Multiplicity distributions in $e^+e^-$ annihilation into hadrons and pure birth branching processes

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Abstract: Recursive solution for a general homogeneous in time pure birth branching process with simultaneous production of any number of “particles” and with continuous evolution parameter is given. Calculational algorithm based on the use of Koenigs function and functional Schröder equation is described. It is shown that multiplicity distributions in $e^+e^-$ annihilation into hadrons for c.m. energies up to 189 GeV are well described by the modified negative binomial distribution, explained by simple pure birth branching process without multiple simultaneous “particle” production. The energy dependence of the evolution parameter is also discussed.
1. Introduction

Recently it has been shown [1-13] that multiplicity distributions in $e^+e^-$ annihilation into hadrons are fairly well described by the modified negative binomial distribution (MNBD). Two scenarios based on the use of the pure birth branching processes (for description of the mathematical formalism see for example [14, 15]) have been proposed to explain the origin of the MNBD. In the scenario advocated in the paper [2] the binomial distribution of particle production sources occurs at some initial stage, each source evolves afterwards according to the pure birth branching process with immigration. In other scenario [3, 5] a fixed number of particle production sources occurs initially and they evolve also according to the pure birth branching process with production of intermediate neutral clusters. The final state hadrons are produced through decay of these clusters. In the branching process used in both scenarios no multiple simultaneous “particle” or cluster production is allowed. In this letter we study the possibility of having simultaneous multiple “particle” production. In section 2 a recursive solution for the pure birth branching process with simultaneous production of any number of “particles” is given together with a calculational algorithm, based on the use of Koenigs function and functional Schröder equation [16] (recent description can be found in [17, 18]). In section 3 results of fits of the charged particle multiplicity distributions in $e^+e^-$ annihilation into hadrons at c.m. energies above 20 GeV [19-28] to the MNBD and to the distribution resulting from the branching process with additional simultaneous pair “particle” production are presented. Our conclusions and discussion are given in last section.

2. Recursive solution for a general pure birth branching process

Let us recall that the branching process with continuous evolution parameter $t$ is determined by the differential probability densities $\alpha_i$ for the transition of one object, “particle”, into $i$ objects, “particles”. All “particles” are assumed to evolve independently. For a homogeneous branching process the $\alpha_i$ do not depend on $t$ and probabilities for transition $1 \rightarrow i$ in an infinitely small interval $\Delta t$ are

$$p_{1 \rightarrow i}(t, t + \Delta t) = \alpha_i \Delta t \quad , \quad i = 0, 2, 3, ...$$  \hspace{1cm} (2.1)

and for transition $1 \rightarrow 1$

$$p_{1 \rightarrow 1}(t, t + \Delta t) = 1 - \alpha \Delta t$$  \hspace{1cm} (2.2)

with

$$\alpha = \sum_{i \neq 1}^{\infty} \alpha_i$$  \hspace{1cm} (2.3)
For the pure birth branching process $\alpha_0 = 0$.

The probability distribution $p_i(t) \equiv p_{1\to i}(0, t)$ for the process having one particle at $t = 0$ can be found from the forward Kolmogorov equation

$$\frac{\partial m}{\partial t} = f(x) \frac{\partial m}{\partial x} \quad \text{with initial condition} \quad m|_{t=0} = x$$

for the probability generating function

$$m(x, t) = \sum_{i=1}^{\infty} p_i(t)x^i$$

with

$$f(x) = \sum_{i=2}^{\infty} \alpha_i x^i - \alpha x$$

Using the Taylor expansion of the equation 2.4 over $x$ one can obtain the following system of differential equations for the probabilities $p_i$

$$\frac{dp_1}{dt} = -\alpha p_1$$

$$\frac{dp_2}{dt} = \alpha_2 p_1 - 2\alpha p_2$$

$$\frac{dp_3}{dt} = \alpha_3 p_1 + 2\alpha_2 p_2 - 3\alpha p_3$$

and for arbitrary $k$

$$\frac{dp_k}{dt} = \sum_{j=1}^{k-1} j\alpha_{k-j+1} p_j - k\alpha p_k$$

with initial condition

$$p_k(0) = \delta_{1k}$$

These equations can be solved one after another, for $p_1$

$$p_1 = \exp (-\alpha t)$$

for $p_2$

$$p_2 = a_2 p_1 (1 - p_1)$$

where $a_i = \alpha_i/\alpha$, and so on.

One can observe that $p_k$ has polynomial dependence of $k$-th power on $p_1$. 

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\[ p_k = \sum_{i=1}^{k} \pi_{ik} p_i \]  

(2.15)

with constant coefficients \( \pi_{ik} \). These coefficients can be found using the following recursions

\[ \pi_{11} = 1, \pi_{12} = -\pi_{22} = a_2, \]  

(2.16)

\[ (k - i)\pi_{ik} = \sum_{j=1}^{k-1} ja_{(k+1-j)}\pi_{ij} = \sum_{j=1}^{k-1} (k - j)a_{(j+1)}\pi_{i(k-j)} \quad , i = 1, 2, ..., k - 1 \]  

(2.17)

and the coefficient \( \pi_{kk} \) can be found from the initial condition 2.12

\[ \pi_{kk} = -\sum_{i=1}^{k-1} \pi_{ik} \]  

(2.18)

This solution satisfies also the backward Kolmogorov equation \( dm/dt = f(m) \).

Solving the backward equation in quadratures one can obtain a functional relation between \( m \) and \( x \). For example, for the branching process with non-zero \( \alpha_2 \) and \( \alpha_3 \) one has:

\[ \varphi(m) = \varphi(x) \exp(-\alpha t) \]  

(2.19)

with

\[ \varphi(x) = \frac{x}{(1-x)^{\gamma+1}(x+\gamma)^{\gamma+1}} \]  

(2.20)

and

\[ \gamma = \frac{\alpha_2 + \alpha_3}{\alpha_3} = \frac{\alpha}{\alpha_3} \]  

(2.21)

In practical calculations the polynomial formulae 2.15 for \( p_k \) have limited numerical precision. More robust calculational algorithm based on the mathematical formalism used by Fatou\[16\] in the studies of the iterations of rational functions is described briefly below. Let us denote

\[ m_n(x, t) = m(x, nt) \]  

(2.22)

The following relations are valid for the probability generating functions \( m_n(x, t) \) (see for example \[14, 15\])

\[ m_n(x, t) = m_{n-1}(m(x, t), t) = m(m_{n-1}(x, t), t) \]  

(2.23)

In analogy with \[16\] one can introduce the Koenigs function for the iterations 2.22 as a limit:
\[ K(x) = \lim_{n \to \infty} \frac{m_n(x,t)}{p_1^n} \quad . \]  

(2.24)

For the pure birth branching process the Koenigs function has the following form

\[ K(x) = \sum_{i=1}^{\infty} \pi_1 x^i \quad . \]  

(2.25)

Let us denote \( \pi_1 = \kappa_i \). The recurrence relations \[2.17\] lead to the following recurrence for \( \kappa_i \)

\[(i - 1)\kappa_i = \sum_{j=1}^{i-1} (i - j)a_{j+1}\kappa_{i-j} = \sum_{j=1}^{i-1} ja_{(i-j+1)}\kappa_j \quad . \]  

(2.26)

The remarkable property of the Koenigs function \[16, 17, 18\] is the functional Schröder equation:

\[ K(m(x,t)) = p_1 K(x) \quad , \]  

(2.27)

it follows as a limit from the functional relations \[2.23\]. Let us denote

\[ m^j = \sum_{i=j}^{\infty} p_i^{(j)} x^j \quad ; j = 2, 3, \ldots \]  

(2.28)

The use of the Taylor expansion of the Schröder equation \[2.27\] results in the following relations:

\[ p_1 \kappa_i = \sum_{j=1}^{i} \kappa_j p_i^{(j)} \quad , \quad i = 1, 2, 3, \ldots \]  

(2.29)

And \( p_i^{(j)} \) can be found using successively the convolutions:

\[ p_j^{(2)} = \sum_{i=1}^{j-1} p_ip_{i-j} \quad j = 2, 3, \ldots \]  

(2.30)

\[ p_j^{(N)} = \sum_{i=1}^{j-1} p_ip_{j-1}^{(N)} \quad j = N, N + 1, \ldots \quad N = 3, 4, \ldots \]  

(2.31)

Using \[2.29\], \[2.30\] and \[2.31\] one can obtain the recurrence for \( p_n \). Indeed, let us assume that we know \( p_i \) for \( i = 1, 2, \ldots, n - 1 \). Then at step \( n \) we calculate

\[ p_n^{(j)} = \sum_{i=1}^{n-j-1} p_ip_{n-i}^{(j-1)} \quad j = 2, \ldots, n - 1 \]  

(2.32)

\[ ^{1}\text{It is worth to note that the Schröder equation and the equation \[2.19\] are identical, therefore the Koenigs function for the branching process with non-zero \( \alpha_2 \) and \( \alpha_3 \) coincides with \( \varphi(x) \) defined in \[2.20\]. In general case the \( K(x) \) is proportional to } \exp(-\alpha \int \frac{du}{f(u)}). \]
and

\[ p_n^{(n)} = p_1^n. \] (2.33)

Finally, the equation (2.29) leads to the following expression

\[ p_n = \kappa_n(p_1 - p_1^n) - \sum_{j=2}^{n-1} \kappa_j p_n^{(j)}. \] (2.34)

The probability to produce \( n_- \) negatively charged particles \( (n_- = n_{ch}/2) \) in the scenario \([3, 5]\) can be found using the formula:

\[ P_{n_-} = \sum_{j=N}^{\infty} p_j^{(N)} q_{n_-}^{(j)} = \sum_{j=N}^{\infty} p_j^{(N)} \frac{j!}{n_-!(j-n_-)!} \varepsilon^{n_-} (1 - \varepsilon)^{j-n_-}. \] (2.35)

Here \( N \) denotes the number of initial sources of particle production, \( q_{n_-}^{(j)} \) is the usual binomial probability to have \( n_- \) negatively charged particles (we remind that \( n_- \) in this scenario is equal to the number of produced charged particle pairs) from decay of \( j \) clusters and \( \varepsilon \) is the probability of cluster decay into pair of charged particles. In practical calculations we stop summation in (2.35) when the difference between 1 and the sum of \( p_j^{(N)} \) becomes smaller than the computer precision.

3. Results of fits

The pure birth branching process with non-zero \( \alpha_2 \) and \( \alpha_3 \) is determined by these differential probability densities and by the evolution parameter \( t \). The probability distribution \( p_1(t) \) depends on some combinations of the \( \alpha_2 \) and \( \alpha_3 \) and on \( p_1(t) \) (2.13). The final multiplicity distribution (2.35) is characterized also by the number of initial sources \( N \) and by the probability \( \varepsilon \). As in the papers \([3, 4]\) we fix \( N \) at the value 7. We use as free parameters in the fits \( p_1, \varepsilon \) and the ratio \( a = \alpha_2 / (\alpha_2 + \alpha_3) \). The parameter \( a \) is equal to one for the MNBD and is equal to zero for the branching process with simultaneous pair “particle” production. Results of the fits are given in Table 1. In the cases when the three-parameter fit has minimum with \( a = 1 \), only the results of the two-parameter fit with fixed \( a = 1 \) are shown.

The influence of the parameter \( a \) is illustrated in Figure 1, where the multiplicity distribution at \( \sqrt{s} = 189 \) GeV is compared with the predictions of the three-parameter fit and with the predictions of the two-parameter fit with fixed \( a = 0.8 \), having \( p_1 = 0.509 \pm 0.007, \varepsilon = 0.861 \pm 0.012 \) and \( \chi^2/NDF = 19.1/(29 - 2) \). From Figure 1 it is seen that the predictions of the fit with \( a = 0.8 \) exceed the predictions of the three-parameter fit at \( n_{ch} \) below 12, the opposite is observed at \( n_{ch} \) above 40.

One can see from Table 1 that the quality of the fits is quite good and that the values of \( \varepsilon \) and \( \chi^2 \) for the MNBD fits are practically the same as in the previous
The values of the parameter $a$ are concentrated near one and the allowed fraction for simultaneous pair production is below a few per cent. We have checked also that the parameter $a$ is near one in the fits with $N$ different from seven, in most cases these fits have higher $\chi^2$ than the fits with $N = 7$ (see also [5]).
Table 1: Results of the fits of the distribution $\frac{2.35}{\sqrt{s}}$ to the charged particle multiplicity distributions in $e^+e^-$ annihilation into hadrons.

4. Discussion and conclusions

In recent publication [13] Biyajima and coworkers have observed that the LEP data at $\sqrt{s} = 133$ GeV [26] favour logarithmic dependence $\sim \log \sqrt{s}$ of the evolution parameter $\hat{t}$ in contradiction with $\log \log Q^2$ prediction given by QCD. For logarithmic dependence the probability $p_1$ should have power law dependence on the c.m. energy $\sqrt{s}$. The energy dependence of the parameter $p_1$ for the MNBD fits is shown in Figure 2. From Figure 2 it is seen that recent LEP measurements at 161 [27] 172, 183
Figure 2: Energy dependence of the probability $p_1$ for $e^+e^-$ annihilation into hadrons. Solid line shows the predictions of the power law fit.

and 189 [28] GeV also favour the logarithmic dependence of the evolution parameter on $\sqrt{s}$. The data are fairly well described by the power law dependence $p_1 = A(\sqrt{s})^{-B}$ with the slope $B = 0.310 \pm 0.005$ and with $\chi^2/NDF = 104.1/(21 - 2)$. As noted in [13], this contradiction with QCD is probably attributed to hadronization effects.

Our conclusions are the following. The recursive solution for the general pure birth branching process is given. The calculational algorithm based on the use of the Koenigs function and the functional Schröder equation is described. The results
of the fits to the charged particle multiplicity distributions in $e^+e^-$ annihilation into hadrons for c.m. energies up to 189 GeV show the validity of the MNBD parametrization and the absence of the component with simultaneous pair production at the level exceeding a few per cent. The latest LEP measurements favour the logarithmic dependence of the evolution parameter $t$ as noted earlier\cite{13}.

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