Scaling Limits of Controlled Branching Processes

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Abstract. In this paper, a special sequence of controlled branching processes is considered. We provide a simple set of sufficient conditions for the weak convergence of such processes to a weak solution to a kind of continuous branching processes with dependent immigration.

Keywords: scaling limits; random controlled branching processes; continuous state branching processes with dependent immigration.

1 Introduction

Suppose that there is a family of random variables \( \{\xi_{n,i} : n, i = 1, 2, \cdots \} \) with values in \( \mathbb{N} := \{0, 1, 2, \cdots \} \), which are mutually independent. Given an \( \mathbb{N} \)-valued random variable \( Z(0) \) independent of \( \{\xi_{n,i}\} \), a Galton-Watson process (GW process) \( \{Z(n) : n \in \mathbb{N}\} \) can be inductively defined by

\[
Z(n + 1) = \sum_{i=1}^{Z(n)} \xi_{n,i}, \quad n = 0, 1, 2, \cdots.
\]

(1.1)

Here, we understand \( \sum_{i=1}^{0} = 0 \). In the classical GW process, \( \xi_{n,i} \) is viewed as the offspring produced by the \( i \)-th individual in \( n \)-th generation. It was proved that the scaling limits of GW processes can be a continuous-state branching process (CB process); see, e.g., [9, 10]. For more theories on scaling limits of generalized GW processes, one can refer to [2, 4, 13]. While each individual is influenced by environments, the mechanism of reproduction may vary in different generations. The mechanism here is understood as a competition or a interaction, and it’s initiated from [15], where Sevastyanov and Zubkov generalized the model of GW processes by considering a constant control on the growth of population size at each generation. Later, Yanev [17] consider the conditions that the controls are random and i.i.d. He introduced a model of controlled branching process with random control function (CBP), which can be formulated as follows. Let \( \{\phi^{(n)}(i) : i = 0, 1, 2, \cdots \}, n = 0, 1, 2, \cdots \) be
a mutually independent random function having the same distribution for each \( n \). A CBP 
\( \{Z(n) : n \in \mathbb{N}\} \) was constructed inductively as follows:

\[
Z(n + 1) = \sum_{i=1}^{\phi^{(n)}(Z(n))} \xi_{n,i}, \quad n = 0, 1, 2, \ldots .
\]  

(1.2)

From equality above, \( \phi^{(n)}(i) \) is viewed as a random control function. The probabilistic theory
on this model was developed by González et al. [7, 8] and so on.

Considering a sequence of such processes \( \{Z_k(n); \ n \geq 0\}_{k \geq 1} \) with offspring and random
control functions \( \{\xi^{(k)}_{n,i}, \phi^{(n)}(i)\} \), we concentrate on how the rescaled processes \( \{Y_k(t) :=
Z_k(\lfloor \gamma_k t \rfloor)/k; \ t \geq 0\}_{k \geq 1} \) converges on the Skorohod space as \( k \to \infty \), where \( \gamma_k \) is a sequence
of positive increasing constants tending to \( \infty \). Such a question was partially answered by
González and del Puerto [6]. They proved the weak convergence of a sequence of CBPs to a
diffusion process under some restrictions on the means and variances of \( \xi^{(k)}_{n,i} \) and \( \phi^{(n)}(i) \). In
fact, the diffusion that they obtained is a Feller branching diffusion with immigration. Their
results are not strange for that if we suppose \( \phi^{(n)}(i) = i + \psi^{(n)}(i) \) and \( \{\psi^{(n)}(i)\} \) is a sequence
of non-negative mutually independent random variables, (1.2) can be viewed as a GW process
with immigration. From a classical result in [9], under mild conditions, \( \{Y_k(t)\} \) converges to
a continuous state branching process with immigration. Inspired by this idea, we assume in
this paper that

\[
\phi^{(n)}(i) = i + \psi^{(n)}(i), \quad n \geq 0, i \geq 1,
\]  

(1.3)

where \( \psi^{(n)}(i) \) takes non-negative integer. Different from the GW process with immigration,
its immigration depends on the current state. Hence, there is a natural conjecture that
the limit process is a continuous state branching process with dependent immigration (CBDI
process).

Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) be a filtered probability space satisfying the usual hypotheses. Let
\( \{B_t\} \) be an \( (\mathcal{F}_t) \)-Brownian motion. Let \( N_0(ds, dz, du) \) and \( N_1(ds, dz, du) \) be \( \mathcal{F}_t \)-Poisson
random measures with intensities \( ds d\mu(dz) du \) on \( (0, \infty)^3 \), respectively, where
\( (z \wedge z^2) \mu(dz) \) is a finite measure and \( \sigma(dz) \) is a \( \sigma \)-finite measure. Denote the compensated
measures of \( \{N_0(ds, dz, du)\} \) by \( \{\bar{N}_0(ds, dz, du)\} \). Let \( Y_0 \) be a non-negative \( \mathcal{F}_0 \)-measurable
random variable satisfying \( \mathbb{E}Y_0 < \infty \). Let \( Y_0, \{B_t\}, \{N_0(ds, dz, du)\} \) and \( \{N_1(ds, dz, du)\} \) be
mutually independent. A CBDI process \( \{Y_t; t \geq 0\} \) is a non-negative solution to the
stochastic integral equation as follows:

\[
Y_t = Y_0 + \int_0^t \sqrt{2cY_s} dB_s + \int_0^t \int_0^{\infty} \int_0^{Y_s-} z\bar{N}_0(ds, dz, du)
+ \int_0^t (\beta(Y_s) - bY_s)ds + \int_0^t \int_0^{\infty} \int_0^{q(Y_s-z)} zN_1(ds, dz, du), \quad t \geq 0.
\]  

(1.4)

where \( c \geq 0, b \) are constants, and \( x \mapsto \beta(x) \) is a Borel function on \( \mathbb{R}_+ := [0, \infty) \), and
\( (x, z) \mapsto q(x, z) \) is a Borel function on \( [0, \infty) \times (0, \infty) \). Moreover, \( \beta(x), q(x, z) \) take non-
negative values. Here and in the sequel, we make the conventions
\[ \int_a^b = \int_{[a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)} \]
for any \( b \geq a \geq 0 \). Let \( C^2_c(\mathbb{R}_+) \) be the set of bounded continuous real functions on \( \mathbb{R}_+ \) with compact support. By Itô’s formula, the generator \( L \) of (1.4) is defined by
\[
Lf(x) = (-bx + \beta(x)) f'(x) + x \int_0^\infty [f(x + z) - f(x) - zf'(x)] m(dz) \\
+ \int_0^\infty [f(x + z) - f(x)] q(x, z) \pi(dz), \quad f \in C^2_c(\mathbb{R}_+). \tag{1.5}
\]
Throughout this paper, the following assumptions are adopted:

- there is a constant \( K \geq 0 \), such that
  \[ |\beta(x)| + \int_0^\infty q(x, z) z \pi(dz) \leq K(1 + x), \quad x \geq 0; \tag{1.6} \]
- there is a non-decreasing and concave function \( r : [0, \infty) \mapsto [0, \infty) \) such that \( \int_0^\infty r(z)^{-1} dz = \infty \) and
  \[ |\beta(x) - \beta(y)| + \int_0^\infty |q(x, z) - q(y, z)| z \pi(dz) \leq r(|x - y|), \quad x, y \geq 0. \]

Under these assumptions, there exists a pathwise unique positive strong solution to (1.4) by [5, Theorem 5.1].

The paper is organized as follows. In Section 2, we make some preparations and give some mild conditions which will be used in main theorems. The conditions here can be achieved through constructing probability generating functions. In section 3, based on the martingale problem approach, some estimates are given and lead to the tightness of rescaled processes. In section 4, the conjecture to be a CBDI process is finally verified by using Skorohod Representative Theorem.

## 2 Preliminaries

Let \( E_k := \{0, k^{-1}, 2k^{-1}, \cdots \} \). Let \( g_k \) be the probability generating function of \( \xi_{m,i}^{(k)} \). In view of (1.3), define the probability generating function of \( \psi^{(n)}_k(i) \) by \( h^{(i)}_k(s) \). For \( 0 \leq \lambda \leq k \), set
\[
R_k(\lambda) = k\gamma_k[g_k(1 - \lambda/k) - (1 - \lambda/k)], \tag{2.1}
\]
\[
F_k(\lambda, x) = \gamma_k[h^{(kx)}_k(1 - \lambda/k) - (1 - \lambda/k)]. \tag{2.2}
\]

We consider conditions as follows:
(A) The sequence \(\{R_k(\lambda)\}\) is uniformly Lipschitz on each bounded interval and converges to a continuous function \(R(\lambda)\) as \(k \to \infty\), where
\[
R(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) m(\mathrm{d}z). \tag{2.3}
\]
(B) For each \(a_1, a_2 > 0\), the sequence \(\{F_k(\lambda, x)\}\) converges to \(F(\lambda, x)\) uniformly on \([0, a_1] \times [0, a_2]\) as \(k \to \infty\), where
\[
F(\lambda, x) = -\beta(x)\lambda + \int_0^\infty (e^{-\lambda z} - 1) q(x, z) \pi(\mathrm{d}z). \tag{2.4}
\]
(C) There exists some positive constant \(K_1\) such that
\[
\left| \frac{\partial F_k}{\partial \lambda}(0, x) \right| = \frac{\gamma_k}{k} \frac{d}{ds} h_k^{\lfloor kx \rfloor}(1) \leq K_1(1 + x), \quad x \geq 0, k \geq 1.
\]
It follows from (C) that there exists some positive constant \(K_2\) such that
\[
|\gamma_k(1 - g_k'(1))| \leq K_2. \tag{2.5}
\]
In fact, condition (A) is from \([12, \text{Condition 2.4}]\), and condition (B) is a generalized form of \([12, \text{Condition 5.3}]\) in the setting of dependent immigration. Condition (C) is about the first moment, which is viewed as a generalization of (2.5).

**Theorem 2.1** For any function \((R, F)\) with representations (2.3) and (2.4), respectively, there are sequences \(\{\gamma_k\}\) and \(\{(R_k, F_k)\}\) as (2.1) and (2.2) satisfying (A-C).

**Proof.** The similar proof was discussed in \([12, \text{Prop 2.6}]\), which should be improved on the present proof. By checking the proof there, it’s sufficient to construct \(\{F_k\}\) satisfying (B-C) and \(\gamma_k\) satisfying
\[
\gamma_k \geq |b| + 2ck + \int_0^\infty u(1 - e^{-ku}) m(\mathrm{d}u). \tag{2.6}
\]
Following this step, let \(\tilde{F}(\lambda, x) = F(\lambda, x) + \beta(x)\lambda\), and \(\tilde{F}_k(\lambda, x) = \tilde{F}(\lambda, [kx]/k) 1_{[0,k]}(\lambda, x)\), which implies that \(\tilde{F}_k(\lambda, x)\) converges to \(\tilde{F}(\lambda, x)\) uniformly on the interval \([0, a_1] \times [0, a_2]\) for each \(a_1, a_2 > 0\). Next, we need only to adjust \(\tilde{\gamma}_k\) to satisfy that for \(k \geq 1\),
\[
\tilde{h}_k^{\lfloor kx \rfloor}(s) = s - \tilde{\gamma}_k^{-1} \tilde{F}(k(1 - s), x), \quad x \in [0, k] \cap E_k
\]
is a probability generating function. Above all, \(s \mapsto \tilde{h}_k^{\lfloor kx \rfloor}(s)\) is an analytic function. By elementary calculations, to ensure that
\[
\frac{d^n}{dz^n} \tilde{h}_k^{\lfloor kx \rfloor}(0) \geq 0, \quad n \geq 0,
\]
it’s sufficient to show that
\[ \tilde{\gamma}_k \geq \int_0^\infty kze^{-kz}q(x,z)m(dz), \quad x \leq k. \]

Consequently, \((1.6)\) implies that there is a sequence \(\{\tilde{\gamma}_k\}\) such that \(\tilde{h}_k^{(kx)}(s)\) is a probability generating function for \(x \in E_k \cap [0,k], k \geq 1\). On the other hand, since \((1.6)\) implies \(\beta(x) \leq \hat{K}\sqrt{k}\) for \(x \leq \sqrt{k}\) and some positive constant \(\hat{K}\) independent of \(k\), we define for \(x \leq \sqrt{k}\),
\[
\tilde{h}_k^{(kx)}(s) = 1 - \frac{2\beta([kx]/k)}{3\sqrt{k}\hat{K}} + \frac{\beta([kx]/k)}{3\sqrt{k}\hat{K}} s + \frac{\beta([kx]/k)}{3\sqrt{k}\hat{K}} s^2;
\]
and for \(x > \sqrt{k}\), let \(\tilde{h}_k^{(kx)}(s) = 1\). For arbitrary \(k \geq 1\), define constant \(\gamma_k = 3k^{3/2}\hat{K}\) and
\[
\hat{F}_k(\lambda, x) := \hat{\gamma}_k[\tilde{h}_k^{(kx)}(1 - \lambda/k) - 1]1_{[0,k] \times [0,\sqrt{k}]}(\lambda, x)
\]
\[= -[\beta([kx]/k)\lambda - \frac{1}{3k}\beta([kx]/k)\lambda^2]1_{[0,k] \times [0,\sqrt{k}]};\]
It’s easy to verify that \(\hat{F}_k^{(kx)}(s)\) tends to \(-\beta(x)\lambda\) uniformly on \([0,a_1] \times [0,a_2]\) as \(k \rightarrow \infty\). In the end, set\[
\gamma_k = \tilde{\gamma}_k + \hat{\gamma}_k, \quad h_k^{(kx)}(s) = \gamma_k^{-1}[\tilde{\gamma}_k \tilde{h}_k^{(kx)}(s) + \hat{\gamma}_k \hat{h}_k^{(kx)}(s)].\]

Consequently, for \(k \geq 1\),
\[
F_k(\lambda, x) = \hat{F}_k(\lambda, x) + \tilde{F}_k(\lambda, x),
\]
\[
= -[\beta([kx]/k)\lambda - \frac{1}{3k}\beta([kx]/k)\lambda^2]1_{[0,k] \times [0,\sqrt{k}]}
+ \int_0^\infty (e^{-\lambda z} - 1)q(x,z)\pi(dz)1_{[0,k]};
\]
satisfies (B). By elementary calculations and combining with \((1.6)\), we can verify (C). Finally, from \((2.7)\), \(\gamma_k\) also satisfies \((2.6)\), which ensures (A). Then we get the desired result. \(\square\)

For convenience, set \(e_\lambda(x) := e^{-\lambda x}\) for \(\lambda \geq 0\) and \(x \geq 0\). Let \(D_0\) be the linear hull of \(\{e_\lambda(x) : \lambda \geq 0\}\). Denote the set of bounded measurable real functions on \(\mathbb{R}_+\) by \(b(\mathbb{R}_+)\).

Next, we introduce an analytical conclusion for the sake of clarity.

**Lemma 2.2** For \(f \in C^2_c(\mathbb{R}^+)\), there exists a sequence of functions \(f_n\) in \(D_0\), such that
\[
f_n \rightarrow f, \quad f_n' \rightarrow f', \quad f_n'' \rightarrow f''
\]
uniformly on \(\mathbb{R}_+\), as \(n \rightarrow \infty\).
Proof. For \( f \in C_c^2(\mathbb{R}^+) \), define a function \( p \) on \([0, 1]\) by
\[
p(x) = \begin{cases} f(-\log(x)), & x > 0, \\ 0, & x = 0. \end{cases}
\]
By simple calculations, \( p \in C^2[0, 1] \). For a real function \( p \) on \([0, 1]\), its Bernstein polynomial is given by
\[
B_n(p, x) = \sum_{r=0}^{n} p(r/n) \binom{n}{r} x^r (1-x)^{n-r}.
\]
Let
\[
f_n(x) = \sum_{k=1}^{n} \binom{n}{k} f(-\log(k/n)) e^{-kx} (1-e^{-x})^{n-k},
\]
which implies that \( f_n \in D_0 \) and
\[
f_n(x) = B_n(p, e^{-x}). \tag{2.8}
\]
By taking derivatives on both sides of (2.8), we have
\[
f'_n(x) = -e^{-x} B'_n(p, e^{-x}).
\]
In fact, as a result of [14, Theorem 7.16], \( \lim_{n \to \infty} f_n = f \) uniformly on \( \mathbb{R}_+ \). Besides, \( B'_n(p, e^{-x}) \) converges to \( p'(e^{-x}) \) uniformly on \( \mathbb{R}_+ \) as \( n \to \infty \). Hence, \( f'_n(x) \to f'(x) \) uniformly on \( \mathbb{R}_+ \) as \( n \to \infty \). The same argument leads to the desired result. \( \square \)

3 Discrete martingale and tightness

In this section, we construct the discrete martingale and prove the tightness of the rescaled processes.

Let \( D([0, \infty), \mathbb{R}_+) \) be the space of càdlàg functions \( \omega : [0, \infty) \mapsto [0, \infty) \). For \( \lambda > 0 \), the distance \( \rho_\lambda \) is defined by
\[
\rho_\lambda(x, y) = |e^{-\lambda x} - e^{-\lambda y}|, \quad x, y \in [0, \infty)
\]
Denote the one-step transition matrix of \( Z_k(n)/k \) to be \( T_k \). Suppose that the process \( \{Z_k(n)/k\}_n \) is adapted to a filtration \((\mathcal{G}_n)_{n \geq 1}\) for each \( k \). Define \( A_k \) as the discrete generator of \( \{Y_k(t)\} \); for more details on discrete generators, see [3, pp 230-233]. Then
\[
A_k f = \gamma_k(T_k - I)f, \quad f \in b(\mathbb{R}_+),
\]
where \( I(x) = x \) for \( x \geq 0 \). Based on discrete generators, we can construct the discrete martingale problem.
Lemma 3.1 For $f \in b(\mathbb{R}_+)$, set

$$M^f_k(n) := \gamma_k f\left(\frac{Z_k(n)}{k}\right) - \gamma_k f\left(\frac{Z_k(0)}{k}\right) - \sum_{i=0}^{n-1} A_k f\left(\frac{Z_k(i)}{k}\right).$$

Then $\{M^f_k(n)\}_n$ is a $(\mathcal{G}_n)$-martingale.

Proof. It follows directly from Markov property. □

It is obvious by elementary calculations that

$$A_k e_{\lambda}(x) = \gamma_k E\left[g_k(e^{-\lambda/k})\phi_k(kx) - e^{-\lambda x}\right].$$

Recall from (1.5). Then for $\lambda \geq 0$,

$$Le_{\lambda}(x) = xe^{-\lambda x} R(\lambda) + e^{-\lambda x} F(\lambda, x), \quad x \geq 0. \quad (3.1)$$

By (1.6), $Le_{\lambda}(x) \to 0$ as $x \to \infty$. Hence $Ae_{\lambda}(x)$ is bounded on $\mathbb{R}_+$. Based on assumptions on probability generating functions, we have the following estimate.

Theorem 3.2 Suppose that (A-C) hold. Then for $\lambda \geq 0$, we have

$$\lim_{k \to \infty} \sup_{x \in E_k} |Le_{\lambda}(x) - A_k e_{\lambda}(x)| = 0.$$

Proof. Observe that

$$A_k e_{\lambda}(x) = \gamma_k E\left[g_k(e^{-\lambda/k})\phi_k(kx) - g_k^{kx}(e^{-\lambda/k})\right] + \gamma_k\left[g_k^{kx}(e^{-\lambda/k}) - e^{-\lambda x}\right].$$

Let

$$B_k(\lambda, x) = \gamma_k E\left[g_k(e^{-\lambda/k})\phi_k(kx) - g_k^{kx}(e^{-\lambda/k})\right]$$

$$= \gamma_k g_k^{kx}(e^{-\lambda/k})[g_k(kx)(e^{-\lambda/k})] - 1, \quad (3.2)$$

and

$$C_k(\lambda, x) = \gamma_k[g_k^{kx}(e^{-\lambda/k}) - e^{-\lambda x}].$$

Similar to [10], Theorem 2.1, there is a more precise approximation:

$$\lim_{k \to \infty} \sup_{x \in E_k} e^{\lambda_0 x}|C_k(\lambda, x) - xe^{-\lambda x} R(\lambda)| = 0, \quad \lambda_0 < \lambda. \quad (3.3)$$

Set

$$u_k(\lambda) := k[1 - g_k(e^{-\lambda/k})].$$
It’s easy to check that for every \( a \geq 0 \), \( u_k(\lambda) \to \lambda \) uniformly on \([0, a]\) as \( k \to \infty \). As a result, 
\[
(1 - u_k(\lambda)/k)^{kx} \to e^{-\lambda x}
\]
uniformly on \( \mathbb{R}_+ \) as \( k \to \infty \). Consequently, combining it with (3.2),
\[
B_k(\lambda, x) = (1 - u_k(\lambda)/k)^{kx} F_k(u_k(\lambda), x),
\]
which follows from (2.B) that \( B_k(\lambda, x) \) converges to \( e^{-\lambda x} F(\lambda, x) \) uniformly on \([0, M]\) as \( k \to \infty \), for \( M > 0 \). On the other hand, (C) yields
\[
|F_k(x)| \leq K_1 \lambda (1 + x), \quad \lambda, x \geq 0, k \geq 1.
\]
(3.4)
Therefore, there is a constant \( \tilde{C} > 0 \), such that
\[
\sup_{k \geq 1} |F(x) - F_k(u_k(\lambda), x)| \leq \tilde{C} e^{\lambda_0 x}.
\]
Define
\[
\epsilon_k(x) := e^{\lambda_0 x} [C_k(\lambda, x) - xe^{-\lambda x} R(\lambda)].
\]
By (3.3), \( \epsilon_k(x) \) converges to 0 uniformly, as \( k \to \infty \). Therefore,
\[
\limsup_{k \to \infty} \sup_{x \in E_k \cap [M, \infty]} k^{kx} (e^{-\lambda/k}) |F(x) - F_k(u_k(\lambda), x)|
\]
\[
\leq \limsup_{k \to \infty} \sup_{x \in E_k \cap [M, \infty]} \left[ \gamma_k^{-1} xe^{-\lambda x} R(\lambda) + \gamma_k^{-1} e^{-\lambda_0 x} \epsilon_k(x) + e^{-\lambda x} \tilde{C} e^{\lambda_0 x} \right]
\]
\[
\leq \tilde{C} e^{(\lambda_0 - \lambda) M},
\]
which yields
\[
\lim_{M \to \infty} \lim_{k \to \infty} \sup_{x \in E_k \cap [M, \infty]} |B_k(\lambda, x) - F(\lambda, x)e^{-\lambda x}| = 0.
\]
That gives the desired result along with the result that \( B_k(\lambda, x) \) converges to \( e^{-\lambda x} F(\lambda, x) \) uniformly on \([0, M]\) as \( k \to \infty \), for \( M > 0 \).

\begin{remark}
Since we can’t prove that \( D_0 \) is a core for \( L \), the result for weak convergence couldn’t be directly obtained from [3, Corollary 8.9] similar to [10, Theorem 2.1]. On the other hand, the result of González and del Puerto [6] is also from another result in [3, Corollary 8.9]. However, it’s not adapted in our scene, for our estimate is restricted on \( f = e_{\lambda} \). In the following pages, we will prove tightness and use Skorohod Representative Theorem to avoid the barrier.
\end{remark}

In the rest of this section, we aim at the proof of tightness of \( \{Y_k(t)\} \). For convenience, we introduce some notations before that. For a fixed constant \( T > 0 \), we consider a sequence of stopping times \( \tau_k \). Let \( \delta_k \) be a sequence of positive constants that tends to 0 as \( k \) tends to infinity. Suppose
\[
0 \leq \tau_k < \tau_k + \delta_k \leq T.
\]
Lemma 3.4 Suppose that (A-C) hold. Then

\[ \lim_{k \to \infty} E[\rho_k^2(Y_k(\tau_k + \delta_k), Y_k(\tau_k))] = 0. \]

Proof. By Lemma 3.1 for \( f \in b(\mathbb{R}_+) \),

\[
E[f(Y_k(t))] = E[f(k^{-1}Z_k(\lfloor \gamma_k t \rfloor))] \\
= E[f(Y_k(0)) + \sum_{i=0}^{\lfloor \gamma_k t \rfloor - 1} \gamma_k^{-1}A_k f(k^{-1}Z_k(i))] \\
= E[f(Y_k(0))] + E\left[ \int_0^{\lfloor \gamma_k t \rfloor/\gamma_k} A_k f(Y_k(s)) \, ds \right].  \tag{3.5}
\]

For \( \lambda > 0 \),

\[
E[|e^{-\lambda Y_k(\tau_k + \delta_k)} - e^{-\lambda Y_k(\tau_k)}|^2] \\
= E[e_{2\lambda}(Y_k(\tau_k)) - 2e_{\lambda}(Y_k(\tau_k) + Y_k(\tau_k + \delta_k)) + e_{2\lambda}(Y_k(\tau_k + \delta_k))] \\
\leq I_1 + I_2 + I_3,  \tag{3.6}
\]

where

\[
I_1 := |E[e^{-2\lambda Y_k(\tau_k + \delta_k)} - e^{-2\lambda Y_k(\tau_k)}]|;  \\
I_2 := |E\left[ 2e^{-\lambda Y_k(\tau_k)} \int_{\lfloor \gamma_k(\tau_k + \delta_k) \rfloor/\gamma_k}^{\lfloor \gamma_k(\tau_k) \rfloor/\gamma_k} A_k e^{-\lambda Y_k(u)} \, du \right]|;  \\
I_3 := |E\left[ 2\gamma_k^{-1}e^{-\lambda Y_k(\tau_k)} [M_k^{\lambda(\cdot)}(\lfloor \gamma_k(\tau_k + \delta_k) \rfloor)] - M_k^{\lambda(\cdot)}(\lfloor \gamma_k(\tau_k) \rfloor)] \right]|.
\]

Then a simple application of (3.5) yields

\[
I_1 = \left| E\left[ \int_{\lfloor \gamma_k(\tau_k + \delta_k) \rfloor/\gamma_k}^{\lfloor \gamma_k(\tau_k) \rfloor/\gamma_k} A_k e_{2\lambda}(Y_k(s)) \, ds \right] \right| \\
\leq E\left[ \int_{\lfloor \gamma_k(\tau_k + \delta_k) \rfloor/\gamma_k}^{\lfloor \gamma_k(\tau_k) \rfloor/\gamma_k} |A_k e_{2\lambda}(Y_k(s)) - A e_{2\lambda}(Y_k(s))| \, ds \right] \\
+ E\left[ \int_{\lfloor \gamma_k(\tau_k + \delta_k) \rfloor/\gamma_k}^{\lfloor \gamma_k(\tau_k) \rfloor/\gamma_k} |A e_{2\lambda}(Y_k(s))| \, ds \right] \\
\leq C_1 \delta_k,  \tag{3.7}
\]

where \( C_1 \) is a positive constant. The same argument implies that

\[
I_2 \leq 2E\left[ \int_{\lfloor \gamma_k(\tau_k) \rfloor/\gamma_k}^{\lfloor \gamma_k(\tau_k + \delta_k) \rfloor/\gamma_k} A_k e_{\lambda}(Y_k(s)) \, ds \right] 
\]
for a positive constant $C_2$. On the other hand,
\[ \Omega = \left\{ \lfloor \gamma_k (\tau_k + \delta_k) \rfloor = \lfloor \gamma_k \tau_k \rfloor + \lfloor \gamma_k \delta_k \rfloor \right\} \bigcup \left\{ \lfloor \gamma_k (\tau_k + \delta_k) \rfloor = \lfloor \gamma_k \tau_k \rfloor + \lfloor \gamma_k \delta_k \rfloor + 1 \right\} \]
\[ := \Omega_1 + \Omega_2. \]

Both $\Omega_1$ and $\Omega_2$ are $\mathcal{G}_{[\gamma_k \tau_k]}$-measurable. Observe that $Y_k(\tau_k)$ is also $\mathcal{G}_{[\gamma_k \tau_k]}$-measurable. Then, from the results of Lemma 3.1 and Doob’s Stopping Theorem, it follows that
\[ E\left\{ e^{-\lambda Y_k(\tau_k)} \left[ M^{\phi_k(\cdot)}_k (\lfloor \gamma_k (\tau_k + \delta_k) \rfloor) - M^{\phi_k(\cdot)}_k (\lfloor \gamma_k \tau_k \rfloor) \right]; \Omega_i \right\} = 0, \quad i = 1, 2. \]
Thus, $I_3 = 0$. Together with (3.6), (3.7) and (3.8), since $\lim_{k \to \infty} \delta_k = 0$, we obtain the desired result. \[ \square \]

In the following, we need the first moment condition for $Y_k(0)$.

\[ (D) \; \sup_{k \geq 1} E[Y_k(0)] < \infty. \]

Based on it, a precise estimate on $E[Y_k(\tau_k)]$ is obtained.

**Lemma 3.5** Suppose that (A), (C) and (D) hold. Then there exists some constant $K_3 \geq 0$, such that
\[ E[Y_k(\tau_k)] \text{ and } E[Y_k(\tau_k + \delta_k)] \leq K_3, \quad k \geq 1. \]

**Proof.** **Step 1.** We prove that there exists some constant $K_4 \geq 0$, such that
\[ E[Y_k(s)] \leq K_4, \quad s \leq T, k \geq 1. \] (3.9)

In fact, by (C) and (2.5),
\[ E[Z_k(n + 1)/k] = g'_k(1-)E[\phi_k(Z_k(n))/k] \]
\[ = g'_k(1-) E\left[ \frac{1}{k} \left( \frac{d}{ds} h_k^{(\cdot)}(n) \right)(1-\gamma_k) + Z_k(n) \right] \]
\[ \leq g'_k(1-) (E[Z_k(n)/k] (1 + K_1/\gamma_k) + K_1/\gamma_k) \]
\[ \leq (1 + K_2/\gamma_k) (E[Z_k(n)/k] (1 + K_1/\gamma_k) + K_1/\gamma_k). \]

By induction,
\[ E[Y_k(t)] \leq \{(1 + K_2/\gamma_k)(1 + K_1/\gamma_k)\}^{[\gamma_k t]} E[Y_k(0)] \]
\[ + K_1/\gamma_k \frac{(1 + K_2/\gamma_k)(1 + K_1/\gamma_k)^{[\gamma_k t]} - 1}{(1 + K_2/\gamma_k)(1 + K_1/\gamma_k) - 1} \quad (3.10) \]

A simple calculation yields (3.9).

**Step 2.** By (C) and (2.5),

\[
|A_k I(x)| = \left| \gamma_k E\left[ \frac{1}{K} \sum_{i=1}^{\phi_k (kx)} \xi_{n,i}^{(k)} - x \right] \right|
\]

\[
= |k^{-1} \gamma_k g_k'(1-\frac{d}{ds} h_k^{(kx)}(1-) + x \gamma_k (g_k'(1) - 1)|
\]

\[
\leq |k^{-1} \gamma_k g_k'(1-\frac{d}{ds} h_k^{(kx)}(1-) + |x \gamma_k (1 - g_k'(1-))|)
\]

\[
\leq (1 + \frac{K_2}{\gamma_k} K_1 (1 + x) + xK_2
\]

\[
\leq K_5 (1 + x), \quad k \geq 1, \ x \geq 0,
\]

for some positive constant \(K_5\), which follows from (3.10) that \(A_k I(Y_k(s))\) is integrable for \(s \leq t, k \geq 1\). The same argument as the proof in Lemma 3.1 for \(f = I\) leads \(M^I(n)\) to be a martingale. By Doob’s Stopping Theorem,

\[
E[Y_k(\tau_k)] = E[Y_k(0)] + E[\int_0^{\tau_k} A_k I d(Y_s)ds]
\]

\[
\leq \sup_{k \geq 1} E[Y_k(0)] + \int_0^T K_5 E[1 + Y_k(s)]ds
\]

\[
\leq \sup_{k \geq 1} E[Y_k(0)] + TK_5 (1 + K_4).
\]

The same argument leads to \(E[Y_k(\tau_k + \delta_k)] \leq K_3\), which completes the proof. \(\square\)

Now, we give the tightness using Aldous’ criterion.

**Theorem 3.6** Suppose that (A-D) hold. Then the process \(\{Y_k(t) : t \geq 0\}_{k \geq 1}\) is tight in \(D([0, \infty), \mathbb{R}_+)\).

**Proof.** Firstly, it follows from (3.9) that \(\{Y_k(t)\}\) is tight for a fix \(t \geq 0\). Next, for a fixed constant \(M > 0\), for \(\epsilon > 0\), \(|a - b| > \epsilon\) and \(0 \leq a, b \leq M\),

\[
|e^{-\lambda a} - e^{-\lambda b}| \geq \lambda e^{-\lambda M} \epsilon.
\]

Hence, by a simple calculation,

\[
P\{ |Y_k(\tau_k + \delta_k) - Y_k(\tau_k)| > \epsilon ; Y_k(\tau_k) \lor Y_k(\tau_k + \delta_k) \leq M \}
\]
\[ \left( \lambda \epsilon \right)^{-2} e^{2\lambda M} E \left[ \left| e^{-\lambda Y_k(\tau_k + \delta_k)} - e^{-\lambda Y_k(\tau_k)} \right|^2 \right]. \]

By Lemma 3.3,
\[ \lim_{k \to \infty} P \{ |Y_k(\tau_k + \delta_k) - Y_k(\tau_k)| > \epsilon; Y_k(\tau_k) \lor Y_k(\tau_k + \delta_k) \leq M \} = 0. \]

On the other hand, by Lemma 3.5,
\[ P(Y_k(\tau_k) \geq M) \leq K_3/M and P(Y_k(\tau_k + \delta_k) \geq M) \leq K_3/M. \]

As a result,
\[ P \{ |Y_k(\tau_k + \delta_k) - Y_k(\tau_k)| > \epsilon \}
\leq P \{ |Y_k(\tau_k + \delta_k) - Y_k(\tau_k)| > \epsilon; Y_k(\tau_k) \lor Y_k(\tau_k + \delta_k) \leq M \}
+ P(Y_k(\tau_k) \geq M) + P(Y_k(\tau_k + \delta_k) \geq M)
\leq P \{ |Y_k(\tau_k + \delta_k) - Y_k(\tau_k)| > \epsilon; Y_k(\tau_k) \lor Y_k(\tau_k + \delta_k) \leq M \} + 2K_3/M. \]

Let \( k \to \infty \) and \( M \to \infty \) following, then we obtain
\[ \lim_{k \to \infty} P \{ |Y_k(\tau_k + \delta_k) - Y_k(\tau_k)| > \epsilon \} = 0. \]

Finally, the tightness of \( \{Y_k(t) : t \geq 0\}_{k \geq 1} \) in \( D([0, \infty), \mathbb{R}_+) \) follows from Aldous’ criterion in [1, Theorem 1]. \( \square \)

### 4 Weak convergence

In this section, we build the relations between the weak solution of (1.4) and its corresponding martingale problem. By an application of Skorokhod Representative Theorem, the weak limit of rescaled processes is proved to be a weak solution of (1.4).

**Theorem 4.1** A positive càdlàg process \( \{Y_t : t \geq 0\} \) is a weak solution of (1.4) with initial value \( Y_0 \) if and only if for every \( f \in C^2_c(\mathbb{R}_+) \),
\[ f(Y_t) = f(x_0) + \int_0^t Lf(Y_s)ds + \text{local mart}, \quad y \geq 0. \quad (4.1) \]

**Proof.** The proof is a modification of that in [5, Proposition 4.2] which needs a stronger condition and that in [11, Theorem 5.1] which consider a different domain of generator. For rigorousness, we give a brief proof for the different part from their proofs. Suppose that (1.4) holds for every \( f \in C^2_c(\mathbb{R}_+) \). We introduce a non-decreasing sequence of functions...
Let \( f_n \in C^2_c(\mathbb{R}_+) \) such that \( f_n(x) = x \), for \( 0 \leq x \leq n \) and \( f'_n(x) \leq 1 \) for \( x \geq 0 \). Let \( \tau_n = \inf\{t > 0, Y_t \geq n\} \). It follows from (4.1) that

\[
M_n(t \land \tau_m) := f_n(Y_{t \land \tau_m}) - f_n(Y_0) - \int_0^t Lf_n(Y_{s \land \tau_m}) \, ds, \quad m \leq n \quad (4.2)
\]
is a martingale. Consequently, taking expectations above, we obtain

\[
E[f_n(Y_{t \land \tau_m})] = E[f_n(Y_0)] + \int_0^t E[-bY_{s \land \tau_m} + \beta(Y_{s \land \tau_m})] \, ds
\]
\[
+ \int_0^t \int_0^\infty E[f_n(Y_{s \land \tau_m} + z) - Y_{s \land \tau_m} - z] ds m(dz)
\]
\[
+ \int_0^t \int_0^\infty E[(f_n(Y_{s \land \tau_m} + z) - Y_{s \land \tau_m})] q(Y_{s \land \tau_m}, z)] ds \pi(dz)
\]
\[
\leq EY_0 + |b|mt + K(1 + m)t.
\]
Hence, by monotone convergence, \( Y_{t \land \tau_m} \) is integrable for \( m \geq 1 \). Letting \( n \to \infty \) in (4.2), by monotone convergence, we obtain that

\[
Y_{t \land \tau_m} - Y_0 - \int_0^t [bY_{s \land \tau_m} - \beta(Y_{s \land \tau_m})] ds - \int_0^t \int_0^\infty z q(Y_{s \land \tau_m}, z) \pi(dz) ds
\]
is a martingale. We omit the rest proof, for it’s the same as that in [5, Proposition 4.2].

Finally, combining all the results above, we can obtain the weak convergence.

**Theorem 4.2** Suppose that (A-D) hold and \( Y_k(0) \) converges in distribution to \( Y_0 \) as \( k \to \infty \). Then \( \{Y_k(t) : t \geq 0\}_k \) converges in distribution on \( D([0, \infty), \mathbb{R}_+) \) to \( \{Y_t : t \geq 0\} \), where \( \{Y_t : t \geq 0\} \) is a weak solution to (1.4) with initial value \( Y_0 \).

**Proof.** The definition of \( M_k(n) \) implies that

\[
f(Y_k(t)) = f(Y_k(0)) + \sum_{i=0}^{\lfloor \gamma_k t \rfloor - 1} A_k f\left( \frac{Z_k(i)}{\gamma_k} \right) + \gamma_k^{-1} M_k^f(\lfloor \gamma_k t \rfloor)
\]
\[
= f(Y_k(0)) + \int_0^{\lfloor \gamma_k t \rfloor / \gamma_k} A_k f(Y_k(s)) ds + \gamma_k^{-1} M_k^f(\lfloor \gamma_k t \rfloor), \quad f \in b(\mathbb{R}_+). \quad (4.3)
\]

Let \( P^{(n)} \) and \( P \) be the distributions of \( Y_n \) and \( Y \) in \( D \), respectively. By Theorem 3.6, \( \{Y_n\}_n \) is relatively compact. Then we can find a probability measure \( Q \) on \( D([0, \infty), \mathbb{R}_+) \) and a subsequence \( P^{(n_i)} \) such that \( Q = \lim_{i \to \infty} P^{(n_i)} \). By Skorokhod Representative Theorem, there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which are defined càdlàg processes \( \{X_t : t \geq 0\} \) and \( \{X_t^{(n_i)} : t \geq 0\} \) such that

(i) the distributions of \( X \) and \( X^{(n_i)} \) on \( D([0, \infty), \mathbb{R}_+) \) are \( P \) and \( P^{(n_i)} \), respectively;
By Theorem 3.2, the first integral on the right side of (4.5) converges to 0 as \( i \to \infty \). Then from [3, p.118], (ii) implies that for each \( t \)
\[
e_\lambda(X_t^{(ni)}) = e_\lambda(X_0^{(ni)}) + \int_0^{[\gamma_n t]/\gamma_n} A_n e_\lambda(X_s^{(ni)}) ds + \gamma_n^{-1} M_n^{\lambda}(\lfloor \gamma_n t \rfloor). \tag{4.4}
\]
Observe that
\[
\int_0^{[\gamma_n t]/\gamma_n} |A_n e_\lambda(X_s^{(ni)}) - L e_\lambda(X_s)| ds
\leq \int_0^{[\gamma_n t]/\gamma_n} |A_n e_\lambda(X_s^{(ni)}) - L e_\lambda(X_s^{(ni)})| ds
+ \int_0^{[\gamma_n t]/\gamma_n} |L e_\lambda(X_s^{(ni)}) - L e_\lambda(X_s)| ds. \tag{4.5}
\]
By Theorem 3.2, the first integral on the right side of (4.5) converges to 0 as \( i \to \infty \). Let
\[
D(x) := \{ t > 0 : \bar{P}(X(t-) = X(t)) = 1 \}.
\]
Then from [3, p.118], (ii) implies that for each \( t \in D(x) \), as \( i \to \infty \), \( X_t^{(ni)} \) converges to \( X_t \). Observe from [3, p.131] that the set \( \mathbb{R}_+/D(x) \) is at most countable. Therefore, the second integral on the right side of (4.5) also converges to 0 as \( i \to \infty \). On the other hand, since \( L e_\lambda(\cdot) \) is bounded, (4.5) also implies that the second term of the right side hand of (4.4) is uniformly bounded for all \( i \geq 1, \omega \in \tilde{\Omega} \). Consequently, taking limits in (4.4) and using bounded convergence theorem, we have
\[
M_t^f = f(X_t) - f(X_0) - \int_0^t L f(X_s) ds, \quad f = e_\lambda \tag{4.6}
\]
is a martingale bounded on each bounded time interval. Next, for \( f \in C_c^2(\mathbb{R}_+) \), let \( f_m \) be given in Lemma 2.2. In view of (1.5), \( L f_m(x) \) converges to \( L f(x) \) uniformly on each bounded interval. As a linear span of \( \{e_\lambda(x)\} \), by (4.6),
\[
f_m(X_t) = f_m(X_0) + \int_0^t L f_m(X_s) ds + M_t^{f_m}. \tag{4.7}
\]
Let \( \tilde{\tau}_M := \inf\{ t > 0, X_t \geq M \} \). Since \( X_t \) is a càdlàg process, \( \tilde{\tau}_M \to \infty \) a.s., as \( M \to \infty \). Replacing \( t \) with \( t \wedge \tilde{\tau}_M \), and taking limits on both sides of (4.7), we use the same argument to obtain that
\[
M_{t \wedge \tilde{\tau}_M}^f = f(X_{t \wedge \tilde{\tau}_M}) - f(X_0) - \int_0^t L f(X_{s \wedge \tilde{\tau}_M}) ds
\]
is a martingale bounded on each bounded time interval. Hence, \( X_t \) solves the martingale problem (1.1). By Theorem 4.1, \( X_t \) is a weak solution of (1.1). By [16, Theorem 137], the pathwise uniqueness of (1.4) implies the uniqueness of distributions. Therefore, \( Q = P \), and \( \lim_{i \to \infty} P^{(ni)} = P \), which completes the desired result. \( \square \)
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