Mirror Symmetry as a Gauge Symmetry

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ABSTRACT

It is shown that in string theory mirror duality is a gauge symmetry (a Weyl transformation) in the moduli space of $N = 2$ backgrounds on group manifolds, and we conjecture on the possible generalization to other backgrounds, such as Calabi-Yau manifolds.

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1 Introduction

Target-space dualities in string theory are symmetries relating backgrounds with different geometries that correspond to the same 2-d Conformal Field Theory (CFT). The simplest example is the $R \to 1/R$ circle duality, that relates a circle of radius $R$ to a circle of radius $1/R$.

This duality is a gauge symmetry in string theory in the following sense. At the self-dual point, $R = 1$, there is an enhanced $SU(2)_L \times SU(2)_R$ affine symmetry. One can deform away from the $R = 1$ point by adding a current-current truly marginal operator, $J\bar{J}$. There is a Weyl rotation in $SU(2)_L$ that takes $J \to -J$, and therefore, this transformation is a symmetry of the self-dual point. However, this transformation relates the conformal deformation $\epsilon J\bar{J}$ to $-\epsilon J\bar{J}$, and on the full modulus line of circle compactifications, this transformation corresponds to the $R \to 1/R$ duality.

In the target space effective action we have the following picture. The worldsheet couplings to operators, perturbing a given 2-d action, become target space fields. There is an $SU(2)_L \times SU(2)_R$ gauge symmetry when the scalar fields get VEVs that correspond to the $R = 1$ point. This gauge symmetry is spontaneously broken to $U(1)_L \times U(1)_R$ when one changes the VEVs of scalar fields. There is a residual $Z_2$ gauge transformation in the spontaneously broken gauge group that relates the VEV corresponding to radius-$R$ compactification to the VEV corresponding to radius-$1/R$ compactification. It is in this sense that the $R \to 1/R$ duality is a gauge symmetry in string theory.

The interpretation of target-space dualities as gauge symmetries was generalized to the duality group $O(d,d,\mathbb{Z})$ of toroidal backgrounds. Moreover, in ref. it was shown that there is a duality in the moduli space of $J\bar{J}$ deformations of $G_k$ WZW models. This duality is a gauge symmetry in the same sense described above. It is called an ‘axial-vector duality’ for reasons that will be clear soon, and it relates curved backgrounds with different geometries, and even with different topologies.

The relation of target space dualities to gauge symmetries shows that they are exact symmetries in string theory (to all orders and interactions).

In this note we will describe a particular target-space duality – mirror symmetry – in the moduli space of $N = 2$ backgrounds on a group manifold $G$. Moreover, mirror duality will be related to a gauge symmetry in string theory.

The structure of the paper is as follows: In section 2, we begin with an $N = 2$ affine construction on a group $G$, and in section 3, we consider the $N = 2$ construction on $SU(2) \times U(1)$. In section 4, we describe the mirror transformation, and in section 5, we discuss mirror duality in the moduli space of $N = 2$ models derived from $SU(2) \times U(1)$ (or $SL(2) \times U(1)$). In section 6, we discuss mirror duality as a gauge symmetry in the moduli space of $N = 2$ models on general groups $G$, and in section 7, we present the $SU(3)$ example. Finally, in section 8, we conjecture that mirror duality is a gauge symmetry in string theory, also for Calabi-Yau compactifications.
2  \( N = 2 \) Affine Construction on a group \( G \)

It is known that any even dimensional group allows an \( N = 2 \) super affine symmetry \([8]\). Following ref. \([9]\), we generate the affine \( N = 2 \) algebra on a group \( G \) at level \( k \). It is sufficient to describe the left-handed part. Let us present currents \( j^a(z) \) and fermions \( \psi^a(z) \) in the adjoint of \( G \) that satisfy the operator product expansion (OPE)

\[
{j^a(z)j^b(w)} = \frac{\hat{k}\delta^{ab}}{(z-w)^2} + \frac{if_{abc}j^c(w)}{z-w} + ..., \tag{2.1}
\]

\[
{\psi^a(z)\psi^b(w)} = \frac{\hat{k}\delta^{ab}}{z-w}, \quad {j^a(z)\psi^b(w)} = 0 + .... \tag{2.2}
\]

Here \( f_{abc} \) are the structure constants of the Lie algebra \( G \), \( \hat{k} \equiv k - C_2(G) \), and dots stand for non-singular terms in the OPE \([3]\). The sigma-model which corresponds to this theory is the level \( k \) \( N = 1 \) WZW Lagrangian on a group \( G \),

\[
S[\hat{g}] = \frac{k}{2\pi} \int d^2zd^2\theta \Tr\left(D\hat{g}^{-1}\partial\partial^{-1}\hat{g} - i \int dt[\hat{g}^{-1}\partial\hat{g}, \partial\hat{g}]\hat{g}^{-1}D\hat{g}\right), \tag{2.3}
\]

where

\[
\hat{g}(z, \bar{z}, \theta, \bar{\theta}) = e^{T_aX^a}, \quad X^a = x^a + \theta\frac{\psi^a}{k} + \frac{1}{k} + \bar{\theta}\Gamma^a, \quad D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}. \tag{2.4}
\]

The chiral \( N = 1 \) supercurrents are

\[
\hat{J}^a = k\Tr(T^aD\hat{g}\hat{g}^{-1}) = \psi^a + \theta \left( j^a - \frac{i}{k}f_{bc}\psi^b\psi^c \right), \tag{2.5}
\]

where the currents \( j^a \) in \((2.1, 2.3)\) are given by

\[
{ j}^a = k\Tr(T^a\partial\hat{g}\hat{g}^{-1}), \quad \hat{g} = e^{T_aX^a}. \tag{2.6}
\]

The central charge of this model is

\[
c = \frac{\hat{k}\dim G}{k + C_2} + \frac{1}{2}\dim G. \tag{2.7}
\]

For the \( N = 2 \) superconformal algebra (SCA), in addition to the stress-tensor and \( N = 1 \) supercurrent,

\[
T(z) = \frac{1}{k}(j^a\psi^a - \psi^a\partial\psi^a), \tag{2.8}
\]

\[
G^0(z) = \frac{2}{k}(\psi^a\psi^a - \frac{i}{3k}f_{abc}\psi^b\psi^c), \tag{2.9}
\]

\(3\) As indices are raised and lowered by \( \delta_{ab} \) we will not be careful about upper and lower indices. Any repetition of indices means a summation. The discussion can be carried out for a general bilinear form \( \eta \) replacing \( \delta \).

we need another $N = 1$ supercurrent which we write as

$$G^1(z) = \frac{2}{k}(h_{ab}\psi^a\psi^b - \frac{i}{3k}S_{abc}\psi^a\psi^b\psi^c).$$  \hspace{1cm} (2.10)$$

We define $G^\pm$ by

$$G^0 \equiv \frac{1}{\sqrt{2}}(G^+ + G^-), \quad G^1 \equiv \frac{1}{\sqrt{2i}}(G^+ - G^-).$$  \hspace{1cm} (2.11)$$

The necessary and sufficient conditions for achieving $N = 2$ SCA are

$$h_{ab} = -h_{ba}, \quad h_{ac}h_{cb} = -\delta_{ab},$$  \hspace{1cm} (2.12)$$

$$f_{abc} = h_{ap}h_{bq}f_{pqc} + h_{bp}h_{cq}f_{pqa} + h_{cp}h_{aq}f_{pqb},$$  \hspace{1cm} (2.13)$$

$$S_{abc} = h_{ap}h_{bq}h_{cr}f_{pqr}.$$  \hspace{1cm} (2.14)$$

When these conditions are satisfied, an $N = 2$ SCA is generated by $T(z), G^+(z), G^-(z), J(z)$ (see for example [9]), where the $U(1)$ current, $J$, is determined from the explicit OPEs:

$$J = h_{ab}\left[\frac{i}{k}\psi^a\psi^b + \frac{1}{k}f^{ab}_c(j^c - \frac{i}{k}f^c_{de}\psi^d\psi^e)\right].$$  \hspace{1cm} (2.15)$$

The condition (2.12) means that $h_{ab}$ is an (almost) complex structure. To see the meaning of conditions (2.13),(2.14), let us introduce the projection operators

$$(P_\pm)_{ab} = \frac{1}{2}\left(\delta_{ab} \pm \frac{1}{i}h_{ab}\right),$$  \hspace{1cm} (2.16)$$

and split the set of the Lie algebra generators $T = \{T^a|[T^a, T^b] = if_{abc}T^c\}$ into two sets $T_+$ and $T_-:

$$T_\pm = \{T^a_\pm|T^a_\pm = (P_\pm)_{ab}T^b\}.$$  \hspace{1cm} (2.17)$$

Then, by a straightforward calculation one finds that (2.13),(2.14) are equivalent to the conditions

$$[T^a_\pm, T^b_\pm] = \frac{i}{2}(f_{abc} \pm iS_{abc})T^c_\pm,$$  \hspace{1cm} (2.18)$$

which can be written schematically as

$$[T_+, T_+] \subset T_+, \quad [T_-, T_-] \subset T_-.$$  \hspace{1cm} (2.19)$$

We may thus summarize the result as follows [9]:

**Theorem:**

Let $T$ be the complexified Lie algebra of $G$. Then the model $G$ has an $N = 2$ structure for every direct sum decomposition $T = T_+ \oplus T_-$ (dim $T_+ = \dim T_-$) such that $T_+$ and $T_-$ separately form a closed Lie algebra, and $T_- = \overline{T_+}$.

For the applications of this result, one must bear in mind the following. Our discussion so far has been purely algebraic. In a geometric context of WZW models, one has both left and
right-moving current algebra, coming from the left and right action of $G$ on itself. Accordingly, two copies of $T$ appear, say $T_L$ and $T_R$ – the generators of the left and right action of $G$, which we will call $G_L$ and $G_R$. In constructing an $N = 2$ structure – by which we mean a structure with $(2,2)$ supersymmetry – with target space $G$, the above theorem must be used twice, once for left-movers and once for right-movers. Accordingly, one actually picks two complex structures on $T$, a left-moving one and a right-moving one.

3 $N = 2$ Construction on $SU(2) \times U(1)$

Let $T = \{T_1, T_2, T_3, T_0\}$, where $\{T_i, |i = 1, 2, 3\}$ are the generators of the $SU(2)$ Lie algebra, $[T_i, T_j] = i\epsilon_{ijk}T_k$, and $T_0$ is the $U(1)$ generator. A complex structure

$$h_{ab} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(3.1)
gives an $N = 2$ SCA, as was shown in [8, 10]. The proof can be done either by a straightforward check that the conditions (2.12),(2.13) are satisfied ((2.14) defines $S_{abc}$ in terms of $h_{ab}$), or by showing that the structure described in the theorem is maintained. Let us do the latter: the projection operators are

$$P_{\pm} = \begin{pmatrix} I \mp i\epsilon & 0 \\ 0 & I \mp i\epsilon \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(3.2)
and therefore,

$$T_+ = \{ T^1_+ = \frac{1}{2}(T_1 - iT_2), \ T^2_+ = \frac{1}{2}(T_3 - iT_0) \},$$

$$T_- = \{ T^1_- = \frac{1}{2}(T_1 + iT_2), \ T^2_- = \frac{1}{2}(T_3 + iT_0) \}.$$  

(3.3)
One finds that

$$[T^1_+, T^2_+] = \frac{1}{2} T^1_+, \quad [T_1, T^2_+] = \frac{1}{2} T^1,$$

(3.4)
and therefore, $[T_+, T_+] \subset T_+$, $[T_- , T_-] \subset T_-.$

4 Mirror Transformation

For simplicity, we first describe the $SU(2) \times U(1)$ model. Combining left-movers and right-movers we have an $N = 2$ affine algebra on $(SU(2) \times U(1))_L \times (SU(2) \times U(1))_R$. A mirror transformation, $m$, is a transformation of $N = 2$ CFT’s that acts as

$$m : \ J \rightarrow -J, \quad \overline{J} \rightarrow \overline{J},$$

(4.1)
where $J (\overline{J})$ is the left- (right-) handed $N = 2 U(1)$ current. From (2.13) it follows that in the present context $m$ acts on the left-handed complex structure as

$$m(h_{ab}) = -h_{ab},$$

(4.2)
while commuting with the right-handed one.

We now arrive to a key point. If the left and right moving complex structures are as described above, then a Weyl rotation in the group $SU(2)_L$ has the right properties to be interpreted as a mirror symmetry. In the realization of the $SU(2) \times U(1)$ model as a WZW model, the field $g$ in $SU(2) \times U(1)$ transforms to $mg$. We pick $m$ to a $\pi$-rotation around the 1-axis,

$$m = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

acting on the Lie algebra as

$$m(T_1, T_2, T_3, T_0) = (T_1, -T_2, -T_3, T_0).$$

Thus, $m$ interchanges $T_+$ with $T_-$, and therefore, it takes the left-handed complex structure to its minus. Thus, the $SU(2) \times U(1)$ model, with the $N=2$ structure under discussion, is equivalent to its own mirror, via the transformation $m$.

5 Mirror Duality in the Moduli Space of $N = 2$ $SU(2) \times U(1)$ (or $SL(2) \times U(1)$)

We now look for a current-current deformation, $W \bar{W}$, where

$$W = \sum_{a=0}^{3} \alpha_a J^a, \quad J^a = \text{Tr} W^a = \text{Tr} \left[ T^a \left( k \partial g g^{-1} - \frac{2}{k} T_b \bar{\psi}^b T_c \psi^c \right) \right], \quad g \in SU(2) \times U(1),$$

$$\bar{W} = \sum_{a=0}^{3} \beta_a \bar{J}^a, \quad \bar{J}^a = \text{Tr} \bar{W}^a = \text{Tr} \left[ T^a \left( k g^{-1} \bar{\partial} g - \frac{2}{k} T_b \bar{\psi}^b T_c \overline{\psi}^c \right) \right]$$

such that the chiral current $J^a$ (antichiral current) is an upper component of the chiral $N=1$ supercurrent in (2.5) (antichiral supercurrent). When $J^a$ and $\bar{J}^a$ are in the Cartan sub-algebra, the $W \bar{W}$ deformation preserves $N=2$ supersymmetry. (This is explained in section 6, in a more general case.)

The deformation $W \bar{W}$ is particularly interesting if $W = J^3$, $\bar{W} = \bar{J}^3$. This deformation is odd under the mirror symmetry $m$,

$$m(W \bar{W}) = -W \bar{W}.$$  

This is true as $m$ anticommutes with $W$ and commutes with $\bar{W}$:

$$\{m, W\} = 0, \quad [m, \bar{W}] = 0.$$  

The first equality is true because $m$ is a rotation around the 1-axis while $W = J^3$ is a rotation around the 3-axis. The second equality is trivially true as $m$ acts purely in the left-handed sector.
The meaning of eq. (5.2) is that under the mirror transformation $m$, the (infinitesimal) perturbation $\epsilon W \bar{W}$ is related to $-\epsilon W \bar{W}$ (as $W \bar{W}$ is mirror odd). Therefore, mirror symmetry is a gauge transformation ($m \in SU(2)_L$) along the $W \bar{W}$ deformation line.

This deformation line was already studied in ref. [6] (although for $N = 0$ WZW models). The perturbation operator

$$W \bar{W} = J^3 \bar{J}^3 = j^3 \bar{j}^3 + \text{(terms with worldsheet fermions)}$$

(5.4)
deforms the $SU(2)$ WZW sigma-model, and generates a one-parameter family of conformal sigma-models parametrized by $0 < R < \infty$ (we refer the reader to ref. [6] for details). Together with the extra $U(1)$ and worldsheet fermions, these sigma-models are $N = 2$ backgrounds.

The mirror duality is nothing but the axial-vector duality of [6], which relates the model $R$ to the model $1/R$. In particular, duality relates the two boundaries of the $R$-modulus ($R \to 0, \infty$) where the conformal sigma-models correspond to $(SU(2)/U(1))_a \times U(1) \times U(1)_{\epsilon \to 0}$ and $(SU(2)/U(1))_v \times U(1) \times U(1)_{\epsilon \to 0}$. Here $U(1)_{\epsilon \to 0}$ denotes a compact, free scalar field at the limit when its compactification radius approaches 0, and $a$ ($v$) denotes the axially gauged (vectorially gauged) $SU(2)/U(1)$. Therefore, mirror symmetry relates the axial Abelian coset to the vector coset. These two (equivalent) descriptions of the parafermionic CFT are related by a $Z_k$ orbifolding [11, 12].

An alternative description of the models along the deformation line (5.4) is the sum of a parafermionic action and the action of a free scalar field with radius $\sqrt{k}R$, up to a $Z_k$ orbifolding which couples the two [13]. The orbifolding acts as a $Z_k$ twist of the parafermionic theory and a simultaneous translation of the free scalar by $2\pi(\sqrt{k}R)/k$. At the boundary $R \to \infty$ the twisted sectors decouple, because a non-zero winding of the scalar field has infinite energy. In the untwisted sector, every $Z_k$-eigenstate of the parafermion combines with a continuum of the free scalar states to form $Z_k$-invariant states. Therefore, at $R \to \infty$ one gets the direct product of an untwisted parafermion with a non-compact ($R \to \infty$) scalar and another scalar field. At the boundary $R \to 0$, since non-zero windings do not carry energy, the $Z_k$ twist acts purely in the parafermionic sector. Thus, at $R \to 0$ one gets the direct product of a $Z_k$-orbifold of a parafermion with an $R \to 0$ scalar and another scalar field. In this description, mirror symmetry acts as a $Z_k$ orbifold on the $N = 2$ minimal model, and as a factorized duality [1] on the two scalar fields.

The discussion above is even more interesting when $SU(2)$ is being replaced by $SL(2)$.

The $SL(2) \times U(1)$ model has an $N = 2$ structure, and mirror duality is a gauge symmetry as $m \in SL(2)$. The two boundaries (related to each other by mirror transformation) correspond to $(SL(2)/U(1))_a \times U(1) \times U(1)_{\epsilon \to 0}$ and $(SL(2)/U(1))_v \times U(1) \times U(1)_{\epsilon \to 0}$. The axial-vector duality in the $SL(2)/U(1)$ case relates backgrounds with different geometries, and even different topologies (the semi-infinite “cigar” and the infinite “trumpet”); this is the 2-d black-hole duality [14].

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4 The terms in $J^3 \bar{J}^3$ which depend on $\psi, \bar{\psi}$ must change the quadratic and quartic fermionic terms in the Lagrangian, in a way compatible with the worldsheet supersymmetry.

5 We define the CFT corresponding to $SL(2)$ to be the one regularized by its Euclidean continuation, see [6].
6 Mirror Duality as a Gauge Symmetry in the Moduli Space of \( N = 2 \) \( G \) Models

The discussion in the previous sections is not limited to the \( SU(2) \times U(1) \) (\( SL(2) \times U(1) \)) case, and can be extended to general groups, \( G \), that admit \( N = 2 \). In fact, the theorem of section 2 can be applied to any group with even rank, \( \text{rank } G = 2n \). To do this, one picks a complex structure on the Cartan subalgebra, that is, we split the generators of the Cartan sub-algebra into two complex-conjugate sets \( H^+, H^- \), such that \( \text{dim } H^+ = \text{dim } H^- = n \), and set

\[
T_+ = \{E_{\alpha^+}, H^+\}, \quad T_- = \{E_{\alpha^-}, H^-\}. \tag{6.1}
\]

Here \( E_{\alpha^+} \) (\( E_{\alpha^-} \)) is the set of generators corresponding to positive (negative) roots. It is obvious that \( \text{dim } T_+ = \text{dim } T_- (= \text{dim } G/2) \) and \( [T_+, T_+] \subset T_+, [T_-, T_-] \subset T_- \). Now, we define \( h_{ab} \) (and therefore, the \( N = 2 \) current \( J \)) in the basis \( \{T_+, T_-\} \) to be

\[
h = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}, \tag{6.2}
\]

namely,

\[
h(T_+) = iT_+, \quad h(T_-) = -iT_. \tag{6.3}
\]

A mirror transformation, \( m \), should take \( h \rightarrow -h \), and from (6.3) it follows that it should interchange \( T_+ \) with \( T_- \):

\[
m(T_+) = T_-, \quad m(T_-) = T_. \tag{6.4}
\]

Such a mirror symmetry can be realized as a symmetry of the \( N = 2 \) model iff \( m \) is a Weyl rotation:

\[
m \text{ is mirror and gauge symmetry } \iff m(h) = -h, \ m \in G_L, \tag{6.5}
\]

namely, when there is a Weyl rotation that takes \( T_+ \leftrightarrow T_- \). When (6.3) is satisfied, the \( N = 2 \) WZW model on \( G \) is \textit{self-mirror}. Moreover, when one allows for marginal deformations as above, the mirror transformation acts non-trivially on the resulting \( N = 2 \) moduli space. (If \( m \) is not a Weyl rotation, then this mirror transformation is not a symmetry of the given \( N = 2 \) structure of the WZW model but maps that structure to another one.)

Let us discuss \( N = 2 \) preserving superconformal deformations, \( WW \), of the \( N = 2 \) \( G \) model. By performing an Abelian duality (for a review, see \[1\]), one finds that a \( G \) WZW model is equivalent to \( [G/U(1)^r] \times U(1)^r \), \( r = \text{rank } G \) (up to an orbifolding by a finite discrete group) \[13, 12\]. Any deformation of the \( U(1)^r \) torus preserves \( N = 2 \). In the \( G \) WZW model, such conformal perturbations correspond to deforming the maximal torus, namely, to \( WW \) in the Cartan subalgebra \( H \). Therefore, any perturbation of the form \( \epsilon_{ij} H^i \overline{\theta} \), \( i, j = 1, ..., r \), \( H^i, \overline{\theta} \in H \), preserves \( N = 2 \).

Now, under mirror transformation, \( m \), \( H^i \rightarrow m(H^i) = m^i_k H^k \), and therefore,

\[
m : \epsilon_{ij} H^i \overline{\theta} \rightarrow (m^i \epsilon)_{ij} H^i \overline{\theta}. \tag{6.6}
\]

As a consequence, the sigma-model backgrounds, corresponding to the deformations \( \epsilon_{ij} \) and \( (m^i \epsilon)_{ij} \), are related by mirror duality, which is a gauge transformation if \( m \in G_L \).
7 The $SU(3)$ Example

Let us choose an orthogonal basis of the Cartan subalgebra

\[ H = \{H_1, H_2\}, \]  

and let $E_\alpha$ be the set of generators corresponding to the six $SU(3)$ roots

\[ \alpha = \{\alpha_+, \alpha_-\}, \quad \alpha_+ = \{(\sqrt{3}/2, 1/2), (0, 1), (-\sqrt{3}/2, 1/2)\}, \quad \alpha_- = -\alpha_+. \]  

(7.8)

Here $\alpha_+$ ($\alpha_-$) are the positive (negative) roots, and $H_i, E_\alpha$ obey

\[ [H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, \quad i, j = 1, 2. \]  

(7.9)

We now decompose the set of generators $T = \{H_i, E_\alpha\}$ into the direct sum $T = T_+ \oplus T_-$, where

\[ T_+ = \{H_+ = H_1 - iH_2, E_{\alpha_+}\}, \quad T_- = \{H_- = H_1 + iH_2, E_{\alpha_-}\}; \]  

(7.10)

these indeed obey the conditions of the theorem in section 2, with a complex structure $h$ given in eq. (6.3). From eq. (7.4) it follows that mirror transformation interchanges $\alpha_+ \leftrightarrow \alpha_-, \quad H_- \leftrightarrow H_+$. Is it a gauge transformation? The answer is yes, because the Weyl reflection that takes $H_2 \to -H_2$ (a reflection of the root $(0,1)$) does the job.

Therefore, mirror symmetry is a gauge symmetry in the $N = 2$ moduli space of the $SU(3)$ model (generated by adding deformations in the Cartan); its action on the moduli space is induced by the transformation $H_2 \to -H_2$, as described in the previous section.

This example can be generalized to the $N = 2$ moduli space of $A_{2n}$ models for all $n$; In these cases mirror symmetry is a Weyl rotation, and therefore, it is a gauge symmetry.

An example where mirror transformation is not a gauge symmetry is the $N = 2$, $SU(2) \times SU(2)$ model. In this case, in order to interchange the positive roots with the negative roots by a Weyl transformation, we need to reflect the Cartans of both $SU(2)$'s: $H_i \to -H_i, \quad i = 1, 2$. Such a transformation fails to interchange $H_+ \leftrightarrow H_-$, and therefore, it is not a mirror transformation.

8 Conjectures: Mirror Duality as a Gauge Symmetry for Calabi-Yau Compactifications

We conclude with some speculations (which are the main motivation for this work). Although we have discussed mirror symmetry as a gauge symmetry in the moduli space of $N = 2$ backgrounds on a group $G$, we speculate that this can be generalized to other examples (such as $N = 2$ cosets $G/\{H \times U(1)\}$). Mirror symmetry is particularly rich in the space of Calabi-Yau (CY) compactifications $G/(H \times U(1)) \times U(1)^2$ at the boundaries, and duality along the deformation line is a mirror transformation.

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6 For Kazama-Suzuki models, $G/(H \times U(1))$, we can deform $(G/H) \times U(1)$ to $G/(H \times U(1)) \times U(1)^2$ at the boundaries, and duality along the deformation line is a mirror transformation.
At this stage, it is not known how to connect mirror pairs of CY backgrounds by marginal deformations of their corresponding $c = 9$ CFTs. This situation is similar to the mirror pair of 2-d ‘black hole’ backgrounds (the ‘cigar’, $SL(2)/U(1)_a$, and the ‘trumpet’, $SL(2)/U(1)_v$); they cannot be connected by a marginal deformation of the $SL(2)/U(1)$ CFT. But in the latter case we understand how to relate them by a mirror duality which is a gauge symmetry: we look at the moduli space of 4-d, $N = 2$ backgrounds connected to $SL(2) \times U(1)$. We then identify mirror symmetry as a gauge symmetry in that moduli space and, in particular, at the boundary it relates the cigar to the trumpet (times free scalars).

The discussion above suggests that a mirror pair of CY backgrounds (times a non-compact space) could appear at the boundary of the moduli space of $d > 6$, $N = 2$ backgrounds. Moreover, it might be possible that there is a self-mirror point (with enhanced symmetry $G$) in the moduli space, and that mirror symmetry is a gauge transformation in $G$.

Let us give some hints that this indeed could be true, at least for particular CY backgrounds. Suppose we start with the $N = 2 \, SU(2)_{k_1}/U(1) \times \prod_{i=2}^5 SU(2)_{k_i}$ model, such that the total central charge is critical, $c = 15$, and make a “GSO projection” by twisting with $\exp(2\pi i J_0)$, where $J$ is the $N = 2 \, U(1)$ current. Now, we deform this model with the four current-current operators at the Cartan sub-algebra of the four $SU(2)$’s, simultaneously. At the boundaries of the deformation line, one gets the product of $\prod_{i=1}^5 SU(2)_{k_i}/U(1)$ (with central charge $c = 9$) with non-compact $U(1)^4$ (with central charge $c = 6$). At one boundary it is twisted only by the GSO projection, and the coset CFT $\prod_{i=1}^5 SU(2)_{k_i}/U(1)$ is related to the CFT sigma-model on a CY manifold in $\mathbb{CP}^4$ [16]. At the other boundary it can be viewed as being twisted by the product of $Z_{k_i}$’s (in the same way it works for a single minimal model); combined with the GSO projection it gives rise to the mirror manifold (when acting on the CY sigma-model corresponding to the product of minimal models).

It should be mentioned that although the duality is not a mirror transformation along the deformation line, it is a mirror transformation acting on the $c = 9$ CY background at the boundary (without acting on the decoupled 4-D non-compact flat space). In the sigma-model description in terms of manifolds admitting $N = 2$, it is therefore suggested that mirror symmetry for CY backgrounds of that type is indeed a gauge symmetry.

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