Quantifying Quantumness and the Quest for Queens of Quantum

Olivier Giraud\textsuperscript{1,2,3,4}, Petr A. Braun\textsuperscript{5,6} and Daniel Braun\textsuperscript{1,2}

\textsuperscript{1} Université de Toulouse; UPS; Laboratoire de Physique Théorique (IRSAMC); F-31062 Toulouse, France
\textsuperscript{2} CNRS; LPT (IRSAMC); F-31062 Toulouse, France
\textsuperscript{3} Université Paris-Sud, LPTMS, UMR8626, Bât. 100, Université Paris-Sud, 91405 Orsay, France
\textsuperscript{4} CNRS, LPTMS, UMR8626, Bât. 100, Université Paris-Sud, 91405 Orsay, France
\textsuperscript{5} Fachbereich Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany
\textsuperscript{6} Institute of Physics, Saint-Petersburg University, 198504 Saint-Petersburg, Russia

Abstract. We introduce a measure of “quantumness” for any quantum state in a finite dimensional Hilbert space, based on the distance between the state and the convex set of classical states. The latter are defined as states that can be written as a convex sum of projectors onto coherent states. We derive general properties of this measure of non-classicality, and use it to identify for a given dimension of Hilbert space what are the “Queen of Quantum” states, i.e. the most non-classical quantum states. In three dimensions we obtain the Queen of Quantum state analytically and show that it is unique up to rotations. In up to 11-dimensional Hilbert spaces, we find the Queen of Quantum states numerically, and show that in terms of their Majorana representation they are highly symmetric bodies, which for dimensions 5 and 7 correspond to Platonic bodies.

PACS numbers: 02.40.Ft, 03.67.-a, 03.67.Mn
1. Introduction

The advent of quantum information theory has led to substantial efforts to understand the resources which are responsible for the enhanced information processing capabilities of quantum systems compared to classical ones. A large part of that research has been directed towards the creation and classification of entanglement [1, 2]. Entanglement plays an important role in quantum teleportation [3] and various quantum communication schemes [4]. It is also known that any pure state quantum computation which does not produce large scale entanglement can be simulated efficiently classically [5]. Physically, entanglement manifests itself as increased correlations between different subsystems compared to what is possible classically [6, 7]. But even for a system consisting of only a single subsystem one may ask how “quantum” a given state is, and what benefits one might draw from its “quantumness”.

In physics there is a wide consensus that the “least quantum” (or “most classical”) pure states are coherent states. These are states which present the smallest possible amount of quantum fluctuations, as defined by a suitable Heisenberg uncertainty relation, evenly distributed over a pair of non-commuting variables. For example, in quantum optics, coherent states have that property of minimal and equal uncertainty for the field quadratures. Moreover, the dynamics of the latter is identical to that given by the classical equations of motion of the harmonic oscillator, and the property of minimal uncertainty is conserved during the time evolution created by the Hamiltonian of the electromagnetic field. The most classical mixed states possible can be obtained as a statistical mixture of coherent states. Any mixed state can be expanded over projectors on coherent states, with real coefficients given by the so-called $P$-function. Consequently, in quantum optics, states with a positive $P$-function are considered as “most classical” [8, 9].

In [10] we extended that definition to systems with Hilbert spaces of finite dimension $d$. These systems are formally equivalent to a spin of size $j$ with $d = 2j + 1$. The $P$-function is not uniquely determined in that case, but leaves a lot of freedom in the specification of the higher spherical harmonics components. We therefore defined the set $C$ of classical states (also called the set of “P-rep” states in [10], for positive $P$-function) as the ensemble of all density matrices $\rho_c$ for which a decomposition in terms of angular momentum coherent states $|\hat{\theta}\phi\rangle$ (see (8) for a precise definition) with positive weights exists,

$$\rho_c = \sum_i \mu_i |\hat{\theta}_i\phi_i\rangle\langle\hat{\theta}_i\phi_i|$$  \hspace{1cm} (1)

with $0 \leq \mu_i \leq 1$ and $\sum_i \mu_i = 1$. At most $d^2 = (2j + 1)^2$ terms in the convex sum are needed [10]. We showed that for $j = 1/2$ all states are classical in that sense. For spin $j = 1$ we found a necessary and sufficient criterion for classicality, and for higher values of $j$ we found “non-classicality witnesses” which allow to easily detect large classes of non-classical states through the violation of necessary conditions for classicality derived from (1). For composite systems, the set of classical states $C$ is in general strictly smaller.
than the set of separable states [10].

The definition (1) of a classical state allows so far just to determine whether a state is classical or not. However, it would be interesting to know how “non-classical” (or, in other words, how “quantum”) a given state is, as one might expect that very non-classical states might be more useful for applications in quantum information processing than states which are only slightly non-classical. The situation is very analogous to the one encountered in the study of entanglement, where one wants to have an entanglement measure in addition to entanglement criteria.

This is the question we are going to pursue in this work. We introduce a measure of quantumness in the next section, study some of its properties, and then apply it to find the “most quantum” states possible for a given Hilbert space dimension. We show that the largest possible quantumness can always be found in a pure state. The states with maximal quantumness turn out to possess remarkable geometrical beauty. We term them “Queens of Quantum” (QQ) states. In the lowest-dimensional non-trivial case \( j = 1 \), i.e. \( d = 3 \) there is a unique Queen of Quantum (up to rotations of the coordinate system), which we determine analytically. In higher dimensions (up to \( j = 5 \)) we find the Queens of Quantum numerically using quadratic optimization. Other attempts to define the “least classical” quantum states were proposed in the literature based on properties of the average value and the variance of the (pseudo-)angular momentum operator \( J \) [11, 12, 13]. We will briefly discuss these results in relation to the QQ states.

2. Measure of Quantumness

2.1. Definition and properties

We define the “quantumness” \( Q(\rho) \) of an arbitrary state \( \rho \) as the distance from \( \rho \) to the convex set of classical states \( \mathcal{C} \). We thus introduce the measure of quantumness by defining

\[
Q(\rho) \equiv \min_{\rho_c \in \mathcal{C}} \|\rho - \rho_c\|,
\]

where the minimum is over all classical states defined in (1), and \( \|A\| \equiv \text{tr}(A^\dagger A)^{1/2} \) denotes the Hilbert-Schmidt norm. Note that our definition of quantumness is very analogous to the entanglement measure based on the distance of a state \( \rho \) from the convex set of separable states [14].

Several consequences follow immediately from (2):

1. For any state \( \rho \), and any dimension \( d \), we have the bounds \( 0 \leq Q(\rho) \leq \sqrt{\text{tr}\rho^2} + \sqrt{\text{tr}\rho_c^2} \leq 2 \). The lower bound is trivially realized for classical states. This implies \( Q(\rho) = 0 \) for all states of a spin 1/2.

2. An improved upper bound on \( Q(\rho) \) that only depends on the purity of \( \rho \) can be found by considering the distance to the maximally mixed state \( 1/(2j+1) \), which is
always classical [10]. This immediately leads to

\[ Q(\rho) \leq \sqrt{\text{tr}\rho^2 - \frac{1}{2j+1}}. \] (3)

For a pure state this bound coincides with the less stringent

\[ Q(\rho) \leq \sqrt{1 - \frac{1}{2j+1}}. \] (4)

3. A different upper bound can be found by minimizing over a single pure coherent state \(|\alpha\rangle\) = |\theta\phi\rangle:

\[ Q(\rho) \leq \min_{\alpha} \||\rho - |\alpha\rangle\langle\alpha| || \leq (1 + \text{tr}\rho^2 - 2\max_{\alpha} H_\rho(\alpha))^{1/2}, \] (5)

where \(H_\rho(\alpha) \equiv \langle\alpha|\rho|\alpha\rangle\) denotes the Husimi function of the state. For a pure state the bound becomes

\[ Q(|\psi\rangle\langle\psi|) \leq \sqrt{2(1 - \max_{\alpha} |\langle\alpha|\psi\rangle|^2)^{1/2}}. \]

4. As the distance to a convex set is a convex function (see e.g. example 3.16 of [15]), \(Q(\rho)\) is a convex function, i.e. for any two states \(\rho_1, \rho_2\), and \(0 \leq p \leq 1\), we have

\[ Q(p\rho_1 + (1-p)\rho_2) \leq pQ(\rho_1) + (1-p)Q(\rho_2). \] (6)

This implies that the quantumness of any mixed state cannot be larger than the largest quantumness of the pure states of which it is a mixture.

5. \(Q(\rho)\) is invariant under rotations of the coordinate system. Indeed, let

\[ R_n = \exp(i\mathbf{n}.\mathbf{J}) \]

be a unitary operator associated with a rotation of the coordinate system about the axis \(\mathbf{n}\) by an angle \(|\mathbf{n}|\). Since \(R_n\) is unitary, we have \(\||\rho - \rho_c|| = ||R_n\rho R_n^\dagger - R_n\rho_c R_n^\dagger||\) for all density matrices \(\rho, \rho_c\). Furthermore, for \(\rho_c \in \mathcal{C}\), we also have \(\tilde{\rho}_c \equiv R_n\rho R_n^\dagger \in \mathcal{C}\), as \(R_n\) only rotates coherent states into other coherent states, and therefore does not change the classicality of \(\rho_c\). Moreover, the map \(\rho_c \to \tilde{\rho}_c\) for given \(R_n\) is an isomorphism \(\mathcal{C} \to \mathcal{C}\). Therefore, we have \(\min_{\rho_c \in \mathcal{C}} \min_{\tilde{\rho}_c \in \mathcal{C}} ||\rho - \rho_c|| = ||\tilde{\rho}_c - \rho_c|| = \tilde{Q}(\tilde{\rho})\) for \(\tilde{\rho} = R_n\rho R_n^\dagger\).

With the same argument one shows that for a composite system consisting of \(s\) subsystems, \(Q(\rho)\) is invariant under independent rotations for all the subsystems, i.e. under transformations \(R = R_{n_1} \otimes R_{n_2} \otimes \ldots R_{n_s}\). In addition, for a system consisting of \(s\) qubits, \(Q(\rho)\) is invariant under all local unitary transformations, as in this case local unitary \(SU(2)\) transformations leave the set \(\mathcal{C}\) invariant. Our measure of classicality shares this property with any measure of entanglement. Indeed, for a multi spin-\(1/2\) system, the set of classical states \(\mathcal{C}\) is identical to the set of totally separable states. Therefore, in this case \(Q(\rho)\) coincides with an entanglement measure. For higher-dimensional subsystems, this is, of course, not true, as for example a \(SU(3)\) transformation of a single spin 1 can transform a coherent state into a non-classical state.
2.2. Simple examples

2.2.1. Spin-1/2 case  As mentioned in [10], any spin-1/2 pure state is classical, thus the set C coincides with the set of all quantum states. All states are thus trivially at distance 0 from C.

2.2.2. Pure spin-1 case  In order to illustrate the behaviour of the quantumness in the simplest non-trivial case, let us consider the family of pure spin-1 states given by

\[ |\psi_x\rangle = \frac{1}{\sqrt{x^2 + 2}} (|1, -1\rangle + x|1, 0\rangle + |1, 1\rangle), \tag{7} \]

where \(|jm\rangle, -j \leq m \leq j\), are eigenvectors of the spin angular momentum operator \(J_z\) with eigenvalue \(m\). Let \(x = \sqrt{2}/\sin \bar{\theta}\). In figure 1 we plot the quantumness \(Q(|\psi_x\rangle\langle \psi_x|)\) obtained numerically using the method described in section 3.4. We see that the largest quantumness is obtained at \(\bar{\theta} = 0\) (or \(x = \infty\)), corresponding to the state \(|1, 0\rangle\). We will prove in section 3.2 that this state is indeed the spin-1 Queen of Quantum state and has \(Q = \sqrt{3}/8\).

2.2.3. Thermal spin states  Consider a density matrix given by the thermal state \(\rho = \exp(-\beta H)/\text{tr}(\exp(-\beta H))\) with inverse temperature \(\beta = 1/k_B T\) and Hamiltonian \(H\). For \(\beta \to 0\) all energy eigenstates are equally populated, so that \(\rho\) is the (properly normalized) identity matrix \(\rho_0\) corresponding to the maximally mixed state. Since \(Q(\rho_0) = 0\), we have the intuitively appealing result that for sufficiently large temperature all thermal states become classical. Indeed, there is a finite temperature where this happens, as there is a finite neighbourhood around \(\rho_0\) (with a finite radius in all
Quantifying Quantumness and the Quest for Queens of Quantum

(6) where all states are classical [10]. For $\beta \rightarrow \infty$ the quantumness depends on the quantumness of the ground state(s) of $H$.

We will illustrate what can happen with two examples for spin 1. In this case we have at our disposal a necessary and sufficient condition of classicality, which has been obtained in [10] (see Sect. 3.2 for a presentation of this criterion).

1. For $H = J_z$ the ground state is the pure coherent state $|1, -1\rangle$ and thus it is classical. Our classicality criterion shows that in this case all thermal states are actually classical.

2. For $H = J_z^2$, the ground state is the non-classical state $|1, 0\rangle$, and the classicality criterion shows that there is a critical temperature $T_c = 1/\beta_c = 1/\ln 2$ above which the quantumness disappears.

Therefore, while the quantumness at low temperature depends on the model and in particular on the quantumness of low-lying states, classicality is a universal property for systems at thermal equilibrium for $T \rightarrow \infty$.

3. Queens of Quantum

We now address the question of finding out which states are the “most quantum” states. This is a highly non-trivial question, as it requires to find a state that maximizes a quantity defined as a minimum over a convex set. We first introduce some definitions.

3.1. Definitions and general properties

We define the “Queen of Quantum” (QQ) states as those states $\rho_{QQ} \in \mathcal{N}$ for which $\rho_{QQ} = \max_{\rho \in \mathcal{N}} Q(\rho)$, where $\mathcal{N}$ is the set of all physical density operators acting on a Hilbert space of given finite dimension. In other words, the Queens of Quantum are the “most quantum” states for a given Hilbert space dimension.

3.1.1. Pure versus mixed

In order to identify the largest quantumness possible for a given Hilbert space dimension, we can restrict ourselves to pure states, as according to (6) the quantumness of a state $\rho$ cannot be larger than the quantumness of the pure states from which it is a mixture. This does not immediately imply, however, that all QQ states are pure. Indeed, suppose that two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ have the same quantumness, $Q(|\psi_1\rangle\langle\psi_1|) = Q(|\psi_2\rangle\langle\psi_2|)$. Then, according to (6), $Q(p|\psi_1\rangle\langle\psi_1| + (1 - p)|\psi_2\rangle\langle\psi_2|) \leq Q(|\psi_1\rangle\langle\psi_1|)$ for $0 \leq p \leq 1$. Equality is possible in principle, unless $Q(\rho)$ is strictly convex, which, however, need not be the case. For example one might imagine that $\mathcal{N}$ has a flat surface containing $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$, in parallel to a flat surface of $\mathcal{C}$ containing the two closest mixed states for $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$. In this case, all states mixed from the two pure QQ states will have the same
maximum quantumness. The problem of degenerate quantumness of the QQ states is in fact generic, as we have seen that all states obtained by rotation of the coordinate system have the same quantumness. Nevertheless, we can start off by determining all pure QQ states, and then try to determine whether states can be mixed from these which would give the same maximum quantumness. For \( j = 1 \) we will show explicitly that all QQ states are pure.

3.1.2. Majorana representation

We now restrict ourselves to the case where \( \rho = |\psi \rangle \langle \psi | \) is a pure state. A spin-\( j \) coherent state can be expressed as [16]

\[
|\vec{\theta} \varphi \rangle = \sum_{m=-j}^{j} \left(\frac{2j}{j+m}\right)^{\frac{1}{2}} \left(\sin \frac{\vec{\theta}}{2}\right)^{j+m} \left(\cos \frac{\vec{\theta}}{2}\right)^{j-m} e^{-i(j+m)\varphi} |jm\rangle.
\]

(8)

Note that with this definition the state \( |j - j\rangle \) corresponds to \( \vec{\theta} = 0 \). We take this state as the South pole, and \( |jj\rangle \) as the North pole. The state \( |\vec{\theta} \varphi \rangle \) can be seen as the state obtained by rotating \( |j - j\rangle \) in a direction specified on the sphere \( S^2 \) by an angle \( \vec{\theta} \) about the \( y \)-axis followed by an angle \( \varphi \) about the \( z \)-axis. This means that the mean value \( \langle \vec{\theta} \varphi |J|\vec{\theta} \varphi \rangle \) of the angular momentum vector points in the direction given by the usual (i.e. counted from the North pole) polar angle \( \theta = \pi - \vec{\theta} \) and azimuth \( \varphi \). We draw the attention of the reader to the fact that in the whole paper \( \vec{\theta} \) will be counted from the South pole. This perhaps unusual convention is adopted here in order to simplify subsequent expressions for the Majorana polynomials.

Any pure state \( |\psi \rangle \) can be represented by its overlap with coherent states

\[
\langle \vec{\theta} \varphi |\psi \rangle = \frac{1}{(1 + |\zeta|^2)^{j}} \sum_{m=-j}^{j} \left(\frac{2j}{j+m}\right)^{\frac{1}{2}} \psi_{m} \zeta^{j+m},
\]

(9)

where \( \zeta \) is a complex number defined by

\[
\zeta = e^{i\varphi} \tan \frac{\vec{\theta}}{2}.
\]

(10)

The scalar product (9), up to a prefactor independent of \( |\psi \rangle \), is a polynomial of degree at most \( 2j \) in the complex variable \( \zeta \). If \( \psi_{j} \neq 0 \), it is a polynomial of degree \( 2j \), and has \( 2j \) complex roots \( \zeta_{1}, \ldots, \zeta_{2j} \). This so-called Majorana polynomial [17, 18] reads

\[
M(\zeta) = \sum_{k=0}^{2j} \binom{2j}{k} \psi_{k-j} \zeta^{k} = \psi_{j} \prod_{k=1}^{2j}(\zeta - \zeta_{k}).
\]

(11)

The inverse stereographic projection of these \( 2j \) roots define \( 2j \) points on the unit sphere \( S^2 \) through (10). If the degree of \( M(\zeta) \) is \( D < 2j \), the Majorana representation is defined as consisting of \( D \) points associated with the \( D \) roots of the polynomial and \( 2j-D \) points at the North pole, and the prefactor \( \psi_{j} \) in (11) is replaced by \( \psi_{D-j} \). This set of points is called the Majorana (or stellar) representation. It entirely characterizes the normalized state \( |\psi \rangle \) up to a global phase. A nice feature of this representation is that all points rotate rigidly if \( |\psi \rangle \) undergoes a rotation \( R_{n} = \exp(i n J) \) of the reference frame. If the state \( |\psi \rangle \) is a coherent state, \( |\psi \rangle = |\vec{\theta} \varphi \rangle \), then the \( 2j \) roots of its Majorana polynomial...
are all equal and given by $\zeta_k = \tan \frac{\pi - \bar{\theta}}{2} e^{i(\varphi + \pi)}$. They correspond to $2j$ points which are antipodal to the vector $\langle \bar{\theta} \phi | J | \bar{\theta} \phi \rangle$. For instance, the state $|j, j\rangle$ (corresponding to $\bar{\theta} = \pi$), which is a coherent state whose mean value is located at the North pole, is represented by $2j$ points at the South pole (once more we recall that we count $\bar{\theta}$ from the South pole).

There is a natural interpretation of the Majorana representation in terms of tensor products of spin states. As a spin-$j$ state, $|\psi\rangle$ can be seen as a fully symmetrized direct product of $2j$ spins $\frac{1}{2}$. Each spin-$\frac{1}{2}$ state is a coherent state of the form (8). As such it can be represented by a Majorana point on the Bloch sphere in the direction antipodal to the coherent state. The state $|\psi\rangle$ corresponds to a symmetrization of $2j$ Bloch spheres and thus can be represented by a set of $2j$ points on a single sphere. The Majorana representation is useful in many contexts: calculation of the Berry phase for pure states [19], proof of Sylvester’s theorem on Maxwell multipoles [20], investigation of states which maximize the variance $(\Delta J)^2 = \langle J, J \rangle - \langle J \rangle^2$ [11], or states such that this variance is uniform over the unit sphere [12].

Since the set $C$ of classical states is invariant under rotation (as coherent states are just rotated to coherent states) the distance from $|\psi\rangle$ to $C$ is the same for any rigid rotation of the $2j$ points on the sphere. The problem of identifying pure QQ states for spin $j$ reduces to identifying optimal distributions of $2j$ points on the unit sphere, up to global rotation.

3.1.3. Pure state as eigenfunction of its classical neighbour  
Before proceeding to the identification of the most quantum states, we derive an important property of QQ states. As we pointed out, the largest $Q(\rho)$ can always be reached with a pure state, $\rho = |\psi\rangle \langle \psi|$. The squared distance from a pure state to $C$ is maximized by the state $\psi_{QQ}$ such that

$$Q^2 (|\psi_{QQ}\rangle \langle \psi_{QQ}|) = \max \min_{|\psi\rangle \in C} \left( 1 - 2 \langle \psi | \rho_c | \psi \rangle + \text{tr} \rho_c^2 \right).$$  

(12)

The necessary condition on state $\psi_{QQ}$ is stationarity of $Q^2 (|\psi\rangle \langle \psi|)$ with respect to variations of $|\psi\rangle$ constrained by the condition $\langle \psi | \psi \rangle = 1$. The first variation of the functional $I[\psi] = 1 - 2 \langle \psi | \rho_c | \psi \rangle + 2E \langle \psi | \psi \rangle$, equal to

$$\delta I = -2 \langle \delta \psi | \rho_c | \psi \rangle + 2E \langle \delta \psi | \psi \rangle + \text{c.c.},$$  

(13)

where $2E$ is the Lagrange multiplier, must be zero for all $|\delta \psi\rangle$. It follows that

$$\rho_c | \psi \rangle = E | \psi \rangle.$$  

(14)

Consequently, the wavefunction of a QQ state is an eigenfunction of (the density matrix of) its nearest classical state. We checked that all QQ states obtained in the following sections indeed do satisfy this property.

3.2. Spin-1 case

We now turn to the analytic investigation of the simplest nontrivial case of spin 1.
3.2.1. Pure states

Let $J_a$, $a = x, y, z$, be the $3 \times 3$ angular momentum matrices for $j = 1$. One can expand any density matrix $\rho$ in the basis of $J_a$ and $(J_a J_b + J_b J_a)/2$, as

$$\rho = \frac{1}{3} 1_3 + \frac{1}{2} u, J + \frac{1}{2} \sum_{a,b=x,y,z} \left( W_{ab} - \frac{1}{3} \delta_{ab} \right) \frac{J_a J_b + J_b J_a}{2}. \quad (15)$$

Here $u$ is a real vector, $W$ a real symmetric matrix with trace 1, and $1_3$ the $3 \times 3$ identity matrix. They are related to $\rho$ by

$$u_a = \text{tr} (\rho J_a), \quad W_{ab} = \text{tr} [\rho (J_a J_b + J_b J_a)] - \delta_{ab}. \quad (16)$$

According to [10], $\rho$ is classical if and only if the real symmetric $3 \times 3$ matrix $Z$ with matrix elements

$$Z_{ab} = W_{ab} - u_a u_b \quad (17)$$

is non-negative.

Let $|\psi\rangle$ be a fixed pure spin-1 state. Its Majorana representation consists of two points on the sphere. Since any state obtained by a global rotation of these two points is at the same distance from $C$, one can without loss of generality consider that these two points are specified by angles $(\bar{\theta}, 0)$ and $(\pi - \bar{\theta}, 0)$. In the $|jm\rangle$ basis, the corresponding state is given by (7), with $\sin \bar{\theta} = \sqrt{2}/x$, since the Majorana polynomial of state $|\psi_x\rangle$ is given by $M(\zeta) = (1 + \sqrt{2}x\zeta + \zeta^2)/\sqrt{x^2 + 2}$. When $\bar{\theta}$ varies from 0 to $\pi/2$, $x$ takes values between $\sqrt{2}$ and $\infty$. Any spin-1 state can thus be brought to the canonical form (7) with $x \in [\sqrt{2}, \infty]$.

Lemma. The state defined by two antipodal points on the Majorana sphere is at distance $\sqrt{3}/8$ from $C$.

Proof. Without loss of generality we take the two points to be at the North and South poles. The corresponding state is $|1, 0\rangle$. Its parameters in the expansion (15) are $u = 0$ and $W = \text{diag}(1, 1, -1)$. A coherent state has parameters $u = n$ and $W_{ab} = n_a n_b$, where $n$ is a three-dimensional unit vector which we parametrize as

$$n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^T. \quad (18)$$

Since $C$ is the convex hull of the set of coherent states, any element $\rho_c(u, W)$ of $C$ can be written as

$$u = \sum_{i=1}^{N} \lambda_i n_i, \quad (19)$$

$$W_{ab} = \sum_{i=1}^{N} \lambda_i (n_i)_a (n_i)_b, \quad (20)$$

where $(n_i)_a, a = x, y, z$, are the components of the vector $n_i$ and $N$ is an integer. The distance $||\rho - \rho_c||^2$ with $\rho = |1, 0\rangle \langle 1, 0|$ is

$$\frac{1}{2} \sum_{i,j} \lambda_i \lambda_j n_i \cdot n_j + \frac{1}{4} \sum_{i,j} \lambda_i \lambda_j |n_i | n_j |^2 - \frac{1}{2} \sum_i \lambda_i \sum_{a,b} W_{ab} (n_i)_a (n_i)_b + \frac{3}{4}. \quad (21)$$
Quantifying Quantumness and the Quest for Queens of Quantum

Using $\sum_i \lambda_i = 1$ and symmetrizing the last term but one in (21) we get

$$||\rho - \rho_c||^2 = \frac{1}{4} \sum_{i,j} \lambda_i \lambda_j (2n_i.n_j + |n_i.n_j|^2)
- \sum_{a,b} W_{ab}(n_i)_a(n_i)_b - \sum_{a,b} W_{ab}(n_j)_a(n_j)_b + 3).$$

We parametrize vectors $n_i$ by angles $\theta_i$ and $\varphi_i$ as in (18). After some trigonometric simplifications we obtain

$$||\rho - \rho_c||^2 = \frac{3}{8} + \frac{1}{4} \sum_{i,j} \lambda_i \lambda_j \left( \frac{1}{2} \sin^2 \theta_i \sin^2 \theta_j \cos 2(\varphi_i - \varphi_j) + 2 \sin \theta_i \sin \varphi_j \right) (1 + \cos \theta_i \cos \theta_j) + \frac{3}{2} \cos \theta_i \cos \theta_j + \frac{3}{4}(2 - \sin \theta_i - \sin \theta_j) + \frac{3}{2} \cos^2 \theta_i \cos^2 \theta_j \right).$$

All terms in (24) are positive or can be expanded as a sum of terms of the form $(\sum_i \lambda_i f(\theta_i))^2$. Thus $||\rho - \rho_c||^2 \geq 3/8$ for all states $\rho_c$. This minimum is reached if all terms in (24) vanish, which implies the conditions $\theta_i = \pi/2$ and

$$\sum_i \lambda_i \cos \varphi_i = 0, \quad \sum_i \lambda_i \cos 2\varphi_i = 0 \quad (24)$$
$$\sum_i \lambda_i \sin \varphi_i = 0, \quad \sum_i \lambda_i \sin 2\varphi_i = 0. \quad (25)$$

These equations admit the solution $\lambda_i = 1/N$ and $\varphi_i = 2\pi i/N$ for $N \geq 3$, which corresponds to state

$$\rho_c = \begin{pmatrix}
\frac{1}{4} & 0 & 0 \\
0 & \frac{1}{2} \\
0 & \frac{1}{4}
\end{pmatrix}. \quad (26)$$

Since $\rho_c \in C$, the minimum $3/8$ is indeed reached. Thus the distance between $|1,0\rangle\langle 1,0|$ and $C$ is $\sqrt{3/8}$. □

**Theorem.** The state $|1,0\rangle$ is the unique pure QQ state up to rotations. Its Majorana representation is given by a pair of antipodal points.

**Proof.** Since $|1,0\rangle$ is at distance $\sqrt{3/8}$ from $C$, it suffices to show that any other pure state is at distance smaller than $\sqrt{3/8}$, for instance by explicitly exhibiting a classical state which is closer. We distinguish two cases. For $x \geq \sqrt{6}$ one can show, using the $Z$-criterion (17), that the state

$$\rho_c(x) = \begin{pmatrix}
\frac{1}{4} & a(x) & b(x) \\
\frac{a(x)}{2} & \frac{1}{2} & a(x) \\
\frac{b(x)}{2} & \frac{a(x)}{2} & \frac{1}{4}
\end{pmatrix} \quad (27)$$

with $a(x) = x/(x^2 + 2)$ and $b(x) = 1/(x^2 + 2)$ is classical. The distance

$$|||\psi_x\rangle\langle \psi_x| - \rho_c(x)||^2 = \frac{3}{8} \frac{(x^2 - 2)^2}{x^2 + 2} \quad (28)$$
is a strictly increasing function of \( x \) on \([\sqrt{6}, \infty[\), and thus any state \( |\psi_x\rangle \) with \( \sqrt{6} \leq x < \infty \) is at distance squared less than \( 3/8 \).

For \( x \leq \sqrt{6} \), one can similarly show that the state given by (27) with \( a(x) = x/(x^2 + 2) \) and \( b(x) = 4(x/(x^2 + 2))^2 - 1/4 \) is classical, and that its squared distance to \( |\psi_x\rangle \),

\[
|\langle \psi_x | \psi_x \rangle - \rho_c(x) |^2 = \frac{(x^2 - 2)^2(x^4 + 12)}{2(x^2 + 2)^4},
\]  

(29)

is a strictly increasing function of \( x \) on \([\sqrt{2}, \sqrt{6}]\), thus bounded by its value at \( \sqrt{6} \), which is \( 3/32 \). Thus, the state with \( x = \infty \) and the states obtained from it by rotations are the only pure states at distance \( \sqrt{3/8} \), all other being closer, which completes the proof. \( \Box \)

Numerical evidence shows that the distance given by (28) is precisely the distance between \( |\psi\rangle \) and \( C \) for all \( x \) (see figure 1).

3.2.2. Mixed states
The above Theorem shows that only states of the form \( R|1,0\rangle \), where \( R \) is a rotation of the coordinate frame, are pure QQ states for spin 1. Since pure QQ states are at a distance \( Q^2 = 3/8 \) from \( C \), any mixed QQ state has to be at least at the same distance. But, as explained in point 4. of section 2.1, convexity implies that mixed states can never be further away from \( C \) than the pure states they are composed of. Therefore, only mixtures \( \rho \) of pure QQ states that verify \( Q(\rho)^2 = 3/8 \) are candidates for mixed QQ states.

Let \( \rho \) be a mixed QQ state. Its most general form of \( \rho \) is

\[
\rho = \sum_i \mu_i R_i |1,0\rangle \langle 1,0| R_i^\dagger,
\]  

(30)

where \( R_i \) represents an arbitrary rotation of the coordinate frame, and the \( \mu_i \) are positive and sum up to 1. The state

\[
\rho_c = \sum_i \mu_i R_i \begin{pmatrix}
\frac{1}{4} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{4}
\end{pmatrix} R_i^\dagger
\]  

(31)

belongs to \( C \) and, by convexity of the norm, satisfies \( ||\rho - \rho_c|| \leq \sqrt{3/8} \). Since \( Q(\rho) = \sqrt{3/8} \), \( \rho_c \) is indeed the classical state closest to \( \rho \), which implies that \( \rho_c \) has to be on the boundary of \( C \). From (16), coordinates \( u \) and \( W \) are linear in \( \rho \). Moreover, they transform respectively as a vector and a 2-tensor under rotations (see e.g. [21]). Using the fact that coordinates for state \( \text{diag}(1/4, 1/2, 1/4) \) read \( u = 0 \) and \( W = \text{diag}(1/2, 1/2, 0) \), for \( \rho_c \) we have \( u(\rho_c) = 0 \) and

\[
W(\rho_c) = \sum_i \mu_i R_i \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 0
\end{pmatrix} R_i^\dagger.
\]  

(32)

The set \( C \) is the set of density matrices such that \( Z \geq 0 \). Since \( \rho_c \) is on the boundary of \( C \) its matrix \( Z \), which is equal to \( W(\rho_c) \), has a vanishing eigenvalue. Thus there exists
some vector \( \mathbf{n} \) such that \( \sum_{a,b} W_{ab} n_a n_b = 0 \). Using (32) one easily concludes that for all rotations, \( R_i \mathbf{n} \) is equal to \((0,0,1)^T\). Thus either all rotations are equal or they have the same rotation axis \((0,0,1)^T\). In the latter case the state \(|1,0\rangle\) is invariant under \( R_i \), and in both cases one concludes from (30) that \( \rho \) is a pure state.

Thus \(|1,0\rangle\) is the unique spin-1 QQ state up to rotation.

### 3.3. Higher values of \( j \)

Again we concentrate on pure states. The problem reduces to finding the maximum over all pure states \(|\psi\rangle\) of the minimum over \( \rho_c \in \mathcal{C} \) of

\[
\text{tr} \left( |\psi\rangle \langle \psi| - \rho_c \right)^2 = 1 - 2 \sum_i \lambda_i |\langle \psi| \alpha_i \rangle|^2 + \sum_{i,k} \lambda_i \lambda_k |\langle \alpha_i | \alpha_k \rangle|^2 ,
\]

where \(|\alpha_i\rangle = |\tilde{\theta}_i \varphi_i \rangle\) are coherent states. The last term in (33) involves the overlap between coherent states

\[
|\langle \alpha | \alpha' \rangle|^2 = \cos^{4j} \frac{\gamma(\alpha, \alpha')}{2},
\]

where \( \gamma(\alpha, \alpha') \) is the angle between the two points corresponding to \( \langle \alpha | J | \alpha \rangle \) and \( \langle \alpha' | J | \alpha' \rangle \). The other sum in (33) involves the overlap between \(|\psi\rangle\) and the coherent states

\[
|\langle \psi | \alpha \rangle|^2 = \frac{|\psi_j|^2}{\prod_{i=1}^{2j} \cos^2 \frac{\theta_i}{2}} \prod_{i=1}^{2j} \sin^2 \frac{\gamma(\alpha, \zeta_i)}{2},
\]

where \( \zeta_i \) are the Majorana points corresponding to state \(|\psi\rangle\) (for simplicity of notation, we assume that \( \psi_j \neq 0 \)) and \( \gamma(\alpha, \zeta_i) \) is the angle between the point \( \langle \alpha | J | \alpha \rangle \) and the Majorana point \( \zeta_i \). Since \( \zeta_i \) are the roots of the Majorana polynomial, whose coefficients depend on the components \( \psi_i \) of \(|\psi\rangle\), it is possible to show, using coefficient-root relations and normalization of \(|\psi\rangle\), that

\[
\frac{|\psi_j|^2}{\prod_{i=1}^{2j} \cos^2 \frac{\theta_i}{2}} = \prod_{i=1}^{2j} \frac{(1 + |\zeta_i|^2)}{\sum_{k=0}^{2j} |\sigma_k|^2 / \binom{2j}{k}}
\]

with \( \sigma_k \) the \( k \)th symmetric polynomial of the \( \zeta_i \) \( (\sigma_0 = 1, \sigma_1 = \sum \zeta_i, \sigma_2 = \sum_{i<j} \zeta_i \zeta_j, \ldots) \).

For the lowest values of \( j \) one can express this quantity as a function of the angles \( \gamma_{ik} \) between points \( \zeta_i \) and \( \zeta_k \). It is equal to 1 for \( j = 1/2 \). For \( j = 1, 3/2 \), we have

\[
\frac{|\psi_j|^2}{\prod_{i=1}^{2j} \cos^2 \frac{\theta_i}{2}} = 1 - \frac{1}{2j} \sum_{1 \leq i < k \leq 2j} \sin^2 \frac{\gamma_{ik}}{2}.
\]

For \( j = 2, 5/2 \),

\[
\frac{|\psi_j|^2}{\prod_{i=1}^{2j} \cos^2 \frac{\theta_i}{2}} = 1 - \frac{1}{2j} \sum_{1 \leq i < k \leq 2j} \sin^2 \frac{\gamma_{ik}}{2} + \frac{1}{2j(2j - 1)} \sum_{\text{all pairwise}} \sin^2 \frac{\gamma_{ik}}{2} \sin^2 \frac{\gamma_{ij}}{2}.
\]
with the last sum running over all ways of pairing $2j$ points into two distinct pairs. These formulae should easily generalize to general $j$. The whole expression (33) can thus be expressed as a function of terms of the form $\sin(\gamma/2)$, which are equal to half the Euclidean distance between a pair of points separated by an angular distance $\gamma$. For instance in the case $j = 1$ the problem corresponds to finding

$$
\max_{\zeta_i} \min_{\lambda_i, \alpha_i} \left( 1 - 2 \sum \lambda_i \sin^2 \frac{\gamma(\alpha_i, \zeta_1)}{2} \sin^2 \frac{\gamma(\alpha_i, \zeta_2)}{2} + \sum \lambda_i \lambda_k \cos^{4j} \frac{\gamma(\alpha_i, \alpha_k)}{2} \right),
$$

(39)

Our quest for pure QQ states thus amounts to finding an optimal arrangement of points on the sphere with two types of particles $\zeta_i$ and $\alpha_i$. The problem of arranging points on a sphere as evenly as possible has a long history. It was known by the ancient Greeks that 4, 6, 8, 12 or 20 points could be arranged in a regular way. About 25 centuries later, by classifying all finite subgroups of the group of rotations in $\mathbb{R}^3$, it was proved that only five regular polyhedra exist. In the framework of electrostatics, one can define a generalized Coulomb potential between $n$ point charges on the sphere as

$$
\frac{1}{\sum \frac{1}{d_{ij}^m}}
$$

(40)

where $d_{ij}$ is the distance between points $i$ and $j$, and $m$ a positive integer. The question of finding a configuration of points on the sphere that minimizes the potential (40) was first investigated by Thomson [22]. Similar questions appear in many fields, from crystallography to biology (see [23] and references therein). Our problem bears some similarity with such questions. However, our potential is more complicated (see e.g. (39)) and two kinds of “particles” are involved. Intuitively, for fixed $\zeta_i$, the minimization problem in (39) corresponds to finding coherent states $\alpha_i$ as remote as possible from the $\zeta_i$ and from each other.

Given the complexity of the minimax problem of the kind of (39) beyond the case $j = 1$, we choose a numerical approach. Many algorithms were devised to numerically obtain optimal configurations of points. Rather surprisingly it turns out that the optimal distribution does not necessarily coincide with regular polyhedra even in the case where these exist (see e.g. [24], where the distribution of point charges that minimizes Coulomb potential (40) is given up to 60 points). In the next subsections we will apply numerical techniques to identify QQ states for the smallest Hilbert dimensions.

3.4. Numerical procedure

The problem of finding the QQ states can be reformulated in terms of convex optimization, and even as an instance of quadratic programming.

3.4.1. Quadratic programming For a fixed state $\rho \in \mathcal{N}$, we represent the matrix $\rho_c \in \mathcal{C}$ minimizing the distance to $\mathcal{C}$ as a linear combination of coherent states whose directions densely and uniformly cover the unit sphere, $\rho_c = \sum_{i=1}^N \lambda_i |\alpha_i\rangle \langle \alpha_i|$ with $N$ large. $Q^2(\rho)$ then becomes a quadratic function of the coefficients $\lambda_i$ which has to be minimized
under the constraints $\lambda_i \geq 0$, $\sum_{i=1}^{N} \lambda_i = 1$. This is a problem of quadratic programming which can be solved by a variety of algorithms. Although the original linear combination contains several thousand coherent states, only a few of them enter the solution with coefficients significantly different from zero. In the search of QQ states, the result of the quadratic minimization was then numerically maximized by variation of the pure state $\rho$. The Majorana configurations were found numerically and then deformed to closest symmetrical figures under the condition that $Q$ increased. It is probably superfluous to stress that maximization and minimization do not commute, so that the maximin and minimax of the squared distance $||\rho - \rho_c||^2$ are different.

The optimization algorithm itself starts from $N$ coherent states randomly distributed on a sphere, on which quadratic programming is performed. This yields an intermediate optimal state which is a combination of a relatively small number $M \simeq 5 - 25$, $M \ll N$, of coherent states. If some of these coherent states point in directions closer than a certain threshold, say, 2 degrees, they are replaced by a single coherent state with a cumulative weight directed along the weighted average direction. This step yields $M'$ coherent states. Then $N - M'$ new random coherent states are generated and the quadratic programming algorithm is run again starting from these $N$ coherent states ($N - M'$ new and $M'$ old). Iterating the process $K$ times (typically $K 1000 - 5000$) with $N 100$ yielded a 7-8 digit precision of the squared distance value.

As expected from the remark of section 3.1.3, the numerically found wavefunction of a QQ state for given $j$ coincides, within numerical accuracy, with an eigenvector of the density matrix of its nearest classical neighbours. In fact, the accuracy with which this property was fulfilled could serve as a measure of accuracy in the search of the maximin. An interesting point is that all QQ states that we obtained were invariably associated with the eigenvector of the density matrix of the classical state corresponding to its largest eigenvalue.

3.5. Results

We carried out a numerical search of the QQ states for $j$ from $1/2$ to 5. The resulting values of the distance are plotted in figure 2 together with the upper bound (4). The distance of the QQ state from $C$ grows monotonically with the increase of $j$; an almost perfect fit is

$$Q_{QQ}^2 \approx 1 - 2/(2j + 1). \quad (41)$$

The resulting arrangements of the Majorana points on the unit sphere that represent the QQ states, as well as the respective sets of coherent states constituting the nearest $\rho_c$ are given in Table 1. The numerically obtained values for the Majorana points and coherent states are listed in the Appendix (Tables A1 and A2,A3). For the first few values of $j$ it is possible to identify regular structures from these numerical results. They can be recognized as highly symmetric figures (see figures 3 and 4) in which $\theta$ and $\varphi$ are typically rational multiples of $\pi$. Table 1 gives the analytical expression that we
obtain for the QQ states in the $|jm\rangle$ basis if we identify the numerical results with these regular structures. We find in particular that for $j = 2$ and $j = 3$ (i.e. 4 and 6 Majorana points), the QQ states correspond to the Platonic bodies (tetrahedron and octahedron, respectively). For $j = 4$, where one would expect to find the next Platonic body, the cube, the symmetry of the Majorana configuration is lower (see figure 3).

The number of local maxima of $Q^2$ with close values rapidly increases with the growth of $j$ while the maxima themselves tend to become more and more shallow. This makes the search of the optimal configuration more and more difficult while the value of the maximin itself can still be reliably established.

It is instructive to compare our results with the optimal distribution of identical point charges on a unit sphere which interact through the standard Coulomb repulsion, plotted e.g. in [22]. The analogy between the problems follows from the possibility to express the optimized quantity in terms of distances between the end points of the Majorana vectors and the vectors of the coherent states (see section 3.3). In the range $j = 1$ – 5 the symmetry of the Majorana configurations of the QQ states coincides with that of the equilibrium configurations of $2j$ charges on a sphere, the only exception being $j = 4$. Indeed, the optimal Coulomb configuration of 8 identical point charges is the square antiprism with symmetry $D_{4d}$. However that latter configuration gives only a local maximum of $Q^2$ equal to 0.76868 which is slightly less than the global maximum 0.77108 realized in a configuration with lower symmetry $D_{2d}$ (Table 1).

It is also interesting to compare our results to the anticoherent spin states introduced in [12]. These states are defined such that $\langle J \rangle = 0$ and $(\Delta J_n)^2 = \langle n.Jn.J \rangle - \langle n.J \rangle \langle n.J \rangle$ is uniform over the unit sphere, i.e. independent of $n$. Platonic states for $j = 2, 3, 4, 6$ and 10 are shown to be anticoherent. In [13] it was shown that multiqubit states with diagonal spin covariance matrix and maximal variances of each
spin component are optimal for reference frame alignment. This property is verified for our QQ states for $j = 2, 3$. QQ states with $j = 2, 3$ are therefore both “anticoherent” and optimal for reference frame alignment.

4. Conclusion

In summary, we have introduced the “quantumness” $Q(\rho)$ for any finite-dimensional quantum state $\rho$. This quantity measures how quantum a state is. $Q(\rho)$ is a real valued, positive, and convex function with a value between 0 and 1 (more precisely, $0 \leq Q(\rho)^2 < 1 - 1/d$, where $d$ is the dimension of the Hilbert space) that measures the
| \( j \) | \( Q^2_{\text{QQ}} \) | Majorana points of \( |\psi\rangle \) on unit sphere | QQ state | Coherent state points of closest classical neighbours |
|---|---|---|---|---|
| 1 | 3/8 | Two antipodal points | \( \frac{1}{\sqrt{2}} (|1, 1\rangle + |1, -1\rangle) \) | Equilateral triangle in equatorial plane |
| 3/2 | 9/19 | Equilateral triangle in equatorial plane | \( \frac{1}{\sqrt{2}} \left( |\frac{3}{2}, \frac{3}{2}\rangle + |\frac{3}{2}, -\frac{3}{2}\rangle \right) \) | Two points on poles, equilateral triangle in equatorial plane |
| 2 | 16/27 | Tetrahedron | \( \sqrt{\frac{3}{2}} |2, 1\rangle + \sqrt{\frac{1}{3}} |2, -2\rangle \) | Overturned tetrahedron |
| 5/2 | 0.645914\(^a\) | Two points on poles, equilateral triangle in equatorial plane | \( \frac{1}{\sqrt{2}} \left( |\frac{5}{2}, \frac{3}{2}\rangle + |\frac{5}{2}, -\frac{3}{2}\rangle \right) \) | Two parallel equilateral triangles symmetric on both sides of equatorial plane |
| 3 | 347/486 | Octahedron | \( \frac{1}{\sqrt{2}} \left( |3, 2\rangle + |3, -2\rangle \right) \) | Cube |
| 7/2 | 0.743138\(^b\) | Two points on poles, regular pentagon in equatorial plane | \( \frac{1}{\sqrt{2}} \left( |\frac{7}{2}, \frac{5}{2}\rangle + |\frac{7}{2}, -\frac{5}{2}\rangle \right) \) | Two parallel regular pentagons |
| 4 | 0.77108 | Four points in plane with line of symmetry; remaining four points obtained by improper \( \pi/2 \)-rotation \( S_4 \) about symmetry line | see Table A1 | Twelve coherent states |
| 9/2 | 0.79676 | Three triangles in parallel planes, central rotated by \( \pi \) | see Table A1 | Two points on poles, two triangles symmetric with respect to equatorial plane and three doublets on equator |
| 5 | 0.81664 | Two points on poles and two squares in parallel planes rotated with respect to each other by \( \pi/4 \) | see Table A1 | Four squares in parallel planes; the two South squares rotated by \( \pi/4 \) with respect to those in North hemisphere |

\( a \) Minimum of \((270286 + 61910 \cos(2x) + 58680 \cos(4x) + 855 \cos(6x) + 1530 \cos(8x) - 45 \cos(10x) - 51200 \sin(x) + 25600 \sin(3x) - 5120 \sin(5x))/262144 \)

\( b \) Minimum of \((68477212 + 10990343 \cos(2x) + 18268726 \cos(4x) + 2030189 \cos(6x) + 845124 \cos(8x) + 25319 \cos(10x) + 26474 \cos(12x) - 91 \cos(14x) - 4014080 \sin(x) + 2408448 \sin(3x) - 802816 \sin(5x) + 114688 \sin(7x))/67108864 \)

**Table 1.** Queens of Quantum: The Majorana points of the QQ states, the QQ states in \( |jm\rangle \) notation, and the set of coherent states of the nearest classical neighbour.
Hilbert-Schmidt distance of $\rho$ to the convex set of classical states, defined as states with positive $P$-function [10]. We have shown that thermal states always become classical ($Q(\rho) = 0$) for temperatures larger than a critical temperature that depends on the dimension of the Hilbert space and the Hamiltonian, whereas the ground state of a system may or may not have non-zero quantumness. We used $Q(\rho)$ in order to find the “Queens of Quantum” states, defined as the states with maximum quantumness for a given Hilbert space dimension. Maximum quantumness can always be reached for pure states, and we have demonstrated that the Queens of Quantum states correspond to beautiful, highly symmetric bodies when expressed in terms of their Majorana representation. For the two lowest dimensions that allow for Platonic bodies ($j = 2$
and 3, with 4 and 6 Majorana points, respectively), they are indeed the corresponding Platonic bodies (tetrahedron and octahedron, respectively). For \( j = 4 \), lowering the symmetry allows to obtain an even larger quantumness compared to the one for the corresponding Platonic body (the cube), and we have identified numerically all other Queens of Quantum states for \( 3/2 \leq j \leq 5 \) using quadratic programming.

Acknowledgments

This work was supported in part by the Agence Nationale de la Recherche (ANR), project QPPRJCCQ. PB is grateful to the Sonderforschungsbereich TR 12 of the Deutsche Forschungsgemeinschaft and to the GDRI-471.

Appendix: QQ states and their closest classical state for \( j \leq 5 \)

This appendix lists the numerical results obtained from the algorithms explained in section 3.4. The Majorana configurations were found numerically and then deformed to the closest symmetrical figures under the condition that \( Q \) was increased. Results are shown in Table A1. The coherent states shown in Tables A2,A3 were numerically found for these symmetrical configurations. Contrary to the Majorana points, the symmetry of the coherent states was not enforced. Along with the numerically found values, we also give the likely exact values for the coherent states and the weights in Tables A2,A3, as far as they can be deduced from the numerically found ones and symmetry considerations.

References

[1] M. Lewenstein, D. Bruss, J. I. Cirac, B. Kraus, M. Kus, J. Samsonowicz, A. Sanpera, and R. Tarrach. *J. Mod. Optics*, 47:2841, 2000.
[2] M. B. Plenio and S. Virmani. *Quantum Information and Computation*, 7:1, April 2005.
[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters. *Phys. Rev. Lett.*, 70:1895, 1993.
[4] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
[5] R. Jozsa and N. Linden. *Proc. R. Soc. Lond. A*, 459:2011–2032, 2003.
[6] J. S. Bell. *Physics*, 1:195, 1964.
[7] N. Gisin. *Phys. Lett. A*, 154(201), 1991.
[8] M. S. Kim, E. Park, P. L. Knight, and H. Jeong. *Physical Review A*, 71(4):043805, April 2005.
[9] L. Mandel. *Physica Scripta T*, 12:34, 1986.
[10] O. Giraud, P. Braun, and D. Braun. *Phys. Rev. A*, 78(4):042112, 2008.
[11] R. I. A. Davis, R. Delbourgo, and P. D. Jarvis. *J. Phys. A: Math. Gen*, 33:1895–1914, 2000.
[12] J. Zimba. *EJTP*, 3:143–156, 2006.
[13] P. Kolenderski and R. Demkowicz-Dobrzański. *Phys. Rev. A*, 78(5):052333, Nov 2008.
[14] V. Vedral and M. B. Plenio. *Phys. Rev. A*, 57(3):1619–1633, Mar 1998.
[15] S. P. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.
[16] G. S. Agarwal. *Phys. Rev. A*, 24:2889, 1981.
[17] E. Majorana. *Nuovo Cimento*, 9:43–50, 1932.
[18] R. Penrose. *The Emperor’s New Mind, Oxford University Press*. Oxford, 1989.
## Table A1.

Numerical coordinates $(\bar{\theta}, \bar{\varphi})$ of the Majorana points for the Queens of Quantum for $1/2 \leq j \leq 5$. The Queens of Quantum are given by $(\bar{\theta}, \bar{\varphi})$ through (10), (11).

| $j$ | $\bar{\theta}$ | $\bar{\varphi}$ | $\bar{\theta}$ | $\bar{\varphi}$ |
|-----|----------------|-----------------|----------------|----------------|
| 1   | 0              | $\pi$           | $\pi/2$        | 4.46095        |
|     | $\pi$          | $\pi$           | $\pi/2$        | 1.82223        |
| 3/2 | $\pi/2$        | $\pi$           | $\pi/2$        | 5.62398        |
|     | $\pi/2$        | $5\pi/3$        | $\pi/2$        | 0.659206       |
|     | $\pi/2$        | $\pi/3$         | 0.251438       | 0              |
| 2   | 0              | $\pi$           | $\pi$          | 2.890154       |
|     | $\pi$          | 0.911591        | $\pi$          | 2               |
|     | 2arccos(1/$\sqrt{3}$) | $\pi$        | 2arccos(1/$\sqrt{3}$) | 5$\pi$/3       |
|     | 2arccos(1/$\sqrt{3}$) | $5\pi$/3       | 2.230002       | $\pi$          |
|     | 2arccos(1/$\sqrt{3}$) | $4\pi$/3       | 0.799772       | $\pi$          |
| 5/2 | 0              | $\pi$           | $\pi$          | 5$\pi$/3       |
|     | $\pi/2$        | $\pi$           | $\pi/3$        | $\pi/3$        |
|     | $\pi/2$        | $5\pi/3$        | $\pi/2$        | 2.0944         |
|     | $\pi/2$        | $\pi/3$         | $\pi/2$        | 4$\pi$/3       |
|     | $\pi/3$        | $\pi$           | $\pi/2$        | $\pi/3$        |
| 3   | 0              | $\pi$           | $\pi$          | 2.341821       |
|     | $\pi/2$        | $\pi$           | $5\pi/3$       | 2.341821       |
|     | $\pi/2$        | $4\pi/3$        | $\pi/3$        | 2.341821       |
|     | $\pi/2$        | 0               | 5$\pi$/3       | $\pi$          |
|     | $\pi/2$        | 4$\pi$/3        | 1.134586       | $\pi$          |
|     | $\pi/3$        | $\pi$           | 1.134586       | 3$\pi$/2       |
| 7/2 | 0              | $\pi$           | $\pi$          | 1.134586       |
|     | $\pi/2$        | $\pi$           | $\pi/2$        | 2.007007       |
|     | $\pi/2$        | 7$\pi$/5        | 5$\pi$/4       | 2.007007       |
|     | $\pi/2$        | 9$\pi$/5        | 7$\pi$/4       | 2.007007       |
|     | $\pi/2$        | $\pi/5$         | $\pi/4$        | 2.007007       |
|     | $\pi/2$        | 3$\pi$/5        | $\pi/4$        | 2.007007       |
|     | $\pi$          | $\pi$           | $\pi$          | $\pi$          |

[19] J. H. Hannay. *J. Phys. A: Math. Gen.*, 31:L53–L59, 1998.

[20] M. R. Dennis. *J. Phys. A: Math. Gen.*, 37:9487, 2004.

[21] W. G. Ritter. *Journal of Mathematical Physics*, 46(8):082103, 2005.

[22] J. J. Thomson. *Philos. Mag.*, 7:237, 1904.

[23] E. L. Altschuler, T. J. Williams, E. R. Ratner, R. Tipton, R. Stong, F. Dowla, and F. Wooten. *Phys. Rev. Lett.*, 78(14):2681–2685, Apr 1997.

[24] J. R. Edmundson. *Acta Cryst. A*, 48:60–69, 1992.
Table A2. Numerical coordinates and weights ($\bar{\theta}_i, \varphi_i, \lambda_i$) of the coherent states for the classical states achieving the minimum distance to the Queens of Quantum for $1/2 \leq j \leq 7/2$. Namely, $\rho_c = \sum_i \lambda_i |\bar{\theta}_i\varphi_i\rangle\langle\bar{\theta}_i\varphi_i|$ with $|\bar{\theta}_i\varphi_i\rangle$ given by (8). The $(\bar{\theta}^e_i, \varphi^e_i, \lambda^e_i)$ are the presumable exact values deduced from the numerical data, based on symmetry considerations. In various cases their entry reads n.a.e., which means that no analytical expression could be extracted from the numerical data. In the case $j = 1$, the exact analytical solution (26) is given.
### Quantifying Quantumness and the Quest for Queens of Quantum

| $j$  | $\theta_i$ | $\varphi_i$ | $\lambda_i$ | $\theta_i^{e}$ | $\varphi_i^{e}$ | $\lambda_i^{e}$ |
|------|------------|-------------|-------------|----------------|----------------|----------------|
| 4    | 0.74844    | 1.95206     | 0.08164     | 1/12           |                 |                |
|      | 0.74558    | 4.34339     | 0.08225     | 1/12           |                 |                |
|      | 0.88777    | 1.24671     | 0.08324     | 1/12           |                 |                |
|      | 0.89612    | 5.02990     | 0.08139     | 1/12           |                 |                |
|      | 1.04909    | 6.27490     | 0.08630     | 1/12           |                 |                |
|      | 1.56825    | 2.62661     | 0.08525     | 1/12           |                 |                |
|      | 1.56244    | 3.66123     | 0.08541     | n.a.e.         | n.a.e.          | 1/12           |
|      | 2.09943    | 6.28277     | 0.08604     | 1/12           |                 |                |
|      | 2.25446    | 1.25118     | 0.08245     | 1/12           |                 |                |
|      | 2.25422    | 5.03059     | 0.08205     | 1/12           |                 |                |
|      | 2.39338    | 1.94823     | 0.08187     | 1/12           |                 |                |
|      | 2.39551    | 4.33750     | 0.08211     | 1/12           |                 |                |
| $9/2$| 0.00150    | 1.06854     | 0.09641     | 0              | irrelevant      | n.a.e.         |
|      | 0.83772    | 4.18728     | 0.06303     | $\arccos(2/3)$ | $4\pi/3$       | n.a.e.         |
|      | 0.82797    | 6.27566     | 0.06297     | $\arccos(2/3)$ | $2\pi$         | n.a.e.         |
|      | 0.82663    | 2.09864     | 0.06302     | $\arccos(2/3)$ | $2\pi/3$       | n.a.e.         |
|      | 1.56795    | 0.83024     | 0.07166     | $\pi/2$       | n.a.e.         |                |
|      | 1.57137    | 1.26468     | 0.07120     | $\pi/2$       | n.a.e.         |                |
|      | 1.57366    | 2.92169     | 0.07336     | $\pi/2$       | n.a.e.         |                |
|      | 1.56980    | 3.37157     | 0.07006     | $\pi/2$       | n.a.e.         |                |
|      | 1.57408    | 5.01531     | 0.07175     | $\pi/2$       | n.a.e.         |                |
|      | 1.57129    | 5.45752     | 0.07132     | $\pi/2$       | n.a.e.         |                |
|      | 2.31140    | 6.28060     | 0.06284     | $\arccos(-2/3)$| $2\pi$         | n.a.e.         |
|      | 2.31289    | 2.08971     | 0.06326     | $\arccos(-2/3)$| $2\pi/3$       | n.a.e.         |
|      | 2.30992    | 4.18140     | 0.06279     | $\arccos(-2/3)$| $4\pi/3$       | n.a.e.         |
|      | 3.13728    | 5.61142     | 0.09634     | $\pi$         | irrelevant      | n.a.e.         |
| 5    | 0.73375    | 0.77724     | 0.05989     | $\pi/4$       | n.a.e.         |                |
|      | 0.75268    | 2.36318     | 0.05537     | $n.a.e.$      | $3\pi/4$       | n.a.e.         |
|      | 0.72910    | 3.91883     | 0.05801     | $5\pi/4$      | n.a.e.         |                |
|      | 0.75488    | 5.50021     | 0.05713     | $7\pi/4$      | n.a.e.         |                |
|      | 1.27503    | 0.78214     | 0.06959     | $\pi/4$       | n.a.e.         |                |
|      | 1.24043    | 2.36002     | 0.07004     | $n.a.e.$      | $3\pi/4$       | n.a.e.         |
|      | 1.26250    | 3.92131     | 0.07053     | $5\pi/4$      | n.a.e.         |                |
|      | 1.25154    | 5.48300     | 0.06869     | $7\pi/4$      | n.a.e.         |                |
|      | 1.88976    | 6.27189     | 0.07205     | $2\pi$        | n.a.e.         |                |
|      | 1.87975    | 1.57367     | 0.06859     | n.a.e.        | $\pi/2$        | n.a.e.         |
|      | 1.89339    | 3.14019     | 0.07050     | $\pi$         | n.a.e.         |                |
|      | 1.87096    | 4.70769     | 0.06717     | $3\pi/2$      | n.a.e.         |                |
|      | 2.42373    | 6.27224     | 0.03774     | $2\pi$        | n.a.e.         |                |
|      | 2.38756    | 1.57823     | 0.05841     | n.a.e.        | $\pi/2$        | n.a.e.         |
|      | 2.40240    | 3.14208     | 0.05641     | $\pi$         | n.a.e.         |                |
|      | 2.38373    | 4.70632     | 0.05988     | $3\pi/2$      | n.a.e.         |                |

**Table A3.** Same as Table A2, but for $j = 4, 9/2, 5$. 

