Dominant Topologies in Euclidean Quantum Gravity

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Abstract

The dominant topologies in the Euclidean path integral for quantum gravity differ sharply according on the sign of the cosmological constant. For \( \Lambda > 0 \), saddle points can occur only for topologies with vanishing first Betti number and finite fundamental group. For \( \Lambda < 0 \), on the other hand, the path integral is dominated by topologies with extremely complicated fundamental groups; while the contribution of each individual manifold is strongly suppressed, the “density of topologies” grows fast enough to overwhelm this suppression. The value \( \Lambda = 0 \) is thus a sort of boundary between phases in the sum over topologies. I discuss some implications for the cosmological constant problem and the Hartle-Hawking wave function.

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It has been forty years since Wheeler first suggested that the topology of spacetime might be subject to quantum fluctuations [1]. We do not yet know whether the resulting picture of “spacetime foam” correctly describes the universe, but the potential implications are clearly important: for example, fluctuations of topology are a key element in Coleman’s proposed wormhole/baby universe solution to the cosmological constant problem [2]. If such fluctuations occur only at the Planck scale, a full-fledged quantum theory of gravity may be necessary to understand their effect. If they can occur at larger scales, however, it may be possible to treat the standard Einstein action as an effective field theory [3] from which we can draw useful conclusions.

To understand the quantum mechanics of spacetime topology, one needs a formalism in which spacetime is treated as a unified entity. Canonical quantum gravity may allow us to investigate changes in spatial topology, but a path integral approach seems more natural if we are interested in the topology of spacetime as a whole. In particular, much of the work on this subject (see, for example, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]) has been based on path integral techniques in Euclidean quantum gravity, that is, general relativity “Wick rotated” to Riemannian (positive definite) metrics.

If the Einstein action is treated as part of an effective field theory for distances larger than the Planck length, one should not worry too much about higher-loop corrections, which will be suppressed by powers of the Planck mass. It is thus sensible to treat the path integral in a saddle point approximation. The purpose of this article is to describe some features of saddle points and to discuss possible implications for spacetime foam. Some of the results presented here are old, but are not widely known among physicists; others are new. A brief report on results for $\Lambda < 0$ has appeared in reference [15].

1 The Euclidean Path Integral

In the Euclidean path integral approach to quantum gravity, the simplest quantity to compute is the partition function [4], which can be written formally as a path integral

$$Z[\Lambda] = \sum_{M} \int [dg] \exp\{-I_E[g]\}. \quad (1.1)$$

The Euclidean action $I_E$ in equation (1.1) is

$$I_E[g] = -\frac{1}{16\pi L_P^2} \int_M (R - 2\Lambda) \sqrt{g} \, d^4x, \quad (1.2)$$

where $g$ is a Riemannian metric on the manifold $M$ and $L_P$ denotes the Planck length. The sum in (1.1) is a “sum over topologies,” that is, a sum over topologically distinct manifolds $M$. It should be noted from the outset that the meaning of such a sum is not entirely clear. Four-manifolds are not classifiable—that is, there is no algorithm that can determine whether two arbitrary four-manifolds are homeomorphic—and the
sum over topologies may yield a number that is noncomputable in the sense of Turing.\textsuperscript{[14]} Geroch and Hartle have discussed the implications of noncomputability for physics \[16\], and argue that it need not be a disaster: even if a quantity like the partition function is not computable, one may be able to obtain approximations to any desired degree of accuracy.

In principle, one might extend the sum (1.1) to objects other than manifolds—for instance, pseudomanifolds \[12\] or “conifolds” \[17, 18\]. Conifolds, in particular, occur at the boundaries of moduli spaces of Einstein metrics, and are classifiable, thus allowing us to evade the problem of noncomputability discussed above. One might also introduce relative phases between terms in the sum. In simpler systems, such phases are restricted by the requirement that amplitudes behave correctly under composition \[19\], but little is known about the case of gravity. For simplicity, I will largely ignore such generalizations, which are unlikely to affect the main conclusions of this paper.

An extremum of the action (1.2) is an Einstein metric, that is, a metric for which

\[ R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{1.3} \]

The classical action for such a metric is

\[ I_E(M, g) = -\frac{\Lambda}{8\pi L^2} Vol(M, g). \tag{1.4} \]

This expression is slightly misleading, however, since the volume of \(M\) depends implicitly on the cosmological constant. We can isolate this dependence by rescaling the metric to set the scalar curvature to \(\pm 12\). (The factor of 12 is conventional; four-manifolds of constant curvature \(\pm 1\) have scalar curvature \(\pm 12\).) To do so, we define

\[ g_{\mu\nu} = \frac{3}{|\Lambda|} \tilde{g}_{\mu\nu}, \tag{1.5} \]

where the rescaled metric \(\tilde{g}\) satisfies (1.3) with \(\Lambda = \pm 3\). The action (1.4) is then

\[ \tilde{I}_E(M, g) = -\frac{9}{8\pi \Lambda L^2} \tilde{v}(M, g), \tag{1.6} \]

where the normalized volume \(\tilde{v}(M, g)\) is the volume with respect to \(\tilde{g}\), and the only dependence on \(\Lambda\) now resides in the overall \(1/\Lambda\) factor.

The normalized volume \(\tilde{v}\) is clearly a geometric quantity, but it is also, in a sense, topological: the set of normalized volumes of Einstein metrics on a manifold \(M\) characterizes the topology of \(M\). In particular, for \(\Lambda < 0\) there is no known example of a manifold that admits two Einstein metrics with different values of \(\tilde{v}\) \[20\]. Roughly

\textsuperscript{*}It is not known whether this is an inherent characteristic of the partition function or merely a problem with the particular representation (1.1). The existence of a “noncomputable” expression for a number is not sufficient to show that the number itself is noncomputable; for example, a sequence of noncomputable numbers can have a computable limit \[16\].

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speaking, \( \tilde{v}(M, g) \) measures the topological complexity of \( M \); for a four-manifold with a constant curvature metric \( g_0 \), for instance, \( \tilde{v}(M, g_0) = 4\pi^2 \chi(M)/3 \), where \( \chi \) is the Euler characteristic. The normalized volume is also closely related to the “minimal volume,” a topological invariant defined as

\[
\text{minvol}(M) = \inf \{ \text{Vol}(M, g) | |K_g| \leq 1 \}.
\]  

(1.7)

Here \( K_g \) is the sectional curvature, that is, the quadratic form

\[
K_g(v, w) = \frac{R_{\mu\nu\rho\sigma} v^\mu w^\nu v^\rho w^\sigma}{(v^\mu v^\mu)(w^\nu w^\nu) - (v^\mu w^\mu)^2}
\]  

(1.8)

and the infimum is over Riemannian metrics on \( M \). Gromov has conjectured that for any manifold \( M \) that admits a hyperbolic metric \( g_0 \), \( \text{minvol}(M) = \tilde{v}(M, g_0) \) [21]; a local version of this conjecture has recently been proven [22].

In the saddle point approximation, the partition function (1.1) is now

\[
Z[\Lambda] = \sum_{(M, g)} \Delta_{(M, g)} \exp \left\{ \frac{9}{8\pi L_P^2} \tilde{v}(M, g) \right\},
\]  

(1.9)

summed over pairs \((M, g)\) of four-manifolds with Einstein metrics. The prefactors \( \Delta_{(M, g)} \) are combinations of Faddeev-Popov determinants coming from gauge-fixing and Van Vleck-Morette determinants coming from small fluctuations around the extrema. Their precise values are not known, but the dependence of \( \Delta_{(M, g)} \) on \( \Lambda \) can be computed from the trace anomaly [11]: up to possible polynomial corrections coming from zero-modes,

\[
\Delta_{(M, g)} \sim \Lambda^{-\gamma/2}, \quad \gamma = \frac{106}{45} \chi(M) - \frac{261}{40\pi^2} \tilde{v}(M, g).
\]  

(1.10)

For our purposes, the crucial observation is that \( \Delta_{(M, g)} \) is no more than exponential in \( \tilde{v} \).

It is useful to rewrite the sum over topologies in equation (1.9) as a sum over normalized volumes,

\[
Z[\Lambda] = \sum_{\tilde{v}} \rho(\tilde{v}) \exp \left\{ \frac{9}{8\pi L_P^2} \tilde{v} \right\}.
\]  

(1.11)

The factor \( \rho(\tilde{v}) \) is a “density of topologies” that counts the number of Einstein manifolds (weighted by \( \Delta_{(M, g)} \)) with a given value of \( \tilde{v} \). For \( \Lambda > 0 \), the exponent in (1.11) picks out the manifold with the largest normalized volume, the four-sphere \( S^4 \). For \( \Lambda < 0 \), manifolds with large normalized volumes—hyperbolic manifolds with large Euler characteristics, for example—are exponentially suppressed. In either case, however, there can be competition between the exponential factor (or “action”) and the density of topologies (or “entropy”). In three spacetime dimensions, it is known that the “entropy” can, in fact, dominate the “action” [14, 23]; one goal of this paper is to see whether the same is true in four dimensions.
Before proceeding further, let us note that not every manifold admits an Einstein metric with any value of the cosmological constant. In fact, as simple a manifold as the connected sum of two tori, $T^4 \# T^4$, admits no Einstein metric [20]. As Hitchin and Thorpe have shown [24,25], in order for a manifold $M$ to admit an Einstein metric, its Euler characteristic and signature must obey the inequality

$$\chi(M) \geq \frac{3}{2} |\tau(M)|.$$  

(1.12)

This condition is necessary, but not sufficient: LeBrun [26] and Sambusetti [27] have constructed infinitely many compact four-manifolds that satisfy the Hitchin-Thorpe inequality but admit no Einstein metric. My philosophy will be that such manifolds can be ignored in the sum over topologies; they are relevant only in higher-loop approximations, which are important at scales at which the Einstein action no longer makes sense as an effective action and the whole approach to quantum gravity must be reconsidered.

## 2 Positive Cosmological Constant

Let us begin by examining the case $\Lambda > 0$. From equation (1.11), the largest individual contribution to the partition function will come from the manifold that admits an Einstein metric with the largest value of $\tilde{v}$. In four dimensions, this is the four-sphere $S^4$ [29], which has a normalized volume

$$\tilde{v}(S^4) = \frac{8\pi^2}{3}.$$  

(2.1)

Relatively few other examples of manifolds admitting Einstein metrics with $\Lambda > 0$ are known explicitly. Examples include $\mathbb{CP}^2$, $S^2 \times S^2$, and the Page metric on an $S^2$ bundle over $S^2$ [20,30]. Tian and Yau have also found a finite-dimensional moduli space of Einstein metrics with $\Lambda > 0$ on the manifolds

$$\mathbb{CP}^2 \# k \mathbb{CP}^2$$

for $5 \leq k \leq 8$ [11]. (For a general discussion of moduli spaces of Einstein metrics, see [20,32].)

To date, very little is understood about the behavior of the “density of topologies” $\rho(\tilde{v})$ for $\Lambda > 0$, and not much can be said about the sum (1.11). We can, however, make some surprisingly strong statements about topologies that do not appear in the partition function in the saddle point approximation. In particular, Myers has shown

†The Euclidean path integral may be misleading in this case, however. Starting with the canonical formalism for simple reparametrization-invariant systems, Marolf has argued that the correct contribution for a Euclidean instanton is $\exp\{-|\bar{I}_E|\}$ [28]. If this is the case, the sign of the exponent in (1.11) should be changed when $\Lambda > 0$, and large values of $\tilde{v}$ will be suppressed.
that any complete Einstein manifold with a positive cosmological constant necessarily has a finite fundamental group [33]; there are no Euclidean wormholes with $\Lambda > 0$. This fact is instrumental in proving the “no multiple birth” theorem for the Hartle-Hawking wave function with $\Lambda > 0$ [34, 35].

The full proof of Myers’ theorem is quite complicated, but a weaker version is rather straightforward: it is easy to show that a closed Einstein manifold with a positive cosmological constant must have vanishing first Betti number [35, 36]. Indeed, suppose the cohomology $H^1(M, \mathbb{R})$ is nontrivial. Any element of this cohomology can be represented by a harmonic one-form $\omega$, that is, a one-form satisfying

$$\nabla_\mu \omega_\nu - \nabla_\nu \omega_\mu = 0, \quad (2.2)$$

$$\nabla_\mu \omega^\mu = 0. \quad (2.3)$$

We can now differentiate (2.2) and use (2.3) to obtain

$$\nabla^\nu \nabla_\mu \omega_\nu = \nabla_\nu \nabla^\nu \omega_\mu = [\nabla_\nu, \nabla^\nu] \omega_\mu = R_\nu^\rho \omega_\rho = \Lambda \omega_\nu. \quad (2.4)$$

Contracting with $\omega^\nu$ and integrating over $M$, we see that

$$0 = \int_M [\Lambda \omega^\nu \omega_\nu - \omega^\nu \nabla^\nu \nabla_\mu \omega_\nu] \sqrt{g} \, d^4x$$

$$= \int_M [\Lambda \omega^\nu \omega_\nu + (\nabla^\nu \omega^\nu)(\nabla_\mu \omega_\nu)] \sqrt{g} \, d^4x. \quad (2.5)$$

But both terms in the last integrand are nonnegative, so (2.3) implies that $\omega_\mu = 0$, and thus $H^1(M, \mathbb{R}) = 0$.

Note that this proof does not really require that we have an Einstein metric: it is enough to demand that the Ricci tensor in (2.4) be positive everywhere. Generalizations have been found for manifolds in which the Ricci tensor is “mostly” positive, with appropriate restrictions on regions of negative curvature [37, 38, 39]. It is tempting to speculate that these results are Euclidean versions of the well-known fact that traversible Lorentzian wormholes require exotic matter [40].

### 3 Negative Cosmological Constant

We now turn to the case $\Lambda < 0$. Observe first that the dominant contributions to the partition function may differ sharply depending on the sign of $\Lambda$. As we have seen, the manifolds that are important when $\Lambda > 0$ have relatively simple fundamental groups. For $\Lambda < 0$, on the other hand, a contribution—a very large contribution, as we shall see below—comes from hyperbolic manifolds, that is, manifolds with constant negative curvature. Hyperbolic manifolds in four dimensions are obtained as quotients $M \approx \mathbb{H}^4/\pi_1$ of hyperbolic four-space $\mathbb{H}^4$; they typically have very complicated fundamental groups, and can have arbitrarily large first Betti numbers [11]. Such manifolds make
no contribution to the $\Lambda > 0$ partition function in the saddle point approximation. Indeed, it has been shown that if a four-manifold $M$ admits a hyperbolic metric, it is the only Einstein metric on $M$ [42].

As in the case of positive cosmological constant, we have nothing like a complete classification of Einstein manifolds with $\Lambda < 0$, but we know a number of interesting examples. In addition to hyperbolic manifolds [13], these include product manifolds $\Sigma_1 \times \Sigma_2$, where $\Sigma_1$ and $\Sigma_2$ are surfaces of genus $h_1, h_2 > 1$ [4], and compact complex manifolds with negative first Chern class, which always admit Kähler-Einstein metrics with $\Lambda < 0$ [45, 46, 20]. In the latter two examples, one typically finds a whole moduli spaces of metrics, with $\tilde{v}$ constant on the moduli space. When this occurs, the prefactor $\Delta_M$ in (1.9) will include the volume of the moduli space, and the density of topologies $\rho(\tilde{v})$ should incorporate this factor.

In contrast to the $\Lambda > 0$ case, we can now say something useful about the function $\rho(\tilde{v})$; while its complete behavior is not understood, it may be shown that $\rho(\tilde{v})$ increases at least factorially with $\tilde{v}$. Indeed, even if we restrict our attention to the special case of hyperbolic manifolds, $\rho(\tilde{v})$ still exhibits at least factorial growth. This means that the “entropy” in the sum (1.11) dominates the action, and the partition function receives large contributions from arbitrarily complicated topologies.

The following proof was explained to me by Lubotzky [47]. Observe first that if we limit our attention to hyperbolic four-manifolds, the number of manifolds with normalized volumes $\tilde{v}(M, g) < V$ is finite for any finite $V$ [48]. We thus need not worry about moduli spaces; it suffices to simply count manifolds. We begin with a hyperbolic “seed manifold” $M$ with fundamental group $G$, metric $g$, and normalized volume $\tilde{v}(M, g) = \tilde{v}_0$. Any subgroup $G_1 \subset G$ determines a covering manifold $M_1$ of $M$ with fundamental group $G_1$, and $M_1$ inherits a hyperbolic metric $g_1$ from the metric $g$ on $M$. Moreover, if $G_1$ has index $n$ in $G$, then $M_1$ is an $n$-fold cover of $M$, and $\tilde{v}(M_1, g_1) = n\tilde{v}_0$. Hence if we can count the number of index-$n$ subgroups of $G$, and if we can avoid double-counting isometric covering spaces, we can obtain a lower limit on the number of hyperbolic manifolds with normalized volumes $n\tilde{v}_0$.

In reference [11], Lubotzky demonstrates the existence of a hyperbolic manifold $M$ whose fundamental group $G$ maps homomorphically onto a free group $F_r$ of rank $r > 1$. This result is useful because the subgroup growth of free groups is well-understood [19]; for large $n$, the number of index-$n$ subgroups of a free group of rank $r$ grows as

$$N(n, r) \sim n(n!)^{r-1}.$$  

The existence of a surjective homomorphism $G \to F_r$ guarantees that any index-$n$ subgroup of $F_r$ determines an index-$n$ subgroup of $G$, so (3.1) gives a lower limit for the number of subgroups of $G$.

This is not yet the whole story, however. Different subgroups $G_1, G_2 \subset G$ may sometimes give isometric covering spaces of $M$.

\[\text{A subgroup } G_1 \text{ has index } n \text{ in } G \text{ if the number of distinct cosets } G_1g \ (g \in G) \text{ is } n.\]
conjugate in $\text{SO}(4, 1)$, the group of isometries of $\mathbb{H}^4$; that is, $G_2 = g^{-1}G_1g$ for some $g \in \text{SO}(4, 1)$. Fortunately, this condition can be simplified: it may be shown that $G_1$ and $G_2$ must be conjugate in the commensurability group $\text{Comm}(G)$ of $G$. By a theorem of Margulis, if $G$ is nonarithmetic, $\text{Comm}(G)$ is a finite extension of $G$ [50]. Since Lubotzky showed in [41] that the “seed manifold” $M$ could be chosen to have a nonarithmetic fundamental group, we can concentrate on this case.

Suppose first that $G_1$ and $G_2$ are conjugate in $G$, that is, $G_2 = g^{-1}G_1g$ for some $g \in G$. Since $G_1$ is an index-$n$ subgroup of $G$, there is a set $X$ of $n$ elements of $G$ such that $g = hx$ for some $x \in X$, $h \in G_1$. Hence $G_2 = x^{-1}G_1x$, $x \in X$, which means there can be at most $n$ subgroups conjugate to $G_1$. Now, $G_1$ and $G_2$ may actually be conjugate in the larger group $\text{Comm}(G)$. But since $\text{Comm}(G)$ is a finite extension of $G$, a similar argument shows that there can be at most $kn$ conjugate index-$n$ subgroups, where $k$ is the index of $G$ in $\text{Comm}(G)$. The estimate (3.1) thus overcounts covering spaces by at most a factor of $kn$.

Combining these results, we obtain a bound

$$
\rho(n\tilde{v}_0) \geq \text{const. } (n!)^{r-1} \sim \exp\{(r-1)n \log n\} \quad \text{(3.2)}
$$

for the density of topologies. It should be clear that this is merely a lower bound—we have considered only hyperbolic manifolds, and only a small subset of hyperbolic manifolds at that. But this result is already sufficient to demonstrate the superexponential growth of $\rho(\tilde{v})$ with $\tilde{v}$, thus showing that the “entropy” dominates the “action” in the sum (1.1).

A further superexponential, contribution to $\rho(\tilde{v})$ comes from manifolds with the product topology $M \approx \Sigma_1 \times \Sigma_2$. Any surface of genus $h > 1$ admits a moduli space $\mathcal{M}_h$ of constant negative curvature metrics [22], and a pair of metrics drawn from $\mathcal{M}_{h_1} \times \mathcal{M}_{h_2}$ determines an Einstein (although not hyperbolic) metric on $\Sigma_1 \times \Sigma_2$. By the Gauss-Bonnet theorem, the volume of a genus $h$ surface with a hyperbolic metric is

$$
\text{Vol}(\Sigma) = \int \sqrt{g} \, d^2x = \frac{1}{2\Lambda} \int R \sqrt{g} \, d^2x = \frac{\pi}{\Lambda} \chi(\Sigma) = \frac{2\pi}{\Lambda}(1-h), \quad \text{(3.3)}
$$

and hence

$$
\tilde{v}(\Sigma_1 \times \Sigma_2) = \frac{4\pi^2}{9}(h_1 - 1)(h_2 - 1). \quad \text{(3.4)}
$$

The number of different product manifolds with normalized volume $V$ is thus roughly equal to the number of factors of $9V/4\pi^2$, which does not grow superexponentially.

In contrast to the hyperbolic case, however, there are many Einstein metrics on each manifold $\Sigma_1 \times \Sigma_2$, and $\rho(\tilde{v})$ must include a factor of the volume of the moduli space.

$^\S$Two subgroups $G$ and $G'$ of a group $\hat{G}$ are commensurable if their intersection $G \cap G'$ has finite index in each of them. The commensurability group $\text{Comm}(G)$ of $G \subset \hat{G}$ is the group $\{g \in \hat{G} : g^{-1}Gg$ is commensurable with $G}\). In the case under consideration here, the fundamental group $G$ is being viewed as a subgroup of $\text{SO}(4, 1)$. That $g \in \text{Comm}(G)$ then follows from the observation that $g^{-1}Gg \cap G \supset G_2$, which has finite index in $G$. 

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of such metrics. Now, the moduli space $\mathcal{M}_h$ for a surface $\Sigma$ has a volume that grows factorially with the genus $h$ [52, 53]. Hence the corresponding volume of the moduli space of Einstein metrics on $\Sigma_1 \times \Sigma_2$ grows at least as fast as $h_1!h_2!$. Product manifolds thus contribute a term

$$\rho(\tilde{v}) \sim \sum_{mn=9\tilde{v}/4\pi^2} (m+1)!(n+1)! \sim \exp \left\{ \frac{9\tilde{v}}{4\pi^2} \log \tilde{v} \right\}$$

(3.5)

to the density of topologies.

4 Implications and Speculations

The clearest implication of the preceding analysis is that the sum over topologies is qualitatively different for $\Lambda > 0$ and $\Lambda < 0$. While there may be some manifolds that admit Einstein metrics with both signs of $\Lambda$, many do not. In particular, the hyperbolic manifolds that lead to the factorial growth (3.2) in $\rho(\tilde{v})$ do not contribute at all to the saddle point approximation for $\Lambda > 0$. The case of vanishing cosmological constant is different still: the classical action is zero, and the partition function is controlled by the one-loop determinants. The value $\Lambda = 0$ thus appears to be a sort of boundary between phases. This does not in itself explain why $\Lambda$ should vanish, of course, but it is suggestive.

For $\Lambda \neq 0$, at least in the saddle point approximation, the key issue is the balance between “action” and “entropy” in the sum (1.11) and similar path integrals. Since the full behavior of $\rho(\tilde{v})$ is not known for either sign of $\Lambda$, the conclusions we can draw are limited. Nevertheless, there is room for some interesting speculation.

For $\Lambda < 0$, it is evident that the factorial growth (3.2) is sufficient to guarantee that the sum (1.11) fails to converge. To say more, we need to understand relative phases: if the terms in (1.11) have identical phases, the series is not even Borel summable, but if the phases differ, it may be possible to define a Borel sum [54]. Relative phases can come from the one-loop determinants $\Delta_M$, or more precisely from negative-eigenvalue modes [55] (and perhaps the zero-modes [6]) of the operators whose determinants appear in this prefactor. The hyperbolic manifolds considered in the preceding section have no global symmetries and no moduli spaces, so no zero-modes are expected. However, the relevant one-loop operator $\Delta^A(1, 1)$, given by

$$\Delta^A(1, 1)\phi_{\mu\nu} = -\nabla_\rho \nabla^\rho \phi_{\mu\nu} - 2R_{\mu\rho\nu\sigma}\phi^{\rho\sigma}$$

(4.1)

acting on symmetric transverse traceless tensors [11, 55], is not positive definite for hyperbolic manifolds, and negative modes may occur. As discussed earlier, additional phases may also be introduced by hand—for example, by adding a term to the action proportional to the Euler characteristic—so the question of Borel summability remains unresolved.

It is worth reemphasizing that the factorial growth (3.2) is only a lower bound; I do not know whether it provides a good estimate for the actual behavior of $\rho(\tilde{v})$. We
may be able to learn more about this question from lattice formulations of quantum gravity. In random triangulation models, the number of geometries on a fixed four-manifold grows exponentially with the number of four-simplices, while the total number of geometries on all manifolds grows factorially [50]. The geometries counted by these models are not the same as those occurring in (1.11)—the metrics need not be Einstein metrics—but the conclusions are similar enough to suggest that the factorial growth (3.2) may describe the actual behavior of \( \rho(\tilde{v}) \), and not merely provide a lower bound.

A very similar divergence in the sum over topologies occurs in string theory [53]. In two dimensions, the divergence can be handled by appealing to matrix models [57], although the cure requires that we abandon any fundamental role for smooth geometries. In four dimensions, we presently know of no comparable solution, but the sum over topologies may ultimately be explained as an expansion in some coupling constant in a more fundamental theory.

Even without such an underlying theory, though, it may be possible to reach some tentative conclusions about the sum over topologies in quantum gravity. The partition function (1.11) is formally identical to that of a thermodynamic system, with \( \tilde{v} \) serving as “energy,” \(-\Lambda\) acting as a “temperature,” and \( \rho(\tilde{v}) \) playing the role of a density of states. As discussed in reference [15], the superexponential growth (3.2) for \( \Lambda < 0 \) is analogous to the behavior of a system with negative heat capacity: just as the microcanonical temperature of such a system is driven to zero as the energy rises, the “microcanonical cosmological constant”

\[
\Lambda_{\text{micro}} = -\frac{9}{8\pi L_P^2} \left( \frac{\partial \ln \rho(\tilde{v})}{\partial \tilde{v}} \right)^{-1}
\]

(4.2)
go to zero as \( \tilde{v} \to \infty \). In a thermodynamic system, this behavior has a straightforward physical origin: rather than increasing the temperature, the addition of energy drives the creation of new states, which are produced so copiously that the energy per state falls. It was argued in [15] that the same may be true in quantum gravity, where processes that would normally increase the absolute value of the vacuum energy might instead merely drive the production of more and more complicated spacetime foam.

For \( \Lambda > 0 \), the thermodynamic analogy also seems to work surprisingly well. From equation (1.11), a positive cosmological constant is analogous to a negative temperature. Negative temperatures typically occur in spin systems, which are characterized by a finite number of states and a maximum energy. It is not known whether the number of manifolds admitting an Einstein metric with \( \Lambda > 0 \) is finite, but it is certainly true that \( \tilde{v} \), the analog of the energy, has a maximum value (2.1), and that it takes that value for only a single topology.

In order to decide whether these analogies are more than coincidences, we need to answer questions like the following: If a phase transition in matter takes place at time \( t_1 \), leading to a nonvanishing vacuum energy density, what is the most likely topology of the universe at time \( t_2 > t_1 \), and what is the probability that the universe will appear to have a nonzero cosmological constant at that time? Questions of this sort
call for a dynamical description of spacetime topology, and such dynamical accounts are notoriously difficult in quantum gravity, requiring us to confront the “problem of time” [58]. For a path integral approach of the kind presented here, the consistent history approach to quantum gravity may offer a fruitful avenue for further research [59].

Finally, it is interesting to consider implications of this work for the Hartle-Hawking wave function of the universe [60]. A manifold with a positive definite Einstein metric is a real tunneling geometry—an instanton for “creation of a universe”—if it can be cut along a codimension-one hypersurface Σ of vanishing extrinsic curvature. The resulting boundary then determines an initial state for a Lorentzian universe [34]. If Σ is separating, this cutting process yields two disjoint pieces, each with a single-component boundary. If Σ is not separating, the resulting manifold has two boundary components, and may represent a “multiple birth” of disconnected universes.

In the construction described in section 3, the “seed manifold” M always contains a three-dimensional submanifold Σ of vanishing extrinsic curvature [11], and can thus be viewed as a real tunneling geometry. Moreover, Σ lifts to n disjoint copies of itself in each of the n-fold covering spaces used in the construction [17]. This means that the covering spaces are themselves real tunneling geometries, each carrying an identical induced metric \( g_Σ \) on a totally geodesic hypersurface Σ. I do not know whether these hypersurfaces are separating. If they are, then the derivation of section 3 demonstrates that the Hartle-Hawking wave function is infinitely peaked at the geometry \( g_Σ \), much as it is in 2+1 dimensions [23]. If they are not, it may still be possible to cap off one of the boundaries in the resulting “multiple birth” geometries, in which case there will again be an infinite peak in the wave function. Work on this question is in progress.

A similar phenomenon occurs for the product manifolds discussed in section 3. For these topologies, a real tunneling geometry can always be obtained by choosing a metric on one of the two surfaces—say Σ₁—that admits an orientation-reversing involution [3]. The resulting Lorentzian universe has the spatial topology \( S^1 \times Σ_2 \), with an initial hypersurface that carries a direct sum metric \( dθ^2 \oplus g_2 \) for some metric \( g_2 \) in the moduli space \( M_{h_2} \).

The Hartle-Hawking wave function is a functional of this boundary metric, and can thus be viewed as a function on \( M_{h_2} \). If we fix a point in \( M_{h_2} \), contributions to the wave function will come from manifolds \( Σ_1 \times Σ_2 \) for every topology \( Σ_1 \) of genus \( h_1 > 1 \), and for every metric \( g_1 \in M_{h_1} \) on \( Σ_1 \) that admits an orientation-reversing involution. For each \( h_1 > 1 \), the corresponding density of topologies will therefore include a factor proportional to the volume of \( M_{h_1}^+ \), the moduli space of metrics that admit orientation-reversing involutions. While the arguments of reference [52] do not apply directly to this moduli space, it is very likely that they can be extended to show that its volume grows superexponentially with \( h_1 \). If this is the case, the Hartle-Hawking wave function will again have infinite peaks, now for the topologies \( S^1 \times Σ_2 \).
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