Systems of four coupled one sided Sylvester-type real quaternion matrix equations and their applications

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Abstract: In this paper, we derive some necessary and sufficient solvability conditions for some systems of one sided coupled Sylvester-type real quaternion matrix equations in terms of ranks and generalized inverses of matrices. We also give the expressions of the general solutions to these systems when they are solvable. Moreover, we provide some numerical examples to illustrate our results. The findings of this paper extend some known results in the literature.

Keywords: Quaternion; Sylvester-type equations; Moore-Penrose inverse; Rank; Solution; Solvability

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1. Introduction

Quaternions were introduced by Irish mathematician Sir William Rowan Hamilton in 1843. It is well known that quaternion algebra is an associative and noncommutative division algebra over the real number field. Quaternions and quaternion matrices have found a huge amount of applications in quantum physics, signal and color image processing, and so on (e.g. [3], [21], [27]-[30]). General properties of quaternions and quaternion matrices can be found in [48]. Quaternion matrix equations play an important role in mathematics and other disciplines, such as engineering, system and control theory. There have been many papers using various approaches to investigate many quaternion matrix equations (e.g. [9]-[11], [38]-[43], [45], [46]).

The Sylvester-type matrix equations have wide applications in neural network [47], robust control ([4], [31]), output feedback control ([25], [26]), the almost noninteracting control by measurement feedback problem ([22], [23]), graph theory [7], and so on. Since Roth [23] first studied the one-sided generalized Sylvester matrix equation

\[ AX - YB = C \]

over the complex field in 1952, there have been many papers to discuss the generalized Sylvester matrix equations (e.g. [1], [2], [8], [13], [17], [18], [20], [24], [31]-[37], [41]). For instance, De Terán et al. ([5], [6]) considered the -Sylvester equation \( AX + X^*B = 0 \) and \( AX + BX^* = 0 \). Quite recently, Dmytryshyn and Kågström [7] presented some solvability conditions of the following

\[ AX - YB = C \]

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systems consisting of Sylvester and $*$-Sylvester equations through the corresponding equivalence relations of the block matrices

$$\begin{cases}
A_i X_k \pm X_j B_i = C_i, & i = 1, \ldots, n_1, \\
F_i' X_{k'} \pm X_j' G_i' = H_i', & i' = 1, \ldots, n_2,
\end{cases}$$

where $k, j, k', j' \in \{1, \ldots, m\}$, each unknown $X_l$ is $r_l \times c_l$, $l = 1, \ldots, m$, and all other matrices are of appropriate sizes. Jonsson and Kågström ([15], [16]) provided some effective approaches for solving one-sided and two-sided triangular Sylvester-type matrix equations.

The study on the coupled generalized Sylvester matrix equations is active in recent years. Lee and Vu [19] derived a consistency condition for the following system of mixed Sylvester matrix equations through the corresponding equivalence relations of the block matrices

$$A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_3 - X_2 B_2 = C_2,$$

(1.1)

where $A_i$, $B_i$ and $C_i$ ($i = 1, 2$) are given matrices over a field, $X_1$, $X_2$ and $X_3$ are unknowns. Wang and He [34] gave some computable necessary and sufficient solvability conditions for the system (1.1), and presented the general solution when (1.1) is solvable. Afterwards, He and Wang [14] provided some necessary and sufficient solvability conditions for the system of mixed Sylvester matrix equations

$$A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_3 - X_2 B_2 = C_2,$$

(1.2)

where $A_i$, $B_i$ and $C_i$ ($i = 1, 2$) are given complex matrices, $X_1$, $X_2$ and $X_3$ are unknowns. They also derived an expression of the general solution to the system (1.2). Recently, Wang and He [35] considered the following three systems of generalized coupled Sylvester matrix equations with four variables

$$\begin{cases}
A_1 X - Y B_1 = C_1, \\
A_2 Z - Y B_2 = C_2, \\
A_3 Z - W B_3 = C_3,
\end{cases}$$

(1.3)

$$\begin{cases}
A_1 X - Y B_1 = C_1, \\
A_2 Y - Z B_2 = C_2, \\
A_3 Z - W B_3 = C_3,
\end{cases}$$

(1.4)

$$\begin{cases}
A_1 X - Y B_1 = C_1, \\
A_2 Y - Z B_2 = C_2, \\
A_3 W - Z B_3 = C_3,
\end{cases}$$

(1.5)

where $A_i$, $B_i$ and $C_i$ ($i = 1, 2, 3$) are given complex matrices, $X$, $Y$, $Z$ and $W$ are unknowns. He, Mauricio, Wang and De Moor [8] considered two sided coupled generalized Sylvester matrix equations with four variables

$$A_i X_i B_i + C_i X_{i+1} D_i = E_i, \quad i = 1, 2, 3,$$

where $A_i$, $B_i$, $C_i$, $D_i$, $E_i$ ($i = 1, 2, 3$) are given complex matrices, $X_i$ are unknowns. Very recently, He and Wang [13] derived the solvability conditions and the general solution to the system of
the periodic discrete-time coupled Sylvester quaternion matrix equations

\[
\begin{cases}
A_k X_k + Y_k B_k = M_k, \\
C_k X_{k+1} + Y_k D_k = N_k,
\end{cases} \quad (k = 1, 2),
\]

where \(A_k, B_k, C_k, D_k, M_k, N_k\) are given matrices, \(X_k\) and \(Y_k\) are unknowns.

To our best knowledge, there has been little information on the solvability and the general solutions to the systems of four coupled one-sided Sylvester-type real quaternion matrix equations with five unknowns. Motivated by the wide applications of generalized Sylvester matrix equations and real quaternion matrix equations and in order to improve the theoretical development of generalized Sylvester real quaternion matrix equations, we in this paper consider the solvability and the expressions of the general solutions to the following systems of four coupled one-sided Sylvester-type real quaternion matrix equations

\[
\begin{cases}
A_1 X_1 - X_2 B_1 = C_1, \\
A_2 X_2 - X_3 B_2 = C_2, \\
A_3 X_3 - X_4 B_3 = C_3, \\
A_4 X_4 - X_5 B_4 = C_4,
\end{cases} \quad (1.6)
\]

\[
\begin{cases}
A_1 X_1 - X_2 B_1 = C_1, \\
A_2 X_2 - X_3 B_2 = C_2, \\
A_3 X_3 - X_4 B_3 = C_3, \\
A_4 X_5 - X_1 B_4 = C_4,
\end{cases} \quad (1.7)
\]

\[
\begin{cases}
A_1 X_1 - X_2 B_1 = C_1, \\
A_2 X_2 - X_3 B_2 = C_2, \\
A_3 X_4 - X_3 B_3 = C_3, \\
A_4 X_5 - X_4 B_4 = C_4,
\end{cases} \quad (1.8)
\]

\[
\begin{cases}
A_1 X_1 - X_2 B_1 = C_1, \\
A_2 X_2 - X_3 B_2 = C_2, \\
A_3 X_3 - X_4 B_3 = C_3, \\
A_4 X_4 - X_5 B_4 = C_4,
\end{cases} \quad (1.9)
\]

\[
\begin{cases}
A_1 X_1 - X_2 B_1 = C_1, \\
A_2 X_3 - X_2 B_2 = C_2, \\
A_3 X_4 - X_3 B_3 = C_3, \\
A_4 X_4 - X_5 B_4 = C_4,
\end{cases} \quad (1.10)
\]

where \(A_i, B_i, C_i, (i = 1, 2, 3, 4)\) are given real quaternion matrices, and \(X_1, \ldots, X_5\) are unknowns.

Note that the \(i\)th equation and \((i + 1)\)th equation in \((1.6)-(1.10)\) have a common variable \(X_{i+1}\).

The given real quaternion matrices \(A_i\) located at the left of the variables and \(B_i\) located at the right of the variables. Systems \((1.1)-(1.5)\) are special cases of systems \((1.6)-(1.10)\).

The remainder of the paper is organized as follows. In Section 2, we provide some known lemmas which are used in this paper. In Section 3, 4, 5, 6, 7, we present some solvability conditions
to the systems of four coupled one sided Sylvester-type real quaternion matrix equations (1.6)-(1.10), respectively. We also derive the general solutions to the systems (1.6)-(1.10), respectively. Moreover, we give some numerical examples to illustrate our results.

Throughout this paper, let $\mathbb{R}$ be the real number fields. Let $\mathbb{H}^{m \times n}$ be the set of all $m \times n$ matrices over the real quaternion algebra $H = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$.

For $A \in \mathbb{H}^{m \times n}$, the symbols $A^*$ and $r(A)$ denote the conjugate transpose and the rank of $A$, respectively. The identity matrix with appropriate size is denoted by $I$. The Moore-Penrose inverse of $A \in \mathbb{H}^{m \times n}$, denoted by $A^\dagger$, is defined to be the unique solution $X$ to the following four matrix equations

\[
\begin{align*}
(1) & \quad AXA = A, & (2) & \quad XAX = X, & (3) & \quad (AX)^* = AX, & (4) & \quad (XA)^* = XA.
\end{align*}
\]

Furthermore, $L_A$ and $R_A$ stand for the two projectors $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ induced by $A$, respectively. It is known that $L_A = L_A^*$ and $R_A = R_A^*$.

2. Preliminaries

In this section, we review some lemmas which are used in the further development of this paper. The following lemma give the solvability conditions and general solution to the mixed Sylvester real quaternion matrix equations (1.1).

Lemma 2.1. [34] Let $A_i, B_i,$ and $C_i (i = 1, 2)$ be given. Set

\[
D_1 = R_B B_2, A = R_{A_2} A_1, B = B_2 L_D_1, C = R_{A_2} (R_{A_1} C_1 B_1^\dagger B_2 - C_2) L_D_1.
\]

Then the following statements are equivalent:

1. The mixed Sylvester real quaternion matrix equations (1.1) is consistent.
2. $R_{A_1} C_1 L_{B_1} = 0, R_{A_1} C = 0, C L_B = 0.$
3. 

\[
\begin{align*}
& r \begin{pmatrix} C_1 & A_1 \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1), \\
& r \begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2), \\
& r \begin{pmatrix} B_2 & B_1 \\ C_2 & C_1 A_1 A_2 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2).
\end{align*}
\]

In this case, the general solution to the mixed Sylvester real quaternion matrix equations (1.6) can be expressed as

\[
\begin{align*}
X &= A_1^\dagger C_1 + U_1 B_1 + L_{A_1} W_1, \\
Y &= -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + V_1 R_{B_1}, \\
Z &= A_2^\dagger (C_2 - R_{A_1} C_1 B_1^\dagger B_2 + A_1 U_1 B_2) + W_1 D_1 + L_{A_2} W_6,
\end{align*}
\]
where

\[ U_1 = A^\dagger CB^\dagger + L_A W_2 + W_3 R_B, \]

\[ V_1 = -R_{A_2} (C_2 - R_{A_1} C_1 B_1^\dagger B_2 + A_1 U_1 B_2) D_1^\dagger + A_2 W_4 + W_5 R_{D_1}, \]

and \( W_1, \ldots, W_6 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

The solvability conditions and general solution to the mixed Sylvester real quaternion matrix equations (1.2) can be found in the following lemma.

**Lemma 2.2.** \([14]\) Let \( A_i, B_i, \) and \( C_i (i = 1, 2) \) be given. Set

\[ A_{11} = R_{(A_2 A_1)} A_2, \quad B_{11} = R_{B_1} L_{B_2}, \quad C_{11} = R_{(A_2 A_1)} (A_2 R_{A_1} C_1 B_1^\dagger + C_2) L_{B_2}. \]

Then the following statements are equivalent:

1. The mixed generalized Sylvester real quaternion matrix equations (1.2) is consistent.
2. \( R_{A_1} C_1 L_{B_1} = 0, \quad R_{A_{11}} C_{11} = 0, \quad C_{11} L_{B_{11}} = 0. \)
3. \( r \left( \begin{array}{cc} C_i & A_i \\ B_i & 0 \end{array} \right) = r(A_i) + r(B_i), \quad r \left( \begin{array}{cc} A_2 A_1 & A_2 C_1 + C_2 B_1 \\ 0 & B_2 B_1 \end{array} \right) = r(A_2 A_1) + r(B_2 B_1). \)

In this case, the general solution to the mixed generalized Sylvester real quaternion matrix equations (1.6) can be expressed as

\[ X_1 = A_1^\dagger C_1 + U_1 B_1 + L_A W_1, \]

\[ X_2 = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + V_1 R_{B_1}, \]

\[ X_3 = -R_{(A_2 A_1)} (C_2 + A_2 R_{A_1} C_1 B_1^\dagger - A_2 V_1 R_{B_1}) B_2^\dagger + A_2 A_1 W_4 + W_5 R_{B_2}, \]

where

\[ V_1 = A_{11}^\dagger C_{11} B_{11}^\dagger + L_{A_{11}} W_2 + W_3 R_{B_{11}}, \]

\[ U_1 = (A_2 A_1)^\dagger (C_2 + A_2 R_{A_1} C_1 B_1^\dagger - A_2 V_1 R_{B_1}) + W_4 B_2 + L_{(A_2 A_1)} W_6, \]

and \( W_1, \ldots, W_6 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

Based on Lemma 2.1, we can solve the following mixed Sylvester real quaternion matrix equations

\[ A_1 X_1 - X_2 B_1 = C_1, \quad A_2 X_1 - X_3 B_2 = C_2. \]
Lemma 2.3. Let $A_i, B_i, C_i$ ($i = 1, 2$) be given. Set

$$A_{11} = R_{(A_2 L_{A_1})} A_2, \quad B_{11} = B_1 L_{B_2}, \quad C_{11} = R_{(A_2 L_{A_1})} (C_2 - A_2 A_1^\dagger C_1) L_{B_2},$$

Then the following statements are equivalent:

1. The mixed Sylvester real quaternion matrix equations (2.1) is consistent.
2. $R_{A_i} C_i L_{B_1} = 0, \quad R_{A_{11}} C_{11} = 0, \quad C_{11} L_{B_{11}} = 0.$
3. 

$$r \begin{pmatrix} C_1 & A_i \\ B_1 & 0 \end{pmatrix} = r(A_1) + r(B_1),$$

$$r \begin{pmatrix} C_2 & A_2 \\ B_2 & 0 \end{pmatrix} = r(A_2) + r(B_2),$$

$$r \begin{pmatrix} C_1 & A_1 \\ C_2 & A_2 \\ B_1 & 0 \\ B_2 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix}.$$ 

In this case, the general solution to the mixed Sylvester real quaternion matrix equations (2.1) can be expressed as

$$X_1 = A_1^\dagger C_1 + U_1 B_1 + L_{A_1} U_2,$$

$$X_2 = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + W_6 R_{B_1},$$

$$X_3 = -R_{(A_2 L_{A_1})} (C_2 - A_2 A_1^\dagger C_1 - A_2 U_1 B_1) B_2^\dagger + A_2 L_{A_1} W_1 + W_3 R_{B_2},$$

where

$$U_1 = A_{11}^\dagger C_{11} B_{11}^\dagger + L_{A_{11}} W_4 + W_5 R_{B_{11}},$$

$$U_2 = (A_2 L_{A_1})^\dagger (C_2 - A_2 A_1^\dagger C_1 - A_2 U_1 B_1) + W_1 B_2 + L_{(A_2 L_{A_1})} W_2,$$

and $W_1, \cdots, W_6$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

The following real quaternion matrix equation

$$A_1 X_1 + X_2 B_1 + C_3 X_3 D_3 + C_4 X_4 D_4 = E_1 \quad (2.2)$$

which play an important role in the construction of the solvability conditions and the general solution to the systems (1.6)-(1.10).

Lemma 2.4. [12, 33] Let $A, B, C, D, E, F$ be given. Set

$$A = R_A C_3, B = D_3 L_{B_1}, C = R_A C_4, D = D_4 L_{B_1},$$

$$E = R_A E_1 L_{B_1}, M = R_A C, N = D L_B, S = C L_M.$$

Then the equation (2.2) is consistent if and only if

$$R_M R_A E = 0, \quad E L_B L_N = 0, \quad R_A E L_D = 0, \quad R_C E L_B = 0.$$
In this case, the general solution can be expressed as

\[ X_1 = A_1^\dagger (E_1 - C_3 X_3 D_3 - C_4 X_4 D_4) - A_1^\dagger T_7 B_1 + L_{A_1} T_6, \]

\[ X_2 = R_{A_1} (E_1 - C_3 X_3 D_3 - C_4 X_4 D_4) B_1^\dagger + A_1 A_1^\dagger T_7 + T_8 R_{B_1}, \]

\[ X_3 = A_1^\dagger E B^\dagger - A_1^\dagger C M_1^\dagger E B^\dagger - A_1^\dagger S C L_1^\dagger E N_1^\dagger D B^\dagger - A_1^\dagger S T_2 R_N D B^\dagger + L_A T_4 + T_3 R_B, \]

\[ X_4 = M_1^\dagger E D^\dagger + S_1^\dagger S C L_1^\dagger E N_1^\dagger + L_M L_S T_1 + L_M T_2 R_N + T_3 R_D, \]

where \( T_1, \ldots, T_8 \) are arbitrary matrices over \( \mathbb{H} \) with appropriate sizes.

The following lemma can be easily generalized to \( \mathbb{H} \).

**Lemma 2.5.** \([22]\) Let \( A \in \mathbb{H}^{m \times n}, B \in \mathbb{H}^{m \times k}, C \in \mathbb{H}^{l \times n}, D \in \mathbb{H}^{m \times p}, E \in \mathbb{H}^{q \times n}, Q \in \mathbb{H}^{m_1 \times k}, \) and \( P \in \mathbb{H}^{l \times n_1} \) be given. Then

1. \( r(A) + r(R_A B) = r(B) + r(R_B A) = r(A, B) \).
2. \( r(A) + r(C L_A) = r(C) + r(AL_C) = r\left( \begin{bmatrix} A \\ C \end{bmatrix} \right) \).

### 3. Some solvability conditions and the general solution to system (1.6)

In this section, we consider the solvability conditions and the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6). For simplicity, put

\[ A_{11} = R_{B_2} B_1, \quad B_{11} = R_{A_1} A_2, \quad C_{11} = B_1 L_{A_{11}}, \quad D_{11} = R_{A_1} (R_{A_2} C_2 B_2^\dagger B_1 - C_1) L_{A_{11}}, \tag{3.1} \]

\[ A_{22} = R_{(A_4 A_3)} A_4, \quad B_{22} = R_{B_3} L_{B_4}, \quad C_{22} = R_{(A_4 A_3)} (A_4 R_{A_2} C_2 B_2^\dagger + C_4) L_{B_4}, \tag{3.2} \]

\[ A_{33} = (L_{A_2}, -L_{A_3}), \quad B_{33} = \begin{pmatrix} R_{C_{11}} B_2 \\ -B_4 B_3 \end{pmatrix}, \quad A_{44} = -L_{(A_4 A_3)}, \tag{3.3} \]

\[ E_1 = A_{33}^\dagger C_3 + (A_4 A_3)^\dagger (C_4 B_3 + A_4 R_{A_2} C_3) - A_{33}^\dagger C_2 - B_4^\dagger D_{11} C_{11}^\dagger B_2, \tag{3.4} \]

\[ A = R_{A_{33}} L_{B_{11}}, \quad B = B_2 L_{B_{33}}, \quad C = R_{A_{33}} A_{44}, \quad D = B_3 L_{B_{33}}, \tag{3.5} \]

\[ E = R_{A_{33}} E_1 L_{B_{33}}, \quad M = R_{A C}, \quad N = D L_B, \quad S = C L_M. \tag{3.6} \]

Now we give the fundamental theorem of this section.

**Theorem 3.1.** Let \( A_i, B_i, \) and \( C_i (i = 1, 2, 3, 4) \) be given. Then the following statements are equivalent:

1. The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) is consistent.
2. \( r\left( \begin{bmatrix} C_i \\ A_i \\ B_i \\ 0 \end{bmatrix} \right) = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \) \tag{3.7}
where matrix equations (1.6) can be expressed as

$$ r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r (A_1, A_2) + r (B_1, B_2), \quad (3.8) $$

$$ r \begin{pmatrix} A_4C_3 + C_4B_3 & A_4A_3 \\ B_1B_3 & 0 \end{pmatrix} = r (A_4A_3) + r (B_4B_3), \quad (3.9) $$

$$ r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & A_4C_3 + C_4B_3 & 0 & A_4A_3 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_4A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_4B_3 \end{pmatrix}, \quad (3.10) $$

$$ r \begin{pmatrix} C_2 & A_2 \\ C_3 & A_3 \\ B_2 & 0 \\ B_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}, \quad (3.11) $$

$$ r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ C_3 & 0 & 0 & A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \quad (3.12) $$

$$ r \begin{pmatrix} C_2 & A_2 \\ A_4C_3 + C_4B_3 & A_4A_3 \\ B_2 & 0 \\ B_4B_3 & 0 \end{pmatrix} = r \begin{pmatrix} A_2 \\ A_4A_3 \end{pmatrix} + r \begin{pmatrix} B_2 \\ B_4B_3 \end{pmatrix}. \quad (3.13) $$

(3)

$$ R_{A_2}C_2L_{B_2} = 0, \quad D_{11}L_{C_{11}} = 0, \quad R_{B_{11}}D_{11} = 0, \quad (3.14) $$

$$ R_{A_3}C_3L_{B_3} = 0, \quad R_{A_{22}}C_{22} = 0, \quad C_{22}L_{B_{22}} = 0, \quad (3.15) $$

$$ R_MR_AE = 0, \quad EL_BL_N = 0, \quad R_AEL_D = 0, \quad R_CEL_B = 0. \quad (3.16) $$

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) can be expressed as

$$ X_1 = A_1^\dagger(C_1 - R_{A_2}C_2B_1^\dagger B_1 + A_2U_1B_1) + W_4A_{11} + L_{A_1}W_6, \quad (3.17) $$

$$ X_2 = -R_{A_2}C_2B_2^\dagger + A_2U_1 + V_1R_{B_2}, \quad X_4 = -R_{A_2}C_3B_3^\dagger + A_3U_2 + V_2R_{B_3}, \quad (3.18) $$

$$ X_5 = -R_{(A_4A_3)}(C_4 + A_4R_{A_3}C_3B_3^\dagger - A_4V_2R_{B_3})B_4^\dagger + A_4A_3T_4 + T_3R_{B_4}, \quad (3.19) $$

$$ X_3 = A_2^\dagger C_2 + U_1B_2 + L_{A_2}W_1, \quad or \quad X_3 = A_3^\dagger C_3 + U_2B_3 + L_{A_3}T_1, \quad (3.20) $$

where

$$ U_1 = B_1^\dagger D_{11}C_{11}^\dagger + L_{B_{11}}W_2 + W_3R_{C_{11}}, \quad (3.21) $$
or from Lemma 2.1 that the system (3.31) is consistent if and only if through the following three steps. In the first step, we consider the system (3.31). It follows can solve the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) Observe that system (3.31) has the form of (1.1), and system (3.32) has the form of (1.2). We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) into two parts We consider the following notations (3.33) and (3.34) respectively, (3.35) and

\begin{align}
V_1 &= -R_{A_1}(C_1 - R_{A_2}C_2B_2^*B_1 + A_2U_1 B_1)A_{11}^* + A_4 W_4 + W_5 R_{A_{11}}, \\
V_2 &= A_{22}^* C_{22} B_{22}^* + L_{A_{22}} T_2 + T_3 R_{B_{22}}, \\
U_2 &= (A_4 A_3)^* (C_4 + A_4 R_{A_3} C_3^* B_3^* - A_4 V_2 R_{B_3}) + T_4 B_4 + L_{(A_4 A_3)} T_6, \\
W_1 &= (I_{p_1}, 0) [A_{33}^* (E_1 - L_{B_{11}} W_2 B_2 - A_4 T_6 B_3) - A_{33}^* Z_7 B_{33} + L_{A_{33}} Z_6], \\
T_1 &= (0, I_{p_2}) [A_{33}^* (E_1 - L_{B_{11}} W_2 B_2 - A_4 T_6 B_3) - A_{33}^* Z_7 B_{33} + L_{A_{33}} Z_6], \\
W_3 &= [R_{A_{33}} (E_1 - L_{B_{11}} W_2 B_2 - A_4 T_6 B_3) B_{33}^* + A_{33} A_{33}^* Z_7 + Z_8 R_{B_{33}}] (I_{p_3}, 0), \\
T_4 &= [R_{A_{33}} (E_1 - L_{B_{11}} W_2 B_2 - A_4 T_6 B_3) B_{33}^* + A_{33} A_{33}^* Z_7 + Z_8 R_{B_{33}}] (I_{p_4}), \\
W_2 &= A^* E B^* - A^* C M^* E B^* - A^* S C^* E N^* D B^* - A^* S Z_1 R_N D B^* + L_A Z_2 + Z_3 R_B, \\
T_6 &= M^* E D^* + S^* C^* E N^* + L_M L_S Z_4 + L_M Z_1 R_N + Z_5 R_D,
\end{align}

the remaining \( W_j, T_j, Z_j \) are arbitrary matrices over \( \mathbb{H} \), \( p_1 \) and \( p_2 \) are the column numbers of \( A_2 \) and \( A_3 \), respectively, \( p_3 \) and \( p_4 \) are the row numbers of \( B_1 \) and \( B_4 \), respectively.

**Proof.** We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) into two parts

\[
\begin{aligned}
A_2 X_3 - X_2 B_3 &= C_2, \\
A_1 X_1 - X_2 B_1 &= C_1,
\end{aligned}
\]

and

\[
\begin{aligned}
A_4 X_3 - X_4 B_3 &= C_3, \\
A_4 X_4 - X_5 B_4 &= C_4.
\end{aligned}
\]

Observe that system (3.31) has the form of (1.1), and system (3.32) has the form of (1.2). We can solve the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) through the following three steps. In the first step, we consider the system (3.31). It follows from Lemma 2.1 that the system (3.31) is consistent if and only if \( r \left( \begin{array}{cc} C_n & A_n \\ B_n & 0 \end{array} \right) = r(A_n) + r(B_n) \), \( (n = 1, 2) \),

\[
\begin{aligned}
r \left( \begin{array}{ccc} C_1 & C_2 & A_1 \\ B_1 & B_2 & 0 \\ 0 & 0 & 0 \end{array} \right) &= r(A_1, A_2) + r(B_1, B_2),
\end{aligned}
\]

or

\[
R_{A_2} C_2 L_{B_2} = 0, \quad R_{B_{11}} D_{11} = 0, \quad D_{11} L_{C_{11}} = 0.
\]
In this case, the general solution to the system \((3.31)\) can be expressed as

\[
X_3 = A_3^\dagger C_2 + U_1 B_2 + L_{A_2} W_1,
\]

\[
X_2 = -R_{A_2} C_2 B_2^\dagger + A_2 U_1 + V_1 R_{B_2},
\]

\[
X_1 = A_1^\dagger (C_1 - R_{A_2} C_2 B_2^\dagger B_1 + A_2 U_1 B_1) + W_4 A_{11} + L_{A_1} W_6,
\]

where

\[
U_1 = B_1^\dagger D_{11} C_{11}^\dagger + L_{B_{11}} W_2 + W_3 R_{C_{11}},
\]

\[
V_1 = -R_{A_1} (C_1 - R_{A_2} C_2 B_2^\dagger B_1 + A_2 U_1 B_1) A_{11}^\dagger + A_1 W_4 + W_5 R_{A_{11}},
\]

and \(W_1, \ldots, W_6\) are arbitrary matrices over \(\mathbb{H}\) with appropriate sizes.

In the second step, we consider the system \((3.32)\). It follows from Lemma 2.2 that the system \((3.32)\) is consistent if and only if

\[
r \begin{pmatrix} C_n & A_n \\ B_n & 0 \end{pmatrix} = r(A_n) + r(B_n), \quad (n = 3, 4),
\]

or

\[
r \begin{pmatrix} A_1 C_3 + C_4 B_3 & A_4 A_3 \\ B_4 B_3 & 0 \end{pmatrix} = r(A_4 A_3) + r(B_4 B_3),
\]

In this case, the general solution to the system \((3.32)\) can be expressed as

\[
X_3 = A_3^\dagger C_3 + U_2 B_3 + L_{A_3} T_1,
\]

\[
X_4 = -R_{A_3} C_3 B_3^\dagger + A_3 U_2 + V_2 R_{B_3},
\]

\[
X_5 = -R_{(A_4 A_3)} (C_4 + A_4 R_{A_4} C_2 B_3^\dagger - A_4 V_2 R_{B_4}) B_4^\dagger + A_4 A_3 T_4 + T_5 R_{B_4},
\]

where

\[
V_2 = A_{22}^\dagger C_{22} B_{22}^\dagger + L_{A_{22}} T_2 + T_3 R_{B_{22}},
\]

\[
U_2 = (A_4 A_3)^\dagger (C_4 + A_4 R_{A_4} C_2 B_3^\dagger - A_4 V_2 R_{B_4}) + T_4 B_4 + L_{(A_4 A_3)} T_6,
\]

and \(T_1, \ldots, T_6\) are arbitrary matrices over \(\mathbb{H}\) with appropriate sizes.

In the third step, equating \(X_3\) in \((3.33)\) and \(X_3\) in \((3.34)\) gives

\[
A_3^\dagger C_2 + (B_1^\dagger D_{11} C_{11}^\dagger + L_{B_{11}} W_2 + W_3 R_{C_{11}}) B_2 + L_{A_2} W_1
= A_3^\dagger C_3 + (A_4 A_3)^\dagger (C_4 + A_4 R_{A_4} C_2 B_3^\dagger) B_3 + T_4 B_4 + L_{(A_4 A_3)} T_6 B_3 + L_{A_3} T_1,
\]

i.e.,

\[
A_{33} \begin{pmatrix} W_1 \\ T_1 \end{pmatrix} + (W_3, T_4) B_{33} + L_{B_{11}} W_2 B_2 + A_4 T_6 B_3 = E_1.
\]
Now we want to solve the matrix equation (3.49). It follows from Lemma 2.4 that the matrix equation (3.49) is consistent if and only if

$$R_M R_A E = 0, \ EL_B L_N = 0, R_A E L_D = 0, R_C E L_B = 0. \quad (3.50)$$

Hence, the general solution to the matrix equation (3.49) can be expressed as

$$\begin{pmatrix} W_1 \\ T_1 \end{pmatrix} = A_{33}^\dagger (E_1 - L_{B_1} W_2 B_2 - A_{44} T_6 B_3) - A_{33}^\dagger Z_7 B_{33} + L_{A_{43}} Z_6, \quad (3.51)$$

$$(W_3, \ T_4) = R_{A_{33}} (E_1 - L_{B_1} W_2 B_2 - A_{44} T_6 B_3) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}, \quad (3.52)$$

$$W_2 = A^\dagger E B^\dagger - A^\dagger C M^\dagger E B^\dagger - A^\dagger S C^\dagger E N^\dagger D B^\dagger - A^\dagger S Z_1 R_N D B^\dagger + L_A Z_2 + Z_3 R_B, \quad (3.53)$$

$$T_6 = M^\dagger E D^\dagger + S^\dagger S C^\dagger E N^\dagger + L_M L_S Z_4 + L_M Z_1 R_N + Z_8 R_D, \quad (3.54)$$

where $Z_1, \ldots, Z_8$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Now we want to prove that (3.50) $\iff$ (3.10)-(3.13). At first, we prove that $R_M R_A E = 0$ is equivalent to (3.10). Applying Lemma 2.5 to $R_M R_A E = 0$ gives

$$R_M R_A E = 0 \iff r(R_A E, \ M) = r(M)$$

$$\iff r(R_A E, \ R_A C) = r(R_A C)$$

$$\iff r(A, \ C, \ E) = r(A, \ C)$$

$$\iff r \begin{pmatrix} E_1 & L_{B_{11}} & A_{44} & A_{33} \\ B_{33} & 0 & 0 & 0 \end{pmatrix} = r(L_{B_{11}}, \ A_{44}, \ A_{33}) + r(B_{33})$$

$$\iff r \begin{pmatrix} E_1 & L_{B_{11}} & L_{(A_{44}A_{33})} & L_{A_2} & L_{A_3} \\ R_{C_{11}} B_2 & 0 & 0 & 0 & 0 \\ B_4 B_3 & 0 & 0 & 0 & 0 \end{pmatrix} = r(L_{B_{11}}, L_{(A_{44}A_{33})}, L_{A_2}, L_{A_3}) + r \begin{pmatrix} R_{C_{11}} B_2 \\ B_4 B_3 \end{pmatrix}$$

$$\iff r \begin{pmatrix} E_1 & I & L_{(A_{44}A_{33})} \\ R_{C_{11}} B_2 & 0 & 0 \\ B_4 B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} I \\ B_{11} \end{pmatrix} + r \begin{pmatrix} R_{C_{11}} B_2 \\ B_4 B_3 \end{pmatrix}$$

$$\iff r \begin{pmatrix} E_1 & I & 0 & 0 \\ B_2 & 0 & 0 & 0 \\ B_4 B_3 & 0 & 0 & 0 \\ 0 & A_2 & A_1 & 0 \\ 0 & 0 & A_{44} A_3 & 0 \end{pmatrix} = r \begin{pmatrix} I & I & 0 \\ A_2 & A_1 & 0 \\ 0 & A_4 A_3 & 0 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ B_4 B_3 \end{pmatrix}$$

$$\iff r \begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ 0 & A_{44} A_3 + C_4 B_3 & 0 & A_{44} A_3 \\ B_1 & B_2 & 0 & 0 \\ 0 & B_4 B_3 & 0 & 0 \end{pmatrix} = r \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 A_3 \end{pmatrix} + r \begin{pmatrix} B_1 & B_2 \\ B_4 B_3 \end{pmatrix}$$

$$\iff (3.10).$$
Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations

Example 1. Given the quaternion matrices:

\[
A_1 = \begin{pmatrix}
1 + k & 1 + i - k & i + j \\
-1 & 2k & 2 + j + k \\
k & 1 + i + k & 2 + i + 2j + k
\end{pmatrix},
B_1 = \begin{pmatrix}
2i + k & 0 & 1 + i + j + k \\
2i - j + k & -1 + j & 1 + j \\
j & 1 - j & i + k
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
1 + k & 2 & i \\
2j & 1 - j & -i + k \\
i + j + k & 1 & k
\end{pmatrix},
B_2 = \begin{pmatrix}
-2 + k & i & 1 - j \\
-j & 1 & i - k \\
-2 & 1 + i + j & 0
\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}
1 + k & i + k & j \\
i - j & -1 - j & k \\
1 + i - j + k & -1 + i - j + k & j + k
\end{pmatrix},
B_3 = \begin{pmatrix}
-1 + j + k & 1 + k & i + j \\
-1 - j + k & -1 - j & -i + k \\
2 & -j + k & j + k
\end{pmatrix},
\]

\[
A_4 = \begin{pmatrix}
j & 1 - j & i + k \\
i & -1 + k & j \\
i + j & -j + k & i + j + k
\end{pmatrix},
B_4 = \begin{pmatrix}
1 + k & -1 - k & i + j \\
j & i + k & 1 \\
1 + j + k & -1 + i & 1 + i + j
\end{pmatrix},
\]

\[
C_1 = \begin{pmatrix}
-1 - i + 5k & -2 - 3j + 5k & 3i + 5j \\
-1 - 7i + j + 2k & 3j + 5k & 4 - 8i + 7j \\
-5 - 3i + k & -4 - 3i + 2j + 5k & 1 - 5i + 9j - 4k
\end{pmatrix},
\]

\[
C_2 = \begin{pmatrix}
4i + 6j + 3k & -2 + 5i + 4j & -3 - i - j \\
2 - i + 9j - 6k & 6 - 5i + 5j - 4k & -5 + 2i - 3j + 2k \\
4 + i + 6j + 3k & -2 - i + 3j - 2k & -2 - i + 3j
\end{pmatrix},
\]

\[
C_3 = \begin{pmatrix}
7j + 6k & 4 + i + j & -1 + 2i + 6j \\
-3 - 5i + 2j + k & -1 + 6j + 5k & 2 - i + 4j \\
-8 + 2i + 3j + 6k & 3 + 4i + 9j - 2k & -2 + i + 3j - 2k
\end{pmatrix},
\]

\[
C_4 = \begin{pmatrix}
-1 - 2i - j - 3k & 3 - 3i - 2j + k & 1 + 2i \\
2 - 3i - 3j - k & -5 - 2i + j + k & 1 - 2i - 2j \\
1 - 5i - 4j - 4k & -2 - 5i - j + 2k & 2 - 2j
\end{pmatrix}.
\]

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6). Check that

\[
r \left( \begin{pmatrix}
C_i \\
A_i \\
B_i
\end{pmatrix} \right) = r(A_i) + r(B_i) = \begin{cases}
4, & \text{if } i = 1 \\
6, & \text{if } i = 2 \\
3, & \text{if } i = 3 \\
4, & \text{if } i = 4
\end{cases}
\]
satisfy the system (1.6).

And

\[
\begin{align*}
&\text{real quaternion matrix equations (1.6) is consistent. Note that All the rank equalities in (3.7)-(3.13) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.6) is consistent. Note that}
\end{align*}
\]

\[
X_1 = \begin{pmatrix} 2 + \text{j} + \text{k} & 1 - 2\text{i} + \text{k} & \text{i} - 2\text{k} \\ -2 & \text{i} + \text{j} & 1 + \text{i} + 2\text{j} \\ \text{j} + \text{k} & 1 - \text{i} + \text{j} + \text{k} & 1 + 2\text{i} + 2\text{j} - 2\text{k} \end{pmatrix},
\]

\[
X_2 = \begin{pmatrix} \text{i} + \text{j} & -\text{i} - \text{j} & \text{k} \\ 1 + 2\text{j} & \text{i} + \text{j} + \text{k} & -1 + \text{i} + \text{j} - \text{k} \\ 2 + \text{k} & -\text{k} & 1 + \text{j} + 2\text{k} \end{pmatrix},
\]

\[
X_3 = \begin{pmatrix} 2\text{i} + \text{k} & 1 + 3\text{i} - \text{k} & \text{j} \\ 1 + \text{j} & -1 & \text{i} - \text{j} \\ 1 + 2\text{i} + \text{j} + \text{k} & 3\text{i} - \text{k} & \text{i} \end{pmatrix},
\]

\[
X_4 = \begin{pmatrix} -2 + \text{j} & -1 + 2\text{i} & \text{k} \\ 0 & 1 + \text{k} & \text{i} + \text{j} \\ -2 + \text{k} & \text{j} - \text{k} & 1 \end{pmatrix},
\]

and

\[
X_5 = \begin{pmatrix} -2 + \text{j} & -1 + 2\text{i} & \text{k} \\ 0 & 1 + \text{k} & \text{i} + \text{j} \\ -2 + \text{k} & \text{j} - \text{k} & 1 \end{pmatrix}
\]

satisfy the system (1.6).
Let \( A_4, B_4, \) and \( C_4 \) vanish in Theorem 3.1. Then we can obtain some necessary and sufficient conditions and general solution to the system of coupled generalized Sylvester real quaternion matrix equations (1.3).

**Corollary 3.2.** [35] Let \( A_i, B_i, \) and \( C_i(i = 1, 2, 3) \) be given. Set

\[
A_4 = A_2L_{A_3}, B_4 = R_{B_1}B_2, A = R_{A_4}A_2, B = B_3L_{B_1},
\]

\[
C = R_{A_4}A_1, D = B_2L_{B_4}, M = R_{A_4}C, N = D_{L_B}, S = CL_M,
\]

\[
C_4 = C_2 - A_2A_3^T C_3 - R_{A_4}C_1 B_1^T B_2, E = R_{A_4}C_4 L_{B_4}.
\]

Then the following statements are equivalent:

1. The system of coupled generalized Sylvester real quaternion matrix equations (1.3) is consistent.
2. \[
    r \left( \begin{array}{cc}
          C_i & A_i \\
          B_i & 0
        \end{array} \right) = r(A_i) + r(B_i), (i = 1, 2, 3),
\]
3. \[
    r \left( \begin{array}{ccc}
          A_1 & A_2 & C_1 \\
          0 & 0 & B_1 \\
          B_2 & 0 & B_2
        \end{array} \right) = r(A_1 A_2) + r(B_1 B_2),
\]
4. \[
    r \left( \begin{array}{ccc}
          B_2 & 0 \\
          B_3 & 0 \\
          C_2 & A_2 \\
          C_3 & A_3
        \end{array} \right) = r \left( \begin{array}{c}
          A_2 \\
          A_3
        \end{array} \right) + r \left( \begin{array}{c}
          B_2 \\
          B_3
        \end{array} \right),
\]
5. \[
    r \left( \begin{array}{ccc}
          C_2 & C_1 & A_1 \\
          C_3 & 0 & 0 \\
          B_2 & B_1 & 0 \\
          B_3 & 0 & 0
        \end{array} \right) = r \left( \begin{array}{cc}
          A_1 & A_2 \\
          0 & A_3
        \end{array} \right) + r \left( \begin{array}{cc}
          B_2 & B_1 \\
          B_3 & 0
        \end{array} \right).
\]

In this case, the general solution to the coupled generalized Sylvester real quaternion matrix equations (1.3) can be expressed as

\[
    X = A_1^T C_1 - U_1 B_1 - L_{A_1} U_2, \quad Y = -R_{A_4} C_1 B_1^T - A_1 U_1 - U_3 R_{B_1},
\]

\[
    Z = A_3^T C_3 + V_1 B_3 + L_{A_3} V_2, \quad W = -R_{A_4} C_3 B_3^T + A_3 V_1 + V_3 R_{B_3},
\]

where

\[
    V_2 = A_1^T (C_4 - A_2 V_1 B_3 - A_1 U_1 B_2) - A_1^T T_7 B_4 + L_{A_4} T_6,
\]

\[
    U_3 = R_{A_4} (C_4 - A_2 V_1 B_3 - A_1 U_1 B_2) B_1^T + A_4 A_3^T T_7 + T_8 R_{B_4},
\]
Theorem 4.1. Let \( A_i, B_i, \) and \( C_i(i = 1, 2, 3, 4) \) be given. Then the following statements are equivalent:

1. The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) is consistent.

2. \[ r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \] (4.1)

3. \[ r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1), \] (4.2)

4. \[ r \begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r(A_3, A_4) + r(B_3, B_4), \] (4.3)

5. \[ r \begin{pmatrix} A_3C_2 + C_3B_2 & C_4 & A_3A_2 & A_4 \\ B_3B_2 & B_4 & 0 & 0 \end{pmatrix} = r(A_3A_2, A_4) + r(B_3B_2, B_4), \] (4.4)

6. \[ r \begin{pmatrix} A_3A_2C_1 + A_3C_2B_1 + C_3B_2B_1 & A_3A_2A_1 \\ B_3B_2B_1 & 0 \end{pmatrix} = r(A_3A_2A_1) + r(B_3B_2B_1), \] (4.5)
In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) can be expressed as

\[ X_1 = A_1^\dagger C_1 + U_1 B_1 + L_{A_1} W_1, \quad X_2 = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + V_1 R_{B_1}, \]

\[ X_4 = -R_{A_3} C_3 B_3^\dagger + A_3 U_2 + V_2 R_{B_3}, \]

\[ X_5 = A_1^\dagger (C_4 - R_{A_3} C_3 B_3^\dagger B_4 + A_3 U_1 B_4) + T_4 A_{22} + L_{A_4} T_6, \]

\[ X_3 = -R_{(A_2 A_4)} (C_2 + A_2 R_{A_1} C_1 B_1^\dagger - A_2 V_1 R_{B_1}) B_2^\dagger + A_2 A_1 W_4 + W_5 R_{B_2}, \]

or

\[ X_3 = A_3^\dagger C_3 + U_2 B_3 + L_{A_3} T_1, \]

where

\[ V_1 = A_1^\dagger C_{11} B_{11}^\dagger + L_{A_{11}} W_2 + W_3 R_{B_{11}}, \]

\[ U_1 = (A_2 A_1)^\dagger (C_2 + A_2 R_{A_1} C_1 B_1^\dagger - A_2 V_1 R_{B_1}) + W_4 B_2 + L_{(A_2 A_4)} W_6, \]

\[ U_2 = B_{22}^\dagger D_{22} C_{22}^\dagger + L_{B_{22}} T_2 + T_3 R_{C_{22}}, \]

\[ V_2 = -R_{A_4} (C_4 - R_{A_3} C_3 B_3^\dagger B_4 + A_3 U_2 B_4) A_{22}^\dagger + A_4 T_4 + T_5 R_{A_{22}}, \]

\[ W_4 = (I_{p_1}, 0) [A_{33}^\dagger (E_1 - A_{11} W_3 A_{44} - B_{44} T_2 B_3) - A_{33}^\dagger Z_7 B_{33} + L_{A_{33}} Z_6], \]

\[ T_1 = (0, I_{p_2}) [A_{33}^\dagger (E_1 - A_{11} W_3 A_{44} - B_{44} T_2 B_3) - A_{33}^\dagger Z_7 B_{33} + L_{A_{33}} Z_6], \]

\[ W_5 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} - B_{44} T_2 B_3) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix}, \]

\[ T_3 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} - B_{44} T_2 B_3) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_3} \end{pmatrix}, \]
Given the quaternion matrices:

\[ W_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SZ_1 R_N DB^\dagger + L_A Z_2 + Z_3 R_B, \]

\[ T_2 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_M L_S Z_4 + L_M Z_1 R_N + Z_5 R_D, \]

the remaining \( W_j, T_j, Z_j \) are arbitrary matrices over \( \mathbb{H} \), \( p_1 \) and \( p_2 \) are the column numbers of \( A_1 \) and \( A_3 \), respectively, \( p_3 \) and \( p_4 \) are the row numbers of \( B_2 \) and \( B_4 \), respectively.

**Proof.** We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.7) into two parts

\[
\begin{cases}
  A_1 X_1 - X_2 B_1 = C_1, \\
  A_2 X_2 - X_3 B_2 = C_2,
\end{cases} \tag{4.8}
\]

and

\[
\begin{cases}
  A_3 X_3 - X_4 B_3 = C_3, \\
  A_4 X_5 - X_4 B_4 = C_4. \tag{4.9}
\end{cases}
\]

Applying the main idea of Theorem 3.1, Lemma 2.2, Lemma 2.4 and Lemma 2.5, we can prove Theorem 4.1.

Now we give an example to illustrate Theorem 4.1.

**Example 2.** Given the quaternion matrices:

\[
A_1 = \begin{pmatrix}
  i & j & 1 + k \\
  k & i + j - 2k & -2 + k \\
  1 + i + j & 2 - i & -j
\end{pmatrix}, \quad
B_1 = \begin{pmatrix}
  1 & k & i + k \\
  i & k & -1 + k \\
  1 + i & 2k & -1 + i + 2k
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
  1 + i + k & 1 - i & 1 + j + k \\
  -1 + i - j & 1 + i & i - j + k \\
  2i - j + k & 2 & 1 + i + 2k
\end{pmatrix}, \quad
B_2 = \begin{pmatrix}
  1 + j - k & -1 + 2j - k & 2 \\
  i + j & 1 + k & 1 + i - j \\
  1 + i + 2j - k & 2j & 3 + i - j
\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}
  2j & 1 - j & i + k \\
  i + j + k & 1 + j & 1 + i + k \\
  i + 3j + k & 2 & -1 + i + k
\end{pmatrix}, \quad
B_3 = \begin{pmatrix}
  -1 & 1 + j & i - j \\
  -i & i + 2j & -1 - 2j \\
  -1 - i & 1 + i + k & -1 + i - k
\end{pmatrix},
\]

\[
A_4 = \begin{pmatrix}
  2 + i + j - 2k & 1 - 2i - j & i \\
  1 + j - 2k & 0 & -i + j \\
  3 + i & 1 - 2i + k & 0
\end{pmatrix}, \quad
B_4 = \begin{pmatrix}
  j & 1 - j & i + k \\
  i & 1 + j & -k \\
  i + j & 2 & i
\end{pmatrix},
\]

\[
C_1 = \begin{pmatrix}
  1 + i - j - 2k & i - 2j + 6k & 1 + i - 2j + 4k \\
  i - 4j + 3k & 3 - 2i + 3j - 6k & 1 - i - 2j + k \\
  4i + j - 6k & 1 - 2i - 4j + 8k & 2 - 7j + 7k
\end{pmatrix},
\]

\[
C_2 = \begin{pmatrix}
  3 - 3i + 3j + 3k & -2 - i + j - 9k & -6 - 6i - 3j + 8k \\
  -2 - 8j + 8k & i + 2j & -4 - 6i - 5j - 6k \\
  1 - 3i - 5j + 11k & -2 + 3j - 9k & -10 - 12i - 8j + 2k
\end{pmatrix}.
\]
All the rank equalities in (4.1)-(4.7) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (7.7). Check that

\[
C_3 = \begin{pmatrix}
3 - 4i + 4j + 3k & -2 + 6i + 2j - k \\
4 - i + 6j - k & -4 - i - 4j + 10k \\
1 - 4i + 8j + 4k & -1 + 8i + 3k
\end{pmatrix},
\]

\[
C_4 = \begin{pmatrix}
2 + i - 3k & 2 + 2j + 3k \\
-1 + 2i - 3j + k & -3 + 6i - 4j + k \\
1 - 3i + 5j - 3k & 1 - 3j + 2k
\end{pmatrix}.
\]

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (7.7). Check that

\[
r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i) = \begin{cases} 5, & \text{if } i = 1 \\ 3, & \text{if } i = 2 \\ 4, & \text{if } i = 3 \\ 5, & \text{if } i = 4 \end{cases},
\]

\[
r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1) = 3,
\]

\[
r \begin{pmatrix} C_3 & C_4 & A_3 & A_4 \\ B_3 & B_4 & 0 & 0 \end{pmatrix} = r(A_3, A_4) + r(B_3, B_4) = 6,
\]

\[
r \begin{pmatrix} A_3C_2 + C_3B_2 & C_4 & A_3A_2 & A_4 \\ B_3B_2 & B_4 & 0 & 0 \end{pmatrix} = r(A_3A_2, A_4) + r(B_3B_2, B_4) = 6,
\]

\[
r \begin{pmatrix} A_3A_2C_1 + A_3C_2B_1 + C_3B_2B_1 & A_3A_2A_1 \\ B_3B_2B_1 & 0 \end{pmatrix} = r(A_3A_2A_1) + r(B_3B_2B_1) = 3,
\]

\[
r \begin{pmatrix} A_3C_2 + C_3B_2 & A_3A_2 \\ B_3B_2 & 0 \end{pmatrix} = r(A_3A_2) + r(B_3B_2) = 3,
\]

\[
r \begin{pmatrix} A_3A_2C_1 + A_3C_2B_1 + C_3B_2B_1 & C_4 & A_4 & A_3A_2A_1 \\ B_3B_2B_1 & B_4 & 0 & 0 \end{pmatrix} = r(A_3A_2A_1, A_4) + r(B_3B_2B_1, B_4) = 6.
\]

All the rank equalities in (4.1)-(4.7) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (7.7) is consistent. Note that

\[
X_1 = \begin{pmatrix} 1 - k & i + j + 2k \\
-1 + j & -i - 2j + k \\
j - k & -j + 3k \end{pmatrix},
\]

\[
X_2 = \begin{pmatrix} 1 + j & i & -1 + k \\ i + j + k & -1 & -i - j + k \\ 1 + i + 2j + k & -1 + i & -1 - i - j + 2k \end{pmatrix},
\]

\[
X_3 = \begin{pmatrix} -2k & 2 + k & i \\ i + 2j & 1 - j & 1 - i \\ i + 2j - 2k & 3 - j + k & 1 \end{pmatrix},
\]

\[
X_4 = \begin{pmatrix} k & i & 1 - k \\ 1 - 2i + j - k & 1 - 3i & 1 + i + 2j + k \\ -1 & 2 + k & i \end{pmatrix}.
\]
\[ X_5 = \begin{pmatrix} i + j & k & 1 + k \\ 1 + 2j & i & 1 + j \\ j + k & k & 1 + 2j \end{pmatrix} \]

satisfy the system \[(1.7)\].

Let \(A_4, B_4,\) and \(C_4\) vanish in Theorem \[4.1\]. Then we can obtain some necessary and sufficient conditions and general solution to the system of coupled generalized Sylvester real quaternion matrix equations \[(1.4)\].

**Corollary 4.2.** \(\text{[5]}\) Let \(A_i, B_i,\) and \(C_i (i = 1, 2, 3)\) be given. Set
\[
\begin{align*}
A &= R_{(A_2A_1)}A_2, \quad B = R_{B_1}L(B_3B_2), \quad C = R_{(A_2A_1)}L_{A_3}, \\
D &= B_2L(B_3B_2), \quad M = R_A C, \quad N = DL_B, \quad S = CL_M, \\
C_4 &= C_2 + A_3^1C_3B_2 + A_2 R_{A_1}C_1B_1^1, \quad E = R_{(A_2A_1)}C_4L(B_3B_2).
\end{align*}
\]
Then the following statements are equivalent:

1. The system of coupled generalized Sylvester real quaternion matrix equations \[(1.4)\] is consistent.
2. \(r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), (i = 1, 2, 3),\)
3. \(r \begin{pmatrix} A_3C_2 + C_3B_2 & A_3A_2 \\ B_3B_2 & 0 \end{pmatrix} = r(A_3A_2) + r(B_3B_2),\)
4. \(r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1),\)
5. \(r \begin{pmatrix} A_3A_2C_1 + A_3C_3B_1 + C_3B_2B_1 & A_3A_2A_1 \\ B_3B_2B_1 & 0 \end{pmatrix} = r(A_3A_2A_1) + r(B_3B_2B_1).\)

In this case, the general solution to the coupled generalized Sylvester real quaternion matrix equations \[(1.4)\] can be expressed as
\[
X = A_1^1C_1 + U_1B_1 + L_{A_1}U_2, \quad Y = -R_{A_1}C_1B_1^1 + A_1U_1 + U_3R_{B_1},
\]
\[
Z = A_3^1C_3 - V_1B_3 - L_{A_3}V_2, \quad W = -R_{A_3}C_3B_3^1 - A_3V_1 - V_3R_{B_3},
\]
where
\[
U_1 = (A_2A_1)^1(C_4 - A_2U_3R_{B_1} - L_{A_3}V_2B_2) - (A_2A_1)^1T_7(B_3B_2) + L_{(A_2A_1)}T_6,
\]
\[ V_1 = R_{(A_2A_1)}(C_4 - A_2U_3R_{B_1} - L_{A_3}V_2B_2)(B_3B_2)^\dagger + (A_2A_1)(A_2A_1)^\dagger T_7 + T_8 R_{(B_3B_2)}, \]
\[ U_3 = A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger ST_2R_NDB^\dagger + L_A T_4 + T_5 R_B, \]
\[ V_2 = M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_M L_S T_1 + L_M T_2 R_N + T_3 R_D, \]

and \(U_2, V_3, T_1, \ldots, T_8\) are arbitrary matrices over \(\mathbb{H}\) with appropriate sizes.

5. Some solvability conditions and the general solution to system (1.8)

In this section, we consider the solvability conditions and the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8). For simplicity, put

\[ A_{11} = R_{(A_2A_1)}A_2, \quad B_{11} = R_{B_1}L_{B_2}, \quad C_{11} = R_{(A_2A_1)}(A_2R_{A_1}C_1B_1^\dagger + C_2)L_{B_2}, \]
\[ A_{22} = R_{(A_3A_4)}A_3, \quad B_{22} = R_{B_4}L_{B_3}, \quad C_{22} = R_{(A_3A_4)}(A_3R_{A_4}C_4B_4^\dagger + C_3)L_{B_3}, \]
\[ A_{33} = (A_2A_1, -A_3A_4), \quad B_{33} = \begin{pmatrix} R_{B_2} \\ -R_{B_3} \end{pmatrix}, \quad A_{44} = R_{B_{11}}R_{B_1}B_2^\dagger, \quad B_{44} = R_{B_{22}}R_{B_4}B_3^\dagger, \]
\[ E_1 = R_{(A_2A_1)}C_2B_2^\dagger + A_{11}R_{A_1}C_1B_1^\dagger B_2^\dagger - C_{11}B_{11}^\dagger R_{B_1}B_2^\dagger - R_{(A_3A_4)}C_3B_3^\dagger - A_{22}R_{A_4}C_4B_4^\dagger B_2^\dagger + C_{22}B_{22}^\dagger R_{B_4}B_3^\dagger, \]
\[ A = R_{A_{33}}A_{11}, \quad B = A_{44}L_{B_{33}}, \quad C = -R_{A_{33}}A_{22}, \quad D = B_{44}L_{B_{33}}, \]
\[ E = R_{A_{33}}E_1L_{B_{33}}, \quad M = R_A C, \quad N = D L_B, \quad S = C L_M. \]

Now we give the fundamental theorem of this section.

**Theorem 5.1.** Let \(A_i, B_i, \) and \(C_i (i = 1, 2, 3, 4)\) be given. Then the following statements are equivalent:

1. The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) is consistent.

2. \[ r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \] (5.1)
\[ r \begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1), \] (5.2)
\[ r \begin{pmatrix} A_3C_4 + C_3B_4 & A_3A_4 \\ B_3B_4 & 0 \end{pmatrix} = r(A_3A_4) + r(B_3B_4), \] (5.3)
\[ r \begin{pmatrix} C_2 & C_3 & A_2 & A_3 \\ B_2 & B_3 & 0 & 0 \end{pmatrix} = r(A_2, A_3) + r(B_2, B_3), \] (5.4)
\[ r \begin{pmatrix} A_2 C_1 + C_2 B_1 & A_3 C_4 + C_3 B_4 & A_2 A_1 & A_3 A_4 \\ B_2 B_1 & B_3 B_4 & 0 & 0 \end{pmatrix} = r(A_2 A_1, A_3 A_4) + r(B_2 B_1, B_3 B_4), \quad (5.5) \]

\[ r \begin{pmatrix} C_2 & A_3 C_4 + C_3 B_4 & A_2 & A_3 A_4 \\ B_2 & B_3 B_4 & 0 & 0 \end{pmatrix} = r(A_2, A_3 A_4) + r(B_2, B_3 B_4), \quad (5.6) \]

\[ r \begin{pmatrix} A_2 C_1 + C_2 B_1 & C_3 & A_2 A_1 & A_3 \\ B_2 B_1 & B_3 & 0 & 0 \end{pmatrix} = r(A_2 A_1, A_3) + r(B_2 B_1, B_3). \quad (5.7) \]

(3)

\[ R_{A_1 C_1 L_{B_1}} = 0, \ R_{A_{11} C_{11}} = 0, \ C_{11} L_{B_{11}} = 0, \]
\[ R_{A_4 C_4 L_{B_4}} = 0, \ R_{A_{22} C_{22}} = 0, \ C_{22} L_{B_{22}} = 0, \]
\[ R_{M R_A E} = 0, \ E L_{B L_N} = 0, \ R_A E L_D = 0, \ R_C E L_B = 0. \]

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.3) can be expressed as

\[ X_1 = A_1^\dagger C_1 + U_1 B_1 + L_{A_1 W_1}, \]
\[ X_2 = -R_{A_1 C_1 B_1^\dagger} + A_1 U_1 + V_1 R_{B_1}, \]
\[ X_4 = -R_{A_4 C_4 B_4^\dagger} + A_4 U_2 + V_2 R_{B_4}, \]
\[ X_5 = A_1^\dagger C_4 + U_2 B_4 + L_{A_4 T_1}, \]
\[ X_3 = -R_{(A_2 A_1)} (C_2 + A_2 R_{A_1 C_1 B_1^\dagger} - A_2 V_1 R_{B_1}) B_2^\dagger + A_2 A_1 W_4 + W_5 R_{B_2}, \]
or

\[ X_3 = -R_{(A_3 A_4)} (C_3 + A_3 R_{A_4 C_4 B_4^\dagger} - A_3 V_2 R_{B_4}) B_3^\dagger + A_3 A_4 T_4 + T_5 R_{B_3}, \]

where

\[ V_1 = A_{11}^\dagger C_{11} B_{11}^\dagger + L_{A_{11} W_2} + W_3 R_{B_{11}}, \]
\[ U_1 = (A_2 A_1)^\dagger (C_2 + A_2 R_{A_1 C_1 B_1^\dagger} - A_2 V_1 R_{B_1}) + W_4 B_2 + L_{(A_2 A_1) W_6}, \]
\[ V_2 = A_{22}^\dagger C_{22} B_{22}^\dagger + L_{A_{22} T_2} + T_3 R_{B_{22}}, \]
\[ U_2 = (A_3 A_4)^\dagger (C_3 + A_3 R_{A_4 C_4 B_4^\dagger} - A_3 V_2 R_{B_4}) + T_4 B_3 + L_{(A_3 A_4) T_6}, \]
\[ W_4 = (I_{p_1}, 0) [A_{33}^\dagger (E_1 - A_{11} W_3 A_{44} + A_{22} T_3 B_{44}) - A_{33}^\dagger Z_7 B_{33} + L_{A_{33} Z_6}], \]
\[ T_1 = (0, I_{p_2}) [A_{33}^\dagger (E_1 - A_{11} W_3 A_{44} + A_{22} T_3 B_{44}) - A_{33}^\dagger Z_7 B_{33} + L_{A_{33} Z_6}], \]
\[ W_5 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} + A_{22} T_3 B_{44}) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix}, \]
\[ T_5 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} + A_{22} T_3 B_{44}) B_{33}^\dagger + A_{33} A_{33}^\dagger Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_4} \end{pmatrix}. \]
Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) into two parts

\[ \begin{align*}
W_3 &= A^\dagger EB^\dagger - A^\dagger CM^\dagger EB^\dagger - A^\dagger SC^\dagger EN^\dagger DB^\dagger - A^\dagger SZ_1R_NDB^\dagger + L_ADZ_2 + Z_3R_B, \\
T_3 &= M^\dagger ED^\dagger + S^\dagger SC^\dagger EN^\dagger + L_MLSZ_4 + L_MZ_1R_N + Z_5R_D,
\end{align*} \]

the remaining \( W_j, T_j, Z_j \) are arbitrary matrices over \( \mathbb{H} \), \( p_1 \) and \( p_2 \) are the column numbers of \( A_1 \) and \( A_4 \), respectively, \( p_3 \) and \( p_4 \) are the row numbers of \( B_2 \) and \( B_3 \), respectively.

**Proof.** We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) into two parts

\[ \begin{align*}
A_1X_1 - X_2B_1 &= C_1, \\
A_2X_2 - X_3B_2 &= C_2,
\end{align*} \]

and

\[ \begin{align*}
A_3X_4 - X_3B_3 &= C_3, \\
A_4X_5 - X_4B_4 &= C_4.
\end{align*} \]

Applying the main idea of Theorem 5.1, Lemma 2.1, Lemma 2.2, Lemma 2.4 and Lemma 2.5, we can prove Theorem 5.1. \( \square \)

Now we give an example to illustrate Theorem 5.1.

**Example 3.** Given the quaternion matrices:

\[
A_1 = \begin{pmatrix}
i + j + k & 2 + i + j - k \\
-1 + j + k & -1 + 2i + j - k
\end{pmatrix}, \quad B_1 = \begin{pmatrix}1 + k & j - k \\
i + 2k & 2j - 2k\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}i & 2 + j \\
1 + i + k & -j\end{pmatrix}, \quad B_2 = \begin{pmatrix}1 + j + 2k & i + 3k \\
j & 1 + i\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}1 + k & i + k \\
1 + i + j + k & -1 + i + j + k\end{pmatrix}, \quad B_3 = \begin{pmatrix}2 + k & -i - 2j \\
2 - 2i - k & -1 - i + 2j\end{pmatrix},
\]

\[
A_4 = \begin{pmatrix}3i + j & 1 + 2j \\
2 + k & 0\end{pmatrix}, \quad B_4 = \begin{pmatrix}-j & 1 + 2j \\
-k & i + 2k\end{pmatrix},
\]

\[
C_1 = \begin{pmatrix}1 - 5i + j + k & 3 + 8i - 7j - 3k \\
3 + 2i - 2j - 7k & -8 + 3i - 4j - 4k\end{pmatrix}, \quad C_2 = \begin{pmatrix}-1 + i + 3j + 2k & 5 + 3i + 3j + 7k \\
1 + 2i + 4j + 5k & -3i - 2j + 5k\end{pmatrix},
\]

\[
C_3 = \begin{pmatrix}-7i - 2j - 2k & -3 - 7i + 4j + 5k \\
-3 + 3i + j + k & -6 - 3i - j + 5k\end{pmatrix}, \quad C_4 = \begin{pmatrix}-2 + 7i - 4j & 2 - 7i + 4j + 8k \\
-6 + i - j + k & 13 - 3i + j + 4k\end{pmatrix}.
\]

Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8). Check that

\[
r \begin{pmatrix}C_i & A_i \\
B_i & 0\end{pmatrix} = r(A_i) + r(B_i) = \begin{cases}
4, & \text{if } i = 1, 2, 3 \\
3, & \text{if } i = 4
\end{cases},
\]
The rank equalities in (5.1)-(5.7) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) is consistent. Note that

$$r\begin{pmatrix} A_2C_1 + C_2B_1 & A_2A_1 \\ B_2B_1 & 0 \end{pmatrix} = r(A_2A_1) + r(B_2B_1) = 4,$$

$$r\begin{pmatrix} A_3C_4 + C_3B_4 & A_3A_4 \\ B_3B_4 & 0 \end{pmatrix} = r(A_3A_4) + r(B_3B_4) = 3,$$

$$r\begin{pmatrix} C_2 & C_3 & A_2 & A_3 \\ B_2 & B_3 & 0 & 0 \end{pmatrix} = r(A_2, A_3) + r(B_2, B_3) = 4,$$

$$r\begin{pmatrix} A_2C_1 + C_2B_1 & A_3C_4 + C_3B_4 & A_2A_1 & A_3A_4 \\ B_2B_1 & B_3B_4 & 0 & 0 \end{pmatrix} = r(A_2A_1, A_3A_4) + r(B_2B_1, B_3B_4) = 4,$$

$$r\begin{pmatrix} C_2 & A_3C_4 + C_3B_4 & A_2 & A_3A_4 \\ B_2 & B_3 & 0 & 0 \end{pmatrix} = r(A_2, A_3A_4) + r(B_2, B_3B_4) = 4,$$

$$r\begin{pmatrix} A_2C_1 + C_2B_1 & C_3 & A_2A_1 & A_3 \\ B_2B_1 & B_3 & 0 & 0 \end{pmatrix} = r(A_2A_1, A_3) + r(B_2B_1, B_3) = 4.$$

All the rank equalities in (5.1)-(5.7) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.8) is consistent. Note that

$$X_1 = \begin{pmatrix} i + j & -1 + k \\ 2 + k & 2i - j \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 + 2i + j & -i + 2j \\ k & 1 + 2k \end{pmatrix},$$

$$X_3 = \begin{pmatrix} i & -1 + j \\ -1 & -i + j + k \end{pmatrix}, \quad X_4 = \begin{pmatrix} -1 + 2j & 1 + 3j \\ -i + 2j + 2k & i + 3j + 3k \end{pmatrix},$$

and

$$X_5 = \begin{pmatrix} k & 1 + 2j \\ i + k & 1 - i + j - k \end{pmatrix}$$

satisfy the system (1.8).

Let $A_4, B_4,$ and $C_4$ vanish in Theorem 5.1. Then we can obtain some necessary and sufficient conditions and general solution to the system of coupled generalized Sylvester real quaternion matrix equations (1.9).

**Corollary 5.2.** Let $A_i, B_i,$ and $C_i (i = 1, 2, 3)$ be given. Set

$$A = R_{(A_2A_1)}A_2, B = R_{B_1L(R_{B_3B_2})}, C = R_{(A_2A_1)}A_3,$$

$$D = B_2L(R_{B_3B_2}), C_4 = C_2 + A_2^1R_{A_1}C_1B_1^1 - R_{A_3}C_3B_3^1B_2,$$

$$E = R_{(A_2A_1)}C_4L(R_{B_3B_2}), M = R_{AC}, N = DLB, S = CLM.$$
Then the following statements are equivalent:

(1) The system of coupled generalized Sylvester real quaternion matrix equations \((1.5)\) is consistent.

(2) 
\[
\begin{array}{c}
\begin{bmatrix}
C_i & A_i \\
B_i & 0
\end{bmatrix} = r(A_i) + r(B_i), (i = 1, 2, 3), \\
\begin{bmatrix}
A_2 & A_3 & C_2 & C_3 \\
0 & 0 & B_2 & B_3
\end{bmatrix} = r(A_2, A_3) + r(B_2, B_3), \\
\begin{bmatrix}
A_2C_1 + C_2B_1 & A_2A_1 \\
B_2B_1 & 0
\end{bmatrix} = r(A_2A_1) + r(B_2B_1), \\
\begin{bmatrix}
A_3 & A_2A_1 & C_3 & A_2C_1 + C_2B_1 \\
0 & 0 & B_3 & B_2B_1
\end{bmatrix} = r(A_3, A_2A_1) + r(B_3, B_2B_1).
\end{array}
\]

(3) 
\[
\begin{array}{c}
R_{A_i}C_iL_{B_i} = 0, (i = 1, 2), R_{M}R_{A}E = 0, \\
E_{L}B_{L}N = 0, R_{A}E_{L}D = 0, R_{C}E_{L}B = 0.
\end{array}
\]

In this case, the general solution to the coupled generalized Sylvester real quaternion matrix equations \((1.5)\) can be expressed as

\[
X = A_1^\dagger C_1 + U_1B_1 + L_{A_i}U_2, 
Y = -R_{A_i}C_1B_1^\dagger + A_1U_1 + U_3R_{B_i}, 
Z = -R_{A_2}C_3B_3^\dagger - A_3V_1 - V_3R_{B_3}, 
W = A_3^\dagger C_3 - V_1B_3 - L_{A_3}V_2,
\]

where

\[
U_1 = (A_2A_1)^\dagger(C_4 - A_2U_3R_{B_1} - A_3V_1B_2) - (A_2A_1)^\dagger T_7(R_{B_3}B_2) + L_{(A_2A_1)}T_6,
\]

\[
V_3 = R_{(A_2A_1)}(C_4 - A_2U_3R_{B_1} - A_3V_1B_2)(R_{B_3}B_2)^\dagger + (A_2A_1)(A_2A_1)^\dagger T_7 + T_8R_{(R_{B_3}B_2)},
\]

\[
U_3 = A^\dagger EB + A^\dagger CM^\dagger EB - A^\dagger SC^\dagger EN^\dagger DB - A^\dagger ST_2R_{M}DB + L_{A}T_3 + T_3R_{B},
\]

\[
V_1 = M^\dagger ED + S^\dagger SC^\dagger EN + L_{M}L_{S}T_1 + L_{M}T_2R_{N} + T_3R_{D},
\]

and \(U_2, V_2, T_1, \ldots, T_8\) are arbitrary matrices over \(\mathbb{H}\) with appropriate sizes.
6. Some solvability conditions and the general solution to system \((1.9)\)

Our goal of this section is to give some necessary and sufficient conditions and the general solution to the system \((1.9)\). Set

\[
A_{j3} = R_{(A_2, A_{2j-1})} A_{2j}, \quad B_{j3} = R_{B_{2j-1}} L_{B_{2j}}, \quad C_{j3} = R_{(A_2, A_{2j-1})} (A_{2j} R_{A_{2j-1}} C_{2j-1} B_{2j}^\dagger + C_{2j}) L_{B_{2j}},
\]

\[(j = 1, 2), \quad A_{33} = (A_2 A_1, -L_{A_3}), \quad B_{33} = \begin{pmatrix} R_{B_2} \\ -B_{4} B_{3} \end{pmatrix}, \quad A_{44} = R_{B_{41}} R_{B_{1} B_{2}^\dagger}, \quad B_{44} = -L_{(A_4 A_3)},
\]

\[E_1 = A_j^\dagger C_3 + (A_4 A_3)^\dagger C_3 B_3 + (A_4 A_3)^\dagger A_4 R_{A_3} C_3 + R_{(A_2 A_1)} C_2 B_2^\dagger + A_{11} R_{A_1} C_1 B_2^\dagger - C_{11} B_{11}^\dagger R_{B_1} B_2^\dagger,
\]

\[
A = R_{A_{33}} A_{11}, \quad B = A_{44} L_{B_{33}}, \quad C = R_{A_{33}} B_{44}, \quad D = B_{3} L_{B_{33}},
\]

\[M = R_{A} C, \quad N = D L_{B}, \quad S = C L_{M}, \quad E = R_{A_{33}} E_1 L_{B_{33}}.
\]

Now we give the fundamental theorem of this section.

**Theorem 6.1.** Let \(A_i, B_i,\) and \(C_i (i = 1, 2, 3, 4)\) be given. Then the following statements are equivalent:

1. The system of one-sided coupled Sylvester-type real quaternion matrix equations \((1.9)\) is consistent.

2. \[
    r \begin{pmatrix} C_i & A_i \\ B_i & 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4), \tag{6.1}
\]

3. \[
    r \begin{pmatrix} A_{k+1} C_k + C_{k+1} B_k & A_{k+1} A_k \\ B_{k+1} B_k & 0 \end{pmatrix} = r(A_{k+1} A_k) + r(B_{k+1} B_k), \quad (k = 1, 2, 3), \tag{6.2}
\]

4. \[
    r \begin{pmatrix} A_{j+2} A_{j+1} C_j + A_{j+2} C_{j+1} B_j + C_{j+2} B_{j+1} B_j & A_{j+2} A_{j+1} A_j \\ B_{j+2} B_{j+1} B_j & 0 \end{pmatrix} = r(A_{j+2} A_{j+1} A_j) + r(B_{j+2} B_{j+1} B_j), \quad (j = 1, 2), \tag{6.3}
\]

5. \[
    r \begin{pmatrix} A_4 A_3 A_2 C_1 + A_4 A_3 C_2 A_1 + A_4 C_3 A_2 A_1 + C_4 A_3 A_2 A_1 & A_4 A_3 A_2 A_1 \\ B_4 B_3 B_2 B_1 & 0 \end{pmatrix} = r(A_4 A_3 A_2 A_1) + r(B_4 B_3 B_2 B_1). \tag{6.4}
\]

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations \((1.9)\) can be expressed as

\[
    X_1 = A_1^\dagger C_1 + U_1 B_1 + L_{A_1} W_1, \quad X_2 = -R_{A_1} C_1 B_1^\dagger + A_1 U_1 + V_1 R_{B_1},
\]
Applying the main idea of Theorem 3.1, Lemma 2.2, Lemma 2.4 and Lemma 2.5, we can prove

\[ X_3 = A_1^t C_3 + U_2 B_3 + L_{A_1} T_1, \quad X_4 = -R_{A_2} C_3 B_3^t + A_3 U_2 + V_2 R_{B_3}, \]

or

\[ X_3 = -R_{(A_2 A_1)} C_2 B_2^t - A_{11} R_{A_1} C_1 B_1^t B_2^t + C_11 B_1^t R_{B_1} B_2^t + A_{11} W_3 A_{44} + A_2 A_1 W_4 + W_5 R_{B_2}, \]

where

\[ A = A_2 A_1 \]

\[ W = W_3 \]

\[ X = X_3 \]

Now we give an example to illustrate Theorem 6.1.

\[ V_1 = A_{11}^t C_{11} B_{11}^t + L_{A_{11}} W_2 + W_3 R_{B_{11}}, \]

\[ U_1 = (A_2 A_1)^t (C_2 + A_2 R_{A_2} C_1 B_1^t - A_2 V_1 R_{B_1}) + W_4 B_2 + L_{(A_2 A_1)} W_6, \]

\[ V_2 = A_{22}^t C_{22} B_{22}^t + L_{A_{22}} T_2 + T_3 R_{B_{22}}, \]

\[ U_2 = (A_4 A_3)^t (C_4 + A_4 R_{A_3} C_3 B_3^t - A_4 V_2 R_{B_3}) + T_4 B_4 + L_{(A_4 A_3)} T_6, \]

\[ W_4 = (I_{p_1}, 0)[A_{33}^t (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) - A_{33}^t Z_7 B_{33} + L_{A_{33}} Z_6], \]

\[ T_1 = (0, I_{p_2})[A_{33}^t (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) - A_{33}^t Z_7 B_{33} + L_{A_{33}} Z_6], \]

\[ W_5 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) B_{33}^t + A_{33}^t A_{33}^t Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} I_{p_3} \\ 0 \end{pmatrix}, \]

\[ T_4 = [R_{A_{33}} (E_1 - A_{11} W_3 A_{44} - B_{44} T_6 B_3) B_{33}^t + A_{33}^t A_{33}^t Z_7 + Z_8 R_{B_{33}}] \begin{pmatrix} 0 \\ I_{p_4} \end{pmatrix}, \]

\[ W_3 = A^t E B^t - A^t C M^t E B^t - A^t S C^t EN^t D B^t - A^t S Z I R_N D B^t + L_{A} Z_2 + Z_3 R_{B}, \]

\[ T_6 = M^t E D^t + S^t C^t E N^t + L_{M} L_{S} Z_4 + L_{M} Z_1 R_N + Z_5 R_{D}, \]

the remaining \( W_j, T_j, Z_j \) are arbitrary matrices over \( \mathbb{H} \), \( p_1 \) and \( p_2 \) are the column numbers of \( A_1 \) and \( A_3 \), respectively, \( p_3 \) and \( p_4 \) are the row numbers of \( B_2 \) and \( B_4 \), respectively.

**Proof.** We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9) into two parts

\[
\begin{align*}
A_1 X_1 - X_2 B_1 &= C_1, \\
A_2 X_2 - X_3 B_2 &= C_2,
\end{align*}
\] (6.5)

and

\[
\begin{align*}
A_3 X_3 - X_4 B_3 &= C_3, \\
A_4 X_4 - X_5 B_4 &= C_4.
\end{align*}
\] (6.6)

Applying the main idea of Theorem 3.1, Lemma 2.2, Lemma 2.4 and Lemma 2.5 we can prove Theorem 6.1. \hfill \Box

Now we give an example to illustrate Theorem 6.1.
Example 4. Given the quaternion matrices:

\[ A_1 = \begin{pmatrix} 1 + k & -1 & 2i + j \\ 0 & i + k & i - 2j \\ 1 + i & 2 - i & 1 + k \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 + k & i + k & j + k \\ -2 - j & 2i - j & -j + k \\ 1 + i - j + k & -1 + i - j + k & 2k \end{pmatrix}, \]

\[ A_2 = \begin{pmatrix} i & j & 1 + 2i + k \\ k & i - j & -1 - 2j + k \\ i + k & i & 2i - 2j + 2k \end{pmatrix}, \quad B_2 = \begin{pmatrix} j & 1 + 2i + j & -i + k \\ i - j & k & 1 + 2j \\ i & 1 + 2i + j + k & 1 - i + 2j + k \end{pmatrix}, \]

\[ A_3 = \begin{pmatrix} 1 + 2i + k & 2 - i - k & 1 + j \\ -1 - 2i - j + k & -2 + i + j - k & -1 + j + k \\ -k & k & j \end{pmatrix}, \quad B_3 = \begin{pmatrix} i + 2j & 1 + 3j & j - 3k \\ i + 2j & 1 + 3j & j - 3k \\ i & 1 & 0 \end{pmatrix}, \]

\[ A_4 = \begin{pmatrix} 2 + 3i + k & 3 - j & i + j + k \\ -3 + 2i - j & 3i - k & -1 - j + k \\ -1 + 5i - j + k & 3 + 3i - j - k & -1 + i + 2k \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & i + k & 1 + 2i - j \\ i & 1 - j & -2 + i - k \\ 1 + i & -1 + i + 2k & -1 + 3i - 2j \end{pmatrix}, \]

\[ C_1 = \begin{pmatrix} -1 + 4i - j - k & -4 + 2i - 5j + 6k & 3 - 2i + 6k \\ 1 - 5i - 6j + k & 5 + i - 2j + k & 3 - 2i + k \\ -6 - 3i + 2j + 3k & -2 - 8i + 3j + 11k & 5j \end{pmatrix}, \]

\[ C_2 = \begin{pmatrix} 2 - 3i & 8 - 3j + 4k & -1 + i - 5j - 8k \\ j - 2k & 6 - 8i - 5j + 2k & -7 - i - 5j - 4k \end{pmatrix}, \]

\[ C_3 = \begin{pmatrix} 3 + 3j - k & -3 + i + 6j - 2k & 1 - 4j - 5k \\ 1 + i + j + 2k & 6 - 4i + 4j + 3k & -6 + 13j - 4k \\ 3 + 4i - j + 6k & 3 - 3i + 5j - 4k & 2 - i + 5j - 7k \end{pmatrix}, \]

\[ C_4 = \begin{pmatrix} -11 - 5i - 6j + k & -1 + 8i - 2j + 7k & -10 + i - 3j + 6k \\ 5 - 11i - 3j - 5k & -6 - 2i - 5j - 3k & -2 - 12i - 3j + 3k \\ -6 - 16i - 5j - 4k & -11 + 6i - 7j & -12 - 7i - 2j + k \end{pmatrix}. \]
Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9). Check that

\[
\begin{align*}
&\begin{bmatrix} C_i & A_i \\ B_i & 0 \end{bmatrix} = r(A_i) + r(B_i) = \begin{cases}
6, & \text{if } i = 1 \\
4, & \text{if } i = 2, 3 \\
3, & \text{if } i = 4 
\end{cases} \\
\begin{bmatrix} A_{k+1}C_k + C_{k+1}A_k & A_{k+1}A_k \\ B_{k+1}B_k & 0 \end{bmatrix} = \begin{cases}
4, & \text{if } k = 1, 2 \\
3, & \text{if } k = 3 
\end{cases} \\
\begin{bmatrix} A_{j+2}A_{j+1}C_j + A_{j+2}C_{j+1}B_j + C_{j+2}B_{j+1}B_j & A_{j+2}A_{j+1}A_j \\ B_{j+2}B_{j+1}B_j & 0 \end{bmatrix} = \begin{cases}
4, & \text{if } j = 1 \\
3, & \text{if } j = 2 
\end{cases} \\
\begin{bmatrix} A_4A_3A_2C_1 + A_4A_3C_2A_1 + A_4C_3A_2A_1 + C_4A_3A_2A_1 & A_4A_3A_2A_1 \\ B_4B_3B_2B_1 & 0 \end{bmatrix} = \begin{cases}
\text{any} & \text{consistent} 
\end{cases}
\end{align*}
\]

All the rank equalities in (6.1)-(6.4) hold. Hence, the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.9) is consistent. Note that

\[
X_1 = \begin{pmatrix} 2i + k & -1 + j + k & 2 + j \\ -2i + k & 1 + j + k & -2 + j \\ 2k & 2j & 2j \end{pmatrix},
X_2 = \begin{pmatrix} 1 & -1 + j & i + k \\ 2 & -2 - j & 2i - k \\ -1 & 1 + 2j & -i + 2k \end{pmatrix},
X_3 = \begin{pmatrix} i + j & 1 + 2i + k & 2k \\ 1 & k & 1 \\ i & 0 & 1 + k \end{pmatrix},
X_4 = \begin{pmatrix} -1 + i + k & 1 + k & i + k \\ -1 - i + k & i + k & -1 + k \\ -2 + 2k & 1 + i + 2k & -1 + i + 2k \end{pmatrix},
\]

and

\[
X_5 = \begin{pmatrix} 1 & -1 + j & i + k \\ 2 & -2 + 2j & 2i + 2k \\ 3 & -3 - j & 3i - k \end{pmatrix}
\]

satisfy the system (1.9).

7. Some solvability conditions and the general solution to system (1.10)

In this section, we consider the solvability conditions and the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10). For simplicity, put

\[
A_{11} = R_{B_2}B_1, \quad B_{11} = R_{A_1}A_2, \quad C_{11} = B_1L_{A_{11}}, \quad D_{11} = R_{A_1}(R_{A_2}C_2B_2)B_1 - C_1)L_{A_{11}}, \\
A_{22} = R_{(A_4L_{A_3})}A_4, B_{22} = B_3L_{B_4}, C_{22} = R_{(A_4L_{A_3})}(C_4 - A_4A_{11}L_{C_3})L_{B_4},
\]

where
Theorem 7.1. Let (7.1)
\[
A_{33} = (L_{A_2}, -A_3 L_{A_2}), \quad B_{33} = \begin{pmatrix} R_{C_{11}} B_2 \\ -R_{B_3} \end{pmatrix},
\]
\[
E_1 = -R_{A_3} C_3 B_3^\dagger + A_3 A_{22} C_2 B_{22}^\dagger - A_2^\dagger C_2 - B_{11}^\dagger D_{11} C_{11}^\dagger B_2,
\]
\[
A = R_{A_{33}} L_{B_{33}}, \quad B = B_2 L_{B_{33}}, \quad C = -R_{A_{33}} A_3, \quad D = R_{B_{33}} L_{B_{33}},
\]
\[
E = R_{A_{33}} E_1 L_{B_{33}}, \quad M = R_{A} C, \quad N = D L_B, \quad S = C L_M.
\]

Theorem 7.1. Let \(A_i, B_i, \) and \(C_i(i = 1, 2, 3, 4)\) be given. Then the following statements are equivalent:

(1) The system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10) is consistent.

(2)
\[
\begin{pmatrix} C_i \\ B_i \\ 0 \end{pmatrix} = r(A_i) + r(B_i), \quad (i = 1, 2, 3, 4),
\]
(7.1)
\[
\begin{pmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{pmatrix} = r(A_1, A_2) + r(B_1, B_2),
\]
(7.2)
\[
\begin{pmatrix} C_3 & A_3 \\ C_4 & A_4 \\ B_3 & 0 \\ B_4 & 0 \end{pmatrix} = r(A_3) + r(B_3),
\]
(7.3)
\[
\begin{pmatrix} C_1 & A_2 C_3 + C_2 B_3 & A_1 & A_2 A_3 \\ B_1 & B_2 B_3 & 0 & 0 \end{pmatrix} = r(A_1, A_2 A_3) + r(B_1, B_2 B_3),
\]
(7.4)
\[
\begin{pmatrix} A_2 C_3 + C_2 B_3 & A_2 A_3 \\ C_4 & A_4 \\ B_2 B_3 & 0 \\ B_4 & 0 \end{pmatrix} = r(A_2 A_3) + r(B_2 B_3),
\]
(7.5)
\[
\begin{pmatrix} A_2 C_3 + C_2 B_3 & A_2 A_3 \\ B_2 B_3 & 0 \end{pmatrix} = r(A_2 A_3) + r(B_2 B_3),
\]
(7.6)
\[
\begin{pmatrix} C_1 & A_2 C_3 + C_2 B_3 & A_1 & A_2 A_3 \\ 0 & C_4 & 0 & A_4 \\ B_1 & B_2 B_3 & 0 & 0 \\ 0 & B_4 & 0 & 0 \end{pmatrix} = r(A_1, A_2 A_3) + r(B_1, B_2 B_3).
\]
(7.7)

(3)
\[
R_{A_2} C_2 L_{B_2} = 0, \quad D_{11} L_{C_{11}} = 0, \quad R_{B_{11}} D_{11} = 0,
\]
where

\[ R_{A_3}C_3L_{B_3} = 0, \ R_{A_2}C_2 = 0, \ C_2L_{B_2} = 0, \]

\[ R_M A_E = 0, \ E_L B_L N = 0, \ R_A E L_D = 0, \ R_C E L_B = 0. \]

In this case, the general solution to the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10) can be expressed as

\[ X_1 = A_1^t(C_1 - R_{A_2}C_2B_2^tB_1 + A_2U_1B_1) + W_4A_{11} + L_{A_1}W_6, \]

\[ X_2 = -R_{A_2}C_2B_2^t + A_2U_1 + V_1R_{B_2}, \ X_4 = A_3^tC_3 + V_2B_3 + L_{A_3}U_2; \]

\[ X_5 = -R_{A_4L_{A_4}}(C_4 - A_4A_3^tC_3 - A_4V_2B_3)B_4^t + A_4L_{A_3}T_1 + T_3R_{B_4}, \]

\[ X_3 = A_3^tC_2 + U_1B_2 + L_{A_2}W_1, \text{ or } X_3 = -R_{A_3}C_3B_3^t + A_3V_2 + T_6R_{B_4}, \]

where

\[ U_1 = B_{11}^tD_{11}C_{11}^t + L_{B_{11}}W_2 + W_3R_{C_{11}}, \]

\[ V_1 = -R_{A_1}(C_1 - R_{A_2}C_2B_2^tB_1 + A_2U_1B_1)A_{11}^t + A_1W_4 + W_5R_{A_{11}}, \]

\[ V_2 = A_{22}^tC_{22}B_{22}^t + L_{A_{22}}T_4 + T_5R_{B_{22}}, \]

\[ U_2 = (A_4L_{A_4})^t(C_4 - A_4A_3^tC_3 - A_4V_2B_3) + T_1B_4 + L_{(A_4L_{A_4})}T_2, \]

\[ W_1 = (I_{p_1}, 0)[A_{33}^t(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}}) - A_{33}^tZ_7B_{33} + L_{A_{33}}Z_6], \]

\[ T_4 = (0, I_{p_2})[A_{33}^t(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}}) - A_{33}^tZ_7B_{33} + L_{A_{33}}Z_6], \]

\[ W_3 = [R_{A_{33}}(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}})B_{33}^t + A_{33}A_{33}^tZ_7 + Z_8R_{B_{33}}] \begin{bmatrix} I_{p_3} \\ 0 \end{bmatrix}, \]

\[ T_6 = [R_{A_{33}}(E_1 - L_{B_{11}}W_2B_2 + A_3T_5R_{B_{22}})B_{33}^t + A_{33}A_{33}^tZ_7 + Z_8R_{B_{33}}] \begin{bmatrix} 0 \\ I_{p_4} \end{bmatrix}, \]

\[ W_2 = A^tEB^t - A^tCM^tEB^t - A^tSC^tEN^tDB^t - A^tSZ_1R_{N}DB^t + L_{A}Z_2 + Z_{3}R_{B}, \]

\[ T_5 = M^tED^t + S^tSC^tEN^t + L_{M}L_{S}Z_4 + L_{M}Z_1R_{N} + Z_{5}R_{D}, \]

the remaining \( W_j, T_j, Z_j \) are arbitrary matrices over \( \mathbb{H} \), \( p_1 \) and \( p_2 \) are the column numbers of \( A_2 \) and \( A_4 \), respectively, \( p_3 \) and \( p_4 \) are the row numbers of \( B_1 \) and \( B_3 \), respectively.
Proof. We separate this system of one-sided coupled Sylvester-type real quaternion matrix equations (7.8) into two parts

\[
\begin{align*}
A_1X_1 - X_2B_1 &= C_1, \\
A_2X_3 - X_2B_2 &= C_2, \\
A_3X_4 - X_3B_3 &= C_3, \\
A_4X_4 - X_5B_4 &= C_4.
\end{align*}
\]

(7.8)

and

(7.9)

Applying the main idea of Theorem 3.1 Lemma 2.1 Lemma 2.3 Lemma 2.4 and Lemma 2.5 we can prove Theorem 7.1.

Now we give an example to illustrate Theorem 7.1.

Example 5. Given the quaternion matrices:

\[
A_1 = \begin{pmatrix} i + j & -j & i + k \\ k & 1 + k & 0 \\ 1 & 0 & 1 + j \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 + j + k & -1 - k & i + j \\ 2k & 1 & 1 + i + j \\ 2 & 2 + i + j & k \end{pmatrix},
\]

\[
A_2 = \begin{pmatrix} 1 & 1 + i + j & 2 + 2i + k \\ 1 - 2i + k & j & 1 \\ i & 1 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & -1 + j & i + k \\ i & -i - j & -1 - k \\ 1 + i & -1 - i & -1 + i \end{pmatrix},
\]

\[
A_3 = \begin{pmatrix} j & i - j & 1 + k \\ 1 + k & 0 & i + j \\ 1 + j + k & i - j & 1 + i + j + k \end{pmatrix}, \quad B_3 = \begin{pmatrix} j + 2k & 1 + j - k & i + j \\ -j - 2k & -1 - j + k & -i - j \\ 2j + 4k & 2j - 2k & 2j \end{pmatrix},
\]

\[
A_4 = \begin{pmatrix} -k & i + j + k & 2i - 2k \\ 1 + k & 1 - j - k & 1 + 2k \\ 1 & 1 + i & 1 + 2i \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 - j & i - k & -i - k \\ i + j & -1 + k & 1 + k \\ 1 + i + 2j & -1 + i + 2k & 1 - i + 2k \end{pmatrix},
\]

\[
C_1 = \begin{pmatrix} -3 - j - 4k & -1 + 2i + 3k & 1 - j \\ 2 - i + j - 3k & -1 + i - k & 2 - i + k \\ 4 + i - 3k & 1 - 2i + j & -2 - 2i - j - k \end{pmatrix}, \quad C_2 = \begin{pmatrix} -2 - 2i - 8j + 5k & 11j + 7k & 2 + i - 8j + 5k \\ -1 - 2i - 2j + 4k & -1 + 2i + 9j - k & 4 + i - 2j + 7k \\ 1 - 2j + k & -3 + i + 5j - k & 2 + 2i + j + 5k \end{pmatrix}, \quad C_3 = \begin{pmatrix} -4 + 9i + 4j - 4k & -8 - 3i + k & -5 + j - 6k \\ 8 + 5i - j + 6k & -2 + 4i + 6j - 2k & -2 + 6i + 2k \\ 4 + 14i + 3j + 2k & -10 + i + 6j + k & -7 + 6i + j - 4k \end{pmatrix},
\]
Now we consider the system of one-sided coupled Sylvester-type real quaternion matrix equations (1.10). Check that

$$r \left( \begin{array}{cc} C_i & A_i \\ B_i & 0 \end{array} \right) = r(A_i) + r(B_i) = \begin{cases} 6, & \text{if } i = 1 \\ 5, & \text{if } i = 2 \\ 4, & \text{if } i = 3, 4 \end{cases}$$

$$r \left( \begin{array}{ccc} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{array} \right) = r(A_1, A_2) + r(B_1, B_2) = 6,$$

$$r \left( \begin{array}{cc} C_3 & A_3 \\ C_4 & A_4 \\ B_3 & 0 \\ B_4 & 0 \end{array} \right) = r(A_3) + r\left( \begin{array}{c} B_3 \\ B_4 \end{array} \right) = 6,$$

$$r \left( \begin{array}{ccc} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ B_1 & B_2 & 0 \end{array} \right) = r(A_1, A_2 A_3) + r(B_1, B_2 B_3) = 6,$$

$$r \left( \begin{array}{cccc} A_2 & A_3 & A_2 & A_3 \\ C_4 & A_4 & C_4 & 0 \\ B_2 B_3 & 0 & B_2 B_3 & 0 \end{array} \right) = r(A_2 A_3) + r(B_2 B_3) = 10,$$

$$r \left( \begin{array}{cccc} C_4 & 0 & A_4 & A_4 \\ B_1 & B_2 B_3 & 0 & 0 \\ 0 & B_4 & 0 & 0 \end{array} \right) = r\left( \begin{array}{cc} A_1 & A_2 A_3 \\ 0 & A_4 \end{array} \right) + r\left( \begin{array}{cc} B_1 & B_2 B_3 \\ 0 & B_4 \end{array} \right) = 4.$$
and
\[ X_5 = \begin{pmatrix} 1 + j & 1 + j & i + k \\ 1 + i + k & 2 - i + k & 3 \\ 1 + 3i + j & k & 1 \end{pmatrix} \]
satisfy the system \([I, J]\).

8. Conclusion

We have provided some necessary and sufficient conditions for the existence and the general solutions to the systems of four coupled one sided Sylvester-type real quaternion matrix equations \([1.6]-[1.10]\), respectively. Moreover, we have presented some numerical examples. It is worthy to say that the main results of this paper can be generalized to an arbitrary division ring with an involutive anti-automorphism.

References

[1] J.K. Baksalary, R. Kala, The matrix equation \(AX - YB = C\), Linear Algebra Appl. 25 (1979) 41-43.
[2] J.K. Baksalary, R. Kala, The matrix equation \(AXB + CYD = E\), Linear Algebra Appl. 30 (1980) 141-147.
[3] N.L. Bihan, J. Mars, Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing, Signal Processing, 84 (7) (2004) 1177-1199.
[4] J. Chen, R. Patton, H. Zhang, Design unknown input observers and robust fault detection filters, Int. J. of control. 63 (1996) 85–105.
[5] F. De Terón, F.M. Dopico, N. Guillery, D. Montealegre, N. Reyes, The solution of the equation \(AX + X^*B = 0\), Linear Algebra Appl. 438 (7) (2013) 2817–2860.
[6] F. De Terón, The solution of the equation \(AX + BX^* = 0\), Linear and Multilinear Algebra 61 (12) (2013) 1605–1628.
[7] A. Dmytryshyn, B. Kågström, Coupled Sylvester-type matrix equations and block diagonalization, SIAM J. Matrix Anal. Appl. 36 (2)(2015) 580–593.
[8] Z.H. He, O.M. Agudelo, Q.W. Wang, B. De Moor, Two-sided coupled generalized Sylvester matrix equations solving using a simultaneous decomposition for fifteen matrices, Linear Algebra Appl. 496 (2016) 549-593.
[9] Z.H. He, Q.W. Wang, A real quaternion matrix equation with with applications, Linear and Multilinear Algebra 61 (2013) 725–740.
[10] Z.H. He, Q.W. Wang, Y. Zhang, Simultaneous decomposition of quaternion matrices involving \(\eta\)-Hermicity with applications, Appl. Math. Comput. 298 (2017) 13–35.
[11] Z.H. He, Q.W. Wang, The \(\eta\)-bihermitian solution to a system of real quaternion matrix equations, Linear and Multilinear Algebra 62 (2014) 1509–1528.
[12] Z.H. He, Q.W. Wang, The general solutions to some systems of matrix equations, Linear and Multilinear Algebra 63 (10) (2015) 2017–2032.
[13] Z.H. He, Q.W. Wang, A system of periodic discrete-time coupled Sylvester quaternion matrix equations, Algebra Colloquium 24 (2017) 169–180.
[14] Z.H. He, Q.W. Wang, A pair of mixed generalized Sylvester matrix equations, Journal of Shanghai University (Natural Science), 20 (2014) 138-156.
[15] I. Jonsson, B. Kågström, Recursive blocked algorithms for solving triangular systems-Part I: One-sided and coupled Sylvester-type matrix equations, ACM Transactions on Mathematical Software. 284 (2002) 392-415.
[16] I. Jonsson, B. Kågström, Recursive blocked algorithms for solving triangular systems-Part II: Two-sided and generalized Sylvester and Lyapunov matrix equations, ACM Transactions on Mathematical Software. 28 (2002) 416-435.
[17] B. Kågström, L. Westin, Generalized Schur methods with condition estimators for solving the generalized Sylvester equation, IEEE Trans. on Automatic Control. 34 (7) (1989) 745–751.
[18] O. Kameník, Solving SDGE Models: A New Algorithm for the Sylvester Equation, *Comput. Econom.* 25 (2005) 167–187.

[19] S.G. Lee, Q.P. Vu, Simultaneous solutions of matrix equations and simultaneous equivalence of matrices, *Linear Algebra Appl.* 437 (2012) 2325-2339.

[20] S.G. Lee, Q.P. Vu, Simultaneous solutions of Sylvester equations and idempotent matrices separating the joint spectrum, *Linear Algebra Appl.* 435 (2011) 2097–2209.

[21] S.D. Leo, G. Scolarici, Right eigenvalue equation in quaternionic quantum mechanics, *J. Phys. A* 33 (2000) 2971-2995.

[22] G. Marsaglia, G.P.H. Styan, Equalities and inequalities for ranks of matrices, *Linear Multilinear Algebra.* 2 (1974) 269–292.

[23] W.E. Roth, The equation $AX - YB = C$ and $AX - XB = C$ in matrices, *Proc. Amer. Math. Soc.* 3 (1952) 392-396.

[24] A. Shahzad, B.L. Jones, E.C. Kerrigan, G.A. Constantinides, An efficient algorithm for the solution of a coupled Sylvester equation appearing in descriptor systems, *Automatica.* 47 (2011) 244–248.

[25] V.L. Syrmos, F.L. Lewis, Output feedback eigenstructure assignment using two Sylvester equations, *IEEE Trans. on Automatic Control.* 38(1993) 495-499.

[26] V.L. Syrmos, F.L. Lewis, Coupled and constrained Sylvester equations in System design, *Circuits Systems Signal Process.* 13 (6) (1994) 663-694.

[27] C.C. Took, D.P. Mandic, Augmented second-order statistics of quaternion random signals, *Signal Processing* 91 (2011) 214-224.

[28] C.C. Took, D.P. Mandic, The quaternion LMS algorithm for adaptive filtering of hypercomplex real world processes, *IEEE Trans. Signal Process.* 57 (2009) 1316-1327.

[29] C.C. Took, D.P. Mandic, Quaternion-valued stochastic gradient-based adaptive IIR filtering, *IEEE Trans. Signal Process.* 58 (7) (2010) 3895-3901.

[30] C.C. Took, D.P. Mandic, F.Z. Zhang, On the unitary diagonalization of a special class of quaternion matrices, *Appl. Math. Lett.* 24 (2011) 1806-1809.

[31] A. Varga, Robust pole assignement via Sylvester equation based state feedback parametrization. *Computer-Aided Control System Design, 2000. CACSD 2000. IEEE International Symposium on.* 57(2000) 13-18.

[32] J.W. van der Woude, Almost noninteracting control by measurement feedback, *Systems Control Lett.* 9 (1987) 7–16.

[33] Q.W. Wang, Z.H. He, Some matrix equations with applications, *Linear Multilinear Algebra.* 60 (2012) 1327–1353.

[34] Q.W. Wang, Z.H. He, Solvability conditions and general solution for the mixed Sylvester equations, *Automatic.* 49 (2013) 2713–2719.

[35] Q.W. Wang, Z.H. He, Systems of coupled generalized Sylvester matrix equations, *Automatica.*50 (2014) 2840–2844.

[36] Q.W. Wang, A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity, *Linear Algebra Appl.* 384 (2004) 43–54.

[37] Q.W. Wang, J.H. Sun, S.Z. Li, Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra, *Linear Algebra Appl.* 353 (2002) 169-182.

[38] Q.W. Wang, J.W. van der Woude, S.W. Yu, An equivalence canonical form of a matrix triplet over an arbitrary division ring with applications, *Sci. China Math.* 54 (5)(2011) 907–924.

[39] Q.W. Wang, X. Zhang, J.W. van der Woude, A new simultaneous decomposition of a matrix quaternity over an arbitrary division ring with applications, *Comm. Algebra.* 40 (2012) 2309-2342.

[40] Q.W. Wang, The general solution to a system of real quaternion matrix equations, *Comput. Math. Appl.* 49 (2005) 665–675.

[41] Q.W. Wang, H.X. Chang, C.Y. Lin, P-(skew)symmetric common solutions to a pair of quaternion matrix equations, *Appl. Math. Comput.* 195 (2008) 721–732.
[42] Q.W. Wang, Bisymmetric and centrosymmetric solutions to system of real quaternion matrix equations, *Comput. Math. Appl.* 49 (2005) 641–650.

[43] Q.W. Wang, J.W. van der Woude, H.X. Chang, A system of real quaternion matrix equations with applications, *Linear Algebra Appl.* 431 (2009) 2291–2303.

[44] H.K. Wimmer, Consistency of a pair of generalized Sylvester equations. *IEEE Trans. on Automatic Control.* 39(1994) 1014-1016.

[45] S.F. Yuan, Q.W. Wang, Two special kinds of least squares solutions for the quaternion matrix equation $AXB + CXD = E$, *Electron. J. Linear Algebra.* 23 (2012) 257–274.

[46] S.F. Yuan, Q.W. Wang, L-structured quaternion matrices and quaternion linear matrix equations, *Linear and Multilinear Algebra* 64 (2016) 321–339.

[47] Y.N. Zhang, D.C. Jiang, J. Wang, A recurrent neural network for solving Sylvester equation with time-varying coefficients. *IEEE Trans. Neural Networks.* 13 (5)(2002) 1053-1063.

[48] F.Z. Zhang, Quaternions and matrices of quaternions, *Linear Algebra Appl.* 251 (1997) 21-57.

[49] Y. Zhang, R.H. Wang, The exact solution of a system of quaternion matrix equations involving $η$-Hermicity, *Appl. Math. Comput.* 222 (2013) 201–209.