V Tree —
Continued Fraction Expansion, Stern-Brocot Tree, Minkowski’s ?(x) Function
In Binary: Exponentially Faster

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Abstract The Stern-Brocot tree and Minkowki’s question mark function ?(x) (or Conway’s box function) are related to the continued fraction expansion of numbers from \( \mathbb{Q} \) with unary encoding of the partial denominators.

We first define binary encodings \( C_I, C_{II} \) of the natural numbers, adapted to the Gauß-Kuz’min measure for the distribution of partial denominators.

Then we define the \( V_I \) tree as analogue to the Stern-Brocot tree, using the binary encodings \( C_I, C_{II} \). We shall see that all numbers with denominator \( q \) are present in the first \( 3.44 \log_2(q) \) levels, instead of \( 1/q \) appearing in level \( q \) in the Stern-Brocot tree. The extension of the \( V_I \) tree, the \( V \) tree, covers all numbers from \( \mathbb{Q} \) exactly once. We also define the binary version of Minkowski’s question mark function, \( ?_V \), and conjecture that it has no derivative at rational points (for the original, \( ?'(x) = 0 \), \( x \in \mathbb{Q}_1 \)).

Keywords: \( V \) tree, Stern-Brocot tree, Minkowski’s question mark function.

“Read the classics” — Edwards [9, S. ix]

Introduction / Motivation: In the theory of stream ciphers, the continued fraction expansion of formal power series from \( F_2[[x^{-1}]] \) leads to an isometry between the coefficient series \( s = (s_k) \) and the encoding of the partial denominators (which are polynomials from \( F_2[x] \) in this case), \( \mathcal{K}: F_2^{\ast} \ni s \mapsto d \in F_2^{\ast} \) is an isometry,

\[
\mathcal{K}: s \mapsto G(s) = \sum_{k \in \mathbb{N}} s_k x^{-k} = [b_1, b_2, b_3, \ldots] \mapsto C_{F_2[x]}(b_1)|C_{F_2[x]}(b_2)|C_{F_2[x]}(b_3)| \cdots = (d_k).
\]

Notation:
\( \mathbb{N} = \{1, 2, 3, \ldots\} \), \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \)
\( \mathbb{D} := \{a/2^k : a \in \mathbb{Z} \text{ odd}, k \in \mathbb{N}_0\} \), dyadic fractions
For \( X \subset \mathbb{R}, X_1 := X \cap (0, 1), X^+ := X \cap (0, \infty) \): \( \mathbb{D}_1, \mathbb{Q}_1, \mathbb{R}_1, \mathbb{D}^+, \mathbb{Q}^+, \mathbb{R}^+ \)
\( A = \{0, 1\} \) is the binary alphabet
\( A^\ast = \{ \varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots \} \) and \( A^\omega \) are the finite, resp. infinite words over \( A \)
For \( v = v_1 v_2 \ldots v_{|v|} \in A^\ast: (v)_2 = \sum_{k=0}^{|v|-1} v_{|v|-k} 2^k \in \mathbb{N}_0 \) in binary, \( (\varepsilon)_2 = 0 \)
0/1-inversion: \( 00 = 1, \ 11 = 0 \), \( 10011 = 01100 \)
\( \varphi := (b + \sqrt{b^2 + 4})/2 \) is the larger root of \( x^2 = x \cdot b + 1 \), eigenvalue/growth rate of \( A_k, B_k \)
if all PDs \( b_i = b \) (\( \lambda_b \) in [5]) \( \varphi_1 = 1.618, \varphi_2 = 2.414, \varphi_4 = 4.236 \)
This aesthetically pleasing result motivated the paper, considering the same problem for $\mathbb{R}$. Since $\mathbb{R}$ is Archimedean, while $\mathbb{F}_2, \mathbb{F}_2[[x^{-1}]]$ are ultrametric, we shall see (Gauß-Kuz’min measure) that an exact isometric result is impossible (with the exception of an ideal context-sensitive “Lévy encoding”, which would though just be the identity on $A^\omega$).

Nevertheless, the presented encodings are a huge step forward in terms of the expected codeword length $H$: From $H_{SB} = \infty$ for the unary, Stern-Brocot case, to $H_{C_I, C_{II}} = 3.507$ for our codes, near the optimum $H_{Lévy} = 3.423$. For more implementation details see [31].

See Berlekamp [1] and Massey [21] for the general solution, Dornstetter [7] and V. [30] for the isometric adaptation, Niederreiter and V. [23], Canales and V. [4] for applications.

I – Definitions

Definition 1. Binary Encodings $C_I, C_{II}$ (see Table I)

Let $C_I: \mathbb{N} \cup \{\omega_0\} \rightarrow A^* \cup \{0^\omega\}$

$$b = \sum_{k=0}^{l} b_k 2^k \mapsto C_I(b) = 0^l 1 \bar{b}_{l-1} \bar{b}_{l-2} \ldots \bar{b}_1 \bar{b}_0,$$

where $l = \lfloor \log_2(b) \rfloor$, and $C_I(\omega_0) := 0^\omega$, be a complete prefixfree code.

Let $C_{II}: \mathbb{N} \cup \{\omega_0\} \rightarrow A^* \cup \{1^\omega\}$

$$b = \sum_{k=0}^{l} b_k 2^k \mapsto C_{II}(b) = 1^l 0 b_{l-1} b_{l-2} \ldots b_1 b_0 = C_I(b),$$

with $l$ as before and $C_{II}(\omega_0) := 1^\omega$, be the 0/1-inverse of $C_I$, again a complete prefix code.

We have $l_{I, II}(b) := 1 + 2 \cdot \log_2(b)$ as length of the codewords for $b$.

Example. $b = 14 = (110)_2 \mapsto 0001001 = C_I(b)$ and $111110 = C_{II}(b)$

Definition 2. Binary V question mark functions $?_V$ and $?^{-1}_V$

(i) For any $p/q \in \mathbb{Q}_1$, let $p/q = [b_1, b_2, \ldots, b_{2l}]$ be its continued fraction expansion (CFE) with an even number of partial denominators (PD) (see Appendix 1).

We define the function $?_V: \mathbb{Q}_1 \rightarrow A^*$ by

$$?_V \left(\frac{p}{q}\right) := C_I(b_1)|C_{II}(b_2)| \ldots |C_I(b_{2l-1})|C_{II}(b_{2l}) \setminus 10^*. $$

The operation \setminus10* removes all, if any, trailing zeroes and then one symbol 1. This affects at most $C_{II}(b_{2l})$ and, only in case of $b_{2l} = 1$, $C_{II}(1) = 0$, also affects $C_I(b_{2l-1})$.

(ii) For any $v \in A^*$, extended to the infinite word $v10^\omega$, let

$$v10^\omega = C_I(b_1)|C_{II}(b_2)| \ldots |C_I(b_{2l-1})|C_{II}(b_{2l})|C_I(\omega_0)$$

be the decomposition of $v10^\omega$ into encodings, starting with $C_I$. Since $C_I, C_{II}$ are complete and prefixfree, this is always possible, in a unique way.
Then \( ?_{V}^{-1} : A^{\ast} \to \mathbb{Q}_{1} \) is defined by

\[
?_{V}^{-1}(v) := \frac{p}{q} = [b_{1}, b_{2}, \ldots, b_{2l}].
\]

By construction, we have \( ?_{V}^{-1}(?_{V}(p/q)) = p/q \) and \( ?_{V}(?_{V}^{-1}(v)) = v \) for all \( v \in A^{\ast} \) and \( p/q \in \mathbb{Q}_{1} \).

(iii) We define real-valued functions \( \overline{\tau}_{V} \) and \( \overline{\tau}_{V}^{-1} \) from \( \mathbb{R}_{1} \) to \( \mathbb{R}_{1} \) by first defining

\[
\overline{\tau}_{V} : \mathbb{Q}_{1} \to \mathbb{D}_{1}, \quad \overline{\tau}_{V}(p/q) := \iota_{AD}(?_{V}(p/q))
\]

with \( \iota_{AD} \) from Appendix 2 on the equivalence \( \mathbb{N} \equiv A^{\ast} \equiv \mathbb{D}_{1} \), and then \( \overline{\tau}_{V} : \mathbb{R}_{1} \to \mathbb{R}_{1} \) by continuous extension. Also, first

\[
\overline{\tau}_{V}^{-1} : \mathbb{D}_{1} \to \mathbb{Q}_{1}, \quad \overline{\tau}_{V}^{-1}(d) := ?_{V}^{-1}(\iota_{DA}(d))
\]

and then continuously extending to \( \mathbb{R}_{1} \).

Table 1: Codes \( C_{I} \) and \( C_{II} \) for partial denominators.

| \( b \) | \( C_{I}(b) \) | \( C_{II}(b) \) | \( l_{II}(b) \) | \( l_{GK} \) | \( \mu_{GK} \) |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1.269 | 0.4150 |
| 2 | 011 | 100 | 3 | 2.557 | 0.1699 |
| 3 | 010 | 101 | 3 | 3.425 | 0.0931 |
| 4 | 00111 | 11000 | 5 | 4.086 | 0.0588 |
| 5 | 00110 | 11001 | 5 | 4.621 | 0.0406 |
| 6 | 00101 | 11010 | 5 | 5.071 | 0.0297 |
| 7 | 00100 | 11011 | 5 | 5.460 | 0.0227 |
| 8 | 0001111 | 1110000 | 7 | 5.802 | 0.0179 |
| 9 | 0001110 | 1110001 | 7 | 6.108 | 0.0144 |
| 10 | 0001101 | 1110010 | 7 | 6.384 | 0.0119 |
| 11 | 0001100 | 1110011 | 7 | 6.636 | 0.0100 |
| 12 | 0001011 | 1110100 | 7 | 7.082 | 0.0073 |
| 13 | 0001010 | 1110101 | 7 | 7.282 | 0.0064 |
| 14 | 0001001 | 1110110 | 7 | 7.468 | 0.0056 |
| 15 | 0001000 | 1110111 | 7 | 7.644 | 0.0050 |
| 16 | 000011111 | 111100000 | 9 | 7.802 | 0.0041 |
| 31 | 000010000 | 111101111 | 9 | 9.471 | 0.0014 |
| 32 | 00000111111 | 11111000000 | 11 | 9.559 | 0.0013 |
| 63 | 00000100000 | 11111011111 | 11 | 11.471 | 0.00035 |
| 64 | 0000001111111 | 1111110000000 | 13 | 11.516 | 0.00034 |
| \( \aleph_{0} \) | 0^\omega | 1^\omega | — | — | — |

(the columns \( l_{GK} \) and \( \mu_{GK} \) are explained in Theorem 8)
Figure 1: $V_{10}$ tree on $Q_1$.

**Definition 3.** $V_{10}$ tree for $Q_1$

We define the $V_{10}$ tree as an infinite binary tree with label $?_{V}^{-1}(v) \in Q_1$ at the node with symbolic address $v$ (see Appendix 4 on trees and addresses).

**Definition 4.** $V_1$ tree and V question mark functions $\hat{?}_V$, $\hat{?}_V^{-1}$ for $Q^+$

From $Q_1$ to $Q^+$ by multiplicative inversion:

(i) For $m \in A, v \in A^*$, let

\[
\hat{?}_V^{-1} : A \times A^* \to Q^+ \\
\hat{?}_V^{-1}(0v) = ?_{V}^{-1}(v) \\
\hat{?}_V^{-1}(1v) = (\hat{?}_{V}^{-1}(v))^{-1}
\]

where $\pi$ is the 0/1-inverted address.

We also set $\hat{?}_V^{-1}(\varepsilon) = 1$. Then $\hat{?}_V^{-1} : A^* \to Q^+$ is defined on all of $A^*$.

(ii) We define the $V_1$ tree as an infinite binary tree with label $\hat{?}_V^{-1}(v)$ at the node with address $v$.

(iii) Let $\hat{?}_V : Q^+ \to A^*$ be the inverse function to $\hat{?}_V^{-1}$,

\[
\hat{?}_V(p/q) = \begin{cases} 
0|?_{V}(\frac{p}{q}), & p < q; \\
\varepsilon, & p = q, \text{ i.e. } p/q = 1; \\
1|?_{V}(\frac{q}{p}), & p > q.
\end{cases}
\]

**Definition 5.** $V$ tree and V question mark functions $\hat{?}_V$, $\hat{?}_V^{-1}$ for $Q$

From $Q^+$ to $Q$ by additive inversion:

(i) For $a \in A, v \in A^*$, let

\[
\hat{?}_V^{-1} : A \times A^* \to Q \\
\hat{?}_V^{-1}(0v) = - (\hat{?}_V^{-1}(v)) \\
\hat{?}_V^{-1}(1v) = \hat{?}_V^{-1}(v).
\]
We also set \( \hat{\gamma}^{-1}(\varepsilon) = 0 \). Then \( \hat{\gamma}^{-1}: A^* \to \mathbb{Q} \) is defined on all of \( A^* \).

(ii) We now define the V tree (see Figure 2) as an infinite binary tree with label \( \hat{\gamma}^{-1}(v) \) at the node with address \( v \).

(iii) Let \( \hat{\gamma}: \mathbb{Q} \to A^* \) be the inverse function to \( \hat{\gamma}^{-1} \),

\[
\hat{\gamma}(p/q) = \begin{cases} 
0 & \text{if } \hat{\gamma}_V((-\frac{p}{q})) = p/q < 0, \\
\varepsilon & \text{if } p/q = 0, \\
1 & \text{if } \hat{\gamma}_V(\frac{p}{q}) = p/q > 0. 
\end{cases}
\]

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**Figure 2:** Full V Tree on \( \mathbb{Q} \) with addresses.

**Definition 6.** Sequences \( V, V_1, V_{10} \)

Reading out the values from the \( V, V_1, \) and \( V_{10} \) trees in the order of the numerical addresses (breadth first), we obtain the following 3 sequences:

\[
V = \left( \frac{p_n}{q_n} \right)_{n=1}^\infty = \left( \frac{0}{1}, \frac{-1}{1}, \frac{2}{1}, \frac{-3}{1}, \frac{4}{1}, \frac{-5}{1}, \frac{6}{1}, \frac{-7}{1}, \frac{8}{1}, \ldots \right) = \mathbb{Q}
\]

\[
V_1 = \left( \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{5}, \frac{4}{5}, \frac{5}{4}, \frac{5}{8}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \ldots \right) = \mathbb{Q}^+
\]

\[
V_{10} = \left( \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{5}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{4}, \frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}, \frac{1}{18}, \frac{1}{20}, \ldots \right) = \mathbb{Q}_1
\]

We show in Theorem 11 that indeed the sequences \( V, V_1, V_{10} \) are a complete ordering of all elements of \( \mathbb{Q}, \mathbb{Q}^+, \) and \( \mathbb{Q}_1 \), respectively, each element appearing exactly once.
II – Rationale

**Definition 7.** Unary Encodings

(i) Let

\[ C_0 : \mathbb{N} \cup \{ \aleph_0 \} \to A^* \cup \{ 0^\omega \}, \quad C_0(b) = 0^{b-1}1, \quad C_0(\aleph_0) = 0^\omega, \]

\[ C_1 : \mathbb{N} \cup \{ \aleph_0 \} \to A^* \cup \{ 1^\omega \}, \quad C_1(b) = 1^{b-1}0, \quad C_1(\aleph_0) = 1^\omega \]

be complete prefixfree codes.

(ii) Let

\[ C_0' : \mathbb{N} \cup \{ \aleph_0 \} \to A^* \cup \{ 0^\omega \}, \quad C_0'(b) = 0^b, \quad C_0'(\aleph_0) = 0^\omega, \]

\[ C_1' : \mathbb{N} \cup \{ \aleph_0 \} \to A^* \cup \{ 1^\omega \}, \quad C_1'(b) = 1^b, \quad C_1'(\aleph_0) = 1^\omega \]

be codes, which are, however, neither complete nor prefixfree.

**Theorem 8.** *(Gauß-Kuz’min-Khinchin-Lévy)*

For almost all values \( r \in \mathbb{R} \), we have:

(i) The probability for a partial denominator \( b \in \mathbb{N} \), its Gauß-Kuz’min measure, is

\[ \mu_{\text{GK}}(b) = -\log_2 \left( 1 - \frac{1}{(b+1)^2} \right) = \log_2 \left( 1 + \frac{1}{b(b+2)} \right). \]

(ii) The geometric average of the partial denominators is Khinchin’s constant

\[ K := \lim_{n \to \infty} \sqrt[n]{b_1 \cdot b_2 \cdots b_n} = 2.68545. \]

(iii) The average gain in precision, per partial denominator in bits, is

\[ \frac{\pi^2}{6 \ln(2)^2} = 3.42371 \ldots =: H_{\text{Lévy}}, \text{ with } 2^{3.42371/2} = 3.27582 \ldots \text{ being Lévy’s constant.} \]

One obtains Khinchin’s constant as geometric average over the Gauß-Kuz’min measure,

\[ K = \prod_{b \in \mathbb{N}} b^{\mu_{\text{GK}}(b)}. \]

**Remark.** The exceptions to this result are

— rational numbers (Euclid, [10] Liber VII, §1+2)
— quadratic-algebraic numbers (Lagrange [17]
— powers \( e^{2/k} \) (Euler [11], Hurwitz [12])
— Liouville numbers (Maillet [20], Liouville [19])
— numbers with bounded PDs (Shallit [26], Jenkinson [13])

and some more, altogether a set of positive Hausdorff dimension, but measure zero. The encodings \( C_I, C_{II} \) are modelled along the Gauß-Kuz’min measure, which suggests “ideal” codeword lengths \( l_{\text{GK}} = -\log_2(\mu_{\text{GK}}(b)) \) (see Table [D]), which are however non-integral.
Remark. Codeword lengths Besides $\mu_{GK}$ from Theorem 8(i) with (non-integral) code-word length $l_{GK}(b) := -\log_2(\mu_{GK}(b))$, we need $l_{I,II}(b) := 1 + 2 \log_2(b)$ from Definition 1 and $l_{SB}(b) := b$ for the unary encoding from Definition 7.

**Proposition 9. Average Codelength**

For each code $X$ with codeword length $l_X$, we define the average codeword length or entropy under the Gauß-Kuz’min distribution as

$$H_X := \sum_{b \in \mathbb{N}} l_X(b) \cdot \mu_{GK}(b).$$

Also, let $H_{\text{Lévy}} = 3.42371 \ldots$ from Theorem 8(iii) as lower bound.

By numerical evaluation, we obtain the average codeword lengths/entropies (Table 2).

| Code $X$ | Lévy  | GK    | $C_1, C_{II}$ | SB    |
|---------|-------|-------|--------------|-------|
| $H_X$   | 3.42371 | 3.43246 | 3.50698     | $\infty$ |

**Theorem 10.** The Stern-Brocot tree and Minkowski’s question mark function $?(x)$

(i) Let $?, ?^{-1}$ be defined analogously to Definition 2 but using codes $C_0, C_1$. Then $?(x)$ is Minkowski’s question mark function (equal to Conway’s box function $\theta$).

$$?(p/q) = \iota_{AD}(C_0(b_1)|C_1(b_2)| \ldots |C_1(b_{2l-1})).$$

(ii) Let $?, ?^{-1}$ be defined analogously to Definition 4 but using codes $C_0, C_1$. The infinite binary tree with $?^{-1}(v)$ as label at node $v$ then is the Stern-Brocot tree (Stern 27, Brocot 3), see Appendix 4.

**Proof.** (i) For $p/q = [b_1, b_2, \ldots, b_{2l}]$, we have

$$?(p/q) = \iota_{AD}(C_0(b_1)|C_1(b_2)| \ldots |C_1(b_{2l})) = 2 \cdot \iota_{AD}(C_0'(b_1)|C_1'(b_2)| \ldots |C_1'(b_{2l})) = 2 \cdot \sum_{k=1}^{l} (-1)^{k+1}2^{-\sum_{i=1}^{k} b_i}.$$

The last representation for $?(x)$ was introduced by Denjoy 4.

(ii) follows from (i) and the known correspondence between Minkowski’s $?(x)$ function and the entries in the Stern-Brocot tree.

**Remark.** The $V_1$ tree is the analogue to the Stern-Brocot tree for binary encoded partial denominators. More on the Stern-Brocot tree and Minkowski’s $?(x)$ function can be found in Salem 25, Viader et al. 29, and Vepstas 28.

7
III – Properties

Theorem 11. Equivalence of the $V_{10}$, $V_1$, $V$ sequences with $\mathbb{Q}_1, \mathbb{Q}^+, \mathbb{Q}$

(i) The sequence $V_{10}$ contains every element from $\mathbb{Q}_1$ exactly once.

(ii) The sequence $V_1$ contains every element from $\mathbb{Q}^+$ exactly once.

(iii) The sequence $V$ contains every element from $\mathbb{Q}$ exactly once.

Proof. (i) Every $p/q \in \mathbb{Q}_1$ has a finite CFE $[b_1, \ldots, b_2]$ (where $b_0 = 0$ can be omitted) with encoding $C(p/q) = C_1(b_1) \ldots |C_\Pi(b_2)$ and resulting address $v = C(p/q)\backslash10^*$.

Hence, $p/q$ is present in the $V_{10}$ tree and the $V_{10}$ sequence at node $v$ and place $n$, respectively, with $n = (1v)_2$, and only there. Different $p/q$ lead to different CFES, since $C_1, C_\Pi$ are prefixfree, and the operation $\backslash10^*$ keeps the node addresses $v$ different.

(ii) By construction, the $V_1$ tree and thus the $V_1$ sequence contain exactly once every element from $\mathbb{Q}_1$ (by (i)), in the left subtree, their multiplicative inverses in the right subtree, and 1 as root or first element, respectively.

Since for every $p/q \in \mathbb{Q}^+$, we either have $p < q$ and thus $p/q \in \mathbb{Q}_1$, or $p > q$ and thus $q/p \in \mathbb{Q}_1$, or $p = q$ and thus $p/q = 1$, and these cases are mutually exclusive, we are done.

(iii) By construction, the V tree and thus the V sequence contain exactly once every element from $\mathbb{Q}^+$ (by (ii)), in the right subtree, their additive inverses in the left subtree, and 0 as root or first element, respectively.

Since for every $p/q \in \mathbb{Q}$, we either have $p/q > 0$ and thus $+p/q \in \mathbb{Q}^+$, or $p/q < 0$ and thus $-p/q \in \mathbb{Q}^+$, or $p/q = 0$, and these cases are mutually exclusive, we are done.

Theorem 12. Monotonicity of Codes $C_1, C_\Pi$

Let $r = [b_1, b_2, \ldots], r' = [b'_1, b'_2, \ldots] \in \mathbb{R}_1$ with $r < r'$, and

$$C(r) = C_1(b_1)|C_\Pi(b_2)|\ldots, \quad C(r') = C_1(b'_1)|C_\Pi(b'_2)|\ldots \in A^\omega$$

their encodings (for $r, r' \in \mathbb{Q}$, terminate with $C_1(\mathbb{N}_0) = 0^\omega$).

Then $C(r) < C(r')$ in lexicographic order.

Proof. Let $C(r)$ and $C(r')$ have identical encodings $C_1(b_1) = C_1(b'_1), C_\Pi(b_2) = C_\Pi(b'_2), \ldots$ until the first $b_l \neq b'_l$.

(i) If $l$ is odd, $b_l > b'_l$ implies $r < r'$, regardless of the further PDs, see [24 Satz 2.9]. Also, $b_l > b'_l$ implies $C_1(b_l) < C_1(b'_l)$ in lexicographical order. For $b_l < b'_l$, all relations are inverted. We thus obtain that $r < r'$ implies $C(r) < C(r')$.

(ii) If $l$ is even, $b_l > b'_l$ implies $r > r'$ and $C_1(b_l) > C_1(b'_l)$. Again for $b_l < b'_l$, all relations are inverted. Therefore, also for even $l$, $r < r'$ implies $C(r) < C(r')$.

\[\square\]
Remark. We are entering the realm of “Experimental Mathematics”
The following two theorems have been proved (or “proved”) by verifying $2^{30}$ cases
(nodes of the respective tree). The author sees no chance of changing circumstances in
levels 31 and below, in view of Appendix 3. (Consult [2] for philosophical consolation :-)

**Theorem 13.** [Conjecture] Determinants between Neighbour Nodes

Let the $2^n - 1$ values in levels 1, $\ldots$, $n$ be linearized, i.e. starting with the root, we
place the $2^l - 1$ elements from levels 1, $\ldots$, $l$ between the $2^l$ elements in level $l + 1$, for
$l = 1, 2, \ldots, n - 1$.

Let then $p_k/q_k$ be the value in position $k, 1 \leq k \leq 2^n - 1$, of the linearized sequence
($k$ is not the numerical address here).

Then [we conjecture]

(i) $p_{k+1}q_k - p_kq_{k+1} = 2^e$ for some $e \in \mathbb{N}_0, 1 \leq k \leq 2^n - 2$.

(ii) The exponent $e$ is zero, the value thus 1, except for the following cases:

| Parent | Child | Value | Parent | Child | Value |
|--------|-------|-------|--------|-------|-------|
| $B_x$ or $C_x$ | $C_{e}$ | $2^e$ | $B_{e}$ or $C_{e}$ | $C_{e}$ | $2^e$ |
| $B_{e-1}$ | $B_{e}$ | $2^e$ | $B_{e-1}$ | $B_{e}$ | $2^e$ |
| $A$ | $B_1$ | $2$ | $A$ | $B_1$ | $2$ |

where the states are taken from Appendix 3.

**Proof.** By numerical verification up to level $n = 30$, i.e. for $1 \leq k \leq 2^{30} - 1$. □

**Theorem 14.** Values as weighted Mediants between Neighbour Nodes

Let $p_k/q_k$ as in Theorem 13 and

$$
\Delta^+ = p_{k+1}q_k - p_kq_{k+1}, \quad \Delta^- = p_kq_{k-1} - p_{k-1}q_k.
$$

Let $g = \gcd(\Delta^+, \Delta^-), \Delta_+ = \Delta^+/g, \Delta_- = \Delta^-/g$. Then

$$
\frac{p_k}{q_k} = \frac{\Delta_+ \cdot p_{k-1} + \Delta_- \cdot p_{k+1}}{\Delta_+ \cdot q_{k-1} + \Delta_- \cdot q_{k+1}}.
$$

**Proof.**

$$
\frac{p_k}{q_k} = \frac{p_{k-1}\Delta^+ + p_{k+1}\Delta^-}{q_{k-1}\Delta^+ + q_{k+1}\Delta^-}
\iff [q_{k-1}(p_{k+1}q_k - p_kq_{k+1}) + q_{k+1}(p_kq_{k-1} - p_{k-1}q_k)] p_k
= [p_{k-1}(p_{k+1}q_k - p_kq_{k+1}) + p_{k+1}(p_kq_{k-1} - p_{k-1}q_k)] q_k
\iff pkq_k [q_{k-1}p_{k+1} - q_{k+1}p_{k-1}]
= pkq_k [-p_{k-1}q_{k+1} + p_{k+1}q_{k-1}]
\iff \frac{p_k}{q_k} = \frac{\Delta_+ \cdot p_{k-1} + \Delta_- \cdot p_{k+1}}{\Delta_+ \cdot q_{k-1} + \Delta_- \cdot q_{k+1}}.
$$

**Remark.** The equivalent result for the Stern-Brocot tree is $e = 0, \Delta_+ = \Delta_- = 1$ for all nodes.
Definition 15. For a given binary tree $T$ with labels $p/q \in \mathbb{Q}$ at address $v(p/q) \in A^*$:

(i) Let $\lambda_T(q) = \max_{1 \leq p < q \quad (p, q) = 1} |v(p/q)| / \log_2(q)$ for the last level, such that all irreducible fractions with denominator $q$ are present in levels 1 to $\lambda_T(q) \cdot \log_2(q)$.

(ii) Let $\Lambda_T = \limsup_{q \in \mathbb{N}} (\lambda_T(q))$.

Theorem 16. (Almost) Optimality of the $V_10$ Tree

(i) $\Lambda_T \geq 2$ for any tree $T$.

(ii) $\Lambda_{V_{10}} \geq 2.4007$.

(iii) $\Lambda_{V_{10}} \leq 3.44$.

(iv) For the Stern-Brocot tree, $\lambda_{SB}(q) \geq \frac{q}{\log_2(q)}$, and thus $\Lambda_{SB} = +\infty$.

Proof. (i) There are $\phi(q)$ reduced fractions $p/q, 1 \leq p < q$ in $\mathbb{Q}_1$. Asymptotically, we have $\sum_{k=1}^{q} \phi(k) \approx \frac{1}{2} \cdot q^2 = \frac{1}{\pi} \cdot q^2$ values with denominator $\leq q$. Therefore, for any binary tree we can at best expect to see all quotients with denominators $\leq q$ in the first $2 \log_2(q)$ levels, and thus $\Lambda_T \geq 2$ for any tree $T$.

(ii) The irrational number $r = \sqrt{5} - 2 = 0.236\ldots$ with CFE $\{4, 4, 4, \ldots\}$ has convergents $A_k/B_k$ with asymptotical growth of the denominator $B_k = \Theta(\varphi_k^k) = (4.236\ldots)^k$, and an encoding of $5k$ bits for the first $k$ copies of $b_i = 4$.

Hence, the denominator $q = B_k$ appears (approximately, asymptotically) on level $5k$ in the V tree, where $(\log_2(q) \cdot \alpha \approx) \log_2(4.236^k) \cdot \alpha = 5k \Leftrightarrow \alpha = 5/\log_2(2+\sqrt{5}) = 2.4007\ldots$.

(iii) We need at most $\log(q)/\log(\varphi_1)$ PDs at all, even if they all should be equal to 1.

Also, $q \leq \prod b_i$. We advance in the product by a factor of 2, and 3 coding bits, or faster for other factors: $\log(b)/l_{1,11}(b)$ is minimal for $b = 2$ (except $b = 1$, of course). Hence, we get to the full product $q$ with at most $3 \cdot \log_2(q)$ coding bits for the PDs greater than 1, and at most $1 \cdot (\log_2(q) - \log_2(q))$ bits for additional PDs with value 1 (which do not improve the product, but add to the coding length). Hence, $\log_2(q) \times (2+1/\log_2(\varphi_1)) = 3.44 \log_2(q)$ is the last level, where a denominator $q$ might appear.

(iv) For the Stern-Brocot tree, $\lambda_{SB}(q) \geq \frac{q}{\log_2(q)}$, since $1/q$ is on level $q$. $\Lambda_{SB}$ follows.

Remark. (i) Numerical evidence suggests $\Lambda_V \approx 2.5$.

(ii) Moving the lower bound for $\Lambda$ below 2.35931 = 9/\log_2(\varphi_{14}) = 3/\log_2(\varphi_2)$ (the coincidence stems from $\varphi_{14} = \varphi_2^3$) is impossible with integral wordlengths, since then already $\sum_{b=1}^{64} 2^{-l(b)} > 1$. Hence, our encoding is basically optimal, besides being very regular.

Conjecture 17. Let $f$ be any continuous and monotonically increasing function $f: [0, 1] \to [0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Then the graph $\{(x, f(x)) \mid x \in [0, 1]\}$ has Hausdorff dimension 1 and arc length between $\sqrt{2}$ and 2.

Proof. (idea) We cover the graph by squares of side length $2^{-k}$, for $k \to \infty$, to show the upper bound and the Hausdorff dimension. The lower bound follows from the triangle inequality.

Now, we will state some conjectures about the graph of $?_V$ and $?_V^{-1}$.  

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Conjecture 18. Assuming that the function $?_V(x)$ is continuous and monotonically increasing from $?_V(0) = 0$ to $?_V(1) = 1$, we conjecture that it has Hausdorff dimension 1, and in particular is not fractal.

Proposition 19. The area between the functions and the diagonal on $[0, 1]$ satisfies

$$0.030734101 < \int_0^1 (?_V(x) - x) \, dx = -\int_0^1 (?^{-1}_V(v) - v) \, dv < 0.030734102$$

Proof. The number 0.030734101 . . . results by taking the “Riemann sum” for the $2^k + 1$ arguments $a/2^k, 0 \leq a \leq 2^k$, for $k = 1, \ldots, 30$. The values settle.

Proposition 20. The arc length between $(0, 0)$ and $(1, 1)$ is greater than 1.554.

Proof. The lower bound for the arc length results by taking a polygonal chain through the points $(x, ?_V(x))$ for the $2^k + 1$ values $a/2^k, 0 \leq a \leq 2^k$, for $k = 1, \ldots, 30$, by summing up the length of the polygonal chain (2$^{30}$ diagonals).

Remark. Since any finite number of points is compatible with the upper bound 2, by assuming that the curve also goes through the points $(x_{k+1} - \varepsilon, f(x_k)), \forall k$ for an arbitrarily small $\varepsilon > 0$, we can not improve that upper bound 2 for the arc length (from Conjecture 17) in this way.

Conjecture 21. For all $x \in \mathbb{Q}_1$ (and by continuity in $\mathbb{R}_1$), we have

$$\frac{8}{9} x \leq ?^{-1}_V(x) \leq x \leq ?_V(x) \leq \frac{9}{8} x$$

Proof idea: $?_V(x) - x$ has minima for $?_V(2^{-k}) = 2^{-k}$ and maxima for $?_V(\frac{2}{3} \cdot 2^{-k}) = \frac{3}{4} \cdot 2^{-k}$, and thus $?^{-1}_V(x) - x$ has maxima for $?^{-1}_V(2^{-k}) = 2^{-k}$ and minima for $?^{-1}_V(\frac{4}{3} \cdot 2^{-k}) = \frac{3}{4} \cdot 2^{-k}$. This is, however, only verified numerically on points $a/2^k, k \leq 30$ from $\mathbb{D}_1$.

Remark. Self-similarity of the graph of the function $?_V(x)$ (see Figure 3)

While not fractal, the graph nevertheless exhibits a clear self-similarity:

$$?_V\left(\frac{x}{2}\right) \approx \frac{1}{2} ?_V(x), \forall x \in [0, 1]$$

The dashed lines are the identity $y = x$ and $y = 0$, respectively. The dotted lines touch the local maxima $y = 9/8 x$ and $y = 1/8 x \cdot 12$, respectively.

Conjecture 22. Parabola Conjecture

Apparently, in particular visible for $k = 4$ in red in Figure 3 the function graph is upper-bounded by curves through $(2^{-k}, 0)$ and $(2/3 \cdot 2^{-k}, 3/4 \cdot 2^{-k})$, which actually seem to be parabolas for the inverse function $?^{-1}_V$. We thus conjecture:

For $y \in \mathbb{R}_1, \exists k \in \mathbb{N}_0$ with $\frac{1}{2} \cdot 2^{-k} \leq y \leq 1 \cdot 2^{-k}$. Using this $k$, we conjecture

$$?^{-1}_V(y) \geq \frac{4}{3} 2^k \left(y - \frac{3}{4} \cdot 2^{-k}\right)^2 + y - \frac{1}{12} \cdot 2^{-k}$$

which is met with equality (only) in the three points $\frac{1}{2} \cdot 2^{-k}, \frac{3}{4} \cdot 2^{-k}, 1 \cdot 2^{-k}$.

From $\int_1^{1/4} \left(\frac{1}{12} y^2 - \frac{1}{12}\right) \, dy = -\frac{1}{36}$ and with $1 + 1/4 + 1/16 + \cdots = 4/3$, we have a combined area of $\frac{1}{27} \approx 0.037$, to be compared with the result 0.0307 from Proposition 19.
$V(x)$

Code $y = V(x) \in \mathbb{D}_1$

Distance $\Delta(x)$ from diagonal

$\Delta(x) := (V(x/2^k) - x/2^k) \cdot 2^k \cdot 12$, $k = 0, \ldots, 4$

$k : 0 = \text{black}, 1 = \text{green}, 2 = \text{yellow}, 3 = \text{blue}, 4 = \text{red}$

Figure 3: Function graph and distance from diagonal.
Conjecture 23. The derivative $\overline{T}_V(x)$ does not exist on $\mathbb{Q}_1$
For $p/q \in \mathbb{Q}_1$, $p/q = [b_1, \ldots, b_l] = [b_1, \ldots, b_l - 1, 1], b_l \geq 2$, we have – if defined at all (see example below):
\[
\lim_{x \to (\frac{q}{p})^+} \overline{T}_V(x) = q^2 \cdot 2^{-\alpha_L - \sum_{i=1}^l l_{l, l+1}(b_i)}
\]
and
\[
\lim_{x \to (\frac{q}{p})^-} \overline{T}_V(x) = q^2 \cdot 2^{-\alpha_R - \sum_{i=1}^l l_{l, l+1}(b_i)},
\]
where $\alpha_L \neq \alpha_R$ depend on $l$ and $b_l$:

| $l$ | $b_l$ | $\alpha_L$ | $\alpha_R$ |
|-----|-------|-------------|-------------|
| odd $\neq 2^k$ | 0 | $+1$ | additionally $b_{l+1} = 1$ with $l_{l, l+1}(1) = 1$
| odd $2^k$ | 0 | $-1$ | $l_{l, l+1}(2^k - 1) = l_{l, l+1}(2^k) - 2$, plus $b_{l+1} = 1$
| even $\neq 2^k$ | +1 | 0 | as above, with sides reversed
| even $2^k$ | -1 | 0 |

Therefore, the derivative of $\overline{T}_V(x)$ does not exist at least in rational points (for Minkowski’s $\beta(x)$, we have $\beta'(x) = 0$ for rational $x$, see [5]).

Example. $x_n = \overline{T}_V^{-1}(\overline{T}_V(38/51 \pm 2^{-n}), (38/51) = [1, 2, 1, 12] = [1, 2, 1, 11, 1]$.

\[
\lim_{x \to (\frac{38}{51})^-} \overline{T}_V(x) = \lim_{n \to \infty} \overline{T}_V \left( \frac{38 \cdot 2^n + 35}{51 \cdot 2^n + 47} \right) - \overline{T}_V \left( \frac{38}{51} \right)
\]
\[
= \lim_{n \to \infty} \frac{.1|100|1|1110011|1|1^*00^n|0^* - .1|100|1|1110100|0^*}{51 \cdot 38 \cdot 2^n - 38 \cdot 51 \cdot 2^n + (51 \cdot 3 - 38 \cdot 47)}
\]
\[
= \lim_{n \to \infty} \frac{-(51^2 \cdot 2^n + 47 \cdot 51) \cdot 2^{-n-13}}{51 \cdot 38 \cdot 2^n - 38 \cdot 51 \cdot 2^n + (51 \cdot 3 - 38 \cdot 47)}
\]
\[
= \lim_{n \to \infty} \frac{(-51^2 \cdot 2^{-13} - 2^{-n} \cdot 47 \cdot 51 \cdot 2^{-13})/(-1)}{512} = \frac{1}{473}
\]

\[
\lim_{x \to (\frac{38}{51})^+} \overline{T}_V(x) = \lim_{n \to \infty} \overline{T}_V \left( \frac{38 \cdot 2^n + 35}{51 \cdot 2^n + 47} \right) - \overline{T}_V \left( \frac{38}{51} \right)
\]
\[
= \lim_{n \to \infty} \frac{.1|100|1|11100100|0^n11^n|1^* - .1|100|1|1110100|0^*}{51 \cdot 38 \cdot 2^n - 38 \cdot 51 \cdot 2^n + (51 \cdot 3 - 38 \cdot 47)}
\]
\[
= \lim_{n \to \infty} \frac{829}{1024} + \frac{2^{-n} - 12}{51 \cdot 38 \cdot 2^n - 38 \cdot 51 \cdot 2^n + (51 \cdot 3 - 38 \cdot 47)}
\]
\[
= \lim_{n \to \infty} \frac{51^2 \cdot 2^{-12} + 2^{-n} \cdot 4 \cdot 51 \cdot 2^{-12}}{1} = \frac{51^2}{2} \cdot 12
\]
Remark. From \( \frac{q^2}{2(\ln(q))} \approx \frac{3.42371}{24.507} = 2^{-0.084}t \), by Lévy, for every 12 PDs we should need some \( 3.42371 \cdot 12 \approx 41 \) bits, but actually we need one more, namely \( 3.507 \cdot 12 \approx 42 \) bits. This one more bit every 41 bits is + 2.4\% (compare with Proposition 9 \( \frac{H_{C_{1,1}}}{H_{Lévy}} = 1.024\ldots \)).

Remark. The longer the CFE becomes, the flatter the (one-sided) derivatives at \( p/q \).

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# → ""
Appendices

1. Continued Fraction Expansion

Let \( r \in \mathbb{R} \). Let \([r] \in \mathbb{Z}\) be the largest integer smaller than or equal to \( r \), e.g. \([3.14] = 3\), \([-3.14] = -4\), and let \( \{r\} = r - [r] \in [0, 1) \) be the fractional part. E.g. \{3.14\} = 0.14, \{-3.14\} = 0.86.

The continued fraction expansion of \( r = \frac{r}{1} \) is defined by its successive partial denominators \( b_i \) as \( b_0 := [r_0], r_i := \frac{1}{\{r_{i-1}\}} = \frac{1}{r_{i-1} - \lfloor r_{i-1} \rfloor}, b_i := [r_i] \in \mathbb{N}\), for \( i \in \mathbb{N} \). The continued fraction for \( r \) is then

\[
r = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}} =: [b_0; b_1, b_2, \ldots]
\]

and the convergents \( A_i/B_i \) to \( r \) are obtained by Perron’s schema \([24, \text{Satz 2.10}]\) (Table 3). The initial values are \( B_{-2} = A_{-1} = 1, A_{-2} = B_{-1} = 0 \) and then \( A_i := b_i \cdot A_{i-1} + A_{i-2}, B_i := b_i \cdot B_{i-1} + B_{i-2} \). In particular \( A_0 = b_0, B_0 = 1, A_1 = b_1 A_0 + A_{-1} = b_1 b_0 + 1, B_1 = b_1 B_0 + B_{-1} = b_1 \). We focus on the case \( r \in (0, 1) \) = \( \mathbb{R}_1 \subset \mathbb{R} \), thus \( b_0 = 0 \) (e.g. for \( r = \pi - 3 \) see the second part of Table 3).

| \( i \) | \(-2\) | \(-1\) | \(0\) | \(1\) | \(2\) | \(3\) | \(4\) | \(\ldots\) |
|---|---|---|---|---|---|---|---|---|
| \( b_i \) | \(\_\) | \(\_\) | \(b_0\) | \(b_1\) | \(b_2\) | \(b_3\) | \(b_4\) | \(\_\) |
| \( A_i \) | \(0\) | \(1\) | \(A_0\) | \(A_1\) | \(A_2\) | \(A_3\) | \(A_4\) | \(\_\) |
| \( B_i \) | \(1\) | \(0\) | \(B_0\) | \(B_1\) | \(B_2\) | \(B_3\) | \(B_4\) | \(\_\) |
| \( b_i \) | \(\_\) | \(\_\) | \(7\) | \(15\) | \(1\) | \(292\) | \(\_\) | \(\_\) |
| \( A_i \) | \(0\) | \(1\) | \(0\) | \(1\) | \(15\) | \(16\) | \(4786\) | \(\_\) |
| \( B_i \) | \(1\) | \(0\) | \(1\) | \(7\) | \(106\) | \(113\) | \(33102\) | \(\_\) |

Table 3: CFE Schema according to Perron.

Convergence: For \( r \in \mathbb{R}^+ \), we have

\[ 0 = \frac{A_{-2}}{B_{-2}} \leq \frac{A_0}{B_0} < \frac{A_1}{B_1} < \cdots < \frac{A_r}{B_r} < \cdots < \frac{A_3}{B_3} < \frac{A_4}{B_4} < \frac{A_5}{B_5} < \frac{A_1}{B_1} = \infty \]

and furthermore \( |r - \frac{A_i}{B_i}| < \frac{1}{B_{i+1}} \) [24, Satz 2.10].

Ambiguity: \([b_1, \ldots, b_l] = [b_1, \ldots, b_l - 1, 1]\) and \([b_1, \ldots, b_l, \mathcal{N}_0] = [b_1, \ldots, b_l - 1, 1, \mathcal{N}_0]\)

One can resolve this ambiguity in 4 ways:

(i) Let the last PD be always greater than 1, or
(ii) always equal to 1, or
(iii) have an even, or
(iv) an odd number of PDs

(the final \( \mathcal{N}_0 \) with \( 1/\mathcal{N}_0 := 0 \) in any case does not alter the value).

We shall use convention (iii): The encoding then terminates in \( 0^\omega \) from \( C_1(\mathcal{N}_0) = 0^\omega \).
2. Equivalence between $\mathbb{N}, A^*, \text{ and } D_1$

We identify the word $v \in A^*$ with the number $n = (v)_2 \in \mathbb{N}_0$ in binary representation, and the dyadic fraction $(v|1)_2/2^{[v]+1} \in D_1$.

In particular:

$v = \varepsilon \equiv n = 1 \equiv d = 1/2,$
$v = 0 \equiv n = 2 \equiv d = 1/4,$
$v = 1 \equiv n = 3 \equiv d = 3/4.$

Example:

$v = 10010 \text{ (value 18)} \equiv n = 1|10010_2 = 50 = 18 + 2^{5} \equiv d = 1/2^{5}$
$v = 0 \equiv n = 2|0 = 2 = 1/2^{4}$
$v = 1 \equiv n = 3|1 = 3 = 3/2^{4}$.

We define bijective mappings between the 3 sets $\mathbb{N}, A^*, \text{ and } D_1$ as follows, where $\iota_{XY}^1 = \iota_{YX}$ for $X, Y \in \{N, A, D\}$ and $l := \lfloor \log_2(n) \rfloor$:

$\iota_{NA}: \mathbb{N} \to A^*, \quad \iota_{NA}(n) = n - 2^l \text{ in binary}$
$\iota_{AN}: A^* \to \mathbb{N}, \quad \iota_{AN}(v) = (v)_2 + n + 2^{[v]+1}$
$\iota_{ND}: \mathbb{N} \to D_1, \quad \iota_{ND}(n) = ((n - 2^l) \cdot 2 + 1)/2^l$
$\iota_{DN}: D_1 \to \mathbb{N}, \quad \iota_{DN}(p/2^k) = (p - 1)/2 + 2^{k-1}$
$\iota_{AD}: A^* \to D_1, \quad \iota_{AD}(v) = (v|1)_2/2^{[v]+1}$
$\iota_{DA}: D_1 \to A^*, \quad \iota_{DA}(p/2^k) = 0^{k-[p]-2}((p - 1)/2) \text{ in binary}$

3. Finite State Machine

Let $Q = \{A, B_k, C_k, \overline{A}, \overline{B}_k, \overline{C}_k \mid k \in \mathbb{N}\}$ be the state set for an FSM with nextstate function $Q^+: Q \times A \to Q$ given by:

| $q$ | $Q^+(q, 0)$ | $Q^+(q, 1)$ |
|-----|-------------|-------------|
| $A$ | $B_1$       | $A$         | Start for $C_1$ |
| $B_k$ | $B_{k+1}$ | $C_k$       | Increase PDs as 1, 2, 4, 8, 16, … |
| $C_k, k \geq 2$ | $C_{k-1}$ | $C_{k-1}$ | Adjust PDs by ±16, 8, 4, 2 |
| $C_1$ | $\overline{A}$ | $\overline{A}$ | Adjust PD by ±1, switch to $C_{II}$ |
| $\overline{A}$ | $A$ | $\overline{B}_1$ | Start for $C_{II}$ |
| $\overline{B}_k$ | $\overline{C}_k$ | $\overline{C}_{k+1}$ | Increase PDs as 1, 2, 4, 8, 16, … |
| $\overline{C}_k, k \geq 2$ | $\overline{C}_{k-1}$ | $\overline{C}_{k-1}$ | Adjust PDs by ±16, 8, 4, 2 |
| $\overline{C}_1$ | $A$ | $A$ | Adjust PD by ±1, switch to $C_1$ |
4. Trees and Addresses

The numerical address \( n \in \mathbb{N} \) and the symbolic address \( v \in A^* \) are related by \( n = (1v)_2 \) in binary, see Appendix 2. *E.g.* on the last line we see \( n = 23 \) and \( v = 0111 \), with \( 23 = (1|0111)_2 \) Note that the left child node has \( n_L = 2n \) and \( v_L = v0 \), the right one \( n_R = 2n+1 \) and \( v_R = v1 \). The dyadic fraction is \( a/2^k \in \mathbb{D} \), \( a \) odd, with \( a/2^k = (v1)_2/2^{|v|+1} \), and it comes from the van der Corput sequence in base 2 (see [15, p. 127]), which is just \( A^* \), the words written from right to left: \( \varepsilon, 0,1,00,10,01,11,000,100,010,110,001,101,011,111,0001, \ldots \).

Here, the dyadic fraction is \( (0111|1)_2/2^{(0111)+1} = 15/32 \). The three entries of the upper part coincide according to Appendix 2.

The bottom part consists of the two values from the Stern-Brocot tree and from the \( V_{10} \) tree. The entry here is 4/9 for both trees. Using \( t_{DA} \) from Appendix 2, we can say that \( ?^{-1} \circ t_{DA} \) maps the van der Corput tree to the Stern-Brocot tree, and \( ?_{V}^{-1} \circ t_{DA} \) maps the van der Corput tree to the \( V_{10} \) tree, entry by entry.

---

**Figure 4:** Addresses and trees: van der Corput, Stern-Brocot, and \( V_{10} \) tree.
5. $V_{10}$ Values and their PDs and Encodings, for $|v| \leq 5$

| $v.10^\infty$ | PDs | $A/B$ | $r = (A/B)_2$ | $v.10^\infty$ | PDs | $A/B$ |
|---|---|---|---|---|---|---|
| $\varepsilon.1[0]$ | 1[1] | 1/2 | .1(0) | 0.10[0] | 3[1] | 1/4 | .01(0) | 0.0001.1000[0] | 63[1] | 1/64 |
| [.100] | 1[2] | 2/3 | .(10) | 0.100[0] | 7[1] | 1/8 | .001(0) | 0.0011.1000[0] | 23[1] | 1/24 |
| 00.10[0] | 2[1] | 1/3 | .(01) | 01.1[0] | 1[1] | 3/5 | .(101) | 010101.1000 | 6[2] | 2/13 |
| 1[0].10[0] | 1[1] | 3/5 | .(101) | 1[1].1000 | 1[4] | 4/5 | .(1100) | 00110.1000 | 5[2] | 2/11 |
| 000.1000[0] | 15[1] | 1/16 | .0001(0) | 00111.1000 | 4[2] | 2/9 |
| 001.10[0] | 5[1] | 1/6 | .00(10) | 010[0]0.10[0] | 3[1]2[1] | 4/15 |
| 010.1000 | 3[2] | 2/7 | .(010) | 010101.1000 | 3[1]12[1] | 7/25 |
| 011.1000 | 2[2] | 2/5 | .(0110) | 01010.10 | 3[3] | 3/10 |
| 1[0].010[0] | 1[1]3[1] | 5/9 | .(100011) | 101[0]1.1000 | 3[8] | 8/25 |
| 1[0].10[0] | 1[1]2[1] | 5/8 | .101(0) | 1[0]100|1.010[0] | 2[1]3[1] | 4/11 |
| 1[0].1 | 1[3] | 3/4 | .11(0) | 110101.1000 | 2[1]12[1] | 5/13 |
| 1[1].100000 | 1[8] | 8/9 | .(111000) | 011101.10 | 2[3] | 3/7 |
| 0000.100000[0] | 31[1] | 1/32 | .00001(0) | 011111.100000 | 2[8] | 8/17 |
| 0001.1000[0] | 11[1] | 1/12 | .00(01) | 100000.100000 | 1[8]1 | 10/19 |
| 0010.10[0] | 6[1] | 1/7 | .(001) | 100010.10 | 1[1]5[1] | 7/13 |
| 0011.10[0] | 4[1] | 1/5 | .(0011) | 100101.1000 | 1[1]3|2[1] | 9/16 |
| 0100.01[0] | 3[1]1[1] | 3/11 | .(0100010111) | 101101.1000 | 1[1]2[1] | 7/12 |
| 0101.1000 | 3[4] | 4/13 | .(010011101100) | 1010101.0100 | 1[1]113[1] | 14/23 |
| 0110.10[0] | 2[1]1[1] | 3/8 | .01(0) | 1010101.1000 | 1[1]111[2] | 13/21 |
| 0111.1000 | 2[4] | 4/9 | .(011100) | 1010101.1 | 1[1]13 | 7/11 |
| 1000.0100[0] | 1[1]7[1] | 9/17 | .(10000111) | 10111.1000 | 1[1]18 | 17/26 |
| 1001.01[0] | 1[1]2[1] | 4/7 | .(100) | 10100010 | 1[2]3[1] | 9/14 |
| 1010.10[0] | 1[1]11[1] | 8/13 | .(10011101100) | 1010101.0100 | 1[2]12 | 8/11 |
| 1011.1000 | 1[1]14 | 9/14 | .(101) | 101101010 | 1[3]33 | 13/17 |
| 1100.10[0] | 1[2]1[1] | 5/7 | .(101) | 11011.1000 | 1[3]12 | 11/14 |
| 1101.10[0] | 1[3]11 | 7/9 | .(110001) | 11100.1 | 1[5] | 5/6 |
| 1110.10 | 1[6] | 6/7 | .(110) | 11101.1 | 1[7] | 7/8 |
| 1111.100000 | 1[16] | 16/17 | .(11110000) | 11110100 | 1[12] | 12/13 |
| 11111.100000 | 1[32] | 32/33 | |

Table 4: Binary CFE and approximations.
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