KISIN MODULES WITH DESCENT DATA AND PARAHORIC LOCAL MODELS

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Abstract. We construct a moduli space \( Y^{\mu,\tau} \) of Kisin modules with tame descent datum \( \tau \) and with fixed \( p \)-adic Hodge type \( \mu \), for some finite extension \( K/\mathbb{Q}_p \). We show that this space is smoothly equivalent to the local model for \( \text{Res}_{K/\mathbb{Q}_p} \text{GL}_n \), cocharacter \( \{\mu\} \), and parahoric level structure. We use this to construct the analogue of Kottwitz-Rapoport strata on the special fiber \( Y^{\mu,\tau} \) indexed by the \( \mu \)-admissible set. We also relate \( Y^{\mu,\tau} \) to potentially crystalline Galois deformation rings.

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1. Introduction

Let \( K/\mathbb{Q}_p \) be a finite extension. Kisin [Kis06] showed that the category of finite flat group schemes over \( \mathcal{O}_K \) killed by a power of \( p \) is equivalent to the category of Breuil-Kisin modules of height \( \leq 1 \). While the former do not naturally live in families, the latter can be made into a moduli space. The landmark paper [Kis09a] uses moduli of Breuil-Kisin modules to construct resolutions of flat deformation rings with stunning consequences for modularity lifting theorems and applications to
the Fontaine-Mazur conjecture. The main result of [Kis09a] is a modularity lifting theorem in the potentially Barsotti-Tate case. One of the key points is a rather surprising connection to the theory of local models of Shimura varieties. Kisin showed that the singularities of the moduli space of Breuil-Kisin modules of rank \( n \) (with fixed \( p \)-adic Hodge type) could be related to the singularities of local models for the group \( \operatorname{Res}_{K/\mathbb{Q}_p} \operatorname{GL}_n \) (with maximal parahoric level) which had been studied by [PR05].

Kisin’s result is globalized in [PR09], where Pappas and Rapoport construct a global (formal) moduli stack \( X^\mu \) of Kisin modules with \( p \)-adic Hodge type \( \mu \in (\mathbb{Z}/p^n)^{\text{Hom}(K, \overline{\mathbb{F}}_p)} \). They link the space \( X^\mu \) via smooth maps with a (generalized) local model \( M(\mu) \). When \( \mu \) is non-minuscule, \( M(\mu) \) is not related to any Shimura variety but is nevertheless known to have nice geometric properties by work of Pappas-Zhu [PZ13] and of the second author [Lev3]. \( M(\mu) \) is constructed inside a mixed characteristic version of the Beilinson-Drinfeld affine Grassmanian. As a result, the nice geometric properties of \( M(\mu) \) transfer to the global moduli stack \( X^\mu \).

While the connection between moduli of Breuil-Kisin modules and local models suffices for proving modularity lifting theorems in the potentially Barsotti-Tate case, it doesn’t seem capture some of the more subtle aspects of the geometry of local deformation rings. These more subtle aspects are connected to the (geometric) Breuil-Mezard conjecture [BM02, EG14], to the weight part in Serre’s conjecture [BDJ10, GHS] and to questions about integral structures in completed cohomology [Bre12, EGS15]. Therefore, there is considerable interest in generalizing the results of Kisin and Pappas-Rapoport. This paper extends the relationship with local models to the case of Breuil-Kisin modules equipped with tame descent data.

We explain the connection to integral structures in completed cohomology. One of the few situations where we have explicit presentations of local deformation rings is the case of tamely Barsotti-Tate deformations rings for \( \operatorname{GL}_2 \). Set \( G_K := \operatorname{Gal}(\overline{K}/K) \) and let \( I_K \subset G_K \) be the inertia subgroup. When \( K/\mathbb{Q}_p \) is unramified and \( \tau : I_K \to \operatorname{GL}_2(\Lambda) \) is a (generic) tame inertial type, then [Bre12, BM14, EGS15] explicitly describe the potentially Barsotti-Tate deformation ring \( \mathcal{R}_{BT,\tau} \) for any \( \mathcal{P} : G_K \to \operatorname{GL}_2(\mathbb{F}) \). These computations provided evidence for the Breuil-Mézard conjecture and led Breuil to several important conjectures [Bre12]. Perhaps the most striking is the precise conjecture about which lattices inside the smooth \( \operatorname{GL}_2(\mathcal{O}_K) \)-representation \( \sigma(\tau) \) (determined by \( \tau \) via inertial local Langlands) can occur globally, in completed cohomology. Breuil’s conjectures were proved by Emerton-Gee-Savitt [EGS15] using the explicit presentations of tamely Barsotti-Tate deformation rings.
In more general situations \((K/\mathbb{Q}_p \text{ ramified or } \mathfrak{p} \text{ non-generic})\), one cannot hope for such an explicit presentation. In this paper, we construct for arbitrary \(K/\mathbb{Q}_p\) and \(GL_n\) resolutions of tamely Barsotti-Tate deformation rings whose geometry is related to that of local models for \(\text{Res}_{K/\mathbb{Q}_p}GL_n\) with parahoric level structure. These resolutions are related to the moduli of Breuil-Kisin modules with descent data. The level structure is determined by the tame inertial type \(\tau\). For example, if \(\tau\) consists of distinct characters, then the local model will have Iwahori level structure, whereas the local models of [Kis09a, PR09], which have trivial descent data, always have maximal parahoric level.

Our perspective in this paper is largely global, in the spirit of [PR09]. Motivated by the moduli stack of finite flat representations of \(G_K\) constructed by [EG], we study moduli stacks \(Y^\mu,\tau\) of Kisin modules with tame descent data and \(p\)-adic Hodge type \(\mu \in (\mathbb{Z}_n)_{\text{Hom}}(K,\mathbb{Q}_p)\).

**Theorem 1.1.** There exists a moduli stack \(Y^\mu,\tau\) of Kisin modules with tame descent data and \(p\)-adic Hodge type \(\mu\), which fits into the diagram

\[
\begin{array}{ccc}
\tilde{Y}^{\mu,\tau} & \xrightarrow{\pi^\mu} & Y^{\mu,\tau} \\
\downarrow & & \downarrow \\
\Psi^\mu & & M(\mu)
\end{array}
\]

where \(M(\mu)\) is the Pappas-Zhu local model [PZ13, Lev3] for \((\text{Res}_{K/\mathbb{Q}_p}GL_n, \mu)\) at parahoric level (determined by \(\tau\)) and both \(\pi^\mu\) and \(\Psi^\mu\) are smooth maps.

**Remark 1.2.** The key step in the construction of the local model diagram is encoded in diagram 3.1. We decompose a Kisin module \((\mathfrak{M}, \phi)\) according to the descent datum and then study the interactions between the images of \(\phi\) on different isotopic pieces. This is reminiscent of the classical definition of local models which involves lattice chains.

In joint work in preparation with Emerton, Gee and Savitt [CEGS], the first author constructs a moduli stack of two-dimensional, tamely potentially Barsotti-Tate \(G_K\)-representations and relates its geometry to the weight part of Serre’s conjecture. In this case, the stack \(Y^{\mu,\tau}\) will be a relatively explicit, partial resolution of the moduli stack of \(G_K\)-representations. The nice geometric properties that \(Y^{\mu,\tau}\) inherits from the local model diagram turn out to be key for understanding the geometry of the latter moduli stack. From this perspective, the present paper and the paper in preparation [CEGS] clarify the geometry which underlies a possible generalization of Breuil’s lattice conjecture in the ramified setting.

In another direction, the local model diagram above allows us to define the analogue of Kottwitz-Rapoport strata inside the special fiber of \(Y^{\mu,\tau}\). For example, if \(K = \mathbb{Q}_p\), we get locally closed
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substacks \( \mathfrak{Y}^{\mu,\tau}_w \) of the moduli space of mod \( p \) Kisin modules with descent datum \( \mathfrak{Y}^{\mu,\tau} \) indexed by certain elements \( w \) in the Iwahori-Weyl group of \( \text{GL}_n \), the so-called \( \mu \)-admissible elements defined by Kottwitz and Rapoport (cf. [PZ13, (9.17)]).

Definition 1.3. A Kisin module \( \mathfrak{M} \in \mathfrak{Y}^{\mu,\tau}_w(\mathbb{F}_p) \) is said to have shape (or genre) \( w \).

This generalizes the notion of genre which is crucial in [Bre12] and more recently [CDM1] in describing tamely Barsotti-Tate deformation rings for \( \text{GL}_2 \).

While Kisin’s resolution was most interesting when \( K/\mathbb{Q}_p \) was ramified, potentially Barsotti-Tate deformation rings have interesting geometry even when \( K = \mathbb{Q}_p \). In addition, when \( n > 2 \), there is an advantage to replacing weight by level and considering potentially crystalline deformation rings in questions related to Serre weight conjectures. This direction is considered in joint work in progress of the second author with B. Le Hung, D. Le and S. Morra which computes tamely crystalline deformations rings with Hodge-Tate weights \((2,1,0)\) for \( K/\mathbb{Q}_p \) unramified with applications to Serre weight conjectures for \( \text{GL}_3 \) [LLLM]. The results of [LLLM] suggest close connections between the strata defined by shapes and Serre weights.

1.1. Overview of the paper. In Section 2, we recall the definition of local models in the sense of Pappas-Zhu, as well as the results of [PZ13, Lev3] on the geometry of local models. In Section 3, we define Kisin modules with decent data, construct the moduli space of Kisin modules with tame descent data (without imposing any conditions related to \( p \)-adic Hodge type) and derive the key diagram \[3.1\]. In Section 4, we construct the local model diagram (again without imposing a \( p \)-adic Hodge type \( \mu \)) and prove that both arrows are (formally) smooth. In Section 5, we construct the stack \( \mathfrak{Y}^{\mu,\tau} \), give a moduli-theoretic description of its generic fiber, describe the Kottwitz-Rapoport stratification of its special fiber and relate it to tamely potentially Barsotti-Tate Galois deformation rings.

1.2. Acknowledgements. The idea of constructing a moduli stack of Breuil-Kisin modules with tame descent data originated in joint work of the first author with M. Emerton, T. Gee and D. Savitt, where this is done for Breuil-Kisin modules corresponding to two-dimensional, tamely Barsotti-Tate Galois representations. The idea that one should be able to relate this moduli stack to local models of Shimura varieties was suggested to us by M. Emerton, whom we thank for many useful conversations. The second author would like to thank B. Bhatt, B. Le Hung, D. Le, S. Morra for many helpful conversations. A. C. was partially supported by the NSF Postdoctoral Fellowship DMS-1204465 and NSF Grant DMS-1501064.
1.3. Notation. Fix a finite extension $K/Q_p$ with $K_0$ the maximal unramified subextension. Let $f := [K_0 : Q_p]$ and $e_K := [K : K_0]$. Let $k$ denote the residue field of $K$, of cardinality $p^f$. Fix a uniformizer $\pi_K$ of $K$. Let $L/K$ be the totally tame extension of degree $p^f - 1$ obtained by adjoining a $(p^f - 1)$st root of $\pi_K$ which we denote by $\pi_L$. Let $W := W(k)$ be the ring of integers of $K_0$.

Let $E(u) \in Z_p[u]$ be the minimal polynomial for $\pi_K$ over $Q_p$ of degree $e := f \cdot e_K = [K : Q_p]$. Note that $P(v) := E(v^{p^f - 1}) \in Z_p[v]$ is the minimal polynomial for $\pi_L$ over $Q_p$.

Set $\Delta := \text{Gal}(L/K)$, which is cyclic of order $p^f - 1$. We take $F$ to be our coefficient field, a finite extension of $Q_p$, with ring of integers $\Lambda$ and residue field $F$. Let $\Delta^* := \text{Hom}(\Delta, \Lambda^\times)$ be the character group. Assume that $K_0$ embeds into $F$ and fix such an embedding $\sigma_0 : K_0 \to F$ which induces an embedding $W \to \Lambda$ and an embedding $k_0 \to F$. We will abuse notation and denote these all by $\sigma_0$.

Let $\tau : \Delta \to GL_n(\Lambda)$ be a tame principal series type, i.e., $\tau \cong \oplus_{i=1}^n \chi_i$ with $\chi_i \in \Delta^*$. We will take $\omega_f : G_K \to W^\times$ to be the fundamental character of niveau $f$ given by $\omega_f(\sigma) = \frac{\sigma(\pi)}{\pi}$.

2. Local models

In this section, we recall the definition and properties of local models for the group $\text{Res}_{K/Q_p}GL_n$, at parahoric level and for general cocharacters. These local models are studied in more detail and for more general groups in [Lev3]. We will review the relevant definitions and the results we will need. One can think of this construction as a mixed characteristic version of the deformation of the affine flag variety used by Gaitsgory in [Gai01]. The strategy in mixed characteristic builds on the work of Pappas and Zhu [PZ13]. For $GL_n$, the construction originates in work of Haines and Ngo [HN02].

Since $K_0$ embeds into $F$, the local models for $\text{Res}_{K/Q_p}GL_n$ decompose as products over the different embeddings of $K_0$ into $\overline{Q_p}$. For now, it is convenient to fix an embedding $\sigma : K_0 \to F$ and let $Q(u) := \sigma(E(u))$, an Eisenstein polynomial over $\Lambda$.

Fix a parabolic subgroup $P$ of $GL_n$ over Spec $\Lambda$. $P$ is the stabilizer of a filtration

$$0 = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_{n-1} \subseteq V_n = \Lambda^n$$

on the free rank $n$ $\Lambda$-module. For any $\Lambda$-algebra $R$ and any rank $n$ projective $R$-module $M$, a $P$-filtration is a filtration $\{\mathcal{F}^i(M)\}$ which is (Zariski) locally isomorphic to $\{V_i \otimes_\Lambda R\}$.

**Definition 2.1.** For any $\Lambda$-algebra $R$, define

$$\text{Gr}^Q(u)(R) := \{\text{isomorphism classes of pairs } (L, \beta)\},$$

where $L$ is rank $n$ projective $R[u]$-module, $\beta : L[1/Q(u)] \cong (R[u]_n)[1/Q(u)]$.
For any \( \Lambda \)-algebra \( R \), define
\[
\text{Fl}_P^{Q(u)}(R) := \{ \text{isomorphism classes of triples } (L, \beta, \varepsilon) \},
\]
where \( (L, \beta) \in \text{Gr}^{Q(u)}(R) \) and \( \varepsilon \) is a \( P \)-filtration on \( L/uL \). There is a natural forgetful morphism \( \text{pr} : \text{Fl}_P^{Q(u)} \to \text{Gr}^{Q(u)} \).

We will also need some variations of these objects. There is a local version of \( \text{Gr}^{Q(u)} \) and a corresponding local version of \( \text{Fl}_P^{Q(u)}(R) \):

**Definition 2.2.** Let \( \hat{R}[u]_{(Q(u))} \) denote the \( Q(u) \)-adic completion of \( R[u] \). For any \( \Lambda \)-algebra \( R \), define
\[
\text{Gr}^{Q(u)}_{\text{loc}}(R) := \{ \text{isomorphism classes of pairs } (\hat{L}, \hat{\beta}) \},
\]
where \( \hat{L} \) is rank \( n \) projective \( \hat{R}[u]_{(Q(u))} \)-module, \( \hat{\beta} \) is a trivialization of \( \hat{L}[1/Q(u)] \).

For any \( \Lambda \)-algebra \( R \), define
\[
\text{Fl}_P^{Q(u)}_{\text{loc}}(R) := \{ \text{isomorphism classes of triples } (\hat{L}, \hat{\beta}, \hat{\varepsilon}) \},
\]
where \( (\hat{L}, \hat{\beta}) \in \text{Gr}^{Q(u)}_{\text{loc}}(R) \) and \( \hat{\varepsilon} \) is a \( P \)-filtration on \( \hat{L}/u\hat{L} \).

**Theorem 2.3.** The natural map \( \text{Gr}^{Q(u)} \to \text{Gr}^{Q(u)}_{\text{loc}} \) given by \( Q(u) \)-adic completion is an isomorphism of functors.

**Proof.** The equivalence is given by the Beauville-Laszlo descent ([BL95], see Proposition 4.1.3 of [Lev3] for more details). \( \square \)

We have in fact a third description of \( \text{Gr}^{Q(u)} \) and \( \text{Fl}_P^{Q(u)} \) when \( p \) is nilpotent in \( R \):

**Definition 2.4.** For any \( \Lambda/p^r\Lambda \)-algebra \( R \), define
\[
\text{Gr}_{\text{alt}}^{Q(u)}(R) := \{ \text{isomorphism classes of pairs } (L', \beta') \},
\]
where \( L' \) is rank \( n \) projective \( R[u] \)-module, \( \beta' : L'[1/Q(u)] \cong (R[u]^n)[1/Q(u)] \). We define \( \text{Fl}_P^{Q(u)}_{\text{alt}} \) in the same way.

**Proposition 2.5.** Let \( R \) be a \( \Lambda/p^r\Lambda \) algebra, there are natural bijections
\[
\text{Gr}^{Q(u)}(R) \cong \text{Gr}_{\text{alt}}^{Q(u)}(R) \text{ and } \text{Fl}_P^{Q(u)}(R) \cong \text{Fl}_P^{Q(u)}_{\text{alt}}.
\]

**Proof.** The equivalence passes through the local versions \( \text{Gr}^{Q(u)}_{\text{loc}}(R) \) and \( \text{Fl}_P^{Q(u)}_{\text{loc}}(R) \). We simply note that when \( p \) is nilpotent, \( u \)-adic and \( Q(u) \)-adic completions of \( R[u] \) are the same since \( Q(u) = u^{e} + pQ'(u) \). \( \square \)
Theorem 2.6. The functors $\text{Gr}^{Q(u)}$ and $\text{Fl}^{Q(u)}$ are represented by ind-schemes which are ind-projective over $\text{Spec } \Lambda$.

Proof. This follows from Proposition 4.1.4 of [Lev3]. □

Let $L_{0,R} := R[\!\!u\!\!]^n \subset (R[\!\!u\!\!]^n)[1/Q(u)]$. In this situation, we can make the ind-structure very concrete.

Definition 2.7. For any integers $a, b$ with $b \geq a$, define

$$\text{Gr}^{Q(u),[a,b]}(R) = \{(L, \beta) \in \text{Gr}^{Q(u)}(R) \mid Q(u)^{-a}L_{0,R} \supset \beta(L) \supset Q(u)^{b}L_{0,R}\}.$$

Similarly, we define $\text{Fl}^{Q(u),[a,b]}_P = \text{Fl}^{Q(u)}_P \times_{\text{Gr}^{Q(u)}} \text{Gr}^{Q(u),[a,b]}$.

Proposition 2.8. The functors $\text{Gr}^{Q(u),[a,b]}$ and $\text{Fl}^{Q(u),[a,b]}_P$ are represented by projective $\Lambda$-schemes.

Proof. See [Lev1, Proposition 10.1.15]. □

In order to describe the geometry of $\text{Fl}^{Q(u)}_P(R)$, we recall the definition of the affine Grassmannian and affine flag varieties.

Definition 2.9. Let $\kappa$ be a field. Let $\text{Gr}_{GL_n}$ be the affine Grassmanian of $GL_n$ over $\kappa$. $\text{Gr}_{GL_n}$ is the ind-scheme parametrizing $R[\![t]\!]$-lattices $L_R$ in $R(\!(t)\!)^n$ for any $\kappa$-algebra $R$.

One can also define the affine Grassmanian $\text{Gr}_G$ for a general connected reductive group $G$ over $\kappa$. This is the fpqc quotient of group functors $G(\!(t)\!)/G[\![t]\!]$, where the loop group $G(\!(t)\!)$ sends a $\kappa$-algebra $R$ to $G(R(\!(t)\!))$. The positive loop group $G[\![t]\!]$ sends a $\kappa$-algebra $R$ to $G(R[\![t]\!])$. The fpqc quotient $\text{Gr}_G$ is representable by an ind-projective ind-scheme over $\kappa$. The affine Grassmanian $\text{Gr}_{GL_n}$ is the ind-scheme parametrizing $GL_n$-bundles on $\text{Spec } R[\![t]\!]$ together with a trivialization on $\text{Spec } R(\!(t)\!)$, where we can think of $GL_n$-bundles in the Tannakian sense as tensor functors from $\text{Rep}_\kappa(G)$ to vector bundles. In particular, one can consider $\text{Gr}_{\text{Res}(K\otimes_{Q_p} F)/FGL_n}$. Over $\overline{F}$, we have product decomposition

$$\text{(Res}(K\otimes_{Q_p} F)/FGL_n)_{\overline{F}} \cong \prod_{K \rightarrow \overline{F}} GL_n.$$

The same then holds for the affine Grassmanian, namely,

$$\text{(Gr}_{\text{Res}(K\otimes_{Q_p} F)/FGL_n})_{\overline{F}} \cong \prod_{K \rightarrow \overline{F}} (\text{Gr}_{GL_n})_{\overline{F}}$$

and so $\text{Gr}_{\text{Res}(K\otimes_{Q_p} F)/FGL_n}$ is a twisted form of $\prod_{K \rightarrow \overline{F}} \text{Gr}_{GL_n}$.
Gr_{GL_n} has a stratification by Schubert cells, as follows. Fix the diagonal torus \( T \) and the upper triangular Borel \( B \). This induces an Bruhat ordering on the set of dominant cocharacters \( \{(d_1, d_2, \ldots, d_n) \mid d_i \geq d_{i+1} \} \) of GL\(_n\). Let \( \mu = (d_1, d_2, \ldots, d_n) \) be a dominant cocharacter. The positive loop group \( GL_n(\kappa[t]) \) acts on the affine Grassmannian \( Gr_{GL_n} \). By the Cartan decomposition for \( GL_n(\kappa[[t]]) \), the orbits of this \( GL_n(\kappa[[t]]) \)-action are indexed by conjugacy classes of cocharacters of \( GL_n \); the orbits are called the affine Schubert cells attached to the (conjugacy classes of) cocharacters. The affine Schubert variety \( S(\mu) \) is defined to be the closure of the open Schubert cell \( S^\circ(\mu) \) corresponding to the conjugacy class of \( \mu \). It is a finite type closed subscheme of \( Gr_{GL_n} \). Concretely, \( S(\mu) \) parametrizes lattices whose position relative to the standard lattice are less than or equal to \( \mu \) for the Bruhat-order. In particular, \( S(\mu) \) is the union of the locally closed affine Schubert cells for all \( \mu' \leq \mu \) [Ric13, Proposition 2.8].

For our chosen parabolic subgroup \( P \subset GL_n \), we recall the definition of the affine flag variety over \( F \); it will be an ind-projective scheme over \( F \). It will depend on our chosen embedding \( \sigma : K_0 \to F \); recall that we’ve defined \( Q(u) := \sigma(E(u)) \).

**Definition 2.10.** The affine flag variety \( Fl_P \) associated to the pair \( (GL_n, P) \) is the moduli space of pairs \( (L, F^\bullet(L/tL)) \) where \( L \) is a lattice in \( R((t))^n \) and \( \{F^\bullet(L/tL)\} \) is a \( P \)-filtration on \( L/tL \) for any \( F \)-algebra \( R \).

We have a forgetful map \( Fl_P \to Gr_{GL_n} \) whose fibers are isomorphic to the flag variety \( GL_n/P \).

**Proposition 2.11.** The functor \( Fl_P^{Q(u)} \) is represented by an ind-projective scheme over \( Spec \Lambda \). Furthermore,

1. The generic fiber \( Fl_P^{Q(u)}[1/p] \) is isomorphic to the product \( GL_n/P_F \times Gr_{Res(K@K_0, F)/GL_n} \) over \( Spec F \);

2. The special fiber \( Fl_P^{Q(u)} \otimes_\Lambda F \) is isomorphic to \( Fl_P \).

**Proof.** See [Lev3 Proposition 2.2.8]. \( \square \)

Fix a geometric cocharacter \( \mu \) of \( Res_K/Q_pGL_n \) which we write as \( (\mu_j) \) where \( \mu_j \) is geometric cocharacter of \( Res_K/K_0GL_n \) for each embedding \( \sigma_j : K_0 \to E \). Furthermore, for each embedding \( \sigma_j \), fix a parabolic subgroup \( P_j \) of \( GL_n \). Define

\[
Fl_K^{E(u)} := \prod_{j \in \mathbb{Z}/f\mathbb{Z}} Fl_{P_j}^{E_j(u)}
\]

where \( E_j(u) = \sigma_j(E(u)) \). Similarly, we define

\[
Fl_K^{[a,b],E(u)} := \prod_{j \in \mathbb{Z}/f\mathbb{Z}} Fl_{P_j}^{[a,b],E_j(u)}.
\]
Remark 2.12. For now, the parabolic subgroups $P_j$ are arbitrary and they are allowed to be distinct. In Section 4, the "shape" of the descent datum on Kisin modules will impose additional conditions on the $P_j$, which will ensure that they determine conjugate parahoric subgroups of $GL_n$.

For the chosen cocharacter $\mu$, we define the reflex field $F[\mu]$. This is the smallest subfield of $\mathbb{F}$ over which the conjugacy class of $\mu$ is defined. Let $\Lambda[\mu]$ denote the ring of integers of $F[\mu]$. Since we have chosen $F$ to contain a copy of $K_0$, this is the union of the reflex fields for each $\mu_j$.

Definition 2.13. Let $S(\mu) \subset (\text{Gr}_{\text{Res}}(K\otimes_{\mathbb{Q}_p}F)/P_{GL_n})F[\mu]$ be the closed affine Schubert variety associated to $\{\mu\}$. For each $j \in \mathbb{Z}/f\mathbb{Z}$, let $I_{GL_n/P_j}$ denote the closed point of $GL_n/P_j$ corresponding to $P_j$. Then the local model $M(\mu)$ associated to $\mu$ is defined to be the Zariski closure of $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} I_{GL_n/P_j} \times S(\mu_j)$ in $\text{Fl}^{E(\nu)}_K$. It is a flat projective scheme over $\text{Spec} \Lambda[\mu]$.

The main theorem on the geometry of local models is:

Theorem 2.14. The local model $M(\mu)$ is normal with reduced special fiber. All irreducible components of $M(\mu) \otimes_{\Lambda} \mathbb{F}$ are normal and Cohen-Macaulay.

Proof. When $\mu$ is minuscule and $P = G$, this is Theorem B of [PR03]. When $\mu$ is a general cocharacter, $M(\mu)$ is the local model in the sense of [PZ13] (though they were originally studied in [HN02]). The above result is Theorem 1.1 of [PZ13] when $K/\mathbb{Q}_p$ is tamely ramified and Theorem 1.0.1 of [Lev3] when $K/\mathbb{Q}_p$ is wildly ramified. The proof uses the coherence conjecture of Pappas and Rapoport proven by [Zhu14].

Remark 2.15. Xuhua He has shown in [He13] that the entire local model $M(\mu)$ is Cohen-Macaulay when the $\lambda_j$ (which are defined below in (2.1)) are all minuscule. The local model is also known to be Cohen-Macaulay when $n = 2$ (via the argument sketched at the end of [Gor01], using the Kottwitz-Rapoport stratification below).

Remark 2.16. In the case when $n = 2$ and $\mu_{j,\psi} = (1,0)$ for all $(j, \psi)$ (which is the case corresponding to tamely Barsotti-Tate Galois representations), it can be shown that the local model coincides with the standard model, defined in terms of a Kottwitz determinant condition. The key point is that the standard model at hyperspecial level is flat, as shown in [PR03]; the same holds at parahoric level and therefore the standard model coincides with the local model in the sense of [PZ13], which is obtained by taking flat closure. The upshot is that in this special case, the entire local model $M(\mu)$ has a moduli interpretation. More details on the moduli interpretation and its relationship with tamely Barsotti-Tate Galois representations will appear in [CEGS].
Although there is no moduli interpretation for $M(\mu)$ in general, we can describe its special fiber in terms of affine Schubert varieties inside the affine flag variety. For each $0 \leq j \leq f - 1$, view $K_0$ as a subfield of $F$ via $\sigma_j = \sigma_0 \circ \varphi^{-j} : K_0 \to F$. Write $\mu_j = (\mu_j, \psi)$, where $\psi$ runs over $K_0$-embeddings $\psi : K \to \bar{F}$ where each $\mu_j, \psi$ is a dominant cocharacter. Define

$$\lambda_j = \sum_{\psi : K \to \bar{F}} \mu_j, \psi.$$  

(2.1)

We recall the definition of the $\lambda_j$-admissible set, which was introduced by Kottwitz and Rapoport; we follow the notation and constructions of Section 2 of [Lev3].

Let $G_0$ be the connected reductive group scheme $\mathrm{Res}_{(\mathcal{O}_K \times \mathcal{O}_K, \sigma_j)}^O \Lambda GL_n$ over $\text{Spec} \Lambda$ whose generic fiber is $G$. Let $G := G_0 \otimes \Lambda[\{u\}]$ be the constant extension. If we set $G^\flat := G_{\mathcal{F}([u])}$, then $G_{\mathcal{F}([u])}$ is a reductive model of $G_{\mathcal{F}([u])}$ and the parabolic $P_j$ determines a parahoric subgroup $P_j \subset G^\flat$.

Let $\widetilde{W}$ be the Iwahori-Weyl group of the split group $G^\flat_{\mathcal{F}([u])}$, defined as $N(\mathcal{F}([u]))/T^\flat_1$, where $N$ is the normalizer of a maximal torus $T^\flat$ in $G^\flat$ and $T^\flat_1$ is the kernel of the Kottwitz homomorphism for $T^\flat$ (see Section 4.1 of [PRS13] for more details). $\widetilde{W}$ sits in an exact sequence

$$0 \to X_*(T^\flat) \to \widetilde{W} \to W \to 0,$$

where $W$ is the absolute Weyl group of $(G^\flat, T^\flat)$. Define

$$\text{Adm}(\lambda_j) := \{w \in \widetilde{W} | w \leq t_\lambda, \lambda \in W \cdot \lambda_j\}.$$  

Let $W_{P_j} \subset W$ be the subgroup corresponding to the parahoric $P_j$. Define

$$\text{Adm}_{P_j}(\lambda_j) := W_{P_j} \text{Adm}(\lambda_j) W_{P_j}.$$  

Note that the $\text{Adm}(\lambda_j)$ only depends on the geometric conjugacy class of $\lambda_j$.

**Theorem 2.17.** The geometric special fiber $\overline{M}(\mu)_{\mathcal{F}}$ can be identified with the reduced union of a finite set of affine Schubert varieties in the affine flag variety $F^K_{E_{\mathcal{F}([u])}}$. Hence we have a stratification

$$\overline{M}(\mu)_{\mathcal{F}} = \bigcup_{(\bar{w}_j) \in \prod_{j=0}^{f-1} \text{Adm}_{P_j}(\lambda_j)} \prod_j S^\circ(\bar{w}_j)$$

by locally closed reduced subschemes, where $S^\circ(\bar{w}_j)$ is an open affine Schubert cell and these are indexed by $j$ and by the admissible set $\text{Adm}_{P_j}(\lambda_j)$.

**Remark 2.18.** The irreducible components of $\overline{M}(\mu)_{\mathcal{F}}$ are indexed by the extremal elements of $\prod_{j=0}^{f-1} \text{Adm}_{P_j}(\lambda_j)$ which are in bijection with the orbit of $(\lambda_j)$ under the Weyl group $\prod_j W_{P_j}$. 
Proof. This follows (by taking a product over the embeddings $\sigma_j$) from Theorem 8.3 of [PZ13] when $K/\mathbb{Q}_p$ is tamely ramified and Theorem 2.3.5 of [Lev3], when $K/\mathbb{Q}_p$ is wildly ramified. \qed

Finally, we recall a generalization of the loop group which acts on $M(\mu_j)$ and on $\text{Fl}^{E_j(u)}_{P_j}$. Define the pro-algebraic group $L^{+,E_j(u)}\text{GL}_n$ over $\text{Spec} \Lambda$ by

$$L^{+,E_j(u)}\text{GL}_n(R) = \lim_{\to} \text{GL}_n(R[u]/E_j(u)^\vee) = \lim_{\to} \text{Res}(\Lambda[u]/E_j(u)^\vee)/\Lambda \text{GL}_n(R).$$

We define a subgroup of $L^{+,E_j(u)}\text{GL}_n$ by

$$L^{+,E_j(u)}P_j(R) := \{ g \in L^{+,E_j(u)}\text{GL}_n(R) \mid g \mod u \in P_j(R) \}.$$

Similarly, for any positive integer $r$, let

$$P_{j,r} := \{ g \in \text{Res}(\Lambda[u]/E_j(u)^r)/\Lambda \text{GL}_n(R) \mid g \mod u \in P_j(R) \}.$$

Proposition 2.19. For any positive integer $r$, the functor $P_{j,r}$ is represented by a smooth, geometrically connected, group scheme of finite type over $\Lambda$. The functor $L^{+,E_j(u)}P_j$ is represented by an affine group scheme (not of finite type) over $\Lambda$ which is formally smooth over $\Lambda$.

Proof. This is a consequence of some general properties about Weil restriction along finite flat morphisms. The fact that $P_{j,r}$ is smooth is a consequence of Proposition A.5.2(4) in [CGP10]. The group scheme $P_{j,r}$ has geometrically connected fibers by Proposition A.5.9 in [CGP10]. \qed

Proposition 2.20. The group $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} L^{+,E_j(u)}P_j$ acts on $\text{Fl}^{E_j(u)}_{K}$. For any cocharacter $\mu$, $M(\mu)$ is stable and the action of $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} L^{+,E_j(u)}P_j$ on $M(\mu)$ factors through $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} P_{j,N}$ for some $N$ sufficiently large.

Proof. Choose $a,b$ such that $M(\mu) \subset \prod_{j \in \mathbb{Z}/f\mathbb{Z}} \text{Fl}^{E_j(u),[a,b]}_{P_j}$. The action of $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} L^{+,E_j(u)}P_j$ on $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} \text{Fl}^{E_j(u),[a,b]}_{P_j}$ is through the group scheme $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} P_{j,r}$ for $r = b - a$. Since $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} P_{j,r}$ is flat (even smooth) over $\Lambda$ by Proposition 2.19, stability of $M(\mu)$ follows from the fact that the generic fiber $S(\mu)$ is a union of orbits for the loop group of $\text{Res}_{K/\mathbb{Q}_p}\text{GL}_n$. \qed

Remark 2.21. An action of pro-algebraic group on a ind-scheme which satisfies the property in Proposition 2.20 is "nice" in the sense of [Gai01].

3. Kisin modules with descent datum

In this section, we will consider moduli of Kisin modules of finite height for the field $K$ together with tame descent datum for $L/K$. We work over the category $\text{Nilp}_\Lambda$ of $\Lambda$-algebras $R$ on which $p^N = 0$ for some $N \gg 0$. 
Recall that $\Delta = \text{Gal}(L/K)$ is a cyclic group of order $p^f - 1$. For any $g \in \Delta$ and any $R \in \text{Nilp}_\Lambda$, we let $\tilde{g}$ be the automorphism of $(W \otimes \mathbb{Z}_p R)[v]$ given by $v \mapsto (g(\varpi)/\varpi \otimes 1)v = (\omega_f(g) \otimes 1)v$, which acts trivially on the coefficients.

We have a decomposition $W \otimes \mathbb{Z}_p \Lambda \simeq \bigoplus_{j=0}^{f-1} \Lambda$, where $\sigma_j = \sigma_0 \circ \varphi^{-j} : W \hookrightarrow \Lambda$ corresponds to the projection onto the $j$th factor in the direct sum decomposition. We will generally consider $j$ modulo $f$. For any $R \in \text{Nilp}_\Lambda$, we get an induced decomposition $(W \otimes \mathbb{Z}_p R)[v] \cong \bigoplus_{j \in \mathbb{Z}/f\mathbb{Z}} R[j].$

Under this isomorphism, we have $\hat{\varpi}v = (\sigma_0 \circ \omega_f(g), \sigma_1 \circ \omega_f(g), \sigma_2 \circ \omega_f(g), \ldots, \sigma_{f-1} \circ \omega_f(g))v$.

Similarly, for any $(W \otimes \mathbb{Z}_p R)[v]$-module $M$, we write $M = \bigoplus_{j \in \mathbb{Z}/f\mathbb{Z}} M(j)$.

for the induced decomposition of $M$. Each $M(j)$ is an $R[v]$-direct summand of $M$.

**Definition 3.1.** Let $\mathcal{M}_R$ be an $(W \otimes \mathbb{Z}_p R)[v]$-module. A semilinear action of $\Delta$ on $\mathcal{M}_R$ is collection of $\tilde{g}$-semilinear bijections $\tilde{g} : \mathcal{M}_R \to \mathcal{M}_R$ for each $g \in \Delta$ such that

$$\tilde{g} \circ \tilde{h} = \tilde{gh}$$

for all $g, h \in \Delta$.

Note that $P(v)$, the minimal polynomial for $\varpi$, is fixed by $\tilde{g}$ for all $g$. Thus, $(W \otimes \mathbb{Z}_p R)[v][1/P(v)]$ inherits a semilinear action of $\Delta$ for any $R \in \text{Nilp}_\Lambda$.

**Definition 3.2.** Let $R$ be any $\Lambda$-algebra. A Kisin module (with bounded height) over $R$ is a finitely generated projective $R[v]$-module $\mathcal{M}_R$, which is Zariski locally on $\text{Spec } R$ finite free of constant rank over $R[v]$, together with an isomorphism $\phi_{\mathcal{M}_R} : \varphi^*(\mathcal{M}_R)[1/P(v)] \cong \mathcal{M}_R[1/P(v)]$.

We say that $(\mathcal{M}_R, \phi_{\mathcal{M}_R})$ has height in $[a, b]$ if

$$P(v)^a \mathcal{M}_R \supset \phi_{\mathcal{M}_R}(\varphi^*(\mathcal{M}_R)) \supset P(v)^b \mathcal{M}_R$$

as submodules of $\mathcal{M}_R[1/P(v)]$.

**Definition 3.3.** A Kisin module with descent datum over $R$ is a Kisin module $(\mathcal{M}_R, \phi_{\mathcal{M}_R})$ together with a semilinear action of $\Delta$ given by $\{\tilde{g}\}_{g \in \Delta}$ which commutes with $\phi_{\mathcal{M}_R}$, i.e., for all $g \in \Delta$,

$$\varphi^*(\tilde{g}) \circ \phi_{\mathcal{M}_R} = \phi_{\mathcal{M}_R} \circ \tilde{g}.$$
Fix integers $[a, b]$ with $a \leq b$ and a positive integer $n$. We take $X^{[a, b]}$ to be the fpqc stack over \Nilp_{\Lambda} such that $X^{[a, b]}(R)$ is the category of Kisin modules over $R$ of rank $n$ with height in $[a, b]$, with pullback defined in the obvious way (see §2.a in [PR09]). Similarly, we define the fpqc stack $Y^{[a, b], \Delta}$, where $Y^{[a, b], \Delta}(R)$ is the category of Kisin modules of rank $n$ with descent datum over $R$ and height in $[a, b]$. We will need some auxiliary spaces as well.

**Definition 3.4.** Fix $N > b - a$. Let $\tilde{X}^{[a, b]}$ be the fpqc stack over \Nilp_{\Lambda} given by

$$\tilde{X}^{[a, b]}(R) := \{(\mathcal{M}_R, \alpha_R) \mid \mathcal{M}_R \in X^{[a, b]}(R), \alpha_R : \mathcal{M}_R \cong R[v]_n \mod P(v)^N\}.$$ 

There is also an infinite version:

$$\tilde{X}^{[a, b], (\infty)}(R) := \{(\mathcal{M}_R, \alpha_R) \mid \mathcal{M}_R \in X^{[a, b]}(R), \alpha_R : \mathcal{M}_R \cong R[v]_n\}.$$ 

We leave out $N$ from the notation $\tilde{X}^{[a, b]}$, though of course the stack does depend on $N$. The natural maps $\tilde{X}^{[a, b]} \to X^{[a, b]}$ (resp. $\tilde{X}^{[a, b], (\infty)} \to X^{[a, b]}$) are formally smooth. For any $r \geq 1$, set

$$X^{[a, b]}_r := X^{[a, b]} \otimes_{A} A/p^r$$

and

$$X^{[a, b], \Delta}_r := X^{[a, b], \Delta} \otimes_{A} A/p^r.$$ 

**Theorem 3.5.** For any $r \geq 1$, $X^{[a, b]}_r$ is representable by an Artin stack of finite type over $\Spec A/p^r$.

Furthermore, $\tilde{X}^{[a, b]}$ is represented by a scheme of finite type over $\Spec A/p^r \Lambda$.

**Proof.** The first statement follows from [PR09, Theorem 2.1] as does the fact that $\tilde{X}_1^{[a, b]}$ is represented by a finite type scheme. Since the inclusion $\tilde{X}_1^{[a, b]} \subseteq \tilde{X}^{[a, b]}_r$ is a nilpotent thickening, $\tilde{X}^{[a, b]}_r$ is also a represented by a scheme by Lemma 87.3.8 of [Stacks] based on the corresponding fact for algebraic spaces and on the fact that a thickening of a scheme in the category of algebraic spaces is a scheme, which is Corollary 8.2 of [Ryd15]. \hfill \Box

We will return to this theorem with descent datum in Theorem 4.6. First, we discuss the Galois type or tame type of a Kisin module with descent datum. Let $\mathcal{M}_R$ be a Kisin module with descent datum over $R$. Write

$$\mathcal{M}_R = \bigoplus_{j \in \mathbb{Z}/f\mathbb{Z}} \mathcal{M}_R^{(j)}.$$ 

We get a semilinear $\Delta$-action on the Frobenius pullback $\mathcal{M}_R^{(j)}$ and $\varphi^*(\mathcal{M}_R^{(j)})$ as well as on the reduction $\varphi^*(\mathcal{M}_R^{(j)})/v\varphi^*(\mathcal{M}_R^{(j)})$.

**Definition 3.6.** Let $\mathcal{M}_R \in Y^{[a, b], \Delta}(R)$ and set $D_{R}^{(j)} := \mathcal{M}_R^{(j)}/v\mathcal{M}_R^{(j)}$. Then we say that $\mathcal{M}_R$ has type $\tau = \oplus_{i=1}^{n} \chi_i$, with $\chi_i \in \Delta^*$, if for all $j \in \mathbb{Z}/f\mathbb{Z}$

$$D_{R}^{(j)} \cong \tau$$

as linear representations of $\Delta$. 
Remark 3.7. In Definition 3.6, we require that the type is the same for all $j \in \{0, \ldots, f-1\}$. If $R = \Lambda$, the fact that $\phi_R$ commutes with the descent datum implies that the type must be the same on each component $D_{R}^{(j)}$, but this does not always hold if $R = \mathbb{F}$. Since we are ultimately interested in relating the Kisin modules with tame descent data to Galois representations over $F$ (see Section 5.3), we do not lose anything from imposing this condition.

Proposition 3.8. If $\mathcal{M}_R$ is a Kisin module with descent datum of type $\tau$, then

$$\varphi D_{R}^{(j)} := \varphi^*(\mathcal{M}_R^{(j)})/v\varphi^*(\mathcal{M}_R^{(j)}) \cong \tau.$$  

Proof. The natural $R$-linear injection $\mathcal{M}_R^{(j)} \hookrightarrow \varphi^*(\mathcal{M}_R^{(j)})$ given by $m \mapsto 1 \otimes m$ is $\Delta$-equivariant and induces an isomorphism modulo $v$. □

Proposition 3.9. Let $\mathcal{M}_R \in Y^{[a,b],\Delta}(R)$. Consider

$$D_{R}^{(j)} = \oplus_{\chi \in \Delta} D_{R,\chi}^{(j)}$$

where $D_{R,\chi}^{(j)}$ is the $\chi$-isotypic piece. Then $D_{R,\chi}^{(j)}$ is a finite projective $R$-module and hence the rank of $D_{R,\chi}^{(j)}$ is locally constant on $\text{Spec } R$.

Definition 3.10. Let $Y^{[a,b],\tau}$ be fpqc stack of Kisin modules with height in $[a,b]$ and descent datum of type $\tau$ over $\text{Nilp}_{\Lambda}$.

Corollary 3.11. The inclusion $Y^{[a,b],\tau} \subset Y^{[a,b],\Delta}$ is a relatively representable open and closed immersion.

Proof. This follows from Proposition 3.9 which says that the type of a Kisin module with descent is Zariski locally constant. □

Define $Y_f^{[a,b],\tau} := Y^{[a,b],\tau} \times_{\Lambda} \Lambda/p^r\Lambda$. In the next section, we will construct a smooth cover of $Y_f^{[a,b],\tau}$ and show that it is representable by an Artin stack of finite type (Theorem 4.6). We will also relate these moduli spaces of Kisin modules with descent datum to the local models from the previous section.

First, we will need a few preliminaries. Recall that $\tau = \bigoplus_{i=1}^n \chi_i$. We can write $\chi_i$ uniquely as

$$\chi_i = (\sigma_0 \circ \omega_f)^{a_i}$$

where $a_i = a_{i,0} + a_{i,1}p + \ldots + a_{i,f-1}p^{f-1}$.

Definition 3.12. Let $a_i$ be as above. For $j \in \mathbb{Z}/f\mathbb{Z}$ define

$$a_{i}^{(j)} = \sum_{k=0}^{f-1} a_{i,f-j+k}p^k$$
where the subscript \( f - j + k \) is taken modulo \( f \).

We have chosen a global ordering on the characters \( \chi_1, \chi_2, \ldots, \chi_n \). However, it will be useful to choose a possibly different ordering at each place \( j \in \mathbb{Z}/f\mathbb{Z} \).

**Definition 3.13.** An orientation of the type \( \tau \) is a set of elements \((s_j \in S_n)_{j \in \mathbb{Z}/f\mathbb{Z}}\) such that
\[
a_j^{(1)} \leq a_j^{(2)} \leq a_j^{(3)} \leq \cdots \leq a_j^{(n)}.
\]

**Remark 3.14.**
1. If the characters \( \chi_i \) are pairwise distinct, then there is a unique orientation for \( \tau \).
2. For a different choice of global ordering, the set of possible orientations changes by diagonal conjugation by \( S_n \).
3. For a non-principal series tame type \( \tau \) over \( \mathbb{Q}_p \) one can consider the base change \( \tau' \) to \( \mathbb{Q}_p/f \) where \( \tau \) becomes a principal series. The orientations for \( \tau' \) reflect what sort of type \( \tau \) was (see Example 3.15 below).

**Example 3.15.** Let \( 0 \leq a < b < p - 1 \). Consider the two dimensional tame types over \( \mathbb{Q}_p \) given by \( \tau_1 = \omega_1^a \oplus \omega_1^b \) and \( \tau_2 = \text{Ind}(\omega_2^{a+pb}) \). The base changes to \( \mathbb{Q}_p^{ab} \) are
\[
\tau_1' = \omega_2^{a+bp} \oplus \omega_2^{b+bp} \quad \text{and} \quad \tau_2' = \omega_2^{a+pb} \oplus \omega_2^{b+ap}
\]
respectively. The unique orientation for \( \tau_1' \) is \((id, id)\), and the unique orientation for \( \tau_2' \) is \((s, id)\) where \( s \) is the non-trivial transposition in \( S_2 \).

Consider the map \( R[u] \to R[v] \) given by \( u \mapsto v^{p-1} \). If \( \mathcal{M}_R \) is a Kisin module over \( R \) with descent datum, then for each \( j \), \( \mathcal{M}_R^{(j)} \) considered as an \( R[u] \)-module has a linear \( \Delta \)-action and so for any \( \chi \in \Delta^* \), we can consider the submodules
\[
\mathcal{M}_R^{(j)} = \{ m \in \mathcal{M}_R^{(j)} | \hat{g}(m) = \chi(g)m \}
\]
for all \( g \in \Delta \). Note that \( \mathcal{M}_R^{(j)} = \bigoplus_{\chi \in \Delta^*} \mathcal{M}_R^{(j)} \) as \( R[u] \)-modules, since the order of \( \Delta \) is prime to \( p \).

Similarly, we can define
\[
\mathcal{M}_R^{(j)} := \{ m \in \mathcal{M}_R^{(j)} | \hat{g}(m) = \chi(g)m \}.
\]
Since the descent datum commutes with the Frobenius action, we get linear maps
\[
\phi^{(j-1)}_{R, \chi} : \mathcal{M}_R^{(j-1)} \to \mathcal{M}_R^{(j)}.
\]

**Remark 3.16.** The \( \chi \)-isotypic piece of \( \varphi^*(\mathcal{M}_R^{(j)}) \) is not isomorphic to \( \varphi^*(\mathcal{M}_R^{(j)}_{\chi}) \). Thus, \( \phi^{(j)}_{R} \) does not define a semilinear Frobenius from \( \mathcal{M}_R^{(j-1)} \) to \( \mathcal{M}_R^{(j)} \). This is why we denote the \( \chi \)-isotypic component by \( \mathcal{M}_R^{(j)} \).
Proposition 3.17. Let $\mathcal{M}_R$ be a Kisin module over $R$ of rank $n$ with descent datum. For any $j \in \mathbb{Z}/f\mathbb{Z}$ and $\chi \in \Delta^*$ the modules $\mathcal{M}^{(j)}_{R,\chi}$ are finite projective $R[1/u]$-modules of rank $n$. Furthermore multiplication by $v$ on $\mathcal{M}^{(j)}_R$ induces an injective $R[1/u]$-module homomorphism

$$\mathcal{M}^{(j)}_{R,\chi} \to \mathcal{M}^{(j)}_{R,\phi(\sigma_j \circ \omega_j)\chi}. \tag{3.1}$$

Proof. The module $\mathcal{M}^{(j)}_{R,\chi}$ is finite projective $R[1/u]$-module because it is a direct summand of the finite projective module $\mathcal{M}^{(j)}_R$; this also implies that multiplication by $v$ on $\mathcal{M}^{(j)}_R$ is injective. By the discussion before Definition 3.1, multiplication by $v$ sends the $\chi$-isotypic piece of $\mathcal{M}^{(j)}_R$ to the $(\sigma_j \circ \omega_j)\chi$-isotypic piece. The rank computation is immediate. \qed

Lemma 3.18. Let $\mathcal{M}_R$ be a Kisin module with descent datum. Let $E_j(u) := \sigma_j(E(u))$. For each $\chi \in \Delta^*$, the Frobenius on $\mathcal{M}_R$ induces an isomorphism $\phi^{(j-1)}_{R,\chi} : \mathcal{M}^{(j-1)}_{R,\chi}[1/E_j(u)] \to \mathcal{M}^{(j)}_{R,\chi}[1/E_j(u)]$ such that

$$E_j(u)^a \mathcal{M}^{(j)}_{R,\chi} \supset \phi^{(j-1)}_{R,\chi} \left( \mathcal{M}^{(j-1)}_{R,\chi} \right) \supset E_j(u)^b \mathcal{M}^{(j)}_{R,\chi}$$

whenever $\mathcal{M}_R$ has $P(v)$-height in $[a,b]$.

Proof. For each $j$, the map $\phi^{(j-1)}_{R} : \mathcal{M}^{(j-1)}_R[1/\sigma_j(P(v))] \cong \mathcal{M}^{(j)}_R[1/\sigma_j(P(v))]$ is a $\Delta$-equivariant isomorphism. Using that $P(v) = E(v^{p^j-1}) = E(u)$, we see that multiplication by $E(u)$ respects the decomposition into isotypic pieces. The height condition is easy to verify. \qed

Choose an orientation $(s_j)$ for $\tau$ as in Definition 3.13. We then have the following commutative diagram for each $j$:

$$\begin{array}{cccccccc}
\phi^{(j-1)}_{R,\chi_{s_j(n)}} & \to & \phi^{(j-1)}_{R,\chi_{s_j(1)}} & \to & \phi^{(j-1)}_{R,\chi_{s_j(2)}} & \cdots & \phi^{(j-1)}_{R,\chi_{s_j(n-1)}} & \to & \phi^{(j-1)}_{R,\chi_{s_j(n)}} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
\mathcal{M}^{(j)}_{R,\chi_{s_j(n)}} & \to & \mathcal{M}^{(j)}_{R,\chi_{s_j(1)}} & \to & \mathcal{M}^{(j)}_{R,\chi_{s_j(2)}} & \cdots & \mathcal{M}^{(j)}_{R,\chi_{s_j(n-1)}} & \to & \mathcal{M}^{(j)}_{R,\chi_{s_j(n)}}
\end{array} \tag{3.1}$$

All the maps in the diagram are injective. The composition across each row is multiplication by $u$. The first horizontal arrow in each row is induced by multiplication $v^{p^j-1-a^{(j)}_{s_j(n)}+a^{(j)}_{s_j(1)}}$. The other horizontal arrows are induced by multiplication by $v^{a^{(j)}_{s_j(k+1)}-a^{(j)}_{s_j(k)}}$ for each $1 \leq k \leq n-1$. If some of the $\{\chi_i\}$ are equal, some of the maps will be the identity.

The diagram should remind one of the diagrams that appear in the classical definition of local models for $\text{GL}_n$ with parahoric level structure, which involve lattice chains (see [PR15] as well as Section 2 of [PRS13]). Once we have chosen an appropriate trivialization of $\mathcal{M}^{(j)}_{R,\chi_{s_j(n)}}$ in the next section the above diagram will determine an $R$-point of an appropriate local model.
4. Smooth modification

We maintain the conventions from the previous section. In particular, we fix an ordering \( \{ \chi_i \}_{i=1}^n \) of the characters appearing in \( \tau \). We would like to package the data of diagram (3.1) in a different way so that the relationship to the local models from §2 becomes clearer. If \( D \) is an \( R \)-module, then by a filtration on \( D \), we always mean by submodules which are direct summands of \( D \). We will work with increasing filtrations.

We continue to work over the category \( \text{Nilp}_A \) of \( A \)-algebra on which \( p \) is nilpotent. We make the following definition:

**Definition 4.1.** Let \( X, X' \) be fpqc stacks on \( \text{Nilp}_A \). A morphism \( f : X \to X' \) is smooth if \( f \mod p^N \) is smooth for all \( N \geq 1 \).

**Definition 4.2.** Let \( \mathcal{M}_R \in Y^{[a,b],\tau}(R) \). An eigenbasis for \( \mathcal{M}_R \) is a collection of bases \( \beta^{(j)} = \{ f_j^{(j)}, f_{j+1}^{(j)}, \ldots, f_n^{(j)} \} \) for each \( \mathcal{M}_R^{(j)} \) such that \( f_j^{(j)} \in \mathcal{M}_R^{(j)} \). An eigenbasis modulo \( P(v)^N \) is a collection of bases \( \{ \beta_N^{(j)} \}_{j \in \mathbb{Z}/f \mathbb{Z}} \) for each \( \mathcal{M}_R^{(j)}/\sigma_j(P(v))^N \mathcal{M}_R^{(j)} \) compatible, as above, with the descent datum.

An eigenbasis exists whenever \( D_R^{(j)} \) is free over \( R \) since one can lift a basis for \( D_R^{(j)} \) to \( \mathcal{M}_R^{(j)} \). In particular, such a basis exists Zariski locally on \( \text{Spec} \ R \) for any \( \mathcal{M}_R \in Y^{[a,b],\tau}(R) \).

**Definition 4.3.** Fix \( N > b - a \). Let \( \tilde{Y}^{[a,b],\tau} \) be the fpqc stack over \( \text{Nilp}_A \) given by

\[
\tilde{Y}^{[a,b],\tau}(R) := \left\{ \mathcal{M}_R, \beta^{(j)}, N \right\} \left| \mathcal{M}_R \in Y^{[a,b],\tau}(R), \beta_N^{(j)} : \mathcal{M}_R^{(j)} \cong R[v]^n \text{ mod } \sigma_j(P(v))^N \right. \\
\text{where } (\beta_N^{(j)}) \text{ is an eigenbasis. We also have an infinite version given by} \\
\tilde{Y}^{[a,b],\tau,\infty}(R) := \left\{ (\mathcal{M}_R, \beta^{(j)}) \right\} \left| \mathcal{M}_R \in Y^{[a,b],\tau}(R), \beta^{(j)} : \mathcal{M}_R^{(j)} \cong R[v]^n \right. \\
\text{where } (\beta^{(j)}) \text{ is an eigenbasis.}
\]

We leave out \( N \) from the notation \( \tilde{Y}^{[a,b],\tau} \), though of course the stack does depend on \( N \). See Proposition 4.4 below for a precise statement.

**Proposition 4.4.** Let \( \mathcal{M}_R \in Y^{[a,b],\tau}(R) \). An eigenbasis \( \{ \beta^{(j)} \} \) for \( \mathcal{M}_R \) induces a trivialization \( \mathcal{M}_R^{(j)} \cong R[u]^n \) for any \( \chi \in \Delta^* \). In particular, we have

\[
\gamma^{(j)} : \mathcal{M}_R^{(j)} \cong R[u]^n.
\]

Similarly, an eigenbasis modulo \( P(v)^N \) induces trivializations of \( \mathcal{M}_R^{(j)} \) modulo \( E(u)^N \).
Proof. An eigenbasis for $\mathcal{M}_R$ induces a $\Delta$-equivariant trivialization

$$\mathcal{M}_R^{(j)} \cong R[v]f_1^{(j)} \oplus \cdots \oplus R[v]f_n^{(j)} \cong R[v] \otimes_{A} \tau.$$ 

We can identify the $\chi$-isotypic component on the right side and see that it is naturally isomorphic to $R[u]^n$. To get the explicit basis for the $\chi$-isotypic component, translate the elements of eigenbasis into the $\chi$-isotypic component by multiplying by the smallest non-negative power of $v$ which is compatible with the descent datum. For example, for $\chi_{sj(n)}$, the basis $\gamma^{(j)}$ will be given by $v^{a_{sj(n)}-a_{sj(1)}} f_{sj(1)}^{(j)}, \ldots, v^{a_{sj(n)}-a_{sj(n-1)}} f_{sj(n-1)}^{(j)}, f_{sj(n)}^{(j)}$. \hfill \qed

Let $(s_j)_{j \in \mathbb{Z}/f\mathbb{Z}}$ be an orientation of $\tau$ (Definition 3.13). Furthermore, define a filtration on $\Lambda^n := \tau$ by

$$\text{Fil}^k(\Lambda^n) = \bigcup_{1 \leq i \leq k} (\Lambda^n)_{\chi_{sj(i)}}.$$ 

Let $P_j \subset \text{GL}_n$ be the parabolic which is the stabilizer of $\{\text{Fil}^k(\Lambda^n)\}$. For example, if all the characters are distinct then $P_j$ is a Borel subgroup for all $j \in \mathbb{Z}/f\mathbb{Z}$.

Recall the group schemes $L^+, E_j(u)P_j$ and $P_{j,N}$ defined before Proposition 2.19 with $P_j$ the parabolic as above. When $p$ is nilpotent in $R$, the $E_j(u)$-adic completion and $u$-adic completions of $R[u]$ coincide and so

$$L^+, E_j(u)P_j(R) = \{ g \in \text{GL}_n(R[u]) \mid g \mod u \in P_j(R) \}.$$ 

Proposition 4.5. The map $\pi(\infty) : \tilde{Y}^{[a,b],\tau,\infty} \to Y^{[a,b],\tau}$ (resp. $\pi(N) : \tilde{Y}^{[a,b],\tau} \to Y^{[a,b],\tau}$) is a torsor (for the Zariski topology) for $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} L^+, E_j(u)P_j$ (resp. $\prod_{j \in \mathbb{Z}/f\mathbb{Z}} P_{j,N}$). In particular, $\pi(N)$ is smooth and $\pi(\infty)$ is formally smooth.

Proof. We observed after Definition 4.2 that an eigenbasis (resp. eigenbasis mod $P(v)^N$) always exists Zariski locally on Spec $R$. We focus on the case of $\pi(\infty)$ since the other case is similar. We want to show that for a given module $\mathcal{M}_R$ with descent datum of type $\tau$ the set of eigenbases at an embedding $\sigma_j$ is a torsor for $L^+, E_j(u)P_j(R)$.

For any two eigenbases $\beta^{(j)} = (f_{sj(i)}^{(j)})$, $\beta'^{(j)} = (e_{sj(i)}^{(j)})$ ordered by $s_j$, consider the matrix $B^{(j)} \in \text{GL}_n(R[v])$ which writes $\beta^{(j)}$ in terms of $\beta'^{(j)}$. Compatibility with descent data imposes the condition that

$$A^{(j)} := D^{-1}_{a_{sj(i)}^{(j)}} B^{(j)} D_{a_{sj(i)}^{(j)}} \in \text{GL}_n(R[u])$$

where $D_{a_{sj(i)}^{(j)}}$ is the diagonal matrix with the $(i,i)$th entry given by $v^{a_{sj(i)}^{(j)}}$. This is just the fact that $v^{a_{sj(k)}^{(j)}-a_{sj(i)}^{(j)}} f_{sj(i)}^{(j)} \in \mathcal{M}_R^{(j)}_{R, \chi_{sj(k)}}$ for any $i, k$. 


Furthermore, if we consider the entries below the diagonal, we have

\[ A^{(j)}_{mk} = \sigma^{(j)}_{m \times k} - \sigma^{(j)}_{s(m) \times s(k)} B^{(j)}_{mk} \]

for \( m > k \) and with our choice of ordering \( \sigma^{(j)}_{s(j)(k)} = \sigma^{(j)}_{s(k)(j)} \geq 0 \) with equality if and only if \( \chi_{s(j)(m)} = \chi_{s(k)(j)} \). Thus, whenever \( \chi_{s(j)(m)} \neq \chi_{s(j)(k)} \) we see that \( A^{(j)}_{mk} \mod u = 0 \). This is exactly the condition the \( A^{(j)} \mod u \in P_j(R) \).

The converse is also true. That is, given \( A^{(j)} \in \GL_n(R[u]) \) with \( A^{(j)} \mod u \in P_j(R) \), the matrix

\[ B^{(j)} := D_{(a^{(j)}_i)} A^{(j)} D_{(a^{(j)}_i)^{-1}} \]

belongs to \( \GL_n(R[v]) \).

Furthermore, for an eigenbasis \( \{ s^{(j)}(i) \} \), \( \{ B^{(j)} s^{(j)}(i) \} \) will again be an eigenbasis. The condition that \( A^{(j)} \mod u \) lies in the parabolic ensures the integrality of \( B^{(j)} \).

\[ \square \]

**Theorem 4.6.** For any \( r \geq 1 \), \( Y^{[a,b],\tau}_r := Y^{[a,b],\tau} \otimes_{\Lambda} \Lambda/p^r \Lambda \) is representable by an Artin stack of finite type over \( \Spec \Lambda/p^r \Lambda \). Furthermore, \( \tilde{Y}^{[a,b],\tau}_r := \tilde{Y}^{[a,b],\tau} \otimes_{\Lambda} \Lambda/p^r \Lambda \) is represented by a scheme of finite type over \( \Spec \Lambda/p^r \Lambda \).

**Proof.** It suffices to prove the second statement, for which we will use a strategy originally employed in [CEGS]. Consider the map

\[ \xi : \tilde{Y}^{[a,b],\tau}_r \to \tilde{X}^{[a,b]}_r \]

given by forgetting the descent datum. It suffices to show that \( \xi \) is relatively representable and finite type by Theorem 3.3.

Given \( (\mathfrak{M}_R, \phi_R, \beta_R) \in \tilde{X}^{[a,b]}_r(R) \) we see that the data of the additive bijections \( \hat{\gamma} : \mathfrak{M}_R \to \mathfrak{M}_R \) for all \( g \in \Delta \), which have to commute with \( \phi_R \), satisfy \( \hat{g}_1 \circ \hat{g}_2 = \hat{g}_1 \circ \hat{g}_2 \), be \( R[v] \)-semilinear and compatible with \( \beta_R \) is representable by a scheme of finite type over \( R \). Indeed, such a bijection \( \hat{\gamma} \) has to induce an \( R((u)) \)-linear automorphism of \( \mathfrak{M}_R[1/v] \) (which can be thought of as an étale \( \varphi \)-module over \( R \) of rank \( n \cdot (p^f - 1) \)). By the proof of Theorem 2.5(b) of [PR09], the data of an \( R((u)) \)-linear automorphism of \( \mathfrak{M}_R[1/v] \) which commutes with \( \phi_R \) is representable by a scheme of finite type over \( R \). Further imposing the the relationships \( \hat{g}_1 \circ \hat{g}_2 = \hat{g}_1 \circ \hat{g}_2 \) and the \( R[v] \)-semilinearity cuts out a closed subscheme. Finally, the requirement that the descent datum preserve the lattice \( \mathfrak{M}_R \subset \mathfrak{M}_R[1/v] \) and the compatibility with \( \beta_R \) are also closed conditions.

We conclude then that \( \tilde{Y}^{[a,b],\tau}_r \to \tilde{X}^{[a,b]}_r \) is relatively representable and finite type and so by Theorem 3.3 \( Y^{[a,b],\tau}_r \) is a scheme of finite type over \( \Spec \Lambda/p^r \Lambda \). Since \( \tilde{Y}^{[a,b],\tau}_r \to Y^{[a,b],\tau}_r \) is a smooth cover we deduce that \( Y^{[a,b],\tau}_r \) is an Artin stack of finite type. \[ \square \]
We are now ready to construct the local model diagram for Kisin modules with descent data:

\[
\begin{array}{ccc}
\tilde{\varphi}(0,h,\tau,(\infty)) & \xrightarrow{\pi(\infty)} & \varphi(0,h,\tau)
\end{array}
\]

To define \( \Psi \), we need to associate to any \( (\mathfrak{M}_R, \phi_R, \{ \tilde{g} \}, \beta) \in \tilde{\varphi}(a,b,\tau,(\infty))(R) \) and each embedding \( \sigma_j \), a triple \((L(j), \alpha(j), \varepsilon(j)) \in \text{Fl}^j_{\psi}(u)(R)\). The pair \((L(j), \alpha(j))\) is straightforward to define and is given by the ‘image’ of Frobenius.

To be precise, we take \( L^{(j)} = \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n) \) and define the trivialization \( \alpha^{(j)} \) by the composition

\[
\varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n)[1/E_j(u)] \xrightarrow{\phi^{(j)}_{R, \chi_{j}}(n)} \mathfrak{M}^{(j)}_{R, \chi_{j}}(n)[1/E_j(u)] \xrightarrow{\gamma^{(j)}} (R[u])^{n}[1/E_j(u)]
\]

where \( \gamma^{(j)} \) is induced by \( \beta^{(j)} \) as in Proposition 4.4. Notice that we are using the alternative description of \( \text{Fl}^j_{\psi}(u) \) from Definition 2.4.

Next, we have to define a filtration \( \varepsilon^{(j)} \) on \( L^{(j)} \mod u \). Let

\[
\varphi D^{(j-1)}_{\chi_{j}} := \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n) \mod u = L^{(j)} \mod u.
\]

The filtration is essentially given by the diagram (3.1). Namely for each \( 1 \leq i \leq n \), let

\[
\omega_i : \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(i) \to \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n)
\]

be the injective map induced by composition along the upper row of (3.1). Then we get the inclusions

\[
u \left( \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n) \right) \subset \omega_1 \left( \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n) \right) \subset \ldots \omega_{n-1} \left( \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n) \right) \subset \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n).
\]

We can then define the filtration \( \varepsilon^{(j)} \) by

\[
\text{Fil}^i \left( \varphi D^{(j-1)}_{\chi_{j}}(n) \right) = \omega_i \left( \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n) \right) / u \left( \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n) \right).
\]

It is not hard to see that the filtration \( \varepsilon^{(j)} \) is a \( P_j \)-filtration for \( P_j \) defined after Proposition 4.4.

In summary, we have

\[
\Psi(\mathfrak{M}_R, \phi_R, \{ \tilde{g} \}, \beta) = \left( \varphi \mathfrak{M}^{(j-1)}_{R, \chi_{j}}(n), \gamma^{(j)} \circ \phi^{(j-1)}_{R, \chi_{j}}(n), \left\{ \text{Fil}^i \left( \varphi D^{(j-1)}_{\chi_{j}}(n) \right) \right\}_{i=1}^{n} \right)_{j \in \mathbb{Z}/f\mathbb{Z}}.
\]

We now come to the main theorem:

**Theorem 4.7.** The morphism \( \Psi \) is formally smooth.
Proof. Roughly, the idea is that image under \( \Psi \) gives the descent datum and the image of Frobenius. What is left is to choose an isomorphism between \( \varphi^*(\mathcal{M}_R) \) and its image (compatible with descent datum) and this will satisfy formal smoothness. We now give the details.

We can twist to reduce the case where \([a,b] = [0,h]\) so that the Frobenius is an honest endomorphism. Let \( R \in \text{Nilp}_A \) and let \( I \) be a square-zero ideal of \( R \). Choose \( (\mathcal{M}_{R/I}, \phi_{R/I}, \{\tilde{\beta}^i\}) \in \tilde{\mathcal{Y}}^{[0,h],[\tau,\infty)}(R/I) \). Assume we are given a lift \((L_R^{(j)}, \tilde{\alpha}^{(j)}, \{\text{Fil}^i(L_R^{(j)} \mod v)\})\) of \( \Psi(\mathcal{M}_{R/I}) \) to \( R \).

Let \( \mathcal{M}_R \) be a free \((W \otimes_{\mathbb{Z}_p} R)[[v]]\)-module of rank \( n \) and choose an isomorphism \( \mathcal{M}_R \otimes_R R/I \cong \mathcal{M}_{R/I} \). By Proposition \ref{prop:5.1}, \( \tilde{\beta}_R^{(j)} \) induces a trivialization \( \tilde{\gamma}_R^{(j)} : \mathcal{M}_R^{(j)} \cong (R/I)[[v]]^n \). We can then choose trivializations \( \tilde{\beta}^{(j)} \) of \( \mathcal{M}_R^{(j)} \) for each \( j \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{M}_R^{(j)} & \xrightarrow{\tilde{\beta}^{(j)}} & R[[v]]^n \\
\downarrow & & \downarrow \\
\mathcal{M}_{R/I}^{(j)} & \xrightarrow{\beta^{(j)}} & (R/I)[[v]]^n
\end{array}
\]

commutes. Let \( f_{s_j(i)} \) be the preimage of the \( i \)th standard basis element under \( \tilde{\beta}^{(j)} \). We define a semilinear \( \Delta \)-action of type \( \tau \) on \( \mathcal{M}_R \) by demanding that \( \Delta \) act on \( f_{s_j(i)} \) through the character \( \chi_{s_j(i)} \). This clearly makes \( \tilde{\beta}^{(j)} \) into an eigenbasis for this descent datum.

The eigenbasis \( \tilde{\beta}^{(j)} \) induces a filtration on \( \varphi \mathcal{D}_{X_{s_j}(n)}^{(j)} \) as in \((4.3)\) (compatible with reduction modulo \( I \)). Choose an isomorphism \( \theta^{(j)} : \varphi \mathcal{M}_{R,X_{s_j}(n)}^{(j-1)} \cong L_R^{(j)} \) compatible with the filtrations on \( \varphi \mathcal{D}_{X_{s_j}(n)}^{(j)} \) and \( L_R^{(j)}/uL_R^{(j)} \) and which reduces to the given isomorphism \( \varphi \mathcal{M}_{R,I,X_{s_j}(n)}^{(j-1)} \cong L_R^{(j)/I} \). Define \( \phi^{(j-1)}_{R,X_{s_j}(n)} \) to be the composition

\[
\varphi \mathcal{M}_{R,X_{s_j}(n)}^{(j-1)} [1/E_j(u)] \xrightarrow{\theta^{(j)}} L_R^{(j)} [1/E_j(u)] \xrightarrow{\tilde{\alpha}^{(j)}} (R[[u]]^n)[1/E_j(u)] \xrightarrow{(\tilde{\gamma}^{(j)})^{-1}} \varphi \mathcal{M}_{R,X_{s_j}(n)}^{(j)} [1/E_j(u)].
\]

Observe that the only map which not an isomorphism without inverting \( E_j(u) \) is \( \tilde{\alpha}^{(j)} \). The “image” of Frobenius is then determined by the image of \( \tilde{\alpha}^{(j)} \).

If \( \mathcal{M}_R \in \tilde{\mathcal{Y}}^{[0,h],[\tau,\infty)}(R) \) then the Frobenius \( \phi_{R/I}^{(j-1)} \) is uniquely determined by \( \phi_{R,X_{s_j}(n)}^{(j-1)} \) by diagram \((5.1)\). Conversely, to construct \( \phi_{R/I}^{(j-1)} \) it suffices to construct \( \phi_{R,X_{s_j}(i)}^{(j-1)} \) for each \( 1 \leq i \leq n-1 \) such that the diagram

\[
\begin{array}{ccc}
\varphi \mathcal{M}_{R,X_{s_j}(i)}^{(j-1)} & \xrightarrow{\omega_i} & \varphi \mathcal{M}_{R,X_{s_j}(i)}^{(j)} \\
\downarrow \phi_{R,X_{s_j}(i)}^{(j-1)} & & \downarrow \phi_{R,X_{s_j}(i)}^{(j)} \\
\mathcal{M}_{R,X_{s_j}(i)}^{(j)} & \xrightarrow{\omega_i'} & \mathcal{M}_{R,X_{s_j}(i)}^{(j)}
\end{array}
\]
commutes. The horizontal arrows are induced by multiplication by $v^{a(j)}_{\pi_j} - a_{j(j)}$, so they are injections by Proposition 3.17. The fact that $\theta^{(j)}$ was chosen to respect filtrations implies that the composition $\phi^{(j-1)}_{R_{X_{j(i)}}} \circ \omega_j$ lies in the image of $\omega'_i$ and so there exists a unique $\phi^{(j-1)}_{R_{X_{j(i)}}}$ which completes the diagram.

We can refine $\Psi$ to a morphism of finite type.

**Proposition 4.8.** Let $N > b - a$. The map $\Psi$ factors through the finite type closed subscheme $\text{Fl}_{K}^{[a,b],E(u)}$. Furthermore, there exists a smooth map $\Psi^N : \tilde{Y}^{[a,b],\tau} \to \text{Fl}_{K}^{[a,b],E(u)}$ such that $\Psi$ is the composition of

$$\tilde{Y}^{[a,b],\tau,\infty} \to \tilde{Y}^{[a,b],\tau} \xrightarrow{\Psi^N} \text{Fl}_{K}^{[a,b],E(u)}.$$ 

**Proof.** Lemma 3.18 says that image of $\Psi$ factors through $\text{Fl}_{P_j}^{[a,b],E_j(u)}$ on each factor and hence through $\text{Fl}_{K}^{[a,b],E(u)}$.

To show that $\Psi$ factors as $\Psi^N$, we have to show that for any $(\mathfrak{M}_R, \beta, \{\tilde{g}\}) \in \tilde{Y}^{[a,b],\tau,\infty}(R)$ the image under $\Psi$ only depends on $\beta$ modulo $P(v)^N$. We see that the trivialization only enters in (4.2).

Furthermore, as we saw in Proposition 3.17 changing the trivialization $\beta^{(j)}$ amounts to changing $\gamma^{(j)} : \mathfrak{M}^{(j)}_{R_{X_{j(i)}}} \cong (R[u])^n$ by an element of $g \in L^{+},E_j(u)\mathcal{P}_j(R)$. On $\text{Fl}_{P_j}^{[a,b],E_j(u)}$, this corresponds to the natural action of $L^{+},E_j(u)\mathcal{P}_j$ defined in Proposition 2.20.

If $\beta^{(j)}$ and $\beta^{(j)}$ are congruent modulo $\sigma_j(P(v))^N$, then $\gamma^{(j)} = g \cdot \gamma^{(j)}$ for $g \in L^{+},E_j(u)\mathcal{P}_j(R)$ with $g \equiv \text{Id}$ mod $E_j(u)^N$. If $g$ is congruent to the identity modulo $E_j(u)^N$, then $g$ acts trivially on $\text{Fl}_{P_j}^{[a,b],E_j(u)}$ (for example, by identifying $\text{Fl}_{P_j}^{[a,b],E_j(u)}$ with lattices as in Definition 2.7). □

**Corollary 4.9.** We get a diagram

$$
\begin{array}{ccc}
\tilde{Y}^{[a,b],\tau} & \xrightarrow{\Psi^N} & \text{Fl}_{K}^{[a,b],E(u)} \\
\pi^N & & \\
\tilde{Y}^{[a,b],\tau} & \xrightarrow{\Psi^N} & \\
\end{array}
$$

where both $\pi^N$ and $\Psi^N$ are smooth.

5. **p-adic Hodge type**

In this section, we define and study a closed substack $Y^{\mu,\tau} \subset Y^{[a,b],\tau}$ which is related to the notion of $p$-adic Hodge type. A similar construction but without descent data was carried out in [PR09 §3]. In $n = 2$ and $\mu \in (\{0, 1\}^n)_{\text{Hom}(K, \overline{\mathbb{Q}})}$ (i.e., $\mu$ minuscule), $Y^{\mu,\tau}$ and the local model diagram are studied in forthcoming work of the first author with Emerton, Gee and Savitt [CEGS].
Let \( \mu \) be a geometric cocharacter of \( \text{Res}_{K/\mathbb{Q}_p} \text{GL}_n \). For each embedding \( \sigma_j : K_0 \to E \), we get a geometric cocharacter \( \mu_j \) of \( \text{Res}_{K/K_0} \text{GL}_n \) such that \( \mu = (\mu_j)_{\sigma_j} \). Assume that \( F = \Lambda[1/p] \) contains the reflex field of the conjugacy class \([\mu]\), i.e., \( \Lambda = \Lambda[\mu] \).

In §2, we defined the local model
\[
M(\mu) = \prod_{j \in \mathbb{Z}/f \mathbb{Z}} M(\mu_j) \subset \prod_{j \in \mathbb{Z}/f \mathbb{Z}} \mathcal{F}_{P_j}^{E_j(u)} = \mathcal{F}_{K}^{E(u)}.
\]

By Theorem 2.14, \( M(\mu) \) is flat and projective over \( \Lambda \) with reduced special fiber. Also, \( M(\mu) \) is stable for the action of “loop group” \( \prod_{j \in \mathbb{Z}/f \mathbb{Z}} \mathbb{L}^{+}_{j,E_j} P_j \) by Proposition 2.20.

Assume that \( a, b \) are integers with \( a \leq b \) such that \( M(\mu) \subset \prod_{i \in \mathbb{Z}/f \mathbb{Z}} \mathcal{F}_{[a,b]}^{E_j(u)} P_j \). For any \( N > a - b \), we saw in Proposition 2.20 that the action of \( \prod_{j \in \mathbb{Z}/f \mathbb{Z}} \mathbb{L}^{+}_{j,E_j} P_j \) on \( M(\mu) \) factors through the action of the smooth connected group scheme \( \prod_{j \in \mathbb{Z}/f \mathbb{Z}} \mathbb{P}_{j,N} \).

**Definition 5.1.** Define the closed subscheme
\[
\tilde{Y}_{\mu,\tau}^{[a,b],\tau} := \tilde{Y}_{r}^{[a,b],\tau} \times_{\mathcal{F}_{K}^{E(u)} \times \mathcal{F}_{\Lambda}/p^{r-1} \mathbb{Z}} M(\mu).
\]

We have an induced smooth map
\[
\Psi^\mu : \tilde{Y}_{\mu,\tau} \to M(\mu).
\]

We would like to show that \( \tilde{Y}_{\mu,\tau} \) descends to a closed substack \( Y_{\mu,\tau} \subset Y_{r}^{[a,b],\tau} \).

**Proposition 5.2.** For any \( r \geq 1 \), there exists a closed substack \( Y_{\mu,\tau}^{r} \subset Y_{r}^{[a,b],\tau} \) such that the diagram
\[
\begin{align*}
\tilde{Y}_{\mu}^{r,\tau} &\longrightarrow \tilde{Y}_{r}^{[a,b],\tau} \\
\pi^\mu \downarrow &\quad \downarrow \pi^{(N)} \\
Y_{\mu}^{r,\tau} &\longrightarrow Y_{r}^{[a,b],\tau}
\end{align*}
\]
is Cartesian. Furthermore, \( Y_{r}^{[a,b],\tau} \times_{\mathbb{Z}/p^r \mathbb{Z}} \mathbb{Z}/p^{r-1} \mathbb{Z} \cong Y_{r-1}^{[a,b],\tau} \).

**Proof.** By Proposition 4.5, \( \pi^{(N)} : \tilde{Y}_{r}^{[a,b],\tau} \to Y_{r}^{[a,b],\tau} \) is a torsor for the smooth group \( \mathcal{G}_r := (\prod_{j \in \mathbb{Z}/f \mathbb{Z}} \mathbb{P}_{j,N})_{\Lambda/p^{r} \Lambda} \). Any \( \mathcal{G}_r \)-stable closed subscheme of \( \tilde{Y}_{r}^{[a,b],\tau} \) descends by faithfully flat descent to a closed substack of \( Y_{r}^{[a,b],\tau} \).

Since \( (M(\mu))_{\Lambda/p^{r} \Lambda} \) is stable under \( \mathcal{G}_r \) so is \( \tilde{Y}_{r}^{[a,b],\tau} \) and we define the desired \( Y_{r}^{[a,b],\tau} \) by descent. This construction is clearly compatible with reduction modulo \( p^{r-1} \).

Since the \( Y_{r}^{[a,b],\tau} \) are compatible with reduction modulo \( p^{r-1} \), we can define a stack \( Y_{\mu,\tau} \) on \( \text{Nilp}_\Lambda \) whose reduction modulo \( p^{r} \) is \( Y_{r}^{[a,b],\tau} \).
**Theorem 5.3.** We have a local model diagram:

\[
\begin{array}{c}
\tilde{Y}^{\mu, \tau} \\
\downarrow \pi^\mu \\
Y^{\mu, \tau} \\
\downarrow \Psi^\mu \\
M(\mu)
\end{array}
\]

where both \(\pi^\mu\) and \(\Psi^\mu\) are smooth maps.

5.1. **Special fiber: Kottwitz-Rapoport strata.** In addition to imposing the \(p\)-adic Hodge type \(\mu\) via the local model diagram (5.1), we can also stratify the special fiber of \(Y^{\mu, \tau}\) by pulling back the stratification in Theorem 2.17. This is the analogue of the Kottwitz-Rapoport stratification in the Shimura variety setting.

Let \(Y^{\mu, \tau}\) denote the special fiber of \(Y^{\mu, \tau}\). As in the discussion before Theorem 2.17, we can write \(\mu_j = (\mu_{j, \psi})\) where \(\psi\) runs over embeddings \(\psi : K \hookrightarrow \overline{F}\) over the embedding \(\sigma_j\) and where each \(\mu_{j, \psi}\) is dominant. We define

\[\lambda_j = \sum_{\psi : K \hookrightarrow \overline{F}} \mu_{j, \psi} \cdot\]

**Proposition 5.4.** For each \(\tilde{w} = (\tilde{w}_j) \in \prod_{j=0}^{f-1} \text{Adm}_{P_j}(\lambda_j)\), there is a locally closed substack \(\overline{Y}^{\mu, \tau}_{\tilde{w}} \subset \overline{Y}^{\mu, \tau}\) such that

\[(\pi^\mu)^{-1}(\overline{Y}^{\mu, \tau}_{\tilde{w}}) = (\Psi^\mu)^{-1}\left(\prod_j S^0(\tilde{w}_j)\right).\]

Furthermore, the closure \(\overline{Y}^{\mu, \tau}_{\overline{w}}\) of \(\overline{Y}^{\mu, \tau}_{\tilde{w}}\) is the union of the strata for all \((\tilde{w}'_j) \in \prod_{j=1}^{f-1} \text{Adm}_{P_j}(\lambda_j)\) such that \(\tilde{w}'_j \leq \tilde{w}_j\) for all \(j\).

**Proof.** The argument is identical to the construction of \(Y^{\mu, \tau}_r\) from Proposition 5.2. We just note that \((L^+, E_i(u)P_j)_F\) is the parahoric subgroup of the loop group \(GL_n(\mathbb{F}[u])\) corresponding to \(P_j\) whose orbits on \(\overline{M}(\mu)\) are in bijection with \(\text{Adm}_{P_j}(\lambda_j)\). The closure relations follow from smoothness of \(\pi^\mu\) and \(\Psi^\mu\). \(\square\)

We now introduce the notion of shape (or genre in French). The genre of Kisin/Breuil module of rank 2 was first introduced in [Bre12] where it is connected to Serre weights for \(GL_2\) over an unramified extension of \(\mathbb{Q}_p\). It also plays an important role in [BM14, EGS15] in computing tamely Barsotti-Tate deformation rings as well as in the recent work of [CDM1, CDM2]. The notion of shape for a rank 3 Kisin modules with \(p\)-adic Hodge type \((2, 1, 0)\) and \(K/\mathbb{Q}_p\) unramified will be used in forthcoming joint work of the second author [LLLM] to compute potentially crystalline deformation rings for \(GL_3\).
Definition 5.5. A Kisin module $\overline{M} \in \overline{Y}^{m,τ}(F_p)$ is said to have shape $\overline{w}$.

Remark 5.6. The shape of Kisin module $\overline{M} \in \overline{Y}^{m,τ}(F)$ has a more concrete interpretation as well. $\overline{M}$ has shape $(\overline{w}_j)$ if the matrix for the Frobenius $\varphi^{(j)}_{F,\chi_{j,\alpha}}$ with respect to any basis compatible with the filtration lies in the double coset $L^+P_j(F)\overline{w}_j L^+P_j(F)$.

5.2. Generic fiber. We would now like to characterize $Y^{\mu,τ}$ so that we can relate it back to potentially crystalline representations and Hodge-Tate weights in the next section. Since $M(\mu)$ is defined by flat closure, this has to be done by working over the “generic” fiber in some suitable sense.

For any complete local Noetherian $\Lambda$-algebra $R$ with finite residue field and maximal ideal $m_R$, we define the $R$-points of $Y^{[a,b],τ}$ as the inverse limit category

$$Y^{[a,b],τ}(R) = \{(M_k, τ_k) \mid M_k \in Y^{[a,b],τ}(R/m_R^j R), τ_k : M_k \otimes R/m_R^k R \cong M_{k-1}\}.$$ 

Similarly, we can define $Y^{\mu,τ}(R)$.

Given $(M_k, τ_k) \in Y^{[a,b],τ}(R)$, the inverse limit $M_R = \varprojlim M_k$ is a module over $(W \otimes_{\mathbb{Z}_p} R)[v]$ equipped with a semilinear Frobenius

$$\phi_R : \varphi^*(M_R)[1/P(v)] \to M_R[1/P(v)]$$

and descent datum of type $τ$.

We now introduce the notion $p$-adic Hodge type first for $\overline{Q}_p$-points and then more generally. Let $F'/F$ be a finite extension with ring of integers $\Lambda'$.

Proposition 5.7. For any Kisin module $M_\Lambda \in Y^{[a,b],τ}(\Lambda')$, let $M_{F'} := M_\Lambda[1/p]$. Then the specialization

$$D_{F'} := \varphi^*(M_{F'})/P(v)\varphi^*(M_{F'})$$

is a finitely generated projective $L \otimes_{\mathbb{Q}_p} F'$-module with a semilinear action of $\Delta$.

Proof. This follows from the fact that $((W \otimes_{\mathbb{Z}_p} \Lambda')[[v]])[1/p]/P(v) \cong L \otimes_{\mathbb{Q}_p} F'$ and that $M_{F'}$ is finitely generated and projective over $((W \otimes_{\mathbb{Z}_p} \Lambda')[[v]])[1/p]$.

We can define a filtration on $D_{F'}$ as in [Kis08].

Definition 5.8. Define

$$\text{Fil}^j(\varphi^*(M_{F'})) := \{m \in \varphi^*(M_{F'}) \mid \phi_{F_{F'},p}(m) \in P(v)^j M_{F'}\}.$$ 

Define $L \otimes_{\mathbb{Q}_p} F'$-submodules

$$\text{Fil}^j(D_{F'}) := \text{Fil}^j(\varphi^*(M_{F'}))/(\text{Fil}^j(\varphi^*(M_{F'})) \cap P(v)\varphi^*(M_{F'})) \subset D_{F'}.$$
Remark 5.9. If $\mathcal{M}_{F'}$ has height in $[a, b]$ then it is a decreasing filtration with $\text{Fil}^n(D_{F'}) = D_{F'}$ and $\text{Fil}^{b+1}(D_{F'}) = 0$.

For $\mathcal{M}_{F'}$ as in Proposition 5.7 and $\chi \in \Delta^*$, we can define $D_{F',\chi}^{(j)} := \varphi(\mathcal{M}_{F',\chi}/E_j(u)^{\varphi} \mathcal{M}_{F',\chi}^{(j-1)})$ together with a filtration defined in an analogous way using $\phi_{\mathcal{M}_{F',\chi}}$ and $E_j(u)$ in place of $\phi_{\mathcal{M}_{F'}}$ and $P(v)$.

Lemma 5.10. Let $\mathcal{M}_{F'}$ be as in Proposition 5.7. Let $D_{F'}^{(j)}$ be the $L \otimes_{K_0,\sigma_j} F'$-submodule of $D_{F'}$ corresponding to $\sigma_j : K_0 \rightarrow F'$. There is a natural isomorphism

$$D_{F'}^{(j)} \cong D_{F',\sigma_j(u)} \otimes_K L$$

of filtered $L \otimes_{K_0,\sigma_j} F'$-modules.

Proof. First, note that we have the isotypic decomposition $D_{F'}^{(j)} = \oplus_{\chi \in \Delta^*} D_{F',\chi}^{(j)}$ as $K \otimes_{K_0,\sigma_j} F'$-modules, which gives an isomorphism $D_{F'}^{(j)} \cong D_{F',\sigma_j(u)} \otimes_K L$ of $K \otimes_{K_0,\sigma_j} F'$-modules, since multiplication by $v$ when $p$ is inverted and $P(v) = 0$ induces isomorphisms $D_{F',\chi}^{(j)} \cong D_{F',\sigma_j(u)}^{(j)} K L$. This can be upgraded to an isomorphism $D_{F'}^{(j)} \cong D_{F',\sigma_j(u)} \otimes_K L$ of $L \otimes_{K_0,\sigma_j} F'$-modules, because multiplication by $v^{p^j-1}$ is multiplication by $u$, which is identified with $\pi_K \otimes 1$ under the isomorphism $((W \otimes_{K_0,\sigma_j} \Lambda')[u])/[1/p]/E_j(u) \cong K \otimes_{K_0,\sigma_j} F'$. This means that $v$ is identified with $\pi_L \otimes 1 \in L \otimes_{K_0,\sigma_j} F'$. The fact that the isomorphism $D_{F'}^{(j)} \cong D_{F',\sigma_j(u)} \otimes_K L$ respects the filtrations on the two sides follows from the commutative diagram 5.11 where all the horizontal maps are now isomorphisms.

Recall that we assume the conjugacy class of $\mu$ is defined over $F$, i.e., $F = F[\mu]$. Associated to $\mu$, we then have a $\mathbb{Z}$-graded $K \otimes_{Q_p} F$-module $V_\mu$ of rank $n$. See for example [Kis08 (2.6)].

Definition 5.11. Let $F'/F$ be a finite extension with ring of integers $\Lambda'$. We say that $\mathcal{M}_{\Lambda'} \in \mathcal{X}^{[a,b],\tau}(\Lambda')$ has $p$-adic Hodge type $\mu$ if

$$\text{gr}^\bullet(D_{F'}) \cong \text{gr}^\bullet(V_\mu \otimes_{K \otimes_{Q_p} F} (L \otimes_{Q_p} F'))$$

as graded $L \otimes_{Q_p} F'$-modules.

We say $\mathcal{M}_{\Lambda'}$ has $p$-adic Hodge type $\leq \mu$ if $\mathcal{M}_{\Lambda'}$ has $p$-adic Hodge type $\mu'$, for some $\mu'$ such that $[\mu'] \leq [\mu]$ in the Bruhat ordering.

Corollary 5.12. Let $F'/F$ be a finite extension and let $\mathcal{M}_{\Lambda'}$ be as above. Set $V_{\mu_j} := V^{(j)}_{\mu_j}$, which is a filtered $K \otimes_{K_0,\sigma_j} F$-module. Then $\mathcal{M}_{\Lambda'}$ has $p$-adic Hodge type $\mu = (\mu_j)_{j \in \mathbb{Z}/f\mathbb{Z}}$ if and only if

$$\text{gr}^\bullet(D_{F',\sigma_j(u)}) \cong \text{gr}^\bullet(V_{\mu_j})$$
for every $0 \leq j \leq f - 1$.

Proof. This follows directly from Lemma 5.10.

Let $\mathfrak{M}_R \in Y^{[a,b],\tau}(R)$. For any finite extension $F'/F$, any homomorphism $x : R \to F'$ factors through the ring of integers $\Lambda'$. We can consider the base change $\mathfrak{M}_x := (\mathfrak{M}_R \otimes_{R,x} \Lambda')[1/p]$ for which we have defined the notion of $p$-adic Hodge type.

We would now like to characterize when $\mathfrak{M}_R \in Y^{[a,b],\tau}(R)$ lies in $Y^{\mu,\tau}(R)$.

**Theorem 5.13.** Let $R$ be a complete Noetherian $\Lambda$-algebra with finite residue field. Assume $R$ is $\Lambda$-flat and reduced. Then $\mathfrak{M}_R \in Y^{[a,b],\tau}(R)$ lies in $Y^{\mu,\tau}$ if and only if for all finite extensions $F'/F$ and all homomorphisms $x : R \to F'$ the base change $\mathfrak{M}_x$ has $p$-adic Hodge type $\leq \mu$.

Proof. Let $N > a - b$. Choose an eigenbasis $\tilde{z}_1 := (\tilde{\beta}^{(j)})_{j \in \mathbb{Z}/f\mathbb{Z}}$ for $\mathfrak{M}_{R,1} \in Y^{[a,b],\tau}(R/mR)$. Since the morphism $\pi^{(N)} : \tilde{Y}^{[a,b],\tau} \to Y^{[a,b],\tau}$ is smooth, we can find a compatible system of points $\tilde{z} := (\tilde{z}_r)_r$ with $\tilde{z}_r \in \tilde{Y}^{[a,b],\tau,\cdot}(R/mR)_{[a,b]}$ such that $\pi^{(N)}(\tilde{z}_r) = \mathfrak{M}_{R,r}$.

We see then that $\mathfrak{M}_R$ is in $Y^{\mu,\tau}(R)$ if and only if $\Psi^N(\tilde{z}_r) \in M(\mu)(R/mR)$ for all $r \geq 1$. The compatible system $\Psi^N(\tilde{z}_r)$ defines a map

$$\Psi^N(\tilde{z}) : \text{Spec } R \to \text{Fl}^{[a,b],E(u)}_K(R).$$

Since $M(\mu)$ is a $\Lambda$-flat closed subscheme of $\text{Fl}^{[a,b],E(u)}_K$, we see that $\Psi^N(\tilde{z})$ factors through $M(\mu)$ if and only if we have a factorization

$$\begin{array}{ccc}
\text{Spec } R[1/p] & \xrightarrow{\Psi^N(\tilde{z})[1/p]} & \text{Fl}^{[a,b],E(u)}_K \\
& \nearrow & \\
& M(\mu)[1/p] = \prod_j (1_{\text{GL}_n/P_j} \times S(\mu_j)).
\end{array}$$

Since $R[1/p]$ is reduced and Jacobson, it suffices to show that we have a factorization at the level of $\mathbb{Q}_p$-points.

We are reduced then to showing that for any $x : R \to F'$, $\mathfrak{M}_x$ has $p$-adic Hodge type $\leq \mu$ if and only if for any choice of eigenbasis $(\beta^{(j)})$ the corresponding $F'$-point $\Psi(x)$ of $\text{Fl}^{[a,b],E(u)}_K$ lies in $\prod_j (1_{\text{GL}_n/P_j} \times S(\mu_j))$. We can enlarge the field if necessary so that $F'$ contains a splitting field for $K/\mathbb{Q}_p$. This ensures that the generic fiber of $\text{Fl}^{[a,b],E(u)}_K$ becomes a product over the embeddings of $\psi : K \to F'$.
We first show that the projection to \( \text{GL}_n/P_j \) is the identity point. Consider the Frobenius map
\[
\phi^{(j-1)}_{x,s_j(n)} : \mathfrak{M}^{(j-1)}_{x,s_j(n)}[1/E_j(u)] \to \mathfrak{M}^{(j)}_{x,s_j(n)}[1/E_j(u)]
\]
which is a map of modules over \((\Lambda'[u])[1/p, 1/E_j(u)]\). Since \( p \) is inverted, reduction mod \( u \) induces an isomorphism
\[
\varphi \mathfrak{M}^{(j-1)}_{x,s_j(n)} \mod u \simeq \mathfrak{M}^{(j)}_{x,s_j(n)} \mod u.
\]
For a choice of eigenbasis \( \beta^{(j)} = (f_1^{(j)}, \ldots) \), we would like to show that the image of the filtration on \( \varphi \mathfrak{M}^{(j-1)}_{x,s_j(n)} \mod u \) is the canonical filtration on \( \mathfrak{M}^{(j)}_{x,s_j(n)} \mod u \) induced by the trivialization (i.e., induced by the eigenbasis). Concretely, this comes down to the fact that
\[
\phi^{(j-1)}_{x,s_j(n)}(a^{(j)} f_1^{(j)} \otimes f_2^{(j)} \otimes f_3^{(j)} \otimes \cdots) \in \text{Fil}'(\mathfrak{M}^{(j)}_{x,s_j(n)} \mod u)
\]
which is equivalent to the commutativity of the (3.1). This shows that \( \Psi(x) \in \prod_j \text{GL}_n/P_j \times \text{Gr}_{\text{Res}(K \otimes K_0 \otimes F')/\text{GL}_n}(F') \).

By twisting by some power of \( E(u) \), we can now reduce to the case of \( [a, b] = [0, h] \). Fix an embedding \( \sigma_j : K_0 \to F' \). We have that \( S(\mu_j) = \prod_{\psi : K \to F'} S(\mu_j, \psi) \), where the product is over embeddings \( \psi \) which extend \( \sigma_j \). Fix such an embedding \( \psi \) and let \( \pi_\psi := \psi(\pi_K) \). We write \( F'[u - \pi_\psi] \) for the completion of \((\Lambda'[u])[1/p]\) at \( u - \pi_\psi \).

Let \( \Psi(x)_\psi \) denote the projection onto the \( \text{Gr}_{\text{GL}_n} \) factor corresponding to the embedding \( \psi \). Then \( \Psi(x)_\psi \in S(\mu_j, \psi) \) if and only if the \((u - \pi_\psi)\)-lattice in \((F'[u - \pi_\psi])^n\) given by the \((u - \pi_\psi)\)-adic completion of
\[
\gamma^{(j)} \circ \phi^{(j-1)}_{x,s_j(n)}(\mathfrak{M}^{(j-1)}_{x,s_j(n)}) \subset ((\Lambda'[u])[1/p])^n
\]
has relative position less than or equal to \( \mu \) (relative to the standard lattice \((F'[u - \pi_\psi])^n\)). For a cocharacter \( \lambda \) of \( \text{GL}_n \), the Bialynicki-Birula decomposition gives a retraction from the open Schubert cell \( S^\circ(\lambda) \subset \text{Gr}_{\text{GL}_n} \) to the flag variety \( \text{GL}_n/P_\lambda \), where
\[
P_\lambda := \left\{ g \in \text{GL}_n | \lim_{(u - \pi_\psi) \to 0} (u - \pi_\psi)^{-\lambda} g(u - \pi_\psi)^\lambda \text{ exists} \right\}
\]
(see the last chapter of [BB73]). Explicitly, this retraction sends a lattice in \((F'[u - \pi_\psi])^n\) to the filtration on \((F')^n\) induced by taking the intersection of the lattice with the preimages of \((u - \pi_\psi)^j (F'[u - \pi_\psi])^n\). Given that
\[
S(\mu_j, \psi) F' = \sqcup_{\lambda \leq \mu_j, \psi} S^\circ(\lambda) F'
\]
for large enough \( F' \), the statement that \( \Psi(x)_\psi \in S(\mu_j, \psi) \) is equivalent to the filtration on \( \mathfrak{M}^{(j-1)}_{x,s_j(n)}/(u - \pi_\psi)^\lambda \mathfrak{M}^{(j-1)}_{x,s_j(n)} \) induced by the preimages of \((u - \pi_\psi)^j (F'[u - \pi_\psi])^n\) being of type \( \leq \mu_j, \psi\).
Let \( D_{x,s_j(n)} := \mathfrak{M}^{(j-1)}_{x,s_j(n)} / E_j(u)^{\varphi} \mathfrak{M}^{(j-1)}_{x,s_j(n)} \). We see then that \( \Psi(x) \in \prod_j \text{GL}_n / \rho_j \times S(\mu_j) \) if and only if \( \text{gr}^s(D_{x,s_j(n)}) \cong V_{\mu_j} \otimes F' \). By Corollary 5.12 this is equivalent to \( \mathfrak{M}_x \) having \( p \)-adic Hodge type \( \leq \mu \).

\( \square \)

Remark 5.14. In the moduli of Kisin modules, one is forced to work with the condition \( \leq \mu \). In \cite[Corollary 2.6.2]{Kis08}, Kisin shows that the \( p \)-adic Hodge type is locally constant in the generic fiber of the semistable deformation ring. However, the proof uses the comparison with \( D_{dR} \) (see also \cite[(A.4)]{Kis09b}). For families of finite height Kisin modules over a complete local ring \( R \) as above, the \( p \)-adic Hodge type need not be locally constant on \( \text{Spec } R[1/p] \).

5.3. Connections to Galois representations. In this subsection, we record two connections to Galois representations in the spirit of \cite{Kis08} and \cite{Kis09a}. This essentially comes down to adding descent datum to the constructions of Kisin in loc. cit.

Let \( R \) be a complete local \( \mathbb{Z}_p \)-algebra. Fix a compatible system of \( p \)-power roots \( \{ \pi_1^{1/p}, \pi_2^{1/p}, \ldots \} \) and let \( L_\infty \) denote the completion of \( L(\pi_1^{1/p}, \pi_2^{1/p}, \ldots) \). We define \( K_\infty \) to be the completion of the field obtained by adjoining the compatible system of \( p \)-power roots of \( \pi_K \) given by \( \{ \pi_1^{1/p}, \pi_2^{1/p}, \ldots \} \).

Note that \( L_\infty \) is Galois over \( K_\infty \) with \( \text{Gal}(L_\infty / K_\infty) \cong \text{Gal}(L/K) = \Delta \).

Definition 5.15. Let \( \mathcal{O}_{E,L} \) be the \( p \)-adic completion of \( (W[v])[1/v] \) equipped with Frobenius and an action of \( \Delta \) in the natural way. An étale \( \varphi \)-module over \( R \) with descent datum is a finite free \( R \otimes \mathbb{Z}_p \mathcal{O}_{E,L} \) module \( \mathcal{M} \) equipped with an Frobenius isomorphism \( \varphi^* (\mathcal{M}) \cong \mathcal{M} \) and a semilinear action \( \{ \tilde{g} \} \) of \( \Delta \) such that \( \phi_{\mathcal{M}} \) and \( \tilde{g} \) commute for all \( g \in \Delta \).

Proposition 5.16. There is a functor \( \mathcal{M}_{\text{dd}} \) from the category of continuous representations of \( G_{K_\infty} := \text{Gal}(\overline{K}/K_\infty) \) on finite free \( R \)-modules to the category of étale \( \varphi \)-modules over \( R \) with descent datum. This functor is an equivalence of categories with quasi-inverse \( T_{\text{dd}} \).

Proof. The main content is the equivalence given by the theory of norm fields over \( L_\infty \) due to Fontaine-Wintenberger (with coefficients \cite[Lemma 1.2.7]{Kis09a}). The addition of descent datum is straightforward (see \cite[§2.1.3]{CDM1} for details). \( \square \)

Let \( F' \) be a finite extension of \( F \) with ring of integers \( \Lambda' \). Let \( V_{F'} \) be a potentially semistable representation of \( G_K \) with Galois type \( \tau \) and \( p \)-adic Hodge type \( \mu \).

Proposition 5.17. Let \( T_{\Lambda'} \) denote a \( G_K \)-stable lattice in \( V_{F'} \). Then there exists \( \mathfrak{M}_{\Lambda'} \in \mathcal{Y}_{\mu,\tau}(\Lambda') \) such that

\[ \mathfrak{M}_{\Lambda'} \otimes_{W[1/v]} \mathcal{O}_{E,L} \cong \mathcal{M}_{\text{dd}}(T_{\Lambda'})_{G_{K_\infty}} \]
Proof. Without descent datum, this is due to Kisin (see Corollary 1.3.15 and Proposition 2.1.5 in [Kis06]). We briefly explain how to extend the result to include decent datum. Let $M = M_{dd}(T_{N}|_{G_{K∞}})$.

Applying Kisin’s results to $T_{Λ}|_{G_L}$, we get a finite height lattice $M_{Λ} \subset M_{Λ}'$. The fact that $M_{Λ}'$ inherits a semilinear action of $Δ$ from $M_{Λ}$ follows from the uniqueness of $M_{Λ}'$ ([Kis06, Lemma 2.1.6]). The fact that $M_{Λ}'$ has type $τ$ follows from the $\text{Gal}(L/K)$-equivariance of the identification $(M_{Λ'}/vM_{Λ'})[1/p] \cong D_{st}(T_{Λ'}[1/p]|_{G_L})$

from [Kis08, §2.5(1)].

Finally, $M_{Λ'}[1/p]$ has $p$-adic Hodge type $μ$ via the identification

$\varphi^*(M_{Λ'}[1/p])/P(v)\varphi^*(M_{Λ'}[1/p]) \cong D_{dR}^*(V_F')$

from the proof of [Kis08, Corollary 2.6.2]. □

Let $ρ : \text{Gal}(\overline{K}/K) \rightarrow \text{GL}_n(Λ')$ be a lattice in a potentially crystalline representation of $\text{Gal}(\overline{K}/K)$ with Galois type $τ$ and $p$-adic Hodge type $μ$. Let $\overline{ρ} : \text{Gal}(\overline{K}/K) \rightarrow \text{GL}_n(Γ')$ denote the reduction of $ρ$ modulo the maximal ideal of $Λ'$.

Corollary 5.18. Let $\overline{ρ}$ be as above. Then there exists $\overline{M} \in Y^μ,τ(Γ')$ such that $T_{dd}(\overline{M}) \cong \overline{ρ}|_{\text{Gal}(\overline{K}/K∞)}$.

We end by considering resolutions of potentially crystalline deformations rings as in [Kis08]. Let $\overline{ρ} : \text{Gal}(\overline{K}/K) \rightarrow \text{GL}_n(Γ)$ be a continuous representations. Let $μ ∈ (\mathbb{Z}^n)^{\text{Hom}(K,\overline{K})}$ be a cocharacter. Let $R_{\overline{ρ}}^{μ,τ,\text{cris}}$ be the (framed) potentially crystalline deformation with $p$-adic Hodge type $μ$ and Galois type $τ$, as constructed by Kisin.

Let $m_{R}$ denote the maximal ideal of $R_{\overline{ρ}}^{μ,τ,\text{cris}}$ and let $ρ_{d}^{\text{univ}} : \text{Gal}(\overline{K}/K) \rightarrow \text{GL}_n(R_{\overline{ρ}}^{μ,τ,\text{cris}}/m_{R}^d)$ be the reduction of the universal deformation. Set $M_{d} := M_{dd}(ρ_{d}^{\text{univ}})$. Define $Y_{\overline{ρ},d}^{μ,τ}$ to be the functor on $R_{\overline{ρ}}^{μ,τ,\text{cris}}/m_{R}^d$-algebras $B$ given by

$Y_{\overline{ρ},d}^{μ,τ}(B) := \{(M_{B}, α) | M_{B} \in Y^μ,τ(B), α : M_{B}[1/u] \cong M_{d} \otimes_{E_{ω,μ,τ,\text{cris}}/m_{R}^d} (W \otimes_{\mathbb{Z}_p} B)((v))\}$

The functors $Y_{\overline{ρ},d}^{μ,τ}$ are relatively represented by projective schemes over $R_{\overline{ρ}}^{μ,τ,\text{cris}}/m_{R}^d$ as subschemes of the affine Grassmannian for $M_{d}$ using the same argument as in [Kis08]. By formal GAGA, there is a projective morphism

$Θ : Y_{\overline{ρ},d}^{μ,τ} \rightarrow \text{Spec } R_{\overline{ρ}}^{μ,τ,\text{cris}}$

reducing to $Y_{\overline{ρ},d}^{μ,τ}$ modulo $m_{R}^d$. 


Theorem 5.19. The projective morphism
\[ \Theta : Y^\mu,\tau \rightarrow \text{Spec } R^\mu,\tau,\text{cris} \]
is an isomorphism on generic fibers.

**Proof.** The proof that $\Theta[1/p]$ is a closed immersion is the same argument as in [Kis08, Proposition 1.6.4] using uniqueness of finite height lattices when $p$ is inverted. The fact that $\Theta[1/p]$ is an isomorphism is then a consequence of Corollary 5.17. □

Corollary 5.20. If $\mu \in \{(0,1)^n\}^{\text{Hom}(K,\mathbb{Q}_p)}$, i.e., $R^\mu,\tau,\text{cris}$ is a potentially Barsotti-Tate deformation ring, then the forgetful map $Y^\mu,\tau \rightarrow Y^\mu,\tau$ is formally smooth.

**Proof.** For $R$ a complete local Noetherian $\Lambda$-algebra, the functor $T_{dd}$ on $Y^{[0,1],\tau}(R)$ canonically extends to a functor $\tilde{T}_{dd}$ valued in representations of $G_K$ (not just $G_{K_{\infty}}$) such that when $R$ is finite flat over $\Lambda$ the representation is potentially crystalline. To construct $\tilde{T}_{dd}$, one first associates to $\mathfrak{M}_R \in X^{[0,1],\tau}(R)$ a strongly divisible module with tame descent as defined in [EGST5 Definition 7.3.1]. The key point is that the monodromy operator is unique and so it commutes with the descent datum. There is a functor $T_{st,L}$ from strongly divisible modules with tame descent to representations of $G_K$ [Sav05, §4].

The difference between $Y^\mu,\tau \rightarrow Y^\mu,\tau$ is then the addition of a framing on the Galois representations which is formally smooth. The details are the same as in [Kis08, Proposition 2.4.6] or [Lev2, Theorem 4.4.1]. □

Corollary 5.21. If $\mu \in \{(0,1)^n\}^{\text{Hom}(K,\mathbb{Q}_p)}$, then $Y^\mu,\tau$ is normal and $Y^\mu,\tau \otimes \mathbb{F}$ is reduced.

**Proof.** This follows directly from Theorem 2.14 and Theorem 5.3. □

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