How Clifford algebra can help understand second quantization of fermion and boson fields

N.S. Mankoč Borštnik

1Department of Physics, University of Ljubljana
SI-1000 Ljubljana, Slovenia

October 13, 2022

Abstract

In the review article in Progress in Particle and Nuclear Physics [4] the authors present a rather detailed review of the achievements so far of the spin-charge-family theory that offers the explanation for the observed properties of elementary fermion and boson fields, if the space-time is higher than \( d = (3 + 1) \), it must be \( d \geq (13 + 1) \), while fermions only interact with gravity. Ref. [4] presents also an explanation for the second quantization postulates for the fermion fields, since the internal space of fermions is in this theory described with the “basis vectors” determined by the Clifford odd objects. The anticommutativity of the “basis vectors” namely transfers to their creation and annihilation operators. This paper shows that the “basis vectors” determined by the Clifford even objects, if used to describe the internal space of boson fields, not only manifest all the known properties of the observed boson fields, but offer as well the explanation for the second quantization postulates for boson fields. Properties of fermion and boson fields with the internal spaces described by the Clifford odd and even objects, respectively, are demonstrated on the toy model with \( d = (5 + 1) \).

1 Introduction

In a long series of works [1] [2] [28] [16] [18] [22] [24] [21] the author has found, together with the collaborators ([14] [15] [20] [25] [8] [7] [4] and the references therein), and in long discussions with participants during the annual workshops ”What comes beyond the standard models”, the phenomenological success with the model named the spin-charge-family theory with the properties:

a. The internal space of fermions are described by the ”basis vectors” which are superposition of odd products of anticommuting objects \( \gamma^a \)’s in \( d = (13 + 1) \) [12] [8] [5] [4]. Correspondingly the ”basis vectors” of one Lorentz irreducible representation in internal space of fermions, together with their Hermitian conjugated partners, anticommute, fulfilling (on the vacuum state) all the requirements for the second quantized fermion fields [12] [8] [14] [15] [4].

b. The second kind of anticommuting objects, \( \tilde{\gamma}^a \)’s, equip each irreducible representation of odd ”basis vectors” with the family quantum number [5] [15].

c. Creation operators for single fermion states — which are tensor products, *\( \mathcal{T} \), of a finite number of odd ”basis vectors” appearing in \( 2^d - 1 \) families, each family with \( 2^d - 1 \) members, and the (continuously) infinite momentum/coordinate basis applying on the vacuum state [5] [4] — inherit anticommutativity.
of "basis vectors". Creation operators and their Hermitian conjugated partners correspondingly anti-commute.

d. The Hilbert space of second quantized fermions is represented by the tensor products of all possible number of creation operators, from zero to infinity \[8\], applying on a vacuum state.

e. In a simple starting action massless fermions carry only spins and interact with only gravity — with the vielbeins and the two kinds of the spin connection fields (the gauge fields of momenta, of \(S^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a)\) and of \(\tilde{S}^{ab} = \frac{1}{2}(\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)\), respectively \[1\]). The starting action includes only even products of \(\gamma^a\)'s and \(\tilde{\gamma}^a\)'s \[12\] and references therein.

f. Spins from higher dimensions, \(d > (3 + 1)\), described by \(\gamma^a\)'s, manifest in \(d = (3 + 1)\) all the charges of the standard model quarks and leptons and antiquarks and antileptons of particular handedness.

g. Gravity — the gauge fields of \(S^{ab}\), \((a,b) = (5,6,...,d)\), with the space index \(m = (0,1,2,3)\) — manifest as the standard model vector gauge fields \[20\]. The scalar gauge fields of \(\tilde{S}^{ab}\) and of some of superposition of \(S^{ab}\), with the space index \(s = (7,8)\) manifest as the scalar higgs and Yukawa couplings \[21\] \[22\] \[16\] \[4\], determining mass matrices (of particular symmetry) and correspondingly the masses of quarks and leptons and of the weak boson fields after (some of) the scalar fields with the space index \((7,8)\) gain constant values. The scalar gauge fields of \(\tilde{S}^{ab}\) and of \(S^{ab}\) with the space index \(s = (9,10,...,14)\) and \((a,b) = (5,6,...,d)\) offer the explanation for the observed matter/antimatter asymmetry \[18\] \[24\] \[25\] \[4\] \[19\] in the universe.

h. The theory predicts the fourth family to the observed three \[30\] \[31\] \[32\] \[33\] \[34\] \[35\] and the stable fifth family of heavy quarks and leptons. The stable fifth family nucleons offer the explanation for the appearance of the dark matter. Due to heavy masses of the fifth family quarks the nuclear interaction among hadrons of the fifth family members is very different than the ones so far observed \[33\] \[36\].

i. The theory offers a new understanding of the second quantized fermion fields (explained in Ref. \[4\]) as well as of the second quantized boson fields. The second quantization of boson fields, the gauge fields of the second quantized fermion fields, is the main topic of this paper \[6\].

j. The theory seems promising to offer a new insight into Feynman diagrams.

The more work is put into the theory the more phenomena the theory is able to explain.

In this paper we shortly overview the description of the internal space of the second quantized massless fermion fields with the "basis vectors" which are the superposition of odd products of the Clifford algebra objects (operators) \(\gamma^a\)'s. Tensor products of "basis vectors" with the basis in ordinary space form the creation operators for fermions which fulfill the anticommutation relations of the Dirac second quantized fermion fields, without postulating them \[9\] \[10\] \[11\]. We kindly ask the reader to read the explanations in Ref. \[4\], Sect. 3.

The "basis vectors", which are the superposition of even products of the Clifford algebra objects \(\gamma^a\)'s and are in a tensor product, \(*_T\), with the basis in ordinary space, have all the properties of the second quantized boson fields, the gauge fields of the corresponding second quantized fermion fields. The main part of this paper discusses properties of the internal space of the second quantized boson fields described by the Clifford even "basis vectors" in interaction among the boson fields themselves and with the second quantized fermion fields with the internal space described by the Clifford odd "basis vectors", Ref. \[4\].

In both cases when describing the second quantized either fermion or boson fields the creation operators are considered to be a tensor, \(*_T\), product of \(2^\frac{d}{2} - 1 \times 2^\frac{d}{2} - 1\) of either anticommuting Clifford odd (in the case of fermion fields) or commuting Clifford even (in the case of boson fields) "basis vectors" and of (continuously) infinite commuting basis of ordinary space.

While in the case of the Clifford odd "basis vectors" the Hermitian conjugated partners belong to another group with \(2^\frac{d}{2} - 1 \times 2^\frac{d}{2} - 1\) members (which is not reachable by either \(S^{ab}\) or \(\tilde{S}^{ab}\)) or both, in the

\[^1\]If there are no fermions present the two kinds of the spin connection fields are uniquely expressible by the vielbeins \[12\].
case of the Clifford even "basis vectors" each of the two groups with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members have their Hermitian conjugated partners among themselves, that is within the group reachable by either $S_{ab}$ or $\tilde{S}_{ab}$.

Subsects 2.1, 2.2, 2.3 of Sect 2 are a short overview of the Clifford odd and the Clifford even algebra, used to described in the spin-charge-family theory the internal space of fermions — as already presented in Ref. (4 in Sect. 3) — and in this paper in particular the internal space of bosons, as the author started in Ref. [6]. In Subsect. 2.1 the anticommuting Grassmann algebra and the two Clifford subalgebras, each algebra with $2 \times 2^{d}$ elements, are presented, and the relations among them discussed.

In Subsect. 2.2 the "basis vectors" of either odd or even character are defined as eigenvectors of all the members of the Cartan subalgebra of the Lorentz algebra for the Grassmann and the two Clifford subalgebras (4, Sect. 3).

The "basis vectors" are products of nilpotents and projectors, each nilpotent and each projector is chosen to be the eigenvector of one member of the Cartan subalgebra. The anticommuting "basis vectors" have an odd number of nilpotents, the commuting "basis vectors" have an even number of nilpotents.

The anticommutation relations (for fermions, with the odd number of nilpotents) and commutation relations (for bosons, with the even number of nilpotents) are presented.

There are obviously only one kind of fermion fields and correspondingly also of their gauge fields observed. There is correspondingly no need for two Clifford subalgebras.

In Subsect. 2.3 this problem is solved by the reduction of the two Clifford subalgebras to only one, what enables also to give the family quantum numbers to the Clifford odd anticommuting "basis vectors", belonging to different irreducible representations of the Lorentz algebra. The reduction enables as well to define to the Clifford even commuting "basis vectors" the generators of the Lorentz transformations in the internal space of bosons.

In Subsect. 2.4 the "basis vectors" for fermions, 2.4.1 and bosons, 2.4.2 are discussed in details for the "toy model" in $d = (5 + 1)$ to make differences in the properties of the Clifford odd and Clifford even "basis vectors" transparent and correspondingly easier to understand. The algebraic application of the Clifford even "basis vectors" on the Clifford odd "basis vectors" is demonstrated, as well as the algebraic application of the Clifford even "basis vectors" on themselves. This subsection is the main part of the article.

In Subsect. 2.5 the generalization of the Clifford odd and Clifford even "basis vectors" to any even $d$ is discussed.

In Sect. 3 the creation operators of the second quantized fermion and boson fields offered by the spin-charge-family theory are studied.

Sect. 4 reviews shortly what one can learn in this article and what remains to study.

In App. A "basis vectors" of one family of quarks and leptons are presented.

In App. B some useful relations are presented.

2 Properties of creation and annihilation operators for fermions and bosons

Second quantization postulates for fermion and boson fields [9, 10, 11] require that the creation and annihilation operators for fermions and bosons depend on finite number of spins and other quantum numbers determining internal space of fermions and bosons and on infinite number of momenta (or coordinates). While fermions carry half integer spins and charges in fundamental representations of the corresponding groups bosons carry integer spins and charges in adjoint representations of the groups. Ref. [4] reports in Subsect. 3.3.1. second quantization postulates for fermions.
The first quantized fermion states are in the Dirac’s theory vectors which do not anticommute. There are the creation operators of the second quantized fermion fields which are postulated to anticommute. The second quantized fermion fields commute with $\gamma^a$ matrices, allowing the second quantization of the Dirac equation which includes the mass term.

Creation and annihilation of boson fields are postulated to fulfill commutation relations.

In the spin-charge-family theory the internal space of fermions and bosons in even dimensional spaces $d = 2(2n + 1)$ is described by the algebraic, $*A$, products of $\frac{d}{2}$ nilpotents and projectors, which are superposition of odd (nilpotents) and even (projectors) numbers of anticommuting operators $\gamma^a$’s. Nilpotents and projectors are chosen to be eigenvectors of $\frac{d}{2}$ Cartan subalgebra members of the Lorentz algebra of $S^{ab} = \frac{i}{4} \{\gamma^a, \gamma^b\}_{-}$, determining the internal space of fermions and bosons.

There are two groups of $2^{\frac{d}{2}-1}$ members appearing in $2^{\frac{d}{2}-1}$ irreducible representations which have an odd number of nilpotents (at least one nilpotent and the rest projectors). The members of one of the groups are (chosen to be) called ”basis vectors”. The other group contains the Hermitian conjugated partners of the ”basis vectors”. The group of these odd ”basis vectors” have all the properties needed to describe internal space of fermions.

There are two groups with an even number of nilpotents, each with $2^{\frac{d}{2}-1}$ $2^{\frac{d}{2}-1}$ members. Each of these two groups have their Hermitian conjugated partners within the same group. Each of the two groups with an even number of nilpotents have all the properties needed to describe the internal space of boson fields, as we shall see in this article.

N.S.M.B. made a choice of $d = (13 + 1)$ since for such $d$ the theory offers the explanation for all the assumptions of the standard model, that is for the charges, handedness, families of quarks and leptons and antiquarks and antileptons, for all the observed vector gauge fields, as well as for the scalar higgs and Yukawa couplings.

A simple starting action ([4] and the references therein) for the second quantized massless fermion and antifermion fields, and the corresponding massless boson fields in $d = 2(2n + 1)$-dimensional space is assumed to be

$$\mathcal{A} = \int d^d x \frac{1}{2} (\bar{\psi} \gamma^a p_{0a} \psi) + h.c. + \int d^d x \ (\alpha R + \tilde{\alpha} \tilde{R}),$$

$$p_{0a} = f^a_{\alpha} p_{\alpha a} + \frac{1}{2 E} \{p_{\alpha}, E f^a_{\alpha}\},$$

$$p_{\alpha a} = p_{\alpha} - \frac{1}{2} S^{ab} \omega_{abc} - \frac{1}{2} S^{ab} \tilde{\omega}_{abc},$$

$$R = \frac{1}{2} \{f^{a[a} f^{b\beta]} (\omega_{abc} - \omega_{cba} \omega^{c} \beta)\} + h.c.,$$

$$\tilde{R} = \frac{1}{2} \{f^{a[a} f^{b\beta]} (\tilde{\omega}_{abc} - \tilde{\omega}_{cba} \tilde{\omega}^{c} \beta)\} + h.c. \quad (1)$$

Here $\mathcal{A}$, $f^{a[a} f^{b\beta]} = f^{a[a} f^{b\beta]} - f^{a[b} f^{\beta a]}. \ The \gamma^a$ operators appear in the Lagrangean for second quantized massless fermion fields in pairs.

Fermions, appearing in families, carry only spins and only interact with gravity, what manifests in $d = (3 + 1)$ as spins and all the observed charges. Vielbeins, $e^a_{\alpha}$ (the gauge field of momenta), and

\[f^a_{\alpha}\] are inverted vielbeins to $e^\alpha_a$ with the properties $e^\alpha_a f^a_b = \delta_\beta^b$, $e^\alpha_a f^a_\beta = \delta^\beta_\alpha$, $E = \text{det}(e^\alpha_a)$. Latin indices $a, b, ... , m, n, ... , s, t, ...$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, ... , \mu, \nu, ... , \sigma, \tau, ...$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index $(a, b, c, ...$ and $\alpha, \beta, \gamma, ...)$, from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ $(m, n, ...$ and $\mu, \nu, ...)$, indexes from the bottom of the alphabets indicate the compactified dimensions $(s, t, ...$ and $\sigma, \tau, ...$. We assume the signature $\eta^{ab} = \text{diag}(1, -1, -1, \cdots, -1)$. 

4
two kinds of the spin connection fields, $\omega_{\alpha\beta}$ (the gauge fields of $S^{ab}$) and $\tilde{\omega}_{\alpha\beta}$ (the gauge fields of $\tilde{S}^{ab}$), manifest in $d = (3 + 1)$ as the known vector gauge fields \cite{20} and the scalar gauge fields taking care of masses of quarks and leptons and antiquarks and antileptons \cite{4} and the weak boson fields \cite{2}

Internal space of fermions is described in any even dimensional space by "basis vectors" with an odd number of nilpotents, the rest are projectors, as mentioned above. Nilpotents, described by an odd number of $\gamma^a$'s, anticommute with $\gamma^a$, projectors, described by an even number of $\gamma^a$'s, commute with $\gamma^a$. Correspondingly the "basis vectors", describing the internal space of fermions, anticommute among themselves.

Internal space of bosons is described in any even dimensional space by "basis vectors" with an even number of nilpotents and of projectors. The "basis vectors" describing bosons therefore commute.

Creation operators for either second quantized fermions or bosons are tensor products, $\ast_T$, of the Clifford odd $2^{d-1} \times 2^{d-1}$ "basis vectors" describing the internal space of fermions or of the Clifford even $2^{d-1} \times 2^{d-1}$ "basis vectors" describing the internal space of bosons, and of an (continuously) infinite number of basis in ordinary momentum (or coordinate) space.

Creation operators for fermions, represented by anticommuting "basis vectors" in a tensor product, $\ast_T$, with the basis in ordinary space, anticommute among themselves and with $\gamma^a$'s, creation operators for bosons, represented by commuting "basis vectors" in a tensor product, $\ast_T$, with the basis in ordinary space, commute among themselves and with $\gamma^a$'s.

"Basis vectors" describing the internal space of fermion and boson fields transfer their anticommutativity or commutativity into creation operators, since the basis in ordinary space commute.

One irreducible representation of "basis vectors", reachable with $S^{ab}$, and determining the internal space of fermions in $d = (13 + 1)$-dimensional space, includes quarks and leptons and antiquarks and antileptons, together with the right handed neutrinos and the left handed antineutrinos, as can be seen in Table \[4\]. There are 64 ($= 2^4 - 1$) members of one irreducible representation, represented as the eigenvectors of the Cartan subalgebra of the $SO(13 + 1)$ group, analysed with respect to the subgroups $SO(3, 1), SU(2), SU(2), SU(3)$ and $U(1)$, with 7 commuting operators \[4\].

These Clifford odd anticommuting "basis vectors" transfer the anti-commutativity to the creation operators for quarks and leptons and antiquarks and antileptons.

The standard model subgroups ($SO(3, 1), SU(2), SU(3), U(1)$) have one $SU(2)$ group less, and correspondingly the right handed neutrinos and the left handed antineutrinos, having no charge, are in the standard model assumed not to exist.

$S^{ab}$'s transform each member of one irreducible representation of fermions into all the members of the same irreducible representation, $\tilde{S}^{ab}$'s transform each member of one irreducible representation to the same member of another irreducible representation. The postulate, presented in Eq. (16), equips each irreducible representation with the family quantum number.

The mass terms appear in the spin-charge-family theory after the scalar fields with the space index $s \geq 5$ ($s = (7, 8)$ indeed), which are the gauge fields of the two kinds of the spin connection fields $\omega_{\alpha\beta}$ and $\tilde{\omega}_{\alpha\beta}$ (the gauge fields of $S^{ab}$ and $\tilde{S}^{ab}$, respectively), gain the constant values, what makes particular charges (hypercharge $Y = \tau^4 + \tau^{23}$ and the weak charge $\tau^{13}$, explained in Table \[4\] non conserved quantities. The appearance of the mass term in $d = (3 + 1)$ is discussed in Subsect. 6.1, 6.2.2, 7.3 and 7.4 in Ref. \[4\].

\[3\]Since the multiplication with either $\gamma^a$'s or $\tilde{\gamma}^a$'s changes the Clifford odd "basis vectors" into the Clifford even "basis vectors", and even "basis vectors" commute, the action for fermions can not include an odd numbers of $\gamma^a$'s or $\tilde{\gamma}^a$'s, what the simple starting action of Eq. (1) does not. In the starting action $\gamma^a$'s and $\tilde{\gamma}^a$'s appear as $\gamma^0 \gamma^a \hat{p}_a$ or as $\gamma^0 \gamma^a S^{ab} \omega_{abc}$ and as $\gamma^0 \gamma^a \tilde{S}^{ab} \tilde{\omega}_{abc}$.

\[4\]The $SO(7, 1)$ part is identical for quarks of any of the three colours and for the colourless leptons, and identical for antiquarks and the colourless antileptons. They differ only in the $SO(6)$ part of the group $SO(13, 1)$.
The "basis vectors", represented by the superposition of even products $\gamma^a$’s (with an even number of nilpotents and the rest of projectors, the eigenvectors of the Cartan subalgebra members) have properties of the internal space of boson fields:

i. $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members of the Clifford even "basis vectors" are Hermitian conjugated to each other or are self adjoint. They appear in two separated groups.

ii. Any member of these two groups of the Clifford even "basis vectors" carries with respect to $S^{ab}(= S^{ab} + \tilde{S}^{ab})$ the quantum numbers in the adjoint representations — either spins of the group $SO(13, 1)$ or spins and charges with respect to the subgroups $SO(3, 1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$ of the group $SO(13, 1)$, if we make this analyses.

iii. The algebraic application of even "basis vectors" of one of the two groups on the odd "basis vectors" (representing the internal space of fermion fields) transforms members of one odd irreducible representation into all the other members of the same representation, keeping the family quantum number unchanged.

iv. The algebraic application of an even "basis vector" to another even "basis vector" of the same group leads to an even "basis vectors" (or zero).

v. The algebraic application of any even "basis vector" to an even "basis vector" of the second group gives or zero.

The properties of even "basis vectors" are demonstrated in the Subsects. (2.4.2, 2.5) of this section.

### 2.1 Grassmann and Clifford algebras

The internal space of fermions and bosons can be described by using either the Grassmann or the Clifford algebras. A part of this section, the one which concerns the Clifford odd "basis vectors" is a short overview of the Subsect. 3.2, of Ref. [4].

In Grassmann $d$-dimensional space there are $d$ anticommuting operators $\theta^a$, and $d$ anticommuting operators which are derivatives with respect to $\theta^a$, $\frac{\partial}{\partial \theta^a}$,

\[
\{\theta^a, \theta^b\} = 0, \quad \{\frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b}\} = 0, \\
\{\theta^a, \frac{\partial}{\partial \theta^b}\} = \delta_{ab}, (a, b) = (0, 1, 2, 3, 5, \cdots, d).
\]  

Defining

\[
(\theta^a)^\dagger = \eta^{ab} \frac{\partial}{\partial \theta^a}, \quad \text{leads to} \quad (\frac{\partial}{\partial \theta^a})^\dagger = \eta^{ab}\theta^a,
\]

with $\eta^{ab} = \text{diag}\{1, -1, -1, \cdots, -1\}$.

$\theta^a$ and $\frac{\partial}{\partial \theta^a}$ are, up to the sign, Hermitian conjugated to each other. The identity is the self adjoint member of the algebra. We make the choice for the following complex properties of $\theta^a$, and correspondingly of $\frac{\partial}{\partial \theta^a}$,

\[
\{\theta^a\}^* = (\theta^0, \theta^1, -\theta^2, \theta^3, -\theta^5, \theta^6, \cdots, -\theta^{d-1}, \theta^d),
\]

$^5$The Clifford odd "basis vectors" presented in Table 4 are orthogonal, that means that the algebraic, $\star_A$, product of any two "basis vectors" is equal to zero. The Hermitian conjugated partners of "basis vectors", contributing to annihilation operators, form a separated group of $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members.

The Clifford odd "basis vectors" form in a tensor product with the basis in ordinary space the creation operators which determine, applying on the vacuum state, the Hilbert space of fermions. The Clifford even "basis vectors", applying algebraically on the Clifford odd "basis vectors", "offer" the interaction among fermions, transforming one "basis vector" into the other, Subsect. 2.4.2, manifesting properties of the boson fields.
\[ \{ \frac{\partial}{\partial \theta_a} \}^\prime = (\frac{\partial}{\partial \theta_0}, \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial \theta_3}, \frac{\partial}{\partial \theta_4}, \frac{\partial}{\partial \theta_5}, \frac{\partial}{\partial \theta_6}, \ldots, \frac{\partial}{\partial \theta_{d-1}}, \frac{\partial}{\partial \theta_d}) \].

The operators \(\theta^a\) are offering \(2^d\) superposition of products of \(\theta^a\), the Hermitian conjugated partners of which are the corresponding superposition of products of \(\frac{\partial}{\partial \theta_a}\).

In \(d\)-dimensional space of anticommuting Grassmann coordinates and of their Hermitian conjugated partners derivatives, Eqs. (2,3), there exist two kinds of the Clifford algebra elements (operators) — \(\gamma^a\) and \(\tilde{\gamma}^a\) — both expressible in terms of \(\theta^a\) and their conjugate momenta \(p^a = i \frac{\partial}{\partial \theta_a}\) [2].

\[ \gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a}), \quad \tilde{\gamma}^a = i (\theta^a - \frac{\partial}{\partial \theta_a}), \]
\[ \theta^a = \frac{1}{2} (\gamma^a - i \tilde{\gamma}^a), \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2} (\gamma^a + i \tilde{\gamma}^a), \]

offering together \(2 \cdot 2^d\) operators: \(2^d\) of those which are products of \(\gamma^a\) and \(2^d\) of those which are products of \(\tilde{\gamma}^a\). Taking into account Eqs. (3,4) it is easy to prove that they form two anticommuting Clifford subalgebras, \(\{\gamma^a, \gamma^b\}_+ = 0\), Refs. (4 and references therein)

\[ \{\gamma^a, \gamma^b\}_+ = 2 \eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+, \]
\[ \{\gamma^a, \tilde{\gamma}^b\}_+ = 0, \quad (a, b) = (0, 1, 2, 3, 5, \ldots, d), \]
\[ (\gamma^a)_\dagger = \eta^{aa} \gamma^a, \quad (\tilde{\gamma}^a)_\dagger = \eta^{aa} \tilde{\gamma}^a. \]

While the Grassmann algebra offers the description of the ”anticommuting integer spin second quantized fields” and of the ”commuting integer spin second quantized fields” [27], the Clifford algebras which are superposition of odd products of either \(\gamma^a\)’s or \(\tilde{\gamma}^a\)’s, offer the description of the second quantized half integer spin fermion fields, and from the point of view subgroups of \(SO(d - 1, 1)\) group manifest spins and charges of fermions in the fundamental representations of the group and subgroups.

The superposition of even products of either \(\gamma^a\)’s or \(\tilde{\gamma}^a\)’s offer the description of the commuting second quantized boson fields with integer spins, as we shall see in this contribution and also in [6], which from the point of the subgroups of the \(SO(d - 1, 1)\) group manifest spins and charges in the adjoint representations of the group and subgroups.

The reduction, Eq. (6) of Subsect. (2.3), of the two Clifford algebras — \(\gamma^a\)’s and \(\tilde{\gamma}^a\)’s — to only one — \(\gamma^a\)’s are chosen — reduces the possibilities to describe either fermions or bosons to only one possibility.

After the decision that only \(\gamma^a\)’s are used to describe the internal space of fermions, the remaining ones, \(\tilde{\gamma}^a\)’s, equip the irreducible representations of the Lorentz group (with the infinitesimal generators \(S^{ab} = \frac{i}{4} \{\gamma^a, \gamma^b\}_\cdot\) with the family quantum numbers (determined by \(\tilde{S}^{ab} = \frac{i}{4} \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_\cdot\)).

The even Clifford algebra objects — which are the superposition of even products of \(\gamma^a\)’s — offer after the reduction of the two subalgebras to only one the description of the second quantized boson fields as the gauge fields of the second quantized fermion fields, the internal space of which are described by the odd Clifford algebra objects. The reduction enables to define the generators of the Lorentz transformations in the internal space of bosons as \(S^{ab} = S^{ab} + \tilde{S}^{ab}\), manifesting the adjoint representation properties of the Clifford even ”basis vectors”.

Both, Clifford odd and Clifford even ”basis vectors”, are discussed in what follows. The Clifford odd ”basis vectors” form \(2^d - 1\) families, each family has \(2^d - 1\) members. Their Hermitian conjugated partners form another group of \(2^d - 1 \times 2^d - 1\) members. The Clifford even ”basis vectors” form two groups of \(2^d - 1 \times 2^d - 1\) members. Members of any of the two groups have the Hermitian conjugated partners within the same group or are self adjoint.
2.2 "Basis vectors" are superposition of either odd or even products of Clifford objects $\gamma^a$'s.

In this subsection the properties of the Clifford odd and the Clifford even "basis vectors" are discussed, the first ones offering the description of the internal space of fermions, the second ones offering the description of the internal space of bosons.

Let us choose the "basis vectors" of either the Clifford odd or of the Clifford even character to be eigenvectors of all the Cartan subalgebra members of the Lorentz algebra. There are $d^2$ members of the Cartan subalgebra in even dimensional spaces. One can choose

$$\mathbf{S}^{03}, \mathbf{S}^{12}, \mathbf{S}^{56}, \ldots, \mathbf{S}^{d-1\, d},$$
$$\tilde{\mathbf{S}}^{03}, \tilde{\mathbf{S}}^{12}, \tilde{\mathbf{S}}^{56}, \ldots, \tilde{\mathbf{S}}^{d-1\, d},$$

$$\mathbf{S}^{ab} = S^{ab} + \tilde{S}^{ab}. \quad (6)$$

Let us look for the eigenvectors of each of the Cartan subalgebra members, Eq. (6), for the Grassmann algebra (with $\mathbf{S}^{ab} = i \left( \theta^a \frac{\partial}{\partial \theta^b} - \theta^b \frac{\partial}{\partial \theta^a} \right)$) and for each of the two kinds of the Clifford subalgebras separately, and let $\mathbf{S}^{ab}, \tilde{\mathbf{S}}^{ab}, \tilde{\mathbf{S}}^{ab}$, be one of the $d^2$ possibilities of $(ab = 03, 12, 56, \ldots, d-1\, d)$.

$$\mathbf{S}^{ab} \frac{1}{\sqrt{2}} (\theta^a + \eta^{aa}_{ik} \theta^b), \quad \mathbf{S}^{ab} \frac{1}{\sqrt{2}} (1 + i \frac{\theta^a \theta^b}{k}) = 0, \quad \text{or} \quad \mathbf{S}^{ab} \frac{1}{\sqrt{2}} i \frac{\theta^a \theta^b}{k} = 0,$$

$$\tilde{\mathbf{S}}^{ab} \frac{1}{2} (\gamma^a + \eta^{aa}_{ik} \gamma^b), \quad \tilde{\mathbf{S}}^{ab} \frac{1}{2} (1 + i \frac{\gamma^a \gamma^b}{k}) = \frac{k}{2} \left(1 + i \frac{\gamma^a \gamma^b}{k}\right),$$

with $k^2 = \eta^{aa} \eta^{bb}$. The proof of Eq. (7) is presented in Ref. [3], App. (I).

Let us use for the two Clifford subalgebras for the nilpotents $\frac{1}{2} (\gamma^a + \eta^{aa}_{ik} \gamma^b), (\frac{1}{2} (\gamma^a + \eta^{aa}_{ik} \gamma^b))^2 = 0$ and projectors $\frac{1}{2} (1 + i \frac{\gamma^a \gamma^b}{k}), (\frac{1}{2} (1 + i \frac{\gamma^a \gamma^b}) = 0$ the notation

$$\gamma^{ab} (k) = \frac{1}{2} (\gamma^a + \eta^{aa}_{ik} \gamma^b), \quad [k] := \frac{1}{2} (1 + i \frac{\gamma^a \gamma^b}{k}), \quad \gamma^{ab} ([k]) = \frac{1}{2} (1 + i \frac{\gamma^a \gamma^b}{k}). \quad (8)$$

One can derive after taking into account Eq. (5) the following useful relations

$$\gamma^{ab} (k) = \eta^{ab} [-k], \quad \gamma^a (k) = -ik \eta^{ab} [-k], \quad \gamma^a [k] = (k), \quad \gamma^b [k] = -ik \eta^{ab} (-k),$$

$$\gamma^{ab} (k) = \eta^{ab} (-k), \quad (\gamma^a (-k))^2 = 0, \quad \gamma^a (k) = \eta^{ab} (k),$$

$$\gamma^{ab} (k) = \eta^{ab} [k], \quad (\gamma^a [k])^2 = \eta^{ab} [k], \quad \gamma^a (k) = \eta^{ab} [k] = [k], \quad [k] = 0,$$

$$\gamma^a (k) = 0, \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0,$$

$$\gamma^{ab} (k) = \eta^{ab} (-k), \quad \gamma^{ab} (k) = \eta^{ab} (k), \quad \gamma^{ab} (k) = \eta^{ab} [k], \quad \gamma^{ab} (k) = \eta^{ab} [k], \quad \gamma^{ab} (k) = \eta^{ab} [k],$$

$$\gamma^{ab} (k) = \eta^{ab} [k], \quad (\gamma^a [k])^2 = [k], \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0, \quad \gamma^a (k) = 0. \quad (9)$$
One can define "basis vectors" to be eigenvectors of all the members of the Cartan subalgebras as either odd or even products of nilpotents and of projectors in any even dimensional space.

A. Clifford odd "basis vectors" members

The Clifford odd "basis vectors" must be products of an odd number of nilpotents, the Clifford even "basis vectors" must be products of an even number of nilpotents. In both cases the rest factors of the "basis vectors" form projectors.

Let us define in \( d \)-dimensional space, \( d = 2(2n + 1) \) and \( d = 4n \), first the Clifford odd "basis vectors", which are eigenvectors of the Cartan subalgebra members, Eq. (6), and are superposition of Clifford odd products of \( \gamma^a \)'s.

Naming the Clifford odd "basis vectors" with \( \hat{b}_m^f \), where \( m \) determines membership in one irreducible representation and \( f \) an irreducible representation it follows

\[
\begin{align*}
\hat{b}_1^\dagger &= 0^{3} \frac{12}{(i)(+)} \cdots (+) , \\
\hat{b}_2^\dagger &= 0^{3} \frac{12}{56} \frac{d-1}{d} (-i)[-] \cdots (+) , \\
\hat{b}_3^\dagger &= (i) (+) (+) \cdots [-] [-] , \\
\hat{b}_4^\dagger &= (i) (+) \cdots (+) ,
\end{align*}
\]

(10)

The first line in \( d = 2(2n + 1) \) case is the product of nilpotents only and correspondingly a superposition of an odd products of \( \gamma^a \)'s. The second one belongs to the same irreducible representation as the first one, since it follows from the first one by the application of \( S_{01} \), for example, the third one follows from the first one by the application of \( S_{d-3d-1} \). One can continue in this way to generate all the \( 2^{d-1} \) members of this irreducible representation.

In \( (d = 4n) \)-dimensional space we start with the maximum odd number of nilpotents, what requires one projector. Then we repeat the same procedure as in the case of \( d = 2(2n + 1) \).

Since \( S_{ab} \) changes two nilpotents into projectors (or one nilpotent and one projector into projector and nilpotent) the number of nilpotents in the whole irreducible representation remains odd.

Let \( \hat{b}_f^m = (\hat{b}_f^m)^\dagger \) denotes the Hermitian conjugated partner of the "basis vector" \( \hat{b}_f^m \).

Hermitian conjugated partners \( \hat{b}_f^m = (\hat{b}_f^m)^\dagger \) follow by taking into account that projectors are self adjoint, while nilpotents change sign according to Eq. (9).

Since \( S_{ab} \) transform two nilpotents into projectors it becomes clear that the group of Hermitian conjugated partners, which have an odd number of nilpotents \( k \) transformed into \( -k \), is separated from the group of "basis vectors".

It is not difficult to prove, just by the algebraic multiplication, that all the Clifford odd "basis vectors" are orthogonal

\[
\hat{b}_f^m \cdot A \hat{b}_f^{m'} = 0 , \quad \hat{b}_f^m \cdot A \hat{b}_f^{m'} = 0 , \quad \forall (m, m', f, f') ,
\]

(11)
while the relation \( \hat{b}_f^m \ast_A \hat{b}_f^{m\dagger} = \delta^{mm'}\delta_{ff'} \) becomes true after each irreducible representation is equipped with the family quantum number.

The orthogonal relations among the "basis vectors" \( \hat{b}_f^m \) and among their Hermitian conjugated partners \( \hat{b}_f^{m\dagger} \) which are superposition of the Clifford odd products of \( \tilde{\gamma}^a \)'s, follow by replacing in Eq. (11) \( \hat{b}_f^{m\dagger} \) by \( \hat{b}_f^m \) and \( \hat{b}_f^m \) by \( \hat{b}_f^{m\dagger} \).

It is not difficult to prove the anticommutation relations of the Clifford odd "basis vectors" and their Hermitian conjugated partners for both Clifford subalgebras ([4, Sect. 5] and references therein). Let us here present the anticommutation relations of only the one of the Clifford subalgebras — \( \gamma^a \)'s.

\[
\begin{align*}
\hat{b}_f^m \ast_A |\psi_{oc}\rangle &= 0 |\psi_{oc}\rangle, \\
\hat{b}_f^{m\dagger} \ast_A |\psi_{oc}\rangle &= |\psi_f\rangle, \\
\{\hat{b}_f^m, \hat{b}_f^{m\dagger}\}_A |\psi_{oc}\rangle &= 0 |\psi_{oc}\rangle, \\
\{\hat{b}_f^{m\dagger}, \hat{b}_f^m\}_A |\psi_{oc}\rangle &= 0 |\psi_{oc}\rangle, \\
\{\hat{b}_f^m, \hat{b}_f^{m\dagger}\}_A |\psi_{oc}\rangle &= \delta^{mm'} |\psi_{oc}\rangle,
\end{align*}
\]

(12)

where \( \ast_A \) represents the algebraic multiplication of \( \hat{b}_f^{m\dagger} \) and \( \hat{b}_f^m \) among themselves and with the vacuum state \( |\psi_{oc}\rangle \) of Eq. (13), which takes into account Eq. (19),

\[
|\psi_{oc}\rangle = \sum_{f=1}^{2^d-1} \hat{b}_f^m \ast_A \hat{b}_f^{m\dagger} |1\rangle,
\]

(13)

for one of the members \( m \), anyone, of the odd irreducible representation \( f \), with \( |1\rangle \), which is the vacuum without any structure — the identity. It follows that \( \hat{b}_f^m |\psi_{oc}\rangle = 0 \).

The relations are valid for both kinds of the odd Clifford subalgebras. To obtain the equivalent relations for the "basis vectors" \( \hat{b}_f^m \) we only have to replace \( \hat{b}_f^m \) by \( \hat{b}_f^{m\dagger} \) and equivalently for the Hermitian conjugated partners.

The Clifford odd "basis vectors" \( \gamma^a \) almost fulfill the second quantization postulates for fermions. There is, namely, the property which the second quantized fermions must fulfill in addition to the relations of Eq. (12) — the following property

\[
\{\hat{b}_f^m, \hat{b}_f^{m\dagger}\}_A |\psi_{oc}\rangle = \delta^{mm'} \delta_{ff'} |\psi_{oc}\rangle,
\]

(14)

---

6Since all the "basis vectors" are orthogonal and so are their Hermitian conjugated partners, Eq. (11), one could conclude that the anticommutation relations could as well be replaced by commutation relations. However, the properties of the Clifford odd and of the Clifford even "basis vectors" (we present the properties of the Clifford even "basis vectors" in what follows) [12] are very different: a. The Clifford odd "basis vectors" have their Hermitian conjugated partners in another group of \( 2^d-1 \) members of each of \( 2^d-1 \) irreducible representations. b. The Clifford odd "basis vectors" have with respect to the eigenvectors of the Cartan subalgebra members, Eq. (9), as well as with respect to the superposition of the Cartan subalgebra members, properties of fermions, that is they manifest the fundamental representations of the \( SO(d − 1, 1) \) group and of subgroups of this group. c. When the Clifford even "basis vectors" apply on the Clifford odd "basis vectors" (in an algebraic product \( \ast_A \)) the Clifford even "basis vectors" transform the Clifford odd "basis vectors" into another Clifford odd "basis vectors", conserving the eigenvalues of the Cartan subalgebras of the internal spaces of Clifford odd and Clifford even "basis vectors" under the recognition that Clifford even "basis vectors" carry integer spins. d. The algebraic products of the Clifford even "basis vectors" make new Clifford even "basis vectors", conserving bosonic quantum numbers (that is integer spins of the Clifford even "basis vectors"). e. The application of the anticommutation relations on the Hilbert space constructed out of the Clifford odd "basis vectors" in the tensor product with the ordinary basis, have the properties of the second quantized fermions ([4, Sect. 5]).

7The Clifford odd "basis vectors" in a tensor product with the ordinary basis form creation operators for any of the fermion states. The Hilbert space is constructed from creation operators of single fermion states of all possible "basis vectors" of all possible momentum \( \vec{p} \) as discussed in Sect. 5 of Ref. [4].
for either $\gamma^a$ or $\tilde{\gamma}^a$. For any $\hat{b}_f^m$ and any $\hat{b}_f^{m\dagger}$ this is not the case. It turns out that besides $\hat{b}_f^{m=1} = (-)\cdots(-)(-)(-i)$, for example, also $\hat{b}_f^{m'} = (-)\cdots(-)[+] [+i]$ and several others give, when applied on $\hat{b}_f^{m=1\dagger}$, non zero contributions. There are namely $2^d - 1$ annihilation operators for each creation operator which give, applied on the creation operator, non zero contribution.

The problem is solvable by the reduction of the two Clifford odd algebras to only one [12, 8, 5, 4] as it is presented in Subsect. 2.3. If $\gamma^a$’s are chosen to determine internal space of fermions the remaining ones, $\tilde{\gamma}^a$’s, determine quantum numbers of each family (described by the eigenvalues of $S^{ab}$ of the Cartan subalgebra members).

Correspondingly the creation operators expressible as tensor products, $*_T$, of the Clifford odd ”basis vectors”, $\hat{b}_f^{m\dagger}$, and the basis in ordinary (momentum or coordinate) space and their Hermitian conjugated partners annihilation operators fulfil the anticommutation relations for the second quantized fermion fields, explaining the postulates of Dirac for the second quantized fermion fields.

**B. Clifford even ”basis vectors”**

Let us denote the Clifford even ”basis vectors” as $i\hat{A}_f^m$, where $i = (I, II)$ points out that there are two groups of Clifford even ”basis vectors” which can not be transformed into one another by the generators of the Lorentz transformation in the space of ”basis vectors”, since each member of the group $I$ has a member in the group $II$ which differ from the member of the group $I$ only in the sign of either one nilpotent or of one projector. All the members within the group, either $i = I$ or $i = II$, are reachable by the generators of the Lorentz transformations in the space of ”basis vectors”. We make a choice correspondingly

$$d = 2(2n + 1)$$

$$I\hat{A}_{1\dagger}^1 = (+i)(+) \cdots [+] ,$$

$$I\hat{A}_{2\dagger}^1 = [-i][-](+) \cdots [+] ,$$

$$I\hat{A}_{3\dagger}^1 = (+)(+) \cdots [-] [-] ,$$

$$I\hat{A}_{1\dagger}^2 = (+i)(+) \cdots (+) ,$$

$$I\hat{A}_{2\dagger}^2 = [-i][-i](+) \cdots (+) ,$$

$$I\hat{A}_{3\dagger}^2 = (+)(+) \cdots [-] [-] ,$$

$$II\hat{A}_{1\dagger}^1 = (-i)(+) \cdots [+] ,$$

$$II\hat{A}_{2\dagger}^1 = [+i][-](+) \cdots [+] ,$$

$$II\hat{A}_{3\dagger}^1 = (-i)(+) \cdots [-] [-] ,$$

$$II\hat{A}_{1\dagger}^2 = (+i)(+) \cdots (+) ,$$

$$II\hat{A}_{2\dagger}^2 = [+i][-i](+) \cdots (+) ,$$

$$II\hat{A}_{3\dagger}^2 = (-i)(+) \cdots [-] [-] .$$

The members of the group $II$ follow from the members of group $I$, Eq. (15), if we replace in $I\hat{A}_{1\dagger}^1$ the nilpotent $(+i)$ by $(-i)$, and in $I\hat{A}_{1\dagger}^2$ the projector $[-i]$ by $[+i]$, and so on.

The Clifford even ”basis vectors” have an even number of nilpotents which change sign under the Hermitian conjugation while projectors are self adjoint according to Eq. (8). Correspondingly the Hermitian conjugated partners of the Clifford even ”basis vectors” ($i\hat{A}_f^{m\dagger}$) belong to the same group $i$: $i\hat{A}_f^{m\dagger}$, with $2^d - 1 \times 2^d - 1$ members. (Because of this property the sign $\dagger$ has no special meaning.)

The creation operators, they are tensor products, $*_T$, of the Clifford even ”basis vectors” (chosen
to be the eigenvectors of the Cartan subalgebra, Eq. (6)) and the basis in ordinary space (momentum or coordinate) carry after the reduction of the two Clifford subalgebras into only one, Subsect. 2.3, the Cartan subalgebra eigenvalues of \( S^{ab} = S^{ab} + \tilde{S}^{ab} \). They fulfill the commutation relations of the second quantized boson gauge fields of the corresponding fermion fields.

Let us say:
After the reduction of the two Clifford subalgebras to only one — the one of \( \gamma^a \)’s — are \( 2^{d/2} - 1 \) members of the Clifford even "basis vectors" \( \hat{A}_m^\dagger \) reachable from any other \( \hat{A}_m^\dagger \) either by \( S^{ab} \)'s or by \( \tilde{S}^{ab} \)'s or by both, and have their Hermitian conjugated partners within the same group \( i \), for ether \( i = I \) or \( i = II \).
The eigenvalues of the Cartan subalgebra are for the Clifford even "basis vectors", after the reduction of the two Clifford subalgebras to only one determined by \( S^{ab} = S^{ab} + \tilde{S}^{ab} \).

Contrary, the Clifford odd \( 2^{d/2} - 1 \) members of each of the \( 2^{d/2} - 1 \) irreducible representations of "basis vectors" have their Hermitian conjugated partners in another set of \( 2^{d/2} \cdot 2^{d/2} - 1 \) "basis vectors".

We shall demonstrate properties of the Clifford odd and Clifford even "basis vectors" on the toy model in \( d = (5 + 1) \) in Subsect. 2.4 in details.

2.3 Reduction of the Clifford space

The creation and annihilation operators of an odd Clifford algebra of both kinds, of either \( \gamma^a \)'s or \( \tilde{\gamma}^a \)'s, turn out to obey the anticommutation relations for the second quantized fermions, postulated by Dirac [4], provided that each of the irreducible representations of the corresponding Lorentz group, describing the internal space of fermions, carry a different quantum number.

But we know that a particular member \( m \) has for all the irreducible representations the same quantum numbers, that is the same "eigenvalues" of the Cartan subalgebra (for the vector space of either \( \gamma^a \)'s or \( \tilde{\gamma}^a \)'s), Eq. (8).

There is a possibility to "dress" each irreducible representation of one kind of the two independent vector spaces with a new, let us call it the "family" quantum number, if we "sacrifice" one of the two vector spaces.

Let us "sacrifice" \( \tilde{\gamma}^a \)'s, using \( \tilde{\gamma}^a \)'s to define the "family" quantum numbers for each irreducible representation of the vector space of "basis vectors" of an odd products of \( \gamma^a \)'s, while keeping the relations of Eq. (19) unchaged:
\[
\{ \gamma^a, \gamma^b \} = 2\eta^{ab} = \{ \tilde{\gamma}^a, \tilde{\gamma}^b \}, \quad \{ \gamma^a, \tilde{\gamma}^b \} = 0, \quad (\gamma^a)^\dagger = \eta^{aa} \gamma^a, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa} \tilde{\gamma}^a,
\]
\((a,b) = (0,1,2,3,5,\cdots,d)\). The proof that relations of Eq. (19) remain valid after the reduction of the internal space of odd "basis vectors" is presented in [4] in App. I, Statements 2. and Statements 3..

We therefore require:
Let \( \tilde{\gamma}^a \)'s operate on \( \gamma^a \)'s as follows [15, 2, 22, 24, 25]
\[
\{ \tilde{\gamma}^a B = (-)^B i B \gamma^a \} |\psi_{oc}>, \tag{16}
\]
with \((-)^B = -1\), if \( B \) is (a function of) an odd products of \( \gamma^a \)'s, otherwise \((-)^B = 1\) [15], \(|\psi_{oc}>\) is defined in Eq. (13).

After the postulate of Eq. (16) the "basis vectors" which are superposition of an odd products of \( \gamma^a \)'s obey all the postulates of Dirac for the second quantized fermion fields, presented in Eqs. (14, 12).

Each irreducible representation of the odd "basis vectors" are after the postulate of Eq. (16) equipped by the quantum numbers of the Cartan subalgebra members of \( S^{ab} \), Eq. (6).
The eigenvalues of the operators $S^{a b}$ and $\tilde{S}^{a b}$ on nilpotents and projectors of $\gamma^a$’s are after the reduction of Clifford space equal to

$$
\begin{align*}
S^{a b}(k) &= \frac{k}{2} \delta^{a b}(k), & \tilde{S}^{a b}(k) &= \frac{k}{2} \delta^{a b}(k), \\
S^{a b}(k)[k] &= \frac{k}{2} \delta^{a b}[k], & \tilde{S}^{a b}(k)[k] &= -\frac{k}{2} \delta^{a b}[k],
\end{align*}
$$

(17)
demonstrating that the eigenvalues of $S^{a b}$ on nilpotents and projectors of $\gamma^a$’s differ from the eigenvalues of $\tilde{S}^{a b}$ on $\gamma^a$’s, so that $\tilde{S}^{a b}$ can be used to equip irreducible representations of $S^{a b}$ with the ”family” quantum number.

After this postulate the vector space of $\tilde{\gamma}^a$’s is ”frozen out”. No vector space of $\tilde{\gamma}^a$’s needs to be taken into account any longer for the description of the internal space of either fermions or bosons, in agreement with the observed properties of fermions. The operators $\tilde{\gamma}^a$’s determine from now on properties of fermion and boson ”basis vectors” written in terms of odd and even numbers of $\gamma^a$’s, respectively: i. The odd products of the Clifford objects $\gamma^a$’s offer the description of the internal space of fermions, ii. The even products of the Clifford objects $\gamma^a$’s offer the description of the internal space of bosons, which are the gauge fields of the fermions.

We shall demonstrate in Sect. 2.4.2 that the Clifford even ”basis vectors”, equipped by the sum of both quantum numbers, $S^{a b}$ and $\bar{S}^{a b}$, $\bar{S}^{a b} = S^{a b} + \tilde{S}^{a b}$, obey the boson second quantized postulates.

Let us present some useful relations following if using Eq. (16),

$$
\begin{align*}
\tilde{\gamma}^a \delta^{a b}(k) &= -i \eta^{a a} \delta^{a b}[k], & \tilde{\gamma}^b \delta^{a b}(k) &= -k \delta^{a b}[k], & \tilde{\gamma}^a \delta^{a b}(k) &= i \delta^{a b}(k), & \tilde{\gamma}^b \delta^{a b}(k) &= -k \eta^{a a} \delta^{a b}(k), \\
\delta^{a b}(k) &= 0, & \delta^{a b}(-k)(k) &= -i \eta^{a a} \delta^{a b}[k], & \delta^{a b}(k)[k] &= i \delta^{a b}(k), & \delta^{a b}(-k)[k] &= 0, \\
\delta^{a b}(k)[k] &= \delta^{a b}(k), & \delta^{a b}(-k)(k) &= 0, & \delta^{a b}(k)[k] &= 0, & \delta^{a b}(-k)[k] &= 0.
\end{align*}
$$

(18)

To point out that the anticommuting properties of $\gamma^a$’s and $\tilde{\gamma}^a$’s remain valid also after the reduction of the two Clifford algebras to only one, Eq. (16), the commutation relations of Eq. (5) are repeated here again

$$
\begin{align*}
\{\gamma^a, \tilde{\gamma}^b\} &= 2 \eta^{a b} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}, \\
\{\gamma^a, \tilde{\gamma}^b\} &= 0, & (a, b) &= (0, 1, 2, 3, 5, \cdots, d), \\
(\gamma^a) &= \eta^{a a} \gamma^a, & (\tilde{\gamma}^a) &= \eta^{a a} \tilde{\gamma}^a, \gamma^a \tilde{\gamma}^a &= \eta^{a a}, & \tilde{\gamma}^a (\gamma^a) &= I, & \tilde{\gamma}^a \gamma^a &= \eta^{a a}, & \tilde{\gamma}^a (\tilde{\gamma}^a) &= I.
\end{align*}
$$

(19)

The proof can be found in (4, App. I, Statements 2. and 3.). Taking into account the anticommuting properties of $\gamma^a$’s and $\tilde{\gamma}^a$’s, Eq. (19), it is not difficult to prove the relations in Eq. (18).

2.4 Properties of Clifford odd and even ”basis vectors” in $d = (5 + 1)$

To clear up the properties of the Clifford odd and Clifford even ”basis vectors” their properties in $d = (5 + 1)$-dimensional space are presented as:

i. The odd products of the Clifford objects $\gamma^a$’s, offering the description of the internal space of fermion fields,

ii. The even products of the Clifford objects $\gamma^a$’s, offering the description of the internal space of boson
fields, which are the gauge fields of the fermion fields, the internal space of which is described by the Clifford odd "basis vectors".

The properties of the Clifford odd and Clifford even "basis vectors" are analysed not only with respect to the group $SO(5, 1)$ but also with respect to the subgroups $SO(3, 1) \times U(1)$ and $SU(3) \times U(1)$ of the group $SO(5, 1)$.

Choosing the "basis vectors" to be eigenvectors of all the members of the Cartan subalgebra of the Lorentz algebra and correspondingly writing them as the products of nilpotents $ab$ and projectors $ab^2$ and projectors $[+] ([+] = [+])$, each of nilpotent or projector is chosen to be the eigenvector of one of the Cartan subalgebra members of the Lorentz algebra in the internal space of fermions (the Clifford odd "basis vectors") or bosons (the Clifford even "basis vectors"), one finds the Clifford odd and the Clifford even "basis vectors" as presented in Table 1. The table presents besides the "basis vectors" and their Hermitian conjugated partners, their eigenvalues of the Cartan subalgebra members of the Lorentz algebra in the internal space of fermions (the Clifford odd "basis vectors") or bosons (the Clifford even "basis vectors"), one finds the Clifford odd and the Clifford even "basis vectors" as presented in Table 1. The table presents besides the "basis vectors" also their eigenvalues of the Cartan subalgebra members of $S^{ab}$ and $\bar{S}^{ab}$ for the group $SO(5, 1)$ and the handedness $\Gamma^{(5+1)}$ and $\Gamma^{(3+1)}$, representing handedness in $d = (5 + 1)$ and $d = (3 + 1)$, respectively.

The odd I group is presenting the "basis vectors" which are products of an odd number of nilpotents (three or one) and of projectors (none or two), offering the description of the internal space of fermion fields, which are the gauge fields of the fermion fields, the internal space of which is described by the Clifford odd "basis vectors". The groups even I and even II present commuting Clifford even "basis vectors", with an even number of nilpotents (none or two) and of projectors (none or two), offering the description of the internal space of boson fields, which are the gauge fields of the corresponding "family" quantum numbers) (the Clifford even "basis vectors") or bosons (the Clifford even "basis vectors"), one finds the Clifford odd and the Clifford even "basis vectors" as presented in Table 1. The table presents besides the "basis vectors" also their eigenvalues of the Cartan subalgebra members of the Lorentz algebra in the internal space of fermions (the Clifford odd "basis vectors") or bosons (the Clifford even "basis vectors"), one finds the Clifford odd and the Clifford even "basis vectors" as presented in Table 1. The table presents besides the "basis vectors" also their eigenvalues of the Cartan subalgebra members of $S^{ab}$ and $\bar{S}^{ab}$ for the group $SO(5, 1)$ and the handedness $\Gamma^{(5+1)}$ and $\Gamma^{(3+1)}$, representing handedness in $d = (5 + 1)$ and $d = (3 + 1)$, respectively.

The even I group is presenting the "basis vectors" which are products of an odd number of nilpotents (three or one) and of projectors (none or two), offering the description of the internal space of fermion fields, which are the gauge fields of the fermion fields, the internal space of which is described by the Clifford odd "basis vectors".

The "basis vectors", and their Hermitian conjugated partners, $\tilde{b}^{m^\dagger}_j$ and $\tilde{b}^m_j$, determining the creation and annihilation operators for fermions, appearing in two separate groups, each with $2^{d-1} \times 2^{d-1}$ members, algebraically anticommute, due to the properties of the Clifford algebra elements $\gamma^a$'s, Eq. (19).

Correspondingly the "basis vectors", determining the creation operators for bosons $\tilde{A}^{\mu}_R$, algebraically commute. They appear in two groups with $2^{d-1} \times 2^{d-1}$ members each.

To illustrate the properties of Clifford odd and even "basis vectors" we present their properties as well with respect to the two subgroups $SO(3, 1) \times U(1)$ and $SU(3) \times U(1)$ of the group $SO(5, 1)$, all with the same number of commuting operators as $SO(5, 1)$.

We use the superposition of members of Cartan subalgebra, Eq. (9), for the subgroup $SO(3, 1) \times U(1)$: $(N^3_\pm, \tau)$ and (for the corresponding operators determining the "family" quantum numbers) $(N^3_\pm, \tilde{\tau})$

\begin{align*}
N^3_\pm := N^3_{(L,R)} := \frac{1}{2}(S^{12} \pm iS^{03}), & \quad \tau = S^{56}, \quad \tilde{N}^3_\pm := \tilde{N}^3_{(L,R)} := \frac{1}{2}(\tilde{S}^{12} \pm i\tilde{S}^{03}), \quad \tilde{\tau} = S^{56}.\quad (20)
\end{align*}

Similarly we use for the subgroup $SU(3) \times U(1)$: $(\tau', \tau^3, \tau^8)$ and (for the corresponding operators

\footnote{The handedness $\Gamma^{(d)}$, one of the invariants of the group $SO(d)$, with the infinitesimal generators of the Lorentz group $S^{ab}$, is defined as $\Gamma^{(d)} = \alpha \epsilon_{a_1a_2...a_d-1a_d} S^{a_1a_2} S^{a_3a_4} \cdots S^{a_{d-1}a_d}$, summed over $(a_1a_2...a_d-1a_d)$, with the constant $\alpha$ chosen so that $\Gamma^{(d)} = \pm 1$. In the case that states are represented by products of $\gamma^a$ s, $\Gamma^{(d)}$ simplifies to $\Gamma^{(d)} = (i)^{(d-1)/2} \prod_a (\sqrt{-\eta_{\alpha\alpha}} \gamma^a)$ for $d = 2n$ and to $\Gamma^{(d)} = (i)^{(d-1)/2} \prod_a (\sqrt{-\eta_{\alpha\alpha}} \gamma^a)$, for $d = 2n + 1$.}
Table 1: $2^{(d=6)} = 64$ "eigenvectors" of the Cartan subalgebra of the Clifford odd and even algebras — the superposition of odd or even products of $\gamma^a$'s — in $d = (5 + 1)$-dimensional space are presented, divided into four groups. The first group, odd $I$, is chosen to represent "basis vectors", named $\tilde{b}_f^m$, appearing in $2^{d-1} = 4$ "families" ($f = 1, 2, 3, 4$), each "family" with $2^{d-1} = 4$ "family" members ($m = 1, 2, 3, 4$). The second group, odd $II$, contains Hermitian conjugated partners of the first group for each family separately, $\tilde{b}_f^m = (\tilde{b}_f^m)^\dagger$. Either odd $I$ or odd $II$ are products of an odd number of nilpotents, the rest are projectors. The quantum numbers of $f$, determined by eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, of $\tilde{b}_f^m$ are for the first odd $I$ and the two last even $I$ and even $II$ groups written above each group of four members with the same $f$. The quantum numbers of each member, determined by eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$, are in these three cases presented in the three columns before the last two columns. For the Hermitian conjugated partners of odd $I$, presented in the group odd $II$, the quantum numbers $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$ are presented above each group of the Hermitian conjugated partners, the three columns before the last two tell eigenvalues of $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$. The two groups with the even number of $\gamma^a$'s, even $I$ and even $II$, have their Hermitian conjugated partners within their own group each. $\Gamma^{(5+1)}$ and $\Gamma^{(3+1)}$ represent handedness in $d = (3 + 1)$ and $d = (5 + 1)$, respectively, defined in the footnote.

| $m$ | $f = 1$ | $f = 2$ | $f = 3$ | $f = 4$ | $\tilde{S}^{03}$ | $\tilde{S}^{12}$ | $\tilde{S}^{56}$ | $p^{(5+1)}$ | $p^{(3+1)}$ |
|-----|---------|---------|---------|---------|----------------|---------------|---------------|-------------|-------------|
| $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$ | $\tilde{b}_f^m$ | | | | | | | | |
| $\text{odd } I$ | $\tilde{S}^{03}$ | $\tilde{S}^{12}$ | $\tilde{S}^{56}$ | $p^{(5+1)}$ | $p^{(3+1)}$ |
| $1$ | $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ | $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | $\tilde{b}_f^m$ | $\tilde{b}_f^m$ | | | | |
| $2$ | $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $3$ | $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ | $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $4$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$ | $\tilde{A}_f^m$ | | | | | | | | |
| $\text{even } I$ | $\tilde{S}^{03}$ | $\tilde{S}^{12}$ | $\tilde{S}^{56}$ | $p^{(5+1)}$ | $p^{(3+1)}$ |
| $1$ | $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ | $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | |
| $2$ | $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $3$ | $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ | $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $4$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $(\tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56})$ | $\tilde{A}_f^m$ | | | | | | | | |
| $\text{even } II$ | $\tilde{S}^{03}$ | $\tilde{S}^{12}$ | $\tilde{S}^{56}$ | $p^{(5+1)}$ | $p^{(3+1)}$ |
| $1$ | $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ | $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | |
| $2$ | $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $3$ | $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ | $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
| $4$ | $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ | $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ | | | | | | | |
determining the "family" quantum numbers) \((\tilde{\tau}', \tilde{\tau}^3, \tilde{\tau}^8)\)

\[
\begin{align*}
\tau^3 &:= \frac{1}{2} (-S^{12} - iS^{03}), \\
\tau^8 &:= \frac{1}{2\sqrt{3}} (-iS^{03} + S^{12} - 2S^{5,6}), \\
\tau' &:= \frac{-1}{3} (-iS^{03} + S^{12} + S^{5,6}).
\end{align*}
\] (21)

The corresponding relations for \((\tilde{\tau}^3, \tilde{\tau}^8, \tilde{\tau}')\) can be read from Eq. \((21)\), if replacing \(S^{ab}\) by \(\tilde{S}^{ab}\).

For the operators \(S^{ab} = S^{ab} + \tilde{S}^{ab}\) the corresponding relations for superposition of the Cartan subalgebra elements \((N^{3,1}_3, \tau)\) and \((\tau', \tau^3, \tau^8)\) follow if in Eqs. \((20, 21)\) \(S^{ab}\) instead of \(S^{ab}\) are used.

2.4.1 "Basis vectors" of odd products of \(\gamma^a\)'s in \(d = (5 + 1)\).

This is a short overview of properties of the Clifford odd "basis vectors" following a lot Ref. [4] and the references within this reference.

To illustrate properties of the Clifford odd "basis vectors" let us analyse them besides with respect to the Cartan subalgebra of \(SO(5,1)\) also with respect to two kinds of the subgroups of \(SO(5,1): SO(3,1) \times U(1)\) and \(SU(3) \times U(1)\) of the group \(SO(5,1)\), with the same number of Cartan subalgebra members in all three cases \((d^2 = 3)\). In the case of \(SO(5,1)\) three eigenvalues of the Cartan subalgebra \(S^{ab}\), Eq. \((6)\), \((S^{03}, S^{12}, S^{5,6})\) for each of the four members, the same for all four families carrying eigenvalues of \((S^{03}, S^{12}, S^{5,6})\) are already presented in Table \([1]\). The superposition of the Cartan subalgebra operators for the subgroups \(SO(3,1) \times U(1)\) and \(SU(3) \times U(1)\) of the group \(SO(5,1)\) are defined in Eqs. \((20, 21)\).

The rest of generators of the two subgroups of the group \(SO(5,1)\) can be found in Eqs. \((50, 52)\) of App. \([13]\).

In Table \([2]\) the Clifford odd "basis vectors" \(\tilde{b}_f^{\nu \dagger}\) are presented (they are odd products of nilpotents and (the rest) of projectors) as the eigenvectors of all \(3\) commuting Cartan subalgebra members, Eq. \((6)\), of the group: i. \(SO(5,1)\) (with 15 generators), ii. \(SO(4) \times U(1)\) (with 7 generators) and iii. \(SU(3) \times U(1)\) (with 9 generators), together with the eigenvalues of the corresponding commuting generators, presented in Eqs. \((20, 21)\). These "basis vectors" are already presented as part of Table \([1]\). They fulfill together with their Hermitian conjugated partners the anticommutation relations of Eqs. \((12, 14)\).

Let us notice that the right handed, \(\Gamma^{(5+1)} = 1\), fourplet of the fourth family of Table \([2]\) can be found in the first four lines of Table \([2]\) if only the \(d = (5+1)\) part is taken into account. It is repeated three more times (in the four lines from 9 to 12, from 17 to 20 and from 25 to 28).

In any chosen \(d = 2n\)-dimensional space there is besides the choice that the "basis vectors" are the right handed Clifford odd objects, like in Table \([1]\) as well the choice that the Clifford odd objects are left handed. Their Hermitian conjugated partners are then correspondingly the right handed Clifford odd objects.

In Fig. \([1]\) the four "basis vectors" of one of the four families, anyone, are presented, demonstrating the eigenvalues of the Cartan subalgebra members \((S^{03}, S^{12}, S^{56})\). We notice two "basis vectors" with \(S^{56} = \frac{1}{2}\) and two with \(S^{56} = -\frac{1}{2}\).

In Fig. \([2]\) the same four family members (the "basis vectors") of any of the four families as the ones on Fig. \([1]\) are presented, this way with respect to the superposition of the Cartan subalgebra members \((S^{03}, S^{12}, S^{56})\), manifesting the \(SU(3) \times U(1)\) subgroups of the \(SO(5,1)\) group. We notice one "colour" triplet with the "fermion" number \(\tau' = \frac{1}{6}\) and one "colourless" singlet with the "fermion" number \(\tau' = -\frac{1}{2}\).

In the case of the group \(SO(6)\) manifesting as \(SU(3) \times U(1)\) and representing the \(SU(3)\) as the colour subgroup and \(U(1)\) as the "fermion" number if embedded into \(SO(13,1)\) the triplet would represent quarks.
Table 2: The “basis vectors” $\hat{b}_f^{m=(ch,s)\dagger}$ are presented for $d=(5+1)$-dimensional case. Each $\hat{b}_f^{m=(ch,s)\dagger}$ is a product of projectors and of an odd number of nilpotents and is the “eigenvector” of all the Cartan subalgebra members, $(S^{03}, S^{12}, S^{56})$ and $(\bar{S}^{03}, \bar{S}^{12}, \bar{S}^{56})$, Eq. (6), with $ch$ (representing the charge, that is the eigenvalue of $S^{56}$) and $s$ (representing the eigenvalues of $S^{03}$ and $S^{12}$) explaining the index $m$ while $f$ determines the family quantum numbers (the eigenvalues of $(\bar{S}^{03}, \bar{S}^{12}, \bar{S}^{56})$). This table presents also the eigenvalues of the three commuting operators $N^3_{L,R} = (S^{12} \pm is^{03})$ and $\tau = S^{56}$ of the subgroups $SU(2) \times SU(2) \times U(1)$ and the eigenvalues of the three commuting operators $\tau^3, \tau^8$ and $\tau'$ of the subgroups $SU(3) \times U(1)$ ($\tau^3 = \frac{1}{2}(-S^{12} - is^{03}), \tau^8 = \frac{1}{2\sqrt{3}}(-is^{03} + S^{12} - 2S^{13,14}), \tau' = -\frac{1}{3}(-is^{03} + S^{12} + S^{56})).$ In these two last cases index $m$ represents the eigenvalues of the corresponding commuting generators. $\Gamma^{(5+1)} = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^5 \gamma^6, \Gamma^{(3+1)} = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. Operators $\hat{b}_f^{m=(ch,s)\dagger}$ and $\hat{b}_f^{m=(ch,s)}$ fulfil the anticommutation relations of Eqs. (12, 14).

| $f$ | $m = (ch,s)$ | $\hat{b}_f^{m=(ch,s)\dagger}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{3+1}$ | $N^3_L$ | $N^3_R$ | $\tau^3$ | $\tau^8$ | $\tau'$ | $S^{03}$ | $S^{12}$ | $S^{56}$ |
|-----|-------------|-------------------------------|-------|--------|--------|----------------|------|--------|-------|-------|-------|--------|--------|--------|
| 1   | $(\frac{1}{2}, \frac{1}{2})$ | $(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $0$ | $\frac{1}{2}$ | $0$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2   | $(\frac{1}{2}, -\frac{1}{2})$ | $[-(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $0$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3   | $(\frac{1}{2}, \frac{1}{2})$ | $[(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 4   | $(\frac{1}{2}, -\frac{1}{2})$ | $[-(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

| $f$ | $m = (ch,s)$ | $\hat{b}_f^{m=(ch,s)\dagger}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{3+1}$ | $N^3_L$ | $N^3_R$ | $\tau^3$ | $\tau^8$ | $\tau'$ | $S^{03}$ | $S^{12}$ | $S^{56}$ |
|-----|-------------|-------------------------------|-------|--------|--------|----------------|------|--------|-------|-------|-------|--------|--------|--------|
| II  | $(\frac{1}{2}, \frac{1}{2})$ | $[(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $0$ | $\frac{1}{2}$ | $0$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2   | $(\frac{1}{2}, -\frac{1}{2})$ | $[-(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $0$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3   | $(\frac{1}{2}, \frac{1}{2})$ | $[(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 4   | $(\frac{1}{2}, -\frac{1}{2})$ | $[-(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

| $f$ | $m = (ch,s)$ | $\hat{b}_f^{m=(ch,s)\dagger}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ | $\Gamma^{3+1}$ | $N^3_L$ | $N^3_R$ | $\tau^3$ | $\tau^8$ | $\tau'$ | $S^{03}$ | $S^{12}$ | $S^{56}$ |
|-----|-------------|-------------------------------|-------|--------|--------|----------------|------|--------|-------|-------|-------|--------|--------|--------|
| IV  | $(\frac{1}{2}, \frac{1}{2})$ | $[(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $0$ | $\frac{1}{2}$ | $0$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 2   | $(\frac{1}{2}, -\frac{1}{2})$ | $[-(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $1$ | $0$ | $\frac{1}{2}$ | $0$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 3   | $(\frac{1}{2}, \frac{1}{2})$ | $[(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 4   | $(\frac{1}{2}, -\frac{1}{2})$ | $[-(\bar{S}^{03})^{12} 56$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
Figure 1: The eigenvalues of the Cartan subalgebra members of the group $SO(5, 1)$ are presented for the "basis vectors", the properties of which are described in Table 2. On the abscissa axis and on the ordinate axis the eigenvalues of the two operators, $S^{12}$ and $-iS^{03}$, are presented, respectively, while the third axis presents the eigenvalues of $S^{56}$. There are two Clifford odd family members "basis vectors" with $S^{56} = \frac{1}{2}$, denoted on the figure by the circle $\bigcirc$, having $(S^{12} = -\frac{1}{2}, -iS^{03} = -\frac{1}{2}, S^{56} = \frac{1}{2})$ and $(S^{12} = \frac{1}{2}, -iS^{03} = \frac{1}{2}, S^{56} = \frac{1}{2})$, respectively, and the two Clifford odd family members "basis vectors" with $S^{56} = -\frac{1}{2}$, denoted on the figure by the square $\square$ having $(S^{12} = -\frac{1}{2}, -iS^{03} = \frac{1}{2}, S^{56} = -\frac{1}{2})$ and $(S^{12} = \frac{1}{2}, -iS^{03} = -\frac{1}{2}, S^{56} = -\frac{1}{2})$, respectively. They all appear in four families.

and the singlet leptons, as can be seen in Table 4.

It is not difficult to generalize what we learned from the case of $d = (5 + 1)$ to any even dimensional space.

Let us summarize the properties of the Clifford odd "basis vectors" in even dimensional spaces:

i. The "basis vectors", described by superposition of the Clifford odd products of $\gamma^a$'s, belonging to either the same or to different families, have their Hermitian conjugated partners in another group of superposition of the Clifford odd products of $\gamma^a$'s.

ii. All the "basis vectors" belonging to the same or different families are algebraically orthogonal and so are algebraically orthogonal among themselves their Hermitian conjugated partners.

iii. The Clifford odd "basis vectors" carry half integer spin (and from the point of view of the lower dimensional space), the charges in the fundamental representations of the corresponding subgroups of the group $SO(d - 1, 1)$, and the half integer family quantum numbers.

iv. The family members of "basis vectors" have the same properties in all the families independently whether one observes the group $SO(d - 1, 1)$ or the subgroups with the same number of commuting operators. In Table 2 presented eigenvalues of the commuting operators ($S^{03}, S^{12}, S^{56}$), describing the properties of the "basis vectors" in $d = (5 + 1)$ (the same for each family carrying different family quantum numbers), are drawn in Fig. 1. In Table 2 presented the $SU(3) \times U(1)$ superposition of ($S^{03}, S^{12}, S^{56}$), that is ($\tau^3, \tau^8, \tau'$) appearing in Eq. (21), (they are the same for each family carrying different family quantum numbers) are drawn in Fig. 2.

---

9 $S^{ab}$ rotate any "basis vector" into any other "basis vector" of the same family and $\tilde{S}^{ab}$ rotate any family member into the same family member of another family.
The eigenvalues of the superposition of the Cartan subalgebra eigenvalues of the subgroups $SU(3)$ and $U(1)$ of the group $SO(5, 1)$ for the "basis vectors", the properties of which appear in Table 2, are presented. On the abscissa axis, on the ordinate axis, and on the third axis the eigenvalues of the superposition of the three Cartan subalgebra members, $\tau^3 = \frac{1}{2}(-S_{12} - iS_{03}), \tau^8 = \frac{1}{\sqrt{3}}(S_{12} - iS_{03} - 2S_{56}), \tau' = -\frac{1}{3}(S_{12} - iS_{03} + S_{56})$ are presented. One notices one triplet, denoted by $\bigcirc$, with the values $(\tau' = \frac{1}{6}, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}), (\tau' = \frac{1}{6}, \tau^3 = \frac{1}{2}, \tau^8 = \frac{1}{2\sqrt{3}}), (\tau' = \frac{1}{6}, \tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}})$, respectively, and one singlet denoted by the square $\Box$ $(\tau_3 = 0, \tau^8 = 0, \tau' = -\frac{1}{2})$. The triplet and the singlet appear in four families.

v. The sum of all the eigenvalues of all the commuting operators over the $2d^2 - 1$ family members is equal to zero for each of $2d^2 - 1$ families, independently whether the group $SO(5, 1)$ or the subgroups of this group are considered. This can be checked in Table 2 for the particular case with $d = (5 + 1)$ for $(S_{03}, S_{12}, S_{56})$, as well as for $(\tau^3, \tau^8, \tau')$.

vi. The sum of the family quantum numbers over all the families is zero, as it can be checked for the $d = (5 + 1)$ case in Table 2.

vii. For a chosen even $d$ there is a choice of either right or left handed "basis vectors". The choice of the handedness of the "basis vectors" determines also the vacuum state for the "basis vectors". The properties of the left handed "basis vectors" differ strongly from the right handed ones. The $d = (5 + 1)$ case demonstrates this difference: When looking at the eigenvalues of the superposition of the Cartan subalgebra members numbers $(\tau^3, \tau^8, \tau')$ of the subgroups $SU(3) \times U(1)$ of the $SO(5, 1)$ group one sees that the right handed realization manifests the "colour" properties of "quarks" and "leptons", Fig. 2, while the left handed would represent the "colour" properties of "antiquarks" and "antileptons" (words "quarks" and "antiquarks" and "leptons" and "antileptons" are put into quotation marks since in the treated case $SU(3)$ and $U(1)$ are subgroups of $SO(5, 1)$ and not of $SO(6)$ as presented in Table 4).

viii. One can define the Hilbert space made of single fermion states which are tensor, $\ast_T$, products of $2^d - 1$ "basis vectors" for each of the $2^d - 1$ families and of (continuously) infinite basis in ordinary space applying on the vacuum state, starting with no fermion state, one single fermion state, continuing with all possible "basis vectors" of all possible momenta (or coordinates) ([4], Sect. 5 and in Refs. cited therein).

ix. The Clifford odd "basis vectors", constructed as superposition of odd products of anticommuting operators $\gamma^a$'s, manifest all the properties desired for the internal space of fermion fields with the anticommutation relations for fermion fields included: When forming creation operators for fermions as the tensor products, $\ast_T$, of the Clifford odd "basis vectors" and the basis in ordinary (momentum or coordinate) space, the single fermion creation operators and (their Hermitian conjugated) annihilation operators fulfil all the requirements postulated by Dirac for the second quantized fermion fields,
explaining correspondingly the postulates of Dirac, Sect. 3.

\[
\hat{b}_f^{m,ab}\psi_{oc} = 0.\psi_{oc},
\hat{b}_f^{m\dagger,ab}\psi_{oc} = |\psi_{oc}\rangle,
\{\hat{b}_f^{m,ab},\hat{b}_f^{m',ab}\}_\ast_A + |\psi_{oc}\rangle = 0.\psi_{oc},
\{\hat{b}_f^{m\dagger,ab},\hat{b}_f^{m',ab}\}_\ast_A + |\psi_{oc}\rangle = 0.\psi_{oc},
\{\hat{b}_f^{m,ab},\hat{b}_f^{m',ab}\}_\ast_A + |\psi_{oc}\rangle = \delta^{mm'}\delta_{ff'}|\psi_{oc}\rangle.
\] (22)

So far we have considered the description of the internal space of fermion fields using the Clifford odd "basis vectors". They demonstrate the half integer spins (and charges from the point of view of (3+1)-dimensional space in the fundamental representation of the charge groups).

In the next subsection the properties of the Clifford even "basis vectors" are discussed. We shall see that the Clifford even "basis vectors" demonstrate the integer spins (and charges from the point of view of (3+1)-dimensional space in the adjoint representation of the charge groups). The Clifford even "basis vectors" obviously manifest properties of boson fields which are the gauge fields of the corresponding fermion fields the internal space of which is described by the Clifford odd "basis vectors".

2.4.2 "Basis vectors" of even products of $\gamma^a$'s in $d = (5+1)$

The Clifford even "basis vectors" are products of an even number of nilpotents, $(k)$, with the rest up to $\frac{d}{2}$ of projectors, $[ab]_k$.

Table 3 repeats the Clifford even "basis vectors" $i\hat{A}_f^{m\dagger}$ from Table 1, pointing out the self adjoint members and the Hermitian conjugated pairs and presenting the eigenvalues of the superposition of the Cartan subalgebra members, manifesting the subgroups $SO(3,1) \times U(1)$ and $SU(3) \times U(1)$, Eqs. (??, 21).

Each of $i\hat{A}_f^{m\dagger}$ is the product of projectors and an even number of nilpotents, and each is the eigenvector of all the Cartan subalgebra members, $(S^{03}, S^{03}, S^{03})$.

To realize that even Clifford "basis vectors" have all the properties needed to describe the internal space of boson fields let us study their properties in the case of $d = (5+1)$-dimensional space, analysing them with respect to the eigenvalues of the Cartan subalgebra members ($S^{03}, S^{12}, S^{56}$), Eq. (4), of the group $SO(5,1)$, as well as of the subgroups $SU(2) \times SU(2) \times U(1)$ and $SU(3) \times U(1)$, the commuting operators of which are presented in Eqs. (20, 21).

The Clifford even "basis vectors" can have in the $d = (5+1)$ case none or two nilpotents. The rest are projectors (three or one). In Table 1 the Clifford even "basis vectors" are denoted by $even_i\hat{A}_f^{m\dagger}$ and $even_{II}\hat{A}_f^{m\dagger}$. They obey commutation relations since even products of (anticommuting) $\gamma^a$'s obey commutation relations.

The Clifford even "basis vectors" $i\hat{A}_f^{m\dagger}$, $i = I, II$ have their Hermitian conjugated partners within the same group, or they are self adjoint, for $i = I$ or $i = II$.

Let us analyse what happens when the Clifford even "basis vectors" apply algebraically on the Clifford odd "basis vectors". We shall see that the spin of the Clifford even "basis vectors" have the integer value, determined by $S^{ab} = S^{ab} + \bar{S}^{ab}$. Correspondingly the subgroups of $SO(d-1,d)$ manifest the adjoint representations.

a. The properties of the algebraic, $*_A$, application of the Clifford even "basis vectors", presented in Table 1 as $i\hat{A}_f^{m\dagger}$, $(i = I, II)$, on the Clifford odd "basis vectors" $\hat{b}_f^{m\dagger}$, presented in Table 2 (as well as in Table 1 under odd $I\hat{b}_f^{m\dagger}$), teaches us that the Clifford even "basis vectors" describe the internal space of the gauge fields of the corresponding $\hat{b}_f^{m\dagger}$. To see this let us evaluate:
Table 3: The Clifford even "basis vectors" \( I\hat{A}_f^{m\dagger} \), each of them is the product of projectors and an even number of nilpotents, and each is the eigenvector of all the Cartan subalgebra members, \( S^{03}, S^{12}, S^{56} \), Eq. (6), are presented for \( d = (5 + 1) \)-dimensional case. Indexes \( m \) and \( f \) determine \( 2^{d-1} \times 2^{d-1} \) different members \( I\hat{A}_f^{m\dagger} \). In the third column the "basis vectors" \( I\hat{A}_f^{m\dagger} \) which are Hermitian conjugated partners to each other are pointed out with the same symbol. For example, with \(*\) are equipped the first member with \( m = 1 \) and \( f = 1 \) and the last member of \( f = 3 \) with \( m = 4 \). The sign \( \hat{\circ} \) denotes the Clifford even "basis vectors" which are self adjoint \( (I\hat{A}_f^{m\dagger})\hat{\circ} = I\hat{A}_f^{m\dagger} \). It is obvious that the sign \( \hat{\dagger} \) looses meaning, since \( I\hat{A}_f^{m\dagger} \) are self adjoint or are (mutually, in pairs) Hermitian conjugated to another \( I\hat{A}_f^{m\dagger} \). This table represents also the eigenvalues of the three commuting operators \( N_{L,R}^3 \) of the subgroups \( SU(2) \times SU(2) \times U(1) \) of the group \( SO(5,1) \) and the eigenvalues of the three commuting operators \( \tau^3, \tau^8 \) and \( \tau' \) of the subgroups \( SU(3) \times U(1) \):

| \( f \) | \( m \) | \( \hat{\circ} \) | \( I\hat{A}_f^{m\dagger} \) | \( S^{03} \) | \( S^{12} \) | \( S^{56} \) | \( N_{L}^3 \) | \( N_{R}^3 \) | \( \tau^3 \) | \( \tau^8 \) | \( \tau' \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| I | 1 | ** | \( [+] [+] [+] \) | 0 | 1 | 1 | \( \frac{1}{2} \) | \( -\frac{1}{2} \) | \( -\frac{1}{2} \) | \( \frac{1}{2} \) | \( -\frac{3}{2} \) |
| 2 | \( \triangle \) | \( + \) | \( [+] [+] [+] \) | 0 | 1 | 1 | \( \frac{1}{2} \) | \( -\frac{1}{2} \) | \( -\frac{1}{2} \) | \( \frac{1}{2} \) | \( -\frac{3}{2} \) |
| 3 | \( \dagger \) | \( [-] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | \( \hat{\circ} \) | \( [+] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| II | 1 | \( \hat{\circ} \) | \( [+] [+] [+] \) | 0 | 1 | 1 | \( \frac{1}{2} \) | \( -\frac{1}{2} \) | \( -\frac{1}{2} \) | \( \frac{1}{2} \) | \( -\frac{3}{2} \) |
| 2 | \( \hat{\circ} \) | \( [-] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | \( \hat{\circ} \) | \( [+] [+] [+] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | \( \hat{\circ} \) | \( [-] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| III | 1 | \( \hat{\circ} \) | \( [+] [+] [+] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | \( \hat{\circ} \) | \( [-] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | \( \hat{\circ} \) | \( [+] [+] [+] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | \( \hat{\circ} \) | \( [-] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| IV | 1 | \( \hat{\circ} \) | \( [+] [+] [+] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | \( \hat{\circ} \) | \( [-] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | \( \hat{\circ} \) | \( [+] [+] [+] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | \( \hat{\circ} \) | \( [-] [-] [-] \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
The algebraic application, \( *_A \), of \( I \hat{A}^{m \dagger}_{f=3} \), \( m = (1, 2, 3, 4) \), presented in Table 1, in the third column of \( \hat{b}^{m=1 \dagger} \), presented as the first Clifford odd \( I \) "basis vector" on both Tables (1, 2), can easily be evaluated by taking into account Eq. (9) for any \( m \).

\[
\begin{align*}
I \hat{A}^{m \dagger}_3 & \ast_A \hat{b}^{1 \dagger}_1 (\equiv (\dagger)(+)) : \\
I \hat{A}^{3 \dagger}_3 (\equiv [+] [+]) & \ast_A \hat{b}^{1 \dagger}_1 (\equiv (\dagger)[+] [+]) \rightarrow \hat{b}^{1 \dagger}_1, \quad (I \hat{A}^{3 \dagger}_3) \ast = I \hat{A}^{3 \dagger}_3, \quad \text{selfadjoint} \\
I \hat{A}^{3 \dagger}_3 (\equiv (-i)(-)[+]) & \ast_A \hat{b}^{1 \dagger}_1 \rightarrow \hat{b}^{2 \dagger}_1 (\equiv [-i][-][+]), \quad (I \hat{A}^{3 \dagger}_3) \ast \rightarrow I \hat{A}^{4 \dagger}_3, \quad *1 \ast \\
I \hat{A}^{3 \dagger}_3 (\equiv (-i)[+] (-)) & \ast_A \hat{b}^{1 \dagger}_1 \rightarrow \hat{b}^{3 \dagger}_1 (\equiv [-i][+][-]), \quad (I \hat{A}^{3 \dagger}_3) \ast \rightarrow I \hat{A}^{2 \dagger}_2, \quad *2 \ast \\
I \hat{A}^{3 \dagger}_3 (\equiv [+] (-)) & \ast_A \hat{b}^{1 \dagger}_1 \rightarrow \hat{b}^{4 \dagger}_1 (\equiv (i)[+][-]), \quad (I \hat{A}^{3 \dagger}_3) \ast \rightarrow I \hat{A}^{1 \dagger}_1, \quad *3 \ast .
\end{align*}
\tag{23}
\]

The arrow sign, \( \rightarrow \), means that the relation is valid up to the constant. The signs selfadjoint and \( *1*, *2*, *3* \) denote whether \( I \hat{A}^{m \dagger}_f \) is selfadjoint or it has its Hermitian conjugated partner within the same group of \( I \hat{A}^{m \dagger}_f \) denoted by the same sign also in Eq. (24).

We conclude that the algebraic, \( *_A \), application of \( I \hat{A}^{m \dagger}_3 \) on \( \hat{b}^{1 \dagger}_1 \) leads to the same or another family member of the same family \( f \) = 1, namely to \( \hat{b}^{m \dagger}_1 \), \( m = (1, 2, 3, 4) \).

The reader can calculate the eigenvalues of the Cartan subalgebra members, Eq. (6), before and after the algebraic multiplication, \( *_A \). Since the Clifford odd "basis vectors" appearing in the algebraic, \( *_A \), multiplication ("interacting") with \( I \hat{A}^{m \dagger}_3 \) carry the half integer eigenvalues of \( \{S^{03}, S^{12}, S^{56}\} \), and so do \( \hat{b}^{m \dagger}_1 \) after the algebraic multiplication, it follows that \( I \hat{A}^{m \dagger}_3 \) carry for all \( m \) the integer eigenvalues of the Cartan subalgebra members, what means that their Lorentz generators \( S^{ab} \) are the sum \( S^{ab} = S^{ab} + \tilde{S}^{ab} \).

We conclude (we shall demonstrate in what follows new examples) that the eigenvalues of the Cartan subalgebra members of all the Clifford even basis vectors \( I, II \hat{A}^{m \dagger}_f \) are equal to \( S^{ab} = S^{ab} + \tilde{S}^{ab} \).

Let us calculate the eigenvalues of the Cartan subalgebra members of \( \hat{b}^{m \dagger}_1 \), for \( m = (1, 2, 3, 4) \). Since each nilpotent and projector determining the "basis vector" \( \hat{b}^{m \dagger}_1 \) is the eigenvector of the Cartan subalgebras of \( S^{ab} \) and \( \tilde{S}^{ab} \), one can read the corresponding eigenvalues from Eqs. (7, 17). These values are written also in Table 2. Since all the members \( \hat{b}^{m \dagger}_1 \) of the family \( f = 1 \) in this case) have the same eigenvalues of \( \{S^{03}, S^{12}, S^{56}\} \), we can conclude that \( I \hat{A}^{m \dagger}_3 \) contributes for \( m = 1 \) the Cartan subalgebra eigenvalues \( (0, 0, 0) \), transforming \( \hat{b}^{m \dagger}_1 \) back into \( \hat{b}^{1 \dagger}_1 \), \( I \hat{A}^{3 \dagger}_3 \) contributes the eigenvalues \( (i, 1, 0) \), transforming \( \hat{b}^{1 \dagger}_1 \) into \( \hat{b}^{2 \dagger}_1 \), \( I \hat{A}^{3 \dagger}_3 \) contributes the Cartan subalgebra eigenvalues \( (i, 0, 1) \), transforming \( \hat{b}^{1 \dagger}_1 \) into \( \hat{b}^{3 \dagger}_1 \), and \( I \hat{A}^{3 \dagger}_3 \) contributes the Cartan subalgebra eigenvalues \( (0, -1, 1) \), transforming \( \hat{b}^{1 \dagger}_1 \) into \( \hat{b}^{4 \dagger}_1 \).
Proceeding like in Eq. (23), while taking into account the fourth line of Eq. (9), one finds

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i(-)[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_1^{2\dagger} (\equiv -i[-]+) = 0,
\]

Eqs. (23, 24) include on the very right hand side of any equation the information whether \( I \hat{A}_f^{m_f} \) is self adjoint or is Hermitian conjugated to another \( I \hat{A}_f^{m_f} \). The pairs, which are Hermitian conjugated to each other, carry the same number: \(*1*\), \(*2*\), and so on up to \(*6*\).

All the rest of \( I \hat{A}_f^{m_f} \), applying on \( \hat{b}_1^{\dagger} \), give zero for any other \( f \) except the one presented in Eqs. (23, 24).

We can repeat this calculation for all four family members \( \hat{b}_f^{m_f} \) of any of families \( f^i = (1, 2, 3, 4) \), concluding

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_f^{m_f} \rightarrow \hat{b}_f^{m_f},
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_f^{m_f} \rightarrow \hat{b}_f^{m_f},
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_f^{m_f} \rightarrow \hat{b}_f^{m_f},
\]

\[
I \hat{A}_4^{m_f} \hat{A}_4^{\dagger} \hat{b}_f^{m_f} \rightarrow \hat{b}_f^{m_f}.
\]

Let us add what we just learned about the Clifford even ”basis vectors” \( I \hat{A}_f^{m_f} \) the properties presented in Table 3:

\[\text{i.} \quad \text{The Clifford even ”basis vectors”} \ I \hat{A}_f^{m_f} \text{ are products of an even number of nilpotents,} \ (k), \text{ with the rest up to} \ \frac{d}{2} \text{ of projectors,} \ [k].\]

\[\text{ii.} \quad \text{Nilpotents and projectors are eigenvectors of the Cartan subalgebra members} \ S_{ab} = S_{ab} + \tilde{S}_{ab}, \text{ Eq. (6), carrying correspondingly the integer eigenvalues of the Cartan subalgebra members.}\]

\[\text{iii.} \quad \text{They have their Hermitian conjugated partners within the same group of} \ I \hat{A}_f^{m_f} \text{ with} \ 2^{d-1} \times 2^{d-1}\]
members.

iv. They have properties of the boson gauge fields as we recognized for the case of \( d = (5 + 1) \)-dimensional space in Eqs. \([23, 24, 25]\); When applying on the Clifford odd ”basis vectors” (offering the description of the fermion fields) they transform the Clifford odd ”basis vectors” into another Clifford odd ”basis vectors”, transferring to the Clifford odd ”basis vectors” the integer spins with respect to the \( SO(d - 1, 1) \) group, while with respect to subgroups of the \( SO(d - 1, 1) \) group they transfer appropriate superposition of the eigenvalues (manifesting the properties of the adjoint representations of the corresponding groups).

v. The sum of all the eigenvalues of each of the Cartan subalgebra members over the 16 members of \( \mathcal{A}_f^m \) is equal to zero, regardless of which subgroups of \( SO(5,1) \) group we are dealing with: \( SU(2) \times SU(2) \times U(1) \) or \( SU(3) \times U(1) \), what can easily generalize to \( 2^{d-1} \times 2^{d-1} \) case.

When looking at the eigenvalues of \( (S^{03}, S^{12}, S^{56}) \), we read from Table 3 that four of 16 ”basis vectors” \( \mathcal{A}_f^m \) are singlets, all with \((S^{03} = 0, S^{12} = 0, S^{56} = 0)\).

Four ”basis vectors” form the fourplet with \( \{(-iS^{03} = 1, S^{12} = 1, S^{56} = 0) \) and \( (-iS^{03} = -1, S^{12} = -1, S^{56} = 0) \) (these two, Hermitian conjugated to each other, are denoted by \( \otimes \otimes \) and \( (-iS^{03} = -1, S^{12} = 1, S^{56} = 0) \) and \( (-iS^{03} = 1, S^{12} = -1, S^{56} = 0) \) (these last two, Hermitian conjugated to each other, are denoted by \( \tilde{\otimes} \)).

Four ”basis vectors” form the fourplet with \( \{(-iS^{03} = 0, S^{12} = 1, S^{56} = 1) \) (denoted by \( \star \star \)), \( (-iS^{03} = 0, S^{12} = -1, S^{56} = 1) \) (denoted by \( \otimes \)), \( (-iS^{03} = -1, S^{12} = 0, S^{56} = 1) \) (denoted by \( \bigtriangleup \)) and \( (-iS^{03} = 1, S^{12} = 0, S^{56} = 1) \) (denoted by \( \bullet \)).

The Hermitian conjugated partners of this last fourplet appear in the third fourplet with \( \{(-iS^{03} = 0, S^{12} = 1, S^{56} = -1) \) (denoted by \( \star \star \)), \( (-iS^{03} = 0, S^{12} = 1, S^{56} = 1) \) (denoted by \( \otimes \)), \( (-iS^{03} = 1, S^{12} = 0, S^{56} = -1) \) (denoted by \( \bigtriangleup \)) and \( (-iS^{03} = -1, S^{12} = 0, S^{56} = -1) \) (denoted by \( \bullet \)).

These 16 Clifford even ”basis vectors” are offering the description of the internal space of the gauge fields of four families of the Clifford odd ”basis vectors” presented in Fig. 1.

Fig. 3 represents the \( 2^{\frac{d}{2} - 1} \times 2^{\frac{d}{2} - 1} \) members \( \mathcal{A}_f^m \) of the Clifford even ”basis vectors” for the case that \( d = (5 + 1) \). The properties of \( \mathcal{A}_f^m \) are presented also in Table 3. There are in this case again 16 members. Manifesting the structure of subgroups \( SU(3) \times U(1) \) of the group \( SO(5,1) \) they are represented as eigenvectors of the superposition of the Cartan subalgebra members \( (S^{03}, S^{12}, S^{56}) \), that is with \( \tau^3 = \frac{1}{2}(-S^{12} - iS^{03}), \tau^8 = \frac{1}{2\sqrt{3}}(S^{12} - iS^{03} - 2S^{56}), \) \( \) and \( \tau' = -\frac{1}{3}(S^{12} - iS^{03} + S^{56}) \). There are four self adjoint Clifford even ”basis vectors” with \( (\tau^3 = 0, \tau^8 = 0, \tau' = 0) \), one sextet of three pairs Hermitian conjugated to each other, one triplet and one antitriplet with the members of the triplet Hermitian conjugated to the corresponding members of the antitriplet and opposite. These 16 members of the Clifford even ”basis vectors” \( \mathcal{A}_f^m \) are the boson ”partners” of the Clifford odd ”basis vectors” \( \hat{b}_f^m \), presented in Fig. 2 for one of four families, anyone. The reader can check that the algebraic application of \( \mathcal{A}_f^m \), belonging to the triplet, transforms the Clifford odd singlet, denoted on Fig. 2 by \( \square \), to one of the members of the triplet, denoted on Fig. 2 by \( \triangle \).

Let us, for example, algebraically apply \( \mathcal{A}_f^3 \) (\( 3 \equiv (-i)(-) [+] \)), denoted by \( \otimes \otimes \), carrying \( (\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{2}{3}) \) on the Clifford odd ”basis vector” \( \hat{b}^1 \), with \( (\tau^3 = 0, \tau^8 = 0, \tau' = -\frac{1}{2}) \) presented in Table 2 and represented on Fig. 2 by \( \square \) as a singlet. \( \mathcal{A}_f^3 \) transforms \( \hat{b}^1 \) (by transferring to \( \hat{b}^1 \) \( (\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = -\frac{2}{3}) \)) to \( \hat{b}^2 \) with \( (\tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}, \tau' = \frac{1}{6}) \), belonging on Fig. 2 to the triplet, denoted by \( \bigcirc \).

We can see that \( \mathcal{A}_f^3 \) with \( (m = 2, 3, 4) \), if applied on the \( SU(3) \) singlet \( \hat{b}^1 \) with \( (\tau' = -\frac{1}{2}, \tau^3 = 0, \tau^8 = 0) \), transforms it to \( \hat{b}^m \) \( (m = 2, 3, 4) \), respectively, which are members of the \( SU(3) \) triplet. All these Clifford even ”basis vectors” have \( \tau' \) equal to \( \frac{2}{3} \), changing correspondingly \( \tau' = -\frac{1}{2} \) into \( \tau' = \frac{1}{6} \) and bringing the needed values of \( \tau^3 \) and \( \tau^8 \).
The Clifford even \( \dagger \) basis vectors \( I \dagger \) in the case that \( d = (5 + 1) \) are presented with respect to the eigenvalues of the commuting operators of the subgroups \( SU(3) \) and \( U(1) \) of the group \( SO(5, 1) \): \( \tau^3 = \frac{1}{2} ( -S^{12} - i S^{03} ) \), \( \tau^8 = \frac{1}{2 \sqrt{3}} ( S^{12} - i S^{03} - 2 S^{56} ) \), \( \tau' = \frac{1}{3} ( S^{12} - i S^{03} + S^{56} ) \). Their properties appear also in Table 3. The abscissa axis carries the eigenvalues of \( \tau^3 \), the ordinate axis carries the eigenvalues of \( \tau^8 \) and the third axis carries the eigenvalues of \( \tau' \). One notices i. four singlets with \( (\tau^3 = 0, \tau^8 = 0, \tau' = 0) \), denoted by \( \bigcirc \), representing four self-adjoint Clifford even \( \dagger \) basis vectors \( I \dagger \), with \( (f = 1, m = 4), (f = 2, m = 3), (f = 3, m = 1), (f = 4, m = 2) \), ii. one sextet of three pairs, Hermitian conjugated to each other, with \( \tau' = 0 \), denoted by \( \triangle \) (\( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = \frac{3}{2 \sqrt{3}}) \)), by \( \triangledown \) (\( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = \frac{3}{2 \sqrt{3}}) \)), by \( \tau^3 = 0, \tau^3 = 1, \tau^8 = 0 \)), and by \( \bigotimes \) (\( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = \frac{3}{2 \sqrt{3}}) \)), \( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = 0, \tau^3 = \frac{1}{2}, \tau^8 = \frac{3}{2 \sqrt{3}}) \)), iii. one triplet, denoted by \( \bigcirc \) (\( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = \frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2 \sqrt{3}}) \)), by \( \bigodot \) (\( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = \frac{2}{3}, \tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}) \)), iv. as well as one antitriplet, Hermitian conjugated to triplet, denoted by \( \bigcirc \) (\( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = -\frac{2}{3}, \tau^3 = -\frac{1}{2}, \tau^8 = \frac{1}{2 \sqrt{3}}) \)), by \( \bigodot \) (\( I \dagger \) \( \times \) \( I \dagger \) with \( (\tau' = -\frac{2}{3}, \tau^3 = 0, \tau^8 = -\frac{1}{\sqrt{3}}) \)).
We conclude:
Algebraic, $*_A$, application of $^I\hat{A}^m_{3}$, with $m = 1, 2, 3, 4$, on $\hat{b}^{m\dagger}_4$ transforms $\hat{b}^{m\dagger}_1$ into $\hat{b}^{m\dagger}_3$, $m = (1, 2, 3, 4)$. Algebraic, $*_A$, application of $^I\hat{A}^m_{1}$, with $m = (1, 2, 3, 4)$ on $\hat{b}^{m\dagger}_2$ transforms $\hat{b}^{m\dagger}_1$ into $\hat{b}^{m\dagger}_2$, $m = (1, 2, 3, 4)$. Algebraic, $*_A$, application of $^I\hat{A}^m_{2}$, with $m = (1, 2, 3, 4)$ on $\hat{b}^{m\dagger}_3$ transforms $\hat{b}^{m\dagger}_3$ into $\hat{b}^{m\dagger}_1$, $m = (1, 2, 3, 4)$.
Algebraic, $*_A$, application of $^I\hat{A}^m_{4}$, with $m = (1, 2, 3, 4)$ on $\hat{b}^{m\dagger}_4$ transforms $\hat{b}^{m\dagger}_4$ into $\hat{b}^{m\dagger}_4$, $m = (1, 2, 3, 4)$.

Let us analyse what happens when the Clifford even "basis vectors" apply algebraically, $*_A$, on the Clifford even "basis vectors".

b. There are 16 $(2^{d+8}-1 \times 2^{d+8}-1)$ Clifford even basis vectors $^I\hat{A}^m_j$, presented in Table 3 and on Fig. 3.

There are six pairs of the Clifford even "basis vectors" $^I\hat{A}^m_j$, which are Hermitian conjugated to each other, denoted in Table 3 by $(\ast\ast, \triangle, \frac{1}{2}, \bullet, \otimes, \odot\odot)$. The algebraic multiplication, $*_A$, of such pairs leads to one of four self adjoint operators, denoted in Table 3 by $\odot\odot$.

One recognizes that $^I\hat{A}^m_j \ast_A ^I\hat{A}^{m\dagger}_j = 0 \forall f, \text{ if } m \neq m_o \text{ and } m \neq m'$, where $^I\hat{A}^{m_o\dagger}_{j}$ represents the self adjoint member for each $f$ separately, denoted on Fig. 3 by $\odot\odot$.

Each $^I\hat{A}^m_j$ has in each group with $f' = f$ as well as $f' \neq f$ only one member $^I\hat{A}^m_{j\dagger}$ for which is $^I\hat{A}^m_j \ast_A ^I\hat{A}^{m\dagger}_j = ^I\hat{A}^m_{j\dagger}$, which is not equal zero. For $f' = f$ the member is the self adjoint $^I\hat{A}^{m_o\dagger}_{j}$ one. All other algebraic multiplication give zero.

Two "basis vectors" $^I\hat{A}^m_j$ and $^I\hat{A}^m_{j\dagger}$ of the same $f$ and of $(m, m') \neq m_o$ are in algebraic multiplication orthogonal (giving zero).

We summarize the above findings in

$$^I\hat{A}^m_j \ast_A ^I\hat{A}^{m\dagger}_j \rightarrow \begin{cases} ^I\hat{A}^{m\dagger}_j, \text{ only one for } \forall f', \text{ or zero} \end{cases} \quad (26)$$

Two "basis vectors" $^I\hat{A}^m_j$ and $^I\hat{A}^m_{j\dagger}$, the algebraic product, $*_A$, of which gives nonzero contribution, "scatter" into the third one $^I\hat{A}^m_{j\dagger}$, like $^I\hat{A}^1_1 \ast_A ^I\hat{A}^{1\dagger}_1 \rightarrow ^I\hat{A}^{1\dagger}_3$ and $^I\hat{A}^2_2 \ast_A ^I\hat{A}^{2\dagger}_2 \rightarrow ^I\hat{A}^{2\dagger}_4$.

Looking at the "basis vectors" of boson fields $^I\hat{A}^m_j$ from the point of view of subgroups $SU(3) \times U(1)$ of the group $SO(5 + 1)$ we recognize in the part of fields forming the octet the colour gauge fields of quarks and leptons and antiquarks and antileptons.

$U(1)$ fields carry no $U(1)$ charges.

Let us write the commutation relations for Clifford even "basis vectors" taking into account Eq. (26).

---

10We can comment the above events, concerning the internal space of fermions and bosons, as:
If the fermion, the internal space of which is described by Clifford odd "basis vector" $\hat{b}^{m\dagger}_1$, absorbs the boson with the "basis vector" $^I\hat{A}^1_1$ (with $S^{03} = 0, S^{12} = 0, S^{56} = 0$), its "basis vector" $\hat{b}^{m\dagger}_1$ remains unchanged.

The fermion with the "basis vector" $\hat{b}^{m\dagger}_1$, absorbing the boson with $^I\hat{A}^1_{2\dagger}$ (with $S^{03} = -i, S^{12} = -1, S^{56} = 0$), changes into fermion with the "basis vector" $\hat{b}^{m\dagger}_4$ (which carries $S^{03} = -\frac{1}{2}, S^{12} = -\frac{1}{2}$, and the same $S^{56} = \frac{1}{2}$ as before).

The fermion with "basis vector" $\hat{b}^{m\dagger}_1$, absorbing the boson with the "basis vector" $^I\hat{A}^3_1$ (carrying $S^{03} = -i, S^{12} = 0, S^{56} = -1$) changes to $\hat{b}^{m\dagger}_3$, (with $S^{03} = -\frac{1}{2}, S^{12} = \frac{1}{2}, S^{56} = -\frac{1}{2}$).

While the fermion with the "basis vector" $\hat{b}^{m\dagger}_1$, absorbing the boson with the "basis vector" $\hat{A}^{3\dagger}_1$ (with $S^{03} = 0, S^{12} = -1, S^{56} = -1$) changes to $\hat{b}^{m\dagger}_4$ (with $S^{03} = \frac{1}{2}, S^{12} = -\frac{1}{2}, S^{56} = -\frac{1}{2}$).

These comments are meaningful when bosons and fermions start to interact, that is when the interaction between fermions and bosons is included.

The spin-charge-family assumes the simple starting action as presented in Eq. (1), in which fermions interact in $d = (13 + 1)$ with gravity only.

11The word "scatter" is used in quotation marks since the "basis vectors" $^I\hat{A}^m_j$ determine only the internal space of bosons, as also the "basis vectors" $\hat{b}^{m\dagger}_j$ determine only the internal space of fermions.
In the case that $I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} \to I \hat{A}_f^{m\dagger}$ and $I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} = 0$ it follows

$$\{ I \hat{A}_f^{m\dagger} , I \hat{A}_f^{m\dagger} \}_* = \begin{cases} I \hat{A}_f^{m\dagger}, & \text{(if } I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} \to I \hat{A}_f^{m\dagger} \text{ and } I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} = 0) \end{cases}, \quad (27)$$

In the case that $I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} \to I \hat{A}_f^{m\dagger}$ and $I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} \to I \hat{A}_f^{m\dagger}$ it follows

$$\{ I \hat{A}_f^{m\dagger} , I \hat{A}_f^{m\dagger} \}_* = \begin{cases} I \hat{A}_f^{m\dagger} - I \hat{A}_f^{m\dagger}, & \text{(if } I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} \to I \hat{A}_f^{m\dagger} \text{ and } I \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger} \to I \hat{A}_f^{m\dagger}) \end{cases}, \quad (28)$$

In all other cases we have

$$\{ I \hat{A}_f^{m\dagger} , I \hat{A}_f^{m\dagger} \}_* = 0. \quad (29)$$

The above findings for the "basis vectors" $I \hat{A}_f^{m\dagger}$ in Eq. (26) are valid as well for $I I \hat{A}_f^{m\dagger}$

$$II \hat{A}_f^{m\dagger} \ast_A II \hat{A}_f^{m\dagger} \to \begin{cases} II \hat{A}_f^{m\dagger}, & \text{only one for } \forall f, \text{ or zero.} \end{cases}, \quad (30)$$

Equivalent figure to Fig. 3 can be drown also for $II \hat{A}_f^{m\dagger}$, as well as the relations presented in Eqs. (27), (28), (29). One only has to replace in these equations $I \hat{A}_f^{m\dagger}$ by $II \hat{A}_f^{m\dagger}$. But we recognize in Table 1 the differences among $I \hat{A}_f^{m\dagger}$ and $II \hat{A}_f^{m\dagger}$, which cause the orthogonality relation presented in Eq. (31).

It remains therefore to see how do the Clifford even "basis vectors" $I \hat{A}_f^{m\dagger}$ and $II \hat{A}_f^{m\dagger}$ algebraically apply ("interact") on each other.

d. Taking into account Eqs. (5), (16) and the third and the fourth line of Eq. (9) one recognizes that in Table 1 any member of even II follows from even I by the application of $\gamma^a \gamma^a$ for any index $a$. We correspondingly find, due to the fact that the two projectors $[k]$ and $[\bar{k}]$ give zero in the algebraic application $[k] \ast_A [\bar{k}] = 0$, the orthogonality relation

$$I \hat{A}_f^{m\dagger} \ast_A II \hat{A}_f^{m\dagger} = 0 = II \hat{A}_f^{m\dagger} \ast_A I \hat{A}_f^{m\dagger}. \quad (31)$$
Correspondingly the application of $\hat{\mathcal{A}}^m_{f} \dagger$ on the Clifford odd ”basis vectors” $\hat{b}^m_{f}$ and their Hermitian conjugated partners $\hat{b}^m_{f}$, differ from the application of $\hat{\mathcal{A}}^m_{f} \dagger$ on the Clifford odd ”basis vectors” $\hat{b}^m_{f}$ studied so far.

The properties of the algebraic application, $*_A$, of $\hat{\mathcal{A}}^m_{f} \dagger$ on the Clifford odd ”basis vectors” $\hat{b}^m_{f}$ and their Hermitian conjugated partners $\hat{b}^m_{f}$ will be studied separately.

The main message of this article to convince the reader that the Clifford algebra provides a description of the inner space of fermion and boson fields, thereby offering a new understanding of the second quantization postulates for fermion and bosonic fields, is fulfilled.

### 2.5 ”Basis vectors” describing internal space of fermions and bosons in any even dimensional space

In Subsect. 2.4 the properties of ”basic vectors”, which describe the internal space of fermions (with the Clifford odd ”basis vectors”) and bosons (with the Clifford even ”basis vectors”) in $d = (5 + 1)=$-dimensional space, are presented in an illustrative way.

The properties of the Clifford odd ”basis vectors” describing the internal space of fermions are discussed in 2.4.1. In 2.4.2 the properties of the Clifford even ”basis vectors” are described, representing the internal space of bosons, manifesting in the algebraic application among themselves and with the Clifford odd ”basis vectors” the properties of the gauge fields of the fermions, the internal space of which is described by the Clifford odd ”basis vectors”.

Generalization to any even $d$ is not difficult. The description of the internal space of fermions follows Ref. [4], of the internal space of bosons started in Ref. [6].

**a.** The ”basis vectors” offering the description of the internal space of fermions, $\hat{\hat{b}}^m_{f}$, must be superposition of odd products of nilpotents $(k)$, $2n' + 1$, in $d = 2(2n + 1)$, $n' = (0, 1, 2, \ldots, \frac{1}{2}(\frac{d}{2} - 1)$ (the minimum number of nilpotents is one and the maximum $2n + 1$), and the rest is the product of $n''$ projectors $[k]$, $n'' = \frac{d}{2} - (2n' + 1)$. (For $d = 4n$ the minimum number of nilpotents is one and the maximum $2n - 1$.)

In even dimensional spaces the nilpotents and projectors are chosen to be ”eigenvectors” of the $\frac{d}{2}$ members of the Cartan subalgebra of the Lorentz algebra describing the internal space of of fermions.

After reducing both types of Clifford subalgebras ($\gamma^a$’s and $\tilde{\gamma}^a$’s) to just one ($\gamma^a$’s are chosen), the generators $S^{ab}$ of the Lorentz transformations in the internal space of fermions described by $\gamma^a$’s determine the $2\frac{d}{2} - 1$ family members for each of $2\frac{d}{2} - 1$ families, while $\tilde{S}^{ab}$’s determine the $\frac{d}{2}$ numbers (the eigenvalues of the Cartan subalgebra members) for the $2\frac{d}{2} - 1$ families.

The Clifford odd ”basis vectors” $\hat{\hat{b}}^m_{f}$ and their Hermitian conjugated partners $\hat{\hat{b}}^m_{f}$ obey the postulates of Dirac for the second quantized fermion fields

$$\begin{align*}
\{\hat{\hat{b}}^m_{f}, \hat{\hat{b}}^{m'}_{f}\} = & \delta^{mn'} \delta_{ff'} \left| \psi_{oc} \right>, \\
\{\hat{\hat{b}}^m_{f}, \hat{\hat{b}}^{m'}_{f}\} = & 0 \cdot \left| \psi_{oc} \right>, \\
\{\hat{\hat{b}}^m_{f}, \hat{\hat{b}}^{m'}_{f}\} = & 0 \cdot \left| \psi_{oc} \right>, \\
\hat{\hat{b}}^m_{f} \dagger \left| \psi_{oc} \right> = & \left| \psi_{m}^{f} \right>, \\
\hat{\hat{b}}^m_{f} \left| \psi_{oc} \right> = & 0 \cdot \left| \psi_{oc} \right>,
\end{align*}$$

(32)
with \((m, m')\) denoting the "family" members and \((f, f')\) denoting the "families" of "basis vectors", \(*_A\), represents the algebraic multiplication of \(\hat{b}_{f}^{\dagger m}\) among themselves and with their Hermitian conjugated objects \(\hat{b}_{f}^{m\dagger}\).

The vacuum state is \(|\psi_{oc}\rangle\), Eq. (13). It is not difficult to prove the above relations if taking into account Eq. (19).

The Clifford odd "basis vectors" \(\hat{b}_{f}^{m\dagger}\)'s and their Hermitian conjugated partners \(\hat{b}_{f}^{m}\)'s appear in two independent groups, each with \(2^{d-1} \times 2^{d-1}\) members, the members of one group have their Hermitian conjugated partners in another group.

It is our choice which one of these two groups with \(2^{d-1} \times 2^{d-1}\) members to take as "basis vectors" \(\hat{b}_{f}^{m}\)’s. Making the opposite choice the "basis vectors" change handedness.

b. The "basis vectors" for bosons, \(\hat{A}_{f}^{m\dagger}\) and \(\hat{A}_{f}^{m}\), must contain in even dimensional space an even number of nilpotents \((k)\), \(2n'\). In \(d = 2(2n + 1), n' = (0, 1, 2, \ldots, \frac{1}{2}(d - 1)),\) the rest, \(n''\), are projectors \([k]\), \(n'' = (\frac{d}{2} - (2n'))\). (In \(d = 4n\) one can have maximally \(\frac{d}{2}\) nilpotents and minimally zero.)

The generators of the Lorentz transformations of \(\hat{A}_{f}^{m\dagger}\) and of \(\hat{A}_{f}^{m}\) are determined by \(S_{ab} = S_{ab} + \tilde{S}_{ab}\). Their properties are (chosen) to be denoted by the Cartan subalgebra members of the Lorentz group with the infinitesimal generators- \(S_{ab}\).

The "basis vectors" are either self adjoint or have the Hermitian conjugated partners within the same group of \(2^{d-1} \times 2^{d-1}\) members.

They do not form families, \(m\) and \(f\) only note a particular "basis vector" \(\hat{A}_{f}^{m\dagger}\). One of the members of particular \(f\) is self adjoint.

The algebraic application, \(*_A\), of the Clifford even "basis vectors" \(\hat{A}_{f}^{m\dagger}\) on the Clifford odd "basis vectors" \(\hat{b}_{f}^{m\dagger}\) can be in general case represented as

\[
\hat{A}_{f}^{m\dagger} *_A \hat{b}_{f}^{m\dagger} \rightarrow \left\{ \begin{array}{ll}
\hat{b}_{f}^{m\dagger}, \\
\text{or zero,}
\end{array} \right.
\]

For each \(\hat{A}_{f}^{m\dagger}\) there are among \(2^{d-1} \times 2^{d-1}\) members of the Clifford odd "basis vectors" (describing the internal space of fermion fields) \(2^{d-1}\) members, \(\hat{b}_{f}^{m\dagger}\), fulfilling the relation of Eq. (33). All the rest \((2^{d-1} \times (2^{d-1} - 1))\), give zero contributions.

The algebraic application, \(*_A\), of the Clifford even "basis vectors" \(\hat{A}_{f}^{m\dagger}\) among themselves fulfil the relation

\[
\hat{A}_{f}^{m\dagger} *_A \hat{A}_{f}^{m\dagger} \rightarrow \left\{ \begin{array}{ll}
\hat{A}_{f}^{m\dagger}, \text{only one for } \forall f', \\
\text{or zero.}
\end{array} \right.
\]

(34)

Eq. (34) means that for each \(\hat{A}_{f}^{m\dagger}\) there are among \(2^{d-1} \times 2^{d-1}\) members of the Clifford even "basis vectors" (describing the internal space of boson fields) \(2^{d-1}\) members, \(\hat{A}_{f}^{m\dagger}\), which lead to \(\hat{A}_{f}^{m\dagger}\). All the rest \(2^{d-1} \times (2^{d-1} - 1)\) of \(\hat{A}_{f}^{m\dagger}\) give zero contributions.

\[12^n\text{"basis vectors" } \hat{A}_{f}^{m\dagger} \text{ and } \hat{A}_{f}^{m\dagger}, \text{the algebraic products, } *_A, \text{of which lead nonzero contributions, "scatter" into the third one } \hat{A}_{f}^{m\dagger}, \text{like it is the case of } d = (5 + 1) \text{ in which } \hat{A}_{1}^{11} *_A \hat{A}_{2}^{21} \rightarrow \hat{A}_{1}^{11} \text{ and the case } \hat{A}_{2}^{21} *_A \hat{A}_{4}^{33} \rightarrow \hat{A}_{4}^{21}. \text{The world "scatter" is used in quotation marks since the "basis vectors" } \hat{A}_{f}^{m\dagger} \text{ determine only the internal space of bosons, as also the "basis vectors" } \hat{b}_{f}^{m\dagger} \text{ determine only the internal space of fermions.} \]
Let us write the commutation relations for Clifford even "basis vectors" taking into account Eq. (34).

i. For the case that $I \hat{A}_f^{m\dagger} *_A I \hat{A}_f^{m\dagger} \rightarrow I \hat{A}_f^{m\dagger}$ and $I \hat{A}_f^{m\dagger} *_A I \hat{A}_f^{m\dagger} = 0$ then

$$\{I \hat{A}_f^{m\dagger}, I \hat{A}_f^{m\dagger}\}_{*_A} \rightarrow I \hat{A}_f^{m\dagger}. \quad (35)$$

ii. For the case that $I \hat{A}_f^{m\dagger} *_A I \hat{A}_f^{m\dagger} \rightarrow I \hat{A}_f^{m\dagger}$ and $I \hat{A}_f^{m\dagger} *_A I \hat{A}_f^{m\dagger} \rightarrow I \hat{A}_f^{m\dagger}$ then

$$\{I \hat{A}_f^{m\dagger}, I \hat{A}_f^{m\dagger}\}_{*_A} \rightarrow I \hat{A}_f^{m\dagger} - I \hat{A}_f^{m\dagger}. \quad (36)$$

iii. In all other cases we have

$$\{I \hat{A}_f^{m\dagger}, I \hat{A}_f^{m\dagger}\}_{*_A} = 0. \quad (37)$$

Let us remind the reader that the note $\rightarrow$ means that relations are fulfilled up to a sign, while the relation $\{I \hat{A}_f^{m\dagger}, I \hat{A}_f^{m\dagger}\}_{*_A}$ means $I \hat{A}_f^{m\dagger} *_A I \hat{A}_f^{m\dagger} - I \hat{A}_f^{m\dagger} *_A I \hat{A}_f^{m\dagger}$.

The equivalent relations, presented in Eqs. (35, 36, 37), are valid also for the Clifford even "basis vectors" $II \hat{A}_f^{m\dagger}$.

Let be repeated that exchanging the role of the Clifford odd "basis vector" $\hat{b}_f^{m\dagger}$ and their Hermitian conjugated partners $\hat{b}_f^m$ (what means in the case of $d = (5 + 1)$ the exchange of odd II, which is right handed, with odd I, which is left handed, Table [1] not only causes the change of the handedness of the new $\hat{b}_f^{m\dagger}$, but also the exchange of the role of the Clifford even "basis vectors" (what means in the case of $d = (5 + 1)$ the exchange of even II with even I).

Let be added that looking at the "basis vectors" of boson fields $I \hat{A}_f^{m\dagger}$ from the point of view of subgroups $SU(3) \times U(1)$ of the group $SO(5 + 1)$ we recognize in the part of "basis vectors" forming the octet, Table [3] and Fig. [3], the colour gauge fields of quarks and leptons and antiquarks and antileptons. $U(1)$ fields carry no $U(1)$ charges.

3 Second quantized fermion and boson fields with internal space described by Clifford algebra

We learned in the previous section that in even dimensional spaces ($d = 2(2n + 1)$ or $d = 4n$) the Clifford odd and the Clifford even "basis vectors", which are the superposition of the Clifford odd and the Clifford even products of $\gamma^a$’s, respectively, offer the description of the internal spaces of fermion and boson fields, after the reduction of the Clifford space of $\gamma^a$’s and $\tilde{\gamma}^a$’s[13] to only the part determined by $\gamma^a$’s.

The Clifford odd algebra offers $2^{\frac{d}{2}-1}$ "basis vectors" $\hat{b}_f^{m\dagger}$, appearing in $2^{\frac{d}{2}-1}$ families (with the family quantum numbers determined by $S_{ab} = \frac{1}{2}(\tilde{\gamma}^a, \tilde{\gamma}^b)_-$), which together with their $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ Hermi-

\textsuperscript{13}$\tilde{\gamma}^a$’s keep the anticommutation relations among themselves and with $\gamma^a$’s, Eqs. (5, 19), as they did before the reduction of the Clifford space, Eq. (5). The proof can be found in Ref. (4, in App. I).
The Clifford even algebra offers $2^{d-1} \times 2^{d-1}$ "basis vectors" of $\mathcal{A}^{\mu \dagger}_f$ (and the same number of $\mathcal{A}^{\mu \dagger}_b$) with the properties of the second quantized boson fields manifesting as the gauge fields of fermion fields described by the Clifford odd "basis vectors" $b_f^{m \dagger}$ (and their Hermitian conjugated partners $b_f^m$), Eqs. (35, 36, 37, 33).

The Clifford odd and the Clifford even "basis vectors" are chosen to be products of nilpotents, $(k)$ (with the odd number of nilpotents if describing fermions and the even number of nilpotents if describing bosons), and projectors, $[k]$. Nilpotents and projectors are (chosen to be) eigenvectors of the Cartan subalgebra members of the Lorentz algebra in the internal space of $S^{ab}$ for the Clifford odd "basis vectors" and of $S^{ab} = S^{ad} + S^{bd}$ for the Clifford even "basis vectors".

To define the creation operators, either for fermions or for bosons besides the "basis vectors" defining the internal space of fermions and bosons also the basis in ordinary space in momentum or coordinate representation is needed. Here Ref. ([4], Subsect. 3.3 and App. J) is overviewed.

Let us introduce the momentum part of the single particle states. The longer version is presented in Ref. (4 in Subsect. 3.3 and in App. J).

\[
\begin{align*}
|\vec{p}\rangle &= \hat{b}_{\vec{p}}^\dagger |0_p\rangle, \quad \langle \vec{p}| = \langle 0_p | \hat{b}_{\vec{p}}, \\
\langle \vec{p}| \vec{p}'\rangle &= \delta(\vec{p} - \vec{p}') = \langle 0_p | \hat{b}_{\vec{p}}^\dagger \hat{b}_{\vec{p}} | 0_p \rangle,
\end{align*}
\]

where the normalization to identity is assumed, $\langle 0_p | 0_p \rangle = 1$. While the quantized operators $\hat{p}$ and $\hat{x}$ commute $\{\hat{p}^i, \hat{p}^j\}_- = 0$ and $\{\hat{x}^k, \hat{x}^l\}_- = 0$, it follows for $\{\hat{p}^i, \hat{x}^j\}_- = i\eta^{ij}$. One correspondingly finds

\[
\begin{align*}
\langle \vec{p}| \vec{x}\rangle &= \langle 0_p | \hat{b}_{\vec{p}}^\dagger \hat{b}_{\vec{x}} \hat{b}_{\vec{x}} \hat{b}_{\vec{p}} | 0_p \rangle = (\langle 0_p | \hat{b}_{\vec{x}}^\dagger \hat{b}_{\vec{p}}^\dagger | 0_p \rangle)^\dagger, \\
\{\hat{b}_{\vec{p}}^\dagger, \hat{b}_{\vec{p}}^\dagger\}_- &= 0, \quad \{\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}}\}_- = 0, \quad \{\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}}\}_- = 0, \\
\{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}}\}_- &= 0, \quad \{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}}\}_- = 0, \quad \{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}}\}_- = 0, \\
\{\hat{b}_{\vec{p}}, \hat{b}_{\vec{x}}\}_- &= e^{i\vec{p} \cdot \vec{x}} \frac{1}{\sqrt{(2\pi)^{d-1}}}, \quad \{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}}\}_- = e^{-i\vec{p} \cdot \vec{x}} \frac{1}{\sqrt{(2\pi)^{d-1}}},
\end{align*}
\]

While the internal space of either fermion or boson fields has the finite number of "basis vectors" (finite number of degrees of freedom), $2^{d-1} \times 2^{d-1}$, the momentum basis is obviously continuously infinite (has continuously many degrees of freedom).

The creation operators for either fermions or bosons must be a tensor product, $*_T$, of both contributions, the "basis vectors" describing the internal space of fermions or bosons and the basis in ordinary, momentum or coordinate, space.

The creation operators for a free massless fermion of the energy $p^0 = |\vec{p}|$, belonging to a family $f$ and to a superposition of family members $m$ applying on the vacuum state $|\psi_{oc} > _T \langle 0_{\vec{p}} >$ can be written as ([4], Subsect.3.3.2, and the references therein)

\[
\tilde{b}_f^s(\vec{p}) = \sum_m c^{sm} f(\vec{p}) \hat{b}_f^s \hat{b}_f^m \hat{b}_f^m \dagger,
\]

where the vacuum state for fermions $|\psi_{oc} > _T \langle 0_{\vec{p}} >$ includes both spaces, the internal part, Eq.(13), and the momentum part, Eq. (38) (in a tensor product for a starting single particle state with zero momentum, from which one obtains the other single fermion states of the same "basis vector" by the
operator $\hat{b}^+_f$, which pushes the momentum by an amount $\vec{p}$.

The creation operators fulfill the anticommutation relations for the second quantized fermion fields

\[
\{\hat{b}^+_f(\vec{p}'), \hat{b}^+_f(\vec{p})\} + |\psi_{oc}\rangle \langle 0_{\vec{p}}| = \delta^{ss'}\delta_{f,f'} \delta(\vec{p}' - \vec{p}) |\psi_{oc}\rangle \langle 0_{\vec{p}}|,
\]

\[
\{\hat{b}^+_f(\vec{p}'), \hat{b}^+_s(\vec{p})\} + |\psi_{oc}\rangle \langle 0_{\vec{p}}| = 0, |\psi_{oc}\rangle \langle 0_{\vec{p}}|,
\]

\[
\{\hat{b}^+_s(\vec{p}'), \hat{b}^+_s(\vec{p})\} + |\psi_{oc}\rangle \langle 0_{\vec{p}}| = 0, |\psi_{oc}\rangle \langle 0_{\vec{p}}|,
\]

\[
\hat{b}^+_s(\vec{p}) |\psi_{oc}\rangle \langle 0_{\vec{p}}| = |\psi^s_s(\vec{p})\rangle > 0, |\psi_{oc}\rangle \langle 0_{\vec{p}}|,
\]

\[
\hat{b}^+_f(\vec{p}) |\psi_{oc}\rangle \langle 0_{\vec{p}}| = 0, |\psi_{oc}\rangle \langle 0_{\vec{p}}|,
\]

\[
|p^0| = |\vec{p}|. \tag{41}
\]

The creation operators $\hat{b}^+_f(\vec{p})$ and their Hermitian conjugated partners annihilation operators $\hat{b}^+_s(\vec{p})$, creating and annihilating the single fermion states, respectively, fulfill when applying on the vacuum state, $|\psi_{oc}\rangle \langle 0_{\vec{p}}|$, the anticommutation relations for the second quantized fermions, postulated by Dirac (Ref. [4], Subsect. 3.3.1, Sect. 5).\(^{17}\)

To write the creation operators for boson fields we must take into account that boson gauge fields have the space index $\alpha$, describing the $\alpha$ component of the boson field in the ordinary space.\(^{16}\) We therefore write

\[
\begin{align*}
1 \hat{A}^{m\dagger}_{fa}(\vec{p}) & = \hat{b}^+_\alpha \ast_T C^m_{fa} 1 \hat{A}^{m\dagger}_f. \tag{42}
\end{align*}
\]

We treat free massless bosons of momentum $\vec{p}$ and energy $p^0 = |\vec{p}|$ and of particular "basis vectors" $1 \hat{A}^{m\dagger}_f$ which are eigenvectors of all the Cartan subalgebra members \(^{17}\) $C^m_{fa}$ determines the vector component of the boson field for a particular $(m, f)$. Creation operators operate on the vacuum state $|\psi_{ocu}\rangle > \ast_T |0_{\vec{p}}\rangle$ with the internal space part just a constant, $|\psi_{ocu}\rangle = |1\rangle$, and for a starting single boson state with a zero momentum from which one obtains the other single boson states with the same "basis vector" by the operator $\hat{b}^+_\alpha$ which push the momentum by an amount $\vec{p}$.

For the creation operators for boson fields in a coordinate representation we find using Eqs. (38 39)

\[
1 \hat{A}^{m\dagger}_{fa}(\vec{x},x^0) = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} 1 \hat{A}^{m\dagger}_f(\vec{p}) e^{-i(p^0 x^0 - \vec{p} \cdot \vec{x})} |p^0 = |\vec{p}|\rangle. \tag{43}
\]

To understand what new does the Clifford algebra description of the internal space of fermion and boson fields, Eqs. (42 43 40), bring to our understanding of the second quantized fermion and boson fields and what new can we learn from this offer, we need to relate $\sum_{ab} C^{ab}_{oc} \omega_{abc}$ and $\sum_{mf} 1 \hat{A}^{m\dagger}_f C^{mf}_{\alpha}$ recognizing that $1 \hat{A}^{m\dagger} C^{mf}_{\alpha}$ are eigenstates of the Cartan subalgebra members, while $\omega_{abc}$ are not.

The gravity fields, the vielbeins and the two kinds of the spin connection fields, $f^a_{\alpha}$, $\omega_{abc}$, $\tilde{\omega}_{abc}$, respectively, are in the spin-charge-family theory\(^{18}\) (unifying spins, charges and families of fermions)

\(^{14}\)The creation operators and their Hermitian conjugated partners annihilation operators in the coordinate representation can be read in [4] and the references therein: $\hat{b}^+_f(\vec{x},x^0) = \sum_m \hat{b}^{m\dagger}_f \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} e^{ms_f(\vec{p})} \hat{p}^\dagger e^{-i(p^0 x^0 - \vec{p} \cdot \vec{x})}$.\(^{14}\)

\(^{15}\)The anticommutation relations of Eq. (41) are valid also if we replace the vacuum state, $|\psi_{oc}\rangle \langle 0_{\vec{p}}|$, by the Hilbert space of Clifford fermions generated by the tensor product multiplication, $\ast_T$, of any number of the Clifford odd fermion states of all possible internal quantum numbers and all possible momenta (that is of any number of $\hat{b}^+_s(\vec{p})$ of any $(s, f, \vec{p})$), Ref. (4, Sect. 5).

\(^{16}\)In the spin-charge-family theory also the Higgs’s scalars origin in the boson gauge fields with the vector index $(7, 8)$, Ref. (4, Sect. 7.4.1, and the references therein).

\(^{17}\)In general the energy eigenstates of bosons are in superposition of $1 \hat{A}^{m\dagger}_f$. One example, which uses the superposition of the Cartan subalgebra eigenstates manifesting the $SU(3) \times U(1)$ subgroups of the group $SO(6)$, is presented in Fig. 3.

\(^{18}\)This is the case for most of the Kaluza-Klein-like theories.
and offering not only the explanation for all the assumptions of the standard model but also for the increasing number of phenomena observed so far) the only boson fields in \( d = (3 + 1) \), observed in \( d = (3 + 1) \) besides as gravity also as all the other boson fields with the Higgs’s scalars included \[20\].

We therefore need to relate

\[
\left\{ \frac{1}{2} \sum_{ab} S^{ab} \omega_{ba} \right\} \sum_m \beta^{m\ell} \tilde{b}_{m\ell}^f (\bar{p}) \text{ relate to } \sum_{m'f} \sum_m \beta^{m\ell} \tilde{b}_{m\ell}^f (\bar{p}),
\]

\[\forall f \text{ and } \forall \beta^{m\ell},\]

\[
S^{cd} \sum_{ab} (c^{ab}_{mf} \omega_{ba}) \text{ relate to } S^{cd} (\hat{A}^{m\ell}_f C^m_{f}),
\]

\[\forall (m, f),\]

\[\forall \text{ Cartan subalgebra member } S^{cd}. \tag{44}\]

Let be repeated that \( \hat{A}^{m\ell}_f \) are chosen to be the eigenvectors of the Cartan subalgebra members, Eq. (6). Correspondingly we can relate a particular \( \hat{A}^{m\ell}_f C^m_f \) with such a superposition of \( \omega_{ba} \)'s which is the eigenvector with the same values of the Cartan subalgebra members as there is a particular \( \hat{A}^{m\ell}_f C^m_f \).

We can do this in two ways:

i. Using the first relation in Eq. (44). On the left hand side of this relation \( S^{ab} \)'s apply on \( \tilde{b}_{m\ell}^f (\bar{p}) \). On the right hand side \( \hat{A}^{m\ell}_f \) apply as well on the same ”basis vector” \( \hat{b}_{m\ell}^f \).

ii. Using the second relation, in which \( S^{cd} \) apply on the left hand side on \( \omega_{ba} \)’s

\[
S^{cd} \sum_{ab} c^{ab}_{mf} \omega_{ba} = \sum_{ab} c^{ab}_{mf} \hat{\omega}_{ba} \eta^{bc} \eta^{ac} + \omega^{abcd} + \omega^{ab} \eta^{bc} - \omega^{ad} \eta^{bc}), \tag{45}
\]

on each \( \omega_{ba} \) separately; \( c^{ab}_{mf} \) are constants to be determined from the second relation, where on the right hand side of this relation \( S^{cd} = S^{cd} + S^{cd} \) apply on the ”basis vector” \( \hat{A}^{m\ell}_f \) of the corresponding gauge field.

Let us demonstrate the first of the two relations on the toy model case of \( d = (3 + 1) \).

The ”basis vectors” \( \hat{b}_{m\ell}^f = 1 \) — can be found in Table 1 as the first and the second ”basis vectors” of the first family of odd \( f \) if only the first and the second factors are taken into account and the third one [+] is neglected. \( \hat{b}_{m\ell}^f = [+] [+] \), \( \hat{b}_{m\ell}^f = [+] [-] \) can be found in the same Table 1 as the first and the second ”basis vectors” of the third family if [+] is neglected.

The corresponding even ”basis vectors” can be found in the same Table 1 in the third and the fourth column under even \( f \) in the first two lines, if neglecting \([+]\), \( \hat{A}_1^{m\ell} = [+] [+] \), \( \hat{A}_1^{m\ell} = (i) (-i) \), \( \hat{A}_2^{m\ell} = (+i) (+) \), \( \hat{A}_2^{m\ell} = (i) (-) \).

Applying \( (S^{01} \omega_{01a} + S^{02} \omega_{02a} + S^{13} \omega_{13a} + S^{23} \omega_{23a} + S^{03} \omega_{03a} + S^{12} \omega_{12a}) \) on \( \hat{b}_{m\ell}^f \), and relating this to \( (\hat{A}_1^{m\ell} C_1^{m\ell} + \hat{A}_2^{m\ell} C_2^{m\ell} + \hat{A}_3^{m\ell} C_3^{m\ell} + \hat{A}_4^{m\ell} C_4^{m\ell} ) \) on \( \hat{b}_{m\ell}^f \) we end up, after taking into account Eqs. (45), (46), with the first relation of the first line of Eq. (46)

\[
C_1^{m\ell} = \frac{1}{2} (i \omega_{02a} + \omega_{12a}), \quad (-) C_2^{m\ell} = \frac{1}{2} (-i \omega_{02a} + \omega_{01a} + i \omega_{13a} - \omega_{23a})),
\]

\[
C_2^{m\ell} = \frac{1}{2} (-i \omega_{02a} - \omega_{12a}), \quad (-) C_2^{m\ell} = \frac{1}{2} (-i \omega_{02a} - \omega_{01a} - i \omega_{13a} - \omega_{23a})).
\tag{46}
\]

The last three relations, of the above equation follow from the equivalent application of \( (S^{01} \omega_{01a} + S^{02} \omega_{02a} + S^{13} \omega_{13a} + S^{23} \omega_{23a} + S^{03} \omega_{03a} + S^{12} \omega_{12a}) \) on \( (\hat{b}_{m\ell}^f, \hat{b}_{m\ell}^f, \hat{b}_{m\ell}^f) \), respectively, after relating them to \( (\hat{A}_1^{m\ell} C_1^{m\ell} + \hat{A}_2^{m\ell} C_2^{m\ell} + \hat{A}_3^{m\ell} C_3^{m\ell} + \hat{A}_4^{m\ell} C_4^{m\ell} ) \) applying on \( (\hat{b}_{m\ell}^f, \hat{b}_{m\ell}^f, \hat{b}_{m\ell}^f) \), respectively.

33
Let us conclude this section by pointing out that either the Clifford odd "basis vectors" $\hat{b}_m^\dagger$, $i = (I, II)$ have in any even $d$-dimensional space $2^d-1 \times 2^d-1$ members, while $\omega_{ab\alpha}$ as well as $\tilde{\omega}_{ab\alpha}$ have each for each $\alpha \frac{d}{2}(d-1)$ members. It is needed to find out what new can this difference bring into the application of the Kaluza-Klein-like theories on elementary physics and cosmology.

4 Conclusions

In the spin-charge-family theory ([4] and references therein) the Clifford odd algebra is used to describe the internal space of fermion fields. The Clifford odd "basis vectors" in the tensor product with a basis in ordinary space form the creation and annihilation operators, in which the anticommutativity of the "basis vectors" is transferred to the creation and annihilation operators for fermions, offering the explanation for the second quantization postulates for fermion fields ([4] and references therein). The Clifford odd "basis vectors" have all the properties of fermions: Half integer spins with respect to the Cartan subalgebra members of the Lorentz algebra in the internal space of fermions in even dimensional spaces ($d = 2(2n + 1)$ or $d = 4n$), as discussed in Subsects. 2.3 and 2.2 and in Sect. 3.

With respect to the subgroups of the $SO(d-1,1)$ group the Clifford odd "basis vectors" appear in the fundamental representations, as illustrated in Subsects. 2.4 and 2.4.1.

In this article, it is demonstrated that the Clifford even algebra is offering the description of the internal space of boson fields. The Clifford even "basis vectors" in the tensor product with a basis in ordinary space form the creation and annihilation operators which manifest the commuting properties of the second quantized boson fields, offering explanation for the second quantization postulates for boson fields [6]. The Clifford even "basis vectors" have all the properties of bosons: Integer spins with respect to the Cartan subalgebra members of the Lorentz algebra in the internal space of bosons, as discussed in Subsects. 2.5 and 2.2 and in Sect. 3.

With respect to the subgroups of the $SO(d-1,1)$ group the Clifford even "basis vectors" manifest the adjoint representations, as illustrated in Subsect. 2.4 and 2.4.2.

There are two kinds of the anticommuting algebras [2]: The Grassmann algebra, offering in $d$-dimensional space $2 \cdot 2^d$ operators ($2^d \theta^a$'s and $2^d \frac{\partial}{\partial \theta^a}$'s, Hermitian conjugated to each other, Eq. (3)), and the two Clifford subalgebras, each with $2^d$ operators called $\gamma^a$'s and $\tilde{\gamma}^a$'s, respectively, [2, 14, 15], Eqs. (2-5), Subsect. 2.1 of this article.

The operators in each of the two Clifford subalgebras appear in two groups of $2^d-1 \times 2^d-1$ of the Clifford odd operators (the odd products of either $\gamma^a$’s in one subalgebra or of $\tilde{\gamma}^a$’s in the other subalgebra), which are Hermitian conjugated to each other. In each Clifford odd group of any of the two subalgebras there appear $2^d-1$ irreducible representation each.

There are as well the Clifford even operators (the even products of either $\gamma^a$’s in one subalgebra or of $\tilde{\gamma}^a$’s in the another subalgebra) which again appear in two groups of $2^d-1 \times 2^d-1$ members each. In the case of the Clifford even objects the members of each group of $2^d-1 \times 2^d-1$ members have the Hermitian conjugated partners within the same group, Subsect. 2.2, Table 1.

The Grassmann algebra operators are expressible with the operators of the two Clifford subalgebras and opposite, Eq. (4). The two Clifford subalgebras are independent of each other, Eq. (5), forming two independent spaces.

Either the Grassmann algebra [13, 8] or the two Clifford subalgebras can be used to describe the internal space of anticommuting objects, if the odd products of operators ($\theta^a$’s or $\gamma^a$’s, or $\tilde{\gamma}^a$’s) are used to describe the internal space of these objects. Describing the commuting objects the even products of operators ($\theta^a$’s or $\gamma^a$’s or $\tilde{\gamma}^a$’s) have to be used.

No integer spin anticommuting objects have been observed so far, and to describe the internal space...
of the so far observed fermions only one of the two Clifford odd subalgebras are needed.

The problem can be solved by reducing the two Clifford subalgebras to only one, the one (chosen to be) determined by $\gamma_{ab}$'s, Subsect. 2.3. The decision that $\tilde{\gamma}^a$ apply on $\gamma^a$ as follows: \( \{\tilde{\gamma}^a B = (-)^B i B \gamma^a\} |\psi_{oc}\rangle, \) Eq. (16), (with \(-)^B = -1\), if $B$ is a function of an odd products of $\gamma^a$'s, otherwise \(-)^B = 1\) enables that $2^{d-1}$ irreducible representations of $S_{ab} = \frac{i}{2} \{\gamma^a, \gamma^b\}_-$ (each with the $2^{d-1}$ members) obtain the family quantum numbers determined by $\tilde{S}_{ab} = \frac{i}{2} \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_-$.

The decision to use in the spin-charge-family theory in $d = 2(2n + 1), n \geq 3$, the superposition of the odd products of the Clifford algebra elements $\gamma^a$'s to describe the internal space of fermions which interact with the gravity only (with the vielbeins, the gauge fields of momenta, and the two kinds of the spin connection fields, the gauge fields of $S_{ab}$ and $\tilde{S}_{ab}$, respectively), Eq. (1), offers not only the explanation for all the assumed properties of fermions and bosons in the standard model, with the appearance of the families of quarks and leptons and antiquarks and antileptons ([4] and the references therein) and of the corresponding vector gauge fields and the Higgs's scalars included [20], but also for the appearance of the dark matter [33] in the universe, for the explanation of the matter/antimatter asymmetry in the universe [18], and for several other observed phenomena, making several predictions [29, 31, 32, 34].

The recognition that the superposition of the even products of the Clifford algebra elements $\gamma^a$'s offers the description of the internal space of boson fields manifesting all the properties of the observed boson fields, as demonstrated in this article, makes clear that the Clifford algebra offers the explanation for the postulates of the second quantized anticommuting fermion and commuting boson fields.

The relations in Eq. (44)

\[
\left\{ \frac{1}{2} \sum_{ab} S_{ab} \omega_{aba} \right\} \sum_{m} \beta^{mf} \hat{b}^{mf}_f(\vec{p}) \text{ relate to } \left\{ \sum_{m,f} \hat{l} \hat{A}^{m\dagger}_f C^{mf}_\alpha \right\} \sum_{m} \beta^{mf} \hat{b}^{mf}_f(\vec{p}), \forall f \text{ and } \forall \beta^{mf},
\]

\[
S^{cd} \sum_{ab} (c_{ab}^{mf} \omega_{aba}) \text{ relate to } S^{cd} (\hat{l} \hat{A}^{m\dagger}_f C^{mf}_\alpha), \forall (m,f), \forall \text{ Cartan subalgebra member } S^{cd},
\]

offers the possibility to replace the covariant derivative $p_{0\alpha}$

\[
p_{0\alpha} = p_\alpha - \frac{1}{2} S_{ab} \omega_{aba} - \frac{1}{2} \tilde{S}_{ab} \omega_{aba}
\]

in Eq. (1) with

\[
p_{0\alpha} = p_\alpha - \sum_{m,f} \hat{l} \hat{A}^{m\dagger}_f C^{m\dagger}_f \alpha - \sum_{m,f} \hat{l} \hat{m}^{\dagger}_f \hat{C}^{m\dagger}_f \alpha,
\]

where the "basis vectors" $\hat{l} \hat{A}^{m\dagger}_f \hat{C}^{m\dagger}_f \alpha$ are related to $\hat{H} \hat{A}^{m\dagger}_f \hat{C}^{m\dagger}_f \alpha$.

In this article the properties of the Clifford even "basis vectors" in relation to the Clifford odd "basis vectors", and correspondingly the transfer of the commutativity and anticommutativity from the Clifford "basis vectors" to the creation and annihilation operators for boson and fermion fields, respectively, are demonstrated, offering the explanation for the second quantization postulates for boson and fermion fields.

These relations need further study to find out what new can the proposed new insight into the internal space of fermions and bosons bring into understanding the second quantized fermion and boson fields.
The Einstein gravity is known as a gauge theory based on the Abelian group of local translations, for which vielbein is the corresponding gauge field. It is also known that that the action of Eq. (1), in which there appear besides vielbeins also the spin connection fields (two kinds of the spin connection fields in the spin-charge-family theory, which are the gauge fields of $S^{ab}$'s and $\tilde{S}^{ab}$'s, respectively, manifest in $d = (3 + 1)$ as ordinary gravity and all the known gauge fields ([20] and references therein) [19]

The study of properties of the second quantized boson fields, the internal space of which is described by the Clifford even algebra, has just started. It is needed to find out whether Eq. (44) is really able to describe gravity and correspondingly unify all the gauge fields, with the scalar fields included.

A One family representation in $d = (13 + 1)$-dimensional space with $2^d - 1$ members representing quarks and leptons and antiquarks and antileptons in the spin-charge-family theory

This appendix illustrates the family members of one family of the Clifford odd "basis vectors", written as products of (odd number of) nilpotents and of projectors, which are chosen to be the eigenvectors of the Cartan subalgebra members, Eq. (7, 6) of the Lorentz algebra in the internal space of fermions. Analysing the group $SO(13, 1)$ with respect to the subgroups $SO(3, 1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$ with the same number of commuting operators as has the group $SO(13, 1)$, one can see in Table 4 that the "basis vectors" of one irreducible representation, one family, of the Clifford odd basis vectors of left handedness, $\Gamma^{(13+1)} = -1$, includes all the quarks and the leptons as well as the antiquarks and antileptons of the standard model, with the right handed neutrino and left handed antineutrino included.

While the starting "basis vectors" can be either left or right handed, the subgroups of the starting group contain left and right handed members, just as required by the standard model [20].

---

19 If there are no fermions present the spin connection fields of both kinds are expressible with the vielbeins ([2], Eq. (103)).
20 The breaks of the symmetries, manifested in Eqs. ([50, 51, 52], are in the spin-charge-family theory caused by the condensate and by the non zero vacuum expectation values (constant values) of the scalar fields carrying the space index (7, 8) (Refs. [21, 15, 4] and the references therein), all originating in the vielbeins and the two kinds of the spin connection fields. The space breaks first to $SO(7, 1) \times SU(3) \times U(1)_{II}$ and then further to $SO(3, 1) \times SU(2)_{I} \times U(1)_{I} \times SU(3) \times U(1)_{II}$, what explains the connections between the weak and the hyper charges and the handedness of spinors.
The needed definitions of the quantum numbers are presented in App. B.
| $\psi$ | | $\psi$ | | $\psi$ | | $\psi$ | | $\psi$ | | $\psi$ | | $\psi$ | | $\psi$ | | $\psi$ | | $\psi$ |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 16 | $u_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 17 | $u_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 18 | $u_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 19 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 20 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 21 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 22 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 23 | $u_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 24 | $u_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 25 | $v_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 26 | $v_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 27 | $v_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 28 | $v_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 29 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 30 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 31 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 32 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 33 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 34 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 35 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 36 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 37 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 38 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 39 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 40 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 41 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 42 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 43 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 44 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 45 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 46 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 47 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 48 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 49 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 50 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 51 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 52 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 53 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 54 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 55 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 56 | $d_0^2$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 57 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 58 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 59 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 60 | $e_L$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 61 | $e_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 62 | $e_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 63 | $e_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
| 64 | $e_R$ | (+|-) | [(+)] | [+] | [+][-] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] | [+] |
The infinitesimal generators of the two invariant subgroups of the group $\text{SU}(2)$ charged $(\tau^1 = \pm \frac{1}{2}, \text{Eq. (51)})$ quarks and leptons and the right handed $(\Gamma(3,1) = 1)$ weak $(\text{SU}(2)_{1f})$ chargeless and $\text{SU}(2)_{1f}$ charged $(\tau^2 = \pm \frac{1}{2})$ quarks and leptons, both with the spin $S^{12}$ up and down ($\pm \frac{1}{2}$, respectively). Quarks distinguish from leptons only in the $\text{SU}(3) \times U(1)$ part: Quarks are triplets of three colours ($\tau^1 = (\tau^3)^2, \tau^2 = (\tau^3)^1$, $(0, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$), carrying the ”fermion charge” ($\tau^4 = \frac{1}{6}$, Eq. (52)). The colourless leptons carry the ”fermion charge” ($\tau^4 = -\frac{1}{2}$). The same multiplet contains also the left handed weak $(\text{SU}(2)_{1f})$ chargeless and $\text{SU}(2)_{1f}$ charged antiquarks and antileptons. The right handed weak $(\text{SU}(2)_{1f})$ charged and $\text{SU}(2)_{1f}$ chargeless antiquarks and antileptons. Antiquarks distinguish from antileptons only in the $\text{SU}(3) \times U(1)$ part: Antiquarks are antitriplets, carrying the ”fermion charge” ($\tau^4 = -\frac{1}{2}$). The anticolourless antileptons carry the ”fermion charge” ($\tau^4 = \frac{1}{6}$). $Y = (\tau^2 + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^1 + Y)$.

B Some useful relations in Grassmann and Clifford algebras, needed also in App. A

The generator of the Lorentz transformation in Grassmann space is defined as follows [2]

$$S^{ab} = (\theta^a \rho^{ab} - \theta^b \rho^{ab}) = S^{ab} + \bar{S}^{ab}, \quad \{S^{ab}, \bar{S}^{cd}\} = 0,$$

(47)

where $S^{ab}$ and $\bar{S}^{ab}$ are the two corresponding generators of the Lorentz transformations of the two Clifford sub algebras, forming orthogonal representations with respect to each other, Eq. (5).

The infinitesimal generators of the Lorentz transformations in the two subspaces, defined with the two Clifford sub algebras are

$$S^{ab} = \frac{i}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a), \quad S^{ab} = \eta^{ab} \bar{S}^{ab},$$

$$\bar{S}^{ab} = \frac{i}{4}(\bar{\gamma}^a \bar{\gamma}^b - \bar{\gamma}^b \bar{\gamma}^a), \quad \bar{S}^{ab} = \bar{\eta}^{ab} \bar{S}^{ab},$$

(48)

where $\gamma^a$ and $\bar{\gamma}^a$ are defined in Eqs. (4, 5). The commutation relations for either $S^{ab}$ or $\bar{S}^{ab}$ or $\bar{S}^{ab}$, $S^{ab} = S^{ab} + \bar{S}^{ab}$, are

$$\{S^{ab}, \bar{S}^{cd}\} = 0,$$

$$\{S^{ab}, S^{cd}\} = i(\eta^{ac} S^{bd} + \eta^{bc} S^{ad} - \eta^{bd} S^{ac})$$

(49)

The infinitesimal generators of the two invariant subgroups of the group $\text{SO}(3,1)$ can be expressed as follows

$$\tilde{N}_\pm = \tilde{N}_{(L,R)} : = \frac{1}{2}(S^{23} \pm i S^{01}, S^{31} \pm i S^{02}, S^{12} \pm i S^{03}).$$

(50)

The infinitesimal generators of the two invariant subgroups of the group $\text{SO}(4)$ are expressible with $S^{ab}, (a, b) = (5, 6, 7, 8)$ as follows

$$\tau^1 = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}),$$

$$\tau^2 = \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}),$$

(51)
while the generators of the $SU(3)$ and $U(1)$ subgroups of the group $SO(6)$ can be expressed by $S^{ab}, (a, b) = (9, 10, 11, 12, 13, 14)$

\[
\tau^3 := \frac{1}{2} \left\{ S^{9 \ 12} - S^{10 \ 11}, S^{9 \ 11} + S^{10 \ 12}, S^{9 \ 10} - S^{11 \ 12}, \\
S^{9 \ 14} - S^{10 \ 13}, S^{9 \ 13} + S^{10 \ 14}, S^{11 \ 14} - S^{12 \ 13}, \\
S^{11 \ 13} + S^{12 \ 14}, \frac{1}{\sqrt{3}} (S^{9 \ 10} + S^{11 \ 12} - 2S^{13 \ 14}) \right\},
\]

\[
\tau^4 := -\frac{1}{3} (S^{9 \ 10} + S^{11 \ 12} + S^{13 \ 14}).
\] (52)

The group $SO(6)$ has $\frac{d(d-1)}{2} = 15$ generators and $\frac{d^2}{2} = 3$ commuting operators. The subgroups $SU(3) \times U(1)$ have the same number of commuting operators, expressed with $\tau^{33}, \tau^{38}$ and $\tau^4$, and 9 generators, 8 of $SU(3)$ and one of $U(1)$. The rest of 6 generators, not included in $SU(3) \times U(1)$, can be expressed as $\frac{1}{2} \left\{ S^{9 \ 12} + S^{10 \ 11}, S^{9 \ 11} - S^{10 \ 12}, S^{9 \ 14} + S^{10 \ 13}, S^{9 \ 13} - S^{10 \ 14}, S^{11 \ 14} + S^{12 \ 13}, S^{11 \ 13} - S^{12 \ 14} \right\}$.

The hyper charge $Y$ can be defined as $Y = \tau^{23} + \tau^4$.

The equivalent expressions for the "family" charges, expressed by $\tilde{S}^{ab}$, follow if in Eqs. (50-52) $S^{ab}$ are replaced by $\tilde{S}^{ab}$.

Acknowledgment

The author thanks Holger Bech Nielsen for very fruitful discussions, Department of Physics, FMF, University of Ljubljana, Society of Mathematicians, Physicists and Astronomers of Slovenia, for supporting the research on the spin-charge-family theory by offering the room and computer facilities and Matjaž Breskvar of Beyond Semiconductor for donations, in particular for the annual workshops entitled "What comes beyond the standard models".

References

[1] N. Mankoč Borštnik, "Spin connection as a superpartner of a vielbein", Phys. Lett. B 292 (1992) 25-29.

[2] N. Mankoč Borštnik, "Spinor and vector representations in four dimensional Grassmann space", J. of Math. Phys. 34 (1993) 3731-3745.

[3] N. Mankoč Borštnik, "Unification of spin and charges in Grassmann space?”, hep-th 9408002, IJS.TP.94/22, Mod. Phys. Lett.A (10) No.7 (1995) 587-595.

[4] N. S. Mankoč Borštnik, H. B. Nielsen, "How does Clifford algebra show the way to the second quantized fermions with unified spins, charges and families, and with vector and scalar gauge fields beyond the standard model", Progress in Particle and Nuclear Physics, http://doi.org/10.1016/j.ppnp.2021.103890.

[5] N.S. Mankoč Borštnik, H.B.F. Nielsen, "Understanding the second quantization of fermions in Clifford and in Grassmann space", New way of second quantization of fermions — Part I and Part II, in this proceedings arXiv:2007.03517, arXiv:2007.03516.

[6] N. S. Mankoč Borštnik, "How do Clifford algebras show the way to the second quantized fermions with unified spins, charges and families, and to the corresponding second quantized vector and scalar
gauge field”, Proceedings to the 24th Workshop “What comes beyond the standard models”, 5 - 11 of July, 2021, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, A. Kleppe, DMFA Založništvo, Ljubljana, December 2021, [arXiv:2112.04378].

[7] N. S. Mankoč Borštnik, ”How Clifford algebra can help understand second quantization of fermion and boson fields”, [arXiv:2112.04378].

[8] N.S. Mankoč Borštnik, H.B.F. Nielsen, ”Understanding the second quantization of fermions in Clifford and in Grassmann space” New way of second quantization of fermions — Part I and Part II, Proceedings to the 22nd Workshop ”What comes beyond the standard models”, 6 - 14 of July, 2019, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2019, [arXiv:1802.05554v4, arXiv:1902.10628].

[9] P.A.M. Dirac Proc. Roy. Soc. (London), A 117 (1928) 610.

[10] H.A. Bethe, R.W. Jackiw, ”Intermediate quantum mechanics”, New York : W.A. Benjamin, 1968.

[11] S. Weinberg, ”The quantum theory of fields”, Cambridge, Cambridge University Press, 2015.

[12] N.S. Mankoč Borštnik, H.B.F. Nielsen, ”New way of second quantized theory of fermions with either Clifford or Grassmann coordinates and spin-charge-family theory” [arXiv:1802.05554v4, arXiv:1902.10628].

[13] D. Lukman, N. S. Mankoč Borštnik, ”Properties of fermions with integer spin described with Grassmann algebra”, Proceedings to the 21st Workshop ”What comes beyond the standard models”, 23 of June - 1 of July, 2018, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2018 [arXiv:1805.06318, arXiv:1902.10628].

[14] N.S. Mankoč Borštnik, H.B.F. Nielsen, J. of Math. Phys. 43, 5782 (2002) [arXiv:hep-th/0111257].

[15] N.S. Mankoč Borštnik, H.B.F. Nielsen, “How to generate families of spinors”, J. of Math. Phys. 44 4817 (2003) [arXiv:hep-th/0303224].

[16] N.S. Mankoč Borštnik, ”Spin-charge-family theory is offering next step in understanding elementary particles and fields and correspondingly universe”, Proceedings to the Conference on Cosmology, Gravitational Waves and Particles, IARD conferences, Ljubljana, 6-9 June 2016, The 10th Biennial Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields, J. Phys.: Conf. Ser. 845 012017 [arXiv:1409.4981, arXiv:1607.01618v2].

[17] N.S. Mankoč Borštnik, ”The attributes of the Spin-Charge-Family theory giving hope that the theory offers the next step beyond the Standard Model”, Proceedings to the 12th Bienal Conference on Classical and Quantum Relativistic Dynamics of Particles and Fields IARD 2020, Prague, 1 – 4 June 2020 Journal of Physics, Conference Series volume 1956 012020, 2021, iopscience.iop.org/issue/1742-6596/1956/1.

[18] N.S. Mankoč Borštnik, ”Matter-antimatter asymmetry in the spin-charge-family theory”, Phys. Rev. D 91 (2015) 065004 [arXiv:1409.7791].

[19] N. S. Mankoč Borštnik, ”How far has so far the Spin-Charge-Family theory succeeded to explain the Standard Model assumptions, the matter-antimatter asymmetry, the appearance of the Dark Matter, the second quantized fermion fields..., making several predictions”, Proceedings to the 23rd Workshop ”What comes beyond the standard models”, 4 - 12 of July, 2020 Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2020, [arXiv:2012.09640]
[20] N.S. Mankoč Borštnik, D. Lukman, ”Vector and scalar gauge fields with respect to \( d = (3 + 1) \) in Kaluza-Klein theories and in the spin-charge-family theory”, Europhys. J. C 77 (2017) 231.

[21] N.S. Mankoč Borštnik, ”The spin-charge-family theory explains why the scalar Higgs carries the weak charge \( \pm \frac{1}{2} \) and the hyper charge \( \mp \frac{1}{2} \)”, Proceedings to the 17th Workshop ”What comes beyond the standard models”, Bled, 20-28 of July, 2014, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2014, p.163-82 [arXiv:1502.06786v1] [arXiv:1409.4981].

[22] N.S. Mankoč Borštnik N S, ”The spin-charge-family theory is explaining the origin of families, of the Higgs and the Yukawa couplings”, J. of Modern Phys. 4 (2013) 823 [arXiv:1312.1542].

[23] N.S. Mankoč Borštnik, H.B.F. Nielsen, ”The spin-charge-family theory offers understanding of the triangle anomalies cancellation in the standard model”, Fortschritte der Physik, Progress of Physics (2017) 1700046.

[24] N.S. Mankoč Borštnik, ”The explanation for the origin of the Higgs scalar and for the Yukawa couplings by the spin-charge-family theory”, J. of Mod. Phys. 6 (2015) 2244-2274, http://dx.org./10.4236/jmp.2015.615230 [arXiv:1409.4981].

[25] N.S. Mankoč Borštnik and H.B. Nielsen, ”Why nature made a choice of Clifford and not Grassmann coordinates”, Proceedings to the 20th Workshop ”What comes beyond the standard models”, Bled, 9-17 of July, 2017, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana, December 2017, p. 89-120 [arXiv:1802.05554v1v2].

[26] N.S. Mankoč Borštnik and H.B.F. Nielsen, ”Discrete symmetries in the Kaluza-Klein theories”, JHEP 04:165, 2014 [arXiv:1212.2362].

[27] N. S. Mankoč Borštnik, Second quantized ”anticommuting integer spin fields”, sent to arXiv.

[28] A. Borštnik, N.S. Mankoč Borštnik, ”Left and right handedness of fermions and bosons”, J. of Phys. G: Nucl. Part. Phys.24(1998) 963-977, hep-th/9707218.

[29] A. Borštnik Bračič, N. S. Mankoč Borštnik, ”On the origin of families of fermions and their mass matrices”, hep-ph/0512062 Phys Rev. D 74 073013-28 (2006).

[30] M. Breskvar, D. Lukman, N. S. Mankoč Borštnik, ”On the Origin of Families of Fermions and Their Mass Matrices — Approximate Analyses of Properties of Four Families Within Approach Unifying Spins and Charges”, Proceedings to the 9th Workshop ”What Comes Beyond the Standard Models”, Bled, Sept. 16 - 26, 2006, Ed. by Norma Mankoč Borštnik, Holger Bech Nielsen, Colin Froggatt, Dragan Lukman, DMFA Založništvo, Ljubljana December 2006, p.25-50, hep-ph/0612250.

[31] G. Bregar, M. Breskvar, D. Lukman, N.S. Mankoč Borštnik, ”Families of Quarks and Leptons and Their Mass Matrices”, Proceedings to the 10th international workshop ”What Comes Beyond the Standard Model”, 17 -27 of July, 2007, Ed. Norma Mankoč Borštnik, Holger Bech Nielsen, Colin Froggatt, Dragan Lukman, DMFA Založništvo, Ljubljana December 2007, p.53-70, hep-ph/0711.4681.

[32] G. Bregar, M. Breskvar, D. Lukman, N.S. Mankoč Borštnik, ”Predictions for four families by the Approach unifying spins and charges” New J. of Phys. 10 (2008) 093002, hep-ph/0606159 hep-ph-07082846.

[33] G. Bregar, N.S. Mankoč Borštnik, ”Does dark matter consist of baryons of new stable family quarks?”, Phys. Rev. D 80, 083534 (2009), 1-16.
[34] G. Bregar, N.S. Mankoč Borštnik, "Can we predict the fourth family masses for quarks and leptons?", Proceedings [arXiv:1403.4441] to the 16th Workshop ”What comes beyond the standard models”, Bled, 14-21 of July, 2013, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2013, p. 31-51, http://arxiv.org/abs/1212.4055.

[35] G. Bregar, N.S. Mankoč Borštnik, ”The new experimental data for the quarks mixing matrix are in better agreement with the spin-charge-family theory predictions”, Proceedings to the 17th Workshop ”What comes beyond the standard models”, Bled, 20-28 of July, 2014, Ed. N.S. Mankoč Borštnik, H.B. Nielsen, D. Lukman, DMFA Založništvo, Ljubljana December 2014, p.20-45 [arXiv:1502.06786v1] [arXiv:1412.5866].

[36] N.S. Mankoč Borštnik, M. Rosina, ”Are superheavy stable quark clusters viable candidates for the dark matter?”, International Journal of Modern Physics D (IJMPD) 24 (No. 13) (2015) 1545003.