Research Article

Dynamic Analysis of a Tapered Composite Thin-Walled Rotating Shaft Using the Generalized Differential Quadrature Method

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A dynamic model of a tapered composite thin-walled rotating shaft is presented. In this model, the transverse shear deformation, rotary inertia, and gyroscopic effects have been incorporated. The equations of motion are derived based on a refined variational asymptotic method (VAM) and Hamilton’s principle. The partial differential equations of motion are reduced to the ordinary differential equations of motion by using the generalized differential quadrature method (GDQM). The validity of the dynamic model is proved by comparing the numerical results with those obtained in the literature and by using ANSYS. The effects of taper ratio, boundary conditions, ply angle, length to mean radius ratios, and mean radius to thickness ratios on the natural frequencies and critical rotating speeds are investigated.

1. Introduction

Composite materials have found numerous applications in many engineering fields, thanks to their outstanding engineering properties, such as high strength, high specific stiffness, light weight, and design ability. Specially, rotating composite shafts are widely used in the aerospace industry, automobile industry, and shipbuilding industry. Research on dynamic characteristics of rotating composite shaft is a focus of composite material structural dynamics. Chen and Peng [1, 2] studied the dynamic behavior of a rotating shaft subjected to axial periodic forces using the finite element method. Song et al. [3] analyzed the vibration and stability behavior of spinning circular cylindrical shafts modeled as thin-walled composite beams considering the effects of transverse shear, rotatory inertias, the axial compressive load, and various boundary conditions. Chang et al. [4] presented a simple spinning composite shaft model based on a first-order shear deformable beam theory. To determine the spinning shaft system’s responses, they used the numerical finite element method to approximate the governing equations by a system of ordinary differential equations. Chang et al. [5] analyzed the vibration behaviors of rotating composite shafts containing randomly oriented reinforcements. Sino et al. [6] developed a simplified homogenized beam theory (SHBT) to analysis the sensitivity of the frequencies and instability thresholds regarding shear effect, stacking order, and fiber orientation. Boukhalfa et al. [7] investigated free vibration analysis of rotating composite shafts on rigid bearings using the p-version hierarchical finite element method with trigonometric shape functions. Boukhalfa and Hadjoui [8] also presented the study of the vibratory behavior of rotating composite shafts. An hp-version of the finite element method was used to model the structure. Ren et al. [9] presented a dynamical model of a rotating composite shaft and studied dynamic characteristics of the shaft for simple support at the ends using the Galerkin method to discretize and solve the governing equations. Arab et al. [10] developed a finite element based on Equivalent Single Layer Theory (ESLT) to model the rotating composite shaft using the Timoshenko beam theory. Furthermore, Layerwise Shaft Theory (LST) is developed based on shaft finite element theory by Arab et al. [11]. They investigated the dynamic analysis of rotating laminated...
shafts including the influences of stacking sequence, fiber orientation, and shear-normal coupling. Zhong et al. [12] conducted the free vibration analysis of a rotating composite thin-walled shaft using the generalized differential quadrature method.

Several authors have focused their attention on the development of tapered shaft rotors. Bauchau [13] treated the optimization of the tapered wall thickness of hollow shafts using Rayleigh. A dynamic model was developed by Kim et al. [14–17] on tapered composite Timoshenko shafts which runs around its axis at a constant speed, which are used by the general Galerkin method. Na et al. [18] evaluated the vibration and stability of a cylindrical shaft modeled as a tapered thin-walled composite rotor made of composite materials by the classical version and also used the Galerkin method to discretize and solve the governing equations. Rachid et al. [20] proposed a theoretical and numerical study on the behavior of a tapered shaft rotor made of composite materials by the classical version $h$ and the version $p$ of the finite element method.

In this paper, a dynamic model for the analysis of a tapered composite thin-walled rotating shaft is presented. Based on a refined variational asymptotic method and Hamilton’s principle, the motion equations of the tapered composite rotating shaft are derived. To solve the motion equations of the shaft, the GDQM is carried here. The effects of taper ratio, ply angle, length to radius of curvature of the mid-line cross-sections and also used the Galerkin method to discretize and solve the free vibration characteristics and stability of variable cross-section composite shaft for cantilever boundary condition. The free vibration characteristics and stability of a cylindrical shaft modeled as a tapered thin-walled composite beam and adopted the extended Galerkin method to solve the eigenvalue problem. Ma et al. [19] studied the behavior of a rotating composite thin-walled rotating shaft including the influences of stacking sequence, fiber orientation, and shear-normal coupling. Zhong et al. [12] conducted the free vibration analysis of a rotating composite thin-walled shaft using the generalized differential quadrature method.

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In this paper, a dynamic model for the analysis of a tapered composite thin-walled rotating shaft is presented. Based on a refined variational asymptotic method and Hamilton’s principle, the motion equations of the tapered composite rotating shaft are derived. To solve the motion equations of the shaft, the GDQM is carried out here. The natural frequencies and critical rotating speeds of the tapered composite thin-walled rotating shaft are then analyzed.

2. Model of Tapered Composite Thin-Walled Rotating Shaft

The model of a tapered composite thin-walled rotating shaft is shown in Figure 1. The shaft rotates along its longitudinal $x$-axis with constant rate $\Omega$. $L$, $h$, $r_R$, and $r_T$ denote the length, the thickness, the radius of curvature of root middle surface, and the radius of curvature of tip middle surface of the shaft, respectively. As shown in Figure 1, $(X, Y, Z)$ is the inertial coordinate system, $(x, y, z)$ represents the rotating reference system, and $(x, s, \xi)$ is a local coordinate system. Associated with the systems $(X, Y, Z)$ and $(x, y, z)$, they have the common origin $O$ at the geometric center and the corresponding unit vectors are $(I, J, K)$ and $(i, j, k)$, respectively.

The linear distribution along the tapered shaft of the radius of curvature of the mid-line cross-sections $r(\xi)$ varies according to the relationship [18]:

$$r(\xi) = [1 - \xi (1 - \sigma)] r_R,$$  \hspace{1cm} (1)

where $\sigma = r_T/r_R$ denotes the taper ratio and $\xi = x/L$ ($0 \leq \xi \leq 1$) is the dimensionless cross-section coordinate.

3. Motion Equations

3.1. Strain Energy and Kinetic Energy of the Composite Rotating Shaft. The displacement function of the composite shaft based on the refined VAM thin-walled beam theory can be assumed as the following form [9]:

$$u(x, y, z, t) = U(x, t) - y(s)\psi_y(x, t) - z(s)\psi_z(x, t) + g(s, x, t),$$

$$v(x, y, z, t) = V(x, t) - z(s)\phi(x, t),$$

$$w(x, y, z, t) = W(x, t) + y(s)\phi(x, t),$$

where $U(x, t)$, $V(x, t)$, and $W(x, t)$ denote the rigid body displacements along the $x$-, $y$-, and $z$-axis and $\phi(x, t)$, $\psi_y(x, t)$, and $\psi_z(x, t)$ are the twist about $x$-axis and rotations about $y$- and $z$-axis, respectively.

In which, $\psi_y(x, t)$ and $\psi_z(x, t)$ can be expressed as follows:

$$\psi_y(x, t) = V'(x, t) - 2y_{xx},$$

$$\psi_z(x, t) = W'(x, t) - 2y_{yx},$$

Figure 1: Coordinate systems and geometry of a tapered composite thin-walled rotating shaft.

where $y_{xx}$ and $y_{yx}$ are the transverse shear strains in the planes $xz$ and $xy$, respectively.

From the classical VAM, the warping displacement function $g(s, x, t)$ is assumed as follows:

$$g(s, x, t) = G(s)\psi'_y(x, t) + g_1(s)U'(x, t) + g_2(s)\psi'_y(x, t),$$

where $G(s)$, $g_1(s)$, $g_2(s)$, and $g_3(s)$ are associated with physical behavior for the torsion twist rate, the axial strain, and the bending curvatures, respectively.

According to equations (2)–(4), the strains of the composite shaft are obtained by
\[ \varepsilon_{xx} = U'(x,t) - y(s)\psi'_y(x,t) - z(s)\psi'_z(x,t), \]
\[ 2\gamma_{sr} = \frac{dg}{ds} + y_r\phi'_y(x,t) + \left( V'(x,t) - \psi'_y(x,t) \right) \frac{dy}{ds}, \]
\[ + (W'(x,t) - \psi'_z(x,t)) \frac{dz}{ds}, \]
\[ 2\gamma_{sx} = \left( V'(x,t) - \psi'_y(x,t) \right) \frac{dz}{ds}, \]
\[ + (W'(x,t) - \psi'_z(x,t)) \frac{dy}{ds}, \]
\[ \text{(5)} \]

where \( r_n \) denotes the normal projection of \( r \) which is the position vector of an arbitrary point on the cross section of the deformed shaft, and \( r_n \) can be expressed as follows:
\[ r_n = y(s) \frac{dz(s)}{ds} - z(s) \frac{dy(s)}{ds}. \]
\[ \text{(6)} \]

The position vector \( r \) can be written as follows:
\[ r = (y + v)i + (z + w)j + (x + u)k. \]
\[ \text{(7)} \]

From the above equation, the velocity of an arbitrary point can be given by
\[ V = \dot{r} = (\dot{v} - \Omega(z + w))i + (\dot{w} + \Omega(y + v))j + \dot{u}k. \]
\[ \text{(8)} \]

The kinetic energy of the composite rotating shaft \( T \) can be written as follows:
\[ T = \frac{1}{2} \int_0^L \int_A \rho(V \cdot V) dA dx. \]
\[ \text{(9)} \]

The strain energy of the composite shaft \( U \) can be expressed as follows:
\[ U = \frac{1}{2} \int_0^L \int_A \alpha_x(x, t) \varepsilon_{xx} + \tau_{xy}(x, t) \gamma_{xy} + \tau_{xz}(x, t) \gamma_{xz} \right) dA dx, \]
\[ \text{(10)} \]

where \( \sigma_{xx} \), \( \tau_{xy} \), and \( \tau_{xz} \) represent the cross-section normal stress, in-plane shear stress, and transverse shear stress, respectively. \( \varepsilon_{xx} \), \( \gamma_{xy} \), and \( \gamma_{xz} \) are associated engineering strains.

### 3.2. Motion Equations

The governing equations of the composite rotating shaft can be derived based on Hamilton’s principle, which is of the following form:
\[ \int_{t_1}^{t_2} (\delta \dot{x} - \delta T) dt = 0. \]
\[ \text{(11)} \]

In the present study, a special ply-angle distribution referred to as circumferentially uniform stiffness (CUS) configuration [21] is considered. By employing Hamilton’s principle and considering the variable cross-section of the tapered shaft, the motion equations involving CUS configuration bending-transverse shear coupling can be written as follows:
\[ -C_{35}(x)\psi'' - C_{55}(x)(V'' - \psi''_y) + m_k\left( V - 2\Omega W - \Omega^2 V \right) = 0, \]
\[ -C_{46}(x)\psi'' - C_{46}(x)(W'' - \psi''_y) + m_k \left( W + 2\Omega V - \Omega^2 W \right) = 0, \]
\[ -C_{46}(x)\psi'' - C_{46}(x)(W'' - \psi''_y) - C_{35}(x)\psi''_y - C_{55}(x)(V' - \psi'_y) - I_z \psi_y = 0, \]
\[ -C_{35}(x)\psi'' - C_{55}(x)(V'' - \psi''_y) - C_{46}(x)\psi''_y - C_{66}(x)(W'' - \psi''_z) - I_y \psi_z = 0, \]
\[ \text{(12)} \]

in which
\[ C_{35} = \int_{\Gamma(x)} \left( A - \frac{B^2}{C} \right) z^2 ds + \left\{ \frac{1}{\Gamma(x)} \left( B/C \right) z ds \right\}^2, \]
\[ C_{35} = -\frac{1}{2} \int_{\Gamma(x)} By \frac{dz}{ds} ds, \]
\[ C_{46} = \frac{1}{2} \int_{\Gamma(x)} Bz \frac{dz}{ds} ds, \]
\[ C_{55} = \int_{\Gamma(x)} \frac{1}{4} \left( \frac{dy}{dz} \right)^2 + D \left( \frac{dz}{ds} \right)^2 ds, \]
\[ C_{66} = \int_{\Gamma(x)} \frac{1}{4} C \left( \frac{dz}{ds} \right)^2 + D \left( \frac{dy}{ds} \right)^2 ds, \]
\[ \text{(13)} \]

where
\[ A = A_{11} - \frac{A_{12}^2}{A_{22}}, \]
\[ B = \left[ A_{16} - \frac{A_{12}A_{26}}{A_{22}} \right], \]
\[ C = \left[ A_{66} - \frac{(A_{66})^2}{A_{55}} \right], \]
\[ D = \left[ A_{44} - \frac{(A_{44})^2}{A_{55}} \right], \]
\[ \text{(14)} \]

\[ A_{ij} = \sum_{k=1}^{N} Q_{ij} (x_k - x_{k-1}), \quad i, j = 1, 2, 6; i, j = 4, 5. \]

Parameters \( A \) and \( B \) denote the reduced axial and coupling stiffness, parameters \( C \) and \( D \) are the reduced shear stiffness, and \( A_{ij} \) is the local stretching stiffness, \( Q_{ij} \) is
transformed stiffness of the kth layer. \( \kappa \) is shear factor of the cross-section which changes with the cross-section and material properties and is of the following form [14]:

\[
\kappa = \frac{6E_{xx}(1-\overline{m}^4)(1+\overline{m}^2)}{G_{xy}v_{xy}(2\overline{m}^6 + 18\overline{m}^4 - 18\overline{m}^2 - 2) - E_{xx}(7\overline{m}^6 + 27\overline{m}^4 - 27\overline{m}^2 - 7)},
\]

where \( \overline{m} = r/R \). \( r \) and \( R \) are the inner and outer radius, respectively. \( E_{xx}, G_{xy}, \) and \( v_{xy} \) are elastic modulus, shear modulus, and Poisson’s ratio:

\[
[Q] = [T]^{-1}[Q][T]^{-T},
\]

in which

\[
[T] = \begin{bmatrix}
  m^2 & n^2 & 0 & 0 & 2mn \\
  n^2 & m^2 & 0 & 0 & -2mn \\
  0 & 0 & m & -n & 0 \\
  0 & 0 & n & m & 0 \\
  -mn & mn & 0 & 0 & m^2 - n^2
\end{bmatrix}
\]

\[
[Q] = \begin{bmatrix}
  Q_{11} & Q_{12} & 0 & 0 & Q_{16} \\
  Q_{21} & Q_{22} & 0 & 0 & Q_{26} \\
  0 & 0 & Q_{44} & 0 & 0 \\
  0 & 0 & 0 & Q_{55} & 0 \\
  0 & 0 & 0 & 0 & Q_{66}
\end{bmatrix}
\]

where \([T]\) is transformation matrix, \([Q]\) is the stiffness matrix, and \( m = \cos \alpha \) and \( n = \sin \alpha \) is the angle made by fiber direction with respect to the x-axis of the coordinate system \((x, s, \xi)\). The stiffness arrays \( Q_{ij} \) are determined according to the material properties of the lamina:

\[
Q_{11} = \frac{E_{11}}{1 - v_{12}v_{21}},
Q_{12} = Q_{21} = \frac{v_{12}E_{22}}{1 - v_{12}v_{21}},
Q_{22} = \frac{E_{22}}{1 - v_{12}v_{21}},
Q_{44} = \kappa G_{23},
Q_{55} = \kappa G_{13},
Q_{66} = G_{12}.
\]

In addition, the mass terms \( m_e \), \( I_z \), and \( I_y \) in (14) are expressed as

\[
\begin{align*}
m_e &= \int_A \rho \, dA, \\
I_z &= \int_A \rho y^2 \, dA, \\
I_y &= \int_A \rho z^2 \, dA.
\end{align*}
\]

The motion equation (12) can be used for a tapered composite thin-walled rotating shaft with arbitrary boundary conditions. In this paper, three boundary conditions for the composite shaft are as follows [12]:

(a) Simply supported-simply supported (S-S):

\[
V = 0, \; W = 0, \; M_y = 0, \; M_z = 0 \quad (x = 0, L).
\]

(b) Clamped-clamped (C-C):

\[
V = 0, \; W = 0, \; \Psi_y = 0, \; \Psi_z = 0 \quad (x = 0, L).
\]

(c) Clamped-free (C-F):

\[
V = 0, \; W = 0, \; \Psi_y = 0, \; \Psi_z = 0 \quad (x = 0),
Q_y = 0, Q_z = 0, M_y = 0, M_z = 0 \quad (x = L).
\]

3.3. Approximate Solution Method. The GDQM is a global numerical approximate method. The derivative of a sufficiently smooth function \( f(x, t) \) at the jth discrete point can be approximated by weighted sums of the function values at all the discrete points. By applying the GDQM, the pth order derivative of \( f(x, t) \) is given by [22]

\[
\frac{\partial^p f(x, t)}{\partial x^p} \bigg|_{x=x_j} = \sum_{k=1}^{N_{GP}} C_{jk}^p f(x_k, t), \quad j = 1, 2, \ldots, N_{GP},
\]

where \( N_{GP} \) is the number of total discrete grid points in the x-direction and \( C_{jk}^p \) is the corresponding weighting coefficient associated to the pth-order derivative which is obtained as below.

For the first-order derivative, the weighting coefficients are given by

\[
C_{jk}^1 = \frac{M^{(1)}(x_j)}{(x_j - x_k)} M^{(1)}(x_k), \quad j \neq k, \; j, k = 1, 2, \ldots, N_{GP},
\]

\[
C_{ii}^1 = - \sum_{k=1}^{N_{GP}} C_{jk}^1, \quad j = 1, 2, \ldots, N_{GP},
\]

where

\[
M^{(1)}(x_k) = \prod_{j=1}^{N_{GP}} \left( x_k - x_j \right).
\]
For the second and higher order derivatives, the weighting coefficients are acquired by using the following recurrence formulation:

\[ C_{jk}^p = p \left( \frac{C_{jk}^1 \cdot C_{jj}^p - C_{jk}^{p-1}}{x_j - x_k} \right), \]

\[ j \neq k, j, k = 1, 2, \ldots, N_{GP}, p = 2, 3, \ldots N_{GP}, \]

and the grid points are chosen as follows:

\[ x_j = \frac{L}{2} \left( 1 - \cos \left( \frac{j - 1}{N_{GP} - 1} \pi \right) \right), \quad j = 1, 2, \ldots, N_{GP}. \]  

(26)

To find the approximate solution of the composite rotating shaft, bending deformation and bending angle are assumed as follows:

\[ v(x, t) = V(x)e^{i\omega t}, \]
\[ w(x, t) = W(x)e^{i\omega t}, \]
\[ \psi_y(x, t) = \Psi_y(x)e^{i\omega t}, \]
\[ \psi_z(x, t) = \Psi_z(x)e^{i\omega t}, \]

where \( V(x), W(x), \Psi_y(x), \) and \( \Psi_z(x) \) are the indefinite functions of the \( x \)-direction, \( \omega \) is the natural frequency, and \( i = \sqrt{-1} \).

Substituting the displacement components (28) into the motion equations (12), a set of ordinary differential equations with variable coefficients toward the \( x \)-direction is derived:

\[ L^*U^* = 0, \]

(29)

where \( U^* = \{V(x), W(x), \Psi_y(x), \Psi_z(x)\} \) is an unknown spatial function vector of mode shape, and \( L^* = [L_{ij}]^{(i, j = 1, \ldots, 4)} \) is a \( 4 \times 4 \) matrix of \( U^* \) and is defined as follows:

\[ L^* = -\omega^2 M_1 + i\omega G_1 + K_1 + K_\Omega, \]  

(30)

where

\[ M_1 = \begin{bmatrix} m_c & 0 & 0 & 0 \\ 0 & m_c & 0 & 0 \\ 0 & 0 & -I_z & 0 \\ 0 & 0 & 0 & -I_y \end{bmatrix}, \]

\[ G_1 = \begin{bmatrix} 0 & -2\Omega m_c & 0 & 0 \\ 2\Omega m_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ K_\Omega = \begin{bmatrix} -m_\Omega^2 & 0 & 0 & 0 \\ 0 & -m_\Omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ K_1 = \begin{bmatrix} -C_{35}(x) \frac{\partial^2}{\partial x^2} & 0 & C_{35}(x) \frac{\partial}{\partial x} & -C_{35}(x) \frac{\partial^2}{\partial x^2} \\ 0 & -C_{46}(x) \frac{\partial^2}{\partial x^2} & -C_{46}(x) \frac{\partial^2}{\partial x^2} & C_{46}(x) \frac{\partial}{\partial x} \\ -C_{55}(x) \frac{\partial}{\partial x} & -C_{46}(x) \frac{\partial^2}{\partial x^2} & -C_{44}(x) \frac{\partial^2}{\partial x^2} + C_{55}(x) & (C_{46}(x) - C_{35}(x)) \frac{\partial}{\partial x} \\ -C_{35}(x) \frac{\partial^2}{\partial x^2} & -C_{66}(x) \frac{\partial}{\partial x} & (C_{35}(x) - C_{46}(x)) \frac{\partial}{\partial x} & -C_{33}(x) \frac{\partial^2}{\partial x^2} + C_{66}(x) \end{bmatrix}. \]
By imposing equations (23) to (29) and rearranging equation (29) according to the orders of derivatives, the approximate governing equations are obtained in the form of linear discrete algebraic equations:

\[ L^{**}U^{**} = L_{4x15}^{**}U_{15x11}^{**} = 0, \quad j = 1, 2, \ldots, N_{GP}, \]

where \( N_{GP} \) is the total discrete grid points in the \( x \)-direction and \( U^{**} \) is written as follows:

\[
U^{**T}_{1x} = \{ V(x_j), V^{(1)}(x_j), V^{(2)}(x_j), \Psi_y(x_j), \Psi_y^{(1)}(x_j), \Psi_y^{(2)}(x_j), \Psi_z(x_j), \Psi_z^{(1)}(x_j), \Psi_z^{(2)}(x_j) \},
\]

where

\[
V^{(p)}(x_j) = \sum_{k=1}^{N_{GP}} C_{jk}^{(p)} V(x_k, t),
\]

\[
W^{(p)}(x_j) = \sum_{k=1}^{N_{GP}} C_{jk}^{(p)} W(x_k, t),
\]

\[
\Psi_y^{(p)}(x_j) = \sum_{k=1}^{N_{GP}} C_{jk}^{(p)} \Psi_y(x_k, t),
\]

\[
\Psi_z^{(p)}(x_j) = \sum_{k=1}^{N_{GP}} C_{jk}^{(p)} \Psi_z(x_k, t),
\]

for \( p = 1, 2 \).

For given boundary conditions, imposing equation (32) on every discrete grid point, the discretized form of the boundary conditions is achieved.

Therefore, the eigenvalue equation is obtained as follows:

\[
(M\omega^2 + G i \omega + K) d = 0,
\]

where \( M, G, \) and \( K \) are the \( 4(N_{GP} - 2) \times 4(N_{GP} - 2) \) numerical coefficient matrices. The dimensions of vector \( d \) is \( 4(N_{GP} - 2) \) which can be expressed as follows:

\[
d = \{ V(x_2), \ldots, V(x_{N_{GP}-1}), W(x_2), \ldots, W(x_{N_{GP}-1}), \Psi_y(x_2), \ldots, \Psi_y(x_{N_{GP}-1}), \Psi_z(x_2), \ldots, \Psi_z(x_{N_{GP}-1}) \}.
\]

For a certain frequency, the eigenvalue equation (35) can be transformed into a standard form as follows [23]:

\[
\begin{bmatrix} 0 & I \\ -K & -G \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \omega \\ d \end{bmatrix} = 0,
\]

where \( I \) is the \( 4(N_{GP} - 2) \times 4(N_{GP} - 2) \) identity matrix.

The vibration characteristics of the tapered composite rotating shaft can be obtained by solving (37).

4. Numerical Results Analysis and Discussion

4.1. Convergence Analysis and Accuracy Verification. The numerical calculations are performed by considering the tapered shaft made of graphite-epoxy whose elastic characteristics are listed in Table 1. The shaft has geometrical characteristics as \( r_p = 0.0635 \text{ m}, \) \( L = 2.47 \text{ m}, \) and \( h = 1.321 \text{ mm}. \) The stacking sequence of the shaft is \([ \pm 30^\circ ]_4.\)

In order to examine the influence of the number of grid points \( N_{GP} \) on the accuracy of the results, a cylindrical thin-walled shaft with S-S boundary condition is first studied. The numerical results of natural frequencies are shown in Table 2 for an increasing number of grid points. It can be seen from Table 2 that no more than 12 grid points are required in order to obtain convergent results for the first three natural frequencies. So, for all results given in this paper, \( N_{GP} = 12 \) unless otherwise noted.

Figures 2–3 show the variation of natural frequencies versus rotating speed for the cantilever composite shaft with different taper ratios from the present model with the ones from [18]. The numerical results are given according to the normalized natural frequencies and rotating speed rate which are defined by \( \Omega^* = \omega/\omega_0 \) and \( \Omega^* = \Omega/\omega_0 \), where the normalizing factor \( \omega_0 = 138.85 \text{ rad/s} \) corresponds to the fundamental frequency of the nonrotating shaft with \( \theta = 0^\circ. \)

From Figures 2 and 3, it clearly appears that when the rotating speed rate \( \Omega^* = 0 \) a single fundamental frequency is obtained. When the rotating speed rate \( \Omega^* > 0 \), a bifurcation of natural frequencies occurs due to the gyroscopic effect. Hence, the natural frequency curve splits into an upper one and a lower one. The upper curve goes up corresponding to the forward whirling (FW or F) mode, while the lower curve goes down corresponding to the backward whirling (BW or B) mode. The minimum rotating rate at which the natural frequency becomes zero valued is called the critical rotating speed, which will cause the dynamically unstable motion of the tapered composite rotating shaft. It also can obviously be seen that the present numerical results agree well with those presented in [18].

In the absence of publications of vibration analysis of tapered composite thin-walled shafts with C-C and S-S boundary conditions, the formulation is verified by using the ANSYS 19.1. The FEA model of the tapered composite thin-walled rotating shaft with \( \sigma = 0.6 \) was shown in Figure 4, and the critical settings in ANSYS are shown in Table 3.

Tables 4 and 5 present the natural frequencies of the tapered composite thin-walled shafts obtained from both present model and ANSYS model. It can be seen from the tables that the numerical results obtained from the present model agree well with those obtained by using ANSYS.

4.2. Taper Ratio Effect. Figure 5 reveals the first three natural frequencies of the tapered composite thin-walled rotating shaft with different boundary conditions versus the taper ratio \((\Omega = 0 \text{ rpm}, \theta = 30^\circ).\) As shown in Figure 5, the natural frequencies increase with the taper ratio of the shaft with C-C and S-S boundary conditions. The first natural frequencies decrease with the taper ratio, while the second and third natural frequencies increase with the taper ratio of the shaft with C-F boundary condition. The tapered composite thin-walled shaft becomes cylindrical thin-walled when the taper ratio \( \sigma = 1 \), which was previously investigated in Ref [12].
Table 1: Mechanical properties of the composite material [2].

| $\rho$ (kg/m$^3$) | $E_{11}$ (GPa) | $E_{22}$ (GPa) | $G_{12}$ (GPa) | $G_{13}$ (GPa) | $G_{23}$ (GPa) | $\nu_{12}$ |
|-----------------|---------------|---------------|---------------|---------------|---------------|-------------|
| 1610            | 192           | 7.24          | 4.07          | 4.07          | 3.0           | 0.24        |

Table 2: Effect of $N_{GP}$ on natural frequencies ($\Omega = 0, \theta = 30^\circ$).

| $N_{GP}$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ | $\omega_6$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| 9        | 41.70       | 135.27      | 281.98      | 516.74      | 845.92      | 4117.67     |
| 10       | 41.70       | 135.03      | 282.79      | 484.27      | 825.67      | 1259.90     |
| 11       | 41.70       | 135.01      | 281.57      | 485.74      | 743.94      | 1231.97     |
| 12       | 41.70       | 135.02      | 281.53      | 481.22      | 746.91      | 1063.69     |
| 13       | 41.70       | 135.02      | 281.53      | 480.95      | 733.65      | 1070.46     |
| 14       | 41.70       | 135.02      | 281.53      | 481.15      | 733.14      | 1038.21     |
| 15       | 41.70       | 135.02      | 281.53      | 481.17      | 733.52      | 1037.62     |

Figure 2: The natural frequency of a tapered composite thin-walled shaft versus rotating speed for different taper ratios ($\theta = 60^\circ$).

Figure 3: The natural frequency of a tapered composite thin-walled shaft versus rotating speed for different taper ratios ($\theta = 90^\circ$).
Figure 6 reveals the first three natural frequencies of the tapered composite thin-walled rotating shaft with different boundary conditions versus the taper ratio ($\Omega = 400 \text{ rpm}, \theta = 30^\circ$). Two natural frequencies occur for each mode because of the gyroscopic effect. The higher value is called the forward mode, and the lower one represents the backward mode. The variation trend of natural frequencies is the same as in Figure 5.

### 4.3. Ply Angle Effect

Figure 7 shows the first three natural frequencies of the tapered composite thin-walled rotating shaft with different boundary conditions versus the ply angle ($\Omega = 0 \text{ rpm}$, $\sigma = 0.5$). Figure 7(a) represents the first mode of the tapered shaft. As shown in the figure, the maximum values of the first natural frequencies occur at $\theta = 80^\circ$ for the shaft with C-C and S-S boundary conditions, while the first natural frequencies of the tapered shaft increase with the ply angle for C-F boundary conditions. It can be seen from Figures 7(b) and 7(c) that the maximum values of the second and third natural frequencies occur at $\theta = 75^\circ$ for the shaft with aforementioned boundary conditions. In addition, it can be seen from Figure 7 that the higher the mode, the larger the variation amplitude.

Figure 8 shows the first three natural frequencies of the tapered composite thin-walled rotating shaft with different boundary conditions versus the ply angle ($\Omega = 1000 \text{ rpm}$, $\sigma = 0.5$). Two natural frequencies occur for each mode because of the gyroscopic effect similar to...
Figure 5: The natural frequencies of the rotating shaft versus the taper for different boundary conditions ($\Omega = 0 \text{ rpm, } \theta = 30^\circ$). (a) The first mode; (b) the second mode; (c) the third mode.

Figure 6: Continued.
Figure 6: The natural frequencies of the rotating shaft versus the taper for different boundary conditions ($\Omega = 400$ rpm, $\theta = 30^\circ$). (a) The first mode; (b) the second mode; (c) the third mode.

Figure 7: The natural frequencies of the rotating shaft versus ply angle for different boundary conditions ($\Omega = 0$ rpm, $\sigma = 0.5$). (a) The first mode; (b) the second mode; (c) the third mode.
Figure 8: The first three natural frequencies of the rotating shaft versus ply angle ($\Omega = 1000$ rpm, $\sigma = 0.5$). (a) C-C; (b) S-S; (c) C-F.

Figure 9: Continued.
Figure 9: Campbell diagrams of the tapered rotating shaft for different taper ratios with $\theta = 60^\circ$. (a) C-C; (b) S-S; (c) C-F.

Figure 10: Campbell diagrams of the tapered rotating shaft for different length to mean radius ratios ($\theta = 75^\circ, \sigma = 0.6$). (a) C-C; (b) S-S; (c) C-F.
Figure 6. It can be seen from Figures 8(a) and 8(b) that the change trend of the natural frequency curve is the same as Figure 7 for the tapered shaft with the ply angle for C-C and S-S boundary conditions. Figure 8(c) shows the change curves of the first three frequencies with increase of the ply angle for C-F boundary condition. It is worth noting that the first backward frequency decreases with increase of the ply angle until decreases to 0 when $\theta = 52^\circ$ (the critical speed of the tapered composite shaft is 1000 rpm at this ply angle) and then the frequency increases with the increase of the ply angle.

4.4. Stability Analysis. The Campbell diagrams for the first bending mode of the tapered composite thin-walled rotating shaft for different boundary conditions and different taper ratios are shown in Figure 9 ($\theta = 60^\circ$). The results show that the critical rotating speeds increase with the taper ratio of the shaft with C-C and S-S boundary conditions, while the critical rotating speeds decrease with the taper ratio of the shaft with C-F boundary condition.

Figure 10 shows the Campbell diagrams of the tapered composite thin-walled rotating shaft for different length to mean radius ratios ($\theta = 30^\circ$, $\sigma = 0.6$). The results show that the critical rotating speeds decrease as the ratio of length to mean radius increase.

The Campbell diagrams of the tapered composite thin-walled rotating shaft for different mean radius to thickness ratios are shown in Figure 11 ($\theta = 30^\circ$, $\sigma = 0.6$). It can be seen from the figure that the critical rotating speeds increase as the ratio of mean radius to thickness increase.

5. Conclusions

This paper studies the dynamical behavior of a tapered composite thin-walled rotating shaft. A dynamic model of the tapered composite shaft considering the transverse shear deformation, rotary inertia, and gyroscopic effects has been
proposed. The motion equations of the tapered composite shaft are derived based on a refined variational asymptotic method and Hamilton’s principle. The GDQM is used to discretize and solve the motion governing equations. The model of the tapered composite thin-walled rotating shaft with C-F boundary condition is verified by comparing the numerical results with those obtained in the literature. In the absence of publications of vibration analysis of tapered composite thin-walled shafts with C-C and S-S boundary conditions, the validity of the model is verified by using the ANSYS. Discussions are made for the influences of the taper ratio, ply angle, length to mean radius ratios, and mean radius to thickness ratios on the frequency characteristics of the tapered composite thin-walled rotating shaft. The results are helpful to improve the understanding about dynamic characteristics of the tapered composite thin-walled rotating shaft and to provide guidance for designing of the composite shaft.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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