Gauge invariance of quantum electrodynamics in the causal approach to renormalization theory

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Received 30 May 2000, revised 5 October 2000, accepted 9 October 2000 by K. Fredenhagen

Abstract. We present an extremely simple solution to the renormalization of quantum electrodynamics based on Epstein-Glaser approach to renormalization theory.

Keywords: renormalization, causality, gauge invariance
PACS: 11.10Gh, 11.15Bt

1 Introduction

The causal approach to renormalization theory pioneered by Epstein and Glaser [22, 23] provides essential simplification at the fundamental level as well as to the computational aspects. This is best illustrated in [39] where quantum electrodynamics is constructed entirely in the framework of the causal approach. Moreover, one can use the same ideas to analyse other theories as for instance, Yang-Mills theories [1, 2, 5–7, 10, 11, 13, 14, 21, 30–33, 36, 38], gravitation [24, 25, 43], etc.

Let us remind briefly the main ideas of Epstein-Glaser-Scharf approach. According to Bogoliubov and Shirkov, the S-matrix is constructed inductively order by order as a formal series of operator valued distributions:

\[ S(g) = 1 + \sum_{n=1}^{\infty} \frac{\pi^n}{n!} \int_{\mathbb{R}^{4n}} \mathrm{d}x_1 \ldots \mathrm{d}x_n \, T(x_1, \ldots, x_n) \, g(x_1) \ldots g(x_n), \]  

(1.0.1)

where \( g(x) \) is a tempered test function in the Minkowski space \( \mathbb{R}^4 \) that switches the interaction and \( T(x_1, \ldots, x_n) \) are operator-valued distributions acting in the Fock space of some collection of free fields. These operator-valued distributions, which are called chronological products should verify some properties which can be argued starting from Bogoliubov axioms. These axioms will be detailed in the next Section. The main point is that one can show that, starting from a convenient interaction Lagrangian \( T(x) \) one can construct the whole series \( T(x_1, \ldots, x_n), \ n \geq 2 \). The interaction Lagrangian must satisfy some requirements such like Poincaré invariance, hermiticity and causality; it is not easy to find a general solution of

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this problem but there are some rather general expressions fulfilling these demands, namely the so-called Wick polynomials. These are expressions operating in Hilbert spaces of a special kind, namely in Fock spaces. A Fock space is a canonical object attached to any single-particle Hilbert and reasonably describes a system of weakly interacting particles.

The physical $S$-matrix is obtained from $S(g)$ taking the adiabatic limit which is, loosely speaking the limit $g(x) \to 1$.

In the old version of renormalization theory, one starts from the naive expressions of the chronological product and sees that they are not properly defined, i.e. some infinities do appear. The main obstacle is to amend the naive expression such that well defined expressions are obtained which do also verify Bogoliubov axioms. In Epstein-Glaser approach, the main problem of the the construction of the chronological products is done recurringly and it is reduced to the problem of distribution splitting. It can be proved that this operation has always solutions consistent with Bogoliubov axioms.

In the case of a gauge theory there is a supplementary property to be verified. The main obstacle in constructing the perturbation series for a gauge field is the fact that, as it happens for the electromagnetic field, one is forced to use non-physical degrees of freedom for the description of the free fields [35, 42, 46] in a Fock space formalism. One must consider an auxiliary Fock space $\mathcal{H}^{gh}$ including, beside the various fields, some fictitious fields, called ghosts, and construct a supercharge that is an operator $Q$ verifying $Q^2 = 0$ such that the physical Hilbert space is $\mathcal{H}_{phys} \equiv \text{Ker}(Q) / \text{Im}(Q)$. The necessity to consider ghost fields comes mainly from the fact that, up to now, there is no other way to construct an interaction Lagrangian. On the other hand, one can construct a convenient interaction Lagrangian in the bigger Hilbert space $\mathcal{H}_{gh}$ and apply the construction of Epstein and Glaser without any change. However, in this case one must impose, beside the usual Bogoliubov axioms, the supplementary condition that the $S$ matrix factorizes to $\mathcal{H}_{phys}$. This condition proves to be too strong and one must replace it by a weaker condition of factorization to the physical Hilbert space in the adiabatic limit:

$$
\lim_{g \to 0} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \left. dx_1 \ldots dx_n \right| g(x_1) \ldots g(x_n) \left[ Q, T(x_1, \ldots, x_n) \right] \right|_{\text{Ker}(Q)} = 0, \quad \forall n \geq 1.
$$

Even this condition seems to be problematic because the adiabatic limit does not exists if zero-mass particles are present, so one must weaken further this requirement as it is done in [10] where one requires that:

$$
\left[ Q, T(x_1, \ldots, x_n) \right] = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^a} T^a_l(x_1, \ldots, x_n), \quad \forall n \in \mathbb{N}^*
$$

for some (auxiliary) time-ordered products $T^a_l(x_1, \ldots, x_n), \quad l = 1, \ldots, n$. This condition leads to the previous one if one formally takes the adiabatic limit and it is called the gauge invariance of the theory. It is impressive that this condition for $n = 1$ and $n = 2$ fixes almost uniquely the possible form of the interaction Lagrangian and leads to the presence of a $r$-dimensional Lie group of symmetries in the case of system composed of $r$ Bosons of spin 1 [1, 26].
This result can be extended to the case of presence of matter fields [2, 27] paving the way to a rigorous understanding of the standard model of elementary particles. In [28] the analysis is pushed to order \( n = 3 \) and the axial anomaly appears in a natural context, as an obstruction to the factorization condition (1.0.3) to the physical Hilbert space. The gauge invariance problem is now to prove that the identities (1.0.3) can be fulfilled for every \( n \in \mathbb{N} \), more precisely to show that one can use the freedom left in the chronological products (the finite renormalizations) to impose gauge invariance in every order of perturbation theory. This problem is addressed in [39] in the case of quantum electrodynamics and in [11, 13] in the Yang-Mills case. The idea is to assume that one has (1.0.3) for \( p = 1, \ldots, n - 1 \) and prove it for \( p = n \). The proof of renormalizability of quantum electrodynamics from [12, 16, 39] is done without using ghost fields; the same is true for the proof of the renormalizability of scalar electrodynamics [20]. This is possible in these two case, but in order to generalize the method to Yang-Mills theories it is better to understand the proof in the ghost quantization formalism.

The main point of this paper is that with a proper formulation of the induction hypothesis one can simplify considerably the proof such that it’s mathematical rigor becomes obvious. The idea is that if one formulates the induction hypothesis in close analogy with the analysis of Epstein and Glaser then one can prove that in the order \( n \) one has instead of (1.0.3) the relation

\[
[Q, T(x_1, \ldots, x_n)] = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l} T^\mu_l (x_1, \ldots, x_n) + P(x_1, \ldots, x_n), \quad \forall n \in \mathbb{N}^*
\]  

(1.0.4)

where \( P(x_1, \ldots, x_n) \) is a quasi-local i.e. a operator-valued distribution verifying

\[
\text{supp} (P(x_1, \ldots, x_n)) \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^{4n} | x_1 = \ldots = x_n\}.
\]  

(1.0.5)

This finite renormalization can be so much restricted from the induction process that it is an elementary matter to show that one can modify appropriately the expressions \( T(x_1, \ldots, x_n) \) and \( T^\mu_l (x_1, \ldots, x_n) \) in such a way that one has \( P(x_1, \ldots, x_n) = 0 \). In this process, the use of some discrete symmetry like charge conjugation is important. This idea was pioneered in [5] and [14].

We will adopt the gauge invariance condition in the form (1.0.3) so we will not touch the adiabatic limit problem in our analysis.

The paper is organized as follows. In the next Section we fix the notations and clarify the setting we use. We will present Bogoliubov axioms of perturbation theory. Because the main point of our paper is to formulate the induction hypothesis in strict analogy to [22] we will summarize the induction argument used for a theory without gauge invariance. Then we present the modification of the setting one must impose to study quantum electrodynamics. In Section 3 we will give the proof of gauge invariance of quantum electrodynamics. In Section 4 we present the same analysis for the case of scalar quantum electrodynamics. The Conclusions are grouped in the last Section.
2 Perturbation theory for QED

2.1 Bogoliubov axioms

We give here the set of axioms imposed on the chronological products following the notations of [22]. We construct a multi-Lagrangian formalism, that is a formal series

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{\mathbb{R}^n} dx_1 \ldots dx_n \; T_{j_1 \ldots j_n}(x_1, \ldots, x_n) \; g_{j_1}(x_1) \ldots g_{j_n}(x_n),$$

(2.1.1)

where $g = (g_j(x))_{j=1, \ldots, p}$ is a multi-valued tempered test function in the Minkowski space $\mathbb{R}^4$ and the expressions $T_{j_1 \ldots j_n}(x_1, \ldots, x_n)$ are some operator-valued distributions acting in a Hilbert space $\mathcal{H}$ with a common dense domain of definition $D_0$; they are called chronological products and are subject to a set of axioms outlined below. The convention of summation over repeating indices of the type $j_1, \ldots, j_n$ is used everywhere. We will frequently use the condensed notation $T_j(X)$ and the convention

$$T_0(\emptyset) \equiv 0.$$  

(2.1.2)

We consider the sets $X$ and $J$ as ordered set. A natural bijection between the elements of $X$ and $J$ obviously exists and extends to subsets of $X$ and $J$. By $XY$ we mean the juxtaposition of the elements of $X$ and $Y$.

Here are Bogoliubov axioms:

- First, it is clear that we can consider them completely symmetrical in all variables:

  $$T_{j_{p(1)} \ldots j_{p(p)}}(x_{p(1)}, \ldots x_{p(p)}) = T_{j_1 \ldots j_n}(x_1, \ldots, x_p), \quad \forall P \in \mathcal{P}_p.$$  

(2.1.3)

- Next, we must have Poincaré invariance. Because we will consider in an essential way Dirac fields, this amounts to suppose that in the in the Fock space we have an unitary representation $(a, A) \mapsto U_{a,A}$ of the group $inSL(2, \mathbb{C})$ (the universal covering group of the proper orthochronous Poincaré group $\mathcal{P}_+^l$ – see [44] for notations) and a finite dimensional representation $A \mapsto D(A)$ of the group $SL(2, \mathbb{C})$ such that:

  $$U_{a,A} T_{j_1 \ldots j_n}(x_1, \ldots, x_p) U_{a,A}^{-1} = D(A^{-1})_{j_1j_1} \ldots D(A^{-1})_{j_nj_n}$$

  $$\times T_{k_1 \ldots k_n}(\delta(A) \cdot x_1 + a, \ldots, \delta(A) \cdot x_p + a),$$

  $$\forall (a, A) \in inSL(2, \mathbb{C}),$$

(2.1.4)

where $SL(2, \mathbb{C}) \ni A \mapsto \delta(A) \in \mathcal{P}_+^l$ is the covering map. In particular, translation invariance is essential for implementing Epstein-Glaser scheme of renormalization. Sometimes it is possible to supplement this axiom by corresponding invariance properties with respect to inversions (spatial and temporal) and charge conjugation. For the standard model only the PCT invariance is available.

- The central axiom seems to be the requirement of causality which can be written compactly as follows. Let us firstly introduce some standard notations. Denote by $V^+ \equiv \{ x \in \mathbb{R}^4 \mid x^2 > 0, x_0 > 0 \}$ and $V^- \equiv \{ x \in \mathbb{R}^4 \mid x^2 > 0, x_0 < 0 \}$ the upper
(lower) lightcones and by $\overline{V^\pm}$ their closures. If $X \equiv \{x_1, \ldots, x_m\} \in \mathbb{R}^{4m}$ and $Y \equiv \{y_1, \ldots, y_n\} \in \mathbb{R}^{4m}$ are such that $x_i - y_j \notin \overline{V^+}$, $\forall i = 1, \ldots, m$, $j = 1, \ldots, n$ we use the notation $X \geq Y$; by definition $X \geq Y$ means $Y \geq X$. If $x_i - y_j \notin \overline{V^+} \cup \overline{V^-}$, $\forall i = 1, \ldots, m$, $j = 1, \ldots, n$ we use the notations: $X \sim Y$. Then the causality axiom writes as follows:

$$T_{J_iJ_j}(X_1X_2) = T_{J_i}(X_1) T_{J_j}(X_2), \quad \forall X_1 \geq X_2.$$  \hfill (2.1.5)

Here the subsets $J_i \equiv J_{X_i}$ of $J$ are associated in an obvious way to the partition $X_1, X_2$ of $X$.

**Remark 2.1:** It is important to note that from (2.1.5) one can derive easily:

$$[T_{J_i}(X_1), T_{J_j}(X_2)] = 0, \quad \text{if} \quad X_1 \sim X_2.$$  \hfill (2.1.6)

- The **unitarity** of the $S$-matrix can be most easily expressed (see [22]) if one introduces, the following formal series:

$$\bar{S}(g) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^{4n}} dx_1 \ldots dx_n \ T_{J_1,\ldots,J_n}(x_1, \ldots, x_n) \ g_{i_1}(x_1) \ldots g_{i_n}(x_n),$$  \hfill (2.1.7)

where, by definition:

$$(-1)^{|X|} \ T_{J}(X) \equiv \sum_{r=1}^{|X|} \ (-1)^r \ \sum_{X_1, \ldots, X_r \in \text{Part}(X)} T_{J_1}(X_1) \ldots T_{J_r}(X_r);$$  \hfill (2.1.8)

here $|X|$ is the cardinal of the set $X$ and the sum runs over all partitions. One calls the operator-valued distributions $T_{J_i}$ **anti-chronological products**. It is not very hard to prove that the series (2.1.7) is the inverse of the series (1.0.1) i.e. we have:

$$\bar{S}(g) = S(g)^{-1}$$  \hfill (2.1.9)

as formal series. Then the unitarity axiom is:

$$\bar{T}_J(X) = T_J(X)^\dagger, \quad \forall X.$$  \hfill (2.1.10)

**Remark 2.2:** The previous relation is consistent because from the definition (2.1.8) it follows that the anti-chronological products have the same symmetry properties as the chronological products. One can show that the following relations are identically verified:

$$\sum_{X_1, X_2 \in \text{Part}(X)} (-1)^{|X_i|} \ T_{J_i}(X_1) \ T_{J_j}(X_2)$$

$$= \sum_{X_1, X_2 \in \text{Part}(X)} (-1)^{|X_i|} \ T_{J_1}(X_1) \ T_{J_j}(X_2) = 0$$  \hfill (2.1.11)

and can be used to define the anti-chronological products in a recursive way. Also one has, similarly to (2.1.5) the following relation:

$$\bar{T}_{J_iJ_j}(X_1X_2) = \bar{T}_{J_i}(X_1) \ T_{J_j}(X_2), \quad \forall X_1 \leq X_2.$$  \hfill (2.1.12)
A renormalization theory is the possibility to construct such a $S$-matrix starting from the first order terms: $T_j(x)$, $j = 1, \ldots, P$ which are Wick polynomial called interaction Lagrangians which should verify the following axioms for all $j = 1, \ldots, P$:

$$U_{a,A}T_j(x) U_{a,A}^{-1} = D(A^{-1})_{jk} T_k(\delta(A) \cdot x + a), \quad \forall A \in SL(2, \mathbb{C}), \quad (2.1.13)$$

$$[T_j(x_1), T_j(x_2)] = 0, \quad \forall x_1, x_2 \in \mathbb{R}^4 \text{ s.t. } x_1 \sim x_2, \quad (2.1.14)$$

and

$$T_j(x) = T_j(x). \quad (2.1.15)$$

In particular, it follows that the Wick monomials $T_j$ have to be Hermitian operators.

Usually, these requirements are supplemented by covariance with respect to some discrete symmetries (like space-time inversions, or PCT), charge conjugations or global invariance with respect to some Lie group of symmetry.

It is not easy to find non-trivial solutions to the set of requirements (2.1.13), (2.1.14) and (2.1.15). In fact, this is a problem of constructive field theory. Fortunately, if one considers that the Hilbert space of the theory is of Fock type, then one has plenty of interesting solutions, namely the Wick polynomials. As underlined in the Introduction, this is one of the main reasons of extending the Hilbert space of a gauge system by including ghost fields: there is no other obvious solution of constructing the interaction Lagrangian without them.

The case of a single Lagrangian corresponds to $P = 1$. Usually it is more convenient to consider that the interaction Lagrangian is of the form:

$$T(x) = c_j T_j(x) \quad (2.1.16)$$

with $c_j$ some real constants. Then, the chronological products of the theory are:

$$T(X) = c_{j_1} \ldots c_{j_n} T_{j_1, \ldots, j_n}(X); \quad (2.1.17)$$

we remind that the convention of summation over dummy indices is used in these formulae.

2.2 Epstein-Glaser induction

In this Subsection we summarize the steps of the inductive construction of Epstein and Glaser [22]. As we have said before, we consider that the Hilbert space is a Fock space. Let us define the canonical dimension of a Wick monomial $\omega(W)$ by assigning to every integer spin field factor and every derivative the value 1, for every half-integer spin field factor the value $3/2$ and summing over all factors.

We suppose that the given interaction Lagrangians $T_j(x)$ are all Wick monomials canonical dimension $\omega_j \leq 4$ ($j = 1, \ldots, P$) acting in a certain Fock space $\mathcal{H}$. Because the Fock space is generated by some free relativistic fields acting on the vacuum $\Omega$ it is easy to see that there always exists a finite dimensional representation of the group $SL(2, \mathbb{C})$ such that we have (2.1.13). Moreover, suppose we have in the Fock space the operators realizing the discrete symmetries (space-time inversions $I_s, I_t$ and charge conjugation) of the free fields $U_{I_s}, U_{I_t}, U_C$. Then one can
argue in the same way that relations of the following type are true:

\[ U_t T_j(x) U_t^{-1} = (S_t)_{jk} T_k(x), \quad U_t T_j(x) U_t^{-1} = (S_t)_{jk} T_k(x), \]

\[ U_C T_j(x) U_C^{-1} = C_{jk} T_k(x), \]

for some constant matrices \( S_t, S, C \). The representation property imposes some constraints between these matrices and the representation \( D(A) \).

If there are non-Hermitean free fields acting in the Fock space, the relation (2.1.15) not generally true, but we have:

\[ T_j(x)^\dagger = T_j(x) \]

where \( j \rightarrow j^* \) is a bijective map of the numbers 1, 2, \ldots, \( P \).

If there are Fermi or ghost fields acting in the Fock space, the causality property (2.1.14) is not true in general but we have:

\[ T_{j_1}(x_1) T_{j_2}(x_2) = (-1)^{\sigma_1 \sigma_2} T_{j_2}(x_2) T_{j_1}(x_1), \quad \forall x_1 \sim x_2. \]

Here \( \sigma_1 \) is the number of Fermi and ghost fields factors in the Wick monomial \( T_j \); if \( \sigma_j \) is even (odd) we call the index \( j \) even (resp. odd).

It is convenient to let the index \( j \) have the value 0 also and we put by definition

\[ T_0 \equiv 1. \]

Moreover, we define a new sum operation of two indices \( j_1, j_2 = 1, \ldots, P \); this summation is denoted by \( + \) but should not be confused with the ordinary sum. By definition we have:

\[ T_{j_1+j_2}(x) = c : T_{j_1}(x) T_{j_2}(x) :, \]

for some positive constant \( c \). We define componentwise the summation for \( n \)-tuples \( J = \{ j_1, \ldots, j_n \} \). The new summation is non-commutative if Fermi or ghost fields are present. We will use the compact notation:

\[ \omega_J \equiv \sum_{j \in J} \omega_j. \]

If \( J = \{ j_1, \ldots, j_n \} \) is given we consider the subset \( J_- \) of odd indices, with the order inherited from \( J \); if \( P \) is a permutation of the set \( J \) then \( P_- \) is the corresponding permutation of \( J_- \) and \( \epsilon(P_-) \) is its signature.

It is a rather complicated combinatorial fact that one can choose the expressions \( T_j \) such that one has:

\[ T_{j_1}(x_1) \ldots T_{j_n}(x_n) = \sum_{K+L=J} \epsilon(P_-) \left( \Omega, T_{k_1}(x_1) \ldots T_{k_n}(x_n) \Omega \right) : T_{l_1}(x_1) \ldots T_{l_n}(x_n) :, \]

where \( \Omega \in \mathcal{H} \) is the vacuum state and \( P \) is the permutation \( P(k_1 l_1 \ldots k_n l_n) = (k_1 \ldots k_n l_1 \ldots l_n) \). This is a compact form of Wick theorem. For instance, if we consider the simplest case of a real scalar field \( \phi(x) \) one can take \( T_j(x) \equiv \frac{1}{n!} : \phi(x)^j ::, \quad j = 0, \ldots, 4. \)

One needs this theorem in an essential way in the Epstein-Glaser induction so, one has to generalize the scheme presented in the previous Subsection to take into account the differences between (2.1.15) + (2.1.14) and (2.2.2) + (2.2.3). The idea
is to replace the symmetry axiom (2.1.3) by skew symmetry. With the notations introduced above, we replace (2.1.3) by:

\[ T_{j_1,\ldots,j_p}(x_{p(1)}, \ldots x_{p(p)}) = \epsilon(P) \ T_{j_1,\ldots,j_n}(x_1, \ldots, x_p), \quad \forall P \in \mathcal{P}_p. \tag{2.2.8} \]

One has to modify appropriately all the relations of Epstein-Glaser analysis. Instead of (2.1.8) we have

\[ (-1)^{|X|} \tilde{T}_J(X) = \sum_{r=1}^{|X|} (-1)^r \sum_{X_1,\ldots,X_r \in \text{Part}(X)} \epsilon(P) \ T_{j_1}(X_1) \ldots T_{j_r}(X_r) \tag{2.2.9} \]

where \( P \) is the permutation mapping \( J \) into \( J_1 J_2 \ldots J_r \). Then, instead of (2.1.10) we have skew-unitarity:

\[ \tilde{T}_J(X) = T_{J'}(X^*)^\dagger, \quad \forall X. \tag{2.2.10} \]

Here we have used the condensed notations

\[ \{x_1, \ldots, x_n\}^* \equiv \{x_n, \ldots, x_1\}, \quad \{j_1, \ldots, j_n\}^* \equiv \{j_n^*, \ldots, j_1^*\}. \tag{2.2.11} \]

Corresponding modifications should appear in the relations (2.1.11). The axioms of Poincaré covariance (2.1.4) and causality (2.1.5) remain unchanged. However, instead of (2.1.6) we have a natural generalization of (2.2.3):

\[ T_{j_1}(X_1) \ T_{j_2}(X_2) = \epsilon(P) \ T_{j_1}(X_2) \ T_{j_2}(X_1), \quad \forall X_1 \sim X_2. \tag{2.2.12} \]

From now on we always insert the factor \( \epsilon(P) \) where \( P \) is the corresponding permutation connecting the order of the indices in the left hand side to the order in the right hand side of any equation. This rule being rather easy to implement, we will not describe in every equation the explicit form of the permutation \( P \). This formulation of the most general setting for the perturbation theory in the sense of Bogoliubov should be compared with the alternative scheme proposed in [3] following the ideas from [9].

The main point of the Epstein-Glaser construction is a careful formulation of the induction hypothesis. We suppose that we have constructed the chronological products \( T_J(X), \ |X| \leq n-1 \) having the following properties: (2.1.4), (2.1.5), (2.2.8) and (2.2.10).

We add to this induction hypothesis the following assumption: the following Wick expansion is valid for all \(|X| \leq n-1\):

\[ T_J(X) = \sum_{K+L=J} \epsilon(P) \ t_K(X) : T_{l_1}(x_1) \ldots T_{l_n}(x_n) : \tag{2.2.13} \]

where \( t_K(X) \) are numerical distributions given by the expressions

\[ t_J(X) = (\Omega, T_J(X) \Omega), \quad \forall J \tag{2.2.14} \]

and with the order of singularity (defined as in [39]) restricted by

\[ \omega(t_K) \leq \omega_K - 4(|X| - 1). \tag{2.2.15} \]
Let us note that the right hand side of (2.2.13) is well defined according to theorem 0 of [22] and from well known properties of such expressions we also have (2.2.12) for \(|X_1|, |X_2| \leq n - 1\). Formula (2.2.15) is the most general form of the power-counting theorem.

We want to construct the operator-valued distributions \(T(X), \ |X| = n\) such that the properties above go from 1 to \(n\). Here are the main steps of the induction proof.

1. One constructs from \(T(X), \ |X| \leq n - 1\) the expressions \(\tilde{T}(X), \ |X| \leq n - 1\) according to (2.2.9) and proves the properties (2.1.11) and (2.1.12) (properly modified) for \(|X| \leq n - 1\).

2. Lemma 2.3: If \(X = \{x_1, \ldots, x_n\}\), let us defines the expressions:

\[
A'_j(X) = \sum_{X_1, X_2 \in \text{Part}(X)} (-1)^{|X_2|} \epsilon(P_-) T_{J_1}(X_1) \tilde{T}_{J_2}(X_2), \tag{2.2.16}
\]

\[
R'_j(X) = \sum_{X_1, X_2 \in \text{Part}(X), X_2 \neq \emptyset, x_n \in X_1} (-1)^{|X_2|} \epsilon(P_-) \tilde{T}_{J_2}(X_2) T_{J_1}(X_1). \tag{2.2.17}
\]

Now, let us suppose that we have a partition \(Y_1, Y_2 \in \text{Part}(X), Y_1 \neq \emptyset, x_n \in Y_2\).

Then:

- If \(Y_2 \geq Y_1\) one has:

\[
A'_j(X) = -\epsilon(P_-) T_{J_2}(Y_2) T_{J_1}(Y_1). \tag{2.2.18}
\]

- and if \(Y_1 \leq Y_2\) one has:

\[
R'_j(X) = -\epsilon(P_-) T_{J_1}(Y_1) T_{J_2}(Y_2). \tag{2.2.19}
\]

The proof is elementary if one uses the causality properties (2.1.5) and (2.1.12).

3. Corollary 2.4. The expression

\[
D_J(X) \equiv A'_j(X) - R'_j(X). \tag{2.2.20}
\]

have causal support i.e. \(\text{supp}(D_J(x_1, \ldots, x_n)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)\) where we use standard notations:

\[
\Gamma^\pm(x_n) \equiv \{(x_1, \ldots, x_n) \in (\mathbb{R}^4)^n \mid x_i - x_n \in V^\pm, \forall i = 1, \ldots, n - 1\}. \tag{2.2.21}
\]

The proof consists of noticing the local character of the support property and reducing all possible cases to typical situations from the preceding lemma.

4. Lemma 2.5: The distribution \(D_J(X)\) can be written as a sum similar to (2.2.13), namely:

\[
D_J(X) = \sum_{K+L=J} d_K(X) : T_{l_1}(x_1) \cdots T_{l_n}(x_n) : \tag{2.2.22}
\]

where \(d_K(X)\) are numerical distributions with causal support: \(\text{supp}(d_K(X)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n)\) and with the degree of singularity restricted by:

\[
\omega(d_K) \leq \omega_K - 4(|X| - 1). \tag{2.2.23}
\]
The proof goes by induction. For the causality property one must prove (see [22]) that we have, in analogy to (2.2.14) the following relation:

\[ d_K(X) = (\Omega, D_K(X) \, \Omega); \]  

(2.2.24)

in this place the use of Wick theorem (2.2.7) is essential. The deriving of the formula (2.2.23) goes like in [39] and is, essentially, the power counting theorem.

5. We define an action of the group \( SL(2, \mathbb{C}) \) one the space of multi-component distributions \( d_j \) according to:

\[ (A \cdot d)_{j_1, \ldots, j_p}(x) \equiv D(A)_{j_1k_1} \cdots D(A)_{j_pk_p} \, d_{k_1, \ldots, k_p}(\delta(A^{-1}) \cdot x) \]  

(2.2.25)

and say that the distribution \( d \) is \( SL(2, \mathbb{C}) \)-invariant iff it verifies:

\[ A \cdot d = d. \]  

(2.2.26)

**Lemma 2.6:** The distributions \( d_K(X) \) defined above are \( SL(2, \mathbb{C}) \)-invariant.

The proof follows from the formula (2.2.24).

6. Now we have the following result from [11, 39]:

**Lemma 2.7:** Let \( d_j \) be a \( SL(2, \mathbb{C}) \)-invariant distribution with causal support. Then, there exists a causal splitting

\[ d_j = a_j - r_j, \quad \text{supp} \, (a_j) \subset \Gamma^+(x_n), \quad \text{supp} \, (r_j) \subset \Gamma^-(x_n) \]  

(2.2.27)

which is also \( SL(2, \mathbb{C}) \)-invariant and such that

\[ \omega(a_j) \leq \omega(d_j), \quad \omega(r_j) \leq \omega(d_j). \]  

(2.2.28)

We outline the proof because the argument is generic and it will also be used for the more general case of gauge invariance. It is known from the general theory of distribution splitting that there exists a causal splitting \( d = a - r \) preserving the order of singularity. Then \( A \cdot d = A \cdot a - A \cdot r \) is a causal splitting of the distribution \( A \cdot d \). Because, by hypothesis, we have \( A \cdot d = d \) it follows that we have

\[ A \cdot a - a = A \cdot r - r. \]  

(2.2.29)

But the left hand side has support in \( \Gamma^+(x_n) \) and the right hand side in \( \Gamma^-(x_n) \) so, the common value, denoted by \( P_A \) have the support in \( \Gamma^+(x_n) \cap \Gamma^-(x_n) = \{ (x_1, \ldots, x_n) \in (\mathbb{R}^4)^n \mid x_1 = \ldots = x_n \} \). But in this case, it is known from the general distribution theory that \( P_A \) is of the form

\[ P_A(x) = p_A(\partial) \, \delta^{n-1}(X), \]  

(2.2.30)

where

\[ \delta^{n-1}(X) \equiv \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n) \]  

(2.2.31)

and \( p_A \) is a polynomial in the derivatives of maximal order \( \omega(d) \). In particular, if \( \omega(d) < 0 \) we have \( p_A = 0 \) and the causal splitting is \( SL(2, \mathbb{C}) \)-covariant. If \( \omega(d) \geq 0 \) then we easily derive that \( P_A \) verifies the following identity:

\[ P_{A_1 \cdot A_2} - A_1 \cdot P_{A_2} + P_{A_1} = 0. \]  

(2.2.32)
This relation says that the map $A \rightarrow P_A$ is a $SL(2, \mathbb{C})$-cocycle with values in the finite dimensional space of polynomials of type (2.2.30). Because $SL(2, \mathbb{C})$ is a connected, simply connected and simple Lie group we can apply Hochschild lemma [44] and obtain that $P_A$ is of the form

$$P_A = A \cdot Q - Q$$  \hspace{1cm} (2.2.33)

for some polynomial $Q$ of order not greater than $\omega(d)$. In particular, we have

$$A \cdot (a - Q) = a - Q$$  \hspace{1cm} (2.2.34)

so we have a $SL(2, \mathbb{C})$-covariant causal splitting $d = (a - Q) - (r - Q)$. \hfill \Box

7. **Corollary 2.8:** There exists a $SL(2, \mathbb{C})$-covariant causal splitting:

$$D_{j}(X) = A_{j}(X) - R_{j}(X)$$  \hspace{1cm} (2.2.35)

with $\text{supp} \ (A_{j}(x_1, \ldots, x_n)) \subset \Gamma^{+}(x_n)$ and $\text{supp} \ (R_{j}(x_1, \ldots, x_n)) \subset \Gamma^{-}(x_n)$. For that reason, the expressions $A_{j}$ and $R_{j}$ are called advanced (resp. retarded) products.

8. **Lemma 2.9:** The following relation is true for any $|Y| = n - 1$:

$$D_{jk}(Y, x)^{\dagger} = (-1)^{n-1} \epsilon(P_{-}) \ D_{j'k}(Y^{*}, x)$$  \hspace{1cm} (2.2.36)

where the permutation $P$ is $J_{k} \rightarrow kJ$. In particular the causal splitting obtained above can be chosen such that

$$A_{jk}(Y, x)^{\dagger} = (-1)^{n-1} \epsilon(P_{-}) \ A_{j'k}(Y^{*}, x).$$  \hspace{1cm} (2.2.37)

The first assertion follows by elementary computations starting directly from the definition (2.2.20) and using the unitarity induction hypothesis (2.1.10) and the relations (2.1.11). This proves that by performing the substitutions:

$$A_{jk}(Y, x) \rightarrow \frac{1}{2} [A_{jk}(Y, x)^{\dagger} + (-1)^{n-1} \epsilon(P_{-}) \ A_{j'k}(Y^{*}, x)]$$  \hspace{1cm} (2.2.38)

and a similar one for the retarded products, we do not affect the relation from the preceding corollary and we obtain a causal splitting verifying the relation from the statement without spoiling the $SL(2, \mathbb{C})$-covariance. \hfill \Box

9. Now we have

**Theorem 2.10:** Let us define

$$T_{j}(X) \equiv A_{j}(X) - A_{j}^{\dagger}(X) \equiv R_{j}(X) - R_{j}^{\dagger}(X).$$  \hspace{1cm} (2.2.39)

Then these expressions satisfy the $SL(2, \mathbb{C})$-covariance, causality and unitarity conditions (2.1.4) (2.1.5) (2.1.6) and (2.1.10) for $p = n$. If we substitute

$$T_{j_{1}, \ldots, j_{k}}(x_{1}, \ldots, x_{n}) \rightarrow \frac{1}{n!} \sum_{p} \epsilon(P_{-}) \ T_{j_{p(1)}, \ldots, j_{p(n)}}(x_{P(1)}, \ldots, x_{P(n)})$$  \hspace{1cm} (2.2.40)

then we also have the symmetry axiom (2.2.8). Moreover, this expression admits a Wick expansion (2.2.13) and verifies the limitation (2.2.15).
The $SL(2, \mathbb{C})$-covariance is obvious. The causality axiom (2.1.5) follows from the two expressions of the definition of $T_J(X)$ if one takes into account the support properties of the advanced and retarded product and also uses lemma 2.3. The unitarity axiom is a result of the definition given above, the property of the advanced products from the preceding lemma, the expressions $A_J'$ and the induction hypothesis (2.2.10) for $p \leq n - 1$. The symmetrization process is obvious as are (2.2.13) and (2.2.14).

As we have mentioned in the Introduction the solution of the renormalization problem is not unique. The non-uniqueness is given by some finite renormalizations $N_J$. There are some restrictions on these finite renormalizations coming from the Poincaré invariance and unitarity but still there remains some arbitrariness. One can restrict even further the arbitrariness requiring the existence of the adiabatic limit. One can prove that this limit does exists if there are no zero-mass particles in the spectrum of the energy-momentum quadri-vector.

2.3 Perturbation theory for zero-mass particles

We remind the basic facts about the quantization of the photon; for more details see [26] and references quoted there. Let us denote the Hilbert space of the photon by $H_{\text{photon}}$; it carries the unitary representation of the orthochronous Poincaré group $H^{[0,1]} \oplus H^{[0,-1]}$ (see [44]).

The Hilbert space of the multi-photon system should be, according to the basic principles of the second quantization, the associated symmetric Fock space $F_{\text{photon}} \equiv F^+(H_{\text{photon}})$. One can construct in a rather convenient way this Fock space in the spirit of algebraic quantum field theory. One considers the Hilbert space $H^{gh}$ generated by applying on the vacuum $\Omega$ the free fields $A^\mu(x), u(x), \bar{u}(x)$ called the electromagnetic potential (reps. ghosts) which are completely characterize by the following properties:

- Equation of motion:
  \[
  \Box A^\mu(x) = 0, \quad \Box u(x) = 0, \quad \Box \bar{u}(x) = 0.
  \] (2.3.1)

- Canonical (anti)commutation relations:
  \[
  [A^\mu(x), A^\rho(y)] = -g^{\mu\rho}D_0(x - y) \times 1, \quad [A^\mu(x), u(y)] = 0, \\
  [A^\mu(x), \bar{u}(y)] = 0, \quad \{u(x), u(y)\} = 0, \quad \{\bar{u}(x), \bar{u}(y)\} = 0, \\
  \{u(x), \bar{u}(y)\} = D_0(x - y) \times 1; \quad \{u(x), u(y)\} = 0,
  \] (2.3.2)

  here $D_m, m \geq 0$ is the Pauli-Jordan distribution:
  \[
  D_m(x) \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{dp}{2\sqrt{p^2 + m^2}} \exp \left( -ix_0 \sqrt{p^2 + m^2} + i\mathbf{x} \cdot \mathbf{p} \right). \] (2.3.3)

- Covariance properties with respect to the Poincaré group. Let $I_s$ and $I_t$ be the space (time) inversion in the Minkowski space $\mathbb{R}^4$. Let $U_{a,A}, U_{t}$ be the unitary operators realizing the $SL(2, \mathbb{C})$ transformations and the space inversion respec-
and \( U_t \) the anti-unitary operator realizing the time inversion; then we require:
\[
U_{a,A} A^\mu(x) U_{a,A}^{-1} = \delta(A^{-1})^\mu \nu A^\nu(\delta(A) \cdot x + a),
\]
\[
U_{a,A} u(x) U_{a,A}^{-1} = u(\delta(A) \cdot x + a), \quad U_{a,A} \bar{u}(x) U_{a,A}^{-1} = \bar{u}(\delta(A) \cdot x + a), \tag{2.3.4}
\]
\[
U_{l,I} A^\mu(x) U_{l,I}^{-1} = (I_l)^\mu \nu A^\nu(I_s \cdot x), \quad U_{l,I} u(x) U_{l,I}^{-1} = -u(I_s \cdot x),
\]
\[
U_{l,I} \bar{u}(x) U_{l,I}^{-1} = -\bar{u}(I_t \cdot x). \tag{2.3.5}
\]

The space-time inversion is: \( U_{l,a} \equiv U_{l} U_{l,I} \).

- Charge invariance. The unitary operator realizing the charge conjugation verifies:
\[
U_C A^\mu(x) U_C^{-1} = -A^\mu(x), \quad U_C u(x) U_C^{-1} = -u(x),
\]
\[
U_C \bar{u}(x) U_C^{-1} = -\bar{u}(x). \tag{2.3.7}
\]
- Moreover, we suppose that these operators are leaving the vacuum invariant:
\[
U_{a,A} \Omega = \Omega, \quad U_{l,I} \Omega = \Omega, \quad U_{l,I} \Omega = \Omega, \quad U_C \Omega = \Omega. \tag{2.3.8}
\]

**Remark 2.11:** One can easily prove that the operators \( U_{a,A}, U_{l,I}, \) and \( U_{l,I} \) are realizing a projective representation of the Poincaré group i.e. they have suitable commutation properties (see [44] rel. (196) from ch. IX. 6). Also the charge conjugation operator commutes with these operators. (As it is well known, there is some freedom in choosing some phases in the definitions of the spatial and temporal inversions [44]; we have made the convenient choice which ensures this commutativity property).

**Remark 2.12:** One can prove that all the operators \( U_{-\nu} \) defined above are leaving the commutation relations invariant. This fact can be used to prove that they are unitary (anti-unitary).

We suppose that in \( \mathcal{H}^{gh} \) we have, beside the scalar product, a sesqui-linear form \( \langle \cdot, \cdot \rangle \) and we denote the conjugate of the operator \( O \) with respect to this form by \( O^\dagger \). One can completely characterize this form by requiring:
\[
A^\mu(x)^\dagger = A^\mu(x), \quad u(x)^\dagger = u(x), \quad \bar{u}(x)^\dagger = -\bar{u}(x). \tag{2.3.9}
\]

Now, we define in \( \mathcal{H}^{gh} \) an important operator called supercharge according to:
\[
Q = \int_{\mathbb{R}^4} d^3x \, \partial^\mu A^\mu(x) \bar{\psi} u(x) \tag{2.3.10}
\]
and one can prove the following properties:
\[
Q \Omega = 0 \tag{2.3.11}
\]
and
\[
\{Q, u(x)\} = 0, \quad \{Q, \bar{u}(x)\} = -i \partial^\mu A^\mu(x), \quad [Q, A^\mu(x)] = i \partial_\mu u(x). \tag{2.3.12}
\]
From these properties one can derive
\[ Q^2 = 0; \]
so we also have
\[ \text{Im}(Q) \subseteq \text{Ker}(Q). \tag{2.3.14} \]

Let us consider the Fock space \( \mathcal{H}^{gh} \) obtained applying to the vacuum \( \Omega \) the fields \( A_\mu(x), u(x) \) and \( \tilde{u}(x) \). We denote by \( \mathcal{W} \) the linear space generated by all Wick monomials acting in this Fock space. Such a linear combinations are called Wick polynomials. If \( M \) is a Wick monomial we define by \( gh_\pm(M) \) the degree in \( u \) (resp. in \( \tilde{u} \)). The ghost number is, by definition, the expression:
\[ gh(M) \equiv gh_+(M) - gh_-(M). \tag{2.3.15} \]

Then we define the operator:
\[ d_QM \equiv: QM : = (-1)^{gh(M)} : MQ : \tag{2.3.16} \]
on monomials \( M \) and extend it by linearity to the whole \( \mathcal{W} \). The operator \( d_Q : \mathcal{W} \to \mathcal{W} \) is called the BRST operator; its properties are following elementary from the properties of the supercharge: beside the Leibnitz rule we have:
\[ d_Qu = 0, \quad d_Q\tilde{u} = -i \partial^\mu A_\mu, \quad d_QA_\mu = i \partial_\mu u. \tag{2.3.17} \]

As a consequence of (2.3.13), it verifies:
\[ d_Q^2 = 0. \tag{2.3.18} \]

Now one can prove that for any Wick monomial \( W \) we have:
\[ U_g(d_QW)U_g^{-1} = d_QU_gWU_g^{-1}, \quad \forall g = (a, A), I_s, I_t, C. \tag{2.3.19} \]
The proof of this relation is elementary if \( W \) is one of the basic fields \( A^\mu, u, \tilde{u} \). Then one extends it by induction for a Wick monomial with an arbitrary number of factors. As a corollary we have:
\[ U_gQ = QU_g, \quad \forall g = (a, A), I_s, I_t, C. \tag{2.3.20} \]

Then we have the central result

**Theorem 2.13:** The sesqui-linear form \( \langle \cdot, \cdot \rangle \) factorizes to a well-defined scalar product on the completion of the factor space \( \text{Ker}(Q)/\text{Im}(Q) \). Then there exists the following Hilbert spaces isomorphism:
\[ \text{Ker}(Q)/\text{Im}(Q) \simeq \mathcal{F}_{\text{photon}}; \tag{2.3.21} \]

The representation of the Poincaré group and the charge conjugation operator are factorizing to \( \text{Ker}(Q)/\text{Im}(Q) \) and are producing unitary operators with the exception of the time (and space-time) inversions which are anti-unitary.

We will need the following relation in the next Section:
\[ (d_QW)^\dagger = -(-1)^{gh(W)} d_Q(W^\dagger) \tag{2.3.22} \]
valid for any Wick monomial. The proof is similar to the proof of (2.3.19): one shows elementary that the relation is true for any of the basic fields \( A^\mu, u, \tilde{u} \) and extends it to an arbitrary product by induction.

We remind that if \( O \) is a self-adjoint operator verifying the condition

\[
d_Q O = 0
\]  (2.3.23)

then it induces a well defined operator \([O]\) on the factor space \( \text{Ker}(Q)/\text{Im}(Q) \simeq \mathcal{F}_\text{photon} \). This kind of observables on the physical space are called \textit{gauge invariant observables}. However, the operators of the type \( d_Q O \) are inducing a null operator on the factor space, so are not interesting. For more general considerations on this point see [9].

Usually one has to add into the game \textit{matter} fields. These are operators for which one has to give separately the corresponding canonical (anti)commutation relations and transformation rules with respect to the Poincaré group and charge conjugation. By definition, we keep the same expression for the supercharge and construct the physical Hilbert space by the same factorization procedure. In particular, this will mean that the BRST operator acts trivially on the matter fields.

We can formulate now what we mean by a perturbation theory of electromagnetism + matter. By definition, this means that we can construct in \( \mathcal{H}^{gh} \) the set of chronological products \( T(X) \) as in the Subsection 2.1 and we impose in addition a factorization condition to the physical Hilbert space. To avoid infra-red divergence problems, we adopt as said in the Introduction the condition (1.0.3) which we prefer to write into the form:

\[
d_Q T(X) = i \sum_{l=1}^{n} \frac{\partial}{\partial x_{l}} T_{l}^{\mu}(X), \quad \forall n \in \mathbb{N}^{*}
\]  (2.3.24)

for some (auxiliary) chronological products \( T_{l}^{\mu}(X), l = 1, \ldots |X| \). In particular, for \(|X| = 1 \) we have

\[
d_Q T(x) = i \frac{\partial}{\partial x^\mu} T^\mu(x).
\]  (2.3.25)

for some Wick polynomials \( T^\mu(x) \).

By definition, this is the \textit{gauge invariance} condition. It can be connected with the usual approaches based on the Ward identities imposed on the (renormalized) Feynman distributions.

Usually, one can extend the definition of the various symmetries of the theory (Poincaré covariance and charge conjugation invariance) to the case of electromagnetism + matter in such a way that all the properties mentioned before remain true.

\section{Renormalizability of quantum electrodynamics}

\subsection{The interaction Lagrangian}

By definition, in this case the matter field is a Dirac field of mass \( m \) denoted by \( \psi(x) = \psi_{a}(x)^4_{\alpha=1} \). To describe this field we need Dirac matrices \( \gamma^\mu, \mu = 0, \ldots, 3 \) for
which we prefer the chiral representation [44]:

\[ \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3; \quad (3.1.1) \]

here \( \sigma_i, \ i = 1, 2, 3 \) are the Pauli matrices. This is a representation in which the matrix \( \gamma_3 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3 \) is diagonal:

\[ \gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.1.2) \]

We denote as usual \( \bar{\psi}(x) = \psi(x)^* \gamma_0 \); it is convenient to consider \( \psi(\bar{\psi}) \) as a column (line) vector. As before, the Dirac field is characterized by:

- Equation of motion (which is, of course the **Dirac equation**):
  \[ (i\gamma \cdot \partial + m) \psi(x) = 0. \quad (3.1.3) \]

- Canonical (anti)commutation relations:
  \[ [\psi(x), A^\mu(y)] = 0, \quad [\psi(x), u(y)] = 0, \quad [\psi(x), \bar{u}(y)] = 0, \]
  \[ \{ \psi(x), \psi(y) \} = 0, \quad \{ \psi(x), \bar{\psi}(y) \} = S_m(x - y) \times 1; \quad (3.1.4) \]
  here \( S_m, \ m \geq 0 \) is a 4 \times 4 matrix given by:
  \[ S_m(x) \equiv (i\gamma \cdot \partial + m) D_m(x). \quad (3.1.5) \]

- Covariance properties with respect to the Poincaré group:
  \[ U_{a,A}\psi(x) U_{a,A}^{-1} = S(A^{-1}) \psi(\delta(A) \cdot x + a), \quad (3.1.6) \]
  \[ U_{I,\mu}\psi(x) U_{I,\mu}^{-1} = i\gamma_0\psi(I_s \cdot x), \quad U_{I,\mu}\psi(x) U_{I,\mu}^{-1} = -i\gamma_4\gamma_3\psi(I_s \cdot x); \]
  here
  \[ S(A) \equiv \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}. \quad (3.1.7) \]

- Charge conjugation invariance. The unitary operator realizing the charge conjugation verifies:
  \[ U_c\psi(x) U_c^{-1} = \gamma_0\gamma_2\bar{\psi}(x)^T. \quad (3.1.8) \]

These relations should be added to the ones from the preceding Subsection. It can be proved that Remark 2.11 stays true.

By definition, the interaction Lagrangian is:

\[ T(x) \equiv e : \bar{\psi}(x) \gamma_\mu \psi(x) : A^\mu(x) = e(\gamma_\mu)_{a,\beta} : \bar{\psi}_a(x) \psi(x)_\beta : A^\mu(x) \quad (3.1.9) \]

(here \( e \) is a real constant: the electron charge) and one can verify easily that we have:

\[ U_{a,A}T(x) U_{a,A}^{-1} = T(\delta(A) \cdot x + a), \quad \forall A \in SL(2, \mathbb{C}), \]
\[ U_{I,\mu}T(x) U_{I,\mu}^{-1} = T(I_s \cdot x), \quad U_{I,\mu}T(x) U_{I,\mu}^{-1} = T(I_s \cdot x), \]
\[ U_cT(x) U_c^{-1} = T(x), \quad (3.1.10) \]
\[ [T(x), T(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad \text{s.t.} \quad x \sim y, \quad (3.1.11) \]

\[ T(x)^\dagger = T(x) \quad (3.1.12) \]

and

\[ gh(T(x)) = 0. \quad (3.1.13) \]

The most important property is (2.3.25) with:

\[ T^\mu(x) \equiv e : \bar{\psi}(x) \gamma^\mu \psi(x) : u(x) = e(\gamma^\mu)_{\alpha, \beta} : \bar{\psi}_\alpha(x) \psi(x)_\beta : u(x). \quad (3.1.14) \]

We list below some obvious properties of the preceding expressions which are similar to the properties above:

\[ U_{a, A} T^\mu(x) U_{a, A}^{-1} = \delta(A^{-1})^\mu_\rho T^\rho(\delta(A) \cdot x + a), \quad \forall A \in SL(2, \mathbb{C}), \]

\[ U_{I, t} T^\mu(x) U_{I, t}^{-1} = (I_s)^\mu_\rho T^\rho(I_t \cdot x), \quad U_{I, t} T^\mu(x) U_{I, t}^{-1} = (I_s)^\mu_\rho T^\rho(I_t \cdot x), \]

\[ U_C T^\mu(x) U_C^{-1} = T^\mu(x). \quad (3.1.15) \]

\[ [T^\mu(x), T^\rho(y)] = 0, \quad [T^\mu(x), T(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad \text{s.t.} \quad x \sim y, \quad (3.1.16) \]

\[ T^\mu(x)^\dagger = T^\mu(x) \quad (3.1.17) \]

and

\[ gh(T^\mu(x)) = 1. \quad (3.1.18) \]

These properties can be easily deduced from the definitions of the various symmetry transformations and the explicit expression (3.1.14).

From the explicit expressions of the second order chronological product obtained in [39] one can verify that we have similar relations in the second order of perturbation theory.

We close by mentioning that properties (2.3.19), (2.3.20) and (2.3.22) remain true.

### 3.2 Gauge invariance of quantum electrodynamics

To be able to use the formalism from Section 2 we choose \( T_j(x) \) to be all Wick monomials of canonical dimension \( \omega_j \leq 4 \). The expressions \( gh_\pm(T_j) \) are the ghost numbers of these monomials. Let us note that we must have formul\( \alpha \)e of the following type:

\[ T(x) = c^j_j T_j(x), \quad T^\mu(x) = c^\mu_j T_j(x) \quad (3.2.1) \]

with \( c^j_j, c^\mu_j \) some real constants. If \( c^j_j \neq 0 \) then, the index \( j \) is even and \( gh(T_j) = 0; \) if \( c^\mu_j \neq 0 \) then the index \( j \) is odd and \( gh_+(T_j) = 1, \) \( gh_-(T_j) = 0. \) The various conditions of covariance listed at the end of the preceding Subsection imply some restrictions on these constants:

\[ D(A)_{kj} c_k = c_j, \quad (S_s)_{kj} c_k = c^s_j, \quad (S_l)_{kj} c_k = c_l_j, \quad C_{kj} c^\mu_k = c^\mu_j, \]

\[ (S_s)_{kj} c'_k = c'_j, \quad (S_l)_{kj} c'_k = c'_{kj}, \quad (S_l)_{kj} c_k = c_l_j, \quad (S_s)_{kj} c'_k = c'_l_j. \quad (3.2.2) \]
In this Subsection we prove that the chronological products can be chosen such that gauge invariance is valid in every order of perturbation theory. We leave aside covariance with respect to space-time symmetries for the moment and prove the main result.

**Theorem 3.1:** Suppose that $T(x)$ is given by (3.1.9) above. Then the distributions $T(X)$ can be constructed such that, beside the conditions of symmetry, $SL(2, \mathbb{C})$-covariance, causality and unitarity (the conditions from Subsection 2.1 for the case $P = 1$) also verify charge conjugation invariance and gauge invariance:

\[
U_C T(x_1, \ldots, x_n) U_C^{-1} = T(x_1, \ldots, x_n),
\]

\[
d_Q T(x_1, \ldots, x_n) = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^i} T_l^\mu(x_1, \ldots, x_n), \quad \forall n \in \mathbb{N}^*
\]

where $T_l^\mu(X)$, $l = 1, \ldots, |X|$ are some (auxiliary) chronological products.

The generic expressions of these chronological products are:

\[
T(X) = \sum_{I, J, K} : \prod_{i \in I} \psi(x_i) t_{i, I, J, K}^{\rho_k}(X) \prod_{j \in J} \psi(x_j) : \prod_{k \in K} A_{\rho_k}(x_k),
\]

and

\[
T_l^\mu(X) = \sum_{I, J, K, \mu} : \prod_{i \in I} \psi(x_i) t_{i, I, J, K}^{\mu; \rho_k}(X) \prod_{j \in J} \psi(x_j) : \prod_{k \in K} A_{\rho_k}(x_k) : u(x_l),
\]

where: a) The sums runs over all distinct triplets $I, J, K \subset \{1, \ldots, n\}$ verifying $|I| = |J|$. (for $T_l^\mu(X)$ one has the restriction $l \notin K$). b) By $\rho_k$ we mean the set $\{\rho_k\}_{k \in K}$. c) The expression $t_{i, I, J, K}^{\rho_k}$ and $t_{i, I, J, K}^{\mu; \rho_k}$ are numerical distributions (in fact, they take values in the matrix space $M_\mathbb{C}(4, 4)^{\otimes |I|}$) and they have corresponding symmetry properties: antisymmetric in the couples $(i, x_i)$, $i \in I$ and also in the couples $(j, x_j)$, $j \in J$ and symmetric in the couples $(k, x_k)$, $k \in K$. d) We have the following limitations on the order of singularity:

\[
\omega(t_{i, I, J, K}^{\rho_k}) \leq 4 - |K| - \frac{3}{2} (|I| + |J|), \quad \omega(t_{i, I, J, K}^{\mu; \rho_k}) \leq 3 - |K| - \frac{3}{2} (|I| + |J|).
\]

\[
(3.2.7)
\]

e) For $I \neq \emptyset$ the matrix-valued distributions $t$ do not contain the matrix $\gamma_5$; for $I = \emptyset$ the numerical-valued distributions $t$ do not contain the completely antisymmetrical tensor $\epsilon^{\mu_1 \mu_2 \mu_3 \mu_4}$.

**Proof:**

(i) The main trick is to extend the induction hypothesis from the Subsection 2.2. We suppose that we have constructed the chronological products $T_j(X)$, $|X| \leq n - 1$ having the properties (2.1.3–2.1.5) and (2.1.10). We will prove inductively that the operators $T(X)$ and $T_l^\mu$ from the statement are some linear combinations of $T_j(X)$.

We supplement the induction hypothesis as follows.

- Ghost number content:

\[
gh_\pm(T_j(X)) = \sum_{j \in J} gh_\pm(T_j).
\]

\[
(3.2.8)
\]
• Charge conjugation invariance:
  \[ U_C T_{j_1, \ldots, j_p} (x_1, \ldots, x_p) U_C^{-1} = c_{j_1 k_1} \ldots c_{j_p k_p} T_{k_1, \ldots, k_p} (x_1, \ldots, x_n), \]
  \[ p = 1, \ldots, n - 1. \]  
  \[ (3.2.9) \]

• Gauge invariance: we define the following expressions \( \forall l = 1, \ldots, p = |X| \leq n - 1: \)
  \[ T(X) = c_{j_1} \ldots c_{j_p} T_{j_1, \ldots, j_p} (X), \quad T_i^\mu (X) = c_{j_1}^\mu \ldots c_{j_p}^\mu T_{j_1, \ldots, j_p} (X); \]  
  \[ (3.2.10) \]
then we require
  \[ d_Q T(X) = i \sum \frac{\partial}{\partial x_l^\mu} T_i^\mu (X). \]  
  \[ (3.2.11) \]

• Wick expansion property: the induction hypothesis (2.2.13) has a more precise
  form for the expressions \( T(X), \quad T_i^\mu (X), \quad |X| \leq n - 1 \) defined above, namely we
  have the relation (3.2.5), (3.2.6); let us note that the formula (3.2.7) is a particular
  case of the general formula (2.2.15).

(ii) We define the expressions the expressions \( T_l \) according to the general formula (2.2.9); then we define \( T(X) \) and \( T_i^\mu (X) \) for \( |X| \leq n - 1 \) by formulae similar
  to (3.10). If we use the restrictions (3.2.2) and these definitions then we easily
  obtain from the induction hypothesis that we have the following properties:

• Symmetry: for all \( P \in \mathcal{P}, \quad p \leq n - 1: \)
  \[ T(x_{P(1)}, \ldots, x_{P(p)}) = T(x_1, \ldots, x_p), \]
  \[ T_{P^{-1}(l)}^\mu (x_{P(1)}, \ldots, x_{P(p)}) = T_{l}^\mu (x_1, \ldots, x_p). \]  
  \[ (3.2.12) \]

• Covariance with respect to \( SL(2, \mathbb{C}) \): for all \( (a, A) \in inSL(2, \mathbb{C}), \quad p \leq n - 1: \)
  \[ U_{a, A} T(x_1, \ldots, x_p) U_{a, A}^{-1} = T(\delta(A) \cdot x_1 + a, \ldots, \delta(A) \cdot x_p + a), \]
  \[ U_{a, A} T_i^\mu (x_1, \ldots, x_p) U_{a, A}^{-1} = \delta(A^{-1})^\mu_\rho T_i^\rho (\delta(A) \cdot x_1 + a, \ldots, \delta(A) \cdot x_p + a). \]  
  \[ (3.2.13) \]

• Causality: for all \( X_1 \geq X_2, \quad |X_1| + |X_2| \leq n - 1: \)
  \[ T(X_1 X_2) = T(X_1) T(X_2), \quad T_i^\mu (X_1 X_2) = T_i^\mu (X_1) T(X_2) + T(X_1) T_i^\mu (X_2). \]  
  \[ (3.2.14) \]

• Unitarity: for all \( |X| \leq n - 1: \)
  \[ \bar{T}(X) = T(X)^\dagger, \quad \bar{T_i^\mu}(X) = T_i^\mu (X)^\dagger. \]  
  \[ (3.2.15) \]

• Charge conjugation: for all \( |X| \leq n - 1: \)
  \[ U_C T(X) U_C^{-1} = T(X), \quad U_C T_i^\mu (X) U_C^{-1} = T_i^\mu (X). \]  
  \[ (3.2.16) \]

• Ghost number: for all \( |X| \leq n - 1: \)
  \[ gh(T(X)) = 0, \quad gh(T_i^\mu (X)) = 1. \]  
  \[ (3.2.17) \]
(iii) From (3.2.4) and the definitions of the antichronological products \( \bar{T}(X) \) and \( \bar{T}^\mu(X) \) we have

\[
d\bar{Q} \bar{T}(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} \bar{T}^\mu_l(X), \quad |X| \leq n - 1.
\]  

(3.2.18)

We define the expressions: \( A'(X), R'(X), A'^\mu_l(X), R'^\mu_l(X), D(X), D'^\mu_l \) for \(|X| = n\) from the expressions \( A'_l(X), R'_l(X) \) given by the formulae (2.2.16) and (2.2.17) in the Subsection 2.2; obviously we must use formulae of the same type as (3.2.10). Let us consider a \( SL(2, \mathbb{C}) \)-covariant causal splitting: \( D_f(X) = A_f(X) - R_f(X) \), with \( \text{supp}(A) \subset \Gamma^+(x_n) \) and \( \text{supp}(R) \subset \Gamma^-(x_n) \) and verifying the restriction (2.2.23). One can show easily that this causal splitting can be also made invariant with respect to charge conjugation. We define the expressions \( A(X), R(X), A^\mu_l(X), R^\mu_l(X) \) again in analogy with (3.2.10).

(iii) Now we investigate the possible obstruction to the extension of the identity (3.2.4) for \(|X| = n\). First we obtain by direct computation that:

\[
d\bar{Q} A'_l(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} A'^\mu_l(X), \quad d\bar{Q} R'_l(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} R'^\mu_l(X), \quad |X| = n;
\]

by substitution we get:

\[
d\bar{Q} D(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} D'^\mu_l(X), \quad |X| = n.
\]  

(3.2.20)

We substitute here the causal decompositions and we get:

\[
d\bar{Q} A(X) - i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} A'^\mu_l(X) = d\bar{Q} R(X) - i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} R'^\mu_l(X).
\]  

(3.2.21)

Now we can reason as in lemma 2.7 –– see formula (2.2.29): the left hand side has support in \( \Gamma^+(x_n) \) and the right hand side in \( \Gamma^-(x_n) \) so the common value, denoted by \( P_n \) should have the support in \( \Gamma^+(x_n) \cap \Gamma^-(x_n) = \{x_1 = \ldots = x_n\} \). This means that we have:

\[
d\bar{Q} A(X) - i \sum_{l \in X} \frac{\partial}{\partial x_l^\mu} A'^\mu_l(X) = P(X),
\]  

(3.2.22)

where \( P(X) \equiv P_n(x_1, \ldots, x_n) \) is a quasi-local operator called anomaly. It has the following structure:

\[
P(X) = \sum_L [p_L(\partial) \delta(X)] W_L(X),
\]  

(3.2.23)

where: a) \( W_L \) are Wick monomials depending on the fields \( u, \psi, \tilde{\psi} \) and \( A_\mu \) and their first order derivatives; b) The dependence on the ghost field is linear; c) \( p_L \) are polynomials in the derivatives with the maximal degree restricted by

\[
\text{deg} (p_L) + \omega_L \leq 5.
\]  

(3.2.24)
Moreover, the anomaly verifies $SL(2, \mathbb{C})$-covariance:

$$U_{a,A} P(x_1, \ldots, x_n) U_{a,A}^{-1} = P(\delta(A) \cdot x_1 + a, \ldots, \delta(A) \cdot x_n + a) ,$$

$$\forall (a,A) \in inSL(2, \mathbb{C}) ,$$

(3.2.25)

charge conjugation invariance:

$$U_C P(X) U_C^{-1} = P(X)$$

(3.2.26)

and gauge invariance:

$$d_Q P(X) = i \sum \frac{\partial}{\partial x_i^q} P^q_l(X)$$

(3.2.27)

for some Wick polynomials $P^q_l(X)$.

(iv) There are a lot of restrictions on the anomaly $P(X)$ and we will be able to prove here that it can be chosen to be equal to 0. The basic idea comes from cohomology theory. We try to exhibit the anomaly in the following form:

$$P(X) = d_Q N(X) - i \sum_{i=1}^n \frac{\partial}{\partial x_i^q} N^q_l(X) + P^q(X)$$

(3.2.28)

with the Wick expressions $N(X)$ (resp. $N^q_l$) having the structure given by (3.2.5) (resp. (3.2.6)). Then we can make the redefinition

$$A \rightarrow A + N , \quad A^l_i \rightarrow A^l_i + N^q_l$$

(3.2.29)

and in this way we replace the anomaly $P$ by a presumably simpler form $P^q$. In this sense, the first two terms from the formula (3.2.28) can be considered as a co-boundary. In fact, these redefining should be made on the expressions $A_j(X)$, $|X| = n$. We note that if we start with some splitting $A_j(X)$, $R_j(X)$ having the same symmetry properties as the couple $A^l_j(X)$, $R^l_j(X)$, the redefinition process might affect these symmetry properties. This is not a problem, because we want to construct the distributions $T_j(X)$, $|X| = n$ having the right symmetry properties; but this can be done at the very end by the symmetrization process (2.2.40). However, we must verify that the generic forms (3.2.5) and (3.2.6) are not spoiled by such redefinitions.

We start with the restrictions (3.2.24)), the $SL(2, \mathbb{C})$-covariance (3.2.25) and the structure (3.2.23) and we have the following generic expression for the anomaly:

$$P(X) = \sum_{i=1}^{17} \mathcal{P}_i(X)$$

(3.2.30)

where the list of the polynomials in the right hand side is:

$$\mathcal{P}_1(X) \equiv \sum_{ijl} : \overline{\psi}(x_i) p^\rho_{ijl}(X) \psi(x_j) : \partial^\rho u(x_l) ,$$

$$\mathcal{P}_2(X) \equiv \sum_{ijl} : [\partial^\rho \overline{\psi}(x_i)] q^\rho_{ijl}(X) \psi(x_j) : u(x_l) ,$$

$$\mathcal{P}_3(X) \equiv \sum_{ijl} : \overline{\psi}(x_i) r^\rho_{ijl}(X) [\partial^\rho \psi(x_j)] : u(x_l) ,$$

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\[ \mathcal{P}_4(X) \equiv \sum_{ijl} : \bar{\psi}(x_i) \ p_{ijl}(X) \ \psi(x_j) : u(x_l) , \]
\[ \mathcal{P}_5(X) \equiv \sum_{ijkl} : \bar{\psi}(x_i) \ p_{ijkl}^{\rho}(X) \ \psi(x_j) : A^\rho(x_k) u(x_l) , \]
\[ \mathcal{P}_6(X) \equiv \sum_{l} p_l(X) \ u(x_l) , \]
\[ \mathcal{P}_7(X) \equiv \sum_{kl} p_{kl}^{\mu}(X) \ \partial_{\mu} u(x_l) , \]
\[ \mathcal{P}_8(X) \equiv \sum_{kl} q_{kl}^{\nu}(X) \ A_{\rho}(x_k) u(x_l) , \]
\[ \mathcal{P}_9(X) \equiv \sum_{kl} p_{kl}^{\rho \lambda}(X) \ A_{\rho}(x_k) \ \partial_{\lambda} u(x_l) , \]
\[ \mathcal{P}_{10}(X) \equiv \sum_{kl} q_{kl}^{\rho \lambda}(X) \ \partial_{\lambda} A_{\rho}(x_k) u(x_l) , \]
\[ \mathcal{P}_{11}(X) \equiv \sum_{klm} p_{klm}^{\rho \lambda \zeta}(X) : A_{\rho}(x_k) A_{\lambda}(x_m) : u(x_l) , \]
\[ \mathcal{P}_{12}(X) \equiv \sum_{klm} p_{klm}^{\rho \lambda \zeta}(X) : A_{\rho}(x_k) A_{\lambda}(x_m) \ \partial_{\zeta} u(x_l) , \]
\[ \mathcal{P}_{13}(X) \equiv \sum_{klm} p_{klm}^{\rho \lambda \zeta}(X) : A_{\rho}(x_k) \ (\partial_{\zeta} A_{\lambda}(x_m)) : u(x_l) , \]
\[ \mathcal{P}_{14}(X) \equiv \sum_{klmr} p_{klmr}^{\rho \lambda \zeta \tau}(X) : A_{\rho}(x_k) A_{\lambda}(x_m) A_{\zeta}(x_r) : u(x_l) , \]
\[ \mathcal{P}_{15}(X) \equiv \sum_{klmr} p_{klmr}^{\rho \lambda \zeta \tau}(X) : A_{\rho}(x_k) A_{\lambda}(x_m) A_{\zeta}(x_r) \ \partial_{\tau} u(x_l) , \]
\[ \mathcal{P}_{16}(X) \equiv \sum_{klmr} q_{klmr}^{\rho \lambda \zeta \tau}(X) : A_{\rho}(x_k) A_{\lambda}(x_m) (\partial_{\tau} A_{\zeta}(x_r)) : u(x_l) , \]
\[ \mathcal{P}_{17}(X) \equiv \sum_{klmrs} p_{klmrs}^{\rho \lambda \zeta \tau \upsilon}(X) : A_{\rho}(x_k) A_{\lambda}(x_m) A_{\zeta}(x_r) A_{\tau}(x_s) : u(x_l) , \]
where the expressions \( p_{\cdots} \) are numerical distributions having natural symmetry properties, \( SL(2, \mathbb{C}) \)-covariant and are restricted by the degree condition (3.2.24). In the preceding sums we have, according to (3.2.23), the restriction that all variables appearing in the factors \( u \) and \( A_{\mu} \) should be distinct. It is obvious that all these polynomials also verify charge conjugation invariance which gives immediately:
\[ \mathcal{P}_i = 0 , \quad i = 6, 7, 11, 12, 13, 17 . \]

We analyse now the other cases. We use a particular form of the redefinition procedure (3.2.28) + (3.2.29), namely “integrations by parts” transferring derivatives from the polynomials on the Wick monomials and we see that we can eliminate some other monomials from the list, namely:
- \( \mathcal{P}_1 \) by redefining \( \mathcal{P}_2 \) and \( \mathcal{P}_3 \);
- \( \mathcal{P}_9 \) and \( \mathcal{P}_{10} \) by redefining \( \mathcal{P}_8 \);
- \( \mathcal{P}_{15} \) and \( \mathcal{P}_{16} \) by redefining \( \mathcal{P}_{14} \).

We analyse now the remaining cases.

(2–3) The structure of the numerical distributions \( q \) and \( r \) must be of the type:
\[ q_{ijl}^{\rho}(X) = K_{ijl}^{\rho} \delta^{n-1}(X) , \]
\[ (3.2.33) \]
If we use in these cases Dirac equation (3.1.3) it follows that the sum of these two contributions is of the form \( P_4 \).

(4) The structure of the numerical distribution \( p \) is

\[
p_{ijl}(X) = \left( K_{ijl} \mathbf{1} + \sum_m K_{ijlm} \gamma \cdot \partial_m \right) \delta^{n-1}(X). \tag{3.2.34}
\]

We integrate by parts, make the redefinition (3.2.29) and conclude that we can take \( P_4(X) \) of the form:

\[
P_4(X) = \delta^{n-1}(X) \times \{ K_1 : [\partial_\mu \bar{\psi}(x_n)] \gamma^\nu \psi(x_n) : + K_2 : \bar{\psi}(x_n) \gamma^\nu [\partial_\mu \psi(x_n)] : u(x_n) \\
+ K_3 : \bar{\psi}(x_n) \gamma^\nu \psi(x_n) : \partial_\mu u(x_n) + K_4 : \bar{\psi}(x_n) \psi(x_n) : u(x_n) \}. \tag{3.2.35}
\]

In the first two terms we use Dirac equation (3.1.3) and we can include these contributions in the last term. Now it follows by elementary computations that charge conjugation invariance imposes \( K_4 = 0 \). We end up with

\[
P_4(X) = -id_\xi \delta^{n-1}(X) K_3 : \bar{\psi}(x_n) \gamma^\nu \psi(x_n) : A_\mu(x_n) \] \tag{3.2.36}

so we can perform the redefinition (3.2.29) and fix \( P_4 = 0 \).

(5) In this case, the numerical distributions \( p \) have the same structure as in the cases (2–3) so we end up with an expression of the type:

\[
P_5(X) = \delta^{n-1}(X) \times K : \bar{\psi}(x_n) \gamma^\nu \psi(x_n) : A_\mu(x_n) u(x_n). \tag{3.2.37}
\]

This contribution is zero because of charge conjugation invariance.

(8) In this case we have

\[
q_\mu(X) = \left( \sum_i C_i \partial^\mu_i + \sum_{ijk} D_{ijk} \partial^\mu_i \partial^\nu_j \partial^\rho_k \right) \delta^{n-1}(X). \tag{3.2.38}
\]

Now we use the gauge invariance of the anomaly (3.2.37). It is not very difficult to prove that this condition implies

\[
\frac{\partial}{\partial x_k^\mu} q_{kl}(X) = k \leftrightarrow l. \tag{3.2.39}
\]

It is sufficient to consider the case \( k = 1, l = 2 \) and we get

\[
C_i = 0, \quad i = 1, 3, 4, \ldots, \\
D_{111} = 0, \quad D_{ijk} = 0, \quad \forall i, j, k \geq 3, \quad D_{222} = 2D_{112} + D_{211}. \tag{3.2.40}
\]

It follows that we have

\[
q_{12}^\mu(X) = \left[ C_2 \partial_2^\mu + D_{122} \partial_1^\mu \partial_2^2 + 2D_{112} (\partial_1^\mu \partial_1 \partial_2 + \partial_2^\mu \partial_2^2) \right. \\
+ D_{211} (\partial_2^2 + \partial_1^2) + 2D_{212} \partial_2^\mu \partial_1 \partial_2 \left. \right] \delta^{n-1}(X). \tag{3.2.41}
\]

We can exhibit the corresponding contribution to \( P_8 \) in the form of a coboundary with

\[
N(X) = \frac{i}{2} q^{\mu
u}(X) : A_\mu(x_1) A_\nu(x_2) : \tag{3.2.42}
\]

\[
N_1^\mu(X) = 0, \quad N_2^\mu(X) = q^{\mu
u}(X) A_\nu(x_1) u(x_2) \tag{3.2.43}
\]
where:

\[ q^{\mu \nu} (X) = \{ g^{\mu \nu} [ C_2 + D_{211} ( \partial_1^2 + \partial_2^2 ) + 2 D_{212} \partial_1 \cdot \partial_2 ] + D_{122} \partial_1^4 \partial_2^2 + 2 D_{112} ( \partial_1^\mu \partial_1^\nu + \partial_2^\mu \partial_2^\nu ) \} \delta^{n-1} (X). \]  

(3.2.44)

So, we can make the redefinition (3.2.29) and put \( P_8 = 0. \)

(14) In this case we have

\[ P^{\mu \nu \rho} (X) = \sum_i ( C_i g^{\mu \nu} \partial_i^\rho + D_i g^{\mu \rho} \partial_i^\nu + E_i g^{\rho \nu} \partial_i^\mu ) \delta^{n-1} (X). \]  

(3.2.45)

As above, we use the gauge invariance condition (3.2.27); we obtain:

\[ \frac{\partial}{\partial x_k} P^{\mu \nu \rho}_{kl} (X) = k \leftrightarrow l. \]  

(3.2.46)

It is sufficient to consider the case \( k = 1, \ l = 2 \) and we get

\[ C_i = D_i = E_i = 0, \quad i = 1, 3, 4, \ldots, \quad C_2 = D_2 \]  

(3.2.47)

so we have:

\[ P^{\mu \nu \rho}_{12\ell 4} (X) = [ C_2 ( g^{\mu \nu} \partial_2^\rho + g^{\mu \rho} \partial_2^\nu ) + E_2 g^{\rho \nu} \partial_2^\mu ] \delta^{n-1} (X). \]  

(3.2.48)

The corresponding contribution to \( P_{14} \) is a coboundary with:

\[ N(X) = \frac{i}{4} (2 C_2 + E_2) \delta^{n-1} (X) : A_\mu (x_n) A^\mu (x_n) A^\nu (x_n) A^\nu (x_n) : \]  

(3.2.49)

\[ N_1^\mu (X) = 0, \quad N_2^\mu (X) = (2 C_2 + E_2) \delta^{n-1} (X) : A_\nu (x_n) A^\nu (x_n) A^\mu (x_n) u (x_n) : \]  

(3.2.50)

so, we can make \( P_{14} = 0 \) as in the case (8).

In conclusion, we can make in (3.2.22) \( P(X) = 0 \) i.e. we have:

\[ d_\partial A(X) = i \sum_{x_i \in X} \frac{\partial}{\partial x_i^\mu} A_i^\mu (X). \]  

(3.2.51)

(v) Now we can implement unitarity as in lemma 2.9. The substitutions (2.2.28) obviously do not affect causality and the \( SL(2, \mathbb{C}) \)-covariance; if we use the relation (2.3.22) we can prove that gauge invariance is also not spoiled.

Now let us define the chronological products \( T_j (X) \), \( |X| = n \) according to the standard formulae (2.2.39). Then we define the expressions \( T(X) \) and \( T^p (X) \) according to the formula (3.2.10). Then these expressions satisfy the Poincaré covariance, causality and unitarity conditions (3.2.13), (3.2.14) and (3.2.15) for \( p = n \). If we make the substitution (2.2.40) then we also have the symmetry axioms (3.2.12) for \( p = n \) and the preceding properties are not affected. One should apply the symmetrization operator in the indices 1, 2, \ldots, \( n \) to the relation (3.2.4) to see that this relation is not affected by the process of symmetrization.

It is now easy to see that the generic expressions (3.2.5) and (3.2.6) are valid for \( |X| = n \) also. This finishes the proof. \( \Box \)
Remark 3.2: One can provide a proof of this theorem working only with the chronological products $T(X)$ and $T_0(X)$ subject to convenient induction hypothesis (essentially the properties outlined in (ii) of the proof). Because there are no derivatives in the first order expressions $T(x)$ and $T_0(x)$ one can avoid the combinatorial argument, based on the compact form of Wick theorem, which leads to (2.2.24).

Remark 3.3: We can fix the covariance properties with respect to the spatial and temporal inversions using a straightforward generalization of the argument from [39] ch. 4.4. In this way we can fix the chronological products such that, beside the properties appearing in the statement of the theorem, we also have:

$$U_t T(x_1, \ldots, x_n) U_t^{-1} = T(I_t \cdot x_1, \ldots, I_t \cdot x_n),$$ (3.2.52)

$$U_l T(x_1, \ldots, x_n) U_l^{-1} = T(I_t \cdot x_1, \ldots, I_t \cdot x_n),$$ (3.2.53)

We now determine the non-unicity of the chronological products $T(X)$. We have:

Proposition 3.4: Suppose that $T(X)$ and $T'(X)$ are two solutions of the renormalization theory for quantum electrodynamics, verifying gauge invariance in the sense of the preceding theorem and the power counting condition (2.2.15). If we have $T_0(X) = T(X)$, $\forall |X| \leq n - 1$ then we have for $|X| = n$:

$$T'(X) - T(X) = d_0 N(X) + i \sum_{l \in X} \frac{\partial}{\partial x_l} N_l'(X) + C_1 \delta^{n-1}(X) : \bar{\psi}(x_n) \gamma^\mu \psi(x_n) : A_\mu(x_n) + C_2 \delta^{n-1}(X) : \bar{\psi}(x_n) \psi(x_n):$$ (3.2.54)

with $i^n C_i \in \mathbb{R}$. In particular, we can absorb the $C_1$-term in the interaction Lagrangian by redefining the coupling constant up to order $n$ and the $C_2$-term by redefining the mass.

Proof: From the gauge invariance condition, the expression $F(X) \equiv T(X) - T'(X)$ verifies:

$$d_0 F(X) = i \sum_{l \in X} \frac{\partial}{\partial x_l} F_l'(X)$$ (3.2.55)

for some Wick polynomials $F_l'(X)$. Now we have from lemma 2.5

$$F(X) = \sum_J [\kappa J(\partial) \delta^{n-1}(X)] W_J(x),$$ (3.2.56)

where $p_J$ are polynomials verifying the restrictions

$$\text{deg}(p_J) + \omega_J \leq 4, \quad \forall i.$$ (3.2.57)

We also have all the properties of symmetry, covariance with respect to $SL(2, \mathbb{C})$ and charge conjugation invariance. We list all polynomials fulfilling these requirements and we obtain the result.

Remark 3.5: The mass renormalization, given by the last term of (3.2.54) destroys the existence of the adiabatic limit, at least in order two of the perturbation theory.
In this case, the matter field is a complex scalar field.

4.1 The interaction Lagrangian

4 Renormalizability of scalar quantum electrodynamics

4.1 The interaction Lagrangian

In this case, the matter field is a complex scalar field \( \phi \). The Hilbert space of the model is generated by applying on the vacuum \( \Omega \) the free fields \( A^\mu(x), u(x), \bar{u}(x), \phi(x) \) and \( \bar{\phi}(x) \). We completely characterize these fields as in Subsection 3.1. The electromagnetic field and the corresponding ghost fields are determined by the same relations as in this Subsection.

- Equation of motion; beside (2.3.1) and (3.1.3) we have Klein-Gordon equations for the fields \( \phi \) and \( \bar{\phi} \):
  \[
  (\Box + m^2) \phi(x) = 0, \quad (\Box + m^2) \bar{\phi}(x) = 0. \tag{4.1.1}
  \]

- Canonical (anti)commutation relations; beside (2.3.2) we require:
  \[
  [A^\mu(x), \phi(y)] = 0, \quad [\phi(x), u(y)] = 0, \quad [\phi(x), \bar{u}(y)] = 0,
  \]
  \[
  [\phi(x), \phi(y)] = 0, \quad [\bar{\phi}(x), \bar{\phi}(y)] = 0, \quad [\phi(x), \bar{\phi}(y)] = D_m(x - y) \times 1. \tag{4.1.2}
  \]
\textbullet{} SL(2, \mathbb{C})-covariance. We keep the corresponding relations (2.3.4) from Subsection 2.3 and we add:

\begin{equation}
U_{a,A} \phi(x) \ U_{a,A}^{-1} = \phi(\delta(A) \cdot x + a) . \tag{4.1.3}
\end{equation}

\textbullet{} Space-time invariance; beside (2.3.5) and (2.3.6) we impose:

\begin{equation}
U_{I} \phi(x) \ U_{I}^{-1} = \phi(I_{s} \cdot x) , \quad U_{I} \phi(x) \ U_{I}^{-1} = \overline{\phi}(I_{t} \cdot x) . \tag{4.1.4}
\end{equation}

\textbullet{} Charge conjugation invariance. The unitary operator realizing the charge conjugation verifies (2.3.7) and

\begin{equation}
U_{C} \phi(x) \ U_{C}^{-1} = \overline{\phi}(x) . \tag{4.1.5}
\end{equation}

\textbullet{} Moreover, we suppose that all these operators are leaving the vacuum invariant i.e. we have (2.3.8).

As in the case of spinorial electrodynamics, one can show that the operators $U_{a,A}$, $U_{I}$, $U_{I}$, and $U_{I}$ are a representation of the Poincaré group; however, in this case the operators realizing the discrete symmetries square to identity. Moreover, the charge conjugation operator commutes with all the preceding operators.

We give as before in $\mathcal{H}^{gh}$ the sesquilinear form $\langle \cdot, \cdot \rangle$ which is completely characterize by requiring beside (2.3.9):

\begin{equation}
\phi(x) ^{\dagger} = \overline{\phi}(x) . \tag{4.1.6}
\end{equation}

The expression of the supercharge remains the same (2.3.10) and one can see that (2.3.11) stays true; to (2.3.12) one must add:

\begin{equation}
[Q, \phi] = 0 , \quad [Q, \overline{\phi}] = 0 . \tag{4.1.7}
\end{equation}

Alternatively if $\mathcal{W}$ is the linear space of all Wick monomials acting in the Fock space $\mathcal{H}^{gh}$ containing the fields $A_{\mu}(x), u(x), \overline{\phi}(x)$ and $\phi(x)$ then the expression of the BRST operator is determined by

\begin{equation}
d_{Q}u = 0 , \quad d_{Q} \overline{u} = -i \overleftarrow{\partial}^{\mu} A_{\mu} , \quad d_{Q}A_{\mu} = i \partial_{\mu} u , \quad d_{Q} \phi = 0 , \quad d_{Q} \overline{\phi} = 0 . \tag{4.1.8}
\end{equation}

The relations (2.3.19) and (2.3.20) are valid in this case also.

As in the case of spinorial QED we give the expression of the interaction Lagrangian.

\begin{equation}
T(x) \equiv ie : \overline{\phi}(x) \overleftarrow{\partial}_{\mu} \phi(x) : A^{\mu}(x) . \tag{4.1.9}
\end{equation}

(here $e$ is a real constant called the electron charge) and can verify easily that the properties (2.1.13), (2.1.14) and (2.1.15) are true. We also have:

\begin{equation}
U_{I} T(x) \ U_{I}^{-1} = -T(I_{s} \cdot x) , \quad U_{I} T(x) \ U_{I}^{-1} = T(I_{t} \cdot x) , \tag{4.1.10}
\end{equation}

As in the case of spinorial electrodynamics we have first order gauge invariance i.e. (2.3.25) with

\begin{equation}
T^{\mu}(x) \equiv ie : \overline{\phi}(x) \overleftarrow{\partial}^{\mu} \phi(x) : u(x) \tag{4.1.11}
\end{equation}
and we have, as in the case of spinorial QED, the corresponding causality, unitarity and covariance properties. We note the change of sign in the relations describing the behaviour of spinorial QED and scalar QED with respect to spatial inversion.

4.2 Gauge invariance of scalar QED

We prove that scalar QED as defined in the previous Subsection is gauge invariant, i.e. the \( S \)-matrix is factorizing (in the adiabatic limit) to the physical Hilbert space; the proof will be extremely similar to the proof from Subsection 3.2 and we will indicate only the appropriate changes.

**Theorem 4.1:** Suppose that \( T_1 \) is given by (4.1.9) above. Then the distributions \( T_n \) can be constructed such that, beside the conditions of symmetry, \( SL(2, \mathbb{C}) \)-covariance, causality and unitarity (2.1.3), (2.1.4), (2.1.5), (2.1.10), verify charge conjugation and gauge invariance:

\[
U_c T(x_1, \ldots, x_n) U_c^{-1} = T(x_1, \ldots, x_n),
\]

\[
d_Q T(x_1, \ldots, x_n) = \sum_{l=1}^n \frac{\partial}{\partial x_l} T_l^\mu(x_1, \ldots, x_n), \quad \forall n \in \mathbb{N}^*,
\]

where \( T_l^\mu(X), l = 1, \ldots, |X| \) are some auxiliary chronological products.

The expressions \( T(X) \) and \( T_l^\mu(X) \) have the following generic form:

\[
T(X) = \sum_{I,J,K} : \prod_{i \in I} \bar{\phi}(x_i) t^{\mu,k}_{I,J,K}(X) \prod_{j \in J} \phi(x_j) : \prod_{k \in K} A_{\mu_k}(x_k) : + \ldots
\]

and

\[
T_l^\mu(X) = \sum_{I,J,I \notin K} : \prod_{i \in I} \bar{\phi}(x_i) t^{\mu,\rho,k}_{I,I,J,K}(X) \prod_{j \in J} \phi(x_j) : \prod_{k \in K} A_{\mu_k}(x_k) : u(x_l) + \ldots,
\]

where by \( \ldots \) we mean contributions where a partial derivative can appear on the scalar fields \( \phi \) or \( \bar{\phi} \). Here the expressions \( t^{\mu,k} \) are numerical distributions, we use the same conventions as in the case of spinorial electrodynamics.

**Proof:** We observe that the induction hypothesis is valid for \( p = 1 \) according to the results from the previous Subsection. We suppose that it is true for \( p \leq n - 1 \) and prove it for \( p = n \). We can proceed in analogy with the proof from Subsection 3.2. Everything stays unchanged with minor modification. The anomaly \( P(X) \) is constructed in the same way and is constrained by the same conditions as in the spinorial case: it has a polynomial structure (order two derivatives on the scalar fields are allowed), it is charge conjugation invariant, \( SL(2, \mathbb{C}) \)-covariant and there are power counting and ghost degree limitations.

(ii) The list of possible anomalies consists of two types of terms: (a) the expressions \( \mathcal{P}_6 - \mathcal{P}_{17} \) from (3.2.31); (b) anomalies containing a pair : \( \bar{\phi}\phi \): with possible derivatives. We list these terms below:

\[
\mathcal{P}'_1(X) = \sum_{ijkl} \mathcal{P}'^{(1)uv}_{ijkl}(X) : \partial_\mu \bar{\phi}(x_i) \phi(x_j) : \partial_\nu u(x_l),
\]

\[
\mathcal{P}'_2(X) = \sum_{ijkl} \mathcal{P}'^{(2)uv}_{ijkl}(X) : \phi(x_i) \partial_\mu \phi(x_j) : \partial_\nu u(x_l),
\]
\[ \mathcal{P}'_3(X) = \sum_{ijl} p^{(3)\mu
u}_{ijl}(X) : \partial_\mu \bar{\phi}(x_i) \partial_\nu \phi(x_j) : u(x_l), \]
\[ \mathcal{P}'_4(X) = \sum_{ijl} p^{(4)\mu
u}_{ijl}(X) : \partial_\mu \partial_\nu \bar{\phi}(x_i) \phi(x_j) : u(x_l), \]
\[ \mathcal{P}'_5(X) = \sum_{ijl} p^{(5)\mu
u}_{ijl}(X) : \bar{\phi}(x_i) \partial_\mu \partial_\nu \phi(x_j) : u(x_l), \]
\[ \mathcal{P}'_6(X) = \sum_{ijl} p^{(6)\mu}_{ijl}(X) : \bar{\phi}(x_i) \phi(x_j) : \partial_\mu u(x_l), \]
\[ \mathcal{P}'_7(X) = \sum_{ijl} p^{(7)\mu}_{ijl}(X) : \partial_\mu \bar{\phi}(x_i) \phi(x_j) : u(x_l), \]
\[ \mathcal{P}'_8(X) = \sum_{ijl} p^{(8)\mu}_{ijl}(X) : \bar{\phi}(x_i) \partial_\mu \phi(x_j) : u(x_l), \]
\[ \mathcal{P}'_9(X) = \sum_{ijl} p^{(9)}_{ijl}(X) : \bar{\phi}(x_i) \phi(x_j) : u(x_l), \]
\[ \mathcal{P}'_{10}(X) = \sum_{ijkl} p^{(10)\mu
u}_{ijkl}(X) : \bar{\phi}(x_i) \phi(x_j) : A_\mu(x_k) \partial_\nu u(x_l), \]
\[ \mathcal{P}'_{11}(X) = \sum_{ijkl} p^{(12)\mu
u}_{ijkl}(X) : \partial_\nu \bar{\phi}(x_i) \phi(x_j) : A_\mu(x_k) u(x_l), \]
\[ \mathcal{P}'_{12}(X) = \sum_{ijkl} p^{(13)\mu
u}_{ijkl}(X) : \bar{\phi}(x_i) \partial_\nu \phi(x_j) : A_\mu(x_k) u(x_l), \]
\[ \mathcal{P}'_{13}(X) = \sum_{ijkl} p^{(10)\mu}_{ijkl}(X) : \bar{\phi}(x_i) \phi(x_j) : A_\mu(x_k) u(x_l). \] (4.2.5)

Degree limitations of the type (3.2.24) are still valid and are easy to write down. The anomalies of the type (a) are treated exactly as in the Subsection 3.2. For the rest, we outline the arguments, which are quite similar to the previous ones.

First, we use integrations by parts to obtain formul\ae of the type (3.2.28) and make the redefinition (3.2.29). In this way we can get rid of the following terms:

- \( \mathcal{P}'_1 \) and \( \mathcal{P}'_2 \) by redefining \( \mathcal{P}'_3 - \mathcal{P}'_5 \);
- \( \mathcal{P}'_6 \) by redefining \( \mathcal{P}'_7 \) and \( \mathcal{P}'_8 \);

We give the analysis of the remaining cases.

(3') In this case the generic form of the distribution: \( p^{\mu\nu}_{ijl} \) is

\[ p^{\mu\nu}_{ijl}(X) = g^{\mu\nu} p_{ijl} \delta^{n-1}(X). \] (4.2.6)

This means that we have:

\[ \mathcal{P}'_3 = \delta^{n-1}(X) : \partial^\nu \bar{\phi}(x_n) \partial_\mu \phi(x_n) : u(x_n). \] (4.2.7)

But one proves rather elementary that from charge conjugation invariance we have \( \mathcal{P}'_3 = 0 \).

(4' − 5') The structure of the numerical distribution is again (4.2.6); we can use in this case Klein-Gordon equation (4.1.1) and we obtain expressions of the type (9'). If we redefine \( \mathcal{P}'_9 \) we eliminate this term.
(7′–8′) In these cases the structure of the numerical distribution is

\[ P_{ij}^\mu(X) = \sum_m p_{ijlm} \partial_m^\mu \delta^{n-1}(X). \]  

By integrations by parts, we get two types of terms: contributions of the type (3′) which disappear because of charge conjugation invariance and contributions of the type (9′). If we redefine \( P'_0 \) we eliminate these terms.

(9′) The generic form of the numerical distribution is

\[ p_{ijl}(X) = c_0 \delta^{n-1}(X) + \sum_m p_{ijlm} \partial_m \delta^{n-1}(X). \]  

After integration by parts and utilization of the equations of motion we get an expression of the type:

\[ P'_0 = \delta^{n-1}(X) [c_1 : \bar{\phi}(x_n) \phi(x_n) : + c_2 : \partial^\mu \bar{\phi}(x_n) \partial_\mu \phi(x_n) :] u(x_n) \]  

which is zero because of charge conjugation invariance.

(10′) In this case the numerical distributions are of the generic form (4.2.6) so we have a coboundary:

\[ P'_{10} = c \delta^{n-1}(X) : \bar{\phi}(x_n) \phi(x_n) : A_\mu(x_n) \partial^\mu u(x_n) \]

\[ = -\frac{ic}{2} d_Q[\delta^{n-1}(X) : \bar{\phi}(x_n) \phi(x_n) : A_\mu(x_n) A^\mu(x_n)] \]  

(11′–13′) The generic expressions of the numerical distributions are of the type (4.2.6) and condition of gauge invariance gives relations of the type (3.2.46). As a result, these contributions are zero.

From now on the proof goes as in the case of spinorial electrodynamics. \( \square \)

**Remark 4.2:** One can implement space-time invariance in the form:

\[ U_{i_r} T(x_1, \ldots, x_n) U_{i_r}^{-1} = (-1)^n T(I_{i_r} \cdot x_1, \ldots, I_{i_r} \cdot x_n), \]  

\[ U_{i_t} T(x_1, \ldots, x_n) U_{i_t}^{-1} = \bar{T}(I_{i_t} \cdot x_1, \ldots, I_{i_t} \cdot x_n), \]  

We note the sign difference in the relation expression spatial inversion covariance.

**5 Conclusions**

We have succeed to give complete proof of the renormalizability of quantum electrodynamics. It is much simpler than the proofs from the literature based on the usual BRST transformation (see [45] and literature quoted there). Its main advantages are the conceptual clearness and the fact that the role of Feynman graphs is minimal (only in writing Wick theorem). It is also clear that the main trick in implementing gauge invariance comes from the the gauge invariance of the anomaly (3.2.27).

It is an interesting problem to extend this analysis to the case of the QCD and the standard model. The QCD case was done in a series of papers [10, 11, 13, 14] using some rather complicated identities – the so-called C-g identities – and
group theoretical analysis. In [5] the case of Yang-Mills fields of zero mass coupled with Dirac fields through vectorial currents is investigated. It this natural to expect that the methods used here will simplify this type of analysis. We will try to extend this analysis to the general case of the standard model, where scalar ghosts and axial currents are present in a forthcoming paper.

It is natural to expect that this approach to renormalization theory gives the same result as the usual procedure of Becchi, Rouet, Stora and Tyutin. A proof of this fact based on the quantum Noether method [34] appears in [4]. However, we can give here a much simpler argument. It is clear that both approaches verify the same set of axioms so one can use the characterization of the non-uniqueness given above.

We close with a comparison with the recent work [9]. First we note that the formalism used in this reference for the construction of the chronological products is more compact: one can get rid of the multi-indices $J$ by constructing in an inductive way expressions of the type $T(A_1(x_1) \ldots A_n(x_n))$ for arbitrary Wick monomials $A_i$; the induction hypothesis are somewhat different, but can be proved to be equivalent to Epstein-Glaser framework [3]. The connection between the two formalism can be understood from the formulae (3.2.10). Next, the emphasis is put on the construction of the interacting fields according to the standard procedure of Epstein-Glaser as a formal power series. From the interacting fields one can construct the local (Haag-Kastler) field algebra without performing the adiabatic limit. The construction of this algebra is done using the interacting version $Q_{\text{int}}$ of the supercharge $Q$. Also, the condition of gauge invariance used in this paper is formulated in a slightly different way in [9] (see also [41]):

$$\frac{\partial}{\partial y^\mu} T(j^\mu(y) (A_1(x_1) \ldots A_n(x_n))) = i \sum_{j=1}^n \delta(y - x_j) T(A_1(x_1) \ldots (\theta A_j)(x_j) \ldots A_n(x_n)). \tag{5.0.14}$$

Here $j^\mu$ is the current and $\theta A$ is the infinitesimal variation of the field $A$ with respect to the usual $U(1)$ gauge transformation; this condition obviously expresses charge conservation. The condition of gauge invariance (3.2.4) of Scharf and collaborators used in this paper can be obtained as a consequence of the preceding condition. One proves in [9] that the condition (5.0.14) can be implemented in every order of perturbation theory.

The author wishes to thank dr. M. Dütsch for many discussions and useful comments.

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