ON BICOMPLEX FOURIER–WIGNER TRANSFORMS

A. EL GOURARI, A. GHANMI, AND K. ZINE

ABSTRACT. We consider the 1- and 2-d bicomplex analogs of the classical Fourier–Wigner transform. Their basic properties, including Moyal’s identity and characterization of their ranges giving rise to new bicomplex–polyanalytic functional spaces are discussed. Particular case of special window is also considered. An orthogonal basis for the space of bicomplex–valued square integrable functions on the bicomplex numbers is constructed by means of the polyanalytic complex Hermite functions.

1 Introduction

The standard Fourier–Wigner (windowed Fourier) transform is the well–defined bilinear mapping $V : (f, g) \mapsto V(f, g)$ on $L^2_c(\mathbb{R}^d) \times L^2_c(\mathbb{R}^d)$ given by the cross–Wigner function \cite{7, 16, 18, 4, 6}

$$V(f, g)(p, q) = \left( \frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} e^{i\langle x - \frac{p}{2}, q \rangle} f(x) g(x - p) dx$$ \hspace{1cm} (1.1)

for every $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$. For fixed window state, it is closely related to Gabor’s transform \cite{8} as well as to the well–known Segal–Bargmann transform \cite{7, 16}. It reduces to the familiar Wigner distribution when $f = g$; see e.g \cite{18, 6}. The transform $V$ has being intensively considered in harmonic analysis \cite{7, 16} and time–frequency analysis \cite{3, 10}. In fact, it is very useful in in the study of nonexisting joint probability distribution of positioned momentum in a given state \cite{18}. It is a basic tool for interpreting quantum mechanics as a form of nondeterministic statical dynamics \cite{11}.

The aim of this paper is to extend this transform to the bicomplex setting, i.e. where $(\mathbb{R} \times \mathbb{R})e_+ + (\mathbb{R} \times \mathbb{R})e_-$ is considered instead of the standard phase (time-frequency) space $\mathbb{R} \times \mathbb{R}$. Although this can be accomplished in a number of different ways, we shall confine our attention to two main natural bicomplex Fourier–Wigner transforms (Sections 3 and 4). We investigate their basic properties such as the corresponding Moyal’s identities (energy preservation principle). We also characterize their ranges leading to new bicomplex–polyanalytic functional spaces. We also provide a new class of four–indices bivariate complex orthogonal polynomials of Hermite type that form an orthonormal basis of the infinite Hilbert space of bicomplex–valued square integrable functions on bicomplex space (see Section 4).

We will rely mostly on the notations and basic tools as given in \cite{9} and relevant to bicomplex numbers $\mathbb{T}$, bicomplex holomorphic functions and bicomplex Hilbert spaces, For further detail, we can refer to \cite{12, 15, 9} and the references therein.

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2 Preliminaries: The rescaled Fourier–Wigner transform.

We begin by reviewing the notion and the basic facts related to the rescaled Fourier–Wigner transform

$$\mathcal{V}^σ(f, g)(p, q) = \left( \frac{\sigma}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\sigma\langle x - \frac{p}{\sigma} \rangle} \hat{g}(x) \overline{\hat{f}(x)} dx.$$

Such transform can be rewritten in terms of the translation operator $T_x$ and the modulation operator $M^σ_x$ given respectively by $T_x g(t) := g(t - x)$ and $M^σ_x g(t) = e^{i\sigma\xi^T t} g(t)$. In fact, we have

$$\mathcal{V}^σ(f, g)(p, q) = \left( \frac{\sigma}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\sigma\langle p, q \rangle} \langle f, M^σ_{-q} T_p \hat{g} \rangle_{L^2_σ(\mathbb{R}^d)}$$

where $\langle p, q \rangle_{\mathbb{R}^d}$ denotes the usual scalar product in $\mathbb{R}^d$ and $L^2_σ(\mathbb{R}^d)$ is the space of $\mathbb{C}$–valued square integrable functions with respect to the Lebesgue measure $dx$ on $\mathbb{R}^d$. Such transform maps $L^2_σ(\mathbb{R}^d) \times L^2_σ(\mathbb{R}^d)$ into $L^2_σ(\mathbb{C}^d)$ (see e.g. [16, 18]). An interesting result satisfied by $\mathcal{V}^σ$ is the Moyal’s formula

$$\langle \mathcal{V}^σ(f, g), \mathcal{V}^σ(\varphi, \psi) \rangle_{L^2_σ(\mathbb{C}^d)} = \langle f, \varphi \rangle_{L^2_σ(\mathbb{R}^d)} \langle \psi, g \rangle_{L^2_σ(\mathbb{R}^d)}$$

for all $f, g, \varphi, \psi \in L^2_σ(\mathbb{R}^d)$. It readily follows from the classical Moyal’s formula for $\mathcal{V}$ ([7, 16, 18, 5]) combined with the fact that $\mathcal{V}^σ(f, g)(p, q) = \sigma^{d/2} \mathcal{V}(f, g)(p, \sigma q)$. It interprets the fact that $\mathcal{V}^σ$ preserves the energy of a signal. Accordingly, it can be shown [18, 5] that the Fourier–Wigner transform $\mathcal{V}^σ$ produces orthonormal bases for the Hilbert space $L^2_σ(\mathbb{C}^d)$ from the ones of $L^2_σ(\mathbb{R}^d)$. More precisely, if $\{ \varphi_k, k \in \mathbb{N} \}$ is an orthonormal basis of $L^2_σ(\mathbb{R}^d)$, then $\{ \varphi_{jk} = \mathcal{V}^σ(\varphi_j, \varphi_k); j, k = 0, 1, 2, \cdots \}$ is an orthonormal basis of $L^2_σ(\mathbb{C}^d)$. This fact will be used, when dealing with the special bicomplex Fourier–Wigner transform discussed in Section 4, in order to obtain bicomplex four–indices orthogonal polynomials of Hermite type that are not tensor product of the Hermite polynomials on $\mathbb{R}$.

The next result is the analog of Theorem 3.1 in [2] for the action of $\mathcal{V}^σ$ on the rescaled Hermite functions

$$h^\sigma_n(t) = -1)^n e^{\frac{\sigma t^2}{2}} \frac{d^n}{dt^n} \left( e^{-\sigma t^2} \right) = \sqrt{\sigma}^n h_n(\sqrt{\sigma} t)$$

that form an orthogonal basis of $L^2_σ(\mathbb{R})$. It asserts that $\mathcal{V}^σ(h^\sigma_m, h^\sigma_n)$ is closely connected to the univariate polyanalytic Hermite function

$$h^{\alpha}_{m,n}(z, \bar{z}) := (-1)^{m+n} e^{\frac{\sigma |z|^2}{2}} \frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} \left( e^{-\alpha |z|^2} \right), \quad \alpha > 0. \quad (2.3)$$

We denote $h_{m,n} = h^\sigma_{m,n}$.

**Proposition 2.1.** We have

$$\mathcal{V}^σ(h^\sigma_m, h^\sigma_n)(p, q) = (-1)^n \frac{2^{m+n}}{\sqrt{2}} h^{\sigma m+n}_{m,n}(z, \bar{z}). \quad (2.4)$$

**Proof.** A straightforward computation starting from the definition of $\mathcal{V}^σ$ and $h^\sigma_n$ shows

$$\mathcal{V}^σ(h^\sigma_m, h^\sigma_n)(p, q) = \sqrt{\sigma}^{m+n} \mathcal{V}(h_m, h_n)(\sqrt{\sigma} p, \sqrt{\sigma} q).$$

Subsequently, by means of Theorem 3.1 in [2] combined with the fact that

$$h^{\alpha}_{m,n}(z, \bar{z}) := \sqrt{\alpha}^{m+n} h_{m,n}(\sqrt{\alpha} z, \sqrt{\alpha} \bar{z}),$$
we obtain
\[ V^\sigma(h^\sigma_m,h^\sigma_n)(p,q) = \sqrt{\sigma^{m+n}}(-1)^n \sqrt{2^{m+n-1}} h_{m,n} \left( \frac{\sqrt{\sigma}}{\sqrt{2}}, \frac{\sqrt{\sigma}}{\sqrt{2}} \right) \]
\[ = (-1)^n \frac{2^{m+n}}{\sqrt{2}} h_{m,n}^{\sigma/2}(z,\bar{z}). \]

\[ \square \]

**Remark 2.2.** The range of \( L^2_\mathbb{C}(\mathbb{R}) \) by the transform \( V^\sigma(\cdot,h^\sigma_n) \) is the Hilbert space spanned by the complex Hermite functions \( h_{m,n}^{\sigma/2}(z,\bar{z}) \) for varying \( m \), which is clearly connected to the so-called true–poly–Fock space \( F^2_{\mathbb{C}}(\mathbb{C}) \) on \( \mathbb{C} \) of level \( n \) in Vasilevski’s terminology \[17, 1]\.

In the sequel, we suggest some natural bicomplex analogs of the Fourier–Wigner transform with input functions belonging to the bicomplex Hilbert spaces \( L^2_\mathbb{C}(X) \) with \( X = \mathbb{R} \) or \( \mathbb{R}^2 \) and output functions in \( L^2_\mathbb{C}(\mathbb{T}) \). We then investigate some of their basic properties, such as the Moyal’s identity, the identification of their ranges, the connection to some bicomplex transforms of Segal–Bargmann type, among others. The central idea in obtaining such bicomplex analogs is basically the idempotent decomposition of any \( \varphi \in L^2_\mathbb{C}(X) \) as \( \varphi = \varphi^+ e_+ + \varphi^- e_- \) with \( \varphi^+, \varphi^- \in L^2_\mathbb{C}(X) \).

Throughout the rest of the paper, the notation \( C_\tau \) (with \( \tau^2 = -1 \)) will be used to mean the complex plane \( C_\tau := \{ z_v := x + \tau y; x, y \in \mathbb{R} \} \) with \( C_i = \mathbb{C} \).

## 3 Unidimensional Bicomplex Fourier–Wigner Transform.

For every given bicomplex number \( Z = z_1 + jz_2 \), where \( z_\ell = x_\ell + iy_\ell \), we associate the hyperbolic numbers \( X_\ell = x_1 e_+ + x_2 e_- \) and \( Y_\ell = y_1 e_+ + y_2 e_- \), and consider the translation operator
\[ T_{X_\ell}(\varphi)(t) := \varphi^+(t-x_1)e_+ + \varphi^-(t-x_2)e_- \]
as well as the modified modulation operator
\[ M_{X_\ell,Y_\ell}^{\sigma,\nu,\mu}(\varphi)(t) := e^{\sigma(\nu e_+ + \mu e_-) - \frac{t}{2}} Y_\ell \varphi(t) \]
for given \( \varphi = \varphi^+ e_+ + \varphi^- e_- \in L^2_\mathbb{C}(\mathbb{R}) \).

**Definition 3.1.** We call unidimensional bicomplex Fourier–Wigner transform the integral transform \( V_{R,T}^{\sigma,\nu,\mu}(\cdot) \) on \( L^2_\mathbb{C}(\mathbb{R}) \times L^2_\mathbb{C}(\mathbb{R}) \) defined by
\[ V_{R,T}^{\sigma,\nu,\mu}(\varphi,\psi)(Z) := \left( \frac{\sigma}{\pi} \right)^{\frac{1}{4}} e^{-\frac{\sigma}{4}(X_1^2 + (Y_1)^2)} \int_{\mathbb{R}} \varphi(t) M_{X_\ell,Y_\ell}^{\sigma,\nu,\mu}(T_{X_\ell}(\varphi(t))^*) \, dt \tag{3.1} \]
with \( X_\ell^* = x_2 e_+ + x_1 e_- \) and \( Z^* = z_1 - jz_2 \).

The following lemmas will play a crucial role in establishing the main results of this section. To this end, we introduce \( z_{1,\nu} = x_1 + \nu y_1 \) and \( z_{2,\mu} = x_2 + \mu y_2 \) for given \( Z = z_1 + jz_2 \) with \( z_\ell = x_\ell + iy_\ell; \ell = 1, 2 \).

**Lemma 3.2.** Let \( \varphi, \psi \in L^2_\mathbb{C}(\mathbb{R}) \). Then, we have the splitting formula
\[ V_{R,T}^{\sigma,\nu,\mu}(\varphi,\psi)(Z) = \left( \frac{2\sigma}{\pi} \right)^{\frac{1}{4}} e^{-\frac{\sigma}{4}|z_{1,\nu}|^2} V_{R,T}^{\sigma,\nu,\mu}(\varphi^+,\psi^+)(z_{1,\nu})e_+ + \left( \frac{2\sigma}{\pi} \right)^{\frac{1}{4}} e^{-\frac{\sigma}{4}|z_{2,\mu}|^2} V_{R,T}^{\sigma,\nu,\mu}(\varphi^-,\psi^-)(z_{2,\mu})e_- \tag{3.2} \]
Moreover, $\nu_{R,T}^{\sigma,v,\mu}(\varphi, \psi)$ belongs to $L^2_T(\mathbb{T})$ and we have the Moyal’s identity

$$\langle \nu_{R,T}^{\sigma,v,\mu}(\varphi_1, \psi_1), \nu_{R,T}^{\sigma,v,\mu}(\varphi_2, \psi_2) \rangle_{L^2_T(\mathbb{T})} = \langle \varphi_1, \varphi_2 \rangle_{L^2_T(\mathbb{R})} \langle \psi_1, \psi_2 \rangle_{L^2_T(\mathbb{R})}$$

(3.3)

for every $\varphi_\ell, \psi_\ell \in L^2_T(\mathbb{R}); \ell = 1, 2$.

Proof. The first assertion follows easily from (3.1) since

$$(X^t)^2 + (Y^t)^2 = |z_2|_t^2 e_+ + |z_1|_t^2 e_-,$$

$$M_{X^t,Y^t}^{\sigma,v,\mu}(t) = e^{\nu(t-\frac{\sigma}{2})y_1} \varphi^+(t) e_+ + e^{\nu(t-\frac{\mu}{2})y_2} \varphi^-(t) e_-$$

and

$$T_{X,x} \psi(t) := T_{x_1} \psi^+(t) e_+ + T_{x_2} \psi^-(t) e_-.$$

In order to prove the second assertion, we notice first that the function $\mathcal{V}_{C}^{\sigma,v}(\varphi^+, \psi^+)(z_1)$ belongs to the Hilbert space $L^2_{C}(C_v)$. Accordingly, the function $e^{-\frac{\sigma}{2} |z_2|_t^2} \mathcal{V}_{C}^{\sigma,v}(\varphi^+, \psi^+)(z_1)$ belongs to $L^2_{C}(C_v \times C_v)$. The same observation holds true for $e^{-\frac{\nu}{2} |z_1|_t^2} \mathcal{V}_{C}^{\sigma,v}(\varphi^-, \psi^-)(z_2)$. This shows that $\nu_{R,T}^{\sigma,v,\mu}(\varphi, \psi)$ belongs to the Hilbert space

$$L^2_{C}(C_v) e_+ + L^2_{C}(C_v) e_- = L^2_{T}(C_v \times C_v) = 4L^2_{T}(\mathbb{T}).$$

Now, by denoting the left-hand side of (3.3) by $M(\varphi_1, \psi_1)$ and making use of (3.2), we obtain

$$M(\varphi_1, \psi_1) = \left( \frac{\sigma}{2\pi} \right) \left( \int_C e^{-\frac{\sigma}{2} |\xi|^2} d\lambda(\xi) \right) \left( \langle \mathcal{V}_{C}^{\sigma,v}(\varphi_1^+, \psi_1^+), \mathcal{V}_{C}^{\sigma,v}(\varphi_2^+, \psi_2^+) \rangle_{L^2_{C}(C_v)} e_+ + \langle \mathcal{V}_{C}^{\sigma,v}(\varphi_1^-, \psi_1^-), \mathcal{V}_{C}^{\sigma,v}(\varphi_2^-, \psi_2^-) \rangle_{L^2_{C}(C_v)} e_- \right).$$

Consequently, from (2.2) we get

$$M(\varphi_1, \psi_1) = \langle \varphi_1^+, \varphi_2^+ \rangle_{L^2_T(\mathbb{R})} \langle \psi_1^+, \psi_2^+ \rangle_{L^2_T(\mathbb{R})} e_+ + \langle \varphi_1^-, \varphi_2^- \rangle_{L^2_T(\mathbb{R})} \langle \psi_1^-, \psi_2^- \rangle_{L^2_T(\mathbb{R})} e_- = \left( \langle \varphi_1^+, \varphi_2^+ \rangle_{L^2_T(\mathbb{R})} e_+ + \langle \varphi_1^-, \varphi_2^- \rangle_{L^2_T(\mathbb{R})} e_- \right) \times \left( \langle \psi_1^+, \psi_2^+ \rangle_{L^2_T(\mathbb{R})} e_+ + \langle \psi_1^-, \psi_2^- \rangle_{L^2_T(\mathbb{R})} e_- \right) = \langle \varphi_1, \varphi_2 \rangle_{L^2_T(\mathbb{R})} \langle \psi_1, \psi_2 \rangle_{L^2_T(\mathbb{R})}.$$

This completes our check of (3.3) and hence the one of Lemma 3.2.

Another needed fact is the action of $\nu_{R,T}^{\sigma,v,\mu}$ on the elementary functions

$$f_{m,n}^{\sigma}(t) := h_m^\sigma(t) e_+ + h_n^\sigma(t) e_-.$$

Namely, we assert

**Lemma 3.3.** We have

$$\nu_{R,T}^{\sigma,v,\mu}(f_{m,n}^{\sigma}, f_{r,s}^{\mu})(Z) = \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} (-1)^{r+s} 2^{m+r} e^{-\frac{\sigma}{2} |z_2|_t^2} h_{m,r}^\sigma / 2(z_1, \bar{z}_1) e_+ + \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} (-1)^{s+r} 2^{m+s} e^{-\frac{\sigma}{2} |z_1|_t^2} h_{m,s}^\sigma / 2(z_2, \bar{z}_2) e_-.$$  

(3.4)
Proof. The result follows making use of (3.2) and Proposition 2.1. Indeed, we have
\[
V_{\mathcal{R},\mathcal{T}}^{\sigma,\nu}(f_{m,n,r,s}^{\sigma})(Z) = V_{\mathcal{R},\mathcal{T}}^{\sigma,\nu}(h_{m}^{\sigma}e_{+} + h_{n}^{\sigma}e_{-}, h_{r}^{\sigma}e_{+} + h_{s}^{\sigma}e_{-})(Z)
\]
\[
= \left(\frac{2\sigma}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{4}(|z_{m}|^{2}e_{+} + |z_{n}|^{2}e_{-})} \left(V_{\mathcal{R},\mathcal{T}}^{\sigma,\nu}(h_{m}^{\sigma}, h_{r}^{\sigma})(z_{v})e_{+} + V_{\mathcal{R},\mathcal{T}}^{\sigma,\nu}(h_{n}^{\sigma}, h_{s}^{\sigma})(z_{2})e_{-}\right)
\]
\[
= \left(\frac{\sigma}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{4}(|z_{m}|^{2}e_{+} + |z_{n}|^{2}e_{-})} \left((-1)^{r}2^{m-r}h_{m,r}^{\sigma/2}(z_{v})e_{+} + (-1)^{s}2^{n+s}h_{n,s}^{\sigma/2}(z_{2})e_{-}\right).
\]

Below, we will discuss the basic properties of the transform $V_{\mathcal{R},\mathcal{T}}^{\sigma,\nu}$ for the special window function $\psi_0(t) := e^{-\frac{1}{2}t^2}$. To this end, we need to associate to a bicomplex number $Z = z_1 + jz_2 \in \mathbb{T}$, its companion $Z_{v,\mu} = z_{v}e_{+} + z_{2}\mu e_{-}$ and perform
\[
T_{v,\mu}^{e} = C_{v}e_{+} - C_{\mu}e_{-} = \{Z_{v,\mu}^{e} = (x_1 + vy_1)e_{+}(x_2 + \mu y_2)e_{-}, x_1, y_1, x_2, y_2 \in \mathbb{R}\}.
\]
Therefore, any bicomplex–valued function $f(Z)$ on $\mathbb{T}$ can be seen as a function on $T_{v,\mu}^{e}$.

**Definition 3.4.** A bicomplex–valued function $f$ on $\mathbb{T}$ is said to be $T_{v,\mu}^{e}$–holomorphic if its companion $f^{e}(Z_{v,\mu}^{e}) := f(Z)$ is $T_{v,\mu}$–holomorphic in the sense that $f^{e}$ satisfies the system of first order differential equations
\[
\frac{\partial f^{e}}{\partial Z_{v,\mu}^{e}} = \frac{\partial f^{e}}{\partial Z_{v,\mu}^{e}} = \frac{\partial f^{e}}{\partial Z_{v,\mu}^{e}} = 0,
\]
where
\[
\frac{\partial}{\partial Z_{v,\mu}^{e}} = \frac{\partial}{\partial z_{v}^{e}} = \frac{\partial}{\partial z_{2}\mu^{e}} = \frac{\partial}{\partial z_{v}^{e}} + \frac{\partial}{\partial z_{2}\mu^{e}} = \frac{\partial}{\partial z_{2}\mu^{e}} + \frac{\partial}{\partial z_{v}^{e}}.
\]
This is clearly equivalent to rewrite $f$ in the form
\[
f(Z) = f^{e}(Z_{v,\mu}^{e}) = F(z_{v}e_{+} + G(z_{2})e_{-})
\]
with $F \in Hol(C_{v})$ (resp $G \in Hol(C_{\mu})$) is a holomorphic function on $C_{v}$ (resp. $C_{\mu}$). A variant bicomplex Bargmann space of the one introduced in [9] is the following.

**Definition 3.5.** We call compagion $T_{v,\mu}$–Bargmann space, the Hilbert space $F^{2,\sigma}(T_{v,\mu}^{e})$ of all bicomplex–valued $T_{v,\mu}$–holomorphic functions $f(Z) = f^{e}(Z_{v,\mu}^{e}) = F(z_{v}e_{+} + G(z_{2})e_{-}$ such that $F \in L_{C_{v}}^{2,\sigma/2}(C_{v})$ and $G \in L_{C_{\mu}}^{2,\sigma/2}(C_{\mu})$. Succinctly,
\[
F^{2,\sigma}(T_{v,\mu}^{e}) = F^{2,\sigma/2}(C_{v})e_{+} + F^{2,\sigma/2}(C_{\mu})e_{-},
\]
where $F^{2,\sigma/2}(C_{\tau})$ denotes the classical complex Bargmann space of weight $\sigma/2$ on $C_{\tau}$.

This functional space is trivially endowed with the bicomplex scaler product
\[
\langle f_{1}, f_{2} \rangle_{T_{v,\mu}^{e}} = \langle F_{1}, F_{2} \rangle_{L_{C_{v}}^{2,\sigma/2}(C_{v})}e_{+} + \langle F_{1}, F_{2} \rangle_{L_{C_{\mu}}^{2,\sigma/2}(C_{\mu})}e_{-}
\]
for given $f_{1}(Z) = F(z_{v}e_{+} + G(z_{2})e_{-}$. Accordingly, it can be seen as subspace of $L_{2}^{2}(C_{v} \times C_{\mu}) = 4L_{2}^{2}(\mathbb{T})$ by considering its range $M_{\sigma/2}(T_{v,\mu}^{e}) := M_{\sigma/2}(F^{2,\sigma}(T_{v,\mu}^{e}))$ by the multiplication operator
\[
M_{\sigma/2}(f) := e^{-\frac{1}{2}|Z_{v,\mu}^{e}|^{2}}f^{e}(Z_{v,\mu}^{e})
\]
\[
= e^{-\frac{1}{2}|z_{v}|^{2} + |z_{2}|^{2}} \left(F(z_{v})e_{+} + G(z_{2})e_{-}\right).
\]
In fact, for given \( Y_\ell = M_\nu f_\ell \in \mathcal{M}^{2,\nu}(\mathbb{T}^e_{v,\mu}); \ell = 1, 2 \), we have
\[
\langle Y_1, Y_2 \rangle_{L^2_{\nu}(\mathbb{T})} = \int_{\mathbb{T}} Y_1(Z)(Y_2(Z))^* d\lambda(Z) = \frac{1}{4} \left( \frac{2\pi}{\sigma} \right) \langle f_1, f_2 \rangle_{\mathbb{T}^e_{v,\mu}}.
\]

The corresponding bicomplex norm is the one given through
\[
\|Y\|^2_{L^2_{\nu}(\mathbb{T})} := \left( \frac{\pi}{4\sigma} \right) \left( \|F\|^2_{L^2_{\nu}^e(C)} + \|G\|^2_{L^2_{\nu}^e(C)} \right).
\]

Thus, we claim the following

**Proposition 3.6.** The space \( \mathcal{M}^{2,\nu}(\mathbb{T}^e_{v,\mu}) \) is a reproducing kernel bicomplex Hilbert space whose kernel function is given by
\[
K_{\sigma}(Z^e_{v,\mu}, W^e_{v,\mu}) = e^{-\frac{\sigma}{2}(|Z^e_{v,\mu}|^2 + |W^e_{v,\mu}|^2)}.
\]

Moreover, we prove

**Theorem 3.7.** The transform \( S_{0,\sigma,v,\mu}^e(\varphi) \) given by
\[
S_{0,\sigma,v,\mu}^e(\varphi) := \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} V_{R,T}^{\sigma,v,\mu}(\varphi, \psi_0)
\]
defines an isometry from \( L^2_{\nu}(\mathbb{R}) \) onto the Hilbert space \( \mathcal{M}^{2,\sigma}(\mathbb{T}^e_{v,\mu}) \). Moreover, the functions
\[
\varphi_n^{\sigma,v,\mu}(Z) = \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} \sigma^n \left( Z^e_{v,\mu} \right)^n e^{-\frac{\sigma}{2}|Z^e_{v,\mu}|^2}
\]
form an orthogonal basis of \( \mathcal{M}^{2,\sigma}(\mathbb{T}^e_{v,\mu}) \) with norm given by
\[
\left\| \varphi_n^{\sigma,v,\mu} \right\|^2_{L^2_{\nu}(\mathbb{T})} = \left( \frac{\pi}{\sigma} \right)^{\frac{1}{2}} 2^n \sigma^nn!.
\]

**Proof.** Notice first that the window state is the Gaussian centred at the origin \( \psi_0(t) := e^{-\frac{\sigma}{2}t^2} = f_{0,0}(t) \) and \( h_n^\sigma = f_{n,n}^\sigma \) for \( e_+ + e_- = 1 \). Thus, from (3.4) and the fact \( h_{n,0}^\sigma(\zeta, \overline{\zeta}) = \alpha^n e^{\frac{\sigma}{2}|\zeta|^2} \), we obtain
\[
V_{R,T}^{\sigma,v,\mu}(h_n^\sigma, \psi_0)(Z) = V_{R,T}^{\sigma,v,\mu}(f_{n,n}^\sigma, f_{0,0})(Z)
\]
\[
= \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} 2^n \left( e^{-\frac{\sigma}{2}|z_{2p}|^2} h_{n,0}^\sigma(\overline{z_{1p}}, z_{1p}) e_+ + e^{-\frac{\sigma}{2}|z_{1p}|^2} h_{n,0}^\sigma(\overline{z_{2p}}, z_{2p}) e_- \right)
\]
\[
= \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} 2^n \sigma \left( z_{1p} e_+ + z_{2p} e_- \right) e^{-\frac{\sigma}{2}(|z_{1p}|^2 + |z_{2p}|^2)}
\]
\[
= \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} 2^n \sigma \left( z_{1p} e_+ + z_{2p} e_- \right) e^{-\frac{\sigma}{2}(|z^e_{v,\mu}|^2)}
\]

These functions form clearly an orthogonal system in the Hilbert space \( L^2_{\nu}(\mathbb{T}) \) in virtue of the Moyal’s identity (3.3) satisfied by \( V_{R,T}^{\sigma,v,\mu} \) and the orthogonality of \( h_n^\sigma \) in \( L^2_{\nu}(\mathbb{R}) \). Indeed, we have
\[
\langle \varphi_m^{\sigma,v,\mu}, \varphi_n^{\sigma,v,\mu} \rangle_{L^2_{\nu}(\mathbb{T})} = \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} \langle V_{R,T}^{\sigma,v,\mu}(h_m^\sigma, \psi_0), V_{R,T}^{\sigma,v,\mu}(h_n^\sigma, \psi_0) \rangle_{L^2_{\nu}(\mathbb{T})}
\]
\[
= \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} \langle h_m^\sigma, h_n^\sigma \rangle_{L^2_{\nu}(\mathbb{R})} \langle \psi_0, \psi_0 \rangle_{L^2_{\nu}(\mathbb{R})}
\]
\[
= \left( \frac{\sigma}{\pi} \right)^{\frac{1}{2}} \|\psi_0\|^2_{L^2_{\nu}(\mathbb{R})} \|h_n^\sigma\|^2_{L^2_{\nu}(\mathbb{R})} \delta_{m,n}
\]
\[
= \|h_n^\sigma\|^2_{L^2_{\nu}(\mathbb{R})} \delta_{m,n}.
\]
This readily follows since \( \| \psi_0 \|_{L^2_1(\mathbb{R})}^2 = \left( \frac{\pi}{\sigma} \right)^{1/2} \) and consequently gives rise to (3.9) for \( \| h_n^2 \|_{L^2_1(\mathbb{R})}^2 = \left( \frac{\pi}{\tau} \right)^{1/2} 2^n \sigma^m n! \). Identity (3.9) can also be handled by direct computation using the explicit expression of \( \phi_n^{\sigma,\nu,\mu} \). The previous result remains valid for any \( \varphi \in L^2_1(\mathbb{R}) \). Indeed, by applying the Moyal’s identity (3.3), we get

\[
\left\| S_0^{\sigma,\nu,\mu}(\varphi) \right\|_{L^2_1(\mathbb{T})}^2 = \left( \frac{\sigma}{\tau} \right)^{1/2} \left| \left\langle \mathcal{V}_{R,1}^{\sigma,\nu,\mu}(\varphi, \psi), \mathcal{V}_{R,1}^{\sigma,\nu,\mu}(\varphi, \psi_0) \right\rangle_{L^2_1(\mathbb{T})} \right| = \left( \frac{\sigma}{\tau} \right)^{1/2} \left| \left\langle \varphi, \varphi \right\rangle_{L^2_1(\mathbb{R})} \left\langle \psi_0, \psi_0 \right\rangle_{L^2_1(\mathbb{R})} \right| = \| \varphi \|_{L^2_1(\mathbb{R})}^2.
\]

This shows in particular that \( S_0^{\sigma,\nu,\mu} \in \mathcal{M}^{2,\sigma}(\mathbb{T}^{\nu,\mu}_v) \). One can conclude for the proof, by noting that the functions \( \phi_n^{\sigma,\nu,\mu} \) (in (3.8)) is a complete orthogonal system in \( \mathcal{M}^{2,\sigma}(\mathbb{T}^{\nu,\mu}_v) \) for the monomials \( (Z_v^{\nu,\mu})^n \) form an orthogonal basis of \( L^2_{\mathbb{T}}^{2,\sigma/2}(\mathbb{C}_v \times \mathbb{C}_\mu) \). Moreover, for any \( \varphi(t) = \sum_{n=0}^\infty c_n h_n^\sigma \in L^2_1(\mathbb{R}) \), we have

\[
S_0^{\sigma,\nu,\mu}(\varphi) = \left( \frac{\sigma}{\pi} \right)^{1/4} \sum_{n=0}^\infty c_n \phi_n^{\sigma,\nu,\mu}(Z)
\]

which follows by means of (3.10) and the continuity of the linear mapping \( S_0^{\sigma,\nu,\mu} \).

**Corollary 3.8.** The transform \( S_0^{\sigma,\nu,\mu} \) is closely connected to the bicomplex Segal–Bargmann transform \( \mathcal{B}^{\sigma,\nu}_{1} \) introduced in [9]. More precisely, we have

\[
S_0^{\sigma,i,i}(\varphi)(Z) = \left( \frac{\sigma}{\pi} \right) e^{-\frac{\xi^2}{2}|Z_v^{\nu,\mu}|^2} \int_{\mathbb{R}} e^{-\sigma (t-Z_v^{\nu,\mu})^2} \xi^2 \varphi(t) dt.
\]

**Proof.** Identity (3.11) which follows by a tedious but straightforward computation. Indeed, we obtain

\[
S_0^{\sigma,\nu,\mu}(\varphi)(Z) = \left( \frac{\sigma}{\pi} \right) e^{-\frac{\xi^2}{2}|Z_v^{\nu,\mu}|^2} \int_{\mathbb{R}} e^{-\sigma (t-Z_v^{\nu,\mu})^2} \xi^2 \varphi(t) dt,
\]

so that for \( \nu = \mu = i \), we recover the bicomplex Segal–Bargmann transform introduced in [9] Eq. (5.6) (with \( \nu = \sigma \) there) from \( L^2_1(\mathbb{R}) \) onto the bicomplex Bargmann space \( \mathcal{C}^2_{\mathbb{C}^{\mathbb{C}_v}}(\mathbb{T}_v) \).

The last result of this section identifies the range \( \mathcal{V}_{R,1}^{\sigma,\nu,\mu}(L^2_1(\mathbb{R}) \times L^2_1(\mathbb{R})) \) as special bicomplex–analytic closed subspace of \( L^2_{\mathbb{T}}(\mathbb{T}) \).

**Definition 3.9.** We call bicomplex \( (n^*, 1^+, 1^+) – \mathbb{T}_v^{\nu,\mu} \)– companion Bargmann space of first kind the Hilbert space \( \mathcal{F}^{2,\sigma,\nu,\mu}_n(\mathbb{T}^{\nu,\mu}_v) \) of all bicomplex–valued functions \( f(Z) = f_\nu(Z_v^{\nu,\mu}) = F(z_1_v)e_+ + G(z_2_v)e_- \) satisfying the system

\[
\frac{\partial^{n+1} f}{\partial (Z_v^{\nu,\mu})^{n+1}} = \frac{\partial f_\nu}{\partial (Z_v^{\nu,\mu})} = \frac{\partial f_\nu}{\partial (Z_v^{\nu,\mu})^\dagger} = 0,
\]

and \( \| F \|_{L^2_{\mathbb{T}}^{2,\sigma/2}(\mathbb{C}_v)}^2 \) and \( \| G \|_{L^2_{\mathbb{T}}^{2,\sigma/2}(\mathbb{C}_v)}^2 \) are finite.

Thus, we claim the following (we omit the proof for its similarity to one provided above in the case \( n = 0 \)).
Lemma 3.10. The spaces $\mathcal{M}_{n}^{2,\sigma}(\mathbb{T}_{v,\mu}) := M_{\sigma}(F_{n}^{2,\sigma,\nu,\mu}(\mathbb{T}_{v,\mu}))$ are closed subspaces of $L^{2}_{\mathbb{T}}(\mathbb{T})$ and we have

$$
\mathcal{M}_{n}^{2,\sigma}(\mathbb{T}_{v,\mu}) = e^{-\frac{Z_{e,\mu}}{2}}\left( F_{n}^{2,\sigma,\nu,\mu}(C_{\nu})e_{+} + F_{n}^{2,\sigma,\nu,\mu}(C_{\mu})e_{-} \right). \quad (3.13)
$$

Moreover, they are pairwise orthogonal in $L^{2}_{\mathbb{T}}$ corresponding to the window function $h_{\sigma}$ of (3.4). Thus, the complex Hermite functions for varying $m$, form an orthogonal basis of the infinite Hilbert space $\mathcal{M}_{n}^{2,\sigma}(\mathbb{T}_{v,\mu})$.

Proof. The proof can be handled in a similar way as for Theorem 3.7 (for $h_{\sigma}$) and the associated bicomplex norm is given by (3.7).

Proposition 3.11. The functions $\psi_{m,n}^{\sigma,\nu,\mu}(Z_{v,\mu}^{e},Z_{v,\mu}^{e} \ast) = e^{-\frac{Z_{e,\mu}}{2}}H_{m,n}(Z_{v,\mu}^{e},Z_{v,\mu}^{e} \ast)$, where

$$
h_{m,n}^{\sigma}(Z_{v,\mu}^{e},Z_{v,\mu}^{e} \ast) := (-1)^{m+n}e^{\frac{\sigma}{\pi}Z_{v,\mu}^{e}Z_{v,\mu}^{e} \ast} \frac{\partial^{m+n}}{(\partial(Z_{v,\mu}^{e} \ast))^{m}\partial(Z_{v,\mu}^{e})^{n}}\left( e^{-\frac{Z_{e,\mu}}{2}}Z_{v,\mu}^{e},Z_{v,\mu}^{e} \ast \right) \quad (3.14)
$$

for varying $m$, form an orthogonal basis of the infinite $\mathbb{T}_{v,\mu}$-Hilbert space $\mathcal{M}_{n}^{2,\sigma}(\mathbb{T}_{v,\mu})$.

Proof. The proof is similar to one provided for $n = 0$, but here we make use of the fact that the complex Hermite functions $h_{m,n}^{\sigma}(x)$ is an orthogonal basis of $L^{2}_{\mathbb{T}}(\mathbb{T})$ and that $H_{m,n}^{\sigma}(z_{v,\mu})e_{+} + H_{m,n}^{\sigma}(z_{v,\mu})e_{-} = \psi_{m,n}(Z_{v,\mu}^{e},Z_{v,\mu}^{e} \ast)$. □

Theorem 3.12. The transform

$$
\mathcal{S}_{n}^{\sigma,\nu,\mu} f := \left( \frac{\sigma}{\pi} \right)^{1/4} \sqrt{2^{n}} Y_{R,\mathbb{T}}^{\sigma,\nu,\mu}(f,h_{n}^{\sigma})
$$

corresponding to the window function $h_{n}^{\sigma}$ defines an isometry from $L^{2}_{\mathbb{T}}(\mathbb{R})$ onto the Hilbert space $\mathcal{M}_{n}^{2,\sigma}(\mathbb{T}_{v,\mu})$.

Proof. The proof can be handled in a similar way as for Theorem 3.7 (for $n = 0$) with $h_{0}^{\sigma} = \psi_{0}$. Let just mention that the expression of the functions $Y_{R,\mathbb{T}}^{\sigma,\nu,\mu}(h_{m,n}^{\sigma},h_{n}^{\sigma})$ is a particular of (3.4).

Thus,

$$
Y_{R,\mathbb{T}}^{\sigma,\nu,\mu}(h_{m,n}^{\sigma},h_{n}^{\sigma})(Z) = \mathcal{V}_{R,\mathbb{T}}^{\sigma,\nu}(f_{m,n},f_{n,n})(Z)
$$

$$
= (-1)^{m+n}e^{\frac{\sigma}{\pi}Z_{v,\mu}^{e}Z_{v,\mu}^{e} \ast} \left( h_{m,n}^{\sigma}(z_{1,\mu})e_{+} + h_{m,n}^{\sigma}(z_{2,\mu})e_{-} \right),
$$

$$
= (-1)^{m+n}e^{\frac{\sigma}{\pi}Z_{v,\mu}^{e}Z_{v,\mu}^{e} \ast} \psi_{m,n}^{\sigma,\nu,\mu}(Z_{v,\mu}^{e},Z_{v,\mu}^{e} \ast) \quad (3.15)
$$

where $\psi_{m,n}^{\sigma,\nu,\mu}$ are as in Proposition 3.11. The range of $L^{2}_{\mathbb{T}}(\mathbb{R})$ by $\mathcal{S}_{n}^{\sigma,\nu,\mu}$ is then spanned by the bicomplex Hermite functions $\psi_{m,n}^{\sigma,\nu,\mu}$ for varying $m$ ($n$ fixed). Thus, one can conclude making use of Proposition 3.11 and the Moyal’s identity (3.3). □

Theorem 3.13. The transform $Y_{R,\mathbb{T}}^{\sigma,\nu,\mu}$ defines an isometry from $L^{2}_{\mathbb{T}}(\mathbb{R}) \times L^{2}_{\mathbb{T}}(\mathbb{R})$ onto the Hilbert space

$$
\mathcal{G}^{2,\sigma}(\mathbb{T}_{v,\mu}) := \bigoplus_{n=0}^{+\infty} \mathcal{M}_{n}^{2,\sigma}(\mathbb{T}_{v,\mu}).
$$
Proof. By Lemma 3.10 it is clear that the bicomplex Hermite functions \( \psi_{m,n}^{\sigma,\nu,\mu} \) for varying \( m \) and \( n \) form an orthogonal basis of the range of \( L_T^2(\mathbb{R}) \times L_T^2(\mathbb{R}) \) by \( \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu} \).

Remark 3.14. The space \( \mathcal{M}^{2,\sigma}(\mathbb{T}_\nu,\mu) \) is strictly contained in \( L_T^2(\mathbb{T}) \) since the functions

\[
\varphi_{m,n}^{\sigma,\nu,\mu}(Z) = \left( \sigma^m z_2^m e_+ + \sigma^n z_1^n e_- \right) e^{-\frac{|z|^2}{2|\nu|^2}}
\]

belong to \( L_T^2(\mathbb{T}) \) whenever \( m \neq n \) but do not belongs to \( G^{2,\sigma}(\mathbb{T}_\nu,\mu) \).

4. Bidimensional bicomplex Fourier–Wigner transform.

In this section, we consider the natural extension to the bicomplex Hilbert space \( L_T^2(\mathbb{R}^2) \) of the operators defined on \( L_T^2(\mathbb{R}^2) \) by

\[
M_{X,Y}^{\nu,\mu} g(U) = e^{\nu,\mu}(U-\frac{1}{2},Y) g(U) \quad \text{and} \quad T_X g(U) := g(U - X)
\]

where \( X, Y \in \mathbb{R}^2 \). Namely, we define

\[
\hat{M}_{X,Y}^{\nu,\mu} \varphi = M_{X,Y}^{\nu,\mu} \varphi^+ e_+ + M_{X,Y}^{\nu,\mu} \varphi^- e_-
\]

and

\[
\hat{T}_X \psi(U) := \psi^+(U - X)e_+ + \psi^-(U - X)e_-
\]

for given \( \varphi = \varphi^+ e_+ + \varphi^- e_- \) and \( \psi = \psi^+ e_+ + \psi^- e_- \) in \( L_T^2(\mathbb{R}^2) \) with \( \varphi^+, \varphi^-, \psi^+, \psi^- \in L_T^2(\mathbb{R}^2) \).

Definition 4.1. We call bidimensional bicomplex Fourier–Wigner transform that we denote by \( \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu} \) the one associated to the "bicomplex time–frequency shift" operator \( \hat{M}_{X,Y}^{\nu,\mu} \hat{T}_X \)

on \( L_T^2(\mathbb{R}^2) \times L_T^2(\mathbb{R}^2) \) and given explicitly by

\[
\mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi,\psi)(Z) = \left( \frac{1}{2\pi} \right)^{\frac{1}{4}} \int_{\mathbb{R}^2} e^{\sigma(v e_+ + \mu e_-)}(U-\frac{1}{2},Y) e^2 \varphi(U) \psi^+(U - X) d\lambda(U) \quad (4.1)
\]

with \( Z = z_1 + j z_2 \in \mathbb{T}, z_\ell = x_\ell + iy_\ell, X = (x_1, x_2) \) and \( Y = (y_1, y_2) \).

By proceeding in a similar way as in the previous section, we can prove the following (we omit the proof).

Lemma 4.2. We have

\[
\mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi,\psi)(Z) = \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi^+,\psi^+)(X,Y) e_+ + \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi^-,\psi^-)(X,Y) e_-
\]

as well as

\[
\left\langle \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi_1,\psi_1), \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi_2,\psi_2) \right\rangle_{L_T^2(\mathbb{T})} = \langle \varphi_1, \varphi_2 \rangle_{L_T^2(\mathbb{C})} \langle \psi_1, \psi_2 \rangle_{L_T^2(\mathbb{C})}. \quad (4.3)
\]

Proposition 4.3. The bicomplex Fourier–Wigner transform \( \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu} \) defines a surjection from \( L_T^2(\mathbb{R}^2) \times L_T^2(\mathbb{R}^2) \) onto \( L_T^2(\mathbb{T}) \).

Proof. Let \( F \in L_T^2(\mathbb{T}) \). Then, we can rewrite \( F \) as \( F = F^+ e_+ + F^- e_- \) for certain \( F^\pm \in L_T^2(\mathbb{C}^2) \). By the surjectivity of \( \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu} \) and \( \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu} \) from \( L_T^2(\mathbb{R}^2) \times L_T^2(\mathbb{R}^2) \) onto \( L_T^2(\mathbb{C}^2) \), we can exhibit \( \varphi^+, \varphi^- \in L_T^2(\mathbb{R}^2) \) such that

\[
F^+(Z) = \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi^+,\psi^+)(X,Y)
\]

and

\[
F^-(Z) = \mathcal{V}_{\nu,\mu}^{\sigma,\nu,\mu}(\varphi^-,\psi^-)(X,Y).
\]
Accordingly,
\[ F(Z) = \mathcal{V}_{\mathbb{R}^2,\mathbb{C}^2}^{\sigma,\nu}(\phi^+,\psi^+)(X,Y)e_+ + \mathcal{V}_{\mathbb{R}^2,\mathbb{C}^2}^{\sigma,\nu}(\phi^-,\psi^-)(X,Y)e_- . \]

In virtue of (4.2) and setting \( \varphi := \phi^+e_+ + \phi^-e_- \) and \( \psi := \psi^+e_+ + \psi^-e_- \), we get
\[ F(Z) = \mathcal{V}_{\mathbb{R}^2,\mathbb{T}}^{\sigma,\nu,\mu}(\varphi^+e_+ + \varphi^-e_- + \psi^+e_+ + \psi^-e_-)(Z) = \mathcal{V}_{\mathbb{R}^2,\mathbb{T}}^{\sigma,\nu,\mu}(\varphi,\psi)(Z). \]

Notice finally that \( \varphi, \psi \in L_2^2(\mathbb{R}^2) \) since \( \varphi^\pm, \psi^\pm \in L_2^2(\mathbb{R}^2) \).

In the sequel, we provide a nontrivial basis for the bicomplex Hilbert space \( L_2^2(\mathbb{T}) \). In fact, the Moyal’s identity (4.3) is an effective tool for constructing orthogonal bases for \( L_2^2(\mathbb{T}) \) from those of \( L_2^2(\mathbb{C}) \). Namely, we assert

**Proposition 4.4.** Let \( (\phi_n) \) be a system in \( L_2^2(\mathbb{R}^2) \) such that \( \phi_n = \phi_n^+e_+ + \phi_n^-e_- \) with \( \phi_n^+, \phi_n^- \in L_2^2(\mathbb{C}) \). If \( (\phi_n^+) \) and \( (\phi_n^-) \) are orthonormal bases of \( L_2^2(\mathbb{C}) \), then the family of functions
\[ \phi_{m,n} := \mathcal{V}_{\mathbb{R}^2,\mathbb{T}}^{\sigma,\nu,\mu}(\phi_m,\phi_n); \quad m, n = 0, 1, 2, \ldots , \]
is an orthonormal basis of \( L_2^2(\mathbb{T}) \).

**Proof.** Under the assumption that \( (\phi_n^+) \) and \( (\phi_n^-) \) are orthogonal in \( L_2^2(\mathbb{C}) \), i.e.,
\[ \langle \phi_n^+, \phi_n^- \rangle_{L_2^2(\mathbb{C})} = \langle \phi_n^-, \phi_n^+ \rangle_{L_2^2(\mathbb{C})} = 0; \quad n \neq n', \]
it follows
\[ \langle \phi_n^+, \phi_{n'}^+ \rangle_{L_2^2(\mathbb{C})} = \langle \phi_n^-, \phi_{n'}^- \rangle_{L_2^2(\mathbb{C})} = 0; \quad \text{for } n \neq n', \text{ and therefore } (\phi_n) \text{ is orthogonal in } L_2^2(\mathbb{C}). \]

Thus,
\[ \langle \phi_m, \phi_{m'} \rangle_{L_2^2(\mathbb{C})} = 0; \quad \text{for } (m, n) \neq (m', n'). \]

Subsequently, the family \( (\mathcal{V}_{\mathbb{R}^2,\mathbb{T}}^{\sigma,\nu,\mu}(\phi_m,\phi_n)) \) is orthogonal in \( L_2^2(\mathbb{T}) \) by means of (4.3). Moreover, the corresponding bicomplex norm is given by
\[ \|\phi_{m,n}\|^2_{L_2^2(\mathbb{T})} = \left| \left( \mathcal{V}_{\mathbb{R}^2,\mathbb{T}}^{\sigma,\nu,\mu}(\phi_m,\phi_n), \mathcal{V}_{\mathbb{R}^2,\mathbb{T}}^{\sigma,\nu,\mu}(\phi_m,\phi_n) \right)_{L_2^2(\mathbb{T})} \right| \]
\[ = \left| \langle \phi_m, \phi_n \rangle_{L_2^2(\mathbb{C})} \langle \phi_m, \phi_n \rangle_{L_2^2(\mathbb{C})} \right| \]
\[ = \left| \langle \phi_m^+, \phi_{n'}^+ \rangle_{L_2^2(\mathbb{C})} \langle \phi_m^-, \phi_{n'}^- \rangle_{L_2^2(\mathbb{C})} e_+ + \langle \phi_m^-, \phi_n^+ \rangle_{L_2^2(\mathbb{C})} \langle \phi_m^+, \phi_n^- \rangle_{L_2^2(\mathbb{C})} e_- \right| \]
\[ = \frac{1}{2} \left( \|\phi_m^+\|^2_{L_2^2(\mathbb{C})} + \|\phi_m^-\|^2_{L_2^2(\mathbb{C})} \right), \]
so that \( \|\phi_{m,n}\|^2_{L_2^2(\mathbb{T})} = 1 \) for \( (\phi_n^+) \) and \( (\phi_n^-) \) being orthonormal in \( L_2^2(\mathbb{C}) \). The fact that \( (\phi_{m,n}) \) is a basis of \( L_2^2(\mathbb{T}) \) follows easily since this is equivalent to \( (\mathcal{V}_{\mathbb{R}^2,\mathbb{C}^2}^{\sigma,\nu}(\phi_m^+, \phi_n^+)) \) and \( (\mathcal{V}_{\mathbb{R}^2,\mathbb{C}^2}^{\sigma,\nu}(\phi_m^-, \phi_n^-)) \) be bases of \( L_2^2(\mathbb{C}) \) in view of the idempotent decomposition (4.2). This holds true since \( (\phi_n^+) \) and \( (\phi_n^-) \) are bases of \( L_2^2(\mathbb{C}) \) and \( \mathcal{V}_{\mathbb{R}^2,\mathbb{C}^2}^{\sigma,\nu} \) is the standard Fourier–Wigner transform mapping orthonormal bases of \( L_2^2(\mathbb{C}) \) to orthonormal bases of \( L_2^2(\mathbb{C}) \). This completes the proof.

**Corollary 4.5.** The functions
\[ h_{m,n,m',n'}^{\sigma}(Z) := \mathcal{V}_{\mathbb{R}^2,\mathbb{T}}^{\sigma,\nu,\mu}(h_{m,n}^{\sigma}, h_{m',n'}^{\sigma})(Z) \]
for varying \( m, n, m', n' = 0, 1, 2, \ldots \), form an orthogonal basis of \( L_2^2(\mathbb{T}) \).
Proof. This is an immediate consequence of Proposition 4.4 since the univariate complex Hermite functions $h^\sigma_{m,n}(\xi, \eta) = h^\sigma_{m,n}(\xi, \overline{\eta}) e^+ + h^\sigma_{m,n}(\xi, \overline{\eta}) e^-$ is an orthogonal basis of $L^2_2(\mathbb{C})$.

\[ \square \]

Remark 4.6. The polynomials associated to $h^\sigma_{m,n,m',n'}$ form a new class of bivariate complex Hermite polynomials which are not a tensor product of four one–dimensional copies of the classical Hermite functions $h^\sigma_{m}$, nor a tensor product of two copies of the complex Hermite functions $h^\sigma_{m,n}$.

Remark 4.7. For the special window function $\psi_0(U) := h^\sigma_0(u) h^\sigma_0(v) = h^\sigma_0(0; U_\tau)$ with $U_\tau = u + \tau v$ and $U^2 = u^2 + v^2$ for $U = (u, v)$, the transform $\varphi \longmapsto V^\sigma_{\mathbb{R}^2, T}(\varphi, \psi_0)$ on is closely connected to the bidimensional Segal–Bargmann transform. Indeed, we have

\[
V^\sigma_{\mathbb{R}^2, T}(\varphi, \psi_0)(Z) = c \int_{\mathbb{R}^2} e^{-\frac{1}{4}(X^2+Y^2)} e^{-\frac{1}{2}(U-(ve_++\mu e_-))^2} \varphi(U) d\lambda(U) = c e^{-\frac{1}{2}|S_{ve_++\mu e_-}|^2} e^{\frac{1}{2}(S_{ve_++\mu e_-})^2} \int_{\mathbb{R}^2} e^{-\frac{1}{2}(U-(S_{ve_++\mu e_-}))^2} \varphi(U) d\lambda(U),
\]

where $c = \sqrt{2\pi}^{-1/2}$ and $S_{ve_++\mu e_-} = (z, w) = X + (ve_++\mu e_-)Y \in \mathbb{C}^2_{ve_++\mu e_-}$ with $X = (x_1, x_2), Y = (y_1, y_2), Z = z_1 + jz_2 \in \mathbb{T}$ and $z_\ell = x_\ell + iy_\ell; \ell = 1, 2$.

5 Concluding remarks

We have considered two bicomplex analogs of the classical (rescaled) Fourier–Wigner transform. This follows using the idempotent decomposition of bicomplex numbers. The standard phase (or time–frequency) space $\mathbb{R} \times \mathbb{R}$ is replaced here by the bicomplex $(\mathbb{R} \times \mathbb{R})e_+ + (\mathbb{R} \times \mathbb{R})e_-$. Thus the concrete description of analytic properties of these transforms are obtained. It gives rise to special generalization of the bicomplex Bargmann space studied in [9]. One of the advantage of this setting is to work simultaneously with two models of the polyanalytic Bargmann space $\mathcal{F}^2_{n'}(\mathbb{C}_T)$, the first one is focused on $e_+$ and the other on $e_-$. This is the case of the first transform and the obtained functional spaces are particular subclasses of the so–called $(n^*, 1^-, 1^+)$–$\mathbb{T}$–polyanalytic functions of first kind. More generally, a bicomplex–valued function $f$ on $\mathbb{T}$ is said to be $(n^*, m^-, k^+)$–$\mathbb{T}$–polyanalytic if it satisfies the system of first order differential equations

\[
\frac{\partial^{n+1} f}{\partial (Z^*)^{n+1}} = \frac{\partial^{m+1} f}{\partial (Z^*)^{m+1}} = \frac{\partial^{k+1} f}{\partial (Z^*)^{k+1}} = 0.
\]

These spaces (and others) will be the subject of a forthcoming paper.

As signaled in Section 3, the range of the first transform is strictly contained in $L^2_2(\mathbb{T})$. This is not the case for the second transform studied in Section 4. In fact, we obtain a Hilbertian orthogonal decomposition of $L^2_2(\mathbb{T})$,

\[
L^2_2(\mathbb{T}) := \bigoplus_{m,n=0}^{+\infty} \mathcal{M}^2_{m,n}(\mathbb{T}^c_{v,h}),
\]

in terms of the ranges $\mathcal{M}^2_{m,n}(\mathbb{T}^c_{v,h}) := \mathcal{S}^\sigma_{m,n}(L^2_2(\mathbb{R}^2))$ of $L^2_2(\mathbb{R}^2)$ by the transforms $\mathcal{S}^\sigma_{m,n} = V^\sigma_{\mathbb{R}^2, T}(\cdot, h^\sigma_{m,n})$ (this is contained in Corollary 4.5). It will be of interest to provide a concrete description of the functions in Corollary 4.5. This will be treated in some detail in a forthcoming paper from a different point of view.
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