Revisiting the Rellich inequality

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Abstract
We revisit the Rellich inequality from the viewpoint of isolating the contributions from radial and spherical derivatives. This naturally leads to a comparison of the norms of the radial Laplacian and Laplace–Beltrami operators with the standard Laplacian. In the case of the Laplace–Beltrami operator, the three-dimensional case is the most subtle and here we improve a result of Evans and Lewis by identifying the best constant. Our arguments build on certain identities recently established by Wadade and the second and third authors, along with use of spherical harmonics.

1 Introduction and main results
We will consider the following Rellich inequality
\[
\left( \frac{n(n-4)}{4} \right)^2 \left\| \frac{f}{|x|^2} \right\|_2^2 \leq \| \Delta f \|_2^2,
\]
where \( \| \cdot \|_2 \) denotes the \( L^2 \)-norm on \( \mathbb{R}^n \) with respect to Lebesgue measure, and \( \Delta \) denotes the standard Laplacian on \( \mathbb{R}^n \). This inequality is ubiquitous in the theory of partial differential equations and spectral theory. The reader may refer to [1, 2, 5, 10, 13–15, 20–23] and references therein for further details, applications and wider perspectives.

If one wishes to treat general functions \( f \) in the Sobolev space \( H^2(\mathbb{R}^n) \), it is necessary to restrict the dimension of the ambient space to \( n \geq 5 \), since the norm on the left-hand side of (1.1) will be infinite for any continuous function with \( f(0) \neq 0 \) when \( n \leq 4 \). Rellich [18,
coefficients. In fact, even if we replace the constant on the left-hand side by any positive constant, (1.1) fails when \( n = 2 \) if we admit all \( f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \).

A fresh perspective on the Rellich inequality (1.1) based on certain identities was recently brought to light by Wadade and the second and third authors in [17]. Here we continue to develop this perspective by examining the contribution arising from radial derivatives and spherical derivatives. It is well-known that the Laplacian decomposes as \( \Delta = \Delta_r + \frac{1}{|x|^2} \Delta_{S^{n-1}} \) where \( \Delta_r \) denotes the radial Laplacian and \( \Delta_{S^{n-1}} \) denotes the Laplace–Beltrami operator. The radial Laplacian is given by

\[
\Delta_r f = \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r} \frac{\partial f}{\partial r},
\]

where \( \partial_r \) is the radial derivative given by \( \partial_r = \frac{x}{|x|} \cdot \nabla \). Also, we introduce the spherical-type derivative \( L_j = \partial_j - \frac{x_j}{|x|} \partial_r \) for each \( j = 1, \ldots, n \), in which case we have the relation

\[
\Delta_{S^{n-1}} = |x|^2 \sum_{j=1}^n L_j^2.
\]

The first of our key identities is the following and, roughly speaking, may be viewed as a means of capturing the radial and spherical contributions to both sides of (1.1). In the following statement, and throughout the paper, we use the notation \( R_n = \frac{1}{4} n(n-4) \).

**Theorem 1** For \( n \geq 2 \) and \( f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) we have

\[
R_n^2 \left\| \frac{f}{|x|^2} \right\|_2^2 = \left\| \Delta_r f \right\|_2^2 - \left\| \Delta_r f + R_n \frac{f}{|x|^2} \right\|_2^2 - 2R_n \left\| f^\# \right\|_2^2
\]

\[
= \left\| \Delta_r f \right\|_2^2 - \left( 1 + \frac{1}{R_n} \right) \left\| \Delta_r f + R_n \frac{f}{|x|^2} \right\|_2^2 + R_n \left\| f^\# \right\|_2^2
\]  

(1.4)

and

\[
\left\| \Delta f \right\|_2^2 = \left\| \Delta_r f \right\|_2^2 + \left( \sum_{j=1}^n \frac{L_j f^2}{|x|^2} \right)_2 + 2R_n \left( \sum_{j=1}^n \frac{L_j f}{|x|^2} \right)_2 + 2 \left( - \sum_{j=1}^n L_j^2 f^\# , f^\# \right),
\]  

(1.5)

where \( f^\# = \partial_r f + \frac{n-4}{2} \frac{f}{|x|} \), \( f^\# = \partial_r^2 f + (n-3) \frac{\partial_r f}{|x|} + \frac{(n-4)^2}{4} \frac{f}{|x|^2} \).

The identity (1.4) was reported in [17, Theorem 1.1] for \( n \geq 5 \). Since \( R_4 = 0 \), it is immediate that it extends to \( n = 4 \). Here we shall verify the cases \( n = 2 \) and \( n = 3 \); although the argument in [17] extends verbatim when \( n = 2 \), a certain amount of care is needed to check the case \( n = 3 \). The identity (1.5) is new and it has the advantage that for \( n = 2, 3 \), both coefficients \(- (1 + \frac{1}{R_n})\) and \( R_n \) are negative. An identity similar to (1.6) was established in [17, Theorem 1.2] (for \( n \geq 5 \)), the only difference being the final term on the right-hand side of (1.6). As we shall see, the form we have presented in Theorem 1 is more convenient for our purposes in the present paper (when handling terms in a decomposition of \( f \) into spherical harmonics), and we also emphasise that \(- \sum_{j=1}^n L_j^2 f^\# \) is a non-negative operator on \( L^2(\mathbb{R}^n) \) (see the forthcoming discussion on spherical harmonics).
Regarding the second and third terms on the right-hand side of (1.6), we have the identities in the forthcoming Theorem 2 in terms of the decomposition of \( f \) into spherical harmonics. The statement is written in terms of the orthogonal projection operator \( P_k \) onto the closed subspace \( \mathcal{H}_k(\mathbb{R}^n) \) of \( L^2(\mathbb{R}^n) \) spanned by spherical harmonics of order \( k \) multiplied by radial functions. When \( k = 0 \), we will often abbreviate \( P_0 \) as \( P \). The spaces \( \mathcal{H}_k(\mathbb{R}^n) \) give rise the standard spherical harmonic decomposition as

\[
L^2(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{R}^n)
\]

and the spherical harmonics of degree \( k \) are well known to be eigenfunctions of the Laplace–Beltrami operator with eigenvalue \(-k(k + n - 2)\), that is

\[
\Delta_{\mathbb{S}^{n-1}} P_k = -\mu_k P_k \quad (\mu_k := k(k + n - 2), k \geq 0).
\]  

**Theorem 2** For \( n \geq 2 \) and \( f \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) \), we have

\[
\left\| \sum_{j=1}^{n} L_j^2 f \right\|_2^2 = \sum_{k=1}^{\infty} \mu_k^2 \left\| \frac{P_k f}{|x|^2} \right\|_2^2, \quad \sum_{j=1}^{n} \left\| L_j f \right\|_2^2 = \sum_{k=1}^{\infty} \mu_k \left\| \frac{P_k f}{|x|^2} \right\|_2^2.
\]  

As we shall see, Theorems 1 and 2 lead to a number of interesting consequences, some of which are new, and some of which provide new and simpler perspectives on known results.

Firstly, we consider a refinement of the Rellich inequality (1.1) in which the Laplacian is replaced by the radial Laplacian. This naturally leads us to also consider a comparison of the size of the norm of the radial Laplacian and the Laplacian. We summarize matters as follows.

**Theorem 3** Let \( n \geq 2 \). Then

\[
\mathbb{R}_n^2 \left\| \frac{f}{|x|^2} \right\|_2^2 \leq \left\| \Delta_r f \right\|_2^2
\]  

holds for all \( f \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) \). If \( n \geq 3 \), then

\[
\left\| \Delta_r f \right\|_2 \leq C \left\| \Delta f \right\|_2
\]  

holds for all \( f \in C^\infty_0(\mathbb{R}^n \setminus \{0\}) \) with \( C = 1 \), and fails for any constant \( C \) when \( n = 2 \). Furthermore, the constants in (1.9) and (1.10) are best possible.

The inequality (1.9) can be found already in work of Evans–Lewis [13, Eq. (2.13)]. We remark that it is possible to quickly derive (1.9) from (1.4) (for \( n \geq 4 \)) and (1.5) (for \( n = 2, 3 \)). The inequality (1.10) with \( C = 1 \) follows immediately from (1.6) for \( n \geq 4 \) since \( \mathbb{R}_n \geq 0 \). When \( n = 2 \), (1.10) must fail for any \( C < \infty \) since otherwise we could combine with (1.9) and obtain an inequality of the form (1.1) with some positive constant on the left-hand side; as we have already pointed out, this is impossible. The case \( n = 3 \) is slightly more subtle since \( \mathbb{R}_3 < 0 \), but we may use Theorem 2 to see that

\[
\left\| \sum_{j=1}^{3} L_j^2 f \right\|_2^2 + 2 \mathbb{R}_3 \sum_{j=1}^{3} \left\| \frac{L_j f}{|x|} \right\|_2^2 = \sum_{k=1}^{\infty} \left( \mu_k (\mu_k - \frac{3}{2}) \right) \left\| \frac{P_k f}{|x|^2} \right\|_2^2 \geq 0.
\]  

For completeness we also note that the optimality of the constant in (1.9) can be shown by arguing as in [9] using the family of functions \( f_\varepsilon(x) = |x|^{\frac{4-n}{2} + \frac{\varepsilon}{2}} \chi(|x|) \). Here \( \chi \) is infinitely
smooth, $\chi(0) = 1$ and $\chi(r)$ vanishes for $r \geq 1$. Then, as $\varepsilon \to 0$, we have $\varepsilon \|\Delta f\|_2^2 \to \mathbb{R}_n^2$ and $\varepsilon \|f\|_2^2 \to 1$. Also, the constant $C = 1$ in (1.10) clearly cannot be improved since equality obviously holds for radial $f$.

Next we present a direct analogue of Theorem 3 for spherical derivatives.

**Theorem 4** Let $n \geq 2$. Then

$$(n-1)^2 \left\| \frac{f}{|x|^2} \right\|^2_2 \leq \sum_{j=1}^{n} L_j^2 f \right\|^2_2 \tag{1.11}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that $ Pf = 0$. If $n \geq 3$, then

$$\left\| \sum_{j=1}^{n} L_j^2 f \right\|^2_2 \leq C_n \|\Delta f\|^2_2 \tag{1.12}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, with $C_n = 1$ when $n \geq 4$, with $C_3 = \frac{64}{25}$ when $n = 3$, and the inequality fails for any $C_2 < \infty$ when $n = 2$. Furthermore, the constants in (1.11) and (1.12) are best possible.

The inequality (1.11) follows immediately from Theorem 2 since we have

$$\left\| \frac{f}{|x|^2} \right\|^2_2 = \sum_{k=1}^{\infty} \left\| \frac{P_k f}{|x|^2} \right\|^2_2$$

whenever $ Pf = 0$, and $\mu_k \geq n - 1$ for $k \geq 1$. It is also clear from this argument that the constant is best possible since equality holds in (1.11) for functions in $\mathcal{H}_1(\mathbb{R}^n)$.

The inequality in (1.12) was considered by Evans–Lewis in [13, Corollary 1]. An examination of their proof yields the explicit constants $C_n = 1$ when $n \geq 4$, and $C_3 = 4$ (see Section 4.1 for details of the three-dimensional case), but there appears to be no discussion on the best constant in [13]. Note that the fact that (1.12) holds with $C_n = 1$ when $n \geq 4$ follows immediately from (1.6). The case $n = 3$ is much more subtle and we shall see that it is possible to obtain the best constant by making use of (1.6) and (1.8).

Finally, in a slightly different direction, we present a refinement of the Rellich inequality (1.1) in which an additional term is added to the left-hand side. For the statement, we introduce the constant

$$\tilde{\mathcal{R}}_n = (n-1)(n-1 + 2 \mathcal{R}_n)$$

and write $P^\perp = I - P$ for the projection operator onto the orthogonal complement of $\mathcal{H}_0(\mathbb{R}^n)$ (closed subspace of $L^2(\mathbb{R}^n)$ consisting of radially symmetric functions).

**Theorem 5** Let $n \geq 2$. For all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\mathcal{R}_n^2 \left\| \frac{f}{|x|^2} \right\|^2_2 + \tilde{\mathcal{R}}_n \left\| \frac{P^\perp f}{|x|^2} \right\|^2_2 \leq \|\Delta f\|^2_2. \tag{1.13}$$

Since $\tilde{\mathcal{R}}_n \geq 0$ for $n \geq 3$ we immediately recover the classical Rellich inequality (1.1) from (1.13). If we instead apply (1.13) for input functions satisfying $Pf = 0$, then we obtain

$$(n-1)^2 \left\| \frac{f}{|x|^2} \right\|^2_2 \leq \|\Delta f\|^2_2 \quad \text{for} \quad Pf = 0, \tag{1.14}$$

In fact, an abstract version of (1.12) was considered in [13], and we shall identify the best constants in such a setting in Section 4.2.
and thus a quantitative improvement in the size of the constant for \( n \geq 3 \). This observation is not new and follows from the abstract result in [13, Theorem 1] (see also work by Caldiroli–Musina [6]); here we give a different perspective via Theorems 1 and 2. The inequality (1.14) extends to the Sobolev space \( H^2(\mathbb{R}^n) \) for \( n \geq 3 \) (with \( Pf = 0 \)) and this follows from an additional density argument (for example, by following [3, Proposition 9]). We also remark that, conversely, (1.13) can be derived from (1.1) and (1.14).\(^2\)

As an application which ties things discussed in the present paper together, we shall see that it is possible to give another proof of (1.12) with the best constant in three dimensions by using (1.14) (see Section 4.1).

The observation that the Hardy inequality on \( L^2(\mathbb{R}^n) \) may be improved by constraining the class of input functions to \( Pf = 0 \) may be found in work of Birman–Laptev [4] and Ekholm–Frank [12] in the context of the spectral theory of Schrödinger operators (see also [3, 11]). The current paper may be viewed as a natural continuation of [3] in the context of the Rellich inequality.

**Organization** In Section 2, we prove the identities in Theorems 1 and 2. The claims in Theorem 3 have already been justified above (in the discussion after the statement of Theorem 3). In Sect. 3 we prove Theorems 4 and 5. Finally, in Sect. 4, we give an alternative proof of (1.12) in three dimensions via (1.14) and using ideas from [13]. This argument extends to the abstract setting considered in [13, Corollary 1] and so we conclude by proving a generalization of Theorem 4 in Sect. 4.2.

## 2 Proofs of Theorems 1 and 2

We write \( H_n = \frac{1}{2} (n - 2) \) for the constant associated with the Hardy inequality on \( L^2(\mathbb{R}^n) \).

**Proof of (1.4)** The argument in [17, Theorem 1.1] is valid for \( n \neq 3 \), so here we give details for the case \( n = 3 \); this case requires some clarification since the proof of [17, Theorem 1.1] proceeds initially using integration by parts to write

\[
\frac{\| f \|_{L^2(|x|^2)}^2}{2} = \frac{2}{(n - 3)(n - 4)} \left( \frac{\| \partial_r f \|_{L^2(|x|)}^2}{2} + \text{Re} \left( \frac{f}{|x|}, \partial_r^2 f \right) \right).
\]

In our current case \( n = 3 \), this is valid in the sense that

\[
\frac{\| \partial_r f \|_{L^2(|x|)}^2}{2} = -\frac{1}{H_3} \text{Re} \left( \frac{f}{|x|^2}, \partial_r f \right),
\]

as can be verified by an integration by parts. Thus, we may proceed along the same line of reasoning as in the proof of [17, Theorem 1.1] as follows. Firstly, an integration by parts gives

\[
\frac{\| f \|_{L^2(|x|^2)}^2}{2} = -\frac{1}{H_3} \text{Re} \left( \frac{f}{|x|^2}, \partial_r f \right).
\]

\(^2\) Indeed, by orthogonality, applications of (1.1) and (1.14), and the commutativity of \( P \) and \( \Delta \), we have

\[
\mathcal{R}_n \frac{f}{|x|^2}^2 + \mathcal{R}_n \frac{P \frac{1}{|x|^2}}{|x|^2}^2 = R_n^2 \left( Pf \right)_{|x|^2}^2 + (R_n + n - 1)^2 \left( \frac{P \frac{1}{|x|^2}}{|x|^2} \right)^2 \leq \| \Delta Pf \|^2 + \| \Delta P \frac{1}{|x|^2} \|^2 = \| \Delta f \|^2.
\]

The observation here was pointed out to us on an earlier version of the present paper.
Therefore, by an application of the (easily verified) equivalence
\[
\|u\|^2 = -c \text{Re}\langle u, v \rangle + b \iff \frac{1}{c^2} \|u\|^2 = \|v\|^2 - \left\| v + \frac{1}{c} u \right\|^2 + \frac{2b}{c^2},
\]
which holds abstractly for \( u, v \) in some Hilbert space, \( b \in \mathbb{R} \) and \( c \in \mathbb{R} \setminus \{0\} \), we obtain
\[
H_3 \left\| \frac{f}{|x|^2} \right\|^2 = \| \partial_r \frac{f}{|x|} \|^2 - \left\| \partial_r \frac{f}{|x|} + H_3 \frac{f}{|x|^2} \right\|^2.
\]
Hence
\[
\left\| \frac{\partial_r f}{|x|} \right\|^2 = (H_3 - 1)^2 \left\| \frac{f}{|x|^2} \right\|^2 + \left\| \partial_r \frac{f}{|x|} + H_3 \frac{f}{|x|^2} \right\|^2.
\]
Yet another integration by parts gives
\[
\text{Re} \left\{ \frac{f}{|x|^2} \partial_r^2 f \right\} = \text{Re} \left\{ \frac{f}{|x|^2}, \partial_r f \right\} + 2(H_3 - 1) \left\| \frac{f}{|x|^2} \right\|^2,
\]
and therefore, from (2.1), we get
\[
\left\| \frac{f}{|x|^2} \right\|^2 = -\frac{1}{R_3} \text{Re} \left\{ \frac{f}{|x|^2}, \partial_r f \right\} - \frac{1}{R_3} \left\| \partial_r \frac{f}{|x|} + H_3 \frac{f}{|x|^2} \right\|^2
\]
and hence, by (2.3) again, we may conclude (1.4) for \( n = 3 \).

\textbf{Proof of (1.5)} For \( a \in \mathbb{R} \), we start by observing that
\[
\left( n - 2 - a \right)^2 \left\| \frac{g}{|x|^{1 + \frac{a}{2}}} \right\|^2 = \left\| \partial_r \frac{g}{|x|^{\frac{a}{2}}} \right\|^2 - \left\| n - 2 - a \frac{g}{|x|^{1 + \frac{a}{2}}} + \partial_r \frac{g}{|x|^{\frac{a}{2}}} \right\|^2
\]
holds for all \( g \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \). Indeed, this follows by expanding the second term on the right-hand side, and then using integration by parts to see that
\[
2\text{Re} \left\{ \frac{g}{|x|^{1 + \frac{a}{2}}}, \partial_r \frac{g}{|x|^{\frac{a}{2}}} \right\} = -(n - 2 - a) \left\| \frac{g}{|x|^{1 + \frac{a}{2}}} \right\|^2.
\]
Now taking \( a = n + 2 \) and \( g \) given by
\[
g(x) = \left( \frac{\partial_r f}{|x|} + (H_n - 1) \frac{f}{|x|^2} \right) |x|^{\frac{n+4}{2}},
\]
we see that (1.5) follows from (1.4) and (2.5).

\textbf{Proof of (1.6)} It was shown in [17, Theorem 1.2] that
\[
\| \Delta f \|^2 = \| \Delta_r f \|^2 + \sum_{j=1}^n L_j f \|^2 = 2R_n \sum_{j=1}^n \left\| \frac{L_j f}{|x|} \right\|^2 + 2 \sum_{j=1}^n \left\| \partial_r L_j f + H_n \frac{L_j f}{|x|} \right\|^2
\]
holds for \( f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) and \( n \geq 5 \), but in fact the proof works just as well for any \( n \geq 2 \). Now note that \( L_j \partial_r = (\partial_r + \frac{1}{|x|}) L_j \) and thus, since we also know that \( L_j \) commutes with multiplication by radial functions, we may write
\[
\sum_{j=1}^n \left\| \partial_r L_j f + H_n \frac{L_j f}{|x|} \right\|^2 = \sum_{j=1}^n \left\| L_j \left( \partial_r f + (H_n - 1) \frac{f}{2|x|} \right) \right\|^2.
\]
By using the integration by parts identity
\[ (g, L_j h) = -(L_j g, h) + (n - 1)\langle g, \frac{x_j}{|x|^2} h \rangle, \]
and the fact that \( \sum_{j=1}^n x_j L_j = 0 \), we obtain (1.6).

**Proof of (1.8)** Using (1.3) and expanding into spherical harmonics, (1.7) yields
\[
\left\| \sum_{j=1}^n L_j^2 f \right\|_2^2 = \left\| \frac{1}{|x|^2} \Delta_{S^{n-1}} f \right\|_2^2 = \sum_{k=1}^\infty \mu_k^2 \int_0^\infty \| (P_k f)(r(\cdot)) \|_{L^2(\mathbb{S}^{n-1})}^2 r^{-5} \, dr
\]
from which we obtain the first identity in (1.8). Next, using (2.6), the fact that \( L_j \) commutes with multiplication by \( |x|^{-2} \), and the identity \( \sum_{j=1}^n x_j L_j = 0 \), we obtain
\[
\sum_{j=1}^n \left\| \frac{L_j f}{|x|^2} \right\|_2^2 = - \left\{ f, \frac{1}{|x|^2} \sum_{j=1}^n L_j^2 f \right\}.
\]
From this we obtain the second identity in (1.8) in a similar manner by expanding into spherical harmonics and using (1.7).

**3 Proofs of Theorems 4 and 5**

As preparation for the proof of (1.12) in Theorem 4, we introduce the weighted one-dimensional version of the Hardy inequality (see, for example, [8, Proposition 2.4]).

**Lemma 6** Let \( t \in \mathbb{R} \setminus \{ 1 \} \). The following inequality holds for all \( g \in C_0^\infty(\mathbb{R} \setminus \{ 0 \}) \) and the constant is best possible:
\[
\int_0^\infty |g'(r)|^2 r^{t+2} \, dr \geq \left( \frac{t + 1}{2} \right)^2 \int_0^\infty |g(r)|^2 r^t \, dr \tag{3.1}
\]
As a consequence, we have the weighted Rellich inequality
\[
\int_0^\infty |g''(r)|^2 r^{t+4} \, dr \geq \left( \frac{t + 3}{2} \right)^2 \left( \frac{t + 1}{2} \right)^2 \int_0^\infty |g(r)|^2 r^t \, dr \tag{3.2}
\]
for \( t \neq -3, -1 \), and it is also known that the constant here is best possible\(^3\).

**Proof of Theorem 4** We have already justified (1.11) after the statement of Theorem 4, so here we prove (1.12) and justify the claim that the derived constants are best possible. First let \( n \geq 4 \). Since \( R_n \geq 0 \) the inequality (1.12) follows immediately from (1.6) with \( C_n = 1 \). To see that this is optimal, test the inequality on \( f \) of the form
\[
f(x) = g(|x|) Y_k(\frac{x}{|x|}) \tag{3.3}
\]
where \( Y_k \) is any spherical harmonic of degree \( k \) with \( L^2(\mathbb{S}^{n-1}) \) norm equal to 1. Then, from \( \Delta = \Delta_r + \frac{1}{|x|^2} \Delta_{S^{n-1}} \), we easily obtain
\[
C_1 \mu_k^2 \leq C_n (C_1 \mu_k^2 + C_2 \mu_k + C_3)
\]
\(^3\) To see this directly, one may argue as in [9] and consider \( g \) of the form \( g_\varepsilon(r) = r^{-\frac{t+1}{2} + \frac{\varepsilon}{2}}\chi(r) \) where \( \chi \) is infinitely smooth, \( \chi(0) = 1 \) and \( \chi(r) \) vanishes for \( r \geq 1 \). Then, as \( \varepsilon \to 0 \), we have \( \varepsilon \int_0^\infty |g_\varepsilon(r)|^2 r^t \, dr \to 1 \) and \( \varepsilon \int_0^\infty |g_\varepsilon'(r)|^2 r^{t+2} \, dr \to \left( \frac{t+3}{2} \right)^2 \left( \frac{t+1}{2} \right)^2 \).
for some constants $C_1$, $C_2$, $C_3$ (independent of $k$) which are all finite and $C_1 \neq 0$ by an appropriate choice of $g$. By taking $k \to \infty$, it is clear that $C_n \geq 1$ is necessary for (1.12) to hold.

The case $n = 2$ can also be handled in an easy manner by testing the inequality (1.12) on $f$ of the form (3.3) with $k = 1$. Indeed, since $-\Delta_{\mathbb{S}^1} Y_1 = Y_1$ we see that (1.12) is equivalent to

$$\left\| \frac{f}{|x|^2} \right\|_2^2 \leq C_2 \| \Delta f \|_2^2$$

for such $f$. On the other hand, we know that the above Rellich inequality cannot hold for such $f$ with any constant $C_2 < \infty$ when $n = 2$ (since (1.2) does not hold for such $f$).

Suppose now that $n = 3$ and $C_3 = \frac{64}{25}$. To prove (1.12), by (1.6), it is equivalent to prove

$$\frac{3}{2} \sum_{j=1}^\infty \frac{L_j}{|x|^2} \left\| \frac{f}{|x|^2} \right\|_2^2 \leq \| \Delta f \|_2^2 + \left( 1 - \frac{1}{C_3} \right) \sum_{j=1}^\infty \mu_j^2 \left\| \frac{P_j}{|x|^2} \right\|_2^2 - 2 \left\langle \sum_{j=1}^3 L_j f_* , f_* \right\rangle .$$

For this, we use Theorem 2 to write the goal as

$$\frac{3}{2} \sum_{k=1}^\infty \mu_k \left\| \frac{P_k}{|x|^2} \right\|_2^2 \leq \| \Delta f \|_2^2 + \left( 1 - \frac{1}{C_3} \right) \sum_{k=1}^\infty \mu_k^2 \left\| \frac{P_k}{|x|^2} \right\|_2^2 - 2 \left\langle \sum_{j=1}^3 L_j f_* , f_* \right\rangle .$$

For all $k \geq 2$ we have $\frac{3}{2} \mu_k \leq (1 - \frac{1}{C_3}) \mu_k^2$ since $C_3 > \frac{4}{3}$. So it suffices to establish

$$\left( \frac{4}{C_3} - 1 \right) \left\| \frac{P_k}{|x|^2} \right\|_2^2 \leq \| \Delta f \|_2^2 - 2 \left\langle \sum_{j=1}^3 L_j f_* , f_* \right\rangle .$$

(3.4)

By expanding $f$ into spherical harmonics, we see that it suffices to prove

$$\left( \frac{4}{C_3} - 1 \right) \int_0^\infty |g(r)|^2 \frac{dr}{r^2} \leq \int_0^\infty \left( g''(r) + 2 g'(r) \right)^2 r^2 dr + 4 \int_0^\infty \left( g'(r) - \frac{1}{2r} g(r) \right)^2 r^2 dr .$$

By expanding the squares on the right-hand side, integration by parts on the cross terms and rearranging, we see that this is equivalent to

$$\frac{4}{C_3} \int_0^\infty |g(r)|^2 \frac{dr}{r^2} \leq \int_0^\infty |g''(r)|^2 r^2 dr + 6 \int_0^\infty |g'(r)|^2 r^2 dr .$$

(3.5)

By double use of (3.1), we see that

$$\int_0^\infty |g''(r)|^2 r^2 dr + 6 \int_0^\infty |g'(r)|^2 r^2 dr \geq \frac{25}{4} \int_0^\infty |g'(r)|^2 r^2 dr \geq \frac{25}{16} \int_0^\infty |g(r)|^2 \frac{dr}{r^2}$$

and hence, by the choice of $C_3$, we obtain (3.5) and thus (1.12).

To see that $C_3 = \frac{64}{25}$ is the best constant, consider $f$ of the form (3.3) with $k = 1$. From the above argument, for such $f$, we see that (1.12) is equivalent to (3.5), and so matters reduce to obtaining the optimal constant in (3.5). One way to argue is to set $g(r) = r^t h(r)$ and note that by choosing $v = \sqrt{3}$ the inequality (3.5) is equivalent to (3.2) with $t = 2v - 2$ but with constant $\frac{4}{C_3} + 6$. The optimal constant in (3.2) for such $t$ is $\frac{121}{16}$, and this allows us to deduce the optimality of $C_3 = \frac{64}{25}$ in (1.12). \(\square\)

\footnote{Note that this obviously holds with $C_3 = 4$ and this recovers what can be obtained using the argument by Evans–Lewis \cite[Corollary 1]{13}.}
Proof of Theorem 5  By (1.4), (1.6), and (1.8), the difference between the left-hand side and right-hand side of (1.13) coincides with
\[
\sum_{k=1}^{\infty} \left( \mu_k (\mu_k + 2R_n) - \tilde{R}_n \right) \left\| \frac{P_k P^\perp f}{|x|^2} \right\|_2^2 \\
+ 2R_n \left\| \frac{f_\star}{|x|^2} \right\|_2^2 + 2 \left\{ - \sum_{j=1}^{n} L_j^2 f_\star, f_\star \right\} + \left\| \Delta f + R_n \frac{f}{|x|^2} \right\|_2^2.
\]
Clearly \( \mu_k (\mu_k + 2R_n) \geq \tilde{R}_n \) for all \( k \geq 1 \), and thus (1.13) follows when \( n \geq 4 \) (since \( R_n \geq 0 \)). For \( n = 2, 3 \) we follow a similar argument, except we use (1.5) rather than (1.4). \( \square \)

4 An abstract version of Theorem 4

4.1 Alternative derivation of \( C_3 = \frac{64}{25} \) in (1.12)

For \( n = 3 \), here we give a different proof of (1.12) with \( C_3 = \frac{64}{25} \) by modifying the argument in [13, Corollary 1]. We begin by expanding
\[
\left\| \Delta f \right\|_2^2 = \left\| \Delta f \right\|_2^2 + \left\| \sum_{j=1}^{3} L_j^2 f \right\|_2^2 + 2 \text{Re} \left\{ \Delta f, \sum_{j=1}^{3} L_j^2 f \right\}.
\]

For the cross term, we use [13, Eq. (2.15)] (which relies on integration by parts and (3.1)) to estimate
\[
2 \text{Re} \left\{ \Delta f, \sum_{j=1}^{3} L_j^2 f \right\} \geq - \frac{3}{2} \left( \frac{f}{|x|^2} \right) \left( \sum_{j=1}^{3} L_j^2 f \right) \geq - \frac{3}{4} \left( \frac{1}{\delta} \left\| \frac{f}{|x|^2} \right\|_2^2 + \delta \left\| \sum_{j=1}^{3} L_j^2 f \right\|_2^2 \right)
\]
for any \( \delta > 0 \). Using this, along with (1.9), we obtain
\[
\left( 1 - \frac{3\delta}{4} \right) \left\| \sum_{j=1}^{3} L_j^2 f \right\|_2^2 \leq \left( \frac{3}{4\delta} - \frac{9}{16} \right) \left\| \frac{f}{|x|^2} \right\|_2^2 + \left\| \Delta f \right\|_2^2.
\]

Up to here we have followed the argument in [13, Corollary 1]. At this point, the argument proceeds in [13] by using the classical Rellich inequality (1.1)\(^5\). However, we observe that in order to prove (1.12) it suffices to consider \( f \) satisfying \( Pf = 0 \) (since \( \sum_{j} L_j^2 f = \sum_{j} L_j^2 P^\perp f \)) and hence we may instead apply the refined version of the Rellich inequality (1.14). For \( 0 < \delta < \frac{4}{3} \), this yields
\[
\left\| \sum_{j=1}^{3} L_j^2 f \right\|_2^2 \leq \frac{16(3 + 4\delta)}{25\delta(4 - 3\delta)} \left\| \Delta f \right\|_2^2
\]
and optimizing (i.e. taking \( \delta = \frac{1}{2} \)) we get (1.12) with \( C_3 = \frac{64}{25} \).

\(^5\) Indeed, using (1.1) and rearranging, we obtain the constant \( \frac{16}{3\delta(4 - 3\delta)} \) for any \( 0 < \delta < 4/3 \). Optimizing in \( \delta \) yields (1.12) with constant 4.
4.2 An abstract version of Theorem 4

The argument in Section 4.1 actually extends to the abstract setting considered in [13, Corollary 1] concerning the generalized Laplacian $\mathcal{L} = \Delta_r + \frac{1}{r^2} \Lambda$. Here, $-\Lambda$ is a non-negative, self-adjoint operator on $L^2(\mathbb{S}^{n-1})$ whose spectrum is purely discrete with isolated eigenvalues $\{\lambda_k\}_{k \in \mathbb{Z}}$ which may only accumulate at infinity. We also assume that zero is an eigenvalue of $-\Lambda$ and write $\lambda_0 = 0$. Then we are able to obtain the following.

**Theorem 7** Let $n = 2, 3$. Then

$$\left\| \frac{1}{|x|^2} \Lambda f \right\|_2^2 \leq (1 - R_n \sqrt{M_n})^2 \| \Delta f \|_2^2,$$

(4.1)

where $M_n = \max_{k \in \mathbb{Z} \setminus \{0\}} (\lambda_k + R_n)^{-2}$. Moreover, when $M_n = (\lambda_{k_0} + R_n)^{-2}$ and $\lambda_{k_0} > -R_n$, the constant in (4.1) is best possible.

One can check that the constant in (4.1) is consistent with Theorem 4 in the sense that it becomes $\frac{64}{27}$ when $n = 3$ and infinite when $n = 2$. When $n \geq 4$ the inequality in (4.1) was obtained in [13, Corollary 1] with constant 1 and, by the obvious extension of the argument in our earlier proof of Theorem 4, this is best possible.

**Proof of Theorem 7** We consider the case $n = 3$ (the argument for $n = 2$ is similar). To see (4.1) we follow the argument in Section 4.1, and simply replace use of (1.14) by

$$\frac{1}{M_3} \left\| f \right\|_2^2 \leq \| \mathcal{L} f \|_2^2 \quad \text{whenever} \quad \int_{\mathbb{S}^2} f(r, \theta) u_0(\theta) \, d\sigma(\theta) = 0 \text{ for all } r > 0,$$

(4.2)

where $u_0$ denotes a normalized eigenfunction of $-\Lambda$ corresponding to the eigenvalue 0. The inequality (4.2) follows from [13, Theorem 1].

To see the optimality of the constant in (4.1) when $M_3 = (\lambda_{k_0} - \frac{3}{4})^{-2}$ and $\lambda_{k_0} > \frac{3}{4}$, we test on functions of the form

$$f(x) = g(|x|) u_{k_0}(\frac{x}{|x|}),$$

where $u_{k_0}$ is a normalized eigenfunction of $-\Lambda$ corresponding to the eigenvalue $k_0$. For such $f$, the inequality $\left\| \frac{1}{|x|^2} \Lambda f \right\|_2^2 \leq C \| \Delta f \|_2^2$ is equivalent to

$$\lambda_{k_0} (2 + (\frac{1}{4} - 1) \lambda_{k_0}) \int_0^\infty |g(r)|^2 \frac{dr}{r^2} \leq \int_0^\infty |g''(r)|^2 r^2 \, dr + 2(1 + \lambda_{k_0}) \int_0^\infty |g'(r)|^2 \, dr.$$  

We now use the fact that, given any $R > 0$, the best constant in the inequality

$$\tilde{C} \int_0^\infty |g(r)|^2 \frac{dr}{r^2} \leq \int_0^\infty |g''(r)|^2 r^2 \, dr + R \int_0^\infty |g'(r)|^2 \, dr$$

(4.3)

is given by $\tilde{C} = \frac{1}{4} (1 + R)$. This follows because the inequality (4.3) is equivalent to

$$\int_0^\infty |h''(r)|^2 r^{2v+2} \, dr \geq (\tilde{C} - \frac{1}{4} R(2 - R)) \int_0^\infty |h(r)|^2 r^{2v-2} \, dr$$

via the relabeling $g(r) = r^v h(r)$ with $v = (\frac{R}{2})^{1/2}$. We then invoke the fact that the constant in (3.2) is best possible. From the above discussion we may conclude that $C \geq (\frac{4 \lambda_{k_0}}{4 \lambda_{k_0} - 3})^2$, and this yields the optimality of the constant in (4.1).

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6 See [13] (and also [7]) for concrete examples of such operators.
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