Loss of superfluidity in the Bose–Einstein condensate in an optical lattice with cubic and quintic nonlinearity

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Abstract

In a one-dimensional shallow optical lattice, in the presence of both cubic and quintic nonlinearity, a superfluid density wave is identified in a Bose–Einstein condensate. Interestingly, it ceases to exist when only one of these interactions is operative. We predict the loss of superfluidity through a classical dynamical phase transition, where modulational instability leads to the loss of phase coherence. In a certain parameter domain, the competition between lattice potential and the interactions is shown to give rise to a stripe phase, where atoms are confined in finite domains. In a pure two-body case, apart from the known superfluid and insulating phases, a density wave insulating phase is found to exist, possessing two frequency modulations commensurate with the lattice potential.

1. Introduction

The mean field Gross–Pitaevskii equation (GP) [1, 2] is known to capture the ground-state properties of the Bose–Einstein condensate (BEC) remarkably well, both in three- and lower-dimensional configurations [3, 4]. In the presence of an optical lattice [5–8], this picture changes drastically when quantum fluctuations become significant [9]. For deep lattice wells, a fluctuation-driven metal–insulator transition has been experimentally confirmed [10–13]. However, in shallow wells, the quantum fluctuations are expected to be small, making the GP equation useful for identifying ground-state structures.

For the discrete nonlinear Schrödinger equation, Smerzi et al in 2002 [14] predicted a dynamical superfluid insulator transition (DSIT), which is different from the fluctuation-driven quantum phase transition. Modulational instability, occurring in the system, leads to the loss of phase coherence. Hence, the DSIT takes place and the superfluid phase transits to the insulating state. Following this prediction, Cataliotti et al in 2003 [15] reported the experimental observation of the destruction of interference patterns and loss of superfluidity through the DSIT. In the same year, Adhikari demonstrated numerically the loss of phase coherence and superfluidity of a BEC [16] with two-body interaction.

In this paper, we demonstrate the existence of a superfluid density wave state of BEC in an optical lattice, in the presence of both cubic and quintic nonlinear interactions. This phase, with twice the periodicity of the lattice, ceases to exist when only cubic and quintic nonlinearities are present. We have considered the two-body interaction to be repulsive and the quintic interaction to be attractive for physical realization. The existence of a stripe phase is shown, where superfluid matter is found only in finite domains. We predict that the DSIT occurs in this system also, where all the atoms transit from the superfluid phase to an insulating phase, with analogous periodicity. The filling fraction in this case varies inversely with the lattice site \( n \) and reaches an asymptotic constant value. It differs for odd and even lattice positions with respect to a given site, taking a constant value for even \( n \). Interestingly, for odd \( n \), it depends on the depth of the lattice potential, as well as the interaction strengths. In the presence of quintic nonlinearity alone, a sinusoidal excitation is found, which exists only in the superfluid phase of similar periodicity as that of the optical lattice. In all these cases, the dispersion plays a sub-dominant role. In the presence of pure two-body interaction, the superfluid density shows a periodicity commensurate with the lattice. However, a density wave insulating phase is found to exist possessing two frequency...
modulations commensurate with the lattice potential. The dynamical phase transition connects the superfluid phase with an insulating phase [17], which in turn is connected with the density wave insulating phase. The solutions are found to be marginally stable, as per the Vakhitov–Kolokolov (VK) criterion [18].

2. BEC with cubic and quintic nonlinearity

The quintic nonlinearity in one dimension, corresponding to an effective three-body interaction, has appeared in a number of works. Muryshev et al [19] have investigated the dynamical stability and the dissipative dynamics of solitons in BEC, where the quintic nonlinearity arises in one dimension due to the interaction between axial and radial degrees of freedom. In [20], the cubic GP equation is used in one dimension when the coupling constant is small, whereas quintic interaction appears in the Tonks–Girardeau regime. The localization of the ground state of BEC in optical lattice and the stability of BEC with quintic nonlinearity have been analyzed by Abdullaev et al [21, 22]. Kolomeisky et al took a different approach and derived the GP functional with cubic and quintic nonlinearity for lower dimensions, using the renormalization group method [23, 24].

The quasi-1D GP equation in an optical lattice, in the presence of both cubic and quintic nonlinearity, takes the form [19–22, 25]

\[ i\psi_t = -\frac{g_1}{2} \psi_{zz} + (g_1 |\psi|^2 + g_2 |\psi|^4) + V(z) - \mu)\psi, \tag{1} \]

where \( V(z) = V_0 \cos^2(z) \) is the optical lattice potential, \( g_1 \) and \( g_2 \) are the appropriately normalized strength of the cubic and quintic nonlinearity, respectively [26]. Since both cubic and quintic terms are functions of scattering length [25], we describe the effect of the interactions in terms of the ratio between cubic and quintic nonlinearity \( \kappa = g_1 / g_2 \). The lattice and the chemical potentials are normalized in terms of the recoil energy \( E_r \). The spatial coordinate and the wavefunction are scaled in the units of wavelength of incident laser light and \( \sqrt{\kappa} \), respectively; here \( k \) is the wave vector.

The ansatz solution of equation (1), \( \psi(z,t) = \sqrt{\sigma(z)} \exp(i\chi(z) - i\omega_t) \), leads to a superfluid phase (SF):

\[ \psi_{SF} = \left[ \frac{g_1}{2g_2} \pm \sqrt{\frac{V_0}{g_2} \cos z} \right] e^{i\chi(z) - i\omega t}, \tag{2} \]

where \( \chi(z) = c \tan^{-1} \left[ \frac{g_1 / 2g_2 - \sqrt{\frac{V_0}{g_2} \cos z}}{g_1 / 2g_2 + \sqrt{\frac{V_0}{g_2} \cos z}} \right] \), \( \omega = \frac{1}{8} (1 - \frac{g_1^2}{g_2^2} - 8\mu) \) and \( c = \pm \frac{1}{4g_2} \sqrt{g_1^2 + 4g_2V_0} \). The periodicity of the density is twice that of the lattice potential, which indicates matter redistribution. Atoms from neighbouring sites have been depleted leading to a density wave behaviour. As one can see from the expression, the phase vanishes where the superfluid density wave attains its maximum value. Since the superfluid is characterized by a definite phase, the number of atoms on each lattice site is unknown, which allows the atom to move freely and it can easily tunnel from one lattice site to another.

![Figure 1. The behaviour of the filling fraction when both cubic and quintic nonlinear interactions are operative. n is the lattice location with respect to a given site. The solid curve shows the filling fraction for the superfluid phase and the dotted curve is for the insulating phase with \( V_0 = 0.002 \) (unit of recoil energy) and \( \kappa = 5 \). (This figure is in colour only in the electronic version)](image)

Denoting \( \frac{g_1}{2g_2} \sqrt{\frac{V_0}{g_2}} \) by \( p \), the solution exists only in the following regimes, when the quintic nonlinear interaction and the lattice potential have opposite signature:

(i) if \( \cos z > p \), then:

(a) The solution exists for all \( z \) for \( p < -1 \), which indicates that the cubic and quintic nonlinearities are of opposite signature.

(b) The solution does not exist for \( p > 1 \).

(c) In the domain where \( -1 < p < 1 \), the solution exists for \( -\pi/2 < z < z_j < \pi/2 \), where \( z_j = \cos^{-1}(p) \). This domain corresponds to a stripe phase, where superfluid matter is found only in finite domains.

(ii) if \( \cos z < -p \), then:

(a) The solution does not exist for \( p > 1 \).

(b) The solution exists for all \( z \) for \( p < -1 \). Therefore, the cubic and quintic nonlinearities are of opposite signature.

(c) When \( -1 < p < 1 \), the solution exists for \( -\pi/2 < z < -z_j < \pi/2 \), where \( z_j = \cos^{-1}(p) \). This also corresponds to a stripe phase.

The average number of atoms per lattice site, the filling fraction, is found to be

\[ \nu_{SF} = \frac{1}{n\pi} \int_{-\pi/2}^{\pi/2} |\psi|^2 dz \]

\[ = \sqrt{\frac{V_0}{g_2}} \left[ -p + \frac{1}{n\pi} (\cos(n\pi) - 1) \right]. \tag{3} \]

Starting from a given lattice point, \( \nu_{SF} \) depends on the lattice location \( n \), as depicted in figure 1. If \( n \) is odd, the filling fraction is a function of the lattice potential and varies inversely with the lattice site number \( n \). If \( n \) is even, the filling fraction takes a constant value depending only on the interactions. The different behaviour of the filling fraction at the neighbouring sites can be traced to the presence of the \( \cos(z) \) term in the density. The odd–even effect arises due to the interplay between the cubic and quintic nonlinear interactions, where a density wave is present. As will be seen
below, for the pure cubic or quintic nonlinearity, this type of behaviour is absent.

The energy of the superfluid component is given by

\[ E_{SF} = \frac{\pi g_1}{6g_2^2} \left[ g_1^2 - 3g_2(2 + V_0 - 2\mu) \right] + 2\pi \left( \frac{1}{2g_2} \sqrt{g_1^2 + 4g_2V_0 - \frac{2g_2^2}{g_1^2 + 4g_2V_0}} \right), \]

which possesses a branch cut at \( V_0 = -\frac{a}{4g} \), where the supercurrent \( J = c \) vanishes. At this value, the phase coherence is lost and the superfluid phase transits into an insulator (I) phase, which is driven by a modulational instability \[14\]. The insulating wavefunctions with identical periodicity as that of the SF phase have the following form:

\[ \psi_{I}(z, t) = \sqrt{\frac{g_1}{g_2}} \cos \frac{z}{2} e^{-i\omega t}, \]

\[ \psi_{I}(z, t) = \sqrt{\frac{g_1}{g_2}} \sin \frac{z}{2} e^{-i\omega t}, \]

with the corresponding energy \( E_I = \frac{\pi g_1}{6g_2^2} \left[ g_1^2 - 3g_2(2 + V_0 - 2\mu) \right] \). In this case, the phase is arbitrary but the number of atoms on each lattice site is fixed. Thus, the atoms stop tunnelling between the lattice sites and the superfluidity is lost. The filling fraction, in this case, is found to be \( v_f = \frac{g_1}{2g_2} \left[ 1 \pm \cos(\pi + \frac{\mu}{g}) \right] \), which is independent of the lattice potential and shows modulations similar to the superfluid phase.

3. BEC with quintic nonlinearity

For tight transverse confinement, strong interactions and low densities, the quintic nonlinear interaction term dominates the GP equation. In this case, we obtain only a superfluid phase with same periodicity of the lattice potential: \( \psi(z, t) = \sqrt{|\sigma_1|} e^{i\theta(z)} \), where \( \sigma_1 = \frac{(-V_0/g_2)^{1/2}}{} \cos \frac{z}{2} \) and \( \omega_1 = \frac{\pi}{g} - \mu \). Interestingly, this superfluid phase does not have any background term. The amplitude of the lattice potential and the three-body interaction strength need to be of opposite sign for the solution to exist. The supercurrent is found to be space-time independent: \( J = \sigma_1 v_f = c_1 \), where \( c_1 = \frac{V_0}{g_2^2} \).

Here, \( v_f \) is the superfluid velocity: \( v_f = \frac{\omega_1}{\omega_2} \). The filling fraction is found to be a constant: \( v_f = \frac{\pi}{2} (-V_0/g_2)^{1/2} \).

4. BEC with cubic nonlinearity

The GP equation, in the presence of pure two-body interactions \( g_1 = g \), is known to exhibit trigonometric solutions of the form \[17\]

\[ \psi(z, t) = \sqrt{\frac{g_1}{g_2}} \cos^{\alpha}(z) e^{i\theta(z)} \]

where the background \( a \) is a free parameter. The non-trivial phase \( \chi_2(z) = \frac{a_2 \tan^{-1}(z) - \frac{1}{2} \tan(z)}{\sqrt{a_2 - \frac{1}{2}}} \), \( a_2 = \frac{1}{2} - \mu + ga \) and the integration constant \( c_2 = \frac{1}{2} (1 - 2a - 2\omega_1) (1 - 2a - 2\omega_2) g^2 \). The average atom number density per lattice site is found to be related to the background \( a' \): \( \nu_{SF} = a - \frac{V_0}{g_2} \).

Here the filling fraction is independent of the number of lattice site \( n \). The non-trivial phase \( (c_2 \neq 0) \) indicates the presence of a superfluid component, with a constant flow density \( J_2 = c_2 \). The potential, density and the phase have the same periodicity in this case, which is very different from the previous cases, with two- and three-body interactions. The ground-state energy of the superfluid state,

\[ E_{SF} = \frac{\pi g}{8} \left( \frac{3V^2}{g^2} - \frac{8aV_0}{g} + 8a^2 \right) - \frac{\pi g_1}{6g_2^2} \left[ g_1^2 - 3g_2(2 + V_0 - 2\mu) \right] + \frac{\pi}{2} \left( \frac{2a - V_0}{g} \right) + \frac{\pi}{2} \sqrt{a(\omega_1^2 - a^2) \left( a - \frac{V_0}{g} \right)} - \frac{\pi}{2} \left( \frac{2a - V_0}{g} \right) + \frac{\pi}{2} \left( \frac{2a - V_0}{g} \right), \]

shows a branch cut at \( a = a_1 = 0 \) and \( a_2 = \frac{V_0}{g_2} \), where \( a_1 \) and \( a_2 \) are the critical values of background. For \( a_1 = 0 \), \( a_2 = \frac{V_0}{g_2} \), and at \( a_2 = \frac{V_0}{g_2} \), the supercurrent \( J_2 = c_2 \) vanishes and the superfluid phase transits to the insulating phase. The energy is continuous; however, the first derivative at the transition point (T) is discontinuous like the previous case, hence, indicating that the phase transition is of first order. The wavefunction in the insulating phase at \( a_1 = T \) is \( \psi(z, t) = \sqrt{\frac{g_1}{g_2}} \cos \frac{z}{2} e^{-i\omega t} \), with the energy \( E^I_1 = \frac{\pi V}{8g} (8\mu - V_0g - 4) \). The corresponding number density is \( v_f^I = \frac{V_0}{g_2} \), indicating that in this insulating phase, the interaction and potential strength have opposite signature. For the second insulating phase at \( a_2 = T \), \( \psi(z, t) = \sqrt{\frac{g_1}{g_2}} \sin \frac{z}{2} e^{-i\omega t} \), with the energy \( E^I_2 = \frac{\pi V}{8g} (5V_0g - 8\mu + 4) \) and the average number density \( v_f^I = \frac{V_0}{g_2} \).

The interaction and potential strength, here, should have the same signature. For a positive \( V_0 \), these two phases respectively belong to the attractive and repulsive regimes. It is worth emphasizing that for a fixed \( V_0 \) and coupling, both these insulating phases only exist for a given number density of atoms. In this case, as well as in the previous cases, interestingly we find that the dispersion affects the density marginally, where it adds a constant term in the phase.

Keeping in mind the possibilities of charge density type of ground states in one dimension, it is natural to investigate if other types of solutions, still commensurate with lattice periodicity, are allowed in this system. We have found such an exact solution exhibiting rational character:

\[ \psi_I(z, t) = \frac{a + b \cos^\delta(z) + c \cos^\delta(z)}{1 + d \cos^\delta(z)} e^{-i\omega t}, \]

with \( \alpha = \beta = 1 \) and \( \delta = 2 \). These solutions do not exist in the superfluid phase. The parameters for this insulating phase are given by

\[ a = \frac{9}{4} \sqrt{\frac{V_0}{g}} + \frac{1}{4} \sqrt{12 - \frac{9V_0}{g}} + \frac{V_0}{g}, \quad b = \sqrt{\frac{V_0}{g}}. \]
\[ c = \pm \frac{V_0}{2} \left(\sqrt{\frac{12}{g} - \frac{9V_0}{g} - 3 \sqrt{-\frac{V_0}{g}}} \right) \quad \text{and} \]
\[ d = \pm \frac{\sqrt{-V_0g}}{2} \left(\sqrt{\frac{12}{g} - \frac{9V_0}{g} + 3 \sqrt{-\frac{V_0}{g}}} \right) \]

with \( \omega_\perp = \frac{\pi}{\lambda} - \mu \). Here, the competition of the lattice potential with the nonlinearity is responsible for the non-perturbative rational solutions with dual frequency character, both commensurate with the lattice potential. It is worth observing that, in the case of solitons, nonlinearity and dispersion compensate each other, leading to stable localized solutions, as well as periodic non-sinusoidal cnoidal waves. Presently, the dispersion affects the character of the solution marginally. For non-singular solutions, \( V_0 \leq -\frac{1}{2} \) in a repulsive interaction regime \( (g > 0) \), and \( V_0 \geq \frac{1}{2} \) in an attractive regime \( (g < 0) \). It should be noted that in contrast to the earlier-found insulating phases, which are independent of the background, the present density wave solution has a background, which depends both on potential and the coupling. We further observe that both the modulations are necessarily present in the presence of the lattice potential.

The energy function is non-analytic as \( d \to 0, \pm 1 \). For \( d = 0 \), the Padé types of solutions representing the insulating phase reduce to the insulating phase exhibiting single frequency behaviour found earlier. In the limit \( d = \pm 1 \), the BEC is pushed gradually to the unstable position on top of the hills of the lattice potential indicating the unphysical nature of this singular point.

5. Experimental realization

In the superfluid phase, the phase coherence between the different sites of the optical lattice in a BEC has been established experimentally through the formation of the interference pattern between the condensates located at the nodes of the laser standing wave. In the insulating phase, the phase coherence is lost, and hence no interference pattern is formed. The phases identified here can be detected in a method used by Cataliotti et al in [15] for a pure two-body interaction. The density wave insulating phase will not show phase coherence; however, unlike the regular insulating phase, the spatial distribution of atoms will show periodic behaviour commensurate with the lattice potential. The stripe phase will show phase coherence between the atoms, where the atoms will occupy only the alternating lattice sites.

To check the parameter regime for the possible existence of these types of density wave and stripe phase, we consider the \(^{87}\text{Rb}\) condensate with \( 10^6 \) number of atoms. The mass of the atom is \( m = 1.44 \times 10^{-25} \text{ kg} \) and the transverse trapping frequency is \( \omega_\perp = 2\pi \times 140 \text{ Hz} \). The s-wave scattering length \( a = 5.4 \text{ nm} \) [26, 27]. The two-body interaction strength is given by \( g_1 = 2a\mu\omega_\perp/\hbar k \). The wave vector \( k = 2\pi/\lambda = 8.06 \times 10^6 \text{ m}^{-1} \). It is worth mentioning that all the above parameters can be changed independently. Larger values of interaction strengths can be achieved by higher density of the condensate, higher \( a \) and for small \( k \). The scattering length can also be tuned by Feshbach resonance. Small \( k \) may be achieved by adjusting the relative angle between the two interfering laser beams. The optical lattice depth is taken as \( V_0 = 0.002 \) (in the units of \( E_r \)), implying that the potential is shallow, where the mean field GP equation is valid. As we increase \( V_0 \), at some point, phase coherence is lost, superfluidity breaks down and there will be no tunnelling of atoms between consecutive lattice sites. Hence, the phenomena discussed in this paper should be observable within the present experimental capability.

6. Conclusion

In summary, a superfluid density wave, having a periodicity twice that of the lattice potential, exists in the presence of both cubic and quintic nonlinear interactions, where the density modulation is over a constant background. A stripe phase is present in a certain parameter regime, where superfluid matter is present in only some domains of the lattice. These phases are absent when only cubic and quintic nonlinear interactions are present. This density wave phase transits to an insulating phase through the DSIT. The phase coherence is lost due to the modulational instability. The stability analysis of the solutions under the VK criterion supports the development of modulational instability in this system. The filling fraction is found to vary inversely with the lattice location from a given site and differs for odd and even \( n \). This different behaviour of the filling fraction in the neighbouring sites can be traced to the presence of both cubic and quintic nonlinearities and is absent when pure cubic or quintic nonlinear interactions are operative. In the pure two-body case, the periodicity of the density is the same as the lattice potential. A density wave insulating phase is observed in this case, having two frequency modulations commensurate with the lattice potential. Explicit calculations show that these types of density wave states and phase transitions can be probed in the present laboratory conditions.

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