Braneworld dynamics with vacuum polarization

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Abstract

We investigate the cosmological dynamics of a brane Universe when quantum corrections from vacuum polarization are taken into account. New vacuum de Sitter points existing on Randall-Sundrum brane are described. We show also that quantum correction can destroy the DGP de Sitter solution on induced gravity brane.

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Investigations of quantum effects in a strong gravitational field and their applications in cosmology have a long story. Since the beginning of 70-th many interesting results modifying the standard Friedmann cosmology due to vacuum polarization and particle production have been obtained. In particular, a vacuum de Sitter solution and an inflationary regime driven solely by vacuum polarization without any matter [11] was described even earlier than a common scalar field inflationary scenario. A detailed analysis of cosmology with vacuum polarization (we will consider only this effect in the present paper) have been done in [2]. After some period of stagnation, this problem begins to attract a considerable attention last several years, mainly due to development of modified gravity models. The form of extra terms in cosmological equations of motion caused by vacuum polarization does not depend on a particular theory of gravity, however, peculiarities of a background metric in these theories could result
in some new dynamical regimes. Recently modifications caused by the vacuum polarization have been studied for regimes with soft future singularities. Such regimes, being impossible in the standard cosmology, are rather typical in some modern cosmological scenarios (see for example, [3]), in particular, they are present in induced gravity brane models [4]. Quantum corrections change the dynamics significantly, leading to a softer singularity or even to non-singular solutions [5, 6, 7].

Another interesting problem is stability of classical solutions with respect to quantum corrections. As quantum terms contain higher time derivatives in comparison with corresponding classical equation of motion, some classical solutions may become unstable. In [7] this instability is described for certain regimes in induced gravity brane cosmology.

It is well known that the vacuum polarization leads to the following vacuum expectation value for the energy density:

$$\rho_q = \langle T_{00} \rangle = k_2 H^4 + k_3(2 \dot{H}H + 6 \dot{H}H^2 - H^2),$$

(1)

where \(H\) is the Hubble parameter, \(k_2, k_3\) depend upon the spin weight of the different fields contributing to the vacuum polarization.

If the classical Friedmann equation has the form of \textit{algebraic} dependence of the Hubble parameter upon the energy density of the Universe \(H = H(\rho)\), substituting \(\rho \rightarrow \rho + \rho_q\) we get a \textit{differential} equation which governs the cosmological evolution with quantum corrections.

It is clear that \(k_2\)-term in \(\rho_q\) can be incorporated into the function \(H(\rho)\). Only \(k_3\)-term containing time derivatives may provide instability. In the paper [7] the case of \(k_2 = 0\) have been studied in braneworld models. It is a good assumption near a soft future singularity where \(\dot{H} \rightarrow \infty\) while \(H\) is finite, and \(k_3\)-term in (1) dominates. In the present paper we describe general features of brane dynamics with quantum corrections in their full form.
We start with modification of classical equations due to $k_2$-term. After that we consider the problem of stability. Plots of the function $H(\rho)$ (which includes $k_2 H^4$ term) presented below can be understood in two different ways. First, the curve $H(\rho)$ can be considered as a track of cosmological evolution in some specific regime when $k_3$-term can be neglected. In this interpretation, $\rho$ is the sum of matter density $\rho_m$ and brane tension $\sigma$. The evolution runs from higher to lower energy (from the right to the left in the plots) till the point $\rho_m = 0, \rho = \sigma$ is reached. On the other hand, these plots can be interpreted as sets of de Sitter fixed points (with $\rho = \sigma$ and $\dot{H} = 0$) of a general dynamics with non-zero $k_3$.

Before we discuss branes it is useful to remember known results in the standard scenario. The modification of the standard Friedmann cosmology caused by the $k_2$-term is shown in Fig.1. The classical cosmology corresponds to the straight line $H^2 \sim \rho$. The case $k_2 < 0$ results only in changing this dependence to $H^2 \sim \sqrt{\rho}$ for large $H$. A positive $k_2$ modifies the situation significantly. In particular, we can see a new vacuum de Sitter point $[8]$. This point is unstable if $k_3 < 0$, on the other hand, the point $(0, 0)$, representing the late-time Friedmann regime is stable for $k_3 < 0$. It means that a trajectory starting in the vicinity of the vacuum de Sitter point has inflationary behavior at the initial stage, then it leaves a neighborhood of de Sitter due to instability. As a result, inflation ends, and finally the trajectory reaches the Friedmann late-time attractor. This scenario of acceleration expansion with the natural exit without any kind of classical matter is called as ”Starobinsky inflation” $[1]$.

We now turn to braneworld models. In order to understand how $k_2$-term modifies the Randall-Sundrum (RS) brane it is necessary to remember about two branches, arising from $\rho^2$ term in the equation of motion. The unmodified
equation for the RS brane is

\[ H^2 = \frac{\Lambda}{6} + \frac{\rho^2}{(9M^3)}. \]  

(2)

Here \( \Lambda \) is a cosmological constant in the bulk, which should be negative in a realistic model, \( M \) is the 5-dimensional Planck mass. The equation (2) when solved with respect to \( \rho \) has two solutions for a given \( H \), one with (+)-sign and the second with (−)-sign before the square root. This equation is represented as a shifted parabola on the plane \((H^2, \rho)\). As the negative brane tension ultimately leads to instability, the second branch of the parabola (which corresponds to negative \( \rho \)) is unphysical. Positive \( k_2 \)-term changes this classical picture in a way, very similar to the standard cosmology case (see the dashed curve in Fig.2) with the (−) branch remaining the unphysical one. A vacuum de Sitter point now corresponds to a positive root of the forth-order equation

\[ (k_2^2/9M^3)H^8 - H^2 + \frac{\Lambda}{6} = 0. \]  

(3)

In the limit \( k_2 \to 0 \) the value of Hubble parameter in this point tends to infinity.

Figure 1: The \( \rho(H) \) dependence in the standard cosmology without \( k_2 \) term of the quantum corrections (solid), with negative \( k_2 \) (dashed) and with positive \( k_2 \) (dot-dashed)
Figure 2: The Randall-Sundrum cosmology without $k_2$-term (solid), with negative $k_2$ (dot-dashed) and with positive $k_2$ (dashed)

The existence of a vacuum solution in the RS brane theory with the quantum corrections was first noticed by Nojiri and Odintsov in [9, 10].

The picture for negative $k_2$ is shown in the dot-dashed curve in Fig.2. The (+) branch is similar to the standard cosmology, however, the (−) branch is also shifted partly to a physical domain $\rho > 0$. In particular, it has a new vacuum de Sitter point. The location the this point is also given by a positive root of (3), however we will see below that stability properties of these two de Sitter points are different.

Finally, we consider induced gravity (IG) brane. The classical equation of motion is

$$m^4(H^2 - \rho/(3m^2))^2 = M^6(H^2 - \Lambda/6).$$

(4)

Here $m$ is the 4-dimensional Planck mass. As the equation of motion is quadratic with respect to $\rho$ there are also two branches of solutions. It should be however noted that in some situations a negative brane tension on an IG brane does not ultimately lead to instability [11], so negative $\rho$ may have a physical sense.
in contrast to a RS brane. The plots in Fig.3 have been done for the case of
Minkowski bulk ($\Lambda = 0$). The (+) branch of the solid curve has a Friedmann-
like limit when the energy density of the brane vanishes ($H \to 0$ if $\rho \to 0$), the
(−) brane has a vacuum de Sitter point, founded first by Dvali, Gabadadze and
Porrati in [12]. We will call it DGP point.

A negative $k_2$ (the dot-dashed curve in Fig.3) does not change the config-
urations of branches. On the other hand, positive $k_2$ can modify the picture
seriously. Large enough positive $k_2$ transforms the diagram so that the lower
branch never enters into the $\rho > 0$ half-plane, and the general picture resembles
the case of RS brane (compare the short-dashed curve in Fig.3 with dashed
curve in Fig.2). It is evident that the DGP point is absent in this case. In
can be easily calculated that this situation is realized if $k_2 > (4/9)(m^6/M^6)$.
Nonzero $\Lambda$ in the bulk does not change this general picture qualitatively, though
the expression for the critical $k_2$ becomes less simple. We can see also that the
upper branch has a vacuum de Sitter point for any positive $k_2$.

As it was shown recently in [7], the effective phantom branch (when $dH/d\rho$
is negative) of IG brane is unstable with respect to $k_3$-term in the vacuum
polarization. Here we will show that this result is rather general and does not
depend on a particular kind of a brane as well as on the value of $k_2$.

We consider a general situation when matter density in the corresponding
effective Friedmann equation is some algebraic (may be multi-value) function of
the Hubble parameter $H$, so the evolution equation for a brane has the form:

$$\rho + \sigma = \tilde{F}_i(H),$$

where the index $i$ marks different branches, $\sigma$ is the brane tension. It is clear
that both RS and IG branes are particular cases of (5).

In the de Sitter point the matter density is diluted, so the equation becomes

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Figure 3: The induced gravity brane cosmology without $k_2$-term (solid), with negative $k_2$ (dot-dashed), with positive $k_2$ which allows the DGP point, denoted as $A$ (long-dashed), and with positive $k_2$ which does not allow the DGP point (short-dashed)

$$\sigma = \tilde{F}_i(H).$$

Now we add the quantum corrections (1). The $k_2$-term can be incorporated into the algebraic functions

$$F_i(H) = \tilde{F}_i(H) + k_2 H^4.$$  

The new de Sitter point is given by

$$\sigma = F_i(H).$$  

The resulting second order differential equation for $H$ can be written in the form of a system of two first order equations

$$\dot{H} = C,$$
$$\dot{C} = -3CH + \frac{c^2}{2m} + \frac{1}{2m k_3} (F_i - \sigma) \equiv f_i(H, C).$$

There is an equilibrium point of this system $(\dot{H}, \dot{C}) = (0, 0)$, which corresponds to the de Sitter solution. In order to investigate stability of this equilibrium
point we need a linearized system:

\[
\dot{C} = \left( \frac{\partial f}{\partial C} \right)_0 C + \left( \frac{\partial f}{\partial H} \right)_0 H, \\
\dot{H} = C.
\]

The eigenvalues of this linearized system have the form:

\[
\mu_{1,2} = \frac{1}{2} \left[ \left( \frac{\partial f}{\partial C} \right)_0 \pm \sqrt{\left( \frac{\partial f}{\partial C} \right)_0^2 + 4 \left( \frac{\partial f}{\partial H} \right)_0} \right].
\]

We can easily see that the first eigenvalue is negative if and only if \( \left( \frac{\partial f}{\partial H} \right)_0 < 0 \).

The second eigenvalue is always negative (because \( \left( \frac{\partial f}{\partial C} \right)_0 = -3H \) is negative in an expanding universe).

Let us now evaluate \( \left( \frac{\partial f}{\partial H} \right)_0 \):

\[
\left( \frac{\partial f}{\partial H} \right)_0 = -\frac{1}{2H^2k_3} [-\sigma + F_i] + \frac{1}{2Hk_3} \frac{\partial F_i}{\partial H}.
\]

In the DeSitter stable point \(-\sigma + F_i = 0\), and we have finally

\[
\left( \frac{\partial f_i}{\partial H} \right)_0 = \frac{1}{2Hk_3} \frac{\partial F_i}{\partial H}.
\]

As a result, if \( \frac{\partial F_i}{\partial H} > 0 \) (a normal branch), the de Sitter solution is stable if \( k_3 < 0 \) (as in the standard cosmology). In the opposite case (a phantom branch) we need \( k_3 > 0 \) for stability.

We can conclude, that if we have \( k_3 < 0 \) in our Universe (which is needed for stability of the Minkowski space), all de Sitter points on phantom branches are unstable.

Note, that we did not specify any particular modified gravity theory, which may even be inspired by some scenario different from brane models. All we need is a function \( H(\rho) \). Of course, this analysis can not be applied to a situation when equation which relates \( H \) and \( \rho \) is a differential one (the most important example of this case is a scalar field with non-zero potential playing the role of matter).

It also should be noted, that, strictly speaking, we have proved the following statement: \textit{A de Sitter point, being a future attractor of a Universe is unstable.}
with respect to vacuum polarization if it located on a phantom branch. However, the results of [7] indicate that an effective phantom regime is unreachable during a cosmological evolution of a IG brane when quantum corrections are taken into account, and it is quite reasonable to suggest that quantum corrections prevent a classical phantom regime from realization in a general situation. This problem requires further investigations.

In any cases, we have obtained new restrictions for possible explanations of a present phantom-like state of ”dark energy” which is often claimed as being favorable by observations [13] [14] [15]. Several years ago it was remarked that a matter with a constant equation of state parameter $\omega < -1$ causes an unwanted Big Rip future singularity [16] [17] [18]. This singularity can be avoided in some scenarios with a phantom scalar field (a scalar field with the wrong sign of the kinetic term) [19] [20], however, the fact that the energy of a phantom scalar field is unbounded below causes severe problem in quantum theory [21] (recently proposed more complicated models for a scalar field phantom see, for example, in [22] [23]). All these problems are absent in modified gravity proposal. In this approach the matter in the Universe remains standard (and, so, there are no problems with matter instabilities), and the effective phantom behavior is achieved due to significant modification of the Friedmann equation. The IG brane is a famous example of such kind of theory [24]. Our results, showing that quantum corrections have significant influence exactly on those branches of cosmological equations which simulates a phantom behavior, make clear that the modify gravity proposal is also not free from instability problems.

Remembering shapes of $H(\rho)$ dependence in braneworld models, we can see that the new branch of RS brane existing for $k_2 < 0$ is stable, as well as new vacuum de Sitter point (3) on this branch. This point has no analog in standard cosmology and is similar to DGP point on IG brane. On the other hand, this point in the case of $k_2 > 0$ is unstable (like in standard cosmology), giving a
possibility of realization of Starobinsky inflation on RS brane. We also can easily see that the Starobinsky inflation on IG brane is always possible (of course, if $k_2 > 0$) on the (+) branch and for $k_2 < (4/9)(m_6/M^6)$ on the (−) branch.

We have studied modifications of brane cosmology caused by vacuum polarization. This effect consists of two different terms in an effective energy density. The term proportional to $H^4$ alters possible fixed points of a cosmological dynamics, while the term containing time derivatives of $H$ may change stability properties of these fixed points. A general condition for the future fixed de Sitter point to be stable have been derived. We should however note that all these results have been obtained when possible quantum corrections in the bulk are neglected. This assumption is reasonable in studies of quiescent future singularities on a brane, because they are singularities of embedding [4], while the bulk remains regular (and, so, far from a quantum regime). Is quantum corrections in the bulk important for a brane dynamics in general situation remains unclear, we leave this problem for future investigations.

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