Homotopy equivalence between Voronoi medusa and Delaunay medusa

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April 13, 2016

Abstract

We trace movements of certain points in space-time along their corresponding continuous path. We partition the space at every moment of time using $\alpha$-Complexes, Voronoi medusa is then the collection or union of restricted Voronoi cells at every moment in time. We can imagine them as a four dimensional structure formed when three dimensional restricted Voronoi cells sweeps continuously through the extra dimension of time. Similarly Delaunay medusa is the collection of corresponding Delaunay triangulations at each moment in time. The Lemma 2 of [1] states that, the Voronoi medusa and corresponding Delaunay medusa are homotopic. We will prove the statement using the Gluing constructions [2, 4G]. Which is the primary ingredient to prove the Nerve theorem as well.

1 Introduction

We start with $k$ points in 3-dimensional space, they move in the space-time along a continuous and non-intersecting path. We trace their movement for a bounded period of time, $t \in [0,1]$, so from here on our time will always be in $[0,1]$. We take a moment in time $t$, and partition the space with $\alpha$-complexes(restricted Voronoi cells). The collection of such restricted Voronoi cells for each moment in time will be called Voronoi medusa $R_m$. If we take the Delaunay triangulations of restricted Voronoi cells at each moment in time, the collection of such Delaunay triangulations will form the Delaunay medusa $A_m$. In this article we aim to prove that these two structure are homotopic.

Next few sections are organized as following, in section 2 we recall some of the concepts from Gluing Constructions [2, 4G], section 3 deals with
basic definitions and terminologies involved in the problem, in section 4 we provide the proof of the main statement.

2 Theoretical Background

In this section we will recall some of the definitions and concepts used in our proof.

A diagram of spaces consists of an oriented graph $\Gamma$ with a space $X_v$ for each vertex $v$ of $\Gamma$ and a map $f_e : X_v \to X_w$ for each edge $e$ of $\Gamma$ from a vertex $v$ to a vertex $w$, the words from and to referring to the given orientation of $e$. Commutativity of the diagram is not assumed. Denoting such a diagram of spaces simply by $X$, we define a space $\sqcup X$ to be the quotient of the disjoint union of all the spaces $X_v$ associated to vertices of $\Gamma$ under the identifications $x \sim f_e(x)$ for all maps $f_e$ associated to edges of $\Gamma$.

There is a more generalized and sophisticated construction, in which one starts with a $\Delta$-complex $\Gamma$ and a diagram of spaces associated to the 1-skeleton of $\Gamma$ such that the maps corresponding to the edges of each $n$-simplex of $\Gamma$, $n > 1$, form a commutative diagram. We call this data a complex of spaces. If $X$ is a complex of spaces, then for each $n$-simplex of $\Gamma$ we have a sequence of maps $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} X_n$, and we define the iterated mapping cylinder $M(f_1, \ldots, f_n)$ to be the usual mapping cylinder for $n = 1$, and inductively for $n > 1$, the mapping cylinder of the composition $M(f_1, \ldots, f_{n-1}) \to X_{n-1} \xrightarrow{f_n} X_n$, where the first map is the canonical projection of a mapping cylinder onto its target end. There is a natural projection $M(f_1, \ldots, f_n) \to \Delta^n$, and over each face of $\Delta^n$ one has the iterated mapping cylinder for the maps associated to the edges in this face. All these iterated mapping cylinders over the various simplices of $\Gamma$ thus fit together to form a space $\Delta X$ with a canonical projection $\Delta X \to \Gamma$. We call $\Delta X$ the realization of the complex of spaces $X$, and we call $\Gamma$ the base of $X$ or $\Delta X$.

For example, a cover $U = \{X_i\}$ of a space $X$, the diagram of spaces $X_U$ whose vertices are the finite intersections of $X_i$’s and whose edges are inclusions is a complex of spaces with $n$-simplices the $n$-fold inclusions. The base $\Gamma$ for this complex of spaces is the barycentric subdivision of the nerve of the cover.

By a map $f : X \to Y$ of complexes of spaces over the same base $\Gamma$ we mean a collection of maps $f_v : X_v \to Y_v$ for all the vertices of $\Gamma$, with commutative squares over all edges of $\Gamma$. There is then an induced map
\[ \Delta f : \Delta X \to \Delta Y. \] If all the maps \( f_v \) are homotopy equivalences then \( \Delta f \) would be a homotopy equivalence as well. [[2], Prop 4G.1].

There is a canonical map \( \Delta X \to \sqcup X \) induced by retracting each mapping cylinder onto its target end. In some cases this is a homotopy equivalence, for example, for a diagram \( X_0 \leftarrow A \leftarrow X_1 \) where the pair \( (X_1,A) \) has the homotopy extension property. Another example is a sequence of inclusions \( X_0 \leftarrow X_1 \leftarrow \ldots X_n \) for which the pairs \( (X_n,X_{n-1}) \) satisfy the homotopy extension property. However, without some conditions on the maps it need not be true that \( \Delta X \to \sqcup X \) is a homotopy equivalence.

When \( X_U \) is the complex of spaces associated to an open cover \( U = \{ X_i \} \) of a paracompact space \( X \), the map \( p : \Delta X \to \sqcup X = X \) is a homotopy equivalence. [[2], Prop 4G.2]

### 3 More Terminology and Definitions

In this section we will provide some more definitions.

#### 3.1 A moment in time

Let \( P \) be the set of \( k \) points in \( \mathbb{R}^3 \). The Voronoi cell of \( p \in P \) consists of all \( x \in \mathbb{R}^3 \) for which \( p \) minimizes the Euclidean distance among all points in \( P \):

\[
vor(p) = \{ x \in \mathbb{R}^3 | \|x - p\| \leq \|x - v\|, v \in P \}
\]  

(1)

The set of Voronoi cells, \( V = V(P) = \{ vor(p) ; p \in P \} \), is the Voronoi tessellation of \( P \). Assuming general position of the points in \( P \), the number of Voronoi cells that can have a non-empty common intersection is at most 4. We represent each set in the nerve by the convex hull of the points that generate its Voronoi cells, which can be a vertex, and edge, a triangle, or a tetrahedron. Together, these convex hulls form a simplicial complex, known as the Delaunay triangulation of \( P \), and denoted as \( D = D(P) \).

The restricted Voronoi cell of \( p \in P \) is defined as below, where \( \alpha_0 \) is any fixed positive constant.

\[
res(p) = \{ x \in vor(p) | \|x - p\| \leq \alpha_0 \}
\]  

(2)

Similar to before, we write \( R = R(P) \) for the set of restricted Voronoi cells. We define the dual alpha complex (nerve), of \( R \), denoting it as \( A \).
3.2 Trajectories

Assuming a continuous trajectory for each data point, we form subsets of space-time by taking unions of restricted Voronoi cells, both in space and in time.

A *trajectory* is a continuous mapping $\tau : [0, 1] \rightarrow \mathbb{R}^3$. We let $T$ be a finite set of trajectories, assuming no two intersects in space-time. At each time $t \in [0, 1]$, we have a finite set of points, $T(t) = \{\tau(t) \mid \tau \in T\}$.

*Restricted Voronoi and Delaunay medusas*. For each point $\tau(t) \in T(t)$, we write $\text{res}(\tau(t))$ for its restricted Voronoi cell in $\mathbb{R}^3 \times \tau$. The restricted Voronoi tessellation at time $t$ is denoted as $R(t) = R(T(t))$. Collecting restricted Voronoi cells in time, we get a 1-parameter family of cells generated by a trajectory:

$$\text{res}(\tau) = \bigcup_{t \in [0,1]} \text{res}(\tau(t)).$$

(3)

Noting that the restricted Voronoi cells on the right hand side of (3) lie in distinct parallel copies of $\mathbb{R}^3$, we call $\text{res}(\tau)$ a stack. While the restricted Voronoi cell in each time-slice is a 3-dimensional convex polyhedron, the stack itself is neither necessarily convex nor necessarily polyhedral. we write $R_m = R(T)$ for the set of stacks, each defined by a trajectory in $T$. We call $R_m$, the restricted Voronoi medusa or simply Voronoi medusa, just to avoid overusing the word restricted.

Similar to stacks of Voronoi cells, we also consider stacks of Delaunay simplices, which we call prisms. The complex of prisms and 4-simplices swept out by the simplices in the alpha complex, $A_m = A(T)$, We call it the restricted Delaunay medusa or simply Delaunay medusa. The readers can refer to [1] for even more detailed definitions.

4 Homotopy of $R_m$ and $A_m$

In this section we will try to establish that the Voronoi medusa and the Delaunay medusa are homotopic and hence they preserve the homology classes.

4.1 Decomposition of $R_m$

We $\epsilon$-thicken each stacks as open stacks with some $\epsilon > 0$, such that their intersections are unchanged and so $R_m$ does not change topologically. We
denote these open stacks as $U_1, U_2, \ldots U_k$. Let $U$ be the set of $U_1, U_2, \ldots U_k$. $U$ will form an open cover of $R_m$.

$$R_m \subset \bigcup_{i \in \{1,2,\ldots k\}} U_i$$

(4)

Using the earlier example in section(2), it directory follows that, all the finite intersection of $U_i \in U$, together with their canonical inclusions will represent a complex of spaces. We state this as our first lemma.

**Lemma 1** The cover $U$ of $R_m$ forms a complex of spaces.

The base of this complex of spaces is the barycentric subdivision $\Gamma$ of the nerve of $U (NU)$.

$R_m$ is closed and bounded, so it is compact and hence paracompact. Using the proposition 4G.2 of [2], we can say that the realization of the complex of spaces ($\Delta R_m$) associated with the cover $U$ and the space $R_m$ are homotopic.

**Lemma 2** $\Delta R_m \simeq R_m$.

### 4.2 Decomposition of $A_m$

The Delaunay medusa $A_m$ can be visualized as constructed with sticks, walls and prisms... glued at their boundaries. Visualizing this with two dimensional Voronoi cells are rather easy. Let’s fix the horizontal plane as an ambient space and vertical direction as dimension of time. Then the corresponding Voronoi medusa and Delaunay medusa can be seen as a three dimensional structures. In this case the stacks of Voronoi medusa can be homotopically sent to vertical sticks. The intersection of two stacks can be horizontal sticks or vertical walls. Similarly intersection of three stacks can be represented as, horizontal sticks, horizontal sheets or prisms or their combinations.

The initial $k$ points will give rise to $k$ sticks, due to their corresponding continuous path in space-time. The intersection of their $\alpha$-complexes at each moment in time will give rise to horizontal sticks(1-simplex), sheets(2-simplex) and tetrahedron(3-simplex) and if the intersection is for continuous period of time, they will sweep a wall(2-cell), prism(3-cell) and 4-cell (swept by tetrahedron) correspondingly. Note that the points are assumed to be in general position at most part of the time. We can take these pieces as basic
building block of $A_m$ glued together in obvious way. Just for the simplicity of notation, let us call these pieces as cover $C^v$ of $A_m$.

4.3 The alternative visualization of $\Delta R_m$

The space $\Delta R_m$, as explained in Proposition 4G.2 of [2], can also be visualized as follows:

Let $A$ denote the index set of $U$. Let $\Delta^n$ denote the standard n-simplex, here $n = \#A - 1$. To each non-empty subset $S$ of $A$ we associate the face $[S]$ of $\Delta^n$ spanned by the elements of $S$, as well as the subspace $U_S = \bigcap_{s \in S} U_s$ of $R_m$.

$\Delta R_m$ is then the subspace of $R_m \times \Delta^n$ defined by:

$$\Delta R_m = \bigcup_{\emptyset \neq S \subset A} U_S \times [S] \quad (5)$$

Here $U_S$ consists of all possible non-empty intersection of $U_i, \forall s \in U$.

As mentioned before $\Delta R_m$ is the realization of the complex of spaces associated with the open cover $U$ of $R_m$.

The product structure on $\Delta R_m$ canonically defines a map $p$ as follows:

$$p : \Delta R_m \rightarrow R_m \quad (6)$$

In Lemma 2 we have already established $p$ is a homotopy equivalence.

4.3.1 Homotopic deformations of $U_S$

We have defined $U_S$ as a subspace of $R_m$, associated to each non-empty subset $S$ of $A$ and $U_S = \bigcap_{s \in S} U_s$. It is easy to see that all these $U_S$ are contractible spatially, however we can not always contract them along the dimension of time. For example when $U_S$ is a single stack of $R_m$, this can be contracted along both space and time, however when $U_S$ is associated with intersection of two single stacks, these can stacks intersect at discrete as well as continuous moment of time. This means they are not contractible along the dimension of time.

If we contract every $U_S$ just spatially, this will be a homotopic deformation. Now these contractions will give us two kinds of object, it could be either points or 1-cells, points for one moment in time and 1-cell for continuous period of time. An intersection between 2 or more stacks can have
multiple points and multiple 1-cells as well, it will depend on the nature of their intersection.

If we take these homotopic deformations of $U_S$ as spaces and their canonical inclusions as maps, we can again represent this data as complex of spaces. This complex of spaces will have the same base $\Gamma$. The realization of this complex of spaces will be the space $A_m$. We will explain this as below.

If we replace all the $U_S$ in equation(5), with their homotopic deformations(spatial contractions), and then take products with corresponding faces $[S]$ of $\Delta^n$, we will get the components(sticks, walls and prisms...) of $A_m$. The resultant union space is indeed the $A_m$. After using proposition 4G.1 of [2] we can write the following lemma.

**Lemma 3** $A_m \simeq \Delta R_m$.

Using lemma 2 and 3 following theorem follows directly.

**Theorem 4** We have $R_m \simeq A_m$

**References**

[1] Herbert Edelsbrunner, Carl-Philipp Heisenberg, Michael Kerber, Gabriel Krens. The Medusa of Spatial Sorting: Topological Construction.

[2] A. Hatcher. Algebraic Topology. Cambridge Univ. Press, 2001.