SPECTRAL FLOW FOR PAIR COMPATIBLE EQUIPARTITIONS

BERNARD HELFFER AND MIKAEL PERSSON SUNDQVIST

Abstract. We show that a recent spectral flow approach proposed by Berkolaiko–Cox–Marzuola for analyzing the nodal deficiency of the nodal partition associated to an eigenfunction can be extended to more general partitions. To be more precise, we work with spectral equipartitions that satisfy a pair compatible condition. Nodal partitions and spectral minimal partitions are examples of such partitions.

Along the way, we discuss different approaches to the Dirichlet-to-Neumann operators: via Aharonov–Bohm operators, via a double covering argument, and via a slitting of the domain.

1. Introduction

1.1. Main goals. We consider the Dirichlet Laplacian $L_{\Omega} = -\Delta$ in a bounded domain $\Omega \subset \mathbb{R}^2$ (and subdomains of $\Omega$), where $\partial \Omega$ is assumed to be piecewise differentiable.

We would like to analyze the relations between the nodal domains of the real-valued eigenfunctions of this Laplacian and the partitions $D$ of $\Omega$ by $k$ open sets $\{D_i\}_{i=1}^k$, which are spectral equipartitions in the sense that in each $D_i$’s the ground state energy $\lambda_1(D_i)$ of the Dirichlet realization of the Laplacian $L_{D_i}$ in $D_i$ is the same. In addition we will consider spectral equipartitions which satisfy a pair compatibility condition (PCC) for any pair of neighbouring $D_i$’s, i.e. for any pair of neighbors $D_i, D_j$ in, there is a linear combination of the ground states in $D_i$ and $D_j$ which is an eigenfunction of the Dirichlet problem in $\text{Int}(D_i \cup D_j)$.

Nodal partitions and minimal partitions are typical examples of these PCC-equipartitions but a difficult question is to recognize which PCC-equipartitions are minimal. This problem has been solved in the bipartite case (which corresponds to the Courant sharp situation) but the problem remains open in the general case.

Our main goal is to extend the construction and analysis of spectral flow and Dirichlet-to-Neumann operators, which was done for nodal partitions in [7], to spectral equipartitions that satisfy the PCC. We describe briefly the construction for nodal domains first:

1.2. The spectral flow construction by Berkolaiko–Cox–Marzuola. We describe shortly the result in [7] that we want to generalize, together with their construction.

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Let $\Omega \subset \mathbb{R}^2$ and let $\lambda_\ast$ be some eigenvalue of the Dirichlet Laplacian $L_\Omega$, with corresponding eigenfunction $\varphi_\ast$. We denote by $\Gamma$ the nodal set of $\varphi_\ast$ inside $\Omega$, i.e. $\Gamma = \{ x \in \Omega : \varphi_\ast(x) = 0 \}$, and introduce also the sets $\Omega_\pm = \{ x \in \Omega : \pm \varphi_\ast(x) > 0 \}$, so that $\varphi_\ast$ is positive in $\Omega_+$ and negative in $\Omega_-$. Also, let $k_\ast$ be the label of the eigenvalue $\lambda_\ast$ if it is simple and the minimal label if $\lambda_\ast$ is degenerate. Also, let $\nu(\varphi_\ast)$ denote the number of nodal domains of $\varphi_\ast$, i.e. the number of connected components of the set $\{ x \in \Omega : \varphi_\ast(x) \neq 0 \}$.

To state the main result of [7], we need to introduce Dirichlet-to-Neumann operators. We only do this at an intuitive level at this point, and refer the reader to [3] for more details. Assume that $E \subset \mathbb{R}^2$ is a bounded domain, and that $\lambda$ is not in the spectrum of $L_E$. Given a sufficiently regular function $g$ on $\partial E$, let $u$ be the unique solution to

$$\begin{cases}
-\Delta u = \lambda u & \text{in } E, \\
u = g & \text{on } \partial E.
\end{cases}$$

Then the Dirichlet-to-Neumann operator $\text{DN}_E(\lambda) : L^2(\partial E) \to L^2(\partial E)$ is defined by

$$\text{DN}_E(\lambda)g := \frac{\partial u}{\partial \nu},$$

where $\nu$ is a unit normal vector pointing out of $E$. For $\lambda$ in the spectrum of $L_E$ one has to be more careful and work in the orthogonal complement of a finite-dimensional subspace of $L^2(\partial E)$. Again, the reader is referred to [3, Section 2] for more details.

**Theorem 1.1 ([7]).** If $\varepsilon > 0$ is sufficiently small, then

$$k_\ast - \nu(\varphi_\ast) = 1 - \dim \ker (L_\Omega - \lambda_\ast) + \text{Mor}(\text{DN}_{\Omega_+}(\lambda_\ast + \varepsilon) + \text{DN}_{\Omega_-}(\lambda_\ast + \varepsilon)), \quad (1.1)$$

where $\text{Mor}$ counts the number of negative eigenvalues of an operator (the so-called Morse index of the operator).

**Remark 1.2.** The number $k_\ast - \nu(\varphi_\ast)$ in the left-hand side above is non-negative due to Courant’s nodal theorem. In [7] this quantity is called the nodal deficiency of the eigenfunction $\varphi_\ast$.

It turns out, that to characterize the negative eigenvalues of the sum $\text{DN}_{\Omega_+}(\lambda_\ast + \varepsilon) + \text{DN}_{\Omega_-}(\lambda_\ast + \varepsilon)$ it is fruitful to study the family of operators $L_{\Omega,\sigma}$, $0 \leq \sigma < +\infty$, induced by the bilinear form

$$\mathcal{B}_\sigma(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \sigma \int_\Gamma u v \, ds, \quad u,v \in H_0^1(\Omega).$$

Also, let $L_{\Omega, +\infty}$ be the Laplacian in $\Omega$ with Dirichlet boundary conditions imposed on $\partial \Omega \cup \Gamma$. Indeed, if we denote by $\{ \lambda_k(\sigma) \}_{k=1}^{+\infty}$ the set of eigenvalues of $L_{\Omega,\sigma}$, in increasing order, then Berkolaiko–Cox–Marzuola shows that if $\varepsilon > 0$ is sufficiently small, then $-\sigma$ is an eigenvalue of $\text{DN}_{\Omega_+}(\lambda_\ast + \varepsilon) + \text{DN}_{\Omega_-}(\lambda_\ast + \varepsilon)$ if, and only if, $\lambda_\ast + \varepsilon = \lambda_k(\sigma)$ for some $k \in \mathbb{N}$.

They also show that each analytic branch of the eigenvalues is increasing with $\sigma$. In fact, either it starts for $\sigma = 0$ with an eigenvalue of $L_{\Omega, +\infty}$, and then it will be constant as $\sigma$ increases, or the eigenvalue will increase strictly with $\sigma$. Moreover, as $\sigma \to +\infty$, the eigenvalues $\lambda_k(\sigma)$ converges to the eigenvalues of $L_{\Omega, +\infty}$.
Due to the construction, the eigenvalue $\lambda_*$ is in fact the lowest eigenvalue of $L_{\Omega, +\infty}$, with multiplicity $\nu(\varphi_*)$. Thus,

$$\lim_{\sigma \to +\infty} \lambda_k(\sigma) = \begin{cases} 
\lambda_*, & \text{if } 1 \leq k \leq \nu(\varphi_*), \\
> \lambda_*, & \text{if } k > \nu(\varphi_*) .
\end{cases}$$

By the definition of $k_*$, the operator $L_{\Omega, 0} = L_{\Omega}$ has exactly $k_* - 1 + \dim \ker(-L_{\Omega} - \lambda_*)$ eigenvalues less than, or equal to $\lambda_*$, and so exactly $k_* - 1 + \dim \ker(-L_{\Omega} - \lambda_*) - \nu(\varphi_*)$ of them will pass $\lambda_* + \varepsilon$, where $\varepsilon > 0$ is sufficiently small.

2. Examples: Equipartitions of the unit circle

Even though we will consider domains in $\mathbb{R}^2$, we start by doing some calculations for the unit circle. We assume that $N$ is odd and consider $N$-equipartitions $D$ (see Figure 2.1 for the cases $N = 3$ and $N = 5$),

$$k(D) = N$$

of the unit circle and the angular part of the Laplacian, $-\frac{d^2}{d\theta^2}$, with Dirichlet conditions at each sub-dividing point. Each interval have length $\Theta = \frac{2\pi}{N}$, and the smallest eigenvalue—the energy of the partition—is given by $\Lambda(D) = (N/2)^2$.

![Figure 2.1. The unit circle with (a) the 3-partition and (b) the 5-partition.](image)

The corresponding magnetic operator on the circle is given by

$$T = -\left( \frac{d}{d\theta} - \frac{i\pi}{2} \right)^2,$$

and its spectrum consist of eigenvalues $\{(\frac{2n-1}{2})^2\}_{n=1}^{+\infty}$, each with multiplicity two,

$$\dim \ker [T - (\frac{2n-1}{2})^2] = 2.$$  

In particular, the minimal label $\ell(D)$ of the eigenvalue $\Lambda(D) = (N/2)^2$ is given by

$$\ell(D) = N .$$

We are going to test the formula

$$\ell(D) - k(D) = 1 - \dim \ker (T - \Lambda(D)) + T(\varepsilon, D) ,$$

(2.1)
where \( T(\varepsilon, D) \) denotes the number of negative eigenvalues of a Dirichlet-to-Neumann operator, discussed below. In fact, since we just saw that \( \ell(D) = N, \ k(D) = N, \ \dim \ker(T - \Lambda(D)) = 2 \), we need to check that
\[
T(\varepsilon, D) = 1.
\]
This is similar to the setting for Quantum graphs. In [20] the number of negative eigenvalues of a Dirichlet-to-Neumann operator of a graph Laplacian corresponding to energy \( \lambda \) is calculated as a difference between the number of eigenvalues of the corresponding Neumann and Dirichlet graph laplacians less than \( \lambda \). But graphs with loops are excluded.

First we compute the Dirichlet-to-Neumann operator and the associated \( 2 \times 2 \) matrix \( M_\lambda \) which associates with the solution \( u \) of
\[
-\frac{d^2}{d\theta^2}u = \lambda u, \ u(0) = u_0, \ u(\Theta) = u_1,
\]
the pair
\[
(v_0, v_1) = (-u'(0), u'((\Theta))).
\]
This leads to
\[
\begin{bmatrix}
v_0 \\
v_1
\end{bmatrix} = M_\lambda \begin{bmatrix}
u_0 \\
u_1
\end{bmatrix},
\]
where \( M_\lambda \) is the matrix
\[
M_\lambda = \begin{bmatrix}
\sqrt{\lambda} \cot(\sqrt{\lambda} \Theta) & -\sqrt{\lambda} \\
-\frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda} \Theta)} & \sqrt{\lambda} \cot(\sqrt{\lambda} \Theta)
\end{bmatrix} = \begin{bmatrix}
\alpha(\lambda) & \beta(\lambda) \\
\beta(\lambda) & \alpha(\lambda)
\end{bmatrix},
\]
and \( \alpha(\lambda) \) and \( \beta(\lambda) \) are defined via the equation above. We continue in the same way along the circle. With \( (u_k, u_{k+1}) = (u(k\Theta), u((k+1)\Theta)) \) and \( (v_k, v_{k+1}) = (-u'(k\Theta), u'((k+1)\Theta)) \), we find that
\[
\begin{bmatrix}
v_k \\
v_{k+1}
\end{bmatrix} = M_\lambda \begin{bmatrix}
u_k \\
u_{k+1}
\end{bmatrix}, \quad 0 \leq k \leq N - 1.
\]
But when we come to \( (u_N, v_N) \) we have walked around the circle, and are back at the point we started. We replace \( (u_N, v_N) \) by \( (-u_0, -v_0) \). Thus, we find that the \( N \times N \) matrix \( M_\lambda \), that associates with \( (u_0, u_1, \ldots, u_{N-1}) \) the \( N \)-tuple \( (v_0, v_1, \ldots, v_{N-1}) \), is given by
\[
M_\lambda := \frac{1}{2} \begin{bmatrix}
2\alpha(\lambda) & \beta(\lambda) & 0 & 0 & \cdots & -\beta(\lambda) \\
\beta(\lambda) & 2\alpha(\lambda) & \beta(\lambda) & 0 & \cdots & 0 \\
0 & \beta(\lambda) & 2\alpha(\lambda) & \beta(\lambda) & \cdots & 0 \\
0 & 0 & \beta(\lambda) & 2\alpha(\lambda) & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \beta(\lambda) \\
-\beta(\lambda) & 0 & 0 & \cdots & \beta(\lambda) & 2\alpha(\lambda)
\end{bmatrix}.
\]
Thus, \( M_\lambda \) has \( \alpha \) on the main diagonal, \( \beta/2 \) on the two subdiagonals, and \( -\beta/2 \) in the corners \((1, N)\) and \((N, 1)\). The eigenvalues of \( M_\lambda \) are given by
\[
\mu_k = \alpha - \beta \cos(2k\pi/N), \quad k = 0, \ldots, N - 1.
\]
Hence the lowest one is \( \mu_0 = \alpha(\lambda) - \beta(\lambda) \), and this eigenvalue is negative if \( \sqrt{\lambda} = N/2 + \varepsilon \), with \( \varepsilon > 0 \) sufficiently small.
To analyze the positivity of the other eigenvalues, it suffices to analyze the sign of $\mu_1$. After division by $\beta(\lambda)$, we have to analyze the sign of
$$\delta_1 := -\cos(2\pi \sqrt{\lambda}/N) - \cos(2\pi/N).$$
If we take $\sqrt{\lambda} = N/2 + \varepsilon$, we have $\delta_1(\varepsilon) = \cos(2\pi \varepsilon/N) - \cos(2\pi/N) > 0$ for $\varepsilon > 0$ small enough.
We conclude that if $\sqrt{\lambda} = N/2 + \varepsilon$, with $\varepsilon > 0$ sufficiently small, then the matrix $M_\lambda$ has exactly 1 negative eigenvalue. This means that Formula (2.1) is indeed true.

Remark 2.1. A more general situation in one dimension, corresponding to an interval, is analyzed in [5].

3. Equipartitions: Notation and definitions

In this section, we describe in which framework we will generalize the results of [7].

3.1. Equipartitions, nodal partitions, and minimal partitions. We consider a bounded connected open set $\Omega$ in $\mathbb{R}^2$. A $k$-partition of $\Omega$ is a family $D = \{D_i\}_{i=1}^k$ of mutually disjoint, connected, open sets in $\Omega$ such that $\Omega = \bigcup_{i=1}^k D_i$. We denote by $O_k(\Omega)$ the set of $k$-partitions of $\Omega$. If $D = \{D_i\}_{i=1}^k \in O_k(\Omega)$ and the eigenvalues $\lambda_1(D_i)$ of the Dirichlet Laplacian in $D_i$ are equal for $1 \leq i \leq k$, we say that the partition $D$ is a spectral equipartition. This is the type of partitions we will work on. We give two examples of how such partitions occur.

We denote by $\{\lambda_j(\Omega)\}_{j=1}^\infty$ the increasing sequence of eigenvalues of the Dirichlet Laplacian in $\Omega$ and by $\{u_j\}_{j=1}^\infty$ some associated orthonormal basis of real-valued eigenfunctions. The ground state $u_1$ can be chosen to be strictly positive in $\Omega$, but the other eigenfunctions $\{u_j\}_{j \geq 2}$ must have zerosets.

For a function $u \in C^0(\overline{\Omega})$, we define the zero set of $u$ as
$$N(u) = \{x \in \Omega \mid u(x) = 0\},$$
and call the components of $\Omega \setminus N(u)$ the nodal domains of $u$. Such a partition of $\Omega$ is called a nodal partition, and we denote the number of nodal domains of $u$ by $\mu(u)$. These $\mu(u)$ nodal domains define a $k$-partition of $\Omega$, with $k = \mu(u)$.

Since an eigenfunction $u_j$, restricted to each nodal domain satisfy the eigenvalue equation $-\Delta u_j = \lambda_j u_j$ together with the Dirichlet boundary condition, it follows that each nodal partition is indeed a spectral equipartition. By the Courant nodal theorem, $\mu(u_j) \leq j$. We also say that the pair $(\lambda_j, u_j)$ is Courant sharp if $\mu(u_j) = j$.

For any integer $k \geq 1$, and for $D$ in $O_k(\Omega)$, we introduce the energy $\Lambda(D)$ of the partition $D$,
$$\Lambda(D) = \max_i \lambda_1(D_i).$$
Then we define
$$\mathcal{L}_k(\Omega) = \inf_{D \in O_k} \Lambda(D).$$
and call $D \in O_k$ a minimal spectral $k$-partition if $\mathcal{L}_k(\Omega) = \Lambda(D)$. 
If $k = 2$, it is rather well known (see [25] or [19]) that $\Sigma_2(\Omega) = \lambda_2(\Omega)$ and that the associated minimal 2-partition is a nodal partition, consisting of the nodal domains of some eigenfunction corresponding to second eigenvalue $\lambda_2(\Omega)$. In general, every minimal spectral partition is an equipartition (see [29]).

### 3.2. Regularity assumptions on partitions.

Attached to a partition $D$, we associate a closed set in $\Omega$, which is called the **boundary set** of the partition:

\[ \mathcal{N}(D) = \bigcup_i (\partial D_i \cap \Omega). \]

$\mathcal{N}(D)$ plays the role of the nodal set (in the case of a nodal partition).

Further, we call a partition $D$ **regular** if its associated boundary set $\mathcal{N}(D)$ is a regular closed set in $\Omega$. In general, a closed set $K \subset \Omega$ is said to be **regular closed** in $\Omega$ if

(i) Except for finitely many distinct critical points $\{x_\ell\} \subset K \cap \Omega$, the set $K$ is locally diffeomorphic to a regular curve. In the neighborhood of each critical point $x_\ell$ the set $K$ consists of a union of $\nu_\ell \geq 3$ smooth half-curves with one end at $x_\ell$.

(ii) The set $K \cap \partial \Omega$ consists of a (possibly empty) finite sets of boundary points $\{z_m\}$. Moreover, in a neighborhood of each boundary point $z_m$, the set $K$ is a union of $\rho_m$ distinct smooth half-curves with one end at $z_m$.

(iii) The set $K$ has the **equal angle meeting property**. By this we mean that the half-curves meet with equal angle at each critical point of $K$, as well as at the boundary (together with the tangent to the boundary).

Nodal sets are regular [8] and in [29] it is proven that minimal partitions are regular (modulo a set of capacity 0).

For our discussion we need a weaker version of regularity which is only expressed on the “boundary set”. The first and second items remain as in the previous definition, but (iii) is changed. Indeed, we say that the closed set $K \subset \Omega$ is **weakly regular** if (i) and (ii) above hold, and further if

(iv) The set $K \cap \partial \Omega$ consists of a (possibly empty) finite set of boundary points $\{z_m\}$. Moreover $K$ is near each boundary point $z_m$ the union of $\rho_m$ smooth half-curves (with distinct tangent vectors at $z_m$) which hit $z_m$ transversally to the boundary $\partial \Omega$.

### 3.3. Odd and even points.

Given a partition $D$ of $\Omega$, we denote by $X^{\text{odd}}(D)$ the set of odd critical points, i.e. points $x_\ell$ for which $\nu_\ell$ is odd. When $\partial \Omega$ has one exterior boundary and $m$ interior boundaries (corresponding to $m$ holes), we should also consider the property (see [28]) that an odd number of lines arrives at some component of the interior boundary (think of the hole as a point). It seems that the assumption that there was only one boundary component was implicitly done in the litterature, or at least we should distinguish between the odd interior boundaries and the even boundaries. This would play a role in the definition of the Aharonov–Bohm operator or in the construction of the double covering.

We define by $\partial \Omega^{\text{odd}}(D)$ the union of the interior components of $\partial \Omega$ for which an odd number of lines of $\mathcal{N}(D)$ arrive. In other words, we will speak
Figure 3.1. Partitions of a set $\Omega$ with three holes. In both cases $\nu_1 = 5$, $\rho_1 = \rho_3 = \rho_4 = \rho_5 = 1$ and $\rho_2 = 3$. (a) A regular partition. Note that the angles between the curves meeting at $x_1$ are $2\pi/5$ and that the angles between the curves meeting at $z_2$ and the boundary is $\pi/4$. At $z_1$, $z_3$, $z_4$ and $z_5$ the curves meet the boundary under a right angle. (b) This partition is weakly regular. The curves meet the boundary transversally, but not necessarily under equal angles. At the critical point $x_1$, two of the curves even meet tangentially.

of odd holes when we are in this case and $\partial \Omega^{\text{odd}}(D)$ corresponds to the union of the boundaries of the odd holes. In Figure 3.2 we have marked $\partial \Omega^{\text{odd}}(D)$ in bold.

Figure 3.2. (a) The partition $D$ of $\Omega$ from Figure 3.1(a), here with the set $\partial R^{\text{odd}}(D)$ in bold. (b) The graph $G(D)$ associated with the partition $D$. Note that it is bipartite.
3.4. **Pair compatibility condition.** Given an partition \( \mathcal{D} = \{D_i\} \) of \( \Omega \), we say that \( D_i \) and \( D_j \) are neighbors, which we write \( D_i \sim D_j \), if the set \( D_{ij} := \text{Int}(D_i \cup D_j) \setminus \partial \Omega \) is connected. We associate with \( \mathcal{D} \) a graph \( G(\mathcal{D}) \) by associating with each \( D_i \) a vertex and to each pair \( D_i \sim D_j \) an edge. We recall that a graph is said to be bipartite if its vertices can be colored by two colors so that all pairs of neighbors have different colors. We say that \( \mathcal{D} \) is admissible if the associated graph \( G(\mathcal{D}) \) is bipartite. Nodal partitions are always admissible, since the eigenfunction changes sign when going from one nodal domain to a neighbor nodal domain.

We turn to a compatibility condition between neighbors in a partition, developed in [25]. Let \( \mathcal{D} = \{D_i\}_{i=1}^k \) be a regular equipartition of energy \( \Lambda(D) \). Given two neighbors \( D_i \) and \( D_j \), \( \Lambda(D) \) is the groundstate energy of both \( L_{D_i} \) and \( L_{D_j} \). There is, however, in general no way to construct a function \( u_{ij} \) in the domain of \( L_{D_{ij}} \) such that \( u_{ij} = c_i u_i \) in \( D_i \) and \( u_{ij} = c_j u_j \) in \( D_j \). For this to be possible, it must hold that the normal derivatives of \( u_i \) and \( u_j \) are proportional on \( \partial D_i \cap \partial D_j \).

We say that the regular partition \( \mathcal{D} = \{D_i\}_{i=1}^k \) satisfies the pair compatibility condition, (for short PCC), if, for some \( \lambda \in \mathbb{R} \), and for any pair \((i,j)\) such that \( D_i \sim D_j \), there is an eigenfunction \( u_{ij} \neq 0 \) of \( L_{D_{ij}} \) such that \( L_{D_{ij}} u_{ij} = \lambda u_{ij} \), and where the nodal set of \( u_{ij} \) is given by \( \partial D_i \cap \partial D_j \). We refer to Figure 4.1 for some 5-partitions of the square that satisfy the PCC. Nodal partitions and spectral minimal partitions satisfy the PCC.

3.5. **Admissible \( k \)-partitions and Courant sharp eigenvalues.** It has been proved by Conti–Terracini–Verzini [17, 18, 19] and Helffer–T. Hoffmann-Ostenhof–Terracini [29], that, for any \( k \in \mathbb{N} \), there exists a minimal regular \( k \)-partition. Other proofs of a somewhat weaker version of this statement have been given by Bucur–Buttazzo–Henrot [15], Caffarelli–F.H. Lin [16]. It is also proven (see [25], [29]) that if the graph of a minimal partition is bipartite, then this partition is nodal. A natural question was to determine how general the previous situation is. Surprisingly this only occurs in the Courant sharp situation.

For any integer \( k \geq 1 \), we denote by \( L_k(\Omega) \) the smallest eigenvalue of \( L_\Omega \), whose eigenspace contains an eigenfunction with \( k \) nodal domains. We set \( L_k(\Omega) = +\infty \), if there are no eigenfunction with \( k \) nodal domains. In general, one can show that

\[
\lambda_k(\Omega) \leq \mathcal{L}_k(\Omega) \leq L_k(\Omega).
\]

The following result gives the full picture of the equality cases:

**Theorem 3.1** ([29]). Suppose that \( \Omega \subset \mathbb{R}^2 \) is smooth and that \( k \in \mathbb{N} \). If \( \mathcal{L}_k(\Omega) = L_k(\Omega) \) or \( \mathcal{L}_k(\Omega) = \lambda_k(\Omega) \) then

\[
\lambda_k(\Omega) = \mathcal{L}_k(\Omega) = L_k(\Omega),
\]

and one can find a Courant sharp eigenpair \((\lambda_k, u_k)\).

4. **The Aharonov–Bohm approach**

4.1. **The Aharonov–Bohm operator.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded connected domain. We recall some definitions and results about the Aharonov–Bohm (AB) Hamiltonian with poles at a finite number of points. These
results were initially motivated by the work of Berger–Rubinstein [6] and further developed in [1, 28, 10, 9]. We begin with the case of one pole.

4.1.1. Simply connected $\Omega$, one AB pole. We assume that there is one AB pole $X$ is located at $X = (x_0, y_0) \in \Omega$ and introduce the magnetic vector potential

\[ A^X(x, y) = (A^X_1(x, y), A^X_2(x, y)) = \frac{\Phi}{2\pi} \left( -\frac{y - y_0}{r^2}, \frac{x - x_0}{r^2} \right). \]

Here $\Phi$ is the intensity of the AB magnetic field, and $r$ denotes the Euclidean distance between $(x, y)$ and $(x_0, y_0)$. We know that in this case the magnetic field vanishes identically in the punctured domain $\bar{\Omega}_X = \Omega \setminus \{ X \}$. We introduce the magnetic gradient $\nabla A^X$ as

\[ \nabla A^X = \nabla - iA^X, \]

and consider the self-adjoint AB Hamiltonian $T^A_X = -(\nabla A^X)^2$. This operator is defined as the Friedrichs extension associated with the quadratic form

\[ C^+_{0,\infty}(\bar{\Omega}_X) \ni u \mapsto \int_{\Omega} \left| \nabla A^X u \right|^2 dx. \]

We introduce next the multi-valued complex argument function

\[ \varphi_X(x, y) = \arg(x - x_0 + i(y - y_0)). \]

This function satisfies

\[ A^X = \frac{\Phi}{2\pi} \nabla \varphi_X. \]

This implies that with the flux condition

\[ \frac{\Phi}{2\pi} = \frac{1}{2}, \]

one has

\[ -A^X = A^X - \nabla \varphi_X, \]

and that multiplication with the function $e^{i\varphi_X}$, uni-valued in $\bar{\Omega}_X$, is a gauge transformation intertwining $T^A_X$ and $T^-A^X_X$.

The anti-linear operator $K^X_X : L^2(\Omega) \to L^2(\Omega)$, defined by

\[ u \mapsto K^X_X u = \exp(i\varphi_X) \tilde{u} \]

becomes a conjugation operator. In particular $(K^X_X)^2$ is the identity operator,

\[ \langle K^X_X u, K^X_X v \rangle = \langle u, v \rangle, \]

and

\[ K^X_X T^A_X = T^-A^X_X K^X_X. \]

We say that a function $u$ is $K^X_X$-real, if it satisfies $K^X_X u = u$. Then the operator $T^A_X$ is preserving the $K^X_X$- real functions. In the same way one proves that the Dirichlet Laplacian admits an orthonormal basis of real valued eigenfunctions or one restricts this Laplacian to the vector space over $\mathbb{R}$ of the real-valued $L^2$ functions, one can construct for $T^A_X$ a basis of $K^X_X$-real eigenfunctions or, alternately, consider the restriction of the AB Hamiltonian to the vector space over $\mathbb{R}$

\[ L^2_{K^X_X}(\bar{\Omega}_X) = \{ u \in L^2(\bar{\Omega}_X), K^X_X u = u \}. \]
4.1.2. Simply connected $\Omega$, several AB poles. We can extend our construction of an Aharonov–Bohm Hamiltonian in the case of a configuration with $\ell$ distinct points $X = \{X_j\}_{j=1}^\ell$ in $\Omega$ (putting a flux $\Phi = \pi$ at each of these points). We can just take as magnetic potential

$$A^X = \sum_{j=1}^\ell A^{X_j}.$$ 

The corresponding AB Hamiltonian $T_{A^X}$ is again defined as the Friedrichs extension, this time via the natural quadratic form in $C_0^+ (\dot{\Omega})$, where $\dot{\Omega} = \Omega \setminus X$.

We can also construct (see [28]) the anti-linear operator $K_X$, where $\varphi_X$ is replaced by a multivalued function $\Phi^X = \sum_{j=1}^\ell \varphi_{X_j}$ which satisfies $\nabla \Phi^X = 2 A^{X_j}$. Moreover

$$\exp(i\Phi^X) = \prod_{j=1}^\ell \exp(i\varphi_{X_j})$$

is uni-valued and belongs to $C^\infty (\dot{\Omega})$. As in the case of one AB pole, we can consider the (real) subspace of the $K^X$-real functions in $L^2_{K^X} (\dot{\Omega})$, and our operator as an unbounded selfadjoint operator in $L^2_{K^X} (\dot{\Omega})$.

4.1.3. Non-simply connected $\Omega$. If $\Omega$ is not simply connected, we also accept some of the Aharonov–Bohm fluxes to be placed in holes in the bounded components of the complement of $\Omega$. If $\Omega$ has $m$ holes $\omega_1, \ldots, \omega_m$, we allow $m'$ “odd” poles $X'_i$ (with $0 \leq m' \leq m$ and $X'_i \in \omega_i$ for $i \in \{1, \ldots, m'\}$).

We call odd holes the holes for which we have introduced the poles $X'_i$. In this case, we can reproduce the same construction. We call the remaining holes in $\Omega$ even.

At the end, for $\hat{X} := X \cup X'$, we have constructed an AB Hamiltonian in $\dot{\Omega}$ associated with a magnetic potential $A^\hat{X}$ with poles at $X$ and half renormalized flux created by the magnetic potential in each odd hole, the flux created in the even hole being 0. Similarly, we can as before use the “conjugation” operator $K_{\hat{X}}$.

It was shown in [28, 1] (inspired by the previous work [6]) that the nodal set of such a $K^\hat{X}$-real eigenfunction has the same structure as the nodal set of a real-valued eigenfunction of the Dirichlet Laplacian except that an odd number of half-lines meet at each pole (see Subsection 3.3), and that the number of lines meeting the interior boundary should be odd (resp. even) at the boundary of an odd (resp. even) hole.

4.2. Equipartitions and nodal partitions of AB Hamiltonians. We start from constructions introduced in [25, 9]. Suppose $\Omega$ is a bounded, simply connected (thus, to simplify, we describe the case without holes), domain and that $\partial \Omega$ is piecewise differentiable. Let $D$ be a regular $k$-equipartition with energy $\Lambda(D) = l_k(\Omega)$ satisfying the PCC.
We further set \( H \) where we use the standard trace for an element of \( \text{tr} \) and the ground state energy of the Dirichlet Laplacian \( H \) when \( D_i \sim D_j \) (here we have extended \( u_i \) and \( u_j \) by 0 outside of \( D_i \) and \( D_j \), respectively, and we recall that \( D_{ij} = \text{Int}(D_i \cup D_j) \)).

In [29] it was proven that there exists a family \( \{ u_i \}_{i=1}^{k} \) of functions such that \( u_i \) is a ground state of \( L_{D_i} \) and \( u_i - u_j \) is a second eigenfunction of \( L_{D_{ij}} \), where \( D_i \sim D_j \). This permits to show that the sesquilinear form

\[
B(u,v) = \int_{\Omega} \nabla_A u \cdot \nabla_A v + \sigma \int_{\Gamma} u \overline{v} \, dS_{\Gamma},
\]

where \( A \) is the magnetic AB potential: \( A = A^X \) and \( dS_{\Gamma} \) is the induced measure on (each arc of) \( \Gamma \).

We consider the family \( \{ B_\sigma \}_{\sigma \in \mathbb{R}} \) of sesquilinear forms defined on the magnetic Sobolev space \( H^1_{0,A}(\Omega) \times H^1_{0,A}(\Omega) \) (see Lénà [32] and also [22]) by

\[
(u,v) \mapsto B_\sigma(u,v) = \int_{\Omega} \nabla_A u \cdot \nabla_A v + \sigma \int_{\Gamma} u \overline{v} \, dS_{\Gamma},
\]

where \( \sigma \) is the magnetic AB potential: \( A = A^X \) and \( dS_{\Gamma} \) is the induced measure on (each arc of) \( \Gamma \).

We should explain how the integral over \( \Gamma \) is to be interpreted. For each arc \( \gamma_i \) in \( \Gamma \cap \overline{\Omega_X} \), we can define in its neighborhood \( V(\gamma_i) \) in \( \overline{\Omega_X} \) a \( C^\infty \) square root \( \sqrt{\exp(i\Phi/2)} \) of \( \exp(i\Phi) \) and we have \( \exp(i\Phi/2)u \in H^1(\gamma_i) \) if \( u \in H^1_A(\Omega) \). We can then define

\[
\int_{\gamma_i} u \overline{v} \, dS_{\gamma_i} := \int_{\gamma_i} (\exp(i\Phi/2)u) \cdot (\exp(i\Phi/2)v) \, dS_{\gamma_i},
\]

where we use the standard trace for an element of \( H^1 \). Note that, with this definition, the “magnetic trace space” on \( \gamma_i \) is identified as

\[
H^{1/2}(\gamma_i) := \exp(-i\Phi/2)H^{1/2}(\gamma_i).
\]

We further set \( H^{1/2}(\Gamma) := \oplus_i H^{1/2}(\gamma_i) \), and writing

\[
\int_{\Gamma} u \overline{v} \, dS_{\Gamma} = \sum_i \int_{\gamma_i} u \overline{v} \, dS_{\gamma_i},
\]

this permits to show that the sesquilinear form \( B_\sigma \) is continuous.

Associated with this sesquilinear form we have the corresponding magnetic-Robin AB Hamiltonian \( L_\sigma \) defined as the Friedrichs extension.

---

1Note that by construction the \( D_i \)'s never contain any point of \( X \). By Euler formula a path \( \gamma \) in \( D_i \) can only contain an even number of points in \( X \). Hence the square root is well defined and the ground state energy of the Dirichlet Laplacian \( L_{D_i} \) is the same as the ground state energy of \( H^X \) in \( D_i \). See [32].
We also define $L_{+\infty}$ as the corresponding AB magnetic Schrödinger operator, with Dirichlet boundary conditions at $\partial \Omega \cup \Gamma$.

We collect some properties of the operators $\{L_\sigma\}$.

**Proposition 4.1.** The self-adjoint operators $\{L_\sigma\}, -\infty < \sigma \leq +\infty$, have compact resolvents.

Moreover, if $\sigma \in \mathbb{R}$, then the domain of $L_\sigma$ consists of all elements $u \in H^1_{0,A}(\Omega)$ such that $(\nabla_A)^2 u \in L^2(\Omega)$, and such that the following transmission conditions are satisfied: If $D_i$ and $D_j$ are two neighbors in the partition $\mathcal{D}$ of $\Omega$, and $\gamma$ is a regular arc in $\partial D_i \cap \partial D_j$, then, on $\gamma$,

$$\nu_i \cdot \nabla_A u_i = \nu_i \cdot \nabla_A u_j = \sigma (u_j - u_i),$$

where $\nu_i$ is the exterior normal to $D_i$ (at a point of $\gamma$) and $u_i$ denotes the restriction of $u$ to $D_i$.

**Proof.** The proof follows in the same manner as for the Laplace operator, with small additions or modifications. We refer to [2, Proposition 2.2] for the characterization of domain, and the compactness of the resolvent. Here, one should note that the magnetic Sobolev space $H^1_{0,A}(\Omega)$ is continuously embedded in the ordinary Sobolev space $H^1(\Omega)$ if the fluxes around the poles are non-integers (see [32, Corollary 2.5]).

For the transmission conditions along the boundary set, we refer to [23]. □

Given $-\infty < \sigma \leq +\infty$, we denote by $\{\hat{\lambda}_n(\sigma)\}_{n \in \mathbb{N}}$ the analytic eigenvalue branches of $L_\sigma$, and we enumerate by $\{\lambda_n(\sigma)\}_{n \in \mathbb{N}}$ the increasing sequence of eigenvalues of $L_\sigma$, counted with multiplicity. As in [7, Lemma 2], a perturbative argument shows that $\sigma \mapsto \hat{\lambda}_n(\sigma)$ is either strictly increasing or equal to $\hat{\lambda}_n(0)$, and the latter case only occurs when $\hat{\lambda}_n(0)$ is an eigenvalue of $L_{+\infty}$.

**Proposition 4.2.** As $\sigma \to +\infty$,

$$\lambda_n(\sigma) \to \lambda_n(+\infty).$$

**Proof.** The resolvents of $L_\sigma$ converge to the resolvent of $L_{+\infty}$ as $\sigma \to +\infty$, see [2, Proposition 2.6] for the proof in the case of the Laplacian. It then follows (see [2, Proposition 2.8]) that the eigenvalues also converge. □

The operator $L_{+\infty}$ can be identified as the direct sum of the AB magnetic Schrödinger operators with vector potential $A$ on each component $D_i$ of the partition $\mathcal{D}$, with Dirichlet boundary conditions on $\partial D_i$. It remains to prove that we can gauge away the magnetic potential. For this we need

**Lemma 4.3.** In each $D_j$, the square root $\exp(i\Phi_X/2)$ can be defined as a univalued function $\exp(i\varphi_j)$ in $C^\infty(D_j)$. Moreover,

$$\nabla_A \exp(i\varphi_j) = \exp(i\varphi_j) \nabla, \quad \text{in } D_j.$$

**Proof.** It suffices to observe that for each $X_\ell$, $\exp(i\Phi_X/2)$ has this property (distinguish the case when $X_\ell \in \partial D_j$ or not). □

We can now construct the magnetic Neumann–Poincaré operator (called $\Lambda_{+}(\varepsilon) + \Lambda_{-}(\varepsilon)$ in [7] in the case without magnetic field). For this we proceed in the following way.
For each $D_j$, we consider $\partial D_j \cap \Omega$. We introduce the magnetic Dirichlet–Neumann operator on $\partial D_j$ which associates, for $\varepsilon > 0$, to a function $h \in H^{1/2}_{\mathbf{A}}(\partial D_j)$, vanishing on $\partial \Omega \cap \partial D_j$ a solution $u$ to

$$
\begin{align*}
T_{\mathbf{A}} u &= (I_k + \varepsilon) u \quad \text{in } D_j, \\
u u &= h \quad \text{on } \partial D_j.
\end{align*}
$$

(4.3)

Assuming first that $\mathbf{A}$ is regular, we define a pairing of elements in $H^{-1/2}_{\mathbf{A}}(\partial D_j)$ and $H^{1/2}_{\mathbf{A}}(\partial D_j)$, inspired by how it is done in the non-magnetic case by the Green–Riemann formula.

If $v_0 \in H^{1/2}_{\mathbf{A}}(\partial D_j)$ there exists $w_0 \in H^1_{\mathbf{A}}(D_j)$ such that

$$
\begin{align*}
&-\langle \nabla \mathbf{A} \rangle^2 w_0 = 0 \quad \text{in } D_j, \\
&w_0 = v_0 \quad \text{on } \partial D_j.
\end{align*}
$$

The mapping $v_0 \mapsto w_0$ is continuous from $H^{1/2}_{\mathbf{A}}(\partial D_j)$ into $H^1_{\mathbf{A}}(D_j)$. Then, we set

$$
\langle \nu_j \cdot \nabla \mathbf{A} u, v_0 \rangle_{H^{-1/2}_{\mathbf{A}}(\partial D_j), H^{1/2}_{\mathbf{A}}(\partial D_j)} := -\langle \nabla \mathbf{A} u, \nabla w_0 \rangle + \langle (\nabla \mathbf{A})^2 u, w_0 \rangle,
$$

(4.4)

where $\nu_j$ is the exterior normal derivative to $\partial D_j$.

Actually, to avoid possible problems with the singularities of $\mathbf{A}$, we can come back to the the case $\mathbf{A} = 0$. We observe that $\exp(i\varphi_j)u$ belongs to $H^{1/2}(\partial D_j)$, where $\varphi_j$ is defined in Lemma 4.3 (this will be our definition of $H^{1/2}_{\mathbf{A}}(\partial D_j)$ in the case of singularities which coincides with the usual one in the regular case). So $\nu_j \cdot \nabla(\exp(i\varphi_j)u)$ belongs to $H^{-1/2}(\partial D_j)$. After multiplication with $\exp(-i\varphi_j)$ we end up in $H^{-1/2}_{\mathbf{A}}(\partial D_j)$.

We then define, for each $D_j$, the reduced magnetic Dirichlet–Neumann operator on $H^{1/2}_{\mathbf{A}}(\partial D_j \cap \Omega)$ by restricting the magnetic Dirichlet–Neumann operator initially defined on $H^{1/2}(\partial D_j)$ and identifying $H^{1/2}_{\mathbf{A}}(\partial D_j \cap \Omega)$ to $\hat{H}^{1/2}_{\mathbf{A}} := \{h \in H^{1/2}_{\mathbf{A}}(\partial D_j), h = 0 \text{ on } \partial \Omega \cap \partial D_j\}$:

$$
\Lambda_{\mathbf{A}}(\varepsilon, I_k)h = \nu_j \cdot \nabla \mathbf{A} u|_{\partial D_j \cap \Omega} := \exp(-i\varphi_j)(\nu_j \cdot \nabla(\exp(i\varphi_j)u)).
$$

We have to verify the compatibility of our definition of $H^{1/2}_{\mathbf{A}}$ in the common boundaries of neighbors $D_i$ and $D_j$, i.e. that the restriction of $H^{1/2}_{\mathbf{A}}(D_i)$ to $\partial D_i \cap \partial D_j \cap \Omega$ coincides with the restriction of $H^{1/2}_{\mathbf{A}}(D_j)$ to $\partial D_i \cap \partial D_j \cap \Omega$. For this it suffices to observe that, for some constant $c_{ij} \neq 0$,

$$
\exp(i\tilde{\varphi}_j) = e_{ij} \cdot \exp(i\tilde{\varphi}_j)
$$

where $\tilde{\varphi}_i$ (respectively $\tilde{\varphi}_j$) is the natural extension of $\varphi_i$ (respectively $\varphi_j$) in a neighborhood of $\partial D_i \cap \partial D_j \cap \Omega_X$ in $\hat{\Omega}_X$ as a solution of $d\tilde{\varphi}_i = \mathbf{A}$ (respectively $d\tilde{\varphi}_j = \mathbf{A}$).

At this point the Neumann–Poincaré operator $\Lambda_{\mathbf{A}}^{NP}(\varepsilon, \mathcal{D})$ is defined as an operator from $H^{1/2}_{\mathbf{A}}(\Gamma)$ into $H^{-1/2}_{\mathbf{A}}(\Gamma)$:

$$
\Lambda_{\mathbf{A}}^{NP}(\varepsilon, \mathcal{D}) = \sum_{j=1}^k \epsilon_j \Lambda_{\mathbf{A}}(\varepsilon, I_k) r_j,
$$

(4.5)
where \( r_j \) is the restriction of \( H^{1/2}_A(\Gamma) \) to \( H^{1/2}_A(\Omega \cap \partial D_j) \) and \( t_j \) is the extension \((by 0)\) of the operator from \( H^{1/2}_A(\Omega \cap \partial D_j) \) to \( H^{-1/2}_A(\Gamma) \).

**Proposition 4.4.** The operator \( \Lambda^{NP}_A(\epsilon, D) \) is self-adjoint.

**Proof.** The proof is similar to the non-magnetic case, and it is based on the corresponding magnetic Green–Riemann formula. We consider one component \( D_1 \). Assume that \( u \) and \( v \) belong to \( H^{1/2}_A(D_1) \) and that \((\nabla_A)^2 u \) and \((\nabla_A)^2 v \) belong to \( L^2(D_1) \). Then we claim that

\[
\langle \nu \cdot \nabla_A u, v|_{\partial D_1} \rangle_{H^{1/2}_1(\partial D_1), H^{-1/2}_2(\partial D_1)} = -\langle \nabla_A u, \nabla_A v \rangle + \langle (\nabla_A)^2 u, v \rangle.
\]

Indeed, according to (4.4), with \( v_0 = v|_{\partial D_1} \),

\[
\langle \nu \cdot \nabla_A u, v|_{\partial D_1} \rangle_{H^{1/2}_1(\partial D_1), H^{-1/2}_2(\partial D_1)} = -\langle \nabla_A u, \nabla_A w_0 \rangle + \langle (\nabla_A)^2 u, w_0 \rangle
\]

\[
= -\langle \nabla_A u, \nabla_A v \rangle + \langle (\nabla_A)^2 u, v \rangle - \langle \nabla_A u, \nabla_A (w_0 - v) \rangle + \langle (\nabla_A)^2 u, (w_0 - v) \rangle
\]

\[
= -\langle \nabla_A u, \nabla_A v \rangle + \langle (\nabla_A)^2 u, v \rangle.
\]

In the last step we used the fact that \( w_0 - v \) satisfies a Dirichlet condition at \( \partial D_1 \), so the terms from the two preceding lines cancel each other.

With the additional condition that \( T_A \) \( u = (\ell_k + \epsilon) u \), we find that

\[
\langle \nu \cdot \nabla_A u, v|_{\partial D_1} \rangle_{H^{1/2}_1(\partial D_1), H^{-1/2}_2(\partial D_1)} = -\langle \nabla_A u, \nabla_A v \rangle + \langle (\nabla_A)^2 u, v \rangle.
\]

If we further assume that \( T_A \) \( v = (\ell_k + \epsilon) v \), then

\[
\langle \nu \cdot \nabla_A u, v|_{\partial D_1} \rangle_{H^{1/2}_1(\partial D_1), H^{-1/2}_2(\partial D_1)} = \langle u|_{\partial D_1}, \nu \cdot \nabla_A v \rangle_{H^{1/2}_1(\partial D_1), H^{-1/2}_2(\partial D_1)}.
\]

The self-adjointness follows.

Following [7], we denote by \( \tau_A(\epsilon, D) \) the number of negative eigenvalues of \( \Lambda^{NP}_A(\epsilon, D) \). We introduce the defect \( \text{Def}(D) \) of the partition \( D \) as

\[
\text{Def}(D) := \ell(D) - k(D),
\]

where \( \ell(D) \) denotes the minimal labelling of the eigenvalue \( \ell_k \) of the AB Hamiltonian \( T_A \), and \( k = k(D) \) is the number of components of the partition \( D \). We are ready to state our main result, and for simplicity we do it for simply connected domains \( \Omega \) (the only change for the general case would be in the definition of the magnetic Aharonov–Bohm potential).

**Theorem 4.5.** Let \( D \) be a regular \( k \)-equipartition of a simply connected domain \( \Omega \) satisfying the PCC with energy \( \ell_k = \ell_k(\Omega) \). Let \( A = A^X \) be the associated Aharonov–Bohm potential. Then, for sufficiently small \( \epsilon > 0 \),

\[
\text{Def}(D) = 1 - \dim \ker(T_A - \ell_k) + \tau_A(\epsilon, D).
\]

**Lemma 4.6.** Assume that \( \sigma > 0 \). Then \( -\sigma \) is an eigenvalue of \( \Lambda^{NP}_A(\epsilon, D) \) if, and only if, \( \ell_k + \epsilon \) is an eigenvalue of \( L_\sigma \). If this is the case, then the multiplicities agree.

**Proof.** This is merely by construction, with the transmission conditions from Proposition 4.1. We refer to [2, Theorem 4.1] and to [7, Lemma 1].

\[\square\]
Proof of Theorem 4.5. The proof is similar to the proof of equation (3) in [7]. The first eigenvalue $\lambda_1(+\infty)$ of $L_{+\infty}$ is also the first Dirichlet eigenvalue on each component of $D$, and hence it has multiplicity $k(D)$. Moreover, it equals $\ell_k$. Hence,

$$
\lim_{\sigma \to +\infty} \lambda_n(\sigma) \begin{cases} 
\ell_k, & 1 \leq n \leq k, \\
> \ell_k, & n > k.
\end{cases}
$$

The operator $L_0 = T_A$, on the other hand, has $\ell(D) + \dim \ker(T_A - \ell_k) - 1$ eigenvalues less than or equal to $\ell_k$, and exactly $k(D)$ of them will converge to $\ell_k$ as $\sigma \to +\infty$. This means that $\ell(D) + \dim \ker(T_A - \ell_k) - 1 - k(D)$ eigenvalues of $L_0$ will cross $\ell_k + \varepsilon$ for some finite $\sigma > 0$, if $\varepsilon > 0$ is sufficiently small.

According to Lemma 4.6, every such crossing gives rise to a negative eigenvalue of $A_X^{NP}(\varepsilon, D)$, including counting multiplicity. \(\square\)

Remark 4.7. It would be interesting to understand, like in the bipartite situation, the link between the zero deficiency property

$$
1 - \dim \ker(T_A - \ell_k) + \tau_A(\varepsilon, D) = 0,
$$

and the minimal partition property.

It is mentioned in [26, Remark 5.2] that if we have a minimal $k$-partition then we are in the Courant sharp situation for the corresponding AB Hamiltonian $T_A$, i.e. it has the zero deficiency property.

The converse is true as recalled above for a bipartite partition but wrong in general. A counterexample is given for the square and $k = 5$ in [9, Fig. 19], which is kindly reproduced in Figure 4.1. We have on the left a 5-partition with one critical odd point which is the nodal partition of the 5-th eigenfunction of its associated AB operator, but is not minimal. We have on the right a 5-partition with four critical odd points which is the nodal partition of the 5-th eigenfunction of its associated AB operator, which is not minimal. It is conjectured that a minimal 5-partition is indeed obtained for the middle configuration with also four odd critical points.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure41.png}
\caption{Three 5-equipartitions satisfying the PCC, with 0 deficiency index. The middle one has minimal energy among these three.}
\end{figure}

4.4. The Berkolaiko–Cox–Marzola construction through double covering lifting. In many of the papers analyzing minimal partitions, the authors refer to a double covering argument. This point of view (which appears
first in [28] in the case of domains with holes) is essentially equivalent to the
Aharanov approach. We just mention the main lines of the argument. One
can, in an abstract way, construct a double covering manifold \( \tilde{\Omega} := \Omega X \)
above \( \Omega X \). One can then lift the initial spectral problem to one for the Laplace
operator on this new (singular) manifold \( \tilde{\Omega} \). In this lifting, the \( K \)
- real eigenfunctions become eigenfunctions which are real and antisymmetric with
respect to the deck map (exchanging two points having the same projection
on \( \Omega X \)).

In the case of the disk, the construction is equivalent to considering the
angular variable \( \theta \in (0, 4\pi) \), and the deck map corresponds to the translation
by \( 2\pi \). The nodal set of the 6-th eigenfunction gives by projection the
Mercedes star and the 11-th eigenvalue (which is the 5-th in the space
of antiperiodic functions) gives by projection the candidate for a minimal
three-partition.

Starting from an eigenfunction \( u \) of \( TA \) with zeroset \( \Gamma \), the idea is now to
apply the construction of Berkolaiko–Cox–Marzola to the Laplacian on
\( \tilde{\Omega} \) and the lifted eigenfunction \( \tilde{u} \), having in mind that this is an antisymmetric
eigenfunction. The zero-set of \( \tilde{u} \) is \( \Pi^{-1}(\Gamma) \). One should then interpret the
quantities for the covering in term of the basis.

Hence we should define the Poincaré-Neumann attached to the Laplacian
on \( \tilde{\Omega} \) and \( \tilde{\Gamma} = \Pi^{-1}(\Gamma) \) and reinterpret it when restricted to antisymmetric
functions on \( \tilde{\Gamma} \). The spectrum of \( TA \) consists of the eigenvalues corresponding
to the antisymmetric eigenfunctions of \( -\Delta \). Hence the labelling of the eigen-
value of \( TA \) corresponds to the labelling of \( -\Delta \) restricted to the antisymmetric
space.

5. The Cutting Construction for General Regular
PCC-equipartitions

5.1. Example: the Mercedes star. We first consider the case when we
have in \( \Omega \) a 3-partition, with only one critical point (which has the topology of the
Mercedes star). We can assume that the critical point is at \( 0 \in \Omega \) and we
denote by \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) the three branches of the star. The partition consists
in three open sets denoted by \( D_3 := \tilde{D}_{12}, D_1 := \tilde{D}_{23} \) and \( D_2 := \tilde{D}_{31} \), where
we have \( \partial \tilde{D}_{ij} \cap \Omega = \Gamma_i \cup \Gamma_j \). One such example is given in Figure 5.1. Starting
from the “formal” Aharanov–Bohm point of view, we try to eliminate the
reference to this operator by using a suitable square root \( \exp(i\varphi X(\cdot)/2) \).

The idea is to look at our Robin form as a sesquilinear form on a functional
space defined on \( \Omega \setminus \Gamma_1 \). We note that \( \exp(i\varphi X(\cdot)/2) \) can be well defined as
a univalued function on \( \Omega \setminus \Gamma_1 \). If \( u \in H_{0,A}^1(\Omega), \tilde{u} := \exp(-i\varphi X/2)u \) belongs
to \( H^1(\Omega \setminus \Gamma_1) \) with boundary condition \( \tilde{u}|_{\partial \Omega} = 0 \) and \( \tilde{u}|_{\Gamma^-} = -\tilde{u}|_{\Gamma^+} \). We
denote this space as \( \tilde{H}^1_{0}(\Omega \setminus \Gamma_1) \).

On this new Sobolev space \( \tilde{H}^1_{0}(\Omega \setminus \Gamma_1) \), we get the sesquilinear form
\[
(\tilde{u}, \tilde{v}) \rightarrow \tilde{B}_\sigma(\tilde{u}, \tilde{v}) = \sum_j \int_{D_j} \nabla \tilde{u} \cdot \nabla \tilde{v} \, dx + \sigma \int_{\Gamma} \tilde{u} \tilde{v} \, dS_{\Gamma} = \int_{\Omega \setminus \Gamma_1} \nabla \tilde{u} \cdot \nabla \tilde{v} \, dx + \sigma \int_{\Gamma} \tilde{u} \tilde{v} \, dS_{\Gamma}.
\]
We next define the operator that replaces $\Lambda$, we can redescribe summing up, we get the map

$$\partial \sigma \lVert D \rVert$$

and $\Gamma$ for each $D_i$ and $\Gamma$, we recover the Dirichlet Laplacian on the disjoint union of the $D_i$’s.

We start from the triple $(\hat{u}_1^0, \hat{u}_2^0, \hat{u}_3^0)$ and use the reduced Dirichlet-to-Neumann operators associated to each $D_i$ (i.e. the Dirichlet-to-Neumann operator restricted to elements with trace 0 on $\partial \Omega$) to associate

- in the case of $D_3$, with a pair $(\hat{u}_1^0 \oplus \hat{u}_2^0)$ in $H^{1/2}(\Gamma_1) \oplus H^{1/2}(\Gamma_2)$ the element $\hat{f}_1 \oplus \hat{f}_2$ in $H^{-1/2}(\Gamma_1) \oplus H^{-1/2}(\Gamma_2)$;
- in the case of $D_1$, to a pair $(\hat{u}_2^0 \oplus \hat{u}_3^0)$ in $H^{1/2}(\Gamma_2) \oplus H^{1/2}(\Gamma_3)$ an element $\hat{g}_2 \oplus \hat{g}_3$ in $H^{-1/2}(\Gamma_2) \oplus H^{-1/2}(\Gamma_3)$;
- in the case of $D_2$, to a pair $(\hat{u}_3^0 \oplus (-\hat{u}_1^0))$ in $H^{1/2}(\Gamma_3) \oplus H^{1/2}(\Gamma_1)$ an element $\hat{h}_3 \oplus \hat{h}_1$ in $H^{-1/2}(\Gamma_3) \oplus H^{-1/2}(\Gamma_1)$.

Summing up, we get the map

$$(\hat{u}_1^0 \oplus \hat{u}_2^0 \oplus \hat{u}_3^0) \mapsto ((\hat{f}_1 + \hat{h}_1) \oplus (\hat{f}_2 + \hat{g}_2) \oplus (\hat{g}_3 + \hat{h}_3)),$$

Here we note that on $\Gamma_1$ the left trace of $\hat{u} \hat{v}$ equals the right trace of $\hat{u} \hat{v}$. The question is now to determine what is the transmission obtained on $\Gamma$, the operator being the standard Laplacian. We write $\hat{u}|_{D_j} = \hat{u}_j$ ($j = 1, 2, 3$) and we can redescribe $\hat{H}_1^0(\Omega \setminus \Gamma_1)$. Then we can express the new transmission relations through $\Gamma_1, \Gamma_2$ and $\Gamma_3$. The transmission conditions are unchanged on $\Gamma_2$ and $\Gamma_3$. We find that

$$-\partial_{\nu_2} \hat{u}_1 - \partial_{\nu_2} \hat{u}_3 = \sigma \hat{u}_3, \quad \hat{u}_3 = \hat{u}_1, \quad \text{on } \Gamma_2,$$

$$-\partial_{\nu_1} \hat{u}_2 - \partial_{\nu_1} \hat{u}_1 = \sigma \hat{u}_2, \quad \hat{u}_2 = \hat{u}_1, \quad \text{on } \Gamma_3,$$

and

$$\partial_{\nu_3} \hat{u}_2 - \partial_{\nu_3} \hat{u}_3 = \sigma \hat{u}_3, \quad \hat{u}_3 = -\hat{u}_2, \quad \text{on } \Gamma_1.$$

We next define the operator that replaces $\Lambda_- (\varepsilon), \Lambda_+ (\varepsilon)$ in [7]. Again, it is an operator from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$.

At this stage we have defined a realization of the Laplacian $\hat{T}_{\Gamma, \Gamma}(\sigma)$ with

$$\hat{\Gamma} = \Gamma_1.$$

For $\sigma = 0$, we get the operator $\hat{L}_{\Omega \setminus \Gamma}$ replacing the Dirichlet realization $L_\Omega$ or the AB Hamiltonian in the magnetic Laplacian. For $\sigma = +\infty$, we recover the Dirichlet Laplacian on the disjoint union of the $D_i$’s.

Figure 5.1. The unit disk with (a) a simple 3-equipartition (b) the boundary of this 3-equipartition.
which acts in $H^{1/2}(\Gamma)$ with the notation introduced in (4.5):

$$\hat{\Lambda}^{NP}(\varepsilon, D, \hat{\Gamma}) = \sum_{i=1}^{3} \iota_{i} \Lambda_{i}(\varepsilon, l_{k}) \hat{r}_{i},$$

where $\hat{r}_{1} = r_{1}$, $\hat{r}_{2}(\hat{a}_{3}^{0}, \hat{u}_{1}^{0}) = (\hat{a}_{3}^{0} - \hat{u}_{1}^{0})$, and $\hat{r}_{3} = r_{3}$.

In this way, we avoid to discuss the artificial singularities introduced with Aharonov–Bohm operators.

In this formalism, we denote by $\hat{\tau}(\varepsilon, D, \hat{\Gamma})$ the number of negative eigenvalues of $\hat{\Lambda}^{NP}(\varepsilon, D, \hat{\Gamma})$. We introduce the defect $\hat{\text{Def}}(D, \hat{\Gamma})$ of the partition $D$ as

$$\hat{\text{Def}}(D) := \hat{\ell}(D, \hat{\Gamma}) - k(D),$$

where $\hat{\ell}(D)$ denotes the minimal labelling of the eigenvalue $\hat{\lambda}_{k}$ of the Hamiltonian $\hat{\tau}_{\Gamma, \hat{\Gamma}}$. We can reformulate Theorem 4.5 in the following way:

**Theorem 5.1.** Let $D$ a regular $k$-equipartition of a simply connected domain $\Omega$ satisfying the PCC with energy $l_{k} = l_{k}(\Omega)$. Then, for sufficiently small $\varepsilon > 0$,

$$\hat{\text{Def}}(D) = 1 - \dim \ker(\hat{L}_{\Omega, \hat{\Gamma}} - l_{k}) + \hat{\tau}(\varepsilon, D, \hat{\Gamma}). \quad (5.1)$$

What we have established above is the validity of the theorem if $\Omega$ is the Mercedes star. Note that due to the symmetries one can have in this case a nicer more explicit expression for $\hat{\Lambda}^{NP}(\varepsilon, D, \hat{\Gamma})$ (recall the analysis on the circle from Section 2). The question is now how to choose $\hat{\Gamma}$ in the general situation and to define $\hat{\Lambda}^{NP}(\varepsilon, D, \hat{\Gamma})$.

5.2. **The general case.** The question is now to extend what we have done for the Mercedes star.

5.2.1. **The choice of $\hat{\Gamma}$.** This is indeed quite analogous to what is done when we want to define the square root of $z \mapsto (z - z_{1})(z - z_{2}) \ldots (z - z_{\ell})$ in a maximal domain of $\mathbb{C}$. By defining branch cuts, we can then recover the double covering by gluing the two sheets along these branch cuts.

In our case, we have in a addition a boundary set $\Gamma$ containing the “odd” points $X_{1}, \ldots, X_{\ell}$ in $\Omega$ and what we have to prove is that $\Gamma$ contains a closed subset $\hat{\Gamma}$ corresponding to the branch cuts, which is minimal, in a sense described below. These branch cuts are either connecting inside $\Gamma$ one odd point to (one point of) the boundary or connecting two odd points.

We should have the property that we can then construct a square root of $\exp(i\theta)$ denoted by $\exp(i\theta/2)$ which is univalued on $\Omega \setminus \hat{\Gamma}$ and maximal in the sense that it can not be extended to a larger open set. The set $\hat{\Gamma}$ will in general not be unique, but all we need is the existence. A natural notion was introduced in [28] called the slitting property and the only change is that holes are replaced here by points (or) poles.

For a given $\mathbf{X} = (X_{1}, \ldots, X_{\ell})$ in $\Omega^{\ell}$ with distinct $X_{i}$, we say that a closed set $N$ slits $\Omega$ with singularities at $\mathbf{X}$ if:

- $N$ is a weakly regular closed set in the sense of Section 3.2;
- $X_{\text{odd}}^{N}(N) = (X_{1}, \ldots, X_{\ell})$;
- $\Omega \setminus N$ is connected.
This definition was introduced to characterize the properties of the nodal domain of the ground state of $T_A$.

Figure 5.2 is inspired by [28, Fig. 1], and shows some examples of regions which are slitting (but replace the holes in the picture by points in our case). Note that crossing points at even points are permitted (see Figure 5.2d). Note also that for the $\ell = 1$ case (Figure 5.2a), a set which slits $\Omega$ consists of one line which joins the outer boundary of $\Omega$ to the pole (Mercedes situation). We have explained above the no-hole situation. The case with holes is treated in the same way, once we have selected some “odd” holes, and placed one pole $X'_j$ in each of these odd holes. If a collection of paths slits a region then no sub- or supercollection of these paths can also slit the region.

In this formalism the main result is

**Proposition 5.2.** If $\Gamma$ is regular with corresponding $X^{\text{odd}}(\Gamma) = X$ and $\partial \Omega^{\text{odd}}(\Gamma)$, then it contains a slitting set $\hat{\Gamma}$ with $X^{\text{odd}}(\hat{\Gamma}) = X^{\text{odd}}(\Gamma) = X$ and $\partial \Omega^{\text{odd}}(\hat{\Gamma}) = \partial \Omega^{\text{odd}}(\Gamma)$.

Then, once we have this slitting property, we have

**Proposition 5.3.** Under the previous assumptions, there exists $\hat{\Gamma}$ such that \( \{X_1, \ldots, X_d\} \subset \hat{\Gamma} \subset \Gamma \) and such that there exists in $\Omega \setminus \hat{\Gamma}$ a univalued regular square root of $\exp(i\Phi_X)$ which is maximal in the sense that it cannot be extended as a univalued regular function in an open set in $\Omega$ containing strictly $\Omega \setminus \hat{\Gamma}$.

The setting has a natural formulation in terms of graph theory. This corresponds indeed in the nodal case to the notion of nodal graph (see for
example [31, Subsection 3.1]). We have a graph contained in $\Omega \subset \mathbb{R}^2$. The boundary points and the singular points of $\Gamma$ are the vertices and the regular arcs of $\Gamma$ are the edges. The vertices of $\partial \Omega$ and the odd vertices (an odd number of edges arrive at the vertex) in $\Omega$ play a special role. We can also define the notion of odd hole by determining the parity of the number of edges arriving at the boundary of the hole. Moreover, $\hat{\Gamma}$ can be considered as a subgraph of $\Gamma$. The graph translation is:

**Lemma 5.4.** If $\Gamma$ is a graph in $\Omega$ with given “odd” set of vertices $X^{\text{odd}}(\Gamma) = (X_1, \ldots, X_\ell)$ and given “odd” holes, then there exists a subgraph $\hat{\Gamma}$ with the same “odd” sets such that $\Omega \setminus \hat{\Gamma}$ is connected.

**Proof (given by G. Berkolaiko).** We first consider the case with no hole. It is better to identify all the points of $\partial \Omega$ and to look at the new graph as a graph $\tilde{\Gamma}$ on the sphere, $\partial \Omega$ being the north pole $P$ and $\Omega$ being $S^2 \setminus \{P\}$. To get the connexity property, it is enough to destroy all the cycles on the graph. It is now enough to observe that if there is a cycle we can delete all the elements of the cycle. But at each vertex of the cycle, only two edges belonging to the cycle arrive. Hence when destroying a cycle, this always preserves the odd vertices and the even vertices. Finally, we observe that no odd vertex can disappear when deleting a cycle. This case is exemplified in Figure 5.3.

![Figure 5.3](image)

**Figure 5.3.** (a) A domain $\Omega$. (b) The constructed graph on the sphere. The boundary of $\Omega$ is mapped to the point $P$. We remove one loop (dashed). (c) Back in $\Omega$, the removed loop $\hat{\Gamma}$ is dashed.

In the case with holes, we identify each component of $\partial \Omega$ with a point and look at a new graph $\tilde{\Gamma}$ on the sphere with $\Omega$ being $S^2 \setminus \{P_1, \ldots, P_m\}$. Each $P_i$ corresponds to a component of $\partial \Omega$. Then it suffices to think in the previous proof that the points $P_\ell$ corresponding to odd boundary components are odd vertices. This situation is exemplified in Figure 5.4. □

5.2.2. **Proof of Theorem 5.1.** With the slitting lemma at hand, we have a general method to define $\hat{\Gamma}$, and then we can complete the proof in the general setting, by following what was done in the case of the Mercedes star. Thus, we introduce a Sobolev space associated with the pair $(\hat{\Gamma}, \Gamma \setminus \hat{\Gamma})$.

We note that $\hat{\Gamma}$ is a union of regular curves $\hat{\gamma}_r$, ending at critical points, and we can choose an orientation of $\hat{\gamma}_r$ so that, locally, in the neighborhood $D(x, r)$
of an interior point $x \in \hat{\gamma}_\ell$, we can write $D(x, r) \setminus \hat{\gamma}_\ell = D^+(x, r) \cup D^-(x, r)$ permitting to define a trace on the left and on the right.

Starting from $H^1(\Omega \setminus \hat{\Gamma})$, we introduce

$$\hat{H}^1_0(\Omega \setminus \hat{\Gamma}) := \{ u \in H^1(\Omega \setminus \hat{\Gamma}) , u|_{\hat{\gamma}_\ell}^+ = -u|_{\hat{\gamma}_\ell}^- , u|_{\partial \Omega} = 0 \} .$$

On this Sobolev space $\hat{H}^1_0(\Omega \setminus \hat{\Gamma})$, we get the sesquilinear form

$$(\hat{u}, \hat{v}) \mapsto \hat{B}_\sigma(\hat{u}, \hat{v}) = \sum_j \int_{D_j} \nabla \hat{u} \cdot \nabla \hat{v} \, dx + \sigma \int_{\hat{\Gamma}} \hat{u} \hat{v} \, dS_{\hat{\Gamma}}$$

$$= \int_{\Omega \setminus \hat{\Gamma}} \nabla \hat{u} \cdot \nabla \hat{v} \, dx + \sigma \int_{\hat{\Gamma}} \hat{u} \hat{v} \, dS_{\hat{\Gamma}} .$$

We can then associate, via the Lax–Milgram theorem, to this sesquilinear form a realization $\hat{T}_{\Gamma, \hat{\Gamma}}(\sigma)$ of the Laplacian in $\Omega \setminus \hat{\Gamma}$ with $\sigma$ transmission properties on $\Gamma \setminus \hat{\Gamma}$, Dirichlet condition on $\partial\Omega$ and $\sigma$-Robin like condition on $\hat{\Gamma}$. As in the Mercedes-star case, $\hat{L}_{\Omega \setminus \hat{\Gamma}}$ corresponds to $\sigma = 0$.

It remains to detail our definition of $\hat{\Lambda}^{NP}$. The only point is to have a clear definition of $\hat{r}_i$ (which is an immediate consequence of the choice of our Sobolev space. We have introduced an orientation on each regular component of $\hat{\Gamma}$.

But $H^{1/2}(\hat{\Gamma})$ can be identified with $\oplus \ell H^{1/2}(\hat{\gamma}_\ell^+)$. So, when defining our Neumann–Poincaré map attached to some $D_i$, and when $\hat{\gamma}_\ell \subset \partial D_i$, we consider the trace $\varepsilon_\ell u_\ell$ with $\varepsilon_\ell = +1$ if $D_i$ is locally on the right side of $\hat{\gamma}_\ell$ and $\varepsilon_\ell = -1$ if it is on the left side. For the other components of $\partial D_i \cap \Omega$, we just proceed like in [7].

5.2.3. Comparison between two constructions. If $\mathcal{D}$ is an equipartition with boundary set $\Gamma$, we can observe that $\Omega \setminus \hat{\Gamma}$ is a bipartite equipartition. Moreover, if it satisfies the PCC, it is a nodal partition. Finally, if $\mathcal{D}$ is a minimal partition then it is a Courant sharp nodal partition in $\Omega \setminus \hat{\Gamma}$ (see [11] where this argument is used for the analysis of the Hexagonal conjecture). It is then natural to compare our construction relative to $\mathcal{D}$ (seen as a partition of $\Omega$) with the Berkolaiko–Cox–Marzuola construction associated with $\mathcal{D}$ seen...
as a nodal partition in $\Omega \setminus \hat{\Gamma}$. The difference is that in the second case, we restrict the first construction to elements which vanish on $\hat{\Gamma}$ and then project on $H^{-\frac{1}{2}}(\Gamma \setminus \hat{\Gamma})$. Coming back to the definitions, it is then immediate to see that an eigenvalue of the Neumann–Poincaré second operator is actually an eigenvalue of the first Neumann–Poincaré operator. It is then interesting to compare the two formulas (5.1) and (1.1) for the pair $(\Omega \setminus \hat{\Gamma}, \Gamma \setminus \hat{\Gamma})$.

Figure 5.5. The slitting example $H_{12}^1$ of [11, Figure 24].

To give a more explicit example, we continue the discussion of the circle from Section 2. Now $\hat{\Gamma}$ is just one point, say $\theta = 0$. The construction in [7] leads to an $(N-1) \times (N-1)$ matrix obtained by taking $u_0 = 0$ and forgetting $v_0$. This leads to the matrix

$$M_0^\lambda := \frac{1}{2} \begin{bmatrix} 2\alpha(\lambda) & \beta(\lambda) & 0 & \cdots & 0 \\ \beta(\lambda) & 2\alpha(\lambda) & \beta(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta(\lambda) & 2\alpha(\lambda) \end{bmatrix}.$$ 

whose spectrum is given by

$$\left\{ \alpha(\lambda) + \beta(\lambda) \cos \frac{k\pi}{N} \right\}_{k=1}^{N-1}.$$ 

Hence $M_0^\lambda$ has the same eigenvalues as $M_\lambda$ except $\alpha - \beta$. All its eigenvalues are positive. Again we can verify for the energy $(N/2)^2$ in this case that (1.1) holds with $\Omega = S^1 \setminus \{0\}$ and the same partition as in Section 2.

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(Bernard Helffer) Laboratoire de Mathématiques Jean Leray, Université de Nantes, 44332 Nantes, France.
E-mail address: Bernard.Helffer@math.univ-nantes.fr

(Mikael Persson Sundqvist) Lund University, Department of Mathematical Sciences, Box 118, 221 00 Lund, Sweden.
E-mail address: mikael.persson_sundqvist@math.lth.se