Analytic Approach for Controlling Realistic Quantum Chaotic Systems

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Abstract. An analytic approach for controlling quantum states, which was originally applied to fully random matrix systems [T. Takami and H. Fujisaki, Phys. Rev. E 75, 036219 (2007)], is extended to deal with more realistic quantum systems with a banded random matrix (BRM). The validity of the new analytic field is confirmed by directly solving the Schrödinger equation with a BRM interaction. We find a threshold of the width of the BRM for the quantum control to be successful.

Keywords: Quantum Control, Rabi Oscillation, Resonant Field, Rotating Wave Approximation

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INTRODUCTION

Theoretical and experimental studies of controlling quantum states have been attracting much attention because of the theoretical progress in the field of quantum computing [1] and of the technical developments in manipulating atomic and molecular systems. Various approaches have been applied to such quantum systems and its target is often "simple" such as one-state-to-one-state control via adiabatic passage [2]. When we consider to design quantum devices with a large number of states interacting with a complex environment or with short-time laser pulses, however, a multi-state-to-multi-state control problem is not an exception but a rule. Optimal control theory [2] and genetic algorithms [3, 4] are most successful methods to solve this kind of complicated problem, but its implementation and interpretation can be still difficult.

From theoretical points of view, a "complex" quantum system may be modeled by fully random matrix systems with generic properties [5] or by quantum chaos systems where their classical limit shows chaotic properties [6]. Gong and Brumer applied the coherent control method [7, 8] to a quantum chaos system [9, 10] and its prediction has been recently confirmed by experiment. We derived an analytic optimal field to control the fully random matrix systems [11] and the results are promising. However, the most realistic quantum systems may be modeled with a banded random matrix [5] and our previous method can not be directly applied to such a more general situation.

In this contribution, we improve our previous approach [11] to deal with more realistic quantum systems with a banded random Hamiltonian. A new analytic field for optimal control is introduced and its validity is evaluated by numerically solving the Schrödinger equation for the multi-state-to-multi-state control problem.

ANALYTIC EXTERNAL FIELD FOR CONTROLLING QUANTUM STATES

In the previous works [11, 12, 13], we studied quantum control dynamics for a Hamiltonian driven by an external field $\varepsilon(t)$,

$$H[\varepsilon(t)] = H_0 + \varepsilon(t)V$$

where $H_0$ and $V$ are fully random matrices. For initial and target states in the eigenstate representation of $H_0$,

$$|\Phi_0\rangle = \sum c_j|\varphi_j\rangle, \quad |\Phi_T\rangle = \sum d_k|\varphi_k\rangle$$

an analytic optimal field was derived as

$$\varepsilon(t) = \frac{\pi\hbar}{T|V|^2} \sum_j \sum_k \text{Re} \left[ c_j^* V_{jk} d_k e^{iw_j k t} \right], \quad V_{jk} = \langle \varphi_j | V | \varphi_k \rangle, \quad w_{jk} = \frac{E_j - E_k}{\hbar}$$

(3)
where $V_{jk}$ are elements for a fully random matrix with $|V|^2 \equiv \langle |V_{jk}|^2 \rangle$. Here $\langle \cdot \rangle$ denotes an ensemble average. Driven by this field, the quantum state $|\psi(t)\rangle$ is shown to be steered from an initial state $|\Phi_0\rangle$ at $t = 0$ to a target state $|\Phi_T\rangle$ at $t = T$ according to

$$|\psi(t)\rangle = \sum_j a_j(t)|\varphi_j\rangle e^{i\varphi_j t} = \hat{U}_0(t, 0)|\Phi_0\rangle \cos \left(\frac{\pi t}{2T}\right) - i\hat{U}_0(t, T)|\Phi_T\rangle \sin \left(\frac{\pi t}{2T}\right),$$  

(4)

where $\hat{U}_0(t_2, t_1)$ is a propagator generated by $H_0$.

We can summarize the reason why the analytic field works well for fully random matrix systems:

- The random phase property of $c_j$ and $d_j$ is used, where the initial and target states are represented in linear combinations of many eigenstates $|\varphi_j\rangle$. If we introduce an approximate number of states, $N \sim \sum |c_j|^2 \sim \sum |d_j|^2$, sums of complex numbers, $\sum c_j^2$, $\sum d_j^2$, $\sum c_j^*d_j$, etc., are quantities with an order $O(1/N)$, while $\sum |c_j|^2 = \sum |d_j|^2 = 1$ from the normalization condition. Thus, quantities of $O(1/N)$ can be ignored compared to those of $O(1)$ for $N \rightarrow \infty$.

- The rotating wave approximation is applicable, which is valid when the control field amplitude is small enough. This situation is satisfied when $T \rightarrow \infty$.

Using these properties, we proved the validity of our analytic control field for fully random matrix systems \cite{11, 12, 13}.

**EXTENSION FOR BANDED RANDOM MATRIX SYSTEMS**

We consider the case that the interaction Hamiltonian $V$ is a banded random matrix in the eigenstate representation of $H_0$. The elements of $V$ are random complex numbers with distribution

$$\langle |V_{jk}|^2 \rangle \equiv \langle |\langle \varphi_j | V | \varphi_k \rangle|^2 \rangle = \exp \left[-\frac{(E_j - E_k)^2}{\Delta_0^2}\right].$$  

(5)

We introduce an analytic optimal field as an extension of the analytic field for fully random matrix systems,

$$\varepsilon(t) = \sum_{jk} \text{Re} \left[A_{jk} c_j^* V_{jk} d_k e^{i\varphi_j t}\right],$$  

(6)

with an extra-amplitude factor $A_{jk}$. The coefficients $a_j(t)$ satisfy the Schrödinger equation

$$i\hbar \frac{d}{dt} a_k(t) = \frac{1}{2} \sum_j \left[A_{jk} c_j^* d_k + A_{kj}^* c_j d_j^* \right] |V_{jk}|^2 \ a_j(t)$$  

(7)

under the rotating-wave approximation. If we assume that the transition is smooth, $a_j(t)$ should be written as

$$a_k(t) = c_k \cos \left(\frac{\pi t}{2T}\right) - i d_k \sin \left(\frac{\pi t}{2T}\right).$$  

(8)

Substituting these coefficients into Eq. (7), we obtain a relation

$$\frac{i\hbar}{2T} \left[-c_k \sin \left(\frac{\pi t}{2T}\right) - i d_k \cos \left(\frac{\pi t}{2T}\right) \right] = \frac{1}{2} \sum_j \left[A_{jk} |c_j|^2 d_k \cos \left(\frac{\pi t}{2T}\right) - i A_{kj}^* c_j |d_j|^2 \sin \left(\frac{\pi t}{2T}\right) \right] |V_{jk}|^2.$$  

(9)

under the assumption of random phases. Finally, we obtain conditions for the coefficients $A_{jk}$

$$A_{jk} = A_{kj}^* \sim \frac{\pi \hbar}{T} \exp \left[-\frac{(E_j - E_k)^2}{\Delta_0^2}\right].$$  

(10)

If we consider the case that those coefficients $c_j$ and $d_k$ have Gaussian distribution functions in the energy space,

$$\langle |c_j|^2 \rangle \propto \exp \left[-\frac{(E_j - E_0)^2}{\Delta^2}\right], \quad \langle |d_k|^2 \rangle \propto \exp \left[-\frac{(E_k - E_0)^2}{\Delta_d^2}\right].$$  

(11)
Steered by an External Field

FIGURE 1. Left: Schematic picture of the multi-state-to-multi-state control process. Right: An example of the interaction Hamiltonian $V$ ($128 \times 128$, $\Delta_0 = 32$), which is assumed to be a banded random matrix in the energy representation of $H_0$.

with centers $E_c$ and $E_d$ and widths $\Delta_c$ and $\Delta_d$, we can define the analytic optimal field as

$$\epsilon(t) = \frac{\pi \hbar}{T} \sum_{jk} \text{Re} \left[ c_j^* V_{jk} d_k e^{i\omega_{jk} t} \right] \exp \left[ \frac{(E_j - E_k)^2}{\Delta_0^2} \right].$$

This field has a finite amplitude only when

$$\Delta_c < \Delta_0 \quad \text{and} \quad \Delta_d < \Delta_0.$$  \hspace{1cm} (13)

If not, the field has an infinite amplitude in the limit of $E_j, E_k \rightarrow \pm \infty$ by the exponential factor $A_{jk}$ in Eq. (12). Thus, the analytic field is refined when the widths of the initial and target states are relatively small compared to the width of the banded random matrix elements.

### Numerical evaluation

We shall confirm the validity of the new optimal field for the system with a banded random-matrix interaction. The numerical test is configured as follows (see Figure 1). The initial and target states are defined as quantum vectors (2) with random complex coefficients $c_j$ and $d_j$ subject to (11). Here, we choose $\Delta_c = \Delta_d = 32$, $E_c = -16$, and $E_d = 16$, where $H_0$ is a $128 \times 128$ random matrix of the Gaussian orthogonal ensemble and is scaled so that its eigenvalues $\{E_j\}$ are distributed in an interval $[-64, 64]$. The interaction Hamiltonian $V$ is also a $128 \times 128$ matrix while its elements obey a banded-random distribution (5) in the eigenstate representation of $H_0$ with $\Delta_0 = 32$.

The optimal field (12) is calculated from those quantities $\{c_j\}$, $\{d_k\}$, $\{V_{jk}\}$, and $\{E_j\}$ with parameters $T$ and $\Delta_0$. In order to check the validity of our optimal field (12), we solve the initial value problems with Hamiltonian (1) driven by the optimal field (12) for various band widths $\Delta_0$ of the interaction Hamiltonian $V$. The results are shown in Figure 2. When we use the original analytic field (3), the performance of the optimal field (dashed curve) decreases for the banded matrices with smaller widths. On the other hand, the final overlaps (solid curve) by the refined analytic field (12) does not change even for the smaller width until the limit $\Delta_0 \approx \Delta_c$ or $\Delta_0 \approx \Delta_d$. 

FIGURE 2. Numerical results for the multi-state-to-multi-state control problem. The solid (dashed) curve represents the result by the new optimal field Eq. (12) (by the previous optimal field with $A_{jk} = 1$). (a) The final overlap is shown as a function of $\Delta_0$ (width of V). (b) The amplitude (defined as a time average of the absolute square of a field) of the optimal field as a function of $\Delta_0$.

CONCLUSION

We extended our previous analytic approach for controlling complex quantum systems to deal with more realistic systems with a banded random matrix. The key ingredient is the amplitude factor $A_{jk}$, which is an exponentially growing function, introduced in the analytic optimal field Eq. (12). We showed that the new analytic optimal field outperforms the previous optimal field for the multi-state-to-multi-state control problem. Interestingly we found a threshold of the width of the banded random Hamiltonian $\Delta_0$ for the control to be successful: $\Delta_0 \approx \Delta_c + \Delta_d$, where the latter are the width of the energy spreading of the initial and final states. In the near future we will apply this optimal field to quantum chaos systems such as quantum kicked rotors (tops) and to more realistic molecular systems [14].

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