A note on the number of terms witnessing congruence modularity

PAOLO LIPPARINI

Abstract. We show that if \( \kappa \) is even and some variety \( \mathcal{V} \) is congruence modular as witnessed by \( k + 1 \) Day terms then, for every \( q \geq 1 \), \( \mathcal{V} \) satisfies the congruence identity
\[
\alpha(\beta \circ \alpha \gamma \circ \beta \circ \ldots) \subseteq \alpha \beta \circ \alpha \gamma \circ \alpha \beta \circ \ldots,
\]
with \( 2^{q+1} - 1 \) factors on the left-hand side and \( \frac{kq}{2^{q-1}} \) factors on the right-hand side. See Theorem 3.3. Here juxtaposition denotes intersection and terms as \( \alpha \gamma \) are counted as a single factor.

If \( \mathcal{V} \) has \( n + 2 \) Gumm terms and \( q \geq 2 \), then the above identity holds with \( 2^q - 1 \) factors on the left and \((2^{q+1} - 2q - 3)n + 3\) factors on the right. See Corollary 4.8.

So far, in general, the best evaluation is obtained by combining the two methods. See Corollary 5.1. It is an open problem whether there is a better way. Other open problems related to similar identities are discussed. The results might shed new light to the problem of the relationship between the number of Day terms and the number of Gumm terms for a congruence modular variety.

By the way, we use Gumm terms also in order to give bounds of the form
\[
\alpha(\beta \circ \gamma \circ \beta \circ \ldots) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ \alpha \beta \circ \alpha \gamma \circ \ldots),
\]
with appropriate numbers of factors. This extends ideas of S. Tschantz. See Section 4. We also slightly improve a result by A. Day, to the effect that if \( n \) is even, then every variety with \( n + 2 \) Jónsson terms has \( 2n + 1 \) Day terms. See Proposition 6.1.

This is a preliminary version, still to be expanded. It might contain inaccuracies (to be precise, it is more likely to contain inaccuracies than subsequent versions).

1. Introduction

By a celebrated theorem by A. Day [D], a variety \( \mathcal{V} \) is congruence modular if and only if there is an integer \( k > 0 \) such that \( \mathcal{V} \) satisfies
\[
\alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha \beta \circ \alpha \gamma \circ \alpha \beta \ldots
\]
where in the above inclusion \( \alpha, \beta, \gamma \) vary among congruences of some algebra in \( \mathcal{V} \) and juxtaposition denotes intersection. When we say that some variety \( \mathcal{V} \) satisfies an identity, we mean that the identity is satisfied in all algebras in \( \mathcal{V} \). Formally, for every algebra \( A \in \mathcal{V} \), the identity has to be satisfied in the structure of all reflexive and admissible relations of \( A \). As we mentioned, \( \alpha, \beta, \ldots \) are intended to vary among congruences; other kinds of

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variables shall be allowed; for example, we shall use the letters \( R, S \) to denote variables for reflexive and admissible relations. The above conventions shall be in charge throughout the paper. A variety satisfying equation (1) is said to be \( k \)-modular.

Day’s actual statement concerns the existence of certain terms; for varieties, this is equivalent to the satisfaction of the above congruence inclusion. In this sense, a variety satisfying equation (1) is said to have \( k + 1 \) Day terms. See below for further details. Stating results in the form of an inclusion is intuitively clearer and notationally simpler, though in certain proofs the terms are what is really used. Cf. Tschantz [T] and below.

We now need a bit more notation. For \( m \geq 1 \), let \( \beta \circ_m \gamma \) denote \( \beta \circ \gamma \circ \beta \ldots \) with \( m \) factors, that is, with \( m - 1 \) occurrences of \( \circ \). In some cases, when we want to mention the last factor explicitly, say, when \( m \) is odd, we shall write, \( \beta \circ \gamma \circ \ldots \circ \beta \) in place of \( \beta \circ_m \gamma \). With this notation, (1) above reads \( \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma \), or even \( \alpha(\beta \circ \alpha \gamma \circ \beta \ldots) \subseteq \alpha \beta \circ_k \alpha \gamma \).

The non trivial point in the proof in [D] is to show that if (1) above holds in the free algebra in \( V \) generated by four elements (and thus we have appropriate terms), then \( \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta + \alpha \gamma \) holds, for every \( m \). Actually, it is implicit in the proof that, for every \( m \), there is some \( D(m) \) which depends only on the \( k \) given by (1) (but otherwise not on the variety at hand) and such that \( \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_{D(m)} \alpha \gamma \). In Section 2 we evaluate \( D(m) \) as given by Day’s proof and then in Section 3 we show that the alternative proof of Day’s result from [L1, Corollary 8] provides a better bound. Other bounds are obtained in Section 4 by using a different set of terms for congruence modular varieties, the terms discovered by H.-P. Gumm. There we also provide a bound for \( \alpha(\beta \circ \alpha \gamma \circ \beta \ldots) \), relying on ideas of S. Tschantz. The methods are combined and compared in Section 5 where we also discuss the (still open) problem of finding the best way for evaluating \( D(m) \). Finally, some connections with congruence distributivity and some further problems are discussed in Sections 6 and 7.

The next section is introductory in character and might be skipped by a reader familiar with congruence modular varieties.

2. Introductory remarks

This section contains a few introductory and historical remarks, with absolutely no claim at exhaustiveness. The reader familiar with congruence modular varieties might want to go directly to the next section.

Obviously, an algebra is congruence modular if and only if it satisfies \( \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta + \alpha \gamma \), for every \( m \geq 3 \) and all congruences \( \alpha, \beta \) and \( \gamma \). The by now standard algorithm by A. Pixley [P] and R. Wille [W] thus shows that a variety \( V \) is congruence modular if and only if, for every \( m \geq 3 \), there is \( k \) such that \( V \) satisfies the congruence identity \( \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma \).
Of course, as we mentioned in the introduction, it is well-known that this is not the best possible result, since A. Day [D] showed that already \( \alpha(\beta \circ \gamma \circ \beta) \subseteq \alpha \beta \circ_k \alpha \gamma \), for some \( k \), implies congruence modularity. That is, the case \( m = 3 \) in the above paragraph is enough. For the reader familiar with the terminology, this implies that congruence modularity is actually a Maltsev condition, rather than a weak Maltsev condition (by the way, Day’s result appeared before the general formulation of the algorithm by Pixley and Wille).

It follows immediately from Day’s result that if a variety \( V \) satisfies the congruence identity \( \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma \), for some \( m \geq 3 \) and \( k \), then \( V \) is congruence modular. Indeed, if \( m \geq 3 \), then \( \alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma \) and we are done by Day’s result.

For \( m \geq 3 \), let us say that an algebra \( A \) is \((m,k)\)-modular if \( \alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma \), for all congruences of \( A \). A variety is \((m,k)\)-modular if all of its algebras are. By the above discussion, a variety \( V \) is congruence modular if and only if \( V \) is \((3,k)\)-modular, for some \( k \), if and only if, for every \( m \geq 3 \), there is some \( k \) such that \( V \) is \((m,k)\)-modular.

In the present terminology, \((3,k)\)-modularity is equivalent to \( k \)-modularity in the sense of [D]. A \( k \)-modular variety is usually said to have \( k + 1 \) Day terms (see Condition (4) in Proposition 2.1 below).

We have given the definition of \((m,k)\)-modularity also in the case when \( m \) is even, but this does not seem to provide a big gain in generality, since, say, \( \alpha(\beta \circ \gamma \circ \alpha \circ \beta) = \alpha(\beta \circ \gamma \circ \beta) \circ \alpha \gamma \). In other words, if we define \( D_V(m) \) to be the least \( k \) such that \( V \) is \((m,k)\)-modular, we have \( D_V(m+1) \leq D_V(m)+1 \), for \( m \) odd.

As now folklore, Day’s arguments can be divided in two steps. The first step can be seen, by hindsight, as a mechanical application of the Pixley and Wille algorithm. Compare the perspicuous discussions in Gumm [G2] and Tschantz [T].

**Proposition 2.1.** For a variety \( V \), the following conditions are equivalent.

1. \( V \) is \( k \)-modular, that is, \((3,k)\)-modular, that is, \( \alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha \beta \circ_k \alpha \gamma \) holds in \( V \).
2. The free algebra in \( V \) over four generators is \((3,k)\)-modular.
3. If \( A \) is the free algebra in \( V \) over the four generators \( a,b,c,d \), \( \alpha \) is \( Cg((a,d),(b,c)) \), \( \beta \) is \( Cg((a,b),(c,d)) \) and \( \gamma \) is \( Cg(b,c) \), then \( (a,d) \in \alpha \beta \circ_k \alpha \gamma \).
4. \( V \) has quaternary terms \( d_0, \ldots, d_k \) such that the following are equations valid in \( V \):
   - \( (a) \) \( x = d_i(x,y,x,y) \), for every \( i \);
   - \( (b) \) \( x = d_0(x,y,z,w) \);
   - \( (c) \) \( d_i(x,x,w,w) = d_{i+1}(x,x,w,w) \), for \( i \) even;
   - \( (d) \) \( d_i(x,y,y,w) = d_{i+1}(x,y,y,w) \), for \( i \) odd, and
   - \( (e) \) \( d_k(x,y,z,w) = w \).
Proof. By now, standard. We shall only use \((1) \iff (4)\), which is implicit in \([D]\). The proof of the full result, again, is implicit in \([D]\) and can be extracted, for example, from the proof of \([CV, \text{Theorem 2.4}]\).

\[\square\]

Theorem 2.2. (Day’s Theorem) A variety \(V\) is congruence modular if and only if, for some \(k\), \(V\) satisfies one (hence all) of the conditions in Proposition 2.1.

The proof of Theorem 2.2 uses the terms given by Proposition 2.1(4). Day’s argument essentially proves a tolerance identity. If an algebra has terms satisfying Proposition 2.1(4), then it satisfies
\[
\alpha(\Delta \circ \alpha \gamma \circ \beta) \subseteq (\alpha \Delta \circ \alpha \Delta) \circ_{\kappa} \alpha \gamma
\]
where, as usual, \(\alpha\) and \(\gamma\) are congruences, but \(\Delta\) is only assumed to be a tolerance containing \(\beta\). Recall that a tolerance is a reflexive and symmetrical compatible relation. Now if we set \(\Delta_h = \beta \circ \alpha \gamma \circ 2^{h+1} \circ \beta\), then \(\Delta_h\) is surely a tolerance, and moreover \(\Delta_{h+1} = \Delta_h \circ \beta \circ \alpha \gamma\), hence, using equation (2) and by an easy induction, it can be proved that, for every \(h\), we have \(\alpha(\beta \circ \alpha \gamma \circ 2^{h+1} \circ \beta) \subseteq \alpha \beta + \alpha \gamma\), hence Theorem 2.2 follows. Actually, the induction shows that, say for \(k\) even, we have \(\alpha(\beta \circ \alpha \gamma \circ 2^{h+1} \circ \beta) \subseteq \alpha \beta \circ p \alpha \gamma\), with \(p = k^h\). In other words, for \(k\) even, \((3, k)\)-modularity implies \((2^{h+1}, k^h)\)-modularity, for every \(h\).

Day’s argument can be modified in order to improve (2) to
\[
\alpha(\Delta \circ \alpha \gamma \circ \Delta) \subseteq (\alpha \Delta \circ \alpha \Delta) \circ_{\kappa} \alpha \gamma
\]
for all congruences \(\alpha\) and \(\gamma\) and tolerance \(\Delta\). Essentially, the above identity is a reformulation of Gumm’s Shifting Principle \([G2]\). Again by induction, we get that, for \(k\) even, \((3, k)\)-modularity implies \((2^{q+1} - 1, k^q)\)-modularity, for every \(q\).

Some further small improvements are possible analyzing Day’s proof, for example one can get
\[
\alpha(\Delta \circ \alpha \gamma \circ \Delta) \subseteq \alpha \Delta \circ (\alpha \gamma \circ_{k-1} (\alpha \Delta \circ \alpha \Delta))
\]
that is, one can “save” an \(\alpha \Delta\) in the first place, and symmetrically this can be done at the end, if \(k\) is odd.

However, we shall show in the next section that equation (3) can be improved further, by using results from \([L1]\).

3. An improved bound

Recall that, for \(m \geq 3\), we say that a variety \(V\) is \((m, k)\)-modular if the congruence inclusion \(\alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma\) holds in \(V\). Thus \((3, k)\)-modularity means having \(k+1\) Day terms or, which is the same, being \(k\)-modular. Moreover, \(D_V(m)\) is the least \(k\) such that \(V\) is \((m, k)\)-modular.

We can use \([L1, \text{Theorem 3 (i) } \Rightarrow (iii)]\) in order to improve the results mentioned in the previous section. Essentially, we evaluate the number of terms
given by the alternative proof of Day’s theorem presented in \[L1\] Corollary 8. If \( R \) is a binary relation on some set \( A \), we set \( R^\sim = \{(b, a) \mid a, b \in A, \text{ and } a R b\} \) (\( R^\sim \) has been denoted by \( R^- \) in \[L1\]).

**Proposition 3.1.** If \( \mathcal{V} \) is a \( k \)-modular variety, that is, \( \mathcal{V} \) is \((3, k)\)-modular, then \( \mathcal{V} \) satisfies

\[
\alpha(\Delta \circ \alpha \gamma \circ \Delta) \subseteq \alpha \Delta \circ \alpha \alpha \gamma,
\]

where \( \alpha \) and \( \beta \) vary among congruences on some algebra \( A \) in \( \mathcal{V} \) and \( \Delta \) is a tolerance on \( A \) such that there exists a reflexive and compatible relation \( R \) for which \( \Delta = R \circ R^\sim \).

**Proof.** In view of Proposition 2.1, this is a special case of \[L1, \text{Theorem 3 (i) } \Rightarrow \text{(iii)}\].

For the reader’s convenience, we present the explicit details in this particular case. By assumption, we have terms as given by Proposition 2.1(4). If \( A \in \mathcal{V} \), \( a, d \in A \) and \( (a, d) \in \alpha(\Delta \circ \alpha \gamma \circ \Delta) \), then \( a \alpha d \) and there are elements \( b, c \in A \) such that \( a \Delta b \alpha \gamma c \Delta d \). Since \( \Delta = R \circ R^\sim \), we have further elements \( b', c' \) such that \( a R b' R^\sim b \alpha \gamma c R c' R^\sim d \), thus \( b R b' \) and \( d R c' \).

Using the terms given by Proposition 2.1(4), we have

1. \( d_i(a, b, c, d) \alpha d_i(a, b, b, a) = a = d_{i+1}(a, b, b, a) \alpha d_{i+1}(a, b, c, d) \), for every \( i \);
2. \( a = d_0(a, b, c, d) \);
3. \( d_i(a, b, c, d) R d_i(b', b', c', c') = d_{i+1}(b', b', c', c') R^\sim d_{i+1}(a, b, c, d) \), for \( i \) even;
4. \( d_i(a, b, c, d) \gamma d_i(a, b, b, d) = d_{i+1}(a, b, b, d) \gamma d_{i+1}(a, b, c, d) \), for \( i \) odd, and
5. \( d_k(a, b, c, d) = d \),

thus the elements \( d_i(a, b, c, d) \), for \( i = 0, \ldots, k \), witness the desired inclusion (notice that item (3) implies \( d_i(a, b, c, d) \Delta d_{i+1}(a, b, c, d) \)). \( \square \)

**Remark 3.2.** Recall from \[L1\] that a tolerance \( \Theta \) is representative if \( \Theta = R \circ R^\sim \), for some reflexive and admissible relation \( R \). It follows from \[L1, \text{Theorem 3 (i) } \Rightarrow \text{(iii)}\] that in Proposition 3.1 it is enough to assume that \( \alpha \) and \( \gamma \) are representative tolerances, rather than congruences.

Actually, a finer result holds! Arguing as in \[CH\], we can relax the assumption that \( \alpha \) is a congruence in Proposition 3.1 to \( \alpha \) being any tolerance (not necessarily representable). The same argument has been applied in \[L3, \text{Proposition 3.3}\] and many times in \[L2\].

**Theorem 3.3.** For \( k \) even and every \( q \geq 1 \), \((3, k)\)-modularity, that is, \( k \)-modularity, implies \((2^{q+1} - 1, \frac{k^q}{2^{q+1}})\)-modularity.

**Proof.** By induction on \( q \geq 1 \).

The induction basis \( q = 1 \) is exactly \((3, k)\)-modularity.
Suppose that the proposition holds for some given \( q \geq 1 \). We have
\[
\alpha(\beta \circ \alpha \gamma \circ 2^{q+1-1} \circ \beta) =
\alpha \left( (\beta \circ \alpha \gamma \circ 2^{q+1-1} \circ \beta) \circ \alpha \gamma \circ (\beta \circ \alpha \gamma \circ 2^{q+1-1} \circ \beta) \right) \subseteq \text{Prop. 3.1}
\]
where the inclusion marked with “ih” follows from the inductive hypothesis and we are allowed to apply Proposition 3.1 since \( \beta \circ \alpha \gamma \circ 2^{q+1-1} \circ \beta = R \circ R' \), for \( R = \beta \circ \alpha \gamma \circ 2^q \circ \beta \circ \alpha \gamma \). We have used several times the fact that \( \alpha \gamma \) is a congruence, thus \( \alpha \gamma \circ \alpha \gamma = \alpha \gamma \). \( \square \)

The proof of Theorem 3.3 applies to a more general context.

**Proposition 3.4.** For \( k \) even, \( h > 1 \) and \( i \geq 0 \), we have that \((2h - 1, k)\)-modularity implies \((2h^2 - 1, \frac{k^2}{2h-1})\)-modularity.

**Proof.** It is enough to show that \((2h - 1, k)\)-modularity implies \((2h^2 - 1, \frac{k^2}{2h-1})\)-modularity; the full result follows then immediately by induction. To prove the above claim, notice that, again by [L1, Theorem 3 (i) \( \Rightarrow \) (iii)], \((2h - 1, k)\)-modularity implies the identity \( \alpha(\Delta \circ \alpha \gamma \circ 2^{h-1} \circ \Delta) \subseteq \alpha \Delta \circ \alpha \gamma \), for \( \Delta \) a representable tolerance. Then the argument in 3.3 applies to get \((2h^2 - 1, \frac{k^2}{2h-1})\)-modularity, taking \( \Delta' = \Delta \circ \alpha \gamma \circ 2^{h-1} \circ \Delta \) in place of \( \Delta \). \( \square \)

By similar arguments one can obtain consequences of the combination of \((2h - 1, k)\)-modularity and of \((2h' - 1, k')\)-modularity. Perhaps this is not particularly interesting.

Of course, analogous results can be proved for \( k \) odd, but we have not worked out the details. At first sight, it appears that in the case \( k \) odd the above arguments provide only a minor improvement with respect to the bounds obtained for (the even) \( k + 1 \). At least, this is confirmed by considering what happens for small values of \( k \).

**Corollary 3.5.** Let \( q \geq 2 \).

1. If \( V \) is 3-modular, then \( V \) is \((2^q - 1, 2^q - 1)\)-modular.
2. If \( V \) is 4-modular, then \( V \) is \((2^q - 1, 2^q)\)-modular.

**Proof.** (1) follows from Proposition 3.1 by an induction similar to the one used in the proof of Theorem 3.3.

(2) is the particular case \( k = 4 \) of Theorem 3.3. \( \square \)

Corollary 3.5 suggests that \((3, 3)\)-modularity and \((3, 4)\)-modularity imply, respectively, \((m, m)\)- and \((m, m + 1)\)-modularity, for every \( m \geq 3 \); however, this is an open problem.

Apart from the above remark, Corollary 3.5(1), within its scope, is the best possible result, as shown by the variety of lattices. Indeed, for \( m \geq 3 \),
every \((m, m - 1)\)-modular variety is \(m - 1\)-permutable: just take \(\alpha = 1\) in the
definition of \((m, m - 1)\)-modularity. Hence lattices are not \((m, m - 1)\)-modular,
while they are \((3, 3)\)-modular, e. g., by Day [D] Theorem on p. 172. See the
discussion at the beginning of Section 6.

4. Employing Gumm terms

Another very interesting characterization of congruence modularity has
been provided by Gumm [G1]. See also [G2] and [FM] for the related commutator
theory for congruence modular varieties.

**Theorem 4.1.** (Gumm [G1]) For a variety \(\mathcal{V}\), the following conditions are
equivalent.

1. \(\mathcal{V}\) is congruence modular.
2. For some \(n\), \(\mathcal{V}\) has ternary terms \(m\) and \(j_1, \ldots, j_{n+1}\) such that the follow-
ing equations are valid in \(\mathcal{V}\):
   a. \(x = j_i(x, y, x)\), for every \(i\);
   b. \(x = m(x, z, z)\);
   c. \(m(x, x, z) = j_1(x, x, z)\);
   d. \(j_i(x, z, z) = j_{i+1}(x, z, z)\), for \(i\) odd, \(i \leq n\);
   e. \(j_i(x, x, z) = j_{i+1}(x, x, z)\), for \(i\) even, \(i \leq n\), and
   f. \(j_{n+1}(x, y, z) = z\).

We say that \(\mathcal{V}\) has \(n + 2\) Gumm terms if Condition (2) above holds for \(\mathcal{V}\)
and \(n\).

Notice that condition (2) as presented here is slightly different from Gumm’s
Corresponding condition [G2, Theorem 7.4(iv)], in which the ordering of the
terms is exchanged. This is not just a matter of symmetry, see, for example,
the comment in [L2, Section 4]. Seemingly, the first formulation of condition
4.1(2) above first appeared in [LTT] and [I].

As already noted by Gumm himself, conditions like 4.1(2) above “compose”
the corresponding conditions for congruence permutability and congruence
distributivity due to A. I. Maltsev [M] and B. Jónsson [J], respectively. In fact,
Gumm terms provide the possibility of merging methods used for congruence
distributivity and congruence permutability, as accomplished in several ways
by several authors. The next lemma is another small result in the same vein.

Notice that the existence of \(n + 2\) Gumm terms is the Maltsev condition
naturally associated with the congruence identity \(\alpha(\beta \circ \gamma) \subseteq \gamma \circ \beta \circ (\alpha \gamma \circ_n \alpha \beta)\), or
even \(\alpha(\beta \circ \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ_n \alpha \beta)\). (Here, to be consistent in the extreme
case \(n = 1\), we conventionally assume \(\alpha \gamma \circ_0 \alpha \beta = 0\); this is another way to see
that having 2 Gumm terms corresponds to congruence permutability.)

**Remark 4.2.** (i) However, we should remark that Gumm terms can be inter-
preted still in another way, besides expressing some condition having the flavor
of “permutability composed with distributivity”. Before explaining the other
possible interpretation of Gumm terms, let us add some details about the 
above expression in quotes. As already noted by Gumm himself, if \( n = 1 \) (or, 
equivalently, if all the \( j_i \)'s are the trivial projections onto the third coordinate), 
then the only nontrivial term is \( m \), which satisfies the Maltsev equations for 
congruence permutability. In this sense \( m \) represents the permutability part. 
On the other hand, if \( m \) is the trivial projection onto the first coordinate, 
then \( m \) is the trivial projection onto the first coordinate, then 
the terms give Jónsson condition characterizing congruence distributivity. If 
this is the case, that is, \( m \) is the trivial projection onto the first coordinate, then 
the terms give Jónsson condition characterizing congruence distributivity. In 
this sense, the \( j_i \)'s represents the distributivity part. More specifically, 
Jónsson terms for \( n + 1 \)-distributivity are terms \( j_0, \ldots, j_{n+1} \) satisfying 
conditions (a), (d)-(f) above as well as \( x = j_0(x, y, z) \) (notice that, if we rename \( m \) 
as \( j_0 \), then (c) becomes a special case of (e)).

(ii) Now give another look from scratch at Condition (2) in [L2]. There is 
no big difference between conditions (b)-(c) for \( m \) and conditions (d)-(f) for 
the \( j_i \)'s. However, in addition, the \( j_i \)'s are supposed to satisfy (a). What if 
we require \( m \) to satisfy the analogue of (a)? That is, what if we ask that \( m \) 
satisfies \( x = m(x, y, x) \), too? At a very first glance, the answer would be that 
we get the condition for \( n + 2 \)-distributivity. It is not precisely so. Formally, 
we have to add another term at the beginning, to be interpreted as the trivial 
projection onto the first coordinate. This term should be numbered as \( j_{-1} \), so 
we have to shift the indices by 1 and what we get is like \( n+2 \)-distributivity 
but with the role of even and odd exchanged. This is sometimes called the 
ALVIN variant of Jónsson condition, since it appeared in [MMM]. See also 
[FV]. We shall sometimes call this condition \( n+2 \)-distributivity\( \sim \), or say that 
\( \forall \) has \( n+3 \) Jónsson\( \sim \) terms.

(iii) Notice that changing the role of odd and even in Jónsson condition 
might appear an innocuous variant, but this is far from being true, if we keep 
\( n \) fixed. Indeed, ALVIN 2-distributivity means having a Pixley term, that is, 
arithmeticity! Let us remark that, however, if \( n \) is odd, the Jónsson and ALVIN 
\( n \)-distributivity conditions are equivalent, just reverse both the order of terms 
and of variables. In a sense, the above arguments show that Jónsson condition 
for \( n \) odd and the ALVIN condition for every \( n \) share some “spurious” aspects 
of permutability, as already evident in the case of Pixley (ALVIN 2) terms. 
We shall make an intense use of this permutability aspect at various places. 
Cf. also the proof of [L2], Theorem 2.3

(iv) For the reader who prefers to see Maltsev conditions expressed in terms 
of congruence identities, \( n \)-distributivity corresponds to the identity \( \alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_n \alpha\gamma \) while ALVIN \( n \)-distributivity corresponds to \( \alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ_n \alpha\beta \). 
As we mentioned, having \( n \) Gumm terms corresponds to \( \alpha(\beta \circ \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_{n-2} \alpha\beta) \).

Of course, however, Gumm’s condition does not require \( x = m(x, y, x) \). 
Hence, in the above sense, Gumm terms can be interpreted as defective ALVIN
(or reversed Jónsson) terms, since one equation of the form \( x = j_i(x, y, x) \), is missing. We shall see that certain arguments which work for congruence distributive varieties can be still applied when one equation of that form is missing. However, it is necessary that the missing equation lies at one end of the chain of terms, i. e., it involves the first (or perhaps sometimes the last) non trivial term. In some cases we might even allow two missing equations, see Proposition \[4,4].

The following results extend \[L2, Lemma 4.1\] and use arguments similar to those used in \[L3, Section 2\]. We describe our main aim. Tschantz \[T\] showed that a congruence modular variety satisfies \( \alpha(\beta + \gamma) \subseteq \alpha(\beta \circ \gamma) \circ (\alpha \beta \circ \alpha \gamma) \). It follows from the general theory of Maltsev conditions that, for every congruence modular variety \( \mathcal{V} \) and every \( m \), there is some \( k \) such that \( \alpha(\beta \circ_m \gamma) \subseteq \alpha(\beta \circ \gamma) \circ (\alpha \beta \circ_k \alpha \gamma) \) holds in \( \mathcal{V} \). Actually, it follows from an accurate analysis of Tschantz proof that \( k \) can be expressed in function only of \( m \) and on the number of Gumm terms, but otherwise \( k \) does not depend on \( \mathcal{V} \). Here we provide a better evaluation than the one given by Tschantz arguments, but we absolutely do not know whether our evaluation gets close to the optimal one.

If \( A \) is an algebra and \( X \subseteq A \), we let \( \overline{X} \) denote the smallest reflexive and admissible relation on \( A \) containing \( X \). Conventionally, let us assume that \( \beta \circ_0 \gamma = 0 \).

**Lemma 4.3.** Suppose that \( \mathcal{V} \) has \( n + 2 \) Gumm terms.

Then \( \mathcal{V} \) satisfies

\[
\alpha(R \circ S) \subseteq \alpha\left(\overline{R} \cup \overline{S}\right) \circ \left((\alpha R \circ \alpha S) \circ_n (\alpha S \circ \alpha R)\right)
\]  

(4)

for every congruence \( \alpha \) and reflexive and admissible relations \( R \) and \( S \).

More generally, if \( \alpha \) is a congruence, \( T_1, \ldots, T_m \) are reflexive and admissible relations and \( R \) and \( S \) are binary relations such that \( R \circ S \supseteq T_1 \circ \cdots \circ T_m \), then

\[
\alpha(T_1 \circ T_2 \circ \cdots \circ T_m) \subseteq \alpha\left(\overline{R} \cup \overline{S}\right) \circ \left((\alpha T_1 \circ \alpha T_2 \circ \cdots \circ \alpha T_m) \circ_n (\alpha T_1^{-1} \circ \cdots \circ \alpha T_m^{-1})\right)
\]

(5)

**Proof.** We first prove (4). Suppose that \( (a, c) \in \alpha(R \circ S) \), thus \( a \circ c \) and there is \( b \) such that \( a R b S c \). Then \( a = m(a, b, b) \overline{R} \cup \overline{S} m(a, a, c) \) and \( a = m(a, a, a) \alpha m(a, a, c) \), thus \( (a, j_1(a, a, c)) = (a, m(a, a, c)) \in \alpha(\overline{R} \cup \overline{S}) \).

The rest is quite standard. We have \( j_i(a, a, c) R j_i(a, b, c) S j_i(a, c, c) \) and also \( j_i(a, a, a) = a = j_i(a, b, a) \alpha j_i(a, b, c) \), thus \( j_i(a, a, c) \alpha R j_i(a, b, c) \) and similarly \( j_i(a, b, c) \alpha S j_i(a, c, c) \), thus, for \( i \) odd, \( j_i(a, a, c) \alpha R \alpha S j_i(a, a, c) = j_{i+1}(a, a, c) \).

For \( i \) even, we compute \( j_i(a, c, c) S^\circ \overline{j_i(a, b, c) R^\circ \overline{j_i(a, a, c)}} \) and, arguing as above, \( j_i(a, c, c) \alpha S^\circ \alpha R^\circ \overline{j_i(a, a, c)} = j_{i+1}(a, a, c) \).
Proposition 4.4. If $V$ has $x = 4.3$ “at both ends”. Remarkably enough, there is no need of the equation $V_{j_{i}}(j_{j}S + 1)$ in place of $In order to prove (6) and (8), apply equation (5) in Lemma 4.3 with $j_{i} = 1, 2, m, b_{0}$ for every odd $m$. Moreover, for every odd $m$, let $x = 4.3.\Box$

Hence the first part of the above proof shows that $(a, m(a, a, c)) \in \alpha(R^{-} \cup S^{-})$.

When $n$ is even, we can apply the trick at the beginning of the proof of 4.3 “at both ends”. Remarkably enough, there is no need of the equation $x = j_{i}(x, y, x)$ in order to carry the argument over. Let us say that a variety $V$ has $n + 2$ defective Gumm terms if $V$ has terms satisfying Condition (2) in Theorem 4.1 except that equation $x = j_{i}(x, y, x)$ is not assumed.

**Proposition 4.4.** If $V$ has $n + 2$ defective Gumm terms and $n$ is even, then $V$ satisfies the following identity.

$$\alpha(R \circ S) \subseteq \alpha(R^{-} \cup S^{-}) \circ ((\alpha R \circ \alpha S) \circ_{n-1} (\alpha S^{-} \circ \alpha R^{-})) \circ \alpha(R \cup S^{-})$$

Moreover, under the assumptions in Lemma 4.3 about $T_{1}, \ldots, T_{m}, R$ and $S$, $V$ satisfies the following identity.

$$\alpha(T_{1} \circ T_{2} \circ \cdots \circ T_{m}) \subseteq \alpha(R^{-} \cup S^{-}) \circ ((\alpha T_{1} \circ \alpha T_{2} \circ \cdots \circ \alpha T_{m}) \circ_{n-1} (\alpha T_{m}^{-} \circ \cdots \circ \alpha T_{2}^{-} \circ \alpha T_{1}^{-})) \circ \alpha(R \cup S^{-})$$

**Proof.** We have $j_{n}(a, c, c) \ R \cup S^{-} j_{n}(b, b, c) = j_{n+1}(b, b, c) = c$, since $n$ is even.

All the rest is like 4.3. \Box

**Theorem 4.5.** Suppose that $V$ has $n + 2$ Gumm terms and $m, h$ are natural numbers.

If $m = 4h + 2$, then the following identities hold through $V$.

$$\alpha(\beta \circ \gamma \circ m^{+1} \circ \beta) \subseteq \alpha(\gamma \circ m^{2h+2} \circ \beta) \circ (\alpha \circ m \circ \alpha \beta) \quad (6)$$

$$\alpha(\beta \circ m^{+1} \circ \beta) \subseteq (\alpha \circ m \circ \alpha \gamma) \circ (\beta \circ m \circ \gamma \circ 2h+2) \circ \gamma) \quad (7)$$

If $m = 4h$, then the following identity holds through $V$.

$$\alpha(\beta \circ \gamma \circ m^{+1} \circ \beta) \subseteq \alpha(\beta \circ \gamma \circ m^{2h+1} \circ \beta) \circ (\alpha \circ m \circ \alpha \beta) \quad (8)$$

**Proof.** In order to prove (6) and (7), apply equation (5) in Lemma 4.3 with $m + 1$ in place of $m$, $T_{1} = T_{3} = \cdots = T_{m+1} = \beta$ and $T_{2} = T_{4} = \cdots = \gamma$ (we
shall soon define $R$ and $S$). Formula (9) in Lemma 4.3 gives $(m+1)n$ factors of the form $\alpha T_j$, but we have $n - 1$ contiguous pairs of the form $\alpha \beta \circ \alpha \beta$, hence, after a simplification, the number of factors of the form $\alpha T_j$ reduces to $(m+1)n - (n-1) = mn + 1$.

Then in the case $m = 4h + 2$, take $R = \beta \circ \gamma \circ 2^h + 1 \circ \beta$ and $S = \gamma \circ \beta \circ 2^h + 2 \circ \beta$. In the case $m = 4h$ take $R = \beta \circ \gamma \circ 2^h \circ \gamma$ and $S = \beta \circ \gamma \circ 2^h + 1 \circ \beta$. Thus in both cases we have $R \circ S = \beta \circ \gamma \circ m+1 \circ \beta \supseteq T_1 \circ \cdots \circ T_{m+1}$ and $R \cup S = S$.

Then, say, $\alpha(\gamma \circ \beta \circ 2^h + 2 \circ \beta) \circ \alpha \beta$ is obviously equal to $\alpha(\gamma \circ \beta \circ 2^h + 2 \circ \beta)$, hence in both cases one more factor is absorbed and we get the conclusions.

Finally, (7) is obtained by taking the converse of both sides of (6), since $m + 1$ is odd and both $2h + 2$ and $mn$ are even. $\square$

**Corollary 4.6.** Suppose that $V$ has $n + 2$ Gumm terms and $V$ satisfies the identity

$$\alpha(\beta \circ \gamma \circ 2^n \circ \beta) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ s \circ \alpha \beta)$$

(9)

Then $V$ also satisfies

$$\alpha(\beta \circ \gamma \circ 4^{n+1} \circ \beta) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ r \circ \alpha \beta)$$

(10)

and, more generally, for $q \geq 1$,

$$\alpha(\beta \circ \gamma \circ r^{2q+1} \circ \beta) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ s \circ \alpha \beta)$$

(11)

**Proof.** Equation (10) is immediate from equation (8) in Theorem 4.5 taking $h = r$ and then using (9).

Equation (11) follows by induction. The case $q = 1$ is the assumption (9). Suppose that (11) holds for some $q \geq 1$. Taking $r' = r2^{q-1}$ and $s' = s + r(n2^{q+1} - 4)$ we have that (9) holds with $r'$, $s'$ in place of, respectively, $r$, $s$. By applying (10), we get an inclusion with parameters $4r' + 1 = r2^{q+1} + 1$ and $s' + 4r'n = s + r(n2^{q+1} - 4) + r2^{q+1} = s + r(n2^{q+2} - 4)$, what we needed. $\square$

**Corollary 4.7.** If $V$ has $n + 2$ Gumm terms, then

$$\alpha(\beta \circ \gamma \circ 2^n \circ \beta) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ k \circ \alpha \beta)$$

and

(12)

$$\alpha(\beta \circ \gamma \circ 2^n \circ \beta) \subseteq (\alpha \beta \circ \alpha \gamma) \circ (\beta \circ \gamma)$$

(13)

hold in $V$, for every natural number $q \geq 1$ and where $k = n(2^{q+1} - 2)$.

**Proof.** By equation (9) in Theorem 4.5 with $h = 0$ we get that equation (9) in 4.4 holds with $r = 1$ and $s = 2n$. Thus (12) follows from (11). By taking the converse of both sides in (12), we get (13), since $2^q + 1$ is odd and $k$ is even. $\square$

As a consequence of Corollary 4.7, replacing $\gamma$ by $\alpha \gamma$ in equation (13), we get that if $V$ has $n + 2$ Gumm terms, then, for every $q \geq 1$, $V$ is $(2^q + 1, n(2^{q+1} - 2) + 2)$-modular, since $\alpha(\beta \circ \alpha \gamma) = \alpha \beta \circ \alpha \gamma$. More generally, we can get the following corollary.
Corollary 4.8. Suppose that $\mathcal{V}$ has $n + 2$ Gumm terms and $q \geq 2$. Then $\mathcal{V}$ is $(2^q - 1, (2^{q+1} - 2q - 2)n + 2)$-modular.

Proof. By induction on $q$.

If $q = 2$, then equation (7) in Theorem 4.5 with $h = 0$ and $\alpha \gamma$ in place of $\gamma$ gives $(3, 2n + 2)$-modularity, since $\alpha(\beta \circ \alpha \gamma) = \alpha \gamma \circ \alpha \beta$. The base step is thus proved. (By the way, the method is essentially known, compare [D, p. 172] and the proofs of [G2, Theorem 7.4(iv) ⇒ (i)], [LTT, Theorem 1 (3) ⇒ (1)]. The main point in the present corollary is the evaluation of some bound $s$ for larger values of $q$.)

Suppose that the corollary holds for some $q \geq 2$, thus we have $\alpha(\beta \circ \alpha \gamma \circ 2^q \cdot \cdot \cdot \circ \beta) \subseteq \alpha \beta \circ_k \alpha \gamma$, with $k = (2^{q+1} - 2q - 2)n + 2$. Apply equation (7) in Theorem 4.5 with $h = 2^q - 1$, thus $2h + 2 = 2^q$ and $m = 2^{q+1} - 2$. We get

$$\alpha(\beta \circ \alpha \gamma \circ 2^q \cdot \cdot \cdot \circ \beta) \subseteq (\alpha \beta \circ_{mn} \alpha \gamma) \circ (\alpha \beta \circ \alpha \gamma \circ 2^q \cdot \cdot \cdot \circ \beta) \circ \alpha \gamma \subseteq \alpha \beta \circ_k \alpha \gamma,$$

since both $mn$ and $k$ are even. Recalling that $k = (2^{q+1} - 2q - 2)n + 2$ and $m = 2^{q+1} - 2$, we get $k + mn = (2^{q+2} - 2(q + 1) - 2)n + 2$, thus the induction step is complete.□

Of course, as in Corollary 4.6, we can provide a version of Corollary 4.8 starting with some fixed identity.

Corollary 4.9. Suppose that $\mathcal{V}$ has $n + 2$ Gumm terms and

$$\alpha(\beta \circ \alpha \gamma \circ 2^{h+1} \circ \beta) \subseteq \alpha \beta \circ_t \alpha \gamma$$

holds in $\mathcal{V}$. Then $\mathcal{V}$ also satisfies

$$\alpha(\beta \circ \alpha \gamma \circ 4^{h+3} \circ \beta) \subseteq \alpha \beta \circ_{t'} \alpha \gamma$$

and, more generally, for every $p \geq 1$, $\mathcal{V}$ satisfies

$$\alpha(\beta \circ \alpha \gamma \circ z \cdot \cdot \cdot \circ \beta) \subseteq \alpha \beta \circ_{t'} \alpha \gamma$$

with $z = 2^p(h + 1) - 1$ and $t' = t + hn(2^{p+1} - 4) + n(2^{p+1} - 2p - 2)$.

Proof. Equation (14) is immediate from equation (7) in Theorem 4.5. Equation (15) follows from an induction similar to the proof of 4.8. □

5. Joining the two approaches and some mystery

Recall that $D_{\mathcal{V}}(m)$ is the smallest $h$ such that $\mathcal{V}$ is $(m, h)$-modular, that is, the smallest $h$ such that $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \beta \circ_h \alpha \gamma$. We have found bounds for $D_{\mathcal{V}}(m)$ using two relatively different methods in Theorem 3.3 and Corollary 4.8. The problem naturally arises to see whether one method is better than the other. So far, this appears a quite mysterious question.
First, notice that there are trivial cases, for example, permutative varieties (here, of course, “trivial” is intended only relative to the problems we are discussing; permutative varieties by themselves are far from being trivial!). As already noticed by Day [D, Theorem 2], congruence permutability corresponds exactly to the case in which \( \mathcal{V} \) has 3 Day terms. In this case, \((m, 2)\)-modularity holds, for every \( m \). This case corresponds exactly also to \( \mathcal{V} \) having 2 Gumm terms. More generally, in \( n \)-permutative varieties, \((m, n)\)-modularity holds, for every \( m \).

In general and, for simplicity, with some approximation, Theorem 3.3 gives a bound roughly of the form \( h = \frac{k^{q-1}}{2q-1} \) for \((2^q, h)\)-modularity, for varieties with \( k + 1 \) Day terms. On the other hand, Corollary 4.8 gives a bound roughly of the form \( h = n2^{q+1} \) for \((2^q, h)\)-modularity, for varieties with \( n + 2 \) Gumm terms. In the following discussion, suppose that \( k > 4 \) (for \( k \leq 4 \), best possible results are probably given by Corollary 3.5). Then \( \frac{k^{q-1}}{2q-1} \) grows asymptotically faster than \( n2^{q+1} \), as \( q \) increases, no matter the value of \( n \). Hence, fixed some variety \( \mathcal{V} \) and for a sufficiently large \( q \), we have that Corollary 4.8 provides a better bound for \( D_\mathcal{V}(2^q) \). By a trivial monotonicity property, this holds also for sufficiently large \( m \), when evaluating \( D_\mathcal{V}(m) \). In other words, for large \( m \), Corollary 4.8 alone provides a better bound for \( D_\mathcal{V}(m) \), in comparison with Theorem 3.3 alone.

However, it might happen that, for some relatively small value of \( m \), Theorem 3.3 gives a better bound for \( D_\mathcal{V}(m) \) rather than Corollary 4.8. As far as \( m \) increases, the evaluation of \( D_\mathcal{V}(m) \) obtained according to Theorem 3.3 becomes worse and worse, in comparison with the one given by Corollary 4.8. However, at a certain point, rather than applying Corollary 4.8 “from the beginning”, we can instead use the related Corollary 4.9 when this becomes more convenient, by using the evaluation given by Theorem 3.3 as the premise in 4.9. In general, the above procedure seems to provide better bounds for \( D_\mathcal{V}(m) \) than Corollary 4.8 alone.

The above arguments furnish the following corollary, which holds for every \( k \), but probably has some advantage only for \( k > 4 \). Notice that, curiously enough, though we are starting from the beginning with the rather complicated and dissimilar formulae given by 3.3 and 4.9, at the end some expressions simplify in such a way that we obtain at least a somewhat readable expression!

**Corollary 5.1.** Suppose that \( k \) is even, \( p, q \geq 1 \), \( \mathcal{V} \) has \( k + 1 \) Day terms and \( n + 2 \) Gumm terms. Then \( \mathcal{V} \) is \((z, w)\)-modular, with \( z = 2^p2^q - 1 \) and \( w = \frac{k^{q}}{2q-1} + n(2^q2^{p+1} - 2^{q+2} - 2p + 2) \).

**Proof.** Because of Theorem 3.3, the assumption in Corollary 4.9 can be applied with \( h = 2^q - 1 \) and \( t = \frac{k^{q}}{2q-1} \). Then equation (15) in 4.9 gives \( z = 2^p2^q - 1 \) and \( w = \frac{k^{q}}{2q-1} + n(2^q - 1)(2^{p+1} - 4) + n(2^{p+1} - 2p - 2) = \frac{k^{q}}{2q-1} + n(2^p2^{p+1} - 2^{p+2} - 2p + 2). \) \( \square \)
Small improvements on Corollary 4.8 and on equations (11), (12) and (15) in Corollaries 4.6, 4.7 and 4.9 can be obtained by employing the trick used in the proof of equation (1) in [L3, Theorem 2.1]. Roughly, we have to write explicitly the nested terms arising from the proofs and “move together” many variables at a time. This seems to provide only small improvements, while, on the other hand, we seem to get more involved proofs and formulae. Probably, similar arguments apply also to Theorem 3.3 and Proposition 3.4. By the above arguments, Corollary 5.1 can be slightly improved, too.

Apart from the possible above small improvements, we do not know whether the situation depicted in Corollary 5.1 is the best possible one. This might be the case, but, on the other hand, it might happen that either Theorem 3.3 or Corollary 4.8 can be improved in such a way that one of them always gives the best evaluation. Or perhaps there is a completely new method which is always the best; or even there is no “best” method working in every situation and everything depends on some particular property of the specific variety at hand.

What lies at the heart of our arguments are the numbers of Day terms in Section 3 and of Gumm terms in Section 4. These numbers are tied by some constraints, as theoretically implicit from the theory of Maltsev conditions. More explicitly, Lakser, Taylor and Tschantz [LTT, Theorem 2] show that if a variety \( V \) has \( k+1 \) Day terms, that is, \( V \) is \( k \)-modular, then \( V \) has \( \leq k^2-k+1 \) Gumm terms (apparently, this can be slightly improved in the case \( k \) even). A folklore result in the other direction is given in the following proposition. See also Propositions 6.1 and 6.3 below for related results.

**Proposition 5.2.** If \( V \) has \( n+2 \) Gumm terms, then \( V \) is \( 2n+2 \)-modular.

**Proof.** This is the case \( q = 2 \) of Corollary 4.8. Alternatively, and more directly, take \( R = \beta \) and \( S = \alpha \gamma \circ \beta \) in Lemma 4.3 and then take converses. □

However, to the best of our knowledge, it is not known whether the above bounds are optimal (in both directions). In a sense, already the basis of the general problems we are considering is filled with obscurity! In any case, whatever the possible methods of proof, we can ask about the actual and concrete situations which might occur.

**Problem 5.3.** (Possible values of the Day spectrum) Which functions (from \( \{ n \in \mathbb{N} \mid n \geq 3 \} \) to \( \mathbb{N} \) ) can be realized as \( D_V \), for some congruence modular variety \( V \)?

Notice that, as we mentioned, if a variety \( V \) is congruence modular and \( r \)-permutable, then \( D_V(m) \leq r \), for every \( m \). Hence \((m, k)\)-modularity has little or no influence, in general, on \( D_V(n) \), for \( n < m \) (of course, monotonicity has to be respected). On the other hand, as we proved in Proposition 3.4 \((m, k)\)-modularity puts some tight constraints on \( D_V(n) \), for \( n > m \). Another perspective to appreciate this aspect is to use Corollaries 4.9 and 5.1 via the above-mentioned result by Lakser, Taylor and Tschantz [LTT].
6. Connections with distributivity

Notions and problems corresponding to Problem 5.3 in the case of congruence distributive varieties have been studied in [L2], though not every problem has been yet completely solved even in this relatively simpler case.

Of course, every congruence distributive variety is congruence modular. This can be witnessed at the level of terms, as already shown by Day [D, Theorem on p. 172]. But the problem he asked shortly after about the optimal number of terms necessary for this is perhaps still open.

Recall that a variety $V$ is $n+1$-distributive if the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_{n+1} \alpha\gamma$ holds in $V$. This is witnessed by the existence of $n+2$ Jónsson terms $j_0, \ldots, j_{n+1}$; the situation is entirely parallel to Proposition 2.1; specific details for the case of congruence distributivity can be found, e.g., in [L2].

Jónsson terms have been briefly recalled in Remark 4.2(i). For short, Jónsson terms for $n+1$-distributivity are terms satisfying condition (2) in Theorem 4.1, with equation (b) replaced by $x = m(x, y, z)$, thus $m$ can be safely relabeled $j_0$ and (c) becomes an instance of (e). Jónsson [J] showed that a variety is congruence distributive if and only if it is $n$-distributive, for some $n$. In a more general context, in [L2] we introduced the following Jónsson distributivity function $J_V$, for a congruence distributive variety $V$. $J_V(m)$ is the least $k$ such that $V$ satisfies the identity $\alpha(\beta \circ m \circ \gamma) \subseteq \alpha\beta \circ_{k+1} \alpha\gamma$. Notice the shift by 1.

The mentioned theorem by Day on [D, p. 172] states that if a variety $V$ is $n+1$-distributive, then $V$ is $(2n+1)$-modular (for notational convenience, here we are again shifting by 1 with respect to [D]). The theorem might be seen as a predecessor to some results from [L2], since the proof of [D, Theorem on p. 172] actually shows that if $V$ is $n+1$-distributive, that is, $J_V(1) = n$, then $J_V(2) = 2n$, the special case $\ell = 2$ of [L2, Corollary 2.2]. Indeed, the terms constructed on [D, p. 172] satisfy $d_i(xyzx) = x$, for every $i$. In terms of congruence identities, they thus witness $\alpha(\beta \circ \gamma \circ \beta) \subseteq \alpha\beta \circ_{2n+1} \alpha\gamma$, rather than simply $\alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha\beta \circ_{2n+1} \alpha\gamma$. Conversely, the identity $\alpha(\beta \circ \gamma \circ \beta) \subseteq \alpha\beta \circ_{2n+1} \alpha\gamma$ can be obtained as a consequence of [L2, Corollary 2.2], which in particular furnishes a proof of Day’s result, taking $\alpha\gamma$ in place of $\gamma$.

We now observe that when $n$ is even Day’s result can be improved by 1.

**Proposition 6.1.** If $n$ is even and $V$ is $n+1$-distributive, then $V$ is $2n$-modular.

**Proof.** For the reader’s convenience, we are going to give an explicit and direct proof below, but first let us observe that the proposition can be obtained as a consequence of Proposition 5.2. Indeed, by Remark 4.2(iii) and since $n + 1$ is odd, then $n + 1$-distributivity is equivalent to ALVIN $n + 1$-distributivity. Hence, by Remark 4.2(iv), terms witnessing ALVIN $n + 1$-distributivity are in particular a set of $n + 1$ Gumm terms. Then Proposition 5.2 (with $n + 1$ in place of $n + 2$) gives the result.
Alternatively, a direct proof can be obtained as in [14] Theorem on p. 172, except for the terms at the end. Given Jónsson terms $j_0, \ldots, j_{n+1}$, we obtain the following terms $d_0, \ldots, d_{2n}$ satisfying condition (4) in Proposition 2.1. The terms are considered as quaternary terms depending on the variables $x, y, z, w$.

$$d_0 = j_0(xyw) \quad d_1 = j_1(xyw) \quad d_2 = j_1(xzw) \quad d_3 = j_2(xzw)$$

$$d_4 = j_2(xyw) \quad d_5 = j_3(xyw) \quad d_6 = j_3(xzw) \quad \ldots$$

$$d_{4i} = j_{2i}(xyw) \quad d_{4i+1} = j_{2i+1}(xyw) \quad d_{4i+2} = j_{2i+1}(xzw) \quad d_{4i+3} = j_{2i+2}(xzw)$$

$$\ldots$$

$$d_{2n} = j_{n+1}(yw) \quad d_{2n-2} = j_{n-1}(xzw) \quad d_{2n-1} = j_n(yzw)$$

Notice the different argument of $j_n$. Notice that $2n - 1 = 4^{\frac{n-2}{2}} + 3$, hence the indices in the penultimate line follow the same pattern of the preceding lines, taking $i = \frac{n-2}{2}$. We can do this since $n$ is assumed to be even.

The fact that $d_0, \ldots, d_{2n}$ satisfy Condition (4) in Proposition 2.1 is easy and is proved as in [14] p. 172. The only differences are that $d_{2n-2}(x, x, w, w) = j_{n-1}(x, w, w) = j_n(x, w, w) = d_{2n-1}(x, x, w, w)$ and that $d_{2n-1}(x, y, y, w) = j_n(y, y, w) = j_{n+1}(y, y, w) = w$. Notice that it is fundamental to have $n$ even!

**Remark 6.2.** We have not used the equation $j_n(x, y, x) = x$ in the above proof. This is another way to see that $n + 1$ Gumm terms imply 2n-modularity, as proved in Proposition 5.2. Put in another way, and taking converses, the congruence identity $\alpha(\beta \circ \gamma) \subseteq (\alpha \beta \circ_{n-1} \alpha \gamma) \circ (\alpha \gamma \circ \beta)$ holding in some variety implies $\alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha \beta \circ_{2n} \alpha \gamma$.

Notice however that the argument in [6,1] does not improve the value of $J_f(2)$ mentioned before Proposition 6.1. We only get from the proof that, for $n$ even, $n + 1$-distributivity, or just having $n + 1$ Gumm terms, imply $\alpha(\beta \circ \gamma \circ \beta) \subseteq (\alpha \beta \circ_{2n-2} \alpha \gamma) \circ (\alpha \beta \circ \gamma)$. Compare the case $q = 1$ in equation (14) in Corollary 4.7. We believe that the point is best seen in terms of congruence identities. In this sense, the argument in [6,1] works for modularity, since in this case we have $\alpha \gamma$ in place of $\gamma$, so $\alpha(\beta \circ \alpha \gamma) = \alpha \beta \circ \alpha \gamma$.

Just as it is possible to define ALVIN or reversed $n$-distributivity, we can consider reversed modularity. In detail, let us say that a variety $\mathcal{V}$ is $k$-modular if $\alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha \gamma \circ k \alpha \beta$ holds in $\mathcal{V}$. Notice that, by taking converses, if $k$ is even, then $k$-modularity and $k$-modularity are equivalent notions. On the other hand, for $k$ odd, we get distinct notions, in general. For example, 3-modularity implies 3-permutability (just take $k = 1$), while 3-modularity does not (e.g., consider the variety of lattices). As a marginal remark, 3-permutability obviously implies 3-modularity, hence 3-permutability and 3-modularity are equivalent notions.

Of course, $k$-modularity can be given a Maltsev characterization as in Proposition 2.1(4), just exchange odd and even.
Recall the definition of defective Gumm terms given right before Proposition 4.4. We can improve Proposition 5.2 in case \( n \) is even if we express the conclusion in terms of reversed modularity.

**Proposition 6.3.** If \( n \) is even and \( V \) has \( n + 2 \) defective Gumm terms, then \( V \) is \( 2n + 1 \)-modular\(^-\).

**Proof.** Take \( R = \beta \) and \( S = \alpha \gamma \circ \beta \) in Proposition 4.4. \( \square \)

Of course, in case \( n \) is even, Proposition 6.3 generalizes Proposition 5.2, since \( 2n + 1 \)-modularity\(^-\) obviously implies \( 2n + 2 \)-modularity. By the way, the observation that one can delete one equation from Gumm conditions, still obtaining a property implying congruence modularity, has appeared before in Dent, Kearnes and Szendrei [DKS, Theorem 3.12] in a different context and with somewhat different terminology and notations.

### 7. Generalized spectra

We now notice that \( D_V \) is not the only spectrum which deserves consideration. Actually, it appears quite natural to introduce a great deal of spectra, as we are going to show. First, along the above lines, we can define also reversed \((m, k)\)-modularity. We define a variety to be \((m, k)\)-modular\(^-\) if 

\[
\alpha(\beta \circ_m \alpha \gamma) \subseteq \alpha \gamma \circ_k \alpha \beta \text{ holds in } V.
\]

Thus \((3, k)\)-modularity\(^-\) is what we have called \( k \)-modularity\(^-\). Of course, if \( V \) is \((m, k)\)-modular\(^-\), then \( V \) is \((m, k+1)\)-modular and, conversely, if \( V \) is \((m, k)\)-modular, then \( V \) is \((m, k+1)\)-modular\(^-\). Put in another way, if we choose the best possible values for \( k \) both in \((m, k)\)-modularity and in \((m, k)\)-modularity\(^-\), these optimal values differ at most by 1. Notice also that, exactly as in the case \( m = 3 \), if \( m \) is odd and \( k \) is even, then \((m, k)\)-modularity is equivalent to \((m, k)\)-modularity\(^-\): just take converses. On the other hand, it follows from the special case \( m = 3 \) treated above that \((m, k)\)-modularity and \((m, k)\)-modularity\(^-\) are distinct notions, in general.

In particular, it is interesting to study the reversed Day spectrum \( D^-_V(m) \) of a congruence modular variety \( V \). We let \( D^-_V(m) \) be the smallest \( k \) such that \( V \) is \((m, k)\)-modular\(^-\). By the above comments, \( D_V \) and \( D^-_V \) differ at most by 1, but are different functions, in general.

Probably more significantly, and inspired by Tschantz [T], we can define the Tschantz modularity function \( T_V \) for a congruence modular variety \( V \) in such a way that, for \( m \geq 2 \), \( T_V(m) \) is the least \( k \) such that the following congruence identity holds in \( V \)

\[
\alpha(\beta \circ_m \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ_k \alpha \beta)
\]

Notice that \( T_V(2) = 0 \) is equivalent to congruence permutability, just take \( \alpha = 1 \) (recall that, by convention we let \( \beta \circ_0 \gamma = 0 \)). More generally, \( T_V(2) = k \) if and only if \( V \) has \( k + 2 \) Gumm terms \( m, j_1, \ldots, j_{k+1} \), but not \( k + 1 \) Gumm
terms. The definition of $T_v$ makes sense for a congruence modular variety, as implicit from \[1]. More explicitly, the existence of $T_v(m)$ follows from Corollary \[4.7]. Further constraints on $T_v(m)$ are provided by Corollary \[4.6]. A rather surprising result, a reformulation of parts of \[L2, Theorem 2.3], asserts that if $V$ has 3 Gumm terms, that is, if $T_v(2) \leq 1$, then $T_v(m + 2) \leq m$, for $m \geq 2$. See, in particular, \[L2, equation (5)].

Of course, we can also define the reverse Tschantz function $T^{-}_v$ as the least $k$ such that the following congruence identity holds in $V$

$$\alpha(\beta \circ_m \gamma) \subseteq \alpha(\beta \circ \gamma) \circ (\alpha \beta \circ_k \alpha \gamma)$$

Notice that, for $m = 2$, the above identity is satisfied by every variety (with $k = 0$), hence in the case of the reverse Tschantz function it is appropriate to start with $m = 3$.

Quite surprisingly, results like Tschantz’ hold even when $\beta$ and $\gamma$ are replaced by reflexive and admissible relations; this relies heavily on Kazda, Kozik, McKenzie, Moore \[AdJt]; see \[L3] for details. Hence we can also define the relational Tschantz functions $T^r_v(m)$ and $T^{-r}_v(m)$ as the least $k$’s such that, respectively, the following congruence identities hold in $V$

$$\alpha(R \circ_m S) \subseteq \alpha(S \circ R) \circ (\alpha S \circ_k \alpha R)$$

$$\quad \quad \quad \alpha(R \circ_m S) \subseteq \alpha(R \circ S) \circ (\alpha R \circ_k \alpha S)$$

Constraints on $T^{-r}_v(m)$ are provided, for example, by \[L3, equation (6) in Corollary 2.2].

Still other characterizations of congruence modular varieties are possible. In \[L3] we have proved that a variety is congruence modular if and only if, for every $m \geq 2$, there is some $k$ such that

$$\alpha(R \circ_m R) \subseteq \alpha R \circ_k \alpha R$$  \[16\]

Thus we can ask about the possible values of the function $R_v$ assigning to $m$ the smallest possible value $k$ such that \[16\] holds in $V$.

By replacing $R$ with $R \circ \alpha S$ in \[16\], it easily follows that a variety is congruence modular if and only if, for every $m \geq 2$ (equivalently, just for $m = 3$), there is some $k$ such that the relational version of modularity

$$\alpha(R \circ_m \alpha S) \subseteq \alpha R \circ_k \alpha S$$  \[17\]

holds in $V$. We let $D^r_v(m)$ be the least $k$ such that \[17\] holds; $D^{-r}_v(m)$ is defined as usual.

Finally, we shall consider an identity dealing with tolerances

$$\Theta^h \Psi^k \subseteq (\Theta \Psi)^\ell, \quad \[18\]$$

where $\Theta^h$ is a shorthand for $\Theta \circ_h \Theta$. This time we get a function $C_v$ depending on two arguments: $C_v(h,k)$ is the least $\ell$ such that \[18\] holds. It seems to be an open problem whether we still get identities equivalent to congruence modularity if we let both $\Theta$ and $\Psi$ vary among reflexive and admissible relations in equation \[18\].
Problem 7.1. (Generalized Day spectra) Which functions can be realized as $T_V$, for some congruence modular variety $V$? The same problem for all the spectra introduced above.

More generally and globally, which 10-tuples of functions can be realized as $(D_V, D_V^\sim, T_V, T_V^\sim, T_V^\rightarrow, T_V^\leftarrow, R_V, D_V^\uparrow, D_V^\downarrow, C_V)$, for some congruence modular variety $V$?

In the special case of a congruence distributive variety (which, in particular, is congruence modular) it is probably interesting to consider simultaneously the spectra introduced in [7,1] and those introduced in [L2]. Some connections between the Jónsson and the modularity spectra have been presented in [L2, Section 4], together with some further problems.

We finally show that all the spectra under consideration are closed under taking pointwise maximum.

Proposition 7.2. For each of the spectra listed in Problem 7.1, the set of realizable functions is closed under pointwise maximum.

In detail, if $V$ and $V'$ are congruence modular varieties, then there is a congruence modular variety $W$ such that, for every $m \geq 3$, $D_W(m) = \max\{D_V(m), D_V'(m)\}$, and the same holds for all the other spectra.

The same holds globally, too; namely, for every pair of realizable 10-uples as in Problem 7.1 there is a congruence modular variety realizing their componentwise maximum.

Proof. This is the same argument as in [L2, Proposition 1.1]. The non-indexed product of two varieties satisfies exactly the same strong Maltsev conditions satisfied by both varieties; and each condition of the form $D_V(m) \leq k$ is a strong Maltsev condition, and the same for all the other spectra.

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We have not yet performed a completely accurate search in order to check whether some of the results presented here are already known. Credits for already known results should go to the original discoverers.

Though the author has done his best efforts to compile the following list of references in the most accurate way, he acknowledges that the list might turn out to be incomplete or partially inaccurate, possibly for reasons not depending on him. It is not intended that each work in the list has given equally significant contributions to the discipline. Henceforth the author disagrees with the use of the list (even in aggregate forms in combination with similar lists) in order to determine rankings or other indicators of, e. g., journals, individuals or institutions. In particular, the author considers that it is highly inappropriate, and strongly discourages, the use (even in partial, preliminary or auxiliary forms) of indicators extracted from the list in decisions about individuals (especially, job opportunities, career progressions etc.), attributions of funds, and selections or evaluations of research projects.

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