Statistics of orbital entanglement production in quantum-chaotic dots

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The production of orbitally entangled electrons in quantum-chaotic dots is investigated from a statistical point of view. The degree of entanglement is quantified through the concurrence and the entanglement of formation. We calculate the complete statistical distributions of the entanglement measures by using random matrix theory. Simple analytical expressions are provided for the concurrence distributions. We identify clear signatures of time-reversal invariance in the production of entanglement at the level of the entanglement-measure distributions, such as the ability of producing maximally entangled (Bell) states, which passed unnoticed in previous works where only the first two moments of the distributions were studied.

PACS numbers: 03.67.Mn, 73.23.-b, 05.45.Pq, 73.63.Kv

In the early times of quantum mechanics, the prediction of entangled quantum states subject to nonclassical correlations was identified as a possible sign of incompleteness of the theory. Despite such original objections, and after many years of debate during which decisive experimental evidence was accumulated, the reality of quantum entanglement is nowadays widely accepted. Besides its fundamental interest, modern research on entanglement has been strongly motivated by its possible application as a resource for quantum information processing. As a consequence, an intensive research activity on the production, manipulation, and detection of quantum entanglement in a variety of physical systems is reflected in an extended literature.

Several efforts have been devoted to the study of the entanglement in electronic systems, with a view on solid-state applications that could lead to a major technological breakthrough in the field of nanoelectronics. As far as the production of entangled electrons is concerned, several proposals based on different interacting and noninteracting electron mechanisms already exist. Recently, Beenakker et al. proposed a ballistic quantum dot as an orbital entangler for pairs of noninteracting electrons; by assuming that the entangler is a chaotic quantum dot, they calculated the average and variance of the concurrence (a standard measure of two-qubit entanglement) and found that these two moments are practically unaffected by the breaking of time-reversal invariance (TRI). This fact is in contrast to other transport properties of ballistic quantum dots such as the conductance, where TRI yields significant weak-localization corrections. More recently, it was shown that signatures of TRI corrections can arise in the average concurrence once spin-orbit interaction is considered. The concurrence fluctuations, however, can be very large (of the order of the average concurrence). This indicates that the first two moments of the concurrence are insufficient for an accurate statistical description of the entanglement production in a quantum dot.

Here we calculate the complete distribution of the degree of orbital entanglement in terms of the concurrence $C$ and the entanglement of formation $E$. We derive simple analytical expressions for the distribution of $C$ for the cases when TRI is preserved and broken, while the distributions of $E$ are calculated via a relation between $C$ and $E$. We find that the degree of produced entanglement is distributed quite differently depending on whether TRI is present or not. In particular, we show that maximally entangled states are produced only when TRI is preserved. Clear signatures of TRI correlations are, however, absent in the first two moments of the distributions, which is in agreement with previous results. We verify our theoretical results for the concurrence distributions by numerical simulations of chaotic quantum dots.

The setup of the orbital entangler as proposed in Ref. is sketched in Fig. 1. It consists of a quantum dot with two attached single-channel leads at the left and right. Each lead is connected to an electron reservoir. The application of a small bias voltage between reservoirs give rise to a coherent current traversing the dot from left to right. Exchange correlations due to scattering within the dot lead to entanglement between transmitted (to the right) and reflected (to the left) electrons, as we show in the following.

We start by considering a separable two-particle state incoming from the left reservoir in Fig. 1:

$$|\Psi_{\text{in}}\rangle = a_1^\dagger a_2^\dagger |0\rangle,$$

where $a_i^\dagger$ creates an incoming electron in the lead $i = 1$, and 2 and $|0\rangle$ denotes the Fermi sea at zero temperature. We disregard spin degeneracy for simplicity (equivalently, one can consider incoming electrons with their spins polarized along the same direction). The outgoing state is a coherent superposition of orbital channels determined by the single-particle scattering matrix $S$, such that

$$|\Psi_{\text{out}}\rangle = \sum_{mn} S_{m1} S_{n2} b_n^\dagger b_m^\dagger |0\rangle.$$

$$\label{eq:1}$$
A widely used measure for quantifying two-qubit entanglement is the concurrence $\mathcal{C}$. It is defined as

$$\mathcal{C}(\rho) \equiv \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

where the $\lambda_i$'s are the eigenvalues (in decreasing order) of the matrix $\rho \tilde{\rho}$, with $\rho$ being a 4 × 4 two-qubit density matrix ($\rho = |1,1\rangle\langle 1,1|/\langle 1,1|1,1\rangle$ in our case) and $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^T \sigma_y \otimes \sigma_y$, where $\sigma_y$ is the second Pauli matrix. The concurrence varies from 0 to 1. The case $\mathcal{C} = 0$ corresponds to separable non-entangled states, while maximally entangled (Bell) states own $\mathcal{C} = 1$. Those states with a $0 < \mathcal{C} < 1$ are non-separable partly entangled states. For our quantum dot entangler in Fig. 1, a finite value of $\mathcal{C}$ would guarantee that the left and right outgoing channels are orbitally entangled.

The concurrence has the advantage of being directly related to the entanglement of formation $\mathcal{E}$, which is one of the most accepted measures of entanglement. Physically, $\mathcal{E}$ quantifies the cost, in terms of Bell states, to prepare a given (pure) state. The entanglement of formation and the concurrence are related through

$$\mathcal{E}(\mathcal{C}) = h \left( 1 + \frac{1 - \mathcal{C}^2}{2} \right),$$

where

$$h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x).$$

On the one hand, $\mathcal{C}$ and $\mathcal{E}$ can be written in terms of the scattering matrix $S$ through the transmission eigenvalues $\tau_1$ and $\tau_2$ as

$$\mathcal{C} = \frac{2\sqrt{\tau_1(1 - \tau_1)\tau_2(1 - \tau_2)}}{\tau_1 + \tau_2 - 2\tau_1\tau_2}.$$

We note that the entanglement is maximum ($\mathcal{C} = 1$) when $\tau_1 = \tau_2$, and minimum ($\mathcal{C} = 0$) when $\tau_1 = 0$ and $\tau_2 = 1$ or $\tau_1 = 1$ and $\tau_2 = 0$. The scattering matrix for the quantum dot of Fig. 1 reads

$$S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix},$$

where $r, r', t,$ and $t'$ are 2 × 2 reflection and transmission matrices, respectively. Thus, $\tau_1$ and $\tau_2$ in Eq. (10) are the eigenvalues of the product $tt'$.

In the presence of TRI, $S$ is unitary and symmetric. If TRI is broken (due to, e.g., the application of a magnetic flux), then $S$ is only unitary.

On the other hand, the chaotic scattering in the quantum dot gives a stochastic character to the entanglement production. Therefore, a statistical analysis of the entanglement is required. Previous works focused only on the mean value and variance of the concurrence. As we have mentioned, here we obtain the complete distribution of the concurrence and entanglement of formation.

In the framework of random matrix theory, the statistical
properties of the $S$ matrix depend on the symmetry class $\beta$. In the presence of TRI, the statistics of an ensemble of unitary and symmetric $S$ matrices is described by the so-called circular orthogonal ensemble ($\beta = 1$). When TRI is absent, the statistical properties of $S$ are described by the Circular Unitary Ensemble ($\beta = 2$). Particular attention has been given in the past to the statistical properties of the transmission eigenvalues $\tau_n$ due to their relevance in quantum transport phenomena. In fact, the joint distribution $p_\beta(\{\tau_n\})$ is known. Here we are interested in the joint distribution of only two eigenvalues, which is given by

$$p_\beta(\tau_1, \tau_2) = c_\beta |\tau_1 - \tau_2|^\beta (\tau_1 \tau_2)^{\beta/2 - 1},$$

where $c_\beta$ is a normalization factor. Thus, the production of maximally or minimally entangled states is eventually determined by the statistical distributions of $\tau_1$ and $\tau_2$ through the relations $\langle \tau_n \rangle$ and $\text{var}(\tau_n)$.

We can now calculate the distribution of the concurrence $P_\beta(C)$ which is defined as

$$P_\beta(C) = \left\langle \delta \left[ C - \frac{2(\tau_1 - \tau_2)}{\tau_1 + \tau_2 - 2\tau_1 \tau_2} \right] \right\rangle_\beta,$$

where $\langle \cdots \rangle$ stands for the ensemble average performed with the probability distribution of Eq. (12). The double integral over the variables $\tau_1$ and $\tau_2$ in Eq. (13) can be performed exactly. After somewhat lengthy but direct algebra, we obtain a surprisingly simple result for the case of preserved TRI ($\beta = 1$):

$$P_1(C) = \frac{2}{(1 + C)^2},$$

while for the variance of the concurrence we have

$$\text{var}(C) = \left\{ \begin{array}{ll}
2[1 - 2(\ln 2)^2] \approx 0.0782 & \text{for } \beta = 1, \\
\frac{e^2}{210}(88 - 21\pi)(21\pi - 40) - 22 \approx 0.0565 & \text{for } \beta = 2.
\end{array} \right.$$
The distribution \( Q_\beta \) related to \( P \) of the transformation \( C \to E \) the entanglement of formation, instead, by calculating the concurrence as a function of transcendental functions in Eq. (9). We proceed numerically, instead, by calculating the concurrence as a function of the entanglement of formation, \( C(E) \). From the Jacobian of the transformation \( C \to E \), we find that \( Q_\beta(E) \) is related to \( P_\beta(C) \) through

\[
Q_\beta(E) = \frac{1}{C(E) \cdot \text{atanh} \left( \sqrt{1 - \frac{C(E)^2}{1 - C(E)^2}} \right)} P_\beta(C(E)).
\]

The distribution \( Q_\beta(E) \) is depicted in Fig. 3. Of course, these curves give us the same information provided by \( P_\beta(C) \) in Fig. 2, and the interpretation of the TRI effects on the entanglement production is similar. However, \( \mathcal{E} \) has the advantage of quantifying the cost of preparing an entangled state in terms of the amount Bell pairs needed.

As regards the mean value and variance of \( \mathcal{E} \), we obtain

\[
\langle \mathcal{E} \rangle = \begin{cases} 
0.285 & \text{for } \beta = 1, \\
0.273 & \text{for } \beta = 2,
\end{cases}
\]

and

\[
\text{var}(\mathcal{E}) = \begin{cases} 
0.078 & \text{for } \beta = 1, \\
0.056 & \text{for } \beta = 2.
\end{cases}
\]

These two moments do not reveal any significant TRI effect, similar to what was obtained for the concurrence.

Summarizing, we present a statistical analysis of the production of orbital entanglement in quantum-chaotic dots by calculating the full distributions of the corresponding entanglement measures. Particularly, we found simple analytical expressions for the concurrence distributions in the presence and absence of TRI. The complete distributions of these measures allowed us to identify significant TRI effects on the entanglement production, which are not reflected at the level of the first and second moments.

We thank S. Montangero for helpful discussions. This work was supported by the “Ramón y Cajal” program of the Spanish Ministry of Education and Science.
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