SAMPLING THEORY WITH AVERAGES VALUES ON THE SIERPINSKI GASKET

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Abstract. In the case of some fractals, sampling with average values on cells is more natural than sampling on points. In this paper we investigate this method of sampling on SG and SG₃. In the former, we show that the cell graph approximations have the spectral decimation property and prove an analog of the Shannon sampling theorem. We also investigate the numerical properties of these sampling functions and make conjectures which allow us to look at sampling on infinite blowups of SG. In the case of SG₃, we show that the cell graphs have the spectral decimation property, but show that it is not useful for proving an analogous sampling theorem.

1. Introduction

A bandlimited function $f$ on $\mathbb{R}$ may be explicitly reconstructed from its values $\{f(k\delta) | k \in \mathbb{Z}\}$ on an appropriately spaced lattice ($\delta = \frac{1}{2B}$ if $B$ is the bandlimit) in terms of the sinc function $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. This is the classical Sampling Theorem of Shannon et al. We are interested in fractal analogs of this theorem, where we replace $\mathbb{R}$ by a self-similar fractal $K$. In this context we will assume that $K$ has an appropriate Laplacian $\Delta$, and bandlimited means that the expansion of $f$ as an infinite series in terms of eigenfunctions $\{u_k\}$ of $\Delta$ with $-\Delta u_k = \lambda_k u_k$ is actually a finite sum where the eigenvalues $\lambda_k$ satisfy $\lambda_k \leq B$, with $B$ being the bandlimit. But also, instead of sampling values of the function at a discrete set of points, we assume that we are given the average values of $f$ over a collection of cells in a natural decomposition of $K$ at a certain level, with the bandlimit $B$ and the cell level $m$ closely...
related. In the context of fractals, these average values are more natural "samples" than the pointwise values.

For the most part we will deal with a single fractal, the Sierpinski gasket \((SG)\) with the standard self-similar measure \(\mu\) and Kigami Laplacian \(\Delta\) (see [3, 7] for details). The spectrum of this Laplacian is describable explicitly by the method of spectral decimation introduced by Fukushima and Shima [2], which relates eigenfunctions and eigenvalues of \(-\Delta\) on \(SG\) with eigenfunctions and eigenvalues of discrete Laplacians \(-\Delta_m\) on graph approximations (denoted here by \(\beta_m\)) to \(SG\). It was shown in [4] how to use this method to obtain a sampling theorem involving point samples on the vertices of \(\beta_m\). On the other hand, it was shown in [6] that the Laplacian on \(SG\) is also definable as a limit of graph Laplacians on the cell graphs \(\Gamma_m\) that also approximate \(SG\) in terms of the average values of functions on \(m\)-cells. The main technical result of this paper is to show that the method of spectral decimation is also valid for the sequence of cell graph approximations. This leads directly to a sampling theorem for average value samples.

It is tempting to speculate that our results should be a special case of a generic result for a larger class of fractals, say, the post-critically finite (PCF) fractals for which spectral decimation holds. But we show that in fact this is not the case for a closely related fractal \(SG_3\) (subdivide the sides of the generating triangle in thirds rather than halves). While \(SG_3\) has an explicitly described spectral decimation (see [1]) and its Laplacian is describable in terms of cell graph approximations (see [8]), we do not have a useful spectral decimation method for the cell graph Laplacians.

The paper is organized as follows. In section 2 we briefly review the theory of the Laplacian on \(SG\) that is needed for explaining our results. The reader may refer to [7] for complete details. In section 3 we describe the spectral decimation results for the cell graph approximations \(\Gamma_m\) of \(SG\). In section 4 we prove the sampling theorem. In section 5 we present numerical data on the sampling functions, and give some conjectures concerning possible exponential localization. This is quite a contrast to the poor localization of the sinc function. We show how the conjectures imply a sampling theorem on infinite blowups of \(SG\). In section 6 we present the negative results for \(SG_3\).

2. \(SG\) Preliminaries

2.1. Definition and Construction. Let \(q_0, q_1, q_2\) denote the vertices of an equilateral triangle. For the purposes of this paper, \(q_0\) will be the
Define $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_i(x) = \frac{1}{2}(x - q_i) + q_i.$$  \hspace{1cm} (2.1)

for $i = 0, 1, 2$. $SG$ is defined to be the unique nonempty compact set satisfying

$$SG = \bigcup_{i=1}^3 F_i(SG).$$

We define a word of length $n$, $(w_1, \ldots, w_n)$ to simply be an element of $\mathbb{Z}_3^n$. Then, we say that $F_w = F_{w_n} \circ \cdots \circ F_{w_1}$. If $T$ is the unit equilateral triangle, the $m$'th level approximation of $SG$ is $\bigcup_{|w|=m} F_w(T)$, where $|w|$ is the number of components in the word $w$.

We let $\beta_m$ denote the usual $m$'th level graph approximation of $SG$, where the vertices of the graph are the vertices of the triangles and the edges of the graph are the edges of the triangles. We can also consider the $m$'th level cell graphs, which we denote $\Gamma_m$. The vertices of this graph represent the right-side-up triangles of the $m$'th order approximation of $SG$. An edge between two vertices indicates that two triangles share a corner in common. It is important to note that no vertices or edges are preserved when going from $\Gamma_m$ to $\Gamma_{m+1}$. This is because the triangles in the level $m$ approximation of $SG$ split into three separate triangles in the level $m+1$ approximation, so no triangles are preserved in subsequent approximations.

In addition, when working with $\Gamma_m$, we assume that all function values are the average values of functions on the smallest triangles on the dyadic point graphs. Specifically, if $a, b, c$ are values on the vertices of a small triangle on the dyadic point graph, then the function value on $\Gamma_m$ for the vertex corresponding to the triangle is $\frac{a+b+c}{3}$. We also assume that if $x$ is the value of a vertex on $\Gamma_m$, and that vertex splits
into three vertices in $\Gamma_{m+1}$ with values $d, e, f$ respectively, then the values must satisfy
\[ x = \frac{d + e + f}{3}. \] (2.2)

2.2. The Laplacian. Let $\mathcal{E}_m(u)$ denote the renormalized graph energy of a function $u$ on $\beta_m$, where $\frac{3}{2}5^m$ is the renormalization constant, and let $\mathcal{E}_m(u, v)$ denote the corresponding bilinear form. We then define the energy $\mathcal{E}(u)$ and the bilinear form $\mathcal{E}(u, v)$ for functions on $SG$ by taking the limit as $m \to \infty$. We say that a function $u$ on $SG$ is in $\text{dom} \mathcal{E}$ if it has finite energy. We then define the Neumann Laplacian on $SG$ by the following:

**Definition 2.1.** Let $u \in \text{dom} \mathcal{E}$. Then $\Delta u = f$ for $f$ continuous if
\[ \mathcal{E}(u, v) = -\int_{SG} fvd\mu \] (2.5)
for all $v \in \text{dom} \mathcal{E}$.

Since it is natural to think of derivatives in terms of limits of difference quotients, we would like to have a similar expression for our Laplacian. We define the renormalized graph Laplacian of a function $u$ on $\Gamma_m$ at vertex $x$ to be
\[ \Delta_m u(x) = \frac{3}{2}5^m \sum_{x \sim y} (u(y) - u(x)). \] (2.6)

We then have the following.

**Theorem 2.2.** Let $\Delta u = f$. Then $\Delta_m u$ converges uniformly to $f$. Conversely, if $u$ is integrable, and $\Delta_m u$ converges uniformly to a continuous function $f$, then $\Delta u$ exists and equals $f$. 
For a proof of this, see [6]. We note that a similar result holds for graph Laplacians defined on the $\beta_m$ graphs ([3, 7]).

3. The Eigenbasis of $\Gamma_m$

In order to proceed with our discussion of sampling, we must first compute the eigenbasis of $\Gamma_m$. Before we can give an explicit construction of the basis, we need to show that we have spectral decimation on the $\Gamma_m$.

**Theorem 3.1** (Spectral Decimation). Let $u$ be an eigenfunction on $\Gamma_m$ with eigenvalue $\lambda_m$. Then, $u$ can be extended to at most two eigenfunctions on $\Gamma_{m+1}$ with eigenvalues $\lambda_{m+1}^{(1)}$ and $\lambda_{m+1}^{(2)}$. Furthermore, for each $\lambda_{m+1}^{(k)}$, the corresponding extension is unique. Conversely, if $u$ is an eigenfunction on $\Gamma_{m+1}$ with eigenvalue $\lambda_{m+1}^{(1)}$ or $\lambda_{m+1}^{(2)}$, then $u'$, the function on $\Gamma_m$ obtained by averaging the values of $u$ up one level is an eigenfunction with eigenvalue $\lambda_m$. The relationship between $\lambda_m$ and $\lambda_{m+1}^{(k)}$ is given in (3.9) and (3.10) below.

**Proof.** Consider Figure 3 below. The picture on the left details a general subgraph of the interior of $\Gamma_m$ for $m > 1$ (The case for $m = 1$ can be obtained by simply deleting the point labeled W and its corresponding edge connecting it to the triangular group of vertices). We extend it to a corresponding general subgraph of $\Gamma_{m+1}$ in the picture adjacent to it.

$$(3 - \lambda_m)u(X) = u(W) + u(Y) + u(Z). \quad (3.1)$$

Now, assume that $u$ extends to an eigenfunction on $\Gamma_{m+1}$. This means that, for vertex $v_1$,

$$3(3 - \lambda_{m+1})u(v_1) = u(v_2) + u(v_3) + u(v_b) \quad (3.2)$$

and similarly for every vertex except possibly for $v_5$, $v_9$, and $v_a$, as one of them might be a boundary vertex. By manipulating these equations for known interior points, and using the mean value property (2.2) of the cell graph, we obtain

$$u(v_1) = \frac{3(4 - \lambda_{m+1})u(X) + 3u(W)}{(3 - \lambda_{m+1})(5 - \lambda_{m+1})} \quad (3.3)$$

and similarly for all of the other interior vertices. In other words, the value of the extended eigenfunction on a vertex is a function of the vertex’s parent, the (different) parent of the nearest neighboring vertex, and the eigenvalue of the extended function. The $(4 - \lambda_{m+1})$
Figure 3. On the left, a general subgraph of $\Gamma_m$ centered for an interior vertex $X$. On the right, that subgraph extended down to $\Gamma_{m+1}$.

term acts as a weighting factor, which makes sense as the value of the function at the parent cell should affect the value more.

To extend an eigenfunction on the boundary, we see in the above figure that if $Y$ is a boundary vertex on $\Gamma_m$, then $v_5$ is a boundary vertex on $\Gamma_{m+1}$. To extend $u$ to an eigenfunction on $\Gamma_{m+1}$, we first observe that such an eigenfunction would satisfy

$$(2 - \lambda_{m+1})u(v_5) = u(v_4) + u(v_6).$$

We then add $u(v_5)$ to both sides and apply (2.2) to get

$$u(v_5) = \frac{3u(Y)}{3 - \lambda_{m+1}}.$$ (3.5)

This is consistent with (3.3) as every vertex adjacent to $v_5$ shares its parent cell, so the only factors that should matter are the current eigenvalue and the value of the function on the parent cell.

All that remains is to figure out what the $\lambda_{m+1}^{(k)}$ are. Adding together the Laplacian relations for $v_1, v_2,$ and $v_3$ and applying (2.2) gives

$$(1 - \lambda_{m+1})3u(X) = u(v_4) + u(v_7) + u(v_b).$$ (3.6)

Doing the same for $v_4, v_7,$ and $v_b,$ and then applying (3.6) gives

$$(2 - 4\lambda_{m+1} + \lambda_{m+1}^2)3u(X) = u(v_5) + u(v_6) + u(v_8) + u(v_9) + u(v_a) + u(v_c).$$ (3.7)
Adding together (3.6) and (3.7) and applying (3.1) gives

\[(3 - \lambda_m)3u(X) = (3 - 5\lambda_m + \lambda_m^2)3u(X)\]  

which simplifies to

\[\lambda_m = \lambda_{m+1}(5 - \lambda_{m+1}),\]  

which has solutions

\[\lambda_{m+1} = \frac{5 \pm \sqrt{25 - 4\lambda_m}}{2}.\]  

The "at most two" in the theorem comes from (3.3) and (3.5). The equation (3.10) produces two new eigenvalues via the quadratic formula as usual. However, if one of these eigenvalues happens to be 3 or 5, the equations for continuation in the interior are no longer valid, so we cannot extend to eigenfunctions in these cases.

With spectral decimation in hand, we now have all of the tools that we need in order to produce the basis. Our construction is recursive.

The basis for \(\Gamma_1\) is easily computed by inspection. It consists of a constant function, a nonconstant function, and a rotation of the nonconstant function. The constant function has eigenvalue 0, and the other two have eigenvalue 3.

Now, consider \(\Gamma_2\). We can create 5 basis elements by using the spectral decimation equations derived above to extend the three elements of the basis of \(\Gamma_1\) down to \(\Gamma_2\). Note that (3.10) implies that the constant eigenfunction can only bifurcate into eigenfunctions with eigenvalues 0 and 5. However, (3.3) shows that 5 is a forbidden eigenvalue, so the constant eigenfunction can only extend to the constant eigenfunction.
Figure 5. The two types of basis elements for $\Gamma_2$ that are not continued from $\Gamma_1$. The left has eigenvalue 5 whereas the right has eigenvalue 3.

The remaining elements of the basis are computed by inspection, and are listed below. The first type is a "battery chain" construction around the hexagon in the graph. Start at a point that lies on the hexagon and place the value $-1$. Then go around the hexagon clockwise, alternating between placing the values 1 and $-1$ on vertices until every point on the hexagon is nonzero. The second type is constructed by placing a 2 at one boundary vertex, $-1$ at its adjacent vertices, $-1$ at the adjacent vertices of the boundary point’s adjacent vertices, and 1 at the two remaining non-boundary vertices. The remaining elements of the basis consist of one function of the first type, and the three rotations of the second type.

In general, consider going from $\Gamma_{m-1}$ to $\Gamma_m$. Using spectral decimation, we take the eigenbasis on $\Gamma_{m-1}$ and extend it to a linear independent set in with cardinality $2 \cdot 3^{m-1} - 1$ on $\Gamma_m$, where every eigenfunction in the basis on $\Gamma_{m-1}$ extends to two eigenfunctions on $\Gamma_m$ except for the constant function, which only extends to the constant function.

We then look at the hexagons on $\Gamma_m$. Each hexagon corresponds to exactly one upside-down triangle on the graph of the dyadic points of $SG$. A simple argument shows that there are $1 + 3 + 3^2 + \ldots + 3^{m-2} = \frac{3^{m-1}-1}{2}$ hexagon cycles on $\Gamma_m$. We proceed as we did for the first type of non-extended eigenfunction on $\Gamma_2$: pick a hexagon and a vertex on it and assign it a value of $-1$, then continue clockwise around the hexagon, alternating between 1 and $-1$ until every vertex on the hexagon has a nonzero value. Do this for every hexagon on $\Gamma_m$ to get $\frac{3^{m-1}-1}{2}$ eigenfunctions with eigenvalue 5.

We now consider the eigenfunctions with eigenvalue 3. First, we take each vertex on the boundary of $\Gamma_m$ and copy the appropriate rotation of the second type of non-extended eigenfunction on $\Gamma_2$ onto the corresponding copy of $\Gamma_2$ on the boundary of $\Gamma_m$, with the 2 being placed on the vertex on the boundary. This yields three eigenfunctions. To
Figure 6. The type of eigenfunctions with eigenvalue 3 on level $m$ that have no support on the boundary.

get the remaining eigenfunctions, we consider the $3^{m-2}$ copies of $\Gamma_2$ in $\Gamma_m$. Take two adjacent copies of $\Gamma_2$, and consider the edge connecting them. Assign a value of 2 to each of the vertices on the edge, and then repeat the construction of the second type of nonextended eigenfunction on $\Gamma_2$ twice. Refer to the figure below for the specific construction. An inductive argument shows that there are $\frac{3^{m-1} - 3}{2}$ eigenfunctions of this type, so we have a total of $\frac{3^{m-1} + 3}{2}$ eigenfunctions at level $m$ with eigenvalue 3.

Counting the eigenfunctions that we have thus far, we see we have

$$2 \cdot 3^{m-1} - 1 + \frac{3^{m-1} - 1}{2} + \frac{3^{m-1} + 3}{2} = 3^m.$$ 

However, we can have at most $3^m$ linear independent basis elements, so we have necessarily constructed the eigenbasis.

4. The Sampling Theorem

We first make precise what we mean by bandlimited.

**Definition 4.1.** A level $m$ bandlimited function on $SG$ is a function that is a linear combination of the first $3^m$ Neumann eigenfunctions.

From this definition, we see that the restriction to $\beta_m$ of a level $m$ bandlimited function on $SG$ is a linear combination of Neumann eigenfunctions without eigenvalue 6. As the dimension of the space of Neumann eigenfunctions on $\beta_m$ is $\frac{3^{m+1} + 3}{2}$ and the dimension of the space generated by the Neumann eigenfunctions with eigenvalue 6 is $\frac{3^m + 3}{2}$, the space of bandlimited functions on $\beta_m$ has dimension $3^m$.

We now define two averages. The discrete average on an $m$-cell $C$ on $\beta_n$ for $n$ at least $m$ is given by:
where $\partial C$ refers to the boundary points to $C$. The continuous average is given by

$$B_C(u) = 3^m \int_C u.$$  

We let $A(u)$ and $B(u)$ refer to the average value functions of $u$ with respect to the above averages. We are now ready to state and prove the sampling theorem.

**Theorem 4.2.** On $SG$, bandlimited functions are uniquely determined by their average values on $\Gamma_m$ cells, where the average values can be taken in either the discrete sense ($A$) or the continuous sense ($B$).

In order to prove the theorem, we need a technical lemma. Recall that the $\beta_m$ graphs all have the spectral decimation property. For a proof of this, see [7]. Spectral decimation on $\beta_m$ yields the same eigenvalue relation as for $\Gamma_m$, namely, (3.10). We can use this to prove the following lemma.

**Lemma 4.3.** Let $u$ be an eigenfunction on $\beta_{m-1}$ with eigenvalue $\lambda_{m-1}$, and let $u'$ be its extension by spectral decimation to an eigenfunction on $\beta_m$ with eigenvalue $\lambda_m$. Then $A(u')$ is an eigenfunction on $\Gamma_m$ with eigenvalue $\lambda_m$.

The proof of this fact is a straightforward computation of the Laplacian at each point of $\Gamma_m$, using the relation $\lambda_{m-1} = \lambda_m(5 - \lambda_m)$.

We first consider the case of the discrete average $A$ on $\beta_m$. Let $X$ denote the space of bandlimited functions on $\beta_m$, which has dimension $3^m$, and let $Y$ denote the space of functions on $\Gamma_m$. Consider the linear map from $X$ to $Y$ defined by taking $u \in X$ to $A(u)$. We now need to show that every eigenfunction on $\Gamma_m$ can be obtained from eigenfunctions on $\beta_m$ under the above map.

It is easy to see that the portion of the eigenbasis of $X$ with eigenvalues 3 and 5 maps bijectively to the portion of the eigenbasis of $Y$ with eigenvalues 3 and 5 respectively. For the remaining portion of the eigenbasis of $X$, which consists of continued eigenfunctions, spectral decimation and the above lemma guarantee that this portion of the eigenbasis of $X$ maps to the remaining portion of the eigenbasis for $Y$, so the mapping is onto, thus bijective.
This proves the theorem in the case of the discrete average $A$. For $B$, the result follows from the above and by observing that for an eigenfunction $u \in X$, $B(u) = cA(u)$ for a nonzero constant $c$.

It is important to note also that this is the broadest definition of bandlimited that we can take, as every eigenfunction with eigenvalue $6$ on $\beta_m$ has $0$ average. Note also that over the course of the proof of the sampling theorem, we proved the following.

**Theorem 4.4.** Let $u$ be an eigenfunction on $SG$ with one of the first $3^m$ eigenvalues. Then the restriction of $u$ to $\Gamma_m$ is an eigenfunction.

5. **Sampling Functions**

5.1. **Numerical Data on $SG$.** We recall that, for a set $A$, $\{\psi_y | y \in A\}$ is a set of sampling function if

$$\psi_y(x) = \delta(y, x)$$

for all $x \in A$. In this paper, we are interested for sampling functions on $m$-cells. Specifically, we say that $\psi$ is a sampling function for the $m$-cell $F_wSG$ if the average of $\psi$ on $F_wSG$ is $1$ and the average of $\psi$ is $0$ on any other $m$-cell in $SG$. By the theorem in the previous section, we can take such functions to be bandlimited, so we can restrict ourselves to looking at functions on the $\Gamma_m$.

As the figures show, there are two major differences between these sampling functions and those on the real line, the translates of the sinc function. A close inspection of the sampling functions shows that they are not symmetric. Looking at the large bump for the non-boundary cell sampling functions, we can see that the bump slightly slants, which seems to reflect its position relative to the boundary. We also see for higher level sampling functions that, unlike the sinc functions, the oscillation dies down rather quickly. After the initial bump, the function oscillates with magnitude around $10^{-9}$.

Having performed some numerical calculations, we can make two conjectures.

**Conjecture 5.1.** The sampling functions are uniformly bounded by some constant (around $1.3$) for all $m$.

This conjecture is consistent with what we see for the sampling functions on the real line. The next conjecture, however, is not.

**Conjecture 5.2.** Let $\psi_m$ be a sampling function on level $m$. Then

$$\int_{SG} |\psi_m|^2 \leq c3^{-m}$$
Figure 7. The unique (up to $D_3$ symmetries) level 1 sampling function. The right column is a zoomed in view of the functions on the left column.

Figure 8. The unique (up to $D_3$ symmetries) level 2 sampling functions. The right column is a zoomed in view of the functions on the left column.

where $c$ is a constant around 1.1.

This further supports the notion that the sampling functions die off much more quickly than the sinc functions.
We can compare these sampling functions on cells with the sampling functions on vertices discussed in [4]. We use the following notation from the aforementioned paper: if $y$ is a vertex on $\beta_m$ such that $y$ is not a boundary point of $SG$, then $\psi^{(m)}_y$ is the function on $SG$ such that, for $x \in \beta_m$, $\psi^{(m)}_y(x) = \delta_{yx}$. For small $m$, the pointwise sampling functions have large oscillations throughout $SG$. For larger $m$, however, we observe that both the pointwise sampling functions and the sampling functions on cells have the same basic oscillatory behavior: there is a big sinusoidal oscillation centered at the chosen point/cell which rapidly dies off. In fact, in [4], we have the following conjecture:

**Conjecture 5.3.** There exist constants $c$ and $\alpha < 1$ (about $\frac{1}{3}$) such that

$$|\phi^{(m)}_y(x)| \leq c\alpha^{d_m(x,y)}$$
for all non-boundary vertices $y$ in $\beta_m$, all $x \in SG$, and all $m$, where $d_m(x,y)$ is the smallest number of $m$-cells separating $x$ and $y$.

This exponential cell distance bound for the sampling functions on vertices matches up with the observed exponential decay of the sampling functions on the cells.

5.2. Sampling on Infinite Blowups. Let $\{i_j\}$ be an infinite sequence such that $i_k = 0, 1, 2$, with the condition that two of the integers 0, 1, and 2 occur infinitely often. Then, for $K = SG$, we define $K_\infty$ by

$$K_\infty = \bigcup_{m=1}^{\infty} F_{i_1}^{-1} \ldots F_{i_m}^{-1} K.$$ 

and let $K_{(m)} = F_{i_1}^{-1} \ldots F_{i_m}^{-1} K$. Note that the sets $K_{(m)}$ are nested.

Fix a cell in $K_\infty$. For simplicity, assume it is in $K$, say $F_w K = F_{w_1} \ldots F_{w_m} K$. Let $\psi_0$ be the sampling function of $F_w K$ in $K$. By the theorem, we can choose $\psi_0$ to be bandlimited. So that the average $A_{w'}(\psi_0)$ of $\psi_0$ on $F_{w'} K$ equals $\delta_{w w'}$ for all $|w'| = m$. 

Figure 9. Some of the level 3 sampling functions. The right column is a zoomed in view of the functions on the left column.
Now, $\psi_0$ has an extension to $K_\infty$ that is $m$-bandlimited and of compact support (modulo constant functions), since each non-constant Neumann eigenfunction on $K$ extends to a Neumann eigenfunction on $K_\infty$ that is supported in $K_{(n)}$ for some $n$ (this $n$ depends on the sequence $i_1, i_2, ...$). Call the extension $\tilde{\psi}_0$. Note that not only is $\tilde{\psi}_0$ bandlimited on the top, but it is also bandlimited on the bottom, because
Figure 11. Some level 4 sampling functions (up to $D_3$ symmetries). The right column is a zoomed in view of the functions on the left column.

we do not add any eigenfunctions with eigenvalue below the smallest nonzero Neumann eigenvalue on $K$ save the constant term.

Note that the average of $\tilde{\psi}_0$ on any $m$-cell $F_w \cdot K$ of $K$ satisfies $\tilde{\psi}_0 = \delta_{ww'}$, but for other $m$-cells in $K_\infty$ we have no information.

Next we want to construct $\tilde{\psi}_1$ on $K_\infty$ that is $m$-bandlimited and of compact support (modulo constants) such that $A_{w'} = \delta_{ww'}$ for every
Figure 12. More level 4 sampling functions (up to $D_3$ symmetries). The right column is a zoomed in view of the functions on the left column.

$m$-cell $F_wK$ in $K$ and the average of $\tilde{\psi}_1$ on any $m$-cell in $K_{(1)}\backslash K$. So we want to take the containment

$$F_wK \subseteq K \subseteq F_{i_3}^{-1}K$$

and apply $F_{i_1}$ to it:

$$F_{i_1}F_wK \subseteq K.$$
So $F_i F_w K$ is an $(m+1)$-cell in $K$, so it has an $(m+1)$-bandlimited sampling function $\varphi_1$, and $A_{w'}(\varphi_1) = \delta(i_1 w)$. Let $\psi_1 = \varphi_1 \circ F_i$ defined on $K_1 = F_i^{-1} K$. Then $A_w(\psi_1)$ on $F_w K = A_{i_1 w}(\varphi_1) = 1$ and the average of $\varphi_1$ is zero on any other $m$-cell in $K(1)$. Also, $\psi_1$ is $m$-bandlimited so there is an extension $\tilde{\psi}_1$ to $K_\infty$ that is bandlimited and compactly supported modulo constants.
Figure 14. More level 4 sampling functions (up to $D_3$ symmetries). The right column is a zoomed in view of the functions on the left column.

We can easily compute the constant contributions to $\tilde{\psi}_0$ and $\tilde{\psi}_1$ since all nonconstant eigenfunctions have total integral zero and $\int_K \psi_0 = \frac{1}{3^m} = \int_{K(j)} \psi_1$, so the constant is $\frac{1}{3^m}$ for $\tilde{\psi}_0$ and $\frac{1}{3^{m+1}}$ for $\tilde{\psi}_1$.

Iterating this argument, we obtain a sequence $\tilde{\psi}_0, \tilde{\psi}_1, \tilde{\psi}_2, ...$ of $m$-bandlimited functions such that the $\tilde{\psi}_j$ are of compact support on $K_\infty$ such that $A_w(\tilde{\psi}_j) = 1$ and the average on any other $m$-cell in $K(j)$ is 0.
Figure 15. The remaining level 4 sampling functions (up to $D_3$ symmetries). The right column is a zoomed in view of the functions on the left column.

Now, assume that the two conjectures of the previous section are true. Conjecture 5.1 says that the sampling functions on $K$ are uniformly bounded, which is clearly equivalent to the $\tilde{\psi}_j$ being uniformly bounded on each $K(j)$. It is also easy to see that Conjecture 5.2 is equivalent to $\int_{K(j)} \Delta \tilde{\psi}_j \leq C$ for some $C$ and for all $j$.

Now, fix $K(n)$. For $j \geq n$, the restriction of $\tilde{\psi}_j$ to $K(j)$ is $m$-bandlimited, so

$$\int_{K(j)} |\Delta \tilde{\psi}_j|^2 \leq M_m \int_{K(j)} |\tilde{\psi}_j|^2$$

where $M_m$ is the highest frequency in the $m$-band. Combining this with the estimate

$$\int_{K(n)} |\Delta \tilde{\psi}_j|^2 \leq \int_{K(j)} |\Delta \tilde{\psi}_j|^2$$
and the aforementioned consequences of Conjecture 5.2, we have the estimate

$$\int_{K(n)} \left| \Delta \tilde{\psi}_j \right|^2 \leq C_n$$

for all $j \geq n$. We can use this to get the Hölder estimate

$$\left| \tilde{\psi}_j(x) - \tilde{\psi}_j(y) \right| \leq c R(x, y) \beta \int_{K(n)} \left| \Delta \tilde{\psi}_j \right|^2$$

for all $x, y \in K(n)$ for some $\beta$, where $R$ is the resistance metric. This gives us uniform equicontinuity on $K(n)$.

As the sequence $\{\tilde{\psi}_j\}$ is uniformly bounded and uniformly equicontinuous, we can apply Arzela-Ascoli to find a subsequence $\{\tilde{\psi}_{j_k}\}$ converging uniformly on each $K(n)$. Let $\tilde{\psi} = \lim_{k \to \infty} \tilde{\psi}_{j_k}$. Then $\tilde{\psi}$ is $m$-bandlimited, since if $\varphi$ is any high frequency eigenfunction, $\int \tilde{\psi} \varphi = \lim_{k \to \infty} \int \tilde{\psi}_{j_k} \varphi = 0$. Also, the average of $\tilde{\psi}$ on $F_w K$ is 1, while the average of $\tilde{\psi}$ on any other $m$-cell is 0, so $\tilde{\psi}$ is a sampling function for $F_w K$ on $K_\infty$.

6. The Case of $SG_3$

Recall that $SG_3$ is the unique nonempty compact set satisfying

$$SG_3 = \bigcup_{i=1}^{6} G_i(SG_3)$$

where the $G_i$ are the six fixed point maps corresponding to the main smaller triangles of $SG_3$. The energy and Laplacian are defined similarly to those for $SG$. We defer to [1] for complete details.

The usual graph approximations of $SG_3$, which we denote $\zeta_m$ are similar to those of $SG$ (see Figure 6.1). The cell graph approximations, which we denote $\xi_m$, are also similar to the $\Gamma_m$. It is easy to see that $\xi_1 = \beta_1$, and that the $\xi_m$ look very similar to the $\Gamma_m$, except the small triangles are replaced by small copies of $\beta_1$.

In [1], it was shown that $SG_3$ and the $\zeta_m$ possess the spectral decimation property. It is then reasonable to ask whether the $\xi_m$, with the usual graph Laplacian, also have the spectral decimation property. We can answer that in the affirmative.

**Theorem 6.1.** Let $u$ be an eigenfunction on $\xi_m$ with eigenvalue $\lambda_m$. Then, $u$ can be extended to at most two eigenfunctions on $\xi_{m+1}$ with
eigenvalues $\lambda^{(1)}_{m+1}$ and $\lambda^{(2)}_{m+1}$. Furthermore, for each $\lambda^{(k)}_{m+1}$, the corresponding extension is unique. Conversely, if $u$ is an eigenfunction on $\xi_{m+1}$ with eigenvalue $\lambda^{(1)}_{m+1}$ or $\lambda^{(2)}_{m+1}$, then $u'$, the function on $\xi_m$ obtained by averaging the values of $u$ up one level is an eigenfunction with eigenvalue $\lambda_m$.

The proof is analogous to that of the previously discussed $\Gamma_m$ case, and will be skipped. We will, however, highlight an important consequence of the calculations. The spectral decimation relation on the $\zeta_m$ requires the eigenvalues between $\zeta_m$ and $\zeta_{m+1}$ to satisfy

$$\lambda_m = \frac{3(\lambda_{m+1} - 5)(\lambda_{m+1} - 4)(\lambda_{m+1} - 3)\lambda_{m+1}}{3\lambda_{m+1} - 14}$$

whereas the corresponding relation between $\xi_m$ and $\xi_{m+1}$ is

$$\lambda_m = \frac{(\lambda_{m+1}^2 - 9\lambda_{m+1} + 19)(\lambda_{m+1} - 4)\lambda_{m+1}}{\lambda_{m+1} - 6}.$$

This is in stark contrast to the $SG$ case, where the relations were the same. Given that the eigenvalue relations being the same was crucial in proving the sampling theorem on $SG$, the fact that the relations are not the same on $\zeta_m$ and $\xi_m$ give little hope in proving such a sampling theorem in this case.
It is also important to note that the average values of the first six Neumann eigenfunctions on $\zeta_1$ are not all eigenfunctions on $\xi_1$. Specifically, no eigenfunction on $\zeta_1$ with eigenvalue $3 \pm \sqrt{2}$ has an average value that is an eigenfunction on $\xi_1$. If we were to have a sampling theorem on $SG_3$ analogous to the one proved in section 4, then the average values of Neumann eigenfunctions with low frequency on $\zeta_m$ must be eigenfunctions on $\xi_m$. Consider the case where $m = 1$. We can attempt to compute a Laplacian matrix that satisfies the eigenfunction condition. However, the computations yield that such a matrix must be a scalar multiple of the identity, which is clearly absurd.

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