Higher-spin quantum and classical Schur-Weyl duality for $\mathfrak{sl}_2$

Steven M. Flores
steven.miguel.flores@gmail.com
Department of Mathematics and Systems Analysis,
P.O. Box 11100, FI-00076, Aalto University, Finland

Eveliina Peltola
eveliina.peltola@hcm.uni-bonn.de
Institute for Applied Mathematics, University of Bonn,
Endenicher Allee 60, D-53115 Bonn, Germany

It is well-known that the commutant algebra of the $U_q(\mathfrak{sl}_2)$-action on the $n$-fold tensor product of its fundamental module is isomorphic to the Temperley-Lieb algebra $\text{TL}_n(\nu)$ with fugacity parameter $\nu = -q - q^{-1}$ (at least in the generic case, i.e., when $q$ is not a root of unity, or $n$ is small enough). Furthermore, the simple $U_q(\mathfrak{sl}_2)$-modules appearing in the direct-sum decomposition of the $n$-fold tensor product module are in one-to-one correspondence with those of the Temperley-Lieb algebra. This double-commutant property is referred to as quantum Schur-Weyl duality.

In this article, we investigate such a duality in great detail. We prove that the commutant of the $U_q(\mathfrak{sl}_2)$-action on any generic type-one tensor product module is isomorphic to a diagram algebra that we call the valenced Temperley-Lieb algebra $\text{TL}_\varsigma(\nu)$. This corresponds to representations with higher spin, which results in the need of valences (or colors) in the Temperley-Lieb diagrams. We establish detailed direct-sum decompositions exhibiting this duality and find explicit bases amenable to concrete calculations, important in applications. We also include a double-commutant type property for homomorphisms between different $U_q(\mathfrak{sl}_2)$-modules, realized by valenced diagrams. The diagram calculus is reminiscent to Kauffman’s recoupling theory and the graphical methods developed among others by Penrose and Frenkel & Khovanov. The results also contain the standard quantum Schur-Weyl duality as a special case, and when specialized to $q \to 1$, imply the classical Frobenius-Schur-Weyl duality for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and a higher-spin version thereof.

We especially aim to provide a self-contained and elementary presentation useful to applications also in areas remote from algebra and representation theory. Only very basic facts about representations are needed to understand this article. For this reason, we also include basic known results as well as results that can be regarded as folklore but are lacking systematic or easy references.
1. INTRODUCTION

Quantum group symmetries, hidden or apparent, are ubiquitous in mathematics and mathematical physics. Traditionally, they have played a central role especially in certain areas of low-dimensional topology, but also in numerous other fields of modern mathematics. The emergence of quantum groups can be argued to originate from observations of hidden symmetries in certain physical quantum systems, deforming the classical Lie algebra symmetries, as pioneered by V. Drinfeld, L. Faddeev, and M. Jimbo in the 1980s. Thereafter, such symmetries have been observed, e.g., in topological quantum field theory, conformal field theory, and integrable models in statistical physics, e.g., [Wit89, MR89, PS90, GRAS96, Bax07, Mar91, Var92]. Quantum groups also immediately gained the interest of mathematicians, with remarkable success. From the Yang-Baxter equation underlying these symmetries, one gets to the theory of tensor categories, higher representation theory, and categorification, see [FKS06, Str10]. Using quantum groups and Hecke algebras, one constructs invariants of knots, tangles, and links [Jon83, Kau90, RT91]. Different formulations of quantum groups led to advances in noncommutative geometry [CK98, Man18], and others in representation theory [Lus89, AKP91, FKK98], for instance. In the relatively new field of random geometry, one finds quantum group symmetries related to probabilistic models. For instance, in [JJK16, KP16, KKP19, KP19, Pei20, FP20a+, FP20b+] explicit highest-weight vectors are used to construct specific correlation functions in conformal field theory, with applications to conformally invariant random geometry. Various quantum group symmetries and dualities have also become useful tools in integrable probability [BP14, BP15], e.g., to analyze non-equilibrium characteristics in asymmetric particle processes and stochastic vertex models (see [CGRS16, KMMO16] and references therein).

Due to the enormous number of applications, various tools and theory on quantum groups have been developed. In the present article, our aim is to focus on concrete and elementary understanding of the simplest, but also most commonly encountered, quantum group associated to the Lie algebra \(\mathfrak{sl}_2(\mathbb{C})\). In order to utilize the full power of these symmetries in applications, detailed knowledge is crucial, and in light of their emergence in many seemingly remote areas, we believe that elementary tools for obtaining concrete information are becoming more and more important. In the present work, we specifically consider the quantum group \(U_q(\mathfrak{sl}_2)\) and its tensor product representations. (Despite the commonly used term “group,” the quantum group that we consider is a non-commutative and non-cocommutative Hopf algebra, whose commutation relations are obtained from those of \(\mathfrak{sl}_2\) by a deformation by a non-zero complex parameter \(q \neq \pm 1\).) The presentation here is mostly self-contained, with little need of prerequisites, and in particular, all used techniques are completely elementary.

It has been known for quite a while, at least since the work [RTW32, Pen71] about 90 years ago, that planar diagrams, now called tangles in the Temperley-Lieb category [TL71, Jon83, KL94, Tur94, CFS95] completely describe homomorphisms between tensor powers \(M^{\otimes n}_{\nu}(1)\), \(M^{\otimes n}_{\nu}(2)\) of the fundamental module \(M^{\otimes n}_{\nu}(1)\) of the Lie algebra \(\mathfrak{sl}_2\). On the other hand, the classical “Schur-Weyl duality” dating back 100 years ago [Sch27, Wey39], shows that the algebra of all endomorphisms of \(M^{\otimes n}_{\nu}(1)\) that commute with the action of the enveloping algebra \(U(\mathfrak{sl}_2)\) is generated by transpositions of the tensor factors (thus being a quotient of the group algebra of the symmetric group acting naturally on \(M^{\otimes n}_{\nu}(1)\)). Interestingly, the latter agrees with the former: transpositions have a natural diagram interpretation, whose action coincides with the planar diagram algebra known as Temperley-Lieb algebra \(TL_n(\nu)\) with (loop) fugacity parameter \(\nu = -2\).

Analogous properties are inherited by the quantum group \(U_q := U_q(\mathfrak{sl}_2)\) [KR81]. Indeed, when \(q\) is not a root of unity, \(U_q\) is semisimple and its representation theory is analogous to that of the classical Lie algebra \(\mathfrak{sl}_2\), and the planar diagram approach generalizes nicely [CFS95, FK97]. In this case, the “quantum Schur-Weyl duality” holds for the Temperley-Lieb algebra \(TL_n(\nu)\) with \(\nu = -2 = -q - q^{-1}\). M. Jimbo [Jim86] (independently, R. Dipper and G. James [DJ89]) observed this from the quasi-triangular structure of \(U_q\): braiding of tensor components in \(M^{\otimes n}_{\nu}(1)\), where \(M^{\otimes n}_{\nu}(1)\) is now the fundamental \(U_q\)-module, gives rise to an action of the \(q\)-deformation of the symmetric group algebra, the Hecke algebra. However, this action is not faithful. Factoring out by its kernel gives the Temperley-Lieb algebra \(TL_n(\nu)\) [Mar91, Mar92]. (A version of such a duality also holds in the non-semisimple case [Mar92, GV13, PSA14], but \(U_q\) has to be extended by so-called divided powers and also the Temperley-Lieb algebra is then non-semisimple.)

The main purpose of the present article is to investigate in detail the commuting actions of the quantum group \(U_q\) and the valenced Temperley-Lieb algebra \(TL_n(\nu)\) on arbitrary tensor product (type-one) modules of \(U_q\). The latter is a planar diagram algebra comprising diagrams of valenced (or colored) tangles, containing the usual Temperley-Lieb type tangles as a special case. Importantly, this algebra can be used to determine all homomorphisms between arbitrary tensor product (type-one) modules of \(U_q\). The well-known standard version of the Temperley-Lieb algebra was also crucial in the development of the theory of knot invariants [Jon83, KL94], and has many historical connections to mathematical physics, e.g., in the field of exactly solvable models in statistical physics [TL71, Mar91, Bax07].

We develop in parallel an explicit algebraic, or combinatorial, description of the structure of these tensor product
modules, and a diagram calculus that facilitates many computations and makes the appearance of the valenced Temperley-Lieb algebra apparent. We have special emphasis on understanding highest-weight vectors, also important in applications, e.g., to conformally invariant random geometry. In addition to the “usual” case of $U_q$, we also include analogous results for its variants obtained from changing the way how the algebra acts on tensor products, or sending the parameter $q$ to its inverse. Such variants and their mutual connections are also needed in applications.

We take the approach of studying directly the quantum group and recover the classical results as a consequence. Traditionally, the classical case is considered more elementary, but the proof of the classical Schur-Weyl duality also relies on quite specific combinatorics. Treating directly the quantum group puts all parameter values $q$ or $\nu$ to equal footing, and also gives a route to go beyond the semisimple case.

A. Background and motivation — the quantum group

In this work, we frequently use the $q$-integers, $q$-factorials, and $q$-binomial coefficients, defined for any integers $k \in \mathbb{Z}$ and $0 \leq \ell \leq m$ and for any $q \in \mathbb{C} \setminus \{0\} =: \mathbb{C}^\times$ respectively as

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [m]! := \prod_{i=1}^{m} [i], \quad \begin{bmatrix} m \\ \ell \end{bmatrix} := \frac{[m]!}{[\ell]![m-\ell]!}. \quad (1.1)$$

When $q \in \{\pm 1\}$, these become the usual integers, factorials, and binomial coefficients. We note that $[m]$ vanishes if and only if $m$ is zero or $q \notin \{\pm 1\}$ and $m$ is an integer multiple of the number $p(q)$ defined as

$$p(q) := \begin{cases} \infty, & q \text{ is not a root of unity}, \\ p, & q = e^{\pi i p'/p} \text{ for coprime } p, p' \in \mathbb{Z}_{>0}. \end{cases} \quad (1.2)$$

Thus, $p(q)$ is the smallest power $p = p(q)$ of $q$ such that $q^p$ equals $+1$ or $-1$.

One of the main characters in this work is the quantum group $U_q := U_q(\mathfrak{sl}_2)$. As a $\mathbb{C}$-algebra, it may be thought of as a deformed version of the universal enveloping algebra $U(\mathfrak{sl}_2)$ of the Lie algebra $\mathfrak{sl}_2$ of traceless $(2 \times 2)$-matrices, with deformation parameter $q \in \mathbb{C}^\times \setminus \{\pm 1\}$. In a suitable sense, the latter is recovered as $q \rightarrow 1$. Explicitly, $U_q$ is the infinite-dimensional associative algebra with unit 1 and generators $E$, $F$, $K$, $K^{-1}$ exclusively satisfying the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \quad (1.3)$$

For each nonnegative integer $s < p(q)$, we let $M_{(s)}$ denote the type-one simple $(s + 1)$-dimensional $U_q$-module defined below, and we let $\rho_{(s)}: U_q \longrightarrow \text{End} M_{(s)}$ denote its corresponding irreducible representation. We often use the shorthand notation $x.v := \rho_{(s)}(x)(v)$ for the action of elements $x \in U_q$ on vectors $v \in M_{(s)}$. This action is completely determined by the action of the generators $\{E, F, K\}$ of $U_q$ on the standard basis $\{e_{\ell}^{(s)} \mid 0 \leq \ell \leq s\}$ for $M_{(s)}$, given by

$$F.e_{\ell}^{(s)} := \begin{cases} e_{\ell+1}^{(s)}, & 0 \leq \ell \leq s - 1, \\ 0, & \ell = s, \end{cases} \quad E.e_{\ell}^{(s)} := \begin{cases} [\ell][s-\ell+1]e_{\ell-1}^{(s)}, & 1 \leq \ell \leq s, \\ 0, & \ell = 0, \end{cases} \quad K.e_{\ell}^{(s)} := q^{s-2\ell}e_{\ell}^{(s)}. \quad (1.4)$$

We call these modules “simple type-one modules.” They are analogues of the highest-weight modules of $\mathfrak{sl}_2$.

The algebra $U_q$ has a bialgebra (and even a Hopf algebra) structure, which enables us to form modules from the ground field $\mathbb{C}$ and from tensor products of its modules. We recommend, e.g., the textbooks [CP94, Kas95, KRT97] for background, although the present work is mostly self-contained. Of particular importance for this article are

$$u_{\varsigma} \otimes V_{\varsigma} := M_{(s_1)} \otimes M_{(s_2)} \otimes \cdots \otimes M_{(s_{d_{\varsigma}})}, \quad \varsigma := (s_1, s_2, \ldots, s_{d_{\varsigma}}) \in \mathbb{Z}_{>0}^{d_{\varsigma}}, \quad \max \varsigma < p(q), \quad (1.5)$$

called “type-one” tensor product modules with “spins” $s_1, s_2, \ldots, s_{d_{\varsigma}}$, carrying the $U_q$-action

$$\rho_{\varsigma} : U_q \longrightarrow \text{End} V_{\varsigma}, \quad \rho_{\varsigma} := (\rho_{(s_1)} \otimes \rho_{(s_2)} \otimes \cdots \otimes \rho_{(s_{d_{\varsigma}})}) \circ \Delta^{(d_{\varsigma})}, \quad (1.6)$$

where $\Delta : U_q \longrightarrow U_q \otimes U_q$ is a coproduct given in (2.11) in section 2 A and $\Delta^{(d_{\varsigma})}$ is its $\Delta_{d_{\varsigma}}$-th iterate given in (2.16). When $d_{\varsigma} = 0$, we use the convention that $\varsigma = (0)$. In this case, $M_{(0)}$ is the trivial one-dimensional module, which we always identify with the ground field $\mathbb{C}$, omitting it from all tensor products.
If $q$ is not a root of unity, $U_q$ is a semisimple algebra. However, when $q$ is a root of unity, the representation theory of $U_q$ is non-semisimple and rather complicated. For small enough spins, the module $U_q \circ V_c$ is semisimple and it has the following direct-sum decomposition into simple $U_q$-submodules with multiplicities $D^{(s)}_{\zeta} \in \mathbb{Z}_{>0}$ given in (2.59):

$$U_q \circ V_c \cong \bigoplus_{s \in E_c} D^{(s)}_{\zeta} M_{(s)} \quad \text{when} \quad n_{\zeta} := s_1 + s_2 + \cdots + s_{d_{\zeta}} < p(q), \quad (1.7)$$

where $M_{(s)}$ are non-isomorphic simple $U_q$-modules, with index set $E_c \ni s$. Analogously to the representation theory of classical Lie algebras, for this semisimple case, each submodule is generated by a highest-weight vector in

$$H_{\zeta} := \{ v \in V_c \mid E.v = 0 \} = \bigoplus_{s \in E_c} H^{(s)}_{\zeta}, \quad (1.8)$$

whose weights $q^s$ are the $K$-eigenvalues: we have $K.v = q^s v$ for highest-weight vectors $v \in H^{(s)}_{\zeta}$. (Note that this structure fails in general when $n_{\zeta} \geq p(q)$.)

B. Concrete understanding of highest-weight vectors

Let us examine the module $U_q \circ V_c$. To understand its submodule structure, the first task is to find linearly independent highest-weight vectors in $V_c$. A particularly useful set of such vectors is indexed by the numbers $D^{(s)}_{\zeta}$, even if $n_{\zeta} \geq p(q)$. We call them (valenced) link-pattern basis vectors. These vectors not only give detailed information about the structure of $U_q \circ V_c$, but are also of independent interest due to their useful properties, which translate to features of other objects in applications of quantum group symmetries (e.g., [BF91, IWWZ13, KP16, KP19, FP20a+, FP20b+]). Similar bases seem to be known, at least implicitly, for the community in diagrammatic representation theory [FK97].

**Proposition 1.1. (Link-pattern basis vectors):** In $U_q \circ V_c$, there exists a collection of linearly independent highest-weight vectors $w_\alpha$ indexed by valenced link patterns $\alpha$ (definition 4.1 in section 4A). Diagrammatically, we have

$$\alpha = \begin{array}{c}
\includegraphics[width=2cm]{link_pattern}
\end{array} \quad \implies \quad \begin{array}{c}
\includegraphics[width=2cm]{highest_weight_vector}
\end{array}, \quad (1.9)$$

and the descendants of $w_\alpha$ also have diagram representations

$$F^{s-\ell}.w_\alpha = \left[ \frac{[s]}{[s-\ell]} \right] \times \begin{array}{c}
\includegraphics[width=2cm]{descendant_vector}
\end{array}. \quad (1.10)$$

These vectors give rise to an embedding of left $U_q$-modules (with multiplicities $D^{(s)}_{\zeta}$ given in (2.59)),

$$\bigoplus_{s \in E_c} D^{(s)}_{\zeta} M_{(s)} \hookrightarrow U_q \circ V_c. \quad (1.11)$$

If $n_{\zeta} < p(q)$, then this embedding is an isomorphism.

**Proof.** The vectors $w_\alpha$ are explicitly constructed in definition 4.1 and lemmas 4.8 and 4.11. Lemma 4.23 gives the diagram representations for the $F$-descendants. Proposition 4.13 gives the embedding or isomorphism (1.11). \hfill \Box

There is another interesting set of highest-weight vectors that yields the direct-sum decomposition (1.7) when $q$ is not a root of unity, and an embedding slightly weaker than (1.11) when $q$ is a root of unity. Despite this weakness, this basis is very useful for instance because it is orthogonal with respect to a standard invariant bilinear pairing $(\cdot | \cdot)$. We call this basis the “conformal-block basis” due to its connections to conformal field theory. (Indeed, these
conformal-block vectors should correspond to intertwiners in a Virasoro VOA [KKP19, section 3].) These highest-weight vectors $u^\varsigma_\varsigma$ are also enumerated by the combinatorial numbers $D^{(s)}_\varsigma$, which count certain walks (see (2.58) in section 2 B). Below, the embedding has multiplicities $\hat{D}^{(s)}_\varsigma$ that can be smaller than $D^{(s)}_\varsigma$ due to a technical condition.

The bilinear pair is defined as $(\cdot | \cdot): \nabla_\varsigma \times \mathbb{C}$,

\[
(\tilde{e}_{\ell_1}^{(s_1)} \otimes \tilde{e}_{\ell_2}^{(s_2)} \otimes \cdots \otimes \tilde{e}_{\ell_d}^{(s_d)} | e_{m_1}^{(s_1)} \otimes e_{m_2}^{(s_2)} \otimes \cdots \otimes e_{m_d}^{(s_d)}) = \prod_{k=1}^{d} \delta_{\ell_k, m_k} [\ell_k]! [s_k]! ,
\]

where

\[
\mathcal{V}_\varsigma = \text{span} \left\{ e_{m_1}^{(s_1)} \otimes e_{m_2}^{(s_2)} \otimes \cdots \otimes e_{m_d}^{(s_d)} \middle| 0 \leq m_1 \leq s_1, 0 \leq m_2 \leq s_2, \ldots, 0 \leq m_d \leq s_d \right\}
\]

is the vector space underlying the left $U_q$-module $u^\varsigma_\varsigma \subset \mathcal{V}_\varsigma$ and

\[
\nabla_\varsigma = \text{span} \left\{ e_{\ell_1}^{(s_1)} \otimes e_{\ell_2}^{(s_2)} \otimes \cdots \otimes e_{\ell_d}^{(s_d)} \middle| 0 \leq \ell_1 \leq s_1, 0 \leq \ell_2 \leq s_2, \ldots, 0 \leq \ell_d \leq s_d \right\}
\]

is a vector space of the same dimension but with a right $U_q$-action $\nabla_\varsigma \circ u^\varsigma_\varsigma$ discussed in section 2.

**Proposition 1.2.** (Conformal-block basis vectors): In $u^\varsigma_\varsigma \subset \mathcal{V}_\varsigma$ and $\nabla_\varsigma \subset u^\varsigma_\varsigma$, there exist collections of linearly independent highest-weight vectors $u^\varsigma_\varsigma$ and $\nabla^\varsigma_\varsigma$ indexed by walks $\rho$ over $\varsigma$, which are also orthogonal:

\[
(\nabla^\varsigma_\varsigma | u^\varsigma_\varsigma) = \delta_{\rho,\rho'} \prod_{j=1}^{d-1} [q-q^{-1}]^{r_j+s_j+1-r_j+1} \left[ \frac{r_j+s_j+1-r_j+1}{2} \right]! [r_j+1]! ,
\]

where $\Theta$ is an explicit constant given by the evaluation of the Theta network (2.133). Diagrammatically, we have

\[
u^\varsigma_\varsigma = \left( \prod_{j=1}^{d-1} \frac{[i q^{1/2}]^{r_j+s_j+1-r_j+1}}{(q-q^{-1})^{r_j+s_j+1-r_j+1}} \right) \times
\]

These vectors give rise to an embedding of left $U_q$-modules (with multiplicities $\hat{D}^{(s)}_\varsigma$ given in (2.86)),

\[
\bigoplus_{s \in \mathcal{E}_\varsigma, s < p(q)} \hat{D}^{(s)}_\varsigma M_{(s)} \hookrightarrow u^\varsigma_\varsigma \subset \mathcal{V}_\varsigma.
\]

If $n_\varsigma < p(q)$, then this embedding is an isomorphism.

**Proof.** The vectors $u^\varsigma_\varsigma$ are given in definition 2.5. Lemma 5.4 gives the diagram representation (1.16), and lemma 5.5 shows that the vectors are orthogonal as in (1.15). Proposition 2.8 gives the embedding or isomorphism (1.17).

We cautiously note that the vectors $u^\varsigma_\varsigma$ and $\nabla^\varsigma_\varsigma$ are not complex conjugates of each other. They represent the holomorphic and anti-holomorphic sectors in conformal field theory.

**C. Quantum Schur-Weyl duality**

More information about the type-one $U_q$-modules can be gained from understanding the commutant algebra $\text{End}_{U_q} \mathcal{V}_\varsigma$, which consists of all $U_q$-homomorphisms from $\mathcal{V}_\varsigma$ to itself. For instance, all projections from $\mathcal{V}_\varsigma$ onto its $U_q$-submodules belong to the commutant, by definition. In fact, the collection of all submodule projectors that act strictly on consecutive pairs of tensorands in $\mathcal{V}_\varsigma$ generate the whole commutant algebra $\text{End}_{U_q} \mathcal{V}_\varsigma$ when $n_\varsigma < p(q)$. For a tensor product of two type-one modules, all multiplicities in the direct-sum decomposition (1.7) equal one:

\[
u^\varsigma_\varsigma \subset \mathcal{V}_{(r,t)} \cong M_{(r-t)} \oplus M_{(r-t)+2} \oplus M_{(r-t)+4} \oplus \cdots \oplus M_{(r+t-2)} \oplus M_{(r+t)} \text{ when } r+t < p(q).
\]
We denote each projection onto the \((s+1)\)-dimensional simple \(U_q\)-submodule \(\pi^{(r,t)}_{(s)}(V_{r,t}) \cong M_{(s)}\) by
\[
\pi^{(r,t)}_{(s)} : V_{r,t} \rightarrow V_{r,t},
\]
\[
\pi^{(r,t)}_{(s)}(\nu_{(r,t)}(p)) := \delta_{p,s} F^p_{(r,t)} for \ \ell \in \{0, 1, \ldots, s\}.
\]

**Proposition 1.3.** Suppose \(n_i < \rho(q)\). Then, the commutant algebra \(\text{End}_{U_q} V_\varsigma\) is generated by the collection of all submodule projectors that act strictly on consecutive pairs of tensorands of vectors in \(V_\varsigma\):
\[
\text{End}_{U_q} V_\varsigma = \langle \pi^{(s_i,s_{i+1})}_{(s)} \mid s \in E_{(s_i,s_{i+1})}, i \in \{1, 2, \ldots, d_\varsigma - 1\}\rangle,
\]
where \(E_{(s_i,s_{i+1})} = \{ |s_{i+1} - s_i|, |s_{i+1} - s_i| + 2, \ldots, s_{i+1} + s_i\}\).

**Proof.** We prove this (known) result in section 6D, identifying the projectors with valenced diagrams, cf. (1.29). \(\square\)

The above fact is no surprise. Indeed, the quantum group is quasi-triangular with R-matrix which gives an action of the braid group on tensor products of \(U_q\)-modules, commuting with the \(U_q\)-action. This braiding can be related to the projectors \(\pi^{(s_i,s_{i+1})}_{(s)}\) (although the explicit connection with general higher-spin representations in \(V_\varsigma\) is somewhat complicated). Such braid representations cannot be faithful (i.e., injective), as the group algebra of the braid group is infinite-dimensional. Alternatively, the commutant algebra is also isomorphic to a quotient of the Hecke algebra, that is, the \(\rho\)-deformation of the symmetric group algebra. This quotient is an explicit planar diagram algebra, the valenced Temperley-Lieb algebra \(\text{TL}_\rho(\nu)\), discussed shortly.

For illustration, let us briefly study the case of the \(n\):th tensor power of the fundamental \(U_q\)-module \(M_{(1)}\),
\[
\mathcal{U}_n \otimes V_n = M_{(1)}^{\otimes n}, \ \ \text{assuming that} \ \ n < \rho(q).
\]

In this case, the submodule projectors
\[
\pi_j := \text{id}^{\otimes (j-1)} \otimes \pi^{(1,1)}_{(1,1)} \otimes \text{id}^{\otimes (n-j-1)} \in \text{End}_{U_q} V_n,
\]
projecting the \(j\):th and \((j+1)\):st tensor components onto the trivial module \(M_{(0)}\), satisfy the relations
\[
\pi_j \pi_{j+1} \pi_j = \frac{1}{2^j} \pi_j, \ \ \text{if} \ 1 \leq i \pm 1 \leq n - 1
\]
\[
\pi_j^2 = \pi_j, \ \ \pi_i \pi_j = \pi_j \pi_i, \ \ \text{if} \ |i - j| > 1.
\]

After normalizing them differently, we see that these relations are exactly the famous Temperley-Lieb relations. Indeed, the operators \(\nu \pi_j\), where we parameterize the loop fugacity \(\nu\) in terms of \(q \in \mathbb{C}^\times\) as
\[
\nu = -q - q^{-1} = -[2] \in \mathbb{C}
\]
FIG. 1.1: Illustration of the quantum Schur-Weyl duality (q-SW). In the rows, adjacent boxes with the same coloring form standard modules \(L^{(s)}_n\), closed under the action of \(V_\varsigma\). The columns form simple type-one modules \(M_{(s)}\), closed under the action of \(U_q\). Arrows indicate the action of \(F \in U_q\), which maps from one copy of \(L^{(s)}_n\) to another.
(the case of \(q = 1\) corresponding to representations of the Lie algebra \(\mathfrak{sl}_2\) discussed in appendix C) satisfy the same relations as the generating set \(\{1_{\text{TL}_n}, U_1, U_2, \ldots, U_{n-1}\}\) of the Temperley-Lieb algebra \(\text{TL}_n(\nu)\) [TL71, Jon83]:

\[
\begin{align*}
U_i U_{i \pm 1} U_i &= U_i, & \text{if } 1 \leq i \pm 1 \leq n - 1, \\
U_i^2 &= \nu U_i, \\
U_i U_j &= U_j U_i, & \text{if } |i - j| > 1
\end{align*}
\]

for all \(i, j \in \{1, 2, \ldots, n - 1\}\). In section 3A, we give a diagram presentation (3.13, 3.14) for these generators.

Conversely, one can show that the image of the algebra \(U_q\) under its representation on \(U_q \otimes V_n\) coincides with the commutant algebra \(\text{End}_{\text{TL}_n} V_n\) of the action of the Temperley-Lieb algebra \(\text{TL}_n(\nu)\) on \(\text{TL}_n \otimes V_n\). This relationship is a well-known fact, observed by M. Jimbo [Jim86] and independently by R. Dipper and G. James [DJ89]. In fact, this property is an easy consequence of the semisimple structure of the module \(U_q \otimes V_n\), essentially following only from Schur’s lemma (see proposition E.6 in appendix E). A slightly more elaborate but still elementary application of Schur’s lemma (theorem E.9 in appendix E) gives a duality decomposition for the bimodule structure of \(V_n\) under both actions of \(U_q\) and \(\text{TL}_n(\nu)\). This is known as the “quantum Schur-Weyl duality” decomposition (see Figure 1.1)

\[
M_{(1)}^{\otimes n} \cong \bigoplus_{s \in E_n} L_{(s)}^{(n)} \otimes M_{(s)}, \quad \text{when } n < p(q),
\]

(q-SW\(_n\))

where \(\{L_{(s)}^{(n)} | s \in E_n\}\) is the complete set of simple \(\text{TL}_n(\nu)\)-modules. Theorem 1.4, discussed shortly, implies this and a more refined statement including a complete description of the general type-one tensor product module \(U_q \otimes V_\varsigma\).

Let us also remark that if \(n \geq p(q)\), the action of the Temperley-Lieb algebra on \(V_n\) still admits a quantum Schur-Weyl duality, but its commutant algebra is strictly larger than just the image of the quantum group \(U_q\), the latter being extended by additional generators called “divided powers” [Lus90, Mar92, MMA92]. The direct-sum decomposition of \(\text{TL} \otimes V_n\) into indecomposable modules is also known [RS07, GV13, PSA14], see also [BFGT09, GST14].

More generally (assuming \(n_\varsigma < p(q)\)), the commutant algebra \(\text{End}_{U_q} V_\varsigma\) on the type-one tensor product module (1.5) is also a diagram algebra, the valenced Temperley-Lieb algebra \(\text{TL}_\varsigma(\nu)\), generated by the valenced tangles [FP18a]
\( \mathbf{TL}_\nu(\nu) \) is an associative unital algebra consisting of valenced diagrams with multiplication defined in (3.4, 3.41) in section 3A, using the Jones-Wenzl projector [Jon83, Wen87]. The special case of \( \zeta = (1, 1, \ldots, 1) \) is the ordinary Temperley-Lieb algebra \( \mathbf{TL}_1(\nu) \). The restriction to \( n_\zeta < p(q) \) should not be essential for \( \{ \mathbf{1}_{\mathbf{TL}_i}, U_1, U_2, \ldots, U_d, -1 \} \) to be the whole generating set of \( \mathbf{TL}_\zeta(\nu) \), and the algebra \( \mathbf{TL}_\zeta(\nu) \) itself is defined whenever \( \max \zeta < p(q) \) [FP18a, FP18b].

The valenced Temperley-Lieb algebra has an action on the vector space \( V_\zeta \), which coincides with the action of projectors as in proposition 1.3. See also Figure 1.2. Indeed, an explicit isomorphism from \( \mathbf{TL}_\zeta(\nu) \) to (1.20) is given by

\[
(-1)^s[s+1] \times \frac{\Theta(s, s+i, s)}{\Theta(s, s+i+1, s)} 
\rightarrow \pi_{(s, s+i+1), (s)}^{(s, s+i, s)},
\]

where \( \Theta \) is again given by the evaluation of the Theta network (2.133), and where the left side of (1.29) is an alternative generating set for \( \mathbf{TL}_\zeta(\nu) \) by [FP18a, proposition 2.10], using Kauffman’s three-vertex notation [KL94, MV94, CFS95]

Now, as the vector space \( V_\zeta \) is endowed with actions of both algebras \( U_q \) and \( \mathbf{TL}_\zeta(\nu) \), with \( \nu = -q - q^{-1} \), we denote the obtained module as

\[
\mathcal{M}_\zeta := U_q \otimes V_\zeta,
\]

with representations \( \mathcal{J}_\zeta: \mathbf{TL}_\zeta(\nu) \rightarrow \text{End} V_\zeta, \rho_\zeta: U_q \rightarrow \text{End} V_\zeta \).

Recalling from proposition 1.1 that the valenced link patterns \( \alpha \) give rise to an explicit direct-sum decomposition (1.7, 1.11) with basis vectors having suggestive diagram representations, it is no surprise that the valenced Temperley-Lieb diagram algebra \( \mathbf{TL}_\zeta(\nu) \) acts naturally on the spaces \( H^{(s)}_\zeta \) of \( U_q \)-highest-weight vectors (1.8). These spaces are graded by the \( K \)-eigenvalues \( q^s \), or alternatively, by a geometric quantity for the link patterns, namely the number \( s \) of “defects.” We denote by \( \mathbf{LP}_\zeta^{(s)} \) the set of valenced link patterns with \( s \) defects and set \( L^{(s)}_\zeta \) := span \( \mathbf{LP}_\zeta^{(s)} \). These spaces also carry a diagram action of the algebra \( \mathbf{TL}_\zeta(\nu) \), explicated in section 3A. Each \( L^{(s)}_\zeta \) is called a \( \mathbf{TL}_\zeta(\nu) \)-standard module. (In fact, \( \mathbf{TL}_\zeta(\nu) \) is a cellular algebra [GL96, GL98, FP18b], and these are its cell modules.)

**Theorem 1.4.** (Higher-spin quantum Schur-Weyl duality): Suppose \( n_\zeta < p(q) \). Then, the following hold:

1. The images of the maps \( \mathcal{J}_\zeta: \mathbf{TL}_\zeta(\nu) \rightarrow \text{End} V_\zeta \) and \( \rho_\zeta: U_q \rightarrow \text{End} V_\zeta \) are semisimple algebras, which equal

\[
\mathbf{TL}_\zeta(\nu) \cong \mathcal{J}_\zeta(\mathbf{TL}_\zeta(\nu)) = \text{End} u_q V_\zeta \quad \text{and} \quad \rho_\zeta(U_q) = \text{End} u_\zeta V_\zeta.
\]

2. The collections \( \{ \mathcal{M}_\zeta^{(s)} | s \in E_\zeta \} \) and \( \{ L^{(s)}_\zeta | s \in E_\zeta \} \) are respectively the complete sets of simple non-isomorphic \( \rho_\zeta(U_q) \)-modules and \( \mathbf{TL}_\zeta(\nu) \)-modules, and we have the direct-sum decomposition

\[
\mathcal{M}_\zeta \cong \bigoplus_{s \in E_\zeta} \mathcal{M}_\zeta^{(s)} \otimes L^{(s)}_\zeta.
\]
3. The linear extension of the following map gives an explicit isomorphism for \((q\text{-SW}_\varsigma)\): with \(w_\alpha\) the explicit \(U_q\)-highest-weight vectors constructed in definition 4.1, \(\alpha \in \text{LP}_\varsigma^{(s)}\), \(s \in \mathbb{E}_\varsigma\), and \(\ell \in \{0,1,\ldots,s\},\)

\[
F^\ell, w_\alpha \mapsto e_\ell^{(s)} \otimes \alpha. \tag{1.33}
\]

**Proof.** We prove this result in section 6D. It is a consequence of the double-commutant theorem (theorem E.9, appendix E) combined with explicit knowledge of the structure of the bi-module \(M_\varsigma\) from proposition 1.1 for the \(U_q\)-structure and proposition 1.8 (see below) for the \(\text{TL}_\varsigma(\nu)\)-structure. In addition, we need a dimension count and an injectivity result, which shows that \(\text{TL}_\varsigma(\nu) \cong \mathcal{K}(\text{TL}_\varsigma(\nu))\) indeed constitutes the whole commutant algebra. \(\Box\)

**Proposition 1.5.** [Special case of proposition 6.1]: The representation \(\mathcal{K} : \text{TL}_\varsigma(\nu) \rightarrow \text{End} \mathcal{V}_\varsigma\) is faithful.

Interestingly, the above proposition applies not only in the (semisimple) case of \(n_\varsigma < p(q)\) but in the complete range \(\max \varsigma < p(q)\) where the algebra \(\text{TL}_\varsigma(\nu)\) is defined. As a special case, we recover the fact that the action of the ordinary Temperley-Lieb algebra \(\text{TL}_n(\nu)\) on the \(n\)th tensor power \((1.21)\) of the fundamental \(U_q\)-module \(M_{(1)}\) is faithful for all values of \(q\), a result proved independently by P. Martin [Mar92, theorem 1] and F. Goodman and H. Wenzl [GW93, theorem 2.4]. The faithfulness is crucial in order to identify \(\text{TL}_\varsigma(\nu)\) as the commutant algebra in item 1 of theorem 1.4 — otherwise, the commutant algebra could be a complicated quotient of \(\text{TL}_\varsigma(\nu)\).

Next, we briefly consider the more general space \(\text{Hom}_{U_q}(\mathcal{V}_\varsigma, \mathcal{V}_\nu)\) of homomorphisms of \(U_q\)-modules between pairs of type-one modules \(\mathcal{U}_\varsigma \subset \mathcal{V}_\varsigma\) and \(\mathcal{U}_\nu \subset \mathcal{V}_\nu\) as in (1.5), with

\[
\varsigma = (s_1, s_2, \ldots, s_d_\varsigma) \in \mathbb{Z}_{>0}^d \quad \text{and} \quad \nu = (p_1, p_2, \ldots, p_d_\nu) \in \mathbb{Z}_{>0}^d, \tag{1.34}
\]

\[
n_\varsigma := s_1 + s_2 + \cdots + s_d_\varsigma \quad \text{and} \quad n_\nu := p_1 + p_2 + \cdots + p_d_\nu, \tag{1.35}
\]

such that \(\max \varsigma < p(q)\) and \(n_\varsigma + n_\nu = 0 \pmod{2}\). It turns out that elements in the spaces \(\text{Hom}_{U_q}(\mathcal{V}_\varsigma, \mathcal{V}_\nu)\) can be realized as \((\varsigma, \nu)\)-tangles, slightly more general than the elements of \(\text{TL}_\varsigma(\nu)\), which are just \((\varsigma, \varsigma)\)-tangles — see section 3A. We denote the space of \((\varsigma, \nu)\)-tangles by \(\text{TL}_\varsigma^\nu\). A generic element in this space looks like

\[
\begin{array}{c}
s_1 \\
\vdots \\
s_d_\varsigma \\
T \\
s_d_\nu \\
\vdots \\
p_1 \\
p_2 \\
p_d_\nu - 1 \\
\end{array} \in \text{TL}_\varsigma^\nu, \tag{1.36}
\]

where \(T\) in the dashed box is a planar network, explicated in sections 3A and 3D. In section 3C, we define an action

\[
\mathcal{F}_\nu : \text{TL}_\varsigma^\nu \rightarrow \text{Hom}(\mathcal{V}_\varsigma, \mathcal{V}_\nu) \tag{1.37}
\]

of the \((\varsigma, \nu)\)-tangles, which actually commutes with the \(U_q\)-action. Moreover, when \(\max(n_\varsigma, n_\nu) < p(q)\), these elements constitute the whole space \(\text{Hom}_{U_q}(\mathcal{V}_\varsigma, \mathcal{V}_\nu)\), analogously to theorem 1.4. For instance, the valenced tangles

\[
\begin{array}{c|c|c|c}
\alpha & \beta & \beta' \\
\hline
\end{array} \xmapsto{\mathcal{F}_\nu} \pi_\alpha^\beta \in \text{Hom}_{U_q}(\mathcal{V}_\varsigma, \mathcal{V}_\nu), \tag{1.38}
\]

where “\(\square\)” is a Jones-Wenzl projector box placed across all through-paths of the tangle, give natural elements in this space, again indexed by valenced link patterns \(\alpha, \beta\) as explained in lemma 5.8 in section 5B.

**Theorem 1.6.** (Double-commutant property): Suppose \(\max(n_\varsigma, n_\nu) < p(q)\). Then, the following hold:

1. Let \(L, R \in \text{Hom}(\mathcal{V}_\nu, \mathcal{V}_\varsigma)\). The diagram

\[
\begin{array}{cccc}
\text{V}_\nu & \xrightarrow{\rho_\nu(x)} & \text{V}_\nu \\
\downarrow L & & \downarrow R \\
\text{V}_\varsigma & \xrightarrow{\rho_\varsigma(x)} & \text{V}_\varsigma
\end{array} \tag{1.39}
\]
commutes for all elements $x \in U_q$ if and only if we have $L = R = \mathcal{F}_\varsigma(T)$ for some valenced tangle $T \in TL_\mathbb{C}$.

2. Let $L \in \text{End} V_m$ and $R \in \text{End} V_\varsigma$. Let $\pi_m^{(s)} \in \text{End} V_m$ and $\pi_\varsigma^{(s)} \in \text{End} V_\varsigma$ be the respective projections onto the $s$-th summand of the direct-sum decompositions (1.7) of $V_m$ and $V_\varsigma$. The diagram

$$
\begin{array}{ccc}
V_m & \xrightarrow{\mathcal{F}_\varsigma(T)} & V_\varsigma \\
L \downarrow & & \downarrow R \\
V_m & \xrightarrow{\mathcal{F}_m(T)} & V_\varsigma
\end{array}
$$

(1.40)

commutes for all valenced tangles $T \in TL_\mathbb{C}$ if and only if we have

$$
L = \rho_m(x) + \sum_{s \in E_m \setminus E_\varsigma} \pi_m^{(s)} \circ L' \quad \text{and} \quad R = \rho_\varsigma(x) + \sum_{s \in E_\varsigma \setminus E_m} R' \circ \pi_\varsigma^{(s)}
$$

(1.41)

for some element $x \in U_q$ and endomorphisms $L' \in \text{End} V_m$ and $R' \in \text{End} V_\varsigma$.

Proof. We prove this result in section 6 B. It is a consequence of a version of the double-commutant theorem (proposition E.5, appendix E), and again, a dimension count and injectivity for $\mathcal{F}_\varsigma$, namely proposition 6.1 in section 6 A. □

Diagram representations for homomorphisms between tensor powers of type (1.21) of the fundamental module of the classical Lie algebra $\mathfrak{sl}_2$ have been known for a long time [RTW32, Pen71, CFS95]. In the semisimple case, this calculus generalizes to the case of $U_q$ and its fundamental modules. These diagrams are called tangles in the “Temperley-Lieb category” [TL71, Jon83, Tur94, KL94, CFS95]. However, the general, valenced tangles as maps between arbitrary type-one tensor product $U_q$-modules $U_q \odot V_\varsigma$ seem to be less explicitly investigated, although implicit in the literature. The main appeal of theorem 1.6 is that it follows from only a few elementary ingredients: Schur’s lemma, leading to general double-commutant tools (completely elementary but still included in appendix E for the sake of exposition), and injectivity of the map $\mathcal{F}_\varsigma$, to recognize the commutant algebra.

D. Interlude — the classical case

In the case $q = 1$, the above results do not make sense as such. Instead, considering the universal enveloping algebra $U := U(\mathfrak{sl}_2)$ of the classical Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, with generators $E$, $F$, and $H$ and relations

$$
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H,
$$

(1.42)

we recover a generalized version of the classical Schur-Weyl duality [Sch27, Wey39].

Theorem 1.7. (Higher-spin Schur-Weyl duality): The following hold:

1. The images of the maps $\mathcal{F}_\varsigma: TL_\mathbb{C}(-2) \longrightarrow \text{End} V_\varsigma$ and $\rho_\varsigma: U \longrightarrow \text{End} V_\varsigma$ are semisimple algebras, which equal

$$
TL_\mathbb{C}(-2) \cong \mathcal{F}_\varsigma(TL_\mathbb{C}(-2)) = \text{End}_U V_\varsigma \quad \text{and} \quad \rho_\varsigma(U) = \text{End}_TL V_\varsigma.
$$

(1.43)

2. The collections $\{M_\varsigma^{(s)} | s \in E_\varsigma\}$ and $\{L_\varsigma^{(s)} | s \in E_\varsigma\}$ are respectively the complete sets of simple non-isomorphic $\rho_\varsigma(U)$-modules and $TL_\mathbb{C}(-2)$-modules, and we have the direct-sum decomposition

$$
M_\varsigma \cong \bigoplus_{s \in E_\varsigma} M_\varsigma^{(s)} \otimes L_\varsigma^{(s)}.
$$

(SW_\varsigma)

3. The linear extension of the following map gives an explicit isomorphism for $(\text{SW}_\varsigma)$: with $w_\alpha$ the explicit $U$-highest-weight vectors constructed in definition 4.1 with $q = 1$, $\alpha \in \text{LP}_\varsigma^{(s)}$, $s \in E_\varsigma$, and $\ell \in \{0,1,\ldots,s\}$,

$$
F^\ell \cdot w_\alpha \longmapsto e_\ell^{(s)} \otimes \alpha.
$$

(1.44)

Proof. We summarize the proof for this result in appendix C. □

The other results discussed above (and below) also have obvious counterparts in the case of $q = 1$ and $\nu = -2$. 
FIG. 1.2: Illustration of the higher-spin quantum Schur-Weyl duality ($q$-SW). In the rows, adjacent boxes with the same coloring form standard modules $L_s$, closed under the action of $T_L(s)$. The columns form simple type-one modules $M_s$, closed under the action of $U_q$. Arrows indicate the action of $F \in U_q$, which maps from one copy of $L_s$ to another. The wide boxes represent Jones-Wenzl projectors and the square nodes represent valenced nodes.

### E. Structure under the valenced Temperley-Lieb algebra

Lastly, we summarize results concerning the “dual” structure of the module $T_L \odot V_s$. Theorem 1.4 already gives an explicit direct-sum decomposition for it when $n_s < p(q)$. However, the module $T_L \odot V_s$ is not always semisimple, and the failure of semisimplicity has gained increasing interest [Mar92, MMA92, RS07, BFGT09, GV13, PSA14, GST14, RSA14, ALZ15]. Non-semisimple cases should correspond to physically relevant applications to conformal field theory (minimal models, logarithmic phenomena), but are also of interest from representation theoretical point of view.

We recall from the above discussion that valenced link patterns $\alpha \in LP_s$ with $s$ defects form a finite set of cardinality $D_s$. For example, the following valenced link pattern has $s = 3$ defects (see section 3 A for the definitions):

\[
\text{Diagram concatenation using the Jones-Wenzl projectors [Jon83, Wen87, KL94], as explained in section 3 A, gives rise to a diagram action of the valenced Temperley-Lieb algebra } T_L(\nu) \text{ on formal linear combinations of valenced link patterns, standard modules (denoted } L_s(\nu) \text{ for each } s). \text{ Proposition 1.1 gives a map } \alpha \mapsto w_\alpha \text{ sending each valenced link pattern to a highest weight vector in } U_q \odot V_s, \text{ and all these vectors } w_\alpha \text{ are linearly independent. The diagram action of } T_L(\nu) \text{ on the link patterns } \alpha \text{ thus gives a natural action on the vectors } w_\alpha. \text{ In fact, this action coincides with the } T_L(\nu)-\text{action appearing in the quantum Schur-Weyl duality (which is via projections as in proposition 1.3).}
\]

When $n_s < p(q)$, items 1 and 3 of the next proposition imply that $T_L \odot H_s(\nu) \cong L_s(\nu)$, i.e., the highest-weight vector spaces are isomorphic to the valenced Temperley-Lieb standard modules. Otherwise, there are additional highest-weight vectors not reachable via the valenced link-pattern basis $w_\alpha$, due to degeneracies in the structure of the representation theory of $U_q$ [Lus90, Kas95, CP94]. In this case, after quotienting out highest-weight vectors which are orthogonal to the vectors $w_\alpha$, we still establish that the quotient $T_L(\nu)$-modules thus obtained are isomorphic to simple $T_L(\nu)$-modules $Q_s(\nu)$ obtained by taking the quotients of the standard modules $L_s(\nu)$ by the radical $\text{rad } L_s(\nu)$ of a natural invariant $T_L(\nu)$-bilinear form on them (section 3 D):

\[
Q_s(\nu) := L_s(\nu) / \text{rad } L_s(\nu) \quad \text{ and } \quad Q_s := \bigoplus_{s \in E_s} Q_s(\nu).
\]

**Proposition 1.8.** (Link-state – highest weight vector correspondence):
1. If \( n_\varsigma < p(q) \), then we have the following isomorphism of left \( \mathbf{TL}_\varsigma(\nu) \)-modules:

\[
\tau_\varsigma \circ V_\varsigma \cong \bigoplus_{s \in E_\varsigma} (s + 1) \tau_\varsigma \circ H_\varsigma^{(s)}. \tag{1.47}
\]

2. The linear map \( \alpha \mapsto w_\alpha \) induces the following embedding of \( \mathbf{TL}_\varsigma(\nu) \)-modules:

\[
L_\varsigma := \bigoplus_{s \in E_\varsigma} L_\varsigma^{(s)} \hookrightarrow \bigoplus_{s \in E_\varsigma} \tau_\varsigma \circ H_\varsigma^{(s)} \subset \tau_\varsigma \circ V_\varsigma. \tag{1.48}
\]

3. The linear map \( \alpha \mapsto w_\alpha \) induces the following embedding of \( \mathbf{TL}_\varsigma(\nu) \)-modules:

\[
\bigoplus_{s \in E_\varsigma \ s < p(q)} (s + 1) L_\varsigma^{(s)} \hookrightarrow \tau_\varsigma \circ V_\varsigma. \tag{1.49}
\]

If \( n_\varsigma < p(q) \), then this embedding (1.49) is an isomorphism.

4. The linear map \( \alpha \mapsto w_\alpha \) induces the following isomorphism of \( \mathbf{TL}_\varsigma(\nu) \)-modules:

\[
Q_\varsigma \cong \frac{\tau_\varsigma \circ H_\varsigma}{\tau_\varsigma \circ \{w_\alpha | \alpha \in L_\varsigma\}^\perp} = \bigoplus_{s \in E_\varsigma} \frac{\tau_\varsigma \circ H_\varsigma^{(s)}}{\tau_\varsigma \circ \{w_\alpha | \alpha \in L_\varsigma^{(s)}\}^\perp} \cong \bigoplus_{s \in E_\varsigma} Q_\varsigma^{(s)}, \tag{1.50}
\]

where \( \{ \cdots \}^\perp \) denotes the orthocomplement as detailed in section 4 B and appendix D.

**Proof.** Item 1 is a part of proposition 3.18 in section 3 C. Items 2 and 3 are respectively parts of proposition 4.12 and proposition 4.16 in section 4 B. Lastly, item 4 is a part of proposition 5.14 in section 5 C. This last result (1.50) is the only part of the present work which is not self-contained, relying on results in [FP18a]. \( \square \)

**Organization of this article**

Section 2 is mainly devoted to preliminaries concerning the quantum group \( \mathbf{U}_q \), its representations, embedding and projection operators, and a canonical invariant bilinear pairing. We also introduce the “conformal-block vectors” \( u_\varsigma^{(s)} \) (recall proposition 1.2), which are highest-weight vectors especially relevant for applications to conformal field theory [KKP19, FP20b]. Appendix A supplements these preliminaries and gathers formulas needed in applications.

In section 3 we introduce and investigate the valenced Temperley-Lieb algebra \( \mathbf{TL}_\varsigma(\nu) \) and its action on the tensor product \( \mathbf{U}_q \)-modules. We also discuss a diagram representation for vectors in these tensor product modules, analogous to the one developed by I. Frenkel and M. Khovanov [FK97], inspired by Kauffman’s Temperley-Lieb recoupling theory [KL94, CFS95]. We show that a natural invariant bilinear pairing defined on valenced link states, which form building blocks for representations of \( \mathbf{TL}_\varsigma(\nu) \), corresponds with the bilinear pairing for the \( \mathbf{U}_q \)-tensor product module.

Section 4 concerns the “link state – highest-weight vector correspondence” (proposition 1.1 and items 2 and 3 of proposition 1.8). Specifically, we explicate the connection between valenced link states \( \alpha \) and the valenced link-pattern basis vectors \( w_\alpha \), and show that these vectors are linearly independent highest-weight vectors in the \( \mathbf{U}_q \)-tensor product module. We also present a graphical calculus for these vectors and their \( F \)-descendants.

In section 5, we find diagram presentations for the conformal-block vectors and show that they are orthogonal (as stated in proposition 1.2). Furthermore, we find diagram presentations for projection operators between various \( \mathbf{U}_q \)-modules (given in lemma 5.8 and proposition 5.9). We also show that certain \( \mathbf{U}_q \)-submodule projectors generate the image of the representation \( \mathbf{F}_\varsigma \) of the valenced Temperley-Lieb algebra (lemma 5.10), which later implies proposition 1.3 in section 6. Finally, we identify quotients of highest-weight vector spaces with simple \( \mathbf{TL}_\varsigma(\nu) \)-modules, interestingly, even in the case when the \( \mathbf{U}_q \)-tensor product module is not semisimple (item 4 of proposition 1.8).

The final section 6 is devoted to the quantum Schur-Weyl duality itself. We prove both theorems 1.4 and 1.6, combining basic general results from appendix E with specific knowledge of the \( \mathbf{U}_q \) and \( \mathbf{TL}_\varsigma(\nu) \)-actions. Importantly, we prove that the \( \mathbf{TL}_\varsigma(\nu) \)-action is faithful whenever \( \max \varsigma < p(q) \) (proposition 1.5).

In appendix B, we briefly discuss the special value \( q = \pm i \) for which the loop fugacity \( \nu \) vanishes (1.24). Appendix C concerns the case of \( q = 1 \), which corresponds to the case of the classical Lie algebra \( \mathfrak{sl}_2 \). In appendix D, we include a few basic facts concerning bilinear pairings and orthogonality. The last appendix E is a self-contained summary of how the double-commutant properties follow from very basic results, combining linear algebra with Schur’s lemma.
2. QUANTUM GROUP ACTION ON ITS TYPE-ONE MODULES

In this section, we discuss the quantum group \( U_q = U_q(\mathfrak{sl}_2) \) and its representation theory. We give needed definitions and notation and introduce basic tools in section 2.1, and gather auxiliary results and details in appendix A. In this article, we only assume very basic familiarity with the concept of a representation. In section 2.2 we introduce the notation. We call \( \mathfrak{sl}_2 \) a (left) representation of some finite-dimensional vector space \( V \). If we have endomorphisms of some finite-dimensional vector space \( V \), then we call \( \mathfrak{sl}_2 \) a (left) representation of \( V \). Also, if a vector \( v \in V \) has a Poincaré-Birkhoff-Witt type basis \( \mathfrak{sl}_2 \), then we call \( \mathfrak{sl}_2 \) a (left) representation of \( V \). If \( 0 < \dim V \) satisfies \( E.v \neq 0 \) and \( K.v \neq 0 \) for some \( \lambda \in \mathbb{C} \), then we call \( \mathfrak{sl}_2 \) a (left) representation of \( V \). Also, if a vector \( v \in V \) satisfies \( E.v = 0 \) and \( K.v = \lambda v \) for some \( \lambda \in \mathbb{C} \), then we call \( v \) a highest-weight vector, and we call the \( \mathfrak{sl}_2 \)-module that generates a highest-weight vector. The following facts about the representation theory of \( \mathfrak{sl}_2 \) are used throughout.

**A. The quantum group \( U_q \) and its type-one modules**

The purpose of this section is to define (variants of) the quantum group \( U_q \) and summarize salient facts about its representation theory. Some basics concerning representations of associative algebras are discussed in appendix E. Throughout this section, we fix \( q \in \mathbb{C} \setminus \{0, \pm 1\} \) and use the notation

\[
p(q) := \begin{cases} 
\infty, & q \text{ is not a root of unity}, \\
p, & q = e^{2\pi i p'/p} \text{ for coprime } p, p' \in \mathbb{Z}_{>0}.
\end{cases}
\]

\( U_q := U_q(\mathfrak{sl}_2) \) is the infinite-dimensional associative \( \mathbb{C} \)-algebra with unit 1, generators \( E, F, K, K^{-1} \), and relations

\[
KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quadKF = q^{-2}FK, \quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}.
\]

As a vector space, \( U_q \) has a Poincaré-Birkhoff-Witt type basis

\[
\{E^kK^mF^\ell \mid k, \ell \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}\}.
\]

A (left) representation of \( U_q \) is a homomorphism \( \rho : U_q \to \text{End} V \) of algebras from \( U_q \) to the algebra \( \text{End} V \) of endomorphisms of some finite-dimensional vector space \( V \). We call \( U_q \) a \( U_q \)-module, emphasizing the action in the notation. We call \( U_q \) simple, and \( \rho \) irreducible, if it is not zero and it contains no non-zero proper submodules. We call \( U_q \) semisimple, and \( \rho \) completely reducible, if \( U_q \) is isomorphic to a direct sum of simple \( U_q \)-modules. If we have \( K.v = \lambda v \) for some \( v \in V \setminus \{0\} \) and \( \lambda \in \mathbb{C} \), then we call \( \lambda \) a weight of \( U_q \)-module. Also, if a vector \( v \in V \) satisfies \( E.v = 0 \) and \( K.v = \lambda v \) for some \( \lambda \in \mathbb{C} \), then we call \( v \) a highest-weight vector, and we call the \( U_q \)-module that generates a highest-weight vector. The following facts about the representation theory of \( U_q \) are used throughout.

**U1.** [Kas95, lemma VI.3.4]: Let \( v_0 \) be a highest-weight vector of weight \( \lambda \in \mathbb{C} \) and \( v_\ell := F^\ell.v_0 \). Then, for all \( \ell \in \mathbb{Z}_{\geq 0} \),

\[
K.v_\ell = \lambda q^{-2\ell}v_\ell, \quad E.v_\ell = \lambda q^{-(\ell-1)} - \frac{q^\ell - \lambda^{-1}}{q^{-1}}[\ell]v_{\ell-1}, \quad F.v_\ell = v_{\ell+1}.
\]

**U2.** [Kas95, theorem VI.3.5]: Let \( N \) be a \( U_q \)-module generated by a highest-weight vector \( v_0 \) of weight \( \lambda \in \mathbb{C} \), and let \( v_\ell := F^\ell.v_0 \). If \( 0 < \dim N = s+1 < p(q) \), then

(a): we have \( \lambda \in \{ \pm q^s \} \),

Acknowledgements

We are very grateful to K. Kytölä for numerous discussions and encouragement. We thank A. Gainutdinov, C. Hongler, J. Jacobsen, V. Jones, R. Kashaev, V. Pasquier, D. Radnell, N. Reshetikin, D. Ridout, Y. Saint-Aubin, and H. Saleur for helpful discussions. E.P. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – GZ 2047/1, Projekt-ID 390685813. She also acknowledges earlier support from the ERC AG COMPASP, the NCCR SwissMAP, and the Swiss NSF. During this work, S.F. was supported by the Academy of Finland grant number 288318, “Algebraic structures and random geometry of stochastic lattice models.”
(b): we have \( v_\ell = 0 \) for \( \ell > s \), and \( \{v_0, v_1, \ldots, v_s\} \) is a basis for \( N \),
(c): we have \( K.v_\ell = \lambda q^{-2\ell}v_\ell \) for all \( \ell \in \{0, 1, \ldots, s\} \),
(d): any other highest-weight vector of \( N \) is a scalar multiple of \( v_0 \), and
(e): \( N \) is simple.

**U3.** [Kas95, proposition VI.5.1]: If \( s + 1 < p(q) \), then any simple \( U_q \)-module of dimension \( s + 1 \) is isomorphic to a highest-weight module \( N_{(s)}(\chi) \) with \( \chi \in \{\pm 1\} \), having basis \( \{v_0, v_1, \ldots, v_s\} \) and \( U_q \)-action

\[
E.v_\ell = \begin{cases} v_{\ell + 1}, & 0 \leq \ell \leq s - 1, \\ 0, & \ell = s, \end{cases} \quad K.v_\ell = \chi q^{s-2\ell}v_\ell.
\]

**U4.** [Kas95, theorem VI.5.5]: If \( s + 1 = p(q) \), then the highest-weight modules \( N_{(s)}(\chi) \) are simple, but these are not all of the simple \( U_q \)-modules of dimension \( s + 1 \).

**U5.** [Kas95, proposition VI.5.2]: If \( s + 1 > p(q) \), then there is no simple \( U_q \)-module of dimension \( s + 1 \).

Throughout, for each \( s \in \mathbb{Z}_{\geq 0} \), we fix a vector space (spin chain) \( V_{(s)} := \text{span} \{e_0^{(s)}, e_1^{(s)}, \ldots, e_s^{(s)}\} \) of dimension \( s + 1 \) and with given basis. We define a left \( U_q \)-module structure on \( V_{(s)} \) via the rules

\[
F.e_\ell := \begin{cases} e_{\ell + 1}, & 0 \leq \ell \leq s - 1, \\ 0, & \ell = s, \end{cases} \quad E.e_\ell := \begin{cases} [\ell][s - \ell + 1]e_{\ell - 1}, & 1 \leq \ell \leq s, \\ 0, & \ell = 0, \end{cases} \quad K.e_\ell := q^{s-2\ell}e_\ell,
\]

and we call the resulting \( U_q \)-module a (left) **type-one** \( U_q \)-module and denote it by

\[
M_{(s)} := U_q \circ V_{(s)}.
\]

Facts U2–U5 imply that, when \( s + 1 \leq p(q) \), the \( U_q \)-module \( M_{(s)} \) is simple and isomorphic to \( N_{(s)}(-1) \). In this article, we do not consider modules of type \( N_{(s)}(-1) \). They can, in fact, be constructed from the simple type-one modules and the one-dimensional module \( N_{(0)}(-1) \) via \( N_{(s)}(-1) \cong N_{(s)}(1) \otimes N_{(0)}(-1) \).

The two-dimensional simple module \( M_{(1)} \), called the **fundamental module**, is the building block of all type-one \( U_q \)-modules. We denote its basis vectors by \( e_0 := e_0^{(1)} \) and \( e_1 := e_1^{(1)} \), so action (2.6) with \( s = 1 \) becomes

\[
F.e_0 = e_1, \quad F.e_1 = 0, \quad E.e_0 = 0, \quad E.e_1 = e_0, \quad K.e_0 = qe_0, \quad K.e_1 = q^{-1}e_1.
\]

In parallel, for each \( s \in \mathbb{Z}_{\geq 0} \), we fix another vector space \( \overline{V}_{(s)} := \text{span} \{e_0^{(s)}, e_1^{(s)}, \ldots, e_s^{(s)}\} \), of dimension \( s + 1 \) and with given basis, we define a right \( U_q \)-module structure on \( \overline{V}_{(s)} \) via the rules

\[
\overline{e}_\ell.F := \begin{cases} e_{\ell + 1}, & 0 \leq \ell \leq s - 1, \\ 0, & \ell = s, \end{cases} \quad \overline{e}_\ell.K := q^{-2\ell}e_\ell,
\]

and we denote the resulting right type-one \( U_q \)-module by

\[
\overline{M}_{(s)} := \overline{V}_{(s)} \circ U_q.
\]

Importantly, the algebra \( U_q \) also has a bialgebra structure [Kas95, CP94, KRT97], given by the coproduct \( \Delta: U_q \to U_q \otimes U_q \) and counit \( \epsilon: U_q \to \mathbb{C} \), the algebra homomorphisms defined by homomorphic extensions of

\[
\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},
\]

\[
\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1.
\]

We use the counit to define a \( U_q \)-action on the ground field \( \mathbb{C} \) by \( x.\lambda = \epsilon(x)\lambda \) for all \( x \in U_q \) and \( \lambda \in \mathbb{C} \). The \( U_q \)-module \( \mathbb{C} \) is isomorphic to the one-dimensional simple module \( M_{(0)} \) and we call it the **trivial module**. Using the coproduct (2.11), we define left and right \( U_q \)-module structures on tensor products of the form

\[
\mathbb{V}_\varsigma := \mathbb{V}_{(s_1)} \otimes \mathbb{V}_{(s_2)} \otimes \cdots \otimes \mathbb{V}_{(s_d)}, \quad \overline{\mathbb{V}}_\varsigma := \overline{\mathbb{V}}_{(s_1)} \otimes \overline{\mathbb{V}}_{(s_2)} \otimes \cdots \otimes \overline{\mathbb{V}}_{(s_d)},
\]

with \( \varsigma := (s_1, s_2, \ldots, s_d) \in \mathbb{Z}_{>0}^d \).
that we call type-one modules and denote by
\[ u_\iota \circ V_\iota := M_{(n)} \otimes M_{(n)} \otimes \cdots \otimes M_{(n)} \quad \text{and} \quad u_\iota \circ V_\iota := M_{(n)} \otimes M_{(n)} \otimes \cdots \otimes M_{(n)}, \]
\[ (2.14) \]
In the special case of \( \iota = \bar{n} \) for some \( n \in \mathbb{Z}_{>0} \), as in \((1.21, 3.1)\), we write \( V_n = V_{\bar{n}} \), and
\[ u_\iota \circ V_n = M(2.15) \]
for the \( n \)-th tensor powers of the left and right fundamental \( U_q \)-modules. The \( U_q \)-actions on these modules are defined by iterating the coproduct \((2.11)\),
\[ \Delta^{(d)} := (\Delta \otimes \text{id}^{(d-2)}) \circ (\Delta \otimes \text{id}^{(d-3)}) \circ \cdots \circ (\Delta \otimes \text{id}) \circ \Delta, \]
\[ (2.16) \]
so that, for all elements \( x \in U_q \) and vectors \( v \in V_\iota \) and \( \varphi \in \overline{V}_\iota \), we set
\[ x.v := \Delta^{(d)}(x).v. \quad \text{and} \quad \varphi.v := \varphi.\Delta^{(d)}(x). \]
\[ (2.17) \]
The coassociativity property of the coproduct \( \Delta \) guarantees that above, the order in which the tensor products are
\[ \text{formed does not matter for the } U_q \text{-module structure.} \]

We implicitly identify the trivial one-dimensional module \( M_{(0)} \) with the ground field \( \mathbb{C} \), and in particular, we omit the factor \( M_{(0)} \) from all tensor products via the canonical isomorphism \( \mathbb{C} \otimes M_{(0)} \cong M_{(0)} \cong M_{(0)} \otimes \mathbb{C} \). With this convention, if \( u_\iota \circ V_{\varpi} \) and \( u_\iota \circ V_\iota \) are two type-one modules, their tensor product is a type-one module too, because
\[ u_\iota \circ V_{\varpi} \otimes u_\iota \circ V_\iota \cong u_\iota \circ V_{\varpi \otimes \iota}, \]
\[ (2.18) \]
where \( \varpi \otimes \iota \) is the multindex obtained by concatenating \( \varpi \) to the left of \( \iota \) and removing possible zero entries. Furthermore, if \( f : u_\iota \circ V_{\varpi} \longrightarrow u_\iota \circ V_\iota \) and \( g : u_\iota \circ V_\iota \longrightarrow u_\iota \circ V_\varphi \) are homomorphisms of \( U_q \)-modules, then the map
\[ f \otimes g : u_\iota \circ V_{\varpi} \otimes u_\iota \circ V_\iota 
\[ u_\iota \circ V_\iota \otimes u_\iota \circ V_\varphi, \quad (f \otimes g)(v \otimes w) := f(v) \otimes g(w), \]
\[ (2.19) \]
extends linearly to a homomorphism of \( U_q \)-modules. (In fact, finite-dimensional \( U_q \)-modules form a monoidal category.)

In applications to conformal field theory [FP20b+], another version of the algebra \( U_q \) is also needed. This algebra \( \bar{U}_q := \bar{U}_q(sl_2) \) differs from \( U_q \) only via its coalgebra structure. Explicitly, \( \bar{U}_q \) is the associative \( \mathbb{C} \)-algebra with unit \( I \) and generators \( \bar{E}, \bar{F}, \bar{K}, K^{-1} \), satisfying the same relations \((2.2)\),
\[ \bar{K}K^{-1} = K^{-1}K = 1, \quad \bar{K}E = q^2KE, \quad \bar{K}F = q^{-2}KF, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}, \]
\[ (2.20) \]
with Poincaré-Birkhoff-Witt type basis \( \{ E^k \bar{K}^m F^\ell | k, \ell \in \mathbb{Z}_{>0}, m \in \mathbb{Z} \} \), but with bialgebra structure given by the following coproduct and counit:
\[ \Delta(E) = \bar{K}^{-1} \otimes \bar{E} + \bar{E} \otimes I, \quad \Delta(F) = F \otimes \bar{K} + I \otimes \bar{F}, \quad \Delta(K) = K \otimes \bar{K}, \quad \Delta(K^{-1}) = \bar{K}^{-1} \otimes K^{-1}, \]
\[ (2.21) \]
\[ \epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1. \]
\[ (2.22) \]
We define a left \( \bar{U}_q \)-module structure on the vector space \( V(\iota) \) via the rules
\[ \bar{F}.e^{(s)}_{\ell} := q^{-1 + (s - 2\ell)} F.e^{(s)}_{\ell}, \quad \bar{E}.e^{(s)}_{\ell} := q^{-1 - (s - 2\ell)} E.e^{(s)}_{\ell}, \quad \bar{K}.e^{(s)}_{\ell} := K.e^{(s)}_{\ell}, \]
\[ (2.23) \]
and we denote by \( V(\iota) \) the resulting left \( \bar{U}_q \)-module. Similarly, we define a right \( \bar{U}_q \)-module structure on \( \bar{V}(\iota) \) via
\[ e^{(s)}_{\ell}.\bar{E} := q^{-1 - (s - 2\ell)} e^{(s)}_{\ell}.E, \quad \bar{E}.e^{(s)}_{\ell} := q^{1 + (s - 2\ell)} e^{(s)}_{\ell}.F, \quad \bar{K}.e^{(s)}_{\ell} := e^{(s)}_{\ell}.K, \]
\[ (2.24) \]
and we denote the resulting right \( \bar{U}_q \)-module by \( \bar{V}(\iota) \otimes \bar{V}(\iota) \). Finally, iterating the coproduct \( \Delta \), we define the tensor product modules \( \bar{V}(\iota) \otimes V(\iota) \) and \( \bar{V}(\iota) \otimes V(\iota) \) with left and right \( \bar{U}_q \)-actions denoted by \( \bar{\iota}.v \) and \( v.\bar{\iota} \), respectively.

The two bialgebras \( U_q \) and \( \bar{U}_q \) are related in a simple way: a calculation shows that the linear map
\[ E^k \bar{K}^m F^\ell \longrightarrow (E^k \bar{K}^m F^\ell)^* := E^\ell \bar{K}^m F^k, \]
\[ (2.25) \]
\[ (E^k \bar{K}^m F^\ell)^* \longrightarrow (E^k \bar{K}^m F^\ell)^* := E^\ell \bar{K}^m F^k, \]
\[ (2.26) \]
gives an anti-isomorphism of associative, unital algebras, as well as an isomorphism of coassociative, counital coalgebras.

The tensor product \( V_\varsigma \) (similarly, \( \overline{V}_\varsigma \)) defined in (2.13) admits the following \( s \)-grading:

\[
V_\varsigma = \bigoplus_{s \in \mathbb{E}^\pm_\varsigma} V_\varsigma^{(s)}, \quad \text{where} \quad \mathbb{E}^\pm_\varsigma := \{-n_\varsigma, -n_\varsigma + 2, \ldots, n_\varsigma - 2, n_\varsigma\}. 
\]  

(2.27)

\[
V_\varsigma^{(s)} := \text{span} \left\{ e_{\ell_1}^{(s_1)} \otimes e_{\ell_2}^{(s_2)} \otimes \cdots \otimes e_{\ell_k}^{(s_k)} \in V_\varsigma \mid n_\varsigma - 2(\ell_1 + \ell_2 + \cdots + \ell_k) = s \right\}. 
\]  

(2.28)

By (2.6, 2.11), all \( K \)-eigenvalues in \( U_v \circ V_\varsigma \) have the form \( q^s \) with \( s \in \mathbb{E}^\pm_\varsigma \). If \( q \) is a root of unity, some of these eigenvalues may coincide. They are all distinct if

\[
n_\varsigma < p(q), \quad \text{where} \quad p(q) := \begin{cases} \infty, & \text{if } q \text{ is not a root of unity}, \\ p, & \text{if } q = e^{\pi ip}/p \text{ for coprime } p, p' \in \mathbb{Z}_{>0}. 
\end{cases} 
\]  

(2.29)

and in this case, we have

\[
n_\varsigma < p(q) \quad \implies \quad V_\varsigma^{(s)} = \{ v \in V_\varsigma \mid K.v = q^s.v \}, \quad \overline{V}_\varsigma^{(s)} = \{ \overline{v} \in \overline{V}_\varsigma \mid \overline{v}.K = q^s.\overline{v} \}.
\]  

(2.30)

The \( s \)-grade of is moved up and down via the generators \( E \) and \( F \), as specified in lemma A.5 in appendix A; e.g.

\[
E.v, \overline{E}.v \in V_\varsigma^{(s+2)}, \quad F.v, \overline{F}.v \in V_\varsigma^{(s-2)}, \quad K^{\pm 1}.v, \overline{K}^{\pm 1}.v \in V_\varsigma^{(s)} \quad \text{for all } v \in V_\varsigma^{(s)}.
\]  

(2.31)

Also, we can trade the generators \( E \) and \( F \) of \( U_q \) with the generators \( \overline{E} \) and \( \overline{F} \) of \( \overline{U}_q \) via

\[
(E^n \overline{K}^m \overline{F}^\ell).v = q^{-ns(s-2\ell+s)+n(s-2\ell)+\ell(s-\ell)}(E^k \overline{K}^m \overline{F}^\ell).v \quad \text{for all } v \in V_\varsigma^{(s)}.
\]  

(2.32)

Next, we consider highest-weight vectors in the left and right \( U_q \)-type-one modules:

\[
H_\varsigma := \{ v \in V_\varsigma \mid E.v = 0 \} \quad \text{and} \quad \overline{H}_\varsigma := \{ \overline{v} \in \overline{V}_\varsigma \mid \overline{v}.F = 0 \}. 
\]  

(2.33)

These spaces are graded as in (2.27–2.28):

\[
H_\varsigma^{(s)} := H_\varsigma \cap V_\varsigma^{(s)} \quad \text{and} \quad \overline{H}_\varsigma^{(s)} := \overline{H}_\varsigma \cap \overline{V}_\varsigma^{(s)};
\]  

(2.34)

with \( K \)-eigenvalues of the form \( q^s \) for integers \( s \in \mathbb{E}^\pm_\varsigma \). Analogously to (2.30), we note that

\[
n_\varsigma < p(q) \quad \implies \quad H_\varsigma^{(s)} = \{ v \in V_\varsigma \mid E.v = 0, K.v = q^s.v \}, \quad \overline{H}_\varsigma^{(s)} = \{ \overline{v} \in \overline{V}_\varsigma \mid \overline{v}.F = 0, \overline{v}.K = q^s.\overline{v} \}.
\]  

(2.35)

**Lemma 2.1.** Suppose \( q \in \mathbb{C}^\times \setminus \{ \pm 1 \} \). The highest-weight vector spaces have the following direct-sum decompositions:

\[
H_\varsigma = \bigoplus_{s \in \mathbb{E}^\pm_\varsigma} H_\varsigma^{(s)} \quad \text{and} \quad \overline{H}_\varsigma = \bigoplus_{s \in \mathbb{E}^\pm_\varsigma} \overline{H}_\varsigma^{(s)}. 
\]  

(2.36)

*Proof.* Definitions (2.34) immediately give the containments \( \subset \subset \) in (2.36), and with \( s \)-gradings (2.27), we see that the sums in (2.36) are direct. To prove the containments \( \supset \supset \), we decompose a vector \( v \in H_\varsigma \) as

\[
v = \bigoplus_{s \in \mathbb{E}^\pm_\varsigma} v^{(s)}, \quad \text{where} \quad v^{(s)} \in V_\varsigma^{(s)},
\]  

(2.37)

and use lemma A.5 from appendix A to obtain

\[
v \in H_\varsigma \quad \overset{(2.33)}{\implies} \quad E.v = 0 \quad \overset{(2.37)}{\implies} \quad E.v^{(s)} = 0 \quad \text{for all } s \in \mathbb{E}^\pm_\varsigma \] 

(2.38)

\[
\overset{(A.24)}{=} \quad v^{(s)} \in H_\varsigma^{(s)} \quad \text{for all } s \in \mathbb{E}^\pm_\varsigma \quad \overset{(2.37)}{\implies} \quad v \in \bigoplus_{s \in \mathbb{E}^\pm_\varsigma} H_\varsigma^{(s)} \]  

(2.39)

which gives the containment \( \supset \supset \) for the left equation of (2.36). The right side of (2.36) follows similarly. \( \square \)
We can describe the highest-weight vectors recursively. For this purpose, and throughout this article, for a multiindex \( \zeta \in \mathbb{Z}_{>0}^d := \mathbb{Z}_{>0} \cup \mathbb{Z}_{>0}^2 \cup \mathbb{Z}_{>0}^3 \cup \cdots \), we denote
\[
\zeta = (s_1, s_2, \ldots, s_d) \implies \zeta := (s_1, s_2, \ldots, s_{d-1}) \quad \text{and} \quad t := s_d.
\]

**Lemma 2.2.** Suppose \( q \in \mathbb{C}^\times \setminus \{\pm 1\} \). The following hold:

1. Write the vector \( v \in \mathcal{V}_\zeta^{(s)} \) in the form
\[
v = \sum_{\ell=0}^{t} v_{\ell} \otimes e_{t-\ell}^{(t)}, \quad \text{where} \quad v_{\ell} \in \mathcal{V}_{\zeta}^{(s+t-2\ell)}.
\]

Then \( v \in \mathcal{H}_\zeta^{(s)} \) if and only if
\[
E.v_{\ell} = \begin{cases} -q^{t-2\ell}[t-\ell+1][\ell] v_{\ell-1}, & \ell > 0, \\ 0, & \ell = 0. \end{cases}
\]

2. Similarly, write the vector \( \bar{v} \in \bar{\mathcal{V}}_\zeta^{(s)} \) in the form
\[
\bar{v} = \sum_{\ell=0}^{t} \bar{v}_{\ell} \otimes e_{t-\ell}^{(t)}, \quad \text{where} \quad \bar{v}_{\ell} \in \bar{\mathcal{V}}_{\zeta}^{(s+t-2\ell)}.
\]

Then \( \bar{v} \in \bar{\mathcal{H}}_\zeta^{(s)} \) if and only if
\[
\bar{v}_{\ell}.F = \begin{cases} -q^{s-t+2\ell-2}[t-\ell+1][\ell] \bar{v}_{\ell-1}, & \ell > 0, \\ 0, & \ell = 0. \end{cases}
\]

**Proof.** Writing a vector \( v \in \mathcal{V}_\zeta^{(s)} \) in the generic form (2.41), we have
\[
E.v \overset{(2.6, 2.11)}{=} q^{-t}(E.v_0) \otimes e_t^{(t)} + \sum_{\ell=1}^{t} \left(q^{2\ell-t}(E.v_{\ell-1}) \otimes e_{t-\ell}^{(t)} + [t-\ell+1][\ell] v_{\ell-1} \otimes e_{t-\ell}^{(t)}\right),
\]
which clearly vanishes if and only if (2.24) holds. This proves item 1, and item 2 can be proven similarly.

**Corollary 2.3.** Suppose \( \max \zeta < p(q) \). Then, for all \( s \in \mathbb{Z} \), we have
\[
\dim \mathcal{H}_\zeta^{(s)} = \dim \bar{\mathcal{H}}_\zeta^{(s)} \leq B^{(s)}_{\zeta},
\]
where the integers \( B^{(s)}_{\zeta} \) are unique solution to the recursion problem
\[
B^{(s)}_{\zeta} = \sum_{r=s-t}^{s+t} B^{(r)}_{\zeta}, \quad B^{(0)}_{\zeta} = \delta_{t,s}.
\]

**Proof.** We first prove that \( \dim \mathcal{H}_\zeta^{(s)} \leq B^{(s)}_{\zeta} \) by induction on \( d_\zeta \in \mathbb{Z}_{>0} \). In the initial case \( d_\zeta = 1 \), we have \( \zeta = (t) \) for some \( t \in \mathbb{Z}_{>0} \), and (2.47) follows from (2.6) and (1.1, 2.1), assuming that \( t < p(q) \). Next, we let \( d \geq 2 \) and assume that \( \dim \mathcal{H}_\zeta^{(s)} \leq B^{(s)}_{\zeta} \) holds for any multiindex \( \zeta \in \mathbb{Z}_{>0}^{d-1} \) with \( \max \zeta < p(q) \). We let \( \mathcal{H}_\zeta^{(s);0} \) denote the vector space
\[
\mathcal{H}_\zeta^{(s);0} := \{ v \in \mathcal{V}_\zeta^{(s)} \mid E^m.v = 0 \}.
\]

Then, taking \( \zeta \in \mathbb{Z}_{>0}^d \) with \( \max \zeta < p(q) \) and using notation (2.40), we consider the linear map
\[
v_{\ell} \mapsto \sum_{\ell=0}^{t} \lambda^{(t)}_{\ell}(E^{t-\ell}.v_{\ell}) \otimes e_{t-\ell}^{(t)}, \quad \text{where} \quad \lambda^{(t)}_{\ell} = (-1)^{t-\ell} q^{(t-\ell)(\ell+1)} [\ell]! [t-\ell]! [t]! \neq 0,
\]
sending \( \mathcal{H}_\zeta^{(s);t+1} \) into \( \mathcal{V}_\zeta^{(s)} \) (where the constants \( \lambda^{(t)}_{\ell} \) are motivated by item 1 of lemma 2.2). We observe that
\[
\sum_{\ell=0}^{t} \lambda^{(t)}_{\ell}(E^{t-\ell}.v_{\ell}) \otimes e_{t-\ell}^{(t)} = 0 \implies v_{\ell} \otimes e_{0}^{(t)} = 0 \implies v_{\ell} = 0,
\]

(2.48)
so this map is injective. Also, repeated application of (2.42) from item 1 of lemma 2.2 implies that the image of this map is $H_\xi^{(s)}$. In conclusion, the map (2.49) is a linear isomorphism from $H_\xi^{(s-t);t+1}$ to $H_\xi^{(s)}$, so we have
\[
\dim H_\xi^{(s)} = \dim H_\xi^{(s-t);t+1}. \tag{2.51}
\]

By definition (2.48) of $H_\xi^{(s);p}$ and (A.24) from lemma A.5, the action of $E$ induces a sequence of $t$ linear maps,
\[
H_\xi^{(s-t);t+1} \xrightarrow{E.} H_\xi^{(s-t+2);t} \xrightarrow{E.} H_\xi^{(s-t+4);t-1} \xrightarrow{E.} \cdots \xrightarrow{E.} H_\xi^{(s+t);t} = H_\xi^{(s+t)}. \tag{2.52}
\]

By definition (2.33), the vector space $H_\xi^{(s-t)}$ contains the kernel of the first (leftmost) map in this chain (2.52), so
\[
\dim H_\xi^{(s-t);t+1} \leq \dim H_\xi^{(s-t)} + \dim H_\xi^{(s-t+2);t}. \tag{2.53}
\]

Iterating this argument $(t-1)$ more times gives
\[
\dim H_\xi^{(s)} (2.51) \leq \dim H_\xi^{(s-t);t+1} (2.53) \leq \sum_{r+s+t=0 \, (\text{mod} \, 2)}^{s+t} \dim H_\xi^{(r)} (2.46) \leq \sum_{r+s+t=0 \, (\text{mod} \, 2)}^{s+t} B_\xi^{(s)} (2.47) = B_\xi^{(s)},
\]

where the rightmost inequality follows from the induction hypothesis. This completes the induction step.

To finish, using the linear isomorphism $(\cdot)^* : V_\xi \to \nabla_\xi$ from lemma A.6 combined again with lemma A.5, we have
\[
v \in H_\xi^{(s)} \quad \xrightarrow{(2.33) \ (2.34)} \quad E.v = 0 \quad \xrightarrow{(A.25) \ (A.37) \ (A.38)} \quad E.v = 0 \quad \xrightarrow{(2.33) \ (2.34)} \quad v^* = 0 \quad \xrightarrow{(2.33) \ (2.34)} \quad v^* \in H_\xi^{(s)}, \tag{2.54}
\]
\[v \in \nabla_\xi^{(s)} \quad \xrightarrow{(2.33) \ (2.34)} \quad v.F = 0 \quad \xrightarrow{(A.27) \ (A.38)} \quad v.F = 0 \quad \xrightarrow{(2.33) \ (2.34)} \quad v^* = 0 \quad \xrightarrow{(2.33) \ (2.34)} \quad v^* \in H_\xi^{(s)}, \tag{2.55}
\]

so the linear map $(\cdot)^*$ restricted to the subspace $H_\xi^{(s)}$ is a linear injection into $\nabla_\xi^{(s)}$, and its inverse $(\cdot)^*$ restricted to the subspace $\nabla_\xi^{(s)}$ is a linear injection into $H_\xi^{(s)}$. This shows that $\dim H_\xi^{(s)} = \dim \nabla_\xi^{(s)}$ and finishes the proof. \hfill $\Box$

### B. Conformal-block vectors and direct-sum decomposition

It is well known that when $n_\varsigma < p(q)$, the type-one $U_q, \tilde{U}_q$-modules $\upsilon_{\varsigma, \tilde{\varsigma}} \circ V_\varsigma$ and $\nabla_\varsigma \circ \upsilon_{\varsigma, \tilde{\varsigma}}$ are semisimple, as explicated in proposition 2.8 below. The main aim of this section is to construct particular highest-weight vectors, that we call “conformal-block vectors,” which give rise to explicit decompositions of these modules into direct sums of simple submodules. The main importance of the conformal-block vectors is their applications to conformal field theory [FP20b⁺], which also explains our choice of terminology. When $n_\varsigma \geq p(q)$, the modules $\upsilon_{\varsigma, \tilde{\varsigma}} \circ V_\varsigma$ and $\nabla_\varsigma \circ \upsilon_{\varsigma, \tilde{\varsigma}}$ are no longer semisimple and also the direct construction of the conformal-block vectors fails, (reflecting degeneracies also in conformal field theory). In section 4, we find other highest-weight vectors, that we call the “(valenced) link-pattern basis vectors,” useful beyond the range $n_\varsigma < p(q)$.

The conformal-block vectors are indexed by certain walks. To define and study them, we first introduce some combinatorial notation and make simple observations. For $r, s, t \in \mathbb{Z}_{\geq 0}$, we set
\[
E_{(s)} = \{s\} \quad \text{and} \quad E_{(r,t)} = \{|r - t|, |r - t| + 2, \ldots, r + t\}. \tag{2.56}
\]

We immediately note the symmetry relations
\[
s \in E_{(r,t)} \iff r \in E_{(t,s)} \iff t \in E_{(s,r)}. \tag{2.57}
\]

For a multiindex $\varsigma = (s_1, s_2, \ldots, s_d) \in \mathbb{Z}^d_{\geq 0}$, we define a walk over $\varsigma$ to be a new multiindex $q = (r_1, r_2, \ldots, r_d)$ whose entries, each called a height, satisfy the following two conditions relative to $\varsigma$:
\[
r_0 = 0, \quad r_{i+1} \in E_{(r_1, r_2, \ldots, r_i)} (2.56) \Rightarrow \{|r_i - s_{i+1}|, |r_i - s_{i+1}| + 2, \ldots, r_i + s_{i+1}\} \quad \text{for all } i \in \{1, 2, \ldots, d_\varsigma - 1\}. \tag{2.58}
\]

As a notation convention, we do not explicitly show the zeroth height $r_0 = 0$ of the walk $q = (r_1, r_2, \ldots, r_d)$ so the length of $q$ matches that of $\varsigma$. We call the last height $d_\varsigma := r_d$ the defect of the walk $q$. Then, we set
\[
D_\varsigma := \#\{\text{walks over } \varsigma \text{ with defect } s\}, \tag{2.59}
\]
\[ E_\varsigma := \{ s \in \mathbb{Z}_{\geq 0} \mid \text{there exists a walk over } \varsigma \text{ with defect } s \}, \]
\[ D_\varsigma := \sum_{s \in E_\varsigma} D_\varsigma^{(s)} = \#\{\text{walks over } \varsigma\}. \]

In the special case of \( \varsigma = \vec{n} \), we simply have
\[ E_n = \{ n \mod 2, (n \mod 2) + 2, \ldots, n \}, \]
and more generally, there are integers \( s_{\min}(\varsigma), s_{\max}(\varsigma) \in \mathbb{Z}_{\geq 0} \) such that (see also [FP18a, lemmas 2.1–2.3])
\[ E_\varsigma = \{ s_{\min}(\varsigma), s_{\min}(\varsigma) + 2, \ldots, s_{\max}(\varsigma) \} \subset E_{n_\varsigma}^+, \quad \text{where} \quad s_{\max}(\varsigma) = n_\varsigma. \]

In fact, the number of walks over \( \varsigma \) can be determined recursively, using notation (2.40):

**Lemma 2.4.** The following hold:

1. The integers \( D_\varsigma^{(s)} \) defined in (2.59), for \( s \in \mathbb{Z} \), are the unique solution to the following recursion problem:
\[ D_\varsigma^{(s)} = \sum_{r \in E_{(s,t)}} D_\varsigma^{(r)} \quad \text{and} \quad D_\varsigma^{(s)} = \delta_{s,t}. \]

2. In particular, we have
\[ s \notin E_\varsigma \iff D_\varsigma^{(s)} = 0, \]
so recursion (2.64) simplifies to
\[ D_\varsigma^{(s)} = \sum_{r \in E_\varsigma \cap E_{(s,t)}} D_\varsigma^{(r)} \quad \text{and} \quad D_\varsigma^{(s)} = \delta_{s,t}, \]

3. The following identity holds:
\[ \sum_{s \in E_\varsigma} (s + 1)D_\varsigma^{(s)} = \prod_{i=1}^{d_\varsigma} (s_i + 1). \]

**Proof.** Item 1 is proved in [FP18a, lemma 4.1, item 4]. Item 2 follows immediately from this and (2.59). We prove item 3 by induction on \( d_\varsigma \in \mathbb{Z}_{>0} \). It is trivial for \( d_\varsigma = 1 \). In the induction step, we need the case \( d_\varsigma = 2 \) as well. In that case, with \( \varsigma = (r,t) \) and for all \( s \in E_{(r,t)} \), we immediately obtain from (2.66) that
\[ D_{(r,t)}^{(s)} = 1 \quad \implies \quad \sum_{s \in E_{(r,t)}} (s + 1)D_{(r,t)}^{(s)} = \sum_{s \in E_{(r,t)}} (s + 1) \overset{(2.56)}{=} (r + 1)(t + 1), \]
which proves (2.67) for the case \( d_\varsigma = 2 \). To finish, we assume that (2.67) holds for \( d_\varsigma = d - 1 \) for some integer \( d \geq 3 \) and show that it holds if \( d_\varsigma = d \) too. Indeed, using notation (2.40), we have
\[ \prod_{i=1}^{d} (s_i + 1) \overset{(2.40)}{=} (t + 1) \sum_{r \in E_\varsigma} (r + 1)D_\varsigma^{(r)} \overset{(2.67)}{=} \sum_{r \in E_\varsigma} \sum_{s \in E_{(r,t)}} (s + 1)D_{(r,t)}^{(s)}D_\varsigma^{(r)} \overset{(2.66)}{=} \sum_{s \in E_\varsigma} (s + 1)D_\varsigma^{(s)}, \]
which completes the induction step. \( \square \)
Our next aim is to define the conformal-block vectors and prove that they are linearly independent highest-weight vectors. To begin, if \( \max \zeta < p(q) \) and \( \zeta = \vartheta + \vartheta \), then we set

\[
\eta^{(s)}_{(r,t)} (v \otimes w) := \sum_{i = 0}^{r+t-s} \sum_{j = 0}^{r+t-s} \delta_{i+j, r+t-s} \frac{(-1)^i q^{i(t+1-j)}}{(q - q^{-1})(r+t-s)!} \frac{[r-i]! [t-j]!}{[j]! [i]! [t]!} F_{i,v} \otimes F_{j,w}.
\]

(2.70)

Similarly, this lemma holds after the symbolic replacements

\[
\rho \mapsto \varepsilon, \ H \mapsto \bar{H}, \ w \mapsto \bar{w}, \ \text{and} \ \eta \mapsto \bar{\eta}.
\]

(2.71)

for all vectors \( v \in H^{(r)}_{\varrho}, \ w \in H^{(t)}_{\vartheta}, \ \bar{v} \in \bar{H}^{(r)}, \) and \( \bar{w} \in \bar{H}^{(t)}_{\vartheta} \), and for all \( s \in E_{(r,t)} \) such that \( \max(r,t) < p(q) \). Also, for \( v = \epsilon^{(r)}_{0}, \ w = \epsilon^{(t)}_{0}, \ \bar{v} = \bar{\epsilon}^{(r)}_{0}, \) and \( \bar{w} = \bar{\epsilon}^{(t)}_{0} \), we simply denote

\[
u^{(s)}_{(r,t)} := \sum_{i = 0}^{r+t-s} \sum_{j = 0}^{r+t-s} \delta_{i+j, r+t-s} \frac{(-1)^i q^{i(t+1-j)}}{(q - q^{-1})(r+t-s)!} \frac{[r-i]! [t-j]!}{[j]! [i]! [t]!} \epsilon^{(r)}_{i} \otimes \epsilon^{(t)}_{j},
\]

(2.72)

\[
\tau^{(s)}_{(r,t)} := \sum_{i = 0}^{r+t-s} \sum_{j = 0}^{r+t-s} \delta_{i+j, r+t-s} \frac{(-1)^i q^{i(t+1-j)}}{(q - q^{-1})(r+t-s)!} \frac{[r-i]! [t-j]!}{[j]! [i]! [t]!} \tau^{(r)}_{i} \otimes \tau^{(t)}_{j}.
\]

(2.73)

In addition to notation (2.40), we frequently use the following notation for a walk \( q \) over a multiindex \( \zeta \in \mathbb{Z}_{d_0}^{d} \):

\[
q = (r_1, r_2, \ldots, r_{d_0}) \quad \Rightarrow \quad \hat{q} = (r_1, r_2, \ldots, r_{d_0-1}), \quad r = r_{d_0-1}, \quad s = r_{d_0}.
\]

(2.74)

**Definition 2.5.** When \( \max(\zeta, \hat{\zeta}) < p(q) \), we recursively define the conformal-block vectors

\[
u^{(t)} := \epsilon^{(t)}_{0} \quad \text{and} \quad \nu^{(s)} := \eta^{(s)}_{(r,t)}(\nu^{(s)}_{(r,t)} \otimes \epsilon^{(t)}_{0}),
\]

(2.75)

\[
\tau^{(t)} := \bar{\epsilon}^{(t)}_{0} \quad \text{and} \quad \tau^{(s)} := \bar{\eta}^{(s)}_{(r,t)}(\tau^{(s)}_{(r,t)} \otimes \bar{\epsilon}^{(t)}_{0}).
\]

(2.76)

In the special case of \( \zeta = \bar{n} \), by definition (2.58), all walks \( q = (r_1, r_2, \ldots, r_n) \) over \( \bar{n} \) have only steps of size one, and the conformal-block vectors can be written very explicitly using definition (2.74–2.75) and formula (2.70):

\[
u^{(t)}_{n} = \nu^{(t)}_{n-1} \otimes \varepsilon_{0}, \quad \nu^{(s)}_{n} = \frac{1}{q-q^{-1}} (-q u^{\hat{\varrho}}_{n-1} \otimes \varepsilon_{1} + \frac{1}{[r_{n-1}]!} F_{1} u^{\hat{\varrho}}_{n-1} \otimes \varepsilon_{0}), \quad r_{n} = r_{n-1} + 1,
\]

(2.77)

where \( \hat{\varrho} = (r_1, r_2, \ldots, r_{n-1}) \). We note that these are essentially the same vectors that were constructed in [KKP19].

**Lemma 2.6.** Suppose \( \max(\varrho, \vartheta, r, t) < p(q) \). Then, for all \( s \in E_{(r,t)} \), we have

\[
\begin{cases}
v \in H^{(r)}_{\varrho} \\
w \in H^{(t)}_{\vartheta}
\end{cases} \quad \Rightarrow \quad \eta^{(s)}_{(r,t)}(v \otimes w) \in H^{(s)}_{\varrho \oplus \vartheta}.
\]

(2.78)

Similarly, this lemma holds after the symbolic replacements \( v \mapsto \bar{v}, \ H \mapsto \bar{H}, \ w \mapsto \bar{w}, \) and \( \eta \mapsto \bar{\eta} \).

**Proof.** We only prove (2.78); the other case can be proven similarly. Without loss of generality, we may assume that

\[
q = (r), \quad \varrho = (t) \quad \Rightarrow \quad \eta^{(s)}_{(t)}(v \otimes w) = u^{(s)}_{(r,t)}.
\]

(2.79)

Then using formula (2.72), it is a straightforward calculation to show that \( E u^{(s)}_{(r,t)} = 0 \) and \( K u^{(s)}_{(r,t)} = q^{s} u^{(s)}_{(r,t)} \).

We next prove that the conformal-block vectors are linearly independent.

**Lemma 2.7.** Suppose \( \max \zeta < p(q) \). The following hold:

1. The collection \( \{u^{(s)}_{\zeta} \mid \max \hat{\zeta} < p(q)\} \) is a linearly independent subset of \( H_{\zeta} \).

2. For each \( s \in E_{\zeta} \), the collection \( \{u^{(s)}_{\zeta} \mid \max \hat{\zeta} < p(q) \text{ and } \text{def}_{\varrho} = s\} \) is a linearly independent subset of \( H_{\zeta}^{(s)} \).
Similarly, this lemma holds after the symbolic replacements \( H \mapsto \bar{H} \) and \( u \mapsto \bar{u} \).

**Proof.** In light of lemma 2.1, item 1 follows from item 2, so it suffices to prove the latter. To this end, by lemma 2.6, we only have to show that the collection \( \{ u_\xi^\varrho \mid \max \hat{\varrho} < p(q) \text{ and } \def_\varrho = s \} \) is linearly independent. We prove this by induction on \( d_\varsigma \in \mathbb{Z}_{>0} \). The case \( d_\varsigma = 1 \) is trivial. For notational simplicity, supplementing notation (2.74), we write

\[
\hat{\varrho}' := \hat{\varrho} = (r_1, r_2, \ldots, r_{d_\varsigma - 2}).
\]

(2.80)

Now, we assume that \( d_\varsigma = d - 1 \) for some integer \( d \geq 2 \) and that the collection

\[
\{ u_\xi^\varrho \mid \max \hat{\varrho} < p(q) \text{ and } \def_\varrho = s \}
\]

(2.81)

is linearly independent. Then, we suppose that

\[
\sum_{g : \text{ max } \hat{g} < p(q)} c_\varrho u_\xi^\varrho = 0
\]

(2.82)

is a vanishing linear combination of vectors in the collection \( \{ u_\xi^\varrho \mid \max \hat{\varrho} < p(q) \text{ and } \def_\varrho = s \} \), where \( c_\varrho \in \mathbb{C} \) are some constants. We write each vector \( u_\xi^\varrho \) in terms of its recursive definition (2.75), to obtain

\[
0 = \sum_{g : \text{ max } \hat{g} < p(q)} c_\varrho \eta_{(r,t)}(u_\xi^\varrho \otimes e^{(i)}) = \sum_{i = 0}^{r + t - s} w_i^{(s)},
\]

(2.83)

where

\[
w_i^{(s)} := \left( \sum_{g : \text{ max } \hat{g} < p(q)} c_\varrho \frac{(-1)^j q^{j(t+1-j)}}{(q - q^{-1})(r + t - s)/2} \frac{[r - i]! [t - j]!}{[j]! [r]! [t]!} F^{i} u_\xi^\varrho \right) \otimes e^{(j)}, \quad \text{and } j = \frac{r + t - s}{2} - i.
\]

(2.84)

Now, by definition 2.5 and lemma 2.6, the \((K \otimes 1)\)-eigenvalue of \( w_i^{(s)} \) is \( q^{-2i} \), and because \( 0 \leq i \leq r < p(q) \), these eigenvalues are distinct. In light of this fact and (2.83), we have \( w_i^{(s)} = 0 \) for all \( i \in \{0, 1, \ldots, \frac{1}{2} (r - s + t)\} \), so

\[
w_0^{(s)} = \left( \sum_{g : \text{ max } \hat{g} < p(q)} c_\varrho \frac{(-1)^j q^{j(t+1-j)}}{(q - q^{-1})(r + t - s)/2} \frac{[r]! [t - j]!}{[j]! [r]! [t]!} u_\xi^\varrho \right) \otimes e^{(j)} = 0, \quad \text{with } j = \frac{r + t - s}{2}.
\]

(2.85)

so the sum in (2.85) vanishes. Then, because collection (2.81) is linearly independent by the induction hypothesis, we have \( c_\varrho = 0 \) for each term in (2.85). This shows that \( \{ u_\xi^\varrho \mid \max \hat{\varrho} < p(q) \text{ and } \def_\varrho = s \} \) is linearly independent.

The statements with \( H \mapsto \bar{H} \) and \( u \mapsto \bar{u} \) can be proven similarly. \( \Box \)

The conformal-block vectors generate submodules enumerated by the integers

\[
\hat{D}_\varsigma^{(s)} := \# \{ \text{walks } g \text{ over } \varsigma \mid \max \hat{\varrho} < p(q) \text{ and } \def_\varrho = s \} \leq \hat{D}_\varsigma^{(s)}.
\]

(2.86)

We show next that these submodules form direct-sum submodules inside \( u_{\varsigma, \bar{\varsigma}} \circ \bar{V}_\varsigma \) and \( \nabla_\varsigma \circ u_{\varsigma, \bar{\varsigma}} \). (See also proposition 4.13 in section 4 C for a refinement of this result.)

**Proposition 2.8.** Suppose \( \max \varsigma < p(q) \). There exists an embedding of left \( U_q \)-modules

\[
\bigoplus_{s \in \mathbb{E}_\varsigma, s < p(q)} \hat{D}_\varsigma^{(s)} M_{(s)} \hookrightarrow u_{\varsigma} \circ \bar{V}_\varsigma
\]

(2.87)

such that the following hold:

1. For each walk \( g \) over \( \varsigma \) with \( \max \hat{\varrho} < p(q) \) and \( \def_\varrho = s \), the collection

\[
\{ F^\ell u_\xi^\varrho \mid 0 \leq \ell \leq s \}
\]

(2.88)

is a basis for the image of a unique direct summand \( M_{(s)} \) in (2.87).
2. The image of each summand $M_{(s)}$ has a unique basis of the form (2.88) with $\max \hat{\rho} < p(q)$ and $\text{def}_q = s$.

3. If $n_c < p(q)$, then (2.87) is an isomorphism of left $U_q$-modules,

$$u_k \circ V_c \cong \bigoplus_{\ell \in E_c} D^{(\ell)}_\varsigma M_{(s)}.$$  \hfill (2.89)

Similarly, this proposition holds for right $U_q$-modules, after the symbolic replacements

$$M \mapsto \overline{M}, \quad u_k \circ V_c \mapsto \overline{V}_c \circ u_k, \quad \text{and} \quad F^\ell \cdot u_k \mapsto \overline{u_k} \cdot F^\ell.$$  \hfill (2.90)

Finally, both the left-action and right-action versions of this proposition hold after replacing $U_q \mapsto \overline{U}_q$ in either.

**Proof.** We first show that, for each walk $q$ over $\varsigma$ with $\max \hat{\rho} < p(q)$, the vector $u^e_k$ with $\text{def}_q = s$ generates a submodule of $u_k \circ V_c$ isomorphic to the simple type-one module $M_{(s)}$ with basis (2.88). By lemma 2.6, each $u^e_k$ with $\text{def}_q = s$ is a highest-weight vector with weight $\mathbb{N}$. The initial case with one summand is trivial. For the induction step, we recall that the

Now, to prove (2.87) and items 1 and 2, it remains to show that this sum is direct. We prove this by induction on the number of summands. The initial case with one summand is trivial. For the induction step, we recall that the intersection $N_1 \cap N_2$ of two submodules $N_1$ and $N_2$ is itself a submodule. Hence, if $N_1$ is simple, then we have

$$N_1 \cap N_2 \neq \{0\} \quad \Rightarrow \quad N_1 = N_1 \cap N_2 \quad \Rightarrow \quad N_1 \subset N_2.$$  \hfill (2.95)

Now we assume that the sum of the first $j - 1$ summands $N_1, N_2, \ldots, N_{j-1}$ in (2.94) is direct for some $j \geq 2$. If the sum of the next summand $N_j$ with the prior sum $N_1 \oplus N_2 \oplus \cdots \oplus N_{j-1}$ is not direct, then with $N_j$ simple, we have

$$N_j \cap (N_1 \oplus N_2 \oplus \cdots \oplus N_{j-1}) \neq \{0\} \quad \Rightarrow \quad N_j \subset N_1 \oplus N_2 \oplus \cdots \oplus N_{j-1}.$$  \hfill (2.96)

Hence, the highest-weight vector space of $N_j$, spanned by a highest-weight vector $u^e_k$ for some walk $q$ over $\varsigma$, must lie inside the highest-weight vector space of the direct sum $N_1 \oplus N_2 \oplus \cdots \oplus N_{j-1}$. However, the latter space is spanned by
a set of highest-weight vectors \( \{ u^g \} \) for other walks \( g \neq g' \) over \( \zeta \). Thus, the former vector \( u^g \) is a linear combination of these latter vectors, which contradicts lemma 2.7. Therefore, the sum of \( N_j \) with \( N_1 \oplus N_2 \oplus \cdots \oplus N_{j-1} \) is direct.

To prove item 3, we note that if \( n_\zeta < p(q) \), then we have \( s \leq s_{\text{max}}(\zeta) = n_\zeta < p(q) \) for any \( s \in E_\zeta \), and also, any walk \( g \) over \( \zeta \) satisfies \( \max g \leq n_\zeta < p(q) \). Therefore, we have \( D_\zeta = \overline{D}_\zeta \) and embedding (2.87) is an isomorphism of left \( U_q \)-modules, because the dimensions of both sides of (2.87) are now equal by item 3 of lemma 2.4. This proves (2.89).

The statements with replacements (2.90) or \( U_q \mapsto U_q \) can be proven similarly.

**Corollary 2.9.** Suppose \( n_\zeta < p(q) \). Then, the following hold:

1. The collection \( \{ u^g \} \) is a basis for \( H_\zeta \).
2. For each \( s \in E_\zeta \), the collection \( \{ u^g \mid \text{def}_v = s \} \) is a basis for \( H_\zeta \).

Similarly, items 1–2 hold after the symbolic replacements \( H \mapsto \overline{H} \) and \( u \mapsto \overline{u} \).

3. We have

\[
\dim H_\zeta = \dim H_\zeta = D_\zeta,
\]

and the direct-sum decompositions

\[
H_\zeta = \bigoplus_{s \in E_\zeta} H^{(s)}_\zeta \quad \text{and} \quad \overline{H}_\zeta = \bigoplus_{s \in E_\zeta} \overline{H}^{(s)}_\zeta.
\]

In particular, we have

\[
s \notin E_\zeta \quad \text{and} \quad n_\zeta < p(q) \implies \dim H^{(s)}_\zeta = \dim \overline{H}^{(s)}_\zeta = 0.
\]

**Proof.** Items 1–2 immediately follow from lemma 2.7 and proposition 2.8. For item 3, equalities (2.97) follow from items 1–2 combined with (2.60, 2.61), and (2.98) and (2.99) then follow from lemma 2.1 with a dimension count.

**C. Submodule projectors and embeddings**

Proposition 2.8 and recursion (2.66) applied to \( \zeta = \bar{n} \) as in (2.15) show that if \( n < p(q) \), then the type-one module \( u_\zeta \otimes V_n \) has a unique simple \((n+1)\)-dimensional submodule isomorphic to \( M_{(n)} \), generated by the highest-weight vector

\[
\theta_0^{(n)} := \xi_0 \otimes \xi_0 \otimes \cdots \otimes \xi_0 \quad \text{(2.100)}
\]

associated to the highest walk \( \varrho_n = (1, 2, \ldots, n) \) over \( \bar{n} \). This submodule has basis

\[
\theta_\ell^{(n)} := F^\ell \theta_0^{(n)} \quad \text{(2.101)}
\]

The corresponding basis for the submodule isomorphic to \( \overline{M}_{(n)} \) in \( \overline{V}_n \otimes u_\zeta \) is

\[
\overline{\theta}_0^{(n)} := \xi_0 \otimes \xi_0 \otimes \cdots \otimes \xi_0,
\]

\[
\overline{\theta}_\ell^{(n)} := \overline{\theta}_0^{(n)} \cdot E^\ell \quad \text{(2.102)}
\]

These basis vectors have explicit formulas:

**Lemma 2.10.** Suppose \( q \in \mathbb{C} \setminus \{ \pm 1 \} \). For all \( \ell \in \{0, 1, \ldots, n\} \), we have

\[
\theta_\ell^{(n)} = q^{\binom{n}{\ell} [\ell]!} \sum_{1 \leq r_1 < \cdots < r_\ell \leq n} q^{\sum_{i=1}^{\ell} (1-r_i)} (\xi_{k_1(\varrho)} \otimes \xi_{k_2(\varrho)} \otimes \cdots \otimes \xi_{k_n(\varrho)})
\]

and similarly,

\[
\overline{\theta}_\ell^{(n)} = q^{-\binom{n}{\ell} [\ell]!} \sum_{1 \leq r_1 < \cdots < r_\ell \leq n} q^{\sum_{i=1}^{\ell} (n-r_i)} (\xi_{k_1(\varrho)} \otimes \xi_{k_2(\varrho)} \otimes \cdots \otimes \xi_{k_n(\varrho)})
\]
where \( q := (r_1, r_2, \ldots, r_\ell) \), and for each \( i \in \{1, 2, \ldots, n\} \),

\[
k_i(q) = \begin{cases} 1, & \text{if } i \in \{r_1, r_2, \ldots, r_\ell\}, \\ 0, & \text{otherwise.} \end{cases}
\]

(2.106)

In particular, we have

\[
\theta^{(n)}_n = [n]! \varepsilon_1 \otimes \varepsilon_1 \otimes \cdots \otimes \varepsilon_1 \quad \text{and} \quad \bar{\theta}^{(n)}_n = [n]! \varepsilon_1 \otimes \varepsilon_1 \otimes \cdots \otimes \varepsilon_1.
\]

(2.107)

**Proof.** Identity (2.104) was proved in [Pel20, lemma B.1] and identity (2.105) can be proven similarly. The observation

\[
- \sum_{i=1}^n (1 - i) = \binom{n}{2} = \sum_{i=1}^n (n - i)
\]

proves identities (2.107) for the special case \( \ell = n \).

We let \( \mathcal{P}_n : V_n \rightarrow V_n \) and \( \mathcal{P}_n : \nabla_n \rightarrow \nabla_n \) denote the respective projections from \( U \), \( \nabla_n \otimes U_n \) onto their unique \((n + 1)\)-dimensional submodules respectively isomorphic to \( \mathrm{M}_n \) and \( \mathrm{M}_n \). Then, we have

\[
\mathcal{P}_n(v) = \begin{cases} v, & v \in \text{span} \{\theta^{(n)}_\ell \mid 0 \leq \ell \leq n\}, \\ 0, & \text{otherwise} \end{cases} \quad \text{for } n < p(q),
\]

(2.109)

\[
\mathcal{P}_n(\bar{v}) = \begin{cases} \bar{v}, & v \in \text{span} \{\bar{\theta}^{(n)}_\ell \mid 0 \leq \ell \leq n\}, \\ 0, & \text{otherwise} \end{cases} \quad \text{for } n < p(q).
\]

(2.110)

Also, using bases (2.101–2.103), we define the embeddings \( \mathcal{J}_n : \mathrm{M}_n \rightarrow \mathcal{V}_n \) and \( \mathcal{J}_n : \mathcal{M}_n \rightarrow \mathcal{V}_n \) by linearly extending

\[
\mathcal{J}_n(e^{(n)}_\ell) := \theta^{(n)}_\ell \quad \text{and} \quad \mathcal{J}_n(e^{(n)}_\ell) := \bar{\theta}^{(n)}_\ell \quad \text{for } \ell \in \{0, 1, \ldots, n\},
\]

(2.111)

and the projectors \( \mathcal{P}_n : V_n \rightarrow \mathrm{M}_n \) and \( \mathcal{P}_n : \nabla_n \rightarrow \mathcal{M}_n \) as inverses of these embeddings on their images:

\[
\mathcal{P}_n(v) := \begin{cases} e^{(n)}_\ell, & v = \theta^{(n)}_\ell \text{ for some } 0 \leq \ell \leq n, \\ 0, & v \notin \text{span} \{\theta^{(n)}_\ell \mid 0 \leq \ell \leq n\} \end{cases} \quad \text{for } n < p(q),
\]

(2.112)

\[
\mathcal{P}_n(v) := \begin{cases} \bar{e}^{(n)}_\ell, & v = \bar{\theta}^{(n)}_\ell \text{ for some } 0 \leq \ell \leq n, \\ 0, & v \notin \text{span} \{\bar{\theta}^{(n)}_\ell \mid 0 \leq \ell \leq n\} \end{cases} \quad \text{for } n < p(q).
\]

(2.113)

More generally, assuming \( \max \zeta < p(q) \), we define the composite embeddings and composite projectors

\[
\mathcal{J}_\zeta := \mathcal{J}_{(s_1)} \otimes \mathcal{J}_{(s_2)} \otimes \cdots \otimes \mathcal{J}_{(s_{d\zeta})}, \quad \mathcal{P}_\zeta := \mathcal{P}_{(s_1)} \otimes \mathcal{P}_{(s_2)} \otimes \cdots \otimes \mathcal{P}_{(s_{d\zeta})}, \quad \mathcal{P}_\zeta := \mathcal{P}_{(s_1)} \otimes \mathcal{P}_{(s_2)} \otimes \cdots \otimes \mathcal{P}_{(s_{d\zeta})},
\]

(2.114)

\[
\mathcal{J}_\zeta := \mathcal{J}_{(s_1)} \otimes \mathcal{J}_{(s_2)} \otimes \cdots \otimes \mathcal{J}_{(s_{d\zeta})}, \quad \mathcal{P}_\zeta := \mathcal{P}_{(s_1)} \otimes \mathcal{P}_{(s_2)} \otimes \cdots \otimes \mathcal{P}_{(s_{d\zeta})}, \quad \mathcal{P}_\zeta := \mathcal{P}_{(s_1)} \otimes \mathcal{P}_{(s_2)} \otimes \cdots \otimes \mathcal{P}_{(s_{d\zeta})}.
\]

(2.115)

In lemma A.2 in appendix A, we gather useful properties of these \( U_q \), \( \bar{U}_q \)-homomorphisms. For instance, the following diagram commutes:

\[
\xymatrix{ V_{n_\zeta} = V_{s_1} \otimes V_{s_2} \otimes \cdots \otimes V_{s_{d\zeta}} \ar[rr]^{\mathcal{J}_\zeta} \ar[dr]_{\mathcal{P}_\zeta} & & \text{im } \mathcal{J}_\zeta = \text{im } \mathcal{P}_\zeta \subset V_{n_\zeta} \ar[dl]_{\mathcal{P}_\zeta} \\
V_\zeta = V_{(s_1)} \otimes V_{(s_2)} \otimes \cdots \otimes V_{(s_{d\zeta})} & & }
\]

(2.116)

In section 3, we give diagram interpretations for the mappings \( \mathcal{J}_\zeta \), \( \mathcal{P}_\zeta \), and \( \mathcal{P}_\zeta \) (lemma 3.11 and corollaries 3.12 and 3.15). Also, in section 5B, we discuss projections onto other submodules in \( U \otimes V_n \) and \( U \otimes V_\zeta \) (lemma 5.8 and proposition 5.9).

A special case of decomposition (2.89) and recursion (2.66) for the multiindex \( \zeta = (r, t) \) together show that

\[
r + t < p(q) \implies U_v \otimes V_{(r,t)} \cong \bigoplus_{s \in E(r,t)} M_{(s)}.
\]

(2.117)
where $E_{(r,t)}$ is the set (2.56). Because no two summands in decomposition (2.117) are isomorphic, for each index $s \in E_{(r,t)}$, we may define the embeddings $\iota^{(s)}_s : V_s \rightarrow V_{(r,t)}$ and $\iota^{(s)}_{\pi(s)} : \bar{V}_s \rightarrow \bar{V}_{(r,t)}$ by linear extension of the rules

\[
\iota^{(s)}_s (e^{(s)}_\ell) := F^{\ell} u^{(s)}_\ell, \\
\iota^{(s)}_{\pi(s)} (\pi^{(s)}_\ell) := \pi^{(s)} E^{\ell}, \\
\ell \in \{0, 1, \ldots, s\},
\]

the projectors $\pi^{(r,t)} : V_{(r,t)} \rightarrow V_{(r,t)}$ and $\pi^{(r,t)} : \bar{V}_{(r,t)} \rightarrow \bar{V}_{(r,t)}$ by linear extension of the rules

\[
\pi^{(r,t)} (F^{\ell} u^{(p)}_{(r,t)}) := \delta_{p,s} F^{\ell} u^{(s)}_s, \\
\pi^{(r,t)} (\pi^{(p)} E^{\ell}) := \delta_{p,s} \pi^{(s)} E^{\ell}, \\
\ell \in \{0, 1, \ldots, s\},
\]

and the maps $\pi^{(r,t)} : V_{(r,t)} \rightarrow V_{(r,t)}$ and $\pi^{(r,t)} : \bar{V}_{(r,t)} \rightarrow \bar{V}_{(r,t)}$ by linear extensions of the rules

\[
\pi^{(r,t)} (F^{\ell} u^{(p)}_{(r,t)}) := \delta_{p,s} e^{(s)}_s, \\
\pi^{(r,t)} (\pi^{(p)} E^{\ell}) := \delta_{p,s} e^{(s)}_s, \\
\ell \in \{0, 1, \ldots, s\}.
\]

In lemma A.3 in appendix A, we gather useful properties of these $U_q, \bar{U}_q$-homomorphisms. For instance, the following diagram commutes:

\[
\begin{array}{ccc}
V_{(r,t)} & \xrightarrow{\pi^{(r,t)}} & V_{(r,t)} \\
\downarrow{\iota^{(s)}_s} & & \downarrow{\pi^{(s)}_s} \\
V_s & \xrightarrow{\iota^{(s)}_{\pi(s)}} & \text{im } \iota^{(s)}_s = \text{im } \pi^{(s)}_s \subset V_{(r,t)}
\end{array}
\]

In proposition 5.9 in section 5B, we give diagram interpretations for these mappings.

### D. Bilinear pairing of type-one modules

Now, we assign a (canonical) bilinear pairing $(\cdot | \cdot)$ to the vector spaces $\bar{V}_\varsigma$ and $V_\varsigma$. The standard tensor product basis vectors are orthogonal with respect to this pairing, and the pairing is invariant under the $U_q$ and $\bar{U}_q$-actions. Furthermore, the conformal-block basis vectors $\bar{u}^{(s)}_\varsigma$ and $u^{(s)}_\varsigma$ are also orthogonal with respect to $(\cdot | \cdot)$, as we prove in lemma 5.5 in section 5A. Later, in section 3D, we provide a diagram interpretation [KL94, FK97] for $(\cdot | \cdot)$, see lemma 3.19.

**Lemma 2.11.** Suppose $\varsigma < p(q)$, and let $(\cdot | \cdot) : \bar{V}_\varsigma \times V_\varsigma \rightarrow \mathbb{C}$ be a bilinear pairing such that

\[
(e^{(s_1)}_0 \otimes e^{(s_2)}_0 \otimes \cdots \otimes e^{(s_k)}_0) (e^{(s_1)}_0 \otimes e^{(s_2)}_0 \otimes \cdots \otimes e^{(s_k)}_0) = 1.
\]

Then, the following statements are equivalent:

1. For all elements $x \in U_q^{\otimes d}$ and $\bar{x} \in \bar{U}_q^{\otimes d}$ and vectors $\bar{v} \in V_\varsigma$ and $w \in V_\varsigma$, we have

\[
(\bar{v} | x.w) = (\bar{v}, x | w) \\
(\bar{v} | \bar{x}.w) = (\bar{v}, \bar{x} | w).
\]

2. We have

\[
(e^{(s_1)}_{\ell_1} \otimes e^{(s_2)}_{\ell_2} \otimes \cdots \otimes e^{(s_k)}_{\ell_k}) (e^{(s_1)}_{\ell_1} \otimes e^{(s_2)}_{\ell_2} \otimes \cdots \otimes e^{(s_k)}_{\ell_k}) = \prod_{k=1}^{d_\varsigma} \delta_{\ell_k, m_k} | \ell_k |^2 | s_k \ell_k |.
\]

**Proof.** Without loss of generality, we consider the case of $d_\varsigma = 1$, when $\varsigma = (t)$ for some $t \in \mathbb{Z}_{\geq 0}$. 
Lemma 2.13. Formulas (2.122, 2.124) in lemma 2.11 uniquely define a 

1 \Rightarrow 2: Assuming that item 1 holds, we derive assertion (2.124) of item 2:

\[
\left( e^{(t)}_\ell | e^{(t)}_m \right) \overset{(2.6)}{=} \left( e^{(0)}_0 | E^\ell | F^m | e^{(0)}_t \right) \overset{(2.123)}{=} \left\{ \begin{array}{ll}
\left( e^{(t)}_0 | E^\ell | F^m | e^{(0)}_t \right) & , \ell \geq m \\
\left( e^{(t)}_0 | E^\ell | F^m | e^{(0)}_t \right) & , \ell \leq m
\end{array} \right.
\]

\[
\overset{(2.6)}{=} \delta_{\ell,m} [\ell]^2 \left[ \frac{t}{\ell} \right] \left( e^{(0)}_0 | e^{(t)}_0 \right) \overset{(2.122)}{=} \delta_{\ell,m} [\ell]^2 \left[ \frac{t}{\ell} \right].
\] (2.25)

2 \Rightarrow 1: To prove the left equality of (2.23), without loss of generality, we assume that \( x \in \{ E, F, K^{\pm 1} \} \), and \( \varpi = e^{(t)}_\ell \) and \( w = e^{(m)}_m \). Then, assuming that item 2 holds, we calculate, e.g., for \( x = E \):

\[
\left( e^{(t)}_\ell | E | e^{(t)}_m \right) \overset{(2.6)}{=} \left[ m \right] [t + 1 - m] \left( e^{(t)}_\ell | e^{(m-1)}_t \right) \overset{(1.1)}{=} \left( e^{(t)}_{\ell+1} | e^{(t)}_m \right) \overset{(2.6)}{=} \left( e^{(t)}_\ell | E | e^{(t)}_m \right),
\] (2.26)

with the convention that \( e^{(t)}_{-1} = 0 \) and \( e^{(t)}_{t+1} = 0 \). The other cases \( x \in \{ F, K^{\pm 1} \} \) and the right equality of (2.23) can be proven similarly. This shows that item 2 implies item 1.

\[\square\]

**Remark 2.12.** Formulas (2.122, 2.124) in lemma 2.11 uniquely define a \( U_q \)-invariant bilinear pairing \( \cdot | \cdot : \overline{\mathcal{V}}_t \times \mathcal{V}_t \rightarrow \mathbb{C} \). See also [Kas95, theorem VII.6.2.].

In appendix A, we consider a bilinear form on \( \mathcal{V}_t \) obtained by taking \( q \mapsto q^{-1} \) instead of \( \overline{\mathcal{V}}_t \) in the first component.

**Lemma 2.13.** Suppose \( \max \varsigma < p(q) \). The following hold:

1. For all vectors \( \varpi_j \in \overline{\mathcal{V}}_{(s,j)} \) and \( w_j \in \mathcal{V}_{(s,j)} \), with \( j \in \{ 1, 2, \ldots, d_\varsigma \} \), we have the factorization

\[
\left( \varpi_1 \otimes \varpi_2 \otimes \cdots \otimes \varpi_{d_\varsigma} | w_1 \otimes w_2 \otimes \cdots \otimes w_{d_\varsigma} \right) = \prod_{k=1}^{d_\varsigma} \left( \varpi_k | w_k \right).
\] (2.27)

2. For all elements \( x \in U_q \) and \( \bar{x} \in \overline{U}_q \) and vectors \( \varpi \in \mathcal{V}_t \) and \( w \in \mathcal{V}_s \), we have

\[
(\varpi | x \cdot w) = (\varpi | \bar{x} \cdot w) \quad \text{and} \quad (\varpi | \bar{x} \cdot w) = (\varpi | \bar{x} \cdot w).
\] (2.28)

3. The subspaces \( \overline{\mathcal{V}}_{(s,j)} \) and \( \mathcal{V}_{(s,j)} \) are orthogonal:

\[
(\varpi | w) = 0 \quad \text{for all} \quad \varpi \in \overline{\mathcal{V}}_{(s,j)} \text{ and } w \in \mathcal{V}_{(s,j)} \text{ with } s \neq t.
\] (2.29)

Also, for all highest-weight vectors \( \varpi \in \overline{\mathcal{R}}_{(s)} \) and \( w \in \mathcal{H}_{(s)} \), we have

\[
(\varpi \cdot E^\ell | F^m \cdot w) = \delta_{s,t} \delta_{\ell,m} [\ell]^2 \left[ \frac{t}{\ell} \right] (\varpi | w) = (\varpi \cdot E^\ell | F^m \cdot w).
\] (2.30)

4. For all vectors \( \varpi \in \mathcal{V}_t \) and \( w \in \mathcal{V}_s \), we have

\[
(\varpi | w) = (\mathfrak{J}_\varsigma(\varpi) | \mathfrak{J}_\varsigma(w)).
\] (2.31)

**Proof.** Using lemma 2.11, we prove items 1–4 as follows:

1. Factorization (2.27) immediately follows from the formula (2.124) for the bilinear pairing.

2. Identities (2.28) readily follow from (2.23) and the definitions of the \( U_q, \overline{U}_q \)-actions on \( \mathcal{V}_t \) and \( \overline{\mathcal{V}}_t \).

3. Orthogonality (2.29) immediately follows from definitions (cf. (2.28)) with (2.124) from item 2 of lemma 2.11. To verify (2.30), we note that the orthogonality of the spaces \( \overline{\mathcal{R}}_{(s)} \) and \( \mathcal{H}_{(s)} \) together with a similar calculation as in (2.25) readily imply the first equality in (2.30), and the second equality follows similarly.
Using definitions (2.72, 2.73) and lemmas 2.11 and 2.13, we have

$$
(\mathcal{C}_k e^{(l)}_m | C_k e^{(l)}_m) = (\theta^{(l)}_m | \theta^{(l)}_m) (2.111) \delta_{\ell,m} [\ell]!^2 \left( \begin{array}{c} t \\ \ell \end{array} \right) = (\epsilon^{(l)}_m | e^{(l)}_m). \tag{2.132}
$$

Asserted identity (2.131) follows from this by linearity and the factorization property (2.127) in item 1 together with the analogous factorization property for the embedding maps \( \mathcal{C}_k \) and \( \mathcal{C}_l \) in (2.114–2.115).

This concludes the proof.

In the next identity, we encounter the evaluation of the \emph{Theta network} [KL94] from [FP18a, lemma A.7]:

$$
\Theta(r,s,t) = \frac{(-1)^{r+s+t+1} [r+s+t+1]! [r+s-1]! [s+t-1]! [t+r-1]!}{[r]! [s]! [t]! [r+s+t]}. \tag{2.133}
$$

**Lemma 2.14.** Suppose \( \max(r,t) < p(q) \). We have

$$
(\pi^{(s)}_{(r,s)} | u^{(s)}_{(r,s)}) = \delta_{s,s'} \frac{\Theta(r,s,t)}{(q - q^{-1})^{r+s+t} [r+s+t]!^2 s! [s+1]}.
$$

**Proof.** Using definitions (2.72, 2.73) and lemmas 2.11 and 2.13, we have

$$
(\pi^{(s)}_{(r,s)} | u^{(s)}_{(r,s)}) = \delta_{s,s'} \frac{(-1)^{r+s+t} q^{-(2r+s+t-1)(t+r-s)/4}}{(q - q^{-1})^{r+s+t} [r+s+t]!^2 [r]! [t]!}
$$

$$
\times \sum_{j=0}^{r+s+t} q^{j(2s+1)} \left( r + t - s \right) j \left( r - \frac{t + s - 2}{2} \right) [t - j]!. \tag{2.135}
$$

By [Pel20, Lemma A.1, item (d)] (with \( \nu_1 = r, \nu_2 = t, \) and \( n = \frac{r+t-s}{2} \) in that lemma), we can explicitly evaluate this sum. Then, using formula (2.133) for the Theta network, we arrive with (2.134).

### 3. TEMPERLEY-LIEB ACTION ON TYPE-ONE \( U_q \)-MODULES

In this section, we define an action of the valenced Temperley-Lieb algebra \( TL_\nu \) on the tensor products \( V_\nu \) and \( \tilde{V}_\nu \). For this purpose, in section 3A we recall definitions and notation from [FP18a] to be used throughout. Then, in sections 3B–3C we define the Temperley-Lieb-actions on \( TL \circ V_\nu \) and \( \tilde{V}_\nu \circ TL \), commuting with the \( U_q \)-action on \( V_\nu \circ V_\nu \) and \( \tilde{V}_\nu \circ \tilde{V}_\nu \). In section 3D, we give a diagram representation for vectors in \( V_n \) and \( \tilde{V}_n \) analogous to the one developed by I. Frenkel and M. Khovanov [FK97] (however, our conventions and purposes are somewhat different). Although such graphical ideas are relatively well known [KL94, CFS95], we present them in detail for the sake of exposition (and because the conventions vary greatly in the literature). We also define a natural invariant bilinear pairing on valenced state diagrams, and relate it to the bilinear pairing discussed in section 2D.

**A. Valenced tangles and link states**

Here, we recall definitions and notation from [FP18a], to be used throughout this article.

\textbf{Temperley-Lieb category.} To begin, we consider planar non-crossing tangles in the Temperley-Lieb category \( TL_1(\nu) \) [KL94, Tur94, Kas95, GL98]. Its object class comprises the special multiindices with all entries equal to one:

$$
\text{Ob } TL_1(\nu) = \left\{ \vec{n} \mid n \in \mathbb{Z}_{\geq 0} \right\}, \quad \text{where } \vec{0} := (0), \quad \vec{n} := (1, 1, \ldots, 1) \text{ for } n \in \mathbb{Z}_{>0}. \tag{3.1}
$$

The morphisms in \( TL_1(\nu) \) are \( (n,m) \)-\emph{tangles}, that is, formal linear combinations \( T \in TL_n^m \) of \( (n,m) \)-\emph{link diagrams}

\[
\text{crossing link } \quad \quad \quad \quad \quad \quad \quad \quad \text{turn-back link }
\]
which consist of two vertical lines with \( n \) (resp. \( m \)) nodes anchored to the left (resp. right) line, and a collection of links between the lines, connecting the nodes pairwise, and specified up to isotopy. The source and target associated with a tangle \( T \in \text{TL}_n^m \) are the objects \( \vec{m} \) and \( \vec{n} \) respectively. In summary, we have

\[
\text{Hom } \text{TL}^1(\nu) = \{ \text{TL}_n^m \mid n, m \in \mathbb{Z}_{\geq 0} \text{ with } n + m = 0 \pmod{2} \}.
\] (3.3)

The composition of two morphisms \( T, U \in \text{Hom } \text{TL}^1(\nu) \) is given by diagram concatenation, where each loop formed by the concatenation is replaced by a factor of the loop fugacity \( \nu \in \mathbb{C} \):

\[
\begin{array}{c}
\includegraphics[width=2cm]{diagram1.png} \\
\end{array} := \nu \times \begin{array}{c}
\includegraphics[width=2cm]{diagram2.png} \\
\end{array}.
\] (3.4)

When \( m = n \), we omit the superscript for \( \text{TL}_n = \text{TL}_n^n \). This space is an associative unital algebra [RSA14, theorem 2.4], the “Temperley-Lieb algebra” \( \text{TL}_n(\nu) \), generated by tangles (3.13, 3.14) given below, and satisfying relations (1.25).

For later use, we determine a minimal collection of generators for the morphism class \( \text{Hom } \text{TL}^1(\nu) \). These constitute the left and right generators (also known as the “evaluation” and “coevaluation” maps), defined as

\[
L_i := \begin{array}{c}
\includegraphics[width=2cm]{diagram3.png} \\
i \end{array} \in \text{TL}^{n-2}_n, \quad R_j := \begin{array}{c}
\includegraphics[width=2cm]{diagram4.png} \\
j \end{array} \in \text{TL}^{n-2}_n (3.5)
\]

for all integers \( n \geq 2 \) and \( i, j \in \{1, 2, \ldots, n - 1\} \). Now, if \( T \) is an arbitrary \((n, m)\)-link diagram with \( s \) crossing links, we can construct \( T \) by an insertion of all \( \ell_L := \frac{n-s}{2} \) left links of \( T \) into the unit diagram \( \mathbbm{1}_{\text{TL}_n} \) by repeated application of the left generators \( L_i \), followed by an insertion of all \( \ell_R := \frac{m-s}{2} \) right links of \( T \) by repeated application of the right generators \( R_j \), that is,

\[
T = L_i_{i_t} L_i_{i_L-1} \cdots L_i_{i_L} \mathbbm{1}_{\text{TL}_n} R_j_{j_L} R_j_{j_R} \cdots R_{j_{j_R}-1} R_{j_{j_R}}.
\] (3.6)

For example,

\[
\begin{array}{c}
\includegraphics[width=8cm]{diagram5.png} \\
\end{array}
\]

for \( i_t = i_L = 1, j_L = j_R = 2 \). This gives the tangle

\[
\begin{array}{c}
\includegraphics[width=2cm]{diagram6.png} \\
\end{array} \in \text{TL}_8.
\] (3.8)
As shown, we include the unit in the middle of the product to emphasize that $L_i$ is an $(s + 2, s)$-link diagram and $R_j$ is an $(s, s + 2)$-link diagram, in spite of the following obvious relations which imply that the unit can be dropped:

$$L_i 1_{TL} = L_i \quad \text{and} \quad 1_{TL} R_j = R_j. \quad (3.9)$$

In (3.6), we order the left generators $L_i$ such that if the upper endpoint of one left link of $T$ is above the the upper endpoint of another left link, then the former is inserted before the latter, and similarly for the $R_i$. This implies that

$$i_1 < i_2 < \ldots < i_\ell \quad \text{and} \quad j_1 < j_2 < \ldots < j_\ell. \quad (3.10)$$

We say that any product of left and right generators of the form in (3.6, 3.10) is in standard form.

**Lemma 3.1.** Each $(n, m)$-link diagram equals a unique product of left and right generators in standard form (3.6, 3.10), and every such product equals a unique $(n, m)$-link diagram.

**Proof.** This is immediate from (3.6, 3.9) and the chosen ordering (3.10).

**Lemma 3.2.** The following is a complete list of independent relations satisfied by the left and right generators:

$$R_j L_i = \begin{cases} 1_{TL}, & i = j \pm 1, \\ \nu 1_{TL}, & i = j, \\ L_i R_{j-2}, & i \leq j - 2, \\ L_{i-2} R_j, & j \leq i - 2, \\ L_i & L_i = L_{i+2} L_j, & j \leq i, \\ R_j & R_{i-2} = R_i R_j, & j \leq i, \end{cases} \quad (3.11)$$

where $s$ is the number of crossing links in $L_i$ and $R_j$.

**Proof.** Each relation (3.11) is easy to verify with a diagram. Also, relations (3.11) allow to write any word formed from the right and left generators in standard form. To see that (3.11) are all of the independent relations, we let

$$\sum_{k_1, k_2, \ldots, k_i} c_{k_1, k_2, \ldots, k_i} T_{k_1} T_{k_2} \cdots T_{k_i} = 0, \quad \text{with} \quad c_{k_1, k_2, \ldots, k_i} \in \mathbb{C} \quad \text{and} \quad T_{k_p} \in \{L_i, R_j \mid i, j \in \mathbb{Z}_{>0}\}, \quad k_p \in \mathbb{Z}_{>0}, \quad (3.12)$$

be a relation where all terms $T_{k_1} T_{k_2} \cdots T_{k_i}$ are in standard form. Because the link diagrams are linearly independent, lemma 3.1 implies that all of the coefficients $c_{k_1, k_2, \ldots, k_i}$ must vanish, so relation (3.12) is trivial.

The Temperley-Lieb algebra $TL_n(\nu)$ is the associative unital algebra with generating set $\{1_{TL_n}, U_1, U_2, \ldots, U_{n-1}\}$ subject to relations (1.25) [TL71, Jon83]. These generators have the following diagrammatic form:

$$L_i R_i = \begin{array}{c|c|c} 1 & 2 & \vdots \\ \hline i - 1 & i & i \\ \hline i + 1 & i & i \\ \hline i + 2 & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots \\ \hline n & n & n \end{array}$$

$$R_{i-1} L_i = \begin{array}{c|c|c} 1 & 2 & \vdots \\ \hline i - 2 & i & i \\ \hline i + 1 & i & i \\ \hline \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots \\ \hline n & n & n \end{array}$$

for all $i \in \{1, 2, \ldots, n - 1\}$, and the unit is

$$1_{TL_n} = \begin{array}{c|c|c} 1 & 2 & \vdots \\ \hline i - 1 & i & i \\ \hline i + 1 & i & i \\ \hline \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots \\ \hline n & n & n \end{array}$$
Relations (1.25) of the Temperley-Lieb algebra $\mathrm{TL}_n(\nu)$ follow from (3.11, 3.13) with the diagram concatenation rules.

**Link states.** Standard modules consisting of link states are building blocks for representations of the Temperley-Lieb algebra $\mathrm{TL}_n(\nu)$. Given an $n$-link diagram with $s$ crossing links, we create an $(n,s)$-link pattern by dividing the link diagram vertically in half, discarding the right half, and rotating the left half by $\pi/2$ radians:

![Diagram](image1)

We call the broken links in the $(n,s)$-link pattern defects. We denote the set of $(n,s)$-link patterns by $\mathrm{LP}_n^{(s)}$, and let

$$\mathrm{LP}_n := \bigcup_{s \in \mathbb{E}_n} \mathrm{LP}_n^{(s)},$$

where $\mathbb{E}_n = \{n \text{ mod } 2, (n \text{ mod } 2) + 2, \ldots, n\}$,

denote the set of $n$-link patterns with any number of crossing links. We call a formal linear combination of $(n,s)$-link patterns with complex coefficients an $(n,s)$-link state, and let $L_n^{(s)}$ denote the complex vector space of $(n,s)$-link states.

We endow the space $L_n^{(s)}$ with a $\mathrm{TL}_n(\nu)$-action via the following diagram concatenation recipe. Given an $n$-link diagram $T$ and an $(n,s)$-link pattern $\alpha \in \mathrm{LP}_n^{(s)}$, the latter rotated $-\pi/2$ radians, we concatenate $T$ to the left of $\alpha$, remove the $k \geq 0$ loops formed by this concatenation, and multiply the result by $\nu^k$:

$$T \triangleright \alpha = \nu \times \alpha.$$  

Importantly, we set diagrams containing turn-back paths to zero, so $\mathrm{TL}_n(\nu)$ preserves the number $s$ of defects:

$$T \triangleright \alpha = 0.$$  

Bilinear extension of this recipe defines a $\mathrm{TL}_n(\nu)$-module structure on $L_n^{(s)}$, and we thus call $L_n^{(s)}$ a $\mathrm{TL}_n(\nu)$-standard module. We also define the $\mathrm{TL}_n(\nu)$-link state module to be the direct sum of all of the standard modules,

$$L_n := \bigoplus_{s \in \mathbb{E}_n} L_n^{(s)}.$$  

The algebra $\mathrm{TL}_n(\nu)$ is semisimple if and only if the parameter $q \in \mathbb{C}^\times$ that determines the fugacity $\nu$ via (1.24) satisfies

$$\text{either } q \in \{\pm 1\}, \quad \text{or } n \leq p(q), \quad \text{or } q \in \{\pm i\} \text{ if } n \text{ is odd},$$

see, e.g., [RSA14, theorem 8.1] and [FP18a, corollary 6.10]. In this case, the collection $\{L_n^{(s)} \mid s \in \mathbb{E}_n\}$ is the complete set of non-isomorphic simple $\mathrm{TL}_n(\nu)$-modules. If $\mathrm{TL}_n(\nu)$ is not semisimple, some of its standard modules $L_n^{(s)}$ are not simple (but still indecomposable), but instead, certain quotients $Q^{(s)}_c$ of the standard modules are simple, see (3.116).

**Valenced tangles and link patterns.** Next, for two multiindices as in (1.34, 1.35), we consider the set of $(\varsigma,\varpi)$-valenced link diagrams,
which consist of two vertical lines with $d_{\varsigma}$ (resp. $d_{\widehat{\varsigma}}$) valenced nodes

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture}
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\node at (0.5,1.5) {$s$};
\node at (0.5,0.5) {$s$};
\end{tikzpicture}
\end{array}
\] (3.22)

anchored to the left (resp. right) line and a collection of links between the lines, connecting the nodes in such a way that, first, for each $i \in \{1, 2, \ldots, d_{\varsigma}\}$ (resp. $j \in \{1, 2, \ldots, d_{\widehat{\varsigma}}\}$), exactly $s_i$ (resp. $p_j$) endpoints reside at the $i$:th left (resp. $j$:th right) node counted from the top, and second, the two endpoints of each link are distinct (that is, we exclude “loop-links” that start and end at the same node; see [FP18a, section 2] for details). Each link is specified up to isotopy. We let $TL_{\varsigma}^{n}$ denote the complex vector space of $(\varsigma, \overline{\varsigma})$-valenced tangles, that is, formal linear combinations of $(\varsigma, \overline{\varsigma})$-valenced link diagrams. In the special case when $\varsigma = \overline{\varsigma}$ and $\varsigma = \overline{\varsigma}$, we omit the arrow in superscripts and subscripts, writing $TL_{\varsigma}^{n}$. Also, when the superscript and subscript are equal, we write $TL_{\varsigma} = TL_{\varsigma}^{\varsigma}$.

For a multiindex $\varsigma$ as in (1.34, 1.35) and an integer $s$, we denote by $LP_{\varsigma}^{(s)}$ the set of $(\varsigma, s)$-valenced link patterns,

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (2,0);
\draw (1,0) -- (1,1);
\draw (2,0) -- (3,0);
\draw (2,0) -- (2,1);
\node at (0.5,1.5) {$s$};
\node at (2.5,1.5) {$s$};
\end{tikzpicture}
\end{array}
\] (3.23)

consisting of $\frac{1}{2}(n_{\varsigma} - s)$ links and $s$ defects (specified up to isotopy) attached to $d_{\varsigma}$ valenced nodes on and above a horizontal line, in such a way that, first, for each $i \in \{1, 2, \ldots, d_{\varsigma}\}$, the $i$:th node counted from the left has exactly $s_i$ strands attached to it, second, the two endpoints of each link are distinct, and third, each defect has one endpoint in the unbounded component of the complement of the links and the horizontal line and another endpoint among the $d_{\varsigma}$ nodes. We let $L_{\varsigma}^{n} := \text{span} LP_{\varsigma}^{(s)}$ denote the complex vector space of $(\varsigma, s)$-valenced link states. We remark that by [FP18a, lemma 4.1], $L_{\varsigma}^{n}$ is nonempty (and thus $L_{\varsigma}^{n}$ is nontrivial) if and only if $s \in E_{\varsigma}$. We also write

\[
\begin{array}{c}
LP_{\varsigma} := \bigcup_{s \in E_{\varsigma}} LP_{\varsigma}^{(s)} \\
L_{\varsigma} := \bigoplus_{s \in E_{\varsigma}} L_{\varsigma}^{(s)}
\end{array}
\] (3.24)

respectively for the collections of $\varsigma$-valenced link patterns and $\varsigma$-valenced link states. When $\varsigma = \overline{\varsigma}$, we omit the arrow in superscripts and subscripts: $L_{\varsigma}^{n}$ and $LP_{\varsigma}^{n}$ stand for the spaces of $(n, s)$-link states and $(n, s)$-link patterns.

We also define an analogous space $\Gamma_{\varsigma}^{(s)}$ of valenced link states, spanned by $(\varsigma, s)$-valenced link patterns $\overline{\sigma} \in CP_{\varsigma}^{(s)}$,

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (2,0);
\draw (1,0) -- (1,1);
\draw (2,0) -- (3,0);
\draw (2,0) -- (2,1);
\node at (0.5,1.5) {$s$};
\node at (2.5,1.5) {$s$};
\end{tikzpicture}
\end{array}
\] (3.25)

We note that space $\Gamma_{\varsigma}^{(s)}$ is naturally isomorphic with $L_{\varsigma}^{(s)}$, via the linear extension of the following reflection map: we let $\alpha^{*} \in CP_{\varsigma}^{(s)}$ denote the reflection of $\alpha \in LP_{\varsigma}^{(s)}$ about a horizontal axis:

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (2,0);
\draw (1,0) -- (1,1);
\draw (2,0) -- (3,0);
\draw (2,0) -- (2,1);
\node at (0.5,1.5) {$s$};
\node at (2.5,1.5) {$s$};
\end{tikzpicture}
\end{array}
\end{array}
\] (3.26)

and we similarly define $\overline{\sigma^{*}} \in LP_{\varsigma}^{(s)}$ for $\overline{\sigma} \in CP_{\varsigma}^{(s)}$. We also define the union $CP_{\varsigma}$ and direct sum $\bigoplus CP_{\varsigma}$ similarly to (3.24).

**Projectors.** Recalling definition (2.1) of $p(q)$, we define the Jones-Wenzl projector of size $s \in \{0, 1, \ldots, p(q) - 1\}$ to be the nonzero tangle $P_{(s)} \in TL_{\varsigma}$ uniquely determined by the following two properties [Jon83, Wen87, KL94]:

**P1.** $P_{(s)}^{2} = P_{(s)}$, and

**P2.** $\nu U_{i} P_{(s)} = P_{(s)} U_{i} = 0$ for all $i \in \{1, 2, \ldots, s - 1\}$.

For example, we have

\[
P_{(0)} = \text{the empty tangle}, \quad P_{(1)} = 1_{TL_{1}}, \quad \text{and} \quad P_{(2)} = 1_{TL_{2}} - \nu^{-1} U_{1}.
\] (3.27)

We represent the tangle $P_{(s)}$ as the empty projector box

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\draw (1,0) -- (2,0);
\draw (1,0) -- (1,1);
\draw (2,0) -- (3,0);
\draw (2,0) -- (2,1);
\node at (0.5,1.5) {$s$};
\node at (2.5,1.5) {$s$};
\end{tikzpicture}
\end{array}
\] (3.28)
We can insert this projector box within any larger link diagram, and abusing notation, we let the symbol $P_s \in \mathcal{T}_n$ also denote this tangle in $\mathcal{T}_n$ with $i = 1$. Then the various projectors $P_1 = 1_{\mathcal{T}_n}$, $P_2, \ldots, P_n \in \mathcal{T}_n$ satisfy the following recursive relation [Wen87, Jon83, KL94]:

$$P_s = 1_{\mathcal{T}_n}, \quad P_{s+1} = P_s + \left( \frac{[s]}{[s+1]} \right) P_s U_s P_s \quad \text{for all } s \in \{1, 2, \ldots, n-1\}. \quad (3.29)$$

In terms of diagrams, recursion relation (3.29) reads

$$s \quad \begin{array}{c} \vdots \end{array} \quad 1 = s \quad \begin{array}{c} \vdots \end{array} \quad + \quad \frac{[s]}{[s+1]} \times \quad \begin{array}{c} \vdots \end{array} \quad 1 \quad \begin{array}{c} \vdots \end{array} \quad . \quad (3.30)$$

With the graphical representation, we can write properties $P_1$ and $P_2$ in the following form (where property $(P1')$ below is slightly stronger than property $P_1$ above)

$$\begin{array}{c} \vdots \end{array} = \begin{array}{c} \vdots \end{array} = \begin{array}{c} \vdots \end{array} = \begin{array}{c} \vdots \end{array} = \begin{array}{c} \vdots \end{array} = 0. \quad (P1')$$

and [FP18b, appendix A]. Also, if $\max \varsigma < p(q)$, then we define the Jones-Wenzl composite projector

$$P_\varsigma := \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_{d-1} \\ s_d \end{array}$$

and the Jones-Wenzl composite embedder and its reflection

$$I_\varsigma := \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_{d-1} \\ s_d \end{array} \quad \text{and} \quad \hat{P}_\varsigma := \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_{d-1} \\ s_d \end{array} \quad (3.32)$$

These valenced tangles have the properties

$$\hat{P}_\varsigma I_\varsigma = 1_{\mathcal{T}_\varsigma}, \quad I_\varsigma \hat{P}_\varsigma = P_\varsigma, \quad P_\varsigma \hat{P}_\varsigma = P_\varsigma \quad \text{(P1)} \quad \hat{P}_\varsigma, \quad \text{and} \quad P_\varsigma I_\varsigma = I_\varsigma. \quad (3.33)$$

**Embeddings.** We identify any $(\varsigma, \omega)$-valenced tangle $T \in \mathcal{T}_\varsigma^{\omega}$ with an $(n_\varsigma, n_\omega)$-tangle $I_\varsigma T \hat{P}_\omega \in \mathcal{T}_{n_\varsigma}^{n_\omega}$, and any $\varsigma$-valenced link state $\alpha \in \mathcal{L}_\varsigma$ with a link state $I_\varsigma(\alpha) \in \mathcal{L}_{n_\varsigma}$, as follows: for each $i \in \{1, 2, \ldots, d_\varsigma\}$ (resp. $j \in \{1, 2, \ldots, d_\omega\}$),

1. we separate the $i$:th (resp. $j$:th) node of $T$ or $\alpha$ into a collection of $s_i$ (resp. $p_j$) adjacent nodes, and
I2. we place a Jones-Wenzl projector box (3.28) across the strands that attach to the node:

\[ \rightarrow s = \text{ } \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \text{ } s. \]  

(3.34)

This identification defines two injective linear maps [FP18a, lemma B.1],

\[ I_\varsigma(\cdot)\hat{P}_\varsigma : \mathbb{TL}_\varsigma^\varsigma \rightarrow \mathbb{TL}_{n_\varsigma}^n \quad \text{and} \quad I_\varsigma : \mathbb{L}_\varsigma \rightarrow \mathbb{L}_{n_\varsigma}, \]  

(3.35)

which can respectively be realized diagrammatically as

\[ I_\varsigma(\cdot)\hat{P}_\varsigma \quad \hat{P}_\varsigma(\cdot)I_\varsigma \quad \hat{P}_\varsigma(\cdot)I_\varsigma \quad \hat{P}_\varsigma(\cdot)I_\varsigma \]  

(3.36)

where \( T \in \mathbb{TL}_{n_\varsigma}^{n_\varsigma} \) (resp. \( \alpha \in \mathbb{L}_{n_\varsigma} \)) is a special tangle (resp. link state) lacking links that would result in loop-links in the valenced tangle (resp. link state); cf. also property (P2) of the Jones-Wenzl projector. These maps preserve the \( s \)-grading 3.24 We summarize them in the following commuting diagrams [FP18a, appendix B]:

\[ \mathbb{TL}_\varsigma^\varsigma \xrightarrow{I_\varsigma(\cdot)\hat{P}_\varsigma} \mathbb{TL}_{n_\varsigma}^n \quad \hat{P}_\varsigma(\cdot)I_\varsigma \quad \hat{P}_\varsigma(\cdot)I_\varsigma \quad \hat{P}_\varsigma(\cdot)I_\varsigma \]  

(3.38)

Valenced Temperley-Lieb category. Valenced tangles can be viewed as morphisms in a category \( \mathbb{TL}(\nu) \), that contains the Temperley-Lieb category \( \mathbb{TL}(1) \). The object class and morphism class of this category are respectively

\[ \text{Ob} \mathbb{TL}(\nu) = \{ \varsigma \in \{ 0 \} \cup \mathbb{Z}_{\geq 0}^\# \mid \max \varsigma < p(q) \} \]  

(3.39)

\[ \text{Hom} \mathbb{TL}(\nu) = \{ \mathbb{TL}_\varsigma^\varsigma \mid \varsigma, \varsigma^\# \in \text{Ob} \mathbb{TL}(\nu) \text{ with } n_\varsigma + n_{\varsigma^\#} = 0 \text{ (mod 2)} \}. \]  

(3.40)

The source and target associated with the tangle \( T \in \mathbb{TL}_\varsigma^\varsigma \) are the objects \( \varsigma^\# \) and \( \varsigma \) respectively. The composition of morphisms of \( \mathbb{TL}(\nu) \) is defined as the following diagram concatenation; see [FP18a, section 2D] for details. Given two valenced tangles \( T \in \mathbb{TL}_\varsigma^\varsigma \) and \( U \in \mathbb{TL}_\psi^\psi \), we utilize the maps \( I_\varsigma(\cdot)\hat{P}_\varsigma \) and \( I_\varsigma(\cdot)\hat{P}_{\psi^\#} \) with properties (3.33) to define

\[ TU := \hat{P}_\varsigma(\cdot)I_\varsigma(\cdot)\hat{P}_\varsigma \quad \hat{P}_{\psi^\#}(\cdot)I_\varsigma(\cdot)\hat{P}_{\psi^\#} \]  

(3.41)

performing on the right side the ordinary diagram concatenation without valenced nodes, as in (3.4). The identity morphism associated with the object \( \varsigma \) is the unit (1.26) of the valenced Temperley-Lieb algebra \( \mathbb{TL}_\varsigma(\nu) = \mathbb{TL}_\varsigma^\varsigma \).
an associative unital algebra with multiplication given by (3.41) with \( \zeta = \varpi = \varepsilon \). By [FP18a, proposition 2.10] (see also [FP18b, theorem 1.1]), when \( n_\zeta < p(q) \), the algebra \( \mathbb{T}_L(n_\zeta) \) is generated by its unit and the generators \( \{ U_1, U_2, \ldots, U_{d_\zeta-1} \} \) given by diagrams (1.26, 1.27). The special case of \( \zeta = \bar{\nu} \) is the Temperley-Lieb algebra \( \mathbb{T}_L(n) \).

We remark that the restriction to \( n_\zeta < p(q) \) in [FP18a, FP18b] should not be essential for \( \{ V_{TL,1}, U_1, U_2, \ldots, U_{d_\zeta-1} \} \) to be the whole generating set of \( \mathbb{T}_L(n) \), but no proof is known to us to date. Also, not all of the relations of this algebra are known to us in general, see [FP18b].

We also let \( T^* \in \mathbb{T}_L^G(n) \) denote the reflection of \( T \in \mathbb{T}_L^G \) about a vertical axis:

\[
T = \begin{array}{c}
0
\end{array} \text{ } \Rightarrow \text{ } \begin{array}{c}
0
\end{array} \quad (3.42)
\]

From (3.26, 3.32, 3.42), we immediately observe that

\[
(I_\zeta \alpha)^* = \alpha^* \hat{P}_\zeta \quad \text{and} \quad \hat{\alpha} \hat{P}_\zeta = (I_\zeta \hat{\alpha}^*)^* \quad (3.43)
\]

Given a valenced tangle \( T \in \mathbb{T}_L^G \) and a valenced link state \( \alpha \in L^{(s)}_\zeta \), we Tutilize the maps \( I_\zeta(\cdot) \hat{P}_\zeta \) and \( I_\zeta(\cdot) \hat{\alpha}^* \) to form the concatenation

\[
T \alpha := \hat{P}_\zeta(I_\zeta T \hat{P}_\zeta)(I_\zeta \alpha) \in L^{(s)}_\zeta, \quad (3.44)
\]

performing on the right side the ordinary diagram concatenation, as in (3.17, 3.18). With \( \varpi = \zeta \), rule (3.44) defines a left \( \mathbb{T}_L(n(\zeta)) \)-action on the space \( L^{(s)}_\zeta \), which we thus call a Temperley-Lieb standard module. We also call the direct sum \( L_\zeta \) defined in (3.24) the link state module. We refer to [FP18a] for details about the \( \mathbb{T}_L(n(\zeta)) \)-standard modules.

We also define analogous right \( \mathbb{T}_L(n(\zeta)) \)-module structures on \( L^{(s)}_\zeta \) and \( \bar{L}_\zeta \), with right \( \mathbb{T}_L(n(\zeta)) \)-action defined by diagram concatenation from the right:

\[
\bar{\alpha} T := \bar{\alpha} \hat{P}_\zeta(I_\zeta T \hat{P}_\zeta)I_\zeta \in \bar{L}^{(s)}_\zeta. \quad (3.45)
\]

We can enumerate the valenced link patterns in terms of the numbers \( D^{(s)}(\zeta) \) appearing in (2.59).

**Lemma 3.3.** The following hold:

1. [FP18a, Lemma 2.8]: We have

\[
\dim L^{(s)}_\zeta = \# \mathbb{L}^{(s)}_\zeta = D^{(s)}(\zeta) \quad \text{and} \quad \dim L_\zeta = \# \mathbb{L}_\zeta = D_\zeta. \quad (3.46)
\]

Similarly, this identity holds after the symbolic replacements \( L \rightarrow \bar{L} \) and \( \mathbb{L} \rightarrow \bar{\mathbb{L}} \).

2. [FP18a, Corollary 2.7]: We have

\[
\dim \mathbb{T}_L^G(n) = \sum_{s \in \bar{E}_\zeta \cap \bar{E}_\omega} D^{(s)}_\omega D^{(s)}_\zeta. \quad (3.47)
\]

**B. Temperley-Lieb representations on fundamental \( U_q \)-modules**

The purpose of this section is to define and study an action of the Temperley-Lieb algebra \( \mathbb{T}_L(n(\nu)) \) on the tensor products \( \mathcal{V}_n \) and \( \bar{\mathcal{V}}_n \). Specifically, we explicitly define a left action \( \tau_L \circ \tau_n \) and a right action \( \tau_r \circ \tau_n \) and prove that the action is via \( U_q, U_q \)-homomorphisms. (In the present section, we mostly consider the case \( q \notin \{ \pm 1, \pm i \} \); the special case \( q = \pm 1 \) when \( \nu = 0 \) is discussed in appendix B and the case \( q = 1 \) in appendix C). Indeed, the Temperley-Lieb generators correspond with multiples of \( U_q, U_q \)-submodule projectors of type (1.22).

For the Temperley-Lieb action, it is useful to define the singlet vectors \( \sigma \in \mathcal{V}_2 \) and \( \bar{J} \in \bar{\mathcal{V}}_2 \) as

\[
\sigma := \begin{pmatrix} q^{1/2} \varepsilon_0 \otimes \varepsilon_1 - q^{-1/2} \varepsilon_1 \otimes \varepsilon_0 \end{pmatrix} \text{ if } q \neq \pm 1 \quad (2.72)
\]

\[
\bar{\sigma} := \begin{pmatrix} q^{-1/2} \varepsilon_0 \otimes \varepsilon_1 - q^{-1/2} \varepsilon_1 \otimes \varepsilon_0 \end{pmatrix} \text{ if } q \neq \pm 1 \quad (2.73)
\]

\[
\bar{J} := \begin{pmatrix} q^{-1/2} \bar{\varepsilon}_0 \otimes \bar{\varepsilon}_1 - q^{-1/2} \bar{\varepsilon}_1 \otimes \bar{\varepsilon}_0 \end{pmatrix} \text{ if } q \neq \pm 1. \quad (3.48)
\]

\[
\bar{\sigma} = \begin{pmatrix} q^{-1} \bar{\varepsilon}_0 \otimes \bar{\varepsilon}_1 - q^{-1} \bar{\varepsilon}_1 \otimes \bar{\varepsilon}_0 \end{pmatrix} \text{ if } q \neq \pm 1. \quad (3.49)
\]
As shown, when \( q \neq \pm 1 \), the vectors \( \mathcal{I} \) and \( \mathcal{I} \) are renormalized versions of the conformal-block vectors \( u_{(1,1)} \) and \( \bar{u}_{(1,1)} \). The choice of normalization will become apparent with diagram representation of the \( \mathcal{TL}_n(\nu) \)-action (section 3D).

Next, for all integers \( n \geq 2 \) and \( i, j \in \{1, 2, \ldots, n-1\} \) we define the (left) actions

\[
\mathcal{S}^{n-2}(L_i) : V_{n-2} \rightarrow V_n \quad \text{and} \quad \mathcal{S}^{n-2}(R_j) : V_n \rightarrow V_{n-2}
\]

by extending linearly the rules

\[
\mathcal{S}_{n-2}(L_i) := \text{id}^{(i-1)} \otimes \mathcal{S}_0^0(L_1) \otimes \text{id}^{(n-i-1)}, \quad \mathcal{S}_0^0(L_1)(\epsilon_0^{(0)}) := \mathcal{I},
\]

\[
\mathcal{S}_{n-2}(R_j) := \text{id}^{(j-1)} \otimes \mathcal{S}_0^0(R_1) \otimes \text{id}^{(n-j-1)}, \quad \mathcal{S}_0^0(R_1)(\epsilon_0^{(0)}) := \begin{cases} 0, & \ell_1 = 0, \ell_2 = 0, \\ iq^{1/2}, & \ell_1 = 0, \ell_2 = 1, \\ -iq^{-1/2}, & \ell_1 = 1, \ell_2 = 0, \\ 0, & \ell_1 = 1, \ell_2 = 1, \end{cases}
\]

where the one-dimensional vector space \( V_0 = \text{span} \{\epsilon_0^{(0)}\} \) is identified with the ground field \( \mathbb{C} \) and dropped from all tensor products. In lemmas 3.6 and B.3, we relate (3.51, 3.52) to left \( U_q, \bar{U}_q \)-submodule embeddings and projectors.

We analogously define the (right) actions (which define homomorphisms of right \( U_q, \bar{U}_q \)-modules)

\[
\mathcal{F}_{n-2}(L_i) : \bar{V}_n \rightarrow \bar{V}_{n-2} \quad \text{and} \quad \mathcal{F}_{n-2}(R_j) : \bar{V}_{n-2} \rightarrow \bar{V}_n,
\]

by extending linearly the rules

\[
\mathcal{F}_{n-2}(L_i) := \text{id}^{(i-1)} \otimes \mathcal{F}_0^0(R_1) \otimes \text{id}^{(n-i-1)}, \quad \mathcal{F}_0^0(R_1)(\epsilon_0^{(0)}) := \mathcal{I},
\]

\[
\mathcal{F}_{n-2}(R_j) := \text{id}^{(j-1)} \otimes \mathcal{F}_0^0(L_1) \otimes \text{id}^{(n-j-1)}, \quad \mathcal{F}_0^0(L_1)(\epsilon_0^{(0)}) := \begin{cases} 0, & \ell_1 = 0, \ell_2 = 0, \\ iq^{1/2}, & \ell_1 = 0, \ell_2 = 1, \\ -iq^{-1/2}, & \ell_1 = 1, \ell_2 = 0, \\ 0, & \ell_1 = 1, \ell_2 = 1, \end{cases}
\]

To begin, we verify that rules (3.50–3.55) for the generator diagrams \( L_i, R_j \) extend naturally to two families of morphisms \( \mathcal{S}_n^m \) and \( \mathcal{F}_n^m \), which in the special case of \( m = n \) give rise to representations of the Temperley-Lieb algebra \( \mathcal{TL}_n(\nu) \): a left \( \mathcal{TL}_n(\nu) \)-action on \( \mathcal{TL}_n(\nu) \)-algebra and a right \( \mathcal{TL}_n(\nu) \)-action on \( \mathcal{TL}_n(\nu) \)-algebra (see corollary 3.9). In general, these maps are not representations per se, as the set of tangles generated by \( L_i, R_j \), and \( 1_{\mathcal{TL}_n(\nu)} \) is not an algebra.

**Lemma 3.4.** (Temperley-Lieb actions): There exists a unique family of maps

\[
\{ \mathcal{S}_n^m : \mathcal{TL}_n^m \rightarrow \text{Hom}(V_m, V_n) \mid n, m \in \mathbb{Z}_{\geq 0}, n + m \equiv 0 \pmod{2} \}
\]

with the following properties:

1. All maps in family (3.56) that are of the form \( \mathcal{S}_{n-2}^n \) and \( \mathcal{S}_{n-2}^n \) are determined by rules (3.51, 3.52).
2. For all maps \( \mathcal{S}_n^k \) and \( \mathcal{S}_k^m \) in family (3.56) and for all tangles \( T \in \mathcal{TL}_n^k \) and \( U \in \mathcal{TL}_k^m \), we have

\[
\mathcal{S}_n^m(TU) = \mathcal{S}_n^k(T) \circ \mathcal{S}_k^m(U).
\]

Similarly, there exists a unique family of maps

\[
\{ \mathcal{F}_n^m : \mathcal{TL}_n^m \rightarrow \text{Hom}(\bar{V}_m, \bar{V}_n) \mid n, m \in \mathbb{Z}_{\geq 0}, n + m \equiv 0 \pmod{2} \}
\]

with the following properties:

3. All maps in family (3.58) that are of the form \( \mathcal{F}_{n-2}^n \) and \( \mathcal{F}_{n-2}^n \) are determined by rules (3.54, 3.55).
4. For all maps \( \mathcal{F}_n^k \) and \( \mathcal{F}_k^m \) in family (3.58) and for all tangles \( T \in \mathcal{TL}_n^k \) and \( U \in \mathcal{TL}_k^m \), we have

\[
\mathcal{F}_n^m(TU) = \mathcal{F}_n^k(U) \circ \mathcal{F}_k^m(T).
\]

**Proof.** By lemma 3.1, any tangle \( T \in \mathcal{TL}_n^m \) equals a polynomial in the generators \( L_i \) and \( R_j \), so rules (3.51, 3.52) and (3.54, 3.55) completely determine families (3.56) and (3.58) via homomorphism properties (3.57) and (3.59). \( \square \)
We often use the following shorthand notation, for all tangles $T \in TL_n^m$ and vectors $v \in V_m$ and $\varpi \in \nabla_n$:

$$\mathcal{F}^m(T)(v) = Tv \quad \text{and} \quad \mathcal{F}^m_n(T)(\varpi) = \varpi T.$$  \hfill (3.60)

**Corollary 3.5.** ($s$-grading preservation): We have

$$v \in V_n^{(a)}, \quad \varpi \in \nabla_n^{(a)} \quad \text{and} \quad T \in TL_n^m \implies T v \in V_n^{(a)} \quad \text{and} \quad \varpi T \in \nabla_n^{(a)}.$$  \hfill (3.61)

**Proof.** Assertion (3.61) immediately follows from rules (3.50–3.55), definitions (3.48), and the $s$-grading (2.28).  \hfill □

Next, we realize actions (3.50–3.55) in terms of $U_q, \tilde{U}_q$-homomorphisms. To this end, in lemma 3.6 and corollary 3.9 we treat the generic case $q \notin \{\pm i\}$, relating $L_i, U_i,$ and $R_j$ to special cases of the embeddings and projectors $(2.118–2.120)$. In the exceptional case $q \in \{\pm i\}$, the embedding is still well-defined but the projectors cannot be defined, for the direct-sum decomposition (3.62), discussed below, fails. We consider this case separately in appendix B.

When $p(q) > 2$, proposition 2.8 with $s = (1, 1)$ gives the following direct-sum decompositions of $U_q, \tilde{U}_q$-modules:

$$u_s \mathcal{G}_s \subset V_2 \quad \text{(2.14)} : \quad M_1(1) \otimes M_1(1) \quad \text{(2.89)} \quad M(0) \oplus M(2) \quad \text{and} \quad \overline{V}_2 \subset u_s \mathcal{G}_s \quad \text{(2.14)} : \quad \overline{M}_1(1) \otimes \overline{M}_1(1) \quad \text{(2.89)} \quad \overline{M}(0) \oplus \overline{M}(2),$$

with bases

$$u_1 \mathcal{G}_1 \subset V_2 \quad \text{for} \quad \mathcal{T}_1 \quad \text{(2.14)} : \quad M_1(1) \otimes M_1(1) \quad \text{(2.89)} \quad M(0) \oplus M(2) \quad \text{and} \quad \overline{V}_2 \subset u_1 \mathcal{G}_1 \quad \text{for} \quad \overline{M}_1(1) \otimes \overline{M}_1(1) \quad \text{(2.89)} \quad \overline{M}(0) \oplus \overline{M}(2),$$

and similarly

$$u_1 \mathcal{G}_1 \subset V_2 \quad \text{for} \quad \mathcal{T}_1 \quad \text{(2.14)} : \quad M_1(1) \otimes M_1(1) \quad \text{(2.89)} \quad M(0) \oplus M(2) \quad \text{and} \quad \overline{V}_2 \subset u_1 \mathcal{G}_1 \quad \text{for} \quad \overline{M}_1(1) \otimes \overline{M}_1(1) \quad \text{(2.89)} \quad \overline{M}(0) \oplus \overline{M}(2),$$

with bases

$$u_1 \mathcal{G}_1 \subset V_2 \quad \text{(2.14)} : \quad M_1(1) \otimes M_1(1) \quad \text{(2.89)} \quad M(0) \oplus M(2) \quad \text{and} \quad \overline{V}_2 \subset u_1 \mathcal{G}_1 \quad \text{(2.14)} : \quad \overline{M}_1(1) \otimes \overline{M}_1(1) \quad \text{(2.89)} \quad \overline{M}(0) \oplus \overline{M}(2),$$

with bases

$$u_1 \mathcal{G}_1 \subset V_2 \quad \text{(2.14)} : \quad M_1(1) \otimes M_1(1) \quad \text{(2.89)} \quad M(0) \oplus M(2) \quad \text{and} \quad \overline{V}_2 \subset u_1 \mathcal{G}_1 \quad \text{(2.14)} : \quad \overline{M}_1(1) \otimes \overline{M}_1(1) \quad \text{(2.89)} \quad \overline{M}(0) \oplus \overline{M}(2),$$

Definitions $(2.118–2.120)$ of the homomorphisms use these bases $(3.63, 3.64)$.

**Lemma 3.6.** Suppose $q \in \mathbb{C}^\times \setminus \{\pm 1, i\}$. Then, for all integers $n \geq 2$ and $i, j \in \{1, 2, \ldots, n - 1\}$, we have

$$\mathcal{F}^n_n^{-2}(L_i) = \left(\frac{q - q^{-1}}{iq^{1/2}}\right) (\text{id} \otimes \mathcal{I}^{(i-1)} \otimes \mathcal{T}_1^{(0)} \otimes \mathcal{T}_1^{(0)}),$$

and similarly,

$$\mathcal{F}^n_n^{-2}(R_j) = \left(\frac{\nu q^{1/2}}{q - q^{-1}}\right) (\text{id} \otimes \mathcal{R}_1^{(j-1)} \otimes \mathcal{R}_1^{(0)} \otimes \mathcal{R}_1^{(0)}),$$

and similarly,

$$\mathcal{F}^n_n^{-2}(L_i) = \left(\frac{\nu q^{1/2}}{q - q^{-1}}\right) (\text{id} \otimes \mathcal{L}_1^{(i-1)} \otimes \mathcal{L}_1^{(0)} \otimes \mathcal{L}_1^{(0)}).$$

Analogous statements also hold for the case of $q \in \{\pm i\}$, given in lemma B.3 in appendix B.

**Proof.** By definitions $(2.118, 2.120)$ of $\mathcal{I}_1^{(0)}$ and $\mathcal{R}_1^{(1)}$, we have

$$\mathcal{I}_1^{(0)}(\mathcal{T}_1^{(0)}) = \mathcal{I}_1^{(0)}(\mathcal{T}_1^{(0)}) = \left(\frac{\nu q^{1/2}}{q - q^{-1}}\right) (\text{id} \otimes \mathcal{R}_1^{(j-1)} \otimes \mathcal{R}_1^{(0)} \otimes \mathcal{R}_1^{(0)}),$$

and similarly,

$$\mathcal{F}^n_n^{-2}(L_i) = \left(\frac{\nu q^{1/2}}{q - q^{-1}}\right) (\text{id} \otimes \mathcal{L}_1^{(i-1)} \otimes \mathcal{L}_1^{(0)} \otimes \mathcal{L}_1^{(0)}).$$

The first identity in (3.69) together with (3.51) imply the first assertion (3.65). The identities

$$R_1 (\mathcal{F}_1^{\ell} \mathcal{U}_1^{(0)}(\mathcal{T}_1^{(0)})) = \left(\frac{\nu q^{1/2}}{q - q^{-1}}\right) \mathcal{R}_1^{(0)}(\mathcal{F}_1^{\ell} \mathcal{U}_1^{(0)}(\mathcal{T}_1^{(0)}))$$

combined with the last identity of (3.69) imply the second assertion (3.66). Assertions (3.67, 3.68) are similar.  \hfill □
Lemma 3.7. (Quantum group homomorphism properties): Suppose \( q \in \mathbb{C}^\times \setminus \{\pm 1\} \).

1. For each map \( \mathcal{F}_n^m \) in family (3.56), we have \( \text{im} \mathcal{F}_n^m \subset \text{Hom}_\nu(V_m, V_n) \). In other words, we have

\[
T(x.v) = x.(Tv) \quad \text{for all tangles } T \in \mathbb{T}_n^m, \text{ elements } x \in U_q, \text{ and vectors } v \in V_m. \tag{3.71}
\]

2. Similarly, for each map \( \mathcal{F}_n^m \) in family (3.58), we have \( \text{im} \mathcal{F}_n^m \subset \text{Hom}_\nu(\overline{V}_n, V_m) \). In other words, we have

\[
(v.x)T = (vT).x \quad \text{for all tangles } T \in \mathbb{T}_n^m, \text{ elements } x \in U_q, \text{ and vectors } v \in \overline{V}_n. \tag{3.72}
\]

Similarly, this lemma holds after the symbolic replacements \( x \mapsto \overline{x} \) and \( U_q \mapsto \overline{U}_q \).

Proof. By lemma 3.1 and homomorphism properties (3.57, 3.59) of lemma 3.4, it suffices to consider \( \mathcal{F}_n^m \) \( \Rightarrow \) \( \mathcal{F}_n^m \). By lemma 3.10.

Corollary 3.8. Suppose \( q \in \mathbb{C}^\times \setminus \{\pm 1\} \). We have

\[
v \in H_{n,m}^{(\nu)}, \quad \overline{v} \in \overline{H}_{n,m}^{(\nu)}, \quad \text{and } T \in \mathbb{T}_n^m \quad \Rightarrow \quad Tv \in H_{n,m}^{(\nu)} \quad \text{and} \quad \overline{v}T \in \overline{H}_{n,m}^{(\nu)}. \tag{3.73}
\]

Proof. Asserted property (3.73) follows from corollary 3.5, lemma 3.7, and definition (2.34).

As a corollary, we also recover the fact that rules (3.50–3.55) induce representations \( \mathcal{I}_n := \mathcal{I}_n^n \) and \( \mathcal{I}_n := \mathcal{I}_n^n \) of the Temperley-Lieb algebra \( \mathbb{T}_n(\nu) \) on \( V_n \otimes V_n \) and \( \overline{V}_n \otimes V_n \), acting as projectors of type (1.22).

Corollary 3.9. Suppose \( q \in \mathbb{C}^\times \setminus \{\pm 1\} \). Then, \( \mathcal{I}_n : \mathbb{T}_n(\nu) \rightarrow \text{End} V_n \) and \( \mathcal{I}_n : \mathbb{T}_n(\nu) \rightarrow \text{End} \overline{V}_n \) are respectively left and right representations, and for all \( j \in \{1, 2, \ldots, n-1\} \), we have

\[
\mathcal{I}_n(U_j) = \nu(id_\otimes (j-1) \otimes \pi_{(1,1)}^{(1,0)} \otimes id_\otimes (n-j-1)), \tag{3.74}
\]

and similarly,

\[
\mathcal{I}_n(U_j) = \nu(id_\otimes (j-1) \otimes \pi_{(1,1)}^{(1,0)} \otimes id_\otimes (n-j-1)). \tag{3.75}
\]

Also, analogous statements hold for the case of \( q \in \{\pm i\} \), given in corollary B.4 in appendix B.

Proof. Item 3 of lemma A.3 gives the relations \( \pi_{(1,1)}^{(1,0)} = \pi_{(1,1)}^{(0,0)} \otimes \pi_{(1,1)}^{(1,1)} \) and \( \pi_{(1,1)}^{(1,0)} = \pi_{(1,1)}^{(0,0)} \otimes \pi_{(1,1)}^{(1,1)} \), and the Temperley-Lieb relation (3.13) gives \( U_j = L_j R_j \). Assertions (3.74, 3.75) now follow from lemmas 3.6 and 3.7. Also, relations (3.11) and rules (3.50–3.55) show that \( \mathcal{I}_n(1_{\mathbb{T}_n}) = id_{V_n} \), and \( \mathcal{I}_n(1_{\mathbb{T}_n}) = id_{\overline{V}_n} \). Therefore, \( \mathcal{I}_n \) and \( \mathcal{I}_n \) are respectively left and right representations of the associative unital algebra \( \mathbb{T}_n(\nu) \) generated by \( 1_{\mathbb{T}_n} \) and \( U_1, \ldots, U_{n-1} \).

In fact, the representations \( \mathcal{I}_n \) and \( \mathcal{I}_n \) are always faithful, as has been proven independently by P. Martin [Mar92, theorem 1] and F. Goodman and H. Wenzl [GW93, theorem 2.4]. We give another proof for this in section 6 A, where we also prove an analogous result for the valenced Temperley-Lieb algebra, in proposition 6.1.

The next lemma shows that the diagram actions on \( V_n \) and \( \overline{V}_n \) work well with tensor products of tangles,

\[
n \otimes \overline{m} \quad := \quad n \oplus m \quad := \quad \begin{array}{ccc} 1, 1, \ldots, 1 \end{array} \quad \text{n+m times} \quad \text{and} \quad \begin{array}{c} T \otimes U \end{array} := \begin{array}{|c|c|} \hline T & \hline \hline \hline U \end{array}. \tag{3.76}
\]

(The Temperley-Lieb category \( \mathbb{T}_n^1(\nu) \) is a monoidal category, with identity object \( \overline{0} \), and the above tensor product.)

Lemma 3.10. (Tensor product): For all tangles \( T \in \mathbb{T}_n(\nu) \) and \( U \in \mathbb{T}_m(\nu) \), we have

\[
\mathcal{I}_n(T) \otimes \mathcal{I}_m(U) = \mathcal{I}_{n+m}(T \otimes U) \quad \text{and} \quad \mathcal{I}_n(T) \otimes \mathcal{I}_m(U) = \mathcal{I}_{n+m}(T \otimes U). \tag{3.77}
\]

Proof. By lemma 3.1 and homomorphism properties (3.57, 3.59) of lemma 3.4, it suffices to consider \( T \in \{L_i, R_j\} \). Then, the assertion follows from the explicit construction of the diagram action in (3.50–3.55).
The Jones-Wenz projector $P_{(n)}$ (3.28) corresponds to the submodule projectors $\mathfrak{P}_{(n)}$ or $\overline{\mathfrak{P}}_{(n)}$, defined in (2.109, 2.110).

**Lemma 3.11.** Suppose $n < p(q)$. Then, we have

$$P_{(n)} \xrightarrow{\mathcal{J}} \mathfrak{P}_{(n)} \quad \quad \text{and} \quad \quad P_{(n)} \xrightarrow{\mathcal{J}} \overline{\mathfrak{P}}_{(n)}.$$  \hspace{1cm} (3.78)

**Proof.** We prove the left side of (3.78); the right side is similar. The assertion is trivial for $n = 1$, as $P_{(1)}$ is just the unit. Hence assuming that $n > 1$, so $2 < p(q)$, corollary 3.9 and the fact that $\nu \neq 0$ (since $p(q) > 2$) imply that

$$(\text{id} \otimes (i-1) \otimes \pi_{(1,1)}(0) \otimes \text{id} \otimes (n-i-1))(P_{(n)}v) \overset{\text{(3.74)}}{=} \nu^{-1}U_{1}P_{(n)}v \overset{\text{(1.2)}}{=} 0$$

for any vector $v \in V_{n}$ and for any indices $i \in \{1, 2, \ldots, n-1\}$. As such [KP16, lemma 2.4] implies that $P_{(n)}v$ is an element of the unique submodule of $\mathfrak{U}_{1} \subset V_{n}$ isomorphic to $\mathfrak{M}_{(n)}$, which by item 2 of corollary 2.9 is generated by the highest-weight vector $\theta_{0}^{(n)}$ corresponding to the unique walk $\theta$ over $\bar{n}$ with $\text{def}_{\bar{n}} = n$. On the other hand, we also have

$$v \in H_{n}^{(s)} \overset{\text{(3.73)}}{=} P_{(n)}v \in H_{n}^{(s)},$$

so item 3 of proposition 2.8 implies that $P_{(n)}v = 0$ for all vectors $v \notin \text{span} \{\theta_{\ell}^{(n)} \mid 0 \leq \ell \leq n\}$. Because $P_{(n)}$ acts as an $U_{q}$-homomorphism by lemma 3.7, it remains to find the value of $P_{(n)}\theta_{0}^{(n)}$. To this end, we only use the case of $n = 1$ being trivial, we assume that $P_{(n-1)}\theta_{0}^{(n-1)} = \theta_{0}^{(n-1)}$ and apply recursion (3.29) along with lemmas 3.4 and 3.10 to obtain

$$P_{(n)}\theta_{0}^{(n)} := \mathcal{J}(P_{(n)})(\theta_{0}^{(n)}) \overset{\text{(3.29)}}{=} \overset{\text{(3.77)}}{=} (\mathcal{J}_{1}(P_{(n-1)}) \otimes \text{id} + \frac{n-1}{n}) \mathcal{J}_{n-1}(P_{(n-1)}U_{n-1}P_{(n-1)})(\theta_{0}^{(n)})

\overset{\text{(3.57)}}{=} \overset{\text{(3.77)}}{=} (\text{id} + \frac{n-1}{n}) \mathcal{J}_{n-1}(P_{(n-1)}U_{n-1})(\mathcal{J}_{n-1}(P_{(n-1)} \otimes \text{id})(\theta_{0}^{(n)})

\overset{\text{(2.100)}}{=} (\text{id} + \frac{n-1}{n}) \mathcal{J}_{n-1}(P_{(n-1)}U_{n-1})(P_{(n-1)}\theta_{0}^{(n-1)} \otimes \varepsilon_{0})

\overset{\text{(3.81)}}{=} (\text{id} + \frac{n-1}{n}) \mathcal{J}_{n-1}(P_{(n-1)}U_{n-1})(\theta_{0}^{(n-1)} \otimes \varepsilon_{0}).$$

After factoring $U_{n-1} = L_{n-1}R_{n-1}$ according to (3.13) and using (3.52) to simplify the last line, we arrive with

$$P_{(n-1)}U_{n-1}(\theta_{0}^{(n-1)} \otimes \varepsilon_{0}) \overset{\text{(3.57)}}{=} \overset{\text{(3.77)}}{=} P_{(n-1)}L_{n-1}R_{n-1}(\theta_{0}^{(n-1)} \otimes \varepsilon_{0}) \overset{\text{(2.100)}}{=} \overset{\text{(3.52)}}{=} 0$$

\[\implies P_{(n)}(\theta_{0}^{(n)}) \overset{\text{(3.81)}}{=} \theta_{0}^{(n-1)} \otimes \varepsilon_{0} \overset{\text{(2.100)}}{=} \theta_{0}^{(n)},\]  \hspace{1cm} (3.82)

so induction on $n \in \mathbb{Z}_{>0}$ shows that $P_{(n)}\theta_{0}^{(n)} = \theta_{0}^{(n)}$. To conclude, we recall that by definition, $\mathfrak{P}_{(n)}$ is the projection

$$\mathfrak{P}_{(n)}(v) = \begin{cases} v, & v \in \text{span} \{\theta_{\ell}^{(n)} \mid 0 \leq \ell \leq n\}, \\ 0, & \text{otherwise} \end{cases} \hspace{1cm} (3.83)$$

from $\mathfrak{U}_{1} \subset V_{n}$ onto its unique submodule isomorphic to $\mathfrak{M}_{(n)}$, thus coinciding with $\mathcal{J}(P_{(n)})$. This finishes the proof. \hfill \square

Similarly, the composite projector $P_{c}$ (3.31) corresponds to the submodule projector $\mathfrak{P}_{c}$ or $\overline{\mathfrak{P}}_{c}$, defined in (2.114).

**Corollary 3.12.** Suppose $\max \zeta < p(q)$. We have

$$P_{c} \xrightarrow{\mathcal{J}_{s}} \mathfrak{P}_{c} \quad \quad \text{and} \quad \quad P_{c} \xrightarrow{\mathcal{J}_{s}} \overline{\mathfrak{P}}_{c}.$$  \hspace{1cm} (3.84)

**Proof.** Lemma 3.11 says that $\mathcal{J}_{s}(P_{(n)}) = \mathfrak{P}_{(s)}$ for any $s \in \mathbb{Z}_{>0}$. Combining with lemma 3.10, we find that

$$\mathfrak{P}_{c} \overset{\text{(2.114)}}{=} \mathfrak{P}_{(s_{1})} \otimes \mathfrak{P}_{(s_{2})} \otimes \cdots \otimes \mathfrak{P}_{(s_{d_{c}})} \overset{\text{(3.78)}}{=} \mathcal{J}_{s_{1}}(P_{(s_{1})}) \otimes \mathcal{J}_{s_{2}}(P_{(s_{2})}) \otimes \cdots \otimes \mathcal{J}_{s_{d_{c}}}(P_{(s_{d_{c}})}) \overset{\text{(3.77)}}{=} \mathcal{J}_{n_{c}}(P_{c}),$$

which proves the left equation in (3.84). The right equation can be proven similarly. \hfill \square
C. Valenced Temperley-Lieb representations on type-one $U_q$-modules

Next, we define actions of the valenced Temperley-Lieb algebra $\mathcal{TL}_\nu(\nu)$ on general tensor products of type $V_\nu$. The main result of this section is proposition 3.18, where we establish an explicit direct-sum decomposition for these $\mathcal{TL}_\nu(\nu)$-modules $\mathcal{TL}_\nu \circ V_\nu$ and $\mathcal{TL}_\nu \circ \mathcal{TL}_\nu$ in terms of $U_q \mathcal{TL}_\nu$-highest-weight vectors. Slightly more generally, in lemma 3.13 we define actions of $(\varsigma, \nu)$-valenced tangles $T \in \mathcal{TL}_\nu$ via the map (3.36) and families (3.56, 3.58). The special case $\nu = \varsigma$ gives rise to left and right representations of the algebra $\mathcal{TL}_\nu(\nu)$. (Again, if $\nu \neq \varsigma$, these maps are not representations per se, although composition of tangles is respected.)

**Lemma 3.13.** (Valenced Temperley-Lieb actions): There exists a unique family of maps

$$\{ \mathcal{J}_\nu^\varsigma : \mathcal{TL}_\nu^\varsigma \rightarrow \text{Hom}(V_\nu, V_\varsigma) \mid \text{max}(\varsigma, \nu) < p(q), n_\varsigma + n_\nu \equiv 0 \text{ (mod 2)} \}$$  \hspace{1cm} (3.86)

with the following properties:

1. All maps in family (3.86) are given by the following rule, for all valenced tangles $T \in \mathcal{TL}_\nu^\varsigma$:

$$\mathcal{J}_\nu^\varsigma(T) := \hat{\mathcal{P}}_\varsigma \circ \mathcal{J}_n^\nu(I_{\varsigma \hat{P} \hat{P}_\nu}) \circ \mathcal{J}_\nu.$$  \hspace{1cm} (3.87)

2. For all maps $\mathcal{J}_\varsigma^\nu$ and $\mathcal{J}_\nu^\nu$ in family (3.86) and for all valenced tangles $T \in \mathcal{TL}_\nu^\nu$ and $U \in \mathcal{TL}_\nu^\nu$, we have

$$\mathcal{J}_\nu^\nu(TU) = \mathcal{J}_\nu^\nu(T) \circ \mathcal{J}_\nu^\nu(U).$$  \hspace{1cm} (3.88)

Similarly, there exists a unique family of maps

$$\{ \mathcal{F}_\nu^\varsigma : \mathcal{TL}_\nu^\varsigma \rightarrow \text{Hom}(\overline{V}_\varsigma, \overline{V}_\nu) \mid \text{max}(\varsigma, \nu) < p(q), n_\varsigma + n_\nu \equiv 0 \text{ (mod 2)} \}$$  \hspace{1cm} (3.89)

with the following properties:

3. All maps in family (3.89) are given by the following rule, for all valenced tangles $T \in \mathcal{TL}_\nu^\varsigma$:

$$\mathcal{F}_\nu^\varsigma(T) := \hat{\mathcal{P}}_\nu \circ \mathcal{F}_n^\nu(I_{\nu \hat{P} \hat{P}_\nu}) \circ \mathcal{J}_\nu.$$  \hspace{1cm} (3.90)

4. For all maps $\mathcal{F}_\varsigma^\nu$ and $\mathcal{F}_\nu^\nu$ in family (3.89) and for all valenced tangles $T \in \mathcal{TL}_\nu^\nu$ and $U \in \mathcal{TL}_\nu^\nu$, we have

$$\mathcal{F}_\nu^\nu(TU) = \mathcal{F}_\nu^\nu(U) \circ \mathcal{F}_\nu^\nu(T).$$  \hspace{1cm} (3.91)

**Proof.** Items 1 and 3 just define all of the maps in families (3.86, 3.89). To prove item 2 (similarly, 4), we use this definition and the corresponding property (3.57) from lemma 3.4 to obtain

$$\mathcal{J}_\nu^\nu(T) \circ \mathcal{J}_\nu^\nu(U) \overset{(3.87)}{=} \hat{\mathcal{P}}_\nu \circ \mathcal{J}_n^\nu(I_{\nu \hat{P} \hat{P}_\nu}) \circ \mathcal{J}_\nu \overset{(3.57)}{=} \hat{\mathcal{P}}_\nu \circ \mathcal{J}_n^\nu(I_{\nu \hat{P} \hat{P}_\nu}) \circ \mathcal{J}_\nu \overset{(3.33)}{=} \hat{\mathcal{P}}_\nu \circ \mathcal{J}_n^\nu(I_{\nu \hat{P} \hat{P}_\nu} \circ \mathcal{J}_\nu \overset{(3.84)}{=} \hat{\mathcal{P}}_\nu \circ \mathcal{J}_n^\nu(I_{\nu \hat{P} \hat{P}_\nu}) \circ \mathcal{J}_\nu \overset{(3.87)}{=} \mathcal{J}_\nu^\nu(TU)$$  \hspace{1cm} (3.92)

for any two maps $\mathcal{J}_\nu^\nu$ and $\mathcal{F}_\nu^\nu$ in family (3.86) and for any two valenced tangles $T \in \mathcal{TL}_\nu^\nu$ and $U \in \mathcal{TL}_\nu^\nu$. \hfill \Box

We often use the following shorthand notation, for all valenced tangles $T \in \mathcal{TL}_\nu^\nu$ and vectors $v \in V_\nu$ and $v \in \overline{V}_\nu$:

$$\mathcal{J}_\nu^\nu(T)(v) = Tv \quad \text{and} \quad \mathcal{F}_\nu^\nu(T)(v) = vT.$$  \hspace{1cm} (3.93)

**Corollary 3.14.** (s-gradation preservation): Suppose $\text{max}(\varsigma, \nu) < p(q)$. We have

$$v \in V_\nu, \quad \overline{v} \in \overline{V}_\nu, \quad \text{and} \quad T \in \mathcal{TL}_\nu^\nu \implies Tv \in V_\nu \quad \text{and} \quad vT \in \overline{V}_\nu.$$  \hspace{1cm} (3.94)
Proof. The case $\varsigma = \hat{\nu}$ and $\varpi = \hat{\nu}$ is the content of corollary 3.5, and using definitions (3.87, 3.90) and lemma A.2 we readily extend (3.61) to (3.94) for general multiindices $\varsigma, \varpi \in \mathbb{Z}_{>0}^2$ as in (1.34, 1.35).

Next, we record an analogue to corollary 3.12 for the Jones-Wenzl composite embedder and projector (3.32).

**Corollary 3.15.** Suppose $\max \varsigma < p(q)$. We have

$$\hat{P}_\varsigma \xrightarrow{\mathcal{J}_\varsigma} \hat{\Psi}_\varsigma \quad \text{and} \quad I_\varsigma \xrightarrow{\mathcal{J}_\varsigma} \mathfrak{J}_\varsigma,$$

and similarly,

$$\hat{P}_\varsigma \xrightarrow{\mathcal{J}_\varsigma} \mathfrak{J}_\varsigma \quad \text{and} \quad I_\varsigma \xrightarrow{\mathcal{J}_\varsigma} \hat{\Psi}_\varsigma.$$

**Proof.** Using the trivial observation that $\hat{P}_{n_\varsigma} = I_{n_\varsigma} = 1_{TL_{\varsigma}}$, with definition (3.87) and corollary 3.12, we have

$$\mathcal{J}_\varsigma^{n_\varsigma}(\hat{P}_\varsigma) = \hat{\Psi}_\varsigma \circ \mathcal{J}_{n_\varsigma}(I_\varsigma \hat{P}_\varsigma) = \mathfrak{J}_\varsigma \circ \mathcal{J}_{n_\varsigma}(P_\varsigma) = \mathfrak{J}_\varsigma = \hat{\Psi}_\varsigma,$$

(3.97)

This proves (3.95), and (3.96) can be proven similarly.

The diagram actions given by lemma 3.13 are also $U_q, \overline{U}_q$-homomorphisms.

**Lemma 3.16.** (Quantum group homomorphism properties): Suppose $q \in \mathbb{C}^\times \setminus \{\pm 1\}$.

1. For each map $\mathcal{J}_\varsigma$ in family (3.86), we have $\im \mathcal{J}_\varsigma \subset \text{Hom}_{U_q}(V_\varpi, V_\varsigma)$. In other words, we have

$$T(x.v) = x.(Tv) \quad \text{for all valenced tangles } T \in TL_\varsigma^\varpi, \text{ elements } x \in U_q, \text{ and vectors } v \in V_\varpi.$$  

(3.99)

2. Similarly, for each map $\overline{\mathcal{J}}_\varsigma$ in family (3.89), we have $\im \overline{\mathcal{J}}_\varsigma \subset \text{Hom}_{\overline{U}_q}(\overline{V}_\varpi, \overline{V}_\varsigma)$. In other words, we have

$$T(x.v) = x.(Tv) \quad \text{for all valenced tangles } T \in TL_\varsigma^\varpi, \text{ elements } x \in U_q, \text{ and vectors } v \in \overline{V}_\varsigma.$$  

(3.100)

Similarly, this lemma holds after the symbolic replacements $x \mapsto \overline{x}$ and $U_q \mapsto \overline{U}_q$.

**Proof.** Lemmas A.2 and 3.7 show that all of the maps $\mathcal{J}_\varpi, \hat{\Psi}_\varsigma, \overline{\Psi}_\varsigma, \mathcal{J}_{n_\varsigma}(I_\varsigma T \hat{P}_\varpi)$ and $\mathcal{J}_{n_\varsigma}(I_\varsigma T \overline{P}_\varpi)$, are $U_q, \overline{U}_q$-homomorphisms. The assertions follow from this by definitions (3.87, 3.90).

**Corollary 3.17.** Suppose $\max(\varsigma, \varpi) < p(q)$. We have

$$v \in H_\varsigma^{(n)}, \quad v \in \mathfrak{R}_\varsigma^{(s)}, \quad \text{and} \quad T \in TL_\varsigma^\varpi \implies Tv \in H_\varsigma^{(n)} \quad \text{and} \quad \varpi T \in \mathfrak{R}_\varsigma^{(s)}.$$  

(3.101)

**Proof.** Asserted property (3.101) follows from corollary 3.14, lemma 3.16, and definition (2.34).

We denote $\mathcal{J}_\varsigma := \mathcal{J}_\varpi$ and $\overline{\mathcal{J}}_\varsigma := \overline{\mathcal{J}}_\varpi$ when $\varpi = \varsigma$. These maps

$$\mathcal{J}_\varsigma : TL_\varsigma(\nu) \rightarrow \text{End} V_\varsigma \quad \text{and} \quad \overline{\mathcal{J}}_\varsigma : TL_\varsigma(\nu) \rightarrow \text{End}^{op} \overline{V}_\varsigma$$

(3.102)

are respectively left and right representations of the valenced Temperley-Lieb algebra, because they send the valenced unit tangle $1_{TL_\varsigma}$ (1.26) to the identity map, by corollary 3.9 and definitions (3.87, 3.90). We next investigate the structure of the $TL_\varsigma(\nu)$-modules $\mathfrak{T}_\varsigma \subset V_\varsigma$ and $\overline{\mathfrak{T}}_\varsigma \subset \overline{V}_\varsigma$ in more detail. In proposition 3.18, we establish a direct-sum decomposition for these $TL_\varsigma(\nu)$-modules in terms of $U_q, \overline{U}_q$-highest-weight vectors, when $n_\varsigma < p(q)$. Theorem 1.4 in section 6C upgrades this direct-sum decomposition into a quantum Schur-Weyl duality. We will also prove in section 6A that these representations are always faithful (even if $n_\varsigma \geq p(q)$). This follows from proposition 6.1, which says that in fact, all maps $\mathcal{J}_\varpi$ and $\overline{\mathcal{J}}_\varpi$ in lemma 3.13 are linear injections.

**Proposition 3.18.** Suppose $\max \varsigma < p(q)$.

1. The vector space $H_\varsigma$ is closed under the left $TL_\varsigma(\nu)$-action on it.
2. For each $s \in E_\zeta$, the vector space $H_\zeta^{(s)}$ is closed under the left $\text{TL}_\zeta(\nu)$-action on it.

3. For each $0 \leq \ell \leq s < p(q)$, the left $\text{TL}_\zeta(\nu)$-modules $\tau_L \otimes H_\zeta^{(s)}$ and $\tau_L \otimes F^\ell H_\zeta^{(s)}$ are isomorphic.

4. If $n_\zeta < p(q)$, then we have the following isomorphism of left $\text{TL}_\zeta(\nu)$-modules:
\[
\tau_L \otimes V_\zeta \cong \bigoplus_{s \in E_\zeta} (s + 1) \tau_L \otimes H_\zeta^{(s)}.
\] (3.103)

Similarly, this proposition holds for right $\text{TL}_\zeta(\nu)$-modules, after the symbolic replacements
\[
H \mapsto \bar{H}, \quad \tau_L \otimes H_\zeta^{(s)} \mapsto \bar{H}_\zeta^{(s)} \otimes \tau_L, \quad F^\ell H_\zeta^{(s)} \mapsto \bar{H}_\zeta^{(s)} E^\ell, \quad \text{and} \quad \tau_L \otimes V_\zeta \mapsto \bar{V}_\zeta \otimes \tau_L.
\] (3.104)

**Proof.** Items 1–2 immediately follow from corollary 3.17. To prove item 3, we note that for each $0 \leq \ell \leq s < p(q)$, the restriction of the action of $F^\ell$ to a map from $H_\zeta^{(s)}$ onto $F^\ell H_\zeta^{(s)}$ is an isomorphism of vector spaces with inverse $[s]![\ell]![s - \ell]^{-1} F^\ell$ and, by lemma 3.16, a homomorphism of $\text{TL}_\zeta(\nu)$-modules. To prove item 4, we note that when $n_\zeta < p(q)$, proposition 2.8 implies the following isomorphism of vector spaces, with $s \leq n_\zeta < p(q)$ by (2.63):
\[
\bigoplus_{s \in E_\zeta} \bigoplus_{\ell = 0}^s F^\ell H_\zeta^{(s)} \cong V_\zeta.
\] (3.105)

Indeed, the $K$-eigenvalues of the different summands $F^\ell H_\zeta^{(s)}$ over $\ell$ for fixed $s$ are distinct, so their sum is direct. These summands are all isomorphic as $\text{TL}_\zeta(\nu)$-modules by item 3, and item 2 shows that the $\text{TL}_\zeta(\nu)$-action on $\tau_L \otimes V_\zeta$ respects the $s$-grading on the left side of (3.103). Asserted isomorphism (3.103) of $\text{TL}_\zeta(\nu)$-modules then follows. Finally, the statements concerning the right $\text{TL}_\zeta(\nu)$-action on $\bar{V}_\zeta \otimes \tau_L$ can be proven similarly. □

We show in proposition 4.12 in section 4B that, when $n_\zeta < p(q)$, the $\text{TL}_\zeta(\nu)$-modules $\tau_L \otimes H_\zeta^{(s)}$ are in fact isomorphic to the standard modules $L_n^{(\nu)}$ (cf. section 3A). In particular, we then know from [FP18a] that in this case, the algebra $\text{TL}_\zeta(\nu)$ is semisimple and $L_n^{(\nu)}$ are all of its simple modules, so proposition 4.12 also implies that $\tau_L \otimes H_\zeta^{(s)}$ are simple. If $n_\zeta \geq p(q)$, the modules $\tau_L \otimes H_\zeta^{(s)}$ might not be simple, and $\text{TL}_\zeta(\nu)$ is generally not semisimple. We discuss this case in proposition 5.14 in section 4B.

**D. Link state bilinear pairing and graphical calculus for the spin chain**

Next, we define a bilinear pairing of valenced link patterns (see also [FP18a, section 3A]). We begin with the special case of $\zeta = \bar{n}$ for some $n \in \mathbb{Z}_{>0}$. To this end, given two link patterns $\bar{\alpha} \in \bar{L}_n$ and $\beta \in L_n$, we concatenate $\bar{\alpha}$ to $\beta$ from below, and delete the overlapping horizontal lines. The resulting diagram is a network $\bar{\alpha} \mid \beta$. For instance,
\[
\bar{\alpha} = \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}, \quad \beta = \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \quad \Rightarrow \quad \bar{\alpha} \mid \beta = \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \bigcirc \\
\bigcirc \\
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\bigcirc
\end{array}
\] (3.106)

Linear extension of the evaluation of the network $\bar{\alpha} \mid \beta$ determines a bilinear pairing $(\cdot \mid \cdot): L_n \times L_n \rightarrow \mathbb{C}$,
\[
(\bar{\alpha} \mid \beta) := \prod \{\text{the weights of all connected components in the network } \bar{\alpha} \mid \beta\}
\]
\[
= \begin{cases} 
\nu^{|\# \text{ loops in } \bar{\alpha} \mid \beta|}, & \text{if the network } \bar{\alpha} \mid \beta \text{ has no turn-back path,} \\
0, & \text{if the network } \bar{\alpha} \mid \beta \text{ has a turn-back path,}
\end{cases}
\] (3.107)

where we assign all loops, through-paths, and turn-back paths the following weights in $\mathbb{C}$:

- loop weight (fugacity):
  \[
  \quad \text{etc.} \quad = \nu, \quad (3.108)
  \]

- through-path weight:
  \[
  \quad \text{etc.} \quad = 1, \quad (3.109)
  \]
(Actually, we have already been using rules (3.108) and (3.110) in diagram concatenation in section 3 A.) Using this, we then define for general multiindices a bilinear pairing \((\cdot \mid \cdot): \mathbb{L}_\varsigma \times \mathbb{L}_\varsigma \to \mathbb{C}\) by
\[
(\bar{\alpha} \mid \beta) := (\bar{\alpha} \hat{P}_\varsigma \mid I_\varsigma \beta).
\]
For instance,
\[
\left(\begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}\right)
\right) = \left(\begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}\right) = (\bar{0}).
\]
(3.112)

The bilinear pairing \((\cdot \mid \cdot)\) is TL\(_\varsigma(\nu)\)-invariant in the sense that \(\{\text{FP18a, lemma 3.1}\}\)
\[
(\alpha \mid T \beta) = (\alpha T \mid \beta)
\]
(3.113)
for all valenced link patterns \(\alpha \in \mathbb{L}_\varsigma\) and \(\beta \in \mathbb{L}_\varsigma\) and for all valenced tangles \(T \in \text{TL}_\varsigma(\nu)\). Using this property, it is straightforward to check that the spaces
\[
\text{rad} \mathbb{L}_\varsigma := \{ \alpha \in \mathbb{L}_\varsigma \mid (\beta \mid \alpha) = 0 \text{ for all } \beta \in \mathbb{L}_\varsigma \} = \bigoplus_{s \in E_\varsigma} \text{rad} \mathbb{L}_\varsigma^{(s)},
\]
(3.114)
\[
\text{rad} \mathbb{L}_\varsigma^{(s)} := \{ \alpha \in \mathbb{L}_\varsigma^{(s)} \mid (\beta \mid \alpha) = 0 \text{ for all } \beta \in \mathbb{L}_\varsigma^{(s)} \},
\]
(3.115)
are TL\(_\varsigma(\nu)\)-submodules of \(\mathbb{L}_\varsigma\) and \(\mathbb{L}_\varsigma^{(s)}\), respectively. We denote the corresponding quotient modules by
\[
Q_\varsigma := \mathbb{L}_\varsigma / \text{rad} \mathbb{L}_\varsigma\quad \text{and} \quad Q_\varsigma^{(s)} := \mathbb{L}_\varsigma^{(s)} / \text{rad} \mathbb{L}_\varsigma^{(s)} = \bigoplus_{s \in E_\varsigma} Q_\varsigma^{(s)}.
\]
(3.116)

We analogously define the right TL\(_\varsigma(\nu)\)-modules rad \(\overline{\mathbb{L}}_\varsigma\), rad \(\overline{\mathbb{L}}_\varsigma^{(s)}\), \(\overline{Q}_\varsigma\), and \(\overline{Q}_\varsigma^{(s)}\).

By [FP18a, proposition 6.7], the collection \(\{Q_\varsigma^{(s)} \mid s \in E_\varsigma, \dim Q_\varsigma^{(s)} > 0\}\) is the complete set of non-isomorphic simple left TL\(_\varsigma(\nu)\)-modules. Also, if \(n_\varsigma < p(q)\), then by [FP18a, theorem 6.9], rad \(\mathbb{L}_\varsigma = \{0\}\), the valenced Temperley-Lieb algebra TL\(_\varsigma(\nu)\) is semisimple, and the collection \(\{L_\varsigma^{(s)} \mid s \in E_\varsigma\}\) is the complete set of all simple TL\(_\varsigma(\nu)\)-modules. Let us also remark that, by [FP18a, corollary 3.8], the link state representation of TL\(_\varsigma(\nu)\) on \(\mathbb{L}_\varsigma\) is faithful if and only if \(\text{rad} \mathbb{L}_\varsigma = \{0\}\). These facts reflect the cellular structure of the valenced Temperley-Lieb algebra [GL96, GL98, FP18b].

Next, we give a diagram representation for vectors in \(V_n\) and \(\overline{V}_n\), which is analogous to the one introduced by I. Frenkel and M. Khovanov [FK97] (but we use different conventions, with applications to conformal field theory in mind). For this purpose, we introduce networks and link states with orientation on through-paths, turn-back paths, and defects, but not on loops or links. We call a collection of nonintersecting, non-self-intersecting planar loops and oriented paths within a rectangle an oriented network. We denote an unspecified orientation by ±:
\[
\begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}\right) = \begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}
\right) \quad \text{or} \quad \begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}
\right)
\]
(3.117)

We define the evaluation of the oriented network \(T\) to be the following complex number:
\[
(T) := \prod\{\text{the weights of all connected components in the oriented network } T\},
\]
(3.118)
where, in addition to the loop weight \(\nu = -q - q^{-1}\) in (3.108) (we emphasize that loops are never oriented), we assign all oriented through-paths and turn-back paths the following weights in \(\mathbb{C}\):
\[
\begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}\right) = \begin{array}{c}
\begin{array}{c}
\hline
\end{array}
\end{array}\right) \quad \text{etc.} = 1,
\]
(3.119)
clockwise turn-back path weight: \[ \text{etc.} = i q^{1/2}, \] (3.120)
counter-clockwise turn-back path weight: \[ \text{etc.} = -i q^{-1/2}, \] (3.121)

and when encountering clashing orientations, we assign the weight zero:

through-path weight: \[ \text{etc.} = 0, \] (3.122)
turn-back path weight: \[ \text{etc.} = 0. \] (3.123)

The choice of these weights becomes apparent soon. We also consider evaluations of formal linear combinations of oriented networks, obtained by linearly extending (3.118). Abusing terminology, we call them oriented networks too.

Analogously, we consider link patterns and link states with oriented defects. As in (3.106), we can produce oriented networks from pairs of such link states:

\[ \alpha = \ \uparrow \quad \beta = \ \uparrow \ \Rightarrow \quad \alpha \ | \ \beta = \cdots \cdots \uparrow. \] (3.124)

Analogously to (3.76), we define the bilinear tensor product \( \alpha \otimes \beta \) of link states (with oriented or non-oriented defects) via bilinear extension of the operation of concatenating \( \alpha \) to the left of \( \beta \):

\[ \alpha \otimes \beta := \begin{array} {c} \alpha \\ \otimes \\ \beta \end{array}. \] (3.125)

Now, for the basis vectors in \( V_{(1)} = \text{span} \{ \varepsilon_0, \varepsilon_1 \} \) and \( \tilde{V}_{(1)} = \text{span} \{ \varepsilon_0, \varepsilon_1 \} \), we use the notation \([CFS95, FK97]\)

\[ \varepsilon_0 = \uparrow, \quad \varepsilon_1 = \downarrow, \quad \varepsilon_0 = \uparrow, \quad \text{and} \quad \varepsilon_1 = \downarrow, \] (3.126)

and the standard bases \( \{ \varepsilon_{\ell_1} \otimes \varepsilon_{\ell_2} \otimes \cdots \otimes \varepsilon_{\ell_n} \mid \ell_1, \ldots, \ell_n \in \{0, 1\} \} \) and \( \{ \varepsilon_{\ell_1} \otimes \varepsilon_{\ell_2} \otimes \cdots \otimes \varepsilon_{\ell_n} \mid \ell_1, \ldots, \ell_n \in \{0, 1\} \} \) of \( V_n \) and \( \tilde{V}_n \) thus obtain a diagram notation via (3.125, 3.126): for instance,

\[ \varepsilon_0 \otimes \varepsilon_1 \otimes \varepsilon_1 = \uparrow \downarrow \downarrow \quad \text{and} \quad \varepsilon_0 \otimes \varepsilon_1 \otimes \varepsilon_1 = \uparrow \downarrow \downarrow \downarrow. \] (3.127)

We then extend this diagram representation linearly to all vectors in \( V_n \) and \( \tilde{V}_n \). For example, the singlet vectors \( s \) and \( \tilde{s} \) defined in (3.48) read

\[ s \quad \text{(3.48)} \quad \text{(3.126)} \quad i q^{1/2} \times \uparrow \downarrow - i q^{-1/2} \times \downarrow \uparrow, \] (3.128)

\[ \tilde{s} \quad \text{(3.48)} \quad \text{(3.126)} \quad i q^{1/2} \times \uparrow \downarrow - i q^{-1/2} \times \downarrow \uparrow. \] (3.129)

By lemmas 2.11 and 2.13, weights (3.119, 3.122) guarantee that evaluations of the oriented networks associated to graphical representations (3.126, 3.127) agree with the bilinear pairing (2.122):

\[ 1 \quad \text{(3.119)} \quad \text{(3.126)} \quad \varepsilon_0 \mid \varepsilon_0 \text{ (2.122)} = 1, \quad 1 \quad \text{(3.119)} \quad \text{(3.126)} \quad \varepsilon_1 \mid \varepsilon_1 \text{ (2.122)} = 1. \] (3.130)
Thanks to lemma 3.4, rules (3.50–3.55) and weights (3.120, 3.121, 3.123) amount to natural graphical rules for the \( \mathsf{TL}^m_n \)-action. Indeed, the left action (3.52) of the right generators is immediately governed by the rules for \( R_1 \):

\[
R_1(\varepsilon_0 \otimes \varepsilon_0) = 0, \quad R_1(\varepsilon_0 \otimes \varepsilon_1) = iq^{1/2}, \quad R_1(\varepsilon_1 \otimes \varepsilon_0) = -iq^{-1/2}, \quad R_1(\varepsilon_1 \otimes \varepsilon_1) = 0.
\]

(3.132)

and analogous rules hold for the right action (3.55) of the left generators. The left action (3.51) of the left generators (resp. right action (3.54) of the right generators) gives rise to additional graphical rules: we have

\[
\mathcal{J} \quad L_1(\varepsilon_0^{(0)}) = \bigotimes_{\bullet} = \bigotimes_{\bullet}.
\]

(3.136)

(3.137)

(We also recall from (3.70) that \( R_1 \mathcal{J} = \nu \), matching the fugacity weight (3.108).) Therefore, in light of lemma 3.4, if we identify the singlet vectors (3.128, 3.129) with simple links as above, then the above \( \mathsf{TL}^m_n \)-action agrees with the \( \mathsf{TL}^m_n \)-action discussed in section 3 B. This also gives a decomposition rule for nested links obtained from repeated application of the left generators from the left (for \( \mathcal{J} \)), or the right generators from the right (for \( \mathcal{J} \)). For example,

\[
\bigotimes_{\bullet} = \bigotimes_{\bullet}.
\]

(3.138)

(3.139)

Next, we verify that the above identifications (3.136–3.139) still respect the bilinear pairing \((\cdot \mid \cdot)\). For instance,

\[
(\mathcal{J} \mid \mathcal{J}) = (\bigotimes_{\bullet} \mid \bigotimes_{\bullet}) = (3.108) = (3.118) = (3.122) = \nu (1.24) = -q-q^{-1} (2.122) (4.48) (\mathcal{J} \mid \mathcal{J}).
\]

(3.140)

We observe that the right side of (3.140) is the bilinear pairing of the singlet vectors \( \mathcal{J} \) and \( \mathcal{J} \), while the left side of (3.140) is the evaluation of an oriented network corresponding to these vectors. Alternatively, we have

\[
(\mathcal{J} \mid \mathcal{J}) = -q \left( \bigotimes_{\bullet} \bigotimes_{\bullet} \right) + \left( \bigotimes_{\bullet} \bigotimes_{\bullet} \bigotimes_{\bullet} \right) - q^{-1} \left( \bigotimes_{\bullet} \bigotimes_{\bullet} \bigotimes_{\bullet} \right)
\]

(3.124)

which equals (3.140). Hence, we see that both choices of representations of \( \mathcal{J} \) and \( \mathcal{J} \) as link states with oriented defects yield the same network evaluation, which coincides with the bilinear pairing (\( \mathcal{J} \mid \mathcal{J} \)).

**Lemma 3.19.** For any two vectors \( \varpi \in \nabla_n \) and \( w \in \nabla_n \), the evaluation of an oriented network \( \varpi \mid w \) equals

\[
(\varpi \mid w) = (\varpi \mid w)
\]

(3.142)

for any choice of representations of the vectors \( \varpi \) and \( w \) as link states with oriented defects.
Proof. We prove (3.142) by induction on \( n \geq 1 \). The initial case \( n = 1 \) is governed by rules (3.130, 3.131). Hence, we assume that identity (3.142) holds for all \( n \in \{1, 2, \ldots, m - 1\} \) for some \( m \geq 2 \), and we prove that it holds for \( n = m \) too. By linearity, it suffices to split the analysis of \( (v \mid w) \) into the following four scenarios:

1. The oriented network \( v \mid w \) has one of the following four forms, for some appropriate oriented sub-network \( T \):

   \[
   \begin{array}{c}
   \begin{array}{c}
   T \\
   \oplus \\
   \oplus \\
   \end{array}
   \end{array},
   \begin{array}{c}
   \begin{array}{c}
   T \\
   \oplus \\
   \oplus \\
   \end{array}
   \end{array}
   \right) \tag{3.143}
   \]

   As an example, we consider the leftmost case of (3.143) with \( \pm \mapsto + \). Then, the vectors \( v \) and \( w \) have the forms

   \[
   v = v' \otimes \varepsilon_0 \quad \text{and} \quad w = w' \otimes \varepsilon_0 \tag{3.144}
   \]

   for some other vectors \( v' \in V_{m-1} \) and \( w' \in V_{m-1} \), and we have \( T = v' \mid w' \). The induction hypothesis gives

   \[
   (v \mid w) = (v' \mid w') \tag{3.147}
   \]

   On the other hand, we see from (3.143) that the evaluation of the oriented network \( v \mid w \) equals that of \( T \) times the evaluation of the rightmost through-path. The latter has weight one by (3.119). Therefore, (3.142) holds:

   \[
   (v \mid w) = (T) = (v \mid w) \tag{3.146}
   \]

   The other cases of (3.143) can be handled similarly.

2. The oriented network \( v \mid w \) has one of the following eight forms, for some appropriate oriented sub-network \( T \):

   \[
   \begin{array}{c}
   \begin{array}{c}
   T \\
   \oplus \\
   \oplus \\
   \end{array}
   \end{array},
   \begin{array}{c}
   \begin{array}{c}
   T \\
   \oplus \\
   \oplus \\
   \end{array}
   \end{array}
   \right) \tag{3.147}
   \]

   Again, we restrict our attention to the leftmost case of (3.147) with \( \mp \mapsto + \), and \( \pm \mapsto - \); the other cases can be handled similarly. Now, the network structure in (3.147) implies that the vectors \( v \) and \( w \) have the forms

   \[
   v = i_q^{1/2} v' \otimes \varepsilon_1 - i_q^{-1/2} v'' \otimes \varepsilon_0 \quad \text{and} \quad w = w' \otimes \varepsilon_1 \tag{3.148}
   \]

   for some other vectors \( v' := v_1 \otimes \varepsilon_0 \otimes v_2 \in \overline{V}_{m-1} \), \( v'' := v_1 \otimes \varepsilon_1 \otimes v_2 \in \overline{V}_{m-1} \), and \( w' \in \overline{V}_{m-1} \). Thus, factorization property (2.127) from lemma 2.13 and the induction hypothesis give

   \[
   (v \mid w) = i_q^{1/2} (v' \otimes \varepsilon_1 \mid w' \otimes \varepsilon_0) - i_q^{-1/2} (v'' \otimes \varepsilon_0 \mid w' \otimes \varepsilon_0) \tag{3.149}
   \]

   Now, the defect exiting \( T \) through its bottom has upward orientation in order to connect to the turnback on the rightmost node with downward orientation. Therefore, we see that \( T = v' \mid w' \), so we obtain

   \[
   (v \mid w) = i_q^{1/2} (T) \tag{3.150}
   \]

   Finally, we observe that deforming the turn-back path of \( v \mid w \) shown in (3.147) into a through-path gives the oriented network \( T \), which completes the induction step for this case:

   \[
   (v \mid w) = i_q^{1/2} (v \mid w) \tag{3.151}
   \]
3. The oriented network $\mathfrak{v} | w$ has one of the following eight forms, for some appropriate oriented sub-network $T$:

\[ \text{Diagram: } T, \text{ and } \text{Diagram: } T, \text{ and } \text{Diagram: } T, \text{ and } \text{Diagram: } T. \tag{3.152} \]

The proof of this case is very similar to that of item 3. We leave it to the reader.

4. The oriented network $\mathfrak{v} | w$ has the following form, for some appropriate sub-diagram $T$:

\[ \text{Diagram: } T. \tag{3.153} \]

where $T$ contains a through-path joining the ends of the exterior path to form a loop. This network structure in (3.153) implies that the vectors $\mathfrak{v}$ and $w$ have the forms

\[ \mathfrak{v} = i^q \frac{1}{2} \mathfrak{v}'' \otimes \mathfrak{v}_0 - i^q \frac{1}{2} \mathfrak{v}'' \otimes \mathfrak{v}_0 \quad \text{and} \quad w = i^q \frac{1}{2} w' \otimes \mathfrak{v}_0 - i^q \frac{1}{2} w'' \otimes \mathfrak{v}_0, \tag{3.154} \]

for some other vectors $\mathfrak{v}' := \mathfrak{v}_1 \otimes \mathfrak{v}_0 \otimes \mathfrak{v}_2 \in \mathcal{V}_{m-1}$, $\mathfrak{v}'' := \mathfrak{v}_1 \otimes \mathfrak{v}_1 \otimes \mathfrak{v}_2 \in \mathcal{V}_{m-1}$, $w' := \mathfrak{v}_0 \otimes \mathfrak{v}_0 \otimes \mathfrak{v}_2 \in \mathcal{V}_{m-1}$, and $w'' := \mathfrak{v}_0 \otimes \mathfrak{v}_1 \otimes \mathfrak{v}_2 \in \mathcal{V}_{m-1}$. Thus, the induction hypothesis gives

\[ \begin{align*}
(\mathfrak{v} | w) &= -q(\mathfrak{v}' \otimes \mathfrak{v}_1 | w' \otimes \mathfrak{v}_1) + (\mathfrak{v}'' \otimes \mathfrak{v}_0 | w' \otimes \mathfrak{v}_1) + (\mathfrak{v}' \otimes \mathfrak{v}_1 | w'' \otimes \mathfrak{v}_0) - q^{-1}(\mathfrak{v}'' \otimes \mathfrak{v}_0 | w'' \otimes \mathfrak{v}_0) \\
(\mathfrak{v}'' | w'') &= -q(\mathfrak{v}' \otimes \mathfrak{v}_1 | w' \otimes \mathfrak{v}_1) - q^{-1}(\mathfrak{v}'' \otimes \mathfrak{v}_0 | w'' \otimes \mathfrak{v}_0). \tag{3.155} \end{align*} \]

Now, the diagrams $\mathfrak{v}' | w'$ and $\mathfrak{v}'' | w''$ only differ by the orientation of the through-path joining the ends of the exterior path in the diagram (3.153):

\[ \begin{align*}
\mathfrak{v}' | w' &= \text{Diagram: } T, \quad \text{and} \quad \mathfrak{v}'' | w'' &= \text{Diagram: } T. \tag{3.156} \end{align*} \]

Therefore, their evaluations $(\mathfrak{v} | w)$ and $(\mathfrak{v}'' | w'')$ are equal by rules (3.118, 3.119), and furthermore, $\nu$ times either evaluation equals $(\mathfrak{v} | w)$ according to rules (3.108, 3.109) and formula (1.24) for $\nu$. Therefore, we have

\[ \begin{align*}
(\mathfrak{v} | w) &= -q(\mathfrak{v}' | w') - q^{-1}(\mathfrak{v}'' | w'') \tag{1.24} \leq \nu(\mathfrak{v}' | w') \tag{3.153} = (\mathfrak{v} | w). \tag{3.157} \end{align*} \]

This concludes the induction step.

The proof is complete.

It follows that the bilinear pairing $(\cdot | \cdot)$ is $\mathcal{TL}_\mathfrak{v}$-invariant in the following sense.

**Corollary 3.20.** Suppose $\text{max} \varsigma < p(q)$. For any valenced tangle $T \in \mathcal{TL}_\mathfrak{v}$ and vectors $\mathfrak{v} \in \mathcal{V}_\mathfrak{v}$ and $w \in \mathcal{V}_\mathfrak{v}$, we have

\[ (\mathfrak{v} | Tw) = (\mathfrak{v}T | w). \tag{3.158} \]

**Proof.** If $\varsigma = \bar{n}$ and $\mathfrak{w} = \bar{n}$ for some $n, m \in \mathbb{Z}_{>0}$, then asserted identity (3.158) follows from the definitions with lemma 3.19: indeed, either expression (3.158) equals the evaluation of an oriented network $\mathfrak{v} | Tw = \mathfrak{v}T | w$; e.g., for

\[ w = \downarrow \downarrow \uparrow \uparrow, \quad \mathfrak{v} = \downarrow \uparrow \downarrow \uparrow, \quad \text{and} \quad T = \text{Diagram: } T, \quad \tag{3.159} \]
the following network (rotated by $-\pi/2$ radians) represents either quantity \(3.158\):

\[
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\]

(3.160)

For the general case, combining the case of $\zeta = \bar{\mu}$ and $\varpi = \bar{\nu}$ with item 4 of lemma 2.13 and corollary 3.15, we have

\[
(v | Tw) \overset{(\ref{2.131})}{=} (v v_{\zeta} | I_{\zeta} Tw) \overset{(3.33)}{=} (v \tilde{P}_{\zeta} | (I_{\zeta} \tilde{P}_{\varpi}) I_{\varpi} w) \overset{(3.158)}{=} (v \tilde{P}_{\zeta} (I_{\zeta} \tilde{P}_{\varpi}) | I_{\varpi} w) \overset{(3.33)}{=} (v T \tilde{P}_{\varpi} | I_{\varpi} w) \overset{(2.131)}{=} (v T | w),
\]

which proves asserted identity \(3.158\).

\[\square\]

**Corollary 3.21.** Suppose $\max \zeta < p(q)$. For any vectors $v \in \mathcal{V}_{n_{\zeta}}$ and $w \in \mathcal{V}_{n_{\varpi}}$, we have

\[
(v \mid T w) = (v \mid \mathcal{P}_{\zeta}(w)) = (v \mid \mathcal{P}_{\varpi}(w)).
\]

(3.162)

**Proof.** The first equality in (3.162) follows by setting $T = \tau_{1}$ in (3.158) and using corollary 3.12. The second equality in (3.162) follows by replacing $w$ with $\mathcal{P}_{\varpi}(w)$ in the first equality and recalling the projection property $\mathcal{P}_{\zeta}^T = \mathcal{P}_{\varpi}$. \[\square\]

## 4. THE LINK STATE – HIGHEST-WEIGHT VECTOR CORRESPONDENCE

The purpose of this section is to explicate the connection between valenced link states $\alpha \in L_{\zeta}$ and certain vectors in $V_{\zeta}$. In particular, we show how valenced link patterns $\alpha \in L_{\zeta} \subset H_{\zeta}^{(s)}$ correspond to linearly independent highest-weight vectors (proposition 4.12). This fact gives rise to an embedding of $T L_{\zeta}(\nu)$-modules,

\[
L_{\zeta} \overset{(3.24)}{=} \bigoplus_{s \in E_{\zeta}} L_{\zeta}^{(s)} \hookrightarrow \bigoplus_{s \in E_{\zeta}} \tau_{L_{\zeta}} \odot H_{\zeta}^{(s)} \overset{(2.36)}{=} \tau_{L_{\zeta}} \odot H_{\zeta}.
\]

(4.1)

In particular, if $n_{\zeta} < p(q)$, then the $T L_{\zeta}(\nu)$-modules $\tau_{L_{\zeta}} \odot H_{\zeta}^{(s)}$ discussed in section 3 C are isomorphic to the standard modules $L_{\zeta}^{(s)}$ discussed in section 3 A. Supplemerting proposition 3.18, we also have an isomorphism of $T L_{\zeta}(\nu)$-modules,

\[
\tau_{L_{\zeta}} \odot V_{\zeta} \overset{(2.89)}{=} \bigoplus_{s \in E_{\zeta}} (s + 1) L_{\zeta}^{(s)} \quad \text{when} \quad n_{\zeta} < p(q).
\]

(4.2)

We thus understand this structure completely, thanks to Temperley-Lieb representation theory (e.g., [RSA14, FP18a]).

Conversely, we recall from item 3 of proposition 2.8 the direct-sum decomposition

\[
u_{\zeta} \odot V_{\zeta} \overset{(2.89)}{=} \bigoplus_{s \in E_{\zeta}} D_{\zeta}^{(s)} M_{(s)} \quad \text{when} \quad n_{\zeta} < p(q)
\]

(4.3)

of the type-one module $\nu_{\zeta} \odot V_{\zeta}$ into a direct sum of simple type-one submodules $M_{(s)}$, whose multiplicities are explicitly given by solving the recursion problem (2.66), also appearing in lemma 3.3 and corollary 2.9,

\[
D_{\zeta}^{(s)} \overset{(3.46)}{=} \dim L_{\zeta}^{(s)} \quad \text{for all} \quad \zeta \in \mathbb{Z}_{\geq 0}^{0},
\]

(4.4)

\[
\overset{(2.97)}{=} \dim H_{\zeta}^{(s)} \quad \text{when} \quad n_{\zeta} < p(q).
\]

(4.5)

Each multiplicity space $H_{\zeta}^{(s)}$ (2.35) comprises highest-weight vectors of weight $q^{s}$, and with $n_{\zeta} < p(q)$, these weights are all distinct (cf. (2.29, 2.30)) and the conformal-block vectors $u_{\zeta}^{(s)}$ spanning $H_{\zeta}^{(s)}$ are all linearly independent. However, if $n_{\zeta} \geq p(q)$, then proposition 2.8 only gives an embedding with multiplicities $\hat{D}_{\zeta}^{(s)}$ (2.86) perhaps smaller than $D_{\zeta}^{(s)}$,}

\[
\bigoplus_{s \in E_{\zeta}, s < p(q)} \hat{D}_{\zeta}^{(s)} M_{(s)} \overset{(2.87)}{=} \nu_{\zeta} \odot V_{\zeta}.
\]

(4.6)
In this case, some of the $K$-eigenvalues $q^\alpha$ become equal, the type-one module $\mathcal{U}_\rho \otimes V_\gamma$ is usually not semisimple, and some of the conformal-block vectors $u^\alpha_\gamma$ become linearly dependent (see appendix B for an example). The embedding (4.6) can nevertheless be strengthened to have multiplicities $D(\lambda)$, see proposition 4.13.

The main aim of this section is to construct a set of highest-weight vectors in $\mathcal{U}_\rho \otimes V_\gamma$ well-defined and linearly independent whenever $\max \varsigma < p(q)$. We call them “(valenced) link-pattern basis vectors” $w_\alpha$, indexed by valenced link patterns $\alpha$. These vectors are particularly amenable to diagram calculations, namely, the Temperley-Lieb action on them coincides with the diagram action of $\mathcal{L}_\gamma(\nu)$ on its standard modules. This observation gives the “link state – highest-weight vector correspondence” (proposition 4.12). Later, in proposition 5.14 in sections 5 C–5 D we use this correspondence to identify quotients of the highest-weight vector spaces with simple $\mathcal{L}_\gamma(\nu)$-modules.

A. From link states to link-state vectors

To begin, we consider the case of $\gamma = \tilde{\gamma}$. We define the link-pattern basis vectors $w_\alpha$ for all $\alpha \in \mathcal{L}_0 \otimes \mathcal{L}_1$, and prove that they are linearly independent highest-weight vectors in $\mathcal{U}_\rho \otimes V_\gamma$. These vectors and their $F$-descendants have a natural graphical representation, already appearing in section 3 D and also discussed in section 4 D. The link-pattern basis vectors are obtained by repeated insertion of (possibly nested) singlet vectors, corresponding to links, standard basis vectors $\varepsilon_0, \varepsilon_0$, corresponding to upward-oriented defects,

$$
\varepsilon_0 \overset{3.126}{=} \uparrow, \quad \varepsilon_0 \overset{3.126}{=} \uparrow.
$$

Precisely, the (nested) singlet vectors are inserted via (a repeated application of) the embedding of the trivial $U_q$-module $M(0)$ into the tensor product $U_q$-module $M(1) \otimes M(1)$ implemented by the left generator tangle $L_1$:

$$
\mathcal{F}_2^0(L_1) : M(0) \longrightarrow M(1) \otimes M(1), \quad \mathcal{F}_2^0(L_1)(\varepsilon_0^{(0)}) \overset{3.51}{=} \mathcal{F}_2^0(L_1)(\varepsilon_0^{(0)}) \overset{3.69}{=} \mathcal{F}_2^0(L_1)(\varepsilon_0^{(0)}).
$$

(If $q = 1$, the right side of (4.10) is not defined, but the map $\mathcal{F}_2^0(L_1)$ is still well-defined: in this case, it embeds the trivial $\mathfrak{sl}_2$-module into the tensor product of two of its fundamental modules, as discussed in appendix C. In the present section, we mainly assume $q \neq 1$, although most results apply verbatim to the case $q = 1$.)

Definition 4.1. For any link pattern $\alpha \in \mathcal{L}_0^{(n)}$, we recursively define the corresponding vector $w_\alpha$ as follows: we set $w_\emptyset := 1 \in \mathcal{L}$ for the empty link pattern $\emptyset \in \mathcal{L}_0$, and we define the other vectors $w_\alpha$ via the following recipe:

1. If $s = n$, then denoting by $\biguplus_n \in \mathcal{L}_0^{(n)}$ the link pattern that consists of $n$ defects,

$$
\biguplus_n := \mathcal{L} \quad \text{n defects},
$$

we set $w_{\biguplus_n} := \varepsilon_0 \otimes \varepsilon_0 \otimes \cdots \otimes \varepsilon_0 \overset{2.100}{=} \varepsilon_0^{(n)} \overset{3.126}{=} \varepsilon_0^{(n)} \overset{3.127}{=} \uparrow \cdots \uparrow$, (4.12)

2. If $s < n$, then assuming that all of the vectors $\{w_\alpha | \beta \in \mathcal{L}_0^{(n)}\}$ with $1 \leq m \leq n - 1$ have been defined, we define $w_\alpha$ for any $\alpha \in \mathcal{L}_0^{(n)}$ as follows. First, we choose a link joining two consecutive nodes in $\alpha$, and we consider the vector $w_\alpha$ associated to the link pattern $\hat{\alpha} \in \mathcal{L}_0^{(n-1)}$ obtained from $\alpha$ by removing the chosen link. Suppose that the removed link joined the $k$:th and $(k + 1)$:st nodes. Then with the vector $w_\hat{\alpha}$ already defined, we set

$$
w_\alpha := L_k w_\alpha \overset{3.65, B.20}{=} \mathcal{F}_2^0(L_1)^{(3.126, 3.69)} (q - q^{-1})^2 \mathcal{L}_{(1,1)}^{(3.65)} \otimes \mathcal{L}_{(1,1)}^{(3.126)} \otimes \mathcal{L}_{(1,1)}^{(3.127)} (w_\hat{\alpha}).
$$

Induction on the number of links in $\alpha$ and the commutation relation for the left generators $L_k$ from (3.11) show that $w_\alpha$ does not depend on the choice of the removed link.
We also linearly extend this to a map \( \alpha \mapsto w_\alpha \) sending any link state \( \alpha \in \mathbb{L}_n^{(s)} \) to a corresponding vector \( w_\alpha \in V_n \).

We similarly define the link-pattern basis vectors \( \mathbb{w}_\alpha \) for all link patterns \( \alpha \in \mathbb{P}_n \),

\[
\mathbb{w}_\alpha := \mathbb{w}_0 \otimes \mathbb{w}_0 \otimes \cdots \otimes \mathbb{w}_0 \quad (\text{n tensors})
\]

\[
\mathbb{w}_\alpha := \mathbb{w}_0 R_k \quad (3.67, B.22)
\]

Recalling the graphical calculus from section 3D, we may identify each vector \( w_\alpha \) (resp. \( \mathbb{w}_\alpha \)) with the diagram of \( \alpha \) (resp. \( \mathbb{P}_n \)) whose defects have an upward orientation: for instance (see also (3.138, 3.139)),

\[
\alpha = \quad \mathbb{w}_\alpha := \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J} = \quad ,
\]

\[
\mathbb{P}_n = \quad \mathbb{w}_\alpha := \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{J} = \quad .
\]

Rules (3.50–3.55) from section 3B amount to natural graphical rules for the \( \mathbb{T}_n \)-action on the vectors \( w_\alpha \) and \( \mathbb{w}_\alpha \), given by the \( \mathbb{T}_n \)-action on the link states \( \alpha \) and \( \mathbb{P}_n \): themselves:

**Lemma 4.2.** For all link states \( \alpha \in \mathbb{L}_m \) and link patterns \( \mathbb{P}_n \), and tangles \( \mathcal{T} \in \mathbb{T}_n \), we have

\[
w_{\mathcal{T} \alpha} = \mathcal{T} w_\alpha \quad \text{and} \quad \mathbb{w}_\alpha \mathcal{T} = \mathbb{w}_{\mathcal{T} \alpha}.
\]

**Remark.** This lemma is rather easy to verify using diagram representations for the link-state vectors via link states with upward-oriented defects (cf. section 3D and above) and the usual tangle action on link states (cf. section 3A).

We invite the reader to check this. Here, we give a proof relying on the formal definition 4.1.

**Proof.** We prove the left equation of (4.18); the right equation can be proven similarly. Recalling from lemma 3.1 that any tangle \( \mathcal{T} \in \mathbb{T}_n \) equals a polynomial in the generators \( L_i \) and \( R_j \), it suffices to show that

\[
w_{L_i \alpha} = L_i w_\alpha \quad \text{and} \quad w_{R_j \alpha} = R_j w_\alpha
\]

for all link patterns \( \alpha \in \mathbb{P}_m \) and indices \( i \in \{1, 2, \ldots, m+1\} \) and \( j \in \{1, 2, \ldots, m-1\} \), by linearity and item 2 of lemma 3.4. The left equation of (4.19) follows from definition 4.1 by choosing \( k = i \) and \( \alpha \mapsto L_i \alpha \), so \( \bar{L}_i \alpha = \alpha \) and

\[
w_{L_i \alpha} = L_i w_{\bar{L}_i \alpha} = L_i w_\alpha.
\]

We prove the right equation of (4.19) by induction on \( m \geq 2 \). In the initial case with \( m = 2 \), we necessarily have \( k = j = 1 \), and we consider two cases as in definition 4.1.

1. If \( \alpha = \alpha_2 \in \mathbb{P}_2^{(2)} \), then \( R_1 \alpha = R_1 \alpha_2 = 0 \) by (3.5, 3.110, 4.11), so \( w_{R_1 \alpha_2} = 0 = R_1 w_{\alpha_2} \) by (3.52, 4.12).

2. If \( \alpha = \alpha_1 \in \mathbb{P}_2^{(0)} \), then \( R_1 \alpha = R_1 \alpha_1 = \nu \) by (3.5, 3.108, 4.7), so \( w_{R_1 \alpha_1} = \nu = R_1 w_{\alpha_1} \) by (3.52, 4.7).

Next, we assume that \( w_{R_j \alpha} = R_j w_\alpha \) for all link patterns \( \hat{\alpha} \in \mathbb{P}_n \) with \( 2 \leq n \leq m-1 \), and we consider a link pattern \( \alpha \in \mathbb{P}_m \). We again consider two cases as in definition 4.1.

1. If \( \alpha = \alpha_{m} \in \mathbb{P}_m^{(m)} \), then definition 4.1 gives

\[
R_j w_{\alpha m} = R_j w_{\alpha_{m+1}} = R_j w_{\alpha_{m+2}} = \cdots = R_j w_{\alpha_{m+k}} = R_j w_{\alpha_{m+k+1}} = R_j w_{\alpha_{m+k+2}} = \cdots
\]

\[
R_j w_{\alpha} = R_j \bar{w}_{\alpha} = \begin{cases} w_{\alpha}, & k = j \pm 1, \\ \nu w_{\alpha}, & k = j, \\ L_k R_{j-2} w_{\alpha}, & k \leq j - 2, \\ L_{j-2} R_k w_{\alpha}, & j \leq k - 2, \end{cases}
\]

recalling relations (3.11) of the left and right generator tangles. Therefore, it remains to verify that the right side of (4.22) coincides with \( w_{R_j \alpha} \). We consider these four cases:
(a): When \( k = j \pm 1 \), we have \( R_{j} \alpha = \hat{\alpha} \), so indeed, \( w_{R_{j} \alpha} = w_{\hat{\alpha}} \).

(b): When \( k = j \), we have \( R_{j} \alpha = \nu \hat{\alpha} \), so indeed, \( w_{R_{j} \alpha} = \nu w_{\hat{\alpha}} \).

(c): When \( k \leq j - 2 \), we have \( \hat{R}_{j} \alpha = R_{j-2} \hat{\alpha} \) and the induction hypothesis applied to \( \hat{\alpha} \) shows that

\[
L_{k} R_{j-2} w_{\hat{\alpha}} \overset{(4.19)}{=} L_{k} w_{R_{j-2} \hat{\alpha}} \overset{(4.13)}{=} w_{R_{j} \alpha}.
\] (4.23)

(d): When \( j \leq k - 2 \), we have \( \hat{R}_{j} \alpha = R_{j} \hat{\alpha} \) and the induction hypothesis applied to \( \hat{\alpha} \) shows that

\[
L_{k-2} R_{j} w_{\hat{\alpha}} \overset{(4.19)}{=} L_{k-2} w_{R_{j} \hat{\alpha}} \overset{(4.13)}{=} w_{R_{j} \alpha}.
\] (4.24)

This finishes the induction step for the right equation of (4.19), thus concluding the proof of the lemma.

We also obtain an explicit formula for the action of the Temperley-Lieb generators \( U_{j} \) on the link-pattern basis vectors. To state this formula, we define for all \( j \in \{1, 2, \ldots, n-1\} \) the \( j \)th cutting map \( \wp_{j} : \text{LP}_{n}^{(s)} \to \text{LP}_{n}^{(s)} \) by linear extension of the following action on link patterns \( \alpha \in \text{LP}_{n}^{(s)} \) (for a formal definition, see [KP16, section 3.3]):

- if a link joins the \( j \)th and \((j+1)\)st nodes of \( \alpha \), then we set \( \wp_{j}(\alpha) := \alpha \),
- otherwise, \( \wp_{j} \) acts as illustrated below:

\[
\begin{align*}
\text{\begin{tabular}{c}
\includegraphics[width=0.3\textwidth]{diagram1.png}
\end{tabular}} & \quad \mapsto \quad \begin{tabular}{c}
\includegraphics[width=0.3\textwidth]{diagram2.png}
\end{tabular} & \quad \mapsto \quad \begin{tabular}{c}
\includegraphics[width=0.3\textwidth]{diagram3.png}
\end{tabular}
\end{align*}
\]

\[
\begin{align*}
\text{\begin{tabular}{c}
\includegraphics[width=0.3\textwidth]{diagram4.png}
\end{tabular}} & \quad \mapsto \quad \begin{tabular}{c}
\includegraphics[width=0.3\textwidth]{diagram5.png}
\end{tabular} & \quad \mapsto \quad \begin{tabular}{c}
\includegraphics[width=0.3\textwidth]{diagram6.png}
\end{tabular}
\end{align*}
\]

\[
\begin{align*}
\text{\begin{tabular}{c}
\includegraphics[width=0.3\textwidth]{diagram7.png}
\end{tabular}} & \quad \mapsto \quad 0
\end{align*}
\]

**Corollary 4.3.** For all \( \alpha \in \text{LP}_{n} \) and for all \( j \in \{1, 2, \ldots, n-1\} \), we have

\[
U_{j} w_{\alpha} = \begin{cases} 
0, & \text{if defects touch both the } j\text{th and } (j+1)\text{st nodes in } \alpha, \\
\nu w_{\alpha}, & \text{if a link joins the } j\text{th and } (j+1)\text{st nodes of } \alpha, \\
w_{\wp_{j}(\alpha)}, & \text{otherwise.}
\end{cases}
\] (4.26)

Similarly, this corollary holds for the right \( U_{j} \)-action after the symbolic replacements \( \alpha \mapsto \overline{\alpha} \), \( \text{LP} \mapsto \overline{\text{LP}} \), and \( w_{\alpha} \mapsto w_{\overline{\alpha}} \).

**Proof.** This immediately follows from setting \( T = U_{j} \) in lemma 4.9.

Our next aim is to prove that the link-pattern basis vectors \( w_{\alpha} \) and \( w_{\overline{\alpha}} \) are highest-weight vectors (lemma 4.4) and linearly independent (lemma 4.7). We also collect other useful properties of these vectors (lemma 4.5 and corollary 4.6).

**Lemma 4.4.** Suppose \( q \in \mathbb{C}^{\times} \setminus \{\pm 1\} \). We have

\[
\{ w_{\alpha} \mid \alpha \in \text{LP}_{n}^{(s)} \} \subset H_{n}^{(s)} \quad \text{and} \quad \{ w_{\overline{\alpha}} \mid \alpha \in \text{LP}_{n}^{(s)} \} \subset \mathcal{R}_{n}^{(s)}. \] (4.27)
Proof. By linearity, we may assume that \( \alpha \) is a link pattern. We prove (4.27) by induction on the number of links in \( \alpha \). In the initial case \( \alpha \) has no links, so \( \alpha = \varnothing \in \mathcal{L}_0^{(s)} \) for some \( s \in \mathbb{Z}_{\geq 0} \), and \( w_{\alpha} = w_{\varnothing} = \theta_0^{(s)} \in H_n^{(s)} \) by (2.100, 4.12).

Next, we let \( \ell \in \mathbb{Z}_{>0} \) and assume that (4.27) holds for all link patterns with at most \( \ell - 1 \) links, and we let \( \alpha \in \mathcal{L}_n^{(s)} \) have \( \ell \) links. Now, if \( \hat{\alpha} \) is a link pattern obtained from dropping a link from \( \alpha \), then \( \hat{\alpha} \) has \( \ell - 1 \) links, so
\[
\alpha \in \mathcal{L}_n^{(s)} \implies \hat{\alpha} \in \mathcal{L}_{n-2}^{(s)} \quad \text{(4.27)}
\]
\[
w_{\alpha} \in H_n^{(s)} \quad \text{and} \quad w_{\hat{\alpha}} = H_{n-2}^{(s)} \quad \text{and} \quad w_{\alpha} = L_{k}w_{\hat{\alpha}} \in H_n^{(s)}, \quad \text{(4.28)}
\]
recalling from corollary 3.8 that the action of \( L_k \) is a \( U_{1}^{(s)} \)-homomorphism that respects the \( s \)-grading. This proves the induction step and thus the first assertion of (4.27). The second assertion can be proven similarly.

As a tool to show that \( w_{\alpha} \) and \( \varpi_{\sigma} \) indexed by link patterns are linearly independent, we consider walk representations for them and the standard basis vectors. For each walk \( \varrho = (r_1, r_2, \ldots, r_n) \) over \( \bar{n} \) as in (2.58), we set
\[
\varepsilon_{\bar{n}}^{(r_1)} := \varepsilon_{\ell_1} \otimes \varepsilon_{\ell_2} \otimes \cdots \otimes \varepsilon_{\ell_n} \quad \text{and} \quad \varepsilon_{\bar{n}}^{(r_2)} := \varepsilon_{\ell_1} \otimes \varepsilon_{\ell_2} \otimes \cdots \otimes \varepsilon_{\ell_n},
\]
where \( \ell_1, \ldots, \ell_n \in \{0, 1\} \) are the unique indices such that the heights \( r_j \) of the walk \( \varrho \) for all \( j \in \{1, 2, \ldots, n\} \), are
\[
r_j = \sum_{i=1}^{j} (1 - 2\ell_i).
\]

By lemma 2.11, these vectors are the useful orthogonality property
\[
(\varepsilon_{\bar{n}}^{(r_1)} | \varepsilon_{\bar{n}}^{(r_2)}) = \delta_{r_1, r_2} \quad \text{(4.31)}.
\]
The set of walks over \( \bar{n} \) has a partial order defined for any two walks \( \varrho = (r_1, r_2, \ldots, r_n) \) and \( \varrho' = (r'_1, r'_2, \ldots, r'_n) \) by
\[
\varrho \preceq \varrho' \quad \text{if and only if} \quad r_i \leq r'_i \quad \text{for all} \quad i \in \{0, 1, \ldots, n\}.
\]
We write \( \varrho < \varrho' \) if \( \varrho \preceq \varrho' \) and \( \varrho \neq \varrho' \), and we say that \( \varrho \) and \( \varrho' \) are incomparable if we have neither \( \varrho \preceq \varrho' \) nor \( \varrho' \preceq \varrho \). For later use, we record the following obvious fact:
\[
\varrho < \varrho' \quad \text{or} \quad \varrho \text{ and } \varrho' \text{ are incomparable} \implies \quad r_i < r'_i \text{ for some } i \in \{0, 1, \ldots, n\}.
\]
Next, we recall from [FP18a, section 4] the notion of a walk representation of a link pattern \( \alpha \in \mathcal{L}_n \) (or \( \sigma \in \mathcal{D}_n \)):
\[
\varrho_{\alpha} := (r_1^{(\alpha)}, r_2^{(\alpha)}, \ldots, r_n^{(\alpha)})
\]
denotes the walk over \( \bar{n} \) associated to \( \alpha \), that is, the walk whose heights \( r_j^{(\alpha)} \) are determined by the links and defects in \( \alpha \) as follows: \( r_1^{(\alpha)} = 1 \), and for \( j \in \{2, 3, \ldots, n\} \),
\[
r_j^{(\alpha)} = \begin{cases} r_{j-1}^{(\alpha)} + 1, & \text{the } j:\text{th node in } \alpha \text{ has a defect or left endpoint of a link,} \\ r_{j-1}^{(\alpha)} - 1, & \text{the } j:\text{th node in } \alpha \text{ has a right endpoint of a link.} \end{cases}
\]

Lemma 4.5. For each \( \alpha \in \mathcal{L}_n \), write the decomposition of \( \varrho_{\alpha} \) over the standard basis as
\[
w_{\alpha} = \sum_{\ell_1, \ell_2, \ldots, \ell_n \in \{0, 1\}} C_{\ell_1, \ell_2, \ldots, \ell_n}^{(\alpha)} \varepsilon_{\ell_1} \otimes \varepsilon_{\ell_2} \otimes \cdots \otimes \varepsilon_{\ell_n}
\]
for some coefficients \( C_{\ell_1, \ell_2, \ldots, \ell_n}^{(\alpha)} \in \mathbb{C} \). Then, the following hold:

1. Suppose a link joins the \( k:\text{th} \) and \( (k+1)\text{st} \) nodes in \( \alpha \), and let \( \hat{\alpha} \in \mathcal{L}_{n-2} \) be the link pattern obtained from \( \alpha \) by removing this link. Then, the coefficients of \( w_{\hat{\alpha}} \) in (4.36) are uniquely determined from those of \( w_{\hat{\alpha}} \) via
\[
\begin{align*}
C_{\ell_1, \ell_2, \ldots, \ell_k, 0, 1, \ell_{k+2}, \ldots, \ell_n}^{(\alpha)} &= iy^{1/2} C_{\ell_1, \ell_2, \ldots, \ell_{k-1}, \ell_{k+2}, \ldots, \ell_n}^{(\alpha)}, \\
C_{\ell_1, \ell_2, \ldots, \ell_{k-1}, 1, 0, \ell_{k+2}, \ldots, \ell_n}^{(\alpha)} &= -iy^{-1/2} C_{\ell_1, \ell_2, \ldots, \ell_{k-1}, \ell_{k+2}, \ldots, \ell_n}^{(\alpha)}, \\
C_{\ell_1, \ell_2, \ldots, \ell_{k-1}, 0, 0, \ell_{k+2}, \ldots, \ell_n}^{(\alpha)} &= C_{\ell_1, \ell_2, \ldots, \ell_{k-1}, 1, 1, \ell_{k+2}, \ldots, \ell_n}^{(\alpha)} = 0.
\end{align*}
\]


2. We have

\[ C_{\ell_1, \ell_2, \ldots, \ell_n}^{(\alpha)} \neq 0 \quad \implies \quad \sum_{i=1}^{j} (1 - 2\ell_i) \leq r_j^{(\alpha)} \quad \text{for all } j \in \{1, 2, \ldots, n\}. \quad (4.38) \]

Similarly, this lemma holds after the symbolic replacements \( \alpha \mapsto \bar{\alpha}, \) \( \mathbb{L}P \mapsto \overline{\mathbb{L}P}, \) \( w_\alpha \mapsto w_{\bar{\alpha}}, \) and \( \varepsilon \mapsto \varepsilon. \)

**Proof.** We prove items 1 and 2 as follows:

1. We can always expand \( w_\alpha \) in the form (4.36), so the task is check that the coefficients have the asserted form:

\[ w_\alpha \overset{4.15}{=} L_k w_{\bar{\alpha}} \overset{3.51}{=} \sum_{\ell_1, \ldots, \ell_k \in (0,1)} C_{\ell_1, \ldots, \ell_k, \ldots, \ell_n}^{(\alpha)} \varepsilon_{\ell_1} \otimes \cdots \otimes \varepsilon_{\ell_{k-1}} \otimes \varepsilon_{\ell_k} \otimes \cdots \otimes \varepsilon_{\ell_n}. \]

By (3.48, 4.36), this proves asserted formulas (4.37) for the coefficients in (4.36).

2. We prove (4.38) by induction on \( n \in \mathbb{Z}_{>0}. \) The initial case with \( n = 1 \) is trivial. Assuming (4.38) holds for all link patterns \( \beta \in \mathbb{L}P_{n-2}, \) we prove that it holds for all link patterns \( \alpha \in \mathbb{L}P_n. \) If \( \alpha \) has no links, then we trivially have

\[ \alpha = \sqcup n \in \mathbb{L}P_n^{(\alpha)} \implies \quad w_\alpha = w_{\sqcup n} = \varepsilon_0 \otimes \varepsilon_0 \otimes \cdots \otimes \varepsilon_0, \quad (4.39) \]

\[ \sum_{j=1}^{n} (1 - 2\ell_j) = j = r_j^{(\alpha)} \quad \text{for all } j \in \{1, 2, \ldots, n\}. \quad (4.40) \]

Therefore, we assume that \( \alpha \in \mathbb{L}P_n^{(s)} \) with \( s < n, \) so \( \alpha \) contains a link joining the \( k:th \) and \((k+1):st\) nodes for some \( k \in \{1, 2, \ldots, n-1\}. \) We let \( \hat{\alpha} \) be the link pattern obtained by removing this link from \( \alpha. \) Then, item 1 gives

\[ C_{\ell_1, \ldots, \ell_n}^{(\alpha)} \neq 0 \quad \implies \quad C_{\ell_1, \ldots, \ell_{k-1}, \ell_{k+2}, \ldots, \ell_n}^{(\hat{\alpha})} \neq 0 \quad \text{and} \quad (\ell_k, \ell_{k+1}) \in \{(0, 1), (1, 0)\}. \quad (4.41) \]

With \( \hat{\alpha} \in \mathbb{L}P_n^{(s)} \), the induction hypothesis shows that

\[ \sum_{1 \leq i \leq j \atop i \neq k, k+1} (1 - 2\ell_i) \leq \begin{cases} \ell_{j-1}^{(\alpha)}, & j \in \{1, 2, \ldots, k-1\}, \\ \ell_{j-2}^{(\hat{\alpha})} + (1 - 2\ell_k) + (1 - 2\ell_{k+1}), & j \in \{k+2, k+3, \ldots, n\}. \end{cases} \quad (4.42) \]

Therefore, we have

\[ \sum_{i=1}^{j} (1 - 2\ell_i) \overset{4.42}{=} \begin{cases} \ell_{j-1}^{(\alpha)} + (1 - 2\ell_k), & j \in \{1, 2, \ldots, k-1\}, \\ \ell_{j-2}^{(\hat{\alpha})} + (1 - 2\ell_k) + (1 - 2\ell_{k+1}), & j \in \{k+2, k+3, \ldots, n\}. \end{cases} \quad (4.43) \]

On the other hand, for the walks \( q_\alpha \) and \( q_{\hat{\alpha}} \) associated to \( \alpha \) and \( \hat{\alpha}, \) we have

\[ r_j^{(\alpha)} \overset{4.35}{=} \begin{cases} r_j^{(\alpha)}, & j \in \{1, 2, \ldots, k-1\}, \\ r_{j-1}^{(\alpha)} + 1, & j = k, \\ r_{j-1}^{(\hat{\alpha})}, & j \in \{k+2, k+3, \ldots, n\}. \end{cases} \quad (4.44) \]

With \( (\ell_k, \ell_{k+1}) \in \{(0, 1), (1, 0)\}, \) we have \((1 - 2\ell_k) \in \{-1\} \) and \((1 - 2\ell_k) + (1 - 2\ell_{k+1}) = 0, \) so (4.43, 4.44) imply

\[ \sum_{i=1}^{j} (1 - 2\ell_i) \overset{4.43}{\leq} \overset{4.44}{\leq} \begin{cases} \ell_{j-1}^{(\alpha)} + 1 = r_k^{(\alpha)}, & j \in \{1, 2, \ldots, k-1\}, \\ \ell_{j-2}^{(\hat{\alpha})} = r_j^{(\hat{\alpha})}, & j \in \{k+2, k+3, \ldots, n\}. \end{cases} \quad (4.45) \]

Hence, (4.38) holds for all \( \alpha \in \mathbb{L}P_n^{(s)} \) with \( s < n \) too. This completes the induction step.
The corresponding assertions for \( \varpi_\sigma \) can be proven similarly.

**Corollary 4.6.** Let \( \alpha \in \text{LP}_n^{(s)} \). Then for any walk \( \varrho \) over \( \vec{n} \) such that \( \varrho_\alpha \) and \( \varrho \) are incomparable or \( \varrho_\alpha \preceq \varrho \), we have

\[
(\pi^{\varrho}_n \mid w_\alpha) = \delta_{\varrho_\alpha, \varrho} (\iota^{1/2})^{n-s}.
\]

(4.46)

Similarly, if \( \varpi \in \overline{\text{LP}}_n^{(s)} \), then for any walk \( \varrho \) over \( \vec{n} \) such that \( \varrho_\alpha \) and \( \varrho \) are incomparable or \( \varrho_\alpha \preceq \varrho \), we have

\[
(\varpi \mid w_\alpha) = \delta_{\varrho_\alpha, \varrho} (\iota^{1/2})^{n-s}.
\]

(4.47)

**Proof.** If \( \varrho_\alpha \preceq \varrho \) or the walks \( \varrho_\alpha \) and \( \varrho \) are incomparable, then property (4.33) and lemma 4.5 imply that the coefficient of \( \epsilon^{\alpha}_n \) in decomposition (4.36) of \( w_\alpha \) equals zero. Claim (4.46) then follows from the orthogonality property (4.31). On the other hand, if \( \varrho = \varrho_\alpha \), then (4.46) is a straightforward calculation. Identity (4.47) can be proven similarly.

**Lemma 4.7.** Suppose \( q \in \mathbb{C}^\times \setminus \{ \pm 1 \} \). The following hold:

1. The collection \( \{ w_\alpha \mid \alpha \in \text{LP}_n \} \) is a linearly independent subset of \( H_n \).

2. For each \( s \in E_n \), the collection \( \{ w_\alpha \mid \alpha \in \text{LP}_n^{(s)} \} \) is a linearly independent subset of \( H_n^{(s)} \).

Similarly, this lemma holds after the symbolic replacements \( w_\alpha \mapsto \varpi_\sigma, \alpha \mapsto \vec{\alpha}, \text{LP} \mapsto \overline{\text{LP}}, \) and \( H \mapsto \overline{H} \).

**Proof.** We prove the lemma for \( w_\alpha \), for the proof in the case of \( \varpi_\sigma \) is similar. In light of lemma 4.4, it suffices to show that the collection \( \{ w_\alpha \mid \alpha \in \text{LP}_n \} \) is linearly independent. For this purpose, we suppose that

\[
\sum_{\alpha \in \text{LP}_n} c_\alpha w_\alpha = 0
\]

(4.48)

is a vanishing linear combination of vectors in this set, where \( c_\alpha \in \mathbb{C} \) are some constants. Taking the bilinear pairing of (4.48) with the vectors \( \pi^{\varrho}_n \) for all walks \( \varrho = \varrho_\vec{n} \), having \( \vec{\beta} \in \overline{\text{LP}}_n \), we obtain the system of equations

\[
\sum_{\alpha \in \text{LP}_n} M_{\beta, \alpha} c_\alpha = 0, \quad \text{where} \quad M_{\beta, \alpha} := (\pi^{\varrho}_n \mid w_\alpha).
\]

(4.49)

By corollary 4.6, the matrix \( M = (M_{\beta,\alpha}) \) is upper-triangular when its column order respects the partial order \( \preceq \) of the walks over \( \vec{n} \). Therefore, we have \( c_\alpha = 0 \) for all \( \alpha \in \text{LP}_n \), so the collection \( \{ w_\alpha \mid \alpha \in \text{LP}_n \} \) is linearly independent.

**B. The link state – highest-weight vector correspondence**

Next, we construct the (valenced) link-pattern basis vectors \( w_\alpha \) for \( \alpha \in \text{LP}_n \) (resp. \( \varpi_\sigma \) for \( \sigma \in \overline{\text{LP}}_n \)). In lemmas 4.8 and 4.11, we prove that these are linearly independent highest-weight vectors in \( \text{H}_n \) (resp. in \( \overline{\text{H}}_n \)). We gather important properties of this “link state – highest-weight vector correspondence” in proposition 4.12.

As a warm-up, let us consider the vectors \( \hat{\text{P}}_n(w_\alpha) \) obtained from \( w_\alpha \) via the projector \( \hat{\text{P}}_n : \text{H}_n \to \mathbb{C} \), defined in (2.114). Because it is a \( \text{U}_q \)-homomorphism, the images of \( w_\alpha \) are highest-weight vectors (cf. lemmas A.2 and 4.4). However, they are not all linearly independent, because the map \( \text{P}_n \) has a nontrivial kernel. In fact, the kernel can be understood explicitly, in terms of “special link patterns” (4.86) discussed in corollary 4.18 below (see also [FP18a, appendix B]). (We recall from item 3 of lemma A.2 that \( \ker \text{P}_n = \ker \text{P}_\text{L} \); corollary 4.18 actually concerns the latter).

To disregard this kernel, we utilize the link-state embedding \( I_\varsigma : \text{L}_\varsigma \to \text{L}_n \) and its reflection \( \check{I}_\varsigma = I_\varsigma \) from (3.32, 3.37, 3.38), satisfying (3.43). For any multiindex \( \varsigma \) as in (1.34, 1.35), we define the vectors \( w_\alpha \) and \( \varpi_\sigma \) as

\[
w_\alpha := \hat{\text{P}}_n(w_{I, \alpha}) \quad \text{and} \quad \varpi_\sigma := \check{\text{P}}_n(\varpi_{\sigma P_\varsigma})
\]

(4.50)

for all valenced link states \( \alpha \in \text{L}_\varsigma \) and \( \sigma \in \overline{\text{L}}_\varsigma \). Recalling corollary 3.15, we can illustrate the vectors \( w_\alpha \) as

\[
\varrho \alpha = \begin{array}{c}
\includegraphics[scale=0.5]{Diagram1}
\end{array} \quad \text{or} \quad \begin{array}{c}
\includegraphics[scale=0.5]{Diagram2}
\end{array}
\]

(3.32, 3.95) \quad \begin{array}{c}
\includegraphics[scale=0.5]{Diagram3}
\end{array}

(4.16)

w_\alpha = \begin{array}{c}
\includegraphics[scale=0.5]{Diagram4}
\end{array}

(4.51)

The vectors \( \varpi_\sigma \) have a similar illustration.
Lemma 4.8. Suppose $\max \varsigma < p(q)$. We have
\[ \{ w_\alpha \mid \alpha \in L_\varsigma^{(i)} \} \subset H_\varsigma^{(i)} \quad \text{and} \quad \{ \varpi_\sigma \mid \sigma \in L_\varsigma^{(i)} \} \subset R_\varsigma^{(i)}. \] (4.52)

Proof. Lemma 4.4 constitutes the special case $\varsigma = \bar{n}$ for some $n \in \mathbb{Z}_{>0}$. From this, we get the general case $\varsigma \in \mathbb{Z}_{>0}^o$.
\[ \alpha \in L_\varsigma^{(i)} \implies I_\varsigma \alpha \in L_\varsigma^{(i)} \implies w_{I_\varsigma \alpha} \in H_\varsigma^{(i)} \quad \text{RespGradeProjhat} \quad w_\alpha \overset{(4.50)}{=} \hat{\Psi}_\varsigma(w_{I_\varsigma \alpha}) \in H_\varsigma^{(i)}, \] (4.53)
as $\hat{\Psi}_\varsigma$ is a $U_q$-homomorphism. This proves the first assertion of (4.52). The second assertion can be proven similarly. \hfill \Box

The next lemma implies that the linear map $\alpha \mapsto w_\alpha$ (resp. $\sigma \mapsto \varpi_\sigma$) respects the action of the valenced Temperley-Lieb algebra on one hand, on its link state module $L_\varsigma$ (resp. $\hat{L}_\varsigma$) and, on the other hand, on $\mathcal{V}_\varsigma$ (resp. $\hat{\mathcal{V}}_\varsigma$). In particular, the diagram action on the valenced link states $\alpha$ and $\bar{\sigma}$ agrees with the graphical rules for the $\mathcal{T}_{\mathcal{V}_\varsigma}$-action on the corresponding vectors $w_\alpha$ and $\varpi_\sigma$ represented as valenced link states with upward-oriented defects (4.51).

Lemma 4.9. Suppose $\max(\varsigma, \varpi) < p(q)$. For all valenced link states $\alpha \in L_\varsigma$ and $\bar{\sigma} \in \hat{L}_\varsigma$ and valenced tangles $T \in \mathcal{T}_{\mathcal{V}_\varsigma}$, we have
\[ w_{T_\alpha} = T w_\alpha \quad \text{and} \quad \varpi_{T_\bar{\sigma}} = \varpi_{T \bar{\sigma}} \] (4.54)

Proof. Lemma 4.2 gives the special case of $\varsigma = \bar{n}$ and $\varpi = \bar{m}$ for some $n, m \in \mathbb{Z}_{>0}$. For general $\varsigma, \varpi \in \mathbb{Z}_{>0}^o$, we have
\[ w_{T_\alpha} \overset{(4.50)}{=} \hat{\Psi}_\varsigma(w_{I_\varsigma T_\alpha}) \overset{(3.33)}{=} \hat{\Psi}_\varsigma(w_{I_\varsigma T \bar{P}_\varpi \bar{I}_\varsigma \alpha}) \overset{(4.54)}{=} \hat{\Psi}_\varsigma(I_\varsigma T \bar{P}_\varpi w_{\bar{I}_\varsigma \alpha}) \overset{(3.95)}{=} \hat{\Psi}_\varsigma(I_\varsigma T w_\alpha) \overset{(3.33)}{=} \hat{\Psi}_\varsigma(I_\varsigma T \bar{P}_\varpi \bar{J}_\varpi(w_\alpha)) \overset{(3.87)}{=} T w_\alpha \] (4.55)
for any valenced tangle $T \in \mathcal{T}_{\mathcal{V}_\varsigma}$ and for any valenced link state $\alpha \in L_\varsigma$. This yields the first equation of (4.54). Similar work shows the second equation of (4.54). \hfill \Box

To prove the linear independence of the valenced link-pattern basis vectors $w_\alpha$ and $\varpi_\sigma$ (lemma 4.11), we begin with an auxiliary observation.

Lemma 4.10. Suppose $\max(\varsigma, \varpi) < p(q)$. We have
\[ \{ w_{I_\varsigma \alpha} \mid \alpha \in L_\varsigma \} = \{ w_{P_\varsigma \alpha} \mid \alpha \in L_\varsigma \} \subset \text{im } \Psi_\varsigma, \] (4.56)
and similarly,
\[ \{ \varpi_{P_\varsigma \bar{\sigma}} \mid \bar{\sigma} \in \hat{L}_\varsigma \} = \{ \varpi_{P_\varsigma \bar{\sigma}} \mid \bar{\sigma} \in \hat{L}_\varsigma \} \subset \text{im } \Psi_\varsigma. \] (4.57)

Proof. Commuting diagram (3.38) shows that $\text{im } I_\varsigma(\cdot) = \text{im } P_\varsigma(\cdot)$, which implies the equality in (4.56). Using homomorphism-like property (4.54) from lemma 4.9 and corollary 3.12, we obtain
\[ \Psi_\varsigma(w_{I_\varsigma \alpha}) \overset{(3.84)}{=} w_{P_\varsigma I_\varsigma \alpha} \overset{(3.33)}{=} w_{I_\varsigma \alpha}. \] (4.58)
This proves the the first asserted inclusion in (4.56). The second asserted formula (4.57) can be proven similarly. \hfill \Box

Lemma 4.11. Suppose $\max(\varsigma, \varpi) < p(q)$. The following hold:

1. The collection $\{ w_\alpha \mid \alpha \in LP_\varsigma \}$ is a linearly independent subset of $H_\varsigma$.
2. For each $s \in E_\varsigma$, the collection $\{ w_\alpha \mid \alpha \in LP_\varsigma^{(s)} \}$ is a linearly independent subset of $H_\varsigma^{(s)}$.

Similarly, this lemma holds after the symbolic replacements $w_\alpha \mapsto \varpi_\sigma$, $\alpha \mapsto \sigma$, $LP \mapsto \hat{LP}$, and $H \mapsto \hat{H}$.

Proof. We prove the lemma for $w_\alpha$, as $\varpi_\sigma$ are similar. By lemma 4.8, it suffices to show that $\{ w_\alpha \mid \alpha \in LP_\varsigma \}$ is linearly independent. Lemma 4.7 gives the special case $\varsigma = \bar{n}$, and the general case then follows using the following facts. First, the map $I_\varsigma(\cdot)$ is a linear isomorphism from $L_\varsigma$ to $P_\varsigma L_\varsigma$ by (3.38). Second, lemma 4.10 gives $w_{I_\varsigma \alpha} \in \text{im } \Psi_\varsigma$ for all $\alpha \in LP_\varsigma$. Third, by lemma A.2, the map $\hat{\Psi}_\varsigma$ is a linear isomorphism from $\text{im } \Psi_\varsigma$ to $\hat{V}_\varsigma$. Combining these facts with lemma 4.7, we conclude that the map $\alpha \mapsto w_\alpha$ defined in (4.50) injectively sends the linearly independent collection $LP_\varsigma$ of valenced link patterns to the collection $\{ w_\alpha \mid \alpha \in LP_\varsigma \}$, which therefore is linearly independent too. \hfill \Box
Proposition 4.12. (Link state – highest-weight vector correspondence): Suppose \( \max \zeta < p(q) \). The map \( \alpha \mapsto w_{\alpha} \) from \( L_\zeta \) to \( \H_\zeta \) has the following properties:

1. It is a homomorphism of \( TL_\zeta(\nu) \)-modules, i.e., \( (4.54) \) holds for all valenced tangles \( T \in TL_\zeta(\nu) \).
2. It is a linear injection. Furthermore, if \( n_\zeta < p(q) \), then it is a linear isomorphism.
3. It respects the \( s \)-grading \( (2.36, 3.24) \), i.e., we have \( \{ w_{\alpha} \mid \alpha \in L_\zeta(s) \} \subset H_\zeta(s) \), with equality if \( n_\zeta < p(q) \).

Similarly, items 1–3 hold for the map \( \bar{\alpha} \mapsto \bar{w}_\bar{\alpha} \) after the symbolic replacements

\[
\alpha \mapsto \bar{\alpha}, \quad w_{\alpha} \mapsto \bar{w}_{\bar{\alpha}}, \quad L \mapsto \bar{L}, \quad \text{and} \quad \H \mapsto \bar{\H}.
\]

Finally, the maps \( \alpha \mapsto w_{\alpha} \) and \( \bar{\alpha} \mapsto \bar{w}_{\bar{\alpha}} \) together preserve the bilinear pairing and the map \((\cdot)^{*} \) \((A.35, 3.26)\):

4. For all valenced link states \( \bar{\alpha} \in \bar{L}_\zeta \) and \( \beta \in L_\zeta \), we have

\[
(\bar{w}_\bar{\alpha} \mid w_\beta) = (\bar{\alpha} \mid \beta).
\]

5. For all valenced link states \( \alpha \in L_\zeta \) and \( \bar{\beta} \in \bar{L}_\zeta \), we have

\[
w_{\alpha}^{*} = \bar{w}_{\bar{\alpha}} \quad \text{and} \quad \bar{w}_{\bar{\beta}} = w_{\beta}.
\]

Proof.Lemma 4.4 implies that the image of the map \( \alpha \mapsto w_{\alpha} \) is a subset of \( \H_\zeta \). We prove items 1–4 as follows:

1. This is a restatement of lemma 4.9 specialized to the case \( \bar{\omega} = \zeta \).

2. Lemma 4.11 shows that the map \( \alpha \mapsto w_{\alpha} \) is a linear injection. Also, lemma 3.3, corollary 2.9, and a dimension count imply that if \( n_\zeta < p(q) \), then \( \alpha \mapsto w_{\alpha} \) is an isomorphism of vector spaces:

\[
n_\zeta < p(q) \implies \dim L_\zeta = D_\zeta = \dim H_\zeta.
\]

3. This follows from lemma 4.8, item 2, and a dimension count: for all \( s \in E_\zeta \), we have

\[
n_\zeta < p(q) \implies \dim L_\zeta(s) = D_\zeta(s) = \dim H_\zeta(s).
\]

The corresponding statements for the map \( \bar{\alpha} \mapsto \bar{w}_{\bar{\alpha}} \) can be proven similarly.

4. We first prove the case \( \zeta = \bar{n} \) for some \( n \in \mathbb{Z}_{>0} \). To this end, definition 4.1 and lemma 3.19 combined with the graphical representation of the vectors \( \bar{w}_{\bar{\alpha}} \) and \( w_{\beta} \) as link states with upward-oriented defects imply that

\[
(\bar{w}_{\bar{\alpha}} \mid w_\beta) = (\bar{\alpha} \mid \beta),
\]

by noticing that the evaluation of the oriented network \( \bar{w}_{\bar{\alpha}} \mid w_\beta \) via rules \((3.119-3.123)\) agrees with the evaluation of the non-oriented network \( \bar{\alpha} \mid \beta \) via rules \((3.108-3.110)\) because all defects in \( \bar{w}_{\bar{\alpha}} \) and \( w_\beta \) have upward orientation. For the general case \( \zeta \in \mathbb{Z}_{>0} \), lemma 2.13 and corollary 3.21 imply that

\[
(\bar{w}_{\bar{\alpha}} \mid w_\beta) = (\bar{\alpha} \mid \beta),
\]

for all valenced link states \( \bar{\alpha} \in \bar{L}_\zeta \) and \( \beta \in L_\zeta \). Then, using the already proven identity \((4.60)\) for the link states \( \bar{\alpha}_\zeta \beta \in \bar{L}_\zeta \) and \( I_\zeta \beta \in L_\zeta \), we obtain the asserted identity:

\[
(\bar{w}_{\bar{\alpha}} \mid w_\beta) = (\bar{\alpha}_\zeta \beta \mid I_\zeta \beta) = (\bar{\alpha} \mid \beta).
\]
5. We first prove the case $\zeta = \vec{n}$ for some $n \in \mathbb{Z}_{\geq 0}$. By linearity, we may assume that $\alpha$ and $\vec{\beta}$ are link patterns. We prove the left equation in (4.61) by induction on the number of links in $\alpha$; the right equation in (4.61) can be proven similarly. In the initial case $\alpha$ has no links, so $\alpha = \cup_{n} \in \text{LP}^{(\nu)}_{\kappa}$ with $\cup_{n}^{*} = \cap_{n}$, and definition 4.1 gives

$$w_{\cup_{n}}^{*} = (4.12) \vartheta_{0}^{(\nu)} \alpha \equiv (A.35, A.36) \vartheta_{0}^{(\nu)} \alpha \equiv (2.100) \vartheta_{0}^{(\nu)} \alpha \equiv (4.14) w_{\cap_{n}}^{*}. \quad (4.67)$$

Next, we let $\ell \in \mathbb{Z}_{>0}$ and assume that the left equation in (4.61) holds for all link patterns with at most $\ell - 1$ links. For the induction step, we let $\alpha \in \text{L}_{n}^{(\nu)}$ have $\ell$ links. Now, if $\hat{\alpha}$ is a link pattern obtained by dropping a link from $\alpha$ joining the $k$:th and $(k + 1)$:st nodes, then $\hat{\alpha}$ has $\ell - 1$ links, so using the induction hypothesis, we find that

$$w_{\hat{\alpha}}^{*} \alpha \equiv (A.17) -q \left( q - q^{-1} \right) (id \otimes (k-1) \otimes \iota_{(1,1)}^{0} \otimes id \otimes (n-k-1)) (w_{\hat{\alpha}}^{*}) \alpha \equiv (4.61) -q \left( q - q^{-1} \right) (id \otimes (k-1) \otimes \iota_{(1,1)}^{0} \otimes id \otimes (n-k-1)) (w_{\hat{\alpha}}^{*}) \alpha \equiv (4.15) -q^{-1} (-q) \bar{\omega}_{\hat{\alpha}}^{*} \alpha \equiv \bar{\omega}_{\hat{\alpha}}^{*}. \quad (4.68)$$

This proves the induction step and thus the left equation in (4.61) for $\zeta = \vec{n}$. For the general case $\zeta \in \mathbb{Z}_{>0}^{\#}$, item 6 of lemma A.2 gives

$$w_{\alpha}^{*} \equiv (4.50) \hat{\omega}_{\zeta}(w_{L_{\alpha}}^{*}) \alpha \equiv (A.9) \hat{\omega}_{\zeta}(w_{L_{\alpha}}^{*}) \alpha \equiv (3.43) \hat{\omega}_{\zeta}(w_{L_{\alpha}}^{*}) \alpha \equiv (4.61) \hat{\omega}_{\zeta}(w_{L_{\alpha}}^{*}) \alpha \equiv (4.50) \hat{\omega}_{\zeta}\alpha. \quad (4.69)$$

for all valenced link states $\alpha \in \text{L}_{\zeta}$. This proves the left equation in (4.61); the right equation is similar.

The proof is complete. 

If $n_{\zeta} \geq p(q)$, then there can exist other $U_{q}$-highest-weight vectors than the ones obtained from the embedding $\alpha \mapsto w_{\alpha}$. This fact is related to degeneracies that arise when the algebra $U_{q}$ is not semisimple and its representation theory does not play well with the usual highest-weight theory of Lie algebras [Lus90, CP94, Kas95]. Indeed, for instance, one can check directly from definitions (2.6) that the element $E^{p(q)}$ acts as zero on any simple type-one module $\text{M}_{(\nu)}$, recalling from (1.1) that $[k \cdot p(q)] = 0$ for any $k \in \mathbb{Z}_{>0}$. In particular, the basis vector $e_{(s)}$ with $s = \text{p}(q)$ is thus a highest-weight vector if $s \geq \text{p}(q)$. More generally, using this and formula A.1 from appendix A for the coproduct of powers of $E$, it follows that $E^{p(q)}$ acts as zero also on any type-one module $u_{\zeta} \circ \text{V}_{\zeta}$. (See also appendix B.)

We show in section 5 C that the highest-weight vectors which are not reachable via the link state – highest-weight vector correspondence are orthogonal to the link-pattern basis vectors $w_{\alpha}$. In particular, we show that certain quotients of the $U_{q}, U_{q}$-highest-weight vector spaces $H_{\zeta}$ and $\hat{H}_{\zeta}$ are isomorphic to simple $\text{TL}_{\zeta}(\nu)$-modules even if $n_{\zeta} \geq p(q)$.

C. Direct-sum decompositions

The next result improves proposition 2.8, with the conformal-block basis replaced by the link-pattern basis.

Proposition 4.13. Suppose $\max \zeta < p(q)$. There exists an embedding of left $U_{q}$-modules

$$\bigoplus_{s \in \text{E}_{\zeta}} D_{s}^{(\nu)} \text{M}_{(\nu)} \leftrightarrow u_{\zeta} \circ \text{V}_{\zeta} \quad (4.70)$$

such that the following hold:

1. For each valenced link pattern $\alpha \in \text{LP}^{(\nu)}_{\zeta}$, the collection

$$\{ F^{\ell} \alpha w_{\alpha} \mid 0 \leq \ell \leq s \} \quad (4.71)$$

is a basis for the image of a unique direct summand $\text{M}_{(\nu)}$ in (4.70).

2. The image of each summand $\text{M}_{(\nu)}$ has a unique basis of the form (4.71) with $\alpha \in \text{LP}^{(\nu)}_{\zeta}$.
3. If \( n_\varsigma < p(q) \), then (4.70) is an isomorphism of left \( U_q \)-modules,

\[
U_\varsigma \circ V_\varsigma \cong \bigoplus_{s \in E_\varsigma} D^{(s)}_\varsigma M_{(s)}. \tag{4.72}
\]

Similarly, this proposition holds for right \( U_q \)-modules after the symbolic replacements

\[
M \mapsto M, \quad U_\varsigma \circ V_\varsigma \mapsto \nabla_\varsigma \circ U_\varsigma, \quad \alpha \mapsto \bar{\alpha}, \quad L^\ell \mapsto L^\bar{\ell}, \quad \text{and} \quad F^\ell \cdot w_\alpha \mapsto \bar{w}_\bar{\alpha} \cdot F^\bar{\ell}. \tag{4.73}
\]

Finally, both the left-action and right-action versions of this proposition hold after replacing \( U_q \mapsto \bar{U}_q \) in either.

**Proof.** After replacing the conformal-block vectors \( u_\varsigma^\ell \) with the link-pattern basis vectors \( w_\alpha \) and instead of lemma 2.6 (resp. lemma 2.7) referencing lemma 4.8 (resp. lemma 4.11), the proof of proposition 2.8 adapts almost verbatim. \( \square \)

**Corollary 4.14.** Suppose \( n_\varsigma < p(q) \). Then, the following hold:

1. The collection \( \{ w_\alpha \mid \alpha \in L^\varsigma \} \) is a basis for \( H_\varsigma \).
2. For each \( s \in E_\varsigma \), the collection \( \{ w_\alpha \mid \alpha \in L^{(s)}_\varsigma \} \) is a basis for \( L^{(s)}_\varsigma \).

Similarly, this corollary holds after the symbolic replacements \( w_\alpha \mapsto \bar{w}_\bar{\alpha}, \alpha \mapsto \bar{\alpha}, L^\ell \mapsto L^\bar{\ell}, \) and \( H \mapsto \bar{H} \).

**Proof.** This immediately follows from lemma 4.11 and proposition 4.13. \( \square \)

The above result gives the dimension of the space \( H^{(s)}_\varsigma \) of highest-weight vectors:

**Corollary 4.15.** Suppose \( \max \varsigma < p(q) \). For all \( s \in E_\varsigma \), we have

\[
D^{(s)}_\varsigma \leq \dim H^{(s)}_\varsigma \leq B^{(s)}_\varsigma \quad \text{and} \quad D^{(s)}_\varsigma \leq \dim \bar{H}^{(s)}_\varsigma \leq B^{(s)}_\varsigma. \tag{4.74}
\]

Furthermore, if \( n_\varsigma < p(q) \), then we have \( \dim H^{(s)}_\varsigma = \dim \bar{H}^{(s)}_\varsigma = D^{(s)}_\varsigma \).

**Proof.** The lower bound follows from lemma 3.3 with item 2 of lemma 4.11, and the upper bound from corollary 2.3. Corollary 4.14 with lemma 3.3 give the dimensions when \( n_\varsigma < p(q) \). \( \square \)

In proposition 3.18 in section 3C, we established a “dual” direct-sum decomposition to that (4.72) of proposition 4.13 (or (2.89) of proposition 2.8): decomposition of the left \( TL_\varsigma(\nu) \)-module \( \tau_\varsigma \circ V_\varsigma \) (resp. \( \nabla_\varsigma \circ \tau_\varsigma \)) into a direct sum of \( TL_\varsigma(\nu) \)-submodules when \( n_\varsigma < p(q) \). However, we did not prove that the summands in this decomposition are simple. Next, we prove a refinement for proposition 3.18, realizing the direct summands \( \tau_\varsigma \circ H^{(s)}_\varsigma \) equivalently as the link state modules \( L^{(s)}_\varsigma \). We note in particular that \( \tau_\varsigma \circ H^{(s)}_\varsigma \cong L^{(s)}_\varsigma \) are simple \( TL_\varsigma(\nu) \)-modules if \( n_\varsigma < p(q) \).

**Proposition 4.16.** Suppose \( \max \varsigma < p(q) \). There exists an embedding of left \( TL_\varsigma(\nu) \)-modules

\[
\bigoplus_{s \in E_\varsigma} \bigoplus_{s < p(q)} (s + 1) L^{(s)}_\varsigma \hookrightarrow \tau_\varsigma \circ V_\varsigma \tag{4.75}
\]

such that the following hold:

1. For each integer \( \ell \in \{0,1,\ldots,s\} \), the collection

\[
\{ F^\ell \cdot w_\alpha \mid \alpha \in L^{(s)}_\varsigma \} \tag{4.76}
\]

is a basis for the image of a unique direct summand \( L^{(s)}_\varsigma \) in (4.75).
2. The image of each summand \( L^{(s)}_\varsigma \) has a unique basis of the form (4.76) with \( \ell \in \{0,1,\ldots,s\} \).
3. If \( n_\varsigma < p(q) \), then (4.75) is an isomorphism of left \( TL_\varsigma(\nu) \)-modules,

\[
\tau_\varsigma \circ V_\varsigma \cong \bigoplus_{s \in E_\varsigma} (s + 1) L^{(s)}_\varsigma. \tag{4.77}
\]
Similarly, this proposition holds for right $\mathrm{TL}_{\zeta}(\nu)$-modules after the symbolic replacements

$$ L \mapsto L, \quad \tau L \odot V_\zeta \mapsto \check{V}_\zeta \odot \tau L, \quad F^\ell \cdot w_\alpha \mapsto \overline{w}_\zeta \cdot E^\ell, \quad \alpha \mapsto \overline{\alpha}, \quad \text{and} \quad \mathrm{LP} \mapsto \overline{\mathrm{LP}}. \quad (4.78) $$

**Proof.** The map $\alpha \mapsto w_\alpha$ in proposition 4.12 gives an embedding of left $\mathrm{TL}_{\zeta}(\nu)$-modules:

$$ L_{\zeta} \mapsto \bigoplus_{s \in E_\zeta} L_{\zeta}^{(s)} \hookrightarrow \bigoplus_{s \in E_\zeta} \tau L \odot H_{\zeta}^{(s)} \subset \tau L \odot V_\zeta. \quad (4.79) $$

By item 3 of proposition 3.18, for each $0 \leq \ell \leq s < p(q)$, the left $\mathrm{TL}_{\zeta}(\nu)$-modules $\tau L \odot H_{\zeta}^{(s)}$ and $\tau L \odot F^\ell \cdot H_{\zeta}^{(s)}$ are isomorphic, and since the $K$-eigenvalues for different $\ell$ are distinct (for fixed $s$), their sum is direct. Furthermore, by fact U2, each of these modules has multiplicity $s + 1 = \dim M(\zeta)$. In conclusion, embedding (4.79) yields

$$ \bigoplus_{s \in E_\zeta} (s + 1) L_{\zeta}^{(s)} \hookrightarrow \bigoplus_{s \in E_\zeta} \bigoplus_{\ell = 0}^s \tau L \odot F^\ell \cdot H_{\zeta}^{(s)} \subset \tau L \odot V_\zeta, \quad (4.80) $$

which gives (4.75). Items 1 and 2 are then immediate from the construction. Item 3 follows by corollary 4.14 and a dimension count, for if $n_\zeta < p(q)$, then the dimensions of both sides of (4.75) are equal by item 3 of lemma 2.4.

To finish, we gather some further implications of proposition 4.12.

**Corollary 4.17.** Suppose $\max \zeta < p(q)$. We have

$$ w_\alpha \in \mathrm{im} \Psi_\zeta \iff \alpha \in \mathrm{im} P_\zeta(\cdot) \quad \text{and} \quad w_\alpha \in \ker \Psi_\zeta \iff \alpha \in \ker P_\zeta(\cdot), \quad (4.81) $$

Similarly, this corollary holds after the symbolic replacements $w_\alpha \mapsto \overline{w}_\zeta$, $\Psi \mapsto \overline{\Psi}$, and $\alpha \mapsto \overline{\alpha}$.

**Proof.** Because $\Psi_\zeta$ and $P_\zeta(\cdot)$ are projections by lemma A.2 and property (P2) of Jones-Wenzl projector, we have

$$ v \in \mathrm{im} \Psi_\zeta \iff \Psi_\zeta(v) = v \quad \text{and} \quad \alpha \in \mathrm{im} P_\zeta(\cdot) \iff P_\zeta \alpha = \alpha. \quad (4.82) $$

Using homomorphism-like property (4.54) from lemma 4.9, identifying the action of the tangle $P_\zeta$ with the projection $\Psi_\zeta$ via corollary 3.12, and recalling that the map $\alpha \mapsto w_\alpha$ is a linear injection by item 2 of proposition 4.12, we obtain

$$ w_\alpha \in \mathrm{im} \Psi_\zeta \overset{(4.82)}{\iff} w_\alpha = \Psi_\zeta(w_\alpha) \overset{(3.84)}{=} \Psi_\zeta(w_\alpha)_\tau \overset{\text{prop. 4.12, item 2}}{=} P_\zeta \alpha = \alpha \overset{(4.82)}{\iff} \alpha \in \ker P_\zeta(\cdot), \quad (4.83) $$

$$ w_\alpha \in \ker \Psi_\zeta \iff 0 = \Psi_\zeta(w_\alpha) = w_{P_\alpha} = 0 \overset{\text{prop. 4.12, item 2}}{\iff} P_\zeta \alpha = 0 \iff \alpha \in \ker P_\zeta(\cdot). \quad (4.84) $$

This proves the first assertion of (4.81), and the second assertion can be proven similarly.

Next, we describe the image and kernel of the map $\Psi_\zeta$. For this purpose, we introduce “special link patterns.” We group the the $n_\zeta$ left nodes of a $n_\zeta$-link pattern into the bins of nodes

left: $\{1, 2, \ldots, s_1\}, \{s_1 + 1, s_1 + 2, \ldots, s_1 + s_2\}, \{s_1 + s_2 + 1, s_1 + s_2 + 2, \ldots, s_1 + s_2 + s_3\}, \ldots$. \quad (4.85)

Then, we define a special link pattern to be a link pattern in $\mathrm{LP}_{n_\zeta}$ that lacks a turn-back link joining two nodes in a common bin of (4.85), and we denote

$$ \mathrm{SP}_{\zeta}^{(s)} := \{\text{special link patterns in } \mathrm{LP}_{n_\zeta}^{(s)}\}, \quad \mathrm{SP}_{\zeta} := \bigcup_{s \in E_{n_\zeta}} \mathrm{SP}_{\zeta}^{(s)} = \{\text{special link patterns in } \mathrm{LP}_{n_\zeta}\}. \quad (4.86) $$

For example, below, the left figure is a special link pattern in $\mathrm{SD}_\zeta^\omega$ with $\zeta = (2, 2, 3, 2)$ and $\omega = (2, 3, 2)$, but the right figure is not such a link pattern:

\[ \text{left: } s_1 = 2, s_2 = 2, s_3 = 3, s_4 = 2 \quad \text{right: } s_1 = 2, s_2 = 2, s_3 = 3, s_4 = 2. \quad (4.87) \]
Corollary 4.18. Suppose \( \max \varsigma < \mathbf{p}(q) \). The following hold:

1. The following collection is a linearly independent subset of \( \operatorname{im} \mathcal{P}_\varsigma \):
   \[
   \{ F^\ell.w_\alpha \mid \alpha \in \mathcal{S} \mathcal{P}_\varsigma^{(s)} \text{, } s \in \mathcal{E}_\varsigma \text{, and } 0 \leq \ell \leq s < \mathbf{p}(q) \}. \tag{4.88}
   \]
   Furthermore, if \( n_\varsigma < \mathbf{p}(q) \), then this set is a basis for \( \operatorname{im} \mathcal{P}_\varsigma \).

2. The following collection is a linearly independent subset of \( \ker \mathcal{P}_\varsigma \):
   \[
   \{ F^\ell.w_\alpha \mid \alpha \in \mathcal{L} \mathcal{P}_n^{(s)} \setminus \mathcal{S} \mathcal{P}_\varsigma^{(s)} \text{, } s \in \mathcal{E}_\varsigma \text{, and } 0 \leq \ell \leq s < \mathbf{p}(q) \}. \tag{4.89}
   \]
   Furthermore, if \( n_\varsigma < \mathbf{p}(q) \), then this set is a basis for \( \ker \mathcal{P}_\varsigma \).

Similarly, this corollary holds after the symbolic replacements 
\( \mathcal{P} \mapsto \overline{\mathcal{P}} \), \( F^\ell.w_\alpha \mapsto w_\alpha \), \( \mathcal{E} \ell \mapsto \alpha \), \( \mathcal{S} \mathcal{P} \mapsto \overline{\mathcal{S} \mathcal{P}} \), and \( \mathcal{L} \mathcal{P} \mapsto \overline{\mathcal{L} \mathcal{P}} \). \tag{4.90}

Proof. To begin, we collect three relevant facts.

- Item 1 of proposition 4.13 implies that the following collection is a linearly independent subset of \( V_{n_\varsigma} \):
   \[
   \{ F^\ell.w_\alpha \mid \alpha \in \mathcal{L} \mathcal{P}_n^{(s)} \text{, } s \in \mathcal{E}_{n_\varsigma} \text{, and } 0 \leq \ell \leq s < \mathbf{p}(q) \}. \tag{4.91}
   \]
   Moreover, proposition 4.13 implies that this set is a basis for \( V_{n_\varsigma} \) if \( n_\varsigma < \mathbf{p}(q) \).

- By item 1 of lemma A.2, the subspaces \( \operatorname{im} \mathcal{P}_\varsigma \) and \( \ker \mathcal{P}_\varsigma \) are closed under the \( U_q \)-action on them.

- By [FP18a, Lemma B.1] the respective collections \( \mathcal{S} \mathcal{P}_\varsigma \) and \( \mathcal{L} \mathcal{P}_n \setminus \mathcal{S} \mathcal{P}_\varsigma \) are bases for \( \operatorname{im} P_\varsigma(\cdot) \) and \( \ker P_\varsigma(\cdot) \).

Now, corollary 4.17 shows that we have \( w_\alpha \in \operatorname{im} \mathcal{P}_\varsigma \) if and only if \( \alpha \in \operatorname{im} P_\varsigma(\cdot) = \operatorname{span} \mathcal{S} \mathcal{P}_\varsigma \), and we have \( w_\alpha \in \ker \mathcal{P}_\varsigma \) if and only if \( \alpha \in \ker P_\varsigma(\cdot) = \operatorname{span} \mathcal{L} \mathcal{P}_n \setminus \mathcal{S} \mathcal{P}_\varsigma \). These facts imply items 1 and 2.

D. Graphical calculus for descendant vectors

In this section, we collect explicit diagram formulas for \( F \)-descendants of the link-pattern basis vectors \( w_\alpha \). Lemma 4.22 concerns the case of a tensor power of fundamental representations and in lemma 4.23, we establish the general case. To begin, we list simple \( q \)-identities.

Lemma 4.19. The following identities hold, for all \( q \in \mathbb{C}^\times \) and \( k, \ell, m \in \mathbb{Z} \):

\[
[k] = q^{k-1} + q^{k-3} + \cdots + q^{3-k} + q^{1-k}, \tag{4.92}
\]

\[
\left[ \begin{array}{c} m \\ \ell \end{array} \right] = q^{\ell-m} \left[ \begin{array}{c} m-1 \\ \ell-1 \end{array} \right] + q^{\ell} \left[ \begin{array}{c} m-1 \\ \ell \end{array} \right]. \tag{4.93}
\]

Proof. The asserted identities are straightforward to verify using definition (1.1).

The next observation (lemma 4.20 and the consequent lemma 4.21 are useful tools in many diagram calculations.

Lemma 4.20. [FK97, page 442] Suppose \( s < \mathbf{p}(q) \). Then within any diagram, we have

\[
\cdots \downarrow \quad \cdots = q \times \quad \cdots \downarrow \quad \cdots, \tag{4.94}
\]

and similarly,

\[
\cdots \downarrow \quad \cdots = q \times \quad \cdots \downarrow \quad \cdots, \tag{4.95}
\]

where the ellipses stand for unspecified parts of the link state which are the same on both sides.
Proof. By property (P2) of the Jones-Wenzl projector, we have

\[ \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} = 0 = \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \] \quad (4.96)\]

After decomposing the link via (3.128, 3.129) and rearranging, we arrive with (4.94) and (4.95).

Lemma 4.21. Suppose \( s < p(q) \). Then, for all \( k, \ell \in \{0, 1, \ldots, s\} \), we have

\[ \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \delta_{k,\ell} q^{k(k-s)} \begin{array}{c} \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} \end{array}^{-1} \quad (4.97) \]

and similarly,

\[ \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \delta_{k,\ell} \begin{array}{c} \begin{array}{c} \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} \end{array}^{-1} \quad (4.98) \]

Proof. Without loss of generality, we assume that \( k \geq \ell \). To evaluate the network of (4.97), we replace the projector box in it with the right side of the recursive identity

\[ \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} + \sum_{r=0}^{s-2} \frac{[s-r-1]}{[s]} \times \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} \quad (4.99) \]

see [MDRR15, equation (2.59)]. After inserting this identity into (4.97), all but two of the resulting terms have components with clashing orientations that cause them to vanish by rule (3.123). With this observation, we find that

\[ \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} + \frac{[\ell]}{[s]} (i q^{1/2})^2 \times \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} \quad (4.100) \]

Replacing the disconnected components with their weights via (3.119, 3.120) and straightening the leftmost defect below the projector box in the right network, thus producing a further factor of \( i q^{1/2} \) according to (3.120), we obtain

\[ \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} + \frac{[\ell]}{[s]} (i q^{1/2})^2 \times \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} \quad (4.101) \]

\[ \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} = \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} + \frac{[\ell]}{[s]} (i q^{1/2})^2 \times \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} \quad (4.102) \]

\[ = \begin{array}{c} \cdots \begin{array}{c} \downarrow \cdots \end{array} \begin{array}{c} \cdots \end{array} \end{array} \]
Repeating this process another \( s - k - 1 \) times, we reduce the original network into one that clearly evaluates to \( \delta_{k,\ell} \):

\[
\begin{array}{c}
\begin{array}{c}
s - \ell \quad \ell \\
\downarrow \quad \downarrow \\
s - k \quad k
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\frac{[s - \ell]}{[s - 1]} q^{-\ell} \frac{[s - \ell - 1]}{[s - 1]} q^{-\ell} \cdots \frac{[k - \ell]}{[k + 1]} q^{-\ell} \times \begin{array}{c}
k - \ell \\
\uparrow \uparrow \\
k
\end{array}
\end{array}
\end{array} = q^{k(s - \ell)} \left[ \frac{k}{s} \right]^{-1} \delta_{k,\ell},
\]

This proves (4.97). Using lemma 4.20 to switch defects with opposite orientations, we obtain (4.98) from (4.97).

The descendants of the highest-weight vectors \( \theta_{0}^{(s)} \) and \( \bar{\theta}_{0}^{(s)} \) defined in (2.101, 2.103) have explicit diagram formulas. These can be proven by induction. Below, we give a proof using diagram calculations, which we find more illuminating.

**Lemma 4.22.** Suppose \( s < p(q) \). Then, for all \( \ell \in \{0, 1, \ldots, s\} \), we have

\[
\theta^{(s)}_{\ell} := F^{\ell} \theta_{0}^{(s)} = \frac{[s]!}{[s - \ell]!} \times \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \\
\ell \quad s - \ell
\end{array}
\end{array},
\]

and similarly,

\[
\bar{\theta}^{(s)}_{\ell} := \bar{\theta}_{0}^{(s)} E^{\ell} = \frac{[s]!}{[s - \ell]!} \times \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \\
\ell \quad s - \ell
\end{array}
\end{array}.
\]

**Proof.** To begin, we show that \( \theta^{(s)}_{\ell} \) and the left diagram in (4.104) are proportional. For this, we observe these facts:

- Thanks to the projector box, lemma 3.12 implies that the left diagram in (4.104) lies in the image of \( \mathcal{P}_{(s)} \).
- With \( \ell \) downward-oriented defects and \( s - \ell \) upward-oriented defects, the left diagram in (4.104) lies also in \( \mathcal{V}_{s}^{(s - 2\ell)} \).
- By definition (2.111) and item 1 of lemma A.2 with (2.28), we have

\[
\text{im } \mathcal{P}_{(s)} \cap \mathcal{V}_{s}^{(s - 2\ell)} = \text{im } \mathcal{J}_{(s)} \cap \mathcal{V}_{s}^{(s - 2\ell)} = \text{span } \{ \theta_{0}^{(s)}, \theta_{1}^{(s)}, \ldots, \theta_{s}^{(s)} \} \cap \mathcal{V}_{s}^{(s - 2\ell)} = \text{span } \{ \theta^{(s)}_{\ell} \}.
\]

It follows that the left diagram in (4.104) equals \( \lambda^{(s)}_{\ell} \theta^{(s)}_{\ell} \) for some constant \( \lambda^{(s)}_{\ell} \in \mathbb{C} \). To compute the constant, we let

\[
\mathcal{V} = \mathcal{E}_{0} \otimes \mathcal{E}_{0} \otimes \cdots \otimes \mathcal{E}_{0} \otimes \mathcal{E}_{1} \otimes \mathcal{E}_{1} \otimes \cdots \otimes \mathcal{E}_{1},
\]

and we use the explicit formula for \( \bar{\theta}^{(s)}_{\ell} \) from lemma 2.10 to compute that

\[
(\mathcal{V} | \lambda^{(s)}_{\ell} \theta^{(s)}_{\ell}) = (2.104) \lambda^{(s)}_{\ell} q^{-\ell(s - \ell)} [\ell]!.
\]

On the other hand, according to lemma 3.19, the bilinear pairing of the vector \( \mathcal{V} \) with the left diagram in (4.104) equals the evaluation of the network in (4.97) of lemma 4.21. Equating this network evaluation with (4.108) gives

\[
\lambda^{(s)}_{\ell} q^{-\ell(s - \ell)} [\ell]! (4.97) q^{-\ell(s - \ell)} \left[ \frac{s}{\ell} \right]^{-1} \implies \lambda^{(s)}_{\ell} = \frac{[s - \ell]!}{[s]!}.
\]

This completes the proof of the first equality in (4.104). We obtain the second equality of (4.104) by using lemmas A.5 and 4.20 to rewrite the former. Identity (4.105) can be proven similarly.

\( \square \)
Lemma 4.23. Suppose \( s < \mathfrak{p}(q) \). Then, for all valenced link states \( \alpha \in \mathcal{L}_s^{(\nu)} \) and for all \( \ell \in \{0, 1, \ldots, s\} \), we have

\[
F^\ell, w_\alpha = \frac{[s]!}{[s-\ell]!} \times \begin{array}{c}
\ell \\
\downarrow \\
\alpha \\
\downarrow \\
\cdots \\
\alpha \\
\downarrow \\
s-\ell
\end{array} 
\]

(4.110)

Similarly, for all valenced link states \( \bar{\alpha} \in \mathcal{L}_s^{(\nu)} \) and for all \( \ell \in \{0, 1, \ldots, s\} \), we have

\[
\bar{\pi}_\alpha, E^\ell = \frac{[s]!}{[s-\ell]!} \times \begin{array}{c}
\ell \\
\downarrow \\
\bar{\alpha} \\
\downarrow \\
\cdots \\
\downarrow \\
s-\ell
\end{array} 
\]

(4.111)

Proof. We prove the first equality in (4.110). As before, we obtain the second equality by using lemmas A.5 and 4.20, and (4.111) can be proven similarly. First, we assume that \( \zeta = \bar{n} \) for some \( n \in \mathbb{Z}_{>0} \). By linearity, we may also assume that \( \alpha \in \mathbb{L}_n^{(\nu)} \). Now, there are some indices \( i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n-1\} \) such that

\[
\alpha = L_{1_{i_k}} L_{1_{i_{k-1}}} \cdots L_{1_{i_2}} L_{1_{i_1}} \cup_j s \quad (4.12)
\]

\[
w_\alpha = L_{1_{i_k}} L_{1_{i_{k-1}}} \cdots L_{1_{i_2}} L_{1_{i_1}} \theta_0^{(\nu)} \quad (4.13)
\]

so

\[
F^\ell, w_\alpha = F^\ell (L_{1_{i_k}} L_{1_{i_{k-1}}} \cdots L_{1_{i_2}} L_{1_{i_1}} \theta_0^{(\nu)}) = L_{1_{i_k}} L_{1_{i_{k-1}}} \cdots L_{1_{i_2}} L_{1_{i_1}} \theta_0^{(\nu)} \quad (4.112)
\]

where the last equality follows from lemma 3.16. Then, lemma 4.22 shows that

\[
F^\ell, w_\alpha = F^\ell (L_{1_{i_k}} L_{1_{i_{k-1}}} \cdots L_{1_{i_2}} L_{1_{i_1}} \theta_0^{(\nu)}) = L_{1_{i_k}} L_{1_{i_{k-1}}} \cdots L_{1_{i_2}} L_{1_{i_1}} \theta_0^{(\nu)} \quad (4.113)
\]

and by (4.112), the right side of (4.114) indeed equals the right side of (4.110).

If \( \zeta \neq \bar{n} \), then with \( I, \alpha \in \mathcal{L}_s^{(\nu)} \) for any valenced link state \( \alpha \in \mathcal{L}_s^{(\nu)} \) by (3.37), we use the above result to write

\[
F^\ell, w_\alpha = F^\ell (\hat{\mathcal{P}}_\zeta (w_{I, \alpha}) = \hat{\mathcal{P}}_\zeta (F^\ell, w_{I, \alpha}) \quad (4.15)
\]

\[
\begin{array}{c}
\ell \\
\downarrow \\
\cdots \\
\downarrow \\
\alpha \\
\downarrow \\
s-\ell
\end{array}
\]

This finishes the proof.

5. GRAPHICAL CALCULUS ON TYPE-ONE MODULES

This section concerns diagram identities and graphical calculus useful both for the analysis of the \( U_q \)-module structure of \( \mathcal{U}_q \otimes \mathcal{V}_\zeta \) and for the \( \mathcal{T}_s^{(\nu)} \)-module structure of \( \mathcal{T}_s^{(\nu)} \otimes \mathcal{V}_\zeta \). In particular, by lemma 3.16, the latter diagram action gives rise to homomorphisms of \( U_q \)-modules between different modules \( \mathcal{V}_\zeta \otimes \mathcal{V}_\zeta \) and \( \mathcal{V}_\zeta \otimes \mathcal{V}_\infty \). The main results of the present section give explicit expressions for such homomorphisms in lemma 5.8 and proposition 5.9 (both in section 5B). We also find diagram presentations for the conformal-block highest-weight vectors (section 5A), establishing that they are orthogonal (lemma 5.5). We also show in lemma 5.10 that certain \( U_q \)-submodule projectors generate the image of the representation \( \mathcal{F}_\zeta \) of the valenced Temperley-Lieb algebra. Lastly, sections 5C–5D concern orthocomplements and quotients of the \( U_q \)-highest-weight vector spaces \( \mathcal{H}_s \) and \( \mathcal{H}_\infty \) with respect to the bilinear pairing \( \langle \cdot | \cdot \rangle \). In particular, we identify in proposition 5.14 certain quotients with simple \( \mathcal{T}_s^{(\nu)} \)-modules via the link state–highest-weight vector correspondence (proposition 4.12).
A. Graphical calculus for conformal-block vectors

In the next auxiliary lemma, we find an explicit sum formula for a valenced link decomposed into oriented defects.

**Lemma 5.1.** Suppose \( \max(r, t) < p(q) \). Within any link state with oriented defects, we have

\[
\begin{align*}
\text{\downdots}_{r} k \downdots_{t} &= \sum_{i=0}^{k} \sum_{j=0}^{k} \delta_{i+j, k} \frac{(-1)^{i}q^{j(k+1-j)}}{(iq^{1/2})^{k}} \left[ \begin{array}{c} k \\bar{1} \end{array} \right] \left[ \begin{array}{c} j \end{array} \right] \times \downdots_{r}^{k-i} \downdots_{t}^{j} \tag{5.1}
\end{align*}
\]

and similarly,

\[
\begin{align*}
\text{\downdots}_{r}^{k} \downdots_{t} &= \sum_{i=0}^{k} \sum_{j=0}^{k} \delta_{i+j, k} \frac{(-1)^{i}q^{j(i-k-1)}}{(-iq^{1/2})^{k}} \left[ \begin{array}{c} k \\bar{1} \end{array} \right] \left[ \begin{array}{c} i \end{array} \right] \times \downdots_{r}^{k-i} \downdots_{t}^{j} \tag{5.2}
\end{align*}
\]

where the ellipses stand for unspecified parts of the link state which are the same on both sides.

**Proof.** We prove formula (5.1) by induction on \( k \in \mathbb{Z}_{\geq 0} \). The initial case \( k = 0 \) is trivial. Assuming that formula (5.1) for the diagram with \( k-1 \) links holds, we consider the left side of (5.1) with \( k \) links. Decomposing those links gives

\[
\begin{align*}
\downdots_{r}^{k} \downdots_{t} &= \frac{iq^{1/2}}{2} \times \downdots_{r}^{k-1} \downdots_{t}^{i-1} - \frac{iq^{-1/2}}{2} \times \downdots_{r}^{k-1} \downdots_{t}^{i+1} \tag{5.3}
\end{align*}
\]

Next, after applying the induction hypothesis and using identity (4.94) from lemma 4.20 to commute the upward-oriented defects to the left of the downward-oriented defects in the second diagram of the result, we obtain

\[
\begin{align*}
\downdots_{r}^{k} \downdots_{t} &= \frac{iq^{1/2}}{2} \times \downdots_{r}^{k-1} \downdots_{t}^{i-1} - \frac{iq^{-1/2}}{2} \times \downdots_{r}^{k-1} \downdots_{t}^{i+1} \tag{5.4}
\end{align*}
\]

Finally, after changing summation indices by \( j + 1 \mapsto j \) in the first term and \( i + 1 \mapsto i \) in the second term, and using the convention that \( \left[ \begin{array}{c} k-1 \\bar{1} \end{array} \right] = 0 \) and \( \left[ \begin{array}{c} k-1 \\bar{1} \end{array} \right] = 0 \), we simplify the right side of (5.4) into the following form:

\[
\begin{align*}
(iq^{1/2})^{2} \frac{(-1)^{j-1}q^{j(k+1-j)}}{iq^{1/2}k}q^{-1} \sum_{i=0}^{k} \sum_{j=0}^{k} \delta_{i+j, k} \left( q^{j} \left[ \begin{array}{c} k-1 \\bar{1} \end{array} \right] + q^{i} \left[ \begin{array}{c} k-1 \\bar{1} \end{array} \right] \right) \times \downdots_{r}^{k-i} \downdots_{t}^{j} \tag{5.5}
\end{align*}
\]

A simplification using identity (4.93) now finishes the proof of (5.1). Identity (5.2) can be proven similarly. \( \square \)

Using the oriented closed three-vertex notation

\[
\text{for } s \in \mathbb{E}_{(r,t)}
\]

\[
\begin{align*}
\text{\downdots}_{r}^{s} \downdots_{t} &:= \frac{r+s-t}{2}, \quad j = \frac{s+t-r}{2}, \quad k = \frac{t+r-s}{2}, \tag{5.6}
\end{align*}
\]

and lemma 5.1, we express the conformal-block vectors \( u_{(r,t)}^{(s)} \) and \( \bar{a}_{(r,t)}^{(s)} \), defined in (2.72, 2.73), in diagram form.
Lemma 5.2. Suppose max(r, t) < p(q). We have

\[
u_{(r, t)}^{(s)} = \frac{(iq^{1/2})^{r+t-s}}{(q - q^{-1})^{r+t-s} \left(\frac{r+t-s}{2}\right)!} \times \begin{array}{c}
\text{Diagram 1}
\end{array}
\tag{5.7}
\]

and

\[
\eta_{(r, t)}^{(s)} = \frac{(-iq^{-1/2})^{r+t-s}}{(q - q^{-1})^{r+t-s} \left(\frac{r+t-s}{2}\right)!} \times \begin{array}{c}
\text{Diagram 2}
\end{array}
\tag{5.8}
\]

Proof. We only prove identity (5.7); identity (5.8) can be proven similarly. Definition (2.72) reads

\[
u_{(r, t)}^{(s)} := \sum_{i=0}^{r+t-s} \sum_{j=0}^{r+t-s} \delta_{i+j, r+s} \frac{(-1)^{j} p^{i+j+1} |r-i]| |t-j|!}{(q - q^{-1})^{(r+t-s)/2} |i+j|! |i|! |j|!} \times \begin{array}{c}
\text{Diagram 3}
\end{array}
\tag{5.9}
\]

and lemma 4.23 gives diagram representations for the vectors \( F^r, e_0^r \) and \( F^j, e_0^0 \):

\[
F^r, e_0^r \otimes F^j, e_0^0 = \frac{|r|! |t|!}{|r-i|! |t-j|!} \times \begin{array}{c}
\text{Diagram 4}
\end{array}, \quad \text{where } p = \frac{r+t-s}{2}, \tag{5.10}
\]

On the other hand, using (5.6) we can write the diagram on the right side of (5.7) in the following form:

\[
\begin{array}{c}
\text{Diagram 5}
\end{array}, \quad \text{where } p = \frac{r+t-s}{2}, \tag{5.11}
\]

where we removed the top projector box thanks to the following simple observation: each internal link diagram of the top projector box with a turn-back link would have clashing orientations and thus weight zero by (3.123).

Next, we use lemma 5.1 to expand the right side of (5.11) into a sum of link patterns only comprising oriented defects, and lemma 4.20 to commute the upward-oriented defects to the left of downward-oriented defects, arriving with

\[
\begin{array}{c}
\text{Diagram 6}
\end{array}, \quad \text{where } p = \frac{r+t-s}{2}. \tag{5.12}
\]

Recalling definition (1.1) of the \( q \)-binomial coefficients, a comparison of (5.12) with (5.9, 5.10) gives (5.7).

Recalling definition (2.70), we also record the following more general result.

Lemma 5.3. Suppose \( \max(q, \vartheta, r, t) < p(q) \). Then, for all valenced link states \( \alpha \in \Gamma_q^{(r)} \) and \( \beta \in \Gamma_q^{(r)} \), we have

\[
\eta_{(r, t)}^{(s)}(w_{\alpha} \otimes w_{\beta}) = \frac{(iq^{1/2})^{r+t-s}}{(q - q^{-1})^{r+t-s} \left(\frac{r+t-s}{2}\right)!} \times \begin{array}{c}
\text{Diagram 7}
\end{array} \tag{5.13}
\]

Similarly, for all valenced link states \( \overline{\alpha} \in \Gamma_q^{(r)} \) and \( \overline{\beta} \in \Gamma_q^{(r)} \), we have

\[
\eta_{(r, t)}^{(s)}(w_{\overline{\alpha}} \otimes w_{\overline{\beta}}) = \frac{(-iq^{-1/2})^{r+t-s}}{(q - q^{-1})^{r+t-s} \left(\frac{r+t-s}{2}\right)!} \times \begin{array}{c}
\text{Diagram 8}
\end{array}. \tag{5.14}
\]
Proof. The proof of this lemma is almost identical to the proof of lemma 5.2. Alternatively, one can use lemma 5.2 to prove this lemma in a manner very similar to the use of lemma 4.22 to prove lemma 4.23 in section 4D.

For the general conformal-block vectors $u_{\varrho}^\varsigma$ and $\overline{u}_{\varrho}^\varsigma$ defined in (2.75, 2.76), we obtain explicit, simple diagram expressions.

**Lemma 5.4.** Suppose $\max \varsigma < p(q)$. For all walks $\varrho = (r_1, r_2, \ldots, r_d)$ over $\varsigma$ with $\max \hat{\varrho} = \max(r_1, \ldots, r_d) < p(q)$, we have

\[
u_{\varrho}^\varsigma = \left( \prod_{j=1}^{d-1} \frac{(iq^{1/2})^{r_j+r_{j+1}+1-r_{j+1}+1}}{(q-q^{-1}) \frac{1}{2} [r_j+s_j+1-r_{j+1}+1]!} \right) \times \]

\[• \ldots \u2022
\]

\[
\text{(5.15)}
\]

and

\[
\overline{u}_{\varrho}^\varsigma = \left( \prod_{j=1}^{d-1} \frac{(-iq^{-1/2})^{r_j+r_{j+1}+1-r_{j+1}+1}}{(q-q^{-1}) \frac{1}{2} [r_j+s_j+1-r_{j+1}+1]!} \right) \times \]

\[
\text{(5.16)}
\]

**Proof.** We prove identity (5.15) by induction on $d_\varsigma \in \mathbb{Z}_>0$; identity (5.16) is similar. The initial case $d_\varsigma = 1$ is trivial. Next, we suppose (5.15) holds for $d_\varsigma = d - 1$ for some integer $d \geq 2$. After inserting the substitutions

\[
\alpha = u_{\varrho}^\varsigma, \quad \beta = e^{(s_\delta)}, \quad r = r_d, \quad t = s_d, \quad s = r_d
\]

into (5.13), with $\hat{\varrho} := (r_1, r_2, \ldots, r_{d-1})$ and $\hat{\varsigma} := (s_1, s_2, \ldots, s_d)$ (so $\alpha$ is given by (5.15) with $d_\varsigma = d - 1$ and $\beta = \chi_{s_d}$ is the link pattern with $s_d$ defects attached to one node), we obtain (5.15) for $d_\varsigma = d$.

Using the diagram simplification formula

\[
\begin{pmatrix}
\begin{array}{c}
\circlearrowleft \\
\uparrow s' \\
\downarrow r \\
\quad t \\
\downarrow s
\end{array}
\end{pmatrix} = \delta_{s,s'} \frac{\Theta(r,s,t)}{(-1)^s [s+1]},
\]

we calculate the bilinear pairing of the conformal-block vectors $u_{\varrho}^\varsigma$ and $\overline{u}_{\varrho}^\varsigma$, finding that they are orthogonal:

**Lemma 5.5.** Suppose $\max \varsigma < p(q)$. For any walks $\varrho = (r_1, r_2, \ldots, r_d)$ and $\varrho' = (r'_1, r'_2, \ldots, r'_d)$ over $\varsigma$ with $\max(\hat{\varrho}, \hat{\varrho}') < p(q)$, we have

\[
(\overline{u}_{\varrho}^\varsigma | u_{\varrho'}^\varsigma) = \delta_{\varrho,\varrho'} \prod_{j=1}^{d-1} \frac{\Theta(r_j, r_{j+1}, s_{j+1})}{(q-q^{-1})^{r_j+r_{j+1}+1-r_{j+1}+1} \left[ r_j+s_j+1-r_{j+1}+1 \right]^2 [r_{j+1}+1]}.
\]

\[
(5.19)
\]
Proof. Using (5.15, 5.16) and lemma 3.19, we have

\[(\varpi_\xi | u_\xi')^{(3.142)} \equiv (5.15, 5.16) \prod_{j=1}^{d_\varsigma-1} \left( \frac{-i q^{-1/2} q_j^{r_j+s_{j+1}-r_{j+1}}}{(q - q^{-1})^{r_j+s_{j+1}-r_{j+1}}!!} \right) \times \prod_{j=1}^{d_\varsigma-1} (q_j - r_j^{r_j+s_{j+1}-r_{j+1}}) \] (5.20)

Then, after using simplification (5.18) \(d_\varsigma - 1\) times to delete each loop in the network, we arrive with (5.19).

Next, we state a corollary to lemma 5.4, relating the conformal-block vectors to the link-pattern basis vectors. For this purpose, generalizing the walk representations of link patterns already encountered in (4.34) in section 4 A, we recall from [FP18a, section 4] that, for any valenced link pattern \(\alpha \in \text{LP}_\varsigma\), the walk representation of \(\alpha\) is a walk

\[\theta_\alpha = (r_1, r_2, \ldots, r_{d_\varsigma})\] (5.21)

over \(\varsigma = (s_1, s_2, \ldots, s_{d_\varsigma})\) as in (2.58), whose heights \(r_j\) are determined by the links and defects in \(\alpha\) as in

\[\alpha = \] (5.22)

where \(\alpha\) is written in the generic form of a trivalent graph with open three-vertices [KL94]

\[\text{for } s \in E_{(r,t)}: \] (5.23)

We also need the (non-oriented) closed three-vertex notation (1.30) Now, we define the following operation \(\alpha \mapsto \alpha\) on valenced link patterns \(\alpha \in \text{LP}_\varsigma\). Writing \(\alpha\) in the generic form (5.22), we define the trivalent link state \(\alpha \in \text{L}_\varsigma\) as

\[\alpha : = \] (5.24)
that is, to obtain \( \alpha \) from \( \alpha \), we replace the \( j \)-th open vertex in the walk representation (5.22) of the valenced link pattern \( \alpha \) with a closed vertex for each step \( j \in \{1, 2, \ldots, d_\alpha - 1\} \) of the walk. We note from (5.23, 1.30) that replacing the \( j \)-th open vertex with a closed vertex inserts three projector boxes on the appropriate cables, which might not exist a priori. However, in [FP18a, definition 4.3] it is shown how the definition makes sense for all \( \max \zeta < p(q) \).

**Corollary 5.6.** Suppose \( \max \zeta < p(q) \). For all valenced link patterns \( \alpha \in \mathcal{L}_\zeta \) such that \( q_\alpha = (r_1, r_2, \ldots, r_{d_\alpha}) \) and \( \max \hat{q}_\alpha < p(q) \), we have

\[
u_\zeta^{\alpha} = \left( \prod_{j=1}^{d_\alpha - 1} \frac{(iq^{1/2})^{|r_j + r_{j+1} - r_{j+1}|}}{(q - q^{-1})^{|r_j + r_{j+1} - r_{j+1}|}} \right) w_\alpha.
\]

Similarly, for all valenced link patterns \( \sigma \in \mathcal{L}_\zeta \) such that \( q_\sigma = (r_1, r_2, \ldots, r_{d_\sigma}) \) and \( \max \hat{q}_\sigma < p(q) \), we have

\[
u_\zeta^{\sigma} = \left( \prod_{j=1}^{d_\sigma - 1} \frac{(-iq^{-1/2})^{|r_j + r_{j+1} - r_{j+1}|}}{(q - q^{-1})^{|r_j + r_{j+1} - r_{j+1}|}} \right) m_\sigma.
\]

**Proof.** This follows immediately from lemma 5.4 and (5.24). \( \square \)

---

**B. Valenced tangle representations of projectors**

Our next aim is to interpret diagrams containing Jones-Wenzl projectors as certain \( U_q \)-submodule projectors. These results generalize lemma 3.11 and corollaries 3.12–3.15 from sections 3B–3C. In lemma 5.8, we show how to use the Jones-Wenzl projectors to project onto any submodule of the tensor product \( \mathcal{V}_\zeta \otimes \mathcal{V}_\zeta \) when \( n_\zeta < p(q) \) (and not just the submodule \( \mathcal{M}_\alpha \) with largest dimension as the Jones-Wenzl projectors do). In proposition 5.9, we interpret the embedding \( \iota^{(r)} \) and the two projectors \( \pi^{(r)}(\zeta) \) and \( \overline{\pi}^{(r)}(\zeta) \), respectively defined in (2.118, 2.119, 2.120), as tangles.

To begin, for each pair \( \alpha \in \mathcal{L}_\zeta^{(s)} \), \( \beta \in \mathcal{L}_\zeta^{(s)} \) of valenced link patterns with the same number \( s \) of defects, we define \( | \alpha \beta | \) to be the \((\zeta, \omega)\)-valenced link diagram obtained by placing \( \alpha \) to the left of \( \beta \) in the plane, rotating both \( \alpha \) and \( \beta \) by \(-\pi/2\) radians, and joining the \( s \) defects of \( \alpha \) and \( \beta \) together pairwise top-to-bottom: for example,

\[
\begin{align*}
\alpha & \in \mathcal{L}_\zeta^{(1,1,1,2)} \\
\beta & \in \mathcal{L}_\zeta^{(2,1,1,1)} \\
\implies | \alpha \beta | & \in \mathcal{T}(2,1,1,1,1,1,2).
\end{align*}
\]

When \( s < p(q) \), we also define \( | \alpha \emptyset \beta | \) to be the \((\zeta, \omega)\)-valenced tangle obtained from \( | \alpha \beta | \) by inserting a vertical projector box across all \( s \) of its crossing links: for example,

\[
\begin{align*}
\alpha & \in \mathcal{L}_\zeta^{(3,1,1,2)} \\
\beta & \in \mathcal{L}_\zeta^{(2,1,1,1)} \\
\implies | \alpha \emptyset \beta | & \in \mathcal{T}(2,1,1,1,1,1,2).
\end{align*}
\]

We remark that tangles as the ones appearing on the right side of either (5.27, 5.28) form a basis for \( \mathcal{T}_\zeta \). The former assertion is clear, and to argue the latter, we see by recursion property (3.29) of the Jones-Wenzl projector that the linear map defined by inserting the vertical projector box into each basis element (5.27) has an upper unitriangular matrix representation (cf. identity (5.39) below), thus being a bijection.

The next lemma gives useful rules for calculating the diagram action in terms of the bilinear pairing \((\cdot, \cdot)\) on \( \mathcal{V}_\zeta \).

**Lemma 5.7.** Suppose \( \max(\zeta, \omega) < p(q) \), and let \( \alpha \in \mathcal{L}_\zeta^{(s)} \) and \( \beta \in \mathcal{L}_\zeta^{(s)} \). Then, the following hold:

1. For all vectors \( v \in \mathcal{V}_\omega^{(s)} \) and \( \overline{v} \in \overline{\mathcal{V}}_\zeta^{(s)} \), we have

\[
| \alpha \beta | v = (m_\beta | v) w_\alpha \quad \text{and} \quad \overline{v} | \alpha \beta | = (\overline{v} | w_\alpha) m_\beta.
\]
2. If \( s < p(q) \), then for all valenced link states \( \gamma \in \mathcal{L}^{(s)}_w \) and \( \delta \in \mathcal{L}^{(s)}_r \), and for all \( \ell \in \mathbb{Z}_{\geq 0} \), we have
\[
| \alpha \; \beta | F^\ell \cdot v \gamma = (w_\gamma | F^\ell \cdot w_\alpha) \quad \text{and} \quad w_\delta E^\ell | \alpha \; \beta | = (w_\delta | w_\alpha) \; w_\beta \cdot E^\ell.
\] (5.30)

Proof. We prove items 1–2 as follows (only the left equations of (5.29) and (5.30); the right ones are similar):

1. To begin, we prove the left equation of (5.29) for the special case \( \zeta = \bar{n} \) and \( \varpi = \bar{m} \). By linearity, we may assume that \( \alpha \in \mathcal{L}^{(s)}_n \) and \( \beta \in \mathcal{L}^{(s)}_m \) are link patterns and \( v \) is a standard basis vector in \( \mathcal{V}_m^{(s)} \) of the form
\[
v = \varepsilon_{\ell_1} \otimes \varepsilon_{\ell_2} \otimes \cdots \otimes \varepsilon_{\ell_m} \quad \text{with} \quad \ell_1, \ell_2, \ldots, \ell_m \in \{0, 1\} \quad \text{and} \quad m - 2(\ell_1 + \ell_2 + \cdots + \ell_m) = s.
\] (5.31)

Writing the link patterns \( \alpha \) and \( \beta \) in terms of left and right generators \( L_i, R_j \) (3.5), definition 4.1 gives
\[
\alpha = L_{i_1} L_{i_{k-1}} \cdots L_{i_2} L_{i_1} \cup \cup \quad (4.13) \quad w_\alpha = L_{i_1} L_{i_{k-1}} \cdots L_{i_2} L_{i_1} \theta_0^{(s)}
\] (5.32)
\[
\beta = \cup \cup R_{j_1} R_{j_2} \cdots R_{j_{l-1}} R_{j_l} \quad (4.15) \quad \tilde{w}_\beta = \theta_0^{(s)} R_{j_1} R_{j_2} \cdots R_{j_{l-1}} R_{j_l},
\] (5.33)
for some \( i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l \in \mathbb{Z}_{>0} \), where \( 2l + s = m \). Also, we have
\[
| \alpha \; \beta | v = (\bar{w}_\beta | v \alpha) = (\theta_0^{(s)} | R_{j_1} R_{j_2} \cdots R_{j_{l-1}} R_{j_l} v \alpha)
\] (5.34)

Now, with the vector \( v \) of the form (5.31) with \( m - \frac{s}{2} \) tensorands equal to \( \varepsilon_1 \) and \( m + \frac{s}{2} \) tensorands equal to \( \varepsilon_0 \), repeated application of rule (3.52) shows that
\[
R_{j_1} R_{j_2} \cdots R_{j_{l-1}} R_{j_l} v \quad (5.35) \quad \bar{R}_{j_1} R_{j_2} \cdots R_{j_{l-1}} R_{j_l} v \quad (5.36)
\]
where, for each \( p \), the factor \( \bar{R}_{j_p} \) arises from the action of the right generator \( R_{j_p} \). Thus, we arrive with
\[
| \alpha \; \beta | v = (\bar{w}_\beta | v \alpha) = (\theta_0^{(s)} | R_{j_1} R_{j_2} \cdots R_{j_{l-1}} R_{j_l} v \alpha)
\] (5.37)

This proves the left equation of (5.29) for the case \( \zeta = \bar{n} \) and \( \varpi = \bar{m} \).

Next, we use the result of the previous paragraph to prove the left equation of (5.29) for general multiindices \( \zeta, \varpi \in \mathcal{L}^{(s)}_w \). Indeed, using lemma 4.9 and corollary 3.15, we obtain
\[
| \alpha \; \beta | v = (\bar{w}_\beta | v \alpha) = (\theta_0^{(s)} | R_{j_1} R_{j_2} \cdots R_{j_{l-1}} R_{j_l} v \alpha)
\] (5.38)
which proves the left equation of (5.29) for the general case.

2. Thanks to lemma 4.9, we may assume that \( \ell = 0 \). To begin, we prove the left equation of (5.30) for the special case \( \zeta = \bar{n} \) and \( \varpi = \bar{m} \). By linearity, we may assume that \( \alpha \in \mathcal{L}^{(s)}_n \), \( \beta \in \mathcal{L}^{(s)}_m \), and \( \gamma \in \mathcal{L}^{(s)}_w \) are link patterns. We decompose the projector box between \( \alpha \) and \( \beta \) via its recursion property (3.29),
\[
| \alpha \; \beta | = \sum_{r \in S_n^<} T_{\alpha,\beta}^{(r)}.
\] (5.39)
\( T^{(r)}_{\alpha, \beta} \in TL_{\alpha}^m \) being tangles with exactly \( r \) crossing links and \( E_{\beta}^m \) the set of all integers \( r \geq 0 \) for which such tangles exist. Then, the action of tangle (5.39) on the vector \( w_\gamma \) reads

\[
| \alpha | \; | \; \hat{\beta} | \; | \; w_\gamma \overset{(5.39)}{=} | \alpha | \; | \; \hat{\beta} | \; | \; w_\gamma + \sum_{r \in E_{\beta}^m \cap (0, s)} T^{(r)}_{\alpha, \beta} w_\gamma \overset{(3.123, 4.16)}{=} (\pi_\beta w_\gamma) w_{\alpha}, \tag{5.40}
\]

as each tangle \( T^{(r)}_{\alpha, \beta} \) appearing in the sum has \( r < s \) crossing links, so all of these tangles necessarily join two defects of \( \gamma \) together in the product \( T^{(r)}_{\alpha, \beta} \gamma \) via turn-back links, so \( T^{(r)}_{\alpha, \beta} \gamma = 0 \) by rules (3.123, 4.16). This proves the left equation of (5.29) for the case \( \zeta = \overline{n} \) and \( \varpi = \overline{m} \). We establish the general case via the same argument as in (5.38).

This concludes the proof. \( \square \)

Assuming that \( \text{rad} L^{(\iota)} = \{0\} = \text{rad} L^{(\iota)} \), and choosing a basis \( B^{(\iota)} \subset L^{(\iota)}(1) \) for the \( TL_{(\iota)} \)-standard module, for each element \( \alpha \in B^{(\iota)} \), we let \( \alpha^\vee \in L^{(\iota)} \) denote the dual of \( \alpha \) with respect to the bilinear pairing \( \langle ;, \; \rangle \),

\[
(\alpha^\vee | \beta) = \delta_{\alpha, \beta} \quad \text{for all } \alpha, \beta \in B^{(\iota)}. \tag{5.41}
\]

Similarly, choosing a basis \( B^{(\iota)} \subset L^{(\iota)} \), for each element \( \bar{\sigma} \in B^{(\iota)} \), we define the dual \( \bar{\sigma}^\vee \in L^{(\iota)} \) of \( \bar{\sigma} \) by the property

\[
(\bar{\sigma} | \bar{\beta}^\vee) = \delta_{\bar{\sigma}, \bar{\beta}} \quad \text{for all } \bar{\sigma}, \bar{\beta} \in B^{(\iota)}. \tag{5.42}
\]

Then, proposition 4.12 implies that the sets \( \{w_\alpha | \alpha \in B^{(\iota)}\} \) and \( \{w_\alpha^\vee | \alpha \in B^{(\iota)}\} \) are dual bases for \( H^{(\iota)} \) and \( \overline{H}^{(\iota)} \):

\[
w_\alpha^\vee := w_{\alpha^\vee} \implies (w_\alpha^\vee | w_\beta) = (w_{\alpha^\vee} | w_\beta) = (\alpha^\vee | \beta) = \delta_{\alpha, \beta}, \tag{5.43}
\]

and similarly, the sets \( \{\pi_\sigma | \sigma \in B^{(\iota)}\} \) and \( \{\pi_\sigma^\vee | \sigma \in B^{(\iota)}\} \) are also dual bases for \( \overline{H}^{(\iota)} \) and \( \overline{H}^{(\iota)} \):

\[
\pi_\sigma^\vee := w_{\pi_\sigma} \implies (\pi_\sigma | \pi_\sigma^\vee) = (\pi_\sigma | \pi_\sigma^\vee) = (\bar{\sigma} | \bar{\sigma}^\vee) = \delta_{\sigma, \bar{\sigma}}. \tag{5.44}
\]

Recalling the \( s \)-gradings of \( L^{(\iota)} \), \( \text{rad} L^{(\iota)} \), and \( H^{(\iota)} \) from (3.24, 3.114, 2.34), assuming that \( \text{rad} L^{(\iota)} = \{0\} = \text{rad} L^{(\iota)} \), any basis \( B^{(\iota)} \) for \( L^{(\iota)} \) or any basis \( B^{(\iota)} \) for \( L^{(\iota)} \) give rise to the above type dual basis pairs for \( L^{(\iota)} \) and \( L^{(\iota)} \) as well as for \( H^{(\iota)} \) and \( \overline{H}^{(\iota)} \).

Recalling from [FP18a, corollary 5.22 and equations (5.106-5.107)] that these assumptions are valid when \( n_\varsigma < p(q) \),

\[
n_\varsigma < p(q) \implies \text{rad} L^{(\iota)} = \bigoplus_{s \in E_\iota} \text{rad} L^{(s)} = \{0\}; \tag{5.45}
\]

and noting that \( \text{rad} L^{(\iota)} \) is isomorphic to \( \text{rad} L^{(\iota)} \) by definition (3.114), the above dual bases are always defined if \( n_\varsigma < p(q) \). In particular, assuming that \( \max(n_\varsigma, n_\varsigma^\iota) < p(q) \), we define the projection operators by homomorphic extensions of

\[
\pi_\alpha^\beta : V_\varsigma \longrightarrow V_\varsigma, \quad \pi_\alpha^\beta(v) := (w_\alpha | v)w_\alpha \quad \text{for } v \in H_\varsigma, \tag{5.46}
\]

so they are homomorphisms of \( U_q^{\iota} \) and \( U_q^{\iota} \)-modules. We also set

\[
\pi_\alpha^\beta = \pi_\alpha^{(\iota)} , \quad \iota_\alpha := \pi_\alpha^{(\iota)} , \quad \pi_\alpha := \pi_\alpha^{(\iota)} \quad \text{and} \quad \pi_\sigma^\beta = \pi_\sigma^{(\iota)} , \quad \iota_\sigma := \pi_\sigma^{(\iota)} , \quad \pi_\sigma := \pi_\sigma^{(\iota)}. \tag{5.48}
\]

To state the next lemma, we also denote by \( \mathcal{W}_s \) the valenced link pattern with \( s \) defects attached to one node.

**Lemma 5.8.** Suppose \( \max(n_\varsigma, n_\varsigma^\iota) < p(q) \). Then, the following hold:

1. With respect to any bases \( B^{(\iota)} \) and \( B^{(\iota)} \) for \( L^{(\iota)} \) and \( L^{(\iota)} \), and for all valenced link states \( \alpha \in B^{(\iota)} \cap L^{(\iota)} \) and \( \beta \in B^{(\iota)} \cap L^{(\iota)} \), the maps (3.87) send the following valenced tangles to the following \( U_q \)-homomorphisms:

\[
| \alpha \; \langle s \; \beta^\vee | \overset{\mathcal{F}_s^{\iota}}{\longrightarrow} \pi_\alpha^\beta, \tag{5.49}
\]

\[
| \alpha \; \langle s \; \beta | \overset{\mathcal{F}_s^{(\iota)}}{\longrightarrow} \iota_\alpha, \tag{5.50}
\]

\[
| \alpha \; \langle s \; \beta^\vee | \overset{\mathcal{F}_s^{\iota}}{\longrightarrow} \pi_\alpha^\beta. \tag{5.51}
\]
2. With respect to any bases $\mathbf{B}_\alpha$ and $\mathbf{B}_\beta$ for $\mathbb{L}_\alpha$ and $\mathbb{L}_\beta$, and for all valenced link states $\pi \in \mathbf{B}_\alpha \cap \mathbb{L}_\alpha^{(s)}$ and $\beta \in \mathbf{B}_\beta \cap \mathbb{L}_\beta^{(s)}$, the maps (3.90) send the following valenced tangles to the following $\mathfrak{U}_q$-homomorphisms:

\[
\begin{align*}
| \alpha \rangle \hat{\otimes} | s \rangle & \xmapsto{\mathcal{F}_\alpha^{(s)}} \tilde{\alpha}_\pi, \quad (5.52) \\
| \lessdot_s \rangle \hat{\otimes} | \beta \rangle & \xmapsto{\mathcal{F}_\beta} \beta, \quad (5.53) \\
| \alpha \rangle \hat{\otimes} | \beta \rangle & \xmapsto{\mathcal{F}_\alpha \mathcal{F}_\beta} \tilde{\alpha}_\beta. \quad (5.54)
\end{align*}
\]

**Proof.** We only prove identities (5.49–5.51), since (5.52–5.54) can be proven similarly. Also, as the first two claims (5.49, 5.50) are special cases of the last (5.51), it suffices to prove only the latter. Furthermore, definition (5.46) of $\pi^{(s)}$ and orthogonality property (2.129) from lemma 2.13 imply that if $v \notin V_\alpha^{(s)}$, then $\pi^{(s)}(v) = 0$. Also, because the link-state vectors $w_\gamma$ with $\gamma \in \mathbf{L}_\alpha^{(s)}$ form a basis for $v \in \mathbb{H}_\alpha$ by item 1 of corollary 4.14, by linearity and since $\pi^{(s)}$ is a homomorphism of $\mathfrak{U}_q$-modules, it actually suffices to prove that $| \alpha \rangle \otimes | \beta \rangle \otimes w_\gamma = \pi^{(s)}(w_\gamma)$ for all $\gamma \in \mathbf{L}_\alpha^{(s)}$: indeed,

\[
| \alpha \rangle \otimes | \beta \rangle \otimes w_\gamma \overset{\text{(5.30)}}{=} (w_\alpha | w_\gamma)w_\alpha \overset{\text{(5.46)}}{=} \pi^{(s)}(w_\gamma)
\]

for any link pattern $\gamma \in \mathbf{L}_\alpha^{(s)}$, by item 2 of lemma 5.7. This proves the lemma. \(\square\)

The next proposition gives explicit, simple diagram presentations for the consecutive-tensorand embedding $\iota^{(s)}$ and projectors $\pi^{(r,t)(s)}$ and $\bar{\pi}^{(r,t)}$, respectively defined in (2.118, 2.119, 2.120).

**Proposition 5.9.** Suppose $r + t < p(q)$, and define

\[
A^{(r,t)} := \frac{\Theta(r,s,t)(iq^{-1})^{s+t-s}}{(-1)^s[s + 1](q - q^{-1})^{\frac{s+t-s}{2}} \left\lfloor \frac{s+t-s}{2} \right\rfloor !} \quad \text{and} \quad B^{(r,t)} := \frac{(q - q^{-1})^{\frac{s+t-s}{2}} \left\lfloor \frac{s+t-s}{2} \right\rfloor !}{(iq^{-1})^{\frac{s+t-s}{2}}}.
\]

Then, the maps (3.87) send the following valenced tangles to the following $\mathfrak{U}_q$-homomorphisms:

\[
\begin{align*}
\begin{tikzpicture}[baseline=-0.5ex, scale=0.7]
\node (a) at (0,0) {$s$};
\node (b) at (1,0) {$t$};
\node (c) at (0.5,1) {$r$};
\draw[thick, ->] (a) -- (c);
\draw[thick, ->] (b) -- (c);
\end{tikzpicture} & \xmapsto{\mathcal{F}^{(r,t)}} A^{(r,t)} \bar{\pi}^{(r,t)}, \quad (5.57) \\
\begin{tikzpicture}[baseline=-0.5ex, scale=0.7]
\node (a) at (0,0) {$s$};
\node (b) at (1,0) {$t$};
\node (c) at (0.5,1) {$r$};
\draw[thick, ->] (a) -- (c);
\end{tikzpicture} & \xmapsto{\mathcal{F}^{(s)}} \iota^{(s)}, \quad (5.58) \\
\begin{tikzpicture}[baseline=-0.5ex, scale=0.7]
\node (a) at (0,0) {$s$};
\node (b) at (1,0) {$t$};
\node (c) at (0.5,1) {$r$};
\draw[thick, ->] (a) -- (c);
\end{tikzpicture} & \xmapsto{\mathcal{F}^{(r,t)}} B^{(r,t)} \iota^{(r,t)}, \quad (5.59)
\end{align*}
\]

Similarly, define

\[
\begin{align*}
\begin{tikzpicture}[baseline=-0.5ex, scale=0.7]
\node (a) at (0,0) {$r$};
\node (b) at (1,0) {$t$};
\node (c) at (0.5,1) {$s$};
\draw[thick, ->] (a) -- (c);
\end{tikzpicture} & \xmapsto{\mathcal{F}^{(r,t)}} A^{(r,t)} \bar{\pi}^{(r,t)}, \quad (5.60) \\
\begin{tikzpicture}[baseline=-0.5ex, scale=0.7]
\node (a) at (0,0) {$r$};
\node (b) at (1,0) {$t$};
\node (c) at (0.5,1) {$s$};
\draw[thick, ->] (a) -- (c);
\end{tikzpicture} & \xmapsto{\mathcal{F}^{(s)}} \iota^{(s)}, \quad (5.61) \\
\begin{tikzpicture}[baseline=-0.5ex, scale=0.7]
\node (a) at (0,0) {$r$};
\node (b) at (1,0) {$t$};
\node (c) at (0.5,1) {$s$};
\draw[thick, ->] (a) -- (c);
\end{tikzpicture} & \xmapsto{\mathcal{F}^{(r,t)}} B^{(r,t)} \iota^{(r,t)}, \quad (5.62)
\end{align*}
\]
Proof. To prove (5.57), we recall from lemma 5.2 that

\[ \beta_s = \begin{array}{c} \vdots \\ \vdots \\ s \end{array} \xrightarrow{(5.7)} \begin{array}{c} r \\ t \\ \vdots \\ \vdots \end{array} \]

and the diagram action on the left side of (5.66) are

\[ w_{\beta_s} = \frac{(q - q^{-1})^{e_{r,s,t}^{u,s}}}{(\Theta(q)^{1/2})^{e_{r,s,t}^{u,s}}} u^{(s)}. \]  

(5.64)

Also, we have the duality

\[ \beta_s = \begin{array}{c} \vdots \\ \vdots \\ s \end{array} \xrightarrow{(5.18)} \begin{array}{c} r \\ t \\ \vdots \\ \vdots \end{array} \]

\[ \beta_s = \frac{(-1)^{s[s + 1]} \Theta(r, s, t)}{\Theta(r, s, t)} \begin{array}{c} \vdots \\ \vdots \\ s \end{array} \xrightarrow{(2.120)} \beta_s = \frac{(\Theta(q)^{1/2})^{e_{r,s,t}^{u,s}}}{(q - q^{-1})^{e_{r,s,t}^{u,s}}} \pi^{(s)}(u_{(r,t)}). \]  

(5.65)

Now, corollary 4.14 implies that the collection \( \{ w_{\beta_u} \mid u \in E_{(r,t)} \} \) is a basis for \( H_{(r,t)} \). Selecting an arbitrary vector \( w_{\beta_s} \) from this set and inserting \( \beta_s \) into (5.49), lemma 5.8 gives

\[ \frac{(-1)^{s[s + 1]} \Theta(r, s, t)}{\Theta(r, s, t)} \begin{array}{c} \vdots \\ \vdots \\ s \end{array} \xrightarrow{(5.49)} \pi^{\beta_s}(w_{\beta_u}) = (w_{\beta_s}, w_{\beta_u}) e_0^{(s)} = \delta_{s,u} e_0^{(s)} \]  

(5.66)

Furthermore, because the map \( \pi^{(r,t)} \) and the diagram action on the left side of (5.66) are \( U_q \)-homomorphisms according to lemma A.2 and item 1 of lemma 3.16 respectively, (5.66) extends for all vectors \( v \in \{ F_r w_{\beta_u} \mid u \in E_{(r,t)} \} \):

\[ \frac{(-1)^{s[s + 1]} \Theta(r, s, t) (\Theta(q)^{1/2})^{e_{r,s,t}^{u,s}}}{(q - q^{-1})^{e_{r,s,t}^{u,s}}} \pi^{(r,t)}(v). \]  

(5.68)

Proposition 4.16 says that this set is a basis for \( V_{(r,t)} \) if \( r + t < p(q) \), so (5.68) holds for all \( v \in V_{(r,t)} \), and (5.57) follows.

To finish, identity (5.58) can be proven similarly as above, (5.59) then follows from (5.57, 5.58) and relation (A.13), and identities (5.61–5.63) can be proven similarly.

The next lemma shows that the image of the representation \( \mathcal{S} \) of the valenced Temperley-Lieb algebra \( TL_q(\nu) \) is generated by the projectors on \( V_\varsigma \) acting on consecutive tensorands. In section 6 A, we show that the representation \( \mathcal{S}_\varsigma \) is in fact faithful, which thus identifies \( TL_q(\nu) \) with its image.

**Lemma 5.10.** Suppose \( n_\varsigma < p(q) \). Then, the image of the representation \( \mathcal{S}_\varsigma : TL_q(\nu) \rightarrow \text{End} V_\varsigma \) is generated by the collection of all submodule projectors that act strictly on consecutive pairs of tensorands of vectors in \( V_\varsigma \):

\[ \mathcal{S}_\varsigma(TL_q(\nu)) = \langle \pi^{(s_i,s_i+1)}_{(s_i,s_i+1)} | s \in E_{(s_i,s_i+1)}, i \in \{1, 2, \ldots, d_\varsigma - 1\} \rangle. \]  

(5.69)

Similarly, this proposition holds after the symbolic replacements \( \mathcal{F} \mapsto \mathcal{F}, \nu \mapsto \nu, \) and \( \pi \mapsto \pi \).
Proof. Identity (5.59) (resp. (5.63)) of proposition 5.9 and lemma 3.10 together imply that

\[
\begin{array}{c}
\quad \\
(\frac{-1}{s} \Theta(s_i, s_{i+1}, s)) \times \\
\mid s_i \\
\mid s_{i+1} \\
\mid s_{i+2} \\
\quad \\
\end{array}
\] \quad (3.77) \Rightarrow (5.59) \quad \pi^{(s_i, s_{i+1})}_{(s)} \quad (5.70)

(resp. for \( \pi^{(s_i, s_{i+1})}_{(s)} \)), for all \( s \in \mathbb{E}(s_i, s_{i+1}) \) and \( i \in \{1, 2, \ldots, d_\varsigma - 1\} \). With \( n_\varsigma < \text{p}(q) \), the prefactor in (5.70) is finite
and nonvanishing. By [FP18a, proposition 2.10] (see also [FP18b, theorem 1.1]), we know that the collection of all
tangles on the left side of (5.70) generates \( TL_\varsigma(\nu) \). Because \( \mathcal{F}_\varsigma \) is a homomorphism of algebras, the claim follows. \( \square \)

Using the previous results, we also obtain a diagrammatic proof for the commuting diagram in [Pel20, Lemma 3.10],
which gives means to calculate compositions of projectors explicitly.

Lemma 5.11. Suppose \( r + t < \text{p}(q) \), and denote \( \hat{\pi} := \hat{\pi}^{(1,1)}_{(0)} \) and \( k := \frac{r + t - s}{2} \). Then, we have

\[
\left( \frac{[r-k][r-t][r+t-k+1]}{2^k [r] [s+1] [t]} \right) \hat{\pi}^{(r,t)}_{(s)} = \hat{\pi}^{(1,1)}_{(0)} \circ \left( \hat{\pi}_{r-k+1,r-k+2} \circ \hat{\pi}_{r-k+2,r-k+3} \circ \cdots \circ \hat{\pi}_{r-1,r} \circ \hat{\pi}_{r,r+1} \right) \circ \mathcal{F}(r,t). \quad (5.71)
\]

Equivalently, the following diagram commutes:

\[
\begin{array}{c}
V_r \otimes V_t \\
\quad \\
\quad \\
\quad \\
V_{r-1} \otimes V_{t-1} \\
\quad \\
\quad \\
V_{r-k+1} \otimes V_{t-k+1} \\
\quad \\
\quad \\
V_{r-k} \otimes V_{t-k} = V_s \\
\quad \\
\quad \\
\quad \\
\quad \\
\end{array}
\] \quad (5.72)

Similarly, with \( \hat{\pi} := \hat{\pi}^{(1,1)}_{(0)} \), we have

\[
\left( \frac{[r-k][r-t][r+t-k+1]}{2^k [r] [s+1] [t]} \right) \hat{\pi}^{(r,t)}_{(s)} = \hat{\pi}^{(1,1)}_{(0)} \circ \left( \hat{\pi}_{r-k+1,r-k+2} \circ \hat{\pi}_{r-k+2,r-k+3} \circ \cdots \circ \hat{\pi}_{r-1,r} \circ \hat{\pi}_{r,r+1} \right) \circ \mathcal{F}(r,t). \quad (5.73)
\]
Proof. Proposition 5.9 and corollary 3.15 show that $\hat{\varpi}^{(r,t)}$ is the image of the following valenced tangle under $\mathcal{S}^{(r,t)}$:

$$\begin{align*}
\frac{1}{A^{(r,t)}} & \quad \begin{tikzpicture}[baseline=-0.5ex]
\node (s) at (0,0) {$s$};
\node (t) at (2,0) {$t$};
\node (r) at (1,1) {$r$};
\draw (s) -- (r) node [midway, above] {\scriptsize $1$};
\draw (t) -- (r) node [midway, above] {\scriptsize $A^{(r,t)}$};
\end{tikzpicture} \\
\overset{(P1')}{(3.32)} \quad & \quad \begin{tikzpicture}[baseline=-0.5ex]
\node (s) at (0,0) {$s$};
\node (t) at (2,0) {$t$};
\node (r) at (1,1) {$r$};
\draw (s) -- (r) node [midway, above] {\scriptsize $1$};
\draw (t) -- (r) node [midway, above] {\scriptsize $I^{(r,t)}$};
\end{tikzpicture}
\end{align*}$$

Decomposing the three-vertex in this tangle, we have

$$\begin{align*}
\begin{tikzpicture}[baseline=-0.5ex]
\node (s) at (0,0) {$s$};
\node (t) at (2,0) {$t$};
\node (r) at (1,1) {$r$};
\draw (s) -- (r) node [midway, above] {\scriptsize $1$};
\draw (t) -- (r) node [midway, above] {\scriptsize $1$};
\end{tikzpicture} & \overset{(1.30)}{=} \begin{tikzpicture}[baseline=-0.5ex]
\node (s) at (0,0) {$s$};
\node (t) at (2,0) {$t$};
\node (A) at (1,1) {$A$};
\node (B) at (3,1) {$B$};
\node (C) at (5,1) {$C$};
\draw (s) -- (A) node [midway, above] {\scriptsize $1$};
\draw (t) -- (A) node [midway, above] {\scriptsize $2$};
\draw (A) -- (B) node [midway, above] {\scriptsize $3$};
\draw (B) -- (C) node [midway, above] {\scriptsize $k$};
\end{tikzpicture}
\end{align*}$$

We consider the image of the valenced tangle (5.75) under the map $\mathcal{S}^{(r,t)}$. We use (3.84) of corollary 3.12 to interpret part C of (5.75) as the projector $\mathcal{P}_{s} = \mathcal{P}_{s} \otimes \mathcal{P}_{s}$, (3.66) of lemma 3.6 to interpret each link in part B of (5.75) as

$$\left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right) \left( \mathcal{I}_{s}^{(r)} \otimes \mathcal{I}_{s}^{(t)} \right) = \left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right) \hat{\varpi}_{j+1}$$

for some appropriate index $j \in \{1, 2, \ldots, r+t\}$, and (3.78) of lemma 3.11 to interpret part A of (5.75) as the projector $\mathcal{P}_{s}$. After combining these three parts, we obtain

$$\begin{align*}
(5.75) & \overset{\mathcal{S}^{(r,t)}}{\underset{(5.74-5.77)}{\longrightarrow}} \begin{align*}
\frac{1}{A^{(r,t)}} \mathcal{P}_{s} \circ \left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right) \hat{\varpi}_{r-k+1,r-k+2} \cdots \hat{\varpi}_{r-1,r} \circ \left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right) \hat{\varpi}_{r,r+1} \circ \mathcal{P}_{s} \circ \mathcal{J}_{s} = \begin{align*}
\frac{1}{A^{(r,t)}} \mathcal{P}_{s} \circ \left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right) \hat{\varpi}_{r-k+1,r-k+2} \cdots \hat{\varpi}_{r-1,r} \circ \mathcal{J}_{s}
\end{align*}
\end{align*}
\end{align*}$$

with labeled parts in (5.77) matching those in (5.75). To finish, using lemma A.2 and corollary 3.15, we identify the two representations (5.74) and (5.77) for the image of $\hat{\varpi}^{(r,t)}$ under the map $\mathcal{S}^{(r,t)}$, to obtain

$$\begin{align*}
\hat{\varpi}^{(r,t)} & \overset{(3.95)}{\underset{(5.74-5.77)}{=}} \frac{1}{A^{(r,t)}} \left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right)^{k} \mathcal{P}_{s} \circ \mathcal{J}_{s} = \frac{1}{A^{(r,t)}} \left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right)^{k} \mathcal{P}_{s} \circ \mathcal{J}_{s} = \frac{1}{A^{(r,t)}} \left( \frac{\imath \nu q^{1/2}}{q - q^{-1}} \right)^{k} \mathcal{P}_{s} \circ \mathcal{J}_{s}.
\end{align*}$$

Simplifying the result to (5.71) via formula (2.133) for the Theta network, we obtain asserted identity (5.71).

\section{C. Semisimple quotient modules of highest-weight vectors}

In this section, we identify quotients of the \(\mathcal{U}_{\mathcal{q}}\)-highest-weight vector spaces $\mathcal{H}_{\mathcal{q}}$ and $\mathcal{H}_{\mathcal{q}}$ with simple $\mathcal{TL}_{\mathcal{q}}(\nu)$-modules via the link state -- highest-weight vector correspondence (proposition 4.12). In general, all simple $\mathcal{TL}_{\mathcal{q}}(\nu)$-modules are obtained from the standard modules $\mathcal{L}_{\nu}$ by taking their quotients by their radicals $\mathcal{L}_{\nu}$ with respect to the link state bilinear pairing $(\cdot, \cdot)$ (recall 3.114--3.116 from section 3D and [FP18a, proposition 6.7]). We verify in lemma 5.13 that this radical embeds into the radical of $\mathcal{H}_{\mathcal{q}}$ with respect to the bilinear pairing $(\cdot, \cdot)$. Then, in proposition 5.14 we prove that such a quotient respects the link state -- highest-weight vector correspondence.

First, using the bilinear pairing $(\cdot, \cdot)$ we can define a notion of orthogonality within $\mathcal{H}_{\mathcal{q}} \times \mathcal{H}_{\mathcal{q}}$. (See also appendix D.) In particular, we define the orthocomplement of $\{w_{\alpha} | \alpha \in \mathcal{L}_{\nu}\}$ in $\mathcal{H}_{\mathcal{q}}$ to be

$$\{w_{\alpha} | \alpha \in \mathcal{L}_{\nu}\}^{ot} := \{v \in \mathcal{H}_{\mathcal{q}} | (\varpi_{\alpha} | v) = 0 \text{ for all } \alpha \in \mathcal{L}_{\nu} \} \subset \mathcal{H}_{\mathcal{q}}.$$  

We similarly define the orthocomplement $\{\varpi_{\alpha} | \alpha \in \mathcal{L}_{\nu}\}^{ot} \subset \mathcal{H}_{\mathcal{q}}$. Also, for each $s \in \mathcal{E}_{\mathcal{q}}$, we define

$$\{w_{\alpha} | \alpha \in \mathcal{L}_{\nu}^{(s)}\}^{ot} := \{v \in \mathcal{H}_{\mathcal{q}} | (\varpi_{\alpha} | v) = 0 \text{ for all } \alpha \in \mathcal{L}_{\nu}^{(s)} \} \subset \mathcal{H}_{\mathcal{q}}^{(s)},$$

and we similarly define $\{\varpi_{\alpha} | \alpha \in \mathcal{L}_{\nu}^{(s)}\}^{ot} \subset \mathcal{H}_{\mathcal{q}}^{(s)}$, where we use the convention that $\mathcal{L}_{\nu}^{(s)}\mathcal{L}_{\nu}^{(s)} = \{0\}$ if $s \notin \mathcal{E}_{\mathcal{q}}$. 

\[\]
Lemma 5.12. Suppose \( \max \varsigma < p(\eta) \).

1. The vector space \( \{ w_\alpha \mid \alpha \in \mathcal{L}_\varsigma \}^\perp \) is closed under the left \( \mathcal{T}_\varsigma(\nu) \)-action on it.
2. For each \( s \in E_\varsigma \), the vector space \( \{ w_\alpha \mid \alpha \in L^{(s)}_\varsigma \}^\perp \) is closed under the left \( \mathcal{T}_\varsigma(\nu) \)-action on it.
3. We have the following direct-sum decomposition of \( \mathcal{T}_\varsigma(\nu) \)-submodules of \( \mathcal{H}_\varsigma \):

\[
\mathcal{H}_\varsigma = \left( \bigoplus_{s \in E_\varsigma} \mathcal{H}_\varsigma \right) \oplus \left( \bigoplus_{s \in E_\varsigma \setminus E_\varsigma} \mathcal{H}_\varsigma \right).
\]

(5.82)

Similarly, this lemma holds for the right \( \mathcal{T}_\varsigma(\nu) \)-action on \( \mathcal{H}_\varsigma \) after the symbolic replacements

\[
w_\alpha \mapsto \overline{w}_\alpha, \quad \alpha \mapsto \overline{\alpha}, \quad \mathcal{L}_\varsigma \mapsto \overline{\mathcal{L}_\varsigma}, \quad \text{and} \quad \mathcal{H}_\varsigma \mapsto \overline{\mathcal{H}_\varsigma}.
\]

(5.83)

Proof. It follows from corollary 3.20 that, for all vectors \( v \in \{ w_\alpha \mid \alpha \in \mathcal{L}_\varsigma \}^\perp \) and valenced tangles \( T \in \mathcal{T}_\varsigma(\nu) \), we have

\[
v \in \{ w_\alpha \mid \alpha \in \mathcal{L}_\varsigma \}^\perp \quad \Rightarrow \quad (\overline{m}_\alpha \mid v) = 0 \quad \text{for all } \overline{\alpha} \in \overline{\mathcal{L}_\varsigma}.
\]

(5.84)

This proves item 1, and a similar argument proves item 2. To prove item 3, we first note that by lemma 2.1, any vector \( v \in \mathcal{H}_\varsigma \) can be written in the form

\[
v = \sum_{s \in E_\varsigma} v^{(s)}, \quad \text{where } v^{(s)} \in \mathcal{H}^{(s)}_\varsigma.
\]

(5.87)

Now, using lemma 4.8, direct-sum decomposition (3.24) of \( \overline{\mathcal{L}_\varsigma} \), and the orthogonality of the spaces \( \overline{\mathcal{H}}^{(s)}_\varsigma \) and \( \mathcal{H}^{(t)}_\varsigma \) for \( s \neq t \) from item 3 of lemma 2.13, we have

\[
v \in \{ w_\alpha \mid \alpha \in \mathcal{L}_\varsigma \}^\perp \quad \Rightarrow \quad \sum_{s \in E_\varsigma} (\overline{m}_\alpha \mid v^{(s)}) = 0 \quad \text{for all } \overline{\alpha} \in \overline{\mathcal{L}_\varsigma}.
\]

(5.88)

\[
v \in \{ w_\alpha \mid \alpha \in \mathcal{L}_\varsigma \}^\perp \quad \Leftrightarrow \quad \sum_{s \in E_\varsigma} (\overline{m}_\alpha \mid v^{(s)}) = 0 \quad \text{for all } \overline{\alpha} \in \overline{\mathcal{L}_\varsigma}, \quad \text{and all } s \in E_\varsigma.
\]

(5.89)

This proves (5.82). The assertions for the right \( \mathcal{T}_\varsigma(\nu) \)-action on \( \overline{\mathcal{H}_\varsigma} \) can be proven similarly. \( \square \)

Using the bilinear pairing, we also define radicals of subspaces of \( \overline{\mathcal{H}_\varsigma} \) and \( \mathcal{H}_\varsigma \). In particular, we define

\[
\operatorname{rad} \{ w_\alpha \mid \alpha \in \mathcal{L}_\varsigma \} := \{ w_\alpha \mid \alpha \in \mathcal{L}_\varsigma \text{ and } (\overline{m}_\alpha \mid w_\alpha) = 0 \text{ for all } \overline{\beta} \in \overline{\mathcal{L}_\varsigma} \}
\]

(5.91)

\[
\overline{\mathcal{H}_\varsigma} \subset \mathcal{H}_\varsigma.
\]

(5.92)

We similarly define \( \operatorname{rad} \{ \overline{m}_\alpha \mid \alpha \in \overline{\mathcal{L}_\varsigma} \} \subset \overline{\mathcal{H}_\varsigma} \). Also, for each \( s \in E_\varsigma \), we define

\[
\operatorname{rad} \{ w_\alpha \mid \alpha \in L^{(s)}_\varsigma \} := \{ w_\alpha \mid \alpha \in L^{(s)}_\varsigma \text{ and } (\overline{m}_\alpha \mid w_\alpha) = 0 \text{ for all } \overline{\beta} \in \overline{L^{(s)}_\varsigma} \}
\]

(5.93)

\[
\overline{L^{(s)}_\varsigma} \subset \overline{L^{(s)}_\varsigma} \subset \overline{\mathcal{H}^{(s)}_\varsigma}.
\]

(5.94)

and we similarly define \( \operatorname{rad} \{ \overline{m}_\alpha \mid \alpha \in L^{(s)}_\varsigma \} \subset \overline{\mathcal{H}^{(s)}_\varsigma} \). As in the proof of lemma 5.12, corollary 3.20 implies that these spaces are closed under their \( \mathcal{T}_\varsigma(\nu) \)-actions, and linearity and similar arguments as in (5.87–5.90) show that

\[
\mathcal{H}_\varsigma = \bigoplus_{s \in E_\varsigma} \operatorname{rad} \{ w_\alpha \mid \alpha \in L^{(s)}_\varsigma \}.
\]

(5.95)
Finally, we define the radical of $H_{\varsigma}$ as
\[
\text{rad } H_{\varsigma} := \{ v \in H_{\varsigma} \mid (w \mid v) = 0 \text{ for all } w \in \overline{H}_{\varsigma} \} \subset H_{\varsigma},
\]
we define $\text{rad} \overline{H}_{\varsigma} \subset \overline{H}_{\varsigma}$ similarly, and for each $s \in E_{\varsigma}^\perp$, we define
\[
\text{rad } H_{\varsigma}^{(s)} := \{ v \in H_{\varsigma}^{(s)} \mid (w \mid v) = 0 \text{ for all } w \in \overline{H}_{\varsigma}^{(s)} \} \subset H_{\varsigma}^{(s)},
\]
and similarly $\text{rad} \overline{H}_{\varsigma}^{(s)} \subset \overline{H}_{\varsigma}^{(s)}$. As above, corollary 3.20 implies that these spaces are closed under their $T L_{\varsigma}(\nu)$-actions, and linearity and similar arguments as in (5.87–5.90) show that
\[
\text{rad } H_{\varsigma} \circ \tau_L = \bigoplus_{s \in E_{\varsigma}^\perp} \text{rad } H_{\varsigma}^{(s)} \circ \tau_L.
\]

The next lemma says that the radical of $L_{\varsigma}$ with respect to the link state bilinear pairing $(\cdot \mid \cdot)$ embeds into the radical of $H_{\varsigma}$ with respect to the bilinear pairing $(\cdot \mid \cdot)$. This fact implies that the complementary subspace of $L_{\varsigma}$ inside $H_{\varsigma}$ is orthogonal to $L_{\varsigma}$, as stated in (5.106–5.107) below.

**Lemma 5.13.** Suppose $\text{max } \varsigma < p(q)$.

1. We have $\text{rad } \{ w_\alpha \mid \alpha \in L_{\varsigma} \} \subset \text{rad } H_{\varsigma}$.
2. For each $s \in E_{\varsigma}$, we have $\text{rad } \{ w_\alpha \mid \alpha \in L_{\varsigma}^{(s)} \} \subset \text{rad } H_{\varsigma}^{(s)}$.

Similarly, this lemma holds after the symbolic replacements $w_\alpha \mapsto w_\alpha, \alpha \mapsto \alpha, L \mapsto \overline{L}$, and $H \mapsto \overline{H}$.

**Proof.** We postpone the proof of this lemma to the next section 5D.

**Proposition 5.14.** Suppose $\text{max } \varsigma < p(q)$. The following hold:

1. The map $\alpha \mapsto w_\alpha$ induces the following isomorphism of $T L_{\varsigma}(\nu)$-modules:
   \[
   Q_{\varsigma} \cong \frac{\text{rad } H_{\varsigma}}{\text{rad } \{ w_\alpha \mid \alpha \in L_{\varsigma} \}^\perp}.
   \]

2. For each $s \in E_{\varsigma}$, the map $\alpha \mapsto w_\alpha$ induces the following isomorphism of $T L_{\varsigma}(\nu)$-modules:
   \[
   Q_{\varsigma}^{(s)} \cong \frac{\text{rad } H_{\varsigma}^{(s)}}{\text{rad } \{ w_\alpha \mid \alpha \in L_{\varsigma}^{(s)} \}^\perp}.
   \]

3. The induced isomorphism of (5.100) respects $s$-grading (2.36, 3.116, 5.82), i.e., we have the following direct-sum decomposition of $T L_{\varsigma}(\nu)$-modules:
   \[
   Q_{\varsigma} \cong \bigoplus_{s \in E_{\varsigma}} \frac{\text{rad } H_{\varsigma}^{(s)}}{\text{rad } \{ w_\alpha \mid \alpha \in L_{\varsigma}^{(s)} \}^\perp} \cong \bigoplus_{s \in E_{\varsigma}} Q_{\varsigma}^{(s)}.
   \]

Similarly, this proposition holds after the symbolic replacements
\[
\alpha \mapsto \alpha, \quad w_\alpha \mapsto w_\alpha, \quad Q \mapsto \overline{Q}, \quad H \mapsto \overline{H}, \quad \text{and } L \mapsto \overline{L}.
\]

**Proof.** We only consider the case of $Q_{\varsigma}$; the assertions for $\overline{Q}_{\varsigma}$ can be proven similarly. The desired induced map is $[\alpha] \mapsto [w_\alpha]$. To see that this map is well-defined, using item 4 of proposition 4.12, we observe that
\[
[\beta] = [\gamma] \overset{(3.116)}{\iff} \beta - \gamma \in \text{rad } L_{\varsigma} \overset{(5.80)}{\iff} \overset{(5.92)}{\beta \in \text{rad } L_{\varsigma}} \quad \iff \quad [w_\beta] = [w_\gamma].
\]
To prove item 1, we must show that the map $[\alpha] \mapsto [w_\alpha]$ is an isomorphism of $T L_{\varsigma}(\nu)$-modules from $L_{\varsigma}$ to the right side of (5.100). First, it is a homomorphism of $T L_{\varsigma}(\nu)$-modules by construction. Second, it is a linear injection because
To prove that the map $[\alpha] \mapsto [w_\alpha]$ is surjective, we use lemma D.3 from appendix D (whose assumptions are guaranteed by item 1 of lemma 5.13) to the subspaces $\{w_\alpha \mid \alpha \in L_\varsigma\} \subset H_\varsigma$ and $\{\bar{w}_\sigma \mid \sigma \in L_\varsigma\} \subset \bar{H}_\varsigma$ to deduce that there exist subspaces $W_1, W_2 \subset H_\varsigma$ such that
\[
\{w_\alpha \mid \alpha \in L_\varsigma\} = W_1 \oplus \text{rad} \{w_\alpha \mid \alpha \in L_\varsigma\} \quad \text{and} \quad \{w_\alpha \mid \alpha \in L_\varsigma\}^\perp = \text{rad} \{w_\alpha \mid \alpha \in L_\varsigma\} \oplus W_2,
\] (5.106)
and in particular,
\[
H_\varsigma = W_1 \oplus \text{rad} \{w_\alpha \mid \alpha \in L_\varsigma\} \oplus W_2.
\] (5.107)

From (5.106, 5.107), we now see that for each vector
\[
[v] \in \frac{H_\varsigma}{\{w_\alpha \mid \alpha \in L_\varsigma\}^\perp},
\] (5.108)
there exists a valenced link state $\alpha \in L_\varsigma$ such that $[v] = [w_\alpha] \in Q_\varsigma$. This shows that the map $[\alpha] \mapsto [w_\alpha]$ is surjective from $L_\varsigma$ to the right side of (5.100) finishing the proof of item 1. Similar work shows item 2, and the $s$-grading preservation asserted in item 3 then immediately follows from items 1–2 combined with item 3 of proposition 4.12 and $s$-grading (2.36) of $H_\varsigma$. The corresponding statements with replacements (5.103) and be proven similarly. □

D. Link state radical inside the highest-weight vector space

In this section, we consider the following images of the highest-weight vector spaces under the embedding $\mathcal{J}_\varsigma$ (2.114):
\[
N_\varsigma^{(s)} := \{\mathcal{J}_\varsigma(v) \mid v \in H_\varsigma^{(s)}\} \quad \text{and} \quad \bar{N}_\varsigma^{(s)} := \{\bar{\mathcal{J}}_\varsigma(\bar{v}) \mid \bar{v} \in \bar{H}_\varsigma^{(s)}\}.
\] (5.109)

By item 4 of lemma 2.13, the embedding $\mathcal{J}_\varsigma$ preserves the bilinear pairing, so we have
\[
\text{rad } N_\varsigma^{(s)} = \{\mathcal{J}_\varsigma(v) \mid v \in \text{rad } H_\varsigma^{(s)}\}.
\] (5.110)

Lemma 5.15. Suppose $\max \varsigma < q(p)$. For each $s \in E_n$, we have
\[
\text{rad } N_\varsigma^{(s)} = N_\varsigma^{(s)} \cap \text{rad } H_\varsigma^{(s)} = \{\bar{\mathcal{J}}_\varsigma(v) \mid v \in \text{rad } H_\varsigma^{(s)}\}.
\] (5.111)

Similarly, this lemma holds after the symbolic replacements $\mathcal{N} \mapsto \bar{\mathcal{N}}$, $H \mapsto \bar{H}$, and $P \mapsto \bar{P}$.

Proof. This lemma can be proven similarly as [FP18a, lemma B.4], by making the symbolic replacements $K \mapsto N$, $L \mapsto H$, and $P \mapsto P$; or $K \mapsto \bar{N}$, $L \mapsto \bar{H}$, and $P \mapsto \bar{P}$. We leave the details to the reader. □

Corollary 5.16. Suppose $\max \varsigma < q(p)$. For each $s \in E_n$, we have
\[
\text{rad } H_\varsigma^{(s)} = \{\bar{\mathcal{J}}_\varsigma(v) \mid v \in \text{rad } H_\varsigma^{(s)}\}.
\] (5.112)

Similarly, this corollary holds after the symbolic replacements $H \mapsto \bar{H}$ and $\bar{P} \mapsto \bar{P}$.

Proof. This corollary can be proven similarly as [FP18a, corollary B.5], by making the symbolic replacements $L \mapsto H$ and $\bar{P} \mapsto \bar{P}$; or $L \mapsto \bar{H}$ and $\bar{P} \mapsto \bar{P}$. We leave the details to the reader. □

Lemma 5.17.

1. We have $\text{rad } \{w_\alpha \mid \alpha \in L_n\} \subset \text{rad } H_n$.

2. For each $s \in E_n$, we have $\text{rad } \{w_\alpha \mid \alpha \in L_n^{(s)}\} \subset \text{rad } H_n^{(s)}$.

Similarly, this lemma holds after the symbolic replacements $w \mapsto \bar{w}$, $\alpha \mapsto \bar{\alpha}$, $L \mapsto \bar{L}$, and $H \mapsto \bar{H}$.
Proof. Item 1 follows from item 2 with direct-sum decompositions (5.95, 5.99). Therefore, it suffices to prove item 2.

To begin, we note that

\[
\{ w_\alpha \mid \alpha \in \text{rad } L_n^{(s)} \} \overset{(5.92)}{=} \text{rad } \{ w_\alpha \mid \alpha \in L_n^{(s)} \},
\]

for any \( s \in E_n \). We see from (5.113) that if \( \text{rad } L_n^{(s)} = \{ 0 \} \), then the right side of (5.113) equals zero, so the assertion in item 2 is trivial. Thus, without loss of generality, we assume that \( \text{rad } L_n^{(s)} \neq \{ 0 \} \). In this case, we have

\[
\text{rad } L_n^{(s)} \neq \{ 0 \} \implies \left\{ \begin{array}{ll}
1 < p(q) < \infty & \text{by [FP18a, corollary 5.2]}, \\
p(q) \mid (s + 1) & \text{by [FP18a, lemma 5.3]}. 
\end{array} \right.
\]

(5.114)

We prove item 2 by induction on \( n \in \mathbb{Z}_{\geq 0} \). In the initial case \( n = 1 \), there is nothing to prove, because both radicals \( \text{rad } \{ w_\alpha \mid \alpha \in L_1^{(1)} \} \) and \( \text{rad } H_1^{(1)} \) are trivial. Thus, we assume that, for some \( n \geq 2 \) and for all \( r \in E_{n-1} \),

\[
\text{rad } \{ w_\beta \mid \beta \in L_{n-1}^{(1)} \} \subseteq \text{rad } H_n^{(1)}.
\]

(5.115)

Recalling from [FP18a, definition 4.3] (see also (5.24) in section 5A) the definition of the trivalent link state \( \alpha \in L_n^{(s)} \) associated to the link pattern \( \alpha \in \text{LP}_n^{(s)} \), we know by [FP18a, proposition 5.7] that \( \text{rad } L_n^{(s)} \) has a basis of the form

\[
\{ \alpha \mid \alpha \in \text{LP}_n^{(s)}, \text{tail}(\alpha) \in \text{R}_n^{(s)} \},
\]

(5.116)

where \( \text{tail}(\alpha) \) is defined in [FP18a, equations (4.46–4.47), section 4A] and the set \( \text{R}_n^{(s)} \) of “radical tails” is defined in [FP18a, beneath equation (5.2), section 5A]. (We refer to [FP18a, sections 4–5] for more details.) In particular, because the map \( \alpha \mapsto w_\alpha \) is a linear injection by item 2 of proposition 4.12, the radical (5.113) has a basis of the form

\[
\{ w_\alpha \mid \alpha \in \text{LP}_n^{(s)}, \text{tail}(\alpha) \in \text{R}_n^{(s)} \}.
\]

(5.117)

Hence, to finish the induction step, it suffices to show that all of the basis vectors \( w_\alpha \) in (5.117) belong to \( \text{rad } H_n^{(s)} \).

To establish this, we will derive a recursive formula for \( \langle \bar{v} \mid w_\alpha \rangle \) for arbitrary vectors \( \bar{v} \in H_n^{(s)} \) and \( w_\alpha \) in the basis (5.117). To begin, using item 2 of lemma 2.2, we write \( \bar{v} \) in the form

\[
\bar{v} \overset{(2.43)}{=} \bar{v}_0 \otimes \varepsilon_1 + \bar{v}_1 \otimes \varepsilon_0, \quad \text{where} \quad \left\{ \begin{array}{l}
\bar{v}_0 \in \overline{H}_n^{(s+1)} , \\
\bar{v}_1 \in \overline{V}_n^{(s+1)} , \\
n \bar{v}_1.F = -q^{-s-1} \varepsilon_0.
\end{array} \right.
\]

(5.118)

Next, we derive a similar recursive formula for the vector \( w_\alpha \) in the set (5.117). In general, by lemma 4.23, we have

\[
\bar{v} \overset{(2.43)}{=} w_\alpha \overset{(4.110)}{=} \bar{w}_\alpha \overset{(2.58)}{=} \bar{w}_\alpha',
\]

(5.119)

for some link pattern \( \alpha' \in L_{n-1}^{(r_{n-1})} \). Furthermore, by (2.58), the penultimate height in (5.119) equals

\[
\text{rad } \{ s - 1, s + 1 \} \subseteq \mathbb{E}_{(t,s)} \overset{(2.56)}{=} \{ s - 1, s + 1 \}.
\]

(5.120)

We consider these two cases separately:

(a): When \( r_n = s = r_{n-1} - 1 \), the rightmost closed three-vertex (1.30) in \( \alpha \) reads

\[
\overset{r_{n-1}}{\overset{r_n}{\text{1}}} = \overset{r_{n-1}}{\overset{r_n}{\text{1}}}
\]

(5.121)
Using property (P1'), we absorb the top projector box of size $r_n$ into the top projector box of size $r_{n-1}$. Bending the resulting diagram, orienting its defects, and using formulas (3.128, 3.136) and lemma 4.23, we arrive with

\[
\begin{align*}
\begin{array}{c}
1 \\
1
\end{array} & \quad \begin{array}{c}
1 \\
1
\end{array}
\end{align*}
\]

Using lemmas 2.11 and 2.13, we calculate (3.128, 3.136) and inserting it into the diagram (5.119) for $w$, we arrive with the resulting diagram, orienting its defects, and using formulas (3.128, 3.136) and lemma 4.23, we arrive with the following recursive formula for $w$:  

\[
w_{\omega}^{(3.126)} = \frac{-iq^{-1/2}(v_{0} \otimes \varepsilon_1 | w_\omega \otimes \varepsilon_1) - \frac{q^{-1}}{[s + 1]} (v_0 \otimes \varepsilon_1 | F.w_\omega \otimes \varepsilon_0)}{(5.118)} - \frac{q^{-1}}{[s + 1]} (v_1 \otimes \varepsilon_0 | F.w_\omega \otimes \varepsilon_0) + \frac{q^{-1}}{[s + 1]} (v_0 | F.w_\omega)
\]

After inserting (5.122) into the diagram (5.119) for $w$, we arrive with the following recursive formula for $w$:  

\[
w_{\omega}^{(3.126)} = \frac{-iq^{-1/2}(v_{0} \otimes \varepsilon_1 | w_\omega \otimes \varepsilon_1) - \frac{q^{-1}}{[s + 1]} (v_0 \otimes \varepsilon_1 | F.w_\omega \otimes \varepsilon_0)}{(5.118)} - \frac{q^{-1}}{[s + 1]} (v_1 \otimes \varepsilon_0 | F.w_\omega \otimes \varepsilon_0) + \frac{q^{-1}}{[s + 1]} (v_0 | F.w_\omega)
\]

Using lemmas 2.11 and 2.13, we now calculate the value of $v | w_\omega$:

\[
\begin{align*}
&\left( v \right | w_\omega)_{(5.126)} = \frac{-iq^{-1/2}(v_{0} \otimes \varepsilon_1 | w_\omega \otimes \varepsilon_1) - \frac{q^{-1}}{[s + 1]} (v_0 \otimes \varepsilon_1 | F.w_\omega \otimes \varepsilon_0)}{(5.118)} - \frac{q^{-1}}{[s + 1]} (v_1 \otimes \varepsilon_0 | F.w_\omega \otimes \varepsilon_0) + \frac{q^{-1}}{[s + 1]} (v_0 | F.w_\omega)
\end{align*}
\]

(b) When $r_n = s = r_{n-1} + 1$, the rightmost closed three-vertex (1.30) in $\omega$ reads

\[
\begin{align*}
\begin{array}{c}
1 \\
r_{n-1} = r_{n-1} + 1
\end{array} & \quad \begin{array}{c}
1 \\
r_{n-1}
\end{array} \\
\begin{array}{c}
1 \\
r_{n-1}
\end{array}
\end{align*}
\]

where we used [FP18a, lemma A.2] to remove the top projector box. Then, after orienting the defects in (5.125) and inserting it into the diagram (5.119) for $w$, we arrive with the following recursive formula for $w$:  

\[
w_{\omega}^{(3.126)} = \frac{-iq^{-1/2}(v_{0} \otimes \varepsilon_1 | w_\omega \otimes \varepsilon_1) - \frac{q^{-1}}{[s + 1]} (v_0 \otimes \varepsilon_1 | F.w_\omega \otimes \varepsilon_0)}{(5.118)} - \frac{q^{-1}}{[s + 1]} (v_1 \otimes \varepsilon_0 | F.w_\omega \otimes \varepsilon_0) + \frac{q^{-1}}{[s + 1]} (v_0 | F.w_\omega)
\]

Using lemmas 2.11 and 2.13, we calculate $v | w$:  

\[
\begin{align*}
&\left( v \right | w_\omega)_{(5.126)} = \frac{-iq^{-1/2}(v_{0} \otimes \varepsilon_1 | w_\omega \otimes \varepsilon_1) - \frac{q^{-1}}{[s + 1]} (v_0 \otimes \varepsilon_1 | F.w_\omega \otimes \varepsilon_0)}{(5.118)} - \frac{q^{-1}}{[s + 1]} (v_1 \otimes \varepsilon_0 | F.w_\omega \otimes \varepsilon_0) + \frac{q^{-1}}{[s + 1]} (v_0 | F.w_\omega)
\end{align*}
\]

To finish, we use induction hypothesis (5.115) to show that (5.124) and (5.127) vanish. We consider two cases:
If \( \text{tail}(\alpha') \in \mathbb{R}^{(s+1)}_{n-1} \), then \( w_{\alpha'} \) belongs to the radical

\[
\text{span} \left\{ w_{\alpha'} \mid \beta \in \mathbb{L}P_n, \text{tail}(\beta) \in \mathbb{R}^{(s+1)}_{n-1} \right\} = \left\{ w_{\beta} \mid \beta \in \text{rad} L^{(s+1)}_{n-1} \right\} \subset \text{rad} H^{(s+1)}_{n-1}. \tag{5.128}
\]

Hence, by induction hypothesis (5.115), we have \((\tau_0 | w_{\alpha'}) = 0\), which implies that \((\sigma | w_{\alpha'}) = 0\) in (5.124) or (5.127).

If \( \text{tail}(\alpha') \not\in \mathbb{R}^{(s+1)}_{n-1} \), then we recall the definition of radical tails from [FP18a, section 5.A], which says that stopping condition 1 of [FP18a, definition 4.3] occurs when forming the trivalent link state \( \alpha \) from \( \alpha \) (as illustrated in [FP18a, figure 4.4]). Together with (5.114), this implies that we necessarily have

\[
r_n = s = (k + 1) p(q) - 2 \quad \text{and} \quad r_{n-1} = r_n + 1 = (k + 1) p(q) - 1. \tag{5.129}
\]

In other words, the above scenario (a) necessarily occurs. Thus, using the fact that \( q^{k p(q)} = q^{-k p(q)} \), we obtain

\[
-1q^{-1/2}(\bar{P} | w_{\alpha'}) = \left(1 + \frac{q^{-(k+1)p(q)}}{[k+1](p(q) - 1)}\right)(\bar{P}_0 | w_{\alpha'}) = \frac{q^{k p(q) - 1} - q^{-k p(q) - 1}}{q^{k p(q) - 1} - q^{-k p(q) + 1}} (\bar{P}_0 | w_{\alpha'}) = 0. \tag{5.129}
\]

In conclusion, we have shown that any basis vector \( w_{\alpha'} \) in the set (5.117) belongs to the radical \( H^{(s)}_{n} \). By linearity, this completes the induction step, thus proving item 2, and item 1 follows.

The statements with \( w \mapsto \bar{\sigma}, \alpha \mapsto \bar{\sigma}, L \mapsto \bar{L}, \) and \( H \mapsto \bar{H} \) can be proven similarly.

\[ \square \]

**Lemma 5.18.** Suppose \( \max \zeta < p(q) \). For each \( s \in \mathbb{E}_n \), we have

\[
\{ \mathcal{J}_z(w_{\alpha}) \mid \alpha \in \text{rad} L^{(s)}_z \} = \{ w_{P_z \alpha} \mid \alpha \in \text{rad} L^{(s)}_{n_z} \}, \tag{5.130}
\]

and similarly,

\[
\{ \mathcal{J}_z(w_{\alpha}) \mid \alpha \in \text{rad} L^{(s)}_z \} = \{ w_{P_z \alpha} \mid \alpha \in \text{rad} L^{(s)}_{n_z} \}. \tag{5.131}
\]

**Proof.** [FP18a, corollary B.5] says that

\[
\text{rad} L^{(s)}_z = \tilde{P}_z \text{rad} L^{(s)}_{n_z}, \tag{5.132}
\]

which together with lemma 4.9, corollary 3.15, lemma A.2, and corollary 3.12 implies that

\[
\{ \mathcal{J}_z(w_{\alpha}) \mid \alpha \in \text{rad} L^{(s)}_z \} \overset{5.132}{=} \{ \mathcal{J}_z(w_{P_z \alpha}) \mid \alpha \in \text{rad} L^{(s)}_{n_z} \} \overset{4.95}{=} \{ (\mathcal{J}_z \circ \tilde{P}_z)(w_{\alpha}) \mid \alpha \in \text{rad} L^{(s)}_{n_z} \} \overset{4.54}{=} \{ w_{P_z \alpha} \mid \alpha \in \text{rad} L^{(s)}_{n_z} \}. \tag{5.133}
\]

This proves (5.130). Identity (5.131) can be proven similarly. \[ \square \]

**Lemma 5.13.** Suppose \( \max \zeta < p(q) \).

1. We have \( \text{rad} \{ w_{\alpha} \mid \alpha \in L^{(s)} \} \subset \text{rad} H^{(s)}_z \).

2. For each \( s \in \mathbb{E}_z \), we have \( \text{rad} \{ w_{\alpha} \mid \alpha \in L^{(s)}_z \} \subset \text{rad} H^{(s)}_z \).

Similarly, this lemma holds after the symbolic replacements \( w_{\alpha} \mapsto w_{\bar{P}_z \alpha}, \alpha \mapsto \bar{\alpha}, L \mapsto \bar{L}, \) and \( H \mapsto \bar{H} \).

**Proof.** Item 1 follows from item 2 with direct-sum decompositions (5.95, 5.99). Therefore, it suffices to prove item 2. Lemma 5.17 already gives the case of \( \zeta = n \). Combining it with lemma 5.18, we obtain

\[
\{ \mathcal{J}_z(w_{\alpha}) \mid \alpha \in \text{rad} L^{(s)}_z \} \overset{5.130}{=} \{ w_{P_z \alpha} \mid \alpha \in \text{rad} L^{(s)}_{n_z} \} \simeq \{ w_{P_z \alpha} \mid \alpha \in \text{rad} L^{(s)}_{n_z} \} \overset{5.18}{\subset} \text{rad} H^{(s)}_{n_z}, \tag{5.135}
\]

where to pass to the second line, we used the observation that, because \( \text{rad} L^{(s)}_{n_z} \) is a \( \mathbb{L}L^{(s)}_{n_z} \)-submodule of \( L^{(s)}_{n_z} \), \( \alpha \in \text{rad} L^{(s)}_{n_z} \) implies \( P_z \alpha \in \text{rad} L^{(s)}_{n_z} \). Thanks to (A.3), applying \( P_z \) to both sides and using corollary 5.16, we obtain

\[
\{ w_{\alpha} \mid \alpha \in \text{rad} L^{(s)}_z \} \overset{5.135}{\subset} \{ \mathcal{P}_z(v) \mid v \in \text{rad} H^{(s)}_{n_z} \} \overset{5.112}{=} \text{rad} H^{(s)}_z. \tag{5.137}
\]

This proves item 2. The case of the subsets of rad \( \bar{H}^{(s)}_z \) can be proven similarly. \[ \square \]
6. HIGHER-SPIN QUANTUM SCHUR-WEYL DUALITY

In this final section, we prove the quantum Schur-Weyl duality theorems 1.4 and 1.6; the latter in section 6B and the former in section 6C. To identify the commutant algebra with the valenced Temperley-Lieb algebra $\mathcal{T}_\nu(V_c)$, we first show (as a special case of proposition 6.1 concerning $\mathcal{T}_\nu(\nu)$) that the representation of $\mathcal{T}_\nu(\nu)$ is faithful whenever defined. Lastly, in section 6D we prove proposition 1.3 concerning a generating set for the commutant algebra of $U_q$: we show that it is obtained from $U_q$-submodule projectors acting on consecutive tensor components of $V_c$. (These operators are also closely related to the R-matrix of $U_q$ [Kas95, CP94], not however discussed in the present work.)

A. On kernels and images of representations

In this section, we prove that the representation $\mathcal{A}_\nu: \mathcal{T}_\nu(\nu) \rightarrow \text{End}V_c$ of the valenced Temperley-Lieb algebra $\mathcal{T}_\nu(\nu)$ on the tensor product $V_c$ is faithful whenever defined (i.e., for all max $\zeta < p(q)$). This follows as a special case of the next proposition. Taking $\zeta = \bar{n}$, we thus also recover a result proved for the Temperley-Lieb algebra $\mathcal{T}_n(\nu)$ independently by P. Martin [Mar92, theorem 1] and F. Goodman and H. Wenzl [GW93, theorem 2.4].

Proposition 6.1. Suppose max$(\zeta, \varpi) < p(q)$. The following maps are injective:

$$\mathcal{A}_\zeta: \mathcal{T}_\zeta(\nu) \rightarrow \text{Hom}(V_{\varpi}, V_{\zeta})$$

and

$$\mathcal{A}_\varpi: \mathcal{T}_\varpi(\nu) \rightarrow \text{Hom}(V_{\zeta}, V_{\varpi}).$$

Proof. We prove the assertion for $\mathcal{A}_\zeta$; the case of $\mathcal{A}_\varpi$ is similar. To begin, we specialize to the case of $\zeta = \bar{n}$ and $\varpi = \bar{m}$ for some $m, n \in \mathbb{Z}_{>0}$. By linearity, to show that the map $\mathcal{A}_{\bar{m}}(\bar{n})$ is injective, it suffices to show that $\mathcal{A}_{\bar{m}}(T) = 0$ implies $T = 0$. For this purpose, we expand each tangle $T \in \mathcal{T}_{\bar{m}}(\bar{n})$ according to the number $r$ of crossing links,

$$T = \sum_{r \in \mathbb{Z}_{\geq 0}} T^{(r)}, \quad \text{where} \quad T^{(r)} = \sum_{\alpha \in \text{LP}_{\bar{n}}, \beta \in \text{LP}_{\bar{m}}} c_{\alpha, \beta}^{(r)} \alpha \otimes \beta,$$

for any vector $v \in V_m$. By considering the action of $T$ on standard basis vectors $e_{m}^\varrho \in V_{m}^{(s)}$ with $\varrho = \varrho_\gamma$, we have

$$0 = T e_{m}^\varrho = \sum_{r \in \mathbb{Z}_{\geq 0}} T^{(r)} e_{m}^\varrho = 0 \quad \text{for all} \quad r < s,$$

because if $r < s$, then for any link pattern $\beta \in \text{LP}_{\bar{m}}$, the oriented network $\beta e_{m}^\varrho$ has a turn-back link of $\beta$ joining two defects of $e_{m}^\varrho$ with identical orientations. Using this observation and item 1 of lemma 5.7, we thus obtain

$$0 = T e_{m}^\varrho = T e_{m}^{\varrho} = 0 \quad \text{with} \quad T^{(r)} e_{m}^{\varrho} = 0 \quad \text{for all} \quad r < s,$$

Now, by lemma 4.7, the collection $\{w_{\alpha} | \alpha \in \text{LP}_{\bar{m}}\}$ is linearly independent, which implies the following system of equations for all link patterns $\alpha \in \text{LP}_{\bar{m}}$ and $\gamma \in \text{LP}_{\bar{n}}$:

$$\sum_{\beta \in \text{LP}_{\bar{m}}} M_{\beta, \gamma} c_{\alpha, \beta}^{(s)} = 0, \quad \text{where} \quad M_{\beta, \gamma} := (\overline{\beta}|e_{m}^\varrho) \text{ and } \varrho = \varrho_\gamma.$$

Corollary 4.6 implies that we can arrange the columns of the matrix $M_{\beta, \gamma}$ in such a way that it is upper-triangular with non-vanishing diagonal elements. Therefore, we conclude that

$$c_{\alpha, \beta}^{(s)} = 0 \quad \text{for all} \quad \alpha \in \text{LP}_{\bar{m}} \quad \text{and} \quad \beta \in \text{LP}_{\bar{m}}.$$
But this contradicts the choice of $s$. Therefore, we have $T = 0$, so the map $\mathcal{I}^m_n$ is injective.

Next, to prove that the map $\mathcal{I}^\infty_\varsigma$ is injective for general multindices $\varsigma, \varpi \in \mathbb{Z}_{>0}$ too, we again show that $\mathcal{I}^\infty_\varsigma(T) = 0$ implies $T = 0$. First, by corollary 3.15, we have

$$\mathcal{I}^\infty_\varsigma(I_\varsigma T \hat{P}_{\varpi})(w) = I_\varsigma T \hat{P}_{\varpi} w \in \text{im } \mathcal{I}_\varsigma$$

(6.8)

for all vectors $w \in V_{n_\varsigma}$. Because $\hat{P}_\varsigma = \mathcal{I}^{-1}_\varsigma$ on $\text{im } \mathcal{I}_\varsigma$ by item 3 of lemma A.2, recalling definition (3.87), we obtain

$$\mathcal{I}^\infty_\varsigma(T) = 0 \quad \Rightarrow \quad (I_\varsigma T \hat{P}_{\varpi}) \mathcal{I}_\varpi(v) = 0 \quad \text{for all vectors } v \in V_{\varpi}. \quad (6.9)$$

Item 3 of lemma A.2 also implies that for each vector $w \in V_{n_\varsigma}$, there exists a vector $v \in V_{\varpi}$ for which we have $\hat{P}_{\varpi}(w) = \mathcal{I}_\varpi(v)$. Furthermore, by corollary 3.12, this pair $v, w$ satisfies

$$\hat{P}_{\varpi} \mathcal{I}_\varpi(v) = \hat{P}_{\varpi} \mathcal{I}_\varpi(w) \quad (3.33) \Rightarrow \quad \hat{P}_{\varpi} \mathcal{I}_\varpi w = (6.10) \quad \text{after inserting this into (6.9) and recalling that we have already shown the injectivity of that } \mathcal{I}^\infty_\varsigma, \text{ we conclude that } \mathcal{I}^\infty_\varsigma(T) = 0 \quad \Rightarrow \quad I_\varsigma T \hat{P}_{\varpi} = 0. \quad (6.11)$$

To finish, we recall from [FP18a, lemma B.1] that the map $T \mapsto I_\varsigma T \hat{P}_{\varpi}$ for valenced tangles $T \in \mathbb{T}L^\varsigma$ is a linear injection. Combining this with (6.11), we see that the property $\mathcal{I}^\infty_\varsigma(T) = 0$ indeed implies that $T = 0$.

The above result also holds for $q \in \{\pm 1\}$, with the same proof.

We remark that if $n_\varsigma < p(q)$, then it alternatively follows from the faithfulness results in [FP18a, corollaries 3.8 and 5.22] combined with proposition 4.12 of the present article that the representation $\mathcal{I}_\varsigma$ for $\mathbb{T}_\varsigma \odot V_\varsigma$ is faithful:

$$n_\varsigma < p(q) \quad \Rightarrow \quad \text{rad } \mathcal{L}_\varsigma = \{0\} \quad \Rightarrow \quad \mathbb{T}_\varsigma \odot V_\varsigma \text{ faithful} \quad (6.12)$$

and

$$\mathbb{T}_\varsigma \odot V_\varsigma \text{ faithful}. \quad (6.13)$$

We recall from lemma 3.16 that the image of $\mathcal{I}^\infty_{\varsigma}$ lies in fact in the commutant space $\text{Hom}_U(V_{\varpi}, V_\varsigma)$. In the next section, we show that if $\max(n_\varsigma, n_{\varpi}) < p(q)$, then this image fills the whole commutant space (theorem 1.6).

Conversely, we consider the left representation $\rho_\varsigma : U_\eta \to \text{End } V_\varsigma$, $\rho_\varsigma := (\rho_{(s_1)} \otimes \rho_{(s_2)} \otimes \cdots \otimes \rho_{(s_{\varsigma+1})}) \odot \Delta^{(d_\varsigma)}$ (6.14)

associated to the module $U_\varsigma \odot V_\varsigma$. We similarly define the right representation $\hat{\rho}_\varsigma$ associated to the module $\hat{V}_\varsigma \odot U_\varsigma$, and the corresponding left and right representations $\tau_\varsigma$ and $\hat{\tau}_\varsigma$ for $\hat{V}_\varsigma \odot V_\varsigma$ and $V_\varsigma \odot \hat{V}_\varsigma$.

Theorem 1.6 also shows that if $\max(\varsigma, n_{\varpi}) < p(q)$, then the image of the representation $\rho_\varsigma : U_\eta \to \text{End } V_\varsigma$ constitutes all operators which commute with the $\mathbb{T}L_\varsigma(\nu)$-action on $\mathbb{T}_\varsigma \odot V_\varsigma$. However, the representation $\rho_\varsigma$ is not faithful (it cannot be, as $U_\eta$ is infinite-dimensional), and its image is isomorphic to a finite-dimensional quotient of the quantum group $U_\eta$, called a $q$-Schur algebra. In the case of $\varsigma = \eta$, the $q$-Schur algebra $\rho_\eta(U_\eta)$ appears, e.g., in [DJ89].

When $\varsigma = (s)$, it is straightforward to find explicit formulas for the image and kernel of $\rho_{(s)}$. We collect these formulas here. For this purpose, we define the projection operators $G^{(s)}_{\ell}$, for $\ell \in \{0, 1, \ldots, s\}$,

$$G^{(s)}_{\ell} := \prod_{0 \leq j \leq s_{\ell}, j \neq \ell} \frac{K - q^{s_{\ell} - 2j}}{q^{s_{\ell} - 2\ell} - q^{s_{\ell} - 2j}}, \quad G^{(s)}_{\ell} e_k^{(s)} = \delta_{k, \ell} e^{(s)}_{\ell}. \quad (6.15)$$

Also, for all $\ell \in \{0, 1, \ldots, s\}$, $k, m \in \mathbb{Z}_{\geq 0}$ with $m \leq k$ and $n \in \mathbb{Z}$, we define the constants

$$A^{(s)}_{\ell, k, n, m} := \frac{1}{[k + m - s]_{q^{-(s_{\ell} - 2\ell - 2m)}} \left[ \begin{array}{c} \ell \\ \ell - k + m \\ \ell + m \\ \ell + m - k \end{array} \right]_{q^{(s_{\ell} - 2\ell)}} \left[ \begin{array}{c} s - \ell + k - m \\ \ell - \ell \\ s - \ell \\ s - \ell - m \end{array} \right]_{q^{s_{\ell} - 2\ell}}. \quad (6.16)$$
\[
B^{(s)}_{\ell,k,n,m} := I\{m \leq \ell + k \leq s\} q^{-n(s-2\ell-2k)} \frac{1}{|m|!^2} \frac{\ell + k}{\ell + k - m} \frac{1}{s - \ell - k + m}.
\]

(6.17)

**Lemma 6.2.** Suppose \( s < p(q) \). Then, the following hold:

1. The following set is a basis for the image \( \text{im} \rho_s \):

\[
\bigcup_{0 \leq \ell \leq s} \{ \rho_s(F^kG^{(s)}_{\ell}) | 0 \leq k \leq s - \ell \} \cup \{ \rho_s(E^kG^{(s)}_{\ell}) | 1 \leq k \leq \ell \},
\]

and similarly, the following set is a basis for the image \( \text{im} \tilde{\rho}_s \):

\[
\bigcup_{0 \leq \ell \leq s} \{ \tilde{\rho}_s(G^{(s)}_{\ell}F^k) | 0 \leq k \leq s - \ell \} \cup \{ \tilde{\rho}_s(G^{(s)}_{\ell}E^k) | 1 \leq k \leq \ell \}.
\]

(6.18)

(6.19)

2. The following set spans the kernel \( \ker \rho_s \subset U_q \):

\[
\bigcup_{0 \leq \ell \leq s, k \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}, 0 \leq m \leq k} \left\{ (F^{k-m} - A^{(s)}_{\ell,k,n,m}E^KK^mF^m)G^{(s)}_{\ell}, E^KK^mF^m(G^{(s)}_0 + G^{(s)}_1 + \cdots + G^{(s)}_s - \mathbf{1}_{U_q}) \right\},
\]

and similarly, the following set spans the kernel \( \ker \tilde{\rho}_s \subset U_q \):

\[
\bigcup_{0 \leq \ell \leq s, k \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}, 0 \leq m \leq k} \left\{ (G^{(s)}_{\ell}F^{k-m} - A^{(s)}_{\ell,k,n,m}E^KK^mF^m), (G^{(s)}_0 + G^{(s)}_1 + \cdots + G^{(s)}_s - \mathbf{1}_{U_q})F^KK^mF^m \right\}.
\]

(6.20)

(6.21)

Similarly, this lemma holds for the left and right representations \( \tau_s \) and \( \tau_{s} \) of \( U_q \).

**Proof.** We prove the assertions concerning \( \rho_s \); the case of \( \tilde{\rho}_s \) is similar. For each element \( x \in U_q \), we let \( [\rho_s(x)]_{i,j} \) denote the entries of the matrix representation with respect to the basis \( \{ e_0^{(s)}, e_1^{(s)}, \ldots, e_s^{(s)} \} \subset M_{s+1} \). Then, we have

\[
0 \leq k \leq s - \ell \quad \implies \quad [\rho_s(F^kG^{(s)}_{\ell})]_{i,j} = \delta_{i,\ell+k}\delta_{j,\ell},
\]

(6.22)

\[
0 \leq k \leq \ell \quad \implies \quad [\rho_s(E^kG^{(s)}_{\ell})]_{i,j} = \frac{[\ell]![s - \ell + 1]}{[\ell-k]![s - \ell + k - 1]} \delta_{i,\ell-k}\delta_{j,\ell},
\]

(6.23)

for all \( \ell \in \{0, 1, \ldots, s\} \). Equations (6.22, 6.23) imply that (6.18) is a basis for \( \rho_s(U_q) \) and that \( \rho_s(U_q) = \text{End} \, V_s \).

To prove item 2, we let \( W' \) denote the subspace of \( U_q \) on the right side of (6.20), and

\[
W := \bigcup_{0 \leq \ell \leq s} \left\{ F^kG^{(s)}_{\ell} | 0 \leq k \leq s - \ell \right\} \cup \left\{ E^kG^{(s)}_{\ell} | 0 \leq k \leq \ell \right\}.
\]

(6.24)

First, a straightforward calculation shows that

\[
W' \subset \ker \rho_s.
\]

(6.25)

It is also straightforward to show that the span of \( W' \cup W \) contains the basis \( \{ E^KK^mF^\ell | k, \ell \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z} \} \) of \( U_q \), and is therefore a spanning set for this algebra. As such, we can write any element \( x \in \ker \rho_s \subset U_q \) in the form

\[
x = y + \sum_i c_i z_i
\]

(6.26)

for some \( y \in W', z_i \in W, \) and \( c_i \in \mathbb{C} \). After acting on both sides of (6.26) with the representation \( \rho_s \), we obtain

\[
x \in \ker \rho_s \quad \implies \quad 0 = \rho_s(x) \overset{(6.26)}{=} \rho_s(y) + \sum_i c_i \rho_s(z_i) \overset{(6.25)}{=} \sum_i c_i \rho_s(z_i).
\]

(6.27)

By item 1, the set \( \{ \rho_s(z) | z \in W \} \) is linearly independent. Thus, identity (6.27) implies that \( c_i = 0 \) for all \( i \), so \( x = y \in W' \). We infer that \( \ker \rho_s \subset W' \), and the equality in (6.20) then follows from this fact combined with (6.25).

Finally, the statements 1–2 for the left and right representations \( \tau_s \) and \( \tau_{s} \) of \( \bar{U}_q \) can also be proven similarly. \( \square \)
B. Double-commutant property for quantum group and Temperley-Lieb actions

Throughout the rest of this section, we assume that \( n_\varsigma < p(q) \). Using the general double-commutant property from proposition E.5 in appendix E and the injectivity of the map \( \mathcal{F}_\varsigma^\varpi \) from proposition 6.1, we establish one of the key results of this section: the \( U_q \)-action and the Temperley-Lieb action are each others’ commutants.

**Theorem 1.6.** (Double-commutant property) Suppose \( \max(n_\varsigma, n_\varsigma) < p(q) \). Then, the following hold:

1. Let \( L, R \in \text{Hom}(V_\varpi, V_\varsigma) \). The diagram

\[
\begin{array}{ccc}
V_\varpi & \xrightarrow{\rho_\varpi(x)} & V_\varpi \\
\downarrow L & & \downarrow R \\
V_\varsigma & \xrightarrow{\rho_\varsigma(x)} & V_\varsigma
\end{array}
\]  

(1.39)

commutes for all elements \( x \in U_q \) if and only if we have \( L = R = \mathcal{F}_\varsigma^\varpi(T) \) for some valenced tangle \( T \in TL_\varsigma^\varpi \).

2. Let \( L \in \text{End}V_\varpi \) and \( R \in \text{End}V_\varsigma \). Let \( \pi_\varpi^{(s)} \in \text{End}V_\varpi \) and \( \pi_\varsigma^{(s)} \in \text{End}V_\varsigma \) be the respective projections onto the \( s \)-th summand of the direct-sum decompositions (1.7) of \( V_\varpi \) and \( V_\varsigma \). The diagram

\[
\begin{array}{ccc}
V_\varpi & \xrightarrow{\mathcal{F}_\varsigma^\varpi(T)} & V_\varsigma \\
\downarrow L & & \downarrow R \\
V_\varpi & \xrightarrow{\mathcal{F}_\varpi^\varsigma(T)} & V_\varsigma
\end{array}
\]  

(1.40)

commutes for all valenced tangles \( T \in TL_\varsigma^\varpi \) if and only if we have

\[
L = \rho_\varpi(x) + \sum_{s \in E_\varpi \setminus E_\varsigma} \pi_\varpi^{(s)} \circ L' \quad \text{and} \quad R = \rho_\varsigma(x) + \sum_{s \in E_\varsigma \setminus E_\varpi} R' \circ \pi_\varsigma^{(s)}
\]  

(1.41)

for some element \( x \in U_q \) and endomorphisms \( L' \in \text{End}V_\varpi \) and \( R' \in \text{End}V_\varsigma \).

Similarly, this theorem holds after the symbolic replacements \( V \mapsto \overline{V} \), \( \rho \mapsto \overline{\rho} \), \( \mathcal{F} \mapsto \overline{\mathcal{F}} \), and \( \pi \mapsto \overline{\pi} \). Finally, the analogue of this theorem holds for the left and right representations \( \tau_\varsigma \) and \( \overline{\tau}_\varsigma \) of \( U_q \).

**Proof.** Lemma 3.16 gives the “if” part of item 1. For the “only if” part, assuming diagram (1.39) commutes, taking \( x = 1 \in U_q \) implies that necessarily \( L = R \in \text{Hom}_{U_q}(V_\varpi, V_\varsigma) \). Proposition 4.13 and lemma E.3 together imply that

\[
\dim \text{Hom}_{U_q}(V_\varpi, V_\varsigma) \overset{(4.72)}{=} \sum_{s \in E_\varpi \cap E_\varsigma} D_{\varpi}^{(s)} D_{\varsigma}^{(s)} \overset{(3.47)}{=} \dim TL_\varsigma^\pi \overset{\text{prop. 6.1}}{=} \dim \mathcal{F}_\varsigma^\varpi(TL_\varsigma^\varpi),
\]  

(6.28)

using the injectivity of \( \mathcal{F}_\varsigma^\varpi \) from proposition 6.1 for the last equality. We conclude that \( L = R = \mathcal{F}_\varsigma^\varpi(T) \) for some valenced tangle \( T \in TL_\varsigma^\varpi \). This proves item 1. Item 2 then follows from item 2 of proposition E.5. Finally, the corresponding statements for the representations \( \overline{\rho}_\varsigma, \overline{\tau}_\varsigma, \) and \( \overline{\tau}_\varsigma \) can be proven similarly. \( \square \)

C. Higher-spin quantum Schur-Weyl duality

The next key theorem gives the complete information about the (bi)-module structure on the vector space \( V_\varsigma \) carrying both the left \( U_q \)-action defined via (2.6, 2.14, 2.16, 2.17) and the left \( TL_\varsigma(\nu) \)-action defined via lemma 3.13,

\[
M_\varsigma := \frac{V_\varsigma}{TL} \otimes V_\varsigma.
\]  

(6.29)

This result contains the well-known quantum Schur-Weyl duality property à la Jimbo \[Jim86\] as a special case.
Theorem 1.4. (Higher-spin quantum Schur-Weyl duality): Suppose \( n_\varsigma < p(q) \). Then, the following hold:

1. The images of the maps \( \mathcal{J}_\varsigma : \mathbb{T}\mathbb{L}_\varsigma(\nu) \rightarrow \text{End} \mathcal{V}_\varsigma \) and \( \rho_\varsigma : \mathbb{U}_q \rightarrow \text{End} \mathcal{V}_\varsigma \) are semisimple algebras, which equal

\[
\mathbb{T}\mathbb{L}_\varsigma(\nu) \cong \mathcal{J}_\varsigma(\mathbb{T}\mathbb{L}_\varsigma(\nu)) = \text{End} \mathbb{U}_q \mathcal{V}_\varsigma \quad \text{and} \quad \rho_\varsigma(\mathbb{U}_q) = \text{End} \mathbb{T}\mathbb{L}_\varsigma. \tag{1.32}
\]

2. The collections \( \{ M_{(s)} | s \in \mathbb{E}_\varsigma \} \) and \( \{ L_{(s)}^{(\varsigma)} | s \in \mathbb{E}_\varsigma \} \) are respectively the complete sets of simple non-isomorphic \( \rho_\varsigma(\mathbb{U}_q) \)-modules and \( \mathbb{T}\mathbb{L}_\varsigma(\nu) \)-modules, and we have the direct-sum decomposition

\[
M_\varsigma \cong \bigoplus_{s \in \mathbb{E}_\varsigma} M_{(s)} \otimes L_{(s)}^{(\varsigma)}. \quad (q\text{-SW}_\varsigma)
\]

3. The linear extension of the following map gives an explicit isomorphism for \( (q\text{-SW}_\varsigma) \): with \( w_\alpha \) the explicit \( \mathbb{U}_q \)-highest-weight vectors constructed in definition 4.1, \( \alpha \in \text{LP}_\varsigma^{(s)} \), \( s \in \mathbb{E}_\varsigma \), and \( \ell \in \{0, 1, \ldots, s\} \),

\[
F^{\ell}.w_\alpha \mapsto e^{(s)}_\ell \otimes \alpha. \tag{6.30}
\]

Similarly, this theorem holds after the symbolic replacements

\[
\mathcal{J} \mapsto \overline{\mathcal{J}}, \quad \mathcal{V} \mapsto \overline{\mathcal{V}}, \quad \rho \mapsto \overline{\rho}, \quad M \mapsto \overline{M}, \quad \mathcal{L} \mapsto \overline{\mathcal{L}}, \quad \text{and} \quad F^{\ell}.w_\alpha \mapsto \overline{w}_\alpha.E^{\ell}. \tag{6.30}
\]

Finally, the analogue of this theorem holds for the left and right representations \( \tau_\varsigma \) and \( \varsigma \) of \( \mathbb{U}_q \).

Proof. Because the representation \( \mathcal{J}_\varsigma \) is faithful by proposition 6.1, we have \( \mathbb{T}\mathbb{L}_\varsigma(\nu) \cong \mathcal{J}_\varsigma(\mathbb{T}\mathbb{L}_\varsigma(\nu)) \). Lemma 3.16 gives \( \mathcal{J}_\varsigma(\mathbb{T}\mathbb{L}_\varsigma(\nu)) \subset \text{End} \mathbb{U}_q \mathcal{V}_\varsigma \), and dimension count (6.28) with \( \varpi = \varsigma \) gives the left equalities in (1.32). The other statements in items 1 and 2 then follow from double-commutant theorem E.9 from appendix E with \( \rho_\varsigma(\mathbb{U}_q) \subset \text{End} \mathcal{V}_\varsigma \). Propositions 4.13 and 4.16 imply item 3. The statements with replacements (6.30), \( \tau_\varsigma \), or \( \varsigma \) follow similarly.

D. Consecutive tensorand projectors as generators for the commutant

The next proposition shows that the commutant algebra \( \text{End} \mathbb{U}_q \mathcal{V}_\varsigma \), isomorphic to the valenced Temperley-Lieb algebra \( \mathbb{T}\mathbb{L}_\varsigma(\nu) \) by theorem 1.4, is generated by the projectors on \( \mathcal{V}_\varsigma \) acting on consecutive tensorands.

Proposition 1.3. Suppose \( n_\varsigma < p(q) \). Then, the commutant algebra \( \text{End} \mathbb{U}_q \mathcal{V}_\varsigma \) is generated by the collection of all submoduler projectors that act strictly on consecutive pairs of tensorands of vectors in \( \mathcal{V}_\varsigma \):

\[
\text{End} \mathbb{U}_q \mathcal{V}_\varsigma = \langle \tau_{(s_{i+1}, s_{(i+1)})}^{\varsigma(i)} | s \in \mathbb{E}_{(s_i, s_{(i+1)})}, i \in \{1, 2, \ldots, d_\varsigma - 1\} \rangle, \tag{1.20}
\]

where \( \mathbb{E}_{(s_i, s_{(i+1)})} = \{ |s_{i+1} - s_i|, |s_{i+1} - s_i| + 2, \ldots, s_{i+1} + s_i \} \).

Similarly, this proposition holds after the symbolic replacements \( \mathbb{U}_q \mapsto \overline{\mathbb{U}_q}, \mathcal{V} \mapsto \overline{\mathcal{V}}, \mathcal{J} \mapsto \overline{\mathcal{J}}, \) and \( \tau \mapsto \overline{\tau} \).

Proof. By theorem 1.4, the map \( \mathcal{J}_\varsigma : \mathbb{T}\mathbb{L}_\varsigma(\nu) \rightarrow \text{End} \mathbb{U}_q \mathcal{V}_\varsigma \) is an isomorphism of algebras, so lemma 5.10 implies that its image is generated by the right side of (1.20). The corresponding statement for \( \overline{\mathbb{U}_q} \) follows similarly.

APPENDICES

A. QUANTUM GROUP — VARIANTS, CALCULATIONS, AND AUXILIARY RESULTS

In this appendix, we gather easy auxiliary results and calculations and some definitions, for usage throughout the present article as well as for future applications.
**Lemma A.1.** Suppose $q \in \mathbb{C}^\times \setminus \{\pm 1\}$. For all integers $0 \leq k, \ell \leq p(q)$ and $m \in \mathbb{Z}$, we have

\[
\Delta(E^k K^m F^\ell) = \sum_{i=0}^k \sum_{j=0}^\ell q^{i(k-i) - j(\ell-j)} \left[ \begin{array}{c} k \\ i \end{array} \right] \left[ \begin{array}{c} \ell \\ j \end{array} \right] E^{k-i} K^{m-\ell+i} F^j \otimes E^i K^{m+k-i} F^{\ell-j}, \quad (A.1)
\]

and similarly,

\[
\Delta(F^k K^m E^\ell) = \sum_{i=0}^k \sum_{j=0}^\ell q^{i(k-i) - j(\ell-j)} \left[ \begin{array}{c} k \\ i \end{array} \right] \left[ \begin{array}{c} \ell \\ j \end{array} \right] F^{k-i} K^{m-\ell+i} E^j \otimes F^i K^{m+k-i} E^{\ell-j}. \quad (A.2)
\]

**Proof.** This follows by relatively straightforward calculations, see, e.g., [Kas95, proposition VII.1.3].

1. Properties of submodule projectors and embeddings

Next, we gather useful properties of the mappings $\mathcal{I}_c$, $\mathcal{P}_c$, and $\hat{\mathcal{P}}_c$ defined in (2.114) in section 2C and used repeatedly throughout this article.

**Lemma A.2.** Suppose $\max c < p(q)$. The following hold:

1. The maps $\mathcal{I}_c$, $\mathcal{P}_c$, and $\hat{\mathcal{P}}_c$ are $\mathcal{U}_q$- and $\mathcal{U}_q$-homomorphisms.
2. $\mathcal{I}_c$ is a linear injection, $\hat{\mathcal{P}}_c$ is a linear surjection, and $\mathcal{P}_c$ is a linear projection.
3. We have $\text{im} \mathcal{I}_c = \text{im} \mathcal{P}_c$, $\ker \hat{\mathcal{P}}_c = \ker \mathcal{P}_c$,
   \[
   \hat{\mathcal{P}}_c \circ \mathcal{I}_c = \text{id}_{\mathcal{P}_c}, \quad \text{and} \quad \mathcal{I}_c \circ \hat{\mathcal{P}}_c = \mathcal{P}_c. \quad (A.3)
   \]
   In other words, the following diagram commutes:

   \[
   \begin{array}{ccc}
   V_{n_c} &=& V_{s_1} \otimes V_{s_2} \otimes \cdots \otimes V_{s_{d_c}} \\
   &\xrightarrow{\mathcal{I}_c} & \text{im} \mathcal{I}_c \subset \text{im} \mathcal{P}_c \\
   &\xrightarrow{\mathcal{P}_c} & \mathcal{P}_c \\
   &\xrightarrow{\hat{\mathcal{P}}_c} & \mathcal{P}_c \\
   \end{array} \quad (A.4)
   \]

   Also, the map $\hat{\mathcal{P}}_c$ restricted to $\text{im} \mathcal{P}_c$ is a $\mathcal{U}_q$- and $\mathcal{U}_q$-module isomorphism, with inverse $\mathcal{I}_c$.

4. We have
   \[
   \hat{\mathcal{P}}_c \circ \mathcal{P}_c = \mathcal{P}_c \quad \text{and} \quad \mathcal{P}_c \circ \mathcal{I}_c = \mathcal{I}_c. \quad (A.5)
   \]

Similarly, items 1–4 hold for right $\mathcal{U}_q$ and $\mathcal{U}_q$-modules, after the symbolic replacements

\[
\mathcal{I} \mapsto \mathcal{I}_c, \quad \mathcal{P} \mapsto \mathcal{P}_c, \quad \hat{\mathcal{P}} \mapsto \hat{\mathcal{P}}_c, \quad V \mapsto \nabla, \quad \text{and} \quad H \mapsto \Pi. \quad (A.6)
\]

6. For all vectors $v \in V_c$ and $\mathfrak{v} \in \nabla_c$, we have
   \[
   \mathcal{I}_c(v)^* = \mathcal{I}_c(\mathfrak{v}^*) \quad \text{and} \quad \mathcal{I}_c(\mathfrak{v})^* = \mathcal{I}_c(v^*). \quad (A.7)
   \]

Similarly, for all vectors $w \in V_{n_c}$ and $\mathfrak{w} \in \nabla_{n_c}$, we have
   \[
   \mathcal{P}_c(w)^* = \mathcal{P}_c(w^*) \quad \text{and} \quad \mathcal{P}_c(w^*) = \mathcal{P}_c(w^*), \quad (A.8)
   \]
   \[
   \hat{\mathcal{P}}_c(w)^* = \hat{\mathcal{P}}_c(w^*) \quad \text{and} \quad \hat{\mathcal{P}}_c(w^*) = \hat{\mathcal{P}}_c(w^*). \quad (A.9)
   \]
Proof. Items 1–3 are immediate from the definitions of the maps in the assertion. Then, item 3 gives item 4:

\[ \mathfrak{P}_c \circ \mathfrak{P}_c \overset{(A.3)}{=} \mathfrak{P}_c \circ \mathfrak{I}_c \circ \mathfrak{P}_c \overset{(A.3)}{=} \mathfrak{P}_c \quad \text{and} \quad \mathfrak{P}_c \circ \mathfrak{I}_c \overset{(A.3)}{=} \mathfrak{I}_c \circ \mathfrak{P}_c \circ \mathfrak{I}_c \overset{(A.3)}{=} \mathfrak{I}_c. \]  

(A.10)

Items 1–4 with replacements (A.6) can be proven similarly.

To finish, we prove item 6. First, because the maps \( \mathfrak{I}_c, \mathfrak{P}_c \), and \((\cdot)^*\) are linear and factor over the tensorands of \( \mathfrak{V}_c \) or \( \overline{\mathfrak{V}}_c \), we may take \( \xi = (s) \) and \( \nu = e^{(s)}_r \) without loss of generality. Then, we have

\[ \mathfrak{I}_c(e^{(s)}_r)^* \overset{(A.22, A.37)}{=} \theta^*_0. \mathfrak{E}^c \overset{(A.35, A.36)}{=} \theta^*_0 \mathfrak{E}^c \overset{(2.103, 2.111)}{=} q^{-\ell(s-t)} \mathfrak{I}_c(e^{(s)}_r)^* \overset{(A.15)}{=} \mathfrak{I}_c(e^{(s)}_r)^*, \]

which proves the left equation of (A.7). The right equation of (A.7) can be proven similarly. Next, because the maps \( \mathfrak{P}_c, \mathfrak{P}_c, \mathfrak{P}_c, \mathfrak{P}_c \), and \((\cdot)^*\) are linear and factor over the tensorands of \( \mathfrak{V}_n \) or \( \overline{\mathfrak{V}}_n \), we may take \( w = \theta^*_r \), so

\[ \mathfrak{P}_c(\theta^*_r)^* \overset{(2.111)}{=} \mathfrak{P}_c(\mathfrak{I}_c(e^{(s)}_r))^* \overset{(A.5)}{=} \mathfrak{I}_c(\mathfrak{I}_c(e^{(s)}_r))^* \overset{(A.5)}{=} \mathfrak{I}_c(\mathfrak{I}_c(e^{(s)}_r))^* \overset{(2.111)}{=} \overline{\mathfrak{I}_c(\mathfrak{I}_c(e^{(s)}_r))^*} \overset{(A.7)}{=} \overline{\mathfrak{I}_c(\mathfrak{I}_c(e^{(s)}_r))^*}, \]

which proves the left equation of (A.8). The other equations in (A.8, A.9) can be proven similarly.

The mappings \( \iota_{(r,t)}^{(s)}, \pi_{(r,t),(s)}^{(r,t)} \), and \( \overline{\pi}_{(s)}^{(r,t)} \) defined in (2.118, 2.119, 2.120) in section 2 C have similar properties.

**Lemma A.3.** Suppose \( r + t < p(q) \). Then for each \( s \in \mathcal{E}_{(r,t)} \), the following hold:

1. The maps \( \iota_{(r,t)}^{(s)}, \pi_{(r,t),(s)}^{(r,t)} \), and \( \overline{\pi}_{(s)}^{(r,t)} \) are homomorphisms of left \( \mathfrak{U}_q \) and \( \overline{\mathfrak{U}}_q \)-modules.
2. \( \iota_{(r,t)}^{(s)} \) is a linear injection, \( \overline{\pi}_{(s)}^{(r,t)} \) is a linear surjection, and \( \pi_{(r,t),(s)}^{(r,t)} \) is a linear projection.
3. We have \( \iota_{(r,t)}^{(s)} \circ \pi_{(r,t)}^{(r,t)} = \text{id}_{\mathfrak{V}_{(r,t)}} \), \( \ker \overline{\pi}_{(s)}^{(r,t)} \circ \iota_{(r,t)}^{(s)} = \ker \pi_{(r,t)}^{(r,t)} \), and

\[ \overline{\pi}_{(s)}^{(r,t)} \circ \iota_{(r,t)}^{(s)} = \text{id}_{\mathfrak{V}_{(r,t)}}, \quad \text{and} \quad \iota_{(r,t)}^{(s)} \circ \overline{\pi}_{(s)}^{(r,t)} = \pi_{(r,t)}^{(r,t)}(s). \]  

(A.13)

Thus, the following diagram commutes:

\[ \begin{array}{ccc}
\mathfrak{V}_{(s)} & \xrightarrow{\iota_{(r,t)}^{(s)}} & \text{im } \iota_{(r,t)}^{(s)} = \text{im } \pi_{(r,t),(s)}^{(r,t)} \subset \mathfrak{V}_{(r,t)} \\
\mathfrak{V}_{(r,t)} & \xrightarrow{\pi_{(r,t),(s)}^{(r,t)}} & \text{im } \pi_{(r,t),(s)}^{(r,t)} = \pi_{(r,t)}^{(r,t)}(s) \\
\end{array} \]  

(A.14)

and the map \( \overline{\pi}_{(s)}^{(r,t)} \) restricted to \( \text{im } \pi_{(r,t),(s)}^{(r,t)} \) is an isomorphism of left \( \mathfrak{U}_q \) and \( \overline{\mathfrak{U}}_q \)-modules, with inverse \( \iota_{(r,t)}^{(s)} \).

4. We have

\[ \overline{\pi}_{(s)}^{(r,t)} \circ \pi_{(r,t),(s)}^{(r,t)} = \pi_{(s)}^{(r,t)} \quad \text{and} \quad \pi_{(r,t),(s)}^{(r,t)} \circ \iota_{(r,t)}^{(s)} = \iota_{(r,t)}^{(s)}. \]  

(A.15)

Similarly, items 1–4 hold for right \( \mathfrak{U}_q \) and \( \overline{\mathfrak{U}}_q \)-modules, after the symbolic replacements

\[ \iota \mapsto \iota, \quad \pi \mapsto \pi, \quad \overline{\pi} \mapsto \overline{\pi}, \quad \mathfrak{V} \mapsto \overline{\mathfrak{V}}, \quad \text{and} \quad H \mapsto H. \]

(A.16)

5. For all vectors \( v \in \mathfrak{V}_{(s)} \) and \( \overline{v} \in \overline{\mathfrak{V}}_{(s)} \), we have

\[ \iota_{(r,t)}^{(s)}(v)^* = (-q)^{(r+t-s)/2} \iota_{(r,t)}^{(s)}(v)^* \quad \text{and} \quad \iota_{(r,t)}^{(s)}(v)^* = (-q)^{(r+t-s)/2} \iota_{(r,t)}^{(s)}(v)^*. \]  

(A.17)

Similarly, for all vectors \( w \in \mathfrak{V}_{(r,t)} \) and \( \overline{w} \in \overline{\mathfrak{V}}_{(r,t)} \), we have

\[ \pi_{(r,t),(s)}^{(r,t)}(w)^* = \pi_{(r,t),(s)}^{(r,t)}(w)^* \quad \text{and} \quad \pi_{(r,t),(s)}^{(r,t)}(w)^* = \pi_{(r,t),(s)}^{(r,t)}(w)^*, \]  

(A.18)

\[ \overline{\pi}_{(s)}^{(r,t)}(w)^* = (-q)^{(r+t-s)/2} \overline{\pi}_{(s)}^{(r,t)}(w)^* \quad \text{and} \quad \overline{\pi}_{(s)}^{(r,t)}(w)^* = (-q)^{(r+t-s)/2} \overline{\pi}_{(s)}^{(r,t)}(w)^*. \]  

(A.19)
Proof. This can be proven similarly as lemma A.2, except that identities (A.17–A.19) are slightly different due to our normalization choice for the embedding maps. For the latter, we first observe that

$$u^{(s)}_{(r,t)} (2.72, 2.73) \equiv (2.36, A.35) (-q)^{(r+t-s)/2} \pi^{(s)}_{(r,t)},$$

(A.20)

Because the maps $u^{(s)}_{(r,t)}$, $\pi^{(s)}_{(r,t)}$, and $(\cdot)^*$ are linear and factor over the tensorands of $V_{(s)}$ or $\overline{V}_{(s)}$, it suffices to calculate

$$u^{(s)}_{(r,t)}(e^{(s)}_{\ell}) (2.118) \equiv (2.22) \equiv (2.27) \equiv (A.20) \equiv (2.28) \equiv (A.27, A.35) \equiv (2.118) (-q)^{(r+t-s)/2} \pi^{(s)}_{(r,t)} E^{(A.20)} \equiv (-q)^{(r+t-s)/2} \pi^{(s)}_{(r,t)} (e^{(s)}_{\ell})^*,$$

(A.21)

which proves the left equation of (A.17). The other equations in (A.17–A.19) can be proven similarly.

2. The star involution

We defined two analogous bialgebras, $U_q$ and $\overline{U}_q$ in section 2 A. There is a simple relation between them:

$$E^k K^m F^\ell \mapsto (E^k K^m F^\ell)^* \equiv E^\ell K^m F^k,$$

(A.22)

extended linearly. This map is an involution in the sense that its inverse is given by the linear extension of

$$E^\ell K^m F^k \mapsto (E^\ell K^m F^k)^* \equiv E^k K^m F^\ell.$$

(A.23)

Lemma A.4. Suppose $q \in \mathbb{C}^x \setminus \{\pm 1\}$. The map $(\cdot)^* : U_q \to \overline{U}_q$ is an anti-isomorphism of associative, unital algebras, as well as an isomorphism of coassociative, counital coalgebras.

Proof. This follows from definition (A.22), its inverse (A.23), and relations (2.2, 2.11, 2.12, 2.20, 2.21, 2.22).

Lemma A.5. Suppose $q \in \mathbb{C}^x \setminus \{\pm 1\}$. If $v \in V_{(s)}$, then we have

$$E.v, \overline{E}.v \in V_{(s+2)}^{(s+2)}, \quad F.v, \overline{F}.v \in V_{(s-2)}^{(s-2)}, \quad K^{\pm 1}.v, \overline{K}^{\pm 1}.v \in V_{(s)}^{(s)},$$

(A.24)

and

$$(E^k K^m F^\ell).v = q^{-k(s-2\ell+k)+m(s-2\ell)+\ell(s-\ell)} (E^k K^m F^\ell).v.$$ (A.25)

Similarly, if $v \in \overline{V}_{(s)}$, then we have

$$\overline{v}.E, \overline{v}.\overline{E} \in \overline{V}_{(s-2)}^{(s-2)}, \quad \overline{v}.F, \overline{v}.\overline{F} \in \overline{V}_{(s+2)}^{(s+2)}, \quad \overline{v}.K^{\pm 1}, \overline{v}.\overline{K}^{\pm 1} \in \overline{V}_{(s)}^{(s)},$$

(A.26)

and

$$\overline{v}.(E^k K^m F^\ell) = q^{-k(s-k)+m(s-2k)-\ell(s-2k+\ell)} \overline{v}.(E^k K^m F^\ell).$$

(A.27)

Proof. Formulas (A.24, A.26) follow from definitions (2.6, 2.24) together with coproduct relations (2.21, 2.21). To prove (A.25), it suffices to check the three special cases $(m, n, p) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, as the general result (A.25) then follows by using formulas (A.24). We prove the three special cases by induction on $d_0 \in \mathbb{Z}_{>0}$. In the initial case $d_0 = 1$, we have $\varsigma = (t)$ for some $t \in \mathbb{Z}_{>0}$, and

$$E.e^{(t)}_{(t-s)/2} = q^{-1-s} E.e^{(t)}_{(t-s)/2}, \quad F.e^{(t)}_{(t-s)/2} = q^{-1+s} F.e^{(t)}_{(t-s)/2}, \quad \text{and} \quad K.e^{(t)}_{(t-s)/2} = K.e^{(t)}_{(t-s)/2},$$

(A.28)

and (A.25) follows because any vector $v \in V_{(s)}^{(s)}$ is proportional to $e^{(t)}_{(t-s)/2}$ by (2.28). Next, we let $d \geq 2$ and assume that for any multiindex $\varsigma \in \mathbb{Z}_{>0}^1$, for any index $s \in \mathbb{E}_\varsigma^{(t)}$, and for any vector $v \in V_{(s)}^{(s)}$, we have

$$E.v = q^{-1-s} E.v, \quad \overline{E}.v = q^{-1-s} F.v, \quad \text{and} \quad \overline{K}.v = K.v.$$ (A.29)

Then, for any $\varsigma = \varsigma \oplus (t)$ with $t \in \mathbb{Z}_{>0}$ as in (2.40), writing a generic vector in the form

$$v \in V_{(s)}^{(s)} \overset{(2.28)}{=} v = \sum_{\ell=0}^{t} v_{\ell} \otimes e^{(t)}_{(t)-\ell}, \quad \text{where} \quad v_{\ell} \in V_{(s)}^{(s+t-2\ell)},$$

(A.30)
it is straightforward to verify (A.29) for \( v \in \mathbb{V}_\varsigma \) using the induction hypothesis: for example,

\[
E.v \equiv_{(A.30)} \sum_{\ell = 0}^{t} (K^{-1}.v_{\ell} \otimes E.e_{\ell}^{(t)} + E.v_{\ell} \otimes e_{\ell}^{(t)})
\]

\[
(2.21)
\]

\[
(2.23)
\]

\[
(\ell.u) = E.v \otimes K.e_{\ell}^{(t)} + v_{\ell} \otimes E.e_{\ell}^{(t)} \equiv_{(A.22)} E.v \otimes e_{\ell}^{(t)}
\]

\[
(2.6)
\]

\[
(2.28)
\]

\[
(2.30)
\]

and \( F.v \) and \( K.v \) are similar. This completes the induction step and, combined with formulas (A.24), finishes the proof of (A.25). Identity (A.27) can be proven similarly, by writing a generic vector with \( \varsigma = (t) \oplus \varsigma \) instead in the form

\[
\varpi \in \mathbb{V}_\varsigma \rightarrow \mathbb{V}_\varsigma \equiv_{(2.28)} \varpi = \sum_{\ell = 0}^{t} e_{\ell}^{(t)} \otimes v_{\ell}, \text{ where } v_{\ell} \in \mathbb{V}_{\varsigma}^{(s+t-2\ell)}.
\]

This concludes the proof.

Next, we define a linear isomorphism \((\cdot)^* : \mathbb{V}_\varsigma \rightarrow \mathbb{V}_\varsigma \) by linear extension of

\[
e_{\ell}^{(s)} := q^{-\ell(s-\ell)} e_{\ell}^{(s)} \quad \text{and} \quad e_{\ell}^{(s)*} := q^\ell(s-\ell) e_{\ell}^{(s)},
\]

and as shown, we also denote its inverse map \((\cdot)^* : \mathbb{V}_\varsigma \rightarrow \mathbb{V}_\varsigma \) by the same symbol. We extend this definition to a linear isomorphism \((\cdot)^* : \mathbb{V}_\varsigma \rightarrow \mathbb{V}_\varsigma \) on the tensor product (2.13), with inverse \((\cdot)^* : \mathbb{V}_\varsigma \rightarrow \mathbb{V}_\varsigma \), via

\[
(v \otimes w)^* := v^* \otimes w^* \quad \text{and} \quad (\varpi \otimes \bar{\varpi})^* := \varpi^* \otimes \bar{\varpi}^*.
\]

This map defines \( U_q, \bar{U}_q \)-homomorphisms in the following sense.

**Lemma A.6.** Suppose \( q \in \mathbb{C} \setminus \{\pm 1\} \). For all elements \( x \in U_q \) and \( \bar{x} \in \bar{U}_q \) and vectors \( v \in \mathbb{V}_\varsigma \) and \( \varpi \in \mathbb{V}_\varsigma \), we have

\[
(x.v)^* = v^*.x^*, \quad (\bar{x}.v)^* = v^*.\bar{x}^*,
\]

and similarly,

\[
(\varpi.x)^* = x^*.*\varpi^*, \quad (\varpi.\bar{x})^* = \bar{x}^*.*\varpi^*.
\]

**Proof.** As the map \((\cdot)^* : U_q \rightarrow \bar{U}_q \) of (A.22) is an antihomomorphism of algebras, it suffices to prove the lemma for \( x \in \{E, F, K^{\pm 1}\} \). Also, we only prove the left equation in (A.37); the other equations in (A.37, A.38) can be proven similarly. We proceed by induction on \( \ell \in \mathbb{Z}_{>0} \). In the initial case \( \ell = 1 \), we have \( \varsigma = (t) \) for some \( t \in \mathbb{Z}_{>0} \), and

\[
(F.e_{\ell}^{(t)})^* = \ell.t+1 = q^{-(t+1)(t-1)} e_{\ell+1}^{(t)} = q^{-(t-1)(\ell-1)} e_{\ell}^{(t)},
\]

\[
(2.6)
\]

\[
(2.9)
\]

\[
(2.24)
\]

\[
(2.23)
\]

\[
(2.35)
\]

\[
(2.21)
\]

\[
(2.11)
\]

\[
(2.22)
\]

\[
(2.30)
\]

\[
(2.28)
\]

\[
(2.27)
\]

\[
(2.27)
\]

\[
(2.27)
\]

\[
(2.27)
\]

\[
(2.27)
\]

\[
(2.27)
\]

\[
(2.27)
\]

\[
(2.27)
\]

The left equation of (A.37) then follows by linearity. Next, we let \( d \geq 2 \) and assume that the left equation in (A.37) holds for any multiindex \( \varsigma \in \mathbb{Z}_{d>0} \). Then, with \( \varsigma = \varsigma \oplus (t) \), by linearity it suffices to consider

\[
v = u \otimes w \in \mathbb{V}_\varsigma, \quad \text{where } u \in \mathbb{V}_\varsigma \text{ and } w \in \mathbb{V}_{(t)}.
\]

Now, by applying the induction hypothesis to \( u \) and \( w \), we obtain

\[
(E.(u \otimes w))^* = (E.\bar{u} \otimes K.w + u \otimes E.w)^* = u^*.E^* \otimes \bar{w}^*.K^* + u^* \otimes w^*.E^*
\]

\[
(2.11)
\]

\[
(A.26)
\]

\[
(A.36)
\]

\[
(A.37)
\]

\[
(A.22)
\]

\[
(A.27)
\]

\[
(A.28)
\]

\[
(A.29)
\]

\[
(A.30)
\]

\[
(A.31)
\]

\[
(A.32)
\]

\[
(A.33)
\]

\[
(A.34)
\]

\[
(A.35)
\]

\[
(A.36)
\]

\[
(A.37)
\]

\[
(A.38)
\]

\[
(A.39)
\]

\[
(A.40)
\]

\[
(A.41)
\]

\[
(A.42)
\]

\[
(A.43)
\]

\[
(A.44)
\]

This proves the left equation of (A.37) for \( x = E \). The cases \( x \in \{F, K^{\pm 1}\} \) can be proven similarly. 

\[\square\]
3. Mapping \( q \mapsto q^{-1} \)

We define the maps \((\cdot)^{\text{op}}: U_{q^\pm} \to \overline{U}_{q^\mp}\) and \((\cdot)^{\text{op}}: \overline{U}_{q^\pm} \to U_{q^\mp}\) by linear extensions of

\[
(E^k K_m F^\ell + \mp \mp \pm E^k) : F^\ell + F^m E^k
\]

These maps are involutions in the sense that, for all elements \( x \in U_{q^\pm} \) and \( \bar{x} \in \overline{U}_{q^\pm} \), we have

\[
(x^{\text{op}})^{\text{op}} = x \quad \text{and} \quad (\bar{x}^{\text{op}})^{\text{op}} = \bar{x}.
\]

**Lemma A.7.** Suppose \( q \in C^s \setminus \{\pm 1\} \). The maps \((\cdot)^{\text{op}}: U_{q^\pm} \to \overline{U}_{q^\mp}\) and \((\cdot)^{\text{op}}: \overline{U}_{q^\pm} \to U_{q^\mp}\) are anti-isomorphisms of associative, unital algebras, as well as isomorphisms of coassociative, counital coalgebras.

**Proof.** This follows from definition (A.45), its inverse (A.46), and relations (2.2, 2.11, 2.12, 2.20, 2.21, 2.22).

Assuming that \( s < p(q) \), we define the maps \((\cdot)^{\text{op}}: V_{(s)} \to \overline{V}_{(s)}\) and \((\cdot)^{\text{op}}: \overline{V}_{(s)} \to V_{(s)}\) by linear extensions of

\[
e^{(s)}_\ell \ := \ \frac{[s]!}{[s-\ell]} e^{(s)}_{s-\ell}, \quad \text{and} \quad e^{(s)}_\ell \ := \ \frac{[s]!}{[s-\ell]} e^{(s)}_{s-\ell},
\]

and extend these maps to tensor products of vectors via

\[
(v \otimes w)^{\text{op}} = v^{\text{op}} \otimes w^{\text{op}} \quad \text{and} \quad (\bar{v} \otimes \bar{w})^{\text{op}} = \bar{v}^{\text{op}} \otimes \bar{w}^{\text{op}}.
\]

Again, we observe that these maps are involutions in the sense that

\[
(v^{\text{op}})^{\text{op}} = v \quad \text{and} \quad (\bar{v}^{\text{op}})^{\text{op}} = \bar{v}.
\]

Furthermore, these maps send the grade subspaces \( V_{(s)} \) and \( \overline{V}_{(s)} \) to the subspaces with opposite grade value:

\[
v \in V_{(s)} \quad \overset{(2.28), (A.47, A.48)}{\Longrightarrow} \quad v^{\text{op}} \in \overline{V}_{(-s)},
\]

\[
\bar{v} \in \overline{V}_{(s)} \quad \overset{(2.28), (A.47, A.48)}{\Longrightarrow} \quad \bar{v}^{\text{op}} \in V_{(-s)}.
\]

**Lemma A.8.** Suppose \( q \in C^s \setminus \{\pm 1\} \). For all elements \( x \in U_{q^\pm}, \bar{x} \in \overline{U}_{q^\pm} \) and vectors \( v \in V_{(s)}, \bar{v} \in \overline{V}_{(s)} \), we have

\[
(x.v)^{\text{op}} = v^{\text{op}}.x^{\text{op}}, \quad (\bar{x}.v)^{\text{op}} = \bar{v}^{\text{op}}.\bar{x}^{\text{op}},
\]

and similarly,

\[
(\bar{v}.x)^{\text{op}} = x^{\text{op}}.\bar{v}^{\text{op}}, \quad (v.\bar{x})^{\text{op}} = \bar{x}^{\text{op}}.v^{\text{op}}.
\]

**Proof.** Checking all four cases is a straightforward computation.

Next, assuming \( s < p(q) \), we define the vectors

\[
\theta^{(s)}_\ell := F^\ell \theta^{(s)}_0, \quad \bar{\theta}^{(s)}_\ell := \theta^{(s)}_0.E^\ell,
\]

the embeddings \( \mathcal{I}_{(s)}: V_{(s)} \hookrightarrow V_{(s)}\) and \( \overline{\mathcal{I}}_{(s)}: \overline{V}_{(s)} \hookrightarrow \overline{V}_{(s)}\) by linear extensions of

\[
\mathcal{I}_{(s)}(e^{(s)}_\ell) \ = \ \theta^{(s)}_\ell, \quad \overline{\mathcal{I}}_{(s)}(e^{(s)}_\ell) = \overline{\theta}^{(s)}_\ell,
\]

and the embeddings \( \mathcal{I}_{(s)}: V_{(s)} \hookrightarrow V_{(s)}\) and \( \overline{\mathcal{I}}_{(s)}: \overline{V}_{(s)} \hookrightarrow \overline{V}_{(s)}\) by

\[
\mathcal{I}_{(s)} := \mathcal{I}_{(s_1)} \oplus \mathcal{I}_{(s_2)} \oplus \cdots \oplus \mathcal{I}_{(s_{d_s})}, \quad \overline{\mathcal{I}}_{(s)} := \overline{\mathcal{I}}_{(s_1)} \oplus \overline{\mathcal{I}}_{(s_2)} \oplus \cdots \oplus \overline{\mathcal{I}}_{(s_{d_s})}.
\]

**Lemma A.9.** Suppose \( \max \xi < p(q) \). We have

\[
\mathcal{I}_{(s)}(v^{\text{op}}) = [s_1]! \cdot [s_2]! \cdots [s_{d_s}]! \mathcal{I}_{(s)}(v^{\text{op}}) \quad \overline{\mathcal{I}}_{(s)}(v^{\text{op}}) = [s_1]! \cdot [s_2]! \cdots [s_{d_s}]! \overline{\mathcal{I}}_{(s)}(v^{\text{op}}).
\]
Proof. In light of \((A.56)\), it suffices to consider the case \(\zeta = (s)\) for some \(s \in \mathbb{Z}_{>0}\) (so \(d_\zeta = 1\)). We will prove the left equation of \((A.57)\), for the right one is similar. To this end, we observe that the map \(f \in \text{End} V(s)\) given by
\[
f(v) := \mathcal{J}_{(s) \mp}^{-1} \left( (\mathcal{J}_{(s) \pm} (v^{(op)})^{op} \right)
\]
(A.58)
is a \(U_q\)-homomorphism: by item 1 of lemma A.2 and properties of the \((\cdot)^{op}\) map, we have
\[
f(x \cdot v) = \mathcal{J}_{(s) \mp}^{-1} \left( (x \cdot v)^{op} \right)^{op} \quad (A.52) = \mathcal{J}_{(s) \mp}^{-1} \left( (\mathcal{J}_{(s) \pm} (v^{(op)}, x^{op})^{op} \right) \quad (A.46) = \mathcal{J}_{(s) \mp}^{-1} \left( x \cdot (\mathcal{J}_{(s) \pm} (v^{(op)})^{op} \right) = x \cdot f(v).
\]
(A.59)
As such, by Schur’s lemma, there exists a constant \(\lambda \in \mathbb{C}\) such that \(f(v) = \lambda v\) for all \(v \in V(s)\), or by \((A.58)\),
\[
\mathcal{J}_{(s) \pm} (v^{(op)}) = \lambda \mathcal{J}_{(s) \mp} (v)^{op}.
\]
(A.60)
After inserting \(\bar{v} = \bar{e}_0^{(s)}\) into \((A.60)\), we finally arrive with
\[
\mathcal{J}_{(s) \pm} (\bar{e}_0^{(s)} )^{op} \overset{(A.66)}{=} \lambda \mathcal{J}_{(s) \mp} (\bar{e}_0^{(s)} )^{op} \overset{(A.55)}{=} \lambda \bar{g}_0^{(s)}^{op}
\]
(A.61)
\[
\mathcal{J}_{(s) \pm} (\bar{e}_0^{(s)} )^{op} \overset{(2.107)}{=} \lambda [s]^{-1} \theta_\ell^{(s)} \quad (A.55) = \lambda [s]^{-1} \mathcal{J}_{(s) \pm} (e_s^{(s)}) \quad (A.47) = \lambda [s]^{-1} \mathcal{J}_{(s) \pm} (e_0^{(s)} )^{op}.
\]
Hence, we have \(\lambda = [s]!\) in \((A.60)\), finishing the proof.

After inserting \(\zeta = (s)\) and \(\bar{v} = \bar{e}_\ell^{(s)}\) into \((A.57)\) and using \((A.47)\) to simplify the result, we obtain
\[
\bar{g}_\ell^{(s)}^{op} = \frac{[\ell]!}{[s-\ell]!} \bar{g}_\ell^{(s)}^{op}.
\]
(A.62)

4. Bilinear forms from the bilinear pairing

Using the maps \((\cdot)^{op}\), we define bilinear forms \(\langle \cdot, \cdot \rangle : V_\zeta \times V_\zeta \to \mathbb{C}\) and \(\langle \cdot, \cdot \rangle : \bar{V}_\zeta \times \bar{V}_\zeta \to \mathbb{C}\) respectively by
\[
\langle v, w \rangle := (v^{(op)}, w) \quad \text{and} \quad \langle \bar{v}, \bar{w} \rangle := (\bar{v}, \bar{w}^{(op)}).
\]
(A.63)

Lemma A.10. Suppose \(\max \zeta < p(q)\). The following hold:

1. We have
\[
\langle e_\ell^{(s_1)} \otimes e_\ell^{(s_2)} \otimes \cdots \otimes e_\ell^{(s_{d_\zeta})}, e_{m_1}^{(s_1)} \otimes e_{m_2}^{(s_2)} \otimes \cdots \otimes e_{m_{d_\zeta}}^{(s_{d_\zeta})} \rangle = \prod_{k=1}^{d_\zeta} \delta_{\ell_k + m_k, s_k}
\]
(A.64)
and similarly,
\[
\langle \bar{e}_\ell^{(s_1)} \otimes \bar{e}_\ell^{(s_2)} \otimes \cdots \otimes \bar{e}_\ell^{(s_{d_\zeta})}, \bar{e}_{m_1}^{(s_1)} \otimes \bar{e}_{m_2}^{(s_2)} \otimes \cdots \otimes \bar{e}_{m_{d_\zeta}}^{(s_{d_\zeta})} \rangle = \prod_{k=1}^{d_\zeta} \delta_{\ell_k + m_k, s_k}.
\]
(A.65)
In particular, the bilinear form is symmetric: for all vectors \(\bar{v}, \bar{w} \in \bar{V}_\zeta\) and \(v, w \in V_\zeta\), we have
\[
\langle v, w \rangle = \langle w, v \rangle \quad \text{and} \quad \langle \bar{v}, \bar{w} \rangle = \langle \bar{w}, \bar{v} \rangle.
\]
(A.66)

2. For all vectors \(\bar{v}_j, \bar{w}_j \in \bar{V}_{(s_j)}\) and \(v_j, w_j \in V_{(s_j)}\), with \(j \in \{1, 2, \ldots, d_\zeta\}\), we have the factorizations
\[
\langle v_1 \otimes v_2 \otimes \cdots \otimes v_{d_\zeta}, w_1 \otimes w_2 \otimes \cdots \otimes w_{d_\zeta} \rangle = \prod_{k=1}^{d_\zeta} \langle v_k, w_k \rangle,
\]
(A.67)
\[
\langle \bar{v}_1 \otimes \bar{v}_2 \otimes \cdots \otimes \bar{v}_{d_\zeta}, \bar{w}_1 \otimes \bar{w}_2 \otimes \cdots \otimes \bar{w}_{d_\zeta} \rangle = \prod_{k=1}^{d_\zeta} \langle \bar{v}_k, \bar{w}_k \rangle.
\]
(A.68)
3. For all elements $x_+ \in U_{q_+}$ and $\bar{x}_- \in \overline{U}_{q_-}$, or $x_\pm \in U_{q_{\pm}}$ and $\bar{x}_\pm \in \overline{U}_{q_{\pm}}$, and vectors $v, w \in \nabla'$, we have

$$\begin{align*}
(v, x_+ + \bar{x}_-) &= \left(x_+^{op}, v - \bar{x}_-\right) \\
(x_+ + \bar{x}_- + v, w) &= \left(x_+^{op}, v - \bar{x}_-\right)
\end{align*}$$

(A.69)

and

$$\begin{align*}
(v, x_+ + \bar{x}_-) &= \left(x_+^{op}, v - \bar{x}_-\right) \\
(x_+ + \bar{x}_- + v, w) &= \left(x_+^{op}, v - \bar{x}_-\right)
\end{align*}$$

(A.70)

4. The pairs $\nabla'^{(i)}$ and $\overline{\nabla'}^{(i)}$, or $\overline{\nabla'}^{(i)}$ and $\overline{\nabla'}^{(i)}$, are respectively orthogonal:

$$\begin{align*}
(v, w) &= 0 \quad \text{for all } v \in \nabla'^{(i)} \text{ and } w \in \nabla'^{(i)} \text{ with } s \neq -t, \\
(v, \bar{w}) &= 0 \quad \text{for all } v \in \overline{\nabla'}^{(i)} \text{ and } \bar{w} \in \overline{\nabla'}^{(i)} \text{ with } s \neq -t.
\end{align*}$$

(A.73)

Also, for all vectors $v \in \nabla'^{(i)}$ with $F_+v = F_+v = 0$ and $w \in \nabla'^{(i)}$ with $E_+w = E_+w = 0$, we have

$$\begin{align*}
(E_+^{op}, v, F_+^{op}, w) &= \delta_{s+t,0} \delta_{\ell,m } \left[ \begin{array}{c} s \\ \ell \end{array} \right] (v, w) = (E_+^{op}, v, F_+^{op}, w),
\end{align*}$$

(A.75)

and for all vectors $v \in \overline{\nabla}'^{(i)}$ with $vF_+ = vF_+ = 0$ and $\bar{w} \in \overline{\nabla}'^{(i)}$ with $\bar{w}E_+ = \bar{w}E_+ = 0$, we have

$$\begin{align*}
(v, E_+^{op}, v, F_+^{op}) &= \delta_{s+t,0} \delta_{\ell,m } \left[ \begin{array}{c} s \\ \ell \end{array} \right] (v, \bar{w}) = (v, E_+^{op}, v, F_+^{op}).
\end{align*}$$

(A.76)

5. For all vectors $v, \bar{w} \in \overline{\nabla}'$ and $v, \bar{w} \in \overline{\nabla}'$, we have

$$\begin{align*}
(v, w) &= \left\{ [s_1]! \cdots [s_d]! \left( \mathcal{J}_{\nabla'}(v), \mathcal{J}_{\nabla'}(w) \right) \right\}
\end{align*}$$

(A.77)

and

$$\begin{align*}
(v, \bar{w}) &= \left\{ [s_1]! \cdots [s_d]! \left( \mathcal{J}_{\nabla'}(v), \mathcal{J}_{\nabla'}(w) \right) \right\}.
\end{align*}$$

(A.78)

6. For all vectors $v, \bar{w} \in \overline{\nabla}'$ and $v, \bar{w} \in \nabla'$, we have

$$\begin{align*}
(v, w) &= (v^{op}, w^{op}) \quad \text{and} \quad (v, \bar{w}) = (v^{op}, \bar{w}^{op}).
\end{align*}$$

(A.79)

Proof. We prove these properties as follows.

1. Properties (A.64, A.65) follow from definition (A.63) of the bilinear form combined with (2.127) of item 1 in lemma 2.13, and distributive property (A.47) of the map $(\cdot)^{op}$. Then, (A.66) follows from this.

2. Properties (A.67, A.68) follow from (A.64, A.65) of item 1 and linearity.

3. We have

$$\begin{align*}
(v, x_+ + \bar{x}_-) &= \left(x_+^{op}, v - \bar{x}_-\right) \quad (A.63) \quad \left(v^{op} | x_+ + \bar{x}_-\right) = \left(v^{op} | x_+ + \bar{x}_-\right) \quad (2.128) \quad \left(v^{op} | x_+ + \bar{x}_-\right) = \left(v^{op} | x_+ + \bar{x}_-\right) \quad (A.49) \quad \left(v^{op} | x_+ + \bar{x}_-\right) = \left(v^{op} | x_+ + \bar{x}_-\right) \quad (A.53) \quad \left(v^{op} | x_+ + \bar{x}_-\right) = \left(v^{op} | x_+ + \bar{x}_-\right),
\end{align*}$$

(A.80)

which proves the left equality of (A.69). The right equality of (A.69) and (A.70–A.72) can be proven similarly.

4. (A.73, A.74) follow from bilinear form definition (A.63), (A.50, A.51), and (2.129) in item 3 of lemma 2.13. Properties (A.75, A.76) then follow from this, (A.45), (A.52, A.53) of lemma A.8, and (2.130) in item 3 of lemma 2.13.

5. Properties (A.77, A.78) follow from (A.57) of lemma A.9, bilinear form definition (A.63), and (2.131) from item 4 in lemma 2.13.

6. Property (A.79) follows from bilinear form definition (A.63) and involution property (A.49).

This completes the proof. □
B. EXCEPTIONAL CASE: \( q = \pm i \)

In lemma 3.6 and corollary 3.9, if \( q \in \{ \pm i \}, \) then \( \nu = 0, \) so (3.66, 3.68, 3.74, 3.75) are identically zero. On the other hand, (3.65, 3.67) are still well-defined, and we can also make sense of analogues of (3.66, 3.68, 3.74, 3.75), as we show in this appendix. To begin, we note that if \( q = \pm i, \) then the vectors in (3.63, 3.64) fail to form bases for the left and right \( U_q \)-modules \( U \circ V_2 \) and \( \overline{V} \circ U. \) For example, the following two vectors are proportional to each other:

\[
F^2.u^{(2)}_{(1,1)} = 0 \quad \text{and} \quad u^{(0)}_{(1,1)} = \mp i 2 F.u^{(2)}_{(1,1)}. \tag{B.1}
\]

However, we can choose for instance the following different bases for \( U \circ V_2 \) and \( \overline{V} \circ U. \):

\[
\begin{align*}
\{ u := u^{(0)}_{(1,1)}, & \quad \theta := u^{(2)}_{(1,1)} = \varepsilon_0 \otimes \varepsilon_0, \quad \zeta := -\frac{1}{4} (1 \pm i) (\varepsilon_0 \otimes \varepsilon_1 + \varepsilon_1 \otimes \varepsilon_0), \quad \mu := \varepsilon_1 \otimes \varepsilon_1 \} \subset U \circ V_2, \tag{B.2} \\
\{ \bar{u} := u^{(0)}_{(1,1)}, & \quad \bar{\theta} := u^{(2)}_{(1,1)} = \bar{\varepsilon}_0 \otimes \bar{\varepsilon}_0, \quad \bar{\zeta} := \frac{1}{4} (1 \mp i) (\bar{\varepsilon}_0 \otimes \bar{\varepsilon}_1 + \bar{\varepsilon}_1 \otimes \bar{\varepsilon}_0), \quad \bar{\mu} := \bar{\varepsilon}_1 \otimes \bar{\varepsilon}_1 \} \subset \overline{V} \circ U. \tag{B.3}
\end{align*}
\]

Then, the left module \( U \circ V_2 \) has the following structure:

- \( u \) generates a trivial \( U_q \)-submodule
  \[ S := U \circ \text{span} \{ u \} \cong M_{(0)} \], with \( K.u = u, \quad E.u = 0, \quad F.u = 0, \tag{B.4} \]
- \( \theta \) and \( \mu \) both generate two-dimensional \( U_q \)-submodules \( U \circ \text{span} \{ \theta, u \} \) and \( U \circ \text{span} \{ \mu, u \} \), respectively, which are non-isomorphic, indecomposable, but not simple, and whose intersection is \( S \):
  \[ K.\theta = -\theta, \quad E.\theta = 0, \quad F.\theta = \pm 2iu, \quad K.\mu = -\mu, \quad E.\mu = \pm 2iu, \quad F.\mu = 0, \tag{B.5} \]
- \( \zeta \) generates the whole module \( U \circ V_2 \), which is indecomposable but not simple:
  \[ K.\zeta = \zeta, \quad E.\zeta = \mp \frac{i}{2} \theta, \quad F.\zeta = \mp \frac{i}{2} \mu, \quad F.E.\zeta = EF.\zeta = u. \tag{B.6} \]

Similarly, the right module \( \overline{V} \circ U \) has the following structure:

- \( \bar{u} \) generates a trivial \( U_q \)-submodule
  \[ \mathbb{S} := \text{span} \{ \bar{u} \} \cong M_{(0)} \], with \( \bar{\sigma}.K = \sigma, \quad \bar{\sigma}.E = 0, \quad \bar{\sigma}.F = 0, \tag{B.7} \]
- \( \bar{\theta} \) and \( \bar{\mu} \) both generate two-dimensional \( U_q \)-submodules \( \text{span} \{ \bar{\theta}, \bar{u} \} \) and \( \text{span} \{ \bar{\mu}, \bar{u} \} \), respectively, which are non-isomorphic, indecomposable, but not simple, and whose intersection is \( \mathbb{S} \):
  \[ \bar{\sigma}.K = -\bar{\sigma}, \quad \bar{\sigma}.E = \pm 2i\bar{\pi}, \quad \bar{\sigma}.F = 0, \quad \bar{\mu}.K = -\bar{\mu}, \quad \bar{\mu}.E = 0, \quad \bar{\mu}.F = \pm 2i\bar{\pi}, \tag{B.8} \]
- \( \bar{\zeta} \) generates the whole module \( \overline{V} \circ U \), which is indecomposable but not simple:
  \[ \bar{\zeta}.K = \bar{\zeta}, \quad \bar{\zeta}.E = \mp \frac{i}{2} \bar{\mu}, \quad \bar{\zeta}.F = \mp \frac{i}{2} \bar{\theta}, \quad \bar{\zeta}.F.E = \bar{\zeta}.EF = \bar{\sigma}. \tag{B.9} \]

Remark B.1. In particular, we see that the quantum Schur-Weyl duality decomposition (q-SW,) in theorem 1.4 cannot hold for this case. However, if \( q \in \{ \pm i \}, \) then the structure (B.2) and a direct calculation shows that any \( U_q \)-homomorphism \( T \in \text{End}_{U_q} V_2 \) must have the form

\[ T(u) = cu, \quad T(\theta) = c\theta, \quad T(\mu) = c\mu, \quad T(\zeta) = au + c\zeta \quad \text{for some} \ a, c \in \mathbb{C}. \tag{B.10} \]

Hence, the space \( \text{End}_{U_q} V_2 \) is two-dimensional, spanned by the identity map and the map \( \psi^{(1,1),(0)}_{(1,1)} : V_2 \rightarrow V_2, \)

\[ \psi^{(1,1),(0)}_{(1,1)}(u) := 0, \quad \psi^{(1,1),(0)}_{(1,1)}(\theta) := 0, \quad \psi^{(1,1),(0)}_{(1,1)}(\mu) := 0, \quad \psi^{(1,1),(0)}_{(1,1)}(\zeta) := u. \tag{B.11} \]

In particular, the Temperley-Lieb algebra is isomorphic to the commutant algebra \( \text{End}_{U_q} V_2 \): we obtain a representation of the Temperley-Lieb algebra on \( TL \circ V_2 \), by setting the generator \( U_1 \) to act as the map (B.11), see corollary B.4.
Similarly to (B.11), we define \( \tilde{\psi}^{(1,1);(0)} : V_2 \rightarrow V_2 \),

\[
\tilde{\psi}^{(1,1);(0)}(\pi) := 0, \quad \tilde{\psi}^{(1,1);(0)}(\theta) := 0, \quad \tilde{\psi}^{(1,1);(0)}(\bar{\theta}) := 0, \quad \tilde{\psi}^{(1,1);(0)}(\zeta) := \zeta.
\]  
(B.12)

We also extend definition (2.118) of the two embeddings \( \iota^{(0)}_{(1,1)} : V_0 \rightarrow V_2 \) and \( \iota^{(0)}_{(1,1)} : V_0 \rightarrow V_2 \) from the range of parameter values \( \{ q \in \mathbb{C}^\times \mid p(q) > 2 \} = \mathbb{C}^\times \setminus \{ \pm 1, \pm i \} \) to the range \( \{ q \in \mathbb{C}^\times \mid p(q) \geq 2 \} = \mathbb{C}^\times \setminus \{ \pm 1 \} \) by linear extensions of

\[
\iota^{(0)}_{(1,1)}(e^{(0)}_0) := u^{(0)}_0, \quad \text{and} \quad \iota^{(0)}_{(1,1)}(e^{(0)}_{0}) := \bar{u}^{(0)}_0.
\]  
(B.13)

and we define the maps \( \hat{\psi}^{(1,1)} : V_2 \rightarrow V_0 \) and \( \hat{\psi}^{(1,1)} : \bar{V}_2 \rightarrow \bar{V}_0 \), analogous to (2.120), by linear extensions of

\[
\hat{\psi}^{(1,1)}(\pi) := 0, \quad \hat{\psi}^{(1,1)}(\theta) := 0, \quad \hat{\psi}^{(1,1)}(\bar{\theta}) := 0, \quad \hat{\psi}^{(1,1)}(\zeta) := e^{(0)}_0.
\]  
(B.14)

\[
\hat{\psi}^{(1,1)}(u) := 0, \quad \hat{\psi}^{(1,1)}(\theta) := 0, \quad \hat{\psi}^{(1,1)}(\bar{\theta}) := 0, \quad \hat{\psi}^{(1,1)}(\zeta) := e^{(0)}_0.
\]  
(B.15)

We record the following obvious properties of these maps, vaguely analogous to lemma A.3:

**Lemma B.2.** Suppose \( q \in \{ \pm 1 \} \) (i.e., \( p(q) = 2 \)). Then, the following hold:

1. The maps \( \iota^{(0)}_{(1,1)}, \psi^{(1,1);(0)}, \) and \( \hat{\psi}^{(1,1)} \) are homomorphisms of left \( U_q \) and \( \bar{U}_q \)-modules.

2. \( \iota^{(0)}_{(1,1)} \) is a linear injection, \( \hat{\psi}^{(1,1)} \) is a linear surjection, but \( \psi^{(1,1);(0)} \) is not a linear projection, as \( \psi^{(1,1);(0)} \circ \psi^{(1,1);(0)} = 0. \)

3. We have \( \text{im} \circ \iota^{(0)}_{(1,1)} = \psi^{(1,1);(0)} = S, \text{ ker } \hat{\psi}^{(1,1)} = \ker \psi^{(1,1);(0)}, \text{ and } \end{verbatim}

\[
\iota^{(0)}_{(1,1)} \circ \psi^{(1,1);(0)} = \psi^{(1,1);(0)} \quad \text{but} \quad \hat{\psi}^{(1,1)} \circ \iota^{(0)}_{(1,1)} = 0.
\]  
(B.17)

Thus, the following diagram commutes:

\[
\begin{array}{ccc}
V_2 & \rightarrow & S \\
\downarrow \hat{\psi}^{(1,1);(0)} & & \\
V_0 & \rightarrow & \text{im} \circ \iota^{(0)}_{(1,1)} = \text{im} \psi^{(1,1);(0)} = S \subset V_2
\end{array}
\]  
(B.18)

Similarly, items 1–3 hold for right \( U_q \) and \( \bar{U}_q \)-modules, after the symbolic replacements

\[
\iota \mapsto \iota, \quad \psi \mapsto \hat{\psi}, \quad \hat{\psi} \mapsto \hat{\psi}, \quad S \mapsto S, \quad \text{and} \quad V \mapsto V.
\]  
(B.19)

**Proof.** In item 1, the \( U_q \)-homomorphism property follows from the definitions of the maps in the assertion, and the \( \bar{U}_q \)-homomorphism property can be verified using this and identity (A.25) from lemma A.5. Items 2–3 are immediate from the definitions of the maps in the assertion. The assertions with replacements (B.19) can be proven similarly. \( \square \)

**Lemma B.3.** Suppose \( q \in \{ \pm i \} \) (i.e., \( p(q) = 2 \)). Then, for all integers \( n \geq 2 \) and \( i, j \in \{ 1, 2, \ldots, n - 1 \} \), we have

\[
\mathcal{A}^{n-2}(L_i) = \left( q - q^{-1} \right) \left( \text{id}^{\otimes(i-1)} \otimes \iota^{(0)}_{(1,1)} \otimes \text{id}^{\otimes(n-i-1)} \right),
\]  
(B.20)

\[
\mathcal{A}^{n-2}(R_j) = \left( q - q^{-1} \right)^{i \otimes \otimes j} \left( \text{id}^{\otimes(j-1)} \otimes \psi^{(1,1);(0)} \otimes \text{id}^{\otimes(n-j-1)} \right),
\]  
(B.21)

and similarly,

\[
\mathcal{F}^{n-2}(R_j) = \left( q - q^{-1} \right)^{i \otimes \otimes j} \left( \text{id}^{\otimes(j-1)} \otimes \iota^{(0)}_{(1,1)} \otimes \text{id}^{\otimes(n-j-1)} \right),
\]  
(B.22)

\[
\mathcal{F}^{n-2}(L_i) = \left( q - q^{-1} \right)^{i \otimes \otimes j} \left( \text{id}^{\otimes(i-1)} \otimes \psi^{(1,1);(0)} \otimes \text{id}^{\otimes(n-i-1)} \right).
\]  
(B.23)
Proof. Assertions (B.20, B.22) are the same as (3.65, 3.67) in lemma 3.6. Assertions (B.21, B.23) can be proven via direct calculations, using (3.52, 3.55, B.15, B.14), and the assumption that \( q \in \{ \pm i \} \), so \( q^{-1} = -q \). □

Corollary B.4. Suppose \( q \in \{ \pm i \} \) (i.e., \( p(q) = 2 \)). Then, \( \mathcal{J}_n : \text{TL}_n(0) \rightarrow \text{End} V_n \) and \( \mathcal{J}_n : \text{TL}_n(0) \rightarrow \text{End}^\text{op} V_n \) are respectively left and right representations, and for all \( j \in \{ 1, 2, \ldots, n - 1 \} \), we have

\[
\mathcal{J}_n (U_j) = (\text{id} \otimes (j-1) \otimes \psi_{(1,1)}^{(0)} \otimes \text{id} \otimes (n-j-1)),
\]

and similarly,

\[
\mathcal{J}_n (U_j) = (\text{id} \otimes (j-1) \otimes \psi_{(1,1)}^{(1)} \otimes \text{id} \otimes (n-j-1)).
\]

Proof. This can be proven similarly as corollary 3.9, but using lemmas B.2 and B.3 instead of lemmas A.3 and 3.6. □

Remark B.5. In agreement with [PSA14], the decomposition of \( V_2 \) as a \( \text{TL}_2(0) \)-module is

\[
\text{TL} \circ V_2 \cong P_2(2) \oplus 2L_2(2),
\]

where \( L_2(2) \) is a simple one-dimensional \( \text{TL}_2(0) \)-module and \( P_2(2) \) is the two-dimensional principal indecomposable \( \text{TL}_2(0) \)-module. As a \( \text{TL}_2(0) \)-module, \( P_2(2) \) is isomorphic to \( \text{TL}_2(0) \) itself, with left multiplication. Explicitly, we have

\[
\text{TL} \circ \text{span} \{ \theta \} \cong L_2(2), \quad \text{TL} \circ \text{span} \{ \mu \} \cong L_2(2), \quad \text{TL} \circ \text{span} \{ u, \zeta \} \cong P_2(2),
\]

and the map \( u \mapsto U_1, \zeta \mapsto 1_{\text{TL}_2} \) gives the isomorphism from \( \text{TL} \circ \text{span} \{ u, \zeta \} \) to \( \text{TL} \circ \text{TL}_2(0) \).

A direct calculation shows that the commutant algebra \( \text{End} \text{TL}_2 V_2 \) is ten-dimensional, as in the generic case. However, not all elements in it are obtained from the image of the \( U_q \)-action on \( \text{TL} \circ V_2 \). Namely, the elements \( E^2 \) and \( F^2 \) act as zero on \( V_2 \), and \( K^2, K^{-2} \) act as the identity on \( V_2 \), so \( K \) and \( K^{-1} \) act as the same element. Therefore, only the following images of the basis elements (2.3) of \( U_q \) give non-zero operators in \( \text{End} \text{TL}_2 V_2 \):

\[
\{ 1, E, K, F, EK, EF, KF, EKF \}.
\]

As was proven by P. Martin [Mar92] (see also [GV13, PSA14]), the remaining two elements in \( \text{End} \text{TL}_2 V_2 \) can be obtained by enlarging \( U_q \) by the additional generators (also known as Lusztig’s divided powers [Lus89])

\[
\lim_{q' \to q} E^2 \text{[2]}_q \quad \text{and} \quad \lim_{q' \to q} F^2 \text{[2]}_q,
\]

with \( q = \pm i \) and the limit \( q' \to q \) taken along a sequence not containing roots on unity. In this case, \( \text{TL}_2(0) \) still remains as the commutant algebra of this larger algebra on \( V_2 \).

C. CLASSICAL CASE: \( q = 1 \)

Theorem 1.7 is a classical version of a “higher-spin Schur-Weyl duality,” which can be thought of as the “\( q \to 1 \)” limit of theorem 1.4. However, such a limit is heuristic rather than literal; we cannot simply set \( q = 1 \) in the relations (2.2) that define the algebra \( U_q \), for the commutator \([E, F]\) in (2.2) is not defined at \( q = 1 \). (We discuss how to make sense of this limit in the end of this appendix.) Instead, we regard \( U_1 = U := U(\mathfrak{sl}_2) \) of as the universal enveloping algebra of the Lie algebra \( \mathfrak{sl}_2 \), with generators \( E, F, \) and \( H \) and relations

\[
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.
\]

This algebra has the following coproduct and counit:

\[
\Delta(E) = E \otimes 1 + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + 1 \otimes F, \quad \Delta(H) = H \otimes 1 + 1 \otimes H,
\]

\[
\epsilon(E) = \epsilon(F) = \epsilon(H) = 0.
\]

The algebra \( U \) is semisimple, and its representation theory is analogous to that of \( U_q \) for \( q \in \mathbb{C}^\times \) not a root of unity. Fixing terminology, as in section 2A, we say that a vector \( v \in V \setminus \{ 0 \} \) in a \( U \)-module \( V \) has weight \( \lambda \in \mathbb{C} \), if we have \( H.v = \lambda v \). If in addition \( v \) satisfies \( E.v = 0 \) and \( K.v = \lambda v \), then we call \( v \) a highest-weight vector, and we call the \( U \)-module that \( v \) generates a highest-weight module. One can show from the definitions that any non-zero \( U \)-module contains a highest-weight vector (see, e.g., [Kas95, proposition V.4.2]). As usual, we use the counit to define a \( U \)-action on the ground field \( \mathbb{C} \) by \( x.\lambda = \epsilon(x)\lambda \) for all \( x \in U \) and \( \lambda \in \mathbb{C} \), and we use the counit \( \Delta \) to define tensor products of modules. Finally, we have the following facts, analogous to (U1–U2):
1. [Kas95, lemma V.4.3]: Let \( v_0 \) be a highest-weight vector of weight \( \lambda \in \mathbb{C} \) and \( v_\ell := F^\ell v_0 \). Then, for all \( \ell \in \mathbb{Z}_{\geq 0} \),
\[
H.v_\ell = (s - 2\ell)v_\ell, \quad E.v_\ell = (s - \ell + 1)v_{\ell-1}, \quad F.v_\ell = v_{\ell+1}. \tag{C.4}
\]

2. [Kas95, Theorem V.4.4]: Let \( \mathcal{N} \) be a \( U \)-module generated by a highest-weight vector \( v_0 \) of weight \( \lambda \in \mathbb{C} \), and let \( v_\ell := F^\ell v_0 \). If \( 0 < \dim \mathcal{N} = s + 1 \), then
   (a): we have \( \lambda = s \),
   (b): we have \( v_\ell = 0 \) for \( \ell > s \), and \( \{v_0, v_1, \ldots, v_s\} \) is a basis for \( \mathcal{N} \),
   (c): we have \( H.v_\ell = (s - 2\ell)v_\ell \) for all \( \ell \in \{0, 1, \ldots, s\} \),
   (d): any other highest-weight vector of \( \mathcal{N} \) is a scalar multiple of \( v_0 \),
   (e): \( \mathcal{N} \) is simple, and
   (f): any finite-dimensional simple \( U \)-module of dimension \( s + 1 \) is isomorphic to \( \mathcal{N} \), having basis \( \{v_0, v_1, \ldots, v_s\} \) and \( U \)-action \( (C.4) \).

On the generic vector space \( \mathcal{V}(s) := \text{span} \{ e_0^{(s)}, e_1^{(s)}, \ldots, e_s^{(s)} \} \), we define a left \( U \)-module structure via the rules
\[
F.e_\ell^{(s)} := \begin{cases} e_{\ell+1}^{(s)}, & 0 \leq \ell \leq s - 1, \\ 0, & \ell = s, \end{cases} \quad E.e_\ell^{(s)} := \begin{cases} \ell(s - \ell + 1)e_{\ell-1}^{(s)}, & 1 \leq \ell \leq s, \\ e_0^{(s)}, & \ell = 0, \end{cases} \quad H.e_\ell^{(s)} := (s - 2\ell)e_\ell^{(s)}, \tag{C.5}
\]
and we denote the corresponding representation by \( \rho(s) : U \rightarrow \text{End} \mathcal{V}(s) \) and the resulting simple \( U \)-module by
\[
\mathcal{M}(s) := U \circ \mathcal{V}(s). \tag{C.6}
\]
We define a left \( U \)-module structure on the tensor product \( \mathcal{V}_\varsigma (2.13) \) by using the coproduct \( (C.2) \),
\[
U \circ \mathcal{V}_\varsigma := \mathcal{M}(s_1) \otimes \mathcal{M}(s_2) \otimes \cdots \otimes \mathcal{M}(s_k), \tag{C.7}
\]
and we denote the corresponding representation by \( \rho_\varsigma : U \rightarrow \text{End} \mathcal{V}_\varsigma \), defined by \( (6.14) \). We also define a right \( U \)-module structure on the generic vector space \( \nabla(s) := \text{span} \{ e_0^{(s)}, e_1^{(s)}, \ldots, e_s^{(s)} \} \) via the rules
\[
\tilde{e}_\ell^{(s)}.F := \begin{cases} e_{\ell+1}^{(s)}, & 0 \leq \ell \leq s - 1, \\ 0, & \ell = s, \end{cases} \quad \tilde{e}_\ell^{(s)}.E := \begin{cases} \ell(s - \ell + 1)e_{\ell-1}^{(s)}, & 1 \leq \ell \leq s, \\ e_0^{(s)}, & \ell = 0, \end{cases} \quad \tilde{e}_\ell^{(s)}.H := (s - 2\ell)e_\ell^{(s)}, \tag{C.8}
\]
and we define a right \( U \)-module structure on the tensor product \( \mathcal{V}_\varsigma (2.13) \) using the coproduct \( (C.2) \),
\[
\mathcal{M}(s) := \nabla(s) \circ U, \quad \mathcal{V}_\varsigma \circ U := \mathcal{M}(s_1) \otimes \mathcal{M}(s_2) \otimes \cdots \otimes \mathcal{M}(s_k). \tag{C.9}
\]
We denote the corresponding representation by \( \tilde{\rho}_\varsigma : U \rightarrow \text{End}^{op} \mathcal{V}_\varsigma \).

We define the spaces of highest-weight vectors in these \( U \)-type-one modules as in \( (2.33) \). These spaces are graded as in \( (2.27, 2.28) \), with \( H \)-eigenvalues of the form \( s \), for integers \( s \in \mathbb{E}_\varsigma \):
\[
H^s := H_\varsigma \cap \mathcal{V}_\varsigma = \{ v \in \mathcal{V}_\varsigma \mid E.v = 0, H.v = s.v \}, \tag{C.10}
\]
\[
\tilde{H}^s := \tilde{H}_\varsigma \cap \nabla(s) = \{ v \in \nabla_\varsigma \mid \tilde{e}.F = 0, \tilde{e}.H = s.v \}. \tag{C.11}
\]

Notably, \( U \) is a semisimple algebra and item 3 of proposition \( 2.8 \) holds for it without restriction on the magnitude of \( n_\varsigma \). Indeed, we have the direct-sum decomposition of \( (C.7) \) into simple left \( U \)-modules as
\[
U \circ \mathcal{V}_\varsigma \cong \bigoplus_{s \in \mathbb{E}_\varsigma} D^s \mathcal{M}(s). \tag{C.12}
\]

1. Classical higher-spin Schur-Weyl duality

Next, we extend the results of sections \( 2 \)–\( 6 \) to the case \( q = 1 \). The key point why we can extend our previous results to the case \( q = 1 \) is the fact that, thanks to the identity
\[
\lim_{q \to 1} [k] = \lim_{q \to 1} \frac{q^k - q^{-k}}{q - q^{-1}} = k \quad \text{for all } k \in \mathbb{Z}. \tag{C.13}
\]
Jones-Wenzl projectors of all sizes exist at \( q = 1 \) and they are given by the recursion (3.29) with \([k] \rightarrow k\) for all quantum integers \([k]\). Therefore, the valenced Temperley-Lieb algebra \( \mathbb{T}_\varsigma(-2) \) is well-defined also with \( q = 1 \) and \( \nu = -2 \). Furthermore, we have the direct-sum decomposition from proposition 4.16 with \( \mathbb{T}_\varsigma(-2) \):

\[
\mathbb{T}_\varsigma \circ \mathbb{V}_\varsigma \cong \bigoplus_{s \in \mathbb{E}_\varsigma} (s + 1) \mathbb{L}^{(s)}_\varsigma,
\]

(C.14)

without restriction on the magnitude of \( n_\varsigma \). The (higher-spin) Schur-Weyl duality theorem 1.7 relates these two decompositions to each other.

**Lemma C.1.** All lemmas, propositions, and corollaries of sections 2–6 hold for the algebras \( U \) and \( \mathbb{T}_\varsigma(-2) \) after making the following replacements:

1. We replace \( q \in \mathbb{C}^\times \) by \( q = 1 \) (and \( \nu = -q - q^{-1} \) by \( \nu = -2 \)).
2. We replace \( p(q) \) by \( \infty \).
3. We replace the \( q \)-integer \([k]\) by the integer \( k \).
4. We replace \( U_q = U_q(\mathfrak{sl}_2) \) by \( U = U(\mathfrak{sl}_2) \), and we consider its modules \( \overline{\mathbb{V}}_\varsigma \circ U \) and \( U \circ \mathbb{V}_\varsigma \).
5. We replace \( \mathbb{U}_q \) by \( U \) and \( \pi \in \mathbb{U}_q \) by \( x \in U \).
6. We omit all factors of the form \( (q - q^{-1})^k \).
7. We have \( \text{rad} \mathbb{L}^{(s)}_\varsigma = \{0\} \) for all \( s \in \mathbb{E}_\varsigma \) in (3.114) and \( \mathbb{Q}^{(s)}_\varsigma = \mathbb{L}^{(s)}_\varsigma \) for all \( s \in \mathbb{E}_\varsigma \) in (3.116), by [FP18a, corollary 5.2].

**Proof.** These replacements do not essentially affect the proofs of the lemmas, propositions, and corollaries of sections 2–6, except for the ones using formula (2.11) for the coproduct for \( U_q \), different from that (C.2) for \( U \). Let us summarize:

1. For lemmas A.4–A.5, proved in appendix A, we replace the generator \( K \) by the generator \( H \) (and omit \( K^{-1} \)), relations (2.2, 2.11, 2.12) by relations (C.1, C.2), and formulas (A.1, A.2) in lemma A.1 by formula

\[
\Delta^{(d)}(x) = \sum_{j=1}^d 1^{\otimes(j-1)} \otimes x \otimes 1^{\otimes(d-j)} \text{ for all } x \in \mathfrak{sl}_2.
\]

(C.15)

Lemma 2.2 readily adapts to the case \( q = 1 \), replacing formula (2.11) for the coproduct for \( U_q \) by that (C.2) for \( U \). The other results in section 2 readily adapt to the case \( q = 1 \).

1. All lemmas, propositions, and corollaries of sections 3, 5, and 6, readily adapt to the case \( q = 1 \).

2. All lemmas, propositions, and corollaries of section 4 readily adapt to the case \( q = 1 \). In particular, proposition 4.12, now with \( q = 1 \), says that the map \( \alpha \mapsto w_\alpha \) is always an isomorphism from \( \mathbb{L}_\varsigma \) to \( \mathbb{H}_\varsigma \).

\[\square\]

**Theorem 1.7.** (Higher-spin Schur-Weyl duality): The following hold:

1. The images of the maps \( \mathcal{I}_\varsigma : \mathbb{T}_\varsigma(-2) \rightarrow \operatorname{End} \mathbb{V}_\varsigma \) and \( \rho_\varsigma : U \rightarrow \operatorname{End} \mathbb{V}_\varsigma \) are semisimple algebras, which equal \( \mathbb{T}_\varsigma(-2) \cong \mathcal{I}_\varsigma(\mathbb{T}_\varsigma(-2)) = \operatorname{End}_U \mathbb{V}_\varsigma \) and \( \rho_\varsigma(U) = \operatorname{End} \mathbb{T}_\varsigma \mathbb{V}_\varsigma \).

2. The collections \( \{\mathbb{M}_{\varsigma} \mid s \in \mathbb{E}_\varsigma\} \) and \( \{\mathbb{L}^{(s)}_\varsigma \mid s \in \mathbb{E}_\varsigma\} \) are respectively the complete sets of simple non-isomorphic \( \rho_\varsigma(U) \)-modules and \( \mathbb{T}_\varsigma \circ \mathbb{L}_\varsigma(-2) \)-modules, and we have the direct-sum decomposition

\[
\mathbb{M}_\varsigma \cong \bigoplus_{s \in \mathbb{E}_\varsigma} \mathbb{M}_{\varsigma} \otimes \mathbb{L}^{(s)}_\varsigma.
\]

(SW\(_\varsigma\))

3. The linear extension of the following map gives an explicit isomorphism for (SW\(_\varsigma\)): with \( w_\alpha \) the explicit \( U \)-highest-weight vectors constructed in definition 4.1 with \( q = 1 \), \( \alpha \in \mathbb{L}^{(s)}_\varsigma \), \( s \in \mathbb{E}_\varsigma \), and \( \ell \in \{0, 1, \ldots, s\} \),

\[
F^\ell \cdot w_\alpha \mapsto e_\ell^{(s)} \otimes \alpha.
\]

(1.44)
The analogue of this theorem holds for the right representation \( \tilde{p} \) of \( U \). after the symbolic replacements
\[
\mathcal{F} \mapsto \tilde{\mathcal{F}}, \quad V \mapsto \tilde{V}, \quad \rho \mapsto \tilde{\rho}, \quad M \mapsto \tilde{M}, \quad L \mapsto \tilde{L}, \quad \text{and} \quad F^q, \omega \mapsto \tilde{\omega} \cdot E^q. \quad \text{(C.16)}
\]

Proof. This follows from the results leading to theorem 1.4 by using lemma C.1. \( \square \)

2. The limit \( q \to 1 \)

As in [Kas95, chapter XVII], we can make sense of the “\( q \to 1 \)” limit of \( U_q \) by considering instead an appropriate topological algebra over the ring \( \mathbb{C}[[h]] \) of formal power series in an indeterminate \( h \) with coefficients in \( \mathbb{C} \). This algebra \( U_h \) is generated by the unit and three generators \( E, F, \) and subject to the relations
\[
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad \text{where} \ q := e^h \quad \text{(C.17)}
\]
and where the expression \( q^X = e^{hX} \) is defined by the limit \( e^{hX} := \lim_{N \to \infty} \sum_{j=0}^{N} \frac{X^j}{j!} h^j \) in the \( h \)-adic topology. Via the embedding
\[
E \mapsto Eq^{H/2}, \quad F \mapsto q^{-H/2}F, \quad K \mapsto q^{H/2}, \quad K^{-1} \mapsto q^{-H/2}, \quad \text{with} \ q := e^h, \quad \text{(C.18)}
\]
the quantum group \( U_q \) can be regarded as a sub-Hopf algebra of \( U_h \) [Kas95, Theorem XVII.4.1]. In the \( h \to 0 \) (i.e., \( q \to 1 \)) limit, we recover the universal enveloping algebra \( U := U(\mathfrak{sl}_2) \), of the classical Lie algebra \( \mathfrak{sl}_2 \), with generators \( E, F, \) and \( H \) and relations (C.1).

The Hopf algebra structure of \( U_q \) in the informal limit \( q \to 1 \) endows \( U \) with a Hopf algebra structure. Indeed, setting \( q \to 1 \) in (C.18), the coproduct (2.11) and counit (2.12) become
\[
\Delta(E) = E \otimes 1 + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + 1 \otimes F, \quad \Delta(1) = 1 \otimes 1, \quad \text{and} \quad \epsilon(E) = \epsilon(F) = 0, \quad \epsilon(1) = 1, \quad \text{(C.19)}
\]
and for the generator \( H = [E, F] \), we obtain
\[
\Delta(H) = \Delta([E, F]) = [\Delta(E), \Delta(F)] = H \otimes 1 + 1 \otimes H, \quad \text{and} \quad \epsilon(H) = \epsilon([E, F]) = 0. \quad \text{(C.20)}
\]
This agrees with (C.2–C.3). Finally, taking the informal limit \( q \to 1 \) of the coproduct \( \Delta \) (2.21) instead, we obtain the same structure (C.19, C.21). Therefore, the Hopf algebra \( \overline{U} \) obtained by the above “\( q \to 1 \)” limit of \( U_q \) is the same as \( U \).

D. ON RADICALS AND ORTHOCOMPLEMENTS

The purpose of this appendix is to gather simple facts from linear algebra, used in the proof of proposition 5.14 in section 5D.

Throughout, we let \( V \) and \( \overline{V} \) be two vector spaces of the same finite dimension \( n \in \mathbb{Z}_{>0} \). We suppose that \( (\cdot, \cdot): V \times \overline{V} \to \mathbb{C} \) is a bilinear pairing of \( V \) and \( \overline{V} \). We say that the bases \( B = \{ e_1, e_2, \ldots, e_n \} \subset V \) and \( \overline{B} = \{ \overline{e}_1, \overline{e}_2, \ldots, \overline{e}_n \} \subset \overline{V} \) are orthogonal if
\[
(\overline{e}_i, e_j) = 0 \quad \text{for all} \ i \neq j, \ \text{with} \ i, j \in \{ 1, 2, \ldots, n \}. \quad \text{(D.1)}
\]

Lemma D.1. There exist orthogonal bases \( \{ e_1, e_2, \ldots, e_n \} \) and \( \{ \overline{e}_1, \overline{e}_2, \ldots, \overline{e}_n \} \) for \( V \) and \( \overline{V} \).

Proof. We perform induction on the dimension \( n = \dim V = \dim \overline{V} \). The initial case \( n = 1 \) is clear. Thus, we assume that \( n \geq 2 \) and that the assertion holds for all pairs of vector spaces of dimension \( 1 \leq m < n \). We consider two cases:
1. If there exists a pair \( e \in V, \bar{e} \in \bar{V} \) of vectors such that \((\bar{e} \mid e) \neq 0\), then we can write
\[
V = \text{span} \{e\} \oplus V', \quad \text{where} \quad V' := \{v \in V \mid (\bar{e} \mid v) = 0\}, \tag{D.2}
\]
\[
\bar{V} = \text{span} \{\bar{e}\} \oplus \bar{V}', \quad \text{where} \quad \bar{V}' := \{\bar{v} \in \bar{V} \mid (\bar{e} \mid \bar{v}) = 0\}. \tag{D.3}
\]
By the induction hypothesis, \( V' \) and \( \bar{V}' \) have orthogonal bases \( \{e_1, e_2, \ldots, e_{n-1}\} \) and \( \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}\} \). Choosing \( e_n = e \) and \( \bar{e}_n = \bar{e} \), we obtain the asserted bases.

2. If \((v \mid \bar{w}) = 0\) for all \( v \in V \) and \( \bar{w} \in \bar{V} \), then any bases \( \{e_1, e_2, \ldots, e_n\} \) and \( \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\} \) for \( V \) and \( \bar{V} \) are orthogonal.

This finishes the induction step. \( \square \)

For all pairs \( W \subset V \) and \( \bar{W} \subset \bar{V} \) of subspaces, we define their orthocomplements as
\[
W^\perp := \{v \in V \mid (\bar{w} \mid v) = 0 \text{ for all } \bar{w} \in \bar{W}\} \subset V, \tag{D.4}
\]
\[
\bar{W}^\perp := \{\bar{v} \in \bar{V} \mid (v \mid \bar{v}) = 0 \text{ for all } v \in W\} \subset \bar{V}, \tag{D.5}
\]
and their radicals as
\[
\text{rad} W := \{w \in W \mid (\bar{w} \mid w) = 0 \text{ for all } \bar{w} \in \bar{W}\} \subset W, \tag{D.6}
\]
\[
\text{rad} \bar{W} := \{\bar{w} \in \bar{W} \mid (w \mid \bar{w}) = 0 \text{ for all } w \in W\} \subset \bar{W}. \tag{D.7}
\]
These definitions readily imply that
\[
\text{rad} W = W \cap W^\perp \quad \text{and} \quad \text{rad} \bar{W} = \bar{W} \cap \bar{W}^\perp. \tag{D.8}
\]

**Lemma D.2.** Let \( W \subset V \) and \( \bar{W} \subset \bar{V} \) have orthogonal bases \( \{e_1, e_2, \ldots, e_m\} \subset W \) and \( \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_m\} \subset \bar{W} \) such that
\[
(\bar{e}_j \mid e_j) \neq 0 \quad \text{for all } 1 \leq j \leq k, \tag{D.9}
\]
\[
(\bar{e}_j \mid e_j) = 0 \quad \text{for all } k + 1 \leq j \leq m, \tag{D.10}
\]
for some \( 1 \leq k \leq m \). Then, the following statements hold:

1. The collections \( \{e_{k+1}, e_{k+2}, \ldots, e_m\} \) and \( \{\bar{e}_{k+1}, \bar{e}_{k+2}, \ldots, \bar{e}_m\} \) are respectively bases for \( \text{rad} W \) and \( \text{rad} \bar{W} \).

2. If \( k = m \), then we have \( V = W \oplus W^\perp \) and \( \bar{V} = \bar{W} \oplus \bar{W}^\perp \).

**Proof.** We prove items 1–2 as follows:

1. We have \( \text{span} \{e_{k+1}, e_{k+2}, \ldots, e_m\} \subset \text{rad} W = W \cap W^\perp \) by (D.1, D.8, D.10). Conversely, if \( e_1, e_2, \ldots, e_m \in \mathbb{C} \) are arbitrary and
\[
v = \sum_{j=1}^m c_j e_j \in W \cap W^\perp \quad \text{and} \quad \bar{v} = \sum_{j=1}^m c_j \bar{e}_j \in \bar{W}, \tag{D.11}
\]
then, we have \((\bar{v} \mid v) = 0\), which is equivalent to
\[
\sum_{j=1}^m c_j^2 (\bar{e}_j \mid e_j) = 0 \quad \iff \quad e_j = 0 \text{ for all } 1 \leq j \leq k \quad \iff \quad v \in \text{span} \{e_{k+1}, e_{k+2}, \ldots, e_m\}. \tag{D.12}
\]
This shows that \( \{e_{k+1}, e_{k+2}, \ldots, e_m\} \) is a basis for \( \text{rad} W \). The case of \( \text{rad} \bar{W} \) can be proven similarly.

2. If \( k = m \), then item 1 shows that \( W \cap W^\perp = \{0\} \), so the sum \( W \oplus W^\perp \) is direct. An analogue of the dimension theorem shows that this inclusion is actually an equality: \( W \oplus W^\perp = V \). Similarly, we have \( \bar{W} \oplus \bar{W}^\perp = \bar{V} \).

This concludes the proof. \( \square \)

**Lemma D.3.** Let \( W \subset V \) and \( \bar{W} \subset \bar{V} \) be subspaces of the same dimension, such that
\[
\text{rad} W \subset \text{rad} V \quad \text{and} \quad \text{rad} \bar{W} \subset \text{rad} \bar{V}. \tag{D.13}
\]
Then, there exist subspaces \( W_1, W_2 \subset V \) such that the following hold:
Similarly, there exist subspaces $\overline{W}_1, \overline{W}_2 \subset \overline{V}$ with properties 1–3, after the symbolic replacements $V \mapsto \overline{V}$ and $W \mapsto \overline{W}$.

**Proof.** By lemma D.1 and item 1 of lemma D.2, we have the direct-sum decompositions
\[ W = W_1 \oplus \text{rad } W, \quad \text{and} \quad \overline{W} = \overline{W}_1 \oplus \text{rad } \overline{W}, \] (D.14)
where $W_1$ and $\overline{W}_1$ respectively have orthogonal bases $\{e_1, e_2, \ldots, e_k\}$ and $\{\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_k\}$, and
\[ (\overline{e}_j | e_j) \neq 0 \quad \text{for all } 1 \leq j \leq k. \] (D.15)
As such, we may apply item 2 of lemma D.2 to the spaces $W_1$ and $\overline{W}_1$ to arrive with the orthogonal decompositions
\[ V = W_1 \oplus W_1^\perp \quad \text{and} \quad \overline{V} = \overline{W}_1 \oplus \overline{W}_1^\perp. \] (D.16)
Also, by definition (D.4) (resp. (D.5)) and direct-sum decomposition (D.14), the set $\text{rad } W$ (resp. $\text{rad } \overline{W}$) is a subspace of $W_1^\perp$ (resp. $\overline{W}_1^\perp$), so there exists a subspace $W_2 \subset W_1^\perp$ (resp. $\overline{W}_2 \subset \overline{W}_1^\perp$) such that
\[ W_1^\perp = \text{rad } W \oplus W_2 \quad \text{and} \quad \overline{W}_1^\perp = \text{rad } \overline{W} \oplus \overline{W}_2. \] (D.17)
Hence, item 1 follows from (D.14), item 2 from (D.17), and item 3 then follows from (D.16, D.17), once we prove that
\[ W^\perp = W_1^\perp \quad \text{and} \quad \overline{W}^\perp = \overline{W}_1^\perp. \] (D.18)
By lemma D.1, we find orthogonal bases $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ and $\{\overline{e}_{k+1}, \overline{e}_{k+2}, \ldots, \overline{e}_n\}$ for the orthocomplements $W_1^\perp$ and $\overline{W}_1^\perp$ in (D.16). In order to prove the left equality in (D.18), we show that $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ is a basis for $W^\perp$.

To begin, we prove that $\{e_{k+1}, e_{k+2}, \ldots, e_n\} \subset W^\perp$. Towards a contradiction, we take
\[ e_j \notin W^\perp \quad \text{for some } j \in \{k + 1, k + 2, \ldots, n\}, \quad \text{and} \quad (\overline{m} | e_j) \neq 0 \quad \text{for some } \overline{m} \in \overline{W}. \] (D.19)
On the other hand, (D.14) shows that $\overline{m} = \overline{m}_1 + \overline{m}_2$ for some $\overline{m}_1 \in \overline{W}_1$ and $\overline{m}_2 \in \text{rad } \overline{W}$. Because $e_j \in W_1^\perp$, we may assume that $\overline{m} \in \text{rad } \overline{W}$. But then, by assumption (D.13), we have
\[ e_j \notin W^\perp \quad \text{and} \quad (\overline{m} | e_j) \neq 0 \quad \text{for some } \overline{m} \in \text{rad } \overline{W} \subset \text{rad } \overline{V}, \] (D.20)
which contradicts definition (D.7) of $\text{rad } \overline{V}$. Therefore, $\{e_{k+1}, e_{k+2}, \ldots, e_n\} \subset W^\perp$ is a linearly independent subset.

To finish, we prove that $W^\perp = \text{span } \{e_{k+1}, e_{k+2}, \ldots, e_n\}$. Indeed, we have
\[ \begin{cases} W^\perp \cap W_1 \subset W_1^\perp, & \text{(D.14)} \quad \Rightarrow \quad W_1^\perp \cap W_1 \subset \text{rad } W \cap W_1 \overset{\text{(D.14)}}{=} \{0\}, \quad \text{(D.21)} \\
W^\perp \cap W_1 \subset W_1 \quad \text{(D.8)} \quad \Rightarrow \quad W^\perp \cap W_1 \subset \text{rad } W \cap W_1 \overset{\text{(D.14)}}{=} \{0\}, \quad \text{(D.21)} \end{cases} \]
and hence, because $\{e_1, e_2, \ldots, e_n\}$ is a basis for $V$ by (D.15, D.16), the collection $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ is a basis for $W^\perp$. This proves the left equality in (D.18), and the right equality can be proven similarly. \qed

### E. DOUBLE-COMMUTANT PROPERTIES

The purpose of this appendix is to show that the double-commutant property and duality discussed in sections 6B–6C follow from very basic results on representations of associative algebras. Here, we only consider finite-dimensional left representations; analogous results also hold for finite-dimensional right representations. Most of the material in this appendix is standard and can be found in one form or another in many textbooks and lecture notes.

Throughout, we consider an associative unital $\mathbb{C}$-algebra $A$. We recall that a (left) representation of $A$ is a homomorphism $\rho : A \to \text{End } V$ of algebras from $A$ to the algebra $\text{End } V$ of endomorphisms of some finite-dimensional vector space $V$. We call $A \subset V$ an $A$-module, emphasizing the action in the notation.
We call $^A \otimes V$ simple, and $\rho$ irreducible, if it is not zero and it contains no non-zero proper submodules. One can show by induction that any finite-dimensional $A$-module has a composition series, i.e., a strictly increasing sequence
\[ ^A \otimes \{0\} = ^A \otimes V_0 \subseteq ^A \otimes V_1 \subseteq ^A \otimes V_2 \subseteq \cdots \subseteq ^A \otimes V_m = ^A \otimes V \quad (E.1) \]
of submodules such that the quotient modules $^A \otimes (V_j/V_{j-1})$, called composition factors of $^A \otimes V$, are simple for all $j = 1, 2, \ldots, m$. It also follows by induction that any two composition series of $^A \otimes V$ have the same length $m$ and the same composition factors up to isomorphism and permutation. This fact is known as the Jordan-Hölder theorem.

Next, we consider the module $^A \otimes A$ with action on itself given by multiplication. The associated representation is called the regular representation of $A$. It follows immediately from the definitions that a subspace $J \subset A$ gives an $A$-submodule if and only if it is a left ideal. Now, if $^A \otimes V$ is a simple $A$-module, we claim that it corresponds to a quotient module $^A \otimes (A/J)$ for some maximal left ideal $J$. Indeed, to see this, we fix a vector $v \in V$ and define a map
\[ \phi : A \rightarrow V, \quad \phi(a) := \rho(a)(v), \quad (E.2) \]
where $\rho : A \rightarrow \text{End} V$ is the representation corresponding to $^A \otimes V$. This map $\phi$ is a non-zero homomorphism of $A$-modules, and since $^A \otimes V$ is simple, we see that
\[ ^A \otimes V \cong ^A \text{im} \phi \cong ^A \otimes (A/\ker \phi), \quad (E.3) \]
where $\ker \phi \subset A$ is the sought maximal left ideal. In particular, all simple $A$-modules appear as composition factors of $^A \otimes A$, so the Jordan-Hölder theorem shows that simple $A$-modules are classified by maximal left ideals of $A$.

We call $^A \otimes V$ semisimple, and $\rho$ completely reducible, if $^A \otimes V$ is isomorphic to a direct sum of simple $A$-modules. We recall that the $A$-action on a direct sum module is diagonal:
\[ ^A \otimes V = \bigoplus_{\lambda} ^A \otimes V_{\lambda} \quad \Rightarrow \quad \rho = \bigoplus_{\lambda} \rho_{\lambda}, \quad (E.4) \]
where $\rho_{\lambda} : A \rightarrow \text{End} V_{\lambda}$ are the representations associated to the summands.

If some of the $A$-modules $^A \otimes V_{\lambda}$ in (E.4) are isomorphic, it is convenient to index them by the same $\lambda$ and write $^A \otimes V$ as a direct sum of mutually non-isomorphic $A$-modules $^A \otimes V_{\lambda}$ with multiplicities $m_{\lambda} \in \mathbb{Z}_{>0}$:
\[ ^A \otimes V \cong \bigoplus_{\lambda} ^A \otimes m_{\lambda} V_{\lambda}, \quad \text{where} \quad ^A \otimes m_{\lambda} V_{\lambda} := \bigoplus_{j=1}^{m_{\lambda}} ^A \otimes V_{\lambda}, \quad (E.5) \]
and where we have implicitly chosen some representatives of the non-isomorphic $A$-modules $^A \otimes V_{\lambda}$.

Of particular interest is the decomposition of a semisimple module $^A \otimes V$ into a direct sum of simple $A$-modules. We cautiously note that such a decomposition is by no means canonical if some submodule has multiplicity greater than one. Indeed, each choice of basis for $V$ in (E.4) and choices of representative modules in (E.5) gives another decomposition, all of which are isomorphic but not equal.

Another important general question is the classification of simple modules. For endomorphism algebras, this is completely understood. If $\dim V = n$, the elements of $A = \text{End} V$ can be viewed as $(n \times n)$-matrices with entries in $\mathbb{C}$. In particular, we know that $A$ has exactly one maximal left ideal, and exactly one simple module, namely $^A \otimes V$.

**Lemma E.1.** Suppose $\{S_{\lambda}\}$ is a finite collection of finite-dimensional vector spaces, and let $A = \bigoplus S_{\lambda}$. Then, $\{^A \otimes S_{\lambda}\}$ is the complete set of non-isomorphic simple $A$-modules, with $A$-action defined as
\[ \bigoplus_{\lambda} L_{\lambda}(v_{\lambda}) := L_{\lambda}(v_{\lambda}) \quad \text{for all elements} \quad \bigoplus_{\lambda} L_{\lambda} \in A \quad \text{and vectors} \quad v_{\lambda} \in S_{\lambda}. \quad (E.6) \]

**Proof.** For any index $\lambda$ and vectors $v_{\lambda}, w_{\lambda} \in S_{\lambda} \setminus \{0\}$, there exists an element $L_{\lambda} \in \text{End} S_{\lambda}$ such that $L_{\lambda}(v_{\lambda}) = w_{\lambda}$. Hence, the modules $^A \otimes S_{\lambda}$ are simple, and they are non-isomorphic by construction. Because the simple $A$-modules are classified by maximal left ideals in $A = \bigoplus S_{\lambda}$, we see that $\{^A \otimes S_{\lambda}\}$ constitute all of the simple $A$-modules.

The main aim of this appendix is to analyze the space $\text{Hom}_A(V, W)$ of $A$-homomorphisms between two finite-dimensional modules $^A \otimes V$ and $^A \otimes W$. Our presentation partially follows [GW09, chapter 4], with statements and proofs presented here tailored to the purposes of the present work. We also recommend, e.g., [Lam91, EGH+11].

All results in this appendix are consequences of complete reducibility and Schur’s lemma, which, being almost obvious, can be regarded as one of the cornerstones of the representation theory of associative algebras.
Lemma E.2. (Schur’s lemma): If $A \odot V$ and $A \odot W$ are finite-dimensional simple $A$-modules, then we have

$$\dim \text{Hom}_A(V, W) = \begin{cases} 1, & V \text{ and } W \text{ are isomorphic,} \\ 0, & \text{otherwise.} \end{cases}$$

(E.7)

In particular, if $V$ is a finite-dimensional simple $A$-module, then the identity map $id_V$ spans $\text{End}_A V$.

Proof. See, e.g., [GW09, lemma 4.1.4]: the key argument is that both the kernel and the image of any homomorphism of $A$-modules is also an $A$-module (and for the latter statement, we consider the algebraically closed field $\mathbb{C}$).

For any $A$-module $A \odot V$, the endomorphism algebra $C := \text{End}_A V$ is called the commutant (i.e., centralizer) of $A$ on $V$. We denote by $C \odot V$ the natural $C$-module structure on $V$. We can also define a representation $\rho_V: A \otimes C \rightarrow \text{End}_V$,

$$\rho_V(a \otimes C) := \rho_V(a) \circ C = C \circ \rho_V(a) \quad \text{for all } a \in A \text{ and } C \in C,$$

(E.8)

since the algebra $C$ commutes with the action of $A$ on $V$. We denote the obtained $(A \otimes C)$-(bi)module by $\hat{C} \odot V$. In theorem E.9, we give a direct-sum decomposition of the $(A \otimes C)$-module, when $A \odot V$ is semisimple.

Conversely, one might ask what is the commutant algebra of $C$ on $V$. Clearly, we have $\rho(A) \subset \text{End}_C V$, and corollary E.6 below shows that if $A \odot V$ is semisimple, then the converse inclusion holds too.

Before proving these results, we note that Schur’s lemma gives a canonical way to decompose a semisimple module $A \odot V$ into a direct sum of simple submodules, thus improving the arbitrary decomposition (E.11). The multiplicity

$$m^V_\lambda = \dim \text{Hom}_A(S_\lambda, V) \in \mathbb{Z}_{\geq 0}$$

(E.9)

of simple submodules isomorphic to $A \odot S_\lambda$ in $A \odot V$ is governed by the space $\text{Hom}_A(S_\lambda, V)$, with $A \odot S_\lambda$ representatives of non-isomorphic simple modules. Indeed, using Schur’s lemma E.2, it is straightforward to check that the map

$$\varphi: \bigoplus_\lambda (A \odot S_\lambda \otimes \text{Hom}_A(S_\lambda, V)) \rightarrow A \odot V,$$

where

$$\varphi_\lambda: A \odot S_\lambda \otimes \text{Hom}_A(S_\lambda, V) \rightarrow A \odot V, \quad \varphi_\lambda(v \otimes L) := L(v),$$

(E.10)

gives a natural isomorphism of $A$-modules. See, e.g., [GW09, proposition 4.1.15] for details.

So far, the multiplicity spaces $\text{Hom}_A(S_\lambda, V)$ have no more structure than that of a vector space. However, they do carry a natural action of the commutant algebra $C$ via left multiplication, and we actually have a similar direct-sum decomposition for $C \odot V$ (cf. equation (E.49)). Furthermore, the $C$-modules $C \odot \text{Hom}_A(S_\lambda, V)$ are also simple, which shows that $C \odot V$ is also semisimple. This observation is key to establish double-commutant theorem E.9.

1. Structure of commutants

To begin, we identify the commutant space $\text{Hom}_A(V, W)$ between two finite-dimensional semisimple modules

$$A \odot V \cong \bigoplus_{\lambda \in E_V} A \odot m^V_\lambda S_\lambda \quad \text{and} \quad A \odot W \cong \bigoplus_{\lambda \in E_W} A \odot m^W_\lambda S_\lambda,$$

(E.11)

where $A \odot S_\lambda$ are non-isomorphic simple modules and $m^V_\lambda, m^W_\lambda \in \mathbb{Z}_{\geq 0}$ non-negative. Only Schur’s lemma E.2 is needed.

Lemma E.3. If $A \odot V$ and $A \odot W$ are semisimple as in (E.11), then the following set is a basis for $\text{Hom}_A(V, W)$:

$$\{ i^V_W \circ \pi^V_{\ell, \lambda} \mid \lambda \in E_V \cap E_W, 1 \leq k \leq m^W_\lambda, \text{ and } 1 \leq \ell \leq m^V_\lambda \},$$

(E.12)

where

$$\pi^V_{\ell, \lambda}(v) := \begin{cases} (i^V_W(\ell))^{-1}(v), & v \in i^V_W(\ell)(S_\lambda), \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } v \in V,$$

(E.13)

and where $\{ i^V_{\ell} \mid 1 \leq \ell \leq m^V_\lambda \}$ and $\{ i^W_k \mid 1 \leq k \leq m^W_\lambda \}$ are bases for $\text{Hom}_A(S_\lambda, V)$ and $\text{Hom}_A(S_\lambda, W)$, respectively.
Proof. To begin, we prove that for all $\lambda \in E_V$ and $\ell, \ell' \in \{1, 2, \ldots, m_V^X\}$ such that $\ell' \neq \ell$, we have

$$i^{(\ell, \lambda)}_V(S_\lambda) \cap i^{(\ell', \lambda)}_V(S_\lambda) = \{0\}. \tag{E.14}$$

Indeed, Schur’s lemma E.2 implies that each map in the basis $\{i^{(\ell, \lambda)}_V | 1 \leq \ell \leq m_V^X\}$ is an isomorphism onto its image, which is a submodule of $A \circ V$ isomorphic to $A \circ S_\lambda$. We form the sum module $i^{(\ell, \lambda)}_V(A \circ S_\lambda) + i^{(\ell', \lambda)}_V(A \circ S_\lambda)$, and note that the intersection $i^{(\ell, \lambda)}_V(A \circ S_\lambda) \cap i^{(\ell', \lambda)}_V(A \circ S_\lambda)$ is an $A$-submodule of either simple module $i^{(\ell, \lambda)}_V(A \circ S_\lambda)$ and $i^{(\ell', \lambda)}_V(A \circ S_\lambda)$. Hence, because both of these submodule are simple, we have

either $i^{(\ell, \lambda)}_V(S_\lambda) \cap i^{(\ell', \lambda)}_V(S_\lambda) = \{0\}$, or $i^{(\ell, \lambda)}_V(S_\lambda) \cap i^{(\ell', \lambda)}_V(S_\lambda) = i^{(\ell, \lambda)}_V(S_\lambda) = i^{(\ell', \lambda)}_V(S_\lambda). \tag{E.15}$

In the latter case, Schur’s lemma again shows that the composed map $(i^{(\ell', \lambda)})^{-1} \circ i^{(\ell, \lambda)}_V$ is a multiple of the identity map $id_{S_\lambda}$, which implies that $i^{(\ell', \lambda)}_V$ and $i^{(\ell, \lambda)}_V$ are proportional to each other, violating the fact that they are distinct basis elements in $\text{Hom}(A \circ S_\lambda, V)$. We conclude that (E.14) indeed holds.

Now, any vector $v \in V$ which belongs to a submodule of $A \circ V$ isomorphic to $A \circ S_\lambda$ lies in the image $i(S_\lambda)$ for some isomorphism $i \in \text{Hom}(A \circ S_\lambda, V)$. Writing $v = i(u)$ for some $u \in S_\lambda$ and expanding $i$ with some $b^{(\lambda)}_\ell \in C$, we have

$$v = i(u) = \sum_{\ell = 1}^{m_V^X} b^{(\lambda)}_\ell i^{(\ell, \lambda)}_V(u) \tag{E.16}$$

$$\implies \sum_{\lambda' \in E_V} \sum_{\ell = 1}^{m_V^X} (i^{(\ell, \lambda')}_V \circ \pi^{\lambda'}_V)(v) = \sum_{\ell = 1}^{m_V^X} b^{(\lambda)}_\ell i^{(\ell, \lambda)}_V(u) = v. \tag{E.17}$$

Therefore, direct-sum decomposition (E.11) combined with a similar analysis for $W$ implies that we can write

$$\text{id}_V \tag{E.17} \sum_{\lambda \in E_V} \sum_{\ell = 1}^{m_V^X} \pi^{\lambda'}_V \circ i^{(\ell, \lambda)}_V \quad \text{and} \quad \text{id}_W \tag{E.15} \sum_{\lambda' \in E_W} \sum_{k = 1}^{m_W^X} \pi^{\lambda'}_W \circ i^{(k, \lambda')}_W. \tag{E.18}$$

Now, we fix an arbitrary map $L \in \text{Hom}(V, W)$ and consider the composed map $\pi^{W}_{k, \lambda'} \circ L \circ i^{(\ell, \lambda)}_V \in \text{Hom}(S_\lambda, S_{\lambda'})$, with $\pi^{W}_{k, \lambda'} := (i^{(k, \lambda')}_W)^{-1}$ defined as in (E.13). As $A \circ S_\lambda$ are simple and non-isomorphic, Schur’s lemma E.2 shows that

$$\pi^{W}_{k, \lambda'} \circ L \circ i^{(\ell, \lambda)}_V \begin{cases} c^{(\lambda)}_{k, \ell}(L) & \lambda' = \lambda, \\ 0 & \lambda' \neq \lambda \end{cases} \tag{E.19}$$

for some constants $c^{(\lambda)}_{k, \ell}(L) \in C$. Therefore, we obtain

$$L = \sum_{\lambda \in E_V \cap E_W} c^{(\lambda)}_{k, \ell}(L) i^{(k, \lambda)}_W \circ \pi^{\lambda'}_V \tag{E.20}$$

This proves that collection (E.12) is a spanning set for $\text{Hom}(V, W)$. Furthermore, we see from (E.19, E.20) that

$$c^{(\lambda)}_{k, \ell}(L) = 0 \quad \text{for all } \lambda \in E_V \cap E_W, \ 1 \leq k \leq m_W^X, \ 1 \leq \ell \leq m_V^X \quad \iff L = 0, \tag{E.21}$$

so the maps in collection (E.12) are also linearly independent, thus forming a basis for $\text{Hom}(V, W)$.

\begin{corollary}
If $A \circ V$ is semisimple as in (E.11), then we have

$$C := \text{End}_AV \cong \bigoplus_{\lambda \in E_V} \text{End}(\text{Hom}(A \circ S_\lambda, V)), \tag{E.22}$$

and the collection $\{c \in \text{Hom}(A \circ S_\lambda, V) | \lambda \in E_V\}$ is the complete set of non-isomorphic simple $C$-modules.
\end{corollary}

\begin{proof}
Using basis (E.12) for $C$ from lemma E.3, we define a homomorphism $\psi : C \rightarrow \bigoplus_{\lambda \in E_V} \text{End}(\text{Hom}(A \circ S_\lambda, V))$ of algebras by homomorphism extension of the rule

$$\psi := \sum_{\lambda \in E_V} \psi_\lambda, \quad \psi_\lambda(i^{(k, \lambda)}_V \circ \pi^{\lambda'}_V)(f) := i^{(k, \lambda)}_V \circ \pi^{\lambda'}_V \circ f \in \begin{cases} \text{Hom}(A \circ S_\lambda, V), & \lambda' = \lambda, \\ \{0\}, & \lambda' \neq \lambda \end{cases} \tag{E.23}$$

\end{proof}
for each element \( f \in \text{Hom}_A(S_\lambda, V) \) and indices \( \lambda \in \mathbb{E}_V \) and \( k, \ell \in \{1, 2, \ldots, n_\lambda^V\} \). Indeed, \( \psi \) is a well-defined injective homomorphism of algebras, because the action \((E.23)\) of it images on the basis elements \( v_i^{(j, \lambda)} \in \text{Hom}_A(S_\lambda, V) \) reads

\[
\psi(v_i^{(j, \lambda)} \circ \pi_{k, \lambda}) = (E.23) \quad v_i^{(j, \lambda)} \circ \pi_{k, \lambda} \quad (E.13) \quad \delta_{\lambda, \lambda'} \delta_{\lambda, j'} v_i^{(k, \lambda')}. \tag{E.24}
\]

Furthermore, \( \psi \) is surjective, because the dimensions of its domain and codomain coincide. Finally, lemma E.1 then implies that the collection \( \{ C \circ \text{Hom}_A(S_\lambda, V) \mid \lambda \in \mathbb{E}_V \} \) is the complete set of non-isomorphic simple \( C \)-modules. \( \square \)

2. General double-commutant property

The next key result is a generalization of \([GW09, \text{theorem 4.1.13}]\) (in the case of associative algebras), and a part of its proof is a straightforward adaptation of the proof given in \([GW09]\). The latter is called the double-commutant theorem, or the double-commutant (double-centralizer) property. We prefer to use the term “double-commutant theorem” for the stronger statement in theorem E.9 below, also including duality of the algebra and its commutant.

**Proposition E.5.** Let \( \rho_V : A \rightarrow \text{End} V \) and \( \rho_W : A \rightarrow \text{End} W \) be finite-dimensional representations of \( A \), and denote the commutant by \( C := \text{Hom}_A(V, W) \). Let \( L \in \text{End} V \) and \( R \in \text{End} W \). Then, the following hold:

1. Suppose the module \( ^A \circ V \) is semisimple. Then, the diagram

\[
\begin{array}{ccc}
V & \overset{C}{\longrightarrow} & W \\
\downarrow L & & \downarrow R \\
V & \overset{C}{\longrightarrow} & W
\end{array}
\]

commutes for all elements \( C \in C \) if and only if there exists an element \( a \in A \) such that, for all \( C \in C \), we have

\[
C \circ \rho_V(a) = C \circ L = R \circ C = \rho_W(a) \circ C. \tag{E.25}
\]

2. Suppose both modules \( ^A \circ V \) and \( ^A \circ W \) are semisimple, with direct-sum decompositions as in \((E.11)\). Then, diagram \((E.25)\) commutes for all elements \( C \in C \) if and only if there exists an element \( a \in A \) such that

\[
L = \rho_V(a) + \sum_{\lambda \in \mathbb{E}_V \setminus \mathbb{E}_W} \pi^V_\lambda \circ L' \quad \text{and} \quad R = \rho_W(a) + \sum_{\lambda \in \mathbb{E}_W \setminus \mathbb{E}_V} R' \circ \pi^W_\lambda \tag{E.26}
\]

for some endomorphisms \( L' \in \text{End} V \) and \( R' \in \text{End} W \), where \( \pi^V_\lambda \in \text{End}_AV \) and \( \pi^W_\lambda \in \text{End}_AW \) denote the projections onto the \( \lambda \)-th summands in \((E.11)\).

**Proof.** We prove items 1 and 2 as follows:

1. (See also \([GW09, \text{theorem 4.1.13}]\).) The “if” part holds by definition of \( C := \text{Hom}_A(V, W) \). To prove the “only if” part, we fix a basis \( \{e_1, e_2, \ldots, e_n\} \) for \( V \), denoting \( n = \dim V \), and consider the fixed vector

\[
w := e_1 \oplus e_2 \oplus \cdots \oplus e_n \in nV = V^{\otimes n}. \tag{E.27}
\]

Let \( \pi \in \text{End}_AV^{\otimes n} \) be the projection onto the submodule \( ^A \circ \{ \rho_V^n(a)(w) \mid a \in A \} \) generated by \( w \), where \( \rho_V^n \) denotes the diagonal action \((E.4)\) on \( nV = V^{\otimes n} \). Its domain being the direct sum \( V^{\otimes n} \), the map \( \pi \) has a matrix representation with entries \( \pi_{ij} \in \text{End}_AV \), with \( i, j \in \{1, 2, \ldots, n\} \). Furthermore, because \( \pi(w) = w \) by definition, we obtain

\[
e_j = \pi_{j,1}(e_1) + \pi_{j,2}(e_2) + \cdots + \pi_{j,n}(e_n) \quad \text{for all } j \in \{1, 2, \ldots, n\}. \tag{E.28}
\]

Denoting \( (C^{\otimes n}(v_1 \oplus v_2 \oplus \cdots \oplus v_n)) := C(v_1) \oplus C(v_2) \oplus \cdots \oplus C(v_n) \) for all \( v_1, \ldots, v_n \in V \) and \( C \in \text{End} V \), we have

\[
(C^{\otimes n} \circ \pi \circ L^{\otimes n})(w) \overset{(E.28)}{=} \bigoplus_{j=1}^n ((C \circ \pi_{j,1} \circ L)(e_1) + (C \circ \pi_{j,2} \circ L)(e_2) + \cdots + (C \circ \pi_{j,n} \circ L)(e_n))
\]
This finishes the proof.

\[ (R \circ C)^{\oplus n}(w) = (\mathcal{C}^{\oplus n} \circ \pi)(L^{\oplus n}(w)) \]

\[ (C \circ \rho_W(a))(e_j) = (R \circ C)(e_j) \quad \text{for all } j \in \{1, 2, \ldots, n\}. \]

In conclusion, because \( \{e_1, e_2, \ldots, e_n\} \) is a basis for \( V \) and \( C \in \mathcal{C} \) is arbitrary, we obtain asserted equalities \( (E.26) \):

\[ C \circ L = R \circ C = C \circ \rho_W(a) = \rho_W(a) \circ C, \quad (E.32) \]

recalling in the last equality that \( C \in \mathcal{C} := \text{Hom}_A(W, V) \) commutes with the \( A \)-action by definition.

2. For the “if” part, Schur’s lemma \( \text{E.2} \) shows that \( \pi^W \circ C \) and \( C \circ \pi^V \) equal zero for all \( \lambda \in E_W \setminus E_V \) and \( \lambda' \in E_V \setminus E_W \), so we readily deduce that diagram \( (E.25) \) commutes:

\[ C \circ L \quad (E.25) \]

\[ = C \circ \rho_V(a) + \sum_{\lambda \in E_V \setminus E_W} C \circ \pi^V \circ L' = \rho_W(a) \circ C + \sum_{\lambda \in E_V \setminus E_W} C \circ \pi^V \circ L' \]

\[ = R \circ C - \sum_{\lambda \in E_W \setminus E_V} R' \circ \pi^W \circ C + \sum_{\lambda \in E_V \setminus E_W} C \circ \pi^V \circ L' = R \circ C, \quad (E.34) \]

For the “only if” part, it suffices to show that \( (E.26) \) in item \( \text{I} \) implies \( (E.27) \) with the same element \( a \in A \). To this end, we first note that for all elements \( C \in \mathcal{C} \), we have

\[ 0 \quad (E.26) \]

\[ = C \circ (L - \rho_V(a)) \quad \text{and} \quad 0 \quad (E.26) \]

\[ = (R - \rho_W(a)) \circ C, \quad (E.35) \]

which implies that the image of \( L - \rho_V(a) \) belongs to the kernel of every map \( C \in \mathcal{C} \), and the image of every map \( C \in \mathcal{C} \) belongs to the kernel of \( R - \rho_W(a) \). To solve for the possibilities for \( L - \rho_V(a) \in \text{End} V \) and \( R - \rho_W(a) \in \text{End} W \), we write arbitrary linear maps \( L'' \in \text{End} V \) and \( R'' \in \text{End} W \) via direct-sum decompositions \( (E.11) \) in the form

\[ L'' = \sum_{\lambda \in E_V} \pi^V \circ L'' \quad \text{and} \quad R'' = \sum_{\lambda \in E_W} R'' \circ \pi^W. \]

Recalling basis \( (E.12) \) of \( C \) from lemma \( (E.3) \), consisting of homomorphisms of \( A \)-modules sending submodules of type \( A \circ S_\lambda \) in \( A \circ \mathcal{V} \) isomorphically to submodules of \( A \circ \mathcal{W} \) of the same type, we see that

\[ \text{im} L'' \in \ker C \quad \text{for all } C \in \mathcal{C} \quad \Rightarrow \quad \pi^V \circ L'' = 0 \quad \text{for all } \lambda \in E_V \cap E_W, \quad (E.37) \]

\[ \text{im} C \in \ker R'' \quad \text{for all } C \in \mathcal{C} \quad \Rightarrow \quad R'' \circ \pi^W = 0 \quad \text{for all } \lambda \in E_V \cap E_W. \quad (E.38) \]

By applying \( (E.37, E.38) \) to \( L'' = L - \rho_V(a) \) and \( R'' = R - \rho_W(a) \), we conclude that

\[ L - \rho_V(a) \quad (E.36) \]

\[ = \sum_{\lambda \in E_V \setminus E_W} \pi^V \circ L' \quad \text{and} \quad R - \rho_W(a) \quad (E.35) \]

\[ = \sum_{\lambda \in E_W \setminus E_V} R' \circ \pi^W, \quad (E.39) \]

which proves \( (E.27) \) with \( L' = L - \rho_V(a) \) and \( R' = R - \rho_W(a) \).

This finishes the proof. \( \square \)

We recover the double-commutant property \([GW09, \text{theorem 4.1.13}]\) as a special case.

**Corollary E.6.** \([GW09, \text{theorem 4.1.13}]\) Suppose \( \mathcal{A} \circ \mathcal{V} \) is semisimple and \( \mathcal{C} := \text{End}_A \mathcal{V} \). Then \( \rho(A) = \text{End}_\mathcal{C} \mathcal{V} \).

**Proof.** This follows from proposition \( \text{E.5} \) by specializing to \( \mathcal{W} = \mathcal{V} \). \( \square \)
Lemma E.7. Suppose the module $^A \odot V$ is semisimple with direct-sum decomposition as in (E.11). Then, we have

$$\rho(A) \cong \bigoplus_{\lambda \in E_V} \text{End} S_\lambda,$$

(E.40)

and the collection $\{S_\lambda \mid \lambda \in E_V\}$ is the complete set of non-isomorphic simple $\rho(A)$-modules.

Proof. We show that the following representation gives rise to the sought isomorphism (E.40) of algebras:

$$\bigoplus_{\lambda \in E_V} \rho_\lambda : A \rightarrow \bigoplus_{\lambda \in E_V} \text{End} S_\lambda.$$

(E.41)

First, (E.41) is injective by (E.11). Second, to prove that (E.41) is surjective, we consider the commutant algebra

$$C := \text{End}_A \left( \bigoplus_{\lambda \in E_V} S_\lambda \right).$$

(E.42)

Because all summands $^A \odot S_\lambda$ are simple and non-isomorphic, Schur’s lemma E.2 shows that each element $C \in C$ must preserve each summand $S_\lambda$, acting as a scalar on it:

$$C \in C \implies \begin{cases} C|_{S_\lambda} : S_\lambda \rightarrow S_\lambda, \\ C|_{S_\lambda} = c_\lambda(C) \text{id}_{S_\lambda} \end{cases}$$

(E.43)

for some constants $c_\lambda(C) \in \mathbb{C}$ and for all $\lambda \in E_V$. On the other hand, using corollary E.6, we obtain

$$\bigoplus_{\lambda \in E_V} \text{End} S_\lambda \overset{(E.43)}{\subset} \text{End} c \left( \bigoplus_{\lambda \in E_V} S_\lambda \right) \overset{\text{cor. E.6}}{=} \bigoplus_{\lambda \in E_V} \rho_\lambda(A) \overset{(E.41)}{\subset} \bigoplus_{\lambda \in E_V} \text{End} S_\lambda.$$  

(E.44)

Lastly, lemma E.1 implies that $\{S_\lambda \mid \lambda \in E_V\}$ constitute the complete set of non-isomorphic simple $\rho(A)$-modules.

$$\square$$

3. Double-commutant theorem

The Jacobson radical of $A$ is the intersection of all of the maximal ideals in $A$. We say that $A$ is semisimple if its Jacobson radical is trivial. Basic examples of semisimple algebras are direct sums of matrix algebras as in lemma E.1:

Lemma E.8. If $A = \bigoplus \text{End} S_\lambda$, with $\{S_\lambda\}$ a finite collection of finite-dimensional vector spaces, then $A$ is semisimple.

Proof. See, e.g., [Lam91, example (7), page 60].

In fact, the celebrated Wedderburn’s structure theorem [Lam91, theorem (3.5)] says that these are all of the finite-dimensional semisimple algebras. Also, there are numerous equivalent notions of semisimplicity. For instance, $A$ is semisimple if and only if all $A$-modules are semisimple [Lam91, theorems (2.5) and (4.14)]. We refer to, e.g., [Lam91, chapters 1–2] for more background and history. For our purposes, semisimplicity of $A$ is only an additional observation.

Without loss of generality, we state the main theorem E.9 only for a unital subalgebra of $\text{End} V$. In general, we can take it to be the image of a finite-dimensional representation of an associative unital algebra.

Theorem E.9. (Double-commutant theorem): Let $A \subset \text{End} V$ be a unital subalgebra of the endomorphism algebra of a finite-dimensional vector space $V$. If the module $^A \odot V$ is semisimple, then the following hold:

1. All three algebras $C := \text{End}_A V$, $A = \text{End}_C V$, and $A \otimes C$ are semisimple.

2. There exists a natural bijection between the collection $\{^A \odot S_\lambda \mid \lambda \in E_V\}$ of all non-isomorphic simple $A$-modules and the collection $\{c \odot L^\lambda \mid \lambda \in E_V\}$ of all non-isomorphic simple $C$-modules, such that

$$^A \odot S_\lambda \cong ^A \odot \text{Hom}_C(L^\lambda, V) \quad \text{and} \quad c \odot L^\lambda \cong c \odot \text{Hom}_A(S_\lambda, V)$$

(E.45)

and we have

$$^A \odot V \cong \bigoplus_{\lambda \in E_V} ^A \odot S_\lambda \otimes_{C} c \odot L^\lambda.$$  

(E.46)

Furthermore, $\{^A \odot S_\lambda \otimes_{C} c \odot L^\lambda \mid \lambda, \lambda' \in E_V\}$ is the complete set of non-isomorphic simple $(A \otimes C)$-modules.
Proof. $C$ is semisimple by corollary E.4 and lemma E.8. Corollary E.6 gives $A = \text{End}_C V$, which is semisimple by lemmas E.7 and E.8. By lemma E.8, it then also follows that $A \otimes C$ is semisimple,

\[
A \otimes C \cong \bigoplus_{\lambda, \lambda'} (\text{End} S_\lambda) \otimes (\text{End} L^{\lambda'}) = \bigoplus_{\lambda, \lambda'} \text{End} (S_\lambda \otimes L^{\lambda'}),
\]

(E.47)

and lemma E.1 then shows that $\{A \otimes C \otimes L^{\lambda'} | \lambda, \lambda' \in E_V\}$ is the complete set of non-isomorphic simple $(A \otimes C)$-modules. Finally, we note that the map $\varphi$ in (E.10) gives rise to a natural isomorphism of $A$-modules

\[
A \otimes V \cong \bigoplus_{\lambda \in E_V} (A \otimes S_\lambda) \otimes \text{Hom}_A(S_\lambda, V),
\]

(E.48)

and a natural isomorphism of $C$-modules

\[
C \otimes V \cong \bigoplus_{\lambda \in E_V} (S_\lambda \otimes c \otimes \text{Hom}_A(S_\lambda, V)),
\]

(E.49)

Thus, direct-sum decomposition (E.46) follows by combining (E.48, E.49). This concludes the proof. \qed

References:

[AKP91] H. H. Andersen, W. Kexin, and P. Polo. Representations of quantum algebras. *Invent. Math.*, 104(1):1–59, 1991.

[ALZ15] H. H. Andersen, G. I. Lehrer and R. B. Zhang. Cellularity of certain quantum endomorphism algebras. *Pacific J. Math.*, 279(1-2):11–35, 2015.

[Bax07] R. J. Baxter. Exactly solved models in statistical mechanics. Originally published by Academic Press, London (1982), reprinted by Dover Publications, 2007.

[BF91] D. Bernard and G. Felder. Quantum group symmetries in two-dimensional lattice quantum field theory. *Nucl. Phys. B*, 365(1):98–120, 1991.

[BP14] A. Borodin and L. Petrov. Integrable probability: From representation theory to Macdonald processes. *Probab. Surveys*, 11, 1–58, 2014.

[BP15] A. Borodin and L. Petrov. Lectures on Integrable Probability: stochastic vertex models and symmetric functions. Stochastic Processes and Random Matrices: Lecture Notes of the Les Houches Summer School: Vol. 104, July 2015.

[BFGT09] P. V. Bushlanov, B. L. Feigin, A. M. Gainutdinov, and I. Yu. Tipunin. Lusztig limit of quantum $\mathfrak{sl}(2)$ at root of unity and fusion of $(1, p)$ Virasoro logarithmic minimal models. *Nucl. Phys. B*, 818(3):179–195, 2009.

[CGRS16] G. Carinci, C. Giardinà, F. Redig, and T. Sasamoto. A generalized asymmetric exclusion process with $q$-symmetries. *Stochastic Processes and Random Matrices: Lecture Notes of the Les Houches Summer School: Vol. 104, July 2016*, 1–58, 2014.

[CP94] V. Chari and A. Pressley. *A guide to quantum groups*. Cambridge University Press, 1994.

[DJ89] R. Dipper and G. James. The $q$-Schur algebra. *Proc. London Math. Soc.*, 59(1):23–50, 1989.

[EGH+11] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina. Introduction to representation theory. *American Mathematical Society (Providence, RI)*, 2011.

[Fel89] G. Felder. BRST approach to minimal models. *Nucl. Phys. B*, 317(1):215–236, 1989. Erratum ibid. B, 324(2):548, 1989.

[FK97] I. B. Frenkel and M. G. Khovanov. Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$. *Duke Math. J.*, 87(3):409–480, 1997.

[FKK98] I. Frenkel, M. Khovanov, and A. Kirillov Jr. Kazhdan-Lusztig polynomials and canonical basis. *Transform. Groups*, 3(4):321–336, 1998.

[FKS06] I. Frenkel, M. Khovanov, and C. Stroppel. A categorification of finite-dimensional irreducible representations of quantum $U_q(\mathfrak{sl}(2))$ and their tensor products. *Selecta Math. (N.S.)*, 12(3-4):379–431, 2006.

[FP18a] S. M. Flores and E. Peltola. Standard modules, radicals, and the valenced Temperley-Lieb algebra. Preprint: arXiv:1801.10003, 2018.

[FP18b] S. M. Flores and E. Peltola. Generators, projectors, and the Jones-Wenzl algebra. Preprint: arXiv:1811.12364, 2018.

[FP20a] S. M. Flores and E. Peltola. Solution spaces of the Benoit & Saint-Aubin partial differential equations. In preparation.

[FP20b] S. M. Flores and E. Peltola. Monodromy invariant CFT correlation functions of first column Kac operators. In preparation.

[GST14] A. M. Gainutdinov, H. Saleur, and I. Yu. Tipunin. Lattice W-algebras and logarithmic CFTs. *J. Phys. A: Math. Theor.,* 47(49):495401, 2014.
[RTW32] G. Rumer, E. Teller, and H. Weyl. Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten. Nachrichten von der Ges. der Wiss. Zu Göttingen. Math.-Phys. Klasse, 498–504, 1932.

[Sch27] I. Schur. Über die rationalen Darstellungen der allgemeinen linearen Gruppe. Sitzungsberichte Akad. Berlin, 58–75, 1927.

[Sha05] I. R. Shafarevich. Basic notions of algebra. Springer-Verlag (Berlin Heidelberg), 2005.

[Str10] C. Stroppel. Schur-Weyl dualities and link homologies. In Proc. ICM 2010, Hyderabad, India, 2010.

[TL71] H. Temperley and E. Lieb. Relations between the ‘percolation’ and ‘colouring’ problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ‘percolation’ problem. In Proc. Roy. Soc. London Ser. A, 322:251–280, 1971.

[Tur94] V. G. Turaev. Quantum invariants of knots and 3-manifolds. Walter de Gruyter (Berlin New York) 1994.

[Var92] A. Varchenko. Multidimensional hypergeometric functions in conformal field theory, algebraic K-theory, algebraic geometry. In Proceedings of the International Congress of Mathematicians, Kyoto 1990, 1992.

[Wan10] Z. Wang. Topological quantum computation. Lecture notes, Microsoft Research, 2010. Available at http://web.math.ucsb.edu/ zhenghwa/data/course/cbms.pdf.

[Wen87] H. Wenzl. On sequences of projections. C. R. Math. Rep. Acad. Sci. Canada, 9(1):5–9, 1987.

[Wey39] H. Weyl. The Classical Groups. Their Invariants and Representations. Princeton University Press, 1939.

[Wit89] E. Witten. Quantum Field Theory and the Jones Polynomial. Comm. Math. Phys., 121:351–399, 1989.