High-jet Relations of the Heat Kernel Embedding Map and Applications

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Abstract

For any compact Riemannian manifold \((M,g)\) and its heat kernel embedding map \(ψ_t : M → l^2\) constructed in \([BBG]\), we study the higher derivatives of \(ψ_t\) with respect to an orthonormal basis at \(x\) on \(M\). As the heat flow time \(t → 0_+\), it turns out the limiting angles between these derivative vectors are universal constants independent on \(g\), \(x\) and the choice of orthonormal basis. Geometric applications to the mean curvature and the Riemannian curvature are given. Some algebraic structures of the \(∞\)-jet space of \(ψ_t\) are explored.

1 Introduction

Let \((M, g)\) be a \(n\)-dimensional compact Riemannian manifold with smooth metric \(g\). In \([BBG]\), Bérard, Besson and Gallot considered the heat kernel embedding

\[ Φ_t : x ∈ M → \left\{ e^{-\lambda_j t/2} \phi_j (x) \right\}_{j≥1} ∈ l^2 \]

where \(\lambda_j\) is the \(j\)-th eigenvalue of the Laplacian \(Δ_g\) of \((M, g)\), \(\{\phi_j\}_{j≥0}\) is the \(L^2\) orthonormal eigenbasis of \(Δ_g\), and \(l^2\) is the Hilbert space of real-valued, square summable series. Let \(\langle , \rangle\) be the standard inner product of \(l^2\), then \(\langle Φ_t (x), Φ_t (y) \rangle\) is the heat kernel

\[ H (t, x, y) = \sum_{j=1}^{∞} e^{-\lambda_j t} \phi_j (x) \phi_j (y) \]

on \(M\). It was proved in \([BBG]\) that the normalized heat kernel embedding

\[ ψ_t : x ∈ M → \sqrt{2} (4π)^{n/4} t^{-n/2} \cdot \left\{ e^{-\lambda_j t/2} \phi_j (x) \right\}_{j≥1} ∈ l^2 \]

(2)

is asymptotically isometric, i.e. the induced metric of the embedded image \(ψ_t (M)\) in \(l^2\) gets closer and closer to the original metric \(g\) as the heat flow time \(t → 0_+\). More precisely, as \(t → 0_+\)

\[ ψ_t^* g_0 = g + \frac{t}{3} \left( \frac{1}{2} S_g ∙ g − \text{Ric}_g \right) + O (t^2), \]

(3)
where \( g_0 \) is the standard metric in \( l^2 \), \( S_g \) is the scalar curvature, and \( \text{Ric}_g \) is the Ricci curvature of \((M, g)\). The embedding \( \psi_t \) is canonical, in the sense that it is constructed by the eigenfunctions of the Laplacian of \((M, g)\).

Given any \( x \in M \), we choose the normal coordinates \((x^1, \cdots, x^n)\) near \( x \) such that \( \{ \frac{\partial}{\partial x^j} \}_{1 \leq j \leq n} \) is orthonormal at \( x \). Let \( \vec{\alpha} \) be the multi-index of the mixed derivative operator \( D^{\vec{\alpha}} \) on this chart. As \( t \to 0^+ \), it turns out these \((l^2\text{-valued})\) coefficients \( \{ D^{\vec{\alpha}} \Phi_t(x) \} \) have a special property: the angles between any two \( D^{\vec{\alpha}} \Phi_t(x) \) and \( D^{\vec{\beta}} \Phi_t(x) \) converge to universal constants independent on the metric \( g \), the point \( x \) on \( M \), and the orthonormal basis \( \{ V_i \}_{1 \leq i \leq n} = \{ \frac{\partial}{\partial x^j} \}_{1 \leq j \leq n} \) at \( x \), but only on \( \vec{\alpha} \) and \( \vec{\beta} \). The asymptote of the length of \( D^{\vec{\alpha}} \Phi_t(x) \) only depends on \( \vec{\alpha} \) and \( n \) too (Theorem 3). These asymptotic relations in the \( k \)-jet space of \( \Phi_t \) (or \( \psi_t \)) have interesting applications:

1. When \( k = 1 \), \[BBG\] used them to construct asymptotically isometric embeddings \( \psi_t = \sqrt{2}(4\pi)^{n/4} t^{\frac{n-2}{2}} \Phi_t : M \to l^2 \) as in \[BB\]. They also used the embedding \( \psi_t \) to define the spectral distance between different metrics on \( M \). When the metrics have a uniform upper bound of the diameter and a uniform lower bound of the Ricci curvature, they established a precompactness theorem on the space of such metrics;

2. When \( k = 2 \), by perturbing the almost isometric embedding \( \psi_t \), \[WZ\] constructed isometric embeddings of compact Einstein manifolds \( M \) into \( \mathbb{R}^{q(t)} \) with controlled second fundamental form and \( q(t) \sim t^{-n/2} \), where the 2-jet bundle \( J^2(\psi_t) \) appeared in the linearized operator of the isometric embedding problem. It was also shown that for any compact Riemannian manifolds, the mean curvature vectors of the image \( \sqrt{t} \psi_t(M) \) converge to constant length \( \sqrt{\frac{n+2}{2n}} \) as \( t \to 0^+ \);

3. When \( k = 3 \), we use them to show the embedded image \( \sqrt{t} \psi_t(M) \subset l^2 \) is asymptotically umbilical in the mean curvature vector direction as \( t \to 0^+ \) (Proposition 14). We also propose to construct constant mean curvature submanifolds in \( \mathbb{R}^q \) by truncating and perturbing \( \sqrt{t} \psi_t(M) \subset l^2 \);

4. When \( k = 2 \), but also considering the secondary leading terms in the asymptote of the limiting angles, we prove the “asymptotic” Gauss formula (Proposition 17) to express the Riemannian curvature tensor of \((M, g)\) as

\[
R(X, Y, Z, W) = \lim_{t \to 0^+} \left[ \langle \nabla_X \nabla_W \psi_t, \nabla_Y \nabla_Z \psi_t \rangle - \langle \nabla_X \nabla_Z \psi_t, \nabla_Y \nabla_W \psi_t \rangle \right].
\]

From this formula we can easily see the symmetry of the Riemannian curvature tensor, including the second Bianchi identity (Lemma 16). We can also express the Levi-Civita connection in terms of \( \psi_t \) (Lemma 14).

For example, the following 2-jet relations were proved in \[WZ\] and played a crucial role to construct isometric embeddings of \( M \) into \( \mathbb{R}^q \). Let \( \langle \cdot, \cdot \rangle \) be the standard inner product in \( l^2 \) and \( |\cdot| \) be the standard metric in \( l^2 \). We have

\[
2
Theorem 1 ([WZ] Theorem 2) For any \( x \in M \), let \( (x^1, \cdots, x^n) \) be the normal coordinates near \( x \) such that \( \{ \frac{\partial}{\partial x^j} \}_{1 \leq j \leq n} \) is orthonormal at \( x \). Then for the \( n \) first derivatives vectors \( \nabla_i \Phi_t \) \((1 \leq i \leq n)\) and the \( n(n+1)/2 \) second derivatives vectors \( \nabla_i \nabla_j \Phi_t \) \((1 \leq i \leq j \leq n)\) of \( \Phi_t \) with respect to \( \{ \frac{\partial}{\partial x^j} \}_{1 \leq j \leq n} \), as \( t \to 0^+ \), we have
\[
\frac{\langle \nabla_i \Phi_t (x) , \nabla_j \Phi_t (x) \rangle}{|\nabla_i \Phi_t (x) | |\nabla_j \Phi_t (x) |} \to \delta_{ij}, \quad \text{and} \quad \frac{\langle \nabla_i \nabla_j \Phi_t (x) , \nabla_k \Phi_t (x) \rangle}{|\nabla_i \nabla_j \Phi_t (x) | |\nabla_k \Phi_t (x) |} \to 0,
\]
and for \( i \neq j \) or \( k \neq l \),
\[
\frac{\langle \nabla_i \nabla_j \Phi_t (x) , \nabla_k \nabla_l \Phi_t (x) \rangle}{|\nabla_i \nabla_j \Phi_t (x) | |\nabla_k \nabla_l \Phi_t (x) |} \to 0,
\]
except when \( \{i,j\} = \{k,l\} \) as sets. Furthermore, for \( i \neq j \),
\[
\frac{\langle \nabla_i \nabla_i \Phi_t (x) , \nabla_j \nabla_j \Phi_t (x) \rangle}{|\nabla_i \nabla_i \Phi_t (x) | |\nabla_j \nabla_j \Phi_t (x) |} \to \frac{1}{3}.
\]
(4)

The convergence is uniform for all \( x \) on \( M \).

In this paper, we generalize Theorem 1 to obtain the limiting angles between any two derivative vectors \( D\overrightarrow{\alpha} \Phi_t (x) \) and \( D\overrightarrow{\beta} \Phi_t (x) \) as \( t \to 0^+ \). It is a refinement of Proposition 15 in [WZ]. The new observation is that the coefficients there can be translated into certain enumeration problems of graphs and can be solved explicitly (Proposition 9).

In the following we denote the disjoint union of two sets by \( \sqcup \), and let \( |\overrightarrow{\alpha}| \) be the total degree of the derivative operator \( D\overrightarrow{\alpha} \).

Definition 2 For any two derivative vectors \( D\overrightarrow{\alpha} \Phi_t (x) \) and \( D\overrightarrow{\beta} \Phi_t (x) \), suppose there are \( s \) distinct indices \( \{j_1, j_2, \cdots, j_s\} \) in \( \{\overrightarrow{\alpha}\} \sqcup \{\overrightarrow{\beta}\} \), such that each \( j_r \) has multiplicity \( a_r \) in \( \overrightarrow{\alpha} \), multiplicity \( b_r \) in \( \overrightarrow{\beta} \), and let the average multiplicity \( \sigma_r := \frac{a_r + b_r}{2} \) for \( r = 1, 2, \cdots, s \). If all \( a_r + b_r \), are even, we define the constants
\[
A(\overrightarrow{\alpha}, \overrightarrow{\beta}) = (-1)^{\frac{|\overrightarrow{\alpha}| - |\overrightarrow{\beta}|}{2}} \prod_{r=1}^{s} \frac{(2\sigma_r)!}{(2\sigma_r)!},
\]
\[
B(\overrightarrow{\alpha}, \overrightarrow{\beta}) = A(\overrightarrow{\alpha}, \overrightarrow{\beta}) \left\{ A(\overrightarrow{\alpha}, \overrightarrow{\alpha}) A(\overrightarrow{\beta}, \overrightarrow{\beta}) \right\}^{1/2}
= (-1)^{\frac{|\overrightarrow{\alpha}| - |\overrightarrow{\beta}|}{2}} \prod_{r=1}^{s} \frac{(2\sigma_r)!}{\sigma_r!} \left\{ \frac{(2a_r)!}{a_r!} \frac{(2b_r)!}{b_r!} \right\}^{1/2}.
\]
(5)
(6)

If some \( a_r + b_r \) is odd, we simply let
\[
A(\overrightarrow{\alpha}, \overrightarrow{\beta}) = 0 = B(\overrightarrow{\alpha}, \overrightarrow{\beta}).
\]
(7)
We have our main

**Theorem 3** *(Asymptotic high-jet relations)* For any two derivative vectors $D^\alpha \Phi_t(x)$ and $D^\beta \Phi_t(x)$, as $t \to 0^+$,

1. If there is an index appeared with odd total multiplicity in $\{\alpha\} \sqcup \{\beta\}$, then
   
   $$\frac{\left\langle D^\alpha \Phi_t(x), D^\beta \Phi_t(x) \right\rangle}{\left| D^\alpha \Phi_t(x) \right| \left| D^\beta \Phi_t(x) \right|} \to 0. \quad (8)$$

   Especially this is the case when $|\alpha| + |\beta|$ is odd;

2. If each index appears with even total multiplicity in $\{\alpha\} \sqcup \{\beta\}$, then
   
   $$\frac{\left\langle D^\alpha \Phi_t(x), D^\beta \Phi_t(x) \right\rangle}{\left| D^\alpha \Phi_t(x) \right| \left| D^\beta \Phi_t(x) \right|} \to B \left( \alpha, \beta \right) \neq 0; \quad (9)$$

3. We have
   
   $$\left| D^\alpha \Phi_t(x) \right|^2 \to \frac{1}{(4\pi t)^{n/2}} \left( \frac{1}{2t} \right)^{|\alpha|} A \left( \alpha, \alpha \right). \quad (10)$$

The convergence is uniform for all $x$ on $M$.

For example, for $n \geq 2$, $i \neq j$, $\alpha = (i, i)$ and $\beta = (j, j)$, we recover Theorem 2 in [WZ], because

$$A \left( \alpha, \beta \right) = \Pi_{r=1}^2 \left[ (-1)^1 \frac{2!}{2^1 \cdot 1!} \right] = 1,$$

$$A \left( \alpha, \alpha \right) = \Pi_{r=1}^1 \left[ (-1)^0 \frac{4!}{2^2 \cdot 2!} \right] = 3 = A \left( \beta, \beta \right),$$

$$B \left( \alpha, \beta \right) = A \left( \alpha, \beta \right) / \left[ A \left( \alpha, \alpha \right) A \left( \beta, \beta \right) \right]^{1/2} = \frac{1}{3}.$$

Theorem 3 is a consequence of the more general

**Proposition 4** As $t \to 0_+$, we have

$$(4\pi t)^{n/2} \left( \frac{\left| |\alpha| - |\beta| \right|}{2t} \right) \cdot D^\beta_y D^\alpha_x H(t, x, y) \bigg|_{x=y} = A \left( \alpha, \beta \right) + O(t), \quad (11)$$

where $\lfloor c \rfloor$ is the largest integer less or equal to a given real number $c$.  


After the completion of this paper, the author received the paper [Ni], which is related to our results. The paper [Ni], motivated by probabilistic questions, describes the small \( \varepsilon \)-asymptotics of the jets along the diagonal of the integral kernel of the smoothing operator \( w ( \varepsilon \sqrt{\Delta g} ) \) for an arbitrary nonnegative even Schwartz function \( w \). For \( w ( x ) = e^{-x^2} \), these reduce to the short time asymptotics of the jets along the diagonal of the heat kernel.

The paper is organized as follows: In Section 2 we first review the heat kernel and its off-diagonal expansion, then introduce certain graphs to aid the computation of the leading terms in (11), and then prove our main Theorem 3. In Section 3 we give applications of the high-jet relations of \( \psi_t \) to the mean curvature of \( \psi_t (M) \subset l^2 \) and the Riemannian curvature of \( M \). In Section 4 we explore more algebraic structures of the high-jet relations, and reformulate them by the lattice geometry on \( \mathbb{Z}_n^+ \), with the motivation for future applications.

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2 High-jet relations

2.1 Heat kernel and derivatives of the distance function

Let \( H (t, x, y) \) be the heat kernel of the Laplacian \( \Delta_g \) on \((M, g)\),

\[
H (t, x, y) := \sum_{j=1}^{\infty} e^{-\lambda_j t} \phi_j (x) \phi_j (y).
\]  
(12)

It is well known that \( H (t, x, y) \) has the Minakshisundaram-Pleijel expansion

\[
H (t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} U (t, x, y)
\]

(e.g. [BeGaM], p.213, or [Ch], p.154), where \( r = r (x, y) \) is the distance function for points \( x \) and \( y \) on \( M \),

\[
U (t, x, y) = u_0 (x, y) + tu_1 (x, y) + \cdots + t^p u_p (x, y) + O (t^{p+1})
\]

(13)
as \( t \to 0_+ \), where the convergence is in \( C^l \) for any \( l \geq 0 \), and

\[
u_0 (x, y) = [\theta (x, y)]^{-1/2} \quad \text{(for } x \text{ and } y \text{ close enough)},
\]

\[
u_1 (x, y) = \frac{S_g (x)}{6},
\]

(14)

where

\[
\theta (x, y) = \text{volume density at } y \text{ read in the normal coordinates around } x
\]

(15)
with \( r = r(x, y) \) (BeGaM, p. 208). Given any \( x \) on \( M \), let \((x^1, \cdots, x^n)\) be the normal coordinates around \( x \) and \( V_j^n = \{ \frac{\partial}{\partial x^j} \}_{j=1}^n \) be an orthonormal basis at \( x \). For any unit vector \( V \in T_xM \), it was derived in [BBC] that for \( x_s = \exp_x(sV) \) and \( x_\tau = \exp_x(\tau V) \) with small \( s \) and \( \tau \),

\[
\theta(x_s, x_\tau) = 1 - \text{Ric}_g(\dot{x}(s), \dot{x}(s)) \frac{(s - \tau)^2}{3!} + O\left(|s - \tau|^3\right),
\]

so

\[
u_0(x_s, x_\tau) = 1 + \frac{1}{12}\text{Ric}_g(\dot{x}(s), \dot{x}(s))(s - \tau)^2 + O\left(|s - \tau|^3\right).
\]

Hence

\[
u_0(x, x) = 1, \quad \nabla_i \nu_0(x, y)|_{x=y} = 0, \quad \nabla_i U(t, x, y)|_{x=y} = O(t), \quad \nabla_i^x \nabla_j^y \nu_0(x, y)|_{x=y} = -\frac{1}{6}\text{Ric}_g(V_i, V_j),
\]

For later computations, we will need the following facts on the derivatives of the squared distance function \( r^2(x, y) \) on the diagonal \( x = y \).

**Lemma 5** For \( r = r(x, y) \) on \( M \times M \), let \( \nabla_i = \nabla_i^x \) be the partial derivative with respect to \( V_j \) in the \( x \) variables, \( \nabla_j = \nabla_j^y \) be the partial derivative with respect to \( V_j \) in the \( y \) variables, \( \nabla_{ij} = \nabla_i^x \nabla_j^y, \nabla_{ij} = \nabla_i^y \nabla_j^x, \nabla_{ijk} = \nabla_i^x \nabla_j^y \nabla_k^x \) and so on. Then we have the following identities

\[
\nabla_i r^2(x, y)|_{x=y} = 0 = \nabla_i r^2(x, y)|_{x=y},
\]

\[
\nabla_{ij} r^2(x, y)|_{x=y} = 2\delta_{ij} = -\nabla_j r^2(x, y)|_{x=y},
\]

\[
\nabla_{ijk} r^2(x, y)|_{x=y} = \nabla_{ijk} r^2|_{x=y} = \nabla_{ijk} r^2|_{x=y} = \nabla_{ijk} r^2|_{x=y} = 0,
\]

\[
\nabla_{ijkl} r^2(x, y)|_{x=y} = -\nabla_{ijkl} r^2|_{x=y} = -\frac{2}{3}(R_{ijkl}(x) + R_{ijkl}(x)),
\]

where \( R_{ijkl}(x) = R(V_i, V_j, V_k, V_l)(x) \) is the Riemannian curvature.

The identities (20) and (21) are well-known (e.g. [BBC]). The identities (22) and (23) can be found in Chapter 16 (p.282) of [DG], where \( \frac{1}{2} r^2(x, y) \) is called the world function, and the limit at \( x = y \) is called the coincidence limit.

### 2.2 Admissible graphs

To compute \( \langle \bar{D}^\alpha \Phi_t, \bar{D}^\beta \Phi_t \rangle \), it is useful to introduce certain graphs.
Definition 6 (Vertex set) Let $S = \{\alpha\} \amalg \{\beta\}$ be the set with $|\alpha| + |\beta|$ vertices, where each vertex is an index $i \in S$, with the sign “+” if it is in $\{\alpha\}$ and sign “-” if it is in $\{\beta\}$. We say the vertex has color $i$ if we do not distinguish its sign.

Definition 7 (Admissible graph) A graph $G$ on the vertex set $S = \{\alpha\} \amalg \{\beta\}$ is called admissible if (i) Each vertex has valent one; (ii) Each edge connects two vertices with the same color; (iii) Each edge has a sign $\pm$ being the negative of the product of the signs of its two end-points. Let $\text{Sign}(G)$ be the product of the signs of all edges of $G$.

Remark 8 The necessary and sufficient condition for $S = \{\alpha\} \amalg \{\beta\}$ to have admissible graph(s) is that each index appears in $S$ with even multiplicity. Especially $|\alpha| + |\beta|$ must be even. If $G$ is not admissible, we simply let $\text{Sign}(G) = 0$.

Proposition 9 Suppose there are $s$ distinct indices $\{j_1, j_2, \ldots, j_s\} \subset \{1, 2, \ldots, n\}$ in $\{\alpha\} \amalg \{\beta\}$, such that each $j_r$ has multiplicity $a_r$ in $\alpha$, and multiplicity $b_r$ in $\beta$. Let $\sigma_r = \frac{a_r b_r}{2}$. Then we have

1. For each admissible graph $G$ of $S = \{\alpha\} \amalg \{\beta\}$, $\text{sign}(G) = (-1)^{\frac{|\alpha| - |\beta|}{2}}$;
2. $\# \{\text{admissible graphs of } S\} = \prod_{r=1}^{s} \left[\frac{(2\sigma_r)!!}{2^r (\sigma_r)!}\right] = |A(\alpha, \beta)|$.

Proof. By the definition of $\text{sign}(G)$, it is the product of the signs of its edges. There are $\frac{|\alpha| + |\beta|}{2}$ edges, each edge connecting vertices $i$ and $i'$ has the sign $(-1)^{\text{sign}(i)\text{sign}(i')}$, and the disjoint union of edges covers all vertices of $G$, so

$$\text{sign} (G) = \prod_{\text{edge } e \text{ of } G} \text{sign}(e) = (-1)^{\frac{|\alpha| - |\beta|}{2}} \prod_{\text{vertex } i \text{ of } G} \text{sign}(i)$$

$$= (-1)^{\frac{|\alpha| - |\beta|}{2}} \cdot (-1)^{|\beta|} = (-1)^\frac{|\alpha| - |\beta|}{2},$$

where the second line is because only vertices in $\beta$ have “-” sign. For each index $j_r$, it has total multiplicity $2\sigma_r$ in $S = \{\alpha\} \amalg \{\beta\}$, i.e. there are $2\sigma_r$ vertices with color $j_r$ in $S$, so the ways to draw disjoint edges on this subset of vertices in $S$ is

$$\frac{(2\sigma_r)!!}{(2\sigma_r - 2)! (2\sigma_r - 4)! \cdots 2!} / \sigma_r!$$

$$= \frac{(2\sigma_r)!!}{(2\sigma_r - 2)! (2\sigma_r - 4)! \cdots 2!} / \sigma_r!$$

$$= \frac{(2\sigma_r)!}{2^{2\sigma_r} (\sigma_r)!}.$$
Considering all distinct indices \( j_1, j_2, \cdots \) and \( j_s \) in \( S \), we see the number of admissible graphs on \( S \) is \( \prod_{r=1}^s \left[ \frac{(2\pi)^{1/2}}{2} \sigma_r \right] \cdot 2 \sigma_r \cdot \sigma_r! \).

**Remark 10** The number \( A\left( \vec{\alpha}, \vec{\beta} \right) \) is related to the moments of the Gaussian \( e^{-|x|^2} \) by

\[
A\left( \vec{\alpha}, \vec{\beta} \right) = 2 \left[ \frac{|\vec{\alpha} + \vec{\beta}|}{2} \right] \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|x|^2} x^{\vec{\alpha}} y^{\vec{\beta}} dx.
\]

Actually, the above combinatorics of the numbers \( A\left( \vec{\alpha}, \vec{\beta} \right) \) is a manifestation of a classical Wick formula (or Isserlis’ theorem) of the Gauss integrals, see e.g. [AT], Section 11.6. The formula is frequently used in the Feynman integral.

### 2.3 High-jet relations from the heat kernel expansion

Given any \( x \) on \( M \), let \( \{V_i\}_{i=1}^n \) be an orthonormal basis at \( x \), \( \nabla_i = \nabla_x V_i \) be the partial derivative with respect to \( V_i \) in the \( x \) variables, and \( \nabla_j = \nabla_y V_j \) be the partial derivative with respect to \( V_j \) in the \( y \) variables, where \( "-\" \) indicates the derivative is with respect to the \( y \)-variables. We also denote \( \nabla_{ij} = \nabla_x V_i \nabla_y V_j \), \( \nabla_{ij} = \nabla_x V_i \nabla_y V_j \), and so on. It is easy to check that for derivative operators \( D^{\vec{\alpha}} \) and \( D^{\vec{\beta}} \),

\[
\left\langle D^{\vec{\alpha}} \Phi_t, D^{\vec{\beta}} \Phi_t \right\rangle (x) = D_y^{\vec{\beta}} D_x^{\vec{\alpha}} H(t, x, y) \big|_{x=y},
\]

(24)

**Proposition 11** Let \( \vec{\alpha} \) and \( \vec{\beta} \) be two multi-indices in the set \( \{1, 2, \cdots n\} \). Then as \( t \to 0_+ \), we have

\[
(4\pi t)^{n/2} \left( \frac{|\vec{\alpha} + |\vec{\beta}|}{2} \right) \cdot D_y^{\vec{\beta}} D_x^{\vec{\alpha}} H(t, x, y) \big|_{x=y} \to A\left( \vec{\alpha}, \vec{\beta} \right),
\]

(25)

where \( [c] \) is the largest integer less or equal to a given real number \( c \). If \( A\left( \vec{\alpha}, \vec{\beta} \right) = 0 \), we further have

\[
(4\pi t)^{n/2} \left( \frac{|\vec{\alpha} + |\vec{\beta}|}{2} \right) \cdot D_y^{\vec{\beta}} D_x^{\vec{\alpha}} H(t, x, y) \big|_{x=y} = O\left( t \right).
\]

(26)

**Proof.** For any multi-index \( \vec{\gamma} \), applying the Leibniz rule to

\[
H(t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{t}{4\pi} U(t, x, y)}
\]

we can write

\[
D^{\vec{\gamma}} H(t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{t}{4\pi} P_{\vec{\gamma}}(t, x, y)},
\]
where $P_\gamma(t, x, y)$ is a polynomial in $D^{\vec{\mu}_j} \left( -\frac{r^2(x,y)}{4t} \right)$ and $D^{\vec{\eta}} U(t, x, y) \bigg|_{x=y}$, i.e. each summand is of the form
\[
\left( \Pi_{j=1}^l D^{\vec{\mu}_j} \left( -\frac{r^2(x,y)}{4t} \right) \right) \bigg|_{x=y} \cdot D^{\vec{\eta}} U(t, x, y) \bigg|_{x=y}
\]
for some multi-indices $\vec{\mu}_j$ and $\vec{\eta}$ with
\[
\Sigma_{j=1}^l |\vec{\mu}_j| + |\vec{\eta}| = |\vec{\gamma}|.
\]
For example, when $\vec{\gamma} = \partial_{x_i}$,
\[
P_\gamma(t, x, y) = \partial_i \left( -\frac{r^2(x,y)}{4t} \right) U(t, x, y) + \partial_i U(t, x, y).
\]
(more examples of $P_\gamma(t, x, y)$ with $|\vec{\gamma}| \leq 4$ are in [WZ]). Now we take
\[
\vec{\gamma} = (\vec{\alpha}, \vec{\beta}), |\vec{\gamma}| = |\vec{\alpha}| + |\vec{\beta}|, D^{\vec{\gamma}} = D_y^{\vec{\beta}} D_x^{\vec{\alpha}},
\]
where $D_y^{\vec{\beta}}$ indicates the derivative $D_y^{\beta_i}$ is with respect to the $y$-variables. We have the following claims:

1. As $t \to 0_+$, the nonzero summands of $P_\gamma(t, x, y)$ involving the highest power of $\frac{1}{t}$ must have $\vec{\mu}_j$'s with $|\vec{\mu}_j| = 2$ as many as possible, such that each $\vec{\mu}_j$ is of the form $(i, i)$, $(i, \overline{i})$ or $(\overline{i}, \overline{i})$ for some $i \in \{1, 2, \cdots, n\}$, and $|\vec{\eta}| = 0$ or 1. We also have
\[
\left| D^{\vec{\gamma}} H(t, x, y) \right|_{x=y} \leq C \frac{1}{(4\pi t)^{n/2}} \cdot t^{-\frac{|\vec{\gamma}|}{2}},
\]
where the constant $C$ depends on $n$ and $|\vec{\gamma}|$.

2. If the total multiplicity of each index in $\{ \vec{\alpha} \} \Pi \{ \vec{\beta} \}$ is even, then we further have $l = \frac{|\vec{\gamma}|}{2}$, and $|\vec{\eta}| = 0$ for the the nonzero summands of $P_\gamma(t, x, y)$ involving the highest power of $\frac{1}{t}$. These terms are of the same sign $(-1)^{|\vec{\gamma}|-|\vec{\beta}|}$, and the total number of such terms is $|A(\vec{\alpha}, \vec{\beta})|$.

For Claim 1, from Lemma [3] we have
\[
\partial_i \left( -\frac{r^2(x,y)}{4t} \right) \bigg|_{x=y} = 0,
\]
\[
\partial_{ij} \left( -\frac{r^2(x,y)}{4t} \right) \bigg|_{x=y} = -\frac{\delta_{ij}}{2t} = -\partial_{ij} \left( -\frac{r^2(x,y)}{4t} \right) \bigg|_{x=y},
\]
\[
\partial_{ijk} \left( -\frac{r^2(x,y)}{4t} \right) \bigg|_{x=y} = 0 = \partial_{ijk} \left( -\frac{r^2(x,y)}{4t} \right) \bigg|_{x=y},
\]
\[
\left| D^{\vec{\mu}_j} \left( -\frac{r^2(x,y)}{4t} \right) \bigg|_{x=y} \right| \leq C \cdot \frac{1}{t}, \text{ for any } \vec{\mu}_j.
\]
So from the equations in (29) and the total degree condition (28), the summand
\[
\left( \Pi_{j=1}^l D^\gamma_j \left( -\frac{r^2 (x, y)}{4t} \right) \right) \Bigg|_{x=y} \cdot D^\gamma U
\]
can not exceed \( \left\lfloor \frac{|\gamma|}{2} \right\rfloor \) copies of factor \( \frac{1}{\gamma} \). The \( \left\lfloor \frac{|\gamma|}{2} \right\rfloor \) copies of factor \( \frac{1}{\gamma} \) can be achieved if \( |\mu_j^*| = 2 \) for all \( j \), and \( |\gamma_j| = 0 \) or \( 1 \) depending on \( |\gamma| \) is even or odd, so
\[
|D^\gamma U| \leq |u_0(x, x)| + |\nabla_i u_0(x, y)|_{x=y} \leq 2
\]
by (16) and (17) as \( t \to 0_+ \). Such summand is nonzero if and only if each \( \mu_j^* \) is of the form \((i, i), (i, \tilde{i})\) or \((\tilde{i}, \tilde{i})\) for some \( i \in \{1, 2, \cdots, n\} \) by Lemma 5, so there are at most \( 3n \) choices of each \( \mu_j^* \). Therefore as \( t \to 0_+ \),
\[
|D^\gamma H(t, x, y)|_{x=y} = \left| \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2}{4t}} P_\gamma (t, x, y) \right|_{x=y} \leq \frac{1}{(4\pi t)^{n/2}} \cdot \left( \frac{1}{2t} \right)^{|\gamma|/2} \cdot (3n)^{\frac{|\gamma|}{2}} \cdot 2.
\]
For Claim 2, since each index appears with even multiplicity in \( \{\alpha\} \Pi \{\beta\} \), it is easy to see the summands (27) with \( l = \left\lfloor \frac{|\gamma|}{2} \right\rfloor \) and \( |\gamma| = 0 \) can have maximal number of \( \mu_j^* \)'s satisfying \( |\mu_j^*| = 2 \), with each \( \mu_j^* \) is of the form \((i, i), (i, \tilde{i})\) or \((\tilde{i}, \tilde{i})\) for some \( i \in \{1, 2, \cdots, n\} \). It is of the form
\[
\left( \Pi_{j=1}^l D^\gamma_j \left( -\frac{r^2 (x, y)}{4t} \right) \right) \Bigg|_{x=y} \cdot U(t, x, x).
\]
Since
\[
\partial_{ij} \left( -\frac{r^2 (x, y)}{4t} \right) |_{x=y} = -\frac{\delta_{ij}}{2t} = -\partial_{ij} \left( -\frac{r^2 (x, y)}{4t} \right) |_{x=y}, \quad (30)
\]
the product
\[
\left( \Pi_{j=1}^l D^\gamma_j \left( -\frac{r^2 (x, y)}{4t} \right) \right) \Bigg|_{x=y} = \Pi_{j=1}^l \text{sign} \left( \mu_j^* \right) \cdot \left( \frac{1}{2t} \right)^{|\gamma|/2}
\]
gives the highest power of \( \frac{1}{\gamma} \), where we let \( \text{sign} \left( \mu_j^* \right) = 1 \) if \( \mu_j^* = (i, i) \) or \((\tilde{i}, \tilde{i})\), and \( \text{sign} \left( \mu_j^* \right) = -1 \) if \( \mu_j^* = (i, \tilde{i}) \). If these summands do not cancel each other, they give the leading term in \( D^\gamma H(t, x, y) |_{x=y} \) as \( t \to 0_+ \), of the form
\[
\frac{1}{(4\pi t)^{n/2}} \cdot \left( \frac{1}{2t} \right)^{|\gamma|/2} \cdot (-1)^a A,
\]
where $A$ is the number of such summands, and $(-1)^a$ is their common sign.

To determine the number $A$, we build a one-to-one correspondence between such summands and admissible graphs in Definition 7. For each $\mu_j = (i, i), (i, t)$ or $(t, t)$ of the summand, we draw an edge between the corresponding vertices in $S = \{ \alpha \} \cup \{ \beta \}$. Drawing $I$ edges on $S$ with no intersecting vertex, we obtain an admissible graph $G$. The sign on each edge of $G$ reflects the sign in (30). Taking product of them we see sign($G$) is exactly the sign of the summand. Conversely, given any admissible graph $G$, we can read $\mu_j$ from its edges, and then can write down the corresponding summand $\left( \prod_{j=1}^r D^\mu_j \left( \frac{-r^2(x, y)}{4t} \right) \right)_{|x=y} \cdot U(t, x, x)$ in $D^\alpha D^\beta H(t, x, y)_{|x=y}$. From Proposition 9 we know sign($G$) = $(-1)^{\frac{|\pi| - |\gamma|}{2}}$, so the summands are of the same sign $(-1)^{\frac{|\pi| - |\gamma|}{2}}$. Proposition 9 also gives the signed count of admissible graphs, which is $A(\alpha, \beta)$. Claim 2 is proved.

If each index appears with even multiplicity in $\{ \alpha \} \cup \{ \beta \}$, then $|\gamma|$ is even, $\left\lfloor \frac{|\gamma|}{2} \right\rfloor = \frac{|\gamma|}{2}$. From the above argument we have as $t \to 0+$,

$$(4\pi t)^{n/2} \left( \frac{|\pi| + |\gamma|}{2} \right) \cdot D^\alpha D^\beta H(t, x, y)_{|x=y}$$

$$\to (4\pi t)^{n/2} \left( \frac{|\pi|}{2} \right) \cdot \left( \frac{1}{2t} \right) \cdot \left( \frac{1}{4t} \right)^{n/2} \cdot \frac{1}{(4\pi t)^{n/2}} A(\alpha, \beta) = A(\alpha, \beta).$$

If some index $i$ appears with odd multiplicity in $\{ \alpha \} \cup \{ \beta \}$, then by our definition $A(\alpha, \beta) = 0$. For a nonzero summand of the form

$$\left( \prod_{j=1}^r D^\mu_j \left( \frac{-r^2(x, y)}{4t} \right) \right)_{|x=y} \cdot D^\eta U,$$

by the equations in (29) and the total degree condition (28), it can have the $\frac{1}{t}$ power at most $\left\lfloor \frac{|\eta| - 1}{2} \right\rfloor$. If the power $\left\lfloor \frac{|\eta| - 1}{2} \right\rfloor$ is achieved, then $|\eta| \leq 2$ by the total degree condition (28), and $|\eta| = 2$ is only possible when $|\gamma|$ is even. If $|\eta| = 0$, there will be a term $D^\sigma (t^2)_{|x=y}$ in the summand, such that $|\sigma| \leq 3$ and $i$ appears with odd multiplicity in $\sigma$, making the summand zero by Lemma 6. So we must have $|\eta| = 1$ when $|\gamma|$ is odd.

Similar to the argument in Claim 2, we have

$$\left| D^\eta H(t, x, y)_{|x=y} \right| \leq C \frac{1}{(4\pi t)^{n/2}} \cdot \left( \frac{1}{t} \right) \left\lfloor \frac{|\eta| - 1}{2} \right\rfloor \left| D^\eta U \right|. \quad (31)$$
When $|\bar{\eta}| = 1$, by (18) we have $|D\bar{\bar{\eta}} U| \leq Ct$. Therefore

$$(4\pi t)^n/2 (2t)^{\frac{|\pi|x|\bar{\bar{\eta}}|}{2}} \cdot D^\bar{\bar{\eta}} \frac{D\bar{\bar{\eta}}}{x} H (t, x, y) \bigg|_{x=y}$$

$$\rightarrow \begin{cases} t^\frac{|\pi|x|\bar{\bar{\eta}}|}{2} \cdot O \left(t^{-\frac{|\pi|x|\bar{\bar{\eta}}|}{2}} \right) \cdot O(t) = O(t) \rightarrow 0 \text{ (if } |\bar{\bar{\eta}}| \text{ is odd)}, \\
(4\pi t)^n/2 (2t)^{\frac{|\pi|x|\bar{\bar{\eta}}|}{2}} \cdot O \left(t^{-\frac{|\pi|x|\bar{\bar{\eta}}|}{2}} \right) \cdot O(1) = O(t) \rightarrow 0 \text{ (if } |\bar{\bar{\eta}}| \text{ is even}). \end{cases} = A \left(\bar{\alpha}, \bar{\beta}\right).$$

Proposition 11 is proved. 

Now we are ready to give the proof of Theorem 3.

**Proof.** Letting $\bar{\alpha} = \bar{\beta}$ in Proposition 11 and noticing

$$\langle D^\bar{\alpha} \Phi_t, D^\bar{\beta} \Phi_t \rangle (x) = D^\bar{\alpha} \frac{D\bar{\alpha}}{x} H (t, x, y) \bigg|_{x=y},$$

we have

$$(4\pi t)^n/2 (2t)^{\frac{|\pi|x|\bar{\beta}|}{2}} \cdot \frac{1}{(4\pi t)^n/2 (2t)^{\frac{|\pi|x|\bar{\beta}|}{2}}} A (\bar{\alpha}, \bar{\alpha}). \quad (32)$$

If $|\bar{\alpha}| + |\bar{\beta}|$ is even, by Proposition 11 we have

$$\lim_{t \rightarrow 0^+} \frac{\langle D^\bar{\alpha} \Phi_t (x), D^\bar{\beta} \Phi_t (x) \rangle}{|D^\bar{\alpha} \Phi_t (x)| |D^\bar{\beta} \Phi_t (x)|}$$

$$= \lim_{t \rightarrow 0^+} \frac{(4\pi t)^n/2 (2t)^{\frac{|\pi|x|\bar{\beta}|}{2}} \cdot \langle D^\bar{\alpha} \Phi_t (x), D^\bar{\beta} \Phi_t (x) \rangle}{(4\pi t)^n/2 (2t)^{\frac{|\pi|x|\bar{\beta}|}{2}} |D^\bar{\alpha} \Phi_t (x)|}$$

$$= \left[ A \left(\bar{\alpha}, \bar{\beta}\right) \right]/ \left[ A (\bar{\alpha}, \bar{\alpha}) A (\bar{\beta}, \bar{\beta}) \right]^{1/2} = B \left(\bar{\alpha}, \bar{\beta}\right).$$
If $|\alpha| + |\beta|$ is odd, then some index must appear with odd multiplicity in $\{\alpha\} \cup \{\beta\}$. By (31) and (32),

$$\lim_{t \to 0^+} \frac{1}{(4\pi t)^{n/2}} \left( \frac{1}{t} \right)^{\left\lfloor \frac{|\beta| - 1}{2} \right\rfloor} \left( \frac{\pi t}{4} \right)^{-\left\lfloor \frac{1}{2} |\alpha| \right\rfloor} A(\alpha, \alpha) \cdot \frac{1}{(4\pi t)^{n/2}} \left( \frac{\pi t}{2} \right)^{\left\lfloor \frac{1}{2} |\beta| \right\rfloor} A(\beta, \beta)$$

$$= O(t) \to 0 = B(\alpha, \beta).$$

Theorem 3 is proved. □

**Remark 12** If we are only concerned with the leading order terms in $\left\langle D^{\alpha} \Phi_t, D^{\beta} \Phi_t \right\rangle (x)$ or $D_y D_x^2 H(t, x, y) |_{x = y}$ as $t \to 0_+$, the ordering of derivatives within $\alpha$ and $\beta$ are not important, and we can regard $\alpha$ and $\beta$ as sets of their indices, respectively. This is because when we switch the ordering of derivatives, say from $D_x^i D_y^j$ to $D_y^j D_x^i$, their difference is a lower order differential operator, and for lower order differential operator $D^\gamma$ with $|\gamma| < |\alpha|$, $|D^\gamma \Phi_t| = o\left(|D^\alpha \Phi_t|\right)$ as $t \to 0_+$ on $M$ by (10).

**Remark 13** We remark that except the constant term $A(\alpha, \beta)$, the next order terms in $t^{\frac{n}{2} + \left\lfloor \frac{|\beta| - 1}{2} \right\rfloor} \cdot D_y D_x^2 H(t, x, y) |_{x = y}$ do depend on $g$ (or curvature), and are important for some geometric applications (e.g. Proposition 17), but to compute them is significantly harder.

## 3 Applications

In this section, we give several applications of the high-jet relations of $\psi_t$ in Theorem 3 and Proposition 11. The applications to the mean curvature and the Riemannian curvature tensor are related to the extrinsic geometry and intrinsic geometry of $\psi_t(M) \subset \mathbb{L}^2$ as $t \to 0_+$ respectively. We also discuss the approximation of the geometry of $(M, g)$ by finitely many eigenfunctions.

### 3.1 Mean curvature of $\psi_t(M)$ in $\mathbb{L}^2$

The first application is about the mean curvature of the embedded image $\psi_t(M) \subset \mathbb{L}^2$. Intuitively speaking, as $t \to 0_+$, the extrinsic geometry of $\psi_t(M) \subset \mathbb{L}^2$ locally looks the same at any point, and is evenly bumpy everywhere. More precisely we have
Proposition 14  For the heat kernel embedding $\psi_t : M \to \mathbb{L}^2$ and $x \in M$, let
\[ A(x,t) \colon T_{\psi_t(x)} \psi_t(M) \times T_{\psi_t(x)} \psi_t(M) \to T_{\psi_t(x)} \psi_t(M)^\perp \]
be the second fundamental form of the embedded image $\psi_t(M) \subset \mathbb{L}^2$ and $H(x,t)$ be the mean curvature vector at $\psi_t(x)$. We have

1. ([WZ] Corollary 22) $\sqrt{t} A(x,t)$ converges to certain normal form for all $x$ on $M$ as $t \to 0_+$. The mean curvature vector $H(x,t)$, after scaled by a factor $\sqrt{t}$, converges to constant length:
\[
\lim_{t \to 0_+} \sqrt{t} |H(x,t)| = \sqrt{\frac{n + 2}{2n}} \tag{33}
\]

2. $\psi_t(M)$ is asymptotically umbilical in the mean curvature vector direction as $t \to 0_+$, in the following sense:
\[
t \cdot \lim_{t \to 0_+} S(x,t) \langle W,H \rangle = -\frac{3}{2} W, \text{ for any } W \in T_{\psi_t(x)} \psi_t(M),
\]

where
\[
S(x,t) : T_{\psi_t(x)} \psi_t(M) \times T_{\psi_t(x)} \psi_t(M)^\perp \to T_{\psi_t(x)} \psi_t(M)
\]
is the second fundamental tensor on $\psi_t(M)$ defined by $S(X,v) = (\nabla_X v)^T$ for any $X \in T_{\psi_t(x)} \psi_t(M)$ and $v \in T_{\psi_t(x)} \psi_t(M)^\perp$, $\mathbf{\perp} : T_{\psi_t(x)} \mathbb{L}^2 \to T_{\psi_t(x)} \psi_t(M)$ is the projection, and "$\perp$" means the orthogonal complement in $\mathbb{L}^2$.

The convergence is uniform for all $x$ on $M$.

Proof. For any $x \in M$, let $(x_1, \ldots, x_n)$ be a local coordinates near $x$ such that the coordinate vectors $\left\{ \frac{\partial}{\partial x_i} \right\}_{i=1}^n := \left\{ V_{ij} \right\}_{i=1}^n$ are orthonormal at $x$. The second fundamental form of the submanifold $\psi_t(M) \subset \mathbb{L}^2$ can be written as
\[
A(x,t) = \sum_{1 \leq i,j \leq n} h_{ij} (x,t) \, dx^i dx^j
\]
where $h_{jk}(x,t) (1 \leq j, k \leq n)$ are vectors in $\mathbb{L}^2$. Then as $t \to 0_+$, we have
\[
\lim_{t \to 0_+} \left( h_{jk} - \nabla_{V_j} \nabla_{V_k} \psi_t \right)(x,t) = 0, \text{ for } 1 \leq j, k \leq n. \tag{34}
\]
This is due to the 2-jet relation
\[
\lim_{t \to 0_+} \left( \langle \nabla_{V_j} \psi_t, \nabla_{V_j} \nabla_{V_k} \psi_t \rangle \right)(x,t) = 0
\]
from [20], so the second order terms in the Taylor expansion of $\psi_t : M \to \mathbb{L}^2$ approximate the second fundamental form of $\psi_t(M) \subset \mathbb{L}^2$ as $t \to 0_+$. Therefore, for the mean curvature $H(x,t) = \frac{1}{n} \sum_{k=1}^n h_{kk}(x)$, we have
\[
\lim_{t \to 0_+} \left( H(x,t) - \frac{\sum_{k=1}^n \nabla_{V_k} \nabla_{V_k} \psi_t}{n} \right)(x,t) = 0. \tag{36}
\]
Item 1 was proved in [WZ] using the 2-jet relations of \( \psi_t \) in Theorem 1. For Item 2, using the 3-jet relations in Theorem 3 for \( \alpha = (i, k, k) \) and \( \beta = (j) \), we have

\[
\frac{\langle \nabla V_i \nabla V_k \nabla V_k \psi_t, \nabla V_j \psi_t \rangle (x, t)}{\langle \nabla V_i \nabla V_k \nabla V_k \psi \mid \nabla V_j \psi \rangle (x, t)} \rightarrow B \left( \vec{\alpha}, \vec{\beta} \right)
\]

\[
= \begin{cases}
-15, & \text{if } i = j = k, \\
-15, & \text{if } i = j \neq k, \\
0, & \text{if } i \neq j,
\end{cases}
\]

and

\[
|\nabla V_i \nabla V_k \nabla V_k \psi_t (x, t)| \rightarrow \sqrt{2} (4\pi)^{n/4} t^{n/4} \left( \frac{1}{4\pi t} \right)^{n/4} \left( \frac{1}{2t} \right)^{3/2} A(\vec{\alpha}, \vec{\alpha})^{1/2}
\]

\[
= \begin{cases}
\frac{1}{2\pi} \sqrt{2/3} = \frac{2}{\pi} \sqrt{6}, & \text{if } i = k, \\
\frac{1}{2\pi} \sqrt{21/22} = \frac{1}{2\pi} \sqrt{3}, & \text{if } i \neq k,
\end{cases}
\]

\[
|\nabla V_j \psi_t (x, t)| \rightarrow \sqrt{2} (4\pi)^{n/4} t^{n/4} \left( \frac{1}{4\pi t} \right)^{n/4} \left( \frac{1}{2t} \right)^{1/2} A(\vec{\beta}, \vec{\beta})^{1/2} = 1.
\]

Therefore as \( t \rightarrow 0^+ \),

\[
\langle \nabla V_i \nabla V_k \nabla V_k \psi_t, \nabla V_j \psi_t \rangle (x, t) \rightarrow \left\{ -\frac{\sqrt{3}}{\pi}, \frac{1}{2\pi} \sqrt{6} = \frac{3}{2\pi}, \text{if } i = j = k, \\
-\frac{\sqrt{3}}{\pi}, \frac{1}{2\pi} \sqrt{3} = \frac{3}{2\pi}, \text{if } i = j \neq k. \right\} = \frac{-3}{2t}.
\]

By the definition of the second fundamental tensor,

\[
2t \cdot \lim_{t \rightarrow 0^+} S(x, t) \langle \nabla V_i \psi_t, H \rangle, \nabla V_j \psi_t \rangle (x, t) = 2t \cdot \lim_{t \rightarrow 0^+} \langle \nabla V_i \psi_t, H (\psi_t (x), t), \nabla V_j \psi_t \rangle (x, t)
\]

\[
= 2t \cdot \lim_{t \rightarrow 0^+} \langle \nabla V_i H (x, t), \nabla V_j \psi_t \rangle (x, t)
\]

\[
= 2t \cdot \lim_{t \rightarrow 0^+} \left( \nabla V_i \left( \frac{1}{n} \sum_{k=1}^{n} \nabla V_k \nabla V_k \psi_t \right), \nabla V_j \psi_t \right) (x, t)
\]

\[
= \frac{1}{n} 2t \cdot \sum_{k=1}^{n} \lim_{t \rightarrow 0^+} \langle \nabla V_i \nabla V_k \nabla V_k \psi_t, \nabla V_j \psi_t \rangle (x, t)
\]

\[
= \begin{cases}
\frac{1}{n} 2t \cdot \sum_{k=1}^{n} \left( -\frac{\delta_{ij}}{2t} \right), & \text{if } i = j, \\
0, & \text{if } i \neq j,
\end{cases}
\]

\[
= -3\delta_{ij}.
\]

Since \( \{ \nabla V_j \psi_t \}_{1 \leq j \leq n} \) span \( T_{\psi(x)} \psi_t (M) \), this means for any \( W \in T_{\psi(x)} \psi_t (M) \),

\[
t \cdot \lim_{t \rightarrow 0^+} S(x, t) (W, H) = -\frac{3}{2} W.
\]
So \( \psi_t(M) \) is asymptotically umbilical in the mean curvature vector direction as \( t \to 0_+ \). □

**Remark 15** In other words, for the image \( \frac{1}{\sqrt{t}} \psi_t(M) \subset \mathbb{R}^2 \), as \( t \to 0_+ \) its mean curvature vector converges to constant length \( \sqrt{\frac{n+2}{2n}} \), and \( \frac{1}{\sqrt{t}} \psi_t(M) \) is asymptotically umbilical in the mean curvature vector direction. If we want to construct constant mean curvature submanifolds in \( \mathbb{R}^3 \), perhaps we can truncate \( \frac{1}{\sqrt{t}} \psi_t(M) \subset \mathbb{R}^q \) by taking the first \( q \) components for large \( q \), and then perturb it by the implicit function theorem.

### 3.2 Riemannian curvature tensor

In the second application we give a **heat kernel embedding** interpretation of the Levi-Civita connection and the Riemannian curvature tensor. The idea is that \( \psi_t : M \to \mathbb{R}^2 \) is an almost isometric embedding, so the “extrinsic” constructions of the Levi-Civita connection and Gaussian curvature of surfaces in \( \mathbb{R}^3 \) using the ambient space can be mimicked here, with \( \mathbb{R}^3 \) replaced by \( \mathbb{R}^2 \). The difficulty is that we need to take the limits as \( t \to 0_+ \) in our interpretation, and to show these limits exist we need the high-jet relations in Proposition 11 and some refinement.

As pointed to the author by L. I. Nicolaescu, there exists a random function interpretation of the Levi-Civita connection and the Riemannian curvature tensor in Section 12 of [AT] (equation (12.2.6) and Lemma 12.2.1, respectively). If the random function \( f \) in [AT] is taken as \( f_t = \sum_{j \geq 1} u_j \phi_j(x) \) with the coefficients \( u_j \) being independent Gaussian random variables with the expectation \( E(u_j) = 0 \) and variance \( \text{Var}(u_j) = e^{-\lambda_j t} \), then Lemma 16 and Proposition 17 follows from the results in [AT] by taking limits as \( t \to 0 \). (For this approach, see [NI], appendix B.)

As before, for any \( p \in M \), we choose the normal coordinates \( (x^1, \ldots, x^n) \) at \( p \), and choose an orthonormal frame field \( \{V_j\}_{j=1}^n \) near \( p \) such that \( \{V_j\}_{j=1}^n \big|_p = \{\frac{\partial}{\partial x^j}\} \). Let \( \nabla \) be the Levi-Civita connection of \((M,g)\), and \( \nabla_{V_j} := \nabla_j \) for \( 1 \leq j \leq n \).

**Lemma 16 (Levi-Civita Connection)** For any \( X,Y \in T_p M \), we have

\[
\nabla_X Y (p) = \lim_{t \to 0_+} \sum_{j=1}^n \left( \nabla_X \nabla_Y \psi_t, \nabla_{V_j} \psi_t \right) (p) V_j (p) \tag{37}
\]

\[
= \lim_{t \to 0_+} d\psi_t^{-1} \left( (\nabla_X \nabla_Y \psi_t (p))^{\mathbb{T}} \right),
\]

where \( \mathbb{T} : T_{\psi_t(p)} \mathbb{R}^2 \to T_{\psi_t(p)} \psi_t(M) \) is the orthogonal projection with respect to the standard metric on \( \mathbb{R}^2 \), and on the right hand side \( \nabla_X \) and \( \nabla_Y \) are the derivatives for \( \mathbb{R}^2 \)-valued functions \( f : M \to \mathbb{R}^2 \).

**Proof.** Since all quantities in (37) are linear in \( X \), we may assume \( X = V_i \). In a neighborhood of \( p \), we write \( Y(x) = \sum_{k=1}^n y^k(x) V_k(x) \). Since \( \nabla_{V_i} V_j (p) = \delta_{ij} \), we have

\[
\nabla_{V_i} Y (p) = \sum_{k=1}^n \nabla_i y^k (p) V_k (p),
\]
where \( \partial_t y^k(x) = \nabla_{V_t} y^k(x) \). On the other hand, by (35) and the 2-jet relation (38), we have

\[
\lim_{t \to 0+} \sum_{i=1}^n \left( \langle \nabla_i \nabla_Y \psi_t, \nabla_j \psi_t \rangle (p) V_j (p) \right)
= \lim_{t \to 0+} \sum_{i=1}^n \left( \langle \nabla_i (\sum_{k=1}^n y^k \nabla_k \psi_t), \nabla_j \psi_t \rangle (p) V_j (p) \right)
= \sum_{i=1}^n \sum_{k=1}^n \lim_{t \to 0+} \left( \langle \nabla_i y^k (p) \nabla_k \psi_t, \nabla_j \psi_t \rangle (p) \right)
+ \langle y^k (p) \nabla_i \nabla_k \psi_t, \nabla_j \psi_t \rangle (p) V_j (p) \right)
= \sum_{i=1}^n \sum_{k=1}^n \left( \partial_t y^k (p) \delta_{kj} V_j (p) + 0 \right)
= \nabla_i y^k (p) V_k (p).
\]

Therefore the first identity

\[
\nabla_X Y (p) = \lim_{t \to 0+} \sum_{i=1}^n \left( \langle \nabla_i \nabla_Y \psi_t, \nabla_j \psi_t \rangle (p) V_j (p) \right)
\]

is proved. Since \( \{ V_j \}_{j=1}^n \) is an orthonormal basis of \( T_p M \), by (35) \( \left\{ \langle \nabla_i \nabla_Y \psi_t (p) \rangle \right\}_{j=1}^n \) is asymptotically an orthonormal basis of \( T_{\psi_t(p)} M (M) \) as \( t \to 0+ \) and we have

\[
\lim_{t \to 0+} \left[ (\langle \nabla_X \nabla_Y \psi_t (p) \rangle)^T - \sum_{j=1}^n \left( \langle \nabla_X \nabla_Y \psi_t, \nabla_j \psi_t \rangle (p) \right) \nabla_j \psi_t (p) \right] = 0.
\]

Applying \( d\psi_t^{-1} \) on both sides, the second identity is proved.  

Suppose \( f : (M, g) \to \mathbb{R}^n \) is an isometric embedding, and \( h \) is the second fundamental form of \( M \). It is well-known that for vectors \( X, Y, Z, W \in TM \), the Riemannian curvature tensor \( R \) of \( M \) satisfies

\[
R (X, Y, Z, W) = \langle h (X, W), h (Y, Z) \rangle - \langle h (X, Z), h (Y, W) \rangle.
\]

This is the Gauss formula, which relates the intrinsic curvature \( R \) of \( (M, g) \) with its extrinsic geometry in the ambient space. In our case \( \psi_t : M \to \mathbb{R}^2 \) is asymptotically isometric as \( t \to 0+ \), and by (34) the second fundamental form \( h (X, Y) \) of \( \psi_t (M) \subset \mathbb{R}^2 \) is approximated by \( \nabla_X \nabla_Y \psi_t \) as \( t \to 0+ \), so it is natural to expect the following “asymptotic” Gauss formula.

**Proposition 17 (Riemannian curvature)** For any \( X, Y, Z, W \) of \( TM \), we have

\[
R (X, Y, Z, W) = \lim_{t \to 0+} \left[ \langle \nabla_X \nabla_W \psi_t, \nabla_Y \nabla_Z \psi_t \rangle - \langle \nabla_X \nabla_Z \psi_t, \nabla_Y \nabla_W \psi_t \rangle \right].
\]  

**Proof.** We only need to prove the above identity when \( X, Y, Z \) and \( W \) are taken from the orthonormal basis \( \{ V_j \}_{j=1}^n \) near \( x \) as above, say they are \( V_i, V_j, V_k \) and \( V_l \) respectively. For general vector fields, we can express \( X = \sum_{j=1}^n a^j (x) V_j (x) \) etc., and plug them in (38). Then by the high-jet relations (35) and (36), as \( t \to 0+ \) the terms of the type \( \langle \nabla_k a^i \nabla_i \psi_t, \nabla_l b^j \nabla_j \psi_t \rangle \) will vanish or cancel each other, and the terms of the type \( \langle \nabla_k a^i \nabla_i \psi_t, \nabla_j \psi_t, V_k \nabla_l \psi_t \rangle \) will converge to 0. For the same reason, switching the ordering of the covariant derivatives in (38), say from \( \nabla_X \nabla_W \psi_t \) to \( \nabla_W \nabla_X \psi_t \) etc., will not affect the limit in (38).
We first refine our estimate of \( \lim_{t \to 0^+} \nabla_{V_i}^x \nabla_{V_i}^y \nabla_{V_j}^y \nabla_{V_k}^y H (t, x, y) \big|_{x=y} \) in Proposition 6. We have proved that

\[
\lim_{t \to 0^+} (4\pi t)^{\frac{3}{2}} (2t)^2 D_x^\gamma D_y^\gamma H (t, x, y) \big|_{x=y} = A \left( \overrightarrow{\alpha}, \overrightarrow{\beta} \right)
\]

for \( \overrightarrow{\alpha} = (i, l) \) and \( \overrightarrow{\beta} = (j, k) \), where \( A \left( \overrightarrow{\alpha}, \overrightarrow{\beta} \right) \) is the leading (constant) term of \( (4\pi t)^{\frac{3}{2}} (2t)^2 D_x^\gamma D_y^\gamma H (t, x, y) \big|_{x=y} \) as \( t \to 0^+ \), but to prove our theorem we will need the next order term, which is linear in \( t \). Recall in Proposition 6 for

\( \overrightarrow{\gamma} = \left( \overrightarrow{\alpha}, \overrightarrow{\beta} \right) \), (where the notation \( \overrightarrow{\beta} \) means the derivative \( D_{\overrightarrow{\beta}} \) is with respect to the \( y \)-variables), we write \( D_{\overrightarrow{\gamma}} H (t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\overrightarrow{\gamma}^2}{4t}} P_{\overrightarrow{\gamma}} (t, x, y) \), where \( P_{\overrightarrow{\gamma}} (t, x, y) \) is a polynomial in \( D_{\overrightarrow{\gamma}} \left( -\frac{r^2(x,y)}{4t} \right) \) and \( D_{\overrightarrow{\gamma}} U (t, x, y) \big|_{x=y} \), i.e. each summand is of the form

\[
\left( \Pi_{i=1}^n D_{\overrightarrow{\mu_i}} \left( -\frac{r^2(x,y)}{4t} \right) \right) \big|_{x=y} \cdot D_{\overrightarrow{\gamma}} U (t, x, y) \big|_{x=y}.
\]

Using Lemma 5 and the total degree condition \( |\overrightarrow{\gamma}| = 4 \), by the same method in Proposition 6 to find the leading terms in \( P_{\overrightarrow{\gamma}} (t, x, y) \) of order \( \left( \frac{1}{t} \right)^2 \), we see the secondary terms of order \( \left( \frac{1}{t} \right)^2 \) must have \( h = 1 \), \( |\overrightarrow{\mu_i}| = 2 \), and \( |\overrightarrow{\eta}| = 2 \), or \( h = 1 \), \( |\overrightarrow{\mu_i}| = 4 \), and \( \overrightarrow{\eta} = 0 \), namely of the form

\[
D_{\overrightarrow{\mu_i}} \left( -\frac{r^2(x,y)}{4t} \right) \big|_{x=y} \cdot D_{\overrightarrow{\gamma}} U (t, x, y) \big|_{x=y}, \text{ with } |\overrightarrow{\mu_i}| \cup |\overrightarrow{\eta}| = |\overrightarrow{\gamma}|, \ |\overrightarrow{\mu_i}| = |\overrightarrow{\eta}| = 2,
\]

or

\[
D_{\overrightarrow{\gamma}} \left( -\frac{r^2(x,y)}{4t} \right) \big|_{x=y} \cdot U (t, x, y) \big|_{x=y} = -\frac{1}{6t} (R_{ijkl} (x) + R_{iklj} (x)),
\]

where we have used \( \text{Eq.} \). So as \( t \to 0^+ \), we have the refined convergence

\[
(4\pi t)^{n/2} (2t)^2 \nabla_{V_i}^x \nabla_{V_i}^y \nabla_{V_j}^y \nabla_{V_k}^y H (t, x, y) \big|_{x=y} = A \left( (i, l), (j, k) \right) - t \sum_{|\overrightarrow{\mu_i}| = |\overrightarrow{\eta}| = 2, \ |\overrightarrow{\mu_i} \cup \overrightarrow{\eta}| = (i, j, k, l)} \left( D_{\overrightarrow{\mu_i}} \left( r^2 (x, y) \right) \big|_{x=y} \cdot D_{\overrightarrow{\gamma}} U (t, x, y) \big|_{x=y} \right)
\]

\[
- \frac{2}{3} t (R_{ijkl} (x) + R_{iklj} (x)) + O (t^2).
\]

For \( \overrightarrow{\gamma} = \left( (i, l), (j, k) \right) \) and \( \overrightarrow{\gamma} = \left( (i, k), (j, l) \right) \), by considering all partitions of \( \overrightarrow{\gamma} \) into \( \overrightarrow{\mu_i} \cup \overrightarrow{\eta} \) with \( |\overrightarrow{\mu_i}| = 2 \), and \( |\overrightarrow{\eta}| = 2 \), and noticing \( \nabla_{x}^a \nabla_{y}^b f (x, y) \big|_{x=y} = \)
\[ \nabla^a_y \nabla^b_x f(x, y) \big|_{x=y} \] for any indices \( a, b \) and functions \( f \) symmetric to \( x \) and \( y \) (here \( r^2(x, y) \)), we have

\[
\sum_{|\vec{m}|=|\vec{n}|=2, \vec{m} \cup \vec{n} = (i, j, \ell, k)} \left( D^{\vec{m}}(r^2(x, y)) \right)_{x=y} \cdot D^{\vec{n}} U(t, x, y)_{x=y}.
\]

We also have \( A \left( \left( \frac{i}{3}, \frac{j}{3}, \frac{k}{3} \right) \right) = A \left( \left( \frac{i}{3}, \frac{k}{3}, \frac{j}{3} \right) \right) \) by the definition of \( A \left( \frac{\alpha}{3}, \frac{\beta}{3} \right) \). Therefore, from (39) we have

\[
\left[ \nabla_{V_i} \nabla_{V_j} \psi_t, \nabla_{V_k} \nabla_{V_l} \psi_t \right] (x) = \left[ \nabla_{V_i} \nabla_{V_j} \psi_t, \nabla_{V_k} \nabla_{V_l} \psi_t \right] (x).
\]

where in the last four identities we have used the symmetries of the Riemannian curvature tensor.

Remark 18 There are several ways to express the curvature tensors by the eigenfunctions and the heat kernel on \( M \), e.g. the Ricci curvature and scalar curvature in [BBG] and [Gil], but the expression [38] has a clear geometric meaning: the Riemannian curvature tensor \( (M, g) \) is computed from the Gauss map of \( \psi_1(M) \to S^2 \) as \( t \to 0_+ \). One more interesting point is that although our construction of the Riemannian curvature tensor via the embedding \( \psi_1 : M \to S^2 \) looks “extrinsic”, it is actually “intrinsic”, since the embedding \( \psi_1 \) is constructed by the eigenfunctions of \( \Delta g \).

If we only want to prove the existence of the limit in (38), we only need the cancellation of the leading terms \( A \left( \left( \frac{i}{3}, \frac{j}{3}, \frac{k}{3} \right) \right) \) and \( A \left( \left( \frac{i}{3}, \frac{k}{3}, \frac{j}{3} \right) \right) \) in (39), but do not need the symmetries of the Riemannian curvature tensor. On the other hand, (38) manifests the symmetries of the Riemannian curvature tensor in a rather direct way. From (38), it is easy to see \( R(X, Y)Z = -R(Y, X)Z \), \( R(X, Y)Z, W = -R(X, Y)W, Z \), \( R(X, Y)Z, W = R(Z, W)X, Y \), and \( R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \) (the first Bianchi identity). (Note
that switching the order of differentiation in the second derivatives, say from \( \nabla_X \nabla_Z \psi_t \) to \( \nabla_Z \nabla_X \psi_t \), will not affect the limit in (38), as remarked in the beginning of the proof of Proposition 17. With a little computation the second Bianchi identity also follows from our formula of \( R \), as follows:

**Lemma 19 (Second Bianchi Identity)** For any coordinates \( (x^1, \ldots, x^n) \) near \( x \), the Riemannian curvature tensor \( R \) satisfies

\[
\frac{\partial}{\partial x^k} R_{klm} + \frac{\partial}{\partial x^l} R_{klm} + \frac{\partial}{\partial x^m} R_{klm} = 0. \tag{40}
\]

**Proof.** Since all terms on the left side are tensors, we only need to prove the identity when \( (x^1, \ldots, x^n) \) is a normal coordinate near \( x \). From (38) we have

\[
\frac{\partial}{\partial x^h} R_{klij} (x) = \nabla_h \lim_{t \to 0^+} \left[ (\nabla_k \nabla_j \psi_t, \nabla_l \nabla_i \psi_t) - (\nabla_k \nabla_i \psi_t, \nabla_l \nabla_j \psi_t) \right] (x)
\]

\[
= \lim_{t \to 0^+} \left[ (\nabla_h k \nabla j \psi_t, \nabla l i \psi_t) + (\nabla h i \nabla j \psi_t, \nabla k l \psi_t)
- (\nabla h i l \nabla j \psi_t, \nabla k \psi_t) - (\nabla h k j \nabla l \psi_t, \nabla i \psi_t) \right] (x),
\]

where \( \nabla_h k : = \nabla_h \nabla k \nabla j \). Similarly

\[
\frac{\partial}{\partial x^k} R_{lhm} (x) = \lim_{t \to 0^+} \left[ (\nabla_k \nabla_h \psi_t, \nabla_l \nabla i \psi_t) + (\nabla_k \nabla i \psi_t, \nabla_l \nabla h \psi_t) \right.
- (\nabla_k \nabla i l \psi_t, \nabla h \psi_t) - (\nabla_k \nabla l i \psi_t, \nabla h \psi_t) \left. \right] (x),
\]

\[
\frac{\partial}{\partial x^l} R_{hki} (x) = \lim_{t \to 0^+} \left[ (\nabla_l \nabla_h \psi_t, \nabla k \nabla i \psi_t) + (\nabla l \nabla k \psi_t, \nabla i \nabla h \psi_t) \right.
- (\nabla l \nabla k i \psi_t, \nabla h \psi_t) - (\nabla l \nabla h k \psi_t, \nabla i \psi_t) \left. \right] (x).
\]

Adding the 3 identities together, (40) follows. There is a convergence issue of limits like \( \lim_{t \to 0^+} \langle \nabla h k j \psi_t, \nabla l i \psi_t \rangle = O \left( \frac{1}{t} \right) \) from Proposition 11 but writing

\[
\frac{\partial}{\partial x^h} R_{klij} + \frac{\partial}{\partial x^l} R_{klij} + \frac{\partial}{\partial x^j} R_{klij} = \frac{1}{2t} \left[ 2t \left( \frac{\partial}{\partial x^h} R_{klij} + \frac{\partial}{\partial x^k} R_{lhij} + \frac{\partial}{\partial x^l} R_{hki} \right) \right]
\]

and applying Proposition 11 easily fix that.

**3.3 Approximation of Riemannian geometry by eigenfunctions**

For the embedding map \( \psi_t : (M, g) \to L^2 \), we can truncate the infinite dimensional \( L^2 \) to \( \mathbb{R}^q \) by taking its first \( q \) components, and denote the resulted map by \( \psi_t^q \). The following truncation lemma in [WZ] gives the bound of \( q \) that can preserve most good properties of \( \psi_t \). It was proved by the Weyl’s asymptotic formula of eigenvalues (see [Ch]) and \( C^0 \)-estimates of eigenfunctions (and their derivatives) on compact Riemannian manifolds.
Lemma 20 (WZ Theorem 6 and Remark 7) Let $\rho > 0$ be a fixed small constant. For integers $q = q(t)$ of the order $t^{-(n^2 + \rho)}$ as $t \to 0_+$, the truncated heat kernel mapping

$$\psi^q(t) : (M, g) \to \mathbb{R}^{q(t)}$$

still satisfies the asymptote

$$\left(\psi^q(t)\right)^* g_0 = g + \frac{t}{3} \left(\frac{1}{2} S_g \cdot g - Ric_g\right) + O(t^2),$$

where $g_0$ is the standard metric in $l^2$. The truncation order $q(t) \sim t^{-(n^2 + \rho)}$ is also valid to preserve all high-jet relations in Theorem 3 for higher derivatives of $\psi_t$.

Using the above truncation lemma, we can approximate the metric $g$, the Levi-Civita connection $\nabla$, and the Riemannian curvature $R$ by $q(t)$, respectively, with the error of certain orders of $t$. This suggests the possibility of approximating the Riemannian geometry of $(M, g)$ by finitely many eigenfunctions. Indeed, if the Schwartz function $\psi$ is chosen with compact support, Theorem 1.6 of [Ni] gives one of such approximations.

It will be useful to get a precise control of the coefficients in the above remainder $O(t^2)$ and their higher derivative counterparts, so we know how large $q(t)$ (i.e. how small $t$) should be in order to yield a given accuracy.

## 4 Algebraic structures in the $\infty$-jet space of $\Phi_t(x)$

In Section 2 we have obtained the angle and length relations of all derivative vectors $D_{\alpha} \Phi_t(x)$ as $t \to 0_+$. These relations are independent on $g$, $x$ and the choice of orthonormal basis, so it is worthwhile to investigate them more closely. The relations are expressed by the constants $A\left(\alpha, \beta\right)$ and $B\left(\alpha, \beta\right)$. In this section we explore more relations between these constants, or in other words study the $\infty$-jet space of $\Phi_t(x)$ as $t \to 0_+$.

### 4.1 Inductive relations of $A\left(\alpha, \beta\right)$ and $B\left(\alpha, \beta\right)$

It turns out the constants $A\left(\alpha, \beta\right)$ and $B\left(\alpha, \beta\right)$ involved in the high-jet relations in Theorem 3 can be defined inductively. Suppose $j_r \in \{1, 2, \ldots, n\}$, but not necessarily in $S = \{\alpha\} \cup \{\beta\}$. Let $\alpha_+ = (\alpha, j_r)$, $\beta_+ = (\beta, j_r)$, $\alpha_{+k} = \left(\alpha, j_r, \ldots, j_\text{k}\right)$. If $j_r \in \beta$, we let $\beta_- = \beta \setminus \{j_r\}$.
is easy to check $A\left(\vec{\alpha}, \vec{\beta}\right)$ satisfies the following properties:

Normalization: \[ A\left(\emptyset, \emptyset\right) = 1, \text{ (where } \emptyset \text{ is the empty set)} \] (42)

Symmetry: \[ A\left(\vec{\alpha}, \vec{\beta}\right) = A\left(\vec{\beta}, \vec{\alpha}\right), \] (43)

Leibniz rule: \[ A\left(\vec{\alpha}_+, \vec{\beta}_-\right) = -A\left(\vec{\alpha}_+, \vec{\beta}\right), \] (44)

Adding index: \[ A\left(\vec{\alpha}_+, \vec{\beta}_+\right) = A\left(\vec{\alpha}, \vec{\beta}\right) (a_r + b_r + 1). \] (45)

These properties completely characterize $A\left(\vec{\alpha}, \vec{\beta}\right)$. Indeed, to obtain a general $A\left(\vec{\alpha}, \vec{\beta}\right)$, we can start from $\vec{\alpha} = \vec{\beta} = \emptyset$, use (45) to obtain $A\left(\vec{\alpha}_{+k}, \vec{\beta}_{+k}\right)$, and then use (44) to obtain $A\left(\vec{\alpha}_{+(k+l)}, \vec{\beta}_{+(k+l)}\right)$ for $0 \leq l \leq k$. Then we can use (45) to add different indices $j$ to $\left(\vec{\alpha}, \vec{\beta}\right)$ to get all $A\left(\vec{\alpha}, \vec{\beta}\right)$ (note if $j' \notin \{\vec{\alpha}\} \Pi \{\vec{\beta}\}$, then $A\left(\vec{\alpha}_{+j'}, \vec{\beta}_{+j'}\right) = A\left(\vec{\alpha}, \vec{\beta}\right)$ by (45)). Similarly, $B\left(\vec{\alpha}, \vec{\beta}\right)$ is characterized by

\[
B\left(\emptyset, \emptyset\right) = 1, \quad B\left(\vec{\alpha}, \vec{\beta}\right) = B\left(\vec{\beta}, \vec{\alpha}\right),
\]

\[
B\left(\vec{\alpha}_+, \vec{\beta}_-\right) = -B\left(\vec{\alpha}, \vec{\beta}\right) \left(\frac{2b_r - 1}{2a_r + 1}\right)^{1/2},
\]

\[
B\left(\vec{\alpha}_+, \vec{\beta}_+\right) = B\left(\vec{\alpha}, \vec{\beta}\right) \frac{a_r + b_r + 1}{(2a_r + 1)(2b_r + 1)^{1/2}}.
\] (46)

### 4.2 Stabilization property from repeated differentials

For some index $j_r \in \{1, 2, \cdots, n\}$ (but not necessarily in $\{\vec{\alpha}\} \Pi \{\vec{\beta}\}$), if we apply $\partial_{j_r}$ repeatedly to $D^{\vec{\alpha}} \Phi_t(x)$ and $D^{\vec{\beta}} \Phi_t(x)$, then the the resulted derivative vectors become “more and more collinear”, and the angle between them has certain stabilization property. More precisely, we have the following

**Proposition 21** \(B\left(\vec{\alpha}_+, \vec{\beta}_+\right)\) and \(B\left(\vec{\alpha}, \vec{\beta}\right)\) always have the same sign, and

\[
\left|B\left(\vec{\alpha}_+, \vec{\beta}_+\right)\right| \geq \left|B\left(\vec{\alpha}, \vec{\beta}\right)\right|,
\] (47)

where the equality holds if and only if $j_r$ appears with the same multiplicity in $\vec{\alpha}$ and $\vec{\beta}$. Further more we have the following limit: if we let the multi-index

\[
\vec{\alpha}_{+k} = \left(\vec{\alpha}, j_r, \cdots, j_r\right)_k, \text{ similarly for } \vec{\beta}_{+k}, \text{ then}
\]

\[
\lim_{k \to \infty} B\left(\vec{\alpha}_{+k}, \vec{\beta}_{+k}\right) = B\left(\vec{\alpha}_+, \vec{\beta}_+\right),
\] (48)
where the multi-indices $\alpha_\ast$ and $\beta_\ast$ are obtained from deleting all $j_r$ indices in $\alpha$ and $\beta$ respectively. We also have

$$\lim_{k \to \infty} B \left( \alpha, \beta_{+k} \right) := 0. \quad (49)$$

**Proof.** Since $|\alpha_\ast| - |\beta_\ast| = |\alpha| - |\beta|$, by definition $B \left( \alpha_\ast, \beta_{+} \right)$ and $B \left( \alpha, \beta_\ast \right)$ always have the same sign. Let us assume $B \left( \alpha, \beta_\ast \right) \geq 0$ (the $B \left( \alpha, \beta_\ast \right) \leq 0$ case is similar). From (46) we have

$$B \left( \alpha_\ast, \beta_{+} \right) = B \left( \alpha_\ast, \beta_\ast \right) \cdot \frac{(a_r + b_r + 1)^2}{(2a_r + 1)(2b_r + 1)} \cdot \frac{(2a_r - b_r)^2}{(2a_r + 1)(2b_r + 1)}^{1/2} \geq B \left( \alpha_\ast, \beta_\ast \right),$$

and the equality holds if and only if $a_r = b_r$. Recall

$$B \left( \alpha, \beta_\ast \right) = \left( -1 \right)^{\frac{|\alpha| - |\beta|}{2}} \prod_{r=1}^{\sigma} \frac{(2\sigma_r)!}{\sigma_r!} \frac{(2a_r)!}{a_r!} \frac{(2b_r)!}{b_r!} \left( \frac{a_r}{b_r} \right)^{\sigma_r}. \quad (50)$$

Since $\sigma_r = \frac{a_r + b_r}{2}$, if we let $l_r = \frac{b_r - a_r}{2} \geq 0$ (assuming $b_r \geq a_r$), then

$$\frac{a_r! b_r!}{\sigma_r! \sigma_r!} = \frac{(\sigma_r + 1) \cdots (\sigma_r + l_r)}{(\sigma_r - 1) \cdots (\sigma_r - l_r)} \quad \text{and} \quad \frac{(2a_r)!}{(2a_r)!} \frac{(2b_r)!}{(2b_r)!} = \frac{(2\sigma_r + 1) \cdots (2\sigma_r + 2l_r)}{(2\sigma_r - 1) \cdots (2\sigma_r - 2l_r)}.$$

So the $r$-th factor in $B \left( \alpha, \beta_\ast \right)$ is

$$\frac{(2\sigma_r)!}{\sigma_r!} \frac{(2a_r)!}{a_r!} \frac{(2b_r)!}{b_r!} \left( \frac{a_r}{b_r} \right)^{\sigma_r} \leq 1.$$

As $k \to \infty$, in the multi-index $(\alpha_{+k}, \beta_{+k})$, $\sigma_r$ is changed to $\sigma_r + k$, but $l_r$ is unchanged, so the factor involving $j_r$ in $B \left( \alpha_{+k}, \beta_{+k} \right)$ becomes

$$\frac{(2\sigma_r + 2k - 1) \cdots (2\sigma_r + 2k - 3)}{(2\sigma_r + 2k + 1) \cdots (2\sigma_r + 2k + 3)} \to 1$$

as $k \to \infty$, and the other factors not involving $j_r$ are unchanged. This gives the limit $B \left( \alpha_\ast, \beta_\ast \right)$.
For $|\alpha| + |\beta|$ is even (otherwise we can replace $\beta$ by $\beta_+$ before we take limit). In the multi-index $(\alpha, \beta_{+2k})$, $\sigma$ and $l_r$ are changed to $\sigma_r + k$ and $l_r + k$ respectively, so the factor involving $j_r$ in $B(\alpha, \beta_{+2k})$ becomes

$$\left[ \frac{(2\sigma_r + 2k - 1)(2\sigma_r + 2k - 3)\cdots(2\sigma_r - 2l_r + 1)}{(2\sigma_r + 2k + 1)(2\sigma_r + 2k + 3)\cdots(2\sigma_r + 4k + 2l_r - 1)} \right]^{1/2} \leq \left[ \frac{2\sigma_r - 2l_r + 1}{2\sigma_r + 4k + 2l_r - 1} \right]^{1/2} \to 0$$

as $k \to \infty$. Therefore $B(\alpha, \beta_{+2k}) \to 0$. By our definition $B(\alpha, \beta_{+(2k+1)}) = 0$. Combining the two cases, $\lim_{k \to \infty} B(\alpha, \beta_{+k}) = 0$. 

**Corollary 22** $-1 < B(\alpha, \beta) \leq 1$, and $B(\alpha, \beta) = 1$ if and only if $\alpha = \beta$.

**Proof.** From (50) we see for any multi-indices $\alpha$ and $\beta$,

$$\left| B(\alpha, \beta) \right| \leq 1. \quad (51)$$

If $B(\alpha, \beta) = 1$ but $\alpha \neq \beta$, then there must be some index $j_r$ appearing with different multiplicities in $\alpha$ and $\beta$. We can construct $\alpha_+^j$ and $\beta_+^j$ by adding the index $j_r$. By (47) (notice the equality can not hold by the condition on $j_r$), we have

$$\left| B(\alpha_+^j, \beta_+^j) \right| > \left| B(\alpha, \beta) \right| = 1,$$

contradicting with (51). Therefore $\alpha = \beta$.

We claim $B(\alpha, \beta) \neq -1$. Otherwise, by the sign of $B(\alpha, \beta)$, we see $|\beta - \alpha|/2$ must be odd, especially $\alpha \neq \beta$. Therefore there is some $l_r \neq 0$. By (50), this makes the $r$-th factor in $B(\alpha, \beta)$ have absolute value strictly less than 1, so $\left| B(\alpha, \beta) \right| < 1$. But this contradicts with $\left| B(\alpha, \beta) \right| = |\beta| = 1$.

**4.3 Lattice geometry on $\mathbb{Z}_+^n$**

In this subsection, we use the high-jet relations of $\Phi_t$ in Theorem 3 to put a metric $d$ on the lattice space

$$\mathbb{Z}_+^n = \{(m_1, \cdots, m_n) | m_j \in \mathbb{Z}, m_j \geq 0 \text{ for } 1 \leq j \leq n\}.$$

The motivation is to visualize these relations by the geometry of $(\mathbb{Z}_+^n, d)$.

For any multi-index $\alpha$ with indices in $\{1, 2, \cdots, n\}$, taking the multiplicity of its each index, we obtain a point $z \in \mathbb{Z}_+^n$; conversely we can use $z$ to recover
\( \vec{\alpha} \). For example, if we let \( z_1, z_2, z_3 \) be the three axes of \( \mathbb{Z}_+^3 \), then the multi-index \( \vec{\alpha} = (1, 1, 2, 3) \) has 1 with multiplicity 2, 2 with multiplicity 1, and 3 with multiplicity 1. So \( \vec{\alpha} \) correspond to \( z = (2, 1, 1) \in \mathbb{Z}_+^3 \). With this identification, we have

**Definition 23** (Angle distance) For any two multi-indices \( \vec{\alpha} \) and \( \vec{\beta} \) in \( \mathbb{Z}_+^n \), we define their angle distance to be

\[
d\left(\vec{\alpha}, \vec{\beta}\right) = \cos^{-1}\left(B\left(\vec{\alpha}, \vec{\beta}\right)\right).
\] (52)

By Theorem 3, \( d\left(\vec{\alpha}, \vec{\beta}\right) \) is the limiting distance of \( \frac{D_{\vec{x}}\varphi_t(x)}{D_{\vec{x}}\varphi_t(x)} \) and \( \frac{D_{\vec{y}}\varphi_t(x)}{D_{\vec{y}}\varphi_t(x)} \) on the unit sphere \( S^\infty \subset \ell^2 \) as \( t \to 0_+ \). Let \( \theta_t\left(\vec{\alpha}, \vec{\beta}\right) \) be the minimal distance of \( \frac{D_{\vec{x}}\varphi_t(x)}{D_{\vec{x}}\varphi_t(x)} \) and \( \frac{D_{\vec{y}}\varphi_t(x)}{D_{\vec{y}}\varphi_t(x)} \) on \( S^\infty \), then \( 0 \leq \theta_t\left(\vec{\alpha}, \vec{\beta}\right) \leq \pi \), and

\[
\lim_{t \to 0_+} \theta_t\left(\vec{\alpha}, \vec{\beta}\right) = d\left(\vec{\alpha}, \vec{\beta}\right).
\] (53)

**Lemma 24** The angle distance \( d \) defined in (52) is a metric on \( \mathbb{Z}_+^n \). For any \( \vec{\alpha} \) and \( \vec{\beta} \) in \( \mathbb{Z}_+^n \), \( 0 \leq d\left(\vec{\alpha}, \vec{\beta}\right) < \pi \).

**Proof.** To prove \( d \) is a metric, we only need to verify that it is nondegenerate, since the triangle inequality of \( \theta_t\left(\vec{\alpha}, \vec{\beta}\right) \) on \( S^\infty \) is preserved when we take the limit in (53) as \( t \to 0_+ \), and \( d\left(\vec{\alpha}, \vec{\beta}\right) = d\left(\vec{\beta}, \vec{\alpha}\right) \) follows from the definition of \( B\left(\vec{\alpha}, \vec{\beta}\right) \). Suppose \( d\left(\vec{\alpha}, \vec{\beta}\right) = 0 \), then \( B\left(\vec{\alpha}, \vec{\beta}\right) = 1 \). From Corollary 22 we have \( \vec{\alpha} = \vec{\beta} \). By Corollary 22 \( B\left(\vec{\alpha}, \vec{\beta}\right) \in (-1, 1] \), so \( 0 \leq d\left(\vec{\alpha}, \vec{\beta}\right) < \pi \).

We can interpret the stabilization property in (48) of Proposition 21 in terms of the distance \( d \) on \( \mathbb{Z}_+^n \):

**Lemma 25** (Stabilization) For any coordinate vector \( v = \frac{\partial}{\partial z^j} \), and any two vectors \( \vec{\alpha}, \vec{\beta} \in \mathbb{Z}_+^n \), we have

\[
\left| d\left(\vec{\alpha} + v, \vec{\beta} + v\right) - \frac{\pi}{2}\right| \geq \left| d\left(\vec{\alpha}, \vec{\beta}\right) - \frac{\pi}{2}\right|,
\]

\[
\lim_{k \to \infty} d\left(\vec{\alpha} + kv, \vec{\beta} + kv\right) = d\left(\vec{\alpha}_*, \vec{\beta}_*\right),
\]

\[
\lim_{k \to \infty} d\left(\vec{\alpha}, \vec{\beta} + kv\right) = \frac{\pi}{2},
\]

where \( \vec{\alpha}_* \) and \( \vec{\beta}_* \) are obtained by setting the \( j \)-th component of \( \vec{\alpha} \) and \( \vec{\beta} \) to be zero.
The stabilization properties in the above lemma make each axis of \((\mathbb{Z}_+^n, d)\) somehow looks like a real projective line. It will be interesting to say more about the geometry of \((\mathbb{Z}_+^n, d)\) as \(|\mathbf{a}| \to \infty\).

In the following we are interested in the orthogonal relations of \(D\mathbf{a} \Phi_t(x)\) as \(t \to 0_+\). From our definition, \(B \left( \mathbf{a}, \mathbf{b} \right) = 0\) if and only if some index appears with odd multiplicity in \(\{ \mathbf{a} \} \cup \{ \mathbf{b} \}\). This can be reformulated as

**Lemma 26 (Orthogonal relations)** \(d \left( \mathbf{a}, \mathbf{b} \right) = \frac{\pi}{2}\) if and only if \(\mathbf{a} - \mathbf{b} \notin (2\mathbb{Z})^n\). Consequently,

1. Any two adjacent lattice points in \(\mathbb{Z}_+^n\) have the angle distance \(\frac{\pi}{2}\);
2. \(\mathbb{Z}_+^n / (2\mathbb{Z}_+)^n\) has \(2^n\) “orthogonal” cosets \(A_1, \cdots, A_{2^n} \subset \mathbb{Z}_+^n\), such that for any two elements in different cosets, their angle distance is \(\frac{\pi}{2}\).

This \(\mathbb{Z}_+^n / (2\mathbb{Z}_+)^n\)-grading of the \(\infty\)-jet space of \(\Phi_t\) as \(t \to 0\) may have future applications.

In the following we give some estimate of \(d \left( \mathbf{a}, \mathbf{b} \right)\). Let \(d_0\) be the “set distance” function on \(\mathbb{Z}_+^n\): for \(\mathbf{a} = (a_1, \cdots, a_n)\) and \(\mathbf{b} = (b_1, \cdots, b_n)\),

\[
d_0 \left( \mathbf{a}, \mathbf{b} \right) = \sum_{j=1}^{n} |a_j - b_j| = |\mathbf{a} \triangle \mathbf{b}|,
\]

where \(\triangle\) is the difference of two sets \(A\) and \(B\): \(A \triangle B = (A \setminus B) \cup (B \setminus A)\).

Intuitively, if \(d_0 \left( \mathbf{a}, \mathbf{b} \right)\) is very large relative to \(|\mathbf{a}| + |\mathbf{b}|\), then \(\mathbf{a}\) and \(\mathbf{b}\) are very different, and we expect the angle \(d \left( \mathbf{a}, \mathbf{b} \right)\) between the derivative vectors \(D\mathbf{a} \Phi_t(x)\) and \(D\mathbf{b} \Phi_t(x)\) to be not too close to 0 or \(\pi\) as \(t \to 0_+\). Indeed, we have

**Proposition 27 (Distance comparison)** There exists a constant \(\delta > 0\), such that

\[
\left| \cos d \left( \mathbf{a}, \mathbf{b} \right) \right| \leq 1 - \delta \frac{|\mathbf{a} \triangle \mathbf{b}|}{|\mathbf{a}| + |\mathbf{b}|},
\]

for any \(\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^n\).

**Proof.** Inequality (54) is equivalent to the following combinatorial inequality

\[
\prod_{r=1}^{n} \frac{(2\sigma_r - 1)(2\sigma_r - 3) \cdots (2\sigma_r - 2l_r + 1)}{(2\sigma_r + 1)(2\sigma_r + 3) \cdots (2\sigma_r + 2l_r - 1)} \leq 1 - \frac{\delta \sum_{r=1}^{n} l_r}{\sum_{r=1}^{n} \sigma_r},
\]

(55)
where $\sigma_r \geq l_r \geq 0$ are integers. We first prove (55) when $s = 1$. Let $f(x) = \ln \left( \frac{1-x}{1+x} \right)$, then $f''(x) \leq 0$ on $[0, 1)$. Therefore

\[
\frac{f \left( \frac{1}{2\sigma_r} \right) + \cdots + f \left( \frac{2l_r - 1}{2\sigma_r} \right)}{l_r} \leq \frac{f \left( \frac{1}{2\sigma_r} + \cdots + \frac{2l_r - 1}{2\sigma_r} \right)}{l_r} = f \left( \frac{l_r}{2\sigma_r} \right), \quad \text{i.e.}
\]

\[
\left( \frac{1 - \frac{1}{2\sigma_r}}{1 + \frac{1}{2\sigma_r}} \right) \cdots \left( \frac{1 - \frac{2l_r - 1}{2\sigma_r}}{1 + \frac{2l_r - 1}{2\sigma_r}} \right) \leq \left( \frac{1 - \frac{l_r}{\sigma_r}}{1 + \frac{l_r}{\sigma_r}} \right)^{l_r} \leq 1 - \frac{l_r}{2\sigma_r} \leq 1 - \frac{l_r}{2\sigma_r}. \quad (56)
\]

For $s > 1$, using (56) we have

\[
\Pi_{r=1}^{s} \left( \frac{(2\sigma_r - 1)(2\sigma_r - 3) \cdots (2\sigma_r - 2l_r + 1)}{(2\sigma_r + 1)(2\sigma_r + 3) \cdots (2\sigma_r + 2l_r - 1)} \right) 
\leq \Pi_{r=1}^{s} \left( 1 - \frac{l_r}{2\sigma_s} \right) \leq \left( \frac{\sum_{r=1}^{s} \left( 1 - \frac{l_r}{\sigma_r} \right)}{s} \right)^{s} \quad \text{(arithmetic-geometric inequality)}
\]

\[
= \left( 1 - \frac{1}{2s} \frac{\sum_{r=1}^{s} l_r}{\sigma_r} \right)^{s} \leq \left( 1 - \frac{1}{2s} \left( \frac{\sum_{r=1}^{s} l_r}{\sum_{r=1}^{s} \sigma_r} \right) \right)^{s} \quad \text{(since $\frac{a}{b} + \frac{c}{d} \geq \frac{a+c}{b+d}$ for $a, b, c, d > 0$)}
\]

\[
\leq 1 - \delta \frac{\sum_{r=1}^{s} l_r}{\sum_{r=1}^{s} \sigma_r},
\]

where the last inequality is because there exists a $\delta > 0$ such that $(1 - \frac{1}{2\sigma x})^s \leq 1 - \delta x$ on $[0, 1]$. \[\Box\]

Inequality (54) gives a quantitative control of the linear independence of the derivative vectors $D^c_\alpha \Phi_t(x)$ and $D^c_\beta \Phi_t(x)$ as $t \to 0^+$. When $|\alpha| = |\beta| = 2$, the uniform linear independence of $D^c_\alpha \Phi_t(x)$ and $D^c_\beta \Phi_t(x)$ as $t \to 0^+$ played an important role in [WZ] to construct isometric embeddings by the implicit function theorem. It will be interesting to see if there are other geometric problems where the higher-jet linear independence property can play role.

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