Regular phase operator and SU(1,1) coherent states of the harmonic oscillator

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Abstract

A new solution is proposed to the longstanding problem of describing the quantum phase of a harmonic oscillator. In terms of an ‘exponential phase operator’, defined by a new ‘polar decomposition’ of the quantized amplitude of the oscillator, a regular phase operator is constructed in the Hilbert–Fock space as a strongly convergent power series. It is shown that the eigenstates of the new ‘exponential phase operator’ are SU(1,1) coherent states associated to the Holstein–Primakoff realization. In terms of these eigenstates the diagonal representation of phase densities and a generalized spectral resolution of the regular phase operator are derived, which are very well suited to our intuitive pictures of classical phase-related quantities.

Keywords: harmonic oscillator, phase operator, SU(1,1) coherent states

(Some figures may appear in colour only in the online journal)

1. Introduction

The search for a well-defined phase operator (or a quantum angle variable) of a harmonic oscillator has a long history. It had already begun in the very first years following the foundation of quantum mechanics. In his fundamental paper on the quantum theory of emission and absorption of radiation, Dirac (1927) introduced the quantized amplitudes of the exponential phase operator

\[ e^{i \theta} = e^{i \phi} \]

as SU(1,1) coherent states associated to the field. Dirac wrote: ‘Resolving the radiation into its Fourier components, we can consider the energy and phase of each component to be dynamical variables describing the radiation field. … In order that an analogous method may be used on the quantum theory, it is necessary to assume that the variables \( E_r, \theta \) are q-numbers satisfying the standard quantum conditions \( \theta E_r = E_r \theta = i \hbar \), etc, where \( \hbar \) is \( (2\pi)^{-1} \) times the usual Planck’s constant, like the other dynamical variables of the problem. This assumption immediately gives light-quantum properties to the radiation.’

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Thus, one is not allowed, for instance, for the action and angle variables, \( J \) and \( w \), respectively, to write

\[ Jw = wJ = h/2\pi \] (5).

That it was possible to derive several correct results from this not-allowed equation (5), according to a remark of Dirac, is to be understood so that for the derivation of these results, in fact, instead of (5) only

\[ e^{2\pi i w} = e^{2\pi i J} = h e^{2\pi i J} \] (6) has been used.

Here \( h \) is Planck’s constant, and in the context of the harmonic oscillator \( J \) (called an index matrix by London) corresponds to the normal modes of the radiation field in polar form, which is analogous to the polar decomposition of complex numbers. In the same year, but earlier than Dirac’s paper appeared, London (1927) published an analysis on the angle variables and canonical transformations in quantum mechanics, and he proved that the angle variable cannot be expressed by a Hermitian matrix. In contrast to the quantum phase (angle), the formal exponential function of it (i.e. the ladder operators, which also appeared in Dirac’s formalism in the polar decomposition) could be represented by well-defined matrices. Dirac (1927) (see footnote 3) quoted London’s paper, which clearly shows that he knew about this discrepancy. In spite of this imperfection, the new theory was capable of giving a correct description of various fundamental processes taking place in light–matter interaction. As Jordan (1927) noted, ‘Thus, one is not allowed, for instance, for the action and angle variables, \( J \) and \( w \), respectively, to write

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\[ e^{2\pi i w} = e^{2\pi i J} = h e^{2\pi i J} \] (6) has been used.’ Here \( h \) is Planck’s constant, and in the context of the harmonic oscillator \( J \) (called an index matrix by London) corresponds to the
number operator $N$, and $w$ to the quantum phase. The formal exponential expression $e^{i\omega z}$ corresponds to the ladder operator $E^\dagger$ (see section 2). The non-unitarity of $E^\dagger$ has also been emphasized by Fock (1932) in the context of second quantization of particle systems. Thus, we may safely say that the founders of quantum theory were well aware of the fact that Heisenberg’s commutation relation $[Q, P] = i\hbar/2\pi$ for a Cartesian coordinate $Q$ and its canonically conjugate momentum $P$, cannot be taken over for action–angle variables (see also additional notes in the reference to Dirac’s quoted paper). It should be noted that, in fact, the first version of field quantization (or in other words, the quantization of a continuous dynamical system) in terms of normal modes was published by Born, Heisenberg and Jordan (1925) in their famous ‘Drei Männer Arbeitle’ (‘Three Men’s Work’). They quantized the field amplitudes in terms of the Cartesian components $Q$ and $P$ by imposing Heisenberg’s commutation relation; thus, they did not encounter the problem of quantum action and phase (for further details see Varró (2006).

In the meantime, there has been much effort devoted to solving the ‘problem of the non-unitarity of the ladder operators and of the non-existence of a Hermitian phase operator.’ Besides the study of the purely mathematical aspects of these problems, the main interest in quantum optics has been to find a well-working formalism for describing the quantal phase properties of electromagnetic radiation, so that these properties could be reliably reconstructed from experimental data. In the frame of the present paper it is not possible even to mention all the important works which have been published on this subject. Still, in the rest of this introduction we shall attempt to briefly summarize the main approaches and results concerning the investigation of the quantum phase problem.

The first period of increasing interest in the nature and description of phase in quantum optics appeared in the 1960s, after the construction of the first lasers. The quantum coherence theory of optical fields was founded primarily by Glauber (1963) and Klauder and Sudarshan (1968) in terms of expectation values of multi-linear expressions of quantized amplitudes, where the action–angle description has no explicit relevance. The non-existence of a Hermitian phase operator of a harmonic oscillator was newly realized by Susskind and Glogower (1964), who seemingly did not know the early discussions by Jordan (1927), London (1927) and Dirac (1927). Susskind and Glogower (1964) introduced formally the Hermitian cosine and sine operators, $C = (E + E^\dagger)/2$ and $S = (E - E^\dagger)/2i$, respectively, whose expectation values for a highly excited coherent state approach the cosine and sine of the phase of the complex parameter of such states. Carrutcrs and Nieto (1968) derived various number-phase uncertainty relations by using this cosine and sine, and Jackiw (1968) constructed a critical state which minimizes one of these uncertainty products. He has also noted that ‘unfortunately these states do not seem to have any physical significance’. It has turned out that the photon number distribution belonging to Jackiw’s minimum-uncertainty state may still be given a quite fundamental physical meaning. Namely, this distribution can also be derived from continuously entangled photon–electron states from the exact states of the jointly interacting subsystems (Varró (2008, 2010). Garrison and Wong (1970) constructed a quantum analogon of the classical periodic phase function (saw-tooth), which satisfies the Heisenberg commutation relation with the number operator on a dense set of the Hardy–Lebesgue space, associated to the oscillator. They also gave an iterative procedure to construct the eigenstates of this phase operator. In our opinion this was the first mathematically correct approach toward the solution of the original problem of the quantum phase. Paul (1974) proposed an alternative description of the phase of a microscopic electromagnetic field, and discussed the possibilities of its measurement.

The second period of interest in the quantum phase problem began around the end of the 1980s, and was initiated by the growing activity in describing non-classical states, such as squeezed states. Pegg and Barnett (1989) constructed a phase operator by using Loudon’s (1973) phase states with equally spaced phase parameters, in a finite-dimensional Hilbert space. We note that the possibility of using a finite-dimensional space in this context has already been discussed by Jordan (1927). The differences of the Garrison–Wong (1970) and Pegg–Barnett (1989) formalism have been thoroughly investigated by Popov and Yarunin (1992) and Gantsog et al (1992). The limit matrix elements of the phase operator in the number representation (as we let the dimension of the Hilbert space go to infinity) obtained by these authors have already been presented by Weyl (1931). Various polar decompositions of the quantized amplitude have been proposed (Zak (1969, Lerner et al 1970, Bergou and Englert 1991) and the non-linear coherent states associated to them have also been investigated for describing real physical processes (de Matos Filho and Vogel 1996, Man’ko et al 1997). Recently Urizar-Lenz and Tóth (2010) have used the old idea of Lévy-Leblond (1976) regarding the ‘spectral extension’ (unfortunately, later denoted by the same letter as the variance) of non-Hermitian operators for quantifying amplitude squeezing without reference to a phase operator. Adam et al (2014) numerically determined the intelligent states minimizing the ‘number-phase uncertainty relation’ associated to the number operator and the annihilation operator. The phase distribution of a highly-squeezed states was determined by Schleich et al (1989), who used the radial integral of the Wigner function (see Schleich 2001) after changing the original (Cartesian) variables to polar coordinates. We note that London’s work (1927) was quoted for the first time in the modern era in this paper. Smith et al (1992) derived the matrix elements of a phase operator directly, on the basis of a windowed quantum phase-space distribution, and Royer (1996) generalized this approach (see also Dubin et al 2000). Lukš and Peřínová (1993) constructed a number-phase Wigner function, depending on a discrete and on a continuous variable. Vaccaro (1995) developed further this approach, and found the satisfactory solution to the problem of fractional photon numbers, which appeared in the former work. Quantum phase measurements have been discussed by Shapiro and Shepard (1991), partly on the basis of
normalizable phase states, which are eigenstates of the usual lowering operator. The completeness integral of these states, however, does not give a meaningful result (see section 4).

No such kind of problem appears when one uses more general SU(1,1) coherent states (see Vourdas 1990, 1992, 1993, Vourdas and Wünsche 1998, Rasetti 2004). Brif and Ben-Aryeh (1994) discussed the coherent states introduced by Barut and Girardello (1971) as an example of intelligent states. Our present study relies, to a considerable extent, on the SU(1,1) coherent states in a Holstein–Primakoff type realization, which was first discussed in this context by Aharonov et al. (1973) (see also Katriel et al 1986, Brif 1995, Gerry and Grobe 1997).

Noh et al. (1991, 1992a, 1992b, 1993) have investigated the quantum phase dispersion on the basis of operationally defined cosine and sine operators, stemming from photon number counts in an eight-port interferometer. Freyberger and Schleich (1993) quite satisfactorily interpreted the experiment by Noh et al (1991). The experimental results of Smithey et al. (1993) confirmed the predictions based on both the Pegg–Barnett formalism (1989) and on the radially integrated Wigner distribution, at least for squeezed vacuum states (Schleich et al 1989). For further reading and references see the topical issue of Physica Scripta, edited by Schleich and Barnett (1993), in which also some historical aspects are summarized by Nieto (1993), the review by Lynch (1995) and the book by Pešinová et al. (1999) on the description of phase in optics. We also refer the reader to the review by Dodonov and Man’ko (2003a) in the book on non-classical states in quantum physics (Dodonov and Man’ko 2003b).

A possible solution to the problem of non-unitarity of the exponential phase operator can be given by a unitary extension (‘dilatation’) to a larger Hilbert space (see Newton 1980, Stenholm 1993). In fact, the general dilatation theorem was already proved by Sz-Nagy in 1953 (see the appendix in Riesz and Sz-Nagy 1965 and references therein). The problem of non-unitarity does not show up in the operational approach developed by Noh et al. (1991, 1992a, 1992b, 1993); neither does it show up in the relative phases of the two modes in their product Hilbert space (Luis and Sánchez-Soto 1996). Concerning further developments of the mathematical description of the quantum phase of a linear oscillator, see e.g. Kastrup (2003, 2006, 2007) and Rasetti (2004). The first systematic studies of the quantum phase (time-shifts) in a general mathematical framework are due to Helstrom (1974, 1976) and Holevo (1973, 1978, 1979). Concerning recent developments, we refer the reader to the thorough analyses by Lahti and Pellonpää (1999, 2000), Busch et al (2001), Pellonpää (2002) and Heinosaari and Pellonpää (2009), in which further development and new results can be found in this direction.

In the present paper a new approach is proposed for treating the quantum phase of the harmonic oscillator. We use the infinite-dimensional Hilbert–Pock space throughout, and will not rely on finite-dimensional subspaces of it. A particular emphasis will be put on the proper formulation and domain of the convergence of operator series which define the quantum-phase–related quantities. In this way we avoid ambiguities and inconsistencies concerning, for instance, the exchange of taking expectation values in finite subspaces and going over to the infinite-dimensional Hilbert space. In section 2 we present the basic definitions concerning the usual polar decomposition and consider some mathematical subtleties of the phase operator introduced by Garrison and Wong (1970). In section 3 we shall introduce an exponential phase operator based on a new polar decomposition of the quantized amplitude of the harmonic oscillator, in terms of which we define the new ‘regular phase operator’. This phase operator is a covariant phase observable, according to the definition used by Lahti and Pellonpää (1999, 2000) and Busch et al (2001).

Section 4 shall be devoted to the introduction of the eigenstates of the new exponential phase operator, which will turn out to be SU(1,1) coherent states in a Holstein–Primakoff realization. These states form a suitable complete set, in terms of which the phase operator and the corresponding phase densities can be expressed in simple and elegant forms that correspond to our classical intuition. In section 5 we shall derive positive operator-valued measures for the generalized spectral resolution of the regular phase operator, and introduce the associated probability distribution and density functions. In section 6 we shall summarize and briefly interpret our results. In order to make our paper possibly self-contained, the details of the mathematical derivations are presented in appendices A and B.

2. The usual polar decomposition of the quantized amplitude, the lack of unitarity of the exponential phase operator and the Garrison–Wong phase operator

The present section will be devoted, on one hand, to the mathematical formulation of the problem we are discussing, and, on the other hand, it is also meant to serve as an explanation of our motivation. We shall present the basic definitions concerning the usual polar decomposition and consider some mathematical subtleties of the phase operator introduced by Garrison and Wong (1970), and thoroughly discussed by Popov and Yaruin (1973, 1992). We shall also deal with the question of convergence concerning the series representation of both the classical and the Garrison–Wong phase.

Since the results of the present paper rely on a special, new polar decomposition of the quantized amplitude of a linear oscillator, first we briefly show a few details that appeared in the original treatments mentioned in the introduction. Dirac (1927) considered the amplitudes \( b \) and \( ib\hbar^2 \) to be canonically conjugate variables, and made the contact transformation by introducing \( N \) as a ‘probable number of systems in the state \( r \)’, as well as a new phase \( \theta \). The amplitudes are expressed in the polar forms \( b_r = e^{-i\theta}N^{1/2} = (N + 1)^{1/2}e^{i\theta/2} \) and \( b_r^* = N^{-1/2}e^{i\theta/2} = e^{i\theta/2}(N + 1)^{1/2} \), and they satisfy the quantum condition \( b_rib_r^* = -ib_r^*b_r = \hbar^2 \). (In Dirac’s notation \( r \) refers to a degree of freedom (it labels a
normal mode of the radiation field), $h$ is Planck’s constant divided by $2\pi$ and $\ast$ denotes Hermitian conjugation. Dirac sometimes explicitly wrote out the symbolic ‘representation’ $e^{i\beta t} \hat{a} = e^{i\beta t}a_{t}$ for the exponential phase operators, with the help of which the shift relations (see equation (2.5) below) can be derived. The quantum condition is formally consistent with the commutation relation $\theta N_{i} - N_{i} \theta t = i\hbar$ (where, interestingly, the phase $\theta$ had a dimension of action). Henceforth we shall use dimensionless quantities, and the following notations (for one mode): $b \rightarrow A$, $b^{\dagger} \rightarrow A^{\ast}$, thus $[A, A^{\ast}] \equiv AA^{\ast} - A^{\ast}A = 1$.

The usual ‘polar decomposition’ of the amplitude is written as

$$A = E \sqrt{N}, \quad A^{\dagger} = \sqrt{N} E^{\dagger}, \quad N = A^{\dagger}A,$$  \hspace{1cm} (2.1)

where the ladder operator $E$ is also called exponential phase operator, and occasionally is denoted by $E = \hat{e}^{i\vartheta}$. The Hilbert space $H$ we are using throughout the present paper is the Fock space spanned by the eigenvectors of the number operator $N$, whose domain $\mathcal{D}(N)$ consists of those vectors for which $N$ has a finite second moment (see e.g. Garrison and Wong 1970 or Busch et al 2001). The method used by Garrison and Wong (1970) relies on the basis of the Hardy class $H^{2}$ of functions analytic inside the unit disk $D = \{z = re^{i\vartheta}, |z| < 1 \}$ of the complex plane.

Garrison and Wong (1970) introduced a phase operator on the basis of the Hardy class $H^{2}$ of functions analytic inside the unit disk $D = \{z = re^{i\vartheta}, |z| < 1 \}$ of the complex plane.

The ladder operator $E$ and its adjoint, $E^{\dagger}$, can be symbolically expressed as

$$E = \sum_{n=0}^{\infty} |n\rangle \langle n + 1|,$$

$$E^{\dagger} = \sum_{n=0}^{\infty} |n + 1\rangle \langle n|,$$

$$EE^{\dagger} = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1,$$

$$E^{\dagger}E = 1 - |0\rangle \langle 0|,$$  \hspace{1cm} (2.3)

where, in the last two equations, we have also displayed their so-called ‘half-unitarity property’, which was first published by London (1927), and later by Fock (1932) in the context of second quantization in configuration space. In the mathematical terminology an operator such as $E^{\dagger}$ is called a partially isometric operator (or simply a partial isometry), because, though the effect of $E^{\dagger}$ preserves the norm, its adjoint $E$ does not do so (it is a true contraction operator). The basic commutators and the shift relations are

$$[E, E^{\dagger}] = EE^{\dagger} - E^{\dagger}E = |0\rangle \langle 0| \equiv P_{0},$$

$$[N, E] = -E, \quad [N, E^{\dagger}] = +E^{\dagger},$$  \hspace{1cm} (2.4)

$$ENE^{\dagger} = N + 1,$$

$$E^{\dagger}NE = N - 1 + P_{0}; \quad E^{\dagger}(N + 1)E = N.$$  \hspace{1cm} (2.5)

We note that the fourth equation in equation (2.4) shows the corresponding commutation relation we quoted from Jordan (1927) in the introduction. The first and the third shift relations shown in equation (2.5) can formally be derived if we write $E = \exp(i\vartheta)$ and $E^{\dagger} = \exp(-i\vartheta)$, by assuming the existence of an Hermitian $\vartheta$, which satisfies the commutation relation $[\vartheta, N] = -i$. Later we shall need the expressions for the effects of the $4$th power of $E$ and $E^{\dagger}$ on a generic state $|\psi\rangle = \sum_{n=0}^{\infty} c_{n} |n\rangle$, i.e.

$$E^{k} |\psi\rangle = \sum_{n=0}^{\infty} |r\rangle k \sum_{n=0}^{\infty} c_{n} |n\rangle$$

$$E^{k} |\psi\rangle = \sum_{n=0}^{\infty} c_{k+n} |n\rangle, \quad (k \geq 0),$$  \hspace{1cm} (2.6)

$$(E^{\dagger})^{k} |\psi\rangle = \sum_{n=0}^{\infty} |r + k\rangle (r \sum_{n=0}^{\infty} c_{n} |n\rangle$$

$$E^{k} |\psi\rangle = \sum_{n=0}^{\infty} c_{n} |k + n\rangle (k \geq 0).$$  \hspace{1cm} (2.7)

From equation (2.7) the isometric property of $E^{\dagger}$ can be easily shown, i.e. $|\langle E^{\dagger} |\psi\rangle| = ||\psi\rangle||$.

Garrison and Wong (1970) introduced a phase operator on the basis of the Hardy class $H^{2}$ of functions analytic inside the unit disk $D = \{z = re^{i\vartheta}, |z| < 1 \}$ of the complex plane.

$$\langle g | \Phi_{GW}^{‘} | f \rangle = \int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi} g(e^{i\theta})^{*} \theta \cdot f(e^{i\theta}) \right \rangle.$$  \hspace{1cm} (2.8)

The basis states of $C$, constructed by Garrison and Wong (1970) are not orthogonal
to each other (they are, of course, linearly independent). In our view, this non-orthogonality is one of the main obstacles which prevents an easy application of these results. This is also the reason why the powers of \( \Phi_{GW} \), expressed in the Fock basis, are very complicated (see Popov and Yarunin 1992). In this context we note that Gantsoo, Miranowicz and Tanaś (1992) performed a thorough analysis of the phase distributions based on the eigenstates of \( \Phi_{GW} \), and made a comparison with the predictions based on the formalism introduced by Pegg and Barnett (1989).

In accord with equation (2.8), the matrix elements of \( \Phi_{GW} \) in the Fock basis are

\[
\langle n | \Phi_{GW} | m \rangle = \begin{cases} 
0, & n = m \\
(-1)^{n-m} \frac{1}{n-m}, & n \neq m .
\end{cases} \tag{2.9a}
\]

By now we have put a prime in the notation of \( \Phi_{GW} \), reminding us that Garrison and Wong (1970) represented the number operator by \( N = -i\hbar d\theta \), in analogy with the \( z \)-component of the angular momentum operator \( L_z \) for motions in the \( x - y \) coordinate plane (\( L_z e^{i m\phi} = m e^{i m\phi} \), \( m = 0, \pm 1, \ldots \)). However, for an oscillator the rotation in phase space is clockwise (if we associate \( q \to x \) and \( p \to y \), see e.g. Schleich 2001). This corresponds to the symbolic expression \( E = \vec{\omega} \Phi_{GW} \) with \( \Phi_{GW} = -\Phi_{GW} \) and to the commutation \( [N, \Phi_{GW}] = i \), so that the correct shift relations in equation (2.5) can also be derived. We note that, by changing the base interval \((-\pi < \theta < \pi)\) to \((0 < \theta < 2\pi)\), the matrix elements of \( \Phi_{GW} \) become

\[
\langle r | \Phi_{GW} | s \rangle = \begin{cases} 
p_n, & r = s \\
p_n + \frac{1}{s-r}, & r \neq s .
\end{cases} \tag{2.9b}
\]

By taking equations (2.6) and (2.7) into account, and introducing an arbitrary reference phase \( \phi_0 \) (corresponding to the base interval \( \phi_0 < \theta < \phi_0 + 2\pi)\), the Garrison–Wong phase operator can be written in the form

\[
\Phi_{GW} \equiv \phi_0 + \pi + \sum_{k=1}^{\infty} \frac{i}{k} \left( E^k e^{-ik\phi_0} - (E^k)^* e^{ik\phi_0} \right) . \tag{2.10}
\]

If we take the references phase \( \phi_0 = 0 \) in equation (2.10), then the matrix elements of \( \Phi_{GW} \) are those shown already in equation (2.9b). The formal operator series can be considered as a quantum analogon of the famous classical Fourier series of the saw-tooth function (see e.g. Gradshteyn and Ryzik 1980).

\[
\Phi_{saw}(e^{i\phi}) = \pi + \sum_{k=1}^{\infty} \frac{i}{k} \left[ e^{ik\phi} - e^{-ik\phi} \right] = \pi - 2 \sum_{k=1}^{\infty} \frac{\sin k\phi}{k} = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin n\phi}{n} \equiv \phi . \tag{2.11}
\]

As is known, though the partial sums of this series are bounded, they suffer from the so-called Gibbs phenomenon, and the series is not absolutely convergent. This is illustrated in figures 1(b) and (c).

It is instructive to compare the usual partial sum \( s_n \) and the Fejér mean \( \sigma_n \equiv \sum_{m=1}^{n} s_m / n \) (which is the arithmetic mean of the first \( n \) partial sums) for the series in equation (2.11), as is shown for \( n = 61 \) in figures 1(c) and (d), respectively. In contrast to the singular behavior of \( s_n \), the Fejér means \( \sigma_n \) behave regularly; in fact, they uniformly reconstruct a function from its Fourier series. Concerning further details on the summability of Fourier series see e.g. Titchmarsh (1939), chapter 13.

On the basis of the conditional convergence of the classical phase function (and the overshoots at the discontinuity points) one would intuitively expect that the convergence of the operator series in equation (2.11) is also conditional, in a sense. More precisely, the series of \( \Phi_{GW} \) is weakly convergent, though the bilinear form \( \langle \chi | \Phi_{GW} | \psi \rangle \) with the kernel matrix (2.9b) is bounded for the Hilbert space elements, as has been emphasized by Popov and Yarunin (1973, 1992),

\[
\left| \langle \chi | (\Phi_{GW} - \pi) | \psi \rangle \right| \leq \pi \| \chi \| \cdot \| \psi \| . \tag{2.12}
\]

One should also take into account that this famous inequality (originally derived by Hilbert for real sequences) results from the symmetric partial double sums

\[
T_n = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{x_n y_m}{n - m} \leq \pi \sqrt{\sum_{n=1}^{\infty} x_n^2} \sqrt{\sum_{m=1}^{\infty} y_m^2} \tag{2.13}
\]

in the \( r \to \infty \) limit (see e.g. the book by Cooke (1950), which has also been quoted by Popov and Yarunin (1992) in this context). Accordingly, the two infinite series on the right-hand side of equation (2.10) separately does not have a well-defined meaning. Such convergence ambiguities are absent in the regular phase operator, to be introduced in the next section. We would also like to point out that in the following considerations we will not rely on any special representation of the Hilbert space of the oscillator (such as the Hardy–Lebesgue space, used by Garrison and Wong 1970).

3. Regular phase operator based on a new polar decomposition

In the present section, on the basis of a new polar decomposition of the quantized amplitude \( A \), we shall introduce a new exponential phase operator \( F \), which depends on a positive parameter \( \nu \) (Varró 2014). In terms of powers of \( F \) (and of its adjoint \( F^+ \)) we shall define a generalization of the Garrison–Wong phase operator (equation (2.10)), which we propose to call the regular phase operator.
We define the new exponential phase operator $F\nu$ by the following polar decomposition of the quantized amplitude $AE\nu N F\nu N$ 

\[ F\nu N (0) , \nu \nu + > = + \]

where the positive parameter $\nu$ is not subject to any further restrictions. The possible physical meaning of $\nu$ will be discussed in section 4. Since both $E\nu$ and $E\nu +$ are contractions (i.e. $\|E\nu\| \leq 1$ and $\|E\nu +\| \leq 1$), and, in general, for any two operators $A$ and $B$ the inequality $\|AB\| \leq \|A\| \cdot \|B\|$ holds, we see from equation (3.1) that $F\nu$ and $F\nu +$ are contractions, too. With the help of equations (2.4) and (2.5) one can easily derive the analogous relations,

\[ \left[ F, F\nu + \right] = \frac{\nu}{(N + 1 + \nu)(N + \nu)} , \]

\[ \left[ N, F\nu k \right] = -kF\nu k , \]

\[ \left[ N, (F\nu +)^k \right] = + k (F\nu +)^k (k \geq 0) , \]  

(3.2)

From the last two equation of (3.2) it follows that the time-evolution of $F\nu$ and $F\nu +$ are the same as that of $E\nu$ and $E\nu +$ (or $A$ and $A\nu +$), respectively.

\[ F\nu F\nu + = (N + 1) \left[ 1 - \frac{\nu}{N + \nu + 1} \right] , \]

\[ F\nu + (N + 1)F\nu = N \left[ 1 - \frac{\nu}{N + \nu} \right] . \]  

(3.3)

\[ e^{i\nu x} F\nu e^{-i\nu x} = F\nu e^{-i\nu x} , \ e^{i\nu x} F\nu + e^{-i\nu x} = F\nu + e^{i\nu x} , \]  

(3.4a)

where $\chi$ may be considered as a dynamical phase $\chi = \omega t$ of an oscillator of circular frequency $\omega$. However, according to equation (3.3), the shift relations (equation (2.5)) valid for $E\nu$ and $E\nu +$ are not preserved for $F\nu$ and $F\nu +$, but the relation $F\nu g(N) = g(N + 1)F\nu$ holds with an arbitrary function $g(N)$. The explicit expressions for the powers of $F\nu$ and $F\nu +$ can simply be calculated by using equation (2.3),

\[ F\nu k = f_k^{1/2} (N)E\nu k , \ (F\nu +)^k = (E\nu +)^k f_k^{1/2} (N) k \geq 0 , \]  

(3.4b)
\[ f_k(n) \equiv \frac{((n+k)(n+(k-1)) \cdots (n+2)(n+1))}{((n+v+k)(n+v+(k-1)) \\
\cdots (n+v+2)(n+v+1))} \]
\[ = \frac{\Gamma(n+1+k)}{\Gamma(n+1+k+v)} \frac{\Gamma(n+1+k)}{\Gamma(n+1)}, \quad (3.5) \]

where \( \Gamma \) denotes the usual gamma function (see e.g. Gradshteyn and Ryzhik 1980).

Notice that for any \( \nu > 0 \), \( f_0(n) = 1 \) for all \( n \geq 0 \), and in the case of \( \nu = 0 \), we have \( f_k(n) = 1 \) for all \( n \geq 0 \) and \( k \geq 0 \). Below we shall need certain inequalities satisfied by the functions \( f_k(n) \), defined in equation (3.5). As is shown in appendix A (see equation (A.8)), for any values \( \nu > 0 \), the following inequalities are valid
\[ f_k(0) < \Gamma(1 + \nu) e^{\nu (1 + \nu)} (k + \nu)^\nu, \]
\[ (k \geq 1, n = 0), \]
\[ f_k(n) < e^{1/4} \sqrt{1 + \nu} (n + \nu)^\nu (k \geq 1, n \geq 1), \]
\[ (\nu > 0), \quad \text{(3.6)} \]

By prescribing the correspondence with the classical (saw-tooth) phase function in equation (2.11), i.e. by using the associations of \( e^{i\Phi} \to F \) and \( e^{-i\Phi} \to F^+ \), we define the new regular phase operator as the infinite series
\[ \Phi = \phi_0 + \pi + \sum_{k=1}^{-\infty} \frac{i}{k} \left[ F^k e^{-ik\phi_0} - (F^+)^k e^{ik\phi_0} \right]. \quad \text{(3.7)} \]

Of course, one has to define the domain of this \( \Phi \), too, by considering the convergence properties of the operator series in equation (3.7). Our aim here is to find a domain in Hilbert space where the right-hand side of equation (3.7) can be represented by a sum of two infinite series, which converge strongly and separately. On the basis of the inequalities in equation (3.6), it is shown in appendix A (see the derivation of equations (A.14a)-(A.14b)), for any generic state \( |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \in H \), the upper bounds of the norms of \( F^k |\psi\rangle \) and \( (F^+)^k |\psi\rangle \) are subject to the uniform estimates
\[ \|F^k |\psi\rangle\|^2 < \frac{\tilde{b}_{\nu}}{(k + \nu)^\nu}, \]
\[ \|(F^+)^k |\psi\rangle\|^2 < \frac{\tilde{b}_{\nu}}{(k + \nu)^\nu} (\nu > 0, k \geq 1), \quad \text{(3.8a)} \]
\[ \tilde{b}_{\nu} \equiv e^{1/4} \Gamma(1 + \nu) + e^{1/4} \sqrt{1 + \nu} (N + \nu)^\nu \phi_0, \]
\[ \{\langle N + \nu \rangle\} = \sum_{n=0}^{\infty} (n + \nu)^\nu |c_n|^2 < \infty. \quad \text{(3.8b)} \]

It is of crucial importance here that the bound parameter \( \tilde{b}_{\nu} \) introduced in equation (3.8b) does not depend on power index \( k \) of \( F^k \) (and of \( (F^+)^k \), either). As is seen in equation (3.8b), the bounds in (3.8a) exist and are non-trivial only for those states for which the \( \nu \)-th moment of \( (N + \nu) \) is finite. We denote the set of such states by \( \mathcal{D}_\nu(N) \),
\[ D_{\nu}(N) = \{ |\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \in H; \sum_{n=0}^{\infty} (n + \nu)^\nu |c_n|^2 < \infty \}. \quad \text{(3.9)} \]

It is clear that for \( \nu \)-values in the range \( 0 < \nu \leq 2 \), the domain \( \mathcal{D}(N) \) (see equation (2.2)) is contained by \( \mathcal{D}_{\nu}(N) \), i.e. \( \mathcal{D}(N) \subseteq \mathcal{D}_{\nu}(N) \), and if \( \nu \geq 2 \), then \( \mathcal{D}_{\nu}(N) \subseteq \mathcal{D}(N) \). On the basis of these considerations, one finds that the infinite series expressions in the defining equation (3.7) of the new phase operator converge strongly in \( \mathcal{D}_{\nu}(N) \), because, for instance, for the first series we have
\[ \left\| \sum_{k=1}^{\infty} \frac{i e^{-ik\phi_0}}{k} |\psi\rangle \right\|^2 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \left\| F^k |\psi\rangle \right\|^2 \leq \sum_{k=1}^{\infty} \frac{b_{\nu}}{(k + \nu)^{1+\nu/2}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\nu/2}} \]
\[ = \left( \tilde{b}_{\nu} \right)^{1/2} \zeta(1 + \nu/2), \quad \text{(3.10)} \]

where we have introduced the Riemann zeta function \( \zeta(s) \) (see e.g. Titchmarsh 1939). Owing to the inequalities in equation (3.8a), the same bound exists for the second series on the right-hand side of equation (3.7). Thus, the two terms of the infinite series in equation (3.7) converge separately and strongly in \( \mathcal{D}_{\nu}(N) \) for any \( \nu > 0 \), which means, at the same time, that we are justified to write equation (3.7) in the alternative form
\[ \Phi = \phi_0 + \pi + \sum_{k=1}^{\infty} \frac{i}{k} F^k e^{-ik\phi_0} - \sum_{k=1}^{\infty} \frac{i}{k} (F^+)^k e^{ik\phi_0} \]
\[ = \phi_0 + \pi - i \left[ \log(1 - F e^{-i\phi_0}) - \log(1 - F^+ e^{i\phi_0}) \right], \quad \text{(3.11)} \]

where we have formally used the sum of the power series
\[ \sum_{k=1}^{\infty} \frac{x^k}{k} = -\log(1 - x), \]
which is convergent in the interval \((-1 < x < 1)\). For \(|z| < 1\) the latter numerical series is absolutely convergent, but, of course, for a complex \( z \) we have to specify the branch of the logarithm, too. In all the cases where we shall use equation (3.11) below, the main branch of the logarithm will be taken in the expectation values. The second equality in equation (3.11) is to be rather considered as the definition of the \( \log \) functions of the operators, which is surely meaningful, at least in the sense of the strong convergence in the domain \( \mathcal{D}_{\nu}(N) \). A completely different situation appears if \( \nu = 0 \) (i.e. \( F = E \) and \( F^+ = E^* \)), in which case the series corresponds to the Garrison–Wong phase operator in equation (2.10)). In this case on the right-hand side of equation (3.10) we encounter with a logarithmically
divergent factor $\sum_{k=0}^{\infty} (1/k)$, and the above estimates cannot be used. The limit of the symmetric double sum in equation (2.13) still exists, and represents the phase operator $\Phi_{GW}$ of Garrison and Wong (1970), as has been emphasized by Popov and Yarunin (1992).

On the basis of the last two equation (3.2), the number-phase commutator can be brought to the form

$$[N, \Phi] = i - 2\pi i P_{\psi},$$

$$P_{\psi} \equiv \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F_k e^{-ik\psi},$$

$$F_k \equiv \begin{cases} F^k, & k \geq 0 \\ (F^*)^k, & k \leq 0, \end{cases} \quad (3.12)$$

where we have introduced the factor $2\pi$ in front of $P_{\psi}$ for later convenience. From equations (3.8a) and (3.9) we see that the series in equation (3.12) converges strongly in $D(\nu)$. Since for the matrix elements of a phase operator, directly on the original Hilbert space of one harmonic oscillator, the expansion coefficients of $F_{\nu}$, defined in equation (3.13b), are $SU(1,1)$ coherent states with Bargmann index $\kappa = \frac{1}{2} (\nu + 1) > \frac{1}{4}$, associated to the $SU(1,1)$ Lie algebra of a type of Holstein–Primakoff realization in the original Hilbert space of one harmonic oscillator. In the last part of the present section we will show that these states serve as a natural basis for representing the new phase operator and other quantum-phase–related quantities.

The normalized solution of the eigenvalue equation $F \mid z \rangle = z \mid z \rangle$ (see appendix B) reads

$$F \mid z \rangle = z \mid z \rangle; \quad z = \left(1 - |\psi|^2\right)^{1/2} \sum_{n=0}^{\infty} \frac{F(\nu + 1 + n)}{F(\nu + 1)(n+1)} z^n \mid n \rangle, \quad z \in D = \{ z, |z| < 1 \}, \quad (4.1)$$

where $z$ is a complex number in the open unit disc $D$. The expansion coefficients of $|z\rangle$ have the same form as those of the $SU(1,1)$ coherent states in the discrete series representation (Perelomov 1986). As is shown in appendix B, these states represent a scaled variant of the states $|\alpha, k\rangle$ derived by Aharonov et al (1973), as eigenstates of a generalized annihilation operator (see equations (B.3d)–(B.3e) in appendix B). The expectation value, variance and the Mandel Q parameter of the power series

$$P_{\psi} \quad (3.13b)$$

is an open question. At any rate, according to equations (3.13b) and (3.14), $\langle \psi \mid 2\pi P_{\psi} | \psi \rangle$ can be considered as a generalization of the London phase distribution $|\langle \psi | \Phi | \psi \rangle|^2$, which has been widely used. We note that the matrix elements $\langle r \mid P_{\psi} | s \rangle$ have the properties (b) and (c) of the phase theorem of Lahti and Pellonpää (1999).

4. Regular phase states and the diagonal representation of the regular phase operator

Our formulae for both the number-phase commutator (equation (3.12)) and for the matrix elements (3.13(a)–(b)) are quite complicated in comparison with those appearing in the Garrison–Wong formalism. In the present section we shall introduce a suitable complete set of states, in terms of which all mathematical objects will receive a simple and elegant form, corresponding to our classical intuition. This correspondence relies on the strong convergence of the defining series of our regular phase operator (expressed by equation (3.11)), i.e. we need not care about convergence questions concerning the rearrangement of infinite (operator) series. We shall prove that the eigenstates of $F$, introduced in equation (3.1), are $SU(1,1)$ coherent states with Bargmann index $\kappa = \frac{1}{2} (\nu + 1) > \frac{1}{2}$, associated to the $SU(1,1)$ Lie algebra of a type of Holstein–Primakoff realization in the original Hilbert space of one harmonic oscillator. In the last part of the present section we will show that these states serve as a natural basis for representing the new phase operator and other quantum-phase–related quantities.

The normalized solution of the eigenvalue equation $F \mid z \rangle = z \mid z \rangle$ (see appendix B) reads

$$F \mid z \rangle = z \mid z \rangle; \quad z = \left(1 - |\psi|^2\right)^{1/2} \sum_{n=0}^{\infty} \frac{F(\nu + 1 + n)}{F(\nu + 1)(n+1)} z^n \mid n \rangle, \quad z \in D = \{ z, |z| < 1 \}, \quad (4.1)$$

where $z$ is a complex number in the open unit disc $D$. The expansion coefficients of $|z\rangle$ have the same form as those of the $SU(1,1)$ coherent states in the discrete series representation (Perelomov 1986). As is shown in appendix B, these states represent a scaled variant of the states $|\alpha, k\rangle$ derived by Aharonov et al (1973), as eigenstates of a generalized annihilation operator (see equations (B.3d)–(B.3e) in appendix B). The expectation value, variance and the Mandel Q parameter of the power series $P_{\psi} \quad (3.13b)$ is an open question. At any rate, according to equations (3.13b) and (3.14), $\langle \psi \mid 2\pi P_{\psi} | \psi \rangle$ can be considered as a generalization of the London phase distribution $|\langle \psi | \Phi | \psi \rangle|^2$, which has been widely used. We note that the matrix elements $\langle r \mid P_{\psi} | s \rangle$ have the properties (b) and (c) of the phase theorem of Lahti and Pellonpää (1999).
We note that Gilles and Knight (1992) have presented an extensive analysis of the statistical properties of the photon number distribution of the type \(|R_\mu \equiv |w_\mu|^2, n = 0,1,\ldots\) of (B.2), in the context of two-mode squeezed states. In this case the physical meaning of the parameter \(\nu\) is the invariant difference of the photon numbers, \(\hat{n}_a - \hat{n}_b\), of the two modes, whose interaction is represented by the parametric Hamiltonian \(\omega(\alpha \hat{a}^\dagger - \alpha^* \hat{a}^\dagger \hat{b}^\dagger)\). Since the regular phase states are one-mode pure states, in spite of common features in the photon number distributions with that of the two-mode squeezed states, we cannot give the parameter \(\nu\) an immediate meaning on this basis.

The eigenstates in equation (4.1) are in fact one-mode SU(1,1) coherent states in a Holstein–Primakoff realization (Aharonov et al. 1973, Katriel et al. 1986, Brif and Ben-Aryeh 1994, Brif 1995, Gerry and Grobe 1996), associated to the Lie algebra of the basis operators \(K_{\pm}\) and \(K_{0}\) (see appendix B),

\[
\begin{align*}
K_{\pm} &= A\sqrt{N + \nu} = F(N + \nu), \\
K_{0} &= \sqrt{N + \nu}A^* = (N + \nu)F^*, \\
K_0 &= N + \frac{1}{2}(\nu + 1) = N + \kappa,
\end{align*}
\]

(4.3a)

where we have introduced the Bargmann index \(\kappa\), in terms of which \(\nu = 2\kappa - 1\). It can also be shown that

\[
[K_{\pm}, K_{\pm}] = 2(K_0 + \kappa), \quad [K_0, K_{\pm}] = \pm K_{\pm},
\]

(4.3b)

from which it follows that the eigenstates of \(F\) are also eigenstates of the commutators (4.3c). On the basis of this property, Aharonov et al. (1973) have shown that the states \(|z\rangle\) are generated from the vacuum state by the unitary transformation,

\[
|z\rangle = \exp(\xi e^{i\eta}K_{\pm} - \xi^* e^{-i\eta}K_{-})|0\rangle, \quad z = \tanh(\xi)e^{i\eta}.
\]

(4.4)

The interaction terms \(\omega K_{\pm}\) represent a kind of intensity-dependent coupling. We note that the coherent states introduced by Barut and Girardello (1971) are eigenstates of \(K_{\pm} = F(N + \nu)\), so they differ from \(|z\rangle\), of course.

The states \(|z\rangle = \rho \exp(i\theta)|0\rangle\) with different \(z=\rho\) are not orthogonal, but they form an overcomplete set in the original Hilbert space \(H\). The completeness can be shown by introducing a suitable weight function, and then integrating with respect to \(\theta\) and \(\rho\) over the open unit disc \(D\) (see appendix B),

\[
\begin{align*}
\frac{1}{\pi} \int_D d^2\mu(z)|z\rangle\langle z| &= 1, \\
\int\! d^2\mu(z) \equiv d^2\zeta(1 - |z|^2)^2, \\
d^2\zeta &\equiv d(\Re z)d(\Im z) = \rho d\rho d\theta, \quad (\nu > 0).
\end{align*}
\]

(4.5)

We note at this point that the so-called normalizable phase states \(|z\rangle_0\), first discussed by Lerner et al. (1970) and later used e.g. by Shapiro and Sheppard (1991) and D’Ariano et al. (1996), represent a special case of \(|z\rangle\) with \(\nu = 0\). They are eigenstates of the usual exponential phase operator \(E\),

\[
E|z\rangle_0 = |z\rangle_0, \quad |z\rangle_0 = \sqrt{1 - |\zeta|^2} \sum_{n=0}^{\infty} z^n|n\rangle,
\]

(4.6)

Unfortunately, the completeness integral (equation (4.5)) does not give a reasonable result in case of \(\nu = 2\kappa - 1 = 0\) (see appendix B); thus one cannot use \(|z\rangle_0\) as basis states. In fact, this circumstance has been one of the roots of our motivations for introducing \(F\) and its regular eigenstates \(|z\rangle\).

Now we are in the position to express the regular phase operator (equation (3.7)) in a diagonal form, in terms of the regular phase states (equation (4.1)). Owing to equation (4.11) and the eigenvalue equation \(E|z\rangle = z|z\rangle\) (and also \(|z\rangle|F^+\rangle = z^\omega|z\rangle\)), by using the completeness relation (equation (4.5)) we receive (see appendix B)

\[
\Phi(z) \equiv \phi(z), \quad \phi(z)^\omega = \phi_0 + \pi - 2\arctan\left[\frac{\rho \sin(\theta - \phi_0)}{1 - \rho \cos(\theta - \phi_0)}\right], \quad (\phi_0 < \phi(z) < \phi_0 + 2\pi),
\]

(4.7a)

where we have used the parametrization \(z = \rho e^{i\phi}\), and introduced the quantum phase function \(\phi(z)\). It is important to note that the quantum phase function does not depend on the parameter \(\nu\). From the original form (equation (3.7)) of the regular phase operator one can show that the diagonal matrix elements of \(\Phi\) between the regular phase states just coincide with the quantum phase function \(\phi(z)\) defined in equation (4.7a),

\[
\langle z | \Phi | z \rangle = \phi(z) = \phi_0 + \pi - 2\sum_{k=1}^{\infty} \rho^k \sin\left[\frac{k(\theta - \phi_0)}{k}\right].
\]

(4.8)

The two forms of \(\phi(z)\) in equations (4.7a) and (4.8) are equivalent, which can be shown by using the Fourier series of the arctan function (see appendix B). Since the Fourier series \(\sum_{k=1}^{\infty} (\sin k\phi)/k\) converges to \(1/2(\pi - \phi)\) in the interval
(0 < φ < 2π), in the limit ρ → 1−0 the quantum phase function ϕ(z) approaches the classical phase function ϕφ(ϕ), given by equation (2.11) (where we have taken the reference phase φφ = 0). This behavior is illustrated in figure 2(a).

5. Generalized spectral resolution of the regular phase operator and phase probability distributions

The regular phase operator Φ defined by equation (3.7) can also be represented as a one-dimensional integral on the base interval (φφ < φ < φφ + 2π) of the set of positive operators Pψψψψ.

\[
Φ = \int_{φφ}^{φφ + 2π} dφψψψψ Pψψψψ = \frac{ν}{π} \int d^2μ(z) |z| Γψψψψ(z) |z|,
\]

(5.1)

where the quantum phase density Pψψψψ(z) is just the well-known Poisson kernel. We have derived it by using the formula 1.447.3 of Gradshteyn and Ryzhik (1980). In fact, we also have

\[
Pψψψψ(z) = \frac{1}{2π} \frac{1 - ρ^2}{1 - 2\rho \cos(θ - φφ - ρ')} + ρ^2, \quad (5.1a)
\]

Like the quantum phase function, the quantum phase density does not depend on the parameter ψ. As a function of θ - φφ, in the limit ρ → 1−0 the Poisson kernel is more and more peaked around φ', as is shown in figure 2(b). The explicit form of the Pψψψψ, in terms of the operators F and F+, can simply be obtained by applying the diagonal representation of Φ and the completeness relation, given by equations (4.7) and (4.4), respectively,

\[
Pψψψψ = \frac{1}{2π} \sum_{k=-∞}^{∞} [F^k e^{-iψψψψ} + (F^*)^k e^{+iψψψψ}]
\]

\[
≡ \frac{1}{2π} \sum_{k=0}^{∞} F_k e^{-iψψψψ}(0 < φ ≤ 2π).
\]

(5.2)

It is worth noting that we have already encountered this positive operator in equation (3.12), where we have calculated the number-phase commutator [N, Φφ] = i - 2πiPψψψψ.

In section 3 we have defined the regular phase operator (equation 3.7)) by using the correspondences eψψψψ → F and e−ψψψψ → F+ in the Fourier series of the classical (periodic sawtooth) phase function (equation (2.11)), which converges (reconstructs) ϕ on the base interval. Similarly, we can associate a set of operators to the periodic functions eψψψψ(φ) (taking on values 1 or 0) defined as

\[
eψψψψ(φ) = \begin{cases} 
1 & \text{if } 2kπ < φ ≤ 2kπ + ψ \\
0 & \text{if } 2kπ + ψ < φ ≤ 2(k+1)π \\
(k = 0, ±1, ±2 \ldots) 
\end{cases}
\]

(5.3)

With eψψψψ(φ) ≡ 0 and eψψψψ(φ) ≡ 1 (Riesz and Sz-Nagy 1965). The function defined in equation (5.3), as a function of ψ, is similar to a degenerate distribution function, in the terminology of probability theory. In the interval (0 < φ ≤ 2π), for instance, it cuts apart (0 < φ ≤ ψ) where it is equal to unity,
and on the rest of the interval it is zero. This function is always equal to its square, i.e. \( [\epsilon_\psi(\phi)]^2 = \epsilon_\psi(\phi) \), and, moreover, \( \epsilon_\chi(\phi)\epsilon_\psi(\phi) = \epsilon_\chi(\phi) \) for \( \chi \neq \psi \). Owing to these properties, we shall call \( \epsilon_\psi(\phi) \) the classical projector function, whose plot is shown in figure 3(a). By the correspondence \( \epsilon_\psi \to F \) (and \( e^{-i\psi} \to F^* \)) we associate a set of operators \( E_\psi \) to the Fourier series representation of \( \epsilon_\psi(\phi) \)

\[
E_\psi(\phi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{i}{k} \left( e^{-ik(\psi - \phi)} - 1 \right) e^{ik(\psi - \phi)}
\]

\[
E_\psi = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{i}{k} \left( e^{-ik(\psi - \phi)} - 1 \right) F_k e^{-ik\phi},
\]

\[
(\phi_0 < \psi < \phi_0 + 2\pi),
\]

\[
(\phi_0 < \psi \leq \phi_0 + 2\pi)
\]

(5.4)

where we have introduced a reference phase \( \phi_0 \), and used the notation \( F_k = F^k \) (\( k \geq 0 \)) and \( F_k = (F^*)^k \) (\( k < 0 \)), as has been already defined in equation (3.12). The operator series (equation (5.4)) converges strongly in \( \mathcal{D}_0(N) \). The diagonal expansion of \( E_\psi \) can be obtained, by using the completeness relation (equation (4.4)) and the eigenvalue equations

\[
E_\psi = \frac{1}{\pi} \int_0^{2\pi} d\psi' \epsilon_\psi(\psi') \epsilon_\psi(\psi) \left( \epsilon_\psi(\psi') \epsilon_\psi(\psi) \right),
\]

\[
E_\psi = \frac{1}{\pi} \int_0^{2\pi} d\psi' \epsilon_\psi(\psi') \epsilon_\psi(\psi) \left( \epsilon_\psi(\psi') \epsilon_\psi(\psi) \right),
\]

where we have introduced the quantum projector function \( \epsilon_\psi(\phi) \), which, similarly to \( \psi(\phi) \) and \( P_\psi(\phi) \), does not depend on the parameter \( \nu \), either. In figure 3 we have compared the classical and quantum projector functions, \( \epsilon_\psi(\phi) \) and \( \epsilon_\psi(\phi = e^{i\phi}) \), defined in equations (5.3) and (5.4b), respectively.

If \( F \) were unitary, then \( \{E_\psi\} \) would form a true spectral set (Riesz and Sz-Nagy 1965) consisting of mutually orthogonal projectors, but this is not the case here. This is just the original problem with the exponential phase operators (with both \( E \) and \( F \)). Still, one can proceed further and derive a 'generalized spectral resolution' of \( \Phi \) (which is essentially equivalent to equation (5.1)). From equations (5.2) and (5.4) it can be seen that \( P_\psi \) is the derivative of \( E_\psi \) with respect to \( \psi \); thus the one-dimensional integral representation of the phase operator (see the first equation in (5.1)) can also be written as

\[
\Phi = \int_{\phi_0}^{\phi_0 + 2\pi} \psi P_\psi d\psi = \int_{\phi_0}^{\phi_0 + 2\pi} \frac{dE_\psi}{d\psi} d\psi
\]

\[
= \int_{\phi_0}^{\phi_0 + 2\pi} \psi dE_\psi,
\]

\[
F_k = \int_{\phi_0}^{\phi_0 + 2\pi} e^{ik\psi} P_\psi d\psi = \int_{\phi_0}^{\phi_0 + 2\pi} e^{ik\psi} dE_\psi. \quad (5.5)
\]

The second set of equations (obtained from equation (5.2) and \( P_\psi = dE_\psi/d\psi \)) expresses the spectral resolution in a wider sense of the contraction operators \( F_k = F^k \) (\( k \geq 0 \)) and \( F_k = (F^*)^k \) (\( k < 0 \)). The last equation is an analogon of the usual spectral resolution of unitary operators (see Riesz and Sz-Nagy 1965). According to equation (5.5) the expectation value of the phase operator in a state represented by a density operator \( \hat{\rho} \) can be brought to the following equivalent forms,

\[
\langle \Phi \rangle = Tr \left( \hat{\rho} \Phi \right) = \int_{\phi_0}^{\phi_0 + 2\pi} \psi d\psi \left[ Tr \left( \hat{\rho} E_\psi \right) \right]
\]

\[
= \int_{\phi_0}^{\phi_0 + 2\pi} \frac{d}{d\psi} \left[ Tr \left( \hat{\rho} E_\psi \right) \right] d\psi
\]

\[
= \int_{\phi_0}^{\phi_0 + 2\pi} \psi Tr \left[ \hat{\rho} P_\psi \right] d\psi. \quad (5.6)
\]
Equation (5.6) shows that \( \text{Tr} [\hat{\rho} E_x] \) can be considered as a phase probability distribution, \( G(\varphi) \),

\[
G(\varphi) \equiv \text{Tr} [\hat{\rho} E_x] = \frac{\nu}{\pi} \int_D d^2\mu(z)c_\varphi(z)\langle \varphi | \hat{\rho} | \varphi \rangle, 
\]

\[
G(\varphi) = \text{Tr} [\hat{\rho} E_x] = \frac{\nu}{\pi} \int_0^1 dr^2 \int_0^{2\pi} d\theta \frac{c_\varphi(z)}{1 - r^2} |z \mid \hat{\rho} \mid z \rangle. 
\]

(5.7a)

where \( c_\varphi(z) \) has been defined in (5.4b). The derivative \( d\text{Tr} [\hat{\rho} E_x] / d\varphi = \text{Tr} [\hat{\rho} P_\varphi] \) (if it exists) defines the phase probability density distribution, \( g(\varphi) \). Owing to the second and third equations of (5.1), the phase probability density distribution \( g(\varphi) \) is an integral of the product of the Q-function \( |z \mid \hat{\rho} \mid z \rangle \) and the Poisson kernel \( (z = r \exp(i\theta)) \),

\[
g(\varphi) \equiv \text{Tr} [\hat{\rho} P_{\varphi + \varphi}] = \frac{\nu}{\pi} \int_D d^2\mu(z)c_{\varphi}(z)\langle \varphi | \hat{\rho} | \varphi \rangle, 
\]

\[
g(\varphi) = \text{Tr} [\hat{\rho} P_{\varphi + \varphi}] = \frac{1}{\pi} \int_0^{2\pi} d\theta |1 - r^2| \int_0^{2\pi} d\theta \frac{c_\varphi(\varphi)}{2\pi} (1 - \varphi - \varphi + r^2). 
\]

(5.8)

We propose to call \( |z \mid \hat{\rho} \mid z \rangle \) the R-function, by referring to the name of the regular phase states \( |z \rangle \), given by equation (4.1), which are one-mode coherent states associated to a Holstein–Primakoff realization of the SU(1,1) Lie algebra. By writing out the explicit form of the weight in \( d^2\mu(z) \) and the integration limits in polar coordinates, we have

\[
g(\varphi) = \text{Tr} [\hat{\rho} P_{\varphi + \varphi}] = \frac{\nu}{\pi} \int_0^1 dr^2 \int_0^{2\pi} d\theta \frac{P_{\varphi + \varphi}(\varphi)}{2\pi} (1 - \varphi + r^2) \langle \varphi | \hat{\rho} | \varphi \rangle, 
\]

(5.8a)

It can be shown by a direct calculation that the definition (equation (5.8a)) of \( g(\varphi) \) is consistent with the form of the matrix elements \( \langle r | P_{\varphi + \varphi} | s \rangle \) of \( P_{\varphi + \varphi} \), as given by equation (3.13b), in Fock representation. This means, at the same time, that, according to equation (3.14), in the limit \( \nu \rightarrow 0 \) (provided this exists) the phase density function \( g(\varphi) \) reduces to the well-known Loudon (1973) phase density,

\[
\lim_{\nu \rightarrow 0} g(\varphi) = \text{Tr} [\hat{\rho} (\varphi_0 + \varphi) (\varphi_0 + \varphi) \langle \varphi_0 + \varphi | \hat{\rho} | \varphi_0 + \varphi \rangle] \quad (0 < \varphi < \pi). 
\]

(5.8b)

As the simplest example for an application of equation (5.8a), let us calculate the phase probability density function of a number state \( |n \rangle \). Since \( \langle z | n \rangle^2 \) does not depend on \( \theta \), the angular integral of the Poisson kernel alone is unity for any values of the radial variable \( r < 1 \), so that

\[
g(\varphi) = \text{Tr} [\hat{\rho} (\varphi_0 + \varphi) (\varphi_0 + \varphi) \langle \varphi_0 + \varphi | \hat{\rho} | \varphi_0 + \varphi \rangle] \quad (0 < \varphi < \pi). 
\]

Thus we have

\[
g(\varphi) = \frac{\nu}{2\pi} \int_0^1 dr^2 \int_0^{2\pi} d\theta \frac{p(\varphi)}{(1 - r^2)^2} \times \left( 1 - r^2 \right)^{n + 1} \frac{(r^2)^n}{B(\nu, n + 1)} = \frac{1}{2\pi}. 
\]

(5.9)

which is a uniform distribution. In calculating the radial integral we have used the definition of the Beta function (equation (B.3c)) in appendix B. This result can be immediately obtained from the first equation of (5.8a), defining equation (5.2) of \( P_{\varphi_0 + \varphi} \). For a regular phase state \( g(\varphi) = |z \mid \hat{\rho} | z \rangle = P_{\varphi_0 + \varphi}(\varphi) \) (according to equations (5.8a) and (5.2)), thus, in fact, figure 2(b) shows the plot of this distribution. We just have to replace the axes labeled ‘phase parameter \( \Theta[2\pi] \)’ with ‘phase variable \( \varphi[2\pi] \),’ and we choose \( \theta = \pi/2 \) in \( z = \rho \exp(i\theta) \). These simple examples illustrate that our regular phase operator \( \Phi \) and \( N \) are value complementary, and that they satisfy the conditions introduced by Busch et al. (2001) on page 5928 of their paper. Since our primary goal in the present paper has been to build up the formalism itself, we leave the discussion of further (non-trivial) examples for future studies.

6. Summary

In the present paper we have given a new description of the quantum phase properties of a harmonic oscillator, which may also represent a quantized mode of the radiation field. The primary new element in our formalism is that, by introducing a new polar decomposition of the quantized amplitude of the oscillator, we have been able to build up a reliable correspondence between the classical Fourier series and the operator series associated to them. Here the word ‘reliable’ means that all the mathematical objects which have been introduced (e.g. phase operator or operator-valued measures) are defined in terms of strongly convergent expressions on a wide enough domain. The new ‘exponential phase operator’ \( F \) contains an (arbitrary) positive parameter \( \nu \), whose presence secures the strong convergence. We think that convergence questions are also of crucial practical importance in defining suitable physical quantities and reconstructing them from measurements. Concerning the mathematical technique itself, it is very convenient that we need not care about convergence questions in rearranging infinite (operator) series, for instance. Moreover, this formalism perfectly suits our classical intuition, though we have exclusively used the abstract space of square-summable sequences (Hilbert–Fock space). We did not rely on lesser known representations, such as the angular representation in the Hardy–Lebesgue space, used by Garrison and Wong (1970). Both approaches have advantages and disadvantages. For instance, we have not determined the eigenstates of the regular phase operator; thus, the positive operator-valued measures, introduced in the generalized spectral resolution of the new phase operator, are not orthogonal projectors.
The main results of the present paper can be summarized as follows. In section 3, by correspondence of the new exponential phase operator $F$ with the classical Fourier series of the periodic phase, we have constructed a regular phase operator $\Phi$ as a strongly convergent series. This phase operator is a covariant phase observable, according to the definition used by Lahti and Pellonpää (1999, 2000) and Busch et al (2001).

In section 4 we have determined the eigenstates of $F$, which turned out to be SU(1,1) coherent states with Bargmann index $\frac{1}{2}(\nu + 1)$, in the Holstein-Primakoff realization. This result is essentially equivalent to that of Aharonov et al (1973); however, they have not applied these states for building up diagonal representations of phase-related operators. Usually, these states are defined by an exponential expression of the SU(1,1) Lie algebra basis elements, acting on the vacuum state, and their eigenstate property is not used. In term of these special SU(1,1) coherent states we have constructed the diagonal representation of the regular phase operator $\Phi$. We note that this expansion is not possible by using the normalized phase states (Shapiro and Shepard 1991), because these latter ones do not form a complete set. The kernels (quantum phase function, phase density and phase projectors) of the diagonal representations are quantum analogons of the corresponding classical Fourier expressions, they perfectly correspond to our intuitive pictures.

In section 5 we have derived the generalized spectral resolution of the phase operator $\Phi$, and from the positive operator-valued measures associated to this resolution, we have derived a new phase probability distribution function and phase probability density function. We have shown that in the $\nu \rightarrow 0$ limit the latter one reduces to the well-known phase density function due to Loudon (1973), which also comes from the Pegg and Barnett (1989) density if we let the dimension of the truncated Hilbert space go to infinity. We have also introduced the positive definite R-function, as the diagonal matrix element of the density operator taken in the SU(1,1) coherent state (regular phase state) basis.

We think that the new mathematical objects we have constructed may be applied in representing and reconstructing measured phase-related physical quantities, like the carrier-envelope phase difference of ultrashort light pulses. The formalism worked out in the present paper may, for example, be useful in describing the quantum phase properties of extreme fields, like few-cycle or attosecond light pulses.

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**Appendix A. Derivation of uniform bounds for $F^k$ and $(F^*)^k$**

In the present appendix we derive the inequalities (equation (3.6)) for the functions $f_\nu(n)$ defined in equation (3.5),

$$f_\nu(n) \equiv \left( \frac{n + \nu}{n + k + \nu} \right)^n \left( 1 + \frac{\nu}{n} \right)^{\nu + k} \cdot \left( 1 + \frac{\nu}{n + k} \right) \cdot \left( 1 + \frac{\nu}{n + k + \nu} \right) \cdot \left( 1 + \frac{\nu}{n + k + \nu} \right) \cdot \exp \left[ \frac{1}{12} \left( \frac{\Theta_{\nu+k}}{n + k} + \frac{\Theta_{n+\nu}}{n + \nu} - \frac{\Theta_{n+k+\nu}}{n + k + \nu} - \frac{\Theta_{\nu}}{n} \right) \right]$$

(A.1)

Since $f_\nu(n) = 1$ for all $n \geq 0$, below we shall consider the nontrivial cases when $k \geq 1$. Our aim is to derive a formula for the upper bound of $f_\nu(n)$, in which we separate the $k$-dependence as a negative power of $k$. By using the Stirling formula (see e.g. Titchmarsh 1939, pp 150–1) for any positive $x$,

$$\Gamma(x + 1) = \left( \frac{x}{e} \right)^x e^{\gamma x} \sqrt{2\pi x} \left( 0 < \gamma < 1 \right).$$

(A.2)

we have

$$f_\nu(n) = \left( \frac{n + \nu}{n + k + \nu} \right)^n \left( 1 + \frac{\nu}{n} \right)^{\nu + k} \cdot \left( 1 + \frac{\nu}{n + k} \right) \cdot \exp \left[ \frac{1}{12} \left( \frac{\Theta_{\nu+k}}{n + k} + \frac{\Theta_{n+\nu}}{n + \nu} - \frac{\Theta_{n+k+\nu}}{n + k + \nu} - \frac{\Theta_{\nu}}{n} \right) \right]$$

(A.3)

where we assumed $n \geq 1$. By using simple algebraic inequalities and keeping in mind the monotonicity of the exponential function, it can be shown that

$$\exp \left[ \frac{1}{12} \left( \frac{\Theta_{\nu+k}}{n + k} + \frac{\Theta_{n+\nu}}{n + \nu} - \frac{\Theta_{n+k+\nu}}{n + k + \nu} - \frac{\Theta_{\nu}}{n} \right) \right] \leq e^{1/4}$$

(k $\geq 1$, $n \geq 1$)

(A.4)

Because $(1 + (\nu/n))^n$ is monotone increasing as $n \rightarrow \infty$ (and converges to $e^\nu$; $(1 + (\nu/n))^n \rightarrow e^\nu$), the second factor on the right hand side of equation (A.3) is smaller than 1 (it converges to 1, from below). The third factor is trivially smaller than the square root of $1 + \nu$, thus we may write these two estimates as

$$\frac{(1 + (\nu/n))^n}{(1 + [\nu/(n + k)])^{\nu + k}} < 1, \quad \frac{\sqrt{1 + \nu/n}}{\sqrt{1 + [\nu/(n + k)]}} < \sqrt{1 + \nu}, \quad (k \geq 1, \ n \geq 1).$$

(A.5)
Thus, according to equations (A.4)–(A.5), in the case \( n, k \geq 1 \) we receive the following estimate for \( f_k(n) \),

\[
f_k(n) < e^{i \theta \sqrt{1 + \nu}} \frac{(n + \nu)^\nu}{(n + k + \nu)^\nu} \\
< e^{i \theta \sqrt{1 + \nu}} \frac{(n + \nu)^\nu}{(k + \nu)^\nu} \quad (k \geq 1, n \geq 1). \tag{A.6}
\]

In the special case \( n = 0 \), again by using the Stirling formula in equation (A.1), we have

\[
f_k (0) = \frac{\Gamma(1 + k)}{\Gamma(1 + \nu + k)} \frac{\Gamma(1 + \nu)}{\Gamma(1)} \\
= \Gamma(1 + \nu) \frac{1}{(k + \nu)^\nu} \frac{1}{(1 + (\nu/k))^\nu} \\
\times \exp \left[ \frac{1}{12} \frac{\theta_k}{k} - \frac{\theta_{k + \nu}}{k + \nu} \right] \sqrt{\frac{k}{k + \nu}},
\]

\[
f_k (0) < \Gamma(1 + \nu) e^{4 + 1/6} \frac{1}{(k + \nu)^\nu} \quad (k \geq 1, n = 0).
\tag{A.7}
\]

We know that \( (1 + (\nu/k))^\nu \to e^\nu \) as \( k \to \infty \), thus it stays finite, and \( 1 < (1 + (\nu/k))^\nu \) is trivially valid for all \( k \geq 1 \). Taking this into account, and the second inequality in equation (A.6), we summarize our results, which are valid for any \( \nu > 0 \) parameter values,

\[
f_k (0) < \Gamma(1 + \nu) e^{4 + 1/6} \frac{1}{(k + \nu)^\nu} \quad (k \geq 1, n = 0),
\]

\[
f_k (n) < e^{i \theta \sqrt{1 + \nu}} \frac{(n + \nu)^\nu}{(k + \nu)^\nu} \quad (k \geq 1, n \geq 1), (\nu > 0).
\tag{A.8}
\]

If \( \nu \geq 1 \), then in the Stirling formula equation (A.2) we have \( \Theta_{1/\nu} \leq \Theta_k < 1 \), and in this case the two estimates in equation (A.8) can be written in a single formula,

\[
f_k(n) < b_k \frac{(n + \nu)^\nu}{(k + \nu)^\nu}, \quad b_k = e^{i \theta \sqrt{2 \pi \nu}}, \quad (k \geq 1, n \geq 0, \nu \geq 1). \tag{A.9}
\]

For a generic state \( \left| \psi \right> = \sum_{n=0}^\infty c_n \left| n \right> \), according to equations (3.4a), (3.4b), (3.5), (6.2) and (2.7), we have

\[
F^k \left| \psi \right> = \sum_{n=0}^\infty c_{n+k} \sqrt{\frac{1}{k+1}} \left| n \right>, \quad (n+k)!
\]

\[
(F^+)^k \left| \psi \right> = \sum_{n=0}^\infty c_{n+k} \sqrt{\frac{1}{k+1}} \left| n \right>. \tag{A.10}
\]

The squared norm of the transformed states are

\[
\left\| F^k \left| \psi \right> \right\|^2 = \sum_{n=0}^\infty f_k(n) |c_{n+k}|^2,
\]

\[
\left\| (F^+)^k \left| \psi \right> \right\|^2 = \sum_{n=0}^\infty f_k(n) |c_n|^2. \tag{A.11}
\]

By using the estimates summarized in equation (A.8), we can derive upper bounds for these norms,

\[
\left\| F^k \left| \psi \right> \right\|^2 = \sum_{n=0}^\infty f_k(n) |c_{n+k}|^2 < \frac{1}{(k + \nu)^\nu} \times \left[ \Gamma(1 + \nu) e^{4 + 1/6} \right] c_k^2 + e^{4 + i \theta \sqrt{1 + \nu}} \\
\times \sum_{n=0}^\infty (n + \nu)^\nu|c_{n+k}|^2.
\tag{A.12a}
\]

\[
\left\| (F^+)^k \left| \psi \right> \right\|^2 = \sum_{n=0}^\infty f_k(n) |c_n|^2 < \frac{1}{(k + \nu)^\nu} \times \left[ \Gamma(1 + \nu) e^{4 + 1/6} \right] c_0^2 + e^{4 + i \theta \sqrt{1 + \nu}} \\
\times \sum_{n=0}^\infty (n + \nu)^\nu|c_n|^2.
\tag{A.12b}
\]

We can write for the infinite sum on the right-hand side of equation (A.12a)

\[
\sum_{n=1}^\infty (n + \nu)^\nu |c_{n+k}|^2 \leq \sum_{n=1}^\infty (n + k + \nu)^\nu |c_{n+k}|^2 \\
= \sum_{r=k+1}^\infty (r + \nu)^\nu |c_r|^2 \leq \sum_{r=1}^\infty (r + \nu)^\nu |c_r|^2.
\tag{A.13}
\]

Moreover, both \( |c_k|^2 \) and \( |c_0|^2 \) are smaller than unity on the right-hand side of equations (A.12a) and (A.12b), respectively. Accordingly, by also taking equation (A.13) into account, we can write a uniform bound for the two norms

\[
\left\| F^k \left| \psi \right> \right\|^2 < \frac{\tilde{b}_{\nu, k}}{(k + \nu)^\nu} \left\| (F^+)^k \left| \psi \right> \right\|^2 < \frac{\tilde{b}_{\nu, k}}{(k + \nu)^\nu}
\tag{A.14a}
\]

\[
(N + \nu)^\nu \tilde{f}_{\nu, k} \equiv e^{4 + 1/6} \Gamma(1 + \nu) e^{4 + i \theta \sqrt{1 + \nu}} \left( (N + \nu)^\nu \right),
\tag{A.14b}
\]

\[
\left\{ (N + \nu)^\nu \right\}_{\nu} \leq \sum_{n=0}^\infty (n + \nu)^\nu |c_n|^2 < \infty.
\tag{A.15}
\]

It is important to note that the bound parameter \( \tilde{b}_{\nu, k} \) introduced in equation (A.14b) does not depend on power index \( k \). The bounds in equation (A.14a) exist and are non-trivial for those states for which the \( \nu \)th moments of \( (N + \nu) \), introduced in equation (A.14b) are finite. We denote the set of such states by \( \mathcal{D}_{\nu}(N) \),

\[
\mathcal{D}_{\nu}(N) = \left\{ \left| \psi \right> = \sum_{n=0}^\infty c_n \left| n \right> \in H; \sum_{n=0}^\infty (n + \nu)^\nu |c_n|^2 < \infty \right\}.
\tag{A.15}
\]
Appendix B. The regular phase states as SU(1,1) coherent states in Holstein–Primakoff realization

The solution of the eigenvalue equation \( F \mid z \rangle = z \mid z \rangle \) is straightforward if we expand \( \mid z \rangle \) in a series of Fock states; \( \mid z \rangle = \sum_{n=0}^{\infty} w_n \mid n \rangle \). By using the definition of \( F \) and \( E \) given by equations (3.1) and (2.3), respectively, we derive a recurrence relation to the unknown coefficients \( w_n \),

\[
F \mid z \rangle = \sum_{k=0}^{\infty} w_k F \mid k \rangle = \sum_{k=0}^{\infty} \mid k \rangle w_{k+1} \sqrt{\frac{k + 1}{k + 1 + \nu}} \\
= z \sum_{n=0}^{\infty} w_n \mid n \rangle,
\]

\[w_{n+1} \sqrt{\frac{n + 1}{n + 1 + \nu}} = zw_n.\]  

(B.1a)

The solution can be immediately written down

\[w_k = z^k \sqrt{\frac{(\nu + 1)(\nu + 2)\cdots(\nu + n)}{1 \cdot 2 \cdots n}} \]

\[w_0 = \sqrt{\frac{\nu + 1}{\nu}} z^n w_0,\]  

(B.1b)

where we have introduced the Pochhammer symbol: 

\[(a)_k \equiv \Gamma(a + n)/\Gamma(a) = a(a + 1)\cdots(a + n - 1).\]

We prescribe unit norm \( || \mid z \rangle || = \sqrt{\langle z \mid z \rangle} = 1 \) for the eigenstate \( \mid z \rangle \), which determines the modulus of the unknown \( w_0 \). By using the formula for the negative binomial series (Gradshteyn and Ryzhik 1980)

\[(1 - x)^{-\nu-1} = 1 + (\nu + 1)x + \frac{(\nu + 1)(\nu + 2)}{2!} x^2 + \cdots + \frac{(\nu + 1)(\nu + 2)\cdots(\nu + k)}{k!} x^k + \cdots,\]

we have

\[
\langle z \mid z \rangle = |w_0|^2 \sum_{n=0}^{\infty} \frac{(\nu + 1)_n}{n!} (|z|^2)^n = |w_0|^2 (1 - |z|^2)^{-\nu-1}.
\]

(B.1d)

Thus, the normalized eigenstates of \( F \) can be brought to the equivalent forms

\[
F \mid z \rangle = z \mid z \rangle; \mid z \rangle = \left(1 - |z|^2\right)^{\nu/2} |z|^\nu \mid n \rangle, \quad z \in D = \{z, |z| < 1\},
\]

(B.2a)

In equation (B.2b) we have written the expansion coefficients \( c_n \) in the usual form of that of the SU(1,1) coherent states in the discrete series representation (Perelomov 1986). Below we shall use the following parametrization

\[\mid z \rangle = \sum_{n=0}^{\infty} w_n \mid n \rangle,\]

\[w_n = \left(1 - \rho^2\right)^{\nu/2} c_n \rho^n e^{i\theta},\]

\[z = \rho e^{i\theta}, \quad 0 \leq \rho < 1, \quad 0 \leq \theta < 2\pi.\]  

(B.2c)

It is interesting to note that the function \( f_{\nu}(n) \), introduced in equation (3.5) (or see equation (A.1)) can be expressed in terms of the factors \( c_n \) in equation (B.2c) as

\[f_{\nu}(n) = c_n/c_{n+k}.\]

The states \( \mid z \rangle \) with different \( z \)-s are not orthogonal, but they form an (over)complete set in the original Hilbert space \( H \). This completeness property has been shown by several authors; here we refer the reader to the recent thorough analysis by Vourdas and Wünsche (1998), and references therein. By introducing the suitable weight function, and using the polar coordinates \( \theta \) and \( \rho \) in the open unit disc \( D \), we shall prove

\[
\frac{\nu}{\pi} \int_D d^2 \mu \mid z \rangle \langle z \mid = \nu \sum_{n=0}^{\infty} |n \rangle \langle n | \rho^2 \int_0^\infty d\theta (1 - \rho^2)^{-\nu-1} e^{i\theta} \sum_{n=0}^{\infty} c_n^n \rho^n |n \rangle \langle n |.
\]

(B.3a)

Having performed the integration with respect to \( \theta \), and introducing the integration variable \( t \equiv \rho^2 \), we obtain the diagonal sum of the diads \( |n \rangle \langle n | \)

\[
\frac{\nu}{\pi} \int_D d^2 \mu \mid z \rangle \langle z \mid = \nu \int_0^1 dt (1 - t)^{\nu-1} \sum_{n=0}^{\infty} c_n^n t^n |n \rangle \langle n |.
\]

(B.3b)

The numerical coefficients \( c_n^2 \) (see definition in equation (B.2c)) can be alternatively expressed in terms of the Beta function \( B(x, y) \) (Euler’s integral of the first kind)

\[c_n^2 = \frac{\Gamma(\nu + n + 1)}{\Gamma(\nu)\Gamma(n + 1)} = \frac{1}{\nu B(x, n + 1)},\]

\[B(x, y) \equiv \int_0^1 t^{x-1}(1 - t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \quad (x > 0, y > 0).\]  

(B.3c)

It can be immediately seen that the radial integral in equation (B.3b) yields just \( B(\nu, n + 1) \), whose product with \( \nu c_n^2 \) equals to unity; thus the completeness relation in equation

\[F \mid z \rangle = z \mid z \rangle; \mid z \rangle = \left(1 - |z|^2\right)^{\nu/2} |z|^\nu \mid n \rangle, \quad z \in D = \{z, |z| < 1\}.
\]

(B.2a)
k is a real parameter. This is the Bargmann index. The coherent states \( |z\rangle \) consist of complex numbers in \( \mathbb{C} \).

The eigenstates derived already by Aharonov et al. (1973), as eigenstates of the generalized destruction operator \( A(k) \equiv \hat{F}(k) \), where \( k \) is a real parameter. This \( \hat{F}(k) \) can be expressed in terms of our \( F \) (defined in equation (3.1)), and an eigenvalue equation analogous to that in equation (4.1) is also satisfied,

\[
F = \hat{F}(k) \left\{ \frac{N + k}{N + \nu} \right\}^{\frac{1}{2}} \frac{1}{\sqrt{k + 1}},
\]

\[
F = \hat{F}(k) \frac{1}{\sqrt{k + 1}} (\nu = k),
\]

\[
\hat{F}(k)|z\rangle = \sqrt{k + 1} F |z\rangle = \alpha |z\rangle. \tag{B.3d}
\]

The eigenstates \( |\alpha, k\rangle \) given by equation (3.28) in the paper by Aharonov et al. (1973) can be expressed in term of the states \( |z\rangle \) as

\[
|z\rangle = \frac{\alpha}{\sqrt{k + 1}} \langle \alpha, k | \equiv \left[ 1 - \frac{|\alpha|^2}{k + 1} \right]^{(k+1)} \times \sum_{n=0}^{\infty} \left[ \frac{\Gamma(\nu + 1 + n)}{\Gamma(\nu + 1)(n!)} \right]^{1/2} \left( \frac{\alpha}{\sqrt{k + 1}} \right)^n |n\rangle, \tag{B.3e}
\]

\( \alpha \in \mathcal{D}_\alpha = \{ \alpha, |\alpha| < \sqrt{k + 1} \}. \]

Since the spectrum of \( F \) consists of complex numbers in the unit disc \( D = \{ |z|, |z| < 1 \} \), the spectrum of \( \hat{F}(k) \) covers \( |\alpha| < \sqrt{k + 1} \). We note that Aharonov et al. (1973) have also proved that, by keeping \( k \) fixed, in the limit \( k \to \infty \) the states \( |\alpha, k\rangle \) converge in norm to the well-known coherent states \( |\alpha\rangle \) (Glauber 1963, Sudarshan 1963),

\[
\lim_{k \to \infty} \||\alpha, k\rangle - |\alpha\rangle\| = 0, \quad |\alpha\rangle = e^{-1/2} |\alpha|^\frac{n}{2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{(n!)}^2} |n\rangle. \tag{B.3f}
\]

This limit behavior has also been recovered by Gerry and Grobe (1997) in a more general frame, in which they discussed intelligent states associated with the Holstein–Primakoff realization for different values of \( \kappa \).

We have seen in section 4 that the eigenstates in equation (4.1) are in fact \( SU(1,1) \) coherent states in a Holstein–Primakoff-type realization, associated to the Lie algebra of basis operators \( K_+ \) and \( K_0 \). The definitions and the basic commutators are

\[
K_- = A \sqrt{N + \nu} = F(N + \nu),
\]

\[
K_+ = (K_-)^* = \sqrt{N + \nu} A^* = (N + \nu) F^* + K_+ K_- = (N + \nu) N, \tag{B.4a}
\]

\[
K_0 = N + \kappa = N + \frac{1}{2}(\nu + 1),
\]

\[
\nu = 2\kappa - 1, \quad [K_-, K_+] = 2(K_0 + \kappa). \tag{B.4b}
\]

By introducing the Hermitian combinations \( K_{1,2} \), we have

\[
K_1 \equiv \frac{1}{2} (K_+ + K_-),
\]

\[
K_2 \equiv \frac{1}{2i} (K_+ - K_-),
\]

\[
K_1^2 + K_2^2 = \frac{1}{2} (K_+ K_+ + K_- K_-) = (N + \kappa)^2 + \kappa - \kappa^2. \tag{B.4c}
\]

The Casimir operator \( C \) takes on the value

\[
C \equiv K_1^2 - K_2^2 = \kappa(\kappa - 1), \quad \kappa = \frac{1}{2}(\nu + 1) > \frac{1}{2}. \tag{B.4d}
\]

where \( \kappa \) is the Bargmann index. The coherent states introduced by Barut and Girardello (1971) are eigenstates of \( K_- = F(N + \nu) \), so they differ from \( |z\rangle \), of course.

The proof of equations (4.7), (4.7a) and (4.8) in section 4 can be carried out as follows. In order to express the regular phase operator (equation (3.7)) in a diagonal form, in terms of the regular phase states (equations (4.1), (B.2b)), we use equation (3.11), the eigenvalue equations \( F|z\rangle = z|z\rangle \) and \( \langle z| F^* = z^* \langle z| \), and the completeness relation (equations (4.4), (B.3a)),

\[
\phi = \phi_0 + \frac{\nu}{\pi} \int_D d^2 \mu (z)(-i) \log \left( 1 - Fe^{-i\phi} \right) \times |z\rangle \langle z| - |z\rangle \langle z| \log \left( 1 - F^* e^{i\phi} \right) \]

\[
= \phi_0 + \frac{\nu}{\pi} \int_D d^2 \mu (z)(-i) \log \left( \frac{1 - ze^{-i\phi}}{1 - z^* e^{i\phi}} \right) |z\rangle. \tag{B.5a}
\]

The argument of the logarithm can be brought to the form

\[
1 - ze^{-i\phi} = \frac{1 - \rho \cos(\theta - \phi_0) - i\rho \sin(\theta - \phi_0)}{1 - \rho \cos(\theta - \phi_0) + i\rho \sin(\theta - \phi_0)} = \frac{1 - ix}{1 + ix}, \quad \text{where } x \equiv \frac{\rho \sin(\theta - \phi_0)}{1 - \rho \cos(\theta - \phi_0)}. \tag{B.5b}
\]

By using the formula 1.622.3 of Gradshteyn and Ryzhik (1980)

\[
\frac{1}{2i} \log \left( \frac{1 + ix}{1 - ix} \right) = \tan x,
\]

i.e.

\[
\frac{1}{i} \log \left( \frac{1 - ix}{1 + ix} \right) = -2 \tan x. \tag{B.5c}
\]

\[
\frac{1}{2i} \log \left( \frac{1 + ix}{1 - ix} \right) = \tan x,
\]

i.e.

\[
\frac{1}{i} \log \left( \frac{1 - ix}{1 + ix} \right) = -2 \tan x. \tag{B.5c}
\]
we have
\[ -i \log \left( \frac{1 - ze^{i\phi_0}}{1 - z^{*}e^{i\phi_0}} \right) = -2 \arctan \left( \frac{\rho \sin(\theta - \phi_0)}{1 - \rho \cos(\theta - \phi_0)} \right). \]
(B.5d)

By taking the second equation of (B.5a) and (B.5d) into account, we have

\[ \phi_0 + \pi - i \log \left( \frac{1 - ze^{i\phi_0}}{1 - z^{*}e^{i\phi_0}} \right) = -i \log \left( \frac{e^{i(\phi_0 + \pi)} - ze^{i\phi_0}}{1 - z^{*}e^{i\phi_0}} \right) \]
\[ = -i \log \left( e^{i\phi(z)} \right) = \phi(z). \]
(B.6a)

\[ e^{i\phi(z)} = e^{i(\phi_0 + \pi)} \frac{1 - ze^{i\phi_0}}{1 - z^{*}e^{i\phi_0}}. \]
(B.6b)

\[ \varphi(z) \equiv \varphi(z, z^*) = \phi_0 + \pi - 2 \arctan \left( \frac{\rho \sin(\theta - \phi_0)}{1 - \rho \cos(\theta - \phi_0)} \right) \]
\[ = (\phi_0 < \varphi(z) < \phi_0 + 2\pi). \]
(B.6c)

The lower and upper bounds of \( \varphi(z) \) can be derived by taking into account the inequality \(-\pi < (-2 \arctan x) < \pi\), valid for any real \( x \) in the interval \((-\infty < x < \infty)\). From equation (3.11) one can also show that the diagonal matrix elements of \( \Phi \) between the regular phase states just coincide with the quantum phase function \( \varphi(z) \) defined in equation (B.6c),

\[ \langle z | \Phi | z \rangle = \varphi(z) = \varphi_0 + \pi - 2 \sum_{k=1}^{\infty} k \sin \left( k \left( \theta - \phi_0 \right) \right). \]
(B.7)

The two forms of \( \varphi(z) \) in equations (B.6c) and (B.7) can be equivalent, which can be shown by using the Fourier series of the arctan function (see e.g. Gradshteyn and Ryzhik 1980, formula 1.448). Since the Fourier series \( \sum_{k=1}^{\infty} (\sin kp)/k \) converges to \( \pi - \varphi \) in the interval \((0 < \varphi < 2\pi)\), in the limit \( \rho \to 1-0 \) the quantum phase function \( \varphi(z) \) approaches the classical phase function \( \Phi_{cl}(e^{i\phi}) \), given by equation (2.11) (where we have taken the reference phase \( \phi_0 = 0 \)).

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