A New Method to Compute the 2-adic Complexity of Binary Sequences

Hai Xiong, Longjiang Qu, and Chao Li

College of Science, National University of Defense Technology, Changsha 410073, China
xiong.hai@163.com, ljqu_happy@hotmail.com, lichao_nudt@sina.com

Abstract. In this paper, a new method is presented to compute the 2-adic complexity of pseudo-random sequences. With this method, the 2-adic complexities of all the known sequences with ideal 2-level autocorrelation are uniformly determined. Results show that their 2-adic complexities equal their periods. In other words, their 2-adic complexities attain the maximum. Moreover, 2-adic complexities of two classes of optimal autocorrelation sequences with period $N \equiv 1 \mod 4$, namely Legendre sequences and Ding-Helleseth-Lams sequences, are investigated. Besides, this method also can be used to compute the linear complexity of binary sequences regarded as sequences over other finite fields.

Index Terms — 2-adic complexity; linear complexity; ideal 2-level autocorrelation sequence; optimal autocorrelation sequence

1 Introduction

Linear feedback shift registers (LFSRs) and feedback with carry shift registers (FCSRs) are two classes of pseudo-random sequence generators. The sequences produced by them could have good randomness, such as low correlation, long period and so on. These pseudo-random sequences are widely used in cryptography and communication systems.

For any binary periodic sequence $s$, it always can be generated by an LFSR or an FCSR. The length of the shortest LFSR resp. FCSR which can generate $s$ is called the linear complexity resp. 2-adic complexity of $s$, symbolically $LC(s)$ resp. $AC(s)$. Since $s$ can be completely determined by the Berlekamp-Massey algorithm [1] resp. rational approximation algorithm [2] with $2LC(s)$ resp. $2AC(s)$ consecutive bits, linear complexity and 2-adic complexity are two of the most important security criteria of binary sequences.

It is of interest to investigate the relationship between linear complexity and 2-adic complexity. However, it may be quite difficult in general since little is known in the literature. Hence a natural tradeoff is to investigate the linear complexity of sequences whose 2-adic complexity is known or the 2-adic complexity of sequences whose linear complexity is known. Until now, there are only a few classes of pseudo-random sequences whose linear complexity and 2-adic complexity both are clear. Seo et. al. [3] and Qi et. al. [4] got a lower bound on the linear complexity of a special class of $l$-sequences respectively. Klapper and Goresky [5] derived a simple result about the 2-adic complexity of $m$-sequences. A breakthrough of this
problem was given by Tian et. al. [6]. They completely determined the 2-adic complexity of $m$-sequences and showed that all the $m$-sequences have optimal 2-adic complexity.

$m$-sequences is a class of ideal 2-level autocorrelation sequences which play a significant role in applications for their optimal autocorrelation. A large amount of ideal 2-level autocorrelation sequences other than $m$-sequences have been constructed, for example Legendre sequences, twin-prime sequences and Hall’s sextic residue sequences [7]. The linear complexities of these sequences have all been determined; see [8] for a survey. However, as far as the authors known, no result about the 2-adic complexities of these sequences other than $m$-sequences is known yet.

In this paper, we will present a new method to compute the 2-adic complexity of binary sequences. According to [5], to determine 2-adic complexity of a binary sequence is equivalent to determine the greatest common divisor of two numbers which are associated with the sequence. Here, we convert this problem to compute the determinant of a circulant matrix and the greatest common divisor of two other integers. Then by using the new method, we prove that all the known sequences with ideal 2-level autocorrelation have the maximum 2-adic complexities, i.e. their 2-adic complexities equal their periods. We also prove that Legendre sequences and Ding-Helleseth-Lam sequences with period $N \equiv 1 \mod 4$ have maximum 2-adic complexities. Hence Legendre sequences, twin-prime sequences and Hall’s sextic residue sequences are nontrivial binary sequences whose linear complexities and 2-adic complexities both could attain the maximum. Finally, as a byproduct, we show that the new method can be used to compute the linear complexity of binary sequences when we regard them as sequences over other finite fields.

The rest of this paper is organized as follows. Section 2 introduces some well-known results and notations. In Section 3, a new method is presented to compute 2-adic complexity of binary sequences. In Section 4, 2-adic complexities of ideal 2-level autocorrelation sequences and two other classes of optimal autocorrelation sequences are determined. Section 5 presents some results on the linear complexity when one regard a binary sequence as a sequence over another finite field. We conclude this paper in Section 6.

2 Preliminaries

In this section, we will introduce some notations and review some well-known results.

2.1 Notations

1. The symbol “+” has a multiple meaning: it stands for the integer addition, or for the addition over $\mathbb{F}_2$, or even for the addition over integer residue rings. But this will not bring confusion in concrete situations.
2. A sequence is called binary if its elements consist of 0 and 1.
3. For a binary sequence \( s = (s_0, s_1, \cdots, s_{N-1}) \), its sequence polynomial is 
\[ P_s(x) = \sum_{i=0}^{N-1} s_i x^i. \]
The complementary sequence of \( s \), denoted by \( \overline{s} \), is defined as 
\( (1-s_0, 1-s_1, \cdots, 1-s_{N-1}) \).
Let \( D_s \) denote the support set of \( s \), which is defined as 
\( D_s = \{0 \leq i \leq N-1 : s_i = 1\} \).
4. Let \( \mathbb{Q}_2 \) denote the complete field of \( \mathbb{Q} \) with respect to the 2-adic absolute value.
5. Assume that \( p = df+1 \) is a prime. Let \( \alpha \) be a primitive element of \( \mathbb{F}_p \). The cyclotomic classes of order \( d \) with respect to \( \mathbb{F}_p \), denoted by \( D_i^{(d,p)} \), \( 0 \leq i \leq d-1 \), are defined as 
\[ D_i^{(d,p)} = \{ \alpha^{i+kd} : 0 \leq k \leq f-1 \}. \]
6. Let \( S \) be a subset of \( \mathbb{Z}/N\mathbb{Z} \) with \( k \) elements. For any integer \( \tau \), define 
\( S + \tau = \{(a + \tau) \mod N : a \in S\} \). If there exists a positive integer \( \lambda \) such that 
\( |S \cap (S + \tau)| = \lambda \) for any \( \tau \neq 0 \mod N \), then \( S \) is called an \( (N, k, \lambda) \) cyclic difference set.

### 2.2 Optimal autocorrelation sequences

Let \( s = (s_0, s_1, \cdots, s_{N-1}) \) be a binary sequence with period \( N \). The autocorrelation function of \( s \) is defined by
\[ C_s(\tau) = \sum_{i=0}^{N-1} (-1)^{s_i+s_i+\tau}, \quad \tau \in \mathbb{Z}/N\mathbb{Z}. \]
Clearly, \( C_s(0) = N \).
We say that \( s \) is an optimal autocorrelation sequence if for any \( \tau \neq 0 \),
1. \( C_s(\tau) = -1 \) and \( N \equiv -1 \mod 4 \); or
2. \( C_s(\tau) \in \{1, -3\} \) and \( N \equiv 1 \mod 4 \); or
3. \( C_s(\tau) \in \{2, -2\} \) and \( N \equiv 2 \mod 4 \); or
4. \( C_s(\tau) \in \{0, -4\} \) and \( N \equiv 0 \mod 4 \).

In Case (1), the sequences are also said to have ideal 2-level autocorrelation. Many classes of such sequences have been reported, such as Legendre sequence, Hall’s sextic residue sequence, twin-prime sequence, and \( m \)-sequence, GMW sequences, Maschiettie’s hyperoval sequences, etc. For a list of such sequences and detailed definitions of these sequences, please refer to [9] or [10]. The following characterization of such sequences is from [7].

**Lemma 1.** [7] Let \( s \) be a binary ideal 2-level autocorrelation sequence with period \( N \). Then \( D_s \), the support set of \( s \), is an \( (N, N+1, 2) \) or \( (N, N-1, 4) \) cyclic difference set. Based on their periods, all the known ideal 2-level autocorrelation sequences can be divided into three classes: (1) \( N = 2^n - 1 \); (2) \( N = p \), where \( p \equiv 3 \mod 4 \) is a prime; (3) \( N = p(p + 2) \), where both \( p \) and \( p + 2 \) are primes.

All the known binary sequences with optimal autocorrelation until 2009 are surveyed by Cai and Ding [10]. Here we only recall the definitions of Legendre sequences and Ding-Helleseth-Lam sequences with period of \( N \equiv 1 \mod 4 \) for later use.
**Legendre Sequences:** Let $p \equiv 1 \mod 4$ be a prime. Let $s$ be a binary sequence defined by

$$s_i = \begin{cases} 
1, & \text{if } i \in D_0^{(2,p)}; \\
0, & \text{otherwise}.
\end{cases}$$

Then $s$ has optimal out-of-phase autocorrelation values $\{1, -3\}$.

**Ding-Helleseth-Lam Sequences:** Let $p \equiv 1 \mod 4$ be a prime. Let $s$ be a binary sequence defined by

$$s_i = \begin{cases} 
1, & \text{if } i \in D_0^{(4,p)} \cup D_1^{(4,p)}; \\
0, & \text{otherwise}.
\end{cases}$$

Then $s$ has optimal out-of-phase autocorrelation values $\{1, -3\}$.

### 2.3 Feedback with carry shift register

A *feedback with carry shift register* (FCSR) consists of a feedback register and a memory cell. It is designed by Klapper and Goresky [11]. The form of an $r$-stage FCSR is presented in Fig. 1 where $q_i$ ($1 \leq i \leq r-1$) $\in \{0, 1\}$, $q_r = 1$. We call $q = \sum_{i=1}^{r} q_i 2^i - 1$ the *connection number* of this FCSR and its operation is defined as follows:

1. Give an initial state $(a_{r-1}, a_{r-2}, \cdots, a_0)$ of the register and $m$ of the memory, where $a_i \in \{0, 1\}$, $m \in \mathbb{Z}$;
2. Compute an integer sum $\sigma = \sum_{i=0}^{r-1} q_i a_i + m$;
3. Shift the register one step to right with outputting the rightmost bit $a_0$;
4. Put $a_r = (\sigma \mod 2)$ into the leftmost of the register;
5. Put $\frac{\sigma - a_r}{2}$ into the memory;
6. Return to Step 2.

The following result about 2-adic complexity of binary sequences was firstly presented by Klapper et. al. [5].
Lemma 2. \[3\] (1) Let \( s \) be a periodic sequence generated by the FCSR with connection number \( q \). Assume that \( s = (s_0, s_1, \cdots) \). Then, in \( \mathbb{Q}_2 \), \( \sum_{i=0}^{\infty} s_i 2^i = \frac{p}{q} \), where \( p \) is an integer such that \(-q \leq p \leq 0\). Particularly, if \( \gcd(p, q) = 1 \) then this FCSR is the shortest one which can produce \( s \) and hence \( AC(s) = \lfloor \log(q + 1) \rfloor \).

(2) Conversely, let \( s = (s_0, s_1, \cdots) \) be a binary periodic sequence. If \( \sum_{i=0}^{\infty} s_i 2^i = \frac{p}{q} \) in \( \mathbb{Q}_2 \), then \( s \) can be produced by the FCSR with connection number \( q \).

Let \( s \) be a periodic sequence and \( \overline{s} \) its complementary sequence. It follows from Lemma \[3\] and the fact \( \sum_{i=0}^{\infty} s_i 2^i + \sum_{i=0}^{\infty} \overline{s}_i 2^i = \sum_{i=0}^{\infty} 2^i = -1 \) that \( s \) and \( \overline{s} \) have the same 2-adic complexity. Hence when we refer to an ideal 2-level autocorrelation sequence, we always assume that, without loss of generality, its support set is an \((N, \frac{N+1}{2}, \frac{N+1}{4})\) cyclic difference set.

2.4 Linear feedback shift register

An \( r \)-stage linear feedback shift register (LFSR) over a finite field \( \mathbb{F}_q \) is given in Fig. 2, where \( q_i (1 \leq i \leq r) \in \mathbb{F}_q, q_r \neq 0 \). We call \( f(x) = \sum_{i=1}^{r} q_i x^i - 1 \) the connection polynomial of this LFSR and its operation is defined as follows:

1. Give an initial state \((a_{r-1}, a_{r-2}, \cdots, a_0)\), where \( a_i \in \mathbb{F}_q \);
2. Compute a sum \( \sigma = \sum_{i=0}^{r-1} q_i a_i \) over \( \mathbb{F}_q \);
3. Shift the register one step to right with outputting the rightmost bit \( a_0 \);
4. Put \( a_r = \sigma \) into the leftmost of the register;
5. Return to Step 2.

The following is a well-known result on the linear complexity of periodic sequences.
Lemma 3. \[7, 12\] (1) Let \( s = (s_0, s_1, \cdots) \) be a periodic sequence generated by the LFSR with connection polynomial \( f(x) \). Then \( \sum_{i=0}^{\infty} s_i x^i = \frac{g(x)}{f(x)} \). Particularly, if \( \gcd(g(x), f(x)) = 1 \), then this LFSR is the shortest one which can produce \( s \) and hence \( LC(s) = \deg(f(x)) \).

(2) Conversely, let \( s = (s_0, s_1, \cdots) \) be a periodic sequence over \( \mathbb{F}_q \). If \( \sum_{i=0}^{\infty} s_i x^i = \frac{g(x)}{f(x)} \), then \( s \) can be produced by the LFSR with connection polynomial \( f(x) \).

2.5 Gauss sums

Let \( p \) be a prime and let \( \psi \) be a multiplicative character of \( \mathbb{F}_p \). Define

\[
G(\psi; \alpha) = \sum_{x \in \mathbb{F}_p^*} \psi(x) w_p^{\alpha x}
\]

and

\[
g(k; \alpha) = \sum_{x \in \mathbb{F}_p} w_p^{\alpha x^k},
\]

where \( k \) is an integer, \( w_p = e^{2\pi i/p} \) is a \( p \)-th primitive unity of \( \mathbb{C} \) and \( \alpha \in \mathbb{F}_p \). Both the above sums are called Gauss sums and they are connected by the following results.

Lemma 4. \[13\] Let \( \psi \) be a multiplicative character of \( \mathbb{F}_p \) with order \( k \). Then,

\[
g(k; \alpha) = \sum_{j=1}^{k-1} G(\psi^j; \alpha) = \sum_{j=1}^{k-1} \psi^j(\alpha^{-1}) G(\psi^j; 1).
\]

Lemma 5. \[13\] Assume that \( p \equiv 1 \mod 4 \). One has

(1) If \( \psi \) is the quadratic character of \( \mathbb{F}_p \), then \( G(\psi; 1) = g(2; 1) = \sqrt{p} \);

(2) If \( \psi \) is a character of order \( 4 \), then

\[
G(\psi; 1) + G(\psi^3; 1) = \pm \left\{ 2 \left( \frac{2}{p} \right) (p + a\sqrt{p}) \right\}^{1/2},
\]

where \( \left( \frac{2}{p} \right) \equiv 2^{p-1} \mod p \) is the Legendre symbol, \( a \) is an integer such that \( a^2 + b^2 = p \), \( a \equiv - \left( \frac{2}{p} \right) \pmod 4 \);

(3) If \( \psi \) is a nontrivial character, then \( |G(\psi; 1)| = \sqrt{p} \).

3 A new method of computing the 2-adic complexity of binary sequences

In this section, we will present a new method of computing the 2-adic complexity of binary sequences. The following is a key lemma of our method.
Lemma 6. Let \( s = (s_0, s_1, \cdots, s_{N-1}) \) be a binary sequence with period \( N \) and let \( P_s(x) = \sum_{i=0}^{N-1} s_i x^i \in \mathbb{Z}[x] \). Let \( A = (a_{i,j})_{N \times N} \) be the matrix defined by \( a_{i,j} = s_{(i-j) \mod N} \), and let us view \( A \) as a matrix over \( \mathbb{Z} \). If \( \det(A) \neq 0 \), then there exist \( u(x), v(x) \in \mathbb{Z}[x] \) such that

\[ u(x)P_s(x) + v(x)(1 - x^N) = \det(A), \]

where \( \deg u \leq N - 1, \deg v \leq N - 2 \).

Proof. It suffices to prove that the following equation system has a solution \((u_0, u_1, \cdots, u_{N-1}, v_0, v_1, \cdots, v_{N-2})^T \in \mathbb{Z}^{2N-1} \), where \( u_i \) and \( v_i \) are the coefficients of \( u(x) \) and \( v(x) \) respectively.

\[
\begin{aligned}
\begin{cases}
 s_0 u_0 + v_0 &= \det(A) \\
 s_1 u_0 + s_0 u_1 + v_1 &= 0 \\
 \vdots &= \vdots \\
 \sum_{i=0}^{N-2} s_{N-2-i} u_i + v_{N-2} &= 0 \\
 \sum_{i=0}^{N-1} s_{N-1-i} u_i &= 0 \\
 \sum_{i=1}^{N-1} s_{N-i} u_i - v_0 &= 0 \\
 \sum_{i=2}^{N-1} s_{N+1-i} u_i - v_1 &= 0 \\
 \vdots &= \vdots \\
 s_{N-1} u_{N-1} - v_{N-2} &= 0 
\end{cases}
\end{aligned}
\]

The coefficient matrix \( C \) of the above equation system is

\[
C = 
\begin{pmatrix}
 s_0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 s_1 & s_0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
 \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 s_{N-2} & s_{N-3} & \cdots & 0 & 0 & 0 & \cdots & 1 \\
 s_{N-1} & s_{N-2} & \cdots & s_0 & 0 & 0 & \cdots & 0 \\
 0 & s_{N-1} & \cdots & s_1 & -1 & 0 & \cdots & 0 \\
 \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
 0 & 0 & \cdots & s_{N-1} & 0 & 0 & \cdots & -1 
\end{pmatrix}.
\]
Adding the last \((N - 1)\) rows of \(C\) on the first \((N - 1)\) rows, we get a new matrix

\[
C' = \begin{pmatrix}
    s_0 & s_{N-1} & \cdots & s_1 & 0 & 0 & \cdots & 0 \\
    s_1 & s_0 & \cdots & s_2 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    s_{N-2} & s_{N-3} & \cdots & s_{N-1} & 0 & 0 & \cdots & 0 \\
    s_{N-1} & s_{N-2} & \cdots & s_0 & 0 & 0 & \cdots & 0 \\
    0 & s_{N-1} & \cdots & s_1 & -1 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & s_{N-1} & 0 & 0 & \cdots & -1
\end{pmatrix}.
\]

Then we have \(\det(C) = \det(C') = \det(A)(-1)^{N-1} = \pm \det(A) \neq 0\). Hence Equation (2) has a unique solution \(\alpha = (u_0, \cdots, u_{N-1}, v_0, \cdots, v_{N-2})^T = C^{-1} \beta\), where \(\beta = (\det(A), 0, \cdots, 0)^T\). Noting that \(C\) is a matrix over \(Z\) and \(\det(C) = \pm \det(A)\), we have \(\alpha = C^{-1} \beta \in Z^{2N-1}\). We finish the proof. 

The following is our first main result on the 2-adic complexity of binary periodic sequence.

**Theorem 1.** Let the symbols be defined as in Lemma 6. If \(\gcd(1 - 2^N, \det(A)) = 1\), then \(AC(s) = N\).

**Proof.** Since \(\gcd(1 - 2^N, \det(A)) = 1\), we have \(\det(A) \neq 0\). According to Lemma 6, there exist \(u(x), v(x) \in \mathbb{Z}[x]\) such that

\[
u(x)P_s(x) + v(x)(1 - x^N) = \det(A).
\]

Substituting \(x = 2\) into the above equation and letting \(M = P_s(2)\), we have

\[
u(2)M + v(2)(1 - 2^N) = \det(A).
\]

Hence we have \(\gcd(M, 1 - 2^N) = 1\) since \(\gcd(1 - 2^N, \det(A)) = 1\). The result then follows from Lemma 2.

Before processing further discussions, we make two remarks on Theorem 1. Firstly, let \(d_1 = \gcd(M, 1 - 2^N)\) and \(d_2 = \gcd(1 - 2^N, \det(A))\). Then it follows from Lemma that \(d_2\) is divided by \(d_1\). Hence by Lemma 2, the smallest connection number \(q\) of \(s\) is \(\frac{1-2^N}{d_1}\), which is lower bounded by \(\frac{1-2^N}{d_2}\). Thus one can get a lower bound on \(q\) and consequently a lower bound on the 2-adic complexity of \(s\) if \(d_2 = \gcd(1 - 2^N, \det(A)) \neq 1\). It is a more general result than Theorem 1. However, for simplicity, we would like to keep Theorem 1 as its present form. Secondly, it is clear that Theorem 1 can be naturally generalized to \(p\)-ary sequences. However, we focus on binary sequences in the present paper.
Theorem 1 provides a new method to compute the 2-adic complexity of binary sequences. The key point of this method is to compute $\det(A)$ and then verify whether $\gcd(2^N - 1, \det(A)) = 1$, where $A$ is the circulant matrix constructed from the sequence. According to linear algebra, $\det(A)$ can be computed as follows.

**Lemma 7.** Let $s$ be a sequence with period $N$ and let $A = (a_{i,j})_{N \times N}$ be the matrix defined by $a_{i,j} = s_{(i-j) \mod N}$. Then $\det(A) = \prod_{j=0}^{N-1} P_s(w_N^j)$, where $w_N = e^{\frac{2\pi i}{N}}$ is an $N$-th primitive unity of $\mathbb{C}$.

It is clear that $P_s(1) = \sum_{i=0}^{N-1} s_i = |D_s|$. For $1 \leq j \leq N - 1$, we have

\[
P_s(w_N^j) = \sum_{i=0}^{N-1} s_i (w_N^j)^i = \sum_{i \in D_s} (w_N^j)^i.
\]

Hence computing $P_s(w_N^j)$ is related to some exponential sums. If the corresponding exponential sums can be computed, then one can compute $\det(A)$ and check whether $\gcd(2^N - 1, \det(A)) = 1$ holds. This is the case of Legendre Sequence, Ding-Helleseth-Lam Sequence and Ding-Helleseth-Martinsen Sequence, as we will see in Section 4.2. On the other hand, if the exponential sums can not be easily computed, we may use other methods to compute $\det(A)$. This is the case of all the known binary sequences with ideal 2-level autocorrelation, as we will see in Section 4.1.

### 4 Determining 2-adic complexities of several binary sequences with Theorem 1

In this section, as applications of our new method, we will determine the 2-adic complexities of many binary sequences. They are examples of the two cases discussed in the last section.

#### 4.1 All the known binary sequences with ideal 2-level autocorrelation

In this subsection, we will use Theorem 1 to uniformly determine the 2-adic complexities of all the known binary ideal 2-level autocorrelation sequences. Two lemmas will be needed. The first one is a well-known result from linear algebra.

**Lemma 8.** Let $B = (b_{i,j})_{n \times n}$ be a matrix defined by

\[
b_{i,j} = \begin{cases} x, & \text{if } i = j; \\ y, & \text{if } i \neq j. \end{cases}
\]

Then $\det(B) = (x + (n-1)y)(x - y)^{n-1}$. 

Lemma 9. (1) Let \( p \) be an odd prime. If \( q \) is a prime factor of \( (2^p - 1) \), then \( q \geq (p + 2) \).

(2) Let \( N = p(p+2) \), where \( p \) and \( p+2 \) both are odd primes. If \( q \) is a prime factor of \( (2^N - 1) \), then \( q \geq (p + 2) \).

Proof. We only give a proof for (2). The proof for (1) is similar and is left to the interested readers. We regard \( 2 \) as an element of \( \mathbb{F}_q \), and denote by \( \text{ord}(2) \) the order of \( 2 \) in \( \mathbb{F}_q^* \).

Since \( 2^{p(p+2)} \equiv 1 \mod q \), we have \( \text{ord}(2) | p(p + 2) \). Noting that \( \text{ord}(2) \neq 1 \), therefore \( \text{ord}(2) = p, p + 2, \) or \( N \). Clearly, we also have \( \text{ord}(2) | (q - 1) \). Thus \( q \geq p + 2 \).

Theorem 2. Let \( s \) be any known ideal 2-level autocorrelation sequence with period \( N \). Then its 2-adic complexity is \( N \).

Proof. By Theorem 1, it suffices to prove that \( \gcd(1 - 2^N, \det(A)) = 1 \), where \( A = (a_{i,j})_{N \times N} \) is the matrix defined by \( a_{i,j} = s_{i-j} \mod N \).

Let \( B = A^T A = (b_{i,j})_{N \times N} \), where \( A^T \) is the transpose of the matrix \( A \). Then
\[
b_{i,j} = \sum_{k=0}^{N-1} s_{k-i}s_{k-j} = \sum_{k=0}^{N-1} s_k s_{k+i-j}.
\]
Thus \( b_{i,j} = |D_s \cap (D_s + (i - j))| \). Noting that \( D_s \) is an \( (N, \frac{N+1}{2}, \frac{N+1}{4}) \) cyclic difference set, we have
\[
b_{i,j} = \begin{cases} 
\frac{N+1}{2}, & \text{if } i = j; \\
\frac{N+1}{4}, & \text{if } i \neq j.
\end{cases}
\]
Hence, by Lemma 8 we have \( \det(B) = \left(\frac{N+1}{2}\right)^2 \left(\frac{N+1}{4}\right)^{N-1} \). Then \( |\det(A)| = \sqrt{\det(B)} = \frac{N+1}{2} \left(\frac{N+1}{4}\right)^{N-1} \).

According to Lemma 1, there are only three cases for \( N \).

If \( N = 2^n - 1 \), then \( |\det(A)| = 2^{n-1}2^{(n-2)\frac{N-1}{2}} \). Since \( 1 - 2^N \) is odd, we have \( \gcd(1 - 2^N, \det(A)) = 1 \).

If \( N = p \), then it follows from Lemma 9 that \( \gcd(2^p - 1, p + 1) = 1 \). Hence \( \gcd(2^N - 1, \det(A)) = 1 \).

If \( N = p(p+2) \), then \( |\det(A)| = \frac{(p+1)^2}{2^p} \left(\frac{p+1}{4}\right)^{\frac{N-1}{2}} \). Similarly, it follows from Lemma 9 that \( \gcd(p+1, 1 - 2^N) = 1 \) and \( \gcd(1 - 2^N, \det(A)) = 1 \).

We are done.

Theorem 2 gives a uniform proof that all the known binary sequences with ideal 2-level autocorrelation have the maximum 2-adic complexities. To the authors’ best knowledge, the 2-adic complexities of all these sequences except \( m \)-sequences are firstly determined. Another consequence of Theorem 2 is that one can say more about the relation of linear complexity...
and 2-adic complexity. As we recalled, \(m\)-sequences are a class of sequences with minimum linear complexity and maximum 2-adic complexity, while some \(l\)-sequences are a class of sequences with minimum 2-adic complexity and maximum linear complexity. Now Legendre sequences, twin-prime sequences and Hall’s sextic residue sequences are examples of the sequences whose linear complexity and 2-adic complexity both attain the maximum.

### 4.2 Legendre sequence and Ding-Helleseth-Lam sequence

In this subsection, we will use Theorem 1 to determine 2-adic complexities of Legendre sequence and Ding-Helleseth-Lam sequence. According to Theorem 1 and the analysis followed, we need to compute \(P_s(w^j)\), which is related to some exponential sums. For Legendre sequence, it is related to quadratic Gauss sum; while for Ding-Helleseth-Lam sequence, it is related to quartic Gauss sum.

**Theorem 3.** Let \(s\) be a Legendre sequence with period \(p \equiv 1 \mod 4\). Then \(AC(s) = p\).

**Proof.** By Theorem 1 it suffices to prove that \(\gcd(1 - 2^p, \det(A)) = 1\), where \(A = (a_{i,j})_{p \times p}\) is the matrix defined by \(a_{i,j} = s(i-j) \mod p\).

Let \(w_p = e^{\frac{2\pi i}{p}}\), \(B_0 = \sum_{x \in D_0^{(2,p)}} w_p^x\) and \(B_1 = \sum_{x \in D_1^{(2,p)}} w_p^x\). According to the definition of Legendre sequence, we have

\[
P_s(w_p^j) = \sum_{i \in D_0^{(2,p)}} w_p^{ij} = \begin{cases} \frac{p-1}{2}, & \text{if } j = 0; \\ B_0, & \text{if } j \in D_0^{(2,p)}; \\ B_1, & \text{if } j \in D_1^{(2,p)}. \end{cases}
\]

By Lemma 5 we have \(1 + 2B_0 = g(2;1) = \sqrt{p}\). Besides, one can easily deduce \(1 + B_0 + B_1 = 0\). Hence, \(B_0 = \frac{\sqrt{p}-1}{2}\) and \(B_1 = -\frac{\sqrt{p}+1}{2}\). Thus, it follows from Lemma 7 that

\[
\det(A) = \prod_{j=0}^{p-1} P_s(w_p^j) = \frac{p-1}{2} \left( \frac{\sqrt{p}-1}{2} \right)^{p-1} \left( -\frac{\sqrt{p}-1}{2} \right)^{p-1} = \frac{p-1}{2} \left( \frac{p-1}{4} \right)^{p-1}.
\]

Similar argument as in Theorem 2 shows that \(\gcd(\det(A), 2^p - 1) = 1\). 

Before introducing the result on the 2-adic complexity of Ding-Helleseth-Lam sequence, we need a lemma.

**Lemma 10.** Let \(p \equiv 1 \mod 4\) be a prime and \(a\) be an odd integer such that \(a^2 + b^2 = p\). Then \(\gcd(1 \pm 2p + a^2p, 2^p - 1) = 1\).
Proof. We only prove that gcd\((1+2p+a^2p, 2^p-1) = 1\) and the other case can be proved similarly.

Assume on the contrary that gcd\((1+2p+a^2p, 2^p-1) = d \geq 1\). Let \(r \geq 0\) be an odd prime factor of \(d\). Then one can deduce that \(r-1 \equiv 0 \pmod{p}\) as in the proof of Lemma 3. Thus \(r = kp+1\), where \(k \geq 2\) is an even integer since both \(r\) and \(p\) are odd. Let \(1+2p+a^2p = ur\). Then \(u\) is an even integer since \(1+2p+a^2p \) is even. Clearly, \(u \equiv 1 \pmod{p}\). On the other hand, \(u = \frac{1+2p+a^2p}{r} < \frac{1+2p+p^2}{r} < p+1\). Thus we get \(u = 1\) which contradicts that \(u\) is an even integer. Hence gcd\((1+2p+a^2p, 2^p-1) = 1\). \(\blacksquare\)

**Theorem 4.** Let \(s\) be a Ding-Helleseth-Lam sequence with period \(p \equiv 1 \pmod{4}\). Then \(AC(s) = p\).

**Proof.** By Theorem 1 it suffices to prove that gcd\((1-2^p, \det(A)) = 1\), where \(A = (a_{i,j})_{p \times p}\) is the matrix defined by \(a_{i,j} = s(i-j) \pmod{p}\).

Let \(\alpha\) be a primitive element of \(\mathbb{F}_p\) and \(w_p = e^{\frac{2\pi i}{p}}\). Let \(\lambda\) be a multiplicative character of \(\mathbb{F}_p\) defined by \(\lambda(\alpha) = i\). Then the order of \(\lambda\) is 4. For \(0 \leq i \leq 3\), let \(B_i = \sum_{x \in D_i^{(4,p)}} w_p^x\). According to the definition of Ding-Helleseth-Lam sequence, we deduce

\[
P_s(w_p^j) = \sum_{i \in D_0^{(4,p)} \cup D_1^{(4,p)}} w_p^{ij} = \begin{cases} 
\frac{p-1}{2}, & \text{if } j = 0; \\
B_0 + B_1, & \text{if } j \in D_0^{(4,p)}; \\
B_1 + B_2, & \text{if } j \in D_1^{(4,p)}; \\
B_2 + B_3, & \text{if } j \in D_2^{(4,p)}; \\
B_3 + B_0, & \text{if } j \in D_3^{(4,p)}. 
\end{cases}
\]  

Hence

\[
\det(A) = \left(\frac{p-1}{2}\right) [(B_0 + B_1)(B_1 + B_2)(B_2 + B_3)(B_3 + B_0)]^{\frac{p-1}{4}}. 
\]

It follows from Lemma 3 that gcd\((\frac{p-1}{2}, 1-2^p) = 1\).

By Lemma 4 we have

\[
\begin{align*}
1 + 4B_0 &= g(4; 1) = G(\lambda; 1) + G(\lambda^2; 1) + G(\lambda^3; 1); \\
1 + 4B_1 &= g(4; \alpha) = \lambda(\alpha^{-1})G(\lambda; 1) + \lambda^2(\alpha^{-1})G(\lambda^2; 1) + \lambda^3(\alpha^{-1})G(\lambda^3; 1); \\
1 + 4B_2 &= g(4; \alpha^2) = \lambda(\alpha^{-2})G(\lambda; 1) + \lambda^2(\alpha^{-2})G(\lambda^2; 1) + \lambda^3(\alpha^{-2})G(\lambda^3; 1); \\
1 + 4B_3 &= g(4; \alpha^3) = \lambda(\alpha^{-3})G(\lambda; 1) + \lambda^2(\alpha^{-3})G(\lambda^2; 1) + \lambda^3(\alpha^{-3})G(\lambda^3; 1).
\end{align*}
\]

One can easily verify that \(G(\lambda^3; 1) = \lambda(-1)G(\lambda; 1)\). Noting that \(\lambda(\alpha) = i\), the above equation can be reduced as

\[
\begin{align*}
1 + 4B_0 &= g(4; 1) = G(\lambda; 1) + G(\lambda^2; 1) + \lambda(-1)G(\lambda; 1); \\
1 + 4B_1 &= g(4; \alpha) = -iG(\lambda; 1) - G(\lambda^2; 1) + i\lambda(-1)G(\lambda; 1); \\
1 + 4B_2 &= g(4; \alpha^2) = -G(\lambda; 1) + G(\lambda^2; 1) - \lambda(-1)G(\lambda; 1); \\
1 + 4B_3 &= g(4; \alpha^3) = iG(\lambda; 1) - G(\lambda^2; 1) - i\lambda(-1)G(\lambda; 1).
\end{align*}
\]
Let $R = \text{Re}(G(\lambda; 1))$ and $I = \text{Im}(G(\lambda; 1))$.

If $p \equiv 1 \mod 8$, then $\lambda(-1) = \lambda(\alpha^{\frac{p-1}{2}}) = i^{\frac{p-1}{4}} = 1$. From Eq. (7), we get

$$
\begin{align*}
2(B_0 + B_1) &= R + I - 1; \\
2(B_1 + B_2) &= I - R - 1; \\
2(B_2 + B_3) &= -R - I - 1; \\
2(B_3 + B_0) &= -I + R - 1.
\end{align*}
$$

Hence

$$
16(B_0 + B_1)(B_1 + B_2)(B_2 + B_3)(B_3 + B_0)
= (1 - (R + I)^2)(1 - (R - I)^2)
= 1 - 2R^2 - 2I^2 + (R^2 - I^2)^2.
$$

It follows from Lemma 5 that $R^2 + I^2 = p$ and $4R^2 = 2(p + a\sqrt{p})$. Hence we deduce $R^2 = \frac{1}{2}(p + a\sqrt{p})$ and $I^2 = \frac{1}{2}(p - a\sqrt{p})$. Thus

$$
16(B_0 + B_1)(B_1 + B_2)(B_2 + B_3)(B_3 + B_0) = 1 - 2p + a^2 p.
$$

It then follows from Eq. (6), $\gcd\left(\frac{p-1}{2}, 1 - 2^p\right) = 1$ and Lemma 10 that $\gcd(\det(A), 1 - 2^p) = 1$.

Similarly, if $p \equiv 5 \mod 8$, then one can deduce

$$
16(B_0 + B_1)(B_1 + B_2)(B_2 + B_3)(B_3 + B_0) = 1 + 2p + a^2 p.
$$

Hence we also have $\gcd(\det(A), 1 - 2^p) = 1$.

The proof is finished.

In this section, by using our new method, the 2-adic complexities of many binary sequences with optimal autocorrelation are determined. We believe that it can be used to determine the 2-adic complexities of more binary sequences. The reader is cordially invited to join this adventure.

On the other hand, we must mention that this method has its own drawback. It can not work for those binary sequences for which one has $\det(A) = 0$, where $A$ is the circulant matrix defined by the sequence. For example, let $s$ be a Ding-Helleseth-Martinsen sequence \cite{10} with period $N = 2q$, where $q \equiv 5 \mod 8$ is a prime. According to the definition of $s$, we have $P_s(w_N^q) = P_s(-1) = 0$, where $w_N = e^{\frac{2\pi i}{N}}$. Then one can deduce that $\det(A) = 0$ from Lemma 7. Similarly, when $s$ is a Sidelnikov-Lempel-Cohn-Eastman sequence \cite{10} with period $N \equiv 0 \mod 4$, one can also prove that $\det(A) = 0$. Other methods may be needed to compute the 2-adic complexities of these sequences.
5 Observe binary sequences from different finite fields

For a binary sequence \( s \), since its elements consist of 0 and 1, it can also be viewed as a sequence over another finite field. Let us denote by \( LC_q(s) \) the linear complexity of \( s \) when we regard it as a sequence over finite field \( \mathbb{F}_q \). Clearly, \( LC_q(s) \) may be different when \( q \) differs. For example, let \( s = 11000, 11000, \cdots \) be a binary sequence with period 5. Then one can verify that \( LC_2(s) = 4 \). However, if we regard \( s \) as a sequence over \( \mathbb{F}_3 \), then \( LC_3(s) = 5 \neq 4 \). It is natural to ask what is the relationship of the different linear complexity of the same binary sequence. In this section, we will investigate this problem and will present some interesting results. To our knowledge, there are only a few results about this problem; see [16].

Firstly, we have the following observation.

**Proposition 1.** Let \( s \) be a binary periodic sequence and \( \mathbb{F}_q \) be a finite field with character \( p \). Then \( LC_q(s) = LC_p(s) \).

**Proof.** Denote by \( N \) the period of \( s \). Then \( \sum_{i=0}^{\infty} s_i x^i = \frac{P_s(x)}{1-x^N} \), where \( P_s(x) \) is the sequence polynomial of \( s \). Since the greatest common divisor of \( P_s(x) \) and \( 1-x^N \) over \( \mathbb{F}_q \) is equal to that of these two polynomials over \( \mathbb{F}_p \), the result then follows from Lemma 3.

Thanks to Proposition 1, we will focus on the odd prime fields in the following. Let \( s \) be a binary sequence with period \( N \). Now, view \( P_s(x) \) and \( 1-x^N \) as polynomials in \( \mathbb{Z}[x] \). Let \( g(x) = \gcd(P_s(x), 1-x^N) \) be a monic polynomial. Then \( g(x) \in \mathbb{Z}[x] \). Clearly, there exist polynomials \( u(x), v(x) \in \mathbb{Z}[x] \) and a nonzero integer \( a \) such that

\[
u(x)P_s(x) + v(x)(1-x^N) = ag(x).
\]

(10)

Note that \( a \neq 1 \) may hold since we are working not on the fields but on the rings. For example, let \( P_s(x) = 1+x \) and \( N = 5 \) as in the before example. It is clear that \( \gcd(P_s(x), 1-x^N) = 1 \) in \( \mathbb{Z}[x] \). Substituting \( P_s(x) = 1+x, N = 5 \) and \( g(x) = 1 \) into (10), one gets \( u(x)(1+x) + v(x)(1-x^5) = a \), which will force \( a \) to be even since both \( 1+x \) and \( 1-x^5 \) are even if \( x \) is an odd integer. Hence \( a \neq 1 \).

**Theorem 5.** Let \( p \) be a prime. The other notations are the same as defined in the above paragraph. Then \( LC_p(s) \leq N - \deg g(x) \). If \( p \nmid a \), then the equality holds.

**Proof.** Let us view \( P_s(x) \) and \( 1-x^N \) as polynomials in \( \mathbb{F}_p[x] \) and denote by \( d(x) \) their greatest common divisor in \( \mathbb{F}_p[x] \). If we also view \( g(x) \) as a polynomial over \( \mathbb{F}_p \), then \( g(x)|d(x) \).

Hence, by Lemma 3 we deduce that \( s \) can be generated by the LFSR over \( \mathbb{F}_p \) with connection polynomial \( (1-x^N)/g(x) \). Therefore, \( LC_p(s) \leq N - \deg g(x) \).

If \( p \nmid a \), then \( a \neq 0 \) in \( \mathbb{F}_p \). By Equation (10), we have \( d(x)|g(x) \), which means \( d(x) = g(x) \).

Thus, it follows from Lemma 3 that \( LC_p(s) = N - \deg g(x) \) at. ■
We should remind the reader that the inequality in the above theorem holds sometimes. For example, let \( s = (11010) \) be a sequence of period 5. Then we have \( g(x) = \gcd(P_{s}(x), 1 - x^{5}) = 1 \) in \( \mathbb{Z}[x] \) while \( d(x) = \gcd(P_{s}(x), 1 - x^{5}) = x - 1 \) in \( \mathbb{F}_{3}[x] \). Hence \( LC_{3}(s) = 4 < 5 = N - \deg g(x) \).

**Corollary 1.** Let \( s \) be a binary ideal 2-level autocorrelation sequence with period \( N \), and let \( p \) be an odd prime.

1. If \( |D_{s}| = \frac{N+1}{2} \) and \( p \nmid (N + 1) \), then \( LC_{p}(s) = N \);
2. If \( |D_{s}| = \frac{N-1}{2} \), \( p \nmid (N + 1) \) and \( p | (N - 1) \), then \( LC_{p}(s) = N - 1 \);
3. If \( |D_{s}| = \frac{N-1}{2} \) and \( p \nmid (N^{2} - 1) \), then \( LC_{p}(s) = N \).

**Proof.** (1) Assume that \( |D_{s}| = \frac{N+1}{2} \). Then \( D_{s} \) is an \( (N, \frac{N+1}{2}, \frac{N+1}{4}) \) cyclic difference set. In the proof of Theorem 2, it is proved that \( \det(A) = \pm (\frac{N+1}{2})(\frac{N+1}{4})^{4} \), where \( A = (a_{ij}) = (s_{(i-j) \mod N}) \) be the matrix defined by \( s \). Comparing (11) and (10), one has \( g(x) = 1 \) and \( a = \det(A) \). Hence \( p \nmid a \) by the assumption \( p \nmid (N + 1) \). The result then follows from Theorem 2.

(2) Assume that \( |D_{s}| = \frac{N-1}{2} \). Then \( D_{s} \) is an \( (N, \frac{N+1}{2}, \frac{N+1}{4}) \) cyclic difference set. According to the result of the first part, we have \( \gcd(P_{s}(x), 1 - x^{N}) = 1 \). Noting that \( P_{s}(x) + P_{s} = (1 - x^{N})/(1 - x) \), one can deduce \( \gcd(P_{s}(x), (1 - x^{N})/(1 - x)) = 1 \). Because \( p | (N - 1) \), we have \( P_{s}(1) = 0 \) which means \( (1 - x)|P_{s}(x) \). Therefore \( \gcd(P_{s}(x)/(1 - x), (1 - x^{N})/(1 - x)) = 1 \). The result then follows from Lemma 3.

(3) One can deduce the result similarly as the second part.

Since \( N \) is finite, the number of primes dividing \( N^{2} - 1 \) is finite. Hence except finitely many cases, the linear complexity of a binary ideal 2-level autocorrelation sequence regarded as a sequence over another prime finite field attains the maximum.

The following interesting result follows immediately from the above corollary.

**Theorem 6.** Let \( s \) be a binary ideal 2-level autocorrelation sequence with period \( N = 2^{n} - 1 \).

Let \( \mathbb{F}_{q} \) be a finite field with an odd character \( p \).

1. If \( |D_{s}| = \frac{N+1}{2} \), then \( LC_{q}(s) = N \);
2. If \( |D_{s}| = \frac{N-1}{2} \) and \( p | (N - 1) \), then \( LC_{q}(s) = N - 1 \);
3. If \( |D_{s}| = \frac{N-1}{2} \) and \( p \nmid (N - 1) \), then \( LC_{q}(s) = N \).

6 Conclusion

To summarize, the contributions of this paper are threefold. Firstly, a new method is presented to compute the 2-adic complexity of binary sequences. Secondly, all the known binary sequences with ideal 2-level autocorrelation are uniformly proved to have the maximum 2-adic complexities, i.e. their 2-adic complexities equal their periods. As far as the
authors known, the 2-adic complexities of all these sequences except \( m \)-sequences are not known before this paper. We also investigated the 2-adic complexities of two classes of optimal autocorrelation sequences with period \( N \equiv 1 \mod 4 \). Thirdly, the new method is used to study the linear complexity of binary sequences taken as sequences over other finite fields. An interesting finding is that, except finitely many cases, the linear complexity of a binary ideal 2-level autocorrelation sequence regarded as a sequence over another prime finite field attains the maximum.

Acknowledgments

The work was supported by the Natural Science Foundation of China (No. 61272484) and Basic Research Fund of National University of Defense Technology (No. CJ 13-02-01).

References

1. J. L. Massery, “Shift-Register Synthesis and BCH Decoding,” IEEE Trans. Inf. Theory, no. 1, vol. 15, pp. 122-127, 1969
2. A. Klapper and M. Goresky, “Cryptanalysis based on 2-adic rational approximation,” in Advances in Cryptology-CRYPTO’95, vol. 963, pp. 262-273, 1995
3. C. Seo, S. Lee, Y. Sung, K. Han and S. Kim, “A lower bound on the linear span of an FCSR,” IEEE Trans. Inf. Theory, vol. 46, pp. 691-693, 2000
4. W. F. Qi and H. Xu, “On the linear complexity of FCSR sequences,” Applied mathematics-A journal of Chinese universities, vol. 18, no. 3, pp. 318-324, 2003
5. A. Klapper and M. Goresky, “Feedback shift registers, 2-adic span, and combiners with memory” J. Cryptology, vol. 10, pp. 111-147, 1997
6. T. Tian and W. F. Qi, “2-adic complexity of binary \( m \) sequences,” IEEE Trans. Inform. Theory, vol. 56, no. 1, pp. 450-454, 2010
7. S. W. Golomb, G. Gong, “Signal Design for Good Correlation,” New York: Cambridge University Press, 2005
8. H. Xiong, L. J. Qu and C. Li, “Linear complexity of binary sequences with interleaved structure,” IET Communications, accepted.
9. Y. Nam, G. Gong, “Crosscorrelation properties of binary sequences with ideal two-level autocorrelation,” SETA06, 2006
10. Y. Cai and C. Ding, “Binary sequences with optimal autocorrelation,” Theretical Computer Science, vol. 410, pp. 2316-2322, 2009
11. A. Klapper and M. Goresky, “2-adic shift register,” in Fast Software Encryption, vol. 809, pp. 174-178, 1993
12. C. Li, L. J. Qu, “Lectures of Cryptography,” Beijing: Science Press, 2010. (In Chinese)
13. B. C. Berndt, R. J. Evans and K. S. Williams, “Gauss and Jacobi Sums,” New York: A Wiley-Interscience Publication, 1997
14. P. J. Davis, “Circulant Matrices,” New York: Chelsea, 1994
15. H. W. Eves, “Elementary Matrix Theory,” Mineola: Dover Publications, 1980.
16. C. Ding, “Cyclic codes from cyclotomic sequences of order four,” Finite Fields and Their Applications, 23, pp. 8-34, 2013