Compatibility and Edge Spaces in Alpha - Topological Spaces

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Abstract. This research presents the concepts of compatibility and edge spaces in $\alpha$-topological spaces, and introduces the $\alpha$-topology combinatorially induced by the $\alpha$-topology. Furthermore, studies the relationship between the $\alpha$-topology on $V \cup E$ and the relative $\alpha$-topology on $V$.

Keywords. compatibility, edge spaces, combinatorial extension.

1. Introduction
The basic concepts of $\alpha$-open sets ($\alpha$-closed sets) are introduced and studied in the mathematical paper [1] which implied to define the $\alpha$-topology, also these terms are generalized by several researchers [2].

In the topological spaces with graphs, many concepts and relations are studied. Also, the concept of compatible (strictly compatible) topology with hypergraph has been introduced in [3]. In addition, compatible (strictly compatible) topological space.

The edge spaces introduced in [3, 4] as an important new topological model of graphs, and these references studied the relationship between the topology on $V$ and the relative one on $V \cup E$. Also, many researches related with these models in [5, 6, 7, 8, 9].

In this research, we introduce the concepts of compatibility and edge spaces in $\alpha$-topological spaces by studying the relationship between the relative $\alpha$-topology on $V$ with the $\alpha$-topology on $V \cup E$ and give an example.

2. Preliminaries and Basic definitions
In this section, any graph $G$ contains the set of vertices $V_G$ and the set of edges $E_G$. The non-empty set $X = V_G \cup E_G$ used to define a topology $\tau$ on it which satisfied the general topological conditions, and for simplest $(X, \tau)$ refers to the topological space that we constructed it. For $A \subseteq X$, the interior and the closure of $A$ in $X$ with respect to $\tau$ are denoted by $Int(A)$ and $Cl(A)$. A sub set $A$ is said to be an $\alpha$-open set, if it satisfied $A \subseteq Int(Cl(Int(A)))$ for all $A \subseteq X$, and the complement of $A$ in $X$ is an $\alpha$-closed set. It is clear that every open set is $\alpha$-open, but in general the reverse is not true. We obtain an $\alpha$-topology $\tau_\alpha$ by taking all $\alpha$-open sets, and $(X, \tau_\alpha)$ is an $\alpha$-topological space. Moreover, the interior of $A$ in this space with respect to $\tau_\alpha$ is denoted by $\alpha-Int(A) = \bigcup \{ B \mid B \subseteq A, B$ is $\alpha$-open$\}$, while the closure of $A$ denoted by $\alpha-Cl(A) = \bigcap \{ B \mid A \subseteq B, B$ is $\alpha$-closed$\}$. In addition, we define the $\alpha-Cl(x)$ as the intersection of all $\alpha$-closed sets contained $x$ [8], where $Cl(x) = \bigcap \{ B \mid x \in B, B$ is closed$\}$ [3].

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An edge space \((X, E)\) is a topological space \(X\) with a subset \(E\) of \(X\) contains points \(e\) which are open (but not closed) with at most two points of its boundaries. All the points contained in \(E\) are edges and any point of its complement \((X \setminus E)\) is a vertex. Since \(e\) is open (but not closed) (that means it has at least one incident vertex), then any incident vertex \(v\) with an edge \(e\), if \(v \in \text{Cl}(e)[4]\).

We recall some basic definitions, remarks and facts in topology and graph.

Definition 2.1 [7]. Let \(G\) be any graph with a vertex \(v\) and an edge \(e\). The set contains \(v\) and all incident edges with \(v\).

Definition 2.2 [3]. An ordered triple \((V, E, f)\) is said to be a hypergraph where \(V, E\) are sets of vertices and edges respectively with an incidence function \(f\) from \(E\) to the set of nonempty subsets of \(V\).

Definition 2.3 [3]. Let \(H\) be a hypergraph, a topology on \(V \cup E\) is said to be compatible with \(H\), if all the edges of \(H\) are open and the set of their endvertices are exactly their boundaries.

Definition 2.4 [3]. Let \(A\) be a subset of a hypergraph \(H\), the symbol \(A^\alpha\) refers to the smallest subset of \(H\) which containing \(A\) and all incident edges for any vertex in \(A\). So, if \(A \subseteq V_H\), then \(A^\alpha = \bigcup_{v \in A \cap V} N_v\).

Moreover, \(A^\alpha = (A \cap E) \cup \bigcup_{v \in A \cap V} N_v\).

Definition 2.5 [3]. A hyperedge space \(\mathcal{H}\) is called an edge space if all the hyperedges are proper edges, and denoted as \((\mathcal{H}, E)\), where \(\mathcal{H} = (X, \tau)\) is a topological space and \(E\) a subset contains a hyperedges.

Definition 2.6 [8]. Let \(X\) be an \(\alpha\)-topological space, then every point in \(X\) which is \(\alpha\)-open but not \(\alpha\)-closed is called \(\alpha\)-hyperedge. Moreover, it is an edge (proper edge) if its \(\alpha\)-boundary points consists of mostly two points, whereas it is a loop if it has exactly one \(\alpha\)-boundary point.

Definition 2.7 [8]. If \(X\) is an \(\alpha\)-topological space, \(A \subseteq X\), then the \(\alpha\)-boundary point of \(A\) is the set of intersection \(\alpha\)-Cl(\(A\)) and \(\alpha\)-Cl(\(X \setminus A\)).

Theorem 2.8 [10]. A closed subset of a compact space is compact.

Definition 2.9 [11]. An \(\alpha\)-topological space \(X\) is said to be \(\alpha\)-compact space if every \(\alpha\)-open cover has a finite subcover.

3. Compatibility and Edge Spaces in Alpha-Topological Spaces

This section introduces a significant model for graphs in \(\alpha\)-topology and study the relationship between the \(\alpha\)-topology on \(V\) and the \(\alpha\)-topology on \(V \cup E\).

Definition 3.1. An \(\alpha\)-topological space \((X, \tau)\) is said to be compatible (strictly compatible) with an \(\alpha\)-hypergraph \(H\) if \(X = V_H \cup E_H\) and \(\{e\}\) is \(\alpha\)-open with \(\tau(e) = f_H(e)\) for all \(e \in E_H\) (also \(\{v\}\) is an \(\alpha\)-closed for all \(v \in V_H\)). Furthermore, \(\tau\) is said to be compatible (strictly compatible) \(\alpha\)-topology on \(X\).

The next theorem gives us the important condition to extending an \(\alpha\)-topology from \(V\) to \(V \cup E\) which is compatible with \(H\). Also, we can see that it is satisfying directly when \((X, \tau)\) is \(T_1\) - space and \(H\) is finitely incident.
Theorem 3.2. A hypergraph \( H = (V, E, f) \) with an \( \alpha \)-topology \( \tau \) on \( V \) such that, \( f(e) \) is \( \alpha \)-closed, \( \forall \ e \in E \). Then the collection of sets \( \tilde{\mathcal{T}} = \{W^\alpha \cup F : W \in \tau, F \subseteq E\} \) is the finest \( \alpha \)-topology on \( V \cup E \) compatible with \( H \) which have the property that \( \tau \) is the relative \( \alpha \)-topology on \( V \).

Proof: Firstly, we must prove that \( \tilde{\mathcal{T}} \) is an \( \alpha \)-topology on \( V \cup E \).

If \( W = F = \emptyset \) then \( \emptyset \) in \( \tilde{\mathcal{T}} \), and if \( W = V, F = E \) then \( V \cup E \) in \( \tilde{\mathcal{T}} \).

To prove \( \tilde{\mathcal{T}} \) is closed under finite intersection, let \( A_1, A_2 \) be two subsets in \( \tau \), then \( (A_1 \cap A_2) \cap V = W_1 \cap W_2 \). So, \( (A_1 \cap A_2) = ( \bigcup_{V \in W} N_v ) \cup \tilde{F} \), where \( \tilde{F} = (A_1 \cap A_2) \cap E \).

Now we must prove that \( \tilde{\mathcal{T}} \) is closed under arbitrary union, let \( \{A_i\}_{i \in I} \) be any subset of \( \tilde{\mathcal{T}} \). Then, \( A_i = ( \bigcup_{V \in W_i} N_v ) \cup F_i \) for all \( i \in I \), where \( F_i \subseteq E \) and \( W_i \in \tau \). Hence, \( \bigcup_{i \in I} A_i = ( \bigcup_{V \in W} N_v ) \cup F \), where \( W = \bigcup_{i \in I} W_i \) and \( F = \bigcup_{i \in I} F_i \) (that means \( F \subseteq E \)). Since \( W \in \tau \) and \( \tau \) is closed under arbitrary union, \( \tilde{\mathcal{T}} \) is closed under arbitrary union. Hence \( \tilde{\mathcal{T}} \) is an \( \alpha \)-topology on \( V \cup E \).

It remains to prove that \( \tau \) concurs with the relative \( \alpha \)-topology on \( V \) to prove later that \( \tilde{\mathcal{T}} \) is the finest \( \alpha \)-topology on \( V \cup E \) compatible with \( H \).

If \( S \) is in the relative topology on \( V \), then the exist \( W \in \tau \) and \( F \subseteq E \) such that \( S = (W^\alpha \cup F) \cap V \). But \( S = W \), hence \( S \in \tau \) (i.e. \( \tau \) is the relative \( \alpha \)-topology on \( V \) inherited from \( \tilde{\mathcal{T}} \)). To prove the converse, let \( S \in \tau \) implies \( \tilde{S} = S^\alpha \), so \( S = \tilde{S} \cap V \) is in the relative \( \alpha \)-topology on \( V \) inherited from \( \tilde{\mathcal{T}} \).

To show that \( \tilde{\mathcal{T}} \) is compatible with \( H \), let \( e \) be the incident edge with the vertex \( v \).

Then all \( \alpha \)-open sets (with respect to \( \tilde{\mathcal{T}} \)) containing \( v \), all of \( N_v \) and \( e \). Hence \( v \) lies in the \( \alpha \)-closure of \( \{e\} \), however \( v \) belongs to its \( \alpha \)-boundaries since it is in its complement. Conversely, assume that \( e \) is not incident with \( v \). Since \( f(e) \) is \( \alpha \)-closed, then there exists an \( \alpha \)-open set \( U \in \tau \) containing \( v \) but distinct from \( f(e) \). Then \( U^\alpha \) is \( \alpha \)-open with respect to \( \tilde{\mathcal{T}} \) contains \( v \) and is distinct from \( e \), means that \( v \) is not in the \( \alpha \)-closure of \( \{e\} \), and hence is not in its boundaries. This implies that \( \tilde{\alpha}(e) = f(e) \). It is clear that \( \{e\} \) is \( \alpha \)-open with respect to \( \tilde{\mathcal{T}} \), which means \( \tilde{\mathcal{T}} \) is compatible with \( H \).

To prove the finest property, suppose that \( \mathcal{T} \) is any \( \alpha \)-topology on \( V \cup E \) compatible with \( H \) and with the property that \( \tau \) is the inherited relative \( \alpha \)-topology on \( V \). We must prove that \( \mathcal{T} \subseteq \tilde{\mathcal{T}} \). Take any \( S \in \mathcal{T} \), then \( S \cap V \) is \( \alpha \)-open with respect to \( \tilde{\mathcal{T}} \) (since \( \tilde{\mathcal{T}} \) is the relative \( \alpha \)-topology inherited from \( \mathcal{T} \)). Also, since \( \mathcal{T} \) is compatible with \( H \), then \( v \in S \Rightarrow N_v \subseteq \mathcal{T} \) (or else, there is some edge \( e \) incident with \( v \) and outside the \( \alpha \)-open set \( S \) containing \( v \), implies that the endvertex \( v \) of \( e \) is not in the \( \alpha \)-closure of \( \{e\} \)). So, we have \( S = \bigcup_{V \in V \cap S} N_v \cup F \) for some set \( F \subseteq E \). Hence \( S \in \tilde{\mathcal{T}} \), that means \( \tilde{\mathcal{T}} \) is the finest \( \alpha \)-topology on \( V \cup E \).

The condition of \( f(e) \) to be \( \alpha \)-closed is necessary in the last theorem, but if it does not satisfy, then the definition of compatible is failed.

Example 3.3. Let \( X = R \) be an \( \alpha \)-topological space, \( Y = V \cup E \) be an \( \alpha \)-subspace such that \( V = \{0\} \cup \{1, \ldots, m\} \) for some positive integer \( m \), and \( E \) consisting of only one incident \( \alpha \)-hyperedge \( \{e\} \) for all vertices in \( V \) except \( \{0\} \). Since \( 0 \in \alpha-Cl(f(e)) \), but it is not an endvertex, therefore the induced \( \alpha \)-topology on \( V \cup E \) is not compatible with \( H \) which contradicting definition (3.1).

The next definition introduce an \( \alpha \)-hyperedge space.

Definition 3.4. An \( \alpha \)-topological space with a subset of \( \alpha \)-hyperedges (which are \( \alpha \)-open but not \( \alpha \)-closed) is called an \( \alpha \)-hyperedge space and denoted as \( \forall \).

The next definition shows when an \( \alpha \)-hyperedge to be an \( \alpha \)-edge space.
Definition 3.5. An $\alpha$-hyperedge space $\mathcal{H}$ is called an $\alpha$-edge space if all the $\alpha$-hyperedges are proper edges, and denoted as $(\mathcal{X}, E)$, where $\mathcal{X} = (X, \tau)$ is an $\alpha$-topological space and $E$ a subset contains an $\alpha$-hyperedges.

Also, another way of specifying the $\alpha$-hyperedge space is to give an $\alpha$-hypergraph with a compatible $\alpha$-topology.

Remark 3.6. An $\alpha$-hyperedge space $\mathcal{H}$ induced a compatible hypergraph $H$ by taking the $\alpha$-hyperedge of $\mathcal{H}$ (for $\alpha$-hyperedge of $H$), the other points for vertices of $H$ and the $\alpha$-boundary points of any edge for its incident vertices.

Definition 3.7. Let $M, N$ be two $\alpha$-hyperedge spaces, then $N$ is said to be an $\alpha$-hyperedge subspace of $M$ if $V_N \cup E_N$ is an $\alpha$-topological subspace of $V_M \cup E_M$, and the underlying $\alpha$-hypergraph of $N$ is a sub underlying $\alpha$-hypergraph of $M$. By another way, $\mathcal{C}_M(e) \subseteq N$ for all $\alpha$-hyperedge $e \in N$. We can say that any $\alpha$-closed subset $F$ of $\mathcal{X}$, $(F, E \cap F)$ is an $\alpha$-hyperedge subspace of $(\mathcal{X}, E)$.

Definition 3.8. Let $H = (V, E, f)$ be an $\alpha$-hypergraph with $\tau$ an $\alpha$-topology on $V$, and $\tilde{\tau}$ the $\alpha$-topology combinatorially induced by $\tau$. Then the combinatorial extension of $V$ dependent on $f$ is the $\alpha$-hyperedge space $H = (V, E, \tilde{\tau})$, where $\tilde{\tau}$ is the $\alpha$-boundary operator dependent on $\tilde{\tau}$.

The next theorem shows the compactness property between $V_H \cup E_H$ and $V$ in $\alpha$-hyperedge space.

Theorem 3.9. Let $H$ be an $\alpha$-hyperedge space, then $V_H \cup E_H$ is $\alpha$-compact if and only if $V_H$ is $\alpha$-compact.

Proof: Assume that $V_H \cup E_H$ is $\alpha$-compact. Since $H$ is $\alpha$-hyperedge space, then $E_H$ is $\alpha$-open. Hence $V_H$ is a $\alpha$-closed subset of a $\alpha$-compact space, so by generalized theorem (2.8), $V_H$ is $\alpha$-compact.

Conversely, assume that $V_H$ is $\alpha$-compact. Let $\tau$ be the $\alpha$-topology on $V_H \cup E_H$, $\tilde{\tau}$ the $\alpha$-topology on $V$, and $\tilde{\tau}$ the $\alpha$-topology combinatorially induced by $\tilde{\tau}$.

We must prove that every $\alpha$-open cover of $V_H \cup E_H$ with elements from $\tau$ has a finite subcover. By theorem (3.2), $\tau$ is compatible with $\alpha$-hypergraph of $H$, therefore $\tau \subseteq \tilde{\tau}$. Hence it is enough to prove that $V_H \cup E_H$ is $\alpha$-compact with respect to $\tilde{\tau}$. So let $U = \{U_I\}_{I \in I} \subseteq \tilde{\tau}$ be an $\alpha$-open cover of $V_H$. Since $\tilde{\tau}$ coincides with the $\alpha$-topology inherited from $\tilde{\tau}$, we have that $\{U_I \cap V_H\}_{I \in I}$ is an $\alpha$-open cover of $V_H$ with respect to $\tilde{\tau}$. Since $V_H$ is $\alpha$-compact, then there exists a finite $J \subseteq I$ such that $\{U_J \cap V_H\}_{J \in J}$ is an $\alpha$-open cover of $V_H$. So, by the definition of $\tilde{\tau}$, we have $\{U_J \cap V_H\} \subseteq (U_J \cap V_H)^\circ \subseteq U_J$ for all $J \in J$.

Furthermore, $\{U_J \cap V_H\}_{J \in J}$ covers $V_H \cup E_H$, since every $\alpha$-hyperedge is incident with some vertices. So $\{U_J\}_{J \in J}$ is the $\alpha$-open subcover of $U$ that we need it. Hence $V_H \cup E_H$ is $\alpha$-compact.

4. Conclusion

In this research, a new model of $\alpha$-topologized hypergraphs framed by studying the concepts of compatibility and edge spaces in $\alpha$-topological spaces. Also, the relationship between the relative $\alpha$-topology on $V$ and the one on $V \cup E$ studied. This work can be open the way to generalize many combinatorial constructions on the same $\alpha$-topological space.

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