Covering stability of Bergman kernels on Kähler hyperbolic manifolds

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Abstract

This paper is a sequel to [32]. In this paper, an estimation of the Bergman Kernel of Kähler hyperbolic manifold is given by the $L^2$ estimate and the Bochner formula. As an application, an effective criterion of the very ampleness of the canonical line bundle of Kähler hyperbolic manifold is given, which is a generalization of Yeung’s result.

1 Introduction

The notion of Kähler hyperbolic is due to Gromov [16]. A non-compact Kähler manifold $(X, \omega, \lambda)$ is called non-compact Kähler hyperbolic if $\omega$ is the exterior differential of a $C^1$ bounded 1-form $\eta$, i.e., $\omega = d\eta$ and $|\eta|^2 \leq \lambda$ on $X$ for some positive constant $\lambda$. A compact Kähler manifold $(X, \omega, \lambda)$ is called Kähler hyperbolic if the lift $\tilde{\omega}$ of $\omega$ to the universal covering $\tilde{X} \to X = \tilde{X}/\Gamma$ is the exterior differential of a $C^1$ bounded 1-form $\eta$, i.e., $\tilde{\omega} = d\eta$ and $|\eta|^2 \leq \lambda$ on $\tilde{X}$ (we say that $X$ is d-bounded by $\lambda$). What’s more, if $\tilde{X}$ is CH (Cartan-Hadamard) manifold (see Appendix II), we call $(X, \omega, \lambda)$ CH Kähler hyperbolic manifold. Let $h^{n,0} = p_1$ be the first plurigenera, i.e., the dimension of the Bergman space of $X$. The $L_2$-Hodge number $\tilde{h}^{n,0} = \tilde{p}_1$ is defined as the integration of the Bergman kernel form of $\tilde{X}$ on a Dirichlet fundamental domain of $X$ in $\tilde{X}$.

Since $(X, \omega, \lambda)$ is compact, its Ricci curvature is bounded, throughout this paper, we shall assume that its Ricci curvature is bounded below by $-1$ unless specified mentioned. Denote by $|X|$ the volume of $X$. Let $\tau = \min_{x \in \tilde{X}} \{\tau(x)\}$, where $\tau(x)$ is the quasi-injectivity radius (see Appendix III). We shall prove that

**Theorem 1.1.** Let $(X, \omega, \lambda)$ be an $n$-dimensional CH Kähler hyperbolic manifold, if $n \geq 2$ and $\tau \geq 2\sqrt{2n}$, then

$$\frac{1}{|X|} |p_1 - \tilde{p}_1| \leq 16\left(\frac{4}{\pi}\right)^n \sqrt{n\lambda \tau^2}. \quad (1.1)$$

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For the compact ball quotients, by the same method as in the proof of Theorem 1.1, one could get a better estimation (see [32]). On the other hand, by a similar argument as in the proof of Theorem 1.1, we get the following generalization of Theorem 1.1.

**Theorem 1.2.** Let \((X, \omega, \lambda)\) be an \(n\) dimensional Kähler hyperbolic manifold, if \(\tau \geq 2\),
\[
\frac{1}{|X|} |p_1 - \tilde{p}_1| \leq 2^{203(2n+\sqrt{2n})} \frac{\lambda}{|B|^2},
\]
where \(|B|\) stands for the minimal volume of the unit ball in \(X\).

If \(\tilde{X}\) is a bounded homogeneous domain \(\Omega\) in \(\mathbb{C}^n\). By the result of Kai-Ohsawa [20] (see also Vinberg-Gindikin-Pjateckii-Šapiro [31]), one may choose suitable globally coordinate \(z\) of \(\Omega\) such that its Bergman kernel \(K(z, \bar{w})dz \otimes d\bar{w}\) satisfies
\[
\tilde{\omega} = i\partial \bar{\partial} \log K(z, z), \quad |\partial \log K(z, z)| \equiv KO_\Omega,
\]
where \(KO_\Omega\) is a positive constant only depends on the complex structure of \(\Omega\). If \(\Omega = \{(u, v) \in \mathbb{C}^p \times \mathbb{C}^q \mid v + \bar{v} - F(u, u) \in V\}\) is a Siegel domain of second kind defined by \(V\) and \(F\), where \(V\) is a convex cone in \(\mathbb{R}^q\) containing no entire straight lines and \(F\) is \(V\)-Hermitian. Ishi [19] proved that
\[
KO_\Omega = \sqrt{p+2q}.
\]
Since \(\Omega\) is homogeneous, its Bergman kernel function \(S_\Omega\) is a constant that only depends on the complex structure of \(\Omega\). Thus its Ricci curvature satisfies
\[
Ric(\tilde{\omega}) = -\tilde{\omega}.
\]
It is well known that the Bergman metric \(\tilde{\omega}\) on every bounded symmetric domain has non-positive sectional curvature. On the other hand, according to D’atri-Miatello [8], a bounded homogeneous domain with negative sectional curvature with respect to \(\tilde{\omega}\) must be symmetric. Thus, according to the above two Theorems, \(p_1\) will be non-vanishing for sufficiently large \(\tau\).

**Theorem 1.3.** The Bergman space of the compact quotient of a bounded symmetric domain \(\Omega\) in \(\mathbb{C}^n\) is nontrivial provided that
\[
\tau > \max\{2^{n+1} \frac{KO_\Omega}{\sqrt{S_\Omega}}, 2\sqrt{2n}\}.
\]

Since \(\frac{1}{|X|} |p_1 - \tilde{p}_1|\) does’t depend on the un-ramified normal covering of \(X\), thus it could be used to investigate the covering behavior of the Bergman kernels. If a Kähler hyperbolic manifold \(X\) has a tower of coverings \(\{X_j\}\), by the result of Chen-Fu [7] (see also [13], [21], [26], [30]), the pull back of the Bergman kernel function on \(X_j\) converges uniformly to that of \(\tilde{X}\) as \(j \to \infty\). Thus it is natural to find an effective estimate of the Bergman kernel function. We shall prove an effective Ramadanov’s Theorem (see [28]) in section 3. Thus, we get an effective estimates of the Bergman kernels on Kähler hyperbolic manifolds, which is also given in section 3.

Inspired by Yeung’s results (see [35], [36], [34]), we shall give an effective ampleness criterion of the canonical line bundle in section 4 by using Calabi’s diastasis function (see [6]) and the Bergman metric instead of the heat kernel.
2 The geometry of Kähler hyperbolic manifolds

The following Theorem on the lower bound of the spectrum of the Laplace operator is due to Donnelly-Fefferman [14] and Gromov [16] (see also Ohsawa [26]). See Appendix-I for the basic notions (such as $\tilde{\partial}$ in (7.1)) of Hodge theory.

**Theorem 2.1.** Let $(X, \omega, \lambda)$ be a non-compact complete Kähler hyperbolic manifold of dimension $n$. If then every $u \in \text{Dom} \tilde{\partial} \cap \text{Dom} (\tilde{\partial})^*$ with degree $p + q \neq n$ satisfies the inequality

$$||\tilde{\partial}u||^2 + ||(\tilde{\partial})^*u||^2 \geq \frac{|n - p - q|}{8\lambda}||u||^2.$$  \hspace{1cm} (2.1)

For $u \in \text{Dom} \tilde{\partial} \cap \text{Dom} (\tilde{\partial})^*$ with degree $p + q = n$, which are orthogonal to the $L^2$ harmonic space,

$$||\tilde{\partial}u||^2 + ||(\tilde{\partial})^*u||^2 \geq \frac{1}{8\lambda}||u||^2.$$  \hspace{1cm} (2.2)

What’s more, if $\omega$ has a global $C^2$ real potential $\psi$ such that

$$\omega = i\partial\bar{\partial}\psi \geq \frac{1}{\lambda} i\partial\psi \wedge \bar{\partial}\psi,$$  \hspace{1cm} (2.3)

the constant in (2.1) could be $\frac{(n-p-q)^2}{8\lambda}$.

**Proof.** By Lemma 7.6, we may assume $u \in C^\infty_0(X, \wedge^{n+k}(T^*_R X \otimes \mathbb{C}))$, $k \geq 1$. Because

$$L^k : \wedge^{n-k}(T^*_R X \otimes \mathbb{C}) \to \wedge^{n+k}(T^*_R X \otimes \mathbb{C})$$

is isomorphism, there exists $\varphi \in C^\infty_0(X, \wedge^{n-k}(T^*_R X \otimes \mathbb{C}))$ such that $u = L^k \varphi$. Thus

$$u = d\theta + u',$$

where

$$\theta = \eta \wedge L^{k-1}\varphi, \ u' = \eta \wedge L^{k-1}d\varphi.$$ 

Since

$$\langle\langle u, u' \rangle\rangle \leq ||u||\sqrt{\lambda} \langle\langle \Delta(L^{k-1}\varphi), L^{k-1}\varphi \rangle\rangle^{1/2},$$

and

$$\langle\langle u, d\theta \rangle\rangle \leq ||d^*u||\sqrt{\lambda} ||L^{k-1}\varphi||.$$

By Lemma 7.5,

$$\langle\langle \Delta(L^{k-1}\varphi), L^{k-1}\varphi \rangle\rangle \leq \frac{1}{k} \langle\langle \Delta u, u \rangle\rangle, \ ||L^{k-1}\varphi||^2 \leq \frac{1}{k} ||u||^2,$$

thus

$$||u||^2 = \langle\langle u, d\theta \rangle\rangle - \langle\langle u, u' \rangle\rangle \leq 2\sqrt{\frac{\lambda}{k}} ||u||\langle\langle \Delta u, u \rangle\rangle^{1/2}.$$ 

(2.1) follows from Lemma 7.3.
By Lemma 7.1 and Lemma 7.2,

\[ L^2(X, \wedge^{p,n-p}T^*X) = H^p_{L^2} \oplus \text{Im}(\tilde{\partial}) \oplus \text{Im}((\tilde{\partial})^*), \]

which shows that every \( u \in \text{Dom}(\tilde{\partial}) \cap \text{Dom}(\tilde{\partial}^*) \) with degree \( p + q = n \), which are orthogonal to the \( L^2 \) harmonic space, could be written as

\[ u = \tilde{\partial}a + (\tilde{\partial})^*b, \quad a \perp \ker \tilde{\partial}, \quad b \perp \ker (\tilde{\partial})^*, \]

thus

\[ ||\tilde{\partial}a||^2 \geq \frac{1}{8\lambda} ||a||^2, \quad ||(\tilde{\partial})^*b||^2 \geq \frac{1}{8\lambda} ||b||^2, \]

which proves (2.2).

If \( \omega \) has a global \( C^2 \) potential \( \psi \), by the Bochner-Kodaira-Nakano identity, for any \( u \in C^\infty_0(X, \wedge^{p,q}T^*X) \),

\[ \int_X (|\tilde{\partial}u|^2_\omega + |\tilde{\partial}_{c\psi}u|^2_\omega) e^{-c\psi} dV_\omega \geq c(p + q - n) \int_X |u|^2_\omega e^{-c\psi} dV_\omega, \tag{2.4} \]

where \( c = \frac{1}{2\lambda}(p + q - n) \). Let \( u = ve^{\frac{c}{2}\psi} \), then

\[ |\tilde{\partial}u|^2_\omega e^{-c\psi} = |\tilde{\partial}v + \frac{c}{2} \tilde{\partial} \psi \wedge v|^2_\omega \leq 2|\tilde{\partial}v|^2_\omega + \frac{\lambda c^2}{2} |v|^2_\omega, \]

and

\[ |\tilde{\partial}_{c\psi}u|^2_\omega e^{-c\psi} = |\tilde{\partial} v + \frac{c}{2} \tilde{\partial} \psi \wedge v|^2_\omega \leq 2|\tilde{\partial} v|^2_\omega + \frac{\lambda c^2}{2} |v|^2_\omega, \]

which proves the Theorem finally. \( \square \)

If \( (X, \omega, \lambda) \) satisfies (2.3), we call it \emph{strongly Kähler hyperbolic}. In application, we also need the following Theorem. \emph{In the following, we shall use the same symbol \( P \) to represent the extension of a differential operator \( P \) in the sense of distribution.}

**Theorem 2.2.** Let \( (X, \omega, \lambda) \) be a weakly pseudoconvex strongly Kähler hyperbolic manifold of dimension \( n \), \( \varphi \) is plurisubharmonic on \( X \). Then for every \( v \in L^2_{\text{loc}}(X, \wedge^{n,q}T^*X) \), \( q \geq 1 \), such that \( \overline{\partial}v = 0 \) in the sense of distribution, and

\[ \int_X |v|^2_\omega e^{-\varphi} dV_\omega < +\infty, \tag{2.5} \]

there exist \( u \in L^2_{\text{loc}}(X, \wedge^{n,q-1}T^*X) \) such that \( \overline{\partial}u = v \) in the sense of distribution, and

\[ \int_X |u|^2_\omega e^{-\varphi} dV_\omega \leq \frac{4\lambda}{q^2} \int_X |v|^2_\omega e^{-\varphi} dV_\omega. \tag{2.6} \]

**Proof.** The general proof is based on the regularization techniques for plurisubharmonic function (see Demailly [10]). We will just explain the proof in the simple case
when $X$ is bounded pseudoconvex domain in $\mathbb{C}^n$. By choosing a smooth plurisubharmonic exhaustion function of $X$ and using the standard smoothing technique, we may assume that $\varphi$ and $\psi$ are smooth on some neighborhood of the closure of $X$ in $\mathbb{C}^n$. Let $u$ be the $L^2$ minimal solution of $\overline{\partial}(\cdot) = v$ in $L^2(X, \wedge^{n,q-1} T^*X, \varphi + \frac{c}{2} \psi)$, where $c = \frac{q}{n}$, then $ue^{\frac{c}{2} \psi}$ is the $L^2$ minimal solution of $\overline{\partial}(\cdot) = \overline{\partial}(ue^{\frac{c}{2} \psi})$ in $L^2(X, \wedge^{n,q-1} T^*X, \varphi + c \psi)$.

Since for every $f \in \text{Dom } \overline{\partial} \cap \text{Dom } (\overline{\partial})^*_{\varphi+c\psi}$ with degree $(n,q)$,

$$
\int_X (|\overline{\partial} f|^2_\omega + |(\overline{\partial})^*_{\varphi+c\psi} f|^2_\omega) e^{-(\varphi+c\psi)} \, dV_\omega \geq c q \int_X |f|^2_\omega e^{-(\varphi+c\psi)} \, dV_\omega,
$$

thus

$$
\int_X (|ue^{\frac{c}{2} \psi}|^2_\omega e^{-(\varphi+c\psi)} \, dV_\omega \leq \frac{1}{cq} \int_X |v + \frac{2}{c} \overline{\partial} \psi \wedge u|^2_\omega e^{-\varphi} \, dV_\omega \leq \frac{2}{cq} ||v||^2 + \frac{c\lambda}{2q} ||u||^2,
$$

we get (2.6). \qed

By the Theorem 2.1, the lower bound of the Laplace operator depends on $\lambda$. But the Proposition below shows that $\lambda$ has a lower bound.

**Proposition 2.3.** Let $(X, \omega, \lambda)$ be a Kähler hyperbolic manifold of dimension $n$, then

$$
\lambda \geq \frac{n}{2n-1}.
$$

**Proof.** By the proof of Lemma 8.4. For any $x \in \tilde{X}$,

$$
r f'(r) \leq (1 + r \sqrt{2n-1} \coth(\frac{r}{\sqrt{2n-1}})) f(r),
$$

where $f(r)$ is the volume of the geodesic ball $B_x(r)$ in $\tilde{X}$. We claim that

$$
f'(r) \geq \sqrt{\frac{n}{\lambda}} f(r),
$$

(which will be proved in the following Lemma). Thus

$$
\frac{r \sqrt{n}}{1 + r \sqrt{2n-1} \coth(\frac{r}{\sqrt{2n-1}})} \leq \sqrt{\lambda},
$$

Since $\tilde{X}$ is complete and non-compact, let $r$ goes to infinity, we get (2.5). \qed

The following Lemma is contained in Gromov’s paper [16].

**Lemma 2.4.** Let $(X, \omega)$ be a Kähler manifold with the volume form satisfies

$$
\frac{\omega^n}{n!} = d\sigma,
$$

where $\sigma$ is a $C^1$ $(2n-1)$-form, such that

$$
|\sigma| \leq C,
$$

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on the geodesic ball $B_x(R)$ for some constant $C > 0$. If the injectivity radius at point $x$ is bigger than $R$, we have

$$Cf'(r) \geq f(r),$$

for any $r \in [0, R]$.

**Proof.** By using the orthogonal decomposition

$$\sigma = \sigma_0 ds + \sigma_1$$

at every point $z \in \partial B_x(r)$, where $ds$ is the volume form of $\partial B_x(r)$. Thus

$$|\sigma| \geq |\sigma_0|.$$

Now

$$f(r) = \int_{B_x(r)} d\sigma = \int_{\partial B_x(r)} \sigma = \int_{\partial B_x(r)} \sigma_0 ds \leq Cf'(r),$$

which proves the Lemma. □

**Remark 2.5.** (2.11) follows from

$$|\eta \wedge \omega^{n-1}/n!| \leq \sqrt{\lambda}/n. \quad (2.16)$$

By Bishop’s volume comparison theorem, the volume of every compact Kähler hyperbolic manifold with Ricci curvature bounded below by some negative constant has a nature upper bound. What we want to show is that Atiyah’s $L^2$ index Theorem will give the nature lower bound of $|X|$, which is optimal in some cases.

**Theorem 2.6.** Let $(X, \omega, \lambda)$ be a Kähler hyperbolic manifold of dimension $n$, then

$$|X| \geq \frac{1}{\sup S_{\tilde{X}}}. \quad (2.17)$$

**Proof.** By (9.8), $\tilde{p}_1$ is an integer. By Gromov’s result (see Theorem 2.5 in [16]), $S_{\tilde{X}}$ is not identically zero. The Theorem follows from the $\Gamma$ invariance of $S_{\tilde{X}}$. □

### 3 Bergman kernels on Kähler hyperbolic manifolds

Denote by $P : \tilde{X} \to X = \tilde{X}/\Gamma$ the universal covering map of a complex manifold $X$. According to (7.5) and (7.6), by choosing local coordinate $(z, \bar{w})$ of $\tilde{X} \times \tilde{X}^*$, one may write

$$B_{\tilde{X}}^{n,0} = \tilde{K}(z, \bar{w})dz \otimes d\bar{w}, \quad (3.1)$$

where $\tilde{K}(z, \bar{w})$ is locally defined function and $dz, d\bar{w}$ is short for $dz^1 \wedge \cdots \wedge dz^n, d\bar{w}^1 \wedge \cdots \wedge d\bar{w}^n$ respectively. Using the projection map $P$, $(z, \bar{w})$ could also be taken as the local coordinate of $X \times X^*$, i.e., one may write

$$P^* B_{\tilde{X}}^{n,0} = K(z, \bar{w})dz \otimes d\bar{w}, \quad (3.2)$$
(here we use the same notion $P : \tilde{X} \times \tilde{X}^* \to X \times X^*$ as the canonical projection induced by $P : \tilde{X} \to X$).

Let’s recall the notion of tower in Riemannian geometry. Let $\tilde{X}$ be a Riemannian manifold and $\Gamma$ a free and properly discontinuous group of isometries of $\tilde{X}$. A tower of subgroups of $\Gamma$ is a nested sequence of subgroups

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \Gamma_j = \{id\},$$

such that $\Gamma_j$ is a normal subgroup of $\Gamma$ of finite index $[\Gamma : \Gamma_j]$ for every $j$. The differential manifolds $X_j = \tilde{X}/\Gamma_j$ are equipped with the push-downs of the Riemannian metric on $\tilde{X}$. The family $\{X_j\}$ is called a tower of coverings on the Riemannian manifold $X = \tilde{X}/\Gamma$ (see [9]). Let

$$\tilde{P}_j : \tilde{X} \to X_j, \; P_j : X_j \to X, \; P = P_j \circ \tilde{P}_j : \tilde{X} \to X,$$  

be the natural projections. Similar as (9.1) and (9.2), we shall define $F_j(x)$ and $\tau_j(x)$ for every $X_j$. Since $\Gamma_j$ is normal in $\Gamma$, $\tau_j(x)$ must be $\Gamma$ invariant. According to [12] (see also [7] and [9]), if $X$ is compact,

$$\tau_j(x) \to \infty,$$

uniformly on $\tilde{X}$ as $j \to \infty$. Throughout this paper, $\tilde{X}$ is assumed to be the universal covering of $X$. It is well known that every Riemannian manifold $X$ with its fundamental group isomorphic to a finitely generated subgroup of $SL(n, \mathbb{C})$ admits a tower of coverings with $\tilde{X}$ being the universal covering (see [5]). Every arithmetic quotient of a bounded symmetric domain satisfies the above condition.

Let $X$ be a Hermitian manifold, similar as (3.1), one may write

$$\tilde{P}_j^* B^0_{X_j} = \tilde{K}_j(z, \bar{w})dz \otimes d\bar{w}, \; B^0_{Bx(\tau_j)} = B_j(z, \bar{w})dz \otimes d\bar{w},$$

and

$$B^0_{F_j(x)} = C_j(z, \bar{w})dz \otimes d\bar{w}. \quad (3.7)$$

By (7.9) and (7.10), we have

$$\text{Trace}(\langle \tilde{P}_j^* B^0_{X_j} - B^0_{F_j(x)}, \tilde{P}_j^* B^0_{X_j} - B^0_{F_j(x)} \rangle)_{F_j(x)}(z) = S_{F_j(x)}(z) - \tilde{P}_j^* S_{X_j}(z). \quad (3.8)$$

Similarly,

$$\text{Trace}(\langle B^0_{\tilde{X}} - B^0_{F_j(x)}, B^0_{\tilde{X}} - B^0_{F_j(x)} \rangle)_{F_j(x)}(z) \leq S_{F_j(x)}(z) - S_{\tilde{X}}(z). \quad (3.9)$$

Therefore

$$\text{Trace}(\langle B^0_{\tilde{X}} - \tilde{P}_j^* B^0_{X_j}, B^0_{\tilde{X}} - \tilde{P}_j^* B^0_{X_j} \rangle)_{F_j(x)} \leq 4(S_{F_j(x)} - S_{\tilde{X}}) + 2(S_{\tilde{X}} - \tilde{P}_j^* S_{X_j}). \quad (3.10)$$

Thus we could prove the following Theorem.
**Theorem 1.4.** Let \((X, \omega, \lambda)\) be an \(n\)-dimensional non-compact CH Kähler hyperbolic manifold with Ricci curvature bounded below by \(-1\). Fix \(x \in X\), \(R > 2\sqrt{2n}\), then for every \(y\) in the geodesic ball \(B_x(R)\) of radius \(R\) around \(x\), such that \(\delta_y := d(y, \partial B_x(R)) \geq 2\sqrt{2n}\), we have

\[
S_{B_x(R)}(y) \leq \sqrt{2}\left(\frac{2}{\pi}\right)^n, \quad (3.11)
\]

and

\[
S_{B_x(R)}(y) - S_X(y) \leq 8\left(\frac{2}{\pi}\right)^n \frac{\sqrt{2}\lambda}{\delta_y}. \quad (3.12)
\]

By a similar argument, one could also get an effective Ramadanov’s Theorem for every Kähler hyperbolic manifold.

**Theorem 3.1.** Let \(\{X_j\}\) be a tower of coverings of a CH Kähler hyperbolic manifolds \((X, \omega, \lambda)\) of dimension \(n\) and diameter \(D\), if \(\tau_j > 4(D + 2^n\sqrt{n})\), we have

\[
\int_{F_j(x) \times F(x)} |\tilde{K}(z, \bar{w}) - K_j(z, \bar{w})|^2 (i^n dz \wedge d\bar{z}) \wedge (i^n dw \wedge d\bar{w}) \leq 95|X|\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j}, \quad (3.13)
\]

\[
\int_{F_j(x) \times F(x)} |K_j(z, \bar{w}) - C_j(z, \bar{w})|^2 (i^n dz \wedge d\bar{z}) \wedge (i^n dw \wedge d\bar{w}) \leq 22|X|\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j}, \quad (3.14)
\]

and

\[
\int_{F_j(x) \times F(x)} |K_j(z, \bar{w}) - C_j(z, \bar{w})|^2 (i^n dz \wedge d\bar{z}) \wedge (i^n dw \wedge d\bar{w}) \leq 26|X|\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j}. \quad (3.15)
\]

**Proof.** By (3.10), the left hand side of (3.11) is no bigger than

\[
I_j := \int_{F(x)} 4(S_{F_j(x)} - S_X) + 2(S_X - \tilde{P}^*_j S_{X_j}). \quad (3.16)
\]

By Theorem 1.1 and Theorem 1.4,

\[
I_j \leq 32\left(\frac{2}{\pi}\right)^n \frac{\sqrt{2}\lambda}{\tau_j - D} |X| + 32\left(\frac{4}{\pi}\right)^n \frac{\lambda\sqrt{n}}{\tau_j^2} |X|. \quad (3.17)
\]

By (2.5),

\[
I_j \leq 32\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j} |X|\left(\frac{8}{3} + \frac{1}{4}\right)
\]

which proves (3.11). Same method works for (3.12) and (3.13). \(\Box\)

Combining (3.13) with (1.11), we could estimate the Bergman kernels of Kähler hyperbolic manifolds.

**Theorem 3.2.** Let \(\{X_j\}\) be a tower of coverings of a CH Kähler hyperbolic manifolds \((X, \omega, \lambda)\) of dimension \(n\) and diameter \(D\), if \(\tau_j > 4(D + 2^n\sqrt{n})\) and \(\tau > \sqrt{2n}\), we have

\[
-8\left(\frac{2}{\pi}\right)^n \frac{\sqrt{2}\lambda}{\tau_j} \leq S_X - \tilde{P}^*_j S_{X_j} \leq 26|X|e^{\frac{2}{3}\sqrt{\tau}} \left(\frac{2}{\pi^2 \tau^2}\right)^n \frac{(2n)! \lambda}{\tau_j}. \quad (3.18)
\]
Proof. By (1.11), for every \( x \in \tilde{X} \),

\[
S_{\tilde{X}}(x) - \tilde{P}_j^* S_{X_j}(x) \geq S_{\tilde{X}}(x) - S_{F_j(x)}(x) \geq -8\left(\frac{2}{\pi}\right)^n \frac{\sqrt{2\lambda}}{\tau_j}. \tag{3.19}
\]

By (3.8) and (3.13),

\[
\int_{F(x)} (S_{F_j(x)} - \tilde{P}_j^* S_{X_j}) \, dV \leq 26|X|\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j}, \tag{3.20}
\]

Since the Bergman space of \( X_j \) is a closed subspace of the Bergman space of \( F_j(x) \), \( S_{F_j(x)} - \tilde{P}_j^* S_{X_j} \) must be a summation of the point-wise norm of some holomorphic \( n \)-forms. By Lemma 8.3,

\[
S_{\tilde{X}}(x) - \tilde{P}_j^* S_{X_j}(x) \leq S_{F_j(x)}(x) - \tilde{P}_j^* S_{X_j}(x) \leq 26|X|\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j} \frac{e^{\tau^2} n!}{(\pi \tau^2)^n} \left(\frac{2n}{n}\right), \tag{3.21}
\]

which proves the theorem. \( \square \)

One may use the same method to get an effective estimation (depends on \( n \), \( |B| \), \( \lambda \), \( D \)) of the Bergman kernels on non-CH Kähler hyperbolic manifolds. We leave it to the interested reader.

The estimation for \( K_j(z, \bar{w}) \) is announced in Yeung [36], we give the details of the proof.

**Theorem 3.3.** Let \( \{X_j\} \) be a tower of coverings of a CH Kähler hyperbolic manifolds \( (X, \omega, \lambda) \) of dimension \( n \) and diameter \( D \), if \( \tau_j > 4(D + 2^n \sqrt{n}) \), we have

\[
|\tilde{K} - K_j|^2(x, y) \leq 190|X|\left(\frac{16n}{e \pi^3}\right)^n \frac{e^{\tau^2}}{\tau^{2n}} \frac{\lambda}{\tau_j}. \tag{3.22}
\]

for every \( x, y \in X_j \) such that \( d(x, y) \leq \frac{\tau_j}{2} + 2D \);

\[
|K_j|^2(x, y) \leq 136|X|\left(\frac{16n}{e \pi^3}\right)^n \frac{e^{\tau^2}}{\tau^{2n}} \frac{\lambda}{\tau_j}, \quad |\tilde{K}|^2(x, y) \leq 128|X|\left(\frac{16n}{e \pi^3}\right)^n \frac{e^{\tau^2}}{\tau^{2n}} \frac{\lambda}{\tau_j}. \tag{3.23}
\]

for every \( x, y \in X_j \) such that \( d(x, y) > \frac{\tau_j}{2} + 2D \), where

\[
|K_j|^2(x, y) = |K_j(z, \bar{w})|^2 |dz|^2 |dw|^2(x, y). \tag{3.24}
\]

**Proof.** If \( d(x, y) \leq \frac{\tau_j}{2} + 2D \), we have \( B_y\left(\frac{\tau_j}{2} - 2D\right) \subset F_j(x) \), by (3.??)

\[
\int_{B_y\left(\frac{\tau_j}{2} - 2D\right) \times F(x)} |\tilde{K} - K_j|^2 \, dV \leq 95|X|\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j}. \tag{3.25}
\]

By (8.3),

\[
\frac{(2\pi n)^n}{(2^n) e^n n!} \frac{(\pi \tau^2)^n}{(2^n) e^{\tau^2} n!} |\tilde{K} - K_j|^2(x, y) \leq 95|X|\left(\frac{2}{\pi}\right)^n \frac{\lambda}{\tau_j}.
\]
Since
\[ \binom{2n}{n} n! = (\frac{2n}{n})! \leq 2^{2n+\frac{1}{2}} (\frac{n}{e})^n, \]
we get (3.20). If \( d(x, y) > \frac{T_j}{4} + 2D \). Consider the subdomain
\[ U_{jxy} := B_x(\frac{T_j}{4} + D) \cup B_y(\frac{T_j}{4} + D) \]
in \( X_j \). Denote by
\[ B^{n,0}_{jxy} := K_{jxy}(z, \bar{w})dz \otimes d\bar{w} \]
its Bergman kernel, thus
\[
\text{Trace} \langle (B^{n,0}_{jxy} - \tilde{P}^*_j B^{n,0}_{X_j}, B^{n,0}_{jxy} - \tilde{P}^*_j B^{n,0}_{X_j}) \rangle_{U_{jxy}} \leq S_{U_{jxy}} - \tilde{P}^*_j S_{X_j}.
\]
Since
\[ B_x(\frac{T_j}{4} + D) \cap B_y(\frac{T_j}{4} + D) = \emptyset, \]
by definition
\[ K_{jxy}(z, \bar{w}) = 0 \]
for every \( z \in B_x(\frac{T_j}{4} + D), \ w \in B_y(\frac{T_j}{4} + D) \). Thus
\[
((\tilde{P}^*_j B^{n,0}_{X_j}, \tilde{P}^*_j B^{n,0}_{X_j}))_{B_y(\frac{T_j}{4} + D)} \leq ((B^{n,0}_{jxy} - \tilde{P}^*_j B^{n,0}_{X_j}, B^{n,0}_{jxy} - \tilde{P}^*_j B^{n,0}_{X_j}))_{U_{jxy} \times F(x)}.
\]
By Theorem 1.1 and (1.12), the right hand side is less than
\[
\int_{F(x)} S_{U_{jxy}} - S_{\tilde{X}} dV + 16|X| (\frac{2}{\pi})^n \frac{\sqrt{n} \lambda}{\tau_j^2} \leq 68|X| (\frac{2}{\pi})^n \frac{\lambda}{\tau_j},
\]
which proves the first inequality in (3.21). The second follows by similar method. □

4 Very ampleness of the canonical line bundle

In this section, we will use Bergman metric and Calabi’s diastasis function to get an effective very ampleness criterion the canonical line bundle on Kähler hyperbolic manifolds.

Let \( X \) be a complex manifold, if \( X \) has the Bergman metric, i.e.
\[ B := i\bar{\partial} \partial \log K(z, \bar{z}) \]
is the fundamental form of a Kähler metric on \( X \), we call \( X \) Bergman hyperbolic.

The notion of diastasis function is due to Calabi (see [6] for definition). The diastasis function for \( B \) is
\[
\mathcal{D}(z, w) := 2 \log \left( \frac{K(z, \bar{z}) K(w, \bar{w})}{|K(z, \bar{w})|^2} \right).
\]  (4.1)
The notion of projective-like is introduced by Loi [24]. We say that $X$ is projective like, if $D(z, w)_{\{z \neq w\}}$ has no zero point.

The following fundamental Theorem is based on Calabi [6].

**Theorem 4.1.** The canonical line bundle $E$ over a compact complex manifold $X$ is very ample if and only if $X$ is Bergman hyperbolic and projective-like.

**Proof.** By definition, $E$ is very ample if and only if all evaluation maps

$$H^0(X, E) \to (J^1 E)_x, \quad H^0(X, E) \to E_x \oplus E_y, \quad x, y \in X, \quad x \neq y,$$

are surjective. Since $\text{Rank}(E) = 1$, $E$ is very ample if and only if the canonical map $\psi$ from $X$ to $\mathbb{P}(H^0(X, E))$ is an embedding. Let

$$\{u_j = u_j(z)dz\}_{j=0, \ldots, N}$$

be a complete orthonormal base of the Bergman space. The canonical map $\psi$ is given by

$$z \rightarrow [u_0(z), u_1(z), \ldots, u_N(z)], \quad (4.3)$$

where $[u_0, u_1, \ldots, u_N]$ is the homogeneous coordinate of $\mathbb{P}(H^0(X, E)) = \mathbb{P}^N$. We shall prove the extremal property of the Bergman metric, i.e., for every $Y \in \mathbb{C}^n \backslash \{0\}$,

$$\sum_{j,k=1}^n \frac{\partial^2 \log K(z, \bar{z})}{\partial z_j \partial \bar{z}_k} Y_j Y_k = \frac{1}{K(z, \bar{z})} \sup_{\|f\|=1, \ f(z)=0} \left\{ |Yf(z)|^2 \mid f = f(z)dz \in \mathcal{H}^{n,0}(X) \right\}, \quad (4.4)$$

where

$$Yf(z) = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z)Y_j.$$

Fix $z_0 \in X$, one may set $H = \mathcal{H}^{n,0}(X)$, choose $u_0 \in H$ such that $\|u_0\| = 1$ and

$$|u_0|(z_0) = \sup\{|u|(z_0) \mid u \in H, \ |u| = 1\}.$$

Set $H_1 = \{u_0\}^\perp \cap H$, choose $u_1^Y$ such that $\|u_1^Y\| = 1$ and

$$|Y u_1^Y(z_0)| = \sup\{|Yu(z_0)| \mid u \in H_1, \ |u| = 1\}.$$

Set $H_2 = \{u_0, u_1^Y\}^\perp \cap H$. Every $u \in H_1$ satisfies $u(z_0) = 0$, every $u \in H_2$ satisfies $Y u(z_0) = 0$ and $u(z_0) = 0$. Thus by choosing a complete orthonormal base $\{u_2, u_3, \ldots\}$ of $H_2$, one has

$$K(z_0, \bar{z}_0) = |u_0(z_0)|^2, \quad \sum_{j,k=1}^n \frac{\partial^2 \log K(z, \bar{z})}{\partial z_j \partial \bar{z}_k} Y_j Y_k = \frac{|Y u_1^Y(z_0)|^2}{|u_0(z_0)|^2}, \quad (4.5)$$

which proves the extremal property of the Bergman metric.
If $\psi$ is not immersion at $z_0$, the rank of $\psi$ is less than $n$ at $z_0$, thus there exists $Y \in \mathbb{C}^n \setminus \{0\}$ such that

$$(Y)_{1 \times n}[\frac{\partial (\frac{u_j}{u_0})}{\partial z_k}(z_0)]_{n \times N} = 0,$$

i.e., $Y\frac{u_j}{u_0}(z_0) = 0$ for every $j = 1, \cdots, N$. Thus

$$Y u_j^Y(z_0) = (Y\frac{u_j}{u_0}(z_0))u_0(z_0) = 0,$$

which shows that $X$ is not Bergman hyperbolic.

On the other hand, if $X$ is not Bergman hyperbolic, i.e., there exists $z_0 \in X$ and $Y \in \mathbb{C}^n \setminus \{0\}$ such that $Y u_j^Y(z_0) = 0$. By definition of $u_j^Y$, $Y u_j^Y(z_0) = (Y u_j^Y(z_0))u_0(z_0) = 0$, for every $j = 1, \cdots, N$. Thus $\psi$ is not immersion at $z_0$. We get that $\psi$ is immersion if and only if $X$ is Bergman hyperbolic.

The injection of $\psi$ is equivalent to the following: For every $z, w \in X$, $z \neq w$, $(u_0(z), \cdots, u_N(z))$ is not parallel to $(u_0(w), \cdots, u_N(w))$, i.e.,

$$\left(\sum_{j=0}^{N} |u_j(z)|^2\right)\left(\sum_{j=0}^{N} |u_j(w)|^2\right) > \left|\sum_{j=0}^{N} u_j(z)\overline{u_j(w)}\right|^2,$$

which is equivalent to $\mathcal{D}(z, w) > 0$. $\square$

By the above Theorem, we could give a very ampleness criterion of the coverings of Kähler hyperbolic manifolds, which is a slightly generalization of Yeung’s result (see [35]).

**Theorem 4.2.** Let $\{X_j\}$ be a tower of coverings of Kähler hyperbolic manifold $(X, \omega, \lambda)$. If $\tilde{X}$ is weakly pseudoconvex and possess a non-positive strictly plurisubharmonic function $\varphi$. Then the canonical line bundle of $X_j$ is very ample for sufficient large $j$.

**Proof.** Step 1: $\tilde{X}$ is Bergman hyperbolic and for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\inf_{p, q \in \tilde{X}, d(p, q) \geq \varepsilon} \tilde{\mathcal{D}}(p, q) \geq \delta_\varepsilon, \quad (4.6)$$

where $\tilde{\mathcal{D}}$ is the diastatic function of $\tilde{X}$.

By Richberg’s Theorem (see [29]), we may assume that $\varphi$ is negative and smooth. Let

$$\omega_0 = i\partial\bar{\partial}(-\log -\varphi).$$

For every $p \in \tilde{X}$, choose a local coordinate $\{z\}$ centered at $p$ such that

$$\{|z| < 1\} \subset \subset \tilde{X}.$$
Take \( v_0 = \overline{\nabla}(\chi(|z|^2)dz), \ v_j = \overline{\nabla}(z_j\chi(|z|^2)dz), \ j = 1, \cdots, n, \)
where \( \chi \in C^\infty(\mathbb{R}, [0, 1]) \) is a cut-off function such that \( \chi = 1 \) on \((-\infty, \frac{1}{2})\); \( \chi = 0 \) on \([1, +\infty)\) and \( |\chi'| \leq 3.\) For every \( k \in \mathbb{N}, \) choose sufficient large \( C_k \) such that
\[
\varphi_k := C_k\varphi + 2k\chi(|z|^2)\log |z|
\]
is plurisubharmonic on \( \tilde{X}. \) Now
\[
\int_{\tilde{X}} |v_0|_{\omega_0}^2 e^{-\varphi_k} dV_{\omega_0} < \infty, \ \int_{\tilde{X}} |v_j|_{\omega_0}^2 e^{-\varphi_{k+1}} dV_{\omega_0} < \infty, \ j = 1, \cdots, n.
\]
Consider the strongly Kähler hyperbolic manifold \((\tilde{X}, \omega_0),\) by Theorem 2.2, for \( j = 0, 1, \cdots, n, \) there exists \( u_j \in L^2_{\text{loc}}(\tilde{X}, \wedge^{n,0}T^*\tilde{X}) \) such that \( \overline{\partial}u_j = v_j \) in the sense of distribution with \( L^2 \) estimates. Thus
\[
u_j(0) = 0, \ j = 0, \cdots, n; \ \frac{\partial u_k}{\partial z_l}(0) = 0, \ k, l = 1, \cdots, n.
\]
Since \( \varphi_k \) is negative and the \( L^2 \) norm of a \((n,0)\)-form not depends on the metric. We get
\[
\chi(|z|^2)dz - u_0, \ z_j\chi(|z|^2)dz - u_j \in \mathcal{H}^{n,0}(\tilde{X}), \ j = 1, \cdots, n.
\]
By the extremal property of the Bergman kernel function and the Bergman metric, we get
\[
S_{\tilde{X}}(p) > 0, \ i\overline{\partial} log \tilde{K}(p) > 0,
\]
which proves that \( \tilde{X} \) is Bergman hyperbolic. By similar method, one could prove that \( \tilde{D}(p, q) > 0 \) for \( p \neq q. \) Since \( S_{\tilde{X}} \) is \( \Gamma \) invariant, there exists \( C > 0 \) such that
\[
\frac{1}{C} < S_{\tilde{X}} < C.
\]
Since
\[
\tilde{D}(p, q) = \tilde{D}(\gamma p, \gamma q)
\]
for every \( \gamma \in Aut(\tilde{X}). \) In order to prove (4.6), it suffices to show that
\[
\inf_{p \in F(x), \ d(p, q) \geq \varepsilon} \tilde{D}(p, q) \geq \delta \varepsilon,
\]
for fixed \( x \in \tilde{X}. \) By using the same method as in the proof of Theorem 3.3, there exist a sufficient large \( G \) such that
\[
|\tilde{K}|^2(p, q) \leq \frac{1}{2C^2},
\]
as long as \( d(p, q) \geq G. \) By definition, if \( d(p, q) \geq G \)
\[
\tilde{D}(p, q) \geq 2 \log 2.
\]
Since
\[ U_{G,\varepsilon} := \{(p, q) \in F(x) \times \tilde{X} \mid \varepsilon \leq d(p, q) \leq G\} \]
is compact,
\[ \delta_\varepsilon := \min\{2 \log 2, \inf_{(p, q) \in U_{G,\varepsilon}} \tilde{D}(p, q)\} > 0 \]
satisfies (4.7).

**Step 2:** There exists an constant \( A(n, D, |X|, \lambda, \tau) \) such that the canonical line bundle of \( X_j \) is very ample for \( \tau_j > A \).

By (3.16), there exists \( A_1(n, D, |X|, \lambda, \tau) \) such that \( X_j \) is Bergman hyperbolic and
\[ \frac{1}{C} < S_{X_j} < C \]
for \( \tau_j > A_1 \). Denote by \( d_j(D_j) \) the Bergman distance (Calabi’s diastatic) function on \( X_j \) respectively. By using the normal coordinate (see Calabi [6]),
\[ D_j(p, q) = d_j(p, q)^2 + O(d_j(p, q)^4). \]
where \( O(d_j(p, q)^4) \) is the curvature terms. Since the Bergman curvature of \( X_j \) is bounded (not depends on \( j \)). There exists \( \varepsilon > 0 \) (not depends on \( j \)) such that
\[ D_j(p, q) \geq \frac{1}{2} d_j(p, q)^2 > 0 \]
for \( 0 < d(p, q) \leq \varepsilon \) and \( \tau_j > A_1 \). Fix such \( \varepsilon \), by Theorem 3.3, there exists \( A_2 \) such that
\[ \frac{S_{X_j}}{S_X} > (1 + \frac{\delta_\varepsilon}{4}) e^{-\frac{\delta_\varepsilon}{4}}, \quad |\tilde{K} - K_j|^2(p, q) \leq \frac{1}{16C^2} \frac{\delta_\varepsilon^2}{e^{\delta_\varepsilon}}, \]
for \( \tau_j > A_2 \). If \( d(p, q) \geq \varepsilon \), we have
\[ \sqrt{\frac{S_{X_j}(p)S_{X_j}(q)}{|K_j|(p, q)}} > 1, \]
for \( \tau_j > A := \max\{A_1, A_2\} \), which proves Step 2. \( \square \)

An elementary method for the effective very ampleness criterion of the compact ball quotients will give in the next section.

## 5 Proof of Theorem 1.1

**Proof of Theorem 1.1.** By Lemma 9.2, we only need to estimate \( M_{B_x(\tau)}^{p,0}(x) \), \( x \in \tilde{X} \) and \( 0 \leq p \leq n - 1 \). Let \( f \) be a holomorphic p-forms on \( B_x(\tau) \) with
\[ \int_{\rho(z) < \tau} |f|^2 \ dV = 1, \]
(5.1)
where $\rho(z) := d\xi(z, x)$. Choose a family of cut-off functions $\chi_{N, \varepsilon} \in C^1(\mathbb{R}, [0, 1])$ such that $\chi_{N, \varepsilon} = 1$ on $(-\infty, 1 - \frac{1}{N})$ and $\chi_{N, \varepsilon} = 0$ on $(1, +\infty)$ with $-N - \varepsilon \leq \chi_{N, \varepsilon} \leq 0$. By Theorem 2.1, for $0 < t < 1$,

$$
\int_{\rho(z)<\tau(1-t)(1-t)} |f|^2 \, dV \leq ||\chi_{N, \varepsilon}(\frac{\rho(z)}{\tau(1-t)})f||^2 \leq \frac{8\lambda}{n-p} ||\phi(\chi_{N, \varepsilon}(\frac{\rho(z)}{\tau(1-t)})f)||^2 \leq \frac{4\lambda(N + \varepsilon)^2}{(n-p)\tau^2(1-t)^2}.
$$

(5.2)

Let $N = 2, \varepsilon, t$ goes to zero, we have

$$
\int_{\rho(z)<\frac{\tau}{2}} |f|^2 \, dV \leq \frac{16\lambda}{(n-p)\tau^2}.
$$

(5.3)

By the Bochner formula for holomorphic tensor field of covariant degree $p$ (see Kobayashi-Horst [22]),

$$
-\partial^i \partial(|f|^2) \geq -p|f|^2.
$$

(5.4)

If $n = 1$, by Lemma 8.1, Lemma 9.2 and (6.3),

$$
\frac{1}{|X|} |p_1 - \tilde{p}_1| \leq \frac{64\lambda}{\pi^4}.
$$

(5.5)

If $n \geq 2$, assume $\tau \geq 2\sqrt{2n}$, by Lemma 8.3 (in case $r = \sqrt{2n}$) and Lemma 9.2,

$$
\frac{1}{|X|} |p_1 - \tilde{p}_1| \leq \sum_{p=0}^{n-1} \frac{16\lambda\pi^p n!}{(n-p)\tau^2(2\pi n)^n} \binom{n+p}{p} \binom{n}{p},
$$

(5.6)

Since

$$
\binom{n+p}{p} = \frac{1}{2} \left( \binom{n+p}{p} + \binom{n+p}{n} \right) \leq 2^{n+p-1}, \binom{n}{p} \leq 2^{n-1},
$$

(5.7)

we get

$$
\frac{1}{|X|} |p_1 - \tilde{p}_1| \leq 4(2^n - 1)(\frac{2e}{n})^n \frac{\lambda}{\pi^n \tau^2} \leq 16(\frac{4}{\pi})^n \sqrt{\frac{\pi}{\tau^2}},
$$

(6.1)

(The second inequality is due to the Stirling’s formula) which proves Theorem 1.1. ∎

By (8.12), Theorem 1.2 follows by using similar method as above.

6 Proof of Theorem 1.4

Proof of Theorem 1.4. Let $f_y$ be a holomorphic n-form on $B_x(R)$ such that

$$
|f_y|^2(y) = S_{B_x(R)}(y), \ |f_y|_{B_x(R)} = 1.
$$

By Lemma 8.3 (in case $r = \sqrt{2n}$) and the Bochner formula for holomorphic tensor field of covariant degree $n$ (see Kobayashi-Horst [22]), we have

$$
S_{B_x(R)}(y) \leq e^n \frac{n!}{(2\pi n)^n} \binom{2n}{n} \leq \left( \frac{2}{\pi} \right)^n \sqrt{2},
$$

(6.1)
(the second inequality is due to the Stirling’s formula). Thus

\[ S_X \leq \left( \frac{2}{\pi} \right)^n \sqrt{2}. \]  

Choosing a family of cut-off function \( \chi_{N,\varepsilon} \) as in the proof of Theorem 1.1. Let \( \rho_y(z) = d(y, z) \), for every \( 0 < t < 1 \), one could solve

\[ \overline{\partial} u_{N,\varepsilon,t,y} = \overline{\partial} \left( \chi_{N,\varepsilon}(\frac{\rho_y}{(1-t)\delta_y})f_y \right) \]  

on \( X \) such that

\[ ||\chi_{N,\varepsilon}(\frac{\rho_y}{(1-t)\delta_y})f_y||^2 = ||u_{N,\varepsilon,t,y}||^2 + ||\chi_{N,\varepsilon}(\frac{\rho_y}{(1-t)\delta_y})f_y - u_{N,\varepsilon,t,y}||^2, \]

(i.e., \( u_{N,\varepsilon,t,y} \) is the \( L^2 \) minimal solution. By Theorem 2.1,

\[ ||u_{N,\varepsilon,t,y}||^2 \leq 8\lambda ||\overline{\partial} \left( \chi_{N,\varepsilon}(\frac{\rho_y}{(1-t)\delta_y})f_y \right)||^2 \leq \frac{4\lambda(N + \varepsilon)^2}{(1-t)^2\delta_y^2}. \]  

Thus

\[ S_X(y) \geq \frac{||\chi_{N,\varepsilon}(\frac{\rho_y}{(1-t)\delta_y})f_y - u_{N,\varepsilon,t,y}||^2(y)}{||\chi_{N,\varepsilon}(\frac{\rho_y}{(1-t)\delta_y})f_y - u_{N,\varepsilon,t,y}||^2} \geq S_{B_x(R)}(y) - 2\sqrt{S_{B_x(R)}(y)\|u_{N,\varepsilon,t,y}\|(y). \]  

Let \( N = 2, t, \varepsilon \) goes to zero, since \( u_{N,\varepsilon,t,y} \) is holomorphic on \( \rho_y \leq \delta_y(1-t)(1-\frac{1}{N}) \). The Theorem follows by using similar method as in the proof of Theorem 1.1. \( \square \)

7 Appendix I

We will fix some basic notions of the Hodge theory on non-compact complex manifolds (see Demailly’s open-book [11] and [4]).

Let \( (X, \omega) \) be a complex manifold of dimension \( n \) and \( F_1, F_2, F_3 \) be Hermitian \( C^\infty \) vector bundles over \( X \). Let

\[ P : C^\infty(X, F_1) \to C^\infty(X, F_2) \]

be a differential operator with smooth coefficients, denote by

\[ P^* : C^\infty(X, F_2) \to C^\infty(X, F_1) \]

the formal adjoint of \( P \), i.e., the unique operator such that for all \( u \in C^\infty(X, F_1) \) and \( v \in C^\infty(X, F_2) \),

\[ \langle \langle Pu, v \rangle \rangle = \langle \langle u, P^*v \rangle \rangle, \]

whenever \( suppu \cap suppiv \) is compact. \( P \) induces a non-bounded operator

\[ \tilde{P} : L^2(X, F_1) \to L^2(X, F_2) \]  

(7.1)
as follows: we say that \( u \in \text{Dom}(\tilde{P}) \), if there exists a \( v(=\tilde{P}u) \in L^2(X,F_2) \), such that
\[
\langle\langle u, P^*g \rangle\rangle = \langle\langle v, g \rangle\rangle
\]
for all \( g \in C_0^\infty(X,F_2) \).

\( P \) could also induce a non-bounded operator
\[
\vec{P} : L^2(X,F_1) \to L^2(X,F_2)
\]
by taking limits. We say that \( u \in \text{Dom}(\vec{P}) \) if there exists a sequence \( \{u_j\} \subset C_0^\infty(X,F_1) \) such that
\[
||u_j - u|| \to 0 (j \to \infty),
\]
and \( \{Pu_j\} \) is a Cauchy sequence of \( L^2(X,F_2) \). Now
\[
\vec{P}u := \lim_{j \to \infty} Pu_j \in L^2(X,F_2).
\]
It follows that both \( \tilde{P} \) and \( \vec{P} \) are densely defined with their graphs closed and
\[
\tilde{P} = \tilde{P}|_{\text{Dom}\tilde{P}}.
\]
By definition, \( C_0^\infty(X,F_1) \) is dense in \( \text{Dom}\tilde{P} \) for the graph norm with respect to \( \tilde{P} \). While \( C_0^\infty(X,F_1) \) is dense in \( \text{Dom}\vec{P} \) for the graph norm with respect to \( \vec{P} \) if and only if \( \vec{P} = \tilde{P} \). Both \( \tilde{P} \) and \( \vec{P} \) have a unique Von-Neumann adjoint. Take \( \tilde{P} \) for example, its Von-Neumann adjoint
\[
(\tilde{P})^* : L^2(X,F_2) \to L^2(X,F_1)
\]
is defined as follows: we say that \( u \in \text{Dom}((\tilde{P})^*) \) if there exists a \( v \in L^2(X,F_1) \), such that
\[
\langle\langle u, \tilde{P}f \rangle\rangle = \langle\langle v, f \rangle\rangle
\]
for all \( f \in \text{Dom}(\tilde{P}) \). By definition we have
\[
(\tilde{P})^* = \tilde{P}^\sharp.
\]
But in general, \( (\tilde{P})^* \) does not coincide with \( \tilde{P}^\sharp \). Actually \( \text{Dom}((\tilde{P})^*) \) consists of all \( u \in \text{Dom}(\tilde{P}) \) satisfying some additional boundary conditions. The following lemma is due to Hörmander [18].

**Lemma 7.1.** The following conditions on \( \tilde{P} \) are equivalent:

1) \( \text{Im}\tilde{P} \) is closed.
2) \( \text{Im}(\tilde{P})^* \) is closed.
3) There is a constant \( C \) such that
\[
||u|| \leq C||\tilde{P}u||, \ u \in \text{Dom}(\tilde{P}) \cap \text{Im}(\tilde{P})^*.
\]
4) There is a constant $C$ such that

$$||v|| \leq C||(\tilde{P})^*v||, \ v \in \text{Dom}((\tilde{P})^*) \cap \text{Im}\tilde{P}.$$  

The best constants above are the same.

If there is another differential operator $Q : C^\infty(X, F_2) \to C^\infty(X, F_3)$ with smooth coefficients, satisfying

$$\tilde{Q} \circ \tilde{P} = 0,$$

we have

$$L^2(X, F_2) = H \oplus \text{Im}(\tilde{Q})^* \oplus \text{Im}\tilde{P}$$

and

$$\text{Ker}\tilde{Q} = H \oplus \text{Im}P,$$

where

$$H = \text{Ker}\tilde{Q} \cap \text{Ker}(\tilde{P})^*.$$  

The following lemma is still due to Hörmander.

**Lemma 7.2.** A necessary and sufficient condition for $\text{Im}(\tilde{Q})^*$ and $\text{Im}\tilde{P}$ both to be closed is that:

$$||u||^2 \leq C^2(||\tilde{Q}u||^2 + ||(\tilde{P})^*u||^2), \ u \in \text{Dom}(\tilde{Q}) \cap \text{Dom}(\tilde{P})^* \cap H^\perp.$$  

What’s more,

$$||u||^2 \leq C^2(||\tilde{Q}u||^2 + ||(\tilde{P})^*u||^2), \ u \in \text{Dom}(\tilde{Q}) \cap \text{Dom}(\tilde{P})^*$$

is equivalent to

$$\text{Im}(\tilde{Q})^* = \text{Ker}(\tilde{P})^*, \ \text{Im}\tilde{P} = \text{Ker}\tilde{Q}.$$  

Let $E$ be a rank $r$ holomorphic vector bundle over $X$, with Hermitian metric $h$. Then we have a unique Chern connection $D$ on $\wedge^{p,q}T^*X \otimes E$ such that:

1) If $D$ is split as a sum of $(1,0)$ and $(0,1)$ connection $D = D' + D'', \ D'' = \bar{\partial}.$

2) $D$ is Hermitian connection with respect to the metric $h.$
Let
\[ \Theta := D^2 : C^\infty(X, \Lambda^{p,q} T^* X \otimes E) \to C^\infty(X, \Lambda^{p+1,q+1} T^* X \otimes E), \]
\[ L := \omega \wedge \cdot : C^\infty(X, \Lambda^{p,q} T^* X \otimes E) \to C^\infty(X, \Lambda^{p+1,q+1} T^* X \otimes E), \]
\[ \Lambda := L^* : C^\infty(X, \Lambda^{p+1,q+1} T^* X \otimes E) \to C^\infty(X, \Lambda^{p,q} T^* X \otimes E), \]
\[ D : C^\infty(X, \Lambda^r(T^*_\mathbb{R} X \otimes \mathbb{C}) \otimes E) \to C^\infty(X, \Lambda^{r+1}(T^*_\mathbb{R} X \otimes \mathbb{C}) \otimes E), \]
\[ D^* : C^\infty(X, \Lambda^{r+1}(T^*_\mathbb{R} X \otimes \mathbb{C}) \otimes E) \to C^\infty(X, \Lambda^r(T^*_\mathbb{R} X \otimes \mathbb{C}) \otimes E), \]
\[ \overline{\partial} : C^\infty(X, \Lambda^{p,q+1} T^* X \otimes E) \to C^\infty(X, \Lambda^{p,q} T^* X \otimes E), \]
\[ D' : C^\infty(X, \Lambda^{p,q} T^* X \otimes E) \to C^\infty(X, \Lambda^{p+1,q} T^* X \otimes E), \]
\[ (D')^* : C^\infty(X, \Lambda^{p+1,q} T^* X \otimes E) \to C^\infty(X, \Lambda^{p,q} T^* X \otimes E), \]
\[ \Delta' := D'(D')^* + (D')^* D' : C^\infty(X, \Lambda^{p,q} T^* X \otimes E) \to C^\infty(X, \Lambda^{p,q} T^* X \otimes E), \]
\[ \Delta'' := \overline{\partial} \overline{\partial} + \partial \partial^* : C^\infty(X, \Lambda^{p,q} T^* X \otimes E) \to C^\infty(X, \Lambda^{p,q+1} T^* X \otimes E), \]
\[ \Delta := DD^* + D^* D : C^\infty(X, \Lambda^r(T^*_\mathbb{R} X \otimes \mathbb{C}) \otimes E) \to C^\infty(X, \Lambda^r(T^*_\mathbb{R} X \otimes \mathbb{C}) \otimes E). \]

The Hermitian metric \( h \) on \( E \) could induce a sesquilinear pairing \( \{ \cdot, \cdot \} \) as follows. For an arbitrary holomorphic trivialization \( \theta : E|_\Omega \to \Omega \times \mathbb{C}^r \).

Let \( H = (h_{\lambda \mu}) \) be the Hermitian matrix with smooth coefficients representing the metric along the fibre of \( E|_\Omega \). For any \( s, t \in C^\infty_\omega(X, E) \) and \( \sigma = \theta(s), \tau = \theta(t) \), one can write
\[ \{ s, t \} = \sum_{\lambda, \mu} h_{\lambda \mu} \sigma^\lambda \wedge \overline{\tau}^\mu. \]

The Hodge-Poincaré-De Rham operator \( * \) is the collection of \( \mathbb{C} \)-linear isometric maps defined by
\[ * : \wedge^{p,q} T^* X \otimes E \to \wedge^{n-q,n-p} T^* X \otimes E, \quad \{ s, *t \} = \langle s, t \rangle dV, \]
where \( \langle \cdot, \cdot \rangle \) is the point-wise Hermitian inner product. The following lemma is classical:

**Lemma 7.3.** \( *\Delta'' = \Delta'*, \; *\Delta' = \Delta''*, \; *\Delta = \Delta * \). What’s more, if \( (X, \omega) \) is Kähler and \( E \) is trivial with trivial metric, \( \Delta = 2\Delta' = 2\Delta'' \) commutes with all operators \( *, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*, L, \Lambda \).

Since the order of \( L, \Lambda \) is zero, they could be point-wisely defined. Let’s recall the notion of primitive: a homogeneous element \( u \in \Lambda^s(T^*_\mathbb{R} X \otimes \mathbb{C}) \) is called primitive if \( \Lambda u = 0 \). The space of primitive elements of total degree \( s \) will be denoted by
\[ \text{Prim}^s T^* X = \oplus_{p+q=s} \text{Prim}^{p,q} T^* X. \]

Now we can state the primitive decomposition formula (see [11] for the proof):
Lemma 7.4. For every \( u \in \wedge^s(T^*_{\mathbb{R}} X \otimes \mathbb{C}) \), there is a unique decomposition

\[
  u = \sum_{r=(s-n)_+}^{\lfloor s/2 \rfloor} L^r u^{(s-2r)}, \quad u^{(s-2r)} \in Prim^{s-2r}T^*X,
\]

where \((s-n)_+ = \max\{0, s-n\}\), \([s/2]\) is the integer part of \(s/2\).

With the help of primitive decomposition formula, we could prove the following result which is crucial in finding the precise lower bound of the spectrum of Laplace-Beltrami operator in Kähler hyperbolic case.

Lemma 7.5. If \((X, \omega)\) is Kähler, \(\varphi \in C_0^\infty(X, \wedge^{n-k}(T^*_{\mathbb{R}} X \otimes \mathbb{C}))\), \(k \geq 1\).

\[
  \langle \triangle(L^k \varphi), L^k \varphi \rangle \geq k \langle \triangle(L^{k-1} \varphi), L^{k-1} \varphi \rangle, \quad ||L^k \varphi||^2 \geq k ||L^{k-1} \varphi||^2.
\]

Proof. For any \( u \in C_0^\infty(X, \wedge^s(T^*_{\mathbb{R}} X \otimes \mathbb{C})) \), by primitive decomposition formula,

\[
  u = \sum_{r=(s-n)_+}^{\lfloor s/2 \rfloor} L^r u^{(s-2r)}.
\]

Since \(u^{(s-2r)}\) is primitive, according to Lemma 7.3,

\[
  \langle \triangle u, u \rangle = \sum_{r=(s-n)_+}^{\lfloor s/2 \rfloor} \left( \prod_{t=1}^r t(n + 2r - s - t + 1) \right) \langle \triangle u^{(s-2r)}, u^{(s-2r)} \rangle,
\]

and

\[
  ||u||^2 = \sum_{r=(s-n)_+}^{\lfloor s/2 \rfloor} \left( \prod_{t=1}^r t(n + 2r - s - t + 1) \right) ||u^{(s-2r)}||^2,
\]

where \(\prod_{t=1}^0 := 1\). Thus for \(0 \leq m \leq k\),

\[
  \langle \triangle(L^m \varphi), L^m \varphi \rangle = \sum_{r=0}^{[n-k/2]} b_{k,m,r} \langle \triangle \varphi^{(n-k-2r)}, \varphi^{(n-k-2r)} \rangle,
\]

and

\[
  ||L^m \varphi||^2 = \sum_{r=0}^{[n-k/2]} b_{k,m,r} ||\varphi^{(n-k-2r)}||^2,
\]

where

\[
  b_{k,m,r} = \prod_{t=1}^{r+m} t(2r + k + 1 - t).
\]

Hence

\[
  ||L^k \varphi||^2 \geq \min_{0 \leq r \leq [n-k/2]} \left\{ \frac{b_{k,k,r}}{b_{k,k-1,r}} \right\} ||L^{k-1} \varphi||^2 = k ||L^{k-1} \varphi||^2,
\]

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and
\[ \langle \triangle (L^k \varphi), L^k \varphi \rangle \geq k \langle \triangle (L^{k-1} \varphi), L^{k-1} \varphi \rangle. \]

The Lemma is proved. \[ \square \]

By the above discussion, one can define \( \tilde{\Theta}, \tilde{L}, \ldots \) as closed and densely defined operators on \( L^2 := \bigoplus_{r=1}^{2n} L^2(X, \wedge^r T^* X \otimes \mathbb{C}) \otimes E = \bigoplus_{p,q=1}^n L^2(X, \wedge^p \alpha^q T^* X \otimes E). \) Because \( \overline{\partial \partial} = 0, \) by definition, \( \overline{\partial \partial} = 0, \) one has the orthogonal decompositions
\[ L^2 = \mathcal{H}_\partial \oplus \text{Im}(\overline{\partial} \partial) \oplus \text{Im}(\overline{(\partial \partial)^*}), \tag{7.2} \]
where
\[ \mathcal{H}_\partial = \text{Ker}(\overline{\partial}) \cap \text{Ker}(\overline{(\partial \partial)^*}) \subset C^\infty, \]
(Similar results and definitions for \( \mathcal{H}_{\partial'}, \mathcal{H}_{\partial''} \) and \( \mathcal{H}_{(\partial')^*} \)). We also define
\[ \mathcal{H}_{\tilde{\partial}} = \text{Ker}(\tilde{\partial}) \cap \text{Ker}(\overline{(\tilde{\partial})^*}) \subset C^\infty. \]
If \( E \) is flat, one also has
\[ L^2 = \mathcal{H} \oplus \text{Im}(\overline{\partial} \partial) \oplus \text{Im}(\overline{(\partial \partial)^*}). \tag{7.3} \]

The space of \( L^2 \) harmonic forms with respect to \( \Delta \) will be defined as \( \mathcal{H}_{\Delta} = \text{Ker}(\overline{\partial \partial}), \) (similar definitions for \( \Delta' \) and \( \Delta'' \)). If \((X, \omega)\) is non-complete, generally \( \mathcal{H}_{\tilde{\partial}} \) is only a subset of \( \mathcal{H}_{\Delta}, \) but if \((X, \omega)\) is complete, one has the following classical result (see Hörmander’s density technique [13], [1] and Demailly’s open book [11]).

**Lemma 7.6.** If \((X, \omega)\) is complete, one has
\[ \tilde{\partial} = \partial, \text{ Dom}(\overline{(\tilde{\partial} \partial)^*}) \subset \text{Dom}(\overline{\partial \partial}) \cap \text{Dom}(\overline{\partial \partial}^*), \tag{7.4} \]
(Similar results for \( \Delta' \) and \( \Delta'' \)) and
\[ \mathcal{H}_{\tilde{\partial}} = \mathcal{H}_\partial = \mathcal{H}_\Delta := \mathcal{H}, \tag{7.5} \]
(Similar definitions for \( \mathcal{H} \) and \( \mathcal{H}_{\Delta} \)).

The Schwartz kernel for the projection onto \( \mathcal{H}_{\tilde{\partial}}^{p,q}(X, E) \) is defined as
\[ B_{X,E}^{p,q} := \sum_j u_j \otimes \overline{u_j} \in H^0(X \times X^*, (\wedge^p \alpha^q T^* X \otimes E) \otimes (\wedge^p \alpha^q T^* (X^*) \otimes \overline{E})) := H_{\overline{\partial \partial}}^{p,q}, \tag{7.6} \]
where \( u_j \) is any complete orthonormal base of the separated Hilbert space \( \mathcal{H}_{\partial \partial}^{p,q}(X, E), \) \( X^* \) the conjugated complex manifold of \( X \) and \( E \) the conjugate of \( E. \) Now let’s recall the notion \( \boxtimes. \) Let \( L_X \) and \( L_Y \) be the vector bundles over the complex manifolds \( X \) and \( Y \) respectively. By definition,
\[ L_X \boxtimes L_Y := P_1^* L_X \otimes P_2^* L_Y, \tag{7.7} \]
where \( P_1 : X \times Y \to X \) and \( P_2 : X \times Y \to Y \) are the canonical projection map.

For any \( u \in \mathcal{H}_{\nu}^{p,q}(X, E) \) and \( P, Q, R \in H^{p,q}_{\nu} \), we shall define

\[
\langle\langle u, P \rangle\rangle = \sum_{j,k} \langle\langle u(\cdot), P_{kj}(\cdot, \bar{w})P_{j}(\cdot) \rangle\rangle P_{k}(w),
\]

(7.8)

\[
\langle\langle P, Q \rangle\rangle = \sum_{j,k,l,m} \langle\langle P_{kj}(\cdot, \bar{w})P_{j}(\cdot), Q_{ml}(\cdot, \bar{z})P_{l}(\cdot) \rangle\rangle P_{m}(z) \otimes P_{k}(w),
\]

(7.9)

\[
\text{Trace}(R) = \sum_{j,k} R_{kj}(z, \bar{z})R_{j}(z),
\]

(7.10)

and

\[
\langle\langle P, Q \rangle\rangle = \int_X \text{Trace}\langle\langle P, Q \rangle\rangle dV,
\]

(7.11)

where

\[
P = \sum_{j,k} P_{kj}(z, \bar{w})P_{j}(z) \otimes P_{k}(w),
\]

(7.12)

Thus, sometimes we call \( B_{X,E}^{p,q} \) the reproducing kernel or generalized Bergman kernel of \( \mathcal{H}_{\nu}^{p,q}(X, E) \).

We call

\[
S_{X,E}^{p,q} := \text{Trace}(B_{X,E}^{p,q})
\]

(7.13)

and

\[
S_{X,E}^{p,q} := \text{Trace}(B_{X,E}^{p,q})dV
\]

(7.14)

the Schwartz kernel functions and forms respectively. When \( E \) is trivial with trivial metric, we shall omit the lower index \( E \), and call

\[
S_X := S_X^{n,0}
\]

(7.15)

and

\[
\mathcal{S}_X := \mathcal{S}_X^{n,0}
\]

(7.16)

the Bergman kernel function and form respectively.

Sometimes, it is convenient to use the following extremal function:

\[
M_{X,E}^{p,q} := \sup \{|u|^2 | u \in \mathcal{H}_{\nu}^{p,q}(X, E), \|u\| = 1\},
\]

(7.17)

(similar definition for \( M_X^{p,q} \)). By Berndtsson’s Lemma [3],

\[
M_{X,E}^{p,q} \leq S_{X,E}^{p,q} \leq r \binom{n}{p} \binom{n}{q} M_{X,E}^{p,q},
\]

(7.18)
where \( \binom{n}{p} \) are binomial coefficients.

If \((X, \omega)\) is a Kähler manifold, one has the Bochner-Kodaira-Nakano identity (see [27] for generalization)
\[
\Delta'' = \Delta' + [i\Theta, \Lambda],
\]  
(7.19)
which implies:

**Lemma 7.7.** If \((X, \omega)\) is a complete Kähler manifold and \(E\) is flat, one has the following orthogonal decomposition
\[
\mathcal{H}^r_{L^2}(M, E) = \oplus_{p+q=r} \mathcal{H}^p_q(M, E),
\]  
(7.20)
where
\[
\mathcal{H}^* = \mathcal{H}^* = \mathcal{H}^* := \mathcal{H}^*_{L^2}.
\]  
(7.21)

### 8 Appendix II

A Cartan-Hadamard (CH) manifold is a complete, simply-connected Riemannian manifold of nonpositive curvature. The following result is due to Green and Wu [15].

**Lemma 8.1.** Let \(X\) be a CH manifold with (real) dimension \(2n\), for any point \(x\) in \(X\), if \(u\) is a non-negative subharmonic function on the geodesic open ball \(B_x(r)\) of radius \(r\) around \(x\) in \(X\), then
\[
\int_{B_x(r)} u \, dV \geq \frac{(\pi r^2)^n}{n!} u(x).
\]  
(8.1)
Another version is due to Li-Schoen [23]. Theorem 2.1 in [23] is slightly different from the following Theorem. But the Theorem below follows easily from Li-Schoen’s result by estimating the constant in Theorem 2.1 [23] carefully.

**Lemma 8.2.** Let \(X\) be a \(m\) real dimensional compact Riemannian manifold with (possibly empty) boundary, for any point \(x\) in \(X\), if \(X\) has no boundary, assume its diameter is no less than \(2r\), otherwise assume that the distance from \(x\) to the boundary of \(X\) is at least \(5r\). Suppose the Ricci curvature of \(X\) is bounded below by \(- (m - 1)k^2\), where \(k \geq 0\). If \(u\) is a non-negative subharmonic function on the geodesic open ball \(B_x(r)\) of radius \(r\) around \(x\) in \(X\), then
\[
\int_{B_x(r)} u \, dV \geq 2^{-200(m-1)(1+kr)} |B_x(r)| \sup_{B_x(r/2)} u,
\]  
(8.2)
where \(|B_x(r)|\) stands for the volume of \(B_x(r)\).

We also have similar results on non-subharmonic functions.

**Lemma 8.3.** Let \(X\) be a CH Kähler manifold with (complex) dimension \(n\). For any non-negative smooth function \(u\) on the geodesic open ball \(B_x(r)\) of radius \(r\) around \(x\) in \(X\), if
\[
-\overline{\partial} \partial u + pu \geq 0,
\]
on $B_x(r)$ for some $p \in \mathbb{Z}^+$, then
\[
u(x) \leq e^{r^2/2} \frac{n!}{(\pi r^2)^n} \left( \frac{n + p}{p} \right) \int_{B_x(r)} u \, dV. \tag{8.3}\]

**Proof.** Consider the complete Kähler metric
\[2 \text{Re} \left( \sum_{j=1}^{p} dt_j \otimes d \bar{t}_j + \sum_{\alpha, \beta=1}^{n} g_{\alpha \beta} dz^\alpha \otimes d \bar{z}^\beta \right), \tag{8.4}\]
on $\mathbb{C}^p \times X$, where $\omega = i \sum_{\alpha, \beta=1}^{n} g_{\alpha \beta} dz^\alpha \wedge d \bar{z}^\beta$. Denote by $\omega_t$, $dV_t$ the associated fundamental form and the volume form respectively. Now our distance function
\[d_t((0, x), (t, z)) = \sqrt{2|t|^2 + \rho(z)^2}, \tag{8.5}\]
where $\rho(z) = d(x, z)$ is the distance function on $(X, \omega)$. Denote by $\partial_t$ the Cauchy-Riemann operator on $\mathbb{C}^p \times X$, we have
\[-\partial_t^* \partial_t (e^{|t|^2} u) \geq |t|^2 e^{|t|^2} u \geq 0, \tag{8.6}\]
by Lemma 8.1, we have
\[\int_{d_t < r} e^{|t|^2} u \, dV_t \geq \frac{(\pi r^2)^{n+p}}{(n+p)!} u(x). \tag{8.7}\]
The Lemma follows from
\[\int_{d_t < r} e^{|t|^2} u \, dV_t \leq \int_{\{2|t|^2 < r^2\} \times B_x(r)} e^{|t|^2} u \, dV_t \]
\[\leq (2^p e^{r^2/2} \int_{2|t|^2 < r^2} d\lambda)(\int_{B_x(r)} u \, dV) = e^{r^2/2} \frac{(\pi r^2)^p}{p!} \int_{B_x(r)} u \, dV, \]
where $d\lambda$ is the Lebesgue measure on $\mathbb{C}^p$. □

One needs the following Lemma to generalize Lemma 8.2.

**Lemma 8.4.** Let $X$ be a $m$ real dimensional compact (without boundary) Riemannian manifold with Ricci curvature bounded below by $-(m-1)k^2$, $k > 0$. Denote by $D$ the diameter of $X$. For any $x \in X$, if $0 < a \leq b \leq D$ and $\partial B_x(b) \neq \emptyset$, we have
\[\frac{|B_x(b)|}{|B_x(a)|} \leq \left( \frac{b}{a} \right) \frac{\sinh(bk)}{\sinh(ak)} \leq \left( \frac{b}{a} \right)^{m-1} e^{(m-1)(b-a)k}. \tag{8.8}\]

**Proof.** Let $\rho(\cdot) := d(\cdot, x)$ be the distance function of $X$. By Laplacian comparison Theorem (see Green and Wu [15]),
\[\Delta \rho \leq (m-1)k \coth(k \rho) \tag{8.9}\]
in the sense of distribution. Thus
\[ \Delta \rho^2 \leq 2 + 2(m - 1)k\rho \coth(k\rho). \tag{8.10} \]
For any \(0 < r \leq b\), integrating the above formula over \(B_x(r)\), we get
\[ \int_{B_x(r)} \Delta \rho^2 \, dV \leq (2 + 2(m - 1)kr \coth(kr))|B_x(r)|. \]
Since
\[ \int_{B_x(r)} \Delta \rho^2 \, dV = \int_{\partial B_x(r)} *d\rho^2 = 2r(\frac{d}{dr}|B_x(r)|), \tag{8.11} \]
we have
\[ \frac{|B_x(b)|}{|B_x(a)|} \leq e^{\int_a^b (1/r + (m-1)k \coth(kr))dr} = \left( \frac{b}{a} \right) \frac{\sinh(bk)}{\sinh(ak)}^{m-1}, \]
which proves the Lemma. \(\Box\)

Consider the product \(X \times \mathbb{R}\), by the above Lemma, we have the similar result as Lemma 8.3.

**Lemma 8.5.** Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n\), with Ricci curvature bounded below by \(-1\) and diameter no less than 2. For any non-negative smooth function \(u\) on the unit ball \(B_x\) with center \(x\) in \(X\), if
\[ -\bar{\partial} \partial u + pu \geq 0, \]
on \(B_x\) for \(0 \leq p \leq n - 1\), then
\[ u(x) \leq 2^{201(2n + \sqrt{n})} \frac{1}{|B_x|} \int_{B_x} u \, dV. \tag{8.12} \]

## 9 Appendix III

Let’s recall some basic facts on covering spaces. Let \((\tilde{X}, \tilde{g})\) be a Riemannian manifold. Let \(\Gamma\) be a subgroup of the isometrics that acts freely and properly discontinuously on \(\tilde{X}\). Let \(X = \tilde{X}/\Gamma\) be the quotient manifold and \(P : \tilde{X} \to X\) be the covering map. Equip \(X\) with the push-down metric \(g\) so that \(P^*g = \tilde{g}\). Denote by \(d_{\tilde{X}}\) and \(d_X\) the distance functions of \((\tilde{X}, \tilde{g})\) and \((X, g)\) respectively. For \(x \in \tilde{X}\), let
\[ F(x) := \{ y \in \tilde{X} \mid d_{\tilde{X}}(y, x) < d_{\tilde{X}}(y, \gamma x), \forall \gamma \in \Gamma \setminus 1 \} \tag{9.1} \]
be the Dirichlet fundamental domain centered at \(x\) and
\[ \tau(x) := \frac{1}{2} \inf \{ d_{\tilde{X}}(x, \gamma x) \mid \gamma \in \Gamma \setminus 1 \} \tag{9.2} \]
be the quasi-injectivity radius of \(P(x)\) in \(X\) with respect to the covering map \(P\). By definition, the geodesic ball \(B_x(\tau(x))\) is contained in \(F(x)\) and if \(\tilde{X}\) has no conjugate
points, \( \tau(x) \) is the injectivity radius of \( P(x) \) in \( X \). In particular, this is the case when \( \tilde{X} \) is CH manifold.

If \((\tilde{X}, \tilde{\omega})\) is the universal covering space of the compact complex manifold \((X, \omega)\), one has the following famous \( L^2 \) index formula due to Atiyah [2].

**Lemma 9.1.** For any compact complex manifold \((X, \omega)\), one has

\[
\chi(X, \Omega^p) = \chi_{L^2}(X, \Omega^p). \tag{9.3}
\]

Here \( \Omega^p \) is the sheaf of germs of holomorphic \( p \)-forms on \( n \) dimensional compact complex manifold \((X, \omega)\). The holomorphic Euler characteristic of \( \Omega^p \) is defined as

\[
\chi(X, \Omega^p) := \sum_{q=0}^{n} (-1)^q h^{p,q}, \tag{9.4}
\]

where the Hodge number \( h^{p,q} \) is defined as

\[
\int_X S^{p,q}_X. \tag{9.5}
\]

The \( L^2 \) Euler characteristic of \( \Omega^p \) is defined as

\[
\chi_{L^2}(X, \Omega^p) := \sum_{q=0}^{n} (-1)^q \tilde{h}^{p,q}, \tag{9.6}
\]

where the \( L^2 \) Hodge number \( \tilde{h}^{p,q} \) is defined as

\[
\int_{\tilde{X}} S^{p,q}_{\tilde{X}}. \tag{9.7}
\]

According to Atiyah’s \( L^2 \) Index Formula, if \( X \) is Kähler hyperbolic, we have the following Lemma.

**Lemma 9.2.** Let \( X \) be a Kähler hyperbolic manifold, one has

\[
\frac{1}{|X|} |p_1 - \tilde{p}_1| \leq \sum_{p=0}^{n-1} \binom{n}{p} \sup_{z \in \tilde{X}} M^{p,0}_{B_\tau}(z). \tag{9.8}
\]

**Proof.** According to Atiyah’s \( L^2 \) Index Formula,

\[
\sum_{p=0}^{n} (-1)^p h^{0,p} = \sum_{p=0}^{n} (-1)^p \tilde{h}^{0,p}. \tag{9.9}
\]

By Lemma 7.7 and Lemma 2.1,

\[
\tilde{p}_1 - p_1 = \sum_{p=0}^{n-1} (-1)^{n+p} h^{p,0}. \tag{9.10}
\]
For any $x \in \tilde{X}$, by definition

$$h^{p,0} = \int_{F(x)} P^* S^p_X \, dV,$$

by Berndtsson’s lemma, for any $z \in \tilde{X}$,

$$P^* S^p_X \leq \left( \frac{n}{p} \right) P^* M^{p,0}_X (z).$$

And by definition, one has

$$P^* M^{p,0}_X (z) \leq P^* M^{p,0}_{F(F(z))}(z) = M^{p,0}_{F(z)}(z),$$

and

$$M^{p,0}_{F(z)}(z) \leq M^{p,0}_{B_{\tau(z)}}(z) \leq M^{p,0}_{B_{\tau}}(z).$$

So the Lemma is proved. □

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