1. Introduction

Developing single-photon sources is an important task in optical quantum technologies [1–3]. In particular, CMOS-compatible on-chip devices are especially in demand for creating scalable and compact quantum photonic circuits [3–5]. In this respect, heralded single-photon sources based on spontaneous four-wave mixing (SFWM) in microring resonators are of great interest since, within the framework of integrated optics, they allow one to achieve high efficiency of the nonlinear process [6–8], obtain generated photons with a narrow spectral width [9], and to approach their deterministic emission using multiplexing techniques [10, 11]. The latter is expected to be quite efficient when using photon number resolving detectors [12]. In addition, the sources can be designed to produce pure single-photon states (transform-limited single-photon wave packets) [13, 14], which are crucially important for observing quantum interference effects and implementing optical quantum computing [15]. It is also worth noting that cryogenic temperatures in this case are not required, in contrast to single-photon sources based on single quantum emitters such as quantum dots or color centers.

In the present paper, we develop a scheme for generating pure single-photon states via SFWM in a system of coupled ring microresonators. In the SFWM process, two pump laser photons are converted into a pair of daughter photons, usually...
called the signal and idler, in a third-order nonlinear optical material. The photon number correlation between the resulting fields can be exploited to herald the existence of one photon by the detection of its partner, which underlies the conditional preparation of single-photon states. Energy conservation requires the signal and idler photons to be generated at frequencies that are symmetrically distributed around the pump frequency. In the general case, due to such a spectral correlation, the heralded photons prove to be in a mixed state. The high purity of the emitted photons is achieved when the joint spectral amplitude (JSA) of the biphoton state is a factorable function in the frequency domain [16], which is possible for a sufficiently broadband pump field. Similar to [14], for the latter to be used we take advantage of a smaller pump quality factor, which makes the linewidth of the microresonator for the pump broader than those for the signal and idler fields. However, instead of using two coupling points via Mach–Zehnder interferometers, we suggest using additional microrings. An important advantage of our scheme is that the microrings can be fabricated in special sizes and tuned in resonance with only three interacting modes, thereby making additional spectral filtering unnecessary. In addition, the present scheme may be easily realized not only with microring resonators but also with other types of resonators such as microspheres and microtoroids.

2. Basic model

In the present paper, we consider a system of three microring resonators coupled to a central one and connected with strait waveguides (buses) (figure 1). The SFWM process occurs in the central ring, while other rings are used for loading the pump field and unloading the generated photons. For simplicity, we consider a degenerate pump scheme. It is assumed that the pump field corresponds to a resonator mode in the zero-dispersion region of the central ring microresonator so that the signal and idler photons can be emitted into the adjacent modes that are separated from the pump mode by equal frequency intervals (the group velocity dispersion is negligible). One of the outer rings is tuned in such a way that one of its resonances coincides with the pump mode of the central ring, whereas other resonances do not coincide with the signal and idler modes. In contrast, other outer rings should be out of resonance with the pump mode but in resonance with the signal and idler modes. When the free spectral range of the outer rings is two times smaller than that of the central ring, the system is similar to that of [14]. However, it is possible to make these rings in other sizes so that only three modes of the central ring prove to be effectively coupled to the strait waveguides.

The Hamiltonian for the system is

\[
\mathcal{H} = \mathcal{H}_{\text{sys}} + \mathcal{H}_{\text{bath}} + \mathcal{H}_{\text{internal}} + \mathcal{H}_{\text{bath}}^\text{int},
\]

where

\[
\mathcal{H}_{\text{sys}} = \hbar \omega_0 x^\dagger x + \sum_{n=m} \hbar \omega_n y^\dagger y_n + \sum_{m=0,l,s} \hbar \omega_{0,m} z^\dagger_m z_m
\]

is the free-field Hamiltonian for the cavities,

\[
\mathcal{H}_{\text{bath}} = \int d\omega \hbar \omega \left[ a_p^\dagger(\omega) a_p(\omega) + \sum_{m=l,s} b_m^\dagger(\omega) b_m(\omega) \right]
\]

is the external bath Hamiltonian,

\[
\mathcal{H}_{\text{internal}} = i \hbar g_{p,y} x^\dagger y + i \hbar g_{s,i} z^\dagger z + h.c.
\]

is the coupling between the rings, and

\[
\mathcal{H}_{\text{bath}}^\text{int} = \frac{i \hbar}{\sqrt{2\pi}} \int d\omega \left[ \sqrt{\kappa_p} x^\dagger x(\omega) + \sqrt{\kappa_s} z^\dagger z(\omega) \right] + h.c.
\]

is the coupling to the external modes. Here \( m = \{i, s\}, n = \{i, s, p\} \), \( \omega_{0,p}, \omega_{0,s}, \omega_{0,m} \) are the central frequencies of the microrings, \( \kappa_{p,s} \) are the coupling parameters between the waveguide and the rings, \( g_{p,s} \) are coupling parameters between the rings, and \( x_p, y_s \) are the annihilation operators for the photons corresponding to the different modes in the rings. The nonzero commutation relations read: \( [x_p, x_s^\dagger] = [y_s, z_p^\dagger] = [z_m, z_s^\dagger] = 1 \) and \( [p_p(\omega), a_i^\dagger(\omega')] = [b_i(\omega), a_i^\dagger(\omega')] = \delta(\omega - \omega') \).

By applying the input–output formalism [17], from equation (1) we obtain the following Heisenberg–Langevin equations:

\[
\begin{align*}
\frac{\partial}{\partial t} + i \omega_{0,p} + \frac{\kappa_p}{2} & \right) x_p - g_p y_p = \sqrt{\kappa_p} a_{\text{inp}}, \\
\frac{\partial}{\partial t} + i \omega_{0,s} + \frac{\kappa_s}{2} & \right) y_s - g_s z_s = 0, \\
\frac{\partial}{\partial t} + i \omega_{0,m} + \frac{\kappa_m}{2} & \right) z_m - g_s y_s = \sqrt{\kappa_m} b_{\text{out}}, \\
\frac{\partial}{\partial t} - i \omega_{0,m} - \frac{\kappa_m}{2} & \right) z_m - g_s y_s = 0,
\end{align*}
\]

where we used the Markov approximation \( \kappa_{p,s}(\omega) \approx \kappa_p = \text{const}, \)

\( \kappa_{m}(\omega) \approx \kappa_m = \text{const} \), \( g_{p,s}(\omega) \approx g_p = \text{const} \) and \( g_{m}(\omega) \approx g_m = \text{const} \).

In what follows, we also assume that the same modes in the different rings are matched with each other: \( \omega_{0,p} = \omega_{0,s} = \omega_{0,m} = \omega_{0} \), which is a natural condition for an efficient energy transfer.

3. Input–output relations

By taking the Fourier transform of (2), we obtain the system of algebraic equations

\[
\begin{align*}
\left[ i \Delta_p + \frac{\kappa_p}{2} \right] x_p(\omega) - g_p y_p(\omega) = \sqrt{\kappa_p} a_{\text{inp}}(\omega), \\
\left[ i \Delta_p y_p(\omega) + g_p x_p(\omega) = 0, \\
\left[ i \Delta_m y_m(\omega) + g_s z_m(\omega) = 0, \\
\right. \\
\left[ i \Delta_m + \frac{\kappa_m}{2} \right] z_m(\omega) - g_s y_m(\omega) = \sqrt{\kappa_m} b_{\text{out}}, \\
\left. a_{\text{inp}}(\omega) - a_{\text{out}}(\omega) = \sqrt{\kappa_p} x_p(\omega), \\
\right. \\
\left. b_{\text{out}}(\omega) - b_{\text{out}}(\omega) = \sqrt{\kappa_m} z_m(\omega),
\right]
\end{align*}
\]
where we introduce $\Delta_n = \omega_{in} - \omega$, and for all the annihilation operators the Fourier transform is defined as $u(t) = \frac{1}{\sqrt{2\pi}} \int dw \, e^{-i\omega t} u(\omega)$.

Let us consider the case when $a_{in,p} = 1$ and $b_{in,m} = 0$, which corresponds to the loading of the pump field. Then we obtain the input–output relations for the pump field operators

$$y_p(\omega) = M_p a_{in,p}(\omega) = \frac{2g_p \sqrt{\kappa_p} a_{in,p}(\omega)}{-2\Delta_p^2 + 2g_p^2 + i\Delta_p \kappa_p}. \quad (4)$$

Similarly, in the case when $a_{in,p} = 0$ and $b_{in,m} = 1$, which corresponds to the loading of the signal and idler fields (and unloading them for the reversed time), we get

$$y_m(\omega) = M_m b_{in,m}(\omega) = \frac{2g_{is} \sqrt{\kappa_{is}} b_{in,m}(\omega)}{-2\Delta_m^2 + 2g_{is}^2 + i\Delta_m \kappa_{is}}. \quad (5)$$

To express the cavity field operators $y_i$ in terms of the output fields $a_{out,p}(\omega)$ and $b_{out,m}(\omega)$, $M_n$ in equations (4) and (5) should be replaced by $M_n^*$.

4. Optimal coupling

To suppress the phase dispersion in the central microring, which is necessary for the effective loading of the pump field at a frequency $\omega_{in}$ into it and unloading the generated photons at the frequencies $\omega_s$, $\omega_i$ from it through the outer microrings, we apply the following condition of the closeness of the frequency dependence of the phase to the linear one:

$$\partial^l_{\omega} \text{Argument}(M_n) \bigg|_{\omega = \omega_{in}} = 0, \quad l = 2, 3, \ldots \quad (6)$$

In our system, we can impose the condition for $l = 3$, which leads to

$$\partial^3_{\omega} \text{Arctan} \left[ \frac{\Delta_p \kappa_p}{2\Delta_p^2 - 2g_p^2} \right] \bigg|_{\omega = \omega_{in}} = 0, \quad (7)$$

and obtain the following optimal ratios between the coupling parameters

$$g_{p,\text{opt}} = \kappa_p/\sqrt{12}, \quad g_{is,\text{opt}} = \kappa_{is}/\sqrt{12}. \quad (8)$$

The remaining conditions for $l > 3$ lead to $\kappa_{p,\text{is}} \to \infty$. Physically, this means that the maximum possible experimental values of $\kappa_{p,\text{is}}$ should be used.

To conveniently visualize the dispersion effects, we introduce the delay function $T_p(\omega) = \text{Argument}(M_p)/(\omega - \omega_{in})$, which shows the difference in the time delay of signals at different frequencies near the central frequency $\omega_{in}$ (the case of unloading the signal and idler fields can be described similarly). Figure 2 demonstrates the difference between the three cases, $g_p = \{0.9 g_{p,\text{opt}}, g_{p,\text{opt}}, 1.1 g_{p,\text{opt}}\}$. It can be seen that the maximum size of the plateau, corresponding to the maximum suppression of negative dispersion effects for the pump field in the central ring, is attained for the ratio (8). Such dispersion suppression is necessary to improve the quality of the heralded photons to the greatest extent possible.
y(α - iω) is the frequency detuning. Now, taking into account the input–output relations. This approach was used for the analysis of the cavity-assisted SFWM in [18, 25, 26].

The SFWM process in the central resonator is described by the effective Hamiltonian

\[ \mathcal{H}_{\text{SFWM}}(t) = \zeta y_p(t)y_p(t)y_s(t), \]

where \( \zeta \) is the effective nonlinearity that takes into account \( \chi^{(3)} \) of the nonlinear material, the microresonator mode functions and other parameters, which are not important for the present analysis.

By applying the first-order perturbation theory, the state vector of the generated biphoton field is calculated as

\[ |\psi\rangle = |0\rangle|\alpha\rangle - \frac{i\zeta}{\hbar(2\pi)^2} \int dt dt' dt'' dt''' e^{i\omega t'} \left( y_p(t')y_p(t'')y_s(t''') y_s(t') + y_p(t')y_p(t'') y_s(t'') y_p(t') + y_p(t) y_p(t') y_s(t') y_s(t') \right) / |\alpha\rangle, \]

where \( |0\rangle = 0 |0\rangle \) is the vacuum state of the signal and idler fields, \( |\alpha\rangle \) is the coherent state of the pump field with a complex amplitude \( \alpha \) (i.e., \( y_p(t)|\alpha\rangle = \alpha(t)|\alpha\rangle \)), and

\[ \Delta \omega = 2\omega_p - \omega_s - \omega_i \]

is the frequency detuning. Now, taking into account the input–output relations (4) and (5) we obtain

\[ |\psi\rangle = |0\rangle|\alpha\rangle - \frac{i\zeta}{\hbar(2\pi)^2} \int \omega d\omega d\omega d\omega' e^{i\Delta \omega t} |\alpha\rangle, \]

where

\[ F(\omega_s, \omega_i) = \mathcal{I}_p(\omega_s, \omega_s)\mathcal{F}(\omega_s, \omega_i) \]

is the convolution of the spectral amplitude of the pump field \( \alpha(\omega_p) \) in the resonator.

To illustrate the spectral correlations between the emitted photons, it is convenient to use the joint spectral intensity (JSI) \( \mathcal{P}(\omega_s, \omega_i) = |\mathcal{F}(\omega_s, \omega_i)|^2 \). In addition, for quantitative analysis we can take advantage of the Schmidt decomposition of the JSI [27, 28], which can be written as

\[ F(\omega_s, \omega_i) = \sum_n \sqrt{\lambda_n} \psi_n(\omega_s) \phi_n(\omega_i), \]

where the Schmidt coefficients satisfy the condition

\[ \sum_n \lambda_n = 1. \]

Then the Schmidt number \( K = 1/\sum \lambda_n^2 \) is usually used as a measure of entanglement in the photon pairs [29, 30], while the purity of the heralded single-photon state is equal to \( \gamma = 1/K. \) A two-photon state for which \( K = 1 \) (the minimum value) represents a factorable state, which exhibits no spectral entanglement and gives rise to pure heralded
single photons. Figure 3 illustrates the JSI distributions calculated numerically for the equal resonator linewidths $\kappa_p = \kappa_{ii}$ and for the broader pump linewidth $\kappa_p \gg \kappa_{ii}$. In both cases, the optimal ratio between the coupling parameters (8) is maintained, and the pump pulse is assumed to be Gaussian, $\alpha(\omega) = (2\pi\sigma)^{-1/4}\exp(-\omega - \omega_p)^2/4\sigma)$, with a spectral width, $\Delta\omega_{1/2} = \sqrt{8\pi}\ln{2}$, optimized for providing the minimum value of the Schmidt number. Similar to [14], the calculations show the near perfect separability of the biphoton minimum value of the Schmidt number. Similar to [14], the calculations show the near perfect separability of the biphoton field in the case of the broad pump linewidth. However, we managed to obtain even smaller Schmidt numbers by optimizing the coupling parameters and spectral width of the pump pulse. In particular, for the ratio of $\kappa_p/\kappa_{ii} = 6.6$ and the optimal spectral width of $\Delta\omega_{1/2} = 0.45 \kappa_p$, we have $K = 1.0003$. A further increase in $\kappa_p/\kappa_{ii}$ to 10 provides $K = 1.00006$, which corresponds to the purity of the heralded photons of $\gamma = 0.9999$.

6. Conclusion

We have shown that a system of optimally coupled ring microresonators is capable of producing almost factorable joint spectral amplitude of the biphoton field, thereby generating near pure heralded single-photon states via spontaneous four-wave mixing. By optimizing the coupling parameters of the system, we present a way of suppressing negative dispersion effects, which, in combination with the optimal spectral width of the pump pulse, provides the highest possible purity of the heralded photons generated in such a scheme. The use of resonant coupling via microrings makes it possible to load and unload only the required field modes, which may simplify the implementation of integrated sources of indistinguishable single photons.

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