THE REFLEXIVE CLOSURE OF THE ADJOINTABLE OPERATORS

E. G. KATSOULIS

ABSTRACT. Given a Hilbert module $E$ over a C*-algebra $A$, we show that the collection $\text{End}_A(E)$ of all bounded $A$-module operators acting on $E$ forms the reflexive closure for the algebra $\mathcal{L}(E)$ of the adjointable operators, i.e., $\text{End}_A(E) = \text{alg lat } \mathcal{L}(E)$. We also make an observation regarding the representation theory of the left centralizer algebra of a C*-algebra and use it to give an intuitive proof of a related result of H. Lin.

1. Introduction

In this note, $A$ denotes a C*-algebra and $E$ a Hilbert C*-module over $A$, i.e., a right $A$-module equipped with an $A$-valued inner product $\langle \cdot, \cdot \rangle$ so that the norm $\|\xi\| \equiv \|\xi, \xi\|^{1/2}$ makes $E$ into a Banach space. The collection of all bounded $A$-module operators acting on $E$ is denoted as $\text{End}_A(E)$. A linear operator $S$ acting on $E$ is said to be adjointable iff given $x, y \in E$ there exists $y' \in E$ so that $\langle Sx, y \rangle = \langle x, y' \rangle$. Elementary examples of adjointable operators are the “rank one” operators $\theta_{\eta, \xi}$, defined by $\theta_{\eta, \xi}(x) \equiv \eta \langle \xi, x \rangle$, where $\eta, \xi, x \in E$. The collection of all adjointable operators acting on $E$ will be denoted as $\mathcal{L}(E)$ while the norm closed subalgebra generated by the rank one operators will be denoted as $\mathcal{K}(E)$.

It is a well known fact that $\mathcal{L}(E) \subseteq \text{End}_A(E)$. However, the reverse inclusion is known to fail in general; this is perhaps the first obstacle one encounters when extending the theory of operators on a Hilbert space to that of operators on a Hilbert C*-module. This problem has been addressed since the beginning of the theory [20, page 447] and has influenced its subsequent development. The first few chapters of the monograph of Manuilov and Troitsky [18] and the references therein provide the basics of the theory and give a good account of what is known regarding that issue. (See also [4, 16].) The purpose of this note is to demonstrate that the inequality between $\mathcal{L}(E)$ and $\text{End}_A(E)$ is intimately related to another area of continuing mathematical interest, the reflexivity of operator algebras.

If $\mathfrak{A}$ is a unital operator algebra acting on a Banach space $\mathfrak{X}$, then $\text{lat } \mathfrak{A}$ will denote the collection of all closed subspaces $M \subseteq \mathfrak{X}$ which are left...
invariant by \( A \), i.e., \( A(m) \in M \), for all \( A \in A \) and \( m \in M \). Dually, for a collection \( \mathcal{L} \) of closed subspaces of \( X \), we write \( \text{alg} \mathcal{L} \) to denote the collection of all bounded operators on \( X \) that leave invariant each element of \( \mathcal{L} \). The reflexive closure of an algebra \( A \) of operators acting on \( X \) is the algebra \( \text{alg} \text{lat} A \); we say that \( A \) is reflexive iff
\[
A = \text{alg} \text{lat} A.
\]

Similarly, the reflexive closure of a subspace lattice \( \mathcal{L} \) is the lattice \( \text{lat} \text{alg} \mathcal{L} \) and \( \mathcal{L} \) is said to be reflexive if \( \mathcal{L} = \text{lat} \text{alg} \mathcal{L} \). A formal study of reflexivity for operator algebras and subspace lattices begun with the work of Halmos \[10\], after Ringrose’s proof \[22\] that all nests on Hilbert space are reflexive. Since then, the concept of reflexivity for operator algebras and subspace lattices has been addressed by various authors on both Hilbert space \[1, 2, 3, 6, 9, 13, 15, 19, 23, 24\] and Banach space \[5, 7, 8\], including in particular investigations on a Hilbert \( C^* \)-module.

The main result of this short note, Theorem \[2.6\], provides a link between the two areas of inquiry discussed above. It shows that the presence of bounded but not adjointable module operators on a Hilbert \( C^* \)-module \( E \) is equivalent to the failure of reflexivity for \( \mathcal{L}(E) \). (Here we think of \( \mathcal{L}(E) \) as an operator algebra acting on \( E \).) Actually, we do more: we completely describe \( \text{lat} \mathcal{L}(E) \) (Theorem \[2.3\]) and we determine that \( \text{alg} \text{lat} \mathcal{L}(E) = \text{End}_A(E) \). This shows in particular that \( \text{End}_A(E) \) is always reflexive. A key step in the proof is a classical result of Barry Johnson \[11, \text{Theorem 1}\]. Actually, our Theorem \[2.6\] can also be thought as a generalization of Johnson’s result, since its statement reduces to the statement of \[11, \text{Theorem 1}\], when applied to the case of the trivial (unital) Hilbert \( C^* \)-module.

Another interpretation for the inequality between \( \mathcal{L}(E) \) and \( \text{End}_A(E) \) comes from the work of H. Lin. Lin shows in \[17, \text{Theorem 1.5}\] that \( \text{End}_A(E) \) is isometrically isomorphic as a Banach algebra to the left centralizer algebra of \( \mathcal{K}(E) \). Furthermore, the isomorphism Lin constructs extends the familiar \( \dagger \)-isomorphism between \( \mathcal{L}(E) \) and the double centralizer algebra of \( \mathcal{K}(E) \). This shows that the gap between \( \mathcal{L}(E) \) and \( \text{End}_A(E) \) is solely due to the presence of left centralizers for \( \mathcal{K}(E) \) which fail to be double centralizers. In Proposition \[3.3\] we observe that the representation theory of the left centralizer algebra of a \( C^* \)-algebra is flexible enough to allow the use of representations on a Banach space. This leads to yet another short proof of Lin’s Theorem, which we present in Theorem \[3.4\]. The only prerequisite for our proof is the existence of a contractive approximate identity for an arbitrary \( C^* \)-algebra. (Compare also with \[4, \text{Proposition 8.1.16 (ii)}\].)
2. THE MAIN RESULT

We begin by identifying a useful class of subspaces of $E$.

**Definition 2.1.** Let $E$ be a Hilbert $C^*$-module over a $C^*$-algebra $A$. If $J \subseteq A$, then we define

$$E(J) := \text{span}\{\xi a \mid \xi \in E, a \in J\}.$$  

The correspondence $J \mapsto E(J)$ of Definition 2.1 is not bijective. Indeed, if $l(J)$ is the closed left ideal generated by $J \subseteq A$, then it is easy to see that $E(l(J)) = E(J)$. Therefore we restrict our attention to closed left ideals of $A$. It turns out that an extra step is still required to ensure bijectivity. First we need the following.

**Lemma 2.2.** Let $E$ be a Hilbert $C^*$-module over a $C^*$-algebra $A$ and let $J \subseteq A$ be a closed left ideal. Then

$$E(J) = \{\xi \in E \mid \langle \eta, \xi \rangle \in J \text{ for all } \eta \in E\}.$$  

**Proof.** The inclusion

$$E(J) \subseteq \{\xi \in E \mid \langle \eta, \xi \rangle \in J \text{ for all } \eta \in E\}$$

is obvious. The reverse inclusion follows from the well known fact [18, Lemma 1.3.9] that

$$\xi = \lim_{\epsilon \to 0} \xi \langle \xi, \xi \rangle \left[\langle \xi, \xi \rangle + \epsilon\right]^{-1}$$

for any $\xi \in E$. □

The following gives now a complete description for the lattice of invariant subspaces of the adjointable operators.

**Theorem 2.3.** Let $E$ be a Hilbert $C^*$-module over a $C^*$-algebra $A$. Then

$$\text{lat}\, \mathcal{L}(E) = \{E(J) \mid J \subseteq \langle E, E \rangle \text{ closed left ideal }\}.$$  

Furthermore, the association $J \mapsto E(J)$ establishes a complete lattice isomorphism between the closed left ideals of $\langle E, E \rangle$ and $\text{lat}\, \mathcal{L}(E)$.

**Proof.** First observe that if $J \subseteq A$ is a closed left ideal, then the subspace $E(J)$ is invariant under $\mathcal{L}(E)$, because $\mathcal{L}(E)$ consists of $A$-module operators.

Conversely assume that $M \in \text{lat}\, \mathcal{L}(E)$ and let

$$J(M) \equiv \text{span}\{\langle \eta, m \rangle \mid \eta \in E \text{ and } m \in M\}.$$  

Clearly, $J(M) \subseteq \langle E, E \rangle$ is a closed left ideal. We claim that $M = E(J(M))$. Indeed, if $m \in M$, then by the definition of $J(M)$ we have $\langle \eta, m \rangle \in J(M)$, for all $\eta \in E$, and so Lemma 2.2 implies that $m \in E(J(M))$. On the other hand, any $\xi a$, with $\xi \in E$ and $a \in J(M)$ is the limit of finite sums of elements of the form $\xi \langle \eta, m \rangle$, where $\eta \in E$ and $m \in M$. However

$$\xi \langle \eta, m \rangle = \theta_{\xi, \eta}(m) \in M$$

and so $M = E(J(M))$. This shows that $J \mapsto E(J)$ is surjective.
In order to prove that \( J \hookrightarrow E(J) \) is also injective we need to verify that \( J = J(E(J)) \), for any closed ideal \( J \subseteq \langle E, E \rangle \). Since \( J \subseteq \langle E, E \rangle \) is a left ideal, \( J(E(J)) \subseteq J \). On the other hand, if \((e_i)_i\) is an approximate unit for \( J \), then any element of \( J \subseteq \langle E, E \rangle \) can be approximated by elements of the form
\[
\sum_k \langle \xi_k, \eta_k \rangle e_k = \sum_k \langle \xi_k, \eta_k e_k \rangle, \quad \xi_k, \eta_k \in E.
\]
However, \( \eta_k e_k \in E(J) \), by Definition 2.1 and so sums of the above form belong to \( J(E(J)) \). Hence \( J \subseteq J(E(J)) \) and so \( J \hookrightarrow E(J) \) is also injective with inverse \( M \mapsto J(M) \).

The proof that \( J \hookrightarrow E(J) \) respects the lattice operations follows from two successive applications of Lemma 2.2. Indeed, if \((J_i)_i\) is a collection of closed ideals of \( \langle E, E \rangle \), then \( \xi \in \bigcap_i E(J_i) \) is equivalent by Lemma 2.2 to \( \langle \eta, \xi \rangle \in \bigcap_i J_i \) which, once again by Lemma 2.2, is equivalent to \( \xi \in E(\bigcap_i J_i) \). Therefore \( \bigcap_i E(J_i) = E(\bigcap_i J_i) \). The proof of \( \vee_i E(J_i) = E(\vee_i J_i) \) is immediate.

The following result was proved by B. Johnson [11] for arbitrary semisimple Banach algebras. As stated below, it can be deduced from the GNS construction and a clever application of Kadison’s Transitivity Theorem.

**Theorem 2.4.** Let \( A \) be a \( C^* \)-algebra and let \( \Phi \) be a linear operator acting on \( A \) that leaves invariant all closed left ideals of \( A \). Then \( \Phi(ba) = \Phi(b)a, \forall a, b \in A \). In particular, if 1 \( \in A \) is a unit then \( \Phi \) is the left multiplication operator by \( \Phi(1) \).

Note that the proof of Theorem 2.3 shows that any bounded \( A \)-module map leaves invariant \( \operatorname{lat} L(E) \). This establishes one direction in the following, which is the main result of the paper.

**Theorem 2.5.** Let \( E \) be a Hilbert module over a \( C^* \)-algebra \( A \). Then
\[
\operatorname{alg lat} L(E) = \operatorname{End}_A(E).
\]
In particular, \( \operatorname{End}_A(E) \) is a reflexive algebra of operators acting on \( E \).

**Proof.** Let \( S \in \operatorname{alg lat} L(E) \) and \( \xi, \eta \in E \). Consider the linear operator
\[
\Phi_{\eta, \xi} : A \ni a \mapsto \langle \eta, S(\xi a) \rangle \in A
\]
We claim that \( \Phi_{\eta, \xi} \) leaves invariant any of the closed left ideals of \( A \). Indeed, if \( J \subseteq A \) is such an ideal and \( j \in J \), then \( \xi j \in E(J) \) and since \( S \in \operatorname{alg lat} L \), \( S(\xi j) \in E(J) \). By Theorem 2.3, we have
\[
\Phi_{\eta, \xi}(j) = \langle \eta, S(\xi j) \rangle \in J
\]
and so \( \Phi_{\eta, \xi} \) leaves \( J \) invariant, which proves the claim. Hence Theorem 2.4 implies now that \( \Phi_{\eta, \xi}(ba) = \Phi_{\eta, \xi}(b)a, \forall a, b \in A \).
Let \((e_i)\) be an approximate unit for \(A\). By the above \(\Phi_{\eta,\xi}(e_i a) = \Phi_{\eta,\xi}(e_i)a\), \(\forall i\), and so

\[
\langle \eta, S(\xi a) \rangle = \lim_i \langle \eta, S(\xi e_i a) \rangle = \lim_i \Phi_{\eta,\xi}(e_i a) = \lim_i \langle \eta, S(\xi e_i) a \rangle = \langle \eta, S(\xi) a \rangle
\]

Hence

\[
\langle \eta, S(\xi a) \rangle = \langle \eta, S(\xi) a \rangle, \quad \forall a \in A,
\]

which establishes that \(S\) is an \(A\)-module map.

The above Theorem can also be thought as a generalization of Theorem 2.4 (Johnson’s Theorem) since its statement reduces to the statement of Theorem 2.4 when applied to the case of the trivial unital Hilbert \(C^*\)-module. In order to incorporate the non-unital case as well, we need to reformulate our main result as follows.

**Corollary 2.6.** Let \(E\) be a Hilbert module over a \(C^*\)-algebra \(A\). Then

\[
\text{alg lat } K(E) = \text{End}_A(E).
\]

**Corollary 2.7.** If \(E\) is a selfdual Hilbert \(C^*\)-module, then \(L(E)\) is reflexive as an algebra of operators acting on \(E\).

In particular, the above Corollary shows that if \(A\) is a unital \(C^*\)-algebra, then \(L(A^{(n)})\), \(1 \leq n < \infty\), is a reflexive operator algebra. This is not necessarily true for \(L(A^{(\infty)})\). Indeed in [18, Example 2.1.2] the authors give an example of a unital commutative \(C^*\)-algebra \(A\) for which \(L(A^{(\infty)}) \neq \text{End}_A(A^{(\infty)})\). By Theorem 2.6, \(L(A^{(\infty)})\) is not reflexive.

### 3. Left Centralizers and a Theorem of H. Lin

An alternative description for the inclusion \(L(E) \subseteq \text{End}_A(E)\) has been given by H. Lin in [17].

**Definition 3.1.** If \(\mathfrak{A}\) is a Banach algebra then a map \(\Phi : \mathfrak{A} \to \mathfrak{A}\) is called a left centralizer if \(\Phi(ab) = \Phi(a)b\), for all \(a, b \in \mathfrak{A}\). If in addition there exists a map \(\Psi : \mathfrak{A} \to \mathfrak{A}\) so that \(\Psi(a)b = a\Phi(b)\), for all \(a, b \in \mathfrak{A}\), then \(\Phi\) is called a double centralizer.

In the case where \(\mathfrak{A}\) has an approximate unit, all centralizers are linear and bounded [12]. In that case, the collection of all left (resp. double) centralizers equipped with the supremum norm forms a Banach space which we denote as \(\text{LC}(\mathfrak{A})\) (resp. \(\text{DC}(\mathfrak{A})\)).

In [17, Theorem 1.5] Lin shows that \(\text{End}_A(E)\) is isometrically isomorphic as a Banach algebra to \(\text{LC}(K(E))\). Furthermore, the isomorphism Lin constructs extends the familiar *-isomorphism of Kasparov [14] between \(L(E)\) and \(\text{DC}(K(E))\). Lin’s proof is similar in nature to that of Kasparov [14] for the double centralizers of \(K(E)\). However it is more elaborate and
also requires some additional results of Paschke [20]. In what follows we
give an elementary proof of Lin’s Theorem. Our argument depends on the
observation that the representation theory for the left centralizers of a $C^*$-
algebra $A$ is flexible enough to allow the use of representations on a Banach
space.

**Definition 3.2.** Let $X$ be a Banach space and let $A$ be a norm closed sub-
algebra of $B(X)$, the bounded operators on $X$. The left multiplier algebra of $A$ is the collection
\[
\text{LM}_X(A) \equiv \{ b \in B(X) \mid ba \in A, \text{ for all } a \in A \}.
\]
If $b \in \text{LM}_X(A)$, then $L_b \in B(A)$ denotes the left multiplication operator
by $b$.

The following has also a companion statement for double centralizers,
which we plan to state and explore elsewhere.

**Proposition 3.3.** Let $A$ be a $C^*$-algebra and assume that $A$ is acting iso-
metrically and non-degenerately on a Banach space $X$. Then the mapping
\[
(1) \quad \text{LM}_X(A) \longrightarrow \text{LC}(A) : b \longmapsto L_b
\]
establishes an isometric Banach algebra isomorphism between $\text{LM}_X(A)$ and
$\text{LC}(A)$.

**Proof.** The statement of this Proposition is a well-known fact, provided
that $X$ is a Hilbert space. In that case, in order to establish the surjectivity
of (1) one starts with a contractive approximate unit $(e_i)_i$ for $A$. If $B \in
\text{LC}(A)$, then the net $(B(e_i))_i$ is bounded and therefore has at least one
weak limit point $b \in B(X)$. The conclusion then follows by showing that
$b \in \text{LM}_X(A)$. (See [21, Proposition 3.12.3] for a detailed argument.)

Bounded nets of operators on Banach space need not have weak limits.
However, the non-degeneracy of the action guarantees that the net $(B(e_i))_i$
is Cauchy in the topology of pointwise convergence and so it converges point-
wise\(^1\) to a bounded operator $b \in B(X)$, even when $X$ is assumed to be a
Banach space. With this observation at hand, the rest of the proof now
goes as in the Hilbert space case.

We are in position now to give the promised proof for Lin’s Theorem.

**Theorem 3.4.** Let $E$ be a Hilbert $C^*$-module over a $C^*$-algebra $A$. Then
there exist an isometric isomorphism of Banach algebras
\[
\phi : \text{End}_A(E) \longrightarrow \text{LC}(\mathcal{K}(E)),
\]
whose restriction $\phi|_{\mathcal{L}(E)}$ establishes a $*$-isomorphism between $\mathcal{L}(E)$ and
$\text{DC}(\mathcal{K}(E))$.

\(^1\)Note that this is not true in general for the right centralizers.
Proof. In light of Proposition 3.3 we need to verify that

$$\text{LM}_E(K(E)) = \text{End}_A(E).$$

Clearly $\text{End}_A(E) \subseteq \text{LM}_E(K(E))$. Conversely, let $S \in \text{LM}_E(K(E))$. If $a \in A$ and $\eta, \xi, \zeta \in E$, then

$$S(\eta \langle \xi, \zeta \rangle a) = S \theta_{\eta, \xi}(\zeta a) = S \theta_{\eta, \xi}(\zeta) a = S(\eta \langle \xi, \zeta \rangle) a.$$  

However vectors of the form $\eta \langle \xi, \zeta \rangle$, $\eta, \xi, \zeta \in E$, are dense in $E$ by [18, Lemma 1.3.9] and so $S$ is an $A$-module map, as desired.

Specializing now the mapping of (1) to our setting, we obtain an isometric isomorphism

$$\phi: \text{End}_A(E) \longrightarrow \text{LC}(K(E)): S \mapsto L_S.$$  

Furthermore, the restriction $\phi|_{\mathcal{L}(E)}$ coincides with Kasparov’s map and the conclusion follows.

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Department of Mathematics, East Carolina University, Greenville, NC 27858, USA

E-mail address: katsoulise@ecu.edu