The robustness of a many-body decoherence formula of Kay under changes in graininess and shape of the bodies

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Abstract. In “Decoherence of macroscopic closed systems within Newtonian quantum gravity” (Kay B S 1998 Class. Quantum Grav. 15 L89-L98) it was argued that, given a many-body Schrödinger wave function \( \psi(x_1, \ldots, x_N) \) for the centre-of-mass degrees of freedom of a closed system of \( N \) identical uniform-mass balls of mass \( M \) and radius \( R \), taking account of quantum gravitational effects and then tracing over the gravitational field amounts to multiplying the position-space density matrix \( \rho(x_1, \ldots, x_N; x'_1, \ldots, x'_N) = \psi(x_1, \ldots, x_N)\psi^*(x'_1, \ldots, x'_N) \) by a multiplicative factor, which, if the positions \( \{x_1, \ldots, x_N; x'_1, \ldots, x'_N\} \) are all much further away from one another than \( R \), is well-approximated by

\[
\left( \prod_K \left( \frac{\|x_K - x'_K\|}{R} \right) \right) \prod_{I<J} \left( \frac{\|x'_I - x'_J\|}{\|x_I - x_J\|} \right) = 24M^2.
\]

Here we show that if each uniform-mass ball is replaced by a grainy ball or more general-shaped lump of similar size consisting of a number, \( n \), of well-spaced small balls of mass \( m \) and radius \( r \) and, in the above formula, \( R \) is replaced by \( r \), \( M \) by \( m \) and the products are taken over all \( Nn \) positions of all the small balls, then the result is well-approximated by replacing \( R \) in the original formula by a new value \( R_{\text{eff}} \). This suggests that the original formula will apply in general to physically realistic lumps – be they macroscopic lumps of ordinary matter with the grains atomic nuclei etc. or be they atomic nuclei themselves with their own (quantum) grainy substructure – provided \( R \) is chosen suitably. In the case of a cubical lump consisting of \( n = (2\ell + 1)^3 \) small balls \( (\ell \geq 1) \) of radius \( r \) with centres at the vertices of a cubic lattice of spacing \( a \) (assumed to be very much bigger than \( 2r \)) and side \( 2\ell a \) we establish the bound

\[
e^{-1/3}(r/a)^{1/n} \ell a \leq R_{\text{eff}} \leq 2\sqrt{3}(r/a)^{1/n} \ell a.
\]

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1. Introduction and Results

In [1], one of us argued that several of the puzzles inherent in our current understanding of quantum and quantum-gravitational physics would appear to find a natural resolution if one were to postulate that quantum gravity is a quantum theory of a conventional type with a total Hilbert space $H_{\text{total}}$ which arises as a tensor product

$$H_{\text{total}} = H_{\text{matter}} \otimes H_{\text{gravity}}$$

of a matter and a gravity Hilbert space, and with a total time-evolution which is unitary, but that, while, as would usually be assumed in a standard quantum theory, one still assumes that there is an ever-pure time-evolving “underlying” state, modelled by a density operator of form

$$\rho_{\text{total}} = |\Psi\rangle\langle\Psi|,$$

$\Psi \in H_{\text{total}}$, at each “instant of time”, one adds the new assumption that the physically relevant density operator is not this underlying density operator, but rather its partial trace, $\rho_{\text{matter}}$, over $H_{\text{gravity}}$. With these assumptions, an initial underlying state of a closed quantum gravitational system with a low degree of matter-gravity entanglement would be expected to become more and more entangled as time increases – equivalently $\rho_{\text{matter}}$ will be subject to an ever-increasing amount of decoherence. In fact the von-Neumann entropy of $\rho_{\text{matter}}$ would be expected to increase monotonically, thus (when the theory is applied to a model for the universe as a whole) offering an explanation for the Second Law of Thermodynamics and (when the theory is applied to a model closed system consisting of a black hole sitting in an otherwise empty universe) offering a resolution to the Information Loss Puzzle. (It offers both of these things in that, by defining the physical entropy of $\rho_{\text{total}}$ to be the von-Neumann entropy of $\rho_{\text{matter}}$, one reconciles an underlying unitary time-evolution on $H_{\text{total}}$ with an entropy which varies/increases in time.)

A second paper, [2], by the same one of us investigated the implications of the theory proposed in [1] for the decoherence of ordinary matter in the non-relativistic, weak-gravitational-field regime. In particular, [2] addressed the question of the appropriate description of the state, at some instant of time, of the center-of-mass degrees of freedom of a system of $N$ bodies, which, for simplicity, were taken to be identical. In ordinary quantum mechanics, such a state would be described by a many-body Schrödinger wave function $\psi(x^1, \ldots, x^N)$ in the usual matter Hilbert space $H_{\text{matter}}$, consisting of the (appropriately symmetrized, according to whether the bodies are treated as fermions or bosons or neither) $N$-fold tensor product of $L^2(\mathbb{R}^3)$. [2] argues that such a $\psi$ needs to be replaced by a total (entangled) matter-gravity state vector $|\Psi\rangle$ in a total matter-gravity Hilbert space which is the tensor product of $H_{\text{matter}}$ (defined as above) with a suitable Hilbert space $H_{\text{gravity}}$ for the modes of the quantized linearized gravitational field. Specifically, thinking of this tensor-product Hilbert space as the set of (appropriately symmetrized) square-integrable functions from $\mathbb{R}^{3N}$ to $H_{\text{gravity}}$, $|\Psi\rangle$ is taken, in [2], to
be the function
\[(x_1, \ldots, x_N) \mapsto \psi(x^1, \ldots, x^N)|\gamma(x^1, \ldots, x^N)\]
where \(|\gamma(x^1, \ldots, x^N)|\) is a certain (non-radiative) quantum coherent state (introduced in [2]) of the linearised gravitational field describing the Newtonian gravitational field due to the simultaneous presence of a body centred on each of the locations \(x^1, \ldots, x^N\).

(We remark that, in [2], for the purposes of calculating these coherent states, the bodies are modelled as classical mass-distributions.) In the non-relativistic, weak-gravitational-field regime, the projector \(\rho_{\text{total}} = |\Psi\rangle\langle\Psi|\) onto this \(|\Psi|\), is thus taken to be, to a very good approximation, the correct description of the “underlying” state of quantum gravity describing our many-body system in the sense described in the previous paragraph. The physically relevant density operator is therefore given, in this approximation, by the partial trace \(\rho_{\text{matter}}\) of this \(\rho_{\text{total}}\) over \(H_{\text{gravity}}\) which, in position space is clearly given by

\[
\rho_{\text{matter}}(x_1, \ldots, x_N; x'_1, \ldots, x'_N) = \psi(x_1, \ldots, x_N)\psi^*(x'_1, \ldots, x'_N)\mathcal{M}(x_1, \ldots, x_N; x'_1, \ldots, x'_N)
\]

where the multiplicative factor \(\mathcal{M}(x_1, \ldots, x_N; x'_1, \ldots, x'_N)\) is defined by

\[
\mathcal{M}(x_1, \ldots, x_N; x'_1, \ldots, x'_N) = \langle \gamma(x_1, \ldots, x_N)|\gamma(x'_1, \ldots, x'_N)\rangle.
\]

(In [2], \(\mathcal{M}\) was written \(e^{-D}\) where \(D\) was called the *decoherence exponent.*) Moreover, it was shown in [2] that, in the case the bodies are taken to be balls with constant mass density and (we work in Planck units where \(G = c = \hbar = 1\)) total mass \(M\) and radius \(R\) (i.e. in the case the classical mass distributions representing the bodies are taken to be such balls) and in the case all the positions \(\{x_1, \ldots, x_N; x'_1, \ldots, x'_N\}\) of their centres of mass are much further away from one another than \(R\) (we shall call this the well-spaced regime)\(\dagger\), the multiplicative factor is well approximated by \(\mathcal{M}_a\) where \(\mathcal{M}_a\) is given by the explicit formula

\[
\mathcal{M}_a(x_1, \ldots, x_N; x'_1, \ldots, x'_N) = \prod_{I=1}^{N} \prod_{J=1}^{N} \left(\frac{|x'_I - x_J|}{|x_I - x'_J|}\right)^{-12M^2}
\]

where it is to be understood that, in the cases \(I = J\), the terms in the denominator \(|x_I - x_J||x'_I - x'_J|\) are to be replaced by \(R^2\).

To summarize, taking into account gravitational effects and then tracing over the gravitational field, as the general theory of [1] prescribes that we should do, has, in the non-relativistic weak-gravitational-field regime, and on the assumption that the bodies are uniform mass balls, an overall effect which is equivalent to multiplying the usual quantum mechanical (pure) position-space density matrix \(\psi(x_1, \ldots, x_N)\psi^*(x'_1, \ldots, x'_N)\)

\(\dagger\) In many applications, one expects the region of (configuration space) \(\times\) (configuration space) where this condition doesn’t hold to be so small in comparison to size of the region where \(\psi^*\psi\) is significantly big that no significant correction would be needed to results calculated on the assumption that the multiplicative factor is well-approximated by the \(\mathcal{M}_a\) in all of (configuration space) \(\times\) (configuration space).
by a multiplicative factor, $\mathcal{M}$, which, in the well-spaced regime, is well-approximated by $\mathcal{M}_a$ given by the formula (1) and it is the product of this pure density matrix with the multiplicative factor which is to be regarded as the physically relevant density operator.

We remark that the formula (1) can also be written

$$
\mathcal{M}_a(x_1, \ldots, x_N; x'_1, \ldots, x'_N) = \left( \prod_K \left( \frac{|x_K - x'_K|}{R} \right) \prod_{I<J} \left( \frac{|x'_I - x'_J||x_I - x'_J|}{|x_I - x_J||x'_I - x'_J|} \right) \right)^{-24M^2}
$$

where the first product is taken over all $K$ from 1 to $N$ and the second product is taken over all $I$ and $J$ from 1 to $N$, which satisfy $I < J$. We also remark that, in the case of a wave function for a single ball state, this prescription amounts to multiplying the density operator $\psi(x)\psi(x')$ by the multiplicative factor

$$
\mathcal{M}_a(x; x') = (|x - x'|/R)^{-24M^2}.
$$

The problems we wish to address in the present paper are that, in the derivation, [2], of (1), the bodies are assumed to be constant mass-density balls whereas in applications, the bodies one is interested in will actually typically be “grainy”, and also not necessarily spherical. In fact, in one application (cf. the discussion of the Schrödinger-cat-like states in [2]) of the above formulae, the balls are interpreted as macroscopic balls of ordinary matter and real ordinary matter is of course grainy in that it is made out of atoms etc. and we may also be interested in lumps of ordinary matter with shapes other than balls. In another application, the above formulae (or rather their obvious generalization to states of many bodies where the masses and radii are not all equal) are interpreted as telling us how the standard non-relativistic quantum mechanics of closed systems of large numbers of atomic nuclei and electrons etc. gets modified according to the theory developed in [1] and [2]. In this latter application, the balls are taken to be models of nuclei and electrons etc. and again, of course, real nuclei and electrons are not actually uniform density balls, but will have their own grainy substructure and also need not be spherical.

However one might hope that if constant-density balls were replaced by grainy, and possibly non-spherical, (classical) “lumps”, then the formula (1) for the multiplicative factor might nevertheless remain approximately correct provided we replace $R$ in (1) by a suitable effective radius $R_{\text{eff}}$. One might further hope that, in the case of a grainy ball, $R_{\text{eff}}$ would turn out to be of the same order of magnitude as $R$ and in the case of a grainy lump of some other shape, to be of the same order of magnitude as some measure of the lump’s typical linear size. (We shall refer below to the lump’s diameter without attempting to give a precise general definition to this notion.) If these hopes were fulfilled, then one might say that the formula (1) for $\mathcal{M}_a$ is robust under (classical) changes in graininess and in shape.

We remark that we are continuing to assume here that our grainy, non-spherical, situations can, as far as their gravitational effects are concerned, still be modelled within the formalism of [2], as classical (albeit now no longer uniform and no longer spherical) mass distributions and we admit that, in principle, we should of course presumably work
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within an extension of the formalism of Kay which allows for truly quantum (and also relativistic) descriptions of graininess (and non-sphericity). (Especially, such a quantum relativistic description may well be important in the second of our applications, to the grainy substructure of the proton etc.) While we shall not attempt to explore such quantum notions of graininess and non-sphericity in this paper, if robustness holds in the above sense under classical changes in graininess and in shape, it might also seem reasonable to guess that our formulae will remain robust even if graininess and changes in shape were taken into account in such a correct quantum way.

In any case, the purpose of the present paper is to investigate whether and/or under what circumstances, these hopes are fulfilled in the case of one specific class of classical grainy models. Namely, models in which every one of the balls of mass \( M \) and radius \( R \) in formula (1) is replaced by a collection of \( n \) little balls, each of mass \( m \) and radius \( r \) fixed at definite positions inside a lump-shaped region (i.e. some smooth closed bounded region of \( \mathbb{R}^3 \)) of diameter approximately equal to \( 2R \) so that its centre of mass is located where the center (i.e. one of the positions \( \mathbf{x}_1, \ldots, \mathbf{x}_N \)) of the ball it replaces was located. We shall assume that the configurations of the balls within each of the lumps are Euclidean-congruent to one-another. However we shall not restrict the relative orientations of the different lumps; they are allowed to be arbitrarily rotated with respect to one another.

We remark that obviously we have \( nm = M \). Here we have in mind choices of values for the various parameters such that

\[
 r \ll \text{any inter-small-ball spacing} \quad \text{and} \quad R \ll \text{any inter-lump spacing},
\]

where, by “any inter-lump spacing” we mean any of the distances between pairs of distinct elements of the set \( \{ \mathbf{x}_1, \ldots, \mathbf{x}_N; \mathbf{x}_1', \ldots, \mathbf{x}_N' \} \) and (assuming from now on that a definite system has been adopted for enumerating the little balls – as \( x_{11}, \ldots, x_{1n} \) etc. – within the 1th lump etc.) by “any inter-small-ball spacing” we mean any of the distances between pairs of distinct elements of the set \( \{ \mathbf{x}_{I1}, \ldots, \mathbf{x}_{In}; \mathbf{x}_{11}', \ldots, \mathbf{x}_{n1}' \} \) for some/any \( I \). Then (in all but an unimportantly small volume of (configuration space) \( \times \) (configuration space) – cf. footnote above) formula (1) will clearly get replaced by

\[
 \mathcal{M}_a^{\text{grainy}}(\mathbf{x}_1, \ldots, \mathbf{x}_N; \mathbf{x}_1', \ldots, \mathbf{x}_N') = \prod_{I=1}^{N} \prod_{i=1}^{n} \frac{\prod_{J=1}^{N} \prod_{j=1}^{n} \left( |\mathbf{x}_{Ii}' - \mathbf{x}_{Jj}|/|\mathbf{x}_{Ii} - \mathbf{x}_{Jj}'| \right)^{-12m^2}}{2} |\mathbf{x}_{Ii} - \mathbf{x}_{Jj}'|/|\mathbf{x}_{Ii}' - \mathbf{x}_{Jj}|(3)
\]

where now it is to be understood that in the cases where \( I = J \) and \( i = j \), the terms in the denominator \( |\mathbf{x}_{Ii} - \mathbf{x}_{Jj}|/|\mathbf{x}_{Ii}' - \mathbf{x}_{Jj}'| \) are to be replaced by \( r^2 \).

In line with (2), if we make the replacements:

\[
 \mathbf{x}_{Ii}' - \mathbf{x}_{Jj} \rightarrow \mathbf{x}_I - \mathbf{x}_J, \quad \text{and} \quad \mathbf{x}_{Ii} - \mathbf{x}_{Ij} \rightarrow \mathbf{x}_I - \mathbf{x}_J \quad \text{except when} \ I = J
\]

(and similarly with primed and unprimed quantities interchanged) in (3), then we clearly expect to get a good approximation\( ^\S \) to \( \mathcal{M}_a^{\text{grainy}} \). Making these replacements, we

\( ^\S \) In fact this will clearly be true in the sense that if we scale the positions of the centres of mass of the lumps while not scaling the lumps themselves, then \( \mathcal{M}_a^{\text{grainy}} / \mathcal{M}_a \) tends to 1 as the scale tends to infinity.
immediately get
\[ M_a^{\text{grainy}}(x_1, \ldots, x_N; x'_1, \ldots, x'_N) \]
\[ \approx \left( \prod_{I} \prod_{j} \frac{|x_I - x'_j|^{n^2}}{|x_I - x_j|^{n^2}} \right)^{-2N} \prod_{i,j} \left( \frac{1}{|x_i - x_j|} \right)^{2N} \]
\[ = \left( \prod_{I \neq J} \frac{|x_I - x_J|^{n^2}}{|x'_I - x'_J|^{n^2}} \right) \prod_{i,j} \left( \frac{1}{|x_i - x_j|} \right)^{2N} \]
where the final product involves all the positions \( x_i \) of little balls inside any single lump – by our assumption of Euclidean congruence, it doesn’t matter which one – and ranges over all values of \( i \) and \( j \) from 1 to \( n \) except that it is to be understood that, in this final product, in the cases \( i = j \) the denominator is to be replaced by \( r^2 \). Clearly another way of saying exactly the same thing as this is that \( M_a^{\text{grainy}} \) is approximately equal to \( M_a \), as given by the formula (1) except that the proviso that, in the cases \( I = J \), the terms in the denominator \( |x_I - x_J| |x'_I - x'_J| \) are to be replaced by \( R^2 \) should be replaced by \( R_{\text{eff}}^2 \) where \( R_{\text{eff}} \) is defined by

\[ R_{\text{eff}} = \left( \prod_{i} \prod_{j} |x_i - x_j| \right)^{\frac{1}{n^2}} \]
where the products each go from 1 to \( n \) and it is to be understood that, in cases where \( i = j \), the terms \( |x_i - x_j| \) are to be replaced by \( r \). Equivalently, we may write

\[ R_{\text{eff}} = \left( r^n \prod_{i<j} |x_i - x_j|^2 \right)^{\frac{1}{n^2}} \]
(4)
(where the product is now over all \( i \) and \( j \) from 1 to \( n \) satisfying \( i < j \)). Thus the first part of our hope (i.e. that replacing our uniform density balls by our grainy lumps can be well-approximated by replacing \( R \) by a suitable \( R_{\text{eff}} \)) is fulfilled.

We now turn to discussing the second part of our hope, namely whether, and/or under what circumstances, \( R_{\text{eff}} \) will turn out to be of the same “order of magnitude” as the diameter of our lump. We have not been able to answer this question in general but, instead, content ourselves with analysing one particularly simple special case of our model. Namely, where each lump is a cube consisting of \( n = (2\ell + 1)^3 \) small balls of radius \( r \) with centres at the vertices of a cubic lattice of spacing \( a \) and side \( 2\ell a \). To spell out precisely what we mean and at the same time set up a useful notation, we assume the ball-centres of any one of these cubes to be coordinatizable so that they lie at the positions \((i_1 a, i_2 a, i_3 a)\) where \( i_1, i_2, \) and \( i_3 \) are integers which each range between 1 and \( 2\ell + 1 \). Clearly the total number of balls, \( n \), in the cube will be \((2\ell + 1)^3\) and we note that there will be a ball at the centre (with coordinates \(((\ell + 1)a, (\ell + 1)a, (\ell + 1)a)\)). In line with (2), we assume

\[ r \ll a \quad \text{and} \quad R \ll \text{any inter-lump spacing}. \]
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(Of course, we require \( r < a/2 \) for the balls to fit into the lattice at all!)

For such a lump, (4) can clearly be rewritten in the form

\[
R_{\text{eff}} = \left( \frac{r}{a} \right)^{1/n} \left( \prod_{i<j} \left| \frac{x_i - x_j}{a} \right|^2 \right)^{1/n} a
\]

(here we continue to assume some system has been adopted for numbering our balls from 1 to \( n = (2\ell + 1)^3 \)) and we notice that, for many values of the pair \((r/a, n)\), the prefactor, \((r/a)^{1/n}\) will itself be of order 1. This will be the case if, as is relevant to a ball or lump of “ordinary matter”, we take \( r/a \) to be of the order of the ratio of the radius of the proton to the Bohr radius (i.e. around \( 10^{-5} \)) even for \( \ell = 1 \) \((n = 27)\) and in fact, if \( \ell = 2 \) \((n = 125)\) or more (and it will be much more if we are taking our cube to be a model for a macroscopic lump of ordinary matter as considered in [2]) then this prefactor will in fact be a number close in value to 1. On the assumption that this prefactor is of order 1 (or very close to 1) our question then reduces to the question whether the quantity

\[
\left( \prod_{i<j} \left| \frac{x_i - x_j}{a} \right|^2 \right)^{1/n} a
\]

is of the same order as \( \ell \). Choosing units such that \( a = 1 \), this amounts to a question about the quantity

\[
Q(\ell) = \left( \prod_{i<j} |x_i - x_j| \right)^{2/n}
\]

where the product is over all pairs of distinct points of the cubic lattice of points with integer coordinates \((i_1, i_2, i_3)\) where \( i_1, i_2 \) and \( i_3 \) range between 1 and \( 2\ell + 1 \) (from now on we shall often simply call such a finite cubic lattice of points a “cube”) and, again, we assume that some system for numbering these points from 1 to \( n = (2\ell + 1)^3 \) has been adopted. We formulate this question in a precise way by asking whether one can find numbers, \( c_1, c_2 \), of order 1 such that

\[
c_1 \ell \leq Q(\ell) \leq c_2 \ell.
\]

It is easy to see that one can satisfy the upper bound by choosing \( c_2 = 2\sqrt{3} \). To see this, notice that each term in the product in (5) is obviously less than or equal to \( 2\sqrt{3} \ell \) since this is the length of the body-diagonal of the cubic lattice and moreover there are \( n(n-1)/2 \) terms in the product. We thus have that \( Q(\ell) \leq (2\sqrt{3}\ell)^{n(n-1)/2n} \) which, since \( \ell \geq 1 \), is clearly less than \( 2\sqrt{3}\ell \).

However, as far as we can see, to show that there exists a (non-zero) positive \( c_1 \) such that \( c_1 \ell \) is a lower bound is not completely trivial because \( Q(\ell) \) is the \( 2/n^2 \) power (i.e. \( 2/(2\ell+1)^3 \) power) of a product of numbers which range in magnitude from 1 to numbers of order \( \ell \) (the largest of the numbers will of course be \( 2\sqrt{3}\ell \)). Nor, to our knowledge, || It is necessary to bear in mind, here and elsewhere, that \( n \) is shorthand for \((2\ell + 1)^3\).
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does it easily follow from any existing standard mathematical result? However, we have succeeded in finding the (albeit probably not best-possible) number \( c_1 = e^{-1/3} \) for which we can prove that it does hold:

**Proposition.**  
\[ e^{-1/3} \ell \leq Q(\ell) \leq 2\sqrt{3} \ell. \]

(Actually, one can show [4], by similar methods to those in the proof of the lower bound below, that if we omit \( \ell = 1 \) the upper bound of \( 2\sqrt{3} \ell \) can be reduced to \( 2.61 \ell \).) We present our proof of the (lower bound in) the proposition in a separate section below.

Putting this proposition together with (5), we thus have the following theorem.

**Theorem.** For a lump consisting of \( n = (2\ell + 1)^3 \) small balls of mass \( m \) and radius \( r \), centred at the vertices of a cubical region (as specified above) of side \( 2\ell a \) of a cubic lattice of spacing \( a \),

\[ e^{-1/3}(r/a)^{1/n} \ell a \leq R_{\text{eff}} \leq 2\sqrt{3}(r/a)^{1/n} \ell a. \]

(Here, we recall that \( a \) is of course assumed to be greater than \( 2r \) and \( R_{\text{eff}} \), when inserted in place of \( R \) in (1), is expected to give a good approximation to the multiplicative factor \( M_a^{\text{grainy}} \) of (3) if \( a \gg 2r \).)

We remark that it is easy to calculate \( Q(\ell) \) numerically for small values of \( \ell \) (say for \( \ell = 1 \ldots 10 \)) and the numerical evidence suggests that, for such small values of \( \ell \), \( Q(\ell) \) is well-approximated by a formula of form \( Q(\ell) \approx A\ell + B \), i.e.

\[ Q(\ell)/\ell \approx A + B/\ell \]  \hspace{1cm} (8)

where \( A \approx 1.2 \) and \( B \approx 0.6 \). If it could be proven that there exist exact values for \( A \) and \( B \) such that this approximate formula holds for all \( \ell \) with an error term which tends to zero as \( \ell \) tends to infinity, then this would of course be a stronger result than our proposition above and would, in particular, tell us that our value for \( c_1 \), \( e^{-1/3} \approx 0.71 \), can be improved to 1.2. However, we have been unable to prove this.

2. Proof of Proposition

We have already proven the upper bound above, so it remains to prove the lower bound.

For this proof, we find it useful to write \( Q(\ell) \) as the product

\[ Q(\ell) = \prod_{i=1}^{n} \omega_i(n) \]

\[ \omega_i(n) = \frac{1}{n(n-1)} \sum_{j \neq i} \frac{1}{r(j-i)} \]

As far as we are aware, the notion of closest relevance to this question in the mathematical literature is the notion [3] of the transfinite diameter of a closed bounded region (which one shows, see again [3], is equivalent to another notion known as the capacity of that region) of the (complex) plane. This is defined to be the supremum over all finite sets of points in the region of the \( 1/n(n-1) \) power of the product of all the distances between pairs of distinct points of the set. (One can show, for example, that the transfinite diameter of a disk is equal to its radius.) If we allow ourselves to generalize this concept of transfinite diameter to closed bounded regions of three-dimensional Euclidean space, then our \( Q(\ell) \) may be seen to resemble one of the terms in the supremum which would define the transfinite diameter of a cube of side \( 2\ell+1 \) (except that the \( 1/n^2 \) power is taken, instead of the \( 1/n(n-1) \) power). However, our interest is in \( Q(\ell) \) itself and no supremum is to be taken.
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where

$$\omega_i(n) = \prod_{j \neq i} (|x_i - x_j|)^{\frac{1}{n^2}}$$

where the product is over all \(j\) from 1 to \(n\) except that the factor with \(j = i\) is omitted. We shall also assume that the numbering of the vertices from \(i = 1\) to \(n\) is such that the centre vertex (with coordinates, as introduced above, \((\ell, \ell, \ell)\)) is numbered \(i = 1\).

**Lemma.**

$$\omega_1 \leq \omega_j \quad (j = 1, \ldots, n).$$

**Proof.** In terms of our coordinatization of our cube, we need to show that the product

$$\mu_{(j_1,j_2,j_3)} = \prod_{(i_1,i_2,i_3) \neq (j_1,j_2,j_3)} |(j_1,j_2,j_3) - (i_1,i_2,i_3)|$$

over all distances from an arbitrary lattice point \((j_1,j_2,j_3)\) to all the other lattice points in our cube is greater than or equal to the product

$$\mu_{(\ell+1,\ell+1,\ell+1)} = \prod_{(i_1,i_2,i_3) \neq (\ell+1,\ell+1,\ell+1)} |(\ell+1,\ell+1,\ell+1) - (i_1,i_2,i_3)|$$

over all distances from the centre point, with coordinates \((\ell+1,\ell+1,\ell+1)\), to all the other lattice points in our cube. (To explain the notation here, \(\omega_j = (\mu_{(j_1,j_2,j_3)})^{1/n^2}\) if \((j_1,j_2,j_3)\) are the coordinates of the point numbered \(j\). The inequality of the lemma will then follow by taking the \(1/n^2\) power of each side of the above inequality.) We will prove this by exhibiting, for each \((j_1,j_2,j_3)\), a one-one onto mapping \(f_{(j_1,j_2,j_3)}\) from (coordinatized) lattice points of our cube to themselves with the properties

$$f_{(j_1,j_2,j_3)}((j_1,j_2,j_3)) = (\ell+1,\ell+1,\ell+1)$$

and

$$|f_{(j_1,j_2,j_3)}((i_1,i_2,i_3)) - (\ell+1,\ell+1,\ell+1)| \leq |(i_1,i_2,i_3) - (j_1,j_2,j_3)|$$ (10)

In fact, given such a mapping we will immediately have

$$\mu_{(j_1,j_2,j_3)} \geq \prod_{(i_1,i_2,i_3) \neq (j_1,j_2,j_3)} |(j_1,j_2,j_3) - (i_1,i_2,i_3)|$$

$$\geq \prod_{(i_1,i_2,i_3) \neq (j_1,j_2,j_3)} |f_{(j_1,j_2,j_3)}((j_1,j_2,j_3)) - f_{(j_1,j_2,j_3)}((i_1,i_2,i_3))|$$

$$= \prod_{(i_1,i_2,i_3) \neq (j_1,j_2,j_3)} |(\ell+1,\ell+1,\ell+1) - f_{(j_1,j_2,j_3)}((i_1,i_2,i_3))|$$

$$= \prod_{(i_1,i_2,i_3) \neq (\ell+1,\ell+1,\ell+1)} |(\ell+1,\ell+1,\ell+1) - (i_1,i_2,i_3)|$$

$$= \mu_{(\ell+1,\ell+1,\ell+1)}$$ (11)

where, in the penultimate equality, we have defined \((\hat{i}_1,\hat{i}_2,\hat{i}_3)\) to be \(f_{(j_1,j_2,j_3)}((i_1,i_2,i_3))\).
It therefore remains to exhibit a mapping, \( f_{(j_1,j_2,j_3)} \), with the above properties. To do this, we define

\[
f_{(j_1,j_2,j_3)}((i_1,i_2,i_3)) = (\ell + 1 - j_1 + i_1, \ell + 1 - j_2 + i_2, \ell + 1 - j_3 + i_3) \pmod{2\ell + 1}
\]

where we use a non-conventional notion of addition modulo \( q \) in which \( q \pmod{q} \) is deemed to be \( q \) rather than 0. One may understand this definition geometrically as a rigid translation of all the points of our cube where, however, if the would-be destination of a point is outside of our cube, the point instead gets mapped to the counterpart-position in our cubic lattice to the position it would arrive at in a neighbouring cube, were our cube to be surrounded by similar neighbours in a cubic lattice arrangement.

Clearly this is a bijection and satisfies (9). To show it satisfies (10) we calculate, using the obvious properties of our notion of “mod”:

\[
|f_{(j_1,j_2,j_3)}((i_1,i_2,i_3)) - (\ell + 1, \ell + 1, \ell + 1)| = \left(\left((\ell + 1 - j_1 + i_1)(\text{mod } 2\ell + 1) - \ell - 1\right)^2 + \left((\ell + 1 - j_2 + i_2)(\text{mod } 2\ell + 1) - \ell - 1\right)^2 + \left((\ell + 1 - j_3 + i_3)(\text{mod } 2\ell + 1) - \ell - 1\right)^2\right)^{1/2}
\]

\[
\leq \left( (j_1 - i_1)^2 + (j_2 - i_2)^2 + (j_3 - i_3)^2 \right)^{1/2} = |(j_1,j_2,j_3) - (i_1,i_2,i_3)|.
\]

\(\square\)

Before proceeding with the proofs of the two bounds in our proposition, it will be useful to define, for a given cube (i.e. cubic lattice of points) of side \( 2\ell + 1 \), certain special sets of lattice points. First, we define the “\( k \)th centre shell” \((k \leq \ell)\) to be the set of lattice points on the surface of the cube of side \( 2k + 1 \) centred on the centre of our given cube (so that a cube of side \( 2\ell + 1 \) would have a total of \( \ell \) centre shells).

**Proposition (Lower Bound).** For a cube of side \( 2\ell + 1 \), we have

\[
\frac{Q(\ell)}{\ell} \geq e^{-\frac{1}{3}} \quad \forall \ell \in \mathbb{N}
\]

**Proof.** For such a cube we have that

\[
n = (2\ell + 1)^3
\]

Let us denote the number of lattice points in the \( k \)th centre shell by \( n_k \). This is given by

\[
n_k = 6(2k + 1)^2 - 12(2k + 1) + 8
\]

We first use the lemma to estimate

\[
Q(\ell) = \prod_{i=1}^{n} \omega_i(n) \geq \omega_1(n)^n
\]

We can rewrite \( \omega_1(n) \) by a relabelling of the lattice points

\[
\omega_1(n) = \left( \prod_{i=1}^{n} \left| x_1 - x_i \right| \right)^{\frac{1}{n^2}} = \left( \prod_{k=1}^{\ell} \prod_{d=1}^{n_k} \left| x_1 - x_{d_k} \right| \right)^{\frac{1}{n^2}}
\]

\(\text{(13)}\)
where $x_{d_k}$ is the coordinate-triple of the $d^{th}$ lattice point in the $k$th centre shell. Each lattice point in the $k$th centre shell is at least a distance $k$ away from the centre lattice point so we have

$$Q(\ell) \geq \alpha(\ell)\frac{1}{n^k}$$

(14)

where

$$\alpha(\ell) = \prod_{k=1}^{\ell} k^{n_k}$$

(15)

From (14), it can be seen that to prove (12) it will be sufficient to prove

$$\alpha(\ell) \geq \left(e^{-\frac{1}{3}\ell}\right)^n = \left(e^{-\frac{1}{3}\ell}\right)^{(2\ell+1)^3}$$

(16)

This can be proved by induction on $\ell$. For all $\ell \in \mathbb{N}$, let $P(\ell)$ be the proposition that (16) is true. We have

$$\alpha(1) = 1 \geq e^{-\frac{1}{3}},$$

so $P(1)$ is true. So, if we can show $P(\ell) \Rightarrow P(\ell + 1)$ ($\forall \ell \in \mathbb{N}$) then the proposition will have been proved to be true. From (15) and (16)

$$\alpha(\ell + 1) = \alpha(\ell)(\ell + 1)^{n_{\ell+1}} \geq \left(e^{-\frac{1}{3}\ell}\right)^n (\ell + 1)^{n_{\ell+1}}$$

Thus to now prove our proposition it will be sufficient to prove that

$$\left(e^{-\frac{1}{3}\ell}\right)^n (\ell + 1)^{n_{\ell+1}} \geq \left(e^{-\frac{1}{3}(\ell + 1)}\right)^{(2\ell+3)^3}$$

This is easily seen to be equivalent to

$$e^{-\frac{1}{3}} \leq \left(1 + \frac{1}{\ell}\right)^{-\frac{1}{3}(\ell + 1)^{\frac{13\ell^2 + 72\ell - 1}{24\ell^2}}}. $$

However the above equation clearly holds so we have proved our inductive proposition (and hence the lower bound in our main proposition).

\[\square\]

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References

[1] Kay B S 1998 Entropy defined, entropy increase and decoherence understood, and some black-hole puzzles solved Preprint \textit{hep-th/9802172}

[2] Kay B S 1998 Decoherence of macroscopic closed systems within Newtonian quantum gravity \textit{Class. Quantum Grav.} \textbf{17} L89-L98 (Preprint \textit{hep-th/9810077})

[3] Goluzin G M 1969 \textit{Geometric Theory of Functions of a Complex Variable} (\textit{Translations of Mathematical Monographs} vol 26) (Providence, Rhode Island: American Mathematical Society/Academic Press)

[4] Abyaneh V 2006 \textit{Gravitationally Induced Decoherence} University of York PhD thesis