THE HYBRID EULER-HADAMARD PRODUCT FORMULA FOR DIRICHLET L-FUNCTIONS IN $\mathbb{F}_q[T]$

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Abstract. For Dirichlet $L$-functions in $\mathbb{F}_q[T]$ we obtain a hybrid Euler-Hadamard product formula. We make a splitting conjecture, namely that the $2k$-th moment of the Dirichlet $L$-functions at $\frac{1}{2}$, averaged over primitive characters of modulus $R$, is asymptotic to (as $\deg R \to \infty$) the $2k$-th moment of the Euler product multiplied by the $2k$-th moment of the Hadamard product. We explicitly obtain the main term of the $2k$-th moment of the Euler product, and we conjecture via random matrix theory the main term of the $2k$-th moment of the Hadamard product. With the splitting conjecture, this directly leads to a conjecture for the $2k$-th moment of Dirichlet $L$-functions. Finally, we lend support for the splitting conjecture by proving the cases $k = 1, 2$. This work is the function field analogue of the work of Bui and Keating. A notable difference in the function field setting is that the Euler-Hadamard product formula is exact, in that there is no error term.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Mean values, or moments, of $L$-functions have many powerful applications in number theory, from the non-vanishing of $L$-functions at certain points, to zero-density estimates, positive lower bounds for the number of zeros of $\zeta(s)$ that lie on the critical line, and the proportion of simple zeros on the critical line (see [12] for a summary). They also have intrinsic interest because results on moments higher than the fourth have not been obtained, and instead we rely on conjectures.

Consider the Riemann zeta-function. For $\Re(s) > 1$,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

Acknowledgments: The author is grateful for an Engineering and Physical Sciences Research Council (UK) DTP Standard Research Studentship (grant number EP/M506527/1). The author would also like to thank Hung Bui and Nigel Byott for various comments and corrections, and Julio Andrade for suggesting this problem and for his comments.

Date: July 6, 2021.

2020 Mathematics Subject Classification. Primary 11M06; Secondary 11M26, 11M50, 11R59.

Key words and phrases. hybrid Euler-Hadamard product, moments, Dirichlet $L$-functions, function fields, random matrix theory.
which can be extended meromorphically to \( \mathbb{C} \) with a simple pole at \( s = 1 \). It was shown by Hardy and Littlewood \([13]\) that

\[
\frac{1}{T} \int_{t=0}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim \log T,
\]

as \( \deg T \to \infty \), and it was shown by Ingham \([16]\) that

\[
\frac{1}{T} \int_{t=0}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \frac{1}{6} \frac{6}{\pi^2} (\log T)^4
\]

as \( \deg T \to \infty \). For higher moments it has been conjectured (see \([19\), equation (4)]) that, for integers \( k \geq 0 \),

\[
\lim_{T \to \infty} \frac{1}{(\log T)^{k^2}} \frac{1}{T} \int_{t=0}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt = f(k)a(k),
\]

where \( f(k) \) is a real-valued function and

\[
a(k) := \prod_p \left( \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{km}} \right).
\]

We have \( a(0) = 1, a(1) = 1, a(2) = \frac{6}{\zeta(2)} = \frac{6}{\pi^2} \), and we have an understanding of \( a(k) \) for higher values of \( k \). The factor \( f(k) \) is more elusive. Clearly, from the results described above, we have \( f(0) = 1, f(1) = 1, f(2) = \frac{1}{12} \). It has been conjectured via number-theoretic means that \( f(3) = \frac{4}{45} \) \([8]\) and \( f(4) = \frac{24024}{1610} \) \([9]\). For higher powers one must look at random matrix theory for conjectures.

We point out that one can also obtain conjectures for higher powers by using the recipe developed by Conrey, Farmer, Keating, Rubinstein, and Snaith \([7]\). Again these conjectures are in agreement with those obtained via random matrix theory. However, in this paper we focus on the latter.

It has been known for some time that there is a relationship between the Riemann zeta-function and eigenvalues of random unitary matrices. In 1972 it was observed by Montgomery and Dyson that the pair correlations of the non-trivial zeros of the Riemann zeta-function appear to behave similarly to the pair correlations of eigenvalues of a random Hermitian matrix \([20]\). Later, Odlyzko produced numerical evidence in support of this \([23]\).

Given that the eigenvalues of a matrix are the zeros of its characteristic polynomial, it is reasonable to expect a relationship between \( \zeta(s) \) on the critical line and the characteristic polynomials of unitary matrices. Keating and Snaith \([19]\) modeled \( \zeta(s) \) at around height \( T \) on the critical line by the characteristic polynomial of a random \( N \times N \) unitary matrix. (Here, \( N \) is chosen such that the mean spacing between the eigenphases of an \( N \times N \) unitary matrix is the same as the mean spacing of the zeros of the Riemann zeta-function at around height \( T \) on the critical line). They obtained the following result for integers \( k \geq 0 \):

\[
\int_{U \in U(N)} |Z(U, \theta)|^{2k} dU \sim f_{\text{CUE}}(k) N^{k^2}
\]

as \( N \to \infty \). Here \( U(N) \) is the set of all unitary \( N \times N \) matrices; for all \( U \in U(N) \), we take \( Z(U, \theta) := \det (I_N - U e^{-i\theta}) \) to be the characteristic polynomial of \( U \); the integral is with respect to the Haar measure on \( U(N) \); and \( f_{\text{CUE}}(k) := \prod_{j=0}^{k-1} \frac{\beta^j}{(1 + j)!} \). (The fact that (2) is independent of \( \theta \) is not immediately obvious, and so we remark that this lack of dependency is not an error). Now, we note that

\[
f_{\text{CUE}}(k) = \begin{cases} 
1 & \text{if } k = 1 \\
\frac{12}{42} & \text{if } k = 2 \\
\frac{44024}{10} & \text{if } k = 3 \\
& \text{if } k = 4.
\end{cases}
\]
That is, $f_{CUE}(k)$ agrees with the established values of $f(k)$, as well as the values that have been conjectured by alternative means. This provides support for the conjecture that $f(k) = f_{CUE}(k)$ for all integers $k \geq 0$. We remark that the results of Keating and Snaith apply to values of $k$ that are not necessarily integer-valued; however, in this paper we are concerned with the integer case.

Note that this conjecture, using random matrix theory, does not introduce the factor $a(k)$ in (1) in any natural way. This was addressed by Gonek, Hughes, and Keating [11] who expressed $\zeta(s)$ as a hybrid Euler-Hadamard product: $\zeta(s) \approx P_X(s)Z_X(s)$, where $P_X(s)$ is a roughly a partial Euler product and $Z_X(s)$ is roughly a partial Hadamard product (a product over the zeros of $\zeta(s)$). The variable $X$ determines the contribution of each factor. They conjectured that, asymptotically, the $2k$-th moment of $\zeta(s)$ on the critical line can be factored into the $2k$-th moment of $P_X(s)$ multiplied by the $2k$-th moment of $Z_X(s)$ (known as the splitting conjecture); and they showed that the former contributes the factor $a(k)$ in (1) and conjectured via random matrix theory that the latter contributes the factor $f(k)$. That is, they obtained a conjecture for the $2k$-th moment of $\zeta(s)$ in a way that the factor $a(k)$ appears naturally. They also lent support for the splitting conjecture by demonstrating that it holds for the cases $k = 1, 2$.

This approach, using an Euler-Hadamard hybrid formula, has been applied to discrete moments of the derivative of the Riemann zeta-function by Bui, Gonek, and Milinovich [4].

The relationship between random matrix theory and the Riemann zeta-function extends to other $L$-functions, particularly certain families of $L$-functions [17]. For example, one aspect of the relationship is that the proportion of $L$-functions of a certain family with conductor $q$ that have $j$-th zero in some interval $[a, b]$ appears to be the same as the proportion of matrices of a certain matrix ensemble (the precise ensemble is dependent on the family) of size $N \times N$ ($N = N(q)$ is chosen so that the mean spacing of the eigenvalues is the same as the mean spacing of the zeros of the $L$-functions of conductor $q$) that have $j$-th eigenvalue in $[a, b]$. At least, this appears to be the case as $q \to \infty$.

Let us consider the family of Dirichlet $L$-functions. The associated ensemble of matrices is the unitary matrices [6, page 887]. By making use of this relationship, Bui and Keating [5] obtained an analogue of [11] where they considered the $2k$-th moment of Dirichlet $L$-functions at $s = \frac{1}{2}$, averaged over all primitive Dirichlet $L$-functions of modulus $q$, instead of the Riemann zeta-function averaged over the critical line. That is, using a hybrid Euler-Hadamard product for the Dirichlet $L$-functions, they conjectured (among other results) that

$$
(3) \quad \frac{1}{\phi^*(q)} \sum_{\chi \mod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \sim a(k) \frac{G^2(k+1)}{G(2k+1)} \prod_{p|q} \left( \sum_{m=0}^{\infty} \frac{d_k(p^m)^2}{p^{km}} \right)^{-1} (\log q)^{k^2}
$$

as $\deg q \to \infty$, where $\phi^*(q)$ is the number of primitive Dirichlet characters of modulus $q$, the star in the sum indicates the sum is over primitive characters only, and $G(z)$ is the Barnes $G$-function. This had been conjectured previously (see [18]), but this approach allows for all the factors to appear naturally.

One can consider the above problems in the function field setting. In fact, it is the function field analogues that give some insight into the relationship between random matrix theory and $L$-functions (be they in number fields or function fields). See [17, Section 3] for details. In function fields, Bui and Florea [3] developed the hybrid Euler-Hadamard product model for the family of quadratic Dirichlet $L$-functions. In this paper we do the same for Dirichlet $L$-functions of any primitive character, which is the function field analogue of the work of Bui and Keating described above. The aim is to provide support for the following conjecture (see [6, page 887]), which is the analogue of (3), in such a way that all factors appear naturally:
Conjecture 1.1. For all non-negative integers \( k \), it is conjectured that

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* |L\left(\frac{1}{2}, \chi\right)|^{2k} \sim f(k)a(k) \prod_{P|R} \left( \sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} \text{deg } R^2,
\]
as \( \text{deg } R \to \infty \), where

\[
a(k) := \prod_{P \in \mathcal{P}} \left( 1 - \frac{1}{|P|} \right)^{k^2} \sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m}.
\]

and

\[
f(k) := \frac{G^2(k + 1)}{G(2k + 1)} = \prod_{i=0}^{k-1} \frac{i!}{(i + k)!},
\]

where \( G \) is the Barnes \( G \)-function.

This conjecture has been verified for the cases \( k = 1, 2 \) by Andrade and Yiasemides [1]:

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* |L\left(\frac{1}{2}, \chi\right)|^2 \sim \frac{\phi(R)}{|R|} \text{deg } R
\]
as \( \text{deg } R \to \infty \), and

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* |L\left(\frac{1}{2}, \chi\right)|^4 \sim \frac{1-q^{-1}}{12} \prod_{P|R} \left( \frac{1-|P|^{-1}}{1 + |P|^{-1}} \right) \text{deg } R^4
\]
as \( \text{deg } R \to \infty \).

We now state our results, but we refer the unfamiliar to reader to the beginning of Section 2 for definitions and notational remarks relating to \( L \)-functions in function fields. In particular, henceforth, the letter \( q \) is reserved for the order of the finite field \( \mathbb{F}_q \). We begin with the Euler-Hadamard hybrid formula, which we prove in Section 3.

Theorem 1.2. Let \( X \geq 1 \) be an integer and let \( u(x) \) be a positive \( C^\infty \)-function with support in \([e, e^{1+q^{-X}}]\). Let

\[
v(x) = \int_{t=x}^{\infty} u(t)dt\]

and take \( u \) to be normalised so that \( v(0) = 1 \). Furthermore, for \( y \in \mathbb{C}\{0\} \) with \( \arg(y) \neq \pi \), we define \( E_1(y) := \int_{w=y}^{y+\infty} \frac{e^{-w}}{w} dw \); and for \( z \in \mathbb{C}\{0\} \) with \( \arg(z) \neq \pi \), we define

\[
U(z) := \int_{x=0}^{\infty} u(x)E_1(z \log x)dx.
\]

Let \( \chi \) be a primitive Dirichlet character of modulus \( R \in \mathcal{M}\{1\} \), and let \( \rho_n = \frac{1}{2} + i\gamma_n \) be the \( n \)-th zero of \( L\left(s, \chi\right) \). Then, for all \( s \in \mathbb{C} \) we have

\[
L\left(s, \chi\right) = P_X\left(s, \chi\right)Z_X\left(s, \chi\right),
\]

where

\[
P_X\left(s, \chi\right) = \exp\left( \sum_{A \in \mathcal{M}}^{\deg A \leq X} \frac{\chi(A) \Lambda(A)}{|A|^s \log |A|} \right)
\]

and

\[
Z_X\left(s, \chi\right) = \exp\left( -\sum_{\rho_n} U\left((s - \rho_n)(\log q)x\right) \right).
\]

Strictly speaking, if \( s = \rho \) or \( \arg(s - \rho) = \pi \) for some zero \( \rho \) of \( L\left(s, \chi\right) \), then \( Z_X\left(s, \chi\right) \) is not well defined. In this case, we take

\[
Z_X\left(s, \chi\right) = \lim_{s_0 \to s} Z_X\left(s_0, \chi\right)
\]
and we show that this is well defined.

**Remark 1.3.** We note that our hybrid Euler-Hadamard product formula, (6), does not involve an error term, unlike the analogous Theorem 1 in [11] and Theorem 1 in [5]. This is due to the fact that we are working in the function field setting.

We also note that $Z_X(s, \chi)$ is expressed in terms of $u(x)$. Whereas, $P_X(s, \chi)$ and $L(s, \chi)$ are independent of $u(x)$. Thus, given the equality (6), we can see that, as long as $u(x)$ satisfies the conditions in the theorem, the value of $Z_X(s, \chi)$ is independent of any further restrictions made on $u(x)$. Ultimately, this is due to the fact that we are working in the function field setting and due to our choice of support for $u(x)$. Indeed, this is why our support for $u(x)$ is not quite the exact analogy to the support of $u(x)$ in Theorem 1 of [5]. We note that in Theorem 1 in [5], $P_X(s, \chi)$ and $L(s, \chi)$ also do not depend on $u(x)$, but this is because the dependency exists in the error term.

We conjecture that the $2k$-th moment of the $L$-functions can be split into the $2k$-th moment of their partial Euler products multiplied by $2k$-th moment of their partial Hadamard products:

**Conjecture 1.4 (Splitting Conjecture).** For integers $k \geq 0$, we have

$$\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| L \left( \frac{1}{2}, \chi \right) \right|^{2k} \sim \left( \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| P_X \left( \frac{1}{2}, \chi \right) \right|^{2k} \right) \cdot \left( \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| Z_X \left( \frac{1}{2}, \chi \right) \right|^{2k} \right)$$

as $X, \deg R \to \infty$ with $X \leq \log q \deg R$.

We then obtain the $2k$-th moment of the partial Euler products in Section 4, and we use a random matrix theory model to conjecture the $2k$-th moment of the Hadamard products in Section 5:

**Theorem 1.5.** For positive integers $k$, we have

$$\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| P_X \left( \frac{1}{2}, \chi \right) \right|^{2k} \sim a(k) \left[ \prod_{P \mid R} \left( \sum_{m=0}^{\infty} \frac{d_k(P^m)}{|P|^{m+1}} \right)^{2k} \right] \left( e^{\gamma X} \right)^{k^2}$$

as $X, \deg R \to \infty$ with $X \leq \log q \deg R$. Here, $\gamma$ is the Euler-Mascheroni constant, and

$$a(k) = \prod_{P \in \mathcal{P}} \left( 1 - \frac{1}{|P|} \right)^{k^2} \frac{\sum_{m=0}^{\infty} d_k(P^m)}{|P|^{m+1}}.$$

**Conjecture 1.6.** For integers $k \geq 0$, we have

$$\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| Z_X \left( \frac{1}{2}, \chi \right) \right|^{2k} \sim \frac{G^2(k+1)}{G(2k+1)} \left( \frac{\deg R}{e^{\gamma X}} \right)^{k^2},$$

as $\deg R \to \infty$, where $\gamma$ is the Euler-Mascheroni constant and $G$ is the Barnes $G$-function. For our purposes, it suffices to note that

$$\frac{G^2(k+1)}{G(2k+1)} = \prod_{i=0}^{k-1} \frac{i!}{(i+k)!}.$$

Note that Conjecture 1.4, Theorem 1.5, and Conjecture 1.6 together reproduce Conjecture 1.1 as desired, but only for certain cases, such as when the largest prime divisor of $R$ has degree less than $X$, or when $P$ is prime.

In Section 6 we rigorously obtain the second moment of the Hadamard product:

**Theorem 1.7.** We have that

$$\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| Z_X \left( \frac{1}{2}, \chi \right) \right|^2 = \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| L \left( \frac{1}{2}, \chi \right) P_X \left( \frac{1}{2}, \chi \right) \right|^{-1}.$$
as $X, \deg R \to \infty$ with $X \leq \log_q \deg R$.

In Section 8 we rigorously obtain the fourth moment of the Hadamard product:

**Theorem 1.8.** We have

$$\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| Z_X\left(\frac{1}{2}, \chi\right) \right|^4 = \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| L\left(\frac{1}{2}, \chi\right) P_X\left(\frac{1}{2}, \chi\right)^{-1}\right|^4 \sim \frac{1}{12} \left(\frac{\deg R}{e^X}\right)^4 \prod_{\deg P > X \atop P \mid R} \left(1 - \frac{1}{|P|}\right)^3 \frac{1}{1 + |P|^{-1}}$$

as $X, \deg R \to \infty$ with $X \leq \log_q \log \deg R$.

We can see that Theorems 1.7 and 1.5, and (4) verify the Splitting Conjecture for the case $k = 1$. This can be seen from the fact that $a(1) = 1$ and

$$\prod_{\deg P \leq X \atop P \mid R} \left(\sum_{m=0}^\infty \frac{d_1(P^m)}{|P|^m}\right)^{-1} = \prod_{\deg P \leq X \atop P \mid R} \left(1 - \frac{1}{|P|}\right).$$

We can also see that Theorems 1.8 and 1.5, and (5) verify the Splitting Conjecture for the case $k = 2$. This can be seen from the fact that $a(2) = 1 - q^{-1}$ and

$$\prod_{\deg P \leq X \atop P \mid R} \left(\sum_{m=0}^\infty \frac{d_2(P^m)}{|P|^m}\right)^{-1} = \prod_{\deg P \leq X \atop P \mid R} \left(1 - \frac{1}{|P|}\right)^3 \frac{1}{1 + |P|^{-1}}.$$

However, in Theorem 1.8 we required the condition $X \leq \log_q \deg R$ which is more restrictive than the condition $X \leq \log_q \deg R$ in the Splitting Conjecture. However, given the results that have been establish in the area of twisted moments (see, for example, [2, 10, 15, 21] for $\zeta(s)$ and [14, 25] for Dirichlet $L$-functions), we expect that one can improve upon this restriction for Theorem 1.8.

### 2. Notation and Background

Let $q$ be a prime power in $\mathbb{N}$ and define $A := \mathbb{F}_q[T]$, the polynomial ring over the finite field of order $q$. We define $\mathcal{M} \subseteq \mathcal{A}$ to be the set of monic polynomials, and we define $\mathcal{P}$ to be the set of monic primes. Henceforth, “prime” shall mean “monic prime”, and the upper-case letter $P$ is reserved for primes even when it is not explicitly stated. In the limits of summations and products, unless otherwise stated, the polynomials appearing therein should be taken to be monic. We define

$$\mathcal{S}(X) := \{A \in \mathcal{A} : P \mid A \Rightarrow \deg P \leq X\},$$

$$\mathcal{S}_\mathcal{M}(X) := \{A \in \mathcal{M} : P \mid A \Rightarrow \deg P \leq X\};$$

and for $\mathcal{B} \subseteq \mathcal{A}$ and integers $n \geq 0$, we define

$$\mathcal{B}_n := \{B \in \mathcal{B} : \deg B = n\}.$$

For $A \in \mathcal{A}\setminus\{0\}$ we define $|A| := q^{\deg A}$, and for the zero polynomial we define $|0| := 0$. For $A, B \in \mathcal{A}$ we define $(A, B)$ to be the greatest common (monic) divisor of $A$ and $B$, and $[A, B]$ is the lowest common (monic) multiple. Suppose $A \in \mathcal{A}$ has prime factorisation $A = P_1^{e_1} \cdots P_n^{e_n}$, then we define the radical of $A$ by $\text{rad}(A) := P_1 \cdots P_n$.

As usual, we write $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$, and we write $f(x) = o(g(x))$ if $\frac{f(x)}{g(x)} \to 0$ as $x \to \infty$. We write $f(x) \ll g(x)$ or $f(x) = O(g(x))$ if there is some constant $c$ such that for all $x$ in the domain of $f$ we have $|f(x)| \leq c|g(x)|$. It may be the case that $f$, and perhaps $g$, are
dependent on some parameter \( k \). For example, we will often have functions where the order of our finite field, \( q \), appears as a constant. If the implied constant above depends on the parameter \( k \), then we write \( f(x) \ll k \) or \( f(x) = O_k(g(x)) \). Otherwise, it is to be understood that the implied constant is independent of \( k \). If \( |f(x)| \leq c|g(x)| \) for all \( x \) greater than some constant \( d = d(k) \), then we write \( f(x) \ll g(x) \) as \( x \to \infty \), or \( f(x) = O(g(x)) \) as \( x \to \infty \).

Let \( a \in \mathbb{C} \) and \( b \in \mathbb{C}\setminus\{0\} \), and let \( f \) be an integrable complex function. The integral \( \int_{t=a}^{a+b} f(t)dt \) is defined to be over the straight line starting at \( a \) and in the direction of \( b \). That is, \( \int_{t=a}^{a+b} f(t)dt = \int_{s=0}^{\infty} f(a + \frac{b}{|b|} s)ds \). If \( a = 0 \) then we will simply write \( \int_{t=0}^{b} f(t)dt \), and if \( b = \pm 1 \) then we will write \( \int_{t=-a}^{t+a} f(t)dt \).

Now we state some standard definitions and results. The prime polynomial theorem tells us that

\[
|\mathcal{P}_n| = \frac{1}{n} \sum_{d|n} \mu(d)q^{\frac{n}{d}}.
\]

which implies

\[
|\mathcal{P}_n| = q^n + \mathcal{O}\left(\frac{q^{\frac{n}{2}}}{n}\right).
\]

The Riemann zeta-function on \( \mathcal{A} \) is defined, for \( \Re(s) > 1 \), by

\[
\zeta_{\mathcal{A}}(s) := \sum_{A \in \mathcal{M}} \frac{1}{|A|^s} = \frac{1}{1 - q^{1-s}}.
\]

We can see that the right side provides a meromorphic continuation for \( \zeta_{\mathcal{A}}(s) \) to \( \mathbb{C} \). Dirichlet characters are defined similarly as in the classical case:

**Definition 2.1** (Dirichlet Characters). A Dirichlet character on \( \mathcal{A} \) with modulus \( R \in \mathcal{M} \) is a function \( \chi : \mathcal{A} \to \mathbb{C}^* \) satisfying the following properties. For all \( A, B \in \mathcal{A} \):

(1) \( \chi(AB) = \chi(A)\chi(B) \);
(2) If \( A \equiv B \pmod{R} \), then \( \chi(A) = \chi(B) \);
(3) \( \chi(A) = 0 \) if and only if \( (A, R) \neq 1 \).

As usual, \( \chi_0 \) represents the trivial character: \( \chi_0(A) = 1 \) if \( (A, R) = 1 \) and is zero elsewhere. A character \( \chi \) is even if \( \chi(a) = 1 \) for all \( a \in \mathbb{F}_q^* \), and otherwise it is odd. Now, suppose \( S \mid R \). We say that \( S \) is an induced modulus of \( \chi \) if there exists a character \( \chi_1 \) of modulus \( S \) such that

\[
\chi(A) = \begin{cases} 
\chi_1(A) & \text{if } (A, R) = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

\( \chi \) is said to be primitive if there is no induced modulus of strictly smaller degree than \( R \). Otherwise, \( \chi \) is said to be non-primitive. \( \phi^*(R) \) denotes the number of primitive characters of modulus \( R \). We denote a sum over all characters \( \chi \) of modulus \( R \) by \( \sum_{\chi \mod R} \), and a sum over all primitive characters \( \chi \) of modulus \( R \) by \( \sum_{\chi \mod R}^* \). The following two results are standard, and proofs can be found in Section 3 of [1].

**Lemma 2.2.** Let \( R \in \mathcal{M} \) and let \( A, B \in \mathcal{A} \). Then,

\[
\sum_{\chi \mod R}^* \chi(A)\overline{\chi}(B) = \begin{cases} 
\sum_{EF=R, \mu(E)\phi(F)} \mu(E)\phi(F) & \text{if } (AB, R) = 1, \\
0 & \text{otherwise;}
\end{cases}
\]

and

\[
\sum_{\chi \mod R}^* \chi(A)\overline{\chi}(B) = \begin{cases} 
\sum_{\chi \text{ even}}_{EF=R, \mu(E)\phi(F)} \mu(E)\phi(F) & \text{if } (AB, R) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]
Corollary 2.3. For all $R \in \mathcal{M}$ we have that
\[ \phi^*(R) = \sum_{EF=R} \mu(E)\phi(F). \]

Definition 2.4 (Dirichlet L-function). Let $\chi$ be a Dirichlet character. The associated L-function, $L(s, \chi)$, is defined for $\text{Re}(s) > 1$ by
\[ L(s, \chi) := \sum_{A \in \mathcal{M}} \frac{\chi(A)}{|A|^s}. \]

If $\chi_0$ is the trivial Dirichlet character of modulus $R$, then
\[ L(s, \chi_0) = \sum_{A \in \mathcal{M}, (A, R) = 1} \frac{1}{|A|^s} = \prod_{P \mid R} \frac{1}{1 - \frac{1}{|P|^s}} = \prod_{P \mid R} \left( 1 - \frac{1}{|P|^s} \right) \frac{1}{1 - q^{1-s}}. \]

We can see that the far right side provides a meromorphic continuation to $\mathbb{C}$ with simple poles at $1 + \frac{2\pi i n}{\log q}$ for $m \in \mathbb{Z}$.

If $\chi$ is a non-trivial character of modulus $R$, then we have some $B \in \mathcal{A}$ with $\deg B < \deg R$ and $(B, R) = 1$ satisfying $\chi(B) \neq 1$. Thus, we have
\[ \sum_{A \in \mathcal{A}, \deg A < \deg R} \chi(A) = \sum_{A \in \mathcal{A}, \deg A < \deg R} \chi(AB) = \chi(B) \sum_{A \in \mathcal{A}, \deg A < \deg R} \chi(A), \]
and so we must have
\[ \sum_{A \in \mathcal{A}, \deg A < \deg R} \chi(A) = 0. \]

This leads to
\[ L(s, \chi) = \sum_{A \in \mathcal{M}} \frac{\chi(A)}{|A|^s} = \sum_{A \in \mathcal{M}, \deg A < \deg R} \frac{\chi(A)}{|A|^s}. \]

Thus, we have a finite polynomial in $q^{-s}$ which provides a holomorphic continuation to $\mathbb{C}$. The Riemann hypothesis for these $L$-functions has been proved in this setting, and so we have that all zeros lie on the critical line. Thus, we can order them and write the $n$-th zero as $\gamma_n = \frac{1}{2} + i\rho_n$ for some $\rho_n \in \mathbb{R}$. Clearly, they are vertically periodic with period $\frac{2\pi}{\log q}$.

Lemma 2.5. Let $\chi$ a primitive character of modulus $R \neq 1$. Then,
\[ \left| L\left(\frac{1}{2}, \chi\right) \right|^2 = 2 \sum_{A, B \in \mathcal{M}, \deg AB < \deg R} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}} + c(\chi), \]
where, if $\chi$ is odd, we define
\[ c(\chi) := - \sum_{A, B \in \mathcal{M}, \deg AB = \deg R - 1} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}}, \]
and if $\chi$ is even we define
\[ c(\chi) := - \frac{q}{(q^\frac{1}{2} - 1)^2} \sum_{A, B \in \mathcal{M}, \deg AB = \deg R - 2} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}} - \frac{2q^\frac{1}{2}}{q^\frac{1}{2} - 1} \sum_{A, B \in \mathcal{M}, \deg AB = \deg R - 1} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}} + \frac{1}{(q^\frac{1}{2} - 1)^2} \sum_{A, B \in \mathcal{M}, \deg AB = \deg R} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}}. \]

Proof. See Lemmas 3.10 and 3.11 in [1].
Lemma 2.6. Let \( R \in \mathcal{M} \) and let \( x \) be a positive integer. Then,
\[
\sum_{\substack{A \in \mathcal{M} \\
\deg A \leq x \\
(A, R) = 1}} \frac{1}{|A|} = \begin{cases}
\frac{\phi(R)}{|R|} x + O\left(\frac{\phi(R)}{|R|} \log \omega(R)\right) & \text{if } x \geq \deg R \\
\frac{\phi(R)}{|R|} x + O\left(\frac{\phi(R)}{|R|} \log \omega(R)\right) + O\left(\frac{2^{\omega(R)} - 1}{q}\right) & \text{if } x < \deg R
\end{cases}
\]

Proof. See Lemma 4.12 in [1]. This result is slightly stronger, but the proof is identical. \( \square \)

Corollary 2.7. If \( a > 0 \) and \( x = a \deg R \), then,
\[
\sum_{\substack{A \in \mathcal{M} \\
\deg A \leq x \\
(A, R) = 1}} \frac{1}{|A|} = \frac{\phi(R)}{|R|} x + O_a\left(\frac{\phi(R)}{|R|} \log \omega(R)\right).
\]

If \( b > 2 \) and \( x = \log_q b^{\omega(R)} \), then
\[
\sum_{\substack{A \in \mathcal{M} \\
\deg A \leq x \\
(A, R) = 1}} \frac{1}{|A|} = \frac{\phi(R)}{|R|} x + O_b\left(\frac{\phi(R)}{|R|} \log \omega(R)\right).
\]

Proof. First consider the case where \( x = a \deg R \). If \( q > e^{\frac{4 \log 2}{a}} \), then
\[
\frac{2^{\omega(R)} x}{q^a} \ll \frac{2^{\omega(R)}}{q^a} \leq \frac{q^{\log q} \deg R - a \deg R}{\deg R} \ll q^{\frac{a}{4} \deg R} \ll a \frac{\phi(R)}{|R|}.
\]

If \( q \leq e^{\frac{4 \log 2}{a}} \), then
\[
\frac{2^{\omega(R)} x}{q^a} \ll \frac{2^{\omega(R)}}{q^a} = q^{\left(\frac{\deg R}{\log \deg R}\right) - \frac{a}{4} \deg R} \ll q^{-\frac{a}{4} \deg R} \ll a \frac{\phi(R)}{|R|},
\]

where the second relation holds for \( \deg R > c_a \), where \( c_a \) is some constant that is dependent on \( a \), but independent of \( q \). Finally, there are only a finite number of cases where \( q \leq e^{\frac{4 \log 2}{a}} \) and \( \deg R \leq c_a \), and so
\[
\frac{2^{\omega(R)} x}{q^a} \ll a \frac{\phi(R)}{|R|}
\]

for these cases too. The proof follows from Lemma 2.6.

Now consider the case where \( x = \log_q b^{\omega(R)} \). We have that
\[
\frac{2^{\omega(R)} x}{q^a} = \frac{2^{\omega(R)} (\log_q b) \omega(R)}{b^{\omega(R)}} \ll_b \frac{2^{\omega(R)} (b + 2) \omega(R)}{\left(\frac{b + 2}{2}\right)^{\omega(R)}} = \left(\frac{4}{b + 2}\right)^{\omega(R)} = \prod_{P \mid R} \left(\frac{4}{b + 2}\right) \ll_b \prod_{P \mid R} \left(1 - \frac{1}{|P|}\right) \ll_b a \frac{\phi(R)}{|R|},
\]

Again, the proof follows from Lemma 2.6. \( \square \)

3. The Hybrid Euler-Hadamard Product Formula

Before proving Theorem 1.2, we prove several lemmas.

Lemma 3.1. For all Dirichlet characters \( \chi \) and all \( \Re(s) > 1 \) we have
\[
\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{A \in \mathcal{M}} \frac{\chi(A) \lambda(A)}{|A|^s}.
\]
Proof. Taking the logarithmic derivative of

\[ L(s, \chi) = \prod_{P \in \mathcal{P}} \left( 1 - \frac{\chi(P)}{|P|^s} \right)^{-1} \]

gives

\[ \frac{L'(s, \chi)}{L(s, \chi)} = - \sum_{P \in \mathcal{P}} \frac{\chi(P) \log |P|}{|P|^s} \left( 1 - \frac{\chi(P)}{|P|^s} \right)^{-1} = - \sum_{A \in \mathcal{M}} \frac{\chi(A) \Lambda(A)}{|A|^s}. \]

\[ \square \]

Lemma 3.2. Let \( \chi \) be a non-trivial character. As \( \Re(s) \to \infty \),

\[ \frac{L'(s, \chi)}{L(s, \chi)} = O_{\chi}(1). \]

Proof. As \( \chi \) is non-trivial, there is some maximal integer \( N \geq 0 \) with \( L_N(\chi) \neq 0 \). Hence,

\[ L(-s, \chi) = \sum_{n=0}^{N} L_n(\chi)q^{ns} \gg q^{N \Re(s)} \]

and

\[ L'(-s, \chi) = - \log q \sum_{n=0}^{N} nL_n(\chi)q^{ns} \ll q^{N \Re(s)}. \]

The proof follows.

\[ \square \]

Lemma 3.3. Let \( X \) be a positive integer, and let \( u(x) \) be a positive \( C^\infty \)-function with support in \([e, e^{1+q^{-X}}]\). Let \( \tilde{u}(s) \) be its Mellin transform. That is,

\[ \tilde{u}(s) = \int_{x=0}^{\infty} x^{s-1}u(x)dx \]

and

\[ u(x) = \frac{1}{2\pi i} \int_{\Re(s)=c} x^{-s}\tilde{u}(s)ds, \]

where \( c \) can take any value in \( \mathbb{R} \) (due to our restrictions on the support of \( u \), we can see that \( \tilde{u}(s) \) is well-defined for all \( s \in \mathbb{C} \), and so, by the Mellin inversion theorem, \( c \) can take any value in \( \mathbb{R} \)). Then,

\[ \tilde{u}(s) \ll \begin{cases} \frac{|s|}{|s|+1} \max_{x} \{|u'(x)| e^{2 \Re(s)} \} & \text{if } \Re(s) > 0 \\ \frac{1}{|s|+1} \max_{x} \{|u'(x)| e^{\Re(s)} \} & \text{if } \Re(s) \leq 0 \end{cases} \]

Proof. We have, by integration by parts, that

\[ \tilde{u}(s) = \int_{x=e}^{e^{1+q^{-X}}} x^{s-1}u(x)dx = -\frac{1}{s} \int_{x=e}^{e^{1+q^{-X}}} x^s u'(x)dx. \]

If \(|s| > 1\), then it is not difficult to deduce that the above is

\[ \ll \begin{cases} \frac{|s|}{|s|+1} \max_{x} \{|u'(x)| e^{2 \Re(s)} \} & \text{if } \Re(s) > 0 \\ \frac{1}{|s|+1} \max_{x} \{|u'(x)| e^{\Re(s)} \} & \text{if } \Re(s) \leq 0 \end{cases} \]

If \(|s| \leq 1\), then, by using the fact that \( \int_{x=e}^{e^{1+q^{-X}}} u'(x)dx = 0 \), we obtain

\[ \tilde{u}(s) = \int_{x=e}^{e^{1+q^{-X}}} \frac{1-x^s}{s} u'(x)dx = -\int_{x=e}^{e^{1+q^{-X}}} \left( \int_{y=1}^{x} y^{s-1}dy \right) u'(x)dx \]

\[ \ll \int_{x=e}^{e^{1+q^{-X}}} |u'(x)|dx \ll \max_{x} \{|u'(x)|\}, \]

from which the result follows.

\[ \square \]
Lemma 3.4. Let $X$ be a positive integer, and let $u(x)$ be a positive $C^\infty$-function with support in $[e, e^{1+q^{-X}}]$, and let $\tilde{u}(s)$ be its Mellin transform. Let

$$v(x) = \int_{1-x}^\infty u(t) \, dt$$

and take $u$ to be normalised so that $v(0) = 1$. Note that its Mellin transform is

$$\tilde{v}(s) = \frac{\tilde{u}(s+1)}{s}.$$

Let $\chi$ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\{1\}$. Then, for $s \in \mathbb{C}$ not being a zero of $L(s, \chi)$, we have

$$\frac{-L' L}{L}(s, \chi) = \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\left(e^{\deg A} A\right) = \frac{1}{2\pi i} \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} \int_{\text{Re}(w)=c} \frac{\tilde{u}(w+1)}{w} |A|^{-w} \frac{\log q}{X} \, dw$$

where $\rho_n = \frac{1}{2} + i\gamma_n$ is the $n$-th zero of $L(s, \chi)$. Note that, by Lemma 3.3, we can see that the sum over the zeros is absolutely convergent.

Proof. Let $c > \max\{0, (1 - \text{Re}(s))(\log q)X\}$. By the Mellin inversion theorem, we have

$$\sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\left(e^{\deg A} A\right) = \frac{1}{2\pi i} \int_{\text{Re}(w)=c} \frac{\tilde{u}(w+1)}{w} \sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} |A|^{-w} \frac{\log q}{X} \, dw$$

The interchange of integral and summation is justified by absolute convergence, which holds because $c > (1 - \text{Re}(s))(\log q)X$ and by Lemma 3.3.

We now shift the line of integration to $\text{Re}(w) = -M$, for some $M > \max\{0, \text{Re}(s)(\log q)X\}$, giving

$$\sum_{A \in \mathcal{M}} \frac{\chi(A)\Lambda(A)}{|A|^s} v\left(e^{\deg A} A\right) = - \frac{L' L}{L}(s, \chi) - \sum_{\rho_n} \frac{\tilde{u}(1 + (\rho_n - s)(\log q)X)}{\rho_n - s}$$

$$- \frac{1}{2\pi i} \int_{\text{Re}(w)=-M} \frac{\tilde{u}(w+1)}{w} \frac{L'}{L} \left(s + \frac{w}{(\log q)X}, \chi\right) \, dw,$$

where the sum over the zeros counts multiplicities. This requires some justification. We make use of the contour that is the rectangle with vertices at

$$c \pm i \left((d - \text{Im}(s))(\log q)X + 2\pi nX\right),$$

$$-M \pm i \left((d - \text{Im}(s))(\log q)X + 2\pi nX\right).$$

Here, $d > 0$ is such that $\frac{1}{2} + id$ is not a pole of $\frac{L'}{L}(s, \chi)$ (that is, not a zero of $L(s, \chi)$). It is clear that as $n \to \infty$ we capture all the poles and the left edge tends to the integral over $\text{Re}(w) = -M$.

Due to the vertical periodicity of $\frac{L'}{L}$, and our choice of $d$, we can see that the top and bottom integrals are equal to $O_{c,M}(n^{-1})$, which vanishes as $n \to \infty$.

By Lemmas 3.2 and 3.3, if we let $M \to \infty$ then we see that the integral over $\text{Re}(w) = -M$ vanishes.

Finally, we note that

$$v\left(e^{\deg A} X\right) = \begin{cases} 1 & \text{if } \deg A \leq X \\ 0 & \text{if } \deg A \geq X(1+q^{-X}). \end{cases}$$
Also, since $X$ is a positive integer, there are no integers in the interval $(X, X(1 + q^{-X})) \subseteq (X, X + \frac{1}{2})$, and so there are no $A \in A$ that have degree in this interval. It follows that

$$\sum_{A \in M} \frac{\chi(A)\Lambda(A)}{|A|^s} v\left(\frac{\log |A|}{e^{(\log q)X}}\right) = \sum_{A \in M} \frac{\chi(A)\Lambda(A)}{|A|^s}.$$

\[ \square \]

**Lemma 3.5.** Suppose $u(x)$ has support in $[e, e^{1+q^{-X}}]$. For all $z \in \mathbb{C}\setminus\{0\}$ with $\arg(z) \neq \pi$ we define

$$U(z) := \int_{x=0}^{\infty} u(x) E_1(z \log x) dx.$$  

(Recall, for $y \in \mathbb{C}\setminus\{0\}$ with $\arg(y) \neq \pi$, we define $E_1(y) := \int_{w=y}^{y+\infty} e^{-w} dw$.) Let $\chi$ be a primitive Dirichlet character of modulus $R \in \mathcal{M}\setminus\{1\}$, and suppose $\rho$ is a zero of $L(s, \chi)$ and $s \in \mathbb{C}\setminus\{\rho\}$ with $\arg(s - \rho) \neq \pi$. Then,

$$\int_{s_0=s}^{s+\infty} \frac{\tilde{u}(1 + (\rho - s_0)(\log q)X)}{\rho - s_0} ds_0 = -U \left( (s - \rho)(\log q)X \right).$$

**Proof.** We have

$$\int_{s_0=s}^{s+\infty} \frac{\tilde{u}(1 + (\rho - s_0)(\log q)X)}{\rho - s_0} ds_0 = \int_{s_0=s}^{s+\infty} \frac{1}{\rho - s_0} \int_{x=0}^{\infty} x^{\rho - s_0}(\log q)X u(x) dx ds_0$$

$$= \int_{x=0}^{\infty} u(x) \int_{s_0=s}^{s+\infty} \frac{x^{\rho - s_0}(\log q)X \log x}{\rho - s_0} ds_0 dx$$

$$= - \int_{x=0}^{\infty} u(x) \int_{w=(s-\rho)(\log q)X \log x}^{x+\infty} e^{-w} dw dx$$

$$= - \int_{x=0}^{\infty} u(x) E_1 \left( (s - \rho)(\log q)X \log x \right) dx$$

$$= - U \left( (s - \rho)(\log q)X \right).$$

The interchange of integration is justified by absolute convergence, which holds for $X > 1$. \[ \square \]

We can now proceed with the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Suppose $s \in \mathbb{C}$ is not a zero of $L(s, \chi)$ and $\arg(s - \rho) \neq \pi$ for all zeros $\rho$ of $L(s, \chi)$. We recall that (8) gives us

$$-\frac{L'(s_0, \chi)}{L(s_0, \chi)} = \sum_{A \in M \atop \deg A \leq X} \frac{\chi(A)\Lambda(A)}{|A|^s} + \sum_{\rho_n} \frac{\tilde{u}(1 + (\rho_n - s_0)(\log q)X)}{\rho_n - s_0},$$

to which we apply the integral $\int_{s_0=s}^{s+\infty} ds_0$ to both sides to obtain

$$\log L(s, \chi) = \sum_{A \in M \atop \deg A \leq X} \frac{\chi(A)\Lambda(A)}{|A|^s \log |A|} - \sum_{\rho} U \left( (s - \rho)(\log q)X \right).$$

For the integral over the sum over zeros, we applied Lemma 3.5, after an interchange of summation and integration that is justified by Lemma 3.3. We now take exponentials of both sides of (9) to obtain

$$L(s, \chi) = \exp \left( \sum_{A \in M \atop \deg A \leq X} \frac{\chi(A)\Lambda(A)}{|A|^s \log |A|} \right) \exp \left( - \sum_{\rho} U \left( (s - \rho)(\log q)X \right) \right)$$

$$= P_X(s, \chi) Z_X(s, \chi).$$

Now suppose we have $s \in \mathbb{C}$, not being a zero of $L(s, \chi)$, but with $\arg(s - \rho) = \pi$ for some zero $\rho$ of $L(s, \chi)$. We can see that $\lim_{s_0 \to s} L(s_0, \chi) = L(s, \chi)$ and $\lim_{s_0 \to s} P_X(s_0, \chi) = P_X(s, \chi) = 0$. The latter is non-zero as $P_X(s, \chi)$ is the exponential of a polynomial. From this, we can deduce
that \( \lim_{s_0 \to s} Z_X(s_0, \chi) = L(s, \chi)(P_X(s, \chi))^{-1} \in \mathbb{C} \). Similarly, if \( s \) is a zero of \( L(s, \chi) \), then we can see that \( \lim_{s_0 \to s} Z_X(s_0, \chi) = L(s, \chi)(P_X(s, \chi))^{-1} = 0 \). This completes the proof.

4. Moments of the Partial Euler Product

We require the following two lemmas before proving Theorem 1.5.

**Lemma 4.1.** For all \( \operatorname{Re}(s) > 0 \) and primitive characters \( \chi \) we define

\[
(10) \quad P_X^*(s, \chi) := \prod_{\deg P \leq X} \left( 1 - \frac{\chi(P)}{|P|^s} \right)^{-1} \prod_{\deg P < X} \left( 1 + \frac{\chi(P)^2}{2|P|^{2s}} \right)^{-1},
\]

and for positive integers \( k \) and \( A \in S_M(X) \) we define \( \alpha_k(A) \) by

\[
P_X^*(s, \chi)^k = \sum_{A \in S_M(X)} \frac{\alpha_k(A) \chi(A)}{|A|^s}.
\]

Then, for positive integers \( k \), we have

\[
(11) \quad P_X \left( \frac{1}{2}, \chi \right)^k = \left( 1 + O_k(X^{-1}) \right) P_X^* \left( \frac{1}{2}, \chi \right)^k = \left( 1 + O_k(X^{-1}) \right) \sum_{A \in S_M(X)} \frac{\alpha_k(A) \chi(A)}{|A|^s}.
\]

We also have that

\[
(12) \quad \alpha_k(A) = d_k(A) \quad \text{if } A \in S_M \left( \frac{X}{2} \right) \text{ or } A \text{ is prime}
\]

\[
0 \leq \alpha_k(A) \leq d_k(A) \quad \text{if } A \notin S_M \left( \frac{X}{2} \right) \text{ and } A \text{ is not prime}.
\]

**Proof.** First we note that

\[
P_X \left( \frac{1}{2}, \chi \right) = \exp \left( \sum_{\deg A \leq X} \frac{\chi(A)\Lambda(A)}{|A|^s \log |A|} \right) = \exp \left( \sum_{\deg P \leq X} \sum_{j=1}^{N_P} \frac{\chi(P)^j}{j|P|^{s_j}} \right),
\]

where

\[
N_P := \left\lfloor \frac{X}{\deg P} \right\rfloor.
\]

Also, by using the Taylor series for \( \log \), we have

\[
P_X^* \left( \frac{1}{2}, \chi \right) = \exp \left( \sum_{\deg P \leq X} \sum_{j=1}^{\infty} \frac{\chi(P)^j}{j|P|^{s_j}} + \sum_{\deg P < X} \sum_{j=1}^{\infty} \frac{(-1)^j \chi(P)^{2j}}{j2^j|P|^{s_j}} \right).
\]

Hence,

\[
P_X \left( \frac{1}{2}, \chi \right) P_X^* \left( \frac{1}{2}, \chi \right)^{-1} = \exp \left( - \sum_{\deg P \leq X} \sum_{j=N_P+1}^{\infty} \frac{\chi(P)^j}{j|P|^{s_j}} - \sum_{\deg P < X} \sum_{j=1}^{\infty} \frac{(-1)^j \chi(P)^{2j}}{j2^j|P|^{s_j}} \right).
\]

We now show that the terms inside the exponential are equal to \( O(X^{-1}) \), from which we easily deduce

\[
P_X \left( \frac{1}{2}, \chi \right)^k = \left( 1 + O_k(X^{-1}) \right) P_X^* \left( \frac{1}{2}, \chi \right)^k.
\]
To this end, using the prime polynomial theorem for the last line below, we have

\[
\sum_{\deg P \leq X} \sum_{j=N_p+1}^{\infty} \frac{\chi(P)^j}{j|P|^\frac{1}{2}} + \sum_{\frac{|P|}{2} < \deg P \leq X} \sum_{j=1}^{\infty} \frac{(-1)^j \chi(P)^{2j}}{j2^j|P|^j} = \sum_{\deg P \leq \frac{|P|}{2}} \sum_{j=N_p+1}^{\infty} \frac{\chi(P)^j}{j|P|^\frac{1}{2}} + \sum_{\frac{|P|}{2} < \deg P \leq X} \sum_{j=2}^{\infty} \frac{\chi(P)^j}{j|P|^j} + \sum_{\frac{|P|}{2} < \deg P \leq X} \sum_{j=1}^{\infty} \frac{(-1)^j \chi(P)^{2j}}{j2^j|P|^j}
\]

\[
\ll \sum_{\deg P \leq \frac{|P|}{2}} |P|^{-\frac{N_p+1}{2}} + \sum_{\frac{|P|}{2} < \deg P \leq X} |P|^{-\frac{1}{2}} \ll q^{-\frac{n}{2}} \sum_{\deg P \leq \frac{|P|}{2}} 1 + \sum_{\frac{|P|}{2} < \deg P \leq X} \frac{q^{-\frac{n}{2}}}{n} \ll \frac{1}{\chi}.
\]

We now proceed to prove (12). The first case is clear, so assume that \( A \notin \mathcal{S}_M \left( \chi \right) \) and \( A \) is not prime. We note that

\[
\left( 1 - \frac{\chi(P)}{|P|^\frac{1}{2}} \right)^{-1} \left( 1 + \frac{\chi(P)^2}{2|P|} \right)^{-1} = \left( 1 + \frac{\chi(P)}{|P|^\frac{1}{2}} + \frac{\chi(P)^2}{|P|^1} + \ldots \right) \left( 1 - \frac{\chi(P)^2}{2|P|} + \frac{\chi(P)^4}{2|P|^2} - \ldots \right) = \sum_{r=0}^{\infty} \left( \sum_{r_1, r_2 \geq 0 \atop r_1 + 2r_2 = r} \left( -\frac{1}{2} \right)^{r_1} \right) \frac{\chi(P)^r}{|P|^\frac{r}{2}} = \sum_{r=0}^{\infty} \frac{2}{3} \left( 1 - \left( -\frac{1}{2} \right)^{\frac{r}{2}} \right) \frac{\chi(P)^r}{|P|^\frac{r}{2}}.
\]

Since

\[
0 \leq \frac{2}{3} \left( 1 - \left( -\frac{1}{2} \right)^{\frac{r}{2}} \right) \leq 1
\]

for all \( r \geq 0 \), the result follows. \( \square \)

**Lemma 4.2** (Mertens’ Third Theorem in \( F_q[T] \)). We have

\[
\prod_{\deg P \leq n} \left( 1 - \frac{1}{|P|} \right)^{-1} \sim e^\gamma n.
\]

**Proof.** The proof is very similar to that of Theorem 3 in [24]. \( \square \)

We can now prove Theorem 1.5.

**Proof of Theorem 1.5.** Throughout this proof, any asymptotic relations are to be taken as \( X, \deg R \xrightarrow{q,k} \infty \) with \( X \leq \log_q \deg R \). By Lemma 4.1 it suffices to prove that

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R} \left| \sum_{A \in \mathcal{S}_M(X)} \frac{\alpha_k(A) \chi(A)}{|A|^\frac{1}{2}} \right|^2 \sim a(k) \prod_{\deg P \leq X} \left( \sum_{m=0}^{\infty} \frac{d_k(P^m)^2}{|P|^m} \right)^{-1} \left( e^\gamma X \right)^{k^2}.
\]

We will truncate our Dirichlet series. This will allow us to bound the lower order terms later. We have

\[
\sum_{A \in \mathcal{S}_M(X)} \frac{\alpha_k(A) \chi(A)}{|A|^\frac{1}{2}} = \sum_{A \in \mathcal{S}_M(X)} \frac{\alpha_k(A) \chi(A)}{|A|^\frac{1}{2}} + O \left( |R|^{-\frac{1}{4}} \right).
\]

(14)
This makes use of the following:

\[
\sum_{A \in S_M(X), \deg A > \frac{1}{4} \deg R} \frac{\alpha_k(A) \chi(A)}{|A|^{\frac{3}{2}}} \leq |R|^{-\frac{1}{16}} \sum_{A \in S_M(X)} \frac{d_k(A)}{|A|^{\frac{3}{2}}} = |R|^{-\frac{1}{16}} \prod_{\deg P \leq X} \left(1 - \frac{1}{|P|^{\frac{3}{4}}}\right)^{-k}
\]

(15)

\[
= |R|^{-\frac{1}{16}} \exp\left(\sum_{\deg P \leq X} -k \log \left(1 - \frac{1}{|P|^{\frac{3}{4}}}\right)\right) = |R|^{-\frac{1}{16}} \exp\left(kO\left(\sum_{\deg P \leq X} \frac{1}{|P|^{\frac{3}{4}}}\right)\right) 
\]

\[
= |R|^{-\frac{1}{16}} \exp\left(kO\left(\frac{\deg R}{\log q \deg R}\right)\right) = O\left(|R|^{-\frac{1}{16}}\right).
\]

By the Cauchy-Schwarz inequality, it suffices to prove that

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| \sum_{A \in S_M(X), \deg A > \frac{1}{4} \deg R} \frac{\alpha_k(A) \chi(A)}{|A|^{\frac{3}{2}}} \right|^2 \sim o(k) \prod_{\deg P \leq X, P \neq R} \left(\sum_{m=0}^{\infty} \frac{d_k(P_m)^2}{|P|^{m}}\right)^{-1} \left(e^7 X\right)^{k^2}.
\]

Now, we have that

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| \sum_{A \in S_M(X), \deg A > \frac{1}{4} \deg R} \frac{\alpha_k(A) \chi(A)}{|A|^{\frac{3}{2}}} \right|^2 = \frac{1}{\phi^*(R)} \sum_{A, B \in S_M(X), \deg A, \deg B > \frac{1}{4} \deg R} \frac{\alpha_k(A) \alpha_k(B)}{|AB|^{\frac{3}{2}}} \sum_{EF=R} \mu(E) \phi(F) 
\]

(16)

\[
= \frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E) \phi(F) \sum_{A, B \in S_M(X), \deg A, \deg B > \frac{1}{4} \deg R} \frac{\alpha_k(A) \alpha_k(B)}{|AB|^{\frac{3}{2}}} 
\]

\[
= \sum_{A \in S_M(X), \deg A > \frac{1}{4} \deg R} \frac{\alpha_k(A)^2}{|A|} + \frac{1}{\phi^*(R)} \sum_{EF=R} \mu(E) \phi(F) \sum_{A, B \in S_M(X), \deg A, \deg B > \frac{1}{4} \deg R} \frac{\alpha_k(A) \alpha_k(B)}{|AB|^{\frac{3}{2}}}.
\]

We first consider the second term on the far right side: The off-diagonal terms. We note that the inner sum is zero if \( \deg F > \frac{1}{4} \deg R \), and we also make use of (12), to obtain

\[
\leq \frac{1}{\phi^*(R)} \sum_{EF=R} \phi(F) \sum_{A, B \in S_M(X), \deg A, \deg B > \frac{1}{4} \deg R} \frac{d_k(A) d_k(B)}{|AB|^{\frac{3}{2}}}
\]

\[
\leq \frac{1}{\phi^*(R)} \prod_{\deg P \leq X} \left(1 - |P|^{-\frac{3}{4}}\right)^{-2k} \sum_{EF=R} \phi(F) \sum_{\deg F \leq 4 \deg R} \frac{d_k(A) d_k(B)}{|AB|^{\frac{3}{2}}}.
\]
where the second relation follows from a contour shift. Similarly, if $X$,  
the fact that $X, R \rightarrow \infty$ with $X \leq \log_q \deg R$. Now we consider the first term on the far right side of (16): The diagonal terms. We required a truncated sum only for the off-diagonal terms, and so we extend our sum using similar means as in (15):

$$\sum_{A \in \mathcal{S}_M(X)} \frac{\alpha_k(A)^2}{|A|} = \sum_{A \in \mathcal{S}_M(X)} \frac{\alpha_k(A)^2}{|A|} + O\left(|R|^{-\frac{1}{2}}\right).$$

Now, using (12) for the first relation below (and part of the second relation), we have that

$$\sum_{A \in \mathcal{S}_M(X)} \frac{\alpha_k(A)^2}{|A|} = \prod_{\deg P \leq X} \left( \sum_{m=0}^\infty \frac{d_k(P^m)^2}{|P|^m} \right)$$

$$= \prod_{\deg P \leq X} \left( \sum_{m=0}^\infty \frac{d_k(P^m)^2}{|P|^m} \right) \prod_{|P| \neq X} \left( 1 + \frac{d_k(P)^2}{|P|^2} + \sum_{m=2}^\infty \frac{d_k(P^m)^2}{|P|^m} \right)$$

$$= \prod_{|P| \neq X} \left( 1 + \frac{1}{|P|^2} + \sum_{m=2}^\infty \frac{d_k(P^m)^2}{|P|^m} \right)$$

$$= (1 + o(1)) a(k) \prod_{|P| \neq X} \left( \sum_{m=0}^\infty \frac{d_k(P^m)^2}{|P|^m} \right) \left( e^{-X} \right)^k.$$

For the last equality, we used Lemma 4.2. The proof follows.

5. Moments of the Hadamard Product

In this section we provide support for the Conjecture 1.6. We require the following lemma.

**Lemma 5.1.** For real $y > 0$ define

$$\text{Ci}(y) := - \int_{t=y}^{\infty} \frac{\cos(t)}{t} dt,$$

and let $x$ be real and non-zero. Then,

$$\text{Re} \, E_1(ix) = - \text{Ci}(|x|).$$

**Proof.** If $x > 0$, then

$$\text{Re} \, E_1(ix) = \text{Re} \int_{w=ix}^{ix+i\infty} \frac{e^{-w}}{w} dw = \text{Re} \int_{w=ix}^{i\infty} \frac{e^{-w}}{w} dw = \text{Re} \int_{t=x}^{\infty} \frac{e^{-it}}{t} dt = - \text{Ci}(|x|),$$

where the second relation follows from a contour shift. Similarly, if $x < 0$, then

$$\text{Re} \, E_1(ix) = \text{Re} \int_{w=ix}^{ix+i\infty} \frac{e^{-w}}{w} dw = \text{Re} \int_{w=ix}^{-i\infty} \frac{e^{-w}}{w} dw = \text{Re} \int_{t=|x|}^{\infty} \frac{e^{it}}{t} dt = - \text{Ci}(|x|).$$
Now, writing $\gamma_n(\chi)$ for the imaginary part of the $n$-th zero of $L(s, \chi)$, we can see that
\begin{equation}
\begin{aligned}
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| Z_X \left( \frac{1}{2}, \chi \right) \right|^{2k} \\
= \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \exp \left( -2k \text{Re} \sum_{\gamma_n(\chi)} U \left( -i\gamma_n(\chi) \left( \log q \right) X \right) \right) \\
= \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \exp \left( -2k \text{Re} \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) E_1 \left( -i\gamma_n(\chi) \left( \log q \right) X \log x \right) dx \right) \\
= \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \exp \left( 2k \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) \text{Ci} \left( |\gamma_n(\chi)| \left( \log q \right) X \log x \right) dx \right).
\end{aligned}
\end{equation}

We note that the terms in the exponential tend to zero as $|\gamma_n(\chi)|$ tends to infinity, and so the above is primarily concerned with the zeros close to $\frac{1}{2}$. As described in Section 1, there is a relationship between the zeros of Dirichlet $L$-functions near $\frac{1}{2}$ and the eigenphases of random unitary matrices near 0: The proportion of Dirichlet $L$-functions of modulus $R$ that have $j$-th zero (that is, its imaginary part) in some interval $[a, b]$ appears to be the same as the proportion of unitary $N(R) \times N(R)$ matrices that have $j$-th eigenphase in $[a, b]$ (at least, this is the case in an appropriate limit). Naturally, one asks what value $N(R)$ should take in terms of $R$. We note that the mean spacing between zeros of Dirichlet $L$-functions of modulus $R$ is $\frac{2\pi}{\log q \deg R}$, while the mean spacing between eigenphases of unitary $N \times N$ matrices is $\frac{2\pi}{N}$. Therefore, we take $N(R) = |\log q \deg R|$. So, we replace the imaginary parts of the zeros with eigenphases of $N(R) \times N(R)$ unitary matrices, and instead of averaging over primitive characters we average over unitary matrices. That is, we conjecture
\begin{equation}
\begin{aligned}
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| Z_X \left( \frac{1}{2}, \chi \right) \right|^{2k} \\
= \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \exp \left( 2k \sum_{\gamma_n(\chi)} \int_{x=0}^{\infty} u(x) \text{Ci} \left( |\gamma_n(\chi)| \left( \log q \right) X \log x \right) dx \right) \\
\sim \int_{A \in \mathcal{U} \left( N(R) \right)} \exp \left( 2k \sum_{\theta_n(A)} \int_{x=0}^{\infty} u(x) \text{Ci} \left( |\theta_n(A)| \left( \log q \right) X \log x \right) dx \right) dA
\end{aligned}
\end{equation}
as $\deg R \to \infty$, where the integral is with respect to the Haar measure, and $\theta_n(A)$ is the $n$-th eigenphase of $A$. The eigenphases are periodic with period $2\pi$, and these periodicised eigenphases are included in the sum. An asymptotic evaluation of the right side can be made identically as in Section 4 of [11]; but we simply replace their $\log X$ with our $(\log q)X$, and we replace their $N = |\log T|$ with our $N(R) = |\log q \deg R|$. This leads us to the conjecture that
\begin{equation}
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| Z_X \left( \frac{1}{2}, \chi \right) \right|^{2k} \sim \frac{G^2(k + 1)}{G(2k + 1)} \left( \frac{\deg R}{e^\gamma X} \right)^k,
\end{equation}
as $\deg R \to \infty$. We note that in [11], their $u(x)$ has a slightly different support than the support of our $u(x)$. However, this does not affect the result.

Remark 5.2. We will provide further justification for one of the steps above, which is not given in [11]. In the middle line of (19) we have a sum over all $\gamma_n(\chi)$. This includes zeros that are far away from $\frac{1}{2}$. We mentioned previously that their contribution is small, but a closer inspection reveals that we cannot dismiss them so easily, and so we must justify replacing them with the eigenphases of our unitary matrices. For the zeros close to $\frac{1}{2}$ (that is, for $\gamma_n(\chi)$ close to 0) we have already provided this justification. For the zeros further away, one can argue that the zeros of a typical Dirichlet $L$-function are equidistributed in some manner, and that the eigenphases of a
typical unitary matrix are also equidistributed in some manner. Thus, we could replace the former with the latter. This is based on the idea that if you sum a function over a set of equidistributed points on some interval \( I \), then the result is roughly equal to the integral over \( I \) of that function multiplied by the reciprocal of the mean spacing of the points. Recall that the mean spacing of our eigenphases is equal to that of our zeros. Naturally, one asks why we do not use the same justification for the zeros close to \( \frac{1}{2} \). The answer is that the function \( C_i(x) \) has a discontinuity at \( x = 0 \), and so we require a stronger justification for the zeros near \( \frac{1}{2} \) (that is, the \( \gamma_n(\chi) \) close to 0). Finally, we remark that we do not provide any rigorous support for the claims on equidistribution above.

6. The Second Hadamard Moment

Before proving Theorem 1.7, we prove several lemmas. First, by (11) we have

\[
P_X\left(\frac{1}{2}, \chi \right) = \left(1 + O(X^{-1})\right) P_X^* \left(\frac{1}{2}, \chi \right).
\]

Rearranging and using (10) gives

\[
P_X\left(\frac{1}{2}, \chi \right)^{-1} = \left(1 + O(X^{-1})\right) P_X^* \left(\frac{1}{2}, \chi \right)^{-1} = \left(1 + O(X^{-1})\right) \prod_{\deg P \leq X} \left(1 - \frac{\chi(P)}{|P|^s}\right)^{-1} \prod_{\frac{X}{2} < \deg P \leq X} \left(1 + \frac{\chi(P)^2}{2|P|^{2s}}\right)^{-1} = \left(1 + O(X^{-1})\right) \sum_{A \in S_M(X)} \frac{\alpha^{-1}(A) \chi(A)}{|A|^\frac{1}{2}},
\]

where \( \alpha^{-1} \) is defined multiplicatively by

\[
\alpha^{-1}(P) := \begin{cases} 
-1 & \text{if } \deg P \leq X \\
0 & \text{if } \deg P > X;
\end{cases}
\]

\[
\alpha^{-1}(P^2) := \begin{cases} 
0 & \text{if } \deg P \leq \frac{X}{2} \\
\frac{1}{2} & \text{if } \frac{X}{2} < \deg P \leq X \\
0 & \text{if } \deg P > X;
\end{cases}
\]

\[
\alpha^{-1}(P^3) := \begin{cases} 
0 & \text{if } \deg P \leq \frac{X}{2} \\
-\frac{1}{2} & \text{if } \frac{X}{2} < \deg P \leq X \\
0 & \text{if } \deg P > X;
\end{cases}
\]

\[
\alpha^{-1}(P^m) := 0 \text{ for } m \geq 4.
\]

Lemma 6.1. For all \( R \in \mathcal{M} \), we have that

\[
\sum_{\substack{HST \in S_M(X) \\
(S,T) = 1 \\
(HST,R) = 1 \\
\deg HS, \deg HT \leq \frac{1}{10} \deg R}} \frac{|\alpha^{-1}(HS)\alpha^{-1}(HT)|}{|HST|} \ll X^3
\]

as \( X \to \infty \).

Proof. Using Lemma 4.2, we have that

\[
\sum_{\substack{HST \in S_M(X) \\
(S,T) = 1 \\
(HST,R) = 1 \\
\deg HS, \deg HT \leq \frac{1}{10} \deg R}} \frac{|\alpha^{-1}(HS)\alpha^{-1}(HT)|}{|HST|} \ll \left( \sum_{H \in S_M(X)} \frac{1}{|H|^s} \right)^3 = \prod_{\deg P \leq X} \left(1 - |P|^{-1}\right)^{-3} \ll X^3
\]

as \( X \to \infty \). \qed
Lemma 6.2. For all \( R \in \mathcal{M} \), we have that
\[
\sum_{HST \in \mathcal{S}_M(X)} \frac{\alpha_1(HS)\alpha_1(HT)}{|HST|} \deg ST \ll X^4
\]
as \( X \to \infty \).

Proof. We have that
\[
\sum_{HST \in \mathcal{S}_M(X)} \frac{\alpha_1(HS)\alpha_1(HT)}{|HST|} \deg ST \ll \sum_{H \in \mathcal{S}_M(X)} \frac{1}{|H|} \sum_{S,T \in \mathcal{S}_M(X)} \frac{\deg ST}{|ST|}
\]
Consider
\[
f(s) := \sum_{S,T \in \mathcal{S}_M(X)} \frac{1}{|ST|^s} = \left( \sum_{T \in \mathcal{S}_M(X)} \frac{1}{|T|^s} \right)^2 \prod_{\deg P \leq X} \left( 1 - |P|^{-s} \right)^{-2}.
\]
Taking the derivative of the above and then evaluating at \( s = 1 \), we obtain
\[
\sum_{S,T \in \mathcal{S}_M(X)} \frac{\deg ST}{|ST|} = 2 \prod_{\deg P \leq X} \left( 1 - |P|^{-1} \right)^{-2} \sum_{\deg P \leq X} \frac{\deg P}{|P| - 1} \ll X^3
\]
as \( X \to \infty \), where we have made use of Lemma 4.2 and the prime polynomial theorem. This, along with the fact that
\[
\sum_{H \in \mathcal{S}_M(X)} \frac{1}{|H|} = \prod_{\deg P \leq X} \left( 1 - |P|^{-1} \right)^{-1} \ll X
\]
as \( X \to \infty \), proves the lemma. \( \square \)

Lemma 6.3. Let \( V \in \mathcal{M} \). \( V \) may or may not depend on \( R \). As \( X, \deg R \to \infty \) with \( X \leq \log_q \deg R \), we have
\[
\sum_{HST \in \mathcal{S}_M(X)} \frac{\alpha_1(HS)\alpha_1(HT)}{|HST|} = \left( 1 + O \left( q^{-\frac{X}{2}} \right) \right) \prod_{\deg P \leq X} \left( 1 - \frac{1}{|P|} \right) + O \left( \frac{1}{|R|^\frac{1}{2}} \right) \sim \prod_{\deg P \leq X} \left( 1 - \frac{1}{|P|} \right).
\]

Proof. The second relation in the Lemma follows easily from Lemma 4.2. We will prove the first. In this proof, all asymptotic relations are to be taken as \( X, \deg R \to \infty \) with \( X \leq \log_q \deg R \).

Similar to (14), we can remove the conditions \( \deg HS, \deg HT \leq \frac{1}{10} \deg R \) from the sum and this only adds an \( O \left( |R|^{-\frac{1}{2}} \right) \) term. Now, writing \( C = HS \) and \( D = HT \), we have
\[
\sum_{HST \in \mathcal{S}_M(X)} \frac{\alpha_1(HS)\alpha_1(HT)}{|HST|} = \sum_{CD \in \mathcal{S}_M(X)} \frac{\alpha_1(C)\alpha_1(D)}{|CD||C,D|}
\]
\[
= \sum_{CD \in \mathcal{S}_M(X)} \frac{\alpha_1(C)\alpha_1(D)}{|CD|} \sum_{G \in \mathcal{S}_M(X)} \phi(G) = \sum_{G \in \mathcal{S}_M(X)} \phi(G) \left( \sum_{C \in \mathcal{S}_M(X)} \frac{\alpha_1(CG)}{|C|} \right)^2.
\]
Before continuing, let us make a definition: For all $A \in \mathcal{M}$ and all $P \in \mathcal{P}$, let $e_{P}(A)$ be the largest integer such that $P^{e_{P}(A)} \mid A$. Continuing, we note that we can restrict the sums to polynomials that are fourth power free. Indeed, $\alpha_{1}(P^{m}) = 0$ for all $P \in \mathcal{P}$ and all $m \geq 4$. Note that if $P \mid G$ then we must have that $0 \leq e_{P}(C) \leq 3 - e_{P}(G)$, while if $P \nmid G$ then $0 \leq e_{P}(C) \leq 3$. So, we have

$$
\sum_{C \leq M(X) \atop (C, V) = 1} \frac{\alpha_{1}(CG)}{|C|} = \prod_{P | G} \left( \sum_{j=0}^{3-e_{P}(G)} \frac{\alpha_{1}(P^{j}+e_{P}(G))}{|P|^{j}} \right) \prod_{P | V} \left( \sum_{j=0}^{3} \frac{\alpha_{1}(P^{j})}{|P|^{j}} \right).
$$

So,

$$
\sum_{G \in S_{M}(X) \atop (G, V) = 1} \phi(G) \left( \sum_{C \in S_{M}(X) \atop (C, V) = 1} \frac{\alpha_{1}(CG)}{|C|} \right)^{2}
= \prod_{\deg P \leq X \atop P \mid V} \left( \sum_{j=0}^{3} \frac{\alpha_{1}(P^{j})}{|P|^{j}} \right)^{2} \prod_{\deg P \leq X \atop P \mid V} \phi(P) \left( \sum_{i=0}^{3} \frac{\alpha_{1}(P^{i})}{|P|^{i}} \right) \left( \sum_{j=0}^{3} \frac{\alpha_{1}(P^{j})}{|P|^{j}} \right)^{2}
= \prod_{\deg P \leq X \atop P \mid V} \left( \sum_{i=0}^{3} \phi(P^{i}) \left( \sum_{j=0}^{3-i} \frac{\alpha_{1}(P^{j+i})}{|P|^{j+i}} \right)^{2} \right)
= \prod_{\deg P \leq X \atop P \mid V} \left( \sum_{i=0}^{3-i} \sum_{j=0}^{3-i} \sum_{k=0}^{i} \phi(P^{i}) \alpha_{1}(P^{j+i}) \alpha_{1}(P^{k+i}) \right) \left( 1 + \frac{1}{|P|} \right)^{2}
= \prod_{\deg P \leq X \atop P \mid V} \left( \frac{1}{|P|} \sum_{X \leq \deg P \leq X} \frac{1}{|P|} \right)
= \left( 1 + O\left( \frac{1}{|P|^{2}} \right) \right) \prod_{\deg P \leq X \atop P \mid V} \left( 1 - \frac{1}{|P|} \right).
$$

The result follows.

**Lemma 6.4.** Let $R \in \mathcal{M}$. Suppose $Z_{1} \leq \deg R$ and $F \mid R$. Further, suppose $C, D \in S_{M}(X)$ with $\deg C, \deg D \leq \frac{1}{10} \deg R$. Then, we have

$$
\sum_{A, B \in \mathcal{M} \atop \deg AB = Z_{1}} \frac{1}{|AB|^{2}} \ll \frac{q^{\frac{Z_{1}}{2}}(Z_{1}+1)|CD|}{|F|}.
$$

**Proof.** Consider the case where $\deg AC > \deg BD$, and suppose that $\deg A = i$. We have that $AC = LF + BD$ for some $L \in \mathcal{M}$ with $\deg L = \deg AC - \deg F = i + \deg C - \deg F$, and
Suppose now that \( \deg B = Z_1 - \deg A = Z_1 - i \). Hence,

\[
\sum_{A, B \in \mathcal{M}} \frac{1}{|AB|^\frac{1}{2}} \leq q^{\frac{Z_1}{2}} \sum_{i=0}^{\deg L = i + \deg C - \deg F} \sum_{B \in \mathcal{M}} \frac{1}{|B|^\frac{1}{2}} \sum_{\deg AC > \deg BD} 1
\]

Similarly, when \( \deg BD > \deg AC \) we have

\[
\sum_{A, B \in \mathcal{M}} \frac{1}{|AB|^\frac{1}{2}} \leq \frac{q^{\frac{Z_1}{2}} (Z_1 + 1) |D|}{|F|}.
\]

Suppose now that \( \deg AC = \deg BD = i \). Then, \( 2i = \deg ABCD = Z_1 + \deg CD \). We have \( \deg B = i - \deg D = \frac{Z_1 + \deg C - \deg D}{2} \), and \( AC = LF + BD \) for some \( L \in \mathcal{A} \) with \( \deg L < i - \deg F = \frac{Z_1 + \deg CD}{2} - \deg F \). Hence,

\[
\sum_{A, B \in \mathcal{M}} \frac{1}{|AB|^\frac{1}{2}} \leq q^{\frac{Z_1}{2}} \sum_{\deg B = \frac{Z_1 + \deg C - \deg D}{2}} \sum_{\deg L = \frac{Z_1 + \deg CD}{2}} 1
\]

The result follows. \( \square \)

We can now prove Theorem 1.7.

**Proof of Theorem 1.7.** Throughout the proof, all asymptotic relations will be taken as \( X, \deg R \to \infty \) with \( X \leq \log_q \deg R \). Now, by (20), we have

\[
(21) \quad \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^* \left| L \left( \frac{1}{2}, \chi \right) P_X \left( \frac{1}{2}, \chi \right)^{-1} \right|^2 \sim \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^* \left| L \left( \frac{1}{2}, \chi \right) P_X \left( \frac{1}{2}, \chi \right)^{-1} \right|^2.
\]

Similar to (14), we truncate our sum:

\[
P_X \left( \frac{1}{2}, \chi \right)^{-1} = \sum_{C \in \mathcal{S}_\mathcal{M}(X)} \frac{\alpha_{-1}(C) \chi(C)}{|C|^\frac{1}{2}} + O \left( |R|^{-\frac{1}{10}} \right).
\]

Using this, the Cauchy-Schwarz inequality, and (4), it suffices to prove that

\[
\begin{aligned}
\sim \frac{1}{\phi^*(R)} \sum_{\chi \bmod R}^* \left| L \left( \frac{1}{2}, \chi \right) \right|^2 & \sum_{C, D \in \mathcal{S}_\mathcal{M}(X)} \frac{\alpha_{-1}(C) \alpha_{-1}(D) \chi(C) \chi(D)}{|CD|^\frac{1}{2}} \\
& \sim \frac{\deg R}{e^\gamma X} \prod_{P > X \atop P | R} \left( 1 - \frac{1}{|P|} \right).
\end{aligned}
\]
Now, by Lemma 2.5, we have

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left| L\left(\frac{1}{2}, \chi \right) \right|^2 \sum_{C,D \in S_M(X)} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \\
= \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \left( a(\chi) + c(\chi) \right) \sum_{C,D \in S_M(X)} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}},
\]

where

\[ a(\chi) := 2 \sum_{A,B \in M} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \]

and \( c(\chi) \) is defined in Lemma 2.5.

We first consider the case with \( a(\chi) \). We have

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* a(\chi) \sum_{C,D \in S_M(X)} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}} \\
= \frac{2}{\phi^*(R)} \sum_{\chi \mod R}^* \sum_{A,B \in M} \frac{\alpha_{-1}(C)\alpha_{-1}(D)\chi(AC)\overline{\chi}(BD)}{|ABCD|^{\frac{1}{2}}} \\
= \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{A,B \in M} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
= 2 \sum_{A,B \in M} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}} \\
+ \frac{2}{\phi^*(R)} \sum_{EF=R} \mu(E)\phi(F) \sum_{A,B \in M} \frac{\alpha_{-1}(C)\alpha_{-1}(D)}{|ABCD|^{\frac{1}{2}}}.
\]

(23)
For the first term on the far right side, the diagonal terms, we can write \( A = GS, B = GT, \)
\( C = HT, D = HS \) where \( G, H, S, T \in M \) and \((S, T) = 1\), giving

\[
\sum_{A, B, C, D \in M, C, D \in S_M(X)} \alpha_1(C) \alpha_1(D) \frac{1}{|ABCD|^2} = 2 \sum_{G \in M, H, S \in S_M(X)} \alpha_1(HT) \alpha_1(HS) \frac{1}{|GHST|}.
\]

By Corollary 2.7 and Lemmas 6.1, 6.2, and 6.3 we obtain the asymptotic relation below. The final equality uses Lemma 4.2.

\[
2 \sum_{A, B, C, D \in M, C, D \in S_M(X)} \alpha_1(C) \alpha_1(D) \frac{1}{|ABCD|^2} \sim \frac{\phi(R)}{|R|} \deg R \prod_{P \mid R} \left(1 - \frac{1}{|P|}\right)
\]

\[
\sim \frac{\deg R}{e^X} \prod_{P > X \mid R} \left(1 - \frac{1}{|P|}\right).
\]

For the second term on the far right side of (23), the off-diagonal terms, we use Lemma 6.4 to obtain

\[
\frac{2}{\phi^*(R)} \sum_{E, F \in R} \mu(E) \phi(F) \frac{\alpha_1(C) \alpha_1(D)}{|ABCD|^2} \leq \frac{2}{\phi^*(R)} \sum_{C, D \in S_M(X)} \alpha_1(C) \alpha_1(D) \frac{1}{|CD|^2} \sum_{E, F \in R} \mu(E) \phi(F) \sum_{A, B \in M, \deg AB < \deg R} \frac{1}{|AB|^2}
\]

\[
\ll \frac{\deg R}{\phi^*(R)} \sum_{C, D \in S_M(X), \deg C, \deg D \leq \frac{1}{10} \deg R} |CD|^2 \sum_{E, F \in R} \mu(E) \left(\frac{\phi(F)}{|F|}\right) \ll \frac{\deg R}{\phi^*(R)} = o(1).
\]

Finally, consider the case with \( c(\chi) \). We recall that if \( \chi \) is odd then it consists of one sum, whereas, if \( \chi \) is even it consists of three sums. We will show that one of the sums for the even \( \chi \) is of lower order. The other sums for the even \( \chi \), and the odd \( \chi \), are similar. We then see that the total
contribution of the case with \( c(\chi) \) is of lower order. We have

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^{*} \sum_{A,B \in \mathcal{M}}^{\chi \text{ even}} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} \sum_{C,D \in \mathcal{S}_M(X)}^{\deg C, \deg D \leq \frac{10}{11} \deg R} \frac{\alpha_1(C)\alpha_1(D)\chi(C)\overline{\chi}(D)}{|CD|^{\frac{1}{2}}}
\]

(27)

where the last relation follows by similar means as the case with \( a(\chi) \).

\[\square\]

7. Preliminary Results for the Fourth Hadamard Moment

In this section we develop the preliminary results that are required for the proof of Theorem 1.8. We begin with two results that will simplify the problem.

**Lemma 7.1.** For \( X \geq 12 \), we have that

\[
P_X \left( \frac{1}{2} \chi \right)^{-2} = (1 + O(X^{-1})) P_X^{**} \left( \frac{1}{2}, \chi \right),
\]

where

\[
P_X^{**} \left( \frac{1}{2}, \chi \right) := \sum_{A \in \mathcal{S}_M(X)} \frac{\beta(A)\chi(A)}{|A|^{\frac{1}{2}}}
\]

and \( \beta \) is defined multiplicatively by

\[
\beta(P) := \begin{cases} 
-2 & \text{if } \deg P \leq X \\
0 & \text{if } \deg P > X 
\end{cases}
\]

\[
\beta(P^2) := \begin{cases} 
1 & \text{if } \deg P \leq \frac{X}{2} \\
2 & \text{if } \frac{X}{2} < \deg P \leq X \\
0 & \text{if } \deg P > X 
\end{cases}
\]

\[
\beta(P^k) := 0 \text{ for } k \geq 3.
\]

**Proof.** By Lemma 4.1 we have

\[
P_X \left( \frac{1}{2} \chi \right)^{-2} = (1 + O(X^{-1})) \prod_{\deg P \leq X} \left( 1 - \frac{\chi(P)}{|P|^{\frac{1}{2}}} \right)^2 \prod_{\frac{X}{2} < \deg P \leq X} \left( 1 + \frac{\chi(P)^2}{2|P|} \right)^2.
\]

By writing \( P_X^{**} \left( \frac{1}{2}, \chi \right) \) as an Euler product, we see that

\[
\prod_{\deg P \leq X} \left( 1 - \frac{\chi(P)}{|P|^{\frac{1}{2}}} \right)^2 \prod_{\frac{X}{2} < \deg P \leq X} \left( 1 + \frac{\chi(P)^2}{2|P|} \right)^2
\]

\[
= P_X^{**} \left( \frac{1}{2}, \chi \right) \prod_{\frac{X}{2} < \deg P \leq X} \left( 1 + \frac{\chi(P)^2}{2|P|} \right)^2
\]

\[
= P_X^{**} \left( \frac{1}{2}, \chi \right) \prod_{\frac{X}{2} < \deg P \leq X} \left( 1 + \frac{\chi(P)^2}{2|P|} \right)^2
\]

\[
= P_X^{**} \left( \frac{1}{2}, \chi \right) \exp \left( O \left( \sum_{\frac{X}{2} < \deg P \leq X} \frac{|P|^{-\frac{3}{2}}}{} \right) \right)
\]

\[
= \left( 1 + O(X^{-1}q^{-\frac{X}{2}}) \right) P_X^{**} \left( \frac{1}{2}, \chi \right).
\]
The result follows. The requirement that $X \geq 12$ is so that the factor $\left( 1 - \frac{2\chi(P)}{|P|^\frac{1}{2}} + \frac{2\chi(P)^2}{|P|} \right)^{-1}$ in the second line is guaranteed to be non-zero.

**Lemma 7.2.** We define

$$\tilde{P}_x^*(\frac{1}{2}, \chi) := \sum_{A \in S_M(X), \deg A > \frac{1}{q} \log_q \deg R} \frac{\beta(A) \chi(A)}{|A|^\frac{1}{2}}.$$

Then, as $X, \deg R \xrightarrow{q} \infty$ with $X \leq \log_q \deg R$,

$$P_x^*(\frac{1}{2}, \chi) = \tilde{P}_x^*(\frac{1}{2}, \chi) + O\left((\deg R)^{-\frac{1}{3}}\right).$$

**Proof.** We have, as $X, \deg R \xrightarrow{q} \infty$ with $X \leq \log_q \deg R$,

$$\sum_{A \in S_M(X), \deg A > \frac{1}{q} \log_q \deg R} \frac{\beta(A) \chi(A)}{|A|^\frac{1}{2}} \ll \frac{1}{(\deg R)^{\frac{1}{4}}} \sum_{A \in S_M(X)} \frac{|\beta(A)|}{|A|^\frac{1}{2}}.$$

Therefore,

$$=(\deg R)^{-\frac{1}{4}} \prod_{\deg P \leq X} \left( 1 + 2|P|^{-\frac{1}{4}} + 2|P|^{-\frac{1}{2}} \right)$$

$$=(\deg R)^{-\frac{1}{4}} \exp \left( O\left( \sum_{\deg P \leq X} |P|^{-\frac{1}{4}} \right) \right)$$

$$=(\deg R)^{-\frac{1}{4}} \exp \left( O\left( \frac{q^{\frac{3}{4}X}}{X} \right) \right) \leq (\deg R)^{-\frac{1}{4}}.$$ 

We now prove several results that will be used to obtain the main asymptotic term in Theorem 1.8.

**Lemma 7.3.** Suppose $A_1, A_2, A_3, B_1, B_2, B_3 \in \mathcal{M}$ satisfy $A_1A_2A_3 = B_1B_2B_3$. Then, there are $G_1, G_2, G_3, V_1, V_2, V_1, V_2, V_2, V_3, V_2, V_3, V_3, V_2, V_3$ in $\mathcal{M}$, satisfying $(V_{i,j}, V_{k,l}) = 1$ when both $i \neq k$ and $j \neq l$ hold, such that

$$A_1 = G_1V_{1,2}V_{1,3} \quad B_1 = G_1V_{2,1}V_{3,1}$$

$$A_2 = G_2V_{1,2}V_{2,3} \quad B_2 = G_2V_{1,2}V_{3,2}$$

$$A_3 = G_3V_{1,3}V_{3,2} \quad B_3 = G_3V_{1,3}V_{2,3}.$$

Furthermore, this is a bijective correspondence. To clarify, $G_i$ is the highest common divisor of $A_i$ and $B_i$; and in $V_{i,j}$ the subscript $i$ indicates that $V_{i,j}$ divides $A_i$ and the subscript $j$ indicates that $V_{i,j}$ divides $B_j$.

**Proof.** Let us write $A_i = G_iS_i$ and $B_i = G_iT_i$, where

$$(29) \quad G_i = (A_i, B_i)$$

$$(S_i, T_i) = 1.$$ 

Since $A_1A_2A_3 = B_1B_2B_3$, we must have that

$$(30) \quad S_1S_2S_3 = T_1T_2T_3.$$ 

First we note that, due to (30) and the coprimality relations in (29), we have that $S_i \mid T_jT_k$ and $T_i \mid S_jS_k$ for $i,j,k$ distinct.

Second, again due to (30) and (29), we must have that $(S_1, S_2, S_3), (T_1, T_2, T_3) = 1.$
Third, for \( i \neq j \), we define \( S_{i,j} := (S_i, S_j) \) and \( T_{i,j} := (T_i, T_j) \). Again due to to (30) and (29), we have \((S_{i,j})^2 | T_k \) and \((T_{i,j})^2 | S_k \) for \( i, j, k \) distinct. Furthermore, \((S_{i,j,1}, S_{i,j,2}) = 1 \) and \((T_{i,j,1}, T_{i,j,2}) = 1 \) for all \( \{i_1, j_1\} \neq \{i_2, j_2\} \), and \((S_{i,j,1}, T_{i,j,2}) = 1 \) for all \( i, j, i_2, j_2 \).

From these three points we can deduce that
\[
S_1 = S_{1,2}S_{1,3}(T_{2,3})^2S_1' \quad T_1 = T_{1,2}T_{1,3}(S_{2,3})^2T_1'
\]
\[
S_2 = S_{1,2}S_{2,3}(T_{1,3})^2S_2' \quad T_2 = T_{1,2}T_{2,3}(S_{1,3})^2T_2'
\]
\[
S_3 = S_{1,3}S_{2,3}(T_{1,2})^2S_3' \quad T_3 = T_{1,3}T_{2,3}(S_{1,2})^2T_3'
\]

for some \( S_i' \) and \( T_i' \) satisfying \((S_i', T_i') = 1 \) for all \( i \) and \((S_i', S_j'), (T_i', T_j') = 1 \) for \( i \neq j \). By (30) we have that \( S_i'S_j'S_3' = T_i'T_2'T_3' \). From these points we can deduce that
\[
S_i' = U_{i,2}U_{1,3} \quad T_i' = U_{2,1}U_{3,1}
\]
\[
S_j' = U_{2,1}U_{2,3} \quad T_j' = U_{2,1}U_{3,2}
\]
\[
S_3' = U_{3,1}U_{3,2} \quad T_3' = U_{1,3}U_{2,3}
\]

where the \( U_{i,j} \) are pairwise coprime. Also, for \( i, j, k \) distinct, because \( U_{i,j} | T_j \) and \((S_j, T_j) = 1 \), we have that \((U_{i,j}, S_j) = 1 \), and hence \((U_{i,j}, S_{j,k}), (U_{i,j}, S_{j,i}) = 1 \). Similarly, for \( i, j, k \) distinct, we have \((U_{i,j}, T_{i,k}), (U_{i,j}, T_{i,j}) = 1 \).

So, by defining
\[
V_{1,2} = S_{1,3}T_{2,3}U_{1,2} \quad V_{2,1} = S_{2,3}T_{1,3}U_{2,1} \quad V_{3,1} = S_{2,3}T_{1,2}U_{3,1}
\]
\[
V_{1,3} = S_{1,2}T_{2,3}U_{1,3} \quad V_{2,3} = S_{1,2}T_{1,3}U_{2,3} \quad V_{3,2} = S_{1,3}T_{2,1}U_{3,2}
\]
we complete the proof for the existence claim.

Uniqueness follows from the following observation: If we have \( G_i \) and \( V_{i,j} \) satisfying the conditions in the Lemma, then we can deduce
\[
G_i = (A_i, B_i) \quad \text{for all } i,
\]
\[
V_{i,j} = \left( V_{i,j}V_{k,j}, \frac{V_{i,j}V_{k,j}V_{i,j}V_{k,j}}{V_{k,j}V_{k,j}} \right) = \left( \hat{B}_j, \frac{\hat{B}_i\hat{B}_j}{A_k} \right) \quad \text{for } i, j, k \text{ distinct},
\]
where we define \( \hat{B}_i, \hat{A}_i \) by \( B_i = G_i\hat{B}_i = (A_i, B_i)\hat{B}_i \) and \( A_i = G_i\hat{A}_i = (A_i, B_i)\hat{A}_i \) for all \( i \). Since the far right side of each line above is expressed entirely in terms of \( A_1, A_2, A_3, B_1, B_2, B_3 \), we must have uniqueness.

\[\square\]

**Lemma 7.4.** Suppose \( V_{1,3}, V_{2,3}, V_{3,1}, V_{3,2} \in \mathcal{M} \), and \((V_{1,3}, V_{3,1}V_{3,2}) = 1 \) and \((V_{2,3}, V_{3,1}V_{3,2}) = 1 \). Then,
\[
\left\{(V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : (V_{1,2}, V_{2,3}V_{3,1}) = 1, (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1 \right\}
\]
\[
= \bigcup_{V \in \mathcal{M}} \left\{(V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : \right.
\]
\[
\left. (V_{1,3}V_{3,1}V_{2,3}V_{3,2}) = 1, V_{1,2}V_{2,1} = V, (V_{1,2}, V_{2,3}V_{3,1}) = 1, (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1 \right\},
\]

and for each such \( V \) we have
\[
\#\left\{(V_{1,2}, V_{2,1}) \in \mathcal{M}^2 : V = (V_{1,2}, V_{2,3}V_{3,1}) = 1, (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1 \right\}
\]
\[
= 2^{\omega(V)-\omega(V_{1,3}V_{2,3}V_{3,1}V_{3,2})}.
\]

**Proof.** For the first claim we note that \((V_{1,2}, V_{2,3}V_{3,1}) = 1 \) and \((V_{2,1}, V_{1,3}V_{3,2}) = 1 \) imply that
\[
(V, V_{1,3}, V_{2,3}) \cdot (V_{3,1}, V_{3,2}) = 1,
\]
and, due to the given coprimality relations of \(V_{1,3}, V_{2,3}, V_{3,1}\), and \(V_{3,2}\) given in Lemma 7.3, we have

\[
(V_{1,3}, V_{2,3}) \cdot (V_{3,1}, V_{3,2}) = (V_{1,3}V_{3,1}, V_{2,3}V_{3,2})
\]

The first claim follows.

We now look at the second claim. For \(A, B \in M\), we define \(A_B\) to be the maximal divisor of \(A\) that is coprime to \(B\), and we define \(A^B\) by \(A = A_B A^B\). We then have that

\[
V = V_{1,3}V_{2,3}V_{3,1}V_{3,2} V^{V_{1,3}V_{2,3}V_{3,1}V_{3,2}} = V_{1,3}V_{2,3}V_{3,1}V_{3,2} V^{V_{1,3}V_{2,3}V_{3,1}V_{3,2}},
\]

where the last equality follows from \((V_i, (V_{1,3}V_{3,1}, V_{2,3}V_{3,2})) = 1\) and the fact that \((V_{1,3}, V_{3,1}) = 1\) and \((V_{2,3}, V_{3,2}) = 1\). Now, \(V = V_{1,2}V_{2,1}\) and by the coprimality relations we must have that \(V^{V_{1,3}V_{2,3}} | V_{1,2} \) and \(V^{V_{1,2}V_{3,1}} | V_{2,1}\). So, we see that

\[
\#\{(V_{1,2}, V_{2,1}) \in M^2 : V_{1,2}V_{2,1} = V, (V_{1,2}, V_{2,3}V_{3,1}) = 1, (V_{2,1}, V_{1,3}V_{3,2}) = 1, (V_{1,2}, V_{2,1}) = 1\}
\]

\[
= \#\{(V_{1,2}, V_{2,1}) \in M^2 : V_{1,2}V_{2,1} = V_{1,3}V_{2,3}V_{3,1}V_{3,2} V^{V_{1,3}V_{2,3}V_{3,1}V_{3,2}},
V^{V_{1,3}V_{3,2}} | V_{1,2}, V^{V_{2,3}V_{3,1}} | V_{2,1}, (V_{1,2}, V_{2,1}) = 1\}
\]

\[
= 2^\omega(V_{1,3}V_{2,3}V_{3,1}V_{3,2}) = 2^\omega(V) - \omega((V_{1,3}V_{2,3}V_{3,1}V_{3,2}))
\]

\[\square\]

We now need to give a definition for the primorials in \(F_q[T]\).

**Definition 7.5 (Primorial Polynomials).** Let \((S_i)_{i \in \mathbb{Z}_{>0}}\) be a fixed ordering of \(P\) such that \(\deg S_i \leq \deg S_{i+1}\) for all \(i \geq 1\) (the order of the primes of a given degree is not of importance here). For all positive integers \(n\) we define

\[
R_n := \prod_{i=1}^n S_i.
\]

We will refer to \(R_n\) as the \(n\)-th primorial. For each positive integer \(n\) we have unique non-negative integers \(m_n\) and \(r_n\) such that

\[
R_n = \left( \prod_{\deg P \leq m_n} P \right) \left( \prod_{i=1}^{r_n} Q_i \right),
\]

where the \(Q_i\) are distinct primes of degree \(m_n + 1\). This definition of primorial is not standard.

**Lemma 7.6.** For all positive integers \(n\) we have that

\[
\log_q \log_q |R_n| = m_n + O(1).
\]

From this we can deduce that

\[
m_n \ll \log_q \log_q |R_n|
\]

for \(n\) satisfying \(m_n \geq 1\). In particular, the implied constant is independent of \(q\).

**Proof.** For the first claim, by (31) and (7), we see that

\[
\log_q |R_n| = \deg R_n \leq \sum_{i=1}^{m_n+1} \left( q^i + O\left(q^{\frac{i}{2}}\right) \right) \ll q^{m_n+1}
\]

and

\[
\log_q |R_n| = \deg R_n \geq \sum_{i=1}^{m_n} \left( q^i + O\left(q^{\frac{i}{2}}\right) \right) \gg q^{m_n}.
\]

By taking logarithms of both equations above, we deduce that

\[
\log_q \log_q |R_n| = m_n + O(1).
\]
For the second claim, if \( m_n \geq 1 \) then \( \log_q \log_q |R_n| \geq 1 \), and so by the first claim we have
\[
\frac{m_n}{\log_q \log_q |R_n|} \ll 1 + \frac{1}{\log_q \log_q |R_n|} \ll 1.
\]
\[\square\]

**Lemma 7.7.** For all \( R \in \mathcal{M} \) with \( \deg R \geq 1 \), non-negative integers \( k \), and \( s \in \mathbb{C} \) with \( \text{Re}(s) > -1 \) we define
\[
f_{R,k}(s) := \prod_{P \mid R} \left(1 - |P|^{-s-1}\right)^k,
\]
\[
h_{R,k}(s) := \prod_{P \mid R} \left(1 + |P|^{-s-1}\right)^{-k}.
\]

Then, for all non-negative integers \( j \) and all integers \( r \) we have
\[
f^{(j)}_{R,k} \left( \frac{2\pi r i}{\log q} \right) \ll_j k^j \left( \log_q \deg R + O(1) \right)^j \prod_{P \mid R} \left(1 - |P|^{-1}\right)^k,
\]
\[
h^{(j)}_{R,k} \left( \frac{2\pi r i}{\log q} \right) \ll_j k^j \left( \log_q \deg R + O(1) \right)^j \prod_{P \mid R} \left(1 + |P|^{-1}\right)^{-k}.
\]

Generally, we could incorporate the \( O(1) \) terms into the relation \( \ll_j \), but for the case \( \deg R = 1 \), where we would have \( \log_q \deg R = 0 \), the \( O(1) \) terms are required.

**Proof.** We will prove only the claim for \( f_{R,k}(s) \) and \( r = 0 \). The proofs for all \( r \) and \( h_{R,k}(s) \) are almost identical. First, we note that
\[
f'_{R,k}(s) = k g_{R}(s) g_{R,k}(s),
\]
where
\[
g_{R}(s) := \sum_{P \mid R} \frac{\log |P|}{|P|^{s+1} - 1}.
\]

We note further that, for integers \( j \geq 1 \),
\[
f^{(j)}_{R,k}(s) = G_{R,k,j}(s) f_{R,k}(s),
\]
where \( G_{R,k,j}(s) \) is a sum of terms of the form
\[
k^m g^{(j_1)}_{R}(s) g^{(j_2)}_{R}(s) \cdots g^{(j_m)}_{R}(s),
\]
where \( 1 \leq m \leq j \) and \( \sum_{r=1}^{m} (j_r + 1) = j \). The number of such terms and their coefficients are dependent only on \( j \).

Now, for all \( R \in \mathcal{M} \), and non-negative integers \( l \), it is not difficult to deduce that
\[
g^{(l)}_{R}(0) \ll_l \sum_{P \mid R} \frac{(\log |P|)^{l+1}}{|P| - 1}.
\]

The function \( \left( \frac{\log x}{x-1} \right)^{l+1} \) is decreasing at large enough \( x \), and the limit as \( x \to \infty \) is 0. Therefore, there exists a constant \( c_l > 0 \) such that for all \( A, B \in \mathcal{A} \) with \( 1 \leq \deg A \leq \deg B \) we have that
\[
c_l \frac{(\log |A|)^{l+1}}{|A| - 1} \geq \frac{(\log |B|)^{l+1}}{|B| - 1}.
\]
Hence, taking \( n = \omega(R) \) and using Definition 7.5, Lemma 7.6, and the prime polynomial theorem, we see that

\[
\sum_{P \mid R} \frac{\log|P|}{|P| - 1} \ll \sum_{P \mid R_{n+1}} \frac{\log|P|}{|P| - 1} \ll \sum_{r=1}^{m+1} \frac{q^r}{r} \ll (m+1)^{l+1}.
\]

The result follows by (33), (34), (35), and (36).

\[\square\]

**Lemma 7.8** (Perron’s Formula). Let \( c \) be a positive real number, and let \( k \geq 2 \) be an integer. Then,

\[
\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^k} ds = \begin{cases} 
0 & \text{if } 0 \leq y < 1; \\
\frac{2\pi i}{(k-1)!} (\log y)^{k-1} & \text{if } y \geq 1.
\end{cases}
\]

If \( k = 1 \), then

\[
\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds = \begin{cases} 
0 & \text{if } 0 \leq y < 1; \\
\pi i & \text{if } y = 1; \\
2\pi i & \text{if } y > 1.
\end{cases}
\]

**Proof.** See [22, 4.1.6, Page 282] \[\square\]

**Lemma 7.9.** Let \( R, M \in \mathcal{M} \) with \( \deg M \leq \deg R \), \( k \) be a non-negative integer, and \( z \) be an integer-valued function of \( R \) such that \( z \sim \deg R \) as \( \deg R \to \infty \). We have that

\[
\sum_{\substack{N \in \mathcal{M} \\
\deg N \leq z \\
(N,R)=1}} \frac{2^{\omega(N) - \omega((N,M))}}{|N|} (z - \deg N)^k
\]

\[
= \frac{(1-q^{-1})}{(k+2)(k+1)} \prod_{P \mid MR} \left( \frac{1}{{1 + |P|^{-1}}} \right) \prod_{P \mid M} \left( \frac{1}{{1 - |P|^{-1}}} \right) \left( z^{k+2} + O_k(z^{k+1} \log \deg R) \right)
\]

as \( \deg R \to \infty \).

**Proof. Step 1:** Let us define the function \( F \), for \( \Re s > 1 \), by

\[
F(s) = \sum_{\substack{N \in \mathcal{M} \\
(N,R)=1}} \frac{2^{\omega(N) - \omega((N,M))}}{|N|^s}.
\]

We can see that

\[
F(s) = \prod_{P \mid MR} \left( 1 + \frac{2}{|P|^s} + \frac{2}{|P|^{2s}} + \ldots \right) \prod_{P \mid M} \left( 1 + \frac{1}{|P|^s} + \frac{1}{|P|^{2s}} + \ldots \right)
\]

\[
= \prod_{P \mid MR} \left( \frac{2}{1 - |P|^{-s}} - 1 \right) \prod_{P \mid M} \left( \frac{1}{1 - |P|^{-s}} \right)
\]

\[
= \prod_{P \in \mathcal{P}} \left( \frac{1 + |P|^{-s}}{1 - |P|^{-s}} \right) \prod_{P \mid MR} \left( 1 + |P|^{-s} \right) \prod_{P \mid M} \left( 1 - |P|^{-s} \right)
\]

\[
= \zeta_A(s)^2 \prod_{P \mid MR} \left( \frac{1}{1 + |P|^{-s}} \right) \prod_{P \mid M \mid P \mid R} \left( \frac{1}{1 - |P|^{-s}} \right).
\]
Now, let $c$ be a positive real number, and define

$$y := \begin{cases} 
q^{s + \frac{1}{2}} & \text{if } k = 0 \\
q^{s} & \text{if } k \neq 0.
\end{cases}$$

On the one hand, we have that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(1 + s) \frac{y^s}{s^{k+1}} ds = \frac{1}{2\pi i} \sum_{N \in \mathcal{M}} \frac{\omega(N) - \omega((N,M))}{|N|} \int_{c-i\infty}^{c+i\infty} \frac{y^s}{|N|^s s^{k+1}} ds$$

(37)

$$= \frac{\log q}{k!} \sum_{N \in \mathcal{M}} \frac{\omega(N) - \omega((N,M))}{|N|} (z - \deg N)^k.$$

For $k \geq 1$, the interchange of integral and summation is justified by absolute convergence, and the second equality follows by Lemma 7.8. For $k = 0$, the above holds by Lemma 7.10 below. We remark that we take $y = q^{s + \frac{1}{2}}$ when $k = 0$ so that $(\frac{1}{|N|}, k + 1) \neq (1, 1)$, which would be a special case of Lemma 7.8 that would be tedious to address.

On the other hand, for all positive integers $n$ define the following curves:

$$l_1(n) := \left[ c - \frac{(2n + \frac{1}{2})\pi i}{\log q}, c + \frac{(2n + \frac{1}{2})\pi i}{\log q} \right];$$

$$l_2(n) := \left[ c + \frac{(2n + \frac{1}{2})\pi i}{\log q}, c - \frac{(2n + \frac{1}{2})\pi i}{\log q} \right];$$

$$l_3(n) := \left[ -\frac{1}{4} + \frac{(2n + \frac{1}{2})\pi i}{\log q}, \frac{1}{4} - \frac{(2n + \frac{1}{2})\pi i}{\log q} \right];$$

$$l_4(n) := \left[ -\frac{1}{4} - \frac{(2n + \frac{1}{2})\pi i}{\log q}, c - \frac{(2n + \frac{1}{2})\pi i}{\log q} \right];$$

$$L(n) := l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n).$$

Then, we have that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(1 + s) \frac{y^s}{s^{k+1}} ds = \frac{1}{2\pi i} \lim_{n \to \infty} \left( \int_{L(n)} F(1 + s) \frac{y^s}{s^{k+1}} ds - \int_{l_1(n)} F(1 + s) \frac{y^s}{s^{k+1}} ds \right.\right.$$

$$- \left. \int_{l_2(n)} F(1 + s) \frac{y^s}{s^{k+1}} ds - \int_{l_3(n)} F(1 + s) \frac{y^s}{s^{k+1}} ds \right).$$

(38)

**Step 2:** For the first integral in (38) we note that $F(1 + s) \frac{y^s}{s^{k+1}}$ has a pole at $s = 0$ of order $k + 3$ and double poles at $s = \frac{2m\pi i}{\log q}$ for $m = \pm 1, \pm 2, \ldots, \pm n$. By applying the residue theorem we see that

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_{L(n)} F(1 + s) \frac{y^s}{s^{k+1}} ds = \text{Res}_{s=0} F(s + 1) \frac{y^s}{s^{k+1}} + \sum_{m \in \mathbb{Z}} \text{Res}_{s=\frac{2m\pi i}{\log q}} F(1 + s) \frac{y^s}{s^{k+1}}.$$

(39)

**Step 2.1:** For the first residue term we have

$$\text{Res}_{s=0} F(s + 1) \frac{y^s}{s^{k+1}}$$

$$= \frac{1}{(k + 2)!} \lim_{s \to 0} \frac{q^{k+2}}{s^{k+2}} \zeta_A(s + 1)^2 \frac{1}{\zeta_A(2s + 2)} \prod_{P \nmid M} \frac{1 - |P|^{-s-1}}{1 + |P|^{-s-1}} \prod_{P \nmid M} \frac{1}{1 - |P|^{-s-1}} y^s.$$
If we apply the product rule for differentiation, then one of the terms will be
\[
\frac{1}{(k+2)!} \lim_{s \to 0} \left( \frac{1}{2s+1} - \frac{1}{2s+2} \right) \prod_{P|MR} \frac{1}{1 + |P|^{-1}} \prod_{P|R} \left( \frac{1}{1 - |P|^{-1}} \right) \frac{d}{ds} |P|^s y^s
\]

\[
= \frac{(1 - q^{-1})(\log q)^k}{(k+2)!} \prod_{P|MR} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|R} \left( \frac{1}{1 - |P|^{-1}} \right) \left( z + O(1) \right)^{k+2}.
\]

The \( O(1) \) term is to account for the case where \( y = q^{z+\frac{1}{2}} \) (when \( k = 0 \)).

Now we look at the remaining terms that arise from the product rule. By using the fact that \( \zeta_A(1+s) = \frac{1}{1-q^{-s}} \), the Taylor series for \( q^{-s} \), and the chain rule, we have, for non-negative integers \( i \), that

\[
\lim_{s \to 0} \frac{1}{(\log q)^{i-1}} \frac{d^i}{ds^i} \zeta(s+1)s = O_i(1).
\]

Similarly, for non-negative integers \( i \),

\[
\lim_{s \to 0} \frac{1}{(\log q)^{i}} \frac{d^i}{ds^i} (2s+2)^{-1} = \frac{1}{(\log q)^{i}} \lim_{s \to 0} \frac{d^i}{ds^i} (1 - q^{-1-2s}) = O_i(1).
\]

By (41), (42), and Lemma 7.7 and the fact that \( \deg M \leq \deg R \), we see that the remaining terms are

\[
\ll k \frac{(\log q)^k}{(k+2)!} \prod_{P|MR} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|R} \left( \frac{1}{1 - |P|^{-1}} \right) z^{k+1} \log \deg R.
\]

Hence,

\[
\text{Res}_{s=0} F(s+1) \frac{y^s}{s^{k+1}} \left( \frac{1 - q^{-1}}{q^s} \right) \prod_{P|MR} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|R} \left( \frac{1}{1 - |P|^{-1}} \right) \left( z^{k+2} + O_k \left( z^{k+1} \log \deg R \right) \right)
\]

as \( \deg R \to \infty \).

**Step 2.2:** Now we look at the remaining residue terms in (39). By similar (but simpler) means as above we can show that

\[
\text{Res}_{s=\frac{2m+1}{\log q}} F(1+s) \frac{y^s}{s^{k+1}} = O_k \left( \frac{(\log q)^k}{m^{k+1}} \prod_{P|MR} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|R} \left( \frac{1}{1 - |P|^{-1}} \right) \frac{1}{m} \right)
\]

as \( \deg R \to \infty \), and so, for \( k \geq 1 \),

\[
\sum_{m \in \mathbb{Z} \atop m \neq 0} \text{Res}_{s=\frac{2m+1}{\log q}} F(1+s) \frac{y^s}{s^{k+1}} = O_k \left( \frac{(\log q)^k}{m^{k+1}} \prod_{P|MR} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|R} \left( \frac{1}{1 - |P|^{-1}} \right) \frac{1}{m} \right)
\]

as \( \deg R \to \infty \). When \( k = 0 \) we look at things more precisely and see that the term \( \frac{1}{m} \) cancels with the term with \( \frac{1}{m} \), and so (44) holds for \( k = 0 \) as well.
Step 2.3: By (39), (43) and (44), we see that
\[
\lim_{n \to \infty} \frac{1}{2\pi i} \int_{L(n)} F(1 + s) \frac{y R^s}{s^3} ds
\]
(45)
\[
= \frac{(1 - q^{-1})(\log q)^k}{(k + 2)!} \prod_{P | MR} \left( 1 - |P|^{-1} \right) \prod_{P | R} \left( 1 - |P|^{-1} \right) \left( z^{k+2} + O_k \left( z^{k+1} \log \deg R \right) \right)
\]
as \deg R \to \infty.

Step 3: We now look at the integrals over \( l_2(n) \) and \( l_4(n) \). For all positive integers \( n \) and all \( s \in l_2(n), l_4(n) \) we have that \( F(s + 1) y^s = O_{q,R,c}(1) \). One can now easily deduce for \( i = 2, 4 \) that
\[
\lim_{n \to \infty} \left| \frac{1}{2\pi i} \int_{l_i(n)} F(1 + s) \frac{y^s}{s^{k+1}} ds \right| = 0.
\]

Step 4: We now look at the integral over \( l_3(n) \). For all positive integers \( n \) and all \( s \in l_3(n) \) we have that
\[
\frac{\zeta_A(s + 1)^2}{\zeta_A(2s + 2)} = O(1)
\]
and
\[
\left| \prod_{P | MR} \left( 1 - |P|^{-s-1} \right) \prod_{P | R} \left( 1 - |P|^{-s-1} \right) y^s \right| \ll \prod_{P | R} \left( 1 - |P|^{-\frac{s}{2}} \right) \prod_{P | M} \left( 1 - |P|^{-\frac{s}{2}} \right) q^{o(\deg R)}
\]
\[
\ll \prod_{P | R} \left( 1 - |P|^{-\frac{s}{2}} \right) \prod_{P | M} \left( 1 - |P|^{-\frac{s}{2}} \right) q^{o(\deg R) - \frac{1}{2} \deg R}
\]
\ll O(1)
\]
as \deg R \to \infty. We now easily deduce that, for \( k \geq 1 \),
\[
(47)
\lim_{n \to \infty} \left| \frac{1}{2\pi i} \int_{l_3(n)} F(1 + s) \frac{y^s}{s^{k+1}} ds \right| = O(1)
\]
as \deg R \to \infty. For the case \( k = 0 \) we must be more careful. Using the fact that \( F(1 + s) \) has vertical periodicity with period \( \frac{2\pi i}{\log q} \) and the fact that \( y = q^{s+\frac{i}{2}} \) where \( z \) is an integer, we have that
\[
\int_{-\frac{1}{2}}^{-\frac{1}{4} + \infty} F(1 + s) \frac{y^s}{s} ds = \sum_{m=0}^{\infty} \int_{-\frac{1}{4} + \frac{2m\pi i}{\log q}}^{-\frac{1}{4} + \frac{(4m+2)\pi i}{\log q}} F(1 + s) \frac{y^s}{s} ds + \int_{-\frac{1}{4} + \frac{(4m+2)\pi i}{\log q}}^{-\frac{1}{4} + \frac{2m\pi i}{\log q}} F(1 + s) \frac{y^s}{s} ds
\]
\[
= \sum_{m=0}^{\infty} \int_{-\frac{1}{4} + \frac{2m\pi i}{\log q}}^{-\frac{1}{4} + \frac{2m\pi i}{\log q}} F(1 + s) \frac{y^s}{s + \frac{4m\pi i}{\log q}} ds - \int_{-\frac{1}{4} + \frac{2m\pi i}{\log q}}^{-\frac{1}{4} + \frac{(4m+2)\pi i}{\log q}} F(1 + s) \frac{y^s}{s + \frac{4m\pi i}{\log q}} ds
\]
\[
= \frac{2\pi i}{\log q} \sum_{m=0}^{\infty} \int_{-\frac{1}{4} + \frac{2m\pi i}{\log q}}^{-\frac{1}{4} + \frac{2m\pi i}{\log q}} \frac{y^s}{s + \frac{4m\pi i}{\log q}} (s + \frac{(4m+2)\pi i}{\log q}) ds
\]
\[
= \frac{2\pi i}{\log q} \sum_{m=0}^{\infty} \int_{-\frac{1}{4} + \frac{2m\pi i}{\log q}}^{-\frac{1}{4} + \frac{(4m+2)\pi i}{\log q}} F(1 + s) \frac{y^s}{s} (s + \frac{2m\pi i}{\log q}) ds
\]
\ll \int_{-\frac{1}{4}}^{-\frac{1}{2}} \frac{1}{|s| \cdot |s + \frac{2\pi i}{\log q}|} ds \ll 1.
A similar result can be obtained for the integral from $-rac{1}{4}$ to $-rac{1}{4} - i \infty$. Hence, we have that

$$\lim_{n \to \infty} \left| \frac{1}{2\pi i} \int_{l_1(n)} F(1+s) \frac{y^s}{s} \, ds \right| = O(1)$$

as $\deg R \to \infty$.

**Step 5:** By (37), (38), (45), (46), (47) and (48), we deduce that

$$\sum_{\substack{N \in M \\ \deg N \leq z \\ (N,R) = 1}} \frac{2^{\omega(N) - \omega((N,M))}}{|N|} (z - \deg N)^k$$

$$= \frac{(1 - q^{-1})}{(k + 2)(k + 1)} \prod_{P|MR} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|M} \left( \frac{1}{1 - |P|^{-1}} \right) \left( z^{k+2} + O_k \left( z^{k+1} \log \deg R \right) \right)$$

as $\deg R \to \infty$. $\square$

**Lemma 7.10.** Let $F(s)$, $z$, and $c$ be as in Lemma 7.9, and let $y = q^{z + \frac{1}{2}}$. Then,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(1+s) \frac{y^s}{s} \, ds = \sum_{\substack{N \in M \\ \deg N \leq z \\ (N,R) = 1}} \frac{2^{\omega(N) - \omega((N,M))}}{|N|^s}.$$

**Proof.** Let $w > z + \frac{1}{2}$ and define

$$F_w(s) := \sum_{\substack{N \in M \\ \deg N > w \\ (N,R) = 1}} \frac{2^{\omega(N) - \omega((N,M))}}{|N|^s}.$$

Then,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(1+s) \frac{y^s}{s} \, ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{\substack{N \in M \\ \deg N \leq w \\ (N,R) = 1}} \frac{2^{\omega(N) - \omega((N,M))}}{|N|^{1+s}} \frac{y^s}{s} \, ds + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_w(s) \frac{y^s}{s} \, ds$$

$$= \sum_{\substack{N \in M \\ \deg N \leq z \\ (N,R) = 1}} \frac{2^{\omega(N) - \omega((N,M))}}{|N|} + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_w(s) \frac{y^s}{s} \, ds,$$

where we have used Lemma 7.8 for the last equality. We must show that the second term on the right side is zero. To this end, we note that

$$F_w(s) \leq \sum_{\substack{N \in M \\ \deg N > w}} \frac{1}{|N|^\Re(s) - 1} \ll q^{w(2 - \Re(s))},$$

and we define the contours

$$l_1(n,m) := [c - ni, c + ni];$$

$$l_2(n,m) := [c + ni, m + ni];$$

$$l_3(n,m) := [m + ni, m - ni];$$

$$l_4(n,m) := [m - ni, c - ni];$$

$$L(n,m) := l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n).$$
We then have that
\[ \int_{L(n,m)} F_w(s) \frac{y^s}{s} ds \leq \frac{2}{m^{\frac{1}{2}}} q^w \left( \frac{y}{q^w} \right)^m \rightarrow 0 \]
as \( m \rightarrow \infty \), since \( q^w > q^{z+\frac{1}{2}} = y \). We also have that
\[ \int_{c+ni}^{c+ni} F_w(s) \frac{y^s}{s} ds \leq \frac{q^w}{n} \int_{c}^{\infty} \left( \frac{y}{q^w} \right)^t dt \ll O_{z,w,c}(n^{-1}) \rightarrow 0 \]
as \( n \rightarrow \infty \), and, similarly,
\[ \int_{c-ni}^{c-ni} F_w(s) \frac{y^s}{s} ds \rightarrow 0 \]
as \( n \rightarrow \infty \). Finally, we note that
\[ \int_{L(n,m)} F_w(s) \frac{y^s}{s} ds = 0 \]
for all positive \( n, m \), by the residue theorem. Hence, we can see that
\[ \int_{c-i\infty}^{c+i\infty} F_w(s) \frac{y^s}{s} ds = 0. \]
as required.

We now give a Corollary to Lemma 7.9.

**Corollary 7.11.** Let \( R, M \in \mathcal{M} \) with \( \deg M \leq \deg R \), \( k \) be a non-negative integer, and \( z \) be an integer-valued function of \( R \) such that \( z \sim \deg R \) as \( \deg R \rightarrow \infty \). We have that
\[
\sum_{N \in \mathcal{M}} \frac{2^{\omega(N)-\omega((N,M))}}{|N|} (\deg N)^k \sum_{i=0}^{k} \binom{k}{i} (-1)^i (z-\deg N)^i z^{k-i},
\]
and let us define
\[ a(R) := (1 - q^{-1}) \prod_{P|M} \left( 1 - \frac{1}{|P|-1} \right) \prod_{P|\mathcal{M}} \left( \frac{1}{1-|P|-1} \right) \]
and let us define
\[ a(R) := (1 - q^{-1}) \prod_{P|M} \left( 1 - \frac{1}{|P|-1} \right) \prod_{P|\mathcal{M}} \left( \frac{1}{1-|P|-1} \right). \]

Then, by Lemma 7.9, we have
\[
\sum_{N \in \mathcal{M}} \frac{2^{\omega(N)-\omega((N,M))}}{|N|} (\deg N)^k = a(R) z^{k+2} \sum_{i=0}^{k} \binom{k}{i} \frac{1}{(i+2)(i+1)} (-1)^i + O_k(a(R) z^{k+1} \log \deg R)
\]
\[ = \frac{a(R) z^{k+2}}{(k+2)(k+1)} \sum_{i=2}^{k+2} \binom{k+2}{i} (-1)^i + O_k(a(R) z^{k+1} \log \deg R) \]
\[ = \frac{a(R) z^{k+2}}{k+2} + O_k(a(R) z^{k+1} \log \deg R). \]
Lemma 7.12. Suppose \( \nu \) is a multiplicative function on \( A \) and that there exists a non-negative integer \( r \) such that \( \nu(P^k) = O(k^r) \) for all primes \( P \) (the implied constant is independent of \( P \)). Furthermore, suppose there is an \( \eta > 0 \) such that \( \nu(A) \ll \eta |A|^\theta \) as \( A \to \infty \).

Let \( R \in \mathcal{M} \) be a variable, \( a, b > 0 \) be constants, and \( X = X(R), y = y(R) \) be non-negative, increasing, integer-valued functions such that \( X \leq a \log_q \log \deg R \) and \( y \geq b \log_q \deg R \) for large enough \( \deg R \).

Let \( c \) and \( \epsilon \) be such that \( c > \epsilon > \max \{0, 1 - \frac{1}{a}\} \) and \( c > \eta \), and let \( \delta > 0 \) be small. Finally, let \( S \in \mathcal{M}; \) \( S \) may depend on \( R \). We then have that

\[
\sum_{\substack{A \in \mathcal{S}_M(X) \\
\deg A \leq y \\
(A,S) = 1}} \frac{\nu(A)}{|A|^c} = \prod_{\deg P \leq X \atop (P,S) = 1} \left( 1 + \frac{\nu(P)}{|P|^c} + \frac{\nu(P^2)}{|P|^{2c}} + \cdots \right) + O_{q,a,b,c,\epsilon,\delta}(\deg R)^{-b(c-\epsilon)(1-\delta)}
\]

as \( \deg R \to \infty \).

Proof. Let \( d \geq 2 \). By similar means as in Lemma 7.10, we have that

\[
\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} ds = \sum_{\substack{A \in \mathcal{S}_M(X) \\
\deg A \leq y \\
(A,S) = 1}} \frac{\nu(A)}{|A|^c}.
\]

Now, let \( n \) be a positive integer and let us define the following contours in \( \mathbb{C} \).

\[
\begin{align*}
l_1(n) &:= \left[ d - \frac{2n\pi i}{\log q}, d + \frac{2n\pi i}{\log q} \right]; \\
l_2(n) &:= \left[ d + \frac{2n\pi i}{\log q}, -c + \epsilon + \frac{2n\pi i}{\log q} \right]; \\
l_3(n) &:= \left[ -c + \epsilon + \frac{2n\pi i}{\log q}, -c + \epsilon - \frac{2n\pi i}{\log q} \right]; \\
l_4(n) &:= \left[ -c + \epsilon - \frac{2n\pi i}{\log q}, d - \frac{2n\pi i}{\log q} \right]; \\
L(n) &:= l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n).
\end{align*}
\]

We can see that

\[
\frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} ds = \\
\frac{1}{2\pi i} \lim_{n \to \infty} \left( \int_{L(n)} \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} ds - \int_{l_2(n)} \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} ds \right. \\
- \int_{l_3(n)} \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} ds - \int_{l_4(n)} \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} ds \right).
\]

For the integral over \( L(n) \) there is a simple pole at \( s = 0 \). So, we have

\[
\frac{1}{2\pi i} \int_{L(n)} \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \frac{q^{(y+\frac{1}{2})s}}{s} ds = \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^c} = \prod_{\deg P \leq X \atop (P,S) = 1} \left( 1 + \frac{\nu(P)}{|P|^c} + \frac{\nu(P^2)}{|P|^{2c}} + \cdots \right).
\]

We can see that for all \( s \in l_2(n) \) and all \( s \in l_4(n) \) we have that \( \sum_{\substack{A \in \mathcal{S}_M(X) \\
(A,S) = 1}} \frac{\nu(A)}{|A|^{s+c}} \) and \( q^{(y+\frac{1}{2})s} \) are uniformly bounded, independently of \( n \). Hence, we can see that the integrals over \( l_2(n), l_4(n) \) tend
to 0 as $n \to \infty$.

Now consider the integral over $I_3(n)$. Suppose $\epsilon < 1$. Then, for all positive integers $n$ and all $s \in I_3(n)$ we have that

$$
\sum_{\substack{A \in S_M(X) \\ (A,S) = 1}} \nu(A) \frac{\nu(A)}{|A|^{s+c}} \ll \prod_{\deg P \leq X} \left( 1 + \frac{|\nu(P)|}{|P|^\epsilon} + \frac{|\nu(P^2)|}{|P|^{2\epsilon}} + \ldots \right) \leq \prod_{\deg P \leq X} \left( 1 + O_{r,\epsilon}\left( \frac{1}{|P|^\epsilon} \right) \right) \leq \exp \left( O_{r,\epsilon} \left( \sum_{\deg P \leq X} \frac{1}{|P|^\epsilon} \right) \right) \leq \exp \left( O_{r,\epsilon} \left( \sum_{\deg P \leq X} \frac{1}{|P|^\epsilon} \right) \right) \leq (\deg R)^{b(c-\epsilon)}
$$

as $\deg R \to \infty$. Now suppose $\epsilon \geq 1$, then we can show that

$$
\sum_{\substack{A \in S_M(X) \\ (A,S) = 1}} \nu(A) \frac{\nu(A)}{|A|^{s+c}} \ll \exp \left( O_{r,\epsilon} \left( a \log_q \log \deg R \right) \right) \ll (\deg R)^{b(c-\epsilon)}
$$

as $\deg R \to \infty$. We also have that

$$
q^{(y+\frac{1}{2})s} \ll (\deg R)^{-b(c-\epsilon)},
$$

from which we deduce that

$$
\frac{1}{2\pi i} \int_{I_3(n)} \sum_{\substack{A \in S_M(X) \\ (A,S) = 1}} \nu(A) \frac{q^{(y+\frac{1}{2})s}}{|A|^{s+c}} \frac{1}{s} ds \ll (\deg R)^{-b(c-\epsilon)(1-\delta)}
$$

as $\deg R \to \infty$. □

We now prove a result that is required to bound the lower order terms in the proof of Theorem 1.8, but first we require two results from [1]:

**Theorem 7.13.** Suppose $\alpha, \beta$ are fixed and satisfy $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Let $X \in M$ and $y$ be a positive integer satisfying $\beta \deg X < y \leq \deg X$. Also, let $A \in A$ and $G \in M$ satisfy $(A,G) = 1$ and $\deg G < (1-\alpha)y$. Then, we have that

$$
\sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A(\mod G)}} d(N) \ll_{\alpha, \beta} \frac{q^y \deg X}{\phi(G)}.
$$

**Proof.** See Theorem 6.1 in [1]. □

**Lemma 7.14.** Let $F, K \in M$, $x \geq 0$, and $a \in \mathbb{F}_q^*$. Suppose also that $\frac{1}{x} < \deg KF \leq \frac{2}{x}$. Then,

$$
\sum_{\substack{N \in M \\ \deg N = x \deg KF \\ (N,F) = 1}} d(N) \frac{d(KF+aN)}{|KF|} \ll q^x x^2 \frac{1}{|KF|} \sum_{\substack{H|K \\ \deg H \leq \frac{1}{2} \deg KF}} \frac{d(H)}{|H|}.
$$

**Proof.** See Lemma 7.7 in [1]. □

**Lemma 7.15.** Let $F \in M$, $K \in M \setminus \{0\}$, and $x \geq 0$ satisfy $\deg KF < x$. Then,

$$
\sum_{\substack{N \in M \\ \deg N = x \\ (N,F) = 1}} d(N) \frac{d(KF+N)}{|KF|} \ll q^x x^2 \sum_{\substack{H|K \\ \deg H \leq \frac{1}{2}}} \frac{d(H)}{|H|}.
$$

**Proof.** See Lemma 7.8 in [1]. □
Lemma 7.16. Let $F \in \mathcal{M}$, $A_3, B_3 \in \mathcal{S}_M(X)$ with $(A_3B_3, F) = 1$, and $z_1, z_2$ be non-negative integers. Also, we define

$$
\widetilde{\deg}(A) := \begin{cases} 
1 & \text{if } \deg A = 0 \\
\deg A & \text{if } \deg A \geq 1.
\end{cases}
$$

Then, for all $\epsilon > 0$ we have the following:

$$
\sum_{A_1, A_2, B_1, B_2 \in \mathcal{M}} 1 \ll \epsilon \left(p^{z_1 + z_2} \frac{|A_3B_3|}{|F|} \right)^{1+\epsilon} \sum_{A_1, A_2, B_1, B_2 \in \mathcal{M}} 1 \ll \epsilon \left(p^{z_1 + z_2} |A_3B_3|(z_1 + z_2 + \deg A_3B_3)^3 \right) \frac{1}{\varphi(F)}
$$

if $z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10} \deg F$; and

$$
\sum_{A_1, A_2, B_1, B_2 \in \mathcal{M}} 1 \ll \epsilon \left(p^{z_1 + z_2} |A_3B_3|(z_1 + z_2 + \deg A_3B_3)^3 \right) \frac{1}{\varphi(F)}
$$

if $z_1 + z_2 + \deg A_3B_3 > \frac{19}{10} \deg F$.

Proof. We can split the sum into the cases $\deg A_1A_2A_3 > \deg B_1B_2B_3$, $\deg A_1A_2A_3 < \deg B_1B_2B_3$, and $\deg A_1A_2A_3 = \deg B_1B_2B_3$ with $A_1A_2A_3 \neq B_1B_2B_3$.

When $\deg A_1A_2A_3 > \deg B_1B_2B_3$, we have that $A_1A_2A_3 = KF + B_1B_2B_3$ where $K \in \mathcal{M}$ and $\deg KF > \deg B_1B_2B_3$. Furthermore,

$$
2 \deg KF = 2 \deg A_1A_2A_3 > \deg A_1A_2A_3 + \deg B_1B_2B_3
$$

$$
= \deg A_1B_1 + \deg A_2B_2 + \deg A_3B_3 = z_1 + z_2 + \deg A_3B_3,
$$

from which we deduce that

$$
a_0 := \frac{z_1 + z_2 + \deg A_3B_3}{2} < \deg KF \leq z_1 + z_2 + \deg A_3 =: a_1.
$$

Also,

$$
\deg KF + \deg B_1B_2 = \deg A_1A_2A_3 + \deg B_1B_2 = z_1 + z_2 + \deg A_3,
$$

from which we deduce that

$$
\deg B_1B_2 = z_1 + z_2 + \deg A_3 - \deg KF.
$$

Similarly, if $\deg A_1A_2A_3 < \deg B_1B_2B_3$, we can show that

$$
b_0 := \frac{z_1 + z_2 + \deg A_3B_3}{2} < \deg KF \leq z_1 + z_2 + \deg B_3 =: b_1
$$

and

$$
\deg A_1A_2 = z_1 + z_2 + \deg B_3 - \deg KF.
$$

When $\deg A_1A_2A_3 = \deg B_1B_2B_3$, we must have that

$$
\deg A_1A_2 = \frac{z_1 + z_2 + \deg B_3 - \deg A_3}{2},
$$

$$
\deg B_1B_2 = \frac{z_1 + z_2 + \deg A_3 - \deg B_3}{2}.
$$
Also, we can write \(A_1A_2A_3 = KF + B_1B_2B_3\), where \(\deg KF < \deg B_1B_2B_3 = \frac{z_1 + z_2 + \deg A_3B_3}{2}\) and \(K \neq 0\) need not be monic.

So, writing \(N = B_1B_2\) when \(\deg A_1A_2A_3 \geq \deg B_1B_2B_3\), and \(N = A_1A_2\) when \(\deg A_1A_2A_3 < \deg B_1B_2B_3\), we have that

\[
\sum_{A_1,A_2,B_1,B_2 \in M}^{\deg A_1B_1 = z_1} \sum_{\deg A_2B_2 = z_2}^{(A_1A_2B_1B_2,F) = 1} \sum_{A_1A_2A_3 = B_1B_2B_3}^{(mod \ F)} \sum_{A_1A_2A_3 \neq B_1B_2B_3}^{1} d(N)d\left((KF + NB_3)A_3^{-1}\right) 
\]

(49)

\[
\leq \sum_{K \in M}^{\deg KF \leq a_0} \sum_{0 < \deg N = z_1 + z_2 + \deg A_3 - \deg KF}^{N \in M} d(N)d\left((KF + NB_3)A_3^{-1}\right) 
\]

\[
+ \sum_{K \in M}^{\deg KF < a_0} \sum_{b_0 < \deg N = z_1 + z_2 + \deg B_3 - \deg KF}^{N \in M} d(N)d\left((KF + NA_3)B_3^{-1}\right) 
\]

\[
+ \sum_{K \in A \setminus \{0\}}^{\deg KF < a_0} \sum_{\deg N = z_1 + z_2 + \deg A_3 - \deg B_3}^{N \in M} d(N)d\left((KF + NB_3)A_3^{-1}\right). 
\]

We must remark that if \(A_3 \mid (KF + NB_3)\) then we define \((KF + NB_3)A_3^{-1}\) by \((KF + NB_3)A_3^{-1}\). If \(A_3 \nmid (KF + NB_3)\), then we ignore the term with \((KF + NB_3)A_3^{-1}\) in the sum; that is, we take the definition \(d\left((KF + NB_3)A_3^{-1}\right) := 0\). We do the same for \((KF + NA_3)B_3^{-1}\).

**Step 1:** Let us consider the case when \(z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10} \deg F\). By using well known bounds on the divisor function, we have that

\[
\sum_{K \in M}^{\deg KF \leq a_1} \sum_{\deg N = z_1 + z_2 + \deg A_3 - \deg KF}^{N \in M} d(N)d\left((KF + NB_3)A_3^{-1}\right) 
\]

\[
\ll \epsilon \left(q^{z_1}q^{z_2}\right)^\frac{1}{2} \sum_{K \in M}^{\deg KF \leq a_1} \sum_{\deg N = z_1 + z_2 + \deg A_3 - \deg KF}^{N \in M} \frac{1}{|K|} 
\]

\[
\leq \left(q^{z_1}q^{z_2}\right)^{1 + \frac{\epsilon}{2}} |A_3| \sum_{K \in M}^{\deg KF \leq a_1} \frac{1}{|K|} 
\]

\[
\ll \epsilon \left(q^{z_1}q^{z_2}\right)^{1 + \frac{\epsilon}{2}} |A_3| \frac{z_1 + z_2 + \deg A_3}{|F|} \ll \epsilon \left(q^{z_1}q^{z_2}\right)^{1 + \epsilon} |A_3| \frac{\deg A_3}{|F|}. 
\]

Similarly,

\[
\sum_{K \in M}^{\deg KF \leq b_1} \sum_{\deg N = z_1 + z_2 + \deg B_3 - \deg KF}^{N \in M} d(N)d\left((KF + NA_3)B_3^{-1}\right) \ll \epsilon \left(q^{z_1}q^{z_2}\right)^{1 + \epsilon} |B_3| \frac{\deg B_3}{|F|}. 
\]

As for the sum

\[
\sum_{K \in A \setminus \{0\}}^{\deg KF < a_0} \sum_{\deg N = z_1 + z_2 + \deg A_3 - \deg B_3}^{N \in M} d(N)d\left((KF + NB_3)A_3^{-1}\right), 
\]
we note that it does not apply to this case where \(z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10} \deg F\) because this would imply \(\deg KF \geq \deg F \geq \frac{20}{19}a_0\), which does not overlap with range \(\deg KF < a_0\) in the sum.

Hence,

\[
\sum_{A_1, A_2, B_1, B_2 \in M} 1 \ll_{\epsilon} \left(q^{z_1} q^{z_2}\right)^{1+\epsilon} |A_3B_3|^{\deg(A_3B_3)} \frac{1}{|F|}
\]

for \(z_1 + z_2 + \deg A_3B_3 \leq \frac{19}{10} \deg F\).

**Step 2:** We now consider the case when \(z_1 + z_2 + \deg A_3B_3 > \frac{19}{10} \deg F\).

**Step 2.1:** We consider the subcase where \(a_0 < \deg KF \leq \frac{3}{2}a_0\). This allows us to apply Lemma 7.14 for the second relation below.

\[
\sum_{K \in M} \sum_{N \in M} d(N)d\left((KF + NB_3)A_3^{-1}\right)
\]

\[
\ll q^{z_1} q^{z_2} |A_3B_3| \left(z_1 + z_2 + \deg A_3B_3\right)^2 \frac{1}{|F|} \sum_{K \in M} \sum_{N \in M} \sum_{H|K} \frac{1}{|K|} \frac{d(H)}{|H|}
\]

\[
\ll q^{z_1} q^{z_2} |A_3B_3| \left(z_1 + z_2 + \deg A_3B_3\right)^2 \frac{1}{|F|} \sum_{H \in M} \frac{d(H)}{|H|} \sum_{K \in M} \frac{1}{|K|}
\]

\[
\ll q^{z_1} q^{z_2} |A_3B_3| \left(z_1 + z_2 + \deg A_3B_3\right)^3 \frac{1}{|F|} \sum_{H \in M} \frac{d(H)}{|H|^2}
\]

Similarly,

\[
\sum_{b_0 < \deg KF \leq \frac{3}{2}b_0} \sum_{N \in M} d(N)d\left((KF + NA_3)B_3^{-1}\right)
\]

\[
\ll q^{z_1} q^{z_2} |A_3B_3| \left(z_1 + z_2 + \deg A_3B_3\right)^3 \frac{1}{|F|}
\]

**Step 2.2:** Now we consider the subcase where \(\frac{3}{2}a_0 < \deg KF \leq a_1\). We have that

\[
\sum_{K \in M} \sum_{N \in M} d(N)d\left((KF + NB_3)A_3^{-1}\right)
\]
We can now apply Theorem 7.13. One may wish to note that $\alpha$ the leading coefficient
\begin{align*}
\frac{1}{2}a_0 < \deg KF \leq \alpha_1 \quad & \frac{1}{2}a_0 < \deg KF \leq \deg N = 2a_0 - \deg KF \\
(N,F) = 1 \\
\leq \sum_{N \in M, \deg N < \frac{N}{2}} \sum_{K \in M, \deg KF = 2a_0 - \deg N} d(N)d(KF + N) \\
\leq \sum_{N \in M, \deg N < \frac{N}{2}} \sum_{M \in M, \deg KF < \deg M} d(M)
\end{align*}
where we define $X_{(N)} := T^{2a_0 - \deg N}$ (The monic polynomial of degree $2a_0 - \deg N$ with all non-leading coefficients equal to $0$).

We can now apply Theorem 7.13. One may wish to note that $y := 2a_0 - \deg N \geq \frac{3}{4}(z_1 + z_2 + \deg A_3 B_3) \geq \frac{3}{4} \frac{19}{10} \deg F$
and so
\[
\deg F \leq \frac{40}{57} y = (1 - \alpha)y
\]
where $0 < \alpha < \frac{1}{2}$, as required. Hence, we have that
\[
\sum_{K \in M, \deg KF < a_1} \sum_{N \in M, \deg N = z_1 + z_2 + \deg A_3 - \deg KF} d(N)d((KF + N)A_3^{-1})
\]
\[
\leq q^{z_1} q^{z_2} |A_3 B_3|(z_1 + z_2 + \deg A_3 B_3) \frac{1}{\phi(F)} \sum_{N \in M, \deg N < \frac{N}{2}} \frac{d(N)}{|N|}
\]
\[
\leq q^{z_1} q^{z_2} |A_3 B_3|(z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{\phi(F)}.
\]
Similarly, if $\frac{3}{2}b_0 < \deg KF \leq b_1$
\[
\sum_{K \in M, \deg KF < b_1} \sum_{N \in M, \deg N = z_1 + z_2 + \deg B_3 - \deg KF} d(N)d((KF + N)B_3^{-1})
\]
\[
\leq q^{z_1} q^{z_2} |A_3 B_3|(z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{\phi(F)}.
\]

**Step 2.3:** We now look at the sum
\[
\sum_{K \in A \setminus \{0\}, \deg KF < a_0} \sum_{N \in M, \deg N = z_1 + z_2 + \deg A_3 - \deg B_3} d(N)d((KF + NB_3)A_3^{-1})
\]
By Lemma 7.15 we have that
\[
\sum_{K \in A \setminus \{0\}, \deg KF < 2a_0} \sum_{N \in M, \deg N = z_1 + z_2 + \deg A_3 - \deg B_3} d(N)d((KF + NB_3)A_3^{-1})
\]
\[
\leq \sum_{K \in A \setminus \{0\}, \deg KF < a_0} \sum_{N \in M, \deg N = a_0} d(N)d(KF + N)
\]
By Lemma 2.5, we have

\[ X \]

Proof of Theorem 1.8.

We can now prove Theorem 1.8. By the Cauchy-Schwarz inequality, (4), and Lemma 4.2, it suffices to prove

\[ \leq q^{z_1 + z_2 - 1} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^2 \frac{1}{|F|} \sum_{K \in A \setminus \{0\}} \sum_{H | K} \frac{d(H)}{|H|} \]

where the second-to-last relation uses the fact that \( a_0 \) is an integer (since \( \deg A_1 A_2 A_3 = \deg B_1 B_2 B_3 \)) and so \( \deg K F < a_0 \) implies \( \deg K F \leq a_0 - 1 \), and the last relation uses a similar calculation as that in Step 2.1.

**Step 2.4:** We apply steps 2.1, 2.2, and 2.3 to (49) and we see that

\[ 1 \ll q^{z_1 + z_2} |A_3 B_3| (z_1 + z_2 + \deg A_3 B_3)^3 \frac{1}{\phi(F)} \]

for \( z_1 + z_2 + \deg A_3 B_3 > 19 \) \( \deg F \). \( \square \)

**8. The Fourth Hadamard Moment**

We can now prove Theorem 1.8.

**Proof of Theorem 1.8.** In this proof, we assume all asymptotic relations are as \( X, \deg R \rightarrow \infty \) with \( X \leq \log_q \deg R \). Using Lemmas 7.1 and 7.2, we have

\[ \frac{1}{\phi^*(R)} \sum_{\chi \mod R} \left| L \left( \frac{1}{2}, \chi \right) P_X \left( \frac{1}{2}, \chi \right)^{-1} \right|^4 \sim \frac{1}{\phi^*(R)} \sum_{\chi \mod R} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 \left| P_X^* \left( \frac{1}{2}, \chi \right) \right|^2 \]

\[ = \frac{1}{\phi^*(R)} \sum_{\chi \mod R} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 \left| P_X^* \left( \frac{1}{2}, \chi \right) \right|^2 + O \left( (\deg R)^{-\frac{3}{2}} \right). \]

By the Cauchy-Schwarz inequality, (4), and Lemma 4.2, it suffices to prove

\[ \frac{1}{\phi^*(R)} \sum_{\chi \mod R} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 \left| P_X^* \left( \frac{1}{2}, \chi \right) \right|^2 \sim \frac{1}{12} (\deg R)^4 \prod_{P | R} \left( 1 \left[ \frac{1}{1 + |P|^{-1}} \right] \prod_{P | R} \left( 1 - |P|^{-1} \right)^4. \right. \]

By Lemma 2.5, we have

\[ \frac{1}{\phi^*(R)} \sum_{\chi \mod R} \left| L \left( \frac{1}{2}, \chi \right) \right|^4 \left| P_X^* \left( \frac{1}{2}, \chi \right) \right|^2 = \frac{1}{\phi^*(R)} \sum_{\chi \mod R} \left( 2a(\chi) + 2b(\chi) + c(\chi) \right)^2 \left| P_X^* \left( \frac{1}{2}, \chi \right) \right|^2, \]

where \( c(\chi) \) is as in Lemma 2.5 and

\[ z_R := \deg R - \log_q 2^{\omega(R)}; \]

\[ a(\chi) := \sum_{A, B \in M, \deg AB \leq z_R} \frac{\chi(A) \overline{\chi}(B)}{|AB|^{1/2}}; \]

\[ b(\chi) := \sum_{A, B \in M, z_R < \deg AB < \deg R} \frac{\chi(A) \overline{\chi}(B)}{|AB|^{1/2}}. \]
Note that, by symmetry in $A, B$, the terms $a(\chi), b(\chi), \text{and } c(\chi)$ are equal to their conjugates and, therefore, they are real. Hence, by the Cauchy-Schwarz inequality, it suffices to obtain the asymptotic main term of

\[
\frac{4}{\phi^*(R)} \sum_{\chi \mod R}^* a(\chi)^2 \left| \overline{P_X^*} \left( \frac{1}{2}, \chi \right) \right|^2
\]

and show that

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* b(\chi)^2 \left| \overline{P_X^*} \left( \frac{1}{2}, \chi \right) \right|^2
\]

and

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* c(\chi)^2 \left| \overline{P_X^*} \left( \frac{1}{2}, \chi \right) \right|^2
\]

are of lower order. The reason we express the sum in terms of $a(\chi)$ and $b(\chi)$ is because the fact that $a(\chi)$ is truncated allows us to bound the lower order terms that it contributes. We cannot do this with $b(\chi)$ but, because $b(\chi)$ is a relatively short sum, we can apply others methods to bound it.

**Step 1; the asymptotic main term of** \[
\frac{4}{\phi^*(R)} \sum_{\chi \mod R}^* a(\chi)^2 \left| \overline{P_X^*} \left( \frac{1}{2}, \chi \right) \right|^2
\]

**By Lemma 2.2 and Corollary 2.3, we have that**

\[
= \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* \sum_{A_1, A_2, B_1, B_2 \in M_{\chi}} \beta(A_3) \beta(B_3) \chi(A_1 A_2 A_3) \overline{\chi(B_1 B_2 B_3)} \frac{1}{|A_1 A_2 A_3 B_1 B_2 B_3|^2}
\]

\[
= \sum_{A_1, A_2, B_1, B_2 \in M_{\chi}} \frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^2}
\]

**Step 1.1:** We consider the first term on the far right side of (51): the diagonal terms. By Lemma 7.3 we have

\[
= \sum_{G_1, G_2, V_1, V_2, V_3 \in M_{\chi}} \frac{\beta(G_3 V_3, 1) \beta(G_3 V_1, 3 V_2, 3)}{|G_1 G_2 G_3 V_1, 3 V_2, 3 V_1, 3 V_2, 3|^2}
\]

(51)
By Lemma 7.4 we have

\[
\begin{align*}
&= \sum_{G_1, G_2 \in M} \frac{1}{|G_1 G_2|} \sum_{r \leq \log q \deg G_1} \sum_{s \leq \log q \deg G_2} \\
&\quad \times \frac{1}{|V_1 V_2|} \sum_{V_1, V_2 \in M} \frac{1}{|V_1 V_2|} \\
&\quad \times \left( \sum_{V \in M} \frac{1}{|V|} \sum_{r \leq \log q \deg G_1} \sum_{s \leq \log q \deg G_2} \\
&\quad \times 2^{-\omega(V) - \omega \left( \left( V_1 V_2, V_3, V_4, V_5, V_6 \right) \right)} \right)
\end{align*}
\]

So, we have

\[
\begin{align*}
&= \sum_{A_1, A_2, B_1, B_2 \in M} \frac{\beta(A_1) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^7} \\
&\quad \times \sum_{G_1, G_2 \in M} \frac{1}{|G_1 G_2|} \sum_{r \leq \log q \deg G_1} \sum_{s \leq \log q \deg G_2} \\
&\quad \times \left( \sum_{V \in M} \frac{1}{|V|} \sum_{r \leq \log q \deg G_1} \sum_{s \leq \log q \deg G_2} \\
&\quad \times 2^{-\omega(V) - \omega \left( \left( V_1 V_2, V_3, V_4, V_5, V_6 \right) \right)} \right)
\end{align*}
\]
Now, by Corollary 2.7, if
\[
\frac{z_R - \deg V V_{1,3} V_{3,1}}{2} \geq \log_q 3^\omega(R)
\]
- that is,
\[
\deg V \leq \deg R - \log_q 18^\omega(R) - \deg V_{1,3} V_{3,1}
\]
- then
\[
\sum_{G_1 \in \mathcal{M}} \frac{1}{|G_1|} = \frac{\phi(R)}{2|R|} (z_R - \deg V V_{1,3} V_{3,1}) + O \left( \frac{\phi(R)}{|R|} \log \omega(R) \right)
\]
\[
\frac{\phi(R)}{2|R|} \left( \deg R - \deg V + O \left( \log \deg R + \omega(R) \right) \right).
\]
If
\[
\deg V > \deg R - \log_q 18^\omega(R) - \deg V_{1,3} V_{3,1},
\]
then
\[
\sum_{G_1 \in \mathcal{M}} \frac{1}{|G_1|} \leq \sum_{G_1 \in \mathcal{M}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|} \omega(R).
\]
Similar results hold for the sum over $G_2$.

So, let us define
\[
m_0 := \min \{ \deg R - \log_q 18^\omega(R) - \deg V_{1,3} V_{3,1}, \deg R - \log_q 18^\omega(R) - \deg V_{2,3} V_{3,2} \},
\]
\[
m_1 := \max \{ \deg R - \log_q 18^\omega(R) - \deg V_{1,3} V_{3,1}, \deg R - \log_q 18^\omega(R) - \deg V_{2,3} V_{3,2} \}.
\]
Then, by (53) and (54), we have
\[
\sum_{V \in \mathcal{M}} \frac{2^{\omega(V) - \omega \left( (V V_{1,3} V_{3,1} V_{3,2}) \right)}}{|V|} \sum_{G_1, G_2 \in \mathcal{M}} \frac{1}{|G_1 G_2|} \phi(R)^2
\]
\[
= \phi(R)^2 \sum_{V \in \mathcal{M}} \frac{2^{\omega(V) - \omega \left( (V V_{1,3} V_{3,1} V_{3,2}) \right)}}{|V|} \left( \deg R - \deg V + O \left( \log \deg R + \omega(R) \right) \right)^2
\]
\[
+ l_1(R, V_{1,3}, V_{3,1}, V_{2,3}, V_{3,2}),
\]
where

\begin{equation}
(56)
\end{equation}

\[
l_1(R, V_{1,3}, V_{3,1}, V_{2,3}, V_{3,2}) \leq \frac{\phi(R)^2 \omega(R) \deg R}{2|R|^2} \sum_{\substack{V \in \mathcal{M} \\
 \deg V \leq \varepsilon_{R} - \deg V_{1,3} - \deg V_{3,1} - \deg V_{2,3} - \deg V_{3,2} \\
 \varepsilon_{R} \leq \deg R \\
 \varepsilon_{R} \leq \deg V \leq \varepsilon_{R} \deg R}} \frac{2^{\omega(V) - \omega(V_{1,3} V_{2,3} V_{3,1} V_{3,2})}}{|V|} \\
+ \frac{\phi(R)^2 \omega(R)^2}{|R|^2} \sum_{\substack{V \in \mathcal{M} \\
 \varepsilon_{R} \leq \deg V \leq \deg R \\
 \varepsilon_{R} \leq \deg V \leq \deg R \\
 \varepsilon_{R} \leq \deg V \leq \deg R}} \frac{2^{\omega(V) - \omega(V_{1,3} V_{2,3} V_{3,1} V_{3,2})}}{|V|}.
\end{equation}

We now apply Corollary 7.11 to both terms on the right side of (55). For the second term, which is (56), it is just two direct applications. For the first term, we must expand \( \deg R - \deg V + O(\log \deg R + \omega(R))^2 \) and use Corollary 7.11 on each of the resulting terms. We obtain

\begin{equation}
(57)
= \frac{1 - q^{-1}}{48} (\deg R)^4 \left( 1 + O\left( \frac{\omega(R) + \log \deg R}{\deg R} \right) \right) \prod_{P \mid R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P \mid |V_{1,3} V_{2,3} V_{3,1} V_{3,2}|} \left( \frac{1}{1 - |P|^{-1}} \right) \prod_{P \mid |V_{1,3} V_{2,3} V_{3,1} V_{3,2}|} \left( \frac{1}{1 - |P|^{-1}} \right) =: l_2(R, V_{1,3}, V_{2,3}, V_{3,1}, V_{3,2}).
\end{equation}

Before proceeding let us make the following definitions: For \( A \in \mathcal{A} \setminus \{0\} \) and \( P \in \mathcal{P} \) we define \( \epsilon_P(A) \) to be the largest non-negative integer such that \( P^{\epsilon_P(A)} \mid A \), and

\begin{equation}
(58)
\gamma(A) := \prod_{P \mid A} \left( 1 + \epsilon_P(A) \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right).
\end{equation}
Then, we can see that
\[
\sum_{V_{1,3}, V_{2,3} \in S_M(X), \deg V_{1,3} V_{2,3} = B_3'} \prod_{P|V_{1,3} V_{2,3}} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|\text{rad}(V_{1,3} V_{2,3})} \left( \frac{1}{1 - |P|^{-1}} \right)
= \prod_{P|B_3'} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \sum_{W_1 W_2 = B_3'} \prod_{P|W_2} \left( \frac{1}{1 - |P|^{-1}} \right) \prod_{P|W_1} \left( e_P(B_3') - 1 \right)
= \prod_{P|B_3'} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|B_3'} \left( \frac{2}{1 - |P|^{-1}} + (e_P(B_3') - 1) \right)
= \prod_{P|B_3'} \left( 1 + e_P(B_3') \right) \frac{1 - |P|^{-1}}{1 + |P|^{-1}} = \gamma(B_3').
\]

Similarly,
\[
\sum_{V_{3,1}, V_{3,2} \in S_M(X), \deg V_{3,1} V_{3,2} = A_3'} \prod_{P|V_{3,1} V_{3,2}} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \prod_{P|\text{rad}(V_{3,1} V_{3,2})} \left( \frac{1}{1 - |P|^{-1}} \right) = \gamma(A_3').
\]

We now substitute (57) to (52) and apply (59) and (60) to obtain
\[
\sum_{A_1, A_2, B_1, B_2 \in M, \deg A_1 B_1, \deg A_2 B_2 \leq R} \frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^{\frac{1}{2}}}
= \sum_{G_3, V_{3,1}, V_{3,2} \in S_M(X), \deg G_3 V_{3,1} V_{3,2} \leq \frac{1}{6} \log_q R} \frac{\beta(G_3 V_{3,1} V_{3,2}) \beta(G_3 V_{1,3} V_{2,3})}{|G_3 V_{1,3} V_{2,3} V_{3,1} V_{3,2}|} l_2(R, V_{1,3}, V_{2,3}, V_{3,1}, V_{3,2})
= \sum_{G_3, A_3', B_3' \in S_M(X), \deg G_3 A_3', \deg B_3' \leq \frac{1}{6} \log_q R} \frac{\beta(G_3 A_3') \beta(G_3 B_3')}{|G_3 A_3' B_3'|} \sum_{V_{5,1}, V_{3,2} \in S_M(X), \deg V_{5,1} V_{3,2} = A_3'} \sum_{V_{1,3} V_{2,3} \in S_M(X), \deg V_{1,3} V_{2,3} = B_3'} l_2(R, V_{1,3}, V_{2,3}, V_{3,1}, V_{3,2})
= 1 - q^{-1} \prod_{P|R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right)^3 \left( \prod_{P|B_3'} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \right)^4 \sum_{G_3, A_3', B_3' \in S_M(X), \deg G_3 A_3' \leq \frac{1}{6} \log_q R} \frac{\beta(G_3 A_3') \beta(G_3 B_3')}{|G_3 A_3' B_3'|} \gamma(A_3') \gamma(B_3')
+ l_3(R),
\]
where

\[(62)\]
\[
l_3(R) \leq \prod_{P|R} \left( \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) \left( \deg R \right)^3 (\omega(R) + \log \deg R) \sum_{\substack{G_3, A'_3, B'_3 \in S_M(X) \\
\deg G_3 A'_3 \leq \frac{1}{6} \log q \deg R \\
\deg G_3 B'_3 \leq \frac{1}{6} \log q \deg R \\
(G_3 A'_3 B'_3, R) = 1 \\
(A'_3, B'_3) = 1}} \frac{\beta(G_3 A'_3) \beta(G_3 B'_3)}{|G_3 A'_3 B'_3|} \gamma(A'_3) \gamma(B'_3).
\]

Consider the first term on the far right side of (61). We recall that $\beta(A) = 0$ if $A$ is divisible by $P^3$ for any prime $P$. Hence, defining $\Pi_{P, X} := \prod_{\deg P \leq X} P$, we may assume that $G_3 = IJ^2$ where $I, J \mid \Pi_{P, X}$, $(IJ, R) = 1$, and $(I, J) = 1$. By similar reasoning, we may assume that $A'_3 = KA''_3$ where $K \mid I$, $(A''_3, R I J) = 1$; and $B'_3 = LB''_3$ where $L \mid I$, $(L, K) = 1$ and $(B''_3, R I J A''_3) = 1$. Then, by the multiplicativity of $\beta$ and $\gamma$, we have

\[(63)\]
\[
\sum_{\substack{G_3, A'_3, B'_3 \in S_M(X) \\
\deg G_3 A'_3 \leq \frac{1}{6} \log q \deg R \\
\deg G_3 B'_3 \leq \frac{1}{6} \log q \deg R \\
(G_3 A'_3 B'_3, R) = 1 \\
(A'_3, B'_3) = 1}} \frac{\beta(G_3 A'_3) \beta(G_3 B'_3)}{|G_3 A'_3 B'_3|} \gamma(A'_3) \gamma(B'_3)
\]
\[
= \sum_{\substack{I \mid \Pi_{P, X} \\
\deg I \leq \frac{1}{6} \log q \deg R \\
(I, R) = 1}} \frac{\beta(I)^2}{|I|} \sum_{\substack{J \mid \Pi_{P, X} \\
\deg J \leq \frac{1}{6} \log q \deg R - \frac{\deg I}{(J, R I J) = 1}} \beta(J^2)^2 \frac{\beta(K^2) \gamma(K)}{|\beta(L^2) \gamma(L)|} \sum_{\substack{L \mid I \\
(L, K) = 1}} \beta(3''^n) \gamma(3''^n) 
\]
\[
\sum_{\substack{A''_3 \mid (\Pi_{P, X})^2 \\
\deg A''_3 \leq \frac{1}{6} \log q \deg R - \frac{\deg IJ^2 K}{(A''_3, R I J) = 1}} \beta(A''_3) \gamma(A''_3) 
\sum_{\substack{B''_3 \mid (\Pi_{P, X})^2 \\
\deg B''_3 \leq \frac{1}{6} \log q \deg R - \frac{\deg IJ^2 L}{(B''_3, R I J A''_3) = 1}} \beta(B''_3) \gamma(B''_3).}
\]

Consider the case where $\deg I > \frac{1}{64} \log q \deg R$ or $\deg J > \frac{1}{64} \log q \deg R$. Without loss of generality, suppose the former. Then, all the sums above, except that over $I$, can be bounded by $O\left((\log q \log \deg R)^c\right)$ for some constant $c > 0$, while the sum over $I$ can be bounded by $O\left((\deg R)^{-\frac{11}{4}}\right)$ (this is obtained in the same way we have done several times before, such as in (15)). So, with these restrictions, we have that the above is $O\left((\deg R)^{-\frac{11}{4}}\right)$.

Now consider the case where $\deg I \leq \frac{1}{64} \log q \deg R$ and $\deg J \leq \frac{1}{64} \log q \deg R$. Then,

\[
\frac{1}{8} \log q \deg R - \deg IJ^2 K \geq \frac{1}{16} \log q \deg R
\]

and

\[
\frac{1}{8} \log q \deg R - \deg IJ^2 L \geq \frac{1}{16} \log q \deg R.
\]
In particular, we can apply Lemma 7.12 to the last two summations of (63):

\[
\begin{align*}
\sum_{\text{deg } A'' \leq \frac{1}{3} \log_q \deg R - \deg IJ^2 K} \frac{\beta(A'')\gamma(A'')}{|A''|} &+ \sum_{\text{deg } B'' \leq \frac{1}{3} \log_q \deg R - \deg IJ^2 L} \frac{\beta(B'')\gamma(B'')}{|B''|} \\
= & \prod_{\text{deg } P \leq X} \left( 1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|} \right) \prod_{P \mid IJ} \left( 1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|} \right)^{-1} \\
& \cdot \prod_{\text{deg } P \leq X} \left( 1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|} \right) \left( 1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|} \right)^{-1} \\
& \cdot \prod_{P \mid IJ} \left( 1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|} \right) \left( 1 + \frac{\beta(P)\gamma(P)}{|P|} + \frac{\beta(P^2)\gamma(P^2)}{|P^2|} \right)^{-1} \\
& + O\left( (\deg R)^{-\frac{2}{3}} \right) \\
= & \prod_{\text{deg } P \leq X} \left( 1 + \frac{2\beta(P)\gamma(P)}{|P|} + \frac{2\beta(P^2)\gamma(P^2)}{|P^2|} \right) \\
& \cdot \prod_{P \mid IJ} \left( 1 + \frac{2\beta(P)\gamma(P)}{|P|} + \frac{2\beta(P^2)\gamma(P^2)}{|P^2|} \right)^{-1} \\
& + O\left( (\deg R)^{-\frac{2}{3}} \right).
\end{align*}
\]

Consider now the two middle summations on the right side of (63). We have

\[
\sum_{K \mid I} \frac{\beta(K^2)\gamma(K)}{\beta(K)|K|} \sum_{L \mid I} \frac{\beta(L^2)\gamma(L)}{\beta(L)|L|} \\
= \prod_{P \mid I} \left( 1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|} \right) \sum_{K \mid I} \frac{\beta(K^2)\gamma(K)}{\beta(K)|K|} \prod_{P \mid K} \left( 1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|} \right)^{-1} \\
= \prod_{P \mid I} \left( 1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|} \right) \prod_{P \mid I} \left( 1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|} \left( 1 + \frac{\beta(P^2)\gamma(P)}{\beta(P)|P|} \right)^{-1} \right) \\
= \prod_{P \mid I} \left( 1 + \frac{2\beta(P^2)\gamma(P)}{\beta(P)|P|} \right).
\]
Applying (64) and (65) to (63), we obtain

\[
\sum_{G_3,A_3',B_3' \in \mathcal{S}_M(X)} \frac{\beta(G_3 A_3') \beta(G_3 B_3')}{|G_3 A_3 B_3|} \gamma(A_3') \gamma(B_3')
\]

\[
= \prod_{\deg P \leq X \atop (P,R)=1} \left( 1 + \frac{2\beta(P) \gamma(P)}{|P|} + \frac{2\beta(P^2) \gamma(P^2)}{|P|^2} \right) \prod_{\deg I \leq \frac{1}{d} \log_q \deg R \atop (I,R)=1} \beta(I)^2 \prod_{\deg J \leq \frac{1}{d} \log_q \deg R \atop (J,R)=1} \beta(J)^2 \frac{1}{|J|^2}
\]

\[
\cdot \prod_{P|I} \left( 1 + \frac{2\beta(P^2) \gamma(P)}{(P)|P|} \right) \prod_{P|J} \left( 1 + \frac{2\beta(P) \gamma(P)}{|P|} + \frac{2\beta(P^2) \gamma(P^2)}{|P|^2} \right)^{-1}
\]

\[
+ O\left( (\deg R)^{-\frac{1}{d}} \right)
\]

\[
= \prod_{\deg P \leq X \atop (P,R)=1} \left( 1 + \frac{2\beta(P) \gamma(P)}{|P|} + \frac{2\beta(P^2) \gamma(P^2)}{|P|^2} + \frac{\beta(P^2)^2}{|P|^2} \right)
\]

\[
\cdot \prod_{I|P,R} \beta(I)^2 \prod_{P|I} \left( 1 + \frac{2\beta(P^2) \gamma(P)}{(P)|P|} \right) \left( 1 + \frac{2\beta(P) \gamma(P)}{|P|} + \frac{2\beta(P^2) \gamma(P^2)}{|P|^2} + \frac{\beta(P^2)^2}{|P|^2} \right)^{-1}
\]

\[
+ O\left( (\deg R)^{-\frac{1}{d}} \right)
\]

Now, recalling the definitions of \(\beta, \gamma\) (equations (28) and (58), respectively) we see that the product above is equal to

\[
\prod_{\deg P \leq X \atop P|I} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \prod_{\deg P > X \atop P|R} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \frac{1}{1+O(|P|^{-2})}
\]

\[
\sim \prod_{P|I} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right)^{-1} \prod_{P > X \atop P|R} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \prod_{\deg P \leq X \atop P|R} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right)
\]

\[
= \prod_{P|I} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right)^{-1} \prod_{P > X \atop P|R} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \prod_{\deg P \leq X \atop P|R} (1-|P|^{-1})^4 \prod_{\deg P \leq X \atop P|R} (1-|P|^{-2})^{-1}
\]

\[
\sim (1-q^{-1})^{-1} \prod_{P|I} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right)^{-1} \prod_{P > X \atop P|R} \left( \frac{(1-|P|^{-1})^3}{1+|P|^{-1}} \right) \left( \frac{1}{e^X} \right)^4,
\]

where we have used Lemma 4.2 for the last equality. Recall that the above is to be applied to the first term on the far right side of (61). We now consider \(l_3(R)\): the second term on the far right side of (61). By means similar to those described in the paragraph after (63), we can show that
there is some constant $c > 0$ such that
\[
\sum_{G_3, A'_3, B'_3 \in S_M(X)} \frac{|\beta(G_3A'_3)\beta(G_3B'_3)|}{|G_3A'_3 B'_3|} \gamma(A'_3)\gamma(B'_3) \ll X^c \ll \left(\log_q \log \deg R\right)^c.
\]

We apply this to (62) to obtain a bound for $l_3(R)$.

Hence, considering all of the above, (61) becomes
\[
\sum_{A_1, A_2, B_1, B_2 \in M} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^\frac{3}{7}} \sim \frac{1}{48} \frac{1}{e^\gamma X} \left(\frac{1-|P|^{-1}}{1+|P|^{-1}}\right)^4 \prod_{P \mid R} \left(1-\frac{1}{P}\right)^3
\]
(66)

**Step 1.2:** We consider the second term on the far right side of (51): the off-diagonal terms. We have
\[
\sum_{E \mid F = R} \mu(E)\phi(F) \sum_{A_1, A_2, B_1, B_2 \in M} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^\frac{3}{7}} \\
\leq \sum_{A_1, B_1 \in S_M(X), \deg A_1, \deg B_1 \leq \frac{1}{4} \log_q \deg R} \frac{|\beta(A_3)\beta(B_3)|}{|A_3B_3|^\frac{3}{7}} \sum_{E \mid F = R} \mu(E)\phi(F) q^{z_1+\frac{z_2}{2}} \sum_{z_1, z_2 = 0}^{z_R} 1.
\]

By Lemma 7.16 we have, for $\epsilon = \frac{1}{10}$,
\[
\sum_{z_1, z_2 = 0}^{z_R} q^{\frac{z_1+\frac{z_2}{2}}{2}} \sum_{A_1, A_2, A_3, B_1, B_2 \in M, \deg A_1 \geq z_1, \deg B_2 = z_2} 1
\]
\[
\ll \frac{|A_3B_3|^{1+\frac{1}{2}}}{|F|^{\frac{1}{10}}} + \frac{|A_3B_3|}{|F|^{\frac{1}{10}}} q^{z_R (\deg R)^3}.
\]

We also have
\[
\sum_{E \mid F = R} \mu(E)\phi(F) \left(\frac{|A_3B_3|^{1+\epsilon}}{|F|^{\frac{1}{10} - \epsilon}} + \frac{|A_3B_3|}{|F|^{\frac{1}{10} - \epsilon}} q^{z_R (\deg R)^3}\right)
\]
\[
= |A_3B_3|^{1+\epsilon} \sum_{E \mid F = R} |\mu(E)|\phi(F) |\phi(F)|^{\frac{1}{10} - \epsilon} + |A_3B_3| q^{z_R (\deg R)^3} \sum_{E \mid F = R} |\mu(E)|
\]
\[
\ll |A_3B_3|^{1+\epsilon} |R| + |A_3B_3R| (\deg R)^3,
\]
where the last relation uses

$$\sum_{EF=R} |\mu(E)| \frac{\phi(F)}{|F|^{\frac{1}{2} \epsilon}} \mathcal{P} \sum_{EF=R} |\mu(E)| \phi(F) = \phi(R) \sum_{EF=R} |\mu(E)| \prod_{P|E} \left( \frac{1}{|P|} \right) \prod_{P|E} \left( \frac{1}{|P| - 1} \right) \leq \phi(R) \sum_{EF=R} |\mu(E)| \prod_{P|E} \left( \frac{1}{|P| - 1} \right) = \phi(R) \frac{|R|}{\phi(R)} = |R|.$$  

Finally, using the fact that

$$\sum_{A_3, B_3 \in S_M(X)} |\beta(A_3)\beta(B_3)| |A_3B_3|^{\frac{1}{2} + \epsilon} \leq \left( \sum_{A \in M} |\beta(A)||A|^{\frac{1}{2} + \epsilon} \right)^2 \leq \left( \sum_{A \in M} d(A)|A|^{\frac{1}{2} + \epsilon} \right)^2 \leq \left( \sum_{A \in M} |A|^{\frac{1}{2} + \epsilon} \right)^4 \leq (\deg R)^{\frac{7}{8}},$$

we see that

$$\frac{1}{\phi^*(R)} \sum_{EF=R} |\mu(E)| \phi(F) \sum_{A_1, A_2, B_1, B_2 \in M} \sum_{A_3, B_3 \in S_M(X)} \beta(A_3)\beta(B_3) \frac{|A_1A_2A_3B_1B_2B_3|}{|A_1B_2A_3B_1B_2B_3|^{\frac{3}{2}}} \ll \frac{|R|}{\phi^*(R)} (\deg R)^{3 + \frac{7}{8}}.$$  

This is indeed of lower order than (66); Section 4 of [1] provides the necessary results to confirm this.

**Step 2:** the asymptotic main term of \( \frac{1}{\phi^*(R)} \sum_{x \mod R} b(x)^2 \left| \mathcal{P}^{*x} X \left( \frac{1}{2}, \chi \right) \right|^2 \): 

We have that

$$\sum_{x \mod R} b(x)^2 \left| \mathcal{P}^{*x} X \left( \frac{1}{2}, \chi \right) \right|^2 \leq \frac{1}{\phi^*(R)} \sum_{x \mod R} b(x)^2 \left| \mathcal{P}^{*x} X \left( \frac{1}{2}, \chi \right) \right|^2$$

\begin{align*}
\leq \frac{1}{\phi^*(R)} \sum_{x \mod R} \sum_{A_1, A_2, B_1, B_2 \in M} & |A_1B_2A_3B_1B_2B_3|^{\frac{3}{2}} \\
& \beta(A_3)\beta(B_3) \chi(A_1A_2A_3) \chi(B_1B_2B_3) \\
\leq \frac{\phi(R)}{\phi^*(R)} \sum_{A_1, A_2, B_1, B_2 \in M} \beta(A_3)\beta(B_3) |A_1A_2A_3B_1B_2B_3|^{\frac{3}{2}} + \frac{\phi(R)}{\phi^*(R)} \sum_{A_1, A_2, B_1, B_2 \in M} \beta(A_3)\beta(B_3) |A_1A_2A_3B_1B_2B_3|^{\frac{3}{2}}.
\end{align*}
Step 2.1: For the diagonal term, by similar means as in (52), we obtain

\[
\sum_{A_1,A_2,B_1,B_2\in M} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^2} \\
\sum_{A_3,B_3\in S_M(X)} z_R < \deg A_1 B_1, \deg A_2 B_2 < \deg R \frac{\beta(G_3 V_3 V_3,3 V_3,1 V_3,2)}{|G_3 V_3 V_3 V_3,1 V_3,2|} \sum_{V \in M} \left( \frac{\omega(V) - \omega(V, V_3 V_3,3 V_3,1 V_3,2)}{|V|} \right) \\
= \sum_{G_3, V_3,3 V_3,1 V_3,2 \in S_M(X)} \deg G_3 V_3,3 V_3,1 V_3,2 \leq \frac{1}{2} \log_q \deg R \frac{\beta(G_3 V_3,3 V_2 V_3,2,3)}{|G_3 V_3 V_3 V_3,1 V_3,2|} \sum_{V \in M} \left( \frac{\omega(V) - \omega(V, V_3 V_3,3 V_3,1 V_3,2)}{|V|} \right) \\
\max \left\{ \frac{z_R - \deg V V_3 V_3,1}{2} \right\} \begin{cases} \leq \deg G_1 < \frac{\deg R - \deg V V_3 V_3,1}{2} \quad (G_1, R) = 1 \\ \leq \deg G_2 < \frac{\deg R - \deg V V_3 V_3,2}{2} \quad (G_1, R) = 1 \end{cases} \\
= \sum_{G_1 G_2 \in M} \frac{1}{|G_1 G_2|}. 
\]

Now, if \( \frac{z_R - \deg V V_3,1}{2} \leq \log_q 3^{2 \omega(R)} \) then
\[
\frac{\deg R - \deg V V_3,1}{2} \leq \log_q 3^{2 \omega(R)} + \frac{1}{2} \log_q 3^{2 \omega(R)} < \log_q 6^{2 \omega(R)},
\]
and so, by Corollary 2.7, we have
\[
\sum_{G_1 \in M} \frac{1}{|G_1|} \leq \sum_{G_1 \in M, \deg G_1 < \log_q 6^{\omega(R)}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|^{\omega(R)}}.
\]

If \( \frac{z_R - \deg V V_3,1}{2} > \log_q 3^{2 \omega(R)} \) then
\[
\sum_{G_1 \in M} \frac{1}{|G_1|} \leq \sum_{G_1 \in M, \deg G_1 < \log_q 6^{\omega(R)}} \frac{1}{|G_1|} \ll \frac{\phi(R)}{|R|^{\omega(R)}},
\]
where we have used Corollary 2.7 twice for the last relation. Similar results hold for the sum over \( G_3 \).

Hence, proceeding similarly as we did for the diagonal terms of \( \frac{\phi(R)}{\phi^s(R)} \sum_{\chi \bmod R} a(\chi) \left| \hat{P}_X \left( \frac{1}{2}, \chi \right) \right|^2 \), we see that there is a constant \( c \) such that
\[
\frac{\phi(R)}{\phi^s(R)} \sum_{A_1,A_2,B_1,B_2\in M} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^2} \ll \frac{\phi(R)}{\phi^s(R)} \sum_{A_1,A_2,B_1,B_2\in M} \frac{\beta(A_3)\beta(B_3)}{|A_1A_2A_3B_1B_2B_3|^2}.
\]
The following is a mathematical text with some LaTeX formulas. It describes the Cauchy-Schwarz inequality and the asymptotic main term of a Dirichlet $L$-function for hybrid Euler-Hadamard product formulas in $\mathbb{F}_q[T]$. The text is divided into steps, with each step providing a different perspective or detail on the main theorem.

### Step 2.2

We now look at the second term on the far right side of (67): the off-diagonal terms. Using Lemma 7.16, we have

\[
\frac{\phi(R)}{\phi^*(R)} \sum_{A_1, A_2, B_1, B_2 \in M} \frac{\beta(A_3)\beta(B_3)}{|A_1 A_2 A_3 A_3 B_1 B_2 B_3|^\frac{1}{2}} \ll \frac{(\deg R)^3}{\phi^*(R)} \sum_{A_1, A_2, B_1, B_2 \in S_M(X)} \frac{|\beta(A_3)\beta(B_3)||A_3 B_3|^\frac{1}{2}}{|A_1 A_2 A_3 B_1 B_2 B_3|^\frac{1}{2}} \ll \frac{|R|(\deg R)^3}{\phi^*(R)}.
\]

### Step 3: the asymptotic main term of \( \frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* c(\chi)^2 \left| P^+ X \left( \frac{1}{2}, \chi \right) \right|^2 \)

We recall that \( c(\chi) \) differs, depending on whether \( \chi \) is even or odd. Furthermore, if \( \chi \) is even, then there are three terms to consider. However, by the Cauchy-Schwarz inequality, it suffices to bound the following for \( i = 0, 1, 2 \):

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* d_i(\chi)^2 \left| P^+ X \left( \frac{1}{2}, \chi \right) \right|^2,
\]

where

\[
d_i(\chi) := \sum_{A, B \in M \atop \deg AB = \deg R - i} \chi(A)\overline{\chi}(B) |AB|^\frac{i}{2}.
\]

We will bound

\[
\frac{1}{\phi^*(R)} \sum_{\chi \mod R}^* d_0(\chi)^2 \left| P^+ X \left( \frac{1}{2}, \chi \right) \right|^2.
\]

The other cases for \( d_i(\chi) \) and the odd case are similar.
Now, we have that

\[
\frac{1}{\phi^*(R)} \sum_{c \mod R}^{*} d_0(\chi)^2 \left| \widetilde{P}_{\chi}(\frac{1}{2}, \chi) \right|^2 \leq \frac{1}{\phi^*(R)} \sum_{c \mod R}^{*} d_0(\chi)^2 \left| \widetilde{P}_{\chi}(\frac{1}{2}, \chi) \right|^2
\]

\[
\leq \frac{1}{\phi^*(R)} \sum_{\chi \mod R} A_{1,2,3}, B_1, B_2 \in M_{A,3, B_3} \in \mathcal{S}_{M}(X) \quad \text{deg } A_1 B_1, \text{deg } A_2 B_2 = \text{deg } R
\]
\[
\text{deg } A_3 B_3 \leq \frac{1}{\chi} \log_q \text{deg } R
\]
\[
(\chi = 1 \quad A_1 A_2 A_3 = B_1 B_2 B_3)
\]
\[
\frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|} + \frac{\phi(R)}{\phi^*(R)} \sum_{\chi \mod R} A_{1,2,3}, B_1, B_2 \in M_{A,3, B_3} \in \mathcal{S}_{M}(X) \quad \text{deg } A_1 B_1, \text{deg } A_2 B_2 = \text{deg } R
\]
\[
\text{deg } A_3 B_3 \leq \frac{1}{\chi} \log_q \text{deg } R
\]
\[
\chi = 1 \quad A_1 A_2 A_3 = B_1 B_2 B_3
\]

For the first term on the far right side of (69), we have, similarly to Step 2.1,

\[
\sum_{G_1, G_2 \in M_{V_1,3}, V_1,3, V_1,3, V_1,2 \in \mathcal{S}_{M}(X)}^{*} \frac{\beta(G_3 V_{1,3} V_{1,2}) \beta(G_3 V_{1,3} V_{2,3})}{|G_3 V_{1,3} V_{2,3} V_{1,3} V_{2,3}|} \sum_{V \in M_{V_1,3, V_1,2} \leq \frac{1}{\chi} \log_q \text{deg } R \quad \text{deg } V \leq \text{deg } R}
\]
\[
\text{deg } V \leq \text{deg } R \quad \text{deg } V = 1
\]
\[
\chi = 1 \quad V(R(V_{1,3}, V_{1,2}) = 1)
\]
\[
\leq \sum_{G_1, G_2 \in M_{V_1,3}, V_1,3, V_1,2 \in \mathcal{S}_{M}(X)} \frac{1}{|G_1 G_2|}
\]
\[
\chi = 1 \quad G_1 G_2 = 1
\]
\[
\leq \sum_{G_1, G_2 \in M_{V_1,3}, V_1,3, V_1,2 \in \mathcal{S}_{M}(X)} \left| \beta(G_3 V_{1,3} V_{1,2}) \beta(G_3 V_{1,3} V_{2,3}) \right| \sum_{V \in M_{V_1,3, V_1,2} \leq \frac{1}{\chi} \log_q \text{deg } R \quad \text{deg } V \leq \text{deg } R}
\]
\[
\text{deg } V \leq \text{deg } R \quad \text{deg } V = 1
\]
\[
\chi = 1 \quad V(R(V_{1,3}, V_{1,2}) = 1)
\]
\[
\leq (\text{deg } R)^2 \prod_{P/R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right)^{\chi} (\log_q \text{deg } R)^c,
\]

for some positive constant c.
For the second term on the far right side of (69), we have, similarly to Step 2.2,

\[
\frac{\phi(R)}{\phi^*(R)} \sum_{A_1, A_2, B_1, B_2 \in M} \frac{\beta(A_3) \beta(B_3)}{|A_1 A_2 A_3 B_1 B_2 B_3|^2}
\]

\[
= \frac{\phi(R)}{|R| \phi^*(R)} \sum_{A_3, B_3 \in S_M(X)} \frac{\beta(A_3) \beta(B_3)}{|A_3 B_3|^2} \sum_{A_1, A_2, B_1, B_2 \in M} \frac{1}{|A_1 A_2 B_1 B_2|^2}
\]

\[
= \frac{|R| (\deg R)^3}{\phi^*(R)} \sum_{A_3, B_3 \in S_M(X)} \frac{\beta(A_3) \beta(B_3)}{|A_3 B_3|^2} \ll \frac{|R| (\deg R)^{3+\frac{3}{2}}}{\phi^*(R)}.
\]

The proof now follows from Steps 1, 2, and 3.

\[
\square
\]

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