Do tensor renormalization group methods work for frustrated spin systems?

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(Dated: March 20, 2019)

No. To illustrate that tensor renormalization group methods are poorly suited for frustrated magnetic systems, we study the thermodynamic properties of the two-dimensional Edwards-Anderson Ising spin-glass model on a square lattice. We show that the limited precision of standard 64-bit data types and not a small cut-off parameter is the main reason for unphysical negative partition function values in spin glasses.

The simulation of spin-glasses [1] with nearest neighbor interactions remains numerical challenge. To date—with only few exceptions [2, 3]—the thermodynamic behavior of spin glasses is best studied using Markov-chain Monte Carlo methods paired with accelerators, such as cluster updates [4] or tempering [5]. Therefore, finding an algorithm that outperforms Monte Carlo for large-scale simulations at low temperatures where the dynamics of spin glasses is exponentially slow is of much interest.

In classical statistical system with local interactions, the Boltzmann weight can be expressed as a tensor product. Moreover, all thermodynamic quantities can be determined by studying an equivalent tensor-network models. The tensor renormalization group method (TRG) is a real-space renormalization group approach initially introduced by Levin and Nave [6] for classical spin systems on two-dimensional regular lattices. Later, the approach was further improved by Xie et al. [7] and successfully applied to the two- and three-dimensional ferromagnetic Ising model.

Unlike the ferromagnetic Ising model, the Edwards-Anderson Ising spin glass is a magnetic system exhibiting both quenched disorder and frustration, i.e., it has no translation symmetry. Whether the TRG method can be applied to the Edwards-Anderson Ising spin glass has recently been investigated by Wang et al. [8]. They find that the TRG method might lead to negative values in the partition function at low temperature and argue that a larger cut-off parameter could be used to alleviate this problem. In this note we argue that the primary reason for negative partition function terms is the limited precision of the data type (double) used, instead of a small cut-off parameter. We show that TRG fails because of the near-cancellation of the positive and negative tensor components in the partition function. A high-precision data type is thus required to study spin glasses using TRG methods.

Model — The Hamiltonian of the Edwards-Anderson Ising spin glass is given by $H(s_i) = -\sum_{(ij)\in\mathcal{N}} J_{ij} s_i s_j$, where the spins $s_i \in \{\pm 1\}$ are on a square lattice and the summation is over nearest neighbors. We use bimodal-distributed interactions between the spins, i.e., $P(J_{ij}) = p\delta(J_{ij} - 1) + (1 - p)\delta(J_{ij} + 1)$, where $p$ is the fraction of ferromagnetic bonds.

Algorithm — The partition function of a classical statistical mechanical model with local interactions can be obtained by taking the product of local tensors at each site and summing over all bond indices, i.e., $Z = \text{Tr} \prod_i T_{x,y,i}$. The local tensor $T_{x,y,i}$ can be defined by tracing out $x_i$ from the product matrices $W$, i.e.,

$$T_{x,y,i} = \sum_{a} W_a x_i W_a y_i \frac{1}{W} W_a x_i W_a y_i \frac{1}{W}$$

where $W^1$ and $W^2$ are $2 \times 2$ matrices defined by

$$W^1 = \begin{pmatrix} \sqrt{\cosh(1/T)} & \sqrt{\sinh(1/T)} \\ \sqrt{\cosh(1/T)} & -\sqrt{\sinh(1/T)} \end{pmatrix}$$

$$W^2 = \begin{pmatrix} \sqrt{\cosh(1/T)} & -\sqrt{\sinh(1/T)} \\ \sqrt{\cosh(1/T)} & -\sqrt{\sinh(1/T)} \end{pmatrix}$$

To coarse grain the network, two neighboring tensors are contracted into one, i.e.,

$$T_{x,y,i}^{(n+1)} = \sum_{a} T_{x,y,i}^{(n)} T_{x,y,i}^{(n)}$$

The lattice size is reduced by a factor of 2 and the contracted tensor $T^{(n)}$ (along $x$ or $y$ axis) has a higher bond dimension $D^2$. The TRG method then truncates tensor $T^{(n)}$ into a lower rank tensor using different strategies [7].

Results — In our C++ implementation all quantities are stored with double or higher precision data types using the GNU Multiple Precision Arithmetic Library (gmplib.org). Without the TRG approximation, the coarse graining scheme based on a tensor-network model should produce the exact partition function of the model with $Z > 0$ for all temperatures $T$.

To demonstrate that a limited data type precision is the primary reason for negative partition function values for spin glasses at low temperature, 960 samples (linear size $L = 4$) with randomly-distributed bimodal interactions ($p = 0.5$) are generated and the partition function of these samples is calculated with double precision ($b = 64$ bits) and a higher precision ($b = 1024$ bits) data type. Figure 1 shows that the failure rate $f(Z)$ of the partition function $Z$ (i.e., when terms in the sum turn negative) as function of temperature $T$. It is clear that with double precision the failure rate increases as temperature decreases. However, the failure rate vanishes when a precision of $b = 1024$ bits is used.

To further show that a higher precision data type can reduce the failure rate of partition function for a two-dimensional spin glass,
we show the failure rate as function of date type’s precision $b$ in Fig. 2. As the precision increases, the failure rate at fixed temperature decreases. These results strongly suggest that negative partition function values are caused by the limited precision of the data type used in simulations instead of a cut-off parameter $D$ [7].

The success of TRG for the ferromagnetic Ising model implies that there is an intrinsic difference between the Ising model and a spin glass. To probe this difference, we plot in Fig. 3 256 tensor components of the contracted tensor $T^{n=1}$ for the two-dimensional ferromagnetic Ising model, as well as a spin glass. Figure 3 shows that for a spin glass near-cancellation of the positive and negative tensor components in the partition function requires a higher precision data type in order to obtain a physical value of the difference between tensor components. In contrast, for the ferromagnetic Ising model all components are positive. Therefore double precision is sufficient to obtain an accurate results.

Summary — By studying the partition function of the two-dimensional Edwards-Anderson Ising spin-glass model on a square lattice using the tensor renormalization group method we demonstrate that the limited precision of the used data type is the culprit for high failure rates at low temperatures and not a small cut-off parameter as surmised in Ref. [7]. To obtain precise partition function values at low temperature, both a high-precision data type and a large cut-off parameter are needed. The high precision requirements result in a sizable numerical overhead when applying tensor renormalization group methods to spin glasses, because the precision requirements grow for decreasing temperature or increasing system size. As such, while TRG, in principle, works for spin glasses, it is an extremely inefficient method for these models.

Acknowledgements — We would like to thank Zhiyuan Xie for fruitful discussions. Z.Z. and H.G.K. acknowledge support from the NSF (Grant No. DMR-1511387). The research is based upon work supported by the Office of the Director of National Intelligence (ODNI), Intelligence Advanced Research Projects Activity (IARPA), via Interagency Umbrella Agreement IA-1198. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the ODNI, IARPA, or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright annotation thereon. We thank the Texas Advanced Computing Center (TACC) at The University of Texas at Austin and Texas A&M University for providing HPC resources.

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