going to show that \( \varphi_t \) uniformly converges to \( \varphi_{KE} \), as \( t \to +\infty \), where \( \omega_{KE} = \theta_0 + dd^c \varphi_{KE} \) with

\[(\theta_0 + dd^c \varphi_{KE})^n = v_Y,
\]

and the bounded \( \theta_0 \)-plurisubharmonic function \( \varphi_{KE} \) is properly normalized. We proceed in several steps.

**Step 1: \( C^0 \)-bounds and normalization.** It follows from the comparison principle that \( (\varphi_t) \) remains uniformly bounded: indeed \( \varphi_{KE} - C \) (resp. \( \varphi_{KE} + C \)) provides a static subsolution (resp. supersolution) to the Cauchy problem if \( C > 0 \) is so large that \( \varphi_{KE} - C \leq \varphi_0 \) (resp. \( \varphi_{KE} + C \geq \varphi_0 \)).

We assume without loss of generality that \( v_Y \) and \( \theta_0 \) are normalized by

\[\int_Y \theta_0^n = v_Y(Y) = 1.\]
The concavity of the logarithm insures that
\[
\int_Y \partial_t \varphi_t \, v_Y = \int_Y \log \left( \frac{(\theta_0 + dd^c \varphi_t)^n}{v_Y} \right) v_Y 
\leq \log \left( \int_Y (\theta_0 + dd^c \varphi_t)^n \right) = 0,
\]
hence \( t \mapsto \int_Y \varphi_t \, v_Y \) is decreasing. We therefore impose the normalization
\[
\int_Y \varphi_{KE} \, v_Y = \lim_{t \to +\infty} \int_Y \varphi_t \, v_Y.
\]

**Step 2: Monotonicity of the Monge-Ampère Energy along the flow.** We now observe that \( t \mapsto E(\varphi_t) \) is increasing, where
\[
E(\varphi_t) := \frac{1}{n+1} \sum_{j=0}^n \int_Y \varphi_t(\theta_0 + dd^c \varphi_t)^j \wedge \theta_0^{n-j}.
\]
More precisely:

**Lemma 5.5.** The function \( t \mapsto E(\varphi_t) \) is differentiable almost everywhere with
\[
\frac{d}{dt} E(\varphi_t) = \int_Y \dot{\varphi}_t(\theta_0 + dd^c \varphi_t)^n \geq 0,
\]
for almost every \( t \in ]0, +\infty[. \)

**Proof.** It is straightforward to check that \( t \mapsto E(\varphi_t) \) is locally Lipschitz, its differentiability almost everywhere thus follows from Rademacher theorem. Our goal is now to compute its derivative.

By the Lipschitz property of \( t \mapsto \varphi_t \) we can find a subset \( I \subset ]0, T[ \) with \( ]0, +\infty[ \setminus I \) having measure zero such that for every \( t_0 \in I \) fixed the function \( t \mapsto \varphi(t, x) \) is differentiable at \( t_0 \) for almost every \( x \in Y \). By the first observation we can also assume that \( t \mapsto E(\varphi_t) \) is differentiable at every \( t \in I \). The semi concavity property of \( \varphi_t \) in \( t \) moreover ensures that for every \( t \in I \) and almost every \( x \in Y \),
\[
\dot{\varphi}_t^+(x) = \dot{\varphi}_t^-(x).
\]

The semi concavity property of \( t \mapsto \varphi_t \) ensures that, for \( x \in Y \) fixed, the function \( t \mapsto \dot{\varphi}_t^+(x) \) is lower semicontinuous, while \( t \mapsto \dot{\varphi}_t^-(x) \) is upper semicontinuous in \( ]0, +\infty[. \) In particular, for \( t_0 \in I \) fixed,
\[
\liminf_{t \to t_0} \dot{\varphi}_t^+(x) \geq \dot{\varphi}_t^0(x) = \dot{\varphi}_t^0(x) \geq \limsup_{t \to t_0} \dot{\varphi}_t^-(x),
\]
for almost every \( x \in Y \).

Fix \( t_0 \in I \) and \( t \in I, t > t_0 \). By concavity of the Monge-Ampère energy (see [BBGZ13, Proposition 2.1]) we obtain
\[
\int_Y \frac{\varphi_t - \varphi_{t_0}}{t - t_0} (\theta_0 + dd^c \varphi_t)^n \leq \frac{E(\varphi_t) - E(\varphi_{t_0})}{t - t_0} \leq \int_Y \frac{\varphi_t - \varphi_{t_0}}{t - t_0} (\theta_0 + dd^c \varphi_{t_0})^n.
\]
Using that $\varphi_{t}$ is a solution to (5.2) and $\dot{\varphi}_{t}(x) = \dot{\varphi}_{0}(x)$ a.e., we get

$$\int_{Y} \varphi_{t} - \varphi_{t_{0}} \dot{\varphi}_{t} \, gdV \leq \frac{E(\varphi_{t}) - E(\varphi_{t_{0}})}{t - t_{0}} \leq \int_{Y} \varphi_{t} - \varphi_{t_{0}} \, e^{\lambda\varphi_{t_{0}}} \, gdV.$$  

Letting $I \ni t \to t_{0}$ and using Lebesgue dominated convergence theorem we arrive at the desired formula for the derivative of $t \mapsto E(\varphi_{t})$ at $t_{0}$.

It remains to check that $\frac{d}{dt} E(\varphi_{t}) \geq 0$. This follows from Jensen inequality,

$$\frac{d}{dt} E(\varphi_{t}) = \int_{Y} \dot{\varphi}_{t}(\theta_{0} + d\varphi_{t})^{n} \geq - \log \int_{Y} \nu_{Y} = 0.$$  

\[\square\]

**Step 3: Asymptotic behavior of $\dot{\varphi}_{t}(x)$**. We claim that there exists a constant $C > 0$ such that for all $t \geq 1$ and $x \in Y$,

$$|\dot{\varphi}_{t}(x)| \leq C.$$  

Indeed, since $t \mapsto \varphi_{t}(x)$ is locally uniformly Lipschitz (away from $t = 0$), there is $C > 0$ such that $|\varphi_{s+1} - \varphi_{1}| \leq C s$, for every $s \in [0, 1]$. Fix such $s$ and consider, for $t > 0$ and $x \in Y$,

$$u_{t}(x) := \varphi(s + t + 1, x) - Cs.$$  

Observe that $u_{0} \leq \varphi_{1}$ and

$$(\theta_{0} + d\varphi_{t})^{n} = e^{\dot{u}_{t}} \nu_{Y}.$$  

Since the function $(t, x) \mapsto \varphi(t + 1, x)$ solves the above equation, it follows from Theorem 4.8 that $u_{t} \leq \varphi_{t+1}$, for all $t > 0$. Thus

$$\varphi_{s+t+1} \leq \varphi_{t+1} + Cs,$$  

and letting $s \to 0$ yields a uniform upper bound for $\dot{\varphi}_{t}$. The lower bound follows similarly.

We now claim that there is a sequence of times $t_{j} \to +\infty$ such that $\dot{\varphi}_{t_{j}}(x) \to 0$ for almost every $x \in Y$. Indeed observe that the functional $t \mapsto \mathcal{F}(\varphi_{t}) := E(\varphi_{t}) - \int_{Y} \varphi_{t} \, \nu_{Y}$ is increasing along the flow : for a.e. $t \geq 1$,

$$\frac{d}{dt} \mathcal{F}(\varphi_{t}) = \int_{Y} \dot{\varphi}_{t}(e^{\dot{\varphi}_{t}} - 1) \, dv_{Y} \geq C^{-1} \int_{Y} |\dot{\varphi}_{t}|^{2} \, dv_{Y} \geq 0.$$  

Since $\mathcal{F}$ is uniformly bounded along the flow there is $t_{j} \to +\infty$ such that

$$\int_{Y} |\dot{\varphi}_{t_{j}}|^{2} \, dv_{Y} \to 0$$  

Since the time derivative $\dot{\varphi}_{t}$ is uniformly bounded for $t \geq 1$, it follows that

$$e^{\dot{\varphi}_{t_{j}}} \to 1$$  

in $L^{q}(Y, dv_{Y})$ for all $1 < q < 2$, and $\dot{\varphi}_{t_{j}}(x) \to 0$ for almost every $x \in Y$ (up to extracting and relabelling). It follows from the elliptic $L^{1}$-$L^{\infty}$ stability [GZ12, Theorem C] that $\varphi_{t_{j}}$ uniformly converges to some $\psi$ which satisfies

$$(\theta_{0} + d\varphi_{t})^{n} = \nu_{Y}$$  

and $\int_Y \psi dv_Y = \int_Y \varphi_{KE} dv_Y$, since $\int_X \varphi_i dv_Y$ decreases to $\int_Y \varphi_{KE} dv_Y$. The uniqueness of the normalized Kähler-Einstein potential [EGZ09] now ensures that $\psi = \varphi_{KE}$, i.e. $\varphi_i$ uniformly converges to $\varphi_{KE}$.

**Step 4: The semi-group property.** The conclusion follows now from the fact that our equation is invariant under translations in time: observe that for all $s > 0$, the function $(t, x) \mapsto \psi(t, x) = \varphi(t + s, x)$ is again a bounded parabolic potential, solution to the equation

$$(\theta_0 + dd^c \psi_t)^n = e^{\partial_t \psi_t} dv_Y.$$ 

Fix $\varepsilon > 0$ and $j$ large enough so that

$$\sup_X |\varphi_{t_j} - \varphi_{KE}| < \varepsilon.$$ 

The function $\psi(t, x) = \varphi_{KE}(x) - \varepsilon$ is a subsolution to the Cauchy problem for the above equation with initial data $\varphi_{t_j}$. Similarly $\varphi_{KE}(x) + \varepsilon$ is a supersolution to the same Cauchy problem. The comparison principle (Theorem 4.8) therefore yields, for all $t \geq 0$ and $x \in X$,

$$\varphi_{KE}(x) - \varepsilon \leq \varphi(t + t_j, x) \leq \varphi_{KE}(x) + \varepsilon.$$ 

Letting $t \to +\infty$ and then $\varepsilon \to 0$ yields the conclusion. \hfill \Box

5.2.3. **Minimal models of intermediate Kodaira dimension.** We finally just say a few words of the more delicate volume collapsing case. We assume here $Y$ is an abundant minimal model of Kodaira dimension $1 < \kappa = \text{kod}(Y) < n$, i.e. $K_Y$ is a semi-ample $\mathbb{Q}$-line bundle with $K_Y = f^* K_{Y_{\text{can}}}$, where $f : Y \to Y_{\text{can}}$ is the Iitaka fibration, with $K_{Y_{\text{can}}}$ ample.

A generic fiber $X_y = f^{-1}(y)$ is a $\mathbb{Q}$-Calabi-Yau variety. We fix $h_A$ a positive hermitian metric of $A$ with curvature form $\theta_A$, and $\eta$ local (multivalued) non-vanishing holomorphic section of $K_{Y_{\text{can}}}$. It occurs that

$$v(h_A) = c_n \frac{\eta \wedge \overline{\eta}}{||\eta||^2_{f^* h_A}}$$

is a globally well defined volume form on $Y$ such that the measure $f_* v(h_A)$ has density in $L^{1+\varepsilon}$ w.r.t to $\theta_A^\kappa$.

Generalizing [ST12, ST17], it has been shown in [EGZ18] that there exists a unique bounded $\theta_A$-psh function $\varphi_{\text{can}}$ on $Y_{\text{can}}$ s.t.

- $(\theta_A + dd^c \varphi_{\text{can}})^\kappa = e^{\varphi_{\text{can}}} f_* v(h_A))$;
- the current $\omega_{\text{can}} = \theta_A + dd^c \varphi_{\text{can}}$ is independent of $h_A$;
- it is smooth in $Y_{\text{can}} \setminus \text{critical values of } f$.
- it satisfies $\operatorname{Ric}(\omega_{\text{can}}) = -\omega_{\text{can}} + \omega_{\text{WP}}$ in $Y_{\text{can}} \setminus \text{critical values}.$

The Weil-Petersson term $\omega_{\text{WP}}$ is a semi-positive $(1,1)$-form which measures the change of complex structures in the fibers of the Iitaka fibration. The current $T_{\text{can}} = f^* \omega_{\text{can}}$ is an important birational invariant s.t.

$$T_{\text{can}}^\kappa \wedge \omega_{SF}^{n-\kappa} = e^{\varphi_{\text{can}} f_* v(h_A)}.$$
Here $\omega_{SF} = \omega_0 + dd^c \rho$ denotes the fiberwise family of Ricci flat KE metrics, $\omega_{SF}|_{X_y} = \text{unique Ricci flat metric in } \{\omega_0\}|_{X_y}$, whose existence has been obtained in [EGZ09].

Extending the main result of [EGZ18], the tools developed in this article allow one to establish the following:

**Theorem 5.6.** If $\dim_{\mathbb{C}} Y \leq 3$ then the normalized Kähler-Ricci flow deforms $\omega_0$ towards the canonical current $T_{\text{can}}$, as $t \rightarrow +\infty$.

**Proof.** For a suitable choice of the normalizing constants, the normalized Kähler-Ricci flow is equivalent to the following parabolic complex Monge-Ampère flow of potentials,

$$\frac{(\omega_t + dd^c \varphi_t)^n}{C_n^{\kappa_t} r^{-(n-\kappa)t}} = e^{\partial_t \varphi_t + \varphi_t v(h)},$$

starting from an initial bounded potential $\varphi_0 \in \text{PSH}(X, \omega_0)$. We have normalized here both sides so that the volume of the left hand side converges to 1 as $t \rightarrow +\infty$. Here $C_n^k$ denotes the binomial coefficient $C_n^k = \binom{n}{k}$. It follows from Theorem 5.2 that this flow admits a unique bounded pluripotential solution.

Once the objects are well defined, the proof is then identical to that in [EGZ18, Theorem D]. The restriction on $\dim_{\mathbb{C}} Y$ is related to a regularity issue for some families of Ricci flat metrics. \hfill $\square$

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