SOME EXPLICIT COCYCLES ON THE FURSTENBERG
BOUNDARY FOR PRODUCTS OF ISOMETRIES OF
HYPERBOLIC SPACES AND $SL(3, \mathbb{K})$

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Abstract. In [11], Nicolas Monod showed that the evaluation map $H^*_m(G \ltimes G/P) \to H^*_m(G)$ between the measurable cohomology of the action of a connected semisimple Lie group $G$ on its Furstenberg boundary $G/P$ and the measurable cohomology of $G$ is surjective with a kernel that can be entirely described in terms of invariants in the cohomology of a maximal split torus $A < G$. In [5] we refine Monod’s result and show in particular that the cohomology of non-alternating cocycles on $G/P$, namely those lying in the kernel of the alternation map, is in general not trivial and lies in the kernel of the evaluation. In this paper we describe explicitly such non-alternating and alternating cocycles on $G/P$ in low degrees when $G$ is either a product of isometries of real hyperbolic spaces or $G = SL(3, \mathbb{K})$, where $\mathbb{K}$ is either the real or the complex field. As a consequence, we deduce that the comparison map $H^*_m(G) \to H^*_m(G)$ from the measurable bounded cohomology is injective in degree 3 for nontrivial products of isometries of hyperbolic spaces.

1. Introduction

Let $G$ be a semisimple Lie group with finite center and finitely many connected components. Its measurable cohomology $H^*_m(G)$ is defined as the cohomology of the complex

$L^0(G^{*+1}, \mathbb{R})^G$

endowed with its usual homogeneous differential. For any homogeneous $G$-space $X$ one can similarly consider the measurable cohomology of the action $H^*_m(G \ltimes X)$ as the cohomology of the complex

$L^0(X^{*+1}, \mathbb{R})^G$

also endowed with its homogeneous differential. Evaluation on a generic base point induces a map

(1) $\text{ev} : H^*_m(G \ltimes X) \to H^*_m(G)$

which is easily shown not to depend on the base point. The measurable cohomology of $G$ is well understood: it is isomorphic to its continuous cohomology [1, Theorem A]. The latter, when $G$ is connected, can be identified by the Van Est isomorphism with the singular cohomology of its compact dual symmetric space. However, in an aim to obtain explicit cocycles representing classes in $H^*_m(G)$, it is desirable to find the smallest/nicest (in terms of dimension, topological properties like connectedness, compactness, etc) homogeneous $G$-space $X$ for which a given class is representable on it (equivalently it is in the image of the evaluation map), or more generally the evaluation map (1) is surjective. For example, it is easy to see that if $K < G$ is a maximal compact subgroup, the evaluation map (1) is in fact an isomorphism for the symmetric space $X = G/K$.

Let now $P < G$ be a minimal parabolic subgroup. Monod remarkably showed that the evaluation map is surjective for the action on the Furstenberg boundary

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1
X \times G/P. Furthermore, he gives a precise description of the kernel in terms of the $w_0$-invariants of the cohomology $H^*_m(A)$ of the maximal split torus $A < P$, where $w_0$ is a representative of the longest element of the Weyl group associated to $A$, or equivalently in terms of $(\wedge m a^*)^w_0$, where the action of $w_0$ on $\wedge m a^*$ is induced by the adjoint representation.

**Theorem 1.** [11 Theorem B] Let $G$ be a semisimple Lie group with finite center and finitely many connected components. The evaluation map

$$H^*_m(G \rtimes G/P) \to H^*_m(G)$$

is surjective and its kernel

$$NH^*_m(G \rtimes G/P) := \text{Ker}(H^*_m(G \rtimes G/P) \to H^*_m(G))$$

fits into an exact sequence

$$0 \to NH^*_m(G \rtimes G/P) \times (\wedge m a^*)^w_0 \to H^*_m(G \rtimes G/P) \times (\wedge m a^*)^w_0$$

when $k \geq 3$, and for $k = 2$, we have an isomorphism

$$NH^2_m(G \rtimes G/P) \cong (a^*)^w.$$

The usual alternation map on $L^0((G/P)^{q+1})$ is idempotent and it induces a splitting into non-alternating and alternating cocycles. Since alternation is a cochain map commuting with the $G$-action, we obtain a decompositon of the cohomomology on the boundary $G/P$ as follows:

$$H^*_m(G \rtimes G/P) \cong H^*_{m,n-\text{alt}}(G \rtimes G/P) \oplus H^*_m(G \rtimes G/P),$$

and the same holds for the kernel:

$$NH^*_m(G \rtimes G/P) \cong NH^*_{m,n-\text{alt}}(G \rtimes G/P) \oplus NH^*_m(G \rtimes G/P).$$

In [3 Theorem 2, Theorem 3] we refine Theorem 1 and show in particular that

$$\left(\wedge^{k-2} a^*\right)^w_0 \cong H^*_{m,n-\text{alt}}(G \rtimes G/P) \cong NH^*_m(G \rtimes G/P),$$

$$\left(\wedge^{k-1} a^*\right)^w_0 \cong NH^*_m(G \rtimes G/P).$$

As a consequence the short exact sequence of Equation 2 corresponds exactly to the decomposition into non-alternating and alternating cocycles on the boundary.

In this paper, we propose to study this kernel in more details following our belief that explicit cohomology classes should be represented by explicit cocycles. We will thus describe the inclusion $i$ and produce a section $s$ of this short exact sequence in low degrees for some families of groups, leading to very explicit $G$-invariant cocycles on the boundary $G/P$. In fact, it is the discovery of such explicit non-alternating cocycles which motivated our paper [5].

In the case of the action of $\text{SL}(3,\mathbb{C})$ on the projective space $P^2(\mathbb{C})$, a remarkable (unbounded) exotic cocycle representing a non-trivial class in degree 5 in the kernel of the evaluation map

$$H^*_m(\text{SL}(3,\mathbb{C}) \rtimes P^2(\mathbb{C})) \to H^*_m(\text{SL}(3,\mathbb{C}))$$

has been exhibited by Goncharov [9].

### 1.1. Products of isometries of real hyperbolic spaces.

Consider first the case of one factor $G = \text{Isom}(\mathbb{H}^n)$. If we look at the upper half space model, we identify $\partial \mathbb{H}^n$ with $\mathbb{R}^{n-1} \cup \{\infty\}$, we fix the stabilizer $P = \text{Stab}(\infty)$ and take for maximal abelian subgroup $A < P$ the group of homotheties, namely $A := \{a_\lambda : x \mapsto \lambda x \mid \lambda \in \mathbb{R}_{>0}\}$. Given a 4-tuple of distinct points $x_0, \ldots, x_3 \in \mathbb{R}^{n-1} \cup \{\infty\} = \partial \mathbb{H}^n$ we define their (positive) cross ratio by

$$l(x_0, \ldots, x_3) = \frac{\|x_2 - x_0\|\|x_3 - x_1\|}{\|x_2 - x_1\|\|x_3 - x_0\|} \in \mathbb{R}_{>0},$$

(3)
Remark. Let $\alpha_0 \in H_2^{\ell}(\Gamma)$ and $\ell \geq 3$. Theorem 2 is obviously also valid for $\Gamma$ where $\alpha_\ell$ denotes the logarithm sending $\alpha_0$ to its Lie algebra $a$. We give explicit versions of these isomorphisms at cocycle level in degree 3 and 4:

**Theorem 2.** A section
\[
s : (\wedge^2 a)^* \rightarrow NH^3_m(G \cap \Pi\partial \mathbb{H}^n)
\]
of the short exact sequence (3) is given by sending an alternating form $\alpha_0 \in (\wedge^2 a)^*$ to the measurable almost everywhere defined $G$-invariant alternating cocycle
\[
s(\alpha_0) : (\Pi\partial \mathbb{H}^n)^k \rightarrow \mathbb{R}
\]
\[
(x_0, \ldots, x_k) \mapsto 4\alpha_0(\log(a_\ell(x_0, x_1, x_2, x_3))^k_{i=1}, \log(a_\ell(x_1, x_2, x_3, x_0))^k_{i=1}),
\]
where $\log$ denotes the logarithm sending $A$ to its Lie algebra $a$.

**Remark.** Let $G^0$ denote the connected component of the identity in $G$. The cocycles in Theorem 2 are obviously also $G^0$-invariant and our proof, relying on the 3-transitivity of $G$ on $G/P$ shows that the same section works for $G^0$ whenever $n_i \geq 3$ for all $i = 1, \ldots, k$. In the case when one of $n_i = 2$, we can only deduce that the value on tuples which are positively oriented in each 2-dimensional factor has the above form. But the yet unknown functoriality of the section prevents us to conclude that the section provided by the spectral sequence takes this form on all configurations of points, even though this is very likely to be the case. The same remark also applies to Theorem 4.

In the case of two factors of dimension 2, this cocycle (on tuples which are positively oriented in each factor) is explicitly given in (11) for the connected component of $G$, i.e. for $G^0 = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$. All other cases are new. The case of several 2-dimensional factors is a straightforward generalization of Monod’s cocycle, but the higher dimensional case, relying on explicitly exploiting the 3-transitivity of the action on the boundary, is more involved as $K \cap P$ is not trivial in dimension $n \geq 3$, where $K = O(n)$ is the maximal compact subgroup.

We will prove Theorem 2 in Section 3 by following the spectral sequence (that we recall in Section 2) introduced by Monod to establish Theorem 4. Note however that in this case it is easy to show retrospectively that $NH^3_m(G \cap \Pi\partial \mathbb{H}^n)$ is generated by the classes given in Theorem 2. Indeed, any such cochain is easily verified to be an alternating cocycle which takes infinitely many and unbounded values. As a consequence, it cannot be the coboundary of a 2-cochain, as those are constant functions by 3-transitivity. Furthermore, it is trivially in the kernel of the evaluation map since $H_2^\ell(G) = 0$. (Indeed, recall that $H^\ast(G)$ is generated (as an algebra) by the cup products of an even number of volume classes). Finally, a similar coboundary argument combined with our Proposition 13 shows that there are no relations among these classes so that they generate a subspace of dimension $\binom{s}{2}$ in the $\binom{s}{2}$-dimensional (by Theorem 1) space $NH^3_m(G \cap \Pi\partial \mathbb{H}^n)$. The point
of our proof is thus merely, beside showing that the natural isomorphism induced by the spectral sequence really is given by the map in Theorem 2, to show how to find these explicit cocycles.

Since by Proposition 18 theses cocycles are unbounded, it is easy to deduce, using the transitivity of $G$ on generic triple of points, that the comparison map from the bounded measurable cohomology of $G$ is injective in this case, which is a conjecture by Monod [10, Problem A] stated more generally for any semisimple Lie group with finite center and finitely many connected components.

**Theorem 3.** Let $G$ be a product of isometry groups of real hyperbolic spaces and $G^0$ be the connected component of the identity. The maps

$$c_G : H^3_{m,b}(G) \to H^3_m(G)$$

and

$$c_{G^0} : H^3_{m,b}(G^0) \to H^3_m(G^0)$$

are injective.

Here bounded cohomology is defined via the complex of essentially bounded functions on the boundary (thanks to [6, Theorem 2]) and the comparison map is simply induced by the inclusion $L^\infty \hookrightarrow L^0$.

Theorem 3 was known in the case of one factor $G = \text{Isom}^+(\mathbb{H}^n)$ [3, 11, 13], but is new for more than one factor. Note also that in contrast to measurable cohomology, no Künneth formula is known for bounded cohomology, so the injectivity cannot be deduced from the factor case.

The classes in degree 4 are given by a surprisingly similar expression, which is all the more astonishing as the intermediate expressions we obtain following the spectral sequence for Theorems 2 and 4 show no similarity until their very last expression.

**Theorem 4.** The isomorphism

$$i : (\wedge^2 a)^* \to NH^4_m(G \times \Pi \partial \mathbb{H}^n_i)$$

of the short exact sequence (2) is given by sending an alternating form $\alpha_a \in (\wedge^2 a)^*$ to the measurable almost everywhere defined $G$-invariant non-alternating cocycle

$$i(\alpha_a) : \left(\prod \partial \mathbb{H}^n_i\right)^5 \to \mathbb{R}$$

$$(x_0, \ldots, x_4) \mapsto \alpha_a(\log(a_{b_i}(x_0,x_1,x_2,x_3))_{i=1}^k, \log(a_{b_i}(x_1,x_2,x_3,x_4))_{i=1}^k),$$

where $\log$ denotes the logarithm sending $A$ to its Lie algebra $a$.

To see that these cocycles are non-alternating, observe that $i(\alpha_a)(x_0, \ldots, x_4) = -i(\alpha_a)(x_4, \ldots, x_0)$. Since there is a factor $-1$, whereas the sign of the permutation $(x_0, \ldots, x_4) \mapsto (x_4, \ldots, x_0)$ is $+1$, the alternation of these cocycles automatically vanishes. In the case of two factors, the computation is carried out in [5, Proposition 18].

While it is easy to show that these cochains are indeed invariant cocycles on $\prod \partial \mathbb{H}^n_i$, we do not know, in contrast to Theorem 2, how to show even their non-triviality in cohomology without going through the spectral sequence. The proof of Theorem 4 will be presented in Section 4.

**Remark.** Our proof also gives an explicit expression in higher degrees for the inclusion and the section. However, the expression is a sum of $(\ell - 1)3^{\ell-1}$, respectively $3^{\ell-2}$, evaluations of $\alpha_a$ on various cross ratios in degree $\ell$ for $\ell$ odd, respectively even, which we were unable to simplify.
1.2. The special linear group \( \text{SL}(3, \mathbb{K}) \) for \( \mathbb{K} = \mathbb{R}, \mathbb{C} \). We give a complete description of the kernel \( NH^*_m(G \curvearrowright G/P) \) for the first cases of irreducible Lie groups of higher rank, namely \( G = \text{SL}(3, \mathbb{K}) \), for \( \mathbb{K} = \mathbb{R}, \mathbb{C} \). In this context we fix \( P \) as the group of upper triangular matrices with \( \mathbb{K} \)-entries and we consider \( A < P \) the subgroup of diagonal matrices with positive real entries. This time, the adjoint action of the longest element \( w_0 \) is given on the 2-dimensional Lie algebra \( \mathfrak{a} \) by a reflection, so there is an eigenspace of fixed vectors and another one associated to the eigenvalue \(-1\). As a consequence, the \( w_0 \)-invariant cohomology of \( A \) is not trivial only in degree one (Lemma 23) and it determines the kernel \( NH^*_m(G \curvearrowright G/P) \) in degree 2 and 3.

In this setting the boundary \( G/P = \mathcal{F}(3, \mathbb{K}) \) parametrizes complete flags in \( P^2(\mathbb{K}) \), namely pairs \((p, \ell)\), where \( p \) is a point on the line \( \ell \). The quotient of the space of tuples of flags with respect to the diagonal action of \( G \) is called configuration space and it has been widely studied so far. In the particular case of triples and 4-tuples, several authors contributed to give an explicit parametrization of a subset of full measure, namely the one of flags in general position (see for instance the works by Bergeron-Falbel-Guilhoux [2] or Dimofte-Gabella-Goncharov [7]). The condition of general position requires, roughly speaking, that the dimension of each possible intersection and each possible subspace spanned by the points and lines of the given tuple of flags matches with the expected dimension. We refer to Section 5.1 for a precise definition.

A complete invariant for the class of a triple \((F_0, F_1, F_2)\) of flags in general position is the triple ratio \( \tau(F_0, F_1, F_2) \in \mathbb{K} \setminus \{0, -1\} \). Despite its name, the triple ratio is a cross ratio computed on the line \( \ell_1 \) in terms of the intersection scheme of the given triple (see Definition 21).

**Theorem 5.** A section

\[
s : (\mathfrak{a}^*)^{w_0} \cong \mathbb{R} \longrightarrow NH^2_m(G \curvearrowright \mathcal{F}(3, \mathbb{K}))
\]

of the short exact sequence (3) is given by sending a \( w_0 \)-invariant linear form \( \alpha \in \mathbb{R} \) to the measurable almost everywhere defined alternating \( G \)-invariant cocycle

\[
s(\alpha) : (\mathcal{F}(3, \mathbb{K}))^3 \longrightarrow \mathbb{R} \\
(F_0, F_1, F_2) \longmapsto -\alpha(\log |\tau(F_0, F_1, F_2)|).
\]

**Remark.** An explicit isomorphism \((\mathfrak{a}^*)^{w_0} \cong \mathbb{R}\) is given in Subsection 5.2 and is used in the proof of Theorem 5. This remark applies also to Theorem 6.

**Remark.** Thanks to personal communications of Elisha Falbel with the first author, we understood that the cocycle realizing the explicit expression of the section can be written as a coboundary when viewed on the space of affine flags. If we write an affine flag as \( F := (v, f) \in \mathcal{F}_{\text{aff}}(3, \mathbb{K}) \), where \( v \) is a non-zero vector of \( \mathbb{K}^3 \) and \( f \) is linear functional vanishing on \( v \), for any \( \alpha \in (\mathfrak{a}^*)^{w_0} \) the function

\[
h_\alpha : (\mathcal{F}_{\text{aff}}(3, \mathbb{K}))^2 \rightarrow \mathbb{R}, \\
h_\alpha(F_0, F_1) := -\alpha \left( \log \frac{f_0(v_1)}{f_1(v_0)} \right)
\]

is measurable and satisfies \( dh_\alpha = s(\alpha) \) by [8 Section 3.3].

The proof of Theorem 5 can be found in Section 5.5 and it relies on a careful study of Monod’s spectral sequence. As is the case for Theorem 2, a simpler proof is possible to show that \( NH^2_m(G \curvearrowright \mathcal{F}(3, \mathbb{K})) \) is generated by one such nonzero cocycle. Indeed, it is immediate to see that the triple ratio is a multiplicative cocycle and hence its logarithm is a cocycle lying in \( L^0((G/P)^3)^G \) which further trivially is in the kernel of the evaluation map, since \( H^2_m(\text{SL}(3, \mathbb{K})) \) vanishes for \( \mathbb{K} = \mathbb{R}, \mathbb{C} \). Alternatively, by the second remark after Theorem 6, it is easy to exhibit an explicit coboundary in \( L^0(G^2)^G \). Additionally, being non-constant, it
determines a non-trivial cohomology class in $H^2_m(G \curvearrowright FL(3, \mathbb{K}))$. Indeed, $G$ acts transitively on pairs of flags in general position, so coboundaries must be constant.

The case $NH^3_m(G \curvearrowright FL(3, \mathbb{K}))$ can be treated similarly. This time configurations of 4-tuples of flags in general position are parametrized by four cross ratios $(z_{i1}, z_{i2}, z_{i3}, z_{i4})$. For each $z_{ij}$ the indices $(i, j)$ refer to the line on which the cross ratio is computed and to the intersection scheme one has to consider. For a precise definition we refer the reader to Definition 23.

**Theorem 6.** The isomorphism
\[ i : (a^*)^m \rightarrow NH^3_m(G \curvearrowright FL(3, \mathbb{K})) \]
of the short exact sequence (3) is given by sending a $\omega_0$-invariant linear form $\alpha_a \in (a^*)^m$ to the measurable almost everywhere defined $G$-invariant non-alternating cocycle
\[ i(\alpha_a) : (FL(3, \mathbb{K}))^4 \rightarrow \mathbb{R}, \quad (F_0, \ldots, F_3) \mapsto -\frac{2}{3} \alpha_a (\log(|z_{i1}|/|z_{i2}|)), \]
where $\log$ denotes the logarithm sending $A$ to its Lie algebra $a$.

**Remark.** Also in this case, for any $\alpha \in (a^*)^m$, we can check that $i(\alpha)$ is a coboundary when viewed on the space of affine flags $FL_{\text{aff}}(3, \mathbb{K})$. If we define
\[ h_\alpha : (FL_{\text{aff}}(3, \mathbb{K}))^3 \rightarrow \mathbb{R}, \quad h_\alpha(F_0, F_1, F_2) := -\frac{2}{3} \alpha \left( \log \frac{f_2(v_0)f_2(v_1)}{\det(v_0, v_1, v_2)} \right), \]
then again $dh_\alpha = i(\alpha)$ by [8, Equation 3.4.1].

The fact that these cocycles are non-alternating follows as for the cocycles in degree 4 on $\Pi \theta^m_{\text{aff}}$; from the fact that $i(\alpha_a)(F_0, \ldots, F_3) = -i(\alpha_a)(F_3, \ldots, F_0)$, which is easily deducible from the definition of the cross ratios of flags.

Again, our proof is based on Monod’s spectral sequence. But in this case we rely on the open source software SageMath to perform some computations.

Here also a more direct approach to show that $NH^3_m(G \curvearrowright FL(3, \mathbb{R}))$ is generated by one such nonzero cocycle is possible: First, it is not hard to verify by hand that $\log(|z_{i1}|/|z_{i2}|)$ is a cocycle lying in $L^0(G/P)^3G$. Second, an explicit coboundary in $L^0(G^3)^G$ is easy to produce (see the remark after Theorem 6) so that these cocycles lie in the kernel of the evaluation maps. Third, a computation based on the relations between cross ratios and triple ratios of a 4-tuple allows to show that the cocycle represents a nontrivial class in $NH^3_m(G \curvearrowright FL(3, \mathbb{R}))$.

**Plan of the paper.** We recall the spectral sequence used by Monod to prove Theorem 1 in Section 2 and give explicit though theoretic descriptions of the nonzero differentials in Section 3. We deal with the case of products of isometry groups of hyperbolic spaces in Section 4 by first recalling in Subsection 4.1 the form of isometries in the upper half space model, where we also compute a little too many $\pi_A$-projections which will be all needed in the proofs of Theorems 2 and 4. In Subsection 4.2 we exploit 3-transitivity to describe the differentials in this case. Finally we prove Theorem 2 in Subsection 4.3, Theorem 3 in Subsection 4.4 and Theorem 4 in Subsection 4.5. In our last Section 5, we start by recalling some generalities like the triple ratio and cross ratio of triples or quadruples of flags in Subsection 5.1. We compute the dimensions of $H^m_m(A)^{\omega_0} = (a^*)^{m*0}$ in Subsection 5.2 and give an expression for the $\pi_A$-projection in Subsection 5.3. Explicit differentials are exhibited in Subsection 5.4. Finally, Theorems 5 and 6 are proved in Subsections 5.5 and 5.6 respectively.

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2. Monod’s spectral sequence

As above, let $G$ be a semisimple Lie group with finite center and finitely many connected components, $P < G$ be a minimal parabolic subgroup, $A < P$ be the maximal split torus, $M = Z_K(A)$ be the centralizer of the Lie algebra $A$ in the maximal compact subgroup $K < G$ and $w_0$ be a representative of the longest element of the Weyl group associated to $A$. The proof of Monod’s Theorem \cite{11} relies on the study of the spectral sequence associated to the double complex

$$C^{p,q} := L^0(G^{p+1}, L^0((G/P)^q))^G,$$

endowed with two differentials

$$d^\Gamma : C^{p,q} \to C^{p+1,q}, \quad d^\gamma : C^{p,q} \to C^{p,q+1}.$$

The vertical differential is the homogeneous differential on $G$, whereas the second one is $(-1)^{p+1}$ times the homogenous differential on $G/P$ (to ensure that $d^\Gamma d^\gamma = d^\gamma d^\Gamma$). This bicomplex leads to two spectral sequences converging to the cohomology of the corresponding total complex.

The spectral sequence of main interest starts at page 1 with

$$E_1^{p,q} = (H^p(C^{p,q}, d^\Gamma), d_1 = d^\gamma).$$

The other spectral sequence, obtained as above by exchanging the roles of $d^\Gamma$ and $d^\gamma$, is easily shown \cite{11} Proposition 6.1 to collapse immediately to 0 (from page 1). Proposition 6.1 in \cite{11} is stated for $G$ connected, but the proof is identical for finitely many connected components. We will see below what can be deduced from knowing that the spectral sequence from (4) converges to 0.

Recall that in general to find the image $[\alpha] \in E_2^{p,q}$ by the differential

$$d_k : E_k^{p,q} \longrightarrow E_k^{p-k+1,q+k}$$

of a class represented by $\alpha \in C^{p,q}$, one needs to find a sequence of $\alpha_i \in C^{p-i,q+i}$, for $0 \leq i \leq k - 1$ such that

$$\alpha_0 = \alpha \quad \text{and} \quad d^\gamma \alpha_{i+1} = d^\Gamma \alpha_i.$$

The image $d_k([\alpha])$ is then simply represented by $d^\gamma \alpha_{k-1}$.

**First page of the spectral sequence.** For the first column (for $q = 0$), we obviously have $E_1^{1,0} = H^p_m(G)$. For the second and third columns, as $P$ and $MA$ are respectively the stabilizers of one and two (generic) point(s) on $G/P$, by the Eckmann-Shapiro isomorphism \cite{12} Theorem 6 we have that $E_1^{1,1} = H^p_m(P)$ and $E_1^{1,2} = H^p_m(MA) \cong H^p_m(A)$. The latter isomorphism holds since $M$ is compact and centralizes $A$. The cohomology of the remaining columns is trivial \cite{11} Proposition 5.1 by the compactness of stabilizers of a generic triple. In fact, the same argument used by Monod \cite{11} Proposition 5.1 still holds when $G$ has a finite number of connected components. In this way we are only left with the $G$-invariant of the coefficient modules $L((G/P)^q)^G$ in degree 0, that is, for $E_1^{0,q}$. The first page is depicted in Figure \cite{11}

**The differential $d_1$.** It is straightforward to check that the differential

$$d_1 : E_1^{0,0} = H^p_m(G) \longrightarrow H^p_m(P) = E_1^{0,1}$$

induced by $d^\gamma$ is, as a map $H^p_m(G) \to H^p_m(P)$, simply $(-1)^{p+1}$ times the map induced by the inclusion $P < G$. What is much less direct is the fact that the restriction map is trivial. When $G$ is connected this is proved in \cite{11} Corollary 3.2. We now adapt the proof for $G$ with finitely many connected components.
$G^0 < G$ be the connected component of the neutral element. Likewise we consider $P^0 < P$ and $M^0 < M$. We have a commutative diagram induced by inclusions

$$
\begin{array}{cccccccc}
p & 3 & H^3_m(G) & \xrightarrow{\partial} & H^3_m(P) & \xrightarrow{\partial} & H^3_m(A) & \xrightarrow{\partial} & 0 \\
2 & H^2_m(G) & \xrightarrow{\partial} & H^2_m(P) & \xrightarrow{\partial} & H^2_m(A) & \xrightarrow{\partial} & 0 \\
1 & H^1_m(G) & \xrightarrow{\partial} & H^1_m(P) & \xrightarrow{\partial} & H^1_m(A) & \xrightarrow{\partial} & 0 \\
0 & \mathbb{R} & \xrightarrow{\partial} & L^0(G/P)^G & \xrightarrow{\partial} & L^0((G/P)^2)^G & \xrightarrow{\partial} & L^0((G/P)^3)^G & \xrightarrow{\partial} & \cdots
\end{array}
$$

The bottom horizontal arrow is zero by [11, Corollary 3.2]. We claim that the vertical arrow on the right is an isomorphism. In fact we can write $P = MAN$ and $P^0 = M^0 AN$, and the cohomology of both $P$ and $P^0$ is isomorphic to the cohomology of the group $B = AN$ (since both $M$ and $M^0$ are compact and they normalize $B$). As a consequence the top horizontal arrow is zero, and the claim is proved.

It is also clear that

$$d_1 : E^{0,q} = L^0((G/P)^q)^G \longrightarrow L^0((G/P)^{q+1})^G = E^{0,q+1}\$$

translates into $(-1)$ times the homogeneous differential $\delta$.

As for

$$d_1 : E^{p,1} = H^p_m(P) \longrightarrow H^p_m(A) = E^{p,2},$$

it is shown in the proof of [11] Proposition 4.1 that it is induced by

$$d^\rightarrow : C^{p,1} = L^0(G^{p+1})^P \longrightarrow L^0(G^{p+1})^A \quad f \quad \mapsto \quad \{ (g_0, \ldots, g_p) \mapsto (-1)^{p+1}[f(w_0^{-1}g_0, \ldots, w_0^{-1}g_p) - f(g_0, \ldots, g_p)] \}.$$
is \((-1)^{p+1}2\) times the projection with kernel the \(w_0\)-invariant cohomology classes in \(H^*_m(A)\) and image the \(w_0\)-equivariant cohomology classes in \(H^*_m(A)\) (for the obvious action by \(\pm 1\) of \(w_0\) on the coefficients \(\mathbb{R}\)). As a consequence the second page \(E_2\) can be depicted as in Figure 2

![Figure 2](image-url)

### The second page \(E_2\)

The differential \(d_2\). The differential

\[
d_2 : E_2^{p,0} = H^p_m(G) \to H^{p-1}_m(A)^{w_0} = E_2^{p-1,2}
\]

vanishes for all \(p \geq 1\). This is proved in [11] only in the case where \(w_0\) acts by \(-1\) on \(a\). We therefore present the general case here, insisting that it is nothing more than a straightforward adaptation of Lemma 4.2 and Proposition 6.3 in [11]. Let thus \(\alpha_0 \in C^{p,0}\) be representing a class \([\alpha] \in E_2^{p,0} = H^p_m(G)\). By construction, there exists \(\alpha_1 \in C^{p-1,1}\) such that \(d_1 \alpha_1 = d_2 \alpha_0\) and

\[
d_2([\alpha_0]) = [d_2 \alpha_1] \in H^{p-1}_m(A)^{w_0} = E_2^{p-1,2}.
\]

In particular, \(d_2 \alpha_1\) represents a cohomology class in \(H^{p-1}_m(A) = E_1^{p-1,1}\) which is \(w_0\)-equivariant. But since \(d_1 = d_2 : H^{p-1}_m(A) = E_1^{p-1,1} \to E_1^{p-1,2} = H^{p-1}_m(A)\) is \((-1)^p2\) times the projection on the \(w_0\)-equivariants of \(H^{p-1}_m(A)\), the class \(d_2 \alpha_1\) lies in its image and hence vanishes in \(E_2^{p-1,2}\).

It follows that the only possibly nonzero differentials \(d_2\), are the two differentials

\[
d_2 : E_2^{1,1} = H^1_m(A)^{w_0} \to H^2_m(G \ltimes G/P) = E_2^{0,2}
\]

and

\[
d_2 : E_2^{1,2} = H^1_m(A)^{w_0} \to H^2_m(G \ltimes G/P) = E_2^{0,3}.
\]

### Nonzero differentials \(d_p\). For \(p \geq 2\), the only possibly nonzero differentials are, trivially, the three starting from the columns 0, 1 or 2 with image on the first row.
Conclusion. It follows that the only way that $E_2$ converges to 0 is that

$$d_{p-1} : H_{m}^{p-2}(A)^{w_0} \rightarrow H_{m}^{p}(G \cap G/P)$$

is an injection,

$$d_p : H_{m}^{p-1}(A)^{w_0} \rightarrow H_{m}^{p}(G \cap G/P)/d_{p-1}(H_{m}^{p-2}(A)^{w_0})$$

is an injection and

$$d_{p+1} : H_{m}^{p}(G) \rightarrow (H_{m}^{p}(G \cap G/P)/d_{p-1}(H_{m}^{p-2}(A)^{w_0}))/d_p(H_{m}^{p-1}(A)^{w_0})$$

is an isomorphism.

A standard diagram chase shows that the inverse of $d_3$ is indeed induced by the evaluation map, finishing the proof of Theorem 1.

3. Explicit differentials

The aim of this section is to exhibit explicit maps for the injection and a section of the short exact sequence (2) presented below and Proposition 10 and in Section 2 we have established that

$$H_{m}^{p}(A)^{w_0} \cong E_{2}^{p,1} \text{ and } H_{m}^{p}(A)^{w_0} \cong E_{2}^{p,2}.$$  

For the explicit injection and section of the short exact sequence (2) presented below we will need to know that classes in $E_{2}^{p,1}$ and $E_{2}^{p,2}$ can be represented by cocycles in $C^{p,1}$ and $C^{p,2}$ with additional symmetries. Denote by $C^{p}(A)$ the cocomplex of continuous $A$-invariant cochains $A^{p+1} \rightarrow \mathbb{R}$ endowed with its homogenous boundary operator and let

$$C^{p}(A)^{w_0} = \{ \alpha \in C^{p}(A) \mid \alpha(w_0a_0w_0^{-1}, \ldots, w_0a_{p}w_0^{-1}) = \alpha(a_0, \ldots, a_p) \}$$

the subspace of $w_0$-invariant cochains.

The aim of this paragraph is to produce explicit maps

$$C^{p+1}(A)^{w_0} \rightarrow C^{p,1} \text{ and } C^{p+1}(A)^{w_0} \rightarrow C^{p,2}$$

inducing the isomorphisms (6).

We fix $\alpha \in C^{p+1}(A)^{w_0}$. Recall that in virtue of the Iwasawa decomposition, any element $g \in G$ can be written uniquely as $g = ank$, where $a \in A, n \in N, k \in K$. Here $N$ is the unipotent radical of the minimal parabolic subgroup and $K$ is a fixed maximal compact subgroup. The Iwasawa decomposition allows us to define a projection of $G$ onto $A$, namely

$$\pi_A : G \rightarrow A \quad \text{g = ank} \quad \mapsto \quad a.$$  

Using this projection, we further define $\alpha_{G} = \pi_A^{\ast}(\alpha) \in L^{0}(G^{p+1})^{P}$ by

$$\alpha_{G}(g_0, \ldots, g_p) = \alpha(\pi_A(g_0), \ldots, \pi_A(g_p)).$$

Via the induction isomorphism $\alpha_{G} \in L^{0}(G^{p+1})^{P} \cong C^{p,1} \ni \alpha_{G}$ we obtain the corresponding cocycle $\overline{\alpha_{G}}$ in $C^{p,1}$ as

$$\overline{\alpha_{G}}(g_0, \ldots, g_p)(hP) = \alpha_{G}(h^{-1}g_0, \ldots, h^{-1}g_p).$$
Now $\alpha_G$ can also be considered as an element in $L^0(GP^{+1})A \cong C^{p,2}$ which corresponds to the cocycle $\overline{\alpha_G} \in C^{p,2}$ given by

$$\overline{\alpha_G}(g_0, \ldots, g_p)(hP, hw_0P) = \alpha_G(h^{-1}g_0, \ldots, h^{-1}g_p).$$

Then, the isomorphisms in (5) are realized at a cochain level by the maps

$$\begin{align*}
C^{p+1}(A)^{w_0}_\alpha & \quad \longrightarrow \quad C^{p,1}_\alpha \\
\alpha & \quad \longmapsto \quad \overline{\alpha_G}^{-1}
\end{align*}$$

and

$$\begin{align*}
C^{p+1}(A)^{w_0}_\alpha & \quad \longrightarrow \quad C^{p,2}_\alpha \\
\alpha & \quad \longmapsto \quad \overline{\alpha_G},
\end{align*}$$

respectively.

### 3.2. Homotopies for $(C^{q, q}_K, d^q)$

In order to follow at cochain level the differentials realizing the injection and a section of the short exact sequence (2), we would ideally want contracting homotopies for the cocomplex $(C^{*,q}, d^q)$, for $q \geq 3$, whose cohomology vanishes by [11, Proposition 5.1]. It is however not so clear how to construct such homotopies even in the two cases we consider in this paper (for $G$ a product of isometries of hyperbolic spaces or $SL(n, K)$). We will see that it will be enough to construct such maps on the subcocomplex of right-$K$-invariant cochains, which we now define:

**Definition 7.** We say that a cochain $\alpha \in C^{p,q}$ is **right-$K$-invariant** if

$$\alpha(g_0, \ldots, g_p)(x_1, \ldots, x_q) = \alpha(g_0gk_0, \ldots, g_pk_p)(x_1, \ldots, x_q)$$

for almost every $g_0, \ldots, g_p \in G$, $x_1, \ldots, x_q \in G/P$ and $k_0, \ldots, k_p \in K$. Furthermore, we define $C^{p,q}_K \subset C^{p,q}$ as the subset of right-$K$-invariant cochains.

**Remark.** It is worth noticing that the cochains $\overline{\alpha_G}$ and $\overline{\alpha_G}$ constructed in the previous section are naturally right-$K$-invariant, since they are defined via induction of the pullback along the projection $\pi_A$ and the latter is trivial on $K$.

It is clear that, if $\alpha \in C^{p,q}_K$, then both $d^+\alpha$ and $d^q\alpha$ are right-$K$-invariant. In particular $(C^{p,q}_K, d^q)$ forms a cocomplex. It is easy to exhibit contracting homotopies

$$h^{p,q}_K : C^{p,q}_K \longrightarrow C^{p-1,q}_K$$

for the cocomplex $(C^{p,q}_K, d^q)$, i.e. maps satisfying

$$(7) \quad h^{p+1,q}d^q + d^qh^{p,q} = \text{Id}.$$ 

This could be done in general with the help of a measurable section of the projection

$$(G/P)^q \overset{\sigma_q}{\longrightarrow} G\backslash(G/P)^q$$

with the additional property that, for $q \geq 3$, the stabilizer of $\sigma(x_1, \ldots, x_q)$ is contained in $K$ for almost every $q$-tuple in the image of $\sigma_q$. In this paper, we follow a different strategy, and we will exhibit contracting homotopies exploiting the transitivity of $G$ on tuples of points in $G/P$. This will be done in Section 4.3 for $G$ a product of isometries of hyperbolic spaces, which acts 3-transitively on $G/P$, and in Section 5.1 for $G = SL(3, K)$, which acts more than 2-transitively on $G/P$, meaning that it acts transitively on generic points in $G/P \times G/P \times P^1(\mathbb{K})$. Indeed recall that in this case $G/P$ is the space of flags in $\mathbb{K}^3$ and $P^1(\mathbb{K})$ parametrizes the points in the flags.

### 3.3. Explicit expression for the injection $i$ of the short exact sequence (2)

**Lemma 8.** Let $\{h^{*, q}\}$ be a family of contracting homotopies for $(C^{q, q}_K, d^q)$. The injection

$$\begin{align*}
i : H^{p-2}_m(A)^{w_0} \cong H^{p-2}(C^{*,2}, d^q) & \hookrightarrow NH^m_p(G \cap G/P) \\
\alpha & \longmapsto d^+h^{1,p}d^+h^{2,p-1} \cdots d^+h^{p-2,3}d^+(\overline{\alpha_G})(e).
\end{align*}$$

induced by $d_{p-1}$ is given explicitly at cochain level as

$$\begin{align*}
C^{p-2}(A)^{w_0}_\alpha & \quad \longrightarrow \quad L^0((G/P)^{p+1})^G \\
\alpha & \quad \longmapsto \quad d^+h^{1,p}d^+h^{2,p-1} \cdots d^+h^{p-2,3}d^+(\overline{\alpha_G})(e).
\end{align*}$$
Note that the evaluation on \( \epsilon \) is well-defined since an almost everywhere defined \( G \)-invariant function on \( G \) is actually defined everywhere. Observe also that this map is defined on all cocycles, but a \( u_0 \)-equivariant cocycle is in the image of the horizontal differential \( d^1 \) and hence will be mapped to 0 by \( d^2 \) and this composition.

**Proof.** We only need to check that the sequence

\[
\begin{align*}
\alpha_2 &:= \left[ \frac{a}{b} \right] \in C^{p,2}_K, \\
\alpha_k &:= h^{p-k+1,k} d^1(\alpha_{k-1}) \in C^{p-k,k}_K,
\end{align*}
\]

for \( 3 \leq k \leq p \), satisfies

\[
\tag{8} d^1 \alpha_{k-1} = d^1 \alpha_k.
\]

We prove inductively that if \( d^1 d^1 \alpha_{k-1} = 0 \), then \( \alpha_k \) is a cocycle for \( d^1 \). We have \( d^1 d^1 \alpha_2 = d^1 d^1 \alpha_2 = 0 \). Suppose that \( d^1 d^1 \alpha_{k-1} = 0 \). We compute

\[
\begin{align*}
d^1 \alpha_k &= d^1 h^{p-k+1,k+1} d^1(\alpha_{k-1}) \\
&= (\text{Id} - h^{p-k+2,k+1} d^1) d^1(\alpha_{k-1}) \\
&= d^1(\alpha_{k-1}),
\end{align*}
\]

where we have used that \( h^{p-k+2,k+1} d^1 + d^1 h^{p-k+1,k+1} = \text{Id} \) and the assumption that \( d^1 d^1(\alpha_{k-1}) = 0 \). Furthermore

\[
d^1 d^1 \alpha_k = d^1(\alpha_k) = d^1(\alpha_{k-1}) = 0.
\]

It follows that the \( d^1 \)-cocycle \( d^1 \alpha_p \in C^{p,p}_K \) represents the image of \( [\alpha] \) under the differential \( d_{p-1} \), which is, as an element in \( L^0((G/P)^{p+1})^G \) given by evaluating the \( G \)-invariant cocycle \( d^1 \alpha_p \) on \( e \) (or any other element of \( G \)). \( \square \)

### 3.4. Explicit expression for a section \( s \) of the short exact sequence \( (2) \).

We follow here the same strategy as for the injection in the previous paragraph. We only need an additional argument for the first step of the differential. We define

\[
\mathcal{H}^{p-1} : C^{p-1}(A)^{w_0} \to L^0(C^{p-1}A) \cong C^{p-2,2}
\]

by

\[
\mathcal{H}^{p-1}(\alpha_G)(g_0, \ldots, g_{p-2}) := (-1)^p \sum_{i=0}^{p-2} (-1)^i \alpha_G(w_0^{-1} \pi_A(g_0), \ldots, w_0^{-1} \pi_A(g_i), w_0^{-1} g_i, \ldots, w_0^{-1} g_{p-2}).
\]

**Lemma 9.** For any \( \alpha \in C^{p-1}(A)^{w_0} \), we have

\[
d^1 \mathcal{H}^{p-1}(\alpha_G) = d^2(\alpha_G).
\]

Furthermore, \( \mathcal{H}^{p-1}(\alpha_G) \) is right-\( K \)-invariant.

**Proof.** The fact that \( \mathcal{H}^{p-1}(\alpha_G) \) is right-\( K \)-invariant is obvious, since it is defined as an alternating sum of evaluations of \( \alpha_G \).

It is a standard computation, based on the fact that \( \alpha_G \) is a cocycle, to check that

\[
d^1 \mathcal{H}^{p-1}(\alpha_G)(g_0, \ldots, g_{p-1}) = (-1)^p \left( \alpha_G(w_0^{-1} g_0, \ldots, w_0^{-1} g_{p-1}) \right.
\]

\[
- \left. \alpha_G(w_0^{-1} \pi_A(g_0), \ldots, w_0^{-1} \pi_A(g_{p-1})) \right).
\]

Now it remains to see that the latter summand is precisely \( \alpha_G(g_0, \ldots, g_{p-1}) \), implying, by the explicit description of the differential \( d^2 \) given in (5), that \( d^2 \mathcal{H}^{p-1}(\alpha_G) = d^2(\alpha_G) \). To see this, write \( a_i = \pi_A(g_i) \), so that we need to check that

\[
\alpha_G(w_0^{-1} a_0, \ldots, w_0^{-1} a_{p-1}) = \alpha(a_0, \ldots, a_{p-1}),
\]
but
\[
\alpha_G(w_0^{-1}a_0, \ldots, w_0^{-1}a_{p-1}) = \alpha(\pi_A(w_0^{-1}a_0), \ldots, \pi_A(w_0^{-1}a_{p-1}))
\]
\[
= \alpha(\pi_A(w_0^{-1}a_0w_0), \ldots, \pi_A(w_0^{-1}a_{p-1}w_0))
\]
\[
= \alpha(w_0^{-1}a_0w_0, \ldots, w_0^{-1}a_{p-1}w_0)
\]
\[
= \alpha(a_0, \ldots, a_{p-1}),
\]
by the \(w_0\)-invariance of \(\alpha\).

\textbf{Proposition 10.} For any \(q \geq 3\), let \(h^{p,q} : C^{p,q}_K \to C^{p-1,q}_K\) be contracting homotopies for the cocomplexes \((C^{p,q}_K, d')\). A section of the projection in the short exact sequence from (2),
\[
H^{p-1}_m(A)^{w_0} \to NH^p_m(G \cap G/P),
\]
is given explicitly at cocycle level by sending a cocycle \(\alpha \in C^{p-1}(A)^{w_0}\) to
\[
d^{-1}h^{1,p}d^{-1}h^{2,p-1}d^{-1} \cdots d^{-1}h^{n-3,p}d^{-1}H^{p-1}(\pi_G)(e) \in L^0((G/P)^{p+1})^G.
\]

\textbf{Proof.} The proof is identical to the proof of Lemma 8 except for the first induction step which is given by Lemma. \(\square\)

\section{The spectral sequence for products of isometries of real hyperbolic spaces}

\subsection{Isometries of \(\mathbb{H}^n\).} As before, we take the upper half space model \(\{x \in \mathbb{R}^n \mid x_n > 0\}\) for \(\mathbb{H}^n\). In this model the isometry group \(G = \text{Isom} (\mathbb{H}^n)\) is identified with the (possibly orientation reversing) conformal transformations of \(\mathbb{R}^n \cup \{\infty\}\) which leave the upper half space, and hence the boundary \(\partial \mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\} \cup \{\infty\}\), invariant. We take for maximal compact subgroup \(K\) the stabilizer of \(e_n = (0, \ldots, 0, 1)\), which is generated by inversions or reflexions by spheres or hyperplanes containing \(e_n\) and orthogonal to the boundary. The minimal parabolic subgroup \(P = \text{Stab}(\infty)\) is given by the similarities of \(\mathbb{R}^{n-1}\) (which naturally extend to the upper half space). More precisely, \(P = \text{ANM}\), where \(A\) consists of the homoteties \(a_\lambda : x \mapsto \lambda x\), for \(\lambda > 0\), \(N\) consists of translations \(n_v : x \mapsto x + v\) by vectors \(v\) in \(\mathbb{R}^{n-1} \times \{0\}\) and \(M = P \cap K = O(n-1)\) are orthogonal transformations (extending to orthogonal transformations of \(\mathbb{R}^n\) which restrict to \(\mathbb{H}^n\)). In contrast, even though \(K \cong O(n)\), a general element of \(K\) is as a composition of inversions harder to describe. We will however only need an explicit description of a representative of the longest element \(w_0 \in K\), which can then be taken as the composition of the inversion with the unit sphere centered at the origin and the reflexion by the hyperplane orthogonal to \(e_1 = (1,0,\ldots,0)\) and thus takes the form
\[
w_0 : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \frac{x-2(x,e_1)e_1}{\|x\|^2}.
\]

We record the following commutativity rules:
\[
a_{\lambda}n_va_{\lambda} = n_{\lambda v}, \quad a_{\lambda}\rho = \rho a_{\lambda}, \quad \rho n_v\rho^{-1} = n_{\rho v},
\]
for \(a_\lambda \in A, n_v \in N\) and \(\rho \in M\). Additionally, setting \(n := n_{e_1}\), we observe the relation
\[
w_0^{-1}n^{-1}w_0^{-1} = n^{-1}w_0^{-1}n^{-1}.
\]

By the Iwasawa decomposition, we have a diffeomorphism between \(G\) and \(\text{ANK}\), which allows us to define the projection
\[
\pi_A : G \to A
\]
\[
g = ank \mapsto a.
\]

The following simple lemma will allow us to compute the \(\pi_A\)-projection of any element of \(G\):
Lemma 11. For any $g \in G$, its $\pi_A$-projection is

$$\pi_A(g) = a_\lambda,$$

where

$$(0, \infty) \ni \lambda = \text{the last coordinate of the vector } g(e_n).$$

Proof. The proof is straightforward. Indeed, since $K$ stabilizes $e_n$, only $N$ and $A$ act non trivially on that point. In particular, if $g = a_\lambda n_v \rho$, for $\rho \in K$, $v \in \mathbb{R}^{n-1} \times \{0\}$ and $\lambda > 0$, then

$$g(e_n) = a_\lambda n_v \rho(e_n) = a_\lambda n_v(e_n) = a_\lambda (v + e_n) = \lambda v + \lambda e_n,$$

which proves the lemma since the last coordinate of $v$ is 0.

We apply this lemma to compute a collection of $\pi_A$-projections needed in the proofs of Theorems 2 and 4 to some specific elements in $G$ which we now define. For $x \in \partial \mathbb{H}^n \setminus \{\infty, 0\}$, denote by $\hat{x}$ the normalized element

$$\hat{x} = \frac{x}{\|x\|}.$$ 

In case $n = 2$, we further assume that $0 < x < \infty$. Now choose a (non unique in dimension $n \geq 3$) isometry $\rho_x \in M$ sending $e_1$ to $\hat{x}$. Set

$$g_{\infty x} := a\|x\|\rho_x,$$

$$g_{\infty e_1} := na\|x-e_1\|\rho_{\hat{x}-e_1},$$

$$g_{0 e_1} := w_0 n^{-1} a\|e_1+wo(x)\|\rho_{\hat{x}+wo(x)}.$$

These isometries are so defined as to send the three points $\infty, 0, e_1$ to the three points listed as indices. For example, $g_{0 e_1}(\infty) = 0, g_{0 e_1}(0) = e_1, g_{0 e_1}(e_1) = x$.

The next proposition will be needed in the proof of Theorem 2.

Proposition 12. For $x \in \partial \mathbb{H}^n \setminus \{\infty, 0\}$ and $g_{\infty x}, g_{\infty e_1}, g_{0 e_1}$ as above we have:

$$\pi_A(g_{0 e_1}) = a_{\frac{\|x\|-e_1\|}{\|x\|}} \pi_A(g_{0 e_1}) = a_{\frac{\|x\|}{\|x\|}} = a_{\frac{\|x\|}{\|x\|}}.$$ 

Note in particular that these projections do not depend on the choice of $\rho_x \in M$.

Proof. We only prove the first line of equations, i.e those involving $g_{0 e_1}$, which are the most tricky. The motivated reader can check the remaining easier equations using the same method. From the definition of $g_{0 e_1}$ we obtain

$$g_{0 e_1} = a_{\frac{\|x\|-e_1\|}{\|x\|}} = a_{\frac{\|x\|}{\|x\|}}.$$ 

where we used twice the commutativity rules given in (10). Since this expression is of the form $ank$ we have that

$$\pi_A(g_{0 e_1}) = a_{\frac{\|x\|-e_1\|}{\|x\|}}.$$
The square of the norm of the element $e_1 + w_0(x)$ is easily computed:

$$
\|e_1 + w_0(x)\|^2 = \|e_1\|^2 + \|w_0(x)\|^2 + 2\langle e_1, w_0(x) \rangle = 1 + \frac{1}{\|x\|^2} - 2\langle e_1, x \rangle = \frac{\|x - e_1\|^2}{\|x\|^2}.
$$

As a consequence we get the first expression

$$
\pi_A(g_{0e_1x}) = a \frac{1}{\|x - e_1\|},
$$

as expected. Starting from the above expression for $g_{0e_1x}$ and using the fact that $w_0^{-1}aw_0 = a^{-1}$ for every $a \in A$, we get that

$$
\pi_A(w_0^{-1}g_{0e_1x}) = a \frac{1}{\|x - e_1\|} \pi_A(w_0^{-1}n_{\rho^{-1}_{e_1+v_0(x)}(e_1)}).
$$

By Lemma \[\text{[11]}\] we know that

$$
\pi_A(w_0^{-1}n_v) = a \frac{1}{\|x + v\|},
$$

for any $v \in \mathbb{R}^{n-1} \times \{0\}$. In this particular context the previous formula allows us to write

$$
\pi_A(w_0^{-1}g_{0e_1x}) = a \frac{1}{\|x - e_1\|} a \frac{1}{\|x + e_1\|} = a \frac{1}{\|x - e_1\|},
$$

which is the second desired expression. We now compute the third term

$$
\pi_A(w_0^{-1}n_{-\|e_1+w_0(x)\|e_1}^{-1}g_{0e_1x}) = \pi_A(w_0^{-1}n^{-1}_{\rho^{-1}_{e_1+v_0(x)}(e_1)})
= a_{\|e_1+w_0(x)\|} \pi_A(w_0^{-1}n_{-\|e_1+w_0(x)\|e_1}^{-1}g_{0e_1x}) (e_1).
$$

where we used again the relations from \[\text{[10]}\] to pull out the dilation term. To conclude the computation, we need to consider the square of the norm of the vector appearing in the translation term. We have that

$$
\left\|\|e_1 + w_0(x)\|e_1 - \rho^{-1}_{e_1+w_0(x)}(e_1)\right\|^2 = \|e_1 + w_0(x)\|^2 + 1 - 2\|e_1 + w_0(x)\|e_1 \| + w_0(x)\|e_1 + w_0(x)\|
= 1 + \frac{1}{\|x\|^2} - 2\langle e_1, w_0(x) \rangle = 1 - 2\langle e_1, w_0(x) \rangle.
$$

Therefore, the last projection is given by

$$
\pi_A(w_0^{-1}n^{-1}_{-\|e_1+w_0(x)\|e_1}^{-1}g_{0e_1x}) = a \frac{1}{\|x - e_1\|} a \frac{1}{\|x + e_1\|} = a \frac{1}{\|x - e_1\|}.
$$

For the proof of Theorem \[\text{[3]}\] we will need the following three propositions:

**Proposition 13.** Take $x \in \partial B^n \setminus \{\infty, 0\}$ and $g_{\infty 0x}, g_{\infty e_1x}, g_{0e_1x}$ as above. For any $y \in \partial B^n \setminus \{\infty, 0, x\}$ set

$$
\mu(x, y) := g_{\infty e_1x}^{-1}(y).
$$

We have

$$
\pi_A(w_0^{-1}g_{0e_1\mu(x,y)}^{-1}) = a \frac{1}{\|x - e_1\|}, \quad \pi_A(w_0^{-1}g_{\infty e_1\mu(x,y)}g_{\infty e_1x}) = a \frac{1}{\|x - e_1\|},
$$

$$
\pi_A(w_0^{-1}g_{\infty 0\mu(x,y)}^{-1}) = a \frac{1}{\|x - e_1\|}, \quad \pi_A(w_0^{-1}g_{\infty 0\mu(x,y)}g_{\infty e_1x}) = a \frac{1}{\|x - e_1\|}.
$$
Proof. For the three left projections, we will use the middle equations of Proposition 12 applied to $\mu(x, y)$. We thus need to understand how we can write the point $\mu(x, y)$. Recall that $\hat{y} = y/\|y\|$ is the normalization of $y$ and the maps $g_{\infty}; g_{\infty}; x, y_{0}; x, y_{0}; x, y_{0}; x, y_{0}; x, y_{0}$ are as in (13). We have

$$\mu(x, y) = g_{\infty}; x(y) = a \frac{1}{x - e_{1}} \rho_{x - e_{1}}^{1} n^{-1} y = a \frac{1}{x - e_{1}} \rho_{x - e_{1}}^{1} (y - e_{1}).$$

Let us further compute the square of the norm of $\mu(x, y) - e_{1}$:

$$\|\mu(x, y) - e_{1}\|^2 = \left\| \frac{y - e_{1}}{x - e_{1}} \rho_{x - e_{1}}^{1} (y - e_{1}) - e_{1} \right\|^2 = \left\| \frac{y - e_{1}}{x - e_{1}} \right\|^2 + 1 - 2 \left\| \frac{y - e_{1}}{x - e_{1}} \right\| \cdot \left\| y - e_{1}, x - e_{1} \right\| \cdot \left\| y - e_{1} \right\| = \left\| y - e_{1} \right\|^2 + \left\| y - e_{1} \right\|^2.$$

Proposition 12 now gives:

$$\pi_{A}(w_{0}^{-1} g_{\infty}; x(y)) = a \frac{1}{x - e_{1}} \rho_{x - e_{1}}^{1} a \frac{1}{x - e_{1}}$$

$$\pi_{A}(w_{0}^{-1} g_{\infty}; x(y)) = a \frac{1}{x - e_{1}}$$

$$\pi_{A}(w_{0}^{-1} g_{\infty}; x(y)) = a \frac{1}{x - e_{1}}.$$

For the projection of the right element of the first line, we have that

$$w_{0}^{-1} g_{\infty}; x(y) g_{\infty}; x(y) = w_{0}^{-1} a \frac{1}{x + w_{0}(\mu(x, y))} \rho_{x + w_{0}(\mu(x, y))} n w_{0}^{-1} a \frac{1}{x + w_{0}(\mu(x, y))} \rho_{x + w_{0}(\mu(x, y))} n^{-1} = a \frac{1}{x + w_{0}(\mu(x, y))} w_{0}^{-1} \rho_{x + w_{0}(\mu(x, y))} n^{-1},$$

where we used Equation (10) to move from the first line to the second one. It follows that

$$\pi_{A}(w_{0}^{-1} g_{\infty}; x(y)) g_{\infty}; x(y) = a \frac{1}{x - e_{1}} \pi_{A}(w_{0}^{-1} \rho_{x + w_{0}(\mu(x, y))} n^{-1} \rho_{x - e_{1}} n^{-1}).$$

To compute the projection of the latter element, we need to consider its evaluation on $e_{n}$ and compute its last coordinate according to Lemma 11

$$\left( w_{0}^{-1} \rho_{x + w_{0}(\mu(x, y))} n^{-1} \rho_{x - e_{1}} n^{-1} \right)(e_{n}) = \left[ w_{0}^{-1} \rho_{x + w_{0}(\mu(x, y))} n^{-1} \rho_{x - e_{1}} n^{-1} \right] \left( \frac{1}{2} e_{n} - \frac{1}{2} \rho_{x - e_{1}}(e_{1}) + \frac{\langle x, e_{1} \rangle}{\|x - e_{1}\|} \right).$$

Since $K$ stabilizes the point $e_{n}$, the last coordinate will be given by $1/2$ divided by the square of the norm of the vector inside the round brackets. If we compute the square of this norm we obtain

$$\left\| \frac{1}{2} e_{n} - \frac{1}{2} \rho_{x - e_{1}}(e_{1}) + \frac{\langle x, e_{1} \rangle}{\|x - e_{1}\|} \right\|^2 = \frac{1}{2} + \frac{\langle x, e_{1} \rangle^2}{\|x - e_{1}\|^2} - \frac{\langle x, e_{1} \rangle}{\|x - e_{1}\|} \cdot \frac{\langle x, e_{1} \rangle}{\|x - e_{1}\|} = \frac{1}{2} + \frac{\langle x, e_{1} \rangle^2}{\|x - e_{1}\|^2} = \frac{\|x\|^2 + 1}{2 \|x - e_{1}\|^2}.$$

Finally we can conclude that

$$\pi_{A}(w_{0}^{-1} g_{\infty}; x(y)) g_{\infty}; x(y) = a \frac{1}{x + w_{0}(\mu(x, y))} a \frac{1}{x - e_{1}} a \frac{1}{\|x - e_{1}\|} = a \frac{1}{\|y - e_{1}\|}. $$

Since the computations of the two remaining projections are entirely similar, we omit the details. \qed
We will need to compute 12 further projections which we simply report now without proof. The details of the computations follow the same line as the proof of Proposition 13.

**Proposition 14.** Take $x \in \partial \mathbb{H}^n \setminus \{\infty, 0\}$ and $g_{\infty 0 x}, g_{\infty 1 x}, g_{0 e 1 x}$ as above. For any $y \in \partial \mathbb{H}^n \setminus \{\infty, 0, x\}$ set

$$\lambda(x, y) := g_{\infty 0 x}^{-1}(y).$$

We have

$$\pi_A(w_0^{-1} g_{0 e 1 \lambda(x, y)}) = a \frac{y - e}{x - e}, \quad \pi_A(g_{\infty 1 \lambda(x, y)}) = a \frac{\|x - y\|}{\|x - e\|},$$

$$\pi_A(w_0^{-1} g_{\infty 0 \lambda(x, y)} g_{\infty 0 x}) = a \frac{\|x - y\|}{\|x - e\|}.$$ 

**Proposition 15.** Take $x \in \partial \mathbb{H}^n \setminus \{\infty, 0\}$ and $g_{\infty 0 x}, g_{\infty 1 x}, g_{0 e 1 x}$ as above. For any $y \in \partial \mathbb{H}^n \setminus \{\infty, 0, x\}$ set

$$\nu(x, y) := g_{0 e 1 x}^{-1}(y).$$

We have

$$\pi_A(w_0^{-1} g_{0 e 1 \nu(x, y)}) = a \frac{y - e}{x - e}, \quad \pi_A(g_{\infty 1 \nu(x, y)}) = a \frac{\|x - y\|}{\|x - e\|},$$

$$\pi_A(w_0^{-1} g_{\infty 0 \nu(x, y)} g_{0 e 1 x}) = a \frac{\|x - y\|}{\|x - e\|}.$$ 

4.2. Contracting homotopies and differentials. Let now $G = G_1 \times \cdots \times G_k$, where $G_i = \text{Isom}(\mathbb{H}^n)$ for some $n_i \geq 2$. A representative for the longest element $w_0 \in G$ can be taken to be on each factor the element $w_0$ defined in [9] in the case of one factor. We hope that the confusing notation

$$w_0 = (w_0, \ldots, w_0) \in K$$

will not confuse the reader. Likewise, we define $n \in G$ as the translation by $e_1$ on each factor, so that here also we write

$$n = (n, \ldots, n) \in P.$$ 

We are more careful with elements in $G/P$ and write

$$\infty = (\infty, \ldots, \infty),$$

$$0 = (0, \ldots, 0),$$

$$1 = (e_1, \ldots, e_1),$$

for our favorite three elements in the boundary $G_1/P_1 \times \cdots \times G_k/P_k = G/P$.

Contracting homotopies for the cocomplexes $(C^q_K, d^q)$, for $q \geq 3$, can obviously be defined as follows:

$$h^{p, q} : C^q_K \rightarrow C^{p-1, q}$$

with

$$h^{p, q}(f)(g_0, \ldots, g_{p-1})(\infty, 0, 1, x_4, \ldots, x_q) := f(e, g_0, \ldots, g_{p-1})(\infty, 0, 1, x_4, \ldots, x_q).$$

Note that this defines a measurable function on a dense subset of $G^p$: Indeed, although $f$ is only defined on a subset of full measure of $G^{p+1}$, the evaluation on $(e, g_0, \ldots, g_{p-1})$ makes sense on a subset of full measure of $G^p$ by the $G$-invariance of $f$. Likewise, the evaluation on $(\infty, 0, 1, x_4, \ldots, x_q)$ is allowed for almost all $(x_4, \ldots, x_q)$ by 3-transitivity. Finally observe that the resulting cochain is $G$-invariant by construction and 3-transitivity.

It is straightforward to check that

$$h^{p+1, q} \circ d^q + d^p \circ h^{p, q} = \text{Id}.$$
Before we proceed, we can again exploit 3-transitivity to give the following explicit form for the differential $d^* : C^{p,2} \to C^{p,3}$:

**Lemma 16.** The differential

$$d^* : C^{p,2} = L^0(G^{p+1})^A \longrightarrow L^0(G^{p+1}) = C^{p,3}$$

is given by

$$d^*(\beta)(g_0, \ldots, g_p) = (-1)^{p+1}[\beta(nw_0^{-1}g_0, \ldots, nw_0^{-1}g_p) - \beta(n^{-1}g_0, \ldots, n^{-1}g_p) + \beta(g_0, \ldots, g_p)].$$

**Proof.** A cochain $\beta \in L^0(G^{p+1})^A$ corresponds to $\overline{\beta} \in C^{p,2} = L^0(G^{p+1}, L((G/P)^2))^G$ in the following way:

$$\overline{\beta}(g_0, \ldots, g_p)(h^{-1}h^{-1}0) := \beta(hg_0, \ldots, hg_p),$$

Likewise the cochain $d^*(\beta) \in L^0(G^{p+1})$ is given by

$$d^*(\beta)(g_0, \ldots, g_p) = d^-\overline{\beta}(g_0, \ldots, g_p)(\infty, 0, 1),$$

where the evaluation on the triple $(\infty, 0, 1)$ makes sense by the $G$-invariance of $d^-\overline{\beta}$. By definition of $d^*$, we have

$$d^*\overline{\beta}(g_0, \ldots, g_p)(\infty, 0, 1) = (-1)^{p+1}[\overline{\beta}(g_0, \ldots, g_p)(0, 1) - \overline{\beta}(g_0, \ldots, g_p)(\infty, 1) + \overline{\beta}(g_0, \ldots, g_p)(\infty, 0)]$$

$$= (-1)^{p+1}[\overline{\beta}(g_0, \ldots, g_p)(w_0n^{-1}\infty, w_0n^{-1}0) - \overline{\beta}(g_0, \ldots, g_p)(n\infty, n0)$$

$$+ \overline{\beta}(g_0, \ldots, g_p)(\infty, 0)]$$

$$= (-1)^{p+1}[\beta(nw_0^{-1}g_0, \ldots, nw_0^{-1}g_p) - \beta(n^{-1}g_0, \ldots, n^{-1}g_p)$$

$$+ \beta(g_0, \ldots, g_p)],$$

which proves the lemma. \[\square\]

### 4.3. Proof of Theorem 2

Let $\alpha_\lambda \in \wedge^2 a^*$ be an alternating form, $\tilde{\alpha} : A^2 \to \mathbb{R}$ be the corresponding inhomogenous 2-cocycle

$$\tilde{\alpha}(a_1, a_2) := \alpha_\lambda(\log a_1, \log a_2)$$

and $\alpha : A^3 \to \mathbb{R}$ be the homogenous $A$-invariant cocycle

$$\alpha(a_0, a_1, a_2) = \tilde{\alpha}(a_0^{-1}a_1, a_1^{-1}a_2).$$

The cocycle

$$\alpha_G : G^3 \to \mathbb{R}$$

defined by

$$\alpha_G(g_0, g_1, g_2) = \alpha(\pi_A(g_0), \pi_A(g_1), \pi_A(g_2)).$$

where $\pi_A : G \to A$ is as above the projection given by the Iwasawa decomposition, is clearly $P$-invariant, so is an element in $L^0(G^3) = C^{2,1}$. Furthermore, its cohomology class in $H^2(C^{2,1}, d^*) \cong H^2_m(P) \cong H^2_m(A) \cong (\wedge^2 a)^*$ clearly corresponds to $\alpha_\lambda$. By Proposition 10 we need to follow $\alpha_G$ through the maps

$$(13) \quad \overline{\alpha_G} \in C^{2,1} \quad d^-\overline{\alpha_G} = d^\beta \in C^{2,2}$$

$$H^2(\overline{\alpha_G}) = : \beta \in C^{1,2} \quad d^-\beta = d^\omega \in C^{1,3}$$

$$h^{1,3}(d^\beta) = : \omega \in C^{0,3} \quad d^-\omega = : \Omega_3 \in C^{0,4}$$

and evaluate $\Omega_3 = d^-\omega$ on $e$. 
Computation of $\beta$. By the definition of $\mathcal{H}^2$, we have
\[
\beta(g_0, g_1) = H^2(\mathcal{H}) (g_0, g_1) \\
= -\alpha_G(w_0^{-1} \pi_A(g_0), w_0^{-1} g_0, w_0^{-1} g_1) + \alpha_G(w_0^{-1} \pi_A(g_0), w_0^{-1} \pi_A(g_1), w_0^{-1} g_1).
\]

Computation of $\omega$. By the description of $d^+$ from Lemma 16 and the definition of $h^{1,3}$ we obtain $\omega \in C^{0,3} = L^0(G, \mathbb{R})$ as
\[
\omega(g) = h^{1,3}(d^+ \beta) = d^+ \beta(e, g) \\
= \beta(nw_0^{-1} g, nw_0^{-1} g) - \beta(n^{-1}, n^{-1} g_0) + \beta(e, g).
\]
Using the expression for $\beta$ above, we can rewrite it as
\[
(14) \quad \omega(g) = -\alpha_G(w_0^{-1} \pi_A(nw_0^{-1}), nw_0^{-1} g, nw_0^{-1} g) \\
+ \alpha_G(w_0^{-1} \pi_A(nw_0^{-1}), w_0^{-1} \pi_A(nw_0^{-1} g), w_0^{-1} nw_0^{-1} g) \\
+ \alpha_G(w_0^{-1} \pi_A(n^{-1}), w_0^{-1} n^{-1}, w_0^{-1} n^{-1} g) \\
- \alpha_G(w_0^{-1} \pi_A(n^{-1}), w_0^{-1} \pi_A(n^{-1} g), w_0^{-1} n^{-1} g) \\
- \alpha_G(w_0^{-1} \pi_A(e), w_0^{-1} g, w_0^{-1} g) \\
+ \alpha_G(w_0^{-1} \pi_A(e), w_0^{-1} \pi_A(g), w_0^{-1} g).
\]
Since $\pi_A(nw_0^{-1}) = \pi_A(n^{-1}) = \pi_A(w_0^{-1}) = e$ the first coordinate of these 6 evaluations of $\alpha_G$ is the identity $e$. Additionally, since $w_0^{-1} nw_0^{-1} = n^{-1} w_0^{-1} n^{-1}$, we have, for $g' = e$ or $g$ (or any element in $G$) that
\[
(15) \quad \pi_A(w_0^{-1} n^{-1} g') = \pi_A(n^{-1} w_0^{-1} n^{-1} g') = \pi_A(w_0^{-1} n^{-1} g'),
\]
where for the last equality we have used that $\pi_A(n' g') = \pi_A(g')$ for any $n' \in N$ and $g' \in G$. It follows that the first and third summands in Equation (14) have precisely the same coordinates and hence cancel. Moreover, the fifth summand vanishes since the evaluation of $\alpha_G$ is zero whenever two of the coordinates are equal (and here the first and second coordinates are $e$). We are thus left with the 2nd, 4th and 6th summands, which, using (15) and the $N$-(left) invariance of $\pi_A$, we rewrite as
\[
\omega(g) = \alpha_G(e, w_0^{-1} \pi_A(w_0^{-1} g), w_0^{-1} n^{-1} g) \\
- \alpha_G(e, w_0^{-1} \pi_A(g), w_0^{-1} n^{-1} g) \\
+ \alpha_G(e, w_0^{-1} \pi_A(g), w_0^{-1} g).
\]
Now recall on the one hand that $\pi_A(w_0^{-1} a) = a^{-1}$ and on the other hand $\alpha(e, a_1, a_2) = -\alpha(e, a_1, (a_2)^{-1})$ and hence
\[
\alpha_G(e, g_1, g_2) = -\alpha_G(e, g_1, \pi_A(g_2)^{-1}),
\]
to conclude that
\[
\omega(g) = -\alpha_G(e, \pi_A(w_0^{-1} g)^{-1}, \pi_A(w_0^{-1} n^{-1} g)^{-1}) \\
+ \alpha_G(e, w_0^{-1} \pi_A(g), \pi_A(w_0^{-1} n^{-1} g)^{-1}) \\
- \alpha_G(e, w_0^{-1} \pi_A(g), \pi_A(w_0^{-1} g)^{-1}) \\
= d^\ast \alpha_G(e, w_0^{-1} \pi_A(g), \pi_A(w_0^{-1} g)^{-1}, \pi_A(w_0^{-1} n^{-1} g)^{-1}) \\
- \alpha_G(w_0^{-1} \pi_A(g), \pi_A(w_0^{-1} g)^{-1}, \pi_A(w_0^{-1} n^{-1} g)^{-1}) \\
= -\alpha_G(g, w_0^{-1} g, w_0^{-1} n^{-1} g)
\]
which corresponds to the value of $\omega(g)$ on our favorite triple of points. Notice that we exploited the $w_0$-invariance of $\alpha_G$ to obtain the last equation. In this way we finally get
\[
(16) \quad \omega(g)(\infty, 0, e_1) = -\alpha_G(g, w_0^{-1} g, w_0^{-1} n^{-1} g).
\]
Computation of $\Omega_3 = d^{-1} \omega(e)$. This final step will finally give us the representative $\Omega_3$ of the image of $\alpha_n$ under the section $s$. By definition of $d^{-1}$ we have
\[
\Omega_3(\infty, 0, e_1, x) = d^{-1} \omega(e)(\infty, 0, e_1, x)
\]
\[
= -\omega(e)(0, e_1, x) + \omega(e)(\infty, e_1, x) - \omega(e)(\infty, 0, x) + \omega(e)(\infty, 0, e_1).
\]
The negative sign is again due to the weight used to define $d^{-1}$. By our computation of $\omega$ in [16], we immediately see that the fourth summand vanishes. In order to compute the remaining three summands, since we only know the value of $\omega(g)$ when evaluated on $(\infty, 0, e_1)$, we need to use transitivity and the invariance of $\omega$ to replace each summand by an appropriate evaluation on this particular triple. More precisely, given any triple $(x, y, z) \in (G/P)^3$, we can choose $g_{xyz} \in G$ to be an element such that
\[
g_{xyz}(\infty, 0, e_1) = x, \quad g_{xyz}(0, e_1) = y \quad \text{and} \quad g_{xyz}(e_1) = z.
\]
In particular, for any such choice we obtain
\[
\Omega_3(\infty, 0, e_1, x) = -\omega(g_{xyz}^{-1}) + \omega(g_{xyz}^{-1}) - \omega(g_{xyz}^{-1}) = 0.
\]
We have already considered choices of such isometries $g_{0e1z}, g_{0e0z}, g_{0e0x}$ in Section 4.1 [12] on each factor and further computed in Proposition 4.2 [12] still on every factor, all the corresponding $\pi_A$-projection of all the group elements appearing in this last expression of $\Omega_3$, that is for each of $g_{0e1z}, g_{0e0z}, g_{0e0x}$ and their left multiplication by $w_0^{-1}$ and $w_0^{-1}n^{-1}$. We can thus conclude that
\[
\Omega_3(\infty, 0, e_1, x) = \alpha \left( \frac{a}{\|x_i\|}, i = 1, k \right) \cdot \frac{a}{\|x_i\|, \|x_i\|^2, i = 1, k}
\]
\[
- \alpha \left( \frac{a}{\|x_i - x_i\|}, i = 1, k \right) \cdot \frac{a}{\|x_i - x_i\|, \|x_i + x_i\|, i = 1, k}
\]
\[
+ \alpha \left( \frac{a}{\|x_i\|}, i = 1, k \right) \cdot \frac{a}{\|x_i\|, \|x_i + x_i\|, i = 1, k}
\]
We now replace the homogeneous cocycle $\alpha$ by its inhomogeneous variant to obtain
\[
\Omega_3(\infty, 0, e_1, x) = \tilde{\alpha} \left( \frac{a}{\|x_i\|}, i = 1, k \right) \cdot \frac{a}{\|x_i\|, \|x_i\|^2, i = 1, k}
\]
\[
- \tilde{\alpha} \left( \frac{a}{\|x_i - x_i\|}, i = 1, k \right) \cdot \frac{a}{\|x_i - x_i\|, \|x_i + x_i\|, i = 1, k}
\]
\[
+ \tilde{\alpha} \left( \frac{a}{\|x_i\|}, i = 1, k \right) \cdot \frac{a}{\|x_i\|, \|x_i + x_i\|, i = 1, k}
\]
where for the last equality we have just used repeatedly that
\[
\tilde{a}(ab, c) = \tilde{a}(a, c) + \tilde{a}(b, c), \quad \tilde{a}(a, bc) = \tilde{a}(a, b) + \tilde{a}(a, c),
\]
\[
\tilde{a}(a^m, b^n) = nm \tilde{a}(a, b), \quad \tilde{a}(a, b) = -\tilde{a}(b, a), \quad \tilde{a}(a, a) = 0,
\]
for any $a, b, c \in A$ and $n, m \in \mathbb{R}$.

Finally observe that
\[
b_i(\infty, 0, e_1, x_i) = \|x_i\|.
\]
and
\[ b_1(0, e_1, x, \infty) = \|x_i\|/\|x_i - e_1\|, \]
so that we can rewrite \( \Omega \) as
\[ \Omega_{3, \infty, 0, e_1, x} = 4\delta \left( (a_{0,0,e_1,x_i})_{i=1}^k (a_{0,0,e_1,x,\infty})_{i=1}^k \right), \]
which by 3-transitivity, the \( G \)-invariance of \( \Omega(e) \) and the Isom(\( \mathbb{H}^n \)) -invariance of the crossratios \( b_i \) finishes the proof of the theorem.

4.4. Proof of injectivity of the comparison map (Theorem 3) for products of isometry groups of real hyperbolic space. Recall that the bounded (measurable) cohomology of a group or its action on its Furstenberg boundary, denoted respectively by \( H^*_m(G) \) and \( H^*_m(G \curvearrowright G/P) \), is defined by considering the co-complexes of essentially bounded functions. The usual inclusion \( L^\infty \hookrightarrow L^0 \) induces comparison maps between the bounded and unbounded measurable cohomology groups, so that we obtain a commutative diagram
\[ H^*_m(G \curvearrowright G/P) \longrightarrow H^*_m(G) \]

Since \( P \) is amenable, the upper horizontal map is an isomorphism by [6] Theorem 2. As for the vertical right arrow, it is still a mysterious open conjecture that it should be an isomorphism [10] Problem A. In this section we are going to prove that such isomorphism holds in degree 3 for products of isometry groups of hyperbolic spaces.

**Lemma 17.** Let \( G \) a product of isometry groups of real hyperbolic spaces and let \( G^0 \) the connected component of the identity. The comparison maps
\[ c_G : H^3_{m,b}(G \curvearrowright G/P) \longrightarrow H^3_m(G^0 \curvearrowright G/P) \]
and
\[ c_G : H^3_{m,b}(G \curvearrowright G/P) \longrightarrow H^3_m(G \curvearrowright G/P) \]
are injective.

**Proof.** If \( f \in L^\infty((G/P)^n) \) is a \( G \)-invariant or \( G^0 \)-invariant cocycle representing a cohomology class in \( H^3_{m,b}(G \curvearrowright G/P) \), respectively \( H^3_{m,b}(G^0 \curvearrowright G/P) \), lying in the kernel of this comparison map, then it is the coboundary \( f = \delta h \) of some not necessarily bounded \( G \)-invariant, respectively \( G^0 \)-invariant \( h \in L^0((G/P)^3) \). But the action of \( G \) on triples of distinct points has one orbit, while the action of \( G^0 \) has a finite number of orbits (since for \( G = \text{Isom}^+((\mathbb{H}^n)) \) there is 1 orbit for \( n \geq 3 \) and 2 orbits for \( n = 2 \)), so any invariant cochain in degree 2, and hence \( h \), is bounded in either cases. \( \square \)

**Proposition 18.** For any \( \alpha \neq 0 \), the cocycle
\[ s(\alpha_a) : (x_0, \ldots, x_3) \mapsto \alpha_a \log(b(x_0, x_1, x_2, x_3)), \log(b(x_1, x_2, x_3, x_0)), \]
where \( s \) is the section exhibited in Theorem 2 is unbounded.

Before proving the unboundedness we observe that this is sufficient to prove the injectivity statement for \( G \), since any cohomology class in \( NH^3_m(G \curvearrowright G/P) \) is represented, by Theorem 2, up to a coboundary, by a cocycle as in the proposition. But we saw in the proof of Lemma 17 that any coboundary in degree 3 is bounded, and the sum of a bounded and unbounded function is clearly unbounded. We thus have:
Corollary 19. The image of the comparison map
\[ H^3_{m,b}(G \rtimes G/P) \rightarrow H^3_{m}(G \rtimes G/P) \]
intersects the kernel \( NH^3_{m}(G \rtimes G/P) \) only in the zero class.

Observe that Lemma [17] and Corollary [19] prove Theorem [3] for \( G \). In fact, by looking at diagram [17], Monod’s conjecture is equivalent to the injectivity of the comparison map
\[ H^3_{m,b}(G \rtimes G/P) \rightarrow H^3_{m}(G \rtimes G/P) \]
and the triviality of the intersection between the image and \( NH^3_{m}(G \rtimes G/P) \).

To complete the proof of Theorem 3 for \( G_0 \), it remains to prove:

Lemma 20. The image of the comparison map
\[ H^3_{m,b}(G^0 \rtimes G/P) \rightarrow H^3_{m}(G^0 \rtimes G/P) \]
intersects the kernel \( NH^3_{m}(G^0 \rtimes G/P) \) only in the zero class.

Proof. Observe that we have a commutative diagram
\[ \begin{array}{ccc}
H^*_{m}(G \rtimes G/P) & \rightarrow & H^*_{m}(G^0 \rtimes G/P) \\
\downarrow & & \downarrow \\
H^*_{m}(G) & \rightarrow & H^*_{m}(G^0),
\end{array} \]
In particular, we get an inclusion of kernels
\[ NH^*_{m}(G \rtimes G/P) \hookrightarrow NH^*_{m}(G^0 \rtimes G/P). \]

By Monod’s Theorem [1] both spaces have the same finite dimension in each degree, so this inclusion is an isomorphism. Now because \( G^0 \) is normal in \( G \), the horizontal injections in the commutative diagram [19] admit left inverses by averaging over \( G/G^0 \). By construction, the corresponding diagram commutes and these left inverses have the obvious property of sending cohomology classes representable by bounded cocycles to cohomology classes representable by bounded classes. The lemma now follows from the corresponding statement for \( G \), established in Corollary [19].

Proof of Proposition 18. The group \( \langle \wedge^2(a) \rangle^* \) has as basis the 2 by 2 determinants on the projections on pairs of factors in \( a = \oplus_{i=1}^k a_i \), which viewed as inhomogeneous cocycles on \( A \) take the form
\[ \alpha_{ij} : A \times A \rightarrow \mathbb{R}, \quad \alpha_{ij}(a,a') := \det \left( \begin{array}{cc} \log |a_i| & \log |a'_j| \\ \log |a_j| & \log |a'_i| \end{array} \right), \]
for \( 1 \leq i < j \leq k \), \( a = (a_1, \ldots, a_k), a' = (a'_1, \ldots, a'_k) \in A \). We denote by \( \Omega_{ij} : (G/P)^3 \rightarrow \mathbb{R} \) the image of \( \alpha_{ij} \) under the section \( s \) of Theorem 2. In particular,
\[ \Omega_{ij}(\infty, 0, e_1, x) = \det \left( \begin{array}{cc} \log \|x_i\| & \log \|e_1 - x_i\| \\ \log \|x_j\| & \log \|e_1 - x_j\| \end{array} \right), \]
for any \( x = (x_1, \ldots, x_k) \in G/P = \Pi G_i/P_i \). Any class \( \Omega \) in the image of the section \( s \) of Theorem 2 is a linear combination \( \Omega = \sum_{i<j} t_{ij} \Omega_{ij} \). We assume that \( \Omega \neq 0 \) so that at least one of the coefficients \( t_{ij} \neq 0 \). By symmetry we can suppose that \( t_{12} \neq 0 \). We claim that there exists a subset of positive measure of \( G_2/P_2 \times \cdots \times G_k/P_k \) satisfying
\[ \sum_{j=2}^k t_{1j} \log \|x_j\| > 0. \]
Indeed, just choose \( x_j \in G_j/P_j = \mathbb{R}^{n_j-1} \) such that \( t_{12} \log \| x_2 \| > 0 \) for \( j = 2 \), and \( t_{1j} \log \| x_j \| \geq 0 \) for \( 3 \leq j \leq k \). This is simply achieved by choosing \( x_j \) such that
\[
\| x_j \| > 1 \quad \text{if} \quad t_{1j} > 0,
\]
\[
\| x_j \| < 1 \quad \text{if} \quad t_{1j} < 0.
\]
Now for any such \((x_2, \ldots, x_k)\), consider \( x = (x_1, \ldots, x_k) \in G/P \) with \( x_1 \in G_1/P_1 = \mathbb{R}^{n_1-1} \). We have
\[
\Omega(x; e_0, e_1, x) = \sum_{j=2}^{k} t_{1j} \det \left( \begin{array}{cc} \log \| x_i \| & \log \| e_1 - x_i \| \\ \log \| x_j \| & \log \| e_1 - x_j \| \end{array} \right) + \text{a constant depending on } x_2, \ldots, x_k \]
\[
= \log \| x_1 \| (\sum_{j=1}^{k} t_{1j} \log \| e_1 - x_j \|) - \log \| e_1 - x_1 \| (\sum_{j=1}^{k} t_{1j} \log \| x_j \|) + C_3
\]
where \( C_1, C_2, C_3 \) depend solely on \( x_2, \ldots, x_k \) which have been chosen so that \( C_2 \neq 0 \). Letting now \( x_1 \) tend to \( e_1 \) shows that \( \Omega \) is unbounded. \( \square \)

4.5. Proof of Theorem 4
As before, let \( \alpha_a \in \wedge^2 \mathfrak{a}^* \) be an alternating form, \( \widetilde{\alpha} : A^2 \to \mathbb{R} \) be the corresponding inhomogenous 2-cocycle
\[
\widetilde{\alpha}(a_1, a_2) := \alpha_a(\log a_1, \log a_2)
\]
and \( \alpha : A^3 \to \mathbb{R} \) be the homogenous \( A \)-invariant cocycle
\[
\alpha(a_0, a_1, a_2) = \widetilde{\alpha}(a_0^{-1}a_1, a_1^{-1}a_2).
\]
The cocycle
\[
\alpha_G : G^3 \to \mathbb{R}
\]
defined by
\[
\alpha_G(g_0, g_1, g_2) = \alpha(\pi_A(g_0), \pi_A(g_1), \pi_A(g_2))
\]
again corresponds to \( \alpha_a \) in \( H^2(C^{2,1}, d^t) \cong H^2_m(P) \cong H^2(\mathbb{A}) \cong (\wedge^2 \mathfrak{a})^* \). According to Lemma 8, we need to follow \( \overline{\alpha_G} \) through the maps
\[
\overline{\alpha_G} \in C^{2,2} \quad d^{-1} \overline{\alpha_G} = d^t \beta \in C^{2,3}
\]
\[
h^{2,3}(d^t \alpha_G) =: \beta \in C^{1,3} \quad d^{-1} \beta = d^t \omega \in C^{1,4}
\]
\[
h^{1,4}(d^{-1} \beta) =: \omega \in C^{0,4} \quad d^{-1} \omega =: \Omega_4 \in C^{0,5},
\]
and evaluate \( \Omega_4 \) on \( e \). As before we are going to subdivide the diagram chase into several steps.

Computation of \( \beta \). By Lemma 16 we know that \( d^{-1} \alpha_G \) is as an element of \( L^0(G^3) = C^{2,3} \) given by
\[
d^{-1} \alpha_G(g_0, g_1, g_2) = - \alpha_G(nw_0^{-1}g_0, nw_0^{-1}g_1, nw_0^{-1}g_2) + \alpha_G(n^{-1}g_0, w_0^{-1}g_1, w_0^{-1}g_2)
\]
\[
- \alpha_G(g_0, g_1, g_2)
\]
\[
= - \alpha_G(w_0^{-1}g_0, w_0^{-1}g_1, w_0^{-1}g_2),
\]
where we used the \( P \)-invariance of \( \alpha_G \). Recall that the signs in the above formula are due to the weight used to define \( d^t \).
Observe that the homotopy $h^{p+1,3} : C^{p,3} \to C^{p-1,3}$ is as a map from $L^0(G^{p+1}) = C^{p,3} \to C^{p-1,3} = L^0(G^p)$ simply given by

$$h^{p+1,3}(\gamma)(g_0, \ldots, g_{p-1}) = \gamma(e, g_0, \ldots, g_{p-1}),$$

for $\gamma \in G^{p+1}$. Applying this to $\gamma = d^\gamma \alpha$ leads to

$$\beta(g_0, g_1) = h^{2,3}(d^\gamma \alpha)(g_0, g_1) = (d^\gamma \alpha G)(e, g_0, g_1) = -\alpha_G(e, w_0^{-1}g_0, w_0^{-1}g_1).$$

**Computation of $\omega$.** Now we want to compute the differential $d^\gamma \beta \in C^{1,4}$. By 3-transitivity any 4-tuple of points in $G/P$ is in the orbit of $\infty, 0, e_1, x$, for some $x \in G/P$. Since $d^\gamma \beta$ is $G$-invariant it will thus be sufficient to evaluate it on such 4-tuples. As above, for any triple of distinct points $x, y, z \in G/P$, we let $g_{xyz}$ be an element in $G$ such that $g_{xyz, \infty} = x, g_{xyz, 0} = y, g_{xyz, e_1} = z$. We can thus express $d^\gamma \beta$ as

$$d^\gamma \beta(g_0, g_1)(\infty, 0, e_1, x) = \beta(g_0, g_1)(0, e_1, x) - \beta(g_0, g_1)(\infty, e_1, x) + \beta(g_0, g_1)((\infty, 0, x) + \beta(g_0, g_1)((\infty, 0, e_1) = \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 1}^{-1}g_1) - \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 1}^{-1}g_1) + \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 1}^{-1}g_1) + \beta(g_0, g_1).$$

Finally we obtain

$$\omega(g)(\infty, 0, e_1, x) = h^{1,4}(d^\gamma \beta)(g)(\infty, 0, e_1, x) = d^\gamma \beta(e, g)(\infty, 0, e_1, x) = \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 1}^{-1}g_1) - \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 1}^{-1}g_1) + \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 1}^{-1}g_1),$$

where we have omitted the last term $\beta(e, g)$ since it vanishes.

**Computation of $d^\gamma \omega$.** We are finally ready to compute our desired cocycle $\Omega_4 = d^\gamma \omega(e)$. Before starting, recall that an oriented 5-tuple of distinct points is in the $G$-orbit of $\infty, 0, e_1, x, y$, for some $x \neq y \in G/P \setminus \{\infty, 0, e_1\}$. For such $x, y$, we set

$$\lambda(x, y) := g_{\infty, x, 0}^{-1}g_0, \mu(x, y) := g_{\infty, x, 1}^{-1}g_0, \nu(x, y) := g_{\infty, x, 1}^{-1}g_0,$$

where $g_{\infty, x, 0}$, $g_{\infty, x, 1}$ and $g_{\infty, x, 1}$ are chosen as above. We have

$$\Omega_4(\infty, 0, e_1, x, y) = d^\gamma \omega(e)(\infty, 0, e_1, x, y) = -\omega(e)(\infty, 0, e_1, x, y) + \omega(e)(\infty, 0, e_1, y) - \omega(e)(\infty, 0, x, y) + \omega(e)(\infty, 0, e_1, y) = -\omega(e)(\infty, 0, e_1, x, y) + \omega(e)(\infty, 0, e_1, y) = -\omega(e)(\infty, 0, e_1, x, y) + \omega(e)(\infty, 0, e_1, y) = -\omega(g_{\infty, x, 0})\Omega_4(\infty, 0, e_1, \lambda(x, y)),$$

where we used $G$-invariance of $\omega$ for the last equality and the fact that both the fourth and the fifth summands vanish. We will now compute the three remaining summands separately.

**Computation of $\omega(g_{\infty, x, 0}^{-1})(\infty, 0, e_1, \nu(x, y))$.** We have that

$$\omega(g_{\infty, x, 0}^{-1})(\infty, 0, e_1, \nu(x, y)) =$$

$$= \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 0}^{-1}g_0) - \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 0}^{-1}g_0) + \beta(g_{\infty, x, 0}^{-1}g_0, g_{\infty, x, 0}^{-1}g_0) = -\alpha_G(e, w_0^{-1}g_{\infty, x, 0}^{-1}g_0, w_0^{-1}g_{\infty, x, 0}^{-1}g_0) + \alpha_G(e, w_0^{-1}g_{\infty, x, 0}^{-1}g_0, w_0^{-1}g_{\infty, x, 0}^{-1}g_0) - \alpha_G(e, w_0^{-1}g_{\infty, x, 0}^{-1}g_0, w_0^{-1}g_{\infty, x, 0}^{-1}g_0).$$
The six different projections on $A$ have been exhibited in Proposition 15 coordinatewise so that we simply obtain

$$\omega(g_{01c,x}^{-1})(\infty, 0, e_1, \nu(x,y)) = -\tilde{\alpha} \left( a \left[ \frac{y-x}{\|y-x\|} \right] \right)_i \left( a \left[ \frac{-2\|x-y\|}{\|x-y\|+1} \right] \right)_i \left( a \left[ \frac{2\|e\|}{\|e\|+1} \right] \right)_i$$

$$+ \tilde{\alpha} \left( d \left[ \frac{y-x}{\|y-x\|} \right] \right)_i \left( d \left[ \frac{-2\|x-y\|}{\|x-y\|+1} \right] \right)_i \left( d \left[ \frac{2\|e\|}{\|e\|+1} \right] \right)_i$$

$$- \tilde{\alpha} \left( d \left[ \frac{y-x}{\|y-x\|} \right] \right)_i \left( d \left[ \frac{-2\|x-y\|}{\|x-y\|+1} \right] \right)_i \left( d \left[ \frac{2\|e\|}{\|e\|+1} \right] \right)_i$$

$$= -\tilde{\alpha} \left( a \left[ \frac{y-x}{\|y-x\|} \right] \right)_i \left( a \left[ \frac{-2\|x-y\|}{\|x-y\|+1} \right] \right)_i \left( a \left[ \frac{2\|e\|}{\|e\|+1} \right] \right)_i .$$

**Computation of $\omega(g_{01c,x}^{-1})(\infty, 0, e_1, \mu(x,y))$.** We have that

$$\omega(g_{01c,x}^{-1})(\infty, 0, e_1, \mu(x,y)) = \beta(g_{01c,10}(x,y)\cdot g_{01c,10}^{-1}) - \beta(g_{01c,10}(x,y)\cdot g_{01c,10}^{-1})$$

$$+ \beta(g_{01c,10}^{-1}(x,y)\cdot g_{01c,10}^{-1}(x,y))$$

$$= -\alpha_G(e, w_0^{-1}g_{01c,10}(x,y)\cdot w_0^{-1}g_{01c,10}^{-1}(x,y))$$

$$+ \alpha_G(e, w_0^{-1}g_{01c,10}^{-1}(x,y)\cdot w_0^{-1}g_{01c,10}^{-1}(x,y))$$

We have already computed these six different projections on $A$ coordinatewise in Proposition 13 so that we can just conclude that

$$\omega(g_{01c,x}^{-1})(\infty, 0, e_1, \mu(x,y))$$

$$= -\tilde{\alpha} \left( a \left[ \frac{y-x}{\|y-x\|} \right] \right)_i \left( a \left[ \frac{-2\|x-y\|}{\|x-y\|+1} \right] \right)_i \left( a \left[ \frac{2\|e\|}{\|e\|+1} \right] \right)_i$$

$$- \tilde{\alpha} \left( a \left[ \frac{y-x}{\|y-x\|} \right] \right)_i \left( a \left[ \frac{-2\|x-y\|}{\|x-y\|+1} \right] \right)_i \left( a \left[ \frac{2\|e\|}{\|e\|+1} \right] \right)_i$$

$$= -\tilde{\alpha} \left( a \left[ \frac{y-x}{\|y-x\|} \right] \right)_i \left( a \left[ \frac{-2\|x-y\|}{\|x-y\|+1} \right] \right)_i \left( a \left[ \frac{2\|e\|}{\|e\|+1} \right] \right)_i .$$

**Computation of $\omega(g_{01c,x}^{-1})(\infty, 0, e_1, \lambda(x,y))$.** We have that

$$\omega(g_{01c,x}^{-1})(\infty, 0, e_1, \lambda(x,y))$$

$$= \beta(g_{0c_1,0}(x,y)\cdot g_{0c_1,0}^{-1}) - \beta(g_{0c_1,0}(x,y)\cdot g_{0c_1,0}^{-1})$$

$$+ \beta(g_{0c_1,0}^{-1}(x,y)\cdot g_{0c_1,0}^{-1}(x,y))$$

$$= -\alpha_G(e, w_0^{-1}g_{0c_1,0}(x,y)\cdot w_0^{-1}g_{0c_1,0}^{-1}(x,y))$$

$$+ \alpha_G(e, w_0^{-1}g_{0c_1,0}^{-1}(x,y)\cdot w_0^{-1}g_{0c_1,0}^{-1}(x,y))$$

We have already recorded these six different projections on $A$ coordinatewise in Proposition 14 so that we can just conclude that
the complex field. For \( G = SL(3, \mathbb{K}) \), the subgroup of upper triangular matrices with entries in \( \mathbb{K} \), then naturally identified with the space of complete flags \( FL \) is a sequence of nested subspaces

\[
\omega(g_{\infty,0})(\infty, 0, e_1, \lambda(x, y)) = -\tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( \frac{a_{x_i, ||x_i||}}{||x_i||} \right)_{i=1}^k \right)
+ \tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( \frac{a_{x_i, ||x_i||}}{||x_i||} \right)_{i=1}^k \right)
- \tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( a_{x_i, ||x_i||} \right)_{i=1}^k \right)
= -\tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( a_{x_i, ||x_i||} \right)_{i=1}^k \right).
\]

**Conclusion.** We can finally compute \( \Omega_4 \) putting everything together:

\[
\Omega_4(\infty, 0, e_1, x, y) = \tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( \frac{a_{x_i, ||x_i||}}{||x_i||} \right)_{i=1}^k \right)
- \tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( a_{x_i, ||x_i||} \right)_{i=1}^k \right)
+ \tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( a_{x_i, ||x_i||} \right)_{i=1}^k \right)
= \tilde{\alpha} \left( \left( \frac{a_{x_i, x_{i+1}}}{x_{i+1} + x_i} \right)_{i=1}^k, \left( a_{x_i, ||x_i||} \right)_{i=1}^k \right).
\]

Now the second summand of this last expression is the coboundary of the function \( \beta : (G/P)^4 \to \mathbb{R} \) defined by

\[
\beta(\infty, 0, e_1, x) = \tilde{\alpha} \left( \left( a_{||x_i||} \right)_{i=1}^k, \left( a_2 \right)_{i=1}^k \right)
\]
and the first summand is exactly the expression claimed in Theorem 4.

5. The spectral sequence for \( SL(3, \mathbb{K}) \) for \( \mathbb{K} = \mathbb{R}, \mathbb{C} \)

5.1. Configurations of flags in general position. Let \( \mathbb{K} \) be either the real or the complex field. For \( G = SL(3, \mathbb{K}) \), we take as minimal parabolic subgroup \( P \) the subgroup of upper triangular matrices with entries in \( \mathbb{K} \). The quotient \( G/P \) is then naturally identified with the space of complete flags \( FL(3, \mathbb{K}) \). Recall that a *complete flag* \( F \in FL(3, \mathbb{K}) \) in \( \mathbb{K}^3 \) is a sequence of nested subspaces

\[
F : (0) = F^0 \subset F^1 \subset F^2 \subset F^3 = \mathbb{K}^3,
\]
where each linear subspace \( F^i \) has dimension \( \dim_{\mathbb{K}} F^i = i \). In this case, the flag is completely determined by the 1 and 2 dimensional subspaces. For this reason, we can alternatively denote a complete flag \( F \) by a pair \( (p, \ell) \), where \( p \in P^2(\mathbb{K}) \) is a point and \( \ell \subset P^2(\mathbb{K}) \) is a line passing through \( p \). The subgroup \( P \) is the stabilizer of the *canonical flag* \( F_{can} \):

\[
F_{can} : (0) \subset (e_1) \subset (e_1, e_2) \subset \mathbb{K}^3,
\]
where \( \{e_1, e_2, e_3\} \) denotes the canonical basis of \( \mathbb{K}^3 \).
The natural action of $\mathrm{SL}(3, \mathbb{K})$ on $\mathcal{F}\mathcal{L}(3, \mathbb{K})$ induces a diagonal action on the product $\mathcal{F}\mathcal{L}(3, \mathbb{K})^{d+1}$. The space of configurations of $(d+1)$-tuples of flags is the quotient
\[ C_{d+1}(\mathcal{F}\mathcal{L}(3, \mathbb{K})) := \mathcal{F}\mathcal{L}(3, \mathbb{K})^{d+1} / \mathrm{SL}(3, \mathbb{K}). \]
Given a $(d+1)$-tuple $(F_0, \ldots, F_d)$ of flags in $\mathcal{F}\mathcal{L}(3, \mathbb{K})$, we denote its configuration class by $[F_0, \ldots, F_d]$.

The cocycles we are going to define will never be defined everywhere but only on the measurable dense subset of flags in general position. A $(d+1)$-tuple of flags $F_0, \ldots, F_d$ is in general position if, roughly speaking, the dimension of any possible intersection or subspace generated by the spaces from the tuple matches the expected dimension. In dimension 3, we can simply (and equivalently) define a $(d+1)$-tuple of flags $F_0, \ldots, F_d$, with $F_i = (p_i, \ell_i)$, to be in general position if

- $p_1, p_2, p_3$ are not aligned whenever $|\{i_1, i_2, i_3\}| = 3$,
- $\ell_{i_1}, \ell_{i_2}, \ell_{i_3}$ do not intersect in a unique point whenever $|\{i_1, i_2, i_3\}| = 3$,
- $p_i \notin \ell_j$ whenever $i \neq j$.

This is precisely the notion of very generic configurations from $\S$. Since the condition of general position is invariant along the $\mathrm{SL}(3, \mathbb{K})$-orbits, it makes sense to speak about configurations of tuples of flags in general position. We denote this space by $C_{d+1}^\text{gen}(\mathcal{F}\mathcal{L}(3, \mathbb{K}))$.

We follow Falbel and Wang to introduce coordinates on the spaces of triples and 4-tuples of flags in general position and refer to $\S$ for more details. In $\S$ the definitions are given for tuples of complex flags, but it should be clear that the same definitions work for tuples of real flags as well.

**Definition 21.** Let $F_0, F_1, F_2 \in \mathcal{F}\mathcal{L}(3, \mathbb{K})$ be a triple of flags in general position, where $F_i = (p_i, \ell_i)$ for $i = 0, 1, 2$. We define the **triple ratio** associated to them as
\[ \tau(F_0, F_1, F_2) := -[\ell_0 \cap \ell_1, \ell_1 \cap \ell_2, p_1, p_1 \cap p_2, p_2]_{\ell_1} \in \mathbb{K}^* \setminus \{-1\}, \]
where $\ell_i \cap \ell_j$ is the intersection point between the lines and $p_0, p_2$ is the line passing through $p_0$ and $p_2$. The notation $[\ldots]$ refers to the usual cross ratio computed on the line $\ell_1$. (See Figure $\S$)

This is, in view of $\S$ Lemma 3.5], equivalent the original definition given by Falbel and Wang. The triple ratio $\tau$ of $(F_0, F_1, F_2)$ changes equivariantly with respect the action of a permutation $\sigma \in S_3$; more precisely, the triple ratio of $(F_{\sigma(0)}, F_{\sigma(1)}, F_{\sigma(2)})$ is given by $\tau^\sigma$, where $\varepsilon(\sigma)$ is the sign of the permutation $\sigma$.

The triple ratio remains constant along $G$-orbits, hence it descends to a well-defined numerical invariant of configuration classes. By $\S$ Proposition 3.1] it determines an identification between the space of configurations in general position $C_{d+1}^\text{gen}(\mathcal{F}\mathcal{L}(3, \mathbb{K}))$ and $\mathbb{K}^* \setminus \{-1\}$. Indeed, since $\mathrm{SL}(3, \mathbb{K})$ acts transitively on the triples given by two generic flags and one generic point in $P^2(\mathbb{K})$, any triple of flags lies in the same orbit as a triple $(F_0, F_1, F_2)$, where
\begin{align}
F_{\text{can}} &= F_0 = \langle e_1 \rangle \subset \mathbb{K}^3, \\
F_1 &= \langle e_3 \rangle \subset \mathbb{K}^3, \\
(F_2)^4 &= \langle e_1 + e_2 + e_3 \rangle.
\end{align}
Now for any $\tau \in \mathbb{K}^* \setminus \{-1\}$ there exists a unique way to complete $\langle e_1 + e_2 + e_3 \rangle$ to a flag $F_2$ such that $\tau(F_0, F_1, F_2) = \tau$. More precisely, the flag $F_2$ is given by
\begin{align}
F_2 &= \langle e_1 + e_2 + e_3 \rangle \subset \mathbb{K}^3, \\
(F_2)^4 &= \langle (\tau + 1)e_1 + \tau e_2 \rangle \subset \mathbb{K}^3.
\end{align}

**Definition 22.** Let $F_0, F_1, F_2, F_3 \in \mathcal{F}\mathcal{L}(3, \mathbb{K})$ be a 4-tuple of flags in general position, where $F_i = (p_i, \ell_i)$ for $i = 0, \ldots, 3$. Let $(i, j, s, t)$ be an even permutation of $(0, 1, 2, 3)$. The **$(i,j)$-cross ratio** is defined as
\[ z_{ij} := [\ell_i, p_i \cdot p_j, p_i \cdot p_s, p_i \cdot p_t]_{p_i}, \]
where \( p_i \cdot p_j \) refers to the line passing through \( p_i \) and \( p_j \). The cross ratio \([\cdot, \cdot, \cdot, \cdot]_{p_i}\) is computed on the line parametrizing all the lines passing through \( p_i \). (See Figure 4)

The family of cross ratios given by Definition 22 is constant along the \( G \)-orbits, hence it descends naturally to configuration classes. By [8, Proposition 3.1] the choice of cross ratios \((z_{01}, z_{10}, z_{23}, z_{32})\) determines an isomorphism between the space of configurations in general position \( C_{\operatorname{gen}}^4(\mathcal{F}_1(3, K)) \) and \((K^* \setminus \{1\})^4\). Indeed, any 4-tuple of generic flags lies in the same orbit as \((F_0, F_1, F_2, F_3)\), where \( F_0, F_1, (F_2)^3 \) are as in (20). Now for any \( a, b, c, d \in K^* \setminus \{1\} \) there exist a unique way to complete \((F_2)^3\) to a flag and a unique flag \( F_3 \) such that the list of cross ratios \((z_{01}, z_{10}, z_{23}, z_{32})\) is given by \((a, b, c, d)\). Indeed, it is sufficient to take

\[
F_0 = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset K^3, \\
F_1 = \langle e_3 \rangle \subset \langle e_3, e_2 \rangle \subset K^3, \\
F_2 = \langle e_1 + e_2 + e_3 \rangle \subset \langle e_1 + e_2 + e_3, (\tau_{012} + 1)e_1 + \tau_{012}e_2 \rangle \subset K^3, \\
F_3 = \langle ab e_1 + a e_2 + e_3, (\tau_{013} + 1)b e_1 + \tau_{013}e_2 \rangle \subset K^3.
\]

Here \( \tau_{ijk} \) is the triple ratio of the configurations \([F_i, F_j, F_k]\). Notice that we can express such triple ratios in terms of \( a, b, c, d \). By Falbel and Wang [8, Equation 3.4.2] we have

\[
\tau_{123} = \frac{d(c - 1)}{b(d - 1)}, \quad \tau_{023} = \frac{a(c - 1)}{c(d - 1)}, \\
\tau_{013} = \frac{b(a - 1)}{d(b - 1)}, \quad \tau_{012} = \frac{c(a - 1)}{a(b - 1)}.
\]

5.2. The invariant cohomology of the maximal torus. In virtue of Theorem 1 we need first to compute the invariant cohomology of the (dual) Lie algebra of a maximal abelian torus \( A \) in \( G \). We can choose \( A \) as the set of diagonal matrices with positive real entries, namely

\[
A = \{ \text{diag}(\lambda, \mu, \nu) \mid \lambda, \mu, \nu > 0, \lambda \mu \nu = 1 \},
\]

with associated Lie algebra given by the space of diagonal traceless matrices with real entries, that is

\[
a := \{ \text{diag}(x, y, z) \mid x + y + z = 0 \}.
\]
Lemma 23. If $w_0$ represents the longest element in the Weyl group of $A$, the $w_0$-invariant cohomology of the dual algebra $\mathfrak{a}^*$ is given by

$$\dim(\wedge^\ell \mathfrak{a}^*)^{w_0} = \begin{cases} 1, & \text{for } \ell = 1, \\ 0, & \text{for } \ell \geq 2. \end{cases}$$

Proof. Since $\mathfrak{a}$ has dimension equal to 2, it is sufficient to check the cases when $\ell$ is equal to either 1 or 2.

We start by noticing that the Weyl group $W$ of $G$ is isomorphic to $W \simeq D_6$. The longest element $w_0$ acts as a reflection on $\mathfrak{a}^*$. As a consequence it admits a one dimensional eigenspace $E(1)$ with eigenvalue 1 and a one dimensional eigenspace $E(-1)$ with eigenvalue $-1$. Thus the dimension of the $w_0$-invariant subspace of $\mathfrak{a}^*$ is one and the statement holds for $\ell = 1$.

When $\ell = 2$, it is immediate to see that there is no invariant elements in $\wedge^2 \mathfrak{a}^*$ because any non trivial vector must be equivariant with respect to the sign of $w_0$. Thus the $w_0$-invariant cohomology vanishes in degree 2 and the statement follows. □

Remark. The proof of the previous lemma can be made more concrete via the choice of a basis for the space $\mathfrak{a}^*$. We denote by $e_i^*$ the linear functional defined on $\mathfrak{a}$ whose value is the $i$-th diagonal component of the matrix. A basis for $\mathfrak{a}^*$ is given by $B = \{e_1^* - e_3^*, e_2^*\}$.

The adjoint action of $w_0$ (whose expression is explicitely given below) on the basis $B$ is given by

$$\text{Ad}(w_0)(e_2^*) = e_2^*, \quad \text{Ad}(w_0)(e_1^* - e_3^*) = -(e_1^* - e_3^*).$$

We immediately see that the only vector fixed by $w_0$ is $e_2^*$. As a consequence

$$\mathfrak{a}^*^{w_0} \simeq \langle e_2^* \rangle.$$

In a similar way, when $\ell = 2$, an invariant element must be the exterior product of two elements of $\mathfrak{a}^*$ that are either both invariant or both equivariant. As a consequence there is no invariant vector with respect to the action of $w_0$ in degree 2.

5.3. The $\pi_A$-projection for $\text{SL}(3, \mathbb{K})$. As we did for products of isometries of real hyperbolic spaces, here we need to give an explicit characterization of the projection map $\pi_A$. As before, we choose $A$ to be the maximal abelian subgroup of diagonal matrices with positive real entries and $P < G$ is the subgroup of upper triangular matrices with entries in $\mathbb{K}$. Furthermore we fix $K = \text{SO}(3, \mathbb{R})$, if $\mathbb{K} = \mathbb{R}$, or $K = \text{SU}(3)$, if $\mathbb{K} = \mathbb{C}$. The unipotent radical of $P$ is the subgroup $N$ of triangular matrices having ones on the diagonal.

Again by the Iwasawa decomposition, we can write any element $g \in G$ in a unique way as a product $a n k$, where $a \in A$, $n \in N$, $k \in K$. Thanks to such decomposition we are allowed to define

$$\pi_A : G \to A, \quad g = a n k \mapsto a.$$

Lemma 24. For any $g \in G$ consider the matrix $s := gg^*$, where either $g^* = g^t$ if $\mathbb{K} = \mathbb{R}$ or $g^* = \overline{g^t}$ if $\mathbb{K} = \mathbb{C}$. If $(s_{ij})_{i,j=1,2,3}$ are the entries of $s$, the $A$-projection of $g$ is given by

$$\pi_A(g) = \text{diag}(a_{11}, a_{22}, a_{33}),$$
where
\[
\begin{align*}
a_{33} &= \sqrt{s_{33}}, \\
a_{22} &= \sqrt{\frac{s_{22}s_{33} - |s_{23}|^2}{s_{33}}}, \\
a_{11} &= \frac{1}{a_{22}a_{33}}.
\end{align*}
\]

Proof. The proof is straightforward. Let \( X \) be the Riemannian symmetric space associated to \( G \). For the convenience of the reader we remind that \( X \) is the set of positive definite either symmetric matrices, when \( K = \mathbb{R} \), or Hermitian matrices, when \( K = \mathbb{C} \), with determinant equal to one.

The natural projection map is given by
\[
p : G \to X, \quad p(g) = gg^* = s.
\]

Let \( g = ank \) be the Iwasawa decomposition of the element \( g \). Recall that \( n \) is an upper triangular unipotent matrix with entries in \( K \) and \( a \) is diagonal with positive real entries. We have that
\[
s = gg^* = (ank)(ank)^* = ankk^*n^*a^* = (an)(n^*a^*),
\]
where we exploited the fact that \( k \) is either orthogonal or unitary. One can verify algebraically that Equation (24) is equivalent to the statement of the lemma. This concludes the proof. \( \square \)

5.4. Contracting homotopies and differentials for \( \text{SL}(3, K) \). We start by fixing as a representative of the longest element \( w_0 \) in the Weyl group the following matrix
\[
w_0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Recall that given a configuration \([F_0, F_1, F_2]\) with triple ratio \( \tau \), it is always possible to write a representative of such class as in Equation (21). We fix such a choice of representative and we call it standard normalization. In this context, a direct computation shows that a matrix fixing \( F_0 \) (the canonical flag) and sending \( F_1 \) to \( F_2 \) is given by
\[
n_\tau := \begin{bmatrix} \frac{1}{\tau} & \tau + 1 & 1 \\ 0 & \tau & 1 \\ 0 & 0 & 1 \end{bmatrix}.
\]

In a similar way, a matrix sending the pair \((F_0, F_1)\) to the pair \((F_1, F_2)\) is given by
\[
g_\tau := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & \tau + 1 & 1 \end{bmatrix}.
\]

Remark. Fixing the standard normalization is equivalent to say that a canonical representative for a configuration \([F_0, F_1, F_2]\) with triple ratio \( \tau \) is given by \((F_{can}, w_0F_{can}, n_\tau w_0F_{can})\), where \( F_{can} \) is the canonical flag and \( w_0 \) represents the longest element in the Weyl group.

As for the case of products, here contracting homotopies for the cocomplexes \((C^{p,q}_K, d^p)\), for \( q \geq 3 \), can easily be defined as follows:
\[
h^{p,q} : C^{p,q}_K \to C^{p-1,q}_K
\]
with
\[
h^{p,q}(f)(g_0, \ldots, g_{p-1})(F_{can}, w_0F_{can}, n_\tau w_0F_{can}, F_4, \ldots, F_q) := f(\epsilon, g_0, \ldots, g_{p-1})(F_{can}, w_0F_{can}, n_\tau w_0F_{can}, F_4, \ldots, F_q).
\]
The previous definition gives back a measurable function on a dense subset of $G^p$: Indeed, although $f$ is only defined on a subset of full measure of $G^{p+1}$, the evaluation on $(e, g_0, \ldots , g_{p-1})$ makes sense on a subset of full measure of $G^p$ by the $G$-invariance of $f$. Likewise, the evaluation on $(F_{can}, w_0 F_{can}, n_w w_0 F_{can}, F_1, \ldots , F_q)$ is allowed for almost all $(F_1, \ldots , F_q)$ by transitivity and by our choice of normalization. The resulting cochain is clearly $G$-invariant by construction and it holds that

$$h^{p+1,q} \circ d^p + d^p \circ h^{q,p} = \text{id}.$$ 

Before computing explicitly the differential $d^p : C^{p,2} \rightarrow C^{p,3}$, we want to point out that the lack a transitivity of the $\text{SL}(3, \mathbb{K})$-action on triples of flags has important consequences on the realization of the space $C^{p,3}$. In fact, since the action is no more transitive on triples (as it were for products of isometries of real hyperbolic spaces), the space of orbits does not boil down to a point, but it is actually not trivial. In fact, we know that it is parametrized by the triple ratio. As a consequence

$$C^{p,3} = L^0((G^{p+1}, L^0((G/P)^3))^G \cong L^0(G^{p+1} \times \mathbb{K}^{**}),$$

where the $\mathbb{K}^{**}$-factor, for $\mathbb{K}^{**} = \mathbb{K}^* \setminus \{-1\}$, is the contribution coming from the triple ratio.

**Lemma 25.** The differential

$$d^p : L^0(G^{p+1})^A \cong C^{p,2} \rightarrow L^0(G^{p+1} \times \mathbb{K}^{**}) \cong C^{p,3}$$

is given by

$$d^p(\beta)(g_0, \ldots , g_p)(\tau) = (-1)^{p+1}[\beta(g_0^{-1}g_0, \ldots , g_p^{-1}g_p) - \beta(n_{\beta}^{-1}g_0, \ldots , n_{\beta}^{-1}g_p) + \beta(g_0, \ldots , g_p)].$$

**Proof.** The proof is analogous to the one of Lemma 16. Any cochain $\beta \in L^0(G^{p+1})^A$ determines a cochain $\overline{\beta} \in C^{p,2} = L^0((G^{p+1}, L((G/P)^3))$ by setting

$$\overline{\beta}(g_0, \ldots , g_p)(h^{-1}F_{can}, h^{-1}w_0 F_{can}) := \beta(hg_0, \ldots , h_0 g_p).$$

Fixing our choice of the standard normalization, we can see that the differential $d^p(\beta) \in L^0(G^{p+1} \times \mathbb{K}^{**})$ is given by

$$d^p(\beta)(g_0, \ldots , g_p)(\tau) = d^p\overline{\beta}(g_0, \ldots , g_p)(F_{can}, w_0 F_{can}, n_w w_0 F_{can}),$$

where the latter evaluation makes sense because of the $G$-invariance of $d^p\overline{\beta}$.

By the definition of $d^p$ we have that

$$d^p\overline{\beta}(g_0, \ldots , g_p)(F_{can}, w_0 F_{can}, n_w w_0 F_{can}) = (-1)^{p+1}[\overline{\beta}(g_0, \ldots , g_p)(w_0 F_{can}, n_w w_0 F_{can}) - \overline{\beta}(g_0, \ldots , g_p)(F_{can}, n_w w_0 F_{can}) + \overline{\beta}(g_0, \ldots , g_p)(F_{can}, w_0 F_{can})].$$

By the way we defined $g_{\tau}$ we know that

$$(w_0 F_{can}, n_{\tau w_0 F_{can}}) = (g_{\tau F_{can}}, g_{\tau w_0 F_{can}}),$$

and thanks to the fact that $n_{\tau F_{can}} = F_{can}$, we can rewrite

$$d^p\overline{\beta}(g_0, \ldots , g_p)(F_{can}, w_0 F_{can}, n_{\tau w_0 F_{can}}) = (-1)^{p+1}[\overline{\beta}(g_0, \ldots , g_p)(g_{\tau F_{can}}, g_{\tau w_0 F_{can}}) - \overline{\beta}(g_0, \ldots , g_p)(n_{\tau F_{can}}, n_{\tau w_0 F_{can}}) + \overline{\beta}(g_0, \ldots , g_p)(F_{can}, w_0 F_{can})]$$

$$= (-1)^{p+1}[\beta(g_0^{-1}g_0, \ldots , g_p^{-1}g_p) + \beta(g_0, \ldots , g_p)].$$

and the statement is proved. \qed
5.5. Proof of Theorem 5. Let $\alpha_a \in \mathfrak{a}^*$ be a $w_0$-invariant linear functional. We consider $\tilde{\alpha} : A \to \mathbb{R}$ the inhomogeneous cocycle defined by

$$\tilde{\alpha}(a) = \alpha_a(\log a),$$

and let $\alpha : A^2 \to \mathbb{R}$ be its homogenized variant

$$\alpha(a_0, a_1) := \tilde{\alpha}(a_0^{-1}a_1).$$

We can consider the $G$-invariant extension of $\alpha$ to $G$ by precomposing with the projection $\pi_A$ of Lemma 24 that is

$$\alpha_G : G^2 \to \mathbb{R}, \quad \alpha(g_0, g_1) = \alpha_G(\pi_A(g_0), \pi_A(g_1)).$$

It is clear that $\alpha_G$ is $P$-invariant, thus it lies in $C^{1,1}$, and its cohomology class in $H^2(C^{1,1}, d') \cong H^1_P(P) \cong H^1_m(A) \cong \mathfrak{a}^*$ corresponds to $\alpha_a$.

By Proposition 10 we have to follow $\pi_G$ through the maps

$$\begin{align*}
\begin{array}{ccc}
\alpha_G & \in & C^{1,1} \\
& \mapsto & d^\ast\alpha_G = d' \beta \in C^{1,2} \\
\end{array}
\end{align*}$$

$$H^1(\alpha_G) =: \beta \in C^{0,2} \mapsto d^{-1} \beta =: \omega \in C^{0,1}.$$

Computation of $\beta$. By the definition of $H^1$, we have

$$\beta(g) = H^1(\alpha_G)(g) = \alpha_G(w_0^{-1}\pi_A(g), w_0^{-1}g).$$

Computation of $\omega$. In this case the computation has only two steps. Thus $\omega(e)$ will give us back the desired cocycle. In virtue of Equation (21) we can fix $F_{can}, w_0 F_{can}, n_\tau w_0 F_{can}$ as triple of flags with triple ratio $\tau$. On such configuration we will evaluate $\omega(e)$. By Lemma 25 we have that

$$\omega(e)(F_{can}, w_0 F_{can}, n_\tau w_0 F_{can}) = d^{-1} \beta(e)(F_{can}, w_0 F_{can}, n_\tau w_0 F_{can})$$

$$= -\beta(g_1^{-1}) + \beta(n_\tau^{-1}) - \beta(e)$$

$$= -\alpha_G(w_0^{-1}\pi_A(g_1^{-1}), w_0^{-1}g_1^{-1})$$

$$+ \alpha_G(w_0^{-1}\pi_A(n_\tau^{-1}), w_0^{-1}n_\tau^{-1}),$$

where we exploited the fact that $\beta(e)$ vanishes.

By Lemma 23 we know that $(\mathfrak{a}^*)^{\omega_0}$ is one dimensional and generated by the functional $e_3^\ast$. This implies that we can write $\alpha_a$ as a multiple of $e_3^\ast$, say $\lambda e_3^\ast$. In an analogous way, the cocycle $\tilde{\alpha}$ is a multiple of the logarithm of the second diagonal coordinate.

Using Lemma 24 one can verify that the second diagonal coordinate of the $A$-projections are given by

$$\pi_A(w_0^{-1}\pi_A(g_1^{-1}))_{22} = \frac{\sqrt{3}}{2|\tau|^2 + 2\text{Re}(\tau) + 2}, \quad (\pi_A(w_0^{-1}g_1^{-1}))_{22} = 1,$$

$$\pi_A(w_0^{-1}\pi_A(n_\tau^{-1}))_{22} = \frac{\sqrt{3}}{2|\tau|^2 + 2\text{Re}(\tau) + 2}, \quad (\pi_A(w_0^{-1}n_\tau^{-1}))_{22} = 1/|\tau|.$$

As a consequence we get that

$$\alpha_G(w_0^{-1}\pi_A(g_1^{-1}), w_0^{-1}g_1^{-1}) = \lambda \log \left| \frac{\sqrt{2|\tau|^2 + 2\text{Re}(\tau) + 2}}{\sqrt{3}} \right|,$$

and similarly for the other term we have

$$\alpha_G(w_0^{-1}\pi_A(n_\tau^{-1}), w_0^{-1}n_\tau^{-1}) = \lambda \log \left| \frac{\sqrt{2|\tau|^2 + 2\text{Re}(\tau) + 2}}{\sqrt{3}|\tau|} \right|.$$
Summing everything up we obtain
\[ \omega(e)(F_{can}, w_0 F_{can}, n_\tau w_0 F_{can}) = -\lambda \log |\tau|, \]
and this concludes the proof.

5.6. Proof of Theorem 6. We will use the same notation we used at the beginning of the previous section. Let \( \alpha_\alpha \) be a \( w_0 \)-invariant linear functional defined on \( \alpha \), let \( \hat{\alpha} \) be the associated inhomogeneous cocycle on \( \Lambda \) and let \( \alpha \) be its homogenization. By precomposing with the projection \( \pi_A \) we obtain the cocycle \( \alpha_G : G^2 \to \mathbb{R} \) defined on \( G \).

This time we view \( \alpha_G \) as an \( \Lambda \)-invariant cocycle lying in \( C^{1,2} \) and its cohomology class represents the element \( \alpha_\alpha \) in \( H^2(C^{1,2}, d^1) \cong H^1_{m}(A) \cong \mathfrak{a}^* \).

By Lemma 6 we have to follow \( \overline{\alpha_G} \) along the path
\begin{equation}
\overline{\alpha_G} \in C^{1,2} \quad d^\tau \overline{\alpha_G} = d^\tau \beta \in C^{1,3}
\end{equation}

\( h^{0,3}(d^\tau \overline{\alpha_G}) =: \beta \in C^{0,3} \quad d^\tau \beta =: \omega \in C^{0,4}. \)

Computation of \( \beta \). We start by computing the differential \( d^\tau \alpha_G \). By Lemma 25 we have that
\[ d^\tau \alpha_G(g_0, g_1)(\tau) = \alpha_G(g_\tau^{-1} g_0, g_\tau^{-1} g_1) - \alpha_G(n_\tau^{-1} g_0, n_\tau^{-1} g_1) + \alpha_G(g_0, g_1) \]
\[ = \alpha_G(g_\tau^{-1} g_0, g_\tau^{-1} g_1), \]
where we moved from the first line to the second one exploiting the \( P \)-invariance (and thus the \( N \)-invariance) of \( \alpha_G \).

To construct \( \beta \) it is sufficient to apply to \( d^\tau \alpha \) the contracting homotopy \( h^{0,3} \), that means
\[ \beta(g)(\tau) = d^\tau \alpha_G(e, g)(\tau) = \alpha_G(g_\tau^{-1}, g_\tau^{-1} g). \]

Computation of \( \omega \). Also in this case the computation has only two steps and \( \omega(e) \) will give back the desired cocycle. Let \( F_0, F_1, F_2, F_3 \) be a 4-tuple of flags in general position. We assume that their configuration class has coordinates
\[ (z_{01}, z_{10}, z_{23}, z_{32}) = (a, b, c, d). \]

We denote by \( \tau_{ijk} \) the triple ratio of the configuration \( [F_i, F_j, F_k] \), where \( i, j, k \in \{0, 1, 2, 3\} \). Again by the transitivity of \( G \), we can suppose that our flags are exactly the ones given by Equation 22.

By the definition of the differential \( d^\tau \) we have that
\[ \omega(e)(F_0, F_1, F_2, F_3) = d^\tau(\beta(e))(F_0, F_1, F_2, F_3) \]
\[ = -\beta(e)(F_1, F_2, F_3) + \beta(e)(F_0, F_2, F_3) \]
\[ -\beta(e)(F_0, F_1, F_3) + \beta(e)(F_0, F_1, F_2). \]

It is easy to see that the last term vanishes. For the other three summands we need to exploit \( G \)-invariance to compute the evaluation, since we know only how to evaluate \( \beta(g) \), for some \( g \), on a standard triple \( (F_{can}, w_0 F_{can}, n_\tau w_0 F_{can}) \).

As a result, we need to find the matrices \( g_{ijk} \in G \) satisfying
\begin{equation}
g_{ijk}(F_{can}, w_0 F_{can}, n_{\tau_{ijk}} w_0 F_{can}) = (F_i, F_j, F_k), \end{equation}
where \( (i, j, k) \) is either \((1, 2, 3), (0, 2, 3) \) or \((0, 1, 3) \). If we set
\[ r = a(1 - b), \quad s = 1 - ab - (\tau_{012} + 1)r, \]
and
\[ u = (a - 1)/\tau_{012}, \quad v = ab - 1 - (1 + \tau_{012})u, \]
we can define
\[
g_{123} := \begin{bmatrix} 0 & 0 & ab \\ 0 & r & ab \\ s & (1 + \tau_{012})r & ab \end{bmatrix}, \quad g_{023} := \begin{bmatrix} v & (\tau_{012} + 1)u & 1 \\ 0 & \tau_{012}u & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_{013} := \begin{bmatrix} ab & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Then the matrices
\[
g_{ijk} = \frac{1}{\sqrt{\det(g'_{ijk})}} g'_{ijk}
\]
satisfy \([27]\).

Exploiting the matrices \(g_{ijk}\) and the \(G\)-invariance, we can rewrite
\[
\omega(\epsilon)(F_0, F_1, F_2, F_3) = -\beta(g^{-1}_{123})(F_{\text{can}}, w_0F_{\text{can}}, n_{\tau_{123}}w_0F_{\text{can}})
+ \beta(g^{-1}_{023})(F_{\text{can}}, w_0F_{\text{can}}, n_{\tau_{023}}w_0F_{\text{can}})
- \beta(g^{-1}_{013})(F_{\text{can}}, w_0F_{\text{can}}, n_{\tau_{013}}w_0F_{\text{can}}).
\]
The explicit computation of \(\beta(g)\) we made in the previous subsection allows us to write the following equation
\[(28)\]
\[
\omega(\epsilon)(F_0, F_1, F_2, F_3) = -\alpha_G(g^{-1}_{123}, g^{-1}_{123}g_{123}) + \alpha_G(g^{-1}_{123}, g^{-1}_{123}g_{123}) - \alpha_G(g^{-1}_{123}, g^{-1}_{123}g_{123}).
\]

Again by Lemma \([23]\) the functional \(\alpha_G\) on \(a^*\) must be a multiple of \(c^*_A\), say \(\lambda e_A^*\). Analogously the cocycle \(\beta\) is a multiple of the logarithm of the second diagonal coordinate.

To evaluate \(\alpha_G\), we thus need to compute the second coordinate of the \(\pi_A\)-projection of each of the six coordinates in the left hand side of \((28)\). In view of Lemma \([24]\) and the fact that these six coordinates are explicit it would be possible to compute those by hand. We however chose to exploit the open source software \texttt{SageMath}, with which we obtained
\[
\alpha_G(g^{-1}_{123}, g^{-1}_{123}g_{123}) = \frac{1}{6} \log \left| \frac{b}{a} \right|^2
\]
\[
\alpha_G(g^{-1}_{123}, g^{-1}_{123}g_{123}) = \frac{27|a|^4|c-1|^2|b-1|^2|d-1|^2}{|a|^4|b-1|^2|c-1|^2|d-1|^2}
\]
\[
\alpha_G(g^{-1}_{123}, g^{-1}_{123}g_{123}) = \frac{4}{3} \log \left| \frac{b}{c} \right|^2
\]

Putting everything together we obtain that
\[
\omega(\epsilon)(F_0, F_1, F_2, F_3) = -\frac{2\lambda}{3} \log \left| \frac{b}{c} \right|^2,
\]
and the statement is proved.

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