AN EXPLICIT WATSON–ICHINO FORMULA WITH CM NEWFORMS

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Abstract. In this paper, we extend the work of Humphries–Khan [HK20] to establish an explicit version of Watson–Ichino formula for \( L(1/2, f \otimes \text{ad} g) \), where \( f \) is a Hecke–Maass form and \( g \) is a CM newform.

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1. Introduction

The purpose of this paper is to establish an explicit Watson–Ichino formula for triple product \( L \)-functions of a special class of Hecke–Maass forms. To begin with, let \( \pi_1, \pi_2, \pi_3 \) be irreducible unitary cuspidal automorphic representations of \( \text{GL}_2(\mathbb{A}) \) with the product of their central characters trivial, where \( \mathbb{A} \) is the adele ring over \( \mathbb{Q} \). One can consider the (complete) triple product \( L \)-function

\[
L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)
\]

associated to them, which was originally defined classically by Garrett [Gar87] and was generalized further by Piatetski-Shapiro and Rallis [PSR87] using an adelic approach. Gross and Kudla [GK92] established an explicit identity relating central \( L \)-values and period integrals (which are finite sums in their case), when the \( \pi_i \)'s correspond to cusp forms of a prime level and weight 2. Watson [Wat02] generalized this identity to higher levels and weights, and Ichino [Ich08] proved an adelic version of this period formula which works for all the cases. In this paper we study the case when the quaternion algebra in [Ich08] is \( \text{GL}_2 \) and the étale cubic algebra is \( \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \) over \( \mathbb{Q} \). Then the Ichino’s formula can be reformulated (cf. [Col20]) as follows: for any \( \varphi_i = \otimes \varphi_{i,v} \in \pi_i, i = 1, 2, 3, \)

\[
\left| \int_{\mathbb{A}^\times \text{GL}_2(\mathbb{Q}) \setminus \text{GL}_2(\mathbb{A})} \varphi_1(g) \varphi_2(g) \varphi_3(g) \ dg \right|^2 \prod_{v=1}^3 \int_{\mathbb{A}^\times \text{GL}_2(\mathbb{Q}) \setminus \text{GL}_2(\mathbb{A})} |\varphi_1(g)|^2 \ dg = \frac{C (\pi_1 \otimes \pi_2 \otimes \pi_3)}{2^6 \cdot L(1, \pi_1, \text{Ad}) L(1, \pi_2, \text{Ad}) L(1, \pi_3, \text{Ad}) \prod_v I_v},
\]

where \( \mathbb{A}^\times \) is diagonally embedded in \( \text{GL}_2(\mathbb{A}) \) as its center, \( C \) is defined so that \( dg = C \prod_v dg_v \) is the Tamagawa measure on \( \mathbb{A}^\times \text{GL}_2(\mathbb{Q}) \setminus \text{GL}_2(\mathbb{A}) \), \( L(s, \pi_i, \text{Ad}) \) is the adjoint \( L \)-function attached to \( \pi_i \), and

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the local constants $I_v'$ are defined as in (2). Moreover, for $v = p < \infty$, $I_v' = \text{vol}(\mathbb{Z}_p^\times \backslash \text{GL}_2(\mathbb{Z}_p))$ when all $\pi_{i,p}$ are unramified, and $\varphi_{i,p} \in \pi_{i,p}$ are unit spherical vectors.

Many authors have derived several explicit versions of Watson–Ichino formula in various cases. For example Nelson [Ne11] extends Watson’s formula and relates

$$\left| \int_{\Gamma_0(q) \backslash \mathcal{H}} y^k f(z) \overline{g(z)} \, d\mu(z) \right|^2 \quad \text{and} \quad L\left(\frac{k}{2}, f \otimes g \otimes \bar{g}\right),$$

where $\mathcal{H}$ is the upper-half plane, $f$ is a Hecke–Maass form of level 1 and $g$ is a newform of squarefree level $q$ ($g$ can be holomorphic of weight $k$ or a Maass form). In this case the triple product $L$-function $L(s, f \otimes g \otimes \bar{g})$ can be factorized as $L(s, f) L(s, f \otimes \text{ad} g)$ by comparing Euler products. Recently Humphries and Khan [HK20] show an exact formula for $L(1/2, f \otimes \text{ad} g)$ (see (1)), where $f$ is a Hecke–Maass form and $g$ is a dihedral Maass newform (which associates to a Grössencharacter on $\mathbb{Q}(\sqrt{D})$ where $D \equiv 1 \pmod{4}$) is a positive squarefree fundamental discriminant. With this formula [HK20] unconditionally gives a proof of the Gaussian moments conjecture for the fourth moment of dihedral Maass newforms. This explicit Watson–Ichino formula also has some applications in studying quantum variance. Huang and Lester [HL23] give an asymptotic formula for the harmonic weighted quantum variance of the family of dihedral Maass forms on $\Gamma_0(D)$ with $D$ restricted by some congruence condition.

In this paper, we continue the work of Humphries–Khan [HK20] and Hu [Hu17] to establish Theorem 1.1 an explicit version of Ichino’s formula for $L(1/2, f \otimes \text{ad} g)$, where $g = g_\Omega$ is a CM newform and $f$ is a Hecke–Maass form. These explicit formulas will have, for example, an application to quantum variance for CM newforms following the ideas of [HL23].

1.1. Main result. Let $q$ be a positive integer and $L^2(\Gamma_0(q) \backslash \mathcal{H})$ be the space of square-integrable functions on the upper-half plane $f : \mathcal{H} \to \mathbb{C}$ such that $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma_0(q)$. The inner product in this space is defined by

$$(f, g)_q := \int_{\Gamma_0(q) \backslash \mathcal{H}} f(z) \overline{g(z)} \, d\mu(z), \quad \text{where } d\mu(z) = y^{-2} \, dx \, dy, \quad z = x + iy.$$ 

But for functions such that $f(\gamma z) = (cz + d)^k f(z)$ with $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ (for example, holomorphic modular forms of weight $k$), $(f, g)_q$ is defined to be the Petersson inner product whenever the integral converges:

$$(f, g)_q := \int_{\Gamma_0(q) \backslash \mathcal{H}} y^k f(z) \overline{g(z)} \, d\mu(z).$$ 

Notice that $y^k f(z) \overline{g(z)}$ is $\Gamma_0(q)$-invariant in the latter case, and the convergency holds if the inner product is defined on cusp forms.

An integer $D$ is a fundamental discriminant if either $D \neq 1$, $D \equiv 1 \pmod{4}$ and $D$ is squarefree, or $4 \mid D$, $\frac{D}{4} \equiv 2, 3 \pmod{4}$ and $\frac{D}{4}$ is squarefree. For each $D$ there exists a quadratic extension $E = \mathbb{Q}(\sqrt{D})$ of $\mathbb{Q}$ such that $E$ has discriminant $D$. One can also define a (quadratic) Dirichlet character modulo $|D|$ by the Kronecker symbol $\chi_D(n) := \left( \frac{D}{n} \right)$. For an imaginary quadratic extension $E/\mathbb{Q}$ with negative fundamental discriminant $D < 0$, consider a Hecke character $\Omega$ on $E^\times \backslash \mathbb{A}_E^\times$ (its associated classical Grössencharacter is also denoted by $\Omega$), whose restriction on $\mathbb{A}_E^\times$ is trivial. Assume that it is unramified everywhere at finite places. At infinity we have $\Omega_\infty = (z/2)^n$ for some $n \in \mathbb{Z}$. Recall that, when $n > 0$ and $\Omega$ does not factor through the norm map $N_E : \mathbb{A}_E^\times \to \mathbb{A}_E^\times$, there is a cuspidal newform (cf. [Rib77] and [Iwa97] Theorem 12.5))

$$g_\Omega(z) := \sum_a \Omega(a)(Na)^n e(zNa) \in S_{2n+1}^\times(\Gamma_0(|D|), \chi_D)$$

with complex multiplication of level $|D|$, weight $2n+1$ and nebentypus $\chi_D$ such that $L(s, g_\Omega) = L(s, \Omega)$. Here the sum is over all integral ideals of $E$, and $Na$ is the norm of $a$. 
All the $L$-functions in this paper are complete without conductor, for example,

$$L(s, \pi) := L_{\infty}(s, \pi)L_{\text{fin}}(s, \pi)$$

is defined as in [Ich08] (instead of the one in [HK20] that $\Lambda(s, \pi) := q(\pi)^{s/2}L(s, \pi)$).

We will prove the following explicit Watson–Ichino formula. See Theorem 2.1 for the more general statement.

**Theorem 1.1.** Let $q_1 \mid q = |D|$ with $D < 0$ a fundamental discriminant, $B^*_0(q_1)$ be the set of (normalized) Hecke–Maass newforms of weight 0 and level $q_1$ with trivial nebentypus. For any CM newform $g = g_1 \in S^*_k(\Gamma_0(q), \chi_D)$ and Hecke–Maass newform $f \in B^*_0(q_1)$ normalized such that the Petersson norms are $\langle g, g \rangle_q = \langle f, f \rangle_q = 1$, we have that, if $4 \nmid q_1$ or $2$ is a Type-2 supercuspidal prime for $f$, then

$$\langle f, y^k|g|^2 \rangle_q^2 = \langle f, y^k|g|^2 \rangle_q^2 = \frac{L(\frac{1}{2}, f)L(\frac{1}{2}, f \otimes \text{ad} g)}{L(1, \text{ad} g)^2 L(1, \text{Sym}^2 f) Sqq_1 \nu_q} \nu_q$$

(notice that $|y^{k/2}g(z)|$ is $\Gamma_0(q)$-invariant), where

$$\nu_q := [\Gamma_0(1) : \Gamma_0(n)] = n \prod_{p \mid n} (1 + p^{-1})$$

otherwise, when $4 \mid q_1$ and $2$ is a Type-1 supercuspidal prime for $f$, $\langle f, y^k|g|^2 \rangle_q^2$ is $L_2(1, \text{Sym}^2 f) = \frac{2}{\pi}$ times above. For oldforms $(\iota_w f)(z) := f(wz)$ we have the same result for

$$\langle \iota_w f, y^k|g|^2 \rangle_q \langle \iota_w f, y^k|g|^2 \rangle_q = \frac{L(\frac{1}{2}, f)L(\frac{1}{2}, f \otimes \text{ad} g)}{L(1, \text{ad} g)^2 L(1, \text{Sym}^2 f)} + \epsilon_f \nu_q$$

for any $w_1, w_2 \mid \frac{q_1}{q_1}$.

**Remark 1.2.** Let $D \equiv 1 \pmod{4}$ be a positive squarefree fundamental discriminant, and $q_1 \mid q = D$. [HK20] Corollary 4.19 shows that, for any dihedral Maass newform $g = g_1 \in B^*_0(\Gamma_0(q), \chi_D)$ and Hecke–Maass newform $f \in B^*_0(q_1)$, and for any $w_1, w_2 \mid \frac{q_1}{q_1}$,

$$\langle \iota_{w_1} f, |g|^2 \rangle_q \langle \iota_{w_2} f, |g|^2 \rangle_q = \frac{L(\frac{1}{2}, f)L(\frac{1}{2}, f \otimes \text{ad} g)}{L(1, \text{ad} g)^2 L(1, \text{Sym}^2 f)} + \epsilon_f \nu_q$$

Theorem 2.1 also leads to the fact that, the above identity holds when $D$ is any positive fundamental discriminant (either $D$ or $D/4$ is squarefree) and $q_1 \mid q = D$, $4 \nmid q_1$ or $2$ is not an “unramified” dihedral supercuspidal prime (i.e. not a Type-1 prime) for $f$, except that $\nu_q/\nu_{q_1} = \nu_{q/q_1}$ does not hold in this case. (If $4 \mid q_1$ and 2 is “unramified supercuspidal” for $f$ then the result is multiplied by $\frac{4}{\pi}$.)

One may notice the extra condition does not show up in [HK20]. The reason is that they assumed $D$ (and therefore $q_1$) squarefree. Otherwise, if $4 \mid q_1$, that is to say, if $\pi_{f,2}$ is supercuspidal (this is the only possible case because $16 \mid D$), the local $L$-factor $L_p(1, \text{Sym}^2 f)$ at $p = 2$ depends on the “type” of $\pi_{f,2}$, which equals either $\frac{1}{2}$ or $\frac{3}{2}$. More details can be found in Section 3.4.

**Remark 1.3.** This explicit Watson–Ichino formula may have some potential applications, for example, in studying quantum variances, following the idea of Huang and Lester [HL23]. To study the distribution of $L^2$-mass for certain forms, for example, dihedral Maass forms or CM forms $g_\Omega$, define

$$\mu_\Omega(\psi) := \langle \psi g_\Omega, g_\Omega \rangle_q \int_{\Gamma_0(q) \backslash \mathcal{H}} y^k \psi(z) |g_\Omega(z)|^2 \, d\mu(z)$$

for any smooth test function $\psi \in \Gamma_0(q) \backslash \mathcal{H} \rightarrow \mathbb{C}$ with mean zero which decays rapidly in the cusp, where $q = |D|$ is the level of $g_\Omega$, and $k$ the weight of $g_\Omega$. The proposed quantum variance corresponding to these CM forms could be defined by a sum of the form

$$Q(\psi; K) := \sum_{k \leq K} |\mu_\Omega(\psi)|^2$$
as $K \to \infty$. Theorem 1.1 gives an explicit formula to write the summands as the central values of certain $L$-functions, when $\psi$ is a Hecke–Maass cuspidal form (old or new). It is possible to establish an asymptotic formula, which relates the (harmonic weighted) quantum variance of the family of CM forms on $\Gamma_0(|D|)$, to its “classical variance” $V(\psi)$. See [LS04, HL23] for more details.

1.2. Organization of the paper. This paper is organized as follows. In Section 2 we fix notations and normalization, recall the Watson–Ichino formula in classical language and show how the local constants can be assembled into the global result.

By definition a CM form $g$ is associated with a Grössencharacter $\Omega$ of an imaginary quadratic extension $E/\mathbb{Q}$ with discriminant $D < 0$. When $D$ is squarefree, the main result is nothing new comparing with Humphries–Khan’s version, except that the archimedean local constants are different (see Proposition 2.2), which has been calculated in [Wat02]. But when $4 \mid D$, the general case cannot be avoided when the levels of $f$ and $g$ are not squarefree. More precisely, for the new case when $\pi_{f,v}$ is supercuspidal, inspired by the work of Hu [Hu16, Hu17], in Section 3.3 we will deal with the Kirillov model of $\pi_{f,v}$ and calculate some special values of Whittaker function of a new vector. Finally, following ideas in [MV10, Hu16, HK20] we will calculate in Section 3 the local constants $I_v$ (Propositions 2.3, 2.4, 2.5) for $\pi_{f,v}$ special, spherical, and supercuspidal respectively, which completes the proof of the Main Theorem 2.1.

2. An explicit version of Watson–Ichino Formula

Let $q$ be a positive integer, $q_1 \mid q$ and $f \in E_0^v(q_1)$ be a Hecke–Maass cuspidal newform or $f \in S^v_k(\Gamma_0(q_1))$ a holomorphic cuspidal newform with trivial nebentypus. Define the adelic lift of $f$ by

$$\varphi_f(g) := ((y^{k/2}f) \mid g)$$

with $g = \gamma g_\infty k_0$ given by the strong approximation where $\gamma \in \text{GL}_2(\mathbb{Q})$, $g_\infty = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{GL}_2^+(\mathbb{R})$, and $k_0 \in K_0(q_1)$. (When $f$ is a newform with nebentypus $\chi_f$, the adelic lift is defined to be the above $\varphi_f(g)$ times $\chi_f(k_0)$, where $\chi_f$ is the character of $K_0(q_1)$ given by applying $\chi_f$ to the lower-right entry.) Let $\pi_f = \oplus_v \pi_{f,v}$ be the cuspidal automorphic representation of $\text{GL}_2(\mathbb{Q})$ generated by $\varphi_f$.

For finite places $v = p$, we know $\pi_{f,p}$ is an unramified principal series representation if $p \nmid q_1$, and a special representation (an unramified twist of the Steinberg representation) if $p \mid q_1$. When $p^2 \mid q_1$, we recall a certain classification of such $\pi_{f,p}$ (cf. [NPS14, Section 2.1.5]):

- **Type 1.** $\pi_{f,p}$ is an “unramified” supercuspidal representation, i.e. $\pi_{f,p} \simeq \pi_{f,p} \otimes \eta_p$, where $\eta_p$ is the unique nontrivial unramified quadratic character of $\mathbb{Q}_p^\times$. Equivalently $\pi_{f,p}$ is a dihedral supercuspidal representation associated with an unramified quadratic field extension $E_p/\mathbb{Q}_p$ and a character of $E_p^\times$ that is not trivial on the kernel of the norm map $N_{E_p/\mathbb{Q}_p} : E_p^\times \to \mathbb{Q}_p^\times$. In this case we call $p$ a Type-1 supercuspidal prime for $f$. (Actually $p$ is called an unramified supercuspidal prime in some other papers, for example, [BM19]. The reason we rename it in this paper is that, to call it “unramified” one might confuse it with the spherical representation.)

- **Type 2.** $\pi_{f,p}$ is supercuspidal satisfying $\pi_{f,p} \neq \pi_{f,p} \otimes \eta_p$, with $\eta_p$ above. Again in this case we call $p$ a Type-2 supercuspidal prime for $f$.

- **Type 3, 4, 5.** $\pi_{f,p}$ is a ramified principal series, or a ramified twist of the Steinberg representation.

In this paper we focus on Type 1 and Type 2, i.e. $\pi_{f,p}$ is supercuspidal whenever $p^2 \mid q_1$. Actually Types 1 and 2 cover all possibilities in Theorem 1.1 when $q_1 \mid |D|$ with $D$ a fundamental discriminant, the only possible $p$ such that $p^2 \mid q_1$ is $p = 2$; but the conductor exponent of $\pi_{f,p}$ is $\geq 4$ for the other three types when $p = 2$ (because any character of $\mathbb{Q}_2^\times$ cannot have conductor exponent 1), while 16 can never divide a fundamental discriminant $D$.

Theorem 2.1. Let $q$ be a positive integer, $q_1 \mid q$, and $\chi$ be a primitive Dirichlet character modulo $q$. Assume that $f \in E_0^v(q_1)$ is a Hecke–Maass newform satisfying that, the corresponding local automorphic
representation $\pi_{f,p}$ is supercuspidal whenever $p^2 \mid q_1$. Then, for any newform $g \in S_k^\chi(q,\chi)$ or $g \in B_0^\chi(q,\chi)$ which is a Hecke eigenform, normalized such that $\langle g, g \rangle = \langle f, f \rangle = 1$, we have that,

$$
\left| \langle f, y^k g \rangle_q^2 \right| = \frac{L\left(\frac{1}{2}, f\right)L\left(\frac{1}{2}, f \otimes \Ad g\right)}{L(1, \Ad g)^2 L(1, \Sym^2 f)} \frac{1}{8q^2} \cdot C_\infty \prod_p C_p,
$$

where

$$
C_p = \begin{cases}
1 & \text{if } p \nmid q, \\
(1 + p^{-1})^{-1} & \text{if } p \mid q, p \nmid q_1, \\
1 & \text{if } p \mid q, p \mid q_1, \\
(1 + p^{-1})^{-1} & \text{if } p \mid q, p^2 \mid q_1, p \text{ is a Type-1 supercuspidal prime for } f, \\
1 & \text{if } p \mid q, p^2 \mid q_1, p \text{ is a Type-2 supercuspidal prime for } f;
\end{cases}
$$

and $C_\infty = \begin{cases} 1 & k > 0 \text{ with } \epsilon_f \in \{\pm 1\} \text{ the parity of } f. \text{ For oldforms } (\iota_w f)(z) := f(wz) \text{ we have the same result, i.e., } \\
k = 0 & \text{for any } w_1, w_2 \mid \frac{q}{q_1}, \text{ any normalized Hecke–Maass newform } f \in B_0^\chi(q_1), \text{ and any normalized Hecke newform } g \in S_k^\chi(q,\chi) \text{ or } g \in B_0^\chi(q,\chi). \end{cases}$

**Proof.** Using the notations in [HK20 Section 4.2], we denote by $\varphi_1, \varphi_2, \varphi_3$ the adelic lifts of $g, \tilde{g}, \iota_w f$ respectively, and by $\pi_1$ the cuspidal automorphic representation of $\GL_2$ generated by $\phi_i$ ($i = 1, 2, 3$). Here we have $\pi_2 = \pi_1$ (the contragredient). Let $\tilde{g} \in S_k^\chi(q,\chi)$ or $B_0^\chi(q,\chi)$ be a Hecke eigenform such that $g$ and $\tilde{g}$ are both associated to the same newform, and $\varphi_2$ be the adelic lifts of $\tilde{g}$ (and define $\varphi_3$ respectively).

We fix a $\GL(2, \Q_v)$-invariant bilinear local pairing $\langle \cdot, \cdot \rangle$ on $\pi_{1,v} \otimes \pi_{1,v}$ for each place $v$ and each $i = 1, 2, 3$, and use this to define a pairing $\langle \cdot, \cdot \rangle$ on $\Pi_v \otimes \Pi_v$ (where $\Pi_v = \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}$) determined on simple tensors $\varphi_v := \varphi_{1,v} \otimes \varphi_{2,v} \otimes \varphi_{3,v}$ and $\tilde{\varphi}_v := \tilde{\varphi}_{1,v} \otimes \tilde{\varphi}_{2,v} \otimes \tilde{\varphi}_{3,v}$ by

$$
\langle \varphi_v, \tilde{\varphi}_v \rangle := \langle \varphi_{1,v}, \tilde{\varphi}_{1,v} \rangle \langle \varphi_{2,v}, \tilde{\varphi}_{2,v} \rangle \langle \varphi_{3,v}, \tilde{\varphi}_{3,v} \rangle.
$$

Note that this is unique up to nonzero scalar. Then we define

$$
I_v(\varphi_v \otimes \tilde{\varphi}_v) := \int_{Z(F_v) \backslash \GL_2(F_v)} \prod_{i=1}^3 (\pi_{1,v}(\psi_v) \varphi_{i,v}, \tilde{\varphi}_{i,v}) \ \text{dg}_v,
$$

(2)

$$
I_v'(\varphi_v \otimes \tilde{\varphi}_v) := \frac{L(1, \pi_{1,v}, \Ad) L(1, \pi_{2,v}, \Ad) L(1, \pi_{3,v}, \Ad)}{\zeta_F(2) L(\frac{1}{2}, \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v})} \langle \varphi_v, \tilde{\varphi}_v \rangle.
$$

We follow the normalization of local Haar measures in [HK20 Section 4.2]. That is, the Haar measure $\text{dg}_v$ on $Z(F_v) \backslash \GL_2(F_v)$ at any non-archimedean place is defined such that, under the decomposition of $\text{dg}_v$ induced by the Iwasawa decomposition, the maximal compact subgroup $\GL_2(Z_p)$ has volume 1; and the Haar measure at any real place is $\text{dg}_v := dx_v \cdot |y_v|_{v^{-1}}^{-1} dx_v \cdot dk_v$ with $g_v = (\begin{smallmatrix} y_v & x_v \\ 0 & 1 \end{smallmatrix}) k_v$, $k_v \in K_v$, where $dk_v$ is the Haar measure on $K_v = \SO(2)$ with volume 1.

The Watson–I IBM suggests forming

$$
\int_{\Gamma_0(q) \backslash \mathcal{H}} y^k |g(z)|^2 \iota_w f(z) \ \text{d}\mu(z) \int_{\Gamma_0(q) \backslash \mathcal{H}} y^k |g(z)|^2 \iota_w f(z) \ \text{d}\mu(z) = \frac{C_{\infty}}{8q} \frac{L(1, f) L(1, f \otimes \Ad g)}{L(1, \Sym^2 f)} \prod_p I_p'(\varphi_p \otimes \tilde{\varphi}_p),
$$

where $C_{\infty} = 1$ if $k > 0$ and $\epsilon_f \in \{\pm 1\}$. **End of Proof.**
where \( \nu_q = q \prod_{p \mid q} (1 + p^{-1}) \). This formula differs with the one given in \([HK20]\) Section 4.3), because the local constant at infinity becomes \( I_\infty(\varphi_\infty \otimes \tilde{\varphi}_\infty) = C_\infty \), given by Proposition 2.2 and also because the \( \mathcal{L} \)-functions in \([HK20]\) are defined with conductors.

Notice that (cf. \([WW02]\) or \([LW12]\)) for \( p \mid q \), the local component \( \pi_{1,p} \) of \( g \) is a unitary ramified principal series representation \( \omega_{1,p} \boxplus \omega_{2,p} \), where the unitary characters \( \omega_{1,p}, \omega_{2,p} \) of \( \mathbb{Q}_p^* \) have conductor exponents
\[
c(\omega_{1,p}) = c(\omega_{2,p}) = \text{ord}_p(q) > 0 \quad \text{and} \quad c(\omega_{3,p}) = 0;
\]
and \( \pi_{2,p} = \tilde{\pi}_1 = \omega_{2,p}^{-1} \boxplus \omega_{1,p}^{-1} \). (Here \( (\chi)_p \) is the local component of the Hecke character corresponding to \( \chi \).) Also, \( \varphi_{1,p}, \varphi_{2,p}, \tilde{\varphi}_{1,p}, \tilde{\varphi}_{2,p} \) are all local newforms in corresponding representations. However, in this paper, the assumption that \( q \) might not be squarefree, leads to more cases for \( \varphi_p \otimes \tilde{\varphi}_p \) than those listed in the proof of \([HK20]\) Corollary 4.19. We list all the cases for \( \pi_{3,p} \) as follows.

(i) When \( p \parallel q_1 \), the local component \( \pi_{3,p} \) of \( f \) is a special representation \( St_{\omega_{3,p}} \), where \( \omega_{3,p} \) is either the trivial character or the unramified quadratic character of \( \mathbb{Q}_p^* \).

(ii) When \( p \nmid q_1 \), the local component \( \pi_{3,p} \) of \( f \) is a unitary unramified principal series representation \( \omega_{3,p} \boxplus \omega_{3,p}^{-1} \), where \( p^{-1/2} < \omega_{3,p}(p) < p^{1/2} \) and \( c(\omega_{3,p}) = 0 \).

(iii) When \( p^2 \mid q_1 \), under our assumption of this theorem, the local component \( \pi_{3,p} \) of \( f \) is a supercuspidal representation with trivial central character and \( c(\pi_{3,p}) = \text{ord}_p(q_1) \).

In all these cases \( \varphi_{3,p} \) and \( \tilde{\varphi}_{3,p} \) are translates of local newforms by \( \pi_{3,p}(\omega_{3,p}^{-1} 0) \) and \( \tilde{\pi}_{3,p}(\omega_{3,p}^{-1} 0) \) respectively. (When \( p \nmid q_1 \), \( \varphi_{3,p} \) is the local newform; and so is it for \( \omega_{3} = 2 \) and \( \tilde{\varphi}_{3,p} \).)

We respectively apply Propositions 2.3 2.4 2.5 with \( F_v = \mathbb{Q}_p, \omega_v = p \) and \( m_v = \text{ord}_p(q) \) to give the local constants \( I_v(\varphi_v \otimes \tilde{\varphi}_v) \). \( \square \)

The following propositions determine all the local constants \( I_v \) we need in the above proof.

**Proposition 2.2** (\([WW02]\) Theorem 3)). For \( F_v \simeq \mathbb{R} \), let \( k(\pi_v) \in \mathbb{Z} \) denote the weight of \( \pi_v \) and let \( \epsilon \in \{ 1, -1, i, -i \} \) denote the local root number. Then
\[
I_v(\varphi_v \otimes \tilde{\varphi}_v) = \begin{cases} 
1 & \text{if } k(\pi_v) = -k(\pi_v) > k(\pi_v) = 0, \\
1 + \epsilon\sqrt{c} & \text{if } k(\pi_v) = k(\pi_v) = 0.
\end{cases}
\]

Now let \( F_v \) be a nonarchimedean local field with uniformizer \( \varpi_v \) and cardinality \( q_v \) of the residue field. The proof of the followings can be found in the next section.

**Proposition 2.3** (cf. \([HK20]\) Proposition 4.16)). Let \( \pi_{1,v} = \omega_{1,v} \boxplus \omega_{2,v} \) and \( \pi_{2,v} = \tilde{\pi}_1 = \omega_{2,v}^{-1} \boxplus \omega_{1,v}^{-1} \) be principal series representations of \( \text{GL}_2(F_v) \) for which the characters \( \omega_{1,v}, \omega_{2,v} \) of \( F_v^* \) have levels (i.e. conductor exponents) \( c(\omega_{1,v}) = m_v > 0 \) and \( c(\omega_{2,v}) = 0 \), and let \( \pi_{3,v} = St_{\omega_{3,v}} \) be a special representation with \( c(\omega_{3,v}) = 0 \) and \( \omega_{2,v}^3 = 1 \). Suppose that \( \pi_{1,v}, \pi_{2,v}, \pi_{3,v} \) are irreducible and unitarizable, so that \( \omega_{1,v}, \omega_{2,v}, \omega_{3,v} \) are unitary. Then if \( \varphi_{1,v}, \varphi_{2,v}, \varphi_{3,v}, \tilde{\varphi}_{1,v}, \tilde{\varphi}_{2,v}, \tilde{\varphi}_{3,v} \) are all local newforms,
\[
I_v(\varphi_v \otimes \tilde{\varphi}_v) = q_v^{m_v} (1 + q_v^{1}).
\]

This also holds if either or both \( \varphi_{3,v} \) and \( \tilde{\varphi}_{3,v} \) are translates of local newforms by \( \pi_{3,v}(\omega_{3,v}^{-1} 0) \) and \( \tilde{\pi}_{3,v}(\omega_{3,v}^{-1} 0) \) respectively, where \( 0 \leq l_1, l_2 \leq m_v - 1 \).

**Proposition 2.4** (cf. \([HK20]\) Proposition 4.17)). Let \( \pi_{1,v} = \omega_{1,v} \boxplus \omega_{2,v}, \pi_{2,v} = \tilde{\pi}_1 = \omega_{2,v}^{-1} \boxplus \omega_{1,v}^{-1} \), and \( \pi_{3,v} = \omega_{3,v} \boxplus \omega_{3,v}^{-1} \) be principal series representations of \( \text{GL}_2(F_v) \) with \( c(\omega_{1,v}) = m_v > 0 \) and \( c(\omega_{2,v}) = c(\omega_{3,v}) = 0 \). Suppose that \( \pi_{1,v}, \pi_{2,v}, \pi_{3,v} \) are irreducible and unitarizable, so that \( \omega_{1,v}, \omega_{2,v}, \omega_{3,v} \) are unitary while \( q_v^{-1/2} < |\omega_{3,v}(\varpi_v)| < q_v^{1/2} \). Then if \( \varphi_{1,v}, \varphi_{2,v}, \varphi_{3,v}, \tilde{\varphi}_{1,v}, \tilde{\varphi}_{2,v}, \tilde{\varphi}_{3,v} \) are all local newforms,
\[
I_v(\varphi_v \otimes \tilde{\varphi}_v) = q_v^{m_v}.
\]

This also holds if either or both \( \varphi_{3,v} \) and \( \tilde{\varphi}_{3,v} \) are translates of local newforms by \( \pi_{3,v}(\omega_{3,v}^{-1} 0) \) and \( \tilde{\pi}_{3,v}(\omega_{3,v}^{-1} 0) \) respectively, where \( 0 \leq l_1, l_2 \leq m_v \).
Proposition 2.5. Let $\pi_{1,v} = \omega_{1,v} \oplus \omega_{2,v}$, $\pi_{2,v} = \tilde{\pi}_{1,v} = \omega_{2,v}^{-1} \oplus \omega_{1,v}^{-1}$ be as above, and $\pi_{3,v}$ be a supercuspidal representation of $GL_2(F_v)$ with $c(\pi_{3,v}) = c_v \leq m_v$. Suppose that $\pi_{1,v}, \pi_{2,v}, \pi_{3,v}$ are irreducible and unitarizable, so that $\omega_{1,v}, \omega_{2,v}, \omega_{3,v}$ are unitary. Then if $\varphi_{1,v}, \varphi_{2,v}, \varphi_{3,v}, \tilde{\varphi}_{1,v}, \tilde{\varphi}_{2,v}, \tilde{\varphi}_{3,v}$ are all local newforms, 

$$I'_v(\varphi_v \otimes \tilde{\varphi}_v) = \begin{cases} q_v^{-m_v} (1 + q_v^{-1}) & \text{if } \pi_{3,v} \not\cong \pi_{3,v} \otimes \eta_v, \\ q_v^{-m_v} & \text{if } \pi_{3,v} \cong \pi_{3,v} \otimes \eta_v, \end{cases}$$

where $\eta_v$ is the (nontrivial) unramified quadratic character of $F_v^\times$. This also holds if either or both $\varphi_{3,v}$ and $\tilde{\varphi}_{3,v}$ are translates of local newforms by $\pi_{3,v}(\begin{smallmatrix} v^{-1} & 0 \\ 0 & 1 \end{smallmatrix})$ and $\pi_{3,v}(\begin{smallmatrix} v^{-1/2} & 0 \\ 0 & 1 \end{smallmatrix})$, respectively, where $0 \leq l_1, l_2 \leq m_v - c_v$. 

3. Local Calculation in the Watson–Ichino Formula

Let $F$ (in this section we drop all the subscripts $v$) be a nonarchimedean local field with ring of integers $\mathcal{O}_F$, uniformizer $\varpi$, and maximal ideal $p = \varpi \mathcal{O}_F$. Let $q := N(p) = |\varpi|^{-1}$, where the norm $|\cdot|$ is such that $|x| = q^{-v(x)}$ for $x \in \mathbb{Z}^{c(x)}$. 

Let $G := GL_2(F), K := GL_2(\mathcal{O}_F)$ and define the congruence subgroup 

$$K_1(p^m) := \left\{ k \in K : k \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{p^m} \right\}$$

for any nonnegative integer $m$. We normalize the additive Haar measure $dx$ on $F$, the multiplicative Haar measure $d^x := \zeta_F(1)|x|^{-1}dx$ on $F^\times$, and the Haar measure $dk$ on $K$ so that 

$$\text{vol}(\mathcal{O}_F; dx) = 1, \quad \text{vol}(\mathcal{O}_F^\times; d^x) = 1, \quad \text{vol}(K; dk) = 1,$$

with $\zeta_F(s) := (1 - q^{-s})^{-1}$. Denote by $Z$ the center of $G$, by $A$ the diagonal subgroup with lower diagonal entry equal to 1, and by $N$ the usual upper triangular unipotent subgroup of $G$. Denote by $B := ZAN$ the usual Borel subgroup of $G$. For $t, y \in F^\times$ and $x \in F$, we set 

$$w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad z(t) := \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \quad a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \quad n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$ 

3.1. Whittaker models. Let $(\pi, V_{\pi})$ be an irreducible admissible smooth representation of $G$. Let $c(\pi)$ be the level (or conductor exponent) of $\pi$, which is the smallest nonnegative integer such that $\pi^{K_1(p^m)} \neq 0$. In this case the invariant space is 1-dimensional, and we call a nontrivial vector in this subspace a newform in $\pi$. In this section $\varphi_{\pi}, \tilde{\varphi}_{\pi}, \pi_{\pi}, \tilde{\pi}_{\pi}$ are newforms unless otherwise specified. 

Fix a nontrivial continuous additive character $\psi$ of $F$. Assume that $\psi$ is unramified in this paper, i.e. the smallest integer $c(\psi)$ such that $\psi$ is trivial on $p^{c(\psi)}$ is 0. Let $W(\psi)$ be the space of all smooth Whittaker functions, i.e. all smooth functions $W(g)$ on $G$ satisfying 

$$W(n(x)g) = \psi(x)W(g) \quad \text{for all } n(x) \in N.$$ 

If $\pi$ is generic, i.e. there is a nontrivial intertwining map $V_{\pi} \to W(\psi)$, we denote the image by $W(\pi, \psi)$ and call it the Whittaker model of $\pi$. 

For generic irreducible unitarizable representations $\pi_1, \pi_2, \pi_3$ with $\pi_1$ a principal series representation, and for $\varphi_1$ in the induced model of $\pi_1$, $W_2 \in W(\pi_2, \psi)$, and $W_3 \in W(\pi_3, \psi)$, we define the local Rankin–Selberg integral by 

$$\ell_{RS}(\varphi_1, W_2, W_3) := \zeta_F(1/2) \int_K \int_{F^\times} \varphi_1(a(y)k) W_2(a(y)k) W_3(a(y)k) \frac{d^x y}{|y|} \, dk.$$ 

Michel and Venkatesh [MV10] show a result that relates $\ell_{RS}$ and the local constants $I(\varphi \otimes \tilde{\varphi})$ in the Watson–Ichino formula.
In particular, the central character of $\Pi = \pi \otimes \omega$ work well. See Lemma 3.14 for more details.

Notice that both $\varphi_\pi$ and $W_\pi$ are $K_1(p^m)$-invariant. The following lemma, together with Lemma 3.14, reduces the calculation of local constants to determining the values of these functions at $g = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$ for $0 \leq j \leq c(\pi).

Lemma 3.2 (cf. [Hu10] Lemma 2.2). Fix an integer $m \geq 0$. For any left $(B \cap K)$-invariant and right $K_1(p^m)$-invariant function $\Theta : K \to \mathbb{C}$, if integrable, we have

$$\int_K \Theta(k) \, dk = \sum_{j=0}^{m} A_j \Theta \left( \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right),$$

where $A_j = \frac{\zeta_F(2)}{\zeta_F(1)} \begin{cases} 1, & \text{if } j = 0, \\ q^{-j} \zeta_F(1)^{-1}, & \text{if } 0 < j < m, \\ q^{-m}, & \text{if } j = m. \end{cases}$

Proof. By the same way of proving [Hu10] Lemma 2.2, one can also show that, for any right $K_1(p^m)$-invariant function $\Theta : G = GL_2(F) \to \mathbb{C}$, if integrable, we have

$$\int_G \Theta(g) \, dg = \sum_{j=0}^{m} A_j \int_B \Theta \left( b \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right) \, db,$$

with $A_j$ defined as in the above lemma, where $dg$ is the normalized Haar measure on $G$ such that $K$ has volume 1, and $db$ is the left Haar measure on $B$ such that $B \cap K$ has volume 1. Lemma 3.2 is a direct corollary of the above formula.

Remark 3.3. The generalization in [HK20] Lemma 5.18 of (4), which says that Lemma 3.2 holds for any right $K_1(p^m)$-invariant function, is wrong. In fact, $\{(\begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} : 0 \leq j \leq m)\}$ is not a complete coset representatives for $K/K_1(p^m)$; one can show that $|K : K_1(p^m)| = \zeta_F(2)^{-1} q^{2m}$ for $m \geq 1$. Luckily, the functions they integrate in Section 5.3 of [HK20] are actually left $(B \cap K)$-invariant, so their calculations work well. See Lemma 3.14 for more details.

We are interested in the following cases: $\pi_1 = \omega_1 \boxplus \omega_2$, $\pi_2 = \omega_1^{-1} \boxplus \omega_1^{-1}$ are principal series representations with $\omega_1, \omega_2$ both unitary, $c(\omega_1) = c(\chi_D)$ and $c(\omega_2) = 0$, so that $c(\pi_1) = c(\pi_2) = c(\chi_D)$; and $\pi_3$ is one of the following cases:

- a special representation $St_{\omega_3}$ with $\omega_3$ unitary and unramified and $\omega_3^2 = 1$, or
- a principal series representation $\omega_3 \boxplus \omega_3^{-1}$ with $q^{-1/2} \leq |\omega_3(\varpi)| < q^{1/2}$ and $c(\omega_3) = 0$ so that $c(\pi_3) = 0$, or
- a supercuspidal representation with trivial central character and $c(\pi_3) \leq c(\chi_D)$.

In particular, the central character of $\Pi = \pi_1 \boxplus \pi_2 \boxplus \pi_3$ is trivial, so $\Pi$ is self dual and $\Pi \simeq \Pi$. One can take the newforms $\tilde{\varphi}_\pi$, so that $\tilde{\varphi}_\pi \otimes \tilde{\varphi}_\pi \otimes \tilde{\varphi}_\pi = \tilde{\varphi}_\pi \otimes \tilde{\varphi}_\pi \otimes \tilde{\varphi}_\pi$ in both the induced and Whittaker models.

Next we will calculate the values of Whittaker functions case by case.

3.2. Whittaker functions for induced representations. For a principal series representation $\pi = \omega \boxplus \omega'$ or a special representation $\pi = St_\omega$, and given a vector $\varphi_\pi$ in the induced model of $\pi$, denote by

$$W_\pi(g) := \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \int_F \varphi_\pi(w \cdot n(x) \cdot g) \psi^{-1}(x) \, dx$$

the corresponding element in the Whittaker model $W(\pi, \psi)$. (This differs with the definition in [HK20] by an inverse of $\psi$, so that $W(n(x)g) = \psi(x)W(g)$ holds.) Here the normalization of $W_\pi$ follows [MV10].
Section 3.2.1] so that the map \( \varphi_\pi \mapsto W_\pi \) is isometric, where the invariant bilinear pairings on \( \pi \otimes \tilde{\pi} \) on the induced model and the Whittaker model are defined respectively by

\[
\langle \varphi_\pi, \tilde{\varphi}_\pi \rangle := \int_K \varphi_\pi(k) \tilde{\varphi}_\pi(k) \, dk, \quad \langle W_\pi, \tilde{W}_\pi \rangle := \int_{F^*} W_\pi(a(y)) \tilde{W}_\pi(a(y)) \, dy
\]

with \( dk \) the Haar measure on \( K \) such that \( \text{vol}(K) = 1 \).

For \( \pi_1 = \omega_1 \boxplus \omega_2, \pi_2 = \omega_2^{-1} \boxplus \omega_1^{-1} \) with \( c(\omega_1) = m > 0 \) and \( c(\omega_2) = 0 \), we recall the following results.

**Lemma 3.4** ([Sch02]). The newform in the induced model of \( \pi_1 \) is given by

\[
\varphi_{\pi_1}(g) = \begin{cases} 
\omega_1(a) \omega_2(d) \frac{a^{1/2}}{|a|} & \text{if } g \in \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} K_1(p^m), \\
0 & \text{if } g \in \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\omega_2^j & 1 \end{pmatrix} K_1(p^m) \text{ for some } 0 < j \leq m.
\end{cases}
\]

Its corresponding Whittaker function has \( W_{\pi_1}(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \) for any \( y \in F^* \),

\[
W_{\pi_1}(a(y)) = \begin{cases} 
\omega_1(y)|y|^{1/2} \psi(y) \epsilon(1, \omega_1 \omega_2^{-1}, \psi^{-1}) & \text{if } v(y) \geq 0, \\
0 & \text{if } v(y) < 0;
\end{cases}
\]

by taking complex conjugates (so that \( W_2 \in \mathcal{W}(\pi_2, \tilde{\psi}) \)) we have

\[
W_{\pi_2}(a(y)) = \begin{cases} 
\omega_2^{-1}(y)|y|^{1/2} \psi(-y) \epsilon(1, \omega_1^{-1} \omega_2, \psi) & \text{if } v(y) \geq 0, \\
0 & \text{if } v(y) < 0.
\end{cases}
\]

Now we work on the values of \( W_{\pi_1}(a(y)\begin{pmatrix} 1 & 0 \\ \omega_1 & 1 \end{pmatrix}) \) for \( 0 \leq j < m \), here \( c(\omega_1) = m > 0, c(\omega_2) = 0 \).

**Lemma 3.5** (cf. [Hu17] [HK20]). We have that

\[
W_{\pi_1} \left( a(y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \begin{cases} 
\omega_1(y)|y|^{1/2} \psi(y) \epsilon(1, \omega_1 \omega_2^{-1}, \psi^{-1}) & \text{if } v(y) \geq -m, \\
0 & \text{if } v(y) < -m.
\end{cases}
\]

\[
W_{\pi_2} \left( a(y) \begin{pmatrix} 1 & 0 \\ \omega_1 & 1 \end{pmatrix} \right) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \begin{cases} 
\omega_1^{-1}(y)|y|^{1/2} \psi(y) \epsilon(1, \omega_1^{-1} \omega_2, \psi) & \text{if } v(y) \geq -m, \\
0 & \text{if } v(y) < -m.
\end{cases}
\]

(Recall that our definition of \( W_\pi \) differs by an inverse with that in [HK20]). And for \( 0 < j < m \),

\[
W_{\pi_1} \left( a(y) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \begin{cases} 
\omega_2(y)|y|^{1/2} \int_{\mathbb{O}_F} \omega_1^{-1} \omega_2(1 + x \omega^j) \psi(-xy) \, dx & \text{if } v(y) = j - m, \\
0 & \text{if } v(y) \neq j - m.
\end{cases}
\]

\[
W_{\pi_2} \left( a(y) \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \begin{cases} 
\omega_2^{-1}(y)|y|^{1/2} \int_{\mathbb{O}_F} \omega_1 \omega_2^{-1}(1 + x \omega^j) \psi(xy) \, dx & \text{if } v(y) = j - m, \\
0 & \text{if } v(y) \neq j - m.
\end{cases}
\]

**Proof.** Let

\[
g = w \cdot n(x) \cdot a(y) \cdot \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} = \begin{pmatrix} -\omega^j & 1 \\ y + x \omega^j & x \end{pmatrix}.
\]

When \( j = 0 \),

\[
g = \begin{pmatrix} y + x & -1 \\ x & 1 \end{pmatrix} = \begin{pmatrix} \frac{y}{x+y} & -1 - \frac{y}{x+y} \\ 0 & y + x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{y}{x+y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{x+y}{y} & 1 \\ x & 0 \end{pmatrix}
\]

if \( v(x+y) \leq v(y) \),

\[
g = \begin{pmatrix} y & -1 \\ x & 1 \end{pmatrix} = \begin{pmatrix} \frac{y}{x+y} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{x+y}{y} & 1 \end{pmatrix}
\]

if \( v(x+y) = r > 0 \).
Notice that \(v\left(\frac{x+y}{y}\right) = r\) if and only if \(x \in y(-1 + \varpi r \mathcal{O}_F^\times)\), and hence \(v\left(\frac{x+y}{y}\right) > 0\) if and only if \(x \in y(-1 + \varpi \mathcal{O}_F)\). In particular \(v\left(\frac{x+y}{y}\right) > 0\) implies \(\frac{x}{y} \in \mathcal{O}_F^\times\). The calculation of \(W_{\pi_1}(a(y)(\frac{1}{1})\) and \(W_{\pi_2}\) follows that in [HK20] Lemma 5.12.

When \(0 < j < m\), let \(r = v\left(\frac{x+y}{y}\right)\). We have that

\[
g = \begin{cases} 
\left(\begin{array}{cccc}
\frac{y}{y + \varpi} & -\frac{y}{y + \varpi} & 0 & \varpi j \\
0 & 1 & 0 & \varpi j \\
\frac{y}{y + \varpi} & -1 & 1 & 0 \\
0 & x & \varpi & 1
\end{array}\right) \begin{cases} 0 & \text{if } r \leq 0, \\
1 & \text{if } r > 0.
\end{cases}
\end{cases}
\]

By the definition of \(W_{\pi_1}\),

\[
W_{\pi_1}(a(y)(\frac{1}{1})) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \int_{v(x) \geq v(y)} \varpi_1^{-1/2} \varpi_2(y + x \varpi)^{1/2} \psi(-x) dx
\]

For \(j > 0\) define \(U_j = 1 + p^j\). Let \(x = (u - 1)y \varpi^{-j}\). We have \(y + x \varpi^{-j} = yu\) and

\[
r \leq 0 \iff v(yu) \leq v((u - 1)y \varpi^{-j}) \iff v\left(\frac{y}{u-1}\right) \geq j \iff u^{-1} \in U_j \iff u \in U_j,
\]

and then

\[
W_{\pi_1}(a(y)(\frac{1}{1})) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \omega_1(y)^{1/2} \int_{U_j} \varpi_1^{-1/2} \omega_2(uy) \psi(-(u - 1)y \varpi^{-j}) |y \varpi^{-j}| du
\]

By the following lemma, \(W_{\pi_1}(a(y)(\frac{1}{1})) = 0\) unless \(v(y) = j - m\), in which case

\[
W_{\pi_1}(a(y)(\frac{1}{1})) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \omega_2(\frac{y}{y^{j-m}})^{2/2} \psi(y \varpi^{-j}) q^m \sum_{b \in U_j / U_m} \omega_1^{-1} \omega_2(b) \psi(-by \varpi^{-j})
\]

Lemma 3.6. Let \(\psi\) be an unramified additive character of \(F\) and \(\omega\) a ramified character of \(F^\times\) with level \(c(\omega)\). For a positive integer \(j\) let \(U_j = 1 + p^j\) be a subgroup of \(\mathcal{O}_F^\times\). Then for \(0 < j < c(\omega)\),

\[
\int_{U_j} \omega(u) \psi(abu) du = \begin{cases} 
q^{-c(\omega)} \sum_{b \in U_j / U(c(\omega))} \omega(b) \psi(ab) & \text{if } v(a) = -c(\omega), \\
0 & \text{if } v(a) \neq -c(\omega).
\end{cases}
\]

Proof. We follow the proof of [Sch12] Lemma 1.1.1. Write \(u = bu'\) for \(b \in U_j / U_r\) and \(u' \in U_r\). Then

\[
\int_{U_j} \omega(u) \psi(abu) du = \sum_{b \in U_j / U_r} \omega(b) \int_{U_r} \omega(u') \psi(abu') du'.
\]

If \(v(a) \leq -c(\omega)\), we take \(r = c(\omega)\) and then \(\omega(u) = 1\). The inner integral becomes

\[
\int_{U_r} \psi(abu') du' = \psi(ab) \int_{F^\times} \psi(abz) dz.
\]

It vanishes when \(v(a) < -c(\omega)\). And when \(v(a) = -c(\omega)\) it equals \(\psi(ab) \int_{F^\times} dz = q^{-c(\omega)} \psi(ab)\).
If \( v(a) > -c(\omega) \), we take \( r = c(\omega) - 1 \) and then \( \psi(ab(u' - 1)) = 1 \) because that \( \psi \) is unramified. The inner integral becomes

\[
\int_{U_r} \omega(u') \psi(ab(u' - 1)) \, du' = \psi(ab) \int_{U_r} \omega(u') \, du' = 0
\]

\[\square\]

To study the values of newforms in \( \pi_3 \) we have the following lemma.

**Lemma 3.7** (Sch02 HK20).

- For \( \pi_3 = \text{St}_{\omega_3} \) with \( \omega_3 \) unitary and unramified, the newform in the induced model is

\[
\varphi_{\pi_3}(g) = \begin{cases} 
\omega_3(ad) \frac{|a|}{d} & \text{if } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} k, \, k \in K_1(p), \\
-q \omega_3(ad) \frac{|a|}{d} & \text{if } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, \, k \in K_1(p).
\end{cases}
\]

Its corresponding Whittaker function has \( W_{\pi_3}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \zeta_F(2)^{-1/2} \); and for any \( y \in F^\times \),

\[
W_{\pi_3}(a(y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \zeta_F(2)^{-1/2} \cdot \begin{cases} 
\omega_3(y)|y| & \text{if } v(y) \geq 0, \\
0 & \text{if } v(y) < 0.
\end{cases}
\]

- For \( \pi_3 = \omega_3 \boxplus \omega_3^{-1} \) with \( \omega_3 \) unitary and unramified, the newform in the induced model is

\[
\varphi_{\pi_3}(g) = \omega_3 \left( \frac{a}{d} \right) \frac{|a|}{d}^{1/2} \text{ for } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, \, k \in K;
\]

\[
W_{\pi_3}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)L(1, \omega_3^2)};
\]

and for any \( y \in F^\times \),

\[
W_{\pi_3}(a(y)) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)L(1, \omega_3^2)} \cdot \begin{cases} 
|y|^{1/2} \sum_{i, i' \geq 0 \atop i + i' = v(y)} \omega_3(\overline{y}^i) \omega_3^{-1}(\overline{y}^{i'}) & \text{if } v(y) \geq 0, \\
0 & \text{if } v(y) < 0.
\end{cases}
\]

Notice that \( (\pi_3(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) W_{\pi_3}) \) is \( K_1(p^{\pi_3 + l}) \)-invariant. To study the oldforms we need the values of

\[
(\pi_3(\begin{pmatrix} \overline{\omega}^{-l} & 0 \\ 0 & 1 \end{pmatrix} W_{\pi_3}) (a(y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \text{ for } 1 \leq l \leq c(\chi_D) - c(\pi_3), \, 0 \leq j \leq c(\pi_3) + l.
\]

Actually, in this paper, only the case when \( j = 0 \) is necessary (see Section 5.5).

**Lemma 3.8.** Let \( l, j \geq 0 \) be two integers.

- For \( \pi_3 = \text{St}_{\omega_3} \), if \( j \leq l \), \( (\pi_3(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) W_{\pi_3}) (a(y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \) is equal to

\[
-\zeta_F(2)^{-1/2} \cdot \begin{cases} 
\psi(y \overline{\omega}^{-j}) \omega_3(y \overline{\omega}^{-l}) |y \overline{\omega}^{-l-j+1}| & \text{if } v(y) \geq 2j - l - 1, \\
0 & \text{if } v(y) < 2j - l - 1.
\end{cases}
\]

if \( j > l \), it is equal to

\[
\frac{\zeta_F(2)^{-1/2}}{\zeta_F(1)L(1, \omega_3^2)} \cdot \begin{cases} 
\omega_3(y \overline{\omega}^{-l}) |y \overline{\omega}^{-l}| & \text{if } v(y) \geq l, \\
0 & \text{if } v(y) < l.
\end{cases}
\]
• For $\pi_3 = \omega_3 \boxplus \omega_3^{-1}$, if $j \leq l$, $(\pi_3(\omega_3^{-1} \nu)W_{\pi_3}) (a(y)(\frac{1}{\omega^j}, 1))$ is equal to

$$\frac{\zeta_F(2)^{1/2}}{\zeta_F(1)L(1, \omega_3^2)} \left\{ \begin{array}{ll}
\psi(y\omega^{-j})|y\omega^{-j-2}|^{1/2} & \omega_3(\omega^i)\omega_3^{-1}(\omega^i') \text{ if } v(y) \geq 2j - l \\
0 & \text{if } v(y) < 2j - l;
\end{array} \right.$$

if $j > l$, it is equal to

$$\frac{\zeta_F(2)^{1/2}}{\zeta_F(1)L(1, \omega_3^2)} \left\{ \begin{array}{ll}
|y\omega^{-l}|^{1/2} & \omega_3(\omega^i)\omega_3^{-1}(\omega^i') \text{ if } v(y) \geq l \\
0 & \text{if } v(y) < l.
\end{array} \right.$$

Recall that [HK20, Lemma 5.17] calculates the spherical case $\pi_3 = \omega_3 \boxplus \omega_3^{-1}$ for $l = 1, j = 0$.

**Proof.** One can verify that

$$a(y) \left( \begin{array}{cc}
\frac{1}{\omega^j} & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
\omega^{-j} & 0 \\
0 & 1
\end{array} \right) = \left( \begin{array}{cc}
y\omega^{-j} & 0 \\
\omega^{-j-1} & 1
\end{array} \right) \left( \begin{array}{cc}
a(y) & 0 \\
\frac{1}{\omega^j} & 1
\end{array} \right).$$

Since $W_{\pi_3}$ is $K_1(p)$-invariant in both cases, we have

$$\left( \begin{array}{cc}
\pi_3(\omega^{-j} & 0 \\
0 & 1
\end{array} \right) W_{\pi_3}(a(y) \left( \begin{array}{cc}
\frac{1}{\omega^j} & 0 \\
0 & 1
\end{array} \right)) = W_{\pi_3}(a(y\omega^{-j})) \text{ when } j > l.$$

When $j \leq l$ we have the following Iwasawa decomposition

$$\left( \begin{array}{cc}
1 & 0 \\
\omega^{-j} & 1
\end{array} \right) = \left( \begin{array}{cc}
\omega^{-j-l} & 1 - \omega^{-j-l} \\
\omega^{-j-l} & 1
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
\omega^{l-j-1} & 0 \\
0 & 1
\end{array} \right),$$

and then

$$\left( \begin{array}{cc}
y & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
\omega^{-j} & 1
\end{array} \right) \left( \begin{array}{cc}
\omega^{-j} & 0 \\
0 & 1
\end{array} \right) = \left( \begin{array}{cc}
y\omega^{-j} & y\omega^{-j} - y\omega^{j-l} \\
\omega^{-j-l} & \omega^{j-l}
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) K_1(p).$$

By the proposition of Whittaker model $W_{\pi_3}(n(x)g) = \psi(x)W_{\pi_3}(g)$ we have

$$\left( \begin{array}{cc}
\pi_3(\omega^{-j} & 0 \\
0 & 1
\end{array} \right) W_{\pi_3}(a(y) \left( \begin{array}{cc}
\frac{1}{\omega^j} & 0 \\
0 & 1
\end{array} \right))
= \omega_{\pi_3}(\omega^{-j-l})\psi(y\omega^{-j})\psi^{-1}(y\omega^{-l-2j})W_{\pi_3}(a(y\omega^{-l-2j}) \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right)) \text{ when } j \leq l,$$

where $\omega_{\pi_3}$ is the central character for $\pi_3$.

At last one can show Lemma 3.8 from Lemma 3.7 noticing that the central character of $St_{\omega_3}$ is $\omega_3^2$,
and that $\psi^{-1}(y\omega^{-l-2j}) = 1$ when $v(y) \geq 2j - l$.

\[\square\]

### 3.3. Whittaker functions for supercuspidal representations.

For a supercuspidal representation $\pi$ of $G = \mathrm{GL}_2(F)$, given the fixed additive character $\psi$, the Kirillov model of $\pi$ is a unique realization on the space of Schwartz functions $\varphi_\pi \in \mathcal{S}(F^{\times})$ such that

$$\left( \begin{array}{cc}
a & b \\
0 & d
\end{array} \right) \varphi_\pi(x) = \omega_\pi(d)\psi(b^{-1}x)\varphi_\pi(ad^{-1}x),$$

where $\omega_\pi$ is the central character for $\pi$ (which is trivial in this paper). The Whittaker function $W_\pi$ corresponding to $\varphi_\pi$ satisfies

$$\varphi_\pi(y) = W_\pi(a(y)), \quad W_\pi(g) = \langle \pi(g)\varphi_\pi, 1 \rangle, \quad \text{and} \quad \langle \varphi_\pi, \hat{\varphi_\pi} \rangle = \langle W_\pi, \hat{W_\pi} \rangle,$$
where the invariant bilinear pairing on $\pi \otimes \hat{\pi}$ on the Kirillov model is given by

$$\langle \varphi_\pi, \tilde{\varphi}_\pi \rangle := \int_{F^*} \varphi_\pi(y) \tilde{\varphi}_\pi(y) \, d^x y.$$ 

In particular we have $W_\pi(a(y)g) = (\pi(g)\varphi_\pi)(y)$ for $y \in F^*$ and $W_\pi(n(x)g) = \psi(x)W(g)$ for $x \in F$.

For any function $\varphi \in \mathcal{S}(F^\times)$ in the Kirillov model of $\pi$ which is supported only at $\varpi^rO_F^\times$, $\varphi(\varpi^r x)$ can be written as a linear combination of characters on $O_F^\times$ by Fourier inversion:

$$\varphi(\varpi^r x) = \sum_{\nu \in O_F^\times} a_\nu(\varphi) \nu(x), \quad \text{where } a_\nu(\varphi) := \int_{O_F^\times} \varphi(\varpi^r x) \nu^{-1}(x) \, d^x x. \quad (9)$$

We say that $\varphi$ contains level $n$ components if $a_\nu(\varphi) \neq 0$ for some level $n$ character $\nu$ (and that it is of level $n$ if it consists of only level $n$ components). Obviously $\varphi$ contains level $n$ components if and only if

$$\int_{\varpi^r O_F^\times} \varphi(x) \nu(\varpi^{-r} x) \, d^x x \neq 0$$

for some level $n$ character $\nu$.

**Lemma 3.9.** Let $\varphi(x) \in \mathcal{S}(F^\times)$ be any function supported only at $\varpi^r O_F^\times$. We have

$$\int_{\varpi^r O_F^\times} \varphi(x) \psi(bx) \, d^x x \neq 0$$

only if $\varphi$ has some level $-r - v(b)$ (and also level $0$ if $v(b) + r \geq -1$) components. In general, if $\varphi(x)$ is of level $n$, then $\varphi(x) \psi(bx)$ consists

- of only level $n$ components if $v(b) > -r - \max\{n, 1\}$,
- of only level $-r - v(b)$ components if $v(b) < -r - \max\{n, 1\}$, and
- of all level $\leq \max\{n, 1\}$ components if $v(b) = -r - \max\{n, 1\}$.

**Proof.** It is sufficient to show the lemma for $\varphi(x) = \chi(\varpi^{-r} x)1_{\varpi^r O_F^\times}(x)$ where $\chi$ is any level $n$ character and $1_{\varpi^r O_F^\times}$ is the characteristic function of $\varpi^r O_F^\times$. [Sch02, Lemma 1.1.1] shows that,

$$\int_{\varpi^r O_F^\times} \psi(x) \, d^x x = \begin{cases} q^{-m} \zeta_F(1)^{-1} & \text{if } m \geq 0, \\ -1 & \text{if } m = -1, \\ 0 & \text{if } m \leq -2; \end{cases} \quad (10)$$

and for any ramified character $\omega$ of $F^\times$,

$$\int_{\varpi^r O_F^\times} \omega^{-1}(x) \psi(x) |x|^s \, d^x x = \begin{cases} \epsilon(s, \omega, \psi) & \text{if } r = -c(\omega), \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Then, for any character $\nu$ of $O_F^\times$ (we extend $\chi, \nu$ to be characters on $F^\times$ by defining $\chi(\varpi) = \nu(\varpi) = 1$),

$$\int_{\varpi^r O_F^\times} \varphi(x) \psi(bx) \nu(\varpi^{-r} x) \, d^x x = \int_{\varpi^r O_F^\times} (\chi \nu)(\varpi^{-r} x) \psi(bx) \, d^x x$$

$$= \zeta_F(1) \nu(b \varpi^r)^{-1} |b \varpi^r|^{-1} \int_{b \varpi^{-r} O_F^\times} \chi \nu(x) \psi(x) \, dx$$

$$= \begin{cases} 1 & \text{if } c(\chi \nu) = 0, \ v(b) + r \geq 0, \\ -q^{-r} & \text{if } c(\chi \nu) = 0, \ v(b) + r = -1, \\ \zeta_F(1) \chi \nu(b \varpi^r)^{-1} \epsilon(1, \chi^{-1} \nu^{-1}, \psi) & \text{if } c(\chi \nu) \geq 1, \ v(b) + r = -c(\chi \nu), \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

This completes the proof, noticing that $c(\chi \nu) \leq \max\{c(\chi), c(\nu)\}$ and that inequality holds only if $c(\chi) = c(\nu)$. 

The Bruhat decomposition says that $G = B \cup BuN$, where $w = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ and $B = ZAN$ is the upper triangular Borel subgroup of $G = \text{GL}_2(F)$. Then the action $\pi(g), g \in G$ in the Kirillov model can be expressed purely in terms of $\pi(w)$ and $\pi|_B$. We recall a fact that shows the operator $\pi(w)$ on some multiplicative characters.

**Fact 3.10** ([Il77], [BH06] Theorem 37.3 and [Hu15] Proposition A.1). Let $\pi$ a supercuspidal representation with trivial central character. Assume that it has conductor $p^{c(\pi)}$. Let $\nu$ be a multiplicative character of $\mathcal{O}_F^\times$ with level $c(\nu)$. The action of $w = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ in the Kirillov model of $\pi$ satisfies

$$\pi(w)(\nu(\varpi^{-r'})1_{\varpi^{-r'}\mathcal{O}_F^\times}) = C_\nu \nu^{-1}(\varpi^{-r'})1_{\varpi^{-r'}\mathcal{O}_F^\times}$$

where $C_\nu = \epsilon(\frac{1}{2}, \pi \otimes \nu^{-1}, \psi)$ is independent of $r$ and $r' = -r - \max\{c(\pi), 2c(\nu)\}$ (except when the residue field of $F$ is of characteristic 2 and $c(\pi) = 2c(\nu) \geq 4$). In particular

$$\pi(w)1_{\varpi\mathcal{O}_F^\times} = C_1 1_{\varpi^{-r-c(\pi)}\mathcal{O}_F^\times}, \text{ where } C_1 = \epsilon(\frac{1}{2}, \pi, \psi) = \pm 1.$$

Next we recall a lemma about the new vector in a supercuspidal representation $\pi_3$ and the values of its corresponding Whittaker function.

**Lemma 3.11** ([Hu16] Lemma 5.10, [Hu17] Corollary 2.18). For a supercuspidal representation $\pi_3$ with trivial central character, the new vector in the Kirillov model is $\varphi_{\pi_3} = 1_{\mathcal{O}_F^\times}$, the characteristic function of $\mathcal{O}_F^\times$. Its corresponding Whittaker function $W_{\pi_3}$ satisfies:

- $W_{\pi_3}(a(y)) = 1_{\mathcal{O}_F^\times}(y)$ for any $y \in F^\times$; and therefore $\langle \varphi_{\pi_3}, \hat{\varphi}_{\pi_3} \rangle = \langle W_{\pi_3}, \hat{W}_{\pi_3} \rangle = 1$.
- For $0 \leq j < c(\pi_3)$, $W_{\pi_3}(a(y)(\begin{smallmatrix} 1 & 0 \\ \varpi^j & 1 \end{smallmatrix}))$ is supported only at $v(y) = \min\{0, 2j - c(\pi_3)\}$, and it consists of only level $c(\pi_3) - j$ (and also level 0 if $j = c(\pi_3) - 1$) components; the exception happens when the residue field of $F$ is of characteristic 2 (the central character is assumed to be trivial in this paper), $c(\pi_3) \geq 4$ is an even number and $j = c(\pi_3)/2$, in which case $W_{\pi_3}(a(y)(\begin{smallmatrix} 1 & 0 \\ \varpi^{-2} & 1 \end{smallmatrix}))$ is supported at $v(y) \geq 0$, consisting of level $c(\pi_3)/2$ components.
- Moreover we have

$$\int_{v(y) = \min\{0, 2j - c(\pi_3)\}} W_{\pi_3}(a(y)(\begin{smallmatrix} 1 & 0 \\ \varpi^j & 1 \end{smallmatrix})) \psi(-\varpi^{-j}y) \, d^x y = \begin{cases} C_1 & \text{if } j = 0, \\ \frac{1}{q-1} C_1 & \text{if } j = 1, \\ 0 & \text{otherwise}, \end{cases}$$

where $C_1 = \epsilon(\frac{1}{2}, \pi_3, \psi) = \pm 1$.

Next we generalize the above lemma.

**Lemma 3.12.** With assumptions and notations in the above lemma,

- for $j \geq c(\pi_3)$,

$$\int_{\mathcal{O}_F^\times} W_{\pi_3}(a(y)(\begin{smallmatrix} 1 & 0 \\ \varpi^j & 1 \end{smallmatrix})) \psi(by) \, d^x y = \begin{cases} 1 & \text{if } b \in \mathcal{O}_F, \\ \frac{1}{q-1} & \text{if } b \in \varpi^{-1}\mathcal{O}_F^\times, \\ 0 & \text{otherwise}, \end{cases}$$

- for $0 \leq j < c(\pi_3)$, in general,

$$\int_{v(y) = \min\{0, 2j - c(\pi_3)\}} W_{\pi_3}(a(y)(\begin{smallmatrix} 1 & 0 \\ \varpi^j & 1 \end{smallmatrix})) \psi(by) \, d^x y$$
vanishes unless \( v(b) = -\min\{j, c(\pi_3) - j\} \) (or \( b \in \mathbb{w}^{-1}\mathcal{O}_F \) when \( j = c(\pi_3) - 1 \)); in the exceptional case when the residue field of \( F \) is of characteristic 2, \( c(\pi_3) \geq 4 \) is even and \( j = c(\pi_3)/2 \), the integral \( \int_{v(y)=r} W_{\pi_3} (a(y)(\frac{1}{\mathbb{w}} \ 1)) \psi(by) \, d^\times y \) vanishes unless \( r \geq 0 \) and \( v(b) = -r - c(\pi_3)/2 \);

- in particular for \( j = 0 \),

\[
\int_{\mathbb{w}^{-c(\pi_3)}\mathcal{O}_F^\times} W_{\pi_3} (a(y)(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})) \psi(by) \, d^\times y = \begin{cases} C_1 & \text{if } b \in -1 + \mathbb{w}^{c(\pi_3)}\mathcal{O}_F, \\ -\frac{C_1}{q} & \text{if } b \in -1 + \mathbb{w}^{c(\pi_3)-1}\mathcal{O}_F, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( C_1 = \epsilon(\frac{1}{q}, \pi_3, \psi) = \pm 1 \).

**Proof.** When \( b = 0 \) or \( b = -\mathbb{w}^{-j} \) the lemma is \([Hu16] \text{ Lemma 5.10}(2)(3)\).

Recall that

\[
W_\pi \left( a(y) \begin{pmatrix} 1 & 0 \\ \mathbb{w}^j & 1 \end{pmatrix} \right) = \left( \pi \begin{pmatrix} 1 & 0 \\ \mathbb{w}^j & 1 \end{pmatrix} \varphi_\pi \right)(y).
\]

When \( j \geq c(\pi_3) \), \( W_{\pi_3} (a(y)(\frac{1}{\mathbb{w}} \ 1)) \) is simply the new vector. The integral is equal to

\[
\int_{\mathcal{O}_F^\times} \psi(by) \, d^\times y = \zeta_F(1)|b|^{-1} \int_{\mathcal{O}_F^\times} \psi(y) \, dy = \begin{cases} \frac{1}{q-1} & \text{if } v(b) \geq 0, \\ 0 & \text{if } v(b) = -1, \\ \frac{1}{q} & \text{if } v(b) \leq 2 \end{cases}
\]

(see \([10]\) for the last step).

When \( 0 \leq j < c(\pi_3) \), Lemma \([8,11]\) shows that, in the general case, \( W_{\pi_3} (a(y)(\frac{1}{\mathbb{w}} \ 1)) \) is supported only at \( v(y) = r' = \min\{0, 2j - c(\pi_3)\} \), consists of only level \( c(\pi_3) - j \) (and also level 0 if \( j = c(\pi_3) - 1 \) components. Then we apply Lemma \([8,9]\) and see that, the integral we study vanishes unless \( v(b) = -\min\{j, c(\pi_3) - j\} \) (or \( v(b) \geq -1 \) when \( j \geq c(\pi_3) - 1 \)). One can study the exceptional case using the same argument.

Recall that for any \( \varphi \in S(F^\times) \) in the Kirillov model of \( \pi \) we have by \([8]\) that

\[
\left( \pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \varphi \right)(x) = \psi(bx)\varphi(x).
\]

Then we can write

\[
W_{\pi_3} \left( a(y) \begin{pmatrix} 1 & 0 \\ \mathbb{w}^j & 1 \end{pmatrix} \right) \psi(by) = \left( \pi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \left( \pi \begin{pmatrix} 1 & 0 \\ \mathbb{w}^j & 1 \end{pmatrix} 1_{\mathcal{O}_F^\times} \right) \right)(y)
= \left( \pi \begin{pmatrix} 1 + b\mathbb{w}^j & b \\ \mathbb{w}^j & 1 \end{pmatrix} 1_{\mathcal{O}_F^\times} \right)(y).
\]

In particular when \( j = 0 \) (and \( v(b) = 0 \)) we can decompose the matrix as

\[
\begin{pmatrix} 1 + b & b \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 + b \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Recall that \([13]\) gives the action of \( w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) in the Kirillov model. Then one can show that

\[
W_{\pi_3} \left( a(y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \psi(by) = \left( \pi_3(\begin{pmatrix} 1 & 1 + b \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) 1_{\mathcal{O}_F^\times} \right)(y)
= \left( \pi \begin{pmatrix} 1 & 1 + b \\ 0 & 1 \end{pmatrix} (\pi_3(w) 1_{\mathcal{O}_F^\times}) \right)(y)
= \left( \pi \begin{pmatrix} 1 & 1 + b \\ 0 & 1 \end{pmatrix} (C_1 1_{\mathbb{w}^{-c(\pi_3)}\mathcal{O}_F^\times}) \right)(y)
= C_1 \psi((1 + b)y) 1_{\mathbb{w}^{-c(\pi_3)}\mathcal{O}_F^\times}(y).
\]
By (1.5) we have

$$\int_{v(y)=-c(\pi_3)} W_{\pi_3} \left( a(y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \psi(by) \, d^\times y$$

$$= C_1 \int_{v(y)=-c(\pi_3)\mathcal{O}_F^\times} \psi((1+b)y) \, d^\times y = \begin{cases} C_1 & \text{if } v(1+b) \geq c(\pi_3), \\ -\frac{C_1}{q-1} & \text{if } v(1+b) = c(\pi_3) - 1, \\ 0 & \text{if } v(1+b) \leq c(\pi_3) - 2. \end{cases}$$

To study the oldforms we need the values of \( \left( \pi_3(\varpi^{-l}0_1)W_{\pi_3} \right)(a(y)(1_00_1)) \) for \( 1 \leq l \leq c(\chi_D) - c(\pi_3) \).

**Lemma 3.13.** With assumptions and notations in the above lemma, for an integer \( l \geq 0 \),

- if \( j \geq c(\pi_3) + l \), then

\[
\left( \pi_3 \left( \varpi^{-l} 0 1 \right) W_{\pi_3} \right) \left( a(y) \left( \frac{1}{\varpi} 0 1 \right) \right) = 1_{\varpi^l\mathcal{O}_F^\times}(y)
\]

and

\[
\int_{v(y)=l} \left( \pi_3 \left( \varpi^{-l} 0 1 \right) W_{\pi_3} \right) \left( a(y) \left( \frac{1}{\varpi} 0 1 \right) \right) \psi(by) \, d^\times y = \begin{cases} \frac{1}{1-q^{-1}} & \text{if } b \in \varpi^{-l}\mathcal{O}_F, \\ 1 & \text{if } b \in \varpi^{-l-1}\mathcal{O}_F^\times, \\ 0 & \text{otherwise}. \end{cases}
\]

- if \( 0 \leq j \leq l \), \( \left( \pi_3(\varpi^{-l}0_1)W_{\pi_3} \right)(a(y)(1_00_1)) \) is supported only at \( v(y) = 2j - l - c(\pi_3) \), and

\[
\int_{v(y)=2j-l-c(\pi_3)} \left( \pi_3 \left( \varpi^{-l} 0 1 \right) W_{\pi_3} \right) \left( a(y) \left( \frac{1}{\varpi} 0 1 \right) \right) \psi(by) \, d^\times y
\]

is equal to

\[
\begin{cases}
C_1 & \text{if } b \in -\varpi^{-j} + \varpi^{c(\pi_3)+l-2j}\mathcal{O}_F, \\
-\frac{C_1}{q-1} & \text{if } b \in -\varpi^{-j} + \varpi^{c(\pi_3)+l-2j-1}\mathcal{O}_F^\times, \\
0 & \text{otherwise},
\end{cases}
\]

where \( C_1 = \epsilon(\frac{1}{q^j}, \pi_3, \psi) = \pm 1 \).

**Proof.** When \( j > l \) we have

$$\begin{pmatrix} 1 & 0 \\ \varpi^{-l} & 1 \end{pmatrix} \begin{pmatrix} \varpi^{-l} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varpi^{-l} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varpi^{-l} & 1 \end{pmatrix}.$$
And by (8) we have
\[
\left( \pi_3 \begin{pmatrix} \varpi^{-l} & 0 \\ 0 & 1 \end{pmatrix} W_{\pi_3} \right) \left( a(y) \begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix} \right) = \pi_3 \left( \begin{pmatrix} \varpi^{-j} & \varpi^{-l} - \varpi^{-j} \\ 0 & \varpi^{j-l} \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ \varpi^{-l} - 1 & 1 \end{pmatrix} \right) W_{\pi_3}(y)
\]
\[
= \pi_3 \left( \begin{pmatrix} \varpi^{-j} & \varpi^{-l} - \varpi^{-j} \\ 0 & \varpi^{j-l} \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 & 0 \\ \varpi^{-l} - 1 & 1 \end{pmatrix} \right) 1_{O_F^\times}(y)
\]
\[
= \psi((\varpi^{-l} - \varpi^{-j}) \varpi^{j-l} y) \left( \pi_3 \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \right) 1_{O_F^\times}(\varpi^{j-l} y)
\]
\[
= W_{\pi_3} \left( a(\varpi^{j-2j} y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \psi((\varpi^{-j} - \varpi^{j-2j}) y).
\]

By Lemma 3.11 it is supported only at \( v(y) = 2j - l - c(\pi_3) \), and
\[
\int_{v(y)=2j-l-c(\pi_3)} \pi_3 \left( \begin{pmatrix} \varpi^{-l} & 0 \\ 0 & 1 \end{pmatrix} \right) W_{\pi_3} \left( a(y) \begin{pmatrix} 1 & 0 \\ \varpi^j & 1 \end{pmatrix} \right) \psi(by) d^\times y
\]
\[
= \int_{v(y)=2j-l-c(\pi_3)} W_{\pi_3} \left( a(\varpi^{-2j} y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \psi((\varpi^{-j} - \varpi^{j-2j} + b) y) d^\times y
\]
\[
= \int_{v(y')=-c(\pi_3)} W_{\pi_3} \left( a(y') \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \psi((\varpi^{j-l} - 1 + b \varpi^{2j-1}) y') d^\times y'
\]
\[
= \begin{cases} C_1 & \text{if } b \in -\varpi^{-j} + \varpi^{c(\pi_3)+l-2j} O_F, \\ \frac{C_3}{q-1} & \text{if } b \in -\varpi^{-j} + \varpi^{c(\pi_3)+l-2j-1} O_F^\times, \\ 0 & \text{otherwise} \end{cases}
\]

by Lemma 3.12. 

3.4. The adjoint lift of a supercuspidal representation. A supercuspidal representation \( \pi \) of \( G = \text{GL}_2(F) \) is called unramified (i.e. of Type-1 as we have defined in Section 2) if \( \pi \simeq \pi \otimes \eta \) for some unramified character \( \eta \neq 1 \) of \( F^\times \). (By comparing central characters one can see that \( \eta \) is quadratic.) Let \( \eta \) be the (nontrivial) unramified quadratic character of \( F^\times \). [GJ78] Corollary 1.3 gives the L-factor of the adjoint lift of a supercuspidal representation:

\[
L(s, \pi, \text{Ad}) = \begin{cases} 1 & \text{if } \pi \not\simeq \pi \otimes \eta, \\ \frac{1}{(1 + q^{-s})^{-1}} & \text{if } \pi \simeq \pi \otimes \eta. \end{cases}
\]

In the case when the residue field of \( F \) has characteristic \( p = 2 \), the “unramification” of a supercuspidal representation \( \pi \) is actually equivalent to its “dihedralness”. Recall that \( \pi \) is called dihedral (cf. [Bum97] Theorem 4.8.6]) if it is associated with a quadratic field extension \( E/F \) and a character of \( E^\times \) that is not trivial on the kernel of the norm map \( N_{E/F} \) from \( E^\times \) to \( F^\times \). (One can also find the construction in [BH06] Section 19.) The Tame Parametrization Theorem (cf. [BH06] Section 20.1) says that every supercuspidal representation \( \pi \) of \( \text{GL}_2(F) \) is dihedral if the residue characteristic of \( F \) is an odd prime; but when the residue characteristic is \( p = 2 \), only the unramified ones have such correspondence: \( \pi \) is supercuspidal and unramified if and only if it is “unramified” dihedral, i.e. it is associated with an unramified quadratic field extension \( E/F \). (The equivalence of “unramified supercuspidal” and “unramified dihedral” is also true when \( p \neq 2 \), cf. [BH06] Section 20.3.) This explains the assumption of the Maass form \( f \) in Theorem 1.1.

3.5. Local constants in the Watson–Ichino formula. To apply Lemma 3.12 to the calculation of the local Rankin–Selberg integral \( \ell_{\text{RS}} \), we need the following result.
Lemma 3.14. Fix an unramified additive character $\psi$ of $F$. Let $\pi_1 = \omega_1 \boxplus \omega_2$, $\pi_2 = \omega_2^{-1} \boxplus \omega_1^{-1}$ be principal series representations of $G = \text{GL}_2(F)$ with $\omega_1, \omega_2$ both unitary, $c(\omega_1) = m > 0$ and $c(\omega_2) = 0$. Let $\pi_3$ be a generic representation of $G$ with trivial central character and $c(\pi_3) \leq m$. Then, for $\varphi_1$ in the induced model of $\pi_1$, $W_2 \in \mathcal{W}(\pi_2, \tilde{\psi})$ and $W_3 \in \mathcal{W}(\pi_3, \psi)$, the function $\Theta : K \rightarrow \mathbb{C}$ defined by

$$
\Theta(k) := \int_{F^*} \varphi_1(a(y)k)W_2(a(y)k)W_3(a(y)k) \frac{d^x y}{|y|}
$$

is left $(B \cap K)$-invariant and right $K_1(p^m)$-invariant.

Proof. Any $b \in B \cap K$ can be decomposed as $b = z(t)a(y')n(x)$ with $t, y' \in \mathcal{O}_F^\times$ and $x \in \mathcal{O}_F$, and we have that

$$a(y)bk = \begin{pmatrix} y & 1 \\ t & 1 \end{pmatrix} \begin{pmatrix} y' & 1 \\ t & 1 \end{pmatrix} = \begin{pmatrix} t & t \\ y'y' & 1 \end{pmatrix} k = z(t)n(y'y')a(y'y')k.
$$

Recall that, the action of $z(t) = (t, 0)$ is given by the central character:

$$\varphi_1(z(t)g) = \omega_1 \omega_2(t) \varphi_1(g), \quad W_2(z(t)g) = \omega_2^{-1} \omega_1^{-1}(t) W_2(g), \quad W_3(z(t)g) = W_3(g);
$$

and the action of $n(x) = (1, x, 1)$ is given by proposition of induced model or Whittaker model respectively:

$$\varphi_1(n(x)g) = \varphi_1(g), \quad W_2(n(x)g) = \tilde{\psi}(x)W_2(g), \quad W_3(n(x)g) = \psi(x)W_3(g).
$$

Therefore

$$\varphi_1(a(y)bk) = \omega_1 \omega_2(t) \varphi_1(a(y'y')k),
$$

$$W_2(a(y)bk) = \omega_2^{-1} \omega_1^{-1}(t) \tilde{\psi}(y'y')W_2(a(y'y')k),
$$

$$W_3(a(y)bk) = \psi(y'y')W_3(a(y'y')k);
$$

and hence

$$\Theta(bk) = \int_{F^*} \varphi_1(a(y'y')k)W_2(a(y'y')k)W_3(a(y'y')k) \frac{d^x y}{|y|} = |y'| \Theta(k) = \Theta(k)
$$

for any $b \in B \cap K$, with $|y'| = 1$ since $y' \in \mathcal{O}_F^\times$.

At last the assumptions on the conductors of these three representations imply the right $K_1(p^m)$-invariance of $\varphi_1, W_2, W_3$, and thus of $\Theta(k)$. $\square$

The above lemma still holds if $\pi_i$, $i = 1, 2, 3$, are all generic with level $c(\pi_i) \leq m$, and with central character $\omega_{\pi_i}$ such that $\omega_{\pi_1}\omega_{\pi_2}\omega_{\pi_3} = 1$.

By the definition of $\ell_{RS}$ together with Lemma 3.2 and Lemma 3.14, $\ell_{RS}(\varphi_{\pi_1}, W_{\pi_2}, W_{\pi_3})$ is equal to

$$\zeta_F(1)^{1/2} \sum_{j=0}^{m} A_j \int_{F^*} \varphi_{\pi_1} \left( a(y) \begin{pmatrix} 1 & 0 \\ \omega_j & 1 \end{pmatrix} \right) W_{\pi_2} \left( a(y) \begin{pmatrix} 1 & 0 \\ \omega_j & 1 \end{pmatrix} \right) W_{\pi_3} \left( a(y) \begin{pmatrix} 1 & 0 \\ \omega_j & 1 \end{pmatrix} \right) \frac{d^x y}{|y|}. \quad (4)
$$

Recall that, by Lemma 3.4, the new vector in the induced model of $\pi_1 = \omega_1 \boxplus \omega_2$, where $c(\omega_1) = m > c(\omega_2) = 0$, satisfies

$$\varphi_{\pi_1} \left( a(y) \begin{pmatrix} 1 & 0 \\ \omega_j & 1 \end{pmatrix} \right) = \begin{cases} \omega_1(y)|y|^{1/2} & \text{if } j = 0, \\
0 & \text{if } 0 < j \leq m. \end{cases}
$$

This means we only need to work on the case with $j = 0$. The integral becomes

$$\ell_{RS}(\varphi_{\pi_1}, W_{\pi_2}, W_{\pi_3}) = \zeta_F(1)^{1/2} \frac{\zeta_F(2)}{\zeta_F(1)} \int_{F^*} \omega_1(y)|y|^{1/2} W_{\pi_2} \left( a(y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) W_{\pi_3} \left( a(y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) \frac{d^x y}{|y|}. \quad (5)
$$
By Lemma 3.3 we have
\[ \ell_{RS}(\varphi_{\pi_1}, W_{\pi_2}, W_{\pi_3}) = \zeta_F(2) \cdot \frac{\zeta_F(1)^{1/2}}{\zeta_F(1)} \sum_{r \geq -m} \int_{v(y) = r} \omega_1(y)|y|^{1/2} \]

\[ = \zeta_F(2)^{3/2} \cdot \frac{\zeta_F(1)^{3/2}}{\zeta_F(1)} \sum_{r \geq -1} \int_{v(y) = r} \psi(-y)\zeta_F(2)^{-1/2}(-q^{-1})\psi(y)\omega_3(y\varpi^{-l})|y\varpi^l| \, d^\times y \]

\[ = -q^{-1} \cdot \frac{\zeta_F(2)}{\zeta_F(1)^{1/2}} \cdot \sum_{r \geq -1} \int_{v(y') = r'} \omega_3(y')|y'| \, d^\times y' \quad (y' = y\varpi^l) \]

\[ = -q^{-1} \cdot \frac{\zeta_F(2)}{\zeta_F(1)^{1/2}} \cdot \sum_{r \geq -1} \int_{\mathbb{O}_F} \omega_3(\varpi^r u) q^{-r} \, d^\times u \]

\[ = -q^{-1} \cdot \frac{\zeta_F(2)}{\zeta_F(1)^{1/2}} \cdot \sum_{r \geq -1} \omega_3(\varpi^{-r}) q^{-r} \]

\[ = -q^{-1} \cdot \frac{\zeta_F(2)}{\zeta_F(1)^{1/2}} \cdot \epsilon(1, \omega_1^{-1} \omega_2, \psi) L(1, \omega_3) \omega_3(\varpi^{-2l-1}), \quad \text{for } 0 \leq l \leq m - 1. \]

By (11) one has
\[ |\epsilon(1, \omega_1^{-1} \omega_2, \psi)| = |\epsilon(1, \omega_1^{-1} \omega_2, \psi) q^{-c(\omega_1^{-1} \omega_2)/2}| = q^{-m/2}. \]

So the numerator in the local constant is
\[ I(\varphi \otimes \tilde{\varphi}) = q^{-m} \frac{\zeta_F(2)^2}{\zeta_F(1)^2} L(1, \omega_3)^2. \]

The denominator in \( I'(\varphi \otimes \tilde{\varphi}) \) is given by
\[ \langle \varphi, \tilde{\varphi} \rangle = \langle W_{\pi_1}, \tilde{W}_{\pi_1} \rangle / \langle W_{\pi_2}, \tilde{W}_{\pi_2} \rangle / \langle W_{\pi_3}, \tilde{W}_{\pi_3} \rangle. \]

By definition we have
\[ \langle W_{\pi_1}, \tilde{W}_{\pi_1} \rangle = \int_{v(y) \geq 0} \left| \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)} \omega_2(y)|y|^{1/2} \right|^2 \, d^\times y = \frac{\zeta_F(2)}{\zeta_F(1)^2} \int_{\mathbb{O}_F} |y| \, d^\times y = \frac{\zeta_F(2)}{\zeta_F(1)} \]

and hence \( \langle W_{\pi_2}, \tilde{W}_{\pi_2} \rangle = \zeta_F(2) / \zeta_F(1) \); also
\[ \langle W_{\pi_3}, \tilde{W}_{\pi_3} \rangle = \int_{v(y) \geq 0} \left| \frac{\zeta_F(2)^{-1/2}}{\zeta_F(1)^2} \omega_3(y)|y| \right|^2 \, d^\times y = \frac{1}{\zeta_F(2)} \int_{\mathbb{O}_F} |y|^2 \, d^\times y = 1. \]
We get that
\[
\frac{I(\varphi \otimes \widehat{\varphi})}{(\varphi, \widehat{\varphi})} = q^{-m} \frac{\zeta_F(2)^2}{\zeta_F(1)} L(1, \omega_3^2)^2 \cdot 1.
\]

The local $L$-factors are defined in the same way as in [HK20]; in this case
\[
L(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = L(s + \frac{1}{2}, \omega_3^2),
\]
\[
L(s, \pi_1, \text{Ad}) = L(s, \pi_2, \text{Ad}) = \zeta_F(s), \quad L(s, \pi_3, \text{Ad}) = \zeta_F(s + 1).
\]

We can simplify that
\[
I(\varphi \otimes \widehat{\varphi}) = q^{-m} \frac{\zeta_F(1)}{\zeta_F(2)} = q^{-m}(1 + q^{-1}).
\]

3.5.2. Proof of Proposition 2.2

For $\pi_3 = \omega_3 \oplus \omega_3^{-1}$ and $0 \leq l \leq m$, we have shown that
\[
\left( \pi_3 \begin{pmatrix} \omega_3^{-l} & 0 \\ 0 & 1 \end{pmatrix} W_{\pi_3} \right) \left( a(y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) = \frac{\zeta_F(2)^{1/2}}{\zeta_F(1)L(1, \omega_3^2)} \begin{cases} \psi(y)|\omega_3 y|^{1/2} \sum_{i+v'}=v(y)+l} \omega_3(\omega_3^3)\omega_3^{-1}(\omega_3^{v'}) & \text{if } v(y) \geq -l, \\
0 & \text{if } v(y) < -l.
\end{cases}
\]

Then
\[
\ell_{RS}(\varphi_{\pi_1}, W_{\pi_2}, \pi_3 \begin{pmatrix} \omega_3^{-l} & 0 \\ 0 & 1 \end{pmatrix} W_{\pi_3})
\]
\[
= \frac{\zeta_F(2)^{3/2}}{\zeta_F(1)^{3/2}} \epsilon(1, \omega_1 \omega_3 \omega_2, \psi) \sum_{r \geq -m} \int_{v(y)=r} \psi(-y) \left( \pi_3 \begin{pmatrix} \omega_3^{-l} & 0 \\ 0 & 1 \end{pmatrix} W_{\pi_3} \right) \left( a(y) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) d^x y
\]
\[
= \frac{\zeta_F(2)^{2}}{\zeta_F(1)^{5/2}} \epsilon(1, \omega_1 \omega_3 \omega_2, \psi) \sum_{r \geq -l} \int_{v(y)=r} L(1, \omega_3^2)^{-1}|\omega_3 y|^{1/2} \sum_{i+v'}=v(y)+l} \omega_3(\omega_3^3)\omega_3^{-1}(\omega_3^{v'}) d^x y
\]
\[
= \frac{\zeta_F(2)^{2}}{\zeta_F(1)^{5/2}} \epsilon(1, \omega_1 \omega_3 \omega_2, \psi) L(1, \omega_3^2)^{-1} \sum_{r \geq 0} q^{-r'/2} \sum_{i+v'=r'} \omega_3(\omega_3^3)\omega_3^{-1}(\omega_3^{v'})
\]
\[
= \frac{\zeta_F(2)^{2}}{\zeta_F(1)^{5/2}} \epsilon(1, \omega_1 \omega_3 \omega_2, \psi) L(1, \omega_3^2)^{-1} \sum_{i \geq 0} \sum_{v' \geq 0} \omega_3(\omega_3^3)q^{-i/2}\omega_3^{-1}(\omega_3^{v'})q^{-v'/2}
\]
\[
= \frac{\zeta_F(2)^{2}}{\zeta_F(1)^{5/2}} \epsilon(1, \omega_1 \omega_3 \omega_2, \psi) L(1, \omega_3^2)^{-1} (1 - \omega_3(\omega_3^{1/2})^{-1}(1 - \omega_3^{-1}(\omega_3^{1/2})^{-1})
\]
\[
= \frac{\zeta_F(2)^{2}}{\zeta_F(1)^{5/2}} \epsilon(1, \omega_1 \omega_3 \omega_2, \psi) L(1, \omega_3^2)^{-1} L(1, \omega_3)^2 L(1, \omega_3)^{-2} L(1, \omega_3)^{-2} L(1, \omega_3)^{-2}.
\]

We now get the numerator
\[
I(\varphi \otimes \widehat{\varphi}) = q^{-m} \frac{\zeta_F(2)^{4}}{\zeta_F(1)^{5}} L(1, \omega_3^2)^{-1} L(1, \omega_3)^{-1} L(1, \omega_3)^{-1} L(1, \omega_3)^{-1} L(1, \omega_3)^{-1}.
\]

For the denominator, the normalization of newform $\varphi_{\pi_3}$ implies $\langle \varphi_{\pi_3}, \widehat{\varphi}_{\pi_3} \rangle = 1$. Then
\[
\langle W_{\pi_1}, \widehat{W}_{\pi_3} \rangle = 1.
\]

Recall that
\[
\langle W_{\pi_1}, \widehat{W}_{\pi_3} \rangle = \langle W_{\pi_2}, \widehat{W}_{\pi_3} \rangle = \frac{\zeta_F(2)}{\zeta_F(1)}.
\]
One can simplify that

\[ I(\varphi \otimes \hat{\varphi}) = q^{-m} \zeta_F(2)^2 L(\frac{1}{2}, \omega_3)^2 L(\frac{1}{2}, \omega_3^{-1})^2 \zeta_F(1)^3 L(1, \omega_3^2)L(1, \omega_3^{-2}). \]

Recall that the local \(L\)-factors are given by

\[ L(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = L(s, \omega_3)^2 L(s, \omega_3^{-1})^2, \]
\[ L(s, \pi_1, \text{Ad}) = L(s, \pi_2, \text{Ad}) = \zeta_F(s), \]
\[ L(s, \pi_3, \text{Ad}) = \zeta_F(s)L(s, \omega_3^2)L(s, \omega_3^{-2}). \]

One can simplify that

\[ I'(\varphi \otimes \hat{\varphi}) = q^{-m}. \]

3.5.3. \textbf{Proof of Proposition 2.2} For \(\pi_3\) supercuspidal (with \(2 \leq c(\pi_3) \leq c(\chi_D) = m\), \(W_{\pi_3}(a(y)(\frac{1}{1} \frac{1}{1}))\) is supported only at \(v(y) = -c(\pi_3)\). By Lemma 3.12 we see

\[ \int_{v(y) = -c(\pi_3)} W_{\pi_3} \left( a(y) \left( \frac{1}{1} \frac{0}{1} \right) \right) \psi(-y) \, d^\times y = C_1, \]

where \(C_1 = c(\frac{1}{2}, \pi_3, \psi) = \pm 1\). Moreover, for the oldforms, we have shown in Lemma 3.13 that \((\pi_3(\omega^{-1} 0) W_{\pi_3}) (a(y)(\frac{1}{1} \frac{0}{1}))\) is supported only at \(v(y) = -l - c(\pi_3)\), and for \(l \geq 0\) we also have

\[ \int_{v(y) = -l - c(\pi_3)} \left( \pi_3 \left( \frac{\omega^{-l}}{0} \frac{0}{1} \right) W_{\pi_3} \right) \left( a(y) \left( \frac{1}{1} \frac{0}{1} \right) \right) \psi(-y) \, d^\times y = C_1. \]

So in both cases we have

\[ \ell_{RS}(\varphi_{\pi_1}, W_{\pi_2}, \pi_3 \left( \frac{\omega^{-l}}{0} \frac{0}{1} \right) W_{\pi_3}) = \frac{\zeta_F(2)^{3/2}}{\zeta_F(1)^{3/2}} \cdot \epsilon(1, \omega_1^{-1}, \omega_2, \psi^{-1}) \cdot (\pm 1). \]

The numerator is

\[ I(\varphi \otimes \hat{\varphi}) = q^{-m} \frac{\zeta_F(2)^3}{\zeta_F(1)^3}. \]

For the denominator, recall that

\[ \langle W_{\pi_1}, \overline{W}_{\pi_1} \rangle = \langle W_{\pi_2}, \overline{W}_{\pi_2} \rangle = \frac{\zeta_F(2)}{\zeta_F(1)}, \quad \langle W_{\pi_3}, \overline{W}_{\pi_3} \rangle = 1; \]

and hence

\[ \frac{I(\varphi \otimes \hat{\varphi})}{\langle \varphi, \hat{\varphi} \rangle} = q^{-m} \frac{\zeta_F(2)}{\zeta_F(1)}. \]

The local \(L\)-factors are given by

\[ L(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = 1; \quad L(s, \pi_1, \text{Ad}) = L(s, \pi_2, \text{Ad}) = \zeta_F(s); \]

let \(\eta\) be the (nontrivial) unramified quadratic character of \(F^\times\). By \cite[Corollary 1.3]{GJ78} we know

\[ L(s, \pi_3, \text{Ad}) = \begin{cases} 1 & \text{if } \pi_3 \not\simeq \pi_3 \otimes \eta, \\ (1 + q^{-s})^{-1} & \text{if } \pi_3 \simeq \pi_3 \otimes \eta. \end{cases} \]

One can simplify that

\[ I'(\varphi \otimes \hat{\varphi}) = q^{-m}(1 + q^{-1})L(1, \pi_3, \text{Ad}) = \begin{cases} q^{-m}(1 + q^{-1}) & \text{if } \pi_3 \not\simeq \pi_3 \otimes \eta, \\ q^{-m} & \text{if } \pi_3 \simeq \pi_3 \otimes \eta. \end{cases} \]
3.5.4. **Direct calculation by matrix coefficients.** One can also calculate the local constants by the methods used in [Hu17], which is to calculate directly the matrix coefficients: by definition,

$$I(\varphi \otimes \tilde{\varphi}) = \frac{1}{\langle \varphi, \tilde{\varphi} \rangle} \int_{Z(F) \setminus GL_2(F)} \Phi_{\pi_1}(g)\Phi_{\pi_2}(g)\Phi_{\pi_3}(g) \, dg,$$

where $\Phi_{\pi}$ denotes the normalized matrix coefficient

$$\Phi_{\pi}(g) = \frac{1}{\langle W_{\pi}, W_{\pi} \rangle} \int_{F^\times} W_{\pi}(a(y)g) W_{\pi}(a(y)) \, d^\times y.$$

We consider the case when $\pi_3$ is supercuspidal for example. Let $b = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = n(x)a(y)$ for some $y \in F^\times$, $x \in F$. For $\pi_1, \pi_2$, one can generalize the results in [HK20] and show that

- $\Phi_{\pi_1}(b)$ is equal to
  $$\begin{cases} \omega_2(y)|y|^{1/2} & v(y) \geq 0, \ v(x) \geq 0 \\ \omega_2(y)|y|^{-1/2} & v(y) \leq 0, \ v(x) \geq v(y) \\ 0 & \text{otherwise.} \end{cases}$$

- $\Phi_{\pi_1}(b(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}))$ is equal to
  $$\begin{cases} \omega_1(y)|y|^{1/2}\omega_1^{-1}\omega_2(x+y)|x+y|^{-1} & \text{if } v(x+y) \leq \min\{-m, v(y)\} \\ 0 & \text{otherwise.} \end{cases}$$

- For $0 < j < m$, $\Phi_{\pi_1}(b(\begin{smallmatrix} 1 & 0 \\ \omega_j & 1 \end{smallmatrix}))$ is equal to
  $$\begin{cases} \omega_2(y)|y|^{-1/2}\omega_1^{-1}\omega_2(1+x\omega^j/y) & \text{if } v(y) \leq j - m, \ v(x) \geq v(y) + m - j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

And $\Phi_{\pi_2}$ is the complex conjugation of $\Phi_{\pi_1}$. For $\pi_3$ supercuspidal, [Hu17] Proposition 2.19 shows:

- $\Phi_{\pi_3}(b)$ is supported on $v(y) = 0$ and $v(x) \geq -l - 1$;
  $$\Phi_{\pi_3}(b) = \begin{cases} 1 & \text{if } v(y) = 0, \ v(x) \geq -l, \\ \frac{1}{q^{-l}} & \text{if } v(y) = 0, \ v(x) = -l - 1; \end{cases}$$

- for $j = c(\pi_3) + l - 1$, $\Phi_{\pi_3}(b(\begin{smallmatrix} 1 & 0 \\ \omega^{-v_1} & 1 \end{smallmatrix}))$ is supported on $v(y) = 0$, $v(x) \geq -l - 1$, and
  $$\Phi_{\pi_3}(b(\begin{smallmatrix} 1 & 0 \\ \omega^{-c(\pi_3)+l-1} & 1 \end{smallmatrix})) = \frac{1}{q^{-l}} \text{ if } v(x) \geq -l;$$

- for $c(\pi_3) < 4$ and $0 \leq j < c(\pi_3) + l - 1$, $\Phi_{\pi_3}(b(\begin{smallmatrix} 1 & 0 \\ \omega^{-c(\pi_3)} & 1 \end{smallmatrix}))$ is supported on $v(y) = \min\{0, 2j - c(\pi_3) - 2l\}$, $v(x) = j - c - 2l$; it is of level $c(\pi_3) + l - j$ as a function in $y$.

Since $W_{\pi}$ is right $K_1(p^m)$-invariant, [Hu17] Lemma 2.2] implies that $I(\varphi \otimes \tilde{\varphi})/\langle \varphi, \tilde{\varphi} \rangle$ is equal to

$$\sum_{j=0}^{m} A_j \int_{Z(F) \setminus B(F)} \prod_{i=1}^{3} \Phi_{\pi_i}(b(\begin{smallmatrix} 1 & 0 \\ \omega_i & 1 \end{smallmatrix})) \, db, \quad A_j = \frac{\zeta_F(2)}{\zeta_F(1)} \cdot \begin{cases} 1 & \text{if } j = 0 \\ \frac{1}{q^{-j}} & \text{if } 0 < j < m \\ \frac{1}{q^{-m}} & \text{if } j = m, \end{cases}$$

where $b = n(x)a(y)$ and $db = |y|^{-1}d^\times y \, dx$. Here

$$\prod_{i=1}^{3} \Phi_{\pi_i}(b) = \begin{cases} |y|\Phi_{\pi_3} & v(y) \geq 0, \ v(x) \geq 0 \\ |y|^{-1}\Phi_{\pi_3} & v(y) \leq 0, \ v(x) \geq v(y) \\ 0 & \text{otherwise.} \end{cases}$$
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So

\[
\int_{Z(F)\setminus B(F)} \prod_{i=1}^{3} \Phi_{\pi_i}(b) \, db = \int_{v(y)=0} \frac{d^x y}{|y| \int_{v(x)\geq 0} dx} = 1.
\]

Next we show that all other terms (when \(0 \leq j < m\)) vanish. For \(j = 0\),

\[
\prod_{i=1}^{3} \Phi_{\pi_i}(b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) = \begin{cases} \frac{|y|}{x+y^2} \Phi_{\pi_3} & \text{if } v(x+y) \leq \min\{-m, v(y)\}, \\ 0 & \text{otherwise}, \end{cases}
\]

with

\[
\Phi_{\pi_3}(b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) = \begin{cases} C_1 & \text{if } v(y) = -c - 2l, \ x \in -y + \mathcal{O}_F, \\ \frac{C_1}{q-1} & \text{if } v(y) = -c - 2l, \ x \in -y + \mathcal{O}_F, \\ 0 & \text{otherwise} \end{cases}
\]

is supported only on \(v(y) = -c - 2l, v(x+y) \geq -l - 1\) (so \(v(x+y) \leq v(y)\) cannot happen). Therefore

\[
\prod_{i=1}^{3} \Phi_{\pi_i}(b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) = 0.
\]

For \(c+l-1 \leq j < m\), \(\Phi_{\pi_3}(b \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix})\) is supported on \(v(y) = 0\). Since \(j - m < 0\),

\[
\prod_{i=1}^{3} \Phi_{\pi_i}(b \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix}) = 0.
\]

For \(0 < j < c+l-1\),

\[
\begin{aligned}
\{v(y) \leq j - m \text{ (which is } < 0) & \} \Rightarrow 2j - c - 2l \leq j - m \Rightarrow j \leq c + 2l - m, \\
v(y) = \min\{0, 2j - c - 2l\} & \}
\end{aligned}
\]

which is possible only if \(l > \frac{1}{2}(m-c)\) (recall that \(0 \leq l \leq m-c\)). Under this assumption \(\Phi_{\pi_3}(b \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix})\) is supported on

\[v(y) = 2j - c - 2l, \quad v(x) = j - c - 2l;\]

but in this case \(v(x) \geq v(y) + m - j - 1\) does not hold. That means, on the support of \(\Phi_{\pi_3}(b \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix})\), \(\Phi_{\pi_1}(b \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix})\) vanishes. So again

\[
\prod_{i=1}^{3} \Phi_{\pi_i}(b \begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix}) = 0.
\]

In conclusion

\[
\frac{I(\varphi \otimes \tilde{\varphi})}{(\varphi, \tilde{\varphi})} = \frac{\zeta_F(2)}{\zeta_F(1)} q^{-m} \int_{Z(F)\setminus B(F)} \prod_{i=1}^{3} \Phi_{\pi_i}(b) \, db = q^{-m} \frac{\zeta_F(2)}{\zeta_F(1)}.
\]

One can continue from (16) and complete the proof.

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