On the curious series related to the elliptic integrals

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Abstract

By using the theory of the elliptic integrals a new method of summation is proposed for a certain class of series and their derivatives involving hyperbolic functions. It is based on the termwise differentiation of the series with respect to the elliptic modulus and integral representations of several of the series in terms of the inverse Mellin transforms related to the Riemann zeta function. The relation with the corresponding case of the Voronoi summation formula is exhibited. The involved series are expressed in closed form in terms of complete elliptic integrals of the first and second kind, and some special cases are calculated in terms of particular values of the Euler gamma function.

Keywords: Series with hyperbolic functions, Elliptic integrals, Mellin transform, Riemann zeta function, Euler gamma function, arithmetic functions, summation formulae

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1 Introduction and auxiliary results

The main goal of this paper is to find closed-form relations for the following series, involving the hyperbolic functions

\[ \sum_{n=1}^{\infty} (\pm 1)^n n^\alpha \frac{\cosh^\beta (\pi anx)}{\sinh^\gamma (\pi anx)}, \quad \sum_{n=1}^{\infty} (\pm 1)^n n^\alpha \frac{\sinh^\beta (\pi anx)}{\cosh^\gamma (\pi anx)}, \]  

\[ \sum_{n=1}^{\infty} n^\alpha [\tanh(\pi anx) - 1], \quad \sum_{n=1}^{\infty} n^\alpha [\coth(\pi anx) - 1], \]  

where \( \alpha, \beta, \gamma \in \{0, 1, 2, 3\}, \gamma > \beta, \ a \in \{1/2, 1, 2\}, \ x > 0, \) being defined by the formula

\[ x \equiv x(k) = \frac{K(k')}{K(k)}, \quad k \in (0, 1), \quad k' = \sqrt{1-k^2}, \]  

and \( K(k) \) is the complete elliptic integral of the first kind [1], [7], Vol. II, [11]

\[ K(k) = \int_{0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \]

The parameter \( k \) is called the elliptic modulus and \( k' \) is the complementary modulus. \( K(k) \) satisfies the Legendre relation

\[ E(k)K(k') + E(k')K(k) - K(k')K(k) = \frac{\pi}{2}, \]  

where \( E(k) \) is the complete elliptic integral of the second kind.
\[ E(k) = \int_0^1 \sqrt{\frac{1-k'^2t^2}{1-t^2}} \, dt. \] (1.6)

Its derivative can be calculated by formula
\[ \frac{dE}{dk} = \frac{E(k) - K(k)}{k}. \] (1.7)

It is known [11], that \( K(k), K(k') \) satisfy the differential equation
\[ \frac{d}{dk} \left( k(k')^2 \frac{du}{dk} \right) = ku \] (1.8)
and \( E(k), E(k') - K(k') \) are, in turn, solutions of the differential equation
\[ (k')^2 \frac{d}{dk} \left( k \frac{du}{dk} \right) + ku = 0. \] (1.9)

The derivative of \( K(k) \) can be calculated by the formula
\[ \frac{dK}{dk} = \frac{E(k) - (k')^2 K(k)}{k(k')^2}. \] (1.10)

In order to express series (1.1), (1.2) in closed form we will proposed a method of termwise differentiation with respect to the elliptic modulus and in some cases we will represent these series, using the inverse Mellin transform related to the Riemann zeta function. We note that this investigation is a continuation of earlier methods proposed by Ling and Zucker (see [6] and [15], respectively), involving Weierstrassian and Jacobian elliptic functions (cf. [1]) and by Glasser et al. [4], basing on the Poisson summation formula. It is also worth to mention that recently some infinite series of the Eisenstein type involving the hyperbolic functions were investigated in [5].

Let \( k_r \) be an elliptic modulus such that \( x(k_r) = \sqrt{r} \) (see (1.3)). In the sequel we will use such values for small \( r \) and the corresponding elliptic integral singular values \( K(k_r) \) (see [2], [3]), namely
\[ k_1 = \frac{1}{\sqrt{2}}, \quad k_2 = \sqrt{2} - 1, \quad k_3 = \frac{1}{4} \sqrt{2} (\sqrt{3} - 1), \quad k_4 = 3 - 2 \sqrt{2}, \] (1.11)
\[ K(k_1) = \frac{\Gamma^2(1/4)}{4\sqrt{\pi}}, \quad K(k_2) = \frac{(\sqrt{2} + 1)^{1/2} \Gamma(1/8) \Gamma(3/8)}{2^{13/4} \sqrt{\pi}}, \] (1.12)
\[ K(k_3) = \frac{3^{1/4} \Gamma^3(1/3)}{2^{7/3} \pi}, \quad K(k_4) = \frac{(\sqrt{2} + 1) \Gamma^2(1/4)}{2^{17/2} \sqrt{\pi}}, \] (1.13)
where \( \Gamma(z) \) is Euler’s gamma function. According to [2] the so-called elliptic alpha function for the integral singular values
\[ \alpha(r) = \frac{E(k'_r)}{K(k_r)} - \frac{\pi}{4[K(k_r)]^2} = \frac{\pi}{4[K(k_r)]^2} + \sqrt{r} \left[ 1 - \frac{E(k_r)}{K(k_r)} \right] \] (1.14)
is calculated, in particular, for small values and we have
\[ \alpha(1) = \frac{1}{2}, \quad \alpha(2) = \sqrt{2} - 1, \quad \alpha(3) = \frac{1}{2} (\sqrt{3} - 1), \quad \alpha(4) = 2(\sqrt{2} - 1)^2. \] (1.15)
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Meanwhile, appealing to relations (2.4.3.1), (2.4.3.3), (2.4.9.2) in [9], Vol. I and the inverse Mellin transform [10], we derive the following integral representations, which will be useful in the sequel

\[
\frac{1}{\sinh(cx)} = \frac{1}{\pi} \int_{\mu-i\infty}^{\mu+i\infty} (2^s - 1)\Gamma(s)\zeta(s)(2cx)^{-s}ds, \quad c > 0, \mu > 1, \tag{1.16}
\]

\[
\frac{1}{\sinh^2(cx)} = \frac{2}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma(s)\zeta(s-1)(2cx)^{-s}ds, \quad c > 0, \mu > 2, \tag{1.17}
\]

\[
\frac{1}{\cosh^2(cx)} = \frac{2}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} (1 - 2^{2-s})\Gamma(s)\zeta(s-1)(2cx)^{-s}ds, \quad c > 0, \mu > 0, \tag{1.18}
\]

\[
\tanh(cx) - 1 = \frac{1}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} (1 - 2^{1-s})\Gamma(s)\zeta(s)(2cx)^{-s}ds, \quad c > 0, \mu > 0, \tag{1.19}
\]

\[
\coth(cx) - 1 = \frac{1}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma(s)(s-1)(2cx)^{-s}ds, \quad c > 0, \mu > 1, \tag{1.20}
\]

where \(\zeta(s)\) is the Riemann zeta function [7], Vol. I, which satisfies the familiar functional equation

\[
\zeta(s) = 2^s\pi^{s-1} \sin \left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s). \tag{1.21}
\]

2 Series with the hyperbolic functions

In order to evaluate series (1.1) we propose the method of termwise differentiation of the series with respect to the elliptic modulus. To do this, we will employ formulae of Sections 5.3.4. and 5.3.6. in [9], Vol. I. Indeed, let us consider the following series (see relation (5.3.4.2) in [9], Vol. I)

\[
\sum_{n=1}^{\infty} \frac{n}{\sinh(n\pi x)} = \frac{1}{\pi^2} K(k) [K(k) - E(k)], \tag{2.1}
\]

where \(x(k)\) is defined by (1.3). It is easy to verify that the function \(x : (0, 1) \to \mathbb{R}_+\) is monotone decreasing and continuously differentiable. This means that equality (2.1) is valid for any \(x > 0\). Moreover, the series (2.1) is differentiated termwise with respect to \(k \in (0, k_0], k_0 < 1\) via the absolute and uniform convergence of its derivative. Hence we have

**Theorem 1.** For all \(x > 0\) the following identities hold

\[
2 \sum_{n=1}^{\infty} \frac{n^2 \cosh(n\pi x)}{\sinh^2(n\pi x)} = \sum_{n=1}^{\infty} \frac{(2n-1) \cosh(\pi(2n-1)x/2)}{\sinh^3(\pi(2n-1)x/2)}
\]

\[
= \frac{4}{\pi^4} K^2(k) \left[k^2 K^2(k) - [K(k) - E(k)]^2\right]. \tag{2.2}
\]

**Proof.** In fact, as we see above \(x(k)\) is a bijective map from \((0, 1)\) to \(\mathbb{R}_+\). Moreover, termwise differentiation with respect to \(k\) in (2.1) gives

\[
-\pi x'(k) \sum_{n=1}^{\infty} \frac{n^2 \cosh(n\pi x)}{\sinh^2(n\pi x)} = \frac{1}{\pi^2} \frac{d}{dk} [K(k) [K(k) - E(k)]]. \tag{2.3}
\]

Meanwhile, since via relation (5.3.4.6) in [9], Vol. I
we find then after differentiation we easily establish the first equality in (2.2). Further, employing twice (1.10), we find

$$x'(k) = -\frac{K(k')}{K^2(k)} \frac{dK(k)}{dk} - \frac{k}{(1-k^2)K(k)} \left[ \frac{E(k')}{k^2} - K(k') \right]$$

and the Legendre identity (1.5) leads us to the final result

$$x'(k) = -\frac{\pi}{2k(1-k^2)K^2(k)}.$$  \hspace{1cm} (2.5)

On the other hand, with the aid of (1.7), (1.9) and (1.10)

$$\frac{d}{dk} [K(k) [K(k) - E(k)]] = \frac{dK(k)}{dk} [K(k) - E(k)] - K(k) \frac{d}{dk} \left( \frac{dE(k)}{dk} \right)$$

$$= [K(k) - E(k)] \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} + \frac{k K(k)E(k)}{1-k^2}$$

$$= \frac{k K^2(k)}{1-k^2} - \frac{[K(k) - E(k)]^2}{k(1-k^2)}.$$  \hspace{1cm} (2.6)

Hence, (2.3) and (2.5) yield the equality

$$\sum_{n=1}^{\infty} \frac{n^2 \cosh(\pi n x)}{\sinh^2(\pi n x)} = \frac{2}{\pi^2} K^2(k) \left[ k^2 K^2(k) - [K(k) - E(k)]^2 \right],$$

which implies the latter equality in (2.2) and completes the proof of Theorem 1.

Appealing to (1.11), (1.12), (1.13), (1.15), we arrive at an immediate corollary.

**Corollary 1.** The following formulae take place

$$2 \sum_{n=1}^{\infty} \frac{n^2 \cosh(\pi n \sqrt{2})}{\sinh^2(\pi n \sqrt{2})} = \sum_{n=1}^{\infty} \frac{(2n - 1) \cosh(\pi (n-1/2))}{\sinh^3(\pi (n-1/2))}$$

$$= \frac{1}{4\pi^2} \left[ \Gamma^4(1/4)/64\pi^4 + \Gamma^4(1/4)/4\pi^2 - 1 \right],$$  \hspace{1cm} (2.6)

$$2 \sum_{n=1}^{\infty} \frac{n^2 \cosh(\pi n \sqrt{3})}{\sinh^2(\pi n \sqrt{3})} = \sum_{n=1}^{\infty} \frac{(2n - 1) \cosh(\pi (2n - 1)/\sqrt{3})}{\sinh^3(\pi (2n - 1)/\sqrt{3})}$$

$$= \frac{1}{8\pi^2} \left[ \Gamma^4(1/8)/512\pi^4 + \Gamma^2(1/8)/(8\sqrt{2}\pi^2) - 1 \right],$$  \hspace{1cm} (2.7)

$$2 \sum_{n=1}^{\infty} \frac{n^2 \cosh(\pi n \sqrt{3})}{\sinh^3(\pi n \sqrt{3})} = \sum_{n=1}^{\infty} \frac{(2n - 1) \cosh(\pi (n-1/2)\sqrt{3})}{\sinh^3(\pi (n-1/2)\sqrt{3})}$$
Next, we will combine relations (5.3.4.1) and (5.3.6.2) in [9], Vol. I, namely,

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1) \cosh(\pi(2n-1)x/2)}{\sinh^{2}(\pi(2n-1)x/2)} = \frac{4k}{\pi^3} K^2(k) E(k), \quad x > 0,
\]

\[
\sum_{n=1}^{\infty} \frac{n \sinh(\pi n x)}{\cosh^{2}(\pi n x)} = \frac{2}{\pi^3} K^2(k) \left[ E(k) - (1 - k^2)K(k) \right], \quad x > 0.
\]

Obtaining particular cases in the same fashion as in Corollary 1, we establish

**Corollary 2.** The following formulae take place

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1) \cosh(\pi(n-1/2))}{\sinh^{2}(\pi(n-1/2))} = \frac{\Gamma^2(1/4)}{4\pi^2 \sqrt{2\pi}} \left[ \frac{\Gamma^4(1/4)}{8\pi^2} + 1 \right],
\]

\[
\sum_{n=1}^{\infty} \frac{n \sinh(\pi n)}{\cosh^{2}(\pi n)} = \frac{\Gamma^2(1/4)}{8\pi^2 \sqrt{2\pi}}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1) \cosh(\pi(2n-1)/\sqrt{2})}{\sinh^{2}(\pi(2n-1)/\sqrt{2})} = \frac{2^{1/4} \Gamma(1/8) \Gamma(3/8)}{16\pi^2 \sqrt{\pi}}
\]

\[
\times \left[ \frac{1}{16\pi^2} + (\sqrt{2} - 1)^{1/2} \right],
\]

\[
\sum_{n=1}^{\infty} \frac{n \sinh(\pi \sqrt{2} n)}{\cosh^{2}(\pi \sqrt{2} n)} = \frac{\Gamma(1/8) \Gamma(3/8)}{16\pi^2 \sqrt{2\pi}}
\]

\[
\times \left[ 1 + \frac{1}{\sqrt{2}} \right]^{1/2} - \frac{2}{32\pi^2} \frac{\Gamma^2(1/8) \Gamma^2(3/8)}{32\pi^2},
\]
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1) \cosh(\pi(n-1/2)\sqrt{3})}{\sinh^2(\pi(n-1/2)\sqrt{3})} = \frac{3^{1/4}}{8\sqrt{2}} \left[ \frac{\Gamma^6(1/3)}{8\pi^3} + \frac{1}{2^{1/3}} \left( 1 - \frac{1}{\sqrt{3}} \right) \right], 
\]

\[
3^{1/4} \frac{\Gamma^3(1/3)}{8\pi^3} = \left[ 2^{-1/3} - \frac{\sqrt{3}}{32\pi^3} \Gamma^6(1/3) \right], 
\]

\[
3^{1/4} \frac{\Gamma^3(1/3)}{8\pi^3} \sum_{n=1}^{\infty} \frac{n \sinh(\pi n \sqrt{3})}{\cosh^2(\pi n \sqrt{3})} = \frac{\Gamma^2(1/4)}{32\pi^2 \sqrt{2}} \left[ \sqrt{2} + 1 - \frac{\Gamma^4(1/4)}{8\pi^2} \right]. 
\]

Taking relations (5.3.4.3), (5.3.4.4), (5.3.4.5), (5.3.4.6), (5.3.6.4), (5.3.6.5), (5.3.6.6) in [9], Vol. I

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\sinh(\pi nx)} = \frac{K(k)}{\pi^2} \left[ E(k) - (1 - k^2)K(k) \right], 
\]

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1) \sinh(\pi(2n-1)x)} = -\frac{1}{8} \log(1 - k^2), 
\]

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi nx)} = \frac{1}{6} - \frac{2K(k)}{\pi^2} \left[ E(k) - \frac{2 - k^2}{3}K(k) \right], 
\]

\[
\sum_{n=-\infty}^{\infty} \frac{1}{\cosh(\pi(2n-1)x/2)} = \frac{2k}{\pi} K(k), 
\]

\[
\sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi nx)} = \frac{2K(k)}{\pi^2} \left[ E(k) - (1 - k^2)K(k) \right], 
\]

respectively, recalling (2.1), (2.4), (2.10) and summing or subtracting one from another, we deduce, for instance, the following equalities

\[
\sum_{n=-\infty}^{\infty} \frac{1}{\cosh(\pi(2n-1)x/2)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\sinh(\pi(2n-1)x/2)} = \frac{k}{\pi} K(k), 
\]

\[
\sum_{n=1}^{\infty} \frac{\cosh(2nx)}{\sinh^2(2\pi nx)} = \frac{2 - k^2}{6\pi^2} K^2(k) - \frac{1}{12}, 
\]

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh(2\pi nx)} = \frac{K(k)}{\pi^2} \left[ \frac{2 - k^2}{6} K(k) - E(k) \right] + \frac{1}{6}. 
\]
An immediate corollary of (2.29), (2.35) is the equality

\[ \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi nx/2)} = \frac{2K(k)}{\pi^2} \left[ 2E(k) - (1 - k^2)K(k) \right] - \frac{1}{2}, \]  

(2.31)

\[ \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi nx)} - \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi(2n-1)x/2)} = \frac{2(1 - k^2)}{\pi^2} K^2(k) - \frac{1}{2}. \]  

(2.32)

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\sinh(\pi nx)} = \sum_{n=1}^{\infty} \frac{1}{2 \cosh^2(\pi(2n-1)x/2)}, \]  

(2.33)

\[ \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi(2n-1)x)} = \frac{K(k)}{\pi^2} \left[ \frac{2 - k^2}{2} K^2(k) - E(k) \right]. \]  

(2.34)

Combining (2.34) with (2.4), we get

\[ \sum_{n=1}^{\infty} \frac{\cosh(\pi(2n-1)x)}{\sinh^2(\pi(2n-1)x)} = \frac{k^2}{2\pi^2} K^2(k). \]  

(2.35)

We note that the same result can be obtained, differentiating (2.23) with respect to \( k \) and invoking (2.29).

An immediate corollary of (2.29), (2.35) is the equality

\[ \sum_{n=1}^{\infty} \frac{\cosh(\pi nx)}{\sinh^2(\pi nx)} = \frac{1 + k^2}{3\pi^2} K^2(k) - \frac{1}{12}. \]  

(2.36)

Now we are ready to apply the method of termwise differentiation with respect to the elliptic modulus to the series (2.22), (2.24), (2.25), (2.26), (2.27), (2.28), (2.29), (2.36). In fact, employing (2.5) and properties of the complete elliptic integrals listed in Section 1, in particular, (1.7), (1.10), after elementary calculations we establish

**Theorem 2.** For all \( x > 0 \) the following formulae hold valid

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2 \cosh(\pi nx)}{\sinh^2(\pi nx)} = \frac{2}{\pi^4} K^2(k) \left[ (E(k) - (1 - k^2)K(k))^2 + k^2(1 - k^2)K^2(k) \right], \]  

(2.37)

\[ \sum_{n=1}^{\infty} \frac{n \cosh(\pi nx)}{\sinh^2(\pi nx)} = \frac{2}{3\pi^4} K^2(k) \left[ E(k) \left( 2(2 - k^2)K(k) - 3E(k) \right) - (1 - k^2)K^2(k) \right], \]  

(2.38)

\[ \sum_{n=1}^{\infty} \frac{(2n-1) \sinh(\pi(2n-1)x/2)}{\cosh^2(\pi(2n-1)x/2)} = \frac{4k}{\pi^3} K^2(k) E(k), \]  

(2.39)

\[ \sum_{n=1}^{\infty} \frac{n \sinh(\pi nx)}{\cosh^3(\pi nx)} = \frac{2}{\pi^4} K^2(k) \left[ E^2(k) - (1 - k^2)K^2(k) \right], \]  

(2.40)

\[ \sum_{n=1}^{\infty} \frac{(2n-1) \sinh(\pi(2n-1)x/2)}{\cosh^3(\pi(2n-1)x/2)} = \frac{4}{\pi^4} K^2(k) \left[ (E(k) - (1 - k^2)K(k))^2 + k^2(1 - k^2)K^2(k) \right] \]  

\[ = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n^2 \cosh(\pi nx)}{\sinh^2(\pi nx)}, \]  

(2.41)
\[ \sum_{n=-\infty}^{\infty} \frac{(2n-1) \sinh(\pi(2n-1)x/2)}{\cosh^2(\pi(2n-1)x/2)} + \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1) \cosh(\pi(2n-1)x/2)}{\sinh^2(\pi(2n-1)x/2)} = \frac{4k}{\pi^4} K^2(k) E(k), \] 

(2.42)

\[ \sum_{n=1}^{\infty} \frac{n(3 + \cosh(4\pi nx))}{\sinh^3(2\pi nx)} = 2 \frac{K^3(k)}{3\pi^4} \left[(2 - k^2)E(k) - 2(1 - k^2)K(k)\right], \] 

(2.43)

\[ \sum_{n=1}^{\infty} \frac{n(3 + \cosh(2\pi nx))}{\sinh^3(\pi nx)} = 8 \frac{K^3(k)}{3\pi^4} \left[K(k)(1 + k^2) - K(k)(1 - k^2)\right], \] 

(2.44)

\[ \sum_{n=1}^{\infty} \frac{n \cosh(2\pi nx)}{\sinh^3(2\pi nx)} = \frac{K^2(k)}{6\pi^4} \left[(2 - k^2)E(k)(K(k) + (1 - k^2)K^2(k) - 3E^2(k))\right]. \] 

(2.45)

**Remark 1.** In the same manner similar equalities can be obtained, differentiating (2.30), (2.31), (2.32), (2.33), (2.34), (2.35) termwise with respect to the elliptic modulus \( k \).

Some particular values of the above series are listed in

**Corollary 3.** *It has a set of identities*

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2 \cosh(\pi n)}{\sinh^2(\pi n)} = \frac{1}{8\pi^2} \left[1 + \frac{\Gamma^2(1/4)}{64\pi^4}\right], \]

\[ \sum_{n=1}^{\infty} \frac{(2n-1) \sinh(\pi(n-1/2))}{\cosh^3(\pi(n-1/2))} = \frac{1}{4\pi^2} \left[1 + \frac{\Gamma^2(1/4)}{64\pi^4}\right], \]

\[ \sum_{n=1}^{\infty} \frac{n \cosh(\pi n)}{\sinh^3(\pi n)} = \frac{1}{8\pi^2} \left[\frac{\Gamma^2(1/4)}{192\pi^4} - 1\right], \]

\[ \sum_{n=1}^{\infty} \frac{(2n-1) \sinh(\pi(n-1/2))}{\cosh^2(\pi(n-1/2))} = \frac{\Gamma^2(1/4)}{4\pi^2 \sqrt{2\pi}} \left[1 + \frac{\Gamma^4(1/4)}{8\pi^2}\right], \]

\[ \sum_{n=1}^{\infty} \frac{n \sinh(\pi n)}{\cosh^3(\pi n)} = \frac{1}{8\pi^2} \left[1 + \frac{\Gamma^4(1/4)}{4\pi^2} - \frac{\Gamma^8(1/4)}{64\pi^4}\right], \]

\[ \sum_{n=1}^{\infty} \frac{\cosh(\pi(2n-1))}{\sinh^2(\pi(2n-1))} = \frac{\Gamma^4(1/4)}{64\pi^3}, \]

\[ \sum_{n=1}^{\infty} \frac{\cosh(\pi n)}{\sinh^2(\pi n)} = \frac{\Gamma^4(1/4)}{32\pi^3} - \frac{1}{12}, \]

\[ \sum_{n=1}^{\infty} \frac{(2n-1) \sinh(\pi(n-1/2))}{\cosh^2(\pi(n-1/2))} + \sum_{n=1}^{\infty} \frac{(-1)^n(2n-1) \cosh(\pi(n-1/2))}{\sinh^2(\pi(n-1/2))} = \frac{\Gamma^2(1/4)}{4\pi^3 \sqrt{2\pi}} \left[\pi + \frac{\Gamma^4(1/4)}{8\pi}\right], \]
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\[ \sum_{n=1}^{\infty} \frac{n(3 + \cosh(4\pi n))}{\sinh^3(2\pi n)} = \frac{\Gamma^4(1/4)}{64\pi^4} \left[ 1 - \frac{\Gamma^4(1/4)}{24\pi^2} \right], \]

\[ \sum_{n=1}^{\infty} \frac{n(3 + \cosh(2\pi n))}{\sinh^3(\pi n)} = \frac{\Gamma^4(1/4)}{48\pi^4} \left[ 3 + \frac{\Gamma^4(1/4)}{8\pi^2} \right], \]

\[ \sum_{n=1}^{\infty} \frac{n \cosh(2\pi n)}{\sinh^3(2\pi n)} = \frac{1}{32\pi^2} \left[ \frac{\Gamma^8(1/4)}{96\pi^4} - \frac{\Gamma^4(1/4)}{8\pi^2} - 1 \right]. \]

**Remark 2.** As a conclusion we stress that employing formulae from Sections (5.3.4) and (5.3.6) in [9], Vol. I, one can make different combinations of the known series and differentiate them termwise any number of times with respect to the elliptic modulus \( k \) to obtain the values of new series, involving powers of hyperbolic functions.

In the meantime, series (2.24), (2.26) can be expressed using integral representations (1.17), (1.18). In fact, substituting the corresponding integrals inside the series, we change the order of integration and summation. This is allowed by virtue of the absolute and uniform convergence with respect to \( x \geq x_0 > 0 \), basing, in turn, on the estimate (it concerns series (2.24) and (2.26) can be treated analogously)

\[ \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi nx)} = \frac{2}{\pi} \sum_{n=1}^{\infty} \left| \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(s)\zeta(s-1)(2\pi nx)^{-s} ds \right|, \]

\[ \leq 2^{1-\mu} \pi^{-\mu-1} x_0^{-\mu} \zeta(\mu-1) \zeta(\mu) \int_{\mu - i\infty}^{\mu + i\infty} \left| \Gamma(s) ds \right| < \infty, \quad \mu > 2, \]

where the convergence of the latter integral can be verified using the Stirling asymptotic formula for the gamma-function when \( |\text{Im}s| \to \infty \) (see [7], Vol. I). Hence, we find the representations

\[ \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi nx)} = \frac{2}{\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \Gamma(s)\zeta(s-1)(2\pi nx)^{-s} ds, \quad (2.46) \]

\[ \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi nx)} = \frac{2}{\pi i} \int_{\mu - i\infty}^{\mu + i\infty} (1 - 2^{-s}) \Gamma(s)\zeta(s-1)(2\pi nx)^{-s} ds. \quad (2.47) \]

Meanwhile, the product of the Riemann zeta-functions can be represented by the Ramanujan identity [13]

\[ \zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}, \quad \text{Re}s > 2, \quad (2.48) \]

where \( \sigma(n) \) is the sum of the divisors of \( n \) [1]. Therefore, substituting this expression in (2.46), (2.47) and changing the order of integration and summation due to the same motivation, and employing the inverse Mellin transform of the gamma-function [10], we obtain the following equalities

\[ \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi nx)} = 4 \sum_{n=1}^{\infty} \sigma(n)e^{-2\pi nx}, \quad (2.49) \]

\[ \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi nx)} = 4 \sum_{n=1}^{\infty} \sigma(n) \left[ e^{-2\pi nx} - 4 e^{-4\pi nx} \right]. \quad (2.50) \]
On the other hand, the Nasim summation formula [8] says

\[
\sum_{n=1}^{\infty} \sigma(n)e^{-2\pi nx} + x^{-2} \sum_{n=1}^{\infty} \sigma(n)e^{-2\pi n/x} = \frac{1}{24} \left( 1 + \frac{1}{x^2} \right) - \frac{1}{4\pi x}, \quad x > 0.
\]

Hence, appealing to (2.49), (2.50), we establish the formulae (x > 0)

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n/x)} + \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n/x)} = \frac{1}{6} \left( 1 + \frac{1}{x^2} \right) - \frac{1}{\pi x},
\]

(2.51)

\[
\sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi n/x)} + \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi n/x)} = \frac{1}{6} \left( 1 + \frac{1}{x^2} \right) - \frac{1}{\pi x}
\]

\[-4 \left( \sum_{n=1}^{\infty} \frac{1}{\sinh^2(2\pi nx)} + \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(2\pi n/x)} \right),
\]

(2.52)

**Theorem 3.** For all \(x > 0\) the following identities hold

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n/x)} = \frac{1}{6} - \frac{2}{\pi^2} K(k') \left[ E(k') - \frac{k^2}{3} + 1 \right],
\]

(2.53)

\[
\sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi n/x)} = \frac{2}{\pi^2} K(k') E(k') - \frac{1}{2}.
\]

(2.54)

**Proof.** Indeed, associating any positive \(x\) with some \(k \in (0, 1)\) by formula (1.3), we appeal to (2.51) and (2.24) to deduce

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n/x)} = \frac{1}{6} - \frac{x}{\pi} + \frac{2}{\pi^2} K(k') \frac{E(k)}{K(k)} - \frac{2(2 - k^2)}{3\pi^2} K^2(k') - \frac{6}{\pi^2} K(k') E(k') - \frac{1}{2}.
\]

But the Legendre identity (1.5) says

\[
\frac{2}{\pi^2} K^2(k') \frac{E(k)}{K(k)} = \frac{x}{\pi} + \frac{2}{\pi^2} K(k')[K(k') - E(k')].
\]

This drives us to (2.53). In order to prove (2.54), we recall (2.4), (2.24) to calculate the series

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n/2x)} = \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n/x)} + \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi(2n-1)x/2)} = \frac{1}{6} - \frac{4}{\pi^2} K(k) E(k) + \frac{2(5 - k^2)}{3\pi^2} K^2(k).
\]

Hence from (2.51), (1.5) we get the value of the series

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh^2(2\pi n / x)} = \frac{1}{6} - \frac{K(k')}{\pi^2} \left[ E(k') - \frac{1 + k^2}{6} K(k') \right].
\]

Consequently, employing identities (2.26), (2.30), (2.52), we establish (2.54), completing the proof of Theorem 3.

\(\square\)
Summing (2.53), (2.54), we arrive at the immediate

**Corollary 4.** It has the equality

\[
\sum_{n=1}^{\infty} \frac{\cosh(2\pi n/x)}{\sinh^2(2\pi n/x)} = \frac{1 + k^2}{6\pi^2} K^2(k) - \frac{1}{12}, \quad x > 0.
\]

In order to treat series (1.2), we recall integral representations (1.19), (1.20) and motivating the interchange of the order of integration and summation in the same manner as in (2.46), (2.47), we use them to derive the identities

\[
\sum_{n=1}^{\infty} n [\tanh(\pi n x) - 1] = \frac{1}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} (2^{1-s} - 1) \Gamma(s) \zeta(s-1)(2\pi x)^{-s} ds
\]

\[= -\frac{1}{4} \left[ \sum_{n=1}^{\infty} \frac{1}{\cosh^2(\pi n x)} + \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n x)} \right], \quad x > 0,
\]

\[
\sum_{n=1}^{\infty} n [\coth(\pi n x) - 1] = \frac{1}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \Gamma(s) \zeta(s) \zeta(s-1)(2\pi x)^{-s} ds
\]

\[= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n x)}, \quad x > 0.
\]

Hence, appealing to (2.4), (2.24), (2.26), (2.29), (2.30), (2.38), (2.43), (2.45), (2.46) and employing again the method of termwise differentiation with respect to the elliptic modulus, we proved the following results.

**Theorem 4.** Let \( x > 0 \). Then

\[
\sum_{n=1}^{\infty} n [1 - \tanh(\pi n x)] = \sum_{n=1}^{\infty} \frac{\cosh(2\pi n x)}{\sinh^2(2\pi n x)} = \frac{1}{12} \left[ \frac{2(2 - k^2)}{\pi^2} K^2(k) - 1 \right], \quad (2.55)
\]

\[
\sum_{n=1}^{\infty} n [\coth(\pi n x) - 1] = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sinh^2(\pi n x)} = \frac{1}{12} - \frac{K(k)}{\pi^2} \left[ E(k) - \frac{2 - k^2}{3} K(k) \right], \quad (2.56)
\]

\[
\sum_{n=1}^{\infty} \frac{n^2}{\sinh^2(\pi n x)} = \frac{2K^2(k)}{3\pi^4} \left[ E(k) \left( 2(2 - k^2) K(k) - 3E(k) \right) - (1 - k^2)K^2(k) \right], \quad (2.57)
\]

\[
\sum_{n=1}^{\infty} \frac{n^2}{\cosh^2(\pi n x)} = \frac{2K^3(k)}{3\pi^4} \left[ (2 - k^2)E(k) - 2(1 - k^2)K(k) \right]. \quad (2.58)
\]

**Theorem 5.** Let \( x > 0 \). Then

\[
\sum_{n=1}^{\infty} n [\coth(2\pi n x) - 1] = \frac{1}{12} - \frac{K(k)}{2\pi^2} \left[ E(k) - \frac{2 - k^2}{6} K(k) \right], \quad (2.59)
\]

\[
\sum_{n=1}^{\infty} n [1 - \tanh(\pi n x/2)] = \frac{1}{12} \left[ \frac{4(1 + k^2)}{\pi^2} K^2(k) - 1 \right], \quad (2.60)
\]

\[
\sum_{n=1}^{\infty} \frac{n^2}{\sinh^2(2\pi n x)} = \frac{K^2(k)}{6\pi^4} \left[ (2 - k^2)E(k)K(k) + (1 - k^2)K^2(k) - 3E^2(k) \right], \quad (2.61)
\]
Corollary 5. The following relations hold

\[
\sum_{n=1}^{\infty} n \left[ 1 - \tanh(\pi n) \right] = \frac{1}{12} \left[ 3 \frac{\Gamma^4(1/4)}{16\pi^3} - 1 \right],
\]

\[
\sum_{n=1}^{\infty} n \left[ 1 - \tanh(\pi n/2) \right] = \frac{1}{12} \left[ 3\frac{\Gamma^4(1/4)}{8\pi^3} - 1 \right],
\]

\[
\sum_{n=1}^{\infty} n \left[ \coth(\pi n) - 1 \right] = \frac{1}{4} \left[ \frac{1}{3} - \frac{1}{\pi} \right],
\]

\[
\sum_{n=1}^{\infty} n \left[ \coth(2\pi n) - 1 \right] = \frac{1}{4} \left[ \frac{1}{3} - \frac{1}{2\pi} \right],
\]

\[
\sum_{n=1}^{\infty} n^2 \left[ \sinh^2(\pi n) \right] = \frac{1}{8\pi^2} \left[ \frac{\Gamma^8(1/4)}{96\pi^4} - \frac{\Gamma^4(1/4)}{8\pi^2} - 1 \right],
\]

\[
\sum_{n=1}^{\infty} n^2 \left[ \cosh^2(\pi n) \right] = \frac{\Gamma^4(1/4)}{64\pi^4} \left[ 1 - \frac{\Gamma^4(1/4)}{24\pi^2} \right].
\]

3 Series of cosech(πnx)

Finally, let us investigate the following series

\[ S(x) = \sum_{n=1}^{\infty} \frac{1}{\sinh(\pi nx)}, \quad x > 0. \] (3.1)

It seems that (3.1) is much simpler that series considered in the previous section. However, its calculation is quite a difficult task. In this section we will deduce a second kind singular integral equation, involving the Hilbert transform \[10\] and whose solution is related to (3.1). Indeed, let represent it in a different form first. To do this, we appeal to integral representation (1.16) for the Riemann zeta-function, substituting it in (3.1) and interchanging the order of integration and summation via the absolute and uniform convergence by the same arguments as above. Hence employing the series for the Riemann zeta-function, we find

\[ S(x) = \frac{1}{\pi i} \int_{\mu-i\infty}^{\mu+i\infty} (2^s - 1) \Gamma(s) \zeta^2(s)(2\pi x)^{-s} ds, \quad \mu > 1. \] (3.2)

Meanwhile, the square of the Riemann zeta-function is represented by another Ramanujan’s identity \[13\]

\[ \zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \quad \text{Res} > 1, \] (3.3)

where \(d(n)\) is the Dirichlet divisor function, i.e. the number of divisors of \(n\), including 1 and \(n\) itself (see \[1\]). Hence from (3.1), (3.2) and straightforward calculations we find
In the meantime, in [12] we established the following particular case of the Voronoi summation formula, related to (3.4), namely

\[ \sum_{n=1}^{\infty} d(n) \left[ e^{-\pi nx} + \frac{2}{\pi x} \left( e^{-4\pi n/x} \text{Ei} \left( \frac{4\pi n}{x} \right) + e^{4\pi n/x} \text{Ei} \left( -\frac{4\pi n}{x} \right) \right) \right] = \frac{1}{4} + \frac{\gamma - \log(\pi x)}{\pi x}, \]  

(3.5)

where \( x > 0 \), \( \gamma \) is Euler’s constant and \( \text{Ei}(z) = \int_{z}^{\infty} \frac{e^t}{t} dt \) is the integral exponential function. Hence with simple substitution (3.5) can be written in the form

\[ \sum_{n=1}^{\infty} d(n) \left[ e^{-\pi nx} + \frac{4}{\pi x} \int_{0}^{\infty} e^{-4\pi nt/x} \frac{dt}{1 - t^2} \right] = \frac{1}{4} + \frac{\gamma - \log(\pi x)}{\pi x}, \]  

(3.6)

where, as usual, the integral in the neighborhood of \( t = 1 \) is understood in the Cauchy principal values sense. So, combining with (3.4), we find

\[ S(x) - \frac{4}{\pi x} \sum_{n=1}^{\infty} d(n) \int_{0}^{\infty} \frac{e^{-2\pi nt/x} - e^{-4\pi nt/x}}{1 - t^2} \, dt + \frac{4}{\pi x} \sum_{n=1}^{\infty} d(n) \int_{0}^{\infty} \frac{e^{-4\pi nt/x}}{1 - t^2} \, dt \]

\[ = \frac{\gamma - \log(\pi x/2)}{\pi x}. \]  

(3.7)

In order to change the order of integration and summation in (3.7), we use the asymptotic behavior of the arithmetic function \( d(n) = O(n^\varepsilon) \) for all \( \varepsilon > 0 \) [1] and split each integral on four integrals over the intervals, containing as end-points \( t = 0, \infty \) and \( 1 \pm \delta, \delta > 0 \). Let us show this on the integral

\[ P.V. \int_{0}^{\infty} \frac{e^{-4\pi nt/x}}{1 - t^2} \, dt = \int_{0}^{1/2} \frac{e^{-4\pi nt/x}}{1 - t^2} \, dt + \lim_{\delta \to 0} \left( \int_{1/2}^{1-\delta} + \int_{1+\delta}^{1} \right) \frac{e^{-4\pi nt/x}}{1 - t^2} \, dt \]

\[ + \int_{2}^{\infty} \frac{e^{-4\pi nt/x}}{1 - t^2} \, dt. \]

Hence for any fixed \( x > 0 \) we have the estimates

\[ \int_{0}^{1/2} \frac{e^{-4\pi nt/x}}{1 - t^2} \, dt \leq 8x^3 \int_{0}^{1/2} \frac{tdt}{6x^3 + (4\pi nt)^3} \]

\[ \leq \frac{(x/\pi)^3/2}{2\sqrt{6}} \int_{0}^{1/2} \frac{dt}{\sqrt{t}} = \frac{(x/\pi)^3/2}{\sqrt{12}} \cdot n^{3/2}, \]

\[ \int_{2}^{\infty} \frac{e^{-4\pi nt/x}}{t^2 - 1} \, dt \leq \frac{1}{8} \left( \frac{x}{\pi n} \right)^2 \int_{2}^{\infty} \frac{dt}{t(t^2 - 1)} = \frac{C_x}{n^2}. \]

Concerning the middle integrals, we appeal to the Lagrange theorem to write
\[ P.V. \int_{1/2}^{2} \frac{e^{-4\pi nt/x}}{1 - t^2} \, dt = \left| \int_{1/2}^{2} \frac{(e^{-4\pi nt/x} - e^{-4\pi n/x})}{1 - t^2} \, dt \right| + e^{-4\pi n/x} \left| P.V. \int_{1/2}^{2} \frac{t}{1 - t^2} \, dt \right| \]

\[ \leq \frac{4\pi n}{x} e^{-2\pi n/x} \left\{ \int_{1/2}^{2} \frac{tdt}{1 + t} + e^{-4\pi n/x} \left| \lim_{\delta \to 0} \left( \int_{1/2}^{1-\delta} \frac{tdt}{1 - t^2} - \int_{1+\delta}^{2} \frac{tdt}{1 - t^2} \right) \right| \right\} \]

\[ = e^{-2\pi n/x} \left[ 2\pi n(3 - 2 \log 2) + e^{-2\pi n/x} \log 2 \right]. \]

Consequently, owing to these estimates the desired interchange is guaranteed, taking \( \varepsilon \in (0, 1/2) \). Hence recalling (3.4), equality (3.7) becomes

\[ S(x) - 2 S(2x) + \frac{2}{\pi x} \int_{0}^{\infty} \left( S \left( \frac{2t}{x} \right) - 2 S \left( \frac{t}{x} \right) \right) \frac{tdt}{1 - t^2} = \frac{\log 2}{\pi x}, \quad x > 0. \]

This is in fact an exceptional case of the Fox second kind integral equation (see [10], Section 11.15), whose solution cannot be written as in [10] in terms of the inverse Mellin transform. Nevertheless, the homogeneous equation (3.9) is solved recently by the author (see details in [14], Corollary 2), using the method of compositions with the Fourier and Hartley transforms. Now, defining the Mellin transform of \( f(x) = S(x) - 2 S(2x) \)

\[ f(x) + \frac{2}{\pi} \int_{0}^{\infty} \frac{f(t)xt \, dt}{1 - (xt)^2} = \frac{\log 2}{\pi x}, \quad x > 0. \]

and taking into account the known formula

\[ \frac{1}{\pi} P.V. \int_{0}^{\infty} \frac{t^{s-1}}{1 - t} = \cot(\pi s), \quad 0 < \text{Res} < 1, \]

the Mellin transform of the left-hand side of (3.9) is equal to

\[ F(s) - F(1 - s) \tan \left( \frac{\pi s}{2} \right). \]

But

\[ F(s) = S^*(s) \left( 1 - 2^{1-s} \right), \]

where \( S^*(s) \) is the Mellin transform of \( S(x) \). Recalling (3.2) and the functional equation (1.21) for the Riemann zeta-function one easily verifies that
Series related to the elliptic integrals

\[ F(s) - F(1-s) \tan \left( \frac{\pi s}{2} \right) = 0, \]

i.e. \( f \) satisfies the homogeneous equation (3.9), which has an infinite number of solutions as it is shown in [14]. We can also get it directly, applying the Hilbert transform to both sides of (3.9) and taking into account that integral (3.10) is zero for \( s = 1/2 \).

Finally, we observe from (1.19), (1.20), (3.2) that the same scheme can be applied to the series (1.2) with \( \alpha = 0 \), because they relate to (3.1), for instance, via identities

\[
\sum_{n=1}^{\infty} \frac{1}{\sinh(\pi nx)} = \sum_{n=1}^{\infty} [\coth(\pi nx/2) - 1] - \sum_{n=1}^{\infty} [\coth(\pi nx) - 1], \quad x > 0,
\]

\[
\sum_{n=1}^{\infty} [\tanh(\pi nx/2) - 1] = \sum_{n=1}^{\infty} [\coth(\pi nx) - 1] - \sum_{n=1}^{\infty} \frac{1}{\sinh(\pi nx)}, \quad x > 0.
\]

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