ON STRUCTURE OF LINEAR DIFFERENTIAL OPERATORS, ACTING IN LINE BUNDLES

VALENTIN LYCHAGIN AND VALERIY YUMAGUZHIN

Abstract. We study differential invariants of linear differential operators and use them to find conditions for equivalence of differential operators acting in line bundles over smooth manifolds with respect to groups of automorphisms.

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V. Yumaguzhin is a corresponding author; phone: +79056368327, e-mail: yuma@diffiety.botik.ru.
1. Introduction

This paper is a continuation of papers [15] and [16], where we analyzed the equivalence of linear differential operators of order \( k = 2, n \geq 3 \), and \( k = 3, n = 2 \), acting in line bundles over a smooth manifolds of dimension \( n \). In this paper we consider the general case, when \( k \geq 3 \) and \( n \geq 2 \).

Possibly Bernard Riemann [22] was the first, who analyzed the equivalence problem for second order scalar differential operators with respect to diffeomorphism group.

He found the curvature of the metric, defined by the symbol of differential operator, as an obstruction to transform differential operators of the second order to operators with constant coefficients.

In the case, when the dimension of the base manifold equals two, Pierre-Simon Laplace [14] found "Laplace invariants" for the second
order hyperbolic differential operators, which are relative invariants, and Lev Ovsyannikov in paper [19] found the corresponding invariants.

All invariants for hyperbolic equations in dimension two were found by Nail Ibragimov in paper [8].

The method, we used in this paper, is very similar to the method, we used in papers [15],[16].

First of all, we are looking for an affine connection on the base manifold $M$ and a linear connection in the line bundle $\xi$, which are naturally associated with our differential operator.

In the case $k = 2$, we used the Levi-Civita connection on $M$, given by the principal symbol of the operator. Then the connection in the line bundle $\xi$ was chosen in such a way that the subsymbol of the operator becomes to be trivial (see [15] for more details).

Remark that all these constructions could be applied for operators of the constant type only. The case of mixed type operators is singular and, as we know, is singular not only for this method.

In the case $k = 3, n = 2$, we’ve used connections, originally found by Viktor Wagner [25] and Shiing-Shen Chern [1],[6] in pure geometrical context, instead of the Levi-Civita connection. The linear connection in the line bundle we found by posed some conditions on the subsymbol of the operator also.

The regularity conditions to use this method, require also that the operator has distinct characteristics and once more, the mixed type operators are excluded from the consideration.

The case of ordinary differential operators, $n = 1$, is also exceptional and we’ll consider it here only as an illustration of the general method.

Remark that the case of ordinary differential operators of the second order was considered by Niky Kamran and Peter Olver in paper [9] and the case of linear ordinary differential equations was investigated by Ernest Wilczynski in book [26].

The second step, when we begin to use these connections, is based on quantization, defining by the connections [15],[16].

The term ”quantization” is taken from the wave mechanics [5], where it was used for reconstruction of differential operators from their symbols.
Necessity of using connections in such type procedures follows directly from standard requirements [17].

To extend this method for the general case, we, first of all, remark that such connections and the corresponding quantizations are exist if our differential operator has the constant type, i.e. GL-orbits of the symbols of the operator do not depend on points of $M$.

In the case of operators of order two it is exactly requirement that operator is elliptic or (ultra) hyperbolic. For the case $k = 3, n = 2$ it is the requirement of distinct characteristics.

In other cases the description of GL-orbits of the symbols is the classical algebraic problem of the GL-classification of $n$-ary forms. We use here slightly modified method, suggested in [3],[4], to find rational invariants and regular orbits of such GL-action.

Thus, for constant type operators, the associated connections exist and the associated quantization allows us to split and represent the differential operator as a sum of symmetric contravariant tensors.

Then, the more or less routine machinery with using of the connections and the splitting, allows us to find the field of rational differential invariants of differential operators.

The third and the last step is based on using of these differential invariants and natural coordinates delivering by differential invariants.

Namely, the values of $n$-invariants on a differential operator in general position could be served as local coordinates on $M$, which we call natural, because coefficients of given operator in these coordinates will not be changed after applying an automorphism to our operator.

Finally, we introduce the notion of the natural atlas and show how to get the global classification of constant type differential operators and the corresponding homogeneous differential equations.

2. Differential operators

2.1. Notations. The notations we use in this paper are similar to notations used in papers [15],[16].

Let $M$ be an $n$-dimensional manifold and let $\pi : E(\pi) \to M$ be a vector bundle.

We denote by $\tau : TM \to M$ and $\tau^* : T^*M \to M$ the tangent and respectively cotangent bundles over manifold $M$, and by $1 : \mathbb{R} \times M \to M$ we denote the trivial line bundle.
The symmetric and exterior powers of a vector bundle $\pi : E(\pi) \to M$ will be denoted by $S^k(\pi)$ and $\Lambda^k(\pi)$.

The module of smooth sections of bundle $\pi$ we denote by $C^\infty(\pi)$, and for the cases tangent, cotangent and the trivial bundles we’ll use the following notations: $\Sigma_k(M) = C^\infty(S^k(\tau))$ — the module of symmetric $k$-vectors and $\Sigma_k^T(M) = C^\infty(S^k(\tau^*))$ — the module of symmetric $k$-forms, $\Omega_k(M) = C^\infty(\Lambda^k(\tau))$ — the module of skew-symmetric $k$-vectors and $\Omega_k^T(M) = C^\infty(\Lambda^k(\tau^*))$ — the module of exterior $k$-forms, $C^\infty(1) = C^\infty(M)$.

**2.2. Jets.** The bundles of $k$-jets of sections of bundle $\pi$ we denote by $\pi_k : J^k(\pi) \to M$ and by $\pi_{k,l} : J^k(\pi) \to J^l(\pi)$ we denote the projection (=reduction) of $k$-jets on $l$-jets, $k \geq l$.

There is the following exact sequence of vector bundles $0 \to S^k(\tau^*) \otimes \pi \to J^k(\pi) \xrightarrow{\pi_{k,k-1}} J^{k-1}(\pi) \to 0$, connecting bundles of $(k - 1)$ and $k$-jets.

The inverse limit of the sequence $J^k(\pi) \xrightarrow{\pi_{k,k-1}} J^{k-1}(\pi)$ is called the bundle of infinite-jets:

$$J^\infty(\pi) = \lim_{\leftarrow} J^k(\pi).$$

The smooth functions on $J^\infty(\pi)$ are just smooth functions on some finite jet bundle. The special vector fields on $J^\infty(\pi)$, we call them horizontal, are extremely important in this paper. Namely, let $X$ be a vector field on manifold $M$. Then by a total lift of $X$ we understand a derivation $\hat{X}$ in the algebra of smooth functions $C^\infty(J^k(\pi))$, where $\hat{X} : C^\infty(J^k(\pi)) \to C^\infty(J^{k+1}(\pi))$, for all $k = 0, 1, \ldots$, and such the following universal property holds:

$$j_{k+1}(S)^* \left( \hat{X}(f) \right) = X(j_k(S)^*(f)),$$

for any section $S \in C^\infty(\pi)$, and function $f \in C^\infty(J^k(\pi))$, and for $k = 0, 1, \ldots$

Shortly, the universal property could be written in the form:

$$j_{k+1}(S)^* \circ \hat{X} = X \circ j_k(S)^*.$$
Linear combinations of total derivations of the form
\[ \sum_{i} \lambda_i \tilde{X}_i, \]
where \( \lambda_i \in C^\infty (J^k (\pi)) \), we’ll call horizontal vector fields on the space \( J^k (\pi) \), and linear combinations of compositions
\[ \sum_{i_1 + \ldots + i_l \leq N} \lambda_{i_1 \ldots i_l} \tilde{X}_{i_1} \circ \ldots \circ \tilde{X}_{i_l} \]
we’ll call total differential operators of order \( \leq N \) on the space \( J^k (\pi) \), if all \( \lambda_{i_1 \ldots i_l} \in C^\infty (J^k (\pi)) \).

We denote by \( \Sigma_N (\pi) \) the module generated by linear combinations
\[ \sum_{i_1 + \ldots + i_l = N} \lambda_{i_1 \ldots i_N} \tilde{X}_{i_1} \cdot \ldots \cdot \tilde{X}_{i_N} \]
of the symmetric products of horizontal vector fields and call them total symbols of degree \( N \) and order \( k \).

Let \( \omega_i \) be exterior (or symmetric) differential \( l \)-forms on manifold \( M \). Then linear combinations of pullbacks \( \pi^* (\omega_i) \) (which we’ll continue to denote by \( \omega_i \)) of the form \( \sum_i \lambda_i \omega_i \), where \( \lambda_i \in C^\infty (J^k (\pi)) \), we’ll call horizontal differential forms of degree \( l \) and order \( k \). The modules of such forms we’ll denote by \( \Omega^k (\pi) \).

2.3. Universal constructions. In the case when the bundle \( \pi \) is a tensor bundle or a bundle of differential operators we’ll need the following universal construction which generalizes the construction of the universal Liouville form on the cotangent bundle.

Let’s consider, for example, the case when \( \pi = \Lambda^l (\tau^*) \) is the bundle of exterior differential forms. Then it is easy to see that there is and unique horizontal \( l \)-form \( \rho^0_l \in \Omega^l (\pi) \) of the order zero and such that
\[ j_0 (\omega)^* (\rho^0_l) = \omega, \]
for all \( \omega \in \Omega^l (M) \).

For the case of contravariant tensors, say \( \pi = \tau \), the universal construction goes in the following way.

We define the universal vector field as a horizontal vector field \( \nu_1 \) of order zero such that
\[ j_1 (X)^* (\nu_1 (f)) = X (f), \]
for all vector fields $X$ on $M$ and all functions $f \in C^\infty(M)$.

**Coordinates**

Let $(x_1, \ldots, x_n)$ be local coordinates on $M$ and $(u^\alpha)$ or $(u_\alpha)$ be induced coordinates in the bundle $\Lambda^l(\tau^*)$ or $\Lambda^l(\tau)$.

Here $\alpha = (\alpha_1 < \alpha_2 < \ldots < \alpha_n)$ are multi indices.

Then $\rho_1 = \sum u^a dx_i$ is the universal Liouville form and

$$\rho^a_1 = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_l} u^a dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_l}.$$ 

For contravariant tensors we have

$$\nu_1 = \sum u_i \frac{d}{dx_i}$$

and

$$\nu^a_i = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_l} u_{\alpha} \frac{d}{dx_{\alpha_1}} \wedge \cdots \wedge \frac{d}{dx_{\alpha_l}},$$

where $\frac{d}{dx_i}$ are the total derivations.

2.4. **Symbols.** We denote by $\text{Diff}_k(\xi)$ the $C^\infty(M)$-module of linear differential operators of order $\leq k$, acting in the vector bundle $\xi$, and by $\pi : \text{Diff}_k(\xi) \to M$ we denote the bundle of differential operators, thus $C^\infty(\pi) = \text{Diff}_k(\xi)$ in this case.

By leading or principal symbol $\sigma_k = \text{smbl}_k(A)$ of operator $A \in \text{Diff}_k(\xi)$ we mean the equivalence class

$$\text{smbl}_k(A) = A \mod \text{Diff}_{k-1}(\xi).$$

It is known that the symbol could be also viewed as a fibre-wise homogeneous polynomial of degree $k$ on the cotangent bundle with values in the bundle of endomorphisms

$$\sigma_k \in S^k(\tau) \otimes \text{End}(\xi)$$

and the following sequence

$$0 \to \text{Diff}_{k-1}(\xi) \to \text{Diff}_k(\xi) \xrightarrow{\text{smbl}_k} \Sigma_k(M) \otimes \text{End}(\xi) \to 0,$$

exact.

From now on the case, that we consider in this paper, $\xi$ is a line bundle. Then the above sequence takes the form

$$0 \to \text{Diff}_{k-1}(\xi) \to \text{Diff}_k(\xi) \xrightarrow{\pi} \Sigma_k(M) \to 0,$$  \quad (1)
and because of this we also call *symbols* elements of $\Sigma_k = \Sigma_k(M)$.

The universal construction, discussed above, in this case gives us a universal symbols $\rho^s_k$ - horizontal symmetric $k$-vector fields of order zero such that

$$j_1 (\sigma)^* ((df_1 \cdot \cdot \cdot df_k] \rho^s_k) = (df_1 \cdot \cdot \cdot df_k] \sigma,$$

for all symbols $\sigma \in \Sigma_k$ on $M$ and all functions $f_i \in C^\infty (M)$.

Here we denoted by $\cdot$ the symmetric product and by $\lozenge$ the hook operator.

In local coordinates the universal symbol $\rho^s_k$ has the same form as $\nu^s_k$, where we changed the exterior product by the symmetric one

$$\nu^s_k = \sum u_\alpha \frac{d}{dx_{\alpha_1}} \cdot \cdot \cdot \frac{d}{dx_{\alpha_l}}.$$

In the case, when line bundle is trivial, $\xi = 1$, and $\pi = Diff_k(1)$ is the bundle of scalar differential operators the universal construction gives us a total differential operator $\Box_k$ on $J^0(\pi)$ of order $k$, $\Box_k : C^\infty (J^0(\pi)) \to C^\infty (J^k(\pi))$, such that

$$A (f) = j_{k+1} (A)^* (\Box_k (f)),$$

for all $A \in Diff_k(1)$ and $f \in C^\infty (M)$.

Remark, that $\Box_k : C^\infty (J^l(\pi)) \to C^\infty (J^{l+k}(\pi))$, for all $l = 1, 2, \ldots$, and $\Box_k$ has the form

$$\Box_k = \sum_{|\alpha| \leq k} u_\alpha \left( \frac{d}{dx} \right)^\alpha,$$

where $(x_1, \ldots, x_n, u_\alpha, 0 \leq |\alpha| \leq k)$ are canonical coordinates in the bundle $\pi : Diff_k(1) \to M$.

2.5. **Total lifts.** Differential operators $\Delta : C^\infty (\alpha) \to C^\infty (\beta)$ of order $k$, acting from a vector bundle $\alpha$ to another vector bundle, say $\beta$, could be lifted to operators $\hat{\Delta} : C^\infty (\hat{\alpha}) \to C^\infty (\hat{\beta})$, where $\hat{\alpha}$ and $\hat{\beta}$ are vector bundles over $J^\infty (\pi)$, induced by the projection $\pi_\infty : J^\infty (\pi) \to M$. The operator $\hat{\Delta}$, which we call *total lift* of $\Delta$, defines also by the universal property:

$$j_{k+1} (h)^* \circ \hat{\Delta} = \Delta \circ j_l (h)^*,$$

for all sections $h \in C^\infty (\pi)$ and $l = 0, 1, \ldots$. 

We'll especially use the following two cases. The total lift of the de Rham operator
\[ d : \Omega^i (M) \to \Omega^{i+1} (M) \]
is the total differential
\[ \hat{d} : \Omega^i (\pi) \to \Omega^{i+1} (\pi). \]

In the another case, when
\[ d\nabla : C^\infty (\alpha) \to C^\infty (\alpha) \otimes \Omega^1 (M) \]
is a covariant differential of a connection \( \nabla \) in the bundle \( \alpha \), operator
\[ \hat{d}\nabla : C^\infty (\hat{\alpha}) \to C^\infty (\hat{\alpha}) \otimes \Omega^1 (\pi) \]
is the total covariant differential.

2.6. Symbols and Quantization. Let \( \Sigma^- = \oplus_{k \geq 0} \Sigma^k (M) \) be the graded algebra of symmetric differential forms and let \( \Sigma^- (\xi) = C^\infty (\xi) \otimes \Sigma^- \) be the graded \( \Sigma^- \)-module of symmetric differential forms with values in bundle \( \xi \).

Assume that we have two connections: connection \( \nabla \) on manifold \( M \) and linear connection \( \nabla^\xi \) in the bundle \( \xi \).

Then the covariant differentials
\[ d\nabla : \Omega^1 (M) \to \Omega^1 (M) \otimes \Omega^1 (M), \]
and
\[ d\nabla^\xi : C^\infty (\xi) \to C^\infty (\xi) \otimes \Omega^1 (M) \]
define two derivations
\[ d\nabla^* : \Sigma^- \to \Sigma^{*-1}, \]
\[ d\nabla^\xi^* : \Sigma^- (\xi) \to \Sigma^{*-1} (\xi), \]
of degree one in graded algebra \( \Sigma^- \) and graded \( \Sigma^- \)-module \( \Sigma^- (\xi) \) respectively.

Namely, all derivations, as well as these derivations, are defined by their actions on generators.

We put
\[ d\nabla^* = d : C^\infty (M) \to \Omega^1 (M) = \Sigma^1, \]
\[ d\nabla^* : \Omega^1 (M) = \Sigma^1 \xrightarrow{d\nabla} \Omega^1 (M) \otimes \Omega^1 (M) \xrightarrow{\text{Sym}} \Sigma^2, \]
and define $d_\xi^*\varphi$ as a derivation over $d_\xi^*\varphi$ such that
\[ d_\xi^* = d_\xi^* : C^\infty (\xi) \to C^\infty (\xi) \otimes \Sigma^1. \]

Let's now $\sigma \in \Sigma_k$ be a symbol. We define a differential operator $\hat{\sigma} \in \text{Diff}_k (\xi)$ as follows:
\[ \hat{\sigma} (h) \overset{\text{def}}{=} \frac{1}{k!} \left\langle \sigma, (d_\xi^* \varphi)^k (h) \right\rangle. \]

Here $h \in C^\infty (\xi)$, $(d_\xi^* \varphi)^k (h) \in C^\infty (\xi) \otimes \Sigma^k$, and $\langle \cdot, \cdot \rangle$ is the natural pairing
\[ \Sigma_k \otimes C^\infty (\xi) \otimes \Sigma^k \to C^\infty (\xi). \]
Remark that the value of the symbol of the derivation $d_\xi^*\varphi$ on a covector $\theta$ equals to the symmetric product by $\theta$ into the module $\Sigma^i (\xi)$ and because the symbol of a composition of operators equals the composition of symbols we get that the symbol of operator $\hat{\sigma}$ equals $\sigma$.

We call this operator $\hat{\sigma}$ a quantization of symbol $\sigma$ and write $\hat{\sigma} = Q_\Sigma (\sigma)$.

By the construction morphism $Q : \Sigma \to \text{Diff}_k (\xi)$ splits sequence (1).

Let's now $A \in \text{Diff}_k (\xi)$ be a differential operator and $\sigma_k (A) \in \Sigma_k$ be its symbol. Then operator $\quad A - Q (\sigma_k (A))$ has order $(k - 1)$, and let $\sigma_{k-1} (A) \in \Sigma_{k-1}$ be its symbol.

Then operator $A - Q (\sigma_k (A)) - Q (\sigma_{k-1} (A))$ has order $(k - 2)$. Repeating this process we get subsymbols $\sigma_i (A) \in \Sigma_i$, $0 \leq i \leq k - 1$, such that $\quad A = Q (\sigma (A))$, where
\[ \sigma (A) = \oplus_{0 \leq i \leq k} \sigma_i (A) \]
is a total symbol of the operator, and $Q (\sigma (A)) = \sum_i Q (\sigma_i (A))$.

**Coordinates**

Let $(x_1, \ldots, x_n)$ be local coordinates in a neighborhood $\mathcal{O} \subset M$ and $e \in C^\infty (\mathcal{O})$ be a nowhere vanishing section of the line bundle $\xi$ over $\mathcal{O}$. Denote by $(x_1, \ldots, x_n, w_1, \ldots, w_n)$ induced standard coordinates in the tangent bundle over $\mathcal{O}$. 
Then, \( d_\nabla (e) = e \otimes \theta \), where \( \theta = \sum \theta_i dx_i \) is the connection form, and \( d_\nabla (dx_k) = - \sum \Gamma^k_{ij} dx_i \otimes dx_j \), where \( \Gamma^k_{ij} \) are the Christoffel symbols of the connection \( \nabla = \nabla^M \).

Thus, in coordinates \((x, w)\) we have \( d_s \nabla (w_k) = - \sum \Gamma^k_{ij} w_i w_j \) and the derivations \( d_s \nabla \) and \( d_s \nabla \xi \) are of the form:

\[
\begin{align*}
    d_s \nabla & = \sum w_i \partial x_i - \sum \Gamma^k_{ij} w_i w_j \partial w_k, \\
    d_s \nabla \xi & = \sum w_i (\partial x_i + \theta_i) - \sum \Gamma^k_{ij} w_i w_j \partial w_k.
\end{align*}
\]

2.7. \textbf{Group actions.} We consider two groups: \( \mathcal{G}(M) \) – the group of diffeomorphisms of manifold \( M \), and \( \text{Aut}(\xi) \) – groups of automorphisms of line bundles \( \xi \) over \( M \).

There is the following sequence of group morphisms

\[
1 \rightarrow F(M) \rightarrow \text{Aut}(\xi) \rightarrow \mathcal{G}(M) \rightarrow 1,
\]

where \( F(M) \subset C^\infty(M) \) is the multiplicative group of smooth nowhere vanishing functions on \( M \).

\textbf{Proposition 1.} A diffeomorphism \( \psi : M \rightarrow M \) admits a lifting to an automorphism \( \tilde{\psi} : E(\xi) \rightarrow E(\xi) \) if and only if \( \psi^* (w_1(\xi)) = w_1(\xi) \), where \( w_1(\xi) \in H^1(M, \mathbb{Z}_2) \) is the first Stiefel-Whitney class of the bundle.

\textbf{Proof.} Remark that a real linear bundle \( \xi \) is trivial if and only if the class \( w_1(\xi) \in H^1(M, \mathbb{Z}_2) \) vanishes ([18]). Therefore, in the case when \( w_1(\xi) = 0 \) the statement of the lemma trivial.

Let now \( w_1(\xi) \neq 0 \) and let \( \psi^*(\xi) \) be the line bundle induced by a diffeomorphism \( \psi \). Then, we have

\[
    w_1(\xi^* \otimes \psi^*(\xi)) = w_1(\xi^*) + w_1(\psi^*(\xi)) = w_1(\xi) + \psi^*(w_1(\xi)) = 0,
\]

if \( \psi^*(w_1(\xi)) = w_1(\xi) \).

Therefore, any nowhere vanishing section of the bundle \( \xi^* \otimes \psi^*(\xi) \) give us an isomorphism between bundle \( \xi \) and \( \psi^*(\xi) \) covering the identity map and then the lift of diffeomorphism \( \psi \). \hfill \Box

\textbf{Corollary 2.} The following sequence of group morphisms

\[
1 \rightarrow F(M) \rightarrow \text{Aut}(\xi) \rightarrow \mathcal{G}_\xi(M) \rightarrow 1,
\]
where
\[ G_\xi (M) = \{ \phi \in G (M) | \phi^* (w_1 (\xi)) = w_1 (\xi) \}, \]
is exact.

**Remark 3.** For the case of complex line bundles this proposition is valid too if we consider the Chern characteristic classes instead of Stiefel-Whitney classes.

For the scalar differential operators \( A \in \text{Diff}_k (1) \) we consider the standard action of the diffeomorphism group:
\[ \phi_* : A \mapsto \phi_* \circ A \circ \phi_*^{-1}, \]
where \( \phi_* = \phi^{*-1} : C^\infty (M) \to C^\infty (M) \) is the induced by \( \phi \) algebra morphism and \( \phi^* (f) = f \circ \phi, f \in C^\infty (M) \).

For general line bundles and operators \( A \in \text{Diff}_k (\xi) \) the action of the automorphism group is the following.

Let \( \tilde{\phi} \) be an automorphism, \( \tilde{\phi} \in \text{Aut}(\xi) \), covering diffeomorphism \( \phi \in G (M) \). Then we define action of \( \tilde{\phi} \) on sections \( s \in C^\infty (\xi) \) as
\[ \tilde{\phi}_* : s \mapsto \tilde{\phi}_* \circ s \circ \tilde{\phi}_*^{-1}, \]
and
\[ \tilde{\phi}_* : A \mapsto \tilde{\phi}_* \circ A \circ \tilde{\phi}_*^{-1}, \]
for differential operators.

3. **Classification of symbols**

In this section we fix a point \( a \in M \) on the manifold \( M \) and consider orbits of symbols \( \sigma \in S^k T_a (M) \), at this point with respect to general linear group \( G = \text{GL} (T_a (M)) \), for \( k \geq 3 \) and \( \dim V \geq 2 \).

The symbols are homogeneous functions of degree \( k \) on the vector space \( V = T^*_a (M) \), or in other words analytical functions \( h \) on the space that satisfy the Euler equations
\[ \delta (h) = kh, \tag{4} \]
where \( \delta \) is the radial vector field on \( V \).
3.1. **Euler equations.** Equation (4) defines a vector subbundle \( \pi_1 : \mathcal{E}_1 \subset J^1 (V_0) \to V_0 \), where \( V_0 = V \setminus 0 \), in the bundle \( \pi_1 : J^1 (V_0) \to V_0 \) of 1-jets of functions on \( V_0 \).

In the canonical coordinates \((x_1, \ldots, x_n, u, u_1, \ldots, u_n)\) on \( J^1 (V) \), where \((x_1, \ldots, x_n)\) are coordinates on the vector space \( V \), \( n = \dim V \), submanifold \( \mathcal{E}_1 \) is given by the equation

\[
x_1 u_1 + \cdots + x_n u_n - ku = 0.
\]

Taking the prolongations of the Euler equations we get subbundles \( \pi_i : \mathcal{E}_i \subset J^i (V_0) \to V_0 \), \( i = 1, 2, \ldots k \).

Remark that submanifolds \( \mathcal{E}_i \) are defined by equations

\[
\sum_i x_i u_{\alpha+1, i} = (k - |\alpha|) u_{\alpha},
\]

for all multi indices \( \alpha = (\alpha_1, \ldots, \alpha_n) \) of the length \( 0 \leq |\alpha| \leq l - 1 \), and where \( \alpha + 1_i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n) \).

We define now subbundles \( \mathcal{E}_i \subset J^i (V_0) \), for \( i \geq k + 1 \) as solutions of the following systems

\[
\sum_i x_i u_{\alpha+1, i} = (k - |\alpha|) u_{\alpha}, \quad |\alpha| \leq k - 1,
\]

\[
u_{\beta} = 0, \quad k + 1 \leq |\beta| \leq i.
\]

In this case we also have inclusion of the first prolongations \( \mathcal{E}_{i}^{(1)} \subset \mathcal{E}_{i+1} \) and therefore the system

\[
x_1 u_1 + \cdots + x_n u_n = ku, \quad u_{\beta} = 0, \quad |\beta| = k + 1,
\]

defines the formally integrable equation in the sense of Spencer [24], [7] and analytical (over \( V \)) solutions of this system are exactly homogeneous polynomials of degree \( k \).

Projections \( \pi_{i, i-1} : J^i (V) \to J^{i-1} (V) \) induce projections of prolongations \( \pi_{i, i-1} : \mathcal{E}_i \to \mathcal{E}_{i-1} \) with kernels (or symbols) \( g_i \subset \mathcal{S}^i \tau^* \).

Thus we have exact sequences of vector bundles

\[
0 \to g_i \to \mathcal{E}_i \xrightarrow{\pi_{i, i-1}} \mathcal{E}_{i-1} \to 0,
\]

for \( i = 1, 2, \ldots \), where we put \( \mathcal{E}_0 = J^0 (V_0) \).

Equation (4) shows that

\[
g_1 = \{ \theta \in T^* (V_0), \delta |\theta| = 0 \},
\]
and
\[ g_i = \{ \theta \in S^i T^* (V_0), \delta \mid \theta = 0 \} \]
for all \( i = 2, \ldots, k \), and \( g_i = 0 \), for \( i = k + 1, \ldots \).

Here we denoted by \( \delta \mid \theta \) the inner product of vector \( \delta \) and tensor \( \theta \).

In other words, if we denote by \( \delta^0 = \text{Ann} (\delta) \subset T^* (V_0) \to V_0 \) the subbundle of the cotangent bundle with fibres \( \text{Ann} (\delta_v) \subset T^*_v \), then
\[ g_i = S^i (\delta^0) \]
for \( i = 1, \ldots, k \).

3.2. Splitting and invariant frame. Let \( \nabla \) be the standard affine connection on space \( V \), considered as the affine manifold. Then the above construction allows us to define tensors
\[ d_l f = \frac{1}{l!} (d_s \nabla)^l (f) \in \Sigma^l (V) \]
for any smooth function \( f \) on \( V \).

The connection \( \nabla \) is \( G \)-invariant and therefore the operators \( d_l \) are also \( G \)-invariants:
\[ A^* (d_l f) = d_l A^* (f) \]
for any affine transformation \( A : V \to V \), and, therefore, for all \( A \in G \).

Applying these operators to homogeneous functions \( H \) of degree \( k \) we get tensors \( d_l H \in g_l \), and because \( j_k (H) = (H, dH, \ldots, d_l H, \ldots, d_k H) \) we get splitting of the Euler bundles
\[ \pi^\mathcal{E}_l : \mathcal{E}_l \to V_0, \]
into the direct sum of symbol bundles \( \gamma_i : g_i \to V_0 \),
\[ \pi^\mathcal{E}_l = 1 \oplus \gamma_1 \oplus \cdots \oplus \gamma_l, \]
l \( \leq k \).

Let now \( u : J^0 (V) = V \times \mathbb{R} \to \mathbb{R} \) be the standard fibre wise coordinate and let
\[ \Theta_l = \frac{1}{l!} (d_s \nabla)^l (u) \in \Sigma^l (\mathcal{E}_l) \],
be horizontal symmetric tensors on the equation.

The, due to the definition of the total lift, we get
\[ j_l (H)^* (\Theta_l) = d_l H, \]
for any homogeneous polynomial $H$.

In other words, tensors $\Theta_l$ are universal differentials of $l$-th order, $l = 0, 1, \ldots$

In the standard jet coordinates $(x, u, \ldots, u_a, \ldots)$ on the jet spaces the universal tensors $\Theta_l$ are of the following form

$$\Theta_l = \sum_{|\alpha|=l} u_\alpha \frac{dx^\alpha}{\alpha!},$$

and it is easy to check that

$$\hat{\delta} \Theta_l = (k - l + 1) \Theta_{l-1},$$

(5)

for all $l = 1, \ldots, k$.

In particular, $\hat{\delta} \Theta_1 = k \Theta_0$, and in the domain, where $\Theta_0 = u \neq 0$, we can represent any horizontal vector field $X$ as sum:

$$X = X_0 + \frac{\Theta_1(X)}{k \Theta_0} \hat{\delta},$$

(6)

where $X_0 \in \ker \Theta_1$ is also horizontal field.

Therefore, due to this splitting, any horizontal 1-form $\lambda$ could be decomposed into the sum

$$\lambda = \lambda^0 + \frac{\lambda \left( \frac{\hat{\delta}}{k \Theta_0} \right)}{\Theta_1},$$

(7)

where the form $\lambda^0$ is considered as a form $\lambda$ restricted on $\ker \Theta_1$.

Applying formula (5), we get

$$\Theta_2 = \Theta_2^0 + \frac{k - 1}{2k \Theta_0} \Theta_1^2,$$

$$\Theta_3 = \Theta_3^0 + \frac{1}{k \Theta_0} \Theta_2^0 \cdot \Theta_1 + \frac{k - 1}{3! (k \Theta_0)^2} \Theta_1^3,$$

where $\Theta_2^0$ and $\Theta_3^0$ are quadratic and cubic differential forms on horizontal vector fields from $\ker \Theta_1$.

These two tensors $\Theta_2^0$ and $\Theta_3^0$ will be important for us. We say that a point $a_2 \in \mathcal{E}_2$ is regular or singular if the quadratic form $\Theta_2$ at this point is regular and $u(a_2) \neq 0$, and singular in the opposite case.
Remark, that regularity $\Theta_2$ is equivalent to regularity of $\Theta_0$. Indeed, assume that $X \in \ker \Theta_2$ and has decomposition (6). Then

$$X\lceil \Theta_2 = X_0\lceil \Theta_2 + \frac{(k - 1) \Theta_1(X)}{k \Theta_0} \Theta_1,$$

and therefore $X \in \ker \Theta_2$ if and only if $\Theta_1(X) = 0$, i.e. $X = X_0$, and $X_0 \in \ker \Theta_0$.

Denote by $E_2^0 \subset \mathcal{E}_2$ the domain of regular points and by $\widetilde{\Theta}_2^0$ the inverse tensor to $\Theta_2^0$.

Then

$$\lambda = \widetilde{\Theta}_2^0 \lceil \Theta_0^0,$$

is a horizontal differential 1-form (on $\ker \Theta_1$) over the regular domain $E_2^0$.

Let $\widehat{\lambda}$ be the horizontal vector field in $\ker \Theta_1$ dual to $\lambda$, i.e. $\lambda = \widehat{\lambda} \lceil \Theta_0^0$, or $\lambda = \lambda \lceil \widetilde{\Theta}_0^0$, and let

$$\Upsilon = \widehat{\lambda} \lceil \Theta_3^0,$$

be a horizontal quadratic form on $\ker \Theta_1$.

Denote by $\Phi$ operator that corresponds to this form i.e. a linear operator acting on horizontal vector fields in $\ker \Theta_1$ and such that

$$\Theta_3^0 \left( \widehat{\lambda}, X, Y \right) = \Theta_2^0 \left( \Phi X, Y \right),$$

for all horizontal vector fields $X, Y$ in $\ker \Theta_1$.

This operator as well as all above constructions well defined over the regular domain only.

We say that a point $a_3 \in \mathcal{E}_3$ is regular if its projection $a_2 = \pi_{3,2}(a_3)$ on the space of 2-jets belongs to the regular domain $E_2^0$ and horizontal vector fields

$$e_1 = \widehat{\lambda}, e_2 = \Phi(e_1), ..., e_{n-1} = \Phi(e_{n-2}),$$

where $n = \dim V$, are linear independent.

3.3. Invariants of homogeneous forms. Let $E_3^0 \subset \mathcal{E}_3$ be the domain of regular 3-jets. This domain is non empty and defined by some number of algebraic inequalities. Therefore, $E_3^0$ is dense into $\mathcal{E}_3$. 
Theorem 4.

(1) Horizontal vector fields \( e_1, \ldots, e_{n-1} \) and \( e_n = \hat{\delta} \) constitute \( G \)-invariant frame over regular domain \( E_0^3 \).

(2) Function \( \Theta_0 = u \) and coefficients of tensors \( \Theta^l_0, l = 2, \ldots, k \) in the frame \( (e_1, \ldots, e_{n-1}) \) in \( \ker \Theta_1 \) are \( G \)-invariants. They are rational functions over the regular domain \( E_0^3 \) and they generate all rational differential \( G \)-invariants of homogeneous forms on \( V \).

The total lift of the symmetric covariant differentials

\[
d_{\nabla^T} : \Sigma^l(V) \xrightarrow{d_{\nabla^T}} \Sigma^l(V) \otimes \Sigma^l(V) \xrightarrow{\text{Sym}} \Sigma^{l+1}(V)
\]

allow us to reconstruct universal tensor \( \Theta_{i+1} \) under condition that we know tensor \( \Theta_i : \overrightarrow{\nabla^T}(\Theta_i) = \Theta_{i+1} \).

We’ll extend the notion of regularity. Namely, we say that a 4-jet \( a_4 \in \mathcal{E}_4 \) is regular if its projection on the space of 3-jets is regular, \( a_3 \in \mathcal{E}_3^0 \), and there are invariants \( J_1, \ldots, J_n = \Theta_0 \) of order \( \leq 3 \), such that \( \hat{d}J_1 \wedge \cdots \wedge \hat{d}J_n \neq 0 \) at point \( a_4 \). Denote by \( \mathcal{E}_4^0 \subset \mathcal{E}_4 \) the domain of regular 4-jets. It is Zariski open and therefore dense domain in \( \mathcal{E}_4 \).

These invariants \( J_1, J_2, \ldots, J_n = \Theta_0 \) are in general position, i.e.

\[
\hat{d}J_1 \wedge \cdots \wedge \hat{d}J_n \neq 0
\]
in an open and dense domain in \( \mathcal{E}_4 \).

Then functions

\[
J_{ab} = e_a (J_b),
\]

where \( a, b = 1, \ldots, n \), are rational \( G \)-invariants and they defined the invariant frame

\[
e_a = \sum_b J_{ab} \frac{d}{dJ_b},
\]

where \( \frac{d}{dJ_b} \) are the Tresse derivatives.

Taking the total covariant derivatives

\[
\hat{\nabla}_{e_a} (e_b) = e_b | d\nabla^T (e_b),
\]

and decomposing them in the invariant frame we get

\[
\hat{\nabla}_{e_a} (e_b) = \sum_c \Gamma^c_{ab} e_c,
\]

where Christoffel symbols \( \Gamma^c_{ab} \) are rational differential \( G \)-invariants too.
This data

$$J = (J_1, J_2, \ldots, J_n = \Theta_0), \quad U = (J_{ab}), \quad \Gamma = (\Gamma_{ab})$$

(8)

completely defines invariant frame by $U$, the total covariant differential $\hat{\nabla}$ by $\Gamma$ and, therefore, all universal tensors $\Theta_I$, because $J_n = \Theta_0$.

**Theorem 5.** For a given data (8) all rational differential invariants of rational functions of degree $k \geq 3, n \geq 2$, are rational functions of invariants $J$ and Tresse derivatives

$$\frac{dU}{dJ}, \quad \frac{d\Gamma}{dJ}.$$ 

Remark also that

$$\dim E_3 = \frac{n(n^2 + 3n + 8)}{6},$$

and therefore codimensions of regular orbits are

$$\dim E_3 - n^2 \geq n,$$

for $n \geq 2$.

3.4. **Orbits of homogeneous forms.** Let $h$ be a homogenous function of order $k$. We say that $h$ is a *regular function* if its 4-jet $j_4 (h)$ has non empty intersection with regular domain $E_4^0$.

Let now $I$ be an invariant of order $l$, then by $I (h) = j_l (h)^* (I)$ we will denote the value of this invariant on the homogeneous function $h$.

Remark that $I (h)$ is a rational function defined on open and dense set in $V$.

Respectively, by $J (h), U (h), \Gamma (h)$ we'll denote the values of data (8) on function $h$.

Invariants $J$ are in general position, therefore there is an open domain $O_h \subset V$, where functions $J (h) = (J_1 (h), \ldots, h)$ are coordinates and, therefore, functions $U (h)$ and $\Gamma (h)$ are functions of $J : U = U (J), \Gamma = \Gamma (J)$.

Geometrically, we’ll consider the following rational map

$$D_h : \quad V \to \mathbb{R}^N,$$

$$D_h : \quad v \in V \mapsto (R = J (h) (v), Q = U (h) (v), S = \Gamma (h) (v)),$$
where
\[ N = \frac{n(n^2 + 2n + 3)}{2}, \]
and the standard coordinates in \( \mathbb{R}^N \) we denoted by \((R, Q, S)\), and \( R = (R_1, ..., R_n) \), \( Q = (Q_{ab}) \), \( S = (S^c_{ab}) \).

Then the image \( \Pi_h \subset \mathbb{R}^N \) of \( D_h \) is an algebraic variety and the image \( D_h (\mathcal{O}_h) \) of the domain \( \mathcal{O}_h \) is the graph of functions \( U (R) \), \( \Gamma (R) \):
\[ Q = U (R) , S = \Gamma (R) . \]

Remark that linear transformations \( A \in G \) leave invariant algebraic relations between differential invariants and therefore do not change the algebraic manifold \( \Pi_h \):
\[ \Pi_{A^* (h)} = \Pi_h. \]

**Theorem 6.** Two regular polynomials \( h \) and \( h' \) of degree \( k \geq 3, n \geq 2 \), are \( G \)-equivalent if and only if \( \Pi_h = \Pi_{h'} \), for the some invariants \( J_1, ..., J_n = \Theta_0 \) in general position.

**Proof.** The above remark shows the necessity of the theorem condition.

Let’s now \( \Pi_h = \Pi_{h'} \). Then domains \( \mathcal{O}_h \) and \( \mathcal{O}_{h'} \) we choose in such a way that \( \Pi (\mathcal{O}_h) = \Pi (\mathcal{O}_{h'}) \) is common for both functions. Over domains \( \mathcal{O}_h \) we have
\[ U_{ab} = U_{ab} (z_1, ..., z_n) , \Gamma^c_{ab} = \Gamma^c_{ab} (z_1, ..., z_n) \]
in coordinates
\[ z_1 = J_1 (h) , ..., z_{n-1} = J_{n-1} (h) , ..., z_n = h, \]
and over domain \( \mathcal{O}_{h'} \) we get the same relations and functions
\[ U_{ab} = U_{ab} (z'_1, ..., z'_n) , \Gamma^c_{ab} = \Gamma^c_{ab} (z'_1, ..., z'_n) \]
in coordinates
\[ z'_1 = J_1 (h') , ..., z'_{n-1} = J_{n-1} (h') , ..., z'_n = h'. \]
Therefore, the diffeomorphism \( A : \mathcal{O}_h \to \mathcal{O}_{h'} \), where \( z \to z' \), transforms \( h' \) to \( h \) and preserves the affine connection. Thus, \( A \in G \) and \( A^* (h') = h \).

**Proposition 7.** Isotropy Lie algebra of an \( \text{GL} (V) \)-orbit of a regular homogeneous \( k \)-form is trivial for \( k \geq 3 \) and \( n \geq 2 \).
Proof. Any vector field $X \in \mathfrak{gl}(V)$ from the isotropy Lie algebra of a regular homogeneous form of degree $k \geq 3$, preserves also invariants determining orbits of 3-jets. Then, $X(J_i(H)) = 0$ in an open domain and, therefore, $X = 0$. □

**Corollary 8.** Let $H$ be a regular homogenous $k$-form, $k \geq 3$. Then the isotropy group of $H$ in $\text{GL}(V)$ is finite.

**Proof.** The isotropy group is discrete, Zariski closed and therefore finite. □

**Corollary 9.** Let $H$ be a regular homogenous $k$-form, $k \geq 3$. Then there is a neighborhood $O_H$ of $H$ in the space of homogeneous forms of degree $k$ such that for any form $H' \in O_H$, which belongs to the $\text{GL}(V)$-orbit of $H$, there is and unique linear transformation $A \in \text{GL}(V)$, such that $A^*(H') = H$.

### 3.5. Example: Binary forms.

To make the general case more transparent we’ll illustrate this approach on example of binary forms. The results in this section are very closed to classification given in [4].

The Euler equation for binary $k$-forms $E_k \subset J^k(V_0)$ has dimension $k + 3$ with coordinates: $x_1, x_2, u, u_1, \ldots, u_k$.

The universal tensors $\Theta_l$ are of the form

$$\Theta_l = \sum_{\alpha_1 + \alpha_2 = l} u_{\alpha_1, \alpha_2} \frac{dx_1^{\alpha_1}}{\alpha_1!} \frac{dx_2^{\alpha_2}}{\alpha_2!},$$

for general $l$, and especially

$$\Theta_0 = u_{0,0}, \quad \Theta_1 = u_{1,0} dx_1 + u_{0,1} dx_2,$$

$$\Theta_2 = \frac{1}{2} (u_{2,0} dx_1^2 + 2 u_{1,1} dx_1 \cdot dx_2 + u_{0,2} dx_2^2),$$

$$\Theta_3 = \frac{1}{6} (u_{3,0} dx_1^3 + 3 u_{2,1} dx_1^2 \cdot dx_2 + 3 u_{1,2} dx_1 \cdot dx_2^2 + u_{0,3} dx_2^3),$$

for small $l$.

To construct the $G$-invariant frame we’ll take two horizontal vector fields

$$\hat{\delta} = x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2},$$

and

$$\eta = u_{0,1} \frac{d}{dx_1} - u_{1,0} \frac{d}{dx_2},$$
in order to have horizontal form $\Theta_1$ as an element of the dual basis.

In splitting (7) we have $\theta^0(\delta) = 0$, therefore the second form in the $G$-invariant coframe should be proportional to form

$$\rho = x_2dx_1 - x_1dx_2.$$ 

Then for the basic forms $dx_1$ and $dx_2$, we get

$$dx_1 = \frac{u_{0,1}}{ku} \rho + \frac{x_1}{ku} \Theta_1, \quad dx_2 = -\frac{u_{1,0}}{ku} \rho + \frac{x_1}{ku} \Theta_1$$

and therefore the quadratic form $\Theta_0^2$ will be the following

$$\Theta_0^2 = \frac{K_2}{(ku)^2} \rho^2,$$

where

$$K_2 = \frac{1}{2} \left( u_{2,0}u_{0,1}^2 - 2u_{1,1}u_{1,0}u_{0,1} + u_{0,2}u_{1,0}^2 \right).$$

In the similar way we get

$$\Theta_3^0 = \frac{K_3}{(ku)^3} \rho^3,$$

where

$$K_3 = \frac{1}{6} \left( u_{3,0}u_{0,1}^3 - 3u_{2,1}u_{0,1}^2u_{1,0} + 3u_{1,2}u_{0,1}u_{1,0}^2 - u_{0,3}u_{1,0}^3 \right).$$

Therefore, the invariant form $\lambda$ equals

$$\lambda = \frac{1}{3} \frac{K_3}{K_2} \frac{\rho}{ku},$$

or substituting $\lambda$ instead of $\rho$ we get

$$\Theta_0^0 = \frac{9K_3^3}{K_2^2} \rho^2, \quad \Theta_3^0 = \frac{27K_3^3}{K_2^2} \rho^3.$$

These forms are $G$-invariants, therefore:

- functions

$$J_0 = u, \quad J_3 = \frac{K_3^2}{K_2^2}$$

are differential invariants of binary forms,
• horizontal forms
\[ \langle \lambda, \Theta_1 \rangle \]
give us \( G \)-invariant coframe, and

• total vector fields
\[ \langle \hat{\lambda}, \hat{\delta} \rangle, \]
where
\[ \hat{\lambda} = \frac{3K_2}{2K_3} \left( u_{0,1} \frac{d}{dx_1} - u_{1,0} \frac{d}{dx_2} \right), \]
form \( G \)-invariant frame.

Applying \( \hat{\lambda}, \hat{\delta} \) to differential invariants we get also invariants.
Thus,
\[ \hat{\delta} (J_0) = kJ_0, \quad \hat{\delta} (J_3) = -kJ_3, \]
\[ \hat{\lambda} (J_0) = 0, \quad \hat{\lambda} (J_3) = J_4, \]
where \( J_4 \) is a new invariant of order 4.

Also, writing down the tensors \( \Theta^0_l \) in invariant coframe we get
\[ \Theta^0_l = \left( \frac{3K_2}{K_3} \right)^l \sum_{\alpha_1+\alpha_2=l} u_{\alpha_1,\alpha_2} \frac{u_{0,1}^{\alpha_1}}{\alpha_1!} \frac{(-u_{1,0})^{\alpha_2}}{\alpha_2!} \lambda^l, \]
and therefore functions
\[ I_l = \left( \frac{3K_2}{K_3} \right)^l \sum_{\alpha_1+\alpha_2=l} u_{\alpha_1,\alpha_2} \frac{u_{0,1}^{\alpha_1}}{\alpha_1!} \frac{(-u_{1,0})^{\alpha_2}}{\alpha_2!} \]
are invariants of order \( l \), for \( l = 4, \ldots, k \).

Remark, that in order 4 we have two invariants \( J_4 \) and \( I_4 \). It is easy to check that there is the following relation between them:
\[ J_4 = 2J_0J_3^2I_4 - 3J_3^2 - \left( 3 - \frac{6}{k} \right) J_3. \]

Moreover, the Christoffel symbols of the standard affine connection in the invariant frame are invariants of the 4th order also.
Computing them we get:
\[ \Gamma^1_{11} = J_3 - \frac{J_4}{2J_3}, \Gamma^2_{11} = \frac{J_3}{K}, \Gamma^1_{12} = 1 - k, \Gamma^2_{12} = 0, \Gamma^2_{22} = 1, \Gamma^1_{22} = 0. \]
Invariant frame can be also written down in terms of Tresse derivatives:
\[
\widehat{\lambda} = J_4 \frac{d}{dJ_3}, \quad \widehat{\delta} = kJ_0 \frac{d}{dJ_0} - kJ_3 \frac{d}{dJ_3}.
\]
Therefore, the discussed above algebraic manifold \( \Pi \) we’ll get in the following way.

Let \( h \) be a binary form of degree \( k \) and let
\[
D_h : \mathbb{R}^2 \rightarrow \mathbb{R}^3
\]
be the following rational mapping
\[
D_h : x \mapsto (R_1 = J_3(h)(x), R_2 = h(x), U = J_4(h)(x)),
\]
where \((R_1, R_2, U)\) are the standard coordinates in \( \mathbb{R}^3 \).

Let \( \Pi_h \) be the image of this mapping. Remark that functions \( hJ_3(h) \) and \( h^2J_4(h) \) are homogeneous functions of degree zero therefore the image of surface \( \Pi_h \) under the rational map
\[
\chi : \mathbb{R}^3 \rightarrow \mathbb{R}^2,
\]
\[
\chi : (R_1, R_2, U) \mapsto (R_1R_2, R_2^2U)
\]
is a characteristic curve \( L_h \subset \mathbb{R}^2 \), given by equation
\[
\text{Res} (P_1 - aQ_1, P_2 - bQ_2) = 0,
\]
where \((P_i, Q_i)\) are polynomials, say in \( t = y/x \), without nontrivial common factor and such that
\[
hJ_3(h) = \frac{P_1}{Q_1}, \quad h^2J_4(h) = \frac{P_2}{Q_2},
\]
\((a, b)\) coordinates on \( \mathbb{R}^2 \), and \( \text{Res} \) is the resultant of polynomials.

Because \( \Pi_h = \chi^{-1}(L_h) \) we get the following result by applying the above theorem.

**Theorem 10.** Two regular binary forms \( h_1 \) and \( h_2 \) of degree \( k \geq 3 \) are \( G \)-equivalent if and only if their characteristic curves coincide:
\[
L_{h_1} = L_{h_2}.
\]

4. **Scalar differential operators**

4.1. **Jets of differential operators and natural invariants.** Denote by \( \chi_k : \text{Diff}_k(M) \rightarrow M \) the vector bundle of scalar linear differential operators of the \( k \)-th order on manifold \( M \).
Sections

\[ S_A : M \to \text{Diff}_k(M) \]

of this bundle will be identified with differential operators \( A \in \text{Diff}_k(M) \).

In this bundle we will use the following canonical local coordinates

\[(x_1,..x_n, u^\alpha),\]

where \((x_1,..x_n)\) are local coordinates on \(M\) and \(u^\alpha\) are fibre wise coordinates in bundle \(\chi_k\). Here \(\alpha = (\alpha_1, ..., \alpha_n)\) are multi indices of length \(0 \leq |\alpha| \leq k\).

In these coordinates the section \(S_A\), that corresponds to operator

\[ A = \sum_{|\alpha| \leq k} a^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \]

has the form

\[ u^\alpha = a^\alpha(x). \]

Exact sequence of modules (1) gives us the exact sequence of vector bundles

\[ 0 \to \chi_{k-1} \to \chi_k \to S^k(\tau) \to 0. \quad (9) \]

Denote by \(\pi_l : J^l(\chi_k) \to M\) the vector bundles of l-jets of sections of bundles \(\chi_k\), or, in other words, bundles of l-jets of the k-th order scalar differential operators.

we’ll denote by \([A]_p^l\) l-jets of operators at a point \(p \in M\).

Bundles \(\chi_k\), as well as bundles \(\pi_l\) are natural in the sense that the action of the diffeomorphism group \(\mathcal{G}(M)\) is lifted to automorphisms of these bundles in the natural way:

\[ \phi^{(l)} : [A]_p^l \mapsto [\phi_*(A)]_p^l, \]

for any diffeomorphism \(\phi \in \mathcal{G}(M)\).

We’ll study orbits (or equivalence) of scalar differential operators under the action of the diffeomorphism group, but first of all we’ll investigate orbits of this action on the finite jet levels.

To this end we’ll consider \(\mathcal{G}(M)\)-action on the jet manifolds \(J^l(\chi_k)\), for \(k = 0, 1, 2, ...\)

Because \(\mathcal{G}(M)\) acts in a transitive way on the base manifold we fix point \(p \in M\) and consider only jets of diffeomorphisms which leave
point \( p \) fixed, i.e. consider the action of differential group \( D_{k+l} \) of order \((k + l)\) on the fibre \( J^l = J^l_p(\chi_k) \).

This is a linear action of the algebraic group \( D_{k+l} \) on vector space \( J^l \), and due to Rosenlicht theorem (see, for example, [[23],[13]]) the regular orbits of this action are separated by rational on \( J^l \) invariants.

We call this invariants by natural differential invariants of order \( \leq l \), for the \( k \)-th order scalar linear differential operators. We also say that an orbit \( O \) is regular if there are \( m = \text{codim} O \) rational invariants \( I_1, ..., I_m \), which are independent in a neighborhood of the orbit and such that

\[
O = \{ I_1 = c_1, ..., I_m = c_m \}
\]

for some constants \( c_1, ..., c_m \).

In other words, regularity of orbit \( O \) means that the quotient space \( J^l / D_{k+l} \) smooth at the point \( O \), and functions \( I_1, ..., I_m \) could be considered as local coordinates in a neighborhood of this point.

The universal construction, discussed above, could be applied for bundles \( \chi_k \) (see, also, [15],[16]).

Namely, the standard reasons show that there is an unique total differential operator of order \( k \):

\[
\Box : C^\infty (J^l(\chi_k)) \to C^\infty (J^{l+k}(\chi_k)),
\]

\( l = 0, 1, ..., \) such that

\[
j_{k+l} (S_A)^* (\Box (f)) = A (j_l (S_A)^* (f)), \quad (10)
\]

for all functions \( f \in C^\infty (J^l(\chi_k)) \) and operators \( A \in \text{Diff}_k(M) \).

It is easy to see that in the standard jet-coordinates in the bundles \( \pi_l \), this operator has the following form

\[
\Box = \sum_{|\alpha| \leq k} u^\alpha \frac{d^{[\alpha]} \ell}{dx^\alpha}.
\]

The main property of this operator is its naturality:

\[
\phi^{(k+l)*} \circ \Box = \Box \circ \phi^{(l)*},
\]

for all diffeomorphisms \( \phi \).

Let \( J = (J_1, ..., J_n) \) be a set of natural differential invariants. We say that they are in general position if

\[
\widehat{d}J_1 \wedge ... \wedge \widehat{d}J_n \neq 0.
\]
Let $I$ be an invariant, then

\[ \hat{d}I = \sum_i I_i \hat{d}J_i, \]

for some rational functions $I_i$, which are called Tresse derivatives. We’ll denote them by $\frac{dI}{dJ}$. Remark that they are invariants by the construction, having, as a rule, higher order then invariant $I$.

Then the principle of $n$-invariants, formulated in [2], gives us the following result.

**Theorem 11.** Let’s natural rational invariants $J_1, \ldots, J_n$ are in general position. Then the field of all natural rational invariants is generated by these invariants and the Tresse derivatives

\[ \frac{d^{\beta}}{dJ^\beta} J_\alpha \]

of invariants

\[ J_\alpha = \square (J_1^{\alpha_1} \cdots J_n^{\alpha_n}), \]

with $0 \leq |\alpha| \leq k$.

**Proof.** Assume that orders of all invariants $J_i$ less then $N - 1$. Then total differentials $\hat{d}J_i$ are linear independent in an open and dense domain $O \subset J^N$. Therefore, for almost all differential operators $A \in \text{Diff}_k (M)$, their values $J_i (A)$ give us local coordinates in a neighborhood of the point $p \in M$. Then functions $J_\alpha (A)$ allow us to find all coefficients of the operator $A$ in this coordinates and values $\frac{d^{\beta}}{dJ^\beta} J_\alpha (A)$ give us all derivatives of these coefficients. Therefore, any rational natural invariant $J$ is a rational function of invariants $J_i$ and $\frac{d^{\beta}}{dJ^\beta} J_\alpha$, by the definition of invariants. \qed

We apply this construction for cases, where we have at least $n$ independent invariants of order zero. To this end we consider the symbol mapping $\text{smbl} : J^0 \to S^k T_p \to 0$. This mapping commutes with the action of the diffeomorphism group and therefore $\text{smbl}^* (J)$ is a natural invariant of order zero for differential operators if $J$ is a $\text{GL} (T_p)$-invariant of homogeneous $k$-forms.

We’ll consider cases, when $n \geq 2$ and $k \geq 3$. Then the above description of invariants of homogeneous $k$-forms shows that stationary
Lie algebras of regular forms are trivial and therefore codimensions of regular orbits are
\[ c(n, k) = \binom{n + k - 1}{k} - n^2. \]
It is easy to check that \( c(n, k) \geq n \) for all \( n \geq 2, k \geq 3 \), with the exception for the following three cases:

\[
\begin{align*}
  n &= 2, k = 3; \\
  n &= 2, k = 4; \\
  n &= 3, k = 3.
\end{align*}
\]

**Theorem 12.** In non exceptional cases the field of natural invariants is generated by invariants of the zero order.

### 4.2. Differential operators of general type

We say that an operator \( A \in \text{Diff}_k(M) \) has a general type at a point \( q \in M \) if there are differential invariants in general position such that
\[ dJ_1(A) \wedge \cdots \wedge dJ_n(A) \neq 0, \]
(11)
at the point \( q \).

Then, as we have seen, functions \( J_1(A), \ldots, J_n(A) \) are local coordinates, we call them natural coordinates, and functions \( J_\alpha(A) = F_\alpha(J_1(A), \ldots, J_n(A)) \) allow us to find all coefficients of the operator \( A \) in terms of the local coordinates.

Therefore, the relations \( J_\alpha = F_\alpha(J_1, \ldots, J_n) \) completely define differential operators locally up to diffeomorphism.

We say also that an operator \( A \in \text{Diff}_k(M) \) has symbol of general type at a point \( q \in M \) if the symbol of operator \( A \) is regular and there are \( \text{GL}(T_q) \)-invariants \( J_1, \ldots, J_n \) of the \( \text{GL}(T_q) \)-action on \( S^k T_q \), i.e. differential invariants of the zero order, which are in general position.

In the last case, relations \( J_\alpha = F_\alpha(J_1, \ldots, J_n) \) also completely define differential operators.

Summarizing, we get the following result.
Theorem 13.

1. The relations $J_\alpha = F_\alpha (J_1, \cdots, J_n)$, $0 \leq |\alpha| \leq k$, for differential invariants $J_1, \cdots, J_n$ in general position, locally define general type differential operators up to local diffeomorphisms.

2. The same relations for differential operators, having symbols of general type, where $J_1, \cdots, J_n$ are $\text{GL}(T_a)$-invariants of symbols being in general position, locally define differential operators up to local diffeomorphisms. In particular, these invariants of the zero order generate the field of rational natural operators.

Remark 14. It follows from the Rosenlicht theorem [23],[13], that the field of rational natural invariants separates regular orbits in the jet spaces of differential operators.

4.3. Symbols of general type. Similar to differential operators, we say that a symbol $\sigma \in \Sigma_k (M)$ has a general type at a point $a \in M$ there are differential invariants of symbols of $k$-th degree, say $J_1, \ldots, J_n$ such that condition (11) holds.

Functions $J_1(\sigma), \cdots, J_n(\sigma)$, as above, are local coordinates in a neighborhood of the point $a \in M$, and we call them also natural.

Let

$$J_\alpha = \left( \frac{\partial J_1^{\alpha_1} \cdots \partial J_n^{\alpha_n}}{\partial \nu_k} \right)$$

be differential invariants, for multi indices $\alpha = (\alpha_1, ..., \alpha_n)$, $|\alpha| = k$.

Then functions $J_\alpha (\sigma) = F_\alpha (J_1(\sigma), \cdots, J_n(\sigma))$ completely defines symbol $\sigma$ in the natural coordinates and we get the following.

Theorem 15. The relations $J_\alpha = F_\alpha (J_1, \cdots, J_n)$, $|\alpha| = k$, for differential invariants $J_1, \cdots, J_n$ of symbols locally define general type symbols up to local diffeomorphisms. The principle of $n$-invariants (11) is also valid for symbols.

Remark 16. In this paper, for obvious reasons, we work only with contravariant symmetric tensors but all results on classification of covariant symmetric tensors, as well as their $\text{GL}$-orbits, are also valid.

5. Differential operators, acting in line bundles

Let $V$ be a vector space of dimension $n (= \dim M)$ and let $\varpi \subset S^4 V$ be a regular $\text{GL}(V)$-orbit. We say that an operator $A \in \text{Diff}_k (\xi)$, acting in a linear bundle $\xi : E(\xi) \to M$, has a constant type
if for any point $q \in M$ and any isomorphism $\phi : T_qM \to V$ the image of the symbol $\phi_*(\text{smbl}_k(A)) \in S^kV$ belongs to $\omega$.

In this case, we are not able to use GL-invariants of symbols as natural coordinates as it was done in the above theorem.

On the other hand, we will see that the description of such operators (and their natural differential invariants) based on existence of connections on the base manifold $M$ as well as in the linear bundle $\xi$, which are naturally related to the differential operators. These connections, as we have seen, allow us to establish isomorphism between differential operators and their total symbols and therefore realize "the dream" of wave mechanics on existence of natural procedure for quantization of differential operators.

Remark that for operators of the second order it was done in [15] and for operators of the third order over 2-dimensional manifolds in [16].

5.1. Wagner and Chern connections. Let $\sigma \in \Sigma_k(M)$ be a symbol of a constant type $\omega$, and let $k \geq 3$ and $n \geq 2$. Then, due to (9), for any point $p \in M$ there is a neighborhood $O_p$ and unique linear isomorphisms $A_{p,p'} : T_pM \to T_{p'}M$, for all $p' \in O_p$, such that $A_{p,p'}^* (\sigma_{p'}) = \sigma_p$.

Therefore, there is and unique linear connection $\nabla^\sigma$ (we call it Wagner connection, see [25],[16]) on manifold $M$ such that isomorphisms $A_{p,p'}$ are operators of the parallel transport along $\nabla^\sigma$. This connection preserves the symbol, i.e.

$$\nabla^\sigma_X (\sigma) = 0,$$

for all vector fields $X$ on $M$.

On the other hand, if symbol $\sigma$ admits such connection then it has a constant type. Summarizing we get the following.

**Theorem 17.**

1. A regular symbol $\sigma \in \Sigma_k(M)$ of constant type has a unique Wagner connection (12), if $k \geq 3, n \geq 2$.
2. The Wagner connection has the trivial curvature tensor, $R^\sigma = 0$.

Keeping in mind differential equations, but not differential operators, we consider conformal classes $[\sigma] = \{ f\sigma, f \in \mathcal{F}(M) \}$ of symbols.
Then the Wagner connections $\nabla^\sigma$ and $\nabla^{f \sigma}$ preserve the conformal class $[\sigma]$ and therefore their covariant differential satisfy the relation
\[ d\nabla^{f \sigma} - d\nabla^\sigma = \omega \otimes \text{Id}, \] (13)
for some differential 1-form $\omega$.

On the other hand relation (12) shows that
\[ \nabla^{f \sigma}_X (\sigma) - \nabla^\sigma_X (\sigma) = \nabla^{f \sigma}_X (f^{-1} f \sigma) = X (f^{-1}) f \sigma = -\frac{X(f)}{f} \sigma. \]

Therefore, the differential 1-form in relation (13) equals:
\[ \omega = -d \ln (|f|). \]

Let $T^\sigma (X,Y)$ be the torsion tensor of the Wagner connection and let $\theta^\sigma$ be the torsion form of this connection, i.e.
\[ \theta^\sigma (X) = \text{Tr} (Y \rightarrow T^\sigma (X,Y)). \]

Then, it is easy to see that,
\[ T^{f \sigma} (X,Y) - T^\sigma (X,Y) = \omega (X) Y - \omega (Y) X \] (14)
and
\[ \theta^{f \sigma} (X) - \theta^\sigma (X) = (n - 1) \omega (X). \] (15)

Let denote by $\tilde{\nabla}^\sigma$ the following deformation of the Wagner connection:
\[ \tilde{\nabla}^\sigma_X = \nabla^\sigma_X + \lambda \theta^\sigma (X), \]
for some $\lambda \in \mathbb{R}$.

Then, due to (13) and (15), we have
\[ \tilde{\nabla}^{f \sigma}_X - \tilde{\nabla}^\sigma_X = \omega (X) + \lambda (n - 1) \omega (X) = (1 + \lambda (n - 1)) \omega (X). \]

Therefore, the following connection, we call it Chern connection,
\[ \nabla^{[\sigma]}_X = \nabla^\sigma_X - \frac{1}{n - 1} \theta^\sigma (X), \]
does not depend on representative of the conformal class $[\sigma]$ and the corresponding parallel transforms preserve this class.

Summarizing, we get the following result.
Theorem 18.

(1) For any conformal class \([\sigma]\) of regular constant type symbol \(\sigma\) of degree \(k \geq 3\), there is and unique Chern connection \(\nabla^{[\sigma]}\), that preserves the conformal class
\[
\nabla^{[\sigma]}_X (\sigma) = -\frac{1}{n-1} \theta^\sigma (X) \sigma.
\]

(2) The torsion form of the Chern connection equals zero.

Remark 19. The Chern connection is a torsion free connection in dimension \(n = 2\).

5.2. Group-type symbols. Let \(M\) be a connected and simply connected manifold and let \(\nabla^\sigma\) be the Wagner connection with torsion tensor \(T^\sigma\). We’ll assume that this connection is complete and its torsion tensor is parallel
\[
d_{\nabla^\sigma} (T^\sigma) = 0.
\] (16)

Then, it is easy to check that the vector space \(g^\sigma\) of all parallel vector fields on \(M\), having dimension \(n = \dim M\), is a Lie algebra with respect to bracket
\[
X, Y \in g^\sigma \rightarrow T^\sigma (X, Y) \in g^\sigma.
\]

Moreover, relations (16) and \(R^\sigma = 0\) implies (see [10]) the following:

(1) \(M\) is an analytic manifold,

(2) \(\sigma\) is an analytic symbol, and

(3) \(\nabla^\sigma\) is an analytic linear connection.

Let \(G^\sigma\) be a connected and simply connected Lie group with Lie algebra \(g^\sigma\), and let \(\bar{\sigma} \in \Sigma_k (G^\sigma)\) be a \(G^\sigma\)-invariant symbol of the constant type \(\varpi\). The following result gives us a classification of such type symbols and it follows from the application of Theorem 7.8 in [10] to our case.

Theorem 20. Let \(M\) be a connected and simply connected manifold and let \(\sigma\) be a regular symbol of degree \(k \geq 3\), having constant type \(\varpi\). Let also the Wagner connection \(\nabla^\sigma\) be complete and the torsion tensor \(T^\sigma\) be parallel. Then any linear isomorphism \(F : T_p M \rightarrow T_e G^\sigma\), such that \(F (\sigma_p) = \bar{\sigma}_e\), could be extended to affine diffeomorphism \(F : M \rightarrow G^\sigma\), such that \(F_* (\sigma) = \bar{\sigma}\) and \(F_{*,p} = F\).
5.3. Connections, associated with differential operators of constant type. In this section we’ll consider operators $A \in \text{Diff}_k(\xi)$, acting in a line bundles $\xi : E(\xi) \to M$ and having a constant type. We’ll show that, under some generality conditions, these operators generate linear connections in the bundle $\xi$, which are in a natural way associated with with operators.

As above, we’ll restrict ourselves by the case $k \geq 3, n \geq 2$, although the case of ordinary differential operators, $n = 1$, we’ll considered separately, as an example.

Thus, let $A \in \text{Diff}_k(\xi)$ be an operator of the constant type and let $\sigma = \sigma_k = \text{smbl}_k(A) \in \Sigma_k(M)$ be its symbol. We’ll assume that $\sigma$ is regular and denote by $\nabla^{\sigma}$ the Wagner connection and by $\theta^{\sigma}$ the torsion form of this connection.

Let $\nabla$ be a linear connection in the line bundle and let $Q_{\nabla,\nabla^{\sigma}}$ be the quantization defined by these two connections, and let

$$A = Q_{\nabla,\nabla^{\sigma}} \left( \sum_i \sigma_i, \nabla \right),$$

be its decomposition, where $\sigma_k, \nabla = \sigma$, and

$$\sigma_{k-1, \nabla} = \text{smbl}_{k-1}(A - Q_{\nabla,\nabla^{\sigma}}(\sigma)).$$

Let $\nabla'$ be another linear connection in the line bundle and let

$$d_{\nabla'} - d_{\nabla} = \theta \otimes \text{Id},$$

for some differential form $\theta \in \Omega^1(M)$.

Then, due to definition (2), we have

$$\text{smbl}_{k-1}(Q_{\nabla',\nabla^{\sigma}}(\sigma) - Q_{\nabla,\nabla^{\sigma}}(\sigma)) = \theta | \sigma \in \Sigma_{k-1}(M), \tag{17}$$

and therefore

$$\sigma_{k-1, \nabla'} - \sigma_{k-1, \nabla} = \theta | \sigma. \tag{18}$$

We’ll say that a regular symbol $\sigma$ is Wagner regular if quadratic symbol

$$g_W = (\theta^{\sigma})^{k-2} | \sigma \in \Sigma_2(M)$$

is non degenerated.
Theorem 21. Let’s a differential operator \( A \in \text{Diff}_k(\xi) \) has a constant type and its symbol is the Wagner regular. Then there exists and unique a linear connection \( \nabla^A \) in the line bundle \( \xi \) such that
\[
(\theta^\sigma)^{k-2} \sigma_{k-1, \nabla} = 0.
\]

Proof. Due to (18), we have \( \nabla' = \nabla^A \) if and only if
\[
\theta]g_W + (\theta^\sigma)^{k-2} \sigma_{k-1, \nabla} = 0,
\]
and the last equation has a unique solution \( \theta \) if \( g_W \) non degenerated. \( \square \)

Remark 22. Connections \( \nabla^A \) and the Wagner connection \( \nabla^{\text{smblk}(A)} \) are natural in the sense that
\[
\phi_*(\nabla^A) = \nabla^{\phi_*(A)}, \phi_*(\nabla^{\text{smblk}(A)}) = \nabla^{\phi_*(\text{smblk}(A))},
\]
for any \( \phi \in \text{Aut}(\xi) \).

5.4. Connections, associated with differential equations. In order to study homogeneous differential equations, associated with differential operators we’ll study conformal classes of differential operators \([A] = \{fA| f \in \mathcal{F}(M)\}\) and geometrical structures associated with them. All operators in this section are assumed to be regular and constant type.

First of all we’ll change connections and will consider Chern connections \( \nabla^C \) \( \text{def} = \nabla[\sigma] \) associated with conformal classes \([\sigma]\) of the symbols instead of the Wagner \( \nabla^\sigma \) ones.

As we have seen (15) the torsion form \( \theta^\sigma \) is not invariant of the conformal class but its differential \( \omega^C = d\theta^\sigma \) does.

In what follows we’ll need some constructions from the linear algebra. To this end we’ll fix a point on \( M \) and denote by \( T \) the tangent space at the point.

Let
\[
\widehat{\omega} : T \rightarrow T^*
\]
be the linear operator, defined by \( \omega^C \), i.e.
\[
\langle \widehat{\omega}(X), Y \rangle = \omega^C(X, Y),
\]
for all vectors \( X, Y \in T \).

Let
\[
\widehat{\omega}_l : S^lT \rightarrow S^lT^*,
\]
be its $l$-th symmetric power for $l = 1, 2, \ldots$,
\[
\tilde{\omega}_l (X_1 \cdots X_l) = \tilde{\omega} (X_1) \cdots \tilde{\omega} (X_l),
\]
where $X_i \in T, i = 1, \ldots, l$.

Remark that $S^l T^* \simeq (S^l T)^*$ and therefore operators $\tilde{\omega}_l$ defines bilinear forms $\omega_l$ on $S^l T$ as follows
\[
\omega_l (X_1 \cdots X_l, Y_1 \cdots Y_l) = \langle \tilde{\omega} (X_1) \cdots \tilde{\omega} (X_l), Y_1 \cdots Y_l \rangle.
\]
Moreover, we have $(\tilde{\omega})^* = -\tilde{\omega}$ and therefore $(\tilde{\omega}_l)^* = (-1)^l \tilde{\omega}_l$, i.e. forms $\omega_l$ are symmetric or skew symmetric when $l$ is even or odd. They also are non degenerated if the initial form $\omega_C$ does.

As we also have seen (18) change of connection in the bundle leads us to change of sub symbol $\sigma_{k-1}$ in the following way
\[
\sigma_{k-1} \mapsto \sigma_{k-1} + \theta | \sigma,
\]
and therefore defines an $T^*$-action in $S^{k-1} T$.

Denote by
\[
L_{[\sigma]} = \{ \theta | \sigma \mid \theta \in T^* \} \subset S^{k-1} T
\]
a subspace in $S^{k-1} T$.

This subspace has dimension $n = \dim T$ and depends on the conformal class of the symbol, and orbits $O (\sigma_{k-1})$ of tensors $\sigma_{k-1} \in S^{k-1} T$ under the $T^*$-action are affine subspaces
\[
O (\sigma_{k-1}) = \sigma_{k-1} + L_{[\sigma]}.
\]

Let $L_{[\sigma]}^0 \subset S^{k-1} T$ be the annihilator of the image $\tilde{\omega}_l (L_{[\sigma]})$.

**Proposition 23.** Let the following conditions hold:

1. \( L_{[\sigma]} \cap \ker \tilde{\omega}_{k-1} = 0 \). \hspace{1cm} (21)
2. \( L_{[\sigma]} \cap L_{[\sigma]}^0 = 0 \). \hspace{1cm} (22)

Then for any tensor $\sigma_{k-1} \in S^{k-1} T$ there is and unique tensor $\sigma_{k-1}^0 \in L_{[\sigma]}^0 \cap O (\sigma_{k-1})$ and covector $\theta \in T^*$ such that
\[
\sigma_{k-1} = \sigma_{k-1}^0 + \theta | \sigma. \hspace{1cm} (23)
\]
Proof. Condition (21) shows that \( \dim(\tilde{\omega}_{k-1}(L_{[\sigma]})) = n \) and therefore \( \text{codim}(L_{[\sigma]}) = \dim(L_{[\sigma]}^{0}) = n \), next condition (22) shows that subspaces \( L_{[\sigma]} \) and \( L_{[\sigma]}^{0} \) are transversal. Therefore any orbit \( O(\sigma_{k-1}) \) has a unique intersection with \( L_{[\sigma]}^{0} \).

To reformulate transversality condition (22) let’s take a tensor \( \gamma \in L_{[\sigma]} \cap L_{[\sigma]}^{0} \). Then \( \gamma \in L_{[\sigma]} \) implies that \( \gamma = \alpha \cdot \sigma \), for some covector \( \alpha \in T^{*} \), and \( \gamma \in L_{[\sigma]}^{0} \) means that \( \omega_{k-1}(\gamma, L_{[\sigma]}) = 0 \), or \( \omega_{k-1}(\alpha \cdot \sigma, \beta \cdot \sigma) = 0 \), for all \( \beta \in T^{*} \). Denote by \( \omega_{\sigma} \) the following characteristic bivector (symmetric, when \( k \) is odd, and skew symmetric for even \( k \))

\[
\omega_{\sigma}(\alpha, \beta) = \omega_{k-1}(\alpha \cdot \sigma, \beta \cdot \sigma).
\]

Proposition 24. Subspaces \( L_{[\sigma]} \) and \( L_{[\sigma]}^{0} \) are transversal if (21) holds and characteristic bivector \( \omega_{\sigma} \) non degenerated.

Corollary 25. Let \( k \) be even and conditions (21, 22) are valid. Then \( \dim M \) is even too.

Definition 26. We say that a regular and constant type operator \( A \in \text{Diff}_{k}(\xi) \) is Chern regular if conditions (21, 22) hold.

Theorem 27. Let \( A \in \text{Diff}_{k}(\xi) \) be Chern regular operator. Then there is and unique linear connection \( \nabla^{[A]} \) in the line bundle \( \xi \), depending on conformal class \([A]\), and such that the subsymbol \( \text{smbl}_{k-1}(A) \), defining by this connection, belongs to \( L_{[\text{smbl}_{k}(A)]}^{0} \).

Proof. Let \( \nabla \) be a connection in the line bundle and let \( \sigma_{k-1} \) be subsymbol of operator \( A \). Then \( d_{\nabla} [A] = d_{\nabla} = \theta \otimes \text{Id} \), and subsymbol of \( A \), defining by \( \nabla^{[A]} \), belong to \( L_{[\text{smbl}_{k}(A)]}^{0} \) if and only if

\[
\sigma_{k-1} - \theta \cdot \sigma \in L_{[\text{smbl}_{k}(A)]}^{0}.
\]

As we have seen this condition uniquely defines covector \( \theta \) if operator \( A \) is Chern regular.

If we take another operator \( fA \) from the conformal class then \( \sigma_{k-1} \) and \( \text{smbl}_{k}(A) \) are multiplied by \( f \) but \( \theta \) will not be changed. \( \square \)

Remark 28. Connections \( \nabla^{[A]} \) and the Chern connection \( \nabla^{[\text{smbl}_{k}(A)]} \) are natural in the sense that

\[
\phi_{*}(\nabla^{[A]}) = \nabla^{[\phi_{*}(A)]}, \phi_{*}(\nabla^{[\text{smbl}_{k}(A)]}) = \nabla^{[\phi_{*}(\text{smbl}_{k}(A))]},
\]

for all automorphisms \( \phi \in \text{Aut}(\xi) \).
5.5. **Example: Ordinary differential operators.** The case of ordinary differential operators in many aspect exceptional from the point of view represented in this paper.

In this section we’ll discuss it in more details.

Let

\[ A = a_k \partial^k + \cdots + a_1 \partial + a_0, \]

be a scalar ordinary differential operator, where \( a_i = a_i (x) \), \( \partial = d/dx \), \( M = \mathbb{R} \).

This operator has the constant type if and only if \( a_k \neq 0 \).

Assume that a function \( \Gamma (x) \) is a Christoffel coefficient of a linear connection \( \nabla \) on \( M \):

\[ \nabla_\partial (\partial) = \Gamma \partial. \]

Then, the quantization \( Q : \Sigma_k \to \text{Diff}_k (M) \), associated with this connection, acts in the following way:

\[ Q (\partial^k) (f) = w^{-k} (w \partial_w - \Gamma w^2 \partial_w)^k (f), \]

for any function \( f = f(x) \).

In particular,

\[
\begin{align*}
Q (1) &= 1, \\
Q (\partial) &= \partial, \\
Q (\partial^2) &= \partial^2 - \Gamma \partial, \\
Q (\partial^3) &= \partial^3 - 3 \Gamma \partial^2 + (\Gamma^2 - \Gamma') \partial.
\end{align*}
\]

This connection is the Wagner connection if \( \nabla_\partial (a_k \partial^k) = 0 \), or

\[ \Gamma = - \frac{a_k'}{ka_k} \]

If operator \( A \) has order 2 and \( \nabla \) is the Wagner connection we get

\[ A = Q (\sigma_2 + \sigma_1 + \sigma_0), \]

where

\[
\begin{align*}
\sigma_2 &= a_2 \partial^2, \\
\sigma_1 &= \left( a_1 - \frac{a_2'}{2} \right) \partial, \\
\sigma_0 &= a_0.
\end{align*}
\]
For operators of the third order the corresponding Wagner connections are of the form
\[ \Gamma = -\frac{a_3'}{3a_3}, \]
and
\[ A = Q (\sigma_3 + \sigma_2 + \sigma_1 + \sigma_0), \]
where
\[
\begin{align*}
\sigma_3 &= a_3 \partial^3, \\
\sigma_2 &= (a_2 - a_3') \partial^2, \\
\sigma_1 &= \left(\frac{4a_3^2}{9a_3} - \frac{a_2a_3'}{3a_3} - \frac{a_3''}{3}\right) \partial, \\
\sigma_0 &= a_0.
\end{align*}
\]

In the case, when \( A \in \text{Diff}_k(\xi) \) and \( \xi \) is a line bundle over \( M = \mathbb{R} \) with a linear connection \( \nabla^\xi \) defined by a differential 1-form \( \theta(x) \, dx \), the quantization \( Q : \Sigma_k(M) \to \text{Diff}_k(\xi) \), associated with the connection \( \nabla \) on \( M \) and the connection \( \nabla^\xi \), has the form
\[ Q (\partial^k) (f) = w^{-k} (w \partial_x - \Gamma w^2 \partial_w + w\theta)^k (f), \]
and for low orders has the form
\[
\begin{align*}
Q (1) &= 1, \\
Q (\partial) &= \partial + \theta, \\
Q (\partial^2) &= \partial^2 + (2\theta - \Gamma) \partial + \theta' - \theta\Gamma + \theta^2, \\
Q (\partial^3) &= \partial^3 + 3(\theta - \Gamma) \partial^2 + (2\Gamma^2 - 6\theta\Gamma + 3\theta^2 - \Gamma' + 3\theta') \partial + \theta'' + 3(\theta - \Gamma) \theta' + ((\Gamma - \theta)(2\Gamma - \theta) - \Gamma') \theta.
\end{align*}
\]

In the case when \( \nabla \) is the Wagner connection for given operator \( A \) we define the associated connection \( \nabla^A \) by a requirement slightly different from the above. Namely, we’ll require that
\[ (\theta dx) |_{\sigma_k} = \sigma_{k-1}. \]  
(27)

Thus, for the second order operators,
\[ A = a_2 \partial^2 + a_3 \partial + a_0. \]
we have
\[ \sigma_2 = a_2 \partial^2, \]
\[ \sigma_1 = (a_1 + a_2 (\Gamma - 2\theta)) \partial. \]

Therefore, in this case we get
\[ \Gamma = -\frac{a_1'}{2a_2}, \theta = \frac{2a_1 - a_2'}{8a_2}, \]
and the invariant quantization \( Q_{\nabla W, \nabla A} \) has the form
\[
Q (1) = 1, \\
Q (\partial) = \partial + \frac{2a_1 - a_2'}{8a_2}, \\
Q (\partial^2) = \partial^2 + \frac{2a_1 + a_2'}{4a_2} \partial - \frac{a_2'' + 2a_1'}{8a_2} + \frac{5}{64} \left( \frac{a_2'}{a_2} \right)^2 + \frac{a_1^2 - 3a_1a_2'}{16a_2^2},
\]
and
\[
\sigma_2 = a_2 \partial^2, \\
\sigma_1 = \left( \frac{a_1}{2} - \frac{a_2'}{4} \right) \partial, \\
\sigma_0 = a_0 + \frac{a_2a_2'' + a_1a_2'}{8a_2} - \frac{a_1^2 + a_2a_1'}{4a_2} - \frac{7}{64} \frac{a_2^2}{a_2}. \tag{28}
\]

Remark that tensors (25 and 26) are invariants of scalar differential operators with respect to the diffeomorphism group and tensors (28) are invariants of the group of automorphisms.

In general case we get the following result (cf. [26]).

**Theorem 29.**

(1) Let \( A \in \text{Diff}_k (\mathbb{R}) \) be an ordinary differential operator of constant type and let \( \nabla W \) be the associated Wagner connection
\[ \nabla_W^W (\text{smbl} A) = 0, \]
and \( Q^w : \Sigma_k (\mathbb{R}) \to \text{Diff}_k (\mathbb{R}) \) be the quantization, defined by \( \nabla^W \). Then the total symbol
\[ \sigma = \sigma_k + \sigma_{k-1} + \cdots + \sigma_0, \]
\[ \sigma_i \in \Sigma_i (\mathbb{R}), \]
defined by the condition

\[ Q^W (\sigma_\cdot) = A, \]

is a tensor invariant of scalar ordinary differential operators with respect to diffeomorphism group \( G(\mathbb{R}) \).

Moreover, functions \( \sigma_0 \) and \( \lambda_i, \sigma_i = \lambda_i \sigma_i^1 \) are scalar differential invariants.

(2) Let \( A \in \text{Diff}_k (\xi) \) be a linear differential operator, acting in a line bundle \( \xi \) over \( \mathbb{R} \) and having the constant type. Let \( \nabla^W \) be the associated Wagner connection, and let \( \nabla^A \) be the linear connection in the line bundle \( \xi \) defined by (27) and \( Q^w, A : \Sigma_k (\mathbb{R}) \to \text{Diff}_k (\xi) \) be the quantization associated with these connections.

Then the total symbol \( \sigma_\cdot, Q^{W,A} (\sigma_\cdot) = A, \) is an invariant tensor of the ordinary differential operators with respect to automorphism group \( \text{Aut}(\xi) \) and the defined above functions \( \sigma_0 \) and \( \lambda_i, i = 2, \ldots, k \) are scalar differential invariants.

**Remark 30.** It is worth to note that these differential invariants are not Wilczynski invariants [26]: they are invariants of operators but not invariants of equations.

5.6. Differential invariants of constant type symbols. Let \( \pi : S^k T(M) \to M \) be the bundle of symmetric \( k \)-vectors (symbols) and let \( \nu_k \in \Sigma_k (\pi) \) be the universal symbol (of order 0). We denote by \( \mathcal{O}_0 \subset J^0 (\pi) \) the domain of regular symbols. The symbols having the constant type \( \varpi \) constitute a subbundle

\[ \pi^\varpi : \mathcal{E}_\varpi \to M \]

of the bundle \( \pi : \mathcal{O}_0 \to M \) of regular symbols.

Then the Wagner connection defines a total covariant differential

\[ \hat{d}^\varpi : \Sigma_1 (\pi^\varpi) \to \Sigma_1 (\pi^\varpi) \otimes \Omega^1 (\pi^\varpi), \]

over the domain of regular symbols, and, by the construction

\[ \hat{d}^\varpi (\nu_k) = 0. \]

Let \( T^\varpi \in \Omega^2 (\pi^\varpi) \otimes \Sigma_1 (\pi^\varpi) \) be the total torsion of the connection and \( \theta^\varpi \in \Omega^1 (\pi^\varpi) \) be the torsion form.
Then, applying the total differential of the dual (to Wagner) connection
\[
\hat{d}_\omega : \Omega^1(\pi^\omega) \to \Omega^1(\pi^\omega) \otimes \Omega^1(\pi^\omega)
\]
we get tensor
\[
\hat{d}_\omega (\theta^\omega) \in \Omega^1(\pi^\omega) \otimes \Omega^1(\pi^\omega).
\]

Taking the symmetric \(g^\omega\) and antisymmetric \(a^\omega\) parts of this we get tensors
\[
g^\omega \in \Sigma^2(\pi^\omega), \quad a^\omega \in \Omega^2(\pi^\omega).
\]
Assuming that tensor \(g^\omega\) is non degenerated we get total operator
\[
A^\omega \in \Sigma_1(\pi^\omega) \otimes \Omega^1(\pi^\omega),
\]
instead of \(a^\omega\), and horizontal 1-forms
\[
\theta^\omega_1, \theta^\omega_2 = A^\omega(\theta^\omega_1), \ldots, \theta^\omega_n = A^\omega(\theta^\omega_{n-1}). \tag{29}
\]

Remark that the torsion \(T^\omega\) and torsion form has order 1 and therefore, tensors \(g^\omega, a^\omega, A^\omega\) and \(\theta^\omega_i\) has order 2.

We say that a domain \(O_2^\pi \subset J^2(\pi^\omega)\) consist of regular 2-jet of symbols if the tensor \(g^\omega\) is non degenerated and
\[
\theta^\omega_1 \wedge \cdots \wedge \theta^\omega_n \neq 0. \tag{30}
\]

Let \((e^\omega_1, \ldots, e^\omega_n)\) be the frame of horizontal vector fields \(e^\omega_i \in \Sigma_1(\pi^\omega)\) dual to coframe \((\theta^\omega_1, \ldots, \theta^\omega_n)\). Then coefficients \(J^\omega_\alpha\) in the decomposition of universal symbol \(\nu_k\) in this frame
\[
\nu_k = \sum_{|\alpha| = k} J^\omega_\alpha (e^\omega_1)^{\alpha_1} \cdots (e^\omega_n)^{\alpha_n}, \tag{31}
\]
are rational functions over regular domain \(O_2^\pi\) and invariants of the diffeomorphism group.

**Theorem 31.** The field of rational natural invariants of symbols having degree \(k\) and constant type \(\pi^\omega\) is generated by invariants \(J^\omega_\alpha, |\alpha| = k\), and invariant derivations \(e^\pi_i, i = 1, \ldots, n\).
5.7. Differential invariants of constant type scalar differential operators. Let $\chi^\omega_k: \text{Diff}^k(M) \rightarrow M$ be the bundle of scalar differential operator of order $k$, having symbols of constant type $\omega$, and let $\text{Diff}^\omega_k(M)$ be its module of smooth sections.

By $\hat{O}_2^\omega \subset J^2(\chi_k^\omega)$ we denote the domain, where 2-jets of symbols are regular in the above sense, i.e. 2-jets of symbols belong to regular domain $O_2^\omega$.

Denote by $\tau^\omega_k: S^k T^\omega \rightarrow M$ bundles of symbols having degree $k$ and constant type $\omega$, and let $\tau_l: S^l T \rightarrow M$ be bundles of symbols of degree $l$, $l = 0, 1, ..., $ and let

$$\tau_{(k)} = \tau^\omega_k \oplus \tau_{k-1} \oplus \cdots \oplus \tau_1 \oplus \tau_0$$

be the bundle of total symbols with principle symbol having of constant type $\omega$.

Consider differential operator

$$\mu_k : J^{k+1}(\chi_k^\omega) \rightarrow \tau_{(k)},$$

which sends differential operators $A \in \text{Diff}^\omega_k(M)$ having regular 2-jet $[A]^2_p \in \hat{O}_2^\omega$ to the total symbol

$$\text{smbl}_{(k)}(A) = (\text{smbl}_k(A), \text{smbl}_{k-1}(A), ..., \text{smbl}_0(A))$$

with respect to the Wagner connection that corresponds to the regular principal symbol $\text{smbl}_k(A)$.

It follows from the construction of the Wagner connection that this operator has order $(k + 1)$ and is natural, i.e. commutes with the action of the diffeomorphism group.

Regularity conditions allow us to construct invariant coframe (29), and then by decomposing (31) the total symbol in this coframe to find natural rational invariants $J_{\alpha}^\omega$, where $|\alpha| \leq k$, on the $(k + 1)$-jet bundle $J^{k+1}(\chi_k^\omega)$.

It follows from (31) that invariants $J_{\alpha}^\omega$ and invariant derivations $e_i^\omega$ generate the field of natural invariants of total symbols.

Therefore, applying the prolongations of $\mu_k$,

$$\mu_k^{(l)} : J^{k+l+1}(\chi_k^\omega) \rightarrow J^l(\tau_{(k)}),$$

we’ll get natural invariants of differential operators of the constant type.
**Theorem 32.** The field of natural differential invariants of linear scalar differential operators of order \( k \geq 3 \), having constant type \( \varpi \), is generated by the basic invariants \( \mu^*_k (J^\varpi_\alpha) \), \( |\alpha| \leq k \), and invariant derivatives \( e_i^\varpi \), \( i = 1, \ldots, n \).

5.8. **Differential invariants of constant type differential operators, acting in line bundles.** In this section we consider \( \text{Aut}(\xi) \)-invariants of linear differential operators, acting in line bundle \( \xi \).

We will consider differential operators of the constant type \( A \in \text{Diff}_k^\varpi (\xi) \subset \text{Diff}_k (\xi) \), i.e. operators such that their principal symbol \( \text{smbl}_k (A) \in \Sigma_k (M) \) belongs to type \( \varpi \).

We denote by

\[
\pi^\varpi_k : \text{Diff}_k^\varpi (\xi) \rightarrow M
\]

the bundle of the constant type operators.

We’ll also assume that the symbols of operators are regular in the previous sense:

\[
j^2 (\text{smbl}_k (A)) \subset \hat{O}^\varpi_2.
\]

Thus the symbol defines the Wagner connection \( \nabla^w \), the invariant coframe and, in addition, the linear connection \( \nabla^A \) in the line bundle \( \xi \), if the symbol is Wagner regular.

From now on we’ll call such symbols simply regular.

Remark that Wagner regularity depends on the second jet of the operator and therefore defines an open subset in the space of second jets of operators \( \mathcal{O}^\varpi_2 \subset J^2 (\pi^\varpi_k) \).

By the construction all of these connections are \( \text{Aut}(\xi) \)-invariants. Therefore, there is the total symbol

\[
\text{smbl}_{(k)} (A) = (\text{smbl}_k (A), \text{smbl}_{k-1} (A), \ldots, \text{smbl}_0 (A)) \in \tau_{(k)},
\]

such that

\[
Q^A (\text{smbl}_{(k)} (A)) = A.
\]

Here we denoted by \( Q^A \) the quantization, defined by connections \( \nabla^{\text{smbl}_k (A)} \) and \( \nabla^A \).

Remark that operator \( Q^A \) is also \( \text{Aut}(\xi) \)-invariant.

Let \( \kappa^A \in \Omega^2 (M) \) be the curvature form of \( \nabla^A \). It depends on the second jets of the operator and by the standard procedure defines a horizontal 2-form \( \kappa \in \Omega^2 (\pi^\varpi_k) \), having the second order and satisfying
the universality condition:
\[ j_2 \left( A \right)^* (\kappa) = \kappa^A, \]
for all differential operators \( A \in \text{Diff}_k^\varpi (\xi) \) with regular symbols.

As above, we consider differential operator
\[ \mu_k : J^{k+1} (\pi_k^\varpi) \to \tau(k), \]
which sends differential operators \( A \in \text{Diff}_k^\varpi (M) \), having regular 2-jet \( [A]_p^2 \in \mathcal{O}_2^\varpi \), to their total symbol and get the following result similar to (32).

**Theorem 33.** The field of natural differential Aut(\( \xi \))-invariants of linear differential operators of order \( k \geq 3 \), having constant type \( \varpi \) and acting in line bundle \( \xi \), is generated by the basic invariants \( \mu_k^* (J_\alpha^\varpi) \), \( |\alpha| \leq k \), \( K_{ij} \), where \( J_\alpha^\varpi \) and \( K_{ij} \) are coordinates of the total symbol and the universal curvature for \( \kappa \) in the invariant frame, and by invariant derivatives \( e_i^\varpi \), \( i = 1, \ldots, n \).

6. **Equivalence of differential operators**

6.1. **Equivalence of scalar differential operators.** Let \( A \) be a linear scalar differential operator over \( M \), i.e. \( A \in \text{Diff}_k(M) \). We’ll say this operator is in **general position** if for any point \( a \in M \) there are natural invariants \( I_1, \ldots, I_n \), where \( n = \text{dim} \, M \), such that their values \( I_i (A), i = 1, \ldots, n \), on this operator are independent in a neighborhood \( U \) of this point, i.e.
\[ dI_1 (A) \wedge \cdots \wedge dI_n (A) \neq 0. \]  
(32)
The principle of \( n \)-invariants states that these invariants and invariants \( I_\alpha = \Box (I_1^{\alpha_1} \cdots I_n^{\alpha_n}) \), \( |\alpha| \leq k \), generate all invariants and dependencies
\[ I_\alpha (A) = F_\alpha (I_1 (A), \ldots, I_n (A)) \]  
(33)
give us coefficients of the operator in these local coordinates
\[ x_1 = I_1 (A), \ldots, x_n = I_n (A). \]  
(34)
We call these coordinates **natural** due to the following their property.

Let \( A' \in \text{Diff}_k(M') \) be another operator and let \( \phi : M \to M' \) be a local diffeomorphism such that
\[ \phi_* (A) = A', \phi (a) = a'. \]
Then we have
\[ \phi_*(I_1(A)) = \phi^{*-1}_i(I_i(A)) = I_i(\phi_*(A)) = I_i(A'), \]
i.e. in natural coordinates the diffeomorphism has the following form
\[ (x_1, ..., x_n) \to (x'_1, ..., x'_n), \]
where
\[ x'_1 = I_1(A'), ..., x'_n = I_n(A'). \]
Therefore functions
\[ F_\alpha = F_\alpha(x_1, ..., x_n) \]
defines the orbit of the germ of operator \( A \) at the point \( a \in M \) with respect to the diffeomorphism group.

We will reformulate this observation in the following way.

For a given operator \( A \in \operatorname{Diff}_k(M) \) in general position we’ll consider:

- **Natural charts** - i.e. local diffeomorphisms
  \[ \phi^I : U^I \to D^I \subset \mathbb{R}^n, \]
on open domains in \( \mathbb{R}^n \) given by such natural invariants \( I = (I_1, ..., I_n) \), that (32) holds in open set \( U^I \).

- **Natural atlas** - i.e. a collection of natural charts \( \{U^I, \phi^I\} \), covering manifold \( M \), and given by distinct natural invariants.

We denote by
\[ D^{IJ} = \phi^I(U^I \cap U^J) \]
and assume that domains \( U^I, U^{IJ} \) are connected and simply connected.

Let \( A_I = \phi^I_*(A|_{U^I}), A_{IJ} = \phi^{*I}_*(A|_{U^I \cap U^J}) \) be the images of the operator \( A \) in natural coordinates.

Then \( \phi^{*I}_*(A_{IJ}) = A^{JI} \), where \( \phi^{JI} : D^{IJ} \to D^{JI} \) are the transition maps.

We call [16] such atlas as natural atlas associated with operator \( A \) and collection the \( (D^I, D^{IJ}, \phi^{IJ}, A^I, A^{IJ}) = D(A) \) we call -natural model of the operator.

**Theorem 34.** Let \( A, A' \in \operatorname{Diff}_k(M) \) be operators in general position. Then these operators are equivalent with respect to group of diffeomorphisms if and only if the following conditions hold:

**Open sets**
\[ U^{II} = (\phi^I)^{-1}(D^I), \]
where \( \phi^{IJ'} = (I_1(A'), ..., I_n(A')) : M \to \mathbb{R}^n \) constitute a natural atlas for operator \( A' \), \( \phi^{IJ'} = \phi^{IJ} : D^{IJ} \to D^{JI} \), and

\[ A_I = \phi^I_{I'}(A'|_{U^I}), A_{IJ} = \phi^I_{I'}(A'|_{U^I \cap U^{J'}}), \]

i.e. when natural models of the operators coincide.

\textbf{Proof.} As we have seen any diffeomorphism, transforming operator \( A \) to \( A' \), in natural coordinates has the form of the identity map. \( \square \)

\textbf{Remark 35.} We have two types of scalar differential operators, where we are able to compute the fields of natural differential invariants: operators with symbols of the general type and the constant type operators.

\section{Equivalence of differential operators, acting in line bundles.}

For the case of operators \( A \in \text{Diff}_k(\xi) \), acting in the line bundle \( \xi \), we’ll restrict ourselves by the regular operators of the constant type only.

In this case we have two connections naturally associated with operator: Wagner connection \( \nabla_{\text{smbl}^k(A)} \) on the manifold and connection \( \nabla^A \) in the line bundle. These connections define also the quantization \( Q^A \), naturally (with respect to the automorphism group \( \text{Aut}(\xi) \)) associated with this operator.

Thus operator \( A \) has well defined total symbol \( \text{smbl}^k(A) \) and accordingly scalar shadow \( A^*_z \in \text{Diff}_k(M) \) of the operator \( A \),

\[ A^*_z = Q^W(\text{smbl}^k(A)). \]

Here \( Q^W \) is the quantization for scalar differential operators given by the Wagner connection \( \nabla_{\text{smbl}^k(A)} \).

Naturality of all these constructions show us that if two regular operators \( A, B \in \text{Diff}_k(\xi) \) of constant type \( \varpi \) are \( \text{Aut}(\xi) \)-equivalent then their scalar shadows \( A^*_1, B^*_1 \in \text{Diff}_k(M) \) should be equivalent with respect to the diffeomorphism group \( G(M) \).

On the other hand, let operators \( A_2, B_2 \in \text{Diff}_k(M) \) are \( G(M) \)-equivalent and let diffeomorphism \( \psi : M \to M \) (see the above construction) sends operator \( A_2 \) to \( B_2 \), \( \psi_*(A_2) = B_2 \).

Then diffeomorphism \( \psi \) has a lift \( \bar{\psi} \in \text{Aut}(\xi) \) if and only if (see, Proposition 1)

\[ \psi^*(w_1(\xi)) = w_1(\xi). \] (35)
Assume that this condition holds, then operators \( \overline{A} = \overline{\psi}^* (A) \) and \( B \) has the same total symbols and scalar shadows.

Let \( \nabla^A = \nabla^\overline{\psi} \) be the image of connection \( \nabla^A \) under the automorphism \( \overline{\psi} \), and let \( \kappa_A \in \Omega^2 (M) \) and \( \kappa_B \in \Omega^2 (M) \) be curvature forms for linear connections \( \nabla^A \) and \( \nabla^B \) respectively. They give us one more condition for diffeomorphism \( \psi : \nabla^B \rightarrow \nabla^A \):

\[
\psi^* (\kappa_B) = \kappa_A, \tag{36}
\]

or equivalently

\[
\kappa_B = \kappa_A.
\]

Then,

\[
d\nabla^\overline{\psi} - d\nabla^B = \theta_\psi \otimes \text{id}
\]

for some differential 1-form \( \theta_\psi \in \Omega^1 (M) \), and therefore

\[
d\theta_\psi = 0,
\]

if condition (36) holds.

Remark, that the different lifts \( \overline{\psi} \) differ on automorphisms given by multiplication on functions \( f \in \mathcal{F} (M) \) that induce transformations \( \theta_\psi \rightarrow \theta_\psi + d \ln |f| \).

Thus the cohomology class

\[
\vartheta_{A,B} \in H^1 (M, \mathbb{R})
\]

of the closed 1-form \( \theta_\psi \) defines a new obstruction for existence of the automorphism \( \psi \).

Summarizing, we get the following.

**Theorem 36.** Two regular operators \( A, B \in \text{Diff}^k (\xi) \) of constant type are \( \text{Aut}(\xi) \)-equivalent if and only if their scalar shadows \( A_\sharp, B_\sharp \in \text{Diff}^k (M) \) are \( \mathcal{G}(M) \)-equivalent and the diffeomorphism \( \psi : M \rightarrow M, \psi_\sharp (A_\sharp) = B_\sharp \), satisfies in addition to the following conditions:

1. It preserves the first Stiefel-Whitney class \( w_1 (\xi) \) of the bundle:

\[
\psi^* (w_1 (\xi)) = w_1 (\xi).
\]

2. It transforms the curvature form of the connection \( \nabla^B \) to the connection form of the connection \( \nabla^A \):

\[
\psi^* (\kappa_B) = \kappa_A.
\]
(3) The obstruction $\vartheta_{A,B} \in H^1(M, \mathbb{R})$ is trivial:

$$\vartheta_{A,B} = 0.$$ 

**Remark 37.** In the above construction of the diffeomorphism $\psi$ in terms of natural atlases we should add the following condition only: components of the curvature forms in the corresponding natural charts should be equal. This condition shall fix the local structure of the diffeomorphism. Topological conditions: $\psi^* (w_1 (\xi)) = w_1 (\xi)$ and $\vartheta_{A,B} = 0$ should be checked additionally.

6.3. **Equivalence of differential equations.** By differential equations in this section we’ll understand homogeneous differential equations given by Chern regular differential operators $A \in \text{Diff}_k (\xi)$, or in another words, by conformal classes $[A]$ of operators.

We’ll use the Chern connection $\nabla^{\text{smbl}_k (A)}$, defining on manifold $M$ by the conformal class $[\text{smbl}_k (A)]$ of the principal symbol, and the linear connection $\nabla^{[A]}$, defining by the conformal class $[A]$ of the operator.

We denote by

$$\text{smbl}_{(k)} (A) = \sum_{i=0}^{k} \text{smbl}_i (A)$$

the total symbol of the operator, given by the quantization, associated with connections $\nabla^{\text{smbl}_k (A)}$ and $\nabla^{[A]}$.

The linear property of the quantization shows us that

$$\text{smbl}_i (fA) = f \text{smbl}_i (A),$$

for all functions $f \in \mathcal{F} (M)$.

In other words, tensors $\text{smbl}_i (A) \in \sum_i (M)$ are relative invariants of operators.

Let $\omega^C = d\theta^{\text{smbl}_k (A)} \in \Omega^2 (M)$ and $\tilde{\omega} : \Omega_1 (M) \to \Omega^1 (M)$ be the operator defined in (20).

Then functions

$$H_i (A) = \alpha^i |\text{smbl}_i (A)|,$$

where $\alpha = \tilde{\omega} (\text{smbl}_1 (A)) \in \Omega^1 (M)$, are also relative invariants

$$H_i (fA) = f^{i+1} H_i (A).$$
Then operators

$$A_{\nu,i} = \lambda_i(A) A \in \text{Diff}_k(\xi),$$

where

$$\lambda_i(A) = \frac{H_i(A)}{H_{i+1}(A)},$$

we call normalizations of operator $A$.

It is easy to check that

$$(fA)_{\nu,i} = A_{\nu,i},$$

and therefore $A_{\nu,i}$ depends on the conformal class only.

**Theorem 38.** Let $[A]$ and $[B]$ be conformal classes of Chern regular differential operators $A, B \in \text{Diff}_k(\xi)$, such that $\lambda_i(A), \lambda_i(B) \in F(M)$, for some $i, i = 0, ..., k - 1$.

Then these classes are $\text{Aut}(\xi)$-equivalent if and only if their normalizations $A_{\nu,i}$ and $B_{\nu,i}$ are $\text{Aut}(\xi)$-equivalent.

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