RATIONAL HOMOLOGY OF THE ORDER COMPLEX OF ZERO SETS OF HOMOGENEOUS QUADRATIC POLYNOMIAL SYSTEMS IN $\mathbb{R}^3$

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Abstract. The naturally topologized order complex of proper algebraic subsets in $\mathbb{R}P^2$, defined by systems of quadratic forms, has rational homology of $S^{13}$.

1. Introduction

Order complexes of zero sets of polynomial systems of various types are an important part of the study of (discriminant or resultant) varieties of non-generic systems, and hence of their complementary sets of generic systems (or other similar objects). For some examples and applications see [9], [10], [3], [6]. In this article we study the (naturally compactified) order complex of proper subsets in $\mathbb{R}P^2$ defined by systems of homogeneous quadratic polynomials; in particular we calculate its rational homology group.

1.1. Main result. Let $W \simeq \mathbb{R}^6$ be the space of all real quadratic forms in $\mathbb{R}^3$. For any $i = 1, \ldots, 5$ consider the Grassmann manifold $G_i(W)$ of $i$-dimensional subspaces in $W$.

Any collection of points in $\mathbb{R}P^2$ defines a vector subspace in $W$ consisting of all quadratic forms vanishing on all corresponding lines in $\mathbb{R}^3$. Denote by $J_i \subset G_{6-i}(W)$ the closure of the union of all points in $G_{6-i}(W)$ corresponding to such subspaces of codimension $i$.

The disjoint union $J_1 \sqcup \cdots \sqcup J_5$ is a partially ordered set, with the order relation defined by the incidence of corresponding subspaces.

Consider the join $J_1 \ast \cdots \ast J_5$, i.e. the union of all simplices of dimensions $\leq 4$, whose vertices correspond to points of different $J_i$, and define the order complex $\Psi \subset J_1 \ast \cdots \ast J_5$ of our partially ordered set as the union of those such simplices all whose vertices are incident to one another. $\Psi$ is a closed subset of the compact space $J_1 \ast \cdots \ast J_5$, in particular is also compact.

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Theorem 1. The rational homology group of the order complex $\Psi$ is equal to $\mathbb{Q}$ in dimensions 0 and 13 and is trivial in all other dimensions.

Remark 1. Similar order complexes related with different spaces of functions participate in many known theories and theorems. For instance, by a version of C. Caratheodory theorem, the order complex of proper subsets in $\mathbb{RP}^1$ specified by systems of homogeneous polynomials of degree $r$ in $\mathbb{R}^2$ is homotopy equivalent to $S^{2r-1}$, see [4]. The order complex of proper vector subspaces in $\mathbb{R}^n$ (= subsets specified by systems of linear equations) is homeomorphic to $S^N$, $N = n(n-1)/2 + n - 2$. Also, the homology groups of certain such complexes related with complex homogeneous polynomials contribute to the homology groups of moduli spaces of complex curves, see [6], [10], [3].

Remark 2. The order complex $\Psi$ is closely related to the geometry of the resultant varieties in the spaces of quadratic polynomial systems in $\mathbb{R}^3$, in particular it is the principal part of the construction of one of possible simplicial resolutions of these varieties. These spaces were studied in [11], where however a different construction of resolutions (the one proposed in [3]) was used.

Problems. There are obvious questions concerning the extension of our calculation to integral homology groups, and also to polynomial systems of higher degrees and/or in spaces of higher dimensions.

1.2. Main spectral sequence. For any $p = 1, \ldots, 5$ and any point $X \subset J_p$, define its subordinate subcomplex $\Psi(X)$ as the union of those simplices in $J_1 \ast \cdots \ast J_p$, all whose vertices correspond to subspaces in $W$ containing the $(6-p)$-dimensional subspace $\{X\}$. By the construction, any $\Psi(X)$ is a compact contractible topological space: it is a cone with vertex $X$.

Define the filtration

$$
\Psi_1 \subset \cdots \subset \Psi_5 \equiv \Psi
$$

in the order complex $\Psi$: its term $\Psi_p$ consists of all simplices, all whose vertices belong to $J_1, J_2, \ldots, J_p$ only; in particular $\Psi_1 = J_1$. In other words, $\Psi_p$ is the union of subcomplexes $\Psi(X)$ over all $X \in J_1, \ldots, J_p$.

Consider the spectral sequence $E_{p,q}^r$, calculating the group $H_*(\Psi, \mathbb{Q})$ and generated by this filtration. Its term $E_{p,q}^1$ is canonically isomorphic to the group $H_{p+q}(\Psi_p, \Psi_{p-1}; \mathbb{Q}) \equiv \overline{\mathcal{P}}_{p+q}(\Psi_p \setminus \Psi_{p-1}, \mathbb{Q})$; here $\overline{\mathcal{P}}_*$ denotes the Borel–Moore homology group, i.e. the homology group of the complex of locally finite singular chains.

Theorem 2. All non-trivial groups $E_{p,q}^1$ of our spectral sequence are shown in Fig. 1. The differential $d^1 : E_{4,5}^1 \to E_{3,5}^1$ is an isomorphism.
Theorem 1 follows immediately from this one. The proof of Theorem 2 takes the rest of the article.

1.3. **Stratification of spaces** $\Psi_p \setminus \Psi_{p-1}$ and $J_p$. There is a natural map $\Psi_p \setminus \Psi_{p-1} \to J_p$: it associates with any point $\varphi \in \Psi_p \setminus \Psi_{p-1}$ the unique point $X \in J_p$ such that $\varphi$ belongs to the cone $\Psi(X)$. Therefore, any space $\Psi_p \setminus \Psi_{p-1}$, $p = 1, \ldots, 5$, splits into several pieces, corresponding to the strata of the natural stratification of the space $J_p$. The latter stratification is elementary: here is the list of its strata.

**Lemma 1.** The varieties $J_i$, $i = 1, \ldots, 5$, consist of the smooth strata, whose points correspond to the subspaces in $W$, formed respectively by polynomials

1. **(J_1)** Vanishing at some point in $\mathbb{P}^2$.
2. **(J_2)**
   1. (a) Vanishing at some two different points in $\mathbb{P}^2$.
   2. (b) Vanishing at some point in $\mathbb{P}^2$ and having zero derivative along a fixed direction at this point.
3. **(J_3)**
   1. (a) Vanishing at some three points in general position (i.e. not in the same line).
   2. (b) Vanishing on some line in $\mathbb{P}^2$.
   3. (c) Vanishing at some two points in $\mathbb{P}^2$ and having zero derivative along a fixed direction at one of these points; this direction is different from the line between these two points.
   4. (d) Vanishing with all partial derivatives at a point in $\mathbb{P}^2$.
4. **(J_4)**
   1. (a) Vanishing at some four points in general position (none three in the same line).
   2. (b) Vanishing on a line and a point outside it.
(c) Vanishing at three points in general position and having zero
derivative along a fixed direction at one of them; this direction should
not coincide with the line connecting this point with either of other two.
(d) Vanishing at two points and having zero derivative along
fixed directions at any of these points; these directions should be differ-
ent from the line connecting these two points.
(e) Vanishing on a line and having a singularity at one fixed
point in it.

\( J_5 \)  
(a) Vanishing on a smooth non-empty conic.
(b) Vanishing on two different lines.
(c) Vanishing on a line with multiplicity 2.

\[ \square \]

**Notation.** In what follows we denote by \( J_p(\alpha) \) the stratum of \( J_p \) listed
in the corresponding item \( (J_p)(\alpha) \) of this lemma. Also, we denote by
\( \Psi_p(\alpha) \) the preimage of \( J_p(\alpha) \) under the natural map
\( \Psi_p \setminus \Psi_{p-1} \to J_p \).

Over any such stratum \( J_p(\alpha) \), this map is a locally trivial fiber bundle
\[ (2) \quad \Psi_p(\alpha) \to J_p(\alpha). \]

The fiber of this bundle over any point \( X \in J_p \) is the order complex
\( \Psi(X) \), from which its link \( \partial \Psi(X) \) (i.e. the union of all simplices not
containing the vertex point \( X \)) is removed (the points of this link cor-
respond to the points of the lower term \( \Psi_{p-1} \) of our filtration).

This difference \( \Psi(X) \setminus \partial \Psi(X) \) is denoted by \( \check{\Psi}(X) \).

By the exact sequence of the pair \( (\Psi(X), \partial \Psi(X)) \) we have
\[ (3) \quad \overline{H}_i(\check{\Psi}(X)) \equiv \tilde{H}_{i-1}(\partial \Psi(X)), \]
where \( \tilde{H}_* \) is the homology group reduced modulo a point.

Before studying these spaces \( \Psi_p(\alpha) \), we need to calculate or remind
the homology groups of several configuration spaces.

### 2. Homology Groups of Some Configuration Spaces

**Notation.** Given a topological space \( M \), denote by \( B(M, p) \) the con-
figuration space, whose points are the subsets of cardinality \( p \) in \( M \).
Denote by \( \tilde{B}(\mathbb{R}P^2, p) \) the space of generic \( p \)-configurations in \( \mathbb{R}P^2 \), i.e.
of those subsets of cardinality \( p \), none three points of which belong to
one and the same line in \( \mathbb{R}P^2 \).

For any manifold \( M \), we denote by \( \mathcal{O}_r \) the rational orientation local
system of groups on \( M \): it is locally isomorphic to the constant \( \mathbb{Q} \)-
system, but the elements of \( \pi_1(M) \) violating the orientation act on
it as the multiplication by \(-1\). For any topological space \( M \) denote
by \( \pm \mathbb{Z} \) the local system on \( B(M, p) \), which is locally isomorphic to
the constant $\mathbb{Z}$-bundle, but the elements of $\pi_1(B(M, p))$ defining odd permutations of $p$ points act on it as multiplication by $-1$. Also, set $\pm \mathbb{Q} \equiv \pm \mathbb{Z} \otimes \mathbb{Q}$.

**Lemma 2** (see e.g. [11]). For any $p$, there is a locally trivial fiber bundle $B(S^1, p) \to S^1$, whose fiber is homeomorphic to $\mathbb{R}^{p-1}$. This fiber bundle is orientable (and hence trivial) if $p$ is odd and is non-orientable if $p$ is even. \hfill $\Box$

**Lemma 3** (see e.g. [7]). A closed loop in the manifold $B(\mathbb{RP}^2, p)$ violates its orientation if and only if the cycle in $\mathbb{RP}^2$ composed by the traces of all $p$ points of our configurations during their corresponding movement defines the non-trivial element of $H_1(\mathbb{RP}^2, \mathbb{Z}_2)$. \hfill $\Box$

**Lemma 4** (see [11]). 1. The Borel–Moore homology groups $\overline{H}_*(B(\mathbb{RP}^2, 2), \mathbb{Q})$ and $\overline{H}_*(B(\mathbb{RP}^2, 2), \pm \mathbb{Q})$ are trivial in all dimensions.

2. The group $\overline{H}_i(B(\mathbb{RP}^2, 2), \mathbb{O}_r)$ is equal to $\mathbb{Q}$ for $i = 4$ and $i = 1$ and is trivial in all other dimensions. \hfill $\Box$

**Lemma 5.** The spaces $\hat{B}(\mathbb{RP}^2, 3)$ and $\hat{B}(\mathbb{RP}^2, 4)$ are homotopy equivalent to one another and to the space of unit cubes centered at the origin in $\mathbb{R}^3$.

**Proof.** Define the function on $\hat{B}(\mathbb{RP}^2, p)$ equal to the minimal angle between $p$ lines corresponding to the points of configurations. It is easy to construct smooth vector fields on $\hat{B}(\mathbb{RP}^2, 3)$ and $\hat{B}(\mathbb{RP}^2, 4)$, which strictly increase this function everywhere except for the manifolds of their maximal points, and thus provide deformation retractions of these spaces to these manifolds. In the case of $\hat{B}(\mathbb{RP}^2, 3)$ such a manifold consists of triples of pairwise orthogonal lines, which can be considered as normals to the faces of a cube in $\mathbb{R}^3$; so they fix a homeomorphism between this manifold and the space of all unit cubes. In the case of $\hat{B}(\mathbb{RP}^2, 4)$ any similar maximal configuration of four lines coincides with the set of main diagonals of a cube, which also fixes such a homeomorphism. \hfill $\Box$

**Corollary 1.** 1. The manifold $\hat{B}(\mathbb{RP}^2, 4)$ is orientable, and $\hat{B}(\mathbb{RP}^2, 3)$ is not.

2. The orientation sheaf $\mathbb{O}_r$ on $\hat{B}(\mathbb{RP}^2, 3)$ coincides with $\pm \mathbb{Q}$.

3. The sheaves $\pm \mathbb{Z}$ (and hence also $\pm \mathbb{Q}$) on $\hat{B}(\mathbb{RP}^2, 3)$ and $\hat{B}(\mathbb{RP}^2, 4)$ are induced from one another by the homotopy equivalence from Lemma 5.

**Proof.** By Lemma 5 the fundamental groups of both these manifolds are generated by $\pi/2$-rotations of a cube around the axes orthogonal to
its faces. Therefore it is enough to check the assertions of our corollary for any such generator of the fundamental groups. □

**Lemma 6.** 1. The group $\Pi_*(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q})$ is trivial in all dimensions.

2. The group $\Pi_*(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q})$ is equal to $\mathbb{Q}$ in dimensions 6 and 3 and is trivial in all other dimensions.

**Proof.** 1. Consider the 24-fold covering over $\hat{B}(\mathbb{R}P^2, 3)$, whose fiber over a triple of lines consists of positively oriented ordered frames of unit vectors in $\mathbb{R}^3$, each of which belongs to one of these lines. The group $\text{GL}_+(3, \mathbb{R})$ is the direct product of the space of this covering and $\mathbb{R}^3$. Therefore this space is an orientable 6-dimensional manifold, homotopy equivalent to $\text{SO}(3, \mathbb{R})$; by Poincaré duality its rational Borel–Moore homology groups are equal to $\mathbb{Q}$ in dimensions 6 and 3 and are trivial in all other dimensions. Our covering is finite, therefore (by Proposition 2 of the Appendix to [5]) the homology group $\Pi_*(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q})$ of its base in any dimension is not greater than that of the total space. So, it is at most 1-dimensional in dimensions 3 and 6 and is trivial in other dimensions. But $\Pi_6(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q}) = 0$ because our space $\hat{B}(\mathbb{R}P^2, 3)$ is non-orientable. Hence also $\Pi_3(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q}) = 0$ because the Euler characteristic of $\hat{B}(\mathbb{R}P^2, 3)$ is equal to 0.

2. Also by Poincaré duality and Corollary 1, $\Pi_*(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q}) \simeq H_{6-*}(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q})$. By the same covering, the latter group can be at most 1-dimensional in dimensions 0 and 3, so the former one is at most 1-dimensional in dimensions 6 and 3 and is trivial in other dimensions. In dimension 6 a non-trivial homology class is realized by the fundamental cycle, and the group $\Pi_3(\hat{B}(\mathbb{R}P^2, 3), \mathbb{Q})$ should also be non-trivial by the Euler characteristic considerations. □

**Corollary 2.** The manifold of all unit cubes with center $0 \in \mathbb{R}^3$ is orientable.

Indeed, by the previous proof and Lemma 5 the 3-dimensional rational homology group of this closed 3-dimensional manifold is non-trivial. □

**Lemma 7.** 1. The group $\Pi_*(\hat{B}(\mathbb{R}P^2, 4), \mathbb{Q})$ is equal to $\mathbb{Q}$ in dimensions 8 and 5 and is trivial in all other dimensions.

2. The group $\Pi_*(\hat{B}(\mathbb{R}P^2, 4), \pm \mathbb{Q})$ is trivial in all dimensions.

**Proof.** 1. By Corollary 1 and Poincaré duality, $\Pi_*(\hat{B}(\mathbb{R}P^2, 4), \mathbb{Q}) \simeq H_{8-*}(\hat{B}(\mathbb{R}P^2, 4), \mathbb{Q})$. By Lemma 3 and Corollary 2 the latter group is equal to $\mathbb{Q}$ in dimensions 0 and 3 and trivial in other dimensions.
2. Let $p : I \rightarrow \hat{B}(\mathbb{RP}^2, 4)$ be the 24-fold covering, induced from the similar covering over $\hat{B}(\mathbb{RP}^2, 3)$, described in the proof of Lemma 6, by the homotopy equivalence from Lemma 5. The space $I$ of this covering is orientable, because the base is. By Poincaré duality the group $H_i(I, \mathbb{Q})$ is equal to $\mathbb{Q}$ in dimensions 8 and 5 and is trivial in other dimensions. The local system $\pm \mathbb{Q}$ is a direct summand of the local system $p_!(\mathbb{Q})$ on $\hat{B}(\mathbb{RP}^2, 4)$. Therefore the homology group $H_i(\hat{B}(\mathbb{RP}^2, 4), \pm \mathbb{Q})$ is never greater than $H_i(I, \mathbb{Q})$, i.e. it can be at most 1-dimensional in dimensions 8 and 5, and is trivial in remaining dimensions. However, $H_8(\hat{B}(\mathbb{RP}^2, 4), \pm \mathbb{Q})$ is trivial because it is Poincaré dual to the 0-dimensional homology group with coefficients in a non-constant local system. Therefore also $H_5(\hat{B}(\mathbb{RP}^2, 4), \pm \mathbb{Q}) = 0$ by the Euler characteristic argument. □

Self-join and Caratheodory theorem. Let us embed a finite cell complex $M$ generically into the space $\mathbb{R}^T$ of a very large dimension, and denote by $M^r$ the union of all $(r - 1)$-dimensional simplices in $\mathbb{R}^T$, whose vertices lie in this embedded manifold (and the “genericity” of the embedding means that these simplices have no interior intersections: if two such simplices have a common point in $\mathbb{R}^T$, then their minimal faces, containing this point, do coincide).

Proposition 1 (C. Caratheodory theorem, see e.g. [8], [4]). For any $r \geq 1$, the space $(S^1)^r$ is homeomorphic to $S^{2r-1}$.

Lemma 8 (see [11]). If $r$ is even, then the sphere $(S^1)^r \sim S^{2r-1}$ has a canonical orientation, and hence a canonical generator of the $(2r-1)$-dimensional homology group. If $r$ is odd, then the homeomorphism of $(S^1)^r$ to itself, induced by any orientation-reversing automorphism of $S^1$, also violates the orientation. □

We will use also the following “homotopy” modification of Caratheodory theorem.

Let $\text{Sym}^i(M)$ be the $i$th symmetric power $M^i/S(i)$ of a manifold $M$. The disjoint union $\text{Sym}^1(M) \sqcup \cdots \sqcup \text{Sym}^r(M)$ is a partially ordered set by incidence of collections of points. Consider the join $\text{Sym}^1(M) * \cdots * \text{Sym}^r(M)$ (i.e. the union of all $(r - 1)$-dimensional simplices, whose vertices correspond to the points of $\text{Sym}^1(M), \ldots, \text{Sym}^r(M)$), and define the topological order complex of this poset as the union of only those simplices, all whose vertices are incident to one another. This space can be identified with the space $\text{Sym}^r(M)$, i.e. with the quotient space $(M * \cdots * M)/S(r)$ of the join of $r$ copies of $M$ by the
obvious action of the symmetric group. Indeed, for any \( j = 1, \ldots, r \) the space \( \text{Sym}^j(M) \) can be realised as the union of barycenters of \((j - 1)\)-dimensional simplices constituting the join \( M \ast \ldots \ast M \).

**Proposition 2** (see [4]). For any \( M \), there is a canonical homotopy equivalence between the order complex \( \text{Sym}^r(M) \) and the space \( M^r \) from the previous lemma. In particular, \( \text{Sym}^r(S^1) \) is canonically homotopy equivalent to \( S^{2r-1} \).

3. Calculation of the spectral sequence

3.1. First column. The first term \( \Psi_1 \) of our filtration of \( \Psi \) is the space \( \mathbb{R}P^2 \), so the column \( p = 1 \) has only one non-trivial cell \( E^{1}_{1,-1} \sim \mathbb{Q} \).

3.2. Second column. The term \( \Psi_2 \setminus \Psi_1 \) consists of two strata, see item \( J_2 \) in Lemma 1. The stratum \( \Psi_2(b) \) is closed in \( \Psi_2 \setminus \Psi_1 \), and we have the exact sequence of the pair

\[
\cdots \rightarrow H_i(\Psi_2(b), \mathbb{Q}) \rightarrow H_i(\Psi_2 \setminus \Psi_1, \mathbb{Q}) \rightarrow H_i(\Psi_2(a), \mathbb{Q}) \rightarrow \cdots
\]

**Lemma 9.** Both groups \( H_*(\Psi_2(b), \mathbb{Q}), H_*(\Psi_2(a), \mathbb{Q}) \) are trivial in all dimensions.

**Proof.** (a) The stratum \( \Psi_2(a) \) is the space of the fiber bundle (2) over the configuration space \( B(\mathbb{R}P^2, 2) \), i.e. the space of unordered pairs \( \{x \neq y\} \subset \mathbb{R}P^2 \). Its fiber \( \tilde{\Psi}(\{x \cup y\}) \) is an open interval whose endpoints are associated with the points \( x \) and \( y \). This interval is constituted by two half-intervals \( (\{x\}, \{x \cup y\}) \) and \( (\{y\}, \{x \cup y\}) \) in \( \Psi_2 \), where \( \{x\} \) and \( \{y\} \) are points of \( J_1 \) corresponding to the points \( x, y \), and \( \{x \cup y\} \) is a point \( J_2 \) from the segment as they belong to the first term \( \Psi_1 \) of our filtration.

Any loop in \( B(\mathbb{R}P^2, 2) \) violates the orientations of fibers of this bundle if and only if it permutes these points \( x \) and \( y \). Thus the \( m \)-dimensional rational Borel–Moore homology group of the total space of this fiber bundle is equal to \( \overline{H}_{m-1}(B(\mathbb{R}P^2, 2), \pm \mathbb{Q}) \). By Lemma 1 all such groups are trivial.

(b) The stratum \( \Psi_2(b) \) also is the space of a fiber bundle. Its base is the the projectivized tangent bundle of \( \mathbb{R}P^2 \), and the fiber over a tangent element at the point \( x \in \mathbb{R}P^2 \) is the half-interval connecting the point of \( J_2 \) corresponding to this tangent element with the point \( \{x\} \in J_1 \). The Borel–Moore homology group of any fiber bundle with such fibers is trivial in all dimensions.

So, both marginal groups in (4) are trivial, hence also the middle group is, and the column \( E^1_{2,q} \) of our spectral sequence is empty.
3.3. **Third column of the spectral sequence.** The space $\Psi_3 \setminus \Psi_2$ consists of four strata, see Lemma 1. Let us filter it by

$$
(5) \Psi_3(d) \subset (\Psi_3(d) \cup \Psi_3(c)) \subset (\Psi_3(d) \cup \Psi_3(c) \cup \Psi_3(b)) \subset (\Psi_3(d) \cup \Psi_3(c) \cup \Psi_3(b) \cup \Psi_3(a))
$$

and study the auxiliary spectral sequence calculating the Borel–Moore homology of $\Psi_3 \setminus \Psi_2$ and generated by this filtration.

**Lemma 10.** The groups $\overline{H}_*(\Psi_3(d), \mathbb{Q})$ and $\overline{H}_*(\Psi_3(c), \mathbb{Q})$ are trivial in all dimensions.

**Proof.** In the case of the stratum $\Psi_3(d)$, the link $\partial \Psi(X)$ of the fiber $\Psi(X)$, $X \in J_3(d)$, is homeomorphic to the closed disc. Indeed, let $x$ be the point of $\mathbb{RP}^2$ defined by $X$, $\{x\}$ the corresponding point of $J_1$, and $PT_x(\mathbb{RP}^2) \sim S^1$ the set of tangent directions at this point, considered as points of the stratum $\Psi_2(b)$ in $J_2$. Then our link $\partial \Psi(X)$ consists of all segments in $\Psi_2$ connecting the point $\{x\}$ with all points of the projective line $PT_x(\mathbb{RP}^2)$. Therefore by (3) the Borel–Moore homology group of the fiber $\hat{\Psi}(X)$ is zero in all dimensions, and the same is true for the entire stratum $\Psi_3(d)$.

For $X \in J_3(c)$, the link $\partial \Psi(X)$ is homeomorphic to a closed line segment. Indeed, the configuration $X \in J_3$ (consisting of two points $x \neq y \in \mathbb{RP}^2$ and a tangent direction $l$ at $x$) has two incident points $\{x\}$ and $\{y\}$ in $J_1$, two incident points $(x, l)$ and $\{x \cup y\}$ in $J_2$, and three segments in $\partial \Psi(X)$ connecting $\{x\}$ with $(x, l)$, $\{x\}$ with $\{x \cup y\}$, and $\{y\}$ with $\{x \cup y\}$. This proves the assertion of Lemma 10 concerning the homology of $\Psi_3(c)$.

**Lemma 11.** $\overline{H}_i(\Psi_3(b), \mathbb{Q}) \simeq \overline{H}_{i-4}(\mathbb{RP}^2, \mathbb{Q})$. In particular, it is equal to $\mathbb{Q}$ if $i = 4$ and is trivial in all other dimensions.

**Proof.** The stratum $\Psi_3(b)$ is fibered over the space $\hat{\mathbb{RP}}^2$ of all lines in $\mathbb{RP}^2$. By Proposition 2 the link $\partial \Psi(L)$ of its fiber over any line $L \in \hat{\mathbb{RP}}^2$ is homotopy equivalent to $S^3$, so that the Borel–Moore homology group of $\hat{\Psi}(X)$ is equal to $\mathbb{Q}$ in dimension 4 and is trivial in all other dimensions. By Lemma 8 the monodromy over loops in $\hat{\mathbb{RP}}^2$ acts trivially on this homology group. Therefore our lemma follows from the spectral sequence of this fiber bundle.

**Lemma 12.** $\overline{H}_i(\Psi_3(a), \mathbb{Q}) = \overline{H}_{i-2}(\hat{B}(\mathbb{RP}^2), 3, \pm \mathbb{Q})$. In particular (by statement 2 of Corollary 4 of Lemma 4 and statement 2 of Lemma 6 it is equal to $\mathbb{Q}$ if $i = 8$ or $i = 5$ and is trivial in all other dimensions.
Proof. The stratum $\Psi_3(a)$ is fibered \([2]\) over the configuration space $\tilde{B}(\mathbb{RP}^2, 3)$. Its fiber over a configuration is equal to the open triangle (whose vertices are associated with the points of the configuration, and the entire triangle is the order complex of non-empty subsets of this configuration). Any generator of $\pi_1(\tilde{B}(\mathbb{RP}^2, 3))$ described in the proof of Corollary \([1]\) (i.e. a rotation of a cube) changes the orientations of this bundle. Therefore our lemma follows from the spectral sequence of this fiber bundle. □

Summing up Lemmas \([10, 11, 12]\) we see that the auxiliary spectral sequence calculating $H^*(\Psi_3 \setminus \Psi_2, Q)$ has reduced to the exact sequence of two right-hand terms in \([5]\). The 8-dimensional generator from Lemma \([12]\) surely gives us a summand, which justifies the cell $E^1_{3,5} \simeq Q$ in Fig. \([1]\). In addition, we have only one suspicious fragment of this exact sequence:

\[(6) \quad 0 \to \overline{H}_5(\Psi_3 \setminus \Psi_2, Q) \to \overline{H}_5(\Psi_3(a), Q) \overset{\partial}{\to} \overline{H}_4(\Psi_3(b), Q) \to \overline{H}_4(\Psi_3 \setminus \Psi_2, Q) \to 0,
\]

where both middle groups are equal to $Q$ by Lemmas \([12, 11]\).

Lemma 13. The differential $\partial : \overline{H}_5(\Psi_3(a), Q) \to \overline{H}_4(\Psi_3(b), Q)$ in \((6)\) is non-trivial.

Proof. Let us construct a cycle generating the group $\overline{H}_5(\Psi_3(a), Q)$. Consider the 3-dimensional submanifold $\mathcal{M}^3 \subset \tilde{B}(\mathbb{RP}^2, 3)$, consisting of triples $\{x, y, z\} \subset \mathbb{RP}^2$ such that $x = (1 : 0 : 0)$, $y \neq x$ is a point in the equatorial line $E \subset \mathbb{RP}^2$ spanned by $x$ and $(0 : 1 : 0)$, and $z$ is an arbitrary point not in this line.

Lemma 14. $\mathcal{M}^3$ is homeomorphic to $\mathbb{R}^3$ and (being oriented in any way) generates the group $\overline{H}_3(\tilde{B}(\mathbb{RP}^2, 3), \pm Q)$.

Proof. The first assertion is obvious. By the second statement of Corollary \([1]\) of Lemma \([5]\) the Poincaré pairing between the groups $\overline{H}_3(\tilde{B}(\mathbb{RP}^2, 3), \pm Q) \equiv \overline{H}_3(\tilde{B}(\mathbb{RP}^2, 3), \mathbb{Z})$ and $H_3(\tilde{B}(\mathbb{RP}^2, 3), Q)$ is well-defined an non-degenerate. By Lemma \([6, 2]\), these groups are one-dimensional, and the latter of them is generated by the fundamental class of the manifold of all possible triples of pairwise orthogonal lines. The cell $\mathcal{M}^3$ has exactly one transversal intersection with this manifold. □

Let $[\mathcal{M}^3] \subset \Psi_3 \setminus \Psi_2$ be the preimage of $\mathcal{M}^3$ under the 2-dimensional fiber bundle $\Psi_3(a) \to \tilde{B}(\mathbb{RP}^2, 3)$. By the Thom isomorphism, it generates the group $\overline{H}_5(\Psi_3(a), Q)$. Its closure in $\Psi_3 \setminus \Psi_2$ contains exactly
those points of the stratum \( \Psi_3(b) \), which constitute the fiber \( \check{\Psi}(E) \) over the point \( E \in \R \P^2 \) (corresponding to the equatorial line) in the fiber bundle \( \Psi_3(b) \to \R \P^2 \). This fiber \( \check{\Psi}(E) \) (supplied with an arbitrary orientation) generates the group \( \check{H}_4(\Psi_3(b), \Q) \), therefore we need only to calculate the incidence coefficient \([\mathcal{M}^3 : \check{\Psi}(E)]\).

Geometrically, any generic point \( T \in \check{\Psi}(E) \) is approached by the points of \( \mathcal{M}^3 \) four times. Indeed, such a point indicates uniquely some two points \( v \neq w \in E \). Namely, by definition \( T \) is some interior point of a triangle in \( \Psi_3 \) spanned by a point of \( J_1 \) (defined by some point of \( \R \P^2 \)), a point of the main stratum \( B(\R \P^2, 2) \) of \( J_2 \), and the point \( \{E\} \in J_3 \). The desired points \( v \) and \( w \) are exactly the points of the 2-configuration defining the second vertex of this triangle. Since \( T \) is generic, we will assume that neither \( v \) nor \( w \) coincides with the fixed point \( x = (1 : 0 : 0) \in E \). Let us fix an orientation of the line \( E \). We will denote by \( v \) (respectively, \( w \)) the point of our 2-configuration which is next to \( x \) along (respectively, against) this orientation. When we approach such a point \( T \) from the side of \( \mathcal{M}^3 \), the point \( y \) can approach either \( v \) or \( w \) along the line \( E \), and \( z \) can approach the remaining point \( w \) or \( v \) in two different ways, in the same way as the one-dimensional cell of the standard cell decomposition of \( \R \P^2 \) is approached from the two-dimensional cell. In total this gives us four possibilities.

**Lemma 15.** All four approaches of \( \check{\Psi}(E) \) from the side of \( \mathcal{M}^3 \) have equal (of the same sign) contributions to their incidence coefficient.

**Proof.** To study the incidence coefficients of these subvarieties in \( \Psi_3 \setminus \Psi_2 \), let us fix their orientations.

The variety \( \check{\Psi}(E) \) is 4-dimensional, and is fibered over the configuration space \( B(E \setminus \{x\}, 2) \); its fiber over the configuration \( (v, w) \) consists of two triangles

\[
(\{v\}, \{v, w\}, \{E\}) \quad \text{and} \quad (\{w\}, \{v, w\}, \{E\}).
\]

Let us orient the first triangle by the indicated sequence of vertices, and the second triangle by the opposite one, so that their sum has no boundary at the edge \( \{v, w\}, \{E\} \). Further, we orient the base by the sequence \( (\partial/\partial v, \partial/\partial w) \), and orient the whole fiber bundle by this pair of orientations of the base and the fibres.

The subvariety \( \mathcal{M}^3 \subset \Psi_3 \setminus \Psi_2 \) is swept out by order complexes \( \Psi(x, y, z) \) for \( (x, y, z) \in \mathcal{M}^3 \). Any such order complex consists of six triangles like \( \{x\}, \{x, y\}, \{x, y, z\} \), \( \{z\}, \{x, z\}, \{x, y, z\} \), etc. So, we have six families of triangles parameterized by configurations \( (x, y, z) \in \mathcal{M}^3 \). Only two of these six 3-parameter families, namely, the ones swept
Fig. 2. Incidence coefficients in the configuration space

out by triangles

\[(8) \quad \{(y, z), \{x, y, z\}\} \text{ and } \{(z, y, z), \{x, y, z\}\},\]

can approach the variety $\tilde{\Psi}(E)$ in its interior points. Therefore it is
enough to consider the 5-dimensional subvariety $[\mathcal{M}^3]^o \subset [\mathcal{M}^3]$ fibered
over $\mathcal{M}^3$, with the fiber over $(x, y, z)$ equal to the sum of such two
simplices. Again, we orient the first of them by the indicated order of
its vertices, and the second by the opposite one, so that these orien-
tations are compatible at the common edges $[[y, z], \{x, y, z\}]$ of these
triangles. Also, we orient the manifold $\mathcal{M}^3$ of configurations $(y, z)$ by
the pair \{the chosen orientation of the line $E \ni y$; the orientation of
the upper hemisphere inducing this orientation on its boundary $E$\}. The
orientation of the entire 5-dimensional cell $[\mathcal{M}^3]^o$ consists of this
orientation of the base and the above-defined orientation of the fibers.

When the triple $(x, y, z) \in \mathcal{M}^3$ degenerates (i.e. $z$ tends to the
equatorial line), the point $(x, y, z) \in J_3$ tends to $\{E\}$, and the sum of
two triangles \[\begin{array}{c}
\end{array}\] (taken with above-described orientations) tends to $\pm$
the sum of triangles \[\begin{array}{c}
\end{array}\], where $v$ and $w$ are the limit positions of $y$ and $z$ (or $z$ and $y$).

To prove Lemma 15 it remains to compare these limit orientations
at the points of $\tilde{\Psi}(E)$ defined by different approaches from the side of
$[\mathcal{M}^3]^o$. Consider a point $(v, w) \in B(E \setminus \{x\}, 2)$ and the sum of oriented
triangles \[\begin{array}{c}
\end{array}\] over it.

First we compare two adjacencies $[\mathcal{M}^3]^o \to \tilde{\Psi}(E)$, when $y = v$ and
$z$ tends to $w$ in two possible ways. All ingredients of these adjacencies
(concerning the behavior of fibers and the point $y$) except for these two
ways of tending $z$ to $w$ are exactly the same. Therefore the contribu-
tions of these adjacencies are equal to one another by the same reasons
as for the two adjacencies of 2- and 1-dimensional cells of the stan-
dard cell decomposition of $\mathbb{R}P^2$. In exactly the same way, two different
adjacencies when $y = w$ and $z$ tends to $v$ also give equal contributions.

It remains to compare the contributions of two adjacencies from dif-
f erent pairs: one with $y$ tending to $v$ and $z$ to $w$, and vice versa for
the other. Some two such adjacencies are shown in Fig. 2. The 2-
configuration shown in the upper left corner of this picture can degen-
erate into the one shown in the lover left corner in two ways shown
in the right part of the picture. Since the orders of two points in E
obtained from y and z in these ways are different, the boundary ori-
entations of the base $B(E \setminus \{x\}, 2)$ induced by the chosen orientation
of $\mathcal{M}^3$ are opposite. At the same time, the fibers (i.e. the sums of
oriented triangles \[8\)) in one case approach the corresponding fiber (7)
respecting its chosen orientation, and in the other case reversing this
orientation. Therefore the boundary orientations of the entire fi-
ber bundle $E \cup \{\beta\}$ induced from the orientation of $\mathcal{M}^3$ in these two cases
differ from one another by the factor $(-1) \times (-1)$ (one difference on
the level of bases, and the other by the comparison of the fibers), hence
they also do coincide.

Lemma 13 is proved, and column $p = 3$ of Fig. 1 is justified. □

3.4. Fourth column of the spectral sequence. The space $\Psi_4 \setminus \Psi_3$
consists of five strata, see Lemma 1. Let us filter it by

\[
\Psi_4(e) \subset (\Psi_4(e) \cup \Psi_4(d)) \subset (\Psi_4(e) \cup \Psi_4(d) \cup \Psi_4(c)) \subset \\
(\Psi_4(e) \cup \Psi_4(d) \cup \Psi_4(c) \cup \Psi_4(b)) \subset \\
(\Psi_4(e) \cup \Psi_4(d) \cup \Psi_4(c) \cup \Psi_4(b) \cup \Psi_4(a))
\]

and study the auxiliary spectral sequence generated by this filtra-
tion and calculating the group $\mathcal{H}_*(\Psi_4 \setminus \Psi_3, \mathbb{Q})$.

Lemma 16. For any point $X \in J_4$, which belongs to one of strata
described in items $(J_4)(e)$, $(J_4)(d)$ or $(J_4)(c)$ of Lemma 1, the reduced
homology group $\tilde{H}_*(\partial \Psi(X), \mathbb{Q})$ of the link of the corresponding order
complex is trivial in all dimensions. In particular, the rational Borel–
Moore homology groups of corresponding pieces $\Psi_4(e)$, $\Psi_4(d)$ and $\Psi_4(c)$
of $\Psi_4 \setminus \Psi_3$ also are trivial in all dimensions.

Proof. Any point $X \in J_4(e)$ is defined by a line $L \subset \mathbb{R}P^2$ and a
singular point $z \in L$. For any such pair $X = (L, z)$, the link $\partial \Psi(X)$
is the union of the order complexes $\Psi(L)$ and $\Psi(z)$, where $L$ and $z$
are considered as some points of $J_3$ listed in items $(J_3)(b)$ and $(J_3)(d)$
of Lemma 1 respectively. The intersection of these two complexes
is the closed segment in $\Psi_2$, connecting the point $\{z\} \in J_1$ and the
point $\{z; \text{the direction of } L \text{ at } z\}$ of the stratum of $J_2(b)$. Therefore
the assertion of our lemma concerning $(J_4)(e)$ follows from the Mayer–
Vietoris sequence of this union.

Any point $X \in J_4(d)$ is specified by two different points $v, w \in \mathbb{R}P^2$
with a generic choice of tangent directions at these points. The link
of $\Psi(X)$ in this case is illustrated in Fig. 3 left. Namely, this link has two vertices in $J_1$ (defined by these points $v$ and $w$ with forgotten directions; in the picture they are shown by dots, while the crosses denote the points supplied with tangent directions), three vertices in $J_2$ (the point $v$ together with the chosen direction, the point $w$ with the direction, and the configuration $\{v, w\}$ with forgotten directions); and two points in $J_3$ (configurations $\{v, w\}$ with the direction at only one of these points). The segments in our picture connect the incident vertices from $J_1$ and $J_2$ or from $J_2$ and $J_3$. It easy to see that this link is a contractible simplicial complex, which proves our lemma for the case of $J_4(d)$.

For $J_4(c)$, the 1-skeleton of the similar link is shown in Fig. 3 right; the point supplied with a tangent direction corresponds to the middle point in the bottom row. Again, it is easy to check that the simplicial complex defined by the poset shown in the picture is contractible. □

The calculation of the fourth column has reduced to the study of only two subspaces, $\Psi_4(b)$ and $\Psi_4(a)$; namely, to the exact sequence

$$
\cdots \to \overline{H}_i(\Psi_4(b), \mathbb{Q}) \to \overline{H}_i(\Psi_4 \setminus \Psi_3, \mathbb{Q}) \to \overline{H}_i(\Psi_4(a), \mathbb{Q}) \to \overline{H}_{i-1}(\Psi_4(b), \mathbb{Q}) \to \cdots
$$

**Lemma 17.** $\overline{H}_i(\Psi_4(b), \mathbb{Q}) \simeq \overline{H}_{i-7}(\mathbb{RP}^2, \mathbb{Q})$, in particular this group is equal to $\mathbb{Q}$ if $i = 9$ and is trivial otherwise.

**Proof.** $\Psi_4(b)$ is fibered over the space $J_4(b)$ of configurations

$$
\{\text{a line } L \in \mathbb{RP}^2; \text{a point } z \in \mathbb{RP}^2 \setminus L\}.
$$

The space of such configurations is obviously diffeomorphic to $T_s\mathbb{RP}^2$, in particular is orientable. Now we need the following lemma.

**Lemma 18.** 1. The link $\partial \Psi(X)$ of $\Psi(X)$, $X = (L, z) \in J_4(b)$, is homotopy equivalent to $S^4$, in particular the group $\overline{H}_i(\Psi(X), \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ if $i = 5$ and is trivial for all other $i$.

2. The fundamental group of the space of configurations (11) acts trivially on the homology bundle associated with the fibers $\hat{\Psi}(X)$.  

\[\text{Fig. 3. Order complexes for strata } J_4(d) \text{ and } J_4(c)\]
Proof. 1. The link \( \partial \Psi(X) \), \( X = \{L, z\} \), consists of two subspaces: \( A = \) the order complex \( \Psi(L) \), and \( B = \) the union of all order complexes \( \Psi(\{v, w, z\}) \), where \( v \) and \( w \) are arbitrary (maybe coinciding) points of the line \( L \). The intersection \( A \cap B \) is the space \( \Psi(L) \cap \Psi_2 \), i.e. the union of similar order subcomplexes \( \Psi(\{v, w\}) \); by Proposition 2 it is homotopy equivalent to \( S^3 \). The space \( A \) is compact and contractible, hence the pair \((A, A \cap B)\) is homotopy equivalent to the pair (cone over \( A \cap B \), its base \( A \cap B \)). As for the space \( B \), any of order complexes \( \Psi(\{v, w, z\}) \) constituting it is canonically PL-homeomorphic to the triangle with vertices \( v, w \) and \( z \), and hence to the cone over \( \Psi(\{v, w\}) \) with vertex \( \{z\} \). These homeomorphisms over all pairs \( \{v, w\} \in \text{Sym}^2(L) \) define a homeomorphism between the union \( B \) of all these triangles and another cone over \( \partial \Psi(L) \cap \Psi_2 \). Therefore our space \( A \cup B \) is homotopy equivalent to the suspension over \( \sim S^3 \).

2. By Lemma 8, the generator of \( \pi_1(\mathbb{RP}^2) \) acts trivially on the group \( H_3(\Psi(L) \cap \Psi_2, \mathbb{Q}) \). Two parts \( A \) and \( B \) of \( \partial \Psi(\{L, z\}) \) are of different origin and cannot be permuted by the monodromy, hence the group \( H_4(\partial \Psi(\{L, z\}), \mathbb{Q}) \) also is preserved by this generator. By the natural boundary isomorphisms (3), the same is true for the groups \( H_\ast(\bar{\Psi}(\{L, z\}), \mathbb{Q}) \).

Lemma 18 is completely proved. Lemma 17 follows from it immediately by the composition of the Thom isomorphism of the corresponding 5-dimensional fiber bundle (2) over the base \( J_4(b) \) and the Thom isomorphism for the non-orientable fibration of this base over \( \mathbb{RP}^2 \).

Lemma 19. Any group \( \overline{H}_\ast(\Psi_4(a), \mathbb{Q}) \) is isomorphic to \( \overline{H}_{\ast-3}(\tilde{B}(\mathbb{RP}^2, 4), \pm \mathbb{Q}) \). In particular, by Lemma 4 \( \overline{H}_\ast(\Psi_4(a), \mathbb{Q}) \equiv 0 \).

Proof. The space \( J_4(a) \) is equal to \( \tilde{B}(\mathbb{RP}^2, 4) \), and the fiber of the fiber bundle \( \Psi_4(a) \to J_4(a) \) over the configuration \( \{x, y, z, u\} \in \tilde{B}(\mathbb{RP}^2, 4) \) is an open 3-dimensional simplex, whose vertices correspond to the points of this configuration. Any element of \( \pi_1(\tilde{B}(\mathbb{RP}^2, 4)) \) defining an odd permutation of four points violates the orientation of the bundle of 3-dimensional simplices. Therefore our lemma follows from the Thom isomorphism for this fiber bundle.

Summing up Lemmas 19, 17 and 16 we obtain that the auxiliary spectral sequence, calculating the group \( \overline{H}_\ast(\Psi_4 \setminus \Psi_3, \mathbb{Q}) \) and generated by the filtration (9), has only one non-trivial cell, isomorphic to \( \mathbb{Q} \) and having total dimension equal to 9. This proves the column \( p = 4 \) of Fig. 11.
3.5. **The differential** $d^1 : E^1_{4,5} \to E^1_{3,5}$ **is an isomorphism.** Recall that the group $E^1_{4,5}$ is generated by the basic cycle of $\overline{H}_9(\Psi_4(b), \mathbb{Q})$, and the group $E^1_{3,5} \equiv \overline{H}_8(\Psi_3 \setminus \Psi_2, \mathbb{Q})$ is generated by the fundamental cycle of the fiber bundle over $\hat{B}(\mathbb{RP}^2, 3)$, whose fiber over the triple $\{u, v, w\} \subset \mathbb{RP}^2$ is an open triangle with vertices associated with these points $u$, $v$ and $w$. Any point of the latter 8-dimensional manifold participates exactly three times in the links $\partial \Psi(X)$ of some configurations $X \in J_4(b)$: namely, of the configurations $(L, z)$ where $z$ is one of our points $u$, $v$ or $w$, and $L$ is the line spanned by the remaining two points. Therefore the incidence coefficient of manifolds generating the groups in question is an odd number, in particular is not equal to zero. □

3.6. **Fifth column of the spectral sequence.** The space $\Psi_5 \setminus \Psi_4$ consists of three strata, see Lemma 1. Let us filter it by

(12) $\Psi_5(c) \subset (\Psi_5(c) \cup \Psi_5(b)) \subset (\Psi_5(c) \cup \Psi_5(b) \cup \Psi_5(a))$

and study the auxiliary spectral sequence generated by this filtration and calculating the Borel–Moore homology of $\Psi_5 \setminus \Psi_4$.

**Lemma 20.** The group $\overline{H}_i(\Psi_5(a), \mathbb{Q})$ is isomorphic to $\overline{H}_{i-11}(\mathbb{RP}^2, \mathbb{Q}_r)$, i.e. is equal to $\mathbb{Q}$ if $i = 13$ and is trivial otherwise.

**Proof.** The space $J_5(a)$ of non-empty non-singular conics in $\mathbb{R}^3$ is diffeomorphic to the space of a 3-dimensional vector bundle over $\mathbb{RP}^2$, namely of the symmetric square of the cotangent bundle: $J_5(a) \simeq S^2 T^* (\mathbb{RP}^2)$, see e.g. Lemma 16 in [11]. In particular, it is an orientable 5-dimensional manifold homotopy equivalent to $\mathbb{RP}^2$.

The space $\Psi_5(a) \subset \Psi_5 \setminus \Psi_4$ is fibered over the space $J_5(a)$ of such conics. For any conic $C \in J_5(a)$, the link $\partial \Psi(C)$ of the fiber $\Psi(C)$ of this fiber bundle coincides with the space $\operatorname{Sym}^4(S^1)$, see Proposition 2. Therefore by Propositions 1 and 2 the Borel–Moore homology groups of the fibers $\Psi(C)$ are isomorphic to $\mathbb{Q}$ in dimension 8, and are trivial in all other dimensions. By Lemma 8 the fundamental group of the base acts trivially on these homology groups.

Now, our lemma follows from the Thom isomorphism $\overline{H}_i(\Psi_5(a), \mathbb{Q}) \simeq \overline{H}_{i-8}(J_5(a), \mathbb{Q})$ of our orientable fiber bundle over the space of conics, and the Thom isomorphism $\overline{H}_j(J_5(a), \mathbb{Q}) \simeq H_{j-3}(\mathbb{RP}^2, \mathbb{Q}_r)$ for the fibration of this space over $\mathbb{RP}^2$. □

Now, let us study $\Psi_5(b)$. The base $J_5(b)$ of the corresponding fiber bundle [2] is the space $B(\mathbb{RP}^2, 2)$ of pairs of different lines in $\mathbb{RP}^2$. 

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Lemma 21. For any point $X \in B(\mathbb{R}P^2, 2)$, the link $\partial \Psi(X)$ of the order complex $\Psi(X)$ is homology equivalent to the sphere $S^7$.

Proof. Let us construct a convenient subdivision of the link $\partial \Psi(X)$. For any of strata $J_i(\alpha), i < 5$, we take all points $Y$ of this stratum subordinate to $X$ and consider the union of the corresponding complexes $\bar{\Psi}(Y)$. This union forms a fiber bundle over the set of such subordinate points $Y$. Then we calculate Borel–Moore homology groups of all these fiber bundles; they form the first term of a spectral sequence calculating the homology of entire $\partial \Psi(X)$ defined by our stratification.

We can ignore all strata for which the homology group of the fiber $\bar{\Psi}(Y)$ is trivial in all dimensions; as was shown previously, it are the strata $J_2(b), J_3(c), J_3(d), J_4(c), J_4(d)$, and $J_4(e)$. The remaining configurations subordinated to the point $X$ (i.e. to a pair of different lines in $\mathbb{R}P^2$) are as follows.

$J_4(a)$ Two points in one line and two points in the other; none of them coincides with the intersection point of these lines. The space of such configurations is equal to $B(\mathbb{R}^1, 2) \times B(\mathbb{R}^1, 2) \simeq \mathbb{R}^4$, the fiber $\bar{\Psi}(Y)$ is a 3-dimensional open simplex. So, the group $\mathbb{H}_i$ of this space is equal to $\mathbb{Q}$ for $i = 7$ and is trivial in all other dimensions.

$J_4(b)$ One of our two lines and one point in the other (different from their intersection point). The space of such configurations is the union of two affine lines, and the fiber $\bar{\Psi}(Y)$ has the Borel–Moore homology group of $\mathbb{R}^5$, see Lemma [18]. So, the Borel–Moore homology group of entire this space is $\mathbb{Q}^2$ in dimension 6 and is trivial in all other dimensions.

$J_3(a)$ The configuration space consists of two three-dimensional cells and one two-dimensional. Indeed, some two points of our 3-configuration should lie in both our lines outside the intersection point, and the third point can lie also in only one of them or to coincide with the intersection point. The first two configuration spaces are equal to $B(\mathbb{R}^1, 2) \times \mathbb{R}^1 \simeq \mathbb{R}^3$, and the third one to $\mathbb{R}^2$. The fiber $\bar{\Psi}(Y)$ in all cases is homeomorphic to an open triangle. Thus, the preimages of the first two pieces in $\Psi_3(a)$ are homeomorphic to $\mathbb{R}^5$, and that of the third one is homeomorphic to $\mathbb{R}^4$.

$J_3(b)$ The configuration space consists of two points (i.e. the choice of one of two lines constituting $X$). The link $\partial \Psi(Y)$ is homotopy equivalent to $S^3$, thus the fiber $\bar{\Psi}(Y)$ has Borel–Moore homology of $\mathbb{R}^4$.

$J_2(a)$ Five different strata: both these two points can lie on one or the other of our two lines (outside the intersection point), or on different
lines, or one of them can coincide with the intersection point, and the other one lie on one of remaining affine lines. The fiber $\tilde{\Psi}(Y)$ in all cases is equal to an open interval.

$J_1$ Three obvious strata equal to $\mathbb{R}^1, \mathbb{R}^1$, and a point; the fiber in all cases consists of one point.

It is easy to calculate, that the incidence coefficients of two 6-dimensional cells listed in item $J_4(b)$ and two 5-dimensional cells listed in $J_3(a)$ form the unit matrix. The same is true for the incidence coefficients of two 4-dimensional cells listed in $J_3(b)$ and two first 3-dimensional cells in $J_2(a)$. The union of remaining cells of dimensions $\leq 4$ is a compact contractible space. Indeed, it is the union of all closed triangles, whose vertices correspond to the crossing point of our two lines, and two other points of these lines (one point on each). All these simplices have a common vertex, hence their union is indeed contractible. So, all cells of our decomposition of $\partial\Psi(X)$ have killed one another, except only for the 7-dimensional cell $\Psi_4(a)$ and the 0-dimensional one. □

**Lemma 22.** The group $\pi_1(B(\mathbb{R}\hat{P}^2, 2))$ acts trivially on the rational homology groups of the fibers $\partial\Psi(X)$ and (which is the same by the natural boundary isomorphisms (3)) on the rational Borel–Moore homology groups of fibers $\tilde{\Psi}(X)$.

**Proof.** Let us define an invariant orientation of the homology bundle with fiber $H_7(\partial\Psi(X), \mathbb{Q})$. This group is generated by the fundamental class of the cell studied in item $J_4(a)$ of the proof of Lemma 21. To orient this cell, we need to choose the orientations of its base $B(\mathbb{R}^1, 2) \times B(\mathbb{R}^1, 2)$ and, for any point of this base (i.e. a configuration of four points), of the fiber over it, i.e. the 3-dimensional simplex whose four vertices correspond to the points of this configuration. To do it, we fix an arbitrary numbering of these four points, and define the $j$th basic tangent vector to the base, $j = 1, \ldots, 4$, as a shift of the $j$th point in the direction from the other point of the configuration in the same line and towards the intersection point of these lines. The orientation of the fiber (i.e. a tetrahedron) is defined by the same numbering of its vertices. A different numbering of points simultaneously changes or preserves the orientations of the base and the fiber, and hence preserves the orientation of the total space.

The orientation thus defined is obviously invariant and defines a trivialization of the homology bundle over $B(\mathbb{R}\hat{P}^2, 2)$ with the fiber $H_7(\partial\Psi(X), \mathbb{Q})$. □
Corollary 3. $\overline{H}_i(\Psi_5(b), \mathbb{Q})$ is equal to $\overline{H}_{i-8}(B(\mathbb{R}P^2, 2), \mathbb{Q})$ for any $i$. In particular, this group is trivial in all dimensions.

Proof. The first statement follows from Lemmas 21, 22 and the spectral sequence of the corresponding fiber bundle [2]; the second statement follows from Lemma 4. □

Lemma 23. For any point $X \in J_5(c)$ (i.e. a projective line $L$ of multiplicity 2) the homology group of the link $\partial\Psi(X)$ of its order complex is trivial in all positive dimensions. In particular, the Borel–Moore homology group of the stratum $\Psi_5(c)$ also is trivial in all dimensions.

Proof. We proceed as in the proof of Lemma 21. There are only the following three strata subordinate to $X$ (except for those consisting of points $Y$ such that $\overline{H}_*(\Psi(Y)) \equiv 0$). In $J_3(b)$, the configuration space of subordinate points consists of only one point: our line $L \sim \mathbb{R}P^1$ taken once, and the fiber is $\hat{\Psi}(L)$. In $J_2(a)$, the configuration space is $B(L, 2)$, and the fiber is equal to an 1-dimensional open interval. In $J_1$, the configuration space is our line $L$, the fiber is one point. Also, we are free not to exclude the stratum $\Psi(X) \cap J_2(b)$, which is fibered over $L$ with the fibers equal to half-open intervals and anyway does not contribute to the homology group of $\partial\Psi(X)$.

The union of these four strata is nothing else than the entire order complex $\Psi(L)$ of the point $L \in J_3$, which is a contractible compact space. □

Summing up Lemmas 20, 23 and Corollary 3, we obtain that the auxiliary spectral sequence, calculating the rational Borel–Moore homology of $\Psi_5 \setminus \Psi_4$ and generated by the filtration (12), has only one non-trivial cell; namely, it is equal to $\mathbb{Q}$ and its total dimension is equal to 13. This justifies column $p = 5$ of Fig. 1. Theorem 2 is completely proved. □

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