A NOTE ON FUNDAMENTAL GROUP LATTICES

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Abstract

The main goal of this note is to provide a new proof of a classical result about projectivities between finite abelian groups. It is based on the concept of fundamental group lattice, studied in our previous papers [8] and [9]. A generalization of this result is also given.

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1 Introduction

The relation between the structure of a group and the structure of its subgroup lattice constitutes an important domain of research in group theory. One of the most interesting problems concerning it is to study whether a group $G$ is determined by the subgroup lattice of the $n$-th direct power $G^n$, $n \in \mathbb{N}^*$. In other words, if the $n$-th direct powers of two groups have isomorphic subgroup lattices, are these groups isomorphic? For $n = 1$ it is well-known that this problem has a negative answer (see [4]). The same thing can be also said for $n = 2$, except for some particular classes of groups, as simple groups (see [5]), finite abelian groups (see [3]) or abelian groups with the square root property (see [2]). In the general case (when $n \geq 2$ is arbitrary) we recall Remark 1 of [2], which states that an abelian group is determined by the subgroup lattice of its $n$-th direct power if and only if it has the $n$-th root property. This follows from some classical results of Baer [1].

The starting point of our discussion is given by papers [8] and [9] (see also Section I.2.1 of [7]), where the concept of fundamental group lattice is introduced and studied. It gives an arithmetic description of the subgroup lattice of a finite abelian group and has many applications. Fundamental group lattices were successfully used in [8] to solve the problem of existence and uniqueness of a finite abelian group whose subgroup lattice is isomorphic to a fixed lattice and in [9] to count some types of subgroups of a finite abelian group. In this paper they will be used to prove that the finite abelian groups are determined by the subgroup lattices of their direct $n$-powers, for any $n \geq 2$. Notice that our proof is more simple than the original one. A more general result will be also inferred.

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Most of our notation is standard and will usually not be repeated here. Basic definitions and results on groups can be found in [6]. For subgroup lattice notions we refer the reader to [4] and [7].

In the following we recall the concept of fundamental group lattice and two related theorems. Let $G$ be a finite abelian group and $L(G)$ be the subgroup lattice of $G$. Then, by the fundamental theorem of finitely generated abelian groups, exist (uniquely determined by $G$) numbers $k \in \mathbb{N}^*$ and $d_1, d_2, ..., d_k \in \mathbb{N} \setminus \{0, 1\}$ satisfying $d_1 | d_2 | ... | d_k$ such that

\[ G \cong \prod_{i=1}^{k} \mathbb{Z}_{d_i}. \]

This decomposition of a $G$ into a direct product of cyclic groups together with the form of subgroups of $\mathbb{Z}^k$ (see Lemma 2.1 of [8]) leads us to the following construction:

Let $k \geq 1$ be an integer. Then, for every $(d_1, d_2, ..., d_k) \in (\mathbb{N} \setminus \{0, 1\})^k$, we consider the set $L(k; d_1, d_2, ..., d_k)$ consisting of all matrices $A = (a_{ij}) \in \mathcal{M}_k(\mathbb{Z})$ which satisfy the conditions:

I. $a_{ij} = 0$, for any $i > j$,
II. $0 \leq a_{1j}, a_{2j}, ..., a_{j-1j} < a_{jj}$, for any $j = 1, k$,
III. 1) $a_{11} | d_1$,
   2) $a_{22} | \left( d_2, d_1 \frac{a_{12}}{a_{11}} \right)$,
   3) $a_{33} | \left( d_3, d_2 \frac{a_{23}}{a_{22}}, d_1 \frac{a_{13}}{a_{12}} \right)$,
   $\vdots$
   k) $a_{kk} | \left( d_k, d_{k-1} \frac{a_{k-1k}}{a_{k-1k-1}}, d_{k-2} \frac{a_{k-2k-1}}{a_{k-2k-2}}, ..., \right.$

\[
\begin{pmatrix}
    a_{12} & a_{13} & \cdots & a_{1k} \\
    a_{22} & a_{23} & \cdots & a_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{k-1k-1} & a_{k-2k-2} & \cdots & a_{k-1k-1} \\
    d_1 & a_{k-1k-1} & a_{k-2k-2} & \cdots & a_{k-1k-1} \\
\end{pmatrix},
\]

where by $(x_1, x_2, ..., x_m)$ we denote the greatest common divisor of numbers $x_1, x_2, ..., x_m \in \mathbb{Z}$. On the set $L(k; d_1, d_2, ..., d_k)$ we introduce the ordering relation " $\leq$ ", defined as follows: for $A = (a_{ij}), B = (b_{ij}) \in L(k; d_1, d_2, ..., d_k)$, put $A \leq B$ if and only if we have

1) $b_{11} | a_{11}$,
2) $b_{22} | \left( a_{22}, \frac{a_{11}}{b_{11}}, \frac{a_{12}}{b_{12}} \right),$
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Theorem A. If \( G \) is a finite abelian group with the decomposition (\( \ast \)), then its subgroup lattice \( L(G) \) is isomorphic to the fundamental group lattice \( L(k; d_1, d_2, ..., d_k) \).

In order to study when two fundamental group lattices are isomorphic (that is, when two finite abelian groups are lattice-isomorphic), the following notation is useful. For every integer \( n \geq 2 \), we denote by \( \pi(n) \) the set consisting of all primes dividing \( n \). Let \( d_i, d'_i \in \mathbb{N} \setminus \{0, 1\} \), \( i = 1, k, i' = 1, k' \), such that \( d_1 | d_2 | ... | d_k \) and \( d'_1 | d'_2 | ... | d'_{k'} \). Then we shall write

\[
(d_1, d_2, ..., d_k) \sim (d'_1, d'_2, ..., d'_{k'})
\]

whenever the next three conditions are satisfied:

a) \( k = k' \).

b) \( d_i = d'_i, \ i = 1, k - 1 \).

c) The sets \( \pi(d_k) \cap \pi \left( \prod_{i=1}^{k-1} d_i \right) \) and \( \pi(d'_k) \cap \pi \left( \prod_{i=1}^{k-1} d'_i \right) \) have the same number of elements, say \( r \). Moreover, for \( r = 0 \) we have \( d_k = d'_k \) and for \( r \geq 1 \), by denoting \( \pi(d_k) \setminus \pi \left( \prod_{i=1}^{k-1} d_i \right) = \{p_1, p_2, ..., p_r\} \), \( \pi(d'_k) \setminus \pi \left( \prod_{i=1}^{k-1} d'_i \right) = \{q_1, q_2, ..., q_r\} \), we have

\[
\frac{d_k}{d'_k} = \prod_{j=1}^{r} \left( \frac{p_j}{q_j} \right)^{s_j},
\]

where \( s_j \in \mathbb{N}^* \), \( j = 1, r \).
The following theorem of [8] will play an essential role in proving our main results.

**Theorem B.** Two fundamental group lattices $L(k; d_1, d_2, \ldots, d_k)$ and $L(k'; d_1', d_2', \ldots, d_k')$ are isomorphic if and only if $(d_1, d_2, \ldots, d_k) \sim (d_1', d_2', \ldots, d_k')$.

## 2 Main results

As we have already mentioned, large classes of non-isomorphic finite abelian groups exist whose lattices of subgroups are isomorphic. Simple examples of such groups are easily obtained by using Theorem B:

1. $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_{10}$ (cyclic groups),
2. $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $H = \mathbb{Z}_2 \times \mathbb{Z}_{10}$ (non-cyclic groups).

Moreover, Theorem B allows us to find a subclass of finite abelian groups which are determined by their lattices of subgroups (see also Proposition 2.8 of [8]).

**Theorem 2.1.** Let $G$ and $H$ be two finite abelian groups such that one of them possesses a decomposition of type (⋆) with $\pi(d_k) = \pi \left( \prod_{i=1}^{k-1} d_i \right)$. Then $G \cong H$ if and only if $L(G) \cong L(H)$.

Next we shall focus on isomorphisms between the subgroup lattices of the direct $n$-powers of two finite abelian groups, for $n \geq 2$. An alternative proof of the following well-known result can be also inferred from Theorem B.

**Theorem 2.2.** Let $G$ and $H$ be two finite abelian groups. Then $G \cong H$ if and only if $L(G^n) \cong L(H^n)$ for some integer $n \geq 2$.

**Proof.** Let $G \cong \bigotimes_{i=1}^{k} \mathbb{Z}_{d_i}$ and $H \cong \bigotimes_{i=1}^{k'} \mathbb{Z}_{d'_i}$ be the corresponding decompositions (⋆) of $G$ and $H$, respectively, and assume that $L(G^n) \cong L(H^n)$ for some integer $n \geq 2$. Then the fundamental group lattices

$$L(k; d_1, d_2, \ldots, d_k, d_k, \ldots, d_k) \quad \text{and} \quad L(k'; d'_1, d'_2, \ldots, d'_k, d'_k, \ldots, d'_k)$$

are isomorphic. By Theorem B, one obtains

$$(d_1, d_1, \ldots, d_k, d_k, \ldots, d_k) \sim (d'_1, d'_1, \ldots, d'_k, d'_k, \ldots, d'_k)$$

and therefore $k = k'$ and $d_i = d'_i$, for all $i = 1, k$. These equalities show that $G \cong H$, which completes the proof.
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Clearly, two finite abelian groups $G$ and $H$ satisfying $L(G^m) \cong L(H^n)$ for some (possibly different) integers $m, n \geq 2$ are not necessarily isomorphic. Nevertheless, a lot of conditions of this type can lead to $G \cong H$, as the following theorem shows.

**Theorem 2.3.** Let $G$ and $H$ be two finite abelian groups. Then $G \cong H$ if and only if there are the integers $r \geq 1$ and $m_1, m_2, ..., m_r, n_1, n_2, ..., n_r \geq 2$ such that $(m_1, m_2, ..., m_r) = (n_1, n_2, ..., n_r)$ and $L(G^{m_i}) \cong L(H^{n_i})$, for all $i = 1, r$.

**Proof.** Suppose that $G$ and $H$ have the decompositions in the proof of Theorem 2. For every $i = 1, 2, ..., r$, the lattice isomorphism $L(G^{m_i}) \cong L(H^{n_i})$ implies that $km_i = k'n_i$, in view of Theorem B. Set $d = (m_1, m_2, ..., m_r)$. Then $d = \sum_{i=1}^{r} \alpha_i m_i$ for some integers $\alpha_1, \alpha_2, ..., \alpha_r$, which leads to

$$kd = k \sum_{i=1}^{r} \alpha_i m_i = \sum_{i=1}^{r} \alpha_i km_i = \sum_{i=1}^{r} \alpha_i k'n_i = k' \sum_{i=1}^{r} \alpha_i n_i.$$ 

Since $d \mid n_i$, for all $i = 1, r$, we infer that $k' \mid k$. In a similar manner one obtains $k \mid k'$, and thus $k = k'$. Hence $m_i = n_i$ and the group isomorphism $G \cong H$ is obtained from Theorem 2.2.

Finally, we indicate an open problem concerning the above results.

**Open problem.** In Theorem 3 replace condition $(m_1, m_2, ..., m_r) = (n_1, n_2, ..., n_r)$ with other connections between numbers $m_i$ and $n_i$, $i = 1, 2, ..., r$, such that the respective equivalence be also true.

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