Global properties of a Hecke ring associated with the Heisenberg Lie algebra

Fumitake Hyodo

Abstract

This study concerns (not necessarily commutative) Hecke rings associated with certain algebras and describes a formal Dirichlet series with coefficients in the Hecke rings, which can be used to generalize Shimura’s series. Considering the case of the Heisenberg Lie algebra, an analog of the identity for Shimura’s series derived employing the rationality theorem, presented by Hecke and Tamagawa, is established. Moreover, this analog recovers the explicit formulae for the isomorphic zeta function and pro-isomorphic zeta function of the Heisenberg Lie algebra shown by Grunewald, Segal and Smith.

1 Introduction

This study concerns Hecke rings introduced by Shimura [12]. A classical study of Hecke rings is the work by Hecke [5] and Tamagawa [15] on the Hecke rings associated with the general linear groups. They showed that these Hecke rings are commutative polynomial rings. Furthermore, they defined formal power series with coefficients in these Hecke rings, and showed their rationality. The results of this work are summarized in [14, Chapter 3], where formal Dirichlet series with coefficients in these Hecke rings were further introduced. Andrianov [1], Hina–Sugano [6], Satake [11], and Shimura [13] studied Hecke rings associated with classical groups, wherein they further

2020 Mathematics Subject Classification. Primary 20C08; Secondary 20G25, 20G30, 11F03, 11M41, 20E07.
developed the work of Hecke [5] and Tamagawa [15]. In addition, other studies were conducted on the Hecke rings associated with Jacobi and Chevalley groups by Dulinsky [3] and Iwahori-Matsumoto [10], respectively.

As mentioned above, various studies have been carried out on Hecke rings. However, the class of Hecke rings defined by Shimura is vast, and only a small part of it has been studied to date.

From now on, an algebra implies an abelian group with a bi-additive product (e.g., an associative algebra, a Lie algebra). Let\( L \) be an algebra that is free of finite rank as an abelian group. Our previous work [8] introduced the Hecke rings \( R_L \) and \( \hat{R}_L \) associated with \( L \). For the definition, see Section 2. In this study, we deal with the formal Dirichlet series \( D_L(s) \) and \( \hat{D}_L(s) \) with coefficients in \( R_L \) and \( \hat{R}_L \), respectively, which are defined in Section 3.

The first result of this study is to show that the Euler product formula for \( \hat{D}_L(s) \) holds, and to give a sufficient condition for \( D_L(s) \) to have the Euler product expansion (cf. Theorems 3.1 and 3.2).

If \( L = \mathbb{Z}^r \) is the ring of the direct sum of \( r \)-copies of the integer ring \( \mathbb{Z} \), the Hecke ring \( R_{\mathbb{Z}^r} \) and \( \hat{R}_{\mathbb{Z}^r} \) coincide with those treated by Hecke [5] and Tamagawa [15]. Further, the formal Dirichlet series \( D_{\mathbb{Z}^r}(s) \) and \( \hat{D}_{\mathbb{Z}^r}(s) \) equal those treated in [14, Chapter 3]. Thus, it can be said that our study generalizes their study. We discuss them in Section 4.

Denote by \( \mathcal{H} \) the Heisenberg Lie algebra, that is, the free nilpotent Lie algebra of class 2 on two generators. The second result of this study is the establishment of identities for \( D_{\mathcal{H}}(s) \) and \( \hat{D}_{\mathcal{H}}(s) \), which is the primary result of this study. Let \( \hat{\theta} = (\hat{\theta}_p) \) be a family of indeterminates indexed by all prime numbers \( p \). The key idea for stating our main theorem involves regarding \( \hat{R}_\mathcal{H} \) as a module over the polynomial ring \( \hat{R}_{\mathbb{Z}^2}[\hat{\theta}] \). The main theorem is as follows:

**Theorem 1.1 (Theorem 5.12).** There exists a formal Dirichlet series \( \hat{I}_2(\hat{\theta}; s) \) with coefficients in \( \hat{R}_{\mathbb{Z}^2}[\hat{\theta}] \) satisfying the following identity:

\[
\hat{I}_2(\hat{\theta}; s) \cdot D_{\mathcal{H}}(s) = \hat{I}_2(\hat{\theta}; s) \cdot \hat{D}_{\mathcal{H}}(s) = 1.
\]

It is worth noting that this theorem is similar to Shimura’s Theorem 4.5 for the case \( r = 2 \). At the conclusion of Section 5.2, we establish that Theorem 1.1 recovers Shimura’s Theorem for \( r = 2 \) via the endomorphism \( \hat{\phi} \) introduced in Definition 5.10.

The proof is essentially done by using some results of our previous study [7] which is described in Section 5.1. There is no great difficulty in proving
the claims stated in this study. Rather, it is important to note a natural
generalization of series of [5], [14], [15], and a concise identity given for a
formal Dirichlet series whose coefficients are not always commutative (cf.
Remark 5.14).

In [7], [8] and this study, the case of the Heisenberg Lie algebra is con-
sidered as a first step. The author expects that many new Hecke r ings will
appear in the class of the Hecke rings of this study. Further study of these
Hecke rings is now in progress by the author. In [9], the author inves-
tigated the Euler factor of \(\hat{D}_L(s)\) at each prime number in the case where \(L\) is a
higher Heisenberg Lie algebra.

Tamagawa [15], by using his Hecke theory, further investigated cer tain
zeta functions, and proved that each of them is an entire function and has
a functional equation. However, even in the Hecke rings associate d with the
Heisenberg Lie algebra, no analogue has been found. Further rese arch is
needed to find such applications to number theory.

It should be mentioned that our series \(D_L(s)\) and \(\hat{D}_L(s)\) are related to the
isomorphic zeta function \(\zeta^i_L(s)\) and the pro-isomorphic zeta function \(\zeta^\wedge_L(s)\)
of \(L\), respectively. They are two of several zeta functions associate d with
\(L\) introduced by Grunewald, Segal and Smith [4].

Let \(\hat{\mathbb{Z}}\) be the profinite completion of \(\mathbb{Z}\), and set \(\hat{L} = L \otimes \hat{\mathbb{Z}}\). Denote by
\(S^i_n(L)\) (resp. \(S^\wedge_n(L)\)) the family of subalgebras \(M\) of \(L\) (resp. \(\hat{L}\)) of index \(n\) such
that there is an isomorphism \(M \cong L\) (resp. \(\hat{L}\)) of algebras over \(\mathbb{Z}\) (resp. \(\hat{\mathbb{Z}}\)). We set \(a^i_n(L) = \#S^i_n(L)\) and \(a^\wedge_n(L) = \#S^\wedge_n(L)\) for each \(n\). The zeta functions
\(\zeta^i_L(s)\) and \(\zeta^\wedge_L(s)\) are defined as follows:

\[
\zeta^i_L(s) = \sum_{n>0} a^i_n(L) n^{-s}, \quad \zeta^\wedge_L(s) = \sum_{n>0} a^\wedge_n(L) n^{-s},
\]

where \(s\) is a complex variable.

As we mention in Section 3, \(\zeta^i_L(s)\) and \(\zeta^\wedge_L(s)\) equal the coefficient-wise
images of \(D_L(s)\) and \(\hat{D}_L(s)\) under the degree maps on \(R_L\) and \(\hat{R}_L\), respec-
tively. For the definition of the degree map on a Hecke ring, see Section
2. Grunewald et al. proved the explicit formulae for \(\zeta^i_H(s)\) and \(\zeta^\wedge_H(s)\) in [4,
Theorem 7.6] as follows:

\[
\zeta^i_H(s) = \zeta^\wedge_H(s) = \zeta(2s-2)\zeta(2s-3),
\]

where \(\zeta(s)\) is the Riemann zeta function. At the end of Section 6, we prove
that our theorem 1.1 also recovers the above formula via the degree map on
\(\hat{R}_L\).
In [4, Theorem 7.6], a generalization of the above formula was established. Precisely, if $L$ is a finitely generated free nilpotent Lie algebra of finite class, then the identity $\zeta_L(s) = \hat{\zeta}_L(s)$ and the explicit formula for them were obtained. For pro-isomorphic zeta functions, Berman, Glazer, and Schein [2] further investigated. The explicit formula for $\zeta_L(s)$ was shown in [2, Section 5], specifically for $L$ belonging to a certain class of Lie algebras over the integer rings of number fields. So far, we have not found any formulae for $D_L(s)$ and $\hat{D}_L(s)$ that recover their formulae except for this study.

2 Hecke rings associated with algebras

First, we briefly recall the definition of Hecke rings and their degree maps. For more details, refer to [14, Chapter 3]. Let $G$ be a group, $\Delta$ be a submonoid of $G$, and $\Gamma$ be a subgroup of $\Delta$. We assume that the pair $(\Gamma, \Delta)$ is a double finite pair; that is, for all $A \in \Delta$, $\Gamma \backslash \Gamma A \Gamma$ and $\Gamma A \Gamma / \Gamma$ are finite sets. Then, one can define the Hecke ring $R = R(\Gamma, \Delta)$ associated with the pair $(\Gamma, \Delta)$ as follows:

- The underlying abelian group is the free abelian group on the set $\Gamma \backslash \Delta / \Gamma$.
- For all $A, B \in \Delta$, the product of $\Gamma A \Gamma$ and $\Gamma B \Gamma$ is defined to be
  \[
  \sum_{\Gamma C \Gamma \in \Gamma \backslash \Gamma / \Gamma} \# \{ \Gamma \beta \in \Gamma \backslash \Gamma B \Gamma \mid C \beta^{-1} \in \Gamma A \Gamma \} \cdot \Gamma C \Gamma.
  \]

For every $A \in \Delta$, write $T_{\Gamma, \Delta}(A)$ for the element $\Gamma A \Gamma$ of $R$. We define the degree map on $R$ to be the additive map $\text{deg}_R : R \to \mathbb{Z}$ such that $T_{\Gamma, \Delta}(A)^{\text{deg}_R} = \# \Gamma \backslash \Gamma A \Gamma$ for every $A \in \Delta$. Notably, it is known that $\text{deg}_R$ forms a ring homomorphism.

Let $p$ be a prime number, and let $L$ be as in Section 1. We next recall the Hecke rings associated with $L$ introduced in [8]. Fix a $\mathbb{Z}$-basis of $L$, and let $r$ be the rank of $L$. Then, $\text{Aut}^{alg}_Q(L \otimes \mathbb{Q})$, $\text{Aut}^{alg}_{\mathbb{Q}_p}(L \otimes \mathbb{Q}_p)$, $\text{End}^{alg}_Z(L)$, and $\text{End}^{alg}_{Z_p}(L \otimes \mathbb{Z}_p)$ are all identified with subsets of $M_r(\mathbb{Q}_p)$. In [8, Section 2], the following notation was introduced:

\[
\begin{align*}
G_L &= \text{Aut}^{alg}_Q(L \otimes \mathbb{Q}), & G_{L_p} &= \text{Aut}^{alg}_{\mathbb{Q}_p}(L \otimes \mathbb{Q}_p), \\
\Delta_L &= \text{End}^{alg}_Z(L) \cap G_L, & \Delta_{L_p} &= \text{End}^{alg}_{Z_p}(L \otimes \mathbb{Z}_p) \cap G_{L_p}, \\
\Gamma_L &= \text{Aut}^{alg}_Z(L), & \Gamma_{L_p} &= \text{Aut}^{alg}_{Z_p}(L \otimes \mathbb{Z}_p).
\end{align*}
\]
The global Hecke rings $R_L$ and the local Hecke ring $R_{L_p}$ are the Hecke rings with respect to $(\Gamma_L, \Delta_L)$ and $(\Gamma_{L_p}, \Delta_{L_p})$, respectively.

The other global Hecke ring $\hat{R}_L$ was introduced in [8, Section 3]. Define the group $\hat{G}_L$ to be the restricted direct product of $G_{L_p}$ relative to $\Gamma_{L_p}$ for all prime numbers $p$, that is, the set of elements $(\alpha_p)_p$ of $\prod_p G_{L_p}$ such that $\alpha_p \in \Gamma_{L_p}$ for almost all $p$. The monoid $\hat{\Delta}_L$ and the group $\hat{\Gamma}_L$ denote $\hat{G}_L \cap \prod_p \Delta_{L_p}$ and $\prod_p \Gamma_{L_p}$, respectively. Then, we write $\hat{R}_L$ for the Hecke ring with respect to $(\hat{\Gamma}_L, \hat{\Delta}_L)$.

Section 3 of [8] described relations among these Hecke rings. The local Hecke ring $R_{L_p}$ is related to the global Hecke ring $\hat{R}_L$ as follows:

**Proposition 2.1 ([8 Proposition 3.1]).** The following assertions hold:

1. The local Hecke ring $R_{L_p}$ is regarded as a subring of $\hat{R}_L$ by the map induced by the natural inclusion of $\Delta_{L_p}$ into $\hat{\Delta}_L$.
2. For each prime number $q$ with $p \neq q$, the local Hecke rings $R_{L_p}$ and $R_{L_q}$ commute with each other in $\hat{R}_L$.
3. $\hat{R}_L$ is generated by the family of local Hecke rings $\{R_{L_p}\}_p$ as a ring.

For simplicity, we set

$$T_{L_p} = T_{\Gamma_{L_p}, \Delta_{L_p}}, \quad T_L = T_{\Gamma_L, \Delta_L}, \quad \hat{T}_L = T_{\hat{\Gamma}_L, \hat{\Delta}_L}.$$

Then, we have $T_{L_p}(\alpha) = \hat{T}_L(\alpha)$ in $\hat{R}_L$ for each $\alpha \in \Delta_{L_p}$. Let us relate the global Hecke rings $\hat{R}_L$ and $R_L$. The map $\eta_L$ denotes the diagonal embedding of $\Delta_L$ into $\prod_p \Delta_{L_p}$. Then, we define the additive map $\eta_L^\star : \hat{R}_L \to R_L$ being $\hat{T}_L(\alpha) \mapsto \sum_\beta T_L(\beta)$, where $\beta$ runs through a complete system of representatives of $\Gamma_L \backslash \eta_L^{-1}(\hat{T}_L(\alpha)\Gamma_L) / \Gamma_L$. Let us denote by $\eta_{L^*} : \Gamma_L \backslash \Delta_L \to \hat{\Gamma}_L \backslash \hat{\Delta}_L$ the map induced by $\eta_L$. Then, the two global Hecke rings are related as follows:

**Lemma 2.2 ([8 Lemma 3.2]).** If the map $\eta_{L^*}$ is bijective, then $\eta_L^\star$ is multiplicative and injective.

From now on, we regard $\hat{R}_L$ as a subring of $R_L$ if $\eta_{L^*}$ is bijective.

In the rest of this section, we relate $\hat{R}_L$ to the automorphism group of $\hat{L}$. 

---

F. HYODO  GLOBAL PROPERTIES OF A HECKE RING

---
Proposition 2.3. Let \( \mathfrak{A}_f \) be the ring of finite adeles over \( \mathbb{Q} \), and set \( \mathbb{Q} = \prod_p \mathbb{Q}_p \). Then, the objects \( \hat{G}_L \), \( \hat{\Delta}_L \), and \( \hat{\Gamma}_L \) satisfy the following identities as subsets of \( M_r(\mathbb{Q}) \):

\[
\hat{G}_L = \text{Aut}^\text{alg}_{\mathfrak{A}_f}(L \otimes \mathfrak{A}_f)(L' \otimes \mathfrak{A}_f) \cap \hat{\Delta}_L, \quad \hat{\Delta}_L = \text{End}^\text{alg}_{\mathbb{Z}}(\hat{L}) \cap \hat{G}_L, \quad \hat{\Gamma}_L = \text{Aut}^\text{alg}_{\mathbb{Z}}(\hat{L}).
\]

Proof. The second and third identities are straightforward consequences of the fact that \( \hat{L} \) equals \( \prod_p (L \otimes \mathbb{Z}_p) \). Let us prove the first identity. Denote by \( GL'_r(\mathbb{Q}) \) the restricted direct product of \( GL_r(\mathbb{Q}_p) \) relative to \( GL_r(\mathbb{Z}_p) \) for all prime numbers \( p \). Then, it is easy to see that \( GL'_r(\mathbb{Q}) \) coincides with \( GL_r(\mathfrak{A}_f) \). And the group \( \hat{G}_L \), by definition, equals the intersection of \( GL'_r(\mathbb{Q}) \) and \( \prod_p G_{L_p} \). Thus, we have

\[
\hat{G}_L = GL_r(\mathfrak{A}_f) \cap \prod_p G_{L_p}.
\]

Since \( L \otimes \mathbb{Q} \) is identified with \( \prod_p (L \otimes \mathbb{Q}_p) \), the group \( \prod_p G_{L_p} \) coincides with \( \text{Aut}^\text{alg}_\mathbb{Q}(L \otimes \mathbb{Q}) \). Hence, we have

\[
GL_r(\mathfrak{A}_f) \cap \prod_p G_{L_p} = GL_r(\mathfrak{A}_f) \cap \text{Aut}^\text{alg}_\mathbb{Q}(L \otimes \mathbb{Q}) = \text{Aut}^\text{alg}_{\mathfrak{A}_f}(L \otimes \mathfrak{A}_f).
\]

This implies the first identity. \( \square \)

3 Formal power series and formal Dirichlet series associated with algebras

Let \( p \) and \( L \) be as in the previous section. We set \( L_p = L \otimes \mathbb{Z}_p \). In this section, the formal series \( P_{L_p}(X) \), \( D_L(s) \), and \( \tilde{D}_L(s) \) are defined. Subsequently, their relationship is described. For a positive integer \( n \) and a nonnegative integer \( k \), we introduce the following notation:

\[
\mathcal{A}_L(n) = \left\{ \alpha \in \hat{\Delta}_L \mid [\hat{L} : \hat{L}^\alpha] = n \right\}, \quad \mathcal{A}_L(n) = \left\{ \alpha \in \Delta_L \mid [L : L^\alpha] = n \right\},
\]

\[
\mathcal{A}_{L_p}(p^k) = \left\{ \alpha \in \Delta_{L_p} \mid [L_p : L_p^\alpha] = p^k \right\},
\]

where \( L_p^\alpha \) is the image of \( L_p \) under the endomorphism \( \alpha \). Additionally, \( \hat{L}^\alpha \) and \( L^\alpha \) are defined in a similar manner. Note that each element of \( \hat{\Delta}_L \) is regarded as an element of \( \text{End}^\text{alg}_\mathbb{Z}(\hat{L}) \) by Proposition 2.3.
Now, the formal power series \( P_{L_p}(X) \) is introduced. We define
\[
T_{L_p}(p^k) = \sum_{\alpha} T_{L_p}(\alpha),
\]
where \( \alpha \) runs through a complete system of representatives of \( \Gamma_{L_p} \setminus A_{L_p}(p^k)/\Gamma_{L_p} \).

The formal power series \( P_{L_p}(X) \) is defined as the generating function of the sequence \( \{T_{L_p}(p^k)\}_k \); that is,
\[
P_{L_p}(X) = \sum_{k \geq 0} T_{L_p}(p^k)X^k.
\]

Next, the formal Dirichlet series \( \hat{D}_L(s) \) and \( D_L(s) \) are defined. We set
\[
\hat{T}_L(n) = \sum_{\hat{\alpha}} \hat{T}_L(\hat{\alpha}), \quad T_L(n) = \sum_{\alpha} T_L(\alpha),
\]
where \( \hat{\alpha} \) (resp. \( \alpha \)) runs through a complete system of representatives of \( \hat{\Gamma}_L \setminus \hat{A}_L(n)/\hat{\Gamma}_L \) (resp. \( \Gamma_L \setminus A_L(n)/\Gamma_L \)). The formal Dirichlet series \( \hat{D}_L(s) \) and \( D_L(s) \) are the generating functions of the sequences of \( \{\hat{T}_L(n)\}_n \) and \( \{T_L(n)\}_n \), respectively; that is,
\[
\hat{D}_L(s) = \sum_{n > 0} \hat{T}_L(n)n^{-s}, \quad D_L(s) = \sum_{n > 0} T_L(n)n^{-s}.
\]

Next, \( P_{L_p}(X) \) is related to \( \hat{D}_L(s) \). For each element \( \hat{\alpha} \) of \( \hat{\Delta}_L \), let \( \alpha_p \) denote its \( \Delta_{L_p} \) component. Then, \( \hat{T}_L(\hat{\alpha}) = \prod_p T_{L_p}(\alpha_p) \) is obtained, where \( p \) runs over all prime numbers. Here, this infinite product is meaningful since its terms commute with each other according to Proposition 2.1 and almost all of them are equal to 1. Consequently, the following theorem is proven:

**Theorem 3.1.** The sequence \( \{\hat{T}_L(n)\}_n \) is multiplicative, and the Euler product formula for \( \hat{D}_L(s) \) holds; that is,
\[
\hat{D}_L(s) = \prod_p P_{L_p}(p^{-s}),
\]
where \( p \) runs through all prime numbers.
Proof. It is easy to see that $T_L(p^k) = \hat{T}_L(p^k)$ in $\hat{R}_L$. Since $\hat{L}$ is isomorphic to $\prod_p L_p$, it follows that $[\hat{L} : \hat{L}_p] = \prod_p [L_p : L_p^\alpha_p]$ for each $\alpha \in \Delta_L$. Hence, we have

$$\hat{A}_L(n) = \prod_p A_{L_p}(p^{v_p(n)}),$$

where $v_p$ is the $p$-adic valuation. This proves the theorem. 

Finally, $\hat{D}_L(s)$ is related to $D_L(s)$ using the additive map $\eta_L^*: \hat{R}_L \to R_L$. It is evident that $\eta_L^*$ maps $\hat{T}_L(n)$ to $T_L(n)$ for each positive integer $n$. Thus, the Euler product formula for $D_L(s)$ is proven.

**Theorem 3.2.** If the map $\eta_{rL}$ is bijective, then the sequence $\{T_L(n)\}_n$ is multiplicative, and the Euler product formula for $D_L(s)$ holds; that is,

$$D_L(s) = \prod_p P_{L_p}(p^{-s}).$$

Proof. By assumption, $\hat{R}_L$ is considered as a subring of $R_L$. Since $T_L(n) = \hat{T}_L(n)$ and $D_L(s) = \hat{D}_L(s)$, Theorem 3.1 implies the desired result. 

**4 Case of the ring $\mathbb{Z}^r$**

Using the notations in Section 3, the theory of the Hecke ring with general linear groups as reported by Hecke [5], Shimura [14], and Tamagawa [15] is considered.

Let $r$ be a positive integer. Clearly, $G_{\mathbb{Z}^r}$ and $G_{\mathbb{Z}_p^r}$ are identified with $GL_r(\mathbb{Q})$ and $GL_r(\mathbb{Q}_p)$, respectively. Similarly, we have $\Delta_{\mathbb{Z}^r} = M_r(\mathbb{Z}) \cap GL_r(\mathbb{Q})$, $\Delta_{\mathbb{Z}_p^r} = M_r(\mathbb{Z}_p) \cap GL_r(\mathbb{Q}_p)$, $\Gamma_{\mathbb{Z}^r} = GL_r(\mathbb{Z})$, and, $\Gamma_{\mathbb{Z}_p^r} = GL_r(\mathbb{Z}_p)$. Thus, the Hecke rings $R_{\mathbb{Z}^r}$ and $R_{\mathbb{Z}_p^r}$ coincide with the Hecke rings treated in [5], [14], and [15]. Furthermore, the Hecke ring $\hat{R}_{\mathbb{Z}^r}$ is identified with $R_{\mathbb{Z}^r}$ as follows:

**Proposition 4.1.** The map $\eta_{\mathbb{Z}^r}^*: \hat{R}_{\mathbb{Z}^r} \to R_{\mathbb{Z}^r}$ is an isomorphism.

Proof. Lemma 3.3 of [8] implies that $\eta_{\mathbb{Z}^r}^*$ is an injective homomorphism. Moreover, the map $\Gamma_{\mathbb{Z}^r}/\Delta_{\mathbb{Z}^r}/\Gamma_{\mathbb{Z}^r} \to \hat{\Gamma}_{\mathbb{Z}^r}/\hat{\Delta}_{\mathbb{Z}^r}/\hat{\Gamma}_{\mathbb{Z}^r}$ induced by $\eta_{\mathbb{Z}^r}$, is bijective according to the elementary divisor theorem. 


Certainly, the formal power series \( \mathcal{P}_{Z_p}(X) \) equals the local Hecke series treated in [5] and [15]. The following theorem was proved:

**Theorem 4.2** ([5 Satz 14], [15 Theorem 3]). Let

\[
T_{r,p}^{(i)} = \Gamma_{Z_p} \text{diag}[1, \ldots, 1, p, \ldots, p] \Gamma_{Z_p}
\]

for each \( i \) with \( 1 \leq i \leq r \). Then, the following assertions hold:

1. \( \mathcal{R}_{Z_p} \) is the polynomial ring over \( \mathbb{Z} \) in variables \( T_{r,p}^{(i)} \) with \( 1 \leq i \leq r \).
2. The series \( \mathcal{P}_{Z_p}(X) \) is a rational function over \( \mathcal{R}_{Z_p} \), more precisely,

\[
f_{r,p}(X) \mathcal{P}_{Z_p}(X) = 1,
\]

where \( f_{r,p}(X) = \sum_{i=0}^{r} (-1)^i p^{i(i-1)/2} T_{r,p}^{(i)} X^i \). Particularly,

\[
f_{2,p}(X) = 1 - T_{2,p}^{(1)} X + p T_{2,p}^{(2)} X^2.
\]

**Remark 4.3.** Theorem 4.2 in the case \( r = 2 \) was proved in [5]. For arbitrary \( r \), it was demonstrated in [15].

The series \( \mathcal{D}_{Z_r}(s) \) is none other than the formal Dirichlet series treated in [14, Chapter 3]. Since \( \eta_{Z_r} \) is bijective, the following theorem is obtained:

**Theorem 4.4.** The Euler product formulae for \( \mathcal{D}_{Z_r}(s) \) and \( \hat{\mathcal{D}}_{Z_r}(s) \) hold; i.e.,

\[
\mathcal{D}_{Z_r}(s) = \hat{\mathcal{D}}_{Z_r}(s) = \prod_{p} \mathcal{P}_{Z_p}(p^{-s}),
\]

where \( p \) runs through all prime numbers.

**Proof.** Obvious. \( \blacksquare \)

Therefore, Theorem 4.2 can be used to derive the following theorem:

**Theorem 4.5** ([14 Theorem 3.21]). Define \( I_r(s) \) to be the infinite product \( \prod_{p} f_{r,p}(p^{-s}) \). Then, the following is obtained:

\[
I_r(s) \mathcal{D}_{Z_r}(s) = I_r(s) \hat{\mathcal{D}}_{Z_r}(s) = 1.
\]

**Proof.** Obvious. \( \blacksquare \)
5 Case of the Heisenberg Lie algebra

This section studies the proposed series in the case of the Heisenberg Lie algebra $H$.

5.1 Local properties

Let us recall the main theorem of [7]. For an element $A$ of $G_{\mathbb{Z}_2}$ and an element $a$ of $\mathbb{Q}_2$, denote by $(A, a)$ the element \( \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix} \) of $GL_3(\mathbb{Q}_p)$, where $|A|$ means the determinant of the matrix $A$. Fix a system $\{x_1, x_2\}$ of free generators of $H$. Then, the set $\{x_1, x_2, [x_1, x_2]\}$ forms a basis of $H$. Hence, the group $G_{H_p}$ is identified with the following subset of $GL_3(\mathbb{Q}_p)$:

\[
\left\{ (A, a) \mid A \in G_{\mathbb{Z}_2}, \ a \in \mathbb{Q}_2^2 \right\}.
\]

In addition, an element $(A, a)$ of $G_{H_p}$ is contained in $\Delta_{H_p}$ (resp. $\Gamma_{H_p}$) if and only if $A$ is in $\Delta_{\mathbb{Z}_2}$ (resp. $\Gamma_{\mathbb{Z}_2}$), and $a$ is in $\mathbb{Z}_2$.

The following three ring homomorphisms $s$, $\phi$, and $\theta$ were introduced in [7, Section 6]:

**Definition 5.1.** For simplicity, we put $\deg = \deg R_{H_p}$. The ring homomorphisms $s : R_{\mathbb{Z}_2} \to R_{H_p}$, $\phi : R_{H_p} \to R_{\mathbb{Z}_2}$, and $\theta : R_{H_p} \to R_{H_p}$ are defined by

\[
T_{\mathbb{Z}_2}(A)^s = T_{H_p}(A, 0) \text{ for each } A \in \Delta_{\mathbb{Z}_2},
\]

\[
T_{H_p}(A, a)^\phi = \frac{T_{H_p}(A, a)^{\deg}}{T_{H_p}(A, 0)^{\deg}} T_{\mathbb{Z}_2}(A) \text{ for each } (A, a) \in \Delta_{H_p},
\]

\[
T_{H_p}(A, a)^\theta = \frac{T_{H_p}(A, a)^{\deg}}{T_{H_p}(A, pa)^{\deg}} T_{H_p}(A, pa) \text{ for each } (A, a) \in \Delta_{H_p}.
\]

**Remark 5.2.** Although the multiplicativity of $s$, $\phi$, and $\theta$ is not obvious by the definition, it was proved in [7, Section 6].

Some relations among the three ring homomorphisms are introduced.

**Proposition 5.3.** The ring homomorphisms $s$, $\phi$, and $\theta$ satisfy the following properties:

\[
\phi \circ s = \text{id}_{R_{\mathbb{Z}_2}}, \quad \theta \circ s = s, \quad \phi \circ \theta = \phi.
\]
Proof. It is an easy consequence of Definition 5.1.

Our previous work [7] defined the element \(T_2(p^k)\) of \(R_\mathcal{H}_p\) for each nonnegative integer \(k\) as follows:

\[
T_2(p^k) = \sum_{(A,a)} T_{\mathcal{H}_p}(A,a),
\]

where \((A,a)\) runs through a complete system of representatives of \(\Gamma_{\mathcal{H}_p} \backslash \Delta_{\mathcal{H}_p} / \Gamma_{\mathcal{H}_p}\) satisfying \(v_p(|A|) = k\). The formal power series \(D_{2,2}(X)\) was defined as the generating function of the sequence \(\{T_2(p^k)\}_k\); that is,

\[
D_{2,2}(X) = \sum_{k \geq 0} T_2(p^k)X^k.
\]

The main theorem of our previous work [7] is as follows:

**Theorem 5.4** ([7, Theorem 7.8]). Let \(T_{2,1}(p)\) and \(T_{2,2}(p)\) be as in Theorem 4.2. For simplicity, let us set \(T_{1,1}(p) = T_{2,1}(p)\) and \(T_{1,2}(p) = T_{2,2}(p)\). Define \(Y = pX\). Then, \(D_{2,2}(X)\) satisfies the following identity:

\[
D_{2,2}(X)^\theta - T_{1,1}(p)^sD_{2,2}(X)^\theta Y + pT_{1,2}(p)^sD_{2,2}(X)Y^2 = 1,
\]

where \(D_{2,2}(X)^\theta\) is the coefficient-wise image of \(D_{2,2}(X)\) under \(\theta\), and \(D_{2,2}(X)^\theta Y\) is defined similarly.

The sequences \(\{T_2(p^k)\}_{k \geq 0}\) and \(\{T_{\mathcal{H}_p}(p^k)\}_{k \geq 0}\) are related as follows:

**Proposition 5.5.** \(T_{\mathcal{H}_p}(p^{2k}) = T_2(p^k)\) and \(T_{\mathcal{H}_p}(p^{2k+1}) = 0\) for each \(k\).

Proof. It is evident that \(v_p([\mathcal{H}_p : \mathcal{H}_p(A,a)]) = 2v_p(|A|)\) for every \((A,a)\in\Delta_{\mathcal{H}_p}\). This completes the proof.

The relation between \(D_{2,2}(X)\) and \(P_{\mathcal{H}_p}(X)\) is described as follows:

**Corollary 5.6.** \(D_{2,2}(X^2) = P_{\mathcal{H}_p}(X)\).

Proof. It is an immediate consequence of the proposition above.
The Hecke ring $R_{\mathcal{H}_p}$ forms a ring over $\mathbb{Z}_p$ via the ring homomorphism $s$. Moreover, owing to the second identity of Proposition 5.3, $\theta$ is a ring homomorphism over $\mathbb{Z}_p$. Thus, $R_{\mathcal{H}_p}$ is a module (not a ring!) over the polynomial ring $\mathbb{Z}_p[\theta]$ in one variable $\theta$. Further, the maps $s$, $\phi$, and $\theta$ depend on $p$. Subsequently, we set $s_p = s$, $\phi_p = \phi$, and $\theta_p = \theta$. Therefore, Theorem 5.4 can be rewritten as follows:

**Theorem 5.7.** Let $f_{2,p}(X)$ be as in Theorem 4.2 and let us keep the notation of Theorem 5.4. Then, $P_{\mathcal{H}_p}(X)$ satisfies the following identity:

$$g_{2,p}(\theta_p; pX^2)P_{\mathcal{H}_p}(X) = 1,$$

where

$$g_{2,p}(\theta_p; X) = \theta_p^2 \cdot f_{2,p}(X/\theta_p) = \theta_p^2 - T_p(1, p)\theta_pX + pT_p(p, p)X^2.$$

**Proof.** Clear.

We have just introduced the three ring homomorphism, of which $\phi_p$ has not been used so far. In fact, it has been shown that $\phi_p$ plays a role establishing the relationship between $P_{\mathcal{H}_p}(X)$ and $P_{\mathbb{Z}_p}(X)$ as follows:

**Theorem 5.8** ([7, Theorem 7.5]), $P_{\mathcal{H}_p}(X)^{\phi_p} = P_{\mathbb{Z}_p}(pX^2)$.

### 5.2 Global properties

In this subsection, the Dirichlet series $D_{\mathcal{H}}(s)$ and $\hat{D}_{\mathcal{H}}(s)$ are considered. Since the bijectivity of $\eta_*\mathcal{H}$ was proved in [8, Lemma 3.4], the map $\eta_*\mathcal{H}$ is an injective ring homomorphism. Moreover, the nonsurjectivity of $\eta_*\mathcal{H}$ was shown in [8, Section 4]. Hence, the global Hecke ring $\hat{\mathcal{H}}$ is a proper subring of $R_{\mathcal{H}}$. However, by Theorems 3.1 and 3.2, the following theorem can be obtained:

**Theorem 5.9.** The Euler product formulae for $D_{\mathcal{H}}(s)$ and $\hat{D}_{\mathcal{H}}(s)$ hold; that is,

$$D_{\mathcal{H}}(s) = \hat{D}_{\mathcal{H}}(s) = \prod_p P_{\mathcal{H}_p}(p^{-s}).$$

**Proof.** Obvious.

The ring homomorphisms $\hat{s}$, $\hat{\phi}$, and $\hat{\theta}_p$ are defined as follows:
Definition 5.10. The ring homomorphisms $\hat{s} : \hat{R}_{\mathbb{Z}^2} \to \hat{R}_H$ and $\hat{\phi} : \hat{R}_H \to \hat{R}_{\mathbb{Z}^2}$ are defined by

$$\hat{T}_{\mathbb{Z}^2}(\hat{A})^{\hat{s}} = \prod_p T_{\mathbb{Z}^2}(A_p)^{s_p} \text{ for each } \hat{A} \in \hat{\Delta}_{\mathbb{Z}^2},$$

$$\hat{T}_H(\hat{\alpha})^{\hat{\phi}} = \prod_p T_{H_p}(\alpha_p)^{\phi_p} \text{ for each } \hat{\alpha} \in \hat{\Delta}_H,$$

where $A_p$ (resp. $\alpha_p$) is $\Delta_{\mathbb{Z}^2}$ (resp. $\Delta_{H_p}$) component of $\hat{A}$ (resp. $\hat{\alpha}$) for each $p$.

The ring homomorphism $\hat{\theta}_p : \hat{R}_H \to \hat{R}_H$ is defined by

$$\hat{T}_H(\hat{\alpha})^{\hat{\theta}_p} = T_{H_p}(\alpha_p)^{\theta_p} \cdot \prod_{q \neq p} T_{H_q}(\alpha_q) \text{ for each } \hat{\alpha} \in \hat{\Delta}_H.$$

Consequently, the following proposition can be obtained according to Proposition 5.3:

Proposition 5.11. The following equalities hold:

1. $\hat{\phi} \circ \hat{s} = id_{\hat{R}_{\mathbb{Z}^2}}$,
2. $\hat{\theta}_p \circ \hat{s} = \hat{s}$ and $\hat{\phi} \circ \hat{\theta}_p = \hat{\phi}$ for each $p$,
3. $\hat{\theta}_p \circ \hat{\theta}_q = \hat{\theta}_q \circ \hat{\theta}_p$ for any two prime numbers $p, q$.

Proof. Obvious. 

From the proposition above, it is evident that the Hecke ring $\hat{R}_H$ is a ring over $\hat{R}_{\mathbb{Z}^2}$ by $\hat{s}$, and that $\hat{\theta}_p$ is a ring homomorphism over $\hat{R}_{\mathbb{Z}^2}$ for each $p$. Set $\hat{\theta} = (\hat{\theta}_p)_p$, and let $\hat{R}_{\mathbb{Z}^2}[\hat{\theta}]$ be the polynomial ring over $\hat{R}_{\mathbb{Z}^2}$ in infinitely many variables $\hat{\theta}$. Then, the Hecke ring $\hat{R}_H$ is an $\hat{R}_{\mathbb{Z}^2}[\hat{\theta}]$-module.

Now, the following theorem is proven, analogous to Theorem 4.5:

Theorem 5.12. $\hat{I}_2(\hat{\theta}; s)$ is defined as the infinite product

$$\prod_p g_{2,p}(\hat{\theta}_p, p^{1-2s}).$$

Then, the following is obtained:

$$\hat{I}_2(\hat{\theta}; s) \cdot D_H(s) = \hat{I}_2(\hat{\theta}; s) \cdot \hat{D}_H(s) = 1.$$
Proof. Theorem 5.9 implies that
\[ \hat{D}_H(s) = \prod_p P_{H_p}(p^{-s}). \]

Let us fix a prime number \( p \). Then, the following is obtained:
\[ \hat{D}_H(s) = \hat{\theta}_p = \prod_{q \neq p} P_{H_q}(q^{-s}). \]

In addition, \( R_{H_p} \) and \( R_{H_q} \) commute with each other in \( \hat{R}_H \) for any prime number \( q \) different from \( p \). Hence, for each element \( a_p \) of \( R_{Z_2^p} \), we have
\[ a_p \cdot (\hat{D}_H(s)) = (a_p \cdot P_{H_p}(p^{-s})) \cdot \prod_{q \neq p} P_{H_q}(q^{-s}). \]

Therefore,
\[ \hat{I}_2(\hat{\theta}; s) \cdot \hat{D}_H(s) = \prod_p (g_{2,p}(\theta_p; p^{1-2s}) \cdot P_{H_p}(p^{-s})). \]

Subsequently, Theorem 5.7 implies that the right-hand side of the equality above is 1, which completes the proof.

In the remainder of this section, it is shown that Theorem 5.12 recovers Shimura’s Theorem 4.5. Let \( \hat{\psi} : \hat{R}_{Z_2}[\hat{\theta}] \to \hat{R}_{Z_2} \) be the ring homomorphism over \( \hat{R}_{Z_2} \) satisfying \( (\hat{\theta}_p)^{\hat{\psi}} = 1 \) for all \( p \). Then \( \hat{\phi} \) and \( \hat{\psi} \) are compatible; that is,
\[ (a \cdot \mathfrak{A})^{\hat{\phi}} = a^{\hat{\psi}} \cdot \mathfrak{A}^{\hat{\phi}} \quad \text{for any} \quad a \in \hat{R}_{Z_2}[\hat{\theta}] \quad \text{and any} \quad \mathfrak{A} \in \hat{R}_H, \]
which follows from Proposition 5.11. Consequently, the following proposition is proven:

**Proposition 5.13.** The following identities hold:

1. \( \hat{I}_2(\hat{\theta}; s)^{\hat{\psi}} = I_2(2s - 1) \),
2. \( \hat{D}_H(s)^{\hat{\phi}} = \hat{D}_{Z_2}(2s - 1) \).

Proof. It is evident that \( g_{2,p}(\theta_p; pX^2)^{\hat{\psi}} = g_{2,p}(1; pX^2) = f_{2,p}(pX^2) \), which implies the first equality. The second one follows from Theorems 5.8, 5.9 and 4.4.
Therefore, it is concluded that Theorem 5.12 recovers Theorem 4.5 when \( r = 2 \): The map \( \hat{\varphi} \) is applied to the equality in Theorem 5.12. Then, Proposition 5.13 and the compatibility of \( \hat{\varphi} \) and \( \hat{\psi} \) imply that

\[
I_2(2s - 1)\hat{D}_2(2s - 1) = 1.
\]

**Remark 5.14.** As shown in Theorem 7.3 of [7], the coefficients of the Hecke series \( D_2(X) \) are not necessarily commutative. Thus, neither are those of \( P_{H_\nu}(X) \) and \( D_{H_\nu}(s) \).

### 6 Zeta functions of algebras

Let us return to the case where \( L \) is as in Section 3. In this section, our series \( D_L(s) \) and \( \hat{D}_L(s) \) are related to the isomorphic zeta function \( \zeta_L^i(s) \) and pro-isomorphic zeta function \( \zeta_L^\wedge(s) \) of \( L \), respectively.

The maps \( \Gamma_L \backslash A_L(n) \rightarrow S_n^i(L) \) being \( \alpha \mapsto L^\alpha \) and \( \hat{\Gamma}_L \backslash \hat{A}_L(n) \rightarrow S_n^\wedge(L) \) being \( \hat{\alpha} \mapsto \hat{L}^\hat{\alpha} \) are both bijective. Hence, for each \( n \), we have

\[
T_L(n)^{\deg R_L} = a_n^i(L), \quad \hat{T}_L(n)^{\deg \hat{R}_L} = a_n^\wedge(L).
\]

Thus, \( D_L(s) \) and \( \hat{D}_L(s) \) are related to \( \zeta_L^i(s) \) and \( \zeta_L^\wedge(s) \) as follows:

\[
D_L(s)^{\deg R_L} = \zeta_L^i(s), \quad \hat{D}_L(s)^{\deg \hat{R}_L} = \zeta_L^\wedge(s).
\]

By definition, we have \( \deg \hat{R}_L|_{R_{L\nu}} = \deg R_{L\nu} \). Moreover, \( \deg \hat{R}_L \) and \( \deg R_L \) are related as follows:

**Proposition 6.1.** If the map \( \eta_{L\nu} \) is bijective, then we have \( \deg \hat{R}_L = \deg R_L \circ \eta_L \), that is, \( \deg R_L|_{\hat{R}_L} = \deg \hat{R}_L \).

**Proof.** Since \( \eta_{L\nu} \) is bijective, so is

\[
\eta_{L\nu}|_{\Gamma_L \backslash \eta^{-1}_L(\hat{\Gamma}_L)^\wedge} : \Gamma_L \backslash \eta^{-1}_L(\hat{\Gamma}_L)^\wedge \rightarrow \hat{\Gamma}_L \backslash \hat{\Gamma}_L\hat{\alpha}\hat{\Gamma}_L.
\]

Thus, we have \( \# \hat{\Gamma}_L \backslash \hat{\Gamma}_L\hat{\alpha}\hat{\Gamma}_L = \# \Gamma_L \backslash \eta^{-1}_L(\hat{\Gamma}_L)^\wedge \), which completes the proof. \( \square \)

The above proposition implies the following identitiy:
Corollary 6.2. If the map \( \eta_* \) is bijective, then \( \zeta^\epsilon_L(s) = \zeta^i_L(s) \).

Proof. Obvious. \( \square \)

Next, the case \( L = \mathbb{Z}^r \) is considered. We make use of the following identity shown by Tamagawa [15]:

**Theorem 6.3 ([15, Corollary]).** The following identity holds:

\[
f_{r,p}(X)^{\deg \hat{R} Z^r} = \prod_{0 \leq k \leq r-1} (1 - p^k X).
\]

This theorem derives the following identity:

**Corollary 6.4.** The following identity holds:

\[
I_2(s)^{\deg \hat{R} Z^r} = \prod_{0 \leq k \leq r-1} \zeta(s - k)^{-1},
\]

where \( \zeta(s) \) is the Riemann zeta function.

Proof. Trivial. \( \square \)

Theorem 4.5 and Corollary 6.4 recover the explicit formulae for \( \zeta^\epsilon_{Z^r}(s) \) and \( \zeta^i_{Z^r}(s) \) proved in [4].

**Corollary 6.5 ([4, Proposition 1.1]).** The following identity holds:

\[
\zeta^i_{Z^r}(s) = \zeta^\epsilon_{Z^r}(s) = \prod_{0 \leq k \leq r-1} \zeta(s - k).
\]

Proof. Clear. \( \square \)

Next, the case \( L = \mathcal{H} \) is investigated. Let \( \psi : \mathbb{Z}[\hat{\theta}] \to \mathbb{Z} \) be the ring homomorphism satisfying \( (\hat{\theta}_p)^\psi = 1 \) for all \( p \). Then, the coefficient-wise images of \( g_{2,p}(\hat{\theta}_p; X) \) and \( \hat{I}_2(\hat{\theta}; s) \) under the homomorphism \( (\deg \hat{R}_H \circ \hat{s}) \otimes \psi : \hat{R}_{\mathcal{H}}[\hat{\theta}] \to \mathbb{Z} \) are as follows:

**Proposition 6.6.** The following identities hold:

\[
g_{2,p}(\hat{\theta}_p; X)^{(\deg \hat{R}_H \circ \hat{s}) \otimes \psi} = (1 - p^2 X^2)(1 - p^3 X^2),
\]

\[
\hat{I}_2(\hat{\theta}; s)^{(\deg \hat{R}_H \circ \hat{s}) \otimes \psi} = \zeta(2s - 2)^{-1} \zeta(2s - 3)^{-1}.
\]
Proof. By the identities (1)-(a) and (2)-(a) of \[7, Proposition 5.4\], we have
\[ T_p(1,p)^\deg \hat{R}_H \circ \hat{s} = p^2(1 + p^{-1}), \quad T_p(p,p)^\deg \hat{R}_H \circ \hat{s} = p^2, \]
which implies the first identity. The second is easily derived by the first one. \qed

Remark 6.7. \( \deg \hat{R}_H \circ \hat{s} \) and \( \deg \hat{R}_{\mathbb{Z}^2} \) are slightly different: For each \( A \in \hat{\Delta}_{\mathbb{Z}^2} \), it follows from \[7, Proposition 5.4\] and \[14, Theorem 3.24\] that
\[ \hat{T}_{\mathbb{Z}^2}(\hat{A})^{\deg \hat{R}_H \circ \hat{s}} = [\hat{\mathbb{Z}}^2 : (\hat{\mathbb{Z}}^2)^A] \cdot \hat{T}_{\mathbb{Z}^2}(\hat{A})^{\deg \hat{R}_{\mathbb{Z}^2}}. \]
Since \( \deg \hat{R}_H \circ \hat{\theta}_p = \deg \hat{R}_H \), the ring homomorphism \( (\deg \hat{R}_H \circ \hat{s}) \otimes \psi \) and \( \deg \hat{R}_H \) are compatible. Therefore, the explicit formulae for \( \zeta_H^i(s) \) and \( \zeta_H^\wedge(s) \) shown in \[4\] are recovered.

Corollary 6.8 \([4, Theorem 7.6]\). The following identity holds:
\[ \zeta_H^i(s) = \zeta_H^\wedge(s) = \zeta(2s - 2)\zeta(2s - 3). \]

Proof. It is an easy consequence of Proposition 6.6 and Theorem 5.12. \qed

References

[1] A. N. Andrianov: Rationality theorems for Hecke series and zeta functions of the groups \( GL_n \) and \( SP_n \) over local fields, Izv. Math. 3(3) (1969), 439–476.

[2] M. N. Berman, I. Glazer, and M. M. Schein: Pro-isomorphic zeta functions of nilpotent groups and Lie rings under base extension, Trans. Am. Math. Soc. 375 (2022), 1051–1100.

[3] J. Dulinski: Jacobi-Hecke algebras and a rationality theorem for a formal Hecke series, Manuscripta Math. 99(2) (1999), 255–285.

[4] F. J. Grunewald, D. Segal, and G. C. Smith: Subgroups of finite index in nilpotent groups, Invent. Math. 93 (1988), 185–223.

[5] E. Hecke: Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung, I, II, Math. Ann. 114(1) (1937), 1–28, 316–351.
[6] T. Hina and T. Sugano: On the local Hecke series of some classical groups over $p$-adic fields, J. Math. Soc. Japan 35(1) (1983), 133–152.

[7] F. Hyodo: A formal power series of a Hecke ring associated with the Heisenberg Lie algebra over $\mathbb{Z}_p$, Int. J. Number Theory 11(8) (2015), 2305–2323.

[8] F. Hyodo: A note on a Hecke ring associated with the Heisenberg Lie algebra, Math. J. Okayama Univ. 64 (2022), 215–225.

[9] F. Hyodo: A formal power series over a noncommutative Hecke ring and the rationality of the Hecke series for $GSp_4$, Preprint, 2022, arXiv:2208.09622v1.

[10] N. Iwahori, H. Matsumoto: On some Bruhat decomposition and the structure of the Hecke rings of $p$-adic Chevalley groups, Inst. Hautes Études Sci. Publ. Math. 25(1) (1965), 5–48.

[11] I. Satake: Theory of spherical functions on reductive algebraic groups over $p$-adic fields, Inst. Hautes Études Sci. Publ. Math 18 (1963), 5–69.

[12] G. Shimura: Sur les intégrales attachées aux formes automorphes, J. Math. Soc. Japan 11(4) (1959), 291–311.

[13] G. Shimura: On modular correspondences for $Sp(n, \mathbb{Z})$ and their congruence relations, Proc. Nat. Acad. Sci. U.S.A. 49(6) (1963), 824–828.

[14] G. Shimura: Introduction to the arithmetic theory of automorphic functions, Princeton University Press, Princeton, 1971.

[15] T. Tamagawa: On the $\zeta$-functions of a division algebra, Ann. of Math. 77(2) (1963), 387–405.

Department of Health Informatics, Faculty of Health and Welfare Services Administration, Kawasaki University of Medical Welfare, Kurashiki, 701-0193, Japan

Email address: fumitake.hyodo@mw.kawasaki-m.ac.jp