Critical level spacing distribution of two-dimensional disordered systems with spin-orbit coupling

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Abstract

The energy level statistics of 2D electrons with spin-orbit scattering are considered near the disorder induced metal-insulator transition. Using the Ando model, the nearest-level-spacing distribution is calculated numerically at the critical point. It is shown that the critical spacing distribution is size independent and has a Poisson-like decay at large spacings as distinct from the Gaussian asymptotic form obtained by the random-matrix theory.

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Recently, the statistical description of electronic properties at the critical region near the Anderson transition in disordered systems has been the subject of great interest. Universal behavior was proposed [1,2] for the variance of the ensemble averaged number of energy levels in a given interval, \( \langle (\delta N)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 \), and for the nearest-level-spacing distribution \( P(s) \). A relation \( \langle (\delta N)^2 \rangle = a \langle N \rangle \) could be extracted from numerical calculations for the three-dimensional Anderson model, while for \( P(s) \) a universal combination of the well known Wigner surmise (small spacings) and the Poisson distribution (large spacings) was proposed. Recent analytical calculations [3,4] suggested, however, another universal relation \( \langle (\delta N)^2 \rangle = a_d \beta \langle N \rangle^{\gamma/\beta} \) near the critical transition (mobility edge) and the corresponding \( P(s) \) was different from both the Poisson and the Wigner limit. Universal means here that the result depends only on the symmetry class \( \beta \), the correlation exponent \( \nu = [d(1-\gamma)]^{-1} \), and the spatial dimension \( d \). The statistics of energy levels are governed by the symmetry of the Hamiltonian belonging to certain universality classes of the corresponding Gaussian ensembles of random matrices: orthogonal (\( \beta = 1 \)), unitary (\( \beta = 2 \)) and symplectic (\( \beta = 4 \)) [5].

In two-dimensional (2D) disordered systems a complete Anderson transition is expected only for the symplectic symmetry class. Two different models for describing the localization problem in two dimensions in the presence of spin-orbit coupling were proposed [6,7]. For the Ando model [8] the critical behavior of the localization length was previously studied by means of the transfer-matrix method [9–11]. The statistics of energy levels for symplectic symmetry in connection with the results of the random-matrix theory (RMT) was considered in the Evangelou-Ziman model [12] and the distribution \( P(s) \) at the transition point for this model has already been mentioned [13]. The multifractal properties of the critical eigenstates in the symplectic case were investigated in [14,15].

In this paper we study the level statistics of electrons with spin-orbit coupling in 2D disordered systems. By using the Ando model we calculate numerically the correlations in the exact one-electron energy spectrum at the metal-insulator transition. Our main result is that the probability density \( P(s) \) of neighboring levels in the critical region is universal, i.e. it does not depend on the size of the system, and has a novel form.

It is known that in the metallic phase \( P(s) \) is very close to the Wigner surmise [16] appropriate for the symplectic symmetry class of the random Hamiltonians [17],

\[
P_{\text{GSE}}(s) = \frac{2^{18}}{3^6 \pi^3} s^4 \exp \left( -\frac{64}{9 \pi} s^2 \right),
\]

where \( s \) is measured in units of the mean level spacing. In the strongly localized regime the energy spectrum is completely uncorrelated, and the distribution of the level spacings obeys the Poisson law

\[
P_p(s) = \exp(-s).
\]

In addition to these two universal distributions one can expect that there is a third form of \( P(s) \), which corresponds exactly to the metal-insulator transition. This new universal statistics was already found for the 3D disordered systems without spin-orbit scattering \((\beta = 1)\) and confirmed numerically [2,18]. An analogous result was recently obtained in the unitary case \((\beta = 2)\) [19], where time-reversal symmetry is broken by the magnetic field.

It is reasonable to suppose that a similar scale-invariant universality of \( P(s) \) also holds at the critical point for 2D disordered electrons with spin-orbit coupling. Thus, we expect that
in the thermodynamic limit there exist three possible limiting situations for \( \beta = 4 \) namely, the Wigner surmise Eq. (1) for the metallic regime, the Poisson law Eq. (2) for the insulating regime, and the critical distribution \( P(s) \) at mobility edge. Therefore, if \( L \to \infty \), the level statistics change discontinuously twice, from the delocalized regime to the transition point and, then, from the transition point to the localized regime.

In order to calculate the critical level statistics we start with the Hamiltonian of the Ando model [8]

\[
H = \sum_{n,\sigma} \epsilon_n c_{n\sigma}^\dagger c_{n\sigma} + \sum_{n,m,\sigma,\sigma'} V(n,\sigma;m,\sigma') c_{n\sigma}^\dagger c_{m\sigma'},
\]

where \( c_{n\sigma}^\dagger \) and \( c_{n\sigma} \) are the creation and annihilation operators of an electron at a lattice site \( n = (x, y) \) with the spin component \( \sigma \), and \( m \) denotes the sites adjacent to the site \( n \). The on-site energy \( \epsilon_n \) is randomly distributed around zero according to a box distribution with a width \( W \). The parameter \( W \) specifies the degree of the disorder. The transfer matrices \( V(n,\sigma;m,\sigma') = V_x, V_y \) depend on the direction

\[
V_{x,y;x+1,y} = \begin{pmatrix} V_1 & V_2 \\ -V_2 & V_1 \end{pmatrix}, \quad V_{x,y;x,y+1} = \begin{pmatrix} V_1 & -iV_2 \\ -iV_2 & V_1 \end{pmatrix}\]

and describe the hopping between the nearest neighbor sites in the lattice. The strength of the spin-orbit coupling is given by the parameter \( S = V_2/V \), where \( V = (V_1^2 + V_2^2)^{1/2} \) is taken as the unit of energy. In what follows we consider \( S = 1/2 \). It was earlier found by the transfer-matrix method [10] that the metal-insulator transition in the middle of the band \( \epsilon/V = 0 \) occurs at a disorder \( W_c/V = 5.74 \).

Applying periodic boundary conditions, the exact discrete eigenvalue spectrum has been obtained from a numerical diagonalization of the Hamiltonian Eq. (3) using a Lanczos algorithm. For square lattices of linear size \( L/a = 50 \) (100) energy intervals \([-1, 0] \) \([-0.5, 0] \) were chosen. Within these intervals the density of states is almost constant, \( \rho = (\Delta L^2)^{-1} \approx 0.127 \). Here, \( a \) is the lattice constant and \( \Delta \) the mean level spacing. The eigenvalues taken from these intervals belong to the critical region where the correlation length \( \xi \) is larger than the system size \( L \). The total numbers of eigenvalues \( N \) were 94672 (300 realizations) and 101744 (160 realizations) for \( L = 50 \) and \( L = 100 \), respectively.

Fig. 1 displays the level spacing distribution function \( P(s) \) calculated for the different sizes of the system at the metal-insulator transition. One can see that independent of the system size the data lie on a common curve. This critical \( P(s) \) is very close to \( P_{\text{GSE}} \) and passes the point \( s_0 \approx 1.63 \), where the two limiting distributions \( P_{\text{GSE}}(s) \) and \( P_P(s) \) cross. For small spacings \( P(s) \propto s^4 \). It is interesting to notice that our results differ from those obtained previously for smaller systems of the Evangelou-Ziman spin-orbit coupling model [13].

We applied the fitting function which was recently proposed for the description of the critical level distribution [3],

\[
P(s) = B s^4 \exp(-A s^{2-\gamma})
\]

and found \( A = 2.77 \pm 0.05 \), \( B = 17.8 \pm 0.8 \) and \( \gamma = 0.28 \pm 0.03 \). Using a confidence level of 95% (\( \alpha = 0.05 \)) the fitting procedure within the range \( 0 < s < 3 \) where the statistics of
our numerical data is rather good yielded $\chi^2 = 29.2$ which is less than the expected value $\chi^2_0 = 63.4$. Hence, the analytical formula (Eq. 3) can be accepted within an approximate relative error of $(\chi^2/N)^{1/2} \approx 2\%$. Although, in the range $0 < s < 3$ the calculated $P(s)$ is in good agreement with Eq. (3), the obtained exponent $\gamma$ gives, however, a different value of the correlation length exponent $\nu = [(1 - \gamma) d^{-1}] \approx 0.7$ as compared to the numerically obtained value $\nu = 2.75$ using the transfer-matrix method [9,10].

Moreover, for large spacings our results deviate markedly from the above Eq. (5) (inset of Fig. 1). Instead, the behavior of $P(s)$ in the range of the spacings $1.5 < s \lesssim 4$ is well described by the Poisson-like asymptotic form

$$P(s) \propto \exp(-\kappa s),$$

with a coefficient $\kappa = 4.0 \pm 0.2$. This decay is much slower than both the Gaussian decay of Eq. (1) and the intermediate decay from Eq. (5), but faster than that for the insulating regime Eq. (2). A similar exponential tail of the critical $P(s)$ was found also in the 3D-case without spin-orbit interactions [20] for which the coefficient $\kappa$ is approximately a factor of 2 less. Such an asymptotic behavior of the critical $P(s)$ is in good agreement with a suggestion about the Gaussian form for the distribution of the number of levels lying within a given energy interval [1,21].

In order to diminish the magnitude of relative fluctuations due to the limited number of realizations and to analyze the asymptotic behavior of the level spacing distribution in detail, it is more convenient to consider a total probability function $I(s) = \int_{s}^{\infty} P(s') ds'$. This quantity implies a portion of spacings which are larger than a given $s$. It is clear that $I(0) = 1$ and $\int_{0}^{\infty} I(s') ds' = 1$ regardless of the disorder. In the strongly localized regime $I_P(s) = \exp(-s)$, and $I_{\text{GSE}}(s)$ can be calculated from the RMT. The results of the numerical calculations for the critical $I(s)$ are shown in Fig. 2. One observes again a discrepancy from both the GSE asymptotic and the $I(s)$ obtained from Eq. (3), particularly when $s$ is large. The form of the critical $I(s)$ is not sensitive to the change of the lattice size, in analogy with $P(s)$. The Poisson tail as described by Eq. (3) is recovered for large spacings: $\ln I(s) \propto -s$.

We have checked that the slope of the linear behavior of $\ln I(s)$ at the transition does not depend on the width of the energy interval from which the levels are taken, as long as they belong to the critical region. In the range of very small probability ($s > 3$) the larger fluctuations observed are due to insufficient statistical data. However, the accuracy of the total calculated $I(s)$ is higher than that of $P(s)$. One should notice that the behavior of the obtained level spacing distribution does obviously not reflect the information on the critical exponent $\nu$ and the dimensionality $d$ in the form as it is expressed by Eq. (3).

In conclusion, we have presented results of computer simulations of the nearest-level-spacing distribution $P(s)$ for 2D disordered systems in the presence of spin-orbit interaction. Exactly at the metal-insulator transition the $P(s)$ and, consequently, the total probability of neighboring spacings $I(s)$ do not depend on the system size and are different from the universal limiting distributions corresponding to the metallic and the insulating regime. They appear to exhibit critical behavior at the disorder $W_c/V = 5.74$ and finite-size scaling properties around the critical point. The obtained large-$s$ parts of $P(s)$ and $I(s)$ are shown to have a Poisson-like decay so that $\ln P(s) = -\kappa s$ where $\kappa \approx 4.0$ is larger as compared to the insulating regime ($\kappa = 1$).
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FIGURES

FIG. 1. The level spacing distribution $P(s)$ for two different system sizes at the critical disorder $W_c/V = 5.74$. Solid lines correspond to $P_{GSE}(s)$ and $P_P(s)$, respectively. The dotted line is a fit of Eq. (5). The inset shows the large-$s$ behavior of $P(s)$. The straight line is a fit according to Eq. (6).

FIG. 2. The integrated probability $I(s)$ at the critical disorder $W_c = 5.74$. The solid curves are $I_{GSE}(s)$ and $I_P(s)$ for the metallic and insulating phases, respectively. The dotted line is obtained from Eq. (5). The inset shows the large-$s$ part of $I(s)$. The straight line fitting the data is $\ln I(s) = -4.0 s + 3.6$. 
\[ P(s) \]

\[ L = 50 - \circ \]
\[ 100 - \bullet \]

Inset:
- Poisson
- GSE
