MILNOR INVARIANTS OF WELDED LINKS

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Abstract. For a classical link, Milnor defined a family of isotopy invariants, called Milnor $\mu$-invariants. In this paper, we show that Milnor $\mu$-invariants can be extended to welded links. The proof contains an alternative and combinatorial proof for the invariance of the original $\mu$-invariants of classical links.

1. Introduction

In [10] [11], Milnor defined a family of isotopy invariants of classical links in the $3$-sphere, called Milnor $\mu$-invariants. Given an $n$-component classical link $L$, the Milnor number $\mu_L(I) \in \mathbb{Z}$ of $L$ is specified by a finite sequence $I$ of indices in $\{1, \ldots, n\}$. This integer is only well-defined up to a certain indeterminacy $\Delta_L(I)$, i.e. the residue class $\pi_L(I)$ of $\mu_L(I)$ modulo $\Delta_L(I)$ is an invariant of $L$. It is known in [11] Theorem 8 that $\pi_L(I)$ is invariant under link-homotopy when the sequence $I$ has no repeated indices. Here link-homotopy is an equivalence relation generated by self-crossing changes and isotopies (cf. [10]). In [4], Habegger and Lin defined Milnor numbers for classical string links in the $3$-ball, and proved that they are well-defined invariants without taking modulo. In this sense, Milnor numbers are suitable for classical string links rather than classical links. These numbers for classical string links are called Milnor $\mu$-invariants.

The notion of virtual links, introduced by Kauffman in [5], is a diagrammatic generalization of classical links in the $3$-sphere. It naturally yields the notion of virtual string links. Virtual (string) links are generalized (string) link diagrams considered up to an extended set of Reidemeister moves. In the virtual context, there are two forbidden moves. Welded (string) links arise from virtual (string) links when we allow to use one of the two forbidden moves, called the over-crossings commute move. The notion of welded objects was first studied by Fenn, Rimányi and Rourke in [3]. The aim of this paper is to give an extension of Milnor $\pi$-invariants to welded links in a combinatorial way.

In [2], Dye and Kauffman first tried to extend Milnor link-homotopy $\pi$-invariants to virtual links. Kotorii pointed out in [6] Remark 4.6] that the extension of Dye and Kauffman is incorrect. In fact, there exists a classical link having two different values of the Dye-Kauffman’s $\pi$. Hence the Dye-Kauffman’s $\pi$ is not well-defined even for classical links (see Remark 5.9).

A successful extension is due to Kravchenko and Polyak in [7]. Using Gauss diagrams, they extended Milnor link-homotopy $\mu$-invariants to virtual tangles, which are slight generalizations of virtual string links. In [6], Kotorii gave an extension of Milnor link-homotopy $\pi$-invariants to virtual links via the theory of nanowords introduced by Turaev in [14]. Both extensions are actually invariants of welded...
objects and combinatorial, but they are restricted to the case of link-homotopy invariants.

In [1], Audoux, Bellingeri, Meilhan and Wagner defined a 4-dimensional version of Milnor $\mu$-invariants. Combining this version of Milnor $\mu$-invariants with the tube map, they extended Milnor isotopy $\mu$-invariants to welded string links. Here, the tube map is a map from welded string links to embedded annuli in the 4-ball (cf. [15, 13]).

The topological approach in [1] to Milnor invariants might be able to extend to welded links. However, the authors believe that it is important to consider a combinatorial approach, since the advantage of virtual/welded objects is that they are combinatorial, and the original Milnor invariants are defined topologically.

In [11], Milnor gave an algorithm to compute $\mu$-invariants for a classical link based on its classical link diagram. This algorithm can be applied to virtual link diagrams and gives us certain values. We can expect that the values are invariants of welded links. That is, it is very likely that there exists a “natural” extension of Milnor isotopy $\mu$-invariants to welded links. In this paper, we show that this is indeed the case by verifying the invariance of the values (Theorem 5.2). The proof of Theorem 5.2 provides an alternative proof for the invariance of the original $\mu$-invariants in terms of link diagrams. As a generalization of [11 Theorem 8], we also establish the invariance of the extended Milnor $\mu$-invariants for non-repeated sequences under self-crossing virtualizations (Theorem 6.1). Our extension also contains the case of welded string links (Corollary 7.1), which coincides with the extension of Audoux, Bellingeri, Meilhan and Wagner in [1]. The approach in this paper is based on virtual link diagrams, purely combinatorial and different from [7, 6, 1].

2. Preliminaries

For an integer $n \geq 1$, an $n$-component virtual link diagram is the image of an immersion of $n$ ordered and oriented circles into the plane, whose singularities are only transverse double points. Such double points are divided into classical crossings and virtual crossings as shown in Figure 2.1.

![Classical and virtual crossings](image)

**Figure 2.1.** Two types of double points

Welded Reidemeister moves consist of Reidemeister moves R1–R3, virtual moves V1–V4 and the over-crossings commute move OC as shown in Figure 2.2. A welded isotopy is a finite sequence of welded Reidemeister moves, and an $n$-component welded link is an equivalence class of $n$-component virtual link diagrams under welded isotopy. We emphasize that all virtual link diagrams and welded links are ordered and oriented.

Let $D$ be an $n$-component virtual link diagram. Put a base point $p_i$ on some arc of each $i$th component, which is disjoint from all crossings of $D$ ($1 \leq i \leq n$). A base point system of $D$ is an ordered $n$-tuple $p = (p_1, \ldots, p_n)$ of base points on $D$. We denote by $(D, p)$ a virtual link diagram $D$ with a base point system $p$. The classical under-crossings of $D$ and base points $p_1, \ldots, p_n$ divide $D$ into a finite number of segments possibly with classical over-crossings and virtual crossings. We call such a segment an arc of $(D, p)$. 
As shown in Figure 2.3, let $a_{i1}$ be the outgoing arc from the base point $p_i$, and let $a_{i2}, \ldots, a_{im_i+1}$ be the other arcs of the $i$th component in turn with respect to the orientation, where $m_i+1$ is the number of arcs of the $i$th component ($1 \leq i \leq n$). In the figure, $u_{ij} \in \{a_k\}$ denotes the arc which separates $a_{ij}$ and $a_{ij+1}$. Let $\varepsilon_{ij} \in \{\pm 1\}$ be the sign of the crossing among $a_{ij}, u_{ij} \text{ and } a_{ij+1}$, and we put $v_{ij} = u_{ij}^{\varepsilon_{ij}} u_{ij+1}^{\varepsilon_{ij}} \cdots u_{ij}^{\varepsilon_{ij}}$ for $1 \leq j \leq m_i$. We call the word $v_{ij}$ a part longitude of $(D, p)$.

Let $A = \langle \alpha_1, \ldots, \alpha_n \rangle$ be the free group of rank $n$, and let $\overline{A}$ be the free group on the set $\{a_{ij}\}$ of arcs. The arcs $a_{ij}$ will be also called letters when they are regarded as elements in $\overline{A}$. For an integer $q \geq 1$, a sequence of homomorphisms

$$
\eta_q = \eta_q(D, p) : \overline{A} \rightarrow A
$$

associated with $(D, p)$ is defined inductively by

$$
\eta_1(a_{i1}) = \alpha_i,
$$

$$
\eta_{q+1}(a_{ij}) = \alpha_i \quad \text{and} \quad \eta_{q+1}(a_{ij}) = \eta_q(v_{ij-1}) \alpha_i \eta_q(v_{ij-1}) \quad (2 \leq j \leq m_i + 1).
$$

Note that our definition of $\eta_q$ is very similar to the original one in [11], but they are not the same because, in [11], $a_{i1} \cup a_{im_i+1}$ is a single arc.

Let $\mathbb{Z}[[X_1, \ldots, X_n]]$ be the ring of formal power series in non-commutative variables $X_1, \ldots, X_n$ with integer coefficients. The Magnus expansion is a homomorphism

$$
E : A \rightarrow \mathbb{Z}[[X_1, \ldots, X_n]]
$$

defined, for $1 \leq i \leq n$, by

$$
E(\alpha_i) = 1 + X_i \quad \text{and} \quad E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots.
$$
For each $1 \leq i \leq n$, let $w_i$ be the sum of the signs of all classical self-crossings of the $i$th component. We call the word $l_i = a_i^{-w_i} v_{im_i}$ the $i$th preferred longitude of $(D, p)$.

**Definition 2.1.** For a sequence $j_1 \ldots j_s$ ($1 \leq s < q$) of indices in $\{1, \ldots, n\}$, the Milnor number $\mu_{(D, p)}(j_1 \ldots j_s)$ of $(D, p)$ is the coefficient of $X_{j_1} \cdots X_{j_s}$ in the Magnus expansion $E$ of $\eta_q(l_i)$.

The remainder of this section gives several lemmas, which will be used in Section 3 and subsequent sections.

Let $q \geq 1$ be an integer. For a group $G$, we denote by $G_q$ the $q$th term of the lower central series of $G$, i.e., $G_1 = G$ and $G_{q+1} = [G, G_q]$ is the normal subgroup generated by all $[g, h] = ghg^{-1}h^{-1}$ with $g \in G$ and $h \in G_q$. For two normal subgroups $N$ and $M$ of $G$, we denote by $NM$ the normal subgroup of $G$ generated by all $nm$ with $n \in N$ and $m \in M$.

The following lemma, which will be used very often in this paper, is easily shown.

**Lemma 2.2 (\cite{11}, page 290).** Let $G$ be a group. For any $g, x, y \in G$, the following hold.

1. If $x \equiv y \pmod{G_q}$, then $x^{-1}gx \equiv y^{-1}gy \pmod{G_{q+1}}$.
2. Let $N$ be a normal subgroup of $G$. If $x \equiv y \pmod{G_q}$, then $x^{-1}gx \equiv y^{-1}gy \pmod{G_{q+1}N}$.

For the $q$th term $A_q$ of the lower central series of the free group $A = \langle \alpha_1, \ldots, \alpha_n \rangle$, we have the following.

**Lemma 2.3 (\cite{11} pages 290 and 291).** The following hold.

1. For any $a_{ij} \in A$, $\eta_q(a_{ij}) \equiv \eta_{q+1}(a_{ij}) \pmod{A_q}$.
2. For any $r_{ij} = a_{ij}^{-1} u_{ij}^{-1} a_{ij} u_{ij}^{-1}$ ($1 \leq j \leq m_i$), $\eta_q(r_{ij}) \equiv 1 \pmod{A_q}$.

Although this is essentially shown in \cite{11}, we give the proof for the readers’ convenience.

**Proof of Lemma 2.3** (1) This is proved by induction on $q$. Since $A_1 = A$, the assertion holds for $q = 1$. Assume that $q \geq 1$. For $j = 1$, we have $\eta_{q+1}(a_{1j}) = a_1 = \eta_{q+2}(a_{1j})$ by definition. For $j \geq 2$, by the induction hypothesis and Lemma 2.2(1), we have

\[
\eta_{q+1}(a_{1j}) = \eta_q(v_{ij}^{-1})a_1\eta_q(v_{ij-1}) \equiv \eta_{q+1}(v_{ij}^{-1})a_1\eta_{q+1}(v_{ij-1}) \pmod{A_{q+1}}
\]

\[
= \eta_{q+2}(a_{1j}).
\]

(2) Put $s_{ij} = a_{ij}^{-1} v_{ij}^{-1} a_{ij} v_{ij}^{-1}$ ($1 \leq j \leq m_i$). Then by (1) and Lemma 2.2(1), it follows that

\[
\eta_q(s_{ij}) = \eta_q(a_{ij}^{-1}) \eta_q(v_{ij}^{-1}) a_1 \eta_q(v_{ij})
\]

\[
= (\eta_{q-1}(v_{ij}^{-1})a_1^{-1}\eta_{q-1}(v_{ij})) \eta_q(v_{ij}^{-1}) a_1 \eta_q(v_{ij})
\]

\[
= \eta_q(v_{ij}^{-1}) a_1^{-1} \eta_q(v_{ij}) \eta_q(v_{ij}^{-1}) a_1 \eta_q(v_{ij}) \pmod{A_q}
\]

\[
= 1.
\]

For $j = 1$, since $r_{11} = s_{11}$, we have $\eta_q(r_{11}) \equiv 1 \pmod{A_q}$. For $j \geq 2$, we have $r_{ij} = s_{ij} u_{ij}^{-1} s_{ij}^{-1} u_{ij}^{-1}$. This implies that

\[
\eta_q(r_{ij}) = \eta_q(s_{ij}) \eta_q(u_{ij}^{-1}) \eta_q(s_{ij}^{-1}) \eta_q(u_{ij}^{-1})
\]

\[
= \eta_q(u_{ij}^{-1}) \eta_q(u_{ij}^{-1}) \pmod{A_q}
\]

\[
= 1.
\]
It is known in [8] that the Magnus expansion $E$ is injective and satisfies the following.

**Lemma 2.4** ([8 Corollary 5.7]). For any $x \in A_q$,

$$E(x) = 1 + (\text{terms of degree } \geq q).$$

By Lemmas 2.3(1) and 2.4, we have the following lemma.

**Lemma 2.5.** If $1 \leq s < q$, then

$$\mu([D, p], q, J_1 \ldots J_s) = \mu([D, p], q+1, J_1 \ldots J_s).$$

Taking the integer $q$ sufficiently large, by this lemma we may ignore $q$ and denote $\mu([D, p], q, J_1 \ldots J_s)$ by $\mu([D, p], J_1 \ldots J_s)$. In the rest of this paper, $q$ is assumed to be a sufficiently large integer.

3. Milnor numbers and welded isotopy relative base point system

A local move relative base point system is a local move on a virtual link diagram with a base point system such that it keeps the positions of base points. A $\varpi$-isotopy is a finite sequence of welded Reidemeister moves relative base point system and a local move as shown in Figure 3.1. We emphasize that in a $\varpi$-isotopy, we do not allow to use two local moves as shown in Figure 3.2. The following theorem gives the invariance of Milnor numbers under $\varpi$-isotopy.

**Theorem 3.1.** Let $(D, p)$ and $(D', p')$ be virtual link diagrams with base point systems. If $(D, p)$ and $(D', p')$ are $\varpi$-isotopic, then $\mu([D, p], I) = \mu([D', p'], I)$ for any sequence $I$.

Let $l_i$ and $l'_i$ be the $i$th preferred longitudes of $(D, p)$ and $(D', p')$, respectively ($1 \leq i \leq n$). To show Theorem 3.1, we observe the difference between $\eta_q(D, p)(l_i)$ and $\eta_q(D', p')(l'_i)$ under $\varpi$-isotopy.

**Proposition 3.2.** If $(D, p)$ and $(D', p')$ are related by a single R1 move relative base point system, then $\eta_q(D, p)(l_i) \equiv \eta_q(D', p')(l'_i) \pmod{A_q}$.

**Proof.** There are four R1 moves depending on orientations of strands. By [12 Theorem 1.1], it is enough to consider the two moves R1a and R1b in a disk $\delta$ as shown in Figure 3.3. Here, the symbol $\circ$ in the figure denotes either a base point or an under-crossing. Without loss of generality, we may assume that the R1a/R1b move is applied to the 1st component.

Let $a_{ij}$ ($1 \leq i \leq n, 1 \leq j \leq m_i + 1$) be the arcs of $(D, p)$ as given in Section 2. Let $a'_{ij}$ and $b$ be the arcs of $(D', p')$ such that each $a'_{ij}$ corresponds to the arc $a_{ij}$ of...
Claim 3.3 that \( l \leq x \)
the R1a/R1b move that relates \((D, p)\) to \((D', p')\).

Proof of Claim 3.3.

(1) By Lemma 2.3(2), we have
\[
\eta_{q}(b^{-1}a_{1h}^{-1}a_{1h'}a_{1h'}^{-1}) \equiv 1 \pmod{A_q} 
\]
(R1a move case),

\[
\eta_{q}'(b^{-1}b^{-1}a_{1h}^{-1}a_{1h'}) \equiv 1 \pmod{A_q} 
\]
(R1b move case).

Hence it follows that
\[
\eta_{q}'(b^{-1}a_{1h}') = \eta_{q}'(b^{-1}(a_{1h})^{-1}a_{1h}a_{1h'}) \equiv 1 \pmod{A_q}
\]
in the R1a move case, and that
\[
\eta_{q}'(b^{-1}a_{1h}^{-1}) = \eta_{q}'(bb^{-1}b^{-1}a_{1h}bb^{-1}) \equiv \eta_{q}'(bb^{-1}) \pmod{A_q}
\]
\[
= 1 
\]
in the R1b move case.

Before showing this claim, we observe that it implies Proposition 3.2.

For \( 2 \leq i \leq n \), the congruence \( \eta_{q}(l_i) \equiv \eta_{q}'(l'_i) \pmod{A_q} \) follows from Claim 3.3 immediately. For \( i = 1 \), in the R1a move case, it follows from Lemma 2.3(1) and Claim 3.3 that
\[
\eta_{q}(l_1) = \alpha_1^{-w_1} \eta_{q}(xy) = \alpha_1^{-w_1} \alpha_1^{-1} \eta_{q}(x) \left( \eta_{q}(x^{-1}) \alpha_1 \eta_{q}(x) \right) \eta_{q}(y) = \alpha_1^{-w_1} \eta_{q}(x) \eta_{q}(a_{1h}) \eta_{q}(y) \equiv \alpha_1^{-w_1} \eta_{q}'((a_{1h}')^{-1}x'a_{1h}y') \pmod{A_q}.
\]
(2) This is proved by induction on $q$. The assertion is certainly true for $q = 1$ or $j = 1$. Assume that $q \geq 1$ and $j \geq 2$, and consider the R1a move case.

Let $v_{ij-1}$ and $v'_{ij-1}$ be part longitudes of $(D, p')$ and $(D, p')$ as given in Section 2 respectively. If $i \neq 1$ or $v'_{ij-1}$ does not contain the word $x' c_{1h}$, then $v_{ij-1}$ is obtained from $v'_{ij-1}$ by replacing $b$ with $a_{1h}$, and $a'_{st}$ with $a_{st}$ for all $s, t$. By (1) and the induction hypothesis, we have

$$
\eta_q(v_{ij-1}) = \eta_q'(v'_{ij-1}) \pmod{A_q}.
$$

Therefore Lemma 2.2(1) implies that

$$
\eta_{q+1}(a_{ij}) = \eta_q(v_{ij-1}) \alpha_i \eta_q(v_{ij-1}) + \eta_q'(v'_{ij-1}) \pmod{A_{q+1}}
$$

If $i = 1$ and $v'_{ij-1}$ contains the word $x' c_{1h}$, then $v_{ij-1}$ and $v'_{ij-1}$ have the forms

$$
v_{ij-1} = xz \quad \text{and} \quad v'_{ij-1} = x' c_{1h} z'$$

for certain words $z \in \mathcal{A}$ and $z' \in \mathcal{A}$ such that $z$ is obtained from $z'$ by replacing $b$ with $a_{1b}$, and $a'_{st}$ with $a_{st}$ for all $s, t$. Combining (1), Lemma 2.3(1) and the induction hypothesis, it follows that

$$
\eta_q(v_{ij-1}) = \alpha_1 x \eta_q(x) \eta_q(x^{-1}) \eta_q(z) \eta_q(z) = \alpha_1 x \eta_q(x) \eta_q(a_{1h}) \eta_q(z) = \alpha_1 x \eta_q(x) \eta_q(a_{1h}) \eta_q(z) \pmod{A_q}
$$

Hence Lemma 2.2(1) implies that

$$
\eta_{q+1}(a_{ij}) = \eta_q(v_{ij-1}) \alpha_1 \eta_q(v_{ij-1}) + \eta_q'(v'_{ij-1}) \pmod{A_{q+1}}
$$

The proof for the R1b move case is similar. \hfill \Box

**Proposition 3.4.** If $(D, p)$ and $(D', p')$ are related by a single R2 move relative base point system, then $\eta_q(D, p)(t_i) \equiv \eta_q(D', p')(t'_i) \pmod{A_q}$.

**Proof.** There are four R2 moves depending on orientations of strands. By [12, Theorem 1.1], it is enough to consider the R2 move in a disk $\delta$ as shown in Figure 3.4.

![Figure 3.4](image)

**Figure 3.4.** An R2 move which relates $(D, p)$ to $(D', p')$
Let \( a_{ij} (1 \leq i \leq n, 1 \leq j \leq m_i + 1) \) be the arcs of \((D, p)\) as given in Section 2. Let \( a'_{ij}, b \) and \( c \) be the arcs of \((D', p')\) such that each \( a'_{ij} \) corresponds to the arc \( a_{ij} \) of \((D, p)\), and \( b \) and \( c \) intersect the disk \( \delta \) as shown in Figure 3.4. Then the domain of \( \eta'_{q} = \eta_{q}(D', p') \) is the free group on \( \{a'_{ij}\} \cup \{b, c\} \).

Note that all the preferred longitudes \( l'_{i} \) of \((D', p')\) do not contain the letter \( c \). Each \( l_{i} \) \( (1 \leq i \leq n) \) is obtained from \( l'_{i} \) by replacing \( b \) with \( a_{gh} \) in Figure 3.4 and \( a'_{st} \) with \( a_{st} \) for all \( s, t \). Hence Proposition 3.3 follows from the claim below. 

\[ \square \]

\textbf{Claim 3.5.} (1) \( \eta'_{q}(a'_{gh}) \equiv \eta'_{q}(b) \pmod{A_{q}} \).

(2) \( \eta_{q}(a_{ij}) \equiv \eta'_{q}(a'_{ij}) \pmod{A_{q}} \) for any \( i, j \).

\textbf{Proof.} (1) By Lemma 3.2, we have

\[ \eta'_{q}(b^{-1}a_{kl}c(a_{kl})^{-1}) \equiv 1 \pmod{A_{q}} \]

and

\[ \eta'_{q}(c^{-1}(a_{kl})^{-1}a_{gh}a_{kl}) \equiv 1 \pmod{A_{q}}. \]

This implies that

\[ \eta'_{q}(b^{-1}a_{gh}) = \eta'_{q}(b^{-1}a'_{kl}c(a'_{kl})^{-1}a'_{gh}a'_{kl}) \equiv \eta'_{q}(a'_{kl})^{-1}(a'_{gh})^{-1}a'_{gh} \pmod{A_{q}}. \]

(2) This is proved by induction on \( q \). The assertion certainly holds for \( q = 1 \) or \( j = 1 \). Assume that \( q \geq 1 \) and \( j \geq 2 \). Let \( v_{i,j-1} \) and \( v'_{i,j-1} \) be part longitudes of \((D, p)\) and \((D', p')\), respectively. Then \( v_{i,j-1} \) is obtained from \( v'_{i,j-1} \) by replacing \( b \) with \( a_{gh} \) in Figure 3.4 and \( a'_{st} \) with \( a_{st} \) for all \( s, t \). By (1) and the induction hypothesis, we have

\[ \eta_{q}(v_{i,j-1}) \equiv \eta'_{q}(v'_{i,j-1}) \pmod{A_{q}}. \]

Hence it follows from Lemma 2.2(1) that

\[ \eta_{q+1}(a_{ij}) = \eta_{q}(v_{i,j-1})a_{ij} \eta_{q}(v_{i,j-1}) \equiv \eta_{q}(v'_{i,j-1})a_{ij} \eta_{q}(v'_{i,j-1}) \pmod{A_{q+1}} \]

\[ = \eta'_{q+1}(a'_{ij}). \]

\[ \square \]

\textbf{Proposition 3.6.} If \((D, p)\) and \((D', p')\) are related by a single R3 move relative base point system, then \( \eta_{q}(D, p)(l_{i}) \equiv \eta_{q}(D', p')(l'_{i}) \pmod{A_{q}}. \)

\textbf{Proof.} There are eight R3 moves depending on orientations of strands. By Theorem 1.1, we may consider the R3 move in a disk \( \delta \) as shown in Figure 3.5. Let \( a_{ij} \) and \( a'_{ij} \) \( (1 \leq i \leq n, 1 \leq j \leq m_{i} + 1) \) be the arcs of \((D, p)\) and \((D', p')\), respectively, such that each \( a'_{ij} \) corresponds to the arc \( a_{ij} \). Then the domain of \( \eta'_{q} = \eta_{q}(D', p') \) is the free group \( \overline{A} \) on \( \{a'_{ij}\} \).

As shown in Figure 3.5 let \( b, c, d, e \in \{a_{kl}\} \) be arcs of \((D, p)\) which intersect \( \delta \). Similarly, let \( b', c', d', e' \in \{a'_{kl}\} \) be arcs of \((D', p')\) which intersect \( \delta \). Note that \( l_{i} \) and \( l'_{i} \) do not contain the letters \( b \) and \( b' \), respectively, for all \( 1 \leq i \leq n \). Without loss of generality, we may assume that the arcs \( b \) and \( b' \) belong to the 1st components of \((D, p)\) and \((D', p')\), respectively. Then \( l_{i} \) and \( l'_{i} \) can be written in the forms

\[ l_{i} = a_{11}^{-w_{1}}x_{i}^{-1}y_{i} \]

and

\[ l'_{i} = (a_{11}')^{-w_{1}}x_{i}'(c')^{-1}y' \]

for certain words \( x, y \in A \) and \( x', y' \in \overline{A} \) such that \( x \) and \( y \) are obtained from \( x' \) and \( y' \), respectively, by replacing \( a'_{st} \) with \( a_{st} \) for all \( s, t \). For \( 2 \leq i \leq n \), each \( l_{i} \) is obtained from \( l'_{i} \) by replacing \( a'_{st} \) with \( a_{st} \) for all \( s, t \). To complete the proof, we need the following.
Lemma 2.2(1) implies that $z$ for certain words $i$ longitudes of $(D, p)$ and $a_{ij} \neq b$ of $(D', p')$.

By the induction hypothesis, we have $xc$ the word $v = 1$ and $\eta$. Since $\eta$, the congruence $\eta(l_i) \equiv \eta'(l'_i) (\text{mod } A_q)$ follows from Claim 3.7 immediately. By Lemma 2.3(2), we have $\eta(e^{-1}c^{-1}dc) \equiv 1 (\text{mod } A_q)$. Hence Claim 3.7 implies that

$$\eta(l_1) = \eta(a_{11}^{-w_1}xe^{-1}dy) = \eta(a_{11}^{-w_1}xe^{-1}dcx^{-1}y) = \eta(a_{11}^{-w_1}xe^{-1}y) \equiv \eta'((a_{11}')^{-w_1}x'e'(c')^{-1}y') (\text{mod } A_q) = \eta'(l'_1).$$

Claim 3.7. For any letters $a_{ij} \neq b$ of $(D, p)$ and $a_{ij}' \neq b'$ of $(D', p')$,

$$\eta(a_{ij}) \equiv \eta'(a_{ij}') (\text{mod } A_q).$$

Before showing this claim, we observe it implies Proposition 3.6. For $2 \leq i \leq n$, the congruence $\eta(l_i) \equiv \eta'(l'_i) (\text{mod } A_q)$ follows from Claim 3.7 immediately. By Lemma 2.3(2), we have $\eta(e^{-1}c^{-1}dc) \equiv 1 (\text{mod } A_q)$. Hence Claim 3.7 implies that

$$\eta(l_1) = \eta(a_{11}^{-w_1}xe^{-1}dy) = \eta(a_{11}^{-w_1}xe^{-1}dcx^{-1}y) = \eta(a_{11}^{-w_1}xe^{-1}y) \equiv \eta'((a_{11}')^{-w_1}x'e'(c')^{-1}y') (\text{mod } A_q) = \eta'(l'_1).$$

Proof of Claim 3.7. This is proved by induction on $q$. The assertion certainly holds for $q = 1$ or $j = 1$. Assume that $q \geq 1$ and $j \geq 2$. Let $v_{ij-1}$ and $v'_{ij-1}$ be part longitudes of $(D, p)$ and $(D', p')$, respectively. If $i \neq 1$ or $v_{ij-1}$ does not contain the word $xe^{-1}d$, then it is obtained from $v'_{ij-1}$ by replacing $a_{st}'$ with $a_{st}$ for all $s, t$. By the induction hypothesis, we have $\eta(l_{ij-1}) \equiv \eta'(l'_{ij-1}) (\text{mod } A_q)$. Therefore Lemma 2.2(1) implies that

$$\eta_{q+1}(a_{ij}) \equiv \eta'(a_{ij}') (\text{mod } A_{q+1}).$$

If $i = 1$ and $v_{ij-1}$ contains $xe^{-1}d$, then $v_{ij-1}$ and $v'_{ij-1}$ can be written in the forms

$$v_{ij-1} = xe^{-1}dz \quad \text{and} \quad v'_{ij-1} = x'e'(c')^{-1}z'$$

for certain words $z \in \mathcal{A}$ and $z' \in \mathcal{A}'$ such that $z$ is obtained from $z'$ by replacing $a_{st}'$ with $a_{st}$ for all $s, t$. Since $\eta(e^{-1}c^{-1}dc) \equiv 1 (\text{mod } A_q)$ by Lemma 2.3(2), it follows that

$$\eta_{q}(v_{ij-1}) = \eta_{q}(xe^{-1}dz) \equiv \eta_{q}(xe^{-1}z) (\text{mod } A_q).$$

By the induction hypothesis we have $\eta_{q}(v_{ij-1}) \equiv \eta'_{q}(v'_{ij-1}) (\text{mod } A_q)$, and hence $\eta_{q+1}(a_{ij}) \equiv \eta'_{q+1}(a_{ij}') (\text{mod } A_{q+1})$. □

Proposition 3.8. If $(D, p)$ and $(D', p')$ are related by one of V1–V4, OC relative base point system and the local move in Figure 3.4, then $\eta(D, p)(l_i) \equiv \eta(D', p')(l'_i) (\text{mod } A_q)$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{An R3 move which relates $(D, p)$ to $(D', p')$}
\end{figure}
Proof. Let $a_{ij} (1 \leq i \leq n, 1 \leq j \leq m_i + 1)$ be the arcs of $(D, p)$. Then the arcs of $(D', p')$ can be uniquely determined as shown in Figure 3.6. Hence we have $\eta_q(D, p) \equiv \eta_q(D', p') \pmod{A_q}$. \hfill $\square$

**Figure 3.6. Proof of Proposition 3.8**

**Proof of Theorem 3.7** By Propositions 3.2, 3.4, 3.6, and 3.8 we have

$$\eta_q(D, p)(l_i) \equiv \eta_q(D', p')(l'_i) \pmod{A_q}.$$  

Then Lemma 2.4 implies that

$$E(\eta_q(D, p)(l_i)) - E(\eta_q(D', p')(l'_i)) = \text{(terms of degree } \geq q).$$

Hence, by definition, $\mu_{(D, p)}(j_1 \ldots j_s i) = \mu_{(D', p')}(j_1 \ldots j_s i)$ for any sequence $j_1 \ldots j_s i$ with $s < q$. \hfill $\square$

**Example 3.9.** Consider the 3-component virtual link diagram $D$ and its base point system $p = (p_1, p_2, p_3)$ in the left of Figure 3.7. Let $a_{ij}$ be the arcs of $(D, p)$. Since $l_1 = a_{21}, l_2 = a_{21}^{-1}(a_{11}a_{23})$ and $l_3 = a_{22}^{-1}a_{22}$, by definition we have

$$\eta_3(l_1) = a_2, \quad \eta_3(l_2) = a_2^{-1}a_1a_2^{-1}a_1^{-1}a_2^{-1}a_1a_2a_1^{-1}a_2a_1a_2^{-1}a_1^{-1}a_2a_1a_2, \quad \eta_3(l_3) = a_2^{-1}a_1^{-1}a_2a_1.$$  

By a direct computation, we have

$$E(\eta_3(l_1)) = 1 + X_2, \quad E(\eta_3(l_2)) = 1 + X_1 + \text{(terms of degree } \geq 3), \quad E(\eta_3(l_3)) = 1 - X_1X_2 + X_2X_1 + \text{(terms of degree } \geq 3).$$

Hence it follows that

$$\mu_{(D, p)}(21) = 1, \quad \mu_{(D, p)}(12) = 1, \quad \mu_{(D, p)}(123) = -1 \quad \text{and} \quad \mu_{(D, p)}(213) = 1,$$

and that $\mu_{(D, p)}(I) = 0$ for any sequence $I$ with length $\leq 3$ except for 21, 12, 123 and 213.
Consider another base point system \( p' = (p'_1, p'_2, p'_3) \) of \( D \) in the right of Figure 3.7. Then we have \( l_1 = a_{22}, l_2 = a_{22}^{-1}(a_{22}a_{11}) \) and \( l_3 = a_{22}^{-1}a_{21} \), and hence
\[
\eta(l_1) = \alpha_2, \quad \eta(l_2) = \alpha_1 \quad \text{and} \quad \eta(l_3) = 1.
\]
This implies that
\[
\mu(D,p')(21) = 1 \quad \text{and} \quad \mu(D,p')(12) = 1,
\]
and that \( \mu(D,p')(I) = 0 \) for any sequence \( I \) with length \( \leq 3 \) except for 21 and 12. Therefore, by Theorem 3.1 (\( D, p \)) and (\( D, p' \)) are not \( \mathcal{W} \)-isotopic.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3_7.png}
\caption{A 3-component link diagram \( D \) with different base point systems \( p = (p_1, p_2, p_3) \) and \( p' = (p'_1, p'_2, p'_3) \)}
\end{figure}

4. Change of a base point system

In this section, we fix an \( n \)-component virtual link diagram \( D \), and observe behavior of \( \eta(l_i) \) under a change of a base point system of \( D \) (Theorem 4.8).

An arc of \( D \) is a segment along \( D \) which goes from a classical under-crossing to the next one, where classical over-crossings and virtual crossings are ignored. We emphasize the definition of arcs of \( D \) is slightly different from that of arcs of \( (D, p) \).

For each \( 1 \leq i \leq n \), we choose one arc of the \( i \)th component and denote it by \( a_{i1} \). Let \( a_{i2}, \ldots, a_{im_i} \) be the other arcs of the \( i \)th component in turn with respect to the orientation, where \( m_i \) denotes the number of arcs of the \( i \)th component. Throughout this section, we fix these arcs \( a_{i1}, \ldots, a_{im_i} \) for \( D \).

Given a base point system \( p = (p_1, \ldots, p_n) \) of \( D \), let \( p(i) \) denote the integer of the second subscript of the arc containing \( p_i \) (\( 1 \leq i \leq n \)). Consider the virtual link diagram \( D \) with a base point system \( p = (p_1, \ldots, p_n) \). For each \( i \)th component of \( (D,p) \), the base point \( p_i \) divides the arc \( a_{ip(i)} \) of \( D \) into two arcs. We assign the labels \( b_p \) and \( a_{ip(i)} \) to the two arcs of \( (D,p) \) as shown in Figure 4.1. The labels of the other arcs of \( (D,p) \) are the same as those of the corresponding arcs of \( D \).

\begin{figure}[h]
\centering
\includegraphics{figure4_1.png}
\caption{Notations for arcs}
\end{figure}

In this setting, the homomorphism \( \eta_p(D,p) \) for \( (D,p) \) is described as follows. We put \( \eta^p_i = \eta_p(D,p) \) for short. The domain of \( \eta^p_i \) is the free group \( \mathcal{A} \) on \( \{a_{ij}\} \cup \{b^p_i\} \). The homomorphism \( \eta^p_i \) from \( \mathcal{A} \) into \( A \) is given inductively by
\[
\eta^p_i(a_{ij}) = \alpha_i, \quad \eta^p_i(b^p_i) = \alpha_i,
\]
\[
\eta^p_{i+1}(a_{ip(i)}) = \alpha_i, \quad \eta^p_{i+1}(a_{ij}) = \eta^p_i((v^p_{i+1})^{-1})\alpha_i\eta^p_i(v^p_{i+1}) \quad (j \neq p(i)),
\]
and
\[
\eta^p_{i+1}(b^p_i) = \eta^p_i((v^p_{ip(i)})^{-1})\alpha_i\eta^p_i(v^p_{ip(i)})^{-1}.
\]
Lemma 2.2(1), it follows that preferred longitudes $p_1, \ldots, p_n$ crossing to the next one, where virtual crossings are ignored. Let $p_\ast$ be the set of all $(p_1, \ldots, p_n) \in P$ such that each $p_i$ lies on a semi-arc which starts at a classical under-crossing. We denote by $P = (p_1, \ldots, p_n) \in P_\ast$ the base point system such that each $p_i$ lies on the arc $a_i$. For the homomorphism $\eta_q$ associated with $(D, p_\ast)$, part longitudes $v_{ij}$ and preferred longitudes $l_i$ of $(D, p_\ast)$, we simply put $\eta_q = \eta_q^P$, $v_{ij} = v_{ij}^P$ and $l_i = l_i^P$.

Let $M^P$ be the normal closure of $\{[\eta_q^P(x, \eta_q^P)]_1 \mid 1 \leq i \leq n\}$ in $A$ and let $M_q = \prod_{p \in P_\ast} M_i^P$. Notice that $M_q = \prod_{p \in P_\ast} \phi_q^P(M_i^P)$.

Lemma 2.4. For any $p \in P$, $M_q^P \subset A_qM^P_{q-1}$. Hence $M_q \subset A_qM_{q+1}$. 

\begin{align*}
v_{ij}^P &= \begin{cases} u_{ij}^P & (p(i) \leq j \leq m_i), \\
u_{ij}^P & (1 \leq j \leq p(i) - 1),
\end{cases} \\
v_{ij}^{P_0} &= v_{ij}^P + l_i^P.
\end{align*}

We now define a word $\lambda^P \in \mathbb{A} (1 \leq i \leq n)$ by

$$\lambda^P = \begin{cases} u_{i1}^P u_{i2}^P \cdots u_{ip(i) - 1}^P & (p(i) \neq 1), \\
1 & (p(i) = 1),\end{cases}$$

and a sequence of homomorphisms $\phi_q^P : A \to A$ by

$$\phi_q^P(\alpha_i) = \alpha_i \quad \text{and} \quad \phi_q^P(\eta_i) = \eta_{q-1}^P(\lambda^P)\alpha_i\eta_{q-1}^P((\lambda^P)^{-1}) \quad (q \geq 2).$$

Notice that the homomorphism $\phi_q^P$ sends each $\alpha_i$ to some conjugate element.

**Lemma 4.1.** For any $x, y \in A$, the following hold.

1. $\phi_q^P(x) \equiv \phi_{q+1}^P(x) \pmod{A_q}$.
2. If $x \equiv y \pmod{A_q}$, then $\phi_q^P(x) \equiv \phi_q^P(y) \pmod{A_q}$.

**Proof.** (1) Since $A_1 = A$, this holds for $q = 1$. For $q \geq 2$, it is enough to show the case $x = \alpha_i$. By Lemma 2.2(1), we have $\eta_q^P = \eta_q^P(\lambda^P) \equiv \eta_q^P(\lambda^P) \pmod{A_q}$. Hence Lemma 2.2(1) implies that

$$\phi_q^P(\alpha_i) = \eta_{q-1}^P(\lambda^P)\alpha_i\eta_{q-1}^P((\lambda^P)^{-1}) \equiv \eta_{q-1}^P(\lambda^P)\alpha_i\eta_q^P((\lambda^P)^{-1}) \pmod{A_q} = \phi_{q+1}^P(\alpha_i).$$

(2) It is enough to show that $\phi_q^P(A_q) \subset A_q$. This is done by induction on $q$. For $q = 1$ it is obvious. Assume that $q \geq 1$. For $x \in A$ and $y \in A_q$, by (1) and Lemma 2.2(1), it follows that

$$\phi_{q+1}^P([x, y]) = [\phi_{q+1}^P(x), \phi_{q+1}^P(y)] \equiv [\phi_{q+1}^P(x), \phi_q^P(y)] \pmod{A_q+1}.$$ 

By the induction hypothesis, we have $\phi_q^P(y) \in A_q$ and hence $\phi_{q+1}^P([x, y]) \in A_q+1$. 

\[\square\]

A semi-arc of $D$ is a segment along $D$ which goes from a classical under-/over-crossing to the next one, where virtual crossings are ignored. Let $P$ be the set of base point systems of $D$. Let $P_0 \subset P$ be the set of all $(p_1, \ldots, p_n) \in P$ such that each $p_i$ lies on a semi-arc which starts at a classical under-crossing. We denote by $P_\ast = (p_1^\ast, \ldots, p_n^\ast) \in P_0$ the base point system such that each $p_i^\ast$ lies on the arc $a_i$. For the homomorphism $\eta_q^P$ associated with $(D, p_\ast)$, part longitudes $v_{ij}^P$ and preferred longitudes $l_i^P$ of $(D, p_\ast)$, we simply put $\eta_q = \eta_q^P$, $v_{ij} = v_{ij}^P$ and $l_i = l_i^P$.

Let $M_q^P$ be the normal closure of $\{[\eta_q^P(x, \eta_q^P)]_1 \mid 1 \leq i \leq n\}$ in $A$ and let $M_q = \prod_{p \in P_\ast} M_i^P$. Notice that $M_q = \prod_{p \in P_\ast} \phi_q^P(M_i^P)$.

**Lemma 4.2.** For any $p \in P$, $M_q^P \subset A_qM^P_{q-1}$. Hence $M_q \subset A_qM_{q+1}$. 

\[\square\]
Proof. Lemma \[2.3\] implies that \( \eta_q(l_i) \equiv \eta_{q+1}(l_i) \pmod{A_q} \). Hence by Lemma \[4.1\] we have
\[
\phi_q^F([\alpha_i, \eta_q(l_i)]) \equiv \phi_q^F([\alpha_i, \eta_{q+1}(l_i)]) \equiv \phi_q^F([\alpha_i, \eta_{q+1}(l_i)]) \pmod{A_q}.
\]

\[
\square
\]

Lemma 4.3. Let \( p_0 \in \mathcal{P}_0 \). For any \( 1 \leq i \leq n \) and \( 1 \leq j \leq m_i \),
\[
\eta_q^{p_0}(a_{ij}) \equiv \phi_q^F(\eta_q(a_{ij})) \pmod{A_qM_{q+1}^{p_0}}.
\]

Proof. This is proved by induction on \( q \). It is obvious for \( q = 1 \). Assume that \( q \geq 1 \).

The induction hypothesis together with Lemma \[4.2\] implies that
\[
(4.1) \quad \eta_q^{p_0}(a_{ij}) \equiv \phi_q^F(\eta_q(a_{ij})) \pmod{A_qM_{q+1}^{p_0}}.
\]

First we consider the case \( 1 \leq j \leq p_0(i) - 1 \). For part longitude \( v_{ij}^{p_0} - 1 \) and \( v_{ij+1} \) of \( (D, p_0) \) and \( (D, p_0) \), respectively, by definition we have
\[
v_{ij}^{p_0} - 1 = (\lambda_i^{p_0})^{-1} u_{ij+1}^{p_0} - 1 = (\lambda_i^{p_0})^{-1} u_{ij+1}^{p_0} - 1.
\]

where \( \alpha_i = (\lambda_i^{p_0})^{-1} u_{ij+1}^{p_0} - 1 \). Then it follows from congruence \[4.1\] and Lemma \[2.2\] (2) that
\[
\eta_q^{p_0}(a_{ij}) \equiv \phi_q^F(\eta_q(a_{ij})) \pmod{A_qM_{q+1}^{p_0}}.
\]

On the other hand, by Lemmas \[2.2\] (1) and \[2.3\] (1) we have
\[
\phi_q^{p_0}(\eta_q(a_{ij})) = \phi_q^{p_0}(\eta_q(a_{ij})) \equiv \phi_q^{p_0}(\eta_q(a_{ij})) \pmod{M_{q+1}^{p_0}}.
\]

Therefore Lemmas \[2.2\] (1) and \[4.1\] (1) imply that
\[
\eta_q^{p_0}(a_{ij}) \equiv \phi_q^{p_0}(\eta_q(a_{ij})) \pmod{A_qM_{q+1}^{p_0}}.
\]

Next we consider the case \( p_0(i) \leq j \leq m_i \). For \( j \neq p_0(i) \), we have \( u_{ij+1} - 1 = \lambda_i^{p_0} v_{ij+1}^{p_0} \).

By congruence \[4.1\] and Lemmas \[2.2\] and \[4.1\] (1), it follows that
\[
\eta_q^{p_0}(a_{ij}) = \eta_q^{p_0}((\lambda_i^{p_0})^{-1}) \alpha_i \phi_q^{p_0}(\eta_q(a_{ij})) \equiv \phi_q^{p_0}(\eta_q(a_{ij})) \pmod{A_qM_{q+1}^{p_0}}.
\]
In the case \( j = p_0(i) \), by substituting 1 for \( v_{ij}^{P_0} \) in the formula above, we have the conclusion.

**Proposition 4.4.** Let \( p_0 \in \mathcal{P}_0 \). For any \( 1 \leq i \leq n \),

\[
\eta_i^{P_0}(l_i^{P_0}) \equiv \phi_i^{P_0}(\eta_i(\lambda_i^{P_0})^{-1}l_i\lambda_i^{P_0})) \pmod{A_4M_i^{P_0}}.
\]

**Proof.** Since \( l_i^{P_0} = a_{i_{P_0}(i)}^{w_i}(\lambda_i^{P_0})^{-1}a_{i_{1i}^{P_0}}l_i\lambda_i^{P_0} \), it follows from Lemmas 4.1 and \ref{lemma-4.3} that

\[
\eta_i^{P_0}(l_i^{P_0}) = \eta_i^{P_0}(a_{i_{P_0}(i)}^{w_i}(\lambda_i^{P_0})^{-1}a_{i_{1i}^{P_0}}l_i\lambda_i^{P_0}) \\
= \alpha_i^{-w_i}\eta_i^{P_0}(\lambda_i^{P_0})^{-1}a_{i_{1i}^{P_0}}l_i\lambda_i^{P_0}) \\
\equiv \alpha_i^{-w_i}\phi_i^{P_0}(\eta_i(\lambda_i^{P_0})^{-1})\phi_i^{P_0}(\eta_i(l_i\lambda_i^{P_0})) \\
= \alpha_i^{-w_i}\phi_i^{P_0}(\eta_i(\lambda_i^{P_0})^{-1})(\phi_i^{P_0}(\eta_i(l_i\lambda_i^{P_0})) \pmod{A_q}) \\
= \alpha_i^{-w_i}\phi_i^{P_0}(\eta_i(\lambda_i^{P_0})^{-1})(\phi_i^{P_0}(\eta_i(l_i\lambda_i^{P_0}))) \\
= \alpha_i^{-w_i}\phi_i^{P_0}(\eta_i(\lambda_i^{P_0})^{-1})\phi_i^{P_0}(\eta_i(l_i\lambda_i^{P_0}))) \\
\times \phi_i^{P_0}(\eta_i(l_i\lambda_i^{P_0})) \pmod{A_4M_i^{P_0}} \\
= \phi_i^{P_0}(\eta_i(\lambda_i^{P_0})^{-1}l_i\lambda_i^{P_0})).
\]

\( \square \)

**Lemma 4.5.** Let \( p_0 \in \mathcal{P}_0 \). For any \( 1 \leq i \leq n \),

\[
\eta_i^{P_0}(a_{i_{P_0}(i)}) \equiv \eta_i^{P_0}(l_i^{P_0}) \pmod{A_4N_i^{P_0}},
\]

where \( N_i^{P_0} \) denotes the normal closure of \( \{[\alpha, \eta_i^{P_0}(l_i^{P_0})] \mid 1 \leq i \leq n \} \) in \( A \).

**Proof.** By Lemmas 2.2 and 2.3, it follows that

\[
\eta_i^{P_0}(a_{i_{P_0}(i)}) = \eta_i^{P_0}(l_i^{P_0})^{-1}[\alpha_i, \eta_i^{P_0}(l_i^{P_0})]\eta_i^{P_0}(a_{i_{P_0}(i)}) \pmod{N_i^{P_0}} \\
= \eta_i^{P_0}(l_i^{P_0})^{-1}\alpha_i\eta_i^{P_0}(l_i^{P_0}) \\
\equiv \eta_i^{P_0}(l_i^{P_0})^{-1}\alpha_i\eta_i^{P_0}(l_i^{P_0}) \pmod{A_4} \\
= \eta_i^{P_0}(l_i^{P_0})^{-1}\alpha_i\eta_i^{P_0}(l_i^{P_0}) \pmod{A_4} \\
= \eta_i^{P_0}(l_i^{P_0})^{-1}\alpha_i\eta_i^{P_0}(l_i^{P_0}) \pmod{A_4} \\
= \eta_i^{P_0}(l_i^{P_0})^{-1}\alpha_i\eta_i^{P_0}(l_i^{P_0}) \pmod{A_4} \\
= \eta_i^{P_0}(l_i^{P_0})^{-1}\alpha_i\eta_i^{P_0}(l_i^{P_0}) \pmod{A_4} \\
= \eta_i^{P_0}(l_i^{P_0})^{-1}\alpha_i\eta_i^{P_0}(l_i^{P_0}) \pmod{A_4}.
\]

\( \square \)

**Lemma 4.6.** Let \( \mathbf{p} \in \mathcal{P} \), and \( p_0 \in \mathcal{P}_0 \) with \( p_0(k) = p(k) \) (1 \( \leq k \leq n \)). For any \( 1 \leq i \leq n \), the following hold.

1. For any \( 1 \leq j \leq m_i \), \( \eta_j^{P_0}(a_{ij}) \equiv \eta_j^{P_0}(a_{ij}) \pmod{A_4N_i^{P_0}} \).
2. \( \eta_j^{P_0}(b^P_0) \equiv \eta_j^{P_0}(b^P_0) \pmod{A_4N_i^{P_0}} \).

**Proof.** This is proved by induction on \( q \). Since \( A_1 = A \), assertions (1) and (2) are obvious for \( q = 1 \). Assume that \( q \geq 1 \).

1. For \( j = p(i) \), we have \( \eta_{q+1}^{P_0}(a_{i_{P_0}(i)}) = \alpha_i = \eta_{q+1}^{P_0}(a_{i_{P_0}(i)}) = \eta_{q+1}^{P_0}(a_{i_{P_0}(i)}) \) by definition. In a way similar to the proof of Lemma 4.2, we have \( N_i^{P_0} \subset A_4N_{i+1}^{P_0} \). Hence for \( j \neq p(i) \), the induction hypothesis implies that

\[
\eta_j^{P_0}(a_{ij}) \equiv \eta_j^{P_0}(a_{ij}) \pmod{A_4N_i^{P_0}} \quad \text{and} \quad \eta_j^{P_0}(b^P_0) \equiv \eta_j^{P_0}(b^P_0) \pmod{A_4N_i^{P_0}}.
\]

Furthermore, by Lemma 4.5 we have

\[
\eta_j^{P_0}(a_{i_{P_0}(i)}) \equiv \eta_j^{P_0}(b^P_0) \pmod{A_4N_i^{P_0}}.
\]
Therefore it follows that
\[ \eta^P_q(v^P_{ij-1}) \equiv \eta^P_q(v^P_{ij-1}) \mod A_qN^p_{q+1}. \]

Therefore by Lemma 2.2 (2) it follows that
\[
\eta^P_{q+1}(i_{j}) = \eta^P_q((v^P_{ij-1})^{-1})a_\eta \eta^P_q(v^P_{ij-1}) \\
\equiv \eta^P_q(v^P_{ij-1})^{-1})a_\eta \eta^P_q(v^P_{ij-1}) \mod A_{q+1}N^p_{q+1} \\
= \eta^P_{q+1}(a_{ij}).
\]

(2) The proof is similar to that of (1).

\[ \Box \]

**Proposition 4.7.** Let \( p \in \mathcal{P} \), and \( p_0 \in \mathcal{P}_0 \) with \( p_0(k) = p(k) \) \((1 \leq k \leq n)\). For any \( 1 \leq i \leq n \), \( \eta^P_q(P^i) \equiv \eta^P_q(P^i) \mod A_qN^p_q \).

**Proof.** By Lemma 4.6, we have \( \eta^P_q(v^P_{ip(i)-1}) \equiv \eta^P_q(v^P_{ip(i)-1}) \mod A_qN^p_q \). This implies that
\[
\eta^P_q(P^i) = \eta^P_q(a_{-w_i}v^P_{ip(i)-1}) = \eta^P_q(a_{-w_i}v^P_{ip(i)-1}) = \eta^P_q(P^i) \mod A_qN^p_q.
\]
Hence, it is enough to show that \( N^p_q \subset A_qM^p_q \), i.e. \([\alpha_k, \eta^P_q(P^i)] \in A_qM^p_q \).

By Lemma 4.1 (1), we have
\[
\phi^P_{q+1}(\alpha_k) \equiv \phi^P_{q+1}(\alpha_k) \mod A_q.
\]
Since \( \phi^P_{q+1}(\alpha_k) = \eta^P_q(\lambda^P_{i}) \equiv \eta^P_q(\lambda^P_{i}) \mod A_q \), we have
\[
\alpha_k \equiv \eta^P_q(\lambda^P_{i}) \equiv \phi^P_{q+1}(\alpha_k) \mod A_q.
\]
Furthermore, Lemma 4.3 and Proposition 4.4 imply that
\[
\eta^P_q(P^i) \equiv \phi^P_{q+1}(\eta^P_q(1,0,\lambda^P_{i})) \mod A_qM^p_q
\]
\[
\equiv \eta^P_q((\lambda^P_{i})^{-1})\phi^P_{q+1}(\eta^P_q(1,0,\lambda^P_{i})) \mod A_qM^p_q
\]
Therefore it follows that
\[
[\alpha_k, \eta^P_q(P^i)] = \phi^P_{q+1}(\lambda^P_{i})(\eta^P_q((\lambda^P_{i})^{-1})\phi^P_{q+1}(\lambda^P_{i})) \mod A_qM^p_q
\]
Combining Propositions 4.4 and 4.7, the following is obtained immediately.

**Theorem 4.8.** Let \( p \in \mathcal{P} \), and \( p_0 \in \mathcal{P}_0 \) with \( p_0(k) = p(k) \) \((1 \leq k \leq n)\). For any \( 1 \leq i \leq n \), \( \eta^P_q(P^i) \equiv \eta^P_q(P^i) \mod A_qM^p_q \). Hence \( \eta^P_q(P^i) \equiv \phi^P_q((\lambda^P_{i})^{-1})\phi^P_{q+1}(\lambda^P_{i}) \mod A_qM^p_q \).

5. Milnor Numbers and Welded Isotopy

Let \( D \) be an \( n \)-component virtual link diagram of a welded link \( L \). As shown in Example 3.9, the Milnor number \( \mu_{(D,p)}(I) \) depends on a choice of a base point system \( p \) of \( D \). Hence it is not an invariant of the welded link \( L \). On the other hand, we show in this section that \( \mu_{(D,p)}(I) \) modulo a certain indeterminacy is an invariant of \( L \) (Theorem 5.2).
Definition 5.1. For a sequence $i_1\ldots i_r$ of indices in $\{1,\ldots,n\}$, the indeterminacy $\Delta_{(D,p)}(i_1\ldots i_r)$ of $(D,p)$ is the greatest common divisor of all $\mu_{(D,p)}(j_1\ldots j_s)$, where $j_1\ldots j_s$ ($2 \leq s < r$) is obtained from $i_1\ldots i_r$ by removing at least one index and permuting the remaining indices cyclicly. In particular, we set $\Delta_{(D,p)}(i_1i_2) = 0$.

Theorem 5.2. Let $D$ and $D'$ be virtual diagrams of a welded link. Let $p$ and $p'$ be base point systems of $D$ and $D'$, respectively. Then $\mu_{(D,p)}(I) \equiv \mu_{(D',p')}(I) \pmod{\Delta_{(D,p)}(I)}$ and $\Delta_{(D,p)}(I) = \Delta_{(D',p')}(I)$ for any sequence $I$.

This theorem guarantees the well-definedness of the following definition.

Definition 5.3. Let $L$ be an $n$-component welded link. For a sequence $I$ of indices in $\{1,\ldots,n\}$, the Milnor $\overline{\mu}$-invariant $\overline{\mu}_L(I)$ of $L$ is the residue class of $\mu_{(D,p)}(I)$ modulo $\Delta_{(D,p)}(I)$ for any virtual diagram $D$ of $L$ and any base point system $p$ of $D$.

Remark 5.4. For a classical link $L$, the above defined Milnor $\overline{\mu}$-invariant of $L$ coincides with the original one in [11] for any sequence.

In the remainder of this section, we fix $D$ and its arcs $a_{ij}$ ($1 \leq i \leq n, 1 \leq j \leq m_i$), and use the same notation as in Section 4. In this setting, the Milnor number $\mu_{(D,p)}(j_1\ldots j_s)$ of $(D,p)$ is given by the coefficient of $X_{j_1}\cdots X_{j_s}$ in $E(\eta_{p}(p)\mu_{(D,p)}(I))$. For short, we put $\mu_{p}(I) = \mu_{(D,p)}(I)$ and $\Delta_{p}(I) = \Delta_{(D,p)}(I)$. In particular, we put $\mu(I) = \mu_{(D,\mu)}(I)$ and $\Delta(I) = \Delta_{(D,\mu)}(I)$.

For each $1 \leq i \leq n$, we define a subset $D_i$ of $\mathbb{Z}\langle X_1,\ldots,X_n \rangle$ to be

$$\left\{ \sum \nu(j_1\ldots j_s)X_{j_1}\cdots X_{j_s} \mid \begin{array}{ll} \nu(j_1\ldots j_s) \equiv 0 \pmod{\Delta(j_1\ldots j_s)} & (s < q), \\
\nu(j_1\ldots j_s) \in \mathbb{Z} & (s \geq q). \end{array} \right\}$$

Although the following three results, Sublemmas 5.5, 5.6 and Lemma 5.7 are essentially shown in [11], we give the proofs for the readers’ convenience.

Sublemma 5.5 (cf. [11] (14) and (16)–(19) on pages 292 and 293). For any $1 \leq i \leq n$, the following hold.

1. $D_i$ is a two-sided ideal of $\mathbb{Z}\langle X_1,\ldots,X_n \rangle$.
2. Let $f_k = E(\eta_{p}(k)) - 1$ ($1 \leq k \leq n$). Then $X_jf_i, f_iX_j \in D_i$ for any $1 \leq j \leq n$.
3. Let $k_1\ldots k_{s+t}$ be a sequence obtained from a sequence $j_1\ldots j_s$ by inserting $t$ ($\geq 1$) indices in $\{1,\ldots,n\}$. Then $\mu((j_1\ldots j_s)k_1\cdots k_{s+t})X_{k_1}\cdots X_{k_{s+t}} \in D_i$.
4. $E(\{\eta_{p}(l_j)\}) - 1 \notin D_i$ for any $1 \leq j \leq n$.

Proof. (1) Let $\nu(j_1\ldots j_s)X_{j_1}\cdots X_{j_s} \in D_i$ and $X_k \in \{X_1,\ldots,X_n\}$. For $s+1 \geq q$, we have $\nu(j_1\ldots j_s)X_kX_{j_1}\cdots X_{j_s} \in D_i$ by definition. For $s+1 < q$, since $\Delta(j_1\ldots j_s)$ is divisible by $\Delta(k_1\ldots j_s)$, $\nu(j_1\ldots j_s)$ is also divisible by $\Delta(k_1\ldots j_s)$. Hence $\nu(j_1\ldots j_s)X_kX_{j_1}\cdots X_{j_s} \in D_i$.

Similarly, we have $\nu(j_1\ldots j_s)X_{j_1}\cdots X_{j_s}X_k \in D_i$.

(2) By definition, it follows that $f_k = E(\eta_{p}(k)) - 1$ has the form

$$\sum \mu((j_1\ldots j_s)k)X_{j_1}\cdots X_{j_s}.$$

Since both $\mu((j_1\ldots j_s)k)$ and $\mu((j_1\ldots j_s)L)$ are divisible by $\Delta(j_1\ldots j_s)$, we have $\mu((j_1\ldots j_s)k)X_{j_1}\cdots X_{j_s} \in D_i$ and $\mu((j_1\ldots j_s)L)X_{j_1}\cdots X_{j_s} \in D_i$. Hence $X_jf_i \in D_i$ and $X_iX_j \in D_i$.

Similarly, we have $f_iX_j \in D_i$ and $f_jX_j \in D_i$.

(3) Since $\mu((j_1\ldots j_s))$ is divisible by $\Delta(k_1\ldots k_{s+t})$, we have the conclusion.
(4) By a direct computation, it follows that
\[ E(\alpha_j, \eta_q(l_j)) - 1 = (E(\alpha_j, \eta_q(l_j)) - E(\eta_q(l_j)\alpha_j)) E(\alpha_j^{-1}\eta_q(l_j)^{-1}) \]
\[ = ((1 + X_j)(1 + f_j) - (1 + f_j)(1 + X_j)) E(\alpha_j^{-1}\eta_q(l_j)^{-1}) \]
\[ = (X_j f_j - f_j X_j) E(\alpha_j^{-1}\eta_q(l_j)^{-1}). \]

Hence, by (1) and (2), we have \( E(\alpha_j, \eta_q(l_j)) - 1 \in \mathcal{D}_1. \)

**Sublemma 5.6** (cf. [11] (12) on page 292). Let \( x \in A \) and \( p \in \mathcal{P} \). If \( E(x) - 1 \in \mathcal{D}_1 \), then \( E(\phi^P_p(x)) - 1 \in \mathcal{D}_1 \).

**Proof.** Put \( E(x) - 1 = \sum \nu(j_1 \ldots j_s)X_{j_1} \cdots X_{j_s} \in \mathcal{D}_1, E(\eta_{q}^{-1}(\lambda^P_p)) = 1 + h_j \) and \( E(\eta_{q-1}^{-1}(\lambda^P_p)^{-1})) = 1 + h_j \). Then we have
\[ E(\phi^P_p(\alpha_j)) - 1 = X_j + X_j h_j + h_j X_j + h_j X_j h_j. \]
This shows that \( E(\phi^P_p(x)) - 1 \) is obtained from \( E(x) - 1 \) by replacing \( X_j \) with \( X_j + X_j h_j + h_j X_j + h_j X_j h_j \). Therefore \( E(\phi^P_p(x)) - 1 \) can be written in the form
\[ \sum \nu(j_1 \ldots j_s)X_{j_1} \cdots X_{j_s} + \sum \kappa(k_1 \ldots k_{s+t})X_{k_1} \cdots X_{k_{s+t}}, \]
where the sequences \( k_1 \ldots k_{s+t} \) are obtained from the sequence \( j_1 \ldots j_s \) by inserting \( t (\geq 1) \) indices in \( \{1, \ldots, n\} \). Since \( \Delta(j_1 \ldots j_s) \) is divisible by \( \Delta(k_1 \ldots k_{s+t}) \), \( \nu(j_1 \ldots j_s) \) is also divisible by \( \Delta(k_1 \ldots k_{s+t}) \). Hence \( \nu(j_1 \ldots j_s)X_{k_1} \cdots X_{k_{s+t}} \in \mathcal{D}_1 \). This implies that \( E(\phi^P_p(x)) - 1 \in \mathcal{D}_1 \).

**Lemma 5.7** (cf. [11] (12)–(15) on page 292). Let \( x, y \in A \) and \( p \in \mathcal{P} \). For any \( 1 \leq i \leq n \), the following hold.

1. \( E(x^{-1} \eta_q(l_i)x - E(\eta_q(l_i)) \in \mathcal{D}_1. \)
2. \( E(\phi^P_p(\eta_q(l_i))) - E(\eta_q(l_i)) \in \mathcal{D}_1. \)
3. \( If x \equiv y \mod A_q M_q, then E(x) - E(y) \in \mathcal{D}_1. \)

**Proof.** (1) Put \( E(\eta_q(l_i)) = 1 + f_i \) and \( E(x) = 1 + h \). Then by Sublemma [5.5.1], (2), we have
\[ E(x^{-1} \eta_q(l_i)x - E(\eta_q(l_i)) \]
\[ = E(x^{-1}) (E(\eta_q(l_i)x) - E(x \eta_q(l_i))) \]
\[ = E(x^{-1}) ((1 + f_i)(1 + h) - (1 + h)(1 + f_i)) \]
\[ = E(x^{-1})(f_i h - h f_i) \in \mathcal{D}_1. \]

(2) Since \( E(\eta_q(l_i)) - 1 = \sum \mu(j_1 \ldots j_s i)X_{j_1} \cdots X_{j_s} \), it follows from the proof of Sublemma [5.6] that
\[ E(\phi^P_p(\eta_q(l_i))) - E(\eta_q(l_i)) = \sum \mu(j_1 \ldots j_s i) \left( \sum \kappa(k_1 \ldots k_{s+t})X_{k_1} \cdots X_{k_{s+t}} \right) \],
where the sequences \( k_1 \ldots k_{s+t} \) are obtained from the sequence \( j_1 \ldots j_s \) by inserting \( t (\geq 1) \) indices in \( \{1, \ldots, n\} \). By Sublemma [5.5.3], we have \( \mu(j_1 \ldots j_s i)X_{k_1} \cdots X_{k_{s+t}} \in \mathcal{D}_1 \), and hence \( E(\phi^P_p(\eta_q(l_i))) - E(\eta_q(l_i)) \in \mathcal{D}_1. \)

(3) It is enough to show that if \( x \in A_q M_q \), then we have \( E(x) - 1 \in \mathcal{D}_1 \). For \( x \in A_q \), it is true by Lemma [2.4]. For \( x \in M_q \), we only need to consider the case \( x = \phi^P_p(\alpha_j, \eta_q(l_j)) \) (1 \( \leq j \leq n \)). By Sublemmas [5.5.4] and [5.6], we have \( E(\phi^P_p(\alpha_j, \eta_q(l_j))) - 1 \in \mathcal{D}_1 \).

**Proposition 5.8.** For any \( p \in \mathcal{P} \), the following hold.

1. \( \mu_p(I) \equiv \mu(I) \mod \Delta(I) \) for any sequence \( I \).
2. \( \Delta_p(I) = \Delta(I) \) for any sequence \( I \).
Proof. (1) For any $1 \leq i \leq n$, it is enough to show that
\[ E(\eta^p_q(I^p_i)) - E(\eta^p_q(l_i)) \equiv 0 \pmod{D_i}. \]

Let $p_0 \in P_0$ with $p_0(i) = p(i)$. By Theorem 4.8 we have
\[ \eta^p_q(I^p_i) \equiv \phi^p_q(\eta_q((\lambda^p_i)^{-1} \lambda^p_i)) \pmod{A_q M_q}. \]

Put $x = \phi^p_q(\eta_q(\lambda^p_i)) \in A$. Then by Lemma 5.7 it follows that
\[
E(\eta^p_q(I^p_i)) - E(\eta_q(l_i)) \equiv E(x^{-1} \phi^p_q(\eta_q(l_i)) x) - E(\eta_q(l_i)) \pmod{D_i}
\equiv E(x^{-1} \phi^p_q(\eta_q(l_i)) x) - E(x^{-1} \eta_q(l_i)) \pmod{D_i}
\equiv E(x^{-1} (E(\phi^p_q(\eta_q(l_i))) - E(\eta_q(l_i)))) E(x)
\equiv 0 \pmod{D_i}.
\]

Since we may assume that $q$ is sufficiently large,
\[ \mu_p(j_1 \ldots j_s i) - \mu(j_1 \ldots j_s i) \equiv 0 \pmod{\Delta(j_1 \ldots j_s i)} \]
for any sequence $j_1 \ldots j_s i$.

(2) This is proved by induction on the length $k$ of $I$. For $k = 2$, we have
\[ \Delta_p(I) = \Delta(I) = 0 \]
by definition. Assume that $k \geq 2$. Let $J_1(I)$ (resp. $J_2(I)$) be the set of all sequences obtained from $I$ by removing exactly one index (resp. at least one index) and permuting the remaining indices cyclically. For any $J \in J_2(I)$, we have $\Delta_p(J) = \Delta(J)$ by the induction hypothesis. Then it follows that
\[
\Delta_p(I) = \gcd \left( \{\mu_p(J) \mid J \in J_2(I)\} \right)
= \gcd \left( \bigcup_{J \in J_1(I)} \{\mu_p(J) \cup \{\mu_p(J') \mid J' \in J_2(J)\} \right)
= \gcd \left( \bigcup_{J \in J_1(I)} \{\mu_p(J) \cup \{\Delta_p(J)\} \right)
= \gcd \left( \bigcup_{J \in J_1(I)} \{\mu_p(J) \cup \{\Delta(J)\} \right).
\]

By (1) we have $\mu_p(J) \equiv \mu(J) \pmod{\Delta(J)}$. This implies that $\Delta_p(I) = \Delta(I)$. \hfill \square

Proof of Theorem 5.2. Since $(D, p)$ and $(D', p')$ are related by $\psi$-isotopies and the two local moves in Figure 3.2, this follows from Theorem 3.1 and Proposition 5.8. \hfill \square

Remark 5.9. In [2], Dye and Kauffman defined Milnor-type “invariants” $\eta^{DK}$ for virtual link diagrams with base point systems. They insisted that $\eta^{DK}$ does not depend on a choice of a base point system. However, it can be seen that for $(D, p)$ and $(D', p')$ given in Example 3.9 their $\eta^{DK}$ have different values. Hence $\eta^{DK}$ is not well-defined even for classical links diagrams.

6. Self-crossing virtualization

A self-crossing virtualization is a local move on virtual link diagrams as shown in Figure 6.1 which replaces a classical crossing involves two strands of a single component with a virtual one. In this section, we show the following theorem as a generalization of [11], Theorem 8.
Theorem 6.1. Let $L$ and $L'$ be welded links, and let $D$ and $D'$ be virtual link diagrams of $L$ and $L'$, respectively. If $D$ and $D'$ are related by a finite sequence of self-crossing virtualizations and welded isotopies, then $\mathcal{P}_L(I) = \mathcal{P}_{L'}(I)$ for any non-repeated sequence $I$.

Let $(D, p)$ be an $n$-component virtual link diagram with a base point system, and let $a_{ij}$ $(1 \leq i \leq n, 1 \leq j \leq m_i + 1)$ be the arcs of $(D, p)$ as given in Section 2. Recall that $A = \langle \alpha_1, \ldots, \alpha_n \rangle$ denotes the free group of rank $n$, and $\overline{A}$ denotes the free group on $\{a_{ij}\}$. For $1 \leq k \leq n$, let $A^{(k)} = \langle \alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n \rangle$ be the free group of rank $n - 1$. We define a homomorphism $\rho_k : A \to A^{(k)}$ by

$$\rho_k(\alpha_i) = \begin{cases} \alpha_i & (i \neq k), \\ 1 & (i = k), \end{cases}$$

and denote by $\eta_{ij}^{(k)} = \eta_{ij}^{(k)}(D, p)$ the composition $\rho_k \circ \eta_{ij} : \overline{A} \to A^{(k)}$.

Lemma 6.2. The following hold.

1. $\eta_{ij}^{(k)}(a_{kj}) = 1$ for any $j$.
2. For any $x \in \overline{A}$, $E(\eta_{ij}^{(k)}(x)) = E(\eta_{ij}(x))|_{X_k=0}$, where $E(\cdot)|_{X_k=0}$ denotes the formal power series obtained from $E(\cdot)$ by substituting 0 for $X_k$.

Proof. (1) Since $\eta_{ij}(a_{kj})$ is a conjugate of $\alpha_k$, $\eta_{ij}^{(k)}(a_{kj})$ is a conjugate of $\rho_k(\alpha_k) = 1$.

(2) It is enough to show the case $x = a_{ij}$. This is proved by induction on $q$. The assertion certainly holds for $q = 1$ or $j = 1$. Assume that $q \geq 1$ and $j \geq 2$. Then we have $E(\eta_{ij}^{(k)}(v_{ij-1})) = E(\eta_{ij}(v_{ij-1}))|_{X_k=0}$ by the induction hypothesis. Hence it follows that

$$E(\eta_{ij+1}^{(k)}(a_{ij})) = E(\rho_k(\eta_{ij}(v_{ij-1}^{-1})\alpha_i\eta_{ij}(v_{ij-1})))$$
$$= E(\eta_{ij}^{(k)}(v_{ij-1}))E(\alpha_i)|_{X_k=0}E(\eta_{ij}^{(k)}(v_{ij-1}))$$
$$= E(\eta_{ij}^{(k)}(v_{ij-1}^{-1})\alpha_i\eta_{ij}(v_{ij-1}))|_{X_k=0}$$
$$= E(\eta_{ij+1}(a_{ij}))|_{X_k=0}.$$

Let $R$ be the normal closure of $\{(\alpha_i, g^{-1} \alpha_i g) \mid g \in A, 1 \leq i \leq n\}$ in $A$, and let $R^{(k)}$ be the normal closure of $\{(\alpha_i, g^{-1} \alpha_i g) \mid g \in A^{(k)}, 1 \leq i \neq k \leq n\}$ in $A^{(k)}$. Note that $[g^{-1} \alpha_i g, h^{-1} \alpha_i h] \in R$ for any $g, h \in A$. In particular, $\eta_{ij}([a_{is}^{(k)}, a_{it}^{(k)}]) \in R$ for any $s, t$ and any $\varepsilon, \delta \in \{\pm 1\}$. Let $A_q^{(k)}$ be the $q$th term of the lower central series of $A^{(k)}$. Then we have the following.

Lemma 6.3. Let $x, y \in A$. If $x \equiv y \pmod{A_q R}$, then

$$\rho_k(x) \equiv \rho_k(y) \pmod{A_q^{(k)} R^{(k)}}.$$

Proof. Since $\rho_k([\alpha_i, g^{-1} \alpha_i g]) \in R^{(k)}$ for any $g \in A$ and $1 \leq i \leq n$, we have $\rho_k(R) \subset R^{(k)}$. Therefore it is enough to show that $\rho_k(A_q) \subset A_q^{(k)}$. 



\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.1}
\caption{Self-crossing virtualization}
\end{figure}
This is proved by induction on \( q \). For \( q = 1 \) the assertion is obvious. Assume that \( q \geq 1 \). For \([x, y] \in A_{q+1} \) with \( x \in A \) and \( y \in A_q \), by the induction hypothesis, we have \( \rho_k ([x, y]) = [\rho_k (x), \rho_k (y)] \in A_{q+1}^{(k)} \).

**Proposition 6.4.** Let \((D, p)\) and \((D', p')\) be \(n\)-component virtual link diagrams with base point systems. For an integer \( k \in \{1, \ldots, n\} \), let \( l_k \) and \( l_k' \) be the \( k\)th preferred longitudes of \((D, p)\) and \((D', p')\), respectively. If \((D, p)\) and \((D', p')\) are related by a self-crossing virtualization, then \( \eta_q^{(k)} (D, p)(l_k) \equiv \eta_q^{(k)} (D', p')(l_k') \pmod{A_q^{(k)} R^{(k)}} \).

**Proof.** Assume that \((D, p)\) and \((D', p')\) are identical except in a disk \( \delta \), where they differ as shown in Figure 6.2. Furthermore, without loss of generality, we may assume that the self-crossing virtualization in the figure is applied to the 1st component.

![Figure 6.2. A self-crossing virtualization which relates \((D, p)\) to \((D', p')\)](image)

Let \( a_{ij} (1 \leq i \leq n, 1 \leq j \leq m_i + 1) \) be the arcs of \((D', p')\) as given in Section 2. Let \( a_{ij} \) and \( b \) be the arcs of \((D, p)\) such that each \( a_{ij} \) corresponds to the arc \( a'_{ij} \) of \((D', p')\), and \( b \) intersects the disk \( \delta \) as shown in Figure 6.2. Note that the domain of \( \eta_q^{(k)} (D, p) \) is the free group \( \mathcal{F} \) on \( \{a_{ij}\} \cup \{b\} \). Let \( \varepsilon \in \{\pm 1\} \) be the sign of the classical crossing among \( a_{1h}, a_{1t} \) and \( b \) in Figure 6.2. Since the self-crossing virtualization is applied to the 1st component, \( l_1 \) and \( l_1' \) can be written in the forms

\[
l_1 = a_{11}^{-1} x a_{11} y \quad \text{and} \quad l_1' = (a_{11})^{-\varepsilon} x' y'
\]

for certain words \( x, y \in \mathcal{F} \) and \( x', y' \in \mathcal{F} \) such that \( x' \) and \( y' \) are obtained from \( x \) and \( y \), respectively, by replacing \( b \) with \( a_{1h}' \), and \( a_{st} \) with \( a_{st}' \) for all \( s, t \). For \( 2 \leq i \leq n \), each \( l_i' \) is obtained from \( l_i \) by replacing \( b \) with \( a_{ih}' \), and \( a_{st} \) with \( a_{st}' \) for all \( s, t \). To complete the proof, we use the following.

**Claim 6.5.** (1) \( \eta_q^{(k)} (D, p)(a_{1h}) \equiv \eta_q^{(k)} (D, p)(b) \pmod{A_q^{(k)} R^{(k)}} \).

(2) \( \eta_q^{(k)} (D, p)(a_{ij}) \equiv \eta_q^{(k)} (D', p')(a'_{ij}) \pmod{A_q^{(k)} R^{(k)}} \) for any \( i, j \).

Before showing this claim, we observe it implies Proposition 6.4.

If \( k \neq 1 \), then the congruence \( \eta_q^{(k)} (D, p)(l_k) \equiv \eta_q^{(k)} (D', p')(l_k') \pmod{A_q^{(k)} R^{(k)}} \) follows from Claim 6.5 directly. If \( k = 1 \), then by Lemma 6.2(1) we have

\[
\eta_q^{(1)} (D, p)(l_1) = \eta_q^{(1)} (D, p)(xy) \quad \text{and} \quad \eta_q^{(1)} (D', p')(l_1') = \eta_q^{(1)} (D', p')(x'y')\]

Hence Claim 6.5 completes the proof.

**Proof of Claim 6.5.** (1) By Lemma 6.3 it is enough to show that

\[
\eta_q (a_{1h}) \equiv \eta_q (b) \pmod{A_q R}.
\]
Lemma 2.3(2) implies that
\[ \eta_q(b^{-1}a_{11}^{-\varepsilon}a_{1h}a_{1h}^{-\varepsilon}) \equiv 0 \pmod{A_q}. \]
Hence it follows that
\[
\eta_q(b^{-1}a_{1h}) = \eta_q(b^{-1}a_{11}^{-\varepsilon}a_{1h}a_{1h}^{-\varepsilon}a_{1h}^{-1}a_{1h}^{-1}a_{1h}) \\
\equiv \eta_q([a_{11}^{-\varepsilon}, a_{1h}^{-1}]) \pmod{A_q} \\
\equiv 1 \pmod{R}.
\]

(2) Put \( \eta'_q = \eta_q(D', p') \) for short. By Lemma 6.3, it is enough to show that
\[ \eta_q(a_{ij}) \equiv \eta'_q(a'_{ij}) \pmod{A_q R}. \]
for any \( 1 \leq i \leq n \) and \( 1 \leq j \leq m_i + 1 \).

This is proved by induction on \( q \). The assertion is certainly true for \( q = 1 \) or \( j = 1 \). Assume that \( q \geq 1 \) and \( j \geq 2 \). Let \( v_{ij-1} \) and \( v'_{ij-1} \) be part longitudes of \((D, p)\) and \((D', p')\), respectively. If \( i \neq 1 \) or \( v_{ij-1} \) does not contain the word \( xa_{11}^{-1} \), then \( v'_{ij-1} \) is obtained from \( v_{ij-1} \) by replacing \( b \) with \( a'_{1h} \), and \( a_{st} \) with \( a'_{st} \) for all \( s, t \). By the proof of (1) and the induction hypothesis, we have \( \eta_q(v_{ij-1}) \equiv \eta'_q(v'_{ij-1}) \pmod{A_q R} \). Hence Lemma 2.2(2) implies that
\[ \eta_{q+1}(a_{ij}) \equiv \eta'_{q+1}(a'_{ij}) \pmod{A_{q+1} R}. \]

If \( i = 1 \) and \( v_{ij-1} \) contains \( xa_{11}^{-1} \), then \( v_{ij-1} \) and \( v'_{ij-1} \) can be written in the forms
\[ v_{ij-1} = xa_{11}^{-1}z \quad \text{and} \quad v'_{ij-1} = x'z' \]
for certain words \( z \in A \) and \( z' \in A' \) such that \( z' \) is obtained from \( z \) by replacing \( b \) with \( a'_{1h} \), and \( a_{st} \) with \( a'_{st} \) for all \( s, t \). Then we have
\[ \eta_q(xz) \equiv \eta'_q(x'z') \pmod{A_q R} \]
by the proof of (1) and the induction hypothesis. Therefore it follows that
\[
\eta_{q+1}(a_{ij}) = \eta_q(v_{ij-1}^{-1})a_1 \eta_q(v_{ij-1}) \\
= \eta_q(z^{-1})\eta_q(a_{ij}^{-1})\eta_q(x^{-1}) \alpha_1 \eta_q(x) \eta_q(a_{ij}^\alpha) \eta_q(z) \\
\equiv \eta_q(z^{-1})\eta_q(a_{ij}^{-1})\eta_q(a_{ij}) \eta_q(x^{-1}) \alpha_1 \eta_q(x) \eta_q(z) \pmod{R} \\
\equiv \eta'_q((z')^{-1}(x')^{-1}) \alpha_1 \eta'_{q+1}(x'z') \pmod{A_{q+1} R} \\
= \eta'_q((v'_{ij-1}^{-1}) \alpha_1 \eta'_{q+1}(v'_{ij-1}) \\
= \eta'_{q+1}(a'_{ij}).
\]

For a sequence \( j_1 \ldots j_s i \) (\( 1 \leq s < q \)) of indices in \( \{1, \ldots, n\} \), we denote by \( \mu_{(D, p)}^{(q,k)}(j_1 \ldots j_s i) \) the coefficient of \( X_{j_1} \ldots X_{j_s} \) in \( E_{(q,k)}(l_i) \). We remark that by Lemma 2.5
\[ \mu_{(D, p)}^{(q,k)}(j_1 \ldots j_s i) = \mu_{(D, p)}^{(q+1,k)}(j_1 \ldots j_s i). \]
Furthermore, by Lemma 6.2(2), if the sequence \( j_1 \ldots j_s \) involves the index \( k \), then \( \mu_{(D, p)}^{(q,k)}(j_1 \ldots j_s i) = 0 \). On the other hand, if \( j_1 \ldots j_s \) does not involve \( k \), then \( \mu_{(D, p)}^{(q,k)}(j_1 \ldots j_s i) = \mu_{(D, p)}^{(q,k)}(j_1 \ldots j_s i) = \mu_{(D, p)}^{(q,k)}(j_1 \ldots j_s i) \).

Theorem 6.6. Let \((D, p)\) and \((D', p')\) be virtual link diagrams with base point systems. If \((D, p)\) and \((D', p')\) are related by a self-crossing virtualization, then \( \mu_{(D, p)}(I) = \mu_{(D', p')}(I) \) for any non-repeated sequence \( I \).
Proof. Let $k$ be the last index of a non-repeated sequence $I$. Then we may put $I = Jk$. Since $J$ does not involve $k$, we have $\mu_{(D,p)}(Jk) = \mu_{(D,D)}^{(q,k)}(Jk)$ and $\mu_{(D',p')}^{(D,p)}(Jk) = \mu_{(D',p')}^{(q,k)}(Jk)$. To complete the proof, we will show that $\mu_{(D,p)}(Jk) = \mu_{(D',p')}^{(q,k)}(Jk)$.

For $x \in A_q^{(k)}R^{(k)}$, we put
\[ E(x) = 1 + \sum \nu(j_1 \ldots j_s)X_{j_1} \cdots X_{j_s}. \]
By Proposition 6.4, it is enough to show that $\nu(j_1 \ldots j_s) = 0$ for any non-repeated sequence $j_1 \ldots j_s$ with $s < q$. If $x \in A_q^{(k)}$, then we have $\nu(j_1 \ldots j_s) = 0$ by Lemma 2.4. If $x \in R^{(k)}$, then we only need to consider the case $x = [\alpha_i, g^{-1}\alpha_i g]$ ($g \in A^{(k)}$, $1 \leq i \neq k \leq n$). Then it follows that
\[
E(x) - 1 = E([\alpha_i, g^{-1}\alpha_i g]) - 1 = (E(\alpha_i g^{-1}\alpha_i g) - E(\alpha_i^{-1}g^{-1}\alpha_i^{-1}g)).
\]
Here we observe that
\[
E(\alpha_i g^{-1}\alpha_i g) - E(\alpha_i^{-1}g^{-1}\alpha_i^{-1}g) = (1 + X_i)(E(g^{-1})(1 + X_i)E(g) - E(g^{-1})(1 + X_i)E(g)(1 + X_i) = X_iE(g^{-1})X_iE(g) - E(g^{-1})X_iE(g)X_i.
\]
This implies that each term of $E(x) - 1$ contains $X_i$ at least twice. Hence we have $\nu(j_1 \ldots j_s) = 0$ for any non-repeated sequence $j_1 \ldots j_s$.

Proof of Theorem 6.7. Let $p$ and $p'$ be base point systems of $D$ and $D'$, respectively. Then $(D, p)$ and $(D', p')$ are related by a finite sequence of self-crossing virtualizations, $w$-isotopies and the two local moves in Figure 3.2. If $(D, p)$ and $(D', p')$ are related by a self-crossing virtualization, then by Theorem 6.6 $\mu_{(D,p)}(I) = \mu_{(D',p')}^{(D,p)}(I)$ for any non-repeated sequence $I$. This implies that $\Delta_{(D,p)}(I) = \Delta_{(D',p')}^{(D,p)}(I)$. If $(D, p)$ and $(D', p')$ are related by a $w$-isotopy or the moves in Figure 3.2, then it follows from Theorems 3.1 and 5.2 that $\mu_{(D,p)}(I) \equiv \mu_{(D',p')}^{(D,p)}(I) \mod \Delta_{(D,p)}(I)$ and $\Delta_{(D,p)}(I) = \Delta_{(D',p')}^{(D,p)}(I)$. This completes the proof.

7. Welded string links

In the previous sections, we have studied Milnor invariants of welded links. Now we address the case of welded string links. Fix $n$ distinct points $0 < x_1 < \cdots < x_n < 1$ in the unit interval $[0, 1]$. Let $[0, 1], \ldots, [0, 1]_n$ be $n$ copies of $[0, 1]$. An $n$-component virtual string link diagram is the image of an immersion
\[
\bigcup_{i=1}^{n} [0, 1] \rightarrow [0, 1] \times [0, 1]
\]
such that the image of each $[0, 1]_i$ runs from $(x_i, 0)$ to $(x_i, 1)$, and the singularities are only classical and virtual crossings. The $n$-component virtual string link diagram $\{x_1, \ldots, x_n\} \times [0, 1]$ in $[0, 1] \times [0, 1]$ is called the trivial $n$-component string link diagram. An $n$-component welded string link is an equivalence class of $n$-component virtual string link diagrams under welded isotopy.

Let $\pi : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be the projection onto the first coordinate. Given an $n$-component virtual string link diagram $S$, an $n$-component virtual link diagram with a base point system is uniquely obtained by identifying points on the boundary of $[0, 1] \times [0, 1]$ with their images under the projection $\pi$. We denote it by $(D_S, p_S)$, where $p_S = (\pi(x_1, 0), \ldots, \pi(x_n, 0)) = (\pi(x_1, 1), \ldots, \pi(x_n, 1))$. We see
that if two virtual string link diagrams $S$ and $S'$ are welded isotopic, then $(D_S, p_S)$ and $(D_{S'}, p_{S'})$ are $\mathcal{W}$-isotopic.

For a sequence $I$ of indices in $\{1, \ldots, n\}$, the Milnor number $\mu_S(I)$ of $S$ is defined to be $\mu_{(D_S, p_S)}(I)$. Theorem 3.1 implies the following directly.

**Corollary 7.1.** Let $S$ and $S'$ be virtual diagrams of a welded string link. Then $\mu_S(I) = \mu_{S'}(I)$ for any sequence $I$.

Combining Theorems 3.1 and 6.6, the following result is obtained immediately.

**Corollary 7.2.** [9, Lemma 9.1]. If two virtual string link diagrams $S$ and $S'$ are related by a finite sequence of self-crossing virtualizations and welded isotopies, then $\mu_S(I) = \mu_{S'}(I)$ for any non-repeated sequence $I$.

We conclude this paper with a classification result of virtual link diagrams with base point systems up to an equivalence relation generated by self-crossing virtualizations and $\mathcal{W}$-isotopies.

**Theorem 7.3.** Let $(D, p)$ and $(D', p')$ be virtual link diagrams with base point systems. Then the following are equivalent.

1. $(D, p)$ and $(D', p')$ are related by a finite sequence of self-crossing virtualizations and $\mathcal{W}$-isotopies.
2. $\mu_{(D, p)}(I) = \mu_{(D', p')}((I))$ for any non-repeated sequence $I$.

**Proof.** (1) $\Rightarrow$ (2): This follows from Theorems 3.1 and 6.6 directly.

(2) $\Rightarrow$ (1): For a small disk $\delta$ which is disjoint from $(D, p)$ (or $(D', p')$), by applying VR2 relative base point system and the local move in Figure 3.1 repeatedly, we can deform $(D, p)$ (or $(D', p')$) such that the intersection between the disk $\delta$ and the deformed diagram is the trivial string link diagram whose each component contains the base point. Hence, $D \setminus \delta$ and $D' \setminus \delta$ can be regarded as string link diagrams $S$ and $S'$ respectively. Since $(D_S, p_S)$ and $(D_{S'}, p_{S'})$ are $\mathcal{W}$-isotopic to $(D, p)$ and $(D', p')$, respectively, it follows from Theorem 3.1 that

$$\mu_S(I) = \mu_{(D, p)}(I) \quad \text{and} \quad \mu_{S'}(I) = \mu_{(D', p')}(I)$$

for any non-repeated sequence $I$. Hence we have $\mu_S(I) = \mu_{S'}(I)$ by assumption. It is known in [11, 9] that Milnor $\mu$-invariants for non-repeated sequences classify welded string links up to self-crossing virtualizations. Therefore $S$ and $S'$ are related by a finite sequence of self-crossing virtualizations and welded isotopies. This implies that $(D_S, p_S)$ and $(D_{S'}, p_{S'})$ are related by a finite sequence of self-crossing virtualizations and $\mathcal{W}$-isotopies. \hfill $\square$

**Remark 7.4.** By Theorem 7.3 two virtual link diagrams with base point systems $(D, p)$ and $(D', p')$ given in Example 3.9 are not related by a finite sequence of self-crossing virtualizations and $\mathcal{W}$-isotopies.

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