Quantum Noise in Bright Soliton Matterwave Interferometry

Simon A. Haine

1Department of Physics and Astronomy, University of Sussex, Brighton BN1 9QH, United Kingdom

There has been considerable recent interest in matterwave interferometry with bright solitons in quantum gases with attractive interactions, for applications such as rotation sensing. We model the quantum dynamics of these systems and find that the attractive interactions required for the presence of bright solitons causes quantum phase-diffusion, which severely impairs the sensitivity. We propose a scheme that partially restores the sensitivity, but find that in the case of rotation sensing, it is still better to work in a regime with minimal interactions if possible.

I. INTRODUCTION

Rotation sensors based on matterwave interferometers have the potential to provide state of the art sensing capabilities [1, 2]. The current pursuits towards fulfilling this potential can be divided into two main approaches: Free-space atom interferometers, which operate in free-fall and use optical transitions between momentum modes to achieve spatial path separation [3–8], or guided configurations which involves the propagation of atoms along some guiding potential to achieve spatial path separation, analogous to an optical fibre [9, 10]. While both approaches have their advantages, one attraction towards guided configurations is the potential for a very large enclosed area [12], and therefore higher per-particle sensitivity. However, guided matterwave interferometry often requires working in a regime where atom-atom interactions are important, leading to complications in the matterwave dynamics [11, 13, 16, 18]. One approach to minimize these effects is to work with atomic gases with attractive interaction in the matterwave wave-packets of equal population before they collide on a narrow barrier, resulting in 50% transmission and 50% reflection. If we consider the full 

$F = \left| \langle \Psi(t) \rangle \right|^2$ 

and 

$F = \left| \langle \Psi(t) \rangle \right|^2$

then the state at some later time $t$ can in general be described by $|\Psi(t)⟩ = U|\Psi⟩$, where $|\Psi⟩$ is the state immediately after the application of the phase shift, $U \propto \exp(-i\hat{H}t/\hbar)$ and $\hat{H}$ is the full $N$-particle Hamiltonian which describes the kinetic energy, potential energy, and arbitrary inter-particle interactions. The QFI of the final state is

$$\mathcal{F}_Q[|\Psi(t)⟩] = 4 \left| \langle \partial_\phi \Psi(t) | \partial_\phi \Psi(t) \rangle - |\langle \Psi(t) | \partial_\phi \Psi(t) \rangle |^2 \right|$$

$$= 4 \left| \langle \partial_\phi \Psi | \hat{U}^\dagger \hat{U} | \partial_\phi \Psi \rangle - |\langle \Psi | \hat{U}^\dagger \hat{U} | \partial_\phi \Psi \rangle |^2 \right|$$

$$= 4 \left| \langle \partial_\phi \Psi | \partial_\phi \Psi \rangle - |\langle \Psi | \partial_\phi \Psi \rangle |^2 \right|$$

$$= \mathcal{F}_Q[|\Psi⟩]$$

where we have used the fact that $\hat{U}$ is independent of $\phi$, and $\hat{U}^\dagger \hat{U} = 1$. That is, $\mathcal{F}_Q$ is unchanged by the subsequent evolution. If the many particle quantum state is initially separable, ie, $|\Psi⟩ = (\hat{a}_\phi^\dagger)^N |0⟩$ where $\hat{a}_\phi = \int \Psi^*(x)\hat{\psi}(x)dx$, where $\hat{\psi}(x)$ is the usual bosonic annihilation operator and $\Psi(x)$ is the single-particle wavefunction, then it can be shown that $\mathcal{F}_Q = N\mathcal{F}_Q$ [10, 35], where

$$F_Q = 4 \left| \int \partial_\phi \Psi^* \partial_\phi \Psi dx - |\int \Psi^* \partial_\phi \Psi dx |^2 \right|$$

is the single-particle QFI. If $\Psi(x,0) = \frac{1}{\sqrt{2}}(\Psi_L + e^{i\phi}\Psi_R)$, where $\Psi_L$ and $\Psi_R$ are orthonormal wave-packets repre-
Of course, the QFI can exceed $N$ when there are non-trivial quantum correlations present. However, the creation of such correlations cannot be modelled by the GPE, which is why models that include the quantum noise should be considered when assessing the metrological usefulness of such devices.

III. MATTERWAVE GYROSCOPE

To demonstrate the role of quantum noise, we consider the example of a gyroscope based on interference of matterwaves confined in a ring shaped potential, described in Fig. (2). Two counter-propagating matterwaves traverse the ring in opposite directions and are interfered, with the goal of estimating the magnitude of a rotational frequency $\Omega$. We consider a Bose gas consisting of two hyperfine components (electronic states $|+\rangle$ and $|-\rangle$), with bosonic annihilation operators $\hat{\psi}_+ (\mathbf{r})$ and $\hat{\psi}_- (\mathbf{r})$ respectively, which obey they usual bosonic commutation relations: $[\hat{\psi}_i (\mathbf{r}), \hat{\psi}_j^\dagger (\mathbf{r}')] = \delta (\mathbf{r} - \mathbf{r}') \delta_{ij}$. An initial state is created with all the atoms in state $|+\rangle$, before implementing an atomic beamsplitter, which performs the
tions. In terms of the coordinate \(\omega\), assuming that the radial and axial confinement is sufficient, the trapping frequencies, and \(\omega_r\) the radius of the torus, \(|\pm\rangle\) propagate around the ring in opposite directions, accumulating a phase-difference due to the external rotation frequency \(\Omega\). (b): After one complete circuit, the two components are interfered via a two-photon Raman transition, and the phase difference is converted into population difference (c).

operation \(\hat{\psi}_\pm \rightarrow \frac{1}{\sqrt{2}}(\hat{\psi}_\pm \mp \hat{\psi}_\mp e^{\pm 2i n \theta})\), where \(\theta\) is the angular coordinate around the ring, coherently transferring 50% of the population to state \(|-\rangle\) while also shifting the angular momentum by \(-2n\hbar\). Such a process could be implemented via a two-photon Raman transition with Laguerre-Gauss beams as described in \[10\] \[14\] \[27\] \[38\]. After time \(T = 2\pi R^2 m / \hbar n\), the two components have each traversed the ring and we apply another Raman coupling pulse to act as a second beamsplitter performing the transformation \(\hat{\psi}_\pm \rightarrow \frac{1}{\sqrt{2}}(\hat{\psi}_\pm - \hat{\psi}_\mp e^{\pm 2i n \theta})\), before the population in each component is measured and used to infer the phase difference accrued, and therefore estimate \(\Omega\). As in \[10\] \[11\] \[14\] \[32\], working in cylindrical coordinates, \(\{r, \theta, z\}\), we assume a trapping potential of the form \(V(r) = \frac{1}{2}m[\omega_\perp^2 z^2 + \omega_r^2 (r - R)^2]\) where \(R\) is the radius of the torus, \(\omega_\perp\) and \(\omega_r\) are the axial and radial trapping frequencies, and \(m\) is the mass of the particles. Assuming that the radial and axial confinement is sufficiently tight, we may ignore the dynamics in these directions. In terms of the coordinate \(\xi = \partial R\), the effective Hamiltonian for the system is

\[
\mathcal{H} = \sum_{j=+,-} \int \hat{\psi}_j^\dagger(\xi) \hat{H}_0 \hat{\psi}_j(\xi) d\xi + \sum_{i,j=+,-} \frac{g_{ij}}{2} \int \hat{\psi}_i^\dagger(\xi) \hat{\psi}_j^\dagger(\xi) \hat{\psi}_j(\xi) \hat{\psi}_i(\xi) d\xi, \tag{3}
\]

where \(\hat{H}_0 = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial \xi^2} - \Omega \hat{L}_z\), and \(\hat{L}_z\) is the \(z\) component of the angular momentum, and we assumed that we are working in a frame rotating around the \(z\)-axis at angular frequency \(\Omega\). \(g_{ij}\) is the two-particle contact potential interaction strength between state \(|i\rangle\) and \(|j\rangle\) atoms. For convenience, we assume that \(g_{++} = g_{--} = g_0 \leq 0\), and \(g_{+-} = 0\). The choice of \(g_{++}\) has very little effect on the results as for most of the duration the two components are not spatially overlapping. \[59\]

A. Noninteracting Case

We begin by examining the simple case where \(g_0 = 0\), as we can obtain an analytic result with which to benchmark the behaviour in the soliton regime. Working in the Heisenberg picture, and expanding our field operators in angular momentum basis \(\psi_\pm(\xi) = \frac{1}{\sqrt{2}} \sum_q \hat{b}_q^\pm e^{iq\xi/R}\), the operators at some time \(t_f\) after the interferometer sequence (beamsplitter/free evolution/beamsplitter) are

\[
\hat{b}_q^\pm(t_f) = \frac{1}{2} \left[ e^{-i\phi_q} (\hat{b}_q^+(0) \mp \hat{b}_q^-(0)) + e^{i\phi_q} (\hat{b}_q^+(0) \pm \hat{b}_q^-(0)) \right], \tag{4}
\]

where \(\phi_q = (\frac{\hbar g_0}{2mR} - \Omega) t_f\). If we use the number difference in each component \(\hat{N}_\pm \equiv \hat{N}_+ - \hat{N}_-\) as our signal, where \(\hat{N}_\pm = \int_{-\pi R}^{\pi R} \psi_\pm^\dagger \psi_\pm d\xi\) then the rotation sensitivity is given by

\[
\Delta \Omega = \sqrt{\text{Var}(\hat{N}_d) / (\partial_\Omega (\langle \hat{N}_d \rangle))^2}. \tag{5}
\]

At \(t_f = T\), \(\phi_q + 2n - \phi_0 = \frac{4\pi n R^2 \Omega}{\hbar} \equiv \phi_0\), where we have subtracted the constant \(4\pi (q + n)/2\pi\) as integer multiples of \(2\pi\) are inconsequential. Importantly, \(\phi_0\) is independent of \(q\), which allows us to greatly simplify \(\hat{N}_d\). Assuming \(\langle \hat{N}_-(0) \rangle = 0\), we obtain \(\text{Var}(\hat{N}_d) = \sin^2 \phi_0 \text{Var}(\hat{N}_+(0)) + \cos^2 \phi_0 \text{Var}(\hat{N}_-(0))\), and \(\langle \hat{N}_d \rangle = \sin \phi_0 \langle \hat{N}_+(0) \rangle\). At the most sensitive point \(\phi_0 = 0\), this simplifies to \(\Delta \Omega = \frac{\hbar}{4\pi R^2 m} \frac{1}{\sqrt{N_s}} \equiv \Delta \Omega_S\), where \(N_s = \langle \hat{N}_+(0) \rangle\) is the total number of atoms. We take \(\Delta \Omega_S\) as our benchmark sensitivity for the device. Importantly, the initial momentum distribution is irrelevant to the sensitivity, indicating that this sensitivity can be obtained regardless of the shape of the initial wave-packet.

B. Soliton Regime

To model the behaviour of our system in the soliton regime we choose the initial state \(|\Psi(0)\rangle = D(\hat{a}_{+})D(\hat{a}_{-})|0\rangle\), where \(D(\hat{a}_{\pm}) = \exp(\alpha \hat{a}_{\pm}^\dagger - \alpha^* \hat{a}_{\pm})\) is the Glauber coherent displacement operator, \(\alpha = \sqrt{N_s}\), where \(N_s = N_t / 2\) is the mean number of atoms in each mode, \(\hat{a}_{\pm} = \frac{1}{\sqrt{2}} \int \Psi_{\pm}^\dagger \psi_{\pm} d\xi\), and \(\Psi_{\pm}(\xi) = B \text{sech}(\sqrt{2m|\mu|/\hbar^2} \xi)e^{\pm ik_0 \xi}\) where \(k_0 = n/R\), and \(B\) is a normalisation constant such that \(\int_{-\pi R}^{\pi R} |\Psi_{\pm}(\xi)|^2 d\xi = 1\). The chemical potential \(\mu\) is related to the number \(N_s\) by \(\mu = -N_s^2 g_0^2 m / 8\hbar^2\). We note that as we have started with our atoms already split between the two components, we forgo the first beamsplitter, allowing us to easily prepare the wave-packets with the correct shape for their occupation numbers. It was previously shown that the dynamics of such systems is reasonably insensitive to the total population statistics, but is sensitive to the
statistics of the population difference \cite{40}. We chose a two-mode Glauber coherent state for our initial state as it reflects the number difference statistics that are obtained from coherent splitting of an ensemble of atoms. Alternatively, we could have used a coherent spin state \cite{41}, which also has this property but for a well-defined total number of atoms. However a Glauber coherent state is much less computationally demanding for the numerical technique employed in this work.

We simulate the dynamics of the system by using a stochastic phase space technique known as the Truncated Wigner (TW) method, which has previously been used to model the dynamics of quantum gases \cite{42-45}, and unlike the GPE, can be used to model non-classical particle correlations \cite{46–48}. The derivation of the TW method has been described in detail elsewhere \cite{42-45,40,41,50}. Briefly, the equation of motion for the Wigner function of the system can be found from the von-Neumann equation by using correspondences between differential operators on the Wigner function and the original quantum operators \cite{51}. By truncating third- and higher-order derivatives (the Truncated Wigner Approximation), a Fokker-Planck equation (FPE) is obtained. The FPE is then mapped to a set of stochastic partial differential equations for complex fields $\psi_j(\xi,t)$, which loosely correspond to the original field operators $\hat{\psi}_j(\xi,t)$, with initial conditions stochastically sampled from the appropriate Wigner distribution \cite{50,52}. The complex fields obey the partial differential equation

$$i\hbar \frac{\partial \psi_j}{\partial t} = \left[ H_0 + g_0(|\psi_j|^2 - \frac{1}{\Delta}) \right] \psi_j,$$

where $\Delta$ is the element that characterises the discretisation of the spatial grid $\xi$. By averaging over many trajectories with stochastically sampled initial conditions, expectation values of quantities corresponding to symmetrically ordered operators in the full quantum theory can be obtained via the correspondence \cite{50,52}

$$\langle \{f(\psi_j^\dagger, \psi_j)\}_{\text{sym}} \rangle = \frac{1}{c_{\text{sym}}} \langle \psi_j^\dagger \psi_j \rangle,$$

where ‘sym’ denotes symmetric ordering and the overline denotes the mean over many stochastic trajectories. The initial conditions for the simulations are chosen as $\psi_\pm(\xi,0) = \sqrt{N_\pm} \phi_\pm(\xi) + \eta_\pm(\xi)$, where $\eta_\pm(\xi)$ are complex Gaussian noises satisfying $\eta^*_\pm(\xi) \eta_\pm(\xi) = \frac{1}{2} \delta_{\text{sym}}(\xi)$, for spatial grid points $\xi$ and $\xi_t$. Equations (6) was solved numerically on a spatial grid with 512 points.

At $t = T$ the wave-packets have completed one circuit of the ring and a beam-splitter implemented via the transformation $\psi_\pm \rightarrow \frac{1}{\sqrt{2}} (\psi_\pm - i \psi_\mp e^{\pm 2ik_0 t})$, before the expectation value and variance of the total number of particles in each component is calculated. We calculate

FIG. 3. (Color Online) Rotation sensitivity as a function of $g_0$ ($g_0$ is expressed in units of $\hbar^2/(mR)$). Red squares: Multi-mode TW (MMTW) model. Red dashed line: Analytic two-mode (TM) model. Blue solid line: (Blue stars): Two-mode (multi-mode) TW model of pre-twisting scheme using $\theta = \theta_\chi$, (TMT, $\theta = \theta_\chi$ and MMT, $\theta = \theta_\chi$, respectively). Black circles (plus symbols): Multi-mode (two-mode) TW model of pre-twisting scheme with a numerically optimised $\theta = \theta_{\text{opt}}$ for each point (TMT, $\theta = \theta_{\text{opt}}$ and MMT, $\theta = \theta_{\text{opt}}$, respectively). The upper black dotted line indicates $\Delta \Omega_S$, and the lower black dotted line indicates $\frac{1}{2} \Delta \Omega_S$, which is the standard sensitivity for matterwaves traversing two revolutions of the ring. Parameters: $N_i = 10^4$ and $k_0 = 80/R$ for all simulations, which corresponds to a maximum interaction parameter of $\chi T = 7.6 \times 10^{-4}$. 

\[ \Delta \Omega = \frac{\Delta \Omega_S}{\Delta \Omega_S} \]
\[ \frac{\partial}{\partial t} \hat{N}_d \] by using finite difference and simulating small variations of \( \Omega \) around \( \Omega = 0 \). Fig. 2 (red squares) shows the rotation sensitivity as a function of the interaction strength \( g_0 \). We see that as \( |g_0| \) increases, the sensitivity is rapidly degraded. We also analysed a single component system where the beam-splitting was performed by quantum reflection/transmission from a narrow barrier as in [32]. Fig. 2 (green diamonds) shows similar behaviour to the two-component system. For comparison, we have also modelled a noninteracting gas, for a variety of initial wave-packet with the same quantum statistics. For the two-component case, the sensitivity was equal to \( \Delta \Omega_S \) in all cases. For the single component system, the sensitivity was also well approximated by \( \Delta \Omega_S \) as long as the final state was still well approximated by two, well separated wave-packets. For gaussian wave-packets, this is achieved when \( \sigma_x k_0 \gtrsim 1 \) and \( \sigma_y \lesssim \pi R \), where \( \sigma_x \) is the initial width of the wave-packet. Outside this regime, the sensitivity decreased when the final width of the wave-packets was of the order of the circumference of the ring, and could no longer be distinguished from each other. We note that making a measurement of the systems angular momentum, rather than position, may relax this constraint further.

IV. TWO-MODE MODEL

The origin of this degradation is the quantum fluctuations in the population difference leading to uncertainty in the energy of each soliton, resulting in phase-fluctuations before the final beam-splitter. For small fluctuations in particle number \( N \) around \( N_s \), the energy of a single soliton is well approximated by

\[ E_N \approx E_{N_1} + \partial_{N_1} E_{N_1} (N - N_s) + \frac{1}{2} \partial_{N_1}^2 E_{N_1} (N - N_s)^2, \]

where \( E_{N_1} \approx \left( \frac{\hbar^2 k_0^2}{2m} + \Omega \hbar k_0 R \right) N_s - \frac{g_0^2 m N_s^3}{24 \hbar} \) is obtained by substituting \( \frac{\hat{N}_d}{\sqrt{N_s}} \Psi_\pm \) into Eq. 3 and making the approximation that the limits of integration are \( \pm \infty \). We can model the effect of the number fluctuations with an effective two-mode Hamiltonian [13, 40, 47, 53–55]:\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}, \]

\[ \mathcal{H}_0 = E_0 \sum_{j=\pm} \hat{a}_j^\dagger \hat{a}_j - \hbar \Omega R k_0 \left( \hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_- \right), \]

\[ \mathcal{H}_{\text{int}} = \frac{\hbar \chi}{2} \left( \hat{a}_+^\dagger \hat{a}_+ \hat{a}_- + \hat{a}_-^\dagger \hat{a}_- \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_- \hat{a}_- + \hat{a}_+^\dagger \hat{a}_+ \hat{a}_- \right), \]

where \( \hbar \chi = \partial_{N_1} E_{N_1} = -g_0^2 m N_s / 4 \hbar^2 \), and \( E_0 = \frac{\hbar^2 k_0^2}{2m} + g_0^2 m N_s^3 / 8 \hbar^2 \). The form of \( E_0 \) is inconsequential as it results in a phase-shift that is common to both modes. Moving to an interaction picture where the operators evolve under \( \mathcal{H}_0 \) and our state evolves under \( \mathcal{H}_{\text{int}} \), and expressing the state in the Fock basis gives

\[ |\Psi(T)\rangle = e^{-|\alpha|^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{\alpha^{n_1}}{\sqrt{n_1!}} \frac{\alpha^{n_2}}{\sqrt{n_2!}} |n_1, n_2\rangle e^{-i\Phi_{n_1, n_2}}, \]

where \( \Phi_{n_1, n_2} = \frac{1}{2} \chi T [n_1(n_1 - 1) + n_2(n_2 - 1)] \). Introducing the pseudo-spin operators \( \hat{J}_z = \frac{1}{2} (\hat{a}_+^\dagger \hat{a}_+^\dagger - \hat{a}_+ \hat{a}_-), \hat{J}_y = \frac{i}{2} (\hat{a}_+ \hat{a}_+^\dagger - \hat{a}_-^\dagger \hat{a}_-), \hat{J}_z = \frac{1}{2} (\hat{a}_-^\dagger \hat{a}_+ - \hat{a}_+^\dagger \hat{a}_-) = \frac{1}{2} \hat{N}_d, \) at the final time \( t = t_f \), evolution under \( \mathcal{H}_0 \) for a period \( T \) followed by the final beamsplitter performs the transformation \( \hat{J}_z(t_f) = -\exp(\cos \phi T \hat{J}_y(t_f)) \). At \( \Omega = 0 \), Eq. 5 becomes \( \Delta \Omega = \frac{\hbar}{4 \pi \Omega} \sqrt{\text{Var}(\hat{J}_y) / \langle \hat{J}_x \rangle^2} \). Using Eq. 10, we obtain

\[ \text{Var}(\hat{J}_y) = \frac{N_t}{4} + \frac{N_t^2}{8} \left( 1 - \exp\left( -2N_t \sin^2(\chi T) \right) \right), \]

\[ \langle \hat{J}_x \rangle = \frac{N_t}{2} \exp\left( -N_t \left( -1 + \cos(\chi T) \right) \right). \]

Fig. 3 (red dashed line) shows that our analytic model gives excellent agreement with both our single-component and two-component multi-mode numeric calculations.

V. PRE-TWISTING TO REDUCE THE EFFECTS OF PHASE DIFFUSION

We can partially restore the sensitivity by implementing a pre-twisting scheme to reverse the effect of \( \mathcal{H}_{\text{int}} \). Fig. 4 shows a quasi-probability distribution formed from individual trajectories from a 2-mode TW simulation evolving under \( \mathcal{H}_{\text{int}} \). Initially, the individual trajectories are spread out in both \( \hat{J}_z \) and \( \hat{J}_y \). However, after a period of evolution, the spread in \( \hat{J}_z \) is converted into a much larger spread in \( \hat{J}_y \), which is the origin of the degradation. By applying a rotation \( \hat{U}_\theta = e^{i \theta \hat{J}_z} \), the state is twisted such that a second period of evolution under \( \mathcal{H}_{\text{int}} \) approximately revives the initial state. However, this process breaks down for larger values of \( \chi T \), as can be seen in the lower panels of fig. 4. This is because for small values of \( \chi T \), the trajectories roughly form an ellipse, which when rotated, is similar in shape to its reflection about the \( \hat{J}_z \) axis. However, for larger values of \( \chi T \), the trajectories form a bent ellipse, which when rotated about the \( \hat{J}_z \) axis, deviates significantly from its reflection about the \( \hat{J}_z \) axis, and thus the second period of nonlinear evolution does not revive the initial state [56, 57]. We note that this process could also have been achieved by simply reversing the sign of \( \chi \) for the second period of evolution. However, this is incompatible with the use of bright solitons as the require a negative interaction constant. The rotation angle that performs the rotation illustrated in fig. 4 is

\[ \theta_\chi = -\cos^{-1}(\gamma), \]
FIG. 4. (Color Online) Quasi-probability distribution for the pre-twisting sequence. Left to right: $|\Psi_0\rangle$, $\hat{U}_\theta|\Psi_0\rangle$, $\hat{U}_\theta \hat{U}_\chi |\Psi_0\rangle$, and $\hat{U}_\chi \hat{U}_\theta \hat{U}_\chi |\Psi_0\rangle$, for $\hat{U}_\chi = \exp(-iH_{int} T/\hbar)$. Top line: $\chi T = -0.03$. Bottom line: $\chi T = -0.06$. For visual clarity, a reduced number of atoms ($N_t = 100$) was used.

with

$$\gamma = \frac{(\exp(2s N_t) - 1)}{\sqrt{(\exp(2s N_t) - 1)^2 + 16s \exp(2N_t (\cos \chi T - \cos 2\chi T))}} \quad (14)$$

and $s = \sin^2 \chi T$, and is derived in appendix (VIII).

The sensitivity that this scheme provides is shown in fig. (3) (blue solid line). There are two factors that influence the sensitivity. The first is the reduction in quantum noise (Var($\hat{J}_y$)) due to this pre-twisting scheme. The second is that the $\theta$ rotation has a non-trivial effect on $d\langle \hat{J}_z \rangle / d\Omega$ due to the interplay between the phase shift accumulated before and after the twisting, with $\theta = 0(\pi)$ leading to perfect addition (cancellation) of this phase. As such, for small values of $\chi T$, $\theta_\chi$ is not the optimum angle, as the reduction in variance is offset by the partial cancellation of phase accumulation. To obtain higher sensitivities, we optimise $\theta$ numerically. The optimum sensitivity is shown in fig. (3) (black crosses). The optimum actually dips slightly below the standard quantum limit (SQL) because the final state in this case has reduced fluctuations in Var($\hat{J}_y$).

We implement the pre-twisting scheme in our multi-mode model by replacing the final 50/50 beamsplitter of the single loop scheme with a variable angle beam splitter performing the transformation $\psi_{\pm} \to \psi_{\pm} \cos \theta - i\psi_{\pm} \sin \theta e^{\pm 2i k_0 \xi}$, and then allowing the solitons to perform a second circuit of the ring before the final 50/50 beamsplitter is implemented. Again, such a transformation is easily implemented via a coherent two-photon Raman transition. However, when assessing the performance of this scheme (fig. 3 blue stars), we see that while there is generally some improvement in sensitivity when compared to the original scheme, there is a significant discrepancy between the 2-mode model and the multi-mode model. In particular, the multi-mode model predicts significantly worse sensitivity than compared to the two-mode model for intermediate values of $|g_0|$. For larger values of $|g_0|$, the multi-mode model still gives about an order of magnitude improvement compared to the original scheme, but this is still worse than what would be obtained by using a non-interacting gas.

VI. DISCUSSION

Our results generally indicate that for the case of rotation sensing with a two-component system, it is better to work in a regime with minimal interactions rather than pursuing the use of bright solitons. If working in a regime where interactions are unavoidable, then one should consider using the pre-twisting scheme presented in this letter. In a single component system, minimising interactions and ensuring that the wave-packet satisfies the conditions for distinguishable wave-packets, is favourable to the use of bright solitons. In situations when these conditions cannot be met, it may be the case.
that bright solitons provide superior performance. As the sensitivity scales with the enclosed area of the device, it is
beneficial to increase the circumference of the ring. However,
when working in the soliton regime, assuming the
magnitude of the momentum kick is held fixed, the time
taken for the solitons to complete a circuit, and therefore
the amount of phase diffusion, increases with the size of
the ring. This will ultimately limit the obtainable sen-
sitivity. In the linear regime however, the expansion of
the wave-packets scales linearly with time, such that the
conditions for wave-packet distinguishability is approxi-
mately independent of the ring circumference (the frac-
tion of the circumference covered by each wave-packet at
the final time is independent of the circumference), so no
such limitations exist.

As the phase-diffusion mechanism investigated in this
manuscript will also be present in any sensing schemes
involving bright-solitons, the results of this paper suggest
that one should always use models that include quantum
noise rather than relying exclusively on mean-field mod-
els to assess the metrological sensitivity.

However, we do not claim that the use of bright solitons
is entirely without benefit. Wave-packet spreading may
prove problematic if beamsplitters that transfer linear
momentum (rather than angular momentum, as consid-
ered in this paper) are used, as a spatially non-localised
source will experience a radial component to the momen-
tum transfer, causing mode-matching issues. Addition-
ally, it may be possible that some detection systems are
less susceptible to imperfections if the matterwaves re-
main spatially localised. It was observed in the experi-
ment of McDonald et al. [18] that the maximum sensitiv-
ity was achieved when the scattering length was tuned to
create a soliton. The reason for this was likely that the
reduction in dispersion reduced various sources of tech-
nical noise such as imperfections in the trapping poten-
tial. Furthermore, for the interrogation times used, the
two soliton wave-packets remained spatially overlapping
for the duration of the experiment, so the system would
not be subjected to the relative phase shearing noise re-
ported in this manuscript. Additionally, the experiment
was not operating at the SQL so it is unlikely that this
noise source would be observed.

Finally, we note that it has been shown that soliton dy-
namics can create non-classical states [26, 58–61]. How-
ever, it has yet to be shown that these states can be used
for enhanced matterwave interferometry, as they will be
subject to the same phase diffusion which is the subject of
this manuscript, and further modelling of these systems
should be pursued.

VII. ACKNOWLEDGEMENTS

The author would like to acknowledge useful discus-
sions with Samuel Nolan, Matthew Davis, Joel Corney,
Murray Olsen, Stuart Szigeti, Michael Bromley, John
Helm, Simon Gardiner, and Nick Robins. The numerical
simulations were performed with XMDS2 [62] on the
University of Queensland School of Mathematics and Physics
computing cluster “Dogmatix”, with thanks to Ian Mor-
timer for computing support. This work was supported
by the European Union’s Horizon 2020 research and in-
novation programme under the Marie Sklodowska-Curie
grant agreement No. 704672.

[1] Alexander D. Cronin, Jörg Schmiedmayer, and David E.
Pritchard, “Optics and interferometry with atoms and
molecules,” Rev. Mod. Phys. 81, 1051–1129 (2009).
[2] F. Riehle, Th. Kisters, A. Witte, J. Helmcke, and Ch. J.
Bordé, “Optical Ramsey spectroscopy in a rotating frame:
Sagnac effect in a matter-wave interferometer,” Phys.
Rev. Lett. 67, 177–180 (1991).

[3] T. L. Gustavson, P. Bouyer, and M. A. Kasevich, “Preci-
sion rotation measurements with an atom interferometer
gyroscope,” Phys. Rev. Lett. 78, 2046–2049 (1997).
[4] Alan Lenef, Troy D. Hammond, Edward T. Smith,
Michael S. Chapman, Richard A. Rubenstein, and
David E. Pritchard, “Rotation sensing with an atom
interferometer,” Phys. Rev. Lett. 78, 760–763 (1997).
[5] T. L. Gustavson, A. Landragin, and M. A. Kasevich,
“Rotation sensing with a dual atom-interferometer
Sagnac gyroscope,” Classical and Quantum Gravity 17,
2385 (2000).
[6] D. S. Durfee, Y. K. Shaham, and M. A. Kasevich, “Long-
term stability of an area-reversible atom-interferometer
sagnac gyroscope,” Phys. Rev. Lett. 97, 240801 (2006).
[7] B. Camuel, F. Leduc, D. Holleville, A. Gauguet, J. Fils,
A. Virdis, A. Clairon, N. Dimarçq, Ch. J. Bordé, A. Lan-
dragin, and P. Bouyer, “Six-axis inertial sensor using
cold-atom interferometry,” Phys. Rev. Lett. 97, 010402
(2006).
[8] Susannah M. Dickerson, Jason M. Hogan, Alex Sugar-
baker, David M. S. Johnson, and Mark A. Kasevich,
“Multiaxis inertial sensing with long-time point source
atom interferometer,” Phys. Rev. Lett. 111, 083001
(2013).
[9] Carlos L. Garrido Alzar, Wenhua Yan, and Arnaud Lan-
dragin, “Towards high sensitivity rotation sensing using
an atom chip,” in Research in Optical Sciences (Optical
Society of America, 2012) p. JT2A.10.
[10] P. L. Halkyard, M. P. A. Jones, and S. A. Gardiner,
“Rotational response of two-component Bose-Einstein con-
densates in ring traps,” Phys. Rev. A 81, 061602 (2010).
[11] M. C. Kandes, R. Carretero-Gonzalez, and M. W. J.
Bromley, “Phase-shift plateaus in the Sagnac effect for
matter waves,” arXiv:1306.1308 (2013).
[12] R. Stevenson, M. R. Hush, T. Bishop, I. Lesanovsky, and
T. Fernholz, “Sagnac interferometry with a single atomic
clock,” Phys. Rev. Lett. 115, 163001 (2015).
[13] M. Kolar, T. Opatrný, and Kunal K. Das, “Criticality
and spin squeezing in the rotational dynamics of a Bose-
Einstein condensate on a ring lattice,” [Phys. Rev. A 92, 043630 (2015)]

[14] Samuel P. Nolan, Jacopo Sabbatini, Michael W. J. Bromley, Matthew J. Davis, and Simon A. Haine, “Quantum enhanced measurement of rotations with a spin-1 Bose-Einstein condensate in a ring trap,” [Phys. Rev. A 93, 023616 (2016)]

[15] T. A. Bell, J. A. P. Glidden, L. Humbert, M. W. J. Bromley, S. A. Haine, M. J. Davis, T. W. Neely, M. A. Baker, and H. Rubinstein-Dunlop, “Bose-Einstein condensation in large time-averaged optical ring potentials,” [New Journal of Physics 18, 035003 (2016)].

[16] Simon A. Haine, “Mean-field dynamics and fisher information in matter wave interferometry,” [Phys. Rev. Lett. 116, 230404 (2016)]

[17] P. Navez, S. Pandey, H. Mas, K. Poulios, T. Fernholz, and W. von Klitzing, “Matter-wave interferometers using TAAP rings,” [New Journal of Physics 18, 075014 (2016)].

[18] G. D. McDonald, C. C. N. Kuhn, K. S. Hardman, S. Betnets, P. J. Everitt, P. A. Altin, J. E. Debs, J. D. Close, and N. P. Robins, “Bright solitonic matter-wave interferometer,” [Phys. Rev. Lett. 113, 013002 (2014)].

[19] Kevin E. Strecker, Guthrie B. Partridge, Andrew G. Truscott, and Randall G. Hulet, “Formation and propagation of matter-wave soliton trains,” [Nature 417, 150–153 (2002)].

[20] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, “Formation of a matter-wave bright soliton,” [Science 296, 1290–1293 (2002)].

[21] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, “Bright matter wave solitons in Bose-Einstein condensates,” [New Journal of Physics 5, 73 (2003)].

[22] Antonio Negretti and Carsten Henkel, “Enhance phase sensitivity and soliton formation in an integrated BEC interferometer,” [Journal of Physics B: Atomic, Molecular and Optical Physics 37, L385 (2004)].

[23] Simon L. Cornish, Sarah T. Thompson, and Carl E. Wieman, “Formation of bright matter-wave solitons during the collapse of attractive Bose-Einstein condensates,” [Phys. Rev. Lett. 96, 170401 (2006)].

[24] N. Veretenov, Yu. Rozhdestvensky, N. Rosanov, V. Smirnov, and S. Fedorov, “Interferometric precision measurements with Bose–Einstein condensate solitons formed by an optical lattice,” [The European Physical Journal D 42, 455–460 (2007)].

[25] Yaroslav V. Kartashov, Boris A. Malomed, and Lluis Torner, “Solitons in nonlinear lattices,” [Rev. Mod. Phys. 83, 247–305 (2011)].

[26] A. D. Martin and J. Ruostekoski, “Quantum dynamics of atomic bright solitons under splitting and recollision, and implications for interferometry,” [New Journal of Physics 14, 043040 (2012)].

[27] J. L. Helm, T. P. Billam, and S. A. Gardiner, “Bright matter-wave soliton collisions at narrow barriers,” [Phys. Rev. A 85, 053621 (2012)].

[28] A. L. Marchant, T. P. Billam, T. P. Wiles, M. M. H. Yu, S. A. Gardiner, and S. L. Cornish, “Controlled formation and reflection of a bright solitary matter-wave,” [Nature Communications 4, 1865 EP – (2013)].

[29] J. Polo and V. Alhufinger, “Soliton-based matter-wave interferometer,” [Phys. Rev. A 88, 053628 (2013)].

[30] J. Cuevas, P. G. Kevrekidis, B. A. Malomed, P. Dyke, and R. G. Hulet, “Interactions of solitons with a Gaussian barrier: splitting and recombination in quasi-one-dimensional and three-dimensional settings,” [New Journal of Physics 15, 063006 (2013)].

[31] J. L. Helm, S. J. Rooney, Christoph Weiss, and S. A. Gardiner, “Splitting bright matter-wave solitons on narrow potential barriers: Quantum to classical transition and applications to interferometry,” [Phys. Rev. A 89, 033610 (2014)].

[32] J. L. Helm, S. L. Cornish, and S. A. Gardiner, “Sagnac interferometry using bright matter-wave solitons,” [Phys. Rev. Lett. 114, 134101 (2015)].

[33] Hitetsugu Sakaguchi and Boris A Malomed, “Matter-wave soliton interferometer based on a nonlinear splitter,” [New Journal of Physics 18, 025020 (2016)].

[34] R. Demkowicz-Dobrzański, M. Jarzyna, and J. Kolodyński, “Quantum limits in optical interferometry,” [Progress in Optics 35 (2015)].

[35] M. Kritsotakis, S. S. Szigeti, J. A. Dunningham, and Haine S. A., “Optimal matterwave gravimetry,” [arXiv:1710.06340 (2017)].

[36] Iva Brezničnova, Lee A. Collins, Katharina Ludwig, Barry I. Schneider, and Joachim Burgdörfer, “Wave chaos in the nonequilibrium dynamics of the Gross-Pitaevskii equation,” [Phys. Rev. A 83, 043611 (2011)].

[37] M. F. Andersen, C. Ryu, Pierre Cladé, Vasant Natarajan, A. Vaziri, K. Helmerson, and W. D. Phillips, “Quantized rotation of atoms from photons with orbital angular momentum,” [Phys. Rev. Lett. 97, 170406 (2006)].

[38] Stuart Moulder, Scott Beattie, Robert P. Smith, Naam Tammuz, and Zoran Hadzibabic, “Quantized supercurrent decay in an annular Bose-Einstein condensate,” [Phys. Rev. A 86, 013629 (2012)].

[39] The only effect of \( g_{\pm} \) is a slight modification of the velocity of the wavepackets, slightly altering the collision time \( T \).

[40] Simon A. Haine and Mattias T. Johnsson, “Dynamic scheme for generating number squeezing in Bose-Einstein condensates through nonlinear interactions,” [Phys. Rev. A 80, 023611 (2009)].

[41] J. M. Radcliffe, “Some properties of coherent spin states,” [Journal of Physics A: General Physics 4, 313 (1971)].

[42] M. J. Steel, M. K. Olsen, L. I. Plimak, P. D. Drummond, S. M. Tan, M. J. Collett, D. F. Walls, and R. Graham, “Dynamical quantum noise in trapped Bose-Einstein condensates,” [Phys. Rev. A 58, 4824–4835 (1998)].

[43] Alice Sinatra, Carlos Lobo, and Yvan Castin, “The truncated Wigner method for Bose-condensed gases: limits of validity and applications,” [Journal of Physics B: Atomic, Molecular and Optical Physics 35, 3599 (2002)].

[44] A. A. Norrie, R. J. Ballagh, and C. W. Gardiner, “Quantum turbulence and correlations in Bose-einstein condensate collisions,” [Phys. Rev. A 73, 043617 (2006)].

[45] Peter D. Drummond and Bogdan Opanchuk, “Truncated wigner dynamics and conservation laws,” [Phys. Rev. A 96, 043616 (2017)].

[46] J. Ruostekoski and A. D. Martin, “The truncated wigner method for Bose gases,” in [Quantum Gasses], edited by The editor (Imperial College Press, 2013) pp. 203–214.

[47] S. A. Haine, J. Lau, R. P. Anderson, and M. T. Johnsson, “Self-induced spatial dynamics to enhance spin squeezing via one-axis twisting in a two-component Bose-Einstein condensate,” [New Journal of Physics 15, 063006 (2013)].

[48] A. D. Martin and J. Ruostekoski, “Quantum dynamics of atomic bright solitons under splitting and recollision, and implications for interferometry,” [New Journal of Physics 14, 043040 (2012)].

[49] J. L. Helm, T. P. Billam, and S. A. Gardiner, “Bright matter-wave soliton collisions at narrow barriers,” [Phys. Rev. A 85, 053621 (2012)].

[50] A. L. Marchant, T. P. Billam, T. P. Wiles, M. M. H. Yu, S. A. Gardiner, and S. L. Cornish, “Controlled formation and reflection of a bright solitary matter-wave,” [Nature Communications 4, 1865 EP – (2013)].

[51] J. Polo and V. Alhufinger, “Soliton-based matter-wave interferometer,” [Phys. Rev. A 88, 053628 (2013)].
condensate,” Phys. Rev. A 90, 023613 (2014)

[48] Stuart S. Szigeti, Robert J. Lewis-Swan, and Simon A. Haine, “Pumped-up su(1,1) interferometry,” Phys. Rev. Lett. 118, 150401 (2017)

[49] P. D. Drummond and A. D. Hardman, “Simulation of quantum effects in raman-active waveguides,” EPL (Europhysics Letters) 21, 279 (1993).

[50] P. B. Blakie, A. S. Bradley, M. J. Davis, R. J. Ballagh, and C. W. Gardiner, “Dynamics and statistical mechanics of ultra-cold Bose gases using c-field techniques,” Advances in Physics, Advances in Physics 57, 363–455 (2008).

[51] C. W. Gardiner and P. Zoller, Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics, 3rd ed. (Springer, Berlin and Heidelberg, 2004).

[52] M.K. Olsen and A.S. Haine, “Numerical representation of quantum states in the positive-P and wigner representations,” Optics Communications 282, 3924 – 3929 (2009).

[53] Mattias T. Johnsson and Simon A. Haine, “Generating quadrature squeezing in an atom laser through self-interaction,” Phys. Rev. Lett. 99, 010401 (2007).

[54] Max F. Riedel, Pascal Böhi, Yun Li, Theodor W. Hänsch, Alice Sinatra, and Christoph Weiss, “Atom-chip-based generation of entanglement for quantum metrology,” Nature 464, 1170–1173 (2010)

[55] S. A. Haine and A. J. Ferris, “Surpassing the standard quantum limit in an atom interferometer with four-mode entanglement produced from four-wave mixing,” Phys. Rev. A 84, 043624 (2011).

[56] Samuel P. Nolan, Stuart S. Szigeti, and Simon A. Haine, “Optimal and robust quantum metrology using interaction-based readouts,” Phys. Rev. Lett. 119, 193601 (2017).

[57] S. S. Mirkhalaf, Nolan S. P., and Haine S. A., “Robustifying twist-and-turn entanglement with interaction-based readout,” arXiv:1803.08789 (2018).

[58] Christoph Weiss and Yvan Castin, “Creation and detection of a mesoscopic gas in a nonlocal quantum superposition,” Phys. Rev. Lett. 102, 010403 (2009).

[59] Alexej I. Streitsova, Offr E. Alon, and Lorenz S. Cederbaum, “Scattering of an attractive Bose-Einstein condensate from a barrier: Formation of quantum superposition states,” Phys. Rev. A 80, 043616 (2009).

[60] Maciej Lewenstein and Boris A Malomed, “Entanglement generation by collisions of quantum solitons in the Born approximation,” New Journal of Physics 11, 113014 (2009).

[61] Bettina Gertjerenken, Thomas P. Billam, Caroline L. Blackley, C. Ruth Le Sueur, Lev Khaykovich, Simon L. Cornish, and Christoph Weiss, “Generating mesoscopic Bell states via collisions of distinguishable quantum bright solitons,” Phys. Rev. Lett. 111, 100406 (2013).

[62] Graham R. Dennis, Joseph J. Hope, and Mattias T. Johnsson, “Xmds2: Fast, scalable simulation of coupled stochastic partial differential equations,” Computer Physics Communications 184, 201 – 208 (2013).

SUPPLEMENTAL MATERIAL: QUANTUM NOISE IN SOLITON MATTER-WAVE INTERFEROMETRY

In this supplemental material we provide further details on the calculations in the main text. Specifically, we derive the rotation angle required for the pre-twisting scheme.

VIII. DERIVATION OF $\theta_\chi$ (EQ. (13))

In this appendix, we provide further details on the calculations in the main text. Specifically, we derive the rotation angle required for the pre-twisting scheme. The angle required for our pre-twisting scheme, $\theta_\chi$, is the angle such that rotation about the $J_z$ axis returns the variance of $J_z$ to its original value, as illustrated in Fig. 5.

The action of the variable angle beamsplitter $\hat{U}_\theta = \exp(i\theta \hat{J}_x)$ on the psuedo-spin operators before $(t = T)$ and after $(t = t_1)$ the rotation is

$$\hat{J}_z(t_1) = \hat{U}_\theta^\dagger \hat{J}_z(T) \hat{U}_\theta = \hat{J}_z(T) \cos \theta_\chi + \hat{J}_y(T) \sin \theta_\chi \tag{15}$$

$$\hat{J}_y(t_1) = \hat{U}_\theta^\dagger \hat{J}_y(T) \hat{U}_\theta = \hat{J}_y(T) \cos \theta_\chi - \hat{J}_z(T) \sin \theta_\chi \tag{16}$$

Therefore, after the rotation, the variance in $\hat{J}_z$ is

$$V(\hat{J}_z(t_1)) = \langle \hat{J}_z^2(t_1) \rangle$$

$$= \langle \hat{J}_z^2(T) \rangle \cos^2 \theta_\chi + \langle \hat{J}_y^2(T) \rangle \sin^2 \theta_\chi$$

$$+ \cos \theta_\chi \sin \theta_\chi \left( \langle \hat{J}_z(T) \hat{J}_y(T) \rangle + \langle \hat{J}_y(T) \hat{J}_z(T) \rangle \right) \tag{17}$$

since the state is chosen such that $\langle \hat{J}_z(t_1) \rangle = \langle \hat{J}_z(T) \rangle = 0$. The evolution under $\hat{U}_\theta = \exp(-i\mathcal{H}_m T)$ commutes with $\hat{J}_z$, so $\langle \hat{J}_z^2(T) \rangle = \langle \hat{J}_z^2(0) \rangle = N_i/4$. The angle $\theta_\chi$ is defined as the angle such that $\langle \hat{J}_z^2(t_1) \rangle = \langle \hat{J}_z^2(T) \rangle = N_i/4$. Expressing
FIG. 5. (Color Online) The rotation angle $\theta_\chi$ about the $J_z$ axis required for re-phasing after a second application of $\hat{U}_\chi$ is the angle that has the same variance in $\hat{J}_z$ before and after rotation.

The pseudo-spin operators in terms of bosonic creation and annihilation operators, and making the substitution $\hat{a}^+_+ \rightarrow \hat{a}$ and $\hat{a}^- \rightarrow \hat{b}$ for ease of notation gives

$$
\hat{J}_z^2 = \frac{1}{4} \left( \hat{a}^+_+ \hat{a}^+_+ \hat{a}^+_+ \hat{a}^+_+ + \hat{b}^+_+ \hat{b}^+_+ + \hat{a}^+_+ \hat{a}^+_+ + \hat{b}^+_+ \hat{b}^+_+ - 2 \hat{a}^+_+ \hat{a}^+_+ \hat{b}^+_+ \hat{b}^+_+ \right)
$$

(18)

$$
\hat{J}_y^2 = \frac{1}{4} \left( 2 \hat{a}^+_+ \hat{b}^+_+ \hat{a}^+_+ \hat{a}^+_+ + \hat{a}^+_+ \hat{a}^+_+ + \hat{b}^+_+ \hat{b}^+_+ - \hat{a}^+_+ \hat{b}^+_+ \hat{a}^+_+ \hat{b}^+_+ \right)
$$

(19)

$$
\hat{J}_z \hat{J}_y + \hat{J}_y \hat{J}_z = \frac{i}{2} \left( \hat{a}^+_+ \hat{a}^+_+ \hat{b}^+_+ + \hat{a}^+_+ \hat{b}^+_+ \hat{b}^+_+ - \hat{a}^+_+ \hat{b}^+_+ \hat{b}^+_+ - \hat{a}^+_+ \hat{b}^+_+ \hat{b}^+_+ \right)
$$

(20)

In order to evaluate these expressions, we need to calculate terms such as $\langle \hat{a}^+_+ \hat{a}^+_+ \hat{b}^+_+ \rangle$ with respect to the state

$$
|\Psi(T)\rangle = e^{-|\alpha|^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{\alpha_{n_1} \alpha_{n_2}}{\sqrt{n_1! \sqrt{n_2!}}} |n_1, n_2\rangle e^{-i\Phi_{n_1, n_2}},
$$

(21)

where

$$
\Phi_{n_1, n_2} = \frac{1}{2} \chi T [n_1(n_1 - 1) + n_2(n_2 - 1)],
$$

(22)

and $\alpha = \sqrt{N_i/2}$. We will explicitly compute one example, and provide the rest of these operator moments in a table.

$$
\langle \hat{a}^+_+ \hat{a}^+_+ \hat{b}^+_+ \rangle = e^{-2|\alpha|^2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_1=0}^{n_1} \sum_{n_2=0}^{n_2} \frac{(\alpha^*)^{m_1} (\alpha^*)^{m_2} \alpha^{n_1} \alpha^{n_2}}{\sqrt{m_1! m_2! n_1! n_2!}} \langle m_1, m_2 | \hat{a}^+_+ \hat{a}^+_+ \hat{b}^+_+ | n_1, n_2 \rangle e^{i(\Phi_{m_1, m_2} - \Phi_{n_1, n_2})}
$$

$$
= e^{-2|\alpha|^2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_1=0}^{n_1} \sum_{n_2=0}^{n_2} \frac{(\alpha^*)^{m_1} (\alpha^*)^{m_2} \alpha^{n_1} \alpha^{n_2}}{\sqrt{m_1! m_2! n_1! n_2!}} n_1 \sqrt{n_1 + 1} \sqrt{n_2} \langle m_1, m_2 | n_1 + 1, n_2 - 1 \rangle e^{i(\Phi_{m_1, m_2} - \Phi_{n_1, n_2})}
$$

$$
= e^{-2|\alpha|^2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_1=0}^{n_1} \sum_{n_2=0}^{n_2} \frac{(\alpha^*)^{m_1} (\alpha^*)^{m_2} \alpha^{n_1} \alpha^{n_2}}{\sqrt{m_1! m_2! n_1! n_2!}} n_1 \sqrt{n_1 + 1} \sqrt{n_2} e^{i(\Phi_{m_1, m_2} - \Phi_{n_1, n_2})} \delta_{m_1, n_1+1} \delta_{m_2, n_2-1}
$$

$$
= e^{-2|\alpha|^2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \sum_{n_1=1}^{n_1+1} \frac{(\alpha^*)^{n_1+1} (\alpha^*)^{m_2} \alpha^{n_1} \alpha^{m_2+1}}{\sqrt{(n_1+1)! m_2! (m_2+1)!}} n_1 \sqrt{n_1 + 1} \sqrt{m_2 + 1} e^{i\chi T n_1} e^{i\chi T (m_2 - m_2)}
$$

$$
= |\alpha|^4 e^{i\chi T} e^{-2|\alpha|^2} \sum_{m_2=0}^{\infty} \sum_{n_1=1}^{\infty} \frac{(1)^{m_2} e^{i\chi T} (n_1 - 1)!}{m_2!} \langle \alpha^2 e^{i\chi T} \rangle^{m_2}
$$

$$
= |\alpha|^4 e^{i\chi T} e^{-2|\alpha|^2} e^{i|\alpha|^2 e^{i\chi T}} e^{i|\alpha|^2 e^{-i\chi T}}
$$

$$
= N_i e^{i\chi T} \exp \left( N_i (\cos \chi T - 1) \right).
$$

(23)
The complete set of moments required to calculate \( \langle \hat{J}_z^2(t_1) \rangle \) is

\[
\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{b}^\dagger \hat{b} \rangle = \frac{N_t}{2} \quad (24a)
\]
\[
\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle = \langle \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \rangle = \frac{N_t^2}{4} \quad (24b)
\]
\[
\langle \hat{a}^\dagger \hat{a} \hat{b} \hat{b} \rangle = \langle \hat{a} \hat{b} \hat{b}^\dagger \hat{a}^\dagger \rangle = \frac{N_t^2}{4} \exp[(N_t(\cos 2\chi T - 1))] \quad (24c)
\]
\[
\langle \hat{a}^\dagger \hat{a} \hat{b} \hat{b} \rangle = \langle \hat{a} \hat{b} \hat{b}^\dagger \hat{a}^\dagger \rangle = \frac{N_t^2}{4} e^{i\chi T} \exp[N_t(\cos \chi T - 1)] \quad (24d)
\]
\[
\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle = \langle \hat{a} \hat{b} \hat{b}^\dagger \hat{a}^\dagger \rangle = \frac{N_t^2}{4} e^{-i\chi T} \exp[N_t(\cos \chi T - 1)] \cdot (24e)
\]

The solution to \( \langle \hat{J}_z^2(t_1) = \langle \hat{J}_z^2(T) \rangle \) for \( \theta_\chi \) gives four non-trivial solutions:

\[
\theta_\chi = \cos^{-1}(\gamma) \quad (25a)
\]
\[
\theta_\chi = \cos^{-1}(-\gamma) \quad (25b)
\]
\[
\theta_\chi = -\cos^{-1}(\gamma) \quad (25c)
\]
\[
\theta_\chi = -\cos^{-1}(-\gamma), \quad (25d)
\]

where

\[
\gamma = \frac{\langle \hat{J}_z^2(T) \rangle - \langle \hat{J}_y^2(T) \rangle}{\sqrt{\langle \hat{J}_z^2(T) \rangle^2 + \langle \hat{J}_y^2(T) \rangle^2 + \langle \hat{J}_z(T) \hat{J}_y(T) + \hat{J}_y(T) \hat{J}_z(T) \hat{J}_z(T) - 2 \langle \hat{J}_z^2(T) \rangle \langle \hat{J}_y^2(T) \rangle \rangle}} \quad (26)
\]

\[
= \frac{(\exp(2N_t \sin^2 \chi T) - 1)}{\sqrt{(\exp(2N_t \sin^2 \chi T) - 1)^2 + 16 \sin^2 \chi T \exp(2N_t(\cos \chi T - \cos 2\chi T))}}.
\]

Of those solutions, the only one that gives better performance than the single loop scheme is Eq. (25d), which is what was used for both the two-mode and multi-mode pre-twisting calculations.