Topological and error-correcting properties for symmetry-protected topological order

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Abstract – We study the symmetry-protected topological (SPT) orders for bosonic systems from an information-theoretic viewpoint. We show that with a proper choice of the onsite basis, the degenerate ground-state space of SPT orders (on a manifold with boundary) is a quantum error-correcting code with macroscopic classical distance, hence is stable against any local bit-flip errors. We show that this error-correcting property of the SPT orders has a natural connection to that of the symmetry-breaking orders, whose degenerate ground-state space is a classical error-correcting code with a macroscopic distance, providing a new angle for the hidden symmetry-breaking properties in SPT orders. We further propose new types of topological entanglement entropy that probe the SPT orders hidden in their symmetric ground states, which also signal the topological phase transitions protected by symmetry. Combined with the original definition of topological entanglement entropy that probes the “intrinsic topological orders”, and the recent proposed one that probes the symmetry-breaking orders, the set of different types of topological entanglement entropy may hence distinguish topological orders, SPT orders, and symmetry-breaking orders, which may be mixed up in a single system.

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Introduction. – Symmetry-protected topological (SPT) orders are gapped phases of matter with certain symmetry and only short-range entanglement. It has been a focus of the recent studies in condensed-matter physics due to the excitement of the new experimental advances in topological insulators and superconductors [1]. The classification of free fermionic SPT phases is well understood [2]. The situation of the interacting systems is more complicated, with extensively recent discussions for both the bosonic case [3–6] and the fermionic case [7–9].

While many recent works are focusing on the symmetry aspects of the SPT orders, we would like to examine more details regarding the topological properties of these systems from an information-theoretic viewpoint. We start with the discussion of bosonic systems in one spatial dimension (1D), where the gapped ground states of local Hamiltonians are extensively studied [10–12].

It is well known that for a 1D gapped Hamiltonian, the ground states obey the entanglement area law [10–15] and can be faithfully represented by the matrix product states (MPS) [16]. When the ground state is unique, the MPS representation has injective matrices and can be adiabatically connected to an isometric form (as shown in fig. 1(a), for periodic boundary conditions) via a renormalization procedure (with possibly blocking of sites) [11,13].

Consider a system with \( n \) sites hence total \( 2n \) (virtual) qubits, and the quantum state of the system in fig. 1(a) can be written as \( |\Psi_n\rangle = \otimes_i |w\rangle_{l,r(i+1)}, \) where the label \( i \) denotes the \( i \)-th site, and the subscript \( l/r \) of the site \( i \) denotes the left/right (virtual) qubit in the site. If one further applies a two-site unitary transformation on each bond, the system can be disentangled to a product state \( |0\rangle^{\otimes 2n} \).

In order to reveal properly the nontrivial topological properties of the system, certain symmetry is needed
ate ground states. This is very different from the state hence the corresponding Hamiltonian has 4-fold degener-
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connected by a line represent a bond, which is given by $|w\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. (a) Each site containing two qubits labelled by $l$ (left) and $r$ (right). (b) Shifting the system by one (virtual) qubit.

to prevent the system from going to a trivial product state, which is the meaning of “symmetry protection”. The distinct topological feature of an SPT state, for instance the state $|\Psi_a\rangle$, is that when putting on a 1D chain with boundary, each boundary carries an unpaired qubit, hence the corresponding Hamiltonian has 4-fold degenerate ground states. This is very different from the state $|\Psi_b\rangle = \otimes_i |w\rangle_{i,A,i}$, as illustrated in fig. 1(b), by shifting $|\Psi_a\rangle$ by a (virtual) qubit. $|\Psi_b\rangle$ essentially is a product state of onsite wave functions, and does not carry any unpaired qubit on a 1D chain with boundary.

$|\Psi_b\rangle$ clearly has the same symmetry as $|\Psi_a\rangle$. However, when certain symmetry is respected (e.g., $D_2 = Z_2 \times Z_2$), $|\Psi_a\rangle$ cannot be adiabatically connected to $|\Psi_b\rangle$ without a phase transition. This phase transition is in this sense topological, which however needs the symmetry protection to happen. It is shown that the underlying reason for $|\Psi_a\rangle$ to be different from $|\Psi_b\rangle$ is that they carry different projective representations of the symmetry group, and theories based on the group cohomology may be used to distinguish different SPT phases [11,13].

In this work, we propose a new approach to the theory of SPT orders from an information-theoretic viewpoint. We show that, under a proper choice of the onsite basis, the degenerate ground-state space is a quantum error-correcting code with a macroscopic classical distance, hence is stable against any local bit-flip errors. This error-correcting property has a natural connection to that of the symmetry-breaking orders, whose degenerate ground-state space is a classical error-correcting code with a macroscopic distance. Our approach hence provides a new angle for the hidden symmetry-breaking properties in SPT orders [17–21].

We further propose new types of topological entanglement entropy, which probe the SPT orders and signal the topological phase transitions protected by symmetry. Our new types of topological entanglement entropy are defined on a manifold with boundary, which can probe the topological properties hidden in the symmetric ground states.

Error-correcting properties. – To examine the error-correcting properties for SPT orders, it is convenient
to transform the MPS isometric form into the cluster state model by an onsite unitary transformation. Notice that the bond state $|w\rangle_{i-\langle,i+1\rangle}$ is a two-qubit stabilizer state with the stabilizer generators $\{X_i, X_{i+1}, Z_i, Z_{i+1}\}$. Now on each site, we apply the transformation

$$U_i = CNOT_{i,i} H_{i,r},$$

where $CNOT_{i,i}$ is the controlled-NOT operation with the $i$-th qubit as the control qubit, and $H_{i,r}$ is the Hadamard transformation on the $i$-th qubit.

After the transformation $\prod_i U_i$, we have

$$X_i X_{i+1} Z_i X_{i+1} Z_{i+1} = Z_i X_i Z_{i+1},$$

which are the stabilizer generators for a 1D cluster state [22] of $2n$ qubits.

We now consider a 1D qubit system of $N$ qubits, with $N$ even. And without confusion we label each qubit by $j$. The 1D cluster state hence corresponds to the stabilizer group with generators $\{Z_{j-1} X_j Z_{j+1}\}$, and the corresponding Hamiltonian

$$H_{clu} = -\sum_j Z_{j-1} X_j Z_{j+1}.$$ (3)

For a 1D ring without boundary, the ground state of $H_{clu}$ is unique. For a chain with boundary, where the summation index $j$ runs from 2 to $N-1$, the ground state is then 4-fold degenerate. We can also view the degenerate ground-state space as a quantum error-correcting code encoding two qubits. As a quantum code, it has only distance 1, as $Z_i$ commutes with all the stabilizer generators.

What we are interested in here is the ability of this code for correcting classical errors (bit flip), which correspond to errors that are tensor products of $X_j$'s. It is straightforward to see that the two logical operators, which are in the form of tensor products of $X_j$'s, are [21,23]

$$\bar{X}_1 = \prod_k X_{2k-1}, \quad \bar{X}_2 = \prod_k X_{2k},$$ (4)

with $k$ running from 1 to $N/2$. This code hence has classical distance $N/2$, which is a macroscopic distance that is half of the system size.

Another way to view $\bar{X}_1$ and $\bar{X}_2$ is that they generate the group $D_2 = Z_2 \times Z_2$ that preserves the topological order of the system [23]. Any local perturbation respecting the symmetry cannot lift the ground-state degeneracy (in the thermodynamical limit) [23–25].

One way to view this symmetry protection is to add a magnetic field along the $X$-direction to the system, and the corresponding Hamiltonian reads

$$H_{clu}(B) = -\sum_j Z_{j-1} X_j Z_{j+1} + B \sum_j X_j.$$ (5)

It is known that there is a phase transition at $B = 1$ for periodic boundary condition [24–26].
It is interesting to compare the system \( H_{\text{clu}}(B) \) with a symmetry-breaking-ordered Hamiltonian
\[
H_{\text{sb}}(B) = -\sum_j Z_{j-1}Z_{j+1} + B\sum_j X_j,
\]
with the same symmetry \( \mathbb{D}_2 \) given by \( X_1, X_2 \). The degenerate ground-state space of \( H_{\text{sb}}(0) \) is a classical error-correcting code with distance \( N/2 \), and is spanned by
\[
|0000\ldots00\rangle, |0101\ldots01\rangle, |1010\ldots10\rangle, |1111\ldots11\rangle.
\]

Denote the symmetric ground state of \( H_{\text{sb}}(B) \) by \( |\psi_{\text{sb}}(B)\rangle \). Then \( |\psi_{\text{sb}}(0)\rangle \) is a stabilizer state stabilized by \( Z_{j-1}Z_{j+1} \) (\( j = 2, \ldots, N - 2 \)) and \( X_1, X_2 \), and is in fact an equal weight superposition of the basis states of the code as given in eq. (7). Similarly, we denote the symmetric ground state of \( H_{\text{clu}}(B) \) by \( |\psi_{\text{clu}}(B)\rangle \). Then \( |\psi_{\text{clu}}(0)\rangle \) is a stabilizer state stabilized by \( Z_{j-1}X_jZ_{j+1} \) (\( j = 2, \ldots, N - 2 \)) and \( X_1, X_2 \). There is no local unitary transformation to transform \( H_{\text{clu}}(B) \) to \( H_{\text{sb}}(B) \). One either needs a nonlocal transformation or a local transformation with an unbounded depth, which reveals the hidden symmetry-breaking property of the SPT order [20,21]. This can also be seen from the fact that the symmetric ground \( |\psi_{\text{sb}}(0)\rangle \) is long-range entangled, and this long-range property does not change even if closing the boundary. However, the state \( |\psi_{\text{clu}}(0)\rangle \), although appears to be long-range entangled for a 1D chain with boundary (characterized by logical operators \( X_1, X_2 \)), is essentially short-range entangled when closing the boundary.

That is, \( |\psi_{\text{clu}}(0)\rangle \) is in fact stabilized by \( Z_{j-1}X_jZ_{j+1} \) with a periodic boundary condition [23]. When the total number of sites \( N \) is even, we have
\[
\prod_k Z_{2k-2}X_{2k-1}Z_{2k} = \bar{X}_1,
\]
\[
\prod_k Z_{2k-1}X_{2k}Z_{2k+1} = \bar{X}_2,
\]
where we interpret \( Z_0 = Z_N \) and \( Z_{N+1} = Z_1 \). This means that, the stabilizer group of the symmetric ground state \( |\psi_{\text{clu}}(0)\rangle \), generated by \( Z_{j-1}X_jZ_{j+1} \) (\( j = 2, \ldots, N - 2 \)) and \( X_1, X_2 \), is also generated by \( Z_{j-1}X_jZ_{j+1} \) with a periodic boundary condition, i.e., \( j = 1, \ldots, N \). Or in terms of error correction, the \( X_1 \) operators are generated as the product of sets of stabilizers, therefore there is in fact no logical operators for the code on a periodic chain and hence no encoded logical qubits (i.e., equivalent to a unique ground state).

In this sense, viewed as a dimension-0 quantum code, \( |\psi_{\text{clu}}(0)\rangle \) also has a macroscopic classical distance (given by the smallest weight element in the stabilizer group which is a tensor product of \( X_i \)'s [27]). Going along the direction respecting the symmetry picks up the symmetric ground state as the exact ground state, which gives rise to the phase transition for both the periodic and open boundary conditions.

\[\text{Fig. 2: (Colour online) (a) Cutting a 1D chain into A, B, C parts; (b) cutting a 1D chain into A, B, C, D parts.}\]

**Topological entanglement entropy.** – Topological entanglement entropy was first proposed to detect topological orders [28,29], and is recently generalized to probe the systems with symmetry-breaking orders [30,31]. In both cases, the degenerate ground-state space as a quantum error-correcting code support logical operators with macroscopic weight. Due to our previous discussion, we know that SPT orders also support this kind of logical operators (e.g., eq. (4)). Inspired by this observation, we introduce new types of topological entanglement entropy to probe the SPT orders.

We consider a 1D chain with boundary. For any gapped ground state, and for the cuttings given in fig. 2, we introduce the topological entanglement entropy
\[
S_{\text{topo}} = S_{AB} + S_{BC} - S_B - S_{ABC},
\]
where \( S_* \) on the right side of equality is the von Neumann entropy of reduced density matrix of the part \(*\).

We propose two kinds of cuttings in fig. 2. Figure 2(a) cuts the system into three parts, and we denote the corresponding topological entanglement entropy by \( S_{\text{topo}}^I \). Figure 2(b) cuts the system into four parts, and we denote the corresponding topological entanglement entropy by \( S_{\text{topo}}^II \). We use \( S_{\text{topo}}^I \) to refer both \( S_{\text{topo}}^I \) and \( S_{\text{topo}}^II \).

Similarly to the topological entanglement entropy introduced previously, \( S_{\text{topo}} \) is an invariant of local unitary transformations and \( S_{\text{topo}} = 0 \) for unique gapped ground states [28,29,31]. We also know that \( S_{\text{topo}} \) is quantized for SPT ordered states due to their degenerate entanglement spectrum [32], hence a nonzero (and quantized in the \( N \to \infty \) limit) \( S_{\text{topo}} \) is a signature of SPT order. We will show that \( S_{\text{topo}} \) also signals topological phase transitions protected by symmetry.

We first examine \( S_{\text{topo}}^I \). For the ideal state of \( B = 0 \), \( S_{\text{topo}}^I = 2 \) for both \( |\psi_{\text{clu}}(0)\rangle \) and \( |\psi_{\text{sb}}(0)\rangle \). When \( B \) increases, for \( |\psi_{\text{clu}}(B)\rangle \), \( S_{\text{topo}}^I \) signals the topological phase transition. To demonstrate this, we perform an exact diagonalization of the Hamiltonian \( H_{\text{clu}}(B) \), and calculate \( S_{\text{topo}}^I \) for the corresponding ground state. We do the calculation with 6, 12, 18, 24 qubits, where each part of \( A, B, C \) contains 2, 4, 6, 8 qubits, respectively. Our results are shown in fig. 3(a).

However, the symmetry-breaking order hidden in the exact symmetric ground state \( |\psi_{\text{sb}}(B)\rangle \) can also be
detected by $S_{\text{topo}}^t$. In fact, for the same calculation with 6, 12, 18, 24 qubits, one gets a very similar figure, as shown in fig. 4(a).

To distinguish SPT orders from a symmetry-breaking one, we can instead use $S_{\text{topo}}^Q$. Since the topological entanglement entropy is only carried in the entire wave function of the exact symmetric ground state [31], computing the wave function of the exact symmetric ground state for any topological phase, as shown in fig. 4(b). Here we do the calculation with 12, 16, 20, 24 qubits, where each part of $A, B, C, D$ contains 3, 4, 5, 6 qubits respectively.

However, $S_{\text{topo}}^Q = 2$ for $|\psi_{\text{clu}}(0)\rangle$, because the “topology” of the SPT states is essentially carried on the boundary, tracing out part of the bulk has no effect on detecting the topological order. For $|\psi_{\text{clu}}(B)\rangle$, $S_{\text{topo}}^Q$ signals the topological phase transition, as shown in fig. 3(b). Similarly here we do the calculation with 12, 16, 20, 24 qubits, where each part of $A, B, C, D$ contains 3, 4, 5, 6 qubits, respectively.

The AKLT State. – As another example, we apply our method to study the SPT orders of the Affleck-Lieb-Kennedy-Tasaki (AKLT) model [3,19]. We start from the AKLT Hamiltonian [33]

$$H_{\text{AKLT}} = \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2,$$

(10)

where $\vec{S}_j = (S_j^x, S_j^y, S_j^z)$ is the spin-1 operator of the $j$-th spin.

On an open chain, the ground state of $H_{\text{AKLT}}$ is four-fold degenerate, and the symmetric ground state (with respect to certain symmetry, for instance, time-reversal symmetry) has $S_{\text{topo}}^t = 2$, similar to the case for the cluster state $|\psi_{\text{clu}}(0)\rangle$. If one adds a perturbation to the system respecting certain (e.g., the time reversal) symmetry, for instance

$$H_{\text{AKLT}}(\lambda) = \cos \lambda \left( \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right)$$

$$+ \sin \lambda \sum_j (\vec{S}_j^z)^2,$$

(11)

for $\lambda \in [0, \pi/2]$, then a topological phase transition protected by symmetry will happen at some $\lambda$. $S_{\text{topo}}^t$ signals this phase transition as demonstrated in fig. 5.

A mixing order of symmetry breaking and SPT. – There could be also systems containing mixing orders of symmetry breaking and SPT, whose symmetric ground states correspond to noninjective matrices in the MPS representations, with isometric forms that couple GHZ states with short-ranged bond states [13].

As an example, we consider a stabilizer group generated by $Z_{j-1}X_jX_{j+1}Z_{j+2}$ with $j$ running from 2 to $N-2$, which is a generalization of the 5-qubit code [34,35] and a special kind of quantum convolutional codes [36]. On a 1D chain with boundary, i.e., for $j = 2, 3, \ldots, N-2$, the Hamiltonian $-\sum_j Z_{j-1}X_jX_{j+1}Z_{j+2}$ has 8-fold ground-state degeneracy. Here we omit the discussion on the slight difference of whether $N$ is a multiple of 3, as these technical details are not essential to our discussion of $S_{\text{topo}}^t$.

The ground state as an error-correcting code has classical distance $[N/3]$, with logical operators $X_1 = \prod_k X_{3k-1}, X_2 = \prod_k X_{3k-2}, X_3 = \prod_k X_{3k}$. Therefore, if one adds a magnetic field along the $X$-direction, i.e.

$$H_{\text{ZXXZ}}(B) = -\sum_j Z_{j-1}X_jX_{j+1}Z_{j+2} + B \sum_j X_j,$$

(12)

the orders of the system (either SPT or symmetry breaking) will be protected.

It turns out that the system combines a $Z_2$ symmetry-breaking order and a $D_2$ SPT order. This can be seen...
state the eigenvalue 1 eigenstate of $X$.

Moreover, the onsite transformation $\prod B$ from the fact that for $B = 0$, the symmetric ground state has $S^1_{\text{topo}} = 3$ and $S^Q_{\text{topo}} = 2$. $S^1_{\text{topo}}$ probes both the symmetry-breaking order and the SPT order, as illustrated in fig. 6(a). $S^Q_{\text{topo}}$ probes only the SPT phase transition, as illustrated in fig. 6(b).

Discussion. – What eq. (1) essentially does is to map the onsite state $|w\rangle_{i, t}$, illustrated in fig. 1(b) to a product state of qubits $|+\rangle_{i} \otimes |+\rangle_{t}$. Here $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is the eigenvalue 1 eigenstate of $X$. Because going from the state $|\Psi_{a}\rangle$ (the state illustrated in fig. 1(a)) to $|\Psi_{b}\rangle$ while respecting symmetry will encounter a phase transition, directly interpolating the cluster state to $|+\rangle^{\otimes 2n}$ (i.e., given by $H_{\text{clu}}(B)$) also undergoes a phase transition. Therefore, the onsite transformation $\prod_i U_i$ transforms $|\Psi_{a}\rangle$ to the symmetric ground state of a quantum error-correcting code with a macroscopic classical distance.

This idea can be generalized to higher spatial dimensions. In a general setting, an SPT-ordered state $|\Phi_{a}\rangle$ is that, when connecting to a product state $|\Phi_{b}\rangle$ with the same symmetry, a phase transition occurs while respecting the symmetry [5]. One can always apply some onsite unitary transformation to transform $|\Phi_{a}\rangle$ to a tensor product of $|+\rangle$, hence at the same time to transform $|\Phi_{b}\rangle$ to the symmetric ground state of some quantum error-correcting code with a macroscopic classical distance (for instance the SPT-ordered 2D cluster state discussed in [23]).

One may also generalize the idea of different types of topological entanglement entropy to higher spatial dimensions. For instance, in 2D, a straightforward way is to replace the chain by a cylinder with boundary, then use the similar cuttings as in fig. 2.

One may also consider a disk with boundary. For any gapped ground state (one may need to avoid the situation of a gapless boundary by adding symmetric local terms to the Hamiltonian), still using $S^1_{\text{topo}}$ as given in eq. (9), one can consider two kinds of cuttings, as given in fig. 7. Similarly as in the 1D case, the cutting of fig. 7(a) probes both the symmetry-breaking orders and the SPT orders, and the cutting of fig. 7(b) probes only SPT orders.

Notice that the topological entanglement entropy proposed in [30,31] is defined on a manifold without boundary (e.g., a 1D ring or a 2D sphere), which detects only symmetry-breaking orders but not SPT orders. Combined with the original definition of topological entanglement entropy [28,29] that probes the “intrinsic topological orders”, and the recent proposed one that probes the symmetry-breaking orders [30,31], the set of different types of topological entanglement entropy may hence distinguish topological orders, SPT orders, and symmetry-breaking orders, which may be mixed up in a single system.

We remark that, although our generalization of topological entanglement entropy detects the SPT orders, it does not require knowing the explicit symmetry of the system. As an example, for the AKLT state discussed in the fourth section, we do not specify the explicit symmetry. This is similar to the case of detecting symmetry-breaking order by the generalized topological entanglement entropy [30,31]. A summary of using (generalized) topological entanglement entropy to detect different orders in gapped system can be found in [30,37].

We hope our discussion adds new ingredients for understanding the microscopic theory of SPT orders.

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