A unified and systematic scheme for construction of differential operator realization of any irreducible representation of $sl(n)$ is developed. The $q$-analogue of this unified scheme is used to construct $q$-difference operator realization of any irreducible representation of $U_q(sl(n))$. Explicit results for $U_q(sl(2))$, $U_q(sl(3))$ and $U_q(sl(n))$ are given.

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1 Introduction

Differential operator realization of Lie algebras are well-known\[1\] and useful in discussing physical systems, both in Quantum Mechanics \[2\] and Field Theory\[3\]. There is a well-known correspondence \[3,4\] between differential operator realization of Lie algebras\[1,3\] and free field representation\[5-9\] of Kac-Moody algebras\[10-11\]. A natural question is whether such a correspondence exists in the case of the deformed algebras too. More precisely, one may ask if there is any relation between free field representation\[13-18\] of quantum affine Kac-Moody algebras(e.g. $U_q(\widehat{sl(n)})$\[19-22\]) and $q$-difference operator realization of quantum algebras(e.g. $U_q(sl(n))$\[23-25\]).

To address this problem, the first step would be to find a $q$-difference operator realization of $U_q(sl(n))$. There has been an attempt for such a realization in ref.\[26\], but it is limited to symmetric representations of $U_q(sl(3))$\[26\]. Our aim in this letter is to present, in a unified scheme, a $q$-difference operator realization of $U_q(sl(n))$. One of the new features of this scheme is that, one is able to construct such a realization, not only for symmetric representations but for any irreducible representations.

The structure of this letter is as follows: In section 2, we will reproduce the differential realization of $sl(2)$, $sl(3)$ and $sl(n)$, for any irreducible representation, by a unified scheme. In section 3, we will go through $U_q(sl(2))$ as a simple example and $U_q(sl(3))$ as a non-trivial example and finally we will present the $q$-difference operator realization of $U_q(sl(n))$, for any irreducible representations.

2 Differential operator realization of $sl(n)$

The simple case of $sl(2)$ will be considered first in order to explain the procedure. The well-known $sl(2)$ finite-dimensional representations with one complex variable will be recovered by a systematic method. Then we shall take $sl(3)$ as a second example of this procedure and next we will generalize them for $sl(n)$.

2.1 $sl(2)$ realization

There are two ways to construct the differential operator representations of $sl(2)$. The first method is to start from the fundamental representation and by tensor multiplying these fundamental irreducible representation, construct one (two) variable(s) differential operator(s) realization of higher dimensional representations, which can be found in \[2\] and \[26\]. In this method one can find realizations of only the symmetric representation. Here we proceed along a different line in order to find such realizations for all kinds of representations (note that this difference, doesn’t show itself, for $sl(2)$.). However, the systematic and generalizable method is

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3 In ref.\[26\], higher dimensional representations have been constructed by taking tensor products of fundamental representations and the coproduct to define the action of the generators on the basis vectors of the algebra. Then $q$-difference operator realization of $U_q(sl(3))$ are constructed. By this method only symmetric irreducible representation can be obtained. Due to its complexity, is not generalizable to higher rank algebras.
the second one, which we will now describe. Let \( X^\pm \) and \( H \) be the Chevalley basis generators of \( \mathfrak{sl}(2) \) with commutation relations

\[
[X^+, X^-] = H \quad [H, X^\pm] = \pm 2X^\pm
\]

and let \( \mu \) be any arbitrary highest weight,

\[
X^+ |\mu> = 0, \quad H |\mu> = 2\mu |\mu> . \tag{2}
\]

Now consider the states

\[
e^{zX^-} |\mu> \tag{3}
\]

in the representation space, where \( z \) is a complex variable. The basis vectors of this vector space are \( |\mu>, X^- |\mu>, ..., (X^-)^{2\mu} |\mu> \).

By the action of \( X^-, X^+ \) and \( H \) on (3) and using the above commutation relations we will have:

\[
X^- e^{zX^-} |\mu> = \partial_z e^{zX^-} |\mu>
\]

\[
X^+ e^{zX^-} |\mu> = (2z\mu - z^2 \partial_z) e^{zX^-} |\mu>
\]

\[
H e^{zX^-} |\mu> = (2\mu - 2z \partial_z) e^{zX^-} |\mu> ; \tag{4}
\]

let define

\[
J^+ = \partial_z \tag{5}
\]

\[
J^- = 2\mu - z^2 \partial_z \tag{6}
\]

\[
H = 2\mu - 2z \partial_z \tag{7}
\]

where \( \partial_z = \frac{\partial}{\partial z} \). One can consider this, as the representation of \( \mathfrak{sl}(2) \) on the sub-space of analytic functions spanned by the monomials, \( \{1, z, z^2, ..., z^{2\mu}\} \). We will find that this new operator representation will satisfy the algebra of (1). The above realization is a one-complex variable differential operator representation of \( \mathfrak{sl}(2) \), when \( z \in \mathbb{C} \). To see the complications involved in the general case, we will take \( \mathfrak{sl}(3) \) as a second example.

### 2.2 \( \mathfrak{sl}(3) \) realization

In this case the states in the representation space will be:

\[
e^{x_1 X^-_1} e^{x_{12} X^-_{12}} e^{x_2 X^-_2} |\mu> \tag{8}
\]

where the generators \( X^-_i \) \( (i = 1, 2) \) correspond to the simple roots \( \alpha^i \), \( X^-_{12} \) is the generator corresponds to the non-simple root

\[
\alpha_{12} = \alpha^1 + \alpha^2
\]

and \( x_1, x_2 \) and \( x_{12} \) are the complex variables. In this type of ordering the factor corresponding to the descendents of two roots lies between them. The defining relations for \( \mathfrak{sl}(3) \) are as follows:

\[
[X^\pm_1, X^\pm_2] \equiv \pm X^\pm_{12} \tag{9}
\]

2
\[ [X_i^+, X_{12}^-] = 0 \quad (10) \]
\[ [X_i^+, X_j^-] = \delta_{ij} H_j, \quad i, j = 1, 2 \quad (11) \]
\[ [H_i, X_j^\pm] = \pm \alpha_{ij} H_i \quad (12) \]
\[ [H_i, X_{12}^\pm] = \pm X_{12}^\pm \quad (13) \]
\[ [H_1, H_2] = 0 \quad (14) \]

where \((a)_{ij}\) is the Cartan matrix. The highest weight \(\mu\) is defined such that:

\[ X_i^+ | \mu >= 0, \quad H_i | \mu >= 2\alpha^i \cdot \mu | \mu >, \quad (15) \]

where \(2\alpha^i \cdot \mu\) is the Dynkin index (i.e. \(\frac{2\alpha^i \cdot \mu}{\alpha^i \cdot \alpha^i} = m^i\)) of the representation, and where we have normalized the roots to \(\alpha^i \cdot \alpha^i = 1\). By considering the action of generators \((X^\pm, H)\) on the states given by (8), we find the following differential operators realization for any irreducible representation with:

\[ J_{\alpha^1}^+ = \partial_{x_1} \quad (16) \]
\[ J_{\alpha^1}^- = -x_1^2 \partial_{x_1} + x_1(x_2 \partial_{x_2} - x_{12} \partial_{x_{12}} + 2\alpha^1 \cdot \mu) - x_{12} \partial_{x_2} \quad (17) \]
\[ J_{\alpha^2}^+ = \partial_{x_2} + x_1 \partial_{x_{12}} \quad (18) \]
\[ J_{\alpha^2}^- = -x_2^2 \partial_{x_2} + x_{12} \partial_{x_1} + 2\alpha^2 \cdot \mu x_2 \quad (19) \]
\[ J_{\alpha^{12}}^+ = \partial_{x_{12}} \quad (20) \]
\[ J_{\alpha^{12}}^- = -x_{12} \partial_{x_{12}} - x_1(x_{12} \partial_{x_1} - x_2^2 \partial_{x_2} + 2\alpha^2 \cdot \mu x_2) - x_{12} x_2 \partial_{x_{12}} + 2\alpha_{12} \cdot \mu x_{12} \quad (21) \]
\[ H_1 = -2x_1 \partial_{x_1} + x_2 \partial_{x_2} - x_{12} \partial_{x_{12}} + 2\alpha^1 \cdot \mu \quad (22) \]
\[ H_2 = x_1 \partial_{x_1} - 2x_2 \partial_{x_2} - x_{12} \partial_{x_{12}} + 2\alpha^2 \cdot \mu \quad (23) \]

in agreement with ref.[2]. It is clear that all of the algebra elements can be constructed by \(J_{\alpha^1}^\pm\) and \(J_{\alpha^2}^\pm\).

The above construction is complete for any representation, with Dynkin indices \((m, n)\)

\[
\begin{array}{c@{\quad}c}
m & n \\
\circ & \circ
\end{array}
\]

such that \(m, n \neq 0\). For any asymmetric representations \((m = 0)\), and symmetric representation \((n = 0)\), the realization must be slightly revised as follows:

For asymmetric \((0, n)\) representations, states in the representation space will be:

\[ e^{x_{12}X_{12}^-} e^{x_2 X_2^-} | \mu > \quad (24) \]

since \(X_{1}^\pm | \mu >= 0\) and \(X_2^+ | \mu >= 0\). After repeating the same procedure as in \((m,n)\) case one finds:

\[ J_{\alpha^1}^+ = -x_2 \partial_{x_{12}} \quad (25) \]
\[ J_{\alpha^1}^- = -x_{12} \partial_{x_2} \quad (26) \]
\[ J_{\alpha^2}^+ = \partial_{x_2} \quad (27) \]
\[ J_{\alpha^2} = -x_2^2 \partial_{x_2} + x_2(-x_{12} \partial_{x_{12}} + 2\alpha^2 \cdot \mu) \] (28)
\[ H_1 = x_2 \partial_{x_2} - x_{12} \partial_{x_{12}} \] (29)
\[ H_2 = -2x_2 \partial_{x_2} - x_{12} \partial_{x_{12}} + 2\alpha^2 \cdot \mu. \] (30)

For symmetric representations \((m, 0)\), states in representation space will be:
\[ e^{x_1 X_i^-} e^{x_{12} X_{12}^-} |\mu > \] (31)

and the generators are found to be:
\[ J_{\alpha^1}^+ = \partial_{x_1} \] (32)
\[ J_{\alpha^1}^- = -x_1^2 \partial_{x_1} + x_1(-x_{12} \partial_{x_{12}} + 2\alpha^1 \cdot \mu) \] (33)
\[ J_{\alpha^2}^+ = x_1 \partial_{x_{12}} \] (34)
\[ J_{\alpha^2}^- = x_{12} \partial_{x_1} \] (35)
\[ H_1 = -2x_1 \partial_{x_1} - x_{12} \partial_{x_{12}} + 2\alpha^1 \cdot \mu \] (36)
\[ H_2 = x_1 \partial_{x_1} - x_{12} \partial_{x_{12}}. \] (37)

So, the minimum number of variables for \((m, n)\), \((0, n)\) and \((m, 0)\) representations are 3, 2, 2 respectively. Comparing expressions (25-30) and (32-37) with (16-23) one finds the following simple prescription for obtaining the differential realization for the special \((m, 0)\) and \((0, n)\) cases from the general \((m, n)\) case.

1- \((m, 0)\) case: set \(x_1\) equal to zero and replace \(\partial_{x_1}\) by \(-x^2 \partial_{x_{12}}\).
2- \((0, n)\) case: set \(x_2\) and \(\partial_{x_2}\) equal to zero.

### 2.3 \(sl(n)\) realization

The Lie algebra \(sl(n)\) defined by the generators \(X^\pm\) and \(H_i\) \((i = 1, ... r = n - 1)\) and the following relations:
\[ [H^+_i, H^-_j] = 0 \] (38)
\[ [X^+_i, X^-_j] = \delta_{i,j} H_i \] (39)
\[ [X^+_i, X^-_j] = 0, \quad if \quad a_{ij} = 0 \] (40)
\[ [H_i, X^\pm_j] = \pm a_{ij} X^\pm_j \] (41)

with Serre relations:
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} q(X^\pm_i)^k X^\pm_j (X^\pm_i)^{n-k} = 0, \quad i \neq j \] (42)

in order to construct differential realization of this algebra, take the following states as states in the representation space:
\[ \prod_{i=1}^{r} \prod_{j=1}^{r} e^{x_{ij} X_i^-} |\mu > \] (43)
where $X^\pm_{ij}$s are the generators correspond to the non-simple roots $\alpha_{ij} = \alpha^i + \alpha^{i+1} + \cdots + \alpha^j$ and $X^\pm_i \equiv X^\pm_i$ corresponds to the simple roots $\alpha_i$.

By similar calculations as in previous cases we obtain the following realization for $sl(n)$ with any arbitrary highest weight $\mu$ in the Chevalley basis:

\begin{equation}
J^+_\alpha^i = \partial_{x_i} + \sum_{j=1}^{i-1} x_{j-i} \partial_{x_j},
\end{equation}

\begin{equation}
J^-_\alpha^i = \sum_{j=1}^{i-1} x_{ji} \partial_{x_{j-i+1}} - x_i (-x_i \partial_{x_i} + \sum_{j=i}^{r} \sum_{k=j}^{r} \sum_{l=j}^{k} a_{il} x_{jk} \partial_{x_{jk}} - 2\alpha^i \cdot \mu) - \sum_{j=i+1}^{r} x_{ij} \partial_{x_{j+1}}
\end{equation}

\begin{equation}
H_{\alpha^i} = -\sum_{j=1}^{r} \sum_{k=j}^{r} \sum_{l=j}^{k} a_{il} x_{jk} \partial_{x_{jk}} + 2\alpha^i \cdot \mu
\end{equation}

where $x_{ii} = x_i$. After straightforward, but lengthy calculation one finds that the above expressions satisfy the $sl(n)$ algebra.

For any representation, if the $i$th Dynkin index is zero ($\alpha^i \cdot \mu = 0$), we must set $x_i$ equal to zero and replace $\partial_{x_i}$ by

\begin{equation}
-\sum_{j=i+1}^{r} x_{j-i} \partial_{x_j}
\end{equation}

in the above expressions. In the case of several zero adjacent Dynkin indices ($\alpha_i \cdot \mu = \alpha_{i+1} \cdot \mu = \cdots = \alpha_j \cdot \mu = 0$), one must set all variables $x_{mm'}$ $i \leq m < m' \leq j$ equal to zero and replace $\partial_{x_{mm'}}$ by

\begin{equation}
-\sum_{k=m'+1}^{r} x_{m'+k} \partial_{x_{mk}}
\end{equation}

For example, in case of fundamental representation where all Dynkin indices but first one are zero, we are left with $n - 1 = r$ variables.

### 3 $U_q(sl(n))$

Let us first set the notations. Consider $q$-exponential function:

\begin{equation}
e^x_q = \sum_{n=0}^{\infty} \frac{(1-q)^n x^n}{(1-q)(1-q^2)\cdots(1-q^n)}
\end{equation}

and the $q$-difference operator

\begin{equation}
D^\pm_x f(x) = \frac{f(q^\pm x) - f(x)}{(q^\pm - 1)x} = \frac{1}{(q^\pm - 1)x} (M_x^\pm 1 - 1) f(x)
\end{equation}

where $M_x$ is a translation operator, defined by $M_x^a f(x) = f(q^a x)$.
3.1 \( U_q(sl(2)) \) and \( U_q(sl(3)) \) realization

Just as in the case of ordinary \( sl(2) \), we take the states

\[
e^\pm_{\mu} | \mu >
\]

in the representation space. The algebra relations are:

\[
[X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}} \quad [H, X^\pm] = \pm 2X^\pm
\]

with the same procedure as of section 2.1 one will find the following \( q \)-difference operator realization for any arbitrary highest weight representation:

\[
J^+ = D^+_z
\]

\[
J^- = \frac{1}{q - q^{-1}}z(M_z^{-2}q^{2\mu} - q^{-2\mu})
\]

\[
H = -2z\partial_z + 2\mu
\]

in agreement with [26].

For \( U_q(sl(3)) \) realization we take

\[
e_{\mu}^{x_1X^\pm} e_{\mu}^{x_2X^\pm} e_{\mu}^{x_3X^\pm} | \mu >
\]

as the states in the representation space. This type of ordering is well-known in quantum algebra, and called ”normal” in [29-30]. The algebra relations are:

\[
[X^+_i, X^-_j] = \delta_{i,j} q^{H_i} - q^{-H_i} = \delta_{i,j} [H_i]_q
\]

\[
[X^\pm_1, X^\pm_2]_{q^{1/2}} = q^{\pm 1/2}X^\pm_1X^\pm_2 - q^{\mp 1/2}X^\pm_2X^\pm_1 = \pm X^\pm_12
\]

\[
[X^\pm_1, X^\pm_2]_{q^{1/2}} = q^{\mp 1/2}X^\pm_1X^\pm_2 - q^{\pm 1/2}X^\pm_2X^\pm_1 = 0
\]

\[
[H_i, X^\pm_{jk}] = \pm(\sum_{l=j}^{k} a_{ij})X^\pm_{jk}
\]

\[
[H_i, H_j] = 0.
\]

Now by the action of \( X^\pm_1 \) and \( H_i \) (\( i, j = 1, 2 \)) on the above states, we will have the following \( q \)-difference operator realization for \( U_q(sl(3)) \):

\[
J^+_{\alpha_1} = D_{x_1}
\]
\[ J_{\alpha^1}^- = \frac{1}{q+1} \left[ \frac{1}{q-1}(1 + M_{x_1}^{+1})x_1(M_{x_1}^{-2}M_{x_1}^{+1}M_{x_2}^{+1}q^{2\alpha^1\mu} - M_{x_1}^{+1}M_{x_2}^{-1}q^{-2\alpha^1\mu}) \right. \]
\[ \left. - q^{-1/2}(1 + M_{x_2}^{+1})x_{12}M_{x_2}^{+1}q^{-2\alpha^1\mu}D_{x_2}^+ \right] \] (59)

\[ J_{\alpha^2}^+ = \frac{q^{1/2}}{q+1}M_{x_1}^{-1}(1 + M_{x_1}^{+1})x_1D_{x_1}^+ + M_{x_1}^{-1}M_{x_2}^{+1}D_{x_2}^+ \] (60)

\[ J_{\alpha^2}^- = \frac{1}{1+q} \left[ q^{1/2}(1 + M_{x_1}^{+1})x_{12}M_{x_1}^{-1}M_{x_2}^{-2}q^{2\alpha^2\mu}D_{x_1}^+ \right. \]
\[ \left. + \frac{1}{q-1}(1 + M_{x_2}^{+1})x_2(M_{x_2}^{-2}q^{2\alpha^2\mu} - q^{-2\alpha^2\mu}) \right] \] (61)

\[ H_1 = -2x_1\partial_{x_1} - x_{12}\partial_{x_{12}} + x_2\partial_{x_2} + 2\alpha^1 \cdot \mu \] (62)

\[ H_2 = x_1\partial_{x_1} - x_{12}\partial_{x_{12}} - 2x_2\partial_{x_2} + 2\alpha^2 \cdot \mu. \] (63)

As before the above elements of the algebra will satisfy the corresponding algebra for any arbitrary representation with non-zero Dynkin indices. For asymmetric case where the first Dynkin index is zero, we should set \( x_1 \) equal to zero and \( D_{x_1}^+ \) must be replace by
\[ -\frac{q^{1/2}}{1+q}M_{x_2}^{-1}(1 + M_{x_2}^{+1})x_2D_{x_2}^+ \]
in the algebra elements. The final result for asymmetric representations \((0,n)\) are as follows:

\[ J_{\alpha^1}^+ = -\frac{q^{1/2}}{1+q}M_{x_1}^{-1}(1 + M_{x_1}^{+1})x_2D_{x_2}^+ \] (64)

\[ J_{\alpha^1}^- = -\frac{q^{3/2}}{1+q}M_{x_1}^{-2}(1 + M_{x_1}^{+1})x_{12}M_{x_2}^{-1}D_{x_2}^+ \] (65)

\[ J_{\alpha^2}^+ = M_{x_2}^{+1}D_{x_2}^+ \] (66)

\[ J_{\alpha^2}^- = \frac{1}{1+q} \left[ -q^{-1}(1 + M_{x_1}^{+1})x_{12}M_{x_1}^{-2}M_{x_2}^{-2}(1 + M_{x_2}^{+1})x_2q^{2\alpha^2\mu}D_{x_1}^+ \right. \]
\[ \left. + \frac{1}{q-1}(1 + M_{x_2}^{+1})x_2(M_{x_2}^{-2}q^{2\alpha^2\mu} - q^{-2\alpha^2\mu}) \right] \] (67)

\[ H_1 = -x_{12}\partial_{x_{12}} + x_2\partial_{x_2} \] (68)

\[ H_2 = -x_{12}\partial_{x_{12}} - 2x_2\partial_{x_2} + 2\alpha^2 \cdot \mu \] (69)

Similarly, where the second Dynkin index is zero, we should set \( x_2 \) equal to zero, and replace \( D_{x_2}^+ \) by zero. So, the final result for symmetric \((m,0)\) representations will be as follows:

\[ J_{\alpha^1}^+ = D_{x_1}^+ \] (70)
\[ J_{\alpha^1}^- = \frac{1}{1 + q q - q^{-1}} (1 + M_{x_1}^{-1}) x_1 (M_{x_1}^{-2} M_{x_1}^{-1} q^{2 \alpha^1 \mu} - M_{x_1}^{-1} q^{-2 \alpha^1 \mu}) \] (71)

\[ J_{\alpha^2}^+ = \frac{q^{1/2}}{1 + q} (1 + M_{x_1}^{-1}) x_1 D_{x_1}^+ \] (72)

\[ J_{\alpha^2}^- = \frac{q^{1/2}}{q + 1} M_{x_1 12}^{-1} (1 + M_{x_1}^{+12}) x_1 D_{x_1}^+ \] (73)

\[ H_1 = -2 x_1 \partial x_1 - x_{12} \partial x_{12} + 2 \alpha^1 \cdot \mu \] (74)

\[ H_2 = x_1 \partial x_1 - x_{12} \partial x_{12} \] (75)

### 3.2 \( U_q(sl(n)) \) realization

For the general case \( U_q(sl(n)) \) we take the following states in representation space:

\[ \prod_{i=1}^r \prod_{j=1}^r c_{q_j} x_{ij} X_{ij}^- |\mu > \] (76)

The algebra of \( U_q(sl(n)) \) in the Chevalley basis can be summarized as follows[23-24]:

\[ [H_i, H_j] = 0 \] (77)

\[ [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm \] (78)

\[ [X_i^+, X_j^-] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} \] (79)

\[ [X_i^+, X_j^+] = 0, \quad \text{if} \quad a_{ij} = 0 \] (80)

with Serre relations,

\[ \sum_{k=0}^n (-1)^k \binom{n}{k}_q (X_i^\pm)^k X_j^\pm (X_i^\pm)^{n-k} = 0, \quad i \neq j \] (81)

with \( a_{ij} = \frac{2 (\alpha^i \cdot \alpha^j)}{(\alpha^i \cdot \alpha^j)} \), Cartan matrix, \( n = 1 - a_{ij} \) and

\[ \binom{n}{k}_q = \frac{[n]_q!}{[k]_q ![n-k]_q!}, \quad [m]_q! = [m]_q [m-1]_q ... [1]_q, \quad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} \] (82)

By the same procedure, difference operator representation of algebra elements in Chevalley basis are as follows:

\[ J_{\alpha^1}^+ = \left( \delta_{i,1} + \prod_{j=1}^{i-1} \prod_{k=j}^r \sum_{l=j}^k a_{li} \right) D_{x_i}^+ \]

\[ + \frac{q^{1/2}}{1 + q} \sum_{j=1}^{i-1} \prod_{k=1}^{i-1} M_{x_{jk}}^{-a_{mi}} \sum_{l=k}^{i-1} a_{ni} (1 + M_{x_{ji}}^{-1}) x_{ji} D_{x_{ji}}^+ \] (83)
\[ J_{\alpha i}^- = \frac{1}{1 + q} \left[ q^{i/2} \sum_{j=1}^{i-1} (1 + M_{x_j}^{+1}) x_{ji} \prod_{k=i}^r M_{x_{jk}}^{+} \sum_{l=j}^{a_{ii}} \prod_{m=j+1}^r \prod_{n=m}^r M_{x_{mn}}^{+} \sum_{p=m}^{a_{ii}} q^{2a_{i\mu}^{+} \mu} D_{x_{j-i}}^{+} \right] + \frac{1}{q - q^{-1}} (1 + M_{x_i}^{+1}) x_i \left( \prod_{j=i}^r \prod_{k=j}^r M_{x_{jk}}^{-} \sum_{l=j}^{a_{ii}} q^{2a_{i\mu}^{+} \mu} - M_{x_{ii}}^{-2} \prod_{k=i}^r \prod_{m=k}^r M_{x_{mk}}^{+} \sum_{p=m}^{a_{ii}} q^{-2a_{i\mu}^{+} \mu} \right) \]

\[ -q^{-1/2} \sum_{j=i+1}^r (1 + M_{x_j}^{+1}) x_{ij} M_{x_{ij}}^{-1} \prod_{k=j}^r M_{x_{ik}}^{+} \sum_{m=i+1}^r \prod_{n=m}^r M_{x_{mn}}^{+} a_{ip} \prod_{s=i+1}^{j-1} M_{x_{is}^{+}}^{(1 - \delta_{i,s}^{+})} q^{-2a_{i\mu}^{+} \mu} D_{x_{i+1}j}^{+} \]

(84)

\[ H_{\alpha i} = - \sum_{j=1}^r \sum_{k=j}^r a_{il} x_{jk} \partial x_{jk} + 2a_{i}^{+} \mu \]

(85)

with \( i = 1, \ldots, r, r = n - 1 \) rank of the group, and \( M_{x_{ii}} = 1 \).

For the representation with some zero Dynkin indices as stated before we must set corresponding variables to zero and corresponding difference operator should be replaced as follows:

\[ D_{x_{mm'}}^{+} \rightarrow -\frac{q^{1/2}}{q + 1} \sum_{i=m'+1}^r \prod_{j=m'+1}^i M_{x_{mj}}^{-1} \prod_{k=m+1}^{m'} M_{x_{ki}}^{+1} \prod_{l=m'+1}^{i-1} M_{x_{lm'}^{+1}}^{-} (1 + M_{x_{m'}}^{+1}) x_{m'+1} D_{x_{m'm}}^{+} \]

(86)

where \( x_{mm'} \) corresponds to \( \alpha_{mm'} = \alpha^m + \alpha^{m+1} + \cdots + \alpha^{m'-1} + \alpha^{m'} \in \Sigma^+ \) and \( \alpha^m \cdot \mu, \ldots, \alpha^{m'} \cdot \mu = 0 \).

Note that for any fundamental representation the number of variables is \( n - 1 = r \).

In the limit of \( q \rightarrow 1 \) all the relations for \( U_q(sl(n)) \) will go to the ordinary \( sl(n) \). To be more specific, when \( q \rightarrow 1 \)

\[ D_{x_{ij}}^{+} \rightarrow \partial x_{ij} \]

(87)

and

\[ M_{x_{ij}}^{\alpha} \rightarrow 1 \]

(88)

equations (83-85) will be exactly the same as the equations (44-46).

We have explicitly checked that the generators defined in eq. (83-85) satify the algebra given in (77-81) for \( n \) up to 6. We have checked the following relations:

\[ [J_{\alpha i}^{+}, J_{\alpha i}^{-}] = \frac{1}{q - q^{-1}} (q_i^{H} - q^{-H_i}) \]

(89)

\[ [H_i, J_{\alpha i}^{\pm}] = \pm 2J_{\alpha i}^{\pm} \]

(90)

\[ [J_{\alpha i}^{\pm}, J_{\alpha i}^{\pm}] = 0 \]

(91)

are satisfied for all \( n \).

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References

[1] Zelobenko D.P. "Compact Lie Groups and Their Representations" (American Mathematical Society Providence, RI, 1973); Barut A.O. and R. Raczka "Theory of Group Representations and Applications" (World Scientific, Singapore, 1986).

[2] Shifman M. A. Int. J. Mod. Phys. A4 (1989) 2897.
[3] Bouwknegt P., J. McCarthy and K. Pilch, Commun. Math. Phys. 131 (1990) 125; Phys. Lett. B234 (1990) 297.
[4] Awata H., A. Tsuchiya and Y. Yamada, Nucl. Phys. B365 (1991) 680.
[5] Wakimoto M., Commun. Math. Phys. 104 (1986) 1605.
[6] Fateev V. A. and Zamolodchikov, A. B., Sov. J. Nucl. Phys. 43(4) (1986) 657.
[7] Bershadsky M. and Ooguri, H., Commun. Math. Phys. 126 (1989) 49.
[8] Gerasimov A., Morozov, A., Olshaneisky, M., Marshakov, A. and Shatashvili, S., Int. J. Mod. Phys. A5 (1990) 2495.
[9] Feigin B. L. and Frenkel, E. V., Commun. Math. Phys. 128 (1990) 161.
[10] Knizhnik V. and A. Zamolodchikov, Nucl. Phys. B247 (1984) 83.
[11] Gepner D. and E. Witten, Nucl. Phys. B278 (1986) 493.
[12] Tsuchiya A. and Y. Kanie, Adv. Stu. Pure Math. 16 (1988) 297 [Erratum: 19 (1989) 675]; Lett. Math. Phys. 13 (1987) 303.
[13] Kimura K. "On Free Boson Representation of the Quantum affine Algebra $U_q(\hat{sl}_2)$" Kyoto preprint, Dec.92.
[14] Lukyanov S. and Shatashvili, S. L. "Free Field Representation for the Classical Limit of Quantum Affine Algebra" IASSNS-Hep-92/62 and Ru-92/37, Sep.92.
[15] Shiraishi J. Phys. Lett. A171 (1992) 243.
[16] Matsuo A. Phys. Lett. B308 (1993) 260.
[17] Abada A. Mod. Phys. Lett. A8 (1993) 715.
[18] Awata H. S. Odake and J. Shiraishi "Free Boson Representation of $U_q(\hat{sl}_3)$" RIMS-920 and YITP/K-1017, May 93; "Free Boson Representation of $U_q(\hat{sl}_n)$" to appear in C.M.P.
[19] Drinfeld V. G., Soviet. math. Doklay 36(1988) 212.
[20] Frenkel I. B. and N. Yu. Reshetikhin, Commun. Math. Phys. 146(1992) 1.
[21] Bernard D. and A. Leclair, preprints CLNS 92/1147 and SPhT-92-054.
[22] Babelon O. and D. Bernard, preprints SPhT-92-062; LPTHE-92-20.
[23] Jimbo M., Lett. Math. Phys. 10 (1985) 63.
[24] Drinfeld V. G., Int. Congr. Math. (1986) 798.
[25] Faddeev L. D., N. Yu. Reshitikhin and L. A. Takhatadjian "Algebraic Analysis" Vol.1 (Academic Press, 1988); Leningrad Math. J. 1 (1990) 193.
[26] Floreanini R. and L. Vinet *Phys. Lett.* **B315** (1993) 299.

[27] Morozov A. and L. Vinet *Mod. Phys. Lett.* **A8** (1993) 2891.

[28] Morozov A. *private communication*.

[29] Rosso M. *Commun. Math. phys.* **124** (1989) 307.

[30] Khrooshkin S. M. and V. N. Tolstoy, *Commun. Math. Phys.* **141** (1991) 1599.