Sasaki manifolds with positive transverse orthogonal bisectional curvature

Hong Huang

Abstract

In this short note we show the following result: Let \((M^{2n+1}, g)\) \((n \geq 2)\) be a compact Sasaki manifold with positive transverse orthogonal bisectional curvature. Then \(\pi_1(M)\) is finite, and the universal cover of \((M^{2n+1}, g)\) is isomorphic to a weighted Sasaki sphere. We also get some results in the case of nonnegative transverse orthogonal bisectional curvature under some additional conditions. This extends recent work of He and Sun. The proof uses Sasaki-Ricci flow.

Key words: Sasaki manifolds, positive transverse orthogonal bisectional curvature, Sasaki-Ricci flow, maximum principle

AMS2010 Classification: 53C44

1 Introduction

In [HS1] and [HS2] He and Sun classified compact Sasaki manifolds with nonnegative transverse bisectional curvature using Sasaki-Ricci flow. In this note we try to extend their results to the case of nonnegative transverse orthogonal bisectional curvature. First we have

**Theorem 1.1.** Let \((M^{2n+1}, g)\) \((n \geq 2)\) be a compact Sasaki manifold with positive transverse orthogonal bisectional curvature. Then \(\pi_1(M)\) is finite, and the universal cover of \((M^{2n+1}, g)\) is isomorphic to a weighted Sasaki sphere.

This extends [HS1, Theorem 1.1], and is a Sasaki analogue of Gu and Zhang [GZ, Corollary 3.2]. Note that a Sasaki manifold \((M^{2n+1}, g)\) \((n \geq 2)\) has positive (resp. nonnegative) transverse orthogonal bisectional curvature if the transverse Kähler metric has positive (resp. nonnegative) orthogonal bisectional curvature.

The following result extends [HS2, Theorem 2], and is a Sasaki analogue of [GZ, Proposition 3.3].

**Theorem 1.2.** Let \((M^{2n+1}, g)\) \((n \geq 2)\) be a compact, simply connected Sasaki manifold with nonnegative transverse orthogonal bisectional curvature. Suppose \(b^{1,1}_B(M) := \dim H^{1,1}_B(M) = 1\). Then either \(M\) is isomorphic to a weighted Sasaki sphere, or \(M\) is a principal \(S^1\)-bundle over an irreducible compact Hermitian symmetric space.
We also have a variant of Theorem 1.2.

**Theorem 1.3.** Let \((M^{2n+1}, g)\) \((n \geq 2)\) be a compact, locally transversely irreducible Sasaki manifold with nonnegative transverse orthogonal bisectional curvature. Then either \(\pi_1(M)\) is finite and the universal cover of \((M^{2n+1}, g)\) is isomorphic to a weighted Sasaki sphere, or \(M\) is locally transversely symmetric.

Recall that a Sasaki manifold is locally transversely irreducible (resp. symmetric) if the transverse Kähler metric is locally irreducible (resp. symmetric). (Compare for example [HS2] and Takahashi [T].)

In Section 2 we study Sasaki-Ricci flow with initial data a compact Sasaki manifolds with positive (nonnegative) transverse orthogonal bisectional curvature, then in Section 3 we prove the theorems. Since the argument (mainly using maximum principle) follows closely that in Chen [Ch], [GZ], [HS1] and [HS2], the presentation will be brief. For some notions not defined here see for example [HS1] and [HS2].

## 2 Sasaki-Ricci flow

Let \((M^{2n+1}, g_0)\) be a compact Sasaki manifold. Recall [SWZ] the Sasaki-Ricci flow

\[
\frac{\partial g^T}{\partial t} = -\text{Ric}^T, \quad g^T(0) = g^T_0.
\]

**Proposition 2.1.** The nonnegativity (positivity) of transverse orthogonal bisectional curvature is preserved along the Sasaki-Ricci flow.

**Proof** The proof is similar to that of Proposition 2.1 in [GZ] by using Hamilton’s maximum principle and a second variation argument originating from Mok [M]. \(\square\)

**Proposition 2.2.** Let \((M^{2n+1}, g)\) \((n \geq 2)\) be a compact Sasaki manifold with nonnegative transverse orthogonal bisectional curvature. Then any real basic harmonic \((1, 1)\)-form on \(M\) is transversely parallel. Moreover, if \(b^{1,1}_B(M) = 1\), then the basic first Chern class \(c^B_1(M) > 0\).

**Proof** The proof is similar to that of Theorem 3.1 in [GZ]. The first conclusion is proved by using the transverse Bochner formula, and the second by using Sasaki-Ricci flow and Proposition 2.1. \(\square\)

**Proposition 2.3.** Let \((M^{2n+1}, g)\) \((n \geq 2)\) be a compact Sasaki manifold with nonnegative transverse orthogonal bisectional curvature. Then \((M^{2n+1}, g)\) is transversely reducible if and only if \(b^{1,1}_B(M) > 1\).

**Proof** With the help of Proposition 2.2, the proof is similar to that of Lemma 2.5 in [HS2]. \(\square\)

Now if \((M^{2n+1}, g_0)\) \((n \geq 2)\) is a compact Sasaki manifold with positive transverse orthogonal bisectional curvature, then it is locally transversely irreducible,
hence transversely irreducible. By Proposition 2.3, $b_B^{1,1}(M) = 1$. (Note that here $b_B^{1,1}(M) = 1$ can also be derived directly from the local transverse irreducibility. Compare Theorem 3.1 (ii) in [GZ].) Then by Proposition 2.2, $c^B_1(M) > 0$. After a homothetic transformation of the initial metric, we consider the normalized Sasaki-Ricci flow

$$\frac{\partial g^T}{\partial t} = -\text{Ric}^T + g^T, \quad g^T(0) = g_0^T,$$

which exists on the time interval $[0, \infty)$.

**Proposition 2.4.** Let $(M^{2n+1}, g_0) \ (n \geq 2)$ be a compact Sasaki manifold with positive transverse orthogonal bisectional curvature. Then along the normalized Sasaki-Ricci flow with initial data $(M^{2n+1}, g_0)$, the infimum of the transverse holomorphic sectional curvature, if non-positive initially, will approach a nonnegative number as $t \to \infty$.

**Proof** The proof is similar to that of Theorem 1.4 in [Ch]. It is again an application of maximum principle.

**Proposition 2.5.** Let $(M^{2n+1}, g_0) \ (n \geq 2)$ be a compact Sasaki manifold with positive transverse orthogonal bisectional curvature. Suppose that (after some time) $\text{Ric}^T \geq cg^T$ (for some positive constant $c$) along the normalized Sasaki-Ricci flow with initial data $(M^{2n+1}, g_0)$. Then along the flow the infimum of the transverse holomorphic sectional curvature, if non-positive initially, will become positive in finite time.

**Proof** The proof is similar to that of Theorem 1.5 in [Ch].

## 3 Proof of Theorems

**Proof of Theorem 1.1** Let $(M^{2n+1}, g_0) \ (n \geq 2)$ be a compact Sasaki manifold with positive transverse orthogonal bisectional curvature. Then as observed above, $b_B^{1,1}(M) = 1$ and $c^B_1(M) > 0$. Now we evolve the metric $g_0$ (after a homothetic transformation if necessary) by the normalized Sasaki-Ricci flow. Using result of Collins [Co, Theorem 1.3] and He [He, Theorem 7.1] and Propositions 2.1, 2.4, we know that as $t \to \infty$, $(M, g^T(t))$ subconverges to a Sasaki-Ricci soliton $(M, g_\infty)$ (on $M$) with positive transverse orthogonal bisectional curvature and nonnegative transverse holomorphic sectional curvature, hence nonnegative transverse bisectional curvature. It follows that the limit soliton has nonnegative transverse Ricci curvature.

**Claim** The limit soliton $(M, g_\infty)$ has positive transverse Ricci curvature.

Otherwise, by using Hamilton’s strong maximum principle, $\text{Ric}^T(g_\infty)$ would have a null eigenvector field $V$ which is transversely parallel. Clearly $J_\infty V$ is also a
transversely parallel null eigenvector field of $Ric^T(g_\infty)$. (Here, $J_\infty$ is the transverse complex structure of $(M, g_\infty)$.) It follows that $(M, g_\infty)$ is locally transversely reducible. On the other hand, $(M, g_\infty)$ has positive transverse orthogonal bisectional curvature, and is locally transversely irreducible. Contradiction.

Then by Proposition 2.5, $g^T(t)$ has positive transverse holomorphic sectional curvature when $t > t_0$ (for some finite $t_0$). Since $g^T(t)$ also has positive transverse orthogonal bisectional curvature, it follows that $g^T(t)$ has positive transverse bisectional curvature when $t > t_0$. (Compare the proof of [Ch, Theorem 1.8].) Using a homothetic transformation we see that $M$ admits a metric with $Ric > 0$, and by Myers’ theorem $\pi_1(M)$ is finite. Now the remaining conclusion in the theorem follows from [HS1, Theorem 1.1].

**Proof of Theorem 1.2** Let $(M^{2n+1}, g_0)$ $(n \geq 2)$ be a compact, simply connected Sasaki manifold with nonnegative transverse orthogonal bisectional curvature and with $b_{11}(M) = 1$. By Proposition 2.3, $(M^{2n+1}, g_0)$ is transversely irreducible. If $(M, g_0)$ is locally transversely symmetric, then since we assume $M$ is simply connected, by [T, Theorems 6.2, 6.1], $M$ is a principal $S^1$-bundle over a Hermitian symmetric space. Now suppose $(M, g_0)$ is not locally transversely symmetric. We evolve the metric $g_0$ by the Sasaki-Ricci flow. Then as in the proof of [GZ, Proposition 3.3], using a strong maximum principle argument originating from Brendle and Schoen [BS], Proposition 3.3 of [HS2], the assumptions on $(M, g_0)$ and Berger’s holonomy theorem, we see that the transverse orthogonal bisectional curvature of $g(t)$ is positive for any $t \in (0, \delta)$ for some $\delta > 0$. Then the desired result follows from Theorem 1.1.

**Proof of Theorem 1.3** The proof is similar to that of Theorem 1.2.

**Acknowledgement** I’m partially supported by NSFC no.11171025.

**Reference**

[BS] S. Brendle, R. Schoen, Classification of manifolds with weakly 1/4-pinched curvature, Acta Math. 200 (2008), 1-13.

[Ch] X.X. Chen, On Kähler manifolds with positive orthogonal bisectional curvature, Advances in Math. 215 (2007), 427-445.

[Co] T. Collins, The transverse entropy functional and the Sasaki-Ricci flow, Trans. Amer. Math. Soc. 365 (2013), 1277-1303.

[GZ] H. Gu, Z. Zhang, An extension of Mok’s theorem on the generalized Frankel conjecture, Sci. China Math. 53 (2010), 1-12.

[He] W. He, The Sasaki-Ricci flow and compact Sasaki manifolds of positive transverse holomorphic bisectional curvature, to appear in J. Geom. Anal.

[HS1] W. He, S. Sun, Frankel conjecture and Sasaki Geometry, arXiv:1202.2589
[HS2] W. He, S. Sun, The generalized Frankel conjecture in Sasaki geometry, arXiv:1209.4026.

[M] N. Mok, The uniformization theorem for compact Kähler manifolds of non-negative holomorphic bisectional curvature, J. Diff. Geom. 27 (1988), 179-214.

[SWZ] K. Smoczyk, G-F. Wang, Y-B. Zhang. The Sasaki-Ricci flow, Internat. J. Math. 21 (2010), no. 7, 951-969.

[T] T. Takahashi, Sasakian $\phi$-symmetric spaces, Tohoku Math. J. 29 (1977), 91-113.

School of Mathematical Sciences, Key Laboratory of Mathematics and Complex Systems,
Beijing Normal University, Beijing 100875, P.R. China
E-mail address: hhuang@bnu.edu.cn