H∞-FUNCTIONAL CALCULUS AND MODELS OF NAGY-FOIAȘ TYPE FOR SECTORIAL OPERATORS

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ABSTRACT. We prove that a sectorial operator admits an H∞-functional calculus if and only if it has a functional model of Nagy- Foiaș type. Furthermore, we give a concrete formula for the characteristic function (in a generalized sense) of such an operator. More generally, this approach applies to any sectorial operator by passing to a different norm (the McIntosh square function norm). We also show that this quadratic norm is close to the original one, in the sense that there is only a logarithmic gap between them.

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1. Introduction

Let H be a separable Hilbert space and let L(H) be the Banach algebra of linear and bounded operators on H. For 0 < ω < π, we put

\[ S_ω := \{ z \in \mathbb{C} ; |\arg(z)| \leq ω \} \cup \{0\}. \]

Let A be a closed operator with domain D(A) and spectrum σ(A). Put ρ(A) := C \setminus σ(A). The operator A is said to be sectorial of type ω if σ(A) ⊂ S_ω and, for each θ with ω < θ < π, there exists C_θ such that

\[ \|(z - A)^{-1}\| \leq \frac{C_θ}{|z|^θ}, \quad z \notin S_θ. \]  

Each operator of this type has a decomposition A = 0 ⊕ A_0 with respect to some direct sum representation, H = Ker(A) ⊕ H_0, where Ker(A_0) = {0} and A_0 has a dense range. From now on, we assume that A itself is one-to-one, which is equivalent to the fact that it has dense range, see Remark 2.6 in [15].

Take θ ∈ (ω, π) and denote by S_θ^0 the interior of the sector S_θ. Following [16], we define the class \( \Psi(S_θ^0) \) as being formed by all holomorphic functions \( f \in \text{Hol}(S_θ^0) \) such that

\[ |f(z)| \leq C\frac{|z|^s}{1 + |z|^{2s}}, \quad \text{for all } z \in S_θ \text{ and some } s, C > 0. \]

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The operator $A$ admits a functional calculus (the so-called Dunford-Riesz calculus), which is constructed on the basis of the Cauchy operator-valued formula, and which provides us with bounded operators $\psi(A)$ for functions $\psi \in \Psi(S^0_\theta)$, $\theta > \omega$, see [16]:

$$(1.3) \quad \psi(A) := \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1}\psi(z) \, dz,$$

where $\gamma$ is the contour defined as $\gamma(t) = -te^{i\theta'}$ if $-\infty < t \leq 0$ and $\gamma(t) = te^{-i\theta'}$, if $0 \leq t < \infty$, for $\theta > \theta' > \omega$. The above calculus can be extended to functions in $H^\infty(S^0_{\theta'})$ (that is, bounded and analytic functions on $S^0_{\theta'}$) by the formula

$$f(A) := \frac{1}{\psi(A)}(f\psi)(A), \quad f \in H^\infty(S^0_{\theta'}) ,$$

although $f(A)$ may well be possibly unbounded [17]. The operator $A$ is said to have a $H^\infty(S^0_{\theta'})$-functional calculus if $f(A) \in L(H)$ for every $f \in H^\infty(S^0_{\theta'})$. In this case, the mapping $f \mapsto f(A)$ is a bounded Banach algebra homomorphism, see [9, Proposition 5.3.4].

We refer to [4, Theorem 3.3] for a discussion of the uniqueness of the continuation of the $H^\infty(\Omega)$-functional calculus from the set of rational functions to different subalgebras of $H^\infty(\Omega)$, where $\Omega$ is a disc or a simply connected domain, and to [23, Theorem 2.6] for a uniqueness result in the context of a functional calculus, which maps analytic functions on $S^0_{\theta'}$ that may have polynomial growth at 0 and at $\infty$ into the set of closed operators on $H$.

For every non-zero $\psi \in \Psi(S^0_{\theta'})$, set $\psi_t(z) := \psi(tz)$ for $t > 0$ and $z \in S^0_{\theta'}$. Define the norm

$$(1.4) \quad \|x\|_A := \left( \int_0^\infty \|\psi_t(A)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

on the linear manifold $H_c$ of those $x \in H$ for which this expression is finite, and let $H_A$ denote the Hilbert space obtained as the completion of $H_c$ with respect to the above norm. Different choices of $\theta$ and $\psi$ give rise to equivalent norms, so to the same space $H_A$ [15]. It is known that

$$(1.5) \quad \|f(A)x\|_A \leq \|f\|_{\infty,\theta} \|x\|_A, \quad x \in H_A, f \in H^\infty(S^0_{\theta'}) ,$$

where $\|f\|_{\infty,\theta}$ denotes the sup-norm of $f$ on $S^0_\theta$, see [17], or [15, Theorem 3.1]. Then the operator $A$ has a bounded $H^\infty(S^0_{\theta'})$-functional calculus if and only if the norm $\| \cdot \|_A$ is equivalent to the norm in $H$ and $H_A = H$.

These questions have much relationship with the Kato problem and the boundedness of the Cauchy integrals on Lipschitz curves; see for example [2, 9] and references therein. We refer also to the survey [25] for a brief introduction to the $H^\infty$ calculus for sectorial operators.

Another setting where the $H^\infty$-functional calculus appears naturally is that of the Nagy-Foiaș functional model of Hilbert space operators [22]. This model is constructed originally for contractive or dissipative operators in the unit disc $\mathbb{D}$ or in the upper half-plane, respectively. Let us explain the above in some more detail. Assume that $T$ is a
contraction on $H$ such that $\lim_{n \to \infty} T^n x = 0$ for all $x \in H$. The existence of a Nagy-Foiaş model for $T$ means that $T$ can be realized as the multiplication operator $\tilde{M}_\delta$ on the quotient space $H^2(\mathbb{D}, E)/\delta H^2(\mathbb{D}, F)$, through a Hilbertian isometry $V : H^2(\mathbb{D}, E)/\delta H^2(\mathbb{D}, F) \to H$. Here $E$ and $F$ are auxiliary Hilbert spaces known as defect spaces for $H$, and $\delta$ in $H^\infty(\mathbb{D}; L(F, E))$ is the (unique) so-called Nagy-Foiaş characteristic function for $T$, see [22]. Then the $H^\infty$-functional calculus for $T$ follows immediately by putting $f(T) := V \circ \tilde{M}_{f(\delta)} \circ V^{-1}$ for all $f \in H^\infty(\mathbb{D})$.

A theory about models of Nagy-Foiaş type in rather general domains $\Omega \subset \mathbb{C}$ has been recently established in [26]. Let us explain its connections with the linear control theory, using the notation of [27]. Recall that a linear observation system is given by

$$\begin{cases}
x'(t) = -Ax(t) \in X, & 0 \leq t < \infty; \\
x(0) = a \in X; \\
y(t) = Cx(t) \in Y, & 0 \leq t < \infty.
\end{cases}$$

Here $Y$ is the output Hilbert space, $X$ is a Hilbert space which has the meaning of the system state space, and $A$ is assumed to be the generator of a $C_0$ semigroup on $X$. The function $y : [0, \infty) \to Y$ is called the output of the system. Then one can define the mapping $\mathcal{O}_{A,C} : X \to \text{Hol}(\rho(A); Y)$ by

$$\mathcal{O}_{A,C}x(z) := C(z - A)^{-1}x, \quad z \in \rho(A).$$

Notice that $-\mathcal{O}_{A,C}x(-z) = \mathcal{L}(y)(z)$ for $\Re z > 0$, where $\mathcal{L}(y)$ is the Laplace transform of $y$.

The mapping $\mathcal{O}_{A,C}$ is called the observation map. It will be shown later that this map gives rise to the observation model of $A$.

A linear control system on $\Omega$ is defined by

$$x'(t) = -Ax(t) + Bu(t), \quad -\infty < t \leq 0$$

for $u \in L^2((-\infty, 0), U)$ of compact support, where $U$ is the input Hilbert space of the system and $B : D(B) \to X$ is a closed operator. Assume that there exists $C_{A,B} : L^2((-\infty, 0), U) \to X$, which is a continuous extension of $u \mapsto x(0)$. Then the controllability map $W_{A,B}$ is defined as $W_{A,B} := C_{A,B} \circ \mathcal{L}^{-1} \circ \text{inv}$, where $\text{inv} f(z) = f(-z)$, see [27], Section 5. This map gives rise to the control model of $A$. Formulae of $W_{A,B}$ type will be given in detail in Section 2 below, for sectorial $A$. These two models are in a certain duality, as is explained in [26], [27]. In fact, one can consider the observation and the control model both for $A$ and for $A^*$, which gives four different models.

The triple $(U, Y, X)$ of Hilbert spaces and corresponding mappings $\mathcal{O}_{A,C}, W_{A,B}$ previously considered can be defined independently of differential equations, and they can be taken as starting point to construct functional models for the operator $A$, in a more abstract context. More precisely, under suitable conditions on $\Omega$ and $A$, such models are
built on the corresponding representation Hilbert spaces $H_{A,\theta}^{\text{obs}}$ and $H_{A,\theta}^{\text{ctr}}$. There is a natural isomorphism between these spaces. It is shown that $A$ is similarly equivalent (but not necessarily unitarily equivalent) to its model, which is, basically, the multiplication operator either on $H_{A,\theta}^{\text{obs}}$ or on $H_{A,\theta}^{\text{ctr}}$.

When implementing this procedure, it is very important to make a good choice of operators $B$ and $C$ for a given operator $A$. The generalized characteristic function $\delta$ need not be unique, and must be well chosen; see [20], where several examples are given. In [27], this theory has been carried out to construct models on certain parabolic domains, which apply to nondissipative perturbations of unbounded self-adjoint operators on a Hilbert space.

The aim of the present paper is to give a concrete construction of control and observation functional models of Nagy-Foiaş type for every sectorial operator $A$ on a Hilbert space of type $\omega$ in sectors $S_\theta^\circ$, $\omega < \theta < \pi$. (To be more precise, the model is constructed for the natural extension of $A$ onto $H_A$.)

The relationships between sectorial operators and control theory are not new, see for instance [1], [14]. In [14, Theorem 4.1], see also [15, p. 204], Le Merdy considers the admissibility of operators $C$ as above in the case that $-A$ is the infinitesimal generator of a bounded analytic $C_0$-semigroup. He shows that $C = \sqrt{A}$ is an admissible operator for $A$ if and only if $A$ has a square function estimate, that is, $\|x\|_A \leq K\|x\|$, $(x \in H)$, for some constant $K > 0$.

Inspired by the above result, we shall prove here that the particular choice

$$ B := 2\sqrt{A}, \quad C := \sqrt{A}, \quad U = Y := H, \quad X := H_A, $$

gives rise to a model of $A$ of Nagy-Foiaş type in $S_\theta^\circ$. Moreover, for $\theta \in (\omega, \pi)$ and $\alpha > 0$ such that $\alpha \theta < \frac{\pi}{2}$, let $\delta_\alpha$ be the operator function defined by

$$ (1.6) \quad \delta_\alpha(z) := \frac{1}{\alpha} \frac{A^\alpha - z^\alpha}{A^\alpha + z^\alpha}, \quad z \in S_\theta. $$

In the definition of $z^\alpha$ and $A^\alpha$ we mean that $z^\alpha$ is the continuous branch of the function $z^\alpha$ on $\mathbb{C} \setminus (-\infty, 0]$ such that $(1)^\alpha = 1$. As it will be explained below, $\delta_\alpha$ is in $H^{\infty}(S_\theta^\circ; L(H))$.

We shall show that for any $\theta$ and $\alpha$ as above, the operator $A$, considered as an operator on $H_A$, possesses a Nagy–Foiaş type model in the open sector $S_\theta^\circ$, and $\delta_\alpha$ is one of its generalized Nagy–Foiaş characteristic functions in the sense of [26] (see Theorems 2.5 and 2.6 below). One of the stages in the proof of this result is to prove equalities

$$ H_A = H_{A,\theta}^{\text{ctr}} = H_{A,\theta}^{\text{obs}}, \quad \forall \theta \in (\omega, \pi) $$

(with the equivalence of the corresponding norms). Thus we obtain that quadratic estimates related to sectorial operators, which on the other hand have revealed significant in the treatment of several analytic
questions \cite{2}, are also in the basis of control theory and functional models of Nagy–Foiaş type.

In particular it follows from the above that for every one-to-one operator $A$ of type $\omega$, and any $\theta \in (\omega, \pi)$, the following three properties are equivalent, see Theorem FC and Theorem \[2.7\] below:

1. There exists a $H^\infty$ functional calculus in $S^\theta_\theta$ for $A$.
2. The equality $H_A = H$ holds.
3. There exists a Nagy–Foiaş type functional model in $S^\theta_\theta$ for $A$, and $\delta_\alpha$ is a generalized characteristic function of $A$.

As a consequence, the existence of a Nagy–Foiaş functional model for $A$ in sectors depends on the equality $H_A = H$ (the equivalence between (1) and (2) above was well known, as it has been already mentioned).

How distant the spaces (and their norms) $H_A$ and $H$ are is a question of independent interest, which we also consider here. In Theorem \[2.7\] we show that the gap between these two norms is, in a certain sense, of logarithmic order and hence is small.

The plan of the paper is as follows. In Section 2 we introduce the main concepts (spaces and operators) associated with a sectorial operator, from the point of view of (abstract) control theory, and state the main theorems of the article. One of the main tools in the paper will be an operator $J_\alpha$ of Hankel type defined on the Hardy-Smirnov class which, in principle, depends on the parameter $\alpha \in (0, \pi/\omega)$. A key point in our arguments is that such an operator does not depend on $\alpha$ in the above range, indeed. To prove this, we need to establish some results about analytic extensions of operator-valued functions $\delta_\alpha$ to sectors $S^\theta_\theta$. Such extensions are studied in Section \[3\]. In Section \[4\] we introduce a Hankel-like operator $J_\alpha$ and prove its basic properties, including a kind of independence with respect to $\alpha$. Control and observation (Hilbert) spaces and operators associated with a sectorial operator are studied in Section \[5\]. We proceed with a collection of preparatory results which eventually allows us to prove the concrete isomorphism results for the control and observation spaces. Section \[6\] is devoted to culminate the proofs of the main theorems, stated in Section \[2\]. Finally, Section \[7\] contains some remarks and comments.

We also would like to mention that \cite{26} contains a construction of a model of Nagy-Foiaş type for an arbitrary generator of a $C_0$ group on a Hilbert space in a suitable vertical strip. It follows, in particular, that these operators always admit an $H^\infty$ calculus in a strip, which has been already known, due to a result by Boyadzhiev and deLaubenfels \cite{3} (the third author, unfortunately, was not aware of this work at that moment).
2. Main results

Let us introduce the following notation. Let $H$ denote a separable Hilbert space. (All Hilbert spaces appearing in this article are supposed to be separable.) For a densely defined closed operator $T$ on $H$ with trivial kernel we define the Hilbert space $TH$ as the set of formal expressions $Tx$, where $x$ ranges over the whole space $H$. Put $\|Tx\|_{TH} := \|x\|_H$ for all $x \in H$. If $T^{-1} \in L(H)$, then the formula $x = T(T^{-1}x)$ allows one to interpret $H$ as a linear submanifold of $TH$. If $T$ is bounded, then, conversely, $TH$ is a a linear submanifold of $H$.

Let $A$ be a sectorial operator on a Hilbert space $H$. We assume concepts and notation of the above section. Our first result concerns the comparison between the norms of the Hilbert spaces $H$ and $H_A$.

Related results can be found in [6], Corollary 4.7 and Theorem 7.1 (b).

In the following result $\text{Log}(z)$ is the principal branch of the logarithm with argument in $[-\pi, \pi)$.

**Theorem 2.1.** Put $\Lambda_k(z) := \text{Log}(z) + 2k\pi i$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. Then for any $r > \frac{1}{2}$ and any $k \in \mathbb{Z}$, $k \neq 0$ we have

$$\Lambda_k(A)^{-r}H \subseteq H_A \subseteq \Lambda_k(A)^rH$$

and there is a constant $C_{k,r} > 0$ such that

$$C_{k,r}^{-1}\|\Lambda_k(A)^{-r}x\| \leq \|x\|_A \leq C_{k,r}\|\Lambda_k(A)^r x\|,$$

for all $x \in \Lambda_k(A)^{-r}H$. The space $\Lambda_k(A)^{-r}H$ is dense in $H_A$.
Functions in $E^2(\Omega; H)$ have strong non-tangential boundary limit values a.e. on $\partial \Omega$, which allow one to identify this space with a closed subspace of $L^2(\partial \Omega; H)$. This gives $E^2(\Omega; H)$ a Hilbert space structure. We refer to [7] and [22] for a background.

Let $\omega$ be in $(0, \pi)$ the type of the sectorial operator $A$. Fix an angle $\theta \in (\omega, \pi)$. Throughout the paper, the class $E^2(\Omega; H)$ will be considered for $\Omega = S^\theta_\theta$ and, in integrals over the boundary $\partial S^\theta_{\theta}$ of $S^\theta_{\theta}$, we give $\partial S^\theta_{\theta}$ the parametrization counterclockwise around $S^\theta_{\theta} \cup \{\infty\}$. As indicated above, we take in this case the operators $B := 2\sqrt{A}$, $C := \sqrt{A}$ and Hilbert spaces $U = Y = H$, $X = H_A$, in the general framework of [26], [27]. Then we get the observation operator $O : H \to \text{Hol}(\rho(A); H)$ associated with $A$, given by

$$\tag{2.2} (Ox)(z) = \sqrt{A}(z - A)^{-1}x, \quad z \in \rho(A), x \in H,$$

and the control operator $W_\theta : E^2(S^\theta_\theta, H) \to H$ for $A$ defined as

$$W_\theta(u) := \frac{1}{\pi i} \int_{\partial S^\theta_{\theta}} (\xi - A)^{-1}\sqrt{A}u(\xi) \, d\xi,$$

for every $u \in E^2(S^\theta_\theta; H)$.

We shall need to show that $O$ and $W_\theta$ are bounded as operators between suitable spaces. For this, set

$$T_A := \frac{\sqrt{A}}{1 + A}.$$

Since $T_A = f(A)$ where $f(z) := \sqrt{z}(1+z)^{-1}$ is in $\Psi(S^\theta_{\theta})$, $T_A$ is bounded. It is also clear that $\ker T_A = 0$ and that $\text{Im} T_A^n$ is dense in $H$ for every $n > 0$, [9, p. 143].

**Definition 2.2.** We define the spaces $H_n := T_A^nH$, $n \in \mathbb{Z}$, by giving them the meaning explained at the beginning of this section.

If $n > 0$, then $H_n$ is a dense linear subset in $H$. If $n < 0$, then $H$ is a dense linear subset of $H_n$. We have a chain of dense imbeddings

$$\cdots \subset H_2 \subset H_1 \subset H_0 = H \subset H_{-1} \subset H_{-2} \subset \cdots .$$

Then the mapping $O$ extends to $H_{-1}$ by putting

$$(Ox)(z) := \frac{\sqrt{A}}{1 + A} \frac{1 + A}{z - A} x \quad (z \in \rho(A), x \in H_{-1}),$$

and $O : H_{-1} \to \text{Hol}(\rho(A); H)$ is clearly continuous. This map is injective. Indeed, if $x \in H_{-1}$ satisfies $Ox(\lambda) = 0$ for some $\lambda \in \mathbb{C}\setminus S^\theta_{\theta}$ then

$$x = (\lambda - A)(\sqrt{A})^{-1}Ox(\lambda) = 0.$$ 

It can be also proven that the linear mappings $W_\theta : E^2(S^\theta_\theta; H) \to H_{-1}$, $O : H_1 \to E^2(\mathbb{C}\setminus S^\theta_{\theta}; H)$ are well-defined and continuous (see Proposition 5.1 below). Then we can introduce the appropriate control and observation spaces for $A$. 


Definitions 2.3. The control space for $A$ is the Hilbert space $H^{\text{ctr}}_{A,\theta}$ obtained as the range of $W_{\theta}$ in $H_{-1}$ with the range norm,

$$
\|x\|_{A,\theta,\text{ctr}} := \min\{\|u\|_{E^2(S^0_{\theta};H)} : x = W_{\theta}(u)\}.
$$

Put $O_{\theta}x := O\{x\}|_{C \setminus S_{\theta}}$. The observation space for $A$ is

$$
H^{\text{obs}}_{A,\theta} := \{x \in H_{-1} : O_{\theta}x \in E^2(C \setminus S_{\theta};H)\},
$$

with the norm $\|x\|_{A,\theta,\text{obs}} := \|O_{\theta}x\|_{E^2(C \setminus S_{\theta};H)}$.

We have the following result.

Theorem 2.4. The spaces $H^{\text{ctr}}_{A,\theta}$ and $H^{\text{obs}}_{A,\theta}$ do not depend on $\theta$ and coincide with $H_{A}$ for any $\theta \in (\omega, \pi)$. The norms in all these spaces are mutually equivalent.

For $\alpha > 0$ such that $\alpha \theta < \pi/2$ let $\delta_{\alpha}$ be the operator-valued function given by (1.6). The fractional power $A^\alpha$ is $\omega\alpha$-sectorial [9], so in particular $\sigma(A^\alpha) \subset S_{\omega\alpha}$. It follows that

$$
\delta_{\alpha}(z) := \frac{1}{\alpha}(1 - 2z^{\alpha}(A^{\alpha} + z^{\alpha})^{-1}), \quad z \in S_{\theta},
$$

is an $L(H)$-valued function on $S^\circ_{\theta}$ of class $H^\infty$. It is also easy to see that $\delta_{\alpha}^{-1}(z)$ exists and is uniformly bounded for $z \in \partial S_{\theta}$ with $z \neq 0$. Hence $\delta_{\alpha}E^2(S^0_{\theta};H)$ is a closed subspace of $E^2(S^0_{\theta};H)$.

For any $x \in H_{A}$, we have $y := (\sqrt{A})^3(1 + A)^{-3}x = \sqrt{A}(1 + A)^{-1}(1 - (1 + A)^{-1})^{2}x \in H$, which means that $Ax = T_{A}^{-3}y$ is defined at least as a member of $H_{-3}$. We define a (possibly unbounded) operator $\tilde{A}$ on $H_{A}$ by

$$
D(\tilde{A}) := \{x \in H_{A} : Ax \in H_{A}\},
$$

and $\tilde{A}x = Ax, x \in D(\tilde{A})$. Then $\tilde{A}$ is an $\omega$-sectorial operator with trivial kernel and $\sigma(A) = \sigma(\tilde{A})$, see Proposition 7.2 below. The following theorem provides a concrete model of Nagy–Foiaş type of $\tilde{A}$, which can be called a control model according to the terminology of [27].

Theorem 2.5. Suppose, as above, that $\theta \in (\omega, \pi), \alpha > 0, \alpha \theta < \frac{\pi}{2}$. Then we have the following.

(i) $\text{Ker } W_{\theta} = \delta_{\alpha}E^2(S^0_{\theta};H)$.

(ii) Consider the quotient space

$$
\mathcal{Q}(S^0_{\theta}, \delta_{\alpha}) := E^2(S^0_{\theta};H)/\delta_{\alpha}E^2(S^0_{\theta};H)
$$

and the corresponding quotient operator $\hat{W}_{\theta} : \mathcal{Q}(S^0_{\theta}, \delta_{\alpha}) \to H_{A}$.

Then $\hat{W}_{\theta}$ is an isomorphism of $\mathcal{Q}(S^0_{\theta}, \delta_{\alpha})$ onto $H_{A}$.

(iii) Let $\hat{M}_z$ be the quotient multiplication operator on $\mathcal{Q}(S^0_{\theta}, \delta_{\alpha})$ given by

$$
\hat{M}_z \rho = \sigma, \quad \text{if } \exists r, s \in E^2(S^0_{\theta};H)
$$

such that $\sigma = [s], \rho = [r], s(z) = zr(z)$.
(here \( [r] \in \mathcal{Q}(S_\theta^o, \delta_\alpha) \) denotes the coset that corresponds to \( r \)).

We put \( D(\hat{M}_z) \) to be the set of all cosets \( [r] \) such that there exist \( r, s \) as above. Then \( \hat{M}_z \) is well-defined. It is a closed operator on \( \mathcal{Q}(S_\theta, \delta_\alpha) \) with dense domain, and \( \hat{W}_\theta \) intertwines \( \hat{M}_z \) with \( \tilde{A} \):

\[
D(\hat{M}_z) \subset \mathcal{Q}(S_\theta^o, \delta_\alpha) \xrightarrow{\hat{M}_z} \mathcal{Q}(S_\theta^o, \delta_\alpha) \\
\downarrow \hat{W}_\theta \quad \quad \quad \quad \quad \downarrow \hat{W}_\theta \\
D(\tilde{A}) \subset H_A \xrightarrow{\tilde{A}} H_A
\]

In the above result, one can always put \( \alpha = \frac{1}{2} \). If \( \theta < \frac{\pi}{2} \), then one can also take \( \alpha = 1 \).

The control model is the closest one to the original Nagy–Foiaş model, because it represents \( \tilde{A} \) as an operator of multiplication by the independent variable on a quotient Hilbert space, see the Introduction. However, to the contrary to the original Nagy–Foiaş setting, we obtain similarity but not unitary equivalence of \( \tilde{A} \) and the multiplication operator. In general, the generalized characteristic function of \( \tilde{A} \) depends on the choice of auxiliary operators \( B \) and \( C \) and is far from being unique, see [26, Section 11] for a discussion.

This theorem implies that the operator \( \tilde{A} \) possesses a unique weak-star continuous \( H^\infty \)-functional calculus in each angle \( S_\theta \), if \( \theta > \omega \). Indeed, the unique weak-star continuous \( H^\infty \)-functional calculus for \( \hat{M}_z \) is

\[
f(\hat{M}_z) = \hat{M}_f(z), \quad f \in H^\infty(S_\theta).
\]

It also implies that there exists a normal dilation of \( \tilde{A} \), whose spectrum is contained in \( \partial S_\theta \). We refer to [8] for dilation results in the context of operators on UMD spaces.

The next result provides us with another explicit realization of the operator \( \tilde{A} \), which is called the observation model for \( \tilde{A} \), as in [27].

For \( \theta \in (\omega, \pi) \) and \( \alpha \) such that \( 0 < \alpha < (\pi/2\theta) \), we introduce the model space

\[
\mathcal{H}(\delta_\alpha, S_\theta^o) := \{ v \in E^2(\mathbb{C} \setminus S_\theta; H) : \delta_\alpha v|_{\partial S_\theta} \in E^2(S_\theta^o; H) \}.
\]

In the remainder of this section we assume \( \theta \) to be fixed, and write \( \mathcal{H}(\delta_\alpha) \) to denote \( \mathcal{H}(\delta_\alpha, S_\theta^o) \). In fact, we shall show that, in a certain sense, the above space does not depend on \( \theta \in (\omega, \pi) \).

**Theorem 2.6.** Under the above conditions \( \mathcal{O}_\theta|_{H_A} \) is an isomorphism of \( H_A \) onto \( \mathcal{H}(\delta_\alpha) \). Moreover, one has the intertwining formula \( \mathcal{O}_\theta D(\tilde{A}) = D(M_z^T) \),

\[
\mathcal{O}_\theta \tilde{A} x = M_z^T \mathcal{O}_\theta x, \quad x \in D(\tilde{A}):
\]
\[
D(\tilde{A}) \subset H_A \quad \xrightarrow{\tilde{A}} \quad H_A \\
D(M_z^T) \subset H(\delta_\alpha) \quad \xrightarrow{M_z^T} \quad H(\delta_\alpha)
\]

where \(D(M_z^T) := \{ f \in H(\delta_\alpha) \mid \exists c \in H \text{ such that } M_z f - c \in H(\delta_\alpha) \}\), and

\[(2.3) \quad M_z^T f := M_z f - c \in H(\delta_\alpha), \quad f \in D(M_z^T).\]

We call \(M_z^T\) the observation model operator and \(H(\delta_\alpha)\) the observation model space. Notice that the above definition \((2.3)\) of \(M_z^T\) is correct, because the only constant \(c \in H\) that belongs to \(H(\delta_\alpha)\) is zero.

The formula for the resolvent of \(M_z^T\) is given by

\[(2.4) \quad ((M_z^T - \lambda)^{-1} f) (w) = \frac{f(w) - f(\lambda)}{w - \lambda}, \quad \lambda \not\in \sigma(\tilde{A}).\]

In order to state our next result, first we reproduce some of the known equivalent conditions for the existence of the \(H^\infty\) calculus for \(A\).

**Theorem FC.** Let \(A\) be a one-to-one operator of type \(\omega\), where \(0 < \omega < \pi\). Then the following statements are equivalent.

(a) \(A\) admits a bounded \(H^\infty(S_\mu^\omega)\)-functional calculus for all \(\mu > \omega\);
(b) \(A\) admits a bounded \(H^\infty(S_\mu^\omega)\)-functional calculus for some \(\mu > \omega\);
(c) \(\{A^{is} \mid s \in \mathbb{R}\}\) is a \(C_0\)-group and for any \(\mu > \omega\), there exists \(c_\mu\) such that \(\|A^{is}\| \leq c_\mu e^{\mu|s|}, \quad s \in \mathbb{R}\);
(d) \(H = H_A\), and the norms in these two spaces are equivalent.

In the case that \(\omega < \pi/2\), the above conditions are equivalent to

(e) for any \(\theta \in (\omega, \pi/2)\), \(A\) is similar to a \(\theta\)-accretive operator.
(f) there exists \(\theta \in (\omega, \pi/2)\) such that \(A\) is similar to a \(\theta\)-accretive operator;

The equivalence of the conditions (a)–(d) was proved by McIntosh in [16]. The fact that (e) and (f) are equivalent to each of the conditions (a)–(d), follows from [9, Theorem 7.3.9].

In [13, Theorem 1.1], Le Merdy proves that \(A\) is similar to a \(\omega\)-accretive operator if and only if there exists and invertible operator \(S \in \mathcal{L}(H)\) such that \(\|S^{-1}A^{it}S\| \leq e^{\omega|t|}, \quad t \in \mathbb{R}\). On the other hand, as shown in [20, Theorem 3], the estimate \(\|A^{it}\| \leq K e^{\omega|t|}\) does not imply the boundedness of \(H^\infty(S_\theta^\omega)\) calculus for \(A\) for \(\theta = \omega\). Our proofs do not use the apparatus of the complete boundedness.

We refer to Theorem 2.4 in [5], Theorem 2.2 in [2], Le Merdy [13, 12], the review [25] and Chapter 7 of the book [9] for additional information.

Our next result shows that this list can be widened.
**Theorem 2.7.** Let $A$ be a one-to-one operator of type $\omega \in (0, \pi)$. Then each of the following conditions is equivalent to conditions (a)–(d) of Theorem FC.

(g) there exist $\theta \in (\omega, \pi)$ and $\alpha$ with $\alpha \omega < \pi/2$ such that the quotient operator $\widehat{W}_\theta$ induces an isomorphism of the quotient space $Q(S^\alpha_\theta, \delta_\alpha)$ onto $H$.

(h) there exist $\theta \in (\omega, \pi)$ and $\alpha$ with $\alpha \omega < \pi/2$ such that the operator $O_\theta$ defines an isomorphism of the space $H$ onto $H(\delta_\alpha)$.

(i) for any $\theta$, $\alpha$ as above, the operator $\widehat{W}_\theta$ induces an isomorphism of $Q(S^\alpha_\theta, \delta_\alpha)$ onto $H$.

(j) for any $\theta$, $\alpha$ as above, the operator $O_\theta$ defines an isomorphism of $H$ onto $H(\delta_\alpha)$.

3. **Analytic extension of inverse characteristic functions**

The class $\Psi(S^\alpha_\mu)$ defined in (1.2) can be represented as the union

$$
\Psi(S^\alpha_\mu) = \bigcup_{s > 0} \Psi_s(S^\alpha_\mu),
$$

where

$$
\Psi_s(S^\alpha_\mu) = \{ f \in \text{Hol}(S^\alpha_\mu) : \| f \|_{\Psi_s} := \sup_{w \in S^\alpha_\mu} \frac{1 + |w|^{2s}}{|w|^s} |f(w)| < \infty \}.
$$

For each fixed $s > 0$, $\Psi_s(S^\alpha_\mu)$ is a Banach algebra with respect to the norm $\| \cdot \|_{\Psi_s}$, and is a subalgebra of $H^\infty(S^\alpha_\mu)$.

The present section deals with analytic extensions of rational expressions involving fractional powers of a sectorial operator. Concretely, let us assume that $\omega < \theta < \pi$ and $0 < \theta \alpha < \pi$. The operator

$$
\tilde{\delta}_{\alpha}(z) := \alpha(I + 2z^\alpha(A^\alpha - z^\alpha)^{-1}) = \alpha \frac{A^\alpha + z^\alpha}{A^\alpha - z^\alpha}; \quad z \in \mathbb{C} \setminus (S_\omega \cup (-\infty, 0]),
$$

is the inverse to the operator $\delta_{\alpha}(z)$ given in (1.6), whenever both expressions are well-defined. Note that $\tilde{\delta}_{\alpha}$ is defined only for $\alpha \in (0, \pi/\theta)$, whereas $\delta_{\alpha}$ is defined only for $\alpha \in (0, \pi/2\theta)$, that is, the range of possible values of $\alpha$ is twice larger in the case of $\tilde{\delta}_{\alpha}$. In particular, for any angle $\theta$, we always can take $\alpha = 1$ in $\tilde{\delta}_{\alpha}$.

Since $A^\alpha$ is $\alpha\theta$–sectorial, we have that $\tilde{\delta}_{\alpha}$ is a bounded operator-valued function on $\partial S_\theta \setminus \{0\}$. Moreover, if $\alpha < \pi/2\theta$, then $\tilde{\delta}_{\alpha}(z) = \delta_{\alpha}^{-1}(z), z \in \partial S_\theta \setminus \{0\}$. In view of the definition of $\tilde{\delta}_{\alpha}$ we consider the scalar functions

$$
\gamma_{\alpha,z}(w) := \alpha \frac{w^\alpha + z^\alpha}{w^\alpha - z^\alpha}, \quad z, w \in \mathbb{C} \setminus (-\infty, 0],
$$

(here $\alpha > 0$). One has

$$(3.1) \quad \tilde{\delta}_{\alpha}(z) = \gamma_{\alpha,z}(A), \quad z \in \mathbb{C} \setminus (S_\mu \cup (-\infty, 0]),$$

if $\alpha \mu < \pi$.

**Lemma 3.1.** Let $\mu < \pi$ and $\alpha > 0$ such that $\alpha \mu < \pi$. 
(i) For any fixed \( z \in S_\mu^0 \), the function \( \xi_z \), given by
\[
\xi_z(w) := \gamma_{\alpha,z}(w) - \gamma_{1,z}(w)
\]
is analytic in \( w \in S_\mu^0 \).
(ii) Put \( \beta = \min(\frac{1}{2}, \alpha) \) and
\[
\eta_z(w) := \xi_z(w) + (1 - \alpha) \frac{w - 1}{w + 1}, \quad \text{for } z, w \in S_\mu^0.
\]
Then the function \( z \mapsto \eta_z \) is analytic from \( S_\mu^0 \) to the space \( \Psi_\beta(S_\mu^0) \) and satisfies
\[
\| \eta_z \|_{\Psi_\beta(S_\mu^0)} \leq C(|z|^{\alpha} + |z|^{-\alpha}), \quad z \in S_\mu^0,
\]
where the constant \( C \) depends only on \( \mu \) and \( \alpha \).

Proof. (i) It is straightforward to check that
\[
\text{Res}(\gamma_{\alpha,z}, z) = 2z
\]
for any \( \alpha \) and any \( z \in S_\mu^0 \). Since \( w = z \) is the only pole of \( \gamma_{\alpha,z}(w) \), which is of order one, assertion (i) follows.

(ii) Take any \( \nu \) such that \( \mu < \nu < \pi \) and \( \alpha \nu < \pi \). Notice that
\[
|w^{\alpha} - z^{\alpha}| \geq C_1 |z|^{\alpha}, \quad |w^{\alpha} - z^{\alpha}| \geq C_1 |w|^{\alpha},
\]
for all \( z, w \) such that \( z \in S_\mu^0 \) and \( w \in \partial S_\nu \), where \( C_1 > 0 \) depends only on \( \mu \) and \( \nu \). We keep the same notation \( C_1 \) although it may be different in each inequality. For \( z \in S_\mu^0 \), \( w \in \partial S_\nu \) with \( |w| \leq 1 \), it follows that
\[
|\gamma_{\alpha,z}(w) - \alpha \frac{w - 1}{w + 1}| = \alpha \left| \frac{2w^{\alpha} - 2w}{w^{\alpha} - z^{\alpha}} \right| \leq C_1 (|w| + |w|^{\alpha}) \leq C_1 (1 + |z|^{-\alpha}) |w|^{\beta}.
\]
Similarly, for \( z \in S_\mu^0 \) and \( w \in \partial S_\nu \) such that \( |w| \geq 1 \),
\[
|\gamma_{\alpha,z}(w) - \alpha \frac{w - 1}{w + 1}| = \alpha \left| \frac{2z^{\alpha} + 2}{w^{\alpha} - z^{\alpha}} \right| \leq C_1 \left( |z|^{\alpha} + \frac{1}{|w|} \right) \leq 2C_1 \frac{|z|^{\alpha} + |z|^{-\alpha}}{|w|^{\beta}}.
\]
These two inequalities give
\[
|\gamma_{\alpha,z}(w) - \alpha \frac{w - 1}{w + 1}| \leq C_1 \frac{|w|^{\beta}}{1 + |w|^{2\beta}} (|z|^{\alpha} + |z|^{-\alpha})
\]
for \( z \in S_\mu^0 \) and \( w \in \partial S_\nu \) where we remind that \( C_1 \) depends only on \( \mu, \nu \) and \( \alpha \).

Note that
\[
C_1 (|w|^{\beta} + |w|^{-\beta}) \leq |w^{\beta} + w^{-\beta}| \leq |w|^{\beta} + |w|^{-\beta}, \quad w \in S_\nu.
\]

Now by applying (3.3) twice (to general \( \alpha \) and to \( \alpha = 1 \)) we find that there exists \( C_2 > 0 \) such that
\[
|(w^{\beta} + w^{-\beta}) \eta_z(w)| \leq C_2 (|z|^{\alpha} + |z|^{-\alpha})
\]
for \( z \in S^0_\mu \) and \( w \in \partial S_\nu \). By (i), for each fixed \( z \) the function \( \eta_z \) is analytic on \( S^0_\mu \). By the Phragmén-Lindelöf theorem, it follows that the latter estimate in fact holds for all \( z \in S^0_\mu \) and \( w \in S^0_\nu \). We conclude that for all \( z \in S^0_\mu, \eta_z \) belongs to \( \Psi_\beta(S^0_\nu) \); in particular, \( \eta_z \in \Psi_\beta(S^0_\mu) \) for \( z \in S^0_\mu \) and \( \text{(3.2)} \) holds true.

Finally, fix \( \lambda, z \in S^0_\mu \). The function
\[
F(w) := w^{-\beta}(1 + w^{2\beta})(\eta_\lambda(w) - \eta_z(w))
\]
is holomorphic in \( w \in S^0_\mu \) and continuous up to the boundary \( \partial S^0_\mu \) (notice that \( \lim_{w \in S^0_\mu, w \to 0} F(w) = 0 \)), so that
\[
\|\eta_\lambda - \eta_z\|_{\Psi_\beta(S^0_\mu)} \leq \sup_{w \in S^0_\mu} \frac{|1 + w^{2\beta}|}{|w|^{\beta}} |(\eta_\lambda - \eta_z)(w)| = \sup_{w \in \partial S^0_\mu} |F(w)|
\]
by the Phragmén-Lindelöf theorem.

Writing the function \( \eta_\lambda - \eta_z \) as
\[
(\eta_\lambda - \eta_z)(w) = \left[ \frac{2\alpha w^\alpha(\lambda^\alpha - z^\alpha)}{(w^\alpha - \lambda^\alpha)(w^\alpha - z^\alpha)} - \frac{2w(\lambda - z)}{(w - \lambda)(w - z)} \right]
\]
we obtain
\[
\|\eta_\lambda - \eta_z\|_{\Psi_\beta(S^0_\mu)} \leq |\lambda^\alpha - z^\alpha| \cdot \sup_{w \in \partial S^0_\mu} \left| \frac{2\alpha(1 + w^{2\beta})w^\alpha}{w^{\beta}(w^\alpha - \lambda^\alpha)(w^\alpha - z^\alpha)} \right| + |\lambda - z| \cdot \sup_{w \in \partial S^0_\mu} \left| \frac{2(1 + w^{2\beta})w}{w^{\beta}(w^\alpha - \lambda^\alpha)(w^\alpha - z^\alpha)} \right|.
\]

From this, and using that \( \beta < \min\{1/2, \alpha\} \), it is readily seen that
\[
\lim_{\lambda \to z} \|\eta_\lambda - \eta_z\|_{\Psi_\beta(S^0_\mu)} = 0.
\]
Thus the function \( \eta : z \mapsto \eta_z, S^0_\mu \to \Psi_\beta(S^0_\mu) \) is continuous. Then a vector-valued version of the Morera theorem follows to obtain that \( \eta \) is analytic. We have done. \( \square \)

**Proposition 3.2.** For every \( \alpha \) such that \( 0 < \alpha < \pi/\theta \), the operator-valued function \( z \mapsto \tilde{\delta}_\alpha(z) - \tilde{\delta}_1(z) \), defined on \( \partial S^0_\theta \setminus \{0\} \), continues to a function on the sector \( S^0_\theta \) of the class \( H^\infty(S^0_\theta; L(H)) \).

**Proof.** Choose \( \beta = \min\{\frac{1}{2}, \alpha\} \). The map \( f \mapsto f(A) \), which goes from \( \Psi_\beta(S^0_\theta) \) to \( L(H) \), is linear and bounded. Hence by \((3.2)\),
\[
\|\eta_z(A)\| \leq C(|z|^\alpha + |z|^{-\alpha}), \quad z \in S^0_\theta.
\]
Therefore a similar estimate holds for \( \xi_z(A) \):
\[
\|\xi_z(A)\| \leq C'(|z|^\alpha + |z|^{-\alpha}), \quad z \in S^0_\theta.
\]
In particular, the map \( z \mapsto \xi_z(A) \) is an analytic continuation of the map \( z \mapsto \tilde{\delta}_\alpha(z) - \tilde{\delta}_1(z) \) to the sector \( S^0_\theta \). As we noted at the beginning of this section, the operator-valued functions \( \tilde{\delta}_\alpha \) and \( \tilde{\delta}_1 \) are bounded on \( \partial S^0_\theta \setminus \{0\} \). Now the assertion of the proposition is obtained by applying Phragmén-Lindelöf theorem to the scalar functions \( z \mapsto \langle \xi_z(A)h_1, h_2 \rangle \), where \( h_1, h_2 \in H \). \( \square \)
4. Hankel–like operators for sectorial operator

In the following result we introduce a Hankel-like operator on the Hardy–Smirnov class which is associated to the inverse characteristic function $\tilde{\delta}_\alpha$, and on which our arguments are based. Thus in principle such an operator depends on the parameter $\alpha \in (0, \pi/\theta)$. We shall see as an application of Proposition 3.2 that indeed it is independent of $\alpha$.

Lemma 4.1. Let $\theta \in (\omega, \pi)$.

(1) Define a Hermitian bilinear pairing by putting

$$\langle f, g \rangle := \frac{1}{2\pi i} \int_{\partial S_\theta} \langle f(\lambda), g(\bar{\lambda}) \rangle_H \, d\lambda,$$

for $f \in E^2(S_\theta^0; H)$ and $g \in E^2(C\setminus S_\theta; H)$. The spaces $E^2(S_\theta^0; H)$ and $E^2(C\setminus S_\theta; H)$ are dual with respect to this pairing.

(2) The space $L^2(\partial S_\theta; H)$ splits into the direct sum

$$L^2(\partial S_\theta; H) = E^2(S_\theta^0; H) \oplus E^2(C\setminus S_\theta; H).$$

This defines parallel continuous projections

$$P_{\text{int}} : L^2(\partial S_\theta; H) \to E^2(S_\theta^0; H), \quad P_{\text{out}} : L^2(\partial S_\theta; H) \to E^2(C\setminus S_\theta; H),$$

given by the Cauchy integrals

$$P_{\text{int}} f(z) := \frac{1}{2\pi i} \int_{\partial S_\theta} \frac{f(\xi)}{\xi - z} \, d\xi, \quad z \in S_\theta^0,$$

$$P_{\text{out}} f(z) := -\frac{1}{2\pi i} \int_{\partial S_\theta} \frac{f(\xi)}{\xi - z} \, d\xi, \quad z \in C\setminus S_\theta.$$

(3) Let $\mathcal{H}(\delta_\alpha, S_\theta^0)$ be the space defined prior to Theorem 2.6. Then $\mathcal{H}(\delta_\alpha, S_\theta^0) = (\delta_\alpha E^2(S_\theta^0; H))^\perp$ for every $\alpha \in (0, \pi/2\theta)$, where the annihilator is calculated with respect to the pairing (4.1).

(4) For $\alpha \in (0, \pi/\theta)$ let us consider the Hankel–like operator

$$\mathcal{J}_{\tilde{\delta}_\alpha} : E^2(S_\theta^0; H) \to E^2(C\setminus S_\theta; H),$$

acting by

$$\mathcal{J}_{\tilde{\delta}_\alpha}(u) := P_{\text{out}}(\tilde{\delta}_\alpha u|_{\partial S_\theta}).$$

Then $\text{Im} \mathcal{J}_{\tilde{\delta}_\alpha} = \mathcal{H}(\delta_\alpha)$ and $\text{Ker} \mathcal{J}_{\tilde{\delta}_\alpha} = \delta_\alpha E^2(S_\theta^0; H)$ for $\alpha \in (0, \pi/2\theta)$. Therefore by factoring $\mathcal{J}_{\tilde{\delta}_\alpha}$ by its kernel we obtain an isomorphism

$$\mathcal{J}_{\tilde{\delta}_\alpha}^* : \mathcal{Q}(S_\theta^0, \delta_\alpha) \to \mathcal{H}(\delta_\alpha, S_\theta^0), \quad \text{whenever } \alpha \in (0, \pi/2\theta).$$

Proof. Statements (1)–(3) are contained in [26, Propositions 2.1 and 2.2], and statement (4) is straightforward.

Now we show that $\mathcal{J}_{\tilde{\delta}_\alpha}$ is independent of $\alpha$. \qed
Proposition 4.2. (1) The Hankel-like operator
\[ \mathcal{J}_{\delta_0} : E^2(S_\theta^0; H) \to E^2(\mathbb{C}\setminus S_\theta; H), \]
does not depend on \( \alpha \) for \( \alpha \in (0, \pi/\theta) \).
(2) The space \( \delta_\alpha E^2(S_\theta^0; H) \), \( 0 < \alpha < \pi/(2\theta) \), does not depend on \( \alpha \).

Proof. (1) By Proposition 3.2, one has
\[ \mathcal{J}_{\delta_0}(f) - \mathcal{J}_{\delta_1}(f) = P_{\text{out}}((\tilde{\delta}_\alpha - \tilde{\delta}_1)f|\partial S_\theta) = 0. \]

(2) By Lemma 4.1, \( \delta_\alpha E^2(S_\theta^0; H) = \text{Ker}(\mathcal{J}_{\delta_0}) \), and the result follows from part (1). \( \Box \)

5. ISOMORPHISM BETWEEN CONTROL AND OBSERVATION SPACES

Let \( A \) be a sectorial operator of type \( \omega \in (0, \pi) \). Our aim in this section is to prove that the control and observation spaces associated with \( A \) as in Definition 2.3 coincide and have equivalent norms. Let us start with the following estimate:

For every \( \theta \in (\omega, \pi) \) and \( \xi \in \partial S_\theta \),

\[
\begin{align*}
(5.1) \quad \left\| \frac{A}{(\xi - A)(1 + A)} \right\| &= \left\| \frac{\xi(1 + A) - (\xi - A)}{(\xi + 1)(\xi - A)(1 + A)} \right\| \\
&\leq \left\| \frac{\xi}{(\xi + 1)(\xi - A)} \right\| + \frac{1}{|\xi + 1|} \left\| (1 + A)^{-1} \right\| \leq C_\theta + \frac{C_\theta}{|\xi + 1|},
\end{align*}
\]

where the constant \( C_\theta \) comes from the condition on \( A \) to be sectorial.

Proposition 5.1. For any \( \theta \in (\omega, \pi) \), the operators
\[ W_\theta : E^2(S_\theta^0; H) \to H_{-1} \quad \text{and} \quad O_\theta : H_1 \to E^2(\mathbb{C}\setminus S_\theta; H) \]
are well-defined, linear and bounded.

Proof. Let \( u \in E^2(S_\theta^0; H) \). Then \( \|W_\theta u\|_{H_{-1}} = \|\sqrt{A}(1 + A)^{-1}W_\theta u\|_H \) whence
\[
\frac{\pi}{2} \|W_\theta u\|^2_{H_{-1}} \leq \|u\|^2_{E^2(S_\theta^0; H)} \int_{\partial S_\theta} \left\| \frac{\sqrt{A}}{\xi - A} \right\|^2 |d\xi|.
\]

By (5.1) it follows that the integral is finite and it proves the boundedness of \( W_\theta : E^2(S_\theta^0; H) \to H_{-1} \). Now, for \( x \in H_1 \), note that
\[
\int_{\partial S_\theta} \|O_\theta x(z)\|_H^2 |dz| = \int_{\partial S_\theta} \|\sqrt{A}(z - A)^{-1}x\|_H^2 |dz| \\
\leq \int_{\partial S_\theta} \left\| \frac{A}{(z - A)(1 + A)} \right\|^2 \left\| \frac{1 + A}{\sqrt{A}} x \right\|^2 H |dz| \leq C\|x\|^2_2,
\]
where the last inequality is obtained again from (5.1). Hence \( O_\theta : H_1 \to E^2(\mathbb{C}\setminus S_\theta; H) \) is well defined and bounded. The proposition is proved. \( \Box \)
Recall Definition 2.3: the control space for $A$ is $H_{A,\theta}^{\text{ctr}} := \text{Im} W_\theta \subset H_{-1}$ endowed with the norm $\|x\|_{A,\theta, \text{ctr}} := \min\{\|u\|_{E^2(S_\theta^0; H)} : x = W_\theta(u)\}$, and the observation space for $A$ is defined as $H_{A,\theta}^{\text{obs}} := \mathcal{O}_\theta^{-1}(E^2(C \setminus S_\theta)) \subset H_{-1}$ with the norm $\|x\|_{A,\theta, \text{obs}} := \|O_\theta x\|_{E^2(C \setminus S_\theta; H)}$. In the next proposition, we collect several basic facts about the spaces $H_{A,\theta}^{\text{ctr}}$ and $H_{A,\theta}^{\text{obs}}$. The arrows “$\hookrightarrow$” mean continuous inclusions. The symbol $\hat{W}_\theta$ denotes the quotient mapping $E^2(S_\theta^0; H)/\text{Ker} W_\theta \twoheadrightarrow \text{Im} W_\theta$.

**Proposition 5.2.** (1) The mapping

$$\hat{W}_\theta : E^2(S_\theta^0; H)/\text{Ker} W_\theta \twoheadrightarrow H_{A,\theta}^{\text{ctr}}$$

is an isometric isomorphism, and $H_1 \hookrightarrow H_{A,\theta}^{\text{ctr}}$. 

(2) The mapping

$$\mathcal{O}_\theta : H_{A,\theta}^{\text{obs}} \twoheadrightarrow E^2(C \setminus S_\theta; H)$$

is an isometry, and $H_1 \hookrightarrow H_{A,\theta}^{\text{obs}} \hookrightarrow H_{-1}$.

(3) The space $H_{A,\theta}^{\text{obs}}$ is complete.

**Proof.** (1) The norm of $x = W_\theta(u)$ in $H_{A,\theta}^{\text{ctr}}$ is exactly the quotient norm of $u + \text{Ker} W_\theta$ in $E^2(S_\theta^0; H)/\text{Ker} W_\theta$, so $\hat{W}_\theta$ is an isometric isomorphism.

For any $\lambda \in C \setminus S_\theta$ and any $x \in H$ the rational function $u_{\lambda,x}(z) := (\lambda - z)^{-1} x$ belongs to $E^2(S_\theta^0; H)$. Then the Dunford-Schwartz calculus gives us

$$W_\theta(u_{\lambda,x}) = 2\sqrt{A}(\lambda - A)^{-1}x \in H_{A,\theta}^{\text{ctr}}.$$ 

On the other hand, for fixed $\lambda$, the vectors $y := 2\sqrt{A}(\lambda - A)^{-1}x$ range over the whole space $H_1$ if $x$ runs over $H$. Hence $H_1 \subset H_{A,\theta}^{\text{ctr}}$. The continuity of this inclusion follows from the estimate

$$\|y\|_{A,\theta, \text{ctr}} \leq \|u_{\lambda,x}\|_{E^2} \leq C_\lambda \|x\|_H \leq C_\lambda^1 \|y\|_1,$$

where $C_\lambda$, $C_\lambda^1$ are constants depending on $\lambda$.

(2) That $\mathcal{O}_\theta : H_{A,\theta}^{\text{obs}} \twoheadrightarrow E^2(C \setminus S_\theta; H)$ is an isometry is clear from the definition of $H_{A,\theta}^{\text{obs}}$, and then $H_1 \hookrightarrow H_{A,\theta}^{\text{obs}}$ is a straightforward consequence of this isometry and Proposition 5.1.

Now for every $\lambda \in C \setminus S_\theta$ and $x \in H_{A,\theta}^{\text{obs}}$ we have that

$$\sqrt{A}(1 + A)^{-1}x = ((\lambda + 1)(1 + A)^{-1} - 1)\mathcal{O}_\theta x(\lambda)$$

whence $\|x\|_{-1} \leq (a|\lambda| + b)\|\mathcal{O}_\theta x(\lambda)\|_H$, and therefore we obtain $c\|x\|_{-1} \leq \|\mathcal{O}_\theta x\|_{E^2(C \setminus S_\theta; H)} = \|x\|_{A,\theta, \text{obs}}^2$, where $a$, $b$, $c$ are positive constants.

(3) Let $(x_n) \subset H_{A,\theta}^{\text{obs}}$ be a Cauchy sequence in $H_{A,\theta}^{\text{obs}}$. By (2) above, there exists $x \in H_{-1}$ and $v \in E^2(C \setminus S_\theta; H)$ such that $x_n \rightarrow x$ in $H_{-1}$ and at the same time $\mathcal{O}_\theta x_n \rightarrow v$ in $E^2(C \setminus S_\theta; H)$. Since the linear map $y \mapsto \mathcal{O}_\theta y(\lambda)$ is continuous in $y \in H_{-1}$ for each fixed $\lambda \in C \setminus S_\theta$, we conclude that $\mathcal{O}_\theta x = v$, hence $x_n \rightarrow x$ in $H_{A,\theta}^{\text{obs}}$. \qed
As it has been seen in Proposition 5.1 (1), the composition mapping
\[ E^2(S_0^\circ; H) \xrightarrow{W_\theta} H_{-1} \xrightarrow{O_\theta} \text{Hol}(\mathbb{C}\backslash S_0^\circ; H) \]
is well defined. Now, via the Hankel-like operator, we prove that in fact the range of \( O_\theta W_\theta \) lies in \( E^2(\mathbb{C}\backslash S_0^\circ; H) \).

**Lemma 5.3.** For any \( \theta \in (\omega, \pi) \) we have \( O_\theta W_\theta = -J_{\delta_1} \). In particular, \( O_\theta W_\theta \) is bounded from \( E^2(S_0^\circ; H) \) to \( E^2(\mathbb{C}\backslash S_0^\circ; H) \).

**Proof.** Take \( \lambda \in \mathbb{C}\backslash S_0^\circ \), and \( u \in E^2(S_0^\circ; H) \). We have
\[
(O_\theta W_\theta u)(\lambda) = \frac{1}{2\pi i} \int_{\partial S_0} \frac{2Au(z)}{(\lambda - A)(z - A)} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\partial S_0} \frac{2Au(z)}{(\lambda - A)(z - \lambda)} \, dz + \frac{1}{2\pi i} \int_{\partial S_0} \frac{(I + A + z) u(z)}{z - \lambda} \, dz
\]
\[
= \frac{1}{\lambda - A} (P_{\text{out}} u)(\lambda) - J_{\delta_1}(u)(\lambda) = -J_{\delta_1}(u)(\lambda),
\]
as we wanted to show. \( \square \)

As a first application of the above result we get the following assertion.

**Proposition 5.4.** There is a continuous embedding \( H_{A,\theta}^{\text{ctr}} \hookrightarrow H_{A,\theta}^{\text{obs}} \).

**Proof.** Take any \( x \in H_{A,\theta}^{\text{ctr}} \). Then \( x = W_\theta u \) for some function \( u \in E^2(S_0^\circ; H) \). By Lemma 5.3 \( O_\theta W_\theta u \in E^2(\mathbb{C}\backslash S_0^\circ; H) \), \( x = W_\theta u \in H_{A,\theta}^{\text{obs}} \), and
\[
\|x\|_{A,\theta,\text{obs}} = \|O_\theta x\|_{E^2(\mathbb{C}\backslash S_0^\circ; H)} = \|O_\theta W_\theta u\|_{E^2(\mathbb{C}\backslash S_0^\circ; H)} \leq C_\theta \|u\|_{E^2(S_0^\circ; H)}.
\]
Therefore \( \|x\|_{A,\theta,\text{obs}} \leq C_\theta \|x\|_{A,\theta,\text{ctr}} \) and \( H_{A,\theta}^{\text{ctr}} \hookrightarrow H_{A,\theta}^{\text{obs}}. \) \( \square \)

Now our aim is to prove the (continuous) reverse inclusion. We shall use the following approximation procedure. For any \( x \in H_{-1} \), we define
\[
(5.2) \quad x_\varepsilon := \frac{1 - \varepsilon^2}{(A + \varepsilon)(1 + \varepsilon A)} x = \frac{\varepsilon^{-1} x}{A + \varepsilon} - \frac{\varepsilon}{A + \varepsilon} x, \quad 0 < \varepsilon < 1.
\]
If \( v = O_\theta x \), then \( O_\theta x_\varepsilon = v_\varepsilon \), where
\[
v_\varepsilon(z) := \varepsilon^{-1} \frac{v(z) - v(-\varepsilon^{-1})}{z + \varepsilon^{-1}} - \varepsilon \frac{v(z) - v(-\varepsilon)}{z + \varepsilon}.
\]
This follows from definitions of \( O_\theta \) and \( x_\varepsilon \).

**Lemma 5.5.** (1) For any \( x \in H \), one has \( x_\varepsilon \to x \) in \( H \) as \( \varepsilon \to 0^+ \).
(2) For any \( v \in E^2(\mathbb{C}\backslash S_0^\circ; H) \), \( v_\varepsilon \to v \) in \( E^2(\mathbb{C}\backslash S_0^\circ; H) \) as \( \varepsilon \to 0^+ \).
(3) \( H_1 \) is dense in \( H_{A,\theta}^{\text{obs}} \).
Proof. Part (1) follows from (5.2) and [9, Proposition 2.1.1]. Part (2) can be deduced in the same way as part (1). Indeed, the multiplication operator $M_z$ on $E^2(S_0^\circ; H)$ given by

$$M_z(u)(z) = zu(z), \quad z \in S_0^\circ, \quad u \in \mathcal{D}(M_z),$$

is the adjoint to $M_z^T$ on $E^2(\mathbb{C}\setminus S_0^\circ; H)$, see for instance [26, Formula (2.7)]. Hence both are $\theta$-sectorial and since

$$(1 - \varepsilon^2)M_z^T(M_z^T + \varepsilon(1 + \varepsilon M_z^T)^{-1}) v,$$

([26, Proposition 1.2]), the assertion follows again from [9, Proposition 2.1.1]. (One can also deduce it from straightforward direct estimates.)

Finally take any $x \in H_{A,\theta}^{\text{obs}}$. By (2) of Proposition 5.2, $H_{A,\theta}^{\text{obs}} \hookrightarrow H_{-1}$. Note that $x_\varepsilon \in H_1$ for any $x \in (0, 1)$. By part (2) of this lemma, we have that $x_\varepsilon \to x$ in $H_{A,\theta}^{\text{obs}}$ as $\varepsilon \to 0^+$. This shows part (3). □

Lemma 5.6. $\mathcal{O}_\theta H_{A,\theta}^{\text{obs}} \subset \mathcal{H}(\delta_\alpha)$ for all $\alpha \in (0, \pi/2\theta)$.

Proof. It suffices to check that $\mathcal{O}_\theta H_1 \subset \mathcal{H}(\delta_\alpha)$, because $\mathcal{O}_\theta : H_{A,\theta}^{\text{obs}} \to E^2(\mathbb{C}\setminus S_0^\circ; H)$ is an isometry, $H_1$ is dense in $H_{A,\theta}^{\text{obs}}$ and $\mathcal{H}(\delta_\alpha)$ is a closed subspace of $E^2(\mathbb{C}\setminus S_0^\circ; H)$. Then, by (1) of Proposition 5.2, it is enough again to show that $\mathcal{O}_\theta H_{A,\theta}^{\text{ctr}} \subset \mathcal{H}(\delta_\alpha)$. Take $x \in H_{A,\theta}^{\text{ctr}}, \ x = W_\theta(u)$ with $u \in E^2(S_0^\circ; H)$. We apply Lemma 5.3 to obtain $\mathcal{O}_\theta(x) = -\mathcal{J}_{\delta_1}(u)$. Finally note that $\mathcal{J}_{\delta_1}(u) \in \mathcal{H}(\delta_\alpha)$ by Proposition 4.2 and Lemma 4.1 (4). □

Proposition 5.7. For every $\theta \in (\omega, \pi)$, $H_{A,\theta}^{\text{obs}} = H_{A,\theta}^{\text{ctr}}$ with equivalent norms.

Proof. It has been shown in Proposition 5.4 that $H_{A,\theta}^{\text{ctr}} \hookrightarrow H_{A,\theta}^{\text{obs}}$. To prove the reverse inclusion, take any $x \in H_{A,\theta}^{\text{obs}}$. By Lemma 5.6 $\mathcal{O}_\theta x \in \mathcal{H}(\delta_\alpha) = \mathcal{J}_{\delta_\alpha}(E(S_0^\circ; H))$. Let consider the quotient map

$$\mathcal{J}_{\delta_\alpha} : \mathcal{Q}(S_0^\circ, \delta_\alpha) \to E^2(\mathbb{C}\setminus S_0^\circ; H),$$

and put $\hat{w} = \mathcal{J}_{\delta_\alpha}^{-1}(\mathcal{O}_\theta x) \in \mathcal{Q}(S_0^\circ, \delta_\alpha)$. Note that $y := W_\theta \hat{w} \in H_{-1}$ by Proposition 5.1. Then

$$\mathcal{O}_\theta y = \mathcal{O}_\theta W_\theta \hat{w} = -\mathcal{J}_{\delta_1} \hat{w} = -\mathcal{J}_{\delta_\alpha} \hat{w} = -\mathcal{O}_\theta x.$$

Since $x \in H_{-1}$ the injectivity of $\mathcal{O}_\theta$ on $H_{-1}$ implies that $x = -y \in H_{A,\theta}^{\text{ctr}}$. □

6. Proofs of main results

Theorem 2.1 is not necessarily associated with functional models, so we give here a self-contained proof of it.

Proof of Theorem 2.7. We are using the Riesz-Dunford calculus. Firstly, we check that

$$\|A_k(A)^{-r}x\|_A \leq C\|x\|, \quad x \in H, k \in \mathbb{Z}\setminus\{0\}.$$
To do this, take $\psi \in \Psi(S^0_0)$. By applying (1.3) to $\Lambda_k(A)^{-r}x$, the Riesz-Dunford formula for $\psi(tA)\Lambda_k(A)^{-r}x$ and (1.1), we get

$$
\|\Lambda_k(A)^{-r}x\|_A^2 \leq C\|x\|^2 \sum_{r=\pm 1} \int_0^{\infty} \left( \int_0^{\infty} |\psi(re^{i\theta})|(1 + |\log(r)|)^{-r} dr \right)^2 \frac{dt}{t} 
\leq C\|x\|^2 \sum_{r=\pm 1} \left( \int_0^{\infty} |\psi(se^{i\theta})|(1 + |\log(s)|)^{-r} ds \right)^2 \frac{dt}{t} 
\leq C\|x\|^2 \sum_{r=\pm 1} \left( \int_0^{\infty} |\psi(se^{i\theta})| \left( \int_0^{\infty} (1 + |\log(u)|)^{-2r} du \right) \frac{1}{s} ds \right)^2 
$$

where we have applied the change of variable $u = s/t$ in the inner integral and the Minkowsky inequality

$$
\left\| \int_0^{\infty} |f(\cdot, s)| \frac{ds}{s} \right\|_2^2 \leq \left( \int_0^{\infty} \|f(\cdot, s)\|_2 \frac{ds}{s} \right)^2.
$$

It follows that

$$
\|\Lambda_k(A)^{-r}x\|_A \leq C \sum_{r=\pm 1} \|x\| \left( \int_0^{\infty} |\psi(se^{i\theta})| \frac{ds}{s} \right) = C_1\|x\|.
$$

In order to prove the first inequality in (2.1), note that the adjoint operator $A^*$ of $A$ on $H$ is also sectorial of type $\omega$. Then by applying the preceding estimate to $A^*$, we obtain that the operator $\Lambda_m(A^*)^{-r}: H \to H_{A^*}$ is bounded for all $m \in \mathbb{Z} \setminus \{0\}$.

Now there is a (natural) duality $(H_A)^* = H_{A^*}$, see [2, Theorem 2.1], and the adjoint operator of $\Lambda_m(A^*)^{-r}: H \to H_{A^*}$ is $\Lambda_m^*(A)^{-r}: H \to H$ where $\Lambda_m^*(z) := \overline{\Lambda_m(z)}$, $z \in \mathbb{C} \setminus (-\infty, 0]$.

For every $y \in H_A$, $x \in H$,

$$
\langle (\Lambda_m(A^*)^{-r})^* y, x \rangle_H = \langle y, \Lambda_m(A^*)^{-r}x \rangle_{(H_A)^*, H_A^*} 
:= \int_0^{\infty} \langle \psi(A)y, \psi(A^*)\Lambda_m(A^*)^{-r}x \rangle_H \frac{dt}{t} 
= \int_0^{\infty} \langle \psi(A)\Lambda_m^*(A)^{-r}y, x \rangle_H \frac{dt}{t} = \langle \Lambda_m^*(A)^{-r}y, x \rangle.
$$

Here we have taken $\psi$ in $\Psi(S^0_0)$ such that $\psi^2 \in \Psi(S^0_0)$, $\int_0^{\infty} \psi^2(\tau) \frac{d\tau}{\tau} = 1$. Finally note that $\Lambda_m^*(z) = \Lambda_k(z)$ for a fixed $k \in \mathbb{Z}$, from which it follows that $\Lambda_k(A)^{-r}: H_A \to H$ is bounded as we wanted to show.

To conclude the proof, let us see that $\Lambda_k(A)^{-r}H$ is dense in $H_A$. Choose a sequence $(f_n)_n$ such that $(\Lambda_k^* f_n)_n \subset \Psi(S^0_0)$ for $r > 1/2$, and $\lim_n f_n(A)x = x$ in $H_A$ for every $x \in H_A$ (for the existence of such a sequence, see for example [3, p. 165] and [17, p. 170]). Note that if $x \in H_c$ (see Introduction) then $\Lambda_k^*(A)f_n(A)x \in H_c$ by (1.5) and

$$
\Lambda_k^{-r}(A) (\Lambda_k^*(A)f_n(A)x) = f_n(A)x \to x.
$$

Now it suffices to apply the density of $H_c$ in $H_A$. □
Proof of Theorem 2.4. We begin by proving that $H_{A,\theta}^{\text{obs}} = H_A$ with equivalent norms. To see this, let first show that for any $\theta_1, \theta_2 \in (\omega, \pi]$ there exists a constant $K = K(\theta_1, \theta_2) > 0$ such that

$$
(6.1) \quad \int_0^\infty \| \sqrt{A}(re^{i\theta_1} - A)^{-1}x \|^2 dr \leq K \int_0^\infty \| \sqrt{A}(re^{i\theta_2} - A)^{-1}x \|^2 dr
$$

for all $x \in H_{-1}$. Denote by $\Gamma_\phi := \{ re^{i\phi}; r > 0 \}$ with $\phi \in (-\pi, \pi]$. Put $\theta = \min(\theta_1, \theta_2)$. The estimate

$$
\| (re^{i\theta_1} - A)^{-1} \sqrt{A}x \| \leq \frac{r}{\sqrt{A}} \| (re^{i\theta_2} - A)^{-1} \sqrt{A}x \| \leq \left( C_0 |e^{i\theta_1} - e^{i\theta_2}| + 1 \right) \| (re^{i\theta_2} - A)^{-1} \sqrt{A}x \|
$$

follows from (1.1). Hence for every two angles $\theta_1, \theta_2 \in (\omega, \pi]$, the quantities $\| \mathcal{O}x \|_{L^2(\Gamma_\phi)}$ and $\| \mathcal{O}x \|_{L^2(\Gamma_\phi)}$ are comparable for all $x \in H_{-1}$. This proves (6.1).

Now it follows from [24, Theorem 2.7] that if $\mathcal{O}x|_{\partial S_{\theta_0}} \in L^2(\partial S_\theta)$ for some $\theta_0 \in (\omega, \pi)$, then $\mathcal{O}x|_{\partial S_\theta} \in E^2(\mathbb{C} \setminus S_\theta; H)$ for any $\theta \in (\omega, \pi)$ (the general case reduces easily to the case $\pi/2 = \theta < \theta_0$). On the other hand, it is readily seen that

$$
(6.2) \quad \int_0^\infty \| \psi_t(A)x \|^2 \frac{dt}{t} = \| \mathcal{O}x \|_{L^2(\Gamma_\phi)}^2
$$

for all $x \in H_{-1}$ if one chooses $\psi(z) = \sqrt{z}(1+z)^{-1} \in \Psi(S_\theta^0)$. Thus $H_{A,\theta}^{\text{obs}} = H_A$ with equivalent norms, for any $\theta \in (\omega, \pi)$.

Also, by Proposition 5.7 $H_{A,\theta}^{\text{obs}} = H_{A,\theta}^{\text{ctr}}$ with equivalent norms. Hence we conclude that $H_{A,\theta}^{\text{obs}} = H_{A,\theta}^{\text{ctr}} = H_A$ for any $\theta \in (\omega, \pi)$ with mutually equivalent norms. \hfill $\square$

Proof of Theorem 2.5. Fix any $\alpha$ such that $0 < \alpha < \pi/2\theta$. By Lemma 5.3 Proposition 4.2 (1) and Lemma 4.1 (4),

$$
\text{Ker } \mathcal{O}_\theta \mathcal{W}_\theta = \text{Ker } J_{\delta_1} = \text{Ker } J_{\delta_0} u = \delta_\alpha E^2(S^0_\theta; H)
$$

for $0 < \alpha < \pi/2\theta$. Further, $\text{Ker } \mathcal{O}_\theta \mathcal{W}_\theta = \text{Ker } \mathcal{W}_\theta$ since $\mathcal{O}_\theta$ is injective and so (i) is proved.

Part (ii) is a consequence of Proposition 5.2 (1) and Theorem 2.4.

Note that the control operator $W_\theta$ intertwines the resolvent of the operator $A$ with the resolvent of the model operator $M^T_\theta$:

$$
(A - \lambda)^{-1} W_\theta = W_{\theta} M^T_{(z-\lambda)^{-1}}, \quad \lambda \in \mathbb{C} \setminus S_\theta,
$$

see [27] formula (5.1)] and for any rational scalar function $q \in H^\infty(S^0_\theta)$, we have

$$
W_\theta(qf) = q(A)W_\theta(f), \quad f \in E^2(S^0_\theta; H).
$$

In particular $M_z(D(M_z) \cap (\text{Ker } W_\theta)) \subset \text{Ker } W_\theta$, and therefore $\hat{M}_z$ is well-defined. Then the operator $\hat{M}_z$ is closed, densely defined and

$$
\hat{A} \hat{W}_\theta = \hat{W}_\theta \hat{M}_z,
$$

as desired.
see a similar proof in [27, Theorem 5.6]. This shows part (iii) and the proof is concluded.

Proof of Theorem 2.6. Since $H_{A,\theta}^\text{obs} = H_A$ then $H_A \hookrightarrow H_{-1}$ by Proposition 5.2. Then $O_\theta|_{H_A}$ is one-to-one and we have $O_\theta|_{H_A} : H_A \hookrightarrow \mathcal{H}(\delta_\alpha)$. Take $v \in \mathcal{H}(\delta_\alpha)$. By Lemma 4.1 (4) there exists $u \in E^2(S_\theta; H)$ such that $\tilde{J}_{\delta_\alpha}(u) = -v$ with $0 < \alpha < \pi/2\theta$. Set $x = W_\theta(u) \in H_A$. By Lemmas 5.3 and 4.2 (1), we obtain that $O_\theta|_{H_A(x)} = O_\theta(W_\theta(u)) = -\tilde{J}_{\delta_\alpha}(u) = v,$ and we conclude that $O_\theta|_{H_A}$ is an isomorphism.

Now we apply the Hilbert identity and the equality (2.4) to show that $O_\theta((\lambda - \tilde{A})^{-1}x) = (\lambda - M_z^T)^{-1}O_\theta(x), \quad x \in H_A,$ for $\lambda \not\in \sigma(\tilde{A})$ whence $O_\theta \tilde{A}x = M_z^T O_\theta x, \quad x \in D(\tilde{A}).$

Proof of Theorem 2.7. The equivalence of (d), (g) and (i) is a direct consequence of Theorem 2.5 (ii) and Theorem 2.4. The equivalence of (d), (h) and (j) is from Theorem 2.6.

7. Comments and final remarks

a. An alternative proof of Theorem 2.1. The functional model allows us to give the following argument: To prove the inclusion and the second inequality, note that the function $\varphi_{r,x}$ given by

$$
\varphi_{r,x}(z) := \frac{\Lambda_k(z)^{-r}}{\sqrt{z}} x, \quad x \in H, \quad k \in \mathbb{Z}\setminus\{0\},
$$

belongs to $E^2(S^\theta_\theta; H)$ for all $\theta > \omega$ and all $r > \frac{1}{2}$. Moreover, for any $x \in H,$

$$
W_\theta(\varphi_{r,x}) = \Lambda_0(A)^{-r}x.
$$

To show this, observe that both parts depend continuously on $x \in H$. So it suffices to check (7.1) for vectors $x$ of the form $x = T_A x_1$ with $x_1 \in H$. For these vectors, the equality follows from the $\Psi(S^\theta_\theta)$-functional calculus. Now take any element $h \in H$ of the form $h = \Lambda_k(A)^{-r}x, x \in H.$ By (7.1) and Proposition 5.7 we have that $h \in H_{A,\theta}^\text{obs} = H_{A,\theta}^\text{ctr}$ and, by Theorem 2.4, that $h \in H_A$. Then by Proposition 5.2 (1) we obtain the inequality

$$
\|\Lambda_k(A)^{-r}x\|_A \leq C\|W_\theta(\varphi_{r,x})\|_{A,\theta,\text{ctr}} \leq C'\|\varphi_{r,x}\|_{E^2(S^\theta_\theta; H)} \leq C_r\|x\|.
$$

The rest of the proof follows the same lines as the proof of Theorem 2.1 which was given in Section 6.

b. Admissibility. Let us assume that $-A$ is the infinitesimal generator of a bounded $C_0$-semigroup $(e^{-tA})_{t>0}$ on $H$. Let $C$ be an observation operator $C : D(A) \rightarrow Y$, for some Hilbert space $Y$, which is continuous
with respect to the graph norm of \( D(A) \). Then \( C \) is called admissible if it satisfies the estimate
\[
\int_0^\infty \|Ce^{-tA}x\|^2 dt \leq K\|x\|^2, \quad x \in D(A),
\]
for some positive constant \( K \). Admissible (and exact) observation operators are important in linear Control Theory, in particular in the linear quadratic optimization problem, see [19] [21] [18] and references therein.

In [14], see also [15, p. 204], admissible operators have been studied in terms of the admissibility of the operator \( \sqrt{A} \) (in this case \( Y = H \)), for bounded analytic semigroups \( (e^{-tA})_{t>0} \) or equivalently when \( A \) is sectorial of type \( \omega < \pi/2 \) (see [15, Proposition 2.2]). In particular \( \sqrt{A} \) is admissible if \( A \) has a \( H^\infty \) functional calculus. Here we obtain the following corollary of Theorem 2.1.

**Corollary 7.1.** Let \( A \) be a sectorial operator such that \( -A \) generates a \( C_0 \)-semigroup \( (e^{-tA})_{t>0} \). Then the operator \( C := \Lambda_k(A)^{-r}\sqrt{A} \) is admissible for \( A \) whenever \( r > 1/2 \) and for all \( k \in \mathbb{Z} \backslash \{0\} \).

**Proof.** This is easy. For every \( x \in D(A) \),
\[
\int_0^\infty \|Ce^{-tA}x\|^2 dt = \int_0^\infty \|\sqrt{tA} e^{-tA}\Lambda_k(A)^{-r}x\|^2 dt = \|\Lambda_k(A)^{-r}x\|^2_A \leq K\|x\|^2
\]
by Theorem 2.1. \( \square \)

c. On \( \omega \)-accretive operators. We recall that a closed operator \( T \) on a Hilbert space \( H \) is called \( \omega \)-accretive if its numerical range \( \{ \langle Tx, x \rangle ; x \in H, \|x\| \leq 1 \} \) is contained in the closed sector \( S_\omega \).

**Proposition 7.2.** For any \( x \in H_A \), consider \( Ax \) as an element of \( H_{-3} \). Define a (possibly unbounded) operator \( \tilde{A} \) on \( H_A \) by
\[
D(\tilde{A}) := \{ x \in H_A ; \ Ax \in H_A \},
\]
and \( \tilde{Ax} = Ax, \ x \in D(\tilde{A}) \). Then the following holds.

(a) \( \tilde{A} \) is similar to an \( \omega \)-accretive operator;

(b) \( \tilde{A} \) has trivial kernel, and \( \sigma(A) = \sigma(\tilde{A}) \).

We notice that (a) follows from [9, Theorem 7.3.9]. We will see that this fact also follows immediately from our main results.

**Proof.** Fix some angle \( \theta \in (\omega, \pi) \) and some \( \alpha \) as in Theorem 2.5. The spectrum \( \operatorname{spec} \delta_\alpha \) of the \( L(H) \)-valued analytic function \( \delta_\alpha \) on \( S_\theta^\circ \) is defined as the set of all points \( \lambda \in S_\theta \) such that \( \delta_\alpha^{-1} \notin H^\infty(S_\theta \cap \mathcal{W}) \) for any neighborhood \( \mathcal{W} \) of \( \lambda \).

By Theorem 2.5 \( \tilde{A} \) is similar to the quotient multiplication operator \( \tilde{M}_\varepsilon \) on \( Q(S_\theta^\circ, \delta_\alpha) \). It is immediate that this operator is \( \omega \)-accretive, which gives (a).
The kernel of $\tilde{M}_z$ is zero. Indeed, if $\tilde{M}_z \rho = 0$, $\rho = [r]$, $r \in E^2(S^\circ_θ, H)$, then $zr(z) = \delta_α(z)h(z)$ for some $h \in E^2(S^\circ_θ, H)$. It follows from the properties of $\delta_α$ that $h(z) = zh_1(z)$, $h_1 \in E^2(S^\circ_θ, H)$, and therefore $\rho = \delta_α \cdot h_1 = 0$. (One can also make use of the fact that the model operator like the one considered here is an analogue of a completely nonunitary contraction in the Nagy–Foiaš theory. Hence it cannot have a point spectrum on $\partial S^\circ_θ$.)

It also follows from well-known results that the spectrum of $\tilde{M}_z$ coincides with the set $\mathrm{spec} \delta_α$, see [22 VI.4.1] for the case of the disc (which transplants easily to any simply connected Jordan domain) or [26 Proposition 2.3]. So it only remains to prove that $\mathrm{spec} \delta_α = \sigma(A)$.

Let us assume first that $\omega < \pi/2$, then we can put $\alpha = 1$ and take $\theta < \pi/2$.

Let $\lambda \in S^\circ_θ$, $\lambda \neq 0$. Then the following properties are equivalent: (i) $\lambda \not\in \sigma(A)$; (ii) $\delta_1(\lambda)$ is invertible; (iii) $\delta_1(z)$ is invertible, with a uniform estimate on the norm of the inverse, for $z$ in some neighborhood of $\lambda$.

It follows that

$$\sigma(A) \setminus \{0\} = \mathrm{spec} \delta_1 \setminus \{0\}.$$ 

Now let us consider the remaining case when $\lambda = 0$. If $0 \not\in \sigma(A)$, then obviously, $\delta_1^{-1}$ exists and is uniformly bounded in a neighborhood of the origin.

Conversely, suppose that $0 \not\in \mathrm{spec} \delta_1$, and let us show that $0 \not\in \sigma(A)$. It follows from the assumption that $(A - z) \Phi(z) = A + z$, for $z$ in a neighborhood $W$ of 0, where $\Phi, \Phi^{-1}$ are functions in $H^\infty(W, L(H))$.

Moreover, $\Phi(z)h \in \mathcal{D}(A)$ for all $z \in W$, $h \in H$. It follows that $(A - z)^{-1} = \Phi(z)(A + z)^{-1}$ for $z \in S^\circ_θ \cap W$. Now (\ref{eq:inverse}) implies the estimate $\| (z - A)^{-1} \| \leq C_1 |z|^{-1}$ for all $z \in W$, $z \neq 0$. This first order estimate of the resolvent implies that 0 either is in the resolvent set of $A$ or is its isolated eigenvalue. The latter contradicts the assumed injectivity of $A$ (see Introduction).

This finishes the proof of the equality $\sigma(A) = \sigma(A)$ for the case when $\omega < \pi/2$.

In the remaining case when $\omega \in [\pi/2, \pi)$, by applying what has been proved already and the results of [9 Section 3.1], it is easy to prove that

$$\sigma(A) = \{ z^2; z \in \sigma(A^{1/2}) \} = \{ z^2; z \in \sigma(A^{1/2}) \} = \sigma(A).$$

This gives the general case. \hfill $\square$

The idea of the following result is that the existence of the $H^\infty$-functional calculus in two half-planes implies automatically the existence of the $H^\infty$-functional calculus in their intersection, if one applies the results by Havin, Nersessian and Ortega-Cerdá.

We refer to [22 Chapter 4, §4] for the definition of purely dissipative operators.
Proposition 7.3. Let $A$ be the generator of an analytic semigroup and assume that $A$ is similar to an $\omega$-accretive operator. Suppose, moreover, that the operators $\pm e^{\pm i\omega}A$ are similar to purely dissipative operators. Then $A$ admits an $H^\infty$-functional calculus in $S^\omega_\omega$.

Proof. Let $\Pi_1$ and $\Pi_2$ be open half-planes such that $\Pi_1 \cap \Pi_2 = S^\omega_\omega$ and let $f \in H^\infty(S^\omega_\omega)$. It follows from [10, Example 4.1] that there is a constant $C$, depending only on $\omega$ and functions $f_j \in H^\infty(\Pi_j)$, $j = 1, 2$, such that

$$f = f_1 + f_2, \quad \|f_j\|_{H^\infty(\Pi_j)} \leq C \|f\|_{H^\infty(S^\omega_\omega)}.$$ 

By the assumption, $A$ has Nagy-Foiaş models in $\Pi_j$. Hence we can write

$$\|f(A)\| \leq \|f_1(A)\| + \|f_2(A)\| \leq \|f_1\|_{H^\infty(\Pi_1)} + \|f_2\|_{H^\infty(\Pi_2)} \leq 2C \|f\|_{H^\infty(S^\omega_\omega)},$$

as we wanted to show. \qed

d. Duality in the models of Nagy-Foiaş type. Note that $A^*$ is also a sectorial operator of type $\omega$, and that $A$ admits an $H^\infty$ calculus iff $A^*$ does, see [2]. This is how four functional models appear: the observation and the control models of $A$ and the observation and the control models of $A^*$. By (1.6), the characteristic function $\delta_{\alpha,A}^*$ of $A^*$ is related with the characteristic function $\delta_{\alpha,A}$ of $A$ via the formula

$$\delta_{\alpha,A^*}(z) = (\delta_{\alpha,A}(\bar{z}))^*.$$ 

It turns out that there is a certain natural Cauchy duality between the functional models of $A$ and the functional models of $A^*$. This point was explained in detail in [26]. The Cauchy pairing between the observation model spaces $\mathcal{H}(\delta_{\alpha,A})$ and $\mathcal{H}(\delta_{\alpha,A^*})$ is given by

$$\langle v, u \rangle_{\delta_{\alpha}} := \frac{1}{2\pi i} \int_{\partial S^\theta_0} \langle \delta_{\alpha}(z)v(z), u(\bar{z}) \rangle \, dz, \quad v \in \mathcal{H}(\delta_{\alpha}), u \in \mathcal{H}(\bar{\delta}_{\alpha}).$$

The following duality formula holds:

$$\langle x, y \rangle_{\mathcal{H}_A} = \langle O_x, O_y \rangle, \quad x \in H_A, y \in H_{A^*},$$

where $O$ is given by (2.2) and $(O_x(y))(z) = \sqrt{A^*(z - A^*)^{-1}}y$, $z \in \rho(A^*)$. See [26, formula (0.1), Proposition 4.2 and §9].

We finish with the following remark. Let $\theta$ be fixed, and take any $\alpha, \beta \in (0, \frac{\pi}{2})$. Theorem 2.6 implies that

$$\mathcal{H}(\delta_{\alpha}) = \mathcal{H}(\delta_{\beta}).$$

In general, suppose that $\Delta, \Delta_1 \in H^\infty(S^\mu_\theta, L(H))$ are two-sided admissible functions (see [26, §2]). Then, by [26, Proposition 11.2], $\mathcal{H}(\Delta) = \mathcal{H}(\Delta_1)$ if and only if there exists an operator-valued function $\psi$
on $S^\theta_0$ with $\psi, \psi^{-1} \in H^\infty(S^\theta_0; L(H))$ such that $\Delta = \psi \Delta_1$. In our situation, the existence of a function $\psi$ such that $\delta_\alpha = \psi \delta_\beta$ can be checked directly.

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