String duality transformations in $f(R)$ gravity from Noether symmetry approach

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Abstract. We select $f(R)$ gravity models that undergo scale factor duality transformations. As a starting point, we consider the tree-level effective gravitational action of bosonic String Theory coupled with the dilaton field. This theory inherits the Busher’s duality of its parent String Theory. Using conformal transformations of the metric tensor, it is possible to map the tree-level dilaton-graviton string effective action into $f(R)$ gravity, relating the dilaton field to the Ricci scalar curvature. Furthermore, the duality can be framed under the standard of Noether symmetries and exact cosmological solutions are derived. Using suitable changes of variables, the string-based $f(R)$ Lagrangians are shown in cases where the duality transformation becomes a parity inversion.

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1 Introduction

In the last thirty years, several shortcomings came out in General Relativity (GR), essentially related to the ultraviolet and infrared behaviors of the theory. In other words, while the theory works very well at intermediate scales (Solar System, up to Galactic scales), it shows problems at quantum and cosmological scales. In the first case, the lack of a Quantum Gravity Theory means that GR cannot be dealt under the standard of the other fundamental theories. In the other case, cosmic dynamics is addressed if huge amounts of dark energy and dark matter are assumed as sources in the Einstein field equations. The shortcoming consists in the fact that such exotic ingredients have not been detected at fundamental level up to now, despite of the fact that their effects are dramatically present at astrophysical and cosmological scales.

A change of perspective in this state of art consists to resort to alternative theories of gravity considered as semi-classical schemes capable of addressing, at least in part, the above mentioned problems and of retaining the positive results of GR. Among these alternatives, one of the most fruitful approach has been the one called Extended Theories of Gravity (ETGs) based on corrections and enlargements of GR [1–4]. The paradigm consists, essentially, in adding minimally and/or non-minimally coupled scalar fields or higher-order curvature invariants into the dynamics. The reason of these additional terms is that any quantum field theory formulated on curved space gives rise to this kind of corrections [5].

Furthermore, any unification scheme, as Superstring, Supergravity or Grand Unified Theories (GUT), takes in account effective actions where non-minimal couplings or higher-order terms in the curvature invariants are present. Specifically, we have to consider terms in the Einstein-Hilbert Lagrangian coming from one-loop or higher-loop corrections in the quantum regime [5–7]. In this framework, the so called $f(R)$ gravity theories [2] are the first straightforward extension of GR as soon as the condition $f(R) \neq R$ is requested.

The main criticism to this approach is that ETGs can be somehow phenomenological and adapted to particular issues coming, e.g., from cosmology or astrophysics (see [8] for a review). However, specific features can be selected in order to match these models with...
some fundamental theory. In particular, \( f(R) \) gravity models showing duality invariance could be useful to relate ETGs with Superstring Theory in the low-energy limit [9–17]. An important output of this approach could be the geometrical interpretation of the dilaton which is a specific feature of String Theory. In general, looking for fundamental symmetries for effective models is a sort of \textit{reconstruction procedure} useful to recover fundamental theories from phenomenology.

The aim of this paper is to show that string-dilaton gravity and \( f(R) \) gravity are related by conformal transformations and that scale factor duality can be recovered for some \( f(R) \) gravity models. This renders possible to select physical \( f(R) \) models at fundamental level.

Another important issue addressed in this paper is related to the fact that duality can be derived from Noether symmetries \cite{18, 19}. In this sense, duality could be considered in relation to conserved quantities emerging in physical theories.

The outline of the paper is as follows. In section 2, the main features of Busher’s duality in bosonic String Theory are summarized. The tree-level effective bosonic String Theory inherits the Busher’s duality transformations of its parent String Theory and holds \( O(n, n) \) symmetry \cite{20}. Section 3 is devoted to \( f(R) \) gravity which is mapped into the tree-level dilaton-graviton string effective action, with the dilaton field and potential being related to the \( f(R) \) functions. Scale factor duality in \( f(R) \) cosmology is discussed in section 4. In particular, by means of Noether symmetries we derive \( f(R) \) models for which the invariance under the duality transformation \( a \to 1/a \) holds for the cosmic scale factor \( a(t) \) at the level of the Lagrangian. We show that cosmological solutions can be derived for such \( f(R) \) models, where Noether symmetries allow the reduction and the exact integration of the dynamical system. In section 5, we draw conclusions and sketch the perspectives. A summary of the Noether Symmetry Approach is reported in the appendix A, giving particular emphasis to \( f(R) \) gravity models.

2 Duality in string theory and its cosmological consequences

Let us start our considerations taking into account duality in String Theory. It is well known that certain classes of two dimensional sigma model exhibits a duality symmetry. It can be shown that two distinct actions, having two geometrically inequivalent target spaces, describe the same physics. In fact, let us consider a two-dimensional sigma model embedded in a \((n + 1)\) Lorentzian Manifold \( \{ \mathcal{M}, G \} \). The action is \cite{21}:

\[
S = \frac{1}{4\pi\alpha'} \int d^2\xi \left( \sqrt{h} h^{\mu\nu} G_{ab}(\partial_\mu X^a)(\partial_\nu X^b) + \epsilon^{\mu\nu} B_{ab}(\partial_\mu X^a)(\partial_\nu X^b) + \alpha' \sqrt{h} R^{(2)}(\phi(X^a)) \right), \tag{2.1}
\]

where \( \alpha' \) is the Regge slope parameter, \( X^a \) are the coordinates on the target space, \( h \) is the absolute value of the determinant of the two-dimensional metric \( h_{\mu\nu} \) on the world-sheet, \( \epsilon_{\mu\nu} \) is the two-dimensional Levi-Civita tensor density, \( B_{ab} \) is an antisymmetric field, \( R^{(2)} \) is the Ricci scalar curvature on the world-sheet, \( \phi(X^a) \) is the dilaton field. Being \( \mathcal{L} \) the Lagrangian in the big parentheses under the sign of integral of eq. (2.1), in order to define the dual Lagrangian \( \mathcal{L} \), there must exist a vector field \( X \) which is an isometry for \( \mathcal{L} \) \cite{22, 23}, that is \( L_X \mathcal{L} = 0 \), where \( L_X \) is the Lie derivative of \( \mathcal{L} \) along \( X \). In the case \( X = \frac{\partial}{\partial X^a} \), the dual action is:

\[
\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2\xi \left( \sqrt{\tilde{h}} \tilde{h}^{\mu\nu} \tilde{G}_{ab}(\partial_\mu \tilde{X}^a)(\partial_\nu \tilde{X}^b) + \epsilon^{\mu\nu} \tilde{B}_{ab}(\partial_\mu \tilde{X}^a)(\partial_\nu \tilde{X}^b) + \alpha' \sqrt{\tilde{h}} R^{(2)}(\phi(X^a)) \right), \tag{2.2}
\]
where
\[ \tilde{G}_{00} = 1/G_{00}, \quad \tilde{G}_{0i} = B_{0i}/G_{00}, \quad \tilde{G}_{ij} = G_{ij} - (G_{0i}G_{0j} - B_{0i}B_{0j})/G_{00}, \quad i, j = 1, \ldots, n, \] (2.3)
and
\[ \tilde{B}_{0i} = -B_{0i} = G_{0i}/G_{00}, \quad \tilde{B}_{ij} = B_{ij} + (G_{0i}B_{0j} - B_{0i}G_{0j})/G_{00}, \quad i, j = 1, \ldots, n. \] (2.4)
Eqs. (2.3) and (2.4) are the so-called Busher’s duality relations for this particular isometry [21, 24].

The dual action is dynamically equivalent to the original one at classical level. At quantum level, the conformal invariance of the model in eq. (2.1), at one loop, implies the imposition of the following conditions [10]:
\[ \frac{1}{a^2} \left( \frac{n - 25}{48\pi^2} \right) + \frac{1}{16\pi^2} \left( 4(\nabla\phi)^2 - 4\nabla^2\phi - R + \frac{1}{12}H^2 \right) = 0, \]
\[ R_{ab} + 2\nabla_a\nabla_b\phi - \frac{1}{4}H_a^{cd}H_{bcd} = 0, \quad \nabla^c H_{cab} - 2(\nabla^c\phi)H_{cab} = 0. \] (2.5)
where \( \nabla \) is the covariant derivative on the target space, \( H_{abc} = \partial_aB_{bc} + \partial_bB_{ca} + \partial_cB_{ab} \), and \( H^2 = H_{abc}H^{abc} \). The eqs. (2.5) can be derived by the Euler-Lagrange equations of the following (effective) action:
\[ S^{n+1} = K \int d^{n+1}x \sqrt{-G} e^{-2\phi} \left( R + 4\nabla_a\phi\nabla^a\phi - \frac{1}{12}H^2 + \Lambda \right), \] (2.6)
where \( K \) is a coupling constant, \( n \) is the number of the spatial dimensions, and the cosmological constant term is given by \( \Lambda = \frac{25-n}{3\alpha^2} \), which is equal to 0 if and only if \( n = 25 \) (the critical dimension for bosonic String Theory).

If the original model (2.1) satisfies the one-loop conformal eqs. (2.5), then so the dualized model (2.2), provided that the dilaton field shifts in the following way [21]:
\[ \phi \rightarrow \tilde{\phi} = \phi - \frac{1}{2} \ln \left( g_{00} \right). \] (2.7)
In case the Lagrangian \( \mathcal{L} \) has \( d \)-Abelian independent spatial isometries, then the invariance group of the action (2.1) and its dual (2.2) is the group \( O(d, d) \) [25–28]. It has been proved [20] that, in case the moduli fields \( G_{ab} \) and \( B_{ab} \) depend only on time \( t \), the effective action (2.6) is \( O(n, n) \) invariant. In ref. [27], it has been shown that if the following spatially flat, homogeneous and isotropic metric (in \( n + 1 \) dimensions)
\[ ds^2 = dt^2 - a^2(t)dx_i^2, \] (2.8)
where \( a(t) \) is the scale factor, is a solution of the field equations, then also the metric with \( a(t) \rightarrow a^{-1}(\pm t) \) is a solution. The effective gravitational action (2.6) exhibits the following duality symmetric transformations among the solutions of its Euler-Lagrange equations (which are nothing else but an application of the Busher’s duality relations in the case of a symmetry made out by \( d \)-Abelian spatial vector fields \( \frac{\partial}{\partial x^i} \))
\[ a \rightarrow \tilde{a} = a^{-1}, \quad \phi \rightarrow \tilde{\phi} = \phi - d\ln(a). \] (2.9)
This is the long-short duality correspondence of the scale factor \( a \) for the string-dilaton cosmology. The duality relations (2.9), between the solutions of the Euler-Lagrange equations derived from the action (2.6), allow to construct the so-called Pre-Big Bang cosmological models [27].
Let us consider now the tree-level dilaton-graviton string effective action

\[ S = \int d^Dx \sqrt{-G} e^{-2\phi} [R + 4\nabla_\mu \phi \nabla^\mu \phi + \Lambda], \]  

(2.10)

that can be achieved from the above results, in the low-energy limit, retaining only the scalar mode (the dilaton) and the tensor mode (the graviton) [9–12, 15, 16, 27, 29, 30]. Here \( D = n + 1 \) is the total number of dimensions, and \( G \) indicates the determinant of the \( D \)-dimensional spacetime metric. In this sense, the effective string action reduce to a scalar-tensor theory (Brans-Dicke-like) [31].

Starting from the action \((2.10)\) we can calculate the field equations for the gravitational field by varying the action with respect to the metric tensor, and the field equations for the dilaton field by varying the action with respect to the dilaton. Since we are interested in cosmological solutions describing the observable universe, we shall consider the case where the space-time has \( D = 4 \) dimensions. In this case, strings are not in critical dimensions and \( \Lambda \neq 0 \). In \( 4D \), we indicate, as usual, the metric tensor with \( g_{\mu\nu} \) and its determinant with \( g \).

So, by varying the action in eq. \((2.10)\) with respect to the metric tensor and the dilaton field we get, respectively, the field equations

\[ G_{\mu \nu} = \frac{1}{2} \Lambda g_{\mu \nu} + 2g_{\mu \nu} \Box \phi - 2g_{\mu \nu} \nabla_\mu \phi \nabla^\mu \phi - 2\nabla_\mu \nabla_\nu \phi, \]  

(2.11)

and

\[ \Box \phi = \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{4} (R + \Lambda). \]  

(2.12)

Taking the trace of eq. \((2.11)\) we get

\[ \Box \phi = -\frac{R}{6} + \frac{4}{3} \nabla_\mu \phi \nabla^\mu \phi - \frac{\Lambda}{3}, \]  

(2.13)

and comparing the latter equation with eq. \((2.12)\) we obtain

\[ \Box \phi = -\frac{1}{2} R, \]  

(2.14)

and also

\[ \nabla_\mu \phi \nabla^\mu \phi = \frac{1}{4} (\Lambda - R). \]  

(2.15)

The latter equation will be useful in the next section.

3 \( f(R) \) gravity as a string-dilaton model

The action of \( f(R) \) gravity in \( 4D \) is given by the action

\[ S = \int d^4x \sqrt{-g} f(R). \]  

(3.1)

Varying the latter with respect to \( g_{\mu \nu} \), one gets the field equations

\[ f'(R) R_{\mu \nu} - \frac{1}{2} f(R) g_{\mu \nu} R = \nabla_\mu \nabla_\nu f'(R) - g_{\mu \nu} \Box f'(R), \]  

(3.2)
which are fourth-order partial differential equations in the metric due to the terms on the right side of the equation above, and the prime indicates the derivative with respect to $R$. By a suitable manipulation, it can be rewritten in an Einstein-like form as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[ f(R) - R f'(R) \right] + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R) \right\},$$

where the contributions due to the higher-order terms can be reinterpreted as a sort of curvature stress-energy tensor given by the form of the $f(R)$ function. Considering also the standard perfect-fluid matter contributions and using physical units $8\pi G = c = 1$, we have

$$G_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[ f(R) - R f'(R) \right] + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R) \right\} + \frac{T^m_{\mu\nu}}{f'(R)} = T_{\mu\nu}^{\text{curv}} + \frac{T^m_{\mu\nu}}{f'(R)},$$

where $T_{\mu\nu}^{\text{curv}}$ is implicitly defined and $T^m_{\mu\nu}$ is the stress-energy tensor of the matter contributions. Moreover, $T_{\mu\nu}^{\text{curv}}$ identically vanishes as soon as $f(R) = R$ and the standard minimal coupling with matter is automatically recovered. The trace of the eq. (3.4) is

$$R f'(R) - 2f(R) + 3\Box f'(R) = T^m,$$

where $T^m = g^{\mu\nu} T^m_{\mu\nu}$. Such an equation dynamically relates $R$ with $T^m$ while in the GR case the algebraic relation $R = -T^m$ holds [1, 4].

The effective string-dilaton Lagrangian can be connected to the $f(R)$ gravity Lagrangian via the conformal transformation

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x).$$

In fact the two actions can be mapped into each other as

$$\sqrt{-g} e^{-2\phi} (R + 4\nabla_\mu \phi \nabla^\mu \phi + \Lambda) = \sqrt{-\tilde{g}} \tilde{f}(\tilde{R}),$$

which, by using eq. (3.6), becomes

$$e^{-2\phi} (R + 4\nabla_\mu \phi \nabla^\mu \phi + \Lambda) = \Omega^4 f(\tilde{R}).$$

Plugging eq. (2.15) into eq. (3.8), and properly arranging the terms, the latter becomes

$$f(\tilde{R}) = 2\Lambda \Omega^{-4} e^{-2\phi},$$

which, by choosing

$$\Omega = e^{-\phi},$$

finally becomes

$$f(\tilde{R}) = 2\Lambda e^{2\phi}.$$
where $a(t)$ is the scale factor and $k$ is the spatial curvature parameter. In this metric, the Ricci scalar reads
\[ R = -6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right]. \tag{4.2} \]

In order to derive the cosmological equations, one can define a canonical Lagrangian $\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R})$, where $Q = \{a, R\}$ is the configuration space and $TQ = \{a, \dot{a}, R, \dot{R}\}$ is the related tangent bundle on which $\mathcal{L}$ is defined. The corresponding point-like action is \cite{1, 18}
\[ S = 4\pi \int dt \mathcal{L}(a, \dot{a}, R, \dot{R}), \tag{4.3} \]
where $a$ and $R$ are independent variables after a suitable Lagrange multiplier is derived by eq. (4.2). The above action can be recast as
\[ S = 4\pi \int dt \left\{ a^3 f(R) - \lambda \left[ R + 6 \left( \frac{\dddot{a}}{a} + \left( \frac{\ddot{a}}{a} \right)^2 + \frac{k}{a^2} \right) \right] \right\}, \tag{4.4} \]
where the Lagrange multiplier $\lambda$ can be obtained by varying the above action with respect to $R$, giving
\[ \lambda = a^3 f'(R). \tag{4.5} \]

After an integration by parts, the Lagrangian $\mathcal{L}$ assumes the form
\[ \mathcal{L} = a^3 \left[ f(R) - R f'(R) \right] + 6a^2 f''(R) \dot{a} \dot{R} + 6f'(R) a \dot{a}^2 - 6ka f'(R), \tag{4.6} \]
from which it is straightforward to obtain the Euler-Lagrange equations
\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{\partial \mathcal{L}}{\partial a}, \tag{4.7} \]
\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{R}} = \frac{\partial \mathcal{L}}{\partial R}, \tag{4.8} \]
which, with some algebra, correspond to the cosmological equations \cite{32}
\[ 2 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right) = -p_{(\text{tot})}, \tag{4.9} \]
and
\[ f''(R) \left\{ R + 6 \left[ \frac{\dddot{a}}{a} + \left( \frac{\ddot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \right\} = 0. \tag{4.10} \]

In particular, eq. (4.10) can be interpreted in terms of the Lagrange multiplier definition, guaranteeing the consistency of the approach. Furthermore, the Hamiltonian constraint
\[ E_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial \dot{a}} \dot{a} + \frac{\partial \mathcal{L}}{\partial \dot{R}} \dot{R} - \mathcal{L} = 0, \tag{4.11} \]
gives
\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{1}{3} p_{(\text{tot})}. \tag{4.12} \]
Here, for the sake of completeness, we added also standard matter that, in the Jordan frame, is minimally coupled to the geometry. By writing
\[ P^{(\text{tot})} = P^{(\text{curv})} + P^{(m)}, \quad \rho^{(\text{tot})} = \rho^{(\text{curv})} + \rho^{(m)}, \]
we put in evidence both curvature and matter contributions, where we have inserted the non-minimal coupling factor \(1/f'(R)\) into the definition of the matter terms. From \(T^{(\text{curv})}_{\mu\nu}\), it is easy to get the curvature pressure
\[ p^{(\text{curv})} = \frac{1}{f'(R)} \left\{ 2 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) + \ddot{R} f''(R) + \dot{R}^2 f'''(R) - \frac{1}{2} \left[ f(R) - R f'(R) \right] \right\}, \]
and the curvature density
\[ \rho^{(\text{curv})} = \frac{1}{f'(R)} \left\{ \frac{1}{2} \left[ f(R) - R f'(R) \right] - 3 \left( \frac{\dot{a}}{a} \right) \dot{R} f''(R) \right\}. \]

Exact solutions for this dynamical system can be achieved by searching for Noether symmetries (see the appendix A for a detailed description of the method) \[33\]. Here we want to show that such symmetries are related to the scale factor duality for the \(f(R)\) Lagrangians discussed in the above section. In other words, there exist models in \(f(R)\) gravity with the same feature of the string-dilaton cosmology at the level of the Lagrangian. The presence of Noether symmetries determines this result. In our further considerations, we will discard the standard matter contribution that is not necessary to the aims of this paper. The contribution of perfect fluid matter to Noether symmetries in \(f(R)\) gravity is discussed in ref. \[33\].

### 4.1 Invariant \(f(R)\) models for \(a \to 1/a\) transformations

Let us use the constant of motion (A.26), obtained from the Noether Symmetry Approach, in order to select a class of Lagrangians where scale factor duality is present. For simplicity reasons let us set \(c_2 = \bar{c} = 0\), then eqs. (A.24) and (A.25) become
\[ \alpha = c_1 a, \]
\[ \beta = -3c_1 \frac{f'(R)}{f''(R)} \].
Inserting these definitions into the constant of motion (A.26) and choosing \(\Sigma_0 = 0\), we obtain:
\[ 6c_1 f''(R)a^3 \dot{R} + 12c_1 f'(R)a^2 \dot{a} - 18c_1 f'(R)a^2 \dot{a} = 0. \]
Immediately we get
\[ f''(R) \dot{R} = f'(R) \left( \frac{\dot{a}}{a} \right). \]

In order to select \(f(R)\) models which undergo scale factor duality transformations, we insert eq. (4.19) into the point-like Lagrangian (4.6). In the following we choose \(k = 0\) (spatially flat metric) for the sake of simplicity.\(^1\) We obtain the simpler Lagrangian
\[ \mathcal{L} = a^3 \left[ f(R) - f'(R) R \right] + 12aa^2 f'(R). \]

\(^1\)Noether symmetries exist also in the cases \(k \neq 0\) as it is shown in ref. [33].
Keeping in mind eq. (3.11), the previous Lagrangian $\mathcal{L}$ can be rewritten as follows:

$$\mathcal{L} = a^3 \left[ 2\Lambda e^{2\phi} - 4\Lambda R e^{2\phi} \phi'(R) \right] + 48 \left( \frac{\dot{a}}{a} \right)^2 \Lambda e^{2\phi} \phi'(R) . \quad (4.21)$$

Notice that, with abuse of notation, we are indicating the Ricci scalar $\tilde{R}$ built with the metric $\tilde{g}_{\mu\nu}$, simply as $R$.

Being $a^3 = e^{3\ln(a)}$, we can rewrite eq. (4.21) as

$$\mathcal{L} = \left[ 2\Lambda e^{2(\phi + \frac{3}{2}\ln(a))} - 4\Lambda R e^{2(\phi + \frac{3}{2}\ln(a))} \phi'(R) \right] + 48 \left( \frac{\dot{a}}{a} \right)^2 \Lambda e^{2(\phi + \frac{3}{2}\ln(a))} \phi'(R) . \quad (4.22)$$

Applying the dilaton shift as similarly done in eqs. (2.7) and (2.9), we have

$$\phi \rightarrow \tilde{\phi} = \phi + \frac{3}{2} \ln(a) , \quad (4.23)$$

and the Lagrangian becomes

$$\mathcal{L} = \left[ 2\Lambda e^{2\tilde{\phi}} - 4\Lambda R e^{2\tilde{\phi}} \tilde{\phi}'(R) \right] + 48 \left( \frac{\dot{a}}{a} \right)^2 \Lambda e^{2\tilde{\phi}} \tilde{\phi}'(R) , \quad (4.24)$$

which can be recast as

$$\mathcal{L} = \left[ f(R) - f'(R)R \right] + 12 \left( \frac{\dot{a}}{a} \right)^2 f'(R) . \quad (4.25)$$

The latter is invariant under the scale factor duality $a \rightarrow 1/a$ and, furthermore, gives rise to the same dynamics as the Lagrangian in eq. (4.20). After these considerations, we point out that the presence of the Noether symmetry gives rise to the scale factor duality invariance.

### 4.2 Cosmological solutions

Exact cosmological solutions can be obtained starting from the previous results. Considering the Ricci scalar curvature in eq. (4.2) for a spatially flat metric, we can compare it with the equation for $R$ which can be obtained using the Lagrangian in eq. (4.25), that is:

$$R = 12 \left( \frac{\dot{a}}{a} \right)^2 , \quad (4.26)$$

which is clearly invariant under scale factor duality. We get

$$a\ddot{a} + 3\dot{a}^2 = 0 , \quad (4.27)$$

obtaining the solution

$$a(t) = \sqrt{c_1 t + c_2} , \quad (4.28)$$

where $c_1$ and $c_2$ are integration constants.

The dynamical equation for $a(t)$ is obtained from the Lagrangian (4.25) as

$$a\ddot{a} f'(R) + a\dot{a} R f''(R) - \dot{a}^2 f'(R) = 0 , \quad (4.29)$$
which, by using eqs. (4.26) and (4.28), can be recast in the form
\[ f'(R) + 2Rf''(R) = 0. \] (4.30)

Moreover, the Hamiltonian constraint in eq. (4.11) gives
\[ f(R) - 2Rf'(R) = 0. \] (4.31)

Finally, by using eqs. (4.30) and (4.31), the functional form of \( f(R) \) is fixed to be
\[ f(R) = f_0 R^2, \] (4.32)
where \( f_0 \) is an integration constant.

4.3 General duality transformations as reflections

It is possible, by using the Noether Symmetry Approach, to obtain large classes of theories where scale factor duality can be represented as a reflection [34–36]. In the specific case of \( f(R) \) gravity, let us consider again the Lagrangian (4.25) and perform the following redefinitions:
\[ R = Ae^{-\phi}, \quad f(R) = e^{-2\phi} F(\phi), \] (4.33)
where \( \phi \) is the dilaton field and \( A \) is an arbitrary constant. Then we obtain
\[ f'(R) = \frac{df}{dR} = \frac{df}{d\phi} \frac{d\phi}{dR} = -A^{-1} e^{-\phi} \left[ F'(\phi) - 2F(\phi) \right], \] (4.34)
where \( F'(\phi) = dF(\phi)/d\phi \). Inserting these transformations into the Lagrangian in eq. (4.25), the latter becomes
\[ \mathcal{L} = \left[ e^{-2\phi} \left( F'(\phi) - F(\phi) \right) \right] - 12A^{-1} e^{-\phi} \left( \frac{\dot{a}}{a} \right)^2 \left[ F'(\phi) - 2F(\phi) \right]. \] (4.35)

We perform now the following transformation:
\[ \phi = g(w), \] (4.36)
where \( g(w) \) is a general function of \( w \) [35, 36]. So, the Lagrangian (4.35) can be rewritten as
\[ \mathcal{L} = e^{-2g(w)} \left[ F'(\phi) - F(\phi) \right] - 12A^{-1} e^{-g(w)} \left( \frac{\dot{a}}{a} \right)^2 \left[ F'(\phi) - 2F(\phi) \right]. \] (4.37)
Assuming that \( [F'(\phi) - F(\phi)] = \gamma \), where \( \gamma \) is a constant, we determine a possible form of the function \( F(\phi) \) and then of \( f(R) \):
\[ F'(\phi) - F(\phi) = \gamma \Rightarrow F(\phi) = \left( \xi e^\phi - \gamma \right), \] (4.38)
where \( \xi \) is an integration constant. Then, from eqs. (4.33), we get the correspondent functional form of \( f(R) \) which is
\[ f(R) = \frac{1}{A} \left( \xi R - \frac{\gamma}{A} R^2 \right). \] (4.39)
So, the Lagrangian (4.37) becomes
\[ \mathcal{L} = \gamma e^{-2g(w)} - 12A^{-1} \left( \frac{\dot{a}}{a} \right)^2 \left( 2\gamma e^{-g(w)} - \xi \right). \] (4.40)

Let us complete the change of variables \((a, \phi) \rightarrow (z, w)\) by using the following transformation:
\[ z = \ln \frac{a}{\sigma} - \mu(w), \] (4.41)
where \(\mu(w)\) is a generic function and \(\sigma\) is a non-zero positive parameter. Immediately we have
\[ a = \sigma e^{\frac{z}{\sigma} e^{\mu(w)}}, \quad \dot{a} = \sigma e^{\frac{z}{\sigma} e^{\mu(w)}} (\dot{z} + \mu w \dot{w}). \] (4.42)

Then, the Lagrangian (4.40) takes the form
\[ \mathcal{L} = \gamma e^{-2g(w)} - 12A^{-1} \left( \dot{z} + \mu w \dot{w} \right)^2 \left( 2\gamma e^{-g(w)} - \xi \right). \] (4.43)

Let us stress that the functions \(g(w)\) and \(\mu(w)\) are completely generic. If we assume that \(g(w)\) and \(\mu(w)\) are, respectively, even and odd functions, that is
\[ g(-w) = g(w), \quad \mu(-w) = -\mu(w), \] (4.44)
then the Lagrangian (4.43) is form invariant under the change
\[ w \rightarrow -w, \quad z \rightarrow -z, \] (4.45)
these transformations being equivalent to \(a \rightarrow 1/a\), if additionally \(\sigma \rightarrow 1/\sigma\). We then conclude that, in the \(\{z, w\}\) space, the duality invariance is a reflection, i.e. an invariance under parity transformation in the considered configurations space. Furthermore the Lagrangian (4.43) represents a class of models, being the two functions \(g(w)\) and \(\mu(w)\) completely arbitrary. Finally the Lagrangian (4.43) is cyclic in \(z\) so that, in the \(\{z, w\}\) space, the dynamics is reduced and a constant of motion exists. In other words, the Noether Symmetry gives rise to a reflection transformation that generalizes scale factor duality.

5 Conclusions and perspectives

Modified theories of gravity are a useful paradigm to cure shortcomings of GR at ultraviolet and infrared scales, due to the lack of a full Quantum Gravity theory. Despite the success of addressing a lot of phenomenology, ranging from inflation [37], accelerated behavior of present universe [1, 2], up to to self-gravitating structures [8], they should be framed into some fundamental theory in order to achieve a self-consistent picture of gravity at all scales.

Due to this state of art, a sort of reconstruction technique is extremely useful. The philosophy is to find out some physically relevant features that allow to relate ETGs to some fundamental theory as Superstring, Supergravity et al.

In this paper, we have considered the straightforward extension of GR, that is \(f(R)\) gravity, where the assumption \(f(R) = R\) of the Hilbert-Einstein action is relaxed. As it is well known, such an extended theory has revealed to be very useful in the last decade in order to address a lot of open problems in astrophysics and cosmology [1]. Here, we have investigated the possibility that scale factor duality, one of the main characteristics of string-dilaton theory, holds also for some class of \(f(R)\) models.
Recasting the tree-level dilaton-graviton effective action of bosonic String Theory into the $f(R)$ theory, we have seen that the dilaton field $\phi$ has a geometrical interpretation in terms of the Ricci scalar curvature. The specific form of the $f(R)$ function is given by the dilaton non-minimal coupling and the dilaton potential by a conformal transformation. Moreover, the possibility to achieve scale factor duality clearly depends on the presence of Noether symmetry in the $f(R)$ Lagrangian.

This result is more general than the case of the tree-level effective bosonic String Theory, since the same scale factor duality (a discrete transformation) can bring back to a Noether symmetry (a continuous transformation). Furthermore, the Noether symmetry allows to reduce the dynamics, to obtain Lagrangians which are invariant under duality transformations and then to achieve exact cosmological solutions.

The issue taken into account in this paper is more general since the same program could be applied to other effective theories and considering other fundamental features in addition to the duality. This will be the argument of further investigations.

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A The Noether symmetry approach

Dynamics given by (4.6) can be reduced and solved searching for Noether symmetries. In principle, the Noether Symmetry Approach allows to select dynamical models where conserved charges come out allowing the reduction [18, 33, 38–41]. The approach can be sketched in the following way.

Let us consider a canonical, non-degenerate point-like Lagrangian $\mathcal{L}(q^i, \dot{q}^i)$ that is

$$\frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0, \quad \det H_{ij} = \det \left| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right| \neq 0, \quad (A.1)$$

where $H_{ij}$ is the Hessian matrix. The dot is the derivative with respect to the affine parameter $\lambda$ (in our case to the cosmic time). In general, $\mathcal{L}$ has the form

$$\mathcal{L} = T(q, \dot{q}) - V(q), \quad (A.2)$$

where $T$ and $V$ are, respectively, the kinetic and potential terms. The energy function associated to $\mathcal{L}$ is

$$E_\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L}, \quad (A.3)$$

which is a constant of motion. Since cosmological problems have a finite number of degrees of freedom, one can take into account point transformations. Invertible coordinate transformations $Q^i = Q^i(q)$ induce transformations of the velocities, that is

$$\dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j, \quad (A.4)$$

and the Jacobian of the transformation $J = \det \left| \frac{\partial Q^i}{\partial q^j} \right|$ is assumed to be non-zero.
In general, an infinitesimal point transformation is represented by a vector field

\[ X = \alpha^i(q) \frac{\partial}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial}{\partial \dot{q}^i}. \]  

(A.5)

A function \( F(q, \dot{q}) \) is invariant under the transformation \( X \) if

\[ L_X F \equiv \alpha^i(q) \frac{\partial F}{\partial q^i} + \left( \frac{d}{d\lambda} \alpha^i(q) \right) \frac{\partial F}{\partial \dot{q}^i} = 0, \]

(A.6)

where \( L_X F \) is the Lie derivative of \( F \). In particular, the condition \( L_X L = 0 \) means that the vector \( X \) is a symmetry for the Lagrangian \( L \). Let us consider now a Lagrangian \( L \) and the related Euler-Lagrange equations

\[ \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} = 0. \]

(A.7)

Considering the vector \( X \) in eq. (A.5) and contracting eq. (A.7) with \( \alpha^j \)'s, gives

\[ \alpha^j \left( \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} \right) = 0. \]

(A.8)

Since

\[ \alpha^j \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}^j} = \frac{d}{d\lambda} \left( \alpha^j \frac{\partial L}{\partial \dot{q}^j} \right) - \left( \frac{d\alpha^j}{d\lambda} \right) \frac{\partial L}{\partial \dot{q}^j}, \]

(A.9)

from eq. (A.8), it follows

\[ \frac{d}{d\lambda} \left( \alpha^j \frac{\partial L}{\partial \dot{q}^j} \right) = L_X L. \]

(A.10)

The consequence is the Noether theorem:

If \( L_X L = 0 \), then the function

\[ \Sigma_a = \alpha^k \frac{\partial L}{\partial \dot{q}^k}, \]

(A.11)

is a constant of motion.

Let us now consider the Lagrangian in eq. (A.2). Since \( X \) is of the form (A.5), \( L_X L \) is a homogeneous polynomial of second degree in the velocities plus an inhomogeneous term in the \( q^i \). This polynomial has to be identically zero and each coefficient must be independently zero. If \( n \) is the dimension of the configuration space, we obtain \( 1 + n(n + 1)/2 \) partial differential equations. This system is overdetermined, therefore the Noether Symmetry Approach can be used to select the functions which assign the models. In the case of modified gravity theories, these functions are couplings and potentials [18].

Considering the specific case which we are discussing, \( f(R) \) cosmology, the configuration space is \( Q = \{ a, R \} \), and the tangent space is \( TQ = \{ a, \dot{a}, R, \dot{R} \} \). The Lagrangian is an application

\[ L : TQ \to \mathcal{R}, \]

(A.12)

where \( \mathcal{R} \) are the real numbers. The generator of symmetry is

\[ X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}}. \]

(A.13)
As discussed above, a symmetry exists if the equation \( L_\chi L = 0 \) has solutions. In other words, a symmetry exists if at least one of the functions \( \alpha \) or \( \beta \) in eq. (A.13) is different from zero. Going to our specific case, the Lagrangian (4.6), and setting to zero the coefficients of the terms \( \dot{a}^2, \dot{R}^2, \dot{a} \dot{R} \), we obtain the following system of equations, linear in \( \alpha \) and \( \beta \),

\[
\begin{align*}
    f'(R) \left( \alpha + 2a \frac{d\alpha}{da} \right) + a f''(R) (\beta + a \partial_a \beta) &= 0, \\
    a^2 f''(R) \partial_R \alpha &= 0, \\
    2f'(R) \partial_R \alpha + f''(R) \left( 2\alpha + a \partial_a \alpha + a \partial_R \beta \right) + a \beta f'''(R) &= 0,
\end{align*}
\]

and, finally, setting to zero the remnant terms, we obtain the constraint

\[
3\alpha \left( f(R) - R f'(R) \right) - a \beta R f''(R) - \frac{6k}{a^2} \left( \alpha f'(R) + a \beta f''(R) \right) = 0. \tag{A.17}
\]

Solutions of eqs. (A.14)–(A.17) exist if explicit forms of \( \alpha \) and \( \beta \) are found. In other words, if at least one of the functions \( \alpha \) and \( \beta \) are different from zero, a Noether symmetry exists. If \( f''(R) \neq 0 \), eq. (A.15) can be immediately solved:

\[\alpha = \alpha(a). \tag{A.18}\]

We do not take into account the case \( f''(R) = 0 \) because it corresponds to standard GR. We can rewrite eqs. (A.14) and (A.16) as follows:

\[
\begin{align*}
    f'(R) \left( \alpha + 2a \frac{d\alpha}{da} \right) + a f''(R) (\beta + a \partial_a \beta) &= 0, \\
    f''(R) \left( 2\alpha + a \frac{d\alpha}{da} + a \partial_R \beta \right) + a \beta f'''(R) &= 0.
\end{align*}
\]

Since \( f = f(R) \), then \( \partial f / \partial a = 0 \); then it is possible to solve eq. (A.20) by writing it as:

\[
\partial_R (\beta f''(R)) = -f''(R) \left( 2 \frac{\alpha}{a} + \frac{d\alpha}{da} \right), \tag{A.21}
\]

and, by integration, its general solution is

\[
\beta = - \left[ 2\alpha + a \frac{d\alpha}{da} \right] f'(R) f''(R) + \frac{h(a)}{f''(R)} \tag{A.22}
\]

Therefore we find that eq. (A.19) gives

\[
f'(R) \left[ \alpha - a^2 \frac{d^2\alpha}{da^2} - a \frac{d\alpha}{da} \right] + a \left[ h + a \frac{dh}{da} \right] = 0, \tag{A.23}
\]

which has the solution

\[
\alpha = c_1 + \frac{c_2}{a} \quad \text{and} \quad h = \frac{\bar{c}}{a}, \tag{A.24}
\]

where, \( a \) being dimensionless, \( c_1 \) and \( c_2 \) have the same dimensions. We can also fix \( \alpha \) to be dimensionless and this fixes the dimensions of \( \beta \) to be \( [\beta] = M^2 \). Then also \( [\bar{c}] = M^2 \), so finally we have:

\[
\beta = - \left[ 3c_1 + \frac{c_2}{a^2} \right] \frac{f'(R)}{f''(R)} + \frac{\bar{c}}{a f''(R)}. \tag{A.25}
\]

This Noether symmetry implies the existence of a constant of motion. From eq. (A.11) and the Lagrangian (4.6) we obtain:

\[
\alpha \left( 6 f''(R) a^2 \dot{R} + 12 f'(R) a \dot{a} \right) + \beta \left( 6 f''(R) a^2 \dot{a} \right) = \Sigma_0, \tag{A.26}
\]

that we have used in the above considerations on duality.
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