Eternal \( k \)-domination on graphs

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April 9, 2021

Abstract

Eternal domination is a dynamic process by which a graph is protected from an infinite sequence of vertex intrusions. In eternal \( k \)-domination, guards initially occupy the vertices of a \( k \)-dominating set. After a vertex is attacked, guards “defend” by each move up to distance \( k \) to form a \( k \)-dominating set containing the attacked vertex. The eternal \( k \)-domination number of a graph is the minimum number of guards needed to defend against any sequence of attacks. The process is well-studied for the \( k = 1 \) situation and we introduce eternal \( k \)-domination for \( k > 1 \).

Determining if a given set is an eternal \( k \)-domination set is in EXP, and in this paper we provide a number of results for paths and cycles, and relate this parameter to graph powers and domination in general. For trees we utilize decomposition arguments to bound the eternal \( k \)-domination numbers, and solve the problem entirely in the case of perfect \( m \)-ary trees.

Keywords – graph theory, domination, eternal domination, trees

1 Introduction

In recent years, researchers have become increasingly interested in dynamic domination processes on graphs and \textit{eternal domination} problems on graphs.
have been particularly well-studied (see the survey [9], for example). In the
all-guards move model for eternal domination, a set of vertices are occupied
by “guards” and the vertices occupied by guards form a dominating set on a
dough. At each step, an unoccupied vertex is attacked and then each guard
may remain at their current vertex or move along an edge to a neighbouring
vertex. The guards aim to occupy a dominating set that contains the attacked
vertex and such a movement of guards is said to “defend against an attack”.

The eternal domination number of a graph, denoted \( \gamma^\infty_{\text{all}} \) is the minimum
number of guards required to defend against any sequence of attacks, where
the subscript and superscript indicate that all guards can move in response
to an attack and the sequence of attacks is infinite. Given the complexity
of determining the eternal domination number of a graph for the all-guards
move model, recent work such as [1, 3, 4, 5, 10, 12, 2], has focused primarily
on bounding or determining the parameter for particular classes of graphs.

In this paper, we extend the notion of eternal domination to that of
eternal \( k \)-domination in the most natural way: suppose at time \( t = 0 \), the
guards occupy a set of vertices that form a \( k \)-dominating set. At each time
step \( t > 0 \) an unoccupied vertex is attacked and every guard moves distance
at most \( k \) so that the guards occupy the vertices of a \( k \)-dominating set that
contains the attacked vertex. We note that for \( k = 1 \), the process is equivalent
to the all-guards move model for eternal domination as described above.
Hence, we focus on results for \( k \geq 2 \).

We present preliminary results in Section 2 that provide general bounds,
the complexity of the associated decision problem, and determine the eternal
\( k \)-domination number exactly for some small classes of graphs. In Section 2.2
we show that the eternal \( k \)-domination number of a graph is bounded above
by the eternal \( k \)-domination number of a spanning tree of the graph, which
motivates us then to focus on trees in Section 3. The eternal domination
number of a tree was characterized in [8] by using two reductions. We ex-
tend the concepts of these reductions to the eternal \( k \)-domination model,
providing reductions that, informally speaking, “trim branches” of trees in
such a way that the change in the eternal \( k \)-domination number is controlled.
However, for the eternal \( k \)-domination model such reductions are insufficient
to characterize the eternal \( k \)-domination number of all trees and we state
some resulting open problems. In Section 3, we provide an upper bound for
the eternal 2-domination number of a tree, which can be extended to an up-
per bound for the eternal \( k \)-domination number of a tree. We also determine
exactly, the eternal 2-domination number for perfect \( m \)-ary trees. Since this

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paper introduces the concept of eternal $k$-domination, we conclude with a
series of open questions in Section 5.

We conclude this section with some formal definitions. Recall, in a
graph $G$, the open distance-$k$ neighbourhood of $x \in V(G)$ is $N_k(x) = \{v \in 
V(G) : d(x, v) = k\}$, and the closed distance-$k$ neighbourhood of $x \in V(G)$
is $N_k[x] = \{v \in V(G) : d(x, v) \leq k\}$.

**Definition 1.** Let $G$ be a graph and $k \geq 1$ an integer. A set $D \subseteq V(G)$ is a
$k$-dominating set if every vertex of $V(G) \setminus D$ is at most distance $k$ from a
vertex in $D$. The minimum cardinality of a $k$-dominating set in graph $G$ is
the $k$-domination number, denoted $\gamma_k(G)$.

**Definition 2.** Let $G$ be a graph. Let $D_{k,q}(G)$ be the set of all $k$-dominating
sets of $G$ which have cardinality $q$. Let $D, D' \in D_{k,q}(G)$. We will say
$D$ transforms to $D'$ if $D = \{v_1, v_2, \ldots, v_q\}$, $D' = \{u_1, u_2, \ldots, u_q\}$, and
$u_i \in N_k[v_i]$ for all $i \in [q]$, the closed $k$th neighbourhood. If we consider $D$ as
the placement of all the guards in a $k$-dominating set, then the set $D'$ is a
permissible movement of all the guards after some attack.

**Definition 3.** An eternal $k$-dominating family of $G$ is a subset $E \subseteq 
D_{k,q}(G)$ for some $q$ so that for every $D \in E$ and each possible attack $v \in V(G)$,
there is a $k$-dominating set $D' \in E$ so that $v \in D'$ and $D$ transforms to $D'$.

A set $D \in D_{k,q}(G)$ is an eternal $k$-dominating set if it is a member
of some eternal $k$-dominating family. Eternal domination is a discrete time-
process, so at each iteration of an “attack” the $k$-dominating set $D$ transforms
into some other set $D'$ within the family.

The eternal $k$-domination number of a graph $G$, denoted $\gamma_{\infty,k}(G)$, is
the minimum $q$ for which an eternal $k$-dominating family of $G$ exists. We use
this notation to indicate that all guards are allowed to move a distance of
at most $k$.

## 2 Eternal $k$-domination on general graphs

### 2.1 Preliminary Results

In this section, we relate the eternal $k$-domination number to known graph
parameters in order to obtain bounds, as well as determine the eternal $k$-
domination number for well-known families of graphs. We then determine
the complexity of computing this number.
By definition, any eternal $k$-dominating set is also a $k$-dominating set, giving us the lower bound in Observation 4. However, we can also bound the eternal $k$-dominating number of a graph by its $\lceil k/2 \rceil$-domination number:

**Observation 4.** For any graph $G$ and integer $k \geq 2$,

$$\gamma_k(G) \leq \gamma_{\text{all}, k}(G) \leq \gamma_{\lceil \frac{k}{2} \rceil}(G).$$

To demonstrate the upper bound, consider a $\lfloor k/2 \rfloor$-dominating set where $\gamma_{\lfloor k/2 \rfloor}(G) = \ell$ and $D = \{v_1, v_2, \ldots, v_\ell\}$ on graph $G$. For each $j \in \{1, 2, \ldots, \ell\}$, the maximum distance between any two vertices in $N_{\lfloor k/2 \rfloor}(v_j)$ is $k$. Specifically, a guard $g_j$ is placed on an arbitrary vertex of $N_{\lfloor k/2 \rfloor}(v_j)$ for each $j \in \{1, 2, \ldots, \ell\}$. The guard $g_j$ will only ever occupy vertices of $N_{\lfloor k/2 \rfloor}(v_j)$.

Then, given an attack at a vertex $x$ in $N_{\lfloor k/2 \rfloor}(v_j)$, guard $g_j$ can move to the attacked vertex and no other guard moves. We note that it is possible that the attacked vertex $x$ is within distance $\lfloor k/2 \rfloor$ from multiple vertices in $D$.

The bounds in Observation 4 are tight. It is easy to see that if graph $G$ has a universal vertex, then $\gamma_{\text{all}, k}(G) = \gamma_k(G) = \gamma_{\lfloor \frac{k}{2} \rfloor}(G) = 1$. However, it is worth noting that the difference between $\gamma_k(G)$ and $\gamma_{\lfloor k/2 \rfloor}(G)$ can be arbitrarily large. Consider $K_{1,n}$ where each edge is subdivided $k - 1$ times and call this graph $S_{n,k}$. Then clearly $\gamma_k(S_{n,k}) = 1$, but $\gamma_{\lfloor k/2 \rfloor}(S_{n,k}) = n + 1$.

Though the bounds of Observation 4 are tight, it is important to note that for some graphs, such as cycles, $\gamma_{\text{all}, k}$ can be much smaller than $\gamma_{\lfloor k/2 \rfloor}$.

**Theorem 5.** For $n \geq 3$ and $k \geq 1$,

$$\gamma_{\text{all}, k}(C_n) = \gamma_k(C_n) = \left\lceil \frac{n}{2k + 1} \right\rceil.$$

**Proof.** Observe $C_n$ can be decomposed into $\left\lceil \frac{n}{2k + 1} \right\rceil$ vertex-disjoint paths of length at most $2k + 1$. Since a center vertex of a path of length at most $2k + 1$ will $k$-dominate the path, $\gamma_k(C_n) = \left\lceil \frac{n}{2k + 1} \right\rceil$.

Next we show that a minimum $k$-dominating set is indeed a minimum eternal $k$-dominating set. Place guards on the vertices of the minimum $k$-dominating set described above. Suppose vertex $v$ is attacked and let $u$ be a vertex within distance $k$ of $v$ that contains a guard. The guard at $u$ moves distance $d(u, v) = x$ to occupy $v$ and all other guards move exactly distance $x$ in the same direction.

As a consequence of Theorem 5 whenever a graph $G$ is Hamiltonian, we obtain the following upper bound, by simply considering the guards moving strictly along the Hamilton cycle.
Corollary 6. Let $G$ be a Hamiltonian graph on $n$ vertices. Then for $k \geq 1$, $\gamma_{\infty, k}(G) \leq \lceil \frac{n}{2k+1} \rceil$.

Although it is easy to see that $\gamma_k(P_n) = \lceil \frac{n}{2k+1} \rceil$, we next show the eternal $k$-domination number of a path is larger.

Theorem 7. For $n \geq 1$ and $k \geq 1$, $\gamma_{\infty, k}(P_n) = \lceil \frac{n}{k+1} \rceil$.

Proof. Let $V(P_n) = \{v_0, v_2, ..., v_{n-1}\}$ where $v_i$ is adjacent to $v_{i+1}$ for $0 \leq i \leq n-2$. By partitioning the path into vertex-disjoint sub-paths, each of length at most $k$ and assigning one guard to each such sub-path, it is easy to see that $\lceil \frac{n}{k+1} \rceil$ guards will suffice to form an eternal $k$-dominating family on $P_n$, by using the reasoning following Observation [4].

Next we prove the lower bound. There must always be a guard within distance $k$ of $v_0$. Thus a guard is always required to be located on the sub-path $v_0, v_1, ..., v_k$. We partition the graph into sub-paths of length at most $k$. Let $P_\ell$ be the sub-path of length $k$ induced by vertices $v_{\ell(k+1)}, ..., v_{\ell(k+1)+k-1}$, for $0 \leq \ell \leq \lceil \frac{n}{k+1} \rceil$, and $P_{\lceil \frac{n}{k+1} \rceil}$ be the sub-path $v_{\lceil \frac{n}{k+1} \rceil k + \lceil \frac{n}{k+1} \rceil}, ..., v_{n-1}$, should it exist.

For some fixed $j$, assume the placement of the guards on $P_n$ are such that for each $i < j$, each sub-path $P_i$ always contains at least 1 guard; i.e. every eternal $k$-dominating set contains exactly one vertex from $P_i$ for each $i < j$. Thus $P_j$ is the lowest-indexed sub-path that does not always contain a guard. Let time $t$ be the first time there is no guard in $P_j$ and assume an attack happens on this path. Furthermore, no guard from $P_i$ can move into $P_j$ since each lower-indexed path must have a guard on it at all times. If there are no guards in $P_{j+1}$ that can move to the attacked vertex in $P_j$, then the guards do not form a $k$-dominating set.

Thus, suppose a guard in $P_{j+1}$ moves to the attacked vertex.

If the guard that moves from the sub-path $P_{j+1}$ to $P_j$ leaves the sub-path $P_{j+1}$ without a guard, a guard from $P_{j+2}$ most move onto $P_{j+1}$, and so on. If there is some sub-path $P_\ell$ such that the guard on that path cannot be replaced by a guard in $P_{\ell+1}$ then the guards do not form an eternal $k$-dominating set, as the subsequent attack can occur within this sub-path and not be guarded. Otherwise, each guard in sub-path $P_{\ell+1}$ moves to sub-path $P_\ell$, eventually leaving $v_{n-1}$ unable to be defended should an attack occur there at time $t+1$. Thus at the end of each time step, there must be a guard in each $P_i$; so at least $\lceil \frac{n}{k+1} \rceil$ guards are required and the result follows. 

\[ \square \]
The previous results show that for some families of graphs, the eternal $k$-domination number grows linearly with the order of the graph. On the other end of the spectrum, we can easily characterize graphs with eternal $k$-domination number 1: $\gamma_{\text{all},k}(G) = 1$ if and only if the diameter of graph $G$ is at most $k$.

An important question to ask when investigating a graph parameter is, how difficult is it to compute? To answer this, we will look at the relationship between the eternal $k$-domination number of a graph and it’s graph power. For a graph $G$, the $k$th power of the graph, $G^k$, is formed by adding an edge $u,v \in E(G)$ whenever $\text{dist}(u,v) \leq k$. Thus, if there exists a path in $G$ from $u$ to $v$ of length at most $k$, then we will witness an edge $uv \in E(G^k)$.

**Theorem 8.** If $G$ is a graph and $k \in \mathbb{N}$, then

$$\gamma_{\text{all},k}(G) = \gamma_{\text{all},1}(G^k).$$

**Proof.** Let $S$ be an eternal $k$-dominating set of $G$ for which $|S| = \gamma_{\text{all},k}(G)$. Suppose a sequence of attacks, $\{a_1,a_2,\ldots,a_\ell\} \subseteq V(G)$ occur. For each guard $g_i$, let $G_i = \{d_1,d_2,\ldots,d_\ell\} \subseteq V(G)$ be the set of corresponding defending moves the guard makes, that is guard $g_i$ moves from $d_{j-1}$ to $d_j$ after attack $a_j$. We now consider the eternal 1-domination of $G^k$ by using corresponding moves of the eternal $k$-domination of $G$. Place the $\gamma_{\text{all},1}(G^k)$ guards in $S$ on the vertices of $G^k$, since $V(G^k) = V(G)$. Suppose in $G^k$ the same sequence of attacks occur on vertices $\{a_1,a_2,\ldots,a_\ell\}$.

For any guard $g_i$ and their sequence $G_i$, moving from $d_{j-1}$ to $d_j$ in $G^k$ is permissible as these vertices have distance at most $k$ in $G$, and thus will be adjacent in $G^k$. Each guard $g_i$, can use the same sequence of moves and still guard $G^k$, thus $\gamma_{\text{all},k}(G) \geq \gamma_{\text{all},1}(G^k)$.

Similarly, let $S'$ be an eternal 1-dominating set of $G_k$ for which $|S'| = \gamma_{\text{all},1}(G^k)$. Suppose a sequence of attacks, $\{a_1,a_2,\ldots,a_\ell\}$ occur. For each guard $g_i$, we define $G_i$ analogously as above. We then consider the eternal $k$-domination of $G$, using moves from the eternal 1-dominination in $G^k$. Place the $\gamma_{\text{all},1}(G^k)$ guards in $S'$ on the vertices of $G$ and consider the eternal $k$-domination process, and again suppose in $G$ the vertices $\{a_1,a_2,\ldots,a_\ell\}$ are attacked in that order. Any guard $g_i$ that moves from $d_{j-1}$ to $d_j$ after an attack $a_j$ in $G^k$ can also move from $d_{j-1}$ to $d_j$ in $G$ since these two vertices are adjacent in $G^k$ and thus are at most distance $k$ in $G$. Thus, $\gamma_{\text{all},k}(G) \leq \gamma_{\text{all},1}(G^k)$, giving the desired result. \hfill $\square$

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In [6] and the subsequent errata [7], it was shown that deciding if a set of vertices of a graph is an eternal domination set is in EXP. Thus, taken with Theorem 8 we have the following complexity result.

**Corollary 9.** Let $G$ be a graph of order $n$, a positive integer $k$ with $0 \leq k \leq n$ and $S \subseteq V(G)$. Deciding if $S$ is an eternal $k$-dominating set for $G$ is in EXP.

### 2.2 Using trees to bound $\gamma_{\text{all},k}^\infty$

In this section, we provide insight to the vertices "covered" or "guarded" by a single guard or a pair of guards, and present bounds on the eternal $k$-domination number for arbitrary graphs based on a partitioning into vertex-disjoint trees.

A subgraph $H$ of a graph $G$ is a *retract* of $G$ if there is a homomorphism $f$ from $G$ to $H$ so that $f(x) = x$ for $x \in V(H)$. The map $f$ is called a *retraction* and we note that since this is an edge-preserving map to an induced subgraph, the distance between any two vertices does not increase in the image.

**Lemma 10.** Let $H$ be a retract of graph $G$. Then $\gamma_{\text{all},k}^\infty(H) \leq \gamma_{\text{all},k}^\infty(G)$.

**Proof.** Let $H$ be a retract of graph $G$ and $f : G \rightarrow H$ be a retraction. We consider two parallel incidences of eternal $k$-domination: one on $G$ and one on $H$. The process on $H$ can be thought of as taking place on $G$, as $H$ is an induced subgraph of $G$. We will restrict the attacks in $G$ to vertices that are also in $H$. Initially, if there is a guard at vertex $v \in V(G)$, then we place a guard at vertex $f(v) \in V(H)$. If a guard in $G$ moves from $x$ to $y$ in response to an attack at vertex $z$ in $H$, we observe that guard may move from $f(x)$ to $f(y)$ in $H$ in response to the attack. Thus, $\gamma_{\text{all},k}^\infty(H) \leq \gamma_{\text{all},k}^\infty(G)$. \hfill $\square$

Below we consider another subgraph that will also prove useful in obtaining an upper bound for $\gamma_{\text{all},k}^\infty(G)$ for an arbitrary graph $G$.

**Lemma 11.** Let $G$ be a graph, $k$ a positive integer and $e \in E(G)$, then

$$\gamma_{\text{all},k}^\infty(G - e) \geq \gamma_{\text{all},k}^\infty(G)$$

where $G - e$ is the subgraph of $G$ obtained by deleting edge $e$.

**Proof.** Let $G$ be a graph and $e \in E(G)$. Consider, $G - e$ the graph with $G$ with edge $e$ removed. Let $S$ be an eternal $k$-dominating set of $G - e$.
of minimum cardinality and suppose a sequence of attacks, \( \{a_1, a_2, \ldots, a_\ell \} \) occur. For each guard \( g_i \), let \( G_i = \{d_1, d_2, \ldots, d_\ell \} \) be the set of corresponding defending moves the guard makes, that is after the attack \( a_j \), the guard \( g_i \) moves from \( d_{j-1} \) to \( d_j \).

Now place guards on the vertices of \( S \) in \( G \) and consider the same sequence of attacks \( \{a_1, a_2, \ldots, a_\ell \} \). Each guard can respond appropriately, moving to \( d_j \) after attack \( a_j \). Since every edge of \( G - e \) is an edge of \( G \), the result follows.

From repeated applications of Lemma 11 we obtain the following.

**Theorem 12.** Let \( G \) be a graph and \( T \) a spanning tree of \( G \), then

\[
\gamma_{\text{alt},k}(T) \geq \gamma_{\text{alt},k}(G).
\]

Theorem 12 suggests that understanding the eternal \( k \)-domination process on trees is important, as it provides an upper bound on the eternal \( k \)-domination number of a general graph. We next consider covering a graph \( G \) with sub-trees with a particular structure, each of which can be guarded by a single guard. By cover, we mean that a guard is assigned to a particular sub-tree and they can respond to any sequence of attacks that occur within that particular sub-tree.

**Definition 13.** A \( k \)-rooted tree with \( k \in \mathbb{Z}^+ \), is a rooted tree where the eccentricity of the root is at most \( k \).

**Definition 14.** Given a graph \( G \), we define a \( k \)-rooted tree decomposition to be a partition of the vertices into sets \( S_i \) for \( 1 \leq i \leq \ell \) such that \( G[S_i] \) contains a spanning subgraph that is \( k \)-rooted tree.

The \( k \)-rooted tree decomposition number of a graph \( G \), denoted \( \tau_k(G) \), is the minimum number of sets \( S_i \) over all possible decompositions.

An easy bound comes from partitioning graph \( G \) into \( \lfloor \frac{k}{2} \rfloor \)-rooted trees as one guard can cover the vertices of a \( \lfloor \frac{k}{2} \rfloor \)-rooted tree.

**Corollary 15.** For any graph \( G \) and \( k \geq 2 \)

\[
\gamma_{\text{alt},k}(G) \leq \tau_{\lfloor \frac{k}{2} \rfloor}(G).
\]

For some graphs, a better bound can be achieved by partitioning graph \( G \) into \( k \)-rooted trees and recognizing that 2 guards can protect the vertices of each \( k \)-rooted tree.
Proposition 16. If $T$ is a $k$-rooted tree for some $k \geq 2$, then $\gamma_{\text{all},k}(T) \leq 2$.

Proof. Let $T$ be a $k$-rooted tree with root $r$. Initially place one guard at $r$ and the second guard at an arbitrary vertex, $u$. After a vertex, $v$ is attacked, the guard at $r$ moves to $v$ and the other guard at $u$ moves to $r$. The resulting $k$-dominating set is equivalent to the original, and the guards can respond to attacks in this manner indefinitely. \hfill \square

Corollary 17. For any graph $G$ and $k \geq 2$,

$$\gamma_{\text{all},k}(G) \leq \min\{2 \cdot \overline{\gamma}_k(G), \overline{\gamma}_{\lfloor k/2 \rfloor}(G)\}.$$ 

In light of Theorem 12 and the fact that we can partition a graph $G$ into $k$-rooted trees in order to find an upper bound for $\gamma_{\text{all},k}(G)$, the next two sections will focus on trees.

3 Eternal $k$-domination on trees

In this section, we consider the conditions for which the eternal $k$-domination number of a tree will equal or be one greater than that of a sub-tree in the aims of working towards determining $\gamma_{\text{all},k}(T)$ for any tree $T$.

In [8], the authors provide a linear-time algorithm for determining the eternal 1-domination number of a tree. Their algorithm consists of repeatedly applying two reductions, $R1$ and $R2$, which we restate here.

$R1$: Let $x$ be a vertex of $T$ incident to at least two leaves and to exactly one vertex of degree at least two. Delete all leaves incident to $x$.

$R2$: Let $x$ be a vertex of degree two in $T$ such that $x$ is adjacent to exactly one leaf, $y$. Delete both $x$ and $y$.

If $T'$ is the result of applying either $R1$ or $R2$ to tree $T$, then $\gamma_{\text{all},1}(T') = \gamma_{\text{all},1}(T) - 1$ [8]. With an aim to characterize the eternal $k$-domination number of trees for $k > 1$, Propositions 18 and 19 generalize the reductions of [8] to arbitrary $k \geq 1$. Figure 1(a) and (b) provide a visualization of the sub-trees described in Propositions 18 and 19, respectively.

Proposition 18. Let $x$ and $y$ be neighbours in $T = (V, E)$ and let $T_x$ be the component of $T$ induced by the deletion of edge $xy$, that contains $x$. If every leaf in $T_x$ is within distance $k$ of $x$ and $\text{diam}(T_x) = 2k$, then

$$\gamma_{\text{all},k}(T[V \setminus D]) = \gamma_{\text{all},k}(T) - 1$$
where $D = V(T_x) \setminus \{x\}$.

**Proof.** It is easy to see $\gamma_{\text{alt},k}^\infty(T) \leq \gamma_{\text{alt},k}^\infty(T[V \setminus D]) + 1$: initially place $\gamma_{\text{alt},k}^\infty(T[V \setminus D])$ guards on an eternal $k$-dominating set of sub-tree $T[V \setminus D]$ and place an additional guard at $x$. Whenever a vertex of $D$ is attacked, a guard moves from $x$ to the attacked vertex and the guards in $T[V \setminus D]$ move as they would if $x$ was attacked in $T[V \setminus D]$. Whenever a vertex of $T[V \setminus D]$ is attacked, the guards currently occupying vertices of $T[V \setminus D]$ move to form an eternal $k$-dominating set on $T[V \setminus D]$ that contains the attacked vertex and the guard in $D$ moves to $x$ (it is possible that there is no guard in $D$, in which case there are two guards on $x$ and when the guards move in response to the attack, one of the two guards remains on $x$).

We next prove that $\gamma_{\text{alt},k}^\infty(T[V \setminus D]) < \gamma_{\text{alt},k}^\infty(T)$. Assume $\gamma_{\text{alt},k}^\infty(T[V \setminus D]) = \gamma_{\text{alt},k}^\infty(T)$ and we will show by way of contradiction that this is not the case. Place the $\gamma_{\text{alt},k}^\infty(T[V \setminus D])$ guards on $T$. Let $\ell_1$ and $\ell_2$ be leaves of $D$ at distance $k$ from $x$ and distance $2k$ from each other. Note that a guard must be on $x$ to ensure that $\ell_1$ and $\ell_2$ are $k$-dominated. We will consider what the eternal $k$-domination process looks like on $T$ and on a copy of $T[V \setminus D]$.

First, any attack in $T[V \setminus D]$ corresponds to the guards moving as required, ensuring a guard is on $x$ after the attack is defended. In $T$ this placement of guards is the same, as vertices in $D$ do not require guards, as $x$ has a guard.
Assume a vertex on the path $x_1 \ldots \ell_1$ is attacked. In $T$ a guard on $x$ must move to the attacked vertex (note: if another guard moves to the attack, it must go through $x$, so we can assume this is the guard that moves). We then need to replace that guard otherwise $\ell_2$ is not $k$-dominated.

On the tree $T[V \setminus D]$ this is equivalent to two guards moving onto $x$, since attacks in $D$ and guards on vertices of $D$ correspond attacks and guards on $x$ on $T[V \setminus D]$.

So we have two situations to consider. The first is if in $T$ an attack occurs at a vertex of $D$. In $T[V \setminus D]$ it corresponds to $x$ having two guards, but one guard is superfluous (as it is not really in this sub-tree).

The second situation is if in $T$ an attack occurs at a vertex of $T[V \setminus D]$. Then a guard on $x$ moves within $T[V \setminus D]$ as required and either a guard moves from $D$ to $x$, or if $x$ had more than one guard, at least one of these guards do not move.

In both situations, the attacks lead to a response in $T[V \setminus D]$ that required one less guard to defend that sub-tree. This means that any sequence of attacks in $T$ result in $T[V \setminus D]$ requiring less than $\gamma_{all,k}^\infty(T[V \setminus D])$ guards to eternally $k$-dominate, a contradiction, so $\gamma_{all,k}^\infty(T[V \setminus D]) = \gamma_{all,k}^\infty(T) - 1$.  

A suspended $i$-end-path in a graph $G$ is a path of length at least $i \geq 2$ such that at least one endpoint of the path is a leaf and all internal vertices of the path have degree exactly 2. The next result generalizes the $R2$ reduction in\cite{[8]} for eternal 1-domination. See Figure 1 (b).

**Proposition 19.** Suppose $T = (V, E)$ contains a suspended $(k + 1)$-end-path $P$ and label the vertices of $P$ as $x_0, x_1, \ldots, x_{k+1}$ where $x_0$ is a leaf and $x_i$ is adjacent to $x_{i+1}$ for $0 \leq i \leq k$. Then

$$\gamma_{all,k}^\infty(T[V \setminus \{x_0, \ldots, x_k\}]) = \gamma_{all,k}^\infty(T) - 1.$$ 

**Proof.** Clearly $\gamma_{all,k}^\infty(T) \leq \gamma_{all,k}^\infty(T[V \setminus \{x_1, \ldots, x_k\}]) + \gamma_{all,k}^\infty(T[\{x_0, \ldots, x_k\})$ and $\gamma_{all,k}^\infty(T[\{x_0, \ldots, x_k\}] = 1$ since it is a path of length $k$. We next show that $\gamma_{all,k}^\infty(T[V \setminus \{x_1, \ldots, x_k\}])$ guards do not suffice to eternally $k$-dominate $T$.

First, we consider eternal $k$-domination on the graph $T[V \setminus \{x_1, \ldots, x_k\}]$. Place the guards and consider a minimal finite sequence of attacks $A = \{a_1, a_2, \ldots, a_q\}$ that requires all $\gamma_{all,k}^\infty(T[V \setminus \{x_1, \ldots, x_k\}]$ guards to defend this sequence of attacks. That is, if there is at least one less guard, this sequence of attacks is not able to be defended.
Now consider the graph $T$. Suppose $\gamma_{\text{all},k}^\infty(T) = \gamma_{\text{all},k}^\infty(T[V\setminus\{x_1,\ldots,x_k\}])$ and place the guards on the vertices of an eternal $k$-dominating set of $T$, note that there is at least one guard on $P$. Consider the sequence of attacks $\{x_0, a_1, x_0, a_2, x_0, a_3, \ldots, x_0, a_q\}$, on tree $T$. Thus there is always a guard in $P$. This means that guard is never able to defend a vertex in $T[V\setminus\{x_1,\ldots,x_k\}]$, hence, there are not enough guards to eternally $k$-dominate $T$, a contradiction.

The two previous results provided a means to “trim branches” off a tree to reduce the eternal $k$-domination number by 1. In each of these propositions, the diameter of $T_x$ is either exactly $k$ or $2k$. When $k < \text{diam}(T_x) < 2k$, there are more interesting interactions between the guards in $T$ and in $D$. It may be possible for guards to move in and out of $D$ while guarding $T$, so $T[(V\setminus D)]$ may or may not decrease by one. In fact, for every diameter in $[k+1, 2k-1]$, we next provide a construction where there exists a tree with $\gamma_{\text{all},k}^\infty(T[(V\setminus D)]) = \gamma_{\text{all},k}^\infty(T)$ and a tree with $\gamma_{\text{all},k}^\infty(T[(V\setminus D)]) + 1 = \gamma_{\text{all},k}^\infty(T)$.

**Example 20.** We first consider a tree $T_x$, rooted at a vertex $x$, with one leaf at distance $k$ from $x$, with $\text{diam}(T_x) = k + \ell$ for some $\ell \in [1, k-1]$. In our first example, to define $T$ we consider $T_x$ and add a star centered at a vertex $y$, with each leaf at distance $k-1$ from $y$, and add the edge $xy$. In this case, $(T[(V \setminus V(T_x) \cup \{x\}]) = \gamma_{\text{all},k}^\infty(T) = 2$, since a guard on a leaf in the star can move to cover $x$, and we can ensure a guard occupies $x$ at all times. Then, whenever there is an attack on the leaf at distance $k$ from $x$, and then a subsequent attack on the vertex in $T_x$ at distance $k + \ell$ from this leaf, we require the guard on $x$ to move into $T_x$ to the new attack, and the previous guard can move back up to $x$.

**Example 21.** For the second example, we define a tree $T$ by considering $T_x$ and adding star centered at a vertex $y$, with each leaf at distance $k$ from $y$, and add the edge $xy$. In this case, $(T[(V \setminus V(T_x) \cup \{x\}]) = 2 < 3 = \gamma_{\text{all},k}^\infty(T)$. A guard on the leaf of the star will not be able to guard $x$, and we have one guard always on $y$ in $T \setminus T_x$, however, in order to guard the leaf at distance $k$ from $x$ in $T_x$, we require a third guard in $T_x$ in order to ensure there is always a guard on $x$ available to guard the leaves as above.

Although the previous results provide insight into a reduction on trees, the two examples above show that additional conditions are needed to characterize sub-trees with diameters between $k$ and $2k$. 

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Our next result provides a way to “trim branches” without changing the eternal $k$-domination number. For $k = 2$, for example, if a vertex is adjacent to at least two leaves, then a leaf can be deleted without changing the eternal 2-domination number. An example of Theorem 22 having been applied seven times to a tree is shown in Figure 4.

**Theorem 22.** Let $T' = (V', E')$ be an induced subgraph of a tree $T = (V, E)$ where $T'$ is a $\lceil \frac{k}{2} \rceil$-rooted tree with root $x$, some leaf $\ell$ distance $\lceil \frac{k}{2} \rceil$ from $x$ in $T'$, and $T[(V \setminus V') \cup \{x\}]$ is connected. Then $\gamma^\infty_{all,k}(T) = \gamma^\infty_{all,k}(T[(V \setminus V') \cup Q])$ where $Q$ is the set of vertices on the $x\ell$-path.

**Proof.** By Lemma 10, $\gamma^\infty_{all,k}(T[(V \setminus V') \cup Q]) \leq \gamma^\infty_{all,k}(T)$. It is easy to see that $\gamma^\infty_{all,k}(T[(V \setminus V') \cup Q]) = g$ guards suffice to eternally $k$-dominate $T$. The guards on $T$ move as they would on graph $T[(V \setminus V') \cup Q]$, with a few exceptions: when a vertex of $T[V \setminus Q]$ is attacked in $T$, the guards of $T$ move to the same vertices guards of $T[(V \setminus V') \cup Q]$ would move to in response to an attack at a vertex on $Q$, with the exception of one guard who moves to the attacked vertex on $V \setminus Q$, rather than a vertex on $Q$. \hfill $\square$

Lemma 23 describes another set of vertices that can be deleted from a tree without changing the eternal $k$-domination number. We first identify two leaves $\ell_1, \ell_2$ that are distance $2k$ apart and let $x$ be the centre of the $\ell_1\ell_2$-path. Informally, we identify all “branches” from $x$ whose leaves are all within distance $k$ of $x$ and remove all vertices on these branches, apart from those on the $\ell_1\ell_2$-path. Lemma 23 proves the sub-tree has the same eternal $k$-domination number as the original tree. An example is shown in Figure 2, where the vertices removed (defined as set $D$ in the theorem) are striped.

The **eccentricity** of vertex $u$ in graph $G$, denoted $\epsilon_G(u)$, is the maximum distance between $u$ and any other vertex in $G$. More formally, $\epsilon_G(u) = \max_{v \in V(G)} d(u, v)$.

**Lemma 23.** Suppose $\ell_1, \ell_2$ are two leaves in tree $T = (V, E)$ such that $d(\ell_1, \ell_2) = 2k$. Let $Q$ denote the $\ell_1\ell_2$-path and let $x$ be the center of $Q$.

Let $N(x) = \{a_1, a_2, \ldots, a_p\}$ for some integer $p$. Let $A_1, A_2, \ldots, A_p$ be the components of $T$ induced by the deletion of $x$ where $a_i \in A_i$. Suppose $\epsilon_{T[A_i]}(a_i) \leq k - 1$ for $i \in \{1, 2, \ldots, q\}$ and $\epsilon_{T[A_i]}(a_i) \geq k$ for $i \in \{q + 1, q + 2, \ldots, p\}$. If

$$D = \left( \bigcup_{1 \leq i \leq q} A_i \right) \setminus Q$$

then $\gamma^\infty_{all,k}(T) = \gamma^\infty_{all,k}(T[V \setminus D])$. 

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Proof. By Theorem 10, $\gamma_{\text{all},k}(T[V\setminus D]) \leq \gamma_{\text{all},k}^\infty(T)$. Suppose $\gamma_{\text{all},k}^\infty(T[V\setminus D]) = s$. We simply modify the movements of the $s$ guards on $T[V\setminus D]$ to defend against attacks on vertices of $D$ in $T$.

We first point out the existence of a particular eternal $k$-dominating family on $T[V\setminus D]$. Each eternal $k$-dominating set on $T[V\setminus D]$ must contain at least one vertex on the $x\ell_1$-path (otherwise $\ell_1$ is not $k$-dominated). Suppose that in response to an attack at a vertex on the $x\ell_1$-path, the guards move to form a $k$-dominating set that does not contain $x$. Clearly, the guards could alternately have moved to form a $k$-dominating set that does contain $x$: after the guards move in response to the attack, there must be a guard on the $x\ell_2$-path (otherwise $\ell_2$ is not $k$-dominated) this guard could have moved to $x$. Thus, there exists an eternal $k$-dominating family on $T[V\setminus D]$ where each eternal $k$-dominating set is of cardinality $s$ and contains $x$. We
now exploit this eternal $k$-dominating family $E_x$ in order to create an eternal $k$-dominating family for $T$ of cardinality $s$.

Initially, place $s$ guards on the vertices of $T$ that correspond to the vertices of some eternal $k$-dominating set $S \subseteq E_x$ on $T[V \setminus D]$. Since this results in a guard on vertex $x$ on $T$, we know $T$ is initially $k$-dominated. If a vertex $y \in V \setminus D$ is attacked in $T$, the guards of $T$ mirror the movements of guards in graph $T[V \setminus D]$ in response to an attack at $y$. If a guard in graph $T[V \setminus D]$ moves from vertex $b$ to $c$, then a guard in graph $T$ will move from vertex $b$ to $c$, with one exception: if there is a guard in $D$, the guard will move to $x$. Note that in $T$, this results in the vertices of $T$ remaining $k$-dominated.

Now suppose that on $T$, vertex $z \in D$ is attacked. If the previous vertex attacked was also in $D$, then a guard moves from $D$ to $x$ and a guard moves from $x$ to $z$. Otherwise, we consider an attack at $\ell_1$ in graph $T[V \setminus D]$. On graph $T[V \setminus D]$, a guard can move from $x$ to $\ell_1$ and the remaining guards move accordingly, to form an eternal $k$-dominating set containing both the attacked vertex and $x$. In $T$, a guard moves from $x$ to $z$ instead of $\ell_1$ and the remaining guards move the same as their counterparts in $T[V \setminus D]$. Thus, the guards on $T$ form a $k$-dominating set.

For $k = 2$, we next consider two examples which illustrate that for some trees, removing $a_1, a_2, \ell_1, \ell_2$ will not change the eternal $2$-domination number, but for other trees, it will. Thus, Examples 24 and 25 illustrate that Lemma 23 cannot be improved by removing vertices of $Q \setminus \{x\}$.

**Example 24.** For $k = 2$, consider the tree $T$ given in Figure 3 (a), where vertex $x$ has been identified. Using Lemma 23, we can “trim branches” of the tree without changing the eternal $2$-domination number. The tree $T[V \setminus D]$ shown in Figure 3 (b) shows the result of applying Lemma 23 to tree $T$ using the vertex identified as $x$ in the figure. We note that though $\gamma_{\text{all,2}}(T) = 6 = \gamma_{\text{all,2}}^\infty(T[V \setminus D]) = 6$, it is also the case that $\gamma_{\text{all,2}}^\infty(T[V \setminus D \setminus \{a_1, a_2, \ell_1, \ell_2\}]) = 6$. (In Figure 3 (a) and (b), this is the part of the graph outside of the bubble). Thus, for this example, “trimming off” $a_1, a_2, \ell_1, \ell_2$ does not change the eternal $2$-domination number.

**Example 25.** For $k = 2$, consider the tree $T'$ given in Figure 3 (c). We may assume that this tree is the result of having applied Lemma 23 to a larger tree. Although $\gamma_{\text{all,2}}^\infty(T') = 4$, we also note that $\gamma_{\text{all,2}}^\infty(T'[V \setminus \{a'_1, a'_2, \ell'_1, \ell'_2\}]) = 3$. Thus, for this example, “trimming” $a'_1, a'_2, \ell'_1, \ell'_2$ does change the eternal $2$-domination number.
In Section 4.2 we present further tree reductions for the case where \( k = 2 \).

4 Eternal 2-domination on trees

4.1 General Results for Trees and m-ary Trees

In this section, we first describe an eternal 2-dominating set for any tree \( T \), which yields an upper bound for \( \gamma_{\text{all,2}}(T) \), and second, determine \( \gamma_{\text{all,2}}(T) \) exactly when \( T \) is a perfect \( m \)-ary tree.

Lemma 26. Let \( T \) be a rooted tree and place guards according to 1.-3. below. The vertices occupied by guards form an eternal 2-dominating set on \( T \).

1. Initially place a guard on each vertex for which the distance to the nearest leaf is even and positive.
2. If 1. results in no guards being placed on the root, then place one guard on the root.
3. Place one guard on an arbitrary leaf.

Proof. It is easy to see that the result holds for any rooted tree of depth 2 or 3. We prove the result by inducting on the depth of the tree. Let \( T' \) be a
tree of depth $d$. Let $T$ be the sub-tree of depth $d - 2$ induced by the deletion
all leaves of $T'$ to get $T''$; then delete all leaves of $T''$ to get $T$ (note: leaves
of $T''$ are the stems in $T'$ with at most one non-leaf neighbour).

Let $f : T' \rightarrow T$ where

$$f(v') = \begin{cases} 
  v & \text{if } v' \text{ is not a leaf in either } T' \text{ or } T''; \\
  z & \text{if } v' \text{ is a leaf in } T' \text{ or } T'' \text{ and } z' \text{ is the nearest vertex that is } \\
  & \text{a grandparent of some leaf in } T'. 
\end{cases}$$

The map above preserves distances in the sub-tree $T \subseteq T'$ and maps
leaves in $T'$ and $T''$ to the nearest vertex that is distance 2 from a leaf, noting
that such a vertex may also happen to be a leaf of $T''$.

Let $D$ be any eternal 2-dominating set on $T$. We create a 2-dominating
set $D'$ on $T'$ that “mirrors” $D$ on $T$. Let $D'$ be the set of vertices on $T'$ where

(a) if $v \in D$ then $f^{-1}(v) = v'$ is in $D'$; and
(b) for each vertex $z'$ of $T'$ that is a grandparent of a leaf, if $z \notin D$ then
$z' \in D'$ and if $z \in D$, then choose an arbitrary child of $z'$ to be in $D'$.

From this construction, $D'$ is a 2-dominating set on $T'$.

Suppose vertex $u' \in V(T')$ is attacked. We consider two cases: (1) when
$u'$ is neither a leaf in $V(T'')$ nor a leaf in $V(T')$ and (2) when $u'$ is a leaf in
$V(T'')$ or $V(T')$.

(1) Suppose $u'$ is neither a leaf in $V(T'')$ nor leaf in $V(T')$. Then $f^{-1}(u') = u$. In this situation, we consider an attack at vertex $u$ in $T$. The guards of
$T$ move from an eternal 2-dominating set $D$ to an eternal 2-dominating set
$D_1$ that contains $u$. We move the guards in $T'$ according to how the guards
moves in $T$. That is, if a guard in $T$ moves from $x \in V(T)$ to $y \in V(T)$, then
in $T'$ the guard at $f^{-1}(x) = x'$ moves $f^{-1}(y) = y'$. Additionally, any guard in
$V(T'')$ or $V(T')$ that is located at a leaf (that is not already a grandparent of
a leaf) moves to the nearest vertex $z'$ that is a grandparent of a leaf. Observe
that $f(z') = z \in V(T)$. Let $D_1'$ be the set of vertices now occupied by guards
in $T'$. Observe that $D_1'$ is a 2-dominating set on $T'$ that contains $u'$. Further
observe that $D_1'$ mirrors $D_1$, just as $D'$ mirrored $D$.

(2) Suppose $u'$ is a leaf in $V(T'')$ or $V(T')$. Then $f^{-1}(u') = z$ for where
$f(z') = z$ and $z'$ is the nearest (to $u'$) grandparent of a leaf in $T'$.

If there is a guard on a vertex that is a child or grandchild of $z'$, then
this guard moves to $z'$ while the guard at $z'$ moves to the attacked vertex.
Call the set of vertices now occupied by the guards $D''$ and note that it is
a 2-dominating set containing the attacked vertex. Observe that $D''$ mirrors $D$ just as $D'$ mirrored $D$.

Next, suppose there is no guard on a vertex that is child or grandchild of $z'$. Then we consider an attack at $z \in V(T)$ and the resulting movements of guards in $T$. In $T$, suppose the guards move from $D$ to the the eternal 2-dominating set $D_1$ that contains $z$.

If a guard in $T$ moves from $x \in V(T)$ to $y \in V(T)$, then the guard at $f^{-1}(x) = x'$ moves $f^{-1}(y) = y'$. The guard at $z'$ moves to the attacked vertex. Finally, any there is a guard on a vertex that is a child or grandchild of $w' \in V(T')$ where $w' \neq z'$ and $w'$ is the grandparent of a leaf, then that guard moves to $w'$. Call the set of vertices now occupied by the guards $D'_1$ and note that it is a 2-dominating set containing the attacked vertex. Observe that $D'_1$ mirrors $D_1$ just as $D'$ mirrored $D$.

We have seen that for any attack, the movements of guard on $T'$ can be guided by the movements of guards on sub-tree $T$. After each attack, if the guards on $T$ can move to an eternal 2-dominating set $D_*$, then the guards on $T'$ can move to a 2-dominating set $D'_*$. \[\square\]

We note that although the result of Lemma 26 is expressed for $k = 2$, the result and proof can easily be extended to arbitrary $k$:

1. Initially place a guard on each vertex for which the distance to the nearest leaf is a positive multiple of $k$.

2. If 1. results in no guards being placed on the root, then place one guard on the root.

3. Place one guard on an arbitrary leaf.

In this paper, we only use Lemma 26 for the $k = 2$ case, so we do not prove the result for arbitrary $k$. However, combining Lemma 26 or the above extension with Theorem 12 will yield an upper bound for the eternal 2-domination number (or eternal $k$-domination number) of any graph.

An $m$-ary tree is a rooted tree where every vertex has at most $m$ children. The depth $d$ of an $m$-ary tree is the eccentricity of the root. A perfect $m$-ary tree is an $m$-ary tree in which every non-leaf vertex has exactly $m$ children and every leaf is distance $d$ from the root where $d$ denotes the depth of the tree.
We next present a lemma that will be helpful in determining the eternal 2-domination number for perfect $m$-ary trees. Recall that an eternal 2-dominating family $\mathcal{E}$ is a set in which the elements are eternal 2-dominating sets, all of the same cardinality, so that: if the guards occupy eternal 2-dominating set $D \in \mathcal{E}$ and there is an attack at vertex $v$, the guards can move from set $D \in \mathcal{E}$ to a set $D' \in \mathcal{E}$ that contains $v$ (i.e. $v \in D'$ and $D$ transforms to $D'$). A minimal eternal 2-dominating family is minimal in terms of the number of eternal 2-dominating sets in the family.

Lemma 27. Let $T$ be a perfect $m$-ary tree of depth $d \geq 2$ for $m \geq 2$. There exists a minimal eternal 2-dominating family in which each eternal 2-dominating set contains the grandparent of every leaf and has cardinality $\gamma_{\infty,2}^\infty(T)$.

Proof. Let $T$ be a perfect $m$-ary tree of depth $d \geq 2$. Suppose there exists no minimal eternal 2-dominating family in which each eternal 2-dominating set contains the grandparent of every leaf and has cardinality $\gamma_{\infty,2}^\infty(T)$. Let $\mathcal{E}$ be a minimal eternal $k$-dominating family in which each eternal 2-dominating set has cardinality $\gamma_{\infty,2}^\infty(T)$. Then there exists some eternal 2-dominating set $D \in \mathcal{E}$ that does not contain the grandparent of some leaf. Suppose $\ell$ is a leaf and $D$ does not contain the grandparent $v$ of $\ell$.

Let $T_v$ be the sub-tree of $T$ rooted at $v$. Observe that every set in $\mathcal{E}$ must contain at least one vertex of $T_v$; else $T_v$ is not 2-dominated. Since $m \geq 2$ and $D$ does not contain $v$, it must be that $D$ contains at least two vertices of $T_v$; otherwise the vertices of $T_v$ are not 2-dominated. Although one of these vertices may be the most recently attacked vertex (so a guard moved to the attacked vertex in $T_v$), let $u \neq v$ be the other vertex on $T_v$ that is in $D$. Since $N(u) \cup N_2(u) \subseteq N(v) \cup N_2(v)$, in response to the last attack, we could have moved a guard to $v$ instead of $u$. So instead of moving to $D$, the guards could have moved to $(D \setminus \{u\}) \cup \{v\}$. We observe if $D$ transforms to $D' \in \mathcal{E}$, then by construction, $(D \setminus \{u\}) \cup \{v\}$ will also transform to $D' \in \mathcal{E}$. Thus, $\mathcal{E} \setminus D \cup (D \setminus \{u\} \cup \{v\})$ is an eternal 2-dominating family. Applying this argument repeatedly results in the desired contradiction. 

We conclude this subsection with the following characterization for $m$-ary trees.
Theorem 28. Let $T$ be a perfect $m$-ary tree of depth $d \geq 2$ and $m \geq 2$. Then
\[
\gamma_{\text{alt},2}(T) = 1 + \frac{m^d - \delta_{\text{oe}}}{m^2 - 1}
\]
where $\delta_{\text{oe}} = \begin{cases} 
1 & \text{if } d \text{ is even} \\
2 & \text{if } d \text{ is odd.}
\end{cases}$

Proof. The upper bound follows from Theorem 26. It is easy to see that the above formula holds for perfect $m$-ary trees of depth 2, 3, and 4. For the sake of contradiction, let $T'$ be a minimum depth perfect $m$-ary tree such that the above formula does not hold. Then $\gamma_{\text{alt},2}(T') \leq (m^d - \delta_{\text{oe}})/(m^2 - 1)$ where $T'$ has depth $d$. In $T'$, let $Z_1$ be the set of leaves, $Z_2$ be the set of parents of leaves, and $Z_3$ be the set of vertices that are grandparents of leaves. We first observe that in any minimum eternal 2-dominating set on $T'$, there is always at least one guard in each sub-tree of $T'$ rooted at a vertex at depth $d - 2$; otherwise, some leaf is not within distance 2 of a guard. Thus, in every minimum eternal $k$-dominating set on $T'$, there are at least $m^{d-2}$ guards on the vertices $Z_1 \cup Z_2 \cup Z_3$ in $T'$.

Let $T$ be a perfect $m$-ary tree of depth $d - 2$. Notice that $T$ is the sub-tree of $T'$ with the leaves and the parents of the leaves of $T'$ removed. Since $T$ has a smaller depth than $T'$, the formula for $\gamma_{\text{alt},2}(T)$ holds. The parity of $d$ does not matter since the parity of the depth of $T$ and $T'$ will always match. We next map the movements of guards in response to attacks at vertices in $T'$ to the movements of guards in $T$.

We will show that $\gamma_{\text{alt},2}(T') - m^{d-2}$ guards suffice to defend against any sequence of attacks on $T$.

Consider any minimum eternal 2-dominating set on $T'$. If a guard occupies $v' \in V(T') \setminus (Z_1 \cup Z_2 \cup Z_3)$, there will be a shadow guard on the corresponding vertex $v \in V(T)$. If there are $q > 1$ guards in the sub-tree rooted at $z'_3 \in Z_3$ then in $T$, then $q - 1$ guards occupy the corresponding vertex $z_3 \in V(T)$, since $Z_3 \subseteq V(T')$. If there is exactly one guard in the sub-tree rooted at $z'_3 \in Z_3$ then no corresponding guard in needed in $T$. We observe, however, that $z_3$ in $T$ will be 2-dominated by a guard in $V(T)$: in $T'$, if the guard in $Z_3$ moved to a grandchild of $z'_3$ in response to an attack, a guard from $V(T)$ must move to $z'_3$. It is clear that the $\gamma_{\text{alt},2}(T') - m^{d-2}$ vertices occupied by guards in $T$ form a 2-dominating set.

We now consider the subsequent attack in two cases.
(1) If a vertex in \(V(T')\backslash(Z_1 \cup Z_2 \cup Z_3)\) is attacked, the guards in \(T\) consider an attack at the corresponding vertex in \(T\).

(2) If a vertex in \(Z_1 \cup Z_2 \cup Z_3\) is attacked, let \(z'_3 \in Z_3\) be the vertex in \(Z_3\) closest to the attacked vertex. By Lemma \[27\] when the guards in \(T'\) move in response to the attack, their positions will form an eternal 2-dominating set that contains \(z'_3\), and the attacked vertex. The guards in \(T\) consider an attack at the corresponding vertex \(z_3 \in V(T)\).

If a guard at vertex \(u' \in V(T')\backslash(Z_1 \cup Z_2 \cup Z_3)\) moves to \(w' \in V(T')\backslash(Z_1 \cup Z_2 \cup Z_3)\) then a guard moves from \(u \in V(T)\) to \(w \in V(T)\).

If a guard moves from vertex of \(z'_3 \in Z_3\) to \(y' \in V(T')\backslash(Z_1 \cup Z_2 \cup Z_3)\) then by Lemma \[27\] there must be another guard in \(Z_1 \cup Z_2 \cup Z_3\) that moves to \(z'_3\). Then in \(T\), there is a guard at \(z_3\) before the attack and this guard moves to \(y \in V(T)\) after the attack.

We note that a guard \(g\) will not move from a vertex of \(Z_1 \cup Z_2\) to a vertex in \(V(T)\backslash(Z_1 \cup Z_2)\) because if there is a guard at a vertex of \(Z_1 \cup Z_2\), then by Lemma \[27\] there is also a guard at the closest vertex in \(z'_3 \in Z_3\). We may assume the guard at \(z'_3\) moves to the vertex \(y \in V(T')\backslash(Z_1 \cup Z_2)\), leaving guard \(g\) in \(Z_1 \cup Z_2\) to move to \(z'_3\). In \(T\), the guard at \(z_3\) moves to vertex \(y \in V(T)\).

If a guard \(g\) moves from a vertex of \(Z_1 \cup Z_2\) to a vertex of \(Z_1 \cup Z_2\) then we observe that by Lemma \[27\] there must be \(q > 1\) guards in the sub-tree rooted at the nearest vertex of \(Z_3\), at least one of whom is located on \(z'_3\). We therefore may assume guard \(g\) is not needed in the eternal 2-dominating set on \(T\).

After the guards have all moved on \(T\), they still form an eternal 2-dominating set in \(T\). Thus, we see that \(\gamma_{alt,2}^{\infty}(T) \leq \gamma_{alt,2}^{\infty}(T') - m^{d-2}\) guards are sufficient to defend against any sequence of attacks in \(T\). Thus,

\[
\gamma_{alt,2}^{\infty}(T) \leq 1 + \frac{m^{d-2} - \delta_{oe}}{m^2 - 1} \leq \gamma_{alt,2}^{\infty}(T') - m^{d-2} \leq \frac{m^d - \delta_{oe}}{m^2 - 1} - m^{d-2}
\]

which provides as contradiction as \(m \geq 2\). Thus, the lower bound has been proven. \(\square\)

### 4.2 Reductions on trees for \(k = 2\)

The results in Section \[3\] provide reductions for trees that control the change in eternal \(k\)-domination number. In this section, we restrict ourselves to \(k = 2\). We show that for trees with certain structure, deleting a portion
of the tree results in the eternal 2-domination number decreasing by 1. We then consider the situations where the eternal 2-domination does not decrease when removing another part of the graph, given further structure.

For the duration of this section, we require the following particular description of tree structure.

**Definition 29.** Let $x$ be a vertex in tree $T$ that is not a leaf and is distance exactly two from a leaf. Define $L$ to be the set of leaves in $N_2(x)$ for $T$. Let $X$ be the set of all vertices that are distance two from vertices of $L$, that is $X = \cup_{\ell \in L} N_2(\ell)$. Let $S$ be the set of vertices adjacent to $x$ that are either leaves themselves or adjacent to a vertex in $L$. We further partition $S$ into two sets: $A \subset S$ is the set of vertices with a least two neighbours in $X$; $B \subseteq S$ is the set of vertices with exactly one neighbour in $X$.

As seen in Figure 4 the sets $L, S, A$ and $B$ are all dependent on the vertex $x$, and we can consider these sets defined for any eligible vertex $x \in V(T)$. Though $L, S, A, B$ each depend on the choice of $x$, we omit any subscripts in an aim to present results and definitions in a more readable way.

![Figure 4: Before and after applying Theorem 22](image)

**Theorem 30.** Let $T$ be a tree and suppose there exists a vertex $x \in V(T)$ that generates sets $L, S, A$ and $B$ as defined in Definition 29 such that $A = \emptyset$ and $|B| > 1$. Then $\gamma_{\infty,2}(T) = \gamma_{\infty,2}(T') + 1$ where $T'$ is the graph induced by the deletion of $L \cup B$ from $T$.

**Proof.** First, we show $\gamma_{\infty,2}(T) \geq \gamma_{\infty,2}(T') + 1$. For the sake of counter example, suppose $\gamma_{\infty,2}(T) = \gamma_{\infty,2}(T')$. Observe that there must always be at least one guard on the vertices $L \cup B \cup \{x\}$. If there is only one guard...
on this set of vertices, it must be located at \( x \). This implies there must always be a guard at vertex \( x \) in \( T \). Suppose a leaf of \( L \) is attacked in \( T \), then a guard \( g \) moves from \( x \) to the leaf. However, if another leaf in \( L \) is attacked subsequently, the guard \( g \) cannot reach this other leaf, thus in \( T \) we require that there must be some new guard that moves onto \( x \) when \( g \) initially moves protect the leaf from the attack. Thus, there is an eternal 2-dominating family for \( T' \) wherein every eternal 2-dominating set contains \( x \). Thus, there is an unnecessary guard in \( T' \), so \( \gamma_{\text{all},2}(T') < \gamma_{\text{all},2}(T) \) as desired.

It is easy to see \( \gamma_{\text{all},2}(T) \leq \gamma_{\text{all},2}(T') + 1 \): we create a guard strategy for \( T \), based on the movements of guards in \( T' \). Initially, suppose guards occupy an eternal 2-dominating set on \( T' \) that contains \( x \). Place guards on the vertices of \( V(T') \) in \( T \) and place an additional guard on an arbitrary vertex of \( B \cup L \). Suppose there is an attack on a vertex of \( V(T') \) in \( T \): if there is a guard in \( B \cup L \), that guard moves to \( x \). The remaining guards move as their counterparts in \( T' \) would move in response to such an attack (this may result in two guards simultaneously occupying \( x \) ). Alternately, suppose there is an attack at a vertex of \( B \cup L \) in \( T \). In \( T' \), we consider an attack at \( x \). The guards in \( T' \) move to occupy an eternal 2-dominating set containing \( x \) (which may result in no guard moving). In \( T \), the guard at \( x \) moves to the attacked vertex in \( B \cup L \); if there is a guard already in \( B \cup L \), that guard moves to \( x \), and the remaining guards move like their counterparts in \( T' \). Thus, the guards form a 2-dominating set on \( T \). In this manner, \( \gamma_{\text{all},k}(T') + 1 \) guards can defend against any sequence of attacks on \( V(T) \). □

Notice that in Theorem 30, we exclude the case where \( |B| = 1 \). This is because there exist trees with \( |B| = 1 \) for which removing set \( L \cup B \) will reduce the eternal 2-domination number, \( T_1 \), and others which will leave the eternal 2-domination number unchanged, \( T_2 \); two such trees are illustrated in Figure 5.

In Theorem 30, we provided a reduction for the case where \( A = \emptyset \). The following result tells us when \( A \) is non-empty.

**Theorem 31.** Let \( T \) be a tree and \( x \) a non-leaf vertex that is a grandparent of a leaf. Then \( A \neq 0 \) iff there exists a leaf \( \ell \) that has two distinct grandparents, one which is \( x \), and another \( x' \).

**Proof.** Let \( T \) be a tree and \( x \) a grandparent of a leaf, and create the sets as defined in Definition 29.
Suppose $A \neq \emptyset$. This means $|X| \geq 2$ and there exists a vertex, $a$ that adjacent to $x$ and another element of $X$. Let $\ell$ be the leaf, $y$ the parent and $x$ the grandparent. $x$ has a neighbour, $a$ that is adjacent to another vertex, $x'$. Since $x'$ need to be a member of $X$ it is within distance two of a leaf, $\ell'$ that is distance two away from $x$ (note it is possible that $\ell = \ell'$). Thus $\ell'$ had two distinct grandparents, one of which is $x$.

Now suppose that there is some leaf $\ell$ that has two distinct grandparents, one of which is $x$, the other $x'$. Note that both $x, x' \in X$. Since $\ell$ is a leaf, there is a common neighbour to $\ell, x, x'$ that we will call $a$. From the definition of set $A$, $a \in A$, so $A \neq \emptyset$.

Though we have settled how identify the case and apply reductions to the case where $A = \emptyset$, the example below demonstrates that the case where $A \neq \emptyset$ is not straightforward.

Figure 6: Tree $T$ with $A \neq \emptyset$

Consider the tree $T$ in Figure 6. The eternal 2-domination number for this tree is 3. When $x$ is $x_2, x_5$ or $x_9$ the eternal 2-domination number and we create the sets as defined in Definition 29 and remove the vertices of $L \cup B$, }

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the does not change, but when \( x \) is \( x_3 \) or \( x_4 \), the eternal 2-domination number decreases by 1.

Thus, the characterization of \( \gamma^\infty_{\text{all},2}(T) \) for all trees remains incomplete and we leave this as an open problem. In Section 5 we present further open problems and concluding remarks.

5 Conclusion and Open Problems

We conclude this paper with a discussion of open problems. In Section 2.1, discussed some graphs for which the parameters \( \gamma_k \) and \( \gamma^\infty_{\text{all},k} \) are equal, and others for which the two parameters are not equal; however the question of for which graphs the parameters equal remains open:

**Question 32.** Can we describe the class of graphs \( (G) \) for which \( \gamma_k(G) = \gamma^\infty_{\text{all},k}(G) \) for all \( G \in (G) \)?

**Question 33.** Let \( G_{n,m} \) be the family of simple graphs on \( n \) vertices and \( m \) edges. For a fixed \( n \) and \( m \), what are the optimal families of graphs, that is, what are the graphs with the smallest eternal \( k \)-domination number? Also what are the least optimal families, that is, the graphs with the largest eternal \( k \)-domination number?

**Question 34.** Given a value of \( k \) and an eternal \( k \)-domination number what is the spectrum of graph orders \( n \) that satisfy the given constraints?

For any given \( k \), if we fix the \( k \)-eternal domination number to be 1, the size of the vertex set, \( n \), can take on any value, just consider the star graph. One interesting result we can obtain from Theorem 7 is the following. Let \( P_{z(k+1),\ell} \) be \( P_n \) with \( \ell \) leaves added to a vertex adjacent to one of the leaves of \( P_n \).

**Corollary 35.** For any given positive integers \( k, z \) and \( n \), with \( n \geq z(k+1) \) there exists a graph on \( n \) vertices whose eternal \( k \)-domination number is \( z \), namely \( P_{z(k+1),\ell} \), where \( \ell = n - z(k+1) \).

**Proof.** For a given \( k \) and any positive integer \( z \) and any positive integer \( n \geq z(k+1) \) consider the path \( P_{z(k+1)} \). This has \( z(k+1) \) vertices and from Theorem 7 we know that the eternal \( k \)-domination number is \( z \). Place a \( k \)-dominating set on this graph so that it is eternally \( k \)-dominated. Label
the vertices of the path $v_1, v_2, \ldots, v_{z(k+1)}$, with $v_1$ and $v_{z(k+1)}$ being the leaves. Add $n - z(k + 1)$ leaves to vertex $v_{z(k+1)-1}$. The guard that is dominating $v_{z(k+1)}$ also dominates these new leaves. Thus, this new graph has $n$ vertices and eternal $k$-domination number $z$. \qed

Though we provide reductions in working towards determining $\gamma_{all,k}^\infty(T)$ and further reductions in working towards determining $\gamma_{all,2}^\infty(T)$ for any tree $T$, we were unable to complete the characterizations and leave them as open problems. We additionally state a few related questions.

**Question 36.** Which graphs $G$ have the property that $\gamma_{all,2}^\infty(G) = \gamma(G)$? Can we characterize the trees with this property?

Clearly $\gamma_{all,2}^\infty(G) = 1$ if and only if $\gamma(G) = 1$ (i.e. $G$ has a universal vertex). If $\gamma(G) = 2$ then $\gamma_{all,2}^\infty(G) = 2$, but the converse is not always true. For example, $\gamma_{all,2}^\infty(C_{10}) = 2 < 4 = \gamma(C_{10})$. In considering trees, we observe that $\gamma_{all,2}^\infty(K_{1,n}) = \gamma(K_{1,n})$ but for caterpillars graphs where there are no degree 2 vertices, $\gamma_{all,2}^\infty$ and $\gamma$ are not equal. However, for every tree $T$ that is formed from a path of $n$ vertices where every internal vertex has a leaf, with $n \geq 3$, we have that $\gamma_{all,2}^\infty(T) \neq \gamma(T)$.

Certainly if there exists a minimum dominating st where each vertex in this set has at least two private neighbours then $\gamma_{all,2}^\infty(T) \leq \gamma(T)$, but the converse remains open.

**Question 37.** Suppose that for every minimum dominating set of a tree $T$, each vertex in the dominating set has at least two private neighbours. Then is $\gamma_{all,2}^\infty(T) = \gamma(T)$?

**Acknowledgements**

M.E. Messinger acknowledges research support from NSERC (grant application 2018-04059). D. Cox acknowledges research support from NSERC (2017-04401) and Mount Saint Vincent University. E. Meger acknowledges research support from Université du Québec à Montréal and Mount Allison University.

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