String Theory : a mere prelude to non-Archimedean Space-Time Structures ?

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Abstract

It took two millennia after Euclid and until in the early 1880s, when we went beyond the ancient axiom of parallels, and inaugurated geometries of curved spaces. In less than one more century, General Relativity followed. At present, physical thinking is still beheld by the yet deeper and equally ancient Archimedean assumption which entraps us into the limited view of ”only one walkable world”. In view of that, it is argued with some rather easily accessible mathematical support that Theoretical Physics may at last venture into the Non-Archimedean realms.

1. A Deep Expression of a Need for Reconsideration

String Theory has for the last nearly three decades known a special status within Theoretical Physics in the pursuit of the grand unification between General Relativity and Quantum Theory. Recently however, it has become the object of a considerable criticism due to a number of reasons which in times to come may, or for that matter, may not turn out to be fully valid.
Be it as it may, even in case String Theory ends up side lined for some
time to come, or possibly for good, it is important not to throw away
with it those of the more fundamental novel ideas which it managed
to bring to the forefront of thinking in Theoretical Physics, ideas that
may actually express a deeper and more generally valid need for recon-
sideration in the realms of our most fundamental physical intuitions,
meanings and concepts.

Two such possible novel ideas worth retaining are the following:

- seeing space-time as involving a number of dimensions quite be-
yond the usual four space-time ones,

- seeing the simplest primary elements of space-time not as mere
dimensionless points, but rather, as entities with a certain struc-
ture.

Remarkably, the second idea above had been familiar in Mathematics,
namely, in Topology, prior to String Theory. It is what rather whims-
sically is called pointless topology, or more appropriately, point-free
topology, see the early survey [1], with references going back into the
1960s.

As for the first above idea, that as well had been suggested in Physics,
prior to String Theory. Indeed, as far back as in the early 1920s,
Kaluza-Klein put forward a five dimensional version of General Rel-
avity. Following that, still higher dimensional versions of General
Relativity were proposed.

In this paper, related to the above two ideas - and at the same time,
going beyond them in ways that have so far mostly escaped the in-
tuition or interest of physicists - we shall draw attention upon, and
argue the possible appropriateness of considering non-Archimedean
structures for space-time, see Appendix 1 for the respective definition.

Here it should be noted that in Theoretical Physics there have been
occasional suggestions for a reconsideration of our usual percep-
tions and conceptions of space-time, and specifically, of its Mathematical
modelling, be it geometric, algebraic, topological, and so on.
Interestingly enough, String Theory itself gave importance to what has lately been called Non-Commutative Geometry.

And on yet more basic and simple levels, it has been suggested in Theoretical Physics that the scalars given by the usual field \( \mathbb{R} \) of real numbers, scalars upon which the field \( \mathbb{C} \) of complex numbers itself is built, should be replaced by other sets of scalars, some of such scalars having a multiplication which is non-commutative, or even non-associative, see [6-9] and the references cited there.

In fact, connected with Quantum Field Theory, there have been suggestions coming from theoretical physicists, [4], for using N-Category Theory, which is a most involved extension of Category Theory introduced in Mathematics in the 1940s, the latter itself being already on such a level of abstraction as to be beyond the present interest of most so called working mathematicians.

A most remarkable latest program for a deep reconsideration of the foundations of Theoretical Physics has just been published in [1]. Here we cite some of its introductory arguments which could hardly be expressed better:

"A striking feature of the various current programmes for quantising gravity including superstring theory and loop quantum gravity is that, notwithstanding their disparate views on the nature of space and time, they almost all use more-or-less standard quantum theory. Although understandable from a pragmatic viewpoint (since all we have is more-or-less standard quantum theory) this situation is nevertheless questionable when viewed from a wider perspective. Indeed, there has always been a school of thought asserting that quantum theory itself needs to be radically changed/developed before it can be used in a fully coherent quantum theory of gravity. This iconoclastic stance has several roots, of which, for us, the most important is the use in the standard quantum formalism of certain critical mathematical ingredients that are taken for granted and yet which, we claim, implicitly assume certain properties of space and time. Such an a priori imposition of spatio-temporal concepts would be a major error if they turn out to be fundamentally incompatible with what is needed for
a theory of quantum gravity. A prime example is the use of the continuum which, in this context, means the real and/or complex numbers. These are a central ingredient in all the various mathematical frameworks in which quantum theory is commonly discussed. For example, this is clearly so with the use of (i) Hilbert spaces and operators; (ii) geometric quantisation; (iii) probability functions on a non-distributive quantum logic; (iv) deformation quantisation; and (v) formal (i.e., mathematically ill-defined) path integrals and the like. The a priori imposition of such continuum concepts could be radically incompatible with a quantum gravity formalism in which, say, space-time is fundamentally discrete: as, for example, in the causal set programme.”

What however appears to have avoided the intuition, awareness or interest of physicists is the use of non-Archimedean space-time structures. And as argued in [9], the fact that, mostly by an omission or default, the mathematical models used in Theoretical Physics have been confined to those which happen to be given by Archimedean structures, has led to an entrapment into a so called ”one single walkable world”, see section 2.4., and entrapment which no one seems to be aware of in Theoretical Physics.

In particular, it is tacitly accepted that time, for instance, extends at most only as far as - but by no means beyond - the realms described by the interval of usual real numbers from $-\infty$ to $+\infty$, that is, as given by the real line $\mathbb{R}$. And such a view of the ranges of all possible time is taken for granted in Classical Mechanics, Relativity and Quantum Theory as well.

By the same token, it is accepted that in Classical Mechanics and Special Relativity there is simply no - and there can never be any - space whatsoever beyond that which is described by $\mathbb{R}^3$.

To a certain extent, such tacit and rather universal assumptions have an historical explanation - however, not necessarily and excuse as well - in the fact that the field $\mathbb{R}$ of real numbers is the only field which is linearly ordered, complete, and Archimedean. And at the beginnings of ancient Mathematics, and specifically Geometry, there was obvi-
ously no use in any non-Archimedean structures.

Certainly, at its ancient origins, Mathematics was confined to the realms directly accessible to our human senses, that is the realms of the physical, and specifically the visible and the palpable. And quantitative measurement of various objects, and also of land, was among its main concerns and applications. In such a context, no doubt, the Archimedean assumption had to arise. After all, without it, there would not be any possibility to choose a unit measure, and then measure by its integer and fractional multiples the visible and palpable entities under consideration.

Remarkably, ever since ancient times and till recently, the Archimedean assumption has never been questioned even in Mathematics, let alone in Physics. On the contrary, it went on and on as an assumption on a level still deeper than those formulated explicitly as various self-evident axioms. It is therefore not so surprising that Theoretical Physics still treats the Archimedean assumption precisely in the same manner.

In this regard, it is rather poignant to recall the development of the views related to the ancient axiom of parallels. Until the early 1800s, no one ever questioned the veracity of that axiom. On the contrary, it was assumed to be true, and all efforts were directed towards proving it from the other axioms. Indeed, the only concern about the axiom of parallels was caused by the fact that, when compared with the other axioms of Geometry, it seemed somewhat complex, and also, it involved the concept of infinity, a concept not directly accessible to our human senses. Thus it was the common perception that the axiom of parallels could not so easily be classified along the other axioms as self-evident.

And until the early 1800s, that was an obvious matter of concern, since it was taken for granted that:

"Axioms can only and only be self-evident statements."

Here in this last statement itself we can see once again a clear and relevant example of the crucial importance of certain axioms which are accepted on a deeper, and less than conscious level. Let us call them
for convenience *deep axioms*. And as it often happens, such deep axioms can easily play a more important role than the usual ones which we are consciously and explicitly formulated and accepted.

And how deep the axiom of parallels managed to penetrate in human awareness is illustrated by the fact that not long before the discovery of non-Euclidean Geometries in which that axiom does no longer hold, such a remarkable philosopher like Immanuel Kant, distinguished with an incisive critical sense, kept considering Euclidean Geometry, and thus the axiom of parallels, as nothing short of being an a priori truth, therefore being the *only* possible Geometry.

Fortunately, in the early 1800s, the axiom of parallels was proved to be independent of the other axioms of Geometry. Furthermore, two different versions of that axiom could clearly be formulated, thus leading to two non-Euclidean Geometries. Needless to say, without that discovery, General Relativity, which got discovered in less than one more century, could not have come about. And so it came to pass that within a few generations, the questioning of the axiom of parallels opened the way to such a revolution in Theoretical Physics, as that brought about by General Relativity.

Yet, today, the Archimedean assumption still keeps physicists enthralled ... Enthralled by, and also entrapped into ”one single walkable world” which it inevitably imposes ... And the resulting limitations, of which no one seems to be aware of in Theoretical Physics, can be tremendous. For instance, in a non-Archimedean space-time structure the very concept of quantum, as well as the finite upper limit on possible velocities cannot but acquire completely new meanings, and possibly, even new formulations or alternatives ...

However, several modern facts of a rather historic proportion in themselves, should have by now drawn attention upon our entrapment into ”one single walkable world” with its Archimedean limitations. Among these facts is the following one:
Back in 1966, Abraham Robinson introduced Nonstandard Analysis. His reason was mainly to place on a rigorous footing the "infinitesimals" used by Leibniz in Calculus, in the late 1600s. However, the point of importance here in Robinson’s construction of the field $\star \mathbb{R}$ of nonstandard reals is the non-Archimedean nature of that field, a property which must of course follow from the uniqueness of the Archimedean field $\mathbb{R}$ of real numbers mentioned above.

To this crucial fact, and being by no means less important, one can add the most intriguing question raised in [5], which can briefly be formulated as follows:

How come that all spaces used so far in Theoretical Physics have as sets a cardinal not larger than that of the continuum, that is, of the real numbers $\mathbb{R}$?

And the relevance of this question is in the fact that, ever since Cantor’s Set Theory, introduced in the late 1800s, we know about sets with cardinality incomparably larger that of the continuum. Furthermore, the cardinal of the continuum is very low among the infinite cardinals. In fact, it is merely the second one, namely, after that of the cardinal of the set $\mathbb{N}$ of nonnegative integers, if one accepts the Continuum Hypothesis.

However, what may be no less important an argument than those above for the need of a reconsideration of the Archimedean mathematical structures used in Theoretical Physics are the surprising and so far not yet explored rich opportunities opened up by non-Archimedean structures.

A well developed example about the actual benefits of such considerable opportunities is given, among others, by the Nonlinear Algebraic Theory of Generalized Solution for large classes of nonlinear partial differential equations, a theory originated in the early 1960s, and by now listed by the American Mathematical Society in their Subject Classification, under 46F30, see:

www.ams.org/msc/46Fxx.html
Some of the relevant such applications of non-Archimedean structures, and not only in the solution of partial differential equations, can be found in [10-33], and in the references cited there.

Returning to [5], my general appreciation of it, expressed succinctly and without much detail, can be found in Appendix 2 which was sent for publication to The Mathematical Intelligencer. Regardless of that view, however, one can note that [5] does not give any attention to the issue of Archimedean versus non-Archimedean structures, and instead, and merely by default, takes the traditional Archimedean view for granted.

Recently, related to Quantum Gravity, so called background free theoretical models have been suggested. As it happens, however, such models are still entrapped into the ”one single walkable world” situation, even if they see themselves free from any usual Geometry, Algebra or Topology. And this entrapment can only one again show how deep some deep axioms can indeed condition one’s whole vision and thinking, even if such deep axioms are by then present only by their, so to say, ghosts ...

The ancient axiom of parallels created a long ongoing controversy which got only solved in the early 1880s. And this solution, in less than one more century, opened the way to General Relativity.

On the other hand, the Archimedean assumption has so far hardly brought with it any comparable controversy. And certainly not in Theoretical Physics ...
And as things so often go with us humans : no controversy means no consideration, let alone reconsideration ...
Thus in Theoretical Physics we remain trapped for evermore into the Archimedean assumption and the corresponding ”one single walkable world” which it imposes ...

Going beyond the ancient axiom of parallels was, no doubt, absolutely necessary for being able to conceive of and achieve General Relativity. Fortunately, once the transition beyond that ancient axiom happened,
in less than one more century, it also proved to be sufficient for that truly revolutionary success in Theoretical Physics to occur.

Nowadays, it is perhaps the time to go beyond the yet deeper ancient Archimedean assumption, and at long last, liberate physical thinking from the limitations of the consequent ”one single walkable world” ...

2. Universes within Universes, Universes next to Universes ...

We shall briefly illustrate here the surprising wealth of structure the non-Archimedean property can bring with itself. A few further details in this regard can be found in [9].

2.1. A simple basic one dimensional and linearly ordered example

For simplicity, we start with one of the most relevant non-Archimedean fields which extend the usual scalar field \( \mathbb{R} \) of real numbers, namely, the field \( ^*\mathbb{R} \) of nonstandard real numbers, introduced by Robinson in the 1960s. And to make more user friendly the presentation of the facts important here - namely, the surprising richness of the respective non-Archimedean structure - we shall recall the corresponding results without proof, since as is well known, such proofs, presented for instance in [3], can be technically rather involved, as it is typical for Nonstandard Analysis.

Let us therefore see

- what are the differences between the Archimedean field of usual real scalars in \( \mathbb{R} \), and on the other hand, their non-Archimedean field extension given by \( ^*\mathbb{R} \) ?

- how much more rich is the structure of \( ^*\mathbb{R} \), when compared with that of \( \mathbb{R} \) ?

As we shall see in the sequel, going from \( \mathbb{R} \) to the much larger \( ^*\mathbb{R} \) involves two expansions, namely, a local one, at each point \( r \in \mathbb{R} \), as
well as a global one, at both infinite ends of the real line $\mathbb{R}$.

A particularly useful way to illustrate it, [3], is by saying that, when we want to go from $\mathbb{R}$ to $\mathbb{R}^\ast$, we need two special instruments which can give us views outside of $\mathbb{R}$, and into the not yet seen, and not yet known worlds situated within $\mathbb{R}^\ast$ beyond the confines of the standard realm of the real line $\mathbb{R}$. Namely, we need

- a so called microscope in order to see the monads which give the new and additional local structure in $\mathbb{R}^\ast$, and we also need
- a so called telescope for being able to look at galaxies giving the new and additional global structure in $\mathbb{R}^\ast$.

In order to understand the structure of $\mathbb{R}^\ast$, the following three things are therefore enough:

- to keep in mind that $\mathbb{R}^\ast$ is a linearly ordered non-Archimedean field,
- to understand how the monads create the local structure of $\mathbb{R}^\ast$, and in this respect, it is enough to understand how they create the local structure of the galaxy of $0 \in \mathbb{R}$, which can be seen as the central galaxy,
- to understand how by uncountably many translations to the right and left, the galaxy of $0 \in \mathbb{R}$, that is, the central galaxy, creates the global structure of $\mathbb{R}^\ast$.

Anticipating and simplifying a more precise description which follows in the sequel, the way $\mathbb{R}^\ast$ is obtained from $\mathbb{R}$ is in summary by the next procedure:
In other words, \( \mathbb{R}^* \) is obtained from \( \mathbb{R} \) as follows:

- at each usual real number \( r \in \mathbb{R} \) there is an expansion by the insertion of the uncountable set \( \text{mon}(r) \), furthermore
- at each of the \( \pm \infty \) ends of \( \mathbb{R} \), there is an expansion in which uncountable unions of uncountable sets \( \text{Gal}(s) \) are joined to \( \mathbb{R} \).

Keisler’s microscope is needed in order to be able to look into \( \text{mon}(0) \), since the infinitesimals cannot be seen from the usual point of view of \( \mathbb{R} \). This also goes of course for each \( \text{mon}(r) \), with \( r \in \mathbb{R} \).

In this way, in order to see every number in \( \text{Gal}(0) \), we need Keisler’s microscope.

Similarly, Keisler’s telescope is needed in order to see what is going on to the left of \( -\infty \), and to the right of \( +\infty \). Indeed, galaxies other than \( \text{Gal}(0) \), we cannot see without Keisler’s telescope.

And now, let us make the above more precise by giving some information on the structure of the uncountable sets \( \text{mon}() \) and \( \text{Gal}() \).

A scalar \( s \in \mathbb{R}^* \) is called \textit{infinitesimal}, if and only if \( |s| \leq r \), for every \( r \in \mathbb{R}, \ r > 0 \). The set of such infinitesimal scalars is denoted by
and following Leibniz, it is called the monad of $0 \in \mathbb{R}$.

For an arbitrary scalar $s \in *\mathbb{R}$ it will be convenient to denote

$$\text{mon}(s) = s + \text{mon}(0)$$

which is but the translate of $\text{mon}(0)$ by $s$, and it is the monad of $s$.

A scalar $s \in *\mathbb{R}$ is called finite, if and only if $|s| \leq r$, for some $r \in \mathbb{R}$, $r > 0$. We denote by

$$\text{Gal}(0)$$

the set of all finite scalars, and call that set the galaxy of $0 \in \mathbb{R}$, or the central galaxy, since it is the only galaxy which contains the real line $\mathbb{R}$.

For an arbitrary scalar $s \in *\mathbb{R}$ it will be convenient to denote

$$\text{Gal}(s) = s + \text{Gal}(0)$$

which is but the translate of $\text{Gal}(0)$ by $s$, and it is the galaxy of $s$.

At last, a scalar $s \in *\mathbb{R}$ is called infinite, if and only if $|s| \geq r$, for every $r \in \mathbb{R}$, $r > 0$.

As follows easily in Nonstandard Analysis, [3], none of the above sets of monads and galaxies is void, and in fact, each of them is uncountably large. Furthermore, as seen in the sequel, both monads and galaxies have surprisingly complex structures which in a way reflect one another, and in addition, each of them is self-similar, recalling one of the well known basic properties of fractal structures.

Now we recall that

$$(2.1) \quad \text{mon}(0) \cap \mathbb{R} = \{ 0 \}$$
thus the only real number $r \in \mathbb{R}$ which is infinitesimal is $r = 0 \in \mathbb{R}$.

Also, for $x, y \in \mathbb{R}$ and $x \neq y$, we have

\[(2.2) \quad \text{mon}(x) \cap \text{mon}(y) = \phi\]

hence we obtain the representation given by a union of \textit{pairwise disjoint} sets

\[(2.3) \quad \text{Gal}(0) = \bigcup_{r \in \mathbb{R}} \text{mon}(r)\]

Further we have

\[(2.4) \quad \mathbb{R} \subsetneq \text{Gal}(0) \subsetneq \ast\mathbb{R}\]

\[(2.5) \quad \ast\mathbb{R} \setminus \text{Gal}(0) \text{ is the set of infinite numbers in } \ast\mathbb{R}\]

It follows that, schematically, we have for $\ast\mathbb{R}$

\[
\begin{array}{ccc}
\bigcup_{s < -\infty} \text{Gal}(s) & \bigcap \text{mon}(r) & \bigcup_{s > +\infty} \text{Gal}(s) \\
\text{infinite} & \text{finite} & \text{infinite} \\
\text{infinite} < 0 & \text{finite} & \text{infinite} > 0
\end{array}
\]

\[
\text{Gal}(0)
\]

And similar with (2.2), for $s, t \in \ast\mathbb{R}$ and $t - s$ infinite, we have

\[(2.6) \quad \text{Gal}(s) \cap \text{Gal}(t) = \phi\]

while for $t - s$ finite, we have

\[(2.7) \quad \text{Gal}(s) = \text{Gal}(t)\]

As a consequence, and similar with (2.3), we have the following representation as a union of \textit{pairwise disjoint} sets

\[(2.8) \quad \ast\mathbb{R} = \bigcup_{\lambda \in \Lambda} \text{Gal}(-s_\lambda) \cup \text{Gal}(0) \cup \bigcup_{\lambda \in \Lambda} \text{Gal}(s_\lambda)\]

where $\Lambda$ is an uncountable set of indices, while $s_\lambda \in \ast\mathbb{R}$ are pos-
itive and infinite for \( \lambda \in \Lambda \), and such that \( s_\lambda - s_\mu \) is infinite for \( \lambda, \mu \in \Lambda, \lambda \neq \mu \).

The use of the term *monad* by Robinson was, as mentioned, inspired by Leibniz who first employed infinitesimals and did so in surprisingly effective manner, even if in a merely intuitive and insufficiently rigorous manner when, parallel with and independently of Newton, started the development of Calculus in the late 1600s. In this regard the fact that

\[
\text{mon}(0) \cap \mathbb{R} = \{ 0 \}
\]

which means that zero is the only real number which is infinitesimal, caused, starting with Leibniz, all sort of difficulties when, prior to the creation of modern Nonstandard Analysis, one tried to deal with infinitesimals. Similarly, the fact that

\[
\ast \mathbb{R} \setminus \text{Gal}(0)
\]

is the set of infinite numbers in \( \ast \mathbb{R} \), thus there are no reals numbers which are infinite, brought with it difficulties when dealing with infinitely large numbers, prior to the modern theory of Nonstandard Analysis.

### 2.2. Global and local structure of \( \ast \mathbb{R} \)

The structure of \( \text{Gal}(0) \), both locally and globally, is presented in (2.3). In view of (2.8), that also gives an understanding of the local structure of \( \ast \mathbb{R} \).

For the understanding of the global structure of \( \ast \mathbb{R} \) we have the order reversing bijective mapping between infinite and infinitesimal numbers in \( \ast \mathbb{R} \), namely

\[
(2.9) \quad ( \ast \mathbb{R} \setminus \text{Gal}(0) ) \ni s \mapsto 1/s \in ( \text{mon}(0) \setminus \{ 0 \} )
\]

Indeed, it is obvious that, given \( s, t \in \ast \mathbb{R} \setminus \text{Gal}(0) \), then
\[ s < t \iff 1/s > 1/t \]

This fact is fundamental, as it shows that the local structure of \( \ast \mathbb{R} \) mirrors its global structure, and vice versa.

It also establishes the link between Keisler’s microscope and telescope, the former letting us see into \( \text{mon}(0) \), while the latter allowing us to look out into \( \ast \mathbb{R} \setminus \text{Gal}(0) \).

In this regard, the properties (2.1) - (2.7) express the local structure of \( \ast \mathbb{R} \), more precisely, within its part given by \( \text{Gal}(0) \), that is, the structure of the finite numbers in \( \ast \mathbb{R} \). This local structure consists of a sort of infinitesimal neighbourhood around every usual real number \( r \in \mathbb{R} \), neighbourhood given by the translate \( \text{mon}(r) = r + \text{mon}(0) \) of \( \text{mon}(0) \). And in view of (2.1), this local structure cannot be seen from \( \mathbb{R} \), thus we have to use Keisler’s microscope.

### 2.3 Self-Similarity in \( \ast \mathbb{R} \)

The local and global structures of \( \ast \mathbb{R} \), as we have seen in subsection 2.2, are closely related, as they are expressed in (2.3), (2.8) and (2.9).

Here we shall point to a self-similar aspect of this interrelation which may remind us of a typical feature of fractals.

In this regard, we recall that the global structure of \( \ast \mathbb{R} \) is given by, see (2.8)

\[
\ast \mathbb{R} = \bigcup_{\lambda \in \Lambda} \text{Gal}(-s_\lambda) \cup \text{Gal}(0) \cup \bigcup_{\lambda \in \Lambda} \text{Gal}(s_\lambda)
\]

while its local structure is described by, see (2.9)

\[
\text{Gal}(0) = \bigcup_{r \in \mathbb{R}} (r + \text{mon}(0))
\]

In this way we obtain the self-similar order reversing bijections, which are expressed in terms of \( \text{mon}(0) \), namely
As we can note, the above bijections in (2.10), (2.11) are given by the very simple algebraic, explicit, and order reversing mapping \( s \mapsto 1/s \) which involves what is essentially a field operation, namely, division. And these two bijections take the place of the much simpler order reversing bijections in the case of the usual real line \( \mathbb{R} \), namely

\[
\begin{align*}
(2.12) \quad ( \mathbb{R} \setminus (-1, 1) ) & \ni r \mapsto 1/r \in ( [-1, 1] \setminus \{ 0 \} ) \\
(2.13) \quad ( [-1, 1] \setminus \{ 0 \} ) & \ni r \mapsto 1/r \in ( \mathbb{R} \setminus (-1, 1) )
\end{align*}
\]

The considerable difference between (2.10), (2.11), and on the other hand, (2.12), (2.13) is obvious. In the former two, which describe the structure of \( ^*\mathbb{R} \), the order reversing bijections represent the set

\[
\text{mon}(0) \setminus \{ 0 \}
\]

through the set

\[
[ \bigcup_{r \in \mathbb{R}, \lambda \in \Lambda} (r - s_\lambda + \text{mon}(0)) ] \cup [ \bigcup_{r \in \mathbb{R}, \lambda \in \Lambda} (r + s_\lambda + \text{mon}(0)) ]
\]

which contains uncountably many translates of the set \( \text{mon}(0) \). And it is precisely this manifestly involved self-similarity of the set \( \text{mon}(0) \)
of monads which is the novelty in the non-Archimedean structure of *R, when compared with the much simpler Archimedean structure of R. This novelty is remarkable since it makes mon(0) having the same complexity with

\[ *R \setminus Gal(0) = \]

\[ = \bigcup_{r \in R, \lambda \in \Lambda} (r - s\lambda + mon(0)) \cup \bigcup_{r \in R, \lambda \in \Lambda} (r + s\lambda + mon(0)) \]

In this way mon(0), which is but the realm of infinitesimals, thus it cannot be seen in terms of R, except with Keisler’s microscope, has the very same complexity as the set *R \setminus Gal(0) of all infinitely large numbers, which again cannot be seen from R, unless Keisler’s telescope is used.

As for (2.12), (2.13), the utter simplicity of the Archimedean trap becomes obvious, since the sets \([-1, 1] \setminus \{0\} \) and \(R \setminus (-1, 1)\) are again in an order reversing bijection. However, no any trace whatsoever of self-similarity, since \([-1, 1]\) does not even once appear in \(\setminus (-1, 1)\).

### 2.4. Walkable worlds …

The local and global nature of a non-Archimedean structure can be illustrated in a more intuitively accessible and geometric manner, the manner which in ancient times was quite likely the one taken in view of its obvious everyday practicality. And to keep things simple, we remain for a while longer with the nonstandard extension *R of the real line R. Indeed, *R is a highly relevant instance of a non-Archimedean structure, thus it is in this regard essentially different from R which is Archimedean. On the other hand, both *R and R have the simplicity of being one dimensional.

Given now any usual real number \(u \in R, u > 0\), let us ask what appears to be an eminently practical question, namely:

How far along the one dimensional world of *R can one walk from any given point \(r \in R\) by taking an arbitrary
large finite number of steps of size \( u \), be it in the positive or negative directions?

The answer given by Nonstandard Analysis is

\[
(2.14) \quad Gal(0)
\]

Needless to say, in view of (2.8), this kind of walk confines us to an utterly small part of \( *\mathbb{R} \). Namely, as follows from (2.3), \( Gal(0) \) only contains the usual real line \( \mathbb{R} \) and the monads around each real number \( r \in \mathbb{R} \), thus the infinitesimal neighbourhoods of such numbers \( r \). Consequently, \( Gal(0) \) does not contain any of the infinite numbers in \( *\mathbb{R} \).

And as is so far mostly the case, it is precisely this limited world which is the one dimensional world of Theoretical Physics. And in fact it is even less, since it is only its strict subset given by the real line \( \mathbb{R} \), and without any of the monads which make up the difference between \( Gal(0) \) and the smaller \( \mathbb{R} \), see (2.3). Indeed, this is the “only one walkable world”

within which the mathematical modelling of Physics tends to be confined ...

And then, let us see

How many other, and even more importantly, what kind of other "walkable worlds" can the non-Archimedean structure of \( *\mathbb{R} \) offer us?

And here may quite likely be a novel and richly promising opportunity to be made use of at last in Theoretical Physics.

The answer to the above general question is very simple. We take an arbitrary step length \( u \in *\mathbb{R}, \, u > 0 \) and an arbitrary starting point \( s \in *\mathbb{R} \), and denote by
(2.15) \( WW_{u,s} \)

that part of \( *\mathbb{R} \) which can be reached from \( s \) by walking any finite number of steps, each of the given length \( u \), and do so either in the positive, or in the negative direction. We call \( WW_{u,s} \) the "walkable world" from \( s \) and with steps \( u \).

To be more precise mathematically, any given \( t \in *\mathbb{R} \) belongs to \( WW_{u,s} \), if and only if there exists \( n \in \mathbb{N}, \ n \geq 1 \), such that

(2.16) either \( s \leq t \leq s + nu \) or \( s - nu \leq t \leq s \)

or equivalently

(2.16*) \( s - nu \leq t \leq s + nu \)

An immediate consequence is that

(2.17) all "walkable worlds" are order isomorphic with \( Gal(0) \)

Indeed, with the above notation, let us define the mapping \( \omega \) by

\[
WW_{u,s} \ni t \mapsto \omega(t) = (t - s)/u \in *\mathbb{R}
\]

then \( \omega \) is obviously injective and strictly increasing, while (2.16*) implies that \( \omega(WW_{u,s}) \subseteq Gal(0) \). However, \( \omega \) is also surjective onto \( Gal(0) \), since for given \( v \in Gal(0) \), we have \( -n \leq v \leq n \), for a suitable \( n \in \mathbb{N}, \ n \geq 1 \), thus \( t = s + vu \in WW_{u,s} \), while obviously \( \omega(t) = v \).

Clearly therefore:

(2.18) A "walkable world" cannot have a smallest or a largest element.

Another immediate consequence is the following self-confinement property of such "walkable worlds", namely

(2.19) \( \forall \ t \in WW_{u,s} : WW_{u,t} = WW_{u,s} \)
In other words, one can never leave, or get out of a ”walkable world” $WW_{u,s}$, no matter at what point $t$ in it one would start walking, and doing so with steps of the same length $u$, or for that matter, with steps of any length $u' = cu$, where $c > 0$ is finite.

In particular it follows that, see (2.14)

\[(2.20) \quad \forall \ u \in \mathbb{R}, \ u \geq 1, \ r \in \mathbb{R} : WW_{u,r} = Gal(0)\]

which, in the case of one dimension, contains what is so far the typical ”only one walkable world” in Theoretical Physics ...

2.5. How are the walkable worlds situated with respect to one another ?

The answer to this question is obtained easily. Suppose given two ”walkable worlds” $WW_{u,s}$ and $WW_{u',s'}$, with $u, \ u', \ s, \ s' \in *\mathbb{R}, \ u, \ u' > 0$, and let us assume that these two ”walkable worlds” are not disjoint, namely

\[(2.21) \quad WW_{u,s} \cap WW_{u',s'} \neq \emptyset\]

Thus we can take a point

\[t \in WW_{u,s} \cap WW_{u',s'}\]

and then in view of (2.16*), for suitable $n, \ n' \in \mathbb{N}, \ n, \ n' \geq 1$, we obtain the inequalities

\[s - nu \leq t \leq s + nu, \quad s' - n'u' \leq t \leq s' + n'u'\]

which imply that

\[(2.22) \quad s - nu \leq s' + n'u', \quad s' - n'u' \leq s + nu\]

Now since $*\mathbb{R}$ is linearly ordered, we can assume without loss of generality that
(2.23) \[ 0 < u \leq u' \]

and then the first inequality in (2.22) gives

\[ s \leq s' + nu + n'u' \leq s' + (n + n')u' \]

hence \( s \in WW_{u',s'} \), which in view of (2.23) means that

(2.24) \[ WW_{u,s} \subseteq WW_{u',s'} \]

In this way, we proved that:

(2.25) Two "walkable worlds" are either disjoint, or one contains the other.

The remarkable fact, however, is that:

(2.26) There are uncountably many pairwise disjoint "walkable worlds". And between any two disjoint "walkable worlds" there are uncountably many other ones.

Also:

(2.27) Non-disjoint "walkable worlds" are nested into one another in uncountable chains.

As for the complex manner in which the "walkable worlds" are situated outside of one another, or on the contrary, nested within one another, the self-similarity property of \( \ast \mathbb{R} \) in subsection 2.3. can offer further insights.

2.6. The sizes and relative sizes of "walkable worlds"

In order to give a further idea about the structure of \( \ast \mathbb{R} \), and in particular, about the sizes and relative sizes of "walkable worlds", we mention the following well known properties of numbers in \( \ast \mathbb{R} \).

Given any nonzero infinitesimal \( \epsilon \in \ast \mathbb{R} \), that is, \( \epsilon \in mon(0) \setminus \{ 0 \} \), there exist nonzero infinitesimals \( \delta, \eta \in mon(0) \setminus \{ 0 \} \), such that \( \epsilon/\delta \)
is again an infinitesimal, while $\epsilon/\eta$ is infinitely large. Similarly, given any nonzero infinitesimal $\epsilon \in mon(0) \setminus \{0\}$, there exist nonzero infinitesimals $\delta', \eta' \in mon(0) \setminus \{0\}$, such that $\delta'/\epsilon$ is again an infinitesimal, while $\eta'/\epsilon$ is infinitely large.

Correspondingly, given any infinitely large $s \in *\mathbb{R}$, that is, $s \in *\mathbb{R} \setminus Gal(0)$, there exist infinitely large $t, v \in *\mathbb{R} \setminus Gal(0)$, such that $s/t$ is infinitesimal, while $s/v$ is infinitely large. Also, given any infinitely large $s \in *\mathbb{R} \setminus Gal(0)$, there exist infinitely large $t', v' \in *\mathbb{R} \setminus Gal(0)$, such that $t'/s$ is infinitesimal, while $v'/s$ is infinitely large.

Furthermore, given any two numbers $v, w \in *\mathbb{R}$, $0 < v \leq w$, then

\begin{equation}
\text{either } w/v \text{ is finite, or } w/v \text{ is infinitely large}
\end{equation}

which equivalently means that

\begin{equation}
\text{either } \exists \ m \in \mathbb{N}, \ m \geq 1 : \ w \leq nv \\
\text{or } \forall \ m \in \mathbb{N}, \ m \geq 1 : \ nv < w
\end{equation}

As an effect of the above, we mention a few surprising facts about "walkable worlds".

First we recall that the Archimedean assumption, when considered within one dimension, has ever since ancient times kept us - and still keeps much of Theoretical Physics - confined within the real line $\mathbb{R}$ which is but a strict subset, see (2.3), (2.20), of the "walkable world" given by the central galaxy, namely

\begin{equation}
Gal(0) = WW_{1,0}
\end{equation}

Compared to this ancient "only one walkable world" however, there are uncountably many infinitesimal, or for that matter, infinitely large "walkable worlds". And on their turn, infinitesimal "walkable worlds" can be infinitely large, or for that matter, infinitesimal when compared with some other infinitesimal "walkable worlds". Similarly, infinitely large "walkable worlds" can be infinitesimal, or on the contrary, in-
Finally large when compared with some other infinitely large "walkable worlds".
In this regard, the central "walkable world" (2.29) has only one distinguishing feature, namely, that it contains both 0 and 1.

Further, let us note that

\[(2.30) \quad \text{mon}(0) \text{ is not a "walkable world"}
\]

although obviously \(0 \in \text{mon}(0)\). Indeed, let us assume that

\[
\text{mon}(0) = WW_{\epsilon, \eta}
\]

for a certain \(\epsilon, \eta \in *\mathbb{R}, \epsilon > 0\). But \(0 \in \text{mon}(0) = WW_{\epsilon, \eta}\), hence (2.19) gives

\[
\text{mon}(0) = WW_{\epsilon, 0}
\]

And then in view of (2.16*), it follows that for every \(\chi \in *\mathbb{R}\), there exists \(n \in \mathbb{N}, n \geq 1\), such that

\[-n\epsilon \leq \chi \leq n\epsilon
\]

which is obviously not true for \(\chi = \sqrt{\epsilon} \in \text{mon}(0)\).

In view of (2.30) it follows that \(\text{mon}(0)\) itself, that is, the set of infinitesimals, is far more large and complex than being "one single walkable world". And this is to be expected from its self-similarity property in (2.10), (2.11).

3. Non-Archimedean structures of higher dimensions

As we have seen, the non-Archimedean nature of \(*\mathbb{R}\) can already bring with it a surprisingly novel and rich structure. Needless to say, the structure of non-Archimedean spaces of higher dimensions can only be yet more rich and surprising.
Such higher, in fact, infinite dimensional non-Archimedean spaces, in fact algebras of generalized functions, have been successfully used in solving large classes of earlier unsolved linear and nonlinear partial differential equations, see [10-33] and the references cited there.

However, there are far simpler non-Archimedean algebras of scalars which can have a relevant use in Theoretical Physics as replacements of the usual Archimedean fields $\mathbb{R}$ or $\mathbb{C}$ of real, respectively, complex scalars, see [6-9] and the references cited there.

What is to note with respect to such non-Archimedean algebras of scalars is that they are in fact infinite dimensional. This is precisely why they are non-Archimedean.

On the other hand, their construction is perfectly similar with the traditional Cauchy-Bolzano construction of the real numbers $\mathbb{R}$. And consequently, their use does not lead to any additional difficulties, except for what still happens to be the surprising novelty and richness of non-Archimedean structures.

The essence of the mentioned traditional construction of the real numbers $\mathbb{R}$ has been formulated in the 1950s as the reduced power construction. And it is one of the main instruments in Model Theory, which is a discipline within Mathematical Logic.

Fortunately, reduced powers can easily be constructed and used without any involvement of Model Theory or Mathematical Logic. And they have actually been used in this way, for instance, in constructing the completion of metric, normed, or uniform topological spaces.

Here for convenience, we recall the main features of the construction of reduced powers, see [6-8] and the references cited there for further detail.

By the way, as noted earlier, reduced powers were extensively used in the construction of large classes of algebras of generalized functions, as well as generalized scalars, both employed in the solution of a large variety of linear and nonlinear PDEs, see [10-33].

The mentioned large class of algebras of scalars, constructed as re-
duced powers, is obtained as follows.

Let $\mathbb{K}$ be any algebra, among others the field $\mathbb{R}$ of real, or alternatively, the field $\mathbb{C}$ of complex numbers, and let $\Lambda$ be any infinite set of indices. Then we consider

$$(3.1) \quad \mathbb{K}^\Lambda$$

which is the set of all functions $x : \Lambda \rightarrow \mathbb{K}$, and which is obviously an algebra with the point-wise operations on such functions. Furthermore, if $\mathbb{K}$ is commutative or associative, the same will hold for the algebra $\mathbb{K}^\Lambda$.

A possible problem with these algebras $\mathbb{K}^\Lambda$ is that they have zero divisors, thus they are not integral domains, in the case of nontrivial $\mathbb{K}$, that is, when $0, 1 \in \mathbb{K}$, and $1 \neq 0$. For instance, if we take $\mathbb{K} = \mathbb{R}$ and $\Lambda = \mathbb{N}$, then

$$(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots) \in \mathbb{K}^\Lambda = \mathbb{R}^\mathbb{N},$$

$$(3.2) \quad (1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots) \neq 0 \in \mathbb{K}^\Lambda = \mathbb{R}^\mathbb{N}$$

and yet

$$(1, 0, 0, 0, \ldots) \cdot (0, 1, 0, 0, \ldots) = (0, 0, 0, \ldots) =$$

$$(3.3) \quad 0 \in \mathbb{K}^\Lambda = \mathbb{R}^\mathbb{N}$$

This issue, however, can be dealt with in the following manner, to the extent that it may prove not to be convenient. Let us take any ideal $\mathcal{I}$ in $\mathbb{K}^\Lambda$ and construct the quotient algebra

$$(3.4) \quad A = \mathbb{K}^\Lambda / \mathcal{I}$$

This quotient algebra construction has four useful features, namely

1. It allows for the construction of a large variety of algebras $A$ in (3.4),

2. The algebras $A$ in (3.4) are fields, if and only if the corresponding ideals $\mathcal{I}$ are maximal in $\mathbb{K}^\Lambda$,
3. The algebras $A$ in (3.4) are without zero divisors, thus are integral domains, if and only if the corresponding ideals $I$ are prime in $\mathbb{K}^\Lambda$.

4. If $\mathbb{K} = \mathbb{R}$, then there is a simple way to construct such ideals $I$ in $\mathbb{K}^\Lambda$, due to their one-to-one correspondence with filters on the respective infinite sets $\Lambda$.

Here it should be mentioned that the above construction of quotient algebras in (3.4) has among others a well known particular case, namely, the construction of the field $^*\mathbb{R}$ of nonstandard reals, where one takes $\mathbb{K} = \mathbb{R}$, while $I$ is a maximal ideal which corresponds to an ultrafilter on $\Lambda$.

Facts 2 and 3 above are a well known matter of undergraduate algebra.

Fact 4 is recalled here briefly for convenience. First we recall that each $x \in \mathbb{R}^\Lambda$ is actually a function $x : \Lambda \to \mathbb{R}$. Let us now associate with each $x \in \mathbb{R}^\Lambda$ its zero set given by $Z(x) = \{ \lambda \in \Lambda \mid x(\lambda) = 0 \}$, which therefore is a subset of $\Lambda$.

Further, let us recall the concept of filter on the set $\Lambda$. A family $\mathcal{F}$ of subsets of $\Lambda$, that is, a subset $\mathcal{F} \subseteq \mathcal{P}(\Lambda)$, is called a filter on $\Lambda$, if and only if it satisfies the following three conditions

\begin{align}
1. & \phi \notin \mathcal{F} \neq \phi \\
2. & J, K \in \mathcal{F} \implies J \cap K \in \mathcal{F} \\
3. & \Lambda \supseteq K \supseteq J \in \mathcal{F} \implies K \in \mathcal{F}
\end{align}

Given now an ideal $\mathcal{I}$ in $\mathbb{R}^\Lambda$, let us associate with it the set of zero sets of its elements, namely

\begin{equation}
\mathcal{F}_\mathcal{I} = \{ Z(x) \mid x \in \mathcal{I} \} \subseteq \mathcal{P}(\Lambda)
\end{equation}

Then

\begin{equation}
\mathcal{F}_\mathcal{I} \text{ is a filter on } \Lambda
\end{equation}
Indeed, $\mathcal{I} \neq \phi$, thus $\mathcal{F}_\mathcal{I} \neq \phi$. Further, assume that $Z(x) = \phi$, for a certain $x \in \mathcal{I}$. Then $x(\lambda) \neq 0$, for $\lambda \in \Lambda$. Therefore we can define $y : \Lambda \rightarrow \mathbb{R}$, by $y(\lambda) = 1/x(\lambda)$, with $\lambda \in \Lambda$. Then however $y.x = 1 \in \mathbb{R}^\Lambda$, hence $\mathcal{I}$ cannot be an ideal in $\mathbb{R}^\Lambda$, which contradicts the hypothesis. In this way condition 1 in (3.5) holds for $\mathcal{F}_\mathcal{I}$.

Let now $x, y \in \mathcal{I}$, then clearly $x^2 + y^2 \in \mathcal{I}$, and $Z(x^2 + y^2) = Z(x) \cap Z(y)$, thus condition 2 in (3.5) is also satisfied by $\mathcal{F}_\mathcal{I}$.

Finally, let $x \in \mathcal{I}$ and $K \subseteq \Lambda$, such that $K \supseteq Z(x)$. Let $y$ be the characteristic function of $\Lambda \setminus K$. Then $x.y \in \mathcal{I}$, since $\mathcal{I}$ is an ideal. But now obviously $Z(x.y) = K$, which shows that $\mathcal{F}_\mathcal{I}$ satisfies as well condition 3 in (3.5).

There is also the converse construction. Namely, let $\mathcal{F}$ be any filter on $\Lambda$, and let us associate with it the set of functions

\[
\mathcal{I}_\mathcal{F} = \{ x : \Lambda \rightarrow \mathbb{R} \mid Z(x) \in \mathcal{F} \} \subseteq \mathbb{R}^\Lambda
\]

Then

\[
\mathcal{I}_\mathcal{F} \text{ is an ideal in } \mathbb{R}^\Lambda
\]

Indeed, for $x, y \in \mathbb{R}^\Lambda$, we have $Z(x+y) \supseteq Z(x) \cap Z(y)$, thus $x, y \in \mathcal{I}_\mathcal{F}$ implies that $x+y \in \mathcal{I}_\mathcal{F}$. Also $Z(x.y) \supseteq Z(x)$, therefore $x \in \mathcal{I}_\mathcal{F}$, $y \in \mathbb{R}^\Lambda$ implies that $x.y \in \mathcal{I}_\mathcal{F}$. Further we note that $Z(cx) = Z(x)$, for $c \in \mathbb{R}$, $c \neq 0$. Finally, it is clear that $\mathcal{I}_\mathcal{F} \neq \mathbb{R}^\Lambda$, since $x \in \mathcal{I}_\mathcal{F} \implies Z(x) \neq \phi$, as $\mathcal{F}$ satisfies condition 1 in (3.5). Therefore (3.9) does indeed hold.

Let now $\mathcal{I}, \mathcal{J}$ be two ideals in $\mathbb{R}^\Lambda$, while $\mathcal{F}, \mathcal{G}$ are two filters on $\Lambda$. Then it is easy to see that

\[
\mathcal{I} \subseteq \mathcal{J} \implies \mathcal{F}_\mathcal{I} \subseteq \mathcal{F}_\mathcal{J}
\]

\[
\mathcal{F} \subseteq \mathcal{G} \implies \mathcal{I}_\mathcal{F} \subseteq \mathcal{I}_\mathcal{G}
\]

We can also note that, given an ideal $\mathcal{I}$ in $\mathbb{R}^\Lambda$ and a filter $\mathcal{F}$ on $\Lambda$, we have by iterating the above constructions in (3.6) and (3.8)
Indeed, in view of (3.6), (3.8), we have for $s \in \mathbb{R}^\Lambda$ the equivalent conditions

$$x \in \mathcal{I} \iff Z(x) \in \mathcal{F}_\mathcal{I} \iff x \in \mathcal{I}_{\mathcal{F}_\mathcal{I}}$$

Further, for $J \subseteq \Lambda$, we have the equivalent conditions

$$J \in \mathcal{F}_{\mathcal{I}_{\mathcal{F}}} \iff J = Z(s), \text{ for some } s \in \mathcal{I}_\mathcal{F}$$

But for $x \in \mathbb{R}^\Lambda$, we also have the equivalent conditions

$$x \in \mathcal{I}_\mathcal{F} \iff Z(x) \in \mathcal{F}$$

and the proof of (3.11) is completed.

In view of (3.11), it follows that every ideal in $\mathbb{R}^\Lambda$ is of the form $\mathcal{I}_\mathcal{F}$, where $\mathcal{F}$ is a certain filter on $\Lambda$. Also, every filter on $\Lambda$ is of the form $\mathcal{F}_{\mathcal{I}_\mathcal{F}}$, where $\mathcal{I}$ is a certain ideal in $\mathbb{R}^\Lambda$.

4. What about the Quanta and the Velocity of Light?

The quantitative aspect of the issue of the quanta and of the velocity of light has so far made perfect sense within the Archimedean structure of the real line $\mathbb{R}$, that is, within its ”only one walkable world” view, being supported by countless physical experiments.

Certainly, within such a view a smallest allowed strictly positive quantity, say, of energy, and in that case called ”energy quantum”, can have a clear meaning. Indeed, outside of that ”only one walkable world” there is supposed to be nothing at all. Therefore, that quantum, or smallest allowed strictly positive quantity - if it really exists - must by necessity be situated within that very same ”only one walkable world”.

And any number of physical experiments support the existence of such quanta.

Similar is, of course, the situation with a largest allowed finite quant-
tity, such as for instance the velocity of light.

However, once that ”only one walkable world” is replaced by a non-Archimedean structure, and in fact, by any non-Archimedean one, a rather unprecedented richness of structure comes into play due to the presence of infinitesimals and infinitely large elements, and consequently, of the corresponding structure of ”walkable worlds”.

Consequently, the very meaning of smallest allowed strictly positive quantity, or alternatively, of largest allowed finite quantity becomes dependent on the specific ”walkable world” within which it is considered. And as seen above even in the simple one dimensional case of ∗R, such ”walkable worlds” are uncountably many, either next to one another, or nested within one another, all of them in a complex self-similar pattern. In this way, it is no longer so clear within which particular ”walkable worlds” should the traditional concepts of quanta and velocity of light be considered. Thus such traditional concepts may possibly need a fundamental reconsideration.

In this regard, here we conclude with a summary for an appropriate consideration of those of the earlier presented facts in the simplest relevant non-Archimedean case, namely, of the one dimensional ∗R, facts which may be relevant for the mentioned possible reconsideration of quanta and/or velocity of light:

1) The self-similarity in (2.10), (2.11), namely

\[ \bigcup_{r \in \mathbb{R}, \lambda \in \Lambda} (r - s \lambda + \text{mon}(0)) \bigcup \bigcup_{r \in \mathbb{R}, \lambda \in \Lambda} (r + s \lambda + \text{mon}(0)) \ni s \mapsto 1/s \in (\text{mon}(0) \setminus \{0\}) \]

and conversely

\( (\text{mon}(0) \setminus \{0\}) \ni s \mapsto 1/s \in [\bigcup_{r \in \mathbb{R}, \lambda \in \Lambda} (r - s \lambda + \text{mon}(0)) \bigcup [\bigcup_{r \in \mathbb{R}, \lambda \in \Lambda} (r + s \lambda + \text{mon}(0)) \]
given by bijective order reversing mappings.

2) The order isomorphism between $Gal(0)$ and any "walkable world" $WW_{u,s}$, with $u, s \in \ast\mathbb{R}, u > 0$, see (2.17).

3) The fact that $mon(0)$ is not a "walkable world", but it has a rich structure made up of them, see (2.30).

4) The fact that two "walkable worlds" are either disjoint, or one contains the other, see (2.25).

**Appendix 1**

**Basic Algebraic Structures**

For convenience, we recall here a few basic concepts from Algebra and Ordered Structures. The respective concepts are introduced step by step, culminating with the ones we are interested in, namely, fields and algebras, and their Archimedean, respectively, non-Archimedean instances.

A group is a structure $(G, \alpha)$, where $G$ is a nonvoid set and $\alpha : G \times G \rightarrow G$ is a binary operation on $G$ which is :

- associative :
  \[ \forall \ x, y, z \in G : \alpha(\alpha(x, y), z) = \alpha(x, \alpha(y, z)) \]

- has a neutral element $e \in G$ :
  \[ \forall \ x \in G : \alpha(x, e) = \alpha(e, x) = x \]

- each element $x \in G$ has an inverse $x' \in G$ :
  \[ \alpha(x, x') = \alpha(x', x) = e \]

The group $(G, \alpha)$ is commutative, if and only if :
∀ x, y ∈ G : α(x, y) = α(y, x)

In such a case the binary operation α is simply denoted by ”+” and called addition, namely

α(x, y) = x + y,  x, y ∈ G

Further, the neutral element is denoted by 0, namely, e = 0, while for every x ∈ G, its inverse x′ is denoted by −x.

It will be useful to note the following. Given any group element x ∈ G and any integer number n ≥ 1, we can define the group element nx ∈ G, by

\[ nx = \begin{cases} 
  x & \text{if } n = 1 \\
  x + x + x + \ldots + x & \text{if } n \geq 2
\end{cases} \]

where the respective sum has n terms. The meaning of this operation is easy to follow. Namely, nx can be seen as n steps of length x each, in the direction x. This interpretation will be particularly useful in understanding the condition defining the Archimedean property, and thus, of the non-Archimedean property as well.

We recall that the usual addition gives a commutative group structure on the integer numbers \( \mathbb{Z} \), rational numbers \( \mathbb{Q} \), real numbers \( \mathbb{R} \), complex numbers \( \mathbb{C} \), as well as on the set \( \mathbb{M}^{m,n} \) of \( m \times n \) matrices, for every \( m, n \geq 1 \).

Now, a ring is a commutative group \((S, +)\) on which a second binary operation \( \beta : S \times S \rightarrow S \), called multiplication, is defined with the properties:

- β is associative
- β is distributive with respect to addition :

\[ \forall \ x, y, z \in S : \beta(x, y + z) = \beta(x, y) + \beta(x, z), \ \beta(x + y, z) = \beta(x, z) + \beta(y, z) \]
Usually, this second binary operation $\beta$ is called *multiplication*, and it is denoted by "," namely

$$\beta(x, y) = x \cdot y, \quad x, y \in S$$

and often, it is denoted even simpler as merely $xy = x \cdot y$, with $x, y \in S$.

The ring $(S, +, \cdot)$ is called *unital*, if and only if there is an element $u \in S$, such that

$$\forall \ x \in S : u \cdot x = x \cdot u = x$$

Usually, the element $u \in S$ is denoted by 1, namely

$$u = 1$$

The ring $(S, +, \cdot)$ is called *commutative*, if and only if

$$\forall \ x, y \in S : x \cdot y = y \cdot x$$

We recall that with the usual addition and multiplication, the integer numbers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ and complex complex numbers $\mathbb{C}$ are commutative unital rings, while the set $\mathbb{M}^{m, n}$ of $m \times n$ matrices, with $m, n \geq 2$, are *noncommutative* unital rings.

An important concept in rings is that of *zero divisor*. Namely, two elements $x, y \in S$ are called zero divisors, if and only if

- $x \neq 0, \ y \neq 0$
- $x \cdot y = 0$

Clearly, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are rings *without* zero divisors, while the set $\mathbb{M}^{m, n}$ of $m \times n$ matrices, with $m, n \geq 2$, has zero divisors, the latter fact being illustrated already in the case of $\mathbb{M}^{2, 2}$ by such a simple example as

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
Commutative rings without zero divisors are called \textit{integral domains}.

The consequence of the above is that in rings with zero divisors one \textit{cannot always} simplify factors in a product. Namely, for \(x, y \in S\), the relation
\[ x \cdot y = 0 \]
need \textit{not always} imply that
\[ x = 0 \quad \text{or} \quad y = 0 \]
just as happens with the above product of two matrices. This further means that, given \(x, y, z \in S\), the relation
\[ x \cdot y = x \cdot z \]
or for that matter, the relation
\[ y \cdot x = z \cdot x \]
need \textit{not always} imply that
\[ y = z \]
even if \(x \neq 0\).

As an effect, in rings with zero divisors \textit{not} every nonzero element has an inverse. Indeed, assuming the contrary, let \(x \cdot y = 0\), with \(x, y \in S\), \(x \neq 0\). Then there exists an inverse \(x' \in S\) for \(x\), which means that \(x \cdot x' = x \cdot x' = 1\). Hence \(x' \cdot (x \cdot y) = x' \cdot 0\), or due to the associativity of the product, we have \((x' \cdot x) \cdot y = 0\), which means \(y = 1 \cdot y = (x' \cdot x) \cdot y = 0\). Thus we obtained that \(x \cdot y = 0\) and \(x \neq 0\) imply \(y = 0\), which gives the contradiction that \(S\) cannot have zero divisors.

An algebraic structure of great importance is that of \textit{fields}. A ring \((F,+,\cdot)\) is a field, if and only if every nonzero element \(x \in F\) has an
inverse $x' \in F$, namely

\[ x \cdot x' = x'.x = 1 \]

It follows that a field cannot have zero divisors.

In this regard, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields, while $\mathbb{Z}$ and $M^{m,n}$, with $m,n \geq 2$, are not fields.

The ring $\mathbb{Z}$ is, as we have seen, an integral domain. But it is not a field, since none of its nonzero elements, except for 1 and $-1$, has an inverse.

On the other hand, as we have seen, the rings $M^{m,n}$, with $m,n \geq 2$, have zero divisors, thus they cannot be fields.

Lastly, a ring $(A, +, \cdot)$ is called an algebra over a given field $K$, if and only if there exists a third binary operation $\gamma : K \times A \rightarrow A$, called multiplication with a scalar, namely, for each scalar $a \in K$, and each algebra element $x \in A$, we have $\gamma(a,x) \in A$.

Usually, this binary operation $\gamma$ is also written as a multiplication ".", even if that may cause confusion. However, one should remember that in an algebra there are two multiplications, namely, one between two algebra elements $x,y \in A$, and which gives the algebra element $x \cdot y \in A$, and another multiplication between a scalar $a \in K$ and an algebra element $x \in A$, giving the algebra element $a \cdot x \in A$.

The properties of this second binary operation, namely, of multiplication with scalars, are as follows. For $a,b \in K$, $x,y \in A$, we have:

- $a \cdot (x + y) = (a \cdot x) + (a \cdot y)$
- $(a + b) \cdot x = (a \cdot x) + (b \cdot x)$
- $(a \cdot b) \cdot x = a \cdot (b \cdot x)$
- $1 \cdot x = x$

Given a field $K$, for instance, $K = \mathbb{R}$, or $K = \mathbb{C}$, a typical and important algebra over $K$ is the set $M^{m,n}_K$ of $m \times n$ matrices with elements which are scalars in $K$, where $m,n \geq 2$.

Here the difference between the two multiplications is obvious. The
first multiplication is that between two matrices in $A, B \in \mathbb{M}^{m,n}_K$. The second multiplication is that between a scalar $a \in K$ and a matrix $A \in \mathbb{M}^{m,n}_K$.

**The Archimedean Property**

The Archimedean property, as much as the property of being non-Archimedean, is essentially related to certain algebraic order structures. The simplest way to deal with the issue is to consider ordered groups. And in fact, we can restrict ourself to commutative groups.

Commutative groups were defined above, therefore, here we briefly recall the definition of partial orders.

A partial order $\leq$ on a nonvoid set $X$ is a binary relation $x \leq y$ between elements $x, y \in X$, which has the following three properties:

- $\leq$ is reflexive:
  $$\forall x \in X : x \leq x$$

- $\leq$ is antisymmetric:
  $$\forall x, y \in X : x \leq y, y \leq x \implies x = y$$

- $\leq$ is transitive:
  $$\forall x, y, z \in X : x \leq y, y \leq z \implies x \leq z$$

In case we have the additional property

$$\forall x, y \in X : \text{ either } x \leq y, \text{ or } y \leq x$$

then $\leq$ is called a linear order on $X$.

Given now a commutative group $(G, +)$, a partial order $\leq$ on $G$ is called compatible with the group structure, if and only if:
∀ x, y, z ∈ G: x ≤ y ⇒ x + z ≤ y + z

A partially ordered commutative group is by definition a commutative group \((G, +)\) together with a compatible partial order \(\leq\) on \(G\). In such a case, for simplicity, we shall use the notation

\((G, +, \leq)\)

In particular, we have a linearly ordered commutative group when the compatible partial order \(\leq\) is linear.

It is easy to see that in the general case of partially ordered commutative group \((G, +, \leq)\), the above condition of compatibility between the partial order \(\leq\) and the group structure can be simplified as follows:

\[ x, y \geq 0 \implies x + y \geq 0 \]

where \(0 \in G\) is the neutral element in \(G\).

We recall that \(\mathbb{Z}, \mathbb{Q}\) and \(\mathbb{R}\) are commutative groups. It is now easy to see that with the usual order relation \(\leq\), each of them is a linearly ordered commutative group.

Examples of partially ordered commutative groups which are not linearly ordered are easy to come by. Indeed, let us consider the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\), with \(n \geq 2\). With the usual addition of its vectors, this space is obviously a commutative group. We can now define on it the partial order relation \(\leq\) as follows. Given two vectors \(x = (x_1, x_2, x_3, \ldots, x_n)\), \(y = (y_1, y_2, y_3, \ldots, y_n) \in \mathbb{R}^n\), then we define \(x \leq y\) coordinate-wise, namely

\[ x \leq y \iff x_1 \leq y_1, \ x_2 \leq y_2, \ x_3 \leq y_3, \ldots, x_n \leq y_n \]

Then it is easy to see that this partial order is compatible with the commutative group on \(\mathbb{R}^n\), but it is not a linear order, when \(n \geq 2\). Indeed, this can be seen even in the simplest case of \(n = 2\), if we take \(x = (1, 0)\) and \(y = (0, 1)\), since then we do not have either \(x \leq y\), or \(y \leq x\).
In particular, $\mathbb{C}$, as well as and $\mathbb{M}_{m,n}^R$, $\mathbb{M}_{m,n}^C$, with $m \geq 2$ or $n \geq 2$, are partially and not linearly ordered commutative groups. Indeed, when it comes to their group structure, each of them can be seen as an Euclidean space. Namely $\mathbb{C}$ is isomorphic with $\mathbb{R}^2$, $\mathbb{M}_{m,n}^R$ is isomorphic with $\mathbb{R}^{mn}$, while $\mathbb{M}_{m,n}^C$ is isomorphic with $\mathbb{R}^{2mn}$.

Finally, we can turn to the issue of being, or for that matter, of not being Archimedean. A partially ordered commutative group $(G, +, \leq)$ is called Archimedean, if and only if:

$$\exists \ u \in G, \ u \geq 0 : \forall \ x \in G, \ x \geq 0 : \exists \ n \in \mathbb{N}, \ n \geq 1 : nu \geq x$$

Obviously, rings, algebras and fields each have, as far as their respective operations of addition are concerned, a commutative group structure as part of their definition. And when a partial order is defined to be compatible with the respective ring, algebra or field structure, it will among other conditions be required to be compatible with the mentioned commutative group structure of addition.

Consequently, the Archimedean condition on rings, algebras and fields can be defined exclusively in terms of the partially ordered commutative group structure of their respective operations of addition.

**Appendix 2**

**Letter to the Editor of The Mathematical Intelligencer**

In the Summer 2006 issue of The Mathematical Intelligencer there are two reviews of Roger Penrose’s book "The Road to Reality" published in 2005. These two reviews recall quite clearly the standard political ways of two party Anglo-Saxon democratic systems as understood and trivialized by journals such as Newsweek, or rather, they may simply recall the ”good cop - bad cop” approach to criminals. One can wonder whether The Mathematical Intelligencer did that by mistake, or
on the contrary, finds it a matter of pride to try to implant such approaches into science. And in the less than fortunate latter case, one can wonder why only two opposing views were presented when commenting upon scientific facts like, say, \(1 + 1 = 2\) ? Why not, indeed, three, or even more opposing views? After all, why should we not have some sort of circus in such rather arid realms like mathematics?

And now back to the mentioned two reviews. The first of them is shorter and quite sparse in detail when it finds the book under review highly meritorious and readable.

The second review recalls an old Jewish story which goes as follows. Somewhere in Medieval Europe, in a place with lingering anti-Semitism, an old Jew is brought to a court of law. Before sentencing, the judge allows the poor man to make a last statement, and this is what he has to say: Your Honour, I only wish I were judged by you as your predecessor did. Yes, when it came to sentencing, he said "He is a Jew, but he is innocent". And now, I am afraid, Your Honour may say "He is innocent, but he is a Jew" . . .

Well, the second review does find quite a number of outstanding features in Penrose's latest book, but all of that is totally and hopelessly drowned in a manifestly vicious overall prejudiced attitude and judgement.

One can only wonder how a third, or perhaps, fourth and so on, review might have looked, had The Mathematical Intelligencer decided to do one better than the trivial Newsweek approach, and present us more than merely the rock bottom two sharply opposing views . . .

And now, may I myself make a brief comment on Penrose's mentioned book, and start by noting that, as it happens, I myself have had some conflicting arguments with him on certain strictly mathematical issue, thus I cannot be counted as one of his unconditional admirers.

First, and above all, the subject of Penrose's latest book is by far the most fundamental and consequential within the sciences of the last few centuries of our modern times.

Second, the way science is pursued for more than half a century by now, scholarship, and even more so a wide ranging and deeply reaching one, is massively discouraged, in favour of narrowly specialized
research production. Penrose happens to be one of the very few scholars, if not in fact the only one nowadays, with a truly impressive depth and breadth in the subject. And in addition to having himself significant research contributions, he has clearly been one of the rather rare breed of distinguished "thinking scientists", and not merely one of the many many merely "working" ones. Consequently, even if his latest book were rather poor, which clearly it is very far from being, one should nevertheless have an extraordinary appreciation for his scholarship and willingness to make the considerable effort to bring that scholarship into the public domain. In this regard it important to point out that it shows a very poor understanding on the part of any reviewer or reader to see Penrose’s latest book as a science popularization one. Indeed, the kind of science it covers simply cannot be popularized too widely. And it is due not only to the mathematics which a more general readership may lack, but also, and in no less measure, to the extremely counterintuitive nature of much of modern physics.

And then, Penrose’s latest book is in fact like a Himalaya he built in the public domain, with a grand and most fascinating view of that fundamental and all important field of science. And that view may, hopefully, tempt many in the future young generations to try to climb that wonder, each according to his or her ability. As for the rest who care to look at it, in this case browse it or read parts of it, it may serve as one outstanding way to connect somewhat better to things beyond, and no less important, than day to day concerns or events. And the fact is that, when it comes to the quality of the book, the first reviewer’s appreciation is far more to the point, and it is only a pity that he did not elaborate on his respective arguments in more abundant detail.

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