Linear perturbations of the Linet - Tian metrics with a positive cosmological constant.

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The Linet - Tian metrics are solutions of the Einstein equations with a cosmological constant, Λ, that can be positive or negative. The linear instability of these metrics in the case Λ < 0, has already been established. In the case Λ > 0, it was found in a recent analysis that the perturbation equations admit unstable modes. The analysis was based on the construction of a gauge invariant function of the metric perturbation coefficients, called here W(y). This function satisfied a linear second order equation that could be used to set up a boundary value problem determining the allowed, real or purely imaginary frequencies for the perturbations. Nevertheless, the relation of these solutions to the full spectrum of perturbations, and, therefore, to the evolution of arbitrary perturbations, remained open. In this paper we consider again the perturbations of the Linet - Tian metric with Λ > 0, and show, using a form of the Darboux transformation, that one can associate with the perturbation equations a self adjoint problem that provides a solution to the completeness and spectrum of the perturbations. This is also used to construct the explicit relation between the solutions of the gauge invariant equation for W(y), and the evolution of arbitrary initial data, thus solving the problem that remained open in the previous study. Numerical methods are then used to confirm the existence of unstable modes as a part of the complete spectrum of the perturbations, thus establishing the linear gravitational instability of the Linet - Tian metrics with Λ > 0.

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I. INTRODUCTION

The Linet - Tian metrics [1], [2], are static solutions of the Einstein equations with a cosmological constant, Λ, that can be positive or negative. They posses also two other commuting Killing vectors: ∂_φ, and ∂_z. The metrics are characterized by two constants: one is κ, associated to the singularities of the metrics, and the other is the cosmological constant Λ [3]. In the limit of vanishing cosmological constant they reduce to a form of the Levi- Civita metric [4], and, therefore, they can be considered as generalizations of the former to include a cosmological constant. We refer to [5] for a recent review and bibliography on these and related types of metrics. Both the Levi - Civita metric, and the Linet - Tian solution with negative cosmological constant, have been found to be gravitationally unstable, [6], [7]. The positive cosmological constant case was analyzed in [8], where it was found that (linear) perturbations that break the symmetry associated to ∂_z (or ∂_φ) contain unstable

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modes. However, because of certain peculiar aspects of the perturbation equations, the relation between those modes and the general evolution of arbitrary perturbations remained unclear. The purpose of this paper is to review and extend the results obtained in [8], and to provide the proof that the existence of unstable modes indeed implies that the positive Λ Linet - Tian space times are unstable.

The plan of the paper is as follows. In the next Section we consider a particular form of the Linet - Tian metric that is suitable for the analysis of the perturbation problem, and review and discuss some of its properties, relevant for the present discussion. In Section III, we consider the general linear perturbation of the Linet - Tian metric, taking into account the existence of its Killing vectors. In the present analysis we restrict the perturbations to the “diagonal” case considered in [8], and review the problems that the presence of gauge ambiguities pose for setting up a meaningful perturbation analysis of the linear stability of the metric. Some details regarding gauge transformations are included for completeness in Appendix D. The problem regarding the gauge ambiguities is solved in Section IV, where we introduce a gauge invariant formulation, directly related to a gauge invariant function ($W(y)$) of the perturbation functions. This not only sidesteps the gauge ambiguities, but, as shown, $W(y)$ provides a “master function”, from which one may compute the metric perturbation coefficients. This, in turn, makes clear the reasons why a formulation directly in terms of the metric perturbation functions will always be subject to gauge ambiguities. Given this situation, it seems appropriate to look directly for the equation satisfied by $W(y)$. This equation, that is derived in Section V, has the form of a linear boundary value problem, with eigenvalues $Ω^2$, where $Ω$ is the (real or pure imaginary) frequency of the resulting perturbation modes. Using a well known procedure, we obtain a related Schrödinger like equation, but with a singular “potential”. The singularity in the “potential” is related to singularities in the coefficients of the equation satisfied by $W(y)$, although, as is shown in Appendix A, $W(y)$ is regular at the singular point of the coefficients. In Section VI, with some details given in Appendix B, and Appendix C, we show that we can relate the analysis of completeness and spectrum of the solutions for $W(y)$ to that of a self adjoint problem, through a Darboux transformation. In Section VII we consider the form that results from the previous analysis for the general evolution of given perturbative initial data, making clear that negative eigenvalues correspond to unstable modes. Numerical results that confirm the existence of unstable modes are given in Section VIII. In Section IX we consider the special cases where the parameter $κ$ takes to values zero or one. The case of purely “radial” perturbations is analyzed in Section X. In Section XI we discuss briefly the results obtained and the conclusions we extract from them.

II. THE LINET - TIAN METRIC WITH A POSITIVE COSMOLOGICAL CONSTANT, AND SOME OF ITS PROPERTIES.

The Linet-Tian metrics with a positive cosmological constant may be (locally) written as [8],

\[
ds^2 = -y^{1/3+p_1/2}(1 - Λy)^{1/3-p_1/2}dt^2 + \frac{1}{3y(1 - Λy)}dy^2
\]
\[+y^{1/3+p_2/2}(1 - Λy)^{1/3-p_2/2}dz^2 + y^{1/3+p_3/2}(1 - Λy)^{1/3-p_3/2}dφ^2\]

(1)
where $\Lambda$ is the cosmological constant, and the parameters $p_i$ are constrained by,

$$p_1 + p_2 + p_3 = 0 \; ; \; p_1^2 + p_2^2 + p_3^2 = \frac{8}{3}$$

They may be given in terms of a single parameter $\kappa$ [9]:

$$p_1 = \frac{2(2\kappa + 2\kappa^2 - 1)}{3(1 + \kappa + \kappa^2)} \; , \; p_2 = -\frac{2(2 + 4\kappa + \kappa^2)}{3(1 + \kappa + \kappa^2)} \; , \; p_3 = \frac{2(2 + 2\kappa - \kappa^2)}{3(1 + \kappa + \kappa^2)}$$

In what follows we restrict $\kappa$ to the range $0 \leq \kappa \leq 1$.

We may naturally assume the range $-\infty < t < \infty$, and $0 \leq y \leq 1/\Lambda$, but the ranges of $z$ and $\phi$ require further consideration [10] [11]. This is because of the symmetrical roles played by $\partial_z$ and $\partial_\phi$. This can be seen as follows. If we change coordinates in (1) as,

$$y = \frac{1}{\Lambda} - x \; ; \; t = \Lambda^{p_1/2}\tilde{t} \; ; \; z = \Lambda^{p_2/2}\tilde{z} \; ; \; \phi = \Lambda^{p_3/2}\tilde{\phi}$$

the metric takes the form,

$$ds^2 = -x^{1/3-p_1/2}(1 - \Lambda x)^{1/3+p_1/2}d\tilde{t}^2 + \frac{1}{3x(1 - \Lambda x)}dx^2$$

$$+ x^{1/3-p_2/2}(1 - \Lambda x)^{1/3+p_2/2}d\tilde{\phi}^2 + x^{1/3-p_3/2}(1 - \Lambda x)^{1/3+p_3/2}d\tilde{z}^2$$

If we now define $\eta$ such that,

$$\kappa = \frac{1 - \eta}{2\eta + 1}$$

and,

$$\tilde{p}_1 = \frac{2(2\eta + 2\eta^2 - 1)}{3(1 + \eta + \eta^2)} \; , \; \tilde{p}_2 = -\frac{2(2 + 4\eta + \eta^2)}{3(1 + \eta + \eta^2)} \; , \; \tilde{p}_3 = \frac{2(2 + 2\eta - \eta^2)}{3(1 + \eta + \eta^2)}$$

we find that [5] can be written in the form,

$$ds^2 = -x^{1/3+\tilde{p}_1/2}(1 - \Lambda x)^{1/3-\tilde{p}_1/2}d\tilde{t}^2 + \frac{1}{3x(1 - \Lambda x)}dx^2$$

$$+ x^{1/3+\tilde{p}_2/2}(1 - \Lambda x)^{1/3-\tilde{p}_2/2}d\tilde{\phi}^2 + x^{1/3+\tilde{p}_3/2}(1 - \Lambda x)^{1/3-\tilde{p}_3/2}d\tilde{z}^2$$

which is identical to (1), but with $\kappa$ replaced by $\eta$ and an interchange of the roles of $\partial_z$ and $\partial_\phi$. This implies that any Linet-Tian metric with a given $\kappa$ and (positive) $\Lambda$ is locally isometric to a Linet-Tian metric with the same $\Lambda$, but with $\kappa$ replaced by $(1 - \kappa)/(2\kappa + 1)$. In particular, this implies that if we find an instability for $\kappa$ in the range,

$$0 \leq \kappa \leq (\sqrt{3} - 1)/2 = 0.366...$$

then that instability will also be present for $(\sqrt{3} - 1)/2 < \kappa \leq 1$. Other implications of this symmetry have been explored in the above mentioned references.
III. SETTING UP THE PROBLEM.

Consider a general linear perturbation of the Linet - Tian metric. This may be written in the form,

\[ g_{\mu\nu}(t, y, z, \phi) = g_{\mu\nu}^{(0)}(y) + \epsilon h_{\mu\nu}(t, y, z, \phi) \]  

(10)

where \( g_{\mu\nu}^{(0)}(y) \) is the (unperturbed) Linet-Tian metric \([1]\), and \( \epsilon \) is an auxiliary parameter, used to keep track of the linearity of the perturbations. The functions \( h_{\mu\nu} \) represent the most general perturbation. Since \( \partial_t, \partial_z, \) and \( \partial_\phi \) are commuting Killing vectors of \( g_{\mu\nu}^{(0)} \), we may restrict our analysis of the perturbation equations to solutions of the form,

\[ h_{\mu\nu}(t, y, z, \phi) = e^{i(\Omega t - k z - \xi \phi)} f_{\mu\nu}(y, \Omega, k, \ell) \]  

(11)

The evolution of a general perturbation would then be expressed as,

\[ h_{\mu\nu}(t, y, z, \phi) = \sum_{\Omega} \sum_{k} \sum_{\ell} C_{\Omega, k, \ell} e^{i(\Omega t - k z - \xi \phi)} f_{\mu\nu}(y, \Omega, k, \ell) \]  

(12)

where \( \sum_a \) stands for either a sum or an integral depending on whether the corresponding variable is discrete or continuous. Implicit in this expansion is the assumption that the set of solutions \( \{ f_{\mu\nu}(y, \Omega, k, \ell) \} \) is “complete” in some appropriate sense, so that (12) is valid for a general perturbation \( h_{\mu\nu}(t, y, z, \phi) \). The expectation here is that this could be achieved by imposing appropriate boundary condition on the solutions of the perturbation equations. There are, however, as indicated in \([8]\), a number of difficulties in establishing if, and in what sense, the expansion (12) has the desired properties. One of these difficulties stems from the fact that, as analyzed in \([8]\), the general expansion (10) is subject to gauge ambiguities, and, therefore it may contain components whose evolution in time is not determined by the evolution equations. Some relevant details regarding this problem are given in Appendix D.

In the present analysis, we consider again the “diagonal” case of \([8]\), where, for simplicity we take \( \ell = 0 \), and the perturbed metric is restricted to the form,

\[ ds^2 = -y^{\frac{1}{4} + \frac{p_1}{2}} \frac{1 + \epsilon e^{i(\Omega t - k z)} F_1(y)}{(1 - \Lambda y)^{p_1 + \frac{1}{2}}} dt^2 + \frac{1 + \epsilon e^{i(\Omega t - k z)} F_2(y)}{3y(1 - \Lambda y)} dy^2 + y^{\frac{1}{4} + \frac{p_2}{2}} \frac{1 + \epsilon e^{i(\Omega t - k z)} F_3(y)}{(1 - \Lambda y)^{p_2 + \frac{1}{2}}} dz^2 + \frac{y^{\frac{3}{4} + \frac{p_3}{2}} (1 + \epsilon e^{i(\Omega t - k z)} F_4(y))}{(1 - \Lambda y)^{p_3 + \frac{1}{2}}} d\phi^2, \]  

(13)

This choice is consistent with the equations of motion but, as indicated in Appendix D, and analyzed below, it is not free from gauge ambiguities.

Consider now the linearized Einstein equations for the metric (13). For \( \Omega \neq 0 \), and \( k \neq 0 \), we have,

\[ F_2(y) = -F_4(y), \]  

(14)

and the remaining equations can be written in the form,

\[ \frac{dF_1}{dy} + \frac{dF_4}{dy} + \frac{p_1 - p_2}{4y(1 - \Lambda y)} F_1 - \frac{8\Lambda y + 9p_2 - 4 + 3p_1}{12y(1 - \Lambda y)} F_4 = 0, \]  

(15)

\[ \frac{dF_3}{dy} + \frac{dF_1}{dy} - \frac{p_1 - p_2}{4y(1 - \Lambda y)} F_3 - \frac{8\Lambda y + 9p_1 - 4 + 3p_2}{12y(1 - \Lambda y)} F_4 = 0, \]  

(16)
and,
\[
\frac{dF_4}{dy} = -\frac{2(1 - \Lambda y)^{-1/3 + p_1/2}(F_3 + F_4) \Omega^2}{y^{1/3 + p_1/2}(-2 - 3p_3 + 4\Lambda y)} + \frac{2(1 - \Lambda y)^{-1/3 + p_2/2}(F_4 + F_1) k^2}{y^{1/3 + p_2/2}(-2 - 3p_3 + 4\Lambda y)}
\]
\[
\frac{(8\Lambda y + 3p_1 - 4)(p_1 - p_2) F_1}{8y(-1 + \Lambda y)(-2 - 3p_3 + 4\Lambda y)}
\]
\[
\frac{(8\Lambda (p_1 - p_2) y + (2 + 3p_1)(p_1 + 2p_2 - 2)) F_3}{8y(-1 + \Lambda y)(-2 - 3p_3 + 4\Lambda y)}
\]
\[
\frac{(32\Lambda^2 y^2 - 8\Lambda(4 + 15p_3)y + 60p_3 + 45p_1p_2 + 44) F_4}{24y(-1 + \Lambda y)(-2 - 3p_3 + 4\Lambda y)}
\]

(17)

Since $4\Lambda y - 3p_3 - 2 = 4\Lambda y - 6(\kappa + 1)/1 + \kappa + \kappa^2 < 0$ in $0 < y < 1/\Lambda$, the $y$-dependent coefficients of the $F_i$ are regular in $0 < y < 1/\Lambda$, but singular, in general, for both $y = 0$ and $y = 1/\Lambda$. Therefore, the general solution of the system (15)-(16)-(17) can be written as a linear combination of three appropriately chosen linearly independent solutions, which are regular, in $0 < y < 1/\Lambda$, but may be singular at either or both $y = 0$, and $y = 1/\Lambda$. In particular, the set (see Appendix D and [8]),

\[
F_1(y) = \frac{(3p_1 + 2 - 4\Lambda y)y^{p_1/4 + p_2/4 - 2/3}(p_1 - p_2)}{(1 - \Lambda y)^{p_1/4 + p_2/4 + 2/3}} + \frac{16y^{p_2/4 - p_1/4} \Omega^2}{(1 - \Lambda y)^{p_1/4 - p_2/4}}
\]

\[
F_3(y) = \frac{(3p_2 + 2 - 4\Lambda y)y^{p_1/4 + p_2/4 - 2/3}(p_1 - p_2)}{(1 - \Lambda y)^{p_1/4 + p_2/4 + 2/3}} + \frac{16y^{p_1/4 - p_2/4}k^2}{(1 - \Lambda y)^{p_1/4 - p_2/4}}
\]

\[
F_4(y) = \frac{(2 - 4\Lambda y + 3p_3)(p_1 - p_2)}{y^{2/3 + p_3/4}(1 - \Lambda y)^{2/3 - p_3/4}}
\]

(18)

is an exact, but pure gauge solution of the system (15)-(16)-(17), that can be removed by an appropriate coordinate transformation. Notice that, as indicated, this solution is regular for $0 < y < 1/\Lambda$, but it is divergent both for $y \to 0$ and $y \to 1/\Lambda$.

As regards the other two independent solutions, we have that one of them, near $y = 0$, behaves as,

\[
F_1(y) \simeq -\frac{2 + 4\kappa + \kappa^2}{\kappa(2 + \kappa)} c_0 + a_1 y^{1 + \kappa + \kappa^2}
\]

\[
F_3(y) \simeq -\frac{\kappa^2 - 2}{\kappa(2 + \kappa)} c_0 + b_1 y^{1 + \kappa + \kappa^2}
\]

\[
F_4(y) \simeq c_0 + c_1 y^{1 + \kappa + \kappa^2}
\]

(19)

plus higher order terms, where $c_0$ is an arbitrary constant, and

\[
a_1 = \frac{(2 + \kappa)(\kappa^2 - 2 + 2\kappa)(1 + \kappa + \kappa^2)^2 \Omega^2 c_0}{3\kappa(\kappa^2 + 2\kappa + 4)}
\]

\[
b_1 = \frac{(\kappa - 2)(\kappa^2 + 2 + 2\kappa)(1 + \kappa + \kappa^2)^2 \Omega^2 c_0}{3\kappa(\kappa^2 + 2\kappa + 4)}
\]

\[
c_1 = \frac{(2 - 2\kappa - \kappa^2)(\kappa^2 + 2 + 2\kappa)(1 + \kappa + \kappa^2)^2 \Omega^2 c_0}{3(2 + \kappa)(\kappa^2 + 2\kappa + 4)}
\]

(20)
and, therefore, the $F_i$ approach a finite limit as $y \to 0$, but with divergent derivatives in that limit, because $(1 + \kappa + \kappa^2)^{-1} < 1$, for $\kappa > 0$.

For the other solution, near $y = 0$, we have,

$$
F_1(y) \simeq - \frac{2 + 4\kappa + \kappa^2}{\kappa(2 + \kappa)} c_2 \ln(y) + \frac{4(1 + \kappa)(1 + \kappa + \kappa^2)}{\kappa^2(2 + \kappa)^2} c_2 - \frac{2 + 4\kappa + \kappa^2}{\kappa(2 + \kappa)} c_3
$$

$$
F_3(y) \simeq \frac{2 - \kappa^2}{\kappa(2 + \kappa)} c_2 \ln(y) + \frac{4(1 + \kappa)(1 + \kappa + \kappa^2)}{\kappa^2(1 + \kappa)^2} c_2 - \frac{\kappa^2 - 2}{\kappa(2 + \kappa)} c_3
$$

$$
F_4(y) \simeq c_2 \ln(y) + c_3
$$

(21)

where $c_2$ and $c_3$ are an arbitrary constant, plus terms that vanish as $y \to 0$, and, therefore, the $F_i$ diverge as $\ln(y)$. The presence of an additional constant, $c_3$, in these expressions is due to the fact that to any solution satisfying the boundary condition (21) we may add an arbitrary solution satisfying (19) without changing the form (21).

Similarly, near $y = 1/\Lambda$, we have a solution that behaves as,

$$
F_1(y) = \frac{(\mu - 9\kappa - 3)}{3\kappa(2 + \kappa)} c_4 + \frac{3\kappa(\kappa - 1)}{9\kappa + \mu - 3} c_5 \left(1 - \Lambda y\right)^{\frac{(1 - \kappa)^2}{4\mu}}
$$

$$
F_3(y) = \frac{9 - 7\mu + 3\kappa}{3\kappa(2 + \kappa)} c_4 + \frac{9(\mu - 1 - \kappa)}{3 + 3\kappa - 5\mu} c_5 \left(1 - \Lambda y\right)^{\frac{(1 - \kappa)^2}{4\mu}}
$$

$$
F_4(y) = c_4 + c_5 \left(1 - \Lambda y\right)^{\frac{(1 + \kappa + 1)^2}{4\mu}}
$$

(22)

plus higher order terms, $c_4$ is an arbitrary constant, $\mu = 1 + \kappa + \kappa^2$, and,

$$
c_5 = \frac{(9\kappa + \mu - 3)(5\mu - 3 - 3\kappa)\mu^2k^2c_4}{\Lambda^{\frac{2\mu + 1}{\kappa}}(2 + \kappa)(2\mu - 3 - 9\kappa)(\kappa - 1)^4}
$$

(23)

For the other independent solution $F_1$, $F_3$ and $F_4$ diverge as $\ln(1 - \Lambda y)$ as $y \to 1/\Lambda$, but we shall not display their leading behaviour for simplicity. Thus, we see that the system has solutions that are well behaved, i.e., do not diverge, at either $y = 0$ or $y = 1/\Lambda$.

What this means is that if we consider a solution that behaves as (19) near $y = 0$, then, in general, as we approach $y = 1/\Lambda$, it will behave as a linear combination of the three linearly independent solutions characterized by their behaviour near $y = 1/\Lambda$, and, therefore, it will diverge for $y \to 1/\Lambda$. As discussed, for instance in [3] or [4], we should, in principle, consider only as appropriate those solutions of the perturbation equations such that the $F_i$, (up to a gauge transformation) do not diverge either at $y = 0$ or $y = 1/\Lambda$. Since solutions of the system can only be obtained numerically, one might then try to impose this condition at say $y = 0$, and, for fixed $\kappa$ and $k$, look for possible values of $\Omega$, such that the solution is also finite as we approach $y = 1/\Lambda$. Unfortunately, because of the gauge ambiguities contained in the system, this simple “shooting” procedure fails to provide the required solutions. What is required here is a gauge invariant function that carries the physical properties of the perturbations, and satisfies the finiteness requirements, while the $F_i$ themselves may still contain gauge dependent divergent components. This problem is considered in the next Section.
IV. GAUGE INVARIANT FORMULATION.

Gauge invariant functions may be constructed in general as a linear combinations of the $F_i(y)$. Let us call $F_i^g(y)$ the solutions given by Eq. (15). Then, a suitable example is the function,

$$W(y) = \mathcal{K}(y) \left[ F_3^g(y) F_4(y) - F_4^g(y) F_3(y) \right]$$  \hspace{1cm} (24)

where $\mathcal{K}$ is an arbitrary function of $y$. If we choose,

$$\mathcal{K}(y) = -\frac{\mu^2}{4} y^{\frac{1+\kappa+\mu}{6}} (1 - \Lambda y)^{-\frac{2\mu - 3 - 3\kappa}{6\mu} (3)}$$  \hspace{1cm} (25)

where here, and in what follows, $\Lambda = 1 + \kappa + \kappa^2$, we get,

$$W(y) = -(2\Lambda y - 3 - 3\kappa) \kappa (2 + \kappa) F_3(y)$$

$$+ \left( \kappa (2 + \kappa) (3\kappa + 2\Lambda y) - 4 (1 - \Lambda y)^{\frac{(\kappa + 1)^2}{6}} \right) \mu^2 \kappa^2$$

$F_4(y)$  \hspace{1cm} (26)

Notice that the coefficients of $F_3$, and $F_4$ are finite both for $y \to 0$ and $y \to 1/\Lambda$. In particular, near $y = 0$, for the solution (19) we have,

$$W(y) \simeq 3 (1 + \kappa + \mu) c_0 - (1 + \kappa + \mu) \mu^2 \Omega^2 c_0 y^4$$  \hspace{1cm} (27)

and, near $y = 1/\Lambda$, for the solution (22) we have,

$$W(y) = \left( \frac{4}{3} \mu (5 - 6) \Lambda - 4 \kappa \mu + 3 \kappa - \mu + 3 \right) c_4$$

$$+ \left( \frac{4}{3} \mu (7\mu - 6 - 6\kappa) \Lambda - 3 \kappa (2 + \kappa) (4\kappa - 1) \right) \kappa^2 \mu^2 c_4$$

$$\Lambda \left( \frac{5}{6} + 1 \right) (7\mu - 3 - 9\kappa) (\kappa - 1)^4 (9\kappa + \mu - 3)^{-1} (1 - \Lambda y)^{\frac{(\kappa + 1)^2}{6} \mu^2 k^2}$$  \hspace{1cm} (28)

plus higher order terms. Thus, $W$ will be well defined and finite for solutions $F_3$, and $F_4$ that satisfy, up to an arbitrary addition of the pure gauge solution, both the finite boundary conditions (19), and (22). Then, the finiteness of $W$ corresponds precisely to the condition for the existence of appropriate solutions of the set (15,16,17).

But the crucial property of $W$ is that it is not only gauge invariant, but it is also a master function, in the sense that the full perturbation can be reconstructed from $W$. This can be seen as follows. First, we solve (26) for $F_3$ in terms of $W$, and, $F_4$,

$$F_3(y) = \frac{\mathcal{K} F_3^g - W}{\mathcal{K} F_4^g}$$  \hspace{1cm} (29)

Replacing (29) in (16), using the fact that the $F_i^g$ are solutions of (16), and rearranging terms we find,

$$\frac{d}{dy} \left( \frac{F_4}{F_4^g} \right) = \frac{W (\kappa + \mu - 1)}{2 \mu F_3^g K (F_3^g + F_4^g) y (1 - \Lambda y)} + \frac{1}{F_3^g + F_4^g} \frac{d}{dy} \left( \frac{W}{F_4^g K} \right)$$  \hspace{1cm} (30)

which implies,

$$F_4(y) = F_4^g \int_0^y \left[ \frac{(\kappa + \mu - 1) W}{2 \mu F_3^g K (F_3^g + F_4^g) y (1 - \Lambda y)} + \frac{1}{(F_3^g + F_4^g)} \frac{d}{dy} \left( \frac{W}{F_4^g K} \right) \right] dy'$$

$$+ C F_4^g (y),$$  \hspace{1cm} (31)
where $C$ is an arbitrary constant, and, therefore, we can express $F_4$ entirely in terms of $W$, and the already known pure gauge solutions.

Using the expressions for $F_3$, and $F_4$ we may also obtain an expression for $F_1(y)$, in terms of $W(y)$, but it turned out to be more useful for the derivations to solve (17) for $F_1(y)$. This is given by,

$$F_1 = \frac{4y (1 - \Lambda y) (2 \Lambda y \mu - 3 - 3\kappa) \mu dF_4}{A_1} - \frac{A_2}{3A_1} F_1$$

$$+ \frac{4 (1 - \Lambda y)^{\frac{\mu-3}{3\nu}} y^\frac{1}{p} \Omega^2 \mu^2 - 4 \mu \Lambda (\mu + \kappa - 1) y + 3 (\mu - 1) (\mu + 1 + 2\kappa) F_3}{A_1}$$

where,

$$A_1 = 4 (1 - \Lambda y)^{\frac{\mu-3}{3\nu}} y^\frac{\kappa+\mu}{p} k^2 \mu^2 - (4 \Lambda y \mu - 3) (\mu + \kappa - 1)$$

and,

$$A_2 = -12 (1 - \Lambda y)^{\frac{\mu-3}{3\nu}} y^\frac{1}{p} \Omega^2 \mu^2 + 12 (1 - \Lambda y)^{\frac{\mu-3}{3\nu}} y^\frac{\kappa+\mu}{p} k^2 \mu^2$$

$$+ 8 \Lambda^2 \mu^2 y^2 + 12 \mu \Lambda (\mu - 5 - 5\kappa) y - 9 \mu^2 + 45\kappa + 45\mu$$

Thus, the full set of diagonal perturbations can be written in terms of the master function $W$. The resulting expressions, nevertheless, still contain gauge ambiguities. In fact, going back to (31), we can see as expected, that $F_4$ reduces to $F_4^g$ when $W(y) = 0$, the pure gauge situation. But, suppose now that we insert in (31) a non trivial $W(y)$, satisfying the boundary condition (27). It is easy to check that if we also set $C = 0$, the resulting $F_4(y)$ satisfies (19) near $y = 0$. But, we can also check that near $y = 1/\Lambda$, since, in general, the integral is finite, $F_4(y)$ approaches, in general, $F_4^g(y)$. There is no contradiction here. It simply means that we cannot choose a simple gauge where $F_4$ is free of $F_4^g(y)$ “contamination”. This suggests that we look directly for the equation that $W(y)$ should satisfy, when the $F_i$ satisfy their corresponding equations. This is derived in the next Section.

V. THE DIFFERENTIAL EQUATION FOR $W(y)$.

We may obtain the equation that $W(y)$ should satisfy if the $F_i(y)$ satisfy the perturbation equations by going back to (31), and taking a new $y$-derivative. Solving for $d^2W/dy^2$, and after several replacements, using the evolution equations for the $F_i$, we finally find that $W$ satisfies the equation,

$$- \frac{d^2W}{dy^2} + Q_1 \frac{dW}{dy} + Q_2 W = \Omega^2 Q_3 W$$

where,

$$Q_1(y) = \frac{4k^2 \mu (2 \Lambda y \mu - 3\mu - 6\kappa) (1 - \Lambda y)^{\frac{\kappa-1}{3\nu}} y^{\frac{1+\kappa}{p}} + 3 (2 + \kappa) \kappa (4 \Lambda y \mu - 6 \Lambda y + 3)}{3 \left(4 (1 - \Lambda y)^{\frac{\kappa-1}{3\nu}} y^{\frac{1+\kappa}{p}} \mu^2 k^2 - \kappa (2 + \kappa) (4 \Lambda y \mu - 3) \right) y(\Lambda y - 1)}.$$
\[ Q_2(y) = \left[ 4k^4 \mu^2 (1 - \Lambda y)^{-\frac{6\kappa+2\mu}{3\mu}} y^{2\kappa+\mu} \right. \]
\[ - k^2 \left( 8\Lambda^2 \mu^2 y^2 - 4\Lambda \mu (3\kappa + 5\mu - 3) y + 3\kappa (2\kappa + 3) (1 + 2\kappa) (2 + \kappa) \right) \]
\[ (1 - \Lambda y)^{-\frac{3\kappa+2\mu}{3\mu}} y^{\kappa} + 6\Lambda \kappa (2 + \kappa) (-3 + 2\mu) \]
\[ \left. \left[ 3y \left( 4 (1 - \Lambda y)^{\frac{(\kappa+1)^2}{3\nu}} y^{\frac{1+2\kappa}{3\nu}} \mu^2 k^2 - \kappa (2 + \kappa) (4\Lambda y \mu - 3) \right) (1 - \Lambda y) \right]^{-1} \right] \]

and,
\[ Q_3(y) = \frac{(1 - \Lambda y)^{-\frac{3+2\mu}{3\nu}} y^{\frac{1-2\mu}{3\nu}}}{3} \]

Thus, imposing appropriate boundary conditions on \( W(y) \), we may consider (35) as a boundary value problem whose solutions determine the allowed values of \( \Omega \). Independently of the details, to analyze an equation of the form (35) it may be useful to put it into a Schrödinger-like form, that possibly leads to an equivalent self adjoint problem. This can be achieved introducing a new coordinate \( r = r(y) \), and two new functions, \( K(y) \), and \( \tilde{W}(r) \), such that,
\[ W(y) = K(y) \tilde{W}(r(y)) \] (39)

Replacing in (35) we get,
\[ - \frac{d^2 \tilde{W}}{dr^2} - \frac{\left( \frac{2dK}{dy} \frac{dr}{dy} + K \frac{d^2 r}{dy^2} - Q_1 \frac{dK}{dy} \right)}{K} \tilde{W} = \Omega^2 Q_3 \frac{r}{2} \tilde{W} \] (40)

If we impose now that \( r(y) \) be a solution of,
\[ \frac{dr}{dy} = \sqrt{Q_3(y)} \] (41)

and also that \( K(y) \) is a solution of,
\[ 2 \frac{dr}{dy} \frac{dK}{dy} + \left( \frac{d^2 r}{dy^2} - Q_1 \frac{dr}{dy} \right) K = 0 \] (42)

replacing in (40), we find that \( \tilde{W} \) satisfies the Schrödinger-like equation,
\[ - \frac{d^2 \tilde{W}}{dr^2} + V \tilde{W} = \Omega^2 \tilde{W} \] (43)

where the “potential” \( V \) is given by,
\[ V = \frac{1}{4Q_3^3} \left( Q_3 \frac{d^2 Q_3}{dy^2} - \frac{5}{4} \left( \frac{dQ_3}{dy} \right)^2 \right) + \frac{1}{4Q_3} \left( Q_1^2 + 4Q_2 - 2 \frac{dQ_1}{dy} \right) \] (44)
and, therefore, it is explicitly given as a function of \( y \), even if we do not have explicit solutions for either (41) or (42). Actually, in our case \( Q_3 \) is given by (38), and the general solution of (41) is,

\[
 r(y) = \frac{2\mu y^{\frac{1}{2\mu}}}{\sqrt{3}} \, 2F_1 \left( \frac{1}{2\mu}, \frac{3+2\mu}{6\mu}, \frac{1+2\mu}{2\mu}; \Lambda y \right) + C_0,
\]

where \( 2F_1(a, b; c; x) \) is a hypergeometric function, with \( C_0 \) an arbitrary constant. In what follows, without loss of generality, we will set \( C_0 = 0 \). The range of \( r \) will be then,

\[
0 \leq r \leq r_1 \tag{46}
\]

where \( r_1 \) is given by,

\[
 r_1 = \int_{0}^{1/\Lambda} \frac{1}{\sqrt{3} (1 - \Lambda y)^{\frac{4\mu-3}{6\mu}} y^{\frac{2\mu-1}{2\mu}}} dy
 = \frac{\Gamma \left( \frac{4\mu-3}{6\mu} \right) \Gamma \left( \frac{1}{2\mu} \right)}{\sqrt{3\Lambda^\mu \pi} \Gamma(2/3)} \tag{47}
\]

We may use (45) to construct a parametric representation of \( V(r) \). This would in principle allow us, as in similar quantum mechanical problems, to carry out a qualitative analysis of the possible spectrum of allowed values of the “eigenvalues” \( \Omega^2 \), and therefore obtain information on the existence of solutions with \( \Omega^2 < 0 \), signalling unstable solutions of the evolution equations. Unfortunately, in our case, irrespective of the value of \( k \), the functions \( Q_1 \) and \( Q_2 \) have vanishing denominators at \( y = y_0 \), where \( y_0 \) is a solution of,

\[
k^2 = \frac{\kappa (\kappa + 2) (4\Lambda y_0 \mu - 3)}{4 (1 - \Lambda y_0)^{\frac{(\kappa-1)^2}{3\mu}} y_0^{\frac{(1+\kappa)^2}{\mu}}} \mu^2 \tag{48}
\]

Notice that, for \( k^2 > 0 \), this equation has always a solution for \( y_0 \) in the range \( 3/(4\Lambda \mu) \leq y_0 < 1/\Lambda \), and, as can be checked, this implies that \( V(y) \) has a double pole at \( y = y_0 \), and, therefore, (43) is not self adjoint, and the analysis fails.

One may ask whether this problem could be solved with a different choice of \( K(y) \) in (24), that would lead to a different equation for the resulting \( W(y) \), and therefore, possibly to a different associated Schrödinger like equation. That this is not the case can be seen as follows. Suppose we introduce a new function \( K_1(y) \), and define \( W_1(y) \) by,

\[
 W(y) = K_1(y) W_1(y) \tag{49}
\]

Replacing in (24) we get,

\[
 - \frac{d^2 W_1}{dy^2} + \tilde{Q}_1 \frac{d W_1}{dy} + \tilde{Q}_2 W_1 = \Omega^2 \tilde{Q}_3 W_1 \tag{50}
\]

where,

\[
 \tilde{Q}_1(y) = Q_1 - \frac{2}{K_1} \frac{dK_1}{dy} \frac{d}{dy} \tag{51}
\]

\[
 \tilde{Q}_2(y) = Q_2 + \frac{Q_1}{K_1} \frac{d^2 K_1}{dy^2} \tag{51}
\]

\[
 \tilde{Q}_3(y) = Q_3
\]
Then, defining \( W_1(y) = K_1(y)\tilde{W}_1(r(y)) \), the same procedure that led to (43), leads now to,

\[
-\frac{d^2\tilde{W}_1}{dr^2} + V_1\tilde{W}_1 = \frac{\Omega^2}{\Lambda}\tilde{W}_1,
\]

(52)

where the “potential” \( V_1 \) is now given by,

\[
V_1 = \frac{1}{4\tilde{Q}_3^3} \left( \tilde{Q}_3 \frac{d^2\tilde{Q}_3}{dy^2} - \frac{5}{4} \left( \frac{d\tilde{Q}_3}{dy} \right)^2 + \frac{1}{4\tilde{Q}_3} \left( \tilde{Q}_1^2 + 4\tilde{Q}_2 - 2\frac{d\tilde{Q}_1}{dy} \right) \right)
\]

(53)

and, since \( \tilde{Q}_3(y) = Q_3(y) \), we have that \( r(y) \) is the same as in (43). But if we replace now (51) in (53), we immediately obtain that \( V_1(y) \equiv V(y) \), and therefore, (43) is invariant under a change \( K(y) \), and the problem cannot be solved by a different choice of this factor.

At this point we must remark that although some coefficients in (35) are singular, the solutions \( W(y) \) must be regular at \( y_0 \), because they are linear combinations of \( F_3 \) and \( F_4 \), with regular coefficients, with \( F_3 \) and \( F_4 \) also regular in a neighbourhood of \( y = y_0 \). In fact one can check, by explicit computation, that near \( y = y_0 \), the general solution of (35) takes the form,

\[
W = a_0 + a_3 (y - y_0)^3 + a_4 (y - y_0)^4 + \ldots
\]

(54)

where \( a_0 \), and \( a_3 \) are arbitrary constants, \( a_4 \) is determined in terms of \( a_0 \) and \( a_3 \), and dots indicate higher order terms, also completely determined in terms of \( a_0 \), and \( a_3 \). This result is, in fact, more general for the type of equations considered here, as is shown in Appendix A.

We have then a situation where Eq. (35) has solutions that are regular in \( 0 < y < 1/\Lambda \), and such that, for appropriate values of \( \Omega \), are also finite at the boundaries \( y = 0 \), and \( y = 1/\Lambda \), but we cannot ascertain whether the solutions form a complete set, or if the eigenvalues \( \Omega^2 \) are, for instance, bounded from below. At this point we recall that there is a well known procedure that may allow us to establish a map between the solutions of (35) and those of a related self adjoint problem. This is the Darboux transformation [12], also considered as the introduction of an “intertwining” operator [13], which will be used here in a manner similar to that analyzed in [14]. The explicit construction is shown in the next Section.

VI. THE DARBOUX TRANSFORMATION.

Consider, in a given domain \( r_0 \leq r \leq r_1 \) of \( r \), the solutions \( \phi = \phi(r) \) of the equation,

\[
-\frac{d^2\phi}{dr^2} + V(r)\phi = \Omega^2\phi
\]

(55)
and a particular solution,
\[ -\frac{d^2\phi_0}{dr^2} + V(r)\phi_0 = \Omega_0^2\phi_0 \quad (56) \]

In correspondence with a given solution \( \phi(r) \), (no particular boundary conditions implied), define a new function \( \chi(r) \), given by,
\[ \chi(r) = \frac{d\phi}{dr} - \frac{\phi}{\phi_0} \frac{d\phi_0}{dr} \quad (57) \]

Then, a simple derivation shows that \( \chi(r) \) is a solution of,
\[ -\frac{d^2\chi}{dr^2} + \tilde{V}(r)\chi = \Omega^2\chi \quad (58) \]

where,
\[ \tilde{V}(r) = -V(r) + 2\Omega_0^2 + 2\frac{\phi_0}{\phi_0^2} \left( \frac{d\phi_0}{dr} \right)^2 \quad (59) \]

Suppose now that \( \tilde{V}(r) \) is such that (58) admits a self adjoint extension. This implies that, after imposing appropriate boundary conditions, there will be a complete set of solutions of (58) (“eigenfunctions”) \( \chi_\alpha \), with corresponding “eigenvalues” \( \Omega^2_\alpha \), and satisfying an orthonormality condition,
\[ \int_{r_0}^{r_1} \chi_\alpha^* \chi_\beta dr = \delta_{\alpha\beta} \quad (60) \]

with \( \delta_{\alpha\beta} \) a Kronecker or Dirac delta function as appropriate.

Next, using (57) we find,
\[ (\Omega_0^2 - \Omega_\alpha^2)\phi_\alpha(r) = \frac{d\chi_\alpha}{dr} + \frac{\chi_\alpha}{\phi_0} \frac{d\phi_0}{dr} \quad (61) \]

This implies that corresponding to every \( \chi_\alpha \), with \( \Omega_\alpha^2 \neq \Omega_0^2 \), we will have a (possibly singular) solution \( \phi_\alpha \) of (55).

Going back to (43), (44) of the previous Section, identifying \( \tilde{W}(r) \) with \( \phi(r) \), and using (44), we see that \( \tilde{V}(r) \), is given implicitly in part by the coefficients \( Q_i \), and in part by the chosen solution \( \phi_0(r) \equiv \tilde{W}^{(0)}(r) \), where \( \tilde{W}^{(0)}(r) \) is a solution of (43) with \( \Omega^2 = \Omega_0^2 \). But, in view of (39) and (41), we have,
\[ \tilde{W}^{(0)}(r(y)) = \frac{W^{(0)}(y)}{K(y)} \quad (62) \]

and,
\[ \frac{1}{\phi_0} \frac{d\phi_0}{dr} = \frac{K}{W^{(0)}} \frac{dy}{dy} \left( \frac{W^{(0)}}{K} \right) = \frac{1}{\sqrt{Q_3}} \left( \frac{1}{W^{(0)}} \frac{dW^{(0)}}{dy} - \frac{1}{K} \frac{dK}{dy} \right) = \frac{1}{\sqrt{Q_3}} \left( \frac{1}{W^{(0)}} \frac{dW^{(0)}}{dy} - \frac{1}{2} Q_1 + \frac{1}{4 Q_3} \frac{dQ_3}{dy} \right) \quad (63) \]
and, therefore, we have,

\[ \tilde{V}(r(y)) = -V(r(y)) + 2\Omega_0^2 \]

\[ + \frac{2}{Q_3} \left( \frac{1}{W^{(0)}} \frac{dW^{(0)}}{dy} - \frac{1}{2} Q_1 + \frac{1}{4Q_3} \frac{dK}{dy} \right)^2 \]

where \( V(r(y)) \equiv V(r) \) is given by (41). As shown in more generality in Appendix A, and Appendix B, given the form of the functions \( Q_i \) near the singular point \( y = y_s \), and considering a general solution \( W^{(0)}(y) \) that does not vanish at \( y = y_s \), the resulting \( \tilde{V}(r(y)) \) is regular in the neighbourhood of \( y = y_s \). Since the functions \( Q_i \) are regular in \( 0 < y < y_s \), and in \( y_s < y < 1/\Lambda \), \( \tilde{V}(r(y)) \) will be regular in \( 0 < y < 1/\Lambda \) if \( W^{(0)}(y) \) has no zeros in that interval. Let us assume that such solution exists, and that the resulting \( \tilde{V}(r) \) is such that (58) admits a self adjoint extension. Then, since the interval \( (r(0), r(1/\Lambda)) \) is finite, the spectrum is entirely discrete, and we may take the functions \( \chi_\alpha \) as real, and normalized to a Kronecker delta function.

We may now take for the solutions \( W^{(\alpha)} \) the set,

\[ W^{(\alpha)}(y) = K(y)\tilde{W}^{(\alpha)}(r(y)) \]

\[ = K(y) \frac{1}{\Omega_0^2 - \Omega_\alpha^2} \left( \frac{dX_\alpha}{dr} + \frac{\chi_\alpha d\phi_0}{\phi_0 dr} \right) \]

\[ = K(y) \frac{1}{\Omega_0^2 - \Omega_\alpha^2} \left( \frac{dX_\alpha}{dr} + \chi_\alpha \left( \frac{1}{W^{(0)}} \frac{dW^{(0)}}{dy} - \frac{1}{K} \frac{dK}{dy} \right) \right) \]

which will also be real, but we must impose the additional requirement that the \( W^{(\alpha)}(y) \) satisfy the boundary conditions (27)-(28). Since from (57) we have,

\[ \chi_\alpha(r(y)) = \frac{1}{KW^{(0)}} \left( W^{(0)} \frac{dW^{(\alpha)}}{dy} - W^{(\alpha)} \frac{dW^{(0)}}{dy} \right) \left( \frac{dr}{dy} \right)^{-1} \]

This implies, as can be checked, that if \( W^{(0)}(y) \) diverges as \( \ln(y) \), or \( \ln(1 - \Lambda y) \), at, respectively, \( y = 0 \), or \( y = 1/\Lambda \), then, \( \chi(r) \) will diverge as \( 1/(\sqrt{\gamma} \ln(y)) \), or \( 1/(\sqrt{(r_1 - r) \ln(r_1 - r)}) \), which leads to complications that are outside the present discussion. The simplest choice, remember that we are interested in the properties of \( W(y) \), and not on those of the \( \chi^{(\alpha)} \), is to impose that \( W^{(\alpha)}(y) \) satisfies also the boundary conditions (27)-(28). This, as shown in Appendix C, leads to a unique self adjoint extension for (58). But notice that in this case, \( \Omega_0^2 \) must be in the spectrum of \( W^{(\alpha)}(y) \), and, therefore, in accordance with (57), not in the spectrum of \( \chi_\alpha \).

Let us consider now \( \alpha \) and \( \beta \) such that \( \Omega_\alpha^2 \neq \Omega_\beta^2 \), and \( \Omega_\beta^2 \neq \Omega_\beta^2 \). We then have,

\[ \int_0^1 \frac{1}{K^2} \left( \frac{dW^{(\alpha)}}{dy} \right) \left( \frac{dW^{(\beta)}}{dy} \right) \left( \frac{dr}{dy} \right)^{-1} dy = \mathcal{N}_\alpha \delta_{\alpha,\beta} \]

where,

\[ \mathcal{N}_\alpha = \int_0^1 \frac{1}{K^2} \left( \frac{dW^{(\alpha)}}{dy} \right)^2 \left( \frac{dr}{dy} \right)^{-1} dy \]

This will be applied in the next Section to obtain the general evolution of an arbitrary perturbation.
VII. GENERAL TIME EVOLUTION OF THE PERTURBATIONS.

Consider again (31). We define the linear (integro-differential) operator \( \mathcal{O} \) as,

\[
\mathcal{O}(W) \equiv \int_0^y \left[ \frac{(\kappa + \mu - 1)W}{2\mu F_4^g K (F_3^g + F_4^g)} y'(1 - \Lambda y') + \frac{1}{(F_3^g + F_4^g)} \frac{d}{dy'} \left( \frac{W}{F_3^g K} \right) \right] dy'.
\]

We now go back to the expansion (12), but restricted to \( \ell = 0 \), and a particular value of \( k \).

We take as the allowed values of \( \Omega \) the full set of \( \Omega_\alpha \) such that \( \Omega_\alpha^2 \) are the eigenvalues of the self adjoint problem associated to \( W \), plus \( \Omega_0 \) such that \( \Omega_0^2 \) is the eigenvalue corresponding to \( W^{(0)} \), which is not included in the set \( \{ \Omega_\alpha \} \). The general expansion (12), in the case of \( F_4^g \), then takes the form,

\[
h_{\phi\phi} = \sum_\alpha \left[ \exp (i\Omega_\alpha t) A_\alpha + \exp (-i\Omega_\alpha t) B_\alpha \right] F_4^{(\alpha)}(y)
\]

\[
+ \left[ \exp (i\Omega_0 t) A_0 + \exp (-i\Omega_0 t) B_0 \right] F_4^{(0)}(y),
\]

in these expressions \( \Omega_\alpha \) (\( \Omega_0 \)) is either the positive real root or positive imaginary root of \( \Omega_\alpha^2 \) (\( \Omega_0^2 \)). \( A_\alpha \), and \( B_\alpha \) are arbitrary constants, and,

\[
F_4^{(\alpha)}(y) = F_4^g(y) \mathcal{O}(W^{(\alpha)}) + C_\alpha F_4^g
\]

where \( a \) stands for either \( \alpha \), or 0, with \( C_\alpha \) an arbitrary constant.

We can see that these expressions are not very useful, because the function \( F_4 \), (and in general all \( F_i \)), contains gauge dependent parts, and these imply that \( h_{\phi\phi} \), (and in general \( h_{\mu\nu} \)), will contain time dependencies that are not fully determined by the initial conditions, in correspondence with the gauge ambiguities present at any given time. Any physically relevant information is therefore contained in the gauge invariant part of the \( h_{\mu\nu} \), and this, in principle, can be extracted precisely from, e.g., \( W \). More explicitly, consider again (70).

In view of the linearity of \( \mathcal{O} \), and the fact that \( F_4^g(y) \) does not depend on \( \Omega \), this can be written as,

\[
h_{\phi\phi}(t, y) = F_4^g(y) \mathcal{O}(W(t, y)) + F_4^g(y) C(t, y)
\]

where,

\[
C(t, y) = \sum_a C_\alpha \left[ \exp (i\Omega_\alpha t) A_\alpha + \exp (-i\Omega_\alpha t) B_\alpha \right],
\]

and,

\[
W(t, y) = \sum_a \left[ \exp (i\Omega_\alpha t) A_\alpha + \exp (-i\Omega_\alpha t) B_\alpha \right] W^{(\alpha)}(y),
\]

where the index \( a \) extends over all the \( \alpha \), plus 0.

Using now (75) we have,

\[
\frac{\partial^2 W}{\partial t^2} = \sum_a \left[ \exp (i\Omega_\alpha t) A_\alpha + \exp (-i\Omega_\alpha t) B_\alpha \right] \left( -\Omega_\alpha^2 W^{(\alpha)} \right)
\]

\[
= \frac{1}{Q_3} \frac{\partial^2 W}{\partial y^2} - \frac{Q_1}{Q_3} \frac{\partial W}{\partial y} - \frac{Q_2}{Q_3} W
\]
which represents the general evolution equation for the gauge invariant $W$. Suppose we are given initial data for (75) in the form of the functions $W(0, y)$, and $(\partial W(t, y)/\partial t)|_{t=0}$. Using (67, 68), for $\alpha$ in the set of eigenvalues of $\chi_\alpha$, we easily obtain,

$$A_\alpha + B_\alpha = \frac{1}{N_\alpha} \int_0^1 \frac{1}{K^2} \left( \frac{dW^{(\alpha)}}{dy} - \frac{W^{(\alpha)}}{W^{(0)}} \frac{dW^{(0)}}{dy} \right) \times \left( \frac{\partial W}{\partial y} - \frac{W}{W^{(0)}} \frac{dW^{(0)}}{dy} \right) \bigg|_{t=0} \left( \frac{dr}{dy} \right)^{-1} dy$$

(76)

and,

$$A_\alpha - B_\alpha = \frac{1}{i\Omega_\alpha N_\alpha} \int_0^1 \frac{1}{K^2} \left( \frac{dW^{(\alpha)}}{dy} - \frac{W^{(\alpha)}}{W^{(0)}} \frac{dW^{(0)}}{dy} \right) \times \left( \frac{\partial^2 W}{\partial t \partial y} - \frac{\partial W}{\partial t} \frac{1}{W^{(0)}} \frac{dW^{(0)}}{dy} \right) \bigg|_{t=0} \left( \frac{dr}{dy} \right)^{-1} dy$$

(77)

from which we can straightforwardly solve for $A_\alpha$, and $B_\alpha$. Finally we may obtain the remaining coefficients $A_0$, and $B_0$, using

$$(A_0 + B_0) W^{(0)}(y) = W(0, y) - \sum_\alpha (A_\alpha + B_\alpha) W^{(\alpha)}(y)$$

(78)

and,

$$(A_0 - B_0) W^{(0)}(y) = \frac{1}{i\Omega_0} \left[ \frac{\partial W}{\partial t} \bigg|_{t=0} - \sum_\alpha i\Omega_\alpha (A_\alpha - B_\alpha) W^{(\alpha)}(y) \right]$$

(79)

Therefore, the expansion (74) is complete, in the sense that it provides the time of evolution of an arbitrary linear perturbation in terms of the “modes”, that is, the solutions of (35) that satisfy the appropriate boundary conditions. Thus, if one, or more, of these modes correspond to eigenvalues $\Omega^2 < 0$, we must conclude that the system is generically unstable.

In the next Section we will provide numerical evidence for the existence of such unstable modes for the system [15, 16, 17].

VIII. NUMERICAL RESULTS.

In this Section we consider numerical solutions of the perturbation equations. Because of their singular nature we cannot directly impose conditions at the boundaries at $y = 0$ or $y = 1/\Lambda$. Instead, we consider, where possible, appropriate expansions corresponding to the chosen type of solution, to impose initial data near the corresponding boundary, and proceed to a numerical integration using a Runge - Kutta integration method, after fixing the values of $\Lambda$, $\kappa$, and $k$, to explore a range of values of $\Omega$, looking for solutions that satisfy the desired boundary conditions at both $y = 0$ or $y = 1/\Lambda$.

More explicitly, let us consider $\kappa = 1/2$. In this case the perturbation equations reduce to,

$$\frac{dF_1}{dy} = -\frac{dF_4}{dy} - \frac{5F_1}{14y(1 - \Lambda y)} + \frac{(28\Lambda y - 51)F_4}{42y(1 - \Lambda y)}$$

(80)

$$\frac{dF_3}{dy} = -\frac{dF_4}{dy} - \frac{5F_3}{14y(1 - \Lambda y)} + \frac{(4\Lambda y - 3)F_4}{6y(1 - \Lambda y)}$$
and,

\[
\frac{dF_4}{dy} = \frac{7(F_3 + F_4)\Omega^2}{2(1 - \Lambda y)^{2/7}(9 - 7\Lambda y)} - \frac{7y^{2/7}(F_1 + F_1)k^2}{2(1 - \Lambda y)^{2/7}(9 - 7\Lambda y)} + \frac{(180 - 420\Lambda y)F_1 + (420\Lambda y - 405)F_3 + (392\Lambda^2 y^2 - 1932\Lambda y + 1179)F_4}{168y(1 - \Lambda y)(7\Lambda y - 9)}
\]

We also have,

\[
W(y) = \frac{45 - 35\Lambda y}{8}F_3(y) + \frac{15 + 35\Lambda y - 98(1 - \Lambda y)^{2/7}}{8}F_4(y)
\]

and we notice that we can derive an expression for \(dW/dy\) entirely in terms of the \(F_i\), (with no derivatives of the \(F_i\)), using (82), and the perturbation equations (80)-(81), although we shall not show it here for simplicity. It is straightforward to derive now an expression for \(\tilde{V}(y)\), entirely in terms of the \(Q_i\), and the \(F_i\) again with no derivatives of the \(F_i\), although the resulting expression is rather too lengthy to be displayed here. It was, nevertheless, used to compute \(\tilde{V}(y)\) numerically, using the results of the numerical integration of the \(F_i\).

To solve the problem of finding numerical solutions of the perturbation equations satisfying appropriate boundary conditions we used a ”shooting” approach, imposing finite boundary conditions at \(y = 0\), and looking for solutions that satisfy the required conditions as we approach \(y = 1/\Lambda\). A straightforward computation shows that near \(y = 0\), for general \(k\), \(\Lambda\), and \(\Omega\), we have a solution that admits an expansion of the form,

\[
F_1(y) = a_0 \left(1 + \frac{175}{816}\Omega^2 y^{4/7} - \frac{50}{459}\Lambda y - \frac{539539}{4543488}\Omega^4 y^{8/7} + \frac{22295}{128061}k^2 y^{3/7} + \frac{424621}{3998808}\Lambda^2 y^2 + \ldots\right)
\]

\[
F_3(y) = a_0 \left(-\frac{7}{17} + \frac{455}{816}\Omega^2 y^{4/7} - \frac{350}{1377}\Lambda y - \frac{717899}{4543488}\Omega^4 y^{8/7} + \frac{28175}{128061}k^2 y^{3/7} + \frac{4667}{16456}\Lambda\Omega^2 y^{14/7} + \frac{47412547}{2504245248}\Omega^6 y^{12/7} - \frac{18865}{33510672}k^2\Omega^2 y^{10/7} - \frac{30800}{169371}\Lambda^2 y^2 + \ldots\right)
\]

and,

\[
F_4(y) = a_0 \left(-\frac{5}{17} + \frac{91}{816}\Omega^2 y^{4/7} - \frac{10}{459}\Lambda y + \frac{343343}{4543488}\Omega^4 y^{8/7} - \frac{14651}{128061}k^2 y^{3/7} - \frac{689}{8712}\Lambda\Omega^2 y^{14/7} - \frac{29059303}{2504245248}\Omega^6 y^{12/7} + \frac{12005}{33510672}k^2\Omega^2 y^{10/7} + \frac{560}{169371}\Lambda^2 y^2 + \ldots\right)
\]
where $a_0$ is a constant and dots indicate higher order terms. This expansion corresponds to the solutions where the $F_i(y)$ approach a finite limit as $y \to 0$.

Using the expansions (83)-(84)-(85), to fix initial values near $y = 0$, we carried out a Runge-Kutta integration of the system (80)-(81), paying attention to the behaviour of $W(y)$, as $y \to 1/\Lambda$. In accordance with the “shooting” idea, for fixed $k$ (and setting $\Lambda = 1$, as already indicated), we looked for values of $\Omega^2$ such that $W(y)$ approached a finite value as $y \to 1/\Lambda$, but showed a divergent behaviour as we changed $\Omega^2$ to slightly larger or smaller values. Setting $\kappa = 1/2$, $\Lambda = 1$, and $k = 1$, we found that for the smaller of those values, which was $\Omega^2 = -0.6107$, ($\Omega = 0.7815i$), we had that $W(y)$ did not vanish in $0 < y < 1/\Lambda$. We, therefore, identified that solution with $W_0(y)$ of the previous Section, and computed $\tilde{V}(y)$.

A plot of $W_0(y)$ as a function of $y$, and of $\tilde{V}$, as a function of $r$ (given by (45), with $\kappa = 1/2$), both for $\Lambda = 1$, $k = 1$, and $\Omega = 0.7815i$, are given in Fig.1, and Fig.2. Notice that $\tilde{V}(r)$ is bounded from below, in fact is positive definite, and diverges to $+\infty$, both at $r = 0$, and $r = r_1$. Therefore, as discussed in the previous Section, the spectrum $\Omega_0^2$ is discrete and positive definite, but the complete spectrum of $W(y)$ contains also the imaginary eigenvalue $\Omega_0 = 0.7815i$. These results imply that for $\kappa = 1/2$, (and, from (6), also for $\kappa = 1/4$), the linear evolution of a general perturbation will contain an unstable mode, and, therefore, the space time will be linearly unstable.

FIG. 1: A plot of $W(y)$, computed using the numerical integration of the $F_i(y)$, as functions of $y$, for $\kappa = 1/2$, $k = 1$, $\Lambda = 1$ and $\Omega^2 = -0.6107$. This is identified with $W_0$, and used to compute $\tilde{V}$. 
FIG. 2: A plot of \( \tilde{V}(r) \), as a function of \( r \), computed using the numerical integration of the \( F_i(y) \), as functions of \( y \), for \( \kappa = 1/2, \, k = 1, \, \Lambda = 1 \) and \( \Omega^2 = -0.6107 \), identifying \( W_0(y) \) with \( W(y) \) computed for the same values of the parameters.

We also computed, as a check, the next higher eigenfunction, which is plotted in Fig.3. obtaining \( \Omega^2 = 3.853, \, (\Omega = 1.963) \). This is in qualitative correspondence to the lowest eigenvalue for \( \tilde{V} \), as shown in Fig. 2., which we would expect to be around \( \Omega^2 \sim 4 \).

FIG. 3: A plot of \( W(y) \), computed using the numerical integration of the \( F_i(y) \), as functions of \( y \), for \( \kappa = 1/2, \, k = 1, \, \Lambda = 1 \) and \( \Omega^2 = 3.853 \).
We have also carried out the integration, again for $\Lambda = 1$, and $k = 1$, and entirely along the same lines, for $\kappa = 1/3$, (isometric to $\kappa = 2/5$), obtaining $\Omega^2 = -0.6545$, ($\Omega = 0.809i$), for the lowest eigenvalue ($\Omega_0$), and $\Omega^2 = 5.128$, for the next eigenvalue, with qualitatively similar graphs for $W$, and $\bar{V}$ as those for $\kappa = 1/2$. Similar results were obtained for $\kappa = 3/4$, (isometric to $\kappa = 1/7$), with $\Omega^2 = -0.7075$, ($\Omega = 0.8411i$) for the lowest eigenvalue $\Omega_0$, and $\Omega^2 = 2.091$ for the next higher eigenvalue, again with graphs for $W$, and $\bar{V}$ similar to those for $\kappa = 1/2$. All these results, therefore confirm the linear instability of the Linet - Tian metrics with a positive cosmological constant.

In the next Section we consider some special values of $\kappa$, where a separate analysis is required.

**IX. SOME SPECIAL CASES.**

In this Section we consider two particular values of $\kappa$. They are $\kappa = 0$, and $\kappa = 1$. The reason why these values are particular can be seen from (19, 20), and (22, 23), which show that there is a qualitative change in the behaviour of the $F_i$, in the limits $y \to 0$, and $y \to 1/\Lambda$, and, therefore, in that of $W(y)$. In what follows we consider these cases separately.

**A. The case $\kappa = 0$.**

If we take $\kappa = 0$ in the perturbation equations (15-17), we get,

\[
\frac{dF_1}{dy} = -\frac{dF_4}{dy} - \frac{3 - 2\Lambda y}{3y(1 - \Lambda y)} F_4,
\]

and,

\[
\frac{dF_3}{dy} = -\frac{dF_4}{dy} - \frac{3 - 2\Lambda y}{3y(1 - \Lambda y)} F_4,
\]

and,

\[
\frac{d^2F_4}{dy^2} = \frac{2\Lambda y - 1}{y(1 - \Lambda y)} \frac{dF_4}{dy} - \frac{\Omega^2 F_4}{3y(1 - \Lambda y)^{5/3}} + \frac{9 - 6\Lambda y - 2\Lambda^2 y^2 + 3k^2 y(1 - \Lambda y)^{1/3}}{9y^2(1 - \Lambda y)^2} F_4
\]

This implies first that there is no gauge ambiguity other than an additive constant in $F_1$, and $F_3$. We also notice that once (88) is solved for $F_4(y)$, the other equations are solved by direct integration. Regarding (88) itself, the coefficients are regular in $0 < y < 1/\Lambda$, and diverge both at $y = 0$, and $y = 1/\Lambda$. We may use the standard procedure to put it in self adjoint (Schroedinger like) form, by changing to a new function $\tilde{F}(r)$, such that,

\[
F_4(y) = K_0(y) \tilde{F}(r(y))
\]

where,

\[
K_0(y) = \frac{1}{y^{1/4}(1 - \Lambda y)^{1/12}}
\]

and,

\[
\frac{dr}{dy} = \frac{1}{\sqrt{3}y^{1/2}(1 - \Lambda y)^{5/6}}
\]
Replacing in (88), we finally obtain,

$$-rac{d^2 \tilde{F}}{dy^2} + \tilde{V} \tilde{F} = \Omega^2 \tilde{F}$$  \hspace{1cm} (92)

where $\tilde{V}(r)$ is given implicitly by,

$$\tilde{V}(r) = \frac{16k^2y(1 - \Lambda y)^{1/3} + 45 - 40\Lambda y}{16y(1 - \Lambda y)^{1/3}}$$  \hspace{1cm} (93)

We can see that the “potential” $\tilde{V}(r)$ is positive definite and continuous in $0 < y < 1/\Lambda$, and that it diverges to $+\infty$, for both $y \to 0$, and $y \to 1/\Lambda$. It is straightforward to prove that (92) admits a unique self adjoint extension, and, on account of the properties of $\tilde{V}$, with a positive definite spectrum. This does not immediately imply that there are no unstable modes for $\kappa = 0$, because we have restricted to $\ell = 0$. In the next Subsection we analyze the case $\kappa = 1$, and find unstable modes. On account of the symmetries between metrics with $\kappa = 0$, and with $\kappa = 1$, this implies that there are also unstable modes for $\kappa = 0$, but for perturbations with $\ell \neq 0$.

**B. The case $\kappa = 1$.**

In the case $\kappa = 1$, the differential equation for $W(y)$ takes the form,

$$-rac{d^2 W}{dy^2} + \frac{\left(4y^{4/3}k^2 (2\Lambda y - 5) + 6\Lambda y + 3 \right)}{3 \left(4y^{4/3}k^2 - 4\Lambda y + 1 \right) y (1 - \Lambda y)} \frac{dW}{dy}$$

$$+ \frac{\left(4k^4y^{5/3} - y^{1/3} (-20 \Lambda y + 15 + 8 y^2 \Lambda^2) k^2 + 6\Lambda (1 - \Lambda y) \right)}{3 (1 - \Lambda y)^2 y (4y^{4/3}k^2 - 4\Lambda y + 1)} W = \frac{\Omega^2}{3 (1 - \Lambda y) y^{5/3} W}$$  \hspace{1cm} (94)

We first notice that if we define a new function $W_1$, such that $W(y) = W_1(z(y))$, where $z(y) = \Lambda y$, then (94) takes the form,

$$-rac{d^2 W_1}{dz^2} + \frac{\left(4z^{4/3}k_1^2 (2z - 5) + 6z + 3 \right)}{3 \left(4z^{4/3}k_1^2 - 4z + 1 \right) z (1 - z)} \frac{dW_1}{dz}$$

$$+ \frac{\left(4k_1^4z^{5/3} - z^{1/3} (-20z + 15 + 8 z^2) k_1^2 + 6 (1 - z) \right)}{3 (1 - z)^2 z \left(4z^{4/3}k_1^2 - 4z + 1 \right)} W_1 = \frac{\Omega_1^2}{3 (1 - z) z^{5/3} W_1}$$  \hspace{1cm} (95)

where $k_1 = k/\Lambda^{2/3}$, and $\Omega_1 = \Omega/\Lambda^{1/6}$. Therefore, since we are interested in the existence of solutions with $\Omega^2 < 0$, without loss of generality, we will set $\Lambda = 1$ in (94) in what follows. Next, we notice that for $k^2 > 3/4$, the coefficients in (94) are regular in $0 < y < 1$. Then, for $k^2 > 3/4$, with the transformation of Section V, we find for equation (44) the potential,

$$V(y) = \left[768y^4k^6 + 48y^{8/3} \left(29 - 56y \right) k^4 + 24y^{4/3} \left(68y - 73 + 32y^2 \right) k^2 \right.$$

$$\left. + 512y^4 - 896y^3 - 1 - 528y^2 + 832y \right] \times \left[48y^{1/3} \left(4y^{4/3}k^2 - 4y + 1 \right)^2 (1 - y) \right]^{-1}$$  \hspace{1cm} (96)
which is regular in $0 < y < 1$. In particular, near $y = 0$ we have,

$$V(y) = -\frac{1}{48y^{1/3}} + \mathcal{O}(y^{2/3}) \quad (97)$$

and, from (45), for $\kappa = 1$ this implies that near $r = 0$ we have,

$$V(r) = -\frac{1}{4r^2} + \mathcal{O}(r^4) \quad (98)$$

Similarly, near $y = 1$ we have,

$$V(y) = \frac{16k^2 - 3}{16(1 - y)} + \mathcal{O}((1 - y)^0) \quad (99)$$

and, therefore, near $r = r_1$, in terms of $r$ we have,

$$V(r) = \frac{16k^2 - 3}{12(r_1 - r)^2} + \mathcal{O}((r_1 - r)^0) \quad (100)$$

The form (98) of $V(r)$ implies that near $r = 0$ the general solution $\tilde{W}(r)$ of (43) takes the form,

$$\tilde{W}(r) \simeq C_1\sqrt{r} + C_2 \ln(r)\sqrt{r} + \ldots \quad (101)$$

where $C_1$, and $C_2$ are arbitrary constants. Similarly, near $r = r_1$ we have, in general,

$$\tilde{W}(r) \simeq C_3(r_1 - r)^{\frac{1}{2} + \frac{2\sqrt{3k}}{3} + C_4(r_1 - r)^{\frac{1}{2} - \frac{2\sqrt{3k}}{3} + \ldots \quad (102)}$$

with $C_3$, and $C_4$ arbitrary constants. As far as obtaining a self adjoint extension for (43), the behaviour of $\tilde{W}$ at the boundary $r = 0$ corresponds to the limit circle case. As discussed in [15], (see also [16]), in this case we impose on the solutions the condition $C_2 = 0$ at the boundary $r = 0$. For the boundary $r = r_1$ we obtain a self adjoint extension only if we impose the condition $C_4 = 0$. These boundary conditions then provide a self adjoint extension for (13). Since the interval $0 \leq r \leq r_1$ is finite, the spectrum is discrete, and, as can be shown, with the condition $C_2 = 0$, it is also bounded from below.

Since with the given boundary conditions the problem (13) is self adjoint, with an appropriate normalization, the set of eigenfunctions $\tilde{W}_\alpha(r)$ corresponding to the eigenvalues $\Omega_\alpha$ is complete and the $\tilde{W}_\alpha(r)$ satisfy the orthonormality conditions,

$$\int_0^{r_1} \tilde{W}_\alpha(r)\tilde{W}_\beta(r)dr = \delta_{\alpha\beta} \quad (103)$$

Recalling that we have $W_\alpha(y) = K(y)\tilde{W}_\alpha(r(y))$, this can be written as,

$$\int_0^1 \frac{1}{K(y)^2}W_\alpha(y)W_\beta(y)\sqrt{Q_3}dy = \sqrt{N_{\alpha}N_{\beta}}\delta_{\alpha\beta} \quad (104)$$

where,

$$N_{\alpha} = \int_0^1 \frac{1}{K(y)^2} (W_\alpha(y))^2 \sqrt{Q_3}dy \quad (105)$$
Therefore, instead of (43) we may solve directly the equation for $W_\alpha(y)$, without having to change to $r$. In this case, the boundary conditions corresponding to the self adjoint extension for $\tilde{W}_\alpha(r)$ imply that for $y \to 0$, $W_\alpha(y)$ admits an expansion of the form,

$$W_\alpha(y) = a_0 \left[ 1 - 3\Omega^2 y^{1/3} + \frac{9}{4}\Omega^4 y^{2/3} + \ldots \right]$$  \hspace{1cm} (106)

where $a_0$ is a constant, while for $y \to 1$, the expansion takes the form,

$$W_\alpha(y) = b_0 (1-y)^{\frac{k}{\sqrt{3}}} \left[ 1 - \frac{6 + 3\Omega^2 + 5\sqrt{3}k - 2k^2}{3(2\sqrt{3}k + 3)} (1-y) + \ldots \right]$$  \hspace{1cm} (107)

where $b_0$ is a constant. These expansions can be extended to arbitrary orders in either $y$, or $1-y$. We have used (106), extended to order $y^2$, and (107), extended to $(1-y)^{\frac{k}{\sqrt{3}}+3}$, to solve (94) numerically, (with $\Lambda = 1$), for different values of $\Omega$, and $k$, to obtain information on the spectrum of $\Omega$, for a given value of $k$. In more detail, let us call $W_\alpha(y, k, \Omega)$ the numerical solution of (94), obtained by imposing (106), for the given values of $k$, and $\Omega$. Similarly, $W_\beta(y, k, \Omega)$ is the numerical solution of (94) for the same $k$, and $\Omega$, but imposing (107). We also (arbitrarily) normalize these solutions so that $W_\alpha(0.5, k, \Omega) = W_\beta(0.5, k, \Omega) = 1$. These solutions are independent in general, but, for fixed $k$, we change the value of $\Omega$ until we find that the plots of $W_\alpha$, and $W_\beta$, as functions of $y$ superimpose as perfectly as the graphic depiction allows. The results, for two particular cases indicated in the captions, are shown in Figures 4 and 5, where the curves plotted are in fact the superposition of the (independent) solutions $W_\alpha$, and $W_\beta$, for the stated values of $k$, and $\Omega$. The reason for using this procedure is that, because of the singular nature of the boundaries, a simple “shooting” from either $y = 0$ or $y = 1$, trying to impose the appropriate boundary condition at the opposite end, turns out to be unreliable, and difficult to implement.

**FIG. 4:** A plot of $W(y)$, as a function of $y$, for $\Lambda = 1$, $k = 1$, and $\Omega^2 = -1.075...$ ($\Omega = 1.037...i$).
FIG. 5: A plot of $W(y)$, as a function of $y$, for $\Lambda = 1$, $k = 2$, and $\Omega^2 = -1.52...$ ($\Omega = 1.23...$).

Going back again to Figures 4 and 5, they correspond in each case to the lowest eigenvalue $\Omega^2$, which, as shown, is negative. Since they show the existence of solutions with $\Omega^2 < 0$, we conclude that, for $\kappa = 1$, we do have unstable solutions that are part of a complete spectrum, and therefore, in this case, i.e., $\kappa = 1$, (and, as indicated for $\kappa = 0$), the space time is also linearly unstable.

X. THE CASE $k = 0$. “RADIAL” PERTURBATIONS.

If we set $k = 0$, $\ell = 0$, the perturbations depend only on $t$, and the “radial” variable $y$. We may analyze this case as the limit as $k \to 0$ of the “diagonal” perturbations of the previous Sections, but it is simpler instead to consider it directly as perturbations that depend only on $y$ and $t$. In this case we may choose a gauge where only $F_1(y), F_2(y),$ and $F_3(y)$ are non vanishing. This gauge is unique up to an additive constant in $F_1(y)$. The resulting perturbation equations may be written in the form:

\[ \frac{dF_1}{dy} = \frac{(8\Lambda^2 y^2 \mu^2 - 12\mu \Lambda (\mu - 1 - \kappa) y - 9\mu - 9\kappa)}{(4 \Lambda y \mu - 3)^2} \frac{dF_3}{dy} + \frac{2\mu \Omega^2 y \frac{1}{\mu}}{(1 - \Lambda y)^{\frac{1}{\mu}}} F_3 + \frac{12\kappa (2 + \kappa) \Lambda \mu}{(4 \Lambda y \mu - 3)^2} F_3, \]

\[ F_2 = \frac{6y (\Lambda y - 1) \mu dF_3}{4 \Lambda y \mu - 3} \frac{dy}{4 \Lambda y \mu - 3} + \frac{3\kappa (2 + \kappa)}{4 \Lambda y \mu - 3} F_3. \]
and,
\[ \frac{d^2 F_3}{dy^2} = \frac{(16 \Lambda^3 \mu^2 y^3 - 8 \Lambda^2 \mu (3 + \mu) y^2 - 6 \Lambda (2 \mu \kappa - 3 \kappa - \mu - 3) y - 9 \kappa - 9) dF_3}{dy} \]
\[ - \frac{y^{\frac{1-2\mu}{2}} \Omega^2}{3 (1 - \Lambda y)^{\frac{2+3\mu}{2}}} F_3 + \frac{6 (2 + \kappa) (2 \kappa + 1) \Lambda y}{y (4 \Lambda y \mu - 3) (1 - \Lambda y) (2 \Lambda y \mu - 3 - 3 \kappa)} F_3 \] (109)

In accordance with (108), this implies that both \( F_2 \) and \( F_1 \), (up to a constant) are given, once we find the solutions for (109). However, instead of \( F_3(y) \), it will be simpler here to consider again the gauge invariant \( W \), which, for \( F_4 = 0 \), reduces to,
\[ W(y) = (3 + 3 \kappa - 2 \Lambda \mu y) F_3(y) \] (110)

Replacing in (109), we find,
\[ \frac{d^2 W}{dy^2} = \frac{(4 \Lambda y \mu - 6 \Lambda y + 3)}{y (1 - \Lambda y) (4 \Lambda y \mu - 3)} \frac{dW}{dy} - \frac{y^{\frac{1-2\mu}{2}} \Omega^2}{3 (1 - \Lambda y)^{\frac{2+3\mu}{2}}} W - \frac{2 \Lambda (2 \mu - 3)}{y (1 - \Lambda y) (4 \Lambda y \mu - 3)} W \] (111)

which is just the limit \( k \to 0 \) of (35). We may, therefore, use the results of Section V and VI to analyze the resulting spectrum of \( \Omega \). We notice here that if we rescale \( y \) to \( y/\Lambda \), and \( \Omega^2 \) to \( \Omega^2 / \Lambda^{1/\mu} \), (111) takes the form,
\[ \frac{d^2 W}{dy^2} = \frac{(4 \mu y - 6 \mu + 3)}{y (1 - y) (4 \mu y - 3)} \frac{dW}{dy} - \frac{y^{\frac{1-2\mu}{2}} \Omega^2}{3 (1 - y)^{\frac{2+3\mu}{2}}} W - \frac{2 (2 \mu - 3)}{y (1 - y) (4 \mu y - 3)} W \] (112)

and, therefore, in what follows we will set \( \Lambda = 1 \), and, without loss of generality, analyze directly (112). There remains the problem of determining the spectrum of \( \Omega \). For this purpose we simply consider again the derivations of Sections III, and Section IV, (in the limit \( k \to 0 \)), noticing that both \( r(y) \) and \( K(y) \), are independent of \( k \), and depend only on \( \mu \). Regarding the construction of \( \tilde{V}(r) \), we notice that in this case, for \( \Omega = 0 \), eq. (112) has the exact solution, [17]
\[ W(y) = C_1 [3 + (4 \mu - 6) y] + C_2 [(3 + (4 \mu - 6) y) \ln(y/(1 - y)) + 6 - 8 \mu y] \] (113)
where \( C_1 \), and \( C_2 \) are arbitrary constants. We, therefore, set,
\[ W^{(0)}(y) = C_1 [3 + (4 \mu - 6) y] \] (114)

and, replacing appropriately in (64), we find,
\[ \tilde{V}(r) = \frac{(1 - y)^{1/\mu - 4/3}}{16 y^{1/\mu} (4 \mu y - 6 y + 3)^2 \mu^2} \left[ 81 - \left( 864 \mu + 1152 \mu^3 + 324 - 2160 \mu^2 \right) y \right. \]
\[ + \left. \left( 324 - 6480 \mu^2 + 3024 \mu + 3072 \mu^3 \right) y^2 + 32 \mu (2 \mu - 3) (4 \mu^2 - 36 \mu + 27) y^3 \right] \] (115)

The term in square brackets in (115) is a polynomial in \( y \) and therefore it is bounded above and below in \( 0 \leq y \leq 1 \). The factor in front of this polynomial is positive definite and continuous in \( 0 < y < 1 \) and diverges to \( +\infty \), both at \( y = 0 \) and \( y = 1 \). The potential \( \tilde{V}(r) \)
is therefore continuous and bounded from below in $0 < y < 1$. To analyze (115) further we notice that near $y = 0$, to leading order, we have,

$$
\tilde{V}(r) = \frac{9}{16\mu^2 y^{1/\mu}} + \ldots
$$

(116)

while, in accordance with (45), again to leading order, we have,

$$
\frac{1}{y^{1/\mu}} = \frac{4\mu^2}{3r^2} + \ldots
$$

(117)

Therefore, also to leading order as $y \to 0$, we have,

$$
\tilde{V}(r) = \frac{3}{4r^2} + \ldots
$$

(118)

and this implies that near $y = 0$, also to leading order, the general solution of (58) is, in this case, of the form,

$$
\chi(r) = C_1 r^{3/2} + C_2 r^{-1/2} + \ldots
$$

(119)

where $C_1$ and $C_2$ are arbitrary constants.

Similarly, near $y = 1$, we have $r \to r_1$, and, to leading order,

$$
(1 - y)^{3-4\mu} = \frac{12\mu^2}{(4\mu - 3)^2(r_1 - r)^2} + \ldots
$$

(120)

and, therefore, again near $y = 1$,

$$
\tilde{V}(r) = \frac{3}{4(r_1 - r)^2} + \ldots
$$

(121)

and this implies that near $r = r_1$, the general solution $\chi(r)$ of (58), to leading order, behaves as,

$$
\chi(r) = C_3(r_1 - r)^{3/2} + C_4(r_1 - r)^{-1/2} + \ldots
$$

(122)

These results imply that we will obtain a self adjoint extension of (58) if we impose $C_2 = 0$, and $C_4 = 0$, as boundary conditions on its solutions. The relation of the solutions of the self adjoint extension of (58) to the general solution of the evolution of arbitrary perturbations follows the lines indicated in the previous Sections, and we shall not repeat it here.
FIG. 6: A plot of $\tilde{V}(y)$, (Eq. (115)), as a function of $r$, for $\Lambda = 1$, and for $\mu = 3/2$, (solid line), and $\mu = 6/5$ (dashed line).

There remains the problem of analyzing the spectrum of this self adjoint extension. From the general properties of $\tilde{V}(r)$, this spectrum will be discrete and bounded from below. However, in general, and as shown in Figure 4 for two examples, $\tilde{V}(r)$ is not positive definite. Therefore, the lowest eigenvalue of (58) might be negative. This can only be analyzed numerically. Here we have the simplifying feature that $\tilde{V}(r)$ is given explicitly as a function of $y$. If we define $\chi_1(y) = \chi(r(y))$, and change coordinates in (58) back to $y$ we find,

$$\frac{d^2 \chi_1}{dy^2} = \frac{(8\mu y + 3 - 6\mu)}{6y(1-y)\mu} \frac{d\chi_1}{dy} - \frac{y^{\frac{1-2\mu}{2\mu}}\Omega^2}{3(1-y)^{\frac{2\mu+3}{2\mu}}} \chi_1$$

$$- \frac{1}{48y^2(1-y)^2(4\mu y - 6y + 3)^2\mu^2} \left[ 81 + 32\mu(2\mu - 3)(4\mu^2 - 36\mu + 27)y^3 + (3024\mu + 3072\mu^3 + 324 - 6480\mu^2)y^2 - (864\mu + 1152\mu^3 + 324 - 2160\mu^2)y \right] \chi_1$$

The boundary conditions for (123) corresponding to the self adjoint extension of (58) are in this case,

$$\chi_1(y) \rightarrow A y^{\frac{3}{4\mu}} \; ; \; \text{for} \; y \rightarrow 0$$

$$\chi_1(y) \rightarrow B (1-y)^{\frac{4\mu-3}{4\mu}} \; ; \; \text{for} \; y \rightarrow 1$$

where $A$ and $B$ are constants. The range of $\mu$ we are interested in is $1 \leq \mu \leq 3$. However, if we take a solution $\chi_1(y)$ of (123), corresponding to the eigenvalue $\Omega^2$, and we define,

$$\chi_2(y) = \chi_1(1-y)$$

one can check that $\chi_2(y)$ is also a solution of (123) for the same $\Omega^2$, but with $\mu$ replaced by,

$$\nu = \frac{3\mu}{4\mu - 3}$$
and satisfying also (124), with the same replacement. Since for \(1 \leq \mu \leq 3/2\), we have \(3 \geq \nu \geq 3/2\), we only need to study the range \(1 \leq \mu \leq 3/2\). We notice that for \(\mu = 1\), \((\kappa = 0)\), and \(\Omega = 0\), we have the exact solution,

\[
\chi_1(y) = C_1 \frac{y^\frac{3}{4}(1-y)^\frac{1}{4}}{2y-3} + C_2 \frac{4y-3}{y^\frac{1}{4}(1-y)^\frac{1}{4}(2y-3)}
\]

where \(C_1\) and \(C_2\) are constants. Clearly, the solution corresponding to the self adjoint extension is obtained by setting \(C_2 = 0\). Since in this case (127) has no nodes, it must correspond to the lowest eigenvalue. Similarly, for \(\mu = 3\), we have the exact solution,

\[
\chi_1(y) = C_1 \frac{(1-y)^\frac{1}{4}y^\frac{1}{4}}{2y+1} + C_2 \frac{4y-1}{(2y+1)(1-y)^\frac{1}{4}y^\frac{1}{4}}
\]

(128)

corresponding again to the eigenvalue \(\Omega^2 = 0\), in agreement with our previous discussion. As a check for the numerical integration procedure, we reobtained numerically the solution (127), (with \(C_2 = 0\)). For the next higher eigenvalue we obtained \(\Omega = 1.635...\), \((\omega^2 = 2.673...)\).

We have also explored numerically the range \(1 < \mu \leq 3/2\), and found that the lowest eigenvalue \(\Omega^2\) was in all cases larger than zero. For instance, for \(\mu = 6/5\) we found \(\Omega = 0.854...\), and for \(\mu = 3/2\) we obtained \(\Omega = 1.05...\). For the next eigenvalue we found \(\Omega = 2.17...\), for \(\mu = 6/5\), and \(\Omega = 2.35...\), for \(\mu = 3/2\).

The general conclusion here is that there are no unstable modes for the “radial” perturbations \((k = 0, \ell = 0)\). We recall that solutions with \(\Omega = 0\), for \(\chi(r)\), do not correspond to solutions for \(W(y)\) with the same eigenvalue.

XI. SUMMARY AND CONCLUSIONS.

In this paper we have analyzed the linear stability of the Linet - Tian space times with a positive cosmological constant \(\Lambda\). These space times contain a particular symmetry relating their two space like Killing vectors that is not present in the case of a negative \(\Lambda\). This symmetry allows for a simplification regarding the parameter space to be analyzed. In a separate study in [8] it was shown that a particular class of perturbations can be restricted to the “diagonal” form (13). This leads to a consistent set of perturbation equations, but it still contain a gauge ambiguity that cannot be removed. To deal with this problem we introduced a gauge invariant function of the metric perturbations, called \(W(y)\) in the text, that satisfies an equation that takes the form of a linear boundary value problem, from which one can extract the allowed values of \(\Omega\), the frequency of the perturbation modes. Although these solutions are well defined, it is not at all clear if, and in what sense, the set of solutions for \(W(y)\) is complete, and in what way they are related to the evolution of arbitrary initial data. This problem is solved by the introduction, using the Darboux transformation, of a related self adjoint problem, with a complete set of eigenvalues and eigenfunctions. This leads to a definite form for the expansion of arbitrary initial data in terms of the modes of \(W(y)\), and its corresponding eigenvalues \(\Omega\). The remaining problem, the existence of solutions of the equation for \(W(y)\) with the required properties is solved numerically, and we display several examples of those solutions. This in turn provides a complete proof of the linear instability of the Linet - Tian metric. It turns out that for
\( \kappa = 0 \), and \( \kappa = 1 \) the perturbation equation require a special treatment that is also given in this paper, again showing that even for those cases the space time is linearly unstable. We have also included a discussion of “radial” perturbations, i.e., those preserving the space like Killing vectors. We find that in this case there are no unstable modes. In conclusion we have shown that linear perturbations of the Linet - Tian metric contain unstable modes, and that these unstable modes are part of a complete set of solutions, and therefore, the linear instability is a generic feature of these metrics.

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[18] Notice that to first order in \( \epsilon \) we may replace \( (T, Y, Z, \Phi) \) by \( (t, y, z, \phi) \) in the equations that follow.
One can straightforwardly check that this choice of gauge leads to a well determined self-adjoint problem, with a complete set of solutions with $\Omega^2 > 0$. Therefore, this sector has only stable modes, and will not be further analyzed here.

### Appendix A: Regularity of $W$ at $y = y_s$.

Consider a function $W(y)$ that is a solution, in a certain range $y_1 \leq y \leq y_2$, of an equation of the general form,

$$-rac{d^2 W}{dy^2} + Q_1 \frac{dW}{dy} + Q_2 W = \Omega^2 Q_3$$  \hspace{1cm} (A1)

where $Q_1$, $Q_2$, and $Q_3$ are functions of $y$. We assume no particular boundary conditions on either $W$ or the functions $Q_i$, but we shall assume that in a neighbourhood of a point $y = y_s$, with $y_s$ interior to the interval $(y_1, y_2)$, the functions $Q_i$ admit Laurent expansions of the form,

$$Q_1(y) = \frac{2}{y - y_s} + a_1 + a_2(y - y_s) + a_3(y - y_s)^2 + a_4(y - y_s)^3 + \ldots$$

$$Q_2(y) = \frac{b_0}{y - y_s} + b_1 + b_2(y - y_s) + b_3(y - y_s)^2 + b_4(y - y_s)^3 + \ldots$$ \hspace{1cm} (A2)

$$Q_3(y) = c_1 + c_2(y - y_s) + c_3(y - y_s)^2 + c_4(y - y_s)^3 + \ldots$$

where dots indicate higher order terms. We also assume $c_1 > 0$. Then, if the coefficients in $[A2]$ satisfy the relations,

$$c_2 = c_1(a_1 + b_0)$$ \hspace{1cm} (A3)

$$b_2 = b_1a_1 - \frac{1}{2}a_1^2b_0 - \frac{3}{4}a_1b_0^2 + \frac{1}{2}a_2b_0 + b_1b_0 - \frac{1}{4}b_0^3$$

a straightforward computation shows that the general solution of $[A1]$ takes the form,

$$W(y) = w_0 + w_1(y - y_s) + w_2(y - y_s)^2 + w_3(y - y_s)^3 + w_4(y - y_s)^4 + \ldots$$ \hspace{1cm} (A4)

where $w_0$, and $w_3$ are arbitrary constants,

$$w_1 = -\frac{1}{2}b_0w_0$$ \hspace{1cm} (A5)

$$w_2 = \frac{1}{4}(b_0^2 - 2b_1 + a_1b_0 + 2c_1\Omega^2)w_0$$

and $w_4$, and higher order terms are (homogeneous) linear functions of $w_0$, and $w_3$. This implies that the general solution of $[A1]$, with the $Q_i$ satisfying $[A3]$, independently of any boundary conditions at either $y = y_1$, or $y = y_2$, is a linear combination of two regular solutions, one that near $y = y_s$ behaves as,

$$W(y) = w_0 \left(1 + \tilde{w}_1(y - y_s) + \tilde{w}_2(y - y_s)^2 + \tilde{w}_3(y - y_s)^3 + \ldots \right)$$ \hspace{1cm} (A6)

and another solution that near $y = y_s$ behaves as,

$$W(y) = w_3 \left[(y - y_s)^3 + \tilde{w}_4(y - y_s)^4 + \ldots \right]$$ \hspace{1cm} (A7)
Appendix B: Regularity of $\tilde{V}(r)$ at $y = y_s$.

We first write (64) in the form,

$$\tilde{V}(r(y)) = -\frac{1}{4Q_3^3} \left[ Q_3 \frac{d^2Q_3}{dy^2} - \frac{5}{4} \left( \frac{dQ_3}{dy} \right)^2 \right] - \frac{1}{4Q_3} \left( Q_1^2 + 4Q_2 - 2 \frac{dQ_1}{dy} \right)$$

$$+ 2\Omega_0^2 + \frac{2}{Q_3} \left( \frac{1}{W_1^{(0)}} \frac{dW_1^{(0)}}{dy} - \frac{1}{2}Q_1 + \frac{1}{4Q_3} \frac{dQ_3}{dy} \right)^2$$

(B1)

Using now the results of Appendix A, where we assume for $W_1^{(0)}$ the general expansion (A4), we find,

$$\tilde{V}(r(y)) = \left( \frac{24a_2 + 19a_1^2 + 14a_1b_0 + 7b_0^2 + 48b_1}{16c_1^2} \right) c_1 - 40c_3$$

$$- 2\Omega_0^2 + O(y - y_s)$$

(B2)

Then, since $dr/dy = \sqrt{Q_3}$, near $y = y_s$ we have,

$$y = y_s + \frac{1}{\sqrt{c_1}} (r - r_s) + O(r - r_s)$$

(B3)

where $r_s = r(y_s)$, and this implies that near $r - r_s$ we have,

$$\tilde{V}(r) = \left( \frac{24a_2 + 19a_1^2 + 14a_1b_0 + 7b_0^2 + 48b_1}{16c_1^2} \right) c_1 - 40c_3$$

$$- 2\Omega_0^2 + O(r - r_s)$$

(B4)

and therefore $\tilde{V}(r)$ is regular at $r = r_s$, irrespective of the choice of $W_1^{(0)}(y)$, provided only that $W_1^{(0)}(y_s) \neq 0$.

Appendix C: The self adjoint problem associated to $W$.

We consider again (B1). As shown in Appendix B, $\tilde{V}(y(r))$ is regular at $y = y_s$ for any solution $W_1^{(0)}(y)$ of (35) that does not vanish at $y = y_s$. Let us now assume that $W_1^{(0)}$ satisfies the boundary conditions (27,28), and that it does not vanish anywhere in $0 \leq y \leq 1/\Lambda$. Then, since $Q_3(y) > 0$ in $0 < y < 1/\Lambda$, and $Q_1(y)$, and $Q_2(y)$ are regular in that interval, away from $y = y_s$, $\tilde{V}(y(r))$ will also be regular in that interval. As we approach $y = 0$, to leading terms, we have,

$$Q_1(y) \approx -\frac{1}{y} - \frac{(8\mu - 3)\Lambda}{3}$$

$$Q_2(y) \approx -\frac{k^2(2\kappa + 3)(2\kappa + 1)y^{3y}}{3y} + \frac{2\Lambda(2\mu - 3)}{3y}$$

$$Q_3(y) \approx \frac{y^{3y}}{3y^2} + \frac{(3 + 2\mu)\Lambda y^{3y}}{9\mu y}$$

(C1)
and, therefore, we have,
\[ \tilde{V}(y(r)) = \frac{9}{16 \mu^2 y^{1/2}} + O(y^0) \]  
(C2)

and, since near \( y = 0 \) we have,
\[ r(y) \approx \frac{2 \mu y^{1/2}}{\sqrt{3}} \]  
(C3)

we finally get that as \( r \to 0 \), we have,
\[ \tilde{V}(r) = \frac{3}{4r^2} + \ldots \]  
(C4)

where dots indicate higher order terms.

Similarly, near \( y = 1/\Lambda \) we have,
\[ Q_1(y) \approx -\frac{\Lambda}{1 - \Lambda y} + \ldots \]
\[ Q_2(y) \approx \frac{(4\kappa - 1)\Lambda^{\frac{4\kappa}{\mu} - 1}}{(2\kappa + 1)^2} (1 - \Lambda y)^{-\frac{3 - 4\mu}{\mu}} + \ldots \]  
(C5)

\[ Q_3(y) \approx \frac{\Lambda^{\frac{2\mu - 1}{\mu}}}{3} (1 - \Lambda y)^{-\frac{3 + 2\mu}{\mu}} + \ldots \]

then, replacing in (B1) we find,
\[ \tilde{V}(y(r)) = \frac{(4\mu - 3)^2 \Lambda^{1/2}}{16 \mu^2} (1 - \Lambda y)^{-\frac{3 - 4\mu}{\mu}} + \ldots \]  
(C6)

Using now (11), near \( y = 1/\Lambda \) we have,
\[ r(y) = r_1 - \frac{2 \sqrt{3} \mu}{\Lambda^{1/2} (4\mu - 3)} (1 - \Lambda y)^{\frac{4\mu - 3}{6\mu}} + \ldots \]  
(C7)

where \( r_1 = r(1/\Lambda) \), and this implies,
\[ \tilde{V}(r) = \frac{3}{4(r_1 - r)^2} + \ldots \]  
(C8)

where in all these expressions dots indicate higher order terms.

Therefore, if \( W(y) \) does not vanish in \( 0 < y < 1/\Lambda \), since the \( Q_i \) are bounded away from \( y = 0, y = y_s \) and \( y = 1/\Lambda \), and \( \tilde{V}(r) \), as already shown, is regular at \( y = y_s \), we have that \( \tilde{V}(r) \) will have a lower bound in \( 0 < r < r_1 \), diverging at \( r = 0 \) and at \( r = r_1 \) as given by (C4), and (C8), respectively. But this, in turn, implies that near \( r = 0 \) the solutions \( \chi(r) \) of (58) behave as,
\[ \chi(r) \approx C_1 r^{3/2} + C_2 r^{-1/2} \]  
(C9)

and near \( r = r_1 \) as,
\[ \chi(r) \approx C_3 (r_1 - r)^{3/2} + C_4 (r_1 - r)^{-1/2} \]  
(C10)

where the \( C_i \) are arbitrary constants, and, therefore, we may construct a self adjoint extension of (58) by imposing the boundary conditions that \( C_2 = 0 \), and \( C_4 = 0 \). The resulting spectrum of \( \Omega^2 \) will be fully discrete and bounded from below.
Appendix D: Gauge transformations of the perturbed Linet-Tian metric.

Consider the perturbation expansion (10), restricted, for simplicity, to functions of the form (11), but with \( \ell = 0 \). We write it in the form,

\[
\begin{align*}
\text{Appendix D: Gauge transformations of the perturbed Linet-Tian metric.} \\
\end{align*}
\]

\[
\begin{align*}
ds^2 &= -\frac{y^{\frac{1}{3}+\frac{p_2}{2}}}{(1-\Lambda y)^{\frac{2}{3}+\frac{1}{2}}} (1 + e^{i(\Omega t-kz)} F_1) \, dt^2 + \frac{1}{3y(1-\Lambda y)} (1 + e^{i(\Omega t-kz)} F_2) \, dy^2 \\
&\quad + \frac{y^{\frac{1}{3}+\frac{p_2}{2}}}{(1-\Lambda y)^{\frac{2}{3}+\frac{1}{2}}} (1 + e^{i(\Omega t-kz)} F_3) \, dz^2 + \frac{y^{\frac{1}{3}+\frac{p_2}{2}}}{(1-\Lambda y)^{\frac{2}{3}+\frac{1}{2}}} (1 + e^{i(\Omega t-kz)} F_4) \, d\phi^2 \\
&\quad + 2e^{i(\Omega t-kz)} (F_5 dt dy + F_6 dt dz + F_7 dz dy + F_8 dt d\phi + F_9 dy d\phi + F_{10} dz d\phi) \\
\end{align*}
\]

where \( F_i = F_i(y) \), and \( \epsilon \) is used to keep track of the perturbation order. Under a first order transformation to new coordinates \( (T, Y, Z, \Phi) \), such that,

\[
\begin{align*}
t &= T + e^{i(\Omega T-kZ)} Q_1(Y) ; \quad y = Y + e^{i(\Omega T-kZ)} Q_2(Y) \\
z &= Z + e^{i(\Omega T-kZ)} Q_3(Y) ; \quad \phi = \Phi + e^{i(\Omega T-kZ)} Q_4(Y) \\
\end{align*}
\]

where the functions \( Q_i \) are arbitrary, the form of the metric (D1) is preserved, but the coefficients \( F_i(y) \) are changed to \( \tilde{F}_i(y) \), where \( [18] \),

\[
\begin{align*}
\tilde{F}_1(y) &= F_1(y) + 2i\Omega Q_1(y) - \frac{4\Lambda y - 2 - 3p_1}{6y(1-\Lambda y)} Q_1(y) \\
\tilde{F}_2(y) &= F_2(y) - \frac{1 - 2\Lambda y}{y(1-\Lambda y)} Q_2(y) + 2 \frac{dQ_2}{dy} \\
\tilde{F}_3(y) &= F_3(y) + \frac{2 + 3p_2 - 4\Lambda y}{6y(1-\Lambda y)} Q_2(y) - 2i k Q_3(y) \\
\tilde{F}_4(y) &= F_4(y) + \frac{2 + 3p_2 - 4\Lambda y}{6y(1-\Lambda y)} Q_2(y) \\
\tilde{F}_5(y) &= F_5(y) + \frac{i\Omega}{3y(1-\Lambda y)} Q_2(y) - y^{\frac{1}{3}+\frac{p_2}{2}} (1-\Lambda y)^{\frac{1}{3}+\frac{p_2}{2}} \frac{dQ_1}{dy} \\
\tilde{F}_6(y) &= F_6(y) + i\Omega y^{\frac{1}{3}+\frac{p_2}{2}} (1-\Lambda y)^{\frac{1}{3}+\frac{p_2}{2}} Q_3(y) + i k y^{\frac{1}{3}+\frac{p_2}{2}} (1-\Lambda y)^{\frac{1}{3}+\frac{p_2}{2}} Q_1(y) \\
\tilde{F}_7(y) &= F_7(y) - \frac{i k}{3y(1-\Lambda y)} Q_2(y) + y^{\frac{1}{3}+\frac{p_2}{2}} (1-\Lambda y)^{\frac{1}{3}+\frac{p_2}{2}} \frac{dQ_3}{dy} \\
\end{align*}
\]

and,

\[
\begin{align*}
\tilde{F}_8(y) &= F_8(y) + i\Omega y^{\frac{1}{3}+\frac{p_2}{2}} (1-\Lambda y)^{\frac{1}{3}+\frac{p_2}{2}} Q_4(y) \\
\tilde{F}_9(y) &= F_9(y) + y^{\frac{1}{3}+\frac{p_2}{2}} (1-\Lambda y)^{\frac{1}{3}+\frac{p_2}{2}} \frac{dQ_4}{dy} \\
\tilde{F}_{10}(y) &= F_{10}(y) - i k y^{\frac{1}{3}+\frac{p_2}{2}} (1-\Lambda y)^{\frac{1}{3}+\frac{p_2}{2}} Q_4(y) \\
\end{align*}
\]

This implies that the sets \( S_1 = \{ F_1, F_2, F_3, F_4, F_5, F_6, F_7 \} \), and \( S_2 = \{ F_8, F_9, F_{10} \} \) transform independently of each other. This is reflected also in the perturbation equations, since, as can be checked, they separate also into two set, one coupling only the functions in \( S_1 \), and the other only those in \( S_2 \).
Going back to (D4), we can, e.g., choose $Q_4$ such $F_8(y) = 0$, and that fixes the gauge, in the sense that $F_9(y)$, and $F_{10}(y)$ are uniquely determined \[19\]. In the case of (D3) we may choose the $Q_i$ such that only $F_1$, $F_2$, $F_3$, and $F_4$, (the “diagonal” terms), are non vanishing, and one can check that this choice leads to a consistent set of perturbation equations (given by (13)) in Section III. This restriction, however, does not determine completely these functions, because, as can be checked, a transformation with the set of functions,\

\[
\begin{align*}
Q_1(y) &= \Omega y^{\frac{p_2-p_1}{4}} (1 - \Lambda y)^{\frac{p_1-p_2}{4}} Q_0 \\
Q_2(y) &= \frac{3}{4} i(p_1-p_2)y^{\frac{1}{4} + \frac{p_1+p_2}{4}} (1 - \Lambda y)^{\frac{1}{4} - \frac{p_1-p_2}{4}} Q_0 \\
Q_3(y) &= -ky^{\frac{p_2-p_1}{4}} (1 - \Lambda y)^{\frac{p_1-p_2}{4}} Q_0 \\
\end{align*}
\]  \\

(D5)

where $Q_0$ is an arbitrary function of $\Omega$, leaves the “diagonal” form invariant. This implies that to any solution of the perturbation equations for $F_1$, $F_2$, $F_3$, and $F_4$, even the trivial one where all $F_i = 0$, we may add a solution of the form (18), and still have a solution.