A reciprocal transformation for the constant astigmatism equation

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Abstract. We construct reciprocal transformations for the constant astigmatism equation and generate some new solutions from the known seeds. One new surface of constant astigmatism is computed explicitly.

1. Introduction

Classical differential geometry of immersed surfaces is a rich source of systems integrable in the sense of soliton theory, see, e.g., [15, 16, 18] and references therein. A prototypical example is the sine-Gordon equation, which describes pseudospherical surfaces (i.e., surfaces of constant negative Gaussian curvature) under a properly chosen parameterization. Classical solution-generating techniques for this equation include the famous Bäcklund transformation as well as the Bianchi nonlinear superposition principle derived from the permutability of Bäcklund transformations, see [13] for the historical account. It is perhaps less known that historical roots of this transformation lie in another class of surfaces, characterized by the constancy of the difference $\sigma - \rho$ between the principal radii of curvature $\sigma, \rho$. A combination of Ribaucour’s results [14] and the Halphén theorem [7] then says that their focal surfaces are pseudospherical, see [3, §129]. Despite this close connection, the literature concerning surfaces whose principal radii of curvature have a constant difference is negligible compared to the vast literature on pseudospherical surfaces. Actually the former have been abandoned and remained largely forgotten for almost a century.

Recently these surfaces reemerged from the systematic search for integrable classes of Weingarten surfaces conducted by Baran and one of us [2]. Although nameless in the nineteenth century, in [2] they have been named the surfaces of constant astigmatism in connotation with the astigmatic interval [17] of the geometric optics. Under parameterization by the lines of curvature (also known as curvature coordinates), surfaces of constant astigmatism equal to 1 correspond to solutions of the constant astigmatism equation

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0. \quad (1)$$
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Other values of the constant astigmatism can be easily obtained by rescaling the Euclidean scalar product.

The only nineteenth-century works on constant astigmatism surfaces we know of are by Lipschitz [10], von Lilienthal [9], and Mannheim [11]. Lipschitz was apparently the first to find a large class of surfaces of constant astigmatism. Within the full class given in terms of elliptic integrals he also pointed out a subclass of surfaces of revolution, further investigated by von Lilienthal. It appears that the Lipshitz surfaces essentially exhaust all surfaces of constant astigmatism explicitly known in the nineteenth century. Of course, one can exploit the aforementioned geometric connection to generate surfaces of constant astigmatism from pseudospherical surfaces or solutions of the sine–Gordon equation, which are known in abundance, see [1, 6, 12] and references therein. However, this has never been done, according to our best knowledge, neither in terms of surfaces nor in terms of solutions of partial differential equations. Formulas for obtaining solutions of the constant astigmatism equation from solutions of the sine-Gordon equations and vice versa can be found in [2]. Yet the only explicit instance of such a relationship we know of is that of the von Lilienthal surfaces to the Beltrami pseudosphere.

In this paper we employ the aforementioned geometric connection with pseudospherical surfaces to establish an auto-transformation of the constant astigmatism equation (1). Our main result is a pair of reciprocal (see [8] transformations each of which generates a three-parametric family of surfaces of constant astigmatism from a single seed. Compared to Bäcklund transformations, reciprocal transformations are easier to apply since taking integrals is easier than solving linear systems of differential equations. On the other hand, attempts to iterate the procedure are seriously hampered by the fact that results are given in parametric form.

2. Point symmetries

A routine computation, see [2], reveals three continuous symmetries of equation (1): the $x$-translation

$$\mathcal{T}_a^x(x, y, z) = (x + a, y, z), \quad a \in \mathbb{R},$$

the $y$-translation

$$\mathcal{T}_b^y(x, y, z) = (x, y + b, z), \quad b \in \mathbb{R},$$

and the scaling

$$\mathcal{S}_c(x, y, z) = (x/c, cy, c^2z), \quad c \in \mathbb{R} \setminus \{0\}.$$

The known discrete symmetries are the $\mathcal{I}$

$$\mathcal{I}(x, y, z) = \left(y, x, \frac{1}{z}\right),$$
the $x$-reversal $\mathcal{R}_x(x, y, z) = (-x, y, z)$, and the $y$-reversal $\mathcal{R}_y(x, y, z) = (x, -y, z)$. Obviously,

$$\mathcal{I} \circ \mathcal{I} = \text{Id},$$

$$\mathcal{I} \circ \mathcal{T}_a^x = \mathcal{T}_a^y \circ \mathcal{I}, \quad \mathcal{I} \circ \mathcal{T}_a^y = \mathcal{T}_a^x \circ \mathcal{I},$$

$$\mathcal{G}_c \circ \mathcal{T}_a^x = \mathcal{T}_{a/c}^x \circ \mathcal{G}_c, \quad \mathcal{G}_c \circ \mathcal{T}_a^y = \mathcal{T}_{c/b}^y \circ \mathcal{G}_c,$$

$$\mathcal{G}_c \circ \mathcal{I} = \mathcal{I} \circ \mathcal{G}_{1/c},$$

$$\mathcal{R}^x \circ \mathcal{G}_{-1} = \mathcal{R}^y.$$

Translations and reversals are mere reparameterizations. The scaling symmetry corresponds to an *offsetting*, i.e., takes a surface to a parallel surface. The involution interchanges $x$ and $y$ (swaps the orientation) and makes a unit offsetting.

It is easy to compute (see, e.g., [5]) the four first-order conservation laws of equation (1). The associated four potentials $\chi, \eta, \xi, \theta$ satisfy

$$\chi_x = z_y + y, \quad \chi_y = \frac{z_x}{z} - x,$$

$$\eta_x = x z_y, \quad \eta_y = \frac{x z_x}{z^2} + \frac{1}{z} - x^2,$$

$$\xi_x = -y z_y + z - y^2, \quad \xi_y = -\frac{y z_x}{z^2},$$

$$\theta_x = x y z_y - x z + \frac{1}{2} x y^2, \quad \theta_y = x y \frac{z_x}{z^2} + \frac{y}{z} - \frac{1}{2} x y^2.$$ (2)

Compatibility of these equations is equivalent to (1).

The involution $\mathcal{I}$ acts on the potentials as follows: $\eta \leftrightarrow \xi$, while $\chi \rightarrow -\chi$ and $\theta \rightarrow -\theta$.

3. Obtaining the reciprocal transformation

Let $z(x, y)$ be a solution of the constant astigmatism equation (1). The corresponding surface of constant astigmatism is determined by its first and second fundamental form

$$\mathbf{I} = u^2 \, dx^2 + v^2 \, dy^2,$$

$$\mathbf{II} = \frac{u^2}{\rho} \, dx^2 + \frac{v^2}{\sigma} \, dy^2,$$

where $x, y$ are properly chosen curvature coordinates and, according to [2],

$$u = \frac{z^2 (\ln z - 2)}{2}, \quad v = \frac{\ln z}{2 z^2}, \quad \rho = \frac{\ln z - 2}{2}, \quad \sigma = \frac{\ln z}{2}.$$

Note that $\sigma$ and $\rho$ are the principal radii of curvature of the surface. Obviously, $\sigma - \rho = 1$.

Let us construct the evolute of the surface. By definition, the evolute has two sheets formed by the loci of the principal centers of curvature. Recall that the evolute of a surface of constant astigmatism is a pseudospherical surface.
Let \( r(x, y) \) be the surface of constant astigmatism corresponding to the solution \( z(x, y) \), let \( n(x, y) \) denote the unit normal vector. Then \( r, n \) satisfy the Gauss–Weingarten system, which takes the form

\[
\begin{align*}
    r_{xx} &= \frac{(\ln z)z_x}{2(\ln z - 2)} r_x - \frac{(\ln z - 2)z_y}{2 \ln z} r_y + \frac{1}{2} (\ln z - 2) n, \\
    r_{xy} &= \frac{(\ln z)z_y}{2(\ln z - 2)} r_x - \frac{(\ln z - 2)z_x}{2 \ln z} r_y, \\
    r_{yy} &= \frac{(\ln z)z_x}{2(\ln z - 2)z^3} r_x - \frac{(\ln z - 2)z_y}{2z \ln z} r_y + \frac{\ln z}{2z} n, \\
    n_x &= -\frac{2}{\ln z - 2} r_x, \\
    n_y &= -\frac{2}{\ln z} r_y.
\end{align*}
\]

Note that \( e_1 = r_x/u, e_2 = r_y/v \), and \( n = e_1 \times e_2 \) constitute an orthonormal frame.

We choose one of the two evolutes, given by

\[
\hat{r} r = r + \sigma n, \quad \hat{n} n = \frac{r_y}{v}.
\]

The first and second fundamental form of the evolute can be written as

\[
\begin{align*}
    \hat{I} &= 4z^3 + \frac{z_x^2}{4z^2} \, dx^2 + \frac{z_x z_y}{2z^2} \, dx \, dy + \frac{z_y^2}{4z^2} \, dy^2, \\
    \hat{II} &= \frac{1}{2} \frac{z^4 z_y}{2z^2} \, dx^2 - \frac{z_y}{2z^2} \, dy^2.
\end{align*}
\]

Next we construct the involute to this pseudospherical surface in order to obtain a new surface of constant astigmatism together with a new solution of the equation (1). Following [3, §136] or [19], we let \( X \) and \( Y \) be parabolic geodesic coordinates on the pseudospherical surface. By definition of the parabolic geodesic coordinates, the first fundamental form should be

\[
\hat{I} = dX^2 + e^{2X} \, dY^2.
\]

Comparing the coefficients, we obtain

\[
X_x^2 + e^{2X} Y_x^2 = z + \frac{z^2_x}{4z^2}, \quad 2X_y X_x + 2e^{2X} Y_y X_x = \frac{z_x z_y}{2z^2}, \quad X_y^2 + e^{2X} Y_y^2 = \frac{z_y^2}{4z^2}.
\]

Solving the last two equations for \( Y_x, Y_y \), we have

\[
Y_x = \frac{z_x z_y - 4z^2 X_x X_y}{2e^{2X} \sqrt{z_y^2 - 4z^2 X_y^2}}, \quad Y_y = \frac{\sqrt{z_y^2 - 4z^2 X_y^2}}{2e^{2X}}, \quad (3)
\]

which converts the remaining equation into

\[
X_x = \frac{z_x X_y + \sqrt{z_y^2 - 4z^2 X_y^2}}{z_y}. \quad (4)
\]
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The compatibility condition for the system (3) is

\[ X_{yy} = -X_y^2 + \frac{z z_{xx} - 2 z_x^2 - z^2 z_y^2 - 2 z^3}{z^3 z_y} X_y + \frac{z_y^2}{4z^3}. \quad (5) \]

Now the system consisting of equations (3), (4) and (5) is compatible by virtue of equation (1).

The general solution of the subsystem (4) and (5) is easily found to be

\[ X = \ln \left( \frac{(x + c_1)^2 z + 1}{z^2} \right) + c_2, \quad (6) \]

apart from the obvious “singular” solution

\[ X = \frac{1}{2} \ln z + c_2. \quad (7) \]

The other unknown \( Y \) is not needed in what follows.

The involute we look for is given by

\[ \tilde{\mathbf{r}} = \mathbf{r} + (a - X) \mathbf{r}_x \]

\[ = \mathbf{r} + \left( \ln z^2 + \frac{2z(a - X) Y_y}{z_y} \right) \mathbf{n} + 2(a - X) \frac{\sqrt{z_y^2 - 4z^2 X_y^2}}{z^3 z_y(2 - \ln z)} \mathbf{r}_x, \quad (8) \]

where \( a \) is an arbitrary constant. The unit normal vector to the involute is

\[ \tilde{\mathbf{n}} = \mathbf{r}_x = \frac{2z_X}{z_y} \mathbf{n} - \frac{2\sqrt{z_y^2 - 4z^2 X_y^2}}{z^3 z_y(\ln z - 2)} \mathbf{r}_x. \]

The singular solution (7) leads to \( \tilde{\mathbf{r}} = \mathbf{r} + (a - c_2) \mathbf{n} \), i.e., we recover the constant astigmatism surface we started with along with all its parallel surfaces. We are left with the solution (6), which can be interpreted as a result of the \( x \)-translation \( \mathfrak{T}_{c_1}^x \) applied to a particular solution

\[ X_0 = \ln \left( \frac{x^2 z + 1}{z^2} \right) + c_2. \]

Since \( \mathfrak{T}_{c_1}^x \) is a reparameterization, we continue with \( X_0 \) in the sequel.

If substituted into formula (8), \( X_0 \) yields the family of involutes

\[ \tilde{\mathbf{r}}_0 = \mathbf{r} + \left( \frac{x^2 z \ln z}{x^2 z + 1} + \frac{(x^2 z - 1)(\ln(x^2 z + 1) + a)}{x^2 z + 1} \right) \mathbf{n} \]

\[ + 2x \frac{2a - 2\ln(x^2 z + 1) + \ln z}{(x^2 z + 1)(2 - \ln z)} \mathbf{r}_x, \quad (9) \]

where \( c_2 \) has been absorbed into \( a \). The corresponding unit normal is

\[ \tilde{\mathbf{n}}_0 = \mathbf{r} + \frac{x^2 z - 1}{x^2 z + 1} \mathbf{n} + \frac{4x}{(x^2 z + 1)(2 - \ln z)} \mathbf{r}_x. \]
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A routine computation shows that \( \mathbf{r}_0 \) has a constant astigmatism, and so has \( \mathbf{r} \), which differs from \( \mathbf{r}_0 \) only by the reparameterization \( \mathbf{x}_c \). The parameter \( a \) corresponds to offsetting, i.e., taking a parallel surface.

However, one more step is required in order to find the corresponding solution of the equation (1). Namely, we have to find the curvature coordinates \( x' \) and \( y' \) on the involute, i.e., coordinates such that \( \mathbf{I}, \mathbf{II} \) are diagonal. If we set
\[
\begin{align*}
x'_x &= -e^{c_2-a}xz \\
x'_y &= -e^{c_2-a}z - x^2z^2 + xzz \\
y' &= c_3 - e^{a-c_2} \frac{xz}{1+x^2z},
\end{align*}
\]
(10)
it is easy to verify that equations (10) are compatible and both \( \mathbf{I}, \mathbf{II} \) are diagonal in terms of coordinates \( x' \) and \( y' \).

4. Properties of the reciprocal transformation and its dual

The computations made in the previous section lead us to the following proposition.

**Proposition 1.** Let \( z(x, y) \) be a solution of the constant astigmatism equation (1), \( \chi, \eta, \xi \) the corresponding potentials (2), and
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]
a real or purely imaginary matrix such that \( \det A = \pm 1 \). Let \( \mathcal{X}_A(x, y, z) = (x'_A, y'_A, z'_A) \) and \( \mathcal{Y}_A(x, y, z) = (x''_A, y''_A, z''_A) \), where
\[
\begin{align*}
x'_A &= \frac{(a_{11} + a_{12}x)(a_{21} + a_{22}y)z + a_{12}a_{22}}{(a_{11} + a_{12}x)^2z + a_{12}^2}, \\
y'_A &= a_{12}^2\eta + a_{11}a_{12}\chi - a_{11}y(a_{11} + a_{12}x), \\
z'_A &= \frac{((a_{11} + a_{12}x)^2z + a_{12}^2)^2}{z},
\end{align*}
\]
and
\[
\begin{align*}
x''_A &= a_{12}^2\xi - a_{11}a_{12}\chi - a_{11}x(a_{11} + a_{12}y), \\
y''_A &= \frac{(a_{11} + a_{12}y)(a_{21} + a_{22}y) + a_{12}a_{22}z}{(a_{11} + a_{12}y)^2 + a_{12}^2z}, \\
z''_A &= \frac{z}{((a_{11} + a_{12}y)^2 + a_{12}^2z)^2}
\end{align*}
\]
Then \( z'_A(x'_A, y'_A) \) and \( z''_A(x''_A, y''_A) \) are solutions of the constant astigmatism equation (1) as well.

**Proof.** The proof consists in a routine computation, which we omit. \( \square \)
Nevertheless, let us explain the origin of the formulas. With the shift $\tilde{\Sigma}_c^x$ reintroduced and parameters renamed, formulas for $x''$ and $y''$ are (10) in terms of the potentials introduced in Sect. 2; $z''$ is obtained as a solution of the equation $\tilde{\sigma} = \frac{1}{2}\ln z''$, where $\tilde{\sigma}$ is one of the principal radii of curvature of the involute $\tilde{r}$. Formulas for $x', y', z'$ follow from these with the help of the involution $\mathcal{I}$.

Observe that $y'_A$ and $x''_A$ are only determined up to an additive constant. It would be more appropriate to write $X_A(x, y, z) = T_{yb}(x', y', z')$ with arbitrary $b$ and analogously for $Y_A$.

The transformations become rather trivial when $a_{12} = 0$.

**Proposition 2.** In the case when $a_{12} = 0$ the transformations reduce to symmetries

$$
X_A = \begin{cases}
\tilde{\Sigma}_a^{x/a_{21}} \circ \mathcal{S}_{-a_{11}} & \text{if } \det A = -1, \\
\tilde{\Sigma}_a^{x/a_{21}} \circ \mathcal{R}^y \circ \mathcal{S}_{-a_{11}} & \text{if } \det A = +1,
\end{cases}
$$

$$
Y_A = \begin{cases}
\tilde{\Sigma}_a^{y/a_{21}} \circ \mathcal{S}_{-1/a_{11}} & \text{if } \det A = -1, \\
\tilde{\Sigma}_a^{y/a_{21}} \circ \mathcal{R}^x \circ \mathcal{S}_{-1/a_{11}} & \text{if } \det A = +1.
\end{cases}
$$

The translations themselves $\Sigma_a^x$ and $\Sigma_a^y$ correspond to the choice

$$
A_a^{\text{transl}} = \begin{pmatrix} i & 0 \\ a_{11} & i \end{pmatrix}.
$$

It follows immediately from the following proposition that transformations $X_A$ form a three-parameter group, and similarly for the transformations $Y_A$.

**Proposition 3.** We have

$$
X_B \circ X_A = X_{BA}, \quad Y_B \circ Y_A = Y_{BA}
$$

for any two real or purely imaginary $2 \times 2$ matrices $A, B$ such that $|\det A| = |\det B| = 1$.

Let us look for a convenient generator of the whole group. Consider the matrix

$$
A_a = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}, \quad a \in \mathbb{R}.
$$

Iterating $A_a^n, n \in \mathbb{N}$, one easily sees that it generates the full three-parameter group. The corresponding transformations are

$$
X_a(x, y, z) = \left( \frac{(ax + 1)xz + a}{x^2 z + 1}, \eta, \frac{(x^2 z + 1)^2}{z} \right),
$$

$$
Y_a(x, y, z) = \left( \xi, \frac{(by + 1)y + bz}{y^2 + z}, \frac{z}{(y^2 + z)^2} \right).
$$

Let $X$ and $Y$ correspond to the choice of $a = 0$, i.e.,

$$
A = A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$
Corollary 1. $\mathcal{X} \circ \mathcal{X} = \text{Id}$, $\mathcal{Y} \circ \mathcal{Y} = \text{Id}$.

Because of this property, $\mathcal{X}$ and $\mathcal{Y}$ are called reciprocal transformations. The explicit formulas are

$$x' = \frac{xz}{x^2z + 1}, \quad y' = \eta, \quad z' = \frac{(x^2z + 1)^2}{z},$$

$$x'' = \xi, \quad y'' = \frac{y}{y^2 + z}, \quad z'' = \frac{z}{(y^2 + z)^2}.$$

Remark 1. Obviously, the transformation $\mathcal{Y}$ admits a restriction to the variables $y, z$, and then becomes equivalent to a circle inversion in the $(y, z^{1/2})$-subspace. Similarly, $\mathcal{X}$ admits a restriction to the variables $x, z$ and becomes equivalent to a circle inversion in the $(x, z^{-1/2})$-subspace. This is not surprising considering the fact that $\mathcal{X}, \mathcal{Y}$ take their origin in a transformation of parabolic geodesic coordinates on a pseudospherical surface, which is a model of Lobachevsky geometry, and that the circle inversion provides a transformation of the Beltrami–Poincaré model of Lobachevsky geometry.

The following identities hold:

$$\mathcal{X} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{Y},$$

$$\mathcal{X} \circ \mathcal{S}_c = \mathcal{S}_{1/c} \circ \mathcal{X}, \quad \mathcal{Y} \circ \mathcal{S}_c = \mathcal{S}_{1/c} \circ \mathcal{Y}$$

Slightly abusing the notation, we have also

$$\mathcal{X} \circ \mathcal{T}_b^y = \mathcal{X} = \mathcal{T}_b^y \circ \mathcal{X},$$

$$\mathcal{Y} \circ \mathcal{T}_c^z = \mathcal{Y} = \mathcal{T}_c^z \circ \mathcal{Y}.$$  

There is no similar identity for $\mathcal{X} \circ \mathcal{T}_d^x$ and $\mathcal{Y} \circ \mathcal{T}_b^y$; instead they generate the three-parameter group considered above.

Nevertheless, we had to leave open the question of whether $\mathcal{X}_A$ and $\mathcal{Y}_B$ commute.

5. Examples

Example 1. Let us apply the transformations $\mathcal{X}$ and $\mathcal{Y}$ to the von Lilienthal solution

$$z = -y^2 + l,$$

where $l > 0$.

Then $\mathcal{X}(x, y, z) = (x', y', z')$, where

$$x' = \frac{x(-y^2 + l)}{x^2(-y^2 + l) + 1}, \quad z' = \frac{(x^2(-y^2 + l) + 1)^2}{-y^2 + l},$$

$$y' = \frac{1}{\sqrt{l}} \text{arctanh}\left(\frac{y}{\sqrt{l}}\right) - x^2y + c_1$$

and $\mathcal{Y}(x, y, z) = (x'', y'', z'')$, where

$$x'' = lx + c_2, \quad y'' = -\frac{y}{l}, \quad z'' = \frac{-y^2 + l}{l^2},$$
c_i being the integration constants. We see that \( z'' = -y'^2 + 1/l \) is another von Lilienthal solution. Nevertheless, \( z'(x', y') \) is a substantially new solution of the equation (1), which, however, is not possible to express explicitly using elementary functions. An implicit formula for this solution is

\[
y' = \frac{1}{\sqrt{l}} \arctan \left( \sqrt{\frac{z' - x'^2 z'^2 - 2x'^2 z' - 1}{l z'}} - \frac{x'^2 z'^2}{(x'^2 + 1)^2} \right) + c_1.
\]

The general von-Lilienthal solution \( z = -y^2 + ky + l \) is related to \( z = -y^2 + l \) by a \( y \)-translation. To obtain its \( \mathcal{X} \)-transformation one can employ the identity \( \mathcal{X} \circ \mathcal{O}_b^y = \mathcal{O}_b^y \circ \mathcal{X} \), while its \( \mathcal{O} \)-transformation is a von Lilienthal surface again.

**Example 2.** Continuing the previous example, where we put \( l = e^{2b} \) for simplicity, we provide a picture of the surface of constant astigmatism generated from the von Lilienthal seed. The von Lilienthal surfaces are obtained by revolving the involutes of tractrix around the asymptote of the latter, see [2] for pictures. We can write

\[
\begin{align*}
\mathbf{r}_1 &= \frac{1}{2} e^{-b} \sqrt{e^{2b} - y^2} [2 - \ln(e^{2b} - y^2)] \cos e^b x, \\
\mathbf{r}_2 &= \frac{1}{2} e^{-b} \sqrt{e^{2b} - y^2} [2 - \ln(e^{2b} - y^2)] \sin e^b x, \\
\mathbf{r}_3 &= \frac{1}{2} (e^{-b} y + 1) \ln(e^{2b} - y^2) - \ln(e^b - y) - e^{-b} y,
\end{align*}
\]

and

\[
\begin{align*}
\mathbf{n}_1 &= e^{-b} \sqrt{e^{2b} - y^2} \cos e^b x, \\
\mathbf{n}_2 &= e^{-b} \sqrt{e^{2b} - y^2} \sin e^b x, \\
\mathbf{n}_3 &= -e^{-b} y.
\end{align*}
\]

From (9) we obtain a formula for \( \mathbf{\tilde{r}} \). For brevity we present it with the offsetting parameter \( a \) set to zero:

\[
\begin{align*}
\mathbf{\tilde{r}}_1 &= \gamma(b, x, y) \left\{ 2e^b \sin(e^b x) - [x^2(e^{2b} - y^2) - 1] \cos(e^b x) \right\}, \\
\mathbf{\tilde{r}}_2 &= -\gamma(b, x, y) \left\{ 2e^b \cos(e^b x) + [x^2(e^{2b} - y^2) - 1] \sin(e^b x) \right\}, \\
\mathbf{\tilde{r}}_3 &= \frac{x^2(e^{2b} - y^2) - 1}{x^2(e^{2b} - y^2) + 1} y \ln[x^2(e^{2b} - y^2) + 1] \\
&\quad + \frac{e^b + y}{2e^b} \cdot \frac{x^2(e^{2b} - y^2) + 1}{x^2(e^{2b} - y^2) + 1} y \ln(e^{2b} - y^2) - \ln(e^b) - y \cdot e^b,
\end{align*}
\]

where

\[
\gamma(b, x, y) = \frac{\sqrt{e^{2b} - y^2}}{2e^b [x^2(e^{2b} - y^2) + 1]} \left\{ 2 \ln[x^2(e^{2b} - y^2) + 1] - \ln(e^{2b} - y^2) + 1 \right\}.
\]
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Obviously, $\tilde{r}$ is real only if $-e^b < y < e^b$. It is easy to check that $\tilde{r}$ develops a singularity (cuspidal edge) if either

$$x^2 = \delta(a, b, y) \quad \text{or} \quad x^2 = \delta(a + 1, b, y),$$

where

$$\delta(a, b, y) = \frac{-1 + e^a \sqrt{e^{2b} - y^2}}{e^{2b} - y^2}.$$ 

A regular part of the surface $\tilde{r}$ for $x > \sqrt{\delta(a + 1, b, y)}$ is shown on Figure 2.

![Figure 1. A transformed von Lilienthal surface.](image)

**Example 3.** Consider the scaling invariant solution

$$z = \frac{1}{2} y \left( -x y + \sqrt{x^2 y^2 - 4} \right).$$ (11)

Observe that the involution $\mathcal{J}$ acts on (11) by changing the sign of the square root. Apparently, no regular surface corresponds to this solution, since no regular surface can be invariant with respect to the offsetting. To apply the transformation $\mathcal{X}_A$, we first obtain the potentials

$$\chi = \sqrt{x^2 y^2 - 4} + \arctan \left( \frac{2}{\sqrt{x^2 y^2 - 4}} \right) + c_1,$$

$$\eta = \frac{1}{2} x \sqrt{x^2 y^2 - 4} - \frac{1}{2} x^2 y + c_2,$$
A reciprocal transformation for the constant astigmatism equation and then the solution

\[
x' = \frac{(a_{12} x + a_{11}) (a_{11} a_{22} + 1) (-xy + \sqrt{x^2 y^2 - 4}) y + 2a_{12} a_{22} x}{(a_{12} x + a_{11})(-xy + \sqrt{x^2 y^2 - 4}) y + 2a_{12}^2},
\]

\[
y' = -a_{11}^2 y + a_{11} a_{12} (-xy + \sqrt{x^2 y^2 - 4} + \arctan \left( \frac{2}{\sqrt{x^2 y^2 - 4}} \right) + c_1)
\]

\[
+ a_{12}^2 \left(-\frac{1}{2} x^2 y + \frac{1}{2} x \sqrt{x^2 y^2 - 4} + c_2 \right),
\]

\[
z' = \frac{(-xy + \sqrt{x^2 y^2 - 4}) (-a_{12}^2 x^2 y - 4a_{11} a_{12} xy - 2a_{11}^2 y + a_{12}^2 x \sqrt{x^2 y^2 - 4})^2}{8xy}.
\]

In particular, it turns out that the solution (11) is also invariant under \( \mathfrak{X} \) and \( \mathfrak{Y} \).

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