Structure-preserving splitting methods for stochastic logarithmic Schrödinger equation via regularized energy approximation

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Abstract. In this paper, we study two kinds of structure-preserving splitting methods, including the Lie–Trotter type splitting method and the finite difference type method, for the stochastic logarithmic Schrödinger equation (SlogS equation) via a regularized energy approximation. We first introduce a regularized SlogS equation with a small parameter $0 < \epsilon \ll 1$ which approximates the SlogS equation and avoids the singularity near zero density. Then we present a priori estimates, the regularized entropy and energy, and the stochastic symplectic structure of the proposed numerical methods. Furthermore, we derive both the strong convergence rates and the convergence rates of the regularized entropy and energy. To the best of our knowledge, this is the first result concerning the construction and analysis of numerical methods for stochastic Schrödinger equations with logarithmic nonlinearities.

1. Introduction

In this paper, we focus on the SlogS equation

$$ du(t) = i\Delta u(t) dt + i\lambda u(t) \log(|u(t)|^2) dt + \tilde{g}(u(t)) \ast dW(t), \quad t > 0, $$

$$ u(0) = u_0, $$

where $\lambda \in \mathbb{R}/\{0\}$ measures the force of the logarithmic nonlinearity, $\Delta$ is the Laplacian operator on $\mathcal{O} \subset \mathbb{R}^d$ with $\mathcal{O}$ being either $\mathbb{R}^d$ or a bounded domain with homogeneous Dirichlet or periodic boundary condition, and $d \in \mathbb{N}^+$ is the spatial dimension. Here $W(t)$ is a $Q$-Wiener process, i.e., $W(t) = \sum_{k \in \mathbb{N}^+} Q_{1/2} e_i \beta_k(t)$ with $\{\beta_k\}_{k \in \mathbb{N}^+}$ being a sequence of independent Brownian motions on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. The operator $Q_{1/2}$ is bounded on $H := L^2(\mathcal{O}; \mathbb{C})$ satisfying $\sum_{i \in \mathbb{N}^+} \|Q_{1/2} e_i\|^2 < \infty$, and $\{e_i\}_{i \in \mathbb{N}^+}$ is an orthonormal basis of $\mathbb{H}$. Here $\tilde{g}$ is a continuous function and

$$ \tilde{g}(u) \ast dW(t) := -\frac{1}{2} \sum_{i \in \mathbb{N}^+} |Q_{1/2} e_i|^2 \left( g(|u|^2) |u|^2 u \right) dt - i \sum_{i \in \mathbb{N}^+} g(|u|^2) g'(|u|^2) |u|^2 u \text{Im}(Q_{1/2} e_i) Q_{1/2} e_i dt + i g(|u|^2) u dW(t) $$

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if \( \tilde{g}(x) = ig(|x|^2)x \) (multiplicative case), and
\[
\tilde{g}(u) \ast dW(t) := dW(t)
\]
if \( \tilde{g} = 1 \) (additive case). The SlogS equation has wide applications in quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum system, Bose-Einstein condensations, etc. (see e.g. \([2, 4, 14, 16, 17, 18]\)).

The logarithmic nonlinearity possesses several features which make the logarithmic Schrödinger equation unique among nonlinear wave equations. For instance, the logarithmic nonlinearity is not locally Lipschitz and causes singularity near vacuum. The large time behavior, like the dispersive phenomenon depending on the sign of \( \lambda \), is totally different from that in the Schrödinger equation with smooth nonlinearity (see e.g. \([6, 7]\)). Moreover, the randomness of the driving noise destroys many well-known conservation laws and structures. Several physical quantities, such as the mass, momentum and energy, may be no longer conservative for the SlogS equation (see e.g. \([3, 12]\)). Only when \( \tilde{g}(x) = ig(|x|^2)x \) with a continuous real-valued function \( g \) and \( W(t) \) is \( L^2(O; \mathbb{R}) \)-valued, the mass conservation law holds. The well-posedness of the SlogS equation driven by the linear multiplicative noise \( \tilde{g}(x) = ix \) has been shown in \([3]\). Recently, the author in \([12]\) obtains the well-posedness of the SlogS equation with general diffusion coefficients (including \( \tilde{g}(x) = ig(|x|^2)x \) and \( \tilde{g}(x) = 1 \)).

Despite various and fruitful numerical results of stochastic Schrödinger equations with smooth nonlinearities (see e.g. \([11, 5, 8, 9, 10, 11, 13, 15]\) and references therein), the numerical analysis of stochastic Schrödinger equations with non-locally Lipschitz nonlinearities, especially for the SlogS equations, is far from being well understood and confronts several challenges. One is that the direct numerical discretization often produces the numerical vacuum which are difficult to deal with when computing the logarithmic nonlinearity. Another challenge lies on the mutual effect of the random noise and the non-locally Lipschitz coefficient, which leads that the existing numerical approach for analyzing the geometric structures and convergence analysis of numerical methods are not available for SlogS equations. To overcome these issues, we will introduce a regularized problem of (1.1) which is used to show the well-posedness of the SlogS equation in \([12]\). Then we show that the regularized energy of the regularized SlogS equation is well-defined, and thus the regularized SlogS equation is a stochastic Hamiltonian partial differential equation whose phase flow preserves the stochastic symplectic structure. The a priori estimates and convergence results of the regularized SlogS equation to (1.1) are also presented.

Furthermore, we propose structure-preserving splitting methods of different types, including the Lie–Trotter splitting methods and finite difference methods, based on the regularized SlogS equation, to inherit the intrinsic properties of original systems. We study several important features of the proposed numerical methods, including the moment estimates of regularized entropy and energy, the mass evolution law and the symplectic structure. Based on the structure-preserving properties of the regularized SlogS equation and numerical methods, error estimates in both strong and weak convergence senses are established between the solutions of (1.1) and the regularized splitting methods. To the best of our knowledge, this is the first result on the numerical methods of stochastic Schrödinger equations of this kind. The proposed numerical methods are even new in the deterministic case, i.e., \( \tilde{g} = 0 \), and all the numerical results in this paper still hold in the deterministic case.

The reminder of this article is organized as follows. In section 2, we introduce some basic notations and present the well-posedness result of the SlogS equation and its regularized version. Section 3 is devoted to constructing and analyzing the Lie–Trotter type splitting method, including \( \epsilon \)-independent a priori estimate, mass evolution law and symplectic structure. In section 4, we propose the finite difference type splitting method, and prove its convergence of the energy
functional and strong convergence. Throughout this article, \( C \) denotes various positive constants which may change from line to line.

2. Regularized SlogS equation

In this section, we introduce some necessary notations, the well-posedness result of SLogS equation \((1.1)\), as well as the properties of the regularized SLogS equation.

2.1. Preliminary. Denote \( \mathbb{H} \) with the product \( \langle u, v \rangle := Re[\int_{\mathcal{O}} uv dx] \), \( u, v \in \mathbb{H} \). Let \( W^{k, p}, k \in \mathbb{N}, p \in \mathbb{N}^+ \) be the classic Sobolev spaces and \( \mathbb{H}^k = W^{k, 2} \). Denote \( L^p := L^p(\mathcal{O}; \mathbb{C}), p \geq 1 \). In order to bound the entropy \( F(\rho) := \int_{\mathcal{O}} (\rho \log \rho - \rho) dx, \rho = |u|^2 \) in the case of \( \mathcal{O} = \mathbb{R}^2 \), we introduce the weighted square integrable space

\[
L_2^\alpha := \{ v \in \mathbb{H} \mid x \mapsto (1 + |x|^2)^\alpha v(x) \in \mathbb{H} \}
\]

with the norm \( \|v\|_{L_2^\alpha} := \|(1 + \cdot |^2)^\alpha v(\cdot)\|_{L^2(\mathbb{R}^2)}, \alpha \geq 0 \). For convenience, we always assume that \( u_0 \in \mathbb{H}^1 \cap L_2^\alpha, \alpha \in (0, 1] \), is \( \mathcal{F}_0 \)-measurable and has any finite \( p \)-moment, \( p \in \mathbb{N}^+ \). The main assumption on \( W \) and \( \tilde{g} \) is stated as follows.

**Assumption 2.1.** The Wiener process \( W \) and \( \tilde{g} \) satisfy one of the following conditions:

1. \( \{W(t)\}_{t \geq 0} \) is \( \mathbb{H} \)-valued, \( \tilde{g} = 1 \) or \( \tilde{g}(x) = ig(|x|^2)x \) with \( g \in C^2_b(\mathbb{R}) \) satisfying the growth condition,

\[
\sup_{x \in [0, \infty)} |g(x)| + \sup_{x \in [0, \infty)} |g'(x)| + \sup_{x \in [0, \infty)} |g''(x)| \leq C_g,
\]

the one-side Lipschitz continuity condition, i.e., for any \( x, y \in \mathbb{C} \),

\[
|(\tilde{g} - \tilde{x})| g(|x|^2)|x|^2 x - g'(|y|^2) g(|y|^2)|y|^2 y| \leq C_g |x - y|,
\]

and

\[
(x + y)(|g(|x|^2)| - g(|y|^2)) \leq C_g |x - y|, \quad x, y \in [0, \infty).
\]

2. \( \{W(t)\}_{t \geq 0} \) is \( L^2(\mathcal{O}; \mathbb{R}) \)-valued and \( \tilde{g}(x) = ig(|x|^2)x \) with \( g \in C^1_b(\mathbb{R}) \) satisfying \((2.2)\) and the growth condition

\[
\sup_{x \in [0, \infty)} |g(x)| + \sup_{x \in [0, \infty)} |g'(x)| \leq C_g.
\]

The functions like \( a, \frac{a}{t + x}, \frac{ax}{t + cx}, \frac{ax}{t + cx^2} \) with \( b, c > 0 \), will satisfy the above conditions on \( g \) in Assumption 2.1.

**Theorem 2.1** (see \cite{[12]}). Let \( T > 0 \) and Assumptions \((2.1)\) hold, \( u_0 \in \mathbb{H}^1 \cap L_2^\alpha, \alpha \in (0, 1] \), be \( \mathcal{F}_0 \)-measurable and have any finite \( p \)-moment with \( p \geq 1 \). Assume that \( \sum_{i \in \mathbb{N}^+} \|Q_i \tilde{e}_i\|_{L_2^\alpha} + \|Q_i \tilde{e}_i\|_{H^1} < \infty \) when \( \tilde{g} = 1 \) and that \( \sum_{i \in \mathbb{N}^+} \|Q_i \tilde{e}_i\|_{L_2^\alpha} + \|Q_i \tilde{e}_i\|_{H^1} < \infty \) when \( \tilde{g}(x) = ig(|x|^2)x \). Then there exists a unique mild solution \( u \) in \( C([0, T]; \mathbb{H}) \) for Eq. \((1.1)\). Moreover, there exists \( C(Q, T, \lambda, p, u_0, \alpha, \tilde{g}) > 0 \) such that

\[
E \left[ \sup_{t \in [0, T]} \|u(t)\|_{L_2^\alpha}^p \right] + E \left[ \sup_{t \in [0, T]} \|u(t)\|_{H^1}^p \right] \leq C(Q, T, \lambda, p, u_0, \alpha, \tilde{g}).
\]
2.2. Regularized energy of regularized SLogS equation. To deal with the logarithmic nonlinearity, we introduce the regularized \$S\text{LogS} \$ equation with \$0 < \epsilon \ll 1\$,
\begin{equation}
du' = i\Delta u' \, dt + i\lambda u' f_\epsilon(|u'|^2) \, dt + \gamma(u') \ast dW(t), \quad u'(0) = u_0.
\end{equation}

The regularized energy is defined by \$H_\epsilon(u) := \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}F_\epsilon(|u|^2)\$, where \$F_\epsilon(x) = \int_0^x f_\epsilon(s) \, dsdx\$ is the regularized entropy and \$f_\epsilon(x)\$ is a suitable approximation of \(\log(\cdot)\). Formally speaking, we expect that \$H_\epsilon(u)\$ approximates the energy of Eq. \((2.1)\). \(H(u) := \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2}F(|u|^2)\). To this end, we impose the following assumption on \$f_\epsilon\$.

**Assumption 2.2.** The regularization function \$f_\epsilon\$ satisfies
\begin{enumerate}[(A1)]
\item \(f_\epsilon(|x|^2) \leq C(1 + \|\log(\cdot)\|) \quad \forall \, x \in \mathbb{C}\).
\item \(\lim_{\epsilon \to 0} f_\epsilon(|x|^2) = \log(|x|^2) \quad \forall \, x \in \mathbb{C}\).
\item \(f_\epsilon(|x|^2) - \log(|x|^2)) \leq C(|x|^2)\delta, \quad \delta \in (0, 1)\) when \(|x| \geq 1\), and \(f_\epsilon(|x|^2) - \log(|x|^2)) \leq C(\frac{1}{\epsilon})\) when \(|x| \leq 1\).
\item \(\frac{\partial}{\partial t} f_\epsilon(|x|^2) \leq C \frac{|x|}{\epsilon + |x|^2} \quad \forall \, x \in \mathbb{C}\).
\end{enumerate}

One typical example satisfying Assumption 2.2 is \(f_\epsilon(|x|^2) = \log(\frac{|x|^2}{1 + |x|^2})\), whose corresponding regularized energy is \(H_\epsilon(u')\) with \(F_\epsilon(|u'|^2) = \int_0^{|u'|^2} \log(\frac{1 + t}{1 + |u|^2}) + \epsilon \log(|u|^2 + \epsilon) - \frac{1}{2} \epsilon \log(\epsilon |u|^2 + 1) + \epsilon \log(\epsilon) \, dt\). Assumption 2.2 is crucial to obtain the strong convergence result and the H"older regularity estimate for Eq. \((2.1)\) (see e.g. \([12]\)) which is illustrated in the following lemma. We would like to remark that \(f_\epsilon(x) = \log(\epsilon + x)\) or \(\log(\epsilon + \sqrt{x})^2\) fails to satisfy \(A1\).

**Lemma 2.1.** Let the condition of Theorem 2.1 and Assumption 2.2 hold. Let \(u^0 := u\) be the mild solution of Eq. \((2.1)\). Then there exists a unique mild solution \(u^\epsilon\) of Eq. \((2.3)\). For \(p \geq 1\), there exist \(C'(Q, T, \lambda, p, u_0, \tilde{g}) > 0\) and \(C'(Q, T, \lambda, p, u_0, \alpha, \tilde{g}) > 0\) such that for \(\epsilon \in [0, 1]\),
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u^\epsilon(t) - u^\epsilon(s)\|^p \right] \leq C'(Q, T, \lambda, p, u_0, \tilde{g}) |t - s|^\frac{p}{2}, \\
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u^\epsilon(t)\|^p \right] \leq C'(Q, T, \lambda, p, u_0, \alpha, \tilde{g}).
\end{align*}

Moreover, for \(p \geq 1\) and \(\delta \in (0, \max(\frac{2}{\max(d - 2, 0)}, 1))\), there exist \(C(Q, T, \lambda, p, u_0, \tilde{g}) > 0\) and \(C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g}) > 0\) such that when \(\mathcal{O}\) is a bounded domain,
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u^0(t) - u^\epsilon(t)\|^p \right] \leq C(Q, T, \lambda, p, u_0, \alpha, \tilde{g}) (\epsilon^\frac{p}{2} + \epsilon^\frac{p}{12}), \\
\mathbb{E} \left[ \sup_{t \in [0,T]} \|u^0(t) - u^\epsilon(t)\|^p \right] \leq C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g}) (\epsilon^\frac{p}{2} + \epsilon^\frac{p}{24}).
\end{align*}

The strong convergence of \(u^\epsilon\) in \((2.2)\) provides a systematic way to study the numerical schemes of the \$S\text{LogS} \$ equation in sections 3 and 4. We present a sketch of the proof of Lemma 2.1 in Appendix. The evolutions of the mass \(M(u) := \|u\|^2\) and the weighted mass \(M_\alpha(u) := \|u\|_{L^\alpha}^\alpha\) of \((2.3)\) are presented in Proposition 5.1 in Appendix. Moreover, when \(\tilde{g} = 1\) or \(\tilde{g}(x) = 1_{\mathbb{R}}\), it can be verified that there is no error between the masses \(M(u^0)\) and \(M(u^\epsilon)\). Below, we show that the regularized energy of \((2.3)\) is well-defined.
Lemma 2.2. Let Assumption 2.2 and the condition of Theorem 2.1 hold. Let $u^0$ and $u^\epsilon$ be the mild solutions of Eq. (1.1) and Eq. (2.3), respectively. The regularized energy is well-defined and satisfies that for $p \geq 1$, 
\[
\mathbb{E}\left[\sup_{t \in [0,T]} H^p_p(u^\epsilon(t))\right] \leq C(Q,T,\lambda,p,u_0,\alpha,\tilde{g}).
\]

Proof. Denote $\rho^\epsilon = |u^\epsilon|^2$ and $\rho^0 = |u^0|^2$. According to Theorem 2.1 and Lemma 2.1 applying the weighted interpolation inequality 
\[
\|v\|_{L^{2-2\eta}} \leq C\|\|v\|_{L^{\frac{2p}{2p-2\eta}}}\|v\|^{-1} \frac{2p-2\eta}{2p},
\]
for $\alpha > \frac{d\eta}{2-2\eta}, \alpha \in (0,1]$, we have that for small $\eta \in [0,1],$
\[
|F_\epsilon(\rho^\epsilon) - F_0(\rho^\epsilon)| \\
\leq \int_{\mathcal{Q}\cap(\rho^\epsilon \geq 1)} \int_0^{\rho^\epsilon} |f_\epsilon(s) - f_0(s)|dsdx + \int_{\mathcal{Q}\cap(\rho^\epsilon \leq 1)} \int_0^{\rho^\epsilon} |f_\epsilon(s) - f_0(s)|dsdx \\
\leq \int_{\mathcal{Q}} \int_0^{\rho^\epsilon} C(\epsilon + |s|)dsdx + \int_{\rho^\epsilon \leq 1} \int_0^{\rho^\epsilon} C\left(\frac{\epsilon}{\epsilon + s}\right)dsdx \\
\leq \int_{\mathcal{Q}} \int_0^{\rho^\epsilon} C\epsilon + |s|dsdx + C\int_{\rho^\epsilon \leq 1} \epsilon(\log(\epsilon + |s|) - \log(\epsilon))dsdx + C\epsilon \|u^\epsilon\|^2 \\
\leq C\epsilon \|u^\epsilon\|^2 + C\epsilon \|u^\epsilon\|_{L^{2+2\delta}}^{2+2\delta} + C\epsilon \eta |\log(\epsilon)| \|u^\epsilon\|_{L^{2-2\eta}}^{2-2\eta}.
\]

Using (2.4) with $\alpha > \frac{d\eta}{2-2\eta}, \alpha \in (0,1]$, the Gagliardo–Nirenberg interpolation inequality, 
\[
\|v\|_{L^{2+2\delta}} \leq C\|\nabla v\|_{L^{\frac{2\delta}{2\delta+d}}} \|v\|^{-1} \frac{2\delta}{2\delta+d}, \delta \in (0, \frac{2}{\max(d-2,0)})
\]
and the uniform boundedness of $u^\epsilon$ in $H^1$, we achieve that 
\[
\mathbb{E}\left[\sup_{t \in [0,T]} |F_\epsilon(\rho^\epsilon) - F(\rho^\epsilon)|\right] \leq C(Q,T,\lambda,p,u_0,\tilde{g}).
\]
Therefore, it suffices to show the uniform boundedness of $F(|u^\epsilon|^2)$. Adopting (2.5) and (2.4), we get 
\[
F(\rho^\epsilon) \leq \|u^\epsilon\|^2 + \int_{|u^\epsilon| \geq 1} |u^\epsilon|^{2+2\delta}dx + \int_{|u^\epsilon| < 1} |u^\epsilon|^{2-2\eta}dx \\
\leq \|u^\epsilon\|^2 + C\|\nabla u^\epsilon\|_{L^{d\delta}}^{d\delta} \|u^\epsilon\|^{2+2\delta-d\delta} + C\|u^\epsilon\|_{L^{\frac{2\delta}{2\delta+d}}} \|u^\epsilon\|^{2-2\eta}.
\]
Using the uniform boundedness of $u^\epsilon$ in $H^1 \cap L^{2\alpha}$ and Young’s inequality, we obtain that 
\[
\mathbb{E}\left[\sup_{t \in [0,T]} |F(\rho^\epsilon)|\right] \leq C(Q,T,\lambda,p,u_0,\alpha,\tilde{g}).
\]
We complete the proof by combining the above estimates. \qed
Remark 2.1. The conditions (A1) and (A2) in Assumption 2.3 are not necessary when proving the boundedness of the regularized energy for Eq. (2.3). However, it is crucial to analyze the strong convergence of numerical methods due to loss of regularity in both time and space of the solutions to both Eq. (1.1) and Eq. (2.3).

Corollary 2.1. Let Assumption 2.2 and the condition of Theorem 2.1 hold. Let $u^0$ and $u^*$ be the mild solutions of Eq. (1.1) and Eq. (2.3), respectively. Then the regularized energy is strongly convergent to the entropy of Eq. (1.1). Furthermore, for $p \geq 1$ and $\delta \in \left(0, \max(\frac{2}{2-d-2p}, 1)\right)$, there exist $C(Q, T, \lambda, p, u_0, \delta, \tilde{\gamma}) > 0$ and $C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{\gamma}) > 0$ such that when $\mathcal{O}$ is a bounded domain,

$$\sup_{t \in [0, T]} \mathbb{E}\left[|F_\epsilon(r'(t)) - F(\rho^0(t))|^p\right] \leq C(Q, T, \lambda, p, u_0, \delta, \tilde{\gamma}) (\epsilon^{\frac{\alpha}{2p}} + \epsilon^\frac{\alpha}{2})$$

and when $\mathcal{O} = \mathbb{R}^d$,

$$\sup_{t \in [0, T]} \mathbb{E}\left[|F_\epsilon(r'(t)) - F(\rho^0(t))|^p\right] \leq C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{\gamma}) (\epsilon^{\frac{\alpha}{2p}} + \epsilon^\frac{\alpha}{2}).$$

Proof. Similar arguments as in the proof of Lemma 2.2 yield that

$$|F_\epsilon(r') - F(\rho^0)| = |F_\epsilon(r') - F(\rho^0)| + |F_\epsilon(r') - F(\rho^0)|$$

$$\leq |F_\epsilon(r') - F(\rho^0)| + |C\epsilon u'_{n}^2| + |C\epsilon^2 u'_{n}^2|_{L_{2+2\delta}^2} + C\epsilon^{\nu}(|\log(\epsilon)|)^\nu |u'|_{L_{2-2\theta}}^{2-2\theta},$$

where $\eta > 0$ is small enough and $\delta(d-2) \leq 2$. Notice that

$$F(\rho^0) - F(\rho^0) = \int_{|u'_{n} > |u'_{\ast}|} (\log(|u'|^2) - \log(|u'|^2)) |u'|^2 dx + \int_{|u'_{n} > |u'_{\ast}|} \log(|u'|^2) (|u'|^2 - |u'|^2) dx$$

$$+ \int_{|u'_{n} < |u'_{\ast}|} (\log(|u'|^2) - \log(|u'|^2)) |u'|^2 dx + \int_{|u'_{n} < |u'_{\ast}|} \log(|u'|^2) (|u'|^2 - |u'|^2) dx$$

$$+ ||u'|^2 - ||u'|^2.$$

The property of the logarithmic function and Hölder’s inequality yield that

$$|F_\epsilon(r') - F(\rho^0)| \leq ||u'|^2 - ||u'|^2| + C||u'_{n} - u'| (||u'_{n} + u'| + ||u'_{n}||_{L_{2+2\delta}} + ||u'_{n}||_{L_{2-2\theta}}^{1+\delta} + ||u'_{n}||_{L_{2-2\theta}}^{1-\frac{n}{2}}).$$

By making use of (2.3) and (2.4), together with a priori estimate of $u'$ in $H^1$ and Lemma 2.1, we complete the proof. 

Since the regularized energy of Eq. (2.3) is well-defined, we further show that Eq. (2.3) is an infinite-dimensional stochastic Hamiltonian system whose phase flow preserves stochastic symplectic structure via a standard argument in [8]. The key of the proof lies on the fact that (2.3) can be rewritten into an infinite-dimensional stochastic Hamiltonian system

$$dP^\epsilon = -\frac{\delta H_\epsilon}{\delta Q^\epsilon} dt + \frac{\delta H_{Sto}}{\delta Q^\epsilon} dW(t), \quad dQ^\epsilon = \frac{\delta H_\epsilon}{\delta P^\epsilon} dt + \frac{\delta H_{Sto}}{\delta P^\epsilon} dW(t)$$

with $P^\epsilon$ (resp., $Q^\epsilon$) being the real (resp., imaginary) part of the solution $u'$. Here $H_{Sto} := \int_0^t \int_0^{(r'(s))^2 + (Q(s))^2} g(s) ds dx$ when $\tilde{g}(x) = ig(|x|^2)x$. For the additive noise case, i.e., $\tilde{g} = 1$, the proof is similar.
Proposition 2.1. Let Assumption 2.2 and the condition of Theorem 2.1 hold. Assume in addition that \( \{W(t)\}_{t \geq 0} \) is \( \mathbb{H} \)-valued, \( \tilde{g} = 1 \) or that \( \{W(t)\}_{t \geq 0} \) is \( L^2(Q; \mathbb{R}) \)-valued, \( \tilde{g}(x) = \text{ig}(|x|^2)x \). The phase flow of Eq. (2.2) preserves the stochastic symplectic structure, i.e.,

\[
\omega(t) = \int_{\mathcal{O}} dP^t(\omega) \wedge dQ^t(\omega) dx = \int_{\mathcal{O}} dP^t(0) \wedge dQ^t(0) dx = \tilde{\omega}(0), \text{ a.s.}
\]

Here, \( \tilde{\omega} \) is the differential 2-form integrated over the space and \( d \) denotes differential in the phase space taken with respect to the initial data. Although the convergence result in Corollary 2.1 gives the approximation error between the entropy of \( u^\epsilon \) and that of \( u \), it is still unknown what is the error between the regularization energy and the original energy due to loss of regularity in both time and space for Eq. (1.1) and Eq. (2.3). To overcome this issue, the weak convergence analysis on the regularization energy is introduced.

Proposition 2.2. Let Assumption 2.2 and the condition of Theorem 2.1 hold. Let \( u^0 \) and \( u^\epsilon \) be the mild solutions of Eq. (1.1) and Eq. (2.3), respectively. Assume in addition that \( \tilde{g}(x) = ix \) and that

\[
|1 - \frac{\partial f_{\epsilon}}{\partial x}(x)| \leq C \left( \frac{1 + \epsilon x + x^2}{(1 + \epsilon x)(1 + \epsilon x)} \right) \quad \forall \ x > 0.
\]

Then the regularized energy is convergent to the energy of Eq. (1.1). Furthermore, for \( p \geq 1 \) and \( \delta \in \left( 0, \max \left( \frac{2}{\max(d_1, d_2, \sqrt{d})}, 1 \right) \right) \), there exist \( C(Q,T,\lambda,p,u_0,\delta) > 0 \) and \( C(Q,T,\lambda,p,u_0,\alpha,\delta) > 0 \) such that for \( t \in [0,T] \), when \( \mathcal{O} \) is a bounded domain,

\[
\mathbb{E} \left[ H_{\epsilon}(u^\epsilon(t)) - H(u^0(t)) \right] \leq C(Q,T,\lambda,p,u_0,\delta)(\epsilon^{\frac{1}{2}} + \epsilon^\delta),
\]

and when \( \mathcal{O} = \mathbb{R}^d \),

\[
\mathbb{E} \left[ H_{\epsilon}(u^\epsilon(t)) - H(u^0(t)) \right] \leq C(Q,T,\lambda,p,u_0,\alpha,\delta)(\epsilon^{\frac{1}{2n+2}} + \epsilon^\delta).
\]

Proof. Applying the Itô formula to \( H_{\epsilon}(u^\epsilon(t)) \) yields that

\[
H_{\epsilon}(u^\epsilon(t)) = H_{\epsilon}(u_0) + \int_0^t \langle -\Delta u^\epsilon, \tilde{g}(u^\epsilon) dW(s) \rangle + \int_0^t \frac{1}{2} \sum_{i \in \mathbb{N}^+} \| \tilde{g}(u^\epsilon) \nabla Q^\frac{1}{2} e_i \|^2 ds
\]

\[- \lambda \int_0^t \left\langle f_{\epsilon}(\|u^\epsilon\|^2) u^\epsilon, \tilde{g}(u^\epsilon) dW(s) \right\rangle
\]

\[- \frac{1}{2} \lambda \int_0^t \sum_{i \in \mathbb{N}^+} \left\langle 2\text{Re}(\tilde{u}^\epsilon \tilde{g}(u^\epsilon) Q^\frac{1}{2} e_i) \frac{\partial f_{\epsilon}}{\partial x}(\|u^\epsilon\|^2) u^\epsilon, \tilde{g}(u^\epsilon) Q^\frac{1}{2} e_i \right\rangle ds.
\]

Then subtracting \( H_{\epsilon}(u^\epsilon(t)) \) from \( H(u^0(t)) \) and taking expectation, we get

\[
\mathbb{E} \left[ H(u^0(t)) - H_{\epsilon}(u^\epsilon(t)) \right] = \mathbb{E} \left[ H(u^0(t)) - H_{\epsilon}(u^0(t)) \right]
\]

\[+ \int_0^t \frac{1}{2} \sum_{i \in \mathbb{N}^+} \mathbb{E} \left( \| \tilde{g}(u^0) \nabla Q^\frac{1}{2} e_i \|^2 - \| \tilde{g}(u^\epsilon) \nabla Q^\frac{1}{2} e_i \|^2 \right) ds
\]

\[- \frac{1}{2} \lambda \int_0^t \sum_{i \in \mathbb{N}^+} \mathbb{E} \left[ \left\langle 2\text{Re}(\tilde{u}^\epsilon \tilde{g}(u^\epsilon) Q^\frac{1}{2} e_i) \frac{\partial f_{\epsilon}}{\partial x}(\|u^\epsilon\|^2) u^\epsilon, \tilde{g}(u^\epsilon) Q^\frac{1}{2} e_i \right\rangle
\]

\[- \left\langle 2\text{Re}(\tilde{u}^\epsilon \tilde{g}(u^\epsilon) Q^\frac{1}{2} e_i) \frac{\partial f_{\epsilon}}{\partial x}(\|u^\epsilon\|^2) u^\epsilon, \tilde{g}(u^\epsilon) Q^\frac{1}{2} e_i \right\rangle \right] ds
\]

\[= \frac{\lambda}{2} \mathbb{E} [F_{\epsilon}(\rho^\epsilon(t)) - F(\rho^0(t))] + I_1 + I_2.
\]
The Hölder inequality yields
\[ |I_1| \leq \int_0^t \frac{1}{2} \sum_{i \in \mathbb{N}^+} E \left[ \int_{\mathbb{O}} |\nabla Q^\frac{1}{2} e_i|^2 (|u' - u_0|^2) dx \right] ds \]
\[ \leq \frac{1}{2} \sum_{i \in \mathbb{N}^+} \|\nabla Q^\frac{1}{2} e_i\|_L^2 T \sup_{t \in [0,T]} E \left[ \|u'(t) - u_0(t)\| \|u(t) + u_0(t)\| \right]. \]

The fact that \( \tilde{g}(x) = \text{ix} \) and (2.4) imply that for \( \delta \in (0, 1], \alpha > \frac{d}{2 - 2\gamma}, \alpha \in (0, 1], \)
\[ |J_2| \leq \int_0^t \sum_{i \in \mathbb{N}^+} E \left[ \int_{\mathbb{O}} |\text{Im}(Q^\frac{1}{2} e_i)|^2 (|u_0|)^2 - \frac{\partial f_0}{\partial x} (|u'(s)|^2) |u'(s)|^4) dx \right] ds \]
\[ \leq CT|\lambda| \sum_{i \in \mathbb{N}^+} \|Q^\frac{1}{2} e_i\|_L^2 \sup_{t \in [0,T]} E \left[ \|u_0(t) - u_0'(t)\| \|u'(t) + u_0(t)\| \right] \]
\[ + CT|\lambda| \sum_{i \in \mathbb{N}^+} \|Q^\frac{1}{2} e_i\|_L^2 \epsilon^{\delta} \left( 1 + \sup_{t \in [0,T]} E \left[ |u'(t)|^{2+2\delta} \right] \right) \]
\[ + CT|\lambda| \sum_{i \in \mathbb{N}^+} \|Q^\frac{1}{2} e_i\|_L^2 \epsilon^{\eta} \sup_{t \in [0,T]} E \left[ \|u'(t)\|_{L_{2\gamma}} \right]. \]

Combining the above estimates with (2.4), Lemma 2.1 and Corollary 2.1, we complete the proof.

\[ \square \]

3. Structure-preserving regularized Lie–Trotter type splitting method

In this section, we propose the regularized Lie–Trotter type splitting methods based on Eq. (2.3), and investigate their strong convergence rates and convergence rates of the regularized entropy.

When considering Eq. (2.3) driven by additive noise, i.e., \( \tilde{g} = 1 \), we apply the following decomposition
\[ dv = i\Delta v dt + dW(t), \quad v = v_0, \]
\[ dw = i\lambda f_0(|w|^2) dt, \quad w = w_0, \]
whose flow satisfies
\[ v_{A,\mathcal{F}_0}(v_0, t) = \Phi_{A,\mathcal{F}_0}^t (v_0) = e^{i\Delta t} v_0 + \int_0^t e^{i\Delta (t-s)} dW(s), \]
\[ w_T(w_0, t) = \Phi_f^t (w_0) = w_0 e^{i\lambda f_0(|w_0|^2) t} \]
with \( e^{i\Delta t} \) being the \( C_0 \)-group generated by \( i\Delta \), where \( v_0, w_0 \) are \( \mathcal{F}_0 \)-measurable. Denote the time step size by \( \tau \) such that \( t_k = k\tau, k = 0, 1, \ldots, N, \) and \( T = N\tau. \) The splitting scheme for the additive noise case is defined by
\[ u_{k+1}^e = \Phi_{\mathcal{F}_k}^t (u_k^e) = \Phi_{\mathcal{A},\mathcal{F}_k}^t (\Phi_{\mathcal{F}_k}^t (u_k^e)), k \geq 0, \quad u_0^e = u_0, \quad \tau > 0. \]

For the multiplicative noise case, i.e., \( \tilde{g}(x) = i g(|x|^2) x \), we proposed two different splitting strategies. When \( W(t) \) is \( \mathbb{H} \)-valued, we use
\[ dv = i\Delta v dt + i g(|v|^2) v \cdot dW(t), \quad v = v_0, \]
\[ dw = i\lambda f_0(|w|^2) v dt, \quad w = w_0. \]
The flow of the first subsystem is
\[ v_{M,\mathcal{F}_0}(v_0, t) = \Phi_{M,\mathcal{F}_0}^t (v_0) \]
For the purpose of performing the numerical method, one may use the exponential Euler method (3.2) the following proposition is omitted since it is similar to that of the exact solution of (2.3).

Following the above strategies, one may construct different kinds of splitting methods by changing (3.4) where another splitting strategy, that is, integral. Denote the flow of the corresponding subsystems by

\[ W_t(s) := W(t_k + s) - W(t_k) \text{ and } v_0 \text{ is } \mathcal{F}_{t_k}-\text{measurable.} \]  

One could also use other discretization, like \( \int_{0}^{\tau} e^{\Delta s} g(|v_0|^2) v_0 d\overline{W}_k(s) \), for the stochastic integral.

For \( \varepsilon \) driven by conservative multiplicative noise, i.e., \( W(t) \) is \( L^2(\Omega; \mathbb{R}) \)-valued, we have another splitting strategy, that is,

\[
\begin{align*}
 dv &= i\lambda v dt, \quad v = v_0, \\
 dw &= i\lambda f_v(|w|^2) w dt + ig(|w|^2) W(t), \quad w = w_0.
\end{align*}
\]

We also remark that in this case, the stochastic integral \( ig(|w|^2) W(t) \) is the Itô integral. Denote the flow of the corresponding subsystems by

\[
\begin{align*}
 v_P(t) &= \Phi_D^\tau(v_0) = e^{i\lambda t} v_0, \\
 w_{t+\tau}(w_0, t) &= \Phi_D^\tau(\mathcal{F}_{t+\tau} u_k)), \quad k \geq 0, \quad u'_0 = u_0, \quad \tau > 0.
\end{align*}
\]

Following the above strategies, one may construct different kinds of splitting methods by changing the order of the splitting or making a composition of different subsystems.

### 3.1. Structure-preserving properties of regularized Lie–Trotter type splitting method.

By making use of the properties of subsystems in the splitting methods, we present the following structure-preserving properties for the proposed numerical methods. The proof of the following propositions is omitted since it is similar to that of the exact solution of (3.4).

**Proposition 3.1.** Let Assumption (3.4) and the condition of Theorem (2.1) hold. Assume in addition that \( \{W(t)\}_{t \geq 0} \) is \( \mathcal{H} \)-valued, \( \overline{g} = 1 \) or that \( \{W(t)\}_{t \geq 0} \) is \( L^2(\Omega; \mathbb{R}) \)-valued, \( \overline{g}(x) = g(x) \).
\[ i\varrho(|x|^{2})x. \] Then the phase flows of the splitting methods (3.1), (3.2) and (3.3) preserve the symplectic structure, i.e.,
\[ \bar{\varphi}_{k+1} := \int_{O} dP_{k+1}^{r} \wedge dQ_{k+1}^{r} dx = \int_{O} dP_{k}^{r} \wedge dQ_{k}^{r} dx = \varphi_{k}, \ a.s., \]
with \( P_{k}^{r} \) (resp., \( Q_{k}^{r} \)) being the real (resp., imaginary) part of the solution \( u_{k}^{r} \).

**Proposition 3.2.** Let Assumption \( \mathcal{A}2 \) and the condition of Theorem \( \mathcal{A}1 \) hold. Assume that \( \bar{g}(x) = ix \) or \( \bar{g}(x) = 1. \) Then the splitting methods (3.1), (3.2) and (3.3) preserve the evolution law of the mass of the exact solution \( u_{k}^{r} \), i.e.,
\[ \mathbb{E}\left[ M(u_{k+1}) \right] = \mathbb{E}\left[ M(u_{k}) \right] + \tau \sum_{i \in \mathbb{N}^{+}} \| Q_{i}^{2} \|^{2} \| \chi(\bar{g} = 1) \|, \]
where \( \chi(\bar{g} = 1) = 1 \) for the additive noise case and \( \chi(\bar{g} = 1) = 0 \) for the multiplicative noise case.

### 3.2. Strong convergence rate of Lie–Trotter type splitting method for RSlogS equation

In the following, we give the strong convergence analysis of the proposed splitting methods.

**Proposition 3.3.** Let Assumption \( \mathcal{A}2 \) and the condition of Theorem \( \mathcal{A}1 \) hold. Let \( \bar{g} = 1. \) Assume in addition that \( f_{t} \) satisfies
\[ |f_{t}(|x|^{2})x - f_{t}(|y|^{2})y| \leq C(1 + |\log(\epsilon)|)|x - y|. \]
Then the numerical solution of the splitting method (3.1) is strongly convergent to the exact one of Eq. (2.3). Moreover, for \( p \geq 2, \) there exists \( C(Q, T, \lambda, p, u_{0}) > 0 \) such that
\[ \sup_{k \leq N} \| u_{k} - u^{\tau} \|_{L^{p}(0, \tau)} \leq C(Q, T, \lambda, p, u_{0})(1 + |\log(\epsilon)|) \tau^{\frac{1}{2}}. \]

**Proof.** For convenience, we illustrate the procedures in the case that \( p = 2. \) For general \( p \geq 2, \) the proof is similar. Assume that \( v \in H^{1} \) is \( F_{\text{inv}} \)-measurable and has any finite moment. Denote the exact flow of (2.3) on \( [t_{k}, t_{k+1}] \) by \( \Psi_{k}^{t}, t \in [0, \tau] \) and the numerical flow by \( \Psi_{k}^{t}, t \in [0, \tau]. \) Then the equation of \( \Psi_{k}^{t}(v) \) and \( \Psi_{k}^{t}(v) = \Phi_{k, t_{k}}(\Phi_{k}^{t}(v)) \) on a small interval \( [0, \tau] \) can be rewritten as
\[ d\Psi_{k}^{t}(v) = i\Delta \Psi_{k}^{t}(v) dt + d\tilde{W}_{k}(t) + i\lambda f_{t}(v) \Psi_{k}^{t}(v) dt, \]
\[ d\Phi_{k}^{t}(v) = i\Delta \Phi_{k}^{t}(v) dt + d\tilde{W}_{k}(t) + i\lambda e^{i\Delta t}[f_{t}(v)\Phi_{k}^{t}(v)] dt. \]
Let \( \varepsilon_{k}^{t} = \Psi_{k}^{t}(v) - \Phi_{k}^{t}(v). \) Then it holds that
\[ d\varepsilon_{k}^{t}(v) = i\Delta \varepsilon_{k}^{t}(v) dt + i\lambda \left( f_{t}(\Psi_{k}^{t}(v)) - e^{i\Delta t}[f_{t}(\Phi_{k}^{t}(v))\Phi_{k}^{t}(v)] \right) dt. \]
Applying the chain rule, integration by parts and (A2), we achieve that
\[ \frac{d}{dt}\| \varepsilon_{k}^{t}(v) \|^{2} \leq 2C\| \varepsilon_{k}^{t}(v) \|^{2} + 2C\| \varepsilon_{k}^{t}(v) \| \left\| f_{t}(\Phi_{k}^{t}(v)) \Psi_{k}^{t}(v) - f_{t}(\Phi_{k}^{t}(v)) \Phi_{k}^{t}(v) \right\| \]
\[ + 2C\| \varepsilon_{k}^{t}(v) \| \left\| (I - e^{i\Delta t})[f_{t}(\Phi_{k}^{t}(v))\Phi_{k}^{t}(v)] \right\| \]
\[ =: 2C\| \varepsilon_{k}^{t}(v) \|^{2} + 2C\| \varepsilon_{k}^{t}(v) \| I_{1} + 2C\| \varepsilon_{k}^{t}(v) \| I_{2}. \]
This leads to \( \frac{d}{dt}\| \varepsilon_{k}^{t}(v) \| \leq C\left( \| \varepsilon_{k}^{t}(v) \| + I_{1} + I_{2} \right). \) The property (A3) of \( f_{t}, \) together with the definition of \( \Phi_{k} \) and \( \| w - e^{i\Delta t}w \| \leq C\sqrt{T}\| w \|_{H^{1}}, \) yields that
\[ I_{1} \leq C(1 + |\log(\epsilon)|)\| \Phi_{k, t_{k}}(\Phi_{k}^{t}(v)) - \Phi_{k}^{t}(v) \| \]
\[ \leq C(|\log(\epsilon)| + 1)T\| w \|_{H^{1}} + C(1 + |\log(\epsilon)|) \int_{0}^{\tau} e^{i\Delta(t-s)}d\tilde{W}(s). \]
In the last inequality, we use the following estimate
\[ \left\| \Phi_t^s(v) \right\|^2_{H^1} \leq \left\| |v| \right\|^2_{H^1} + \left\| |v| \right\|^2 + 4\lambda^2 s^2 \left| \frac{\partial f}{\partial x}(|v|^2) Re(v \nabla v) \right|^2 \]
\[ \leq \left\| |v| \right\|^2_{H^1} + \left\| |v| \right\|^2 + 4C|\lambda|s^2 \left\| \nabla v \right\|^2 \leq \left\| |v| \right\|^2_{H^1} + Cs^2 \left\| v \right\|^2_{H^1}, \ s \in [0, \tau]. \]
For the term \( I_2 \), similar arguments lead to
\[ I_2 \leq C\sqrt{t} \left\| f_c(|\Phi_t^s(v)|^2) \Phi_t^s(v) \right\|_{H^1} \leq C\sqrt{t} \left( (1 + |\log(\epsilon)|)(1 + Ct)\left\| v \right\|_{H^1} \right). \]
Combining the above estimates on \( I_1 \) and \( I_2 \), we have that
\[ \left\| \epsilon_k^s(v) \right\| \leq C(1 + |\log(\epsilon)| + 1) \int_0^t \left( \sqrt{r} \left\| v \right\|_{H^1} + \left\| \int_0^t e^{i\Delta(s-r)} d\tilde{W}_k(r) \right\| \right) ds. \]
This, together with the Burkholder inequality, implies that
\[ \left\| \epsilon_k^s(v) \right\|_{L^2(\Omega; H^1)} \leq C(1 + |\log(\epsilon)|)(1 + \left\| v \right\|_{L^p(\Omega; H^1)}). \]
Next we show the stability of \( \Phi_t^s \) to get the global error estimate. Direct calculations, together with the chain rule and (A2), yield that
\[ \left\| \Phi_t^s(v) - \Phi_t^s(w) \right\|^2 \]
\[ = \left\| v - w \right\|^2 + 2 \int_0^t \langle \Phi_t^s(v) - \Phi_t^s(w), f_c(|\Phi_t^s(v)|^2) \Phi_t^s(v) - f_c(|\Phi_t^s(w)|^2) \Phi_t^s(w) \rangle ds \]
\[ \leq \left\| v - w \right\|^2 + 4 \int_0^t \left| \Phi_t^s(v) - \Phi_t^s(w) \right|^2 ds. \]
Therefore, it holds that \( \left\| \Phi_t^s(v) - \Phi_t^s(w) \right\|^2 \leq \exp(Ct) \left\| v - w \right\|^2. \) This, together with the fact that \( g = 1, \) implies \( \left\| \Phi_t^s(v) - \Phi_t^s(w) \right\| \leq \exp(Ct) \left\| v - w \right\|. \) The similar arguments lead to the stability estimate of \( \Phi_t^s, \)
\[ \left\| \Phi_t^s(v) \right\|_{L^p(\Omega)} \leq \exp(Ct)(1 + \left\| v \right\|^2_{L^2p(\Omega; H^1)}), \]
\[ \left\| \Phi_t^s(v) \right\|_{H^1} \leq \exp(Ct)(1 + \left\| v \right\|^2_{L^2p(\Omega; H^1)}). \]
Taking \( p \)-moment and using stability estimates of \( \Phi_t \) and \( \Psi_t \), and decomposing the global error as
\[ \left\| u^s(t_N) - u_N^s \right\| \leq \sum_{k=0}^{N-1} \left\| \prod_{j=k+1}^{N-1} \left( \frac{\Psi_j^s - \Phi_j^s}{\Phi_j^s(\sqrt{u_0})} \right) \right\| \left( \prod_{j=0}^{k-1} \Phi_j^s(\sqrt{u_0}) \right) \cdot \left( 1 + |\log(\epsilon)| + \left\| \Phi_j^s(\sqrt{u_0}) \right\|_{H^1} \right), \]
we complete the proof. \( \square \)

The method of proving the strong convergence in additive noise case can not be directly used to the multiplicative noise case due to the existence of diffusion terms. Below we present the convergence analysis of the proposed methods in the multiplicative noise case.

**Proposition 3.4.** Let Assumption \( 2.2 \) and the condition of Theorem \( 2.1 \) hold. Let \( \tilde{g}(x) = ig(|x|^2)x. \) Assume in addition that \( f_c \) satisfies (3.5). Then the numerical solution of (3.2) is strongly convergent to the exact solution of Eq. (3.5). Moreover, for \( p \geq 2, \) there exists \( C(Q, T, \lambda, p, u_0, \tilde{g}) > 0 \) such that
\[ \sup_{k \leq N} \left\| u_k^s - u^s(t_k) \right\|_{L^p(\Omega; H^1)} \leq C(Q, T, \lambda, p, u_0, \tilde{g})(1 + |\log(\epsilon)|) T^{\frac{p}{2}}. \]
Proof. We show the details of the proof in the case that \( p = 2 \). Assume that \( v, w \in \mathbb{H}^1 \) are \( \mathcal{F}_t \)-measurable and have any finite moment. Denote the exact flow of (2.3) on \( \{0 \leq s \leq t \} \). Fix \( s \in [0, \tau] \), define the auxiliary flow \( \Phi_k^v(s) = \Phi_{s, k}^v(\Phi_j^v(w)) \). Then the equations of \( \Psi_k^v(t) \) and \( \Phi_k^v(w) \) can be rewritten as

\[
\begin{align*}
\frac{d}{dt} \Psi_k^v(t) &= \text{i} \Delta \Psi_k^v(t) + \text{Im} \left( Q_k^v \right) \Psi_k^v(t) \, dW_k(t) \\
&\quad - \text{i} g(\|\Phi_k^v(t)\|^2) \Psi_k^v(t) \, d\lambda(t) \, dt - \text{i} g'(\|\Phi_k^v(t)\|^2) \Psi_k^v(t) \, d\lambda(t) \, dt
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} \Phi_k^v(w) &= \text{i} \Delta \Phi_k^v(w) + \text{Im} \left( Q_k^v \right) \Phi_k^v(w) \, dW_k(t) \\
&\quad - \text{i} g'(\|\Phi_k^v(w)\|^2) \Phi_k^v(w) \, d\lambda(t) \, dt - \text{i} g(\|\Phi_k^v(w)\|^2) \Phi_k^v(w) \, d\lambda(t) \, dt
\end{align*}
\]

Using the Itô formula, letting \( v = u^v(t_k) \) and \( w = u^w_k \), taking expectation and using the conditions (2.1) and (2.2) on \( g \), as well as the condition (A2) on \( f_c \), we have that

\[
\mathbb{E} \left[ \left\| \Psi_k^v(t) - \Phi_k^v(w) \right\|^2 \right] \leq \mathbb{E} \left[ \left\| v - w \right\|^2 \right] + C \int_0^t \mathbb{E} \left[ \left\| \Psi_k^v(t) - \Phi_k^v(w) \right\|^2 \right] \, ds + C \int_0^t \mathbb{E} \left[ \left\| \Phi_k^v(t) - \Phi_k^v(w) \right\|^2 \right] \, ds
\]

Using the conditions (2.1) and (2.2) on \( g \) and Itô’s formula, as well as \( \|v - e^{i\Delta t}w\| \leq C\sqrt{t}\|v\|_{\mathbb{H}^1} \), we arrive at

\[
\mathbb{E} \left[ \left\| \Phi_k^v(t) - \Phi_k^v(w) \right\|^2 \right] \leq C \mathbb{E} \left[ \left\| \Phi^v_j(w) - \Phi^v_j(w) \right\|^2 \right] \leq C\tau^2 \sup_{s \in [0, t]} \left( 1 + \frac{1}{2} \log(\epsilon) \right) \mathbb{E} \left[ \left\| \Phi^v_j(w) \right\|^2 \right].
\]

It suffices to estimate

\[
\Phi_{s, k}^v(\Phi_j^v(w)) - \Phi_j^v(w) = (e^{i\Delta t} - I) \Phi_j^v(w) + \int_0^{s} e^{i\Delta (s-r)} II_{mod, k}(\Phi_j^v(w)) \, dr
\]

\[
\quad + \int_0^{s} e^{i\Delta (s-r)} g(\|\Phi_{s, k}^v(\Phi_j^v(w))\|^2) \Phi_{s, k}^v(\Phi_j^v(w)) \, dW_k(r)
\]

for \( s \in [0, \tau] \), where

\[
II_{mod, k}(v_0) := -\frac{1}{2} \sum_{i \in \mathbb{N}^+} \left| Q_k^v \right|^2 g(\|\Phi_{s, k}^v(v_0)\|^2) \Phi_{s, k}^v(v_0)
\]

\[
- \text{i} \sum_{i \in \mathbb{N}^+} \text{Im} \left( Q_k^v \right) g(\|\Phi_{s, k}^v(v_0)\|^2) \Phi_{s, k}^v(v_0) \Phi_{s, k}^v(v_0) \Phi_{s, k}^v(v_0),
\]

for \( s \in [0, \tau] \).
for any $\mathcal{F}_k$-measurable function $v_0$. By taking second moment and using the continuity estimate of $e^{\Delta t}$ and the boundedness of the flow $\Phi_{M,F_k}$, i.e. $E[|\Phi_{M,F_k}(v_0)|^2] \leq C|v_0|^2$, we get

$$E\left[|\Phi_{M,F_k}(\Phi^r_{f}(w)) - \Phi^r_{f}(w)|^2\right]$$

$$\leq 2C\tau E\left[|\Phi^r_{f}(w)|^2\right] + 2E\left[\int_0^\tau |II_{mod,k}(\Phi^r_{f}(w))|^2 dr\right]$$

$$+ 2E\left[\int_0^\tau e^{\Delta(s)}g(|\Phi^r_{M,F_k}(\Phi^r_{f}(w))|^2)\Phi^r_{M,F_k}(\Phi^r_{f}(w))dW_k(r)\right]^2$$

$$\leq C\tau \left(1 + E[|\Phi^r_{f}(w)|^2]\right).$$

Using the Gronwall inequality and similar arguments in proving the estimates of $I_1$ and $I_2$ in the proof of Proposition 3.3, we have

$$E\left[|\Psi^r_{f}(v) - \Phi^r_{f}(v)|^2\right] \leq e^{C\tau}E\left[|v - w|^2\right] + C\tau^2(1 + |\log(\epsilon)|^2) + E\left[|w|^2\right].$$

Based on the definition of $\Phi^r_{f}$ and $\Phi^r_{k}$, the stability estimate in $\mathbb{H}^1$

$$\|\Phi^r_{f}(w)\|_{\mathbb{H}^1} \leq \|w\|_{\mathbb{H}^1} + C\tau^2\|w\|_{\mathbb{H}^1},$$

$$E[\|\Phi^r_{k}(w)\|_{\mathbb{H}^1}] \leq \exp(C\tau)E\left[\|w\|_{\mathbb{H}^1} + C\tau^2\|w\|_{\mathbb{H}^1}\right]$$

can be shown. Taking $t = \tau$, and using the a priori estimate of $|\Phi^r_{f}(w)|_{\mathbb{H}^1}$ and $|\Phi^r_{f}(w)|_{\mathbb{H}^1}$, we have

$$E\left[\|u^r(t_k) - u^r_{k+1}\|^2\right] \leq e^{C\tau}\left(E\left[\|u^r(t_k) - (u^r_{k})\|^2\right] + C\tau^2(1 + |\log(\epsilon)|^2)\right).$$

By repeating the above procedures, we conclude

$$E\left[\|u^r(t_k) - u^r_{k+1}\|^2\right] \leq C(Q,T,\lambda, u_0, p, \bar{g})(1 + |\log(\epsilon)|^2)\tau,$$

which completes the proof. \qed

**Proposition 3.5.** Under the condition of Proposition 3.4, the splitting scheme (3.3) is strongly convergent. Moreover, for $p \geq 2$, there exists $C(Q,T,\lambda, p, u_0, \bar{g}) > 0$ such that

$$\sup_{k \leq N} \|u^r_k - u^r(t_k)\|_{W_2(Q,T,\mathbb{H})} \leq C(Q,T,\lambda, p, u_0, \bar{g})(1 + |\log(\epsilon)|)^{+}\tau^{+}.$$

**Proof.** The proof is similar to that of Proposition 3.3. We present the details for $p = 2$. The main difference is that for an $\mathcal{F}_k$-measurable $w$, $\Phi^r_{f}(w)$ is replaced by $\tilde{\Phi}^r_{f}(w)$ which satisfies

$$d\tilde{\Phi}^r_{f}(w) = i\Delta \tilde{\Phi}^r_{f}(w) dt + i\lambda |\tilde{\Phi}^r_{f}(w)|^2 \tilde{\Phi}^r_{f}(w) dW_k(t) + e^{\Delta t} II_{mod}(\Phi^r_{f}(w)) dt$$

$$+ i\epsilon |\tilde{\Phi}^r_{f}(w)|^2 \tilde{\Phi}^r_{f}(w) dt,$$

where

$$II_{mod}(\Phi^r_{f}(w)) := -\frac{1}{2} \left(g(|\tilde{\Phi}^r_{f}(w)|^2) + \sum_{i \in \mathbb{N}^+} |Q_i^\frac{1}{2} e_i|^2\right)$$

$$- i\epsilon g(|\tilde{\Phi}^r_{f}(w)|^2) |\tilde{\Phi}^r_{f}(w)|^2 \tilde{\Phi}^r_{f}(w) \sum_{i \in \mathbb{N}^+} \text{Im}(Q_i^\frac{1}{2} e_i) Q_i^\frac{1}{2} e_i.$$
for any \( q \geq 2 \), via similar steps in the proof of Proposition 3.4. Using the Itô formula and letting \( v = u'(t_k) \) and \( w = u_k' \), then taking expectation and exploiting the conditions (2.1) and (2.2) on \( g \), as well as the condition (\( \Delta_2 \)) on \( f \), yield that

\[
\begin{align*}
\mathbb{E} \left[ \| \Phi^t_k(v) - \tilde{\Phi}^t_k(w) \|^2 \right] \\
\leq \mathbb{E} \left[ \| v - w \|^2 \right] + CE \int_0^t \mathbb{E} \left[ \| \tilde{\Phi}^t_k(w) - \Phi^t_k(w) \|^2 \right] ds + C \int_0^t \mathbb{E} \left[ \| \Psi^t_k(v) - \tilde{\Phi}^t_k(w) \|^2 \right] ds \\
+ C \int_0^t \mathbb{E} \left[ \| f_s(\tilde{\Phi}^t_k(w)) - e^{1t} f_s(\|\Phi^t_k(w)\|^2)\Phi^t_k(w) \|^2 \right] ds \\
+ C \int_0^t \mathbb{E} \left[ \| I_{mod}(\tilde{\Phi}^t_k(w)) - e^{1t} I_{mod}(\Phi^t_k(w)) \|^2 \right] ds.
\end{align*}
\]

Similar to the proof of Proposition 3.4, it suffices to estimate \( \tilde{\Phi}^t_{M,T,k}(\Phi^t_f(w)) - \Phi^t_f(w) \) for and \( \Phi^t_f(w) - \Phi^t_f(w) \) for \( t \in [0, \tau] \). By the definition of \( \Phi_f \) and (A1), \( \mathbb{E} \left[ \| \Phi^t_f(w) - \Phi^t_f(w) \|^2 \right] \leq C\tau^2 (1 + |log(\epsilon)|^2)\|\Phi^t_f(w)\|^2 \). The definition of \( \tilde{\Phi}^t_{M,T,k} \) yields that

\[
\begin{align*}
\tilde{\Phi}^t_{M,T,k}(\Phi^t_f(w)) - \Phi^t_f(w) \\
= (e^{1t} - 1)\Phi^t_f(w) + \int_0^t e^{1t} I_{mod}(\Phi^t_f(w)) ds + \int_0^t e^{1t} g(\|\Phi^t_f(w)\|^2)\Phi^t_f(w) d\tilde{W}_k(s).
\end{align*}
\]

By taking the second moment and using the continuity estimate of \( e^{1t} \), we get

\[
\begin{align*}
\mathbb{E} \left[ \| \tilde{\Phi}^t_{M,T,k}(\Phi^t_f(w)) - \Phi^t_f(w) \|^2 \right] \\
\leq 2\tau \mathbb{E} \left[ \| \Phi^t_f(w) \|^2 \right] + 2\tau \mathbb{E} \left[ \int_0^t \| I_{mod}(\Phi^t_f(w)) \|^2 ds \right] \\
+ 2\tau \mathbb{E} \left[ \int_0^t e^{1t} g(\|\Phi^t_f(w)\|^2)\Phi^t_f(w) d\tilde{W}_k(s) \right] \leq C\tau \left( 1 + \mathbb{E} \left[ \| \Phi^t_f(w) \|^2 \right] \right).
\end{align*}
\]

Substituting the above estimates into the estimate \( \mathbb{E} \left[ \| \Psi^t_k(v) - \tilde{\Phi}^t_k(w) \|^2 \right] \), and using the priori estimate of \( \Phi^t_f(w) \) and \( \Phi^t_k(w) \), we get that for \( t = \tau \), we have

\[
\mathbb{E} \left[ \| u'(t_{k+1}) - u'_{k+1} \|^2 \right] \leq e^{C\tau \left( \mathbb{E} \left[ \| u'(t_k) - (u_k') \|^2 \right] + C\tau^2 (1 + |log(\epsilon)|^2) \right)}.
\]

By repeating the above procedures, we conclude

\[
\mathbb{E} \left[ \| u'(t_{k+1}) - u'_{k+1} \|^2 \right] \leq C(Q, T, \lambda, u_0, p, \bar{g})(1 + |log(\epsilon)|^2) \tau,
\]

which completes the proof. \( \square \)

**Proposition 3.6.** Let the condition of Proposition 3.4 hold. Assume that \( W(t) \) is an \( L^2(\mathcal{O}; \mathbb{R}) \)-valued process. Then the splitting scheme (3.4) is strongly convergent. Moreover, for \( p \geq 2 \), there exists \( C(Q, T, \lambda, p, u_0, \bar{g}) > 0 \) such that

\[
\sup_{k \leq N} \| u_k' - u'(t_k) \|_{L^p(\mathcal{O}; \mathbb{R})} \leq C(Q, T, \lambda, p, u_0, \bar{g})(1 + |log(\epsilon)|)^{\frac{1}{2}}.
\]

**Proof.** We only present the details of the case that \( p = 2 \). Assume that \( v, w \in \mathcal{H}^1 \) are \( \mathcal{F}_{t_k} \)-measurable and have any finite moment. Denote the exact flow of Eq. (2.3) on \( [t_k, t_{k+1}] \) by
\[ d\Phi^t_k(w) = 1 \Delta \Phi^t_k(w) dt + 1 \epsilon^{\Delta t} g([\Phi^t_{f+g,F_{\tau_k}}(w)]^2) \Phi^t_{f+g,F_{\tau_k}}(w) dW_k(t) \]

\[ - \frac{1}{2} \epsilon^{\Delta t} (g([\Phi^t_{f+g,F_{\tau_k}}(w)]^2))^2 \Phi^t_{f+g,F_{\tau_k}}(w) \sum_i |Q^i e_i|^2 \]

\[ + 1 \epsilon^{\Delta t} |f_i([\Phi^t_{f+g,F_{\tau_k}}(w)]^2)\Phi^t_{f+g,F_{\tau_k}}(w)| dt. \]

Denote \( v = u^t(t_k) \) and \( w = u^t_k \). Following the similar procedures in the proof of Proposition 3.3 and using the chain rule, the growth condition on \( g \), (2.2), (3.5) and (A2), we get that for \( t \in [0, \tau] \),

\[
\begin{align*}
E \left[ \left\| \Phi^t_k(w) - \Phi^t_k(w) \right\|^2 \right] & \leq E \left[ \left\| v - w \right\|^2 \right] + C \int_0^t E \left[ \left\| \Phi^s_k(w) - \Phi^s_k(w) \right\|^2 \right] ds \\
& + C \int_0^t E \left[ \| (I - \epsilon^{\Delta t}) f_i([\Phi^s_{f+g,F_{\tau_k}}(w)]^2) \Phi^s_{f+g,F_{\tau_k}}(w) \| \right] ds \\
& + C (1 + |\log(\epsilon)|^2) \int_0^t E \left[ \left\| \Phi^s_k(w) - \Phi^s_{f+g,F_{\tau_k}}(w) \right\|^2 \right] ds \\
& + C \int_0^t E \left[ \left\| (I - \epsilon^{\Delta s}) g([\Phi^s_{f+g,F_{\tau_k}}(w)]^2) \Phi^s_{f+g,F_{\tau_k}}(w) \right\|^2 \right] ds.
\end{align*}
\]

By the similar procedures in the proof of Proposition 3.3 it is not hard to obtain the a priori estimate of the numerical solution, that is \( \sup_{k \leq N} E \left[ \left\| u^t_k \right\|_{H^1} \right] \leq C \). The property of \( \epsilon^{\Delta s} \), the growth condition of \( g \), (A1) and (A5), yield that

\[
\begin{align*}
E \left[ \left\| \Phi^t_k(w) - \Phi^t_{f+g,F_{\tau_k}}(w) \right\|^2 \right] & \leq C E \left[ \left\| \Phi^t_k \Phi^t_{f+g,F_{\tau_k}}(w) - \Phi^t_{f+g,F_{\tau_k}}(w) \right\|^2 \right] \\
& \leq C \tau (1 + |\log(\epsilon)|^2) E \left[ \left\| w \right\|_{H^1} \right].
\end{align*}
\]

and

\[
\begin{align*}
E \left[ \left\| (I - \epsilon^{\Delta s}) f_i([\Phi^t_{f+g,F_{\tau_k}}(w)]^2) \Phi^t_{f+g,F_{\tau_k}}(w) \right\|^2 \right] \\
+ E \left[ \left\| (I - \epsilon^{\Delta s}) g([\Phi^t_{f+g,F_{\tau_k}}(w)]^2) \Phi^t_{f+g,F_{\tau_k}}(w) \right\|^2 \right] \leq C \tau (1 + |\log(\epsilon)|^2) E \left[ \left\| w \right\|_{H^1} \right].
\end{align*}
\]

Then the Gronwall inequality, together with the above estimates, yields that

\[
\begin{align*}
E \left[ \left\| \Phi^t_k(w) \right\|^2 \right] & \leq C \tau \left( E \left[ \left\| v - w \right\|^2 \right] + t^2 (1 + |\log(\epsilon)|^2) E \left[ \left\| w \right\|_{H^1} \right] \right) \\
& \leq C \tau \left( E \left[ \left\| v - w \right\|^2 \right] + t^2 (1 + |\log(\epsilon)|^2) \right).
\end{align*}
\]

Making use of an iteration argument and the a priori estimates of \( u^t_k \), we obtain

\[
E \left[ \left\| u^t_{k+1} - u(t_{k+1}) \right\|^2 \right] \leq C \tau \left( E \left[ \left\| u^t_k - u(t_k) \right\|^2 \right] + t^2 (1 + |\log(\epsilon)|^2) \right) \\
\leq \cdots \leq C \tau (1 + |\log(\epsilon)|^2),
\]

which completes the proof. \( \square \)

With slight modification of our approach, one can obtain the same strong convergence rate for the exponential Euler method or the accelerated exponential Euler method for RSlogS equations. Combining the approximation error between Eq. 1.1 and Eq. 2.3 in Lemma 2.1 and the strong convergence result in Propositions 3.3, 3.6 we obtain the following convergence result.
This scheme reads of finite difference methods for SlogS equations in terms of the regularized mid-point scheme. Indeed, \( \tau \) to the exact one of Eq. (1.1). Moreover, for \( p \geq 2 \) and \( \delta \in (0, \max(\frac{2}{\max(T-2,0)}, 1)) \), there exist \( C(Q, T, \lambda, p, u_0) > 0 \) and \( C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g}) > 0 \) such that when \( \mathcal{O} \) is a bounded domain,

\[
\sup_{k \leq N} \| u_k^\tau - u(t_k) \|_{L^p(\Omega, \mathbb{R})} \leq C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g})((1 + |\log(\epsilon)|)\tau^\frac{1}{2} + \epsilon^\frac{1}{2} + \epsilon^\frac{1}{4}),
\]

and when \( \mathcal{O} = \mathbb{R}^d \),

\[
\sup_{k \leq N} \| u_k^\tau - u(t_k) \|_{L^p(\Omega, \mathbb{R})} \leq C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g})((1 + |\log(\epsilon)|)\tau^\frac{1}{2} + \epsilon^\frac{1}{2} + \epsilon^\frac{1}{4}).
\]

**Corollary 3.1.** Let the condition of Theorem 3.1 hold. Then the regularized entropy of (3.1) is strongly convergent to the entropy of Eq. (1.1). Furthermore, for \( p \geq 2 \) and \( \delta \in (0, \max(\frac{2}{\max(T-2,0)}, 1)) \), there exist \( C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g}) > 0 \) and \( C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g}) > 0 \) such that when \( \mathcal{O} \) is a bounded domain,

\[
\| F(|u(t_k)|^2) - F(|u_k|^2) \|_{L^p(\Omega)} \leq C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g})((1 + |\log(\epsilon)|)\tau^\frac{1}{2} + \epsilon^\frac{1}{2} + \epsilon^\frac{1}{4}),
\]

and when \( \mathcal{O} = \mathbb{R}^d \),

\[
\| F(|u(t_k)|^2) - F(|u_k|^2) \|_{L^p(\Omega)} \leq C(Q, T, \lambda, p, u_0, \alpha, \delta, \tilde{g})((1 + |\log(\epsilon)|)\tau^\frac{1}{2} + \epsilon^\frac{1}{2} + \epsilon^\frac{1}{4}).
\]

We have known that \( H(u^\tau) \) is an approximation of the original energy \( H(u) \) in Proposition 2.2. One may expect that the splitting regularized scheme is also an approximation of the energy functional. However, the analysis is even more intricate than expected. Some new techniques are needed to get the convergence of the energy for splitting scheme due to loss of the regularity in time and space of the mild solution. This will be studied in the future.

4. Structure-preserving regularized finite difference type splitting scheme

In the section, we will propose several regularized finite difference schemes, including the regularized Crank–Nilcoson scheme and the regularized mid-point scheme, to study the error between the regularized energy and original one. Throughout this section, we assume that there exists a small \( \tau_0(T, \lambda, g, Q, u_0, \epsilon) > 0 \) such that for \( \tau < \tau_0 \), there exists a numerical solution for the proposed scheme. Indeed, \( \tau_0 \) will be depending on \( \log(\epsilon) \) according to (A1). Let \( \tau < \tau_0 \) be the time step size such that \( T = N\tau \).

4.1. Regularized mid-point scheme. We present the framework analyzing the properties of finite difference methods for SlogS equations in terms of the regularized mid-point scheme. This scheme reads

\[
\Phi_{S, \mathcal{R}_k}(u_k^\tau) = u_k^\tau + \int_{t_k}^{t_{k+1}} \tilde{g}(\Phi_{S, \mathcal{R}_k}(u_k^\tau)) \ast dW(s),
\]

(4.1)

\[
\begin{align*}
&u_{k+1}^\tau = \Phi_{\Delta+2}^\tau(\Phi_{S, \mathcal{R}_k}(u_k^\tau)) := \Phi_{S, \mathcal{R}_k}^\tau(u_k^\tau) + \frac{u_{k+1}^\tau - u_k^\tau}{\tau} + 1\lambda \tau f_c((\Phi_{S, \mathcal{R}_k}^\tau(u_k^\tau) + u_{k+1}^\tau)^2) + \frac{\Phi_{S, \mathcal{R}_k}^\tau(u_k^\tau) + u_{k+1}^\tau}{2},
\end{align*}
\]

where \( u_k^\tau, k \leq N \), is the numerical solution at \( k \)th step and \( u_0^\tau = u_0 \).

It can be verified that \( \Phi_{S, \mathcal{R}_k} \) has the analytic solution if one of the following cases holds:

Case 1. \( W(t) \in \mathbb{H} \) and \( \tilde{g} = 1 \); Case 2. \( W(t) \in \mathbb{H} \) and \( \tilde{g} = i\epsilon \); Case 3. \( W(t) \in L^2(\mathcal{O}; \mathbb{R}) \) and \( \tilde{g}(x) = ig(|x|^2)x \). In these three cases, (4.1) becomes a numerical scheme. Otherwise, some
numerical solver $\tilde{\Phi}_{S,F_{\eta_k}}^\tau$ is needed to discretize $\Phi_{S,F_{\eta_k}}^\tau$. For example, one may use the Euler method and get
\[
\tilde{\Phi}_{S,F_{\eta_k}}^\tau(u_k) := u_k - \tau \sum_{i \in \mathbb{N}^+} |Q \partial_x e_i|^2 [g(|u_k|^2)]^2 u_k(t - \tau_k) + i \int_{t_k}^{t} g(|u_k|^2) u_k^* dW(s) - 1 \sum_{i \in \mathbb{N}^+} \text{Im}(Q \partial_x e_i)Q^* e_i(t - \tau_k) g'(|u_k|^2) g(|u_k|^2) u_k^2 u_k^*.
\]

For simplicity, let us deal with the case that $\Phi_{S,F_{\eta_k}}^\tau$ has an analytic solution since the numerical analysis of other discrete scheme with the numerical solver $\tilde{\Phi}_{S,F_{\eta_k}}^\tau$ is similar.

First, we would like to present the structure-preserving properties, including the symplectic structure and the mass evolution law, of (4.1), which is summarized as follows.

**Proposition 4.1.** Let Assumption 2.2 and the condition of Theorem 2.1 hold. Assume that $\{W(t)\}_{t \geq 0}$ is $\mathcal{B}$-valued, $\tilde{g} = 1$ or that $\|W(t)\|_{t \geq 0}$ is $L^2(\mathcal{O}, \mathbb{R})$-valued, $\tilde{g}(x) = \text{ig}(|x|^2)x$. Then the phase flow of (4.1) preserves the stochastic symplectic structure, i.e., $\bar{\omega}_{k+1} = \bar{\omega}_k$.

Assume that $\tilde{g}(x) = kx$ or $\tilde{g}(x) = 1$. Then (4.1) preserves the mass evolution law of the mass of the exact solution, i.e.,
\[
\mathbb{E} \left[ M(u_{k+1}) \right] = \mathbb{E} \left[ M(u_k) \right] + \tau \sum_{i \in \mathbb{N}^+} \|Q \partial_x e_i\|^2 \chi(\tilde{g} = 1),
\]
where $\chi(\tilde{g} = 1) = 1$ for the additive noise case and $\chi(\tilde{g} = 1) = 0$ for the multiplicative noise case.

Due to the discretization of $e^{\Lambda t}$ and loss of regularity in both time and space, the strong convergence order of (4.1) is less than that of splitting schemes (3.3)–(3.4).

**Proposition 4.2.** Let Assumption 2.2 and the condition of Theorem 2.1 hold. Assume in addition that $f_k$ satisfies (3.3). Then the numerical solution of (4.1) is strongly convergent to the exact one of Eq. (2.3). Moreover, for $p \geq 2$, there exists $C(Q,T,\lambda,p,u_0,\tilde{g}) > 0$ such that
\[
\sup_{k \leq N} \|u_k - \bar{u}^\tau(t_k)\|_{L^p(\Omega; L^1)} \leq C(Q,T,\lambda,p,u_0,\tilde{g}) \epsilon^{-1}(1 + |\log(\epsilon)|)\tau^\frac{3}{2} + \tau^\frac{1}{2}.
\]

**Proof.** We only give the details for the multiplicative noise case since the proof of the additive noise case is analogous. First, we show the uniform boundedness in $H^1$ of $u_{k+1}^\tau$. Multiplying $u_{k+1}^\tau = \frac{\Phi_{S,F_{\eta_k}}^\tau(u_k) + u_{k+1}^\tau}{2}$ on the second equation of $u_{k+1}^\tau$ in (4.1), using integration by parts and the fact that $f_k$ is real-valued, we have that $\|u_{k+1}^\tau\|^2 = \|\Phi_{S,F_{\eta_k}}^\tau(u_k)\|^2$. Then from the Burkholder inequality, the growth condition on $g$ and Gronwall’s inequality, it follows that
\[
\|u_{k+1}^\tau\|_{L^p(\Omega; L^1)} \leq \exp(C\tau)\|\Phi_{S,F_{\eta_k}}^\tau(u_k)\|_{L^p(\Omega; L^1)} \leq \exp(C\tau)\|u_k\|_{L^p(\Omega; L^1)}.
\]
Similarly, multiplying $\Delta u_{k+1}^\tau$ on the equation of $u_{k+1}^\tau$ in (4.1), repeating the above procedures and using (A5), we get that
\[
\|\nabla u_{k+1}^\tau\| \leq \|\nabla \Phi_{S,F_{\eta_k}}^\tau(u_k)\|^2 + C\tau\|\nabla u_{k+1}^\tau\| \leq \frac{1 + C\tau}{1 - C\tau}\|\nabla \Phi_{S,F_{\eta_k}}^\tau(u_k)\|^2 \leq \exp(C\tau)\|\nabla \Phi_{S,F_{\eta_k}}^\tau(u_k)\|^2,
\]
\[
\|\Phi_{S,F_{\eta_k}}^\tau(u_k)\|_{L^p(\Omega; L^1)} \leq \exp(C\tau)\|u_k\|_{L^p(\Omega; L^1)}.
\]
Combining the above estimates, we conclude that
\[
\sup_{k \leq N} \|u_k\|_{L^p(\Omega; L^1)} + \sup_{k \leq N-1} \sup_{t \in [0,\tau]} \|\Phi_{S,F_{\eta_k}}^\tau(u_k)\|_{L^p(\Omega; L^1)} \leq C(Q,T,\lambda,p,u_0,\tilde{g}).
\]
According to the definition, it is not hard to see that \( \hat{u} \) right-continuous with left limit, and thus a predictable process. Then the mild form of \( \hat{u}(t) \) is

\[
\hat{u}(t) = \sum_{k=1}^{N-1} \left( \Phi_{S,\tau_{i+1}, \Phi_{S,\tau_i}} \right) u_0 = \Phi_{S,\tau_{k}, \Phi_{S,\tau_{k+1}}} u_k', \quad \text{if } t \in [t_k, t_{k+1}), \quad k \leq N - 1,
\]

\[
\hat{u}(t) = \Phi_{S,\tau} \lim_{t \to \tau_{k+1}} \hat{u}(t) = u_{k+1}', \quad \hat{u}(0) = u_0.
\]

Now we are in a position to present the error estimate of (1.1). We introduce the following auxiliary process \( \hat{u} \),

\[
\hat{u}(t) = \sum_{k=1}^{N-1} \int_0^t S_{k,t}(s) \left( -\frac{1}{2} \left( g(|\hat{u}(s)|^2) \right) \partial_s u(s) + I \right) ds + \int_0^t S_{k,t}(s) g(|\hat{u}(s)|^2) \hat{u}(s) dw(s) + \int_0^t S_{k,t}(s) \left( f_s(|\hat{u}(s)|^2) \hat{u}(s) - f_s(|\hat{u}_1|^{1/2}) \hat{u}_1 + \frac{1}{2} \right) ds,
\]

where \( S_{k,t} = S_{t-k} \), \( T_r = \frac{r}{t-k} \), and \( S_{k,t} = \sum_{j=1}^k \chi_{[t_{j-1}, t_j]}(s) \), \( [s] \) is integer part of \( \frac{s}{r} \) and \( \hat{u}_{[s]} \)

\[
\hat{u}'(t) = (e^{\Delta t} - S_t^+ u_0) + \int_0^t S_{k,t}(s) \left( f_s(|u'|^2) u'(s) - f_s(|\hat{u}_1|^{1/2}) \hat{u}_1 + \frac{1}{2} \right) ds
\]

by means of Fourier transform and Plancherel’s equality (see e.g. [13] Appendix), it holds that

\[
\| e^{\Delta t} - S_t^{+} \|_{L^2(H^{1/2})} \leq C(f_T) \chi, \| \hat{u} - I \|_{L^2(H^{1/2})} \leq C \chi \|
\]

Thus, we have \( || I_1 || \leq C \chi \| u_0 \|_{H^1} \). To bound \( I_2 \), we adopt the continuity estimate of \( \hat{u}(s) \) on each small interval. More precisely, for \( s \in [t_{j-1}, t_j], \ j \leq k \),

\[
\| \hat{u}(s) - \hat{u}_1^{(s)} \| \leq \frac{1}{2} \| \hat{u}(s) - \phi_{S_j \tau_{j-1}}(u_1) \| + \frac{1}{2} \| \hat{u}_1^{(s)} - u_1 \|
\]

Taking th moment, using the growth condition of \( f \) and \( g \), as well as an a priori estimate of \( \hat{u} \) in \( H^1 \), we obtain

\[
\| \hat{u}(s) - \hat{u}_1^{(s)} \|_{L^p(\Omega)} \leq C \chi \log(|\epsilon|) \chi. \]
Theorem 4.1. Let the condition of Proposition 4.2 hold. Then the numerical solution of (3.5) is strongly convergent to the exact solution of Eq. (1.1). Moreover, for $p \geq 2$ and $\delta \in (0, \max(\frac{3}{4} - \frac{3}{2} - 2\mu_0, 1))$, there exist $C(Q, T, \lambda, p, u_0, \alpha, \gamma) > 0$ and $C(Q, T, \lambda, p, u_0, \alpha, \gamma) > 0$ such that when $\mathcal{O}$ is a bounded domain,

$$\sup_{k \leq N} \| u_k^e - u(t_k) \|_{L^p(\Omega; \mathbb{H})} \leq C(Q, T, \lambda, p, u_0, \gamma)(\epsilon^{-1}((1 + |\log(\epsilon)|)\tau^{\frac{1}{2}} + \tau^{\frac{3}{2}}) + \epsilon^\tau + \epsilon^{\frac{3}{2}}),$$

and when $\mathcal{O} = \mathbb{R}^d$,

$$\sup_{k \leq N} \| u_k^e - u(t_k) \|_{L^p(\Omega; \mathbb{H})} \leq C(Q, T, \lambda, p, u_0, \alpha, \gamma)(\epsilon^{-1}((1 + |\log(\epsilon)|)\tau^{\frac{1}{2}} + \tau^{\frac{3}{2}}) + \epsilon^\tau + \epsilon^{\frac{3}{2}}).$$
4.2. Regularized Crank–Nicolson scheme. Based on the study of the regularized midpoint scheme, we are in a position to study the properties of the regularized Crank–Nicolson scheme and show that it is a good approximation of the energy of Eq. (1.1). The regularized splitting Crank–Nicolson type scheme reads

\[
\Phi_{S,F_{\tau}}^{n}(u_{k}^{n}) := u_{k}^{n} + \int_{t_{k}}^{t_{k+1}} \tilde{g}(\Phi_{S,F_{\tau}}^{n}(u_{k}^{n})) \ast dW(s),
\]

where

\[
u_{k+1}^{\varepsilon} = \Phi_{\Delta+f}^{\varepsilon}(\Phi_{S,F_{\tau}}^{n}(u_{k}^{n})) := \Phi_{S,F_{\tau}}^{n}(u_{k}^{n}) + i\Delta \frac{\Phi_{S,F_{\tau}}^{n}(u_{k}^{n}) + u_{k+1}^{n}}{2} + i\lambda \int_{0}^{1} f_{\theta}(\Phi_{S,F_{\tau}}^{n}(u_{k}^{n}))^{2} + (1 - \theta)|u_{k+1}^{n}|^{2} d\theta \frac{\Phi_{S,F_{\tau}}^{n}(u_{k}^{n}) + u_{k+1}^{n}}{2},
\]

where \(u_{k}^{n}, k \leq N\), is the numerical solution at the \(k\)th step and \(u_{0}^{n} = u_{0}\). By the chain rule, one can verify that

\[
\int_{0}^{1} f_{\theta}(\Phi_{S,F_{\tau}}^{n}(u_{k}^{n}))^{2} + (1 - \theta)|u_{k+1}^{n}|^{2} d\theta \frac{\Phi_{S,F_{\tau}}^{n}(u_{k}^{n}) + u_{k+1}^{n}}{2} = \frac{\tilde{F}_{\varepsilon}(u_{k+1}^{n}) - \tilde{F}_{\varepsilon}(\Phi_{S,F_{\tau}}^{n}(u_{k}^{n}))^{2}}{2} \Phi_{S,F_{\tau}}^{n}(u_{k}^{n}) + u_{k+1}^{n},
\]

where \(\tilde{F}_{\varepsilon}\) is the integrand in the regularized entropy \(F_{\varepsilon}\).

In the following, we focus on the uniform a priori estimate of (4.2) when \(\tilde{g}(x) = ix\) and \(\mathcal{O}\) is bounded. For convenience, assume that \(u_{0}\) is a deterministic function. We will study the convergence of (4.2) for general diffusion coefficient case in the future.

**Proposition 4.3.** Let the condition of Proposition 4.2 hold, \(\tilde{g}(x) = ix\), \(\mathcal{O}\) be a bounded domain. Then the modified energy of the numerical solution of the Crank–Nicolson type method (4.2) is well-defined. Moreover, for \(p \geq 2\), there exists \(C(Q,T,\lambda,\mu,\nu) > 0\) such that

\[
\sup_{k \leq N} \|\nabla u_{k}^{\varepsilon}\|_{L^{p}(\Omega;\mathbb{R})} \leq C(Q,T,\lambda,\mu,\nu).
\]

**Proof.** The proof is similar to that of Proposition 4.2. The main difference lies on the a priori estimate of the auxiliary process \(\tilde{u}\) defined by

\[
\tilde{u}(t) = \Phi_{\Delta+f}^{t} \prod_{i=0}^{k-1} \Phi_{\Delta+f,\tau}^{t_{i}} u_{0}, \text{ if } t \in [t_{k}, t_{k+1}), \ k \leq N - 1,
\]

\[
\tilde{u}(t_{k+1}) = \Phi_{\Delta+f}^{t_{k+1}} \lim_{t \to t_{k+1}} \tilde{u}(t) = u_{k+1}^{\varepsilon}, \quad \tilde{u}(0) = u_{0}.
\]

Following the same steps as in the proof of Proposition 4.2, we obtain

\[
\sup_{k \leq N-1} \sup_{t \in [0,\tau]} H_{\varepsilon}(\Phi_{S,F_{\tau}}^{n}(u_{k}^{n})) \|_{L^{p}(\Omega;\mathbb{R})} \leq C(Q,T,\lambda,\mu,\nu).
\]

By the Gagliardo–Nirenberg interpolation inequality (2.5), the procedures in the proof of Lemma 2.2 Young’s and Hölder’s inequalities, it can be verified that for a small enough \(\varepsilon > 0, \eta > 0\) and \(\delta < \frac{\varepsilon}{4} \),

\[
\frac{1}{2} \|\nabla v\|^{2} \leq |H_{\varepsilon}(v)| + \delta \|\nabla v\|^{2} + C(\delta')(1 + \|v\|^{2} + \|v\|^{4+4d-2d}) + \|v\|^{2-2\eta}.
\]

Since \(\mathcal{O}\) is bounded, by (4.3), we obtain

\[
\sup_{k \leq N-1} \sup_{t \in [0,\tau]} \|\Phi_{S,F_{\tau}}^{n}(u_{k}^{n})\|_{L^{p}(\Omega;\mathbb{R})} \leq C(u_{0}, T, Q, p),
\]

which completes the proof. \(\Box\)
Compared to (4.1), (4.2) fails to preserve the stochastic symplectic structure. However, it preserves the mass evolution law of the exact solution.

**Proposition 4.4.** Let the condition of Theorem 4.2 hold and \( \tilde{g}(x) = ix \) or \( \tilde{g}(x) = 1 \). Then (4.2) preserves the evolution law of the mass of the exact solution, that is,

\[
\mathbb{E} \left[ M(u_{n+1}^\epsilon) \right] = \mathbb{E} \left[ M(u_n^\epsilon) \right] + \tau \sum_{i \in \mathbb{N}^+} \| Q_i^\tau e_i \|^2 \chi(\tilde{g} = 1),
\]

where \( \chi(\tilde{g} = 1) = 1 \) for the additive noise case and \( \chi(\tilde{g} = 1) = 0 \) for the multiplicative noise case.

In the following, we give the error estimate of (4.2) by making use of (4.1) and solve its convergence problem in temporal direction. The convergence analysis of (4.2) is more complicated than that of the splitting type scheme since the boundedness of energy of numerical solution may not imply the boundedness of the numerical solution under \( H^1 \)-norm. Generally speaking, the a priori estimate of (4.2) may be not uniform with respect to \( \epsilon \).

**Theorem 4.2.** Let \( \tilde{g} = 0,1 \) or \( ix \), and \( \mathcal{O} \) be bounded. Assume that \( f_\epsilon(x) = \log(\frac{e^x + x}{x}) \), \( x > 0 \). Then the numerical solution of (4.2) is strongly convergent to the exact one of Eq. (4.1). Moreover, for \( p \geq 1 \) and \( \delta \in (0, \max(1, 1)) \), there exists \( C(Q,T,\lambda, u_0, \delta, \tilde{g}, p) > 0 \) such that

\[
\sup_{k \leq N} \| u_k^\epsilon - u(t_k) \|_{L^p(\Omega; H)} \leq C(Q,T,\lambda, u_0, \delta, \tilde{g}, p) \left( \epsilon^{-1} (1 + |\log(\epsilon)|) \tau^{\frac{3}{2}} + \tau^{\frac{3}{2}} + \epsilon^2 + \epsilon^2 + \epsilon^{-1} \tau^{\frac{3}{2}} |\log(\epsilon)| \right).
\]

**Proof.** Thanks to Proposition 4.3 and the continuity of \( e^{i\Delta t} \), it holds that \( \| u_{k+1}^\epsilon - \Phi^\tau_{\Delta + f}(u_k^\epsilon) \|_{L^p(\Omega; H)} \leq C(\tau^{\frac{3}{2}} + \tau |\log(\epsilon)|) \). By expanding the flow \( \tilde{\Phi}_{\Delta + f} \) of (4.2) via \( \Phi_{\Delta + f} \) of (4.1), and following the same procedures as in the proof of Theorem 4.2, we obtain

\[
u^\tau(t) - \tilde{u}(t) := II_1 + II_2 + II_3 + II_4 + II_5 + II_6 + II_7 + II_8,
\]

where \( II_1 - II_7 \) are presented in the proof of Theorem 4.2. Here \( II_8 \) is the expansion error \( \tilde{\Phi}_{\Delta + f} \) given by

\[
II_8 := \lambda \int_0^t S_{k,t}(s) \left( f_\epsilon(\tilde{u}_s, \tilde{u}_s) \right) ds - \int_0^t f_\epsilon(\theta |\Phi_{\Delta + f}(u_k^\epsilon)|^2 + (1 - \theta)|u^\epsilon_{k+1}|^2) \tilde{u}_s^\epsilon |d\theta| \tilde{u}_s^\epsilon \] ds.
\]

In order to estimate \( II_8 \), let us consider the event that \( |\Phi_{\Delta + f}(u_k^\epsilon)|^2 \geq |u^\epsilon_{k+1}|^2 \). The estimate on the event that \( |\Phi_{\Delta + f}(u_k^\epsilon)|^2 \leq |u^\epsilon_{k+1}|^2 \) is similar.

The convexity of \( |\cdot|^2 \) implies that \( |\tilde{u}_{k+\frac{1}{2}}^\epsilon|^2 \leq \frac{1}{2} |\Phi_{\Delta + f}(u_k^\epsilon)|^2 + \frac{1}{2} |u^\epsilon_{k+1}|^2 \). Assume that \( \theta_0 \in (0, \frac{1}{2}) \) is the largest number such that \( |\tilde{u}_{k+\frac{1}{2}}^\epsilon|^2 = \theta_0 |\Phi_{\Delta + f}(u_k^\epsilon)|^2 + (1 - \theta_0) |u^\epsilon_{k+1}|^2 \). Otherwise, the proof of the desired estimate is simple by choosing one of the following estimates. Then when \( \theta \geq \theta_0 \) it holds that for \( \delta_1^\epsilon \in (0, 1) \),

\[
\left| f_\epsilon(\tilde{u}_{k+\frac{1}{2}}^\epsilon) - f_\epsilon(\theta |\Phi_{\Delta + f}(u_k^\epsilon)|^2 + (1 - \theta)|u^\epsilon_{k+1}|^2) \right| \delta_1^\epsilon \]

\[
\leq C \left( \frac{\theta |\Phi_{\Delta + f}(u_k^\epsilon)|^2 + (1 - \theta)|u^\epsilon_{k+1}|^2 - |\tilde{u}_{k+\frac{1}{2}}^\epsilon|^2}{\epsilon + |\tilde{u}_{k+\frac{1}{2}}^\epsilon|^2} \right) \delta_1^\epsilon \]

\[
+ C \epsilon \delta_1^\epsilon \left( \frac{\theta |\Phi_{\Delta + f}(u_k^\epsilon)|^2 + (1 - \theta)|u^\epsilon_{k+1}|^2 - |\tilde{u}_{k+\frac{1}{2}}^\epsilon|^2}{1 + |\tilde{u}_{k+\frac{1}{2}}^\epsilon|^2} \right) \delta_1^\epsilon.
\]
When $\theta \leq \theta^0$, the dominant part of $f_\epsilon(\hat{u}_{k+1}^\epsilon)^2 - f_\epsilon(\Phi_{S,T}^{\epsilon} u_k^\epsilon)^2 + (1 - \theta)|u_{k+1}^\epsilon|^2$ is a concave function over $\theta$. Using Jensen’s inequality, we obtain that
\[
\int_0^{\theta^0} \left( \log(\epsilon + |\hat{u}_{k+1}^\epsilon|^2) - \log(\epsilon + \theta |\Phi_{S,T}^{\epsilon} u_k^\epsilon|^2 + (1 - \theta)|u_{k+1}^\epsilon|^2) \right) d\theta 
\leq \theta^0 \left( \log(\epsilon + |\hat{u}_{k+1}^\epsilon|^2) - \log(\epsilon + \frac{\theta^0}{2} |\Phi_{S,T}^{\epsilon} u_k^\epsilon|^2 + (1 - \frac{\theta^0}{2})|u_{k+1}^\epsilon|^2) \right)
\leq C \theta^0 \frac{\hat{u}_{k+1}^\epsilon|^2 - \frac{\theta^0}{2} |\Phi_{S,T}^{\epsilon} u_k^\epsilon|^2 - (1 - \frac{\theta^0}{2})|u_{k+1}^\epsilon|^2}{\epsilon + \frac{\theta^0}{2} |\Phi_{S,T}^{\epsilon} u_k^\epsilon|^2 + (1 - \frac{\theta^0}{2})|u_{k+1}^\epsilon|^2}
\leq C \theta^0 \frac{\hat{u}_{k+1}^\epsilon|^2 - \frac{\theta^0}{2} |\Phi_{S,T}^{\epsilon} u_k^\epsilon|^2 - (1 - \frac{\theta^0}{2})|u_{k+1}^\epsilon|^2}{\epsilon + \frac{\theta^0}{2} |\Phi_{S,T}^{\epsilon} u_k^\epsilon|^2 + (1 - \frac{\theta^0}{2})|u_{k+1}^\epsilon|^2}
\]
Combining the above estimates, using the boundedness of $S_{k,t}$ and $T_\tau$, we achieve that for $\delta_1 = \frac{1}{2}$,
\[
\|I_{S}\|_{L^p(\Omega, \mathcal{B}, \mathcal{F}, H^1)} 
\leq C \|u_{k+1}^\epsilon - \Phi_{S,T}^{\epsilon} u_k^\epsilon\|_{L^p(\Omega, \mathcal{B}, \mathcal{F}, H^1)} (\|u_{k+1}^\epsilon\|_{L^p(\Omega, \mathcal{B}, \mathcal{F}, H^1)} + \|\Phi_{S,T}^{\epsilon} u_k^\epsilon\|_{L^p(\Omega, \mathcal{B}, \mathcal{F}, H^1)} + \log(\epsilon))^{\frac{1}{2}}
\leq C \|u_{k+1}^\epsilon - \Phi_{S,T}^{\epsilon} u_k^\epsilon\|_{L^p(\Omega, \mathcal{B}, \mathcal{F}, H^1)} + C \|u_{k+1}^\epsilon - \Phi_{S,T}^{\epsilon} u_k^\epsilon\|_{L^p(\Omega, \mathcal{B}, \mathcal{F}, H^1)} + C \|u_{k+1}^\epsilon\|_{L^p(\Omega, \mathcal{B}, \mathcal{F}, H^1)}.
\]
Based on the estimates of $I_1-I_3$ in the proof of Proposition 4.2 and the above estimate of $I_3$, we complete the proof.

4.3. Weak convergence of the regularized energy. In order to show that $\Phi_{S,T}^{\epsilon}$ is a suitable scheme which is convergent to the exact solution in terms of energy, we take $f_\epsilon(x) = \log(\frac{x + \epsilon}{1 + \epsilon})$ to illustrate the main strategy. One could generalize the choices of $f_\epsilon$ according to the proof of the following proposition.

**Proposition 4.5.** Let the condition of Theorem 4.2 hold and $\tilde{g}(x) = ix$. Assume that $f_\epsilon = \log(\frac{x + \epsilon}{1 + \epsilon})$, $x > 0$. Then the regularized energy of $\Phi_{S,T}^{\epsilon}$ is convergent to the energy of $(1.1)$.

Furthermore, for $p \geq 1$, and $\delta \in (0, \max(\frac{2}{\max(d - 2, 0)}, 1))$, there exists $C(Q, T, \lambda, p, u_0) > 0$ such that
\[
\left| E \left[ H_\epsilon(u_k^\epsilon) - H_0(u_0^\epsilon) \right] \right| \leq C(Q, T, \lambda, p, u_0) \left( (\epsilon^{-1} (1 + |\log(\epsilon)|)^\frac{1}{\delta} + \frac{\epsilon}{\delta}) + \epsilon^{-1} |\log(\epsilon)|^{\frac{1}{\delta}} \right).
\]

**Proof.** According to Lemma 2.2, it suffices to estimate $E[H_\epsilon(u_k^\epsilon) - H_\epsilon(u^\epsilon(t_k))]$. By analyzing the expansion of the energy of $u_k^\epsilon$ and $u^\epsilon(t_k)$, we obtain
\[
E \left[ H_\epsilon(u^\epsilon(t_k)) - H_\epsilon(u_k^\epsilon) \right] = E \left[ H_\epsilon(u^\epsilon(t_{k-1})) - H_\epsilon(\Phi(u_k^\epsilon)) \right]
+ \int_{t_k}^{t_{k+1}} \sum_i \left( \|u^\epsilon(s)\| Q^\frac{1}{2} e_i \|^2 - \|u^\epsilon(s)\| Q^\frac{1}{2} e_i \|^2 \right) ds
+ \frac{\lambda}{2} \int_0^t \sum_i \left( f_\epsilon(|u^\epsilon|^2) u^\epsilon, -\frac{1}{2} \frac{\partial}{\partial x} |Q^\frac{1}{2} e_i |^2 \right) ds
- \frac{\lambda}{2} \int_0^t \sum_i \left( f_\epsilon(|u^\epsilon|^2) u^\epsilon Q^\frac{1}{2} e_i, u^\epsilon Q^\frac{1}{2} e_i \right) ds
- \frac{\lambda}{2} \int_0^t \sum_i \left( 2Re(\bar{u}^\epsilon u^\epsilon Q^\frac{1}{2} e_i) \frac{\partial}{\partial x} (|u^\epsilon|^2) u^\epsilon, iu^\epsilon Q^\frac{1}{2} e_i \right) ds.
where \( \hat{u} \) is defined in the proof of Proposition 4.3. By applying the similar estimates in the proof of Lemma 2.2 and Hölder’s inequality, it follows that for \( \delta \leq \min(1, \frac{2}{\max(\delta, 1, 2, 0)}) \) and \( \eta \in (0, 1) \),

\[
\mathbb{E}\left[ H_{\varepsilon}(u^\varepsilon(t_k)) - H_{\varepsilon}(u^\varepsilon_k) \right] - \mathbb{E}\left[ H_{\varepsilon}(u^\varepsilon(t_{k-1})) - H_{\varepsilon}(u^\varepsilon_{k-1}) \right] \\
\leq C\tau \sup_{t \in [t_k, t_{k+1}]} \mathbb{E}\left[ \|u^\varepsilon(t) - \hat{u}(t)\| \|u^\varepsilon(t) + \hat{u}(t)\| \right] \\
+ C\tau \| \sum_{i \in \mathbb{N}^+} \|Q_\varepsilon^\varepsilon e_i\|_{L^2}^2 e^{\eta^2} \left( 1 + \sup_{t \in [0, T]} \mathbb{E}\left[ \|u^\varepsilon(t)\|^{2-2\eta} \right] + \sup_{t \in [0, T]} \mathbb{E}\left[ \|\hat{u}(t)\|^{2-2\eta} \right] \right) \\
+ C\tau \| \sum_{i \in \mathbb{N}^+} \|Q_\varepsilon^\varepsilon e_i\|_{L^2}^2 \sup_{t \in [0, T]} \mathbb{E}\left[ \|u^\varepsilon(t)\|^{2+\frac{2\delta}{1-\frac{2\delta}{2}} + \|\hat{u}(t)\|^{2+\frac{2\delta}{1-\frac{2\delta}{2}}} \right].
\]

Using Theorem 4.2, Proposition 2.2 and iteration arguments, we finish the proof. \( \square \)

5. Appendix

**Proposition 5.1.** Let the condition of Theorem 2.4 and Assumption 2.1 hold. Let \( u^\varepsilon \) be the mild solution of Eq. (2.3) and \( u^0 \) be the mild solution of Eq. (1.1). When \( \tilde{g} = 1 \), the mild solution \( u^\varepsilon \) is shown to satisfy the following evolution laws,

\[
M(u^\varepsilon(t)) = M(u^0_0) + \int_0^t \sum_{i \in \mathbb{N}^+} \|Q_\varepsilon^\varepsilon e_i\|^2 ds + 2 \int_0^t \langle u^\varepsilon(s), dW(s) \rangle,
\]

\[
M_\alpha(u^\varepsilon(t)) = M_\alpha(u^0_0) + \int_0^t 4\alpha((1 + |x|^2)^{\alpha-1} x u^\varepsilon(s), i\nabla u^\varepsilon(s)) ds \\
+ \int_0^t \sum_{i \in \mathbb{N}^+} \|Q_\varepsilon^\varepsilon e_i\|^2 ds + \int_0^t 2((1 + |x|^2)^\alpha u^\varepsilon(s), dW(s)).
\]

When \( \tilde{g} = ig(|x|^2)x \), the mild solution \( u^\varepsilon \) satisfies the following evolution laws,

\[
M(u^\varepsilon(t)) = M(u^0_0) + 2 \int_0^t \langle u^\varepsilon(s), ig(|u^\varepsilon(s)|^2)u^\varepsilon(s) dW(s) \rangle \\
+ \int_0^t \langle u^\varepsilon(s), -i \sum_{k \in \mathbb{N}^+} \text{Im}(Q_\varepsilon^\varepsilon e_k)Q_\varepsilon^\varepsilon e_k g(|u^\varepsilon(s)|^2)g(|u^\varepsilon(s)|^2)u^\varepsilon(s)u^\varepsilon(s) \rangle ds,
\]

\[
M_\alpha(u^\varepsilon(t)) = M_\alpha(u^0_0) + \int_0^t 4\alpha((1 + |x|^2)^{\alpha-1} x u^\varepsilon(s), i\nabla u^\varepsilon(s)) ds \\
- 2 \int_0^t ((1 + |x|^2)^\alpha u^\varepsilon(s), i \sum_{k \in \mathbb{N}^+} \text{Im}(Q_\varepsilon^\varepsilon e_k)Q_\varepsilon^\varepsilon e_k g(|u^\varepsilon(s)|^2)g(|u^\varepsilon(s)|^2)u^\varepsilon(s)u^\varepsilon(s) \rangle ds \\
+ 2 \int_0^t ((1 + |x|^2)^\alpha u^\varepsilon(s), ig(|u^\varepsilon(s)|^2)u^\varepsilon(s) dW(s)).
\]

**Sketch Proof of Lemma 2.1.** The Hölder regularity estimate is a consequence of [12] Corollary 3.2. The proof of the strong convergence is similar to that of [12] Theorem 1.1. By using Proposition 5.1 and assumptions on \( g \) and \( f_\varepsilon \), it is not hard to establish the following a priori estimate

\[
\mathbb{E}\left[ \sup_{t \in [0, T]} \|u^\varepsilon(t)\|_{H}^{p+1} \right] + \mathbb{E}\left[ \sup_{t \in [0, T]} \|u^\varepsilon(t)\|_{L^2}^{p+2} \right] \leq C(Q, T, \lambda, p, u_0, \alpha).
\]

Denote \( f_0(x) = \log(x) \). Then by applying Itô formula to \( \|u^\varepsilon - u\|^2 \), using integration by parts and the properties in Assumption 2.1 and 2.2, we obtain the following error estimates. In the
In the case of \( \widetilde{g} = 1 \), for \( \eta'(d - 2) \leq 2 \), it holds that

\[
\|u(t) - u^{n}(t)\|^2 \\
= \int_0^t \|u - u^{n}\|^{2} \lambda f_{0}(\|u\|^{2}) u - f_{\epsilon_{n}}(\|u^{n}\|^{2})u^{n}\) \(ds\) \\
\leq \int_0^t 4\lambda \|u(s) - u^{n}(s)\|^2 \(ds\) + 4\lambda \|\mathbb{I} m(u(s) - u^{n}(s), f_{0}(\|u\|^{2}) - f_{\epsilon_{n}}(\|u\|^{2})u)\| \(ds\) \\
\leq \int_0^t 6\lambda \|u^{m}(s) - u^{n}(s)\|^2 \(ds\) + 4\lambda \|\mathbb{I} m(u^{m}(s) - u^{n}(s))L_{1}\| \left(\frac{\epsilon_{m} - \epsilon_{n}}{\epsilon_{m} + \|u^{n}\|^{2}}\right) \|u^{n}\| \(ds\) \\
+ 2\lambda |C| \epsilon_{n}^\eta \int_0^t \|u^{m}\|^{2+2\eta} \(ds\).
\]

In the case of \( \widetilde{g}(x) = i\varpi(\|x\|^{2})x \), for \( \eta'(d - 2) \leq 2 \),

\[
\|u^{m}(t) - u^{n}(t)\|^2 \\
\leq \int_0^t \left( 4\lambda + C(g, Q) \right) \|u^{m}(s) - u^{n}(s)\|^2 \(ds\) + 4\lambda \epsilon_{n}^\eta \int_0^t \|u^{m}(s) - u^{n}(s)\| \(ds\) \\
+ 2\lambda |C| \epsilon_{n}^\eta \int_0^t \|u^{m}\|^{2+2\eta} \(ds\) \int_0^t (u^{m} - u^{n}, 1\{ (\|u^{m}\|^{2})u^{m} - g(\|u^{n}\|^{2})u^{n} \}dW(s)).
\]

Combining the above error estimates with the a priori estimates of \( u^{t} \) and following the steps in the proof of [12, Theorem 1.1], we obtain the desired convergence rate. \( \square \)

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