LOCALIZATION FOR $\mathcal{N} = 2$ SUPERSYMMETRIC GAUGE THEORIES IN FOUR DIMENSIONS

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ABSTRACT. We review the supersymmetric localization of $\mathcal{N} = 2$ theories on curved backgrounds in four dimensions using $\mathcal{N} = 2$ supergravity and generalized conformal Killing spinors. We review some known backgrounds and give examples of new geometries such as local $T^2$-bundle fibrations. We discuss in detail a topological four-sphere with generic $T^2$-invariant metric. This review is a contribution to the special volume on recent developments in $\mathcal{N} = 2$ supersymmetric gauge theory and the 2d-4d relation.

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1. Introduction

Non-perturbative exact results in interacting quantum field theories (QFTs) are rare and precious and usually we explain them using non-trivial symmetries of QFT, such as quantum groups in the theory of quantum integrable systems [1–3]. Another instrumental symmetry for exact results in QFTs is supersymmetry. For example, Seiberg-Witten solution [4] of four-dimensional $\mathcal{N} = 2$ supersymmetric field theories is explained by rigid constraints imposed by $\mathcal{N} = 2$ supersymmetry on the low-energy effective Lagrangian and certain assumptions on electric-magnetic duality. For a review of $\mathcal{N} = 2$ four-dimensional theory from the modern angle of view see contributions [V:1] and [V:2] in this volume.

Exact non-perturbative results in supersymmetric QFTs are suited for strong tests of non-perturbative dualities between QFTs that have different microscopic description, they give a practical approximation to interesting physical phenomena in non-supersymmetric QFTs, and they open new perspectives on fascinating geometrical spaces such moduli spaces of instantons, monopoles, complex structures, flat connections, and others.

A fruitful non-perturbative approach is supersymmetric localization. In finite dimensional geometry localization appeared in the Lefchetz fixed-point formula, Duistermaat-Heckman and Atiyah-Bott formula for integration of equivariantly closed differential forms [5]. In [6] Witten generalized the localization formula for the infinite-dimensional geometry of the path integral of supersymmetric quantum mechanics. Similar approach was proposed to two-dimensional sigma models [7], four-dimensional gauge theories [8] and others. The similarity of these constructions is the topological twist of a given supersymmetric QFT. The topological twist introduces a certain background connection for the local internal R-symmetry of the theory. Usually, this connection is such that there exists a scalar fermionic supersymmetry generator $Q$ for a QFT coupled to a generically curved background metric. In topologically twisted theories the stress-energy tensor is $Q$-exact, and, consequently, the theory is metric independent. A further twist to the supersymmetric localization of gauge theories, called $\epsilon$-equivariant deformation or $\Omega$-background $\mathbb{R}^{4}_{\epsilon_1,\epsilon_2}$, has been added by Nekrasov [9] based on the considerations of [10–13]. The construction of the gauge theory instanton partition function is reviewed in [V:3] of this volume. The $\epsilon$-equivariant partition function $Z_{\epsilon_1,\epsilon_2}$, referred as Nekrasov’s function, turned out to be a fascinating object of mathematical physics, with profound connections to other branches of research such as topological strings (see review [V:12, V:13] in this volume), matrix models (see review [V:7] in this
volume), quantum groups \([2, 3]\) and integrable systems \([14]\). For a recent study of the instanton partition function \(Z_{\epsilon_1, \epsilon_2}\) for a large class of quiver theories see \([15, 16]\). A profound connection between four-dimensional gauge theory supersymmetric objects (BPS) and two-dimensional conformal field theories (CFT), called BPS/CFT correspondence in \([17]\), was a subject of long research \([9, 13, 18–22]\).

Another version of localization was used in \([23]\) for \(\mathcal{N} = 2\) supersymmetric gauge theory on a four-sphere \(S^4\) with an insertion of a Wilson operator \([23]\) or ’t Hooft operators \([24]\). The topological twist is not necessary because of rich \(OSp(2|4)\) symmetry that \(\mathcal{N} = 2\) QFT on \(S^4\) has. A similar localization was later performed for gauge theories on \(S^3\) \([25]\), on \(S^2\) \([26, 27]\), on \(S^5\) \([28, 29]\), on squashed \(S^3\) \([30, 31]\), on squashed \(S^4_{\epsilon_1, \epsilon_2}\) \([32]\) and other geometries. For a review of 3d localization in this volume see \([V:5]\), for a review of line operators (such as Wilson and ’t Hooft operators) in 4d gauge theory see review \([V:6]\) in this volume, and for review of surface operators see \([V:8]\). The four-sphere partition function of the \(\mathcal{N} = 2\) gauge theory of class \(\mathcal{S}_g\) (see \([V:1]\) in this volume) turned out to be equal to the correlation function of the 2d conformal \(g\)-Toda theory, the statement known as AGT conjecture \([33]\), which provided explicit beautiful realization of the 4d/2d BPS/CFT correspondence. For a review of AGT conjecture (4d/2d BPS/CFT correspondence) in this volume see \([V:9]\), for review of the superconformal index see \([V:11]\) and for a review of the 3d/3d version of the BPS/CFT correspondence see \([V:10]\).

A general procedure to construct a QFT on a curved manifold with some amount of supersymmetry is to couple QFT with supergravity, choose the supergravity background fields in such a way that there exists a non-trivial supersymmetric variation under which these background fields are invariant and then freeze the supergravity fields. This construction was explored for \(\mathcal{N} = 1\) supersymmetric four-dimensional theories in \([34]\).

In this note we will partially analyze off-shell \(\mathcal{N} = 2\) supersymmetry backgrounds suitable for localization, and review the case of the four-sphere \([23]\) with a generic \(T^2\)-invariant deformation of the metric. We employ the formalism of \(\mathcal{N} = 2\) supergravity known as superconformal tensor calculus, see \([35–38]\) and reviews \([39, 40]\). For alternative analysis of \(\mathcal{N} = 2\) supergravity localization backgrounds see \([41]\).

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### 2. \(\mathcal{N} = 2\) Supergravity

#### 2.1. Gravity multiplet

A way to construct \(\mathcal{N} = 2\) Poincare supergravity is to promote the \(\mathcal{N} = 2\) superconformal symmetry to local gauge symmetry and introduce associated gauge fields. The gauge fields are combined with auxiliary fields to form Weyl multiplet. The notations are collected in the table 2.1.

The gauge fields \((\omega^\mu_\nu, f^\mu_\nu, \phi^i_\mu)\) associated to the rotation, the special conformal symmetry and the special conformal supersymmetry are expressed in terms of the
Table 2.1. The Weyl multiplet

| symmetry                        | gauge field | constraint | parameter |
|---------------------------------|-------------|------------|-----------|
| translation                     | $e_{\mu}^{\hat{\mu}}$ | $\hat{R}(e_{\mu}^{\hat{\mu}})$ |           |
| rotation                        | $\omega_{\mu}^{\hat{\mu}\hat{\nu}}$ | $\hat{R}(\omega_{\mu}^{\hat{\mu}\hat{\nu}})$ |           |
| special conformal               | $f_{\mu}^{\hat{\mu}}$ | $\hat{R}(\omega_{\mu}^{\hat{\mu}\hat{\nu}})$ |           |
| dilatation                      | $b_{\mu}$ | $\hat{R}(\omega_{\mu}^{\hat{\mu}\hat{\nu}})$ |           |
| translational supersymmetry $Q$ | $\psi_{\mu}^{j}$ | $\varepsilon^{i}$ |           |
| conformal supersymmetry $S$     | $\phi_{\mu}^{j}$ | $\hat{R}(\psi_{\mu}^{j})$ | $\eta^{j}$ |
| $SU(2)_R$-symmetry              | $V_{\mu j}^{i}$ | $\hat{A}_{\mu}$ |           |
| $U(1)_{\tilde{R}}$-axial symmetry |             |             |           |

auxiliary fields

|            |         |
|------------|---------|
| tensor     | $T_{\mu \nu a}$ |
| spinor     | $\chi^{i}$ |
| scalar     | $M$      |

other fields from the constraints on superconformal covariant curvatures $\hat{R}$ for the fields $e_{\mu}^{\hat{\mu}}, \omega_{\mu}^{\hat{\mu}\hat{\nu}}, \psi_{\mu}^{j}$.

It is convenient to use 6d spinorial notations for the spinors of 4d $\mathcal{N} = 2$ theories under dimensional reduction. We use index conventions from appendix A and chirality conventions as in 2.2.

Table 2.2. 6d chirality

| Weyl multiplet | field | variation | 6d chirality |
|----------------|-------|-----------|--------------|
|                | $\psi_{\mu}^{j}$ | $\varepsilon^{i}$ | +1           |
|                | $\phi_{\mu}^{j}$ | $\eta^{j}$ | -1           |
|                | $\chi^{i}$ | | +1           |
|                | $T_{\mu \nu a}$ | | -1           |

vector multiplet

| $\lambda^{i}$ | +1 |

To find the action of the $\mathcal{N} = 2$ Poincare supergravity interacting with $n_V$ vector multiplets and $n_H$ hypermultiplets one considers Weyl multiplet coupled with $n_V + 1$
vector multiplets and \( n_H + 1 \) hypermultiplets and then uses one vector multiplet and one hypermultiplet as auxiliary fields to gauge fix the non-Poincare superconformal gauge symmetries and to integrate out non-Poincare supergravity fields. Finally, one gets the on-shell physical fields of the Poincare \( \mathcal{N} = 2 \) supergravity: the frame \( \hat{e}^\mu_\mu \), the gravitino doublet \( \psi^i_\mu \) and the graviphoton \( A_\mu \) together with \( n_v \) vector multiplets and \( n_h \) hypermultiplets. (See more details in the diagram [42], page 81).

To construct gauge theories on fixed curved backgrounds with partially preserved off-shell supersymmetry the full machinery described above is not necessary. It is sufficient to consider the off-shell action and the supersymmetry transformation for the vector multiplets and hypermultiplets coupled to the Weyl gravity multiplet and then freeze the fields of the Weyl multiplet to a supersymmetry invariant background [32, 34, 43].

The supersymmetry transformation is a linear superposition of the Poincare supersymmetry variation \( \varepsilon^i \) and the conformal supersymmetry variation \( \eta^i \). Since variation of bosonic fields in Weyl multiplet is proportional to fermionic fields of Weyl multiplet and they are set to zero in the background, the supersymmetric equation is the vanishing variation of the independent fermions \( \psi^i_\mu \), \( \chi^i \). The field \( \phi^i_\mu \) is expressed in terms of \( \psi^i_\mu \) and \( \chi^i \) through the curvature constraints, and the vanishing variation of \( \psi^i_\mu \) and \( \chi^i_\mu \) automatically implies vanishing variation of \( \phi^i_\mu \).

We quote the variation of gravitino \( \psi^i_\mu \) and auxiliary field \( \chi^i_\mu \) under the Poincare and conformal supersymmetries \( \varepsilon^i \) and \( \eta^i \) from [40], page 429. The equations are\(^1\):

\[
\begin{align*}
\delta_{\varepsilon,\eta} \psi^i_\mu &= D_\mu \varepsilon^i - \frac{1}{16} \gamma^i_\mu \varepsilon^i - \gamma^i_\mu \eta^i = 0 \\
\delta_{\varepsilon,\eta} \chi^i &= -\frac{1}{24} [D_\mu \varepsilon^i + \frac{1}{6} \gamma^i_\mu \varepsilon^i] + \frac{1}{12} \gamma^i_\mu \eta^i + \frac{1}{2} \mathcal{M} \\
&= 0
\end{align*}
\tag{2.1}
\]

From the first equation the conformal supersymmetry parameter \( \eta \) can be expressed in terms of the Poincare supersymmetry parameter \( \varepsilon \) as

\[
\eta^i = \frac{1}{4} \mathcal{D} \varepsilon^i
\tag{2.2}
\]

where we used (A.33). Later we use this relation to substitute \( \eta^i \) with \( \frac{1}{4} \mathcal{D} \varepsilon^i \) and vice versa. The second equation, called the auxiliary equation, can be transformed using the Lichnerowicz formula for \( \mathcal{D}^2 \) (A.39) and the divergence of the first equation, see appendix (A.41):

\[
\begin{align*}
D_\mu \varepsilon^i - \frac{1}{16} \gamma^i_\mu \varepsilon^i - \frac{1}{4} \gamma^i_\mu \mathcal{D} \varepsilon^i &= 0 \\
\mathcal{D} \eta &= -\frac{1}{2} (\frac{1}{6} R + \mathcal{M}) \varepsilon + \frac{1}{16} [D^\mu \varepsilon] \gamma^i_\mu \varepsilon \\
&= (\frac{1}{4} \mathcal{D}^2 \varepsilon)
\end{align*}
\tag{2.3}
\]

Here \( R \) denotes the scalar curvature (A.38) of the background metric.

The equations (2.3) are called generalized conformal Killing spinor equations, and the spinor \( \varepsilon \) is called generalized conformal Killing spinor.

The generalized conformal Killing spinor equations transform covariantly with respect to local Weyl transformation

\[
g_{\mu\nu} \mapsto e^{2\Omega} g_{\mu\nu}
\tag{2.4}
\]

\(^1\)In some \( \mathcal{N} = 2 \) supergravity literature the auxiliary scalar field \( \mathcal{M} \) in Weyl multiplet is denoted \( D \). For the conventions on the Clifford algebra see Appendix A.3; the slash symbol on tensors denotes Clifford contraction as in equation (A.10).
with the weights
\[ \varepsilon \mapsto e^{\frac{1}{2}\Omega}\varepsilon, \quad M \mapsto e^{-2\Omega}M, \quad T \mapsto e^{-\Omega}T \]  
(2.5)

Therefore the solutions can be classified by their conformal class.

The generalized conformal Killing spinor equations, similarly to the conformal Killing equations, can be rewritten as the generalized parallel transport equations on the section of doubled spinorial bundle
\[ \mathcal{D}_\mu \left( \begin{array}{c} \varepsilon^i \\ \eta^i \end{array} \right) = 0 \]  
(2.6)

for certain \( \mathcal{D}_\mu \). This representation could be useful to classify the solutions.

The solution to the generalized conformal Killing equations is particularly simple for conformally flat background with vanishing auxiliary field \( T_{\mu\nu a} \), flat R-symmetry gauge connection and vanishing auxiliary scalar \( M \). In the flat \( \mathbb{R}^4 \) coordinates \( x^\mu \), the solution is simply
\[ \varepsilon^i(x) = \hat{\varepsilon}^i + \hat{\eta}^i \]  
(2.7)

where \( \varepsilon^i \) and \( \eta^i \) are arbitrary constant spinor parameters associated with translational and special conformal supersymmetry respectively. The maximal dimension of the space of solutions to the parallel transport equation in a bundle is the rank of this bundle. We see that the conformally flat background with flat R-symmetry connection and vanishing \( T_{\mu\nu a} \) is maximally supersymmetric. The 16 sections are generated by 8 components of \( \hat{\varepsilon} \) and 8 components of \( \hat{\eta} \).

It would be interesting to find the complete classification of the solutions to the generalized conformal Killing equation with various amounts of supersymmetry. In this note we will focus on particular backgrounds interesting for the localization of gauge theories.

2.2. Vector multiplet. The 4d \( \mathcal{N} = 2 \) vector multiplet \( (A_m, \lambda^i, Y^{ij}) \) includes the gauge field \( A_\mu \) and two real scalar fields \( \Phi_a \) combined into the reduction of 6d gauge field \( (A_m) = (A_\mu, \Phi_a) \), the \( SU(2)_R \)-doublet of gaugino fermions \( \lambda^i \), and the \( SU(2)_R \)-triplet of auxiliary fields represented by the matrix \( Y^{ij} \) symmetric in \( (ij) \). The gaugino \( \lambda^i \) is the reduction of the \( SU(2)_R \)-doublet of 6d Weyl spinors of chirality +1 for \( \gamma^6_{\text{6d}} \). The spinor fields from the \( SU(2)_R \)-doublet enter into the Lagrangian and supersymmetry variation holomorphically, their complex conjugates never appear in the Euclidean formulation of the theory.

The supersymmetry variation for the vector multiplet is
\[
\begin{align*}
\delta A_m &= \frac{1}{2}\lambda^i \gamma_m \varepsilon_i \\
\delta \lambda^i &= -\frac{1}{4} F_{mn} \gamma^{mn} \varepsilon^i + Y^i \varepsilon^j + \Phi_a \gamma^a \eta + \frac{1}{8} T_{\mu\nu a} \Phi^a \gamma^{\mu\nu} \varepsilon \\
\delta Y^{ij} &= -\frac{1}{2} \varepsilon^i (\mathcal{D}_j \lambda^i)
\end{align*}
\]  
(2.8)

where there are two extra two terms for \( \delta \lambda^i \) compared to the standard translational supersymmetry. In our conventions the supersymmetry parameters \( \varepsilon^i, \eta^i \) are bosonic and \( \delta_{\varepsilon, \eta} \) is fermionic, for a field \( \phi \) the field \( \delta_{\varepsilon, \eta} \phi \) has opposite statistics of \( \phi \).
Table 2.3. The symmetry action of $\delta_{\epsilon, \eta}^2$

| $\delta_{\epsilon, \eta}^2$ acts by | parameter                     |
|-----------------------------------|-------------------------------|
| $\mathcal{L}_v$                  | $v^m = \frac{1}{4}(\epsilon \gamma^m \epsilon)$ |
| $SO(2)_{\tilde{R}}$             | $\tilde{R}^{ab} = -\frac{1}{2}(\eta \gamma^{ab} \epsilon)$ |
| $SU(2)_{R}$                     | $R^{ij} = (\eta^{(i} \epsilon^{j)}) \lambda_j$ |
| dilatation                      | $(\eta \epsilon)$            |

If $\epsilon$ and $\eta = \frac{1}{4} \mathcal{D} \epsilon$ solve the generalized conformal Killing spinor equations (2.3), the supersymmetry transformations (2.8) closes off-shell:

$$
\delta_{\epsilon, \eta}^2 A_\mu = \frac{1}{4}(\epsilon \gamma^\nu \epsilon) F_{\nu \mu} + \frac{1}{4}[\epsilon \gamma^a \Phi_a, D_\mu] \\
\delta_{\epsilon, \eta}^2 \Phi_a = \frac{1}{4}[(\epsilon \gamma^m \epsilon) D_m \Phi_a] - \frac{1}{2}(\eta \gamma_{ab} \epsilon) \Phi^b + \frac{1}{2}(\eta \epsilon) \Phi_a \\
\delta_{\epsilon, \eta}^2 \lambda^i = \frac{1}{4} \left( (\epsilon \gamma^m \epsilon) D_m \lambda^i + \frac{1}{4} D_\mu (\epsilon \gamma_{\nu} \epsilon) \gamma^{\mu \nu} \lambda^i \right) - \frac{1}{8}(\eta \gamma_{ab} \epsilon) \gamma^{ab} \lambda^i + \frac{3}{4}(\eta \epsilon) \lambda^i + (\eta^{(i} \epsilon^{j)}) \lambda_j \\
\delta_{\epsilon, \eta}^2 Y^{ij} = \frac{1}{4}[(\epsilon \gamma^m \epsilon) D_m Y^{ij}] + (\eta \epsilon) Y^{ij} + (\eta^{(k} \epsilon^{l)}) Y^{ij}_{kl} + (\eta^{(k} \epsilon^{l)}) Y^{ij}_{kl} (2.9)
$$

The variation $\delta_{\epsilon, \eta}^2$ contains the Lie derivative action by the 6d reduced vector field

$$
v^m = \frac{1}{4}(\epsilon \gamma^m \epsilon) (2.10)
$$

The scalar components $m \equiv a$ generate the gauge transformation by $v^a \Phi_a$.

The Lagrangian of 4d $\mathcal{N} = 2$ vector multiplet coupled to the Weyl gravity multiplet can be found in [37–39], or [40] page 433

$$
S = -\frac{1}{g_\text{YM}} \int \sqrt{g} d^4x \text{tr} \left( \frac{1}{2} F_{mn} F^{mn} + \lambda^i \gamma^m D_m \lambda_i + (\frac{1}{6} R + M) \Phi_a \Phi^a - 2 Y_{ij} Y^{ij} + F^{\mu \nu} T_{\mu \nu a} \Phi^a + \frac{1}{4} T_{\mu \nu a} T^{\mu \nu b} \Phi^a \Phi_b \right) (2.11)
$$

Provided $\epsilon$ and $\eta = \frac{1}{4} \mathcal{D} \epsilon$ satisfy (2.1) the action $S$ is invariant under $\delta_{\epsilon, \eta}$

$$
\delta_{\epsilon, \eta} S = 0. (2.12)
$$

3. Generalized conformal Killing spinor

Presently the complete classification of the solutions to the generalized conformal Killing spinor equation (2.1) is not available. We list some known examples. In all these examples the $U(1)_{\tilde{R}}$ connection is set to zero, the square of the supersymmetry transformation $\delta_{\epsilon}^2$ generates isometry transformation and possibly $SU(2)_{R}$ transformation but without dilatation and $U(1)_{\tilde{R}}$ transformation.
3.1. **Topologically twisted theories.** One simple class of solutions which exists on any smooth 4-manifold is the Donaldson-Witten topological twist [8]. One sets $R$-symmetry $SU(2)_R$ connection to compensate right component of the $Spin(4) = SU(2)_L \times SU(2)_R$ spin-connection. In the twisted theory the 8 components of the 4d $\mathcal{N} = 2$ spinor generators transform as a one-form, self-dual two-form and scalar. The scalar component yields the scalar supersymmetry charge defined on any smooth 4-manifold. The theory localizes to the instanton configurations $F_A^+ = 0$.

3.2. **Omega background.** Another example is the equivariant twist of the topologically twisted theory on any manifold with $U(1)$ isometry. To construct such theory, one uses a combination of the scalar supersymmetry of the topologically twisted theory and the one-form supercharge contracted with the vector field that generates $U(1)$ isometry. Localization of such theory on $\mathbb{R}^4$ counts equivariant instantons and gives Nekrasov partition function [9–12, 44], with two equivariant parameters $\epsilon_1, \epsilon_2$, each associated to the rotation of the $\mathbb{R}^2$ planes in the decomposition $\mathbb{R}_{\epsilon_1, \epsilon_2}^4 = \mathbb{R}_{\epsilon_1}^2 \oplus \mathbb{R}_{\epsilon_2}^2$.

For a review of instanton counting see contribution [V:9] of this volume.

3.3. **Conformal Killing spinor.** Another example is conformally flat and $SU(2)_R$-flat metric with $T_{\mu \nu} = 0$ and conformal Killing spinor. A spinor of this type has been used to localize the physical $\mathcal{N} = 2$ gauge theory on $S^4$ [23]. The isometry vector field has two fixed points: the north and the south poles of $S^4$. In the neighborhood of the north pole the theory is locally isomorphic to the theory in the Omega-background with parameters $\epsilon_1 = \epsilon_2 = r^{-1}$ where $r$ is the radius of $S^4$, counting equivariant instantons $F_A^+ = 0$. In the neighborhood of the south pole the theory is conjugate to the theory in Omega-background, and it counts equivariant anti-instantons $F_A^- = 0$. The complete partition function on $S^4$ is the fusion of the Nekrasov partition function and its conjugate:

$$Z_{S^4} = \int [da] |Z_{\mathbb{R}^4, r^{-1}, r^{-1}}(ia)|^2$$

where $Z_{\mathbb{R}^4, \epsilon_1, \epsilon_2}(a)$ is the complete partition function in Omega background including the classical and perturbative factors, and $a$ is the gauge Lie algebra equivariant argument of $Z_{\mathbb{R}^4, \epsilon_1, \epsilon_2}$ that physically is interpreted as the electrical type special coordinate on the Coulomb moduli space of the $\mathcal{N} = 2$ theory or boundary conditions at infinity of $\mathbb{R}_{\epsilon_1, \epsilon_2}^4$ for the scalar field $\Phi_a$ of the vector multiplet. In the formula (3.1) we omit from the arguments of the partition function the parameters of the Lagrangian.

More generally, the cases of $\Omega$-background and conformal Killing spinor could be viewed as the specialization of local $T^2$-bundle geometry.

3.4. **Local $T^2$-bundles.** Consider a manifold $X_4$ endowed with the metric structure of the warped product $X_4 = T^2_{w_1, w_2} \times \Sigma_2$ where $\Sigma_2$ is a Riemann surface, possibly with boundaries, and $T^2_{w_1, w_2}$ is a flat 2-torus with basis cycles of length $(2\pi w_1, 2\pi w_2)$. Here $(w_1, w_2)$ are locally arbitrary functions on $\Sigma_2$. This geometry generalizes the Omega background on $\mathbb{R}^4$ and the standard conformal Killing spinor geometry on $S^4$. The ellipsoid solution [32] is a special example of $X_4$. Another case was studied in [45].
We denote the coordinates along the two circles on $T^2$ by $(\phi_1, \phi_2)$. We pick two real parameters $(\epsilon_1, \epsilon_2)$ with the aim to get $\delta^2_{\epsilon, \eta}$ action by the vector field

$$v = \epsilon_1 \partial_{\phi_1} + \epsilon_2 \partial_{\phi_2}$$

(3.2)

We assume that $\epsilon_i$ are such that $(\epsilon_1 w_1)^2 + (\epsilon_2 w_2)^2 \leq 1$ everywhere on $\Sigma$. For any generic functions $(w_1, w_2)$ on $\Sigma$ we can always find local coordinates $(\theta, \rho)$ such that

$$\cot \theta := \frac{\epsilon_1 w_1}{\epsilon_2 w_2} \hspace{1cm} \Leftrightarrow \hspace{1cm} w_1 = \epsilon_1^{-1} \sin \rho \cos \theta$$

$$\sin^2 \rho := (\epsilon_1 w_1)^2 + (\epsilon_2 w_2)^2 \hspace{1cm} w_2 = \epsilon_2^{-1} \sin \rho \sin \theta$$

(3.3)

After we have fixed special coordinates $(\theta, \rho)$ on $\Sigma$, the metric components $g_{\mu\nu}(\rho, \sigma)$ are parametrized by three arbitrary functions $g_{\theta\theta}(\theta, \rho)$, $g_{\theta\rho}(\theta, \rho)$ and $g_{\rho\rho}(\theta, \rho)$.

Next we choose the frame on $X^4$ of the form

$$e^1 = w_1(\theta, \rho) d\phi_1 \quad e^3 = e^3_\rho(\theta, \rho) d\theta + e^3_\rho(\theta, \rho) d\rho$$

$$e^2 = w_2(\theta, \rho) d\phi_2 \quad e^4 = e^4_\rho(\theta, \rho) d\rho$$

(3.4)

Three functions $e^3_\rho(\theta, \rho), e^3_\rho(\theta, \rho), e^4(\theta, \rho)$ generically parametrize 2d metric by the relations $g_{\theta\theta} = e^3_\theta e^3_\theta$, $g_{\theta\rho} = e^3_\theta e^3_\rho$ and $g_{\rho\rho} = e^3_\rho e^3_\rho + e^4 e^4$. It is convenient to denote

$$e^3_\rho(\theta, \rho) \equiv \sin \rho f_1(\theta, \rho) \quad e^3_\rho(\theta, \rho) \equiv f_3(\theta, \rho) \quad e^4(\theta, \rho) \equiv f_2(\theta, \rho)$$

(3.5)

and present solution for the background fields $T_{\mu\nu}$ and $V_{\mu i}$ in terms of $f_1, f_2, f_3$.²

In the $\gamma$ matrix basis (A.45) we choose the $SU(2)_R$-doublet spinor $(\varepsilon^1, \varepsilon^2)$ in the frame (3.4) to be given by

$$\varepsilon^1 = e^{\frac{1}{2} (i\phi_1 + i\phi_2)}(e^{-i\frac{\theta}{2}} \sin \frac{\rho}{2}, -e^{i\frac{\theta}{2}} \sin \frac{\rho}{2}, ie^{-i\frac{\theta}{2}} \cos \frac{\rho}{2}, -ie^{i\frac{\theta}{2}} \cos \frac{\rho}{2})$$

$$\varepsilon^2 = e^{-\frac{1}{2} (i\phi_1 + i\phi_2)}(e^{-i\frac{\theta}{2}} \sin \frac{\rho}{2}, e^{i\frac{\theta}{2}} \sin \frac{\rho}{2}, -ie^{-i\frac{\theta}{2}} \cos \frac{\rho}{2}, -ie^{i\frac{\theta}{2}} \cos \frac{\rho}{2})$$

(3.6)

Notice that this spinor satisfies the standard reality condition $\varepsilon^2 = \bar{\varepsilon}^1$ where bar is the complex conjugation and $\varepsilon$ is the Majorana bilinear matrix (A.45), and that this spinor is the transformation to the $(\phi_1, \phi_2, \rho, \theta)$ frame of the standard conformal Killing spinor $\varepsilon(x) = \varepsilon_\sigma + x^\mu \gamma_\mu \varepsilon_c$ that was used in [23] for $S^4$, where $x^\mu$ are stereographic projection coordinates on $S^4$.

²In these notations the solution can be easily specialized to the Hama-Hosomichi ellipsoid [32] metrically defined by the equation in $\mathbb{R}^5$ with the standard metric

$$r_1^{-2}(X_1^2 + X_2^2) + r_2^{-2}(X_3^2 + X_4^2) + r^{-2}X_5^2 = 1$$

by taking

$$X_1 + iX_2 = r_1 \sin \rho \cos \theta e^{i\phi_1}, \quad X_3 + iX_4 = r_2 \sin \rho \sin \theta e^{i\phi_2}, \quad X_5 = r \cos \rho$$

and

$$f_1(\theta, \rho) = f_{\text{HH}}(\theta, \rho) = \sqrt{r_1^2 \sin^2 \theta + r_2^2 \cos^2 \theta} \quad f_2(\theta, \rho) = g_{\text{HH}}(\theta, \rho) = \sqrt{r_2^2 \sin^2 \rho + r_1^2 r_2^2 f_1(\theta)^{-2} \cos^2 \rho}$$

$$f_3(\theta, \rho) = h_{\text{HH}}(\theta, \rho) = (-r_2^2 + r_2^2) f_1(\theta)^{-1} \cos \theta \sin \theta \cos \rho$$

In the case of round sphere $S^4$ we set

$$f_1(\theta, \rho) = r \quad f_2(\theta, \rho) = r \quad f_3(\theta, \rho) = 0$$
For the spinor (3.6) we find the bilinear vector field

\[ v^m = \frac{1}{4} \varepsilon^i \gamma^m \varepsilon_i = \frac{1}{2} \varepsilon^1 \gamma^m \varepsilon^2 : \quad v^\mu |_{\mu \in (\phi_1, \phi_2, \theta, \rho)} = (\epsilon_1, \epsilon_2, 0, 0) \]

\[ v^a |_{a \in (5, 6)} = (-\cos \rho, -i) \]

(3.7)

This vector field is the natural isometry of the \( T^2 \)-bundle \( X_4 \).

Under the ansatz (3.6), the equations on the background fields \( V_{ij}^\mu, T^\mu_{\nu}, M \) are inhomogeneous ordinary linear equations, which can be directly solved. Though the system is overdetermined, as there are \( 32 + 8 \) linear equations from \( \delta \psi_\mu^i \) and from \( \delta \chi^i \) on \( 12 + 6 + 1 = 19 \) components for \( V, T, M \), we find that solution always exists for any \( T^2 \)-bundle. Moreover, the solution is not unique; the space of solutions forms a vector bundle of rank three. This is completely analogous to the case of the Hama-Hosomichi ellipsoid [32].

Below \( T_{\hat{\mu} \hat{\nu}} \) denote the components of \( T \) in the frame (3.4) \( v^\mu_\mu \), so that \( T_{\mu \nu} = T_{\hat{\mu} \hat{\nu}} e_\mu^\mu e_\nu^\nu \). The \( V \) denotes the connection one-form of the \( SU(2)_R \) gauge field \( D = d + V \). The components of the \( T \) do not depend on \( (\phi_1, \phi_2) \), and the components of \( V = i \sigma_I V^I \), where \( \sigma_I \) are the standard Pauli matrices, depend on \( (\phi_1, \phi_2) \) as

\[ \begin{pmatrix} V^1 \\ V^2 \end{pmatrix} = \begin{pmatrix} \cos(\phi_1 + \phi_2) & \sin(\phi_1 + \phi_2) \\ -\sin(\phi_1 + \phi_2) & \cos(\phi_1 + \phi_2) \end{pmatrix} \begin{pmatrix} \hat{V}^1 \\ \hat{V}^2 \end{pmatrix}, \quad V^3 = \hat{V}^3 \]

(3.8)

where \( \hat{V} \) is constant in \( (\phi_1, \phi_2) \).

The particular solution is

\[ T_{12} = 0 \quad T_{34} = 0 \]

\[ T_{13} = 2 \sin \theta \left( \frac{1}{f_1} - \frac{1}{f_2} \right) \quad T_{23} = -2 \cos \theta \left( \frac{1}{f_1} - \frac{1}{f_2} \right) \]

\[ T_{14} = -\frac{2 \sin \theta f_3}{f_1 f_2} \quad T_{24} = \frac{2 \cos \theta f_3}{f_1 f_2} \]

(3.9)

In the equation (3.7) \( \varepsilon^i \) denotes the +1 chiral 6d spinors and \( \gamma^m \) for the 6d gamma-matrices, while in the equation (3.6) the components of the spinor \( \varepsilon^i \) are presented with respect to the 4d Clifford algebra representation (A.45).
for $T$ components and

$$\dot{V} = \left( -\frac{1}{4\epsilon_1} \sin 2\theta \cos \rho \left( \frac{1}{f_1} - \frac{1}{f_2} \right) + \frac{\sin^2 \theta f_3}{2\epsilon_1 f_1 f_2} \right) i\sigma_2 d\phi_1 +$$

$$\left( \frac{1}{4\epsilon_2} \sin 2\theta \cos \rho \left( \frac{1}{f_1} - \frac{1}{f_2} \right) + \frac{\cos^2 \theta f_3}{2\epsilon_2 f_1 f_2} \right) i\sigma_2 d\phi_2 +$$

$$\left( \frac{1}{2} + \frac{\sin^2 \theta}{2\epsilon_1 f_1} + \frac{\cos^2 \theta r_1}{2\epsilon_1 f_2} + \frac{\sin 2\theta \cos \rho f_3}{4\epsilon_1 f_1 f_2} \right) i\sigma_3 d\phi_1 +$$

$$\left( \frac{1}{2} + \frac{\cos^2 \theta}{2\epsilon_2 f_1} + \frac{\sin^2 \theta}{2\epsilon_2 f_2} - \frac{\sin 2\theta \cos \rho f_3}{4\epsilon_2 f_1 f_2} \right) i\sigma_3 d\phi_2 +$$

$$\left( \frac{(f_1 - f_2) \cos \rho + \sin \rho \partial_\rho f_1 - \partial_\theta f_3}{2f_2} \right) i\sigma_1 d\theta$$

$$+ \left( \frac{f_3 (\sin \rho \partial_\rho f_1 + f_1 \cos \rho - \partial_\theta f_3) - f_2 \partial_\theta f_2}{2f_1 f_2 \sin \rho} \right) i\sigma_1 d\rho$$  (3.10)

are the $V$ components. This particular solution can be deformed by three-parametric family

$$\delta T_{12} = c_3$$  \hspace{1cm} $$\delta T_{34} = c_3 \cos \rho$$

$$\delta T_{13} = -c_1 \cos \rho \sin \theta - c_2 \cos \theta$$  \hspace{1cm} $$\delta T_{23} = c_1 \cos \rho \cos \theta - c_2 \sin \theta$$  \hspace{1cm} $\delta T_{14} = c_1 \cos \theta - c_2 \cos \rho \sin \theta$  \hspace{1cm} $$\delta T_{24} = c_1 \sin \theta + c_2 \cos \theta \cos \rho$$  \hspace{1cm} $\delta T_{33} = c_3 \cos \rho$  \hspace{1cm} $\delta T_{34} = c_3$  \hspace{1cm} (3.11)

together with

$$\dot{\delta V} = \left( -\frac{1}{4} c_1 \sin^2 \rho f_1 d\theta - \frac{1}{4} \sin \rho c_1 f_3 d\rho - \frac{1}{4} c_2 \sin \rho f_2 d\rho \right) i\sigma_1 +$$

$$\left( -\frac{1}{8\epsilon_1} c_1 \sin 2\theta \sin^2 \rho \partial_\rho f_1 + \frac{1}{8\epsilon_2} c_2 \sin 2\theta \sin^2 \rho \partial_\rho f_2 - \frac{1}{4} c_3 \sin \rho f_3 d\rho \right) i\sigma_2 +$$

$$\left( -\frac{1}{8\epsilon_1} c_2 \sin 2\theta \sin^2 \rho \partial_\rho f_1 + \frac{1}{8\epsilon_2} c_2 \sin 2\theta \sin^2 \rho \partial_\rho f_2 + \frac{1}{4} c_3 \sin^2 \rho f_1 d\theta + \frac{1}{4} c_3 \sin \rho f_3 d\rho \right) i\sigma_3$$  \hspace{1cm} (3.12)

where $c_1, c_2, c_3$ are arbitrary functions on $\Sigma$. The background auxiliary scalar $M$ is

$$-\frac{1}{2} \left( \frac{1}{6} R + M \right) = \frac{1}{4f_1^2} - \frac{1}{4f_2^2} + \frac{f_3^3}{4f_1^2 f_2^2}$$

$$+ c_1 \left( -\frac{\cos \rho}{4f_1} + \frac{3 \cos \rho}{4f_2} - \frac{\cot 2\theta f_3}{2f_1 f_2} + \frac{\sin \rho \partial_\rho f_1}{4f_1 f_2} - \frac{\partial_\theta f_3}{4f_1 f_2} \right) + \frac{\sin \rho \partial_\rho c_1}{4f_2}$$

$$+ c_2 \left( -\frac{\cot 2\theta}{2f_1} + \frac{\cos \rho f_3}{4f_1 f_2} - \frac{\partial_\theta f_2}{4f_1 f_2} \right) - \frac{1}{4f_1} \partial_\theta c_2 - \frac{1}{16} \sin^2 \rho \left( c_1^2 + c_2^2 + c_3^2 \right)$$  \hspace{1cm} (3.13)
3.5. **Four-sphere.** A topological four-sphere $X_4 = S^4_{e_1, e_2}$ with $T^2$ invariant metric can be presented as a local $T^2_{w_1, w_2}$ bundle fibered over a two-dimensional digon $\Sigma_2$. One of the cycles of $T^2_{w_1, w_2}$ collapses at one edge of the digon, and the other cycle collapses at the other edge:

![Diagram of S^4 and T^2 with \Sigma_2]

The coordinates $(\theta, \rho)$ on the base $\Sigma_2$ are in the range $(\theta, \rho) \in [0, \frac{\pi}{2}] \times [0, \pi]$. The $w_1$ cycle collapses at $\theta = \frac{\pi}{2}$ and the $w_2$ cycle collapses at $\theta = 0$. Both circles collapse in the corners of the digon. The corner $\rho = 0$ will be called the north pole, and the corner $\rho = \pi$ will be called the south pole. The metric on $S^4_{e_1, e_2}$ is smooth at the cusps and the edges of the digon $\Sigma_2$ if the functions $f_i(\theta, \rho)$ satisfy asymptotically

\[
\begin{align*}
  f_1(\theta, \rho)|_{\theta=0} &= \epsilon_2^{-1}, & f_1(\theta, \rho)|_{\theta=\pi} &= \epsilon_1^{-1} \\
  f_1(\theta, \rho)|_{\rho=0,s} &= (\epsilon_1^{-2} \sin^2 \theta + \epsilon_2^{-2} \cos^2 \theta)^{\frac{1}{2}} \\
  f_2(\theta, \rho)|_{\rho=0,s} &= (\epsilon_1 \epsilon_2 f_1(\theta, \rho))^{-1} \\
  f_3(\theta, \rho)|_{\rho=0,s} &= \pm(\epsilon_2^{-2} - \epsilon_1^{-2}) f_1(\theta, \rho)^{-1} \cos \theta \sin \theta
\end{align*}
\]

(3.14)

The metric along $\Sigma_2$ is arbitrary in the interior. In particular, taking $f_2(\theta, \rho)$ very large in the interior, it is possible to stretch $S^4_{e_1, e_2}$ to a very long cylinder with two hemispherical caps attached at the ends. Localization on this geometry presumably is related to the convolution of the ground state topological wave functions with its conjugate by cutting the $S^4_{e_1, e_2}$ in the middle at $\rho = \frac{\pi}{2}$, as in the AGT correspondence [33] with Liouville theory and quantum Teichmüller theory [46, V:9], and Nekrasov-Witten construction [47].

The background fields (3.9)(3.10) and the spinor (3.6) with generic smooth functional parameters $c_1, c_2, c_3$ (3.11)(3.12) is a supersymmetric background only in the interior of $\Sigma_2$. The coordinates are singular at the north and the south poles $\rho = 0$ and $\rho = \pi$. We need to ensure that the spinor $\epsilon$ and the background fields $V$ and $T$ are smooth in a proper coordinate system around the poles. At the north pole $\rho = 0$ we choose approximately Cartesian coordinates

\[
\begin{align*}
  x_1 &= 2\epsilon_1^{-1} \tan \frac{\theta}{2} \cos \theta \cos \phi_1 \\
  x_2 &= 2\epsilon_1^{-1} \tan \frac{\theta}{2} \cos \theta \sin \phi_1 \\
  x_3 &= 2\epsilon_2^{-1} \tan \frac{\theta}{2} \sin \theta \cos \phi_2 \\
  x_4 &= 2\epsilon_2^{-1} \tan \frac{\theta}{2} \sin \theta \sin \phi_2
\end{align*}
\]

(3.15)

with the standard frame $e^\mu_\mu = \delta^\mu_\mu$. In the $x$-frame the spinor (3.6) becomes

\[
  x_\xi = e^{\text{-}\frac{\gamma_{11} e^{\phi_{11}}}{\epsilon_1} - \frac{\gamma_{12} e^{\phi_{12}}}{\epsilon_2} - \frac{\gamma_{14} e^{\phi_{14}}}{\epsilon_1} - \frac{\gamma_{24} e^{\phi_{24}}}{\epsilon_2} \gamma_{34} \xi}, \quad \text{with} \quad \sin \beta = \frac{\epsilon_2^{-1} \sin \theta}{\epsilon_1 \sin^2 \theta + \epsilon_2^{-2} \cos^2 \theta}
\]

(3.16)
The spinor \( x\varepsilon \) is not smooth in the \( x \) frame but is \( SU(2)_R \) gauge equivalent to the conformal class of the standard smooth spinor in the Omega-background [9] (up to the Weyl transformation \( x\varepsilon_\Omega \to x\varepsilon_\Omega \cos \frac{\theta}{2} \)):

\[
x\varepsilon_\Omega := (\varepsilon_\varsigma - \frac{1}{2} \Omega_{\mu\nu} x^\mu \gamma^\nu \varepsilon_\varsigma) \cos \frac{\theta}{2}
\]

(3.17)

where non-zero components of \( \Omega \) are \( \Omega_{12} = -\Omega_{12} = \epsilon_1 \) and \( \Omega_{34} = -\Omega_{43} = \epsilon_2 \). Namely, for \( \varepsilon_\varsigma = (1 + i)(0, 0, 1, 0) \) we find

\[
x\varepsilon_\Omega^1 = (1 + i)(\cos \frac{\theta}{2})(-\tan \frac{\theta}{2} \sin \theta e^{i\phi_2}, -\tan \frac{\theta}{2} \cos \theta e^{i\phi_1}, 1, 0) \\
x\varepsilon_\Omega^1 = (1 + i)(\cos \frac{\theta}{2})(-\tan \frac{\theta}{2} \cos \frac{\beta + \theta}{2} e^{i\phi_2}, -\tan \frac{\theta}{2} \cos \frac{\beta + \theta}{2} e^{i\phi_1}, \cos \frac{\beta - \theta}{2}, -\sin \frac{\beta - \theta}{2} e^{i(\phi_1 + \phi_2)})
\]

with \( \varepsilon^2 \) found by Majorana conjugation \( \varepsilon^2 = \varepsilon^* \).

The \( SU(2)_R \) gauge transformation relating spinor \( x\varepsilon \) (3.6) and the Omega-background spinor \( \varepsilon_\Omega \) near the north pole is

\[
\varepsilon_\Omega^i = U_j^i \varepsilon^j,
\]

\[
U = \begin{pmatrix}
\cos \frac{\theta - \beta}{2} & \frac{i e^{i(\phi_1 + \phi_2)} \sin \frac{\beta - \theta}{2}}{\cos \frac{\beta - \theta}{2}} \\
\frac{i e^{-i(\phi_1 + \phi_2)} \sin \frac{\beta - \theta}{2}}{\cos \frac{\beta - \theta}{2}} & \cos \frac{\theta - \beta}{2}
\end{pmatrix}
\]

(3.18)

Requiring that \( SU(2)_R \) gauge field \( UV = UdU^{-1} + UVU^{-1} \) (3.8)(3.10)(3.12) is smooth at the origin, and that components \( T_{\mu\nu} \) are well defined in the \( x \) frame, we find the parameters

\[
c_1 = \left(\frac{1}{f_1} - \frac{1}{f_2}\right) \varphi(\rho), \quad c_2 = -\frac{f_3}{f_1 f_2} \varphi(\rho), \quad c_3 = 0
\]

(3.19)

where \( \varphi(\rho) \) is any smooth function such that \( \varphi(\rho)_{\rho=0} = 1 + O(\rho^2) \) and \( \varphi(\rho)_{\rho=\pi} = -1 + O(\rho^2) \). Then the gauge field \( UV \) is smooth everywhere and \( T_{\rho=0} = T^-, T_{\rho=\pi} = T^+ \).

In our conventions the spinor \( \varepsilon \) is of positive chirality at the north pole (transforms under self-dual spacial rotations) and negative-chirality at the south pole (transforms under anti-self-dual rotation). In the zeroth order approximation the theory around the north pole is topological Donaldson-Witten theory that localizes to configurations \( F^+ = 0 \), and the theory around the south pole is conjugated and localizes to configurations \( F^- = 0 \). In the first order approximation the theory around poles is equivalent to the theory in the Omega-background, and localizes respectively to the equivariant instantons \( F^+ = 0 \) at the north pole and equivariant anti-instantons \( F^- = 0 \) around the south pole.

With the choice (3.19) at \( \rho = 0 \) we find that non-zero components of \( T = T^- \) are

\[
T_{12} = -T_{34} = \epsilon_1 - \epsilon_2
\]

(3.20)

If the geometry in the neighborhood of the north pole is approximated by the embedded ellipsoid in \( \mathbb{R}^5 \) with radius \((r_1, r_1, r_2, r_2, r)\) for \( r_1 = \epsilon_1^{-1}, r_2 = \epsilon_2^{-1} \) as in [32] then the
curvature of the \( SU(2)_R \) background field at \( \rho = 0 \) is particularly simple

\[
F_V = \left( \frac{-2r^2 + r_1^2 + r_2^2}{4r_1^2 r_2^2} (dx_1 \wedge dx_3 - dx_2 \wedge dx_4) \right) i\sigma_1 \\
+ \left( \frac{-2r^2 + r_1^2 + r_2^2}{4r_1^2 r_2^2} (dx_1 \wedge dx_4 + dx_2 \wedge dx_3) \right) i\sigma_2 \\
+ \left( \frac{r_1^2 - r_2^2}{2r_1^4} dx_1 \wedge dx_2 + \frac{r_2^2 - r_1^2}{2r_2^4} dx_3 \wedge dx_4 \right) i\sigma_3
\]  

(3.21)

One can compare \( F_V \) with the metric curvature at the north pole and notice the difference: the \( SU(2)_R \) background field differs from the usual topologically twisted theory. The non-zero metric curvature components in the \( x \)-frame at the north pole are

\[
R_{12}^{12} = \frac{r^2}{r_1^2}, \quad R_{34}^{34} = \frac{r^2}{r_2^2}, \quad R_{24}^{24} = R_{13}^{13} = R_{14}^{14} = R_{23}^{23} = \frac{r^2}{r_1^2 r_2^2} \quad (3.22)
\]

3.6. Superconformal Index. For this geometry the base is the product of an interval and the circle \( \Sigma_2 = I_\langle(\theta)\rangle \times S^1_\langle(\rho)\rangle \). At the ends of the interval \( I \) the two circles of \( T \) collapse. The slice of \( X_4^{\epsilon_1,\epsilon_2} \) at fixed \( \rho \) is topologically an \( S^3_{\epsilon_1,\epsilon_2} \), and then \( X_4 = S^3_{\epsilon_1,\epsilon_2} \times S^1 \). A suitable \( SU(2)_R \) background field ensures existence of unbroken supercharge. The partition function on \( S^3_{\epsilon_1,\epsilon_2} \times S^1 \) computes the superconformal index \([48–52]\) and \([V:5]\) section 4.1 and \([V:11]\) in this volume.

3.7. Other geometries. It would be interesting to study more general four-manifolds with the structure of local \( T^2 \) bundle such as \( S^2 \times S^2 \) or \( T^2 \times \Sigma_2 \) where \( \Sigma_2 \) is a Riemann surface.

4. Localization

Often a supersymmetric quantum field theory with a particular choice of the supercharge can be interpreted as infinite-dimensional version of the Cartan model for \( G \)-equivariant cohomology on the space of fields of the theory, see e.g. \([8, 53]\). The supercharge \( Q \) plays the role of the equivariant differential. The path integral is interpreted as the infinite-dimensional version of Mathai-Quillen form for the Thom class of the BPS equations bundle over the space of fields \([54, 55]\). For example, in the Donaldson-Witten topological gauge theory \([8]\), the space of fields is the infinite-dimensional affine space of connections \( A \) in a given principal \( G \)-bundle on a four-manifold \( X_4 \) for a compact Lie group \( G \), the group \( \mathcal{G} \) of the equivariant action is the infinite-dimensional group of gauge transformations, and the fibers of the equation bundle over \( A \) is the space of self-dual adjoint valued two-forms. The Mathai-Quillen form for the Thom class, with a choice of section \( F_A^+ \), localizes to the zeroes of the section: instanton configurations. The construction is equivariant with respect to the \( \mathcal{G} \) action on \( A \). The path integral over \( \mathcal{A}/\mathcal{G} \) reduces to the integration over the instanton moduli space \( \mathcal{M}_{\text{inst}} = \{ A|F_A^+ = 0\}/\mathcal{G} \).
4.1. **Omega background.** See [9–12, 44] and [V:3] in this volume. A conventional 4d $\mathcal{N} = 2$ theory with Lagrangian formulation is specified by the choice of a compact semi-simple Lie group $G$ for the gauge group and a representation $R$ of $G$ for the hypermultiplet matter. The automorphism group of the representation $R$ is the flavor group $F$. The path integral of $\mathcal{N} = 2$ theory in Omega background $\mathbb{R}^4_{t_1,t_2}$ localizes to the equivariant form on moduli space of instantons, further integration over moduli space is localized to the fixed points the equivariant group action. The equivariant group $G = L \times G \times F$ is the product of the isometry of the space-time $L = SO(4)$, the gauge group $G$ that acts on the framing at infinity, and the flavour group $F$. Let $\mathcal{T}$ be the maximal torus of the equivariant group $T = T_k \times T_G \times T_F$. The coordinates on the complexified Lie algebra of $T$ are $(\epsilon, a, m)$. Physically, the parameters $a$ are the asymptotics at the space-time infinity of the scalar field $\Phi$ in the gauge vector multiplet, the parameters $m$ are the matter fields masses, and the parameters $\epsilon = (\epsilon_1, \epsilon_2)$ are the equivariant space-time rotation angular momenta, the $\Omega$-background parameteres. In our conventions the subscript $\mathcal{T}$ denotes the dependence on $(\epsilon, a, m)$.

The partition function $Z$ in the Omega background can be represented as a product of the classical, perturbative and non-perturbative contributions:

$$Z_{\mathcal{T}}(q) = Z_{\mathcal{T}}^{\text{tree}} Z_{\mathcal{T}}^{1\text{-loop}} Z_{\mathcal{T}}^{\text{inst}}(q)$$

Formally,

$$Z_{\mathcal{T}}(q) = \sum_k q^k \int_{\mathcal{A}/G_{\text{gauge}}} \text{eu}_\mathcal{T}(\Omega^{2+} \otimes g) \text{eu}_\mathcal{T}((S^- \otimes S^+) \otimes R)$$

where $\mathcal{A}$ is the infinite-dimensional space of $G$-connections on a principal $G$-bundle $E \to M$ with fixed trivialization at infinity, $G_{\text{gauge}} = \text{Aut}(E)$ is the group of gauge transformations equal to identity at the space-time infinity, $\Omega^{2+} \otimes g$ is the infinite-dimensional vector bundle over $\mathcal{A}$ with the fiber being the space of self-dual $g$-valued two-forms, $S^{\pm} \otimes R$ is the infinite-dimensional vector bundle over $\mathcal{A}$ with the fiber being the space of positive/negative chirality $R$-valued spinors.

Mathematically, the instanton partition function is

$$Z_{\mathcal{T}}^{\text{inst}}(q) = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}_k} \text{eu}_\mathcal{T}(\mathcal{E}_R).$$

Here we are assuming that $G = \times_{i \in I} G_i$ where $G_i$ are simple factors and $I$ denotes the set of labels for the simple gauge group factors, $q = \{q_i | i \in I\}$ is the $|I|$-tuple of the exponentiated complexified gauge coupling constants $q_i = \exp(2\pi i \tau_i)$, $k = \{k_i | i \in I\}$ is an $n$-tuple of non-negative integers, $k_i$ is the instanton charge (second Chern class)$^4$

$^4$For a generic compact simple Lie group $G$ the integer $k$ classifies the topology of $G$-bundle on $S^4$ by $\pi_3(G) = \mathbb{Z}$. The instanton number $k$ can be computed as

$$k = \frac{1}{8\pi^2} \int_M \langle F, \wedge F \rangle = -\frac{1}{16\pi^2 h^\vee} \int_M \text{Tr}_{\text{adj}} F \wedge F$$

in the conventions where $F$ is $g$-valued two-form, $\langle , \rangle$ is the invariant positive definite bilinear form on $g$ induced from the standard bilinear form on $\mathfrak{h}^*$ in which long roots have length squared 2, the $\text{Tr}_{\text{adj}}$ is the trace in adjoint representation, and $h^\vee$ is the dual Coxeter number for $g$. For $G = SU(n)$ the instanton charge $k$ is the second Chern class $k = c_2$. 
of $G_i$-bundle on the space-time $M = \mathbb{R}^4 = S^4 \setminus \infty = \mathbb{CP}^2 \setminus \mathbb{CP}^1$ with fixed framing at infinity, the $\mathcal{M}_k = \times_{i \in I} \mathcal{M}_{G_i,k_i}$ is the instanton moduli space: $\mathcal{M}_{G_i,k_i}$ is moduli space of the anti-self-dual $G_i$-connections on $\mathbb{R}^4$ with the second Chern class $k_i$. The integration measure $\text{eu}_T(\mathcal{E}_R)$ is the $T$-equivariant Euler class of the matter bundle $\mathcal{E}_R \to \mathcal{M}_k$ where a fiber of $\mathcal{E}_R$ is the space of the virtual zero modes for the Dirac operator: $\Gamma(S^- \otimes R) \to \Gamma(S^+ \otimes R)$ associated to the hypermultiplet.

Here the classical contribution is

$$Z_T^{\text{tree}}(q) = \exp \left( -\frac{1}{2\epsilon_1 \epsilon_2} \sum_{i \in I} 2\pi i \tau_i \langle a_i, a_i \rangle \right) \quad (4.4)$$

where $\langle \rangle$ is the standard bilinear form on the Lie algebra of $G_i$ normalized such that the long root length squared is 2, and $\tau$ is the complexified coupling constant

$$\tau = \frac{4\pi \imath}{g^2_{YM}} + \frac{\theta}{2\pi} \quad (4.5)$$

Let

$$R = \bigoplus \ell \ R_\ell \otimes M_\ell \quad (4.6)$$

be the decomposition of the matter representation onto the irreducible, with respect to $G$, components, with the multiplicity spaces $M_\ell \simeq \mathbb{C}^{N_\ell}$, on which the masses have the value $m_{\ell,1}, \ldots, m_{\ell,N_\ell}$.

The one-loop contribution is expressed in terms of the special function related to Barnes double gamma function

$$G_{\epsilon_1,\epsilon_2}(x) = \text{Reg}[ \prod_{n_1,n_2 \geq 0} (x + n_1 \epsilon_1 + n_2 \epsilon_2) ] \quad (4.7)$$

where $\text{Reg}[\cdots]$ denotes regularization of the infinite product with Weierstrass multipliers.

We find the one-loop factors for the theory in the Omega background for vector multiplet and hypermultiplet to be given by

$$Z_T^{\text{1-loop;vec}}(a;m) = \prod_i \prod_{\alpha \in \Delta_i^+} G_{\epsilon_1,\epsilon_2}(\alpha \cdot a_i) G_{\epsilon_1,\epsilon_2}(\epsilon_1 + \epsilon_2 - \alpha \cdot a_i) \quad (4.8)$$

$$Z_T^{\text{1-loop;hyper}}(a;m) = \prod_\ell \prod_{f=1}^{N_\ell} \prod_{w \in P(R_\ell)} G_{\epsilon_1,\epsilon_2}(w \cdot a_i + m_{ij} + \frac{1}{2}(\epsilon_1 + \epsilon_2))^{-1}$$

Here $\Delta_i^+$ denotes the set of positive roots for the $i$-th gauge group factor $G_i$ and $P(R_\ell)$ denotes the set of weights for the irreducible representation $R_\ell$. These expressions follow from the equivariant index computation by Atiyah-Singer formula for the self-dual complex and the Dirac complex respectively. The Aityah-Singer formula for the equivariant index of complex $C$ evaluated at the group element $g \in \mathbb{G}$

$$\text{ind}(C;g) = \sum_{f \in \text{fixed points}} \frac{\text{tr}_{C_f}(g)}{\det_T(1 - g)} \quad (4.9)$$
For the self-dual complex
\[
\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2^+ \quad (4.10)
\]
on \mathbb{R}^4 \cong \mathbb{C}^2(\mathbb{C},\mathbb{C}) under the equivariant action \( z_1 \rightarrow t_1 z_1, \quad z_2 \rightarrow t_2 z_2 \) we find that each two-dimensional weight space of root \( \alpha \) and its conjugate \( -\alpha \) contributes as
\[
\omega + \bar{\omega} + t_1 t_2 \omega + \bar{t}_1 \bar{t}_2 \bar{\omega} - (t_1 \omega + \bar{t}_1 \bar{\omega} + t_2 \omega + \bar{t}_2 \bar{\omega}) \left( \frac{1}{(1 - t_1)(1 - t_1)(1 - t_2)(1 - t_2)} \right) = \omega \left( \frac{1}{(1 - t_1)(1 - t_1)} + \bar{\omega} \frac{1}{(1 - t_1)(1 - t_2)} \right) \quad (4.11)
\]
where
\[
\omega = e^{i\alpha \cdot a}, \quad t_1 = e^{i\epsilon_1}, \quad t_2 = e^{i\epsilon_2} \quad (4.12)
\]
Taking \( \bar{t}_1 = t_1^{-1}, \bar{t}_2 = t_2^{-1} \) and expanding the index in positive powers of \( t_1, t_2 \) one finds the Chern character (the sum). Converting the Chern character (the product) into the Euler character (the product) we find (4.8).

For the Dirac complex associated with the hypermultiplet of mass \( m \), the weight space \( w \) contributes as
\[
\frac{\omega^1 \omega^2}{(1 - t_1)(1 - t_2)} \quad (4.13)
\]
where \( \omega = e^{iw \cdot a}, \mu = e^{im} \). This can be seen from Atiyah-Singer formula for the Dirac complex \( S^+ \xrightarrow{D} S^- \) with numerator \( t_1^1 t_2^1 + \bar{t}_1^1 \bar{t}_2^1 - \bar{t}_1^1 \bar{t}_2^1 - t_1^1 \bar{t}_2^1 \) or from the fact that Dirac complex is the twist of Dolbeault complex twisted by the square root of the canonical bundle. Again, expanding in positive powers of \( t_1, t_2 \) and converting the sum to the product we find the equivariant Euler character, or the one-loop determinant (4.8) for the hypermultiplet.

The explicit expression for \( Z_{\text{inst}}(q) \) can be found for example in [22], [44] and in Y. Tachikawa’s review [V:3] in this volume.

4.2. Supersymmetric configurations on \( S^4_{t_1, t_2} \). The path integral for the partition function of QFT with an action \( S \) invariant under a fermionic symmetry \( \delta S = 0 \) localizes near the supersymmetric configurations, which are the field configurations invariant under \( \delta \). In other words the supersymmetric configurations are the zeroes of the odd vector field \( \delta \) in the space of all field configurations [8]. The localization theorem is the infinite-dimensional generalization of the Atiyah-Bott formula [5] for the integration of the equivariantly closed differential forms over a manifold on which a compact Lie group \( G \) acts
\[
\int_M \alpha = \int_F \frac{i_F \alpha}{e(N_F)} \quad (4.14)
\]
where \( F \subset M \) is the fixed point locus of \( G \) action on \( M \) and \( e(N_F) \) is the equivariant Euler class of the normal bundle to \( F \).

From the analysis of the equations (2.8), similar to the \( S^4 \) [23] and the ellipsoid case [32] we expect that the only smooth field configurations that satisfy \( \delta \lambda = 0 \) for the topologically trivial gauge bundle is the trivial gauge field, vanishing scalar \( \Phi^5 \),
constant scalar $\Phi^6 = \text{const} = \Phi_6$ and a suitable auxiliary field $Y^i_j$ proportional to $\Phi^6$. It is easy to see that such a solution exists. Under the ansatz $F_{mn} = 0, \Phi_5 = 0$ the equations turn into an overdetermined algebraic linear system of equations on $\Phi_6$ and $Y^i_j$, and this system has one-dimensional kernel corresponding to the zero mode of $\Phi_6$. What is more difficult to show is the absence of other solutions, and presumably this can be shown similarly to the analysis in [56].

With the ansatz $F_{mn} = 0, \Phi_5 = 0$ and all fermions set to zero, we find the explicit supersymmetric configuration invariant under the $\delta_{\epsilon, \eta}$ (2.8)

$$Y^i_j = \hat{Y}^i_j \Phi_6, \quad \hat{Y}^i_j = \left( \left( \frac{1}{2f_1} - \frac{\varphi'(\rho) \cos \rho}{4f_1} + \frac{\varphi'(\rho) \cos \rho}{4f_2} \right) \sigma_3^i \right)_j + \left( \frac{f_3}{2f_1f_2} - \frac{\varphi'(\rho) \cos \rho f_3}{4f_1f_2} \right) \left( \epsilon_2^2(i\phi_1 + i\phi_2) \sigma_2 \epsilon - \frac{1}{2}(i\phi_1 + i\phi_2) \right)_j (4.15)$$

It is straightforward to evaluate the classical action on the supersymmetric configuration (4.15) and find

$$S|_{\text{susy conf}} = -\frac{1}{g_{YM}^2} \text{tr} \Phi_6^2 \int_{S^4_{\epsilon_1, \epsilon_2}} \sqrt{g} d^4 x \left( \frac{1}{6} R + M + 2\hat{Y}^i_j \hat{Y}^j_i - \frac{1}{16} T_{\mu\nu} T^{\mu\nu} \right) (4.16)$$

From the explicit solution for the $T_{\mu\nu} (3.9)(3.11)(3.19)$, the $\hat{Y}^i_j (4.15)$ and $M (3.13)$ we find that most terms in the action combine into total derivative

$$\sqrt{g}((\frac{1}{6} R + M) + 2\hat{Y}^i_j \hat{Y}^j_i - \frac{1}{16} T_{\mu\nu} T^{\mu\nu}) =$$

$$= f_1 f_2 r_1 r_2 \sin^3 \rho \sin \theta \cos \theta \left( \frac{\varphi f_3 \partial_\rho f_2}{2f_1 f_2^3} - \frac{\varphi \partial_\rho f_1}{2f_1 f_2^3} + \frac{\varphi \partial_\rho f_3}{2f_1 f_2^3} - \frac{\varphi \partial_\rho f_2 \sin \rho}{2f_2} \right)$$

$$+ \frac{\varphi f_3 \tan \theta}{2f_1 f_2^2} - \frac{\varphi f_3 \cot \theta}{2f_1 f_2^2} - \frac{\varphi \partial_\rho \sin \rho}{2f_1 f_2} + \frac{2 \varphi \cos \rho}{f_1 f_2} + \frac{3 \varphi \cos \rho}{f_2} + \frac{2 \varphi \cos \rho}{f_2}$$

$$= -\partial_\rho \left( r_1 r_2 \varphi \sin 2\theta \sin^3 \rho \frac{f_3}{4f_2} \right) + \partial_\rho \left( r_1 r_2 \varphi \sin 2\theta \sin^4 \rho \frac{f_3}{4f_2} \right) + \frac{3}{2} \varphi r_2 \sin 2\theta \sin^3 \rho$$

The last term is the only term non-vanishing after integration over $S^4_{\epsilon_1, \epsilon_2}$. It gives

$$S|_{\text{susy conf}} = -\frac{8\pi^2 r_1 r_2}{g_{YM}^2} \text{tr} \Phi_6^2 (4.18)$$

Therefore, the contribution from the smooth configuration of the localization locus for the partition function is

$$Z_{\text{pert}}^{\text{susy conf}} = \int d\Phi_6 e^{-S|_{\text{susy conf}} Z_{1\text{-loop}}(\Phi_6)} = \int d\Phi_6 e^{-\frac{1}{g_{YM}^2} \frac{8\pi^2}{g_{YM}^2} \langle \Phi_6, \Phi_6 \rangle Z_{1\text{-loop}}(\Phi_6)}$$

\[5\]In our conventions $\Phi_6$ is an element of the Lie algebra of the gauge group. For $U(N)$ gauge group $\Phi_6$ is represented by anti-Hermitian matrices. The bilinear form $\langle , \rangle$ is the positive definite invariant metric on the Lie algebra normalized such that the length squared of the long root is 2. For $U(N)$ group $\text{tr} \Phi^2 = -\langle \Phi, \Phi \rangle$. 




where $Z_{1\text{-loop}}(\Phi_6)$ needs to be computed from the fluctuations of the quantum fields around the supersymmetric background. Since mathematically such determinant is the same as a certain infinite-dimensional equivariant Euler class as in the equation (4.14), it can be computed [23] using equivariant Atiyah-Singer index theorem for the transversally elliptic operators [57]. The Atiyah-Singer index theorem computes the index as the sum of the contributions from the fixed points: the north and the south pole of the $S^4_{\epsilon_1,\epsilon_2}$. The result is that the $Z_{1\text{-loop}}$ factorizes into the product of two factors, each related to the one-loop factor $Z^{1\text{-loop}}_T$ (4.8) of the gauge theory partition function in the Omega background coming from the north or the south pole of the $S^4_{\epsilon_1,\epsilon_2}$. Careful application of Atiyah-Singer index theorem for the transversally elliptic operator shows that the north pole contributes the factor $Z^{1\text{-loop}}_T$ obtained from the expansion of the index in the positive powers of the equivariant parameters $t_1 = e^{i\epsilon_1}, t_2 = e^{i\epsilon_2}$ (4.8). The contribution of the south pole is obtained from the expansion of the index in the negative powers of the equivariant parameters.

The argument of $Z^{1\text{-loop}}_T$, the equivariant parameter $a$ of the gauge theory in the Omega background, relates to the scalar fields on $S^4_{\epsilon_1,\epsilon_2}$ in the way

$$a = v^a \Phi_a$$  \hspace{0.5cm} (4.20)$$

where $v^a$ is the vector field (3.7). At the north pole for the supersymmetric configuration we find

$$a = -i\Phi_6 - \Phi_5 = -i\Phi_6$$  \hspace{0.5cm} (4.21)$$

On $X^4$ it is natural to assume the mass parameter pure imaginary, since the mass can be thought as the fixed background value of the scalar field $\Phi_6$ in the vector multiplet of gauged flavour symmetry, so for convenience we set that mass parameters on $X^4$ are $im$ where $m$ is real. Then, up to an overall phase, and assuming that the arguments $\epsilon_1$ and $m_{H}$ in (4.8) are pure imaginary we find

$$Z_{1\text{-loop}} S^4_{\epsilon_1,\epsilon_2} = Z_{1\text{-loop}, T}(ia, im)$$  \hspace{0.5cm} (4.22)$$

The classical contribution also factorizes, and using (4.4), (4.5) we find the partition function (4.19) can be rewritten as

$$Z_{S^4_{\epsilon_1,\epsilon_2}}^{\text{pert}} = \int |da| Z_{T}^{\text{pert}}(\epsilon_1, \epsilon_2; ia, im)^2$$  \hspace{0.5cm} (4.23)$$

The above formula for the partition function takes into account only the perturbative contribution in the localization computation around the smooth solution of the supersymmetric equations. However, the complete partition function on $S^4_{\epsilon_1,\epsilon_2}$ is also contributed by the point like instanton/anti-instanton configurations, with point instantons supported at the north pole and the point anti-instantons supported at the south pole [23]. This follows from the analysis of the asymptotics of the localization equations near the north and south poles: the supersymmetric theory on $S^4_{\epsilon_1,\epsilon_2}$ near the north pole is approximated by the gauge theory in the Omega-background, and the supersymmetric theory on $S^4_{\epsilon_1,\epsilon_2}$ near the south pole is approximated by the conjugated version of the gauge theory in the Omega-background. This argument leads to
the complete formula

$$Z_{S^{1,1}, \epsilon} = \int [da] |Z_T(\epsilon_1, \epsilon_2; ia; im; q)|^2$$  \hspace{1cm} (4.24)

4.3. **Hypermultiplets.** The treatment of conformal massless hypermultiplets is straightforward and is similar to [32]. The mass-term are added by gauging the flavour symmetry, introducing the vector-multiplet for the flavour-symmetry group and then freezing all the fields of this flavour-symmetry vector field to zero except the constant scalar field \((\Phi_0)_{\text{flavour}}\) which then plays the role of the mass parameter.

4.4. **Open problem.** It should be possible to classify all possible \(T^2\)-bundle solutions to generalized conformal Killing equations, construct the supersymmetric theories on such backgrounds and localize the partition function generalizing the result (4.24).

**Appendix A. Conventions and useful identities**

A.1. **Indices.** For the 4d theories with 8 supercharges (\(N = 2\) supersymmetry in 4d) we use the notations of the \((0,1)\) 6d supersymmetric theories under the dimensional reduction. The table A.1 summarizes the index notations

| Type of indices | symbol | range | fields |
|----------------|--------|-------|--------|
| 4d space-time vectors | \(\mu, \nu, \rho, \sigma\) | \([1, \ldots, 4]\) | gauge field \(A_\mu\) |
| \(U(1)_R = SO(2)_R\) vectors | \(a, b\) | \([5, 6]\) | scalar field \(\phi_a\) |
| 6d vectors (the sum of above) | \(m, n, p, q\) | \([1, \ldots, 6]\) | \(A_m = (A_\mu, \phi_a)\) |
| \(SU(2)_R\) | \(i, j\) | 1, 2 | gaugino doublet \(\lambda^i\); auxiliary triplet \(Y^i_j\) |

The symmetrization and anti-symmetrization of tensors

\[
t_{(m_1 \ldots m_r)} = \frac{1}{r!} \sum_{\sigma \in \text{Perm}(r)} t_{m_{\sigma(1)} \ldots m_{\sigma(r)}}
\]

\[
t_{[m_1 \ldots m_r]} = \frac{1}{r!} \sum_{\sigma \in \text{Perm}(r)} (-1)^{\sigma} t_{m_{\sigma(1)} \ldots m_{\sigma(r)}} \hspace{1cm} (A.1)
\]

A.2. **Spinors.** The spinors \(\lambda\) and \(\varepsilon\) in the \((0,1)\) Euclidean supersymmetric 6d theory are the holomorphic \(SU(2)_R \simeq Sp(1)_R\) doublets of Weyl four-component spinors, of Weyl chirality +1, for the 6d Clifford algebra over complex numbers \(\mathbb{C}\). We take \(\lambda \equiv (\lambda^i)_{i=1,2}\) where each \(\lambda^1\) and \(\lambda^2\) is 6d Weyl fermion. In total the spinor \(\lambda\) has 8 complex components.\(^6\)

\(^6\)We construct the Lagrangian and supersymmetry algebra using only holomorphic/algorithmic dependence on the spinorial components. In other words, the complex conjugate of gaugino \((\lambda^i)\) never appears neither in the Lagrangian, nor in the measure of the path integral, nor in the supersymmetry transformations. The fermionic analogue of the *contour of integration* in the path integral or...
A.3. **Clifford algebra.** The $8 \times 8$ complex matrices $\gamma_m$ represent the 6d Clifford algebra
\[ \{\gamma_m, \gamma_n\} = 2g_{mn} \]  \hspace{1cm} (A.2)
The chirality operator $\gamma^\ast_6$ anticommuting with all $\gamma_m$ is
\[ \gamma_s = i\gamma_1 \ldots \gamma_6; \quad \{\gamma_s, \gamma_m\} = 0; \quad \gamma_s^2 = 1. \]  \hspace{1cm} (A.3)
The chirality of the spinors is the eigenvalue of $\gamma_s$. The projection operators that split $S = S^+ \oplus S^-$ are
\[ \gamma_\pm = \frac{1}{2}(1 \pm \gamma_s), \quad \varepsilon_\pm = \gamma_\pm \varepsilon_\pm = \pm \gamma_s \varepsilon_\pm \]  \hspace{1cm} (A.4)
Explicit form of $\gamma_m$ matrices is not needed, but for concreteness one can recursively define the $\gamma_m^{(d)}$ matrices of size $2^{d/2} \times 2^{d/2}$ in even dimension $d$ in terms of $\gamma_m^{(d-2)}$ as follows (see e.g. [58])
\[ \gamma_m^{(d)} = \sigma_3 \otimes \gamma_m^{(d-2)}, \quad m \in [1, \ldots, d-2] \]
\[ \gamma_{d-1}^{(d)} = \sigma_1 \otimes 1, \quad \gamma_d^{(d)} = \sigma_2 \otimes 1 \]
\[ \gamma_s^{(d)} = \sigma_3 \otimes \gamma_s^{(d-2)} \]  \hspace{1cm} (A.5)
where $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ are the $2 \times 2$ Pauli matrices
\[ (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}). \]  \hspace{1cm} (A.6)
We use antisymmetric multi-index notations
\[ \gamma_{m_1 \ldots m_r} = [\gamma_{m_1} \ldots \gamma_{m_r}], \]  \hspace{1cm} (A.7)
and we use underline notation for the multi-index
\[ \gamma_\underline{r} \]  \hspace{1cm} (A.8)
In the contraction of multi-index we use non-repetitive summation
\[ A^p B_\underline{r} \equiv \sum_{m_1 < \ldots < m_p} A^{m_1 \ldots m_p} B_{m_1 \ldots m_p} = \frac{1}{p!} A^{m_1 \ldots m_p} B_{m_1 \ldots m_p} \]  \hspace{1cm} (A.9)
For the forms we use component and slashed notation
\[ \omega \equiv \frac{1}{r!} \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}, \quad \phi = \omega_{\mu_1 \ldots \mu_r} \gamma^{\mu_1 \ldots \mu_r} \]  \hspace{1cm} (A.10)
Contraction identity
\[ \gamma^m \gamma_\underline{r} \gamma^m = (-1)^r (d-2r) \gamma_\underline{r} \]  \hspace{1cm} (A.11)
Multi-index contraction identity
\[ \gamma^2 \gamma_\underline{r} \gamma^p = \Delta(d, r, p) \gamma_\underline{r} \]  \hspace{1cm} (A.12)
with
\[ \Delta(d, r, p) = (-1)^p (p-1)^{1/2} (-1)^p \sum_{q=\max(p+r-d,0)}^{\min(r,p)} (-1)^q \binom{r}{q} \binom{d-r}{p-q} \]  \hspace{1cm} (A.13)
the reality condition is not necessary since evaluation the Pfaffian or top degree form is an algebraic operation.
The contraction formula and the completeness of \((\gamma^p)_{p \in [0, \ldots, d]}\) for \(d \in 2\mathbb{Z}\) in the matrix algebra of \(2^{d/2} \times 2^{d/2}\) matrices implies the Fierz identity
\[
(\gamma^L)_{\alpha_2}^{\alpha_1}(\gamma^L)_{\alpha_4}^{\alpha_3} = \sum_{k=0}^{d} \tilde{\Delta}(d, r, k)(\gamma^L)_{\alpha_4}^{\alpha_1}(\gamma^L)_{\alpha_2}^{\alpha_3}
\]
where
\[
\tilde{\Delta}(d, r, k) = (-1)^{k(k-1)/2} 2^{-d/2} \Delta(d, r, k)
\]
(A.15)

The terms with \(k > d/2\) in the Fierz identity are conveniently represented as
\[
\gamma_k \gamma^* = (-1)^{r(r-1)/2} i^{-n/2} \gamma_k^\vee
\]
(A.16)
where \(\gamma_k^\vee\) is complementary in indices of \(\gamma_k\) with a proper permutation sign. The Fierz identity is
\[
(\gamma^L)_{\alpha_2}^{\alpha_1}(\gamma^L)_{\alpha_4}^{\alpha_3} = \sum_{k=0}^{d/2} \tilde{\Delta}(d, l, k)(\gamma^L)_{\alpha_4}^{\alpha_1}(\gamma^L)_{\alpha_2}^{\alpha_3} + (-1)^{d/2} \sum_{k=0}^{d/2-1} \tilde{\Delta}(d, l, d-k)(\gamma^L\gamma^*)_{\alpha_4}^{\alpha_1}(\gamma^L\gamma^*)_{\alpha_2}^{\alpha_3}.
\]
(A.17)

This form is useful when applied to the chiral spinors.

A.4. Spinor bilinears. The spinor representation space \(S\) can be equipped with an invariant complex bilinear form \((,): S \otimes S \to \mathbb{C}\). In components we write
\[
(\eta\varepsilon) := \eta^\alpha C_{\alpha\beta} \varepsilon^\beta
\]
(A.18)
where \(C\) is a matrix representing the bilinear form.

All operators \(\gamma^\vee\) are symmetric or antisymmetric with respect to \(C\). The symmetry of \(C\gamma^\vee\) depends on the dimension \(d\) and is summarized in the table A.4

| \(d \mod 8\) | \(2\) | \(4\) | \(6\) | \(8\) |
|---|---|---|---|---|
| \(C_1\) | ++-- | -+-+ | [+-+] | ++-- |
| \(C_2\) | --++ | [+-+] | ++-- | ++-- |

The entries \(s_0s_1s_2s_3\) with \(s_r = \pm 1\) denote the transposition symmetry of \(C\gamma^\vee\) for \(r \mod 4\). There are two choices of \(C\) denoted by \(C_1\) and \(C_2\) in the table. In representation (A.5) one can take
\[
C_1 = \cdots \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1
\]

\[
C_2 = \cdots \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2.
\]
(A.19)

\(^7\)Often in the physics literature the dual spinor \(\eta_\beta = \eta^\alpha C_{\alpha\beta}\) (an element of the dual space \(S^\vee\)) is denoted \(\bar{\varepsilon}\) and is called Majorana conjugate to \(\varepsilon\). We have chosen here to avoid the bar notation to avoid confusion with complex conjugation.
The bilinear form $C_2$ for spinors in even dimension $d$ can be also used as the bilinear form for spinors in $d+1$-dimensions. For the theories with 8 supercharges in $d = 4, 5, 6$ dimensions we are using $C$ highlighted in the table A.4.

The matrices $C\gamma_\xi$ represent bilinear forms on $S$ valued in $r$-forms, in other words, for spinors $\eta$ and $\varepsilon$ the

$$\omega_\xi = (\eta \gamma_\xi \varepsilon)$$

(A.20)

transform covariantly as the rank $r$ form. Since

$$\gamma_\xi C = (-1)^{\frac{d}{2}} C \gamma_\xi$$

(A.21)

it follows that

$$\begin{align*}
(\eta \gamma_\xi \varepsilon) &= (\eta_+ \gamma_\xi \varepsilon_+) + (\eta_- \gamma_\xi \varepsilon_-), \\
&\quad \frac{d}{2} + r \in 2\mathbb{Z}
\end{align*}$$

(A.22)

In $d = 6$ the bilinears in the spinors of the same chirality transform as forms of odd rank; while the bilinears in the spinors of opposite chirality transform as forms of even rank.

$$d = 6 : \begin{cases} (\varepsilon_+ \gamma_\xi \varepsilon'_+ \neq 0 \text{ only for } r \in \{1, 3, 5\}) \\ (\eta_- \gamma_\xi \varepsilon_+) \neq 0 \text{ only for } r \in \{0, 2, 4, 6\} \end{cases}$$

(A.23)

The bilinear form valued in 1-forms is antisymmetric in $d = 6$ for either choice of $C$. To construct the standard fermionic action $(\lambda \gamma^m D_m \lambda)$ we need the symmetric 1-form valued bilinear form. For the minimal 6d (0, 1) supersymmetry we introduce a $SU(2)_R$ doublet of Weyl fermions $(\lambda_i^\pm)$ and then use $C \otimes \varepsilon$, where $\varepsilon = \epsilon_{ij}$ is the standard $2 \times 2$ antisymmetric symbol, as the symmetric bilinear form on the $S^+ \otimes \mathbb{C}^2$. The resulting 1-form valued bilinear is symmetric and there is a proper fermionic kinetic action

$$(\lambda^i \bar{D} \lambda_i) \equiv (\lambda^i \bar{D} \lambda^j) \epsilon_{ji}$$

(A.24)

We use the standard antisymmetric $2 \times 2$ tensor $\epsilon_{ij}$ to raise and lower the $SU(2)_R$ indices $i, j$ in the pattern $^i_i$:

$$\begin{align*}
\lambda^i := \epsilon_{ij} \lambda_j, \\
\lambda_j := \lambda^i \epsilon_{ij}, \\
\epsilon_{ij} \epsilon_{ik} &= \delta^j_k, \\
(\varepsilon_{ij} \eta^i) &= \frac{1}{2} \epsilon_{ij} (\varepsilon^k \eta_k)
\end{align*}$$

(A.25)

When the $SU(2)_R$ indices are omitted, the contraction $^i_i$ is assumed

$$(\varepsilon \gamma_\xi \varepsilon') \equiv (\varepsilon^i \gamma_\xi \varepsilon'_i)$$

(A.26)

A.5. $d = 6$ Fierz identities. For $d = 6$ and $l = 1$ we find

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $\tilde{\Delta}(6, 1, k)$ | $\frac{3}{4}$ | $-\frac{1}{2}$ | $-\frac{1}{4}$ | 0 | $-\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ |

$^8$Consistent with the fact that for $\frac{d}{2} \in 2\mathbb{Z}$ the tensor product $S^+ \otimes S^-$ contains odd rank forms; and $S^+ \otimes S_-, S_- \otimes S_- \otimes S_- \otimes S_-$ contains even rank forms; in particular for $\frac{d}{2} \in 2\mathbb{Z}$ the representation $S^\pm$ is dual to $S^\mp$; while for $\frac{d}{2} \in 2\mathbb{Z} + 1$ the representation $S^\pm$ is dual to $S^\mp$. 
Notice that \( \tilde{\Delta}(6,1,k) = (-1)^k \tilde{\Delta}(6,1,6-k) \). Therefore if we project Fierz identity (A.17) with \( \gamma_+ \) applied to the \( \alpha_2 \) and \( \alpha_4 \) indices, we find that terms with even \( k \) vanish. In addition the middle term \( k = 3 \) vanishes too. Finally

\[
(\gamma^4)_{\alpha_2} (\gamma_1)_{\alpha_3\alpha_4} = - (\gamma^4)_{\alpha_4} (\gamma_1)_{\alpha_3\alpha_2} \quad \text{projected by} \quad (\gamma_\pm)_{\alpha_2} (\gamma_\pm)_{\alpha_4} \tag{A.27}
\]

A frequently used form of the above identity involves cyclic permutation of three ++-chiral spinor doublets \( \epsilon^i, \kappa^i, \lambda^i \). Taking the sum of

\[
(\epsilon^i \gamma_m \kappa^i) \gamma^m \lambda^i = - (\epsilon^i \gamma_m \lambda^i) \gamma^m \kappa^i \\
(\lambda^i \gamma_m \kappa^i) \gamma^m \epsilon^i = - (\lambda^i \gamma_m \epsilon^i) \gamma^m \kappa^i \tag{A.28}
\]

we find

\[
(\epsilon \gamma_m \kappa^i) \gamma^m \lambda^i + (\lambda \gamma_m \kappa^i) \gamma^m \epsilon^i + (\epsilon \gamma_m \lambda^i) \gamma^m \kappa^i = 0 \tag{A.29}
\]

Now we consider projection of 6d Fierz identity at \( p = 1 \) on spinors of opposite chirality. Take ++-chiral doublet \( \epsilon^i \) and --chiral doublet \( \eta^i \). We find

\[
(\epsilon^j \gamma_1 \kappa_j) \gamma^i \eta^i = \frac{3}{2} (\epsilon^j \eta^i) \epsilon_i - \frac{1}{2} (\epsilon^j \gamma_2 \eta^i) \gamma_2 \epsilon_j \\
(\epsilon^j \gamma_2 \eta^i) \gamma_2 \epsilon_j = - \frac{5}{4} (\epsilon^j \gamma_1 \epsilon_i) \gamma_2 \eta^i \tag{A.30}
\]

where at \( l = 2 \) the explicit coefficients in (A.17) are given as follows:

| \( k \) | 0   | 1   | 2   | 3   | 4   | 5   | 6   |
|--------|-----|-----|-----|-----|-----|-----|-----|
| \( \tilde{\Delta}(6,2,k) \) | \(-\frac{15}{8}\) | \(-\frac{5}{8}\) | \(-\frac{3}{8}\) | \(+\frac{1}{8}\) | \(-\frac{5}{8}\) | \(+\frac{5}{8}\) |

Hence, from the equations (A.30) we find another useful 6d Fierz identity

\[
(\epsilon^j \gamma^m \epsilon_i) \gamma_m \eta^i = 4(\epsilon^j \eta^i) \epsilon_j, \quad (\epsilon = \gamma_+ \epsilon, \quad \eta = \gamma_- \eta) \tag{A.31}
\]

A.6. 6d (0, 1) theory conventions. The spinor \( \epsilon \) is ++-chiral, the spinor \( \eta \) is -- chiral

\[
\epsilon = \gamma_+ \epsilon, \quad \eta = - \gamma_+ \eta = \gamma_- \eta \tag{A.32}
\]

The tensor field \( T_{\mu \nu a} \) is 6d anti-self-dual, \( \ast_{6d} T = - T \). Useful contraction identities

\[
\gamma^\mu \hat{F}^\nu = \hat{F}^\nu \gamma^\mu - 4 F_{\mu \nu} \gamma^m, \quad \hat{T} = T_{\mu \nu a} \gamma^\mu \gamma^\nu \gamma^a \tag{A.33}
\]

\[
\gamma^\mu \hat{\gamma}_\mu = \gamma^a \hat{T}_{\gamma_a} = \gamma^m T_{\gamma_m} = 0 \tag{A.34}
\]

\[
T_{\mu \nu a} \gamma^{\mu \nu} = \frac{1}{2} \{ \hat{T}, \gamma_\alpha \}, \quad T_{\mu \nu a} \gamma^{\mu \nu} = \frac{1}{4} \{ \hat{T}, \gamma_\mu \} \tag{A.35}
\]

The Bianchi identity on the field strength

\[
D_m F_{pq} + D_q F_{mp} + D_p F_{qm} = 0, \quad \gamma^{mpq} D_m F_{pq} = 0 \tag{A.36}
\]
Positive chirality of $\varepsilon \equiv \varepsilon^+$ and negative chirality of $T \equiv T^-$ implies

$$T \gamma_2 \varepsilon = 0$$

$$(\varepsilon \gamma_2 \varepsilon) = 0, \quad r \mod 4 \in \{2, 3\}; \quad (\varepsilon^{(i)} \gamma_2 \varepsilon^{(j)}) = 0, \quad r \mod 4 \in \{0, 1\}$$

$$(A.37)$$

The spin-connection and the metric curvatures

$$\gamma_\mu \{ T, \gamma_\alpha \} = \{ T, \gamma_\rho \} \gamma_\alpha \varepsilon, \quad T_{\mu \nu \alpha} \gamma_{\rho} \gamma^{\mu \nu \varepsilon} = T_{\rho \mu \nu} \gamma^{\mu \nu \varepsilon}$$

$$(A.8).$$

$$(A.38)$$

The spin-connection and the metric curvatures

$$\gamma^{\mu} T_{\mu \nu \alpha} \gamma^{\mu \nu \varepsilon} = T_{\rho \mu \nu} \gamma^{\mu \nu \varepsilon}, \quad \gamma_\alpha T_{\mu \nu \beta} \gamma^{\mu \nu \varepsilon} = T_{\mu \nu \alpha} \gamma^{\mu \nu \varepsilon}$$

$$(A.7).$$

The bilinear form for the 6d Clifford algebra of type $(++)$, and the bilinear form in 4d and 6d notations agree:

$$D_{\mu} \{ V, \gamma_\alpha \} = 4 T_{\mu \alpha \beta} \gamma^{\mu \nu \varepsilon} = 4 T_{\mu \alpha \beta} \gamma^{\mu \nu \varepsilon}$$

The divergence of the first equation in the system $(2.1)$ implies

$$D^\mu D_\mu \varepsilon - \frac{1}{16} [D^\mu , T] \gamma_\mu \varepsilon - \frac{1}{2} T \eta = \frac{1}{4} \phi^2 \varepsilon$$

$$(A.40)$$

The spin-connection and the metric curvatures

$$\gamma^{\mu} T_{\mu \nu \alpha} \gamma^{\mu \nu \varepsilon} = T_{\rho \mu \nu} \gamma^{\mu \nu \varepsilon}, \quad \gamma_\alpha T_{\mu \nu \beta} \gamma^{\mu \nu \varepsilon} = T_{\mu \nu \alpha} \gamma^{\mu \nu \varepsilon}$$

$$(A.7).$$

The spin-connection and the metric curvatures

$$D_\mu \varepsilon = \partial_\mu \varepsilon + \frac{1}{4} \omega^{\mu \sigma \rho \varepsilon} \gamma_\rho \varepsilon + (V^R)^{i}_{j} \varepsilon^{j}$$

$$(A.39)$$

where $(V^R)^{i}_{j}$ is the $SU(2)$-connection.

A.7. Supersymmetry equations. The divergence of the first equation in the system $(2.1)$ implies

$$D^\mu D_\mu \varepsilon - \frac{1}{16} [D^\mu , T] \gamma_\mu \varepsilon - \frac{1}{2} T \eta = \frac{1}{4} \phi^2 \varepsilon$$

$$(A.40)$$

which together with Lichnerowicz formula $(A.39)$ produces

$$\frac{1}{4} \phi^2 \varepsilon + \frac{1}{3} \left( \frac{1}{4} R \varepsilon - \frac{1}{2} F^R \varepsilon - \frac{1}{16} [D^\mu , T] \gamma_\mu \varepsilon - \frac{1}{4} T \eta \right) = 0$$

$$(A.41)$$

and the linear combination with the second equation in $(2.1)$ produces

$$\frac{1}{4} \phi^2 \varepsilon = - \frac{1}{2} \left( \frac{1}{6} R + M \right) \varepsilon + \frac{1}{16} [D^\mu , T] \gamma_\mu \varepsilon$$

$$(A.42)$$

A.8. The 6d and 4d spinor conventions. As in $(A.5)$ we take

$$\gamma^{(6)}_\mu = (\gamma^{(4)}_\mu 0 0 - \gamma^{(4)}_\mu), \quad \gamma^{(6)}_5 = (0 1 0), \quad \gamma^{(6)}_6 = (0 i 0)$$

$$(A.43)$$

where $\varepsilon^{(4)}_\pm$ denote the $\pm$-chiral spinors of the 4d Clifford algebra with respect to $\gamma^{(4)}_\pm$, the $C^{(6)}$ is the bilinear form for the 6d Clifford algebra of type $(---++)$, and $c^{(4)}$ is the bilinear form of the 4d Clifford algebra of type $(---++)$. In these conventions the bilinears computed in 4d and 6d notations agree:

$$\varepsilon^{(6)}_+ C^{(6)} \eta^{(6)}_- = \varepsilon^{(4)}_+ c^{(4)} \eta^{(4)}_- + \varepsilon^{(4)}_- c^{(4)} \eta^{(4)}_+$$

$$(A.44)$$
For the explicit form of spinors we use 4d gamma-matrices, the 4d chirality operator and 4d bilinear form in terms of (A.6)

\[
(\gamma_i, \gamma_4) = (\sigma_2 \otimes \sigma_i, \sigma_1 \otimes \sigma_0),
\]

\[
\gamma^{(4)}_* = -\gamma_1 \ldots \gamma_4 = -\sigma_3 \otimes \sigma_0
\]  
(A.45)

\[
c^{(4)} = -i\sigma_0 \otimes \sigma_2
\]

We decompose

\[
T_{\mu\nu} \gamma^a = T_{\mu\nu -} \gamma^- + T_{\mu\nu +} \gamma^+
\]  
(A.46)

in terms of

\[
T_{\mu\nu -} = (T_{\mu\nu 5} - iT_{\mu\nu 6}) \quad \gamma^- = \frac{1}{2}(\gamma^5 + i\gamma^6) \quad \gamma^- \gamma^{56}_* = -\gamma^-
\]

\[
T_{\mu\nu +} = (T_{\mu\nu 5} + iT_{\mu\nu 6}) \quad \gamma^+ = \frac{1}{2}(\gamma^5 - i\gamma^6) \quad \gamma^+ \gamma^{56}_* = +\gamma_+
\]  
(A.47)

with \(\gamma^{56}_* = -i\gamma^{56}\). Since \(T_{\mu\nu a}\) is of negative 6d chirality, the \(T_{\mu\nu \pm}\) has \(\mp 4d\) chirality.

We define

\[
T^{(4)}_{\mu\nu} \equiv (T_{\mu\nu +} - T_{\mu\nu -}) = 2iT_{\mu\nu 6}
\]  
(A.48)

In terms of the 4d spinors the generalized conformal Killing equation (2.1) takes form

\[
D_\mu \varepsilon - \frac{1}{16} T^{(4)}_{\rho\sigma} \gamma^{\rho\sigma} \gamma_\mu \varepsilon = \gamma_\mu \eta
\]  
(A.49)

Other 6d - 4d notational definitions are

\[
\Phi^\pm = \frac{1}{2}(\Phi^5 \mp i\Phi^6), \quad \Phi_\pm = (\Phi^5 \pm i\Phi^6)
\]

\[
T_{\mu\nu a} \gamma^{(4)}_a \varepsilon^6_+ = T^{(4)}_{\mu\nu} \gamma^{(4)}_\mu \varepsilon^6_+ - \Phi^- \varepsilon^6_+, \quad \Phi^a \gamma_a \eta = 2\Phi^- \eta^6_- - 2\Phi^+ \eta^6_+
\]  
(A.50)

APPENDIX B. SUPERSYMMETRY ALGEBRA

B.1. The off-shell closure of the supersymmetry on the vector multiplet. Here we explicitly compute \(\delta^2\) on vector multiplet for \(\delta\) defined by:

\[
\delta A_m = \frac{1}{2} \lambda^i \gamma_m \varepsilon_i
\]

\[
\delta \lambda^i = -\frac{1}{4} F_{mn} \gamma^{mn} \varepsilon^i + Y^i \varepsilon^j + \Phi^a \gamma^a \eta^i + \frac{1}{8} T_{\mu\nu a} \Phi^a \gamma^{\mu\nu} \varepsilon^i
\]  
(B.1)

\[
\delta Y^{ij} = -\frac{1}{4} \varepsilon^i (\slashed{D} \lambda^j)
\]

provided that spinors \((\varepsilon, \eta)\) with \(\eta = \frac{1}{2} \slashed{D} \varepsilon\) satisfy generalized conformal Killing equations (2.3).

We find a contribution of several terms in \(\delta^2 \lambda^i\). In the flat space we drop the terms proportional to \(D_\mu \varepsilon, \eta\) and \(T\) and find

\[
\delta^2_{\text{flat}} \lambda^i = \delta \left( -\frac{1}{4} F_{mn} \gamma^{mn} \varepsilon^i \right) + \delta \left( Y^i \varepsilon^j \right)
\]  
(B.2)
with
\[ \delta(-\frac{1}{4} F_{mn} \gamma^{mn} \varepsilon^i) = -\frac{1}{4} D_p (\lambda^j \gamma_q \varepsilon^j) \gamma^{pq} \varepsilon^i = \frac{1}{4} (\varepsilon^j D_p \lambda_j) \gamma^{pq} \varepsilon^i (A.27) \]
\[ = \frac{1}{4} (\varepsilon^j D_p \lambda_j) \varepsilon^i + \frac{1}{4} (\varepsilon^q \varepsilon^i) D_q \lambda^i - \frac{1}{8} (\varepsilon^i D_p \lambda^j) \gamma^{pq} \varepsilon^i (A.31) \]
and together with the \( \delta(Y^i \varepsilon^i) \) we find
\[ \delta^2_{\text{flat}} \lambda^i = \frac{1}{4} (\varepsilon^q \varepsilon^i) D_q \lambda^i + \frac{1}{4} (\varepsilon^j D_p \lambda^j) \varepsilon^i - \frac{1}{2} (\varepsilon^j D_p \lambda^i) \varepsilon^i \] (B.3)
Next we account for \( D_\mu \varepsilon \) and \( \eta \) terms, still keeping \( T = 0 \). The transformation would be complete on conformally flat space. The \( \delta^2_{\text{cflat}} \lambda \) acquires new contributions
\[ \delta^2_{\text{cflat}} \lambda^i = \delta^2_{\text{flat}} \lambda^i + \text{term}_c \] (B.5)
where
\[ \text{term}_c = -\frac{1}{4} (\lambda^\gamma \gamma_{\mu} \eta) \gamma^{\mu q} \varepsilon^i + \frac{1}{2} (\lambda^\gamma \gamma_{\mu} \eta) \gamma^{\mu q} \varepsilon^i \] (A.29) on \( \gamma^\gamma \gamma^q \)
\[ = -\frac{1}{4} (\varepsilon^q \varepsilon^i) (\varepsilon^q \lambda) + \frac{1}{4} (\eta \varepsilon^q \varepsilon^i) - \frac{3}{4} (\eta \varepsilon^q \varepsilon^i) \] (B.6)
Then we expand the middle term in 4d indices
\[ \frac{1}{4} (\eta \varepsilon^q \varepsilon^i) = \frac{1}{4} (\varepsilon^q \varepsilon^i) + \frac{1}{4} (\eta \varepsilon^q \varepsilon^i) \] (A.14) to the first and the last term in (B.6) to find
\[ -\frac{1}{2} (\varepsilon^q \varepsilon^i) = -\frac{1}{4} (\varepsilon^q \varepsilon^i) + \frac{1}{4} (\eta \varepsilon^q \varepsilon^i) \] (B.8)
All \( \gamma_{pq} \gamma^{pq} \) terms are cancelled using (A.25) and the scalar terms are simplified as
\[ (\eta \varepsilon^i) \lambda - \frac{2}{4} (\varepsilon^q \varepsilon^i) \lambda_j = \frac{3}{4} (\eta \varepsilon^i) \lambda + (\eta \varepsilon^i \varepsilon^i) \lambda_j \] (B.9)
and the contribution from the non-flat but conformally flat terms is
\[ \text{term}_c = +\frac{1}{8} (\eta \varepsilon^q \varepsilon^i) \gamma^{\mu \lambda} - \frac{1}{8} (\eta \varepsilon^q \varepsilon^i) \gamma^{\mu \lambda} + \frac{3}{4} (\eta \varepsilon^i \varepsilon^i) \lambda_j \] (B.10)
Then we compute the \( T \)-terms in
\[ \delta^2 \lambda^i = \delta^2_{\text{cflat}} \lambda^i + \text{term}_T \] (B.11)
and find
\[ \text{term}_T = \frac{1}{64} (\varepsilon \gamma_{\mu} \gamma_{\nu} \varepsilon^i) + \frac{1}{16} T_{\mu \nu} (\varepsilon \gamma^\alpha \lambda) \gamma^{\mu \nu} \varepsilon \] (A.29) on \( \gamma^\gamma \gamma^q \) (A.35)
\[ = -\frac{1}{64} (\varepsilon \gamma_{\mu} \gamma_{\nu} \varepsilon^i) \gamma^{\mu \lambda} + \frac{1}{64} (\varepsilon \gamma_{\nu} \lambda) \gamma^{\mu \gamma} \varepsilon \gamma_{\mu} \varepsilon + \frac{1}{32} (\varepsilon \gamma_{\alpha} \lambda) \varepsilon \] (A.29)
\[ = \frac{1}{64} (\varepsilon \gamma_{\mu} \gamma_{\nu} \varepsilon^i) \gamma^{\mu \lambda} + \frac{1}{32} (\varepsilon \gamma_{\nu} \lambda) \varepsilon \] (A.29)
\[ = \frac{1}{64} (\varepsilon \gamma_{\mu} \gamma_{\nu} \varepsilon^i) \gamma^{\mu \lambda} - \frac{1}{64} (\varepsilon \gamma_{\nu} \lambda) \varepsilon \] (A.35)
The \( \text{term}_T \) can be combined with the \( \text{term}_c \):
\[ \frac{1}{8} (\eta \varepsilon^q \varepsilon^i) \gamma^{\mu \lambda} + \frac{1}{32} T_{\mu \nu} (\varepsilon \gamma^\alpha \varepsilon) \gamma^{\mu \nu} \lambda = \frac{1}{16} D_\mu (\varepsilon \gamma_{\nu} \varepsilon) \gamma^{\mu \nu} \lambda \] (B.13)
so that finally
\[ \delta^2_{\varepsilon, \eta} \lambda^i = \frac{1}{4} (\varepsilon \gamma^m \varepsilon) D_m \lambda^i + \frac{1}{16} D_\mu (\varepsilon \gamma_\mu \varepsilon) \gamma^{\mu \nu} \lambda^i - \frac{1}{8} (\eta \gamma_{ab} \varepsilon) \gamma^{ab} \lambda^i + \frac{3}{4} (\eta \varepsilon) \lambda^i + (\eta (i \varepsilon^j)) \lambda_j \] (B.14)

The variation \( \delta^2_{\varepsilon, \eta} Y_{ij} \) and \( \delta^2_{\varepsilon, \eta} A_m \) are computed similarly.

B.2. The invariance of the Lagrangian. The 4d \( \mathcal{N} = 2 \) supersymmetric Lagrangian for vector multiplet in curved background for vanishing fermionic fields of Weyl multiplet is proportional to (2.11)
\[ \frac{1}{2} F_{mn} F^{mn} + \lambda^i \gamma^m D_m \lambda_i + (\frac{1}{6} R + M) \Phi_a \Phi^a - 2 Y_{ij} Y^{ij} - F^{\mu \nu} T_{\mu \nu a} \Phi^a + \frac{1}{4} T_{\mu \nu a} T^{\mu \nu b} \Phi^a \Phi_b \] (B.15)

The trace is implicitly implied in all terms. To check the invariance under (2.8) we first consider the flat background with \( D_\mu \varepsilon = 0, \eta = 0, T = 0, M = 0 \). After that we will add the variational terms in conformally flat background, and finally we will add the remaining \( T \)-terms. We find modulo total derivative
\[ \delta_{\text{flat}} (\frac{1}{2} F^{mn} F_{mn}) = - (\varepsilon \gamma^n \lambda) D^m F_{mn} \]
\[ \delta_{\text{flat}} (\lambda \gamma^m D_m \lambda) = \frac{1}{2} (\lambda \gamma^m D_m F_{pq} \gamma^{pq} \varepsilon) + 2 Y_{ij} (\varepsilon^j D^i \lambda_i) \] (A.36)
\[ = (\lambda \gamma^n \varepsilon) D^m F_{mn} - 2 Y_{ij} (\varepsilon^j D^i \lambda_i) \]
\[ \delta_{\text{flat}} (-2 Y_{ij} Y^{ij}) = 2 Y_{ij} (\varepsilon^j D^i \lambda_i) \] (B.16)
that all terms add to zero. In conformally flat background the new terms appear in the variation of fermionic kinetic term and the coupling of scalars to the curvature
\[ \delta_{\text{cflat}} (\frac{1}{6} R + M) \Phi_a \Phi^a = (\frac{1}{6} R + M) (\lambda \gamma^a \Phi_a \varepsilon) \]
\[ \delta_{\text{cflat}} (\lambda \gamma^m D_m \lambda) = \delta_{\text{flat}} (\lambda \gamma^m D_m \lambda) + \text{term}_c \] (B.17)

where
\[ \text{term}_c = -2 (\lambda [D^i \gamma_j \Phi_a \eta^j]) + \frac{1}{2} (\lambda \gamma^a F_{pq} \gamma^{pq} D_\mu \varepsilon) \] (2.1)
\[ = -2 (\lambda \gamma^a F_{ma} \eta^i) + 2 \Phi_a (\lambda \gamma^a D^i \eta^j) + 2 (\lambda F_{ma} \gamma^a \eta^j) = 2 (\lambda \gamma^a \Phi_a D^i \eta^j) \] (2.1)
\[ = - (\frac{1}{6} R + M) (\lambda \gamma^a \Phi_a \varepsilon) \] (B.18)

so all terms in (B.17) cancel when added together.

Next we consider the remaining \( T \)-terms for a generic background. We set
\[ \mathcal{L} = \mathcal{L}_{\text{cflat}} + \mathcal{L}_T \] (B.19)
where
\[ \mathcal{L}_T = - F^{\mu \nu} T_{\mu \nu a} \Phi^a + \frac{1}{4} T_{\mu \nu a} T^{\mu \nu b} \Phi^a \Phi_b \] (B.20)
and we find
\[
\delta (\mathcal{L}_T) = (\lambda \gamma^\nu \varepsilon) [D^\mu T_{\mu \nu a}] \Phi^a + (\lambda \gamma^\nu \varepsilon) T_{\mu \nu a} F_{\mu a}^{\nu a} - \frac{1}{2} (\lambda \gamma^\nu \varepsilon) F_{\mu \nu a}^{\nu a} + \frac{1}{4} (\lambda \gamma^a \varepsilon) T_{\mu \nu a} T_{\mu \nu b} \Phi^b
\]
(B.21)

In the variation of the fermionic action the new terms are
\[
\delta (\lambda \gamma^m D_m \lambda) = \delta_{\text{cflat}} (\lambda \gamma^m D_m \lambda) + \text{term}_{T_1} + \text{term}_{T_2}
\]
(B.22)

where \( \text{term}_{T_1} \) comes from \( T \)-terms in generalized conformal Killing equation (2.3) and \( \text{term}_{T_2} \) comes from the \( T \)-term in the variation \( \delta_{\varepsilon, \eta, \lambda} \) (2.8)

\[
\text{term}_{T_1} = 2 \Phi_a (\lambda \gamma^a D_\eta)_{T} + \frac{1}{32} (\lambda \gamma^\mu F T \gamma_\mu \varepsilon)
\]
(B.23)

Then we find
\[
\frac{1}{32} (\lambda \gamma^\mu F_{\mu \nu a} T \gamma_\mu \varepsilon) = \frac{1}{4} (\lambda \gamma^\nu \varepsilon) F_{\mu \nu a}^{\nu a} - \frac{1}{2} (\lambda \gamma^\nu \varepsilon) T_{\mu \nu a} F_{\mu a}^{\nu a} + \frac{1}{4} (\lambda \gamma^\nu \varepsilon) T_{\mu \nu a} T_{\mu \nu b} \Phi^b
\]
(B.24)

and
\[
\text{term}_{T_2} = - \frac{1}{4} \lambda D_\rho (T_{\mu \nu a} \Phi^a \gamma^{\mu \nu} \varepsilon) = - \frac{1}{4} (\lambda \gamma^\rho \gamma^{\mu \nu} \varepsilon) [D_\rho T_{\mu \nu a}] \Phi^a - \frac{1}{2} (\lambda \gamma^\nu \varepsilon) T_{\mu \nu a} F_{\mu a} - \frac{1}{4} (\lambda \gamma^\rho \gamma^{\mu \nu} \varepsilon) T_{\mu \nu a} T_{\mu \nu b} \Phi^b
\]
(B.25)

Using (A.37) all \( TF \) terms cancel between (B.25) and (B.21) and finally the \( [DT] \) \( \Phi \) terms in (B.21)(B.23)(B.25) cancel as well as

\[
\text{(B.21)} : \quad (\lambda \gamma^\nu \varepsilon) [D^\mu T_{\mu \nu a}] \Phi^a
\]
\[
\text{(B.23)} : \quad \frac{1}{2} (D^\mu T_{\mu \nu a}) \Phi_b (\lambda \gamma^\rho \gamma^{\nu a} \varepsilon) = - \frac{1}{2} (\lambda \gamma^\nu \varepsilon) D^\mu T_{\mu \nu a} \Phi^a + \frac{1}{2} (\lambda \gamma^{\nu a} \varepsilon) \Phi^b D^\mu T_{\mu \nu a}
\]
\[
\text{(B.25)} : \quad - \frac{1}{4} \lambda D_\rho (T_{\mu \nu a}) \Phi^a \gamma^{\mu \nu} \varepsilon = - \frac{1}{4} (\lambda \gamma^\nu \varepsilon) D^\mu T_{\mu \nu a} \Phi^a - \frac{1}{4} (\lambda \gamma^\rho \gamma^{\mu \nu} \varepsilon) \Phi^a D_\rho T_{\mu \nu a}
\]
(B.26)
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