RUNNING AFTER A NEW K"AHLER-EINSTEIN METRIC

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Abstract. We deal with compact K"ahler manifolds $M$ which are acted on by a semisimple compact Lie group $G$ of isometries with codimension one regular orbits. We provide an explicit description of the standard blow-ups of such manifolds along complex singular orbits, in case $b_1(M) = 0$ and the regular orbits are Levi non-degenerate. Up to very few exceptions, all the nonhomogeneous manifolds in this class are shown to admit a $G$-invariant K"ahler-Einstein metric, giving completely new examples of compact K"ahler-Einstein manifolds.

1. Introduction.

We consider the class of compact K"ahler manifolds $M$ with the following two properties: (a) the first Betti number $b_1(M) = 0$; (b) a compact semisimple Lie group $G$ of (holomorphic) isometries of $M$ acts with codimension one regular orbits. Such manifolds, which we will call $K$-manifolds throughout the following, have been already considered by several authors: many facts on the structure of $K$-manifolds have been so far discovered and successfully used to provide interesting new examples of K"ahler-Einstein manifolds (see amongst others [HS], [Sa], [KS], [DW], [PS], [GC]).

Besides some new general results on $K$-manifolds, which are contained in first sections of this paper, we may summarize our main result in the following Theorem.

Main Theorem. Let $M$ be a non-homogeneous, $K$-manifold acted on by the semisimple Lie group $G$, with only one complex singular $G$-orbit $S$.

If $S$ has complex codimension one and if the regular $G$-orbits are Levi non-degenerate then:

(1) $M$ is a $G$-homogeneous holomorphic bundle $G \times_{G_Q, \rho} F$ over the flag manifold $G/G_Q$, where $G$, $F$, $G_Q$ and the group $Q = \rho(G_Q)$, given by the action $\rho$ of $G_Q$ on the fiber $F$, are as in following table:

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when $F$ is the complex quadric $Q^r = SO_{r+2}/SO_2 \times SO_r$, the group $Q = SO_{r+1}$ acts as the standard subgroup of $SO_{r+2}$; when $F$ is the projective space $\mathbb{C}P^r = SU_{r+1}/S(U_1 \times U_r)$, the group $Q = SO_{r+1}/\text{Center}$ acts as the standard subgroup of $PSU_{r+1}$;

(2) if $M$ is one of the manifolds described in the Table, with the exception of case 1 with $G = SU_3$ and $F = \mathbb{C}P^2$, case 2 with $G = SU_p \times SU_2, p > 2$, case 4 with $F = Q^7$ and case 5 with $F = Q^9$, then it is Kähler-Einstein with positive first Chern class.

We remark that we can always suppose, up to blow up, that the complex singular orbits of a K-manifold are of complex codimension one; moreover the case when both singular orbits of a K-manifold are complex can be completely described along the lines developed in [PS] (see also [Sp]).

At the best of our knowledge all these manifolds are new examples of nonhomogeneous Kähler-Einstein manifolds, with the only exception of the manifold listed in n.1 with $F = Q^2$, which has been first discussed by Guan and Chen in [GC]. It remains to be checked which of these manifolds admits a Kähler-Einstein blow down; the present paper already contains several results which could be used for such further investigation.

The paper is organized as follows. In section 2 we review some basic facts on K-manifolds and compact Levi nondegenerate homogeneous CR-manifolds. In section 3, starting from known results in [HS] and [PS], we give a fine description of the canonical blow-up of a K-manifold in Theorem 3.1, while in Corollary 3.5 we give the full list of such blow-ups when the regular $G$-orbits are Levi nondegenerate.

In section 4 we describe a generic $G$-invariant Kähler metric in terms of a suitable curve in the Lie algebra of $G$ and we write down the Einstein equation for a $G$-invariant Kähler metric for any K-manifold. When the regular $G$-orbits are Levi nondegenerate and there exists only one complex singular $G$-orbit of complex codimension one, we also give necessary and sufficient conditions in order that a $G$-invariant Kähler-Einstein metric on the regular part of $M$ extends as a smooth metric on the whole $M$ (Theorem 4.2).
We conclude the paper with section 5, where we prove the existence of a $G$-invariant Kähler-Einstein metric on each of the manifolds listed in the main Theorem; this is achieved proving the existence of a solution of the Einstein equation with the appropriate boundary conditions determined in section 4. The proof of this last fact has been inspired by the methods used by Guan and Chen in their paper [GC].

At the moment the authors are not able to see whether the condition of Levi non degeneracy is essential for the existence of Kähler-Einstein metric on K-manifolds with only one complex singular orbit of complex codimension one. We also stress the fact that our theorem does not state that every excluded case does not admit a Kähler-Einstein metric.

As for notation, if $G$ is a Lie group acting isometrically on a Riemannian manifold $(M, g)$ and $X \in \mathfrak{g}$, we will adopt the symbol $\hat{X}$ to denote the corresponding Killing vector field on $M$.

The Lie algebra of a Lie group will be always denoted by the corresponding gothic letter. For a group $G$ and a Lie algebra $\mathfrak{g}$, $Z(G)$ and $\mathfrak{z}(\mathfrak{g})$ denote the center of $G$ and of $\mathfrak{g}$, respectively. For any subset $A$ of $G$ or of the Lie algebra $\mathfrak{g}$, $C_G(A)$ and $C_{\mathfrak{g}}(A)$ are the centralizer of $A$ in $G$ and $\mathfrak{g}$, respectively.

Finally, for any subspace $n \subset \mathfrak{g}$, $n^\perp$ denotes the orthogonal complement of $n$ in $\mathfrak{g}$ w.r.t. the Cartan-Killing form.

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2. Fundamentals of K-manifolds.

2.1 K-manifolds, KO-manifolds and KE-manifolds.

A $K$-manifold is a pair formed by a compact Kähler manifold $(M, J, g)$ and by a compact semisimple Lie group $G$ acting almost effectively and isometrically (and hence also biholomorphically) on $M$, such that:

i) $b_1(M) = 0$;

ii) $G$ acts by cohomogeneity one, i.e. the regular orbits of the $G$-action are of codimension one in $M$.

We will also use the notation $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$ for the Kähler fundamental form of $(M, g, J)$ and $\rho = r(\cdot, J\cdot)$ for the corresponding the Ricci form.

For the general properties of cohomogeneity one manifolds and of K-manifolds, see e.g. [AA], [AA1], [BR], [HS], [PS]. At this moment, we only need to recall the concept of normal geodesic.

If $p \in M$ is a regular point, let us denote by $L = G_p$ the corresponding isotropy subgroup. Since $M$ is orientable, every regular orbit $G \cdot p$ is orientable. Hence we may consider a unit normal vector field $\xi$, defined on the subset of regular points $M_{\text{reg}}$, which is orthogonal to every regular orbit. It is known (see [AA1]) that any integral curve of $\xi$ is a geodesic. Any geodesic $\gamma : \mathbb{R} \rightarrow M$ which is an integral curve of $\xi$ on $M_{\text{reg}}$ is called normal geodesic and it crosses every $G$-orbit orthogonally.
The following Proposition will be a basic tool for the sequel.

**Proposition 2.1.** [PS] Let \((M, J, g)\) be a K-manifold acted on by the compact semisimple Lie group \(G\). Let also \(p \in M_{\text{reg}}\) and \(L = G_p\) the isotropy subgroup at \(p\). Then:

1. there exists an element \(Z\) (determined up to scaling) so that
   \[
   \mathbb{R}Z \in C_g(\mathfrak{l}) \cap \mathfrak{l}^\perp, \quad C_g(\mathfrak{l} + \mathbb{R}Z) = \mathfrak{z}(\mathfrak{l}) + \mathbb{R}Z;
   \]  
   in particular, the connected subgroup \(K \subset G\) with subalgebra \(\mathfrak{k} = \mathfrak{l}^\perp + \mathfrak{a}\) is the isotropy subgroup of a flag manifold \(F = G/K\);

2. the dimension of the subspace \(\mathfrak{a} = C_g(\mathfrak{l}) \cap \mathfrak{l}^\perp\) is either 1 or 3; in case \(\dim_{\mathbb{R}} \mathfrak{a} = 3\), then \(\mathfrak{a}\) is a 3-dimensional subalgebra isomorphic to \(\mathfrak{su}_2\) and there exists a Cartan subalgebra \(\mathfrak{t} \subset \mathfrak{l}^\mathbb{C}\) so that \(\mathfrak{a}^\mathbb{C} = \mathbb{C}H_\alpha + \mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha}\) for some root \(\alpha\) of the root system of \((\mathfrak{g}^\mathbb{C}; \mathfrak{t}^\mathbb{C})\).

Note that if \(\dim_{\mathbb{R}} \mathfrak{a} = 1\) (or \(\dim_{\mathbb{R}} \mathfrak{a} = 3\)) at some regular point \(p\), then the same occurs at any other regular point. So we may consider the following definition.

**Definition 2.2.** Let \((M, J, g)\) be a K-manifold and \(L = G_p\) the isotropy subgroup of a regular point \(p\). We say that \(M\) is a K-manifold with ordinary action (or, shortly, KO-manifold) if \(\dim_{\mathbb{R}} \mathfrak{a} = \dim_{\mathbb{R}} (C_g(\mathfrak{l}) \cap \mathfrak{l}^\perp) = 1\).

In all other cases, we say that \(M\) is with extra-ordinary action (or, shortly, KE-manifold).

2.2 The structure of K-manifolds. Standard and non-standard K-manifolds.

The following is another crucial property of K-manifolds for studying their structure as G-manifolds. It is essentially a corollary of the results in [HS].

**Proposition 2.3.** Let \((M, J, g)\) be a K-manifold acted on by the compact semisimple Lie group \(G\). Then it has exactly two singular orbits, one of which is complex.

**Proof.** It is known that a compact cohomogeneity one manifold has either two singular orbits or no singular orbit at all. On the other hand, if there is no singular orbit, it is known that the orbit space \(\Omega = M/G\) is diffeomorphic to \(S^1\) (see e.g. [AA], [Br]). But this cannot occur, because \(b_1(M) = 0\).

Consider now an Iwasawa decomposition of \(G^\mathbb{C}\), so that we may write \(G^\mathbb{C} = G \cdot S\), where \(S\) is a solvable subgroup of \(G^\mathbb{C}\). We recall that, by compactness, the complexified Lie group \(G^\mathbb{C}\) acts on \(M\) with one open orbit and that, by a result of Ahiezer (see [Ah]), \(M\) is projective algebraic. From Borel’s fixed point theorem, it follows that there exists a point \(p_0\) which is fixed by \(S\). Then the G-orbit \(G \cdot p_0\) is complex since it coincides with \(G^\mathbb{C} \cdot p_0\) and it is singular by dimensional reasons. □

As we pointed out in the proof of Proposition 2.3, a K-manifold \(M\) is acted on by the complexified Lie group \(G^\mathbb{C}\) which has an open orbit. According to the definition of Huckleberry and Snow in [HS], \(M\) is called almost homogeneous.
In that paper, the authors give several information on a class of almost homogeneous manifolds, which contains all the K-manifolds.

In the following theorem, we collect some basic features and the information which are immediately implied by the results of Huckleberry and Snow.

In the following statement and in the rest of the paper, a fixed point for the $G$-action is considered as a singular complex orbit.

**Theorem 2.4.** Let $(M, J, g)$ be a K-manifold acted on by the compact semisimple Lie group $G$. Let also $p \in M_{\text{reg}}$ and $\Omega = G^C \cdot p = G^C / H$ be the open orbit.

1. There exists a unique K-manifold $\tilde{M}$ acted on by $G$, whose singular complex orbits have complex codimension one, and admitting a $G$-equivariant holomorphic map $\tilde{\pi} : \tilde{M} \to M$, which is a blow-down map along singular complex orbits.

2. If $M$ has two singular complex orbits, then $\tilde{M}$ can be $G$-equivariantly and holomorphically fibered onto a flag manifold $\tilde{\pi} : \tilde{M} \to G^C / P$,

$$ \tilde{\pi} : \tilde{M} \to G^C / P $$

where: a) the standard fiber is $\mathbb{CP}^1$; b) the isotropy of the flag manifold $G^C / P$ is the minimal parabolic subgroup $P \subset G^C$ which contains $H$; c) the intersection of $\Omega = G^C \cdot p = G^C / H$ with the fiber of $\tilde{\pi}$ is $\mathbb{C}^*$.

3. If $M$ has exactly one singular complex orbit, then $\tilde{M}$ can be $G$-equivariantly and holomorphically fibered onto a flag manifold

$$ \tilde{\pi} : \tilde{M} \to G^C / P $$

where:

a) the standard fiber is $\mathbb{CP}^n$, $Q^n = \{ [z] \in \mathbb{CP}^{n+1}, tzz = 0 \}$, $\mathbb{CP}^n \times \mathbb{CP}^n$, $G_{2,2m}(\mathbb{C})$ or $EIII = E_6 / \text{Spin}_{10} \times \text{SO}_2$;

b) the isotropy of the flag manifold $G^C / P$ is a minimal parabolic subgroup of $G^C$ containing $H$;

c) the intersection of $\Omega = G^C \cdot p = G^C / H$ with the fiber $P / H$ is $\mathbb{C}^n$, $Q^{(n)} = \{ [z] \in \mathbb{CP}^{n+1}, tzz = 1 \}$, $\mathbb{CP}^n \times \mathbb{CP}^n \setminus \{ [z], [w] \mid tzw = 0 \}$, $\mathbb{CP}^n \setminus Q^{n-1}$, $\text{Sp}_n(\mathbb{C}) / \text{Sp}_{n-1}(\mathbb{C})$ or $F_4(\mathbb{C}) / \text{Spin}_9(\mathbb{C})$.

In each of these cases, $P / H$ is the tangent space of a compact rank one symmetric space.

After the results in [HS], it is convenient to introduce the following definitions concerning the different types of K-manifolds.

**Definition 2.5.** For any K-manifold $M$, we will call the manifold $\tilde{M}$ defined in Theorem 2.4 (1) the canonical blow up of $M$.

We will say that a K-manifold $M$ is standard if it has two singular complex orbits and non-standard otherwise.

A K-manifold $M$ whose canonical blow up $\tilde{M}$ admits a $G$-equivariant holomorphic fibration $\pi : \tilde{M} \to Q = G^C / P$ onto a flag manifold with standard fiber $\mathbb{CP}^1$, will be called projectable.
Remarks 2.6.

(i) If \( \pi : \tilde{M} \to Q = G^C/P \) is a \( G \)-equivariant holomorphic fibration onto a flag manifold with fiber \( \mathbb{C}P^1 \), then \( P \) is a minimal parabolic subgroup containing \( H \subset G^C \).

(ii) In [PS], we introduced the concept of \( K \)-manifolds with projectable complex structure. It can be proved that such \( K \)-manifolds are necessarily standard and hence that being projectable is not the same of having projectable complex structure.

On the other hand, if we restrict ourselves to the class of \( K \)-manifolds with ordinary action, \( M \) is projectable if and only if it has projectable complex structure (see later).

From Definitions 2.2 and 2.5, the class of \( K \)-manifolds is naturally subdivided into four families: standard \( KO \)-manifolds and standard \( KE \)-manifolds, on one side, and non-standard \( KO \)-manifolds and non-standard \( KE \)-manifolds on the other side.

A complete description of the standard \( KO \)-manifolds has been reached in [PS]. An analogous description of standard \( KE \)-manifolds can be performed following the same line of arguments used in [PS].

2.3 The structural decomposition associated with the CR structure of a regular orbit.

We recall that a CR structure of codimension \( r \) on a manifold \( N \) is a pair \((D, J)\) formed by a distribution \( D \subset TN \) of codimension \( r \) and a smooth family \( J \) of complex structures \( J_x : D_x \to D_x \) on the spaces of the distribution. The CR structure \((D, J)\) is called integrable if the distribution \( D^{10} \subset T^C N \), given by \(+i\)-eigenspaces \( D^{10}_x \subset D_x^C \) of the complex structure \( J \) verifying

\[ [D^{10}, D^{10}] \subset D^{10}. \]

With this definition, we have that any complex structure \( J \) on \( N \) can be classified as integrable CR structure of codimension zero.

An integrable CR structure \((D, J)\) of codimension one is called Levi non-degenerate if the underlying distribution \( D \) is a contact distribution. This means that any local (contact) 1-form \( \theta \), which defines the distribution (i.e. such that \( \ker \theta = D \)) is maximally non-degenerate, that is \((d\theta)^n \wedge \theta \neq 0\).

A smooth map \( \phi : N \to N' \) between two CR manifolds \((N, D, J)\) and \((N', D', J')\) is called CR map (or holomorphic map) if: a) \( \phi_*(D) \subset D' \); b) for any \( x \in N \), \( \phi_*(J_x) = J_{\phi(x)} \circ \phi_*|_{D_x} \). A CR transformation of \((N, D, J)\) is a diffeomorphism \( \phi : N \to N \) which is also a CR map.

Any submanifold \( S \) of a CR manifold \((N, D, J)\) is endowed by the family of subspaces \( D_x^S \subset T_x S \) and a family of complex structures defined by

\[ D_x^S = \{ u \in (T_x S \cap D_x) : Jv \in (T_x S \cap D_x) \} \quad J_x = J|_{D_x^S}, \quad x \in S. \]

If \( D^S = \bigcup_{x \in S} D_x^S \) is a distribution, we call \((D^S, J)\) induced CR structure. Note that if \( S \) is a hypersurface of a complex CR manifold \((N, J)\), then \( D^S \) is always a distribution and \((D^S, J)\) is an integrable CR structure of codimension one.
Let \((G/L, D, J)\) be a homogeneous CR manifold of a compact semisimple Lie group \(G\) and assume that \(D\) is of codimension one. Then \(g\) has the \(B\)-orthogonal decomposition \(g = l + n\), where \(n\) can be identified with the tangent space \(T_{p_0}(G/L)\), \(p_0 = eL\), via the linear isomorphism

\[
\phi : n \to T_{p_0}(G/L), \quad \phi(X) = \hat{X}|_{p_0}.
\]

If we denote by \(m\) the subspace

\[
m = \phi^{-1}(D_{p_0}) \subset n,
\]

we get the following orthogonal decomposition of \(g\):

\[
g = l + n = l + \mathbb{R}Z_D + m.
\]

(2.3)

where \(Z_D \in (l + m)^\perp\). Notice that, since the decomposition is \(\text{ad}_l\)-invariant, we have that the element \(Z_D\) is always in \(C_g(l) \cap l^\perp\).

Using again the identification map \(\phi : n \to T_{p_0}(G/L)\), we may consider the complex structure

\[
J : m \to m, \quad J \overset{\text{def}}{=} \phi^*(J_{p_0}).
\]

(2.4)

Note that \(J\) is uniquely determined by the direct sum decomposition

\[
m^C = m^{10} + m^{01}, \quad m^{01} = \overline{m^{10}},
\]

(2.5)

where \(m^{10}\) and \(m^{01}\) are the \(J\)-eigenspaces with eigenvalues \(+i\) and \(-i\), respectively.

**Definition 2.7.** Let \((N = G/L, D, J)\) be a compact homogeneous CR manifold with an invariant CR structure \((D, J)\) of codimension one. Then:

a) we call the **structural decomposition of \(g\)** associated with \(D\) the orthogonal decomposition (2.3), with \(m \simeq D_{eL}\);

b) we call the **holomorphic (resp. anti-holomorphic) subspace associated with \((D, J)\)** the subspace \(m^{10} \subset m^C\) (respectively \(m^{01} = \overline{m^{10}}\)) defined by (2.5).

We also recall that a \(G\)-invariant CR structure \((D, J)\) on \(G/L\) is integrable if and only if the associated holomorphic subspace \(m^{10} \subset m^C\) is so that

\[
f^C + m^{10} \quad \text{is a subalgebra of} \quad g^C.
\]

(2.6)

We will refer to (2.6) as the **integrability condition for the holomorphic subspace** \(m^{10}\).

Let us now consider the regular orbits of a K-manifold. Note that if \(G/L = G \cdot p_0\) is a regular orbit of \(M\) and if \((D, J)\) is the induced CR structure on \(G/L\), then \((G/L, D, J)\) is a compact homogeneous CR manifold. Therefore any regular point \(p_0\) determines a structural decomposition for \(g = l + \mathbb{R}Z_D(p_0) + m(p_0), l = g_{p_0}\), and a holomorphic subspace \(m^{10}(p_0)\), which are those associated with the induced CR structure of \(G \cdot p_0 = G/L\).
3. Non-standard K-manifolds with Levi non-degenerate $G$-orbits.

3.1 The global structure of a non-standard K-manifold.

The first main result of this section is the proof of the following fact.

**Theorem 3.1.** Let $(M, J, g)$ be a non-standard K-manifold acted on by the compact semisimple Lie group $G$. Then the canonical blow-up $\tilde{M}$ is $G$-diffeomorphic to a manifold of the form $G \times_{G_Q} F$ where:

- a) $F$ is a $G_Q$-equivariant compactification of the tangent space $T(G_Q/N)$ of a non-trivial compact rank one symmetric space $G_Q/N$;
- b) $G/G_Q$ is a flag manifold;
- c) if $M$ is a KE-manifold, $G_Q$ is $SU_2$ and $F$ is either $\mathbb{CP}^2$ (= compactification of $T\mathbb{RP}^2 = T(\text{SO}_3/\text{O}_2)$) or $\mathbb{CP}^1 \times \mathbb{CP}^1$ (= compactification of $TS^2 = T(\text{SO}_3/\text{SO}_2)$); if $M$ is a KO-manifold, $F$ is the standard fiber of the $G$-equivariant holomorphic bundle $\tilde{\pi} : \tilde{M} \to G^C/P$ given in Theorem 2.4 (3).

In order to prove this theorem, we first need to make some observations. Let $\tilde{M}$ be the canonical blow-up of a non-standard K-manifold $M$ and let $\tilde{\pi} : \tilde{M} \to G^C/P$ the holomorphic projection given in Theorem 2.4 (3). It is clear that if the fiber $F$ is not equal to $\mathbb{CP}^1$ (that is $M$ is non-projectable), then the claim of Theorem 3.1 is immediately verified. Therefore, we prove Theorem 3.1 if we can show the following two facts:

i) a non-standard K-manifold $M$ is projectable only if it is a KE-manifold;
ii) if $M$ is a non-standard KE-manifold, then it admits a $G$-equivariant holomorphic fibration $\tilde{\pi}' : \tilde{M} \to G^C/P'$ with fiber equal to $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

The proof of these two facts will be the content of the remaining part of this section, where we prove Proposition 3.2 and Theorem 3.3. The content of Theorem 3.3 is practically the claim i). The claim ii) is an immediate consequence of the claims of Proposition 3.2 and of Theorem 3.3.

Let $M$ be a KE-manifold, $p$ a regular point and $L = G_p$. Consider also the connected component $F$ of the fixed point set $\text{Fix}(L)$ which passes through $p$. Note that $F$ is a closed, 4-dimensional, totally geodesic submanifold, that $T_pF$ is $J$-invariant and that it coincides with the fixed point set of the isotropy representation of $L$.

In particular, it follows that $F$ is a complex submanifold and hence Kähler. Moreover the group $A = \exp(\mathfrak{a}) \subset G$, where $\mathfrak{a} = C_{\g}(\mathfrak{l}) \cap \mathfrak{l}^\perp = \mathfrak{su}_2$, acts on $F$ by cohomogeneity one (see Proposition 2.1 (2)). If one could check that $b_1(F) = 0$, then we could conclude that $F$ is a K-manifold.

In the following Proposition, we show that this is actually true and we give a detailed description of all the possibilities for $F$ and for the action of $A$ on it.
Proposition 3.2. Let $M$ be a KE-manifold, $p$ a regular point and $F$ the connected component of $\text{Fix}(L)$ through $p$, where $L = G_p$.

Then $F$ is a 4-dimensional KE-manifold acted on by the compact semisimple Lie group $A \simeq (N_G(L)/L)^o$. In particular, the action $\rho$ of $A$ on $F$ is one of the following:

1. $\rho(A) = SU_2$, $F \cong CP^2 = SU_3/U_2$ and the action is the standard action of $SU_2$ given by the embedding $SU_2 \subset SU_3$.
2. $\rho(A) = SU_2$, $F$ is a non-trivial $CP^1$-bundle over $CP^1$ and the group acts on $F$ by bundle automorphisms.
3. $\rho(A) = SO_3$, $F \cong CP^2 = SU_3/U_2$ and the action is given by the standard embedding $SO_3 \subset SU_3$.
4. $\rho(A) = SO_3$, $F \cong CP^1 \times CP^1$ and $A$ acts diagonally on $F$.

In cases (1) and (2), $F$ is standard (actually, in case (1) one of the two singular complex orbits is a fixed point); in cases (3) and (4) $F$ is non-standard.

Proof. By the previous remarks, $F$ is a 4-dimensional Kähler manifolds acted on by the 3-dimensional Lie group $A$ with cohomogeneity one and and extra-ordinary action; note that $A$ is either $SU_2$ or $SO_3$. We have to show that $b_1(F) = 0$. Indeed we recall that the Albanese map $\alpha : F \rightarrow Alb(F)$ is equivariant; moreover $A^C$, being semisimple, acts trivially on $Alb(F)$. On the other hand, $A^C$ has an open orbit in $F$ and therefore $Alb(F) = \{0\}$, i.e. $b_1(F) = 0$.

We now show that only cases (1), (2), (3) and (4) may occur.

If $A$ has a fixed point, then by the Cone Theorem (see [HO]), $F \cong CP^2$ and case (1) occurs.

Suppose now that $F$ has no fixed point and that the canonical blow-up of $F$ coincides with $F$.

If $F$ is standard, by Theorem 2.4 (2), $F$ is a $CP^1$-bundle over $CP^1$ and the group $A = SU(2)$ acts on $F$ by bundle automorphisms. This bundle cannot be nontrivial, since otherwise it would be $A$-equivariantly biholomorphic to $CP^1 \times CP^1$ with the diagonal action of $A = SU_2$ and at least one singular orbit would not be complex.

Assume now that $F$ is non-standard. By Theorem 2.4 (3), there is an $A$-equivariant holomorphic bundle $F \rightarrow A^C/P$, with fiber equal to $CP^1$, $CP^2$ or $Q_2$. If the base $A^C/P$ is trivial, the possibilities for $F$ are either $CP^2$ or $Q_2 \cong CP^1 \times CP^1$. But the second case cannot occur because $Q_2$ admits an $A$-equivariant holomorphic fibration over the diagonal $CP^1$ and $F$ would not be the smallest parabolic subgroup containing $H$.

If the base $A^C/P$ is not trivial, then $F$ must be a $CP^1$-bundle over $CP^1$. Any such bundle is $A$-equivariantly diffeomorphic to a homogeneous bundle of the form

$E_k := SU_2 \times_{T^1, \rho_k} CP^1$,

for some $k \in \mathbb{N}$, where $T^1 \subset SU_2$ acts on $CP^1$ by means of the homomorphism

$\rho_k : T^1 \rightarrow SU_2$, \quad $\rho_k(e^{i\theta}) = \text{diag}(e^{ik\theta}, e^{-ik\theta})$.

Notice that for $k = 1$, $E_1$ is actually $CP^1 \times CP^1$, where $SU(2)$ acts diagonally.

The proof is concluded if we can show that actually the cases $k > 1$ do not occur. Given a singular point $q \in E_k$, we may suppose that the singular isotropy
at $q$ is given by $T^1$. Notice that the isotropy representation of $T^1$ at $q$ decomposes into the sum of the standard isotropy representation of $T^1$ on $SU(2)/T^1$ and the representation $\rho_k$ on $CP^1$. When $k > 1$, these two representations are not equivalent and hence the complex structure on $T_pE_k$ has to preserve the tangent space to the singular orbit $A \cdot q = SU(2)/T^1$. It follows that when $k > 1$ any singular orbit is complex and this contradicts our hypothesis. □

**Theorem 3.3.** Let $M$ be a non-standard $K$-manifold, $G$ the compact semisimple Lie group acting on $M$ and $L = G_p$, $p \in M$, a regular isotropy subgroup. Then:

a) $M$ is projectable only if the action of $G$ is extra-ordinary;

b) assume that $M$ is projectable: if $F$ is the connected component of $\text{Fix}(L)$ through $p$, then $F \cong CP^1 \times CP^1$; furthermore, $M$ fibers holomorphically and $G$-equivariantly onto a $G^C/\tilde{P}$, with $\tilde{P}$ parabolic in $G^C$ and with standard fiber equal to $F = CP^1 \times CP^1$.

**Proof.** a) We will show that if $M$ is projectable with ordinary action, then it is standard. With no loss of generality, we will assume that $M = \tilde{M}$, where $\tilde{M}$ is the canonical blow-up.

Let $\gamma$ be a normal geodesic through $p$ and $\pi : M \to G^C/P$ a $G$-equivariant holomorphic fibration onto a flag manifold, with fiber $CP^1$. Let also $K = P \cap G$, so that we may write $G^C/P = G/K$.

For any regular point $\gamma_t$, the fibration $\pi$ induces a CRF map $\tilde{\pi} : G \cdot \gamma_t = G/L \to G/K$ with fiber $K/L = S^1$. In particular, we have that $\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z$ for some $Z \in C_\mathfrak{g}(l) \cap l^\perp$.

On the other hand, by Prop. 2.1 in [PS], the moment map $\mu : M \to \mathfrak{g}^*$ induces a $G$-equivariant map from $G/L = G \cdot \gamma_t$ onto a flag manifold $G/K_t$

$$\mu_t : G/L = G \cdot \gamma_t \to G/K_t$$

where $L$ is of codimension one in $K_t$. In particular

$$\mathfrak{k}_t = \mathfrak{l} + \mathbb{R}Z_t$$

for some $Z \in C_\mathfrak{g}(l) \cap l^\perp$. Since $C_\mathfrak{g}(l) \cap l^\perp$ is 1-dimensional we conclude that $\mathfrak{k}_t = \mathfrak{k}$ and that $G/K_t = G/K$ for any $t$.

Furthermore, if $g = l + \mathbb{R}Z_D(t) + \mathfrak{m}_t$ is the structural decomposition associated with the induced CR structure of $G \cdot \gamma_t = G/L$, using once again the fact that $\dim \mathbb{R}C_\mathfrak{g}(l) \cap l^\perp = 1$, we conclude that $\mathfrak{k}_t = \mathfrak{k} = \mathfrak{l} + \mathbb{R}Z_D(t)$, for any regular point $\gamma_t$.

To conclude, consider the anti-holomorphic subspace $\mathfrak{m}^{01}_t$ of the orbit $G \cdot \gamma_t = G/L$. By Proposition 2.8 a) and b), we have that

$$\mathfrak{p} = \mathfrak{t}^C + \mathfrak{n} = (\mathfrak{t}^C + \mathbb{C}Z_D(t)) + \mathfrak{n}, \quad \mathfrak{t}^C + \mathfrak{m}^{01}_t \subset \mathfrak{p}$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. Since $\dim \mathbb{C}m^{01} = \dim \mathbb{C}n$ and they are both $\mathcal{B}$-orthogonal to $\mathfrak{t}^C + \mathbb{C}Z_D$, it follows that $\mathfrak{m}^{01} = \mathfrak{n}$. But $\mathfrak{n}$ is the nilradical and hence

$$[\mathfrak{k}, \mathfrak{m}^{01}] \subset \mathfrak{m}^{01}, \quad [\mathfrak{k}, \mathfrak{m}^{10}] = [\mathfrak{k}, \mathfrak{m}^{01}] \subset \mathfrak{m}^{10}.$$
This means that the induced CR structure \((D, J)\) of \(G \cdot \gamma_t\) is \(\text{ad}_F\)-invariant, that is it is \textit{projectable} in the sense of [PS]. The conclusion follows from the fact that in this case, by Prop. 4.1 in [PS], both singular orbits are complex.

b) First of all we prove that \(F\) has only one complex \(A\)-orbit. We recall that \(F\) contains a normal geodesic and that a singular \(A\)-orbit in \(F\) is contained in a singular \(G\)-orbit in \(M\). We will show that a complex \(A\)-orbit in \(F\) is contained in a complex \(G\)-orbit and therefore \(F\) has only one complex orbit since \(M\) is non-standard. Let \(p \in F\) such that \(A \cdot p\) is a complex orbit; we denote by \(N'\) and \(N\) the normal spaces to the \(A\)- and \(G\)-orbits respectively and by \(v \in N \subset N'\) the tangent vector of a normal geodesic through \(p\). Now let \(w\) be a unit vector in \(N\); since the isotropy \(G_p\) acts transitively on the unit sphere of \(N\), we may find \(g \in G_p\) with \(gw = v\) and since the normal space \(N'\) is complex, we may find \(h \in A_p\) such that \(hv = Jv\). Then \(Jw = gJv = ghv\), meaning that \(Jw \in N\) and therefore \(N\) is complex.

It follows from Proposition 3.1 that \(F\) has no fixed point and it is either \(F \cong \mathbb{C}P^1 \times \mathbb{C}P^1\) or \(F \cong \mathbb{C}P^2\). This implies that the canonical blow up of \(F\) coincides with \(F\) itself. Hence, we may again assume that \(M = \tilde{M}\) with no loss of generality.

Since \(M\) is projectable, there exists a \(G\)-equivariant holomorphic fibration \(\pi : M \to G^C/P = G/Q\) with fiber \(\mathbb{C}P^1\) onto the flag manifold \(G^C/P = G/Q\), where \(Q = G \cap P\); for any \(x \in M\) we denote by \(Z_x\) the fiber \(\pi^{-1}(\pi(x))\). Without loss of generality, we may assume that \(L \subset Q\) and since \(\dim Q/L = 1\), we may write the Lie algebra \(q\) of \(Q\) as \(q = l + \mathbb{R} \cdot a\) for some \(a \in \alpha\); this means that \(\hat{a}_p \in T_p F \cap T_p Z_p\) and since both \(F\) and the the fiber \(Z_p\) are complex, \(T_pZ_p \subset T_pF\). This argument applies actually to any point \(y\) along the normal geodesic through \(p\), which is contained in \(F\); by the \(G\)-equivariance of \(\pi\) we have that \(T_xZ_x \subset T_xF\) for all regular point \(x \in F\), hence for all \(x \in F\). This means that \(Z_x \subset F\) for all \(x \in F\). Therefore \(F \cong \mathbb{C}P^1 \times \mathbb{C}P^1\), since \(\mathbb{C}P^2\) is not the total space of a \(\mathbb{C}P^1\)-fibration.

Let us now consider the holomorphic subspace \(m^{10}\) associated with the induced CR structure of \(G/L = G \cdot p\) and put

\[ a^{10} = a^C \cap m^{10}, \quad a^{01} = \overline{a^{10}} = a^C \cap m^{01}. \]

Let also \(m^{01} = m^{10} \cap (a^{10})^\perp\).

Notice that the complex isotropy subalgebras \(h = \mathfrak{g}^C_p\) and \(h^a = \mathfrak{a}^C_p\) of the actions of \(G^C\) and \(A^C\), respectively, are equal to

\[ h = l^C + m^{01} = l^C + (a^{01} + m^{01}), \quad h^a = a^{01}. \]

Note that \(p = \text{Lie}(P)\) is a minimal parabolic subalgebra of \(\mathfrak{g}^C\) properly containing \(h\). Since \(F\) contains every fiber \(Z_x\) of \(p\) for \(x \in F\), we have that \(\dim \mathbb{C} p \cap a^C = 2\); moreover since \(a^{01} \subset p \cap a^C\) is generated by a regular element of \(a^C\), as it can be checked directly using the explicit action of \(A\) on \(F = \mathbb{C}P^1 \times \mathbb{C}P^1\), we conclude that \(p \cap a^C\) is a parabolic subalgebra of \(a^C\).

Since \(\dim \mathbb{C} p = \dim \mathbb{C} h + 1\), it follows that \(p\) is equal to

\[ p = (p \cap a^C) + m^{01}. \]
In particular \( m^{01} \subset p \). We now claim that the subspace
\[
\tilde{p} = p + a^c = t^c + a^c + m^{01}
\]
is also a parabolic subalgebra.

First of all, consider a Cartan subalgebra \( t^c \) of \( t^c \). By Proposition 2.1, for any regular element \( Z \in a^c \), the subalgebra \( t^c + \mathbb{R}Z \) is a Cartan subalgebra for \( g^c \). We already remarked that \( a^{01} \) is generated by a regular element of \( a^c = sl_2(\mathbb{C}) \) and therefore \( t^c = t^c_1 + a^{01} \subset p \) is a Cartan subalgebra for \( g^c \) included in \( p \).

Now, consider the root system \( R \) of \( g^c \) determined by \( t^c \) and denote by \( S \) and \( P \) the closed subsystems of \( R \) defined by
\[
S = \{ \beta \in R : E_\beta \in h \} , \quad P = \{ \beta \in R : E_\beta \in p \} .
\]
We recall that \( a^c = a^{01} + C \cdot E_\alpha + C \cdot E_- \alpha \) for some root \( \alpha \in R \). Since \( a^c \cap h = a^{01} \), it follows that \( \pm \alpha \notin S \). On the other hand, since \( P \) is a parabolic subsystem, for any root \( \gamma \in R \), either \( \gamma \) or \( -\gamma \) is in \( P \). Hence we may assume \( P = S \cup \{ \alpha \} \).

To prove that \( \tilde{p} = p + a^c \) is a parabolic subalgebra, we have only to check that the subset of \( \tilde{P} = S \cup \{ \alpha, -\alpha \} \) is closed. This reduces to show that \( (S + \{ -\alpha \}) \cap R \subset \tilde{P} \); suppose not, then there exists a root \( \beta \in S \) so that \( \beta - \alpha \in R \setminus \tilde{P} \). In particular \( \beta - \alpha \notin P \) and since \( P \) is parabolic, this implies that \( \alpha - \beta \in P \), actually \( \alpha - \beta \in S \).

But then \( \alpha = (\alpha - \beta) + \beta \in (S + S) \cap R \subset S \) and this is a contradiction.

If \( \tilde{P} \) denotes the parabolic subgroup of \( G^c \) with Lie algebra \( \tilde{p} \), then \( \tilde{M} \) fibers holomorphically and \( G \)-equivariantly onto \( N = G^c/\tilde{P} \), with a complex 2-dimensional fiber \( \mathcal{F} \). Since \( A \subset \tilde{P} \cap G \), we get that \( A \) acts almost effectively on \( \mathcal{F} \) and hence that \( F \cap \mathcal{F} \) is at least three dimensional. Since \( \mathcal{F} \) and \( F \) are both complex, we conclude that \( \mathcal{F} = F \). \( \square \)

### 3.2 Non-standard K-manifold with Levi non degenerate orbits.

We now reduce to consider only K-manifold with regular \( G \)-orbits, which are Levi non-degenerate. Notice that if one regular \( G \)-orbit is Levi non-degenerate, then all regular \( G \)-orbits are Levi non-degenerate.

The complete list of all simply connected, compact homogeneous manifolds \( G/L \) of a compact semisimple Lie group, which admit a \( G \)-invariant Levi non-degenerate integrable CR structure \((D, J)\) of codimension one, has been obtained in [AS] (see also [AHR]). According to the results in [AS], any such simply connected homogeneous CR manifold falls in one of the following three families:

a) \((G/L, D, J)\) is a homogeneous \( S^1 \)-bundles \( \pi : G/L \to F = G/K \) over a flag manifold \( F = G/K \) with invariant complex structure \( J_F \), and \((D, J)\) is the unique CR structure such that the map \( \pi \) is holomorphic;
b) \((G/L, D, J)\) is a sphere bundles \( G/L = S(N) \subset TN \) of a compact rank one symmetric space \( N = G/H \), with the CR structure \((D, J)\) is induced by the natural complex structure of \( TN = G^c/H^c \);
c) \(G/L\) is one of the following manifolds: \( SU_n/T^1 \cdot SU_{n-2}, SU_p \times SU^q/T^1 \cdot U_{p-2} \cdot U_{q-2}, SU_n/T^1 \cdot SU_2 \cdot SU_2 \cdot SU_{n-4}, SO_{10}/T^1 \cdot SO_6, E_6/T^1 \cdot SO_8 \); these
manifolds admit canonical holomorphic fibrations over a flag manifold \((F, J_F)\) with typical fiber \(S(S^k)\), where \(k = 2, 3, 5, 7, 9\) or \(11\), respectively; the CR structure is determined by the invariant complex structure \(J_F\) on \(F\) and by an invariant CR structure on the typical fiber, depending on one complex parameter.

From Theorem 2.4, Theorem 3.1 and the above quoted classification of compact homogeneous CR manifolds, the following Corollary is obtained.

**Corollary 3.5.** Let \((M, J, g)\) be a non-standard \(K\)-manifold with one regular \(G\)-orbit \(G/L = G \cdot x\), which is Levi non-degenerate. Then only one of the following cases may occur (in what follows, \(\tilde{M}\) is the blow up of \(M\) along the unique singular \(G^\mathbb{C}\)-orbit):

i) \(M = \tilde{M}\) and it is \(G\)-equivariantly biholomorphic to one of the following compactifications of the tangent space of a compact rank one symmetric space \(S\):

a) \(\mathbb{C}P^n\); in this case \(S = \mathbb{R}P^n\) and \(G = \text{SO}_{n+1}\) or \(G = \text{Spin}_7\) if \(n = 7\);

b) \(Q^n = \{(z) \in \mathbb{C}P^{n+1}, \overline{1}zz = 0\}\); in this case \(S = S^n\) and \(G = \text{SO}_{n+1}\) or \(G = \text{Spin}_7\) if \(n = 7\);

c) \(\mathbb{C}P^n \times \mathbb{C}P^n\); in this case \(S = \mathbb{C}P^n\) and \(G = \text{SU}_{n+1}\);

d) \(G_{2,2n}(\mathbb{C})\); in this case \(S = \mathbb{H}P^n\) and \(G = \text{Sp}_n\);

e) \(E_{6}/(\text{SO}_2 \times \text{Spin}_{10})\); in this case \(S = \mathbb{O}P^2\) and \(G = F_4\);

ii) \(\tilde{M}\) is biholomorphically \(G\)-equivalent to a manifold of the form \(G \times_{G_Q, \rho} F\), where \(G, F, G_Q\) and the group \(Q = \rho(G_Q)\), given by the action \(\rho\) of \(G_Q\) on \(F\), are as in one of the cases of Table 1. The cases in no. 1 of Table 1 are the only possibilities for non-standard KE-manifolds; all other cases correspond to non-standard KO-manifolds.

**Proof.** The content of the corollary follows directly from the above remarks. In particular, Table 1 has been obtained as follows: the groups \(G\) and \(G_Q\) are determined by the list given at point c) at the beginning of this section; the groups \(Q\) are determined as the only groups, which are homomorphic images of \(G_Q\) and acting non-standardly on one of the manifolds listed in (3.a) of Theorem 2.4, and the fibers \(F\) are determined accordingly. \(\square\)

**Remark 3.6.** We observe that each manifold \(M = G \times_{G_Q, \rho} F\), where \(G, G_Q, Q = \rho(G_Q)\) and \(F\) are as in Table 1, and for each of the two \(G\)-invariant complex structure \(J_o\) on the flag manifold \(G/G_Q\), there exists a \(G\)-invariant complex structure \(J\) on \(M\), such that the canonical projection \(\pi: (M, J) \to (G/G_Q, J_o)\) is holomorphic.

We indicate how to prove this claim in case 1 of Table 1. The flag manifold \(G/G_Q\) can be written as \(G^\mathbb{C}/P = \text{SL}_n(\mathbb{C})/P\), where \(P\) is a parabolic subgroup. \(P\) admits a holomorphic projection onto \(S = \text{SL}_2(\mathbb{C})/Z\), \(Z\) being the center and \(S\) acts holomorphically onto \(Q^2\) and onto \(\mathbb{C}P^2\) in a standard way, with an action \(\tilde{\rho}\), which extends the action \(\rho\) of \(G_Q\) on \(F\). Therefore, the manifold \(M\) is \(G\)-equivariantly diffeomorphic to \(\text{SL}_n(\mathbb{C}) \times \tilde{P}, \tilde{\rho}\) \(F\); this last manifold can be shown to be \(G^\mathbb{C}\)-homogeneous, holomorphic bundle over \(G^\mathbb{C}/P = G/G_Q\), proving our claim.
The cases 2, 3 and 4 can be checked similarly. In case 5, it is enough to check that the parabolic subgroup $P$ such that $E_6(\mathbb{C})/P = E_6/\text{SO}_2 \times \text{Spin}_{10}$ admits a projection onto $\text{Spin}_{10}(\mathbb{C})$; this can be achieved considering that the isotropy action of $P$ on the tangent space of the symmetric space $E_6/\text{SO}_2 \times \text{Spin}_{10}$ coincides with the action of $\mathbb{C}^* \times \text{Spin}_{10}(\mathbb{C})$ (see Table 2 in [Be], p. 313).

Notice also that the claim implies that each of the fiber bundles described in Table 1 does correspond to a K-manifold, since each of them can be realized as an algebraic variety.

**Remark 3.7.** The manifold $M = G \times_{G_{Q,F}} F$ described in case 1 of Table 1, with $F = Q^2$, is the manifold discussed in [GC].

4. The Einstein equation for a non-standard K-manifold.

4.1 Optimal transversal curves and the algebraic representatives of closed 2-forms.

By the results of [Sp], it is known that on any K-manifold $M$ of real dimension $2n$, there exists a family of curves $\eta : \mathbb{R} \to M$ which verify the following properties:

1. the points $\eta_t$ are of the form
   $$\eta_t = \exp(itZ) \cdot p_o$$
   for some $p_o \in M$, which is regular for the $G^\mathbb{C}$-action and for some $Z \in g$; more precisely, in case $M$ is standard, $p_o = \eta_0$ is any $G$-regular point; in case $M$ is non-standard, $p_o$ is a point of the singular $G$-orbit, which is not complex;
2. $\eta$ intersects any regular $G$-orbit; in particular, in case $M$ is standard, then $\eta_t \in M_{\text{reg}}$ for any $t$; in case $M$ is non-standard, $\eta_t \in M_{\text{reg}}$ if and only if $t \neq 0$;
3. for any point $\eta_t \in M_{\text{reg}}$, the tangent vector $\eta_t' = J\hat{Z}_{\eta_t}$ is transversal to the regular orbit $G \cdot \eta_t$;
4. any element $g \in G$ which is in a stabilizer $G_{\eta_t}, \eta_t \in M_{\text{reg}}$, fixes pointwise the whole curve $\eta_t$; in particular, all regular orbits $G \cdot \eta_t$ are $G$-equivalent to the same homogeneous space $G/L$;
5. the structural decompositions
   $$g = l + \mathbb{R}Z_D(t) + m(t)$$
   associated with the CR structure of the regular orbits $G/L = G \cdot \eta_t$ do not depend on $t$; furthermore, $Z_D(t) = Z$ for any $\eta_t \in M_{\text{reg}}$;
6. there exists a basis $\{F_1, G_1, \ldots, F_{n-1}, G_{n-1}\}$ for $m$ such that for any $\eta_t \in M_{\text{reg}}$ the complex structure $J_t : m \to m$, induced by the complex structure of $T_{\eta_t}M$, is of the following form:
   $$J_tF_j = \lambda_j(t)G_j, \quad J_tG_j = -\frac{1}{\lambda_j(t)}F_j; \quad (4.1)$$
   where the function $\lambda_j(t)$ is either one of the functions $-\tanh(t), -\tanh(2t), -\coth(t)$ and $-\coth(2t)$ or it is identically equal to 1.
Any curve $\eta$ which verifies (1) - (5) is called *optimal transversal curve*.

Consider now a closed $G$-invariant, $J$-invariant 2-form, which is bounded and defined on the regular points subset $M_{\text{reg}}$.

If $\eta : \mathbb{R} \to M$ is an optimal transversal curve, since $g$ is semisimple and $\varpi$ is $G$-invariant, then for any $t \in \mathbb{R}$ there exists a unique $\text{ad}_t$-invariant endomorphism $F_{\varpi, t} \in \text{Hom}(g, g)$ such that:

$$B(F_{\varpi, t}(X), Y) = \varpi_{\eta_t}(\hat{X}, \hat{Y}), \quad X, Y \in g.$$  \hspace{1cm} (4.2)

Using the fact that $\varpi$ is $G$-invariant and closed, it is not difficult to realize that for any $X, Y, W \in g$ the following holds

$$F_{\varpi, t}([X, Y], W) = [F_{\varpi, t}(X), Y] + [X, F_{\varpi, t}(Y)] .$$

This means that $F_{\varpi, t}$ is a derivation of $g$ and hence of the form

$$F_{\varpi, t} = \text{ad}(Z_{\varpi}(t)) \hspace{1cm} (4.3)$$

for some $Z_{\varpi}(t) \in \mathfrak{a} = C_{g}(l) \cap \mathfrak{t}^\perp$.

We call the curve $Z_{\varpi} : \mathbb{R} \to C_{g}(l) = \mathfrak{z}(l) + \mathfrak{a}$, $t \mapsto Z_{\varpi}(t)$ the *algebraic representative of the 2-form $\varpi$ along the optimal transversal curve $\eta$.*

Note that the 2-form $\varpi$ can be completely recovered by its algebraic representative $Z_{\varpi}(t)$. In fact, the following proposition holds.

**Proposition 4.1.** ([PS],[Sp]) Let $(M, J, g)$ be a $K$-manifold acted on by the compact semisimple Lie group $G$. Let also $\eta_t = \exp(tZ_D) \cdot p_o$ be an optimal transversal curve and $Z_{\varpi} : \mathbb{R} \to \mathfrak{z}(l) + \mathfrak{a}$ the algebraic representative of a bounded, $G$-invariant, $J$-invariant closed 2-form $\varpi$ along $\eta$. Then:

1. if $M$ is a standard $K$-manifold or a non-standard KO-manifold (equivalently, if either $\mathfrak{a} = \mathbb{R}Z_D$, or $\mathfrak{a} = \mathfrak{su}_2$ and $M$ is standard), then there exists an element $I_{\varpi} \in \mathfrak{z}(l)$ and a smooth function $f_{\varpi} : \mathbb{R} \to \mathbb{R}$ so that

$$Z_{\varpi}(t) = f_{\varpi}(t)Z_D + I_{\varpi}; \hspace{1cm} (4.4)$$

2. if $M$ is non-standard KE-manifold (equivalently, if $\mathfrak{a} = \mathfrak{su}_2$ and $M$ is non-standard), then there exists a Cartan subalgebra $\mathfrak{t}^C \subset \mathfrak{t}^C + \mathfrak{a}^C$ and a root $\alpha$ of the corresponding root system, such that $Z_D \in \mathbb{R}(iH_\alpha)$ and $\mathfrak{a} = \mathbb{R}Z_D + \mathbb{R}F_\alpha + \mathbb{R}G_\alpha$, where $F_\alpha$ and $G_\alpha$ are as in (4.9') below; furthermore there exists an element $I_{\varpi} \in \mathfrak{z}(l)$, a real number $C_{\varpi}$ and a smooth function $f_{\varpi} : \mathbb{R} \to \mathbb{R}$ so that

$$Z_{\varpi}(t) = f_{\varpi}(t)Z_D + \frac{C_{\varpi}}{\cosh(t)}G_\alpha + I_{\varpi} . \hspace{1cm} (4.4')$$
Conversely, if \( Z_{\omega} : \mathbb{R} \to C_{\mathfrak{g}}(t) \) is a curve in \( C_{\mathfrak{g}}(t) \) of the form (4.4) or (4.4'), then there exists a unique closed \( J \)-invariant, \( G \)-invariant 2-form \( \varpi \) on \( M_{\text{reg}} \), having \( Z_{\omega}(t) \) as algebraic representative; such 2-form is the unique \( J \)- and \( G \)-invariant form which verifies the following at all points of \( \eta_t \in M_{\text{reg}} \) and for any \( V, W \in \mathfrak{m} \)

\[
\varpi_{\eta_t}(V, W) = B(Z_{\omega}(t), [V, W]), \quad \varpi_{\eta_t}(J\tilde{Z}_D, \tilde{Z}_D) = -f_{\omega}'(t)B(Z_D, Z_D) . \quad (4.5)
\]

We say that the 2-form \( \varpi \) is \textit{tame} if there exists an optimal transversal curve \( \eta_t \), along which the algebraic representative \( Z_{\omega}(t) \) of the form (4.4). By the previous proposition, in case \( M \) is a non-standard KE-manifold, \( \varpi \) is tame in case if and only if the constant \( C_{\omega} \), which should appear in the expression (4.4') of \( Z_{\omega}(t) \), is equal to 0. For any other kind of K-manifold, a bounded \( G \)-invariant, \( J \)-invariant closed 2-form \( \varpi \) is always tame.

Due to Proposition 4.1, a \( G \)-invariant Kähler metric \( g \) on a K-manifold \( M \) is Einstein, with Einstein constant \( c > 0 \), if and only if the algebraic representative \( Z_{\omega}(t), Z_{\rho}(t) \) of the Kähler form \( \omega \) and the Ricci form \( \rho \), along an optimal transversal curve \( \eta_t = \exp(itZ_D) \cdot p_o \), verify \( Z_{\rho}(t) = cZ_{\omega}(t) \) for any \( t \in \mathbb{R} \), which is equivalent to

\[
f_{\rho}(t) = cf_{\omega}(t) \quad , \quad I_{\rho} = cI_{\omega} \quad , \quad C_{\rho} = cC_{\omega} \quad , \quad (4.6)
\]

the last equation appearing only if \( M \) is a non-standard KE-manifold. We call (4.6) \textit{the Einstein equations of a K-manifold}.

### 4.2 The Einstein equations of a non-standard K-manifold.

We now want to write down the Einstein equations for a \( G \)-invariant Kähler metric \( g \) on a non-standard K-manifold \( M \). Since we are interested in the manifolds listed in Corollary 3.5, we will always assume that the canonical blow up \( \tilde{M} \) of \( M \) is of the form \( \tilde{M} = G \times_{G_Q, \rho} F \) for a subgroup \( G_Q \) and an action \( \rho \) of \( G_Q \) on \( F \).

In what follows we will always denote by \( N_F \)

\[
N_F = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{m}_F = \frac{1}{2}(\dim_{\mathbb{R}}(\rho(G_Q) \cdot x) - 1) ,
\]

for any regular point \( x \in F_{\text{reg}} \) in the standard fiber.

Let \( \eta_t = \exp(itZ_D) \cdot p_o \) be an optimal transversal curve and \( \mathfrak{g} = \mathfrak{l} + \mathbb{R}Z_D + \mathfrak{m} \) the structural decomposition given by the CR structure of all regular orbits \( G \cdot \eta_t = G/L \). By Th. 3.2 in [Sp], we may consider the following orthogonal decomposition

\[
\mathfrak{g} = (\mathfrak{l}_o + \mathfrak{l}_F) + \mathbb{R}Z_D + (\mathfrak{m}_F + \mathfrak{m}') , \quad \mathfrak{l} = \mathfrak{l}_o + \mathfrak{l}_F , \quad \mathfrak{m} = \mathfrak{m}_F + \mathfrak{m}'
\]

where \( \mathfrak{g}_Q = (\mathfrak{l}_o + \mathfrak{l}_F) + \mathbb{R}Z_D + \mathfrak{m}_F \) is the Lie algebra of the group \( G_Q \) and where \( \mathfrak{l}_o \) is the kernel of non-effectivity on \( F \), so that \( \mathfrak{g}_F := \text{Lie}(\rho(G_Q)) \simeq \mathfrak{l}_F + \mathbb{R}Z_D + \mathfrak{m}_F \). In [Sp], the pair \((\mathfrak{g}_F, \mathfrak{l}_F)\) is called \textit{Morimoto-Nagano pair of the orbits} \( G \cdot \eta_t \).
By the definition of optimal transversal curve, we know that there exists a basis for \( \mathfrak{m} \), with respect to which the complex structure \( J_i : \mathfrak{m} \to \mathfrak{m} \) induced by the CR structure of the orbit \( G \cdot \eta_t \) assumes a particularly simple expression. Using the results in [Sp], one can check that we may always consider a basis for \( \mathbb{R}Z_D + \mathfrak{m} \)

\[
\{ F_0 = Z_D, F_1, G_1, \ldots, F_{n-1}, G_{n-1} \}
\]

where the following properties hold:

1. The elements \( F_i, G_i \), with \( 1 \leq i \leq N_F \) are a basis for \( \mathfrak{m}_F \);
2. the elements \( F_i, G_i \) with \( N_F + 1 \leq i \leq n - 1 \) are a basis for \( \mathfrak{m}' \);
3. for any \( t \neq 0 \) (that is when \( \eta_t \in M_{reg} \)),

\[
J_i F_i = \begin{cases} 
\left[ -\tanh((1)^t) \ell(t) \right] G_i & \text{if } 1 \leq i \leq N_F \\
G_i & \text{if } N_F + 1 \leq i \leq n - 1
\end{cases}
\tag{4.7}
\]

where \( \ell(t) \) is 2 if \( F_i \in [\mathfrak{m}_F, \mathfrak{m}_F]^\C \cap \mathfrak{m}_F^\C \) and 1 otherwise.

There also exists a Cartan subalgebra \( \mathfrak{t}^\C \subset \mathfrak{t}^\C + \mathbb{C}Z_D \) for \( \mathfrak{g}_F^\C \), so that \( \mathfrak{t}_F^\C = \mathfrak{t}^\C \cap \mathfrak{g}_F^\C \)

is a Cartan subalgebra for \( \mathfrak{g}_F^\C \) and the root systems \( \mathcal{R} \) and \( \mathcal{R}_F \) of \( (\mathfrak{g}_F^\C, \mathfrak{t}_F^\C) \) and \( (\mathfrak{g}_F^\C, \mathfrak{t}_F^\C) \) decompose into

\[
\mathcal{R}_F = \mathcal{R}_F^o + \mathcal{R}_F', \quad \mathcal{R} = \mathcal{R}_o \cup \mathcal{R}_1 = (\mathcal{R}_1^o \cup \mathcal{R}_F^o) \cup (\mathcal{R}_F' \cup \mathcal{R}_r) ,
\]

where

\[
\mathcal{R}_1^o = \{ \alpha , \ E_\alpha \in \mathfrak{t}_o^\C \} , \quad \mathcal{R}_F^o = \{ \alpha , \ E_\alpha \in \mathfrak{t}_F^\C \} ,
\]

\[
\mathcal{R}_F' = \{ \alpha , \ E_\alpha \in \mathfrak{m}_F^\C \} , \quad \mathcal{R}_r = \{ \alpha , \ E_\alpha \in \mathfrak{m}^\C \} .
\]

Moreover, the Cartan subalgebra can be chosen so that the elements \( F_i, G_i \) are expressed as follows in terms of the root vectors.

If \( 1 \leq i \leq N_F \), then two cases may occur: either there exists a pair of roots \( \{ \alpha_i, \alpha_i^d \} \in \mathcal{R}_F' \) and an integer \( \epsilon_i = \pm 1 \) so that

\[
F_i = \frac{E_{\alpha_i} - E_{-\alpha_i}}{2} + (-1)^{i+1} \epsilon_i \left( E_{\alpha_i^d} + E_{-\alpha_i^d} \right),
\tag{4.8}
\]

\[
G_i = i \frac{E_{\alpha_i} + E_{-\alpha_i}}{\sqrt{2}} ,
\tag{4.9}
\]

or there exists one root \( \alpha_i \in \mathcal{R}_F' \) such that

\[
F_i = \frac{E_{\alpha_i} - E_{-\alpha_i}}{\sqrt{2}} , \quad G_i = i \frac{E_{\alpha_i} + E_{-\alpha_i}}{\sqrt{2}} .
\tag{4.9'}
\]

This second case may occur only when \( N_F \) is odd and, in this case, there is only one such vector \( F_i \) and we may assume it is the vector \( F_{N_F} \).

If \( N_F + 1 \leq i \leq n - 1 \), then there exists a root \( \beta_i \in \mathcal{R}_r \) so that

\[
F_i = \frac{E_{\alpha_i} - E_{-\alpha_i}}{\sqrt{2}} , \quad G_i = i \frac{E_{\alpha_i} + E_{-\alpha_i}}{\sqrt{2}} .
\tag{4.10}
The ordering of the roots can be chosen so that the root vectors \( E_\alpha \), which define the elements \( F_i, i \geq N_F + 1 \), by (4.10) are those corresponding to the roots \( \alpha_i \in R'_+ = R^+ \cap R' \) (see [Sp]).

The explicit expression of the element \( Z_D \) for each manifold \( M \) of Table 1 is given (up to scaling) in the Table of Def. 1.7 in [AS]. To determine the factor to obtain the exact expression of \( Z_D \), it is possible to consult Table 1 in [Sp], where, for each possibility of the Morimoto-Nagano pair \((g_F, l_F)\), the elements \( Z_o \), such that \( \exp(itZ_o)\cdot p_o \) is an optimal transversal curve, are given. For convenience of the reader, we list the expressions for \( Z_D \), derived in this way, in Table A1 in Appendix.

Let us now write down the explicit expression for the Einstein equations (4.6).

In the following we will also assume that the Ricci form \( \rho \) and the Kähler form \( \omega \) are tame, i.e. that the constants \( C_\rho \) and \( C_\omega \) in the expression for \( Z_\rho(t) \) and \( Z_\omega(t) \) are both equal to 0, even in case \( M \) is a non-standard KE-manifold.

We will see a posteriori that, for all the K-manifold of considered in Main Theorem (2), the Ricci form is tame; hence we have no loss of generality with this assumption.

By looking at Table 1 in [Sp], one can check that for any \( 1 \leq i \leq N_F \), the bracket \([F_i,G_i] \) is orthogonal to \( \mathfrak{z}(0) \). Set \( i = 1 \). Then, since \( I_\omega, I_\rho \in \mathfrak{z}(0) \), it follows that (4.6)_1 holds if and only if for any \( t \in \mathbb{R} \)

\[
\mathcal{B}(Z_\rho(t), [F_1, G_1]) - c\mathcal{B}(Z_\omega(t), [F_1, G_1]) = \frac{\rho_{\eta_t}(\hat{F}_1, J\hat{F}_1) - c\omega_{\eta_t}(\hat{F}_1, J\hat{F}_1)}{-\coth(\ell_i t)} = 0. \tag{4.11}
\]

Using the expression for \( \rho_{\eta_t} \) given in Thm. 5.1 and Prop. 5.2 in [Sp], assuming that \( F_1 \) is chosen so that \([F_1, G_1]_{l+m} \) is orthogonal to \([l+m, l+m] \) (there is no loss of generality in this assumption) and by (5.13) and (5.13') in [Sp], it follows that (4.11) is equal to (we used the fact that for any \( 1 \leq j \leq N_F \), \( \mathcal{B}(Z_D, [F_j, G_j]) = \ell_j \) - see (5.13) in [Sp])

\[
h'(t) - 2 \sum_{i=1}^{N_F} \ell_i \tanh(-1)^i \ell_i t + c f_\omega(t) + 4\mathcal{B}(Z, Z_D) = 0, \tag{4.12}
\]

where

\[
h(t) = \log \det|g_{ij}(t)|, \quad g_{ij}(t) = g_{\eta_t}(\hat{F}_i, \hat{F}_j) = -\omega_{\eta_t}(\hat{F}_i, J\hat{F}_j), \tag{4.13}
\]

\[
c = -2\mathcal{B}(Z_D, Z_D)c > 0, \quad Z^\kappa = \sum_{k=N_F+1}^{n-1} iH_{\beta_k}. \tag{4.14}
\]

From the properties of the chosen adapted basis, it follow that the vectors \( \hat{F}_i|_{\eta_t}, J\hat{F}_i|_{\eta_t} \) are \( g \)-orthogonal for any \( t \neq 0 \) and that for \( 1 \leq 2s, 2s + 1 \leq N_F \) and \( N_F + 1 \leq i \leq n - 1 \):

\[
g_{\eta_t}(\hat{F}_0, \hat{F}_0) = -\mathcal{B}(Z_D, Z_D)f'_\omega(t), \tag{4.15}
\]
$g_{\eta_i}(\tilde{F}_{2s+1}, \tilde{F}_{2s+1}) = \coth(\lambda_{2s+1}) f_\omega(t) \lambda_{2s+1}$, $g_{\eta_i}(\tilde{F}_{2s}, \tilde{F}_{2s}) = \tanh(\lambda_{2s}) f_\omega(t) \lambda_{2s}$.

$$g_{\eta_i}(\tilde{F}_i, \tilde{F}_i) = -\beta_i(iZ_D) f_\omega(t) - \beta_i(i\omega) .$$

(4.17)

In case $\beta_i(Z_D) \neq 0$ (which is always true if the regular $G$-orbits are Levi non-degenerate), we may write

$$g_{\eta_i}(\tilde{F}_i, \tilde{F}_i) = -\beta_i(iZ_D)(f_\omega(t) + a_i) , \quad \text{where} \quad a_i = \frac{B(iH_{\beta_i}, J_\omega)}{B(iH_{\beta_i}, Z_D)} . \quad (4.17')$$

Now, if $N_F$ is odd,

$$h'(t) = f''_\omega(t) f'_\omega(t) + N_F f'_\omega(t) f'_\omega(t) + f''_\omega(t) + \sum_{i=N_F+1}^{n-1} \frac{1}{f_\omega(t) + a_i} + \ell_{N_F}(\tanh(\ell_{N_F}t) - \coth(\ell_{N_F}t)) .$$

In case $N_F$ is even, we have

$$h'(t) = f''_\omega(t) f'_\omega(t) + N_F f'_\omega(t) f'_\omega(t) + \sum_{i=N_F+1}^{n-1} \frac{1}{f_\omega(t) + a_i} .$$

So, (4.12) becomes

$$f''_\omega(t) f'_\omega(t) + f'_\omega(t) \left( \frac{N_F}{f_\omega(t)} + \sum_{j=N_F+1}^{n-1} \frac{1}{f_\omega(t) + a_j} \right) + \ell f_\omega(t) - N_F^{(1)}(\tanh(t) + \coth(t)) - 2N_F^{(2)}(\tanh(2t) + \coth(2t)) + 4B(Z^\kappa, Z_D) = 0$$

(4.18)

where $N_F^{(1)}$ denotes the number of vectors $F_i$ with $\ell_i = 1$ and $N_F^{(2)}$ the number of those vectors with $\ell_i = 2$.

Let us now consider the condition (4.6)2. Recall that $g_Q = (l_o + l_F) + \mathbb{R}Z_D + m_F$ is the isotropy subalgebra of the flag manifold $G/G_Q$ and that the restrictions $\beta_m|_{\mathfrak{g}(0)}$ of the roots $\beta_m \in \mathfrak{r}'$, corresponding to highest weight vectors of $\mathfrak{m}'^\mathbb{C}$, generate the entire dual space $3(1)^{C*}$ (see e.g. [Al]).

There is no loss of generality if we assume that for any such roots $\beta_m$, the vector $F_m = \frac{E_{\beta_m} - E_{-\beta_m}}{\sqrt{2}}$ coincides with one of the vectors $F_j$, $N_F + 1 \leq j \leq n - 1$, which belong to the basis for $\mathfrak{m} = \mathfrak{m}_F + \mathfrak{m}'$ we have chosen.

Consider one of these vectors $F_m$ and the equation

$$\rho_{\eta_m}(\tilde{F}_m, J\tilde{F}_m) = c\omega_{\eta_m}(\tilde{F}_m, J\tilde{F}_m) = 0 .$$

(4.19)

Note that equation (4.19) is equivalent to any other equation, which is obtained by replacing $F_m$ by any other vector $F_i$ in the same irreducible $l$-module in $\mathfrak{m}'$ (which is clearly included in some irreducible $g_Q$-module).
From Thm. 5.1 and Prop. 5.2 in [Sp], (4.19) is equivalent to

\[ h'(t) - 2 \sum_{i=1}^{N_F} \ell_i \tanh^{-1}(\ell_i t) + \hat{c} f + 2 \mathcal{B}(Z^\kappa, Z_D) + \]

\[ + \frac{2 \mathcal{B}(Z_D, Z_D) \mathcal{B}(Z^\kappa, iH_{\beta_m})}{\mathcal{B}(Z_D, iH_{\beta_m})} - \frac{2c \mathcal{B}(Z_D, Z_D) \mathcal{B}(I_\omega, iH_{\beta_m})}{\mathcal{B}(Z_D, iH_{\beta_m})} = 0, \]  

(4.20)

Subtracting (4.12) from (4.20), we get that

\[ 0 = -2 \mathcal{B}(Z^\kappa, Z_D) + \frac{2 \mathcal{B}(Z_D, Z_D) \mathcal{B}(Z^\kappa, iH_{\beta_m})}{\mathcal{B}(Z_D, iH_{\beta_m})} - \frac{2c \mathcal{B}(Z_D, Z_D) \mathcal{B}(I_\omega, iH_{\beta_m})}{\mathcal{B}(Z_D, iH_{\beta_m})}, \]

so that

\[ \mathcal{B}(c I_\omega, iH_{\beta_m}) = \mathcal{B}(Z^\kappa - \frac{\mathcal{B}(Z^\kappa, Z_D)}{\mathcal{B}(Z_D, Z_D)} Z_D, iH_{\beta_m}) \]  

(4.21)

From the previous remarks, it follows that (4.21) determines \( I_\omega \) uniquely and we may write

\[ I_\omega = \frac{1}{c} \left( Z^\kappa - \frac{\mathcal{B}(Z^\kappa, Z_D)}{\mathcal{B}(Z_D, Z_D)} Z_D \right) = \frac{1}{c} Z^\perp_1 \]  

(4.22)

where by \( Z^\perp_1 \) we denote the orthogonal projection of \( Z^\kappa \) in \( (Z_D)^\perp \).

It also implies that the coefficients \( a_m \), which appear in (4.17’), are equal to

\[ a_m = \frac{1}{c} \frac{\mathcal{B}(iH_{\beta_m}, Z^\kappa)}{\mathcal{B}(iH_{\beta_m}, Z_D)} \]  

(4.23)

Equation (4.18) and (4.22) are the Einstein equations we were looking for.

The explicit expressions for \( Z^\kappa \) and the values of the coefficient \( a_m \) for all K-manifolds of Table 1 are listed in Table A1 (see Appendix). As remarked in Appendix, for all those cases \( Z^\kappa = Z^\perp_1 \) and hence (4.22) reduces to \( I_\omega = \frac{1}{c} Z^\kappa \).

4.3 The differential problem which characterizes the Einstein-Kähler metrics.

We are now ready for the main result of this section. The following Theorem gives the differential problem that one has to solve in order to determine the Kähler-Einstein metrics (if any) on the non-standard K-manifolds of Table 1.

In all the statements of this subsection, we will assume the same hypothesis on \( \tilde{M} \) and the same optimal transversal curve \( \eta_t \) used in §4.2.
Theorem 4.2. Let $\tilde{M} = G \times_{G_{Q,\rho}} F$ be one of the manifolds given in Table 1. Then $\tilde{M}$ admits a Kähler-Einstein metric with Einstein constant $c > 0$ and tame Ricci form $\rho$ if there exists a smooth function $f : ]0, +\infty[ \subset \mathbb{R} \to ]0, +\infty[ \subset \mathbb{R}$ which verifies the following conditions:

1. for any $t \in ]0, +\infty[$ and for any root $\beta_m \in R'_+$
   
   $$f(t) > 0, \quad f'(t) > 0, \quad -(f(t) + a_m)\beta_m(iZ_D) > 0,$$

   where the coefficient $a_m$ are defined by (4.23);

2. $f$ verifies on $(0, +\infty)$ the differential equation (4.18) with $\tilde{c}$ and $Z^\kappa$ as defined in (4.14);

3. $-B(iH_{\beta_m}, I_\omega) = -a_m\beta_m(iZ_D) > 0$ for any $\beta_m \in R'_+$, $\lim_{t \to 0} f(t) = 0 = \lim_{t \to 0} f''(t)$ and the following limits exist and are finite
   
   $$\lim_{t \to 0} f'(t) = C_1, \quad \lim_{t \to 0} f'''(t) = C_2$$

   with $C_1 > 0$;

4. the limits $\lim_{t \to +\infty} f(t)$ and $\lim_{t \to +\infty} -(f(t) + a_m)\beta_m(Z_D)$ are finite and positive for any $\beta_m \in R'_+$ and
   
   $$\lim_{t \to +\infty} e^{2\tilde{c} 2t} f'(t) = C_3, \quad \lim_{t \to +\infty} e^{\tilde{c} 2t} \left(1 + \frac{1}{\tilde{c} F} f''(t) \right) = 0,$$

   $$\lim_{t \to +\infty} e^{2\tilde{c} 2t} \left(1 + \frac{5 f''(t)}{\tilde{c} 6 f'(t)} + \frac{1}{\tilde{c} F} f'''(t) \right) = C_4,$$

   for some finite values $C_3 > 0, C_4$ and where $\tilde{c} F = \begin{cases} 2 & \text{if } F = \mathbb{C} \mathbb{P}^r \\ 1 & \text{if } F = \mathbb{Q}^r \end{cases}$.

If (1) to (4) are satisfied, then the Kähler form $\omega$ of the Kähler-Einstein metric is given by the algebraic representative

$$Z_\omega(t) = f(t)Z + \frac{1}{\tilde{c} Z^\kappa},$$

where $f$ has to be meant as extended over the whole real axis as a continuous odd function.

Proof. We claim that a $G$-invariant metric $g$ on $M_{\text{reg}} (= \tilde{M}_{\text{reg}})$ with tame Kähler form $\omega$ is Kähler-Einstein if and only if the algebraic representative $Z_\omega(t) = f(t)Z_D + I_\omega$, with $t > 0$, verifies (1), (2) and $I_\omega = \frac{1}{\tilde{c}} Z^\kappa$. In fact, by the previous discussion, we know that (4.18) and $I_\omega = \frac{1}{\tilde{c}} Z^\kappa$ are necessary and sufficient conditions for the algebraic representative $Z_\omega(t) = f(t)Z_D + I_\omega$ to represent a Kähler-Einstein metric on $M_{\text{reg}}$. Furthermore, by (4.15), (4.16) and (4.17), $g$ is positive definite on $M_{\text{reg}}$ if and only if the conditions (1) are verified.

To check that $f$ has to be extended as an odd function over $\mathbb{R}$, we recall that, in case $\tilde{M}$ is a KO-manifold, $\eta$ coincides with a re-parameterization of a normal
geodesic for some $G$-invariant Kähler metric (see [Sp]). Let $\gamma_s$ be one of these normal geodesics, with

$$\gamma_0 = \eta_0 = p_o, \quad \gamma_s = \eta_{t(s)}, \quad \frac{ds}{dt} = ||\tilde{Z}_D||_{\eta_t}.$$  

By definition of geodesic symmetry $\sigma$ at $p_o$ (see [AA], [AA1] for the definition) we have that $\sigma_s(\tilde{Z}_D|_{p_o}) = -\tilde{Z}_D|_{p_o}$. Then

$$f(t(-s))Z_D + I_\omega = Z_\omega(t(-s)) = Ad(\sigma_{p_o})Z_\omega(t(s)) = -f(t(s))Z_D + I_\omega, \quad (4.25)$$

and therefore $f(t(-s)) = -f(t(s))$. Now, by considering that $\frac{dt}{ds} = \frac{1}{||\tilde{Z}_D||_{\eta_{t(s)}}}$ is an even function of $s$ (and hence that $t(s)$ is an odd function), it follows that $f : \mathbb{R} \to \mathbb{R}$ is an odd function. When $\tilde{M}$ is a KE-manifold, we observe that $\eta$ lies in $\text{Fix}(L) = \mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2$ and we may argue in a similar way, replacing $\tilde{M}$ with $\text{Fix}(L)$.

By a result of De Turk and Kazdan (see e.g. [Be]), any $C^2$ Einstein metric is real analytic in a geodesic normal coordinate system. Therefore it remains to prove that, if $f : [0, +\infty[ \to \mathbb{R}$, verifies (1) and (2), then the Kähler-Einstein metric $g$ on $M_{\text{reg}}$, associated with $Z_\omega = f(t)Z_D + \frac{1}{c}Z^\kappa$, for $t > 0$, extends to a $C^2$ metric on $\tilde{M}$ if (3) and (4) hold.

A key ingredient to prove the $C^2$-extendibility at the noncomplex singular orbit is given by the following Lemma.

**Lemma 4.3.** Let $g$ be the Kähler-Einstein metric on $M_{\text{reg}}$ associated with the Kähler form $\omega$ given by the algebraic representative $Z_\omega(t) = f(t)Z_D + I_\omega$, with $I_\omega = Z^\kappa$ and $f : \mathbb{R} \to \mathbb{R}$ an odd function which is smooth on $\mathbb{R} \setminus \{0\}$ and verifies (1) and (2) of Theorem 4.2. Define $\Lambda : [0, +\infty[ \to \mathbb{R}$ as

$$\Lambda(t) = -\mathcal{B}(Z_D, Z_D) \int_0^t f(u) du. \quad (4.26)$$

Let $U$ be a $G$-invariant neighborhood of the noncomplex singular $G$-orbit in $\tilde{M}$. If the smooth tensor field $dd^c\Lambda$ where

$$\hat{\Lambda} : M_{\text{reg}} \cap U \to \mathbb{R}, \quad \hat{\Lambda}(p) = \hat{\Lambda}(g \cdot \eta_t) = \Lambda(t), \quad (4.27)$$

extends as a $C^2$ tensor field on the whole $U$, then $\omega$ extends to a $C^2$ 2-form on $U$. In particular $\omega$ extends if $\hat{\Lambda}$ extends as a $C^4$ function.

**Proof.** Consider the unique $G$-invariant $J$-invariant closed 2-forms $\varpi_f$ and $\varpi_{I_\omega}$ on $M_{\text{reg}}$ with associated algebraic representatives $Z_{\varpi_f}(t) \overset{\text{def}}{=} f(t)Z_D$ and $Z_{\varpi_{I_\omega}} \overset{\text{def}}{=} I_\omega$. From definitions, the Kähler form $\omega$ on $M_{\text{reg}}$, determined by $Z_\omega$, coincides with

$$\omega = \varpi_f + \varpi_{I_\omega}.$$  

We claim that $\varpi_{I_\omega}$ extends to a smooth $G$-invariant 2-form on $\tilde{M}$. In fact, we recall that $\frac{1}{c}Z^\kappa$ is the element in the center $Z(G_Q)$, which corresponds to the $G$-invariant
Kähler-Einstein metric $\hat{g}$ on the flag manifold $G/G_Q$, with Kähler form defined by (see [Be], Ch.8)

$$\hat{\omega}_o(\hat{X}', \hat{Y}') = \frac{1}{c}B(Z^\kappa, [X', Y'])$$,

where $o = eG$ and for any $X', Y' \in m'$. So, by definitions and Proposition 4.1, at any point $\eta_t \in M_{\text{reg}}$ and for any vectors $X, Y \in \mathbb{R}Z_D + m_F \subset g_Q$ and $X', Y' \in m'$

$$\varpi_{L_\eta}(\hat{X}, \hat{Y})_{\eta_t} = \frac{1}{c}B(Z^\kappa, [X, Y]) = 0, \varpi_{L_\eta}(\hat{X}, \hat{X}')_{\eta_t} = \frac{1}{c}B(Z^\kappa, [X, X']) = 0,$$

$$\varpi_{L_\eta}(\hat{Z}_D, J\hat{Z}_D)_{\eta_t} = 0, \varpi_{L_\eta}(\hat{X}', J\hat{Y}')_{\eta_t} = \pi^{*}(\hat{\omega})(\hat{X}', \hat{Y}')_{\eta_t},$$

where $\pi$ is the projection $\pi: \tilde{M} \to G/G_Q$. This implies that on $M_{\text{reg}}$, $\varpi_{L_\eta} = \pi^{*}(\hat{\omega})$ and this proves that $\varpi_{L_\eta}$ can be extended smoothly on the entire $\tilde{M}$. Hence, $\omega$ extends to a $C^2$ 2-form on $\mathcal{U}$ if and only if $\varpi_f$ does.

Now, notice that at any point $p \in M_{\text{reg}}$ and for any two vector fields $V, W$

$$\varpi_f(V, W)|_p = (dd^{c}\hat{\Lambda})(V, W)|_p \overset{\text{def}}{=} \left[-V(JW(\hat{\Lambda})) + W(JV(\hat{\Lambda})) + JVW(\hat{\Lambda})\right]_p,$$

(4.28)

Due to $G$-invariance, (4.28) needs to be checked only at the points $\eta_t \in M_{\text{reg}}$, with vector fields $V, W$ of the form $\hat{Z}_D, J\hat{Z}_D$ or $\hat{X}, \hat{Y}$ with $X, Y \in m$. Since $\hat{\Lambda}$ is constant along $G$-orbits, it follows that

$$(dd^{c}\hat{\Lambda})(\hat{X}, \hat{Y})|_{\eta_t} = -J[\hat{X}, \hat{Y}](\hat{\Lambda})_{\eta_t} = \frac{B(Z_D, [X, Y])}{B(Z_D, Z_D)}J\hat{Z}_D(\hat{\Lambda})_{\eta_t} =$$

$$= B(f(t)Z_D, [X, Y]) = \varpi_f(\hat{X}, \hat{Y})|_{\eta_t}$$

$$(dd^{c}\hat{\Lambda})(\hat{X}, \hat{Z}_D)|_{\eta_t} = -J[\hat{X}, Z_D](\hat{\Lambda})_{\eta_t} = 0 = \varpi_f(\hat{X}, \hat{Z}_D)|_{\eta_t},$$

$$(dd^{c}\hat{\Lambda})(\hat{Z}_D, J\hat{Z}_D)|_{\eta_t} = J[\hat{Z}_D, X](\hat{\Lambda})_{\eta_t} = 0 = \varpi_f(\hat{X}, \hat{Z}_D)|_{\eta_t},$$

$$(dd^{c}\hat{\Lambda})(\hat{Z}_D, J\hat{Z}_D)|_{\eta_t} = J\hat{Z}_D(J\hat{Z}_D(\hat{\Lambda})|_{\eta_t} = -B(Z_D, Z_D)f'(t) = \varpi_f(\hat{Z}_D, J\hat{Z}_D)|_{\eta_t},$$

and this proves (4.28). Our claim follows immediately. $\square$

End of proof of Theorem 4.2. Let us now prove that (3) is a sufficient condition for the Kähler-Einstein metric $\hat{g}$ on $M_{\text{reg}}$ to be $C^2$ extendible on a neighborhood of the noncomplex singular orbit $G \cdot \eta_0$.

First of all, we claim that we can find a slice $S \ni \eta_0$, containing $\eta_t$ for small $t$. Indeed, if $\tilde{M}$ is KO-manifold, then $\eta_t$ is reparameterization of a normal geodesic for any $G$-invariant Kähler metric and therefore one can take $S$ to be $\exp_{\eta_0}(V)$ for some suitable open neighborhood of 0 in the normal space to the singular orbit $G \cdot \eta_0$; when $\tilde{M}$ is a KE-manifold, then $S$ can be constructed along the same line inside the fixed point set $F \subset Fix(L)$.

We also know that there exists a local section $\chi: \mathcal{V}_{\eta_0} \subset G/G_{\eta_0} \to G$ such that the map $(v, s) \to \chi(v) \cdot s$ is a diffeomorphism between $\mathcal{V} \times S$ and an open neighborhood of $\eta_0$ in $G \cdot S \subset \tilde{M}$ (see Lemma 2.2 in [Pa]).
From this and from the fact that the function \( \hat{\Lambda} \), defined in (4.27), is constant along \( G \)-orbits, it follows that it extends as a \( C^4 \) function on a neighborhood of \( \eta_0 \) if and only if \( \hat{\Lambda}|_{S \setminus \{\eta_0\}} : S \setminus \{\eta_0\} \to \mathbb{R} \) extends as a \( C^4 \) function over \( S \).

We identify \( S \) with a suitable ball of radius \( r \) in \( \mathbb{R}^{2n-N_F-1} \), via some \( G_{\eta_0} \)-equivariant identification. Let us also denote by \( T : \ ] - r, r[ \to \mathbb{R} \) the odd smooth function defined on the positive values by the relation

\[
g \cdot x = \eta T(|x|) , \quad \text{for } x \in S , \ g \in G_{\eta_0} .
\]

Then

\[
\hat{\Lambda} : S \setminus \{0\} \to \mathbb{R} , \quad \hat{\Lambda}(x) = \Lambda(T(|x|))
\]

by the \( G_{\eta_0} \)-invariance. So, necessary and sufficient condition for \( \hat{\Lambda} \) to extend to a \( C^4 \) function on \( S \) is that \( \Lambda : \ ]0, r[ \to \mathbb{R} \), extends to a \( C^4 \) even function over \( ] - r, r[ \). This occurs if (3) (without the restriction on the sign of \( C_1 \)) is verified. Moreover, using (4.15) - (4.17), one can check that if (1), (2) and (3) holds with \( C_1 > 0 \), then the extension of \( g \) on \( G \cdot \eta_0 \) is positive definite at \( \eta_0 \) and hence on the whole orbit.

We now deal with the extendability of the metric on a \( G \)-invariant neighborhood of the complex singular orbit \( G \cdot \rho_o \), where \( \rho_o = \lim_{t \to \infty} \eta_t \). Note that \( G_{\rho_o} \) is connected and its Lie algebra is given by \( \mathfrak{g}_{\rho_o} = I + \mathbb{R} \cdot \mathfrak{z}_D \).

As before, we consider a 2-dimensional slice \( S \) through \( \rho_o \), which contains the curve \( \eta_t \) for large \( t \), and, using the Lemma 2.2 in [Pa], we determine a neighborhood \( U_0 \) of \( \rho_o \) in \( G \cdot \rho_o \) and a local section \( \chi : U \to G \) such that the map \( U \times S \ni (x, s) \mapsto \chi(x) \cdot s \) is a diffeomorphism onto its image.

Now notice that, since \( S \) is \( G_{\rho_o} \)-invariant, the vector field \( \hat{Z}_S \) is tangent to \( S \). So any tangent space \( T_{\eta} S \) is spanned by the vectors \( \hat{Z}_S|_{\eta} \) and \( J \hat{Z}_S|_{\eta} \) for \( \eta \) and hence it is \( J \)-invariant; in particular, \( S \) is a \( G_{\rho_o} \)-invariant complex submanifold of \( M \).

By the Riemann Mapping Theorem, by choosing \( S \) sufficiently small, we may assume that there exists a biholomorphism \( \varphi : S \to \Delta \subset \mathbb{C} \) which maps \( S \) onto the unit disc \( \Delta = \{|z| < 1\} \subset \mathbb{C} \) and so that \( \varphi(\rho_o) = 0 \). If we use \( \varphi \) to identify \( S \) with \( \Delta \), it follows from Schwarz Lemma that the 1-parameter group \( \exp(\mathbb{R} \mathfrak{z}_D) \subset G_{\rho_o} \) acts on \( \Delta \) as a closed group of rotations. In particular, using the standard polar coordinates \( (r, \theta) \) for \( \Delta \), we have that

\[
\hat{Z}_S|_{S} = k \frac{\partial}{\partial \theta} , \quad J \hat{Z}_S|_{S} = -kr \frac{\partial}{\partial r} , \quad \eta_t = (r(t) = Ae^{-kt}, \theta = 0) , \quad (4.29)
\]

for some positive real constant \( A \) and where \( \frac{1}{k} \) is the smallest positive rational number such that \( \exp(\frac{t}{k} \mathfrak{z}_D) \cdot \eta_t = \eta_0 \) for any \( t > 0 \).

Let us now prove that \( k = \varepsilon F \) as defined in the statement of the Theorem. Using Table 1 in [Sp], where the explicit expressions of \( \rho_o(\mathfrak{z}_D) \) are given, one can check that the adjoint action of \( \rho(\exp(\pi \mathfrak{z}_D)) \) on the tangent space \( T_{\rho_o}(\rho(\mathfrak{g}_Q) \cdot \rho_o) \) of the singular orbit \( G \cdot \rho_o \subset F \) is equal to \(-I\) and that for no value \( 0 < k < 1 \), \( \rho(\exp(\pi k \mathfrak{z}_D)) \) acts on \( T_{\rho_o}(\rho(\mathfrak{g}_Q) \cdot \rho_o) \) as a multiple of the identity. On the other hand, by direct inspection of the explicit action of \( \rho(\mathfrak{g}_Q) = SO_{r+1}, SO_{r+1}/\mathbb{Z}_2 \) on \( F = Q', \mathbb{C}P^{r} \) (see e.g. [Uc], p. 157-158), the action of \( \rho(\exp(\pi \mathfrak{z}_D)) \) on \( S \) coincides
with the geodesic symmetry if \( F = Q^r \) and with identity map in case \( F = \mathbb{C}P^r \). In particular, it follows that \( k = \varepsilon_F \).

From the fact that \( \|\hat{Z}_D\|^2_{\hat{g}} = -B(Z_D, Z_D)f'(t) \), we get that the restriction of \( g \) on the vectors tangent to \( S \setminus \{p_o\} \) is

\[
g|_{TS \times TS} = -\frac{Bf'(t(r))}{\varepsilon_F r^2} (dr^2 + r^2 d\theta^2). \tag{4.30}
\]

By the results in [Ve] (see also [Ber]), we have that \( g|_{TS \times TS} \) extends to a \( C^2 \) Riemannian metric over the whole \( S \) if and only if the function \( \sqrt{f'(t(r))} \) extends as a \( C^2 \) even positive function on \( ]-1,1[ \). This is equivalent to conditions (4).

It remains to prove that if (4) is verified then the metric \( g \) extends to \( C^2 \) Riemannian metric over the entire \( \chi(U) \cdot S \simeq U \times S \).

Consider now the following frames field \( \{X_1, \ldots, X_n\} \). Let \( X_1, X_2 \) be any two smooth vector fields tangent to \( S \) and linearly independent at any point. Then, consider the vectors \( E_i \in g \), \( 3 \leq i \leq 2n \), which are equal to the linearly independent vectors \( F_j \) or \( G_j \) defined in (4.8) - (4.10), and for any \( x \in S \) let \( X_i|_x = \hat{E}_i|_x \). By choosing \( S \) sufficiently small, we may always suppose that all the \( X_i \)'s are linearly independent at any point of \( S \). Finally, let us extend the field of frames \( \{X_i\} \) on \( S \) to a field of frames on \( \chi(U) \cdot S \simeq U \times S \), by setting

\[
\{X_1, \ldots, X_n\}_{\chi(u)x} = \{\chi(u)_x(X_1|_x), \ldots, \chi(u)_x(X_1|_x)\}.
\]

Now, \( g \) extends to a \( C^2 \) tensor if and only if the functions \( g_{ij} = g(X_i, X_j) \) extends as \( C^2 \) functions. By the \( G \)-invariance of \( g \) and by the construction of the frames \( \{X_i\} \), these conditions are equivalent to the conditions for the extendability of the restrictions \( g_{ij}|_{S \setminus \{p_o\}} \) to \( C^2 \) functions over \( S \).

Observe that, by definitions from the fact that \( [Z_D, Z_\omega(t)] = 0 \) for any \( t \), we have for \( i, j \geq 3 \) and for any point \( x = (r, \theta) \in S \setminus \{p_o\} \)

\[
g_{ij}|(r, \theta) = \left[ \exp\left(-\frac{\theta}{\varepsilon_F}Z_D^*\right)g\right](\hat{E}_i, \hat{E}_j)|_{(r, \theta)} =
\]

\[
= -B\left(Z_\omega(t(r)), [Ad_{\exp\left(-\frac{\theta}{\varepsilon_F}Z_D\right)}(E_i), J_t Ad_{\exp\left(-\frac{\theta}{\varepsilon_F}Z_D\right)}(E_j)]\right) =
\]

\[
= -B\left(Ad_{\exp\left(-\frac{\theta}{\varepsilon_F}Z_D\right)}(Z_\omega(t(r))), [E_i, J_t^0(E_j)]\right) = -B\left(Z_\omega(t(r)), [E_i, J_t^0(E_j)]\right), \tag{4.31}
\]

where

\[
J_t^0 = Ad_{\exp\left(-\frac{\theta}{\varepsilon_F}Z_D\right)} \circ J_t \circ Ad_{\exp\left(-\frac{\theta}{\varepsilon_F}Z_D\right)}. \tag{4.32}
\]

From the fact that the projection \( \pi : \tilde{M} \to G/G_Q \) is holomorphic, it follows

\[
\text{ad}_{Z_D} \circ J_t|_{m'} = J_t \circ \text{ad}_{Z_D}|_{m'} \text{ for any } t \text{ and hence that } J_t^m|_{m'} \equiv J_t \text{ using the explicit expressions of } Z_D \text{ given in Table 1 in [Sp]}, \text{ one can check that, for any } 1 \leq i \leq N_F
\]

\[
Ad_{\exp\left(\frac{\theta}{\varepsilon_F}Z_D\right)}(F_i) = \cos\left(\frac{\theta}{\varepsilon_F}\right) F_i + \sin\left(\frac{\theta}{\varepsilon_F}\right) G_i,
\]
\[ Ad_{\exp(\frac{\theta}{\varepsilon_f}Z_D)}(G_i) = \sin \left( \frac{\theta}{\varepsilon_f} \right) F_i + \cos \left( \frac{\theta}{\varepsilon_f} \right) G_i , \]
and hence that
\[
J_i^\theta(F_i) = (-1)^i \left( \frac{\tanh(t) - \coth(t)}{2} \sin \left( \frac{2\theta}{\varepsilon_f} \right) F_i - \right)
- \left( \tanh(-1)^i(t) \cos^2 \left( \frac{\theta}{\varepsilon_f} \right) + \tanh(-1)^{i+1}(t) \sin^2 \left( \frac{\theta}{\varepsilon_f} \right) \right) G_i ,
J_i^\theta(G_i) = \left( \tanh(-1)^i(t) \cos^2 \left( \frac{\theta}{\varepsilon_f} \right) + \tanh(-1)^{i+1}(t) \sin^2 \left( \frac{\theta}{\varepsilon_f} \right) \right) F_i -
- \left( -(-1)^i \frac{\tanh(t) - \coth(t)}{2} \sin \left( \frac{2\theta}{\varepsilon_f} \right) G_i . \right)
\]

From this, (4.15) and (4.16), we conclude that the functions \( g_{ij}(r, \theta) \), with \( 3 \leq i, j \leq 2N_F \) and which are not identically equal to 0, are proportional to functions of the form
\[ f(t(r)) + a_i , \quad \tanh(t(r)) - \coth(t(r)) \sin \left( \frac{2\theta}{\varepsilon_f} \right) f(t(r)) , \quad \left( \tanh(-1)^i(t(r)) \cos^2 \left( \frac{\theta}{\varepsilon_f} \right) + \tanh(-1)^{i+1}(t(r)) \sin^2 \left( \frac{\theta}{\varepsilon_f} \right) \right) f(t(r)) . \]

Now, by the proof of Prop. 2.7 in [KW], the functions \( g_{ij} \) of the form (4.33) and (4.34) extend as \( C^2 \) functions on the origin if and only if

i) the value \( \lim_{r \to 0} g_{ij}(r, \theta) \) does not depend on \( \theta \);

ii) \( g_{ij} \) extends as a \( C^2 \) function over \( ]-1, 1[ \times \mathbb{R} \) such that \( g_{ij}(-r, \theta) = g_{ij}(r, \theta + \pi) \) for any \( \theta \) and any \( r > 0 \);

iii) for \( k = 1, 2 \), \( \lim_{r \to 0} r^k \frac{\partial^k g_{ij}(r, \theta)}{\partial r^k} \) is a homogeneous polynomial of degree \( k \) in the variables \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \).

First of all, note that (i) is verified by all functions (4.33), (4.34). Moreover, by some straightforward computations, one can check that all those functions verify (ii) if and only if \( f(t(r)) \) extends as a \( C^2 \) even function on a neighborhood of 0, which turns out to be equivalent to the conditions

\[ \lim_{t \to \infty} f(t) = C_5 , \quad \lim_{t \to \infty} e^{2\varepsilon_x t} f'(t) = 0 , \quad \lim_{t \to \infty} e^{2\varepsilon_x t} \left( f'(t) + \frac{f''(t)}{\varepsilon_f} \right) = C_6 \]
for some finite values \( C_5 \) and \( C_6 \); but these conditions are immediately satisfied if \( f \) verifies (4). In a similar way, a simple computation shows that also (iii) is verified by all functions (4.33), (4.34) since \( f \) verifies (4).

Finally, we recall that, by the previous remarks, if conditions (4) are satisfied, then the functions \( g_{ij}|_{S \setminus \{p_\ell\}} \) with \( i, j = 1, 2 \) are immediately extendible on \( S \), while the functions \( g_{i\ell}|_{S \setminus \{p_\ell\}} \) with \( i = 1, 2 \) and \( \ell \geq 3 \) are identically vanishing.

The positivity of the extension of the metric on the points of the singular complex orbit is assured by the positivity of the limits \( \lim_{t \to +\infty} f(t) \) and \( \lim_{t \to +\infty} -(f(t) + a_m)\beta_m(Z_D) \). \( \Box \)
5. New examples of non-standard Kähler-Einstein K-manifolds.

We have now all ingredients to determine new examples of non-standard K-manifolds that belong to the class described by Corollary 3.5 (ii). The main result is the following.

**Theorem 5.1.** Let \( \check{M} = G \times_{G_\varrho} F \) be one of the manifolds given in Table 1, but not in case 1 with \( G = SU_3 \) and \( F = \mathbb{C}P^2 \); not in case 2 with \( G = SU_p \times SU_2, p > 2 \), not in case 4 with \( F = Q^7 \), nor in case 5 with \( F = Q^9 \). Then it is Kähler-Einstein with positive first Chern class.

The crucial step for the proof of Theorem 5.1 is the following proposition. The meaning of any adopted notation is the same of §4.

**Proposition 5.2.** Let \( \check{M} = G \times_{G_\varrho} F \) be one of the manifolds given in Table 1 and adopt the same notation of §4.2; in particular, set \( N_F = N_F^{(1)} \) and \( \varepsilon_F = \begin{cases} 2 & \text{if } F = \mathbb{C}P^r \\ 1 & \text{if } F = Q^r \end{cases} \).

Consider a positive real number \( \dot{c} > 0 \), denote by \( V_c = \left( \frac{N_F + 1}{\dot{c}^2} \right)^2 \) and let \( V : [0,1] \to [0,V_c] \), be a smooth map such that:

a) \( \lim_{\theta \to 0} V(\theta) = 0, \lim_{\theta \to 0} \dot{V}(\theta) = \dot{V}_0 > 0 \) and \( \lim_{\theta \to 1} V(\theta) = V_c \) (here we set \( \dot{V} = \frac{dV}{d\theta}, \dot{V} = \frac{d^2V}{d\theta^2} \) and \( \ddot{V} = \frac{d^3V}{d\theta^3} \));

b) the limits \( \lim_{\theta \to 1} \dot{V}(\theta) = \dot{V}_1 > 0, \lim_{\theta \to 1} \ddot{V}(\theta) \) and \( \lim_{\theta \to 1} \dddot{V}(\theta) \) exist and are finite;

c) \( V \) verifies on \( ]0,1[ \) the equation

\[
\ddot{V} + \frac{\dot{V}^2}{2V} \left( \frac{N_F - 1}{\sqrt{V}} + \sum_{m=N_F+1}^{n-1} \frac{1}{\sqrt{V} + a_m} \right) + \dot{V} \left( \dot{c} \frac{\sqrt{V} - N_F}{1 - \varepsilon_F} - 1 \right) = 0
\]

(5.1)

where \( a_m = a_{\beta_m} \), with \( a_{\beta_m} \) as defined in Theorem 4.2.

Suppose also that the K-manifold \( \check{M} \) is so that

d) for any \( \beta_m \in R'_+ \) the following inequalities hold:

\[
\left| B(Z_D, Z_D) \cdot \frac{B(Z^\alpha, iH_{\beta_m})}{B(Z_D, iH_{\beta_m})} \right| > N_F + \varepsilon_F , \quad B(Z^\alpha, iH_{\beta_m}) < 0 .
\]

Then the curve \( Z_\omega(t) = f(t)H + I_\omega \in z(t) + \mathbb{R}Z_D \), where \( I_\omega \in z(t) \) is as in (4.22) and where

\[
f(t) = \sqrt{V(\tanh^2(\varepsilon_F t))} ,
\]

(5.2)

is the algebraic representative of a Kähler-Einstein metric on \( \check{M} \) with Einstein constant \( \dot{c} = -\dot{c} \frac{\varepsilon_F}{B(Z_D, Z_D)} \).

**Proof.** Let \( f(t) = \sqrt{V(\tanh^2(\varepsilon_F t))} \) and \( \theta(t) = \tanh^2(\varepsilon_F t) \). We claim that a) - d) imply that \( f \) verifies the hypothesis 1) - 4) of Theorem 4.2.
From definitions
\[ \frac{d}{dt} \theta(t) = \frac{2\varepsilon_F \tanh(\varepsilon_F t)}{\cosh^2(\varepsilon_F t)} = 2\varepsilon_F \sqrt{\theta}(1 - \theta) \]
and hence
\[ f'(t) = \frac{\varepsilon_F}{\sqrt{V}} \dot{V} \sqrt{\theta}(1 - \theta), \]
(5.3)

\[ f''(t) = -\varepsilon_F^2 (1 - \theta)(-V\ddot{V} + 3\theta V\dddot{V} + \theta\dot{V}^2 - \theta^2\dddot{V}^2 - 2\theta V\ddot{V} + 2\theta^2 V\dddot{V}) \]
(5.4)

\[ f'''(t) = \varepsilon_F^3 \left[ -\frac{8(1 - \theta)^2 \sqrt{\theta} V}{V^{3/2}} + \frac{4(1 - \theta)\theta^{3/2} \dot{V}}{\sqrt{V}} - \frac{3(1 - \theta)^3 \sqrt{\theta} \dddot{V}^2}{V^{3/2}} + \frac{6(1 - \theta)^2 \theta^{3/2} \dddot{V}^2}{V^{3/2}} + \frac{3(1 - \theta)^3 \sqrt{\theta} \dddot{V}}{\sqrt{V}} - \frac{12(1 - \theta)^2 \theta^{3/2} \dddot{V}}{\sqrt{V}} - \frac{6(1 - \theta)^3 \theta^{3/2} \dddot{V}\dot{V}}{\sqrt{V}^{3/2}} + \frac{4(1 - \theta)\theta^{3/2} \dddot{V}}{\sqrt{V}^{3/2}} \right]. \]
(5.5)

It follows that
\[ \frac{f''(t)}{f'(t)} = \varepsilon_F \left[ 2\frac{\dddot{V}}{\dot{V} - \dddot{V}} \right] (1 - \theta)\sqrt{\theta} + \frac{1}{\sqrt{\theta}}(1 - 3\theta) \]
(5.6)

\[ f'(t) \left( \frac{N_F}{f(t)} + \sum_{m=N_F+1}^{N} \frac{1}{f(t) + a_m} \right) = \varepsilon_F \sqrt{V} \dot{V} \sqrt{\theta}(1 - \theta) \left( \frac{N_F}{\sqrt{V}} + \sum_{m=N_F+1}^{N} \frac{1}{\sqrt{V} + a_m} \right) \]
(5.7)

\[ \tilde{c}f(t) - N_F \tanh(t) - N_F \coth(t) = \tilde{c}f(t) - 2N_F \coth(2t) = \]
\[ = \tilde{c}\sqrt{V} - N_F(2 - \varepsilon_F)\sqrt{\theta} - \frac{\varepsilon_F N_F}{\sqrt{\theta}}. \]
(5.8)

After some simple algebraic manipulation, it follows immediately that \( V \) verifies (5.1) if and only if \( f(t) \) verifies (2) of Theorem 4.2 with \( \tilde{c} = 2\varepsilon_F\tilde{c} \).

Moreover, if a) and b) are verified, from (5.6) - (5.8) we get that
\[ \lim_{t \to 0} f(t) = \lim_{\theta \to 0} \sqrt{V}(\theta) = 0 \]
\[ \lim_{t \to 0} f'(t) = \dot{V}_0 \sqrt{\lim_{\theta \to 0} \frac{\theta}{V}} = \dot{V}_0 > 0 \]

Similarly, one can check that \( \lim_{t \to 0} f''(t) = 0 \) and that \( \lim_{t \to 0} f'''(t) \) is finite, so that all parts of condition (3) of Theorem 4.2 is verified, with the only exception of the inequalities \( \mathcal{B}(Z^n, iH_{\beta_m}) < 0 \).
By some tedious but direct computations, one can check in the same way that also all parts of condition (4) of Theorem 4.2 are verified, with the only exception of the positivity of the limits \( \lim_{t \to +\infty} f(t) \) and \( \lim_{t \to +\infty} -(f(t) + a_m) \beta_m(Z_D) \).

So, in order to conclude the proof, we need to check that if d) is true and \( V \) verifies the boundary conditions a), then the condition (1) and the remaining parts of (3) and (4) in Theorem 4.2 are automatically verified by \( f(t) = \sqrt{V(t \tanh^2(t))} \).

For this we need a pair of preparatory lemmata.

Let us introduce the notation \( \alpha = \frac{N_F - 1}{2} \geq 0 \) and \( \alpha' = \frac{N_F - 1}{2} \geq -\frac{1}{4} \); note that \( \hat{c} = 4 \left( \alpha' + 1 \right)^2 > 0 \).

Let us also rewrite (5.1) as

\[
\frac{d}{d\theta} \left[ \log \left( \frac{1}{n-1} \prod_{m=N_F+1}^n \left| \sqrt{V} + a_m \right| \right) \right] = \frac{2(1+\alpha') - \hat{c} \sqrt{V}}{1-\theta} + \frac{\alpha}{\theta} \tag{5.9}
\]

Remark 5.3. Note that, since \( \hat{c} = -c B(Z_D, Z_D) \epsilon_F \), if we assume that for \( 0 < \theta < 1 \)

\[
0 < V(\theta) < \hat{c} = 4 \left( \frac{\alpha' + 1}{\epsilon} \right)^2
\]

and that hypothesis d) holds true, then the sign of \( a_m \) and \( a_m + \sqrt{V(\theta)} \) does not depend on \( \theta \) (to check this, use Table A1 in Appendix); in particular, the value of \( a_m + \sqrt{V(\theta)} \) is never equal to 0.

Now, from (5.9) and the above remark, we immediately obtain the following lemma.

**Lemma 5.4.** Assume that d) of Proposition 5.2 is true and let \( V : [a, b] \subset [0,1[ \to \mathbb{R} \) be a \( C^1 \)-solution of (5.1) with \( 0 < V(\theta) < \hat{c} = 4 \left( \frac{\alpha' + 1}{\epsilon} \right)^2 \) for any \( \theta \in [a, b] \). Then, for \( \theta_1 \leq \theta_2 \in [a, b] \),

\[
\left( \frac{\theta_2}{\theta_1} \right)^{\alpha} \frac{1 + \sqrt{\theta_1}}{1 + \sqrt{\theta_2}} < \frac{V(\theta_2)}{V(\theta_1)} \left( \frac{V(\theta_2)}{V(\theta_1)} \right)^{\alpha} \prod_{m=N_F+1}^n \left( \frac{a_m + \sqrt{V(\theta_2)}}{a_m + \sqrt{V(\theta_1)}} \right) < \left( \frac{\theta_2}{\theta_1} \right)^{\alpha} \frac{1 - \theta_1}{1 - \theta_2} \tag{5.10}
\]

**Proof.** By hypothesis,

\[
2(1 + \alpha') \frac{1 - \sqrt{\theta}}{1 - \theta} < \frac{2(1+\alpha') - \hat{c} \sqrt{V}}{1-\theta} < \frac{2(1 + \alpha')}{1 - \theta}
\]

From these inequalities and from (5.9), the claim follows immediately by integration. □
Theorem 5.6. Under the hypothesis d) of Proposition 5.2 and assuming that $\prod_{m}a_{m} > 0$, it follows that $V$ is monotone increasing on $[a, b]$ and $0 < V(\theta) < 4 \left( \frac{1 + \alpha'}{e} \right)^2$ for any $\theta \in [a, b]$.

Proof. By hypothesis, there exists at least one point $\theta_{o} \in [a, b]$ such that $0 < V(\theta_{o}) < V(b) < 4 \left( \frac{1 + \alpha'}{e} \right)^2$ and with $\dot{V}(\theta_{o}) > 0$. Moreover, by Remark 5.3, we also have that $\prod_{m} \left( a_{m} + \sqrt{V(\theta_{o})} \right) \neq 0$.

Now, consider the set $K = \{ \theta \in [\theta_{o}, b] : \dot{V}(\theta) > 0, V(\theta) < 4 \left( \frac{1 + \alpha'}{e} \right)^2 \}$. It is clear that $K$ is open in $[\theta_{o}, b]$. Moreover, if $\theta_{1} = \sup K$, from the left hand side of (5.10) it follows that

$$M_{\theta_{o}} \left( \frac{1 + \sqrt{\theta_{o}}}{1 + \sqrt{\theta_{1}}} \right)^{4(\alpha'+1)} \left( \frac{\theta_{1}}{\theta_{o}} \right)^{\alpha} \dot{V}(\theta_{1}) V(\theta_{1})^{\alpha} \prod_{m} |a_{m} + \sqrt{V(\theta_{1})}|$$

(5.11)

with $M_{\theta_{o}} = \dot{V}(\theta_{o}) V(\theta_{o})^{\alpha} \prod_{m} |a_{m} + \sqrt{V(\theta_{o})}| \neq 0$. This shows that $\dot{V}(\theta_{1}) > 0$ and hence that it belongs to $K$. In particular, it follows that $K$ is open and closed and hence it coincides with $[\theta_{o}, b]$, proving the claim for any $\theta \in [\theta_{o}, b]$.

A similar argument, which uses the right hand side of (5.10) in place of the left hand side, brings to the same conclusion for any $\theta \in [a, \theta_{o}]$. $\square$

End of Proof of Proposition 5.2. To conclude the proof, it suffices to observe that if $V : [0, 1] \to [0, V_{b}]$, with $V_{b} = 4(\alpha'+1)^2/\bar{c}^2$, then Lemma 5.5 applies on $V|_{[a, b]}$ for any interval $[a, b] \subset [0, 1]$. In particular, $V$ is strictly increasing and $f(t) = V(\tanh^{2}(t))$ is monotone increasing and positive on $[0, \infty[$. So it remains to check that the positivity of $-(f(t) + a_{m}) \beta(t \cdot Z_{D})$ for any $t \in [0, \infty]$ and of its limits for $t \to 0$ and $t \to +\infty$; this is an immediate consequence of the hypothesis d) and Remark 5.3. $\square$

The following theorem shows that, if d) of Proposition 5.2 holds true, then a solution for the Kähler-Einstein equation exists with only one possible exception. The proof of this fact is practically an adaptation to our more general case of the arguments used in §6 and §7 of [GC].

Theorem 5.6. Under the hypothesis d) of Proposition 5.2 and assuming that $\tilde{M} \neq SO_{10} \times (T^{1} \times SO_{8}) \cdot Q^{7}$, for any constant $\bar{c} > 0$, there exists a real analytic solution $V : [0, 1] \to [0, V_{c}]$ of the differential problem given by conditions a), b) and c) of Proposition 5.2

Proof. The proof is subdivided into several claims.

Claim 1. For any interval $[a, b] \subset [0, 1]$ and any positive constant $0 < c_{a} < V_{c}$ there exists a monotone increasing solution $V : [a, b] \to [0, V_{c}]$ of (5.1) with $V_{a} = c_{a}$.

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1 We are indebted to D. Guan for pointing us a serious mistake in a previous version of the proof of Theorem 5.6
In order to prove this claim, we consider the Cauchy problem \((C_k)\) given by the differential equation (5.1) together with initial conditions \(V(a) = c_a\) and \(\dot{V}(a) = k > 0\). Let us denote by \([a, b_k] \subset [0, 1]\) the maximal interval on which there exists a solution \(V_k\) of \((C_k)\), satisfying \(V_k(\theta) < V_{\tilde{c}}\) for all \(\theta \in [a, b_k]\). By Lemma 5.4, the solution \(V_k\) is monotone increasing. We also have that \(\lim_{\theta \to b_k} V_k(\theta) = V_{\tilde{c}},\) otherwise the interval \([a, b_k]\) would not be maximal.

We claim that \(\sup_{k > 0} b_k = 1\). Indeed, suppose that \(\sup_{k > 0} b_k = b^* < 1\). It would imply that for every \(k > 0\), there exists some \(\theta_k \in ]a, b_k[\) so that

\[
\dot{V}_k(\theta_k) = \frac{V_{\tilde{c}} - c_a}{b_k - a} > \frac{V_{\tilde{c}} - c_a}{b^* - a}.
\]

However, from Lemma 5.4, it would follow that there exists some positive constant \(H\) so that

\[
0 < \frac{V_{\tilde{c}} - c_a}{b^* - a} < kH \left(\frac{1 - a}{1 - b^*}\right)^{2(1+\alpha')} \left(\frac{b^*}{a}\right) \alpha
\]

for any \(k\). This leads to a contradiction when we let \(k \to 0\).

**Claim 2.** Let \(V : [a, b] \subset [0, 1] \to [0, V_{\tilde{c}}]\) be a monotone increasing solution of (5.1) with \(V(a) > 0\). Then for any positive number \(c_b\) with \(V(b) < c_b < V_{\tilde{c}}\), there exists a monotone increasing solution \(\tilde{V}\) of (5.1) with \(\tilde{V}(a) = V(a)\) and \(\tilde{V}(b) > c_b\).

As before, let us denote by \((C_k)\) the Cauchy problem given by the differential equation (5.1) together with initial conditions \(V(a) = c_a\) and \(\dot{V}(a) = k > 0\). Let also \(K \subset [0, +\infty[\) be the set of all \(k \geq 0\) such that the Cauchy problem \((C_k)\) has a solution \(V_k\) defined on \([a, b]\) and with \(V_k(b) < V_{\tilde{c}}\).

The claim is proved if we can show that \(\sup_{k \in K} V_k(b) = V_{\tilde{c}}\). Suppose not and let \(\sup_{k \in K} V_k(b) = \lambda < V_{\tilde{c}}\). In particular, we have that for any \(k \in K\), any solution \(V_k\) of \((C_k)\) on \([a, b]\) verifies \(V_k(b) \leq \lambda < V_{\tilde{c}}\) and hence, using (5.10), one can easily check that \(K\) is open and closed in \([0, +\infty[\) and hence that \(K = [0, +\infty[\). But then, using the same arguments as in the previous claim, we obtain that

\[
k \left(\frac{b}{a}\right)^{\alpha} \left(\frac{1 + \sqrt{a}}{1 + \sqrt{b}}\right)^{4(1+\alpha')} \left(\frac{c_a}{\lambda}\right)^{\alpha} \prod_{m=N+1}^{n-1} \left|\frac{a_m + \sqrt{c_a}}{a_m + \sqrt{\lambda}}\right| \leq \frac{\lambda - c_a}{b - a} ,
\]

for any \(k > 0\), which is impossible.

**Claim 3.** For any \(k \in ]0, 1[\) there exists a monotone increasing real analytic solution \(V : [0, k] \to [0, V_{\tilde{c}}]\) of (5.1) with

\[
\lim_{\theta \to 0} V(\theta) = 0 , \quad \lim_{\theta \to k} V(\theta) = V_{\tilde{c}}
\]

Using Claim 1 and 2, we may construct a sequence \(\{V^{(n)}\}\) of monotone increasing solutions of (5.1) defined on the intervals \([\frac{1}{n}, k - \frac{1}{n}]\), with values in \([0, V_{\tilde{c}}]\) and such that

\[
\lim_{n \to \infty} V^{(n)}(\frac{1}{n}) = \lim_{n \to \infty} \frac{1}{n} = 0 , \quad \lim_{n \to \infty} V^{(n)}(k - \frac{1}{n}) = V_{\tilde{c}} .
\]
Note that, for any interval \([a, b] \subset ]0, k\[, there exists an integer \(N_{a,b}\) such that for any \(n > N_{a,b}\),
\[\{a, b\} \subset \left[\frac{1}{n}, k - \frac{1}{n}\right].\]
Now, fix a positive value \(0 < \varepsilon < a\) and notice that, for any \(\theta \in [a, b]\) and any function \(V^{(n)}\), with \(n\) large enough that \([a - \varepsilon, b] \subset \left[\frac{1}{n}, k - \frac{1}{n}\right]\), there exists a value \(a - \varepsilon < \theta_n < \theta\) so that
\[
\dot{V}^{(n)}(\theta_n) = \frac{V^{(n)}(\theta) - V^{(n)}(a - \varepsilon)}{\theta - a + \varepsilon} < \frac{V^{(n)}(b) - V^{(n)}(a - \varepsilon)}{\varepsilon} < \frac{V_{\varepsilon}}{\varepsilon}.
\]
From (5.10) (where we replace \(\theta_1\) and \(\theta_2\) with \(\theta_n\) and \(\theta\), respectively) and from the fact that for any \(n\), \(\frac{V^{(n)}(\theta_n)}{\dot{V}^{(n)}(\theta_n)} < 1\), it follows that there exists a constant \(C\), independent of \(n\) and the interval \([a, b] \subset ]0, k\[, so that \(\dot{V}^{(n)}(\theta) < C\) for any \(n\) sufficiently large and for any \(\theta \in [a, b]\).
In particular, we conclude that the functions \(\dot{V}^{(n)}\big|_{[a, b]}\) are uniformly bounded and hence that the functions \(V^{(n)}\big|_{[a, b]}\) converge, up to a subsequence, to a \(C^0\)-function, say \(V\).
Consider now an exhausting sequence of intervals \([a_n, b_n] \subset ]0, k\[, on each of these intervals we may define the function \(V : [a_n, b_n] \to ]0, V_{\varepsilon}[\) as limit of the sequence \(V^{(n)}\); since the limit function \(V\) coincides on the intersection of two intervals \([a_n, b_n], [a_{n_2}, b_{n_2}]\), we obtain a nondecreasing continuous function \(V\) defined on the open interval ]0, \(k\[. Moreover, since
\[
\lim_{\theta \to 0} V(\theta) = \inf_{\theta \in ]0, k[} V(\theta) \leq \inf_{\theta \in ]\frac{1}{n}, k - \frac{1}{n}[} V^{(n)}(\theta) = \frac{1}{n},
\]
we get \(\lim_{\theta \to 0} V(\theta) = 0\). Now fix \(\varepsilon > 0\) and \(t_o > k - \frac{\varepsilon}{3C}\); take also \(N_o\) large enough so that \(t_o \leq k - \frac{1}{n}\) and \(|V^{(n)}(k - \frac{1}{n}) - V_{\varepsilon}| < \varepsilon/3\) for all \(n \geq N_o\). Then, choosing a suitable large \(n\)
\[
|V(t_o) - V_{\varepsilon}| \leq |V(t_o) - V^{(n)}(t_o)| + |V^{(n)}(t_o) - V^{(n)}(k - \frac{1}{n})| + |V^{(n)}(k - \frac{1}{n}) - V_{\varepsilon}| \leq |V(t_o) - V^{(n)}(t_o)| + C|t_o - k + \frac{1}{n}| + \frac{\varepsilon}{3} \leq \varepsilon
\]
Since \(V\) is non-decreasing, from this we have that \(\lim_{\theta \to k} V(\theta) = V_{\varepsilon}\).
To conclude the proof of the claim, it remains to check that \(V\) is actually a solution of (5.1), and then it will also follow that it is real analytic.
First of all, notice that for any \(\theta \in ]0, k[\), the value \(V(\theta) \neq 0\).
In fact, assume that there is some \(\theta_o\) so that \(\lim_{n \to 0} V^{(n)}(\theta_o) = V(\theta_o) = 0\). On the other hand, since \(\lim_{\theta \to k} V(\theta) > 0\), there has to be some point \(\theta_o + \delta \in [\theta_o, k[\) so that \(V(\theta_o + \delta) > 0\). So, we may consider the functions \(V^{(n)}\big|_{[\theta_o - \delta', \theta_o + \delta]}\) for some fixed \(0 < \delta' < \theta_o\). Fix \(\varepsilon > 0\); since each function \(V^{(n)}\) is positive and strictly increasing, we have that for all \(n\) sufficiently large, there exists some \(\theta_n \in [\theta_o - \delta', \theta_o[\) and some \(\xi_n \in [\theta_o, \theta_o + \delta[\) such that
\[
V^{(n)}(\theta_o) < \varepsilon,
\]
\[
\dot{V}^{(n)}(\theta_n) V^{(n)}(\theta_n)^\alpha = \frac{V^{(n)}(\theta_o) - V^{(n)}(\theta_o - \delta)}{\delta'} V^{(n)}(\theta_n)^\alpha < \frac{V^{(n)}(\theta_o)^{1+\alpha}}{\delta'} ,
\]
\[
\dot{V}^{(n)}(\xi_n) V^{(n)}(\xi_n)^\alpha = \frac{V^{(n)}(\theta_o + \delta) - V^{(n)}(\theta_o)}{\delta} V^{(n)}(\xi_n)^\alpha >
\]
\[
\frac{V(\theta_o + \delta) - \varepsilon}{\delta} V^{(n)}(\theta_o)^\alpha .
\]

From these inequalities, we get that for any \( \varepsilon > 0 \) sufficiently small, there exists some \( \theta_o - \delta' < \theta_n < \xi_n < \theta_o + \delta \) such that
\[
\frac{\dot{V}^{(n)}(\xi_n) V^{(n)}(\xi_n)^\alpha}{\dot{V}^{(n)}(\theta_n) V^{(n)}(\theta_n)^\alpha} > \frac{\delta' V(\theta_o + \delta) - \varepsilon}{\varepsilon} \geq \frac{1}{2}
\]
which brings to an immediate contradiction with the inequalities (5.10).

At this point, it remains to observe that, being \( V(\theta) = \lim_{n \to \infty} V^{(n)}(\theta) \neq 0 \) for any \( \theta \in ]0, k]\) and since, for any closed interval \([a, b] \subset ]0, k]\), the first derivatives of the functions \( V^{(n)}|_{[a, b]} \) are uniformly bounded, it follows from (5.1) that also the second derivatives of \( V^{(n)}|_{[a, b]} \) are uniformly bounded. So, for any interval \([a, b]\), the sequence \( V^{(n)}|_{[a, b]} \) converges to \( V|_{[a, b]} \) in \( C^1 \) and we get that the limit function \( V \) is a solution of (5.1), by well known results on the smooth dependence of solutions from the initial data.

**Claim 4.** In case \( \tilde{M} \neq SO_{10} \times (SO_2 \times SO_8)^T \), \( E_6 \times (SO_2 \times Spin_{10})^Q \), there exists a monotone increasing real analytic solution \( V : ]0, 1[ \to ]0, V_c[ \) of (5.1) with
\[
\lim_{\theta \to 0} V(\theta) = 0 , \quad \lim_{\theta \to 1} V(\theta) = V_c .
\]

For any \( k \in \mathbb{N} \) denote by \( V_k : ]0, 1 - \frac{1}{k}[, 0, V_c[ \) the solution of (5.1) given by Claim 3. Note that the same arguments used in the proof of Claim 3 show that the values \( \dot{V}_k(\theta) \) are uniformly bounded on intervals \([0, b]\) for \( b < 1 \). This means that the sequence \( \{V_k\} \) converges uniformly on compacta of \([0, 1]\) to a nondecreasing solution of (5.1) with \( V : ]0, 1[ \to ]0, V_c[ \) and that \( \lim_{\theta \to 1} V(\theta) = 0 \). It remains to check that \( \lim_{\theta \to 0} V(\theta) = \sup_{\theta \in [0, 1]} V(\theta) \) is equal to \( V_c \).

We fix \( 0 < \varepsilon < \min\{(\alpha' + 1)/3\} \) and for any \( k \) we choose \( \theta_k \in ]0, 1 - \frac{1}{k}[ \) with
\[
V_k(\theta_k) = \left( \frac{2\alpha' + 2 - 2\varepsilon}{\varepsilon} \right)^2 .
\]
Note that 0 is not a limit point for \( \{\theta_k\} \); this follows from the mean value theorem together with the fact that \( V_k \) are uniformly bounded on the compact subsets of \([0, 1/2]\) (see proof of Claim 3).

**Lemma 5.7.** In case \( \tilde{M} \neq SO_{10} \times (T^1 \times SO_8)^Q \), for any sufficiently small \( \varepsilon > 0 \), we have that \( \sup_k \{\theta_k\} < 1 \).

**Proof.** We consider the change of variable \( \theta(s) = 1 - e^{-s} \). Since \( \frac{d\theta}{ds} = e^{-s} = 1 - \theta \), for any function \( F_\theta \) we have that
\[
F_s \overset{\text{def}}{=} \left. \frac{dF}{ds} \right|_s = \hat{F}(\theta(s))(1 - \theta(s)) .
\]
From (5.9), we obtain that
\[ \frac{d}{ds} \left[ \log \left( V_k^n V_k^\alpha \prod_{m=N+1}^{n-1} (|\sqrt{V_k} + a_m|) \right) \right] = \alpha \left( \frac{1 - \theta(s)}{\theta(s)} \right) + 1 + 2\alpha' - \hat{c} \sqrt{\frac{V_k}{\theta(s)}} \]
\[ (5.14) \]

We consider the points \( s_k \in \mathbb{R} \) such that \( \theta(s_k) = \theta_k \) and, up to a subsequence, we suppose that the sequence of points \( \theta(s_k) \) tends to 1, that is \( \lim_{k \to +\infty} s_k = +\infty \), aiming to obtain a contradiction.

We consider the functions \( \tilde{V}_k(s) \equiv V_k(\theta(s + s_k)) \). First of all, we want to prove that, for every compact \( K \subset \mathbb{R} \), the derivatives \( \tilde{V}_k'|_K \) are uniformly bounded for all \( k \) large enough. Observe that the functions \( \tilde{V}_k \) satisfy the equation
\[ \tilde{V}_k''(s) = (\tilde{V}_k'(s))^2 \left( \sum_{l} \frac{1}{a_l^2 - \tilde{V}_k} - \frac{\alpha}{\tilde{V}_k} \right) + \tilde{V}_k'(s) \left( \frac{\alpha(1 - \theta)}{\theta} \right)_{s+s_k} + 1 + 2\alpha' - \hat{c} \frac{\sqrt{\tilde{V}_k(s)}}{\sqrt{\theta(s + s_k)}} \]
\[ (5.15) \]

where we denote by \( a_l \) the positive coefficients which appear in the collection of coefficients \{a_l\}; indeed, we recall that the coefficients \( a_l \) appear in pairs of opposite signs (see Table A1 in Appendix).

Note that \( \theta(s + s_k) \geq \theta(s_k) \) for \( s \geq 0 \); hence, for \( k \) large enough, we have that for all \( s \geq 0 \)
\[ \tilde{V}_k''(s) \leq C(\tilde{V}_k'(s))^2 + \tilde{V}_k'(s)(-1 + 3\varepsilon) \]
where \( C = \sum_{l} \frac{1}{a_l^2 - \varepsilon} \). By integration, since \( \tilde{V}_k(0) \) does not depend on \( k \), we have for all \( s \geq 0 \)
\[ \tilde{V}_k'(s) \leq C_1 \tilde{V}_k'(0) e^{(3\varepsilon - 1)s} \]
\[ (5.16) \]
for some positive constant \( C_1 \).

We claim now that the sequence of values \{\tilde{V}_k'(0)\} is bounded from above and from below by two positive constants.

Indeed, integrating (5.16), we get that
\[ \tilde{V}_k(s) - \tilde{V}_k(0) \leq C_1 \tilde{V}_k'(0) \int_0^s e^{(3\varepsilon - 1)t} dt \]
and therefore \( V_\varepsilon - \left( \frac{2\alpha' + 2 - 2\varepsilon}{3\varepsilon} \right)^2 \leq C_1 \tilde{V}_k'(0) \), showing that \( \inf_k \tilde{V}_k'(0) \) is positive.

We now show that \( \tilde{V}_k''(0) \) are also bounded from above. Note that
\[ \frac{\alpha(1 - \theta)}{\theta} \left| s+s_k \right| + 1 + 2\alpha' - \hat{c} \frac{\sqrt{\tilde{V}_k(s)}}{\sqrt{\theta(s + s_k)}} \geq 1 + 2\alpha' - \hat{c} \sqrt{\frac{\tilde{V}_k}{\theta(s + s_k)}} \geq \frac{3}{2} \]
for all \( s \geq 0 \) and all large enough \( k \). Hence
\[ \tilde{V}_k''(s) \geq -\frac{\alpha}{\tilde{V}_k(s)} \tilde{V}_k'(s)^2 - \frac{3}{2} \tilde{V}_k'(s) \]
and by integration
\[ \hat{V}'_k(s)\hat{V}_k(s)\alpha \geq C_2\hat{V}'_k(0)e^{-3s/2}, \]
where \( C_2 = \hat{V}_k(0)^\alpha \) does not depend on \( k \). Again, by integration, for all \( s \geq 0 \)
\[ \frac{1}{\alpha + 1}V_{\hat{c}}^{\alpha + 1} \geq \frac{1}{\alpha + 1}(\hat{V}_k(s)^{\alpha + 1} - \hat{V}_k(0)^{\alpha + 1}) \geq \frac{2C_2}{3}\hat{V}'_k(0)(1 - e^{-3s/2}) \]
and this shows that the sequence \( \{\hat{V}'_k(0)\} \) is also bounded from above.

Let us now fix a compact interval \( I = [-A^2, 0] \). Since \( s \geq -A^2 \) for any \( s \in I \), we have \( \theta(s + s_k) \geq \theta(s_k - A^2) \to 1 \) for \( k \to +\infty \). Hence for \( k \) large enough and for all \( s \in I \), the right hand side of (5.14) is bigger or equal than \(-2\) and, by integration,
\[ \frac{\hat{V}'_k(0)(\prod I a_l^2 - \hat{V}_k(0))\hat{V}_k(0)^\alpha}{\hat{V}'_k(s)(\prod I a_l^2 - \hat{V}_k(s))\hat{V}_k(s)^\alpha} \geq e^{-2\int_0^s dt} = e^{2s}. \]
From this we conclude that, for some positive constants \( C_3, C_4, \)
\[ C_3\hat{V}'_k(0)e^{-2s} \geq \hat{V}'_k(s)(\sum I a_l^2 - \hat{V}_k(s))\hat{V}_k(s)^\alpha \geq C_4\hat{V}'_k(s)\hat{V}_k(s)^\alpha. \tag{5.17} \]
From (5.17), it follows that the functions \( (\hat{V}_k(s)^{\alpha + 1})' \) are uniformly bounded on every compact subset in \([-\infty, 0[\). Now, from (5.16), the fact that the sequence \( \{\hat{V}'_k(0)\} \) is bounded from above and that the functions \( \hat{V}_k(s) \) are uniformly bounded, it follows also that the functions \( (\hat{V}_k(s)^{\alpha + 1})' \) are uniformly bounded on \([0, +\infty[\). We conclude that, up to a subsequence, \( \hat{V}_k(s)^{\alpha + 1} \) converge to a continuous, nondecreasing function \( W : \mathbb{R} \to [0, \hat{V}_c] \). Note that \( W(0) \neq 0 \) and that, being \( \hat{V}'_k(0) \) bounded from below by a positive constant, the function \( W \) is not constant.

We now prove that \( W \) never vanishes; indeed, for any \( s \in [-A^2, 0] \), the right hand side of (5.14) does not exceed a constant \( M \) and therefore
\[ \frac{\hat{V}'_k(0)(\prod I a_l^2 - \hat{V}_k(0))\hat{V}_k(0)^\alpha}{\hat{V}'_k(s)(\prod I a_l^2 - \hat{V}_k(s))\hat{V}_k(s)^\alpha} \leq e^{Ms}. \]
Hence there are constants \( C_5, C_6 \) such that for all \( s \in I \),
\[ C_5\hat{V}'_k(0) \leq e^{Ms}\hat{V}'_k(s)\hat{V}_k(s)^\alpha \prod I (a_l^2 - \hat{V}_k(s)) \leq C_6\hat{V}'_k(s)\hat{V}_k(s)^\alpha. \]
Since \( \inf_k \hat{V}'_k(0) > 0 \), we see that \( (\hat{V}_k(s)^{\alpha + 1})' \) are bounded from below by a positive constant which depends only on the interval \( I \). In particular \( W' \) is never zero. On the other hand, if there exists some \( t_o < 0 \) such that \( W(t_o) = 0 \), then \( W(s) = 0 \) for all \( s < t \) and this is not possible. So, \( W \) never vanishes and the sequence \( \hat{V}_k \) converges uniformly on compacta to a differentiable function \( \hat{V} \) \( \overset{\text{def}}{=} \) \( (W)^{1/(\alpha + 1)} \).
Note that the limit function \( \hat{V} \) is not constant and nondecreasing.
From (5.16) and the bounds on the sequence \( \{ \tilde{V}'(0) \} \), we see that \( \tilde{V}'(s) \leq C_7 e^{(3\varepsilon - 1)s} \) for some constant \( C_7 \) and for all \( s \geq 0 \). Therefore \( \lim_{s \to +\infty} \tilde{V}'(s) = 0 \).

The function \( \tilde{V} \) satisfies the equation

\[
\left( \log(\tilde{V}'\tilde{V}^\alpha \prod (\varpi_l^2 - \tilde{V})) \right)' = 1 + 2\alpha' - \sqrt{\tilde{V}}. \tag{5.18}
\]

If we denote by \( U(s) \equiv \tilde{V}'\tilde{V}^\alpha \prod (\varpi_l^2 - \tilde{V}) \), the equation (5.18) becomes

\[
U' = (1 + 2\alpha' - \sqrt{\tilde{V}})\tilde{V}^\alpha \prod (\varpi_l^2 - \tilde{V}) \tilde{V}'. \tag{5.19}
\]

Let us consider the function

\[
g(x) = (1 + 2\alpha' - \sqrt{x})x^\alpha \prod (\varpi_l^2 - x)
\]

for \( x \in [0, +\infty] \) and let \( G(x) = \int_0^x g(t)dt \) for \( x \geq 0 \). Integrating (5.19) we get that for any \( s_1, s_2 \in \mathbb{R} \)

\[
U(s_2) - U(s_1) = G(\tilde{V}(s_2)) - G(\tilde{V}(s_1)). \tag{5.20}
\]

Let us denote by

\[
a = \lim_{s \to -\infty} \tilde{V}(s), \quad b = \lim_{s \to +\infty} \tilde{V}(s),
\]

and note that \( a, b \in [0, \tilde{V}_c] \).

We already noticed that \( \lim_{s \to +\infty} \tilde{V}'(s) = 0 \) and therefore \( \lim_{s \to +\infty} U(s) = 0 \). Hence, from (5.20) it follows that \(-U(s) = G(b) - G(\tilde{V}(s))\). This implies also that \( \lim_{s \to -\infty} U(s) \) exists and it is finite. Indeed, we claim that \( \lim_{s \to -\infty} U(s) = 0 \).

In fact, in case \( \lim_{s \to -\infty} U(s) = \lambda \neq 0 \), from the existence of \( \lim_{s \to -\infty} \tilde{V}(s) \) and the definition of \( U \), we infer that also \( \lim_{s \to -\infty} \tilde{V}'(s) \) exists; but this implies that \( \lim_{s \to -\infty} \tilde{V}'(s) = 0 \) and it contradicts the hypothesis that \( \lambda \neq 0 \).

From (5.20) and the previous remarks, we conclude that

\[
0 = G(b) - G(a) = \int_a^b g(x)dx. \tag{5.21}
\]

Now, if we can prove that \( \int_0^{\tilde{V}_c} g(x)dx < 0 \), we immediately get a contradiction and we conclude the proof of the lemma. Indeed, notice that if we choose \( \varepsilon \) small enough, we have that the value

\[
b = \lim_{s \to +\infty} \tilde{V}(s) > \tilde{V}(0) = V_k(\theta_k) = 4 \left( \frac{\alpha' + 1 - \varepsilon}{\hat{c}} \right)^2
\]

can be made arbitrarily close to \( \tilde{V}_c = 4 \left( \frac{\alpha' + 1}{\hat{c}} \right)^2 \); in particular, we can choose \( \varepsilon \) so that \( \int_0^{b} g(x)dx < 0 \). Now, since \( g(x) \geq 0 \) if and only if \( x \geq \left( \frac{1 + 2\alpha'}{\hat{c}} \right)^2 \), we have that
in case \( a \geq \left( \frac{1+2\alpha'}{\epsilon} \right)^2 \), the integral \( \int_a^bg(x)dx \) is negative, which is contradictory with (5.21); when \( a < \left( \frac{1+2\alpha'}{\epsilon} \right)^2 \), we have that \( \int_a^bg(x)dx < \int_0^bg(x)dx < 0 \) and again this contradicts (5.21).

In the Appendix we estimate the sign of the integral \( \int_0^{V_s}g(x)dx \) for all cases of Table 1. The reader can check that for all possibilities, except the cases \( \tilde{M} = \text{SO}_{10} \times (T^1 \times \text{SO}_8) Q^7, E_6 \times (SO_2 \times \text{Spin}_{10}) Q^9 \), this integral is negative. This concludes the proof. \( \square \)

We may now conclude the proof of Claim 4. For a given small \( \epsilon > 0 \), by Lemma 5.7, we may suppose that \( \lim_{k \to \infty} \theta_k = \theta_o < 1 \). Therefore for \( \theta > \theta_o \), we have that \( V_k(\theta) \) is bigger than \( \left( \frac{2\alpha' + 2 - 2\epsilon}{\epsilon} \right)^2 \) for \( k \) large enough. So \( \lim_{\theta \to 1} V(\theta) = \sup_{\theta \in [0,1]} V(\theta) \geq \left( \frac{2\alpha' + 2 - 2\epsilon}{\epsilon} \right)^2 \). By the freedom on the choice of \( \epsilon > 0 \), we have \( \lim_{\theta \to 1} V(\theta) = V_c \).

**Claim 5.** Let \( V : ]0,1[ \to ]0, V_c[ \) be the solution of (5.1) given in Claim 4. Then, for any \( \epsilon > 0 \) there exist a positive constants \( M_\epsilon \) and a point \( \theta_\epsilon \in ]0,1[ \) so that, for any \( \theta \in [\theta_\epsilon,1[ \)

\[
\dot{V}(\theta)(1 - \theta) < M_\epsilon(1 - \theta)^{1 - 2\epsilon} .
\]

(5.22)

We use the same change of variable \( \theta(s) = 1 - e^{-s} \) we used in the proof of Lemma 5.7. Since \( \lim_{\theta \to 1} \hat{c} \sqrt{\frac{V}{\theta}} = \hat{c} \sqrt{V_c} = 2(\alpha' + 1) \), it follows that for any given \( \epsilon > 0 \), there exists some \( s_o \) such that for all \( s \geq s_o \) (and hence for all \( \theta(s) \geq \theta_\epsilon = \theta(s_o) \)), we have that

\[
\frac{d}{ds} \left[ \log \left( V'V^\alpha \prod_{m=N+1}^{n-1} |\sqrt{V} + a_m| \left( \frac{1}{\theta(s)} \right)^\alpha \right) \right] < 1 + 2\alpha' - 2(\alpha' + 1) + 2\epsilon = -1 + 2\epsilon .
\]

By integration, we get that for any \( \theta_\epsilon \leq \theta < 1 \)

\[
\log \left( \frac{V'(\theta)V^\alpha(\theta)}{V'(\theta_\epsilon)V^\alpha(\theta_\epsilon)} \prod_{m=N+1}^{n-1} |\sqrt{V(\theta)} + a_m|^{\theta_\epsilon^\alpha} \right) < (-1 + 2\epsilon)(s - s_o) ;
\]

therefore

\[
\frac{V'(\theta)}{V'(\theta_\epsilon)} < \frac{\dot{M}_\epsilon}{V(\theta_\epsilon)^\alpha \prod_{m=N+1}^{n-1} |\sqrt{V(\theta)} + a_m|^{\theta_\epsilon^\alpha} e^{-(1-2\epsilon)s_o}} e^{-(1-2\epsilon)s_o} = \dot{M}_\epsilon \left( \frac{1 - \theta}{1 - \theta_\epsilon} \right)^{1 - 2\epsilon} .
\]

for some suitable positive constant \( \dot{M}_\epsilon \). Since for any \( s \), \( V'(\theta(s)) = \dot{V}(\theta)(1 - \theta) \), it follows that

\[
\frac{\dot{V}(\theta)(1 - \theta)}{V(\theta_\epsilon)(1 - \theta_\epsilon)} < \dot{M}_\epsilon \left( \frac{1 - \theta}{1 - \theta_\epsilon} \right)^{1 - 2\epsilon}
\]

and this implies the claim.
We have now all ingredients to complete the proof of Theorem 5.5. In fact, consider the solution $V$ obtained by Claim 3. If we can show that
\[
\lim_{\theta \to 0} \dot{V}(\theta) = \dot{V}_o > 0
\] (5.23)
and that $\lim_{\theta \to 1^-} \dot{V}(\theta)$, $\lim_{\theta \to 1^-} \ddot{V}(\theta)$ and $\lim_{\theta \to 1^-} \dddot{V}(\theta)$ exist and are finite, we are done.

First of all, it is not hard to check that, by (5.10), $\log \left( \frac{\dot{V}(\theta)}{V(\theta)} \right)^\alpha$ verifies the Cauchy condition for $\theta$ tending to 0. Hence $\lim_{\theta \to 0} \log \left( \frac{\dot{V}(\theta)}{V(\theta)} \right)^\alpha$ exists and, using again (5.10), one can check that this limit is positive. Since $\lim_{\theta \to 0} \dot{V}(\theta) = 0$, by de L'Hôpital Theorem, we also have that $\lim_{\theta \to 0} \left( \frac{\dot{V}(\theta)}{V(\theta)} \right)^{\alpha+1}$ exists and that it is positive. From these facts, we conclude that (5.23) is verified.

Consider now a value $\varepsilon < \frac{1}{2}$ and let $M_\varepsilon$ be the constant given in Claim 5. In this case,
\[
0 < \lim_{\theta \to 1^-} \dot{V}(1 - \theta)^{\frac{1}{2} + \varepsilon} = \left[ \lim_{\theta \to 1^-} \dot{V}(1 - \theta) \right] (1 - \theta)^{-\frac{1}{2} - \varepsilon} < M_\varepsilon \lim_{\theta \to 1^-} (1 - \theta)^{\frac{1}{2} - \varepsilon} = 0 .
\] (5.24)

From (5.24) and de L'Hôpital theorem, we get
\[
\lim_{\theta \to 1^-} \frac{2(1 + \alpha') - \hat{c}\sqrt{\frac{\dot{V}}{V}}}{(1 - \theta)^{\frac{1}{2} + \varepsilon}} = \lim_{\theta \to 1^-} \frac{2(1 + \alpha')\sqrt{\theta} - \hat{c}\sqrt{V}}{(1 - \theta)^{\frac{1}{2} - \varepsilon}} = \\
= \lim_{\theta \to 1^-} \frac{\frac{2(1 + \alpha')}{2\sqrt{\theta}} - \frac{\hat{c}\dot{V}}{2\sqrt{V}}}{(\frac{1}{2} + \varepsilon)(1 - \theta)^{-(\frac{1}{2} + \varepsilon)}} = \\
= \lim_{\theta \to 1^-} \frac{2(1 + \alpha')}{2(\frac{1}{2} + \varepsilon)\sqrt{\theta}}(1 - \theta)^{\frac{1}{2} + \varepsilon} - \frac{\hat{c}^2}{4(1 + \alpha')(\frac{1}{2} + \varepsilon)} \lim_{\theta \to 1^-} \dot{V}(1 - \theta)^{\frac{1}{2} + \varepsilon} = 0 .
\] (5.25)

Now, using (5.24), (5.25) and (5.1), we obtain that for all $\theta > \theta_o$, for some suitable $\theta_o$, there exists a positive constant $C_1 > 0$ such that
\[
\left| \dddot{V}(1 - \theta) \right| = \left| -\frac{\ddot{V}^2(1 - \theta)}{2\sqrt{V}} \left( \frac{2\alpha}{\sqrt{V}} + \sum_{m=N+1}^{n-1} \frac{1}{\sqrt{V} + a_m} \right) + \\
+ \left( 2(1 + \alpha') + \frac{\alpha}{\theta} - \hat{c}\sqrt{\frac{V}{\theta}} \right) \dot{V} \left( C_1(1 - \theta)^{\frac{1}{2} - \varepsilon} + \frac{\alpha}{\theta} \right) \right| < \dot{V} \left( C_1(1 - \theta)^{\frac{1}{2} - \varepsilon} + \frac{\alpha}{\theta} \right)
\] (5.26)

Therefore, by integration, we get that for any two $\theta_o < \theta_1 < \theta_2 < 1$
\[
\log \left( \frac{\dot{V}_{\theta_2}}{\dot{V}_{\theta_1}} \right) < C_1 \int_{\theta_1}^{\theta_2} \frac{1}{(1 - \theta)^{\frac{1}{2} + \varepsilon}} d\theta + \int_{\theta_1}^{\theta_2} \frac{\alpha}{\theta} d\theta = 
\]
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\[ C_1 = 2\left(1 - \theta_1\right)^{\frac{1}{2} - \varepsilon} - \log(\theta_1^\alpha) = C_2\left(1 - \theta_1\right)^{\frac{1}{2} - \varepsilon} - \log(\theta_1^\alpha). \quad (5.27) \]

and

\[ \log \left( \frac{\bar{V}_{\theta_2}}{V_{\theta_1}} \right) > -C_2\left(1 - \theta_1\right)^{\frac{1}{2} - \varepsilon} + \log(\theta_1^\alpha). \quad (5.28) \]

From (5.27) and (5.28), it follows that \( \log(\bar{V}_\theta) \) verifies the Cauchy condition for \( \theta \) tending to 1 and hence that \( \lim_{\theta \to 1} \bar{V}_\theta \) exists and it is finite. Moreover by (5.10) this limit is positive as required.

The claims that \( \lim_{\theta \to 1} \bar{V}(\theta) \) and \( \lim_{\theta \to 1} \bar{V}(\theta) \) exist and are finite can be checked immediately from (5.1), with a straightforward application of de L'Hôpital theorem. \( \square \)

\textit{Proof of Theorem 5.1.} One can check that for any K-manifold described in Table 1 of Corollary 3.5, with the only exceptions of the manifold in n.1 with \( G = SU_3 \) and \( F = \mathbb{C}P^2 \) and those in n.2 with \( G = SU_p \times SU_2, \ p > 2, \) the condition d) of Proposition 5.2 is satisfied. Then the conclusion follows as direct corollary of Proposition 5.2 and Theorem 5.6. \( \square \)

\section*{APPENDIX}

In the following table, we adopt the same notation for simple root system adopted in [GOV] (see also [AS]).

For any case of Table 1, we list the Lie algebra \( \mathfrak{g} \), the Morimoto-Nagano algebra \( \mathfrak{g}_F \), the value of \( N_F \) (which in all considered cases coincides with \( N_F^{(1)} \), being \( N_F^{(2)} = 0 \)), the element \( \theta_D \) in the dual space \( \mathfrak{h}^* \) of a Cartan subalgebra of \( \mathfrak{g} \), which is \( \mathcal{B} \)-dual to the element \( -iZ_D \), the set of roots in \( \mathcal{R}_+ \), the element \( \theta^\kappa \in \mathfrak{h}^* \), which is \( \mathcal{B} \)-dual to \( -iZ^\kappa \) and all values for \( \text{ca}_{\beta_m} = \frac{\mathcal{B}(Z_\kappa^\gamma, iH_\beta)}{\mathcal{B}(Z_D, iH_\beta)} \)

which occurs for \( \beta_m \in \mathcal{R}_+ \). Notice that for all considered cases \( Z^\kappa \) is orthogonal to \( Z_D \) and hence that \( Z_\perp^\kappa = Z^\kappa \).
| $n^0$ | $\mathfrak{g}$ | $\mathfrak{g}_F$ | $N_F$ | $\theta_D$ | $R'_{\kappa}$ | $\theta^\kappa$ | $B(Z^\kappa_m, \mathbb{H}_d)$ \\|-----|---------------|---------|-----|---------|---------------|---------------|-----------------|
| 1   | $\mathfrak{su}_{\ell+1}$ | $\mathfrak{su}_2$ | 1   | $-\frac{1}{2}(\varepsilon_1 - \varepsilon_2)$ | $\varepsilon_1 - \varepsilon_a$, $\varepsilon_2 - \varepsilon_a$ | $(\ell-1)(\varepsilon_1 + \varepsilon_2) - 2\sum_{a=3}^\ell \varepsilon_a$ | $\pm 2(\ell+1)$ |
| 2   | $\mathfrak{su}_{p+1} \oplus \mathfrak{su}_{q+1}$ | $\mathfrak{su}_2 \oplus \mathfrak{su}_2$ | 2   | $-\frac{1}{2}(\varepsilon_1 - \varepsilon_2)$ | $\varepsilon_1 - \varepsilon_a$, $\varepsilon_2 - \varepsilon_a$ | $(p-1)(\varepsilon_1 + \varepsilon_2) + (q-1)(\varepsilon'_1 + \varepsilon'_2) - 2\sum_{a=3}^{p+q} \varepsilon_a - 2\sum_{b=3}^{q+1} \varepsilon'_b$ | $\pm 2(p+1)$, $\pm 2(q+1)$ |
| 3   | $\mathfrak{su}_{\ell+1}$ | $\mathfrak{so}_6$ | 4   | $-\frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$ | $\varepsilon_1 - \varepsilon_a$, $1 \leq i \leq 4$ | $(\ell-1)\sum_{i=1}^4 \varepsilon_i - 4\sum_{a=3}^\ell \varepsilon_a$ | $\pm 2(\ell-1)+8$ |
| 4   | $\mathfrak{so}_{10}$ | $\mathfrak{so}_8$ | 6   | $\frac{1}{2}\sum_{i=2}^5 \varepsilon_i$ | $\varepsilon_1 \pm \varepsilon_i$, $2 \leq i, j \leq 5$ | $8\varepsilon_1$ | $\pm 16$ |
| 5   | $\mathfrak{e}_6$ | $\mathfrak{so}_{10}$ | 8   | $-\frac{1}{2}(2\varepsilon_1 + \varepsilon_6 + \varepsilon)$ | $\varepsilon_1 - \varepsilon_a$, $\varepsilon_1 + \varepsilon_i + \varepsilon_k + \varepsilon$ | $12(-\varepsilon_6 + \varepsilon)$ | $\pm 24$ |

Table A1

We now list the explicit expressions for the integrals $\int_0^{V_c} g(x) dx$, which appear in the proof of Lemma 5.7, for all cases of Table A1. We normalize $V_c = 1$, i.e. we assume that $\varepsilon = 2(1 + \alpha')$ and $c = -\frac{\varepsilon \mathbb{E}}{B(Z_D, Z_D)}$.

If $\tilde{M}$ is as in n.1 with $\varepsilon_F = 1$, we have

$$\frac{1}{4^{l-1}} \int_0^1 (l+1)^2 - 4x)^{l-1}(1 - 2\sqrt{x}) dx,$$

which is negative, as it is proved in [GC]. With $\varepsilon_F = 2$, we obtain

$$\frac{1}{2 \cdot 9^{l-1}} \int_0^1 (l+1)^2 - 9x)^{l-1}(1 - 3\sqrt{x}) dx .$$

Now,

$$\int_0^1 (l+1)^2 - 9x)^{l-1}(1 - 3\sqrt{x}) dx <$$

$$< (l+1)^{2(l-1)} \left( \int_0^{1/9} 1 - 3\sqrt{x} dx \right) + ((l+1)^2 - 9)^{l-1} \left( \int_1^{1/9} 1 - 3\sqrt{x} dx \right) =$$

$$= \frac{1}{27}((l+1)^{2(l-1)} - 28((l+1)^2 - 9)^{l-1}) ,$$
which is negative for all \( l \geq 3 \), by the same arguments in [GC].

If \( \tilde{M} \) is as in n.2, the only possibility for \( \varepsilon_F \) is \( \varepsilon_F = 2 \) and the integral is

\[
\frac{1}{4p-1} \int_0^1 \sqrt{x}((p + 1)^2 - 4x)^{p-1}((q + 1)^2 - 4x)^{q-1}(1 - 2\sqrt{x})dx .
\]

Now,

\[
\int_0^1 \sqrt{x}((p + 1)^2 - 4x)^{p-1}((q + 1)^2 - 4x)^{q-1}(1 - 2\sqrt{x})dx < \\
< (p + 1)^{2(p-1)}(q + 1)^{2(q-1)} \left( \int_0^{1/4} \sqrt{x}(1 - 2\sqrt{x})dx \right) + \\
+ ((p + 1)^2 - 4)^{p-1}((q + 1)^2 - 4)^{q-1} \left( \int_{1/4}^1 \sqrt{x}(1 - 2\sqrt{x})dx \right) = \\
= \frac{1}{48} \left( (p + 1)^{2(p-1)}(q + 1)^{2(q-1)} - 17((p + 1)^2 - 4)^{p-1}((q + 1)^2 - 4)^{q-1} \right) ,
\]

which is again negative for all \( p, q \geq 2 \).

If \( \tilde{M} \) is as in case n.3, we have only the possibility \( \varepsilon_F = 2 \) and the integral is

\[
\frac{1}{92l-6} \int_0^1 x^{3/2}((l + 3)^2 - 9x)^{2(l-3)}(2 - 3\sqrt{x})dx,
\]

which can be proved to be negative for all \( l \geq 4 \).

If \( \tilde{M} \) is as in case n.4, with \( \varepsilon_F = 2 \) we have that the integral is

\[
\int_0^1 x^{5/2}(4 - x)^4(3 - 4\sqrt{x})dx < 0,
\]

as it can be verified using e.g. some symbolic manipulation computer program like Mathematica. If we assume that \( \varepsilon_F = 1 \), the integral is

\[
\frac{1}{78} \int_0^1 x^{5/2}(256 - 49x)^4(6 - 7\sqrt{x})dx ,
\]

which can be checked to be positive.

If \( \tilde{M} \) is as in n.5, with \( \varepsilon_F = 1 \), the integral is

\[
\int_0^1 x^{7/2}(\frac{64}{9} - x)^8(8 - 9\sqrt{x})dx ,
\]

which can be checked to be positive; in case \( \varepsilon_F = 2 \), the integral is

\[
\int_0^1 x^{7/2}(\frac{144}{25} - x)^8(4 - 5\sqrt{x})dx
\]

which can be checked to be negative.
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