String Backgrounds in String Geometry

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Abstract

String geometry theory is a candidate of the non-perturbative formulation of string theory. In order to determine the string vacuum, we need to clarify how string backgrounds are described in string geometry theory. In this paper, we show that arbitrary configurations of the string backgrounds are embedded in configurations of the fields of string geometry theory. Especially, we show that the action of the string backgrounds is obtained by a consistent truncation of the action of the string geometry model; the configurations of the fields of string geometry theory satisfy their equations of motion if and only if the embedded configurations of the string backgrounds satisfy their equations of motion.

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1. Introduction

Superstring theory is a promising candidate of a unified theory including gravity. However, superstring theory is established at only the perturbative level as of this moment. The perturbative superstring theory lacks predictability because it has many perturbatively stable vacua.

String geometry theory is proposed as a candidate of non-perturbative formulation of superstring theory [1]. From string geometry theory, the author of Ref. [1] derived the partition function and thus the all order amplitudes of the perturbative superstring theory in a flat background\(^1\). The procedure is as follows:

step. 1. we define the framework of *Riemannian superstring manifolds*,
step. 2. we consider a *string geometry model*\(^2\) based on it and derive its equations of motion,
step. 3. we find a perturbative vacuum solution for the equations of motion and
step. 4. we can obtain a partition function of the perturbative superstring theory from fluctuations around the solution.

\(^1\)A perturbative topological string theory is also derived from the topological sector of string geometry theory [2].
\(^2\)The action of string geometry theory is not determined as of this moment. On this stage, we should consider various possible actions. Then, we call each action a *string geometry model* and call the whole formulation *string geometry theory*. In [1], the perturbative string theory on the flat spacetime is derived from a gravitational model coupled with a u(1) field on a Riemannian string manifold, whereas in [3], it is derived from gravitational models coupled with arbitrary fields on a Riemannian string manifold.
A part of the perturbatively stable vacua is described by expectation values of the massless modes of the string theory as backgrounds. Actually, bosonic closed strings around the backgrounds are described perturbatively by the nonlinear sigma model [4–6],

\[
S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[ (g^{ab}G_{\mu\nu}(X(\sigma_1, \sigma_2)) + i\epsilon^{ab}B_{\mu\nu}(X(\sigma_1, \sigma_2))) \partial_\mu X^a \partial_\nu X^b + \alpha' \phi(X(\sigma_1, \sigma_2)) \right].
\]

The backgrounds satisfy the equations of motion,

\[
R_{\mu_1\mu_2} - \frac{1}{4} H_{\mu_1\nu_1\mu_2} H_{\mu_2}^{\nu_1} + 2 \nabla_{\mu_1} \nabla_{\mu_2} \phi = 0,
\]
\[
R - 4 \nabla_{\mu_1} \phi \nabla^\mu_1 \phi + 4 \nabla_{\mu_1} \nabla^\mu_1 \phi - \frac{1}{2} |H|^2 = 0,
\]
\[
\nabla_{\mu_1} (e^{-2\phi} H^{\mu_1\nu_1\nu_2}) = 0,
\]

which are derived from the action,

\[
S = \frac{1}{2\kappa_1^2} \int d^10 x \sqrt{G} e^{-2\phi} \left[ R + 4 \nabla^\mu_1 \phi \nabla_{\mu_1} \phi - \frac{1}{2} |H|^2 \right],
\]

where $|H|^2 := \frac{1}{\pi} G^{\mu_1\nu_1} G^{\mu_2\nu_2} G^{\mu_3\nu_3} H_{\mu_1\mu_2\mu_3} H_{\nu_1\nu_2\nu_3}$. In this paper, we discuss only at the tree level, i.e., without any quantum effects like anomalies. Therefore, we do not consider a torsion or higher derivative terms.

As a first step to determine the string vacuum, we clarify how arbitrary configurations of the string backgrounds are embedded in configurations of the fields of a string geometry model in this paper. This work corresponds to the step. 2 and a part of the step. 3 in a general background\(^3\). In the following, we focus on the bosonic string background since our essential argument lies in the bosonic case. A supersymmetric extension will be given in [7].

The organization of this paper is as follows. In Sec. 2, we introduce a string geometry model. In Sec. 3, we derive the equations of motion of the string background from the equations of motions of the string geometry model defined in Sec. 2 by a consistent truncation. In Sec. 4, we conclude and discuss our results.

\(^3\) The step. 1 is completed in Ref. [1] and does not depend on backgrounds.
2. String geometry model

We define a string geometry model by an action,

\[ S = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}X_{\mathcal{D}r}(\bar{\tau}) \mathcal{D}\bar{\tau} \sqrt{G} e^{-2\phi} \left[ R + 4\nabla_I \Phi \nabla^I \Phi - \frac{1}{2} |H|^2 \right], \tag{6} \]

where \( G_N \) is a constant, \( I = \{d, (\mu \bar{\sigma})\} \), \( |H|^2 := \frac{1}{2} G^{I_1J_1} G^{I_2J_2} G^{I_3J_3} H_{I_1I_2I_3} H_{J_1J_2J_3} \), and we use the Einstein notation for the index \( I \). The action (6) consists of a metric \( G_{I_1I_2} \), a scalar field \( \Phi \) and field strengths \( H_{I_1I_2I_3} \) of a two-form field \( B_{I_1I_2} \). They are defined on a Riemannian string manifold. It is an infinite dimensional Riemannian manifold \([1]\) parametrized by coordinates \( (\Sigma, X_{\bar{D}r}(\bar{\tau}), \bar{\tau})^4 \), where \( \Sigma \) is a worldsheet with a metric \( h(\bar{\sigma}, \bar{\tau}) \), \( \bar{\tau} \) is the global time on \( \bar{\Sigma} \) and \( X_{\bar{D}r}(\bar{\tau}) \) is a map from \( \Sigma_\nu \) to the 26-dimensional Euclidean space \( \mathbb{R}^{26} \). \( \bar{D}_T \) represents all the backgrounds except for the target metric, that consist of the B-field and the dilaton. The cotangent space is spanned by \( dX^\mu_{\bar{D}r} := d\bar{\tau} \) and \( dX_{\mu(\bar{\sigma})} := dX^\mu(\bar{\sigma}, \bar{\tau}) \), where \( \mu = 1, \ldots, 26 \). The summation over \( \bar{\sigma} \) is defined by \( \int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau}) \), where \( \bar{e}(\bar{\sigma}, \bar{\tau}) := \sqrt{h_{\bar{\sigma} \bar{\sigma}}} \). This summation is transformed as a scalar under \( \bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X_{\bar{D}r}(\bar{\tau})) \) and invariant under \( \bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma}) \). For example, an explicit form of the line element is given by

\[ ds^2(h, X_{\bar{D}r}(\bar{\tau}), \bar{\tau}) = G_{dd}(h, X_{\bar{D}r}(\bar{\tau}), \bar{\tau})(d\bar{\tau})^2 + 2d\bar{\sigma} \int d\bar{\sigma}' \bar{e}(\bar{\sigma}, \bar{\tau}) \sum_{\mu} G_{d(\mu \bar{\sigma})}(h, X_{\bar{D}r}(\bar{\tau}), \bar{\tau}) dX^\mu(\bar{\sigma}, \bar{\tau}) \]

\[ + \int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau}) \int d\bar{\sigma}' \bar{e}(\bar{\sigma}', \bar{\tau}) \sum_{\mu, \mu'} G_{(\mu \bar{\sigma})(\mu' \bar{\sigma}')}(h, X_{\bar{D}r}(\bar{\tau}), \bar{\tau}) dX^\mu(\bar{\sigma}, \bar{\tau}) dX^{\mu'}(\bar{\sigma}', \bar{\tau}). \]

The inverse metric \( G^{IJ}(h, X_{\bar{D}r}(\bar{\tau}), \bar{\tau}) \) is defined by \( G_{IJ} G^{JK} = G^{iK} G^{iJ} = \delta^K_I \), where \( \delta^K_I = 1 \) and \( \delta^{(\mu \bar{\sigma})}_{(\mu' \bar{\sigma}')}(\bar{\tau}) = \frac{1}{\bar{e}(\bar{\sigma}, \bar{\tau})} \delta^{\mu'}_\mu \delta(\bar{\sigma} - \bar{\sigma}') \).

3. Consistent truncation

In this section, we consider a consistent truncation between eq. (5) and eq. (6).
From eq. (6), the equations of motion are given by

\[ R_{IJ} - \frac{1}{4} H_{IL_1 L_2} H_{JL_1 L_2} + 2 \nabla_I \nabla_J \Phi = 0, \]  

(7)

\[ R - 4 \nabla_I \Phi \nabla^I \Phi + 4 \nabla_I \nabla^I \Phi - \frac{1}{2} |H|^2 = 0, \]  

(8)

\[ \nabla_I \left( e^{-2\Phi} H^{L_1 L_2} \right) = 0, \]  

(9)

where we use the equation of motion of the scalar to the Einstein equation.

We consider the following ansatz,

**Metric:**

\[
G_{dd} \left( X^d_{D_T}, X^d_{\bar{D}_T} \right) = \alpha,
\]

\[
G_{(\mu_1 \bar{\sigma}_1)(\mu_2 \bar{\sigma}_2)} \left( X^d_{D_T}, X^d_{\bar{D}_T} \right) = \beta(\bar{\sigma}_1) \delta_{\bar{\sigma}_1 \bar{\sigma}_1} G_{\mu_1 \mu_2} \left( X^d_{\bar{D}_T}(\bar{\sigma}_1) \right) \delta_{\bar{\sigma}_1 \bar{\sigma}_2},
\]

the others = 0,

**Scalar field:**

\[
\Phi \left( X^d_{D_T}, X^d_{\bar{D}_T} \right) = \int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau}) f(\bar{\sigma}) \delta_{\bar{\sigma} \bar{\sigma}} \phi \left( X^d_{\bar{D}_T}(\bar{\sigma}) \right),
\]

**2-form field:**

\[
B_{(\mu_1 \bar{\sigma}_1)(\mu_2 \bar{\sigma}_2)} \left( X^d_{D_T}, X^d_{\bar{D}_T} \right) = b(\bar{\sigma}_1) \delta_{\bar{\sigma}_1 \bar{\sigma}_1} B_{\mu_1 \mu_2} \left( X^d_{\bar{D}_T}(\bar{\sigma}_1) \right) \delta_{\bar{\sigma}_1 \bar{\sigma}_2},
\]

the others = 0,

where \( \delta_{\bar{\sigma}_1 \bar{\sigma}_2} = \frac{1}{e(\bar{\sigma}, \bar{\tau})} \delta(\bar{\sigma}_1 - \bar{\sigma}_2) \), \( G_{\mu_1 \mu_2} \) is a symmetric tensor field, \( \phi \) is a scalar field and \( B_{\mu_1 \mu_2} \) is an anti-symmetric tensor field on a 26-dimensional spacetime.

We remark that the fields in the ansatz depend not only on the string zero modes \( x^{\mu} \) but also on the other modes of \( X^\mu_{D_T}(\bar{\sigma}) \) since the fields will be identified with the backgrounds in eq. (1) when we derive a partition function. We also remark that the

\[ G^{(\mu_1 \bar{\sigma}_1)(\mu_2 \bar{\sigma}_2)} \left( X^d_{D_T}, X^d_{\bar{D}_T} \right) = \beta^{-1}(\bar{\sigma}_1) \delta^{-1}_{\bar{\sigma}_1 \bar{\sigma}_1} \delta_{\bar{\sigma}_1 \bar{\sigma}_2} G_{\mu_1 \mu_2} \left( X^d_{\bar{D}_T}(\bar{\sigma}_1) \right). \]
ansatz has a non-trivial dependence on the worldsheet. The consistent truncation will be ensured due to the relation between the worldsheet dependence of the fields and of the indices of the string geometry fields. For example, see $\bar{\sigma}_1$ dependence on the ansatz for the metric. In addition, the factor $\delta_{\bar{\sigma}\bar{\sigma}}$ reflects that the point particle limit is a field theory.

By substituting the ansatz, eqs. (7) $\sim$ (9) become

$$
\left[R_{\mu\nu} \left( X_{\bar{D}_a} (\bar{\sigma}_1) \right) - \frac{1}{4} \beta^{-2}(\bar{\sigma}_1) b^2(\bar{\sigma}_1) H_{\mu_1\mu_2} \left( X_{\bar{D}_a} (\bar{\sigma}_1) \right) H^{\mu_1\mu_2}_\nu \left( X_{\bar{D}_a} (\bar{\sigma}_1) \right) + 2 f(\bar{\sigma}_1) \nabla_\mu \nabla_\nu \phi \left( X_{\bar{D}_a} (\bar{\sigma}_1) \right) \right] \times \delta_{\bar{\sigma}_1 \bar{\sigma}_2} \delta_{\bar{\sigma}_1 \bar{\sigma}_1} = 0,
$$

\hspace{1cm} (10)

$$
\int d\bar{\sigma} \bar{e}(\bar{\sigma}, \bar{\tau}) \delta_{\bar{\sigma}\bar{\tau}} \beta^{-1}(\bar{\sigma}) \left[ R \left( X_{\bar{D}_a} (\bar{\sigma}) \right) - 4 f^2(\bar{\sigma}) \partial_{\mu_1} \phi \left( X_{\bar{D}_a} (\bar{\sigma}) \right) \partial^{\mu_1} \phi \left( X_{\bar{D}_a} (\bar{\sigma}) \right) 
+ 4 f(\bar{\sigma}) \nabla_{\mu_1} \nabla^{\mu_1} \phi \left( X_{\bar{D}_a} (\bar{\sigma}) \right) - \frac{1}{2} \beta^{-2}(\bar{\sigma}) b^2(\bar{\sigma}) |H \left( X_{\bar{D}_a} (\bar{\sigma}) \right)|^2 \right] = 0,
$$

\hspace{1cm} (11)

$$
\left[ - 2 f(\bar{\sigma}_1) \partial_{\mu_1} \phi \left( X_{\bar{D}_a} (\bar{\sigma}_1) \right) H^{\mu_1\mu_2}(X(\bar{\sigma}_1)) + \nabla_{\mu_1} H^{\mu_1\mu_2}(X_{\bar{D}_a} (\bar{\sigma}_1)) \right] \times \delta_{\bar{\sigma}_1 \bar{\sigma}_2} \delta_{\bar{\sigma}_1 \bar{\sigma}_1} = 0,
$$

\hspace{1cm} (12)

and the other components are automatically satisfied. We use the Einstein notation for only the index $\mu$.

Eqs. (10) $\sim$ (12) are satisfied if and only if $G_{\mu_1\mu_2}(x)$, $\phi(x)$ and $B_{\mu_1\mu_2}(x)$ satisfy eqs. (2) $\sim$ (4) and the following relations are satisfied:

$$
f(\bar{\sigma}) = 1, \quad \beta^{-2}(\bar{\sigma}) b^2(\bar{\sigma}) = 1.
$$

Therefore, we conclude that the string backgrounds can be embedded into the string geometry model in the sense of the consistent truncation. In addition, the above discussion is valid without taking $\alpha' \to 0$ limit, which corresponds to $X(\bar{\sigma}) \to x$. This fact will be important to derive eq. (1) since the backgrounds in eq. (1) depend not only on the string zero modes $x$ but also on the other modes of $X(\bar{\sigma})$.

### 4. Conclusion and Discussion

In this paper, we have shown that arbitrary configurations of the string backgrounds are embedded in configurations of fields of string geometry theory. Especially, the action of
the string background is obtained from the action of the string geometry model by the consistent truncation. This result strongly indicates that string geometry theory does not depend on string backgrounds.

We expect that we can derive the partition functions of the strings on arbitrary backgrounds from string geometry theory, because of the discussion in the last paragraph in the previous section. We are also interested in supergravity. With a little alteration, due to the existence of the RR fields, the gauge field and the Grassmann coordinates, we may show that the consistent truncation is valid for supergravity [7].

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