Abstract—The Conti-Boston factorization theorem (CBFT) for linear tail-biting trellis realizations is extended to group realizations with a new and simpler proof, based on a controller granule decomposition of the behavior and known controllability results for group realizations. Further controllability results are given; e.g., a trellis realization is controllable if and only if its top (controllability) granule is trivial.

I. INTRODUCTION

Tail-biting trellis realizations are the simplest class of realizations of codes on cyclic graphs. Decoding is generally simpler than for conventional trellis realizations [11]. Koetter and Vardy [8], [9] developed the foundations of the theory of linear tail-biting trellis realizations. Their key result was a factorization theorem (KVFT), which shows that every reduced realization has a factorization into elementary trellises. Recently, Conti and Boston [2] have proved a stronger unique factorization theorem (CBFT): the behavior (“label code”) of a reduced linear tail-biting trellis realization factors uniquely into quotient spaces of “span subcodes.” This work was the main stimulus for the work reported here.

Our main result is a generalization of the CBFT to group realizations, with a new proof that we feel is even simpler and more insightful. [2, Remark III.3] notes that such a generalization is not straightforward.

In Section II we introduce a granule decomposition along the lines of the controller granule decomposition of minimal conventional trellis realizations of Forney and Trott [5], [6], and the span subcode decomposition of [2].

In Section III on the controllability of group realizations, we show that this granule decomposition yields a unique factorization of a group trellis behavior $B$. We develop other controllability properties not considered in [2]: e.g., the trellis diagram of an uncontrollable group trellis realization is disconnected [4]. We show that the controller canonical realization based on this factorization is one-to-one, minimal and group-theoretic, but possibly nonhomomorphic.

Our development uses only elementary group theory, principally the fundamental theorem of homomorphisms (FTH) and the correspondence theorem (CT). For a brief introduction to the necessary group theory and our notation, see [3].

A. Preliminaries

A (tail-biting) trellis realization $R$ of length $n$ is defined by a set of $n$ symbol alphabets $\{A_j, j \in \mathbb{Z}_n\}$, a set of $n$ state alphabets $\{S_j, j \in \mathbb{Z}_n\}$, and a set of $n$ constraint codes $\{C_j \subseteq S_j \times A_j \times S_{j+1}, j \in \mathbb{Z}_n\}$, where index arithmetic is in $\mathbb{Z}_n$; e.g., $S_{n-1} = S_{n-1} \times A_{n-1} \times S_0$.

The configuration universe $U = \prod_{j \in \mathbb{Z}_n} C_j$ is thus a subset of $S \times A \times S$, where $A = \prod_{j \in \mathbb{Z}_n} A_j$ and $S = \prod_{j \in \mathbb{Z}_n} S_j$.

In a linear trellis realization, each symbol or state alphabet is a finite-dimensional vector space over some field $F$, and each $C_j$ is a subspace of $S_j \times A_j \times S_{j+1}$, so $U$ is a subspace of $S \times A \times S$. (In [2] and [4], it is assumed that $A_j = F$ always.) In a group trellis realization, each symbol or state alphabet is a finite abelian group, and each $C_j$ is a subgroup of $S_j \times A_j \times S_{j+1}$, so $U$ is a subgroup of $S \times A \times S$.

The extended behavior $B$ of $R$ is the set of configurations $(s, a, s') \in U$ such that $s = s'$; i.e., such that the constraints of $U$ and the equality constraints $s = s'$ are both satisfied [3].

Its behavior $\mathcal{B}$ is the projection of $B$ onto $A \times S$, which is an isomorphism. The code $C$ realized by $R$ is the projection of $\mathcal{B}$ or $B$ onto $A$. The (normal) graph of $R$ [3] is the single-cycle graph with $n$ vertices corresponding to the constraint codes $C_j$, $n$ edges corresponding to the state variables $S_j$, where edge $S_j$ is incident on vertices $C_{j-1}$ and $C_j$, and $n$ half-edges corresponding to the symbol variables $A_j$, where half-edge $A_j$ is incident only on vertex $C_j$.

II. GRANULE DECOMPOSITION

A. Partial ordering of fragments

A proper fragment of a trellis realization $R$ corresponds to a circular interval $[j, k]$, $j \in \mathbb{Z}_n$, $k \in \mathbb{Z}_n$, and will be denoted by $F^{[j, k]}$. $F^{[j, k]}$ includes the constraint codes $\{C_j', j' \in [j, k]\}$ and the internal state variables $\{S_j', j' \in (j, k)\}$, and has boundary $\{S_j, S_k\}$. Accordingly, we define its vertex set as $V(F^{[j, k]}) = [j, k]$, and its edge set as $E(F^{[j, k]}) = (j, k)$. The (normal) graph of every proper fragment is cycle-free.

We define the level of $F^{[j, k]}$ as the number $\ell = |E(F^{[j, k]})|$ of its internal state variables; i.e., $\ell = k - j - 1 \mod n$. 

G. David Forney, Jr.
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, MA 02139 USA
Email: forney@mit.edu

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Thus $|V(F^{(j,k)})| = \ell + 1$. We may denote a level-$\ell$ fragment $F^{(j,\ell+1)}$ by $F^{(j,\ell+1)}$. A level-$(n-1)$ fragment $F^{(j,i)}$ is obtained from $R$ by cutting the edge $S_j$ into two half-edges; it contains all $n$ constraint codes and $n-1$ internal state variables. A level-0 fragment $F^{(j,1)} = F^{(j)}$ contains one constraint code $C_j$ and no internal state variables.

We also regard the entire realization $R$ as a fragment, whose level is $n$. $R$ contains $\ell = |E(R)| = n$ internal state variables, and $\ell = |V(R)| = n$ (not $\ell + 1$) constraint codes.

As observed in [2], the set $\mathcal{G}(R)$ of fragments of a tail-biting trellis realization $R$ is partially ordered by set inclusion. The maximum fragment $R$ includes all proper fragments $F^{(j,k)}$. The partial ordering of proper fragments corresponds to the partial ordering of the circular intervals $[j,k]$ by set inclusion; i.e., $F^{(j',k')} \subseteq F^{(j,k)}$ iff $|j',k'| \subseteq [j,k]$. The minimal fragments are the level-0 fragments $F^{(j,1)}$.

The partial ordering of $\mathcal{G}(R)$ may be illustrated by a Hasse diagram, as follows. A fragment $F' \in \mathcal{G}(R)$ is said to be covered by another fragment $F \in \mathcal{G}(R)$ if $F' \subset F$ and there is no fragment $F'' \in \mathcal{G}(R)$ such that $F' \subset F'' \subset F$. In our setting, $F'$ is covered by $F$ if $F' \subset F$ and the level of $F'$ is one less than the level of $F$. The set $\mathcal{G}(R)$ is thus said to be graded by level (number of internal state variables).

The Hasse diagram of $\mathcal{G}(R)$ is illustrated in Figure 1 for a tail-biting trellis realization $R$ of length $n = 4$.

As numerous authors have observed (e.g., [9, 2]), a conventional trellis realization may be viewed as a special case of a tail-biting trellis realization in which $S_0$ is trivial. Correspondingly, the Hasse diagram of a conventional trellis realization is a subdiagram of the Hasse diagram for a tail-biting trellis realization $R$ of the same length, comprising the fragments $\{F \in \mathcal{G}(R) \mid F \not\subset F^{(0,0)}\}$. By cyclic rotation of the index set $\mathbb{Z}_n$, any level-$(n-1)$ fragment $F^{(j,i)}$ may be regarded as a conventional trellis realization.

**B. Subbehaviors**

For every proper fragment $F = F^{(j,k)} \in \mathcal{G}(R)$, we define the subbehavior $\mathcal{B}_F = \mathcal{B}^{(j,k)}$ as the set of $(a,s) \in \mathcal{B}$ that are all-zero on or outside the boundary of $F$. For example, $F^{(0,0)}$ is the behavior of a conventional trellis realization of length $n$. We also define $\mathcal{B}_R = \mathcal{B}$.

Evidently if $F' \leq F$, then $\mathcal{B}_{F'} \subseteq \mathcal{B}_F$. Thus the set $\{\mathcal{B}_F, F \in \mathcal{G}(R)\}$ has the same partial ordering as $\mathcal{G}(R)$.

For a level-0 fragment $F^{(j,0)}$, we have

$$\mathcal{B}^{(j,0)} = \{ (a,0) \mid a_j \in \{C_j\} \land a_{j'} = 0 \text{ if } j' \neq j \},$$

where $\{C_j\}_{A_j} = \{ a_j \in A_j \mid \{0,a_j\} \in C_j \}$ is the cross-section of $C_j$ on $A_j$. As in [3], $\{C_j\}_{A_j}$ will be denoted by $A_j^c$, and called the nondynamical symbol alphabet of $C_j$.

**C. Granules**

For non-level-0 fragments, we define $\mathcal{B}_{\leq F}$ as the behavior generated by all $\mathcal{B}_{F'}$ such that $F' < F$, as in [2]. In other words, $\mathcal{B}_{\leq F} = \sum_{F' < F} \mathcal{B}_{F'}$. Evidently $\mathcal{B}_{\leq F} \subseteq \mathcal{B}_F$.

We define the controller granule $\Gamma_F$ as the quotient $\mathcal{B}_F/\mathcal{B}_{\leq F}$. In the linear case, $\mathcal{B}_F$ and $\mathcal{B}_{\leq F}$ are vector spaces, and their quotient $\Gamma_F$ is a vector space of dimension $\dim \Gamma_F = \dim \mathcal{B}_F - \dim \mathcal{B}_{\leq F}$. In the group case, $|\Gamma_F| = |\mathcal{B}_F|/|\mathcal{B}_{\leq F}|$.

For a level-0 fragment $F^{(j,1)}$, we define the nondynamical granule $\Gamma_F$ as $\mathcal{B}^{(j,1)} \cong A_j^c$. The set $\{\Gamma_F, F \in \mathcal{G}(R)\}$ thus consists of nondynamical granules at level $\ell = 0$, and controller granules at levels $\ell > 0$.

At level $n$, where $F = R$, we will call $\Gamma_R = \mathcal{B}/\mathcal{B}_{\leq R}$ the top granule of $\mathcal{R}$, or the controllability granule of $\mathcal{R}$, since as we will see, $\Gamma_R$ governs the controllability properties of $\mathcal{R}$.

Note that $\mathcal{B}_{\leq R} = \sum_{j} \mathcal{B}^{(j,1)}$, the behavior generated by all level-$(n-1)$ subbehaviors $\mathcal{B}^{(j,1)}$. We will call $\mathcal{B}_{\leq R}$ the controllable subbehavior $\mathcal{B}^c$ of $\mathcal{B}$.

At levels $1 \leq \ell \leq n-1$, a proper fragment $F^{(j,k)}$ covers precisely two fragments, namely $F^{(j,k-1)}$ and $F^{(j+1,k)}$. Thus $\mathcal{B}_{\leq F^{(j,k)}} = \mathcal{B}^{(j,k-1)} + \mathcal{B}^{(j+1,k)}$, and the corresponding controller granule is

$$\Gamma^{(j,k)} = \frac{\mathcal{B}^{(j,k)}}{\mathcal{B}^{(j,k-1)} + \mathcal{B}^{(j+1,k)}}.$$  

Forney and Trott [5, 6] define a controller granule for a conventional group trellis realization similarly as $\Gamma^{(j,k)} = C^{(j,k)}/(C^{(j,k-1)} + C^{(j+1,k)})$, where the subcode $C^{(j,k)} \subseteq C$ is the set of $a \in C$ that are all-zero outside the boundary of $F^{(j,k)}$. The two definitions turn out to be equivalent for minimal conventional trellis realizations.

**D. $\ell$-controllable behaviors**

For $0 \leq \ell \leq n-1$, we define the $\ell$-controllable behavior $\mathcal{B}_\ell$ as the behavior generated by all level-$\ell$ subbehaviors $\mathcal{B}^{(j,\ell)}$. In other words, $\mathcal{B}_\ell = \sum_{j} \mathcal{B}^{(j,\ell)}$. Note that $\mathcal{B}_{\ell-1} = \mathcal{B}^c$, the controllable subbehavior of $\mathcal{B}$. We also define $\mathcal{B}_n = \mathcal{B}$.

Evidently $\mathcal{B}_{\ell-1} \subseteq \mathcal{B}_\ell$ for $1 \leq \ell \leq n$. Moreover, $\mathcal{B}_0 = \sum_j \mathcal{B}^{(j,0)} = A^c \times \{0\}$, where $A^c = \{a \in A \mid (a,0) \in \mathcal{B}\} = \prod_j A_j^c$. We call $\mathcal{B}_0$ the nondynamical behavior of $\mathcal{B}$.

We thus have a chain of subgroups $\mathcal{B}_0 = A^c \times \{0\} \subseteq \mathcal{B}_1 \subseteq \ldots \subseteq \mathcal{B}_n = \mathcal{B}$, which is a normal series since all groups are abelian. We denote the factor groups of this chain by $Q_\ell = \mathcal{B}_\ell/\mathcal{B}_{\ell-1,1}$, $0 \leq \ell \leq n$, plus $Q_0 = \mathcal{B}_0$.

By elementary group theory, we have $|\mathcal{B}| = \prod_{\ell} |Q_\ell|$; or, in the linear case, $\dim \mathcal{B} = \sum_{\ell} \dim Q_\ell$. If we define sets $|Q_\ell|$ of coset representatives for the cosets of $\mathcal{B}_{\ell-1}$ in $\mathcal{B}_\ell$, then every $(a,s) \in \mathcal{B}$ may be uniquely expressed as a sum of coset representatives; or, in the linear case, if we define a
basis $B_\ell$ for each quotient $Q_\ell$, then every \((a, s) \in B\) may be uniquely expressed as a linear combination of basis elements.

Since $Q_\ell$ is generated by the elements of $B_\ell$ that are not in $B_{\ell-1}$, and every element of $B_\ell$ is an element of some level-$\ell$ subbehavior $B^{[j,j+\ell]}$, the nonzero coset representatives in $Q_\ell$ may all be taken as elements of some $B^{[j,j+\ell]} \setminus B_{\ell-1}$. We note that if $(a, s) \in B^{[j,j+\ell]} \setminus B_{\ell-1}$, then the support of $s$ must be precisely the length-$\ell$ circular interval $[j+1, j+\ell]$, else $(a, s) \in B_{\ell-1}$.

The level-$\ell$ subbehaviors $B^{[j,j+\ell]}$ thus comprise a sufficient set of representatives for $Q_\ell$. We say that unique factorization holds if every element of every level-$\ell$ behavior $B_\ell$ is a unique sum of elements of level-$\ell$ subbehaviors $B^{[j,j+\ell]}$, modulo $B_{\ell-1}$; i.e., if $B_\ell$ modulo $B_{\ell-1}$ is the (internal) direct sum

$$B_\ell = \bigoplus_{j \in Z_n} B^{[j,j+\ell]} \mod B_{\ell-1}.$$  

III. CONTROLLABILITY AND UNIQUE FACTORIZATION

In previous work \cite{4, 3}, we have defined controllability as the property of “having independent constraints,” since we have proved that a realization is observable if and only if its dual realization has this property.

We now show that for a linear or group tail-biting trellis realization $R$, controllability in this sense is equivalent to the property that the top granule $\Gamma_R$ is trivial. Simultaneously, we obtain an easy proof that unique factorization holds for $R$, under the proviso (as in \cite{3, 2}) that $R$ is reduced; that is, $R$ is state-trim—i.e., $B_{|S_j} = S_j$ for all $j$—and $R$ is branch-trim—i.e., $B_{|S_j} \times A_j \times S_{j+1} = C_j$ for all $j$.

(Notation: in this section, we will use notation appropriate to the group case—i.e., we use sizes rather than dimensions; the reader may translate to the linear case if desired.)

A. Controllability

In \cite{4, 3}, a realization $R$ is called controllable if the he constraints of $U$ and the equality constraints $s = s'$ are independent. More concretely, $R$ is controllable if the image $S'$ of the syndrom-former homomorphism $U \to S$ defined by $(s, a, s') \mapsto s - s'$ is equal to $S$. Since the kernel of this homomorphism is the extended behavior $B$, we have $U/B \cong S' \subseteq \subseteq B$, by the FTH. This yields the following controllability test: $|U|/|B| \leq |S'|$, with equality if and only if $R$ is controllable \cite{3}. In other words, since $B \cong B$, a realization is uncontrollable if and only if its constraints are dependent in the following sense:\footnote{This result may be understood as follows. Ignoring state equality constraints, there are $|U| = \prod_j |C_j|$ possible configurations. If the state equality constraints $\{s_j = s'_j : j \in Z_n\}$ are all independent of the set of code constraints $\{C_j : j \in Z_n\}$, then each state equality constraint $s_j = s'_j$ reduces the number of possible configurations by a factor of $|S_j|$, so $|B| = |U|/|S'|$.

where $|S| = \prod_j |S_j|$. If the constraints are dependent—i.e., if $R$ is not controllable—then the reduction is strictly less, and $|B| > |U|/|S'|$.}

$$|B| > |U|/|S'| = \prod_j |C_j| \prod_j |S_j|.$$  

B. Disconnected trellis realizations

We now show that if the top granule $\Gamma_R = B/B^c$ is nontrivial, then $B$ consists of $|\Gamma_R|$ disconnected subbehaviors, namely the cosets of the controllable subbehavior $B^c = \bigoplus_j B_{|S_j}$ in $B$. Similar results were proved in \cite{4} and \cite{7} Appendix A); the proof here is simpler, and does not rely on duality.

Lemma. For a linear or group trellis realization $R$ with behavior $B$ and controllable subbehavior $B^c$, for any $j \in Z_n$:

(a) $B_{|S_j}/(B^c)_{|S_j} \cong \Gamma_R$;

(b) $B_{|S_j} \times A_j \times S_{j+1}/(B^c)_{|S_j} \times A_j \times S_{j+1} \cong \Gamma_R$.

Proof. (a) The projections of $B$ and $B^c$ onto $S_j$ have a common kernel $B_{|[j,j]} = \{(a, s) \in B \mid s_j = 0\}$. Thus $B_{|S_j}/(B^c)_{|S_j} \cong B/B^c = \Gamma_R$, by the CT.

(b) The projections of $B$ and $B^c$ onto $S_j \times A_j \times S_{j+1}$ have a common kernel $B_{|[j,j+1]} = \{(a, s) \in B \mid (s_j, a_j, s_{j+1}) = (0, 0, 0)\}$, so (b) follows also from the CT.

If $R$ is reduced, as we assume, then $B_{|S_j} = S_j$ and $B_{|S_j} \times A_j \times S_{j+1} = C_j$. Moreover, we may regard $B^c$ as the behavior of the controllable subrealization $R^c$ of $R$, defined as the reduced tail-biting trellis realization with state spaces $(S_j)^c = (B^c)_{|S_j}$, symbol spaces $A_j$, and constraint codes $(C_j)^c = (B^c)_{|S_j} \times A_j \times S_{j+1}$. This lemma then states that $S_j/(S_j)^c \cong \Gamma_R$ and $(C_j)/(C_j)^c \cong \Gamma_R$.

More concretely, (a) implies that, if $\Gamma_R$ is nontrivial, then for each $j$, each coset $B^c + (a, s)$ of $B^c$ in $B$ passes through a distinct corresponding coset $(S_j)^c + (s_j)$ of $(S_j)^c$ in $S_j$. Similarly, $C_j$ partitions into $|\Gamma_R|$ disjoint cosets of $(C_j)^c$, each representing state transitions within one coset of $B^c$ in $B$. The trellis diagram of $R$ thus consists of $|\Gamma_R|$ disjoint subdiagrams, one representing each coset of $B^c$ in $B$. Thus for any $j, j'$, there is no trajectory $(a, s)$ connecting any state $s_j$ in a given coset of $(S_j)^c$ in $S_j$ to a state $s_{j'}$ in a coset of $(S_{j'})^c$ in $S_{j'}$, unless the two cosets correspond to the same coset of $B^c$ in $B$.

C. First-state chain

We now show that the controller granules of $R$ are isomorphic to factor groups of certain normal series.

Lemma (first-state chain). For $j \in Z_n$, $1 \leq \ell \leq n - 1$,

$$|\Gamma_{[j,j+\ell]}| \cong \begin{cases} (B_{|S_j} \times A_j \times S_{j+1})/(B^c)_{|S_j} \times A_j \times S_{j+1} & \text{if } (B^c)_{|S_j} \times A_j \times S_{j+1} \neq 0, \\ 1 & \text{otherwise}. \end{cases}$$

Proof. We have $\Gamma_{[j,j+\ell]} = B_{|S_j} \times A_j \times S_{j+1}$ and $B_{|[j,j+\ell]} + \{a, s\} \subseteq B_{|[j,j+\ell]}$ onto $S_j \times A_j \times S_{j+1}$ are $B_{|S_j} \times A_j \times S_{j+1}$ and $B_{|[j,j+\ell]} \times A_j \times S_{j+1}$, respectively, and their common kernel is $B_{|S_j} \times A_j \times S_{j+1}$. Similarly, the projections of $B_{|[j,j+\ell]} \times A_j \times S_{j+1}$ and $B_{|[j,j+\ell]} \times A_j \times S_{j+1}$ onto $S_{j+1}$ are $B_{|S_{j+1}}$ and $B_{|S_{j+1}}$, respectively, and their common kernel is $B_{|S_{j+1}} \times A_j \times S_{j+1}$.

Both isomorphisms follow from the CT.
It follows from the first isomorphism that for each $C_j$ there is a normal series $(B^{(j)})_{|S_j \times A_j \times S_j+1|} = \{0\} \times A_j \times \{0\} \subseteq (B^{(j+1)})_{|S_j \times A_j \times S_j+1|} \subseteq \cdots \subseteq (B^{(j)})_{|S_j \times A_j \times S_j+1|}$, whose factor groups are isomorphic to the granules $\Gamma^{(j+1)}$, $0 \leq \ell \leq n-1$. This chain implies that

$$|(B^{(j)})_{|S_j \times A_j \times S_j+1|}| = \prod_{\ell=0}^{n-1} |\Gamma^{(j+\ell)}|.$$ 

This result will be useful in the next section.

It follows from the second isomorphism that for each state space $S_{j+1}$ there is a normal series $(B^{(j)})_{|S_{j+1}|} = \{0\} \subseteq (B^{(j+1)})_{|S_{j+1}|} \subseteq \cdots \subseteq (B^{(j)})_{|S_{j+1}|}$, whose factor groups are isomorphic to the granules $\Gamma^{(j+1)}$, $1 \leq \ell \leq n-1$. We call this normal series the first-state chain at $S_{j+1}$, since $S_{j+1}$ is the first possibly nonzero state in the trajectories in $B^{(j+\ell)}$, $1 \leq \ell \leq n-1$. This chain implies that

$$|(B^{(j)})_{|S_{j+1}|}| = \prod_{\ell=1}^{n-1} |\Gamma^{(j+\ell)}|.$$ 

**D. Controllability and unique factorization**

We will now show that $R$ is controllable if and only if $B = B^c$; i.e., if and only if the top granule $\Gamma_R$ is trivial. Moreover, the controller granule decomposition gives a unique factorization of both $B^c$ and $B$.

We first state a technical lemma that shows that in the controllable subrealization $\mathcal{R}_c$, the number of transitions $(s_j, a_j, s_{j+1}) \in (C_j)^c$ is the number of states $s_j \in (S_j)^c$ times the number of transitions $(0, a_j, s_{j+1}) \in (B^{(j)})_{|S_j \times A_j \times S_{j+1}|}$.

**Lemma.** For all $j$, $|(C_j)^c| = \prod_{\ell=1}^{n-1} |\Gamma^{(j+\ell)}|$.

**Proof.** The projection of $B^c$ on $S_j$ is $(S_j)^c$, and its kernel is $B^{(j)}$, so $(S_j)^c \cong B^c/B^{(j)}$ by the FTH. The projections of $B^c$ and $B^{(j)}$ on $S_j \times A_j \times S_{j+1}$ are $(C_j)^c$ and $(B^{(j)})_{|S_j \times A_j \times S_{j+1}|}$, respectively, and $B^{(j+1)}$ is their common kernel, so $B^{(j)}/B^{(j+1)} \cong (C_j)^c/(B^{(j)})_{|S_j \times A_j \times S_{j+1}|}$ by the CT.

Next, we define $P_c$ as the product of the sizes of all controller granules up to level $n-1$, i.e., $P_c = \prod_{\ell=0}^{n-1} \prod_{s_j \in S_j} |\Gamma^{(j+\ell)}|$, and $P = |\Gamma_R|/P_c$ as the product of the sizes of all controller granules. We observe that since $P$ is the number of possible sums of granule representatives, we have $|B| \leq P$, with equality if and only if unique factorization holds for $B$. Similarly, we have $|B^c| \leq P_c$, with equality if and only if unique factorization holds for $B^c$.

**Theorem (controllability and unique factorization).** Let $B$ and $B^c$ be the behaviors of a reduced linear or group tail-biting trellis realization $R$ and its controllable subrealization $\mathcal{R}_c$, respectively. Then:

(a) $\mathcal{R}_c$ is controllable.
(b) Unique factorization holds for $B^c$; i.e., $|B^c| = P_c$.
(c) $\mathcal{R}$ is controllable if and only if $B = B^c$; i.e., iff the top granule $\Gamma_R$ is trivial.
(d) Unique factorization holds for $B$; i.e., $|B| = P$.

**Proof.** (a-b) From the previous lemma, $\prod_{j} |(C_j)^c| = \prod_{F \subseteq R, j \in E(F)} |\Gamma_F|$; hence, $\prod_{j} |(C_j)^c| = \prod_{F \subseteq R, j \in E(F)} |(B^{(j)})_{|S_j \times A_j \times S_{j+1}|}|$. By Section II-C, we have $|(B^{(j)})_{|S_j \times A_j \times S_{j+1}|}| = \prod_{i=0}^{n-1} |\Gamma^{(j+\ell)}|$, so $\prod_{j} |(C_j)^c|/\prod_{j} |(S_j)^c| = \prod_{F \subseteq R, j \in E(F)} |\Gamma_F| = |\mathcal{R}|$. Therefore, by our controllability test, we have $|B^c| \geq P_c$, with equality if and only if unique factorization holds for $B^c$. Thus $|B^c| = P_c$, $\mathcal{R}_c$ is controllable, and unique factorization holds for $B^c$.

(c) By Section II-B, $B$ is the disjoint union of $|\Gamma_R|$ disconnected cosets of $B^c$. Thus we have $|B| = |\Gamma_R||B^c|$, $|C_j| = |\Gamma_R|||C_j)^c|$, and $|S_j| = |\Gamma_R|(|S_j)^c|$. Therefore $\prod_{j} |(C_j)^c|/\prod_{j} |(S_j)^c| = P_c = |B^c| = |B|/|\Gamma_R|$, so $\mathcal{R}$ is controllable if and only if $|\Gamma_R| = 1$.

(d) By Section II-B every element of $B$ is uniquely expressible as the sum of an element of $B^c$ and a coset representative in $|\Gamma_R|$, so since unique factorization holds for $B^c$, it holds also for $B$.

**E. State space and constraint code sizes**

Unique factorization of $B$ implies unique factorization of $B_F$ for any fragment $F \subseteq R$. It follows that the size of each state space $S_j$ and each constraint code $C_j$ may be determined in terms of granule sizes as follows:

**Corollary (state space and constraint code sizes).** If $R$ is a reduced linear or group tail-biting trellis realization with state spaces $S_j$ and constraint codes $C_j$, then:

(a) $S_j \cong B/B^{(j)}$, and $|S_j| = \prod_{F \subseteq R, j \in E(F)} |\Gamma_F|$; 
(b) $C_j \cong B/B^{(j+1)}$, and $|C_j| = \prod_{F \subseteq R, j \in E(F)} |\Gamma_F|$.

**Proof.** (a) If $R$ is state-trim at $S_j$, then $S_j = B^{(j)}$. Moreover, the kernel of the projection of $B$ onto $S_j$ is $B^{(j)}$. Thus $S_j \cong B/B^{(j)}$ by the FTH, so $|S_j| = |B|/|B^{(j)}| = P/P_{F \subseteq R, j \in E(F)} |\Gamma_F| = \prod_{F \subseteq R, j \in E(F)} |\Gamma_F|$, since $F \subseteq F^{(j)}$ if $j \not\in E(F)$.

(b) If $R$ is branch-trim at $C_j$, then $C_j = B^{(j+1)}$ and the kernel of the projection of $B$ onto $C_j$ is $B^{(j+1)}$. Thus $C_j \cong B/B^{(j+1)}$ by the FTH, so $|C_j| = |B|/|B^{(j+1)}| = P/P_{F \subseteq R, j \in E(F)} |\Gamma_F| = \prod_{F \subseteq R, j \in E(F)} |\Gamma_F|$, since $F \subseteq F^{(j+1)}$ if $j \not\in E(F)$.

In other words, assuming trimness, $S_j$ factors into components isomorphic to those granules $\Gamma_F$ such that $S_j \in E(F)$ (i.e., $S_j$ is “active” during $F$). Also, $C_j$ factors into components isomorphic to those granules $\Gamma_F$ such that $C_j \in E(F)$ (i.e., $C_j$ is “active” during $F$).
The unique factorization result of Section III-E implies that every reduced linear or group trellis realization is equivalent to a controller canonical realization, which we define as follows.

For each $\mathcal{F} \leq \mathcal{R}$, we have a one-to-one map $\Gamma_{\mathcal{F}} \to [\Gamma_{\mathcal{F}}]$ from the granule $\Gamma_{\mathcal{F}}$ to the set of coset representatives $[\Gamma_{\mathcal{F}}] = [\mathcal{B}_{\mathcal{F}}/\mathcal{B}_{\mathcal{F}}]$. We may thus map each element of the Cartesian product $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}}$ to the sum $(a, s) = \sum_{\mathcal{F} \leq \mathcal{R}} (a_{\mathcal{F}}, s_{\mathcal{F}})$ of the corresponding coset representatives $(a_{\mathcal{F}}, s_{\mathcal{F}}) \in [\Gamma_{\mathcal{F}}]$, which is an element of $\mathcal{B}$. By unique factorization, the map so defined from $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}}$ to $\mathcal{B}$ is one-to-one.

More concretely, the map $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}} \to \mathcal{B}$ may be implemented as follows. We generate the trajectories in $[\Gamma_{\mathcal{F}}]$ by an atomic trellis realization whose state spaces $S_{\mathcal{F}}$ are equal to $\Gamma_{\mathcal{F}}$ when $\mathcal{S}_{\mathcal{F}} \in \mathcal{E}(\mathcal{F})$, and trivial otherwise. An element of $\Gamma_{\mathcal{F}}$ determines the state value $(s_{\mathcal{F}}, \mathcal{S}_{\mathcal{F}})$ when $\mathcal{S}_{\mathcal{F}} \in \mathcal{E}(\mathcal{F})$, and the symbol value $(a_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}})$ when $\mathcal{E}_{\mathcal{F}} \in \mathcal{V}(\mathcal{F})$. The state value $s_{\mathcal{F}}$ is thus the sum $\sum_{\mathcal{F} \leq \mathcal{R}} (s_{\mathcal{F}}, \mathcal{S}_{\mathcal{F}})$, and the symbol value $a_{\mathcal{F}}$ is the sum $\sum_{\mathcal{F} \leq \mathcal{R}} (a_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}})$. The size of the aggregate state space $S_{\mathcal{F}}$ is thus $|S_{\mathcal{F}}| = \prod_{\mathcal{F} \leq \mathcal{R}} |\mathcal{E}(\mathcal{F})| |\Gamma_{\mathcal{F}}|$, as in our state space size result. Thus the controller canonical realization is a minimal realization of $\mathcal{B}$. (We can also show that the number of possible transitions $(s_j, a_j, s_{j+1})$ is $\prod_{\mathcal{F} \leq \mathcal{R}} |\mathcal{E}(\mathcal{F})| |\Gamma_{\mathcal{F}}|$, as in our constraint code size result.)

If $\mathcal{B}$ is linear, then the controller canonical realization of $\mathcal{B}$ is easily seen to be linear. However, for a group realization $\mathcal{R}$, although the map $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}} \to \mathcal{B}$ yields a one-to-one, group-theoretic, and minimal realization of $\mathcal{B}$, it may well not be isomorphic, even when $\mathcal{R}$ is conventional $\mathcal{S}$. This issue was raised in [2, Remark III.3] via the following example, in which the controller canonical realization is nonhomomorphic.

Example (Conventional group trellis realization over $\mathbb{Z}_4$). Let $\mathcal{R}$ be a conventional group trellis realization of length 3 with behavior $\mathcal{B} = \{(112, 0120) \subseteq \mathbb{Z}_4^3 \times \mathbb{Z}_4^4\}$, i.e., $\mathcal{B} = \{(000, 0000), (112, 0120), (220, 0200), (332, 0320)\} \cong \mathbb{Z}_4$. Its $\ell$-controllable subbehaviors are $\mathcal{B}_1 = \{(000, 0000)\}$, $\mathcal{B}_2 = \mathcal{B} = \mathcal{B}/\mathcal{B}_1 \cong \mathbb{Z}_4$, and $\mathcal{B}_3 = \mathcal{B} \cong \mathbb{Z}_4$. Its nontrivial controller granules are $\Gamma_{[0,1]} = \mathcal{B}/\mathcal{B}_1 \cong \mathbb{Z}_4$, which is realized by a 2-state atomic trellis realization that is active during $[0,1]$, and $\Gamma_{[0,2]} = \mathcal{B}/\mathcal{B}_2 \cong \mathbb{Z}_4/2\mathbb{Z}_4 \cong \mathbb{Z}_2$, which is realized by a 2-state atomic trellis realization that is active during $[0,2]$.

Figure 2 depicts the controller canonical realization of $\mathcal{B}$ via trellis diagrams for the atomic trellis realizations of $\Gamma_{[0,1]} = \mathcal{B}/\mathcal{B}_1$ and $\Gamma_{[0,2]} = \mathcal{B}/\mathcal{B}_2$, plus a trellis diagram for $\mathcal{B}$.

IV. Conclusion

We have generalized the CBFT to group trellis realizations, with a proof based on a controller granularity decomposition of $\mathcal{B}$ and our controllability test for general group realizations.

It would be natural to dualize these results, using a dual observer granularity decomposition. However, as discussed in [6], such a dualization is not straightforward, even for minimal conventional trellis realizations. Developing a nice dual observer granularity decomposition for linear and group tail-biting trellis realizations is a good goal for future research.

It would be nice also to extend these results to non-trellis realizations. However, it is known (see [4, Appendix A]) that unique factorization generally does not hold for nontrellis linear or group realizations, even simple cycle-free realizations. New ideas will therefore be needed.

Finally, we would like ultimately to redevelop all of the principal results of classical discrete-time linear systems theory using a purely group-theoretic approach. However, the classical theory generally assumes an infinite time axis. One possible approach would be to regard a time-invariant or periodically time-varying linear or group system on an infinite time axis as the “limit” of a sequence of covers of a linear or group tail-biting trellis realization on a sequence of finite time axes of increasing length. Such an approach would hopefully be purely algebraic, and thus might avoid the subtle topological issues discussed in [6].

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