SPECTRAL CURVES FOR ALMOST-COMPLEX TORI IN $S^6$

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To each non-isotropic almost-complex immersion of a 2-torus into $S^6$ we associate an algebraic curve, called the spectral curve, and a linear flow in the intersection of two Prym varieties on this spectral curve. We show that generically the spectral curve is smooth and compute the dimension of the moduli space of such curves and of the torus in which the eigenline bundles lie.

1. INTRODUCTION

The six-sphere has a natural almost-complex structure $J_x : T_x S^6 \to T_x S^6$ given by $J_x(y) = x \cdot y = x \times y$, where we consider $S^6$ as a subspace of the imaginary octonions and $\cdot$ denotes octonionic multiplication. An immersion $f : M^2 \to S^6$ of a Riemann surface $M^2$ is almost-complex if $df \circ J_M = J \circ df$, where $J_M$ denotes the complex structure on $M$. This can be interpreted in terms of a calibrated geometry: $f$ is almost-complex precisely when the cone $C$ over $f(M^2)$ is an associative (singular) submanifold of $\text{Im } O$. This means that $C$ is calibrated with respect to the 3-form $\alpha'(x, y, z) = \langle x \cdot y, z \rangle$

(and hence volume minimising in its homology class) or equivalently that $1 \oplus T_y(C)$ is an associative subalgebra of $O$.

Let $f : M^2 \to S^6$ be an almost-complex immersion. Such an $f$ is necessarily minimal, and the language of harmonic sequences can be used to classify the types of almost-complex curves in $S^6$ [BVW94]. The isotropic (or super-minimal) immersions can be described in terms of holomorphic data. Either they are totally geodesic and thus given by the intersection of $S^6$ with an associative 3-plane or have been shown by Bryant to possess a Weierstrass type representation [Bry82]. We show that when $M^2$ is a torus, the non-isotropic immersions also give holomorphic data, by means of a spectral curve construction. These immersions are all superconformal, and either linearly full in $S^6$ or linearly full in a totally geodesic $S^5 \subset S^6$. In the latter case, the almost-complex condition says precisely that $f$ is a minimal Legendrian immersion. In the case of tori, various authors [Sha91, MM01, McI02] have described these immersions in terms of algebro-geometric (spectral curve) data and the periodicity problem has also been solved [CM04]. This showed
that minimal Legendrian tori are abundant in the sense that for every positive integer n, there are countably many real n-dimensional families of them. of every real dimension.

We shall describe the spectral data for superconformal almost-complex tori \( f : M^2 \to S^6 \). The condition that \( f \) is almost-complex is equivalent to the existence of a primitive lift into the 6-symmetric space \( G_2/T^2 \), where \( T^2 \) denotes the maximal torus of \( G_2 \). A superconformal immersion which is merely minimal possesses a primitive lift into \( SO(7)/T^3 \), for \( T^3 \) the maximal torus of \( SO(7) \). The main task is to characterise the subspace \( G_2/T^2 \subset SO(7)/T^3 \) in terms of symmetries of the spectral data. The group \( G_2 \) is the connected component of the subgroup of \( GL(7, \mathbb{R}) \) preserving a generic 3–form \( \alpha' \) [Bry87], for example the calibration form given above. In his study of Langlands duality for \( G_2 \) Higgs bundles [Hit07], Hitchin used this to give a beautiful description of \( G_2 \) Higgs bundles in terms of spectral curve data. He found that the abelian variety for a \( G_2 \) Higgs bundle is not a Prym variety but rather is given by the intersection of two Prym varieties, the symmetries of which together characterise \( G_2 \). We show that a similar picture persists for superconformal almost-complex tori \( f : M^2 \to S^6 \); the eigenline bundles give a linear flow in the intersection of two Prymians. The eigenline bundles also satisfy a reality condition since we complexify the group \( G_2 \) in order to obtain the spectral data, and there is also a natural sixfold symmetry by which we quotient.

2. INTEGRABLE SYSTEMS FORMULATION

We begin by describing how a superconformal almost-complex immersion \( f : M^2 \to S^6 \) yields a primitive lift into the 6-symmetric space \( G_2/T^2 \). A non-isotropic branched minimal immersion \( \mathbb{R}^2 \to S^n \) of finite type is in fact an immersion (this for example follows from the similar result proven in [McI95] for maps into complex projective space) and so it is no restriction to ask \( f \) to be an immersion. Recall that \( G_2 \subset SO(7) \) is the automorphism group of the octonion algebra, given explicitly by those matrices whose (orthonormal) columns \( f_1, \ldots, f_7 \) satisfy

\[
\begin{align*}
  f_3 &= f_1 \cdot f_2, \\
  f_5 &= f_1 \cdot f_4, \\
  f_6 &= f_2 \cdot f_4, \\
  f_7 &= f_3 \cdot f_4.
\end{align*}
\]

The maximal torus \( T^2 \) of \( G_2 \) consists of matrices of the form

\[
\text{diagonal}(1, R_{\theta}, R_{\phi}, R_{\theta+\phi})
\]

where the \( R_{\theta} \) denotes the standard rotation matrix with angle \( \theta \). Taking \( f_1 = f \) and \( f_2 \) to be a unit tangent vector then for any unit-length choice of \( f_4 \perp \text{span}_{\mathbb{R}} \{ f, f_2, f_3 \} \) the above equations define a \( G_2 \) framing of the almost-complex immersion \( f \). We shall make use of a particular such framing.

One may define an order six involution of \( G_2 \) by \( \tau = \text{Ad}_C \), where

\[
C := (1, R_{\frac{\pi}{6}}, R_{\frac{2\pi}{6}}, R_{\pi}).
\]
Writing $g^C := g \otimes \mathbb{C}$, $\tau$ gives the decomposition

$$g^C = \bigoplus_{j=0}^5 g_j$$

where $g_j$ is the exp $2\pi\sqrt{-1}j/6$-eigenspace of $\tau$. The maximal torus $T = T^2$ of $G_2$ is preserved by $\tau$, and hence this also yields the decomposition

$$T(G_2/T)^C = \bigoplus_{j=0}^5 [g_j].$$

Recall that the harmonic sequence of a harmonic map from a surface into a complex projective space is a sequence of line bundles (or equivalently of maps into $\mathbb{CP}^n$) obtained by taking successive derivatives

$$f_{j+1} = \partial_j (\frac{\partial}{\partial z} \otimes f_j),$$

and that if some $k$ consecutive elements of this sequence are mutually orthogonal then so are all sets of $k$ consecutive elements. The harmonic sequence of $f : M^2 \to S^6$ is defined to be that of its natural lift into $\mathbb{CP}^6$ and each of the maps in the sequence is harmonic. This sequence enables one to give a useful classification of harmonic maps from a surface into a sphere or complex projective space. The map $f$ is said to be isotropic if its harmonic sequence is the Frénet frame of a linearly full holomorphic curve; this is necessarily the case if seven consecutive elements of the sequence are orthogonal [BW92] and such maps are by definition described by holomorphic data. We call $f : M^2 \to S^6$ superconformal if precisely six consecutive members of its harmonic sequence are orthogonal, and every almost-complex $f : M^2 \to S^6$ is either isotropic or superconformal [BVW94]. For further information on harmonic sequences in this context we refer the reader to [BW92, BPW95, Bur95].

Given a superconformal $f : M^2 \to S^6$ we may then use harmonic sequences to define a natural lift

$$F = (f, f_2, f_3, f_4, f_5, f_6, f_7)$$

so that

$$[F] : \tilde{M}^2 \to G_2/T$$

is well-defined. In fact we may write

$$F^{-1}dF = (u_0 + u_1)dz + (\bar{u}_0 + \bar{u}_1)d\bar{z}, \quad u_j \in g_j.$$  

This says that the lift $[F]$ is primitive with respect to $\tau$; it satisfies this condition precisely when $f$ is a superconformal almost-complex immersion [BPW95]. Here $\tilde{M}^2$ denotes the universal cover of $M^2$.

$G_2^C$ additionally possesses the antiholomorphic involution $\rho(g) = \bar{g}$, which commutes with $\tau$. The form $\varphi = F^{-1}dF$ satisfies the Maurer-Cartan equation

$$d\varphi + \frac{1}{2} [\varphi, \varphi] = 0$$

and one easily checks that

$$\varphi_\zeta := (u_0 + u_1\zeta)dz + (\bar{u}_0 + \bar{u}_1\zeta^{-1})d\bar{z}$$
does also for all $\zeta \in \mathbb{C}^*$. These forms define a family

$$\nabla_\zeta = \nabla + \varphi_\zeta$$

of flat connections in a trivial principal $G^C_2$ bundle $P$ over $M^2$.

The complex Lie group $G^C_2$ is the connected component of the subgroup of $GL(7, \mathbb{C})$ preserving a generic 3–form $\alpha'$ [Bry87]. The action of the general linear group on the space of 3–forms on a seven dimensional complex vector space $V$ has a natural open orbit and by “generic” we mean that $\alpha'$ lies in this open orbit. More precisely (see [Hit07]) the $\Lambda^7 V^*$–valued symmetric bilinear form

$$q(v, w) := -\frac{1}{6}(v \lrcorNER{+} \alpha') \wedge (w \lrcorNER{-} \alpha') \wedge \alpha'$$

when viewed as a map $V \to V^* \otimes \Lambda^7 V^*$ has determinant $\kappa^3$, where $\kappa \in (\Lambda^7 V^*)^3$ is a polynomial in $\alpha'$. The open orbit is given by $\kappa(\alpha') \neq 0$. One sees in this context that $G^C_2 \subset SO(7, \mathbb{C})$ by showing that the connected component of the subgroup preserving $\alpha'$ also preserves the metric defined by

$$g = \frac{q}{\kappa^{1/3}}.$$

Here we have taken the positive root with respect to the orientation given by $\kappa^3$. When $\alpha'$ has the usual normal form

$$\alpha' = (\theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4) \wedge \theta_5 + (\theta_1 \wedge \theta_3 - \theta_4 \wedge \theta_2) \wedge \theta_6 + (\theta_1 \wedge \theta_4 - \theta_2 \wedge \theta_3) \wedge \theta_7 + \theta_4 \wedge \theta_6 \wedge \theta_7$$

then $q$ is the standard Euclidean metric. Thus we may equivalently view the connections $\nabla_\zeta$ as acting on a rank seven holomorphic vector bundle $\mathbf{V}$ on $M$ with $G^C_2$–structure given by a 3–form $\alpha'$ on each $V_z$ with $\kappa(\alpha') \neq 0$. The family of flat connections $\nabla_\zeta$ respects these symmetries in the sense that

$$\tau(\varphi_\zeta) = \varphi_{\zeta^{-1}}, \rho(\varphi_\zeta) = \varphi_{\zeta^{-1}},$$

where $\epsilon = \exp \pi \sqrt{-1}/3$.

To summarise, a non-isotropic almost-complex immersion $f : \tilde{M} \to S^6$ is equivalent to a $\mathbb{C}^*$–family of flat connections $\nabla + \varphi_\zeta$ in a trivial $G^C_2$ bundle $\mathbf{V}$ over $M$, where $\varphi_\zeta$ is symmetric with respect to $\tau, \rho$ as specified above. Consider now the twisted loop algebrae given by

$$\Lambda^\rho_\tau \mathfrak{g}_2^C := \{ A_\zeta = \sum_{j=-d}^d A_j \zeta^j | A_j \in \mathfrak{g}_j, A_{-j} = \rho(A_j), \zeta \in \mathbb{C}^* \}, \quad d \in \mathbb{N}.$$

These symmetries can also be expressed as

$$\rho(A_\zeta) = A_{\zeta^{-1}} \quad \text{and} \quad \tau(A_\zeta) = A_{\zeta^1}.$$

If $A = \sum_{j=-d}^d A_j \zeta^j : \tilde{M} \to \Lambda^\rho_\tau \mathfrak{g}_2^C$ satisfies the Lax pair equation

$$dA = [A, \varphi_\zeta]$$

with $\varphi_\zeta = (A_{d-1} + A_d \zeta) dz + (A_{1-d} + A_{-d} \zeta^{-1}) d\bar{z}$ we call it a polynomial Killing field. Note that the Lax equation is equivalent to the requirement that $A$ be
a parallel section of the pullback of $\text{Ad}(P)$ to $\tilde{M}$. A polynomial killing field clearly yields a superconformal almost-complex immersion $\tilde{M} \to S^6$, and if appropriate periodicity conditions are satisfied than this immersion descends to $M$. We shall now restrict our attention to almost-complex immersions $f : \mathbb{C} \to S^6$ obtained from polynomial Killing fields; these maps are said to be of finite type, and include all doubly periodic examples [BPW95]. The Lax equation clearly forces $d = 6k + 1$.

3. Spectral data for the Lax equations

We show that a non-isotropic almost-complex immersion $f : \mathbb{C} \to S^6$ of finite type gives rise to an algebraic curve, which we call the spectral curve. There are a number of different constructions of spectral curves which have been used in other geometric settings. Beginning with a commutative algebra of polynomial Killing fields, one can use the characteristic polynomial, as in [AvM80], the eigenline curve, as in [Hit90] or take Spec, as in [McI95]. Alternatively, one could take a more Lie-theoretical approach, as in [KP94, Kan89]. The characteristic polynomial is the simplest definition, but can for some surface classes have unnecessary singularities which do not reflect the underlying geometry. We shall prove that for non-isotropic almost-complex immersions of the plane into $S^6$, the characteristic polynomial is generically smooth, justifying our choice.

We may pull the trivial bundle $V$ and the three form $\alpha'$ back to $\mathbb{C} \times \mathbb{P}^1$, and write $V, \alpha'$ for the restrictions to $\{z\} \times \mathbb{P}^1$ (so $V$ is the trivial rank seven holomorphic vector bundle on $\mathbb{P}^1$). The polynomial Killing fields $A'_\zeta(z) \in \Lambda^{0,7} \mathfrak{g}_2^C$ are holomorphic sections of $\mathcal{O}(2(6k + 1)) \otimes \text{End } V$. We shall realise the characteristic polynomial of $A'_\zeta(z)$ inside the total space of a line bundle over $\mathbb{P}^1$. The eigenvalues of an element of $\mathfrak{g}_2^C$ are of the form $0, \mu_1, \mu_2, \mu_3, -\mu_1, -\mu_2, -\mu_3$, and satisfy $\mu_1 + \mu_2 + \mu_3 = 0$. Hence the characteristic polynomial has the form

$$\det(\mu \cdot \text{id}_V - A'_\zeta(z)) = \mu(\mu^6 - a_1 \mu^4 + \frac{a_2}{4} \mu^2 - a_2)$$

with

$$a_1(\zeta) = \mu_1^2 + \mu_2^2 + \mu_3^2, \quad a_2(\zeta) = (\mu_1 \mu_2 \mu_3)^2,$$

We let $\tilde{X}$ be the complex surface which is the total space of the line bundle $\mathcal{O}(2(6k + 1))$ and $\pi_X$ its projection onto $\mathbb{P}^1$. The pull-back of $\mathcal{O}(2(6k + 1))$ to $\tilde{X}$ has a tautological section $\mu$. The characteristic polynomial is thus a section of $H^0(\tilde{X}, \pi_X^* \mathcal{O}(14(6k + 1)))$, and defines a reducible algebraic curve $\tilde{\Sigma}'$ in $\tilde{X}$, one of whose components is $\mathbb{P}^1$. We denote the other component by $\tilde{\Sigma}$. The matrices $A'_\zeta(z)$ for different $z$ are related by conjugation, and hence $\tilde{\Sigma}'$ is independent of $z$. 
The $\tau$-symmetry of (2.1) gives that $a_j(\zeta) = a_j(\zeta)$ so we may write $a_j(\zeta) = b_j(\lambda)|_{\lambda=\zeta^6}$. Notice that this forces the degrees of $a_1, a_2$ to be divisible by six so that $a_1$ cannot be an element of $H^0(\mathbb{P}^1, \mathcal{O}(4(6k + 1)))$ of full degree; we have instead that $a_1 \in H^0(\mathbb{P}^1, \mathcal{O}(4(6k)))$. Clearly $a_2 \in H^0(\mathbb{P}^1, \mathcal{O}(12(6k + 1)))$. The $\rho$-symmetry of (2.1) yields $b_j(\lambda) = b_j(\lambda^{-1})$. Let $X$ denote the total space of the line bundle $\mathcal{O}(2(k + 1))$ and $\pi_X$ its projection onto $\mathbb{P}^1$. Writing $\eta$ for the tautological section of $\pi_X^*\mathcal{O}(2(k + 1))$ over $X$, then again $\Sigma' := \Sigma'/\tau$ decomposes into a copy of $\mathbb{P}^1$ and the algebraic curve $\Sigma$ given by the divisor of

$$\eta^6 - b_1 \eta^4 + \frac{b_2}{4} \eta^2 - b_2 \in H^0(X, \pi_X^*\mathcal{O}(12(k + 1))).$$

We call $\Sigma'$ the spectral curve of the almost-complex immersion $f : \mathbb{C} \rightarrow S^6$, and we term $\Sigma$ the main component of $\Sigma'$.

**Theorem 3.1.** The main component $\Sigma$ of a generic spectral curve $\Sigma'$ is smooth, with genus $5(6k + 1)$. The real dimension of the moduli of such curves $\Sigma$ is $(16k + 4)$.

**Proof.** We begin by proving smoothness, for which we use an argument similar to that employed by [KP94] in their study of spectral curves of $G_2$ Higgs bundles. Set $B := H^0(\mathbb{P}^1, \mathcal{O}(4k)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(12k + 2))$, and write $B_\mathbb{R}$ for the real slice given by $b_j(\lambda) = b_j(\lambda^{-1})$. The main component $\Sigma$ of the spectral curve is given by $(b_1, b_2) \in B_\mathbb{R}$, and we wish to show that those points which yield singular spectral curves are contained in a lower dimensional subvariety of $B_\mathbb{R}$. Recall that the complete linear system $|D|$ defined by a divisor $D$ on $X$ is the space of all effective divisors linearly equivalent to $D$, a linear system $l$ is any linear subspace of $|D| = \mathbb{P}H^0(X, \mathcal{O}(D))$ and that $p \in X$ is a base point of $l$ if $p$ is contained in all members of $l$. We denote the base locus of $l$ on $X$ by $B_X l$. Bertini’s theorem [Ber07] says in part that a generic element of a linear system is smooth away from the base locus of the system. For each fixed $b_1 \in H^0(\mathbb{P}^1, \mathcal{O}(4k))$, as $c \in \mathbb{R}$ and $b_2 \in \mathcal{O}(12k + 2)$ vary,

$$c(\eta^6 - b_1 \eta^4 + \frac{b_2}{4} \eta^2) - b_2$$

defines a linear subsystem $l(b_1)$ of $|\pi_X^*\mathcal{O}(12k + 2)|$. This contains the affine subspace $a(b_1) \simeq \mathbb{P}(\pi_X^*H^0(\mathbb{P}^1, \mathcal{O}(12k + 2)))$ defined by setting $c = 1$. We will show that $a(b_1)$ is base point free, and hence the same is true for $l(b_1)$. In particular $a(b_1)$ contains the divisor of $\eta^6 - b_1 \eta^4 + (\frac{b_2}{4}) \eta^2 = \eta^2(\eta^2 - b_1/2)^2$ and so the base locus $B_X a(b_1)$ of $a(b_1)$ is a subset of

$$B_{(\eta=0)} a(b_1) \cup B_{(\eta^2=b_1/2)} a(b_1).$$

The restriction of $a(b_1)$ to the projective line $\{\eta = 0\}$ clearly has no base points. The hyperelliptic curve $C(b_1)$ defined by $\eta^2 = b_1/2$ is smooth for a
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generic choice of $b_1$, and writing $\pi$ for the projection to $\mathbb{P}^1$, we have

$$B_{C(b_1)}(\pi^*H^0(\mathbb{P}^1, \mathcal{O}(12k+2))) = B_{C(b_1)}\pi^*|\mathcal{O}(12k+2)|.$$  

But the projection of this base locus to $\mathbb{P}^1$ is then contained in the base locus for $|\mathcal{O}(12k+2)|$, which is empty.

We have then that for a generic $b_1$ the linear system $a(b_1)$ has no base points. Since dividing (3.1) by the constant $c$ does not change the resulting curve, Bertini’s theorem implies that a generic $(b_1, b_2) \in B$ defines a smooth curve. Taking real points yields the analogous result for $B_R$, which has real dimension $(4k+1) + (12k+3) = 16k+4$, and parameterises the space of spectral curves.

We now compute the degree of the ramification divisor $R$ of the projection $\lambda : \Sigma \to \mathbb{P}^1$. The cover is clearly totally branched over 0 and $\infty$, and the ramification points for $\lambda \in \mathbb{C}^*$ are given by the vanishing of the discriminant $\Delta = b_2\left(\frac{1}{2}b_1^3 - 27b_2\right)$. At the $12k+2$ points for which $b_2(\lambda) = 0$, the equation for $\Sigma$ factors as

$$\eta^2(\eta^2 - \frac{b_1}{2})^2 = 0,$$

and we may assume that three pairs of eigenvalues come together as $\mu_1 = -\mu_2 = -\mu_3 = \sqrt{b_1/2}$ and $\mu_3 = -\mu_2 = -\sqrt{b_1/2}$. At the $12k+2$ points where $27b_2 - (1/2)b_1^3 = 0$, we have

$$\left(\frac{\eta^2 - \frac{b_1}{6}}{6}\right)^2 \left(\frac{\eta^2 - 2b_1}{3}\right) = 0$$

and we may assume that two pairs of eigenvalues come together as $\mu_1 = \mu_2 = \sqrt{b_1/6}$, $-\mu_1 = -\mu_2$ (thus $\mu_3 = -2\sqrt{b_1/6}$). Hence

$$|R| = (12k+2) \cdot 3 + (12k+2) \cdot 2 + 2 \cdot 5 = 20(3k+1)$$

so by the Riemann-Hurwitz formula,

$$2 - 2g = 6 \cdot 2 - |R| = -60k - 8$$

and the genus of the smooth algebraic curve $\Sigma$ is $g = 5(6k + 1)$. Since arithmetic genus is constant in families, this is also the arithmetic genus of any singular curves $\Sigma$. \hfill \square

**Remark 3.2.** We hence also obtain that generically the curve $\hat{\Sigma}$ is smooth, the ramification divisor $\hat{R}$ of the map $\zeta : \hat{\Sigma} \to \mathbb{P}^1$ has degree $|\hat{R}| = 60(6k + 1) + 10$ and the genus of $\hat{\Sigma}$ is $30(6k + 1)$.

We explain now how the spectral curve $\Sigma'$ can be realised as the characteristic polynomial of a polynomial killing field depending only on $\lambda = \zeta^6$. Let

$$C_\zeta = \text{diagonal}(1, S_\zeta, S_{\zeta^2}, S_{\zeta^3})$$
where

\[ S_\xi = \begin{pmatrix} \frac{1}{2}(\zeta + \zeta^{-1}) & -\frac{i}{2}(\zeta - \zeta^{-1}) \\ \frac{1}{2}(\zeta - \zeta^{-1}) & \frac{i}{2}(\zeta + \zeta^{-1}) \end{pmatrix} \]

and define

\[ A'_\xi(z) = \text{Ad}_{C_\xi^{-1}} A'_\xi(z). \]

**Lemma 3.3.** For each \( z \in T^2 \), the endomorphism \( A'_\xi(z) \) is a polynomial Killing field depending only on \( \lambda = \xi^6 \). As a function of \( \lambda \) it satisfies \( A'_\lambda(z) \in \Lambda^{6}\mathfrak{g}_{2}^{\mathbb{C}} \) whenever \( A'_\xi(z) \in \Lambda^{6}\mathfrak{g}_{2}^{\mathbb{C}} \).

**Proof.** By the rigidity of holomorphic functions, it suffices to prove these results for \( \zeta = e^{i\theta} \in S^1 \). Then \( S_\xi = R_\theta \) and so \( A'_\xi = \text{Ad}_{C_\xi^{-1}} \text{Ad}_{C_\xi} A'_\xi = A'_\xi \); and so \( A'_\xi \) depends only on \( \lambda = \xi^6 \). Also \( S^{-1}_\xi = R^{-1}_\theta = R_{-\theta} = S^{-1}_{\xi^{-1}} \) and so \( A'_\xi \) is polynomial in \( \lambda \). Since \( C_\xi \) and \( C_{\xi^{-1}} \) are each of the form \( \sum_{j=-3}^{3} C_j \xi^j \), *a priori* we would expect that \( A'_\xi(z) \in \mathcal{O}(12k + 14) \otimes \text{End} \mathbf{V} \). However we have just shown that \( A'_\xi \) is a function of \( \lambda = \xi^6 \) so we must have \( A'_\xi \in \mathcal{O}(12(k + 1)) \otimes \text{End} \mathbf{V} \) or \( A'_\lambda \in \mathcal{O}(2(k + 1)) \otimes \text{End} \mathbf{V} \). The \( \rho \)-symmetry is inherited from \( A'_\xi \). \( \square \)

### 4. Symmetries of the Linear Eigenline Flow

We now identify geometric structures on the spectral curve resulting from the fact that the immersion \( f \) is almost-complex and show that the immersion gives rise to a linear flow in the intersection of two Prymians of \( \Sigma \). In his study of \( G_2 \) Higgs bundles \([\text{Hit}07]\), Hitchin encoded the \( G_2 \) structure of that problem in terms of spectral curve data and in particular in terms of a linear flow in a similar sub-torus of the Jacobian. Our approach here is an adaption of his work, and can be viewed as another aspect of the similarity between the spectral curve approaches to finite-type harmonic maps and to Higgs bundles.

We shall eventually express these symmetries in terms of the spectral curve \( \Sigma' \) and its main component \( \Sigma \). However not all of the geometric structures that we shall utilise (for example the symplectic form introduced below) naturally descend to the quotient picture and so we shall first work with the unquotiented curves \( \Sigma' \) and \( \Sigma \). Essentially this is necessitated by the fact that the degree of the polynomial killing field \( A'_\xi(z) \) is not in general divisible by six and so an attempt to define a symplectic form as below using \( A_\lambda(z) \) would not yield an everywhere non-degenerate form. As explained earlier, the generic 3-form \( \alpha' \) on the trivial rank seven holomorphic vector bundle \( \mathbf{V} \) defines a non-degenerate symmetric bilinear form \( g \), and we will now investigate the symmetries on the eigenline bundles resulting from this special orthogonal structure. The form \( g \) induces on \( \mathbf{V} \) for each \( z \in \mathbb{C} \) a skew-symmetric bilinear form \( \omega'_z \) valued in \( \mathcal{O}(2(6k + 1)) \)

\[ \omega'_z(v_1, v_2) = g(A'_\xi(z)v_1, v_2). \]
Let \( V_0 = V_0^z \) be the line bundle contained in \( \ker A'_\zeta(z) \) and define \( V_1 = V_1^z \) as the quotient
\[
0 \rightarrow V_0 \rightarrow V \rightarrow V_1 \rightarrow 0.
\]
Here and henceforth we suppress the dependence on \( z \); we shall show that the holomorphic line bundles \( V_0^z \) are all isomorphic.

The forms \( g \) and \( \omega_z \) are clearly well-defined on the quotient \( V_1 \). The dual bundle \( V_1^* \) is the annihilator of \( V_0 \), and writing
\[
\gamma : V^* \simeq V
\]
for the isomorphism given by \( g \), then for each \( \zeta \in \mathbb{P}^1 \), the subspace \( \gamma(V_1^z(\zeta)) \) is the orthogonal complement of \( V_0(\zeta) \). Away from the finitely many \( \zeta \in \mathbb{P}^1 \) satisfying \( a_2(\zeta) = (\mu_1\mu_2\mu_3)^2 = 0 \), the restriction \( A_\zeta(z) \) of the polynomial killing field to the orthogonal complement of \( V_0 \) has no kernel. Setting \( E = V_1^z \otimes O(-6(6k+1)) \), the obvious analog of (4.1) gives a skew-symmetric form \( \omega_z \) on \( E_z \), clearly non-degenerate when \( a_2(\zeta) \neq 0 \).

**Lemma 4.1** (c.f. [Hit07]). For each \( z \in \mathbb{C} \), the form \( \omega_z \) is everywhere non-degenerate, so the vector bundle \( E_z \) is symplectic and hence holomorphically trivial. The restriction \( A_\zeta(z) \) of the polynomial killing field \( A'_\zeta(z) \) acts as a symplectic endomorphism of \( E_z \).

In light of this, we shall again suppress the \( z \) and simply write \( E \) since holomorphically all these bundles are isomorphic.

**Proof.** The polynomial killing field \( A_\zeta(z) \) is a section of \( \mathcal{O}(2(6k+1)) \otimes g_2^C \) and so \( \omega_z \in H^0(\Lambda^2 V^* \otimes \mathcal{O}(2(6k+1))) \). Thus \( \omega_z^2 \) is a section of \( \Lambda^6 V^* \otimes \mathcal{O}(6(6k+1)) \). Furthermore, the metric \( g \) gives an isomorphism
\[
\psi : \Lambda^6 V^* \rightarrow V
\]
characterised by \( \alpha' = 6\psi(\alpha') \), vol, where vol denotes the volume form. A straightforward computation yields
\[
(4.2) \quad \omega_z(A_\zeta(z)v_1, v_2) + \omega_z(v_1, A_\zeta(z)v_2) = 0
\]
and from this we see that under the induced action on \( \Lambda^2 V^* \) the skew-symmetric form \( \omega_z \) satisfies \( A_\zeta(z)(\omega_z) = 0 \). Hence for each \( z \) the section of \( V \) defined by
\[
v_0^z = \psi(\omega_z^2)
\]
is a zero-eigenvector of \( A_\zeta(z) \). Let \( \epsilon_0, \ldots, \epsilon_6 \) be an orthonormal basis for \( H^0(\mathbb{P}^1, V) \), with \( \epsilon_0 \) valued in \( V_0 \). Then
\[
v_0(z) = -i\mu_1\mu_2\mu_3\epsilon_0
\]
and we see that when regarded as a section of \( H^0(\mathbb{P}^1, V_0 \otimes \mathcal{O}(6(6k+1))) \), the zero-eigenvector section \( v_0^z \) is nowhere–vanishing, so defines an isomorphism \( \mathcal{O}(-6(6k+1)) \simeq V_0 \subset V \). The isomorphism \( \psi \) gives \( \Lambda^6(V_1) \simeq V_0^* \) and we have
\[
\Lambda^6(V_1) \simeq V_0^* \simeq \mathcal{O}(6(6k+1))
\]
and thus
\[ \Lambda^6 E \simeq \mathcal{O}. \]

The form \( \omega_z \) must then be non-degenerate everywhere and so \( E \) is a symplectic vector bundle. The symplectic form \( \omega_z \) defines an isomorphism \( E \simeq E^* \) and holomorphically \( E \) is the trivial rank 6 bundle. From (4.2) we see that \( A_\zeta(z) \) acts as a symplectic endomorphism of \( E \). \( \square \)

When \( a_2(\xi) \neq 0 \), \( \omega_z \) provides an isomorphism \( V_1^* \simeq V_1 \), which allows us to identify \( V_1 \) with the orthogonal complement \( \omega_z(V_1^*) \) of \( V_0 \) at such points. For such \( \zeta \), the restriction of the polynomial killing field \( A'_\zeta(z) \) to \( \omega_z(V_1(\xi)) \subset V(\xi) \) agrees with its action on \( V_1(\xi) \) as a quotient, and (4.2) similarly still holds. For \( \zeta \) satisfying \( a_2(\zeta) = 0 \), \( V_0(\zeta) \) is contained in its orthogonal complement \( \gamma(V_1^*) \).

**Definition 4.2.** For each \( z \in \mathbb{C} \) define \( \hat{E}_z \to \Sigma \) to be the unique line bundle contained in the sub-sheaf \( \ker(\mu \cdot \text{id} - \zeta^* A_\zeta)(z) \subset \zeta^* E \). We call \( \hat{E}_z \) the eigenline bundles.

We note that one could instead look at the eigenlines of the action of \( A'_\zeta \) on \( \zeta^* V \) and indeed to reconstruct the original almost-complex immersion \( f \) we will need to consider glueing information on the points of \( \hat{\Sigma}' \) where the two components intersect, given by \( a_2(\xi) = 0 \) (or their equivalent on the quotient curve \( \Sigma' \)). For now though we are interested in the symmetries of the bundles \( \hat{E}_z \). The eigenline bundles satisfy
\[ \zeta_\# \hat{E}_z^* \simeq E^* \simeq E. \]

We have on \( \hat{\Sigma} \) the involution
\[ \sigma : (\zeta, \mu) \mapsto (\zeta, -\mu), \]
which defines a Prym variety in which, as we shall see, our eigenline bundles lie. Let \( \hat{C}_1 \simeq \hat{\Sigma}/\sigma \) be the curve defined by
\[ y^3 - a_1(\xi)y^2 + \frac{a_1(\xi)^2}{4}y - a_2(\xi) = 0 \]
and write \( \hat{\pi}_1(\mu, \zeta) = (\mu^2, \zeta) \) for the natural projection \( \hat{\Sigma} \to \hat{C}_1 \). Define
\[ \text{Nm}_{1} \quad \text{Pic}(\hat{\Sigma}) \to \text{Pic}(\hat{C}_1) \]
\[ \quad \mathcal{O}(D) \mapsto \mathcal{O}(\hat{\pi}_1(D)). \]

The Prym variety \( P(\hat{\Sigma}, \hat{C}_1) \) is defined to be the connected component of \( \ker(\text{Nm}_1)_* \) containing the identity. When \( \hat{\pi}_1^* \) is injective, \( (\text{Nm}_1)_* \) is given by the dual of \( \hat{\pi}_1 \) and so \( \ker(\text{Nm}_1)_* \) is itself connected \[ \text{Mum} \]. The criterion given in [BNR89] [3.10] for the injectivity of \( \hat{\pi}_1^* \) is here easily seen to be satisfied.
**Theorem 4.3.** The eigenline bundles $\hat{E}_z$ have degree $-30(6k + 1) - 5$ and lie in a constant translate of the Prym variety defined by $\hat{\pi}_1 : \Sigma \to \hat{C}_1$:

$$\hat{E}_z \otimes \mathcal{O}(\frac{1}{2}\hat{R}) \in P(\Sigma, \hat{C}_1).$$

They also satisfy

$$\tau^*\hat{E}_z \simeq \hat{E}_z$$

and the reality condition

$$\rho^*\hat{E}_z \simeq \hat{E}_z.$$

**Proof.** The push-forward of the sheaf of sections of each eigenline bundle $\zeta_*\mathcal{O}(\hat{E}_z)$ is the sheaf of sections of a vector bundle [Gun67], and the fibres of this bundle are

$$(\zeta_*\hat{E}_z)_\zeta = \oplus_{(\zeta, \mu) \in \Sigma} J_r(\hat{E}_z)_{(\zeta, \mu)},$$

where $J_r(\hat{E}_z)_{(\zeta, \mu)}$ denotes the jet space of local sections of $\hat{E}_z$ of degree $r = \deg \hat{R}_{(\zeta, \mu)}$ and $\hat{R}$ is the ramification divisor of the cover $\zeta : \hat{\Sigma} \to \mathbb{P}^1$.

Note that $J_0(\hat{E}_z)_{(\zeta, \mu)}$ is simply $\hat{E}_z(\zeta, \mu)$. The natural projection of the jet space to the bundle fibre gives

$$0 \to \oplus_{(\zeta, \mu) \in \Sigma} J_r(\hat{E}_z)_{(\zeta, \mu)}/(\hat{E}_z)_{(\zeta, \mu)} \to (\zeta_*\hat{E}_z)_{(\zeta, \mu)} \to \oplus_{(\zeta, \mu) \in \Sigma} \hat{E}_z_{(\zeta, \mu)} \to 0.$$

Dualising, we have the exact sequence of sheaves (c.f. [HIt87])

$$(4.3) \quad 0 \to \mathcal{O}(E^*) \to \mathcal{O}(\zeta_*\hat{E}_z)^* \to \mathcal{R} \to 0,$$

where $\mathcal{R}$ is the skyscraper sheaf with stalk $\oplus_{(\zeta, \mu) \in \Sigma} J_r(\hat{E}_z)_{(\zeta, \mu)}/(\hat{E}_z)_{(\zeta, \mu)}$ on the ramification divisor of $\zeta$. The degree of the push-forward bundle is given by

$$d(\zeta_*\hat{E}_z) = d(\hat{E}_z) + 1 - g_\Sigma - \deg(\zeta)(1 - g_{\mathbb{P}^1}) = d(\hat{E}_z) - (30(6k + 1) + 5),$$

where the first equality is a standard result (see e.g. [HSW99]) and the second uses Remark 3.2. In the same remark we computed that the degree of the ramification divisor $R$ is $60(6k + 1) + 10$, so using the exactness of (4.3) we obtain $d(\hat{E}_z) = -30((6k + 1) + 5)$ as claimed. Note that we could also have used the relation $\zeta_*\hat{E}_z^* \simeq E^* \simeq E$ to compute the degree of the eigenline bundles.

Under the involution $\sigma$ the eigenline bundles satisfy

$$\sigma^*\hat{E}_z(\zeta, \mu) = \hat{E}_z(\zeta, -\mu)$$

and by definition $\omega_z$ defines a non-trivial section of

$$\hat{E}_z^* \otimes \sigma^*\hat{E}_z^* \otimes \mathcal{O}(\hat{R}).$$

Since we have shown that this line bundle has zero degree we conclude that it must be the trivial bundle. This shows that the bundles $\hat{E}_z^* \otimes \mathcal{O}(\frac{1}{2}\hat{R})$ lie in the Prym variety defined by the involution $\sigma$, where we have fixed a square root of the bundle $\mathcal{O}(\hat{R})$. Using (2.1), the symmetry of the eigenline bundles with respect to $\rho$ and $\tau$ is clear. \qed
So far we have utilised only the involution $\sigma$ which could be described purely in terms of the symplectic structure. To encode the fact that the frame to our almost-complex torus lies in $G_2 \subset SO(7)$ we return our attention to the $3$–form $\alpha' \in H^0(\mathbb{P}^1, V)$ satisfying $\kappa(\alpha') \neq 0$ which defines the $G_2$–structure on $V$.

Denote by $\alpha$ the restriction of $\alpha'$ to the orthogonal complement $V_0^\perp \simeq \gamma(V_1^*)$ of $V_0$. We defined earlier the symplectic vector bundle $E = V_1 \otimes O(-6(6k+1))$, and so we may write $\alpha \in H^0(\mathbb{P}^1, \Lambda^3 E^* \otimes O(3(6k+1)))$. Define as in [Hit07, Hit00]

$$K_\alpha : E \rightarrow E \otimes O(6(6k+1)) \quad v \mapsto (v, \alpha) \wedge \alpha$$

where we have used $\Lambda^5 E^* \simeq E$. It is shown in [Hit00] that the entire space $E$ is an eigenspace of $K_\alpha^2$, and we write the eigenvalue as

$$s(\alpha) := \frac{1}{6} \text{trace} K_\alpha^2 \in H^0(\mathbb{P}^1, O(12(6k+1))).$$

Furthermore, if $a_2(\zeta) = 0$, then from the definition of $g$, the vector $v_0 \in V_0^\perp \simeq V_1^*$ satisfies $(v_0, \alpha') \wedge (v_0, \alpha') \wedge \alpha' = 0$ and so the eigenvalue $s(\alpha)(\zeta)$ must vanish. Hence $a_2$ divides $s(\alpha)$ and the eigenvalues $\sqrt{s(\alpha)}$ are well-defined on the hyperelliptic curve $\hat{C}_2$ given in the total space of $O(-6(6k+1))$ by

$$z^2 = a_2(\zeta).$$

Since $a_2(\zeta)$ has degree $12((6k+1))$, the curve $\hat{C}_2$ has genus $36k + 10$. Write $\hat{p}_2 : \hat{C}_2 \rightarrow \mathbb{P}^1$ for the natural projection and observe that

$$\hat{\pi}_2 : \hat{\Sigma} \rightarrow \hat{C}_2 \quad (\zeta, \mu) \mapsto (\zeta, \mu - \frac{a_1}{2})$$

exhibits $\hat{\Sigma}$ as a three-to-one cover of $\hat{C}_2$.

**Theorem 4.4.** The constant translate $\hat{E}_\zeta \otimes O(\frac{1}{2}R)$ of the eigenline bundles also lies in the Prym variety $P(\hat{\Sigma}, \hat{C}_2)$ defined by $\hat{\pi}_2 : \hat{\Sigma} \rightarrow \hat{C}_2$.

**Proof.** The map $p_2^* K_\alpha$ has for each $\zeta$ two 3-dimensional eigenspaces which define vector bundles $E^+$ and $E^-$ on $\hat{C}_2$. If $W$ is any six-dimensional complex vector space, $GL(6)$ acts on $\Lambda^3 W^*$ with stabiliser $SL(3) \times SL(3) \times Z_2$. It is shown in [Hit00] that if we choose a basis $v_1, \ldots, v_6$ for $W$ and write the dual basis as $\theta_1, \ldots, \theta_6$ then the three forms $\alpha$ satisfying $s(\alpha) \neq 0$ have normal form

$$\alpha = \theta_1 \wedge \theta_2 \wedge \theta_3 + \theta_4 \wedge \theta_5 \wedge \theta_6,$$

and that

$$K_\alpha v_i = v_i(\theta_1 \wedge \ldots \wedge \theta_6), \quad i = 1, 2, 3; \quad K_\alpha v_i = -v_i(\theta_1 \wedge \ldots \wedge \theta_6), \quad i = 4, 5, 6.$$
We label our eigenspaces so that away from the divisor $D$ on $\mathbb{P}^1$ given by $a_2(\zeta) = 0$, the span of $v_1, v_2, v_3$ in this normal form is $E^+$. The restriction of $\alpha$ to $E^+$ is thus non-vanishing away from $\hat{p}_2^*(D)$. Returning momentarily to the vector space situation, the hypersurface within the space of three forms $\Lambda^3 W^+$ satisfying $s = 0$ itself has an open orbit under the action of $GL(6)$, in which the three-form has normal form \cite{Hit07}.

$$\alpha = \theta_1 \wedge \theta_3 \wedge \theta_5 + \theta_2 \wedge \theta_4 + \theta_3 \wedge \theta_4 \wedge \theta_5.$$ 

With respect to both of these normal forms, as in \cite{Hit07} the symplectic form \(\omega \) is given by $\theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_5 + \theta_3 \wedge \theta_6$ and the eigenspaces $E^+, E^-$ are Lagrangian.

A straightforward calculation shows that the endomorphisms $K_\alpha$ and $A_\zeta$ commute, and hence $\hat{p}_2^* A_\zeta$ restricts to $A_\zeta^+ \in \text{End}(E^+) \otimes \hat{p}_2^* \mathcal{O}(2(6k + 1))$.

The curve $\hat{\Sigma}$ is then biholomorphic to the curve given in the total space of $H^0(\hat{C}_2, \mathcal{O}(2(6k + 1)))$ by the characteristic polynomial of $A_\zeta^+$, and the eigenline bundles $\hat{E}_z$ are also given by the eigenlines of the action of $A_\zeta^+$ on $E^+$. Then using also that $E^+$ is Lagrangian,

\begin{equation}
(\hat{\pi}_2)_* E^*_z = (E^+)^* \simeq \rho^* \mathcal{E}/E^+.
\end{equation}

Thus $E^+$ is the kernel of the natural surjection $\hat{p}_2^* \mathcal{E} \to (\hat{\pi}_2)_* E^*_z \simeq (E^+)^*$.

For a generic almost-complex immersion $f : T^2 \to S^6$, at all points $\zeta \in D$ the three-form $\alpha(\zeta)$ will lie in the open orbit described above. We assume then that on $\hat{p}_2^*(D)$, the three form $\alpha(\zeta)$ lies in this open orbit, and then as in \cite{Hit07} using the normal form one sees that the restriction of $\alpha$ to $E^+$ vanishes on $\hat{p}_2^*(D)$ with multiplicity two. Thus $\alpha$ defines an isomorphism

\begin{equation}
\Lambda^3(E^+) \simeq \hat{p}_2^* \mathcal{O}(-9(6k + 1)).
\end{equation}

Once again, a straightforward calculation gives that for any line bundle $\mathcal{L}$ on $\Sigma$,

\begin{equation}
\text{Nm}_2\mathcal{L} = \det(\hat{\pi}_2, \mathcal{L}) \otimes \det(\hat{\pi}_2, \mathcal{O}(\hat{R}_2)),
\end{equation}

where now $\hat{R}_2$ denotes the ramification divisor of $\hat{\pi}_2$. Then from (4.4), (4.5) and (4.6) we have that

\begin{equation}
\text{Nm}_2(\hat{E}_z^* \otimes \mathcal{O}(-\frac{1}{2} \hat{R})) = \hat{p}_2^* \mathcal{O}(9(6k + 1)) \otimes \det(\hat{\pi}_2, \mathcal{O}(\hat{R}_2)) \otimes \text{Nm}_2(\mathcal{O}(\frac{1}{2} \hat{R})).
\end{equation}

The ramification divisors can naturally be expressed in terms of the respective canonical divisors

$$\mathcal{O}(\hat{R}_2 - \hat{R}) = K_{\Sigma} \otimes \hat{\pi}_2^* K_{\hat{C}_2}^* \otimes (K_{\Sigma})^* \otimes \zeta^* K_{\mathbb{P}^1}.$$ 

Then since $\text{Nm}_2 \cdot \hat{\pi}_2^* = 3I$ and $\zeta = \hat{p}_2 \cdot \hat{\pi}_2$, we have

\begin{equation}
\text{Nm}_2(\mathcal{O}(\frac{1}{2}(\hat{R}_2 - \hat{R}))) = K_{\hat{C}_2}^{-3/2} \otimes \hat{p}_2^* K_{\mathbb{P}^1}^{3/2} = K_{\hat{C}_2}^{-3/2} \otimes \hat{p}_2^* \mathcal{O}(-3).
\end{equation}
But the ramification divisor \( S \) for the degree two map \( \hat{p}_2 \) satisfies \( S \simeq \hat{p}^*_2\mathcal{O}(6(6k + 1)) \), so
\[
K_{\hat{C}_2}^{-3/2} \simeq \hat{p}^*_2(\mathcal{O}(6(6k + 1) - 2)^{-3/2} \simeq \hat{p}^*_2(\mathcal{O}(-9(6k + 1) + 3)).
\]
We have then shown that \( \text{Nm}_2(\mathcal{E}_z^* \otimes \mathcal{O}(-\frac{1}{2}\hat{R})) = 0 \), so that \( \mathcal{E}_z^* \otimes \mathcal{O}(-\frac{1}{2}\hat{R}) \in P(\hat{\Sigma}, \hat{C}_1) \cap P(\hat{\Sigma}, \hat{C}_2). \)

\[\square\]

We shall now quotient by the six-fold symmetry \( \tau \), which clearly descends to the curves \( \hat{C}_1 \) and \( \hat{C}_2 \). Define
\[
C_j = \frac{\hat{C}_j}{\tau},
\]
so that \( C_1 \) is given explicitly by
\[y^3 - b_1(\lambda)y^2 + \frac{b_1(\lambda)^2}{4}y - b_2(\lambda) = 0\]
and \( C_2 \) by \( z^2 = b_2(\lambda) \). We have the following commutative diagram
\[
\begin{array}{ccc}
\Sigma & \xrightarrow{\pi_1} & C_1 \\
\downarrow{\pi_2} & & \downarrow{p_1} \\
C_2 & \xrightarrow{p_2} & P^1
\end{array}
\]
with \( \pi_1(\eta, \lambda) = (\eta(\eta^2 - \frac{b_1}{2}), \lambda) \) and \( \pi_2(\eta, \lambda) = (\eta^2, \lambda) \).

Let \( P(C_2, \mathbb{P}^1), P(\Sigma, C_1) \) be the Prym varieties of the 2-1 covers \( p_2 \) and \( \pi_1 \). Notice that the norm map
\[N_{\pi_1}: \left[ \sum n_ip_1 \right] \mapsto \left[ \sum n_i\pi_1(p_i) \right]\]
on equivalence classes of divisors is well defined on \( J(\Sigma)/\pi^*_2J(C_1) \). Thus we may define \( \text{Tur}(\Sigma, \mathbb{P}^1) \) by the exact sequence
\[0 \to \text{Tur}(\Sigma, \mathbb{P}^1) \to P(\Sigma, C_1) \to P(C_2, \mathbb{P}) \to 0.\]

Alternatively \( \text{Tur}(\Sigma, \mathbb{P}^1) \subset P(\Sigma, C_2) \) consists of those line bundles satisfying \( \sigma^*L \otimes L \simeq \mathcal{O} \), where \( \sigma(\eta, \lambda) = (-\eta, \lambda) \) is the involution corresponding to the two-sheeted cover \( \pi_2: \Sigma \to C_2 \), if \( W \) is a subspace of \( \text{Jac}(\Sigma) \) we denote the set of elements of \( W \) satisfying \( \sigma^*\mathcal{L} \simeq \mathcal{L} \) by \( W^- \). We will employ similar notation for subspaces of \( \text{Jac}(C_2) \) which are anti-symmetric with respect to the hyperelliptic involution. We have in particular that
\[\text{Tur}(\Sigma, \mathbb{P}^1) = P(\Sigma, C_2)^- = P(\Sigma, C_2) \cap P(\Sigma, C_1).\]
We denote by \( \text{Tur}_R(\Sigma, \mathbb{P}^1) \) the real subspace given by the condition \( \rho^*\mathcal{E}_z \simeq \mathcal{E}_z \). Theorem 4.3 and Theorem 4.4 then clearly yield a characterisation of the symmetry of the eigenline bundles. Griffiths’s has given a criterion for when the eigenline flow resulting from a Lax equation is linear [Gri85], which
Burstall showed what was satisfied for harmonic maps into symmetric spaces \cite{Bur92}. Hence we have

**Corollary 4.5.** The eigenline bundles \( l \) define a linear flow in a constant translate of the intersection of the two Prym varieties defined by (4.7):

\[
\begin{align*}
\mathbb{C} & \rightarrow \text{Tur}_\mathbb{R}(\Sigma, \mathbb{P}^1) \\
z & \mapsto E_z \otimes \mathcal{O}\left(\frac{1}{2}R\right)
\end{align*}
\]

We now compute the dimension of this subvariety.

**Proposition 4.6.** The intersection of the two Prym varieties \( \text{Tur}(\Sigma, \mathbb{P}^1) \) in which the eigenline bundles flow has dimension \( 12k + 3 \).

**Proof.** The discriminant of the polynomial defining \( C_1 \to \mathbb{P}^1 \) is the same as that for the cover \( \Sigma \to \mathbb{P}^1 \) but in this case each zero of the discriminant yields only one ramification point and so by the Riemann-Hurwitz formula,

\[
2 - 2g_{C_1} = 3 \cdot 2 - 2(12k + 2) - 4
\]

and

\[
g_{C_1} = 12k + 2.
\]

Clearly,

\[
g_{C_2} = 6k.
\]

Thus from

\[
H^1(\Sigma, \mathcal{O})^- = \pi^*_C H^1(C_2, \mathcal{O})^- \oplus TP(\Sigma, C_2)^-
\]

we calculate that

\[
\dim (\text{Tur} \Sigma, \mathbb{P}^1) = P(\Sigma, C_2)^- = (g_{\Sigma} - g_{C_1}) - (g_{C_2} - g_{\mathbb{P}^1}) = 12k + 3.
\]

\[\square\]

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