In this work we shall exhaustively study the effects of modified gravity on the energy spectrum of the primordial gravitational waves background. S. Weinberg has also produced significant works related to the primordial gravitational waves with the most important one being the effects of neutrinos on primordial gravitational waves. With this sort review, our main aim is to gather all the necessary information for studying the effects of modified gravity on primordial gravitational waves in a concrete and quantitative way and in a single paper. After reviewing all the necessary techniques for extracting the general relativistic energy spectrum, and how to obtain in a WKB way the modified gravity damping or amplifying factor, we concentrate on specific forms of modified gravity of interest. The most important parameter involved for the calculation of the effects of modified gravity on the energy spectrum is the parameter $a_M$ which we calculate for the cases of $f(R, \phi)$ gravity, Chern-Simons-corrected $f(R, \phi)$ gravity, Einstein-Gauss-Bonnet-corrected $f(R, \phi)$ gravity, and higher derivative extended Einstein-Gauss-Bonnet-corrected $f(R, \phi)$ gravity. The exact forms of $a_M$ is presented explicitly for the first time in the literature. With regard to Einstein-Gauss-Bonnet-corrected $f(R, \phi)$ gravity, and higher derivative extended Einstein-Gauss-Bonnet-corrected $f(R, \phi)$ gravity theories, we focus on the case that the gravitational wave propagating speed is equal to that of light’s in vacuum. We provide expressions for $a_M$ expressed in terms of the cosmic time and in terms of the redshift, which can be used directly for the numerical calculation of the effect of modified gravity on the primordial gravitational wave energy spectrum.

INTRODUCTION

To date, the current perception of our Universe indicates that the Universe went through four distinct evolutionary eras, the inflationary era [1–4], the radiation domination, the matter domination era and the dark energy eras. Our knowledge is limited though, since we known very well only the physics beyond the recombination era, nearly at redshift $z \sim 1100$ where the Cosmic Microwave Background (CMB) modes exited the Hubble horizon. Concerning the dark energy era, we do not know what is the origin of this late-time acceleration era, and the same applies for the inflationary era and the most mysterious of all, the reheating and the subsequent radiation domination eras. With regard to the inflationary era, we do not even know whether it even occurred. Modified gravity in its various forms [5–10] can play a prominent role toward consistently describing inflation and the dark energy era without the need for scalar fields. In some cases, it is possible to describe inflation and the dark energy era within the same theoretical framework, see [11] for the first attempt toward this research line, and also Refs. [12–20] for some more recent works. However, admittedly the radiation domination era, and specifically, its early stages, remains quite mysterious and to date inaccessible by any currently undergoing experiment. Hopefully though, all the future interferometric gravitational wave experiments like the LISA laser interferometer space antenna [21, 22], the dHz probing DECIGO [23, 24], the Hz-kHz frequencies probing Einstein Telescope [25], and the future BBO (Big Bang Observatory) [26, 27], are expected to probe the frequency range corresponding to the reheating and radiation domination era. Specifically, the frequency range of the future gravitational waves experiments corresponds to way higher frequencies compared to the CMB ones, and will probe modes that became subhorizon right after the inflationary era, during the early stages of the reheating era. At intermediate frequencies, very promising results may be obtained by the Square Kilometer Array (SKA) [28] and the NANOGrav collaboration [29, 30], which are based on measurements of pulsar timing.
arrays. In the literature there exist several theoretical attempts to efficiently predict the energy spectrum of the primordial gravitational waves, see Refs. [31] [33] and references therein. Notable is also the work of S. Weinberg in the field, see for example Refs. [84] [55], especially notable is the effect of neutrino on the primordial gravitational waves, which was first discussed in [84]. The primordial gravitational waves form a stochastic background and on this background, important information during and after the inflationary era is imprinted. The stochastic primordial gravitational wave background is a unique tool that will probe directly the inflationary and short-post-inflationary era, since these gravitational waves are superadiabatic amplifications of the gravitational field’s zero-point fluctuations. More importantly, the evolution of the primordial gravitational waves is described by linear differential equations since the coupling of the gravitational waves with matter is tiny, contrary to the CMB modes, which obey non-linear evolution equations for wavelengths larger than 10 Mpc. There are three vital effects on the primordial gravitational waves spectrum, first the effects of the first horizon crossing during inflation, secondly the post-inflationary second horizon crossing, where the primordial tensor modes become subhorizon modes again, and thirdly, post-inflationary effects on the spectrum, caused by several sources, like matter content of the Universe, supersymmetry breaking or even modified gravity. All these effects are encoded on the stochastic background of primordial gravitational waves and will certainly offer insights on the physics that stretch back from the electroweak phase transition to the inflationary era.

If the outcome of future interferometric experiments is the discovery of a primordial gravitational wave stochastic background, modified gravity seems to be a compelling description of the primordial era of our Universe, and specifically of the inflationary and short post-inflationary ones. This is due to the fact that standard descriptions of the inflationary era, like scalar field theories, fail to produce an observable energy spectrum of primordial gravitational waves [56], unless tachyons are used. In the context of modified gravity on the other hand, blue tilted inflation is predicted, and a blue tilt can even explain recent observations on pulsar timing arrays, see for example [48] [52]. To be specific, a blue tilted tensor spectral index or an abnormal reheating era can produce an observable gravitational wave spectrum. In view of the above, in this paper we shall thoroughly study how to calculate the modified gravity effects on the energy spectrum of primordial gravitational waves. We shall use a WKB method firstly introduced in Ref. [53], which offers the possibility to quantify the overall effect of modified gravity on an integral of a single parameter $a_M$. After reviewing how to calculate the energy spectrum of the primordial gravitational waves, including the modified gravity effects, we shall calculate the parameter $a_M$ for several modified gravity theories of interest. Our aim is to offer all the information needed for the calculation of modified gravity effects for redshifts stretching from present time up to the end of the inflationary era.

This article is organized as follows: In section II we present all the formalism necessary for the extraction of the energy spectrum of the primordial gravitational waves. We review standard features of primordial gravitational waves and we extract the differential equation which governs the evolution of the primordial gravitational waves. We also show explicitly how the effects of modified gravity are encoded on a single parameter and we discuss how to calculate the overall effect of modified gravity on the primordial gravitational waves. In section III, we calculate and present formulas for the parameter $a_M$ which quantifies the effect of modified gravity on primordial gravitational waves. We shall calculate it for several modified gravities of phenomenological interest, and specifically for $f(R, φ)$ gravity, for Chern-Simons-corrected $f(R, φ)$, for Einstein-Gauss-Bonnet-corrected $f(R, φ)$ gravity and for higher derivative Einstein-Gauss-Bonnet-corrected $f(R, φ)$ gravity. Finally, the conclusions of this work are presented at the end of the paper.

I. THE SPECTRUM OF PRIMORDIAL GRAVITATIONAL WAVES IN GENERAL RELATIVITY AND MODIFIED GRAVITY EFFECTS

In this section we shall review the general features of primordial gravitational waves in the context of general relativity (GR) and we shall also quantify the way that modified gravity affects the spectrum. The analysis shall be based on Refs. [34] [46] [47] [54] [55] [57] [76] and references therein, and more details can be found in [54].

The primordial tensor perturbations are basically perturbations of a flat Friedmann-Robertson-Walker (FRW) metric,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 ,$$

where $a(t)$ is the scale factor as usual. Using the conformal time $τ$, the perturbed FRW metric is,

$$ds^2 = a^2[-dt^2 + (δ_{ij} + h_{ij})dx^i dx^j],$$

with $x'$ being the comoving spatial coordinates, and $h_{ij}$ denotes the gauge-invariant metric tensor perturbation, which is symmetric $h_{ij} = h_{ji}$, traceless $h_{ii} = 0$ and transverse $∂^i h_{ij} = 0$ conditions. The reason for traceless, transverse and
symmetric is that every tensor mode describing a gravitational wave should actually have these features. The second order Lagrangian corresponding to the tensor perturbation $h_{ij}(\tau, \mathbf{x})$ is,

$$S = \int d\tau d\mathbf{x} \sqrt{-g} \left[ -\frac{\partial^{\mu\nu}}{64\pi G} \partial_{\mu} h_{ij} \partial_{\nu} h_{ij} + \frac{1}{2} \Pi_{ij} h_{ij} \right],$$  \hspace{1cm} (3)

where the tensor part of the anisotropic stress $\Pi_{\mu\nu}$ is,

$$\Pi_{ij} = T_{ij} - \rho \delta_{ij}$$  \hspace{1cm} (4)

and satisfies $\Pi_{ii} = 0$, $\partial^i \Pi_{ij} = 0$, while it acts as an external source in the action \(\Box\). Upon varying the action \(\Box\) with respect to $h_{ij}$, we obtain,

$$h_{ij}'' + 2 \frac{a'(\tau)}{a(\tau)} h_{ij}' - \nabla^2 h_{ij} = 16\pi G a^2(\tau) \Pi_{ij}(\tau, \mathbf{x}),$$  \hspace{1cm} (5)

with the “prime” denoting differentiation with respect to the conformal time. Upon Fourier transforming Eq. (5), we get,

$$h_{ij}(\tau, \mathbf{x}) = \sum_r \sqrt{16\pi G} \int \frac{dk}{(2\pi)^{3/2}} \epsilon_{ij}^r(k) h_k^r(\tau) e^{ik\mathbf{x}},$$  \hspace{1cm} (6a)

$$\Pi_{ij}(\tau, \mathbf{x}) = \sum_r \sqrt{16\pi G} \int \frac{dk}{(2\pi)^{3/2}} \epsilon_{ij}^r(k) \Pi_k^r(\tau) e^{ik\mathbf{x}},$$  \hspace{1cm} (6b)

with $r = \{ + \}$ or $\{ \times \}$ denoting the polarization of the tensor perturbation, and the polarization tensors satisfy $[\epsilon_{ij}^r(k) = \epsilon_{ji}^r(k)]$, and also $\epsilon_{ij}^r(0) = 0$, and $h_k \epsilon_{ij}^r(k) = 0$. Eq. (5) in conjunction with Eq. (3) yields,

$$S = \sum_r \int d\tau dk \frac{a^2}{2} \left[ h_k^{r'} h_k^{r''} - k^2 h_k^r h_k^r + 32\pi G a^2 \Pi_k^r h_k^r \right],$$  \hspace{1cm} (7)

which is the action for the Fourier transformed gravitational tensor perturbations. The resulting theory can easily be quantized, with $h_k^r$ playing the role of the canonical variable, and the corresponding conjugate momentum is,

$$\pi_k^r(\tau) = a^2(\tau) h_k^{r\dagger}(\tau),$$  \hspace{1cm} (8)

hence the theory is quantized if the following equal time commutation relations hold true,

$$[\hat{h}_k^r(\tau), \hat{\pi}_k^s(\tau)] = i \delta^{rs} \delta(3)(k - k'),$$  \hspace{1cm} (9a)

$$[\hat{h}_k^r(\tau), \hat{h}_k^s(\tau)] = [\hat{\pi}_k^r(\tau), \hat{\pi}_k^s(\tau)] = 0.$$  \hspace{1cm} (9b)

The Fourier components of $\hat{h}_{ij}(\tau, \mathbf{x})$ satisfy the relation $\hat{h}_{ij}^r = \hat{h}_{ij}^{r\dagger}$, since $\hat{h}_{ij}(\tau, \mathbf{x})$ is a Hermitian operator. Therefore we have,

$$\hat{h}_k^r(\tau) = h_k(\tau) \hat{a}_k^r + h_k^*(\tau) \hat{a}_k^{r\dagger},$$  \hspace{1cm} (10)

where $\hat{a}_k^r$ and $\hat{a}_k^{r\dagger}$ being the creation and annihilation operators respectively, which satisfy,

$$[\hat{a}_k^r, \hat{a}_k^{s\dagger}] = \delta^{rs} \delta(3)(k - k'),$$  \hspace{1cm} (11a)

$$[\hat{a}_k^r, \hat{a}_k^s] = [\hat{a}_k^{r\dagger}, \hat{a}_k^{s\dagger}] = 0.$$  \hspace{1cm} (11b)

More importantly, each mode $h_k(\tau)$ satisfies the equation,

$$h_k'' + 2 \frac{a'(\tau)}{a(\tau)} h_k' - k^2 h_k = 16\pi G a^2(\tau) \Pi_k(\tau).$$  \hspace{1cm} (12)

The modes $h_k(\tau)$ depend both on the conformal time and on the wavenumber $k = |\mathbf{k}|$, but do not depend on the polarization and on the direction, as it is apparent from Eq. (11). The Wronskian normalization condition,

$$h_k(\tau)\dot{h}_k^r(\tau) - h_k^*(\tau)\dot{h}_k^r(\tau) = \frac{i}{a^2(\tau)},$$  \hspace{1cm} (13)
makes compatible the commutation relations \( [\hat{h}_i, \hat{h}_j] \), and also the initial condition for the modes is,

\[
\hat{h}_k(\tau) \rightarrow \exp(-ik\tau) a(\tau) \sqrt{2k} \quad \text{as} \quad \tau \rightarrow -\infty,
\]

which corresponds to the Bunch-Davies vacuum and describes modes which are initially at subhorizon scales and satisfy Eq. (13).

Let us proceed to the spectrum of the primordial gravitational waves \( \Omega_{gw}(k, \tau) \), and an inherent quantity to the energy spectrum, the tensor power spectrum \( \Delta^2_h(k, \tau) \). The latter can be obtained by considering,

\[
\langle 0 | \hat{h}_{ij}(\tau, \mathbf{x}) \hat{h}_{ij}(\tau, \mathbf{x}) | 0 \rangle = \int_0^\infty 64\pi G \frac{k^3}{2\pi^2} |h_k(\tau)|^2 \frac{dk}{k},
\]

and thus, the inflationary tensor power spectrum is obtained,

\[
\Delta^2_h(k, \tau) = \frac{d\langle 0 | \hat{h}_{ij}^2(0) | 0 \rangle}{dk} = 64\pi G \frac{k^3}{2\pi^2} |h_k(\tau)|^2.
\]

The energy spectrum of the primordial gravitational waves evaluated at present day \( \Omega_{gw}(k, \tau) \) is equal to,

\[
\Omega_{gw}(k, \tau) = \frac{1}{\rho_{crit}(\tau)} \frac{d\langle 0 | \hat{\rho}_{gw}(\tau) | 0 \rangle}{dk}.
\]

By definition, the energy spectrum of the primordial gravitational waves is the energy density of the gravitational waves, evaluated per logarithmic wavenumber interval. For the evaluation of \( \Omega_{gw}(k, \tau) \), we can treat the tensor perturbation \( h_{ij} \) as a quantum field in an unperturbed FRW geometric background, the stress-energy tensor of which is,

\[
T_{\alpha\beta} = -2 \frac{\delta L}{\delta g^{\alpha\beta}} + \bar{g}_{\alpha\beta} L,
\]

as it is obtained by the action \( \mathcal{L} \), with \( L \) denoting the Lagrangian function in Eq. (3). Since the future laser interferometers will seek for primordial gravitational waves in high frequencies compared to the CMB ones, we can omit the anisotropic stress couplings, and thus the gravitational wave energy density is,

\[
\rho_{gw} = -T_0 = \frac{1}{64\pi G} \frac{(h_{ij}')^2 + (\nabla h_{ij})^2}{a^2},
\]

and the corresponding vacuum expectation value of it is,

\[
\langle 0 | \rho_{gw} | 0 \rangle = \int_0^\infty \frac{k^3}{2\pi^2} \frac{|h'_{ij}(\tau)|^2 + k^2 |h_k(\tau)|^2}{a^2} \frac{dk}{k},
\]

hence the gravitational wave energy spectrum reads,

\[
\Omega_{gw}(k, \tau) = \frac{8\pi G}{3H^2(\tau)} \frac{k^3}{2\pi^2} \frac{|h'_{ij}(\tau)|^2 + k^2 |h_k(\tau)|^2}{a^2(\tau)}. \]

Moreover, using \( |h'_{ij}(\tau)|^2 = k^2 |h_k(\tau)|^2 \) the gravitational wave energy spectrum evaluated at present day can be rewritten as follows,

\[
\Omega_{gw}(k, \tau) = \frac{1}{12} \frac{k^2 \Delta^2_h(k, \tau)}{H_0^2(\tau)},
\]

where \( H_0 \) is the Hubble rate at present day, and we also assumed that the present day scale factor is equal to unity, in order for comoving quantities (frequencies and wavelengths) to be identical with physical quantities. The interest in primordial gravitational wave searches is for large frequency modes that became subhorizon during the dark era of reheating and radiation domination era, thus for modes which entered the Hubble horizon first after the end of the inflationary era. Let us now be more concrete on the calculation of the energy spectrum of the primordial gravity waves, and first we consider the effects on it, caused by horizon re-entry of a mode \( k \), in which case \( 34, 40, 47, 54, 55, 57 \),

\[
h^{\lambda}_{k}(\tau) = h^{\lambda}_{k}(\nu) \left( \frac{3j_1(k\tau)}{k\tau} \right),
\]
where \( j_\ell \) denotes the \( \ell \)-th spherical Bessel function. The Fourier transformation of the primordial tensor perturbation during a power-law cosmological evolution \( a(t) \propto t^p \) is,

\[
h_k(\tau) \propto a(t)^{\frac{1-3p}{3p}} J_{3p-1}(k\tau),
\]

(24)

with \( J_n(x) \) being the Bessel function. Moreover, taking into account the damping caused by the relativistic degrees of freedom in the early Universe which do not remain constant, the following factor for \( h_k(\tau) \) is obtained \[10\],

\[
\left( \frac{g_*(T_{in})}{g_{*0}} \right) \left( \frac{g_{*0}}{g_{**}(T_{in})} \right)^{4/3} \left( \frac{3g_{**0}(k\tau_0)}{k\tau_0} \right)^2 T_{eq}^2(x_{eq}) T_2^2(x_R),
\]

(25)

where the scale factor evolves as \( a(t) \propto T^{-1} \) assuming adiabatic evolution which follows by the fact that the entropy of the Universe \( S \) is constant,

\[
S = \frac{2\pi^2}{48} g_{**}(aT)^3 = \text{const}.
\]

(26)

Above, \( T_{in} \) denotes the temperature of the Universe at horizon re-entry,

\[
T_{in} \simeq 5.8 \times 10^6 \text{ GeV} \left( \frac{g_{**}(T_{in})}{106.75} \right)^{-1/6} \frac{k}{10^{14}\text{ Mpc}^{-1}}.
\]

(27)

Also it is vital to take into account the damping factor caused by the current acceleration of the Universe \( \sim (\Omega_m/\Omega_\Lambda)^2 \) \[34,35\]. The full expression for the present day measured primordial gravity waves energy spectrum per log frequency, includes all the aforementioned damping effects and in addition and more importantly, it contains the transfer functions which are calculated numerically by integrating the evolution differential equation of the Fourier transformed tensor perturbation. The full expression for the energy density of the primordial gravitational waves is,

\[
\Omega_{gw}(f) = \frac{k^2}{12H_0^2} \Delta_h^2(k),
\]

(28)

with the detailed form of \( \Delta_h^2(k) \) being \[34,46,47,54,57\],

\[
\Delta_h^2(k) = \Delta_{h}^{(p)}(k)^2 \left( \frac{\Omega_m}{\Omega_\Lambda} \right)^2 \left( \frac{g_*(T_{in})}{g_{*0}} \right) \left( \frac{g_{*0}}{g_{**}(T_{in})} \right)^{4/3} \left( \frac{3g_{**0}(k\tau_0)}{k\tau_0} \right)^2 T_{eq}^2(x_{eq}) T_2^2(x_R),
\]

(29)

where \( g_*(T_{in}(k)) \) in Eq. \[29\] is \[42\],

\[
g_*(T_{in}(k)) = g_{*0} \left( \frac{A + \tanh \left[ -2.5 \log_{10} \left( \frac{k/2\pi}{2.5 \times 10^{-19} \text{Hz}} \right) \right]}{A + 1} \right) \left( B + \tanh \left[ -2.5 \log_{10} \left( \frac{k/2\pi}{6 \times 10^{-19} \text{Hz}} \right) \right] \right),
\]

(30)

with the parameters \( A \) and \( B \) being equal to,

\[
A = \frac{-1 - 10.75/g_{*0}}{-1 + 10.75g_{*0}},
\]

(31)

\[
B = \frac{-1 - g_{max}/10.75}{-1 + g_{max}/10.75},
\]

(32)

with \( g_{max} = 106.75 \) and \( g_{*0} = 3.36 \). Also by replacing \( g_{*0} = 3.36 \) with \( g_{*s} = 3.91 \), we can calculate \( g_*(T_{in}(k)) \), using the same formulas, namely, Eqs. \[30\], \[41\] and \[42\]. Note that the “bar” above the Bessel function in Eq. \[29\] indicates averaging over many integration periods. Also, the term \( \Delta_{h}^{(p)}(k)^2 \) in Eq. \[29\] denotes the inflationary era’s primordial tensor spectrum, the analytic form of which is \[34,46,47,54,55,57\],

\[
\Delta_{h}^{(p)}(k)^2 = A_T(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T},
\]

(33)
where the scale \( k_{\text{ref}} = 0.002 \text{ Mpc}^{-1} \) is the CMB pivot scale. Also \( A_T(k_{\text{ref}}) \) denotes the amplitude of the primordial tensor perturbations, \( n_T \) denotes the tensor spectral index. Replacing,

\[
A_T(k_{\text{ref}}) = r \mathcal{P}_\zeta (k_{\text{ref}}),
\]

where \( \mathcal{P}_\zeta (k_{\text{ref}}) \) is the amplitude of the primordial scalar perturbations, we have finally,

\[
\Delta_h^{(p)}(k)^2 = r \mathcal{P}_\zeta (k_{\text{ref}}) \left( \frac{k}{k_{\text{ref}}} \right)^{n_T}.
\]

Regarding the transfer functions \( T_1(x_{\text{eq}}) \) and \( T_2(x_R) \), the analytic form of the first one is \([34, 46, 47, 54, 55, 57]\),

\[
T_1^2(x_{\text{eq}}) = [1 + 1.57x_{\text{eq}} + 3.42x_{\text{eq}}^2],
\]

and it basically characterizes the modes that re-entered the Hubble horizon approximately during the matter-radiation equality with cosmic time instance \( t = t_{\text{eq}} \). About the notation of several parameters in Eq. (36),

\[
x_{\text{eq}} = \frac{k}{k_{\text{eq}}}, \quad \text{and} \quad k_{\text{eq}} \equiv a(t_{\text{eq}})H(t_{\text{eq}}) = 7.1 \times 10^{-2} \Omega_m h^2 \text{ Mpc}^{-1}.
\]

Now regarding the transfer function \( T_2(x_R) \) appearing in Eq. (29), it is related with modes that entered the Hubble horizon during the reheating era, with the mode \( k \) being \( k > k_R \). Its analytic form is,

\[
T_2^2(x_R) = (1 - 0.22x^{1.5} + 0.65x^2)^{-1},
\]

where \( x_R = \frac{k}{k_R} \), and also the \( k_R \) wavenumber is equal to,

\[
k_R \simeq 1.7 \times 10^{13} \text{ Mpc}^{-1} \left( \frac{g_* (T_R)}{106.75} \right)^{1/6} \left( \frac{T_R}{10^6 \text{ GeV}} \right),
\]

with \( T_R \) being the reheating temperature. Also the reheating frequency is,

\[
f_R \simeq 0.026 \text{ Hz} \left( \frac{g_* (T_R)}{106.75} \right)^{1/6} \left( \frac{T_R}{10^6 \text{ GeV}} \right).
\]

Having presented the GR primordial gravitational wave energy density, in the next section we shall consider the effects caused by modified gravity of various forms on the GR waveform. Thus we shall quantify the effects of modified gravity on the GR waveform in an explicit way.

### A. The Modified Gravity Effect on the Energy Spectrum of the Primordial Gravitation Waves: A WKB Approach

In this section we shall quantify the effect of an arbitrary modified gravity on the GR waveform of primordial gravitational waves. Let us recall for convenience at this point the differential equation that is obeyed by the Fourier transformation of the tensor perturbation \( h_{ij} \),

\[
\ddot{h}(k) + (3 + \alpha_M) H \dot{h}(k) + \frac{k^2}{a^2} h(k) = 0,
\]

with \( \alpha_M \) being defined as follows,

\[
\alpha_M = \frac{Q_i}{Q_i H},
\]

where \( Q_i \) is unique for every distinct modified gravity. The overall effect of modified gravity is encoded on the parameter \( a_M \), and the functional form of this parameter is different for distinct modified gravities. In addition, the evolution differential equation \([40]\) characterizes all the distinct polarizations of the gravitational waves. In order to extract in a consistent way the overall modified gravity effect, we shall use Nishizawa’s approach \([54, 55]\), which is basically a WKB approach. Expressed in terms of the conformal time, the differential equation \([41]\) becomes,

\[
\dddot{h}(k) + (2 + \alpha_M) H \dot{h}'(k) + k^2 h(k) = 0,
\]
FIG. 1. Post-inflationary subhorizon modes for which the WKB solution of the evolution equation (42) is justified. These subhorizon modes, became subhorizon immediately after inflation, so during the early stages of the dark ages, the reheating and the subsequent radiation domination eras. These early subhorizon modes will be probed by the future interferometer gravitational wave experiments.

with the “prime” in the above equation indicating differentiation with respect to the conformal time \( \tau \), and also we defined \( \mathcal{H} = \frac{a'}{a} \). We shall extract the WKB solution by taking into account only subhorizon modes satisfying Eq. (42) (see Fig. 1). This is highly justified for primordial gravitational waves studies, since the high frequency interferometers will probe modes which became subhorizon modes immediately after the inflationary era, so during the early reheating era. Considering a solution of the form \( h_{ij} = Ae^{iB}h_{ij}^{GR} \), for theories in which the speed of the gravitational wave is equal to unity in natural units, the WKB solution for subhorizon modes is [54, 55], (42) is of the form,

\[
h = e^{-D}h_{GR},
\]

where \( h_{ij} = h e^{i\eta} \), with \( h_{GR} \) denoting the GR waveform which is the solution to the differential equation (42) by taking \( a_M = 0 \). More importantly, the parameter \( D \) is equal to,

\[
D = \frac{1}{2} \int^\tau a_M \mathcal{H} d\tau_1 = \frac{1}{2} \int_0^z a_M \frac{dz'}{1+z'} dz',
\]

and mainly quantifies the direct effect of modified gravity on the GR waveform of primordial gravitational waves. Now if one wishes to calculate the energy spectrum of the primordial gravitational waves, it is vital to calculate the damping/amplification factor of Eq. (42) starting from redshift \( z = 0 \) which corresponds to present time up to redshifts corresponding deeply in the reheating era. The latter redshifts correspond to primordial modes that became subhorizon modes immediately after the inflationary era, so basically we are interested in extreme subhorizon modes at present day, with significantly small wavelength compared to the CMB scale modes. These subhorizon modes will be probed in fifteen years from now, from LISA, BBO, DECIGO and other gravitational waves experiments. Thus, by also taking intro account the effects of modified gravity, the energy spectrum of the primordial gravitational waves at present day is,

\[
\Omega_{gw}(f) = e^{-2D} \times \frac{k^2}{12H_0^2} \mathcal{P}_\xi(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T} \left( \frac{\Omega_m}{\Omega_\Lambda} \right)^2 \left( \frac{g_{*s}(T_{in})}{g_{*s0}} \right) \left( \frac{g_{*s0}}{g_{*s}(T_{in})} \right)^{4/3} \left( \frac{3j_1(k\tau_0)}{k\tau_0} \right)^2 T_1^2(x_{eq}) T_2^2(x_R).
\]

Depending on the specific form of the modified gravity, the parameter \( D \) might be positive or negative. Therefore, the GR energy spectrum might be damped or amplified due to the overall modified gravity effects. In the next subsections we shall calculate in detail all the possible forms of the parameter \( a_M \) appearing in Eqs. (41) and (42) corresponding to various modified gravities of interest. We shall give the expressions of \( a_M \) both with respect to the cosmic time and with respect to the redshift, for the calculational convenience of the reader. We shall also present the results in a table also for reading convenience of the reader.
II. PRIMORDIAL GRAVITY WAVES IN MODIFIED GRAVITY IN ITS VARIOUS FORMS

Let us first consider the calculation of the parameter $a_M$ appearing in Eq. (41) for the case of pure $f(R, \phi)$ gravity, in which case the gravitational action is,

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right).$$  \hfill (45)

For this form of modified gravity, the parameter $Q_t$ is equal to $Q_t = \frac{1}{\kappa^2} \frac{\partial f(R, \phi)}{\partial R}$, where $\kappa = \frac{1}{M_p}$, with $M_p$ being the reduced Planck mass. Hence, for pure $f(R, \phi)$ gravity, the parameter $a_M$ reads,

$$a_M = \frac{\partial^2 f}{\partial R \partial \phi} \phi + \frac{\partial f}{\partial R} R. \hfill (46)$$

We can express the above formula in terms of the redshift in order to have an expression ready for the integral in Eq. (43) in terms of the redshift. Using the following formula,

$$\frac{d}{dt} = -H(1+z) \frac{d}{dz},$$ \hfill (47)

the parameter $a_M$ expressed in terms of the redshift reads,

$$a_M = \frac{-\partial^2 f}{\partial R \partial \phi} H(z)(1+z) \frac{d\phi}{dz} - \frac{\partial f}{\partial R} H(z)(1+z) \frac{dR}{dz}. \hfill (48)$$

Let us consider the case of a Chern-Simons corrected $f(R, \phi)$ gravity, in which case the gravitational action reads,

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) + \frac{1}{8\nu} \nu(\phi) R \tilde{R} \right), \hfill (49)$$

where $R \tilde{R} = \epsilon^{abcd} R_{ab}^c R_{de}^f$ and $\epsilon^{abcd}$ stands for the totally antisymmetric Levi-Civita tensor. In the literature, terms of the form $\nu(\phi) R \tilde{R}$ are known as Chern-Simons terms. We need to note though that the term $\nu(\phi) R \tilde{R}$ is formally the Chern-Pontryagin density, which connected to an actual three dimensional Chern-Simons term via the exterior derivative $\nu(\phi) R \tilde{R} = d(\text{Chern} - \text{Simons})$. Notably the Chern-Pontryagin density is the analogue of the quantity $*F_{\mu\nu} F^{\mu\nu}$ which is constructed by the curvature $F_{\mu\nu}$ on a principal bundle with connection $A_\mu$, but abusively in the literature it is called Chern-Simons term, due to the analogy we pointed out. In the Chern-Simons corrected $f(R, \phi)$ gravity, the term $Q_t$ is equal to \[87\],

$$Q_t = \frac{1}{\kappa^2} \frac{\partial f}{\partial R} + \frac{2\lambda k}{a}, \hfill (50)$$

where $\lambda_t$ denotes the polarization of the gravitational wave and takes the values $\lambda_t = 1$ for right handed gravity waves, and $\lambda_L = -1$ for left handed gravity waves, while $k$ is the wavenumber of the tensor mode. Thus for the Chern-Simons corrected $f(R, \phi)$ gravity, the $a_M$ term in terms of the cosmic time reads,

$$a_M = \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R \partial \phi} \phi + \frac{1}{\kappa^2} \frac{\partial^2 f}{\partial R^2} \tilde{R} + \frac{2\lambda k}{a} - \frac{2\lambda \nu k H}{a} \left( \frac{1}{\kappa^2} \frac{\partial f}{\partial R} + \frac{2\lambda \nu k}{a} \right) H, \hfill (51)$$

where the last term numerator is found by differentiating $Q_t$ in Eq. (50). Expressing $a_M$ in terms of the redshift, we have,

$$a_M = \frac{-\partial^2 f}{\partial R \partial \phi} H(z)(1+z) \frac{d\phi}{dz} - \frac{\partial f}{\partial R} H(z)(1+z) \frac{dR}{dz} + \frac{2\lambda \nu k (1+z) H}{a} + \frac{2\lambda \nu \nu_d(z) k}{a} \left( \frac{1}{\kappa^2} \frac{\partial f}{\partial R} - \frac{2\lambda (1+z) H^2}{a} \right) H. \hfill (52)$$

where $\nu_{dd}(z)$ is equal to,

$$\nu_{dd}(z) = -(1+z) H \left( \frac{dH}{dz} (1+z) \frac{d\nu}{dz} + \frac{d\nu}{dz} H + H (1+z) \frac{d^2 \nu}{dz^2} \right). \hfill (53)$$
Now let us consider the case of $f(R, \phi)$ gravity with an Einstein-Gauss-Bonnet term, in which case the gravitational action reads,

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) - \frac{1}{2} \xi(\phi) g \right).$$  \hspace{1cm} (54)

For these theories, the speed of gravitational tensor perturbations is not equal to that of light’s, but it is equal to $87$,

$$c_T^2 = 1 - \frac{4\left(\dot{\xi} - \ddot{\xi} H\right)}{\frac{\partial f}{\partial R} - 4\dot{\xi} H}. \hspace{1cm} (55)$$

Since we shall consider only theories for which the gravitational wave speed is equal to that of light’s, the condition $c_T^2 = 1$ can be satisfied only if the Gauss-Bonnet scalar coupling function $\xi(\phi)$ satisfies the differential equation $\ddot{\xi} - \dot{\xi} H = 0$. The inflationary phenomenology of this class of Einstein-Gauss-Bonnet theories have been thoroughly studied in the literature, see $88$-$91$. For Einstein-Gauss-Bonnet corrected $f(R, \phi)$ theories, the parameter $Q_\ell$ reads $87$,

$$Q_\ell = \frac{\partial f}{\partial R} - 4\dot{\xi} H,$$  \hspace{1cm} (56)

therefore, the parameter $a_M$ for the Einstein-Gauss-Bonnet corrected $f(R, \phi)$ theory reads,

$$a_M = \frac{\frac{1}{x^2} \frac{\partial^2 f}{\partial \phi^2} \phi + \frac{1}{x^2} \frac{\partial^2 f}{\partial R \partial \phi} R - 4\dot{\xi} H - 4\dot{\xi} H}{\left(\frac{1}{x^2} \frac{\partial f}{\partial R} - 4\dot{\xi} H\right) H}, \hspace{1cm} (57)$$

and $\ddot{\xi}$ must be replaced with $\ddot{\xi} = \dot{\xi} H$ due to the gravitational wave speed constraint $c_T^2 = 1$. In terms of the redshift, the parameter $a_M$ has the following form,

$$a_M = \frac{-\frac{1}{x^2} \frac{\partial^2 f}{\partial \phi^2} H(z)(1 + z) d\phi dz - \frac{1}{x^2} \frac{\partial^2 f}{\partial R \partial \phi} H(z)(1 + z) dR dz + 4H^2(1 + z) d\phi dz - 4H^2(1 + z)^2 d\phi dz dH dz}{\left(\frac{1}{x^2} \frac{\partial f}{\partial R} + 4H^2(1 + z) d\phi dz\right) H}, \hspace{1cm} (58)$$

Finally, let us consider a generalized Einstein-Gauss-Bonnet-corrected $f(R, \phi)$ theory with higher order derivative couplings, in which case the gravitational action is,

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) - \frac{1}{2} \xi(\phi) G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right),$$  \hspace{1cm} (59)

where $G^{\mu\nu}$ is the Einstein tensor and $c_1$, $c_2$ are dimensionless constants. For this theory, the gravitational wave speed is again non-trivial and different from that of light’s in vacuum. Specifically the gravitational wave speed is $87$,

$$c_T^2 = 1 - \frac{8c_1 \left(\ddot{\xi} - \dot{\xi} H\right) + 2c_2 \dot{\xi}^2}{\frac{\partial f}{\partial R} - 4\dot{\xi} H}. \hspace{1cm} (60)$$

As in the in Einstein-Gauss-Bonnet-corrected theory we presented previously, we are interested for theories with gravitational wave speed equal to that of light’s in vacuum, and for the case at hand, the theory can have $c_T^2 = 1$ when the Gauss bonnet scalar coupling $\xi(\phi)$ satisfies the differential equation

$$8c_1 \left(\ddot{\xi} - \dot{\xi} H\right) + 2c_2 \dot{\xi}^2 = 0.$$  \hspace{1cm} (61)
With this constraint satisfied, the only parameter that needs to be calculated for evaluating the energy spectrum of the primordial gravitational waves spectrum is \( a_M \) given in Eq. (41). In the case at hand, the parameter \( Q_t \) is equal to \( \xi H \),

\[
Q_t = \frac{\partial f}{\partial R} - 4c_1 \xi H - \frac{c_2}{2} \dot{\phi}^2,
\]

therefore, the parameter \( a_M \) for the Einstein-Gauss-Bonnet-corrected \( f(R, \phi) \) theory with higher derivative coupling terms reads,

\[
a_M = \frac{1}{\xi c_1} \frac{\partial^2 f}{\partial R \partial \phi} \dot{\phi} + \frac{1}{\xi c_1} \frac{\partial^2 f}{\partial \phi^2} R - 4c_1 \xi H - 4c_1 \xi \dot{H} + c_2 \xi \dot{\phi}^2 + c_2 \xi \ddot{\phi} \left( \frac{1}{\xi c_1} \frac{\partial f}{\partial R} - 4c_1 \xi H + \frac{c_2}{2} \xi \dot{\phi}^2 \right) H.
\]

The above concludes the most complicated extension of \( f(R, \phi) \) modified gravity with \( c_T^2 = 1 \), for which the parameter \( a_M \) of Eq. (41) can be calculated. It is always a computational challenge to calculate numerically the parameter \( D \) appearing in Eq. (44) for redshifts corresponding to modes which became subhorizon after the inflationary era, during the reheating and the radiation domination era. In Table IV we gather all the various forms of the parameter \( a_M \) for the various forms of modified gravity which were considered in this section.

III. CONCLUSIONS

In this work we studied the way that various forms of modified gravity affect the energy spectrum of the primordial gravitational waves. We presented the standard features of the energy spectrum of the primordial gravitational waves in GR, and how modified gravity affects the spectrum in a quantitative way. The critical effect of modified gravity on the energy spectrum of the primordial gravitational waves is quantified on a single parameter denoted \( a_M \) and its integral for redshifts extending from present day up to redshifts corresponding to the early post-inflationary era. We calculated the parameter \( a_M \) for several modified gravities of interest, and specifically for the \( f(R, \phi) \) gravity, for Chern-Simons-corrected \( f(R, \phi) \) gravity, for Einstein-Gauss-Bonnet-corrected \( f(R, \phi) \) gravity and for higher derivative extensions of Einstein-Gauss-Bonnet-corrected \( f(R, \phi) \) gravity. The motivation for studying several modified gravity effects on primordial gravitational waves is based on the possible verification of the stochastic primordial gravitational waves background by future experiments. The actual verification will stir things up significantly in the context of GR, the abnormal reheating effects could be minor and could not amplify the spectrum significantly to be detectable, see for example the \( w \) EoS post-inflationary parameter \( C_2 \) of [34]. Thus, it is rather compelling to study in detail and thoroughly all the effects of modified gravity in its various forms, on the energy spectrum of primordial gravitational waves. With this paper we presented a rigid overview of the method needed for calculating in a formally correct way the overall effect of several modified gravities of interest on the energy spectrum of the primordial gravitational waves.

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