Dynamical response near quantum critical points

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We study high frequency response functions, notably the optical conductivity, in the vicinity of quantum critical points (QCPs) by allowing for both detuning from the critical coupling and finite temperature. We consider general dimensions and dynamical exponents. This leads to a unified understanding of sum rules. In systems with emergent Lorentz invariance, powerful methods from conformal field theory allow us to fix the high frequency response in terms of universal coefficients. We test our predictions analytically in the large-\( N \) \( O(N) \) model and using the gauge-gravity duality, and numerically via Quantum Monte Carlo simulations on a lattice model hosting the interacting superfluid-insulator QCP. In superfluid phases, interacting Goldstone bosons qualitatively change the high frequency optical conductivity, and the corresponding sum rule.

A quantum critical point (QCP) is a zero-temperature phase transition, driven by quantum fluctuations, reached by tuning a non-thermal parameter such as a magnetic field [1], as shown in Fig. 1. Proximity to a QCP alters many observables, even if the (detuned) ground state is otherwise conventional. Of particular importance are dynamical response functions such as the optical conductivity \( \sigma(\omega) \) [1–14], where changing the frequency probes physics at different energy scales set by the non-thermal detuning and by the temperature. What often complicates the analysis of the real-time dynamics, especially on short time scales, is the destruction of quasi-particles at the QCP, and the corresponding abundance of incoherent excitations at finite but small detuning.

In this letter, we focus on a large family of non-metallic QCPs [1] found in magnetic insulators, Dirac semimetals, cold atomic gases in optical lattices [15–17], thin film superconductors or arrays of Josephson junctions [2]. This will serve as comparison ground for the more intricate metallic QCPs occurring in heavy fermion materials for example [18]. Specifically, we study how the detuning of the non-thermal parameter from its critical value, as well as temperature, modify the optical conductivity. In particular, our analysis at large frequencies is not restricted to the quantum critical fan. We derive sum rules for the conductivity that generalize the standard \( f \)-sum rule [19] to the scaling regime near QCPs. Our methods are not perturbative in any interaction strength. We test our predictions using large-scale quantum Monte Carlo simulations of an interacting superfluid-insulator QCP. While our focus is on the portions of the phase diagram smoothly connected to the critical fan, we also point out the qualitative changes to \( \sigma(\omega) \) and the resulting sum rules which result from interacting Goldstone bosons in broken-symmetry phases.

**Setup:** Let us consider a system near a QCP that is reached by tuning a non-thermal parameter \( g \) to zero. We work in the universal scaling regime, at frequencies smaller than microscopic (UV) scales, and assume that hyperscaling is obeyed. Such a system is described by the following low-energy action in \( d \) spatial dimensions:

\[
S = S_{\text{critical}} - g \int dt \, d^d x \, O(t, x),
\]

where \( O \) is the only relevant operator whose coupling \( g \) necessitates fine-tuning; it has (spatial) scaling dimension

\[
\Delta = d + z - 1/\nu,
\]

where \( \nu > 0 \) is the correlation length critical exponent, and \( z \) is the dynamical exponent. The equal-time 2-point function of \( O \) at the QCP is thus \( \langle O(0, x)O(0, 0) \rangle \propto 1/|x|^{2\Delta} \). For example at the superfluid-insulator QCP in 2d belonging to the Wilson-Fisher universality class, \( O \sim \phi_{s}\phi_{s} \) is the “mass” term of the 2-component order parameter field \( \phi_{s} \). At \( T = 0 \), the correlation length diverges as \( \xi \sim g^{-\nu} \) on the insulator side.

We are interested in probing the properties of the nearly critical system by studying dynamical response functions such as the optical conductivity: \( \sigma(\omega) = \frac{1}{\omega} \langle J_x(-\omega)J_x(\omega) \rangle_{g,T} \), where \( \omega \) is the frequency, and \( J \) is

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the current operator that enters in the retarded correlator. Near the QCP, the conductivity will obey scaling:
\[
\sigma(\omega) = \omega^{d-2/z} f_{\pm} \left( \frac{\omega}{|g|^\Delta T}, \frac{\omega}{T} \right),
\]
where \( f_{\pm} \) is a dimensionless scaling function that depends on which side of the transition the system is poised. We have set \( h = k_B = 1 \), and the charge \( Q = 1 \) and \( c = 1 \), where \( c \) appears in the energy scale \( c|k|^2 \). Other response functions such as order parameter susceptibilities or the shear viscosity will have an analogous structure.

**Large frequencies:** In this letter we focus on the behavior of the conductivity at high frequencies \( \omega \gg T, |g|^{1/\nu} \), which allows us to controlably study the deviations away from criticality. The resulting asymptotics will also serve as the key ingredient in the derivation of sum rules for the response functions. Our first main result is that the asymptotic behavior is
\[
\sigma(\omega) = (i\omega)^{(d-2)/z} \left( \sigma_\infty + c_1 \frac{g}{(\omega^{d+z-\Delta})^z} + c_2 \frac{O(1)}{\omega^{\Delta/z} + \cdots} \right), \tag{3}
\]
where \( \sigma_\infty, c_{1,2} \) are real constants fixed by the universality class, independent of detuning and \( T \). The \( \sigma_\infty \) term is the conductivity of the critical theory; the \( c_{1,2} \) terms arise from deviations from the QCP due to detuning and temperature. Note that the \( c_1 \) term in brackets simply scales as \( \omega^{-1/\nu} \), by virtue of (2). In odd \( d \), the imaginary part of \( \sigma \) can have a non-universal logarithmic contribution, not written here. For simplicity, we consider the generic case where the \( c_{1,2} \) power-laws are not equal, and more generally do not differ by \( 2n/z \) (\( n \) being an integer), i.e. \( 2\Delta \neq d + z + 2n \). [20]

When \( z = 1 \), recent work has derived [11] the \( c_2 \) term in Eq. (3) at \( T > 0 \) but zero detuning \( g = 0 \). Here, we identify the new effects coming from detuning, and their interplay with temperature. In particular, the \( c_1 \) term purely arises from \( g \) and can have important consequences on the dynamics. Its existence was glimpsed deep in the quantum critical fan, \( T \gg |g|^{1/\nu} \), in a specific AdS/CFT calculation [14], and in fact holds much more broadly. For CFTs \( (z = 1) \) we will derive Eq. (3), present a universal expression for \( c_1/c_2 \), and confirm our predictions with two independent computations in non-trivial CFTs. For \( z \neq 1 \), we provide a general scaling argument for the \( c_1 \) term, and confirm that Eq. (3) is satisfied by a class of strongly interacting QC theories described by the gauge-gravity duality.

Working at general \( z \), we first explain the origin of the \( c_1 \) term by using a scaling argument. Let us imagine that the system is at \( T > 0 \) in the QC fan. Since there is no phase transition in the fan, the conductivity will receive a correction \( \delta \sigma \) that is analytic in the coupling \( g \) about \( g = 0 \), which generally will be linear. Further, by using the scaling dimension of \( g \), and the fact that \( \omega \gg T \) is the dominant energy scale, we get \( \delta \sigma \sim g/\omega^{(d+z-\Delta)/z} \). We stress that this term does not depend on \( T \). A more precise and general argument can be made by first expressing the dynamical conductivity as \( \sigma(\omega) = \frac{1}{2\pi} \langle J_x J_x e^{-i\omega t} O(0) \rangle_T Z_{0,T}/Z_{g,T} \), using Eq. (1), where \( Z_{g,T} \) is the full partition function. The expectation value is taken using the \( g = 0 \) action, and temperature \( T \geq 0 \). We expand \( e^{-i\omega t} O \) to first order in \( g \), and evaluate the resulting 3-point function \( \langle J_x(\omega) J_x(-\omega) O(\omega \rightarrow 0) \rangle_T = \omega^{(d+z-\Delta)/z} F(T/\omega) \), for a scaling function \( F \) (note that spatial momenta are set to zero). Generally, \( F(0) \neq 0 \) and is a property of the QCP at \( T = 0 \). Hence, as \( \omega \gg T \), \( c_1 = F(0) \) and is \( T \)-independent. If there is no phase transition as we vary \( T \) at fixed \( g \neq 0 \), by adiabaticity \( c_1 \) must remain unchanged all the way to, and including, \( T = 0 \).

In contrast to the \( c_1 \) term, the \( c_2 \) term depends on both \( g \) and \( T \) through the expectation value of \( O \), and was previously identified at finite temperature but zero detuning \( g = 0 \) (and \( z = 1 \)) [11]. Let us recall the main idea of that derivation, focusing on the case \( z = 1 \), and see how it generalizes to \( g \neq 0 \). The Kubo formula for the conductivity states that we need to evaluate the current-current correlation function. Since we are interested in short times (large-frequencies) we consider the operator product expansion (OPE) of \( J_x(t,0) J_x(0,0) \) in the \( t \rightarrow 0 \) limit. Crucially, by spacetime locality the product can be replaced by a sum of local operators evaluated at \( t = 0 \), with increasing scaling dimensions. The first non-trivial term in the sum will generally arise due to the leading relevant operator at the QCP, \( O \), and will be \( \sim t^{\Delta-2d} O(t = 0) \). We can take the expectation value of the OPE at finite \( g \) and \( T \) since we work at short times, \( t \ll |g|^{-\nu z}, T^{-1} \). Fourier transforming then leads to the \( c_2 \) term in Eq. (3). \( c_2 \) itself depends on neither \( g \) nor \( T \); it is related to a coefficient in the OPE. In contrast, the \( z \neq 1 \) case is not as simple due to the lack of a sharp notion of spacetime locality needed to constrain the OPE. The \( c_2 \) term at \( z \neq 1 \) is allowed by scaling, and below we will confirm its existence in a class of interacting Lifshitz theories.

The perturbative expansion used to derive the \( c_1 \) term is different from the commonly used perturbative expansions about a free (Gaussian) theory: it uses the structure of the generally interacting QCP itself to determine the corrections at finite detuning. The expansion should hold when the detuned system has a finite correlation length, but can fail in regions separated from the “fan” by a phase transition, where potentially new gapless modes can arise. We will see an example of this failure later.

We have obtained the asymptotic expansion Eq. (3) near generic QCPs. In the context of classical critical phenomena, similar expansions for short-distance spatial correlators of the order parameter have been found for thermal Wilson-Fisher fixed points in 3D (where \( z = 1 \) [21, 22]. The coefficients in the expansion for these spatial correlators have recently been computed for the strongly-coupled Ising critical point [23]. These classical results are most similar to Eq. (3) analytically continued [24, 25] to imaginary time, when \( z = 1 \) and \( T = 0 \). In this limit, the asymptotic behavior of short-distance correlators contains both analytic and non-analytic terms in the
thermal detuning parameter \((T - T_c)\), since \(\langle O \rangle \sim |g|^\nu \Delta\) where \(g\) is interpreted as \((T - T_c)\) under the quantum-to-classical mapping. This highlights that Eq. (3), just as in the classical case, cannot be derived via a single perturbative expansion. Our derivation indeed illustrates the different mechanisms behind the \(c_1\) and \(c_2\) terms, and is valid near QCPs at finite \(g\) and \(T\), as well as when \(z \neq 1\).

**Universal ratios:** For QCPs described by conformal field theories (\(z = 1\), the expansion described above to get the \(c_1\) term is called conformal perturbation theory, and is very powerful because the 3-point function \(\langle J(x_1)J(x_2)O(x_3) \rangle_{\text{QCP}}\) is fixed by conformal symmetry and operator dimensions up to a single theory-dependent constant. (This is not the case for general \(z\).) The conformal symmetry thus allows us to show that for all CFTs the ratio \(c_1/c_2\) is universal and only depends on \(\Delta\) and the normalization of \(O\):

\[
\frac{c_1}{c_2} = \frac{-\Gamma(4 - \Delta)\Gamma(\Delta - \frac{3}{2})}{2^{7 - 4\Delta}\Gamma(1 + \Delta)\Gamma(\frac{3}{2} - \Delta)}, \quad c_2 = C_{\text{JJJO}}, \tag{4}
\]

where we have given the answer in 2d. \(\Gamma(z)\) is the gamma function, and \(\mathcal{C}_{\text{OO}}\) appears in the correlator \(\langle O(-p)O(p)\rangle_{\text{QCP}} = \mathcal{C}_{\text{OO}}p^{2\Delta - 3}\) expressed in frequency-momentum space. The real constant \(C_{\text{JJJO}}\) enters in the 3-point function \(\langle J\phi J\phi \rangle_{\text{QCP}}\). The detailed derivation of Eq. (4) and its generalization to \(d \neq 2\) is given in App. A.

In order to get insight about the generic \(z\) case, we employ the holographic gauge-gravity duality [26–28] to study charge transport in a class of interacting large-\(N\) matrix field theories. Such theories are dual to gravitational theories existing in a \((d + 2)\)-dimensional curved spacetime whose isometries are in correspondence with the Lifshitz symmetries of the matrix field theories at general \(z\). This approach is useful because techniques such as conformal perturbation theory, which are non-perturbative in interaction strength and robust against large \(N\) for the validity of (6) are trivially satisfied. For general \(d, \Delta = 2\) or \(z + d - 2\) constitute special cases since the rhs of Eq. (6) can be finite and non-zero (see the \(O(N)\) model calculation below). Again, (6) holds in the same regime as the asymptotic expansion, i.e. for points in the \((g,T)\) phase diagram that can be reached from the QC region without crossing phase transitions. Knowledge about the expansion is needed to ensure that \(\sigma(\omega)\) decays sufficiently fast at large frequencies. The other ingredient is the analyticity of \(\sigma\) in the upper half-plane of complex frequencies (causality), which allows us to prove the sum rule by contour integration (App. D).

**\(O(N)\) model:** We now examine the physics described above in the context of the interacting QCPs in the \(O(N)\) model in 2d, which have \(z = 1\) and are CFTs. We focus on 2 cases: \(N = \infty\) (which is solvable), and \(N = 2\) which describes an interacting superfluid-insulator QCP.

These QCPs are described by a relativistic \(\phi^4\)-theory for an order parameter field \(\phi_a\) with \(N\) real components [32]:

\[
S = -\int d^3x \left( \frac{1}{2} \partial_a \phi_a \partial^a \phi_a + r \phi_a \phi_a + \frac{u}{2N} (\phi_a \phi_a)^2 \right) \tag{7}
\]

This action is written in real time. When \(r\) is large, this model yields a gapped phase with unbroken \(O(N)\) symmetry; when \(r\) is small, \(O(N)\) is spontaneously broken and the low energy effective theory contains Goldstone bosons if \(N > 1\). There are conserved currents \(j_{\mu}^{ab} = \partial_a \phi^b - \phi^b \partial_a \phi\), and our goal is to compute the corresponding conductivity. When \(1 < d < 3\), dimensional analysis suggests that this QCP has a relevant operator \(\phi_a \phi_a\) with detuning parameter \(g \sim r\). This is qualitatively correct; in App. C, we precisely identify \(O\) and \(g\) in terms of slightly different variables.

When \(N = \infty\), this model is exactly solvable through large-\(N\) techniques [32]. The resulting QCP has \(\nu = 1\) and is thus distinct from the Gaussian fixed point at \(u = 0\). Let us begin by studying the disordered phase, which occupies the entire phase diagram except the broken symmetry state at \(T = 0\) and \(g < 0\). We obtain the following asymptotic expansion via an explicit computation of the conductivity (App. C)

\[
\sigma(\omega) = \frac{1}{16} + \frac{4g}{i\omega} \left( \frac{\langle O \rangle_{\Delta T}}{4N\omega^2} + \ldots \right), \tag{8}
\]
interacting superfluid-insulator QCP, where quasiparticle excitations have been destroyed by fluctuations. We analyze its imaginary time conductivity numerically using large-scale lattice quantum Monte Carlo (QMC) simulations. We work with the action Eq. (7) in Euclidean spacetime (devoid of the sign problem), discretized on a 512 × 512 × 512 cubic lattice. Details of the numerical methods are in App. E. Fig. 2 shows the universal part of the imaginary frequency conductivity in the disordered phase at different values of the detuning, near the QCP. We plot the conductivity relative to its groundstate value \( \sigma_\infty \) as a function of the frequency rescaled by the single-particle gap \( m \propto g^\nu \). In order to do so, we must subtract off a non-universal lattice correction to \( \sigma \), and employ \( \sigma_\infty = 0.355(5) \), found with recent conformal bootstrap calculations [33] along with numerical simulations [8–12]. The resulting data collapses to a single universal curve. The large-\( \omega \) field theory prediction (solid line) for the subleading term, which scales as \( c_1 \omega^{-1/\nu} \), with \( \nu = 0.67 \), is also shown. At \( N = 2 \), in contrast to the \( N = \infty \) case Eq. (8), the next subleading term \( \propto c_2 \omega^{1/\nu - 3} \) comes with nearly the same exponent, so that in practice we combine both the \( c_{1,2} \) terms into a single one. By looking at the high frequency limit, we see that \( c_1 \) is negative, in agreement with our result at \( N = \infty \), Eq. (8). The numerical data is also consistent with our predicted scaling \( \sigma - \sigma_\infty \propto \omega^{-1/\nu} \), but due to the need to subtract off a large background conductivity to extract \( c_1 \) and \( \nu \), we presently cannot perform a more quantitative analysis.

In the superfluid phase, both the numerical and field theory analyses become complicated by the presence of the broken symmetry and the associated strongly coupled Goldstone boson(s) (at finite \( T < T_c \), the order becomes algebraic). In order to analytically understand the asymptotic behavior of \( \sigma(\omega) \), and the associated sum rule Eq. (6), one would need to use methods beyond what we have discussed so far. It will be interesting to see whether the result will be similar to the \( N = \infty \) case, Eq. (9), with the associated breakdown of the sum rule. We leave this important question for the future.

**Outlook:** We have determined the large-frequency optical conductivity near a QCP for a wide class of theories, Eq. (3), in general dimensions. Our analysis incorporates non-thermal detuning and temperature, and thus extends beyond the QC fan which facilitates comparison with experiments. This has led to a unified understanding of sum rules in the phase diagram near such QCPs. Interestingly, we have found that in certain superfluid phases, interacting Goldstone bosons can qualitatively change the results. It will be of interest to analyze such effects more broadly. Our findings can potentially be tested at QCPs in superconductor-insulator systems or Josephson junction arrays [2], and in ultra-cold atomic gases. In the latter case, the physics of the superfluid-insulator QCP has already been realized [15–17], and proposals for measuring the optical conductivity exist (e.g. by periodic phase-modulation of the optical lattice [34]). Finally, although this letter focused on the optical con-

\[
C \times (m / \Omega n / m)^{1/\nu}
\]

FIG. 2. Log-log plot of the asymptotic behavior of \( \sigma(\Omega n / m) \) at imaginary frequencies, in the disordered phase of the O(2) model, computed using QMC in the limit \( T \to 0 \). Each set of colored dots represents a different detuning \( g \). \( m \propto g^\nu \) is the single particle gap. The line is the field theory prediction (3) at large \( \Omega n / m \), with \( \nu = 0.67 \).

where \( \langle O \rangle_{g,T} = N m^2 \), with \( m(g,T) \) being the detuning and temperature induced mass, given in App. C. Using the previously derived values \( \sigma_\infty = \frac{16}{\pi^4}, \Delta = 2, C_{JJ} = \frac{1}{167} \) and \( C_{OO} = -16 N \) [11], we find exact agreement with Eq. (4). Now, the \( g \)-linear term, although purely imaginary, alters the sum rule Eq. (6) from its \( g = 0 \) form because we have the special situation \( \Delta = 2 \). Indeed, we find that the rhs of Eq. (6) becomes finite, \(-2 \pi g\), which is independent of temperature, and changes sign across \( g = 0 \), see App. D.

The conductivity in the ordered phase at \( N = \infty \), which occurs when \( T = 0 \) and \( g < 0 \), is qualitatively distinct. When the condensate is along the 1-direction \( \langle \phi_1 \rangle \neq 0 \), the asymptotic conductivity for \( J_{12}^a \) reads

\[
\sigma(\omega) = \frac{1}{16} + \frac{64}{3 \pi^2} \frac{|g|}{|g|} \ln \frac{\omega}{|g|} + O \left( \frac{1}{\omega} \right).
\]

We find disagreement with (3), which can be understood as follows: conformal perturbation theory was based around the convergence of the \( g \)-expansion of \( \langle J J e^{-igf} \rangle_{QCP} \). When \( g < 0 \), this expansion can lead to IR divergences associated with the instability of the symmetric vacuum: \( \phi_\alpha \) has obtained an expectation value in the true vacuum. At \( N = \infty \), logarithmic corrections to \( \sigma \) are a consequence of the coupling to Goldstone bosons, as we show in App. C. Deviations from Eq. (3) hence follow from the superfluid instability of the symmetric vacuum when \( g < 0 \). We also note that the new logarithmic enhancement in Eq. (9) makes the sum rule Eq. (6) ill-defined because the integral diverges. Further, the logarithmic contribution in Eq. (9) is present when \( 2 < d < 3 \), for all temperatures at which long range order exists, with a proportionality coefficient related to the superfluid density (see App. C).

When \( N = 2 \), the model Eq. (7) describes a strongly
ductivity, our general techniques apply to other correlation functions.

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\[
\langle J_z(p_1)J_z(p_2)O(p_3) \rangle = A_{JJOO} \cdot \left[ I\left( \frac{D}{2}, \frac{D}{2} - 1, \frac{D}{2} - 1, \Delta - \frac{D}{2} + 1 \right) + \frac{\Delta}{D}(D - 2 - \Delta) I\left( \frac{D}{2} - 1, \frac{D}{2} - 1, \frac{D}{2} - 1, \Delta - \frac{D}{2} \right) \right]
\]

where \( p_{1,2,3} \) are chosen to lie in the time direction:

\[
p_1 = (\Omega, 0), \quad (A5a)
\]
\[
p_2 = (-\Omega - p, 0), \quad (A5b)
\]
\[
p_3 = (p, 0), \quad (A5c)
\]

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Appendix A: Asymptotics in Conformal Field Theory

In this appendix we use techniques from conformal field theory (CFT) to derive Eq. (3) near a QCP with \( z = 1 \). Conformal field theories have an enhanced symmetry group containing Lorentz transformations and scale invariance [35], and describe many \( z = 1 \) QCPs of physical relevance. As we will see, this symmetry group is powerful enough to completely fix \( c_1 \) and \( c_2 \), for any CFT, in terms of a few simple numbers (operator dimensions and operator product expansion coefficients). We denote with \( D \) the spacetime dimension, \( D = d + 1 \). In this appendix, we shall work in Euclidean (imaginary) time.

In a CFT, a (Lorentz) scalar operator of dimension \( \Delta \) has a two-point function

\[
\langle O(x)O(0) \rangle = \frac{C_{OO}}{x^{2\Delta}}. \quad (A1)
\]

The only free parameters are the operator normalization \( C_{OO} \) and scaling dimension \( \Delta > (D - 1)/2 \) [36]. For the purposes of this work, it is convenient to work in frequency-momentum space:

\[
\langle O(p)O(-p) \rangle = C_{OO}p^{2\Delta - D}. \quad (A2)
\]

(In special cases like \( \Delta = D/2 \), logarithms can also appear.) From the Fourier transform, one finds

\[
C_{OO} = C_{OO} \times \frac{2^{D-2\Delta} \pi^{D/2} \Gamma\left(\frac{D}{2} - \Delta\right)}{\Gamma(\Delta)}. \quad (A3)
\]

As \( C_{OO} > 0 \), for many operator dimensions \( \Delta \) of interest (including \( \Delta \simeq 1.51 \) for the relevant \( O(2) \)-invariant scalar operator in the \( N = 2 \) Wilson-Fisher QCP in \( D = 3 \)), we see that \( C_{OO} < 0 \).

When detuning the system away from criticality, we want to understand how the sourced scalar field \( O \) modifies the conductivity. As explained in the main text, it will prove useful to know the momentum-space 3-point correlator [37]

\[
I(a,b,c,d) \equiv \int_0^\infty dx \ x^a p_1^b p_2^c p_3^d K_a(p_1x)K_c(p_2x)K_d(p_3x)
\]

(A6)
Here $K_a$ is the modified Bessel function of the second kind. Once again, we see that up to an overall normalization $A_{JJ\mathcal{O}}$, the form of Eq. (A4) is completely fixed by conformal invariance. In what follows, we will focus on the limit $p \ll \Omega$, relevant for the computation of the high frequency conductivity.

The presence of the scalar operator $\mathcal{O}$ modifies the operator product expansion (OPE) associated with a conserved current. In momentum space, the OPE of the operator product expansion (OPE) associated with a conserved current operator (obtained by Fourier transforming the real space form) contains the non-analytic term

$$J_x(\Omega) J_x(-\Omega) = C_{JJ\mathcal{O}} \Omega^{D-2} \frac{\mathcal{O}(p)}{\Omega^\Delta} + \cdots . \tag{A7}$$

in the limit $p \ll \Omega$. The OPE coefficient $C_{JJ\mathcal{O}}$ can be related to $A_{JJ\mathcal{O}}$ by contracting both sides of Eq. (A7) with $\langle \cdots \mathcal{O}(p) \rangle$, and then taking the limit $p \to 0$. In doing so, one finds that the leading order singular contribution in $p$ is

$$\langle J_x(\Omega) J_x(-\Omega - p) \mathcal{O}(p) \rangle = C_{JJ\mathcal{O}} \Omega^{D-2} \frac{\mathcal{O}(p) \mathcal{O}(\Delta-D)}{\Omega^\Delta} + \cdots + \text{(regular as } p \to 0). \tag{A8}$$

Using the small-$x$ Taylor expansion of the Bessel function:

$$K_b(x) = (2^{b-1} \Gamma(b)x^{-b} + \cdots) + (2^{-b-1} \Gamma(-b)x^b + \cdots), \tag{A9}$$

we find that the $p^{2\Delta-D}$ contribution in Eq. (A4) arises from the second term in the above expansion:

$$\int \left( \frac{D}{2} - 1, \frac{D}{2} - 1, \frac{D}{2} - 1, \Delta - \frac{D}{2} \right) = p^{2\Delta-D} \int_0^\infty dx \ x^{\Delta-1} \Omega^{D-2} \left[ K_{D/2-1}(\Omega x) \right]^2 2^{D/2-\Delta-1} \Gamma \left( \frac{D}{2} - \Delta \right) + \cdots \tag{A10}$$

with the function

$$\Psi(a;b) = \int_0^\infty dx \ x^{a-1} K_b(x)^2 = \frac{\sqrt{\pi} \Gamma \left( \frac{D}{2} \right) \Gamma \left( \frac{D}{2} + b \right) \Gamma \left( \frac{D}{2} - b \right)}{4 \Gamma \left( \frac{D}{2} \right)} \tag{A11}$$

Hence, we find the relation

$$C_{JJ\mathcal{O}} = - \frac{A_{JJ\mathcal{O}}}{\mathcal{O}(0)} \Delta \left( 1 - \frac{D}{2} - \Delta \right) 2^{D/2-\Delta-1} \Gamma \left( \frac{D}{2} - \Delta \right) \Psi \left( \frac{D}{2} - 1 \right) . \tag{A12}$$

## 1. Conductivity

Given the CFT data described above, we are now ready to use conformal perturbation theory to compute the asymptotic behavior of the two-point function $\langle J_x(\Omega) J_x(-\Omega) \rangle$ when we detune away from criticality, by a finite temperature $T$, and by a coupling constant to a (relevant) scalar operator $\mathcal{O}$. For simplicity, we assume that $\Delta \neq D/2 + n$, where $n$ is an integer. Assuming that conformal perturbation theory is well behaved, we find

$$\langle J_x(\Omega) J_x(-\Omega) \rangle_g = \frac{Z_{g=0}}{Z_g} \langle J_x(\Omega) J_x(-\Omega) e^{-g\mathcal{O}(0)} \rangle_{g=0} = \langle J_x(\Omega) J_x(-\Omega) [1 - g\mathcal{O}(0)] \rangle_{g=0} + \cdots \tag{A13}$$

where $\int d^{D+1}x \mathcal{O}(x) = \mathcal{O}(0)$ is the $p \to 0$ limit of the Fourier transform of $\mathcal{O}(x)$, and the superscript “$c$” denotes the connected correlation function, which we will omit from now on for simplicity. Here, we have omitted the temperature $T$ as the correction to the groundstate conductivity that we study first will not depend on $T$ at all. This correction is linear in $g$ and arises from the finite expectation value

$$\lim_{p \to 0} \langle J_x(\Omega) J_x(-\Omega) \mathcal{O}(p) \rangle = \text{finite} , \tag{A14}$$

which is evaluated at the QCP, namely in the groundstate of the CFT. Whenever $\Delta > D/2$, it is in fact the leading order contribution in the $p \to 0$ limit. Indeed, when comparing $A_{JJ\mathcal{O}}$ to $C_{JJ\mathcal{O}}$, we Taylor expanded a Bessel function in Eq. (A4). If we focused on the first term in Eq. (A9), we find that the $p$-dependence of the correlator drops out.
So we take the $p \to 0$ limit, and combining Eqs. (A4), (A6) and (A9), we find at $g = 0$:

\[
\langle J_x(\Omega)J_x(-\Omega)O(0) \rangle = -A_{JJO} \frac{\Omega^{D-2}}{\Omega^{D-\Delta}} \cdot (D-\Delta) \left(1 - \frac{\Delta}{2}\right) 2^{\Delta-1-D/2} \Gamma \left(\Delta - \frac{D}{2}\right) \psi \left(D - \Delta; \frac{D}{2}\right).
\]  

(A15)

This is one singular contribution to the asymptotic expansion of the current two-point function, and another comes simply from the OPE itself, as discussed above:

\[
\langle J_x(\Omega_x)J_x(-\Omega_x) \rangle_{g,T} = \left\{ \sigma_\infty \Omega_x^{D-2} + \Omega_x^{D-2}C_{JJO} \frac{O(0)}{\Omega_x^{2}} + \cdots \right\}_{g,T}
\]  

(A16)

where we have explicitly restored the temperature; $\Omega_x = 2\pi \ell T$ is a Matsubara frequency and $\ell \geq 0$ an integer. $\langle O \rangle_{g,T}$ will generally depend on both $g$ and $T$. This latter contribution to the conductivity is local, coming from the OPE (in contrast, Eq. (A15) is a non-local contribution). Putting these two equations together, we obtain

\[
\sigma(i\Omega) = \Omega^{D-3} \left[ \sigma_\infty + \frac{c_1 g}{\Omega^{D-\Delta}} + \frac{C_{JJO} \langle O \rangle_{g,T}}{\Omega^{\Delta}} + \cdots \right],
\]  

(A17)

which, upon analytic continuation to real frequencies [24, 25], gives Eq. (3) from the main text, but with $z = 1$. Recall that $c_2 = C_{JJO}$. Combining Eqs. (A12) and (A15), we see that the ratio $c_1/c_2$ is independent of $C_{JJO}$, and depends only on $C_{OJO}$, $D$ and $\Delta$:

\[
c_1/c_2 = -\frac{C_{OJO}}{2^{D-2\Delta}} \frac{\Gamma(1+\frac{D-2\Delta}{2})\Gamma(2-\frac{\Delta}{2})\Gamma(D-1+\frac{\Delta}{2})\Gamma(\Delta - D/2)\Gamma(D)}{\Gamma(1+\frac{\Delta}{2})\Gamma(2-\frac{D-2\Delta}{2})\Gamma(D-1+\frac{D-2\Delta}{2})\Gamma(\Delta - D/2)\Gamma(D)}
\]  

(A18)

In the special case $D = 3$, Eq. (A18) simplifies to Eq. (4) using $\Gamma$ function identities.

### Appendix B: Results from Lifshitz Holography

In this appendix, we summarize the field theoretic results obtained by studying a special class of interacting Lifshitz field theories accessible through the gauge-gravity duality. Gauge-gravity duality maps the correlation functions of certain field theories with large $N$ matrix degrees of freedom to classical computations in various curved spacetimes in one higher spacetime dimension. In the simplest case of the correspondence, a large $N$ CFT is dual to a classical gravity theory on anti-de Sitter (AdS) space [26], but the correspondence is now believed to be far more generic [27, 28]. In particular, there is a “Lifshitz” geometry, with metric

\[
ds^2 = \frac{dr^2}{r^2} + \frac{dt^2}{r^{2z}} + \frac{dx^2}{r^2}
\]  

(B1)

where $r$ is the extra holographic dimension, and $(t, x)$ represent the $D$ dimensional spacetime of the Lifshitz QFT. The isometries of Eq. (B1) (symmetries of the metric) may be interpreted as the symmetries (translation, spatial rotation and dilatation) of a Lifshitz field theory [38]. Classical gravity computations in such a background are believed to reproduce the correlation functions of an unknown Lifshitz field theory. Note that in the special case $z = 1$, the metric reduces to that of AdS, and the dual QFT is conformal, and relatively well-understood [26].

The details of the gravity computation are beyond the scope of this letter and will be reported in [29]. The computation proceeds somewhat similarly to [14]. Here, we focus merely on the final results, and compare them to our prediction for the high frequency conductivity, Eq. (3). In order to fix $c_1$ and $c_2$, we must carefully study the three-point function $\langle J_x(\Omega_1)J_x(\Omega_2)O(\Omega_3) \rangle$, just as we did when $z = 1$. What we find in our holographic model is that this three-point function takes a very similar form to Eq. (A4):

\[
\langle J_x(\Omega_1)J_x(\Omega_2)O(\Omega_3) \rangle = A_{JJO} \left\{ zI \left( \frac{d+z}{2z}, \frac{d+z-2}{2z}, \frac{d+z-2}{2z}, \frac{2\Delta-d-z}{2z} \right) - \frac{\Delta}{2} \left( \Delta + 2 - d - z \right) I \left( \frac{d-z}{2z}, \frac{d+z-2}{2z}, \frac{d+z-2}{2z}, \frac{2\Delta-d-z}{2z} \right) \right\}. 
\]  

(B2)

with $I(a, b, c, d)$ defined in Eq. (A6). We stress that this formula is only valid when the three momenta in $I(a, b, c, d)$ are entirely in the $t$ direction. The constant $A_{JJO}$ can also be computed in terms of certain parameters of the bulk
gravity description, but its value is not relevant here.

As before, we consider the limit

\[
\Omega_1 = \Omega, \quad \Omega_2 = -\Omega - p, \quad \Omega_3 = p, \quad \text{(B3)}
\]

with \( p \ll \Omega \). The correlator Eq. (B2) has a term regular in \( p \) as \( p \to 0 \), given by

\[
\langle J_\mu(x) J_\nu(-\Omega) \mathcal{O}(0) \rangle = -A_{JJ\mathcal{O}} \Omega^{\frac{d-2}{2}} \left( 1 - \frac{\Delta}{2} \right) \left( d + z - \Delta \right) \Gamma \left( \frac{2\Delta - d - z}{2z} \right) \psi \left( \frac{d + z - \Delta}{2z} ; \frac{d + z - 2}{2z} \right).
\]

Similarly, we find a non-analytic contribution in \( p \):

\[
\langle J_\mu(x) J_\nu(-\Omega) \mathcal{O}(p) \rangle = -A_{JJ\mathcal{O}} \Omega^{\frac{d-2}{2}} \left( \frac{p}{\Omega} \right)^{\frac{2\Delta - d - z}{2}} \Delta + \frac{2}{2} \left( d + z - 2 \Delta \right) \Gamma \left( \frac{d + z - 2\Delta}{2z} \right) \psi \left( \frac{d + z - 2}{2z} \right).
\]

We attribute this non-analytic contribution to the presence of the operator \( \mathcal{O} \) in the OPE of \( J_\mu J_\nu \):

\[
J_\mu(\Omega) J_\nu(-\Omega) = \cdots + \frac{C_{JJ\mathcal{O}}}{\Omega^{\frac{d-2}{2}}} \mathcal{O}(0) + \cdots \quad \text{(B6)}
\]

where

\[
C_{JJ\mathcal{O}} \equiv -A_{JJ\mathcal{O}} \Omega^{\frac{d-2}{2}} \frac{(2 - \Delta)(d + z - \Delta)}{\Delta(2 + \Delta - d - z)} \left( \frac{2\Delta - d - z}{2z} \right) \Gamma \left( \frac{d + z - 2\Delta}{2z} \right) \psi \left( \frac{d + z - 2}{2z} \right).
\]

The extent to which such an OPE is well-behaved for general non-conformal theories is not well understood [39–41]. Our holographic results are consistent nonetheless with this non-analytic contribution emerging from an OPE. Following the logic of conformal perturbation theory, we hence fix the ratio

\[
\frac{c_1}{c_2} = -\frac{C_{O\mathcal{O}}}{2(d+z-2\Delta)/z} \frac{(2 - \Delta)(d + z - \Delta)}{\Delta(2 + \Delta - d - z)} \left( \frac{2\Delta - d - z}{2z} \right) \Gamma \left( \frac{d + z - 2\Delta}{2z} \right) \psi \left( \frac{d + z - 2}{2z} \right)
\]

In a separate calculation, we can compute the high-frequency expansion of the conductivity of the theory dual to Eq. (B1), and find that it exactly matches the result of the three-point function calculation, Eq. (B8). In \( d = 2 \), this reduces to Eq. (5).

**Appendix C: Conductivity of the O(N) Model at \( N = \infty \)**

The (Euclidean time) action of the O(N) model is

\[
S = \frac{1}{2} \int d^{d+1}x \left[ \partial_\mu \phi^a \partial^\mu \phi^a + \frac{i}{\sqrt{N}} \left( \phi^a \phi^a - \frac{N}{g} \right) + \frac{\lambda^2}{4u} \right],
\]

with \( \mu \) indices running over spacetime coordinates, and \( a = 1, \ldots, N \). There are \( \binom{N}{2} \) conserved currents associated with the O(N) global symmetry:

\[
J_\mu^{ab} = \phi^a \partial_\mu \phi^b - \phi^b \partial_\mu \phi^a.
\]

and it is the two point correlator of this current which we will compute. In the limit \( N = \infty \), this model becomes exactly solvable for any \( d \) [1, 32, 42]. The solution is made manifest by performing a Hubbard-Stratonovich transformation to Eq. (C1):

\[
S = \frac{1}{2} \int d^{d+1}x \left[ \partial_\mu \phi^a \partial^\mu \phi^a + \frac{i}{\sqrt{N}} \left( \phi^a \phi^a - \frac{N}{g} \right) + \frac{\lambda^2}{4u} \right],
\]

and taking the \( u \to \infty \) limit, which imposes the constraint \( \phi_a \phi_a = N/g \) (the theory is then a sigma model). For spatial dimensions \( 1 < d < 3 \), this model has an interacting QCP obtained by tuning \( g \), and distinct from the Gaussian one at \( u = 0 \). We shall study correlation functions in the vicinity of this fixed point.

The relevant scalar operator of interest here, \( \mathcal{O} \), is often crudely thought of as \( \phi^2 = \phi^a \phi^a \). However, one finds more precisely that at \( N = \infty \) [11]:

\[
\mathcal{O} = i \sqrt{N} \lambda.
\]
(Our normalization of $\mathcal{O}$ differs by a factor of $\sqrt{N}$ from that in [11].) We will split our discussion from henceforth into two parts, depending on whether the model is in a disordered phase where $\langle \phi^a \rangle = 0$, or an ordered phase where $\langle \phi^a \rangle \neq 0$. Let us note that in all $d$, the dimension of $\mathcal{O}$ is $\Delta = 2$, and the fixed point has $z = 1$.

1. Disordered Phase

In the disordered phase, the saddle point equations imply that $\langle \phi^a \rangle = 0$, and

$$\langle \mathcal{O} \rangle = Nm^2,$$  \hspace{1cm} (C5)

where $m$ is an effective mass that depends on $g$ and temperature $T$, and scales as $N^0$. We perform the Gaussian path integral over $\phi^a$ in Eq. (C3). Keeping only the leading order terms at $N = \infty$, one finds [43]

$$S_{\text{eff}} = \int d^{d+1}x \left[ ig\sqrt{N}\lambda + \frac{1}{2} \lambda \Pi(-\partial^2) \lambda \right]$$  \hspace{1cm} (C6)

where

$$g = \frac{1}{2g} + \frac{1}{2g_c},$$  \hspace{1cm} (C7a)

$$\Pi(p^2) = \frac{1}{4\mu} + \frac{1}{2} \int \frac{d^{D}q}{(2\pi)^D} \frac{1}{(p + q)^2q^2}.$$  \hspace{1cm} (C7b)

Hence we have identified the detuning parameter $g$ in terms of the deviation of $g$ from its critical value, $g_c$. Let us review the explicit relation between $g$, $m$ and $T$ in the disordered phase:

$$\frac{1}{g} - \frac{1}{g_c} = \int \frac{d^d k}{(2\pi)^d} T \sum_{\omega_n} \frac{1}{|k|^2 + \omega^2_n + m^2} - \int \frac{d^{d+1} p}{(2\pi)^{d+1}} \frac{1}{p^2}$$

$$= \int \frac{d^d k}{(2\pi)^d} \left[ \coth(\epsilon(|k|)/2T) - \frac{1}{2|k|} \right] = \int \frac{d^d k}{(2\pi)^d} \left[ \frac{n_b(\epsilon(|k|))}{\epsilon(|k|)} + \frac{1}{2\sqrt{k^2 + m^2} - \frac{1}{2|k|}} \right]$$

$$= \int \frac{d^d k}{(2\pi)^d} \left[ \epsilon(|k|) / \epsilon(|k|) + \frac{m^{d-1}}{(4\pi)^{(d+1)/2}} \Gamma\left(1 - \frac{d}{2}\right) \right]$$

$$= \frac{1}{2^{d-1} \pi^{d/2} \Gamma\left(\frac{d}{2}\right)} \int_m^\infty \frac{d\epsilon}{\epsilon^2 - m^2} \left( \epsilon^{d-1} \frac{n_b(\epsilon)}{\epsilon} + \frac{m^{d-1}}{(4\pi)^{(d+1)/2}} \Gamma\left(1 - \frac{d}{2}\right) \right).$$  \hspace{1cm} (C8)

where, here and below, we have defined the single particle dispersion relation

$$\epsilon(|k|) = \sqrt{|k|^2 + m^2},$$  \hspace{1cm} (C9)

as well as the Bose-Einstein distribution, $n_b(\epsilon) = 1/(\epsilon^{2}/T - 1)$. $\omega_n = 2\pi n T$ is a bosonic Matsubara frequency with $n$ being an integer. In $d = 2$, one finds the closed form solution [1]

$$m(g, T) = 2T \sinh^{-1} \left( \frac{1}{2} e^{4\pi g/T} \right).$$  \hspace{1cm} (C10)

We must also compute the normalization of the two point function $C_{\mathcal{O}\mathcal{O}}$, in order to compare our direct computation of $c_{1,2}$ with Eq. (A18). From Eq. (C7b), it is straightforward to see

$$- \frac{N}{C_{\mathcal{O}\mathcal{O}}} \equiv p^{d-D} \int \frac{d^D q}{(2\pi)^D} \frac{1}{2q^2 (q - p)^2} = \frac{\Gamma\left(\frac{d-1}{2}\right) \Gamma\left(\frac{3-d}{2}\right)}{2^{d-1} \pi^{d/2} \Gamma\left(\frac{3}{2}\right)}.$$  \hspace{1cm} (C11)

Standard Feynman tricks may be used to compute this integral.

The conductivity follows directly from coupling the action Eq. (C1) to a gauge field $A^\mu$ and computing $\delta^2 S / \delta(A^\mu)^2$, as described in [3]:

$$\sigma(\Omega_\ell) = -\frac{T}{\Omega_\ell} \sum_{\omega_n} \int \frac{d^d k}{(2\pi)^d} \left[ \frac{4}{d} |k|^2 G_{\phi_2 \phi_2}(\Omega_\ell - \omega_n, k) G_{\phi_1 \phi_1}(\omega_n, -k) - G_{\phi_1 \phi_1}(\omega_n, k) - G_{\phi_2 \phi_2}(\omega_n, k) \right]$$  \hspace{1cm} (C12)

where $\Omega_\ell = 2\pi \ell T$ is a Matsubara frequency ($\ell \geq 0$ is an integer), and the $\phi$ Green’s function in the disordered phase
is

\[ G_{\phi_1, \phi_2}(\omega_n, k) = G_{\phi_2, \phi_2}(\omega_n, k) = \frac{1}{\omega_0^2 + |k|^2 + m^2}. \]  

(C13)

As it stands, the integral Eq. (C12) is divergent. In the disordered phase, it may be regulated by multiplying the last two terms by \( \partial k_x/\partial k_x \), and integrating by parts on \( k_x \). The sum over Matsubara frequencies may subsequently be performed explicitly, as in [11], along with the angular integral over \( k \):

\[
\sigma(\Omega_\ell) = -\frac{1}{2^{d-3}\pi^{d/2}d\Gamma(d/2)\Omega_\ell} \int_m^\infty \text{d} \epsilon \left( \epsilon^2 - m^2 \right)^{d/2} \left[ \frac{2n_\ell(\epsilon)}{\Omega_\ell^2 + 4\epsilon^2} - \frac{\Omega_\ell^2}{4\epsilon(4\epsilon^2 + \Omega_\ell^2)} - \frac{n_\ell(\epsilon)^2}{2T\epsilon} - \frac{(1 + \epsilon/T)n_\ell(\epsilon)}{2\epsilon^2} \right].
\]  

(C14)

To analyze the asymptotics, we proceed in two steps. We first begin with the second term in the above integral:

\[
\int_m^\infty \text{d} \epsilon \left( \epsilon^2 - m^2 \right)^{d/2} \frac{\Omega_\ell}{4\epsilon(4\epsilon^2 + \Omega_\ell^2)} = -\frac{m^{d-1}}{4d\pi\Omega_\ell^2} \left( \frac{d}{2} \right) \left( \frac{1 + d}{2} \right) \left\{ \Omega_\ell^2 \left( 1 + O \left( \frac{m^2}{\Omega_\ell^2} \right) \right) \right\}
\]  

(C15)

This expression contains all contributions to this integral, up to subleading polynomial contributions in \( m/\Omega_\ell \), as denoted explicitly. The first line of this equation contains a contribution to the conductivity at \( O(\Omega_\ell^{-1}) \), which will be a part of the \( c_1 \) term; the second line contains the \( \sigma_\infty \) and \( c_2 \) terms respectively. There is a second contribution to the conductivity of importance in our asymptotic expansion, which will arise from the last two terms in Eq. (C14), and also contributes to the \( c_1 \) term:

\[
\int_m^\infty \text{d} \epsilon \left( \epsilon^2 - m^2 \right)^{d/2} \frac{n_\ell(\epsilon)^2}{2T\epsilon} = -\frac{1}{2} \int_m^\infty \text{d} \epsilon \left( \epsilon^2 - m^2 \right)^{d/2} \frac{\partial n_\ell(\epsilon)}{\partial \epsilon} \epsilon
\]  

(C16)

Combining Eqs. (C8), (C15) and (C16), and using \( \Gamma \)-function identities, we obtain

\[
\sigma(\Omega_\ell) = \frac{\pi^{1-\frac{d}{2}}}{2^{2d}\Gamma(1 + \frac{d}{2})\sin(\frac{\pi}{2}(d-1))} \left[ 1 + \frac{2d}{\Omega_\ell^2} m^2 \right] + \frac{2}{\Omega_\ell} \left[ \frac{1}{8} - \frac{1}{\Omega_\ell^2} \right] + \cdots
\]  

\[
= \frac{\pi^{1-\frac{d}{2}}}{2^{2d}\Gamma(1 + \frac{d}{2})\sin(\frac{\pi}{2}(d-1))} \left[ 1 + 2d \frac{g}{N\Omega_\ell^2} \right] - \frac{4g}{\Omega_\ell} + \cdots.
\]  

(C17)

We emphasize that the last term, \(-4g/\Omega_\ell\), does not depend on temperature, which is a non-trivial consequence of the self-consistency equation for the mass \( m(g, T) \), (C8).

Let us now compare Eq. (C17) to the CFT formalism developed previously. From Eq. (C17) we have

\[
\frac{c_1}{c_2} = -\frac{2^{2d}\Gamma(1 + \frac{d}{2})\sin(\frac{\pi}{2}(d-1))}{\pi^{1-\frac{d}{2}}}
\]  

(C18)

On the other hand from Eq. (A18), using \( \Delta = 2 \), along with Eq. (C11), we predict from general CFT methods that

\[
\frac{c_1}{c_2} = \frac{2^{d+3}\pi^{d/2}d\Gamma(d/2)\Gamma(\frac{d+1}{2})\Gamma(\frac{d-1}{2})\Gamma(\frac{d+1}{2})\Gamma(\frac{d-1}{2})\Gamma(\frac{d+1}{2})\Gamma(\frac{d-3}{2})}{\Gamma(\frac{d+1}{2})\Gamma(\frac{d-1}{2})\Gamma(\frac{d+1}{2})\Gamma(\frac{d-3}{2})} = \frac{2^{d+2}\pi^{\frac{d+1}{2}}\Gamma(\frac{d}{2})\Gamma(d-1)\sin(\frac{\pi}{2}(d-3))}{\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})}
\]  

(C19)

Applying a few more \( \Gamma \)-function identities, one finds that Eqs. (C18) and (C19) are exactly the same. This serves as a highly non-trivial check of our CFT formalism. In \( d = 2 \), it was computed in [11] that

\[
C_{JJ0} = \frac{1}{4N} + O \left( N^{-2} \right).
\]  

(C20)
and so in fact from Eq. (C17) (in \( d = 2 \)) we see that \( c_{1,2} \) agree precisely with CFT predictions.

In \( d = 2 \), it is rather non-trivial that Eq. (C17) agrees with Eq. (3). The reason is that, a priori, one might have expected two contributions to the \((m/\Omega)^2\) contribution to Eq. (3): one from \( \langle O \rangle/\Omega^2 \), and one from \((g/\Omega)^2\). Evidently, the latter contribution vanishes, implying that there is no contribution to \( \sigma(\omega) \) from conformal perturbation theory at second order. It would be interesting if there is a deep reason why this must occur in the large \( N \) limit.

### 2. Ordered Phase

Let us briefly discuss the nature of the conductivity in the ordered phase, which requires \( g < 0 \). For simplicity, we work at \( T = 0 \), and will comment on the extension to \( T > 0 \) briefly at the end of this subsection. We also assume that the symmetry breaking is oriented along the \( a = 1 \) direction,

\[
\langle \phi_1 \rangle = \sqrt{N} \phi_0 \\
\phi_0 = (-2g)^{1/2} = \left( \frac{1}{g} - \frac{1}{g_c} \right)^{1/2}, \quad g < g_c,
\]

so that we can write

\[
\phi_a(x) = \left( \sqrt{N} \phi_0 + \phi(x), \phi_{a>1}(x) \right)
\]

Our key result will be the emergence of logarithmic corrections to \( \sigma(\omega) \) (corresponding to a current that mixes with the \( a = 1 \) direction), which goes beyond the result given in Eq. (3). Extending the derivation in [43] to general \( d \), we find that the \( \phi \) Green’s function is

\[
\frac{1}{G_{\phi\phi}(k)} = k^2 + \frac{2\phi_0^2}{\Pi(k)} = k^2 + M^{d-1}k^{3-d}, \quad M^{d-1} = \frac{|\text{CFT}|\phi_0^2}{N},
\]

where we have introduced a mass scale \( M \) that is associated with amplitude fluctuations of \( \phi_a \) (along \( a = 1 \)). For instance in \( d = 2 \), \( G_{\phi\phi} = k(k + M) \). Using Eq. (C12) at \( T = 0 \) to compute the conductivity associated with \( J_{12}^2 \), we find at large \( \Omega \):

\[
\sigma = -\frac{1}{\Omega} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} 4 k^2 G_{\phi\phi}(-k_0, -k)G_{\phi\phi}(k_0 - \Omega, k) + O \left( \frac{1}{\Omega} \right).
\]

\( G_{\phi\phi}(k) = 1/k^2 \) is simply the free massless Goldstone propagator. This integral is divergent, as was Eq. (C12) in the disordered phase. The simple method that we used to regulate Eq. (C12) in the disordered phase fails in the ordered phase, and so we resort to a hard momentum cutoff \( \Lambda \). This leads to UV divergences in \( \Lambda \) which must be subtracted away; although such a regulator cannot unambiguously fix \( \sigma \) at \( O(\Omega^{-1}) \), we will be able to determine exactly the leading logarithmic correction to \( \sigma \). The UV divergent part of Eq. (C24) can be identified using asymptotic techniques:

\[
\sigma = -\frac{\pi^{d-1}}{(2\pi)^d \Gamma \left( \frac{1+d}{2} \right) \Omega} \int_0^\Lambda dk \frac{4k^d}{d+1} \frac{1}{k^2 + M^{d-1}k^{3-d}} + O \left( \frac{1}{\Omega} \right)
\]

\[
= -\frac{4\pi^{d-1}}{(2\pi)^d (d^2 - 1) \Gamma \left( \frac{1+d}{2} \right) \Omega} \left[ \Lambda^{d-1} - (d-1)M^{d-1} \log \frac{\Lambda}{\Omega} \right] + O \left( \frac{1}{\Omega} \right).
\]

Upon regularization, which involves subtracting a function \( \Lambda^{d-1} F(\Lambda/M) \) from \( \langle J_{12}^2 J_{12}^2 \rangle \) (the precise form of \( F \) is not necessary for the present computation), we obtain a logarithmic correction to \( \sigma \):

\[
\sigma(\omega) = \cdots - \frac{4\pi^{d-1}}{(2\pi)^d (d+1) \Gamma \left( \frac{1+d}{2} \right) \Omega} \left( M^{d-1} \log \frac{\Omega}{M} + \cdots \right).
\]

Let us evaluate these coefficients explicitly in the special case \( d = 2 \). First, we can relate \( M \) to \( g \) explicitly [43]:

\[
M = 32|g|.
\]
Using $G_{\phi\phi} = 1/[k(k + M)]$:
\[
\sigma = \frac{1}{\Omega} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{4k_x^2}{k(k + M)(k_x^2 + k_y^2 + (k_0 - \Omega)^2)} - \frac{1}{k^2} - \frac{1}{k(k + M)} \right].
\]  
(C28)

After doing angular integrals (integrate over $\cos \theta$ variable for first term) we find
\[
\sigma = \frac{1}{\Omega} \int \frac{dk}{2\pi^2} \left[ \frac{(k^2 - \Omega^2)^2 \log\left(\frac{k + \Omega}{k - \Omega}\right)}{8\Omega^3(k + M)} - 2k\Omega(k^2 + \Omega^2) \right] + 1 + \frac{k}{k + M}.
\]  
(C29)

Taylor expanding this integrand, it is straightforward to identify a linear and logarithmically divergent contribution in $\Lambda$. After regularization, one finds
\[
\sigma = \cdots - \frac{2M}{3\pi^2\Omega} \log \frac{\Omega}{M} + \cdots,
\]  
(C30)
in agreement with Eq. (C26), and leading to Eq. (9) in the main text.

For a current such as $J_{23}^0$ – which does not mix with the direction of broken symmetry – both propagators in Eq. (C12) are $1/k^2$. Hence, we find that the conductivity at all frequencies is
\[
\sigma^{(23)} = \frac{1}{16}.
\]  
(C31)

This again disagrees with Eq. (3), and is a consequence of the breakdown of the conformal perturbative expansion in the symmetry broken phase, as discussed in the main text.

Finally, let us briefly mention the generalization of (C26) to finite temperature $T$. We will consider $2 < d < 3$. Then it is well-known that superfluidity exists at finite temperature $T < T_c$, and within the superfluid phase $\rho_\varphi \propto M^{d-1}$ will acquire temperature dependence, i.e. $\varrho_0(g, T)$. The explicit form could be computed from (C8) and (C21b). In order to compute $\sigma(\Omega)$, we must first compute the polarization function $\Pi(p, \Omega_n)$. Generalizing (C7b) to $T > 0$, we find
\[
\Pi(p, \Omega_n) = \frac{1}{2} \sum_{\ell \neq 0} \int \frac{d^dq}{(2\pi)^d} G_{\phi\phi, \ell} \varrho(p - q, \Omega, \Omega) G_{\phi\phi, \ell}(q, \Omega, \Omega) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left( 1 + 2n_\varrho(|q|) \right) \left( \Omega_n^2 + |p - q|^2 - |q|^2 \right) \left( 4|q|^2\Omega_n^2 + (\Omega_n^2 + |p - q|^2 - |q|^2)^2 \right)
\]  
(C32)
The integral over $q$ is convergent, but we will not find it necessary to compute it analytically. As we expect, upon sending $T \to 0$ (so that $n_\varrho \to 0$), we may analytically recover the $T = 0$ result (C11). At finite $T$, when the momenta $p$ and $\Omega_n$ are large, we find
\[
\Pi(p, \Omega_n) = \frac{|C_{\varrho\varrho}|}{N} \left( \Omega_n^2 + p^2 \right)^{d-1} + \frac{2\pi^2 g^2 \Gamma(d - 1)}{\Gamma(\frac{d}{2})} \frac{T^{d-1}}{\Omega_n^2 + p^2} + O(T^{d+1}), \quad (p^2 + \Omega_n^2 \gg T^2).
\]  
(C33)

Hence, we may approximate $\Pi(p)$ with its $T = 0$ form, so long as the argument $p$ is large; corrections will arise at $O(T/p)^{d-1})$. From (C23), we also conclude that $G_{\phi\phi}(p, \Omega_n)$ is well approximated by its $T = 0$ form so long as $p^2 + \Omega_n^2 \gg T^2$:
\[
G_{\phi\phi}(p, \Omega_n) = k^2 \left[ 1 + \left( \frac{M}{k} \right)^{d-1} - \frac{|C_{\varrho\varrho}|}{N} \frac{2\pi^2 g^2 \Gamma(d - 1)}{\Gamma(\frac{d}{2})} \left( \frac{MT}{k^2} \right)^{d-1} + \cdots \right], \quad k^2 \equiv p^2 + \Omega_n^2,
\]  
(C34)
where we recall that $M$ now depends on both $g$ and $T$. An asymptotic analysis reveals that the $\Lambda$-dependence of (C25), appropriately generalized to $T > 0$, is unchanged. Hence, after regulation, we conclude that the coefficient of the logarithmic divergence in (C26) is unchanged. The vanishing of the logarithm at the critical temperature $T = T_c$ is due to the vanishing of the superfluid density, i.e. $\varrho_0, M \to 0$ as $T \to T_c$.

**Appendix D: Sum Rules at $\Delta = 2$ or $d + z - 2$**

We derive the sum rule for the case $\Delta = 2$ or $\Delta = z + d - 2$ in general dimensions based on the asymptotic expansion and the causal properties of the current corre-
where the contour is shown in Fig. 3. Changing variables to \( \omega \), where \( z = \omega^2 \), and using \( F(|z| \to \infty) = 0 \), we find
\[
\int_0^{\infty} d\omega \frac{2\omega}{\pi} \frac{\text{Im}[\delta C_R(\omega) - \delta C^\infty]}{\omega^2 + \alpha^2} = F(-\alpha^2) \tag{D7}
\]
Taking the \( \alpha \to 0 \) limit, we have \( F(0) = \delta C_R(0) - \delta C^\infty = C_R(0) - \delta C^\infty \), which leads to
\[
\int_0^{\infty} d\omega \frac{\text{Im}[\delta C_R(\omega)]}{\omega} = \frac{\pi}{2} [C_R(0) - \delta C^\infty] \tag{D8}
\]
where we used the fact that \( \delta C^\infty \) is real. If \( C_R(0) \) doesn’t vanish, it will contribute a delta function \( \delta(\omega) \) to \( \text{Re} \sigma \). Moving \( C_R(0) \) to the l.h.s., we thus obtain the sum rule:
\[
\int_0^{\infty} d\omega \left[ \left. \text{Re} \sigma(\omega) - \sigma(\omega) \right|_{g=T=0} \right] = -\frac{\pi}{2} \delta C^\infty = \frac{\pi}{2} c_1 g, \tag{D9}
\]
which is our main result. In the second equality, we have specialized the general result to the \( \Delta = 2 \) case.

1. **O(N) model at \( N = \infty \)**

We now apply the sum rule Eq. (D9) to the O(N) model at \( N = \infty \) in \( 1 < d < 3 \):
\[
\int_0^{\infty} d\omega \text{Re} \left[ \left. \sigma(\omega) - \sigma(\omega) \right|_{g=T=0} \right] = -2\pi g, \tag{D10}
\]
which holds everywhere in the phase diagram except in the ordered phase, \( g < 0 \) at \( T \leq T_c \). We have used the fact that for all dimensions \( 1 < d < 3 \): \( \Delta = 2 \), and \( c_1 = 4 \), Eq. (C17). The special case of Eq. (D10) at \( g = 0 \) and \( d = 2 \) was first derived in [7].

Let us briefly comment on \( C_R(0) \). It will vanish at \( T = 0 \) when \( g > 0 \) because the DC conductivity vanishes. However, at \( T > 0 \) in the O(N) model at \( N = \infty \), \( \text{Re} \sigma(\omega) \) will receive a \( \delta(\omega) \) contribution due to thermally activated charge carriers, with weight proportional to \( C_R(0) \neq 0 \). This delta function is a peculiarity of the \( N = \infty \) limit, where quasiparticles exist, and is not expected at finite \( N \) or more generally in interacting QCPs.

Appendix E: Monte Carlo simulations

1. **Model and observables**

For numerical simulations, we study a complex scalar field theory regularized on a cubic lattice. Explicitly, we consider the classical partition function \( Z = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{-S[\psi, \psi^*]} \) with lattice action,
\[
S = \sum_{\langle i, j \rangle} [\psi_i \psi_j^* + c.c + 2r \sum_i |\psi_i|^2 + 4u \sum_i |\psi_i|^4. \tag{E1}
\]
Here, $\psi_i$ is a complex scalar field residing on the sites of a cubic lattice, which corresponds to a $D = 2 + 1$ dimensional discretized Euclidean space-time. We study lattices with space-time volume $V = \beta \times L \times L$. Throughout, we set the inverse temperature $\beta = L$. The lattice model has a global U(1) symmetry and hence it is expected to be described at long distances by the $\phi^4$ theory in Eq. (7) with $N = 2$.

At a critical coupling $u = u_c$ the system undergoes a phase transition between a disordered, $u > u_c$, and a broken symmetry phase, $u < u_c$. We define the dimensionless detuning parameter as $\delta u = \frac{u - u_c}{u_c}$. Here, we focus only on the disordered phase, i.e. $\delta u > 0$. Our main observable is the dynamical conductivity $\sigma(i\Omega_n)$, evaluated at Matsubara frequency $\Omega_n = 2\pi n / \beta$ with $n \in \mathbb{Z}$. To define the conductivity, we introduce an external U(1) gauge field $A_{i,i+\eta}$ through a Peierls substitution $\psi_i \psi_{i+\eta}^* \rightarrow \psi_i \psi_{i+\eta}^* e^{iA_{i,i+\eta}}$. The bond current is then $J_{i,i+\eta} = \frac{\delta \sigma_{xx}}{\delta A_{i,i+\eta}}$ and the conductivity is defined as,

$$\sigma(i\Omega_n) = -\frac{1}{\Omega_n} \Pi_{xx}(\Omega_n) \quad (E2a)$$

$$\Pi_{xx}(\Omega_n) = \frac{1}{\beta L^2} \sum_{i,j} e^{i\Omega_n \tau_{i,j}} \frac{\delta \langle J_{i,i+\eta} \rangle}{\delta A_{i,j}} \quad (E2b)$$

where $\tau_{i,j}$ is the discrete imaginary time distance between the lattice points $i,j$. We measure the conductivity in units of $Q^2 / h$, which amounts to multiplying the conductivity in Eq. (E2b) by $2\pi$. We study lattices with linear size $L = 512$ and we set the microscopic parameter $r = -5.89391$. The critical coupling is then $u_c = 7.70285(5)$ as was determined in a previous study [12]. We made sure that the correlation length, $\xi$, satisfies $\xi < L/2$. In Fig. 4(a) we give the Matsubara conductivity in the disordered phase for a set of detuning parameters $\delta u$ in close vicinity to the phase transition.

2. Fitting procedure

As discussed in the main text, in two spatial dimensions the conductivity is a universal amplitude and hence near criticality it is expected to follow a scaling form $\sigma(i\Omega_n, \delta u) = f_+(i\Omega_n/m)$, where $m$ is the single particle gap in the disordered phase. Near criticality the gap vanishes following a power law form $m = m_0(\delta u)^\nu$, with $\nu$ being the correlation length exponent and $m_0$ a non-universal coefficient.

To compute the scaling function from the numerical Monte Carlo data we rescale the Matsubara frequency axis by the single particle gap. We found that at low frequency all curves collapse to a single universal curve, whereas at high frequency we observe significant deviation from the scaling form.

To understand the origin of these non-universal corrections, we note that lattice discretization inevitably introduces a UV cutoff scale $\Lambda \sim 1/a$ where $a$ is the lattice constant. At large frequency $\Omega_n > \Lambda$, the numerical result deviate from the continuum limit as lattice scale effects become sizable. The cutoff scale corrections are expected to be smooth both in $\Omega_n$ and $\delta u$ and we model them using a simple cubic polynomial ansatz

$$\sigma(i\Omega_n, \delta u, \Lambda) \approx f_+(i\Omega_n/m) + \sum_{l=1}^3 \alpha_l \Omega_n^l \quad (E3)$$

We further assume that since we study a small range of detuning parameters, the coefficients $\alpha_l$ have a weak dependence on $\delta u$ and we therefore take them to be constants.

Our main task now is to compare the Monte Carlo data with the asymptotic large frequency behavior of the optical conductivity predicted in Eq. (3). For $N = 2$, the correlation length exponent was estimated in previous high precision Monte Carlo studies to be $\nu = 0.6717(3)$.
such that the power law exponents in Eq. (3) equal $(d + z - \Delta)/z = 1/\nu = 1.48987$ and $\Delta/z = 3 - 1/\nu = 1.51013$. We see that the two exponents are nearly identical and hence cannot be resolved within our numerical accuracy. We, therefore, combine them to a single exponent, and consider the following large frequency form for the optical conductivity,

$$f_+(x \gg 1) \sim \sigma_\infty + C \times x^{-1/\nu} \quad (E4)$$

For the infinite frequency conductivity, we take the high precision bootstrap estimate $\sigma_\infty = 0.3554(6) [33]$. This leaves us with four free fitting parameters $C, \alpha_i=1,2,3$ that we determine using least square minimization. Since the expression in Eq. (E4) is valid only in the high frequency limit, in the numerical fit we only use data points that satisfy $\Omega_c < \Omega_n$. We performed the fit on a range of lower cutoff frequencies $\Omega_c/m_0 = 50, 100, 150, 200$. We find that the coefficients $\alpha_i$ are nearly independent of $\Omega_c$, and equal $\alpha_1 \approx -0.1, \alpha_2 \approx 0.01$ and $\alpha_3 \approx -0.001$, working in units where the UV cutoff $\Lambda$ (inverse lattice spacing) has been set to unity. We subtract the cutoff scale corrections and plot the universal scaling function $f_+$ in Fig. 4(b). Our estimate for the power-law coefficient is $C = -5.0(5)$. The quoted numerical error is dominated by variations with respect to the lower cutoff frequency $\Omega_c$. For curve plotted in Fig. 2 in the main text we used $\Omega_c/m = 100$, for which $C = -4.83$ and the reduced goodness of fit equals $\chi^2 = 0.93$.

As a final remark, we wish to emphasize that although our results are consistent with the predicted scaling form, our numerical analysis involves subtraction of a non-universal background signal that is relatively large compared to the high frequency component of the universal scaling function. As a consequence, we did not manage to extract an independent estimate for the predicted power law exponents. Improving the numerical scheme for eliminating the large frequency corrections to scaling is an interesting line of research that we intend to study in the future.
When this condition is not satisfied, additional logarithms appear in Eq. (3).

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