The Lorenz Renormalization Conjecture

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ABSTRACT
Universality in low-dimensional dynamical systems is the remarkable phenomenon that the geometry of a system at a phase transition, as well as the bifurcation patterns of nearby systems, is determined by its topology. This can be explained by an associated renormalization operator being a contraction on topological conjugacy classes. For more physically realistic systems this paradigm is too simple. In the context of Lorenz dynamics, which is associated with homoclinic bifurcations of flows, the renormalization behavior is much richer. Depending on topology, one finds either the traditional universality or two more intricate classes of dynamics. A conjecture describing the dynamics of the Lorenz renormalization operator in terms of these three classes of behavior is stated and the consequences each case has on the dynamics of Lorenz maps is discussed. Numerical evidence supporting the conjecture is provided.

KEYWORDS
renormalization; rigidity; universality

AMS SUBJECT CLASSIFICATION
37E20; 37E05

1. Introduction
Renormalization in low-dimensional dynamical systems is characterized by hyperbolic horseshoe dynamics where topological classes are contracted and instability is associated with changes in topology. There are an abundance of low-dimensional systems which adhere to this paradigm, such as unimodal maps [Sullivan 92, McMullen 96, Lyubich 99], critical circle maps [de Faria 92, Yampolsky 03] and circle maps with breaks [Khanin and Teplinsky 13]; as well as partial results for dissipative Hénon-like maps [De Carvalho et al. 05], area-preserving maps [Eckmann et al. 84, Gaidashev et al. 16] and higher-dimensional analogs of unimodal maps [Collet et al. 81]. In particular, these phenomena are not only limited to one-dimensional maps and dissipative maps in higher-dimensions. This research springs from the question: in what way does the renormalization paradigm need to be modified as its scope is expanded to include more physically realistic systems coming from flows and maps in higher dimensions?

Renormalization was introduced to dynamics by Coullet and Tresser [78] and independently by Feigenbaum [78]. Working numerically, they discovered that unimodal maps on the boundary of chaos exhibit universal phenomena and showed how this could be explained by an associated dynamical system acting on these maps, called the period-doubling operator, having a hyperbolic fixed point. The former authors also predicted that these phenomena could be measured in physical experiments and this has since been confirmed in many different settings [Maurer and Libchaber 79, Linsay 81]. The stable manifold of the period-doubling fixed point coincides with its topological conjugacy class. In particular, the unstable direction is associated with changes in topology. This is the underlying mechanism behind universality and it has been found to hold for the more general renormalization operator in many settings as mentioned above. Due to this and the fact that universality has been measured for physical systems it was natural to think that this mechanism would hold in some generality. Surprisingly, Martens and Winckler [17] showed that even in the one-dimensional setting of Lorenz maps (see Figure 1), instability of renormalization is not only associated with changes in topology; the dynamics of the renormalization operator inside topological classes is not necessarily a contraction. This has drastic consequences on the dynamics of Lorenz maps, leading to the phenomena of coexistence and dimensional discrepancy, see §1.6. It also causes universality and rigidity phenomena to become more intricate than they are in for example unimodal and critical circle dynamics (see §1.1 for an overview).

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1For a more complete list of references, see e.g. [Martens and Winckler 17].

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Note that the phenomena discussed here are visible in the large-scale dynamics of individual maps; in particular, the maps under consideration have attractors whose basin has full measure and it is the geometry of these attractors that occupy most of our attention. Compare this with Levin and Swiatek [02] who study of these attractors that occupy most of our attention. In particular, the maps under consideration have attractors associated with simplified models of convection in the atmosphere, so there is a tangible connection between this research and physical systems which would be interesting to investigate further.

The purpose of this article is to state a conjecture which classifies the dynamics of the Lorenz renormalization operator and to discuss the consequences it has on the dynamics of infinitely renormalizable Lorenz maps. The conjecture is supported with numerical experiments as well as partial results from previous research on Lorenz renormalization. We hope that our conjecture will act as a focus for what should aim to be proven for these systems. More generally, we wish to provide an indication of what kind of renormalization phenomena to expect from flows and maps in higher dimensions.

The article is organized as follows: First, we give an informal overview of renormalization in the classical settings of critical circle maps and unimodal maps, see §1.1. There are many similarities between the renormalization operator on Lorenz maps and these classical systems and we hope that this is highlighted by this informal discussion. In §1.2 and §1.3 we formally define the space of Lorenz maps and the renormalization operator acting thereupon. We want to emphasize the connection between renormalization and rigidity and for this reason we spend some time discussing rigidity in §1.4. In §1.5 we state the Lorenz Renormalization Conjecture and then go on to discuss the consequences it has on the dynamics of Lorenz maps in §1.6. This concludes the first part of the article; the second part in §2 is dedicated to the numerical experiments performed to support the conjecture. In §2.1 and §2.2 we discuss how to represent Lorenz maps and the renormalization operator on a computer. The algorithms used for locating periodic points of the renormalization operator are described in §2.3 and §2.4. Finally, a discussion of our implementation of these algorithms is in §2.5 and the results of the numerical experiments we performed are in §2.6. The source code, together with instructions on how to reproduce the results, are freely available online [Winckler 18].

1.1. Informal overview of renormalization

In this section, we give an informal overview of the theory of renormalization for critical circle maps and unimodal maps. The purpose is to emphasize the similarities with the renormalization of Lorenz maps, which are formally introduced starting in §1.2.

Let $\mathcal{X}(I)$ be a class of dynamical systems on an interval $I$ with one critical point and smooth branches. We will consider critical circle maps (lifted to $I$), unimodal maps and Lorenz maps (see Figure 2). The order of the critical point will be fixed throughout and we normalize $\mathcal{X}(I)$ so that no proper subinterval is invariant by $f \in \mathcal{X}(I)$, i.e. the critical value(s) of $f$ lie in the boundary of $I$, as in Figure 2. We write “critical value(s)” since critical circle maps and Lorenz maps have two critical values, whereas unimodal maps have one critical value. Note that the turning point of a unimodal map $f \in \mathcal{X}(I)$ is allowed to be either a maximum or a minimum.

Renormalization is based around the idea that in order to understand the dynamics of $f \in \mathcal{X}(I)$ it may be fruitful to instead look at the first-return map to some judiciously chosen subinterval $J \subseteq I$. It may happen that $J \subseteq I$ can be chosen so that $g$ is in the same class as $f$ up to rescaling, i.e. $g \in \mathcal{X}(J)$; in this situation $f$ is said to be renormalizable. (Note that for unimodal maps $g$ may look upside-down compared to $f$ as happens in Figure 2.) Of course, in general $f$ need not be renormalizable, but then usually something else can be said about its dynamics, e.g. that it has a periodic attractor or an absolutely continuous invariant measure.

Figure 2 illustrates the concept of renormalizable maps. In this figure, $J$ contains the critical point but it is important to realize this choice is in no sense canonical; one could instead take any of the intervals

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2Recall that the first-return map $g$ to $J$ (induced by $f$) is defined as follows: let $J'$ consist of all $x \in J$ for which $\exists k \geq 1$ such that $f$ is defined at $\hat{t}(x)$, $v_j = 0$, ..., $k-1$, and such that $\hat{t}(x) \in J$; given $x \in J'$ let the first-return time $\tau(x)$ be the smallest $k \geq 1$ such that $\hat{t}(x) \in J$; then $g: J' \rightarrow J$ is defined by $g(x) = f^\tau(x)$. 

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in the (finite) orbit of \( J \) (or subintervals of these in case \( f \) is more than once renormalizable). However, in what follows we force the choice of \( J \) to be unique by always insisting that \( J \) is the largest possible interval containing the critical point such that the first-return map to \( J \) is in \( \mathcal{X}(J) \) and \( J \neq I \). Note that the first-return time is constant on \( J \) for unimodal maps, whereas for circle maps and Lorenz maps it can assume different values on either side of the critical point. The first-return time tells us how many steps to iterate \( J \) before it returns, but it does not necessarily say anything about how the intervals on the orbit of \( J \) are ordered in the real line. This is something that needs to be considered because two renormalizable maps may share a return time, but still have different orderings of the return interval (for circle maps this is not an issue). The ordering of the intervals in the orbit of \( J \) in the real line is called the type of the renormalization. Going back to the examples in Figure 2, the unimodal map is of period-doubling type and the Lorenz map is of monotone (2, 1)–type.

The set \( \mathcal{X}(I) \) can naturally be partitioned into two classes: one consisting of at most finitely renormalizable maps and the other consisting of infinitely renormalizable maps. Let us explain what this means. Given a renormalizable map \( f \in \mathcal{X}(I) \), we know that there is some subinterval \( J_1 \) on which the first-return map \( f_1 \) is a map in \( \mathcal{X}(J_1) \). We can now ask if \( f_1 \) is renormalizable as well; if so, there is a subinterval \( J_2 \subseteq J_1 \) on which the first-return map \( f_2 \) (induced by \( f_1 \)) is a map in \( \mathcal{X}(J_2) \). If this process can be repeated indefinitely for some choice of intervals \( J_1 \supseteq J_2 \supseteq J_3 \supseteq \ldots \), then \( f \) is said to be infinitely renormalizable; otherwise \( f \) is at most finitely renormalizable. In the latter case, there is a minimal interval \( J \) to which the first-return map \( g \) is in \( \mathcal{X}(J) \) but \( g \) itself is not renormalizable; the dynamics of \( f \) on the orbit of \( J \) (which is a finite collection of intervals) can then be inferred from the dynamics of \( g \). The focus here is on the case of infinitely renormalizable maps, so we will not say anything more about at most finitely renormalizable maps. Infinitely renormalizable maps typically appear at the boundary of a phase transition; by this we mean that there are nearby maps with stable dynamics as well as nearby maps with chaotic dynamics. An amazing feature of infinitely renormalizable maps is that their geometry and the bifurcation patterns of nearby maps are to a large extent determined by their topology. We will now describe what we mean by this.

First, we need to some setup. Let \( f \in \mathcal{X}(I) \) be infinitely renormalizable so that there is a sequence of maximal intervals \( I = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots \), each containing the critical point of \( f \), and such that the first-return map to \( I_k \) is a map \( f_k \in \mathcal{X}(I_k) \) (with the convention \( f_0 = f \)). By definition each \( f_k \) is renormalizable so it has a type, i.e. the ordering of the orbit of \( I_k+1 \) under \( f_k \); this ordering can be described by a permutation \( \pi_k \). The sequence \( (\pi_0, \pi_1, \pi_2, \ldots) \) is known as the combinatorial type of \( f \); if the length of \( \pi_k \) is bounded as a function of \( k \) then \( f \) is said to be of bounded type, and if furthermore \( \pi_k = \pi_{k+1} \) for every \( k \) then \( f \) is said to be of stationary type. For the rest of this discussion we will always assume bounded type.

We can now describe the geometry of \( f \). Let \( \Lambda_k \) be the set of intervals given by the orbit of \( I_k \) under \( f \) up to the (largest) first-return time and let \( \Lambda = \bigcap \bigcup \Lambda_k \); this is illustrated in Figures 3 and 5. When we talk about the “geometry” of \( f \) we usually refer to \( \Lambda \) or its presentation \( \{ \Lambda_k \} \). In the case of unimodal maps (and

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Footnote: For unimodal \( f \), we define \( \pi_0(j) = j \) if \( f^i(J_{k+1}) \) is the \( i \)-th interval as they are embedded in the real line counting from the critical value; i.e. \( \pi_0(1) = 1, \pi_0(2) = i \) where \( f^i(J_{k+1}) \) is the interval adjacent to \( f_k(J_{k+1}) \); \( \pi_0(3) = j \) where \( f^j(J_{k+1}) \) is the interval adjacent to \( f^i(J_{k+1}) \) which is not \( f(J_{k+1}) \), and so on.
A family of unimodal maps we can look at the parameters $F$ the geometric arrangement of the parameters $g$ almost a dynamical system, except for the nuisance depends on the topology of the (infinitely renormalizable) versality condition on the family). This constant only $f_0$ makes up $\Lambda_0$. The intersection of all the intervals in $\Lambda_1, \Lambda_2, \ldots$ is the Cantor attractor of $f$.

Lorenz maps, see Remark 1.18) $\Lambda$ is a minimal Cantor attractor whose basin has full Lebesgue measure, and in the case of critical circle maps $\Lambda$ is the entire circle. In both cases the sets $\Lambda$ are rigid, i.e. if $g \in \mathcal{X}(I)$ is topologically conjugate to $f$, then the conjugacy maps $\Lambda(f)$ onto $\Lambda(g)$ diffeomorphically. A diffeomorphism looks affine on small scales and hence rigidity implies that the microscopic geometry of $f$ and $g$ must look the same. This is similar to, but stronger than, the notion of universality in phase space which, under the additional assumption of stationary combinatorics, states that $|j_{k+1}|/|j_k|$ converges as $k \to \infty$ to a universal constant which only depends on the topology of the map $f$.

Another kind of universality occurs in the parameter space of families of maps undergoing a phase transition. To explain this, let $\lambda \mapsto F(\lambda) \in \mathcal{X}(I)$ be a family of maps in $\mathcal{X}(I)$ with parameter $\lambda$. By the “bifurcation patterns” of the family we are referring to the geometric arrangement of the parameters $\lambda$ where $F(\lambda)$ undergoes a bifurcation. For example, for a family of unimodal maps we can look at the parameters $\{\lambda_k\}$ where a period-doubling bifurcation of period $2^k$ to $2^{k+1}$ takes place. Incredibly, the geometry of these bifurcations is asymptotically independent of the family itself, e.g. the ratios $|\lambda_{k+1} - \lambda_k|/|\lambda_k - \lambda_{k-1}|$ converge to a universal constant as $k \to \infty$ (under some transversality condition on the family). This constant only depends on the topology of the (infinitely renormalizable) map at the phase transition, i.e. $\lim_k F(\lambda_k)$.

We now turn to an explanation of rigidity and universality but first we need to introduce the renormalization operator. Let $f \in \mathcal{X}(I)$ be renormalizable so it has a maximal interval $J \subseteq f$ which contains the critical point of $f$ and such that the first-return map to $J$ is some $g \in \mathcal{X}(J)$. The operator which sends $f$ to $g$ is almost a dynamical system, except for the nuisance that $g$ is not defined on the same interval as $f$. To fix this, $g \in \mathcal{X}(I)$ is rescaled to a map $G \in \mathcal{X}(I)$; i.e. $G = A^{-1} \circ g \circ A$, where $A : I \to J$ is affine and onto. The dynamical system on $\mathcal{X}(I)$ defined by sending $f$ to $G$ is called the renormalization operator $R$. As we will see, rigidity and universality are consequences of $R$ being a contraction on topological classes (of infinitely renormalizable maps). First, there are some subtleties involved in how to rescale the first-return map; without going into details, for circle maps $A$ is chosen to be orientation-reversing and for unimodal maps $A$ is chosen so that the critical point of $G$ and the critical point of $f$ both are maxima or minima (for Lorenz maps $A$ is always orientation-preserving, see also Remark 1.14). Intuitively, this rescaling is the equivalent of zooming in on the dynamics of $f$ that happens on the interval $J$; in particular, if $f$ has an attractor, then renormalization is a microscope which looks closer on the part of the attractor which lies inside $J$.

Having introduced the renormalization operator, we can now give an intuitive explanation of rigidity and universality. The following discussion is illustrated in Figure 4 so it may be helpful to keep that figure in mind. Let $f_0 \in \mathcal{X}(I)$ be a critical circle map or unimodal map of stationary combinatorial type and let $\lambda \mapsto F(\lambda)$ be a family which contains $f_0$. The crux of renormalization is that the topological conjugacy class $T$ of $f_0$ contains a hyperbolic fixed point $f_* \in \mathcal{X}(I)$ of

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4 This should hold independently of the order of the critical point, but has so far only been proved under the assumption that the order is an integer (even for unimodal maps, odd for circle maps).
and that the stable manifold of this fixed point coincides with $T$. This is an extremely strong statement about the dynamics of $R$ and proving it is not easy. An immediate consequence is that the iterated forward images of the family $F(\lambda)$ under $R$ (restricted to the renormalization type of $f_0$) accumulate on the unstable manifold of the fixed point. In particular, assuming the family crosses the stable manifold transversally at $f_0$, for $n$ large there is a segment of this family which after a $n$ iterations will be arbitrarily close to the unstable manifold. As a consequence, the bifurcation patterns on this segment will resemble the bifurcation patterns of the unstable manifold. Furthermore, the attractor of the $n$-th renormalization $f_n = R^n f_0$ will be very similar to the attractor of $f$. Since $R$ is a microscope this amounts to saying that the attractor of $f_0$ looks like the attractor of $f$, on small scales. This is the reason why the geometry of $f_0$ and the bifurcation patterns of $F(\lambda)$ near $f_0$ only depend on the topology of $f_0$. As we will see later, for Lorenz maps stable manifolds of renormalization fixed points can be strictly smaller than the topological class of the fixed point, leading to more complicated phenomena.

The above discussion can be generalized to periodic combinatorics (i.e. $\pi_{j+k} = \pi_j$ for all $j$ and some fixed period $k$) by considering periodic points of the renormalization operator. Generalizing further to bounded combinatorics, it can be shown that the renormalization operator restricted to its limit set is conjugate to a full shift on infinitely many symbols (with each symbol corresponding to a distinct type of renormalization); this is colloquially referred to as the renormalization horseshoe.

### 1.2. Lorenz maps

In this section, we define the set of Lorenz maps that we will consider. See Figure 1 for an illustration of the graph of a Lorenz map.

**Definition 1.1.** Let $I = [l, r]$ be a closed interval. A Lorenz map $f$ on $I$ (see Figure 1) is a monotone increasing function which is continuous except at a critical point, $c \in (l, r)$, where it has a jump discontinuity, and $f(I \setminus \{c\}) \subset I$. Consequently, $f$ has two (monotone) branches, $f_0 : [l, c] \to I$ and $f_1 : [c, r] \to I$.

The branches are assumed to satisfy the following conditions: (i) $f_0(c) = r$ and $f_1(c) = l$; and (ii) $f_k(x) = \phi_k(|c-x|^z)$, for some real number $z>0$, and $C^2$-diffeomorphisms $\phi_k$; for $k = 0, 1$. The numbers $f_0(c)$ and $f_1(c)$ are called the critical values, and $z$ is called the critical exponent of $f$.

The set of all Lorenz maps on $[0,1]$ satisfying the above conditions is denoted $L$.

**Convention.** Unless the domain of definition of a Lorenz map $f$ is mentioned, it is implicitly assumed to be the unit interval $[0,1]$ minus some $c \in (0,1)$.

**Remark 1.2.** It bears pointing out that the critical point $c$ is not fixed, but depends on the map $f$. Later on we will see that the critical point moves under renormalization. This is an essential feature of Lorenz maps which has very strong consequences on the dynamics and results in new renormalization phenomena not present in unimodal and circle dynamics [Martens and Winckler 17].

**Remark 1.3.** We have chosen to normalize $f \in L$ so that the critical values of $f$ coincide with 0 and 1. Another common choice is to normalize so that 0 and 1 are fixed points of $f$. Which is appropriate depends on context; for the purpose of numerics our choice is more convenient. Be aware that this choice does affect which maps are renormalizable or not; e.g. being renormalizable in the minimal dynamical interval normalization that we use is a weaker condition than it is to be renormalizable in the fixed boundary normalization.

**Remark 1.4.** The second condition on the branches ensures that the behavior of $f$ near the critical point is like that of the power map $x^z$ near 0. In particular, this precludes the possibility of a flat critical point (i.e. where derivatives of all orders vanish) and

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![Figure 5. Illustration of the first two levels in the presentation of the Cantor attractor of an infinitely renormalizable Lorenz map $f$. The first-return intervals $J_1 \supset J_2 \supset ...$ are mapped forward by $f$ as indicated by arrows. The finite collection of intervals on the orbit of $J_0$ makes up $\Lambda_k$. The intersection of all the intervals in $\Lambda_1, \Lambda_2, ...$ is the Cantor attractor of $f$.](image-url)
ensures that the order of the critical point is the same when approaching from the left and from the right. Flat critical points and distinct left and right critical orders are expected to lead to degenerate renormalization phenomena in general (e.g. limits of renormalization are no longer Lorenz maps) which is why we avoid treating them here. The critical order is invariant under renormalization so it is natural to fix it once and for all. Here we assume \( x > 1 \) as it ensures that the first derivative vanishes at the critical point which in turn leads to a non-trivial renormalization theory.

**Convention.** The critical exponent \( x \in \mathbb{R} \) is fixed and \( x > 1 \).

**Remark 1.5.** Lorenz maps were introduced by Guckenheimer and Williams [79] (see also Afraimovich et al. [81]) in order to describe the dynamics of threedimensional flows geometrically similar to the well-known Lorenz system [Lorenz 63]. The flows they consider have a saddle with a one-dimensional unstable manifold which exhibits recurrent behavior. Their construction is to take a two-dimensional transversal section to the stable manifold and assume that the associated first-return map has an invariant foliation whose leaves are exponentially contracted. Taking a quotient over the leaves results in a one-dimensional map as described by Definition 1.1.

In the above construction, the critical exponent \( x \) naturally comes out as the absolute value of the ratio between two eigenvalues of the linearized flow at the singularity. In particular, it is important for Lorenz theory to be able to handle any real critical exponent \( x \geq 0 \) (as opposed to unimodal theory where it may be possible to get away with saying something like “the critical exponent is generically two”). Guckenheimer and Williams [79] considered \( x \in (0, 1) \); the first to investigate \( x > 1 \) were Arneodo et al. [81].

**Remark 1.6.** In more generality, Lorenz maps can be thought of as the underlying dynamical model for a large class of higher dimensional flows undergoing a homoclinic bifurcation. Hence there are very strong reasons why Lorenz dynamics needs to be further explored. We can only guess that this theory is still so largely underdeveloped, as compared to unimodal and circle dynamics, because of the fact that the holomorphic tools developed in these other theories are not suitable for adaptation to discontinuities and arbitrary real critical exponents. New ideas and tools are desperately needed!

**Remark 1.7.** There is a genuine problem relating to smoothness that needs mentioning. Even if the invariant foliation mentioned in Remark 1.5 is smooth, the holonomy map need not be [Milnor 97, Hirsch et al. 77]. As a consequence, the associated Lorenz map need not have \( C^2 \) branches, regardless of how smooth the initial flow is. Without \( C^2 \)-smoothness the renormalization apparatus breaks down [Chandramouli et al. 09]. In transferring results about maps to flows this problem needs to be addressed.

### 1.3. Renormalization

In this section we define the renormalization operator on Lorenz maps following Martens and de Melo [01]. This definition shares similarities with the renormalization operators on circle maps and unimodal maps discussed in §1.1.

**Definition 1.8.** A Lorenz map \( f \in \mathcal{L} \) is renormalizable iff there exist \( n_0, n_1 \geq 2 \) such that \( J = [f^{n_1-1}(0), f^{n_1-1}(1)] \) is contained in \((0, 1)\) and contains \( c \) in its interior, and such that the first-return map to \( J \) is again a Lorenz map (on \( J \)). The graph of a renormalizable Lorenz map is illustrated in Figure 2.

**Remark 1.9.** Let \( f \) be renormalizable with \( J \) as above. Then for \( k = 0, 1 \)

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C_k = \{ J, f_k(J), f(f_k(J)), ..., f^{n_k-2}(f_k(J)) \}
\]

are two pairwise disjoint sequences of intervals and \( f^{n_k-1}(f_k(J)) \subset J \). In particular, the first-return time to \( J \) is \( n_0 \) to the left and \( n_1 \) to the right of the critical point. The collection \( C_0 \cup C_1 \) need not be pairwise disjoint a priori (but in the case of monotone combinatorics discussed below it always is). See Figure 5 for an illustration of the action of \( f \) on \( C_0 \cup C_1 \) for a twice renormalizable map (defined below).

Next, we will define the renormalization operator roughly by sending a renormalizable map to a first-return map. There are two issues that need to be addressed: (i) a first return is a Lorenz map on some \( J \subset (0, 1) \) but we want a Lorenz map on \([0, 1)\) to get a well-defined operator on \( \mathcal{L} \); (ii) a renormalizable map has more than one interval on which to take a first return, so we need to make a choice as to which interval to use. The first issue is taken care of by rescaling and the second by taking first returns around the largest possible interval containing the critical point (more on this below).

**Remark 1.10.** It is important to realize that the property of being renormalizable is intrinsic to the maps...
themselves; regardless what choices we make in defining a renormalization operator this does not affect the dynamics of infinitely renormalizable maps, see also Remark 1.14.

**Definition 1.11.** Let \( A_I : [0, 1] \to I \) denote the increasing affine map taking \([0, 1]\) onto the interval \( I \), and let \( g : U \setminus E \to V \) be an interval map which is allowed to be discontinuous on some exceptional set \( E \). The *rescaling* to \([0, 1]\) of \( g \) (synonymously, \( g \) *rescaled to \([0, 1]\))* is the map \( G : [0, 1] \setminus A_U^{-1}(E) \to [0, 1] \) defined by \( G = A_U^{-1} \circ g \circ A_U \). In this situation we also conversely say that \( g \) is a rescaling of \( G \).

**Definition 1.12.** Let \( f \) be a renormalizable Lorenz map so that the first-return map to \( J = [f^{n_0\!-\!1}(0), f^{n_0\!-\!1}(1)] \) is a Lorenz map on \( J \). The rescaling to \([0, 1]\) of the first-return map to \( f \) is called a renormalization of \( f \). The type (or combinatorics) of the renormalization is the pair of words, \( w = (w_0, w_1) \), defined by the itineraries of \( J_0 = [f^{n_0\!-\!1}(0), c] \) and \( J_1 = (c, f^{n_0\!-\!1}(1)] \) up to the first-return times. Explicitly, define \( w_k \) to be the word on symbols \( \{0, 1\} \) of length \( n_k \) such that \( f^j(J_k) \subset [w_k(j), c] \) for \( j = 0, \ldots, n_k - 1 \) and \( k = 0, 1 \).

In this case we also say that \( f \) is \( w \)-renormalizable and call the rescaled first-return map a \( w \)-renormalization.

The second issue mentioned above is that there may be many choices for the return interval \( J \) around the critical point (corresponding to different choices of \( n_0 \) and \( n_1 \)). However, such intervals must be nested so we can always pick the maximal interval; this is the same as choosing the \( w \)-renormalization of \( f \) for which the sum of the return times \(|w| = |w_0| + |w_1| \) is minimal.\(^6\)

**Definition 1.13.** Define the renormalization operator, \( \mathcal{R} \), by sending a renormalizable \( f \) to the \( w \)-renormalization of \( f \) for which \(|w| \) is minimal.

Maps for which \( \mathcal{R}^j f \) is \( w_j \)-renormalizable for every \( j \geq 0 \) are called infinitely renormalizable; if \( \sup |w_j| < \infty \), then \( f \) is said to be of bounded type; if furthermore \( w_j = w \) for all \( j \) (i.e. \( w_j \) does not depend on \( j \)), then \( f \) is said to be of stationary type. In the latter case we will also say that \( f \) is infinitely \( w \)-renormalizable. The maps in the orbit of \( \mathcal{R}, \{f, \mathcal{R}f, \mathcal{R}^2f, \ldots\} \), are called the successive renormalizations of \( f \).

**Remark 1.14.** The dynamics of the Lorenz renormalization operator is richer than its counterpart on unimodal maps and circle maps due to the fact that the two branches of a Lorenz map are completely independent. In particular, the critical point of \( \mathcal{R}f \) is in general not the same as the critical point of \( f \). This may seem like an artificial problem stemming from how we normalize \( \mathcal{L} \) and from the fact that we rescale first-return maps using affine rescalings. Let us show why it is in fact an essential problem that cannot be circumvented by normalizing or rescaling differently.

Say \( \mathcal{L} \) is normalized so that the critical point is always at zero. Then you immediately run into the issue that the domains of the branches of the first-return map in general have different lengths; for deep renormalizations one branch can even disappear asymptotically. This issue can be circumvented by rescaling in such a way that the branches of the renormalization have the same length, e.g. by using a Möbius map. However, then you end up introducing distortion into the rescaling and this distortion cannot always be controlled. In fact, the critical point will approach the boundary in our choice of normalization and rescaling if and only if the distortion of such Möbius rescalings blows up. The phenomena in §1.6 will appear regardless how the rescaling is chosen since they reflect intrinsic properties of the maps and have nothing to do with how the renormalization operator is defined.

As a philosophical remark, for any renormalization operator we always want to rescale affinely if at all possible. The reason for this is because affine maps exactly preserve geometry and one of the main points of renormalization is to study microscopic geometry. However, sometimes it simply is not possible to use affine rescalings and as a result controlling the distortion of the rescalings becomes a central problem; this is the case for Hénon maps [De Carvalho et al. 05].

A particularly simple type of renormalization that we often talk about is given by the following definition:

**Definition 1.15.** The type \( w = (w_0, w_1) \) is said to be of monotone combinatorics if \( w_0 = 011 \cdots 1 \) and \( w_1 = 100 \cdots 0 \); more succinctly, it is also called \((a, b)\)-type, where \( a = |w_0| - 1 \) and \( b = |w_1| - 1 \).

**Remark 1.16.** The reason for introducing this class is twofold: it is the simplest class to describe and analyze but at the same time it is not too restrictive, since it

\(^{6}\)In the notation \([x, c)\) for a half-open interval \( x > c \) is allowed.

\(^{6}\)In particular, if \( f \) is both \( w \)-renormalizable and \( w' \)-renormalizable (and \( w \neq w' \)), then \( w_0 = w_0w_1 \cdots \) and \( w_1 = w_1w_0 \cdots \), or vice versa, see [Martens and de Melo 01].
turns out that the phenomena we are introducing appear even for stationary monotone types.

The dynamics of an infinitely renormalizable Lorenz map resembles that of an infinitely renormalizable unimodal map in that they both have a Cantor attractor. The presentation of the Cantor attractor is constructed in the same way as outlined in §1.1 by taking the intersection of the orbits of the first-return intervals, see Figure 5.

**Conjecture 1.17.** The closure of the post-critical set of an infinitely renormalizable Lorenz map of bounded type is a minimal Cantor attractor whose basin has full Lebesgue measure.

**Remark 1.18.** For Lorenz maps, the post-critical set is the union of the $\omega$–limit sets of the critical values. This conjecture is a theorem for a large class of monotone combinatorics [Martens and Winckler 14, 17].

### 1.4. Rigidity

In this section, we define the notions related to rigidity that we will need. Rigidity was informally overviewed in §1.1.

The fundamental topic of this paper is the structure of topological conjugacy classes. Intuitively, the following conjecture describes the “external” structure of conjugacy classes and the rigidity discussion below is focused on their “internal” structure.

**Conjecture 1.19.** The set $T_w \subset \mathcal{L}$ of infinitely $w$–renormalizable Lorenz maps coincides with the topological conjugacy class of any $f \in T_w$. Furthermore, $T_w$ is a manifold of codimension two.

**Remark 1.20.** The first statement would follow if it were shown that there are no wandering intervals for $f \in T_w$. This is known for a large class of monotone combinatorics [Martens and Winckler 14, 17] but the general problem of when Lorenz maps do not support wandering intervals is still wide open. The codimension of $T_w$ must be two since topologically full families of Lorenz maps are two-dimensional [Martens and de Melo 01].

The notion of rigidity in its most general form states that given that two objects are weakly equivalent they must in fact be strongly equivalent. In dynamics the weak equivalence is typically described by topological conjugacy and the stronger equivalence simply imposes more regularity on the conjugacy, e.g. quasi-symmetry or smoothness. We will use the following definition:

**Definition 1.21.** The (classical) notion of rigidity is when two topologically conjugate maps are automatically smoothly conjugate on their attractors.

**Remark 1.22.** Differentiable maps look affine on small scales, so in the presence of rigidity two maps have attractors which on a large scale may look very different but when zoomed in on a particular spot they start to look the same. In this sense rigidity is a strong form of universality in phase space as discussed in §1.1.

**Remark 1.23.** Two crucial ingredients in proving classical rigidity is first to prove that successive renormalizations converge and then to control the rate of convergence. Typically, these ingredients come from the fact that there is a hyperbolic renormalization fixed point which attracts both maps as described in §1.1.

It is worth pointing out that the study of rigidity in dynamics was initiated by Herman [79], answering a conjecture by Arnold [61], but the close connection between rigidity and renormalization was only later realized.

The classical definition of rigidity is too coarse for our purposes so we are led to studying rigidity classes and the partition of topological classes into rigidity classes.

**Definition 1.24.** The $C^1$–rigidity class (or rigidity class) of $f \in T_w$ is defined as the set of $g \in T_w$ such that $f$ and $g$ are $C^1$–smoothly conjugate on their attractors.

**Remark 1.25.** With this terminology we may characterize classical rigidity as the statement that a topological class coincides with a rigidity class. From Martens and Winckler [17] we know that $T_w$ may, depending on $w$, consist of more than one rigidity class. Hence, the classical concept of rigidity is too restrictive, see also Martens and Palmisano [17]. Instead, the correct notion should be to describe the arrangement of a topological class into rigidity classes [Martens et al. 17].

Even in the classical cases of critical circle maps and unimodal maps there is already a natural foliation into codimension–1 rigidity classes determined by the value of the critical exponent. This is however a trivial observation compared to the above-mentioned articles which concern far more subtle phenomena.
1.5. Main conjecture

In this section we formulate a conjecture describing the dynamics of the Lorenz renormalization operator. The dynamics of this operator differs from its unimodal and circle counterparts in that it may degenerate in the manner described by the following definition:

**Definition 1.26.** The successive renormalizations of \( f \in \mathcal{L} \) are said to **degenerate** iff they have no convergent subsequence in \( \mathcal{L} \). Furthermore, they are said to **degenerate by flipping** iff the successive critical points approach the boundary \( \{0, 1\} \) and flip between being close to 0 and being close to 1; i.e. \( c(R^k f) \to 0 \) and \( c(R^{k+1} f) \to 1 \) as \( k \to \infty \), or vice versa.

The purpose of the following conjecture is to describe the “internal” structure of a topological class \( T_w \) in terms of \( \mathcal{C}^1 \)-rigidity classes (see §1.4) and to describe the dynamics of \( \mathcal{R} \) on \( T_w \). The conjecture is illustrated in Figure 6. The consequences on the dynamics of Lorenz maps is discussed in §1.6.

**The Lorenz Renormalization Conjecture.** Let \( T_w \subset \mathcal{L} \) be the set of infinitely \( w \)-renormalizable Lorenz maps. For each \( w \) (such that \( T_w \neq \emptyset \)) exactly one of the following statements about \( T_w \) holds:

(A) \( T_w \) is a rigidity class and the stable manifold of a hyperbolic renormalization fixed point. [stable]

(B) \( T_w \) is separated into two nonempty open components by the codimension–1 stable manifold of a hyperbolic renormalization fixed point. These two components are foliated by finite codimension rigidity classes and the successive renormalizations of any \( f \) in these components degenerate by flipping. [saddle]

(C) \( T_w \) is separated into three nonempty open components by the codimension–1 stable manifold of a hyperbolic renormalization period-two point. One component is a rigidity class and the stable manifold of a hyperbolic renormalization fixed point. The remaining two components are foliated by finite codimension rigidity classes and the successive renormalizations of any \( f \) in these components degenerate by flipping. [period-two saddle]

Conversely, each of the above three cases do occur.

**Remark 1.27.** The Lorenz Renormalization Conjecture can be generalized from stationary to periodic combinatorics by considering iterates of \( \mathcal{R} \). With some loss in readability it could be stated for bounded type (there are still only the same three possible cases). For unbounded combinatorics it is not clear what the right conjecture should be as it is possible to force successive renormalizations to not be relatively compact by choosing larger and larger return times for one branch. This leads to Lorenz maps whose attractor do not have a physical measure [Martens and Winckler 18].

**Remark 1.28.** A very surprising feature of Lorenz maps is that the dimension of the unstable manifold of a renormalization fixed point depends on the combinatorics; in cases (A) and (C) the dimension is two and in case (B) it is three. Two of the unstable directions are always related to moving the two critical values; a third unstable direction is gained when the movement of the critical point under renormalization becomes unstable (see Figure 8). In the confounding case (C) there is a mix of both: the fixed point has two unstable directions, whereas the period–2 point has three unstable directions. This situation occurs e.g. for monotone \((8, 2)\)-type when the critical exponent is two (see Figure 7).

**Remark 1.29.** Evidence for case (A) is supported by Martens and Winckler [14]. More recent is Martens and Winckler [17] where the unstable behavior of the renormalization operator within topological classes was discovered; it supports case (B). An unpleasant deficiency in these articles is that no upper bound on the dimension of the unstable manifold is given but

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8Just as the one unstable direction for unimodal renormalization is related to moving the one critical value.
the numerics in this article provides evidence of the conjectured dimension. Case (C) is so far only supported by this article. Numerically no other cases seem to occur, see §2.6 for examples of each case.

**Remark 1.30.** Fixed points, $f$, of monotone $(a, a)$–type are symmetric and they are in one-to-one correspondence with unimodal renormalization fixed points; it is an exercise to verify that the unimodal map $g(x) = f(\min\{x, 1-x\})$, with $g(0.5) = 1$, is a fixed point of the unimodal renormalization operator. In particular, the monotone $(1, 1)$–type Lorenz renormalization fixed point corresponds to the well-known fixed point of the unimodal period-doubling operator.

It seems reasonable to expect all of these “unimodal fixed points” to be dynamically similar, but curiously they are not; conjecturally, for $a>\max\{2x-1, 2\}$ they belong to case (B), else they belong to case (A). For example, when $x = 2$ this “bifurcation” occurs for $a = 4$, see §2.6.

**Remark 1.31.** Compare the Lorenz Renormalization Conjecture with the classical systems of unimodal maps, critical circle maps, etc. In these systems only case (A) can occur and the set of renormalization, $\mathcal{A}$, is a **horseshoe**: that is, $\mathcal{A}$ is hyperbolic and the restriction $\mathcal{R}|_{\mathcal{A}}$ is conjugate to a full shift on infinitely many symbols. Furthermore, orbits of the renormalization operator (where defined) are exponentially contracted to $\mathcal{A}$ [Avila and Lyubich 11].

As a counterpoint, the limit set of Lorenz renormalization cannot be a horseshoe due to case (C); instead, it seems to strictly contain a horseshoe which because of case (B) does not attract all orbits of renormalization.

### 1.6. Consequences on the dynamics of Lorenz maps

In this section, we discuss the consequences the Lorenz Renormalization Conjecture has on the dynamics of infinitely renormalizable Lorenz maps. Recall that on infinitely renormalizable unimodal maps and critical circle maps (of bounded type) renormalization always contracts, as in case (A). As a consequence, there is universality in phase space and parameter space as discussed in §1.1.

For the rest of this section assume that $f$ and $g$ are two topologically conjugate infinitely renormalizable Lorenz maps of bounded type. Regardless of which case $f$ and $g$ belong to they should have a Cantor attractor according to conjecture 1.17. In the stable case (A), $f$ and $g$ are automatically in the same rigidity class (see §1.4) so the conjugacy maps the attractors diffeomorphically onto each other. In particular, they have the same Hausdorff dimension. On the other hand, in cases (B) and (C) if $f$ is in the stable manifold of the fixed point $f$, and if the successive renormalizations of $g \in T_w$ degenerate by flipping, then the Hausdorff dimension of $g$ is zero whereas for $f$ it will be positive. This is called the **coexistence** phenomenon; i.e. that one topological class may contain maps with bounded geometry as well as maps with degenerate geometry. Note that in case (C) we can even pick $f$ and $g$ so that the Hausdorff dimension of their attractors are positive but distinct by choosing $f$ in the stable manifold of the fixed point and $g$ in the stable manifold of the period-two point. In fact, $f$ and $g$ are not even quasi-symmetrically conjugate, i.e. there is no quasi-symmetric rigidity.

In case (A) one finds universality in phase space identical to the case of circle maps and unimodal maps. However, in cases (B) and (C), the scalings of $f$ and $g$ are necessarily asymptotically the same only if they belong to the same rigidity class. If $f$ and $g$ are in different rigidity classes then the conjugacy is not differentiable on their attractors. For example, it is possible to choose topologically conjugate $f$ and $g$ such that their Hausdorff dimension is the same (zero) but the conjugation is not differentiable on their attractors (i.e. by picking them in distinct rigidity classes where the successive renormalizations degenerate).

To understand universality in parameter space, consider a topologically full family of maps $\lambda \mapsto F(\lambda)$ which hits a given topological class $T_w$. By Conjecture 1.19 such a family can be taken to have dimension two. In case (A) the dynamics of $\mathcal{R}$ is contractive so there is universality in parameter space analogous to the case of circle maps and unimodal maps. On the other hand, in cases (B) and (C) the bifurcation patterns the family $F$ depend on which rigidity class the family hits. In order for a family to exhibit all possible bifurcation patterns associated with the class $T_w$ it needs to hit all rigidity classes. This can only happen for a family of dimension at least three. This is called **dimensional discrepancy**; i.e. the dimension of a topologically full family is too small to exhibit all possible bifurcation patterns. Another way to state this is that there is a difference between topologically full families (of dimension two) and geometrically full families (of dimension three).

To summarize, the appearance of directions of instability of renormalization within a topological class $T_w$ leads to a stratification of $T_w$ into several rigidity...
As follows: given \((c, v, \phi)\), where \(v = (v_0, v_1)\) and \(\phi = (\phi_0, \phi_1)\), define \(F(c, v, \phi)\) to be the Lorenz map \(f : [0, 1] \times [c] \to [0, 1]\) whose branches \(f_0 : [0, c] \to [v_0, 1]\) and \(f_1 : [c, 1] \to [0, v_1]\) are the rescalings of \(\phi_0(1-(1-x)^3)\) and \(\phi_1(x^2)\), respectively (see Definition 1.11). The parameters \(v = (v_0, v_1)\) are called boundary values.

**Remark 2.2.** It is clear that \(F\) is injective; furthermore, its image is renormalization invariant by Lemma 2.5.

**Definition 2.3.** Let \(D \subset \mathcal{D}\) be a finite-dimensional subset of diffeomorphisms together with a projection \(\text{proj}_D : \mathcal{D} \to D\). Let

\[
L = (0, 1) \times [0, 1) \times (0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} 
\]

denote the set of truncated Lorenz maps.

**Remark 2.4.** For simplicity of implementation, we choose \(D\) to be a set of piecewise linear homeomorphisms. Of course, this is not a subset of diffeomorphisms but for the purpose of the numerics it empirically does not matter.

To address the issue of smoothness, cubic interpolation could be used instead of linear interpolation, but then care has to be taken that the interpolation is monotone. Another idea is to linearly interpolate functions on \([0, 1]\) and taking the inverse of the nonlinearity operator; this would ensure monotonicity as well as \(C^2\)-smoothness. A third idea is to use finite pure internal structures, which ensures monotonicity and \(C^\infty\)-smoothness [Martens and Winckler 14].

We choose not to pursue these paths here as the implementation would become more involved and since it would not give qualitatively different results.

**2.2. Truncated renormalization**

**Lemma 2.5.** Let \(f = F(c, v, \phi)\) as in Definition 2.1. If \(f\) is \(w\)-renormalizable, then \(\mathcal{R}f = F(c', v', \phi')\) for some \((c', v', \phi')\). Explicitly, let

\[
k_w = |w_k|, p_0 = f^{n_0-1}(0), p_1 = f^{n_0-1}(1), \phi_0(x) = v_0 + (1 - v_0)\phi_0(x), \text{ and } \phi_1(x) = v_1\phi_1(x);
\]

then

\[
c' = \frac{c - p_0}{p_1 - p_0}, \quad v'_0 = \frac{f^{n_1}(p_0) - p_0}{p_1 - p_0}, \quad v'_1 = \frac{f^{n_1}(p_1) - p_0}{p_1 - p_0},
\]

and \(\phi'_0, \phi'_1\) are the respective rescalings of...
f^{n_0-1} \circ \tilde{\phi}_0 : [\tilde{\phi}_0^{-1} \circ f(p_0), 1] \to [f^{n_0}(p_0), p_1],
\quad f^{n_1-1} \circ \tilde{\phi}_1 : [0, \tilde{\phi}_1^{-1} \circ f(p_1)] \to [p_0, f^{n_1}(p_1)].

**Proof.** Denote the first-return map associated with the renormalization by \( g : J \setminus \{c\} \to J \), where \( J = [p_0, p_1] \). Then \( c' \) is the relative position of \( c \) in \( J \), \( v_0 \) is the relative length of \( g([p_0, c]) \) in \( J \), and \( v_1' \) is the relative length of \( g((c, p_1]) \) in \( J \); written out this is (2–1). The statement for \( \phi_0', \phi_1' \) is just saying that they are the branches of \( g \) without the initial folding \( x^0 \) that comes from \( f J \). Since \( g \) is a first return to \( J \) the \( f \)-images of \( J \) do not meet the critical point before they return; this means that \( \phi_k' \) are diffeomorphisms.

**Definition 2.6.** Let \( F \) and \( (D, \text{proj}_D) \) be as in Definitions 2.1 and 2.3, respectively, and let \( P(c, v, \phi) = (c, v, \text{proj}_D(\phi_0), \text{proj}_D(\phi_1)) \). For every renormalizable \( F(c, v, \phi) \), define the **truncated renormalization operator**, \( R \), by

\[
R(c, v, \phi) = P \circ F^{-1} \circ \mathcal{R} \circ F(c, v, \phi).
\]

This is well-defined by Remark 2.2.

**Remark 2.7.** For a class of monotone combinatorics with \( |w| \) large the renormalization operator is close to having finite dimensional image, in the sense that the diffeomorphisms \( \phi_k' \) in lemma 2.5 are close to being linear [Martens and Winckler 17]. In other words, \( R \) can automatically be a good approximation of \( \mathcal{R} \), depending on the combinatorics.

**Remark 2.8.** Taking the above remark to its extreme, it even makes sense to consider the trivial set \( D = \{\text{id}\} \) of diffeomorphisms, and looking at the corresponding truncated renormalization operator; it is explicitly defined by (2–1) with \( \phi = (\text{id}, \text{id}) \). This is the operator we used to estimate the eigenvalues in Figure 8.

Empirically, it exhibits all the dynamics of the Lorenz Renormalization Conjecture and seems to be a remarkably good approximation of the full renormalization operator as far as qualitative behavior is concerned. This should not come as a great surprise as one method of proving existence of fixed points for \( \mathcal{R} \) involves homotoping to this three-dimensional truncation and proving it has a fixed point [Martens and Winckler 14, 17].

**Definition 2.9.** For every renormalizable \( F(c, v, \phi) \), define the **modified renormalization operator**, \( \tilde{R} : (c, v, \phi) \rightarrow (c', v') \), in the same way as the truncated renormalization operator, except changing (2–1) to

\[
c' = p_0 - c + (p_1 - p_0)c,
\quad v_0' = p_0 - f^{n_0}(p_0) + (p_1 - p_0)v_0,
\quad v_1' = p_0 - f^{n_1}(p_1) + (p_1 - p_1)v_1.
\]

Note that the image of \( \tilde{R} \) is contained in \( \mathbb{R}^3 \).

**Remark 2.10.** The idea of the above operator is to improve the numerical behavior of \( R \) by not dividing by the length of the return interval in (2–1). From the same equation it can be seen that the set of zeros of \( \tilde{R} \) coincide with the set of \( (c, v, \phi) \) for which \( (c, v) \) are fixed by \( R \). We found that the Newton method on \( \tilde{R} \) has better convergence properties than the Newton method on \( R \). Given \( w \), we use it to determine what the right value for \( c \) should be for a truncated renormalization fixed point (see the fixed-point algorithm in the next section).

### 2.3. Locating fixed points

The perhaps simplest idea for locating fixed points of the truncated renormalization operator is to use a Newton iteration. This is feasible for short combinatorics, but for longer combinatorics it is practically impossible to find starting guesses for which it converges.

The method we employ can be thought of as acting on the two-dimensional families \( v \rightarrow F(c, v, \phi) \) (see Definition 2.1). It consists of three separate algorithms: one which determines a \( v \) such that \( F(c, v, \phi) \) is renormalizable, followed either by an algorithm which takes \( F(c, v, \phi) \) and produces a new \( c \), or one which takes \( F(c, v, \phi) \) and produces a new \( \phi \). Combined, these methods empirically behave like a contraction toward a family which contains a renormalization fixed point and for which the first algorithm is a contraction toward this fixed point.

**Definition 2.11.** (Renormalization fixed point algorithm). Input: the combinatorics \( w \).

1. Pick an initial guess for \( c \) and \( \phi \).
2. Apply the modified Thurston algorithm to \( v \rightarrow F(c, v, \phi) \) to get new boundary values \( v' \) (see §2.4 and Remark 2.15).
3. Take a Newton step with the operator \( \tilde{R} \) on \( F(c, v', \phi) \) to get a new critical point \( c' \).
4. Apply the modified Thurston algorithm to \( v \rightarrow F(c', v, \phi) \) to get new boundary values \( v'' \).
5. Apply \( R \) to \( F(c', v'', \phi) \) to get new diffeomorphisms \( \phi' \).
6. Stop if \( (c, v, \phi) = (c', v'', \phi') \), else set \( c = c' \), \( \phi = \phi' \) and go back to step (2).
Output: the Lorenz map $F(c, v, \phi)$ (supposedly a renormalization fixed point).

**Remark 2.12.** The above algorithm empirically seems to converge for the initial guesses $\phi = (\text{id}, \text{id})$ and a large set of $c$. Theoretically, there is no guarantee for the output to be a renormalization fixed point, but practically we observe that it is (as long as the algorithm converges).

### 2.4. The Thurston algorithm

The Thurston algorithm is a fixed-point method that realizes any periodic combinatorics in a full family of maps. It originates in Douady and Hubbard [93] and is also known as the Spider Algorithm in the complex setting [Hubbard and Schleicher 94]. In real dynamics it is usually employed to prove the full family theorem [Martens and de Melo 01, de Melo and van Strien 93]. We use it to locate renormalizable maps within the two-dimensional families $v \to F(c, v, \phi)$ (see definition 2.1).

**Definition 2.13.** (The Thurston Algorithm). Input: a critical point $c$, diffeomorphisms $\phi = (\phi_0, \phi_1)$, and combinatorics $w = (w_0, w_1)$.

1. Pick an initial guess of shadow orbits
   \[
   \{x_k(0) = k, x_k(1), \ldots, x_k(m-1) = c\},
   \]
   Let $W_k$ be the concatenation of $w_k$ followed by $w_{1-k}$, for $k = 0, 1$.
2. Set $v = (x_0(1), x_1(1))$, and let $f = F(c, v, \phi)$ with branches $f_0$ and $f_1$.
3. Pull back $x_k$ with $f$ according to the combinatorics $W_k$:
   \[
y_k(j-1) = f_{W_k(j)}^{-1}(x_k(j)), \quad j = 1, \ldots, m-1, k = 0, 1.
   \]
4. Set $y_k(m-1) = c$, $k = 0, 1$.
5. Stop if $y_k = x_k$, else set $x_k$ to $y_k$, $k = 0, 1$, and go back to (2).

Output: the map $f$ which is a realization of the combinatorics $w$ in the family $v \to F(c, v, \phi)$.

**Remark 2.14.** As long as the initial guess is chosen consistently (i.e. if the shadow orbits are ordered according to $w$) this algorithm is guaranteed to stop; in this case, the realization $f$ is renormalizable and the boundary values of $Rf$ equal the critical point of $Rf$.

In practice, the algorithm converges if the initial guess consists of uniformly spaced points $x_0(0) < \ldots < x_0(m-1)$ and $x_1(0) > \ldots > x_1(m-1)$ even though these are not ordered according to the combinatorics $w$.

**Remark 2.15.** We modify the above algorithm so that the realization $f$ fixes its boundary values under renormalization; i.e. $Rf(k) = f(k)$, for $k = 0, 1$. This is convenient as we are interested in renormalization fixed points. The modification is to replace step (4) with:

(4') Let $p_0 = x_0(|w_1| - 1)$ and $p_1 = x_1(|w_0| - 1)$ and set

\[
y_k(m-1) = p_0 + (p_1 - p_0)v_k, \quad k = 0, 1.
\]

Note that $[p_0, p_1]$ is the return interval of $f$ if $y_k = x_k$, so what this step does is to set the relative boundary values of the first-return map. Replacing $v_k$ with parameters $t_k$ varying in $[0, 1]$, it is possible to find the whole domain of $w$-renormalizability in the family.

**Remark 2.16.** There is a relationship between the modified Thurston algorithm from the previous remark and the renormalization operator—if the modified Thurston algorithm is applied to a family which contains a renormalization fixed point then the output of the algorithm will be the renormalization fixed point. So, the renormalization fixed point is also the fixed point of a contractive “Thurston operator.”

### 2.5. Implementation

The source code for an implementation of the fixed point algorithm of §2.3 is freely available online [Winckler 18]. It compiles to three executables which were used to produce the results of §2.6; see the accompanying README for instructions on how to reproduce the results.

The Eigen library [Guennebaud and Jacob 10] is used for linear equation solvers and eigenvalue estimation; we also use its bindings to the multiple precision library MPFR [Fousse et al. 07, Holoborodko 08] as well as its automatic differentiation routines. Standard double precision arithmetic is only sufficient for short combinatorics, which is why the implementation needs multiple precision. Automatic differentiation is used to evaluate the derivative of $R$. Note that this is not the same thing as numerical differentiation (taking finite differences); instead it uses the chain-rule to exactly (up to numerical precision) evaluate derivatives.
It is known that

Remark 2.17. It is known that \( a \) and \( b \) sufficiently large implies case (B) [Martens and Winckler 17].

2.6. Results

The experiments in this section were performed using a truncation of \( R \) in dimension three up to dimension 1000. Higher dimensions were needed only when evaluating the renormalization of period-2 points, such as in Figure 7, otherwise the three-dimensional truncation gave qualitatively accurate results. Results are only stated for monotone combinatorics; some non-monotone combinatorics were tested as well but it is harder to present these in a clear manner so they are not included. The programs also work with arbitrary \( z \) but experiments investigating the \( z \)-dependence have been left out to keep this section focused.

The following table shows which of case (A), (B) or (C) of the Lorenz Renormalization Conjecture the first few monotone \((a, b)\)-types fall under for \( z = 2 \):

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | \( (a, b) \) |
|---|---|---|---|---|---|---|---|---|----------------|
| A | A | A | A | A | A | A | A | A | \( 1 \) |
| A | A | A | A | C | C | C | \( 2 \) | |
| A | B | B | B | B | B | B | \( 3 \) | |
| B | B | B | B | B | B | \( 4 \) | |
| B | B | B | B | B | \( 5 \) | |
| B | B | B | B | \( 6 \) | |
| B | B | B | \( 7 \) | |
| B | B | \( 8 \) | |
| B | \( 9 \) | |

For example, the above table shows that \((a, a)\)-type has a two-dimensional unstable manifold for \( a = 1, 2, 3 \), and a three-dimensional unstable manifold for \( a \geq 4 \); \((a, 2)\)-types with \( a \geq 7 \) has both a fixed point and a period-2 point. Note that the complete table is symmetric about the diagonal.

Figure 8. Dependence of the eigenvalue associated with movement of the critical point on monotone type \((a, b)\) for \( z = 2 \).

It is not clear exactly when case (C) occurs; from the above table only \((a, 1)\)-type and \((a, 2)\)-type seem viable, but a test with increasing \( a \) did not reveal any \((a, 1)\)-types of case (C). Note that we are only discussing stationary combinatorics and \( z = 2 \) here.

In creating the above table, we performed roughly the following steps:

1. Locate a fixed point for the three-dimensional truncated renormalization operator (see Remark 2.8), using \( c = 0.5 \) as an initial guess for the critical point; if it doesn’t converge, try other values for \( c \) until it does.

The derivative of the three-dimensional truncation of \( R \) at the fixed point has three eigenvalues. Denote the eigenvalue with the smallest magnitude by \( \lambda_c \); this is the eigenvalue associated with moving the critical point (the other two eigenvalues are associated with changing the boundary values). If \( \lambda_c \in (0, 1) \) then we must be in case (A); if \( \lambda_c \in (-1, 0] \) we go to the next step; if \( |\lambda_c| > 1 \) we must be in case (B). The behavior of \( \lambda_c \) is illustrated in Figure 8.

2. Try to locate a period-2 orbit of \( R \) by looking for a fixed point of twice \((a, b)\)-renormalizable type. We observe in this situation that one of three things happen:

i. the algorithm diverges by \( c \uparrow 1 \) (most common case),

ii. the algorithm converges to the fixed point found in the previous step (only seems to happen if \( c \) is picked close to the \( c \) of the fixed point),

iii. the algorithm converges and \( c \) is different from that of the fixed point.

In the first two situations we are in case (A) and in the last situation we are in case (C). In the first two situations this step is repeated with different guesses for \( c \) to make sure the last situation was not missed due to a bad initial guess.

The graphs of the fixed point and period-2 orbit for \((8,2)\)-type can be found in Figure 7.

3. Increase the dimension of the truncation of \( R \) to see if it affects the above classification; in all cases we tried the eigenvalues changed slightly in value but not enough to affect the classification.

For example, once \((2, 1)\)-renormalizable type is given by \((011, 10)\) and twice \((2, 1)\)-renormalizable is given by \((0111010, 10011)\).
Notation

\( f, f_0, f_1 \)  
Lorenz map \( f \) with branches \( f_0, f_1 \)

\( c, c(f) \)  
the critical point of \( f \)

\( \alpha \)  
critical exponent

\( \mathcal{L} \)  
set of Lorenz maps

\( w = (w_0, w_1) \)  
type of renormalization

\( \mathcal{R} \)  
renormalization operator

\( T_w \)  
topological class

\( v = (v_0, v_1) \)  
boundary values, \( v_k = f(k) \)

\( \phi = (\phi_0, \phi_1) \)  
diffeomorphisms

\( F \)  
family of Lorenz maps \( F(c, v, \phi) \)

\( D, \text{proj}_D \)  
finite-dimensional diffeomorphism, projection

\( L \)  
set of truncated Lorenz maps

\( R, \hat{R} \)  
truncated renormalization operators

Funding

This project has received funding from the European Union’s Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 743959.

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