A unified theory for the SU(4) spin-orbital model on the triangular lattice

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By combining the density matrix renormalization group (DMRG) method with Gutzwiller projected wave functions, we provide clear evidence that the ground state of the SU(4) Kugel-Khomskii spin-orbital model on the triangular lattice can be well described by a “single” Gutzwiller projected wave function with an emergent parton Fermi surface, despite it exhibits strong finite size effect and even-odd discrepancy in quasi-one-dimensional cylinders. This ground state preserves SU(4) symmetry, but spontaneously breaks translational symmetry by doubling the unit cell along one of the lattice vector directions. The finite size effect and even-odd discrepancy can be resolved by the fact that the parton Fermi surface consists of open orbits in the reciprocal space. Thereby, a nematic spin-orbital liquid state is expected in the two-dimensional limit. Furthermore, our DMRG results indicate that the fluctuating stripes are critical and the central charge of each stripe is $c = 3$, in agreement with the SU(4)$\otimes$Wess-Zumino-Witten conformal field theory. All these results are consistent with the Lieb-Schultz-Mattis-Oshikawa-Hastings theorem.

Introduction.— The search of quantum spin liquids is one of central issues in condensed matter physics [1–4]. The quantum fluctuations and geometric frustration are two paramount mechanisms for suppressing magnetic order. With both mechanisms in play, the $S = 1/2$ Heisenberg antiferromagnet on the triangular lattice was regarded as a natural candidate hosting a quantum spin liquid. While Anderson proposed a disordered ground state for this model [5], such scenario was, however, ruled out by later studies [6, 7]. By now its has been established that the ground state of the $S = 1/2$ Heisenberg antiferromagnet on the triangular lattice exhibits a three-sublattice 120° magnetic order [8]. In a sense, $S = 1/2$, the most quantum case for SU(2) spins, is still not “quantum” enough to melt the classical order on the triangular lattice.

Apart from additional competing interactions [9–16] and stronger geometric frustration [17–20], orbital degeneracy provides an alternative mechanism to enhance quantum fluctuations [21]. In certain transition metal oxides, spin and orbital degrees of freedom might play a symmetric role and be promoted to an SU(4) symmetry. The SU(4) Kugel-Khomskii (KK) model is a minimal model for describing such spin-orbital materials [22–28], where the larger symmetry amplifies quantum fluctuations and might stabilize spin-orbital liquid ground states. The SU(N) quantum magnetism has also been studied in the platform of ultracold atoms in optical lattices [29, 30]. Meanwhile, it was proposed that a Hubbard-type model with an approximate SU(4) symmetry is feasible to depict correlated moiré materials [31–35].

From theoretical side, much effort has been devoted to investigating quantum phases in SU(4) quantum magnets. In one dimension, the SU(4) KK model is integrable and has gapless excitations [36, 37], whose low-energy effective theory is the SU(4)$\otimes$ Wess-Zumino-Witten (WZW) conformal field theory (CFT) [38–40]. The ground state of the SU(4) KK model on a two-leg ladder breaks the translational symmetry with the formation of SU(4)-singlet plaquettes [41, 42]. With extra interactions, the SU(4)-singlet plaquette state becomes the exact ground state on the ladder [43]. The situation in two dimensions is less well studied: Several complementary methods point to a Dirac-type spin-orbital liquid for the SU(4) KK model on the honeycomb lattice [44, 45], whereas a few candidate ground states, such as the plaquette order [23, 46], a $\mathbb{Z}_2$ spin-orbital liquid [47], and simultaneous dimerization with broken SU(4) symmetry [48], have been proposed for the SU(4) KK model on the square lattice.

The ground state of the SU(4) KK model on the triangular lattice also remains elusive. Exact diagonalization and variational calculations on finite-size clusters indicate an SU(4)-singlet plaquette liquid [49]. Recently, a density matrix renormalization group (DMRG) study, in combination with field theory analysis, suggested a gapless liquid ground state with an emergent parton Fermi surface [50]. On the contrary, another DMRG study [51] and a self-consistent mean-field theory [52] favor a stripe ordered state.

In this work, we reexamine the ground state of the SU(4) KK model on the triangular lattice by combining the DMRG method [53, 54] with Gutzwiller projected wave function approach. Our recent methodology development [55–57] provides a useful tool for converting Gutzwiller projected wave functions into matrix product states (MPSs) and exploiting them to boost DMRG calculations and characterize ground states. With two complementary methods and extensive numerical efforts on cylinders with circumference up to $L_y = 6$, we have gathered clear evidence for unveiling a critical stripey ground state in the two-dimensional (2D) limit. This state preserves SU(4) symmetry, but spontaneously breaks translational symmetry by doubling the unit cell along one of the lattice vector directions. It is worth emphasizing that there is a
forms of partons, dubbed “effective Hamiltonian”, which determines mean-field ground states and low-energy excitations. As an example, the uniform $\pi$-flux state, suggested in Ref. [50], is described by the following effective Hamiltonian:

$$H_{\varphi=\pi} = -\sum_{m=1}^{4} \sum_{(ij)} \left( f^\dagger_{i,m} f_{j,m} + f^\dagger_{j,m} f_{i,m} \right).$$

This parton Hamiltonian (3) preserves SU(4) rotational and lattice translational symmetries, and gives rise to a fourfold degenerate band structure. The single-occupancy or 1/4 filling is indicative of a circular Fermi surface in the parton band [see Fig. 1(a)] and a gapless state. In this work, we propose a stripy state as a result of the instability of this uniform $\pi$-flux state, which breaks the lattice translational symmetry by doubling the unit cell along the one of the lattice vector directions. Without loss of generality, we take $x$ direction to be the symmetry breaking direction [see Figs. 2(a) and 2(b)]. The mean-field Hamiltonian for such a stripy state reads

$$H_{\text{stripy}}(\delta) = -\sum_{m=1}^{4} \sum_{(ij)} [1 + (-1)^\nu \delta] \left( f^\dagger_{i,m} f_{j,m} + f^\dagger_{j,m} f_{i,m} \right),$$

where $r_i$ is the column index of lattice site $i$, $r_j \geq r_i$ is assumed for NN bond $(ij)$, and $\delta \in [-1, 1]$ is the stripy strength. A finite $\delta$ deforms the original circular Fermi surface, which recovers the uniform $\pi$-flux state at $\delta = 0$. When $\delta > \delta_c = 0.09$, the deformed Fermi surface consists of open orbits in the reciprocal space [see Fig. 1(b)].

Thanks to the newly developed method [55, 56], the ground states of Hamiltonians (3) and (4) can be represented as MPSs, upon which the Gutzwiller projection can be easily implemented to impose the single-occupancy constraint and obtain many-body wave functions in the physical spin-orbital Hilbert space. These trial wave functions, in their MPS form, can be used to initialize DMRG calculations, which is dubbed as “Gutzwiller-boosted DMRG” [57]. Consequently, both DMRG-obtained ground states and Gutzwiller projected trial wave functions can be flexibly studied with powerful MPS techniques.

To perform DMRG calculations on triangular lattices, we consider cylinder geometries, for which the periodic boundary condition (PBC) for spin-orbital degrees of freedom is imposed along the $y$ direction, while the $x$ direction is left open. Following the common practice of DMRG, the 2D model is mapped to a one-dimensional (1D) model with long-range interactions. In choosing the 1D path, we consider two different mappings, labeled as XCL$_y$ and YCL$_x$, where $L_y$ is the circumference of the cylinder [see Figs. 2(a) and 2(b)].

During our numerical calculations, we have made explicit use of the $U(1)^{\otimes 3}$ quantum numbers corresponding to the Cartan generators $\{x^i, \tau^i, \sigma^a \tau^i \}$ [60]. Our DMRG calculations have been performed on cylinders with circumference up to $L_y = 6$. The number of states kept in DMRG calculations is as large as $D = 18000$, resulting in a truncation error $\epsilon_{\text{D}} \approx 10^{-6}$.
The colors of the bonds represent the expectation value $\Lambda_{ij} \equiv \langle A_i \cdot A_j \rangle$ between neighboring sites for the central columns in a cylinder of circumference $L_x = 4$ and length up to $L_y = 24$. The red dashed lines represent the 1D path in DMRG calculations, and plaquette SU(4)-singlet projectors are defined on the rhombuses. (c) The SU(4) spin correlation function $B(r) = \Lambda_{i,i+r}$ versus distance $r$ for XC and YC cylinders with length $L_y = 48$. The dashed line indicating an exponential decay of $|B(r)|$ is a guide to the eye. Inset: $B(r)$ versus $r$ for an XC3 cylinder with length $L_y = 60$. The dashed line indicates an power-law decay of $|B(r)|$. (d) The two-plaquette SU(4)-singlet correlation function $D(r) = \langle \mathbf{P}_r \mathbf{P}_{r+2z} \rangle = \langle \mathbf{P}_r \mathbf{P}^{\dagger}_{r+2z} \rangle$ versus distance $r$ in cylinders of length $L_y = 24$ (XC4 and YC4) and of length $L_y = 16$ (XC6 and YC6).

(\$\epsilon_1 \approx 10^{-4}\$) for $L_y = 4$ ($L_y = 5$ and 6). In the processes of converting Gutzwiller projected wave functions to MPS, we have obtained the ground states of the KK model (1) with different cylindrical boundary conditions and various system sizes. As mentioned earlier, the Gutzwiller projected wave function $|\Psi_{\text{stripy}}(\delta)\rangle$ is determined from the effective Hamiltonian (4) at 1/4 filling. As illustrated in Fig. 3(a), its variational energy per site $\epsilon_g(\delta)$ (on the cylinder) reaches its minimum at $\delta \approx 0.15$, suggesting that the uniform $\pi$-flux state ($\delta = 0$) is unstable against the stripy deformation. The wave-function fidelity between $|\Psi_{\text{stripy}}(\delta)\rangle$ and the DMRG-obtained ground state $|\Psi_{\text{DMRG}}\rangle$, $F(\delta) = |\langle \Psi_{\text{stripy}}(\delta) | \Psi_{\text{DMRG}} \rangle|$, is also maximized at $\delta \approx 0.15$, in good agreement with the energy comparison. Here $|\Psi_{\text{DMRG}}\rangle$ is obtained by using Gutzwiller-boosted DMRG with $D = 15000$ for both (a) and (b).

**Energetics and fidelity of the stripy state.—** By the combinatorial method of DMRG and Gutzwiller projected wave functions, we have obtained the ground states of the KK model (1) with different cylindrical boundary conditions and various system sizes. As mentioned earlier, the Gutzwiller projected wave function $|\Psi_{\text{stripy}}(\delta)\rangle$ is determined from the effective Hamiltonian (4) at 1/4 filling. As illustrated in Fig. 3(a), its variational energy per site $\epsilon_g(\delta)$ (on the cylinder) reaches its minimum at $\delta \approx 0.15$, suggesting that the uniform $\pi$-flux state ($\delta = 0$) is unstable against the stripy deformation. The wave-function fidelity between $|\Psi_{\text{stripy}}(\delta)\rangle$ and the DMRG-obtained ground state $|\Psi_{\text{DMRG}}\rangle$, $F(\delta) = |\langle \Psi_{\text{stripy}}(\delta) | \Psi_{\text{DMRG}} \rangle|$, is also maximized at $\delta \approx 0.15$, in good agreement with the energy comparison. Here $|\Psi_{\text{DMRG}}\rangle$ is obtained by using Gutzwiller-boosted DMRG with $D = 15000$ for both (a) and (b).

As is well known, boundary effects due to the cylinder geometry might lead to biased results. To eliminate boundary effects, we also consider the Gutzwiller projected wave function $|\Psi_{\text{stripy}}(\delta)\rangle$ on an XC torus with $L_x \times L_y = 6 \times 4$. It is found that a lower variational energy is achieved when partons assume APBCs along both directions. The numerical results for the variational energy $\epsilon_g(\delta)$ and the fidelity $F(\delta)$ are shown in Fig. 3(b). At first glance, there is a contradiction between $\epsilon_g(\delta)$ and $F(\delta)$: the lowest energy $\epsilon_g(\delta)$ is achieved at $\delta \approx \pm0.25$, while the highest fidelity $F(\delta)$ appears at $\delta = 0$. However, this seemingly contradiction can be resolved as follows: On a torus, $|\Psi_{\text{stripy}}(\delta)\rangle$ and its dual stripy state $|\Psi_{\text{stripy}}(-\delta)\rangle$ are energetically degenerate. For $|\delta| \geq 0.25$, these two states are found to be almost orthogonal to each other, with an overlap less than 0.05. Thus, one can construct cat states, $|\Psi_{\text{cat}}(\delta)\rangle \propto |\Psi_{\text{stripy}}(\delta)\rangle \pm |\Psi_{\text{stripy}}(-\delta)\rangle$ and compute the fidelity $F_{\text{cat}}(\delta) \equiv |\langle \Psi_{\text{cat}}(\delta) | \Psi_{\text{DMRG}} \rangle|^2$ (defined by the larger overlap with one of the two cat states). As shown in Fig. 3(b), the fidelity $F_{\text{cat}}(\delta)$ reaches its maximum at $\delta \approx \pm0.25$, which is indeed consistent with the behavior of $\epsilon_g(\delta)$.

It is remarkable that the unit-cell doubling in the stripy state does not open an energy gap on the parton Fermi surface. This remaining parton Fermi surface is due to the 1/4 filling of the Fermi sea, and is quite different from the usual gap opening by the dimerization in SU(2) quantum spin-1/2 systems, on which the half-filled spinon band acquires an energy gap by the dimerization.

**Correlation functions.—** To characterize the stripy state, we have calculated a variety of correlation functions. The leading-order correlation consists of two SU(4) spins, i.e., the
SU(4) spin correlation function $\Lambda_{ij} \equiv \langle \mathbf{I}_i \cdot \mathbf{I}_j \rangle$, whose values on NN bonds, $\Lambda_{ij} \in [-2.5, 1.5]$, give rise to the local energy density up to a constant of 1/2. It can be seen from Figs. 2(a) and 2(b) that, on both types of cylinders, XC4 and YC4, $\Lambda_{ij}$ exhibits sizable twofold oscillations along the x direction, indicating translational symmetry breaking [61]. Meanwhile, the translational symmetry is preserved along the periodic y direction, which allows us to define a y-independent correlation function, $B(r) = \Lambda_{i+j+z}$. Numerical results from DMRG suggest an exponential decay of $B(r)$ on cylinders up to $L_y = 48$, as demonstrated in Fig. 2(c), which is consistent with the scenario of a parton Fermi surface consisting of open orbits in the reciprocal space. This is due to the fact that none of the allowed momenta in these finite $L_y$ cylinders cuts the Fermi surface [see Fig. 1(b)]. Note that the APBC (along y direction) for partons leads to lower energy states on $L_y = 4$ cylinders, while the PBC is energetically favored by $L_y = 6$ cylinders. We have also performed DMRG calculations on odd circumference ($L_y = 3$) cylinders [60], the correlation function $|B(r)|$ exhibits a power-law decay on a $L_y = 3$ cylinder because some of the allowed momenta do cut the Fermi surface.

In addition to the two-site SU(4) spin correlation function $\Lambda_{ij}$, two-plaquette SU(4)-singlet correlation functions have been investigated as well. To form an SU(4)-singlet, at least four SU(4) spins are required. In particular, we are interested in four-site plaquettes on strong double-columns, which have stronger bonds or lower energy density $\Lambda_{ij}$. Thus, the two-plaquette SU(4)-singlet correlation function can be defined on strong double-columns as $D(r) \equiv \langle P_p P_{p+2z} \rangle - \langle P_p \rangle \langle P_{p+2z} \rangle$, where $P_p$ projects a state in the plaquette-p onto the SU(4) singlet. The striped structure breaks the $C_3$ lattice rotational symmetry and the cylindrical geometry breaks the mirror symmetry, resulting in two distinct types of plaquettes on a single XC or YC cylinder, i.e., $P_0$ and $P_1$ illustrated in Figs. 2(a) and 2(b), respectively. The expectation values of SU(4)-singlet projectors on strong double-columns (in the bulk) are given by $\langle P_0 \rangle \approx \langle P_1 \rangle \approx 0.32 (0.29)$ on XC4 (XC6) and YC4 (YC6) cylinders. The two-plaquette correlation function $D(r)$ is shown in Fig. 2(d), which exhibits a rapid decay as the distance $r$ increases.

Entanglement entropy.— The gapless nature of the striped state and the emergence of a parton Fermi surface can be further revealed and characterized by the von Neumann entanglement entropy (EE). Intuitively, the striped state can be thought of as $L_x/2$ weakly coupled 1D chains, i.e., zigzag SU(4) spin chains on the aforementioned strong double-columns [see Fig. 4(b)]. We shall focus on YC cylinders, on which each zigzag chain consists of 2$L_x$ sites.

Within each strong double-column, the entanglement between one subsystem with $l_y$ spins and the remaining $2L_x - l_y$ spins is measured by the von Neumann EE $S(l_y)$. For 1D critical systems described by CFT, $S(l_y)$ follows the scaling law [62–64]

$$S(l_y) = \frac{c}{3} \log \left( \frac{2L_x \sin \frac{\pi l_y}{2L_x}}{\pi} \right) + \gamma,$$

where $c$ is the central charge of the CFT and $\gamma$ is a nonuniversal constant. We have evaluated the EE $S(l_y)$ for the Gutzwiller projected state $|\Psi_{\text{SR}}(\delta = 0.15)\rangle$ as well as the DMRG-obtained state $|\Psi_{\text{DMRG}}\rangle$ on a strong double-column in the middle of YC cylinders. As shown in Fig. 4(a), the EE of both states are well fitted by Eq. (5), and the extracted central charge is $c \approx 3$. Although many CFTs have this central charge, the SU(4)$_1$ WZW model seems to be the most natural candidate, since it is compatible with the SU(4) symmetry and often shows up in SU(4) critical chains. The SU(4)$_1$ criticality is also consistent with the description of 1D four-component gapless fermions subject to Gutzwiller projection [65]. Although not shown, we have also obtained same results for other zigzag chains in the bulk of YC cylinders.

Treating a cylinder as an effective 1D chain with a big unit cell of $L_y$ spins and dividing the cylinder into two parts, the first $l_y$ big unit cells and the remaining $L_x - l_y$ ones, we are able to calculate the EE $S(l_y)$ between these two subsystems. It is found that as $l_y$ increases, $S(l_y)$ quickly saturates with twofold oscillations on even circumference cylinders [60], which is consistent with the short-range correlations along the x direction and the translational symmetry breaking. All these results on EE point to an emergent parton Fermi surface consisting of open orbits in the SU(4) KK model on the triangular lattice.

Lieb-Schultz-Mattis-Oshikawa-Hastings argument.— The scenario of a critical striped ground state is also consistent with the Lieb-Schultz-Mattis-Oshikawa-Hastings (LSM-OH) theorem [66–69]: The SU(4) KK model on the triangular lattice, with a fundamental representation 4 on each site, cannot have a nondegenerate gapped ground state preserving both SU(4) and translational symmetries. When considering the Young tableaux representing SU(4) representations within each unit cell, such trivial gapped state can only be achieved with the number of boxes in the Young tableaux being an integer multiple of 4. With the striped deformation in the ground state, the unit cell is doubled and has a Young tableau of two
boxes (symmetric representation 10 or antisymmetric representation 6). According to the LSMOH argument, this stripy state would either be gapless or have topological order, ruling out the possibility of a trivial gapped state. As our numerical results provide strong evidence of criticality on each strong double-column, a gapped state with topological order is unlikely, which leaves the critical stripy state as the only possibility.

**Conclusion and discussion.**— In summary, we have studied the SU(4) KK model on the triangular lattice by combining the DMRG method with Gutzwiller projected wave functions. With extensive numerical results and analytical arguments, we gather strong evidence of a critical stripy ground state, which spontaneously breaks translational symmetry along one of the lattice vector directions and has an emergent parton Fermi surface. Our proposal of a Gutzwiller projected wave function has been shown to describe well this critical stripy state and to resolve the even-odd discrepancy in a simple and unified manner. Thus, we expect that this nematic spin-orbital liquid state is stable in the 2D limit and serves as an excellent ground-state candidate for the SU(4) KK model on the triangular lattice beyond the quasi-1D geometry. For future works, it would be interesting to understand different instabilities of the parton Fermi surface in the SU(4) KK model. For instance, a chiral spin liquid, which emerges in the presence of three-body interactions [51, 70], is one of such instabilities. This would require to map out the phase diagram near the SU(4) KK model, where other exotic phases of matter might appear. Along this line of research, it would certainly be important to identify suitable experimental realizations of such SU(4) quantum magnets.

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**Conclusion and discussion.**— In summary, we have studied the SU(4) KK model on the triangular lattice by combining the DMRG method with Gutzwiller projected wave functions. With extensive numerical results and analytical arguments, we gather strong evidence of a critical stripy ground state, which spontaneously breaks translational symmetry along one of the lattice vector directions and has an emergent parton Fermi surface. Our proposal of a Gutzwiller projected wave function has been shown to describe well this critical stripy state and to resolve the even-odd discrepancy in a simple and unified manner. Thus, we expect that this nematic spin-orbital liquid state is stable in the 2D limit and serves as an excellent ground-state candidate for the SU(4) KK model on the triangular lattice beyond the quasi-1D geometry. For future works, it would be interesting to understand different instabilities of the parton Fermi surface in the SU(4) KK model. For instance, a chiral spin liquid, which emerges in the presence of three-body interactions [51, 70], is one of such instabilities. This would require to map out the phase diagram near the SU(4) KK model, where other exotic phases of matter might appear. Along this line of research, it would certainly be important to identify suitable experimental realizations of such SU(4) quantum magnets.

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Supplemental Material for “A unified theory for the SU(4) spin-orbital model on the triangular lattice”

In this Supplemental Material, we provide further details about (1) the convergence diagnosis for DMRG simulations, (2) the efficiency of Gutzwiller-boosted DMRG, (3) the properties of DMRG-obtained ground states, and (4) the boundary effect in the projected uniform $\pi$-flux state.

CONVERGENCE DIAGNOSIS FOR DMRG SIMULATIONS

To verify the convergence of DMRG-obtained ground states, we study the scaling behavior of the ground-state energy per site $\epsilon_g$ with the DMRG truncation error $\epsilon_D$ on $L_x = 6$ cylinders, the widest cylinder we have considered in this work. As shown in Fig. S1, $\epsilon_g$ scale linearly with $\epsilon_D$ when bond dimension $D \geq 12000$, suggesting a faithful convergence in our DMRG simulations initialized with random MPSs. Notice that here we have also shown the results for the Alternative XC (AXC) cylindrical geometry. The details of AXC cylinders will be discussed below.

![Fig. S1. Ground-state energy per site $\epsilon_g$ as a function of DMRG truncation error $\epsilon_D$ for (a) XC6 with $L_x = 16$, (b) AXC6 with $L_x = 16$, and (c) YC6 with $L_x = 16$. The orange dash lines being a guide to the eye are obtained by a linear fit of the last three data points with bond dimension $D = 12000, 15000, 18000$.](image)

EFFICIENCY OF THE GUTZWILLER-BOOSTED DMRG

![Fig. S2. The ground-state energy (minus a constant) for a specific DMRG sweep with the total wall time cost after the sweep. The open orange star marks the variational energy of $|\Psi_{\text{stripy}}(\delta = 0.15)\rangle$, which has been converted to an MPS with bond dimension $\tilde{D} = 2000$. The time cost of preparing such an initial MPS of $|\Psi_{\text{stripy}}(\delta = 0.15)\rangle$ can be negligible comparing to the time cost of DMRG calculations. Red dots and blue triangles show the energy decreasing of the DMRG simulations initialized with $|\Psi_{\text{stripy}}(\delta = 0.15)\rangle$ at $\tilde{D} = 2000$ and a randomly generated MPS, respectively. The lattice geometry is XC4 with $L_x = 24$ and the truncation error is fixed to $\epsilon_D = 1.0 \times 10^{-6}$.](image)

In order to investigate the quality of the Gutzwiller projected wave function generated from $H_{\text{stripy}}(\delta)$ [see Eq. (4) in the main text], we compare the performance of DMRG simulations with initial MPSs prepared from i) a randomly generated product state
and ii) a Gutzwiller projected states $|\Psi_{\text{stripy}}(\delta = 0.15)\rangle$, respectively. In Fig. S2, we show the optimized ground-state energies as a function of the total wall time for these two DMRG calculations with different initial ansatz. At any specific time during DMRG simulations, the MPS obtained by Gutzwiller-boosted DMRG always has a lower energy than that obtained by DMRG initialized with a random MPS. Moreover, a smoother energy descent curve of Gutzwiller-boosted DMRG also indicates a better variational optimization path in the Hilbert space. The wave-function fidelity between the states obtained by these two kinds of DMRG simulations is about $F \approx 1 - 10^{-6}$.

**GROUND-STATE PROPERTIES OF THE SU(4) KUGEL-KHOMSKII MODEL ON THE TRIANGULAR LATTICE**

**Entanglement entropy scaling along the $x$ direction**

By viewing a $L_x \times L_y$ cylinder as an effective 1D chain with a unit cell of $L_y$ sites, we have calculated the bipartite entanglement entropy (EE) $S(l_x)$ for this effective 1D chain with $L_y$ unit cells. The bipartite EE $S(l_x)$ versus the subsystem length $l_x$ for different circumferences $L_y$ are shown in Fig. S3. We find that as $l_x$ increases, $S(l_x)$ quickly saturate (in the presence of oscillations), independent of cylindrical geometries, suggesting short-range correlations along the $x$ direction in which the translational symmetry is broken. Besides, the twofold oscillation for cylinders with $L_y = 4$ [Fig. S3(b)] and $L_y = 6$ [Fig. S3(d)] and the fourfold oscillation for cylinders with $L_y = 5$ [Fig. S3(c)] indicate the translational symmetry breaking in the $x$ direction.

![Graph](image)

**FIG. S3.** The bipartite EE $S(l_x)$ versus subsystem length $l_x$ on the $L_y = (a) 3$, (b) 4, (c) 5, and (d) 6 cylinders. The type of the cylinders is shown in the legends.

**Alternative XC geometry**

In addition to the XC and YC cylinders [see Figs. 2(a) and (b) in the main text], we have also considered a 1D chain mapping labeled as AXC $L_y$ [see Fig. S4(a)] for carrying out DMRG simulations. We find that $\langle \lambda_i \cdot \lambda_j \rangle$, the expectation value of $\lambda_i \cdot \lambda_j$ on the nearest-neighbor bond $\langle ij \rangle$, exhibit an almost uniform pattern in the bulk of an AXC4 cylinder [see Fig. S4(a)], but we still have evidence for a stripy state on AXC cylinders: i) The EE $S(l_x)$ for the AXC geometry also exhibits sizable twofold oscillations, as shown in Figs. S3(b) and (d), indicating a translational symmetry breaking in the $x$ direction; ii) The plaquette-plaquette correlation $D(r)$ also shows a twofold oscillation [see Fig. S4(b)].
FIG. S4. (a) The mapping of the triangular lattice to a 1D chain on an AXC cylinder. The colors of the bonds represent the expectation value $\Lambda_{ij}$ between neighboring sites for the central columns in an AXC cylinder of length $L_x = 24$. The red dashed lines represent the 1D path in DMRG calculations, and plaquette SU(4)-singlet projectors are defined on the rhombuses. (b) The two-plaquette SU(4)-singlet correlation function $D(r)$ as a function of distance on AXC4 cylinders. Two plaquettes $P_p$ and $P_{p+1}$ are set on the same row.

$\Lambda_{ij}$ on odd circumference cylinders

We further remark that the behaviors of the ground state of the SU(4) KK model on odd circumference cylinders are very sensitive to the type and width of the cylinder. For $L_y = 3$ cylinders, previous study in XC3 geometry found that the nearest-neighbor correlation $\Lambda_{ij}$ is quite uniform [50]. However, for the YC3 geometry, our results indicate that the pattern of $\Lambda_{ij}$ still spatially fluctuates despite of good convergence in DMRG simulations [see Fig. S5(a)]. For a wider YC5 cylinder, $\Lambda_{ij}$ is also strongly fluctuating and has fourfold oscillations [see Fig. S5(b)]. Although this fluctuation is suppressed on the XC5 cylinder, a small inhomogeneity of $\Lambda_{ij}$ is clearly visible [see Fig. S5(c)].

![Central columns in an AXC cylinder](image)

(a) YC3 with $L_x = 16$, (b) YC5 with $L_x = 12$, and (c) XC5 with $L_x = 20$. The bond dimension used in DMRG calculations is set up to 18000 ~ 20000.

PROJECTED UNIFORM $\pi$-FLUX STATE ON CYLINDER AND TORUS

Although the Gutzwiller projected uniform $\pi$-flux state exhibits “translational symmetry breaking” on cylinders in viewing from $\Lambda_{ij}$, the twofold oscillation is smeared out in the central region of the cylinder [see Fig. S6(a)]. This smearing also appears in the oscillation of the bipartite EE $S(l_x)$, as shown in Fig. S6(c). Notice that the bipartite $S(l_x)$ of the projected uniform $\pi$-flux state does not saturate, which is sharply different from those shown in Fig. (S3). We further place the projected uniform $\pi$-flux state on a $L_x \times L_y = 6 \times 4$ torus and measure the nearest-neighbor correlation $\Lambda_{ij}$. The uniform pattern of $\Lambda_{ij}$ illustrated in Fig. S6(b) indicates that the translational symmetry is not broken. Thus, the “translational symmetry breaking” observed on cylinders is more likely an open-boundary effect.

![Projected uniform $\pi$-flux state](image)
FIG. S6. The nearest-neighbor correlation $\Lambda_{ij}$ for the projected uniform $\pi$-flux state on XC4 (a) cylinder with $L_x = 16$ and (b) torus with $L_x = 6$. (c) The bipartite EE $S(l_x)$ versus subsystem length $l_x$ for the projected uniform $\pi$-flux state on an XC4 cylinder with length $L_x = 24$. 