Asymptotic normality for estimators of the additive regression components under random censorship

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Abstract

We establish asymptotic normality for estimators of the additive regression components under random censorship. To build our estimators, we couple the marginal integration method Newey (1994) with an initial Inverse Probability of Censoring Weighted estimator of the multivariate censored regression function introduced by Carbonez \textit{et al.} (1995) and Kohler \textit{et al.} (2002). Asymptotic confidence bands are derived from our result.

\textit{Key words:} additive model, censored data, censored regression, marginal integration

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1 Introduction

Censored data arise in many statistical application domains, especially in epidemiology and reliability. When studying the relationship between a dependent variable and covariates, nonparametric estimates are of particular interest in the presence of censored data, because no scatter plots can be drawn to detect the possibly complex form of this relationship. Several estimators have already been proposed to estimate the regression function. The key idea is to transform the observed data and derive so-called \textit{synthetic data} estimators (see Buckley and James (1979)). For instance, Fan and Gijbels (1994) developed a local version of the parametric Buckley and James estimator,

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while Carbonez et al. (1995) and Kohler et al. (2002) studied a nonparametric Inverse Probability of Censoring Weighted [I.P.C.W.] estimator (see also Brunel and Comte (2006)).

However, in most situations, the encountered variables (typically the onset of a disease in epidemiology) are related to numerous factors or predictors. Consider the particular case of the construction of a disease risk prediction tool, that is, the estimation of the probability of developing the disease, given a number (about ten generally) of predictors. In this setting, classical nonparametric estimators, such as kernel-type estimators, are unsuitable because of the well-known curse of dimensionality; for instance, the rate of convergence for nonparametric estimators of the conditional survival function is increasing in the regressor dimension in a censored setting (see Dabrowska (1995) or Deheuvels and Derzko (2007)), just like it is the case in an uncensored one (see, e.g., Härdle (1990) and the relevant references therein). One common solution to get round this issue is to work under the additive model assumption, when possible. In the uncensored case, several methods have been proposed to estimate the additive regression function. We shall evoke, among others, the methods based on $B$-splines (see Stone (1985)), on the backfitting algorithm (see Hastie and Tibshirani (1990)) and on marginal integration (see, e.g., Newey (1994), Tjøstheim and Auestad (1994) or Linton and Nielsen (1995)). In Fan and Gijbels (1994), it is shown that the backfitting ideas also applies to censored data. Here, following the ideas introduced in Debbarh and Viallon (2007a), we make use of the marginal integration method, coupled with initial multivariate nonparametric I.P.C.W. estimators to provide an estimator for the additive censored regression function. At this point, it is noteworthy that the developments we propose here for I.P.C.W.-type estimators shall apply with minor modifications to cope with other synthetic data estimators.

In former works, we established the mean-square convergence rate (Debbarh and Viallon (2007a)), the uniform consistency rate (Debbarh and Viallon (2007b)) and a uniform law of the logarithm (Debbarh and Viallon (2007c)) for such estimators of the additive regression function in the presence of censored data. In Debbarh and Viallon (2007c), we also proposed a method to construct simultaneous almost certainty bands, that is confidence bands which contain the true value of the additive component with asymptotical probability one, uniformly over the predictor domain. Obviously, those kinds of confidence bands may be very conservative, and classical confidence intervals derived from an asymptotical normal law may be desirable. To construct such intervals is one of the aims of the present paper, which is organized as follows. After having recalled how to construct estimators for the additive components in Section 2, we establish their asymptotic normality in Section 3. This limit law completes the one obtained by Camlong-Viot et al. (2000) in the uncensored case (see also Linton and Nielsen (1995) and Sperlich et al. (2002)). Then, in Section 4, we show how to obtain confidence intervals from the aforementioned
convergence in law. Finally, Section 5 is devoted to the proof of our result.

2 Notations

Consider the triple \((Y, C, X)\) defined in \(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d\), \(d \geq 1\), where \(Y\) is the variable of interest, \(C\) the censoring variable and \(X = (X_1, \ldots, X_d)\) a vector of concomitant variables. Throughout, we work with a sample \(\{(Y_i, C_i, X_i)_{1 \leq i \leq n}\}\) of independent and identically distributed replicae of \((Y, C, X)\). Actually, in the right censorship model, \(Y_i\) and \(C_i\) are not observed and only \(Z_i = \min\{Y_i, C_i\}\) and \(\delta_i = I\{Y_i \leq C_i\}\), \(1 \leq i \leq n\), are at our disposal, \(I_E\) standing for the indicator function of \(E\). Accordingly, the observed sample is \(D_n = \{(Z_i, \delta_i, X_i), i = 1, \ldots, n\}\), and for all \(t \in \mathbb{R}^+\), we set \(F(t) = P(Y > t), G(t) = P(C > t)\) and \(H(t) = P(Z > t)\) the right continuous survival functions pertaining to \(Y, C\) and \(Z\) respectively.

Further denote by \(\psi\) a given real measurable function. In this paper, we are concerned with the regression function of \(\psi(Y)\) evaluated at \(X = x\), in the particular case where this function is additive,

\[
m_{\psi}(x) = \mathbb{E}(\psi(Y) | X = x), \quad \forall \ x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]

\[
= \mu + \sum_{\ell=1}^{d} m_{\ell}(x_{\ell}). \quad \quad (2.1)
\]

In view of (2.2), the so-called additive components \(m_{\ell}, \ell = 1, \ldots, d\), as well as the constant \(\mu\) are defined up to an additive constant. Therefore, it is quite common to work under the identifiability assumption \(\mathbb{E}m_{\ell}(X_{\ell}) = 0, \ell = 1, \ldots, d\), which ensures that \(\mu = \mathbb{E}\psi(Y)\).

Let \((h_n)_{n \geq 1}\) and \((h_{\ell,n})_{n \geq 1}, \ell = 1, \ldots, d\), be \(d+1\) sequences of positive constants and denote by \(f\) the density function of the covariate \(X\). Introduce \(\hat{f}_n\) the Akaike-Parzen-Rosenblatt (Akaike (1954), Parzen (1962), Rosenblatt (1956)) estimator of \(f\) pertaining to \(K\) and \(h_n\),

\[
\hat{f}_n(x) = \frac{1}{nh_n^d} \sum_{j=1}^{n} K\left(\frac{x - X_j}{h_n}\right),
\]

where \(K\) is a given convolution kernel in \(\mathbb{R}^d\). We further denote by \(G_n^*\) the Kaplan-Meier estimator of \(G\) (Kaplan and Meier (1958)). Namely, adopting
the convention \( \prod_\varnothing = 1 \) and \( 0^0 = 1 \), we have, for all \( y \in \mathbb{R} \),

\[
G^*_n(y) = 1 - \prod_{1 \leq i \leq n} \left( \frac{N_n(Z_i) - 1}{N_n(Z_i)} \right)^{\beta_i},
\]

(2.3)

with \( \beta_i = \mathbb{I}_{\{Z_i \leq y\}}(1 - \delta_i) \) and \( N_n(y) = \sum_{j=1}^n \mathbb{I}_{\{Z_j \geq y\}} \).

To estimate the regression function defined in (2.1) (at this point, we do not work under the additive assumption (2.2) yet), we propose the following estimator (see Carbonez et al. (1995), Debbarh and Viallon (2007a), Jones et al. (1994), Kohler et al. (2002) and Maillot and Viallon (2007)). Denoting by \( K_1, \ldots, K_d \) \( d \) given kernels defined in \( \mathbb{R} \), we introduce

\[
\hat{m}^*_{\psi,n}(x) = \sum_{i=1}^n W_{n,i}(x) \frac{\delta_i \psi(Z_i)}{G^*_n(Z_i)} \quad \text{where} \quad W_{n,i}(x) = \frac{\prod_{\ell=1}^d \frac{1}{h_{\ell,n}} K_{\ell}(\frac{x-x_{i,\ell}}{h_{\ell,n}})}{n f_n(x_i)}.
\]

(2.4)

Adopting the convention \( 0/0 = 0 \), \( \hat{m}^*_{\psi,n} \) is properly defined since \( G^*_n(Z_i) \neq 0 \) if and only if \( Z_i = Z(n) \) and \( \delta(n) = 0 \), where \( Z(k) \) is the \( k \)-th ordered statistic associated to the sample \( (Z_1, \ldots, Z_n) \) for \( k = 1, \ldots, n \) and \( \delta(k) \) is the \( \delta_j \) corresponding to \( Z(k) = Z_j \).

For all \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and every \( \ell = 1, \ldots, d \), further set \( x_{-\ell} = (x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_d) \). Under the assumption (2.2), we will estimate the additive components via the marginal integration method (Linton and Nielsen (1995), Newey (1994)). Towards this aim, we introduce \( d \) given density functions defined in \( \mathbb{R} \), \( q_1, \ldots, q_d \). Then, setting \( q(x) = \prod_{\ell=1}^d q_{\ell}(x_\ell) \) and, for \( \ell = 1, \ldots, d \),

\[
q_{-\ell}(x_{-\ell}) = \prod_{j \neq \ell} q_j(x_j),
\]

consider, for \( \ell = 1, \ldots, d \), the quantities

\[
\eta_{\ell}(x_{\ell}) = \int_{\mathbb{R}^{d-1}} m_{\psi}(x) q_{-\ell}(x_{-\ell}) dx_{-\ell} - \int_{\mathbb{R}^d} m_{\psi}(x) q(x) dx.
\]

(2.5)

It is straightforward that the two following equalities hold,

\[
\eta_{\ell}(x_{\ell}) = m_{\ell}(x_{\ell}) - \int_{\mathbb{R}} m_{\ell}(z) q_{\ell}(z)dz,
\]

(2.6)

\[
m_{\psi}(x) = \sum_{\ell=1}^d \eta_{\ell}(x_{\ell}) + \int_{\mathbb{R}^d} m_{\psi}(z) q(z) dz.
\]

(2.7)

In view of (2.6) and (2.7), \( \eta_{\ell} \) and \( m_{\ell} \) are equal up to an additive constant, in such a way that \( \eta_{\ell} \) is actually an additive component too, which fulfils an alternative identifiability condition.

**Remark 2.1** Observe that \( \eta_{\ell} = m_{\ell} \) for the particular choice \( q_{\ell} = f_{\ell} \), \( f_{\ell} \) denoting the density function pertaining to \( X_{\ell} \), \( \ell = 1, \ldots, d \). However, \( f_{\ell} \) being unknown in most situations, we most often have \( \eta_{\ell} \neq m_{\ell} \).
From (2.4) and (2.5), a natural estimator of the \( \ell \)-th additive component \( \eta_\ell \) evaluated at \( x_\ell \) is given, for \( \ell = 1, \ldots, d \), by

\[
\hat{\eta}_\ell^*(x_\ell) = \int_{\mathbb{R}^{d-1}} \hat{m}^*_{\psi,n}(x)q_{-\ell}(x_{-\ell})dx_{-\ell} - \int_{\mathbb{R}^d} \hat{m}^*_{\psi,n}(x)q(x)dx,
\]

(2.8)

from which an estimator \( \hat{m}^*_{\psi,\text{add}} \) of the additive regression function can easily be deduced (see (2.7)),

\[
\hat{m}^*_{\psi,\text{add}}(x) = \sum_{\ell=1}^d \hat{\eta}_\ell^*(x_\ell) + \int_{\mathbb{R}^d} \hat{m}^*_{\psi,n}(x)q(x)dx.
\]

(2.9)

3 Hypotheses and Results

These preliminaries being given, we introduce the assumptions to be made to state our result. First, consider the hypotheses pertaining to \((Y, C, X)\).

\( (C.1) \) \( C \) and \((X, Y)\) are independent.
\( (C.2) \) \( G \) is continuous on \( \mathbb{R}^+ \).
\( (C.3) \) There exists a finite constant \( M \) such that \( \sup_y |\psi(y)| \leq M \).
\( (C.4) \) \( m_\psi \) is \( k \)-times continuously differentiable, \( k \geq 1 \), and

\[
\sup_x \left| \frac{\partial^k}{\partial x_\ell^k} m_\psi(x) \right| < \infty; \quad \ell = 1, \ldots, d.
\]

As mentioned in Gross and Lai (1996), functionals of the (conditional) law can generally not be estimated on the complete support when the variable of interest is right-censored. Accordingly, we will work under the assumption \((A)\) that will be said to hold if either \((A)(i)\) or \((A)(ii)\) below holds. Denote by \( T_L = \sup \{ t : L(t) > 0 \} \) the upper endpoint of the distribution of a random variable with right continuous survival function \( L \).

\( (A)(i) \) There exists a \( \tau_0 < T_H \) such that \( \psi = 0 \) on \((\tau_0, \infty)\).
\( (A)(ii) \) (a) For a given \( k/(2k + 1) < p \leq 1/2 \), \( \left| \int_{T_R} F^{-p/(1-p)}dG \right| < \infty \);
(b) \( T_R < T_G \);
(c) \( n^{2p-1}h_{\ell,n}^{-1} |\log(h_{\ell,n})| \to \infty \), as \( n \to \infty \), for every \( \ell = 1, \ldots, d \).

Remark 3.1 (i) It is noteworthy that assumption \((A)(ii)\) allows for considering the estimation of the “classical” regression function, which corresponds to the choice \( \psi(y) = y \). On the other hand, normality for estimators of functionals such as the conditional distribution function \( \mathbb{P}(Y \leq \tau_0 | X) \) can be obtained under weaker conditions, when restricting ourselves to \( \tau_0 < T_H \).

(ii) When working under \((A)(ii)\), assumption \((C.3)\) can be weakened. In this
setting, it is indeed sufficient to work under (C.3) below.

(C.3) There exists a finite constant $M$ such that $\sup_{y \leq \tau_0} |\psi(y)| \leq M$.

(iii) It is also noteworthy that condition (C.1) is stronger than the conditional independence of $C$ and $Y$ given $X$, under which Beran (1981) worked to build an estimator of the conditional survival function (see also Dabrowska (1993) and Deheuvels and Derzko (2007)). Note, however, that the two assumptions coincide if $C$ and $X$ are independent. In other respect, to use Beran’s local Kaplan–Meier estimator, the censoring has to be locally fair, that is $\mathbb{P}[C \geq t \mid X = x] > 0$ whenever $\mathbb{P}[Y \geq t \mid X = x] > 0$. Here, we basically only suppose that $G(t) > 0$ whenever $F(t) > 0$, which is, on its turn, a weaker assumption. For a nice discussion on the difference between Beran’s estimator and Carbonez et al.’s estimator, we refer to Carbonez et al. (1995).

Denote by $C_1, \ldots, C_d$ compact intervals of $\mathbb{R}$ and set $C = C_1 \times \ldots \times C_d$. For every subset $E$ of $\mathbb{R}^d$, and any $\alpha > 0$, introduce the $\alpha$-neighborhood $E_\alpha$ of $E$, namely $E_\alpha = \{x : \inf_{y \in E} |x - y|_{\mathbb{R}^d} \leq \alpha\}$, $|\cdot|_{\mathbb{R}^d}$ standing for the Euclidean norm on $\mathbb{R}^d$.

We will work under the following regularity assumptions on $f$ and $f_\ell$, $\ell = 1, \ldots, d$ ($f_\ell$ denoting the density function of $X_\ell$, as in Remark 2.1). These functions are supposed to be continuous. Moreover, we assume the existence of a constant $\alpha > 0$ such that the following assumptions hold.

(F.1) $\forall x_\ell \in C_\alpha^\ell, f_\ell(x_\ell) > 0$, $\ell = 1, \ldots, d$, and $\forall x \in C_\alpha, f(x) > 0$.

(F.2) $f$ is $k'$-times continuously differentiable on $C_\alpha$, with $k' > kd$.

Regarding the kernels $K$ and $K_\ell$, $\ell = 1, \ldots, d$, defined in $\mathbb{R}^d$ and $\mathbb{R}$ respectively, they are assumed to be bounded, integrable to 1, with compact support and such that,

(K.1) $K_\ell$ is of order $k$, $\ell = 1, \ldots, d$.

(K.2) $K$ is of order $k'$.

In addition, we impose the following assumptions on the given integrating density functions $q_{-\ell}$ and $q_\ell$, $\ell = 1, \ldots, d$.

(Q.1) $q_{-\ell}$ is bounded and continuous, $\ell = 1, \ldots, d$.

(Q.2) For $\ell = 1, \ldots, d$, $q_\ell$ has a compact support included in $C_\ell$ and has $(k + 1)$ continuous and bounded derivatives.

Finally, turning our attention to the sequences $(h_n)_{n \geq 1}$ and $(h_{\ell,n})_{n \geq 1}$, $\ell = 1, \ldots, d$, we will work under the conditions below.

(H.1) $h_n = c' \left(\frac{\log n}{n}\right)^{1/(2k'+d)}$, for a fixed $0 < c' < \infty$. 

6
Some more notation is needed for the statement of our results. Set, for all \( x \in C \) and every \( \ell = 1, \ldots, d \),

\[
b_\ell(x_\ell) = \frac{c^k}{k!} \int_R u^k K_\ell(u)du \left( (-1)^k m_\ell^{(k)}(x_\ell) - \int_R m_\ell(z) q_\ell^{(k)}(z)dz \right),
\]

(3.1) and

\[
\sigma_\ell^2(x_\ell) = \frac{\int_R K_\ell^2(u)du}{cf_\ell(x_\ell)} \int_{R^{d-1}} H(x) \frac{q_\ell^2(x_\ell)}{f(x_{-\ell}|x_\ell)} dx_{-\ell},
\]

(3.2)

where

\[
H(x) = \mathbb{E} \left[ \frac{\psi^2(Y)}{G(Y)} \mid X = x \right].
\]

(3.3)

We have now all the ingredients to state our main result in Theorem 3.1 below.

**Theorem 3.1** Under the conditions (C.1-2-3-4), (F.1-2), (K.1-2), (Q.1-2) and (H.1-2), we have, for every \( \ell = 1, \ldots, d \) and all \( x_\ell \in C_\ell \),

\[
\frac{n^{k/(2k+1)}}{\sigma_\ell(x_\ell)} \left\{ \hat{\eta}_\ell(x_\ell) - \eta_\ell(x_\ell) \right\} - b_\ell(x_\ell) \overset{\mathcal{L}}{\longrightarrow} \mathcal{N}(0, 1).
\]

(3.4)

This result naturally implies the following corollary, which correspond to a refinement of Theorem 3.1 in Debbarh and Viallon (2007a).

**Corollary 3.1** Under the conditions of Theorem 3.1, we have, for every \( \ell = 1, \ldots, d \) and all \( x_\ell \in C_\ell \),

\[
n^{2k/(2k+1)} \mathbb{E} \left( \hat{\eta}_\ell(x_\ell) - \eta_\ell(x_\ell) \right)^2 = b_\ell^2(x_\ell) + \sigma_\ell^2(x_\ell) + o(1).
\]

The proof of Theorem 3.1 is postponed to Section 5. A rough outline of our arguments is as follows. First, we consider the case where both the density function \( f \) of \( X \) and the function \( G \) are known (see Lemma 5.1 below). In this setting, we show that the estimator defined in (2.4) can be written as an uncensored estimator of the regression function, and the result of Lemma 5.1 follows from similar arguments as those developed in Camlong-Viot et al. (2000) in the uncensored setting. Then, from the result of Ango-Nze and Rios (2000), we can extend Lemma 5.1 to the case where only \( G \) is known. Finally,
the uniform consistency of $G_n$ (see, for instance, Földes and Rejtő (1981)) enables us to conclude the demonstration of Theorem 3.1 in the general case.

4 Application: construction of confidence intervals

4.1 Construction of confidence intervals

Under the assumption of Theorem 3.1, it is straightforward that, for $\ell = 1, \ldots, d$, the interval

$$\left[ \hat{\eta}_\ell^*(x_\ell) \pm n^{-k/(2k+1)}(1.96\sigma_\ell(x_\ell) - b_\ell(x_\ell)) \right]$$

is a confidence interval for $\eta_\ell(x_\ell)$, at an asymptotic 95% level, for all $x_\ell \in C_\ell$. However, as can be seen in the definition (3.1), the bias term $b_\ell$ is a "complex" function involving $m_\ell$ and $m_\ell^{(k)}$. Bias estimates could be built in by using estimates of these quantities, but this would result in quite complex algorithms to derive the confidence bands. Following the ideas which have been proposed to construct confidence bands for kernel-type estimators of the regression function (see, e.g., Section 4.2 in Härdle (1990)), our aim is now to make the bias term vanish. A close look into the proof of Theorem 3.1 reveals that, if the bandwidth $h_{\ell,n}$ is chosen proportional to $n^{-1/(2k+1)}$ times a sequence that tends slowly to 0 then the bias vanishes asymptotically. In this case, for $\ell = 1, \ldots, d$, the interval

$$\left[ \hat{\eta}_\ell^*(x_\ell) \pm 1.96\hat{\sigma}_\ell(x_\ell)n^{-k/(2k+1)} \right]$$

provides a confidence interval for $\eta_\ell(x_\ell)$, at an asymptotic 95% level, for all $x_\ell \in C_\ell$. Then, given any consistent estimator $\hat{\sigma}_\ell$ of $\sigma_\ell$ (making use, for instance, of kernel estimators for the density functions involved in $\sigma_\ell$), we conclude by Slutsky’s Theorem that

$$\left[ \hat{\eta}_\ell^*(x_\ell) \pm 1.96\hat{\sigma}_\ell(x_\ell)n^{-k/(2k+1)} \right]$$

(4.1)

provides a confidence interval for $\eta_\ell(x_\ell)$, at an asymptotic 95% level, for all $x_\ell \in C_\ell$.

4.2 Illustration: a simple simulation study

In the following paragraph, we present some results from a simulation study, which especially enables to compare the just given confidence bands with the almost certainty bands we proposed in Debbarh and Viallon (2007c).
We worked with a sample size \( n = 1000 \), and considered the case where \( \mathbf{X} = (X_1, X_2) \in \mathbb{R}^2 \) (i.e. \( d = 2 \)) was such that \( X_1 \sim \mathcal{U}(-1, 1) \) and \( X_2 \sim \mathcal{U}(-1, 1) \), where \( \mathcal{U}(a, b) \) stands for the uniform law on \((a,b)\). Set \( m_1(x) = 0.5 \times \cos^2(x) \) and \( m_2(x) = 0.5 \times \sin^2(x) \). We selected \( \psi = \mathbb{I}_{[-0.9]} \), and considered the model \( \mathbb{E}[\psi(Y)|X_1 = x_1, X_2 = x_2] = m_1(x_1) + m_2(x_2) \). Under this model, the variable \( Y \) was simulated as follows. For each integer \( 1 \leq i \leq n \), let \( p_i = m_1(x_{1,i}) + m_2(x_{2,i}) \) where \( x_{j,i} \) is the \( i \)-th observed value of the variable \( X_j \), \( j = 1, 2 \). Note that \( 0 < p_i < 1 \) for every \( 1 \leq i \leq n \). Each \( Y_i \) was then generated as one \( \mathcal{U}(0.9 - p_i, 1+0.9 - p_i) \) variable. Following this proceed ensured that \( \mathbb{P}(Y_i \leq 0.9|X_i = x_i) = p_i = m_1(x_{1,i}) + m_2(x_{2,i}) \). Regarding the censoring variable, we generated an i.i.d. sample \( C_1, \ldots, C_n \) such that \( C_i \sim \mathcal{U}(0, 1) \). This choice yielded, \textit{a posteriori}, \( \mathbb{P}(\delta = 1) \approx 0.2 \). We used Epanechnikov kernels (for \( K_1 \) and \( K_2 \)) and selected \( q_1 = q_2 = 0.5 \times \mathbb{I}_{[-1,1]} \) (in such a way that the additive component to estimate were \( \eta_{\psi,j} = m_j - 0.25 \), \( j = 1, 2 \)). As for the bandwidth choice, we opted \textit{a priori} for \( h_{1000} = h_{1,1000} = h_{2,1000} = 0.2 \).

Graphical representations of the results are provided in Figure 1. It can be seen that the confidence intervals derived from the asymptotic normality are less conservative than the ones obtained from the uniform law of the logarithm. The price to pay is however that the true function does not belong to the former intervals at every \( x \in [0, 1] \). Therefore, in most applications, we recommend the construction of both confidence bands to assess the form of the relationship between the dependant variable and covariates.

![Graphical representations of the results](image-url)

(a) First component  
(b) Second component

Figure 1. Results of the simulation study for (a) the first additive component and (b) the second additive component: true additive components (blue solid line), their estimates (black dashed line), the 95% confidence intervals (black dotted lines) and the almost certainty bands (red dotted lines).
Here we present the detailed proof of Theorem 3.1. Only the proof for \( \ell = 1 \) is presented, the proof for the \( d - 1 \) remainder components being similar and then omitted.

### 5.1 The case where both \( f \) and \( G \) are known

Recall the definitions (2.4) and (2.8) of \( \tilde{m}_{\psi,n}^* \) and \( \hat{\eta}_{\ell}^* \) respectively. Further denote by \( \tilde{m}_{\psi,n}(x) \) and \( \hat{\eta}_{\ell}(x_\ell) \) the versions of \( \tilde{m}_{\psi,n}^* \) and \( \hat{\eta}_{\ell}^* \) respectively, in the case where both \( f \) and \( G \) are known. Namely, we have

\[
\tilde{m}_{\psi,n}(x) = \sum_{i=1}^{n} \tilde{W}_{n,i}(x) \frac{\delta_{i\psi}(Z_i)}{G(Z_i)} \quad \text{with} \quad \tilde{W}_{n,i}(x) = \frac{\prod_{\ell=1}^{d} \frac{1}{h_{\ell,n}} K_{\ell}(x_\ell - X_{i,l})}{nf(X_i)},
\]

and

\[
\hat{\eta}_{\ell}(x_\ell) = \int_{\mathbb{R}^{d-1}} \tilde{m}_{\psi,n}(x) q_{-\ell}(x_{-\ell}) dx_{-\ell} - \int_{\mathbb{R}^d} \tilde{m}_{\psi,n}(x) q(x) dx.
\]

Consider the function \( \Psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that,

\[
\Psi(y, c) = \frac{\mathbb{I}_{\{y \leq c\}} \psi(y \wedge c)}{G(y \wedge c)}, \quad \text{for all} \quad (y, c) \in \mathbb{R}^2.
\]

In view of (5.1) and (5.3), we have

\[
\tilde{m}_{\psi,n}(x) = \sum_{i=1}^{n} \tilde{W}_{n,i}(x) \Psi(Y_i, C_i).
\]

In the sequel we will make frequent use of a conditional argument, along with the independence assumption (C.1), that especially enables us to obtain the following kind of result.

\[
m_{\psi}(X) := \mathbb{E}(\Psi(Y, C) \mid X) = \mathbb{E}\left\{ \frac{\mathbb{I}_{\{Y \leq C\}} \psi(Z)}{G(Z)} \mid X \right\} = \mathbb{E}\left\{ \frac{\psi(Y)}{G(Y)} \mathbb{E}\left[ \mathbb{I}_{\{Y \leq C\}} \mid X \right] \mid X \right\} = m_{\psi}(X).
\]
function $m_\psi$. This property enables to treat the particular case where $G$ is known with arguments similar to those used in the uncensored case.

We will first establish the following result, which correspond to Theorem 3.1 in the case where $f$ and $G$ are known.

**Lemma 5.1** Assume (C.1-2-3-4), (K.1), (Q.1-2) and (H.2) hold. Then, for every $\ell = 1, \ldots, d$ and all $x_\ell \in C_\ell$,

$$n^{k/(2k+1)} \left\{ \hat{\eta}_\ell(x_\ell) - \eta_\ell(x_\ell) \right\} - b_\ell(x_\ell) \xrightarrow{L} \mathcal{N}(0, 1).$$

(5.6)

**Proof.** In view of the discussion above, we will mostly borrow the arguments developed in the uncensored case by Camlong-Viot et al. (2000). Towards this aim, introduce the following quantities.

\[
\tilde{\Psi}_n(y_i, c_i) = \Psi(y_i, c_i) \int_{\mathbb{R}^{d-1}} \prod_{\ell=2}^d \frac{1}{h_{\ell,n}} K_\ell \left( \frac{x_\ell - Y_{i,\ell}}{h_{\ell,n}} \right) q_{-1}(x_{-1}) f_{X_{i,1}} d x_{-1},
\]

(5.7)

\[
G(u_{-1}) = \int_{\mathbb{R}^{d-1}} \prod_{\ell=2}^d \frac{1}{h_{\ell,n}} K_\ell \left( \frac{x_\ell - u_\ell}{h_{\ell,n}} \right) q_{-1}(x_{-1}) d x_{-1},
\]

(5.8)

\[
\hat{\alpha}_1(x_1) = \frac{1}{n h_{1,n}} \sum_{i=1}^n \tilde{\Psi}_n(y_i, c_i) \int_{\mathbb{R}^{d-1}} \frac{1}{f_{X_{i,1}}(x_{i,1})} K_1 \left( \frac{x_{1,1} - X_{i,1}}{h_{1,n}} \right),
\]

(5.9)

\[
\tilde{m}(x_1) = \mathbb{E} \left( \tilde{\Psi}_n(y_i, c_i) \big| X_{i,1} = x_1 \right),
\]

(5.10)

\[
C_n = \mu + \int_{\mathbb{R}^{d-1}} \sum_{j=2}^d m_j(u_j) G(u_{-1}) d u_{-1},
\]

\[
\hat{C}_n = \int_{\mathbb{R}^{d}} \tilde{m}_n(x) q(x) d x,
\]

(5.12)

Next observe that

\[
\hat{\eta}_1(x_1) - \eta_1(x_1) = \left\{ \hat{\alpha}_1(x_1) - \tilde{m}(x_1) \right\} + \mathbb{E} \left( \hat{C}_n - C_n - C \right),
\]

(5.14)

and set

\[
\beta_1(x_1) = (-1)^k c^k \eta_1^{(k)}(x_1) \int_{\mathbb{R}} \frac{v^k}{k!} K_1(v) d v_1,
\]

(5.15)

\[
\beta_2 = c^k \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \frac{v^k}{k!} \tilde{m}_n^{(k)}(x_1 + v_1 h_1) K_1(v_1) d v_1 \right\} q_1(x_1) d x_1.
\]

(5.16)
From (5.14)-(5.15)-(5.16), and because of Slutsky’s theorem, the proof of Lemma 5.1 will be completed as soon as the four following results will be established.

\[ n^{k/(2k+1)} (\hat{\alpha}_1(x_1) - \tilde{m}(x_1) - \beta_1(x_1)) \xrightarrow{L} \mathcal{N}(0, \sigma_1^2(x_1)), \]  
\[ n^{k/(2k+1)} \mathbb{E}\left( \hat{C}_n - C_n - C \right) = \beta_2 + o(1), \]  
\[ n^{2k/(2k+1)} \text{Var}(\hat{C}_n) = o(1), \]  
\[ n^{2k/(2k+1)} \text{Cov}(\hat{C}_n, \hat{\alpha}_1(x_1)) = o(1). \]

**Proof of (5.17):** In a first step, our aim is to show that

\[ n h_{1,n} \{ \hat{\alpha}_1(x_1) - \mathbb{E}(\hat{\alpha}_1(x_1)) \} \longrightarrow \mathcal{N}(0, \sigma_1^2(x_1)). \]  
\[ (5.21) \]

We claim that

\[ n h_{1,n} \text{Var}(\hat{\alpha}_1(x_1)) \rightarrow \sigma_1^2(x_1) \text{ as } n \rightarrow \infty, \]  
\[ (5.22) \]

where \( \sigma_1^2(x_1) \) is as in (3.2). Recalling (5.9), note that

\[
\text{Var}(\hat{\alpha}_1(x_1)) = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E}\left\{ \frac{\tilde{\Psi}_n(Y_i, C_i)}{f_1(X_i,1)h_{1,n}} K_1 \left( \frac{x_1 - X_{i,1}}{h_{1,n}} \right) \right\}^2 \\
- \frac{1}{n^2} \sum_{i=1}^{n} \left\{ \mathbb{E}\left( \frac{\tilde{\Psi}_n(Y_i, C_i)}{f_1(X_i,1)h_{1,n}} K_1 \left( \frac{x_1 - X_{i,1}}{h_{1,n}} \right) \right) \right\}^2 \\
= \frac{1}{n} \left[ \frac{1}{h_{1,n}} \Phi_{1,n}(x_1) - [\Gamma_{1,n}(x_1)]^2 \right],
\]

with

\[
\Gamma_{1,n}(x_1) = \mathbb{E}\left\{ \frac{1}{h_{1,n}} K_1 \left( \frac{x_1 - X_{i,1}}{h_{1,n}} \right) \tilde{\Psi}_n(Y_i, C_i) \right\},
\]

\[
\Phi_{1,n}(x_1) = \mathbb{E}\left\{ \frac{1}{h_{1,n}} K_1^2 \left( \frac{x_1 - X_{i,1}}{h_{1,n}} \right) \tilde{\Psi}_n^2(Y_i, C_i) \right\}.
\]

Using classical conditioning arguments and recalling the definition (5.10) of \( \tilde{m} \), it is straightforward that

\[
\Gamma_{1,n}(x_1) = \int_{\mathbb{R}} \frac{1}{h_{1,n}} K_1 \left( \frac{x_1 - u_1}{h_{1,n}} \right) \tilde{m}(u_1) du_1,
\]

\[
\Phi_{1,n}(x_1) = \int_{\mathbb{R}} \frac{1}{h_{1,n}} K_1^2 \left( \frac{x_1 - u_1}{h_{1,n}} \right) \mathbb{E}\left( \tilde{\Psi}_n^2(Y_i, C_i) \mid X_{i,1} = u_1 \right) f_1(u_1) du_1.
\]
In other respect, the definitions (5.7) and (5.10), when combined with the argument used in (5.5), yield

\[ |\tilde{m}(u_1)| = |E(\tilde{\Psi}_n(Y_i, C_i) \mid X_{i,1} = u_1)| \\
= \left| E\left( \frac{\delta \psi(Z_i)}{G(Z_i)} \frac{G(X_{i,-1})}{f(X_{i,-1} \mid X_{i,1})} \mid X_{i,1} = u_1 \right) \right| \\
= \left| \int_{\mathbb{R}^{d-1}} m_\psi(u)G(u_{-1})du_{-1} \right| < \infty, \]

in such a way that

\[ \frac{1}{n} \Gamma^2_{1,n}(x_1) \to 0 \text{ as } n \to \infty. \tag{5.23} \]

Moreover, using once again the argument used to derive (5.5) and keeping in mind the definition (3.3) of $H$, it is easy to derive that, under (C.1),

\[ E(\tilde{\Psi}_n^2(Y_i, C_i) \mid X_{i,1} = x_1) \\
= E\left\{ \left( \frac{\delta \psi(Z_i)}{G(Z_i)} \right)^2 \left( \frac{G(X_{i,-1})}{f(X_{i,-1} \mid X_{i,1})} \right)^2 \mid X_{i,1} = x_1 \right\}, \\
= E\left\{ \left[ \frac{\delta \psi(Z_i)}{G(Z_i)} \right]^2 X_i \left( \frac{G^2(X_{i,-1})}{f^2(X_{i,-1} \mid X_{i,1})} \right) \mid X_{i,1} = x_1 \right\}, \\
= \int_{\mathbb{R}^{d-1}} H(u) \frac{G^2(u_{-1})}{f^2(u_{-1} \mid x_1)} f(u_{-1} \mid x_1)du_{-1}, \\
= \int_{\mathbb{R}^{d-1}} H(u) \frac{G^2(u_{-1})}{f(u_{-1} \mid x_1)} du_{-1}. \tag{5.24} \]

Next, making use of the classical change of variable $v_\ell h_{\ell,n} = u_\ell - x_\ell$ along with a Taylor expansion of order $k$ (which is rendered possible by (Q.2)), we readily have by (K.1), for a given $0 < \theta < 1$,

\[ \int_{\mathbb{R}^{d-1}} \prod_{\ell=2}^d \frac{1}{h_{\ell,n}} K_{\ell} \left( \frac{x_\ell - u_\ell}{h_{\ell,n}} \right) q_\ell(x_\ell)dx_{-1} - q_{-1}(u_{-1}) \\
= \int_{\mathbb{R}^{d-1}} \prod_{\ell=2}^d \left( K_{\ell}(v_\ell) \left[ \frac{v_\ell^{k} h_{\ell,n}^{k}}{k!} q_\ell^{(k)}(\theta v_\ell h_{\ell,n} + u_\ell) \right] \right)dv_{-1} \\
= o(1), \tag{5.25} \]

Combining (5.24) and (5.25), we get

\[ E(\tilde{\Psi}_n^2(Y_i, C_i) \mid X_{i,1} = x_1) = \int_{\mathbb{R}^{d-1}} H(x) \frac{q_{-1}(x_{-1})}{f(x_{-1} \mid x_1)} dx_{-1} + o(1). \tag{5.26} \]
In addition, setting $\Phi(x_1) = \int_{\mathbb{R}^{d-1}} H(x) \frac{q_1^2(x_1)}{f(x_1 \mid x_1)} \, dx_1$ and using once again the change of variable $v_1 h_{1,n} = x_1 - u_1$, we obtain

$$
\Phi_{1,n}(x_1) = \int_{\mathbb{R}} K_1^2(v_1) \frac{f_1(x_1 - h_1 v_1)}{f_1(x_1)} d\Phi(x_1) + \Phi(x_1) \int_{\mathbb{R}} K_1^2(v_1) d\Phi(x_1).
$$

But, by $(C.3), (K.1), (F.1)$ and $(Q.1-2)$, it is easily shown that the quantity $|E(\tilde{\Psi}(Y_i, C_i) \mid X_{i,1} = u_1)/f_1(u_1) - \Phi(x_1)/f_1(x_1)|$ is bounded. Therefore, when combined with Lebesgue’s dominated convergence Theorem enables us to conclude that

$$
\Phi_{1,n}(x_1) \rightarrow \frac{\Phi(x_1)}{f_1(x_1)} \int_{\mathbb{R}} K_1^2(v_1) d\Phi(x_1). \quad (5.27)
$$

From (5.23) and (5.27), the claim (5.22) is proved.

Now, we set

$$
\tilde{T}_{i,n} = \frac{1}{h_{1,n}} \frac{\tilde{\Psi}_n(Y_i, C_i)}{f_1(X_{i,1})} K_1 \frac{x_1 - X_{i,1}}{h_{1,n}}, \quad T_{i,n} = \tilde{T}_{i,n} - E\tilde{T}_{i,n},
$$

and $s_n^2 = \text{Var}(T_{i,n}) = n \text{Var}(\hat{\alpha}_1(x_1)). \quad (5.28)$

For all $\varepsilon > 0$, we have

$$
\mathbb{E} \left\{ \frac{T_{i,n}^2}{ns_n^2} \mathbb{I}_{\left\{ \frac{T_{i,n}}{\sqrt{ns_n}} \geq \varepsilon \right\}} \right\} \leq \frac{M_1}{h_{1,n}^2 ns_n^2} \mathbb{E} \left( \frac{T_{i,n}}{\sqrt{ns_n}} \right) \geq \varepsilon \leq \frac{M_1}{h_{1,n}^2 ns_n^2} \mathbb{E}(T_{i,n}^2)
$$

$$
\leq \frac{M_1}{h_{1,n}^2 ns_n^2} E(T_{i,n}^2) \leq \frac{M_1}{h_{1,n}^2 ns_n^2} \frac{\sigma_1^2(x_1)}{\varepsilon^2 n^4} \leq M_2 \frac{\varepsilon}{\varepsilon^2 h_{1,n}^2 n^4}, \quad (5.29)
$$

where $M_1$ and $M_2$ are two finite and positive constants. Combining (5.29) with the fact that $T_{i,n}/s_n^2 \rightarrow 0$ (which follows from (5.22) and (5.28)), we can apply the normal convergence criterion (see, e.g. Loève (1963), p.295) to obtain
Finally, (5.21) readily comes from (5.9), (5.22), (5.28) and (5.30).

Now, our aim is to evaluate the term $|\tilde{m}(x_1) - \mathbb{E}\hat{\alpha}_1(x_1)|$. First, from (5.7), (5.8) and (5.10), note that

$$\tilde{m}(x_1) = \mathbb{E}\left(\Psi_n(Y_i, C_i) \mid X_{i,1} = x_1 \right)$$

Then, using a conditioning argument along with the independence assumption (C.1), we get

$$\tilde{m}(x_1) = \mathbb{E}\left\{ \mathbb{E}\left(\frac{\Psi(Y_i, C_i)}{f(X_{i,-1} | X_{i,1})} G(X_{i,-1}) \right) \mid X_{i,1} = x_1 \right\}$$

Thus, by (K.1) and (C.4), a Taylor expansion yields

$$\mathbb{E}(\hat{\alpha}_1(x_1)) - \tilde{m}(x_1) = \int_R \frac{1}{h_{1,n}} \tilde{m}(u_1) K_1(x_1 - u_1) du_1 - \tilde{m}(x_1)$$

By combining this last result with (5.21), we conclude to (5.17).\[\square\]

**Proof of (5.18)**. Keep in mind the definitions (5.11) and (5.12) of $C_n$ and $\hat{C}_n$. Then, according to Fubini’s Theorem and under the additive model assumption,
\[
\mathbb{E}(\hat{C}_n - C_n) = \mathbb{E}\left\{ \int_{\mathbb{R}^d} \tilde{m}_{\psi,n}(x)q(x)dx - \mu - \int_{\mathbb{R}^{d-1}} \sum_{j=2}^{d} m_j(u_j)G(u_{-1})du_{-1} \right\}
\]

\[
- \int_{\mathbb{R}^{d-1}} \sum_{j=2}^{d} m_j(u_j)G(u_{-1})du_{-1}
\]

\[
= \sum_{j=1}^{d} \int_{\mathbb{R}^d} \frac{1}{h_{1,n}} m_j(u_j)G(u_{-1}) \int_{\mathbb{R}} K_1\left(\frac{x_1 - u_1}{h_{1,n}}\right)q_1(x_1)dx_1du
\]

\[
- \int_{\mathbb{R}^{d-1}} \sum_{j=2}^{d} m_j(u_j)G(u_{-1})du_{-1}
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{h_{1,n}} m_1(u_1)K_1\left(\frac{x_1 - u_1}{h_{1,n}}\right)q_1(x_1)dx_1du_1.
\]

But, by (C.4) and (K.1) and using a Taylor expansion, we get,

\[
\mathbb{E}(\hat{C}_n - C_n) - C
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} q_1(x_1)m_1(x_1 + h_{1,n}v_1)K_1(v_1)dv_1dx_1 - C
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} q_1(x_1)[m_1(x_1 + h_{1,n}v_1) - m_1(x_1)]K_1(v_1)dv_1dx_1
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} q_1(x_1)\left[\frac{h_{1,n}v_1^k}{k!}m_1^{(k)}(x_1 + v_1 h_{1,n})\right]K_1(v_1)dv_1dx_1 + o(h_{1,n}^k)
\]

\[
= h_{1,n}^k \beta_2 + o(h_{1,n}^k), \quad (5.31)
\]

which allows us conclude to (5.18). \( \square \)

**Proof of (5.19).** In view of the definitions (5.1), (5.7), (5.8) and (5.12), using the boundedness of \( G \) (which is ensured by (Q.1-2)) and \( \Psi \) (which is ensured by (C.3)) along with the fact that, by (F.1), \( f \) is bounded away from 0, we have, for a given \( M_3 > 0 \),

\[
\text{Var}(\hat{C}_n) \leq \frac{M_3}{n h_{1,n}^2} \mathbb{E} \left( \int_{\mathbb{R}} K_1\left(\frac{x_1 - X_{i,1}}{h_{1,n}}\right)q_1(x_1)dx_1 \right)^2 = O\left(\frac{1}{n}\right), \quad (5.32)
\]

which naturally implies (5.19). \( \square \)

**Proof of (5.20).** Using the Cauchy-Schwartz inequality, we infer from (5.22) and (5.32) that,

\[
\text{Cov}(\hat{C}_n, \hat{\alpha}_1(x_1)) \leq \text{Var}(\hat{C}_n)^{1/2} \text{Var}(\hat{\alpha}_1(x_1))^{1/2} = O\left(n^{-1/(2k+1)}\right),
\]

which implies (5.20). \( \square \)
As already pointed out, the proof of Lemma 5.1 is readily completed by combining (5.17), (5.18), (5.19) and (5.20).

5.2 The case where \( f \) is unknown but \( G \) is known

The key idea in this case is to use the uniform consistency of \( \hat{f}_n \) (see, e.g., Ango-Nze and Rios (2000)) along with the following decomposition,

\[
\frac{1}{\hat{f}_n} = \frac{1}{\hat{f}} - \frac{\hat{f}_n - \hat{f}}{\hat{f}_n f}.
\] (5.33)

When the density \( f \) is unknown and \( G \) is known, the additive components estimates are defined as follows, for \( \ell = 1, \ldots, d \),

\[
\hat{\eta}_\ell(x_\ell) = \int_{\mathbb{R}^{d-1}} \tilde{m}_{\psi,n}(x) q_{-\ell}(x_{-\ell}) dx_{-\ell} - \int_{\mathbb{R}^d} \tilde{m}_{\psi,n}(x) q(x) dx. \tag{5.34}
\]

where

\[
\tilde{m}_{\psi,n}(x) = \sum_{i=1}^n W_{n,i}(x) \frac{\delta_i \psi(Z_i)}{G(Z_i)}. \tag{5.35}
\]

We will establish the following result.

Lemma 5.2 Under the hypotheses of Theorem 3.1, we have

\[
\sup_{x_\ell \in C_\ell} |\hat{\eta}_\ell(x_\ell) - \hat{\eta}_\ell(x_\ell)| = O \left( \sqrt{\log n} \cdot \frac{1}{nh_d \gamma_n} \right) \text{ a.s.}. \tag{5.36}
\]

Proof. First note that the term \( (nh_{1,n})^{-1} \sum_{i=1}^n |K((x - X_i)/h_{1,n})| \) is almost surely uniformly bounded on \( C \) under the assumptions we made on \( K \) and \( f \). Moreover, \( f \) and then \( f_n \) (for \( n \) large enough) are bounded away from 0 (see (F.1)). Then, in view of the definitions (2.4) and (5.1), along with the decomposition (5.33), we get, by (F.2) and (H.1), that, for a given \( C_1 > 0 \),

\[
\sup_{x \in C} \sum_{i=1}^n |W_{i,n}(x) - \bar{W}_{i,n}(x)| \leq C_1 \sup_{x \in C} |\hat{f}_n(x) - f(x)|
\]

\[
= O \left( \sqrt{\log n} \cdot \frac{1}{nh_d \gamma_n} \right) \text{ a.s.}, \tag{5.37}
\]

where we used the following result, due to Ango-Nze and Rios (2000),
sup x∈C | f̂(x) − f(x)| = O \left( \sqrt{\frac{\log n}{nh_n^d}} \right) \text{ a.s.}, \text{ under (H.1), (F.2) and (K.2)}. 

Next, under the assumptions (A), (C.2) and (C.3), we have max; ψ(Z_i)/G(Z_i) < ∞. Thus, from (5.2), (5.34) and (5.35), we conclude that, for a given C_2 > 0,

\[ \sup_{x \in C} | \hat{\eta}_\ell(x) - \hat{\eta}_\ell(x)| \leq 2 \sup_{x \in C} | \hat{m}_\psi,n(x) - \hat{m}_\psi,n(x)| \]
\[ \leq 2C_2 \sup_{x \in C} \sum_{i=1}^{n} |W_{i,n}(x) - \hat{W}_{i,n}(x)| \]
\[ = O \left( \sqrt{\frac{\log n}{nh_n^d}} \right) \text{ a.s.}, \]

which is Lemma 5.2. □

5.3 The case where both \( f \) and \( G \) are unknown

We have the following decomposition.

\[ \sup_{x \in C} | \hat{\eta}_\ell(x) - \eta(x)| \leq \sup_{x \in C} | \hat{\eta}_\ell(x) - \hat{\eta}_\ell(x)| + \sup_{x \in C} | \hat{\eta}_\ell(x) - \eta(x)| \]

Lemma 5.3 Under the assumptions of Theorem 3.1, we have

\[ \sup_{x \in C} | \hat{\eta}_\ell(x) - \hat{\eta}_\ell(x)| = o\left(n^{-k/(2k+1)}\right) \text{ a.s.}. \] (5.38)

PROOF. Observe that

\[ \sup_{x \in C} | \hat{\eta}_\ell(x) - \hat{\eta}_\ell(x)| \leq 2 \sup_{x \in C} | \hat{m}_\psi,n(x) - \hat{m}_\psi,n(x)|. \] (5.39)

First consider the case where (A)(i) holds. Under the assumptions of Theorem 3.1 we have

\[ | \hat{m}_\psi,n(x) - \hat{m}_\psi,n(x)| \]
\[ \leq M \sum_{i=1}^{n} |W_{i,n}(x)| \sup_{y \leq \tau_0} \left| \frac{1}{G(y)} \right| - \frac{1}{G_n^*(y)} \]
\[ \leq M \sum_{i=1}^{n} |W_{i,n}(x)| \sup_{y \leq \tau_0} \left| G(y) - G_n^*(y) \right| \sup_{y \leq \tau_0} \frac{1}{G_n^*(y)G(y)}, \] (5.40)
where $M$ is as in (C.3). Since $\tau_0 < T_H$, the iterated law of the logarithm of $\log n$ ensures that

$$\sup_{y \leq \tau} |G(y) - G_n^*(y)| = O\left(\sqrt{\frac{\log \log n}{n}}\right) \text{ a.s. } . \quad (5.41)$$

Besides, by the conditions imposed on $K$ and $f$, the term $\sum_{i=1}^{n} |W_{n,i}(x)|$ is almost surely uniformly bounded. Combining this last result with (5.41), it follows that

$$\sup_{x \in \mathcal{C}} |\tilde{m}_{\psi,n}^* (x) - \tilde{m}_{\psi,n} (x)| = O\left(\sqrt{\frac{\log \log n}{n}}\right) \text{ a.s. } . \quad (5.42)$$

From (5.39) and (5.42), we readily conclude to the result of Lemma 5.3 in the case where (A)(i) holds.

In the case where (A)(ii) holds, the proof follows from the same lines as above, making use of either the iterated law of the logarithm of Gu and Lai (1990) (if (A)(ii) holds with $p = 1/2$) or Theorem 2.1 of Chen and Lo (1997) (if (A)(ii) holds with $k/(2k + 1) < p < 1/2$) instead of the iterated law of the logarithm of Földes and Rejtő (1981). The details are omitted. \Box

Finally, putting the results of Lemma 5.1, Lemma 5.2 and Lemma 5.3 all together achieves the demonstration of Theorem 3.1.

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