Simple approach to the chaos-order contributions and symmetry breaking in nuclear spectra

A.G. Magner, A.I. Levon and S.V. Radionov
Institute for Nuclear Research, 03680 Kyiv, Ukraine
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The simple one-parameter nearest neighbor-spacing distribution (NNSD) is suggested for statistical analysis of nuclear spectra. This distribution is derived within the Wigner-Dyson approach in the linear approximation for the level repulsion density of quantum states. The obtained NNSD gives the individual information on the Wigner and Poisson contributions in agreement with that of the statistical experimental distributions of collective states in deformed nuclei. Using this NNSD, one finds that the symmetry breaking due to the fixing of projections of the angular momentum of collective states enhances a chaos as a shift of the NNSD from the Poisson to Wigner distribution behavior.

I. INTRODUCTION

Statistical analysis of the quantum energy spectra for complex many-body systems such as atomic nuclei is in fruitful progress [1, 2]. Different statistical methods have been proposed to obtain information on the chaoticity versus regularity in nuclear spectra [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The short-range fluctuation properties in experimental spectra are analyzed usually in terms of the nearest-neighbor spacing distributions (NNSDs). For a quantitative measure of the degree of chaoticity of the many-body dynamics, the statistical probability density \( p(s) \) as a function of spacings \( s \) between the nearest neighboring levels can be derived within the general Wigner-Dyson (WD) approach based on the level repulsion density \( g(s) \) (the units will be specified below) [8, 9, 10, 13, 14].

\[
p(s) = g(s) \exp \left( - \int_0^s g(s') \, ds' \right).
\]  

The order is approximately associated with the Poisson dependence of \( p(s) \) on the spacing \( s \) variable for \( g(s) \), that is independent of \( s \). The chaoticity can be related to the Wigner distribution, as clearly follows for \( g(s) \propto s \) [15]. An intermediate nature of spectra between these two limit statistics should be expected, see for instance Refs. [6, 7, 12, 21, 22].

The estimated values of parameters of the NNSD shed light on the intermediate statistical situation with considered spectra. The Brody NNSD [23] is based on the expression for the level repulsion density that interpolates between the Poisson and the Wigner distribution. Berry and Robnik [24] derived the NNSD starting from the microscopic expression for the density of levels of a system through its classical Hamiltonian. Other one-parameteric distribution NNSDs were suggested in Refs. [7, 25, 26].

For further studies of the order-chaos properties of nuclear systems, it might be worthwhile to apply a simple analytical approximation to the level repulsion density \( g(s) \) in Eq. (1). For analysis of the statistical properties in terms of the mixed Poisson and Wigner distributions, the linear WD (LWD) approximation to the level repulsion density \( g(s) \) was suggested in Refs. [28, 29]. It is the two-parameter LWD (LWD2, see Appendix); in contrast, e.g., to the one-parameter Brody approach [10]. The LWD (LWD2) approximation, as based on a smooth analytical (linear) function \( g(s) \), can be justified within the WD theory, see also Ref. [14]. Moreover, it gives more proper information on the separate Poisson order-like and Wigner chaos-like contributions. The LWD2 NNSD \( p_2(s) \) was applied recently [29] for a statistical study of experimental data on the collective states in deformed nuclei with a given angular momentum \( I \) and parity \( \pi \), and compared with the Brody distribution [10]. These results are in accordance with the works of Shriner et al. [12, 14]. They are alternative to that for the nuclear states of a single-particle nature; see, e.g., Ref. [40]. To derive the NNSD with one parameter from Eq. (1) and, at the same time, keep the same quantitative individual information of their order and chaos contributions is still an open question. In addition, the attractive subject of the research is to learn statistical properties of the new symmetry breaking phenomenon [7, 11, 12, 13], in particular, a violation of the isospin symmetry and pair correlation breaking. Another attractive subject is related to a symmetry breaking due to a fixed projection of the angular momentum \( K \) in a nuclear collective motion [7, 12, 41].

In the present work, we obtain the probability distribution \( p_1(s) \) with a single parameter on the basis of the linear approximation (LWD1) to a level repulsion density \( g(s) \) in Eq. (1) and compare with the previously presented (Brody [23] and LWD2 [29]) approaches, and with the new experimental data for a symmetry breaking observation. The statistical properties of the nuclear collective states obtained by the NNSD \( p_1(s) \) are tested below using the experimental results from Refs. [30, 31, 32] for NNSDs fitted by the LWD1. The LWD1 NNSD is applied also for studying the new symmetry breaking phenomenon [7, 31, 53].
II. WIGNER-DYSON LWD APPROACH

Key quantity in Eq. (1) is the level repulsion density $g(s)$. It is convenient to consider $s$ in units of the average $D$ of distances between levels, $s = S/D$, where $S$ is the distance between neighboring levels and $D$ is locally a mean distance between neighboring levels in usual energy units.

The experimental data are always known within the finite spacing interval, and both normalization conditions (3) and (4) can be dependent on the upper integration limit $s_{\text{max}}$. Assuming, however, a good convergence over spacing variable $s$, one can approximately simplify these conditions for the probability distribution $p(s)$ as function of the dimensionless variable $s$ by expanding $s_{\text{max}}$ to the infinity,
\begin{align}
\int_0^\infty p(s) \, ds &= 1 \label{eq:2} , \\
\int_0^\infty s \, p(s) \, ds &= 1 \label{eq:3} .
\end{align}

For the Poisson and Wigner limits, from Eq. (1) one has the corresponding well known distributions, which obey Eqs. (2) and (3).
\begin{equation}
 p_P(s) = \exp(-s), \quad p_W(s) = \left(\frac{\pi s}{2}\right) \exp\left(-\frac{\pi s^2}{4}\right) . \label{eq:4}
\end{equation}

Keeping a link with the analytical properties of the level repulsion density $g(s)$, it is convenient to define the probability $p(s)$ [Eq. (1)] with a general smooth density $g(s)$, that is a polynomial of not too a large power. As shown in Refs. [28, 29], this density smoothness is essentially used in the derivation of Eq. (1). For the simplest statistical analysis in terms of the Poisson- and Wigner-like distribution contributions, one can use the expansion of $g(s)$ in series of a few powers of $s$,
\begin{equation}
 g(s) \approx a + bs \label{eq:5} ,
\end{equation}
where $a$ and $b$ are fitting parameters. Substituting this expansion into the general Wigner-Dyson formula (1), one obtains explicitly the analytically simple distribution
\begin{equation}
 p_{\text{LWD}}(s) = (a + bs) \exp\left(-as - \frac{b}{2}s^2\right) . \label{eq:6}
\end{equation}

Taking the limits $a \to 1$, $b \to 0$ and $a \to 0$, $b \to \pi/2$ in Eq. (6), one simply arrives relatively at the standard Poisson $p_P(s)$ and Wigner $p_W(s)$ distributions (1). In this way, a linear approximation (5) unifies analytically these two limit cases through a smooth level repulsion density $g(s)$. Its parameters $a$ and $b$ in Eq. (5) are associated with the Poisson and Wigner distribution contributions.

The two-parameter distribution LWD2 (11) can be simplified approximately by reducing it to one parameter. As Eq. (11) obeys identically the normalization condition (2), one satisfies only the approximate normalization condition (3). Thus, one obtains a relation between the parameters $a$ and $b$ [marked by low index one in $p_{\text{LWD}}(s)$]:
\begin{equation}
 \int_0^\infty s \, p_1(s) \, ds = \sqrt{\frac{\pi}{2b}} \, e^{w^2} \operatorname{erfc}(w) = 1 \label{eq:7} ,
\end{equation}
where
\begin{equation}
 w = a/\sqrt{2b} . \label{eq:8}
\end{equation}
Here, $\operatorname{erfc}(w)$ is the standard error function,
\begin{equation}
 \operatorname{erfc}(w) = 1 - \operatorname{erf}(w) = 1 - \frac{2}{\sqrt{\pi}} \int_0^w dx \exp(-x^2) . \label{eq:9}
\end{equation}

Solving Eq. (7) with respect to $b = b(w)$ and using Eq. (8) for $a = a(w)$, one finds
\begin{equation}
 p_1(s) = [a(w) + b(w)s] \exp\left[-a(w)s - \frac{b(w)}{2}s^2\right] , \label{eq:10}
\end{equation}
where
\begin{equation}
 a = \sqrt{\pi} \, w \, e^{w^2} \operatorname{erfc}(w) , \quad b = \frac{\pi}{2} \, e^{2w^2} \operatorname{erfc}^2(w) . \label{eq:11}
\end{equation}

The probability distribution which obeys both normalization conditions (2) and (3) is given by Eq. (10), where $a(w)$ and $b(w)$ are functions of only one parameter $w$ through Eq. (11). Eq. (10) approaches the Wigner limit for $w \to 0$ and the Poisson limit for $w \to \infty$, respectively,
\begin{equation}
 a(w) = \sqrt{\pi w} + O(w^2) , \quad b(w) = \frac{\pi}{2} - 2\sqrt{\pi w} + O(w^2) \label{eq:12}
\end{equation}
and
\begin{equation}
 a(w) = 1 - \frac{1}{2w^2} + O\left(\frac{1}{w^3}\right) , \quad b(w) = \frac{1}{2w^2} + O\left(\frac{1}{w^3}\right) . \label{eq:13}
\end{equation}
Thus, the probability density (10) is a simple analytical continuation from the Poisson $p_P(s)$ to Wigner $p_W(s)$ limit distributions (1) through a smooth linear level-repulsion density $g(s)$ [Eq. (5)], and both equations (2) and (3) are satisfied.

III. DISCUSSIONS OF RESULTS

Fig. 1 shows the results of testing the LWD1 [Eq. (10)] by fitting the NNSDs with a good statistics: Numerical quantum spectra in the circular (a) and heart (b) billiards, and for the nuclear data ensemble [NDE, (c)]. The NDE includes 1726 neutron and proton resonance energies [46]. The LWD1 (10) is in good agreement with both numerical (a,b) and experimental NDE (c) NNSDs, as well with the corresponding Poisson (a) and Wigner (b,c) limits, see Eqs. (4), (12), (13) and Table I. The sampling intervals for building the NNSDs (after the unfolding procedure [28]) in Fig. 1 are given by $\gamma_s = 0.1$. In all other figures, one finds the reliable parameter $\gamma_s = 0.2$. They are taken from the condition of the stable smoothed NNSD values without sharp jumps between the neighbor energies.

Experimental NNSDs for the collective states with different angular momenta $I^\pi = 0^+, 2^+, 4^+$ are excited in
FIG. 1. NNSDs \( p(s) \) as functions of a dimensionless spacing variable \( s \) for (a) Poisson- and (b) Wigner-like numerical calculations and (c) experimental Wigner-like results (see text) by staircase lines. LWD1 NNSDs (10) are shown by solid lines. Dots present the Poisson (a) and Wigner (b,c) curves (4).

Table I. Parameters \( a_i, b_i \) and \( w \) of one- and two-parameter LWDi approximations \( (i = 1,2) \) [Eqs. (10) and (A1) for \( i = 1 \) and 2, respectively] for the exemplary cases and collective excited states in several nuclei. Results: Fig. 1(a) and (b) are compared with the symmetry breaking by setting \( K^+ = 0 \) (b), 2 (c) and 4 (d). Averaged \( s \) values \( \langle s \rangle \) [Eq. (A5)] for the LWD2 are shown in the 9th column. The standard accuracies found by \( \chi^2 \) of least-squares fittings (in percent) are shown in the 6th and 10th columns.

| Figure | system | \( a_1 \) | \( b_1 \) | \( w \) | \( \chi^2 \) | \( a_2 \) | \( b_2 \) | \( \langle s \rangle \) | \( \chi^2_a \) |
|--------|--------|--------|--------|--------|----------|--------|--------|----------|----------|
| 1a     | circle | 0.98   | 0.02   | 4.79   | 0.99     | 0.98   | 0.03   | 0.99     | 0.9      |
| b      | heart  | 0.08   | 1.41   | 0.05   | 3.6      | 0.02   | 0.98   | 0.99     | 2.1      |
| c      | NDE    | 0.07   | 1.44   | 0.04   | 0.99     | 0.00   | 1.03   | 0.99     | 5.6      |
| 2a     | 0\(^+\) | 0.32   | 0.98   | 0.23   | 11.4     | 0.26   | 1.02   | 0.85     | 9.2      |
| b      | 2\(^+\) | 0.58   | 0.54   | 0.56   | 11.8     | 0.53   | 0.74   | 0.82     | 10.2     |
| c      | 4\(^+\) | 0.68   | 0.40   | 0.76   | 9.1      | 0.66   | 0.42   | 0.89     | 8.5      |
| 3a     | 4\(^+\) | 0.50   | 0.67   | 0.43   | 11.8     | 0.50   | 0.56   | 0.91     | 11.5     |
| b      | \( K = 0 \) | 0.32   | 0.97   | 0.23   | 11.0     | 0.31   | 0.84   | 0.89     | 9.9      |
| c      | \( K = 2 \) | 0.07   | 1.44   | 0.04   | 11.3     | 0.02   | 0.86   | 1.06     | 10.6     |
| d      | \( K = 4 \) | 0.14   | 1.30   | 0.08   | 12.3     | 0.07   | 0.77   | 1.08     | 11.1     |

Several actinide nuclei. They are fitting by the LWD1 (10) and LWD2 approximations (A1) in Fig. 2, see also the parameters of these fittings given in Table I. All spectra in the same energy interval 0–3 MeV demonstrate an intermediate structure between an order and a chaos with varying dominance of the Wigner to the Poisson contribution for increasing the angular momentum from 0\(^+\) to 4\(^+\). With increasing angular momentum, one can see a shift of the NNSD to the Poisson limit. As shown in Ref. 29, e.g., spectra 0\(^+\) in the energy interval 0–3 MeV are intermediate between an order and a little more pronounced chaos structure, while ordered nature is dominant for the experimental spectra in the extended energy interval about 0–4 MeV.

Taking actinides as example, see Ref. 29, one shows a good agreement between experimental 31,32 and theoretical 47 results that confirms the collective nature of the desired states and, finally, completeness of the level sequences. At the same time, the theoretical distribution for the extended energy interval 0–4 MeV is shifted to the Poisson law as compared to the experimental and theoretical distributions in the interval 0–3 MeV.

Fig. 3 shows the typical example of symmetry breaking phenomena 33. Experimental NNSDs are obtained from the analysis of sequences for the 4\(^+\) states performed for three actinides 228,230\(^{\text{Th}}\) and 232\(^{\text{U}}\). The rotational bands were identified in this analysis. The identification of states that is associated with rotational bands was performed on the following conditions. i) The angular distribution for the state with a given spin as a band member candidate is assigned by the DWBA calculation. (This state is necessarily included into the band.) ii) The transfer cross section in the (p,t) reaction to states in the potential band has to be decreased with increasing spin. iii) Energies of the states in this band can be evaluated approximately by the expression for a rotational band \( E = E_0 + A I(I + 1) \) with a small and smooth variation of the inertial parameter \( A \). In such a way new sequences with the angular momentum 4\(^+\) were formed, separately for the \( K = 0, 2 \) and 4 states. The latter can be considered as pure sequences. As seen from Fig. 3, if we fix the projection of the angular momentum \( K \), the NNSD is changed toward the chaos (Wigner distribution) for each of the cases \( K = 0 \) [Fig. 3(b)] versus 2 (c) and 4 (d) (see also Table I). This effect observed in both LWD1 and LWD2 approximations is more en-
FIG. 2. The same as in Fig. 1 but for different experimental states in the actinide nuclei: (a-c): for $0^+$, $2^+$, and $4^+$, respectively. The fits by the LWD1 [10], LWD2 [11] and Brody [23] approach are respectively shown by solid, dashed and dotted lines (staircase lines in Fig. 2 are taken from Ref. 29).

FIG. 3. NNSDs for full spectrum (a) and symmetry breaking by fixed $K = 0$ (b), 2 (c) and 4 (d) projections of the angular momentum $4^+$ for the actinides which are included in Fig. 2. Solid, dashed and dotted lines are fits by the LWD1 [10], LWD2 [11] and Brody [23] NNSDs, respectively.

hanced for the $K = 2$ (c) [or 4 (d)] case than for $K = 0$ (b). Thus, we found a similar effect of the symmetry breaking like in the single-particle spectra [28], that was explained by decreasing the number of single-valued integrals of motion. This analysis confirms also a proposed explanation of NNSD shifts to the Poisson limit with increasing the angular momentum $0^+$, $2^+$ and $4^+$ (Fig. 2) by mixing the sequences with different symmetries (different $K$). An extended version of the proper discussions of these phenomena will be presented in the forthcoming work.

As seen from Figs. 2 and 3 and Table I results of the fitting of experimental NNSDs [30–39] by the LWD1 [10], LWD2 [11] and Brody approach [23] are basically close, though some differences are visible. Their main features, - the position of maxima and the Poisson $a$ and Wigner $b$ distribution contributions, - are approximated within good accuracy of calculations. In the LWD1 approximation we related these parameters by satisfying the normalization condition (3) which is idealized as compared to Eq. (A5) with respect to the upper integration limit in the LWD2 approach (Sec. II and Appendix). As a result, the LWD1 has one parameter for fitting as the Brody NNSD. In these LWD1 derivations we assumed a fast convergence of the normalization integral in Eq. (A5) as function of a maximal spacing value $s_{\text{max}}$. On the other hand, in the LWD2 case [Eq. (A1)] we keep $a$ and $b$ independent in the fitting procedure and check, then, the accuracy of Eq. (A5) for $\langle s \rangle$ (Table I). The upper integration limit $s_{\text{max}}$ must be larger than all of energy spacings in a given experimental spectrum, and this has to be checked too. The LWD2 approximation (A1) and Brody formula (besides of small values of $s$) look visually better fitted with the improved accuracy (see Figs. 2 and 3 and Table I), especially near maxima of the experimental data. The LWD1 is better fitted on the right of the distribution maximum in a wider spacing interval. This provides explicitly the normalization condition (3). A simpler one-parameter fitting has obvious analytical advantages. In particular, the LWD1 is preferable for calculations in Fig. 2(b) where the LWD2 average $\langle s \rangle$ (A5) differs notably from one (Table I).
Thus, sometimes, the LWD1 and LWD2 NNSDs can be helpful as those giving a complementary information on statistical properties of quantum spectra.

IV. CONCLUSIONS

We derived the simple one-parameter NNSD approximation to the Wigner-Dyson probability distribution. Several exemplary problems were demonstrated: standard circular (Poisson) and cordial (Wigner) billiards, and famous experimental neutron-resonance states in many nuclei (Fig. 1). Using this approximation we provide statistical analysis of the nuclear collective excitations with several spins (Fig. 2): 0+, 2+, and 4+ in a number of actinides to show the good agreement with the one-parameter LWD1, as well as with the two-parameter LWD2 (Table I). For the linear approximation to level repulsion densities, one obtains a clear information on the quantitative measure of the Poisson order and Wigner chaos contributions in the experimental spectra, separately, in contrast to the heuristic Brody approach. However, one finds in our calculations that the Brody formula agrees largely well with the LWD probability-distribution results (again, apart from small values of s). The precision of fitting for the experimental data by the two-parameter LWD2 is improved but the full analytical one-parameter LWD1 approximation has an obvious advantage. Simplifying analytically the normalization condition for the spacing average we do not need to check its precision.

We confirm the intermediate structure between the Poisson and Wigner statistical peculiarities of the experimental spectra for nuclear collective states by evaluating their separate contributions (Fig. 2). Also, one finds that the Wigner contribution dominates in the NNSD for 0+ states and the Poisson contribution is enhanced with increasing the angular momentum. All considered nuclear spectra are collective and complete for a given angular momentum I and parity π. In accordance with Ref. [2], for collective states of a wider energy interval in deformed nuclei, the statistical distributions are closer to the Poisson distribution, and in other cases the situation is intermediate (see also Ref. [10]), in contrast to the single-particle states [40]. It has been shown that the symmetry breaking due to the fixing of the projection K of angular momentum I enhances the chaos by a shift of the NNSD toward the Wigner distribution. This property is common for the collective and single-particle states. In perspective, it will be worthwhile to study more systematically the influence of symmetry breaking phenomena on these distributions of the collective states in deformed nuclei.

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Appendix: The LWD2 NNSD approach

For the comparison, let us present also the LWD2 NNSD approximation [22],

\[ p_2(s) = \frac{a + bs}{n} \exp \left(-\frac{b}{2} s^2 \right), \]  

(A1)

where

\[ n \equiv \int_0^{s_{\text{max}}} ds \exp \left(-\frac{b}{2} s^2 - as \right) = a n_0 + b n_1, \]  

(A2)

\[ n_0 = \sqrt{\frac{\pi}{2b}} \exp \left(\frac{a^2}{2b^2} \right) \left[ \text{erf} \left( \frac{a + bs_{\text{max}}}{\sqrt{2b}} \right) - \text{erf} \left( \frac{a}{\sqrt{2b}} \right) \right], \]  

(A3)

with independent parameters a and b. As referred to quantum spectra given in a finite integration limit \( s_{\text{max}} \), the LWD2 distribution \( p_2(s) \) obeys the following normalization conditions:

\[ \int_0^{s_{\text{max}}} p_2(s) \, ds = 1 \]  

(A4)

and

\[ \langle s \rangle \equiv \int_0^{s_{\text{max}}} s \, p_2(s) \, ds = 1. \]  

(A5)

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