Bézier curves and surfaces based on modified Bernstein polynomials

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Abstract

In this paper, we use the blending functions of Bernstein polynomials with shifted knots for construction of Bézier curves and surfaces. We study the nature of degree elevation and degree reduction for Bézier Bernstein functions with shifted knots for \( t \in \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right] \). Parametric curves are represented using these modified Bernstein basis and the concept of total positivity is applied to investigate the shape properties of the curve. We get Bézier curve defined on \([0,1]\) when we set the parameter \(\alpha, \beta\) to the value 0. We also present a de Casteljau algorithm to compute Bernstein Bézier curves and surfaces with shifted knots. The new curves have some properties similar to Bézier curves. Furthermore, some fundamental properties for Bernstein Bézier curves and surfaces are discussed.

Keywords and phrases: Degree elevation; Degree reduction; de Casteljau algorithm; Bernstein operators with shifted knots; Bézier curve; Tensor product; Shape preserving; Total positivity.

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1 Introduction

It was S.N. Bernstein [1] in 1912, who first introduced his famous operators \(B_n : C[0,1] \to C[0,1]\) defined for any \( n \in \mathbb{N} \) and for any function \( f \in C[0,1]\) where \( C[0,1] \) denote the set of all continuous functions on \([0,1]\) which is equipped with sup-norm \( \| \cdot \|_{C[0,1]} \)

\[
B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left( \frac{k}{n} \right), \quad x \in [0,1].
\] (1.1)

and named it Bernstein polynomials to prove the Weierstrass theorem [9].

Bernstein showed that if \( f \in C[0,1] \), then \( B_n(f; x) \Rightarrow f(x) \) where \( \Rightarrow \) represents the uniform convergence. One can find a detailed monograph about the Bernstein polynomials in [10].

Later it was found that Bernstein polynomials possess many remarkable properties and has various applications in areas such as approximation theory [9], numerical analysis, computer-aided geometric design, and solutions of differential equations due to its fine properties of approximation [18].

In computer aided geometric design (CAGD), Bernstein polynomials and its variants are used in order to preserve the shape of the curves or surfaces. One of the most important curve in CAGD [21] is the classical Bézier curve [2] constructed with the help of Bernstein basis functions. Other
works related to different generalization of Bernstein polynomials and bezier curves and surfaces can be found in \[3, 4, 5, 7, 8, 11, 12, 13, 14, 15, 16, 18, 19, 20\]

In 1968 Stancu \[22\] showed that the polynomials

\[
P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right)\]

(1.2)

converge to continuous function \(f(x)\) uniformly in \([0,1]\) for each real \(\alpha, \beta\) such that \(0 \leq \alpha \leq \beta\). The polynomials (1.2) are called as a Bernstein-Stancu polynomials.

In 2010, Gadjiev and Gorhanalizadeh \[6\] introduced the following construction of Bernstein-Stancu type polynomials with shifted knots:

\[
S_n^{(\alpha, \beta)}(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha_1}{n+\beta_1}\right)\]

(1.3)

where \(\frac{\alpha_1}{n+\beta_1} \leq x \leq \frac{n+\alpha_2}{n+\beta_2}\) and \(\alpha_k, \beta_k\) \((k = 1, 2)\) are positive real numbers provided \(0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2\). It is clear that for \(\alpha_2 = \beta_2 = 0\), then polynomials (1.3) turn into the Bernstein-Stancu polynomials (1.2) and if \(\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0\) then these polynomials turn into the classical Bernstein polynomials.

In recent years, generalization of the Bézier curve with shape parameters has received continuous attention. Several authors were concerned with the problem of changing the shape of curves and surfaces, while keeping the control polygon unchanged and thus they generalized the Bézier curves in \[7, 8, 17, 18\].

The outline of this paper is as follow: Section 2 introduces a modified Bernstein functions with shifted knots \(G_n^{(\alpha, \beta)}\) and their Properties. Section 3 introduces degree elevation and degree reduction properties for these modified Bernstein functions. Section 3.2 introduces a de Casteljau algorithm for \(G_n^{(\alpha, \beta)}\). In Section 4 we define a tensor product patch based on algorithm 1 and its geometric properties as well as a degree elevation technique are investigated. Furthermore tensor product of Bézier surfaces on \([\alpha_n + \beta, n+\alpha \beta]\times[\alpha_n + \beta, n+\alpha \beta]\) for Bernstein polynomials with shifted knots are introduced and its properties that is inherited from the univariate case are being discussed.

In next section, we construct basis functions with shifted knots with the help of (1.3).

## 2 Bernstein functions with shifted knots

The Bernstein functions with shifted knots is defined as follows

\[
G_n^{(\alpha, \beta)}(t) = \binom{n}{k} \left(\frac{n+\beta}{n}\right)^n \left(\frac{t-\alpha}{n+\beta}\right)^k \left(t-\frac{n+\alpha}{n+\beta}\right)^{n-k}\]

(2.1)

where \(\frac{\alpha}{n+\beta} \leq t \leq \frac{n+\alpha}{n+\beta}\) and \(\alpha, \beta\) are positive real numbers provided \(0 \leq \alpha \leq \beta\).

### 2.1 Properties of the Bernstein functions with shifted knots

**Theorem 2.1** The Bernstein functions with shifted knots possess the following properties:

1. Non-negativity: \(G_n^{(\alpha, \beta)}(t) \geq 0 \quad k = 0, ..., n, \quad t \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]\).
(2.) Partition of unity:

\[ \sum_{k=0}^{n} G_{n,\alpha,\beta}^k (t) = 1, \quad \text{for every } t \in \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right]. \]

(3.) End-point interpolation property holds:

\[ G_{n,\alpha,\beta}^k \left( \frac{\alpha}{n+\beta} \right) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases} \]

\[ G_{n,\alpha,\beta}^k \left( \frac{n+\alpha}{n+\beta} \right) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases} \]

Clearly both sided end point interpolation property holds.

(4.) Reducibility: when \( \alpha = \beta = 0 \) formula (2.1) reduces to the classical Bernstein bases on \([0,1]\).

**Proof:** All these property can be deduced easily from equation (2.1).

![Cubic Bezier blending functions with shifted knots](image)

Figure 1: ‘Cubic Bezier blending functions with shifted knots’

Fig. 1 shows the modified Bernstein basis functions of degree 3 with shifted knots for \( \alpha = 4, \beta = 6 \). Here we can observe that sum of blending functions is always unity and also satisfies end point interpolation property. In case \( \alpha = \beta = 0 \), it turns out to be classical Bernstein basis on \([0,1]\) which is shown in Fig. 2.

Apart from the basic properties above, the Bernstein functions with shifted knots also satisfy some recurrence relations, as for the classical Bernstein basis.
3 Degree elevation and reduction for Bernstein functions with shifted knots

Technique of degree elevation has been used to increase the flexibility of a given curve. A degree elevation algorithm calculates a new set of control points by choosing a convex combination of the old set of control points which retains the old end points. For this purpose, the identities (3.1), (3.2) and Theorem (3.1) are useful.

Degree elevation

\[
\left( \frac{n + \alpha}{n + \beta} - t \right) G_{n,\alpha,\beta}^k (t) = \left( \frac{n + \alpha}{n + \beta} - t \right) \left( \frac{n + \beta}{n} \right)^n \left( t - \frac{\alpha}{n + \beta} \right)^k \left( \frac{n + \alpha}{n + \beta} - t \right)^{n-k}
\]

and

\[
\left( t - \frac{\alpha}{n + \beta} \right) G_{n,\alpha,\beta}^k = \left( \frac{n}{n + \beta} \right) \left( \frac{n + \alpha}{n + \beta} - t \right) \left( \frac{n + 1}{n + \beta} \right) \left( t - \frac{\alpha}{n + \beta} \right)^k \left( \frac{n + \alpha}{n + \beta} - t \right)^{n-k+1}
\]

Proof:
Consider

\[
\left( \frac{n + \alpha}{n + \beta} - t \right) G_{n,\alpha,\beta}^k = \left( \frac{n + \alpha}{n + \beta} - t \right) \left( \frac{n}{n + \beta} \right) \left( \frac{n + \alpha}{n + \beta} - t \right)^k \left( \frac{n + \alpha}{n + \beta} - t \right)^{n-k}
\]

Similarly for
where

Bernstein functions with shifted knots of degree \( n \) is a linear combination of two Bernstein functions with shifted knots of degree \( n + 1 \):

\[
G_{n,\alpha,\beta}^k (t) = (n + 1 - k) \frac{n + \alpha}{n + 1} G_{n+1,\alpha,\beta}^k (t) + \left( \frac{k + 1}{n + 1} \right) G_{n+1,\alpha,\beta}^{k+1} (t)
\]

(3.3)

where

\[
\frac{\alpha}{n+\beta} \leq t \leq \frac{n+\alpha}{n+\beta} \quad \text{and} \quad \alpha, \beta \quad \text{are positive real numbers satisfying} \quad 0 \leq \alpha \leq \beta.
\]

Proof:

\[
\left( \frac{n}{n + \beta} \right) G_{n,\alpha,\beta}^k (t) = G_{n,\alpha,\beta}^k \left( \frac{n + \alpha}{n + \beta} - t + \frac{t - \alpha}{n + \beta} \right)
\]

\[
\left( \frac{n}{n + \beta} \right) G_{n,\alpha,\beta}^k (t) = \left( \frac{n + \alpha}{n + \beta} - t \right) G_{n,\alpha,\beta}^k + \left( t - \frac{\alpha}{n + \beta} \right) G_{n,\alpha,\beta}^k
\]

on using equation (3.1), (3.2), we can easily get

\[
G_{n,\alpha,\beta}^k (t) = \left( \frac{n + 1 - k}{n + 1} \right) G_{n+1,\alpha,\beta}^k (t) + \left( \frac{k + 1}{n + 1} \right) G_{n+1,\alpha,\beta}^{k+1} (t)
\]

Theorem 3.2 Each Bernstein functions with shifted knots of degree \( n \) is a linear combination of two Bernstein functions with shifted knots of degree \( n - 1 \):

\[
G_{n,\alpha,\beta}^k (t) = \frac{n + \beta}{n} \left( t - \frac{\alpha}{n + \beta} \right) G_{n-1,\alpha,\beta}^{k-1} (t) + \frac{n + \beta}{n} \left( \frac{n + \alpha}{n + \beta} - t \right) G_{n-1,\alpha,\beta}^k (t)
\]

(3.4)

where

\[
\frac{\alpha}{n+\beta} \leq t \leq \frac{n+\alpha}{n+\beta} \quad \text{and} \quad \alpha, \beta \quad \text{are positive real numbers satisfying} \quad 0 \leq \alpha \leq \beta.
\]
Proof On using Pascal’s type relation i.e., we get

\[ G_{k,n,\alpha,\beta}^{(t)} = \binom{n}{k} \left( \frac{n+\beta}{n} \right)^n \left( t - \frac{\alpha}{n+\beta} \right)^k \left( \frac{n+\alpha}{n+\beta} - t \right)^{n-k} \]

\[ = \left\{ \frac{n-1}{(k-1)} + \binom{n-1}{k} \right\} \left( \frac{n+\beta}{n} \right)^n \left( t - \frac{\alpha}{n+\beta} \right)^k \left( \frac{n+\alpha}{n+\beta} - t \right)^{n-k} \]

\[ + \left( \frac{n-1}{k} \right) \left( \frac{n+\beta}{n} \right)^n \left( t - \frac{\alpha}{n+\beta} \right)^k \left( \frac{n+\alpha}{n+\beta} - x \right)^{n-k} \]

\[ = \frac{n+\beta}{n} \left( t - \frac{\alpha}{n+\beta} \right) G_{n-1,\alpha,\beta}^{k-1}(t) + \frac{n+\beta}{n} \left( \frac{n+\alpha}{n+\beta} - t \right) G_{n-1,\alpha,\beta}^k(t) \]

Theorem 3.3 The end-point property of derivative:

\[ \frac{d}{dt} \left( \frac{\alpha}{n+\beta} \right) = (n+\beta)(P_1 - P_0) \left( \frac{n-1+\alpha}{n-1} \right)^{n-1} \left( \frac{n-1+\alpha}{n-1+\beta} - \frac{\alpha}{n+\beta} \right)^{n-1-k} \quad (3.5) \]

\[ \frac{d}{dt} \left( \frac{n+\alpha}{n+\beta} \right) = (n+\beta)(P_n - P_{n-1}) \left( \frac{n-1+\alpha}{n-1} \right)^{n-1} \left( \frac{n+\alpha}{n+\beta} - \frac{\alpha}{n-1+\beta} \right)^{n-1} \quad (3.6) \]

i.e. Bernstein-Bézier curves with shifted knots are tangent to fore-and-aft edges of its control polygon at end points.

Proof: Let

\[ P(t) = \sum_{k=0}^{n} P_k G_{n,\alpha,\beta}^k(t) \]

\[ = \sum_{k=0}^{n} P_k \binom{n}{k} \left( \frac{n+\beta}{n} \right)^n \left( t - \frac{\alpha}{n+\beta} \right)^k \left( \frac{n+\alpha}{n+\beta} - t \right)^{n-k} \]

or

\[ P(t) = V(t) \]

then on differentiating both hand side with respect to ‘t’, we have

\[ \frac{d}{dt} P(t) = \frac{d}{dt} V(t) \]

Let

\[ A_{\alpha}^{n}(t) = \binom{n}{k} \left( \frac{n+\beta}{n} \right)^n \left( t - \frac{\alpha}{n+\beta} \right)^k \left( \frac{n+\alpha}{n+\beta} - t \right)^{n-k} \]

then

\[ V(t) = \sum_{k=0}^{n} P_k A_{\alpha}^{n}(t) \]

6
\[(A_k^n)'(t) = \binom{n}{k} \left(\frac{n + \beta}{n}\right)^{k} \left(\frac{t - \alpha}{n + \beta}\right)^{k-1} \left(\frac{n + \alpha - t}{n + \beta - t}\right)^{n-k} \]
\[\binom{n}{k} \left(\frac{n + \beta}{n}\right)^{k} \left(\frac{t - \alpha}{n + \beta}\right)^{k} (n-k) \left(\frac{n + \alpha - t}{n + \beta - t}\right)^{n-k-1} \]
\[= (n + \beta)\{A_k^{n-1}(t) - A_k^{n-1}(t)\} \]

which implies

\[V'(t) = \sum_{k=0}^{n} P_k(A_k^n)'(t).\]

Now

\[V\left(\frac{\alpha}{n + \beta}\right) = P\left(\frac{\alpha}{n + \beta}\right) = (n + \beta)(P_1 - P_0)A_0^{n-1}(t)\]

and

\[P\left(\frac{\alpha}{n + \beta}\right) = (n + \beta)(P_1 - P_0)\left(\frac{n - 1 + \beta}{n - 1}\right)^{n-1} \left(\frac{n - 1 + \alpha}{n - 1 + \beta - \alpha}\right)^{n-1-k} \]

Similarly after some computation, we have

\[V\left(\frac{n + \alpha}{n + \beta}\right) = P\left(\frac{n + \alpha}{n + \beta}\right) = (n + \beta)(P_n - P_{n-1})A_{n-1}^{n-1}(\frac{n + \alpha}{n + \beta})\]

\[P\left(\frac{n + \alpha}{n + \beta}\right) = (n + \beta)(P_n - P_{n-1})\left(\frac{n - 1 + \beta}{n - 1}\right)^{n-1} \left(\frac{n + \alpha - \alpha}{n + \beta - \alpha}\right)^{n-1} \]

3.1 Degree elevation for Bézier curves with shifted knots

Bézier curves with shifted knots have a degree elevation algorithm that is similar to that possessed by the classical Bézier curves. Using the technique of degree elevation, we can increase the flexibility of a given curve.

\[P(t) = \sum_{k=0}^{n} P_k G_{n,\alpha,\beta}^k (t) \]

\[P(t) = \sum_{k=0}^{n+1} P_k^* G_{n+1,\alpha,\beta}^k (t), \]

where

\[P_k^* = \left(1 - \frac{n + 1 - k}{n + 1}\right) P_{k-1} + \left(\frac{k}{n + 1}\right) P_k \]  \hspace{1cm} (3.7)

The statement above can be derived from Theorem 3.1. When \(\alpha = \beta = 0\) formula 3.7 reduce to the degree evaluation formula of the Bézier curves. If we let \(P = (P_0, P_1, ..., P_n)^T\) denote the vector of control points of the initial Bézier curve of degree \(n\), and \(P^{(1)} = (P_0^*, P_1^*, ..., P_{n+1}^*)\) the vector of
control points of the degree elevated Bézier curve of degree \( n + 1 \), then we can represent the degree elevation procedure as:

\[
P^{(1)} = T_{n+1} P,
\]

where

\[
T_{n+1} = \frac{1}{n+1} \begin{bmatrix} n+1 & 0 & \ldots & 0 & 0 \\ n+1-n & n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & n+1-2 & 2 & 0 \\ 0 & 0 & \ldots & n+1-1 & 1 \\ 0 & 0 & \ldots & 0 & n+1 \end{bmatrix}_{(n+2) \times (n+1)}
\]

For any \( l \in \mathbb{N} \), the vector of control points of the degree elevated Bézier curves of degree \( n+l \) is: \( P^{(l)} = T_{n+l}T_{n+2}...T_{n+1}P \). As \( l \to \infty \), the control polygon \( P^{(l)} \) converges to a Bézier curve.

### 3.2 de Casteljau algorithm:

Bézier curves with shifted knots of degree \( n \) can be written as two kinds of linear combination of two Bézier curves with shifted knots of degree \( n - 1 \), and we can get the two selectable algorithms to evaluate Bézier curves with shifted knots. The algorithms can be expressed as:

**Algorithm 1.**

\[
\begin{align*}
P_0^0(t) &\equiv P_i^0 \equiv P_i \quad i = 0, 1, 2, \ldots, n, \\
P_r^1(t) &= \frac{n+\beta}{n} \left( t - \frac{\alpha}{n+\beta} \right) P_{r+1}^r(t) + \frac{n+\beta}{n} \left( \frac{n+\alpha}{n+\beta} - t \right) P_{r-1}^r(t) \quad (3.8) \\
r &= 1, \ldots, n, \quad i = 0, 1, 2, \ldots, n - r, \quad \frac{n+\alpha}{n+\beta} \leq t \leq \frac{n+\beta}{n+\beta}, \quad 0 \leq \alpha \leq \beta.
\end{align*}
\]

Then

\[P(t) = \sum_{i=0}^{n-1} P_i^1(t) = \ldots = \sum_{i=0}^{n-1} P_i^1(t) b_{r,n-r}^i(t) = \ldots = P_0^n(t) \quad (3.9)\]

It is clear that the results can be obtained from Theorem (3.2). When \( \alpha = \beta = 0 \), formula (3.8) and (3.9) recover the de Casteljau algorithms of classical Bézier curves. Let \( P^0 = (P_0, P_1, \ldots, P_n)^T \), \( P^r = (P_0^r, P_1^r, \ldots, P_n^r)^T \), then de Casteljau algorithm can be expressed as:

**Algorithm 2.**

\[
P^r(t) = M_r(t)P_0^0 \quad (3.10)
\]

where \( M_r(t) \) is a \((n-r+1) \times (n-r+2)\) matrix and

\[
M_r(t) = \frac{n+\beta}{n} \begin{bmatrix} \frac{n+\alpha}{n+\beta} - t & \frac{t-\alpha}{n+\beta} & \ldots & 0 & 0 \\ 0 & \frac{n+\alpha}{n+\beta} - t & \frac{t-\alpha}{n+\beta} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \frac{n+\alpha}{n+\beta} - t & \frac{t-\alpha}{n+\beta} & 0 \\ 0 & 0 & \ldots & \frac{n+\alpha}{n+\beta} - t & \frac{t-\alpha}{n+\beta} \end{bmatrix}
\]
4 Tensor product Bézier surfaces with shifted knots

on \( \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right] \times \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right] \)

We define a two-parameter family \( P(u,v) \) of tensor product surfaces of degree \( m \times n \) as follow:

\[
P(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} G^i_{m,\alpha,\beta}(u) G^j_{n,\alpha,\beta}(v), \quad (u,v) \in \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right] \times \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right],
\]

where \( P_{i,j} \in \mathbb{R}^3 \) (\( i = 0, 1, ..., m, j = 0, 1, ..., n \)), and \( G^i_{m,\alpha,\beta}(u) \), \( G^j_{n,\alpha,\beta}(v) \) are modified Bernstein functions respectively. We refer to the \( P_{i,j} \) as the control points. By joining up adjacent points in the same row or column to obtain a net which is called the control net of tensor product Bézier surface.

4.1 Properties

1. Geometric invariance and affine invariance property: Since

\[
\sum_{i=0}^{m} \sum_{j=0}^{n} G^i_{m,\alpha,\beta}(u) G^j_{n,\alpha,\beta}(v) = 1,
\]

\( P(u,v) \) is an affine combination of its control points.

2. Convex hull property: \( P(u,v) \) is a convex combination of \( P_{i,j} \) and lies in the convex hull of its control net.

3. Isoparametric curves property: The isoparametric curves \( v = v^* \) and \( u = u^* \) of a tensor product Bézier surface are respectively the Bézier curves with shifted knots of degree \( m \) and degree \( n \), namely,

\[
P(u,v^*) = \sum_{i=0}^{m} \left( \sum_{j=0}^{n} P_{i,j} G^j_{n,\alpha,\beta}(v^*) \right) G^i_{m,\alpha,\beta}(u), \quad u \in \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right];
\]

\[
P(u^*,v) = \sum_{j=0}^{n} \left( \sum_{i=0}^{m} P_{i,j} G^i_{m,\alpha,\beta}(u^*) \right) G^j_{n,\alpha,\beta}(v), \quad v \in \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right].
\]

The boundary curves of \( P(u,v) \) are evaluated by \( P(u, \frac{\alpha}{n+\alpha}) \), \( P(u, \frac{n+\alpha}{n+\beta}) \), \( P(\frac{\alpha}{n+\alpha}, v) \) and \( P(\frac{n+\alpha}{n+\beta}, v) \).

4. Corner point interpolation property: The corner control net coincide with the four corners of the surface. Namely, \( P(\frac{\alpha}{n+\alpha}, \frac{\alpha}{n+\alpha}) = P_{0,0}, P(\frac{\alpha}{n+\alpha}, \frac{n+\alpha}{n+\beta}) = P_{0,n}, P(\frac{m+\alpha}{m+\beta}, \frac{\alpha}{n+\alpha}) = P_{m,0}, P(\frac{m+\alpha}{m+\beta}, \frac{n+\alpha}{n+\beta}) = P_{m,n} \).

5. Reducibility: When \( \alpha = \beta = 0 \) formula (4.1) reduces to a classical tensor product Bézier patch.

4.2 Degree elevation and de Casteljau algorithm

Let \( P(u,v) \) be a tensor product Bézier surface with shifted knots of degree \( m \times n \). As an example, let us take obtaining the same surface as a surface of degree \( (m+1) \times (n+1) \). Hence we need to find new control points \( P'_{i,j} \) such that
Given the control net \( P \), we set the parameter \( \alpha \) for the intermediate point \( P_{i,j} \).

Then

\[
P_{i,j} = \alpha_i \beta_j P_{i-1,j-1} + \alpha_i (1 - \beta_j) P_{i-1,j} + (1 - \alpha_i) (1 - \beta_j) P_{i,j}\]

which can be written in matrix form as

\[
\begin{bmatrix}
1 - \frac{m+1-i}{m+1} & \frac{m+1-i}{m+1}
\end{bmatrix}
\begin{bmatrix}
P_{i-1,j-1} & P_{i-1,j} & P_{i,j}
\end{bmatrix}
\begin{bmatrix}
1 - \frac{n+1-j}{n+1}
\end{bmatrix}
\]

The de Casteljau algorithms are also easily extended to evaluate points on a Bézier surface. Given the control net \( P_{i,j} \in \mathbb{R}^3, i = 0, 1, ..., m, \ j = 0, 1, ..., n. \)

\[
P_{0,0}^0(u, v) = P_{i,j} \quad i = 0, 1, 2, ..., m; \ j = 0, 1, 2, n.
\]

\[
P_{r,t}^r(u, v) = \begin{bmatrix}
\frac{m+\beta}{m} & \frac{m+\beta}{m+\gamma} - t \\
\frac{m+\beta}{m+\gamma} & t - \frac{n}{m+\beta}
\end{bmatrix}
\begin{bmatrix}
P_{i,j}^r & P_{i,j+1}^r & P_{i+1,j}^r
\end{bmatrix}
\begin{bmatrix}
\frac{n+\gamma}{n+\beta} & \frac{n+\gamma}{n+\beta} & \frac{n+\gamma}{n+\beta}
\end{bmatrix}
\]

or

When \( m = n \), one can directly use the algorithms above to get a point on the surface. When \( m \neq n \), to get a point on the surface after \( k \) applications of formula (4.5), we perform formula (3.10) for the intermediate point \( P_{i,j}^k \).

Note: We get classical Bézier curves and surfaces for \( (u, v) \in \left[ \frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta} \right] \times \left[ \frac{\alpha}{n+\gamma}, \frac{n+\alpha}{n+\gamma} \right] \) when we set the parameter \( \alpha = \beta = 0. \)

5 Future work

In near future, we will construct \( q \)-analogue of Bézier curves and surfaces with shifted knots and we will also study de Casteljau algorithm and degree evaluation properties for curves and surfaces.

References

[1] S. N. Bernstein, Constructive proof of Weierstrass approximation theorem, *Comm. Kharkov Math. Soc.* (1912)

[2] P.E. Bézier, Numerical Control-Mathematics and applications, *John Wiley and Sons, London*, 1972.

[3] Cetin Disibuyuk and Halil Oruc, Tensor Product \( q \)-Bernstein Polynomials, *BIT Numerical Mathematics*, Springer 48 (2008) 689-700.

[4] Cetin Disibuyuk. "Tensor Product \( q \)Bernstein Bézier Patches", *Lecture Notes in Computer Science*, 2009.

[5] Rida T. Farouki, V. T. Rajan, Algorithms for polynomials in Bernstein form, *Computer Aided Geometric Design*, Volume 5, Issue 1, June 1988.
[6] A.D. Gadjiev, A.M. Ghorbanalizadeh, Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables, *Appl. Math. Comput.* 216 (3) (2010) 890-901.

[7] Khalid Khan, D.K. Lobiyal, Adem Kilicman, A de Casteljau Algorithm for Bernstein type Polynomials based on $(p, q)$-integers, *arXiv* 1507.04110.

[8] Khalid Khan, D.K. Lobiyal, Bezier curves based on Lupas $(p, q)$-analogue of Bernstein polynomials in CAGD, *arXiv*:1505.01810.

[9] P. P. Korovkin, Linear operators and approximation theory, *Hindustan Publishing Corporation, Delhi*, 1960.

[10] G.G. Lorentz, Bernstein Polynomials, *Univ. of Toronto Press*, Toronto, 1953.

[11] A. Lupaş, A $q$-analogue of the Bernstein operator, *Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca*, 9(1987) 85–92.

[12] M. Mursaleen, K. J. Ansari, A. Khan, On $(p, q)$-analogue of Bernstein Operators, *Applied Mathematics and Computation*, 266 (2015) 874-882.

[13] M. Mursaleen, K. J. Ansari and Asif Khan,, Some Approximation Results by $(p, q)$-analogue of Bernstein-Stancu Operators, *Applied Mathematics and Computation*, 264,(2015), 392-402.

[14] M. Mursaleen, Asif Khan, Generalized $q$-Bernstein-Schurer Operators and Some Approximation Theorems, *Journal of Function Spaces and Applications* Volume 2013, Article ID 719834, 7 pages [http://dx.doi.org/10.1155/2013/719834](http://dx.doi.org/10.1155/2013/719834).

[15] G. M. Phillips. "A survey of results on the $q$-Bernstein polynomials", *IMA Journal of Numerical Analysis*, 2009.

[16] G.M. Phillips, Bernstein polynomials based on the $q$-integers, *The heritage of P.L.Chebyshev, Ann. Numer. Math.*, 4 (1997) 511–518.

[17] Li-Wen Hana, Ying Chua, Zhi-Yu Qiu, Generalized Bézier curves and surfaces based on Lupaş $q$-analogue of Bernstein operator, *Journal of Computational and Applied Mathematics* 261 (2014) 352-363.

[18] Halil Oruk, George M. Phillips, $q$-Bernstein polynomials and Bézier curves, *Journal of Computational and Applied Mathematics* 151 (2003) 1-12.

[19] N. I. Mahmudov and P. Sabancgil, Some approximation properties of Lupaş $q$-analogue of Bernstein operators, *arXiv*:1012.4245v1 [math.FA] 20 Dec 2010.

[20] Sofiya Ostrovska, On the Lupaş $q$-analogue of the bernstein operator, *Rocky mountain journal of mathematics* Volume 36, Number 5, 2006.

[21] Thomas W. Sederberg, Computer Aided Geometric Design Course Notes, Department of Computer Science Brigham Young University, October 9, 2014.

[22] D.D. Stancu, Approximation of functions by a new class of linear polynomial operators, *Rev. Roumaine Math. Pure Appl.* 13 (1968) 1173-1194.