SYMPLECTIC SURGERY AND THE SPIN$^c$-DIRAC OPERATOR

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Abstract. Let $G$ be a compact connected Lie group, and $(M, \omega)$ a compact Hamiltonian $G$-space, with moment map $J : M \to \mathfrak{g}^*$. Under the assumption that these data are pre-quantizable, one can construct an associated Spin$^c$-Dirac operator $\hat{\theta}_C$, whose equivariant index yields a virtual representation of $G$. We prove a conjecture of Guillemin and Sternberg that if $0$ is a regular value of $J$, the multiplicity $N(0)$ of the trivial representation in the index space $\text{ind}(\hat{\theta}_C)$, is equal to the index of the Spin$^c$-Dirac operator for the symplectic quotient $M_0 = J^{-1}(0)/G$. This generalizes previous results for the case that $G = T$ is abelian, i.e. a torus.

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1. Introduction

Let $(M, \omega)$ be a compact symplectic manifold, with integral symplectic form $[\omega] \in H^2(M, \mathbb{Z})$, and $L \to M$ a Hermitian line bundle whose first Chern class is $c_1(L) = [\omega]$. It is then possible to choose a Hermitian connection $\nabla$ on $L$ with curvature $F(L) = 2\pi i \omega$. Choose an $\omega$-compatible, positive almost complex structure on $M$, and let

$$\Lambda^* T^* M \otimes \mathbb{C} = \bigoplus_{i,j} \Lambda^{i,j} T^* M$$

be the associated bigrading of the bundle of forms. Using a Hermitian connection on the canonical bundle, one has the twisted Spin$^c$-Dirac operator

$$\hat{\theta}_C : \mathcal{A}^{0,\text{even}}(M, L) \to \mathcal{A}^{0,\text{odd}}(M, L)$$
where

\[ \mathcal{A}^{i,j}(M, L) = C^\infty(M, \Lambda^{i,j} T^*M \otimes L) \]

is the space of \(L\)-valued forms of type \((i, j)\). The Riemann-Roch number of \(L \to M\) is defined to be the dimension of the virtual vector space

\[ \text{ind}(\partial_C) = [\ker(\partial_C)] - [\ker(\partial_C^*)]. \]

By the index theorem of Atiyah-Singer,

\[ \text{RR}(M, L) = \int_M \text{Td}(M) \text{Ch}(L), \]

where \(\text{Td}(M)\) and \(\text{Ch}(L)\) are the Todd class of \(M\) and Chern character of \(L\), respectively. Since \(\text{Ch}(L) = e^\omega\), and since any two compatible almost complex structures are homotopic, \(\text{RR}(M, L)\) is a symplectic invariant of \(M\).

If \(M\) is in fact Kähler, \(L\) a holomorphic Hermitian line bundle, and \(\nabla\) its canonical connection, then \(\partial_C\) is the Dirac operator \(\sqrt{2(\bar{\partial} + \partial^*)}\) associated to the twisted Dolbeault complex. If \(L\) is moreover sufficiently positive, \(\text{RR}(M, L)\) is simply the dimension of the space of holomorphic sections. This happens for instance for coadjoint orbits \(O = G.\lambda\) of compact Lie groups \(G\), where \(\lambda\) is a dominant weight. Here \(\omega\) is the canonical invariant Kähler form on \(O\), and \(L\) the pre-quantum line bundle. By the Borel-Weil-Bott theorem, the space of holomorphic sections of \(L\) is just the irreducible representation space with highest weight \(\lambda\).

Suppose now that a compact Lie group \(G\) acts on \(M\) in a Hamiltonian fashion, with equivariant moment map \(J : M \to \mathfrak{g}^*\). That is,

\[ \iota(\xi_M)\omega = d\langle J, \xi \rangle \]

for all \(\xi \in \mathfrak{g}\) where \(\xi_M\) is the fundamental vector field. Assume also that the action lifts to an action on \(L\), in such a way that the fundamental vector fields on \(m\) and \(L\) are related by Kostant’s formula

\[ \xi_L = \text{Lift}(\xi_M) + 2\pi\langle J, \xi \rangle \frac{\partial}{\partial \phi}, \]

where “Lift” is the horizontal lift with respect to the connection and \(\frac{\partial}{\partial \phi}\) the generating vector field for the scalar \(S^1\)-action on \(L\). By making the above choices \(G\)-invariant, we obtain \(G\)-representations on \(\ker(\partial_C)\) and \(\ker(\partial_C^*)\), hence a virtual representation on \(\text{ind}(\partial_C)\). The character \(\text{RR}(M, L) = \chi \in R(G)\) of this representation is called the equivariant Riemann-Roch number, and the equivariant index theorem of Atiyah-Segal-Singer expresses \(\chi(g)\) as an integral of characteristic classes over the fixed point manifold \(M^g\). Consider the decomposition of \(\chi\) into irreducible characters \(\rho_\mu\) for \(G\),

\[ \chi = \sum_{\mu \in \hat{G}} N(\mu) \rho_\mu, \quad N(\mu) \in \mathbb{Z}. \]

Let \(T\) be the maximal torus of \(G\), \(\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{g}]\) the corresponding decomposition of the Lie algebra, and \(\Lambda \subset \mathfrak{t}\) the integral lattice. Let \(\mathfrak{t}_+^* \subset \mathfrak{g}^*\) be some choice of a positive Weyl chamber, \(\mathcal{R}_+\) the associated system of positive roots, and label the irreducible representations by the set of dominant weights \(\Lambda_+^* = \Lambda \cap \mathfrak{t}_+^*\). If 0 is a regular value of \(J\), the action of \(G\) on \(J^{-1}(0)\) is locally free, and therefore the symplectically reduced space \(M_{\text{red}} = J^{-1}(0)/G\) is a symplectic orbifold. Moreover, \(L_{\text{red}} := (L|J^{-1}(0))/G\) is a pre-quantum orbifold-line bundle. One has an associated
Spin$^c$-Dirac operator, with index $\text{RR}(M_{\text{red}}, L_{\text{red}})$ given by the orbifold-index theorem of Kawasaki [24]. This paper will be concerned with the proof of a conjecture of Guillemin and Sternberg [14], that the multiplicity $N(0) =: \text{RR}(M, L)^G$ of the trivial representation is exactly the Riemann-Roch number $\text{RR}(M_{\text{red}}, L_{\text{red}})$.

Since orbifolds will play an essential role in this paper, we will allow that $M$ itself is an orbifold, and prove the following:

**Theorem 1.1.** Let $G$ be a compact connected Lie group, and $(M, \omega)$ a quantizable Hamiltonian $G$-orbifold, with moment map $J : M \to \mathfrak{g}^*$. If $0$ is a regular value of $J$, the multiplicity of the trivial representation is given by

$$\text{RR}(M, L)^G = \text{RR}(M_{\text{red}}, L_{\text{red}}).$$

If a compact, connected Lie group $H$ acts on $L \to M$, such that the action commutes with the action of $G$, this equality holds as an equality of virtual characters for $H$.

By the “shifting-trick”, one has pre-quantum orbifold-bundles $L_\mu$ for all reduced spaces $M_\mu = J^{-1}(\mu)/G_\mu$, whenever $\mu \in \Lambda^*_+ \cap \mathfrak{t}^*$ is a regular value of $J$, and obtains the following

**Corollary 1.2.** If $\mu \in \Lambda^*_+$ is a regular value of $J$, $N(\mu) = \text{RR}(M_\mu, L_\mu)$. In particular, the support of the multiplicity function is contained in the moment polytope $\Delta = J(M) \cap \mathfrak{t}^*$.

For the Kähler case, a variant of Theorem 1.1 was proved by Guillemin and Sternberg in their 1982 paper [16]. For the case that $G$ is abelian, but $M$ not necessarily Kähler, Theorem 1.1 was proved by Guillemin [13] under some additional assumptions and by Vergne [33] and Meinrenken [29] in general. For $G$ nonabelian, it was shown in [29] that Theorem 1.1 is true if one replaces $L$ by a suitable high tensor power, $L^m$. This was deduced from a simple stationary phase argument, for which it is in fact irrelevant that $L$ is a pre-quantum line bundle or that $M$ is symplectic. For $G = SU(2)$ and under an additional condition on the moment map, Theorem 1.1 was proved in Jeffrey-Kirwan [21]. We will show in this paper that neither this extra condition nor taking tensor powers is necessary, and that Theorem 1.1 holds for any compact group $G$, without additional hypotheses besides $0$ being a regular value.

Our approach is based on the symplectic cutting technique of E. Lerman [26], which was already used in [10] to simplify the proof for the abelian case. The first step will be to cut $M$ into smaller pieces, and to prove a gluing formula which expresses the Riemann-Roch number of $M$ in terms of the Riemann-Roch number of the cut spaces. (Indeed, these gluing formulas are valid for arbitrary Hermitian vector bundles, not only pre-quantum line bundles.) By combining this result with a “cross section theorem”, we prove the excision property of $N(0)$, i.e. that $N(0)$ depends only on local data near $J^{-1}(0)$. This allows us to replace $M$ by a new quantizable space $M_{\text{cut}}$, such that $M_{\text{cut}}$ is isomorphic to $M$ near $J^{-1}(0)$, and such that the moment polytope of $M_{\text{cut}}$ is contained in a small neighborhood of zero. It can be arranged that $M_{\text{cut}}$ has has very few $T$-fixed point manifolds. By explicitly writing down the fixed point formula for $\text{RR}(M_{\text{cut}}, L_{\text{cut}})$, we can compute the multiplicity of $0$ for $M_{\text{cut}}$, and find that it is equal to $\text{RR}(M_{\text{red}}, L_{\text{red}})$.

For the group $G = SU(2)$, an alternative and much easier proof of Theorem 1.1 is available. We sketch this proof in the appendix, which is independent of the rest of the paper.
If 0 is not a regular value of the moment map, the reduced space $M_0 = \Phi^{-1}(0)/G$ is in general not an orbifold, but has more serious singularities. The extension of Theorem 1.1 to this case will be discussed in a separate paper with Reyer Sjamaar.

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2. Orbifolds

In this section, we review some background material on orbifolds, and describe the orbifold version of the Berline-Vergne localization formula. Let us briefly recall some basic definitions (for details, see Satake [31] or Kawasaki [23, 24]).

Let $M$ be a paracompact Hausdorff topological space. An orbifold chart for $M$ is a triple $(U, V, H)$ consisting of an open subset $U$ of $M$, a finite group $H$, an open subset $V$ of some manifold, and a homeomorphism $U = V/H$. An orbifold structure on $M$ is a collection of orbifold charts $\{(U_i, V_i, H_i)\}$, such that the $U_i$ cover $M$ and are subject to appropriate compatibility conditions. In particular, one assumes that for any two orbifold charts $(U_i, V_i, H_i)$ and $x \in U_1 \cap U_2$, there exists an open neighborhood $U \subset U_1 \cap U_2$ of $x$ and an orbifold chart $(U, V, H)$, together with injections $\rho_i : H \to H_i$ and $\rho_i$-equivariant embeddings $\phi_i : V \to V_i$, with the following property:

$$h_i, \phi_i(V) \cap \phi_i(V) \neq \emptyset \Rightarrow h_i \in \rho_i(H).$$

Our definition differs slightly from [23], in that we do not assume that the actions of the $H_i$ are effective. The compatibility conditions ensure that on each connected component of $M$, the generic stabilizers $H_{i,*}$ for the $H_i$-actions are isomorphic. Their order therefore defines a locally constant function on $M$, which is called the multiplicity function $d_M : M \to \mathbb{N}$.

For all $x \in M$, there is a compatible orbifold chart $(U_x, V_x, H_x)$ such that the preimage of $x$ in $V_x$ is a fixed point. The corresponding $H_x$ is uniquely determined up to isomorphism, and is called the isotropy group of $x$. Given any finite group $H$, the connected components of the set of all $x$ such that $H_x \sim H$ are smooth manifolds. In this way, one obtains a decomposition $M = \cup M_i$, such that $M_i$ is a connected manifold and consists of points of a fixed isotropy group, $H_i$. Moreover, for all $i \neq j$, $M_i$ meets the closure of $M_j$ only if $H_j$ is an (abstract) subgroup of $H_i$.

On each connected component of $M$, there is unique open, dense, connected stratum (called principal stratum) $M_*$ on which the corresponding isotropy group $H_*$ is minimal, i.e. $\#H_* = d_M$.

If $G$ is a compact Lie group and $G \times P \to P$ a locally free action on a manifold $P$, the orbit space $M = P/G$ is an orbifold\(^1\). Indeed, by the slice theorem a neighborhood of any orbit $G.p = x$ is equivariantly diffeomorphic to a neighborhood of the zero section of the associated bundle $G \times_{G_p} N_p$, where $N_p \subset T^*_p P$ is the conormal space to $G.p$. Therefore, for some open neighborhood $V_p$ of $0 \in N_p$,

\(^1\)In fact, any orbifold is of this form; for example one can take $P$ to be the $O(n)$ frame bundle with respect to some Riemannian metric on $M$, and $H = O(n)$. 


one has a homeomorphism $V_p/G_p \cong U_p$ onto some neighborhood of $x$ in $M$. The multiplicity of $M$ is simply the rank of a generic stabilizer for the $G$-action.

Given any orbifold $M$, orbifold fiber bundles $\pi : E \to M$ are defined by $H$-equivariant fiber bundles $Z \to E_V \to V$ in orbifold charts $(U, H, V)$, together with suitable compatibility conditions. Notice that the fibers $\pi^{-1}(x)$ are in general not diffeomorphic to $Z$, but only to some quotient of $Z$ by the action of the isotropy group $H_x$. For example, the tangent bundle of $M$ is an orbifold vector bundle, with fiber at $x \in M$ equal to $T_x V_x/H_x$, where $H_x$ is the isotropy group and $(U_x, V_x, H_x)$ an orbifold chart around $x$. Sections of an orbifold vector bundle $E$ are defined by $H$-invariant sections in orbifold charts. It is not true that any orbifold mapping $\sigma : M \to E$ with $\pi \circ \sigma = \text{id}_M$ gives rise to a section of $E$. For example, $\mathbb{C}/\mathbb{Z}_2 \to \text{pt}$ does not have any non-vanishing sections.

Continuing in this fashion, almost all constructions for manifolds can be generalized to orbifolds, by taking the corresponding $H$-equivariant version in local orbifold charts. One thus defines suborbifolds, orbifold-principal bundles, de Rham theory, characteristic classes, Riemannian (complex, symplectic, Spin) structures, and so on. An orbifold principal bundle $H \to P \to M$ over $M$ is an orbifold $P$, together with a locally free action of some Lie group $H$, such that $M = P/H$. If $X$ is any $H$-space, one can form an associated bundle $\pi \bigoplus H X = (P \times X)/H$. If only some finite cover $\tilde{H} \to H$ acts on $X$, one can still form the associated bundle, by merely regarding $P$ as a $\tilde{H}$-orbifold principal bundle. Notice that if $H$ is connected, this does not depend on the choice of the cover.

Suppose a compact, connected Lie group $G$ acts on $M$, in other words, $G \times M \to M$ is an orbifold mapping. Since $G$ is connected, the components $F$ of the fixed point set $M^G$ are suborbifolds of $M$. If $G$ is abelian the normal bundle $\nu_P$ is even dimensional, and admits an invariant Hermitian structure (see e.g. [6], p. 217). If $M$ is oriented, any choice of such a Hermitian structure equips $F$ with an orientation.

Consider an orbifold chart $(U_x, H_x, V_x)$ around a point $x \in F$. It is not always true that the $G$-action on $U_x$ lifts to $V_x$, but some finite covering $\tilde{G} \to G$ does. In particular, the weights for the action on $\nu_F(x)$ are weights for $\tilde{G}$, but not necessarily for $G$. In [27], these are called “orbi-weights” of $G$.

Let $\mathcal{A}_G(M)$ be the space of equivariant differential forms, i.e. polynomial $G$-equivariant mappings $\alpha : \mathfrak{g} \to \mathcal{A}(M)$. Let $\partial_\mathfrak{g}$ be the equivariant differential

$$d_\mathfrak{g} : \mathcal{A}_G(M) \to \mathcal{A}_G(M), \quad d_\mathfrak{g}(\alpha)(\xi) = d\alpha(\xi) - 2\pi i \alpha(\xi_M)\alpha(\xi),$$

where $\xi_M$ is the fundamental vector field corresponding to $\xi$. Since $\partial_\mathfrak{g}^2 = 0$, $(\mathcal{A}_G(M), d_\mathfrak{g})$ is a complex, and its cohomology $H^*_G(M)$ is called the equivariant cohomology of $M$. More generally, one can consider the complex $\mathcal{A}_G^\ast(M)$ of equivariant differential forms $\alpha$ which are defined and analytic around $0 \in \mathfrak{g}$, and define its cohomology $H^*_G(M)$.

An example for an equivariantly closed form is the equivariant Euler form of the normal bundle $\nu_F$. Suppose $G = T$ is abelian. If $\nu_F$ splits into a direct sum of invariant orbifold-line bundles $L_j$, with Chern classes $c_j$ and orbi-weights $\alpha_j$, one defines

$$\text{Eul}_i(\nu_F, \xi) = \prod_j (c_j + 2\pi i \langle \alpha_j, \xi \rangle).$$

In the general case, this equation defines $\text{Eul}_i(\nu_F, \cdot)$ by means of the splitting principle.
Suppose now that $M$ is compact, connected, and oriented, and consider the integration mapping $\int : \mathcal{A}_c^\omega(G(M)) \to \mathcal{A}_c^\omega(pt)$. Since $\int \circ d_g = 0$, this descends to a mapping on $H_c^\omega(G(M))$. Let $\iota_F : F \to M$ be the embedding of the fixed point orbifolds.

**Theorem 2.1.** (Orbifold version of the Berline-Vergne Localization Formula.) Suppose $G = T$ is abelian, and let $\alpha \in \mathcal{A}_c^\omega(M)$ be $d_t$-closed. Then

$$\frac{1}{d_M} \int_M \alpha(\xi) = \sum_F \frac{1}{d_F} \int_F \iota_F^* \alpha(\xi) \text{Eul}(\nu_F, \xi)$$

for all $\xi$ in sufficiently small neighborhood of $0 \in \frak{t}$. Here, the sum is over the fixed point orbifolds, and $d_F$ is the multiplicity of $F$ as a suborbifold of $M$.

The proof of this result proceeds exactly as in the manifold case, see e.g. [7]. The multiplicities occur because the proof involves an integration over the fibers of $\nu_F$, to be performed in local orbifold charts. In passing to the quotient, one has to divide by the “twist” of $\nu_F$, which gives a factor $d_M/d_F$.

If $M$ is a $T$-manifold resp. orbifold with an (oriented) $T$-invariant boundary $Z = \partial M$, the localization formula includes additional boundary terms. Let us just consider the case $T = S^1$, and suppose for simplicity that the action of $S^1$ on $\partial M$ is locally free. Let $j : Z \to M$ be the inclusion, and $\theta \in \mathcal{A}^1(Z)$ a connection. Then the boundary contribution is given by

$$-\frac{1}{d_M} \int_{\partial M} j^* \alpha(\xi) \text{dR}(\theta(\xi)) \wedge \theta.$$

For a proof (in the manifold case), see e.g. Kalkman [22], or Vergne [34].

3. Riemann-Roch Theorems for Orbifolds

Suppose $M$ is a compact orbifold, equipped with a positive Kähler structure, and that $E \to M$ is a holomorphic Hermitian orbifold vector bundle over $M$. Just as in the manifold case, one has a twisted Dolbeault complex of $E$-valued forms,

$$\bar{\partial} : \mathcal{A}^{i,j}(M, E) \to \mathcal{A}^{i,j+1}(M, E),$$

and a corresponding Spin$^c$-Dirac operator

$$\bar{\partial}_C = \sqrt{2}(\bar{\partial} + \bar{\partial}^*) : \mathcal{A}^{0,even}(M, E) \to \mathcal{A}^{0,odd}(M, E).$$

If $M$ is only almost Kähler, it is still possible to construct $\bar{\partial}_C$, using Hermitian connections on $E$ and on the canonical line bundle, $\wedge^{0,n} T^* M$. For the details of this construction see e.g. Duistermaat [4]. The dimension of the index space

$$\text{ind}(\bar{\partial}_C) = [\ker(\bar{\partial}_C)] - [\ker(\bar{\partial}_C)]$$

is independent of all choices, and will be called the Riemann-Roch number $\text{RR}(M, E)$ of $E \to M$.

**Remark 3.1.** If $M$ is a symplectic orbifold and $E \to M$ a complex (or symplectic) vector bundle, one can always choose a compatible positive almost complex structures on $M$ and Hermitian structure on $E$. Since any two choices are homotopic, $\text{RR}(M, E)$ does not depend on these choices.
If a compact Lie group $G$ acts on the above data and if all choices are made $G$-equivariant, one has $G$-actions on $A^{i,j}(M, E)$, and the index space carries a virtual representation of $G$. In this case, we let $\text{RR}(M, E) = \chi \in R(G)$ be the equivariant Riemann-Roch number, i.e. $\chi(g) = \text{tr}(g \cdot \text{ind}(\mathfrak{g} \mathfrak{c}))$. The orbifold version of the equivariant index theorem expresses $\chi$ as an integral of certain characteristic classes over an associated orbifold $\tilde{M}$. Let us digress on how $\tilde{M}$ is defined. As a set,

$$\tilde{M} = \bigcup_{x \in M} \text{Conj}(H_x),$$

where $\text{Conj}(H_x)$ is the set of conjugacy classes in the isotropy group $H_x$. Given an atlas for $M$, orbifold charts for $\tilde{M}$ can be constructed as follows. For each $V_i$, let

$$\tilde{V}_i = \{(v, h) \in V_i \times H_i \mid h.v = v\}.$$

Then $H_i$ acts on $\tilde{V}_i$ via $a.(v, h) = (a.v, a.h.a^{-1})$, and by definition $\tilde{U}_i := \tilde{V}_i/H_i$. Notice that the action of $H_i$ on the preimage of a given connected component need not be effective, even if the action on $V_i$ was.

One can show that the orbifold charts $(\tilde{U}_i, \tilde{V}_i, H_i)$ inherit the compatibility conditions from the $(U_i, V_i, H_i)$, and thus define an orbifold structure on $\tilde{M}$. Usually, $\tilde{M}$ has various components of different dimension.

**Remark 3.2. (Properties of $\tilde{M}$):**

a. Let $G \times P \to P$ be a locally free action of a compact Lie group $G$ on an orbifold $P$. Then $M = P/G$ is an orbifold, and $\tilde{M} = \tilde{P}/G$, where

$$\tilde{P} = \{(p, g) \in \tilde{P} \times G \mid g.p = p\}.$$

b. Consider the natural mapping $\tau : \tilde{M} \to M$. Since all mappings $\tilde{V} \to V$ in the above orbifold charts are $H$-equivariant immersions (on all connected components), it follows that $\tau$ is an immersion. (If $M$ is a quotient of a manifold by a locally free action of an abelian group, $\tau$ restricts to orbifold embeddings of the connected components of $M$. This is however false in general, because the orbifold isotropy groups are only defined up to isomorphism, and may be glued together globally in a nontrivial way.)

Let $N_{\tilde{M}} \to \tilde{M}$ denote the normal bundle of this immersion. In a local orbifold chart $(U, V, H)$, $N_{\tilde{M}}$ is obtained from the normal bundle $N_V$ of the immersion $\tilde{V} \to V$ by dividing out the action of $H$.

c. Let $\tilde{E} \to \tilde{M}$ be an orbifold vector bundle, given in local orbifold charts $(U, V, H)$ by $H$-equivariant vector bundles $\tilde{E}_V \to \tilde{V}$. Observe that for all $(v, h) \in \tilde{V}$, there is an action of $h$ on the fiber of $\tilde{E}_V$ over $(v, h)$. These actions glue together to give a canonical section $A$ of the automorphism bundle $\text{Aut}(\tilde{E})$. (Notice however that the action of $A$ on sections of $\tilde{E}$ is trivial!) Suppose that $\tilde{E}$ is a complex Hermitian orbifold bundle with connection. Let $F(\tilde{E}) \in \mathcal{A}^2(\tilde{M}, \text{End}(\tilde{E}))$ be the curvature. We define twisted characteristic forms $\text{Ch}^{\tilde{M}}(\tilde{E})$ and $D^{\tilde{M}}(\tilde{E})$ by

$$\text{Ch}^{\tilde{M}}(\tilde{E}) = \text{tr}(A e^{\frac{1}{2}F(\tilde{E})}) \in \mathcal{A}(\tilde{M}).$$

and

$$D^{\tilde{M}}(\tilde{E}) = \det \left(1 - A^{-1} e^{-\frac{1}{2}F(\tilde{E})}\right) \in \mathcal{A}(\tilde{M}).$$
d. If $F$ is a connected suborbifold of $M$, the principal stratum of $F$ is contained in some stratum of $M$, and the multiplicity $d_F$ is the order of the isotropy group corresponding to that stratum. Moreover, the associated orbifold $\tilde{F}$ is a suborbifold of $\tilde{M}$.

Let us now turn to the case where $M$ is a symplectic orbifold. Then $\tilde{M}$ is a symplectic orbifold, and the normal bundle $N_{\tilde{M}}$ is a symplectic orbifold bundle. Choose a compatible positive almost complex structure $J$ on $M$, thereby making the tangent bundles of $M$ and $\tilde{M}$ into Hermitian vector bundles. Consider a Hermitian connection on $TM$, with curvature $F(M) \in \mathcal{A}^2(M, \text{End}(TM))$, and let

$$\text{Td}(M) = \det \left( \frac{i}{2\pi} F(M) \right)$$

and $\text{Td}(\tilde{M})$ be the corresponding Todd forms.

Let $E \to M$ be an orbifold vector bundle over an almost complex orbifold, and let $\tilde{E} = \tau^* E$.

**Theorem 3.3** (Kawasaki). The Riemann-Roch number $\text{RR}(M, E) := \dim(\text{ind}(\tilde{\phi}_C))$ is given by the formula

$$\text{RR}(M, E) = \int_{\tilde{M}} \frac{1}{d_{\tilde{M}}} \frac{\text{Td}(\tilde{M}) \text{Ch}_{\tilde{M}}(\tilde{E})}{D^M(N_{\tilde{M}})}.$$

(24)

(For orbifolds $M$ that can be represented as quotients of manifolds by locally free actions of abelian groups, the orbifold index theorem is due to Atiyah \[1\].) Let now $G$ be a compact, connected Lie group acting on $E \to M$, and suppose all the above choices have been made $G$-invariant. By lifting the fundamental vector fields, one obtains an action on $\tilde{M}$ of some finite cover of $G$ (due to the holonomy phenomenon mentioned in Remark 3.2 (b), $G$ itself need not act on $\tilde{M}$). In \[33\], M. Vergne has proved the following orbifold version of the equivariant index theorem of Atiyah-Segal-Singer \[4\]. Denote by $\text{Td}_g(\tilde{M}, \xi), \text{Ch}_g(\tilde{E}, \xi)$ etc. the equivariant characteristic classes for the respective bundles on $\tilde{M}$.

**Theorem 3.4** (Vergne). For $\xi \in g$ sufficiently small, the equivariant Riemann-Roch number $\chi = \text{RR}(M, E)$ is given by the formula

$$\chi(e^\xi) = \int_{\tilde{M}} \frac{1}{d_{\tilde{M}}} \frac{\text{Td}_g(\tilde{M}, \xi) \text{Ch}^\tilde{M}_g(\tilde{E}, \xi)}{D^M(N_{\tilde{M}}, \xi)}.$$

(25)

More generally, Vergne has proved a cohomological formula for $\chi(g e^\eta)$, with $\eta$ in the Lie algebra of the stabilizer of $g$, as an integral over $\tilde{M}^g$. If we take $\eta = 0$, and $g = \exp(\xi) \in T$ generic in the sense that $g$ generates $T$, this includes the following special case:

**Theorem 3.5.** (Fixed point formula for orbifolds.) The equivariant index $\chi(e^\xi)$ of $\tilde{\phi}_C$ is given by the formula $\chi|T = \sum_F \chi_F$, where the sum is over the connected components $F$ of the fixed point orbifold $M^T$, and $\chi_F$ is a meromorphic function given by the formula

$$\chi_F(e^\xi) = \int_{\tilde{F}} \frac{1}{d_{\tilde{F}}} \frac{\text{Td}(\tilde{F}) \text{Ch}^{\tilde{F}}(\tilde{F}, \xi)}{D^{\tilde{F}}(N_{\tilde{F}}) D^\xi(\tilde{\nu}_F, \xi)}.$$

(26)

Here $\nu_F$ is the normal bundle of $F$ in $M$, and $\tilde{\nu}_F$ its pullback to $\tilde{F}$. 
orbifold singularity of order $k$ $G$.

Let $\chi$ is smooth, the character of the action on the normal bundle is +1, and so the line bundle. The fixed point set consists of two points, $z$ coordinate $w$ coordinate $c$.

The teardrop orbifold $M$ is obtained by gluing a copy of $C/\mathbb{Z}_k$, with coordinate $w^k$, with a copy of $C$, with coordinate $z$, via the mapping $z \mapsto w^{-k}$ for $z \neq 0$. Note that $M$ is a disjoint union of a copy of $M$ and $(k - 1)$ points. Let $G = S^1$ act by rotation, i.e. $z \mapsto e^{i\phi} \cdot z$, and let $E = M \times C$ be the trivial line bundle. The fixed point set consists of two points, $z = 0, \infty$. At $z = 0$, $M$ is smooth, the character of the action on the normal bundle is +1, and so the fixed point contribution becomes $\chi(0)(e^{i\phi}) = (1 - e^{i\phi})^{-1}$. At $z = \infty$, there is an orbifold singularity of order $k$, hence $\tilde{F}$ consists of $k$ points. The normal bundle is isomorphic to $C/\mathbb{Z}_k$, and $S^1$ acts by the orbi-weight $-\frac{1}{k}$. Thus

$$\chi(\infty)(e^{i\phi}) = \frac{1}{k} \sum_{l=0}^{k-1} \frac{1}{1 - e^{-\phi}}$$

where $c = \exp(-\frac{2\pi i}{k})$. Using the identity

$$\frac{1}{k} \sum_{l=0}^{k-1} \frac{1}{1 - e^{l\phi}} = \frac{1}{1 - u^k},$$

we can carry out the summation and find $\chi(\infty)(e^{i\phi}) = (1 - e^{-i\phi})^{-1}$. The two contributions add up to give $\chi(e^{i\phi}) = (1 - e^{i\phi})^{-1} + (1 - e^{-i\phi})^{-1} = 1$.

We will now give a simple application of the fixed point formula in connection with holomorphic induction. Let $G$ be a compact connected Lie group, with maximal torus $T$, and $t^*_+ \subset t^* = (\mathfrak{g})^T$ some choice of a positive Weyl chamber. For each face $\Sigma$ of $t^*_+$, there is a compact, connected subgroup $G_{\Sigma} \subset G$ with the property that $G_\alpha = G_{\Sigma}$ for all $\alpha \in \text{int}(\Sigma)$. Write $t^*_{\Sigma,+} \supset t^*_+$ for the positive Weyl chamber of $G_{\Sigma}$, $W_{\Sigma} \subset W$ for the Weyl group, $\mathfrak{R}_{\Sigma,+} \subset \mathfrak{R}_+$ for the positive roots, and $\Lambda^*_{\Sigma,+} = \Lambda^* \cap t^*_{\Sigma,+}$ for the dominant weights. Let $G/G_{\Sigma}$ be equipped with its canonical $G_{\Sigma}$-invariant complex structure, corresponding to the interpretation as a coadjoint orbit for $G$. If $Y_{\Sigma}$ is a compact, almost complex $G_{\Sigma}$-orbifold, the associated bundle $G \times_{G_{\Sigma}} Y_{\Sigma}$ has a canonically induced almost complex structure, and given a $G_{\Sigma}$-equivariant orbifold-vector bundle $E_{\Sigma} \to Y_{\Sigma}$, one can form the associated bundle $G \times_{G_{\Sigma}} E_{\Sigma} \to G \times_{G_{\Sigma}} Y_{\Sigma}$. In particular, one can apply this to the case where $Y_{\Sigma} = G_{\Sigma} \cdot \mu$ is an integral coadjoint orbit through a $G_{\Sigma}$-weight $\mu \in \Lambda^*_{\Sigma,+}$, and $E_{\Sigma} = G_{\Sigma} \times T C_{\mu}$ the corresponding pre-quantum line bundle. Then $\text{RR}(G_{\Sigma} \cdot \mu, G_{\Sigma} \times_{G_{\Sigma}} C_{\mu})$ is the irreducible $G_{\Sigma}$-representation $\chi_{\Sigma,\mu}$ labelled by $\mu$. The associated bundle $G \times_{G_{\Sigma}} Y_{\Sigma}$ is just the coadjoint orbit $G \cdot \mu$, and the Riemann-Roch number of $G \times_{G_{\Sigma}} E_{\Sigma} = G \times_{G_{\Sigma}} C_{\mu}$ is by definition the holomorphic induction of $\chi_{\Sigma,\mu}$. Let

$$\text{Ind}_{G_{\Sigma}}^G : R(G_{\Sigma}) \to R(G)$$

denote the holomorphic induction map.

**Theorem 3.6.** Let $Y_{\Sigma}$ be a compact almost complex $G_{\Sigma}$-orbifold, and $E_{\Sigma} \to Y_{\Sigma}$ a $G_{\Sigma}$-equivariant orbifold vector bundle. The $G$-equivariant Riemann-Roch number
of $G \times G_\Sigma$ \(E_\Sigma\) is related to the \(G_\Sigma\)-equivariant Riemann-Roch number of \(E_\Sigma\) by holomorphic induction:

\[(28) \quad \text{RR}(G \times G_\Sigma Y_\Sigma, G \times G_\Sigma E_\Sigma) = \text{Ind}_{G_\Sigma}^G \text{RR}(Y_\Sigma, E_\Sigma)\]

Proof. The proof is by applying the fixed point formula to both sides. We write \(\chi = \text{RR}(G \times G_\Sigma Y_\Sigma, G \times G_\Sigma E_\Sigma)\) and \(\chi_\Sigma = \text{RR}(Y_\Sigma, E_\Sigma)\), and denote by \(N_\Sigma(\mu)\) the multiplicity of a \(G_\Sigma\)-weight \(\mu \in \Lambda^*_\Sigma\) in \(\chi_\Sigma\). Since \((G/G_\Sigma)^T = W/W_\Sigma, \)

\[M^T = (G \times G_\Sigma Y_\Sigma)^T = W \times W_\Sigma Y_\Sigma^T.\]

We will think of \(W/W_\Sigma\) as the set of all \(w \in W\) such that \(w \cdot t^*_\Sigma \subset t^*_{\Sigma,+}\), so that

\[\chi(e^\xi) = \sum_{F \subset M^T} \chi_F(e^\xi) = \sum_{w \in W/W_\Sigma} \sum_{F \subset Y_\Sigma^T} \chi_F(w^{-1}(\xi)).\]

Consider the \(T\)-action on \(T_x M\), for \(x \in Y_\Sigma^T \subset M^T\). The weights for the action on \(T_x M/T_x Y_\Sigma\) are just \(\mathfrak{g}_+ - \mathfrak{g}_{\Sigma,+}\), the set of positive roots of \(G\) that are not positive roots for \(G_\Sigma\). It follows that for all \(F \subset Y_\Sigma^T \subset M^T\), the fixed point contributions \(\chi_{\Sigma,F}(\xi), \chi_F(\xi)\) with respect to \(Y_\Sigma, M\) are related by

\[\chi_F(\xi) = \frac{\chi_{\Sigma,F}(\xi)}{\prod_{\beta \in \mathfrak{g}_+ - \mathfrak{g}_{\Sigma,+}} (1 - e^{-2\pi i \beta,\xi})}.\]

Summation over all fixed point contributions in \(Y_\Sigma^T \subset M^T\) gives, by another application of the fixed point formula,

\[\chi(e^\xi) = \sum_{w \in W/W_\Sigma} \sum_{F \subset Y_\Sigma^T} \chi_{\Sigma,F}(w^{-1}(\xi)) = \sum_{w \in W/W_\Sigma} \prod_{\beta \in \mathfrak{g}_+ - \mathfrak{g}_{\Sigma,+}} (1 - e^{-2\pi i \beta,\xi}) \sum_{\mu \in \Lambda^*_\Sigma} N_\Sigma(\mu) \sum_{w \in W/W_\Sigma} \frac{\rho_{\Sigma,\mu}(w^{-1}(\xi))}{\prod_{\beta \in \mathfrak{g}_+ - \mathfrak{g}_{\Sigma,+}} (1 - e^{-2\pi i \beta,w^{-1}(\xi)})}.\]

The same formula with \(N_\Sigma(\mu) = 1\) gives an expression for \(\text{Ind}_{G_\Sigma}^G \rho_{\Sigma,\mu}\). Thus

\[\chi(e^\xi) = \sum_{\mu \in \Lambda^*_\Sigma} N_\Sigma(\mu) \text{Ind}_{G_\Sigma}^G \rho_{\Sigma,\mu}(e^\xi) = \text{Ind}_{G_\Sigma}^G \chi_\Sigma(e^\xi).\]

\[\square\]

4. SYMPLECTIC SURGERY

Let \((M, \omega)\) be a symplectic orbifold, and \(S^1 \times M \to M\) a Hamiltonian action with moment map \(\Phi : M \to \mathbb{R}\). Suppose that 0 is a regular value of \(\Phi\), and let \(M_{\text{red}} = \Phi^{-1}(0)/S^1\) be the symplectic quotient. The symplectic cutting construction of Lerman \cite{L} yields two new symplectic orbifolds \(M_-, M_+\), such that, as a set,

\[(29) \quad M_- = M_{\text{red}} \cup \{\Phi < 0\}, \quad M_+ = M_{\text{red}} \cup \{\Phi > 0\},\]

and the embeddings of \(M_{\text{red}}, \{\Phi < 0\}, \{\Phi > 0\}\) are symplectic.

The structure of symplectic orbifolds on these spaces is obtained as follows. Consider the product \(M \times \mathbb{C}\), with symplectic form \(\omega - \frac{i}{2} dz \wedge d\bar{z}\), and the diagonal circle action \(e^{i\phi} \cdot (x, z) = (e^{i\phi} \cdot x, e^{-i\phi} z)\), with moment map \(\psi(x, z) = \Phi(x) - |z|^2\).
Then 0 is a regular value of \( \psi \), and so \( \psi^{-1}(0)/S^1 \) is a symplectic orbifold. The level set \( \psi^{-1}(0) \) consists of two components:

\[
\psi^{-1}(0) = \{(x,z)| \Phi(x) > 0, \; |z|^2 = \Phi(x)\} \cup (\Phi^{-1}(0) \times \{0\}).
\]

It is now easy to see that the map

\[
\alpha : \{\Phi > 0\} \to \psi^{-1}(0) \subset M \times \mathbb{C}, \; x \mapsto (x, \sqrt{\Phi(x)})
\]

satisfies \( \alpha^*(\omega - \frac{1}{2} dz \wedge d\bar{z}) = \omega \). This identifies \( \psi^{-1}(0)/S^1 = M_+ \). For \( M_- \), one simply takes the opposite circle action on \( \mathbb{C} \).

**Example:** Take \( M = \mathbb{C}P(1) \), with the Fubini-Study form, and consider the natural action of \( S^1 \), \( e^{i\theta} [z_0 : z_1] = [e^{i\theta} z_0 : z_1] \). A moment map for this action is given by \( \Phi([z_0 : z_1]) = -\frac{|z_0|^2}{|z_1|^2} + \frac{1}{2} \). Zero is a regular value of \( \Phi \), the action of \( S^1 \) on \( \Phi^{-1}(0) \) is free, and the cut spaces are spheres with half the volume of \( \mathbb{C}P(1) \). Consider, on the other hand, the moment map \( \Phi(k) = k \Phi \), for \( k \in \mathbb{N} \), which lets \( S^1 \) rotate \( \mathbb{C}P(1) \) with \( k \)-fold speed. For this case, the \( S^1 \) action on the zero level set is only locally free, and the cut spaces are teardrop-orbifolds with a \( \mathbb{Z}/k\mathbb{Z} \)-singularity at \( M_{\text{red}} = \{pt.\} \).

For the cutting construction, the Hamiltonian \( S^1 \) action needs only be locally defined, in some neighborhood of the hypersurface \( Z = \Phi^{-1}(0) \). The corresponding cut space \( M_{\text{cut}} \) (equal to \( M_+ \cup M_- \) for a globally defined \( S^1 \) action) has one or two connected components, depending on whether or not \( Z \) disconnects \( M \). We denote the two copies of \( M_{\text{red}} \) in \( M_{\text{cut}} \) by \( M_{\text{cut}}^{\pm} \).

Consider for example the torus \( M = T^2 = \mathbb{R}^2/\mathbb{Z}^2 \), with the action of \( S^1 \) defined by \( e^{2\pi i t} [x,y] = [x+t,y] \). In a neighborhood of the circle \( e^{2\pi i t}, [0,0] \), one can take \( \Phi([x,y]) = y \) as a moment map, and \( M_{\text{cut}} \) is symplectomorphic to \( S^2 \).

**Remark 4.1.**

a. If a Lie-group \( G \) acts on \( M \) in a symplectic fashion, and if this action commutes with the action of \( S^1 \), one obtains a symplectic \( G \)-action on \( M_{\text{cut}} \). If the action on \( M \) has a moment map, the restriction of this moment map to \( M - Z \) extends smoothly to a \( G \)-moment map on \( M_{\text{cut}} \).

b. Consider \( Z \to M_{\text{red}} \) as an orbifold principal \( S^1 \)-bundle. The normal orbifold-bundle of \( M_{\text{red}} \), in \( M_{\text{cut}} \), is equal to the associated bundle \( Z \times \mathbb{S}^1, \mathbb{C} \), where \( S^1 \) acts on \( \mathbb{C} \) by the character \( e^{\pm i\phi} \). In particular, they have opposite Chern classes.

c. The cutting construction does not use nondegeneracy of \( \omega \).

Let us return to the case where the \( S^1 \) action is globally defined, and consider an \( S^1 \)-equivariant Hermitian orbifold vector bundle \( E \to M \). Let \( \pi_1 : M \times \mathbb{C} \to M \) denote projection to the first factor, and define \( E_{\text{red}} = (E|_{\Phi^{-1}(0)})/S^1 \) and \( E_{\pm} = (\pi_1^* E|_{\psi^{-1}(0)})/S^1 \), with the induced Hermitian structure. The mapping (31) induces a mapping

\[
E|_{\{\Phi > 0\}} \times S^1 \to \pi_1^* E,
\]

which descends to an isomorphism of Hermitian vector bundles \( E_+|_{\{\Phi > 0\}} \cong E|_{\{\Phi > 0\}} \), and similarly of course \( E_-|_{\{\Phi < 0\}} \cong E|_{\{\Phi < 0\}} \). It follows that cutting is local on the level of Hermitian vector bundles, in the following sense: Suppose that the \( S^1 \) action on \( E \to M \) is only defined near a given \( S^1 \)-invariant hypersurface \( Z = \Phi^{-1}(0) \). Then one can define a reduced Hermitian bundle \( E_{\text{red}} \to ...
\(M_{\text{red}}\) and a cut bundle, \(E_{\text{cut}} \to M_{\text{cut}}\). If a compact Lie group \(G\) acts on \(E \to M\) and if all data are \(G\)-invariant, one obtains smooth \(G\)-actions on \(E_{\text{red}}\) and \(E_{\text{cut}}\). Given this situation, let us now study the relation of the \((G\text{-equivariant})\) Riemann-Roch numbers of \(M\) and its cut- and reduced spaces.

**Theorem 4.2** (Gluing Formula). The \((G\text{-equivariant})\) Riemann-Roch numbers of \(E\), \(E_{\text{cut}}\), \(E_{\text{red}}\) are related by

\[
\text{RR}(M, E) = \text{RR}(M_{\text{cut}}, E_{\text{cut}}) - \text{RR}(M_{\text{red}}, E_{\text{red}}).
\]

**Proof.** Let us assume \(G = \{1\}\) for simplicity of notation; for the general case, one simply replaces characteristic classes by \((G\text{-equivariant})\) characteristic classes. Let \(B \subset M\) be a tubular neighborhood of \(Z\), such that the Hamiltonian \(S^1\) action is defined and locally free on \(B\). Pick compatible almost complex structures of \(M\) and \(M_{\text{cut}}\) in such a way that they agree over some neighborhood of \(M - B\). Also, choose Hermitian connections \(\nabla, \nabla_{\text{cut}}\) on \(E, E_{\text{cut}}\) that agree over some neighborhood of \(M - B\) and are \(S^1\)-invariant over \(B\).

Let \(\tilde{B}\) be the preimage of \(B\) in \(\tilde{M}\). Over \(\tilde{B}\), we can introduce \(S^1\)-equivariant characteristic classes, and write

\[
\text{RR}(M, E) = \int_{\tilde{M} - \tilde{B}} \frac{1}{d_{\tilde{M}}} \text{Td}(\tilde{M}) \text{Ch}_{\tilde{M}}(\tilde{E}) + \int_{\tilde{B}} \frac{1}{d_{\tilde{M}}} \text{Td}_{\mathbb{R}}(\tilde{M}, \xi) \text{Ch}_{\mathbb{R}}(\tilde{E}, \xi) \bigg|_{\xi = 0},
\]

To the second term, we apply the localization formula for manifolds with boundary, and rewrite it in the form

\[
\int_{\partial B} \alpha(\xi) \bigg|_{\xi = 0},
\]

for some equivariant differential form \(\alpha \in \mathcal{A}_G(\partial B)\). Let us do the same computation for \(M_{\text{cut}}\), with \(B\) replaced by \(B_{\text{cut}}\). \(M_{\text{cut}}\) contains two copies of \(M_{\text{red}}\) as codimension 2, \(S^1\)-fixed suborbifolds, but with opposite normal bundles, and the complement is symplectomorphic to \(M - Z\). We hence obtain the same terms as above, plus two additional terms corresponding to the two copies of \(M_{\text{red}}\):

\[
\int_{M_{\text{red}}} \frac{1}{d_{M_{\text{red}}}} \text{Td}(\tilde{M}_{\text{red}}) \text{Ch}_{\mathbb{R}}(\tilde{E}_{\text{red}}, \xi) + \int_{M_{\text{red}}} \frac{1}{d_{M_{\text{red}}}} \text{Td}_{\mathbb{R}}(\tilde{M}_{\text{red}}, \xi) \text{Ch}_{\mathbb{R}}(\tilde{E}_{\text{red}}, \xi),
\]

evaluated at \(\xi = 0\). But

\[
D_{\mathbb{R}}(\tilde{M}_{\text{red}}, \tilde{\nu}, \tilde{\nu}^*)^{-1} + D_{\mathbb{R}}(\tilde{M}_{\text{red}}, \tilde{\nu}, \tilde{\nu}^*)^{-1} = 1,
\]

(using the formal identity \((1 - z)^{-1} + (1 - z^{-1})^{-1} = 1\), so the sum of these terms is just \(\chi_{\text{red}}\).

**Remark 4.3.**

a. Notice that in the above proof, the Chern character may be replaced by any characteristic class of \(\tilde{E}\).

b. For any symplectic orbifold \(M\), let \(\tau(M) = \int_M \text{Td}(M)\). The above Theorem, applied to the trivial bundle \(E = M \times \mathbb{C}\), greatly simplifies the computation of \(\tau(M)\).

i) Consider for example the cutting of \(M = \mathbb{C}P(n)\) along the equator sphere \(Z = S^{2n-1}\). Then \(M_{\text{cut}}\) consists of two copies of \(\mathbb{C}P(n)\), and the reduced space is \(\mathbb{C}P(n-1)\). Hence \(\tau(\mathbb{C}P(n)) = 2\tau(\mathbb{C}P(n)) - \tau(\mathbb{C}P(n - 1))\), or \(\tau(\mathbb{C}P(n)) = \tau(\mathbb{C}P(n - 1)) = \ldots = \tau(\mathbb{C}P(0)) = 1\).
(ii) Consider next a Riemann surface $M$ of genus $g$. By cutting along $g$ circles, $M$ can be made into a sphere, hence $\tau(M) = \tau(S^2) - g = 1 - g$.

(iii) Let $M$ be any symplectic manifold, and $p \in M$. The symplectic analogue of the blow-up $Bl_p(M)$ of $M$ at $p$ (see (23)) replaces $p$ by a “small” $\mathbb{C}P(n-1)$, and may be regarded as a cutting operation (see (25)), with $M_+ = Bl_p(M)$, $M_- = \mathbb{C}P(n)$, and $M_{\text{red}} = \mathbb{C}P(n-1)$. Since $\tau(\mathbb{C}P(k))$ is equal to one, $\tau(Bl_p(M)) = \tau(M) - \tau(\mathbb{C}P(n)) + \tau(\mathbb{C}P(n-1)) = \tau(M)$. More generally, one can consider symplectic blow-ups $Bl_N(M)$ along a symplectic submanifold $N \subset M$, and a similar argument shows that $\tau(Bl_N(M)) = \tau(M)$.

(iv) Let $M_k$ be the teardrop-orbifold of order $k$. Recall from the first example in this section that $M_k$ can be constructed by cutting the sphere $\mathbb{C}P(1) = S^2$, with $S^1$ acting with $k$-fold speed. The gluing formula $RR(\mathbb{C}P(1), \mathbb{C}) = 2 RR(M_k, \mathbb{C}) - RR(pt, \mathbb{C})$ shows (using Kawasaki’s formula) that

$$1 = 2(\tau(M_k) + \frac{1}{k} \sum_{j=1}^{k-1} \frac{1}{1 - e^{2\pi i j/k}}) - 1$$

$$= 2(\tau(M_k) + \frac{1}{2}(1 - \frac{1}{k})) - 1.$$ 

Hence $\tau(M_k) = \frac{1}{2}(1 + \frac{1}{k})$. More generally, if $M$ is a 2-dimensional symplectic orbifold of genus $g$, with $r$ orbifold-singularities of order $k_1, \ldots, k_r$, one finds

$$\tau(M) = 1 - g - \frac{1}{2} \sum_{i=1}^{r} (1 - \frac{1}{k_i}).$$

(35)

c. As pointed out in (29), the original space can be recovered from the cut space by means of the symplectic gluing procedure of Gompf (13): Let $N$ be a symplectic orbifold, and let $\iota_i : N \to M_i$ be symplectic embeddings into two given symplectic orbifolds with $\dim M_i = \dim N + 2$. If the corresponding normal bundles $\nu_i$ are opposite, $\nu_1^* \cong \nu_2^*$. Gompf’s method yields a new symplectic manifold $M$, which as a set is a union of $M_1 - N$, $M_2 - N$ and the unit circle bundle of $\nu_1 = \nu_2^*$. One can show that the gluing procedure is quantizable as well, i.e. given symplectic orbifold vector bundles $E_i$ with the same restriction to $N \subset M_i$, then there is a symplectic orbifold vector bundle $E$ on $M$, and the $E_i$ are obtained from $E$ by cutting.

Let $Z$ be the generic stabilizer for the action of $S^1$ on $\Phi^{-1}(0)$. Then the action of $H = S^1/Z$ on $\Phi^{-1}(0)$ is generically free. We may use $H$ instead of the original $S^1$ for the cutting construction, and obtain a cut space $M_+$ which is again the disjoint union of $M_{\text{red}}$ and $\{ \Phi > 0 \}$, but they are glued together in a different way. Conversely, starting from a generically free action on $\Phi^{-1}(0)$, with corresponding cut space $M_+$, obtains new cut spaces $M^{(l)}_+$ ($l \in \mathbb{N}$) by replacing $S^1$ by its $l$-fold cover. The normal bundle $\nu^{(l)}$ of $M_{\text{red}}$ in $M^{(l)}_+$ is equal to the quotient of $\nu$ by the action of $Z$, and the equivariant Chern class (where we consider the action of the original $G = S^1$ on $M^{(l)}_+$) gets divided by a factor of $l$. The space $M^{(l)}_{\text{red}}$ consists simply of $l$ copies of $M_{\text{red}}$, in particular $d_{\tilde{M}^{(l)}_{\text{red}}} = l d_{\tilde{M}_{\text{red}}}$. Therefore, the fixed point
contribution $\chi_{M_{\text{red}}}^{(l)}$ becomes

$$\chi_{M_{\text{red}}}^{(l)}(e^{i\xi}) = \frac{1}{l} \int_{M_{\text{red}}} \frac{1}{\text{Td}_R(M_{\text{red}})} \frac{1}{\text{Ch}_R(M_{\text{red}})} \frac{1}{\text{D}_R^{M_{\text{red}}}} \sum_{j=0}^{l-1} \frac{1}{\text{D}_R^{M_{\text{red},j}}} \left( \hat{\nu}^{(l)} + 0 \right),$$

where $\hat{M}_{\text{red},j}$ denotes the $j$th copy of $M_{\text{red}}$. The sum over $j$ can be computed, using Equation (27):

$$\frac{1}{l} \sum_{j=0}^{l-1} \frac{1}{\text{D}_R^{M_{\text{red},j}}} \left( \hat{\nu}^{(l)} + 0 \right) = \frac{1}{\text{D}_R^{M_{\text{red}}}} \left( \hat{\nu}, \xi \right).$$

This leads to the following observation:

**Proposition 4.4.** The fixed point contributions $\chi_{M_{\text{red}}}^{\alpha}(e^{i\xi})$ to $\text{RR}(M_{\pm}, E_{\pm})$ depend only on the cutting hypersurface $Z$, not on how the projection $Z \rightarrow M_{\text{red}}$ is made into an orbifold $S^1$ bundle.

We will now consider the special case that $E = L$ is a pre-quantum line bundle, with Hermitian connection satisfying the pre-quantum condition

$$\frac{i}{2\pi} \text{curv}(\nabla) = \omega,$$

and such that the $S^1$-action on $M$ lifts to $L$ according to (31). It is well known that $L_{\text{red}} = (L|\Phi^{-1}(0))/S^1$ has a unique Hermitian connection, such that the pullback to $\Phi^{-1}(0)$ is equal to the restriction of $\nabla$.

The trivial line bundle $L_C = C \times C$, with Hermitian fiber metric

$$\langle w_1, w_2 \rangle = \exp(-\pi |z|^2) \bar{w}_1 \cdot w_2,$$

$(w_i \in (L_C)_{\bar{z}})$ and connection 1-form $A = -\pi z d\bar{z}$ is a pre-quantum bundle for the symplectic structure $-\frac{i}{\pi} d\bar{z} \wedge d\bar{z}$. The pre-quantum lift (31) of the $S^1$ action on $C$ to $L_C$ is simply the trivial action. Hence, $L \boxtimes L_C$ is a pre-quantum line bundle for $M \times C$, and the above prescription gives a pre-quantum bundle for $M_{+}$,

$$L_{+} = (L \boxtimes L_C|\psi^{-1}(0))/S^1.$$

Notice that over $\{\Phi > 0\}$, we have now two pre-quantum line bundles: One coming from the embedding into $M$, the other from the embedding into $M_{+}$.

**Theorem 4.5.** There exists a canonical isomorphism of pre-quantum line bundles with connection, $L_{+}|\{\Phi > 0\} \cong L|\{\Phi > 0\}$, and $L_{+}|M_{\text{red}}^{-1} \cong L_{\text{red}}$.

**Proof.** Since $L_C$ is trivial, $L \boxtimes L_C \cong \pi_{1}^{L}$ as a complex vector bundle, but the fiber metric is multiplied by a factor $e^{-\pi |z|^2}$, and the connection is

$$\nabla \boxtimes \nabla C = \pi_{1}^{L} \nabla - \pi z d\bar{z}.$$

The map

$$\alpha : \{\Phi > 0\} \rightarrow \psi^{-1}(0) \subset M \times C, \ x \mapsto (x, \sqrt{\Phi(x)})$$

in (31) is covered by a bundle map

$$\beta : L|\{\Phi > 0\} \rightarrow \pi_{1}^{L}, \ \lambda \mapsto \lambda e^{\frac{\Phi(x)}{2}}$$

which preserves the fiber metric, and satisfies

$$\beta^{*} (\nabla) = \nabla + \frac{\pi}{2} d\Phi, \ \beta^{*} (\pi z d\bar{z}) = \frac{\pi}{2} d\Phi.$$

This proves the first isomorphism, and the second is obvious. \[\square\]
A similar result holds of course for $L_− \rightarrow M_−$. As a consequence of this Theorem, cutting is local even on the level of pre-quantum line bundles.

It is possible to cut at nonzero levels $α ∈ \mathbb{R}$ of the moment map, by simply redefining the moment map used for the cutting construction as $Φ′ = Φ − α$. Given a $S^1$-invariant complex vector bundle $E → M$, one again obtains cut bundles $E_+$ and $E_−$. Notice however that if $L$ is a pre-quantum line bundle for $M$, the cut bundles obtained in this way are not pre-quantum line bundles for $M_±$. Similarly, $L|Φ^{-1}(α)/S^1$ is not a pre-quantum bundle for $M_α = Φ^{-1}(α)/S^1$, since the $S^1$ action on $L$ satisfies the pre-quantum condition with respect to $Φ$, but not with respect to $Φ′$. If $α$ is an integer, the correct lift is obtained by tensoring $L$ with the trivial line bundle $\mathbb{C}$, with $S^1$ acting by the character $e^{-2πi(α,ξ)}$, before applying the cutting construction. More generally, if $α$ is a rational number, one can choose $k ∈ \mathbb{N}$ such that $kα$ is an integer, and cut with respect to the $k$-fold cover of the original $S^1$. This will introduce extra orbifold singularities in $M_±$, thereby making $M_±$ quantizable by means of orbifold-line bundles.

To conclude this section, let us briefly explain (following [10]) why the gluing formula implies the Guillemin-Sternberg conjecture for the case $G = S^1$ (and therefore also for the abelian case $G = T$, using reduction in stages). It can be deduced from the fixed point formula that the multiplicity of 0 is equal to $RR(M_0, L_0)$ if zero is a maximum or minimum of $Φ$, since in that case $M_0$ is contained in $M$ as a fixed point manifold, and $L_0 = L|M_0$. Let $N_±$ be the multiplicity functions for the cut bundles $L_± → M_±$. By the gluing formula,

$$N(μ) = N_+(μ) + N_-(μ) − RR(M_0, L_0)δ_μ,0.$$ 

Since 0 is a maximum of the moment map for $M_−$ and a minimum for $M_+$, we have $N_+(0) = N_−(0) = RR(M_0, L_0)$. Hence $N(0) = RR(M_0, L_0)$, q.e.d. In particular, a weight does not occur if it is not in the image of the moment map.

**Corollary 4.6.** (See [10]) Let $(M, L)$ be a quantizable Hamiltonian $S^1$ space, with multiplicity function $N(μ)$, and suppose 0 is a regular value of the moment map. Then the multiplicity function for the cut bundle $L_+ → M_+$ is given by $N_+(μ) = N(μ)$ if $μ ≥ 0$, and $N_+(μ) = 0$ if $μ < 0$.

5. **Multiple Cutting**

5.1. **Cutting with respect to polytopes.** Let $T$ be a $k$-torus, and $M$ a compact connected Hamiltonian $T$-orbifold, with moment map $Φ : M → \mathfrak{t}^*$. Denote

$$M_θ = \{ x ∈ M | t_x = θ\}.$$  

(37)

The connected components of $M_θ$ are symplectic suborbifolds of $M_θ$. Let $M = \bigcup M_i$ be the corresponding decomposition of $M$, where the $M_i$ are the connected components of the various $M_θ$, and let $\mathfrak{h}_i$ denote the isotropy group corresponding to $M_i$.

By the convexity theorem of Atiyah [2] and Guillemin-Sternberg [13], and its extension to the orbifold case by Lerman and Tolman [27], the image $Δ = Φ(M)$ is equal to the convex hull of the image of the fixed point set, $Φ(M^T)$. More precisely, the closure of $Φ(M_i)$ is a convex polytope $Δ_i$ of codimension $\dim \mathfrak{h}_i$.

---

4Given an action of a compact Lie group $G$ on an orbifold $M$, the set $M_H$ of points with stabilizer $G_x$ equal to some fixed group $H$ is in general not a suborbifold, because the action of $G_x$ does not always lift to an action in orbifold charts around $x$. 


which is contained in an affine space of the form \( \{ \alpha \} + h_0^0 \subset t^* \). It turns out that the \( \Phi(M_i) \) define a subdivision of \( \Delta \) into rational convex polytopes. In particular, the connected components of the set \( \Delta_* \subset \Delta \) of regular values are convex open subpolytopes of \( \Delta \).

Let \( R \) be a rational, convex polytope \( R \subset t^* \), i.e. a polytope defined by a finite number of inequalities and equalities,

\[
\begin{align*}
\{ \langle \alpha, v_i \rangle & \geq \mu_i : i = 1, \ldots, r \\
\langle \alpha, v_i \rangle & = \mu_i : i = r + 1, \ldots, N
\end{align*}
\]

where \( v_i \in \Lambda \) and \( \mu_i \in \mathbb{R} \). Denote by \( R_\mathbb{R} \) the affine subspace generated by \( R \), and by \( T_R \subset T \) the torus perpendicular to \( R_\mathbb{R} \). We will call \( R \) admissible, if it satisfies the following conditions:

(A) The affine hyperplanes \( \langle \alpha, v_i \rangle = \mu_i \), \( i = 1, \ldots, N \), are all transversal.

(B) The faces of \( R \) are transversal to all (interior and exterior) faces of \( \Delta \).

Let \( \alpha \in R \cap \Delta \), and \( x \in \Phi^{-1}(\alpha) \). Let \( R_\alpha \subset R \) be the unique face that contains \( \alpha \) in its interior. Condition (B) is equivalent to \( t_\mathbb{R} + t_{R_\alpha} = t^* \), or to

(B') For all \( \alpha \in R \cap \Delta \) and \( x \in \Phi^{-1}(\alpha) \), \( t_\mathbb{R} \cap t_{R_\alpha} = \{0\} \).

For the pre-quantization setting, we will need the additional condition

(Q) For all \( i = 1, \ldots, N \), \( \mu_i \) is an integer.

Let \( T_i \) be the circle group with generator \( v_i \), and \( T' = \prod T_i \). Let \( T' \) act on the product \( M \times \mathbb{C}^r \), with moment map

\[
\psi_i(x, z) = \begin{cases} 
\langle \Phi(x), v_i \rangle - \mu_i - |z_i|^2 & : i = 1, \ldots, r \\
\langle \Phi(x), v_i \rangle - \mu_i & : i = r + 1, \ldots, N.
\end{cases}
\]

For all \( I \subset \{1, \ldots, N\} \), let \( I' \) be the complementary set, \( R_I \) the subpolytope of \( R \) defined by \( \langle \alpha, v_i \rangle = \mu_i \) for all \( i \in I \), and \( T_I = \prod_{i \in I} T_i \). Then

\[
\psi^{-1}(0) \cong \bigcup_I \Phi^{-1}(\text{int}(R_I)) \times T_I.
\]

Assumption (A) implies that for all \( R_I \neq \emptyset \), \( T_I \) is a finite cover of \( T_{R_I} \), and by assumption (B), it acts locally freely on \( \psi^{-1}(\text{int}(R_I)) \). Hence \( T' \) acts locally freely on \( \psi^{-1}(0) \), and the cut space

\[
M_R := \psi^{-1}(0)/T'
\]

is a symplectic orbifold. Moreover, we have a decomposition of \( M_R \) into symplectic suborbifolds,

\[
M_R = \bigcup_{R' \subset R} W_{R'}
\]

where

\[
W_{R'} = \Phi^{-1}(\text{int}(R'))/T_{R'}.
\]

**Remark 5.1.** a. For all faces \( R_I \subset R \), there is a natural embedding \( M_{R_I} \hookrightarrow M_R \) as a symplectic suborbifold of codimension \( 2(\dim R - \dim R_I) \). The normal bundle \( N_I \) of \( \Phi^{-1}(\text{int}(R_I))/T_I \) in \( M_R \) is given by the associated bundle \( \Phi^{-1}(\text{int}(R_I)) \times_{T_I} \mathbb{C}^{|I|} \), with the standard action of \( T_I \) on \( \mathbb{C}^{|I|} \). Let us rewrite this in terms of \( T_{R_I} \). The covering mapping \( T_I \rightarrow T_{R_I} \) gives rise to an isomorphism \( t_I \rightarrow t_{R_I} \). Let \( \Lambda_I \subset \Lambda \) be the lattice generated by the \( v_i, i \in I \), and
\[ N_I = \bigoplus_{i \in I} N_{-\alpha_i}, \]

where \( N_{-\alpha_i} \) is the associated bundle

\[ \Phi^{-1}(\text{int}(R_I)) \times_{T_{R_I}} \mathbb{C} \]

for the orbi-character \( \exp(-2\pi i (\alpha_i, \xi)) \).

b. Given a symplectic action of a Lie group \( G \) on \( M \) which commutes with the \( T \)-action, one obtains a symplectic \( G \)-action on \( M_{R} \). If the action is Hamiltonian, with moment map \( J : M \to \mathfrak{g}^* \), the induced action on \( M_{R} \) is Hamiltonian, and the moment map \( J_{R} \) is obtained from the \( T_{I} \)-invariant restrictions \( J|\Phi^{-1}(\text{int}(R_I)) \). In particular, one has a Hamiltonian \( T \)-action on \( M_{R} \), with moment polytope \( \Delta_{R} = \Phi_{R}(M_{R}) = \Delta \cap R \).

c. Cutting is local, in the sense that for all open subsets \( U \subset t^* \), there is a canonical symplectic isomorphism

\[ \Phi^{-1}(U) \cong \Phi^{-1}(U)_{R}. \]

In particular, one does not always need a global \( T \)-action to define the cut space.

d. Given a \( T \times G \)-equivariant symplectic (resp. Hermitian) orbifold-vector bundle \( E \to M \), one obtains a \( T \times G \)-equivariant symplectic (resp. Hermitian) orbifold-vector bundle \( E_{R} \to M_{R} \), by letting

\[ E_{R} = (E \boxtimes \mathbb{C})|\Psi^{-1}(0)/T', \]

where we use the action of \( T' \) on \( E \) induced by the canonical map \( T' \to T \), and the trivial action on \( \mathbb{C} \). More generally, one can let \( T' \) act on \( \mathbb{C} \) by a nontrivial character. (The gluing formulas proved in this paper won’t depend on this choice.) In particular, if \( E = L \) is a pre-quantum line bundle, one will only obtain a pre-quantum bundle \( L_{R} \) for \( M_{R} \) if condition (Q) is satisfied and if one uses the character \( \exp(-2\pi i (\mu, \xi)) \) for \( T' \). In the sequel, we will always make this choice for pre-quantum line bundles, without mentioning this explicitly.

For each face \( R_{I} \subset R \), there is a natural identification

\[ E_{R}|(\Phi^{-1}(\text{int}(R_I)))/T_{I} = (E|\Phi^{-1}(\text{int}(R_I)))/T_{I}. \]

If \( E = L \) is a pre-quantum line bundle, this identification preserves the Hermitian structure and the connection.

e. The \( T \times G \)-equivariant Riemann-Roch numbers \( \text{RR}(M_{R}, E_{R}) = \chi_{R} \) depend only on the polytope \( R \), not on the choice of the \( v_{i}, \mu_{i} \). In fact, all \( T \)-fixed point contributions \( \chi_{R,F} \) are independent of this choice.

**Example:**
A compact Hamiltonian \( T \)-space \( M \) is called a symplectic toric orbifold if \( \dim M = 2 \dim T \) and the \( T \)-action is effective. By a theorem of Delzant, symplectic toric manifolds are completely classified by their moment polytopes \( \Phi(M) = \Delta \). Lerman and Tolman have shown that in the orbifold case, any rational simple polytope \( \Delta \subset t^* \) can occur as a moment map image. To specify \( M \), one needs in addition a positive integer attached to each facet \( \Delta_{i} \); this corresponds to choosing some lattice vector \( v_{i} \) perpendicular to \( \Delta_{i} \). It was mentioned in [26] that every symplectic toric
orbifold can be obtained by symplectic cutting: Let $M = \mathbb{C}P(1)^k$, with symplectic form on $\mathbb{C}P(1)$ equal to $l \in \mathbb{R}_{>0}$ times the Fubini-Study form. The moment map 

$$\Phi_j(w_1, \ldots, w_k) = -l \frac{|w_j|^2 - 1}{|w_j|^2 + 1}$$

$(w_j \in \mathbb{C} \cup \{\infty\})$ defines a Hamiltonian $(S^1)^k$ action on $M$, with moment map image the cube $P_l = \{\alpha \in (\mathbb{R}^k)^* | -l \leq \alpha_j \leq l\}$. Consider a compact polytope $R$ as above, and choose $l$ large enough such that $R \subset \text{int}(P_l)$. The corresponding cut space $M_R$ does not depend on $l$, and is the symplectic toric orbifold associated to $R$. If $l \in \mathbb{N}$, the $l$th power of the hyperplane bundle is a pre-quantum bundle for $(\mathbb{C}P(1), l\omega_{F.S.})$, and by taking exterior tensor products one obtains a pre-quantum bundle $L$ for $M$. The equivariant character $\chi = \text{RR}(M, L)$ for the $T$-action on $M$ is given by 

$$\chi(z) = \prod_{j=1}^k \left( \sum_{\nu_j = -l}^l z_{\nu_j} \right) = \sum_{\alpha \in P_l} z^\alpha,$$

for $z = (z_1, \ldots, z_k) \in T$. Hence, by applying the equivariant version of Corollary 4.6 in stages, one recovers the well-known result that the equivariant Riemann-Roch number for $M_R$ is given by 

$$\text{RR}(M_R, L_R)(z) = \sum_{\alpha \in R} z^\alpha. \quad (48)$$

Remark 5.2. Letting $D_R$ be the symplectic toric orbifold associated to $R$, and $M$ any Hamiltonian $T$-space, one can actually define $M_R$ to be the reduction at 0 of $M \times D_R^{-}$ with respect to the diagonal action; here $D_R^{-}$ denotes $D_R$ with the opposite symplectic structure.

5.2. The abelian gluing formula. Consider now a finite collection $\mathcal{R} = \{R\}$ of admissible polytopes, such that $\Delta \subset \bigcup_{R \in \mathcal{R}} R$, and such that for each polytope in $\mathcal{R}$, all faces are also in $\mathcal{R}$, and for all $R_1, R_2 \in \mathcal{R}$, the intersection $R_1 \cap R_2$ is a face of each. For each $R \in \mathcal{R}$, let $W_R = \Phi^{-1}((\text{int}(R))/T_R$. By (42), each cut space $M_R$ is a disjoint union of symplectic orbifolds 

$$M_R = \bigcup_{R' \subset R} W_{R'} \quad (49)$$

Although the gluing of the $W_{R'}$ depends on the choice of the $v_i$, the Riemann-Roch numbers $\text{RR}(M_R, E_R)$ are independent of this choice.

Theorem 5.3. (Gluing Formula) The Riemann-Roch numbers satisfy the gluing rule 

$$(-1)^{\text{dim}\Delta} \text{RR}(M, E) = \sum_{R \in \mathcal{R}} (-1)^{\text{dim}\Delta} \text{RR}(M_R, E_R). \quad (50)$$

If a compact Lie group $G$ acts on $E \to M$ and this action commutes with the action of $T$, the gluing rule holds for the corresponding $G$-equivariant Riemann-Roch numbers.

We can actually prove a slightly stronger, local result. Write each $\text{RR}(M_R, E_R) = \chi_R$ as a sum over fixed point contributions $\chi_{R,F}$, where $F$ ranges over the connected components of $M_R^T$. 

Suppose $F \subset W_S$ is a connected component of a $T$-fixed point orbifold for some $S \in \mathcal{R}$. If $\dim S = \dim T$, $W_S$ is an open subset of $M$, so $F$ is also a fixed point orbifold for $M$, and clearly $\chi_F = \chi_{S,F}$. Suppose, on the other hand, that $\dim S < \dim T$. Then $F$ is not a fixed point orbifold of $M$, and is a fixed point orbifold for $M_R$ if and only if $\Phi_S(F) \in R$.

**Theorem 5.4.** *(Local gluing formula)* Let $F \subset \Phi^{-1}(\text{int}(S))/T_S$ be a fixed point orbifold, where $S \in \mathcal{R}$ is a polytope with $\dim S < \dim T$. Then

\begin{equation}
\sum_{R \ni \mu} (-1)^{\dim R} \chi_{R,F} = 0,
\end{equation}

where $\mu = \Phi_S(F)$.

Recall that for the $S^1$ case, we needed the identity $(1 - z)^{-1} + (1 - z^{-1})^{-1} = 1$ in the proof of the gluing formula. To prove Theorem 5.4 we need a somewhat more sophisticated version of this identity.

Consider a $k$-dimensional simplicial cone $C \subset \Lambda^*_R := t^*$, and let $\alpha_j \in \Lambda^* \otimes \mathbb{Z} \mathbb{Q}$ be linearly independent generating vectors for $C$, with the property that the lattice $\Lambda_C^*$ generated by the $\alpha_j$ contains $\Lambda^*$. Let $\Lambda_C$ be the dual lattice, and $\Gamma_C = \Lambda/\Lambda_C$. For all $\gamma \in \Gamma_C$, let $c_{\nu}(\gamma) = e^{\nu(\alpha_j, \gamma)}$, and define the following meromorphic function on the complexified torus $T^C$:

\begin{equation}
f_C(z) = \frac{1}{\#\Gamma_C} \sum_{\gamma \in \Gamma_C} \left( \prod_{j=1}^{k} (1 - c_{\nu}(\gamma)z^{\alpha_j}) \right)^{-1},
\end{equation}

where $z^\nu = \exp(2\pi i (\mu, \xi))$ if $z = \exp(\xi) \in T^C$. Notice that the individual summands on the right hand side are only functions on the covering torus $(\Lambda_R/\Lambda_C)^C$, but the sum is $\Gamma_C$-invariant and therefore descends to $T^C$. If $C$ is a lower dimensional simplicial polytope, $f_C$ is defined similarly, by considering the lattice $(\mathbb{R}^C \cap \Lambda^*)_C$.

**Lemma 5.5.** On the set of all $z \in T^C$ such that $|z^\mu| < 1$ for all $\mu \in C - \{0\}$,

\begin{equation}
f_C(z) = \sum_{\mu \in C} z^\mu.
\end{equation}

**Proof.** By Taylor’s expansion,

\[
f_C(z) = \frac{1}{\#\Gamma_C} \sum_{\nu_j \geq 0} \left( \sum_{\gamma \in \Gamma_C} e^{2\pi i \sum_{j} \nu_j (\alpha_j, \gamma)} \right) z^{\sum j \nu_j \alpha_j}.
\]

But the sum over $\Gamma_C$ equals $\#\Gamma_C$ if $\sum_j \nu_j \alpha_j \in \Lambda^*$, 0 otherwise. \hfill \square

Recall now that a simplicial fan in $\Lambda^*_R$ is a collection $\mathcal{C} = \{ C \}$ of simplicial cones, with the property that for each cone $C$ in $\mathcal{C}$, all faces are also in $\mathcal{C}$, and the intersection of any two cones in $\mathcal{C}$ is a face of each. The fan is called complete if the $C$’s cover all of $\Lambda_R$. A fan $\mathcal{C}'$ is called a refinement of $\mathcal{C}$ if any cone in $\mathcal{C}$ is a union of cones in $\mathcal{C}'$. We define

\begin{equation}
f_{\mathcal{C}}(z) = \sum_{C \in \mathcal{C}} (-1)^{\dim C} f_C(z).
\end{equation}

It follows from Lemma 5.5 that $f_{\mathcal{C}}(z)$ is invariant under (simplicial) refinements.
Lemma 5.6. If \( \mathcal{C} \) is a complete simplicial fan,
\[
(55) \quad \sum_{C \in \mathcal{C}} (-1)^{\dim C} f_C(z) = 0.
\]

Proof. Let \( \beta_1, \ldots, \beta_k \in \Lambda^* \) be a lattice basis, and \( \mathcal{C}_1 \) the simplicial fan whose cones are generated by all subsets of \( \pm \beta_1, \ldots, \pm \beta_k \) not containing both \( \pm \beta_j \), for any \( j \). Let \( \mathcal{C}_2 \) be the fan whose cones are all intersections of cones in \( \mathcal{C}, \mathcal{C}_1 \), and let \( \mathcal{C}_3 \) be a simplicial refinement of \( \mathcal{C}_2 \). Since \( \mathcal{C}_3 \) refines both \( \mathcal{C}, \mathcal{C}_1 \), it follows that \( f_{\mathcal{C}} = f_{\mathcal{C}_3} = f_{\mathcal{C}_1} \). But
\[
f_{\mathcal{C}_1}(z) = (-1)^k \prod_{j=1}^{k} \left( \frac{1}{1 - z^{\beta_j}} + \frac{1}{1 - z^{-\beta_j}} - 1 \right) = 0,
\]
q.e.d.

Proof of Theorem 5.4. Without loss of generality, we may assume \( \mu = 0 \). We will also assume \( G = \{ e \} \) for simplicity, the general case is obtained by working with \( T \times G \)-equivariant characteristic classes.

For each polytope \( R \ni 0 \), let \( C = C_R \) be the cone \( \mathbb{R}_{\geq 0} R \), and let \( \mathcal{C} = \{ C \} \) be the fan obtained in this way. Since the statement of the theorem depends only on local data in a neighborhood of \( \Phi^{-1}(0) \), we can assume without loss of generality that \( \mathcal{R} = \mathcal{C} \). Let us first consider the case \( \dim S = 0 \), which implies that \( S = \{ 0 \} \) is a vertex of all \( C \in \mathcal{C} \). This means in particular that \( 0 \) is a regular value of \( \Phi \), and that \( F = M_0 = \Phi^{-1}(0)/T \). By assumption (A), \( \mathcal{C} \) is a complete simplicial fan in \( t^* \).

We now have to investigate how the local contributions from the fixed point formula add up. Notice that the multiplicity of the fixed point manifold \( F \subset M_C \) depends on \( C \). In fact, the generic isotropy group of \( F \) as a suborbifold of \( M_C \) is a \( \# \Gamma_C \)-fold cover of the isotropy group of \( F \) as identified with \( M_S \), since this is the “twist” of the normal bundle \( N_C \) of \( F \) in \( M_C \). It follows that the associated orbifold \( \tilde{F}_C \) of \( F \) considered as a suborbifold of \( M_C \) is a \( \# \Gamma_C \)-fold cover of \( \tilde{F} := \tilde{F}_S \).

By (44), there is an explicit description of \( N_C \): Suppose the cone \( C \) is generated by the orbi-weights \( \alpha_j \). For each orbi-weight \( \alpha \in \Lambda^* \otimes \mathbb{Q} \), let \( N_\alpha \) be the associated orbifold bundle \( N_\alpha = \Phi^{-1}(0) \times_T C \), where \( T \) acts by the (orbi-) character \( \exp(2\pi i (\alpha, \xi)) \). Then \( N_C = \oplus_\alpha N_{-\alpha} \).

Consider now \( E' = \Phi^{-1}(0) \times_T (t \otimes \mathbb{C}) \) as a \( T \)-equivariant orbifold bundle over \( F \), and let \( F_1(E', \xi) \) be its equivariant curvature. Let \( \tilde{E}' \) be the pullback of \( E' \) to \( \tilde{F} \), and \( A \in C^\infty(\text{Aut}(\tilde{E}')) \) as in section 3. Notice that the equivariant curvature of \( N_\alpha \) is \( F_1(N_\alpha, \xi) = (\alpha, F_1(E', \xi)) \).

By performing the summation over the fibers of \( \tilde{F}_C \to \tilde{F} \), the fixed point contribution of \( F \subset M_C \) becomes
\[
\chi_{C,F}(e^x) = \int_{\tilde{F}} \frac{1}{d\tilde{F}} \frac{Td(\tilde{F}) Ch_i(\tilde{F}; N_\xi)}{D(\tilde{F})} f_C(A^{-1} e^{\sqrt{-1} F_1(E', \xi)}).
\]
The local glueing formula now follows directly from (53).

If \( S \neq \{ 0 \} \), each \( C \) contains \( S \) as a subspace, and the collection of quotient cones \( C/S \) defines a complete rational simplicial fan in \( t^*/S \). The normal bundle \( N_C \) of \( F \) in \( M_C \) splits into the direct sum of its part in \( M_S \) and its symplectic orthogonal, which we denote by \( N_{C/S} \):
\[
N_C = N_S \oplus N_{C/S}.
\]
Hence
\[
\chi_{C,F}(e^t) = \int_{\tilde{F}} \frac{1}{|\det(\tilde{F})|} \left( \frac{1}{D^F(N_F)} \right) f_{C/S}(A^{-1} e^{\pi i F}(E',\xi)),
\]
where now \( E' = \Phi^{-1}(\text{int}(S)) \times_{T_S} (t_S \otimes \mathbb{C}) \). Again, the claim follows from (53).

6. Nonabelian Cutting

In this section, we will prove a generalization of Theorem 5.3 to nonabelian groups. Let \( G \) be a compact connected Lie group, with maximal torus \( T \), and \( G = K A \) its decomposition into its semisimple part, \( K = (G,G) \), and its connected abelian part. By general properties of orbit type decompositions, the stabilizer into semisimple and abelian part. The Lie-algebras are given by \( k \), \( \mathfrak{g}^\infty \) submanifold of \( \mathfrak{g} \) where now \( \mathfrak{g}^\infty \) of \( \Sigma \), with equality if and only if \( \Sigma = \Sigma' \). Let \( G_\Sigma = K_\Sigma A_\Sigma \) be the decomposition into semisimple and abelian part. The Lie-algebras are given by \( \mathfrak{g}_\Sigma = [\mathfrak{g}_\Sigma,\mathfrak{g}_\Sigma] \) and \( a_\Sigma = (\mathfrak{g}_\Sigma)^{G_\Sigma} \), respectively. For all \( \alpha \in \Sigma \),
\[
T_\alpha(\Sigma) = (\mathfrak{g}^\infty)^{G_\Sigma} \cap t^* = a_\Sigma^* = [\mathfrak{g}_\Sigma,\mathfrak{g}_\Sigma]^0 \cap t^*.
\]
Consider now a compact connected Hamiltonian \( G \)-orbifold \( M \), with moment map \( J : M \to \mathfrak{g}^* \). By a theorem of Kirwan \( \Delta = J(M) \cap t^*_\Sigma \) is a compact convex polytope. (To be precise, Kirwan’s theorem only covers the manifold case, the extension to Hamiltonian orbifolds is proved in [28].)

Given a face \( \Sigma \) of \( t^*_\Sigma \), let \( U_\Sigma \) be the open set
\[
U_\Sigma = \{ \alpha \in t^*_\Sigma | G_\alpha \subset G_\Sigma \} = \bigcup_{\Sigma \subset \Sigma'} \text{int}(\Sigma'),
\]
and define
\[
Y_\Sigma = J^{-1}(G_\Sigma U_\Sigma), \quad M_\Sigma = J^{-1}(G U_\Sigma) = G \times_{G_\Sigma} Y_\Sigma.
\]
Let \( \pi_\Sigma : t^* \to a_\Sigma^* \) denote the projection, and \( q : \mathfrak{g}^* \to t^*_\Sigma \) the mapping that sends \( \alpha \) to the unique point of intersection of the coadjoint orbit \( G_\alpha \) with \( t^*_\Sigma \). Notice that the restriction of \( q_\Sigma := \pi_\Sigma \circ q \) to \( G_\Sigma U_\Sigma \) is smooth.

**Theorem 6.1.** (Symplectic cross section theorem.) \( Y_\Sigma \) is a connected symplectic submanifold of \( M \), and \( M_\Sigma \) is a Hamiltonian \( G_\Sigma \)-space, with the restriction of \( J \) serving as a moment map. The action of \( A_\Sigma \subset G_\Sigma \) on \( Y_\Sigma \) extends in a unique way to an action on \( M_\Sigma \) which commutes with the \( G \)-action. Moreover, this action is Hamiltonian, with moment map \( \Phi_\Sigma = q_\Sigma \circ J \).

For a proof of the first part, see [19], p. 194. The second part is obvious since \( \Phi_\Sigma = J \) on \( Y_\Sigma \).

The idea to use the local \( A_\Sigma \)-actions for symplectic cutting of Hamiltonian \( G \)-spaces is due to Chris Woodward \([16]\). Consider a simple polytope \( R \subset t^* \) of the form (58). Suppose that for all faces \( S \) of \( R \), and all \( \Sigma \) such that \( S \cap \Sigma \neq 0 \), the torus \( T_S \) is a subtorus of \( A_\Sigma \). By taking perpendiculars, the condition
\[
S \cap \Sigma \neq 0 \implies T_S \subset A_\Sigma
\]
for all \( S, \Sigma \) is equivalent to the following assumption:

(C) For all faces \( S \) of \( R \) meeting a face \( \Sigma \) of \( t^*_\Sigma \), the tangent space to \( S \) contains the space perpendicular to \( \Sigma \) (i.e. the space \( t^*_\Sigma \cap t^* \)).
For all $\Sigma$, choose a neighborhood $V_\Sigma \subset U_\Sigma$ such that $V_\Sigma \cap R = V_\Sigma \cap \pi_{\Sigma}^{-1}(R_\Sigma)$ where $R_\Sigma = R \cap \alpha_{\Sigma}^\ast$, and define the cut space $M_R$ by gluing the cut spaces with respect to the local $A_\Sigma$-actions, $(J^{-1}(G.V_\Sigma))_{R_\Sigma}$. $M_R$ is a symplectic orbifold if for all $\Sigma$, $R_\Sigma$ is admissible for $J^{-1}(G.V_\Sigma)$, in other words if the following condition is satisfied:

(B’) For all faces $S$ of $R$ and all $x \in J^{-1}(S \cap t^*_+)$, $g_x \cap t_S = \{0\}$.

To summarize, in the nonabelian case we will call a polytope $R \subset t^*$ admissible if it satisfies conditions (A), (B’) and (C), plus the extra condition (Q) if we are in the pre-quantization setting. Each admissible polytope $R$ defines a cut space $M_R$.

The discussion from the abelian case goes through with no essential changes: We have a decomposition into symplectic suborbifolds,

\[ M_R = \bigcup_{S \subset R} (q \circ J)^{-1}(\text{int}(S))/T_S, \]

and for each face $S \subset R$, there is a canonical embedding $M_S \to M_R$ as a symplectic suborbifold of codimension $2(\dim R - \dim S)$. The induced action of $G$ on $M_R$ is Hamiltonian, and the corresponding moment polytope is

\[ \Delta_R = J_R(M_R) \cap t^*_+ = \Delta \cap R. \]

If $E \to M$ is a $G$-equivariant complex vector bundle, one obtains a cut-bundle $E_R \to M_R$ with a canonically induced $G$-action.

**Remark 6.2.** A more geometric description of condition (B’) can be obtained as follows. For all subalgebras $h \subset g$, let $M_{(h)} \subset M$ be the set of points $x$ with infinitesimal stabilizer $g_x$ conjugate to $h$. It is well-known that only finitely many conjugacy classes ($h$) occur as stabilizers, and that each $M_{(h)}$ has a finite number of connected components. For each representative $h$ for ($h$), let $M_h$ be the set of all $x$ with $g_x = h$; then $M_{(h)} = G.M_h$. By equivariance of the moment map,

\[ J(M_h) \subset (g^*)^h = \{ \alpha \in g^* | \text{ad}^*(\xi)\alpha = 0 \}. \]

Let $z = g^b$ be the centralizer of $h$ in $g$, and $Z = \exp(z)$. Then $Z \subset G$ acts on $M_h$ in a Hamiltonian fashion, and by [24] the restriction $J| M_h$ serves as a moment map.

Since every moment map $J$ has the property

\[ \text{im}(T_x J) = g_x^0 \text{ for all } x \in M, \]

this shows that the image under $J$ of each connected component of $M_h$ is an open subset of an affine space of the form $\alpha + (h^0)^b$, where $\alpha \in z^*$. By a suitable choice of $h$, we can assume that $t_1 = z \cap t$ is a Cartan subalgebra of $z$. Then $J(M_h) \cap t^*_+ = J(M_h) \cap t^*_+$, and therefore

\[ J(M_{(h)}) = G.J(M_h) = G.Z.(J(M_h) \cap t^*_+) = G.(J(M_h) \cap t^*). \]

It follows that the image under $\Phi$ of each connected component of $M_{(h)}$ is an open connected subset of $W.(\alpha + (h^0 \cap t^*)) \cap t^*_+$ for some $\alpha \in z^* \cap t^*$. Using this result, condition (B’) can be formulated as follows:

(B) The faces of $R$ intersect all sets $q \circ J(M_{(h)})$ transversally.

Notice that this condition is generically fulfilled.
Theorem 6.3. (Nonabelian gluing formula) Suppose that $\mathcal{R} = \{R\}$ is a finite collection of admissible polytopes such that the $R$'s cover $\Delta$, the faces of each $R \in \mathcal{R}$ are also in $\mathcal{R}$, and for all $R_1, R_2 \in \mathcal{R}$, their intersection is a face of each. For each $G$-equivariant vector bundle $E \to M$, one has the following gluing formula for $G$-equivariant Riemann-Roch numbers:

\[
(-1)^{\dim \Delta} \text{RR}(M, E) = \sum_{R \in \mathcal{R}} (-1)^{\dim R} \text{RR}(M_R, E_R).
\]

Proof. This follows again from a local result: Suppose that $F$ is a $T$-fixed point manifold for some $M_R$, and $\alpha = q \circ J_R(F)$. If $\dim R = \dim T$ and $\alpha \in \text{int}(R)$, then $F$ is a $T$-fixed point manifold for $M$, and $\chi_F = \chi_{R.F}$. Otherwise, let $S \in \mathcal{R}$ be the unique polytope with $\alpha \in \text{int}(S)$. Since $M_S \subset M_R$ if $S \subset R$, if $F$ is $T$-fixed for all $M_R$ with $\alpha \in R$. We have to show that

\[
\sum_{R \ni \alpha} (-1)^{\dim R} \chi_{R,F}(e^\xi) = 0.
\]

The normal bundle $\nu_{R,F}$ of $F$ in $M_R$ is a direct sum of the normal bundle of $F$ in $M_S$, and the restriction to $F$ of the normal bundle of $M_S$ in $M_R$. Hence (65) follows from the abelian result, Theorem 5.4.

We will now explicitly describe a decomposition $\mathcal{R}$ of $t^*$ into admissible polytopes, which we will use in the following section. Assume first that $G$ is semisimple. Let $\mathfrak{S}_+ = \{\beta_1, \ldots, \beta_k\} \subset \mathfrak{R}_+$ be the set of simple positive roots, and $\alpha_1, \ldots, \alpha_k \in \Lambda^* \otimes \mathbb{Z} \mathbb{Q}$ generating vectors for the edges of $t^*_+$ such that $\beta_i$ is perpendicular to the facet spanned by the $\alpha_j$, $j \neq i$. Let $\mathfrak{C}$ be the complete simplicial fan in $t^*_+$ generated by $\alpha_1, \ldots, \alpha_k, -\beta_1, \ldots, -\beta_k$. The cones in this fan are generated by all subsets which do not contain both $\alpha_i$ and $-\beta_i$, for any $i$. For generic choices $\gamma \in \text{int}(t^*_+) \cap \Lambda^* \otimes \mathbb{Z} \mathbb{Q}$, the polytopes $R_C = \gamma + C$ are admissible. Notice that for the polytope $R_0$ that contains $0$, the intersection $R_0 \cap t^*_+$ is equal to the intersection of the convex hull of $W.\gamma$ with $t^*_+$. The following picture shows the decomposition for the Weyl chamber of $G = SU(3)$, the intersection $R_0 \cap t^*_+$ is shaded.

If $G$ is a general compact connected group, let $G = K A$ be the decomposition into semisimple and abelian part. We can apply the above construction to the semisimple part, to obtain a decomposition $\mathcal{R}^{(1)}$ of $t^* \cap t^*$ into polytopes. Now take any decomposition $\mathcal{R}^{(2)}$ of $a^*$ into rational polytopes satisfying assumption (A), and let $\mathcal{R}$ be the set of all products $R^{(1)} \times R^{(2)}$, $R^{(i)} \in \mathcal{R}^{(i)}$. Again, these polytopes will be admissible for generic choices $\gamma \in t^* \cap \text{int}(t^*_+)$. The important point about this decomposition is that for all polytopes $R = R^{(1)} \times R^{(2)}$ with $0 \not\in R^{(1)}$, there exists a face $\Sigma \neq a^*$ of $t^*_+$ such that the intersection
In this situation, Theorem 3.6 shows that the computation of \( RR(M_R, E_R) \) reduces to that of \( RR(Y_{R, \Sigma}, E_R|Y_{R, \Sigma}) \).

7. Quantization

Up to this point, we have been dealing with arbitrary \( G \)-equivariant vector bundles \( E \). Let us now focus on the special case that \( E = L \to M \) is a pre-quantum line bundle, as explained in the introduction. Let \( \chi = RR(M, L) \in R(G) \) denote the equivariant Riemann-Roch number, and \( N : \Lambda^*_+ \to \mathbb{Z} \) the multiplicity function. In this section, we will prove the Guillemin-Sternberg conjecture, Theorem [1.1]. We will use that Theorem 1.1 is already proved in the abelian case (see the remarks at the end of section [1]). Using induction on \( \dim G \), we can also assume that Theorem [1.1] (hence also Corollary [1.2]) holds for all proper subgroups of \( G \).

Choose a decomposition \( R = R^{(1)} \times R^{(2)} \) as described at the end of the previous section, in such a way that 0 is contained in the interior of a unique polytope \( R_0 = R^{(1)}_0 \times R^{(2)}_0 \). We claim that for any polytope \( R = R^{(1)} \times R^{(2)} \) with \( 0 \not\in R \), we have \( RR(M_R, L_R)^G = 0 \). Indeed, if \( 0 \not\in R^{(2)} \) this follows from the abelian result since already \( RR(M_R, L_R)^A = 0 \). If \( 0 \not\in R^{(1)} \), we have a global symplectic cross-section \( Y_{R, \Sigma} \), and the result follows from Theorem 3.6 because \( RR(Y_{R, \Sigma}, E_R|Y_{R, \Sigma})^{G_\Sigma} = 0 \) since \( \Phi(Y_{R, \Sigma}) \not\in 0 \).

We may choose \( R_0 \cap t^*_+ \) arbitrarily small, this proves the following excision property for \( RR(M, L)^G \):

**Proposition 7.1.** If \( R \subset t^* \) is any admissible polytope containing 0, \( RR(M, L)^G = RR(M_R, L_R)^G \).

In particular, \( RR(M, L)^G \) depends only upon local data near \( J^{-1}(0) \). Let us now suppose that \( 0 \in J(M) \) is a regular value of \( J \), and denote \( P = J^{-1}(0) \). Then \( \pi : P \to M_{\text{red}} \) is an orbifold-principal \( G \)-bundle. We will use the normal form theorem of Gotay to describe a neighborhood of \( P \) in \( M \). Choose a connection \( \theta \in \mathcal{A}^1(P, g) \) on \( P \), and equip the product \( P \times g^* \) with the closed two-form

\[
\sigma = \pi^* \omega + d(pr_2, \theta),
\]

where \( pr_2 : P \times g^* \to g^* \) denotes projection to the second factor. The diagonal action of \( G \) makes \( P \times g^* \) into a Hamiltonian \( G \)-space, with moment map equal to \( \pi \).

**Theorem 7.2.** (Normal form theorem [14] ) There exists a \( G \)-equivariant symplectomorphism from a neighborhood of \( \bar{P} \) in \( P \times g^* \) to a neighborhood of \( P \) in \( M \).

A well-known consequence of the normal form theorem is the following description of reduced spaces, for \( \alpha \) close to 0. Let \( F^\theta = d\theta \in \mathcal{A}^2(P, g) \) be the curvature of \( \theta \). Let \( O_\alpha \) be the coadjoint orbit through \( \alpha \), equipped with its canonical symplectic structure \( \sigma_\alpha \). The \( G \)-action on \( O_\alpha \) is Hamiltonian, with moment map the embedding \( J_\alpha : O_\alpha \to g^* \). By the shifting trick, the reduced space \( M_\alpha = J^{-1}(\alpha)/G_\alpha \) is symplectomorphic to the reduced space at zero of \( M \times O_\alpha^{-} \), where the superscript "-" indicates that one takes the opposite symplectic form. By doing this calculation in the canonical model, one finds:
Corollary 7.3. There is a neighborhood $U \ni 0$ in the set of regular values of $J$, such that for all $\alpha \in U$, the reduced space $M_\alpha$ is symplectomorphic to the symplectic fiber bundle

$$M_\alpha = P \times_G O_\alpha \xrightarrow{\phi_\alpha} M_0,$$

with symplectic form given by the minimal coupling recipe of Sternberg fiber bundle. Then there is a neighborhood $\Delta_{\alpha} \ni (\alpha,\beta) = 0$.

(67) $\pi^*_\alpha \omega_\alpha = \pi^*_0 \omega_0 + d(J_\alpha, \theta) + \sigma_\alpha$.

Here $\pi_\alpha : P \times_G O_\alpha \to M_\alpha$ denotes the projection.

Similarly, one can express the associated orbifold $\hat{M}_\alpha$ as a symplectic fiber bundle

(68) $\hat{M}_\alpha \cong \hat{P} \times_G O_\alpha \xrightarrow{\hat{\phi}_\alpha} \hat{M}_0$,

where $\hat{P}$ was defined in (20). Notice that $N_{\hat{M}_\alpha} = \hat{\phi}_\alpha^* N_{\hat{M}_0}$, and hence

(69) $D^{\hat{M}_\alpha}(N_{\hat{M}_\alpha}) = \hat{\phi}_\alpha^* D^{\hat{M}_0}(N_{\hat{M}_0})$.

Proof of Theorem 1.1: Choose $R_0 = R_0^{(1)} \times R_0^{(2)}$ as above, in such a way that $R_0 \cap t_0^*$ is contained in the neighborhood $U$ from Corollary 7.3. The moment polytope for $M_{R_0}$ is then simply $\Delta_{R_0} = t_0^* \cap R_0$, and the set of regular values of $J_{R_0}$ is $\text{int}(R_0) \cap t_0^*$. Notice that 0 is a regular value for the action of $A \subset G$ on $M_{R_0}$, since $t_0^* \oplus \{0\}$ is transversal to all faces of $R_0$. From the result for the abelian case, we have

$$\text{RR}(M, L)^G = \text{RR}(M_{R_0}, L_{R_0})^G = \text{RR}((M_{R_0})_{A}, (L_{R_0})_{A})^K,$$

where the subscript $A$ denotes the reduced space with respect to the $A$-moment map. Using reduction in stages, it is therefore sufficient to prove Theorem 1.1 for the semisimple case.

Let us assume for the rest of this proof that $G$ is semisimple. Let \{ $\beta \mid \beta \in \mathfrak{g}_+$ \} $ \subset A \otimes \mathbb{Q}$ be the dual basis to $\mathfrak{g}_+$. By definition, $R_0$ is given by inequalities

(70) $\langle \alpha - \gamma, \beta \rangle \leq 0$ for all $\beta \in \mathfrak{g}_+$.

Choose $k \in \mathbb{N}$ such that for all $\beta \in \mathfrak{g}_+$, $v_\beta := -k \beta \in A$ and $\mu_\beta := -k(\gamma, \beta) \in \mathbb{N}$. Thus $R_0$ is given by

(71) $\langle \alpha, v_\beta \rangle \geq \mu_\beta$ for all $\beta \in \mathfrak{g}_+$.

To compute $N(0)$, we may replace $L \to M$ by the cut bundle $L_{M_{R_0}} \to M_{R_0}$, which we continue to denote by $L \to M$. The components of the $T$-fixed point set for $M$ are then simply the preimages

(72) $J^{-1}(w, \gamma) \cong M_w, \gamma \cong M_\gamma \cong P/T$,

for $w \in W$. We can write $\chi(e^\xi)$ as a sum over fixed point contributions $\chi_\gamma$ of $M_\gamma$ considered as a fixed point manifold in the symplectic cross-section $J^{-1}(\text{int}(t_0^*))$:

(73) $\chi(e^\xi) = \sum_{w \in W} \det(w) e^{2\pi i (w(\delta - \Delta, \xi))} \chi_\gamma(w^{-1}(\xi)) / \prod_{\beta \in \pi_0}(1 - e^{-2\pi i (\beta, \xi)})$.

The fixed point formula for $\chi_\gamma$ involves the normal bundle $\nu_\gamma$ of $M_\gamma$ in $J^{-1}(\text{int}(t_0^*))$. By (13), $\nu_\gamma = \oplus N_{-\alpha, \beta}$, where the orbiformal weights $\alpha_\beta := -\frac{1}{\beta} \beta$ form the dual basis to
\( \nu_{\beta} \). The equivariant Chern class of \( N_{-\alpha_{\beta}} \) is simply \( c_{\beta}(\xi) = -\langle \alpha_{\beta}, F^{\theta} + 2\pi i \xi \rangle \). We have

(74)

\[
\chi_{\gamma}(w^{-1}(\xi)) = \int_{\tilde{M}_\gamma} \frac{1}{d_{\tilde{M}_\gamma}} \text{Td}(\tilde{M}_\gamma) \frac{\text{Ch}_{\tilde{M}_\gamma}(\tilde{L}_{w(\gamma)}, \xi)}{D_{\tilde{M}_\gamma}(N_{\tilde{M}_\gamma})} \prod_{\beta \in \Phi^+} D_{\tilde{M}_\gamma}(N_{-w(\alpha_{\beta})}, \xi)^{-1}.
\]

Let us consider (74) as an equality of meromorphic functions on \( t \otimes \mathbb{C} \). For all \( w(\alpha_{\beta}) \), we can expand

(75)

\[
D_{\tilde{M}_\gamma}(N_{-w(\alpha_{\beta})}, \xi)^{-1} = (1 - e^{2\pi i w(\alpha_{\beta}), \xi}) \text{Ch}_{\tilde{M}_\gamma}(N_{w(\alpha_{\beta}))}^{-1}
\]

into a geometric series with respect to \( e^{2\pi i w(\alpha_{\beta}), \xi} \). Of course, this will only converge if \( \langle w(\alpha_{\beta}), \Im(\xi) \rangle > 0 \). Since we want to expand all factors in (74) simultaneously we polarize the weights: Let

(76)

\[
l_{\beta}^w = \begin{cases} 0 & : \langle w(\alpha_{\beta}), \eta \rangle > 0 \\ 1 & : \langle w(\alpha_{\beta}), \eta \rangle < 0 \end{cases} \quad \text{for all } \eta \in \text{int}(t_+),
\]

where \( t_+ \) is the positive Weyl chamber in \( t \), and write \( \alpha_{\beta}^w = (-1)^{l_{\beta}^w} w(\alpha_{\beta}) \), and \( \epsilon_w = (-1)^{\sum l_{\beta}^w} \). Then we may rewrite the denominator of the second term in (74), for \( \Im(\xi) \in \text{int}(t_+) \), as

\[
D_{\tilde{M}_\gamma}(N_{-w(\alpha_{\beta})}, \xi)^{-1} = (-1)^{l_{\beta}^w} D_{\tilde{M}_\gamma}(N_{-\alpha_{\beta}^w}, \xi)^{-1} \text{Ch}_{\tilde{M}_\gamma}(N_{\alpha_{\beta}^w}, \xi)
\]

\[
= (-1)^{l_{\beta}^w} \sum_{l_{\beta} \geq 0} \text{Ch}_{\tilde{M}_\gamma}(N_{(l_{\beta}+l_{\beta}^w) \alpha_{\beta}^w}, \xi).
\]

Notice also that \( L_\gamma \equiv \phi^* L_0 \otimes N_\gamma \) by the pre-quantum condition and (57), thus \( \text{Ch}_{\tilde{M}_\gamma}(L_\gamma, \xi) = \phi^*_\gamma \text{Ch}_{\tilde{M}_\gamma}(L_0) \text{Ch}_{\tilde{M}_\gamma}(N_\gamma, \xi) \). We hence obtain the formula

\[
\chi(e^\xi) \prod_{\beta \in \Phi^+} (1 - e^{-2\pi i \langle \beta, \xi \rangle}) = \sum_{w \in W} \sum_{l_{\beta} \geq 0} \epsilon_w \det(w) \times
\]

\[
\times e^{2\pi i w(\delta - \delta_{-\delta, \xi})} \int_{\tilde{M}_\gamma} \frac{1}{d_{\tilde{M}_\gamma}} \text{Td}(\tilde{M}_\gamma) \phi^*_\gamma \text{Ch}_{\tilde{M}_\gamma}(L_0) \text{Ch}_{\tilde{M}_\gamma}(N_{w(\gamma)+\sum(l_{\beta}+l_{\beta}^w) \alpha_{\beta}^w}, \xi).
\]

On the other hand, Weyl's character formula

(77)

\[
\chi(e^\xi) \prod_{\beta \in \Phi^+} (1 - e^{-2\pi i \langle \beta, \xi \rangle}) = \sum_{\mu \in \Lambda^+_+} N(\mu) \sum_{w \in W} e^{2\pi i w(\delta + \mu - \delta_{-\delta, \xi})}.
\]

shows that \( N(\mu) \) is the coefficient of \( e^{2\pi i \langle \mu, \xi \rangle} \) in \( \chi(e^\xi) \prod(1 - e^{-2\pi i \langle \beta, \xi \rangle}) \) ( because, for all \( \nu \in \Lambda^+_+ \), \( \nu(\delta + \nu) - \delta_{-\delta, \xi} \) if and only \( w = 1 \)). To find \( N(0) \), we thus have to solve the equation

(78)

\[
w(\delta + \gamma) - \delta = -\sum(l_{\beta} + l_{\beta}^w) \alpha_{\beta}, \quad l_{\beta} \geq 0.
\]

Let us apply \( w^{-1} \) to both sides, and take the scalar product with \( \beta^\xi \), using that \( \alpha_{\beta} = -\frac{1}{k} \beta; \)

(79)

\[
\langle \delta + \gamma - w^{-1}(\delta), \beta^\xi \rangle = (-1)^{l_{\beta}^w} (l_{\beta} + l_{\beta}^w) k^{-1}, \quad l_{\beta} \geq 0
\]

The left hand side of (79) is strictly positive, since \( \delta - w^{-1}(\delta) \) is a sum of positive roots. If \( w \neq 1 \), there is no solution of (78), because at least one \( l_{\beta}^w = 1 \), which
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makes the right hand side negative. If \( w = 1 \), we have \( l_\beta^w = 0 \) for all \( \beta \in \mathcal{S}_+ \), and therefore \( l_\beta = k(\gamma, \beta^2) \). But then

\[
\sum_{\beta \in \mathcal{S}_+} l_\beta \alpha_\beta = \sum_{\beta \in \mathcal{S}_+} \langle \gamma, \beta^2 \rangle \beta = \gamma.
\]

We have thus shown:

\[
N(0) = \int_{\tilde{M}_0} \frac{1}{\phi^* D_{\tilde{M}_0}(N_{\tilde{M}_0})} \text{Td} (\tilde{M}_0) \hat{\phi}_{\gamma}^* \text{Ch} (\tilde{M}_0). \tag{80}
\]

Let us finally integrate over the fibers of \( \tilde{\phi}_{\gamma} \), using that \( \int_O \text{Td} (O) = 1 \) for every coadjoint orbit \( O \):

\[
N(0) = \int_{\tilde{M}_0} \frac{1}{D_{\tilde{M}_0}(N_{\tilde{M}_0})} \text{Td} (\tilde{M}_0) \text{Ch} (\tilde{M}_0) = \text{RR}(M_0, L_0),
\]

q.e.d.

8. Appendix: A Short Proof for \( G = SU(2) \)

In this section, we will give a short proof of Theorem 1.1 for the case \( G = SU(2) \), modeled after the proof for \( G = S^1 \) in [10]. The main idea will be to construct a Hamiltonian \( S^1 \)-space, which has the same multiplicities and the same reduced spaces. Let \((M, \omega)\) be a quantizable Hamiltonian \( G \)-space, and suppose that 0 is a regular value of the moment map \( J \).

For simplicity, we will assume that \( M \) is a manifold, and that the action on \( J^{-1}(0) \) is free, although the proof is easily adaptable to the orbifold case. Let \( T = S^1 \) be the maximal torus of \( SU(2) \). The dominant weights of \( G \) are labeled by nonnegative integers, \( \Lambda^+ = \mathbb{Z}_{\geq 0} \), and the positive root is equal to 2. By Weyl’s character formula,

\[
\chi(e^{i\phi}) = \sum_{\mu \in \mathbb{Z}_{\geq 0}} N(\mu) \left( \frac{e^{i\mu\phi}}{1 - e^{-2i\phi}} + \frac{e^{-i\mu\phi}}{1 - e^{2i\phi}} \right), \tag{82}
\]

This shows that the restriction of \( \chi \) to \( S^1 \) extends to a rational function on the Riemann sphere, \( \mathbb{C} \cup \{\infty\} \), and

\[
N(0) = \text{res}_{z=\infty} \left( \frac{(1 - z^{-2})\chi(z)}{z^{-1}} \right). \tag{83}
\]

On the other hand, we can use the equivariant index theorem [26] to express \( \chi|_{S^1} \) as a sum over fixed point contributions, \( \chi|_{S^1} = \sum_F \chi_F(z) \). By analytic continuation, this becomes an equality of rational functions on \( \mathbb{C} \cup \{\infty\} \).

Since 0 is a regular value of \( J \), all \( S^1 \)-fixed point components \( F \) will have \( J_F \neq 0 \). By the symplectic cross section theorem, \( Y_+ = J^{-1}(\mathbb{R}_{>0}) \) is a symplectic submanifold of \( M \), equipped with a Hamiltonian action of \( T = S^1 \) whose moment map is simply the restriction of \( J \). The restriction of \( L \) to \( Y_+ \) is a pre-quantum line bundle. For all \( J_F > 0 \), we can view \( F \subset Y_+ \) as a fixed point manifold for the \( S^1 \) action on \( Y_+ \). Let \( \chi_{+ , F}(e^{i\phi}) \) be the corresponding fixed point contribution. Then

\[
\chi_F(z) = \frac{\chi_{+, F}(z)}{1 - z^{-2}}.
\]
Note that the Weyl group \( W = \{ e, g \} \cong \mathbb{Z}_2 \) of \( G \) acts effectively on the set of connected components of \( M^F \), and \( \chi_{g,F}(z) = \chi_F(z^{-1}) \). Hence

\[
\chi(z) = \sum_{J_F > 0} \left( \frac{\chi_{+,F}(z)}{1 - z^{-2}} + \frac{\chi_{+,F}(z^{-1})}{1 - z^2} \right)
\]

or

\[
(1 - z^{-2}) \chi(z) = \sum_{J_F > 0} \left( \chi_{+,F}(z) - z^{-2} \chi_{+,F}(z^{-1}) \right).
\]

Since \( \chi_{+,F}(z^{-1}) = O(z^{-J_F}) \) for \( z \to \infty \), it follows that only the first term will contribute to the residue at \( z = \infty \), and we obtain the formula

\[
N(0) = \sum_{J_F > 0} \text{res}_{z=\infty} \frac{\chi_{+,F}(z)}{z-1}.
\]

We will now use the following trick, which is due to Eugene Lerman and may be regarded as the \( SU(2) \) version of symplectic cutting. Let \( \mathbb{C}^2 \) be equipped with its standard symplectic structure and the standard action of \( U(2) \). Let \( \phi : M \times (\mathbb{C}^2)^{-} \to \mathfrak{su}(2)^* \) be the moment map for the diagonal \( SU(2) \subset U(2) \) action, and define \( M_+ \) to be the reduced space, \( M_+ = \phi^{-1}(0)/SU(2) \). It is easy to check the following properties of \( M_+ \):

a. As a set, \( M_+ \) is equal to the disjoint union \( M_0 \cup Y_+ \), and the embeddings of \( M_0 \) and \( Y_+ \) are symplectic.

b. The normal bundle of \( M_0 \) in \( M_+ \) is isomorphic to the associated bundle

\[
\nu = J^{-1}(0) \times_{SU(2)} \mathbb{C}^2.
\]

c. From the action of the center \( U(1) \subset U(2) \) on \( \mathbb{C}^2 \) we get an induced Hamiltonian \( U(1) \) action on \( M_+ \), which fixes \( M_0 \) and is equal to the action of the maximal torus \( T \subset SU(2) \) on \( Y_+ \). The weights for the \( U(1) \) action on \( \nu \) are \((-1, -1)\).

d. Let \( L_{\mathbb{C}^2} \) be the trivial line bundle over \( \mathbb{C}^2 \), with fiber metric \( \exp(-\pi |z|^2) \). Then \( L_+ = (L \boxtimes L_{\mathbb{C}^2}^{-}|\phi^{-1}(0))/SU(2) \) is a pre-quantum line bundle for \( M_+ \). There is a natural lift of the \( U(1) \) action on \( M_+ \) to \( L_+ \).

Let \( \chi_+ = \text{RR}(M_+, L_+) \) be the equivariant Riemann-Roch number of \( M_+ \) with respect to this \( U(1) \) action, and \( N_+(\mu) \) the multiplicity function. By the fixed point formula for \( \chi_+ \), we have

\[
\chi_+(z) = \sum_{J_F > 0} \chi_{+,F}(z) + \int_{M_0} \frac{\text{Td}(M_0) \text{Ch}(L_0)}{\det(1 - z e^{\frac{\pi}{2} F(\nu)})}.
\]

The second term is \( O(z^{-2}) \) for \( z \to \infty \) since \( \dim_{\mathbb{C}} \nu = 2 \). Hence

\[
N_+(0) = \text{res}_{z=\infty} \frac{\chi_+(z)}{z-1} = \sum_{J_F > 0} \text{res}_{z=\infty} \frac{\chi_{+,F}(z)}{z-1} = N(0).
\]

On the other hand, we see from (85) that \( \chi_+(z) \) is holomorphic for \( z \to 0 \), and

\[
N_+(0) = \chi_+(0) = \int_{M_0} \text{Td}(M_0) \text{Ch}(L_0) = \text{RR}(M_0, L_0),
\]

q.e.d.
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