Three new weak formulations of the problem of American call options valuation

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Abstract. Three new weak formulations of the problem of American call options valuation are given. The first of these is a parabolic obstacle problem in a finite domain. The second is a parabolic variational inequality with a convex and Lipschitz-continuous functional and the last one is a semilinear parabolic equation with a discontinuous spatial operator. All these problems are equivalent – they have the same unique solution. Different formulations can be used both for theoretical research and for constructing numerical methods.

1. Introduction
The option is one of the most important financial derivatives, and a wide variety of options are traded in exchanges. Its gives to its owner the right to buy (call option) or to sell (put option) a fixed quantity of assets of a specified stock at a fixed price (exercise or strike price). There are two major types of traded options. One is the American option that can be exercised at any time prior to its expiry date, and the other option, which can only be exercised on the expiry date, is called the European option. It was shown by Black and Scholes that the value of an European option is governed by a second-order differential equation of degenerate parabolic type [1]. This is now referred to as the Black–Scholes equation. The value of an American option is determined by a linear complementarity problem involving the Black–Scholes differential operator and a constraint on the value of the option (cf. e.g. [2]). In the case of put options this complementarity problem can also be formulated in a weak form as a variational inequality with an obstacle inside a domain (cf. e.g. [3]).

In this paper we consider call options problem and propose three new formulations of it: a parabolic variational inequality in a finite domain with an obstacle inside a domain, a parabolic variational inequality with a convex and Lipschitz-continuous functional and a semilinear parabolic equation with a discontinuous spatial operator. The existence of a unique solution is proved, the same for all the constructed problems. Different formulations of the problem are useful from the point of view of investigating the properties of its solution, and also from the point of view of constructing various approximations [4], [5], [6].

1.1. Formulation of the problem
Let \( u \) denote the value of an American call option with strike price \( K \), expiry date \( T \), and let \( x \) be the price of the underlying asset of the option. It is known that \( u \) satisfies the following linear complementarity problem almost everywhere (a.e.) in \( Q_T = \mathbb{R}_+ \times (0,T) \), \( \mathbb{R}_+ = \{ x \in \mathbb{R} : x > 0 \} \) [2]:

\[
u'(x,t) + A(t)u(x,t) \geq 0,
\]

(1)
$u(x,t) - \psi(x) \geq 0,$  
(2)  
$(u'(x,t) + A(t)u(x,t))(u(x,t) - \psi(x)) = 0,$  
(3)  
with Cauchy data

$u(x,0) = \psi(x), \quad x \in \mathbb{R}.$  
(4)  

Here $t$ denote the time until expiry, $u' = \partial u / \partial t$, $\psi(x)$ is a payoff function, $A(t)$ is the Black–Scholes operator defined by the equality

$A(t)u = -\sigma^2(x,t)x^2 \frac{\partial^2 u}{\partial x^2} + (d(t) - r(t))x \frac{\partial u}{\partial x} + r(t)u,$  
(5)  

where $\sigma$ is a local volatility, $d$ is a variable dividend yield and $r$ is an interest rate. We focus on the case of a vanilla call, when the payoff function is

$\psi(x) = (x - K)^+, \quad v^+ = \max\{0,v\}.$  

Let $u$ be a solution of the complementarity problem (1)-(4). The set

$Q_\psi = \{(x,t) \in Q_T : u(x,t) = \psi(x)\}$

is called the region of exercise and its boundary is called the exercise boundary or free boundary. Let it admits a parameterization $x = \gamma(t), t \in [0,T]$. The function $\gamma$ (early exercise curve) is a hidden unknown of the considered problem. Further we assume that the region of exercise has nonzero Lebesgue measure, since otherwise due to (3 the complementarity problem reduces to Black–Scholes equation $u'(x,t) + A(t)u(x,t) = 0$ in $Q_T$ with Cauchy data (4).

We make some assumptions on $\sigma,d,r$, which ensures the existence of a weak solution of the problem (1)-(4), its regularity and also the non-emptiness of the region of exercise:

$$(H_1) \text{ there exists constants } \underline{\sigma}, \overline{\sigma}, \underline{r}, \overline{r}, d, \overline{d}, C_{\sigma}, \text{ such that a.e. on } Q_T \quad \underline{\sigma} \leq \sigma(x,t) \leq \overline{\sigma}, \quad \left| x \frac{\partial \sigma(x,t)}{\partial x} \right| \leq C_{\sigma}, \quad 0 \leq \underline{r} \leq r(t) \leq \overline{r}, \quad 0 \leq d \leq d(t) \leq \overline{d}. \quad$$

We note separately the case of constant parameters. It is well known that in the case of constant $\sigma,d,r$, the early exercise curve $x = \gamma(t), t \in [0,T]$, of an American call option is a continuous increasing function for $t > 0$, and (cf. e.g. [6, p. 257])

$$\lim_{t \to 0^+} \gamma(t) = K \max\left\{ \frac{L}{d}, 1 \right\}, \quad \underline{\sigma} \leq \gamma(t) \leq \frac{\mu}{\mu - 1} K.$$  

Here a constant $\mu = \mu(\sigma,d,r) \geq 1$ is such that $\mu = 1$ if and only if $d = 0$, so the exercise boundary is bounded by explicit bounds. This property allows us to consider (1)-(4) only in a bounded domain $Q_{\gamma T} = (0,L) \times (0,T)$ with the boundary condition $u(L,t) = \psi(L), t \in [0,T]$. Further we shall prove that this remains true in the general case under the assumption $(H_1)$ (see the theorem 1 below).

1.2. Function spaces

Let $L > 0, I = (0,L),$ $H = L_2(I)$ with the norm and the inner product

$$\|v\|_H^2 = \int_I v^2(x) \, dx, \quad \langle u,v \rangle = \int_I u(x)v(x) \, dx.$$

The space $D(I)$ of the infinitely smooth functions with compact support in $I$ is dense in $H$. By $V$ we denote the weighted Sobolev space

$$V = \left\{ v \in H : x \, dv / dx \in H \right\}, \quad \|v\|_V^2 = \int_I \left( v^2 + (x \, dv / dx)^2 \right) \, dx,$$
where the derivative is understood in the sense of the distributions on $I$. It is known that $V$ is the separable Hilbert space, the functions of $V$ are continuous on $I$, however, the limit $\lim_{x \to 0} v(x)$ may not exist for $v \in V$. An example is the function $v = \sin(x^\alpha)$, where $-1/2 < \alpha < 0$. We note also that the embedding $V \subset H$ is not compact.

We introduce the subspace $V_0 = \{ v \in V : v(L) = 0 \}$ and denote by $V^*$ the dual space of $V_0$. The inclusions $V_0 \subset H = H^* \subset V^*$ are continuous and dense, so, we can extend the inner product of $H$ to the duality pairing between $V^*$ and $V_0$. Then $(u, v)$ is the scalar product in $H$ provided that $u \in H$. We call $K_\nu$ the subset of $V$

$$K_\nu = \{ v \in V : \nu(x) \geq \psi(x) \ \text{in} \ I, \nu(L) = \psi(L) \}.$$

We also use well-known Sobolev spaces $H^m(I)$ and the spaces of the functions from interval $(0, T)$ to a Banach space $B$, namely, $C((0, T); B)$, $L^p((0, T); B)$ and $W^m_p((0, T); B)$. In particular, $L_1((0, T); B)$ is the space of square integrable functions on $(0, T)$ with values in $B$; $H^1((0, T); B) = W^1_1((0, T); B)$. We also define the space $W = \{ w \in L_2((0, T), V_0) : w' \in L_2((0, T), V^*) \}$ equipped with the norm $\| w \|_{W} = \| w \|_{L_2((0, T), V_0)} + \| w' \|_{L_2((0, T), V^*)}$.

2. A first formulation: obstacle problem

2.1. Perpetual call options

Let us define the operator

$$A_{\nu} u = -\frac{\sigma^2 x^2}{2} \frac{\partial^2 u}{\partial x^2} + (d - \bar{r}) x \frac{\partial u}{\partial x} + \bar{r} u, \quad x \in R_+.$$

Consider the following stationary free boundary problem.

**Problem (P$_\nu$)** Find $L_\nu > 0$ and bounded function $u_\nu = u_\nu(x)$, satisfying the relations

$$A_{\nu} u_\nu(x) = 0, \quad x \in (0, L_\nu), \quad u_\nu(L_\nu) = \psi(L_\nu), \quad u_\nu'(L_\nu) = \psi'(L_\nu).$$

The solution of this problem is well known (cf. e.g. \cite[p. 259]{7}). It has the form

$$u_\nu(x) = \left( \frac{L_\nu}{\mu} \right)^{\frac{1}{\mu}} \frac{\mu x}{L_\nu}, \quad L_\nu = \frac{\mu}{\mu - 1} K,$$

where $\mu = (a + (a^2 + 2 \sigma^2 \bar{r})^{1/2}) / \sigma^2 > 1, \quad a = \sigma^2 / 2 + d - \bar{r}$.

We fix an arbitrary $L \geq L_\nu$ and we will assume that the function $u_\nu(x)$ is defined on $[0, L]$, by setting $u_\nu(x) = \psi(x)$ for all $x \in [L_\nu, L]$. Note that $A_{\nu} \psi = d_\nu - \bar{r} K > 0$ on $[L_\nu, L]$. We take $u_\nu(x)$ as a bounded function and

$$x^2 \frac{\partial^2 u_\nu}{\partial x^2} \geq 0, \quad x \frac{\partial u_\nu}{\partial x} - u_\nu \geq 0, \quad x \frac{\partial u_\nu}{\partial x} \geq 0 \quad \text{on} \quad [0, L]. \quad (6)$$

It is easy to verify that $u_\nu(x)$ is a solution of the following complementarity problem on $(0, L)$:

$$A_{\nu} u_\nu(x) \geq 0, \quad u_\nu(x) - \psi(x) \geq 0, \quad A_{\nu} u_\nu(x)(u_\nu(x) - \psi(x)) = 0. \quad (7)$$

Next we prove (see theorem 1) that

$$\psi(x) \leq u(x, t) \leq u_\nu(x), \quad \text{for all} \quad (x, t) \in [0, L] \times [0, T], \quad (8)$$
where \( u \) the solution of (1)-(4). It follows from (8) that \( u(x,t) = \psi(x) \) on \([L_0,L] \times [0,T]\). The property (8) allows us to consider (1)-(4) only in a bounded domain \([0,L] \times [0,T]\), \( L \geq L_0 \), with the boundary condition \( u(L,t) = \psi(L) \), \( t \in [0,T] \). In this regard, we consider

**Problem** (\( P_\psi \)) Find \( u \) satisfying relations (1)-(3) a.e. in \( Q_{LT} = (0,L) \times (0,T), \ L \geq L_0, \) and initial and boundary conditions \( u(x,0) = \psi(x), \ x \in [0,L], \ u(L,t) = \psi(L), t \in [0,T] \).

### 2.2. Variational inequality with an obstacle inside a domain

Further the Black–Scholes operator will be considered as a linear operator from \( V \) to \( V' \), with the bilinear form, having for a.e. \( t \in (0,T) \) the form

\[
(A(t)u,v) = \int_I \left( \frac{\sigma^2(x,t)u_x^2}{2} \frac{\partial v}{\partial x} + r(t)uv \right) dx
\]

\[
+ \int_I \left( d(t) - r(t) + \sigma^2(x,t) + \sigma(x,t) \frac{\partial \sigma(x,t)}{\partial x} \right) x \frac{\partial u}{\partial x} v dx.
\]

Note that if \( A(t)u \in H \), then \( A(t)u \) has representation (5). We define operator \( A : L_2(0,T;V) \rightarrow L_2(0,T;V') \) by relation \( (Au)(t) = A(t)u(t) \) a.e. on \((0,T)\). We note some important structural properties of these operators.

**Lemma 1.** Under assumptions \( (\mathcal{H}_i) \) there exists a positive constants \( M, \mu, \lambda \), depending on only the constants in \((\mathcal{H}_i)\), such that for all \( w \in V, v \in V_0 \)

\[
| (A(t)w,v) | \leq M \| w \|_V \| v \|_{V'}, \tag{9}
\]

\[
(A(t)v,v) + \lambda \| v \|_{H}^2 \geq \mu \| v \|_{V'}^2. \tag{10}
\]

In addition, let \( w \in L_2(0,T;V) \cap H^1(0,T;V'), \ w^- \in L_2(0,T;V_0) \). Then

\[
w^-(0) = 0, \quad (w'(t) + A(t)w(t),w'-(t)) \geq 0 \text{ a.e. on } (0,T) \implies w \geq 0. \tag{11}
\]

Here \( u \geq v \ (u \leq v) \) means that \( u(x,t) \geq v(x,t) \ (u(x,t) \leq v(x,t)) \) a.e. on \( Q_{LT} \).

**Proof.** The estimate (9) and Gårding’s inequality (10) are well-known (cf. e.g. cite[3, pp. 31, 32]). We have a.e. on \((0,T)\)

\[
(A(t)w(t),w'-(t)) = -(A(t)w'-(t),w'-(t)). \tag{12}
\]

Since \( w^- = 0 \), similarly, for \( 0 < t' \leq T \)

\[
\int_0^{t'} (w'(t),w'-(t)) dt = -\int_0^{t'} ((w^-)'(t),w'-(t)) dt = -\frac{1}{2} \| w'-(t') \|_{H}^2. \tag{13}
\]

Taking into account (12), (13), after integration first inequality (11) we get

\[
\frac{1}{2} \| w'-(t') \|_{H}^2 + \int_0^{t'} (A(t)w(t),w'-(t)) dt \leq 0.
\]

Using Gårding’s inequality (10) leads to the estimate

\[
\| w'-(t') \|_{H}^2 \leq 2 \lambda \int_0^{t'} \| w-(t) \|_{H}^2 dt.
\]

whence \( w^- = 0 \) by the Gronwall lemma.

Next, for brevity, we use the notation \( X(0,T;S) \) also for the sets \( S \subset V \), assuming

\[
X(0,T;S) = \{ v \in X(0,T;V): v(t) \in S \text{ a.e. on } (0,T) \}.
\]

Standard reasoning leads to the following equivalent variational formulation of the problem \( (P_\psi) \) (cf. e.g. cite[3, pp. 186]).
**Problem** (P) Find $u \in L^2(0,T;K_\psi)$ such that $u' \in L^2(0,T;V^*)$, $u(0) = \psi$ and a.e. on $(0, T)$

$$(u'(t) + A(t)u(t), v - u(t)) \geq 0 \quad \forall v \in K_\psi.$$  \hspace{1cm} (14)

**Theorem 1.** Under the assumption $(H_i)$ there exist a unique solutions $u$ of the problem (P). Moreover $\psi(x) \leq u(x,t) \leq u_\psi(x)$ a.e. $(x,t) \in Q_{LT}$, and, as a consequence, $u | x \in L_\infty(Q_{LT})$, $u = \psi$ on $[L_\eta, L] \times [0,T]$.

**Proof.** The existence of a unique solution of the problem (P) can be proved by a penalty method in the same way as in [8] for the case of put options. Let us prove that $u \leq u_\psi$. We have $u_\psi \in K_\psi$ and

$$(A_\omega u_\psi, v - u_\psi) \geq 0 \quad \forall v \in K_\psi.$$  \hspace{1cm} (15)

Define the function $u_\psi(t) = u_\psi$ on $[0,T]$. It is the solution of the inequality

$$(u_\psi'(t) + A(t)u_\psi(t), v - u_\psi(t)) \geq (F(t), v - u_\psi(t)) \quad \forall v \in K_\psi,$$  \hspace{1cm} (15)

with initial condition $u_\psi(0) = \psi$, and $F(t) = (A(t) - A_\omega)u_\psi \in H$. It is easy to verify that

$$(F(t), v) = \int_0^T \left[ \frac{\sigma^2}{2} \frac{\partial^2 u_\psi}{\partial x^2} \left( x \right) - \nu(t) \left( x \right) u_\psi \right] dx \geq 0, \quad v \in K_\psi,$$

since all terms on the right-hand side are nonnegative due to the assumption $(H_i)$ and (6). We now compare the inequalities (14) and (15). Let $w = u_\psi - u$. Summing the inequality (14) with trial function $v = u - w^* = \min\{u, u_\psi\} \in K_\psi$, and inequality (15) with trial function $v = u_\psi + w^* = \max\{u, u_\psi\} \in K_\psi$, we obtain $(w^*(t) + A(t)w(t), w^*(t)) \geq (F(t), w^*(t)) \geq 0$. Since $w^*(0) = 0$, it follows from (11) that $w \geq 0$, so, $u \leq u_\psi$.

3. **A second formulation: variational inequality with a Lipschitz-continuous functional**

Let $L_\omega$ be the characteristic function of the set $\omega$. Let us define functions

$$g(x,t) = (d(t)x - r(t)K)1_{[K,L]}(x), \quad G(x,t) \geq g^*(x,t),$$

and $\phi: (0,T) \times H \to R$ as follows:

$$\phi(t,v) = \int_K G(t,x)v^* \, dx.$$  

The possible choices are $G(x,t) = g^*(x,t)$ or $G = \text{ess sup}\{g^*(x,t), (x,t) \in Q_{LT}\}$. It is clear that $\phi$ is convex and Lipschitz-continuous in $H$. The function $g$ is chosen so that the functional $g^*(t) - A(t)\psi$ is positive, that is,

$$(g^*(t) - A(t)\psi, v) \geq 0 \quad \forall v \in K_\omega, \quad t \in (0,T).$$

This follows from the following representation ($\delta$ is the Dirac function):

$$A(t)\psi = -\frac{\sigma^2(K(t)K^2}{2} \delta_{x=K} + g(x,t), \quad t \in (0,T).$$

By $\partial \phi(t,u)$ we denote the subdifferential of $\phi(t,u)$ at a point $u: \partial \phi(t,u) = \{\chi \in H : \phi(t,v) - \phi(t,u) \geq (\chi(t), v - u) \forall v \in H\}$. Let $V_\psi = \{v \in V : v(L) = \psi(L)\}$.

**Problem** (P2) Find $u \in L^2(0,T;V_\psi)$ such that $u' \in L^2(0,T;V^*)$, $u(0) = \psi$ and a.e. on $(0,T)$
\[(u'(t) + A(t)u(t), v - u(t)) + \phi(t, v - \psi) - \phi(t, u(t) - \psi) \geq 0 \quad \forall v \in V_{\psi}. \quad (16)\]

We use also an equivalent to (16) writing of this variational inequality:
\[u'(t) + A(t)u(t) + \chi(t) = 0 \quad \text{in} \quad V^*, \quad (17)\]
\[\chi(t) \in \partial \phi(t, u(t) - \psi). \quad (18)\]

It is not difficult to prove that
\[
\partial \phi(t, u) = G(x, t)1_{[K, t]}(x)h(u), \quad h(s) = \begin{cases} -1, & s < 0, \\ [-1, 0], & s = 0, \\ 0, & s > 0. \end{cases}
\]

**Theorem 2.** Under the assumption \((H_1)\) there exists a unique solution \(u\) of the problem \((P_\phi)\).

**Moreover, the problems** \((P)\) **and** \((P_\phi)\) **are equivalent.**

**Proof.** (a) The existence of a unique solution to problem \((P_\phi)\) can be proved using the regularization method: \(u_\varepsilon(t) \in V_{\psi}\),
\[u'_\varepsilon(t) + A(t)u_\varepsilon(t) + \beta_\varepsilon(t, u_\varepsilon(t) - \psi) = 0 \quad \text{in} \quad V^*, \quad t \in (0, T),
\]
where for \(\varepsilon > 0\) operators \(\beta_\varepsilon(t, \cdot) : H \to H\) are defined as
\[\beta_\varepsilon(t, v) = G(x, t)1_{[K, t]}(x)h_\varepsilon(v), \quad h_\varepsilon(s) = \begin{cases} -1, & s < 0, \\ (s - \varepsilon)/\varepsilon, & 0 \leq s \leq \varepsilon, \\ 0, & s > 0. \end{cases}
\]

We omit the proof. It is based on the properties of the operators \(A(t)\) indicated in Lemma 1, as well as on the following properties: operators \(\beta_\varepsilon(t, \cdot)\) are Lipschitz-continuous, monotone, uniformly bounded, and
\[-(h_\varepsilon(s) - h_\varepsilon(s'))(s - s') \leq (\varepsilon + \nu)/4, \quad s, s' \in \mathbb{R}.
\]

(b) Let \(u\) be a solution of \((P_\phi)\). We prove that \(u(t) \in K_{\psi}\) a.e. on \((0, T)\). To this end, we choose \(v = u + e^-\), \(e = u - \psi\) in (16). In this case \(v - \psi = e^+\), so, we have \(\phi(t, v - \psi) = 0\) and \((u'(t) + A(t)u(t), e^-(t)) \geq \phi(t, e(t))\), or
\[e^-(t) + A(t)e(t), e^-(t)) \geq -(A(t)\psi, e^-(t)) + \phi(t, e(t)). \quad (19)\]

Omitting some technical calculations, we obtain
\[-(A(t)\psi, e^-(t)) + \phi(t, e(t)) = \frac{\sigma^2(K, t)K^2}{2} e^-(K, t) + (G - g, e^-(t)) \geq 0,
\]
since \(G - g = (G - g^+) + g^- \geq 0\). Thus, from (19) we have
\[(e^-(t) + A(t)e(t), e^-(t)) \geq 0, \quad e(0) = 0,
\]
and \(e^- = 0\) due to (11), that is \(u(t) \in K_{\psi}\).

(c) Let \(u\) be a solution of \((P_\phi)\). Since \(u(t) \in K_{\psi}\), and \(\phi(t, u(t) - \psi) = 0\), then
\[u'(t) + A(t)u(t), v - u(t)) + \phi(v - \psi) \geq 0 \quad \forall v \in V_{\psi}. \quad (20)\]

It follows from (20) that \(u\) also satisfies the inequality (14), that is, \(u\) is the solution of problem \((P)\).

By the uniqueness of the solutions of problems \((P)\) and \((P_\phi)\), they coincide.

4. Regularity of the solution
Theorem 3. Let the assumptions \( (H_i) \) hold and let \( u \) be a solution of (P). Then
\[
u(t) / x^k \in H^1(0, T; \mathcal{H}) \cap C([0, T]; H^1), \quad k = 0, 1, \ u \in L_2(0, T; H^2),
\]
so, as a consequence, \( u \in C([0, L] \times [0, T]) \).

Proof. We introduce a smooth function \( \eta = \eta(x) : \eta = 0 \) on \([0, K]\), \( \eta = \psi \) on \([L_0, L]\). For the new unknown function \( w = (u - \eta) / x^k \) equation (17) takes the form: find \( w \in W \), satisfying initial condition \( w(0) = u_0 \), and a.e. on \((0, T)\) the equality
\[
\frac{\partial w}{\partial t} - \frac{\sigma^2(x, t)x^2}{2} \frac{\partial^2 w}{\partial x^2} + \rho(x, t)x \frac{\partial w}{\partial x} + g(t)w = f(x, t).
\] (21)

Here \( u_0 = (\psi(x) - \eta(x)) / x^k \), \( f(x, t) = (\chi(x, t) - A(t)\eta) / x^k \), \( \rho = d - r - k \sigma^2 \), \( g = r + k(d - r) \). By theorem 1 we have \( w = 0 \) on \([L_0, L] \times [0, T]\). Since \( \chi = 0 \) on \([0, K] \times [0, T]\), then
\[
f(t) = 0 \quad \text{on} \quad ([0, K] \cup [L_0, L]) \times [0, T], \quad f \in L_\infty(Q_{LT}), \quad u_0 \in V_0.
\] (22)

It is clear, that \( F(t) = \partial f(t) / \partial x \in V^* \), \( v_0 = du_0 / dx \in L_1(I) \). Setting \( q = \partial w / \partial x \) and differentiating (21) with respect to \( x \), we obtain the following problem for \( q : \) find \( q \in W \), satisfying initial condition \( q(0) = v_0 \), and a.e. on \((0, T)\) the equality
\[
\frac{\partial q}{\partial t} - \frac{\partial}{\partial x} \left( \frac{\sigma^2(x, t)x^2}{2} \frac{\partial q}{\partial x} \right) + \frac{\partial}{\partial x} \left( \rho(x, t)xq \right) + g(t)q = F(t) \quad \text{in} \quad V^*.
\] (23)

We write (23) in a form \( q'(t) + B(t)q(t) = F(t), t \in (0, T), \) \( q(0) = v_0 \). The existence of a unique solution of this problem can be derived from the results on abstract parabolic equations, since \( B(t) \) satisfies assumptions (9), (10) with some constants, and \( F \in L_2(0, T; V^*) \), \( v_0 \in H \). Since \( q \in W \subset C([0, T]; H) \), then
\[
w \in C([0, T]; H^1); \quad \partial w / \partial x \in L_2(0, T; V) \quad \text{or} \quad x \partial^2 w / \partial x^2 \in L_2(0, T; H),
\] (24)

and a fortiori \( x^2 \partial^2 w / \partial x^2 \in L_2(0, T; H) \). Consequently, equation (21) is valid almost everywhere in \( Q_{LT} \) and
\[
\partial w / \partial t \in L_2(0, T; H).
\] (25)

By definition of \( w \) assertions (24) and (25) are also valid for function \( u / x^k \). So, \( u / x^k \in H^1(0, T; H) \cap C([0, T]; H^1) \). Due to (24) and the identity
\[
\frac{d^2u}{dx^2} = \frac{d^2(u / x)}{dx^2} + 2 \frac{d(u / x)}{dx},
\]
we obtain \( u \in L_2(0, T; H^2) \).

Corollary 1. Let \( u \) be a solution of (P). Then
\[
u(t) / x^k \in L_2(0, T; H), \quad Au \in L_2(0, T; H).
\]

5. A third formulation: semilinear parabolic equation
Let us define weakly nonlinear operator \( \beta : (0, T) \times H \to H \) by the equality
\[
\beta(t, u) = g^+(x, t)1_{[K, L]}(x)h_b(u), \quad h_b(s) = \begin{cases} -1, & s \leq 0, \\ 0, & s > 0, \end{cases}
\]
and consider the following semilinear parabolic problem:

Problem (P_\beta) Find \( u \in L_2(0, T; V_\psi) \) such that \( u' \in L_2(0, T; V^*) \), \( u(0) = \psi \) and a.e. on \((0, T)\)
\[ u'(t) + A(t)u(t) + \beta(t,u(t) - \psi) = 0 \text{ in } V'. \quad (26) \]

**Theorem 4.** Under the assumption \((H_1)\) there exists a unique solution \(u\) of the problem \((P_{\beta})\). Moreover, problem \((P_{\beta})\) is equivalent to problem \((P_{\phi})\) with \(G = g^+\) in the definition of the functional \(\phi\).

**Proof.** The solution of Problem \((P_{\beta})\) is unique. This is ensured by Gårding’s inequality (10) and the monotonicity of the operator \(\beta(t,\cdot)\) for all \(t\). By the uniqueness of the solutions to problems \((P_{\phi})\) and \((P_{\beta})\), it remains to prove that the solution of Problem \((P_{\beta})\) is also a solution of Problem \((P_{\phi})\).

Let \(u\) be a solution of problem \((P_{\phi})\). It satisfies equation (17) and also inclusion (18) holds, i.e.

\[ \chi(x,t) \in g^+(x,t)1_{[K,L]}(x)h(u(x,t) - \psi(x)). \quad (27) \]

We need to make sure that

\[ \chi(x,t) = \beta(t,u(x,t) - \psi(x)) \text{ a.e. in } Q_{LT}. \quad (28) \]

Let us calculate the function \(\chi\). We have,

\[ \bar{Q}_{LT} = Q_{\psi} \cup Q_{\phi}, \quad Q_{\phi} = \{(x,t) \in \bar{Q}_{LT} : u(x,t) > \psi(x)\}. \]

The function \(h(s)\) is unique for \(s > 0\). Therefore, it follows from (27), that \(\chi(x,t) = 0, \ (x,t) \in Q_{\phi}\), as well as \(\beta(t,u(x,t) - \psi(x)) = 0, \ (x,t) \in Q_{\phi}\). In this way, (28) is satisfied at points \((x,t) \in Q_{\phi}\), as well as at the points of the set \((0,K) \times [0,T]\).

Now let us calculate \(\chi\) a.e. on \(Q_{\psi} \cap ((K,L) \times [0,T])\). According to Theorem 3 we have \(u \in H^1(0,T;H) \cap L_2(0,T;H^2)\), so the equation (17) is valid almost everywhere in \(Q_{LT}\), and \(\chi(x,t) \leq 0\) by (27). Consequently, it follows from (17) that a.e. on \(Q_{\psi}\)

\[ \chi(x,t) = -A(t)\psi(x,t) = -g(x,t) = -g(x,t)1_{[K,L]}(x) = g^+(x,t)1_{[K,L]}(x)h_0(u(x,t) - \psi(x)). \]

Thus, (28) is satisfied a.e. in \(Q_{LT}\).

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