Intersecting families with covering number three

Peter Frankl\textsuperscript{1}, Jian Wang\textsuperscript{2}

\textsuperscript{1}Rényi Institute, Budapest, Hungary
\textsuperscript{2}Department of Mathematics
Taiyuan University of Technology
Taiyuan 030024, P. R. China

E-mail: \textsuperscript{1}frankl.peter@renyi.hu, \textsuperscript{2}wangjian01@tyut.edu.cn

Abstract

We consider k-graphs on n vertices, that is, \( F \subset \binom{[n]}{k} \). A k-graph \( F \) is called intersecting if \( F \cap F' \neq \emptyset \) for all \( F, F' \in F \). In the present paper we prove that for \( k \geq 7, n \geq 2k \), any intersecting k-graph \( F \) with covering number at least three, satisfies \( |F| \leq \binom{n-k}{k-1} - \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + \binom{n-k}{k-2} + \binom{n-k-2}{k-2} + 3 \), the best possible upper bound which was proved in \([4]\) subject to exponential constraints \( n > n_0(k) \).

1 Introduction

Let \([n] = \{1, \ldots, n\} \) be the standard n-element set, \( \binom{[n]}{k} \) the collection of its k-subsets, \( 0 \leq k \leq n \). A subset \( F \subset \binom{[n]}{k} \) is called a k-uniform family or simply k-graph. The family \( F \) is said to be intersecting if \( F \cap F' \neq \emptyset \) for all \( F, F' \in F \). Similarly two families \( F, G \) are called cross-intersecting if \( F \cap G \neq \emptyset \) for all \( F \in F, G \in G \). Set \( \cap \mathcal{F} = \bigcap_{F \in F} F \). If \( \cap \mathcal{F} = \emptyset \) then \( \mathcal{F} \) is called a star. Stars are the simplest examples of intersecting families. The quintessential Erdős-Ko-Rado Theorem shows that they are the largest as well.

**Theorem 1.1** \([2]\). Suppose that \( n \geq 2k > 0, F \subset \binom{[n]}{k} \) is intersecting, then

\[
|F| \leq \binom{n-k}{k-1}.
\]

In the case \( n = 2k \), \( \binom{n-k}{k-1} = \frac{1}{2} \binom{n}{k} \) and being intersecting is equivalent to \( |F \cap \{[n] \setminus F\}| \leq 1 \) for every \( F \in \binom{[n]}{k} \). Consequently there are \( 2^{n-k-1} \) intersecting families \( F \subset \binom{[2k]}{k} \) attaining equality in \([13]\). However for \( n > 2k \geq 4 \) there is a strong stability.

**Theorem 1.2** (Hilton-Milner Theorem \([13]\)). Suppose that \( n > 2k \geq 4, F \subset \binom{[n]}{k} \) is intersecting and \( F \) is not a star, then

\[
|F| \leq \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + 1.
\]

Let us define the Hilton-Milner Family

\[
\mathcal{H}(n, k) = \left\{ F \in \binom{[n]}{k} : 1 \in F, F \cap [2, k+1] \neq \emptyset \right\} \cup \{[2, k+1]\},
\]

showing that \((1.2)\) is best possible.

For an intersecting family \( F \subset \binom{[n]}{k} \), define the family of transversals (or covers), \( \mathcal{T}(F) \) by

\[
\mathcal{T}(F) = \{ T \subset [n] : T \cap F \neq \emptyset \text{ for all } F \in F \}.
\]
Define the covering number $\tau(\mathcal{F}) = \min\{|T| : T \in \mathcal{T}(\mathcal{F})\}$. Note that $\tau(\mathcal{F}) - 1$ is the minimum number of vertices whose deletion results in a star.

Obviously, $\tau(\mathcal{F}) = 1$ iff $\mathcal{F}$ is a star. The Hilton-Milner Family $\mathcal{H}(n,k)$ provides the maximum of $|\mathcal{F}|$ for intersecting $k$-graphs with $\tau(\mathcal{F}) \geq 2$.

**Example 1.3.** Define

$$\mathcal{B} = \{[2,k+1], [2] \cup [k+2,2k], [3] \cup [k+2,2k]\}$$

and

$$\mathcal{A} = \left\{ A \in \binom{[n]}{k} : 1 \in A \text{ and } A \cap B \neq \emptyset \text{ for each } B \in \mathcal{B} \right\}.$$  

Set $\mathcal{G}(n,k) = \mathcal{A} \cup \mathcal{B}$.

It is easy to check that for $n \geq 2k$, $\mathcal{G}(n,k)$ is an intersecting $k$-graph with $\tau(\mathcal{G}) = 3$. The 3-element transversals are $\{1,2,3\}$ and $\{1,u,v\}$ with $u \in [2,k+1]$, $v \in [k+2,2k]$. Consequently, for $k$ fixed and $n \to \infty$

$$|\mathcal{G}(n,k)| = (k^2 - k + 1)\binom{n-3}{k-3} + O\left(\binom{n-4}{k-4}\right).$$

One can compute $|\mathcal{G}(n,k)|$ exactly by using inclusion-exclusion

$$|\mathcal{G}(n,k)| = \binom{n-1}{k-1} - \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + \binom{n-2k}{k-1} + \binom{n-k-2}{k-3} + 3.$$

Note that if $n,k \to \infty$ with $k^2/n \to \infty$ then

$$|\mathcal{G}(n,k)| = (1 - o(1))\binom{n-1}{k-1}.$$  

Let us define the function

$$f(n,k,s) = \max\left\{ |\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k} \text{ is intersecting and } \tau(\mathcal{F}) \geq s \right\}.$$  

With this terminology the Erdős-Ko-Rado and Hilton-Milner Theorems can be stated as

$$f(n,k,1) = \binom{n-1}{k-1}, \quad f(n,k,2) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \quad \text{for } n \geq 2k.$$  

Erdős and Lovász [3] proved that $|k!(e-1)| \leq f(n,k,k) \leq k^k$ and $f(n,3,3) = 10$. Lovász [15] conjectured that $|k!(e-1)|$ is the exact bound. In [9], Lovász’s conjecture was disproved for $k \geq 4$ and the lower bound of $f(n,k,k)$ was improved to $\left(\frac{k}{2} + 1\right)^{k-1}$ for even $k$ and $\left(\frac{k+1}{2}\right)^{(k-1)/2} \left(\frac{k+1}{k+1}\right)^{(k-1)/2}$ for odd $k$.

The first author proved in [4] that for $k \geq 4$, $n > n_0(k)$

$$f(n,k,3) = |\mathcal{G}(n,k)|, \quad k \geq 4, \quad n > n_0(k)$$

and up to isomorphism $\mathcal{G}(n,k)$ is the only optimal family. The methods of that paper can be improved to yield $n_0(k) = ck^2$ for some absolute constant $c$.

We should mention that $f(n,k,4)$ was determined for $k \geq 9$ and $n > n_0(k)$ in [8] and the cases of $k = 4,5$ were solved by Chiba, Furuya, Matsubara, Takatou [1] and Furuya, Takatou [12], respectively.

In the present paper, we prove [15] for $k \geq 7$ and $n \geq 2k$.

**Theorem 1.4.** For $k \geq 7$ and $n \geq 2k$,

$$f(n,k,3) = |\mathcal{G}(n,k)|.$$
This is a considerable improvement on the results of Kupavskii [14] which prove the existence of a large constant $D$ such that $[1,k]$ holds for $k > D$ and $n > Dk$.

An important tool to tackle intersecting families is shifting that can be tracked back to Erdős-Ko-Rado [2]. We are going to give the formal definition in Section 2 but let us define here the “end product” of shifting. Let $(x_1, \ldots, x_k)$ denote the set $\{x_1, \ldots, x_k\}$ where we know or want to stress that $x_1 < \ldots < x_k$. Define the shifting partial order $\prec$ by setting $(a_1, \ldots, a_k) \prec (b_1, \ldots, b_k)$ iff $a_i \leq b_i$ for $1 \leq i \leq k$. Then $A \subset \binom{[n]}{k}$ is called initial iff for all $A, B \in \binom{[n]}{k}$, $A \prec B$ and $B \in A$ imply $A \in A$.

Let us mention that while the full star and the Hilton-Milner Family are initial, $G(n, k)$ is not. Define the function

$$g(n, k, s) = \max \left\{|F| : F \subset \binom{[n]}{k} \text{ is intersecting, initial and } \tau(F) \geq s \right\}.$$ 

The clique number $\omega(F)$ for $F \subset \binom{[n]}{k}$ is defined as

$$\omega(F) = \max \left\{q : \exists Q \in \binom{[n]}{q}, (Q) \subset F \right\}.$$ 

Clearly, $\tau(F) \geq \omega(F) - k + 1$. Let

$$K(n, k, s) = \left\{K \in \binom{[n]}{k} : 1 \in K, |K \cap [2, k + s - 1]| \geq s - 1 \right\} \cup \binom{[2, k + s - 1]}{k}.$$ 

It is easy to see that $K(n, k, s)$ is intersecting, initial and $\tau(F) = \omega(F) - k + 1 = s$. Thus $g(n, k, s) \geq |K(n, k, s)|$.

Let us show that $g(n, k, s)$ can be deduced from a result in [6] concerning intersecting families with clique number at least $q$.

**Theorem 1.5 ([6]).** Let $n > 2k$. Suppose that $F \subset \binom{[n]}{k}$ is intersecting and $\omega(F) \geq k + s - 1$. Then

$$|F| \leq |K(n, k, s)|.$$ 

Define

$$F(i) = \{F \setminus \{i\} : i \in F \in F\}, \quad F(i) = \{F : i \notin F \in F\}$$

and note that $|F| = |F(i)| + |F(i)|$. For $P \subset Q \subset [n]$, let

$$F(P, Q) = \{F \setminus Q : F \cap Q = P, F \in F\} \subset 2^{[n] \setminus Q}.$$ 

We also use $F(Q)$ to denote $F(\emptyset, Q)$. For $F(\{i\}, Q)$ we simply write $F(i, Q)$.

The following theorem is an easy consequence of Theorem 1.5.

**Theorem 1.6.** For $n > 2k$,

$$g(n, k, s) = |K(n, k, s)|.$$ 

**Proof.** Let $F \subset \binom{[n]}{k}$ be intersecting, initial and $\tau(F) \geq s$. Since $\tau(F) \geq s$, there exists an edge $F$ in $F([s-1])$. Then by initiality we see that $[s, k + s - 1] \subset F$. It follows that $\binom{[k+s-1]}{k} \subset F$. That is, $\omega(F) \geq k + s - 1$. Noting that $\tau(K(n, k, s)) = s$, the statement of the theorem follows from Theorem 1.5. 

Two cross-intersecting families $A, B$ are called non-trivial if none of them is a star, i.e., $\tau(A) \geq 2$ and $\tau(B) \geq 2$. Define $A_0 = \{U, V\}$ where $U$ and $V$ are two disjoint $a$-sets in $[n]$. Let

$$B_0 = \left\{B \in \binom{[n]}{b} : B \cap U \neq \emptyset, B \cap V \neq \emptyset \right\}.$$ 

Clearly, $A_0, B_0$ are non-trivial cross-intersecting.

Recently, the first author [7] proved the following result concerning non-trivial cross-intersecting families, which is essential for the proof of Theorem 1.4.
Theorem 1.7 ([7]). Let $2 \leq a \leq b$, $n \geq a + b$. Suppose that $A \subset \binom{[n]}{a}$ and $B \subset \binom{[n]}{b}$ are non-trivial and cross-intersecting. Then

$$
|A| + |B| \leq \binom{n}{b} - 2\binom{n-a}{b} + \binom{n-2a}{b} + 2. 
$$

Moreover, if $n > a + b$, then up to symmetry $A_0, B_0$ are the only families achieving equality in (1.8) unless $a = b = 2$.

It is easy to check that (1.8) also holds for $a = 1$ and $b \geq 2$. By using (1.8) and establishing two analytical inequalities, we prove the following two extensions of (1.8), which are also needed in the proof of Theorem 1.3.

Proposition 1.8. Let $A \subset \binom{[n]}{a}$ and $B \subset \binom{[n]}{b}$ be cross-intersecting families, $n \geq a + b$, $b > a \geq 1$. Suppose that $A$ is non-trivial and $B$ is non-empty. Then

$$
|A| + |B| \leq \binom{n}{b} - 2\binom{n-a}{b} + \binom{n-2a}{b} + 2. 
$$

Moreover, for $n > a + b$ and $a \geq 2$, $A_0, B_0$ are the only families achieving equality in (1.9).

Proposition 1.9. Let $A \subset \binom{[n]}{a}$ and $B \subset \binom{[n]}{b}$ be cross-intersecting families, $n \geq a + b$, $b \geq a + 2 \geq 3$. If $A$ is non-trivial, then

$$
|A| + |B| \leq \binom{n}{b} - 2\binom{n-a}{b} + \binom{n-2a}{b} + 2. 
$$

Moreover, for $n > a + b$ and $a \geq 2$, $A_0, B_0$ are the only families achieving equality in (1.10).

Let us present some more results that are needed in our proofs. We need the Frankl-Tokushige inequality [10, 7] as follows.

Theorem 1.10 ([10, 7]). Let $A \subset \binom{X}{a}$ and $B \subset \binom{X}{b}$ be non-empty cross-intersecting families with $n = |X| \geq a + b$, $a \leq b$. Then

$$
|A| + |B| \leq \binom{n}{b} - \binom{n-a}{b} + 1. 
$$

Moreover, unless $n = a + b$ or $a = b = 2$ the inequality is strict for $|A| > 1$ and $|B| > 1$.

For $\mathcal{F} \subset \binom{[n]}{k}$ and $A \subset D \subset [n]$, define $\alpha(A, D) = \frac{|\mathcal{F}(A,D)|}{\binom{n}{k} \cdot |A|}$. Note that $D \in \mathcal{T}(\mathcal{F})$ implies $\mathcal{F}(\emptyset, D) = \emptyset$.

The next statement can be deduced using an old argument of Sperner [16].

Lemma 1.11 ([7, 16]). Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family and let $D \in \mathcal{T}(\mathcal{F})$. If $A, B \subset D$, $A \cap B = \emptyset$ then

$$
\alpha(A, D) + \alpha(B, D) \leq 1. 
$$

We need also the following three inequalities concerning binomial coefficients.

Proposition 1.12 ([11]). Let $n, k, i$ be positive integers. Then

$$
\binom{n-i}{k} \geq \left(\frac{n-k-(i-1)}{n-(i-1)}\right)^i \binom{n}{k}. 
$$

Proposition 1.13. For $n \geq 4k$ and $\ell + i \leq 13$,

$$
\binom{n-i-\ell}{k-3} \geq \left(\frac{3}{4}\right)^\ell \binom{n-i}{k-3} 
$$
Thus, \[
\frac{n-k-\ell-i+4}{n-\ell-i+1} \geq \frac{n-k}{n}.
\]
Thus,
\[
\frac{(n-i-\ell)}{(n-k-\ell)} \geq \left( \frac{n-k-\ell-i+4}{n-\ell-i+1} \right)^{\ell} \geq \left( \frac{n-k}{n} \right)^{\ell} \geq \left( \frac{3}{4} \right)^{\ell}.
\]

Proposition 1.14. Let \( n, k, p, i \) be positive integers with \( i \geq 3 \). Then
\[
\frac{(n-i)}{(k-2)} - \frac{(n-p-i)}{(k-2)} \geq \left( \frac{n-k-i+3}{n-i+1} \right)^{i-2} \left( \frac{n-2}{k-2} - \frac{(n-p-2)}{(k-2)} \right).
\]

Proof. Note that
\[
\frac{(n-k)}{(n-2)} \frac{(n-k-i)}{(n-i+1)} \frac{(n-k-i+3)}{(n-i+1)} \frac{(n-2)}{(k-2)} \frac{(n-k-p-2)}{(k-2)} \geq \left( \frac{n-k-i+3}{n-i+1} \right)^{i-2} \left( \frac{n-2}{k-2} - \frac{(n-p-2)}{(k-2)} \right).
\]

2 Shifting ad extremis and 2-cover graphs

In this section, we define a new technique called shifting ad extremis that is essential for the present paper.

Recall the shifting operation as follows. Let \( 1 \leq i < j \leq n, \mathcal{F} \subset \binom{[n]}{k} \). Define
\[
S_{ij}(\mathcal{F}) = \{ S_{ij}(F) : F \in \mathcal{F} \},
\]
where
\[
S_{ij}(F) = \begin{cases} 
(F \setminus \{j\}) \cup \{i\}, & j \in F, i \notin F \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}; \\
F, & \text{otherwise}.
\end{cases}
\]

Let \( \mathcal{F} \subset \binom{[n]}{k} \), \( \mathcal{G} \subset \binom{[n]}{r} \) be families having certain properties (e.g., intersecting, cross-t-intersecting) that are maintained by simultaneous shifting and certain properties (e.g., \( \tau(\mathcal{F}) \geq s \)) that might be destroyed by shifting. Let \( \mathcal{P} \) be the collection of the latter properties that we want to maintain.

Define the quantity
\[
w(\mathcal{F}) = \sum_{F \in \mathcal{F}} \sum_{i \in F} i.
\]
Obviously \( w(S_{ij}(\mathcal{F})) \leq w(\mathcal{F}) \) for \( 1 \leq i < j \leq n \) with strict inequality unless \( S_{ij}(\mathcal{F}) = \mathcal{F} \).

Definition 2.1. Suppose that \( \mathcal{F} \subset \binom{[n]}{k} \), \( \mathcal{G} \subset \binom{[n]}{r} \) are families having property \( \mathcal{P} \). We say that \( \mathcal{F} \) and \( \mathcal{G} \) have been shifted ad extremis with respect to \( \mathcal{P} \) if \( S_{ij}(\mathcal{F}) = \mathcal{F} \) and \( S_{ij}(\mathcal{G}) = \mathcal{G} \) for every pair \( 1 \leq i < j \leq n \) whenever \( S_{ij}(\mathcal{F}) \) and \( S_{ij}(\mathcal{G}) \) also have property \( \mathcal{P} \).

Let \( \mathcal{F} \subset \binom{[n]}{k} \) be an intersecting family and \( \mathcal{P} = \{ \tau(\mathcal{F}) \geq 3 \} \). Then \( \mathcal{F} \) is shifted ad extremis if \( S_{ij}(\mathcal{F}) = \mathcal{F} \) for all \( 1 \leq i < j \leq n \) unless \( \tau(S_{ij}(\mathcal{F})) = 2 \). A pair \((i, j)\), \( 1 \leq i < j \leq n \), is called a shift-resistant pair if \( \tau(S_{ij}(\mathcal{F})) = 2 \). Define the shift-resistant graph \( \mathcal{H} = H_\mathcal{P}(\mathcal{F}) \) of shift-resistant pairs \((i, j)\) such that
\[
\tau(S_{ij}(\mathcal{F})) = 2 \text{ for } (i, j) \in \mathcal{H} \text{ and } S_{ij}(\mathcal{F}) = \mathcal{F} \text{ for } (i, j) \notin \mathcal{H}.
\]
Note that $\mathcal{F}$ is initial if and only if $\mathcal{H}$ is empty.

We can obtain shifted ad extremis families by the following shifting ad extremis process. Let $\mathcal{F}$, $\mathcal{G}$ be cross-intersecting families with property $\mathcal{P}$. Apply the shifting operation $S_{ij}$, $1 \leq i < j \leq n$, to $\mathcal{F}, \mathcal{G}$ simultaneously and continue as long as the property $\mathcal{P}$ is maintained. Recall that the shifting operation preserves the cross-intersecting property (cf. [5]). By abuse of notation, we keep denoting the current families by $\mathcal{F}$ and $\mathcal{G}$ during the shifting process. If $S_{ij}(\mathcal{F})$ or $S_{ij}(\mathcal{G})$ does not have property $\mathcal{P}$, then we do not apply $S_{ij}$ and choose a different pair $(i', j')$. However we keep returning to previously failed pairs $(i, j)$, because it might happen that at a later stage in the process $S_{ij}$ does not destroy property $\mathcal{P}$ any longer. Note that the quantity $w(\mathcal{F}) + w(\mathcal{G})$ is a positive integer and it decreases strictly in each step. Eventually we shall arrive at families that are shifted ad extremis with respect to $\mathcal{P}$.

We say that cross-intersecting families $\mathcal{A}, \mathcal{B}$ are saturated or form a saturated pair if adding an extra $k$-set to either of the families would destroy the cross-intersecting property.

**Lemma 2.2.** Let $\mathcal{F} \subset \binom{[n]}{\ell}$ be an intersecting family of the maximal size with $\tau(\mathcal{F}) \geq 3$. Suppose that $\mathcal{F}$ is shifted ad extremis for $\tau(\mathcal{F}) \geq 3$ with the shift-resistant graph $\mathcal{H}$, and $\mathcal{T}^{(3)}(\mathcal{F})$ is a star with center $a$. Let $\mathcal{A} = \mathcal{F}(a)$, $\mathcal{B} = \mathcal{F}(\bar{a})$. Then $\mathcal{A}, \mathcal{B}$ are saturated cross-intersecting families and $\mathcal{B}$ is intersecting. Moreover, $\mathcal{A}, \mathcal{B}$ are shifted ad extremis for $\tau(\mathcal{B}) \geq 2$ with the shift-resistant graph $\mathcal{H} \cap \binom{[n]}{\ell+1}$.

**Proof.** Clearly $\mathcal{A}, \mathcal{B}$ are cross-intersecting and $\mathcal{B}$ is intersecting. Since $\mathcal{F}$ is of maximum size, $\mathcal{A}, \mathcal{B}$ form a saturated pair. Moreover, $\tau(\mathcal{F}) \geq 3$ implies the non-triviality of $\mathcal{B}$, i.e., $\tau(\mathcal{B}) \geq 2$.

Fix an arbitrary $(i, j) \in \binom{[n]}{\ell+1}$. If $(i, j) \notin \mathcal{H}$, then $S_{ij}(\mathcal{F}) = \mathcal{F}$ and thereby $S_{ij}(\mathcal{A}) = \mathcal{A}$ and $S_{ij}(\mathcal{B}) = \mathcal{B}$. If $(i, j) \in \mathcal{H}$, then $\tau(S_{ij}(\mathcal{F})) = 2$. Since every $T \in \mathcal{T}^{(3)}(\mathcal{F})$ contains $a$, it follows that \{a, i\} is a transversal of $S_{ij}(\mathcal{F})$. Then $S_{ij}(\mathcal{B})$ is a star. Hence, the shift-resistant graph for the shifted ad extremis cross-intersecting families $\mathcal{A}, \mathcal{B}$ is $\mathcal{H} \cap \binom{[n]}{\ell+1}$. \hfill $\blacksquare$

Let $\mathcal{A} \subset \binom{[n]}{\ell}, \mathcal{B} \subset \binom{[n]}{k-\ell}$ be saturated cross-intersecting families with $n > \ell + k$ and let $\mathcal{B}$ be non-trivial. Suppose that $\mathcal{A}, \mathcal{B}$ are shifted ad extremis with respect to $\tau(\mathcal{B}) \geq 2$. That is, for each $(i, j)$, $1 \leq i < j \leq n$, either (a) or (b) occurs.

(a) $S_{ij}(\mathcal{A}) = \mathcal{A}, S_{ij}(\mathcal{B}) = \mathcal{B}$;

(b) $S_{ij}(\mathcal{B})$ is a star.

Let us define the 2-cover graph $\hat{\mathcal{H}}$ for $\mathcal{B}$ on the vertex set $[n]$ where for $1 \leq i < j \leq n$

$$(i, j) \in \hat{\mathcal{H}} \text{ iff } \mathcal{B}(i, j) = \emptyset.$$ 

If for $D \subset [n]$, $\mathcal{F}(D) = \binom{\binom{[n]}{\ell}}{k-|D|}$ then we say that $D$ is full in $\mathcal{F}$ or $\mathcal{F}(D)$ is full. By saturatedness, we see that $\hat{\mathcal{H}}$ is the graph formed by all full pairs in $\mathcal{B}$. Note that if $(i, j)$ is of type (b) then $(i, j) \in \hat{\mathcal{H}}$ but not necessarily vice versa. Let $\mathcal{H}$ be the corresponding shift-resistant graph. It is easy to see that $\mathcal{H}$ is a subgraph of $\hat{\mathcal{H}}$.

For convenience, we often use $\{i, j\} \in \mathcal{H}$ to denote that $(i, j) \in \hat{\mathcal{H}}$ for $i < j$ and $(j, i) \in \mathcal{H}$ for $i > j$. It is easy to check that $\mathcal{G}(n, k)$ is shifted ad extremis for $\tau(\mathcal{G}(n, k)) \geq 3$ with the shift-resistant graph

$$\mathcal{H}_0 = \{(i, j) : i \in [2, k + 1], j \in \{1, k + 2, k + 3, \ldots, 2k\}\}.$$

Let $\mathcal{A} = \mathcal{G}(n, k)(1)$ and $\mathcal{B} = \mathcal{G}(n, k)(\bar{1})$. Clearly, $\mathcal{A}, \mathcal{B}$ are shifted ad extremis for $\tau(\mathcal{B}) \geq 2$ and

$$\hat{\mathcal{H}}_0 = \{(i, j) : i \in [2, k + 1], j \in \{k + 2, k + 3, \ldots, 2k\}\} \cup \{(2, 3)\}.$$

**Definition 2.3.** We say that $\hat{\mathcal{H}}$ is partially shifted if $(i, j) \in \hat{\mathcal{H}}$ and $(x, j) \notin \hat{\mathcal{H}}$ imply $S_{xj}(i, j) \in \hat{\mathcal{H}}$. I.e., if $i < x$ then $(i, x) \in \hat{\mathcal{H}}$; if $i > x$ then $(x, i) \in \hat{\mathcal{H}}$.

The reason to consider $\hat{\mathcal{H}}$ instead of $\mathcal{H}$ is that $\hat{\mathcal{H}}$ is partially shifted.
Proposition 2.4. Suppose that \( A \subset \binom{[n]}{\ell} \), \( B \subset \binom{[n]}{k} \) are cross-intersecting that were shifted ad extremis with respect to \( \tau(B) \geq 2 \), form a saturated pair, but not both initial. Then \( \tilde{H} \) is partially shifted.

Proof. Suppose for contradiction that \((x, y) \in \tilde{H}, (z, y) \notin \tilde{H} \) but \( S_{2y}(x, y) \notin \tilde{H} \). By the definition, we have \( B(x, y) = \emptyset \). Since \((A, B)\) forms a saturated pair, it follows that \( A(x, y) \) is full, that is, \( A(x, y) = \binom{[n]}{\ell-2} \). Since \((z, y) \notin \tilde{H} \), by the definition \( S_{2y}(A) = A \), it follows that \( A(x, z) \) is also full. Then by \( \{x, z\} \notin \tilde{H} \) we infer \( B(\tilde{x}, \tilde{z}) \neq \emptyset \), contradicting the cross-intersecting property. Thus the proposition is proven. \( \blacksquare \)

Lemma 2.5. If \( \tilde{H} \) is partially shifted and triangle-free, then \( \tilde{H} \) is a complete bipartite graph on partite sets \( X \) and \( Y \), \( X \cup Y = |X| + |Y| \).

Proof. Let us pick \((i, j) \in \tilde{H} \) with \( i + j \) minimal.

Claim 2.6. \( i = 1 \).

Proof. Indeed, if \( i \geq 2 \) then both \((1, i), (1, j) \notin \tilde{H} \). By partial-shiftedness, \( S_{1i}(1, j) = (1, j) \in \tilde{H} \) and \( S_{ij}(1, j) = (1, i) \in \tilde{H} \), a contradiction. \( \blacksquare \)

Let \( Y = \{y_1, \ldots, y_p\} \) \((y_1 < y_2 < \cdots < y_p)\) be the neighbors of \( 1 \) in \( \tilde{H} \), \( y_1 = j \) and let \( X = \{x_1, \ldots, x_q\} \) be the neighbors of \( j = y_1, x_1 = 1 \).

Claim 2.7. \( \{x_u, y_v\} \in \tilde{H} \) for all \( 1 \leq u \leq q, 1 \leq v \leq p \).

Proof. By symmetry assume \( x_u < y_v \). If \( x_u = 1 \), the statement holds by the definition. Assume \( 1 < x_u \). If \((x_u, y_v) \notin \tilde{H} \), then by partial-shiftedness \( S_{x_u y_v}(1, y_v) = (1, x_u) \in \tilde{H} \), implying that \((1, x_u, y_v)\) is a triangle of \( \tilde{H} \), contradiction. \( \blacksquare \)

Since \( \tilde{H} \) is triangle-free, \( X \cap Y = \emptyset \) and we proved that the graph \( \tilde{H} \) is complete on \( X \times Y \). Let us show that \( X \cup Y = [p + q] \). Indeed, otherwise we can find \( z, w \) with \( 1 < z < w \) and \( z \notin X \cup Y, w \in X \cup Y \). Suppose by symmetry \( w \in Y \). Then \((1, z) \notin \tilde{H}, (y_1, z) \notin \tilde{H} \). Note also that \( \{y_1, w\} \notin \tilde{H} \). If \( y_1 > w \) then \((1, y_1) \in \tilde{H} \) implies \( S_{wy}(1, y_1) = (1, w) \in \tilde{H} \), a contradiction. Thus we may assume \( y_1 < w \). Should \((z, w) \in \tilde{H} \) hold we infer \( S_{zy}(1, w) = (y_1, z) \in \tilde{H} \), a contradiction. Thus \((z, w) \notin \tilde{H} \) and thereby \( S_{zw}(1, w) = (1, z) \in \tilde{H} \), a contradiction.

Let us show next that there are no edges \((u, v) \in \tilde{H} \) outside \( X \times Y \). Since any pair \((u, v) \subset Z \) for \( Z = X \) or \( Y \) would create a triangle, we may assume \( v > p + q \). Now \( 1 < y_1 \leq p + q < v \) and the partial-shiftedness imply that \( S_{1v}(u, v) = (1, u) \in \tilde{H} \) and \( S_{y_1 v}(u, v) = \{u, y_1\} \in \tilde{H} \). It follows that \((1, y_1, u)\) forms a triangle, contradiction. Thus \( \tilde{H} \) is a complete bipartite graph on partite sets \( X \) and \( Y \), \( X \cup Y = [p + q] \). \( \blacksquare \)

Proposition 2.8. Suppose that \( A \subset \binom{[n]}{\ell} \), \( B \subset \binom{[n]}{k} \) are cross-intersecting that were shifted ad extremis with respect to \( \tau(B) \geq 2 \), form a saturated pair, but not both initial. If \( \tilde{H} \) is triangle-free, then \( \tilde{H} \) is a complete bipartite graph on partite sets \( X \) and \( Y \), \( X \cup Y = |X| + |Y| \), \( 2 \leq |X| \leq k \), \( 2 \leq |Y| \leq k \).

Proof. By Lemma 2.5 and Proposition 2.4, we are left to show that \( 2 \leq |X| \leq k \), \( 2 \leq |Y| \leq k \).

Let us prove \( \min\{|X|, |Y|\} \geq 2 \) first. The opposite would mean that \( \tilde{H} \) is a star. Thus it is sufficient to prove that \( \tilde{H} \) contains two independent edges. To this end fix an edge \((i, j)\) of type (b). Here is the place that we use that \( A, B \) are not both initial.

Since \( S_{ij}(B) \) is a star but \( B \) is not, we may choose \( K, L \in B \) with \( K \cap \{i, j\} = \{i\}, L \cap \{i, j\} = \{j\} \). Fix such a pair \( K, L \) with \( |K \cap L| \) maximal. Note that \( B(i) \cap B(j) = \emptyset \) implies \( |K \cap L| \leq k - 2 \). Pick \( x \neq i, x \in K \setminus L \), \( y \neq j, y \in L \setminus K \). We claim that \((x, y)\) is not of type (a). Indeed, if \( x < y \) and \((x, y)\) is of type (a), then \( S_{xy}(B) = B \) implies \( L \setminus \{y\} \cup \{x\} =: L' \in B \). But \( |K \cap L'| = |K \cap L| + 1 \), a contradiction.
Similarly, \( y < x \) and \( S_{xy}(B) = B \) would imply \( (K \setminus \{x\}) \cup \{y\} = K' \in B \). Again, \( |K' \cap L| = |K \cap L| + 1 \) provides the contradiction. Consequently, \( \{x, y\} \) is of type (b), whence an edge of \( \tilde{H} \). Thus \( \tilde{H} \) contains two independent edges and therefore \( \min\{|X|, |Y|\} \geq 2 \).

Next we show that \( \max\{|X|, |Y|\} \leq k \).

**Claim 2.9.** For every \( B \in B \) either \( X \subset B \) or \( Y \subset B \).

**Proof.** In the opposite case we may choose \( x \in X \setminus B, y \in Y \setminus B \). Since \( \{x, y\} \in \tilde{H}, A(x, y) \) is full. Now \( B \cap \{x, y\} = \emptyset \) contradicts the cross-intersecting property. \( \square \)

Using the non-triviality of \( B \), there are members \( B, B' \in B \) with \( X \subset B, Y \subset B' \). In particular, \( |X| \leq k \) and \( |Y| \leq k \). This concludes the proof of the proposition. \( \square \)

### 3 The case that \( \mathcal{T}^{(3)}(\mathcal{F}) \) is non-trivial

In this section, we prove an important special case of our main result.

**Theorem 3.1.** Let \( \mathcal{F} \subset \binom{\mathbb{N}}{k} \) be an intersecting family with \( n > 2k \), \( \tau(\mathcal{F}) \geq 3 \) and \( |\mathcal{F}| \) maximal. Suppose that \( \mathcal{F} \) is shifted and extremal with a non-empty shift-resistant graph \( H \), and \( \mathcal{T}^{(3)}(\mathcal{F}) \) is non-trivial. Then \( |\mathcal{F}| < |\mathcal{G}(n, k)| \) for \( k \geq 7 \).

For the proof of Theorem 3.1, we need the following computational lower bounds for \( |\mathcal{G}(n, k)| \).

**Lemma 3.2.** For \( n \geq 4k \) and \( k \geq 7 \),

\[
\begin{align*}
(3.1) & \quad |\mathcal{G}(n, k)| > 3 \left( \binom{n-6}{k-2} - \binom{n-k-6}{k-2} \right) + \binom{n-3}{k-3} + 3 \binom{n-4}{k-3} + 6 \binom{n-5}{k-3}, \\
(3.2) & \quad |\mathcal{G}(n, k)| > 4 \left( \binom{n-5}{k-2} - \binom{n-k-5}{k-2} \right) + \binom{n-3}{k-3} + 3 \binom{n-4}{k-3} + 4 \binom{n-5}{k-3}.
\end{align*}
\]

**Proof.** For \( k \geq 7 \), we have

\[
|\mathcal{G}(n, k)| > \binom{n-1}{k-1} - \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + \binom{n-2k}{k-1} = \sum_{i=2}^{k} \left( \binom{n-i}{k-2} - \binom{n-k-i}{k-2} \right)
\geq \sum_{i=2}^{7} \left( \binom{n-i}{k-2} - \binom{n-k-i}{k-2} \right).
\]

First we prove (3.1). The RHS of (3.1) is equal to

\[
\sum_{2 \leq i \leq 4} \left( \binom{n-i}{k-2} - \binom{n-k-i}{k-2} \right) + \binom{n-4}{k-3} + 3 \binom{n-5}{k-3} - 3 \binom{n-6}{k-3}
+ \binom{n-k-3}{k-3} + 3 \binom{n-k-4}{k-3} + 3 \binom{n-k-5}{k-3} + 3 \binom{n-k-6}{k-3}.
\]

By \( k \geq 7 \), it is at most

\[
\sum_{2 \leq i \leq 4} \left( \binom{n-i}{k-2} - \binom{n-k-i}{k-2} \right) + \binom{n-4}{k-3} + 3 \binom{n-5}{k-3} - 3 \binom{n-6}{k-3} + \binom{n-10}{k-3}
+ 2 \binom{n-11}{k-3} + 3 \binom{n-12}{k-3} + 3 \binom{n-13}{k-3}
\leq \sum_{2 \leq i \leq 4} \left( \binom{n-i}{k-2} - \binom{n-k-i}{k-2} \right) + \binom{n-4}{k-3} + 3 \binom{n-5}{k-3} + 3 \binom{n-12}{k-3} + 3 \binom{n-13}{k-3}.
\]
By (3.3), it suffices to show that
\[
\binom{n - 4}{k - 3} + 3 \binom{n - 5}{k - 3} + 3 \binom{n - 12}{k - 3} + 3 \binom{n - 13}{k - 3} \leq \sum_{5 \leq i \leq 7} \left( \binom{n - i}{k - 2} - \binom{n - k - i}{k - 2} \right).
\]

For \( n \geq 4k \), by (1.14) we have
\[
\binom{n - 6}{k - 3} + 3 \binom{n - 9}{k - 3} \geq \left( \frac{3}{4} \right)^2 \left( \frac{3}{4} \right)^5 \binom{n - 4}{k - 3} > \binom{n - 4}{k - 3}
\]
and
\[
2 \binom{n - 7}{k - 3} + 3 \binom{n - 8}{k - 3} + 3 \binom{n - 10}{k - 3} \geq \left( \frac{3}{4} \right)^2 \left( \frac{3}{4} \right)^3 \left( \frac{3}{4} \right)^5 \binom{n - 5}{k - 3} > 3 \binom{n - 5}{k - 3}.
\]
Adding the above two inequalities, we get
\[
\binom{n - 4}{k - 3} + 3 \binom{n - 5}{k - 3} < \binom{n - 6}{k - 3} + 2 \binom{n - 7}{k - 3} + 3 \binom{n - 8}{k - 3} + 3 \binom{n - 9}{k - 3} + 3 \binom{n - 10}{k - 3}.
\]
Therefore,
\[
\binom{n - 4}{k - 3} + 3 \binom{n - 5}{k - 3} + 3 \binom{n - 12}{k - 3} + 3 \binom{n - 13}{k - 3} \leq \binom{n - 6}{k - 3} + 2 \binom{n - 7}{k - 3} + 3 \binom{n - 8}{k - 3} + 3 \binom{n - 9}{k - 3} + 3 \binom{n - 10}{k - 3} + 3 \binom{n - 11}{k - 3} + 3 \binom{n - 12}{k - 3} + 3 \binom{n - 13}{k - 3}
\]
\[
\leq \sum_{5 \leq i \leq 7} \sum_{1 \leq j \leq 7} \binom{n - i - j}{k - 3} \leq \sum_{5 \leq i \leq 7} \left( \binom{n - i}{k - 2} - \binom{n - k - i}{k - 2} \right)
\]
and (3.1) follows.
For (3.2), by \( k \geq 7 \) the RHS of (5.2) is at most
\[
\sum_{2 \leq i \leq 5} \left( \binom{n - i}{k - 2} - \binom{n - k - i}{k - 2} \right) + \binom{n - 4}{k - 3} + \binom{n - 5}{k - 3} + \binom{n - 12}{k - 3} + 2 \binom{n - 13}{k - 3}
\]
\[
+ 3 \binom{n - 5}{k - 3}
\]
\[
\leq \sum_{2 \leq i \leq 5} \left( \binom{n - i}{k - 2} - \binom{n - k - i}{k - 2} \right) + \binom{n - 4}{k - 3} + \binom{n - 5}{k - 3} + \binom{n - 10}{k - 3} + 2 \binom{n - 11}{k - 3} + 3 \binom{n - 12}{k - 3}.
\]
Then it suffices to prove that
\[
\binom{n - 4}{k - 3} + \binom{n - 5}{k - 3} + \binom{n - 10}{k - 3} + 2 \binom{n - 11}{k - 3} + 3 \binom{n - 12}{k - 3} \leq \sum_{6 \leq i \leq 7} \left( \binom{n - i}{k - 2} - \binom{n - k - i}{k - 2} \right).
\]
Adding the above three inequalities, we get
\[ \frac{4}{7} \left( \frac{n-7}{k-3} \right) + \frac{n-8}{k-3} + \frac{n-9}{k-3} + \frac{n-10}{k-3} + \frac{4}{9} \left( \frac{n-13}{k-3} \right) \geq \left( \frac{4}{7} \times \frac{3}{4} \right)^3 + \left( \frac{3}{4} \right)^4 + \left( \frac{3}{4} \right)^5 + \left( \frac{4}{9} \times \frac{3}{4} \right)^9 \left( \frac{n-4}{k-3} \right) > \left( \frac{n-4}{k-3} \right), \]
\[ \frac{3}{7} \left( \frac{n-7}{k-3} \right) + \frac{n-8}{k-3} + \frac{n-9}{k-3} + \frac{2}{9} \left( \frac{n-13}{k-3} \right) \geq \left( \frac{3}{7} \times \frac{3}{4} \right)^2 + \left( \frac{3}{4} \right)^3 + \left( \frac{3}{4} \right)^4 + \frac{2}{9} \left( \frac{3}{4} \right)^8 \left( \frac{n-5}{k-3} \right) > \left( \frac{n-5}{k-3} \right), \]
\[ \frac{4}{3} \left( \frac{n-13}{k-3} \right) \geq \frac{4}{3} \times \frac{3}{4} \left( \frac{n-12}{k-3} \right) = \left( \frac{n-12}{k-3} \right). \]

Adding the above three inequalities, we get
\[ \left( \frac{n-4}{k-3} \right) + \left( \frac{n-5}{k-3} \right) + \left( \frac{n-12}{k-3} \right) < \left( \frac{n-7}{k-3} \right) + 2 \left( \frac{n-8}{k-3} \right) + 2 \left( \frac{n-9}{k-3} \right) + 2 \left( \frac{n-10}{k-3} \right) + 2 \left( \frac{n-13}{k-3} \right). \]

Thus,
\[ \left( \frac{n-4}{k-3} \right) + \left( \frac{n-5}{k-3} \right) + \left( \frac{n-10}{k-3} \right) + 2 \left( \frac{n-11}{k-3} \right) + 3 \left( \frac{n-12}{k-3} \right) \leq \left( \frac{n-7}{k-3} \right) + 2 \left( \frac{n-8}{k-3} \right) + 2 \left( \frac{n-9}{k-3} \right) + 2 \left( \frac{n-10}{k-3} \right) + 2 \left( \frac{n-11}{k-3} \right) + 2 \left( \frac{n-12}{k-3} \right) + 2 \left( \frac{n-13}{k-3} \right) \leq \sum_{6 \leq i \leq 7} \sum_{1 \leq j \leq 7} \left( \frac{n-i-j}{k-3} \right) \leq \sum_{6 \leq i \leq 7} \left( \frac{n-i}{k-2} - \frac{n-k-i}{k-2} \right). \]

and (3.3) follows. \( \square \)

**Lemma 3.3.** For \( n \geq 2k \) and \( k \geq 7, \)
\[ (3.4) \prod_{2 \leq i \leq k-1} \frac{t+k-6+i}{t+i} > \left( \frac{2t+3k-11}{2t+k+1} \right)^{k-2}. \]
\[ (3.5) \prod_{2 \leq i \leq k-1} \frac{t+k-5+i}{t+i} > \left( \frac{2t+3k-9}{2t+k+1} \right)^{k-2}. \]

**Proof.** Note that for \( m > d > i > 0, \)
\[ (3.6) \frac{m-d-i}{m-i}, \frac{m-d+i}{m+i} < \left( \frac{m-d}{m} \right)^2. \]

Equivalently,
\[ \frac{(m-d)^2-i^2}{(m-d)^2} < \frac{m^2-i^2}{m^2}, \] that is, \( \left( \frac{i}{m} \right)^2 < \left( \frac{i}{m-d} \right)^2, \)

which is true for \( m > d > 0. \) Applying (3.6) repeatedly with \( m = t + \frac{3k}{2} - \frac{11}{2} \) and \( d = k - 6, \) we obtain
\[ \frac{(t+2)(t+3) \ldots (t+k-1)}{(t+k-4)(t+k-3) \ldots (t+2k-7)} < \left( \frac{t+\frac{7}{2}+\frac{1}{2}}{t+\frac{3k}{2}-\frac{11}{2}} \right)^{k-2} = \left( \frac{2t+k+1}{2t+3k-11} \right)^{k-2}. \]

and (3.3) follows. Similarly, we can obtain (3.5). \( \square \)
Lemma 3.4. For $2k + 1 \leq n \leq 4k$ and $k \geq 7$,

\[
\binom{n-6}{k-1} + 2 \binom{n-6}{k-2} - 2 \binom{n-6}{k-4} - \binom{n-6}{k-5} > \binom{n-k}{k-1} + \binom{n-k-1}{k-1},
\]

\[
\binom{n-5}{k-1} - \binom{n-5}{k-4} > \binom{n-k}{k-1} + \binom{n-k-1}{k-1}.
\]

Proof. Let us prove (3.7) first. Set $t = n - 2k$. Then $1 \leq t \leq 2k$. Note that

\[
\binom{n-6}{k-2} = \frac{k-1}{n-k-4} \binom{n-6}{k-1} = \frac{k-1}{t+k-4} \binom{n-6}{k-1},
\]

\[
\binom{n-6}{k-4} = \frac{(k-1)(k-2)(k-3)}{(n-k-2)(n-k-3)(n-k-4)} \binom{n-6}{k-1}.
\]

Moreover,

\[
\binom{n-k}{k-1} = \frac{n-2k+1}{n-k} \binom{n-k}{k-1} = \frac{t+1}{t} \binom{n-k}{k-1}
\]

and

\[
\frac{(n-6)}{(n-k)} = \prod_{0 \leq i \leq k} \frac{n-6-i}{n-k-i} = \prod_{2 \leq i \leq k} \frac{t+k-6+i}{t+i}.
\]

Since

\[
\frac{2(k-1)(k-2)(k-3)}{(t+k-2)(t+k-3)(t+k-4)} + \frac{(k-1)(k-2)(k-3)(k-4)}{(t+k-1)(t+k-2)(t+k-3)(t+k-4)}
\]

\[
= \frac{(k-1)(k-2)(k-3)}{(t+k-2)(t+k-3)(t+k-4)} \left( 2 + \frac{k-4}{t+k-1} \right)
\]

\[
= \frac{(k-1)(k-2)(k-3)(2t+3k-6)}{(t+k-1)(t+k-2)(t+k-3)(t+k-4)},
\]

we see that (3.7) is equivalent to

\[
\left( 1 + \frac{2(k-1)}{t+k-4} - \frac{(k-1)(k-2)(k-3)(2t+3k-6)}{(t+k-1)(t+k-2)(t+k-3)(t+k-4)} \right) \prod_{2 \leq i \leq k} \frac{t+k-6+i}{t+i} > \frac{2t+k+1}{t+k}.
\]

By moving out the first term $\frac{t+k-4}{t+k}$ and the last term $\frac{t+2k-6}{t+k}$ from $\Pi$, we get

\[
\left( t+3k-6 - \frac{(k-1)(k-2)(k-3)(2t+3k-6)}{(t+k-1)(t+k-2)(t+k-3)(t+k-4)} \right) \prod_{3 \leq i \leq k-1} \frac{t+k-6+i}{t+i} > \frac{(t+2)(2t+k+1)}{t+2k-6}.
\]

By further moving out the terms $\frac{t+k-3}{t+k-4}$, $\frac{t+k-2}{t+k-4}$, $\frac{t+k-1}{t+k}$ from $\Pi$, we have

\[
((t+3k-6)(t+k-1)(t+k-2)(t+k-3) - (k-1)(k-2)(k-3)(2t+3k-6))
\]

\[
\cdot \prod_{6 \leq i \leq k-1} \frac{t+k-6+i}{t+i} > \frac{(t+2)(t+3)(t+4)(t+5)(2t+k+1)}{t+2k-6}.
\]

By factorization, we have

\[
(t+3k-6)(t+k-1)(t+k-2)(t+k-3) - (k-1)(k-2)(k-3)(2t+3k-6)
\]

\[
eq t(t+2k-5)(t+2k-4)(t+2k-3).
\]
It follows that

$$\prod_{6 \leq i \leq k-1} \frac{t + k - 6 + i}{t + i} \geq \frac{(t + 2)(t + 3)(t + 4)(t + 5)(2t + k + 1)}{t(t + 2k - 6)(t + 2k - 5)(t + 2k - 4)(t + 2k - 3)}$$

Equivalently,

$$\prod_{2 \leq i \leq k-1} \frac{t + k - 6 + i}{t + i} > \frac{(t + 2)(t + 3)(t + 4)(t + 5)(2t + k + 1)}{t(t + 2k - 6)(t + 2k - 5)(t + 2k - 4)(t + 2k - 3)}.$$

For $k = 7, 8, 9, 10, 11$ and $t \leq 2k$, it can be checked directly that (3.9) holds. Thus we may assume $k \geq 12$.

Since $\frac{2t + k + 1}{t} = 2 + \frac{k + 1}{t} \leq k + 3$ and by $k \geq 12$

$$\frac{(t + k - 4)(t + k - 3)(t + k - 2)(t + k - 1)}{(t + 2k - 6)(t + 2k - 5)(t + 2k - 4)(t + 2k - 3)} \leq \frac{(3k - 4)(3k - 3)(3k - 2)(3k - 1)}{(4k - 6)(4k - 5)(4k - 4)(4k - 3)}$$

$$\leq \frac{32 \times 33 \times 34 \times 35}{42 \times 43 \times 44 \times 45} \approx 0.3514 < \frac{2}{5},$$

we see that the RHS of (3.9) is less than $\frac{2(k+3)}{5}$. By (3.4), $t \leq 2k$ and $k \geq 12$, we infer that the LHS of (3.9) is greater than

$$\left(\frac{2t + 3k - 11}{2t + k + 1}\right)^{k-2} \geq \left(\frac{7k - 11}{5k + 1}\right)^{k-2} \geq \left(\frac{73}{61}\right)^{k-2}.$$

We prove $\left(\frac{73}{61}\right)^{k-2} > \frac{2(k+3)}{5}$ for $k \geq 12$ by induction. For $k = 12$ we have $\left(\frac{73}{61}\right)^{10} \approx 6.0246 > 6 = \frac{2 \times (12+3)}{5}$. For $k + 1 \geq 13$, by induction hypothesis

$$\left(\frac{73}{61}\right)^{k-1} = \left(\frac{73}{61}\right)^{k-2} + \frac{12}{61} \left(\frac{73}{61}\right)^{k-2} > \frac{2(k + 3)}{5} + \frac{12}{61} \left(\frac{73}{61}\right)^{k-2} > \frac{2(k + 4)}{5}.$$

Now let us prove (3.8). Similarly, set $t = n - 2k$ and then $1 \leq t \leq 2k$. Note that

$$\frac{(n - 5)(n - 4)}{(n - k - 1)(n - k - 2)(n - k - 3)} = \frac{(k - 1)(k - 2)(k - 3)}{(t + 1)(t + 2)(t + 3)} = \frac{(k - 1)(k - 2)(k - 3)}{(t + 1)(t + 2)(t + 3)} \frac{n - 5}{k - 1}.$$

Then (3.8) is equivalent to

$$\left(1 - \frac{(k - 1)(k - 2)(k - 3)}{(t + 1)(t + 2)(t + 3)}\right) \prod_{2 \leq i \leq k} \frac{t + k - 5 + i}{t + i} > \frac{2t + k + 1}{t + k}.$$

By moving the last term $\frac{t + 2k - 5}{2t + k + 1}$ from $\prod$ to the RHS, we obtain

$$\left(1 - \frac{(k - 1)(k - 2)(k - 3)}{(t + 1)(t + 2)(t + 3)}\right) \prod_{2 \leq i \leq k-1} \frac{t + k - 5 + i}{t + i} > \frac{2t + k + 1}{t + 2k - 5}.$$

For $k = 7, 8, 9, 10$ and $t \leq 2k$, it can checked directly that (3.10) holds. Thus we may assume $k \geq 11$.

Note that

$$1 - \frac{(k - 1)(k - 2)(k - 3)}{(t + 1)(t + 2)(t + 3)} \geq 1 - \frac{k - 3}{k} = \frac{3}{k}$$

and by (3.5) and $k \geq 11$

$$\prod_{2 \leq i \leq k-1} \frac{t + k - 5 + i}{t + i} \geq \left(\frac{2t + 3k - 9}{2t + k + 1}\right)^{k-2} \geq \left(\frac{7k - 9}{5k + 1}\right)^{k-2} \geq \left(\frac{17}{14}\right)^{k-2}.$$

Moreover, $t \leq 2k$ and $k \geq 11$ imply

$$\frac{2t + k + 1}{t + 2k - 5} = 2 - \frac{3k - 11}{t + 2k - 5} \leq 2 - \frac{3k - 11}{4k - 5} \leq 2 - \frac{22}{39} < \frac{3}{2}.$$
Thus, for \( k \geq 11 \) it suffices to show \((\frac{17}{14})^{k-2} > \frac{k}{2}\). We prove it by induction on \( k \). For \( k = 11 \), we have \((\frac{17}{14})^{9} \approx 5.7397 > \frac{11}{2}\). For \( k + 1 \geq 12 \), by induction hypothesis we have
\[
\left(\frac{17}{14}\right)^{k-1} > \left(\frac{17}{14}\right)^{k-2} + \frac{3}{14} \left(\frac{17}{14}\right)^{k-2} > \frac{k}{2} + \frac{3}{14} \left(\frac{17}{14}\right)^{k-2} > \frac{k+1}{2}.
\]

Now we are ready to prove Theorem 3.51.

**Proof of Theorem 3.51.** By the maximality of \(|F|\), we infer that \( T \) is full in \( F \) for all \( T \in \mathcal{T}^{(3)}(F) \). It follows that \( \mathcal{T}^{(3)}(F) \) is intersecting.

**Claim 3.5.** There exists \( \{a, b, c\}, \{d, e, f\} \in \mathcal{T}^{(3)}(F) \) such that \( (a, b), (d, e) \in H \), \( \{a, c\} \) is a transversal of \( S_{ab}(F) \), \( \{d, f\} \) is a transversal of \( S_{de}(F) \), \( \{a, b, c\} \cap (d, e) = \emptyset \) and \( f \in \{a, b, c\} \).

**Proof.** Since \( H \neq \emptyset \), let \( (a, b) \in H \). Then \( \tau(S_{ab}(F)) \leq 2 \). Using \( \tau(F) \geq 3 \) we must have equality.

Fix \( c \) so that \( (a, c) \) is a transversal of \( S_{ab}(F) \) and note that \( \{a, b, c\} \) is a transversal of \( F \). Hence the maximality of \(|F|\) implies that \( \{a, b, c\} \) is full, i.e., \( \{a, b, c\} \subset F \in \binom{[n]}{k} \) implies \( F \in F \).

Since \( \tau(F) > 2 \), \( (a, b), (a, c), (b, c) \) are not transversals of \( F \). Thus for each \( x \in \{a, b, c\} \) we can fix \( F_x \in F \) with \( F_x \cap \{a, b, c\} = \{x\} \).

We claim that there is some \( (d, e) \in H \) with \( \{a, b, c\} \cap (d, e) = \emptyset \). In the opposite case \( F \) is initial on \( \binom{[n]}{k-1} \setminus \{a, b, c\} \). Let \( E \in \binom{[n]\setminus\{a,b,c\}}{k-1} \). Then \( E \cup \{x\} \not\subset F_x \) implies \( E \cup \{x\} \not\subset F \). Now \( E \in F(a), E \in F(b) \) imply \( E \cup \{b\} \subset S_{de}(F) \).

However \( E \cup \{b\} \cap \{a, c\} = \emptyset \), i.e., \( \{a, c\} \) is not a transversal of \( S_{ab}(F) \), a contradiction.

Now \( (d, e) \in H \) implies \( \tau(S_{de}(F)) = 2 \). Let \( \{d, f\} \) be a transversal of \( S_{de}(F) \). Since \( \mathcal{T}^{(3)}(F) \) is intersecting, \( \{a, b, c\} \cap (d, e) = \emptyset \) implies \( f \in \{a, b, c\} \). Thus the claim holds.

**Claim 3.6.** Suppose that \( (x, y) \in H \) and \( (x, z) \in \mathcal{T}^{(2)}(S_{xy}(F)) \). If \( E \in F(x, y) \cap F(x, y) \) then \( z \in E \).

**Proof.** Suppose that \( z \notin E \). Then \( E \in F(x, y) \cap F(x, y) \) implies \( E \cup \{y\} \subset S_{xy}(F) \). However, \( (E \cup \{y\}) \cap (x, z) = \emptyset \), contradicting the fact that \( (x, z) \in \mathcal{T}^{(2)}(S_{xy}(F)) \).

**Fact 3.7.** Let \( F \subset \binom{[n]}{k} \) be an intersecting family with \( \tau(F) \geq 3 \). Then for every \( P \in \binom{[n]}{2} \) and \( Q \subset [n] \setminus P \),
\[
\left| \mathcal{F}(P, P \cup Q) \right| \leq \left(n - \left| Q \right| - 2\right) - \left(n - k - \left| Q \right| - 2\right).
\]

**Proof.** Since \( \tau(F) \geq 3 \), there exists \( F(P) \in F \) such that \( P \cap F(P) = \emptyset \). By the intersection property of \( F \), we infer
\[
\left| \mathcal{F}(P, P \cup Q) \right| \leq \left(n - \left| Q \right| - 2\right) - \left(n - \left| Q \right| - 2 - |F(P) \setminus Q|\right)
\leq \left(n - \left| Q \right| - 2\right) - \left(n - k - \left| Q \right| - 2\right).
\]

We first prove the theorem for \( n \geq 4k \).

**Proposition 3.8.** If \( \mathcal{T}^{(3)}(F) \) is non-trivial, then \( |F| < |G(n, k)| \) for \( n \geq 4k \) and \( k \geq 7 \).

**Proof.** By Claim 3.5 there are \( \{a, b, c\}, \{d, e, f\} \in \mathcal{T}^{(3)}(F) \) such that \( (a, b), (d, e) \in H \), \( \{a, c\} \) is a transversal of \( S_{ab}(F) \), \( \{d, f\} \) is a transversal of \( S_{de}(F) \), \( \{a, b, c\} \cap (d, e) = \emptyset \) and \( f \in \{a, b, c\} \).

Let \( (v, w) = \{a, b, c\} \setminus \{f\} \). Since \( \mathcal{T}^{(3)}(F) \) is non-trivial, there exists \( T \in \mathcal{T}^{(3)}(F) \) such that \( f \notin T \). Clearly \( T \cap (v, w) \neq \emptyset \neq T \cap (d, e) \).

There are essentially two possibilities for \( T \).

**Case 1.** \( T = \{w, d, g\} \) with \( g \notin \{v, w, d, e, f\} \).
Let $\mathcal{P} = \{\{v, w, f\}, \{d, e, f\}, \{w, d, g\}\} \subset T^{(3)}(\mathcal{F})$ and $U = \{v, w, d, e, f, g\}$. Then for any $R \in \binom{[n]}{k-2}$ and $S \in \binom{[n]}{k-2}$ with $R \cup S \subset \mathcal{F}$, we have $S \in T^{(2)}(\mathcal{P})$. It is easy to see that

$$T^{(2)}(\mathcal{P}) = \{\{v, d\}, \{w, d\}, \{w, e\}, \{w, f\}, \{d, f\}, \{f, g\}\}.$$  

Note that the non-triviality of $\mathcal{P}$ implies $|F \cap U| \geq 2$ for all $F \in \mathcal{F}$. Define

$$\mathcal{F}_i = \{F \in \mathcal{F} : |F \cap U| = i\}, \quad i = 2, 3, \ldots, 6.$$

**Claim 3.9.**

\begin{equation}
|\mathcal{F}_2| \leq 3 \left( \binom{n-6}{k-2} - \binom{n-k-6}{k-2} \right). \tag{3.12}
\end{equation}

**Proof.** Note that $\mathcal{F}(\{w, d\}, U)$ and $\mathcal{F}(\{f, g\}, U)$ are cross-intersecting. If one of them is empty, then by (3.11) we infer

$$|\mathcal{F}(\{w, d\}, U)| + |\mathcal{F}(\{f, g\}, U)| \leq \binom{n-6}{k-2} - \binom{n-k-6}{k-2}.$$  

If both of them are non-empty, then by (3.11)

$$|\mathcal{F}(\{w, d\}, U)| + |\mathcal{F}(\{f, g\}, U)| \leq \binom{n-6}{k-2} - \binom{n-k-6}{k-2} + 1 < \binom{n-6}{k-2} - \binom{n-k-6}{k-2}.$$  

Similarly, we have

$$|\mathcal{F}(\{w, e\}, U)| + |\mathcal{F}(\{d, f\}, U)| \leq \binom{n-6}{k-2} - \binom{n-k-6}{k-2} - \binom{n-6}{k-2} - \binom{n-k-6}{k-2}.$$  

Adding these inequalities, we obtain (3.12). \hfill \square

Let $T \in \binom{[n]}{3}$. By (3.12) we have

$$|\mathcal{F}(T, U)| + |\mathcal{F}(U \setminus T, U)| \leq \binom{n-6}{k-3}.$$  

It follows that $|\mathcal{F}_3| \leq \frac{1}{6} \binom{6}{3} \binom{n-6}{k-3} = \binom{n-6}{k-3}$. Hence, by (3.11) we obtain that

$$|\mathcal{F}| = |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4| + |\mathcal{F}_5| + |\mathcal{F}_6|$$

$$\leq 3 \left( \binom{n-6}{k-2} - \binom{n-k-6}{k-2} \right) + 10 \binom{n-6}{k-3} + 15 \binom{n-6}{k-4} + 6 \binom{n-6}{k-5} + \binom{n-6}{k-6}$$

$$= 3 \left( \binom{n-6}{k-2} - \binom{n-k-6}{k-2} \right) + \binom{n-3}{k-3} + 3 \binom{n-4}{k-3} + 6 \binom{n-5}{k-3}$$

$$< |\mathcal{G}(n, k)|.$$  

**Case 2.** $T = \{w, d, e\}$.

Let $\mathcal{P} = \{\{v, w, f\}, \{d, e, f\}, \{w, d, e\}\} \subset T^{(3)}(\mathcal{F})$ and $U = \{v, w, d, e, f\}$. Then for any $R \in \binom{[n]}{k-2}$ and $S \in \binom{[n]}{k-2}$ with $R \cup S \subset \mathcal{F}$, we have $S \in T^{(2)}(\mathcal{P})$. Note that $|U| = 5 = 3 + 2$ implies $T^{(2)}(\mathcal{P}) = \binom{[n]}{k-2} \setminus \{U \mid P \in \mathcal{P}\}$. That is,

$$T^{(2)}(\mathcal{P}) = \{\{v, d\}, \{v, e\}, \{v, w, d\}, \{v, e\}, \{w, f\}, \{d, f\}, \{e, f\}\}.$$  

Define $\mathcal{F}_i = \{F \in \mathcal{F} : |F \cap U| = i\}, \quad i = 2, 3, \ldots, 6$. Note that $(\mathcal{F}(\{w, d\}, U), \mathcal{F}(\{e, f\}, U)), (\mathcal{F}(\{w, e\}, U), \mathcal{F}(\{d, v\}, U)), (\mathcal{F}(\{w, f\}, U), \mathcal{F}(\{v, e\}, U))$ are three cross-intersecting pairs. By a
Consequently, R

at most one of

similar argument as in the proof of Claim 3.9, we obtain that

\[ |F(\{w, d\}, U)| + |F(\{e, f\}, U)| \leq \binom{n-5}{k-2} - \binom{n-k-5}{k-2}, \]
\[ |F(\{w, e\}, U)| + |F(\{v, d\}, U)| \leq \binom{n-5}{k-2} - \binom{n-k-5}{k-2}, \]
\[ |F(\{w, f\}, U)| + |F(\{v, e\}, U)| \leq \binom{n-5}{k-2} - \binom{n-k-5}{k-2}. \]

Moreover, by (3.11) we have

\[ |F(\{d, f\}, U)| \leq \left( \binom{n-5}{k-2} - \binom{n-k-5}{k-2} \right). \]

Thus,

\[ (3.13) \quad |F_2| = \sum_{P \in T^{(2)}(P)} |F(P, U)| \leq 4 \left( \binom{n-5}{k-2} - \binom{n-k-5}{k-2} \right). \]

Recall that \{a, c\} is a transversal of \( S_{ab}(F) \) and \{d, f\} is a transversal of \( S_{de}(F) \). We claim that for any \( R \in \binom{[n]\setminus\{a\}}{k-4} \), at most one of \( R \cup \{a, d, c\} \), \( R \cup \{b, d, e\} \) is in \( F \). Indeed, otherwise if \( R \cup \{a, d, c\} \), \( R \cup \{b, d, e\} \) is in \( F \) then \( R \cup \{a, d\} \in F(a, b) \cap F(a, b) \) but \( c \notin R \cup \{d, e\} \), contradicting Claim 3.6. Thus for any \( R \in \binom{[n]\setminus\{a\}}{k-3} \), at most one of \( R \cup \{a, d, e\} \), \( R \cup \{b, d, e\} \) is in \( F \). Similarly, at most one of \( R \cup \{d, v, w\} \), \( R \cup \{e, v, w\} \) is in \( F \). It follows that

\[ |F(\{a, d, e\}, U)| + |F(\{b, d, e\}, U)| \leq \binom{n-5}{k-3}, \]
\[ |F(\{d, v, w\}, U)| + |F(\{e, v, w\}, U)| \leq \binom{n-5}{k-3}. \]

Consequently,

\[ |F_3| = \sum_{B \in \binom{[n]}{5}} |F(B, U)| \leq 8 \binom{n-5}{k-3}. \]

Therefore, by (3.2) we conclude that

\[ |F| = |F_2| + |F_3| + |F_4| + |F_5| \]
\[ \leq 4 \left( \binom{n-5}{k-2} - \binom{n-k-5}{k-2} \right) + 8 \binom{n-5}{k-3} + 5 \binom{n-5}{k-4} + \binom{n-5}{k-5} \]
\[ = 4 \left( \binom{n-5}{k-2} - \binom{n-k-5}{k-2} \right) + \binom{n-3}{k-3} + 3 \binom{n-4}{k-3} + 4 \binom{n-5}{k-3} \]
\[ \leq |\mathcal{G}(n, k)|. \]

In Proposition 3.8 we proved that if \( T^{(3)}(F) \) is non-trivial then \( |F| < |\mathcal{G}(n, k)| \) for \( n \geq 4k \) and \( k \geq 7 \). Let us prove that \( |F| < |\mathcal{G}(n, k)| \) for \( k \geq 7 \) in the full range.

Proposition 3.10. If \( T^{(3)}(F) \) is non-trivial, then \( |F| < |\mathcal{G}(n, k)| \) for \( n \geq 2k + 1 \) and \( k \geq 7 \).

Proof. By Proposition 3.8 we may assume that \( 2k < n < 4k \). By Claim 3.6 let \( \{a, b, c\}, \{d, e, f\} \) be the two transversals of \( F \) with \( \{a, b, e\} \cap \{d, e, f\} = \{f\} \) and let \( \{v, w\} = \{a, b, c\} \setminus \{f\} \). Since \( T^{(3)}(F) \) is non-trivial, there exists \( T \in T^{(3)}(F) \) such that \( f \notin T \). Clearly \( T \cap \{v, w\} \neq \emptyset \neq T \cap \{d, e\} \).

We distinguish two cases as above. However the computation is very much different for \( 2k < n < 4k \).

Case 1. \( T = \{w, d, g\} \) with \( g \notin \{v, w, d, e, f\} \).
Let $\mathcal{P} = \{\{v, w, f\}, \{d, e, f\}, \{w, d, g\}\} \subset \mathcal{T}^{(3)}(\mathcal{F})$ and $D = \{v, w, d, e, f, g\}$. Then for any $R \in \binom{[n] \setminus D}{k-2}$ and $S \in \binom{D}{k-2}$ with $R \cup S \in \mathcal{F}$, we must have $S \in \mathcal{T}^{(2)}(\mathcal{P})$. Recall that

$$\mathcal{T}^{(2)}(\mathcal{P}) = \{\{v, d\}, \{w, d\}, \{w, e\}, \{w, f\}, \{d, f\}, \{f, g\}\}.$$ 

Note that $\mathcal{T}^{(2)}(\mathcal{P})$ can be partitioned into three disjoint pairs: $(\{v, d\}, \{w, f\})$, $(\{w, d\}, \{f, g\})$ and $(\{w, e\}, \{d, f\})$. For any disjoint pair $\{A, A'\} \subset \mathcal{T}^{(2)}(\mathcal{P})$, by (1.12) we have

$$\alpha(A, D) + \alpha(A', D) \leq 1.$$ 

Set $B = D \setminus A$ and $B' = D \setminus A'$. Using (1.12) again we infer

$$\alpha(A, D) + \alpha(B, D) \leq 1, \quad \alpha(A', D) + \alpha(B', D) \leq 1.$$ 

It follows that

$$|\mathcal{F}(A, D)| + |\mathcal{F}(A', D)| + |\mathcal{F}(B, D)| + |\mathcal{F}(B', D)|$$

$$= (\alpha(A, D) + \alpha(A', D)) \binom{n-6}{k-2} + (\alpha(B, D) + \alpha(B', D)) \binom{n-6}{k-4}$$

$$\leq \binom{n-6}{k-2} - (1 - \alpha(A, D) - \alpha(A', D)) \binom{n-6}{k-2} + (2 - \alpha(A, D) - \alpha(A', D)) \binom{n-6}{k-4}$$

$$= \binom{n-6}{k-2} + \binom{n-6}{k-4} - (1 - \alpha(A, D) - \alpha(A', D)) \left( \binom{n-6}{k-2} - \binom{n-6}{k-4} \right)$$

$$\leq \binom{n-6}{k-2} + \binom{n-6}{k-4}.$$ 

For any $P \in \binom{D}{k-2} \setminus \mathcal{T}^{(2)}(\mathcal{P})$, $\mathcal{F}(P, D) = \emptyset$ implies

$$|\mathcal{F}(P, D)| + |\mathcal{F}(D \setminus P, D)| \leq \binom{n-6}{k-4}.$$ 

For $A \in \binom{D}{k-2}$ just use $\alpha(A, D) + \alpha(D \setminus A, D) \leq 1$. Eventually we obtain

$$|\mathcal{F}| = \sum_{Q \subseteq D} |\mathcal{F}(Q, D)| \leq 3 \left( \binom{n-6}{k-2} + \binom{n-6}{k-4} \right) + \left( \binom{6}{4} - 6 \right) \left( \binom{n-6}{k-4} \right) + \frac{1}{2} \left( \binom{6}{3} \right) \left( \binom{n-6}{k-3} \right)$$

$$+ 6 \left( \binom{n-6}{k-5} + \binom{n-6}{k-6} \right)$$

$$\leq 3 \binom{n-6}{k-2} + 10 \binom{n-6}{k-3} + 12 \binom{n-6}{k-4} + 6 \binom{n-6}{k-5} + \binom{n-6}{k-6}.$$ 

Note that

$$\binom{n-1}{k-1} = \binom{n-6}{k-1} + 5 \binom{n-6}{k-2} + 10 \binom{n-6}{k-3} + 10 \binom{n-6}{k-4} + 5 \binom{n-6}{k-5} + \binom{n-6}{k-6}.$$ 

Then by (3.7)

$$\binom{n-1}{k-1} - |\mathcal{F}| \geq \binom{n-6}{k-1} + 2 \binom{n-6}{k-2} - 2 \binom{n-6}{k-4} - \binom{n-6}{k-5} > \binom{n-k}{k-1} + \binom{n-k-1}{k-1}$$

and thereby $|\mathcal{F}| < |\mathcal{G}(n, k)|$.

**Case 2.** $T = \{w, d, e\}$.

Set $\mathcal{P} = \{\{v, w, f\}, \{d, e, f\}, \{w, d, e\}\} \subset \mathcal{T}^{(3)}(\mathcal{F})$ and $D = \{v, w, d, e, f\}$. Then for any $R \in \binom{[n] \setminus D}{k-2}$ and $S \in \binom{D}{k-2}$ with $R \cup S \in \mathcal{F}$, we have $S \in \mathcal{T}^{(2)}(\mathcal{P})$. Recall that

$$\mathcal{T}^{(2)}(\mathcal{P}) = \{\{v, d\}, \{v, e\}, \{w, d\}, \{w, e\}, \{w, f\}, \{d, f\}, \{e, f\}\}.$$ 

16
Note that \( \{w, d\}, \{e, f\}, \{w, e\}, \{d, f\}, \{w, f\}, \{v, e\} \) are three disjoint pairs. For a disjoint pair \( \{A, A'\} \), set \( B = D \setminus A \) and \( B' = D \setminus A' \). By a similar argument as in Case 1, we obtain

\[
|F(A, D)| + |F(A', D)| + |F(B, D)| + |F(B', D)| \leq \binom{n - 5}{k - 2} + \binom{n - 5}{k - 4}.
\]

For \( \{d, f\} \) just use \( \alpha(\{d, f\}, D) + \alpha(D \setminus \{d, f\}, D) \leq 1 \). This leads to

\[
|F| = \sum_{Q \subseteq D} |F(Q, D)| \leq 4 \binom{n - 5}{k - 2} + 6 \binom{n - 5}{k - 3} + 5 \binom{n - 5}{k - 4} + \binom{n - 5}{k - 5}.
\]

Note that

\[
\binom{n - 1}{k - 1} = \binom{n - 5}{k - 1} + 4 \binom{n - 5}{k - 2} + 6 \binom{n - 5}{k - 3} + 4 \binom{n - 5}{k - 4} + \binom{n - 5}{k - 5}.
\]

Then by (3.8)

\[
\binom{n - 1}{k - 1} - |F| \geq \binom{n - 5}{k - 1} - \binom{n - 5}{k - 4} > \binom{n - k}{k - 1} + \binom{n - k - 1}{k - 1}
\]

and \( |F| < |G(n, k)| \) follows.

By Propositions 3.8 and 3.11, we conclude that Theorem 3.1 holds.

\[\square\]

### 4 Proofs of Propositions 1.8 and 1.9

In this section, we prove Propositions 1.8 and 1.9 and some inequalities that are needed in the proof of Theorem 1.4.

For the following lemma we need a simple inequality.

\[(4.1) \quad \left(\frac{3}{2}\right)^{d} > 2d - 3 \quad \text{for} \quad d \geq 1.\]

**Proof.** The statement is easily checked for \( d = 1, 2, 3 \) and 4. For \( d + 1 \geq 5 \) we apply induction. As

\[
\left(\frac{3}{2}\right)^{d+1} = \frac{3}{2} \left(\frac{3}{2}\right)^{d} > 2d - 3 + \frac{1}{2}(2d - 3) > 2d - 3 + 2 = 2(d + 1) - 3,
\]

by the induction hypothesis and \( 2d - 3 > 4 \) (for \( d \geq 4 \)), the proof of (4.1) is complete. \( \square \)

Let us prove two analytic inequalities.

**Lemma 4.1.** Suppose that \( n, b, a \) are positive integers, \( n \geq a + b \). Then for \( b \geq a + 1 \),

\[
\binom{n}{b} - 2 \binom{n - a}{b} + \binom{n - 2a}{b} + 2 \geq \binom{n - 1}{a - 1} + \binom{n - 1}{b - 1} - \binom{n - a - 1}{b - 1} + 1,
\]

with equality holding iff \( n = a + b \) or \( a = 1 \). For \( b \geq a + 2 \),

\[
\binom{n}{b} - 2 \binom{n - a}{b} + \binom{n - 2a}{b} + 2 \geq \binom{n}{a},
\]

with equality holding iff \( n = a + b \) or \( a = 1, b = 3 \).

**Proof.** Let us prove (4.2) first. For \( n = a + b \), using \( \binom{a+b-1}{a-1} = \binom{a+b-1}{b} \) one sees that both sides of (4.2) are equal to \( \binom{a+b}{b} \). In case of \( a = 1 \), by substituting \( a = 1 \) into (4.2) and using

\[
\binom{n}{b} - 2 \binom{n - 1}{b} + \binom{n - 2}{b} = \binom{n - 2}{b - 2},
\]

17
we infer that both sides are equal to \((\frac{n}{b} - \frac{n-2}{b-1}) + 2\).

From now on we assume \(n \geq a + b + 1\) and \(a \geq 2\). We prove (4.2) with strict inequality, that is,

\[
(\frac{n}{b}) - 2\left(\frac{n-a}{b}\right) + \left(\frac{n-2a}{b}\right) \geq \left(\frac{n-1}{a-1}\right) + \left(\frac{n-1}{b-1}\right) - \left(\frac{n-a-1}{b-1}\right).
\]

We distinguish two cases.

**Case 1.** \(b = a + 1\).

Rearranging (4.4) yields

\[
\left(\frac{n}{a+1}\right) - \left(\frac{n}{a}\right) \geq \left(\frac{n-a}{a+1}\right) + \left(\frac{n-a-1}{a+1}\right) - \left(\frac{n-2a}{a+1}\right)
\]

By using \(\binom{n}{a+1} = \frac{n-a}{a+1}\binom{n-a-1}{a}\) and \(\binom{n-a-1}{a+1} = \frac{n-2a-1}{a+1}\binom{n-a-1}{a}\), it suffices to show

\[
\frac{n-2a-1}{a+1} \left(\frac{n}{a}\right) \geq \frac{2n-3a-1}{a+1} \left(\frac{n-a-1}{a}\right) - \left(\frac{n-2a}{a+1}\right).
\]

Plugging in \(n = 2a + 2\) yields (note that \(\binom{n-2a}{a+1} = 0\))

\[
\left(\frac{2a+2}{a}\right) \geq (a+3) \left(\frac{a+1}{a}\right) = (a+1)(a+3).
\]

Equivalently, \((2a+2)! \geq (a+1)!(a+3)!\). This is true for \(a = 2\). Therefore \((2a+4)(2a+3) > (a+2)(a+4)\) implies it for all \(a > 2\) as well.

From now on we assume that \(n \geq 2a+3\) and for convenience set \(t = n - 2a - 1\). There are two subcases according the size of \(t\).

**Subcase 1.1** \(2 \leq t \leq a\).

By (4.3), it suffices to show that

\[
\prod_{1 \leq i \leq a} \frac{a+1+t+i}{t+i} \geq \frac{a+2t+1}{t}.
\]

In the case \(a = t = 2\) we have \(\frac{6 \times 7}{3 \times 4} \geq \frac{7}{2}\) and the equality holds. In the case \(a = 3\) we need

\[
\frac{(t+5)(t+6)(t+7)}{(t+1)(t+2)(t+3)} \geq \frac{2(t+2)}{t},
\]

which is easily verified for both \(t = 2\) and \(3\). For the case \(a \geq 4, 2 \leq t \leq a\) note that the RHS of (4.6) is maximal for \(t = 2\) when its value is \(\frac{a+5}{2}\). On the LHS, by \(t \leq a\)

\[
\frac{a+1+t+i}{t+i} \geq \frac{2a+1+t}{a+t} > \frac{3}{2}.
\]

Thus (4.6) follows from \(\left(\frac{1}{2}\right)^a \geq \frac{a+5}{2}\). By (4.1) and \(a \geq 4\), we infer

\[
\left(\frac{3}{2}\right)^a > 2a - 3 > \frac{a+5}{2}
\]

and (4.6) holds.

**Subcase 1.2** \(t \geq a + 1\), i.e. \(n \geq 3a + 2\).

For \(a = 2\), the LHS of (4.3) equals

\[
\left(\frac{n}{3}\right) - 2\left(\frac{n-2}{3}\right) + \left(\frac{n-4}{3}\right) = (n-2) + 2(n-3) + (n-4) = 4n - 12.
\]

The RHS of (4.3) equals

\[
\left(\frac{n-1}{1}\right) + \left(\frac{n-1}{2}\right) - \left(\frac{n-3}{2}\right) = (n-1) + (n-2) + (n-3) = 3n - 6.
\]
Using \( n \geq a + b + 1 \geq 6 \), we see that (4.4) holds. Now assume \( a \geq 3 \). Recall the formula

\[
\binom{n - p}{q} - \binom{n - p - a}{q} = \sum_{1 \leq j \leq a} \binom{n - p - j}{q - 1}.
\]

Applying it twice we obtain

\[
\binom{n}{a + 1} - 2\binom{n - a}{a + 1} + \binom{n - 2a}{a + 1} = \sum_{1 \leq i \leq a} \left( \binom{n - i}{a} - \binom{n - a - i}{a} \right)
\]

(4.7)

implying (4.4). Thus (4.2) follows from (4.8).

Using (4.8) implies \( n - a \leq 6 \), by (1.13) and \( n \geq 3a \) we infer

\[
\frac{n - a}{n - \ell + 2} \geq \frac{n - a - \ell + 2}{n - \ell + 1} \geq \left( \frac{n - a}{n} \right)^{\ell - 1} \geq \left( \frac{2}{3} \right)^{\ell - 1}, \quad \ell = 3, 4.
\]

Since \( n \geq (\ell - 1)a \) implies \( \frac{n - a - \ell + 2}{n - \ell + 1} \geq \frac{a - \ell + 2}{a - 1} \), by (4.13) and \( n \geq 3a \) we infer

\[
\binom{n - 1}{a} - \binom{n - a - 1}{a} + \binom{n - 1}{a - 1},
\]

implying (1.3).

**Case 2.** \( b \geq a + 2 \geq 4, n \geq b + a + 1 \).

We apply induction on \( n \) in which we assume that (1.3) holds for the triples \((n - 1, b, a)\) and \((n - 1, b - 1, a)\) to prove it for the triple \((n, b, a)\).

Let \( \delta(x, y) \) be the Kronecker symbol,

\[
\delta(x, y) = \begin{cases} 
1, & \text{if } x = y; \\
0, & \text{if } x \neq y.
\end{cases}
\]

The equality we want to prove is

\[
\binom{n}{b} - 2\binom{n - a}{b} + \binom{n - 2a}{b} + \delta(n, a + b) \geq \binom{n - 1}{a - 1} + \binom{n - 1}{b - 1} - \binom{n - a - 1}{b - 1}.
\]

By the induction hypothesis and \( b \geq a + 2 \) we may use the instances \((n - 1, b, a)\) and \((n - 1, b - 1, a)\):

\[
\binom{n - 1}{b} - 2\binom{n - 1 - a}{b} + \binom{n - 1 - 2a}{b} + \delta(n - 1, a + b) \geq \binom{n - 2}{a - 1} + \binom{n - 2}{b - 1} - \binom{n - a - 2}{b - 1},
\]

\[
\binom{n - 1}{b - 1} - 2\binom{n - 1 - a}{b - 1} + \binom{n - 1 - 2a}{b - 1} + \delta(n - 1, a + b - 1) \geq \binom{n - 2}{a - 1} + \binom{n - 2}{b - 2} - \binom{n - a - 2}{b - 2}.
\]

Adding them we infer \( (n > a + b) \) implies \( \delta(n - 1, a + b - 1) = 0 \)

\[
(4.8) \quad \binom{n}{b} - 2\binom{n - a}{b} + \binom{n - 2a}{b} + \delta(n - 1, a + b) \geq 2\binom{n - 2}{a - 1} + \binom{n - 1}{b - 1} - \binom{n - a - 1}{b - 1}.
\]

Since \( n \geq a + b > 2a \), we infer

\[
2\binom{n - 2}{a - 1} \geq \frac{2(n - a)}{n - 1} \binom{n - 1}{a - 1} > \frac{2(n - a)}{n} \binom{n - 1}{a - 1} > \binom{n - 1}{a - 1}.
\]

Thus (4.2) follows from (4.8).
Now we prove \(1.3\) by a similar approach. For \(n = a + b\), both sides of \(1.3\) equal \((-a-b)\). For \(a = 1\), by substituting \(a = 1\) into \(1.3\) we get \(\binom{n-2}{b-2} + 2 \geq n\). Since \(b \geq a + 2 \geq 3\) and \(n \geq a + b\) imply \(\binom{n-2}{b-2} \geq n - 2\), \(1.3\) follows. From now on we assume \(n \geq a + b + 1\), \(a \geq 2\) and we prove \(1.3\) with strict inequality. We distinguish two cases.

Case 1. \(b = a + 2\).

Then \(1.3\) is equivalent to
\[
\binom{n}{a+2} - 2\binom{n-a}{a+2} + \binom{n-2a}{a+2} + 2 \geq \binom{n}{a}.
\]

Note that \(n \geq a + b + 1 \geq 2a + 3\). For convenience set \(t = n - 2a - 2\). There are two subcases according the size of \(t\).

Subcase 1.1 \(1 \leq t \leq a - 1\).

Then by \(\binom{n-2a}{b} = \binom{t+2}{b} = 0\), we need to show that
\[
(4.9) \quad \binom{n}{a+2} - \binom{n}{a} \geq 2\binom{n-a}{a+2}.
\]

Note that
\[
\binom{n}{a} = \frac{(a+1)(a+2)}{(n-a)(n-a)} \binom{n}{a+2}.
\]

After rearranging (4.9) is equivalent to
\[
(4.10) \quad \prod_{1 \leq i \leq a+2} \frac{a + t + i}{t + i} \geq \frac{2(a + t + 2)(a + t + 1)}{(a + t + 2)(a + t + 1) - (a + 2)(a + 1)}.
\]

For \(a = 2\) and \(t = 1\), we have \(\frac{4 \times 5 \times 6 \times 7}{2 \times 3 \times 4} = 7 > 5 = \frac{2 \times 4 \times 5}{2 \times 3 \times 4} + 2\) and (4.10) holds. In case of \(a = 3\) we need
\[
\frac{(t+4)(t+5)(t+6)(t+7)(t+8)}{(t+1)(t+2)(t+3)(t+4)(t+5)} \geq \frac{2(t+5)(t+4)}{(t+5)(t+4) - 20},
\]

which is easily verified for both \(t = 1\) and \(2\). Note that the RHS of (4.10) is maximal for \(t = 1\) when its value is \(a + 3\). On the LHS, by \(t \leq a - 1\) we infer for \(i \leq a + 1\)
\[
\frac{a + t + i}{t + i} \geq \frac{2a + 1 + t}{a + 1 + t} \geq \frac{3}{2}.
\]

Thus (4.10) will follow from \((\frac{3}{2})^{a+1} > a + 3\). By (4.1) and \(a \geq 4\),
\[
(\frac{3}{2})^{a+1} > 2(a + 1) - 3 = 2a - 1 \geq a + 3.
\]

Subcase 1.2 \(t \geq a\), i.e. \(n \geq 3a + 2\).

Note that for \(a \geq 2\)
\[
(4.11) \quad \binom{n}{a+2} - 2\binom{n-a}{a+2} + \binom{n-2a}{a+2} = \sum_{1 \leq i \leq a} \sum_{1 \leq j \leq a} \binom{n-i-j}{a} > \binom{n-2}{a} + 2\binom{n-3}{a}.
\]

By (4.13) and \(n \geq 3a + 2\), we infer for \(\ell \leq 3\)
\[
\frac{n-\ell}{\binom{n}{a}} > \frac{n-a-\ell+1}{\binom{n-a}{a}} > \frac{n-a-2}{\binom{n-2}{a}} > \left(\frac{2}{3}\right)^{\ell}, \quad \ell = 2, 3.
\]

Using \(\left(\frac{3}{2}\right)^{2} + 2\left(\frac{3}{2}\right)^{3} = \frac{27}{8} > 1\), we obtain that (4.11) is greater than \(\binom{n}{a}\) and (4.13) holds.

Case 2. \(b \geq a + 3 \geq 5\), \(n \geq b + a + 1\).

We apply induction on \(n\) in which we assume that (4.13) holds for the triples \((n-1, b, a)\) and \((n-1, b-1, a)\) to prove it for the triple \((n, b, a)\). The equality we want to prove is
\[
\binom{n}{b} - 2\binom{n-a}{b} + \binom{n-2a}{b} + 2b(n, a + b) \geq \binom{n}{a}.
\]

20
By the induction hypothesis and $b \geq a + 3$ we may use the instances $(n - 1, b, a)$ and $(n - 1, b - 1, a)$:

\[
\frac{n - 1}{b} - 2 \frac{n - 1 - a}{b} + \frac{n - 1 - 2a}{b} + 2\delta(n - 1, a + b) \geq \frac{n - 1}{a},
\]

\[
\frac{n - 1}{b - 1} - 2 \frac{n - 1 - a}{b - 1} + \frac{n - 1 - 2a}{b - 1} + 2\delta(n - 1, a + b - 1) \geq \frac{n - 1}{a}.
\]

Since $n \geq a + b + 1$ implies $\delta(n - 1, a + b) = 0$, adding them we infer

\[(4.12) \quad \frac{n}{b} - 2 \frac{n - a}{b} + \frac{n - 2a}{b} + 2\delta(n - 1, a + b) \geq 2 \frac{n - 1}{a}.
\]

Since $n \geq a + b > 2a + 2$ and $a \geq 2$, we infer

\[
\frac{1}{a + 1} \frac{n}{a} \geq \frac{1}{a + 1} \left( \frac{2a + 2}{2} \right) = 2a + 1 > 2.
\]

It follows that

\[
2 \frac{n - 1}{a} \geq 2 \frac{n - a}{n} \frac{n}{a} \geq \frac{2(a + 2)}{2a + 2} \frac{n}{a} \geq \left(1 + \frac{1}{a + 1}\right) \frac{n}{a} \geq \frac{n}{a} + 2.
\]

Thus $4.13$ follows from $4.12$ with strict inequality. \qed

**Proof of Proposition 1.8.** If $B$ is non-trivial then $1.9$ follows from $1.8$. Consequently we may assume by symmetry that $1 \in B$ for all $B \in \mathcal{B}$. Consider the cross-intersecting families $A(1)$ and $B(1)$. They are both non-empty. Indeed, $A(1) \neq \emptyset$ because of $\cap A = \emptyset$ and $B(1) \neq \emptyset$ because of $|B(1)| = |B|$. Applying $1.11$ and the obvious inequality $|A(1)| \leq \binom{n - 1}{a - 1}$ we infer

\[
|A| + |B| = |A(1)| + |A(1)| + |B(1)| \leq \left( \frac{n - 1}{a - 1} \right) + 1 + \left( \frac{n - 1}{b - 1} \right) - \left( \frac{n - a - 1}{b - 1} \right).
\]

By $4.2$ we conclude that $1.9$ holds with the equality holding iff $n = a + b$ or $a = 1$. \Box

**Proof of Proposition 1.9.** If $B$ is non-empty, then $1.10$ follows from $1.9$. If $B$ is empty, then

\[
|A| + |B| = |A| \leq \frac{n}{a}.
\]

By $4.3$ we obtain $1.10$ with the equality holding iff $n = a + b$ or $a = 1, b = 3$. \Box

**Corollary 4.2.** For $k \geq 5$ and $n \geq 2k + 1$,

\[(4.13) \quad \left( \frac{n - 4}{k - 1} \right) - 2 \left( \frac{n - k - 2}{k - 1} \right) + \left( \frac{n - 2k}{k - 1} \right) > \left( \frac{n - 5}{k - 2} \right) - \left( \frac{n - k - 3}{k - 2} \right) + \left( \frac{n - 5}{k - 4} \right).
\]

**Proof.** Applying $4.2$ for $(n - 4, k - 1, k - 2)$, we obtain that

\[
\left( \frac{n - 4}{k - 1} \right) - 2 \left( \frac{n - k - 2}{k - 1} \right) + \left( \frac{n - 2k}{k - 1} \right) \geq \left( \frac{n - 5}{k - 2} \right) + \left( \frac{n - 5}{k - 2} \right) - \left( \frac{n - k - 3}{k - 2} \right) - 1.
\]

Since $k - 4 < k - 3 < n - 5$ and $(n - 5) > 2(k - 3)$, we infer

\[
\left( \frac{n - 5}{k - 3} \right) - \left( \frac{n - 5}{k - 4} \right) = \frac{n - 2k + 2}{k - 3} \frac{n - 5}{k - 4} \geq \frac{3(n - 5)}{k - 3} > 1.
\]

That is, $\left( \frac{n - 5}{k - 3} \right) > \left( \frac{n - 5}{k - 4} \right) + 1$ and $4.13$ follows. \Box

**Lemma 4.3.** For $n > 2k$, $k \geq 5$ and $2 \leq q \leq k - 1$,

\[(4.14) \quad \left( \frac{n - q - 1}{k - 1} \right) - 2 \left( \frac{n - k - 1}{k - 1} \right) + \left( \frac{n - 2k + q - 1}{k - 1} \right) + 2 > \left( \frac{n - q - 1}{k - q} \right) - \left( \frac{n - k - q - 1}{k - q} \right).
\]
Proof. For $3 \leq q \leq k - 1$, by applying (4.3) to the triple $(n - q - 1, k - 1, k - q)$ and noting $n - q - 1 > (k - 1) + (k - q)$, we obtain
\[
\binom{n - q - 1}{k - 1} - 2 \binom{n - k - 1}{k - 1} + \binom{n - 2k + q - 1}{k - 1} + 2 > \binom{n - q - 1}{k - q}
\]
and (4.14) follows.

Now we assume that $q = 2$ and (4.14) is equivalent to
\[
(4.15) \quad \binom{n - 3}{k - 1} - 2 \binom{n - k - 1}{k - 1} + \binom{n - 2k + 1}{k - 1} + 2 \geq \binom{n - 3}{k - 2} - \binom{n - k - 3}{k - 2}.
\]

Let
\[
h(n, k) = \binom{n - 3}{k - 1} - 2 \binom{n - k - 1}{k - 1} + \binom{n - 2k + 1}{k - 1}.
\]

We distinguish two cases.

Case 1. $n \geq 3k$.

Note that for $k \geq 5$
\[
h(n, k) = \sum_{1 \leq i \leq k - 2} \left( \binom{n - 3 - i}{k - 2} - \binom{n - k - i - 1}{k - 2} \right) \geq \binom{n - 4}{k - 2} - \binom{n - k - 2}{k - 2} + \binom{n - 5}{k - 2} - \binom{n - k - 3}{k - 2} + \binom{n - 6}{k - 2} - \binom{n - k - 4}{k - 2}.
\]

Since $n \geq (\ell + 2)k/2$ implies $\frac{n - k - \ell}{n - 2 - \ell} \geq \frac{n - k - \ell}{n}$, by (4.15) we infer
\[
\frac{(n - 3 - \ell)}{k - 2} - \frac{(n - k - 1 - \ell)}{k - 2} \geq \frac{n - k - \ell}{n - 2 - \ell} \geq \frac{n - k - \ell}{n}, \quad \ell = 1, 2.
\]

Thus, for $n \geq 3k$ and $k \geq 5$ we have
\[
h(n, k, q) \geq \left( \frac{2}{3} + \left( \frac{2}{3} \right)^2 \right) \left( \binom{n - 3}{k - 2} - \binom{n - k - 1}{k - 2} \right) + \binom{n - 6}{k - 2} - \binom{n - k - 4}{k - 2}
\]
\[
\geq \binom{n - 3}{k - 2} - \binom{n - k - 1}{k - 2} + \binom{n - 6}{k - 2} - \binom{n - k - 4}{k - 2}.
\]

Using \(\binom{n - 6}{k - 2} \geq \binom{n - k - 1}{k - 2}\) and \(\binom{n - k - 4}{k - 2} \leq \binom{n - k - 3}{k - 2}\), we conclude that
\[
h(n, k) > \binom{n - 3}{k - 2} - \binom{n - k - 1}{k - 2} + \binom{n - k - 1}{k - 2} - \binom{n - k - 3}{k - 2} = \binom{n - 3}{k - 2} - \binom{n - k - 3}{k - 2}
\]
and (4.15) holds.

Case 2. $2k < n \leq 3k - 1$.

To prove (4.15), it suffices to show that
\[
\binom{n - 3}{k - 1} - 2 \binom{n - k - 1}{k - 1} \geq \binom{n - 3}{k - 2} - \binom{n - k - 3}{k - 2}.
\]

Since \(\binom{n - 3}{k - 2} = \frac{k - 1}{n - k - 1} \binom{n - 3}{k - 1}\) and \(\binom{n - k - 3}{k - 2} = \frac{(k - 1)(n - 2k)}{(n - k - 1)(n - k - 2)} \binom{n - k - 1}{k - 1}\), the equality is equivalent to
\[
\frac{n - 2k}{n - k - 1} \binom{n - 3}{k - 1} \geq \frac{2}{(n - k - 1)(n - k - 2)} \binom{n - k - 1}{k - 1}.
\]

Let $t = n - 2k$. Then $1 \leq t \leq k - 1$. Substituting $t = n - 2k$ and expanding the binomial coefficients, we get the equivalent version
\[
\frac{t}{t + k - 1} \prod_{1 \leq i \leq k - 1} t + i + k - 2 \geq 2 - \frac{(k - 1)t}{(t + k - 1)(t + k - 2)}.
\]
By moving the first term \(\frac{t+k-1}{t^k-1}\) in front of the product, it changes to:

\[
(4.16) \quad \frac{t}{t + 1} \prod_{2 \le i \le k} \frac{t + i + k - 2}{t + i} \ge 2 - \frac{(k-1)t}{(t + k - 1)(t + k - 2)}.
\]

Note that 2 \(\leq\) \(i\) \(\leq\) \(k - 3\) and \(t \leq k - 1\) imply \(\frac{t+k-1}{t^k-1}\) \(\geq\) \(\frac{3k-6}{2k-4}\). Moreover, \(\frac{t}{t+1} \geq \frac{1}{2}\) and \(\frac{t+k-2}{t^k-2}\) \(\geq\) \(\frac{3k-5}{2k-3}\). Consequently for \(k \geq 5\)

\[
\frac{t}{t + 1} \frac{t + 2k - 4}{t^k - 2} \frac{t + 2k - 3}{t^k - 2} \geq \frac{3k-5}{2k-3} = \frac{55}{56} > \frac{55}{56} \left(\frac{3}{2}\right)^{k-4}.
\]

It follows that the LHS of (4.16) is greater than \(\frac{55}{56} \left(\frac{3}{2}\right)^{k-4}\). On the other hand, the RHS of (4.16) is less than 2. For \(k \geq 6\) we have \(\frac{55}{56} \left(\frac{3}{2}\right)^{k-4} \geq \frac{55}{56} \left(\frac{3}{2}\right)^2 = \frac{495}{224} > 2\).

Thus (4.16) holds for \(k \geq 6\). For \(k = 5\) and \(1 \leq t \leq 4\), one can check directly that (4.16) holds. Thus the lemma is proven. \(\square\)

5 The proof of Theorem 1.4

In this section we use the method of shifting ad extremis, introduced in Section 2 to obtain the exact value of \(f(n,k,3)\) for \(n > 2k\) and \(k \geq 7\). Before that let us prove the following two inequalities.

**Lemma 5.1.** For \(n > 2k \geq 6\),

\[
(5.1) \quad g(n,k,3) < |\mathcal{G}(n,k)|.
\]

**Proof.** Note that

\[
g(n,k,3) = \binom{n-1}{k-1} - \binom{n-k-2}{k-1} - (k+1)\binom{n-k-2}{k-2} + k + 1.
\]

By (1.4) we have

\[
|\mathcal{G}(n,k)| - g(n,k,3) = k\binom{n-k-2}{k-2} - \binom{n-k}{k-1} - \binom{n-k}{k-1} + \binom{n-k-2}{k-3} - k + 2
\]

\[
= (k-2)\left(\binom{n-k-2}{k-2} - 1\right) - \binom{n-k-3}{k-2} - \cdots - \binom{n-2k}{k-2}
\]

\[
= \sum_{1 \leq i \leq k-2} \left(\binom{n-k-2}{k-2} - 1 - \binom{n-k-2-i}{k-2}\right) > 0. \quad \square
\]

**Lemma 5.2.** For \(n > 2k \geq 6\),

\[
(5.2) \quad |\mathcal{G}(n,k)| > 3\left(\binom{n-4}{k-2} - \binom{n-k-2}{k-2} + 1\right) + 4\binom{n-4}{k-3} + \binom{n-4}{k-4}.
\]

**Proof.** Let \(R = (5,6,\ldots,k+2)\). Consider the following construction.

\[
\mathcal{F}_R = \left\{ P \cup R : P \in \left\{\binom{[2,3,4]}{2}\right\} \right\} \bigcup \left\{ F \in \binom{[n]}{k} : |F \cap [4]| \geq 3 \right\}
\]

\[
\bigcup \left\{ F \in \binom{[2,n]}{k} : F \cap [4] = (1,2) \text{ or } (1,3) \text{ or } (1,4), \quad F \cap R \neq \emptyset \right\}.
\]

It is easy to check that \(\mathcal{F}_R\) is intersecting, initial and \(\tau(\mathcal{F}) = 3\). Moreover,

\[
|\mathcal{F}_R| = 3\left(\binom{n-4}{k-2} - \binom{n-k-2}{k-2} + 1\right) + 4\binom{n-4}{k-3} + \binom{n-4}{k-4}.
\]

Then by (5.1) we conclude that

\[
|\mathcal{F}_R| \leq g(n,k,3) < |\mathcal{G}(n,k)|. \quad \square
\]
Now we are in a position to prove our main theorem.

**Proof of Theorem 1.4.** Assume that \( F \subseteq \binom{[n]}{k} \) is an intersecting family, \( n > 2k, \tau(F) \geq 3 \) and \( |F| \) is maximal. Without loss of generality, assume further that \( F \) is shifted ad extremis with respect to \( \tau(F) \geq 3 \) and let \( \mathcal{H} \) be the corresponding shift-resistant graph. If \( F \) is initial, then by (5.1) \( |F| \leq g(n, k, 3) < |\mathcal{G}(n, k)| \) and we are done. Thus we may assume \( \mathcal{H} \neq \emptyset \).

By Theorem 2.3 we assume that \( T(n) \) is a star. Without loss of generality, suppose that all \( T \in T^{(3)}(F) \) contain \( a \). Let \( \mathcal{A} = F(a), \mathcal{B} = F(a) \) and set \( V = [n] \setminus \{a\} \). By Lemma 2.2 \( \mathcal{A}, \mathcal{B} \) are shifted ad extremis with respect to \( \tau(\mathcal{B}) \geq 2 \) and \( \mathcal{H} \cap (\binom{v}{2}) \) is the corresponding shift-resistant graph. Define the 2-cover graph \( \mathcal{H} \) as

\[
\hat{\mathcal{H}} = \left\{ (i, j) \in \binom{V}{2} : B(i, j) = \emptyset \right\}.
\]

By Claim 3.3 \( \mathcal{H} \cap (\binom{v}{2}) \) is non-empty. Since \( \mathcal{H} \cap (\binom{v}{2}) \) is a subgraph of \( \hat{\mathcal{H}}, \hat{\mathcal{H}} \) is non-empty as well.

Now by Proposition 2.3 either \( \hat{\mathcal{H}} \) contains a triangle or \( \hat{\mathcal{H}} \) is a complete bipartite graph on partition sets \( X \) and \( Y \) where \( X \cup Y \) is the set of the first \(|X| + |Y|\) elements in \( V \), \( 2 \leq |X| \leq k \), \( 2 \leq |Y| \leq k \).

We deal with the two cases separately.

**Proposition 5.3.** If \( \hat{\mathcal{H}} \) contains a triangle, \( n \geq 2k + 1 \) and \( k \geq 5 \), then

\[
|F| = |\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{G}(n, k)|.
\]

Moreover, the equality holds iff \( F = \mathcal{G}(n, k) \) up to isomorphism.

**Proof.** Let \( (u, v, w) \subseteq V \) be a triangle in \( \hat{\mathcal{H}} \) with \( u + v + w \) minimal. By the definition of \( \hat{\mathcal{H}} \), we know that

\[
B(u, v) = B(u, w) = B(v, w) = \emptyset.
\]

By saturatedness \( (u, v), (v, w), (u, w) \) are full in \( \mathcal{A} \).

**Claim 5.4.** For every \( x \in \{u, v, w\} \) there is \( A_x \in \mathcal{A} \) satisfying

\[
(5.3) \quad A_x \cap \{u, v, w\} = \{x\}.
\]

**Proof.** Suppose that (5.3) fails for \( x \). Since \( \{u, v, w\} \setminus \{x\} \) is a cover of \( \mathcal{B} \), \( \tau(F) > 2 \) implies that \( A \cap \{u, v, w\} = \emptyset \) for some \( A \in \mathcal{A} \).

We claim that \( \{z, x\} \) is shiftable for every \( z \in A \). If \( \{z, x\} \) is a shift-resistant pair, then \( B(z, x) = \emptyset \). Since \( (A, B) \) forms a saturated pair, it follows that \( A(z, x) \) is full. Then there are many \( A_x \in \mathcal{A} \) satisfying (5.3). Thus \( \{z, x\} \) must be shiftable.

Should \( z < x \) hold, \( S_{xz}(A) = \mathcal{A} \) and the fullness of each edge of \( \{u, v, w\} \setminus \{x\} \) implies that \( \{u, v, w\} \setminus \{x\} \) is also a triangle. This contradicts the minimality of \( u + v + w \). Thus \( z > x \) for every \( z \in A \).

Now fix \( z \in A \). As \( \{z, x\} \) is shiftable, \( S_{xz}(A) = (A \setminus \{z\}) \cup \{x\} \) is in \( \mathcal{A} \) and satisfies (5.3), concluding the proof. \( \square \)

By the cross-intersection property \( |B \cap \{u, v, w\}| \geq 2 \) for all \( B \in \mathcal{B} \). For notational convenience, set

\[
B_x = B(\{u, v, w\} \setminus \{x\}, \{u, v, w\}) \subseteq \binom{V \setminus \{u, v, w\}}{k - 2}
\]

and

\[
A_x = A(\{x\}, \{u, v, w\}) \subseteq \binom{V \setminus \{u, v, w\}}{k - 2}, \quad x \in \{u, v, w\}.
\]

Note that \( A_x, B_x \) are cross-intersecting. Since \( \mathcal{B} \) is non-trivial, \( B_x \) is non-empty. By (5.3) \( A_x \) is also non-empty. Thus by (1.11) we have

\[
(5.4) \quad |A_x| + |B_x| \leq \binom{n - 4}{k - 2} - \binom{n - k - 2}{k - 2} + 1.
\]
Set

\[ \mathcal{A}_0 = \{ A \in \mathcal{A} : |A \cap \{u, v, w\}| \geq 2 \}. \]

By fullness, we have

\[ |\mathcal{A}_0| = 3\binom{n-4}{k-3} + \binom{n-4}{k-4}. \]

If \( \mathcal{A}(\{u, v, w\}) = \emptyset \), then

\[ |\mathcal{A}(\{u, v, w\})| + |\mathcal{B}(\{u, v, w\})| = |\mathcal{B}(\{u, v, w\})| \leq \frac{n-4}{k-3}. \]

Adding (5.5), (5.6) and (5.4) for \( A \)

**Case 1.**

We distinguish two cases.

If \( \mathcal{B}(\{u, v, w\}) = \emptyset \) and thereby

\[ |B(u, v, w) = \emptyset \] and thereby

\[ |B(u, v, w) = \emptyset \]

Then let \( \mathcal{B}^* \subseteq \mathcal{B} \). The non-triviality of \( \mathcal{B} \) implies the non-triviality of \( \mathcal{B}^* \) and thereby \( |\mathcal{B}^*| \geq 2 \). By (1.8), we obtain that

\[ \sum_{u, v, w} (|A_u| + |B_v|) \leq 4\binom{n-4}{k-3} + \binom{n-4}{k-4} + 3\left( \binom{n-4}{k-2} - \binom{n-k-2}{k-2} + 1 \right) \]

and we are done. Thus in the rest of the proof we may assume that \( \mathcal{A}(\{u, v, w\}) \neq \emptyset \).

**Claim 5.5.** If \( \mathcal{A}(\{u, v, w\}) \neq \emptyset \), then

\[ |\mathcal{A}(\{u, v, w\})| + |\mathcal{B}(\{u, v, w\})| \leq \frac{n-4}{k-1} - 2\frac{n-k-2}{k-1} + \frac{n-k-3}{k-1}. \]

Moreover, the equality holds iff there exist disjoint sets \( S, T \in \binom{\{u, v, w\}}{k-2} \)

such that \( \mathcal{B}_u \cup \mathcal{B}_v \cup \mathcal{B}_w = \{S, T\} \) and \( \mathcal{A}(\{u, v, w\}) = \{E \in \binom{\{u, v, w\}}{k-1} : E \cap S \neq \emptyset, E \cap T \neq \emptyset\} \).

**Proof.** We distinguish two cases.

**Case 1.** \( \mathcal{A}(\{u, v, w\}) \) is non-trivial.

If \( \mathcal{B}(\{u, v, w\}) = \emptyset \) then let \( \mathcal{B}^* = \mathcal{B}_u \cup \mathcal{B}_v \cup \mathcal{B}_w \). The non-triviality of \( \mathcal{B} \) implies the non-triviality of \( \mathcal{B}^* \) and thereby \( |\mathcal{B}^*| \geq 2 \). By (1.8), we obtain that

\[ |\mathcal{A}(\{u, v, w\})| \leq |\mathcal{A}(\{u, v, w\})| + |\mathcal{B}^*| - 2 \leq \left( \frac{n-4}{k-1} - 2\frac{n-k-2}{k-1} + \frac{n-k-3}{k-1} \right). \]

Moreover, the equality holds iff there exist disjoint sets \( S, T \in \binom{\{u, v, w\}}{k-2} \)

such that \( \mathcal{B}^* = \{S, T\} \) and \( \mathcal{A}(\{u, v, w\}) = \{E \in \binom{\{u, v, w\}}{k-1} : E \cap S \neq \emptyset, E \cap T \neq \emptyset\} \).

If \( \mathcal{B}(\{u, v, w\}) \neq \emptyset \) then let

\[ \mathcal{B}^* = \left\{ D \in \binom{V \setminus \{u, v, w\}}{k-2} : \exists B \in \mathcal{B}(\{u, v, w\}), B \subseteq D \right\} \] and \( \hat{\mathcal{B}} = \mathcal{B}^* \cup \hat{\mathcal{B}}. \)

Clearly \( |\mathcal{B}^*| \geq n-4-(k-3) \). We claim that \( |\hat{\mathcal{B}}| \geq |\mathcal{B}(\{u, v, w\})| + 2 \). For \( n \geq 2k + 1 \), we have \( n-4-(k-3) \geq k \). Consequently for \( |\mathcal{B}(\{u, v, w\})| \leq k-2 \), we have

\[ |\hat{\mathcal{B}}| \geq |\mathcal{B}^*| \geq k \geq |\mathcal{B}(\{u, v, w\})| + 2. \]

From now on assume \( |\mathcal{B}(\{u, v, w\})| \geq k-1 \). By Sperner’s argument [16],

\[ |\mathcal{B}^*| \geq |\mathcal{B}(\{u, v, w\})| \frac{n-4-(k-3)}{k-2}. \]

Set \( t = |\mathcal{B}(\{u, v, w\})| \). Since \( n-4-(k-3) \geq k \), it is sufficient to show

\[ \frac{tk}{k-2} \geq t + 2 \] or equivalently \( 2t \geq 2(k-2) \).
which follows from \( t \geq k - 1 > k - 2 \). Thus \( |B| \geq |B(\{u, v, w\})| + 2 \geq 3 \).

Note that \( \mathcal{A}(\{u, v, w\}) \) and \( \mathcal{B} \) are cross-intersecting and both non-trivial. By (1.8) and \( |B| \geq 3 \), we obtain

\[
|\mathcal{A}(\{u, v, w\})| + |\mathcal{B}(\{u, v, w\})| \leq |\mathcal{A}(\{u, v, w\})| + |B| - 2 < \left( \frac{n - 4}{k - 1} \right) - 2 \left( \frac{n - k - 2}{k - 1} \right) + \left( \frac{n - 2k}{k - 1} \right).
\]

Case 2. \( \mathcal{A}(\{u, v, w\}) \) is a star.

Let \( z \in \cap \mathcal{A}(\{u, v, w\}) \). Clearly, \( |\mathcal{B}(\{u, v, w, z\})| \leq \left( \frac{n - 5}{k - 2} \right) \). If \( \mathcal{B}(\{u, v, w, z\}) \neq \emptyset \), then by (1.11) we have

\[
|\mathcal{A}(\{u, v, w\}, z)| + |\mathcal{B}(\{u, v, w\}, z)| \leq \left( \frac{n - 5}{k - 2} \right) - \left( \frac{n - 5 - (k - 3)}{k - 2} \right) + 1.
\]

It follows that

\[
|\mathcal{A}(\{u, v, w\})| + |\mathcal{B}(\{u, v, w\})| = |\mathcal{A}(\{u, v, w\}, z)| + |\mathcal{B}(\{u, v, w, z\})| \leq \left( \frac{n - 5}{k - 2} \right) - \left( \frac{n - 5 - (k - 3)}{k - 2} \right) + 1 + \left( \frac{n - 5}{k - 4} \right).
\]

If \( \mathcal{B}(\{u, v, w, z\}) = \emptyset \), then \( |\mathcal{B}(\{u, v, w\})| = |\mathcal{B}(\{u, v, w, z\})| \leq \left( \frac{n - 5}{k - 2} \right) \). Since \( \mathcal{B} \) is non-trivial, we may choose \( B_z \in \mathcal{B} \) such that \( z \notin B_z \). By the cross-intersection property, \( A \cap B_z \neq \emptyset \) for all \( A \in \mathcal{A}(\{u, v, w\}) \). Note that \( |B_z \cap \{u, v, w\}| \geq 2 \). Thus,

\[
|\mathcal{A}(\{u, v, w\})| + |\mathcal{B}(\{u, v, w\})| = |\mathcal{A}(\{u, v, w\}, z)| + |\mathcal{B}(\{u, v, w, z\})| \leq \left( \frac{n - 5}{k - 2} \right) - \left( \frac{n - 5 - |B_z \setminus \{u, v, w\}|}{k - 2} \right) + \left( \frac{n - 5}{k - 4} \right) \leq \left( \frac{n - 5}{k - 2} \right) - \left( \frac{n - 5 - (k - 3)}{k - 2} \right) + \left( \frac{n - 5}{k - 4} \right) \leq \left( \frac{n - 4}{k - 1} \right) - 2 \left( \frac{n - k - 2}{k - 1} \right) + \left( \frac{n - 2k}{k - 1} \right).
\]

Adding (5.13), (5.14) for \( x = u, v, w \) and (5.17), we arrive at

\[
|A| + |B| = |A_0| + \sum_{x \in \{u, v, w\}} (|A_x| + |B_x|) + |\mathcal{A}(\{u, v, w\})| + |\mathcal{B}(\{u, v, w\})| \leq 3 \left( \frac{n - 4}{k - 3} \right) + \left( \frac{n - 4}{k - 4} \right) + 3 \left( \frac{n - 4}{k - 2} \right) - 3 \left( \frac{n - k - 2}{k - 2} \right) + 3 + \left( \frac{n - 4}{k - 1} \right) - 2 \left( \frac{n - k - 1}{k - 1} \right) + \left( \frac{n - 2k}{k - 1} \right) + \left( \frac{n - 4}{k - 1} \right) - \left( \frac{n - k - 1}{k - 1} \right) - \left( \frac{n - k - 2}{k - 1} \right) + \left( \frac{n - 2k}{k - 1} \right) + \left( \frac{n - 2k}{k - 3} \right) + 3 = |G(n, k)|.
\]

The equality holds if the equalities hold in (5.7) and (5.14) for \( x = u, v, w \). By Claim 5.3, \( \mathcal{B}(\{u, v, w\}) = \emptyset \) and there exist disjoint sets \( S, T \in \binom{V \setminus \{u, v, w\}}{k - 2} \) such that \( B_u \cup B_v \cup B_w = \{S, T\} \) and \( \mathcal{A}(\{u, v, w\}) = \).
\( \{E \in \binom{V \setminus \{u,v,w\}}{k-1} : E \cap S \neq \emptyset, E \cap T \neq \emptyset \} \). By Theorem 1.10 and \(|\mathcal{B}_u \cup \mathcal{B}_v \cup \mathcal{B}_w| = 2\), we infer that 
\(|\mathcal{B}_u| = |\mathcal{B}_v| = |\mathcal{B}_w| = 1\). Without loss of generality, assume that
\[ \mathcal{B} = \{S \cup \{u,v\}, S \cup \{u,w\}, T \cup \{v,w\}\}. \]

Then
\[ \mathcal{A} = \left\{ A \in \binom{V}{k-1} : A \cap B \neq \emptyset \text{ for all } B \in \mathcal{B} \right\}. \]

Thus the proposition is proven. \( \square \)

**Proposition 5.6.** Suppose that \( n > 2k \) and \( k \geq 5 \). If \( \hat{H} \) is a complete bipartite graph on partite sets \( X \) and \( Y \) where \( X \cup Y \) consists of the first \( |X| + |Y| \) elements in \( V \), \( 2 \leq |X| \leq k, 2 \leq |Y| \leq k \), then
\[ |\mathcal{F}| = |\mathcal{A}| + |\mathcal{B}| < |\mathcal{G}(n, k)|. \]

**Proof.** Assume that \( X = \{x_1, \ldots, x_p\}, Y = \{y_1, \ldots, y_q\} \) and \( \{x_1, \ldots, x_p, y_1, \ldots, y_q\} \) is the set of first \( p + q \) elements in \( V \). Define
\[ \mathcal{A}_0 = \left\{ A \in \binom{V}{k-1} : A \cap X \neq \emptyset, A \cap Y \neq \emptyset \right\}. \]

By Claim 2.9 we know that for any \( B \in \mathcal{B} \) either \( X \subseteq B \) or \( Y \subseteq B \). By saturatedness, we have
\[ |\mathcal{A}_0| = \binom{n-1}{k-1} - \binom{n-p-1}{k-1} - \binom{n-q-1}{k-1} + \binom{n-p-q-1}{k-1}. \]

**Claim 5.7.** \( p + q \geq k + 2 \).

**Proof.** Suppose for contradiction that \( p + q \leq k + 1 \). Let \( K \) be the set of first \( k + 1 \) elements in \( V \). We first show that \( \binom{K}{k} \subseteq B \). Since \( B \) is shifted on \( V \setminus X \) and \( V \setminus Y \), it is sufficient to show that \( K \setminus \{x_1\} \) and \( K \setminus \{y_1\} \) are in \( B \), where \( x_1 \) (\( y_1 \)) is the smallest element of \( X \) (\( Y \)), respectively. By symmetry consider \( x_1 \). By non-triviality of \( B \), there exists \( B \in \mathcal{B} \) with \( x_1 \notin B \). This implies \( B \cap Y = \emptyset \). Now \( B \cap Y \in \binom{V \setminus Y}{k-1} \). Set \( B_1 = K \setminus \{x_1\} \). By \( Y \subseteq X \cup Y \subseteq K \), \( Y \subseteq B_1 \), we infer that \( B_1 \setminus Y \in \binom{V \setminus Y}{k-q} \) and \( B_1 \setminus Y \subset B \). By shiftedness on \( V \setminus Y \), \( B_1 \in \mathcal{B} \). Combining with \( K \setminus \{y_1\} \in B \), \( \binom{K}{k} \subseteq B \) follows.

Recall that \( H \cap \binom{V}{2} \neq \emptyset \). Let \( (i, j) \in H \cap \binom{V}{2} \). By Claim 5.5 we know that \( S_{ij}(B) \) is a star. Since \( H \cap \binom{V}{2} \) is a subgraph of \( \hat{H} \), \( (i, j) \in \hat{H} \). It follows that \( (i, j) \in X \times Y \), i.e., either \( i \in X, j \in Y \) or \( i \in Y, j \in X \). In any case, \( i < j \) and \( i, j \in X \cup Y \subseteq K \). Consequently \( K \setminus \{j\} \) and \( K \setminus \{i\} \) are in \( B \). Thus \( K \setminus \{i, j\} \in B \cap B(i) \cap B(j) \), contradicting the fact that \( S_{ij}(B) \) is a star. \( \square \)

**Claim 5.8.** If \( p \leq k - 1 \) then \( \mathcal{B}(X) \) is non-trivial. If \( q \leq k - 1 \) then \( \mathcal{B}(Y) \) is non-trivial.\( \square \)

**Proof.** Suppose the contrary and let \( z \in \hat{B} \) for all \( B \in \mathcal{B}(X) \). Note that \( z \notin Y \). Indeed the opposite would mean \( z \in B \) for all \( B \in \mathcal{B} \), i.e., \( \cap \mathcal{B} \neq \emptyset \), contradiction.

Thus \( z \notin Y \). Consequently \( \{z, y\} \) is a transversal of \( B \) for all \( y \in Y \). That is, \( (X \cup \{z\}) \times Y \) should be the 2-cover graph \( \hat{H} \), contradiction. \( \square \)

Now we distinguish two cases.

**Case 1.** \( p, q \leq k - 1 \).

Consider \( \mathcal{A}(\overline{X}) \subset \binom{V \setminus X}{k-1} \) and \( \mathcal{B}(X) \subset \binom{V \setminus X}{k-1} \). Using \( p, q \leq k - 1 \), Claim 5.7 implies \( p, q \geq 3 \).

By Claim 5.8 both \( \mathcal{B}(X) \) and \( \mathcal{B}(Y) \) are non-trivial. Note that \( \mathcal{B}(X) \) is \((k-p)\)-uniform, \( \mathcal{A}(\overline{X}) \) is \((k-1)\)-uniform. Since \( (k-1) \geq (k-p) + 2 \) and \( n - p - 1 \geq (k-1) + (k-p) \), by Proposition 1.10 we infer
\[ |\mathcal{A}(\overline{X})| + |\mathcal{B}(X)| \leq \binom{n-p-1}{k-1} - 2\binom{n-k-1}{k-1} + \binom{n-2k+p-1}{k-1} + 2, \]
Similarly, we have
\[(5.10) \quad |\mathcal{A}(\overline{Y})| + |\mathcal{B}(Y)| \leq \binom{n - q - 1}{k - 1} - 2\binom{n - k - 1}{k - 1} + \binom{n - 2k + q - 1}{k - 1} + 2.\]

By adding (5.8), (5.9) and (5.10), we arrive at
\[|\mathcal{A}| + |\mathcal{B}| \leq |\mathcal{A}_0| + |\mathcal{A}(\overline{X})| + |\mathcal{B}(X)| + |\mathcal{A}(\overline{Y})| + |\mathcal{B}(Y)| \leq \binom{n - 1}{k - 1} + \binom{n - p - q - 1}{k - 1} - 4 \binom{n - k - 1}{k - 1} + \binom{n - 2k + p - 1}{k - 1} + \binom{n - 2k + q - 1}{k - 1} + 4.\]

Let \(p + q = x\) and define
\[f(x, p) = \binom{n - 1}{k - 1} + \binom{n - x - 1}{k - 1} - 4 \binom{n - k - 1}{k - 1} + \binom{n - 2k + p - 1}{k - 1} + \binom{n - 2k + x - p - 1}{k - 1} + 4.\]

Since
\[
\binom{x}{\ell - 1} = \frac{(x - 1)(x - 2)\cdots(x - \ell + 1)}{(\ell - 1)!} < \frac{x(x - 1)\cdots(x - \ell + 2)}{(\ell - 1)!} = \binom{x}{\ell},
\]
we see \(\binom{x}{\ell - 1} < \binom{x}{\ell}\) for \(x > \ell - 2\). It follows that \(2\binom{x}{\ell} < \binom{x}{\ell - 1} + \binom{x}{\ell + 1}\). Hence \(2f(x, p) < f(x, p + 1) + f(x, p - 1)\) for fixed \(x\) and \(x - k + 1 < p < k - 1\). By symmetry we have \(f(x, k - 1) = f(x, x - k + 1)\). Therefore,
\[f(x, p) \leq \max\{f(x, k - 1), f(x, x - k + 1)\} = f(x, k - 1).\]

Note also that \(2f(x, p) < f(x - 1, p) + f(x + 1, p)\) for fixed \(p\) and \(k + 1 < k + 2 \leq x = p + q \leq 2k - 2\), we deduce that
\[f(x, p) \leq \max\{f(k + 1, k - 1), f(2k - 2, k - 1)\}.\]

Note that
\[f(k + 1, k - 1) = \binom{n - 1}{k - 1} + \binom{n - k - 2}{k - 1} - 4 \binom{n - k - 1}{k - 1} + \binom{n - 2k + 1}{k - 1} + 4
= \binom{n - 1}{k - 1} + \binom{n - k - 2}{k - 1} - \binom{n - k - 1}{k - 1} + \binom{n - 2k}{k - 1} - 2 \binom{n - k - 1}{k - 1} + \binom{n - 2k}{k - 1} + \binom{n - k - 2}{k - 2} + 4
= |\mathcal{G}(n, k)| + \binom{n - 2k}{k - 2} + 1 - \binom{n - k - 2}{k - 2} < |\mathcal{G}(n, k)|\]
and
\[f(2k - 2, k - 1) = \binom{n - 1}{k - 1} + \binom{n - 2k + 1}{k - 1} - 4 \binom{n - k - 1}{k - 1} + 2 \binom{n - k - 2}{k - 1} + 4
= \binom{n - 1}{k - 1} + \binom{n - 2k}{k - 1} + \binom{n - 2k}{k - 2} - \binom{n - k - 1}{k - 1} + \binom{n - k}{k - 1}
+ \binom{n - k - 1}{k - 2} - 2 \binom{n - k - 1}{k - 1} + 2 \binom{n - k - 2}{k - 1} + 4
= |\mathcal{G}(n, k)| + \binom{n - 2k}{k - 2} + 1 - \binom{n - k - 2}{k - 2} < |\mathcal{G}(n, k)|.\]

Thus we conclude that \(|\mathcal{A}| + |\mathcal{B}| < |\mathcal{G}(n, k)|\).

Case 2. \(p = k\).

Then \(X \in \mathcal{B}\) and the intersection property of \(\mathcal{B}\) implies that \(q \leq k - 1\). Since \(\mathcal{A}, \mathcal{B}\) are cross-intersecting, we infer \(\mathcal{A}(\overline{X}) = \emptyset\). It follows that
\[(5.11) \quad |\mathcal{A}(\overline{X})| + |\mathcal{B}(X)| = 1.\]
Claim 5.9.

\[(5.12) \quad |A(Y)| + |B(Y)| \leq \binom{n-q-1}{k-1} - 2 \binom{n-k-1}{k-1} + \binom{n-2k+q-1}{k-1} + 2. \]

Proof. Since 2 \leq q \leq k - 1, by Claim 5.8, we see that \( B(Y) \) is non-trivial. Note that \( B(Y) \) is \((k-q)\)-uniform and \( A(Y) \) is \((k-1)\)-uniform. If \( A(Y) \neq \emptyset \), then by (1.18)

\[ |A(Y)| + |B(Y)| \leq \binom{n-q-1}{k-1} - 2 \binom{n-k-1}{k-1} + \binom{n-2k+q-1}{k-1} + 2. \]

Moreover, the equality holds iff \( B(Y) \) consists of two disjoint sets \( S, T \in \binom{V \setminus Y}{k-q} \). Then \( B = \{ X, S \cup Y, T \cup Y \} \). By the intersection property of \( B \), there exist \( x_i \in S \cap X \) and \( x_j \in T \cap X \). But then \( \{ x_i, x_j \} \) is a 2-cover of \( B \), contradicting \( \hat{H} = X \times Y \). Thus the inequality is strict and \( 5.12 \) holds.

If \( A(Y) = \emptyset \), since \( B \) is intersecting, then

\[ |A(Y)| + |B(Y)| = |B(Y)| \leq \binom{n-q-1}{k-1} - \binom{n-k-q-1}{k-q} \leq \binom{n-q-1}{k-1} - \binom{n-k-q-1}{k-q} \leq \binom{n-q-1}{k-1} - \binom{n-k-q-1}{k-q} + 2. \]

By adding (5.8) with \( p = k \), (5.11) and (5.12), we arrive at

\[ |A(X)| + |B(Y)| \leq |A_0| + |A(Y)| + |B(X)| + |A(Y)| + |B(Y)| \leq \binom{n-1}{k-1} + \binom{n-k-q-1}{k-1} - 3 \binom{n-k-1}{k-1} + \binom{n-2k+q-1}{k-1} + 3. \]

Let

\[ f(q) = \binom{n-1}{k-1} + \binom{n-k-q-1}{k-1} - 3 \binom{n-k-1}{k-1} + \binom{n-2k+q-1}{k-1} + 3. \]

Note that \( 2f(q) < f(q+1) + f(q-1) \) for \( 2 < q < k - 1 \). We infer \( |A(X)| + |B(Y)| \leq \max\{ f(2), f(k-1) \} \).

Let us show that for \( k \geq 4 \), \( f(2) < |G(n,k)| \).

\[ f(2) = \binom{n-1}{k-1} + \binom{n-k-3}{k-1} - 3 \binom{n-k-1}{k-1} + \binom{n-2k+1}{k-1} + 3 \]

\[ = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-1}{k-1} + \binom{n-k-3}{k-1} - \binom{n-k-1}{k-1} \]

\[ + \binom{n-2k}{k-1} + \binom{n-2k}{k-2} + 3 \]

\[ = |G(n,k)| + \binom{n-k-1}{k-2} + \binom{n-k-3}{k-1} - \binom{n-k-1}{k-1} + \binom{n-2k}{k-2} - \binom{n-k-2}{k-3} \]

\[ = |G(n,k)| + \binom{n-2k}{k-2} - \binom{n-k-3}{k-2} < |G(n,k)|. \]

Let us show next \( f(k-1) = |G(n,k)| \).

\[ f(k-1) = \binom{n-1}{k-1} + \binom{n-2k}{k-1} - 3 \binom{n-k-1}{k-1} + \binom{n-k-2}{k-1} + 3 \]

\[ = \binom{n-1}{k-1} + \binom{n-2k}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-1}{k-1} + \binom{n-k-2}{k-1} - \binom{n-k-1}{k-1} \]

\[ + \binom{n-k-2}{k-1} + 3 \]

\[ = |G(n,k)| + \binom{n-k-1}{k-2} - \binom{n-k-1}{k-1} + \binom{n-k-2}{k-1} - \binom{n-k-2}{k-3} \]

\[ = |G(n,k)| + \binom{n-k-2}{k-2} - \binom{n-k-1}{k-1} + \binom{n-k-2}{k-1} = |G(n,k)|. \]
Thus we obtain that $|A| + |B| < |G(n, k)|$ and the proposition is proven. □

Combining Propositions 5.3 and 5.6 we conclude that the theorem holds. □

References

[1] S. Chiba, M. Furuya, R. Matsubara, M. Takatou, Covers in 4-uniform intersecting families with covering number three, Tokyo Journal of Mathematics (1) 35 (2012), 241–251.

[2] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 12 (1961), 313–320.

[3] P. Erdős, L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, in “Infinite and Finite Sets,” Proc. Colloquium Math. Society Janos Bolyai, Vol. 10, (A. Hajnal et al., Eds.), pp. 609–627, North-Holland, Amsterdam, 1975.

[4] P. Frankl, On intersecting families of finite sets, Bull. Austral. Math. Soc. 21 (3) (1980), 363–372.

[5] P. Frankl, The shifting technique in extremal set theory, Surveys in Combinatorics 123 (1987), 81–110.

[6] P. Frankl, Erdős-Ko-Rado theorem for a restricted universe, Electron. J. Combin. (2) 2020: #18.

[7] P. Frankl, On the maximum of the sum of the sizes of non-trivial cross-intersecting families, arXiv:2209.01826, 2022.

[8] P. Frankl, K. Ota, N. Tokushige, Uniform intersecting families with covering number four, J. Combin. Theory, Ser. A 71 (1) (1995), 127–145.

[9] P. Frankl, K. Ota, N. Tokushige, Covers in uniform intersecting families and a counterexample to a conjecture of Lovász, J. Comb. Theory, Ser. A 74(1) (1996), 33–42.

[10] P. Frankl, N. Tokushige, Some best possible inequalities concerning cross-intersecting families, J. Comb. Theory, Ser. A 61 (1992), 87–97.

[11] P. Frankl, J. Wang, A product version of the Hilton-Milner Theorem, arXiv:2206.07218, 2022.

[12] M. Furuya, M. Takatou, Covers in 5-uniform intersecting families with covering number three, Australas. J Comb. 55 (2013), 249–262.

[13] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, Q. J. Math. 18 (1967), 369–384.

[14] A. Kupavskii, Structure and properties of large intersecting families, arXiv:1810.00920, 2018.

[15] L. Lovász, On minimax theorems of combinatorics, Math. Lapok 26 (1975), 209–264.

[16] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Zeitschrift 27 (1928), 544–548.