Review

Computational performance of Free Mesh Method applied to continuum mechanics problems

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Abstract: The free mesh method (FMM) is a kind of the meshless methods intended for particle-like finite element analysis of problems that are difficult to handle using global mesh generation, or a node-based finite element method that employs a local mesh generation technique and a node-by-node algorithm. The aim of the present paper is to review some unique numerical solutions of fluid and solid mechanics by employing FMM as well as the Enriched Free Mesh Method (EFMM), which is a new version of FMM, including compressible flow and sounding mechanism in air-reed instruments as applications to fluid mechanics, and automatic remeshing for slow crack growth, dynamic behavior of solid as well as large-scale Eigen-frequency of engine block as applications to solid mechanics.

Keywords: finite element method (FEM), free mesh method (FMM), enriched free mesh method (EFMM), meshless method, solid and fluid mechanics, parallel computing

1. Introduction

Computational Mechanics has been widely employed since the middle of the last century in various fields of science and engineering with great success because of the inherent versatility of the method.1) One of the most serious concerns of the method, however, is the cost of the meshing. As well known, the computer simulation analyst spends much time in creating the mesh in such computational mechanics as the finite element method (FEM), which becomes a major part of the cost of a simulation process. Since the cost of central processing unit (CPU) time is drastically decreasing, the issue is more the man time, and less the computer time. It is natural that the concept of meshless method has evolved,2)–6) which is the idea of reducing the reliance on the elements and more flexible ways to make use of mesh.

The free mesh method (FMM)7) is one of the earliest FEM-based meshless methods intended for particle-like finite element analysis of problems that are difficult to handle using global mesh generation. FMM is a node-based finite element method that employs a local mesh generation technique and a node-by-node algorithm for the finite element calculations.

The theoretical background, the algorithm and the accuracy of the method have been reviewed by the present author.8) Following this, we review, here, some unique numerical examples of FMM and the Enriched Free Mesh Method (EFMM), which is a new version of FMM, selected from among fluid and solid mechanics.9)–27)

In the following section, two kinds of solutions for fluid mechanics are discussed: one is the rather basic example on the shock wave, which is intended to study the preliminary parallel performance, and the other is more practical one for the clarification of the mechanism of sound generation and the primary factor that determines tone. Here, the sounding mechanism in air-reed instruments is investigated based on the computational fluid dynamics, assuming a sound source based on the acoustic theory. As well known, air-reed instrument is a kind of the aerophone such as pipe organs, flutes, recorders or bamboo flutes that neither have mechanically oscillating parts nor sound by human
lip like a brass instrument. Instead, these instruments have a wedge-shaped structure called “edge”. The blown jet causes self-excited oscillation by the edge and a sound is generated. The mechanism of the sound-generation has been studied from an acoustic point of view, and precise experiments have revealed the oscillating sheet behavior of the air jet and the feedback mechanism between the jet and the resonance tube. However, since studies of sound generation based on acoustical theory have assumed a fluid behavior, which is essential of air-reed instruments sounding, being wave, analysis results become inaccurate due to the reason that it is hard to analyze the oscillation with large amplitude and the start of sounding as only single frequency with small amplitude has been treated.

To reveal the mechanism of sound generation in air-reed instruments in detail, the computational fluid dynamics could be used, which will clarify the effects of the performance conditions and the shape of instruments in addition to the precise harmonic makeup of the generated sound. Moreover, as the sound of an instrument changes rapidly in the first ten milli-seconds when the standing wave is established and this change is audible to humans as “sound quality”, it is necessary to perform numerical simulations with high accuracy, particularly for the onset of sound.

Therefore, the purpose of the present study is to perform a numerical analysis of sound generation in a air-reed instrument, and to investigate the mechanism of sound generation and primary factor that determines “tone”. It is considered that the quantitative numerical simulation of musical instrument sound will lead to the establishment of a scientific measure for designing musical instruments. Also, this study will be directly applied to the design of rapid transit railway, car, airplane, electric device, etc., in which the fluid noise is of primary concern.

Section 3 discusses three kinds of numerical solutions applying FMM and EFMM to solid mechanics problems, which include analyses of slow crack propagation, dynamic behavior of elastic body and eigen-value of practical structure with rather large degrees of freedom.

In the slow crack growth analysis, the finite element mesh is robustly created during the propagation process of crack without any special treatment for crack tip. Namely, automatic simulation of crack propagation is realized here only with predistributed nodes within the domain. Surface patch data is given to define the boundaries of the object. Next, internal nodes are distributed within the analysis domain, where the nodes in the neighborhood of the crack tip are arranged with the special nodal pattern, which is moved as the crack propagates. The local mesh is generated around each node. Since the satellite elements associated with each node can be determined in a local manner using the coordinates of the neighboring nodes and surface patch, the local mesh generation is easily parallelized in terms of nodes. The process of local mesh generation is repeated as the crack propagates.

FMM and EFMM are, then, applied to a simple three-dimensional (3-D) and dynamic elasticity problem for the purpose of accuracy check of these methods, where the solutions are compared between the present solutions and available theoretical ones, and also to a rather large scale eigen-value problem of practical structure. As well known, eigen-value problems occur naturally in the vibration analysis of structures with many degrees of freedom. The eigen-values are used to determine the natural frequencies (or eigen-frequencies) of vibration, and the eigenvectors determine the shapes of these vibrational modes. The accurate computation of eigen-frequencies of elastic domain is of high engineering importance because the smallest frequencies have to be higher compared with any frequency of a dynamically applied load. Otherwise, resonance may occur, resulting in the amplification of the eigen-mode and finally the destruction of the structure. This is of particular importance in 3-D thin domains, such as 3-D plates, where the smallest eigen-frequencies are proportional to the thickness. Due to the complexity of a 3-D eigen-analysis, much attention has been given historically to the derivation of plate models. In engineering practice, the Reissner–Mindlin (R–M) plate model is frequently used as an approximation of the 3-D thin plate domain, and is assumed to be valid for thicknesses of plates under 5% compared with other dimensions.

In this context, an investigation of the high quality 3-D analysis of eigen-frequencies is naturally one of the most important research areas in solid mechanics. We discuss here a 3-D eigen-frequency analysis by FMM and EFMM with the Lanczos method. It is well known that the Lanczos method can obtain a limited number of eigen-frequency in high accuracy with less CPU time and is considered as a suitable method for solving the large-scale eigen-value problems for practical problems.

In the final section, the present review paper is concluded with some remarks.
2. Applications to fluid mechanics

2.1. Compressible flow analysis. The compressible Euler equation for two-dimensional flows can be written as

$$\frac{\partial U}{\partial t} + \frac{\partial F_i}{\partial x_i} = 0 \quad (i = 1, 2) \quad [1]$$

where

$$U = [\rho, \rho u_1, \rho u_2, \rho e]^T$$

$$F_i = [\rho u_i, \rho u_i u_1 + \rho \delta_{i1}, \rho u_i u_2 + \rho \delta_{i2}, u_i (\rho e + p)]^T$$

Here, the subscript $i$ specifies the co-ordinate axis, the summation convention over repeated subscripts is assumed, $u_1$ and $u_2$ denote the velocity components with respect to the Cartesian co-ordinate system, $\rho$ the density, $p$ the pressure, $e$ the specific total energy and $\delta_{ij}$ the Kronecker’s delta, respectively. Assuming that the fluid is an ideal gas, the constitutive relation is the equation of state, which can be written as

$$p = (\gamma - 1) \rho \left( e - \frac{1}{2} u_i^2 \right) \quad [2]$$

where $\gamma$ is the ratio of specific heats.

Solution algorithm. The governing equation above is discretized by the two-step Taylor–Galerkin method. The shock wave, which is a characteristic phenomenon of compressible flows and a cause of instability in numerical solutions, is stabilized by adding artificial viscosity based on the Lapidus model as follows.

First step: From the Taylor expansion of unknown values $U$ about time $t = t_n$, we obtain

$$U_{n+1/2}^E \int_{\Omega_E} d\Omega = \sum_\beta \left( \int_{\Omega_E} N_\beta d\Omega \right) U_{n, \beta}^n - \frac{1}{2} \Delta t \sum_\beta \left( \int_{\Omega_E} \frac{\partial N_\beta}{\partial x_i} d\Omega \right) F_{n, \beta}^{n, 1/2} \quad [3]$$

where $\Omega_E$ indicates the domain of element $E$, $N_\beta$ is the piecewise linear shape function associated with node $\beta$ and $\Delta t$ is the timestep. Since integration of the right-hand side of Eq. [3] involves only the element $E$, the intermediate unknown value $U_{n+1/2}^E$ can be calculated from the nodal values of the element $E$ alone. Therefore, the calculation can be carried out independently for each element.

Second step: Using the intermediate unknown value $U_{n+1/2}^E$ obtained in the first step, the change in unknown values from time $t = t_n$ to time $t = t_{n+1}$ can be evaluated as

$$\sum_\beta \left( \int_{\Omega} N_\beta N_\beta d\Omega \right) \delta U_\beta
= \Delta t \sum_\varepsilon \left( \int_{\Omega} \frac{\partial N_\beta}{\partial x_i} P_\varepsilon d\Omega \right) F_{n+1/2}^{n+1/2}
= \Delta t \sum_\varepsilon \left( \int_{\Gamma} l_i N_\beta P_\varepsilon d\Gamma \right) F_{n+1/2}^{n+1/2} \quad [4]$$

Here, $\delta U_\beta = U_{n+1}^\beta - U_n^\beta$ and the divergence theorem has been employed, introducing the integral over the surface $\Gamma$ of the analysis domain $\Omega$ with $(l_i, l_i)$ being the direction cosines of the outward normal to $\Gamma$. $P_\varepsilon$ is the piecewise constant shape function associated with the element $\varepsilon$. Equation [4] can be expressed in a matrix form as

$$M \delta U = \Delta t (f_1 - f_1) \quad [5]$$

where $M$ is a mass matrix, $\delta U = U^{n+1} - U^n$ is a vector of the increment of nodal values $U$, and $f_1$ and $f_1$ are vectors due to the domain and surface integrals, respectively. Because both integrals on the right-hand side of Eq. [4] contain the shape function $N_\alpha$ as a weighting function, only elements including node $\alpha$ have non-zero values, all other elements contributing nothing to the summations in the right-hand side. Therefore, the components of the right-hand vector associated with the node $\alpha$ can be calculated from nodal values of the central node $\alpha$ and the associated satellite nodes.

Equation [5] is solved explicitly in the present study. Using the standard lumped mass matrix $M_L$, the equation is solved as

$$\delta U = M_L^{-1} (f_1 - f_1) \quad [6]$$

Then, the nodal values at time $t = t_{n+1}$ are obtained as

$$U^{n+1} = U^n + \delta U \quad [7]$$

In order to calculate the value of central node $\alpha$ at the next time step $t = t_{n+1}$, only the value of central node $\alpha$ at time step $t = t_n$ is required. From viewpoint of parallel computing, this means the new values of the central nodes can be calculated independently by each node.

Stabilized unknown values $U_{n+1}^E$ can be calculated based on the Lapidus model as

$$M_{L\alpha} (U_{n+1}^{\alpha} - U_n^{\alpha})
= \Delta t \sum_i \int_{\Omega} \frac{\partial}{\partial x_i} \left( V_i^{n+1} \frac{\partial U^{n+1}}{\partial x_i} \right) N_\alpha d\Omega \quad [8]$$

where
\[ V_{n+1} = C \text{Lap} h_e^2 \left( \frac{\partial u_{n+1}^{(i)}}{\partial x_i} \right) \]  

Here, \( C \text{Lap} \) is the Lapidus constant and \( h_e \) is the representative length of the element \( e \). Stabilized unknown values of the central node \( \alpha \) can be calculated from the values of the central node \( \alpha \) and the associated satellite nodes, owing to the shape function \( N_\alpha \) in Eq. [8].

Numerical example. A numerical example is shown to demonstrate the performance of the presented method in parallel computing. The model under consideration is a supersonic flow over a forward facing step\(^{43}\) as shown in Fig. 1.

A total number of nodes used for the calculation is 25,601, which are distributed uniformly within the wind tunnel. The fluid is assumed to be air and a ratio of specific heats \( \gamma = 1.339 \) is taken. Inflow speed is set to Mach 3. Initial conditions similar to those used by Woodward and Colella\(^{43}\) are applied. The boundary at the solid wall is assumed to be the slip condition. The Lapidus constant is set to \( C \text{Lap} = 2.0 \), which is equal to that of the study by Löhrner and others.\(^{42}\)

Local meshes around each node are first generated in a parallel way, with allocations of computational loads to each processor then optimized by renumbering operations using the ParMETIS library.\(^{44}\) Flow simulation using the aforementioned solution algorithm is then performed. The density contours obtained are shown in Fig. 2, whereas Fig. 3 shows the speed-up versus number of processors recorded using the Hitachi SR8000. Note that a high parallel efficiency is achieved by simply dividing computational loads between the processors.

2.2. Analysis of sounding mechanism in air-reed instruments.\(^{19,22}\) Governing equations and solution algorithm. As the variation in pressure due to the passage of sound is as small as the order of \( 10^{-3} \)

Fig. 1. Model and boundary conditions.

Fig. 2. Time variation of density contours: (a) \( t = 0.5 \), (b) \( t = 1.0 \), and (c) \( t = 2.0 \).

Fig. 3. Parallel performance on Hitachi SR8000.
to $10^{-5}$ times the static pressure of the ambient fluid, it is difficult to capture both the static pressure of the fluid and the sound pressure simultaneously with satisfactory accuracy. Furthermore, the simulation of sound propagation as a compressible fluid phenomenon in real-world requires enormous computational power. Solving a low Mach number flow using the compressible Navier–Stokes equation numerically also involves the stiffness problem and the numerical solution may not converge or accuracy may be degraded.

In this study, a hybrid solution method is adopted, in which sound generation and propagation are analyzed separately: The behavior of air is computed assuming incompressible flow using the computed pressure change, and generated sound is computed by aerodynamic sound prediction based on the Lighthill theory.\(^{45}\) FMM with the fractional step method\(^ {46}\) is adopted as the numerical analysis scheme of the fluid.

The Navier–Stokes equation assuming negligible external force is given as follows,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u}$$  \[10\]

$$\nabla \cdot \mathbf{u} = 0$$  \[11\]

where \(u\) and \(p\) are normalized velocity and pressure, respectively, and \(\text{Re}\) is the Reynolds number. Discretizing temporally Eqs. [10] and [11] by the fractional step method, we have

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n = -\nabla p^{n+1} + \frac{1}{\text{Re}} \Delta \mathbf{u}^n$$  \[12\]

$$\nabla \cdot \mathbf{u}^{n+1} = 0$$  \[13\]

where \(n\) is the time step. Then, the intermediate flow velocity \(\mathbf{u}\) is defined as follows,

$$\mathbf{\tilde{u}} = \mathbf{u}^n - \Delta \mathbf{u} = \left\{ (\mathbf{u}^n \cdot \nabla) \mathbf{u}^n - \frac{1}{\text{Re}} \Delta \mathbf{u}^n \right\}$$  \[14\]

where \(\Delta t\) is the time increment. By taking the divergence of Eq. [12] with Eq. [13], we obtain the following Poisson’s equation for pressure,

$$\Delta p^{n+1} = \frac{1}{\Delta t} \nabla \mathbf{\tilde{u}}$$  \[15\]

Then, by subtracting Eq. [14] from Eq. [12], the equation for velocity is obtained as follows,

$$\mathbf{u}^{n+1} = \mathbf{\tilde{u}} - \Delta t \nabla p^{n+1}$$  \[16\]

The above procedures are summarized as follows. First, the intermediate flow velocity \(\mathbf{\tilde{u}}\) is calculated using Eq. [14], and pressure \(p^{n+1}\) by substituting \(\mathbf{\tilde{u}}\) for Eq. [15]. Then, velocity \(\mathbf{u}^{n+1}\) is calculated by substituting the pressure for Eq. [16]. Both velocity and pressure fields are interpolated using linear basis. In air-reed instruments, the Mach number of air jets around the edge is at most 0.06. Under this condition, if the distance between the sound source and the observation point is sufficiently large, the Curle’s equation,\(^ {37}\) a solution of the Lighthill’s theory, can be given as

$$p_s(x, t) = \frac{1}{4\pi c_0} \frac{x}{x^2} \int_S n_i p(y, t - x/c_0) dS$$  \[17\]

where \(x\) is the coordinate of the observation point, \(p\) is the pressure against the object surface, \(S\) is the object surface, \(x = |x|\), \(y\) is the coordinate in the analysis domain, \(n_i\) is the outward normal vector on \(S\), and \(c_0\) is the speed of sound in the static fluid. If the pressure of edge surface is found, the sound pressure at the observation point can be determined using Eq. [17].

Large-scale computational grids are required to capture microscopic phenomenon such as sound generation. Due to the extremely heavy computations involved, parallel computing is indispensable. In the present study, we employ, effectively, a node-based parallelization of FMM.\(^ {7,11,17}\)

**Numerical analysis of edge tone.** Figure 4 shows the three-dimensional analysis model of fluid around the reed-edge, which is divided into 4-node tetrahedral elements, a total of 163,510 nodes or 918,580 elements (see Fig. 5). Nodes are densely concentrated around the nozzle and the edge in order to perform stable and accurate fluid analysis. The inflow condition is given at the left hand side of the nozzle, a free outflow condition \((p = 0)\) is given at the top, the bottom and the right hand side of the model, and a no-slip condition is assumed for the rest of the model. Analysis is performed varying velocities of air jets injected into the nozzle \((3.0, 5.0\) and \(7.0\) [m/s], respectively). The conditions common to all the cases are listed in Table 1, and the detailed analysis conditions for each case in Table 2, respectively. The representative length is set to be \(1.0^{-3}\) [m], which is equal to the thickness of the nozzle.

Employed are 16 processors of the Hitachi SR8000 supercomputer, conducting 100,000 steps analysis with approximately 70 CPU hours. Figure 6 shows the \(|u| = 1.0\) iso-surface, which is a non-dimensional velocity in Case 2. Figure 7 shows the absolute value of the velocity on the plane \(y = 7.5\) for Case 2, which shows that the air jet flows in from nozzle and oscillates on the upper and lower sides of...
the reed edge. This oscillation is also seen in both Case 1 and Case 3. The sound pressure is computed from the pressure distribution at the reed edge surface with Eq. [17] and the power spectrum by FFT analysis. Equation [17] is integrated by taking the summation of the product of an area of surface patch of tetrahedral element facing the edge and an average of pressure of three nodes composing the surface patch. On the other hand, the FFT analysis is performed with the Hanning window, where no smoothing technique is employed.

Figures 8 to 13 show the time histories of sound pressure and the power spectrum at static state for the above three Cases, where Figs. 8, 10 and 12 depict that the faster the inlet velocity of jet is, the shorter the time until the self-excited oscillation becomes. Brown\(^4\)) has performed a detailed experiment about the edge tone, achieving the following experimental formula,
Fig. 6. Iso-surface of |u| = 1.0 (Case 2).

Fig. 7. Velocity distributions at middle section (Case 2), upper: t = 0.0388 [s], lower: t = 0.0396 [s].

Fig. 8. Sound pressure vs. time (Case 1).

Fig. 9. Power spectrum (Case 1).

Fig. 10. Sound pressure vs. time (Case 2).

Fig. 11. Power spectrum (Case 2).

Fig. 12. Sound pressure vs. time (Case 3).
where \( f \) [Hz] is the frequency of the edge tone, \( U \) [m/s] is the velocity of the air jet, and \( l \) [m] is the distance from the edge. The parameter \( j \) is the oscillation mode, which is determined from \( U \) and \( l \). If the distance from the edge is constant, \( j \) increases with the flow velocity. Figure 14 shows the relationship between \( U \), \( l \) and \( f \) after Brown, where the sudden changes of frequencies represent the points at which the value of \( j \) changes. Figure 15 is a comparison between the Brown’s experimental equation and the present numerical results, where the two straight lines are the Brown’s formula for \( j = 1.0 \) and 2.3, respectively, and the three marks on the straight lines are the present results for \( v = 3.0, 5.0 \) and 7.0 m/s, respectively. The mid mark among the three marks at each velocity shows the frequency with the highest intensity, and the other two marks have half the intensity. Figure 15 implies that the result for \( v = 3.0 \) m/s corresponds to a value of \( j = 1.0 \) in the Brown’s formula, and the results for \( v = 5.0 \) and 7.0 m/s correspond to \( j = 2.3 \), respectively. Therefore, it is considered that the change in the oscillating mode occurs between \( v = 3.0 \) m/s and \( v = 5.0 \) m/s.

The harmonic components mentioned above are observed in the present three-dimensional analysis, which are never seen in the two-dimensional analysis.\(^{12} \) These harmonic components might be generated due to the three-dimensional behavior of the air around the edge. However, these two analyses are unable to be compared in a straightforward manner, because resolution of computational grids and other conditions are not the same between the two models, although both models have the same shape of edge at a cross section. Further investigation will be required in this respect. For example, it has been discussed that the air-reed instruments have the feedback mechanism called the acoustic feedback between the edge and the resonance tube.\(^{35},36 \) The acoustic effect, namely compressibility effect of fluid, may play the essential role in this feedback mechanism. On the other hand, the edge tone also has the feedback mechanism that pressure change at the edge surface affects the oscillation of the air jet, which is called the hydrodynamic feedback and is thought to have no coupled effects with sound.\(^{49} \) There are, however, various arguments on whether the hydrodynamic feedback is related to the compressibility of fluid or not. We presume that the phenomenon could be captured by extending the present incompressible fluid model somehow.
3. Applications to solid mechanics

3.1. Automatic remeshing for slow crack propagation analysis. Two numerical examples of crack propagation are shown here. Figure 16 shows nodal distribution for the analysis: (a) boundary nodes; and (b) nodes around crack tip. Figure 17 shows two models: Models (a) and (b). In the latter, the direction of crack propagation is determined by

\[ \sin \theta + (3 \cos \theta - 1) \cot \beta = 0 \]  \[19\]

where \( \beta \) is the crack angle before its propagation, measured from the direction of the load and \( \theta \) the direction of crack propagation\(^{50}\) (see Fig. 18). Figures 19 and 20 are the results of Models (a) and (b), respectively.

Figure 21 shows another example of remeshing for crack propagation analysis, where local meshes are generated only where the connectivity between central nodes and the associated satellite nodes needs to be renewed, then local mesh is renewed only around the crack tip. It is noted that the CPU time for this remeshing is much reduced by this local process.

3.2. Dynamic analysis. First, the static deformation of a three dimensional cantilever model under the bending load is considered as shown in Fig. 22(a). The mesh (number of elements: 11,090 and number of nodes: 2,605) employed is shown in Fig. 22(b). Figure 23 depicts the normalized displacement at the loaded point versus the degrees of freedom, showing much faster convergence of EFMM against FMM, which is equivalent to the FEM of constant strain or linear displacement (linear FEM) in terms of accuracy.

Next, the smoothing effect of EFMM is evaluated quantitatively by means of comparing the eigen-value distribution of FMM and EFMM stiffness matrices. The eigen-value distributions for both FMM and EFMM are depicted in Fig. 24, respectively, showing that the range of the eigen-value obtained by EFMM stiffness matrix is narrower, when compared with that by FMM. In addition, the distribution of eigen-value of EFMM tends to concentrate in lower frequency range, compared with that of FMM. From this analysis, it is considered that the condition number of EFMM is better than that of

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Fig. 16. Nodal distribution for slow crack propagation analysis: (a) Boundary nodes, and (b) Nodes around crack tip.

Fig. 17. Analysis models.

Fig. 18. Initial crack and direction of crack propagation.

Fig. 19. Results of Models (a) and (b).

Fig. 20. Results of Models (a) and (b).

Fig. 21. Another example of remeshing for crack propagation analysis.
FMM, resulting in faster convergence of EFMM solutions.

In this context, the measurement of the number of iterations to the convergence of the CG method is conducted for the above bending problem of a cantilever model shown in Fig. 22(a). The relationship between the degrees of freedom and the number of iterations is shown in Fig. 25, which demonstrates that the number of iterations of the CG method to a converged solution is smaller in EFMM, compared to that in FMM. This tendency seems grow as the number of degrees of freedom increases, which is an extremely important advantage in solving large scale problems and a by-product in addition to the improvement of accuracy in EFMM.

Next, we discuss the application of EFMM to dynamic problems in elasticity. The discretized equation for the dynamic problems in elasticity can be given as

\[
[M] \{a\} + [C] \{v\} + [K] \{u\} = \{f\}
\]

where \{a\}, \{v\}, and \{u\} are the acceleration, the velocity and the displacement vectors, respectively. \([M]\), \([C]\), and \([K]\) are the mass, the damping and the stiffness matrices, respectively, and \{f\} the load vector. Here, we use the Newmark \(\beta\) method (\(\beta = 1/4\)) as the time integration technique. Analysis is conducted on the cantilever beam assuming 2 different geometries of cross section as shown in Fig. 26. \([M]\) employed here is the concentrated mass matrix, and the damping is not considered. The step loading is applied to the edge of the beam at \(t = 0\), and the material constants are, respectively, the Young modulus: \(2.1 \times 10^{11}\) [Pa], the Poisson’s ratio: 0.3, and the density: \(7.86 \times 10^3\) [kg/m³]. The time histories of displacement at the loading point of EFMM and FMM (or linear FEM) are compared to the theoretical solution (see Fig. 27). Table 3 shows the comparison of natural frequency. From the figure and the table, it is apparent that the solutions of EFMM agree very well with theoretical solutions compared with those of FMM (or FEM). The error is only 1% with EFMM, while 10% with FMM (or linear FEM).
3.3. Large-scale Eigen-frequency analysis of engine block.\textsuperscript{23,27} The Lanczos method is able to solve the standard eigen-value problem as
\[ \mathbf{A} \mathbf{y} = \lambda \mathbf{y} \] [21]
with high accuracy, sequentially from the eigen-value with the maximum absolute value by a series of iterative computations. The eigen-frequencies of structural problems are obtained by solving the equation consisting of the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ as
\[ \mathbf{K} \mathbf{x} = \omega^2 \mathbf{M} \mathbf{x} \] [22]
where $\omega$ is the vibration frequency and $\mathbf{x}$ is the displacement vector.

Here, the mass matrix is decomposed into the product of triangular matrices as
\[ \mathbf{M} = \mathbf{B} \mathbf{B}^T \] [23]
where $\mathbf{M}$ is assumed to be a lumped mass matrix, and $\mathbf{B}^T$ the lower triangular matrix.

Substituting Eq. [23] into Eq. [22], we have
\[ \frac{1}{\omega^2} \mathbf{B}^T \mathbf{x} = \mathbf{B}^T \mathbf{K}^{-1} \mathbf{B} \mathbf{B}^T \mathbf{x} \] [24]
By assuming
\[ \mathbf{A} = \mathbf{B}^T \mathbf{K}^{-1} \mathbf{B}, \quad \mathbf{y} = \mathbf{B}^T \mathbf{x}, \quad \lambda = \frac{1}{\omega^2} \] [25]
Equation [22] is transformed into the standard eigen-value problem of Eq. [21]. Comparing Eq. [22] with Eq. [25], we can calculate sequentially from the eigen-value with the minimum absolute value.

Employed is an iterative method for matrix inversion in order to reduce the amount of necessary memory size. It is advantageous to use EFMM in this context, as the stiffness matrix discretized by EFMM converges more quickly compared to FEM in iterative calculations. In the following, we compare among EFMM, FEM using linear tetrahedral elements and FEM using quadratic tetrahedral elements. The material constants employed here are as follows, the
Fig. 21. Local remeshing method: (a) Initial mesh, and (b) through (d) Meshes for growing cracks (remeshed parts are depicted by red color).

(a) Cantilever beam under bending load
(b) 3-D mesh

Fig. 22. Cantilever beam and its 3-D mesh.
Young’s modulus: $70.6 \times 10^3$ [MPa], the Poisson’s ratio: 0.33, and the density: $2.7 \times 10^{-3}$ [kg/m³].

First, the accuracy of the solution of the mode 1 eigen-frequency for a cantilever beam as shown in Fig. 28 is studied, comparing the relative errors of EFMM (EFMM Tri3), FEM (linear or FEM Tri3) and FEM (quadratic or FEM Tri6) with the theoretical value51) (see Fig. 29). It is noted that the accuracy of EFEM, which is based on simple linear elements, is almost equivalent to that of FEM (quadratic) when the number of nodes increases.

Next, in order to study the accuracy in practical applications, natural or eigen-frequency analysis is performed on an engine block. Figure 30 is a CAD model of an engine block, from which meshes are created: the model with 228,192 elements shown in Fig. 31(a) and that with 1,679,426 elements shown in Fig. 31(b). These show half of the structure beyond the Y–Z section including the point P in Fig. 30.

Table 4 shows the combinations of analysis parameters and resulted solutions. It is noted that the number of degrees of freedom of FEM (quadratic) calculation using Fig. 31(a) and that of FEM (linear) and EFMM calculations using Fig. 31(b) are almost the same size of 1 million degrees of freedom. The eigen-frequencies of the modes 1 to 5 are calculated by FEM (linear), FEM (quadratic), and EFMM. The mesh model shown in Fig. 31(a) is used to compare all the three methods. FEM (linear) and EFMM also use the mesh model shown in Fig. 31(b). Presuming that the results of FEM (quadratic) of 1 million degrees of freedom are exact, Fig. 32 shows the relative errors of the solutions of other four combinations given in Table 4. From the figure, it is seen that the accuracy of EFMM is excellent compared to FEM (linear) for both cases of smaller and larger degrees of freedom. It is also clear that EFMM can
obtain accuracy very close to FEM (quadratic) in the case of 1 million degrees of freedom. The deformations and equivalent strain distributions (depicted by color gradation) for the mode 1 of FEM (quadratic) model using Fig. 31(a) and those of EFMM using Fig. 31(b) are, respectively, shown in Fig. 33. It is

Table 3. Comparison of natural frequency

| Cross-section | Natural frequency | Theoretical solution | EFMM | FMM |
|---------------|-------------------|----------------------|------|-----|
| Square        | 8.353             | 8.237                | 8.948|
| Circle        | 7.234             | 7.297                | 8.069|

Fig. 29. Relative error vs. number of nodes.
noted here that the figure in the left hand side shows the results of FEM (quadratic) with number of degrees of freedom being 1,069,542, whereas that in the right hand side shows the results of EFMM with number of degrees of freedom 1,001,874. These show that not only eigen-values but eigen-modes are compared very well between FEM (quadratic) and EFMM of the similar degrees of freedom. It is also noted that the strain distributions, shown in different color, are almost the same between the two.

4. Concluding remarks

In the present review paper, the several numerical demonstrations are presented as the applications
of FMM and EFMM as well, emphasizing on fluid and solid mechanics. First, the three-dimensional incompressible fluid simulations of the edge tone, which is the sound source of the air-reed instruments, are presented in order to study the mechanism of sound generation in terms of fluid mechanics. As the result, the process from initiation of the air jet to self-excited oscillation and stabilization of the edge tone is clarified through the three-dimensional numerical fluid analysis, confirming that the strongest frequency component corresponds to the frequency derived by the Brown’s experimental equation. Furthermore, the existence of two or more minor frequency components in addition to the main frequency is confirmed. These minor components have not been observed in the two-dimensional fluid analysis, and are assigned to harmonic components of the edge tone. Lastly, it is shown that the transfer of oscillation mode, which is one of the features of the edge tone, is captured under the assumption of the incompressible fluid. However, the extent of the influence of compressibility of the fluid in an edge tone, and the acoustic effect of a resonance tube remain to be clarified in the future.

Some fundamental performances of FMM and FEMM on the static and dynamic problems are reviewed with the applications to solid mechanics. The eigen-value distribution of the stiffness matrix created by EFMM concentrates in low frequency range, which could be a reason that the solution for the static problem by FEMM converges faster than that by FMM. When EFMM applied to a dynamic and time-dependent problem, the eigen-frequency coincides very well with the theoretical solution. It is also found that EFMM can provide highly accurate eigen-frequencies without increasing the number of degrees of freedom for a rather realistic model of engine block.

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