Appendix A: Model parameters used in simulations, definitions of quantities in figures

For convenient reference, this Appendix includes the parameters for all simulations. The model is given in Eq. (1). All $B_{i,u} = 1$ and all $D_{i,uv} = d/ (M - 1)$. The $A_{ij,u}$ are independent for different $(i,j)$ pairs (except in Appendix G).

In Fig. 2, the probability of $A_{ij,u}$ to be non-zero is $c = 1/8$, and the non-zero elements are sampled from a normal distribution with mean $A_{ij,u} = 0.3$, std $A_{ij,u} = 0.45$. The same elements $A_{ij,u}$ are non-zero across all patches $u$. The correlation coefficient between non-zero $A_{ij,u}$ in different patches is $\rho = \text{corr} [A_{ij,u}, A_{ij,v}] = 0.95$ for $u \neq v$. (The correlation is 0.964 when interactions with $A_{ij,u} = 0$ are also counted.) The initial (pool) diversity is $S = 250$. In Fig. 2(A), $M = 1$. In Fig. 2(B), $M = 8$ patches and $d = 10^{-3}$. The cutoff is $N_c = 10^{-15}$. For each $i,j$, the $M$ values $A_{ij,u=1..M}$ are drawn simultaneously from a multi-variate normal distribution with correlation matrix $C_{uv} = \rho + (1 - \rho) \delta_{uv}$, using standard numerical methods (e.g., as implemented in the Matlab function mvnrnd).

Fig. 3(bottom), uses the same parameters as Fig. 2, but with a range of values for $d$, $S$ and $N_c$.

Fig. 4 uses the runs shown in Fig. 2(B). Standard deviation and mean are estimated from 1600 time points during the time period $t = [10^4, 2 \cdot 10^5]$. Fig. 6 uses multiple runs, with the same parameters as 2, except for $d = 10^{-4}$ and the values of $N_c$ that are detailed in the figure legend.

Fig. 5(a) uses the same parameters as 6, and $N_c = 10^{-15}$. Fig. 5(c), shows the line where half of the runs are fixed points, and half continue to fluctuate until $t = 2 \cdot 10^5$. It uses same parameters as 6, except with $D = d/ (M - 1) = 10^{-4}$.

In Fig. 5(a), the size of the fluctuations are calculated from $\text{var} (\xi_u) = \langle \xi_u^2 (t) \rangle = \sigma^2 C_{N_u} (t,t)$, with $C_{N_u} (t,t) = (N_u^2 (t)) - \text{lim}_{t \to \infty} \langle N_u(t) N_u(t') \rangle$. For more details on the averaging, see Appendix B, Fig. A.

Fig. 5(a) shows the strength of noise at different diversities. Extinctions beyond the time shown in simulations ($t = 2 \cdot 10^5$) take extremely long times to happen, so reaching these lower diversities in simulations is unfeasible. Instead, we remove species that are most likely to go extinct. Recalling that the time to extinction is $\tau (1/N_c)^{2M N_{\text{eff}}/W}$, we remove species with the lowest $N_{\text{eff}}$. This is done by running the system for time $\Delta t = 15 \cdot 10^3$, calculating $N_{\text{eff}}$ as for Fig. 6, and removing the 5 species with the lowest values of $N_{\text{eff}}$. This process is repeated. Other protocols for species removal where attempted, such as increasing $N_c$ in time; they give similar results. The results are averaged over 3 runs, with independent sampling of interactions and initial conditions.

Fig. 7 shows the linear stability $\lambda_{\text{stab}}$ of fixed points. To calculate this, the equations of motion, Eq. (1), are linearized around a fixed point $N^*_u$. This gives the matrix equation $d \vec{\theta}/dt = \mathbf{G} \vec{\theta}$, where $\vec{\theta}$ is a vector with one entry for each patch of a surviving species, with $n_{i,u} = N_{i,u} - N^*_{i,u}$. $\mathbf{G}$ is a square matrix, which depends on the equilibrium abundances $N^*_{i,u}$. For a state with fluctuating abundances, we use the time average of $N_{i,u} (t)$ for $N^*_{i,u}$. Let $\{ \lambda_j \}$ be all the eigenvalues of $\mathbf{G}$. Then $\lambda_{\text{stab}} = \max_j | \Re (\lambda_j) |$, where $\Re (\cdot)$ denotes the real part. Note that a fixed point is stable if $\lambda_{\text{stab}} < 0$. The results are averaged over the same 3 runs as in Fig. 5(a).

Appendix B: DMFT equations

In this section, we present the full DMFT equations, and explain how they can be reduced to the steady-state equations quoted in the main text.

We consider as a starting point equation Eq. (1). For the sake of clarity, we derive DMFT under simplifying assumptions, but the result is much more robust and could be applied to different ecology models as well as real data [1]. DMFT for ecological models has a double valency analogous to the one of mean-field theories in physics: it is at the same time an exact theory for some simple models, and a powerful approximation largely applicable to a broad range of systems. For the sake of clarity, the derivation assumes a fully connected model (all interactions are non-zero), but the results hold for any connectivity $C$ as long as $C \gg 1$, see remark at the end of this Appendix.

The assumptions which make DMFT exact are the following: all constants $N_{i,u}(0), B_{i,u}, D_{i,uv}$ and $A_{ij,u}$ are random variables, sampled from known distributions. More precisely:

- In each patch $u$ and for all species $i$, the parameters $X^u = \{ N_{i,u}(0), B_{i,u}, D_{i,uv} \}_{i=1}^M$ are drawn from a probability distribution $P$ which is a product measure $\mathbb{P}_u (X^u) = \prod_{i=1}^M \mathbb{P}(X^u_i)$;

- The interaction matrix can be decomposed as $A_{ij,u} = \mu/S + \sigma/\sqrt{S} a_{ij,u}$. $a_{ij,u}$ are standard random variables with mean zero, variance one, and correlation:

$$E [a_{ij,u} a_{kl,v}] = \delta_{ik} \delta_{jl} \rho_{uv}$$

where we used the Kronecker symbol $\delta_{ik}$, and $\rho_{uv} = \rho + (1 - \rho) \delta_{uv}$ is a uniform correlation $\rho$ between patches.
With these conventions, we rewrite Eq. (1) in the following way:

\[
\frac{d}{dt} N_{i,u} = N_{i,u} [B_{i,u} - N_{i,u} - \mu m_u(t) + \eta_{i,u}(t)] \\
+ \sum_v D_{i,uv} (N_v - N_{i,u})
\]

where \( m_u(t) = S^{-1} \sum_i S N_{i,u}(t) \) is the mean abundance in patch \( u \), and \( \eta_{i,u}(t) = -\sigma S^{-1/2} \sum_j a_{ij} N_{j,u}(t) \) is a species-and-patch-dependent noise.

The DMFT equation can be obtained by following Ref. [2]; in the large-\( S \) limit, it can be shown that the statistics of this multi-species deterministic process corresponds to the following one-species stochastic process, for each patch.

\[
\frac{d}{dt} N_u = N_u [B_u - N_u - \mu m_u(t) + \eta_u(t)] \\
+ \sum_v D_{uv} (N_v - N_u)
\]  

(B1)

where \( \{ N_u(0), B_u, D_{uv} \} \) are sampled from the distribution \( \mathbb{P}(X^u) \), \( m_u(t) \) is a deterministic function, and \( \eta_u(t) \) is a zero-mean Gaussian noise. The variability from one species to another becomes in the DMFT setting the randomness contained in \( \{ N_u(0), B_u, D_{uv} \} \) and \( \eta_u(t) \).

To make this point crystal clear, let us introduce two different averages:

- \( \mathbb{Y} \) averages over the stochastic process in Eq. (B1): over the stochastic noise \( \eta_u \) and over the distribution \( \mathbb{P}(X^u) \);

- \( \mathbb{E}_S(Y) \) denotes the statistical average over the deterministic multi-species system. \( \mathbb{E}_S(Y) = \sum_i S Y_i \), and therefore also includes sampling of \( X^i \).

DMFT represents in terms of a stochastic process the deterministic dynamical system governing the dynamics of the \( S \) species in the ecosystem. In consequence, averages over the stochastic process coincide with average over species [2–4]: for a given observable \( Y \): \( \mathbb{Y} = \lim_{S \to \infty} \mathbb{E}_S(Y) \). This is analogous to the representation of the environment of an open physical system in terms of thermal noise, as it is done e.g. in the case of the Langevin equation.

The second important aspect of DMFT is self-consistency. This is related to the fact that the noise is induced by the dynamics of the species themselves, so its properties can be obtained from dynamical averages:

\[
\begin{align*}
\langle m_u(t) \rangle &= \mathbb{Y} N_u(t) \\
\langle \eta_u(t) \eta_u(t') \rangle &= \sigma^2 \rho_{uv} \mathbb{Y} N_u(t) N_v(t')
\end{align*}
\]

where we used a last average \( \mathbb{Y} \) over the stochastic noise only, in order to define its covariance. Henceforth we use the notation \( C_{uv}^N(t, t') = \rho_{uv} \mathbb{Y} N_u(t) N_v(t') \). These relations exactly take into account the correlations that emerge between the abundances and the interactions.

We now show how DMFT equations simplify for a time-translationally-invariant state of the system, which is in general reached after some transient time. In this state, all one-time observables become constant in time, and two-time observables become functions of the time difference only.

\[
\begin{align*}
\langle z_u z_v \rangle &= \sigma^2 \lim_{t - t' \to \infty} C_{uv}^N(t, t') \\
\langle z_u \xi_{u'} \rangle &= \sigma^2 \lim_{t - t' \to \infty} \mathbb{Y} C_{uv}^N(t, t') \\
\langle \xi_u \xi_{u'} \rangle &= \delta_{uv} \sigma^2 \left[ C_{uv}^N(t, t') - \lim_{t' \to \infty} C_{uv}^N(t, t') \right]
\end{align*}
\]

As explained above, DMFT can be implemented as an approximation for a large variety of systems. In this case one has to infer the average \( \mu \), the standard deviation \( \sigma \) of interactions, and the distribution \( \mathbb{P}(X^u) \) from the data (we remind that \( X^u = \{ N_u(0), B_u, D_{uv} \} \)) and use them as an input to define an effective model. The generalization to patch-dependent cumulants \( \mu_u \) and \( \sigma_u \) is
The abundance covariance within a patch and \( Y(t' = 0) \) is very small. Right: The covariance in flat. In other words, the correlation of abundances across patches is mainly static: dotted lines are reasonably flat. In other words, the covariance abundance within a patch \( u = v \), and across patches \( u \neq v \) in dotted lines. The correlation in abundances across patches is mainly static: dotted lines are reasonably flat.

In addition, the paper focuses on the case where migration connects all patches to one another. But the basic DMFT framework, Eq. (B1), is valid even if only certain patches are connected, and migration is zero otherwise.

### Appendix C: Extinction probability of a species

Here the probability of extinction of a species is presented, at the limit \( N_e \ll D \ll 1 \). More specifically, we assume that \( N_e \) is small compared to the typical fluctuations of the abundances. In addition, simulations we see that it is reasonable to assume complete lack of synchrony, namely that the noise \( \xi_u \) is uncorrelated between different patches, see Appendix B, Fig. A. We will therefore assume that in the following calculation. Finally, we assume that for at least one patch, \( N^*_u > 0 \), otherwise the species goes quickly extinct.

Within DMFT, the problem thus becomes one of calculating the extinction probability of a meta-population (single species), under environmental fluctuations, that are uncorrelated between the different patches. We only present the result here; a full account will be given elsewhere.

Let \( x_u = \ln N_u \). The equations of the DMFT, Eq. (B1), become

\[
\partial_t x_u = N^*_u e^{x_u} - \sigma_x u + D \sum_{v} (e^{x_v - x_u} - 1) .
\]

Here \( D = d/(M - 1) \). We look for a rare realization of \{\( \xi_u \)\} that makes all the \( x_u \) reach \( x_c = \ln N_e \), in the case when the cut-off is low, \( x_c \to -\infty \). The calculation proceeds within the framework of large-deviation theory [6]. First, one defines the “action”

\[
J = \frac{1}{2W} \int_{t' = 0}^{t_f} dt \sum_{u=1}^{M} \xi_u^2 ,
\]

with \( \xi_u \) substituted with its value from Eq. (C1), and \( W \) defined as in the main text. Here we approximated the noise correlations by white noise, which is justified here as the extinction event takes a time which is long compared to the correlation time. We assume that \( D \) is small.

Then the mean time to the occurrence of such an event scales as \( P \sim e^{-J_{\text{min}}} \) with \( J_{\text{min}} \) the action minimized over all population trajectories \( \{x_u(t)\}_{t=1,M} \) that start at \( t = -\infty \) at the typical value of \( x_u \), obtained by the zero-noise fixed point of Eq. (C1), and terminate at \( t_f \) at \( x_c = \ln N_e \).

We first describe the result for \( M = 1 \). In this case there is only one patch, \( u = 1 \), with \( N^*_1 \). If \( N^*_1 < 0 \) the species is extinct. On the other hand, if \( N^*_1 > 0 \), then we obtain the known result [7, 8]

\[
J_{\text{min}} = \frac{2e_c}{W} N^*_1 .
\]

The result for all \( M \) is a generalization of this result, of the form

\[
J_{\text{min}} = \frac{2e_c}{W} MN^*_{\text{eff}} .
\]
To describe the calculation of $N_{\text{eff}}^*$, order the patches so that $N_1^* \geq N_2^* \geq \ldots \geq N_M^*$. Then there exists $1 \leq m \leq M$ such that

$$w \equiv -\sqrt{\frac{1}{m} \sum_{u=1}^{m} (N_u^* - w)^2 / w},$$

and where $w$ satisfies: $w \leq N_u^*$ for all $u \leq m$, and $w > N_u^*$ for all $u > m$. Such a partition can be shown to always exist. Then

$$N_{\text{eff}}^* = \frac{1}{4M} \sum_{\{u\}_+} (N_u^* - w)^2 / w.$$

The derivation will be given elsewhere. We illustrate the result by considering two cases. First, in the $M = 1$ example, since $N_1^* > 0$, the partition is $\{u\}_+ = \{1\}$ and $\{u\}_- =$ the empty set. Indeed, this gives $w = -N_1^*$, so $w \leq N_1^*$. Then $N_{\text{eff}}^* = N_1^*$ and $J_{\text{min}} = 2N_1^* = 2N_1^* N_{\text{eff}}^*$, so the result for $M = 1$ is reproduced. Another simple case is when there are $M$ patches with identical carrying capacities $N_u = N^*$. Here $\{u\}_+ = \{1, \ldots, M\}$ and $w = -N^*$. Then $N_{\text{eff}}^* = N^*$, and $J_{\text{min}} = 2M N_{\text{eff}}^* = 2M N^*$. This result is intuitively clear: to go extinct, the species must go extinct in all patches at once, so the probability is $P \sim \exp \left( \frac{1}{M} M N^* \right) \sim (P_1)^M$, where $P_1$ is the result for $M = 1$.

Appendix D: Diversity and stability at low migration rates

We use notations from Appendix B. Within the time-translational-invariant state:

$$\frac{1}{N_u} \frac{dN_u}{dt} = N_u^* - N_u + \xi_u(t) + \sum_{v \sim u} D_{uv} \left( \frac{N_v^*}{N_u^*} - 1 \right).$$

Consider the case of low migration, $D \to 0^+$. We now develop a theory assuming that the amplitude of the endogenous fluctuations,

$$W \equiv \int dt C_\xi(t,t'),$$

remains finite in the limit $D \to 0^+$. Assume the species survives, i.e. there is at least one patch with $N_u^* > 0$. If $N_u^* < 0$ then $N_u = O(D)$. If $N_u^* > 0$ then $N_u = O(1)$ and therefore $\sum_{v \sim u} D_{uv} \left( \frac{N_v^*}{N_u^*} - 1 \right) = O(D)$. Taking the time average of the above equation

$$0 = \frac{1}{N_u} \frac{dN_u}{dt} = N_u^* - N_u + O(D)$$

and therefore $\overline{N_u} = N_u^* + O(D)$.

The previous arguments lead to the conclusion that in the $D \to 0^+$ limit $\overline{N_u} = N_u^*$ if $N_u^* > 0$ and is equal to zero otherwise. In the following we provide more detail more this argument and its possible limitations. For this last equality to be valid, we need that $\sum_{v \sim u} \left( \frac{N_v^*}{N_u^*} - 1 \right)$ will be finite, so that $D \sum_{v \sim u} \left( \frac{N_v^*}{N_u^*} - 1 \right)$ will indeed be small. This might break if $N_u$ can be small while some other $N_v$ remains $O(1)$. An estimate for that proceeds by noting that the carrying capacity of patch $u$ in the presence of other patches is larger or equal to $N_u^* - MD \approx N_u^*$, its carrying capacity alone. If patch $u$ fluctuates alone, then

$$\frac{dN_u}{dt} = N_u^* + \xi(t) \Rightarrow P(x) \sim e^{\frac{2N_u^*}{W}}$$

This gives for $1/N_u$

$$e^{-x_u} \sim \int_0^\infty e^{\frac{W}{W_0} (N_u^* - 1)} dx = N_u^* + 1 + \frac{W}{W_0} = 1 + \frac{W}{N_u^*}.$$

For any given $N_u^*$ this is finite. It diverges as $N_u^* \to 0$. Therefore the migration term is negligible only if $\frac{2DW}{D} \ll N_u^*$. (Note that migration itself would limit $N_u$ going below much below $DN_u$, which would make this term smaller.) The main approximation (or limitation) of our approach is the assumption that $W$ remains finite in the small $D$ limit. This is shown to hold in simulations presented in Appendix B. It breaks down if the noise develops long-lasting correlations in time. Our approximation will be nevertheless good for large $|N_u^*|$ and for weak endogenous fluctuations.

We now used the relationship discussed above between $\overline{N_u}$ and $N_u^*$ to determine the statistics of $N_u^*$. We shall use the term “source” for patches where $N_u^* > 0$, and “sink” otherwise. In order to understand the correlation between the sources in the different communities, we unpack $N_u^*$ using Appendix B. Taking the time-average is equivalent to averaging over the dynamical noise $\xi$. Therefore, in patch $u$ for species $i$, $z_{i,u} = -\sigma S^{-1/2} \sum_j a_{ij,u} N_{j,u}^* = -\sigma S^{-1/2} \sum_j + a_{ij,u} N_{j,u}^*$. The sum $\sum_j$ means that we only sum over $N_{j,u}^* > 0$. Here, we recall that $a_{ij,u}$ are standard random variables with mean zero, variance one, and correlation between patches:

$$\mathbb{E}[a_{ij,u} a_{kl,v}] = \delta_{ik} \delta_{jl} \rho_{uv},$$

where we used the Kronecker symbol $\delta_{ik}$.

Therefore:

$$N_{i,u}^* = 1 - \mu m_u - \sigma S^{-1/2} \sum_{j,+} a_{ij,u} N_{j,u}^* \quad \text{(D1)}$$

1 The term “source” is used here so as to include patches (sometimes referred to as pseudo-sinks) where a species might still receive migration from patches with even larger $N_u^*$. But the contribution of this migration is small and not required for its persistence.
where we recall $m_u = \langle N^*_{i,u} \rangle = \langle N^*_{i,u} \rangle_+$. We can now compute the different moments of the multivariate Gaussian random variable $N^*_u$, using equation (D1). We obtain the closure:

\[
\begin{align*}
\text{mean} [N^*_u] &= 1 - \mu \langle N^*_{i,u} \rangle_+ \\
\text{covariance} [N^*_u, N^*_v] &= \sigma^2 \rho_{uv} \langle N^*_{i,u}N^*_{i,v} \rangle_+
\end{align*}
\]

When $u = v$, as $\rho_{uu} = 1$, we find the expected single community result. In particular, mean $[N^*_u]$ and variance $[N^*_u]$ do not depend on the patch $u$.

We numerically solve the closure in a self-consistent way: start with a guess for $\langle N^*_{i,u}N^*_{i,v} \rangle_+$ and then (1) produce many samples of the vector $N^*_u=1..M$ and (2) calculate the next estimate for $\langle N^*_{i,u}N^*_{i,v} \rangle_+$, by averaging only over $N^*_u$ and $N^*_v$, that are both positive. For stability of this numerical scheme, we only replace half the samples at each iteration. We use $10^5$ samples and 1000 iterations. The algorithm is always found to converge to the same solution.

Given covariance $[N^*_u, N^*_v]$, the distribution of $N^*_i$ is completely specified: it is a multivariate Gaussian in $u$, has the single-patch statistics of a single community, and a known covariance between patches. The solution can then also give the distribution of the number of sourcing patches.

In addition, we can compute the correlation coefficient $\rho_{N,i}$ of the $N^*_u$’s. We use here our simple case of a uniform correlation $\rho_{a}$ between patches $\rho_{uv} = \rho_{a} + (1 - \rho_{a})\delta_{uv}$. We introduce the notation $\rho_{a}$ instead of ‘$\rho$’ in this section in order to avoid confusion with $\rho_{N,i}$.

$$
\rho_{N,i} \equiv \frac{\text{covariance} [N^*_u, N^*_v]}{\text{variance} [N^*_u]} = \rho_{a} \frac{\langle N^*_{i,u}N^*_{i,v} \rangle_+}{\langle N^*_i \rangle^2_+}
$$

The results are surprising: even when $\rho_{a} \to 1$, the overlap between communities is not perfect ($\rho_{N,i} < 1$), so the total diversity is larger than the one in each patch. This happens exactly at the transition to chaos at $\sigma_{c} = \sqrt{2}$, see Fig. B.

On Fig. D, we compare the theory predictions to simulations. In terms of diversity, the theory appears to give an upper bound to the simulations. The difference becomes larger at higher values of $\sigma_c$ and for $\rho_{a}$ closer to one. To look further into this difference, it is useful to study diversity as a function of the value of $N^*_c$ of each species. As shown in Fig. 6 in the main text, most of the difference in diversity is due to low values of $N^*_c$, which are precisely the species that are more likely to go extinct, with good agreement with theory at higher values of $N^*_c$. This is demonstrated in Fig. C, which shows that the theoretical prediction for the number of species with $N^*_c > 0.2$ is closer to simulation results than the predictions for total diversity. At the moment we do not know if remaining differences are because the theoretical argument is only approximate, or whether principle, with exceedingly low values of $N^*_c$ and $D$, it could be approached by simulations for any $\sigma_c$.

To find the boundary of parameter space where fixed points lose their stability and the system becomes chaotic, we look at the linear stability of persistent species. When $\sigma$ is small, the species that are not sourced in each patch do not affect the stability, and so the question simplifies to single patch stability, which when $\text{corr} [A_{ij}, A_{ij}] = 0$, results in $\sigma_{c} = \sqrt{2}$ and with $1/2$ of the species being sourced in each patch [9].

![Figure B](image1.png) Theoretical predictions for the diversity as a function of $\sigma$ for $M = 1, 8$ patches, $\rho_{a} = 0, 0.5, 0.95$ and $\rho_{a} \to 1$.

![Figure C](image2.png) The fraction of persistent species $S^*/S$ (circles) is compared to theoretical bound (blue dashed line), for different values of $N^*_c$. Also shown is the fraction of species above $N^*_c > 0.2$, compared to the theoretical bound for that (red dotted line), showing better agreement than for the full diversity. Simulations use the same parameters as in Fig. 6, but with a range of values for $N^*_c$ (Fig. 6 corresponds to the points at $N^*_c = 10^{-15}$).
Appendix E: Single patch ($M = 1$)

Here we show that in principle a single patch can reach and maintain a dynamically fluctuating state. However, this requires prohibitively large $S$, not attainable in practice. In Fig. E and Fig. F we show results of a numerical solution [2] to the DMFT equations detailed in Appendix B. At extremely low values of $N_c$ the system appears to reach a final diversity above the May bound and, hence, to be chaotic. DMFT however describes the behavior in the $S \gg 1$ limit. When full simulations of the model in Eq. (1) are carried out at finite $S$, they diversity falls somewhat below the DMFT final diversity, leading to a fixed point, rather than a chaotic state, see Fig. F.
Figure E. DMFT numerics for a single patch, \( M = 1 \), showing that chaos is in principle possible here, although for unrealistic values of model parameters. (A) The fraction of species above different values of \( N_0 \), \( P(N > N_0) \) is plotted as a function of time, for different values of \( N_0 \). (B) The curves for different \( N_0 \) collapse when \( P(N > N_0) - \phi_\infty(N_0) \sim |\ln N_0|/t \). Here \( \phi_\infty(N_0) \) is a fitted parameter, the extrapolated value of \( P(N > N_0) \) at long times. (C) The values of \( \phi_\infty(N_0) \) are well above the linear stability bound ("May bound"), and at (very) low \( N_0 \) come quite close to the theoretical maximal value for \( \phi_\infty(N_0) \), predicted in Appendix D. Here \( \sigma = 2, \mu = 10, N_c = 10^{-120} \).

Figure F. The DMFT solution and the simulations only agree up to times \( t \sim 10^2 \), after which the diversity in the simulations reduces more rapidly and reaches a fixed point. This means that the convergence to the DMFT solution is slow with \( S \).

Finite-size correction to the DMFT result are important since they show that maintaining a dynamically fluctuating state for realistic values of \( S \) is not possible for \( M = 1 \).

Appendix F: Correlations of interactions in a pair of species

In the main text we assumed that \( A_{ij,u} \) is sampled independently from \( A_{ji,u} \). Here we show that the long-lived endogenous fluctuations can be found even if this assumption is relaxed. For this purpose, we consider a symmetric network of non-zero \( A_{ij,u} \) specifically \( A_{ij,u} \neq 0 \) if and only if \( A_{ji,u} \). We define \( \gamma \) the correlation of the non-zero elements \( \gamma = \text{corr}[A_{ij,u}, A_{ji,u}]_{A_{ij,u}} \). Fig. G shows two simulations, one with \( \gamma > 0 \) and the other with \( \gamma < 0 \). In both cases the system relaxes to a long-lived state with fluctuating abundances, without further loss of diversity up to time \( 2 \cdot 10^8 \). They are intended solely to demonstrate that conditions with \( \gamma \neq 0 \) exist, rather than a systematic exploration of such cases.

The parameters for the simulations (using the notation of Appendix A) are the following:

- Run with positive \( \gamma \): \( \gamma = 1/4, S = 350, \text{mean}(A_{ij,u}) = 0.075, \text{std}(A_{ij,u}) = 0.175, c = 0.357, M = 8, d = 10^{-3}, \rho = 0, N_c = 10^{-15} \).
- Run with negative \( \gamma \): \( \gamma = -1/2, S = 250, \text{mean}(A_{ij,u}) = 0.075, \text{std}(A_{ij,u}) = 0.358, c = 0.5, M = 8, d = 10^{-3}, \rho = 0, N_c = 10^{-15} \).

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