The universal features of a Fermi liquid and their physical consequences continue to attract the attention of the condensed-matter community. In a generic Fermi liquid, the imaginary part of the retarded fermionic self-energy $\Sigma_R(k, \omega)$ on the mass shell is determined solely by fermions in a narrow ($\sim \omega$) energy range around the Fermi surface and behaves as $\Sigma'' \propto \omega^2 + (\pi T)^2$. This regular behavior of the self-energy has a profound effect on such observables as the specific heat and uniform spin and charge susceptibilities, which have the same functional dependences as for free fermions, i.e., the specific heat is linear in $T$ and the susceptibilities approach finite values at $T = 0$. A regular behavior of the fermionic self-energy is also in line with a general reasoning that turning on the interaction in $D > 1$ should not affect drastically the low-energy properties of a system, unless special circumstances, e.g., a proximity to a quantum phase transition, interfere.

The subject of this paper is the analysis of non-analytic corrections to the Fermi-liquid behavior in a generic interaction $U(q)$ and show that only $U(0)$ and $U(2p_F)$ matter.

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Our motivation to pursue a further study of non-analytic corrections to the Fermi-liquid behavior is twofold. First, we want to verify that power counting arguments by carrying out an explicit analytic calculations of several observable quantities: specific heat, effective mass, and spin- and charge susceptibilities. That power counting may be misleading is seen, e.g., from the example of the free-fermion susceptibility: according to power counting, it should also have a non-analytic momentum dependence of the effective mass $m^*(T)$ and to the $T$-dependence of the subleading term in the specific heat, $\delta C(T)$, with extra logarithms in 3D and 1D.

Misawa [11] argued that the non-analyticity in the fermionic self-energy should gives rise to a non-analytic momentum expansion of the particle-hole susceptibility $\chi(Q, T)$. For non-interacting fermions, $\chi(Q, 0)$ is given by the Lindhard function and is analytic in $Q$ for small $Q$ in all $D$. Diagrammatically, corrections to the Lindhard function are obtained by self-energy and vertex-corrections insertions into the particle-hole bubble (cf. Fig.2). Diagrams with self-energy insertions are readily estimated by power counting, and the result is that the non-analytic, $\omega^D$ term in $\Sigma$ gives rise to $\delta \chi(Q) \propto Q^{D-1},$ with extra logs for $D = 3$ and $D = 1$. Power counting also suggests that the non-analyticity in $\Sigma$ should affect the temperature dependence of the uniform susceptibility $\chi(0, T)$ and gives rise to $\delta \chi(0, T) \propto T^{D-1},$ with extra logarithms in $D = 3$ and $D = 1$. By the same arguments, non-analyticity in $\Sigma$ should lead to the $T^{D-1}$-dependence of the effective mass $m^*(T)$ and to the $T$-dependence of the subleading term in the specific heat, $\delta C(T)$, with extra logarithms in 3D and 1D.

The non-analytic $\omega^D$ terms (as well as non-analytic vertex corrections) are of fundamental interest as they may give rise to anomalous temperature and momentum dependences of observable quantities. A well-known example is the $T^3 \ln T$ term in the specific heat of a 3D Fermi liquid, caused by the anomalous term $\Sigma''(\omega) \propto \omega^3$ (and hence $\Sigma'(\omega) \propto \omega^3 \log \omega$) [4]. A related example is the linear-in-$T$ correction to the conductivity of a weakly disordered 2D system [7, 14]. Non-analytic corrections are also important for the theory of quantum critical phenomena in itinerant ferromagnets [11], as a non-analyticity of the static spin susceptibility changes the nature of the phase transition [10].

Belitz, Kirpatrick and Vojta (BKV) [12] and, later,
not been performed. Hirashima and Takahashi \textsuperscript{18} computed $\chi(0,T)$ in 2D numerically, but could not draw any definite conclusions about the $T$-dependence because of numerical difficulties. Chitov and Mills (CM) \textsuperscript{12} found analytically that the leading term in $\chi_s(0,T)$ in 2D scales as $T$, in agreement with power counting \textsuperscript{13}. At the same time, CM found that different contributions to the non-analytic $T$ term in $m^*(T)$ and to the $T^2$ term in $\delta C(T)$ (both predicted by power counting), cancel each other, and only analytic corrections survive. Meanwhile, Bedell and Coffey \textsuperscript{15} reported a $T^2$ term in $\delta C(T)$. Very recently, Das Sarma et al. found a linear-in-$T$ term in the effective mass for the Coulomb interaction in $D=2$ \textsuperscript{21}.

In this paper, we present analytic results for the specific heat, effective mass, and spin and charge susceptibilities of an interacting 2D Fermi system, up to second order in the short-range interaction $U(q)$. We found that power counting arguments are generally valid, i.e., $\chi_s(Q) \propto |Q|$, $\chi_s(T) \propto T$, $m(T) \propto T$, and $\delta C(T) \propto T^2$. These results agree with Refs. \textsuperscript{13} \textsuperscript{14} \textsuperscript{15}; the form of $\chi_s(T)$ agrees with that found by CM but the forms of $m(T)$ and $\delta C(T)$ disagree with those by CM. Still, our prefactors for non-analytic terms differ from that found in \textsuperscript{15} \textsuperscript{19}. We also verified that in 3D, $\chi_s(Q,0) \propto Q^2 \ln Q$ while $\chi_s(0,T) \propto T^2$, in agreement with BKV. Finally, in agreement with CM, we found no non-analytic terms in the charge susceptibility, $\chi_c$.

Another motivation for this study is to clarify the origin of the non-analytic corrections in the Fermi-liquid theory. We found that these corrections originate from the singularities in the dynamic particle-hole response function, $\Pi(q,\Omega)$, near $q=0$ and $q=2k_F$, where $\Pi(q,\Omega)$ is non-analytic. For $D=2$, near $q=0$

$$\Pi_{ph}^{<q=0}(q,\Omega_m) = \frac{m}{2\pi} \left( 1 - \frac{|\Omega_m|}{\sqrt{(vFq)^2 + \Omega_m^2}} \right) .$$

whereas near $q=2k_F$,

$$\Pi_{ph}^{<q=2k_F}(q,\Omega_m) = \frac{m}{2\pi} \times \left( 1 - \frac{\hat{q}}{2k_F} + \left[ \frac{\Omega_m}{2v_Fk_F} \right]^2 + \left( \frac{\hat{q}}{2k_F} \right)^2 \right)^{1/2} ,$$

where $\hat{q} \equiv q - 2k_F$ and $\hat{q} \ll 2k_F$. Physically, these two singularities give rise to a zero-sound mode and Friedel oscillations, respectively. The singularity near $q=0$ is entirely dynamic, while the one near $2k_F$ is also present in the static limit for $q > 2k_F$. We found that the singularities in $\Pi(q,\Omega)$ are necessary ingredients which make power counting arguments valid. Furthermore, we found that the singular pieces in the effective mass, specific heat and $\chi_s(Q,T)$ originate exclusively from the scattering amplitude with zero momentum transfer and zero total momentum: $\Gamma_{\alpha,\beta;\gamma,\delta}(k,-k;k,-k) = U(0)\delta_{\alpha\gamma}\delta_{\beta\delta} - U(2k_F)\delta_{\alpha\delta}\delta_{\beta\gamma}$. This implies that (i) non-analytic terms depend only on $U(0)$ and $U(2k_F)$ but not on the interaction averaged over the Fermi surface, and (ii) up to overall sign, the non-analyticities at $q \approx 0$ and $q \approx 2k_F$ contribute equally to individual diagrams for the fermionic self-energy and $\chi_s$, i.e., singular corrections to a Fermi liquid can be viewed equivalently as coming either from the singularity in the dynamical particle-hole bubble at $q \approx 0$ or at $q = 2k_F$. As the $q = 0$ singularity is entirely dynamical, the non-analytic corrections to a Fermi liquid are dynamical in nature as well.

Effective mass and specific heat. To find the effective mass, $m^*(T)$, and the correction to the specific heat, $\delta C(T)$, one needs to know the real part of the fermionic self-energy, $\Sigma'(k,\omega)$, on the mass shell, i.e., at $\epsilon_k \equiv (k^2 - k_F^2) / 2m = \omega$. The two nontrivial second-order diagrams for the Fermi energy are presented in Fig. 1. We evaluated $\Sigma'(k,\omega)$ first, using the spectral representation, and then obtained $\Sigma'(\omega)$ on the mass shell via Kramers-Kronig transformation. The imaginary part of the self-energy reduces to well-known forms \textsuperscript{2} $\Sigma''(k,\omega) \propto \omega^2 \ln \omega$ and $\Sigma''(k,\omega) \propto T^2 \ln T$ for $k$ near the Fermi surface and in the limits of $T \to 0$ and $\omega \to 0$, respectively. We obtained $\Sigma''(k,\omega)$ at arbitrary $\omega/T$ and $\omega/\epsilon_k$. The full expressions are, however, rather involved and will be presented elsewhere \textsuperscript{21}. Using full $\Sigma''(k,\omega)$, we find for the real part of the self-energy on the mass shell, neglecting a regular, Fermi-liquid type $\omega$ term

$$\Sigma'(\omega) = -\frac{mU^2}{16\pi^2v_F^2} \omega |g(\omega/T)| ,$$

where $U^2 = U^2(0) + U^2(2p_F) - U(0)U(2p_F)$ and

$$g(x) = 1 + \frac{4}{x^2} \left[ \frac{\pi^2}{12} + \text{Li}_2 \left( -e^{-|x|} \right) \right]$$

where $\text{Li}_2(x)$ is a polylogarithmic function.

In the two limits, $g(\infty) = 1$ and $g(x \ll 1) \approx 4 \ln 2/x$. The first limit corresponds to $T = 0$ in which case Eq. \textsuperscript{43} gives $\Sigma'(\omega) \propto \omega |\omega|$. This non-analytic form agrees with power counting. For small $\omega/T$, i.e., for $x \ll 1$, the $1/x$ form of $g(x)$ leads to the $\omega T$ term in $\Sigma'(\omega)$ for $\omega \ll T$. This in turn implies that the quasiparticle mass

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Non-trivial second-order diagrams for the self-energy.}
\end{figure}
$m^*(T)$ acquires a linear-in-$T$ correction

$$m^*(T) = m \left( 1 - m^2 \bar{U}^2 \frac{\ln \frac{2}{8\pi^2}}{T} \right).$$  \hspace{1cm} (5)

Using Eqs. (4,1), we find a correction to the specific heat

$$\delta C(T) = \frac{2m}{T} \left[ \int_{-\infty}^{\infty} d\omega \frac{\partial n}{\partial \omega} \Sigma'(\omega, \epsilon_k = \omega) \right] = 0.174 C_{FL} m^2 \bar{U}^2 \left( \frac{T}{E_F} \right),$$  \hspace{1cm} (6)

where $C_{FL} = \pi T^3 m/3$ is the Fermi-gas result. We see that a non-analyticity in the fermionic self-energy gives rise to the $T^2$ term in the specific heat. This term comes only from fermions in a near vicinity of the Fermi surface and from this perspective is model-independent.

We also verified that the non-analytic part of $\Sigma'(\omega)$ stems exclusively from the scattering process in which two internal momenta in the self-energy diagram are near $-\mathbf{k}$, whereas the third one is near $\mathbf{k}$, i.e., when both the total and transferred momentum are near zero. As an independent check, we obtained the non-analytic part of $\Sigma'(\omega)$ by re-expressing the self-energy in terms of the particle-hole bubble—Eq. (3) then results from the (logarithmic) singularity of the particle-particle bubble at small total momentum and frequency. This should indeed be the case if only the scattering amplitude with zero total momentum is important.

Spin susceptibility. The relevant diagrams for the spin susceptibility are presented in Fig. 2. Evaluation of the diagrams is rather tedious but straightforward. We calculated all diagrams in two ways: i) explicitly, by exploring the non-analyticities in the particle-hole bubble near $q = 0$ and $2k_F$, and ii) by retaining only vertices in which both total and transferred are small. In the second approach, we expanded in total and transferred momenta and extracted universal contributions, which are independent of the upper cutoff for the expansion. We obtained identical results in both methods, which proves that only a single scattering amplitude is relevant.

The non-analytic contributions to the spin susceptibility from individual diagrams are as follows

$$\chi_1(Q,T) = \chi_0 K(Q,T) [U^2(0) + U^2(2k_F)];$$
$$\chi_2(Q,T) = -\chi_0 K(Q,T) U(0) U(2k_F);$$
$$\chi_3(Q,T) = \chi_0 K(Q,T) [U^2(2k_F) - U^2(0)];$$
$$\chi_4(Q,T) = \chi_0 K(Q,T) U(0) U(2k_F);$$
$$\chi_5 = \chi_6 = \chi_7 = 0,$$  \hspace{1cm} (7)

where $\chi_0 = m/\pi$, and $K(Q,0)$ and $K(0,T)$ are given by

$$K(Q,0) = \frac{2}{3\pi} \left( \frac{mU}{4\pi} \right)^2 \frac{|Q|}{k_F}; \quad K(0,T) = \left( \frac{mU}{4\pi} \right)^2 \frac{T}{E_F}.$$  \hspace{1cm} (8)

Collecting all contributions, we find

$$\chi_s(Q,T) = \sum_{i=1}^{7} \chi_i(Q,T) = \chi_0 \left[ 1 + 2K(Q,T)U^2(2k_F) \right].$$  \hspace{1cm} (9)

We see that all non-analytic contributions with $U(0)$ cancel out, and the final result depends only on $U(2k_F)$.

We also performed a similar calculation in 3D and reproduced the BKV’s result—the analog of Eq. (9) but with $K_{3D}(Q,0) = (1/18)(mk_F U/4\pi^2)^2 (Q/k_F)^2 \ln (k_F/Q)$. In agreement with BKV, we also found that $K(0,T) \propto T^2$ with no logarithmic corrections.

We now look more deeply into how the non-analytic contributions to susceptibility emerge. The power-counting argument does not rely on the singularity in the particle-hole bubble. Indeed, near $q = 0$, the singular $\Omega_m/\sqrt{(v_F q)^2 + \Omega_m^2}$ piece in $\Pi_{ph}(q,\Omega_m)$ has the scaling dimension of one and hence can be treated as a constant in power counting. However, we found that for each diagram, a replacement of $\Pi_{ph}(q,\Omega_m)$ by a constant does not give rise to a linear-in-$|Q|$ term in $\chi_s(Q,0)$ because the prefactor for the $|Q|$ term contains the integral over $q$, in which all poles are located in the same half-plane. The $q$ integral then obviously vanishes. This vanishing could not be detected in power counting. The substitu-
tion of the full $\Pi_{ph}(q, \Omega_m)$ into the susceptibility makes power-counting arguments valid as $\Pi_{ph}(q, \Omega_m)$ contains a branch-cut singularity which is present in both half-planes of $q$. With this term present, the location of the poles in a complex $q$ plane becomes unessential, and the $q$ integral does not vanish.

The linear-in-$T$ dependence of $\chi(0, T)$ is also associated with the singularity in $\Pi_{ph}(q, \Omega)$, but the way it emerges is different from $T = 0$ and is similar to an anomaly. Consider for example the $q = 0$ contribution to $\chi_1(0, T)$. Leaving the integration over $q$ and summation over $\Omega_m$ as the last operations, we obtain

$$\chi_{1=0}(0, T) = -2\chi_0 \left( \frac{mU(0)}{4\pi} \right)^2 \frac{T}{E_F} J,$$

where

$$J = \sum_m \int dq F^2 q \frac{\Omega_m^2 (2\Omega_m^2 - (v_F q)^2)}{(\Omega_m^2 + (v_F q)^2)^3}.$$

Contrary to the $T = 0$ case, the result for $J$ at finite $T$ depends on the order of the integration and summation. Indeed, integrating first over $q$ in infinite limits one obtains $1/4$ for all values of $\Omega_m$, and a subsequent frequency summation does not yield a universal piece confined to low energies. On the other hand, performing the summation over $\Omega_m$ first, keeping $q$ finite, and then integrating over $q$, one obtains a universal $1/4$ piece in $J$ which gives rise to a linear-in-$T$ piece in $\chi_s(0, T)$. The correct way to compute $J$ is the second one, because the $\Omega_m$-independent result of the integration over $q$ is inconsistent with the fact that the integrand in (11) obviously vanishes at $\Omega_m = 0$ for all finite $q$. In reality, $q$ is always bounded from below by the inverse system size.

We see that despite a formal analogy between the $Q$- and $T$- dependences of $\chi_s$, the mechanism behind $\chi_s(0, T) \propto T$ is very different from the one that leads to $\chi_s(Q, 0) \propto |Q|$ as in the latter case the order of integration is unessential. The factor $J$ in (11) can be readily found for arbitrary $D$. For $D \geq 2$, we have

$$J = -\frac{(D-2)(4-D)}{8} \left( \frac{T}{E_F} \right)^{D-2} \int_0^\infty \frac{dzz^{D-2}}{e^z-1}.$$  

For $D \to 2$, we reproduce $J \to -1/4$. For arbitrary $2 < D < 3$, $J \propto T^{D-2}$, i.e., $\chi_s(T) \propto T^{D-2}$. For $D = 3$, however, the integral is regular, and $J \propto T$, i.e., $\chi_s(T) \propto T^2$ without logarithmic corrections.

**Charge susceptibility.** For the charge susceptibility, we have two additional contributions given by diagrams 6 and 7 in Fig. 2. We find

$$\chi_6(Q, T) = -\chi_0 K(Q, T) \left[ U^2(0) - U^2(2k_F) \right];$$

$$\chi_7(Q, T) = -\chi_0 K(Q, T) \left[ U^2(0) + U^2(2k_F) \right]$$

where $K(Q, T)$ is given by (12). Combining this last result with Eq. (11), we find that all non-analytic terms from individual diagrams cancel out, i.e., the charge susceptibility is regular, in agreement with Ref. [1].

To conclude, in this paper we demonstrated that the universal singularities in the bosonic response functions of a Fermi liquid give rise to universal non-analytic corrections to the Fermi-liquid forms of the self-energy and thermodynamic variables. We obtained explicit results in 2D for $\delta C(T) \propto T^2$, $\chi_s(Q, T = 0) \propto |Q|$, $\chi_s(Q = 0, T) \propto T$. We demonstrated that these non-analytic terms come from the processes with both transferred and total momentum close to zero. We also demonstrated that thermal ($\propto T$) and quantum ($\propto |Q|$) corrections to the spin susceptibility are of different origin. This explains why in 3D, $\chi_s(Q, 0) \propto Q^2 \ln Q$, while $\chi_s(0, T) \propto T$.

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