Parameter analysis in continuous data assimilation for three-dimensional Brinkman-Forchheimer-extended Darcy model

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Abstract

In this paper, we study analytically the long-time behavior of three-dimensional Brinkman-Forchheimer-extended Darcy model, in the context that the parameters related to the damping nonlinear term are unknown. This work is inspired by the approach firstly introduced for two-dimensional Navier-Stokes equations by Carlson, Hudson and Larios. We show estimates in \(L^2\) and \(H^1\) for large-time error between the true solution and the assimilated solution, which is constructed with the unknown damping parameters and observational measurements obtained continuously in time from a continuous data assimilation technique proposed by Azouani, Olson and Titi.

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1 Introduction

In an attempt to determinate a mathematical model as accurate as possible which represents the dynamics of any physical phenomena, parameters related to natural features of the involved elements of the system frequently arise, for instance, the external force and the kinematic viscosity of an incompressible fluid on analysis of its velocity through Navier-Stokes equations, or thermal viscosity if the temperature of the fluid is an object of interest. Thus, the development of data-driven techniques for the purpose of establishing accurate values of the parameters, namely parameter learning, becomes a fundamental point of the analysis of the dynamical system, in order to improve the model.

Recently, parameter estimation algorithms were applied to two dimensional Navier-Stokes equations (see [9]) and three dimensional Lorenz system (see [10]). In a sensitivity-type analysis, results in [9] and [10] exhibited the large-time error between the true solution of the model and the assimilated solution due to the deviation between the approximate and physical parameters.
Based on ideas of [9] and [10], we consider the three-dimensional Brinkman-Forchheimer-
extended Darcy model, also named as three-dimensional Navier-Stokes with damping equations
\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p + a|u|^{2\alpha}u = f, \\
\nabla \cdot u = 0,
\]  
(1.1)

where \(u = (u_1(x,t), u_2(x,t), u_3(x,t))\) is the spatial velocity field, \(p = p(x,t)\) is a scalar pressure
field, \(f = f(x,t)\) is a given external force and \(\nu > 0\) is the kinematic viscosity. The parameters
of interest in this work are \(\alpha > 1\) and \(a > 0\), i.e., the coefficients related to the damping nonlinear
term \(a|u|^{2\alpha}u\), which we suppose being both unknown. Notice that in the limit case \(a = 0\),
we obtain the classical Navier–Stokes system. Equation (1.1), as well as most fluid models in porous
media recently studied, arised from Darcy’s Law, that describes a proportional relation between
the instantaneous rate of discharge through a porous medium, the viscosity of the fluid, and the
pressure drop over a given distance.

Darcy’s law is generally valid for flows with Reynold’s number \(Re \leq 1\), i.e., laminar flows. For
the treatment of Darcy’s Law deviation cases, a more suitable model is obtained by coupling a
quadratic term in seepage velocity to account for the increased pressure drop, namely the Darcy-
Forchheimer equation (see [3]):
\[
\mu k v + \beta \rho |v|^2 v = -\nabla p,
\]

where \(\mu\) is the kinematic viscosity, \(k\) is the permeability of the porous medium, \(\beta\) is the inertial
factor and \(\rho\) is the density of the fluid flowing through the medium. For the system (1.1), \(a|u|^{2\alpha}u\)

is a drag term related to the pore dimension, shape and porosity (see [19], [20], [25], [29] and [36]).

Although the most widely values used for \(\alpha\) are \(0, \frac{1}{2} \) and \(1\), extrapolations to others real numbers
have appeared in literature and the suitable range for modeling purpose is still source of uncertainty
(see [3], [26], [27], [28] and [34]). Due to the mathematical restriction related with existence and
uniqueness of solutions, in this paper we will consider the case \(\alpha > 1\).

The aim of this work is to present a parameter analysis with respect to errors on the damping

- term parameters \(\alpha\) and \(a\), based on the technique of a continuous data assimilation algo-

- rithm (named here as AOT algorithm) proposed in [4]. This algorithm, initially applied to
two-dimensional Navier-Stokes equations, was designed to work for general linear and nonlinear
dissipative dynamical systems, based on a feedback control that works inserting the large

- scale observations into the physical model through of a linear interpolant operator constructed

- from these observational measurements. This feedback control is basically used for relaxing
the solution of the constructed system towards the real-state solution of the original model.
In past recently years, it has been analyzed for several important 1D, 2D and 3D physical models
(see [5], [1], [2], [6], [7], [11], [12], [13], [16], [17], [18]).

We prove that under certain hypothesis, one can estimate \(u\) asymptotically in \(H^1\)-norm for
observable interpolants accurate enough in \(L^2\)-norm (see Theorems 3.4 and 5.0). For this purpose,

we consider the assimilated system given by
\[
\frac{\partial w}{\partial t} + (w \cdot \nabla)w - \nu \Delta w + \nabla p + b|w|^{2\beta}w = f + \eta(I_h(u) - I_h(w)), \\
\nabla \cdot w = 0,
\]

where \(\eta > 0\) is the relaxation (nudging) parameter, \(u\) is the solution of (1.1), \(b > 0\) is a guess for
\(a\) and \(\beta > 1\) is a guess for \(\alpha\), \(I_h\) is a linear interpolation operator satisfying certain conditions (see
[15] below) and \(h > 0\) is a parameter related with spatial resolution of the operator.

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We consider the systems (1.1) and (1.2) subjected to the periodic boundary condition on a box domain \( \Omega = [0, l]^3 \), namely,
\[
    u(x,t) = u(x + le_i, t), \quad \forall (x,t) \in \mathbb{R}^3 \times [0, T),
\]
where \( e_1, e_2 \) and \( e_3 \) are the canonical basis of \( \mathbb{R}^3 \) and \( l > 0 \) the fixed period. For the assimilated operator, we assume that
\[
    I_h : (H^j([0, l]^3))^3 \longrightarrow (L^2([0, l]^3))^3
\]
is a linear operator where there exist dimensionless constants \( c_0 \) and \( c_1 \) such that \( c_1 = 0 \) and \( c_0 > 0 \) if \( j = 1 \), and \( c_0 > 0 \) if \( j = 2 \), and the following inequality is satisfied:
\[
    \|I_h(g) - g\|_{L^2}^2 \leq c_0 h^2 \|\nabla g\|_{L^2}^2 + c_1 h^4 \|\Delta g\|_{L^2}^2, \quad \forall g \in (H^j([0, l]^3))^3.
\]
The outline of this paper is as follows: In Section 2, we start stating the mathematical setting, notations and classical inequalities used for obtaining all the results. Furthermore, weak and strong solutions for systems of interest (1.1) and (1.2) are defined. The known results of global well-posedness for these systems are also presented.

In Section 3, we enunciate the results obtained: Theorems 3.1, 3.4 and 3.6 assert that considering the observational measurements of the system (1.1) joined with the technique of AOT algorithm, the true solution of the system can be recovered with only approximated values of the parameters related to damping term, namely \( a \) and \( \alpha \), except for a remaining error of the difference between the real parameter and approximated inserted one.

In Section 4, estimates in \( L^2 \) and other appropriate spaces for the solution of the system (1.1) are performed, as well as for space and time derivatives; Such results are applied on proofs of the theorems established in Section 3. Finally, Sections 5, 6 and 7 contain the proofs of Theorems 3.1, 3.4 and 3.6 respectively.

2 Functional setting and results

2.1 Functions spaces, functionals and inequalities

For \( \Omega = [0, l]^3 \) and \( m \in \mathbb{N} \), let \( L^m(\Omega) \) be the usual \( l \)-periodic Lebesgue \( m \)-integrable space, \( H^m(\Omega) \) the usual \( l \)-periodic trigonometric Sobolev space (see [14]) and \( H \) and \( V \) the vector subspaces given by
\[
    H = \{ u \in (L^2(\Omega))^3; \nabla \cdot u = 0 \}, \quad V = \{ u \in (H^1(\Omega))^3; \nabla \cdot u = 0 \}
\]
and endowed with the product structures. For each fixed \( \alpha > 1 \), let us also define the Banach space \( Y_\alpha = V \cap (L^{2\alpha+2}(\Omega))^3 \) and, then \( Y'_\alpha = V' + (L^{2\alpha+2}(\Omega))^3 \).

Let \( A : Y_\alpha \rightarrow Y'_\alpha \) be the operator defined by
\[
    \langle A(u), v \rangle_{Y'_\alpha, Y_\alpha} = \int_\Omega \nabla u \cdot \nabla v \, dx,
\]
and, for each \( 0 \leq \gamma \leq \alpha \), \( G_\gamma : Y_\alpha \rightarrow Y'_\alpha \) the operator given by
\[
    \langle G_\gamma(u), v \rangle_{Y'_\alpha, Y_\alpha} = \int_\Omega |u|^{2\gamma} u \cdot v \, dx.
\]
We also define the bilinear operator \( B : Y_\alpha \times Y_\alpha \rightarrow Y'_\alpha \) as
\[
    \langle B(u, v), w \rangle_{Y'_\alpha, Y_\alpha} = \int_\Omega (u \cdot \nabla v) \cdot w \, dx.
\]
For \( u, v, w \in Y \), the term \( B \) has the property
\[
(B(u, v), w)_{Y_0^\ast, Y_0} = -(B(u, w), v)_{Y_0^\ast, Y_0},
\]
and hence
\[
(B(u, w), w)_{Y_0^\ast, Y_0} = 0.
\]

Moreover, we have that
\[
-(B(u, u) - B(v, v), u - v)_{Y_0^\ast, Y_0} = \frac{1}{2}(B(u - v, u - v), u + v)_{Y_0^\ast, Y_0}.
\]

Let \( P : (L^2(\Omega))^3 \to H \) be the classical Helmholtz-Leray orthogonal projection (see [14]). If \( u, v, \in V \cap (H^2(\Omega))^3 \), then
\[
Au = -P(\Delta u) = -\Delta u \quad \text{and} \quad B(u, v) = P(u \cdot \nabla v),
\]
and an equivalent norm in \( V \cap (H^2(\Omega))^3 \) is given by
\[
(||u||_{L^2}^2 + ||Au||_{L^2}^2)^{\frac{1}{2}},
\]
where, hereafter, the domain \( \Omega = [0,l]^3 \) is omitted in the expressions involving norms.

We recall some particular three-dimensional cases of the Gagliardo-Nirenberg inequality (see [15] and [33]):
\[
\begin{align*}
&\|g\|_{L^\infty} \leq C_\infty \left( \|\nabla g\|_{L^2}^\frac{1}{2} \|A g\|_{L^2}^\frac{1}{2} + \frac{1}{l^\frac{1}{2}} \|g\|_{L^2} \right), \quad \forall g \in V \cap (H^2(\Omega))^3; \\
&\|g\|_{L^p} \leq C_p \left( \|\nabla g\|_{L^2}^{\frac{np}{n-\gamma}} \|A g\|_{L^2}^{\frac{np}{n-\gamma}} + \frac{1}{l^{\frac{np}{n-\gamma}}} \|g\|_{L^2} \right), \quad \forall 6 < p < \infty \quad \text{and} \quad g \in V \cap (H^2(\Omega))^3; \\
&\|g\|_{L^p} \leq C_p \left( \|g\|_{L^2}^{\frac{n-p}{n-\gamma}} \|\nabla g\|_{L^2}^{\frac{np}{n-\gamma}} + \frac{1}{l^{\frac{np}{n-\gamma}}} \|g\|_{L^2} \right), \quad \forall 2 < p \leq 6 \quad \text{and} \quad g \in V,
\end{align*}
\]
where \( C_p \) are dimensionless constants.

Furthermore, we use the following inequality to deal with the nonlinear damping term (see [23]):
\[
(|x|^\gamma - |y|^\gamma) \cdot (x - y) \geq \frac{1}{2} |x - y|^2 (|x|^\gamma + |y|^\gamma) \quad \forall x, y \in \mathbb{R}^3 \quad \text{and} \quad \gamma \geq 0,
\]
and, by Mean Value Theorem, there exists a dimensionless constant \( \kappa(\gamma) > 0 \) such that
\[
|x|^\gamma - |y|^\gamma \leq \kappa(\gamma) |x - y| (|x| + |y|)^\gamma \quad \forall x, y \in \mathbb{R}^3 \quad \text{and} \quad \gamma \geq 0.
\]

### 2.2 Weak and strong solutions

In order to define and analyze weak and strong solutions for equation (1.1), we rewrite it using functional settings as
\[
\begin{align*}
\frac{du}{dt} + \nu Au + B(u, u) + aG_\alpha(u) &= P(f), \\
u(0) &= u_0,
\end{align*}
\]
and (1.2) as
\[
\begin{align*}
\frac{dw}{dt} + \nu Aw + B(w, w) + bG_\beta(w) &= P(f) + \eta P(I_h(u) - I_h(w)), \\
w(0) &= w_0.
\end{align*}
\]
Definition 2.1 (Weak solution). Suppose $\alpha > 1$ and $a > 0$. Let $f \in L^2((0,T),H)$ and $u(0) = u_0 \in H$. A local weak solution for system $(2.7)$ is a function $u \in L^\infty((0,T),H) \cap L^2((0,T),V) \cap L^{2a+2}(0,T), (L^{2a+2}(\Omega))^3)$ such that satisfies $(2.7)$ in $L^1((0,T),Y_{a})$. We say that $u$ is a global weak solution if $u$ is local weak solution for each $T > 0$.

Remark 2.2. If $u$ is a weak solution for system $(2.7)$, then
\[
\frac{du}{dt} \in L^2((0,T),V') + L^{\frac{2a+2}{\alpha+1}}((0,T),(L^{\frac{2a+2}{\alpha+1}}(\Omega))^3),
\]
$u \in C([0,T),H)$ and $\left\langle \frac{du}{dt}, u \right\rangle_{Y_{a},Y_{a}} = \frac{1}{2} L^2_u(u(0))$ (see $(2.7)$ and $(3.8)$).

Definition 2.3 (Strong solution). Suppose $\alpha > 1$ and $a > 0$. Let $f \in L^2((0,T),H)$ and $u(0) = u_0 \in V$. A local strong solution for system $(2.7)$ is a weak solution such that $u \in L^\infty((0,T),V) \cap L^2((0,T),(H^2(\Omega))^3)$. We say that $u$ is a global strong solution if $u$ is local strong solution for each $T > 0$.

Remark 2.4. If $u$ is a strong solution for system $(2.7)$, then
\[
\frac{du}{dt} \in L^2((0,T),H) + L^{\frac{2a+2}{\alpha+1}}((0,T),(L^{\frac{2a+2}{\alpha+1}}(\Omega))^3)
\]
and $u \in C([0,T),V)$ (see $(2.3)$).

Theorem 2.5 (Global existence and uniqueness of weak and strong solutions). Suppose $\alpha > 1$, $a > 0$ and $f \in L^2_{\text{loc}}(\mathbb{R}^+,H)$. If $u(0) = u_0 \in H$, then the system $(2.7)$ has a unique global weak solution, which is continuously dependent on the initial data in the $H$-norm. Furthermore, if $u(0) = u_0 \in V$, then the global weak solution is strong and it is also continuously dependent on the initial data in the $V$-norm.

The proofs of the above result can be found [20] (see also [8], [23], [24], [35], [37], [38], and [40]).

Concerning to the assimilated system $(2.8)$, similar definitions of weak and strong solutions can be stated. Likewise, there are the following results:

Theorem 2.6 (Existence and uniqueness of weak and strong solutions for $(2.8)$ with $c_1 = 0$). Suppose $\alpha, \beta > 1$, $a, b > 0$ and $f \in L^2_{\text{loc}}(\mathbb{R}^+,H)$, $u$ a global weak solution of $(2.7)$ and $I_h$ a linear operator that satisfies $(2.5)$ with $c_1 = 0$. If $w(0) = w_0 \in H$, then the system $(2.8)$ has an unique global weak solution, which is continuously dependent on the initial data in the $H$-norm. Furthermore, if $w(0) = w_0 \in V$, then the global weak solution of $(2.7)$ is strong and also continuously dependent on the initial data in the $V$-norm.

Theorem 2.7 (Existence and uniqueness of strong solutions for $(2.8)$ with $c_1 > 0$). Suppose $\alpha, \beta > 1$, $a, b > 0$ and $f \in L^2_{\text{loc}}(\mathbb{R}^+,H)$, $u$ a global strong solution of $(2.7)$ and $I_h$ a linear operator that satisfies $(2.5)$. If $w(0) = w_0 \in V$, and $u_1 \geq \eta^2 c_1 h^4$, then the system $(2.8)$ has an unique global strong solution and there is continuous dependence on the initial data in the $V$-norm.

The proof of the above result can be also found in [20].

3 Statements of main theorems

In this work, we are considering dimensional equations in $(2.7)$ and $(2.8)$ and all the results and inequalities presented here are in a correct balance of units. Since $\alpha$ and $\beta$ are dimensionless.
Theorem 3.1. Let \( I < \alpha, \beta < 3 \) and \( \nu > 0 \) given. Suppose that the linear interpolation operator \( J_h \) satisfies (3.5) with \( c_0 > 0 \) and \( c_1 = 0 \). Consider \( \eta \) and \( h \) large and small enough, respectively, such that

\[
\eta > \frac{8(\beta - 1)}{\beta h^{\frac{1}{2}} + \nu^{\frac{1}{2}}}, \quad \text{and} \quad \nu > 4\eta c_0 h^2,
\]

(3.2)

Moreover, let \( u \) be a global strong solution of (2.7), \( M > 0 \) such that

\[
\| \nabla u(0) \|_{L^2}^2 + \frac{1}{12} \| u(0) \|_{L^2}^2 \leq M.
\]

(3.3)

and \( w \) a global weak solution of (2.8). If \( 1 < \alpha, \beta < 2 \), we have

\[
\| w(t) - u(t) \|_{L^2}^2 \leq e^{-\frac{4}{
u t}}\| w(0) - u(0) \|_{L^2}^2 + |\alpha - \beta|^2 \left[ \frac{32\alpha^2 \nu^2}{\eta^4} M_1 + \frac{64\alpha^2 C_{14}^2 A_0}{(2 - \max\{\alpha, \beta\})^2} \right] + \left| \tilde{a} - \tilde{b} \right|^2 \left[ \frac{2 \nu^2}{\eta^4} M_1 + 2 C_{10}^6 A_0 \right],
\]

(3.4)

for all \( t \geq 0 \), where

\[
A_0 = \left( \frac{\nu^2 + 2\nu}{\eta^2 \nu} \right) \left[ M_2 + \frac{1}{12} M_1 \right]^5
\]

(3.5)

and \( M_1 \) and \( M_2 \) are constants given in Corollary 4.2 and uniform estimates for norms of \( u \) and \( \nabla u \) in \( L^2 \), respectively.

Furthermore, if \( 2 \leq \alpha < 3 \) or \( 2 \leq \beta < 3 \), we have

\[
\| w(t) - u(t) \|_{L^2}^2 \leq e^{-\frac{4}{\nu t}}\| w(0) - u(0) \|_{L^2}^2 + |\alpha - \beta|^2 \left[ \frac{2 \alpha^2 \nu^2}{\eta^4} M_1 + \frac{2^{22} \alpha^2 C_0^2 C_{14}^{14}}{(3 - \max\{\alpha, \beta\})^2} A_1 \right] + \left| \tilde{a} - \tilde{b} \right|^2 \left[ \frac{2 \nu^2}{\eta^4} M_1 + 2^{16} C_{14}^{14} A_1 \right],
\]

(3.6)

for all \( t \geq 0 \), with

\[
A_1 = \frac{\nu^8}{\nu^{10}} \left( \frac{1}{\nu} + \frac{2}{\eta \nu \alpha} \right) \left[ \frac{1}{\nu} M_2^2 + \frac{4}{\eta \nu \alpha - \nu} M_2^2 + \frac{8}{\eta \nu \alpha} \| f \|_{L^2}^2 M_2^2 + \frac{2}{\eta \nu} M_1^2 \right].
\]

(3.7)
Remark 3.2. Note that the relaxation parameter $\eta$ does not appear on $A_1$ term $\frac{\nu M|^2}{\nu^2}$, while it is present in fraction denominator of each term of $A_0$. Therefore, for the case $1 < \alpha, \beta < 2$, it is possible that the error of approximation $\|\tilde{u}(t) - u(t)\|_{L^2}$ be small enough in the asymptotic sense by choosing $\eta$ large enough (and consequently $h$ small enough, namely, $I_h$ accurate enough), regardless of the parameter errors $|\alpha - \beta|$ and $|\tilde{a} - \tilde{b}|$.

Remark 3.3. Since $u$ is a strong solution to (2.7), by using (6.10) with $c_1 = 0$ and (3.9), we obtain

$$\|I_h(u(t)) - u(t)\|_{L^2}^2 \leq \frac{\nu}{4\eta} M_2, \text{ for all } t \geq 0,$$

and thus $I_h(u)$ can also be used directly as an approximation to $u$. For many cases, combining approaches to obtain a better approximation is more suitable. For instance, if $I_h$ is the projection onto low Fourier modes, we can consider

$$u(t) \approx I_h(u(t)) + (I - I_h)(w(t)),$$

with $I$ the identity operator. Here, the low modes values are extracted from $I_h(u)$ while the high modes values are from $w$.

If we consider only strong solutions of (2.8), we have results in $V$-norm. In this case, we need also an estimate for the initial data of the assimilated system (2.8), as given in (3.3). Besides, we also have the restriction $1 < \alpha, \beta < 2$, but $c_1 \geq 0$.

Theorem 3.4. Suppose $f \in L^\infty(\mathbb{R}^+; H), 1 < \alpha, \beta < 2$ and $a, b > 0$. Let $u$ be a global strong solution of (2.7). Consider the linear interpolation operator $I_h$ satisfying (1.5) and $M > 0$ satisfying (3.3), as well as $M_3$ and $\tilde{Z}_1$ constants given in (6.19) and Corollary 4.2. Consider also $\eta$ and $h$ large and small enough, respectively, such that (6.22) is satisfied and

$$\eta > \frac{32\nu^2 c_0 h^2}{\nu} + 4\tilde{Z}_1 + 2^{14}\kappa^2(2\beta)C_0^2C_{\tilde{a}}^4 |a| M_3 \text{ and } \nu^2 > \frac{32c_1}{7}h^4 \eta \left(\nu + 8\eta^2\right)^{l^2}.$$ (3.8)

Moreover, let $w$ be a global strong solution of (2.8) with

$$\|\nabla w(0)\|_{L^2}^2 + \frac{1}{T^2} \|w(0)\|_{L^2}^2 \leq M.$$ (3.9)

Then, for $B, C$ and $D$ constants given in (6.20), (6.27) and (6.28), we have for all $t \geq 0$,

$$\|\nabla(w(t) - u(t))\|_{L^2}^2 + \frac{1}{T^2} \|w(t) - u(t)\|_{L^2}^2 \leq Be^{-\frac{\eta t}{2}} \left(\|\nabla(w(0) - u(0))\|_{L^2}^2 + \frac{1}{T^2} \|w(0) - u(0)\|_{L^2}^2 \right) + C|\alpha - \beta|^2 + D|\tilde{a} - \tilde{b}|^2.$$ (3.10)

Remark 3.5. Observe that $\tilde{Z}_1$ given in (6.19) depends on constant $H$ appeared in (6.17), which in turn contains $\eta$ only on fraction denominators, that implies the same for $\tilde{Z}_1$. However, inequality (3.8) can always be obtained for $\eta$ and $h$ large and small enough, respectively.

Once again, note that in $C$ and $D$ given (6.20) and (6.28), the term $\frac{\nu M_3^2}{\nu^2}$ has no relaxation term $\eta$ on its denominator. In the next result, we present an estimate where $\eta$ appears on fraction denominator of each term that multiplies $|\alpha - \beta|^2$ and $|\tilde{a} - \tilde{b}|^2$, thus, theoretically, being possible to make the error of approximation $\|w(t) - u(t)\|_{H^1}$ be small enough, regardless of the parameter errors, by having $\eta$ large enough. We restrict the analysis to the case where $f \in L^\infty(\mathbb{R}^+; H)$ with $f_t \in L^\infty(\mathbb{R}^+; H)$. Furthermore, the estimates are given from the time $\frac{T^2}{\nu}$. 

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Theorem 3.6. Let $f \in L^\infty(\mathbb{R}_+; H)$ with $f_t \in L^\infty(\mathbb{R}_+; H)$, $u$ the global strong solution of (2.7), with $1 < \alpha < 2$ and $a > 0$. Consider $I_h$ a linear operator that satisfies (1.5) and $w$ a global strong solution of (2.7) with $1 < \beta < 2$, $b > 0$ that satisfies (3.9). Also, suppose $M > 0$ satisfying (3.3), $\eta$ large enough and $h$ small enough such that (2.8) and (3.9) are valid. Then, for $B$, $C$ and $D$ constants given in (3.2), (7.3) and (7.3), we have

$$\|\nabla(w(t) - u(t))\|_{L^2}^2 + \frac{1}{K^2}\|w(t) - u(t)\|_{L^2}^2 \leq Be^{-\frac{\eta}{\nu}}(1 - \frac{\eta^2}{\nu^2})^\beta \left( \|\nabla(w(\frac{\eta^2}{\nu^2}) - u(\frac{\eta^2}{\nu^2}))\|_{L^2}^2 + \frac{1}{K^2}\|w(\frac{\eta^2}{\nu^2}) - u(\frac{\eta^2}{\nu^2})\|_{L^2}^2 \right) + C|\alpha - \beta|^2 + D|a - \bar{b}|^2,$$

for all $t \geq \frac{2t^2}{\nu}$.

4 Estimates to the system (2.7)

Henceforth we present auxiliary results that will be useful in the proofs of Theorems 3.1 and 3.4 and they are based on energy-type estimates. To overcome eventual lack of regularities, these estimates are initially obtained for approximate solutions coming from the Galerkin’s procedure. Then, via a limit process, they are also obtained for the exact solutions. Since this procedure is standard, we will present the estimates directly on the exact system.

Lemma 4.1. Suppose $f \in L^\infty(\mathbb{R}_+; H)$, $\alpha > 1$, $a > 0$ and let $u$ be a global strong solution of (2.7). Consider $K$ given by

$$K = \frac{l^2}{\nu} \|f\|_{L^\infty L^2}^2 + \frac{4\nu^{\frac{a+1}{2}}}{a^{\frac{a+1}{2}}}.$$

Then, we have the following estimates:

$$\|u(t)\|_{L^2}^2 \leq e^{\frac{2a}{\nu}}\|u(0)\|_{L^2}^2 + \frac{l^2}{\nu}K, \forall t \geq 0;$$

$$\int_r^t \|u(s)\|_{L^\infty L^2}^{2\nu + 1}ds \leq \frac{1}{\nu}\|u(t)\|_{L^2}^2 + \frac{2(2l^2/\nu)\|f\|_{L^\infty L^2}^{2\nu + 1}}{2\nu + 1 + \frac{4a}{\nu}l^2/\nu^2} \left( t - r \right), \forall t \geq r \geq 0;$$

$$\|\nabla u(t)\|_{L^2}^2 \leq \left( \frac{1}{\nu} + \frac{1}{\nu^{\frac{1}{a}} + \frac{1}{\nu^{\frac{1}{a}}}} \right) e^{-\frac{2a}{\nu}}\|u(0)\|_{L^2}^2 + K \left( \frac{3}{\nu} + \frac{3l^2}{\nu^{\frac{a}{2}} + \frac{1}{\nu^{\frac{a}{2}}}} \right) + \frac{l^2}{\nu^2}\|f\|_{L^\infty L^2}^2, \forall t \geq \frac{l^2}{\nu};$$

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(0)\|_{L^2}^2 + \frac{1}{\nu} \left( \frac{\nu}{\nu - l^2/\nu^2} \right) \left( \frac{\nu}{\nu - l^2/\nu^2} \|f\|_{L^\infty L^2}^2 + \frac{4l^2 - 1}{\nu^{\frac{a}{2}} + \frac{1}{\nu^{\frac{a}{2}}}} \right) + \frac{l^2}{\nu^2}\|f\|_{L^\infty L^2}^2, \forall 0 \leq t \leq \frac{l^2}{\nu};$$

$$\int_r^t \|Au(s)\|_{L^2}^2ds \leq \frac{2}{\nu}\|\nabla u(0)\|_{L^2}^2 + \frac{2}{\nu^{\frac{1}{a}} + \frac{1}{\nu^{\frac{1}{a}}}} \int_r^t \|\nabla u(s)\|_{L^2}^2ds + \frac{4(t - r)}{\nu^2}\|f\|_{L^\infty L^2}^2, \forall t \geq r \geq 0.$$
4.0.1 Proof of Lemma 4.1

Multiplying the system (2.7) by the strong solution $u(t)$, integrating in $\Omega$, performing integration by parts and using (2.2), we obtain the following equality:

$$
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 + a\|u\|_{L^{2,2}^{\alpha+2}}^2 = (f, u)_{L^2}.
$$

Using Young and Hölder’s inequalities, we obtain

$$
|\langle f, u \rangle_{L^2}| \leq \|u\|_{L^{2,2}^{\alpha+2}} \|f\|_{L^{2,2}^{\alpha+1}} \leq \frac{a}{2} \|u\|_{L^{2,2}^{\alpha+2}}^2 + \left( \frac{2}{a} \right) \|f\|_{L^{2,2}^{\alpha+1}}^2.
$$

Therefore, if we integrate (4.9) over $[r, t]$ and use (4.2), we obtain

$$
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 \leq \frac{t^2}{\nu} \|f\|_{L^2}^2 + 2 \frac{\nu^a}{2} t^{\alpha+1}.
$$

4.0.2 Proof of Lemma 4.2

Besides, if we integrate (4.10) over $[r, t]$, we obtain (4.3). Now, integrating (4.10) over $[c, d]$ and using (4.2), we obtain

$$
\int_c^d \|\nabla u(s)\|_{L^2}^2 ds \leq \frac{1}{2\nu} \|u(c)\|_{L^2}^2 + (d-c) \frac{t^2}{\nu^2} \|f\|_{L^2}^2 + (d-c) a \frac{\nu^a}{2} t^{\alpha+1}.
$$

Considering $c = t$ and $d = t + \frac{t^2}{\nu}$, we have

$$
\int_t^{t+\frac{t^2}{\nu}} \|\nabla u(s)\|_{L^2}^2 ds \leq \frac{1}{2\nu} e^{-\frac{2\nu}{\alpha+1}} \|u(0)\|_{L^2}^2 + \frac{3t^2}{2\nu^2} K, \forall t \geq 0.
$$
Multiplying the system (2.7) by Au, integrating over \( \Omega \) and performing integration by parts, we obtain the following equality:

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|_{L^2}^2 + \nu \| Au \|_{L^2}^2 + a \| u^\alpha \|_{L^2}^2 + \frac{a \alpha}{2} \| \nabla |u|^{\alpha-1} u^2 \|_{L^2}^2 = -(u \cdot \nabla u, Au)_{L^2} + (f, Au)_{L^2}.
\]

Using Young and Hölder’s inequalities of the right side above, we get for \( \epsilon > 0 \),

\[
| (f, Au)_{L^2} | \leq \| f \|_{L^2} \| Au \|_{L^2} \leq \nu \epsilon \| Au \|_{L^2}^2 + \frac{1}{4 \nu \epsilon} \| f \|_{L^2}^2,
\]  

\[
|(u \cdot \nabla u, Au)_{L^2}| \leq \int_{\Omega} |u| |\nabla u|^{\frac{1}{2}} |\nabla u|^{\frac{1}{2}} |Au| \, dx \leq \| u \|_{L^{2(\alpha-1)}} \| \nabla u \|_{L^{2\alpha}} \| \nabla u \|_{L^{\frac{2\alpha}{\alpha-1}}} \| Au \|_{L^2} \leq \frac{1}{2\nu} \| u \|_{L^2}^\alpha \| \nabla u \|_{L^2}^\frac{2(\alpha-1)}{\alpha} + \nu \| Au \|_{L^2}^2 \leq \frac{\alpha}{2} \| u \|_{L^2}^\alpha \| \nabla u \|_{L^2}^2 + \left( \frac{1}{2\nu} \right)^{\alpha-1} \left( \frac{2}{\alpha} \right) \| \nabla u \|_{L^2}^2 + \frac{\nu}{2} \| Au \|_{L^2}^2.
\]  

Hence we have

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \nu \left( \frac{1}{2} - \epsilon \right) \| Au \|_{L^2}^2 + a \| u \|_{L^2}^\alpha \| \nabla u \|_{L^2}^2 + \frac{a \alpha}{2} \| \nabla |u|^{\alpha-1} u^2 \|_{L^2}^2 \leq \frac{1}{2\nu^{\frac{\alpha-1}{\alpha}}} \| \nabla u \|_{L^2}^2 + \frac{1}{4 \nu \epsilon} \| f \|_{L^2}^2.
\]  

Fix \( t \geq \frac{L^2}{\nu} \) and \( r \) such that \( t - \frac{L^2}{\nu} \leq r \leq t \). Choosing \( \epsilon = \frac{1}{2} \), integrating (4.13) over \([r, t]\) and using (4.14) properly, we have

\[
\frac{l^2}{\nu} \frac{1}{2} \| \nabla u(t) \|_{L^2}^2 \leq \frac{1}{2} \| \nabla u(r) \|_{L^2}^2 + \frac{1}{2\nu^{\frac{\alpha-1}{\alpha}}} \| \nabla u(s) \|_{L^2}^2 ds + \frac{1}{2\nu} \int_{t - \frac{L^2}{\nu}}^{t} \| f(s) \|_{L^2}^2 ds \leq \frac{1}{2} \| \nabla u(r) \|_{L^2}^2 + \frac{l^2}{2\nu} \| f \|_{L^\infty L^2}^2 + \frac{1}{2\nu^{\frac{\alpha-1}{\alpha}}} \left( \frac{1}{2 \nu} \right)^{\alpha-1} \left( \frac{2}{\alpha} \right) \| u(0) \|_{L^2}^\alpha + \frac{3l^2}{2\nu} K.
\]  

Now, integrating the above inequality with respect to \( r \) over \([t - \frac{L^2}{\nu}, t]\) and using again (4.14) appropriately, we obtain

\[
\frac{l^2}{\nu} \frac{1}{2} \| \nabla u(t) \|_{L^2}^2 \leq \frac{1}{2} \int_{t - \frac{L^2}{\nu}}^{t} \| \nabla u(r) \|_{L^2}^2 dr + \frac{l^2}{2\nu} \| f \|_{L^\infty L^2}^2 + \frac{l^2}{2\nu^{\frac{\alpha-1}{\alpha}}} \left( \frac{1}{2 \nu} \right)^{\alpha-1} \left( \frac{2}{\alpha} \right) \| u(0) \|_{L^2}^\alpha + \frac{3l^2}{2\nu} K,
\]  

\[
\leq \frac{1}{2\nu} \left( \frac{1}{2} e^{-2s \left( t - \frac{L^2}{\nu} \right)} \right) \| u(0) \|_{L^2}^2 + \frac{3l^2}{2\nu} K + \frac{l^4}{2\nu^{\frac{3\alpha-2}{\alpha}} a^{\frac{\alpha}{\alpha}}} \left( \frac{1}{2 \nu} \right)^{\alpha-1} \left( \frac{2}{\alpha} \right) \| u(0) \|_{L^2}^\alpha + \frac{3l^2}{2\nu} K.
\]

Therefore, we get (4.14). Moreover, integrating (4.10) over \([0, \frac{L^2}{\nu}]\), we obtain

\[
\int_{0}^{\frac{L^2}{\nu}} \| \nabla u(s) \|_{L^2}^2 ds \leq \frac{l^4}{\nu^\alpha} \| f \|_{L^\infty L^2}^2 + \frac{4l^2}{\nu^{\frac{3\alpha-2}{\alpha}}} a^{\frac{\alpha}{\alpha}}.
\]  

(4.16)
Let $0 \leq t \leq \frac{t^2}{\nu}$. Integrating (4.15) with $\epsilon = \frac{1}{4}$ over $[0, t]$, we obtain

$$\|\nabla u(t)\|_{L^2}^2 \leq \|\nabla u(0)\|_{L^2}^2 + \frac{1}{\nu^{\frac{3}{4}} a^{\frac{1}{4}}} \left( \frac{t^4}{\nu^2} \|f\|_{L^\infty}^2 L^2 + \frac{4l^2}{\nu} a^{\frac{1}{2}} \right) + \frac{t^2}{\nu^2} \|f\|_{L^\infty}^2 L^2. \tag{4.17}$$

and thus (4.15) is valid.

Finally, integrating (4.15) over $[r, t]$ choosing $\epsilon = \frac{1}{4}$ we obtain (4.6).

We summarize Lemma 4.1 below:

**Corollary 4.2.** Suppose $f \in L^\infty(\mathbb{R}_+; H)$, $\alpha > 1$ and $\alpha > 0$. Let $u$ be a global strong solution of (2.7). Consider $K$ as given in (4.7) and $M > 0$ satisfying (3.3). Define the following

$$M_1 = l^2 M + \frac{l^2 K}{\nu},$$

$$\tilde{M} = \max \left\{ \frac{1}{\nu^{\frac{3}{4}} a^{\frac{1}{4}}} \left( \frac{t^4}{\nu^2} \|f\|_{L^\infty}^2 L^2 + \frac{4l^2}{\nu} a^{\frac{1}{2}} \right), \frac{l^2}{2\nu^{\frac{3}{4}} a^{\frac{1}{4}}} M + K \left( \frac{3}{2\nu} + \frac{3l^2}{2\nu^{\frac{3}{4}} a^{\frac{1}{4}}} \right) \right\},$$

$$M_2 = \frac{l^2}{\nu^2} \|f\|_{L^\infty}^2 L^2 + M + \tilde{M},$$

$$M_3 = \frac{2}{\nu^{\frac{3}{4}} a^{\frac{1}{4}}} M_2 + \frac{4}{\nu^2} \|f\|_{L^\infty}^2 L^2,$$

$$M_4 = \frac{2l^2}{\nu} \|f\|_{L^\infty}^2 L^2 + \frac{2l^2}{\nu} a^{\frac{1}{2}} \|f\|_{L^\infty}^2 L^2.$$

Then,

$$\|u(t)\|_{L^2}^2 \leq M_1, \forall t \geq 0,$$

$$\|\nabla u(t)\|_{L^2}^2 \leq M_2, \forall t \geq 0,$$

$$\int_r^t \|Au(s)\|_{L^2}^2 ds \leq \frac{2}{\nu} M_2 + (t - r) M_3, \forall t \geq r \geq 0,$$

$$\int_r^t \|u(s)\|_{L^{2\alpha + 2}}^2 ds \leq \frac{1}{a} M_1 + (t - r) M_4, \forall t \geq r \geq 0.$$

**Corollary 4.3.** Suppose $f \in L^\infty(\mathbb{R}_+; H)$, $\alpha > 1$, $\alpha > 0$ and $u$ be a global strong solution of (2.7). Consider $M > 0$ that satisfies (3.3) and $M_2$ as in Corollary 4.2 Then, fixed $\eta > 0$, we have for all $t \geq 0$,

$$\int_0^t e^{-\frac{\eta}{2}(s-t)} \|Au(s)\|_{L^2}^2 ds \leq \frac{4}{\nu} M_2 + \frac{16}{\eta \nu^{\frac{3}{4}} a^{\frac{1}{4}}} M_2 + \frac{32}{\eta \nu^2} \|f\|_{L^\infty}^2 L^2.$$
4.0.2 Proof of Corollary 4.3

Using inequality (4.15) with \( \varepsilon = \frac{1}{4} \) we obtain
\[
\frac{d}{ds} \left\| \nabla u(s) \right\|_{L^2}^2 + \frac{\nu}{2} \left\| A u(s) \right\|_{L^2}^2 \leq \frac{1}{\nu \varepsilon^{\alpha}} \frac{1}{a^{\alpha + 1}} M_2 + \frac{2}{\nu} \left\| f \right\|_{L^\infty L^2}^2.
\]

Multiply the above inequality by \( e^{\frac{2}{\alpha} (s-t)} \), we have
\[
\frac{d}{ds} \left( e^{\frac{2}{\alpha} (s-t)} \left\| \nabla u(s) \right\|_{L^2}^2 \right) + \frac{\nu}{2} e^{\frac{2}{\alpha} (s-t)} \left\| A u(s) \right\|_{L^2}^2 \leq e^{\frac{2}{\alpha} (s-t)} \left( \frac{1}{\nu \varepsilon^{\alpha}} \frac{1}{a^{\alpha + 1}} M_2 + \frac{\eta}{8} M_2 + \frac{2}{\nu} \left\| f \right\|_{L^\infty L^2}^2 \right).
\]

Integrating over \([0, t]\), we obtain
\[
\left\| \nabla u(t) \right\|_{L^2}^2 + \frac{\nu}{2} \int_0^t e^{\frac{2}{\alpha} (s-t)} \left\| A u(s) \right\|_{L^2}^2 ds \leq e^{\frac{2}{\alpha} t} \left\| \nabla u(0) \right\|_{L^2}^2 + \frac{8}{\eta} \left( \frac{1}{\nu \varepsilon^{\alpha}} \frac{1}{a^{\alpha + 1}} M_2 + \frac{\eta}{8} M_2 + \frac{2}{\nu} \left\| f \right\|_{L^\infty L^2}^2 \right).
\]

Then, we have the result.

\[\blacksquare\]

**Lemma 4.4.** Suppose \( f \in L^\infty(\mathbb{R}_+; H) \) with \( f_i \in L^\infty(\mathbb{R}_+; H) \), \( 1 < \alpha < 2 \), \( \alpha > 0 \) and \( a \) be a global strong solution of (2.7). Consider \( M > 0 \) such that (4.3) is satisfied and \( M_1, M_2, M_3 \) and \( M_4 \) as given in Corollary 4.3. Let
\[
M_5 = \frac{\nu}{a^2} M_1 + M_4 + \frac{\nu(\alpha + 1)}{a} M_2 + \frac{(2\alpha + 2)l^2}{a \nu} \left\| f \right\|_{L^\infty L^2}^2 + \frac{4C_2^2}{a} \frac{(2\alpha + 2)}{l} M_2 + \frac{2\nu M_2}{a} \left( \frac{1}{l^2 a} \right),
\]
\[
M_6 = \nu M_2 + \frac{a}{\alpha + 1} M_5 + \frac{2l^2}{\nu} \left\| f \right\|_{L^\infty L^2}^2 + 8C_2^2 \frac{4l^2}{\nu^3} \left( \frac{4l^2}{\nu^3} M_2^3 + 2\nu M_2 + l^2 \nu M_3 + \frac{1}{l^2 \nu} M_1 M_2 \right),
\]
\[
M_7 = M_6 \left[ \frac{3 \nu}{2l^2} + \frac{108(C_3 C_6)^4}{\nu^4} M_2^2 + \frac{6(C_3 C_6)^2}{\nu l^2} M_2^2 + \frac{4C_2^2 C_6^2}{\nu l} M_2 + \frac{2C_3 C_6^2}{l^2} M_2^2 \right] + \frac{2l^4}{\nu^2} \left\| f \right\|_{L^\infty L^2}^2,
\]
\[
M_8 = \frac{2}{\nu} M_7 + 2^{\alpha} \frac{C_3^2}{\nu^2} \left( M_1^2 M_2^2 + \frac{1}{\nu^2} M_1^2 \right) M_2 + \frac{2^{\alpha} a C_3^2}{\nu^{2\alpha}} \frac{1}{a^{2\alpha}} M_2 \left( \frac{C_3^2}{\nu^{2\alpha}} M_2 \right) M_2 + \frac{2^{\alpha} a C_3^2}{\nu^{2\alpha}} \frac{1}{a^{2\alpha}} M_1^2 + \frac{2}{\nu} \left\| f \right\|_{L^\infty L^2}^2.
\]

(4.18)
Then,

$$\|u(t)\|_{L^{2\alpha+2}}^{2\alpha+2} \leq M_5, \ \forall \ t \geq \frac{l^2}{\nu};$$

(4.19)

$$\int_t^{t+\frac{L}{2}} \|u_t(s)\|_{L^2}^2 ds \leq M_6, \ \forall \ t \geq \frac{l^2}{\nu};$$

(4.20)

$$\|u_t(t)\|_{L^2}^2 \leq M_7, \ \forall \ t \geq \frac{2l^2}{\nu};$$

(4.21)

$$\|Au(t)\|_{L^2} \leq M_8, \ \forall \ t \geq \frac{2l^2}{\nu}. $$

(4.22)

### 4.0.3 Proof of Lemma 4.4

Multiplying the system (2.7) by $u_t$, integrating in $\Omega$ and performing integration by parts, we obtain the following:

$$\|u_t\|_{L^2}^2 + \frac{\nu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{a}{(2\alpha + 2)} \frac{d}{dt} \|u^{\alpha+1}\|_{L^2}^2 = -(u \cdot \nabla u, u_t)_{L^2} + (f, u_t)_{L^2}. $$

(4.23)

By (2.4) and Hölder’s inequality, we have

$$|(u \cdot \nabla u, u_t)_{L^2}| \leq \|u_t\|_{L^2} \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2} \leq C_\infty \|u_t\|_{L^2} \left( \|\nabla u\|_{L^2}^2 \|Au\|_{L^2} + \frac{1}{l^2} \|u\|_{L^2}^2 \right) \|\nabla u\|_{L^2},$$

and

$$|(f, u_t)_{L^2}| \leq \epsilon \|u_t\|_{L^2}^2 + \frac{1}{4\epsilon} \|f\|_{L^2}^2.$$ 

Thus,

$$(1 - 2\epsilon) \|u_t\|_{L^2}^2 + \frac{\nu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{a}{(2\alpha + 2)} \frac{d}{dt} \|u^{\alpha+1}\|_{L^2}^2 \leq \frac{1}{4\epsilon} \|f\|_{L^2}^2 + \frac{C_\infty^2}{\epsilon} \left( \|\nabla u\|_{L^2}^2 \|Au\|_{L^2} + \frac{1}{l^2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \right). $$

Choosing $\epsilon = \frac{1}{4}$, Hölder’s inequality implies

$$\frac{1}{2} \|u_t\|_{L^2}^2 + \frac{\nu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \frac{a}{(2\alpha + 2)} \frac{d}{dt} \|u^{\alpha+2}\|_{L^{2\alpha+2}}^2 \leq \|f\|_{L^2}^2 + 4C_\infty^2 \left( \frac{1}{4l^2} \|\nabla u\|_{L^2}^2 + \frac{1}{4l^2} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \right).$$

(4.24)
Fix now $t \geq 0$. Considering $s$ such that $t < s < t + \frac{t^2}{\nu}$, integrating over $[s, t + \frac{t^2}{\nu}]$ and using Corollary 1.19, we obtain

$$
\frac{a}{(2\alpha + 2)} \left\| u \left( t + \frac{t^2}{\nu} \right) \right\|_{L^{2\alpha+2}}^{2\alpha+2} \leq \frac{a}{(2\alpha + 2)} \left\| u(s) \right\|_{L^{2\alpha+2}}^{2\alpha+2} + \frac{\nu}{2} \left\| \nabla u(s) \right\|_{L^2}^2 + \frac{l^2}{\nu} \left\| f \right\|_{L^\infty}^2 \left( \int_t^{t + \frac{t^2}{\nu}} \right) \left( 4 \frac{\nu}{M^2} \right) \left\| \nabla u(r) \right\|_{L^2}^2 + \frac{1}{l^2} \left\| u(r) \right\|_{L^2}^2 \left\| \nabla u(r) \right\|_{L^2}^2 \right) \, dr
$$

Integrating above over $[t, t + \frac{t^2}{\nu}]$ in $s$, we get

$$
\frac{a}{(2\alpha + 2)} \frac{l^2}{\nu} \left\| u \left( t + \frac{t^2}{\nu} \right) \right\|_{L^{2\alpha+2}}^{2\alpha+2} \leq \frac{a}{(2\alpha + 2)} \int_t^{t + \frac{t^2}{\nu}} \left\| u(s) \right\|_{L^{2\alpha+2}}^{2\alpha+2} ds + \frac{l^2}{2} M_2 + \frac{l^4}{\nu^2} \left\| f \right\|_{L^\infty}^2 \\
+ 4C_2^2 \left( \frac{4l^2}{\nu^2} M_3^2 + 2\nu M_2 + l^2 M_3 + \frac{1}{l^2} M_1 M_2 \right)
$$

Then, we obtain (4.19).

Consider $t \geq \frac{\nu}{l^2}$. Integrating (2.24) over $[t, t + \frac{t^2}{\nu}]$, we have

$$
\frac{1}{2} \int_t^{t + \frac{t^2}{\nu}} \left\| u(t) \right\|_{L^2}^2 ds \leq \frac{\nu}{2} \left\| \nabla u(t) \right\|_{L^2}^2 + \frac{a}{2\alpha + 2} \left\| u(t) \right\|_{L^{2\alpha+2}}^{2\alpha+2} + \frac{l^2}{\nu} \left\| f \right\|_{L^\infty}^2
$$

Therefore we conclude (4.20).

We now differentiate (2.27) with respect to $t$ and take the inner product with $u_t$ to obtain

$$
\frac{1}{2} \frac{d}{dt} \left\| u_t \right\|_{L^2}^2 + \frac{\nu}{2} \left\| \nabla u_t \right\|_{L^2}^2 + a(2\alpha + 1) \left\| u_t \right\|_{L^{2\alpha+2}}^2 \left( \int u_t \cdot \nabla u \right)_{L^2} + \left( f_t, u_t \right)_{L^2} = -(u_t \cdot \nabla u, u_t)_{L^2} + (f_t, u_t)_{L^2}. \tag{4.25}
$$

Using (2.3) and Hölder’s inequality,

$$
\left( f_t, u_t \right)_{L^2} \leq \frac{\nu}{4l^2} \left\| u_t \right\|_{L^2}^2 + \frac{l^2}{\nu} \left\| f_t \right\|_{L^2}^2,
$$

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Then, by (4.25),

\[ \| (u_t \cdot \nabla u, u_t) \|_{L^2} \leq \| u_t \|_{L^2} \| \nabla u \|_{L^2} \| u_t \|_{L^6} \]

\[ \leq C_3 C_6 \left( \| u_t \|_{L^2}^2 \| \nabla u_t \|_{L^2}^2 + \frac{1}{l^2} \| u_t \|_{L^2} \right) \| \nabla u \|_{L^2} \left( \| \nabla u_t \|_{L^2} + \frac{1}{l} \| u_t \|_{L^2} \right) \]

\[ \leq \left( \frac{54(C_3 C_6)^2}{\nu^3} \| \nabla u \|_{L^2}^2 + \frac{3(C_3 C_6)^2}{\nu^2} \| u_t \|_{L^2} \| \nabla u \|_{L^2}^2 + \frac{2C_3^2 C_6^2}{\nu l} \| \nabla u \|_{L^2}^2 + \frac{C_3 C_6}{l^2} \| \nabla u \|_{L^2} \right) \| u_t \|_{L^2}^2 + \frac{3\nu}{8} \| \nabla u_t \|_{L^2}^2. \]

Then, by \[ \frac{d}{dt} \| u_t \|_{L^2}^2 \leq \left( \frac{\nu}{4l^2} + \frac{54(C_3 C_6)^2}{\nu^3} M_2^2 + \frac{3(C_3 C_6)^2}{\nu^2} M_2^2 + \frac{2C_3^2 C_6^2}{\nu l} M_2^2 + \frac{C_3 C_6}{l^2} \right) \| u_t \|_{L^2}^2 + \frac{l^2}{\nu} \| f_t \|_{L^2}^2. \]

Fixed \( t \geq \frac{l^2}{\nu} \), consider \( s < t < t + \frac{l^2}{\nu} \). Integrating the above inequality over \([s, t + \frac{l^2}{\nu}]\), using (4.20) and defining

\[ K_2 = \frac{\nu}{4l^2} + \frac{54(C_3 C_6)^2}{\nu^3} M_2^2 + \frac{3(C_3 C_6)^2}{\nu^2} M_2^2 + \frac{2C_3^2 C_6^2}{\nu l} M_2^2 + \frac{C_3 C_6}{l^2} \]

we get

\[ \left\| u_t \left( t + \frac{l^2}{\nu} \right) \right\|_{L^2}^2 \leq \| u_t(s) \|_{L^2}^2 + 2K_2 \| u_t(r) \|_{L^2}^2 dr + \frac{\nu}{2l^2} \| f_t \|_{L^2}^2. \]

Integrating the above inequality over \([t, t + \frac{l^2}{\nu}]\) in \( s \) and using again (4.20), we obtain

\[ \| u_t \left( t + \frac{l^2}{\nu} \right) \|_{L^2}^2 \leq \int_t^{t + \frac{l^2}{\nu}} \| u_t(s) \|_{L^2}^2 ds + \frac{2l^2 K_2 M_2}{\nu} + \frac{\nu}{2l^2} \| f_t \|_{L^2}^2. \]

Then, we conclude (4.21). Finally, we take the \( L^2 \)-norm in (2.7) and get

\[ \nu \| Au \|_{L^2} \leq \| u_t \|_{L^2} + \| (u \cdot \nabla) u \|_{L^2} + \frac{1}{2} \| \nabla u \|_{L^2} + \| f \|_{L^2}. \]

Using (2.4) and Hölder and Young’s inequalities, we have

\[ \| (u \cdot \nabla) u \|_{L^2} \leq \| u \|_{L^4} \| \nabla u \|_{L^4} \leq C_4^2 \left( \frac{\| u \|_{L^2}^2 \| \nabla u \|_{L^2}^2 + \frac{1}{l^2} \| u \|_{L^2} \right) \| \nabla u \|_{L^2} \| Au \|_{L^2} \]

\[ \leq 432 \left( \| u \|_{L^2} \| \nabla u \|_{L^2}^2 + \frac{1}{l^2} \| u \|_{L^2} \right). \]
Since $1 < \alpha < 2$, using (2.4) and (4.19), we obtain
\[
\|u\|^\alpha \|u\|_{L^2} \leq \|u\|^{\alpha+1} \|u\|_{L^2} \leq \alpha \|u\|_{L^{\alpha+2}} \bigg( \|\nabla a\|_{L^2} + \frac{1}{L^2} \|u\|_{L^2} \bigg) \leq 2^\alpha a M_2 C_\infty \bigg( \|\nabla u\|_{L^2} + \frac{1}{L^2} \|u\|_{L^2} \bigg)
\]
\[
\leq 2^\alpha a M_2 C_\infty \bigg( \|\nabla u\|_{L^2} + \frac{1}{L^2} \|u\|_{L^2} \bigg)
\]
\[
\leq \frac{2^{3\alpha-1} a^{3/2} M_2^{3/2} C_\infty^{3/2}}{L^{\alpha-1}} \|\nabla u\|_{L^2} + \frac{2^{3\alpha-1} a M_2^{3/2} C_\infty^{3/2}}{L^{\alpha-1}} \|\nabla u\|_{L^2} + \frac{L^2}{16} \|u\|_{L^2},
\]
for all $t > \frac{L^2}{16}$. Next, using above estimates in (4.20), we conclude that
\[
\|Au\|_{L^2} \leq M_7 + 432 C_\infty \bigg( M_1 \int M_2^{3/2} + \frac{1}{L^2} M_1 \bigg) + \frac{2^{3\alpha-1} a^{3/2} M_2^{3/2} C_\infty^{3/2}}{L^{\alpha-1}} M_1 \|f\|_{L^\alpha L^2}.
\]
Then, we have (4.22).

5 Proof of Theorem 3.1

Let $u$ be the strong solution to (2.7) and $w$ the weak solution of (2.8). Denoting $g = w - u$, we have
\[
\frac{d}{dt} g + \nu g + B(w, w) - B(u, u) + bG_\beta(w) - aG_\alpha(u) = -\eta \mathcal{P}(I_\beta(g)).
\]
(5.1)

Multiplying the system (5.1) by $g$, integrating in space, using integration by parts and (2.3), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|g\|_{L^2}^2 + \nu \|\nabla g\|_{L^2}^2 + b \langle |w|^{2\beta} w - |u|^{2\beta} u, g \rangle_{Y_\beta, Y_\beta} = \nu \left\langle \left( \frac{1}{\nu} u \right)^{2\alpha} \right. u - \left. \left( \frac{1}{\nu} u \right)^{2\beta} u, g \right\rangle_{Y_\beta, Y_\beta} + \frac{1}{2} \left( g \cdot \nabla g, w + u \right)_{Y_\beta, Y_\beta} - \eta \left( I_\beta(g) - g \right)_{L_2} - \eta \|g\|_{L^2}^2.
\]
(5.2)

We now estimate the terms of (5.2). Using (2.1)–(2.6) along Young and Hölder’s inequalities, we have the following estimates:
\[
|\eta(I_\beta(g) - g, g)_{L_2}| \leq \eta \|I_\beta(g) - g\|_{L_2} \leq \eta \|g\|_{L_2}^2 + \frac{1}{4\nu \eta} \|g\|_{L_2}^2 \quad \text{and} \
\left( \frac{\eta}{\nu} \right)^{\beta-1} \|g\|_{L_2}^\beta + \eta \|g\|_{L_2}^\beta \quad \text{and} \
\left( \frac{\eta}{\nu} \right)^{\beta-1} \|g\|_{L_2}^\beta + \eta \|g\|_{L_2}^\beta.
\]
(5.3)

\[
\frac{1}{2} \left( g \cdot \nabla g, w + u \right)_{Y_\beta, Y_\beta} \leq \frac{1}{2} \int_\Omega |g|^{\beta} |g|^{\frac{\beta-1}{2}} |u + w| \|\nabla g\| dx
\]
\[
\leq \frac{1}{2} \|g\|^{\beta} \|u + w\|_{L^{2\beta}} \|g\|^{\frac{\beta-1}{2}} \|\nabla g\|_{L^2} = \frac{1}{2} \|g\| \|u + w\|^{\beta} \|g\|^{\frac{\beta-1}{2}} \|\nabla g\|_{L^2}
\]
\[
\leq \frac{1}{16\nu \eta} \|g\| \|u + w\|^{\beta} \|g\|^{1/2} \|\nabla g\|_{L^2}
\]
\[
\leq \frac{1}{\nu^{\beta} (\eta \nu)^{\beta-1}} \|g\| \|u + w\|^{\beta} \|g\|^{1/2} + \frac{(\beta - 1) \nu^{\beta}}{\beta (16\nu \eta)^{\beta-1}} \|g\|^{1/2} \|\nabla g\|_{L^2}.
\]
(5.4)
By Mean Value Theorem, we have
\[
\left| \frac{l}{\nu} u^{2\alpha} - \frac{l}{\nu} u^{2\beta} \right| \leq 2|\alpha - \beta| \left( \frac{l}{\nu} u^{2\alpha + 1} + \frac{l}{\nu} u^{2\beta + 1} \right) \left| \ln \frac{l}{\nu} u \right| .
\]

Henceforward, we divide in two cases for \( \alpha \) and \( \beta \):

**Case 1** \( 1 < \alpha, \beta < 3 \):

\[
\mathcal{a}_{\nu} \int_{\Omega} \left( \left| \frac{l}{\nu} u^{2\alpha} - \frac{l}{\nu} u^{2\beta} \right| \right) \leq 2\mathcal{a}_{\nu} \left( \frac{l}{\nu} u^{2\alpha + 1} + \frac{l}{\nu} u^{2\beta + 1} \right) \left| \ln \frac{l}{\nu} u \right| dx
\]

\[
\leq 4\mathcal{a}_{\nu} \left( \frac{l}{\nu} \right)^{2\alpha} \left| \frac{l}{\nu} \right| \left( \frac{1}{\nu} \right)^{2\beta} \left| \frac{l}{\nu} \right| \left( 1 - \beta \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

\[
\leq \frac{4\mathcal{a}_{\nu} \left( \frac{l}{\nu} \right)^{2\alpha} \left| \frac{l}{\nu} \right| \left( \frac{1}{\nu} \right)^{2\beta} \left| \frac{l}{\nu} \right| \left( 1 - \beta \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

Finally,
\[
\mathcal{a} - \mathcal{b} \int_{\Omega} \left( \left| \frac{l}{\nu} u^{2\alpha} - \frac{l}{\nu} u^{2\beta} \right| \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

\[
\leq \mathcal{a} - \mathcal{b} \left( \frac{l}{\nu} \right)^{2\alpha} \left( \frac{l}{\nu} \right)^{2\beta} \left( 1 - \beta \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

\[
\leq \mathcal{a} - \mathcal{b} \left( \frac{l}{\nu} \right)^{2\alpha} \left( \frac{l}{\nu} \right)^{2\beta} \left( 1 - \beta \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

\[
+ \frac{2}{\epsilon l} \left( \frac{l}{\nu} \right)^{2\beta} \left( \frac{l}{\nu} \right)^{2\beta} \left( 1 - \beta \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

\[
\leq 2^{14} \mathcal{a} - \mathcal{b} \left( \frac{l}{\nu} \right)^{2\alpha} \left( \frac{l}{\nu} \right)^{2\beta} \left( 1 - \beta \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

\[
+ \frac{2}{\epsilon l} \left( \frac{l}{\nu} \right)^{2\beta} \left( \frac{l}{\nu} \right)^{2\beta} \left( 1 - \beta \right) \left( \frac{l}{\nu} u \right)^{2\beta + 1} \left| \ln \frac{l}{\nu} u \right| dx
\]

(5.7)
From Corollary 4.2, we have
\[
\frac{1}{2} \frac{d}{dt} \|g(t)\|^2_{L^2} + \left[ \nu \left( \frac{1}{2} - \epsilon \right) - \frac{1}{(\beta(\epsilon + h)^2) \nu} \right] \|\nabla g\|^2_{L^2} + \left[ \frac{b}{2^2} \nu - \frac{1}{(\beta(\epsilon + h)^2) \nu} \right] \|g\|(\|w\| + |u|)^2_{L^2}
\leq \eta \left[ \frac{(\beta - 1) \epsilon}{\beta(\epsilon + h)^2 \nu} \right] \|g\|^2_{L^2} + \frac{2^{10} \alpha^2 C^2_0 [\alpha - \beta]^2}{(3 - \max\{\alpha, \beta\})^2} \nu^{10} \left( \frac{1}{\nu} + \frac{1}{\eta \nu^2} \right) \frac{C^{14}}{4^4} \left( \|\nabla u\|^2_{L^2} + \frac{1}{\nu} \|u\|^4_{L^2} \right) + \frac{32 \alpha^2 [\alpha - \beta]^2 \nu^2}{\eta \nu^2} \|u\|^2_{L^2} + 2^{14} |\tilde{a} - \tilde{b}|^2 \frac{8^{10}}{\nu^{10}} \left( \frac{1}{\nu} + \frac{1}{\eta \nu^2} \right) \frac{C^{14}}{4^4} \left( \|\nabla u\|^2_{L^2} + \frac{1}{\nu} \|u\|^4_{L^2} \right) + \frac{2}{\eta} \|\tilde{a} - \tilde{b}\|^2 \frac{\nu^2}{\nu^2} \|M_1\|.
\] (5.8)

Choosing, \( \epsilon = \frac{1}{4}, \tilde{\epsilon} = \frac{4\sqrt{\alpha}}{8(\beta - 1)}, \) \( \eta \) and \( h \) as in the statement of Theorem 3.1, \( M_1 \) and \( M_2 \) as given in Corollary 4.2, we have

\[
\frac{1}{2} \frac{d}{dt} \|g(t)\|^2_{L^2} \leq \frac{e^{-\frac{2\pi t}{\eta \nu^2}} \|g(0)\|^2_{L^2} + \frac{\alpha^2 C^2_0 [\alpha - \beta]^2}{(3 - \max\{\alpha, \beta\})^2} \nu^{10} \left( \frac{1}{\nu} + \frac{1}{\eta \nu^2} \right) \frac{C^{14}}{4^4} \left( \frac{M_0^6}{\nu} \int_0^t e^{\frac{\pi}{\nu} (s-t)} \|Au(s)\|^2_{L^2} ds + \frac{8}{\eta \nu^6} M_1^7 \right)}{\frac{2^{10} \alpha^2 [\alpha - \beta]^2 \nu^2}{\eta \nu^2} \|M_1\| + \frac{2^{14}}{\eta \nu^2} \|\tilde{a} - \tilde{b}\|^2 \frac{\nu^2}{\nu^2} \|M_1\|}
\]

for all \( t \geq 0 \). Using Gronwall’s inequality, we conclude that

\[
\|g(t)\|^2_{L^2} \leq e^{-\frac{2\pi t}{\eta \nu^2}} \|g(0)\|^2_{L^2} + \frac{\alpha^2 C^2_0 [\alpha - \beta]^2}{(3 - \max\{\alpha, \beta\})^2} \nu^{10} \left( \frac{1}{\nu} + \frac{1}{\eta \nu^2} \right) \frac{C^{14}}{4^4} \left( \frac{M_0^6}{\nu} \int_0^t e^{\frac{\pi}{\nu} (s-t)} \|Au(s)\|^2_{L^2} ds + \frac{8}{\eta \nu^6} M_1^7 \right)
\]

Thus, the desired inequality in \( H \)-norm stated in Theorem 3.1 for the case \( 2 \leq \alpha < 3 \) or \( 2 \leq \beta < 3 \) is obtained.

Case 1 \( \alpha, \beta < 2 \):

We choose \( \tilde{\epsilon} = 1, \epsilon = \frac{1}{4} \) and \( \tilde{\epsilon} = \frac{4\sqrt{\alpha}}{8(\beta - 1)} \) in inequalities \( 5.3 - 5.4 \). Since \( 1 < \alpha, \beta < 2 \), we
replace inequalities (5.6) and (5.7) by the following ones:

\[
\bar{a} \frac{\nu}{t^2} \left( \left| \frac{l}{\nu} \right|^{2\alpha} u - \left| \frac{l}{\nu} \right|^{2\beta} u, g \right)_{Y_\nu^2, Y_\beta^2} \leq 2 \bar{a} \frac{\nu^2}{t^2} (\alpha - \beta) \int_\Omega |g| \left( \left| \frac{l}{\nu} \right|^{2\alpha+1} + \left| \frac{l}{\nu} \right|^{2\beta+1} \right) \ln \left| \frac{l}{\nu} \right| \, dx \\
\leq 4 \bar{a} \frac{\nu^2}{t^2} (\alpha - \beta) \int_\Omega |g| \left( \frac{1}{2 - \max\{\alpha, \beta\}} \left| \frac{l}{\nu} \right|^5 + \frac{1}{e} \left| \frac{l}{\nu} \right| \right) \, dx \\
\leq \frac{4\bar{a} |\alpha - \beta|}{2 - \max\{\alpha, \beta\}} \frac{\nu^2}{t^2} \|g\|_{L^\infty} \|u\|^5_{L^2} + \frac{4\bar{a} |\alpha - \beta| \nu}{t^2} \|g\|_{L^2} \|u\|_{L^2} \\
\leq \frac{4\bar{a} C_0 |\alpha - \beta|}{2 - \max\{\alpha, \beta\}} \frac{\nu^2}{t^2} \left( \|\nabla g\|_{L^2} + \frac{1}{t} \|g\|_{L^2} \right) \|u\|^5_{L^2} \\
+ \frac{4\bar{a} |\alpha - \beta| \nu}{t^2} \|g\|_{L^2} \|u\|_{L^2} \\
\leq \frac{64\bar{a}^2 C_0^4 |\alpha - \beta|^2}{(2 - \max\{\alpha, \beta\})^2} \frac{\nu^4}{t^2} \left( \frac{1}{\nu} + \frac{1}{\nu \eta^2} \right) C_6^{10} \left( \|\nabla u\|^2_{L^2} + \frac{1}{t^2} \|u\|^2_{L^2} \right)^5 \\
+ \frac{32\bar{a}^2 |\alpha - \beta|^2 \nu^2}{\nu \eta^2} \frac{1}{t^2} \|u\|^2_{L^2} + \frac{\nu}{4} \|\nabla g\|_{L^2} + \frac{\nu \eta}{4} \|g\|_{L^2}^2; \quad (5.9)
\]

and

\[
\left| \bar{a} - \bar{b} \right| \frac{\nu}{t^2} \left( \left| \frac{l}{\nu} \right|^{2\beta} u, g \right)_{Y_\nu^2, Y_\beta^2} \leq \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \|g\|_{L^\infty} \|u\|^5_{L^2} + \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \|g\|_{L^2} \|u\|_{L^2} \\
\leq \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \left( \|\nabla g\|_{L^2} + \frac{1}{t} \|g\|_{L^2} \right) \|u\|^5_{L^2} + \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \|g\|_{L^2} \|u\|_{L^2} \\
\leq \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \left( \frac{1}{\nu} + \frac{2}{\nu \eta^2} \right) \|u\|_{L^2}^{10} + \frac{2}{\nu \eta^2} \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \|u\|_{L^2}^2 + \frac{\nu}{4} \|\nabla g\|_{L^2} + \frac{\nu \eta}{4} \|g\|_{L^2}^2 \\
\leq 2 \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \left( \frac{1}{\nu} + \frac{2}{\nu \eta^2} \right) C_6^{10} \left( \|\nabla u\|^2_{L^2} + \frac{1}{t^2} \|u\|^2_{L^2} \right)^5 + \frac{2}{\nu \eta^2} \left| \bar{a} - \bar{b} \right| \frac{\nu^2}{t^2} \|u\|_{L^2}^2 \\
+ \frac{\nu}{4} \|\nabla g\|_{L^2} + \frac{\nu \eta}{4} \|g\|_{L^2}^2. \quad (5.10)
\]

Then, choosing \( \bar{\epsilon} = \epsilon = 1 \), using (5.2), hypothesis (5.2), \( M_1 \) and \( M_2 \) as given in Corollary 4.2, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|g\|^2_{L^2} + \frac{\eta}{8} \|g\|_{L^2}^2 \leq \frac{64\bar{a}^2 C_0^4 |\alpha - \beta|^2 \nu^2 (\eta^2 + 2 \nu)}{(2 - \max\{\alpha, \beta\})^2} \left( M_2 + \frac{1}{t^2} M_1 \right)^5 \\
+ \frac{32\bar{a}^2 |\alpha - \beta|^2 \nu^2}{\eta^2} M_1 \\
+ 2 C_6^{10} \left| \bar{a} - \bar{b} \right| \frac{\nu^2 (\eta^2 + 2 \nu)}{\eta \nu^2} \left( M_2 + \frac{1}{t^2} M_1 \right)^5 + 2 \left| \bar{a} - \bar{b} \right| \nu^2 \frac{\nu^2}{\eta \nu^2} M_1, \forall t \geq 0. \quad (5.11)
\]

By Gronwall’s inequality, we conclude the inequality stated in Theorem 3.1 in \( H \)-norm, for the case \( 1 < \alpha, \beta < 2 \).
6 Proof of Theorem 3.4

We add a convenient term in inequality (5.3) and get
\[
|\eta(Ih(g) - g, g)_L^2| \leq \eta \tilde{\epsilon} ||Ih(g) - g||^2_L^2 + \frac{3\eta}{8} ||g||^2_L^2 \leq \eta \epsilon, \quad \eta \epsilon, \quad \eta \epsilon, \quad \eta \epsilon.
\]

Choose now \( \epsilon = \frac{1}{4} \) and \( \tilde{\epsilon} = \frac{\epsilon^3}{8\max(\alpha, \beta)} \) in inequalities (5.3)-(5.4), \( \tilde{\epsilon} = \epsilon = \frac{3}{4\nu} \) in (5.9)-(5.10) and \( \tilde{\epsilon} = 1 \) in (6.1). By (5.2), hypotheses (3.2) and division by \( t^2 \), we obtain
\[
\frac{1}{2} \frac{d}{dt} ||g||^2_L^2 \leq \left[ \frac{3\eta}{8} + \frac{\nu}{2t^2} \right] ||g||^2_L^2 + \eta \epsilon \frac{h^2}{t^2} ||Ag||^2_L^2 + \left| \frac{\alpha - \beta}{2} \right|^2 \frac{128\tilde{\epsilon}^2 C_0^2}{(2 - \max(\alpha, \beta))^2} \frac{L^2}{\nu'} \left( ||\nabla u||^2_L^2 + \frac{1}{t^2} ||u||^2_L^2 \right)^5 \right] + \frac{32\tilde{\epsilon}^2 \nu}{t^2} ||u||^2_L^2.
\]

By multiplying the system (5.1) by \( Ag \), integrating over \( \Omega \), we have
\[
\frac{1}{2} \frac{d}{dt} ||g||^2_L^2 + \nu ||Ag||^2_L^2 + b \left( ||w||^{2\beta} - ||u||^{2\beta} \right) ||Ag||^2_L^2 = \tilde{\beta} \left[ \frac{l}{\nu} ||u||^{2\beta} - \frac{l}{\nu} ||u||^{2\beta} \right] + (w \cdot \nabla u, Ag)_{L^2} - (w \cdot \nabla g, Ag)_{L^2} - \eta ||Ih(g) - g, Ag||_{L^2} - \eta ||\nabla g||^2_L^2.
\]

In order to estimate the terms of (6.3), we use (1.5), (2.4), (2.6) and Hölder and Young’s inequalities:
\[
|\eta(Ih(g) - g, Ag)_{L^2}| \leq \eta \left( \sqrt{c_0 h^2 ||g||^2_L^2 + c_1 h^4 ||Ag||^2_L^2} \right) ||Ag||_{L^2} \leq \frac{8\eta c_0 h^2}{\nu} ||g||^2_L^2 + \frac{8\eta c_1 h^4}{\nu} ||Ag||^2_L^2 + \frac{\nu}{32} ||Ag||^2_L^2;
\]
\[
||(w \cdot \nabla u, Ag)_{L^2}|| \leq ||g||_{L^\infty} ||\nabla u||_{L^2} ||Ag||_{L^2} \leq C_\infty \left( ||\nabla g||^2_L^2 + \frac{l}{t^2} ||u||_{L^2} \right) ||\nabla u||_{L^2} ||Ag||_{L^2} = C_\infty ||\nabla g||^2_L^2 ||Ag||^2_L^2 + C_\infty \frac{1}{l^2} ||g||_{L^2} ||\nabla u||_{L^2} ||Ag||_{L^2} \leq \frac{432}{l^3} C_\infty^2 ||\nabla g||^2_L^2 ||\nabla u||^2_L^2 + \frac{4}{\nu} C_\infty^2 ||g||^2_L^2 ||\nabla u||^2_L^2.
\]
\[
\|(w \cdot \nabla g, Ag)_{L^2}\| \leq \|w\|_{L^6}\|\nabla g\|_{L^2}\|Ag\|_{L^2} \\
\leq C_6 C_3 \left( \|\nabla w\|_{L^2} + \frac{1}{\beta} \|w\|_{L^2} \right) \left( \|\nabla g\|_{L^2} \frac{1}{\beta} \|Ag\|_{L^2} + \frac{1}{\beta^2} \|\nabla g\|_{L^2} \right) \|Ag\|_{L^2} \\
= C_6 C_3 \left( \|\nabla w\|_{L^2} + \frac{1}{\beta} \|w\|_{L^2} \right) \frac{1}{\beta^2} \|\nabla g\|_{L^2} \|Ag\|_{L^2} \\
+ C_6 C_3 \left( \|\nabla w\|_{L^2} + \frac{1}{\beta} \|w\|_{L^2} \right) \|\nabla g\| \frac{1}{\beta^2} \|Ag\|_{L^2}^2 \\
\leq \frac{8}{\nu} C_6^2 C_3^2 \left( \|\nabla w\|_{L^2} + \frac{1}{\beta^2} \|w\|_{L^2} \right) \frac{1}{\beta^2} \|\nabla g\|_{L^2}^2 + \frac{12^3}{\nu} C_6^4 C_3^4 \left( \|\nabla w\|_{L^2} + \frac{1}{\beta^2} \|w\|_{L^2} \right)^2 \|\nabla g\|_{L^2}^2 \\
+ \frac{\nu}{8} \|Ag\|_{L^2}^2.
\]

Since \(1 < \beta < 2\), denoting \(J_{10} = b \|(w)^{2\beta} w - |u|^{2\beta} u, Ag)_{L^2}\|\) and using (2.4) and (2.6), we get

\[
J_{10} \leq |b(\beta)| \int_{\Omega} |g| |Ag|(|u| + |v|)^{2\beta} dx \leq |b(\beta)| \|g\|_{L^6} \|Ag\|_{L^2} \|(u + w)^{2\beta}\|_{L^3} \\
\leq 8 \|b(\beta)| C_6 C_6^{2\beta} \left( \|\nabla g\|_{L^2} + \frac{1}{\beta} \|g\|_{L^2} \right) \|Ag\|_{L^2} \left( \|\nabla (u + w)^{\frac{1}{\beta}}\|_{L^2} \|Ag\|_{L^2} \left( \frac{1}{\beta^2} \|u + w\|_{L^2}^{2\beta} + \|\nabla (u + w)^{\frac{1}{\beta}}\|_{L^2} \|Ag\|_{L^2}^{2\beta} \right) \right) \\
\leq 8 \|b(\beta)| C_6 C_6^{2\beta} \left( \|\nabla g\|_{L^2} + \frac{1}{\beta} \|g\|_{L^2} \right) \|Ag\|_{L^2} \left( \frac{1}{\beta^2} \|u + w\|_{L^2}^{2\beta} + \|\nabla (u + w)^{\frac{1}{\beta}}\|_{L^2} \|Ag\|_{L^2}^{2\beta} \right) \\
\leq 8 \|b(\beta)| C_6 C_6^{2\beta} \left( \|\nabla g\|_{L^2} + \frac{1}{\beta} \|g\|_{L^2} \right) \|Ag\|_{L^2} \left( \frac{1}{\beta^2} \|u + w\|_{L^2}^{2\beta} + \|\nabla (u + w)^{\frac{1}{\beta}}\|_{L^2} \|Ag\|_{L^2}^{2\beta} \right) \\
\leq \frac{212}{\nu} |b|^2 \kappa^2 (\beta) C_6^2 C_6^{4\beta} \left( \|\nabla g\|_{L^2} + \frac{1}{\beta^2} \|g\|_{L^2} \right) \left( \frac{1}{\beta^2} \|u + w\|_{L^2}^{4\beta} + \|\nabla (u + w)^{\frac{1}{\beta}}\|_{L^2} \|Ag\|_{L^2}^{4\beta} \right) \\
+ \left( \frac{16}{\nu} \right) \frac{212}{\nu} |b|^2 \kappa^2 (\beta) C_6^2 C_6^{4\beta} \left( \|\nabla g\|_{L^2} + \frac{1}{\beta^2} \|g\|_{L^2} \right) \|\nabla (u + w)\|_{L^2}^{2\beta} + \frac{\nu}{8} \|Ag\|_{L^2}^2.
\]
Finally, we have
\[
\left\| (\tilde{a} - \tilde{b}) \frac{\nu}{L^2} \left( \frac{l}{\nu} \right)^{2\beta} u, Ag \right\|_{L^2} \leq |\tilde{a} - \tilde{b}| \frac{l^2}{\nu^2} \|u\|_{L^2(t)}^2 + \|Ag\|_{L^2} \|u\|_{L^2(t)}
\]
\[
\leq 2^{11} |\tilde{a} - \tilde{b}| \frac{l^2}{\nu^2} C^{10}_4 \left( \|\nabla u\|_{L^2}^2 \|u\|_{L^2(t)}^2 + \frac{1}{l^2} \|u\|_{L^2}^2 \right) + 4 |\tilde{a} - \tilde{b}| \frac{\nu}{L^2} \|u\|_{L^2}^2 + \frac{\nu^2}{8} \|Ag\|_{L^2}^2.
\]

Considering \(M_1\) and \(M_2\) given in Corollary 4.2, we define the following quantities:

\[
Z_1(t) = 432C^1_6 \frac{1}{l^2} M_2 + 4C^2_6 \frac{1}{l^2} M_2 + 8C^2_6 C^2_3 \frac{1}{l^2} \left( \|\nabla w(t)\|_{L^2}^2 + \frac{1}{l^2} \|w(t)\|_{L^2}^2 \right)
\]
\[
+ 12^2 C^2_6 C_6^3 \frac{1}{l^2} \left( \|\nabla w(t)\|_{L^2}^2 + \frac{1}{l^2} \|w(t)\|_{L^2}^2 \right) + 2^{14} \kappa^2 (2\beta) C^2_6 C^4_6 \frac{b^2}{l^2} \left( M_{1,2}^{2\beta} + \|w(t)\|_{L^2}^{2\beta} \right)
\]
\[
+ 2^{24+2\beta} \kappa^2 (2\beta) C^2_6 C^4_6 \frac{b^2}{l^2} \left( M_{2,2}^{2\beta} + \|w(t)\|_{L^2}^{2\beta} \right)
\]
\[
+ 2^{24+2\beta} \kappa^2 (2\beta) C^2_6 C^4_6 \frac{b^2}{l^2} \left( \|\nabla g(t)\|_{L^2}^2 + \frac{1}{l^2} \|g(t)\|_{L^2}^2 \right)
\]
\[
+ \frac{\nu}{2L^2};
\]

\[
Z_2(t) = 2^{12} \kappa^2 (2\beta) C^2_6 C^4_6 \frac{l}{l^2} \|Ag(t)\|_{L^2}^2;
\]

\[
Z_3 = \frac{64\tilde{a}^2 \nu}{e^2} \frac{1}{l^2} M_1 + \frac{2^{15} \tilde{a}^2 (C^1_4 + C^{10}_4) l^2}{(2\max{\{\alpha, \beta\}})^2 \nu^2} \left( M_2 + \frac{1}{l^2} M_1 \right)^5;
\]

\[
Z_4(t) = \frac{2^{14} \tilde{a}^2 C^{10}_4 l^4}{(2\max{\{\alpha, \beta\}})^2 \nu^2} \|Ag(t)\|_{L^2}^2;
\]
\[
Z_5 = 6\frac{\nu}{l^2}M_1 + 2^{12}C_6^{10} + C_1^{10}\frac{l^2}{\nu} \left( M_2 + \frac{1}{l^2}M_1 \right)^5; \quad (6.14)
\]

\[
Z_6(t) = 2^{11}C_1^{10} \frac{l^4}{\nu^2}M_2^4 \| Au(t) \|_{L^2}^2. \quad (6.15)
\]

Then, taking into account (6.2), (6.3), (6.4), (6.5), (6.6), (6.7), (6.8) and (6.9) yield
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla g(t)\|_{L^2}^2 + \frac{1}{l^2} \| g(t) \|_{L^2}^2 \right) \leq \left( \|\nabla g(t)\|_{L^2}^2 + \frac{1}{l^2} \| g(t) \|_{L^2}^2 \right) \left( -\frac{3\eta}{8} + \frac{8\nu^2r_0h^2}{\nu} + Z_1(t) + Z_2(t) \right) + |\alpha - \beta|^2 (Z_3 + Z_4(t)) + |\tilde{\alpha} - \tilde{\beta}|^2 (Z_5 + Z_6(t)), \quad (6.16)
\]

for all \( t \geq 0 \). Besides, note that
\[
\|\nabla g(0)\|_{L^2}^2 + \frac{1}{l^2} \| g(0) \|_{L^2}^2 \leq 2 \|\nabla u(0)\|_{L^2}^2 + \frac{2}{l^2} \| u(0) \|_{L^2}^2 + 2 \|\nabla w(0)\|_{L^2}^2 + \frac{2}{l^2} \| w(0) \|_{L^2}^2 \leq 4M.
\]

Next, consider
\[
H = M_2 + \frac{1}{l^2}M_1 + e^{23\kappa^2(2\beta)}C_6^{12}C_6^{4\beta} \frac{1}{l^2}M_2 \left\{ 4M \right\}
\]
\[
+ 2|\alpha - \beta|^2 \left[ \frac{23}{\eta} \left[ \frac{64\nu^2}{\mu}M_1 + \frac{2^{15}\nu^2(C_6^{12} + C_1^{10}) l^2}{(2 - \max\{\alpha, \beta\})^2 \nu^2} \left( M_2 + \frac{1}{l^2}M_1 \right)^5 \right] \right.
\]
\[
+ \frac{2^{14}\nu^2C_1^{10}}{(2 - \max\{\alpha, \beta\})^3 \nu^2} M_2^4 \left( \frac{4}{\nu}M_2 + \frac{16}{\eta\nu^{\frac{\nu}{2(\alpha - \beta)}}} M_2 + \frac{32}{\eta\nu^2} \| f \|_{L^2}^2 L^2 \right) \right]
\]
\[
+ 2|\tilde{\alpha} - \tilde{\beta}|^2 \left[ \frac{23}{\eta} \left[ \frac{6\nu^2}{\mu}M_1 + 2^{12}(C_6^{10} C_1^{10}) l^2 \left( M_2 + \frac{1}{l^2}M_1 \right)^5 \right] \right.
\]
\[
+ \frac{2^{11}C_1^{10}}{\nu^2} M_2^4 \left( \frac{4}{\nu}M_2 + \frac{16}{\eta\nu^{\frac{\nu}{2(\alpha - \beta)}}} M_2 + \frac{32}{\eta\nu^2} \| f \|_{L^2}^2 L^2 \right) \right\}. \quad (6.17)
\]

and
\[
T^* = \sup \left\{ t^* > 0; \|\nabla g(t)\|_{L^2}^2 + \frac{1}{l^2} \| g(t) \|_{L^2}^2 \leq H, \forall 0 \leq t \leq t^* \right\}. \quad (6.18)
\]

Then,
\[
\|\nabla w(t)\|_{L^2}^2 + \frac{1}{l^2} \| w(t) \|_{L^2}^2 \leq 2 \|\nabla g(t)\|_{L^2}^2 + \frac{1}{l^2} \| g(t) \|_{L^2}^2 + 2 \|\nabla u(t)\|_{L^2}^2 + \frac{1}{l^2} \| u(t) \|_{L^2}^2 \leq 2H + 2M_2 + \frac{1}{l^2}M_1 \leq 4H,
\]

for all \( 0 \leq t < T^* \). Defining
\[
\check{Z}_1 = 432C_6^{11} M_2^2 + 4C_6^{12} \frac{1}{\nu} M_2 + 8C_6^{12} C_6^{2\beta} \frac{1}{l^2} \nu H
\]
\[
+ 2^{\frac{3\beta}{1-2\beta}} \kappa^2(2\beta) C_6^{2\beta} C_6^{2\beta} \frac{1}{l^2} \nu H \left( M_2^{2\beta} + (l^2H)^{2\beta} \right)
\]
\[
+ 2^{\frac{7\beta}{1-2\beta}} \kappa^2(2\beta) C_6^{2\beta} C_6^{2\beta} \frac{1}{l^2} \nu H \left( M_2^{4\beta} + (4H)^{4\beta} \right) \quad (6.19)
\]
\[
+ 2^{\frac{7\beta}{1-2\beta}} \kappa^2(2\beta) C_6^{2\beta} C_6^{2\beta} \frac{1}{l^2} \nu H \left( M_2^{4\beta} + (4H)^{4\beta} \right) + \frac{\nu}{2l^2},
\]

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Therefore
\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla g(t) \|_{L^2}^2 + \frac{1}{T^2} \| g(t) \|_{L^2}^2 \right) \leq \left( \| \nabla g(t) \|_{L^2}^2 + \frac{1}{T^2} \| g(t) \|_{L^2}^2 \right) \left( -\frac{3\eta}{8} + \frac{8\eta^2 c_0 h^2}{\nu} + \tilde{Z}_1 + Z_2(t) \right) + |\alpha - \beta|^2 (Z_3 + Z_4(t)) + |\tilde{a} - \tilde{b}|^2 (Z_5 + Z_6(t)),
\]
for all \(0 \leq t < T^*\). Thus, using Gronwall's inequality,
\[
\| \nabla g(t) \|_{L^2}^2 + \frac{1}{T^2} \| g(t) \|_{L^2}^2 \leq e^\left( -\frac{3\eta}{8} + \frac{8\eta^2 c_0 h^2}{\nu} + \tilde{Z}_1 \right) \int_0^t e^\left( -\frac{3\eta}{8} + \frac{8\eta^2 c_0 h^2}{\nu} + \tilde{Z}_1 \right) (t-r) ds \left( \| \nabla g(0) \|_{L^2}^2 + \frac{1}{T^2} \| g(0) \|_{L^2}^2 \right) (Z_3 + Z_4(r)) dr + 2|\alpha - \beta|^2 \int_0^t e^\left( -\frac{3\eta}{8} + \frac{8\eta^2 c_0 h^2}{\nu} + \tilde{Z}_1 \right) (t-r) ds \left( \| \nabla g(0) \|_{L^2}^2 + \frac{1}{T^2} \| g(0) \|_{L^2}^2 \right) (Z_5 + Z_6(r)) dr,
\]
0 \leq t < T^*. Using again estimates given in Corollary [22] we have
\[
\left( -\frac{3\eta}{8} + \frac{8\eta^2 c_0 h^2}{\nu} + \tilde{Z}_1 \right) (t-r) + \int_r^t Z_2(s) ds \leq \left( -\frac{3\eta}{8} + \frac{8\eta^2 c_0 h^2}{\nu} + \tilde{Z}_1 + 2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3 \right) (t-r) + 2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_2.
\]
Since, by hypothesis,
\[
-\frac{3\eta}{8} + \frac{8\eta^2 c_0 h^2}{\nu} + \tilde{Z}_1 + 2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3 < -\frac{\eta}{8},
\]
it follows that
\[
\| \nabla g(t) \|_{L^2}^2 + \frac{1}{T^2} \| g(t) \|_{L^2}^2 \leq e^{-\frac{\eta}{8} + 2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3} \left( \| \nabla g(0) \|_{L^2}^2 + \frac{1}{T^2} \| g(0) \|_{L^2}^2 \right) + 2|\alpha - \beta|^2 \int_0^t e^{-\frac{\eta}{8} + 2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3} (Z_3 + Z_4(r)) dr + 2|\alpha - \beta|^2 \int_0^t e^{-\frac{\eta}{8} + 2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3} (Z_5 + Z_6(r)) dr \leq e^{-\frac{\eta}{8} + 2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3} \left( \| \nabla g(0) \|_{L^2}^2 + \frac{1}{T^2} \| g(0) \|_{L^2}^2 \right) + |\alpha - \beta|^2 \frac{2^4}{\eta} e^{2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3} Z_3 + 2|\alpha - \beta|^2 \frac{2^4}{\eta} e^{2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3} Z_5 + 2|\alpha - \beta|^2 \frac{2^4}{\eta} e^{2^{13}\kappa^2 (2\beta) C_6^2 C_6^{\frac{4}{5}} \frac{l}{\nu^2} M_3} Z_6,\]
Therefore, the result asserted in Theorem 3.4 follows from (6.25).
7 Proof of Theorem 3.6

In the proof of Theorem 3.4 we obtained that $T^* = \infty$. Then, using Gronwall in (6.20), we obtain

$$\|\nabla g(t)\|_{L^2}^2 + \frac{1}{t^2} \|g(t)\|_{L^2}^2 \leq e^{-\frac{3\eta}{2} (t - \frac{2\eta}{\nu})} \left( \frac{2\eta}{\nu} \right)^\frac{3}{4} M_2 \left( \left\| \nabla g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 + \frac{1}{t^2} \left\|g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 \right)$$

for all $t \geq \frac{2\eta}{\nu}$, where $Z_2, Z_3, Z_4, Z_5, Z_6$ and $\tilde{Z}_1$ are given in (6.14)-(6.15) and (6.19). Then, using (6.21) and (6.22), we have

$$\|\nabla g(t)\|_{L^2}^2 + \frac{1}{t^2} \|g(t)\|_{L^2}^2 \leq e^{-\frac{3\eta}{2} (t - \frac{2\eta}{\nu})} + 2^{13} \int_{\frac{2\eta}{\nu}}^t e^{-\frac{3\eta}{2} (t - \sigma)} ds \left( \left\| \nabla g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 + \frac{1}{t^2} \left\|g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 \right)$$

for all $t \geq \frac{2\eta}{\nu}$. By Lemma 4.4 we get

$$\int_{\frac{2\eta}{\nu}}^t e^{-\frac{3\eta}{2} (t - \sigma)} Z_4(\sigma) d\sigma = \frac{2^{14} a_2 C_{10}^0}{(2 - \max\{\alpha, \beta\})^2 \nu^2 M_2^2} \int_{\frac{2\eta}{\nu}}^t e^{-\frac{3\eta}{2} (t - \sigma)} \left\| A u(\sigma) \right\|_{L^2}^2 d\sigma \leq \frac{2^{14} a_2 C_{10}^0}{(2 - \max\{\alpha, \beta\})^2 \nu^2 M_2^2} \frac{t^4}{\eta^2} M_2^4 \frac{8}{\eta} M_2^2;$$

$$\int_{\frac{2\eta}{\nu}}^t e^{-\frac{3\eta}{2} (t - \sigma)} Z_5(\sigma) d\sigma = 2^{11} C_{10}^0 \frac{t^4}{\nu^2} M_2^4 \int_{\frac{2\eta}{\nu}}^t e^{-\frac{3\eta}{2} (t - \sigma)} \left\| A u(\sigma) \right\|_{L^2}^2 d\sigma \leq 2^{11} C_{10}^0 \frac{t^4}{\nu^2} M_2^4 \frac{1}{\eta} M_2^2.$$

Consequently,

$$\|\nabla g(t)\|_{L^2}^2 + \frac{1}{t^2} \|g(t)\|_{L^2}^2 \leq e^{2^{13} C_{10}^{44} \frac{4}{\nu^2} M_2} \left( e^{-\frac{3\eta}{2} (t - \frac{2\eta}{\nu})} \left\| \nabla g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 + \frac{1}{t^2} \left\|g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 \right)$$

$$+ 2^{13} \int_{\frac{2\eta}{\nu}}^t e^{-\frac{3\eta}{2} (t - \sigma)} \left\| A u(\sigma) \right\|_{L^2}^2 d\sigma \leq e^{2^{13} C_{10}^{44} \frac{4}{\nu^2} M_2} \left( e^{-\frac{3\eta}{2} (t - \frac{2\eta}{\nu})} \left\| \nabla g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 + \frac{1}{t^2} \left\|g \left( \frac{2\eta}{\nu} \right) \right\|_{L^2}^2 \right)$$

$$+ 2^{13} \int_{\frac{2\eta}{\nu}}^t e^{-\frac{3\eta}{2} (t - \sigma)} \left\| A u(\sigma) \right\|_{L^2}^2 d\sigma.$$
for all \( t \geq \frac{2\ell^2}{\nu} \). Let \( B \) be given in (6.26), and

\[
\begin{align*}
\tilde{C} &= \frac{2B}{\eta} \left\{ 2^3 \left[ 6\alpha^2 \frac{\nu}{\ell^4} M_1 + \frac{2\alpha^2 (C_{10} + C_{12})}{(2 - \max\{\alpha, \beta\})^2 \nu^2} \left( M_2 + \frac{1}{\ell^2} M_1 \right)^5 \right] \\
&\quad + \frac{2^{14}r^2 C_{10}^4}{(2 - \max\{\alpha, \beta\})^2 \nu^2} \frac{1}{\nu^4} \left( M_2 M_8 \right) \right\}, & (7.2)
\end{align*}
\]

\[
\begin{align*}
\tilde{D} &= \frac{2B}{\eta} \left\{ 2^3 \left[ 6 \nu^2 \frac{1}{\ell^4} M_1 + 2^{12} (C_{6} + C_{10}) \nu^2 \left( M_2 + \frac{1}{\ell^2} M_1 \right)^5 \right] \\
&\quad + 2^{11} C_{10}^4 \frac{1}{\nu^4} \left( M_2 M_8 \right) \right\}. & (7.3)
\end{align*}
\]

Finally, the result stated in Theorem 3.6 follows from (7.1).

8 Conclusion

In this paper, we proved that under suitable conditions, it is possible to approximate physical solutions of the three-dimensional Brinkman-Forchheimer-extended Darcy model even with the parameters related to the damping term unknown. Computational experiments, as well as analysis of algorithms to recovery the physical parameters \( \alpha \) and \( a \) based in continuous data assimilation techniques will be subject of forthcoming work.

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