Comparison theorem for neutral stochastic functional differential equations driven by $G$-Brownian motion

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Abstract

In this paper, we investigate sufficient and necessary conditions for the comparison theorem of neutral stochastic functional differential equations driven by $G$-Brownian motion ($G$-NSFDE). Moreover, the results extend the ones in the linear expectation case [1] and nonlinear expectation framework [8].

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1 Introduction

Since the order preservation has been introduced for studying complex stochastic models by compared with a simpler ones, this property also called as “comparison theorem” in the literature. Two types of comparison theorems have been extensively investigated: the distribution (weak) sense and the pathwise (strong) sense respectively, such as Chen-Wang [3] for multidimensional diffusion processes, Wang [16] for superprocesses in the distribution (weak) sense; Ikeda-Watanabe [10] for stochastic differential equations, Yang-Mao-Yuan [17] for one-dimensional stochastic hybrid systems in the pathwise (strong) sense. Moreover, the comparison theorem in the pathwise (strong) sense implies the
distribution (weak) sense. We refer readers to see [7, 10, 13, 17], and the references therein.

Recently, there are some results for comparison theorem on stochastic differential delay equations, such as, Bao-Yuan [2] established a comparison theorem for stochastic differential delay equations with jumps, Bai-Jiang [1] proved the comparison theorem for stochastic functional differential equations whose drift term satisfies the quasimonotone condition and diffusion term is independent of delay. Moreover, Huang-Yuan [9] provided sufficient and necessary conditions such that the order preservation holds for distribution-dependent neutral stochastic functional differential equations.

Furthermore, under the framework of nonlinear expectation, some sufficient condition is presented in [11, Theorem 7.1] for comparison theorem of one-dimensional stochastic differential equations driven by $G$-Brownian motion ($G$-SDEs). In addition, Luo-Wang [12, 13] prove a comparison theorem for multidimensional $G$-SDEs. There are some results for comparison theorem of stochastic functional differential equations driven by $G$-Brownian ($G$-SFDEs); see [8], where the representation of the $G$-expectation (2.3) introduced in [4, 6] plays important role in the proof of necessary condition of the comparison theorem.

Motivated by the above significant works, we shall investigate the order preservation for neutral stochastic functional differential equations driven by $G$-Brownian ($G$-NSFDEs). To achieve this, thanks the work completed by He-Han [5], the authors studied existence and stability of solutions to $G$-NSFDEs.

The paper is organized as follows. In Section 2, we recall some preliminaries on $G$-Brownian motion and its related stochastic calculus. In Section 3, we prove the comparison theorem for the $G$-NSFDEs.

2 $G$-Expectation and $G$-Brownian motion

For a matrix $A$, let $A^*$ be its transpose and $|A| = \sqrt{\text{trace}(AA^*)}$. Let $\mathcal{M}_m$ be the collection of all $m \times m$ matrices and $\mathcal{S}_m$ ($\mathcal{S}_m^+$) be the set of the symmetric (symmetric and positive definite) ones in $\mathcal{M}_m$. For $M, \bar{M} \in \mathcal{M}_m$, define $\langle M, \bar{M} \rangle = \sum_{k,l=1}^m M_{kl}\bar{M}_{kl}$. For $X, \bar{X} \in \mathcal{S}_m$, the notation $X \geq \bar{X}$ (resp., $X > \bar{X}$) means that $X - \bar{X}$ is non-negative (resp., positive) definite. Fix two positive constants $\sigma$ and $\bar{\sigma}$ with $\sigma < \bar{\sigma}$, define

$$G(X) = \frac{1}{2} \sup_{\gamma \in \mathcal{S}_m^+ \cap [\sigma^2I_{m \times m}, \bar{\sigma}^2I_{m \times m}]} \langle \gamma, X \rangle, \quad X \in \mathcal{S}_m. \quad (2.1)$$

From the definition of the function $G$, it has the following properties ([14, Chapter 2]):

(a) (Positive homogeneity) $G(\lambda X) = \lambda G(X)$, $\lambda \geq 0, X \in \mathcal{S}_m$.

(b) (Sub-additivity) $G(X + \bar{X}) \leq G(X) + G(\bar{X})$, $G(X) - G(\bar{X}) \leq G(X - \bar{X})$, $X, \bar{X} \in \mathcal{S}_m$. 

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Remark 2.1. (b) and (c) imply that $G$ defined by (2.1) is continuous under $| \cdot |$.

For convenience, we list some notations and spaces appearing in this paper as follows.

- $\Omega = C_0([0, \infty); \mathbb{R}^m)$, the $\mathbb{R}^m$-valued and continuous functions on $[0, \infty)$ vanishing at zero, which is endowed with the metric

$$
\rho(\omega^1, \omega^2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \left( \max_{t \in [0,n]} |\omega^1_t - \omega^2_t| \wedge 1 \right), \omega^1, \omega^2 \in \Omega.
$$

- $C_{b,\text{lip}}(\mathbb{R}^m)$ is the set of bounded and Lipschitz continuous functions on $\mathbb{R}^m$.

- $L_{ip}(\Omega_T) = \{ \varphi(\omega_{t_1}, \ldots, \omega_{t_n}), n \in \mathbb{N}^+, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,\text{lip}}((\mathbb{R}^m)^n) \}$.

- $L_{ip}(\Omega) = \bigcup_{T > 0} L_{ip}(\Omega_T)$.

- $\mathbb{E}^G$ is the nonlinear expectation on $\Omega$ such that coordinate process $(B(t))_{t \geq 0}$, i.e. $B(t)(\omega) = \omega_t, \omega \in \Omega$ is an $m$-dimensional $G$-Brownian motion.

- $\langle B \rangle$ is the quadratic variation process of $B$.

- $L^p_{G}(\Omega)$ is the completion of $L_{ip}(\Omega)$ under the norm $(\mathbb{E}^G| \cdot |^p)^{\frac{1}{p}}, p \geq 1$.

- $M_{G}^{p,0}([0, T]) = \left\{ \sum_{j=0}^{N-1} \eta_j I_{[t_j, t_{j+1})}(t); \eta_j \in L^p_{G}(\Omega_{t_j}), 0 = t_0 < t_1 < \cdots < t_N = T \right\}$.

- $M_{G}([0, T])$ is the completion of $M_{G}^{p,0}([0, T])$ under $\| \xi \|_{M_{G}^{p,0}([0, T])} = \left( \mathbb{E}^G \int_0^T |\xi_t|^p dt \right)^{\frac{1}{p}}$.

- $\mathcal{M}$ is the collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$.

For the construction of nonlinear expectation $\mathbb{E}^G$, one can refer to [14] for more details. According to [1, 6], there exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}$ such that

$$
(2.2) \quad \mathbb{E}^G[X] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_\mathbb{P}[X], \ X \in L^1_{G}(\Omega),
$$

where $\mathbb{E}_\mathbb{P}$ is the linear expectation under probability measure $\mathbb{P} \in \mathcal{P}$. $\mathcal{P}$ is called a set that represents $\mathbb{E}^G$. To see (2.2) more clear, let $W^0$ be an $m$-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and define

$$
\mathbb{H} := \{ \theta : \theta \text{ is an } \mathbb{M}^m\text{-valued progressively measurable}
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$$
\mathbb{H} := \{ \theta : \theta \text{ is an } \mathbb{M}^m\text{-valued progressively measurable}
$$
stochastic process, \( \theta_s \theta_s^* \in [\sigma^2 I_{m \times m}, \bar{\sigma}^2 I_{m \times m}], \ s \geq 0 \).

For any \( \theta \in \mathbb{H} \), let \( \mathbb{P}_\theta \) be the law of \( \int_0^t \theta_s dW_s^0 \). By \([14] \), taking \( \mathcal{P} = \{\mathbb{P}_\theta, \theta \in \mathbb{H}\} \), then

\[
E^G[X] = \sup_{\theta \in \mathbb{H}} \mathbb{E}_{\mathbb{P}_\theta}[X], \ X \in L^1_G(\Omega).
\]

Define \( C(A) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(A) \), \( A \in \mathcal{B}(\Omega) \) the associated Choquet capacity to \( \mathbb{E}^G \). We call a set \( A \in \mathcal{B}(\Omega) \) is polar if \( C(A) = 0 \), and a property holds \( C \)-quasi-surely (\( C \)-q.s.) if it holds outside a polar set. Moreover, for the quadratic variation process \( \langle B \rangle \), by \([14] \) Corollary 5.7 and the property (d) of the function \( G \), we have \( C \)-q.s.

\[
\sigma^2 I_{m \times m} \leq \frac{d}{dt} \langle B \rangle(t) \leq \bar{\sigma}^2 I_{m \times m}.
\]

### 3 G-NSDE

For a fixed constant \( r_0 \geq 0 \), let \( \mathcal{C} = C([-r_0, 0]; \mathbb{R}^d) \), which equipped with uniform norm \( \| \cdot \|_\infty \). For any continuous map \( f : [-r_0, \infty) \to \mathbb{R}^d \) and \( t \geq 0 \), let \( f_t \in \mathcal{C} \) be such that \( f_t(s) = f(s + t) \) for \( s \in [-r_0, 0] \). Generally, \( (f_t)_{t \geq 0} \) is called the segment of \( (f(t))_{t \geq -r_0} \). Let \( x = (x^1, \cdots, x^d), \ y = (y^1, \cdots, y^d) \in \mathbb{R}^d \), we call \( x \leq y \) if \( x^i \leq y^i \) holds for all \( 1 \leq i \leq d \). Now we introduce the partial-order on \( \mathcal{C} \). Let \( \xi = (\xi^1, \cdots, \xi^d), \ \eta = (\eta^1, \cdots, \eta^d) \in \mathcal{C} \).

- We call \( \xi \leq \eta \) if \( \xi^i(s) \leq \eta^i(s) \) holds for all \( s \in [-r_0, 0] \) and \( 1 \leq i \leq d \).
- We call \( \xi \leq_N \eta \) if \( \xi \leq \eta \) and \( \xi(0) - N(\xi) \leq \eta(0) - N(\eta) \).
- For any \( t \in [-r_0, 0] \), define \( (\xi \wedge \eta)^i(t) = \min\{\xi^i(t), \eta^i(t)\}, \ 1 \leq i \leq d \), then \( \xi \wedge \eta \in \mathcal{C} \).

Consider the following G-NSFDE:

\[
\begin{cases}
d[Y(t) - N(Y_t)] = b(t, Y_t) dt + \langle h(t, Y_t), d\langle B \rangle(t) \rangle + \sigma(t, Y_t) dB(t), \\
d[\bar{Y}(t) - N(\bar{Y}_t)] = \bar{b}(t, \bar{Y}_t) dt + \langle \bar{h}(t, \bar{Y}_t), d\langle B \rangle(t) \rangle + \bar{\sigma}(t, \bar{Y}_t) dB(t),
\end{cases}
\]

where \( N : \mathcal{C} \to \mathbb{R}^d \) is called neutral term, \( h = (h^{ij})_{1 \leq i, j \leq m} \) with \( h^{ij} = h^{ji}, \ \bar{h} = (\bar{h}^{ij})_{1 \leq i, j \leq m} \) with \( \bar{h}^{ij} = \bar{h}^{ji} \), and \( b, \bar{b}, N : [0, \infty) \times \mathcal{C} \to \mathbb{R}^d; h^{ij}, \bar{h}^{ij} : [0, \infty) \times \mathcal{C} \to \mathbb{R}^d; \sigma, \bar{\sigma} : [0, \infty) \times \mathcal{C} \to \mathbb{R}^d \otimes \mathbb{R}^m \) are measurable.

**Definition 3.1.** For any \( s \geq 0 \) and \( \xi, \bar{\xi} \in \mathcal{C} \), we call \((Y(t), \bar{Y}(t))_{t \geq s}\) solve \((3.1)\) if for all \( t \geq s \),

\[
\begin{align*}
Y(t) - N(Y_t) &= \xi(0) - N(\xi) + \int_s^t b(r, Y_r) dr + \int_s^t \langle h(r, Y_r), d\langle B \rangle(r) \rangle + \int_s^t \sigma(r, Y_r) dB(r), \\
\bar{Y}(t) - N(\bar{Y}_t) &= \bar{\xi}(0) - N(\bar{\xi}) + \int_s^t \bar{b}(r, \bar{Y}_r) dr + \int_s^t \langle \bar{h}(r, \bar{Y}_r), d\langle B \rangle(r) \rangle + \int_s^t \bar{\sigma}(r, \bar{Y}_r) dB(r),
\end{align*}
\]

where \((Y_t, \bar{Y}_t)_{t \geq s}\) is the segment process of \((Y(t), \bar{Y}(t))_{t \geq s-r_0}\) with \((Y_s, \bar{Y}_s) = (\xi, \bar{\xi})\).
Throughout the paper, we make the following assumptions.

(H1) There exists an increasing function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $t \geq 0, \xi, \eta \in \mathcal{C}$,

$$
|b(t, \xi) - b(t, \eta)|^2 + |\dot{b}(t, \xi) - \dot{b}(t, \eta)|^2 + |h(t, \xi) - h(t, \eta)|^2 + |\dot{h}(t, \xi) - \dot{h}(t, \eta)|^2
+ |\sigma(t, \xi) - \sigma(t, \eta)|^2 + |\dot{\sigma}(t, \xi) - \dot{\sigma}(t, \eta)|^2 \leq \alpha(t)\|\xi - \eta\|_\infty^2.
$$

(H2) There exists a constant $K$ such that

$$
|b(t, 0)|^2 + |\dot{b}(t, 0)|^2 + |h(t, 0)|^2 + |\dot{h}(t, 0)|^2 + \|\sigma(t, 0)\|_{HS}^2 + \|\dot{\sigma}(t, 0)\|_{HS}^2 \leq K, \quad t \geq 0.
$$

(H3) $N(0) = 0$ and $N(\xi) \leq N(\eta)$ for $\xi \leq \eta$.

(H4) There exists a constant $\kappa \in (0, 1)$ such that

$$
|N(\xi) - N(\eta)| \leq \kappa \max_{1 \leq i \leq d} \|\xi^i - \eta^i\|_{\infty}, \quad \xi, \eta \in \mathcal{C}.
$$

Remark 3.1. According to [9, Lemma 2.1], under (H1)-(H4), for any $s \geq 0$ and $\xi, \bar{\xi} \in \mathcal{C}$, $G$-NSFDE (3.1) has a unique solution $\{Y(s, \xi; t), \dot{Y}(s, \xi; t)\}_{t \geq s - r_0}$ with $(Y_s, \dot{Y}_s) = (\xi, \bar{\xi})$. Moreover, the segment processes $\{Y(s, \xi)_t, \dot{Y}(s, \xi)_t\}_{t \geq s}$ satisfy

$$
\mathbb{E}^G \sup_{t \in [s, T]} (\|Y(s, \xi)_t\|_{\infty}^2 + \|\dot{Y}(s, \xi)_t\|_{\infty}^2) < \infty, \quad T \in [s, \infty).
$$

Definition 3.2. The $G$-NSFDE (3.1) is called $N$-order-preserving, if for any $s \geq 0$ and $\xi, \bar{\xi} \in \mathcal{C}$ with $\xi \leq_N \bar{\xi}$, it holds $\mathcal{C}$-q.s.

$$
Y(s, \xi)_t \leq_N Y(s, \bar{\xi})_t, \quad t \geq s.
$$

We first state the following sufficient conditions for the order preservation, which reduce back to the corresponding ones in [1] when the noise is an $m$-dimensional standard Brownian motion and in [9] when the system with distribution independent.

Theorem 3.2. Assume (H1)-(H4) and the following two conditions are satisfied:

(A1) For any $1 \leq i \leq d$, $\xi, \eta \in \mathcal{C}$ with $\xi \leq_N \eta$ and $\xi^i(0) = N^i(\xi) = \eta^i(0) = N^i(\eta)$,

$$
b'(t, \xi) - \bar{b}'(t, \eta) + 2G(h^i(t, \xi) - \bar{h}^i(t, \eta)) \leq 0, \quad a.e. \quad t \geq 0.
$$

(A2) The diffusion terms $\sigma = (\sigma^{ij})$ and $\bar{\sigma} = (\bar{\sigma}^{ij})$ satisfy $\sigma = \bar{\sigma}$. Moreover, $\sigma^{ij}(t, \xi)$ only depends on $t$ and $\xi^i(0) = N^i(\xi)$, for any $1 \leq i \leq d$, $1 \leq j \leq m$, $\xi \in \mathcal{C}$.

Then the $G$-NSFDE (3.1) is $N$-order-preserving.
Before proceeding to the proof of Theorem 3.2, we give the following proposition. Without loss of generality, we assume \( s = 0 \) and omit the subscript \( s \). Denote \( \bar{Y}(t) = Y(s, \xi; t) \), \( \dot{Y}(t) = \dot{Y}(s, \xi; t) \), \( Y^N(t) = (Y^{i,N}(t), \cdots, Y^{d,N}(t)) = Y(t) - N(Y_t) \) and \( \dot{Y}^N(t) = (\dot{Y}^{i,N}(t), \cdots, \dot{Y}^{d,N}(t)) = \dot{Y}(t) - N(\dot{Y}_t) \). Let

\[
\tau^i = \inf \{ t > 0 : Y^i(t) > \bar{Y}^i(t) \}, \quad i = 1, 2, \cdots, d, \\
\tau^i_N = \inf \{ t > 0 : Y^{i,N}(t) > \dot{Y}^{i,N}(t) \}, \quad i = 1, 2, \cdots, d.
\]

**Proposition 3.3.** ([9, Proposition 4.3]) Let \( \tau = \min\{\tau^1, \cdots, \tau^d\} \) and \( \tau_N = \min\{\tau^1_N, \cdots, \tau^d_N\} \). Assume (H3) and (H4), then \( \tau_N \leq \tau \) on \( \Omega \).

**Proof.** As we set conditions on the neutral term \( N \) is same with [9], we omit the proof. \( \square \)

**Remark 3.4.** ([9, Remark 4.4]) The following condition (C1) and (C2) are equivalent.

(C1) (H1) together with Theorem 3.2 (A2).

(C2) \( \sigma = \bar{\sigma} \), and there exists a constant \( L > 0 \) such that for any \( i = 1, \cdots, n \),

\[
\sum_{j=1}^{m} (|\sigma_{ij}(t, \xi) - \sigma_{ij}(t, \eta)|^2 + |\bar{\sigma}_{ij}(t, \xi) - \bar{\sigma}_{ij}(t, \eta)|^2) \\
\leq L|\xi^i(0) - N^i(\xi) - \eta^i(0) + N^i(\eta)|^2, \quad t \geq 0, \quad \xi, \eta \in \mathcal{C}.
\]

**Proof of Theorem 3.2** We first prove the result under the following condition (A1′) instead of (A1).

(A1′) For any \( 1 \leq i \leq d \), \( \xi, \eta \in \mathcal{C} \) with \( \xi \leq_N \eta \) and \( \xi^i(0) - N^i(\xi) = \eta^i(0) - N^i(\eta), \)

\[
b^i(t, \xi) - \bar{b}^i(t, \eta) + 2G(h^i(t, \xi) - \bar{h}^i(t, \eta)) < 0, \quad \text{a.e. } t \geq 0.
\]

Assume (H1)-(H4), and let conditions (A1′) and (A2) hold. We aim to prove that \( \mathcal{C}\{Y_t \leq_N \bar{Y}_t, \quad t \in [0, T]\} = 1 \), which equal to prove that for any \( 1 \leq i \leq d \),

\[
\mathcal{C}\{Y^{i,N}(t) > \dot{Y}^{i,N}(t), \quad t \in [0, T]\} = 0.
\]

By the definition of \( \tau_N^i \), this equal to prove that for any \( 1 \leq i \leq d \), \( \mathcal{C}\{\tau_N^i = \infty\} = 1 \). It suffices to prove \( \mathcal{C}\{\tau_N^i = \infty\} = 1 \). In fact, if this holds, by Proposition 3.3 it holds that \( \mathcal{C}\{\tau = \infty\} = 1 \), i.e., for any \( 1 \leq i \leq d \),

\[
\mathcal{C}\{Y^i(t) > \dot{Y}^i(t), \quad t \in [0, T]\} = 0.
\]

Combining with (3.4) and (3.5), so the \( N \)-order preservation holds. To this end, let us assume the contrary. If there exists some \( M > 0 \) such that \( \mathcal{C}\{\tau_N^i < M\} > 0 \). It follows the definition of \( \tau_N^i \) that there exists an \( i \in \{1, \cdots, d\} \) such that \( \mathcal{C}(A) > 0 \), for \( A = \{\tau_N^i = \infty\} \).
\( \tau_N < M \). For any \( t \geq 0 \), let \( Y_t^Z = (Y_t^1, \ldots, Y_t^{i-1}, Y_t^i - (Z_t^{i,N})^+, Y_t^{i+1}, \ldots, Y_t^d) \), where \( (Z_t^{i,N})^+ = \{(Y_t^{i,N} - \bar{Y}_t^{i,N}) \vee 0 \} \). Set

\[
(3.6) \quad \tau_H^i = \inf\{ t > \tau_N^i : H^i(t) > 0 \},
\]

where \( H^i(t) = b^i(t, Y_t^Z) - \bar{b}^i(t, \bar{Y}_t) + 2G[h^i(t, Y_t^Z) - \bar{h}^i(t, \bar{Y}_t)] \). We claim that \( \tau_H^i > \tau_N^i \). In fact, by the definition of \( \tau_H^i \), we know that \( \tau_H^i \geq \tau_N^i \). If \( \tau_H^i = \tau_N^i \), it follows the definition of \( \tau_N^i \) that

\[
Y_{i,N}^i(\tau_N^i) - \bar{Y}_{i,N}^i(\tau_N^i) = 0, \quad Y_{i,N}^i(r) - \bar{Y}_{i,N}^i(r) < 0, \quad r \in [-r_0, 0),
\]

thus, \( (Z_t^{i,N})^+ \equiv 0 \). Moreover, by Proposition 3.3, we have \( \tau_N \leq \tau^i \), \( i = 1, \ldots, d \), thus \( Y_{\tau_N}^i \leq \bar{Y}_{\tau_N}^i \) and \( Y_N^i(\tau_N^i) \leq \bar{Y}_N^i(\tau_N^i) \). Letting \( t = \tau_N^i, \xi = Y_t^Z(= Y_{\tau_N}^i), \eta = \bar{Y}_{\tau_N}^i \) in the condition \( \text{(A1')} \), we have \( H^i(\tau_N^i) = H^i(\tau_H^i) < 0 \). This would be in contradiction with 3.6! Therefore, the claim holds true. Furthermore, \( H^i(t) \leq 0 \) for all \( t \in [\tau_N^i, \tau_H^i] \).

Now, we take the following \( C^2 \)-approximation of \( s^+ \) as \cite{7} Theorem 1.1. For any \( n \geq 1 \), construct \( \psi_n : \mathbb{R} \to [0, \infty) \) as follows: \( \psi_n(s) = \psi_n'(s) = 0 \) for \( s \in (-\infty, 0] \), and

\[
\psi''_n(s) = \begin{cases} 
4n^2s, & s \in [0, \frac{1}{4n}], \\
-4n^2(s - \frac{1}{n}), & s \in [\frac{1}{2n}, \frac{1}{n}], \\
0, & \text{otherwise}.
\end{cases}
\]

One can see that

\[
(3.7) \quad 0 \leq \psi''_n \leq 1_{(0, \infty)}, \quad \text{and as } n \uparrow \infty : 0 \leq \psi_n(s) \uparrow s^+, \quad s\psi''_n(s) \leq 1_{(0, \frac{1}{n})}(s) \downarrow 0.
\]

Noting that \( \psi_n(Y_{i,N}^i(0) - \bar{Y}_{i,N}^i(0)) = \psi_n(\xi^i(0) - \eta^i(0) - N(\xi^i) + N(\eta^i)) = 0, \sigma = \bar{\sigma}, \) and

\[
\begin{aligned}
d(Y_{i,N}^i(t) - \bar{Y}_{i,N}^i(t)) &= (b^i(t, Y_t) - \bar{b}^i(t, \bar{Y}_t))dt + (h^i(t, Y_t) - \bar{h}^i(t, \bar{Y}_t), d(B)(t)) \\
&\quad + \sum_{j=1}^{m} (\sigma^i_j(t, Y_t) - \sigma^i_j(t, \bar{Y}_t))dB^j(t),
\end{aligned}
\]
we then have from Itô’s formula that
\[
\psi_n(Y^{i,N}(t \wedge \tau^i_H) - \bar{Y}^{i,N}(t \wedge \tau^i_H))^2
\]
\[
= 2 \sum_{j=1}^{m} \int_{0}^{t \wedge \tau^i_H} (\sigma^{ij}(s, Y_s) - \sigma^{ij}(s, \bar{Y}_s)) \{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) dB^j(s)
\]
\[
+ 2 \int_{0}^{t \wedge \tau^i_H} \{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) \langle h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s), dB(s) \rangle
\]
(3.8)
\[
+ 2 \int_{0}^{t \wedge \tau^i_H} \{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) ds
\]
\[
+ \sum_{j=1, k=1}^{m} \int_{0}^{t \wedge \tau^i_H} \{\psi_n \psi''_n + \psi''_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s))
\]
\[
\times (\sigma^{ij}(s, Y_s) - \sigma^{ij}(s, \bar{Y}_s)) (\sigma^{ik}(s, Y_s) - \sigma^{ik}(s, \bar{Y}_s)) dB_{jk}(s)
\]
\[
= M_i(t \wedge \tau^i_H) + \bar{M}_i(t \wedge \tau^i_H) + I_1 + I_2
\]
for any \(n \geq 1\), \(1 \leq i \leq d\) and \(t \geq 0\), where
\[
M_i(t \wedge \tau^i_H) := 2 \sum_{j=1}^{m} \int_{0}^{t \wedge \tau^i_H} (\sigma^{ij}(s, Y_s) - \sigma^{ij}(s, \bar{Y}_s)) \{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) dB^j(s),
\]
\[
\bar{M}_i(t \wedge \tau^i_H) := 2 \int_{0}^{t \wedge \tau^i_H} \{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) \langle h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s), dB(s) \rangle
\]
\[
- 4 \int_{0}^{t \wedge \tau^i_H} G[\{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) \langle h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s) \rangle] ds,
\]
\[
I_1 := 2 \int_{0}^{t \wedge \tau^i_H} (b^i(s, Y_s) - \bar{b}^i(s, \bar{Y}_s)) \{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) ds
\]
\[
+ 4 \int_{0}^{t \wedge \tau^i_H} G[\{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) \langle h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s) \rangle] ds,
\]
\[
I_2 := \sum_{j=1, k=1}^{m} \int_{0}^{t \wedge \tau^i_H} \{\psi_n \psi''_n + \psi''_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s))
\]
\[
\times (\sigma^{ij}(s, Y_s) - \sigma^{ij}(s, \bar{Y}_s)) (\sigma^{ik}(s, Y_s) - \sigma^{ik}(s, \bar{Y}_s)) dB_{jk}(s).
\]

For simplicity, let \(\Phi^N_n(s) = \{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)), s \in [0, T]\). Due to \(\Phi^N_n(s) \geq 0, s \in [0, T]\) and the property (a) of \(G\), for any \(n \geq 1, s \in [0, T]\), we have
(3.9) \(G[\{\psi_n \psi'_n\} (Y^{i,N}(s) - \bar{Y}^{i,N}(s)) \langle h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s) \rangle] = \Phi^N_n(s) G(h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s))\).

Moreover, note that \(0 \leq \psi'_n(Y^{i,N}(s) - \bar{Y}^{i,N}(s)) \leq 1_{\{Y^{i,N}(s) > \bar{Y}^{i,N}(s)\}}\). By (3.6), it holds that
(3.10) \(H^i(s) \Phi^N_n(s) \leq 0, s \in [0, \tau^i_H]\).
Step 1. Estimate the term $I_1$: Combining (3.9) and (3.10), (H1), $0 \leq \psi'_n \leq 1$ and properties (b) and (c) of $G$, there exists a constant $C(T, \bar{\sigma}) > 0$ depends on $T$ and $\bar{\sigma}$ such that for any $n \geq 1$, $t \in [0, T]$

$$I_1 = 2 \int_0^{t \wedge \tau_H^i} \left[ b^i(s, Y_s) - b^i(s, \bar{Y}_s) + 2G(h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s)) \right] \Phi_N^*(s) ds$$

$$= 2 \int_0^{t \wedge \tau_H^i} \left[ b^i(s, Y_s) - b^i(s, Y_s^Z) + 2G(h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s)) \right] \Phi_N^*(s) ds$$

$$+ 2 \int_0^{t \wedge \tau_H^i} \left[ b^i(s, Y_s^Z) - b^i(s, \bar{Y}_s) + 2G(h^i(s, Y_s^Z) - \bar{h}^i(s, \bar{Y}_s)) \right] \Phi_N^*(s) ds$$

$$\leq 2 \int_0^{t \wedge \tau_H^i} \left[ b^i(s, Y_s) - b^i(s, Y_s^Z) + 2G(h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s)) \right] \Phi_N^*(s) ds$$

$$\leq \int_0^{t \wedge \tau_H^i} \left[ b^i(s, Y_s) - b^i(s, Y_s^Z) + 2G(h^i(s, Y_s) - \bar{h}^i(s, \bar{Y}_s)) \right]^2 ds$$

$$+ \int_0^{t \wedge \tau_H^i} \psi_n(Y^{i,N}(s) - \bar{Y}^{i,N}(s))^2 ds$$

$$\leq \int_0^{t \wedge \tau_H^i} C(T, \bar{\sigma})|(Z_i^{i,N})^+|^2 ds + \int_0^{t \wedge \tau_H^i} \psi_n(Y^{i,N}(s) - \bar{Y}^{i,N}(s))^2 ds$$

$$\leq \int_0^{t \wedge \tau_H^i} C(T, \bar{\sigma})|(Y^{i,N} - \bar{Y}^{i,N})^+|^2 ds.$$

Step 2. Estimate the term $I_2$: From (3.7), (H1), Remark 3.4 and (2.4), we get

$$I_2 \leq \int_0^{t \wedge \tau_H^i} \left( 1_{\{Y^{i,N}(s) - \bar{Y}^{i,N}(s) \in (0, \frac{1}{n})\}} + 1_{\{Y^{i,N}(s) - \bar{Y}^{i,N}(s) \in (0, \infty)\}} \right)$$

$$\times \sum_{j=1, k=1}^m (\sigma^{ij}(s, Y_s) - \sigma^{ij}(s, \bar{Y}_s))(\sigma^{ik}(s, Y_s) - \sigma^{ik}(s, \bar{Y}_s))d \langle B \rangle_{jk}(s)$$

$$\leq \int_0^{t \wedge \tau_H^i} C(T, \bar{\sigma})|(Y^{i,N}(s) - \bar{Y}^{i,N}(s))|^2 ds, \quad n \geq 1, \quad t \in [0, T].$$

Step 3. Estimate the term $M_i(t \wedge \tau_H^i)$: Since $M_i$ is a $G$-martingale, we have

$$\mathbb{E}^G M_i(t \wedge \tau_H^i) = 0, \quad t \in [0, T].$$

Step 4. Estimate the term $\bar{M}_i(t \wedge \tau_H^i)$: Note that $\bar{M}_i$ is a non-increasing $G$-martingale, we obtain from (3.9) that $\mathbb{E}^G \bar{M}_i(t \wedge \tau_H^i) \leq 0, \quad t \in [0, T].$
Therefore, for any \( n \geq 1 \) and \( t \in [0, T] \), there exists a constant \( C > 0 \) such that

\[
\mathbb{E}^G[1_A \psi_n(Y^{i,N}(t \wedge \tau^i_H) - \bar{Y}^{i,N}(t \wedge \tau^i_H))^2] \leq C\mathbb{E}^G 1_A \int_0^{t \wedge \tau^i_H} |(Y^{i,N}(s) - \bar{Y}^{i,N}(s))^+|^2 ds
\]

\[
\leq C \int_0^{t \wedge \tau^i_H} \mathbb{E}^G(1_A(Y^{i,N}(s) - \bar{Y}^{i,N}(s))^+)^2 ds.
\]

Letting \( n \uparrow \infty \), by the monotone convergence theorem in [14, Theorem 6.1.14], we have

\[
\mathbb{E}^G \{1_A((Y^{i,N}(t \wedge \tau^i_H) - \bar{Y}^{i,N}(t \wedge \tau^i_H))^+)^2\} \leq C\int_0^{t \wedge \tau^i_H} \mathbb{E}^G[1_A(Y^{i,N}_s - \bar{Y}^{i,N}_s)^+|^2 ds.
\]

By Gronwall’s inequality and (3.2), we arrive at

\[
\mathbb{E}^G[1_A|(Y^{i,N}(t \wedge \tau^i_H) - \bar{Y}^{i,N}(t \wedge \tau^i_H))|+|^2] = 0, \ t \in [0, T].
\]

This yields that \( Y^{i,N}(t \wedge \tau^i_H) \leq \bar{Y}^{i,N}(t \wedge \tau^i_H) \) on \( A = \{ \tau^i_N = \tau^i_N < M \} \) for all \( t \in [\tau^i_N, T] \subset [0, T] \), which is contradicts with the definition of \( \tau^i_N \) in (3.3) for \( \tau^i_H > \tau^i_N \). Therefore, we prove that for every \( M > 0 \), \( C\{ \tau^i_N < M \} = 0 \), i.e., \( C\{ \tau^i_N = \infty \} = 1 \).

In general, if (A1) in Theorem 3.2 holds, let \( \bar{b}_e = \bar{b} + \bar{c} \), where \( \bar{c} = (\epsilon, \cdots, \epsilon) \in \mathbb{R}^d \), \( \epsilon > 0 \). Let \( \bar{Y}^\epsilon(t) \) solve (3.1) with \( \bar{Y}^\epsilon_0 = \bar{Y}_0 \) and the drift term \( \bar{b}_e \) instead of \( \bar{b} \). It is easy to deduce under the same conditions in Theorem 3.2 that

\[
\lim_{\epsilon \to 0^+} \mathbb{E}^G \sup_{t \in [0, T]} |\bar{Y}^\epsilon(t) - \bar{Y}(t)| = 0.
\]

Note that \( \bar{b}_e \) satisfies condition (A1'), by the above discussion, we have \( C\)-q.s.

\[
Y_t \leq_N \bar{Y}_t^\epsilon, \ t \in [0, T].
\]

Letting \( \epsilon \to 0 \), it follows from (3.12) and the continuity of \( N \) that \( C\)-q.s. \( C\)-q.s.

\[
Y_t \leq_N \bar{Y}_t, \ t \in [0, T].
\]

The following result shows that the sufficient conditions appeared in Theorem 3.2 are also necessary if all coefficients are continuous on \( [0, \infty) \times \mathcal{C} \).

**Theorem 3.5.** Let (H1)-(H4) hold and (3.1) be \( N \)-order-preserving. Moreover, if \( b, h, \sigma \) and \( \bar{b}, \bar{h}, \bar{\sigma} \) are continuous on \( [0, \infty) \times \mathcal{C} \), then conditions (A1) and (A2) in Theorem 3.2 hold.
Proof of (A1). We use the representation theorem as explained in [8, Theorem 3.3]. Let $1 \leq i \leq d$ be fixed. For any $t_0 \geq 0$ and $\xi, \eta \in \mathcal{G}$ with $\xi \leq_\mathcal{N} \eta$ and $\xi^i(0) - N^i(\xi) = \eta^i(0) - N^i(\eta)$, it holds $\mathcal{C}$-q.s.

\begin{equation}
Y(t_0, \xi)_t \leq_\mathcal{N} Y(t_0, \eta)_t, \quad t \geq 0.
\end{equation}

Denote $Y(t) = Y(t_0, \xi; t)$, $\bar{Y}(t) = \bar{Y}(t_0, \eta; t)$, $Y^N(t) = Y(t) - N(Y_t)$ and $\bar{Y}^N(t) = \bar{Y}(t) - N(\bar{Y}_t)$. Recalling the expression (2.1), for any $\gamma \in \mathbb{S}^m \cap [\sigma^2 \mathbf{I}_{m \times m}, \sigma^2 \mathbf{I}_{m \times m}]$, taking $\theta_s = \sqrt{\gamma}, s \geq 0$, we have $\mathbb{P}_{\gamma}$-a.s. $\langle B \rangle(r) = r\gamma$. Accordingly, by (3.1) and (3.13), for any $s \geq 0$, we obtain $\mathbb{P}_{\gamma}$-a.s.

\begin{align}
0 & \geq Y^{i,N}(t_0 + s) - Y^N(t_0 + s) = \xi^i(0) - N(\xi^i) - \eta^i(0) + N(\eta^i) \\
& + \int_{t_0}^{t_0 + s} [b^i(r, Y_r) - \bar{b}^i(r, \bar{Y}_r)] \, dr + \int_{t_0}^{t_0 + s} \langle h^i(r, Y_r) - \bar{h}^i(r, \bar{Y}_r), d\langle B \rangle(r) \rangle \\
& + \sum_{j=1}^{m} \int_{t_0}^{t_0 + s} [\sigma^{ij}(r, Y_r) - \bar{\sigma}^{ij}(r, \bar{Y}_r)] \, dB^j(r) \\
& = \int_{t_0}^{t_0 + s} [b^i(r, Y_r) - \bar{b}^i(r, \bar{Y}_r)] \, dr + \int_{t_0}^{t_0 + s} \langle h^i(r, Y_r) - \bar{h}^i(r, \bar{Y}_r), \gamma \rangle \, dr \\
& + \sum_{j=1}^{m} \int_{t_0}^{t_0 + s} [\sigma^{ij}(r, Y_r) - \bar{\sigma}^{ij}(r, \bar{Y}_r)] \, dB^j(r).
\end{align}

Letting $E_{\mathbb{P}_{\gamma}} = E_{\mathbb{P}_{\gamma}}$ in (2.3), taking expectation in (3.14) under $\mathbb{P}_{\gamma}$, by Remark 3.1 we have

\begin{equation}
\frac{1}{s} \int_{t_0}^{t_0 + s} \mathbb{E}_{\mathbb{P}_{\gamma}} \{ [b^i(r, Y_r) - \bar{b}^i(r, \bar{Y}_r)] + \langle h^i(r, Y_r) - \bar{h}^i(r, \bar{Y}_r), \gamma \rangle \} \, dr \leq 0, \quad s > 0.
\end{equation}

Letting $s \downarrow 0$ in (3.15), it follows from (2.3), (3.2), the continuity of $b, \bar{b}, h, \bar{h}$ and dominated convergence theorem that

\[ [b^i(t_0, \xi) - \bar{b}^i(t_0, \eta)] + \langle h^i(t_0, \xi) - \bar{h}^i(t_0, \eta), \gamma \rangle \leq 0. \]

In terms of the definition of $G$ in (2.1), we get

\[ [b^i(t_0, \xi) - \bar{b}^i(t_0, \eta)] + 2G(h^i(t_0, \xi) - \bar{h}^i(t_0, \eta)) \]

\[ = [b^i(t_0, \xi) - \bar{b}^i(t_0, \eta)] + 2 \sup_{\gamma \in \mathbb{S}^m \cap [\sigma^2 \mathbf{I}_{m \times m}, \sigma^2 \mathbf{I}_{m \times m}]} \frac{1}{2} \langle h^i(t_0, \xi) - \bar{h}^i(t_0, \eta), \gamma \rangle \leq 0. \]

Then (A1) holds.

\[ \Box \]
Proof of (A2). For any $\xi, \bar{\xi} \in \mathcal{C}$ with $\xi \leq_N \bar{\xi}$, it holds $\mathcal{C}$-q.s.

$$Y_t \leq_N \bar{Y}_t, \quad t \geq 0.$$ 

Now, letting $\theta_s = \bar{\sigma}$, by (2.3), we have $\mathbb{P}_\theta$-a.s. $Y_t \leq_N \bar{Y}_t, t \geq 0$. Noting that $\mathbb{P}_\theta$-a.s. $\langle B \rangle(r) = \bar{\sigma}^2 r$ and $\mathbb{P}_\theta$ is the law of $\int_0^t \theta_s dW_s$, thus (3.1) comes back to the SDE driven by classic Brownian motion under $\mathbb{P}_\theta$. According to the necessary condition of order preservation for functional SDEs with distribution independent in [9, Theorem 4.8], we prove (A2). □

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