Identities involving 3-variable Hermite polynomials arising from umbral method

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Abstract
In this paper, we employ an umbral method to reformulate the 3-variable Hermite polynomials and introduce the 4-parameter 3-variable Hermite polynomials. We also obtain some new properties for these polynomials. Moreover, some special cases are discussed and some concluding remarks are also given.

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1 Introduction
The multi-variable Hermite polynomials have been used in the analysis of charged-beam transport problems in classical mechanics as well as in the formulation of quantum-phase-space mechanics. Umbral methods have been largely exploited to study the properties of the Hermite polynomials. Recently Dattoli et al. applied the method of umbral to obtain certain results for the Hermite polynomials [8]. The study of umbral formalism provides a fairly helpful tool in many topics of practical nature concerning physics of free electron laser. In this paper, we extend the umbral treatment of the Hermite polynomials from two variables to three variables.

We begin with some umbral results on the 2-variable Hermite polynomials (2VHP) $H_n(x,y)$. We recall that 2VHP $H_n(x,y)$ are defined by means of the following generating function and series definition [2]:

$$
\sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!} = e^{xt+y^2t^2} \tag{1}
$$

and

$$
H_n(x,y) = n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{x^{n-2k}y^k}{k!(n-2k)!} \tag{2}
$$

respectively.
The boundary conditions for 2VHP $H_n(x, y)$ are as follows [8, 21]:

$$H_n(x, 0) = x^n$$  \hspace{1cm} (3)

and

$$H_n(0, y) = n! \frac{\sqrt{\pi} y^n}{\Gamma(\frac{n}{2} + 1)} \left| \cos \left( \frac{n \pi}{2} \right) \right|$$  \hspace{1cm} (4)

respectively.

In this paper, we employ the umbral method to the 3-variable Hermite polynomials. Also, we exploit the umbral method to obtain several extensions of the 3-variable Hermite polynomials. Recently, Dattoli et al. gave an umbral method for 2VHP $H_n(x, y)$, which plays an important role in the field of special functions and applied mathematics to obtain all the relevant properties of the other special polynomials as well as special functions [8].

In [8], Dattoli considered the idea of umbral, denoted by $\hat{b}_y$, for 2VHP $H_n(x, y)$ as follows:

$$\hat{b}_y \phi_0 = \frac{\sqrt{\pi} r!}{\Gamma(\frac{r}{2} + 1)} \left| \cos \frac{r \pi}{2} \right|, \quad (\phi_0 \neq 0),$$  \hspace{1cm} (5)

where $\phi_0$ is known as polynomial vacuum and $\hat{b}_y$ acting on the state $\phi_0$ yields 2VHP $H_n(x, y)$.

The exponential of umbral $\hat{b}_y$ is particularly important to derive the generating functions for 2VHP $H_n(x, y)$. The exponential of umbral $\hat{b}_y$ is as follows [8]:

$$e^{\hat{b}_y} \phi_0 = e^{r^2}.$$  \hspace{1cm} (6)

In view of equation (5), 2VHP $H_n(x, y)$ can be reduced binomially as follows:

$$H_n(x, y) = (x + \hat{b}_y)^n \phi_0,$$  \hspace{1cm} (7)

see [8].

Dattoli [8] introduced the 2-parameter 2-variable Hermite polynomials 2P2VHP $H_n(x, y|\beta, \alpha)$:

$$H_n(x, y|\beta, \alpha) = \hat{b}_y^n (x + \hat{b}_y)^n \phi_0.$$  \hspace{1cm} (8)

The generating function of 2P2VHP $H_n(x, y|\beta, \alpha)$ is as follows:

$$\sum_{n=0}^{\infty} H_n(x, y|\beta, \alpha) \frac{t^n}{n!} = e^{r^2} e^{(\alpha, \beta)}(y^\beta t),$$  \hspace{1cm} (9)

where

$$e^{(\alpha, \beta)}(x) = \sum_{r=0}^{\infty} \frac{\Gamma(\alpha r + \beta + 1)x^r}{\Gamma(\frac{\alpha r + \beta}{2} + 1)r!} \left| \cos \left( \frac{\alpha r + \beta}{2} \pi \right) \right|.$$  \hspace{1cm} (10)
see [8], which is a generalisation of the exponential function. It is worthy to note that

$$e_{(1,0)}(x) = e^x.$$  

(11)

Now, we recall that the 3-variable Hermite polynomials (3VHP) $H_n(x, y, z)$ are defined by means of the following generating function and series definition [6]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z) = \exp\left(xt + yt^2 + zt^3\right)$$  

(12)

and

$$H_n(x, y, z) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{H_{n-3\ell}(x, y) x^k}{k!(n - 3k)!},$$  

(13)

respectively.

The operational definition of 3VHP $H_n(x, y, z)$ is as follows [6]:

$$H_n(x, y, z) = e^{D_x^3 + yD_x^2} x^n,$$  

(14)

where

$$D_x := \frac{d}{dx}.$$  

The Gould–Hopper polynomials (GHP) $H_n^{(m)}(x, y)$ are defined by means of the following generating function and series definition [13]:

$$\sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!} = e^{xt + ytm}$$  

(15)

and

$$H_n^{(m)}(x, y) = n! \sum_{r=0}^{[\frac{n}{m}]} \frac{x^{n-mp} y^r}{r!(n - mr)!},$$  

(16)

respectively.

Since

$$m \hat{\beta}_y \phi_0 = \frac{y^{\frac{m}{p}} r!}{\Gamma\left(\frac{m}{p} + 1\right)} A_{m,r},$$  

(17)

$$A_{m,r} = \begin{cases} 1 & r = mp, p \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$  

(18)

Dattoli [8] defined GHP $H_n^{(m)}(x, y)$ in terms of the $n$th power of the binomial given by

$$H_n^{(m)}(x, y) = (x + m \hat{\beta}_y)^n \phi_0.$$  

(19)
The 3-variable generalised Hermite polynomials (3VgHP) \( H_n^{(s,m)}(x, y, z) \) are defined by means of the following generating function [11]:

\[
\sum_{n=0}^{\infty} H_n^{(s,m)}(x, y, z) \frac{t^n}{n!} = e^{x(tz^{\frac{1}{3}} + y(tz^{\frac{1}{3}}) + tz^{\frac{1}{3}})}
\]

and equivalently by

\[
H_n^{(s,m)}(x, y, z) = n! \sum_{r=0}^{\lfloor \frac{n}{3} \rfloor} \frac{H_n^{(m)}(x, y) z^r}{(n - sr)! r!}.
\]

The operational definition of 3VgHP \( H_n^{(s,m)}(x, y, z) \) are as follows [11]:

\[
H_n^{(s,m)}(x, y, z) = \exp(zD_x + yD_y) x^n
\]

and

\[
H_n^{(s,m)}(x, y, z) = \exp(zD_x) H_n^{(m)}(x, y).
\]

In this paper, motivated by the work of Dattoli on the umbral behaviour of the Hermite polynomials [8, 10], we extend the umbral formalism to the 3-variable Hermite polynomials. In Sect. 2, we define an umbra for the 3-variable Hermite polynomials and obtain umbral definition for 3-variable Hermite polynomials and 3-variable generalised Hermite polynomials. In Sect. 3, we introduce an extension of 3-variable Hermite polynomials to 4-parameter 3-variable Hermite polynomials by using the umbral formalism and establish certain results involving these polynomials. In Sect. 4, we discuss some special cases of 4-parameter 3-variable Hermite polynomials. Some concluding remarks are given in Sect. 5.

2 Umbra and 3-variable Hermite polynomial

In [3, 7, 8, 16], it is established that the umbral method serves as an important tool to deal with certain properties of special functions. In this paper, by making use of their method, we introduce the umbral definition of the 3-variable Hermite polynomials \( H_n(x, y, z) \). In this section, we also obtain the umbra for 3VHP \( H_n(x, y, z) \) and study some of its new properties.

Taking \( x = 0 \) and \( y = 0 \) in equation (13), we obtain the boundary condition for the 3-variable Hermite polynomials \( H_n(x, y, z) \):

\[
H_n(0, 0, z) = \frac{z^n n!}{\Gamma(\frac{n}{3} + 1)} \left( \left\lfloor 2 \cos \frac{n\pi}{3} \right\rfloor - |\cos n\pi| \right).
\]

In view of equation (24), we introduce the following umbra:

\[
\hat{c}_z \psi_0 = \frac{z^{\frac{n!}{3}}}{\Gamma(\frac{n}{3} + 1)} \left( \left\lfloor 2 \cos \frac{n\pi}{3} \right\rfloor - |\cos n\pi| \right) \psi_0 \neq 0,
\]

where \( \hat{c}_z \) acts on the vacuum \( \psi_0 \).
It follows from Eq. (25) that
\[ e^{\hat{c}_zt} \psi_0 = e^{rt^2}. \] (26)

Using equations (6) and (26) in equation (12), we get the following umbral form of the generating function of 3VHP \( H_n(x, y, z) \):
\[ \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!} = e^{(x + \hat{b}_y + \hat{c}_z)t} \phi_0 \psi_0, \] (27)
which on expanding the exponential function in the right-hand side and then comparing the equal powers of \( t \) from both sides of the resultant equation gives the following umbral definition of the 3-variable Hermite polynomials \( H_n(x, y, z) \):
\[ H_n(x, y, z) = (x + \hat{b}_y + \hat{c}_z)^n \phi_0 \psi_0 \]
\[ = e^{\hat{c}_z D_x^n} \phi_0 \psi_0 \]
where \( \hat{b}_y \) is acting on \( \phi_0 \) and \( \hat{c}_z \) is acting on \( \psi_0 \).

The use of the above equation allows a significant simplification of the theory of 3VHP \( H_n(x, y, z) \), and it would be largely exploited in the field of special functions. We note that such a point of view has opened new avenues in the derivation of lacunary generating functions and for the relevant combinatorial interpretation [12].

Now, we obtain the umbral definition and umbral operational definition of the 3-variable generalised Hermite polynomials (3VgHP) \( H_n^{(s,m)}(x, y, z) \).

In view of equation (25), we introduce the following generalised form of umbra \( \hat{c}_z \):
\[ s \hat{c}_z \psi_0 = \frac{z^r r!}{\Gamma(z + 1)} A_{s,r}, \] (28)
where
\[ A_{s,r} = \begin{cases} 1 & r = sp, p \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases} \] (29)
If we take \( A_{3,r} = (|2 \cos r \frac{\pi}{3}| - |\cos r \pi|) \), then for \( s = 3 \) equation (28) gives (25) and \( s \hat{c}_z \psi_0 \) reduces to \( \hat{c}_z \psi_0 \).

By equation (28), we have
\[ e^{s \hat{c}_zt} \psi_0 = e^{rt^2}. \] (30)

Using equations (23) and (30), we get
\[ H_n^{(s,m)}(x, y, z) = \exp(\hat{c}_z D_x^n) H_n^{(s,m)}(x, y) \psi_0, \] (31)
which on further simplification gives umbral operational definition of $3VgHP$ as follows:

$$H_n^{(s,m)}(x,y,z) = e^{(\tilde{c}_x+b_y)D_x}x^{\alpha}y^{\beta}z^{\gamma} \phi_0 \psi_0,$$

where $\tilde{c}_x$, $b_y$, $\phi_0$ and $\psi_0$ are defined in equations (17) and (28), respectively.

By using Crofton identity given in [9] we obtain $3VgHP$ binomially as follows:

$$H_n^{(s,m)}(x,y,z) = (x + \tilde{c}_x y + \tilde{c}_x z)^n \phi_0 \psi_0.$$ (33)

In the next section, we generalise the 3-variable Hermite polynomial to 4-parameters 3-variables Hermite polynomials arising from umbral method.

### 3 An extension of the 3-variable Hermite polynomials

It is realised that the advantage of umbral method is that this method serves as an important extension of certain special functions that cannot be extended by using classical operational method; see for example [14, 15]. In this section, by using the fact that the power of these umbras can be any real numbers, we extend the 3-variable Hermite polynomials to 4-parameter 3-variable Hermite polynomials by using the Hermite umbras given as $\tilde{b}_y$ and $\tilde{c}_z$ in equations (5) and (25), respectively.

Further, we study the properties of the 4-parameter 3-variable Hermite polynomials $H_n(x,y,z|\beta,\alpha;p,q)$ and apply the umbral method to aforementioned polynomial.

We introduce the 4-parameter 3-variable Hermite polynomials (4P3VHP) $H_n(x,y,z|\beta,\alpha;p,q)$ given by

$$H_n(x,y,z|\beta,\alpha;p,q) = \tilde{b}_y^\beta \tilde{c}_z^\gamma (x + \tilde{b}_y^\alpha + \tilde{c}_z^\alpha)^n \phi_0 \psi_0,$$ (34)

where $\alpha$, $\beta$, $p$ and $q \in \mathbb{N} \cup \{0\}$.

By equation (34), we have the following generating function for 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$.

**Theorem 3.1** The generating function of 4P3VHP $H_n(x,y,z|\beta,\alpha;p,q)$ is given by

$$\sum_{n=0}^{\infty} H_n(x,y,z|\beta,\alpha;p,q) \frac{t^n}{n!} = e^{(x+y+z)t} e^{(p+q)\frac{t^3}{3}} \mathcal{E}_{p,q}(z^\frac{q}{3} t),$$ (35)

where

$$\mathcal{E}_{p,q}(x) = \sum_{r=0}^{\infty} \frac{\Gamma(p+q+1)x^r}{\Gamma\left(\frac{p+q+1}{3}\right)r!} \left( 2\cos \left( \frac{(p+q)\pi}{3} \right) - \left| \cos \left( \frac{(p+q)\pi}{3} \right) \right| \right).$$ (36)
Proof. From equation (34), we have

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z|\beta, \alpha; p, q) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{b}_n \hat{c}_n \left( x \hat{c}_n + \hat{c}_n^q \right)^n \phi_0 \psi_0
\]

which, on expanding the exponentials in the right-hand side, gives

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z|\beta, \alpha; p, q) = \hat{b}_n \hat{c}_n x^n \left( e^{\hat{c}_n^q} + e^{-\hat{c}_n^q} \right) \phi_0 \psi_0.
\]

Since it is obvious that \([x + \hat{b}_n^q, \hat{c}_n] = 0\) and \([x, \hat{b}_n^q] = 0\) and using the Weyl decoupling identity [9]

\[
e^{\hat{A} \hat{B}} = e^{\hat{A}} e^{\hat{B}}, \quad k = [\hat{A}, \hat{B}], (k \in \mathbb{C})
\]  

in the above equation, we find

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z|\beta, \alpha; p, q) = \hat{b}_n \hat{c}_n x^n e^{\hat{c}_n^q} e^{\hat{c}_n^{q q}} \phi_0 \psi_0
\]

which, on expanding the exponentials in the right-hand side, gives

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z|\beta, \alpha; p, q) = e^{x^n \left( e^{\hat{c}_n^q} + e^{-\hat{c}_n^q} \right) \phi_0 \psi_0}.
\]

Now, using equations (5) and (25), we get

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y, z|\beta, \alpha; p, q) = x^n \sum_{r=0}^{\infty} z^r \sum_{s=0}^{\infty} \frac{\Gamma(p+qs+1)\Gamma(\frac{p+qs}{3}+1)\Gamma(\frac{p+qs}{3}+1)!}{\Gamma(p+qs+1)!} \left( 2 \cos \pi \frac{(p+qs)}{3} - |\cos (p+qs)\pi| \right).
\]

Using equations (10) and (36) in the right-hand side of the above equation, we get assertion (35).

Remark 3.1. The function \(E_{p, q}(x)\) is a generalisation of \(e^x\), as for \(p = 0\) and \(q = 1\) in equation (36), we get \(E_{0, 1}(x) = e^{x^3}\).

Next, we obtain the following series definition for 4P3VHP \(H_n(x, y, z|\beta, \alpha; p, q)\).

Theorem 3.2. The series definition for 4P3VHP \(H_n(x, y, z|\beta, \alpha; p, q)\) is given by

\[
H_n(x, y, z|\beta, \alpha; p, q) = n! \sum_{r=0}^{n} z^{\frac{p+qr+1}{3}} \Gamma(p+qr+1) \frac{\Gamma(\frac{p+qr}{3}+1)\Gamma(\frac{p+qr}{3}+1)!}{\Gamma(p+qr+1)!} \left( 2 \cos \pi \frac{(p+qr)}{3} - |\cos (p+qr)\pi| \right),
\]  

(39)
where \( H_{n, \alpha}(x, y|\beta, \alpha) \) denotes 2P2VHP given by means of the following generating function:

\[
\sum_{n=0}^{\infty} H_{n}(x, y|\beta, \alpha) \frac{t^n}{n!} = e^{\beta y} e^{\alpha y} t^\frac{n}{2}.
\]

**Proof** From equation (38), we have

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n}(x, y, z|\beta, \alpha; p, q) = \hat{\beta}_y^p \hat{c}_z e^{(\alpha y + \beta z)} t^\frac{p}{2} \phi_0 \psi_0,
\]

which, on expanding exponentials in the right-hand side of the above equation, gives

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n}(x, y, z|\beta, \alpha; p, q) = \sum_{n=0}^{\infty} \frac{\hat{c}_z^n}{n!} \sum_{r=0}^{n} \frac{\hat{c}_z^n}{r!} \phi_0 \psi_0.
\]

Using equations (8) and (25), we get

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n}(x, y, z|\beta, \alpha; p, q) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{\Gamma(p + qr + 1) \beta^{p + qr}}{\Gamma(\frac{p}{3} + 1)} (n - r)! \phi_0 \psi_0.
\]

Comparing the equal powers of \( t \) from both sides of the above equation, we get assertion (39). □

Further, we discuss an alternative formulation of the theory of the generalised Hermite polynomials using umbral formalism, which will be embedded with the technique developed in this paper.

Now, we obtain the following result.

**Theorem 3.3** The following formula for 4P3VHP \( H_n(x, y, z|\beta, \alpha; p, q) \) holds:

\[
H_{n+k}(x, y, z|\beta, \alpha; p, q)) = \sum_{r=0}^{k} \sum_{s=0}^{r} \binom{k}{r} \binom{r}{s} x^s H_n(x, y, z|\beta, \alpha; q(k - r) + p, q).
\] (40)

**Proof** From equation (34), we have

\[
H_{n+k}(x, y, z|\beta, \alpha; p, q) = \hat{\beta}_y^p \hat{c}_z e^{(\alpha y + \beta z)} n^k \phi_0 \psi_0
\]

\[
= \hat{\beta}_y^p \hat{c}_z e^{(\alpha y + \beta z)} (x + \hat{\alpha}_y^p + \hat{c}_z) n^k \phi_0 \psi_0.
\]
Expanding the first bracket of the right-hand side of the above equation binomially, we have

\[ H_{n+k}(x, y, z|\beta, \alpha; p, q) = \hat{b}_y^\beta \hat{c}_z^\gamma \sum_{r=0}^{k} \binom{k}{r} \phi_0 \psi_0. \]

Again, expanding the first bracket of the right-hand side of the above equation binomially, we find

\[ H_{n+k}(x, y, z|\beta, \alpha; p, q) = \sum_{r=0}^{k} \sum_{s=0}^{r} \binom{n}{s} \binom{r}{s} \binom{s}{u} \frac{1}{2^{s-u}} \times H_{n-s}(x, y, z|\alpha(r-u) + \beta, \alpha; q(s-r) + p, q). \]  

Using equation (34) in the right-hand side of the above equation, we get assertion (40).

For \( k = n \), Theorem 3.3 gives the following result.

**Corollary 3.1** The following index duplication formula for 4P3VHP

\[ H_n(x, y, z|\beta, \alpha; p, q) \]

holds:

\[ H_{2n}(x, y, z|\beta, \alpha; p, q) = \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{s} \binom{r}{s} \binom{s}{u} \frac{1}{2^{s-u}} \times H_{n-s}(x, y, z|\alpha(r-u) + \beta, \alpha; q(s-r) + p, q). \]  

Further, we obtain the following result.

**Theorem 3.4** The following argument duplication formula for 4P3VHP \( H_n(x, y, z|\beta, \alpha; p, q) \) holds:

\[ H_n(2x, y, z|\beta, \alpha; p, q) = \sum_{s=0}^{n} \sum_{r=0}^{s} \sum_{u=0}^{r} \binom{n}{s} \binom{s}{u} \frac{1}{2^{s-u}} \times H_{n-s}(x, y, z|\alpha(r-u) + \beta, \alpha; q(s-r) + p, q). \]

**Proof** From equation (34), we have

\[ H_n(2x, y, z|\beta, \alpha; p, q) = \hat{b}_y^\beta \hat{c}_z^\gamma \left[ x + \hat{b}_y^\beta \frac{z}{2} + \hat{c}_z^\gamma \frac{y}{2} \right]^n \phi_0 \psi_0, \]

which on simplification gives

\[ H_n(2x, y, z|\beta, \alpha; p, q) = \hat{b}_y^\beta \hat{c}_z^\gamma \sum_{s=0}^{n} \binom{n}{s} \left( x + \hat{b}_y^\beta \frac{z}{2} + \hat{c}_z^\gamma \frac{y}{2} \right)^s \phi_0 \psi_0. \]
Expanding the second bracket in right-hand side of the above equation binomially, we find

\[
H_n(2x, y|\beta, \alpha; p, q) = \hat{b}_y^{\beta} \hat{b}_z^{\epsilon} \sum_{s=0}^{n} \sum_{r=0}^{s} \left( \begin{array}{c} n \\ s \end{array} \right) \left( \begin{array}{c} s \\ r \end{array} \right) \left( \begin{array}{c} r \\ u \end{array} \right) \psi_0 \phi_0,
\]

which on further simplification gives

\[
H_n(2x, y, z|\beta, \alpha; p, q) = \hat{b}_y^{\beta} \hat{b}_z^{\epsilon} \sum_{s=0}^{n} \sum_{r=0}^{s} \left( \begin{array}{c} n \\ s \end{array} \right) \left( \begin{array}{c} s \\ r \end{array} \right) \left( \begin{array}{c} r \\ u \end{array} \right) \psi_0 \phi_0.
\]

Using equation (34) in the right-hand side of the above equation, we get assertion (42). □

Now, we find the following series representation of the 4-parameter 2-variable Hermite polynomials in terms of the 4-parameter 3-variable Hermite polynomials.

**Theorem 3.5** The series definition of 4P2VHP \( H_n(x, y|\beta, \alpha; p, q) \) is given by

\[
H_n(x, y|\beta, \alpha; p, q) = \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-1)^r H_{n-r}(x, y, z|\beta, \alpha; p + qr, q).
\] (43)

**Proof** From equation (34), we have

\[
H_n(x, y|\beta, \alpha; p, q) = \hat{b}_y^{\beta} \hat{b}_z^{\epsilon} (x + \hat{b}_y^{\beta} + \hat{c}_z^{\epsilon} - \hat{c}_z^{\epsilon}) \psi_0 \phi_0,
\] (44)

from which, on expanding binomially, we get

\[
H_n(x, y|\beta, \alpha; p, q) = \hat{b}_y^{\beta} \hat{b}_z^{\epsilon} \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-1)^r (x + \hat{b}_y^{\beta} + \hat{c}_z^{\epsilon}) \psi_0 \phi_0.
\] (45)

Using equation (34) in the above equation we get assertion (43). □

**Remark 3.2** For taking \( p = 0 \) and \( q = 1 \) in equation (44) of Theorem 3.5, we get the following series representation of 2P2VHP \( H_n(x, y|\beta, \alpha) \):

\[
H_n(x, y|\beta, \alpha) = \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-1)^r H_{n-r}(x, y, z|\beta, \alpha; r, 1).
\] (46)

**Remark 3.3** For taking \( \beta = 0, \alpha = 1, p = 0 \) and \( q = 1 \) in equation (44) of Theorem 3.5, we get the following series representation of 2VHP \( H_n(x, y) \):

\[
H_n(x, y) = \sum_{r=0}^{n} \left( \begin{array}{c} n \\ r \end{array} \right) (-1)^r H_{n-r}(x, y, z|r, 1).
\] (47)
Now, we obtain the operational definition of 4P3VHP \( H_n(x, y, z|\beta, \alpha; p, q) \). Since \( D_yH_n(x, y) = nH_{n-1}(x, y) \) and \( D_zH_n(x, y, z) = nH_{n-1}(x, y, z) \), it can be verified that

\[
H_n(x, y, z|\beta, \alpha; p, q) = y^\beta e_{(\alpha, \beta)}(y^2D_y)z^\gamma e_{(\alpha, \beta)}(z^2D_z) x^\alpha,
\]

and for taking \( \alpha = 1 \) and \( \beta = 0 \), we have

\[
H_n(x, y, z|-, 1; p, q) = z^\gamma e_{p,q}(z^2D_z)H_n(x, y).
\]

In the next section, we consider some special cases of the results established in this section.

## 4 Special cases

In this section, we obtain certain new as well as known special polynomials by using suitable choices for parameters and variable z in equations (34), (35) and (39) as special cases of 4-parameter 3-variable Hermite polynomials.

In the following table, the umbral definitions, generating functions and series definitions of certain polynomials are listed.

For the same choices of parameters \( \alpha, \beta, p \) and \( q \) considered in Table 1, equations (41) and (42) give the index duplication and argument duplication formulas for the special polynomials mentioned in the same table. The respective formulas are listed in Table 2.

In the concluding remarks, we present further argument supporting the effectiveness of the umbral method.

### Table 1: Some new and known special polynomials

| No. | Parameters | Polynomials | Umbral definition | Generating function | Series definition |
|-----|------------|-------------|-------------------|-------------------|------------------|
| I.  | \( q = 1 \) | \( H_n(x, y, z|\beta, \alpha, p, 1) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
| II. | \( q = 1 \) | \( H_n(x, y, z|\beta, 1, p, 1) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
| III. | \( q = 1 \) | \( H_n(x, y, z|\beta, 1, 1, p) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
| IV. | \( q = 1 \) | \( H_n(x, y, z|\beta, 1, 1, 1) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
| V.  | \( q = 1 \) | \( H_n(x, y, z|\alpha, 1, 1, 1) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
| VI. | \( q = 1 \) | \( H_n(x, y, z|\beta, 1, 1, 1) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
| VII. | \( q = 1 \) | \( H_n(x, y, z|\beta, 1, 1, 1) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
| VIII. | \( q = 1 \) | \( H_n(x, y, z|\beta, 1, 1, 1) \) | \( y^\beta e_{(\alpha, \beta)}(y^2D_y) \) | \( \Gamma(1+q/p) \) | \( \Gamma(1+q/p) \) |
Table 1 (Continued)

| S. No. | Parameter | Polynomials | Umbral definition | Generating function | Series definition |
|--------|-----------|-------------|-------------------|--------------------|------------------|
| IX.    | $\beta = 0$; $\alpha = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| X.     | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XI.    | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XII.   | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XIII.  | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XIV.   | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XV.    | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XVI.   | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XVII.  | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |
| XVIII. | $\alpha = 0$; $\beta = 0$; $\gamma = 1$ | $\beta_0(x)$ | $e^{\alpha x} | x \rangle$ | $\frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta$ | $\sum_{n=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} x^\beta \frac{\pi^{\gamma+1}}{\Gamma(\gamma+1)} x^\gamma$ |

5 Concluding remarks

Gaussian integral representation of Hermite polynomials as well as specific umbral methods play an important role in classical problems arising in quantum optics, quantum mechanics, biomathematics and engineering (see for example [1, 17–20]). They are exploited to calculate the optical mode overlapping and transition rates between quantum eigenstates of the harmonic oscillator. A general method allowing the direct evaluation of these integrals has not been developed. Babusci et al. described a unifying method, flexible for generalisation, which provides a direct method for the evaluation of this class of integrals [4, 5].
We consider the following integral:

\[ I_n = \int_{-\infty}^{\infty} H_n(ax + b, y, z|\beta, \alpha; p, q) e^{-c x^2 + d x} \, dx. \]  

(48)

By equation (48), we have

\[ \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} H_n(ax + b, y, z|\beta, \alpha; p, q) \frac{t^n}{n!} e^{-c x^2 + d x} \, dx = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \]
In view of equations (48) and (52), we get the following result:

\[ \sum_{n=0}^{\infty} I_n t^n n! = e^{\frac{\beta}{2}} e_{(\alpha,\beta)}(y^2 t) z^\frac{\pi}{2} \int_{-\infty}^{\infty} e^{(a t + t^2) x^2} dx. \]  

(49)

Since

\[ \int_{-\infty}^{\infty} e^{b x - a x^2 + c} dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{c^2}{a}}, \]

see [5], presenting the Gaussian integral, we find

\[ \sum_{n=0}^{\infty} I_n t^n n! = e^{\frac{\beta}{2}} e_{(\alpha,\beta)}(y^2 t) z^\frac{\pi}{2} \frac{\sqrt{\pi}}{\sqrt{c}} \exp \left( \frac{\alpha^2}{4c} + \frac{\xi^2}{4c} + \frac{a^2}{2c} t \right). \]  

(50)

Using equations (1) and (35) in the right-hand side of the above equation, we obtain

\[ \sum_{n=0}^{\infty} I_n t^n n! = \sqrt{\frac{\pi}{\sqrt{c}}} \exp \left( \frac{\xi^2}{4c} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} H_n(b, y, z; \beta, \alpha; p, q) H_r \left( \frac{a^2}{2c}, \frac{\alpha^2}{4c} \right) \frac{t^{n+r}}{n!}. \]

Next, comparing the equal powers of \( t \) from both sides of the above equation, we get

\[ I_n = \sqrt{\frac{\pi}{\sqrt{c}}} \exp \left( \frac{\xi^2}{4c} \right) \sum_{r=0}^{n} \binom{n}{r} H_n(b, y, z; \beta, \alpha; p, q) H_r \left( \frac{a^2}{2c}, \frac{\alpha^2}{4c} \right). \]  

(51)

In view of equations (48) and (51), we get the following result:

\[ \int_{-\infty}^{\infty} H_n(ax + b, y, z; \beta, \alpha; p, q) e^{-x^2 + \xi x} dx \]

\[ = \sqrt{\frac{\pi}{\sqrt{c}}} \exp \left( \frac{\xi^2}{4c} \right) \sum_{r=0}^{n} \binom{n}{r} H_n(b + \frac{a^2}{2c}, y, z; \beta, \alpha; p, q) \frac{a^{2r}}{(4c)^r} \frac{t^{n+r}}{n!}. \]

Again, using equation (35) in the right-hand side of equation (50), we find

\[ \sum_{n=0}^{\infty} I_n t^n n! = \sqrt{\frac{\pi}{\sqrt{c}}} \exp \left( \frac{\xi^2}{4c} \right) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} H_n \left( b + \frac{a^2}{2c}, y, z; \beta, \alpha; p, q \right) \frac{a^{2r}}{(4c)^r} \frac{t^{n+r}}{n!}. \]

Comparing the equal powers of \( t \) from both sides of the above equation, we get

\[ I_n = \sqrt{\frac{\pi}{\sqrt{c}}} \exp \left( \frac{\xi^2}{4c} \right) \sum_{r=0}^{n} \binom{n}{r} \frac{1}{(n-2r)!} H_{n-2r} \left( b + \frac{a^2}{2c}, y, z; \beta, \alpha; p, q \right) \frac{a^{2r}}{(4c)^r}. \]  

(52)

In view of equations (48) and (52), we get the following result:

\[ \int_{-\infty}^{\infty} H_n(ax + b, y, z; \beta, \alpha; p, q) e^{-x^2 + \xi x} dx \]

\[ = \sqrt{\frac{\pi}{\sqrt{c}}} \exp \left( \frac{\xi^2}{4c} \right) \sum_{r=0}^{n} \binom{n}{r} \frac{1}{(n-2r)!} H_{n-2r} \left( b + \frac{a^2}{2c}, y, z; \beta, \alpha; p, q \right) \frac{a^{2r}}{(4c)^r}. \]
Similarly, for the same choices of parameters $\alpha, \beta, p$ and $q$ considered in Table 1, we can evaluate the integrals involving the special polynomials mentioned in the same table.

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References
1. Andrews, L.C.: Special Functions for Applied Mathematics and Engineering. MacMillan, New York (1985)
2. Appell, P., Kampé de Fériet, J.: Fonctions Hypergéométriques et Hypersphériques: Polynômes d’Hermite. Gauthier-Villars, Paris (1926)
3. Araci, S.: Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus. Appl. Math. Comput. 233, 599–607 (2014)
4. Babusci, D., Dattoli, G., Del Franco, M.: Lectures on mathematical methods for physics. Internal Report ENEA RT/2010/5837
5. Babusci, D., Dattoli, G., Quattromini, M.: On integrals involving Hermite polynomials. Appl. Math. Lett. 25(8), 1157–1160 (2012)
6. Dattoli, G.: Generalized polynomials, operational identities and their applications. Higher transcendental functions and their applications. J. Comput. Appl. Math. 118(1–2), 111–123 (2000)
7. Dattoli, G., Germano, B., Licciardi, S., Martineili, M.R.: On umbral treatment of Gegenbauer, Legendre and Jacobi polynomials. Int. Math. Forum 12(11), 531–551 (2017)
8. Dattoli, G., Germano, B., Martineili, M.R., Ricci, P.E.: Lacunary generating functions of Hermite polynomials and symbolic methods. Ilirias J. Math. 4, 16–23 (2015)
9. Dattoli, G., Ottavini, P.L., Torre, A., Vázquez, L.: Evolution operator equations: integration with algebraic and finite-difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory. Riv. Nuovo Cimento Soc. Ital. Fis. (4) 20(2) (1997) 133 pp.
10. Dattoli, G., Ricci, P.E., Cesarano, C.: Monomial polynomials and the associated formalism. Integral Transforms Spec. Funct. 13(2), 155–162 (2002)
11. Dattoli, G., Ricci, P.E., Khomasuridze, I.: On the derivation of new families of generating functions involving ordinary Bessel functions and Bessel-Hermite functions. Math. Comput. Model. 46(3–4), 410–414 (2007)
12. Gessel, I.M., Jayawant, P.: A triple lacunary generating function for Hermite polynomials. Electron. J. Combin. 12 (2005) Research Paper, 30, 14 pp.
13. Gould, H.W., Hopper, A.T.: Operational formulas connected with two generalizations of Hermite polynomials. Duke Math. J. 29, 51–63 (1962)
14. Jang, L.-C., Kim, T., Kim, D.S., Kim, H.Y.: Extended $r$-central Bell polynomials with umbral calculus viewpoint. Adv. Differ. Equ. 2019, Article ID 202 (2019)
15. Kim, T., Kim, D.S.: Some identities of extended degenerate $r$-central Bell polynomials arising from umbral calculus. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 114(1), Paper No. 1 (2020)
16. Kim, T., Kim, D.S., Kwon, H.-I., Kwon, J.: Umbral calculus approach to $r$ Stirling number of the second kind and $r$ Bell polynomials. J. Comput. Anal. Appl. 27, 173–188 (2019)
17. Lebedev, N.N.: Special Functions and Their Applications, Revised edn. Dover, New York (1972) xii+308 pp. Translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication
18. Li, T., Viglialoro, G.: Analysis and explicit solvability of degenerate tensorial problems. Bound. Value Probl. 2018, Article ID 2 (2018)
19. Louisell, W.H.: Quantum Statistical Properties of Radiation. Wiley Classics Library. A Wiley-Interscience Publication, Wiley, New York (1990) xvii+528 pp. Reprint of the 1973 edition
20. Viglialoro, G., Woolley, T.E.: Boundedness in a parabolic-elliptic chemotaxis system with nonlinear diffusion and sensitivity and logistic source. Math. Methods Appl. Sci. 41(5), 1809–1824 (2018)

21. Widder, D.V.: The Heat Equation. Pure and Applied Mathematics, vol. 67, xiv+267 pp. Academic Press [Harcourt Brace Jovanovich, Publishers], New York (1975)