Are there local hidden variables models with time correlated detection violating the Bell inequality?

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Abstract
Explicit local hidden variables models are exhibited that assume a correlation between detection events produced in the same detector at different times. It is shown that some models give predictions closer to the Bell limit than models without time correlation.

1 Introduction

Many experiments have been performed for the test of the Bell inequalities\cite{1}. Amongst the experiments, those using photons are relevant because they may allow closing the locality loophole more easily than those using massive particles (like e. g. atoms\cite{2}.) Until recently all tests involving optical photons suffered from the “detection loophole”, that is the fact that entangled photons, when measured with low-efficiency detectors, give results that may be reproduced by local hidden variables models\cite{3}. However the progress in photon detectors, now available with high efficiency and low noise, have allowed recent experiments free from the detection loophole\cite{4}, \cite{5}.

The loophole-free violation of a Bell inequality would be of paramount importance. Indeed it would mean that no local hidden variables model (or “local realistic theory”) exists compatible with empirical evidence\cite{6}. Consequently it is relevant to search for any possible loophole in the empirical tests. The purpose of this paper is to study whether some time correlations
between the production and/or detection of photons might give rise to new loopholes. Such correlations may be more relevant for entangled photons produce by parametric down conversion than for photons produced in atomic cascades. In the latter it is likely that different atomic decays and detections are uncorrelated. In contrast parametric down conversion produces a beam where entangled photons appear spontaneously at random times, and most probably with bunching due to the Bose character of photons.

If there are no time correlations, for a given pair of photons emerging from the source I will label $p_a, p_b$, the single detection probabilities by Alice and Bob, respectively, and $p_{ab}$ the coincidence detection probability. Here we will consider the possibility that the detection of a photon may be either enhanced or inhibited by a previous detection. In order to make the study I will consider models where, after the detection by Alice (Bob) of a photon belonging to the first “photon pair” (that I will name “event” for simplicity of writing) produced in the source, it is enhanced or inhibited the detection probability of the Alice (Bob) photon belonging to the second event. We assume similar correlations between the detections of the third event and the fourth event, and so on excluding any other correlation. In the experiment by Christensen et al.\[5\] the polarizer’s settings are chosen at random, but only once every second. Many photon pairs are produced during that time interval, so that time correlations cannot be excluded.

To begin with we revisit a well known local hidden variables model. It has one hidden variable, $\lambda$, with a homogeneous probability density, i. e.

$$\rho (\lambda) = \frac{1}{\pi}; \lambda \in [0, \pi],$$

(1)

and the detection probability by Alice, given $\lambda$ and $\alpha$, is assumed to be

$$P(\lambda, \alpha) = \frac{1}{6} \left[ 1 + \sqrt{2} \cos (\lambda - \alpha) \right]^2$$

$$\equiv \frac{1}{3} \left[ 1 + \sqrt{2} \cos (2\lambda - 2\alpha) + \frac{1}{2} \cos (4\lambda - 4\alpha) \right],$$

(2)

which may be checked to fulfil

$$0 < P(\lambda, \alpha) < 1,$$

as it should. Similarly for Bob

$$P(\lambda, \beta) = \frac{1}{3} \left[ 1 + \sqrt{2} \cos (2\lambda + 2\beta) + \frac{1}{2} \cos (4\lambda + 4\beta) \right].$$

(3)
Hence we get the following coincidence and single probabilities

\[ p_{ab}(\phi) = \frac{1}{9} \left[ 1 + \cos(2\phi) + \frac{1}{8} \cos(4\phi) \right], \]

\[ \phi \equiv \alpha - \beta, \quad p_a(\alpha) = p_b(\beta) = 1/3, \quad (4) \]

where \( \alpha \) and \( \beta \) are the angles of the polarizer’s settings with, say, the vertical. The Bell inequality in the form of Clauser and Horne may be written

\[ B \equiv \frac{3p_{ab}(\phi) - p_{ab}(3\phi)}{p_a + p_b} \leq 1, \quad (5) \]

and it is fulfilled by the local model predictions for any \( \phi \), as it should. In particular for the usually measured angles \( \phi = \pi/8, 3\phi = 3\pi/8 \) we get

\[ B = \frac{3 \times \frac{1}{9} \left( 1 + \frac{\sqrt{2}}{2} \right) - \frac{1}{9} (1 - \frac{\sqrt{2}}{2})}{2 \times \frac{1}{3}} = \frac{3 \times 0.190 - 0.032}{0.667} = 0.805 < 1. \quad (6) \]

The predictions of this model are close to the quantum predictions for experiments with detectors having efficiency about 67%. In particular the value of the parameter \( B \), eq. (6) exactly reproduces the quantum prediction with detector efficiency 2/3. This suggests that a similar model for experiments involving photons not maximally entangled, similar to the recent detection-loophole-free ones [4], [5], might provide values much closer to the Bell limit (that is the corresponding parameter similar to \( B \) closer to unity). However such models would be more involved than the one given by eqs. (1) to (3).

2 Model with inhibited detection

Now we consider a modification of the model by assuming that Alice’s detection of her photon in the second event is inhibited if she detected her photon in the first event, and similar for Bob. Thus the probability of two detections from two events will be zero, for a single detection from the first event (and no detection in the second event) will be \( p_a \), the probability of a single detection from the second event will be \( (1 - p_a) p_a \). Finally the probability of zero detections will be \( (1 - p_a)^2 \). Therefore the mean probability of detection per event will be \( (2 - p_a) p_a \)

\[ p'_a = \frac{1}{2} (2 - p_a) p_a = p_a - \frac{1}{2} p_a^2. \]
The same is true for Bob.

Now we shall study coincidences. In the first event the probability of one or zero coincidences will be \( p_{ab} \) or \( 1 - p_{ab} \), respectively, and the probability of a coincidence in the second event is zero if either Alice or Bob or both had one detection in the first event. Thus the probability of having one coincidence in the second event will be \( (1 - 2p_a + p_{ab}) p_{ab} \). (Remember that we assume \( p_a = p_b \)). The mean probability of coincidence per event will be

\[
p_{ab}' = \frac{1}{2} [p_{ab} + (1 - 2p_a + p_{ab}) p_{ab}] = p_{ab} - p_a p_{ab} + \frac{1}{2} p_{ab}^2.
\]

Hence the parameter eq.(5) becomes

\[
B' = \frac{3p_a (\phi) - p_{ab} (3\phi)}{2p_a'} = \frac{(1 - p_a) [3p_{ab} (\phi) - p_{ab} (3\phi)] + \frac{3}{2} p_{ab} (\phi)^2 - \frac{1}{2} p_{ab} (3\phi)^2}{2 (p_a - \frac{1}{2} p_a^2)}
\]

\[
= \frac{1 - p_a}{1 - \frac{1}{2} p_a} B + \frac{3p_{ab} (\phi)^2 - p_{ab} (3\phi)^2}{2p_a (2 - p_a)}.
\]

Inserting the values of eqs.(4) and (6) we get

\[
B' = 0.805 \times \frac{2/3}{5/6} + 3 \times 0.190^2 - 0.032^2 = 0.644 + 0.096 = 0.740.
\]

3 Model with enhanced detection

In this case, if Alice detects one photon from the first event, she will detect another one in the second event with certainty. The probability of this situation is \( p_a \). If she does not detect in the first event (probability \( 1 - p_a \)), she may detect in the second event. The total probability of this is \( p_a (1 - p_a) \). In summary the mean single probability per event is

\[
p_a'' = \frac{3}{2} p_a - \frac{1}{2} p_a^2.
\]

If there is a coincidence count in the first event (probability \( p_{ab} \)) there will be another in the second event with certainty. If there is a single detection by Alice in the first pair (probability \( p_a - p_{ab} \)) there will be another one in the second event. The probability that also Bob detects in the second event will be \( p_a \), so the probability of one single count in the first event and one
coincidence in the second event will be $2 \left( p_a - p_{ab} \right) p_a$. If there is no detection in the first event (probability $1 - 2p_a + p_{ab}$) the probability of coincidence in the second event will be $p_{ab}$. In summary, the mean probability of coincidence count per event will be

$$p''_{ab} = \frac{1}{2} \left[ 2p_{ab} + 2 \left( p_a - p_{ab} \right) p_a + \left( 1 - 2p_a + p_{ab} \right) p_{ab} \right]$$

$$= \frac{3}{2}p_{ab} - 2p_a p_{ab} + \frac{1}{2}p_{ab}^2 + p_a^2.$$

The parameter eq.(5) becomes

$$B'' = \frac{\left( \frac{3}{2} - 2p_a \right) \left[ 3p_{ab} (\phi) - p_{ab} (3\phi) \right]}{\left( \frac{3}{2} - \frac{1}{2}p_a \right) 2p_a} + \frac{3p_{ab}^2 (\phi) - p_{ab}^2 (3\phi)}{(3 - p_a) 2p_a} + \frac{2p_a}{3 - p_a}$$

$$= \frac{3 - 4p_a}{3 - p_a} B + \frac{3p_{ab} (\phi)^2 - p_{ab} (3\phi)^2}{2p_a (3 - p_a)} + \frac{2p_a}{3 - p_a}.$$ 

Putting the values of eqs.(4) and (6) we get

$$B'' = \frac{5}{8} \times 0.805 + \frac{3 \times 0.190^2 - 0.032^2}{16/9} + \frac{1}{4} = 0.503 + 0.060 + 0.25 = 0.813.$$

The enhancement in detection increases but slightly the parameter $B$.

### 4 Conclusion

The time correlation in detection is able to increase the value of the Bell parameter eq.(5) in the model eqs.(1) to (3). The effect is too small to allow a violation of the Bell inequality. However a small increase might be enough in models of the experiments involving photons not maximally entangled, similar to the recent detection-loophole-free ones[4], [5]. Indeed in this case a model without time correlations could provide values close to the Bell limit, as mentioned at the end of Section 2. This possibility will be studied elsewhere.

### References

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