Abstract

Consider a bandit learning environment. We demonstrate that popular learning algorithms such as Upper Confidence Band (UCB) and $\varepsilon$-Greedy exhibit risk aversion: when presented with two arms of the same expectation, but different variance, the algorithms tend to not choose the riskier, i.e. higher variance, arm. We prove that $\varepsilon$-Greedy chooses the risky arm with probability tending to 0 when faced with a deterministic and a Rademacher-distributed arm. We show experimentally that UCB also shows risk-averse behavior, and that risk aversion is present persistently in early rounds of learning even if the riskier arm has a slightly higher expectation. We calibrate our model to a recommendation system and show that algorithmic risk aversion can decrease consumer surplus and increase homogeneity. We discuss several extensions to other bandit algorithms, reinforcement learning, and investigate the impacts of algorithmic risk aversion for decision theory.

1 Introduction

Online learning algorithms are used widely in the digital economy—their applications range from recommendation systems to trading algorithms. These algorithms learn about the economic environment while interacting within it, making them a useful tool to deploy in settings of incomplete information. As they are used in more and more settings of economic interest, it becomes important to understand the economic implications of their use. For example, Calvano et al. (2020) show that...
pricing algorithms can learn to collude without being explicitly told to do so, achieving close to perfect collusion.

We are interested in the behavioral implications of using particular learning algorithms, primarily in terms of inherent risk preferences. To be more precise, take the simple example of providing the learning algorithm with two options—either pull lever A and get a certain payoff of 0, or pull lever B and get a stochastic payoff of either 1 or −1, distributed uniformly. At any point in time, contingent on past observations of payoffs from pulled action (the bandit setting), an algorithm specifies a distribution on actions it takes next. What risk attitude does this implied choice function have? Typical notions of bandit convergence in terms of regret do not make predictions in cases where algorithms need to choose among options of the same expected action, or in early rounds of learning, when regret guarantees do not give strong guarantees yet. In these cases, other features of algorithms might determine how algorithms choose among actions.

These questions have an increasing range of implications today due to the importance of bandit algorithms, for example in recommendation systems. The analysis for these systems is asymptotic, but transient behaviour is predominant in a world where data is big and changes quickly—the amount of content that a user can engage with is increasing, which raises questions on whether asymptotic rates of convergence can ensure intended behavior by algorithms.

A mismatch between intended behavior of an algorithm can have real world consequences. As mechanism design Maskin and Riley (1984) shows, risk aversion can significantly impact the terms of business that are determined. A preference for safer, lower value options when riskier but slightly higher value options are available can introduce inefficiency in platform design.

Our contributions are threefold. First, we introduce a relevant definition of risk aversion for bandit algorithms. Our analysis of risk aversion combines theory with empirics.

Second, we establish theoretically that the $\varepsilon$-Greedy algorithm chooses the less risky arm with high probability when the arms have equal expected value. The tractability of this analysis relies on the fact that $\varepsilon$-Greedy is an index policy with expected value for each of the actions being the corresponding index. This structure allows us to describe the learning dynamics by a one-dimensional random walk, whose asymptotic behavior is well-understood. Many other algorithms also use an index (usually reward estimates for actions) such as EXP3, Thompson Sampling and Upper Confidence Band algorithms (compare Lattimore and Szepesvári 2020), and we expect
that our approach can be suitably generalized to analyze this class of algorithms.

Finally, we complement our theoretical analysis by running simulations in practically relevant settings, such as the recommendation system environment, for widely studied algorithms such as UCB. We also discuss the economic import of our results, in particular towards a decision theory for algorithms.

1.1 Main Intuition for the Results

The main intuition for our results is that many algorithms undersample actions for which they received low rewards. Riskier actions that get a low reward are “trapped” in pessimistic estimates of reward. This leads to a behaviour consistent with risk aversion over long time spans of learning. This is represented in Figure [1]. In the region of advantage for the risky arm above the x-axis, the estimate moves around more, and since expected values are the same, the advantage can quickly dissipate and becoming negative, at which point the advantage is updated less often, which means that the risky arm is undersampled. This can persist for quite long, and for perfectly risky algorithms such as the ε-Greedy, it is the dominant effect.
2 Related Work

Algorithmic collusion: A closely related strand of literature studies algorithmic collusion, for example in Calvano et al. (2020); Brown and MacKay (2021); Asker et al. (2021); Hansen et al. (2021) show that algorithms can learn to charge supracompetitive prices, and even learn punishment strategies that enforce these prices in equilibrium. Hansen et al. (2021) show that misspecified algorithms can lead to higher prices because they overestimate their own price sensitivity. Our analysis of risk preferences of algorithms is motivated by similar implicit behavioral implications of using algorithms.

Algorithmic confounding: The literature on algorithmic confounding, for example in Chaney et al. (2018), shows that recommendation systems trained on data from users already exposed to recommendation systems can increase homogeneity and decrease utility. In essence, the algorithms fail to take into account that their data reflects both user preferences and what the users were shown by the system. This leads it to homogenize towards popular options, which can be interpreted as the algorithm deciding in favour of “safer” options. We give a definition of risk aversion which is independent of internal reward estimates and hence confounding. This means our definition is more broadly applicable.

Exploration-Exploitation Tradeoffs: Our results also relate to work on the explore vs exploit tradeoff (see Auer et al. (2002)). Algorithms which explore less can get stuck for longer in a bad reputation phase, hence exhibiting more aversion to risk. However, our analysis shows that risk aversion is a more foundational property of how the algorithm makes choices, so making an algorithm explore more cannot completely resolve its risk attitude.

Strategic exploration: Bandits exploring together can have strategic implications, as shown by Bolton and Harris (1999). Agents may free-ride in equilibrium, leading to lower levels of exploration. This is related to the literature on incentivizing exploration, for example in Sellke and Slivkins (2021). Agents can be incentivized to explore more by controlling the rate at which they get information.
3 Model

A decision maker repeatedly takes one of \( k \) actions, which give her a stochastic payoff sampled identically and independently distributed from distributions \( F_i \in \Delta(\mathbb{R}) \), \( i \in [k] \). The \textit{strategy} or \textit{algorithm} used by the decision maker can be abstractly represented by a function \( \pi : ([k] \times [0,1])^* \rightarrow \Delta([k]) \), which we also call \textit{policy}. Different algorithms imply different \( \pi \).

A general algorithm takes the following form. For each \( t \in \mathbb{N} \), repeatedly, she chooses an action \( A_t \sim \pi(A_1, r_1, A_2, r_2, \ldots, A_{t-1}, r_{t-1}) \) and gets a reward \( r_t \sim F_{A_t} \).

For the sake of exposition, we consider two algorithms primarily in this article.

\textbf{Example} (\( \epsilon \)-Greedy). The \( \epsilon \)-Greedy algorithm chooses the empirically best action with probability \( 1 - \epsilon \), and randomizes between all the actions with probability \( \epsilon \). Thus the strategy function can be written as:

\[
\pi_i(A_1, r_1, A_2, r_2, \ldots, A_{t-1}, r_{t-1}) = \begin{cases} 
\frac{1-\epsilon}{\left| \arg \max_i \sum_{t,A_t=i} r_t \right|} & \text{if } i \in \arg \max_i \frac{1}{\left| \left\{ t \mid A_t=i \right\} \right|} \sum_{t,A_t=i} r_t \\
\epsilon & \text{otherwise}.
\end{cases}
\]

\textbf{Example} (UCB). The Upper Confidence Band (UCB) algorithm derives an “optimistic” estimate (the upper limit of a confidence band) of the mean from the empirical mean, and then maximizes this estimate. Given that \( T_i(t-1) \) samples have been observed with empirical mean \( \hat{\mu}_i(t-1) \) from action \( i \), the estimate is:

\[
\text{UCB}_i(t-1, \delta) = \begin{cases} 
\infty & \text{if } T_i(t-1) = 0 \\
\hat{\mu}_i(t-1) + \sqrt{\frac{2\log(2)}{T_i(t-1)}} & \text{otherwise}.
\end{cases}
\]

The associated policy function is:

\[
\pi_i(A_1, r_1, A_2, r_2, \ldots, A_{t-1}, r_{t-1}) = \begin{cases} 
\frac{1}{\left| \arg \max_i \text{UCB}_i \right|} & \text{if } i \in \arg \max_i \text{UCB}_i(t-1, \delta) \\
0 & \text{otherwise}.
\end{cases}
\]
We will also denote by the *Gittins algorithm* the theoretically optimal policy $\pi$ maximizing
\[
\sum_{t=1}^{\infty} \delta^t r_t.
\]

We will consider at several instances 2-armed bandits and call the arms $n$ (for *non-risky*) and $r$ (for *risky*). In this case, $[k] = \{n, r\}$.

The main quantities of interest is the last-iterate probability of choosing the non-risky arm
\[
P[A_t = n].
\]
The probability is taken with respect to randomness in both reward and the algorithm, if the algorithm is randomized.

## 4 Cautious Algorithms

In this section, we show theoretically that the $\varepsilon$-Greedy algorithm tends to choose less risky arms.

**Theorem 1.** For any exploration rate $(\varepsilon_t)_{t \in \mathbb{N}}$ such that $\varepsilon_t \to 0$ and $\sum_{t=0}^{T} \varepsilon_t \to \infty$, $P[A_t = n] \to 1$.

We note that this convergence result implies the same convergence for the expected number of times the non-risky arm is chosen,
\[
\frac{1}{T} \mathbb{E} [\{t | A_t = n\}] \to 1.
\]

**Proof.** First, observe that $\varepsilon$-Greedy can be written as a stochastic process of a particularly simple form if it uses arms $n, r$:
\[
\arg \max_{i \in [k]} \frac{1}{\{t | A_t = i\}} \sum_{t: A_t = i} r_i = \begin{cases} 
\{r\} & \text{if } \sum_{t: A_t = i} r_i > 0 \\
\{n\} & \text{if } \sum_{t: A_t = i} r_i < 0 \\
\{r, n\} & \text{if } \sum_{t: A_t = i} r_i = 0
\end{cases}
\]

Also, note that $\sum_{t: A_t = i} r_i = \sum_{t=1}^{T} r_t$. Hence, the quantities $\sum_{t=1}^{T} r_t$ are sufficient as a state for the computation of $\varepsilon_t$-Greedy.
Next, consider the transition distribution of \( X_T = \sum_{t=1}^{T} r_t \). The transition distribution of \((X_t)_{t \in \mathbb{N}}\) is

\[
X_0 = 0 \\
X_{t+1} = \begin{cases} 
X_t & \text{w.p. } \frac{\varepsilon_t}{2} + (1 - \varepsilon_t)(1_{X_t < 0} + \frac{1}{2}1_{X_t = 0}) \\
X_t + x_t & \text{w.p. } \frac{\varepsilon_t}{2} + (1 - \varepsilon_t)(1_{X_t > 0} + \frac{1}{2}1_{X_t = 0})
\end{cases}
\]

where \( x \sim \text{Rademacher} \) independently across time. We will call the process \((X_t)_{t \in \mathbb{N}}\) “the” lazy random walk. We first observe that the probability that this process is positive is related to \( \varepsilon \)-Greedy choosing the non-risky arm.

**Claim 1.** \( \mathbb{P}[X_t \leq 0] \to 1 \text{ as } t \to \infty \implies \mathbb{P}[A_t = n] \to 1 \).

**Proof.** First, observe that as \( \sum_{t=0}^{T} \varepsilon_t \to \infty \), the lazy random walk steps infinitely often almost surely, which means that \( \mathbb{P}[X_t = 0] \to 0 \).

Furthermore, note that \( P[A_t = n|X_t < 0] \geq 1 - \varepsilon_t/2 \). Thus, \( P[A_t = n] \geq P[A_t = n, X_t < 0] = P[A_t = n|X_t \leq 0]P[X_t < 0] \geq (1 - \varepsilon_t/2)P[X_t < 0] \). Since \( \varepsilon_t \to 0 \) and \( P[X_t < 0] - P[X_t \leq 0] \to 0 \), the claim follows. \( \square \)

It is hence sufficient to show \( \mathbb{P}[X_t > 0] \to 0 \).

Define the time since the last passing time of zero as \( \tau^T_0 := \max\{t \leq T | X_t = 0\} \). Define \( S'_0 := \sum_{t' = \tau^T_0}^{t} S_{t'} \), where \( S_{t'} \sim \text{Bernoulli}(1 - \frac{1}{2} \varepsilon_{t'}) \) the number of times the lazy random walk steps if it is positive.

Let \((H_t)_{t \in \mathbb{N}}\) be a standard random walk.

**Claim 2.** \( \mathbb{P}[X_t > 0] = \mathbb{P}[H_{t'} > 0, t' = 1, 2, \ldots, S'_0] \).

**Proof.** We have that

\[
X_t > 0 \iff X_{t'} > 0, t' = \tau^T_0 + 1, \tau^T_0 + 2, \ldots, t \\
\iff H_{t'} > 0, t' = 1, 2, \ldots, S'_0.
\]

The first line comes from the definition of \( \tau^T_0 \). For the second line, note that \( X_{t'} > 0 \) implies that it steps \( S'_0 \) times from \( t' = \tau^T_0 \) to \( t' = t \). This is because the risky arm is favored in this region and
hence the safe arm is chosen with $\frac{1}{2}\varepsilon_t'$ probability, which is when the process does not step. This is equivalent to a standard random walk remaining above 0 for $S_t^t$ periods.

**Claim 3.** $t - \tau_t^0 \xrightarrow{P} T_{t\to\infty} \infty$

**Proof.** We would like to show:

$$\forall c > 0, \delta > 0 : \exists t \in \mathbb{N} : t \geq t : P[t - \tau_t^0 \leq c] \leq \delta.$$

Fix any $c$ and any $\delta > 0$. Then:

$$P[t - \tau_t^0 \leq c] = P[\exists t' \in \{t - c, t - c + 1, ..., t\} : X_{t'} = 0]$$

$$\leq \sum_{t' = t - c}^{t} P[X_{t'} = 0]$$

$$\leq \sum_{t' = t - c}^{t} \left\{ P\left[X_{t'} = 0 \mid \{s \in [\tau_t^0, t'] \mid X_{s+1} - X_s \neq 0\}\geq \kappa\right] + P[\{|s \in [\tau_t^0, t']|X_{s+1} - X_s \neq 0\}| < \kappa] \right\}$$

$$\leq c \max_{l \in [\kappa, T]} \left( \frac{l}{\kappa} \right) 2^{-l}$$

$$+ cP\left[ \exists m \geq \frac{t - \tau_t^0}{\kappa} \land n \in [\tau_t^0, t - m] \mid X_n = X_{n+1} = ... = X_{n+m} \right]$$

For the first inequality, we use the fact that the probability can be split into two, one conditioning on an event $A$ and the other conditioning on its complement $A^c$, and then replace the latter with the probability of $A^c$. For the second, the first term just replaces each term in the sum with the largest element of the sum. The second term uses the pigeonhole principle, since the event that $X_t$ steps at most $\kappa$ times for $t - \tau_t^0$ periods is the same as saying that there is at least one continuous sequence of length $\frac{t - \tau_t^0}{\kappa}$ that does not step.

By Stirling’s approximation, the first term is approximately $c \max_{l \in [\kappa, T]} \frac{1}{\sqrt{l\pi}} = \frac{c}{\sqrt{\kappa\pi}} \leq \frac{c}{\sqrt{(t-c)\pi}} \rightarrow 0$.

For the second term,
\[ c \mathbb{P} \left[ \exists m \geq \frac{t - \tau_0}{\kappa} \wedge n \in [\tau_0, t - m] \mid X_n = X_{n+1} = \ldots = X_{n+m} \right] \]
\[ \leq c \prod_{s' \in \{s, s+1, \ldots, s+\frac{1}{\kappa}\}} (1 - \varepsilon_{s'}) \]
\[ \leq c \exp \sum_{s' \in \{s, s+1, \ldots, s+\frac{1}{\kappa}\}} \varepsilon_{s'} \]
which goes to zero given that \( \sum_{t} \varepsilon_{t} = \infty \). \( \square \)

**Claim 4.** \( S_0^t \xrightarrow{P_{t \to \infty}} \infty \).

**Proof.** Fix \( c, \delta > 0 \). By Claim 3 we can choose \( t' \) such that for any \( t \geq t, t - \tau_0^t > 2c \) with probability at least \( \delta/2 \). Choose \( t \) large enough such that \( \varepsilon_{t''}/2 \leq \kappa := 2\sqrt{\delta/(2c)} \) for \( t' \geq t - 2c \), which is possible as \( \varepsilon_t \to 0 \). Then, as at most \( c \) zero draws of the Bernoulli random variable need to happen between \( t - c \) and \( t \), whose probability is bounded by \( \kappa \), we can bound
\[
\mathbb{P}[S_0^t \leq c] \leq \delta/2 + \mathbb{P}[S_0^t \leq c | t - \tau_0^t > 2c] \]
\[ \leq \delta/2 + \left(\frac{2c}{c}\right)^\kappa \]
\[ = \delta/2 + \left(\frac{2c}{c}\right)^\kappa \]
\[ = \delta. \]
This concludes the proof. \( \square \)

**Claim 5.** For any \( x \in \mathbb{N}_{\geq 0} \)
\[
\mathbb{P}[H_t = y, H_{t'} > 0, t' = 1, 2, \ldots, t] = \frac{y \mathbb{P}[|H_t| = y]}{t}. 
\]

and therefore
\[
\mathbb{P}[H_{t'} > 0, t' = 1, 2, \ldots, t] = \frac{\mathbb{E}[|H_t|]}{t}. 
\]

**Proof.** Suppose that \( H_t = y > 0 \). Let \( N_t(x, y) \) be the number of ways to get from \((0, x)\) to \((t, y)\). Note that the event \( E = \{H_t = y, H_{t'} > 0, t' = 1, 2, \ldots, t\} \) has happened iff the random walk
stays on the same side of 0 in the interval \([1, t]\). Let \(N\) denote the number of ways to do this, and \(\pi = \mathbb{P}(E|S_t = y)\). Then \(\pi = \frac{N}{N_t(0, y)}\), but also \(\pi = \frac{y}{t}\) by the Reflection Principle. As a result, the total number of ways is \(\frac{y}{t} N_t(0, y)\), and each has \(\frac{1}{2}(t + y)\) rightward steps and \(\frac{1}{2}(t - y)\) leftward steps. Therefore

\[
\mathbb{P}[H_t = y, H_{t'} > 0, t' = 1, 2, \ldots, t] = \frac{y}{t} N_t(0, y) p_{\frac{1}{2}}(t + y) q_{\frac{1}{2}}(t - y) = \frac{y\mathbb{P}[|H_t| = y]}{t}
\]

Summing over \(y\) gives the second equation.

The asymptotics of the absolute value of a random walk are well understood:

\[
\lim_{t \to \infty} \frac{\mathbb{E}[|H_t|]}{\sqrt{t}} = \sqrt{\frac{2}{\pi}}.
\]

See, e.g., [Weisstein (2002)](https://www.wolframalpha.com/input/?i=absolute+value+of+a+random+walk), and references therein. This implies for large enough \(t\) that

**Claim 6.** There is a constant \(C > 0\) such that \(\mathbb{P}[H_{t'} > 0, t' = 1, 2, \ldots, t] \leq Ct^{-\frac{1}{2}}\).

Let \(\delta > 0\). Set \(c := \frac{\sqrt{2}}{\sqrt{\delta}}\) and \(\delta' := \frac{\delta}{2}\). We find that with probability at least \(1 - \delta' = \frac{\delta}{2}\), \(S_0^t > c = (2/\varepsilon)^{1/2}\). In particular,

\[
\mathbb{P}[X_t > 0] \leq \mathbb{P}[t - \tau_0^t - S_0^t \leq (2/\varepsilon)^{1/2}] + \mathbb{P}[t - \tau_0^t - S_0^t > (2/\varepsilon)^{1/2}] \mathbb{E}[X_t \leq 0|t - \tau_0^t - S_0^t > (2/\varepsilon)^{1/2}]
\]

\[
\leq \frac{\varepsilon}{2} + (1 - \varepsilon) \mathbb{E}[H_t - \tau_0^t - S_0^t \leq 0|t - \tau_0^t - S_0^t > (2/\varepsilon)^{1/2}]
\]

\[
= \frac{\varepsilon}{2} + (1 - \varepsilon) \mathbb{E}[H_1, H_2, \ldots, H_t - \tau_0^t - S_0^t \leq 0|t - \tau_0^t - S_0^t > (2/\varepsilon)^{1/2}]
\]

\[
\leq \frac{\varepsilon}{2} + (1 - \varepsilon) \mathbb{E}[H_1, H_2, \ldots, H_{\lfloor(2/\varepsilon)^{1/2}\rfloor} \leq 0]
\]

\[
\leq \frac{\varepsilon}{2} + \mathbb{E}[H_1, H_2, \ldots, H_{\lfloor(2/\varepsilon)^{1/2}\rfloor} \leq 0]
\]

\[
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This demonstrates convergence.

Several comments are in order:

First, the theorem only considers a 2-armed bandit with one deterministic and one Rademacher-distributed arm. The extension to a non-deterministic arm with a more general distributions is possible but requires different techniques, in particular in the application of the reflection principle.
On the other hand, the proof does not straightforwardly generalize to two ordered arms, e.g. in an order of second-order stochastic dominance, or for normally distributed reward distributions with identical expectation, but different standard deviation. The reason for this is that the proof technique relies on the fact that the unnormalized sum of rewards is a sufficient state for the algorithm.

Failure modes of the theorem play into when the algorithm explores too much or too little. If $\varepsilon_T$ does not go to zero, full risk aversion is impossible. However, also the difference in the “laziness” identified in the algorithm becomes not arbitrarily strong. Hence, large exploration making risk-aversion weaker. Too little exploration might lead to cases, for example, where the algorithm commits to an arm after a finite time.

Still, the reduction to a random walk gives clear intuition for algorithms that rely on estimates of action quality. The estimate of the advantage of the risky arm is a lazy random walk which is “lazier” when the non-risky arm has advantage. This abstract quality can be observed in more complex environments, which we discuss in subsection 6.2.

5 Experiments

We run simulations to support our theory.

First, we consider convergence rates of two popular bandit algorithms facing a deterministic and a stochastic arm. Then, we calibrate an experiment to a recommendation system application, and derive economic inefficiencies arising in this environment.

Our experiments use $\varepsilon$-Greedy (with exploration rate $t^{-\frac{1}{2}}$) and UCB (with $\delta$ parameter growing at $\log(1 + t \log^2(t))$).

5.1 Synthetic data

First, we illustrate the risk aversion of $\varepsilon$-Greedy and UCB in a simple bandit setting. In our experiments, we use a deterministic reward of 0 and a Rademacher distributed reward. We estimate the mean probability of choosing a particular arm by averaging 1,000 runs of $\varepsilon$-Greedy and UCB for rounds $t = 1$ to 1,000 rounds. We use a Savitzky-Golay filter to smoothen the outcomes. The results are represented in Figure 2.
As established by our theoretical result, $\epsilon$-Greedy behaves perfectly risk averse, choosing the risky arm with a small and decreasing probability. We find that UCB has an imperfect risk aversion, plateauing at a choice of about 46% for the risky arm.

Second, we show that this risk behaviour exists transiently even when the arms do not have the same expected reward. We consider biases favoring the risky arm from $b = 0.1$ to $1$. This means that we are comparing an arm with deterministic reward $-b$ to a Rademacher distributed arm. Figures 3 and 4 show the results. For small biases, risk aversion can persist for quite a large number of time periods. For bigger biases, the effect is small, as we would expect.

Figures 3 and 4 also give us an idea of the certainty equivalents for the algorithms for different times. The heatmaps identify the probability of choosing the risky arm for each pair $(b, t)$, except at $(b, t)$ such that $\mathbb{P}[A_t = r] \approx \frac{1}{2}$, where that color is black. At such $(b, t)$, the algorithm is indifferent between the two arms, so that the bias at that specific time period corresponds to a notion of certainty equivalent for the algorithm for that time period. As can be seen from the heatmap, lower time periods correspond to a higher certainty equivalent, while as time progresses the certainty equivalent decreases towards zero, as would be predicted by no regret. But the certainty equivalent is persistently positive, which is what our theory of risk aversion predicts.
Figure 3: $\epsilon$-Greedy with bias

Figure 4: UCB with bias
5.2 Calibration to a recommendation system

A major application of bandit algorithms is in recommendation systems, for example on streaming sites where they make content recommendations based on user profile and content desirability. Since deployed recommendation systems and their data are proprietary, we simulate a simple recommendation system that illustrates the points that we would like to illustrate. The system faces a large number of users, each of whom interact with the system for several time periods. In our main run, we choose that they interact for $T = 10,000$, which is a conservative upper bound on the pieces of content that, for example, a user in spotify can interact with.\footnote{Considering data from 2020\cite{houl2020}, users on netflix watch 10 movies and 2 series per month. Assuming that each was selected from a roster of 100 pieces of content, this leads to about 1,000 pairwise comparisons.} We consider two types of content, which we call $m$ movies and $s$ series. The system maximizes a continuous-valued measure of content consumption, which is heterogenous among users.

We consider users that have a personal preference for a type of content, $x_{i,j} \sim N(0, 1)$ independently across users and content types. The system gets a feedback $x_{i,j} + \epsilon_{i,j,t}$ from serving content $j$ to user $i$ in time period $t$. Users also have a person-and-content specific content preference, which we assume to be independently $N(0, \text{diag}(\sigma_s, \sigma_m))$-distributed, where $\sigma_s, \sigma_m > 0$. We operationalize that series give a more divisible and faster feedback by assuming that $\sigma_s < \sigma_m$. Hence, movies are riskier content than series.

This can be expanded to a random utility function

$$u(j; i) = x_{i,j} + \epsilon_{i,j}$$

where agents are assumed to choose content of type $m$ if and only if $u(m; i) > u(s; i)$.

We run our simulations for $T = 10,000$ rounds, and estimate population demand for 1,000 users. Our results are shown in Figure 5. Series, which is the less risky content, is shown 5% more to users than are movies.

5.3 Economic Interpretation

The algorithm’s risk bias can artificially reduce consumer surplus, even if it is not in the interest of the deployer to do so. Since there is no inherent bias in consumer preference for either content in our model, if users are free to make their choice of content, the market shares of each content would
be 0.5. However in our simulations we find that the market share of series content is 10% larger. This means that the algorithm is choosing an outcome misaligned with consumer welfare, which is maximized when market share is in expectation the same across both content. This consumer surplus effect can be even larger in the long term if dissatisfied consumers were to leave the service, especially in extremely concentrated industries such as the streaming industry. Our calibration is not rich enough to determine the effects on consumer surplus of such selection, but we view this as an attractive area for future research.

This also raises question of algorithmic fairness. The algorithm’s risk bias is skewing the market share towards content that it has an inherent bias for, i.e. it is creating its own market rather than catering to the existing market. As with other questions tackled by the field of algorithmic fairness, this bias can perpetuate one section of society and culture at the expense of others. If it is the case that less risky content is also culturally dominant and over-represented, then over time, the fact that the algorithm prefers one content over the other can lead to exit by agents who consume the marginalized content, or they could even be forced to modify their tastes to match the algorithm’s preference. In either case, this would reinforce the algorithm’s risk bias and lead to an even larger bias towards the less risky content. In the long run, this can artificially create and perpetuate homogeneity amongst users, often at the cost of marginalized sections.
6 Extensions

6.1 Decision theory for Algorithms

It is natural to think about algorithm behaviour directly through preferences first rather than risk attitudes, since the former are more fundamental in standard theory. However, any attempt to formulate a decision theory for algorithms runs into immediate complications. Consider two lotteries $l_1 = 0.21_{(1)} + 0.61_{(0)} + 0.21_{(-1)}$ and $l_2 = 0.21_{(2)} + 0.61_{(0)} + 0.21_{(-2)}$. Clearly $l_2 >_{SOSD} l_1$, but most algorithms like $\epsilon$-Greedy will not differentiate between the two, due to normalization of the reward estimates.

Defining a revealed preference of some sort can in fact run into fundamental contradictions, in particular with transitivity. For example, suppose we define it in the following manner:

**Definition.** We say that an algorithm (strictly) prefers arm $i$ over arm $j$ at time $t$, i.e. $j >_{\pi,t} i$, if

$$\mathbb{P}[A_t = j] > \mathbb{P}[A_t = i]$$

Then we can run into trouble quite quickly:

**Proposition.** When $\pi = \epsilon$-Greedy for small $\epsilon$ and $t = 3$, $>_{\pi,t}$ is intransitive.

**Proof.** Consider three lotteries $l_1 = 1_{(-0.01)}$, $l_2 = 0.511_{(1)} + 0.491_{(-1)}$ and $l_3 = 0.341_{(1)} + 0.331_{(-0.02)} + 0.331_{(-1)}$. It is straightforward to check that with this definition, $l_1 > l_3 > l_2 > l_1$ for $\epsilon$-Greedy when $\epsilon$ is small. □

Moreover, even if it is possible to derive a preference ordering, it could fail to satisfy the implications of standard theories. For example, consider Expected Utility Theory. Independence requires that mixing another lottery to two lotteries shouldn’t change their relative ordering. However mixing a complicated lottery to the comparison between a simple lottery and a fixed reward could make it harder for the algorithm to distinguish quickly between the two, hence making it possible to reverse the ordering between them. As a consequence of these complications we focus on the narrower topic of risk attitudes to draw up a cleaner theory, while a decision theory for algorithms is a valuable area for future research.
6.2 Contextual Bandits and Reinforcement Learning

In this work, we consider bandit algorithms. In many environments with autonomous agents, however, the environment has a richer structure.

First consider a generalization to the contextual bandit environment. In a contextual bandit problem, a policy is given by

$$\pi: ([l] \times [k] \times \mathbb{R})^* \times [l] \to [k].$$

In each round, the agent receives one of $l$ contexts $c_t$ from a distribution $F \in \Delta([l])$. The agent chooses an arm $A_t \in [k]$ and receives a reward $r_t \sim F_{c_t, A_t}$. In this environment, Theorem 1 implies that the contextual bandit algorithm running separate greedy algorithms for each contexts for each context converges to choosing a deterministic over a stochastic arm of the same expectation depending on context. The frequency at which higher-variance actions are taken in expectation over contexts is an empirical measure of risk aversion.

Even more generally, risk aversion can be defined for reinforcement learning. In a simulation-based environment, agents receive a state together with their environment. This is modelled as

$$\pi: ([o] \times [k] \times \mathbb{R})^* \times [s] \to [k]$$

where agents choose an action based on a observation-action-reward history and a new state. The observation in this case is driven by a Markov decision process. Often, the state of the decision process can be returned to a fixed state. In this case, one immediate way to operationalize risk aversion of the learning policy is to consider for a fixed history $h = (o_1, A_1, r_1, o_2, A_2, r_2, \ldots, o_t)$ the probability

$$\mathbb{P}[A_t = r|h]$$

for some action $r$ that is “risky”. One of the most popular policies $\pi$ for this problem is Upper Confidence Trees. For such trees, the same intuition as in our main section will apply: parts of the tree that had a sample with low reward will be undersampled (in the language of upper confidence trees, subtrees are “purged”) and risk aversion is to be expected. We leave the construction of environments to benchmark risk aversion for future work.
6.3 Gittins Index

The Gittins index policy is the foremost Bayesian algorithm, known to be optimal in a wide class of problems. In a Bayesian formulation of our problem, it would in fact address many of the issues we raise in this paper. With a correctly specified prior, i.e. with a prior that has support only among distributions with the same expected value on both arms, it would indeed be truly indifferent between the two arms, hence being risk neutral in the weak sense that we define. Further, even a small bias in favor of the risky arm, with the prior again being correctly specified to have support only on distributions with that bias as the expected value, the policy would more often than not choose the risky arm, hence also being risk neutral in the strong sense. However it is crucial to assume that the prior is exactly specified, since even a vanishing error in prior specification can make the policy behave arbitrarily. The reason many online learning algorithms are deployed is because specifying such priors is hard.

7 Conclusion

We propose a theory of risk aversion of algorithms, defined as a preference for the safer option when faced with two options with the same expected reward. We theoretically show that the $\epsilon$-greedy algorithm is perfectly risk-averse. We also empirically show the risk aversion of the Upper Confidence Band algorithm. We then calibrate to a recommendation system environment and show economic consequences of the identified bias.

There are multiple avenues for future research. Often the environment features multiple bandits non-cooperatively, for example by different firms using different price-setting algorithms. In multi-agent environments, the risk behaviour we identify might lead to additional emergent behaviors. Another direction is the design of risk-neutral bandit algorithms. Our paper includes simulated results to test empirical validity of the theory, but a more in-depth empirical study of deployed recommendation systems online-learners could provide further validation of the theory we propose. Our extension to decision theory and Reinforcement Learning are direct avenues for further theoretical study.
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