Sublinear Time Estimation of Degree Distribution Moments: The Degeneracy Connection

(Full Version)

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Abstract

We revisit the classic problem of estimating the degree distribution moments of an undirected graph. Consider an undirected graph $G = (V, E)$ with $n$ (non-isolated) vertices, and define (for $s > 0$) $\mu_s = \frac{1}{n} \sum_{v \in V} d_v^s$. Our aim is to estimate $\mu_s$ within a multiplicative error of $(1 + \varepsilon)$ (for a given approximation parameter $\varepsilon > 0$) in sublinear time. We consider the sparse graph model that allows access to: uniform random vertices, queries for the degree of any vertex, and queries for a neighbor of any vertex. For the case of $s = 1$ (the average degree), $\tilde{O}(\sqrt{n})$ queries suffice for any constant $\varepsilon$ (Feige, SICOMP 06 and Goldreich-Ron, RSA 08). Gonen-Ron-Shavitt (SIDMA 11) extended this result to all integral $s > 0$, by designing an algorithms that performs $\tilde{O}(n^{1-1/(s+1)})$ queries. (Strictly speaking, their algorithm approximates the number of star-subgraphs of a given size, but a slight modification gives an algorithm for moments.)

We design a new, significantly simpler algorithm for this problem. In the worst-case, it exactly matches the bounds of Gonen-Ron-Shavitt, and has a much simpler proof. More importantly, the running time of this algorithm is connected to the degeneracy of $G$. This is (essentially) the maximum density of an induced subgraph. For the family of graphs with degeneracy at most $\alpha$, it has a query complexity of $\tilde{O}\left(n^{1-1/s} \left(\alpha^{1/s} + \min\{\alpha, \mu_{1/s}\}\right)\right) = \tilde{O}(n^{1-1/s} \alpha/\mu_{1/s})$. Thus, for the class of bounded degeneracy graphs (which includes all minor closed families and preferential attachment graphs), we can estimate the average degree in $\tilde{O}(1)$ queries, and can estimate the variance of the degree distribution in $\tilde{O}(\sqrt{n})$ queries. This is a major improvement over the previous worst-case bounds. Our key insight is in designing an estimator for $\mu_s$ that has low variance when $G$ does not have large dense subgraphs.

1 Introduction

Estimating the mean and moments of a sequence of $n$ integers $d_1, d_2, \ldots, d_n$ is a classic problem in statistics that requires little introduction. In the absence of any knowledge of the moments of the sequence, it is not possible to prove anything non-trivial. But suppose these integers formed the degree sequence of a graph. Formally, let $G = (V, E)$ be an undirected graph over $n$ vertices,
and let \( d_v \) denote the degree of vertex \( v \in V \), where we assume that \( d_v \geq 1 \) for every \( v \).

Feige proved that \( O^*(\sqrt{n}) \) uniform random vertex degrees (in expectation) suffice to provide a \((2+\varepsilon)\)-approximation to the average degree [24]. (We use \( O^*(\cdot) \) to suppress \( \text{poly}(\log n, 1/\varepsilon) \) factors.) The variance can be as large as \( n \) for graphs of constant average degree (simply consider a star), but the constraints of a degree distribution allow for non-trivial approximations. Classic theorems of Erdős-Gallai and Havel-Hakimi characterize such sequences [29, 22, 27].

Again, the star graph shows that the \((2+\varepsilon)\)-approximation cannot be beaten in sublinear time through pure vertex sampling. Suppose we could also access random neighbors of a given vertex. In this setting, Goldreich and Ron showed it is possible to obtain a \((1+\varepsilon)\)-approximation to the average degree in \( O^*(\sqrt{n}) \) expected time [25].

In a substantial (and complex) generalization, Gonen, Ron, and Shavitt (henceforth, GRS) gave a sublinear-time algorithm that estimates the higher moments of the degree distribution [26]. Technically, GRS gave an algorithm for approximating the number of stars in a graph, but a simple modification yields an algorithm for moments estimation. For precision, let us formally define this problem. The degree distribution is the distribution over the degree of a uniform random vertex. The \( s \)-th moment of the degree distribution is \( \mu_s \triangleq \frac{1}{n} \sum_{v \in V} d_v^s \).

The Degree Distribution Moment Estimation (DDME) Problem. Let \( G = (V, E) \) be a graph over \( n \) vertices, where \( n \) is known. Access to \( G \) is provided through the following queries. We can (i) get the id (label) of a uniform random vertex, (ii) query the degree \( d_v \) of any vertex \( v \), (iii) query a uniform random neighbor of any vertex \( v \). Given \( \varepsilon > 0 \) and \( s \geq 1 \), output a \((1+\varepsilon)\)-multiplicative approximation to \( \mu_s \) with probability\(^2 > 2/3 \).

The DDME problem has important connections to network science, which is the study of properties of real-world graphs. There have been numerous results on the significance of heavy-tailed/power-law degree distributions in such graphs, since the seminal results of Barabási-Albert [6, 11, 23]. The degree distribution and its moments are commonly used to characterize and model graphs appearing in varied applications [8, 36, 15, 37, 9]. On the theoretical side, recent results provide faster algorithms for graphs where the degree distribution has some specified form [7, 10]. Practical algorithms for specific cases of DDME have been studied by Dasgupta et al and Chierichetti et al. [18, 14]. (These results requires bounds on the mixing time of the random walk on \( G \).)

1.1 Results

Let \( m \) denote the number of edges in the graph (where \( m \) is not provided to the algorithm). For the sake of simplicity, we restrict the discussion in the introduction to case when \( \mu_s \leq n^{s-1} \). As observed by GRS, the complexity of the DDME problem is smaller when \( \mu_s \) is significantly larger. GRS designed an (expected) \( O^* \left(n^{1-1/(s+1)} / \mu_s^{1/(s+1)} + n^{1-1/s}\right) \)-query algorithm for DDME and proved this expression was optimal up to \( \text{poly}(\log n, 1/\varepsilon) \) dependencies. (Here \( O^*(\cdot) \) also suppresses additional factors that depend only on \( s \).) Note that for a graph without isolated vertices, \( \mu_s \geq 1 \) for every \( s > 0 \), so this yields a worst-case \( O^*(n^{1-1/(s+1)}) \) bound. The \( s = 1 \) case is estimating the average degree, so this recovers the \( O^*(\sqrt{n}) \) bounds of Goldreich-Ron. We mention a recent result by Aliakbarpour et al. [3] for DDME, in a stronger model that assumes additional access to uniform random edges. They get a better bound of \( O^*(m/(n\mu_s)^{1/s}) \) in this stronger model, for \( s > 1 \) (and \( \mu_s \leq n^{s-1} \)). Note that the main challenge of DDME is

\(^1\)The assumption on there being no isolated vertices is made here only for the sake of simplicity of the presentation, as it ensures a basic lower bound on the moments.

\(^2\)The constant 2/3 is a matter of convenience. It can be increased to at least \( 1 - \delta \) by taking the median value of \( \log(1/\delta) \) independent invocations.
in measuring the contribution of high-degree vertices, which becomes substantially easier when random edges are provided. In the DDME problem without such samples, it is quite non-trivial to even detect high degree vertices.

All the bounds given above are known to be optimal, up to poly(log n, 1/ε) dependencies, and at first blush, this problem appears to be solved. We unearth a connection between DDME and the \textit{degeneracy} of $G$. The degeneracy of $G$ is (up to a factor 2) the maximum density over all subgraphs of $G$. We design an algorithm that has a nuanced query complexity, depending on the degeneracy of $G$. Our result subsumes all existing results, and provides substantial improvements in many interesting cases. Furthermore, our algorithm and its analysis are significantly simpler and more concise than in the GRS result.

We begin with a convenient corollary of our main theorem. A tighter, more precise bound appears as Theorem 3.

\textbf{Theorem 1.} Consider the family of graphs with degeneracy at most $\alpha$. The DDME problem can be solved on this family using

$$O^\ast \left( \frac{n^{1-1/s}}{\mu_s^{1/s}} \left( \alpha^{1/s} + \min\{\alpha, \mu_s^{1/s}\} \right) \right)$$

queries in expectation. The running time is linear in the number of queries.

Consider the case of bounded degeneracy graphs, where $\alpha = O(1)$. This is a rich class of graphs. Every minor-closed family of graphs has bounded degeneracy, as do graphs generated by the Barabási-Albert preferential attachment process [6]. There is a rich theory of bounded expansion graphs, which spans logic, graph minor theory, and fixed-parameter tractability [32]. All these graph classes have bounded degeneracy. For every such class of graphs, we get a $(1 + \varepsilon)$-estimate of $\mu_s$ in $O^\ast(n^{1-1/s}/\mu_s^{1/s})$ time. We stress that bounded degeneracy does not imply any bounds on the maximum degree or the moments. The star graph has degeneracy 1, but has extremely large moments due to the central vertex.

Consider any bounded degeneracy graph without isolated vertices. We can accurately estimate the average degree ($s = 1$) in poly(log $n$) queries, and estimate the variance of the degree distribution ($s = 2$) in $\sqrt{n} \cdot \text{poly}(\log n)$ queries. Contrast this with the (worst-case optimal) $\sqrt{n}$ bounds of Feige and Goldreich-Ron for average degree, and the $O^*(n^{2/3})$ bound of GRS for variance estimation. For general $s$, our bound is a significant improvement over the $O^*(n^{1-1/(s+1)/\mu_s^{1/(s+1)}})$ bound of GRS.

The algorithm attaining Theorem 1 requires an upper bound on the degeneracy of the graph. When an degeneracy bound is not given, the algorithm recovers the bounds of GRS, with an improvement on the extra poly(log $n$)/$\varepsilon$ factors. More details are in Theorem 3. We note that the degeneracy-dependent bound in Theorem 1 cannot be attained by an algorithm that is only given $n$ as a parameter. In particular, if an algorithm is only provided with $n$ and must work on all graphs with $n$ vertices, then it must perform $\Omega(\sqrt{n})$ queries in order to approximate the average degree even for graphs of constant degeneracy (and constant average degree). Details are given in Subsection 7.1 in the full version of the paper.

The bound of Theorem 1 may appear artificial, but we prove that it is optimal when $\mu_s \leq n^{s-1}$. (For the general case, we also have optimal upper and lower bounds.) This construction is an extension of the lower bound proof of GRS.

\textbf{Theorem 2.} Consider the family of graphs with degeneracy $\alpha$ and where $\mu_s \leq n^{s-1}$. Any algorithm for the DDME problem on this family requires $\Omega\left( \frac{n^{1-1/s}}{\mu_s^{1/s}} + \left( \alpha^{1/s} + \min\{\alpha, \mu_s^{1/s}\} \right) \right)$ queries.
1.2 From degeneracy to moment estimation

We begin with a closer look at the lower bound examples of Feige, Goldreich-Ron, and GRS. The core idea is quite simple: DDME is hard when the overall graph is sparse, but there are small dense subgraphs. Consider the case of a clique of size $100\sqrt{n}$ connected to a tree of size $n$. The small clique dominates the average degree, but any sublinear algorithm with access only to random vertices pays $\Omega(\sqrt{n})$ for a non-trivial approximation. GRS use more complex constructions to get an $\Omega(n^{1-1/(s+1)})$ lower bound for general $s$. This also involves embedding small dense subgraphs that dominate the moments.

Can we prove a converse to these lower bound constructions? In other words, prove that the non-existence of dense subgraphs must imply that DDME is easier? A convenient parameter for this non-existence is the degeneracy.

But the degeneracy is a global parameter, and it is not clear how a sublinear algorithm can exploit it. Furthermore, DDME algorithms are typically very local; they sample random vertices, query the degrees of these vertices and maybe also query the degrees of some of their neighbors. We need a local property that sublinear algorithms can exploit, but can also be linked to the degeneracy. We achieve this connection via the degree ordering of $G$. Consider the DAG obtained by directing all edges from lower to higher degree vertices. Chiba-Nishizeki related the properties of the out-degree distribution to the degeneracy, and exploited this for clique counting [13]. Nonetheless, there is no clear link to DDME. (Nor do we use any of their techniques; we state this result merely to show what led us to use the degree ordering).

Our main insight is the construction of an estimator for DDME whose variance depends on the degeneracy of $G$. This estimator critically uses the degree ordering. Our proof relates the variance of this estimator to the density of subgraphs in $G$, which can be bounded by the degeneracy. We stress that our algorithm is quite simple, and the technicalities are in the analysis and setting of certain parameters.

1.3 Designing the algorithm

Designate the weight of an edge $(u, v)$ to be $d_u^{s-1} + d_v^{s-1}$. A simple calculation yields that the sum of the weights of all edges is exactly $M_s \triangleq \sum_v d_v^s = n \cdot \mu_s$. Suppose we could sample uniform random edges (and knew the total number of edges). Then we could hope to estimate $M_s$ through uniform edge sampling. The variance of the edge weights can be bounded, and this yields an $O^*(m/(n\mu_s)^{1/s}) = O^*(n^{1-1/s})$ algorithm (when no vertex is isolated). Indeed, this is very similar to the approach of Aliakbarpour et al. [3]. Such variance calculations were also used in the classic Alon-Matias-Szegedy result of frequency moment estimation [5].

Our approach is to simulate uniform edge samples using uniform vertex samples. Suppose we sampled a set $R$ of uniform random vertices. By querying the degrees of all these vertices, we can select vertices in $R$ with probability proportional to their degrees, which allows us to uniformly sample edges that are incident to vertices in $R$. Now, we simply run the uniform edge sampling algorithm on these edges. This algorithmic structure was recently used for sublinear triangle counting algorithms by Eden et al. [20].

Here lies the core technical challenge. How to bound the number of random vertices that is sufficient for effectively simulating the random edge algorithm? This boils down to the behavior of the variance of the “vertex weight” distribution. Let the weight of a vertex be the sum of weights of its incident edges. The weight distribution over vertices can be extremely skewed, and this approach would require a forbiddingly large $R$.

A standard technique from triangle counting (first introduced by Chiba-Nishizeki [13]) helps reduce the variance. Direct all edges from lower degree to higher degree vertices, breaking ties consistently. Now, set the weight of a vertex to be the sum of weights on incident out-
edges. Thus, a high-degree vertex with lower degree neighbors will have a significantly reduced weight, reducing overall variance. In the general case (ignoring degeneracy), a relatively simple argument bounds the maximum weight of a vertex, which enables us to bound the variance of the weight distribution. This yields a much simpler algorithm and proof of the GRS bound.

In the case of graphs with bounded degeneracy, we need a more refined approach. Our key insight is an intimate connection between the variance and the existence of dense subgraphs in $G$. We basically show that the main structure that leads to high variance is the existence of dense subgraphs. Formally, we can translate a small upper bound on the density of any subgraph to a bound on the variance of the vertex weights. This establishes the connection to the graph degeneracy.

1.4 Simplicity of our algorithm

Our viewpoint on DDME is quite different from GRS and its precursor [25], which proceed by bucketing the vertices based on their degree. This leads to a complicated algorithm, which essentially samples to estimate the size of the buckets, and also the number of edges between various buckets (and “sub-buckets”). We make use of buckets in our analysis, in order to obtain the upper bound that depends on the degeneracy $\alpha$ (in order to achieve the GRS upper bound, our analysis does not use bucketing).

As explained above, our main DDME procedure, Moment-estimator is simple enough to present in a few lines of pseudocode (see Figure 1). We feel that the structural simplicity of Moment-estimator is an important contribution of our work.

Moment-estimator takes two sampling parameters $r$ and $q$. The main result Theorem 3 follows from running Moment-estimator with a standard geometric search for the right setting of $r$ and $q$. In Moment-estimator we use $id(v)$ to denote the label of a vertex $v$, where vertices have unique ids and there is a complete order over the ids.

| Moment-estimator \(_s(r, q)\) |
|--------------------------------|
| 1. Select $r$ vertices, uniformly, independently, at random and let the resulting multi-set be denoted by $R$. Query the degree of each vertex in $R$, and let $d_R = \sum_{v \in R} d_v$. |
| 2. For $i = 1, \ldots, q$ do: |
| (a) Select a vertex $v_i$ with probability proportional to its degree (i.e., with probability $d_{v_i}/d_R$), and query for a random neighbor $u_i$ of $v_i$. |
| (b) If $d_{v_i} < d_{u_i}$ or $d_{v_i} = d_{u_i}$ and $id(v_i) < id(u_i)$, set $X_i = (d_{v_i}^{n-1} + d_{u_i}^{n-1})$. Else, set $X_i = 0$. |
| 3. Return $X = \frac{1}{r} \cdot \frac{d_R}{q} \cdot \sum_{i=1}^{q} X_i$. |

Figure 1: Algorithm Moment-estimator for approximating $\mu_s$.

1.5 Other related work

As mentioned at the beginning of this section, Aliakbarpour et al. [3] consider the problem of approximating the number of $s$-stars for $s \geq 2$ when given access to uniformly selected edges. Given the ability to uniformly select edges, they can select vertices with probability proportional to their degree (rather than uniformly). This can be used to get an unbiased estimator of $\mu_s$ (or the $s$-star count) with low variance. This leads to an $O(m/(n\mu_s)^{1/s})$ bound, which is optimal (for $\mu_s \leq n^{s-1}$).
Dasgupta, Kumar, and Sarlos give practical algorithms for average degree estimation, though they assume bounds on the mixing time of the random walk on the graph \cite{DGK14}. A recent paper of Chierichetti et al. builds on these methods to sample nodes according to powers of their degree (which is closely related to DDME) \cite{CGG14}. Simpson, Seshadhri, and McGregor give practical algorithms to estimate the entire cumulative degree distribution in the streaming setting \cite{SSM15}. This is different from the sublinear query model we consider, and the results are mostly empirical.

In \cite{EDJK16}, Eden et al. present an algorithm for approximating the number of triangles in a graph. Although this is a very different problem than DDME, there are similar challenges regarding high-degree vertices. Indeed, as mentioned earlier, the approach of sampling random edges through a set of random vertices was used in \cite{EDJK16}.

The degeneracy is closely related to other “density” notions, such as the arboricity, thickness, and strength of a graph \cite{Alo78}. There is a rich history of algorithmic results where run time depends on the degeneracy \cite{BMR07, BM10, CBN01, GGG12}.

Other sublinear algorithms for estimating various graph parameters include: approximating the size of the minimum-weight spanning tree \cite{Che12, CLR93, CDLP17}, maximum matching \cite{LM10, SM14} and of the minimum vertex cover \cite{LM10, SM14, Che12, GGM10, GGM11}.

2 The main theorem

**Theorem 3.** For every graph $G$, there exists an algorithm that returns a value $Z$ such that $Z \in [(1 - \epsilon)\mu_s(G), (1 + \epsilon)\mu_s(G)]$ with probability at least $2/3$. Assume that algorithm is given $\alpha$, an upper bound on the degeneracy of $G$. (If no such bound is provided, the algorithm assumes a trivial bound of $\alpha = \infty$.) The expected running time is the minimum of the following two expressions.

\[
O\left(2^s \cdot n^{1 - 1/s} \cdot \log^2 n \cdot \left(\frac{\alpha}{\mu_s}\right)^{1/s} + \min \left\{\frac{n^{1 - 1/s} \cdot \alpha}{\mu_s^{1/s}}, \frac{n^{s - 1} \cdot \alpha}{\mu_s}\right\} \cdot \frac{s \log n \cdot \log(s \log n)}{\epsilon^2}\right) \tag{1}
\]

\[
O\left(\frac{n^{1 - 1/(s+1)}}{\mu_s^{1/(s+1)}} + \min \left\{\frac{n^{1 - 1/s} \cdot n^{s - 1 - 1/s}}{\mu_s^{1 - 1/s}}, \frac{n^{s - 1 - 1/s}}{\mu_s}\right\} \cdot \frac{s \log n \cdot \log(s \log n)}{\epsilon^2}\right) \tag{2}
\]

Equation (2) is essentially the query complexity of GRS (albeit with a better dependence on $s$, $\log n$, and $1/\epsilon$). Thus, our algorithm is guaranteed to be at least as good as that. If $\alpha$ is exactly the degeneracy of $G$, then we can prove that Equation (1) is less than Equation (2). Within each expression, there is a min of two terms. The first term is smaller iff $\mu_s \leq n^{s - 1}$.

The mechanism of deriving this rather cumbersome running time is the following. The algorithm of Theorem 3 runs Moment-estimator for geometrically increasing values of $r$ and $q$, which is in turn derived from a geometrically decreasing guess of $\mu_s$. It uses this guess to set $r$ and $q$. There is a setting of values depending on $\alpha$, and a setting independent of it. The algorithm simply picks the minimum of these settings to achieve the smaller running time.

3 Sufficient conditions for $r$ and $q$ in Moment-estimator

In this section we provide sufficient conditions on the parameters $r$ and $q$ that are used by Moment-estimator (Figure 1), in order for the algorithm to return a $(1 + \epsilon)$ estimate of $\mu_s$. First we introduce some notations. For a graph $G = (V, E)$ and a vertex $v \in V$, let $\Gamma(v)$ denote
the set of neighbors of \( v \) in \( G \) (so that \( d_v = |\Gamma(v)| \)). For any (multi-)set \( R \) of vertices, let \( E_R \) be the (multi-)set of edges incident to the vertices in \( R \). We will think of the edges in \( E_R \) as ordered pairs; thus \((v, u)\) is distinct from \((u, v)\), and so \( E_R \triangleq \{(v, u) : v \in R, u \in \Gamma(v)\} \). Observe that \( d_R \), as defined in Step 1 of \textbf{Moment-estimator} equals \(|E_R|\). Let \( M_s = M_s(G) \triangleq \sum_{v \in V} d_v^s \), so that \( \mu_s = M_s/n \). In the analysis of the algorithm, it is convenient to work with \( M_s \) instead of \( \mu_s \).

A critical aspect of our algorithm (and proof) is the \textit{degree ordering on vertices}. Formally, we set \( u \prec v \) if \( d_u < d_v \) or, \( d_u = d_v \) and \( id(u) < id(v) \). Given the degree ordering, we let \( \Gamma^+(v) \triangleq \{ u \in \Gamma(v) : u \prec v \} \), \( d^+_v \triangleq |\Gamma^+(v)| \), and \( E^+ \triangleq \{(v, u) : v \in V, u \in \Gamma^+(v)\} \). Here and elsewhere, we use \( \sum_v \) as a shorthand for \( \sum_{v \in V} \).

**Definition 4.** We define the weight of an edge \( e = (v, u) \) as follows: if \( v \prec u \) define \( wt(e) \triangleq (d_v^s - 1 + d_u^s - 1) \). Otherwise, \( wt(e) \triangleq 0 \). For a vertex \( v \in V \), \( wt(v) \triangleq \sum_{u \in \Gamma(v)} wt((v, u)) = \sum_{u \in \Gamma^+(v)} wt((v, u)) \), and for a (multi-)set of vertices \( R \), \( wt(R) \triangleq \sum_{v \in R} wt(v) \).

Observe that given the above notations and definition, \textbf{Moment-estimator} selects uniform edges from \( E_R \) and sets each \( X_i \) (in Step 2b) to \( wt((v_i, u_i)) \). Based on Definition 4, we obtain the next two claims, where the first claim connects between \( M_s \) and the weights of vertices.

**Claim 5.** \( \sum_v wt(v) = M_s \).

**Proof:** By the definition of the weights:

\[
\sum_{v \in V} wt(v) = \sum_{(v, u) \in E^+} (d_v^s - 1 + d_u^s - 1) = \sum_{(v, u) \in E} (d_v^s - 1 + d_u^s - 1) = \sum_{v \in V} \sum_{u \in \Gamma(v)} d_u^s - 1 = \sum_{v \in V} d_v^s = M_s ,
\]

and the claim is established.

**Claim 6.** \( \text{Exp}[X] = \mu_s \), where \( X \) is as defined in Step 3 of the algorithm.

**Proof:** Recall that \( wt(R) \triangleq \sum_{v \in R} wt(v) \). Note that \( X_i \) (as defined in Step 2b of the algorithm) is exactly \( wt((v_i, u_i)) \). Conditioning on \( R \),

\[
\text{Exp}[X|R] = \frac{1}{|E_R|} \cdot \sum_{v \in R} \sum_{u \in \Gamma^+_v} wt((v, u)) = \frac{1}{|E_R|} \cdot \sum_{v \in R} wt(v) = \frac{1}{|E_R|} \cdot wt(R) .
\]

By the definition of \( X \) in the algorithm (see Step 3),

\[
\text{Exp}[X|R] = \frac{1}{r} \cdot \frac{|E_R|}{q} \cdot \sum_{i=1}^{q} \text{Exp}[X_i|R] = \frac{1}{r} \cdot wt(R) .
\]

Now, let us remove the conditioning. Since \( wt(R) \triangleq \sum_{v \in R} wt(v) \), by linearity of the expectation,

\[
\text{Exp}[wt(R)] = \frac{r}{n} \cdot \sum_{v \in V} wt(v) ,
\]

and thus (using Claim 5),

\[
\text{Exp}[X] = \text{Exp}[\text{Exp}[X|R]] = \frac{1}{n} \sum_{v \in V} wt(v) = \mu_s ,
\]

which completes the proof.
3.1 Conditions on the parameters \( r \) and \( q \).

We next state two conditions on the parameters \( r \) and \( q \), which are used in the algorithm, and then establish several claims, based on the conditions holding. The conditions are stated in terms of properties of the graph as well as the approximation parameter \( \varepsilon \) and a confidence parameter \( \delta \).

1. The vertex condition:
   \[
   r \geq \frac{30 \cdot n \cdot \sum_v \text{wt}(v)^2}{\varepsilon^2 \cdot \delta \cdot M_s^2}.
   \]

2. The edge condition:
   \[
   q \geq \frac{200 \cdot m \cdot M_{2s-1}}{\varepsilon^2 \cdot \delta^3 \cdot M_s^2}.
   \]

**Lemma 7.** If Condition 1 holds, then with probability at least \( 1 - \delta/2 \), all the following hold.

1. \( \text{wt}(R) \in [(1 - \frac{\delta}{3}) \cdot \frac{r}{n} \cdot M_s, (1 + \frac{\delta}{3}) \cdot \frac{r}{n} \cdot M_s] \).
2. \(|E_R| \leq \frac{42 \delta}{3} \cdot \frac{r}{n} \cdot m\).
3. \( \sum_{(v, u) \in E^+} \text{wt}((v, u))^2 \leq \frac{18 \delta}{3} \cdot \frac{r}{n} \cdot M_{2s-1} \).

**Proof:** First, we look at the random variable \( \text{wt}(R) \). By the definition of \( \text{wt}(R) \) and Claim 5, \( \text{Exp}[\text{wt}(R)] = (r/n) \cdot \sum_v \text{wt}(v) = (r/n) \cdot M_s \). Turning to the variance of \( \text{wt}(R) \), since the vertices in \( R \) are chosen uniformly at random,

\[
\text{Var}_R[\text{wt}(R)] = r \cdot \text{Var}_v[\text{wt}(v)] = r \left( \frac{1}{n} \sum_v \text{wt}(v)^2 - \left( \frac{1}{n} \sum_v \text{wt}(v) \right)^2 \right) \leq \frac{r}{n} \sum_v \text{wt}(v)^2.
\]

By Chebyshev’s inequality,

\[
\Pr \left[ \left| \text{wt}(R) - \text{Exp}[\text{wt}(R)] \right| \geq \frac{\varepsilon}{2} \cdot \text{Exp}[\text{wt}(R)] \right] \leq \frac{4 \text{Var}[\text{wt}(R)]}{\varepsilon^2 \cdot \text{Exp}[\text{wt}(R)]^2} = \frac{4(r/n) \sum_v \text{wt}(v)^2}{\varepsilon^2 \cdot (r/n)^2 \cdot M_s^2} = \frac{4n \sum_v \text{wt}(v)^2}{\varepsilon^2 \cdot M_s^2 \cdot r}.
\]

Applying the lower bound on \( r \) from Condition 1, this probability is at most \( \delta/6 \). (Indeed, Condition 1 was defined as such to get this bound.)

The other bounds follow simply from Markov’s inequality. Observe that \( \text{Exp}[|E_R|] = (r/n)(2m) \), and so \( \Pr[|E_R| > (12/\delta)(r/n)m] \leq \delta/6 \).

The random variable \( Y = \sum_{(v, u) \in E^+} \text{wt}((v, u))^2 \) (which is non-negative), satisfies

\[
\text{Exp}[Y] = \frac{r}{n} \cdot \sum_{(v, u) \in E^+} \text{wt}((v, u))^2 = \frac{r}{n} \cdot \sum_{(v, u) \in E^+} (d_v^{a-1} + d_u^{a-1})^2 \\
\leq 3 \cdot \frac{r}{n} \cdot \sum_{(v, u) \in E^+} (d_v^{2s-2} + d_u^{2s-2}) = 3 \cdot \frac{r}{n} \cdot \sum_v d_v^{2s-1} = 3 \cdot \frac{r}{n} \cdot M_{2s-1}. \tag{3}
\]

By Markov’s inequality, \( \Pr[Y \geq (18/\delta)(r/n)M_{2s-1}] \leq \delta/6 \). We apply a union bound to complete the proof. \( \blacksquare \)
Theorem 8. If Conditions 1 and 2 hold, then \( X \in [(1-\varepsilon)\mu_s, (1+\varepsilon)\mu_s] \) with probability at least \( 1-\delta \).

Proof: Condition on any choice of \( R \). We have \( \text{Exp}[X|R] = (1/r)\text{wt}(R) \). Turning to the variance, since the edges \((v_i, u_i)\) are chosen from \( E_R \) uniformly at random,

\[
\text{Var}[X|R] = \left( \frac{1}{r} \right)^2 \cdot \left( \frac{|E_R|}{q} \right)^2 \cdot \text{Var} \left[ \sum_{i=1}^{q} X_i \mid R \right] = \left( \frac{1}{r} \right)^2 \cdot \frac{|E_R|^2}{q} \cdot \text{Var}[X_1|R] \\
\leq \left( \frac{1}{r} \right)^2 \cdot \frac{|E_R|^2}{q} \cdot \text{Exp}[(X_1|R)^2] \\
= \left( \frac{1}{r} \right)^2 \cdot \frac{|E_R|}{q} \cdot \sum_{(v,u) \in E_R^+} \text{wt}((v,u))^2 \\
= \frac{1}{q} \cdot \frac{|E_R|}{r} \cdot \sum_{(v,u) \in E_R^+} \frac{\text{wt}((v,u))^2}{r}.
\]

Let us now condition on \( R \) such that the bounds of Lemma 7 hold. Note that such an \( R \) is chosen with probability at least \( 1-\delta/2 \). We get

\[
\text{Var}[X|R] \leq \frac{250}{\delta^2} \cdot \frac{1}{q} \cdot \frac{m}{n} \cdot \frac{M_{2s-1}}{n}.
\]

We apply Chebyshev’s inequality and invoke Condition 2:

\[
\text{Pr} \left[ \left| (X|R) - \text{Exp}[X|R] \right| \leq \frac{\varepsilon}{2} \cdot \mu_s \right] \leq \frac{4 \cdot \text{Var}[X|R]}{\varepsilon^2 \cdot \mu_s^2} \leq \frac{1}{q} \cdot \frac{4 \cdot (250/\delta^2) \cdot m \cdot M_{2s-1}}{\varepsilon^2 \cdot M_s^2} \leq \frac{\delta}{2}.
\]

By Lemma 7, \( \text{Exp}[X|R] = (1/r)\text{wt}(R) \in [(1-\varepsilon/2)\mu_s, (1+\varepsilon/2)\mu_s] \). By taking into account both the probability that \( R \) does not satisfy one (or more) of the bounds in Lemma 7 and the probability that \( X \) (conditioned on \( R \) satisfying these bounds) deviates by more than \((\varepsilon/2)\mu_s\) from its expected value, we get that \( |X - \mu_s| < \varepsilon\mu_s \) with probability at least \((1-\delta/2)^2 > 1-\delta \).

\[\blacksquare\]

3.2 The Algorithm with Edge Samples

As an aside, the above analysis can be slightly adapted to prove the result of Aliakbarpour et al. [3] on estimating moments using random edge queries. Observe that we can then simply set \( R = V \) and \( r = n \) in Moment-estimator. This immediately gives \( \text{wt}(R) = M_s, |E_R| = 2m, \) and \( \sum_{(v,u) \in E_R^+} \text{wt}((v,u))^2 \leq 3M_{2s-1} \) (as shown in Equation (3)). Similarly to what is shown in the proof of Theorem 8, \( \text{Var}[X] = O(q^{-1} \cdot m \cdot M_{2s-1}) \) (where \( X \) is as defined in Step 3 of the algorithm). As shown in Equation (10), \( M_{2s-1} \leq M_s^{2-1/s} \). Thus, if we set \( q \geq \frac{c_q}{\varepsilon^2 \mu_s^2} \cdot \frac{m}{M_s^{2-1/s}} \) (for a sufficiently large constant \( c_q \)), we satisfy Condition 2. This is exactly the bound of Aliakbarpour et al.

4 Satisfying Conditions 1 and 2 in general graphs

We show how to set \( r \) and \( q \) to satisfy Conditions 1 and 2 in general graphs. Our setting of \( r \) and \( q \) will give us the same query complexity as [26] (up to the dependence on \( 1/\varepsilon \) and \( \log n \), on
which we improve, and the exponential dependence on \( s \) in [26], which we do not incur. In the next section we show how the setting of \( r \) and \( q \) can be improved using a degeneracy bound.

For \( c_r \) and \( c_q \) that are sufficiently large constants, we set

\[
    r = \frac{c_r}{\varepsilon^2 \cdot \delta} \cdot \frac{n}{M_s^{1/(s+1)}}, \quad q = \frac{c_q}{\varepsilon^2 \cdot \delta^3} \cdot \min \left\{ n^{1-1/s}, \frac{n^{s-1/s}}{M_s^{1-1/s}} \right\}.
\]  

(4)

This setting of parameters requires the knowledge of \( M_s \), which is exactly what we are trying to approximate (up to the normalization factor of \( n \)). A simple geometric search argument alleviates the need to know \( M_s \). For details see Section 6.

In what follows (and elsewhere) we make use of Hölder’s inequality:

**Theorem 9** (Hölder’s inequality). For values \( p \) and \( q \) such that \( p, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \),

\[
    \sum_{i=1}^{k} |x_i \cdot y_i| \leq \left( \sum_{i=1}^{k} |x_i|^p \right)^{1/p} \left( \sum_{i=1}^{k} |y_i|^q \right)^{1/q}.
\]

We refer to \( p \) and \( q \) as the conjugates of the formula.

In order to assert that \( r \) as set in Equation (4) satisfies Condition 1, it suffices to establish the next lemma.

**Lemma 10** (Condition 1 holds).

\[
    \sum_v \text{wt}(v)^2 \leq 4M_s^{2-\frac{1}{s+1}}.
\]

**Proof:** Let \( \theta = M_s^{1/(s+1)} \) be a degree threshold. We define

\[
    H \triangleq \{ v : d_u > \theta \} \quad \text{and} \quad L \triangleq V \setminus H.
\]

This partition into “high-degree” vertices \( (H) \) and “low-degree” vertices \( (L) \) will be useful in upper bounding the maximum weight \( \text{wt}(v) \) of a vertex \( v \), and hence upper bounding \( \sum_v \text{wt}(v)^2 \). Details follow.

We first observe that \( |H| \leq M_s^{1/(s+1)} \). This is true since otherwise, \( \sum_{v \in H} d_v^+ > M_s^{1/(s+1)} \). \( M_s^{\frac{1}{s+1}} = M_s \), which is a contradiction. We claim that this upper bound on \( |H| \) implies that

\[
    \max_u d_v^+ \leq M_s^{1/(s+1)}.
\]

(5)

To verify this, assume, contrary of the claim, that for some \( v \), \( d_v^+ > M_s^{1/(s+1)} \). But then there are at least \( M_s^{1/(s+1)} \) vertices \( u \) such that \( d_u \geq d_v^+ > M_s^{1/(s+1)} \). This contradicts the bound on \( |H| \).

It will also be useful to bound \( \sum_{u \in H} d_u^{s-1} \). By Hölder’s inequality with conjugates \( s \) and \( s/(s-1) \) and the bound on \( |H| \),

\[
    \sum_{u \in H} d_u^{s-1} = \sum_{u \in H} \cdot 1 \cdot d_u^{s-1} \leq |H|^{1/s} \left( \sum_{u \in H} d_u^s \right)^{\frac{s-1}{s}} \leq M_s^{\frac{1}{s+1}} \cdot M_s^{\frac{1}{s-1}} \leq M_s^{\frac{s}{s+1}}.
\]

(6)

We now turn to bounding \( \max_v \{ \text{wt}(v) \} \). By the definition of \( \text{wt}(v) \) and the degree ordering,

\[
    \text{wt}(v) = \sum_{u \in \Gamma^+(v)} (d_u^{s-1} + d_u^{-1}) \leq 2 \sum_{u \in \Gamma^+(v)} d_u^{s-1} = 2 \sum_{u \in \Gamma^+(v) \cap L} d_u^{s-1} + 2 \sum_{u \in \Gamma^+(v) \cap H} d_u^{s-1}.
\]

(7)
For the first term on the right-hand-side of Equation (7), recall that $d_u \leq M_1^{s/(s+1)}$ for $u \in L$. Thus, by Equation (5),

$$
\sum_{u \in \Gamma^+(v) \cap L} d_u^{s-1} \leq d_v^{s-1} \cdot M_1^{\frac{s}{s+1}} \leq M_1^{\frac{s}{s+1}}.
$$

(8)

For the second term, using $\Gamma^+(v) \cap H \subseteq H$ and applying Equation (6),

$$
\sum_{u \in \Gamma^+(v) \cap H} d_u^{s-1} \leq \sum_{u \in H} d_u^{s-1} \leq M_s^{1/(s+1)}.
$$

(9)

Finally,

$$
\sum_v \text{wt}(v)^2 \leq \max_v \{\text{wt}(v)\} \cdot \sum_v \text{wt}(v) \leq M_s^{2-1/(s+1)},
$$

where the second inequality follows by combining Equations (7)–(9) to get an upper bound on $\max_v \{\text{wt}(v)\}$ and applying Claim 5.

The next lemma implies that Condition 2 holds for $q$ as set in Equation (4).

Lemma 11 (Condition 2 holds).

$$
\min \left\{ n^{1-1/s}, \frac{n^{s-1/s}}{M_1^{1-1/s}} \right\} \geq 2m \cdot \frac{M_2-1}{M_s^2}.
$$

Proof: We can bound $M_{2s-1}$ in two ways. First, by a standard norm inequality, since $s \geq 1$,

$$
M_{2s-1} = \sum_v d_v^{2s-1} \leq \left( \sum_v d_v^s \right)^{(2s-1)/s} = M_s^{2-1/s}.
$$

(10)

We can also use the trivial bound $d_v \leq n$ and get $M_{2s-1} \leq n^{s-1} \cdot M_s$. Thus, $M_{2s-1} \leq \min\{M_2^{2-1/s}, n^{s-1} \cdot M_s\}$. By applying Hölder’s inequality with conjugates $s/(s-1)$ and $s$ we get that

$$
2m = \sum_v 1 \cdot d_v \leq n^{(s-1)/s} \cdot \left( \sum_v d_v^s \right)^{1/s} = n^{1-1/s} \cdot M_s^{1/s}.
$$

(11)

We multiply the bound by $M_{2s-1}$ to complete the proof.

5 The Degeneracy Connection

The degeneracy, or the coloring number, of a graph $G = (V, E)$ is the maximum value, over all subgraphs $G'$ of $G$, of the minimum degree in $G'$. In this definition, we can replace “minimum” by “average” to get a 2-factor approximation to the degeneracy (refer to [2]: Theorem 2.4.4 and Corollary 5.2.3 of [19]). Abusing notation, it will be convenient for us to define $\alpha(G) = \max_{S \subseteq V} \left\{ \frac{|E(S)|}{|S|} \right\}$.

We also make the following observation regarding the relation between $\alpha(G)$ and $M_s(G)$.

Claim 12. For every graph $G$, $\alpha(G) \leq M_s(G)^{\frac{1}{s+1}}$. 

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Proof: Let $S$ be a subset of vertices that maximizes $\frac{|E(S)|}{|S|-1}$, and let $\overline{d}(S)$ denote the average degree in the subgraph induced by $S$. Then $\overline{d}(S) = \frac{2|E(S)|}{|S|} \geq \frac{|E(S)|}{|S|-1} = \alpha(G)$. Hence, $|S| \geq \alpha(G)$, and by Hölder’s inequality (Theorem 9) with conjugates $s/(s-1)$ and $s$, 

$$\alpha(G) \cdot |S| \leq \sum_{v \in S} d_v \leq \left( \sum_{v \in S} d_v^s \right)^{1/s} \cdot |S|^{1-1/s},$$

implying that $\alpha(G) \cdot |S|^{1/s} \leq M_s(G)^{1/s}$. Since $|S| \geq \alpha(G)$, we get that $\alpha(G) \leq M_s(G)^{1/s+1}$. 

In this section, we show that the following setting of parameters for Moment-estimator$_s$ satisfies Conditions 1 and 2, for every graph $G$ with degeneracy at most $\alpha$ (i.e., $\alpha(G) \leq \alpha$), and for appropriate constants $c_r$ and $c_q$. 

$$r = \frac{c_r}{\varepsilon^2 \cdot \delta} \cdot \min \left\{ \frac{n^2}{M_s^{1/(s+1)}}, \frac{2\alpha \cdot \log^2 n}{M_s^{1/s}} \right\}, \quad (12)$$

$$q = \frac{c_q}{\varepsilon^2 \cdot \delta^3} \cdot \min \left\{ \frac{n \cdot \alpha}{M_s^{1/s}}, \frac{n^s \cdot \alpha}{M_s}, \frac{n^{1-1/s}}{M_s^{1/s}} \cdot \frac{n^{s-1/s}}{M_s^{1-1/s}} \right\}. \quad (13)$$

Clearly the setting of $r$ and $q$ in Equation (12) and Equation (13) respectively, can only improve on the setting of $r$ and $q$ for the general case in Equation (4) (Section 4).

Our main challenge is in proving that Condition 1 holds for $r$ as set in Equation (12) (when the graph has degeneracy at most $\alpha$). Here too, the goal is to upper bound $\sum_v \text{wt}(v)^2$. However, as opposed to the proof of Lemma 10 in Section 4, where we simply obtained an upper bound on $\max_v \{\text{wt}(v)\}$ (and bounded $\sum_v \text{wt}(v)^2$ by $\max_v \{\text{wt}(v)\} \cdot M_s$), here the analysis is more refined, and uses the degeneracy bound. For details see the proof of our main lemma, stated next.

Lemma 13 (Condition 1 holds). For a sufficiently large constant $c$, 

$$\sum_v \text{wt}(v)^2 \leq c \cdot 2^s \cdot \alpha^{1/s} \cdot M_s^{2-1/s} \cdot \log^2 n.$$ 

In order to prove the lemma we first introduce the following definitions and claim.

Definition 14. For a set $S$ and a vertex $u$, let $\Gamma_S(u)$ denote the set $\Gamma(u) \cap S$, and let $\Gamma^+_S(u)$ denote the set $\Gamma^+(u) \cap S$. For two sets of vertices $S$ and $T$ (which are not necessarily disjoint), let $E^+(S, T) \triangleq \{(u, v) : (u, v) \in E^+, u \in S, v \in T\}$.

Definition 15. We partition the vertices (with degree at least 1) according to their degree. For $0 \leq i \leq \log n$, let 

$$U_i \triangleq \{u \in V : d_u \in (2^{i-1}, 2^i]\}.$$ 

For each $i$ we partition the vertices in $V$ according to the number of outgoing edges that they have to $U_i$. Specifically, for $1 \leq j \leq \log(n/\alpha)$, define 

$$V_{i,j} \triangleq \{v \in V : |\Gamma_{U_i}^+(v)| \in (2^{j-1} \alpha, 2^j \alpha]\}.$$ 

Also define $V_{i,0} \triangleq \{v \in V : |\Gamma_{U_i}^+(v)| \leq \alpha\}$. Hence, $\{V_{i,j}\}_{j=0}^{\log(n/\alpha)}$ is a partition of $V$ for each $i$. 

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A central building block in the proof of Lemma 13 is the next claim. This claim establishes an upper bound on the number of edges going from vertices in \( V_{i,j} \) to vertices in \( U_i \), for every \( i \) and \( j \) (within the appropriate intervals). In the proof of this claim we exploit the degeneracy bound.

**Claim 16.** Let \( V_{i,j} \) and \( U_i \) be as defined in Definition 15, and let \( E^+(V_{i,j}, U_i) \) be as defined in Definition 14. For every \( 0 \leq i \leq \lceil \log n \rceil \) and every \( 2 \leq j \leq \lceil \log(n/\alpha) \rceil \),

\[
|E^+(V_{i,j}, U_i)| \leq M_s \cdot 2^{-(i-1)(s-1)+j-1}.
\]

**Proof:** Since \( G \) has degeneracy at most \( \alpha \),

\[
|E^+(V_{i,j}, U_i)| = |E^+(V_{i,j}, U_i)| = |E(V_{i,j}, U_i)| \leq \alpha \cdot (|V_{i,j}| + |U_i|).
\]

On the other hand, by the definition of \( V_{i,j} \),

\[
|E^+(V_{i,j}, U_i)| \geq 2^{j-1} \cdot \alpha \cdot |V_{i,j}|.
\]

Combining the above two bounds (and because \( 2^{j-1} - 1 \geq 1 \) for \( j \geq 2 \)), we get that \( |U_i| \geq |V_{i,j}| \), and we obtain the following bound on the number of edges in \( E^+ \) between \( V_{i,j} \) and their neighbors in \( U_i \):

\[
|E^+(V_{i,j}, U_i)| \leq 2 \cdot \alpha \cdot |U_i|.
\]  \( (14) \)

We next upper bound \( |U_i| \). By the definition of \( V_{i,j} \), for every \( v \in V_{i,j} \) there exists a vertex \( u \in U_i \) such that \( d_u \geq d_v \). By the definition of \( U_i \), for every \( u', u'' \in U_i \), \( d_{u'} \geq d_{u''}/2 \), implying that for every \( v \in V_{i,j} \) and every \( u' \in U_i \) it holds that \( d_u \geq d_v/2 \). Hence, for every \( u \in U_i \), we have that \( d_u \geq 2^{j-2} \alpha \). Also by the definition of \( U_i \), for every \( u \in U_i \) it holds that \( d_u \geq 2^{i-1} \).

Therefore,

\[
M_s \geq \sum_{u \in U_i} d_u^s \geq |U_i| \cdot 2^{(i-1)(s-1)} \cdot 2^{j-2} \alpha,
\]

which directly implies that

\[
|U_i| \leq M_s \cdot 2^{-(i-1)(s-1)+j-2} \cdot \alpha^{-1}.
\]  \( (15) \)

The proof follows by plugging in Equation (15) in Equation (14).

We are now ready to prove Lemma 13.

**Proof of Lemma 13:** By the definition of \( \text{wt}(v) \), and since \( d_v \leq d_u \) for every \( v \) and \( u \in \Gamma^+(v) \),

\[
\sum_v \text{wt}(v)^2 = \sum_v \left( \sum_{u \in \Gamma^+(v)} (d_v^{s-1} + d_u^{s-1}) \right)^2 \leq 4 \cdot \sum_v \left( \sum_{u \in \Gamma^+(v)} d_u^{s-1} \right)^2.
\]  \( (16) \)

In order to bound the expression on the right-hand-side of Equation (16) we consider each \( U_i \) (as defined in Definition 15) separately: By applying Hölder’s inequality with conjugates \( s \) and \( s/(s-1) \) we get the following bound for every \( v \) and \( 0 \leq i \leq \lceil \log n \rceil \).

\[
\sum_{u \in \Gamma_i^+(v)} 1 \cdot d_u^{s-1} \leq |\Gamma_i^+(v)|^{1/s} \cdot \left( \sum_{u \in \Gamma_i^+(v)} d_u^{s} \right)^{(s-1)/s} \leq |\Gamma_i^+(v)|^{1/s} \cdot M_s^{(s-1)/s}.
\]  \( (17) \)
For a vertex $u$, let $\Gamma^-(u) \triangleq \{ v : u \in \Gamma^+(v) \}$. By applying Equation (17) (to one term of the square $\left( \sum_{u \in \Gamma^+_i(v)} d_u^{s-1} \right)^2$),
\[
\sum_v \left( \sum_{u \in \Gamma^+_i(v)} d_u^{s-1} \right)^2 \leq \sum_v |\Gamma^+_i(v)|^{1/s} \cdot M_s^{(s-1)/s} \cdot \sum_{u \in \Gamma^+_i(v)} d_u^{s-1} \\
= M_s^{(s-1)/s} \cdot \sum_{u \in U_i} d_u^{s-1} \cdot \sum_{v \in \Gamma^-(u)} |\Gamma^+_i(v)|^{1/s} \\
= M_s^{(s-1)/s} \cdot \sum_{j=0}^{[\log n]} \left( \sum_{u \in U_i} d_u^{s-1} \cdot \sum_{v \in \Gamma^-(u) \cap V_{i,j}} |\Gamma^+_i(v)|^{1/s} \right).
\]

Equation (18) follows by the definition of $\Gamma^-(u)$ (and switching the order of the summations), and Equation (19) follows from splitting the sum in Equation (18) based on the partition of $V$ into the subsets $V_{i,j}$ (recall that $i$ is fixed for now and $V_{i,j}$ is as defined in Definition 15).

From this point on we only consider $j$’s such that $V_{i,j}$ is not empty. For each $j$ (and $i$), by the definition of $V_{i,j}$,
\[
\sum_{u \in U_i} d_u^{s-1} \cdot \sum_{v \in \Gamma^-(u) \cap V_{i,j}} |\Gamma^+_i(v)|^{1/s} \leq \sum_{u \in U_i} d_u^{s-1} \cdot \sum_{v \in \Gamma^-(u) \cap V_{i,j}} \left( 2^j \cdot \alpha \right)^{1/s} \\
\leq 2^{j/s} \cdot \alpha^{1/s} \cdot \sum_{u \in U_i} d_u^{s-1} \cdot \left| \Gamma^-(u) \cap V_{i,j} \right|.
\]

For $j < 2$, we trivially upper bound $|\Gamma^-(u) \cap V_{i,j}|$ by $d_u$. Thus,
\[
\sum_{u \in U_i} d_u^{s-1} \cdot \sum_{v \in \Gamma^-(u) \cap V_{i,j}} |\Gamma^+_i(v)|^{1/s} \leq 2^{1/s} \cdot \alpha^{1/s} \cdot \sum_{u \in U_i} d_u^{s} \leq 2 \cdot \alpha^{1/s} \cdot M_s \quad \text{(for $j < 2$)}.
\]

Turning to $j \geq 2$, since all vertices in $U_i$ have degree at most $2^i$ and by Equation (20), we get:
\[
\sum_{u \in U_i} d_u^{s-1} \cdot \sum_{v \in \Gamma^-(u) \cap V_{i,j}} |\Gamma^+_i(v)|^{1/s} \leq 2^{j/s} \cdot \alpha^{1/s} \cdot 2^{(s-1)} \cdot |E^+(V_{i,j}, U_i)|.
\]

Hence, by Claim 16,
\[
\sum_{u \in U_i} d_u^{s-1} \cdot \sum_{v \in \Gamma^-(u) \cap V_{i,j}} |\Gamma^+_i(v)|^{1/s} \leq 2^{j/s-j+s} \cdot \alpha^{1/s} \cdot M_s.
\]

By using the bound in Equation (21) for $j < 2$, the bound in Equation (22) for $j \geq 2$ and plugging them in Equation (19) we get
\[
\sum_v \left( \sum_{u \in \Gamma^+_i(v)} d_u^{s-1} \right)^2 \leq 4 \cdot \alpha^{1/s} \cdot M_s^{2-1/s} + 2^{s} \cdot \alpha^{1/s} \cdot M_s^{2-1/s} \cdot \sum_{j=3}^{[\log n]} 2^{j/s-j} \\
\leq 2^{s+1} \cdot \alpha^{1/s} \cdot M_s^{2-1/s} \cdot \log n,
\]
where the last inequality holds because $s \geq 1$ (in fact, for $s > 1$ we can save a $\log n$ factor). Using the inequality $(\sum_{i=1}^{\ell} x_i)^2 \leq (2\ell - 1) \cdot \sum_{i=1}^{\ell} x_i^2$, Equation (23) implies that
\[
\sum_v \left( \sum_{u \in \Gamma^+_i(v)} d_u^{s-1} \right)^2 \leq 2^{s+2} \cdot \alpha^{1/s} \cdot M_s^{2-1/s} \cdot \log^2 n.
\]
The lemma follows by combining Equation (16) with Equation (24).

It remains to establish Condition 2.

**Lemma 17** (Condition 2 holds),

\[
\min \left\{ \frac{n \cdot \alpha}{M_s^{1/3}}, \frac{n^s \cdot \alpha}{M_s}, n^{1-1/s}, \frac{n^{s-1}}{M_s^{1-1/s}} \right\} \geq m \cdot \frac{M_{2s-1}}{M_s^2}.
\]

**Proof:** Since \( G \) has bounded degeneracy \( \alpha \), it follows that \( m \leq n \cdot \alpha \). By Equation (11), \( m \leq n^{1-1/s} \cdot M_s^{1/s} \). As shown in the proof of Lemma 11, \( M_{2s-1} \leq \min\{M_s^{2-1/s}, n^{s-1} \cdot M_s\} \). The proof follows from the above two bounds.

### 5.1 The case of \( s = 1 \): estimating the average degree

When \( s = 1 \) (so that \( M_s = M_1 = 2m \) and \( \mu_s = \mu_1 \) is the average degree), there is a very simple analysis of a slight variant of **Moment-estimator**. Observe that for \( s = 1 \), by Definition 4, for every edge \( e \), \( \text{wt}(e) = 2 \), and \( \text{wt}(v) = 2 \cdot d_v^+ \). For a degree threshold \( \theta = 2\alpha/\varepsilon \), let \( H \triangleq \{ v : d_v > \theta \} \), and \( L \triangleq \mathcal{V} \setminus H \). By the definition of \( H \) we have that \( |H| < M_1/\theta = \varepsilon M_1/(2\alpha) \).

Since the graph has degeneracy at most \( \alpha \), \( \sum_{v \in H} d_v^+ \leq \alpha \cdot |H| \leq \varepsilon M_1/2 \). This implies that \( \sum_{v \in L} \text{wt}(v) \geq (1 - \varepsilon) M_1 \).

Suppose that we modify the algorithm so that \( X_i \) is set to \( d_{v_i}^{s-1} + d_{u_i}^{s-1} = 2 \) only if \( d_{v_i} \leq \theta \) (as well as \( v_i < u_i \)), and is otherwise set to 0. Under this modification, \( \text{Exp}[X] \in [(1 - \varepsilon) M_1, M_1] \).

Since \( \text{wt}(v) \leq 2\theta \) for each \( v \in L \), we get that \( \sum_{v \in L} \text{wt}(v)^2 \leq 2\theta \cdot M_1 = (4\alpha/\varepsilon) \cdot M_1 \). Therefore, in order to satisfy Condition 1, it suffices to set \( \alpha = \frac{c_0}{\varepsilon^2 \cdot s} \).

Thus, as compared to the setting in Equation (12), we save a \( \log^2 n \) factor (at the cost of factor of \( 1/\varepsilon \)), but, more importantly, the analysis is very simple (as compared to the proof of Lemma 13). The setting of \( q \) is as in Equation (13), which for \( s = 1 \) gives \( q = \frac{c_0}{\varepsilon^2 \cdot s} \).

### 6 Wrapping things up

The proof of our final result, Theorem 3, follows by combining Theorem 8, Lemma 10, Lemma 13 and Lemma 17, with a geometric search for a factor-2 estimate of \( M_s \) (which determines the correct setting of \( r \) and \( q \) in the algorithm).

For convenience, we restate the bounds of Theorem 3 in terms on \( M_s \).

\[
O\left(2^s \cdot n \cdot \log^2 n \cdot \left(\frac{\alpha}{M_s^{1/3}}\right)^{1/s} + \min\left\{ \frac{n \cdot \alpha}{M_{s}^{1/3}}, \frac{n^s \cdot \alpha}{M_s}, n^{1-1/s}, \frac{n^{s-1}}{M_{s}^{1-1/s}} \right\} \cdot \frac{s \log n \cdot \log(s \log n)}{\varepsilon^2} \right)
\]

**Proof of Theorem 3:** Recall that the setting of \( r \) and \( q \) in Equation (12) and Equation (13), respectively, equals the setting in Equation (4) when \( \alpha \) is set to its maximum possible value \( M_s^{1/(s+1)} \). Hence, it suffices to prove the theorem under the assumption that the algorithm is provided with \( \alpha \) (which upper bounds \( \alpha(G) \)). It follows from Theorem 8, Lemma 10, Lemma 13 and Lemma 17, that when **Moment-estimator** is invoked with parameters \( r \) and \( q \) as set in Equation (12) and Equation (13), respectively, the algorithm returns a value \( X \) such
that $X \in [(1-\varepsilon)\mu_s,(1+\varepsilon)\mu_s]$ with probability at least $1-\delta$. However, these settings require the knowledge of $M_s$, which is the parameter we are trying to approximate (up to the normalization factor $n$). Hence, we use the following search algorithm.

We start with a guess $\hat{M}_s = n^{s+1}$ (the maximum possible value of $M_s$), and compute $r$ according to Equation (12) and $q$ according to Equation (13) assuming $M_s = \hat{M}_s$, with the given approximation parameter $\varepsilon$ and with $\delta = 1/3$. We then invoke $\text{Moment-estimator}_s$, $\Theta(\log(s \log n))$ times and let $Z$ be the median of the returned values. If $Z \geq \hat{M}_s$ then we stop and return $Z$. Otherwise we halve $\hat{M}_s$ and repeat.

Observe that $r$ and $q$ are decreasing functions of $M_s$. Hence, the running time of each invocation of $\text{Moment-estimator}_s$ is at most the running time of the last invocation. By Claim 6 and Markov’s inequality, for each invocation of $\text{Moment-estimator}_s$, $\Pr[X \geq 3M_s] \leq 1/3$ (where $X$ is the value returned by the algorithm). We stress that this holds regardless of whether $r$ and $q$ satisfy Conditions 1 and 2 (respectively). By the definition of $Z$, for values $\hat{M}_s > 3M_s$, the probability that we will stop in each step is $O(1/(s \log n))$. Therefore, with high constant probability we will not stop before $\hat{M}_s \leq 3M_s$.

Once $\hat{M}_s \leq 3M_s$, we will satisfy Conditions 1 and 2. By Theorem 8, in each invocation of $\text{Moment-estimator}_s$, $X \in [(1-\varepsilon)\mu_s,(1+\varepsilon)\mu_s]$ with probability at least $1-\delta = 2/3$, so that $Z \in [(1-\varepsilon)\mu_s,(1+\varepsilon)\mu_s]$ with probability at least $1 - O(1/(s \log n))$. Thus, once $\hat{M}_s \leq M_s/2$, the algorithm will stop and return a value in $[(1-\varepsilon)\mu_s,(1+\varepsilon)\mu_s]$ with probability at least $1 - O(1/(s \log n))$. By a union bound over all iterations, the algorithm returns such a value with high constant probability.

In order to bound the expected running time, we first observe that the running time of an invocation of $\text{Moment-estimator}_s$, with $\hat{M}_s \in [M_s/2^{i+1}, M_s/2^i)$ is at most $2^{i+1}$ times the running time of an invocation with $\hat{M}_s = M_s$. On the other hand, the probability that the algorithm does not stop before we reach $\hat{M}_s \in [M_s/2^{i+1}, M_s/2^i)$ for any $i \geq 1$, is $O(\log(n)\cdot i)$. The bound on the expected running time follows.

## 7 Lower Bounds for Bounded Degeneracy

The lower bounds given in this section hold for algorithms that are allowed degree and neighbor queries, as well as pair queries (that is, queries of the form: “is there an edge between $u$ and $v$”). These lower bound show that the complexity of the algorithm, as stated in Theorem 3 for graphs with degeneracy at most $\alpha$, is tight up to the dependence on $1/\varepsilon$ and polylogarithmic factors in $n$, for any constant $s$ (or even $s = O(\log \log n)$).

**Theorem 18.** Any constant-factor approximation algorithm for $M_s$ must perform
$$\Omega \left( \frac{n-\alpha(G)^{1/s}}{M_s^{1/s}(G)} \right)$$
queries.

**Proof:** For every $n$, $\tilde{M}_s$ and $\tilde{M}_s/n^s \leq \tilde{\alpha} \leq (\tilde{M}_s/c)^{1/(s+1)}$, we next define two families of graphs: $G_1$ and $G_2$. Each graph in $G_1$ consists of a clique $C_1$, over $\tilde{\alpha}$ vertices, and an independent set $C_2$, over $n - \tilde{\alpha}$ vertices. For each graph in $G_2$, the vertices are partitioned into three sets: $C_1, C_2, C_3$, of sizes $\tilde{\alpha}, (\tilde{M}_s/\tilde{\alpha})^{1/s} - \tilde{\alpha},$ and $n - (\tilde{M}_s/\tilde{\alpha})^{1/s}$, respectively. The set $C_1$ is a clique, and is connected by a complete bipartite graph to $C_2$, where there are no edges within $C_2$. The set $C_3$ is an independent set. Within each family, the graphs differ only

---

3To be precise, in order to satisfy the conditions for values $M_s \leq 3\hat{M}_s$, we invoke $\text{Moment-estimator}_s$ with $3r$ and $3q$.

4Recall that by Claim 12, $\alpha(G) \leq M_s(G)^{1/s}$, and note that $M_s(G) = \sum_v d_v^s \leq \sum_v d_v \cdot n^{s-1} \leq 2\alpha(G)n^s$. 

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in the labeling of the vertices. By construction, in both families, all graphs have degeneracy \( \Theta(\tilde{\alpha}) \). For each \( G \in \mathcal{G}_1 \) we have that \( M_s(G) = O(\tilde{\alpha}^{s+1}) \leq \tilde{M}_s/c \), and for each \( G \in \mathcal{G}_2 \), \( M_s(G) = \Theta(\tilde{\alpha} \cdot (\tilde{M}_s/\tilde{\alpha}) + ((\tilde{M}_s/\tilde{\alpha})^{1/2} - \tilde{\alpha}) \cdot \tilde{\alpha}^s) = \Theta(M_s) \).

Clearly, unless the algorithm “hits” a vertex in the clique \( C_1 \) of a graph belonging to \( \mathcal{G}_1 \) or a vertex in \( C_1 \cup C_2 \) in a graph belonging to \( \mathcal{G}_2 \), it cannot distinguish between a graph selected randomly from \( \mathcal{G}_1 \) and a graph selected randomly from \( \mathcal{G}_2 \). The probability of hitting such a vertex is \( O\left(\left(\frac{M_s/\tilde{\alpha}}{n}\right)^{1/s}\right) \). Thus, in order for this event to occur with high constant probability, 

\[
\Omega\left(\frac{\tilde{M}_s^{1/s}}{M_s}\right) \text{ queries are necessary.}
\]

**Theorem 19.** Any constant-factor approximation algorithm for \( M_s \) must perform

\[
\Omega\left(\min\left\{ \frac{n \cdot \alpha(G)}{M_s(G)^{1/s}}, \frac{n^s \cdot \alpha(G)}{M_s(G)}, \frac{n^{1-1/s} \cdot (n^{1-1/s}/M_s(G))^{1-1/s}}\right\}\right)
\]

queries.

**Proof:** The proof of the theorem is based on simple modifications to the lower bound constructions in [26, Thm. 5]. We note that this theorem of [26] was stated explicitly for algorithms that perform degree and neighbor queries since such was the algorithm presented in [26] (as well as the current paper). However, it was noted in [26, Sec. 7] that they also hold when pair queries are allowed (as they are essentially based on “hitting special vertices”).

The theorem is proved by considering two cases that are defined according to the relation between \( M_s(G) \) and \( n \) (namely, \( M_s(G)^{1/s} \leq n - c \) and \( M_s(G)^{1/s} > n - c \) for a constant \( c \), which determines the hardness of approximation). Within each case there are two sub-cases that are defined according to the relation between \( \alpha(G) \) and \( (M_s(G)/n)^{1/s} \) (namely, \( \alpha(G) < (M_s(G)/n)^{1/s} \) and \( \alpha(G) \geq (M_s(G)/n)^{1/s} \)). Each of the four sub-cases gives us one of the terms in the lower bound (within the \( \min\{\cdot\} \) expression). For each of the sub-cases we consider the values \( \tilde{\alpha} \) and \( \tilde{M}_s \) that correspond to the sub-case, and we construct two families of graphs: \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \). In both families all graphs have degeneracy \( \Theta(\tilde{\alpha}) \). Every graph \( G \in \mathcal{G}_1 \) satisfies \( M_s(G) \leq \tilde{M}_s \), while every graph \( G \in \mathcal{G}_2 \) satisfies \( M_s(G) \geq c \cdot \tilde{M}_s \). The lower bound is based on the difficulty of distinguishing between a random graph selected from \( \mathcal{G}_1 \) and a random graph selected from \( \mathcal{G}_2 \), with a specified number of queries. In all cases we modify the construction in [26, Thm. 5] either by decreasing the degeneracy or increasing it.

**The case \( \tilde{M}_s^{1/s} \leq n - c \).** For this case we modify the construction described in [26, Item (2) of Thm. 5]. In both families, the vertices are partitioned into two subsets, \( S \) and \( V \setminus S \), where the size of \( S \) is \( c \). For each graph in \( \mathcal{G}_1 \), the set \( S \) is an independent set, while for each graph in \( \mathcal{G}_2 \), each vertex in \( S \) has \( d' \) neighbors in \( V \setminus S \), for an appropriate setting of \( d' \). In both families, the vertices in \( V \setminus S \) have degree \( d \), for an appropriate setting of \( d \) (where in \( \mathcal{G}_1 \) their neighbors are all in \( V \setminus S \) while in \( \mathcal{G}_2 \) some of their neighbors are in \( S \)). Observe that the graphs in the family \( \mathcal{G}_1 \) can be viewed as obtained from the graphs of \( \mathcal{G}_2 \) by replacing pairs of edges between \( S \) to \( V \setminus S \) by a single edge in \( V \setminus S \). We refer to the edges between \( S \) and \( V \setminus S \) in the graphs of \( \mathcal{G}_2 \), and the edges replacing them in the graphs of \( \mathcal{G}_1 \) as “special edges”. In [26], the settings of \( d' \) and \( d \) are such that the number of stars in graphs belonging to \( \mathcal{G}_2 \) is (roughly) a factor \( c \) larger than the number of stars in graphs belonging to \( \mathcal{G}_1 \). The difficulty of distinguishing between a random graph selected from \( \mathcal{G}_1 \) and a random graph selected from \( \mathcal{G}_2 \) is based on upper bounding the following two very similar events: (1) “Hitting” a vertex \( v \) in \( S \) when querying a graph in \( \mathcal{G}_2 \) by either performing a degree/neighbor query on \( v \) or by performing a neighbor query on \( u \in V \setminus S \) and receiving \( v \) as an answer. (2) “Hitting” a vertex
The case $\tilde{M}_s^{1/s} > n - c$. In this case we may assume, without loss of generality, that $\tilde{M}_s < n^{s+1}/c'$ for a sufficiently large constant $c'$, or else the lower bound $\Omega(n^{s-1/s}/\tilde{M}_s^{1/s-1/s})$ is trivial, and similarly that $\tilde{M}_s < n^s \cdot \tilde{\alpha}/c'$, or else the lower bound $\Omega(n^s \cdot \tilde{\alpha}/\tilde{M}_s)$ is trivial. We may also assume that $\tilde{\alpha} \leq \tilde{M}_s^{1/(s+1)}$ since $\alpha(G) \leq M_s(G) \cdot (s+1)$ for every graph $G$ (by Claim 12).

Here we modify the construction described in [26, Item (3) of Thm. 5]. The construction is similar to the one described for $\tilde{M}_s^{1/s} \leq n - c$, except that the size of the set $S$ needs to be increased. Specifically, we let $|S| = b$ for $b = [c\tilde{M}_s/n^s]$, and in the graphs in $G_2$ each vertex in $S$ is connected to every other vertex in the graph. Therefore, for each $G \in G_2$, $M_s(G) = \Omega(\tilde{M}_s)$.

If $\tilde{\alpha} < (\tilde{M}_s/n)^{1/s}$, then in both families each vertex in $V \setminus S$ has degree $d = \tilde{\alpha}$. Since $b < \tilde{\alpha}$ (due to $\tilde{M}_s < n^s \cdot \tilde{\alpha}/c'$ for a sufficiently large constant $c'$), we get that $\alpha(G) = \Theta(\tilde{\alpha})$ for all graphs in $G_2$. The difference between the families is that in the graphs belonging to $G_2$, each vertex in $V \setminus S$ has $b = |S|$ neighbors in $S$ and $d - b$ neighbors in $V \setminus S$, while in the graphs belonging to $G_1$, each vertex in $V \setminus S$ has $d$ neighbors in $V \setminus S$ (and each vertex in $S$ only neighbors each other vertex in $S$). This implies that for each $G \in G_1$, $M_s(G) \leq n\tilde{\alpha}^2 < \tilde{M}_s$ (where we have again used $b < \tilde{\alpha}$). Since the number of edges between $S$ and $V \setminus S$ in each graph in $G_2$ (the number of corresponding special edges within $V \setminus S$ in each graph in $G_1$) is $O(b \cdot n) = O(\tilde{M}_s/n^{s-1})$, and the total number of edges is $\Omega(n \cdot \tilde{\alpha})$, we get a lower bound of $\Omega(n^s \cdot \tilde{\alpha}/\tilde{M}_s)$.

If $\tilde{\alpha} \geq (\tilde{M}_s/n)^{1/s}$, then we use the same construction as above only with $d = (\tilde{M}_s/n)^{1/s}$, and we “plant” a clique of size $\tilde{\alpha}$ in $V \setminus S$. The degeneracy is hence increased to $\Theta(\tilde{\alpha})$, and the value of $M_s(G)$ for $G \in G_1$ is increased to at most $2\tilde{M}_s$ (due to the clique). Since the number of edges (not including those in the clique) is $\Omega(n \cdot d) = O(M_s^{1/s} \cdot n^{1-1/s})$, we get a lower bound of $\Omega(n^{s-1/s}/\tilde{M}_s^{1-1/s})$.

7.1 On knowing the degeneracy bound

One may wonder if the query complexity of Theorem 3 can be obtained without knowledge of the degeneracy $\alpha$. The ideal situation would be one where an algorithm has the complexity of Moment-estimator, without knowing $\alpha$. The worst-case lower bound of [26] do not preclude

\[ v \in S \text{ or a special edge between vertices in } V \setminus S \text{ when querying a graph in } G_1, \text{ where the number of special edges is } |S| \cdot d'/2. \] 

In what follows we modify the settings of $d'$ and $d$ (as defined in [26, Item (2) of Thm. 5]), and in the case of large $\alpha$ perform an additional small modification.

In both the sub-case $\tilde{\alpha} < (\tilde{M}_s/n)^{1/s}$ and the sub-case $\tilde{\alpha} \geq (\tilde{M}_s/n)^{1/s}$, we set $d' = \lceil \tilde{M}_s^{1/s} \rceil$. This ensures that for each graph $G \in G_2$, $M_s(G) \geq c\tilde{M}_s$.

If $\tilde{\alpha} < (\tilde{M}_s/n)^{1/s}$, then we set $d = \tilde{\alpha}$. Hence, $\alpha(G) = \Theta(\tilde{\alpha})$ for graphs in both families, and $M_s(G) \leq n \cdot ((\tilde{M}_s/n)^{1/s})^s = \tilde{M}_s$ for graphs in $G_1$. Since the number of edges between $S$ and $V \setminus S$ in graphs belonging to $G_1$ (which is of the same order as the number of special edges in graphs belonging to $G_1$) is $O(\tilde{M}_s^{1/s})$ while the total number of edges is $\Omega(n \cdot \tilde{\alpha})$, we get a lower bound of $\Omega(n \cdot \tilde{\alpha} / \tilde{M}_s^{1/s})$ (this of course requires formalizing, as done in [26]).

If $\tilde{\alpha} \geq (\tilde{M}_s/n)^{1/s}$, then we set $d = \lceil (\tilde{M}_s/n)^{1/s} \rceil$, so that it still holds that $M_s(G) \leq n \cdot ((\tilde{M}_s/n)^{1/s})^s = \tilde{M}_s$ for graphs in $G_1$. In both families, within $V \setminus S$ we add edges so as to form a clique on a subset of size $\tilde{\alpha}$, thus increasing the degeneracy to $\Theta(\tilde{\alpha})$. Since this modification is the same in both families, it does not affect the ability to distinguish between the two families. Since the number of edges (not including those in the clique) is $(n - c) \cdot d = \Omega(\tilde{M}_s^{1/s} \cdot n^{1-1/s})$, we get a lower bound of $\Omega(n^{1-1/s})$. 

The worst-case lower bound of $\tilde{M}_s^{1/s}$ when $\tilde{\alpha} < (\tilde{M}_s/n)^{1/s}$ is $\Omega(M_s^{1/s} \cdot n^{1-1/s})$. Since the number of edges (not including those in the clique) is $\Omega(n \cdot \tilde{\alpha})$, we get a lower bound of $\Omega(M_s^{1/s} \cdot n^{1-1/s})$. 

The degeneracy bound is increased to $\Theta(\tilde{\alpha})$, and the value of $M_s(G)$ for $G \in G_1$ is increased to at most $2\tilde{M}_s$ (due to the clique). Since the number of edges (not including those in the clique) is $\Omega(n \cdot d) = O(\tilde{M}_s^{1/s} \cdot n^{1-1/s})$, we get a lower bound of $\Omega(n^{s-1/s}/\tilde{M}_s^{1-1/s})$. 

The case $\tilde{M}_s^{1/s} > n - c$. In this case we may assume, without loss of generality, that $\tilde{M}_s < n^{s+1}/c'$ for a sufficiently large constant $c'$, or else the lower bound $\Omega(n^{s-1/s}/\tilde{M}_s^{1/s-1/s})$ is trivial, and similarly that $\tilde{M}_s < n^s \cdot \tilde{\alpha}/c'$, or else the lower bound $\Omega(n^s \cdot \tilde{\alpha}/\tilde{M}_s)$ is trivial.
this possibility, since it uses graphs with high degeneracy. Nonetheless, a slight adaptation of
those arguments shows that the bound holds even for bounded degeneracy graphs, if the
algorithm must work on all graphs. We will focus solely on estimating average degree, since
that suffices to make our point.

**Definition 20.** For a constant c, an algorithm A is called c-valid if: given query access to a
graph G, with probability 2/3, A outputs a c-approximation to the average degree of G.

Note that Moment-estimator, with access to a degeneracy bound, is not valid. This is
because it is only required to be accurate for graphs with the given degeneracy bound, not all
graphs.

**Theorem 21.** Let n be a sufficiently large integer. Consider the class of graphs on n vertices
with degeneracy at most 2. For any constant c, any c-valid algorithm must perform \( \Omega(\sqrt{n}) \)
queries on these graphs.

**Proof:** Similarly to previous lower-bound proofs, for each sufficiently large n, we define two
distributions over labeled graphs. Consider a graph consisting of two connected components,
a cycle on \( \lfloor n - 4c\sqrt{n} \rfloor \) vertices and a cycle on \( \lceil 4c\sqrt{n} \rceil \) vertices (where c is the constant in the
statement of the theorem). The distribution \( \mathcal{G}_1 \) is generated by labeling the vertices using a
uniform random permutation in \([n]\). For the second distribution, take the graph that consists of
a cycle on \( \lfloor n - 4c\sqrt{n} \rfloor \) vertices and a clique on \( \lceil 4c\sqrt{n} \rceil \) vertices. The distribution \( \mathcal{G}_2 \) is generated
by labeling the vertices according to a uniform random permutation.

Consider any (possibly randomized) algorithm for deciding, given query access either to
a graph generated by \( \mathcal{G}_1 \) or to a graph generated by \( \mathcal{G}_2 \), according to which distribution was
the graph generated. As long as the algorithm does not perform a query on a vertex that
belongs to the small cycle (if the graph is generated by \( \mathcal{G}_1 \)) or the small clique (if the graph is
generated by \( \mathcal{G}_2 \)), answers to its queries are identically distributed under the two distribution.
This implies that any such decision algorithm must perform \( \Omega(\sqrt{n}) \) queries in order to succeed
with probability at least 2/3.

Now consider a c-valid algorithm \( \mathcal{A} \) that makes at most s queries on any graph with n
vertices and degeneracy at most 2. We use this algorithm in order to construct an algorithm
\( \mathcal{B} \) that distinguishes \( \mathcal{G}_1 \) from \( \mathcal{G}_2 \) in \( O(s) \) queries. It works as follows. Given query access to
\( \mathcal{G}_1 \), \( \mathcal{B} \) runs 20 independent runs of \( \mathcal{A} \). If any run makes more than s queries, \( \mathcal{B} \) terminates and
outputs “\( \mathcal{G}_2 \)”. At the end of the runs, all of them have provided some estimate for the average
degree. If the median is at most 2c, then \( \mathcal{B} \) outputs “\( \mathcal{G}_1 \)”. Otherwise, it outputs “\( \mathcal{G}_2 \)”. 

Suppose \( \mathcal{G} \sim \mathcal{G}_1 \), and recall that the average degree of \( \mathcal{G} \) is exactly 2. All runs are guaranteed
to make at most s queries. Thus, all of them will output an estimate. By the validity of \( \mathcal{A} \) and
a Chernoff bound, the median estimate will be at most 2c with probability at least 2/3, and
\( \mathcal{B} \) gives a correct output. Now suppose that \( \mathcal{G} \sim \mathcal{G}_2 \), and recall that The average degree of \( \mathcal{G} \)
is at least 3c^2. If any run takes more than s queries, \( \mathcal{B} \) outputs correctly. Otherwise, by
a similar argument to the one made for \( \mathcal{G} \sim \mathcal{G}_1 \), the median estimate will be at least 3c with
probability at least 2/3. Thus, \( \mathcal{B} \) is correct.

The query complexity of \( \mathcal{B} \) is \( O(s) \) and it distinguishes \( \mathcal{G}_1 \) from \( \mathcal{G}_2 \) with probability at least
2/3. Therefore, s must be \( \Omega(\sqrt{n}) \).

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