Sedentary Quantum Walks

Chris Godsil*
Combinatorics & Optimization
University of Waterloo

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Abstract

Let $X$ be a graph with adjacency matrix $A$. The continuous quantum walk on $X$ is determined by the unitary matrices $U(t) = \exp(itA)$. If $X$ is the complete graph $K_n$ and $a \in V(X)$, then

$$1 - |U(t)_{a,a}| \leq \frac{2}{n}.$$ 

In a sense, this means that a quantum walk on a complete graph stay home with high probability. In this paper we consider quantum walks on cones over an $\ell$-regular graph on $n$ vertices. We prove that if $\ell^2/n \to \infty$ as $n$ increases, then a quantum walk that starts on the apex of the cone will remain on it with probability tending to 1 as $n$ increases. On the other hand, if $\ell \leq 2$ we prove that there is a time $t$ such that local uniform mixing occurs, i.e., all vertices are equally likely.

We investigate when a quantum walk on strongly regular graph has a high probability of “staying at home”, producing large families of examples with the stay-at-home property where the valency is small compared to the number of vertices.

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1 Introduction

Let $X$ be a graph with adjacency matrix $A$. A continuous quantum walk on $X$ is defined by the 1-parameter family of unitary matrices

$$U(t) = \exp(itA).$$

We view these matrices as operating on the Hilbert space $\mathbb{C}^{V(X)}$. If the state of the corresponding system at time 0 is given by a unit vector $z$, then the state at time $t$ is $U(t)z$.

We use $M \circ N$ to denote the Schur product of matrices $M$ and $N$ (of the same order). The questions concerning a quantum walk of physical interest can always be expressed as questions concerning the mixing matrices

$$M(t) := U(t) \circ U(t) = U(t) \circ U(-t)$$

The entries of $M(t)$ are non-negative and each row and each column sums to 1. Thus they specify probability densities on $V(X)$, and these describe the outcome of measurements on an associated quantum system.

If $X$ is the complete graph on $n$ vertices, then continuous quantum walks on $X$ have a very surprising property: for large $n$, for any time $t$, the mixing matrix $M(t)$ is close to $I$. Informally we say that $K_n$ has the “stay-at-home-property”.

In this paper we do two things. A graph $X$ on $n+1$ vertices is a cone over the graph $Y$ on $n$ vertices if $Y$ is an induced subgraph of $X$ and the vertex in $V(X) \setminus V(Y)$ has valency $n$, that is, it is adjacent to each vertex in $Y$. Thus $K_{1,n}$ is the cone over the empty graph on $n$ vertices and $K_n$ is the cone over $K_{n-1}$. We investigate when a cone over a regular graph has the stay at home property.

The second question is which strongly regular graphs have the stay-at-home property. A graph is strongly regular if it is regular (but neither complete nor empty) and there are integers $a$ and $c$ such that each pair of adjacent vertices has exactly $a$ common neighbours and each pair of distinct non-adjacent vertices has exactly $c$ common neighbours. The graphs $mK_n$ formed by $m$ vertex disjoint copies of $K_n$ is strongly regular when $m, n > 1$; more interesting examples are $C_5$ and the Petersen graph. Since strongly regular graph are the most regular graphs we have (complete and empty graphs aside), considering them in this context is very natural.
2 Complete Graphs

We establish the stay-at-home property for the complete graphs. The result is not new, but the machinery we develop to prove the result will be useful.

Recall that if $A$ is real symmetric matrix with distinct eigenvalues $\theta_1, \ldots, \theta_m$, then there are spectral projections $E_1, \ldots, E_m$ such that

$$A = \sum_r \theta_r E_r.$$ 

Hence $E_r$ represents orthogonal projection onto the $\theta_r$-eigenspace. Hence

$$E^T_r = E_r = E_r^2$$

and $E_r E_s = 0$ if $r \neq s$. Further $I = \sum_r E_r$ and, more generally, if $f$ is a function defined on the eigenvalues of $A$, then

$$f(A) = \sum_r f(\theta_r) E_r.$$ 

The eigenvalues of $K_n$ are $n - 1$ (with multiplicity 1) and $-1$ (with multiplicity $n - 1$); the corresponding spectral idempotents are

$$E_0 = \frac{1}{n}J, \quad E_1 = I - \frac{1}{n}J.$$ 

Therefore

$$U(t) = e^{(n-1)it} E_0 + e^{-it} E_1 = e^{-it} \left( I + \frac{e^{nit} - 1}{n} J \right)$$

and it follows immediately that the entries of $U(t) - e^{-it} I$ are bounded above in absolute value by $\frac{2}{n}$, while the diagonal entries are bounded below in absolute value by $1 - \frac{2}{n}$.

Note that if we can establish that there is a bound $c/n$ for all off-diagonal entries of $U(t)$ then, since $M(t)$ is doubly stochastic, we have

$$|M(t)_{a,a}| \geq 1 - (n - 1) \frac{c^2}{n^2} = 1 - \frac{c^2}{n} + \frac{1}{n^2},$$

whence it follows that $X$ is stay-at-home. We will make free use of this observation.
2.1 Lemma. If $X$ is a regular graph on $n$ vertices, then for any two vertices $a$ and $b$,
\[
\left| (U_X(t) - e^{-it}U_X(t))_{a,b} \right| \leq \frac{2}{n}.
\]

Proof. Assume $X$ has valency $k$. Since $\bar{A} = J - I - A$, we see that
\[
U_X(t) = \exp(-itA) \exp(itI) \exp(itJ) = e^{-it}U_X(-t) \exp(itJ)
\]
Using the spectral decomposition of $J$ we have
\[
\exp(itJ) = e^{nit \frac{1}{n} J} + I - \frac{1}{n} J = \frac{e^{nit} - 1}{n} J + I.
\]
and since
\[
U_X(-t)J = \sum_m \frac{(it)^m}{m!} (-A)^m J = e^{-ikt} J,
\]
we conclude that
\[
U_X(-t) \exp(itJ) = \frac{e^{nit} - 1}{n} e^{-ikt} J + U_X(-t).
\]
This result implies that if $X$ is regular, then it is stay-at-home if and only if $\bar{X}$ is. In particular it yields another proof that $K_n$ has the stay-at-home property.

3 Eigenvalues and Eigenvectors of Joins

Let $X$ be a $k$-regular graph on $m$ vertices and let $Y$ be an $\ell$-regular graph on $n$ vertices. In this section we describe the spectral decomposition of their join. Note that the join has an equitable partition $\pi$ with cells $(V(X), V(Y))$. Set $A = A(X)$ and $B = A(Y)$ and let $\hat{A}$ denote the adjacency matrix of $Z$. If $Z := X + Y$ then the adjacency matrix for $Z$ is
\[
\hat{A} = \begin{pmatrix} A & J \\ J^T & B \end{pmatrix}
\]
and the adjacency matrix of the quotient $Z/\pi$ is
\[
Q = \begin{pmatrix} k & n \\ m & \ell \end{pmatrix}.
\]
Its eigenvalues are the zeros of the quadratic

\[ t^2 - (k + \ell)t + (k\ell - mn), \]

thus they are

\[ \frac{1}{2}(k + \ell \pm \sqrt{(k - \ell)^2 + 4mn}). \]

We denote them by \( \mu_1 \) and \( \mu_2 \), with \( \mu_1 > \mu_2 \).

Since

\[ (Q - \mu_1 I)(Q - \mu_2 I) = 0 \]

we see that the columns of \( Q - \mu_2 I \) are eigenvectors for \( Q \) with eigenvalue \( \mu_1 \), and the columns of \( Q - \mu_1 I \) are eigenvectors for \( Q \) with eigenvalue \( \mu_2 \). Hence the eigenvectors of \( Z \) belonging to \( \mu_1 \) and \( \mu_2 \) respectively can be written in partitioned form:

\[ \begin{pmatrix} (k - \mu_2)1 \\ m1 \end{pmatrix}, \begin{pmatrix} (k - \mu_1)1 \\ m1 \end{pmatrix}. \]

The remaining eigenvectors of \( Z \) can be taken to be orthogonal to these two vectors, and therefore such eigenvectors must sum to zero on \( V(X) \) and \( V(Y) \). If \( x \) is an eigenvector for \( X \) orthogonal to \( 1 \) with eigenvalue \( \lambda \), then

\[ \begin{pmatrix} x \\ 0 \end{pmatrix} \]

is an eigenvector for \( Z \) with eigenvalue \( \lambda \). Similarly if \( y \) is an eigenvector for \( Y \) orthogonal to \( 1 \), then

\[ \begin{pmatrix} 0 \\ y \end{pmatrix} \]

is an eigenvector for \( Z \) (with the same eigenvalue as \( y \)).

4 Spectral Idempotents for Joins

We are going to construct a refinement of the spectral decomposition of the join \( Z \) of \( X \) and \( Y \). (If \( X \) and \( Y \) are connected and have no eigenvalue in common, this will be the actual spectral decomposition of \( Z \).) This decomposition will, in large part, be built from the spectral decompositions of \( X \) and \( Y \).
If $X$ is connected, we will use the spectral decomposition of $A$:

$$A = \sum_r \theta_r E_r$$

where $\theta_r = k$ and $E_r = \frac{1}{m}J$. If $X$ is not connected, then $k$ has multiplicity greater than one, and we may decompose the eigenspace belonging to $k$ as the sum of the span of the constant vectors and the span of the vectors that are constant on components and sum to zero. The idempotent belonging to $k$ can then be written as the sum of $\frac{1}{m}J$ and a second idempotent. Now we have a refinement of the spectral decomposition of $X$, with one extra term. We will still write this in the form above, with the understanding that $\theta_2 = \theta_1$. A similar fuss can be made if $Y$ is not connected; we write its decomposition as

$$B = \sum_s \nu_s F_s.$$ 

If $Az = \theta z$ and $1^T z = 0$, then

$$\hat{A} \begin{pmatrix} z \\ 0 \end{pmatrix} = \theta \begin{pmatrix} z \\ 0 \end{pmatrix}.$$ 

Similarly if $Bz = \theta z$ and $1^T z = 0$, then

$$\hat{A} \begin{pmatrix} 0 \\ z \end{pmatrix} = \theta \begin{pmatrix} 0 \\ z \end{pmatrix}.$$ 

We see that $n + m - 2$ of the eigenvalues of $X + Y$ are eigenvalues of $X$ and eigenvalues of $Y$.

Define

$$\hat{E}_r = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{F}_s = \begin{pmatrix} 0 & 0 \\ 0 & F_s \end{pmatrix}$$

and let $N_1$ and $N_2$ be the projections belonging to the eigenvalues $\mu_1$ and $\mu_2$ of $Z$. Then we have a decomposition for $Z$:

$$\hat{A} = \mu_1 N_1 + \mu_2 N_2 + \sum_{r>1} \theta_r \hat{E}_r + \sum_{s>1} \nu_s \hat{F}_s. \quad (4.1)$$

We determined $\mu_1$ and $\mu_2$ in the previous section.

Since $\sum_r E_r = I$ and $\sum_s F_s = I$ we have

$$\sum_{r>1} \hat{E}_r = \begin{pmatrix} I - \frac{1}{m}J & 0 \\ 0 & 0 \end{pmatrix}, \quad \sum_{s>1} \hat{F}_s = \begin{pmatrix} 0 & 0 \\ 0 & I - \frac{1}{n}J \end{pmatrix}.$$
and since the sum of the idempotents in (4.1) is \( I \), it follows that
\[
N_1 + N_2 = I - \begin{pmatrix}
I - \frac{1}{m}J & 0 \\
0 & I - \frac{1}{n}J
\end{pmatrix}.
\]

The idempotent \( N_1 \) represents projection onto the span of the eigenvector
\[
\begin{pmatrix}
(k - \mu_2)1_m \\
m1_n
\end{pmatrix}
\]
and consequently
\[
N_1 = c \begin{pmatrix}
(k - \mu_2)^2J_{m,m} & m(k - \mu_2)J_{m,n} \\
m(k - \mu_2)J_{n,m} & m^2J_{n,n}
\end{pmatrix}
\]
where \( c \) is determined by the constraint \( \text{tr}(N_1) = 1 \). This means that
\[
c^{-1} = m(k - \mu_2)^2 + m^2n = m((k - \mu_2)^2 + mn).
\]

If we set
\[
\Delta = (k - \ell)^2 + 4mn
\]
then, after some calculation, we find that
\[
(k - \mu_2)^2 + mn = \sqrt{\Delta}(k - \mu_2).
\]
Hence \( c^{-1} = m\sqrt{\Delta}(k - \mu_2) \). We can carry out similar calculations for \( N_2 \), with the result that
\[
N_1 = \frac{1}{m\sqrt{\Delta}(k - \mu_2)} \begin{pmatrix}
(k - \mu_2)^2J_{m,m} & m(k - \mu_2)J_{m,n} \\
m(k - \mu_2)J_{n,m} & m^2J_{n,n}
\end{pmatrix},
\]
\[
N_2 = \frac{1}{m\sqrt{\Delta}(\mu_1 - k)} \begin{pmatrix}
(k - \mu_1)^2J_{m,m} & m(k - \mu_1)J_{m,n} \\
m(k - \mu_1)J_{n,m} & m^2J_{n,n}
\end{pmatrix}.
\]

5 The Transition Matrix of a Join

Suppose \( Z \) is the join \( X + Y \) and \( a \) and \( b \) are two vertices in \( X \). We want to determine when we have perfect state transfer from \( a \) to \( b \) in \( Z \). We note that if we do have perfect state transfer from \( a \) to \( b \) at time \( t \), then \( U_Z(t)_{a,y} = 0 \) for all vertices \( u \) of \( Y \) and \( U_Z(t)_{a,a} = 0 \).
5.1 Lemma. Assume $Z$ is the join of graphs $X$ and $Y$. If $a, b \in V(X)$ and $y \in V(Y)$, then

$$U_{X+Y}(t)_{a,y} = \frac{1}{\Delta} (\exp(i\mu_1 t) - \exp(i\mu_2 t))$$

and

$$U_{X+Y}(t)_{a,b} - U_X(t)_{a,b} = \frac{1}{m} \left( \frac{k - \mu_2}{\sqrt{\Delta}} \exp(i\mu_1 t) - \frac{k - \mu_1}{\sqrt{\Delta}} \exp(i\mu_2 t) - \exp(ikt) \right).$$

Proof. Since $(\hat{E}_r)_{a,y} = (\hat{F}_r)_{a,y} = 0$ we have

$$U_{X+Y}(t)_{a,y} = \exp(i\mu_1 t)(N_1)_{a,y} + \exp(i\mu_2 t)(N_2)_{a,y} = \frac{1}{\sqrt{\Delta}} (\exp(i\mu_1 t) - \exp(i\mu_2 t)).$$

From our spectral decomposition,

$$U_{X+Y}(t)_{a,b} = \frac{k - \mu_2}{m\sqrt{\Delta}} \exp(i\mu_1 t) - \frac{k - \mu_1}{m\sqrt{\Delta}} \exp(i\mu_2 t) + \sum_{r>1} (E_r)_{a,b} \exp(i\theta_r t).$$

Since $X$ is regular,

$$U_X(t) = \frac{1}{m} \exp(ikt)J_n + \sum_{r>1} \exp(i\theta_r t)E_r,$$

from which our second expression follows. \qed

Therefore $U_{X+Y}(t)_{a,y} = 0$ if and only if $\exp(i(t(\mu_1 - \mu_2))) = 1$, that is, if and only if for some integer $c$,

$$t = \frac{2c\pi}{\mu_1 - \mu_2} = \frac{2c\pi}{\sqrt{\Delta}}.$$

Since $(k - \mu_2) - (k - \mu_1) = \sqrt{\Delta}$, we find that for these values of $t$ we have

$$U_{X+Y}(t)_{a,b} - U_X(t)_{a,b} = \frac{1}{m} (\exp(i\mu_1 t) - \exp(ikt)).$$
6 Irrational Periods and Phases

We divert from our main theme to consider some consequences our the results in the previous section. If \(a\) and \(b\) are distinct vertices in \(X\), we have perfect state transfer from \(a\) to \(b\) if there is a time \(t\) and a complex scalar \(\gamma\) such that

\[
U(t)e_a = \gamma e_b.
\] (6.1)

The scalar \(\gamma\) is called a phase factor. Since \(U(t)\) is unitary, \(\|\gamma e_b\| = 1\) and therefore \(|\gamma| = 1\). If (6.1) holds, then

\[
\gamma^{-1}e_a = U(-t)e_b
\]

and, taking complex conjugates of both sides, we obtain

\[
\gamma e_a = U(t)e_b
\]

This show that if we have perfect state transfer from \(a\) to \(b\) at time \(t\), we have perfect state transfer from \(b\) to \(a\) at the same time, and with the same phase factor. We also see that

\[
U(2t)e_a = \gamma^2 e_a.
\]

We say that \(X\) is periodic at the vertex \(a\) if there is a positive time \(t\) such that \(U(t)e_a\) is a scalar times \(e_a\). Any vertex involved in perfect state transfer is necessarily periodic.

In all known cases of perfect transfer, the phase factor \(\gamma\) is a root of unity. We use the results of the Section 5 to construct examples of periodic vertices where the period is irrational and the associated phases are irrational.

A graph \(X\) on \(n\) vertices is a cone over a graph \(Y\) if there is a vertex \(u\) of \(X\) with degree \(n - 1\) such that \(X \setminus u \cong Y\). (Equivalently \(X\) is isomorphic to \(K_1 + Y\).) We say \(u\) is the apex of the cone.

6.1 Lemma. Suppose \(Y\) is an \(\ell\)-regular graph on \(n\) vertices and let \(Z\) be the cone over \(Y\). The \(Z\) is periodic at the apex with period \(2\pi/\sqrt{\ell^2 + 4n}\).

Proof. We view \(Z\) as the join of \(X\) and \(Y\), with \(X = K_1\). If \(a\) is the apex vertex, then \(|U_Z(t)_{a,a}| = 1\) if and only if \(U_Z(t)_{a,y} = 0\) for all \(y\) in \(V(Y)\). This holds if and only if \(t/2\pi\) is an integer multiple of \(\Delta\), where \(\Delta = \sqrt{\ell^2 + 4n}\).
We have
\[ U(t)_{a,a} = \sum_r e^{i \theta r} (E_r)_{a,a} \]
where \((E_r)_{a,a} \geq 0\) and \(\sum_r (E_r)_{a,a} = 1\). Hence \(|U(t)_{a,a}| = 1\) if and only if \(e^{i \theta r} = e^{i \theta_1}\) for all \(r\), and therefore
\[ U(t)_{a,a} = e^{i \theta_1}. \]
So for a periodic cone the phase factor at the period is \(e^{it \mu_1}\), where
\[ \mu_1 = \frac{1}{2}(\ell + \sqrt{\ell^2 + 4n}). \]
Now it follows that for some integer \(c\)
\[ t \mu_1 = \frac{2c\pi}{\sqrt{\ell^2 + 4n}} \frac{1}{2}(\ell + \sqrt{\ell^2 + 4n}) = \frac{c\pi \ell}{\sqrt{\ell^2 + 4n}} + c\pi. \]
In all currently known cases where we have perfect state transfer, the phase factor is a root of unity. The calculations we have just completed show that if \(\ell^2 + 4n\) is not a perfect square, we have periodicity on the cone over \(Y\) with phase factor not a root of unity.

7 Walks on Cones

We say a complex vector or matrix is flat if all its entries have the same absolute value. We use \(e_a\) for \(a\) in \(V(X)\) to denote the standard basis vectors for \(\mathbb{C}^{V(X)}\). We have uniform mixing relative to the vertex \(a\) if there is a time \(t\) such that \(U(t)e_a\) is flat. If we have uniform mixing relative to each vertex at the same \(t\), we say that \(X\) admits uniform mixing. This holds if and only if the matrix \(U(t)\) is flat. Clearly uniform mixing is the antithesis of the stay-at-home property.

There are many examples of graphs that admit uniform mixing, including the hypercubes, but very few examples of graphs that admit uniform mixing at a vertex at time \(t\), but do not have uniform mixing at time \(t\). Carlson et al. [2] showed that there is uniform mixing on the star \(K_{1,n}\), starting from the vertex of degree \(n-1\). We add to the list of examples. We say a graph \(X\) on \(n+1\) vertices is a cone over the graph \(Y\) on \(n\) vertices if \(Y\) is an induced subgraph of \(X\) and the vertex in \(V(X) \setminus V(Y)\) has valency \(n\), that is, it is adjacent to each vertex in \(Y\). Thus \(K_{1,n}\) is the cone over the empty graph on \(n\) vertices.
7.1 Lemma. If \( Y \) is a regular graph with valency at most two and \( Z \) is the cone over \( Y \), then \( Z \) admits local uniform mixing starting from the apex.

Proof. Assume \( n = |V(Y)| \) and that \( Y \) is \( \ell \)-regular. Denote the cone over \( Y \) by \( Z \) and let \( a \) denote the apex. We set

\[
\Delta = \sqrt{\ell^2 + 4n}
\]

and recall from Section 3 that the eigenvalues in the eigenvalue support of \( a \) are

\[
\frac{1}{2}(\ell \pm \Delta).
\]

We denote these by \( \mu_1 \) and \( \mu_2 \), assuming that \( \mu_1 > \mu_2 \).

If \( y \in V(Y) \) then Lemma 5.1 yields that

\[
U_Z(t)_{a,y} = \frac{1}{\Delta} (e^{it\mu_1} - e^{it\mu_2}) = \frac{e^{it\mu_2}}{\Delta} (e^{it\Delta} - 1).
\]

We note that this is independent of the choice of \( y \) in \( Y \), and conclude that we have uniform mixing from \( a \) if and only if there is a time \( t \) such that

\[
\frac{1}{\Delta} |e^{it\Delta} - 1| = \frac{1}{\sqrt{n+1}};
\]

equivalently we need

\[
|e^{it\Delta} - 1| = \frac{\sqrt{\ell^2 + 4n}}{\sqrt{n+1}}.
\]

As \( \ell^2 + 4n = \ell^2 - 4 + 4n + 4 \), the ratio on the right lies in the interval \([0, 2]\) if and only if \( \ell \leq 2 \). Hence in these cases we can find a time \( t \) for which satisfies this equation, and then we have uniform mixing starting from \( a \). \( \square \)

By taking Cartesian powers, we get further examples of cones with uniform mixing starting from one vertex of the graph. It seems plausible that, in most cones, we do not get uniform mixing starting from a vertex in the base, but we do not have a proof in general. We leave it as an exercise to show that we do get uniform mixing in \( K_{1,n} \) starting from a vertex of degree one if and only if \( n = 3 \). This was first observed by Hanmeng Zhan (private communication), Cartesian powers of \( K_{1,3} \) are our only known examples of graphs that admit uniform mixing and are not regular.
7.2 Corollary. Assume $Z$ is the cone over an $\ell$-regular graph $Y$ on $n$ vertices, with apex $a$. Then

$$|U(t)_{a,a}| \geq \frac{\ell^2}{\ell^2 + 4n}.$$ 

Proof. If $y \in V(Y)$ then from the proof of the theorem,

$$U_Z(t)_{a,y} = \frac{e^{it\mu_2}}{\Delta} (e^{it\Delta} - 1).$$

and consequently

$$|U_Z(t)_{a,y}| \leq \frac{2}{\sqrt{\ell^2 + 4n}}.$$ 

Hence

$$M(t)_{a,y} \leq \frac{4}{\ell^2 + 4n}$$

and (since $M(t)$ has row sum 1),

$$M(t)_{a,a} \geq 1 - \frac{4n}{\ell^2 + 4n} = \frac{\ell^2}{\ell^2 + 4n}. \quad \square$$

Consider a sequence of $\ell$-regular graphs on $n$ vertices, where $\ell/n \to \infty$ as $n$ increases. Then, for large $n$, the corresponding cones have the stay at home property at the apex.

We note also that if $\ell/\sqrt{n} \to c$ as $n \to \infty$, then $|U(t)_{a,a}| \geq c/(c+4)$ while the off-diagonal entries $U(t)_{a,y}$ are bounded above by

$$\frac{2}{\sqrt{c+4}} n^{-1/2}.$$

So a relaxed form of the stay-at-home property holds in this case.

8 Mixing

The average mixing matrix $\hat{M}$ of a walk is defined by

$$\hat{M} = \lim_{T \to \infty} \frac{1}{T} \int_0^T M(t) \, dt.$$ 

From [5] we have that

$$\hat{M} = \sum_r E_r^2;$$
it follows that $\hat{M}$ is positive semidefinite and doubly stochastic. 

For $K_n$ we have

$$M(t) = \left(I + \frac{e^{nit} - 1}{n}J\right) \circ \left(I + \frac{e^{-nit} - 1}{n}J\right)$$

$$= I + 2\cos(nt) - \frac{1}{n}I + 2 - 2\cos(nt)J$$

$$= \left(1 - 2\frac{\cos(nt)}{n}\right)I + 2\frac{1 - \cos(nt)}{n^2}J$$

while

$$\hat{M} = \left(1 - \frac{2}{n}\right)I + \frac{2}{n^2}J.$$  

8.1 Lemma. We have $I \succ M(t) \succ 2\hat{M} - I$;

Proof. First

$$M(t) = U(t) \circ U(-t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r \circ E_s$$

and since $E_r \circ E_s = E_s \circ E_r$, it follows that

$$M(t) = \sum_r E_r^{\circ 2} + 2 \sum_{r<s} \cos((\theta_r - \theta_s)t). \quad (8.1)$$

Now

$$I = \left(\sum_r E_r\right)^{\circ 2} = \sum_{r,s} E_r \circ E_s$$

and consequently

$$I - M(t) = \sum_{r<s} (2 - 2\cos((\theta_r - \theta_s)t)) E_r \circ E_s \quad (8.2)$$

The matrices $E_r \circ E_s$ are positive semidefinite and the above shows that $I - M$ is a non-negative linear combination of positive semidefinite matrices, whence it is positive semidefinite.

For the second inequality, we note first that

$$I = I \circ I = \sum_r E_r^2 + 2 \sum_{r<s} E_r \circ E_s$$
and, adding this to (8.1) yields that

\[ M(t) + I = 2 \sum_r E_r^2 + 2 \sum_{r<s} (1 + \cos((\theta_r - \theta_s)t)) E_r \circ E_s. \]  

(8.3)

Appealing again to the fact that the Schur products \( E_r \circ E_s \) are positive semidefinite, we deduce that \( M(t) - 2\hat{M} + I \succeq 0 \). \( \square \)

8.2 Corollary. If \( \hat{M}_{a,a} = 1 - c \), then \( M(t)_{a,a} \geq 1 - 2c \) for all \( t \).

Proof. If \( M(t) - (2\hat{M} - I) \) is positive semidefinite, then \( M(t)_{a,a} \geq 2\hat{M}_{a,a} - 1 \). \( \square \)

If \( M(t) > 1 - 2c \) for all \( t \), then clearly the diagonal entries of \( \hat{M} \) are at least \( 1 - 2c \). So to prove that a graph is stay-at-home, it suffices to show that \( \hat{M} \) is close to \( I \).

We consider what happens when one of the bounds in Lemma 8.1 is tight. A graph \( X \) is periodic if \( U_X(t) \) is a periodic function of \( t \). Clearly \( X \) is periodic if its eigenvalues are integers; in particular \( K_n \) is periodic.

8.3 Lemma. If \( X \) is connected and \( I = M(t) \), then \( X \) is periodic.

Proof. If \( M(t) = I \), then \( U(t) \) is diagonal. Since \( U(t) \) commutes with \( A \) and \( X \) is connected, \( U(t) = \gamma I \) for some \( \gamma \). Now \( \det(U(t)) = 1 \) because \( \text{tr}(A) = 0 \), and therefore if \( n = |V(X)| \), we have have \( \gamma^n = 1 \). Hence

\[ I = U(t)^n = U(nt) \]

and therefore \( X \) is periodic. \( \square \)

8.4 Lemma. If \( X \) is connected and \( M(t) = 2\hat{M} - I, \) then \( X \) is a complete graph.

Proof. Assume \( M(t) = 2\hat{M} - I \). By (8.3) we have \( \cos(t(\theta_r - \theta_s)) = -1 \), which implies that \( \cos(2t(\theta_r - \theta_s)) = 1 \) and now (8.2) yields that \( M(2t) = I \).

If \( \cos(t(\theta_r - \theta_s)) = -1 \), then there is an odd integer \( m_{r,s} \) such that

\[ t(\theta_r - \theta_s) = m_{r,s}\pi \]

and consequently

\[ \frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \]

is the ratio of two odd integers. Since

\[ \theta_r - \theta_s = (\theta_r - \theta_q) - (\theta_q - \theta_s), \]

we see \( X \) has at most two distinct eigenvalues; therefore it has diameter one and it is a complete graph. \( \square \)
From our expressions at the start of this session, if $X = K_n$ and $\cos(nt) = -1$ then $M(t) = 2\hat{M} - I$.

9 Absolute Value Bounds

We derive an upper bound on the absolute values of the entries of $U(t)$. If $a \in V(X)$, the eigenvalue support of $a$ is the set of eigenvalues $\theta_r$ such that $E_r e_a \neq 0$. Since the idempotents $E_r$ are positive semidefinite, we have $E_r e_a = 0$ if and only if

$$(E_r)_{a,a} = e_a^T E_r e_a = 0.$$ 

We say that a set $S$ of eigenvalues satisfies the ratio condition if whenever

$$\theta_k, \ \theta_t, \ \theta_r, \ \theta_s \in S$$

and $\theta_k \neq \theta_t$, we have

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_t} \in \mathbb{Q}.$$ 

9.1 Lemma. If $a, b \in V(X)$, then

$$|U(t)_{a,b}| \leq \sum_r (E_r)_{a,b}.$$ 

If equality holds at time $t$, the intersection of eigenvalue supports of $a$ and $b$ satisfies the ratio condition and $U(t)_{a,b}$ is a periodic function of $t$.

Proof. Let $S$ denote the intersection of the eigenvalue supports of $a$ and $b$.

We have

$$U(t)_{a,b} = \sum_r e^{it\theta_r} (E_r)_{a,b}$$

and applying the triangle inequality, we find that

$$|U(t)_{a,b}| \leq |(E_r)_{a,b}|.$$ 

This is the stated bound.

Equality holds in our bound if and only if the complex numbers

$$e^{it\theta_r} \text{sign}((E_r)_{a,b})$$
are equal for all $r$ such that $(E_r)_{a,b} \neq 0$. In this case, $e^{2it(\theta_r - \theta_s)} = 1$ for all eigenvalues $\theta_r$ and $\theta_s$ in $S$. Hence we have

$$U(2mt)_{a,b} = \sum_{r,s} (E_r)_{a,b} = 0$$

for all $m$. We also see that there are integers $m_{r,s}$ such that

$$t(\theta_r - \theta_s) = m_{r,s}\pi$$

and therefore, if $\theta_k \neq \theta_\ell$ and $\theta_k, \theta_\ell, \theta_r, \theta_s$ lie in $S$, then

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_\ell} \in \mathbb{Q}.$$

From the proof of [4, Theorem 6.1] we have that either all elements of $S$ are integers, or they are all of the form $\frac{1}{2}(a + b_i\sqrt{\Delta})$, where $\Delta$ is a square-free integer and $a$ and $b_i$ are integers.

10 Strongly Regular Graphs: Spectral Decomposition

A graph $X$ is strongly regular if there are parameters $k$, $a$ and $c$ such that

(a) $X$ is $k$-regular.

(b) Each pair of adjacent vertices has exactly $a$ common neighbours.

(c) Each pair of distinct non-adjacent vertices have exactly $c$ common neighbours.

(d) $0 < k < n - 1$.

If $n = |V(X)|$ with $k$, $a$ and $c$ as given, we refer to the 4-tuple $(n, k; a, c)$ as the parameters of $X$. We use $\ell$ to denote $n - 1 - k$, the valency of the complement of $X$.

In this section we review some of the properties of strongly regular graphs, including the spectral decomposition of the adjacency matrix.

If $m, n > 1$, then the disjoint union $mK_n$ of $m$ copies of $K_n$ is strongly regular. The Petersen graph provides a more interesting example. The complement of a strongly regular graph is strongly regular. A strongly regular
graph is primitive if it is not isomorphic to \(mK_n\) or its complement. For more information of strongly regular graphs see [1, 3, 6].

We denote the adjacency matrix of the complement of \(X\) by \(\overline{A}\). A graph \(X\) is strongly regular with parameters \((n, k; a, c)\) if and only if

\[
A^2 = kI + aA + c\overline{A}
\]
or, equivalently

\[
A^2 - (a - c)A - (k - c)I = cJ.
\]

It follows that a primitive strongly regular graph has exactly three eigenvalue: its valency \(k\) and the two roots of the quadratic

\[
t^2 - (a - c)t - (k - c).
\]

These are

\[
\frac{1}{2} \left( a - c \pm \sqrt{(a - c)^2 + 4k - 4c} \right)
\]
where the larger root is \(\theta\) and the smaller \(\tau\). We denote the multiplicity of \(\theta\) and \(\tau\) respectively by \(m_\theta\) and \(m_\tau\). These multiplicities can be expressed in terms of \(n, k, \theta\) and \(\tau\):

\[
m_\theta = \frac{(n - 1)(-\tau) - k}{\theta - \tau}, \quad m_\tau = \frac{(n - 1)\theta + k}{\theta - \tau}.
\]

The adjacency matrix \(A\) has spectral decomposition

\[
A = kE_0 + \theta E_1 + \tau E_2
\]
where \(E_0 = \frac{1}{n}I\),

\[
E_1 = \frac{m_\theta}{n} \left( I + \frac{\theta}{k}A - \frac{\theta + 1}{\ell} \overline{A} \right)
\]
and

\[
E_2 = \frac{m_\tau}{n} \left( I + \frac{\tau}{k}A - \frac{\tau + 1}{\ell} \overline{A} \right).
\]

11 Orthogonal Arrays, Steiner Designs

With the exception of the so-called conference graphs, the eigenvalues of strongly regular graphs are integers. Neumaier [5] proves a deep and remarkable result: for each positive integer \(k\), there are only finitely many strongly
regular graphs with least eigenvalue $-k$ that do not arise from either block graphs of Steiner 2-designs or orthogonal arrays.

An orthogonal array $OA(k, n)$ is an $n^2 \times k$ array with entries from $N = \{1, \ldots, n\}$ such that each ordered pair from $N \times N$ occurs exactly once in each pair of columns. We define the graph of the array to be the graph with the rows of the array as its vertices, with two rows adjacent if they agree on exactly one coordinate. The graph of an $OA(k, n)$ is strongly regular with parameters

$$n^2, \ k(n - 1), \ n - 2 + (k - 1)(k - 2), \ k(k - 1).$$

The eigenvalues are

$$k(n - 1), \ n - k, \ -k$$

with respective multiplicities

$$1, \ k(n - 1), \ (n - 1)(n + 1 - k).$$

We note that $k \leq n + 1$ and if $k = n + 1$, the graph is complete. We assume implicitly that $2 \leq k \leq n$. For more information, see e.g [3, Section 10.4].

11.1 Lemma. If $X$ is the graph of an orthogonal array $OA(k, n)$ and $k = o(n)$, then $X$ has the stay-at-home property.

Proof. The diagonal entries of $\hat{M}$ are equal to

$$\frac{1}{n^2} + \left(\frac{k(n - 1)}{n^2}\right)^2 + \left(\frac{(n - 1)(n + 1 - k)}{n^2}\right)^2.$$

If $k = o(n)$, then $\hat{M} \circ I$ converges to $I$ as $n$ increases, and Corollary 8.2 now yields the conclusion. \qed

The parameters of the block graph of a Steiner 2-$(v, k, 1)$ design are

$$\frac{v(v - 1)}{k(k - 1)}, \ \frac{v - k}{k - 1}, \ \frac{v - 1}{k - 1} - 2 + (k - 1)^2, \ k^2;$$

its eigenvalues are

$$\frac{k}{k - 1}, \ \frac{v - k}{k - 1}, \ \frac{k - k}{k - 1}, \ -k$$

with respective multiplicities

$$1, \ v - 1, \ \frac{v(v - 1)}{k(k - 1)} - v - 1.$$

Arguing as above, we conclude that $k = o(v)$, then the block graph is stay-at-home.
12 Other Strongly Regular Graphs

We show that, for a large strongly regular graph, the off-diagonal entries of $M(t)$ are small.

12.1 Theorem. If $X$ is a strongly regular graph on $n$ vertices, the off-diagonal entries of $U(t)$ are bounded in absolute value by $d/n^{1/4}$ for some $d$.

Proof. Assume $X$ is strongly regular. From Section 6.1.8 we have

$$m_\theta \theta = \frac{(n-1)(-\tau)\theta - k\theta}{nk(\theta - \tau)} = \frac{(n-1)(k-c)}{nk(\theta - \tau)} - \frac{\theta}{n(\theta - \tau)}.$$

Since $\tau < 0$,

$$\frac{\theta}{n(\theta - \tau)} < \frac{1}{n},$$

while

$$\frac{(n-1)(k-c)}{nk(\theta - \tau)} = \frac{n-1}{n} \frac{k-c}{k} \frac{1}{\theta - \tau} < \frac{1}{\theta - \tau}.$$

From [1, Section 6.1.8] or [6, Lemma 10.3.1] we have that

$$(\theta - \tau)^2 = n \frac{k\ell}{m_\theta m_\tau}.$$  

Since $X$ is $k$-regular with diameter two, $n \leq k^2 + 1$. Therefore

$$(\theta - \tau)^2 \geq n \frac{(n - \sqrt{n})\sqrt{n}}{n^2/4}$$

and we conclude that, for some constant $d$,

$$\frac{1}{\theta - \tau} < \frac{d}{n^{1/4}}.$$  

We now consider the diagonal elements of $M(t)$. From our expressions for the spectral idempotents at the end of Section 6.1 it follows that the diagonal entries of $\hat{M}$ are equal to

$$1/n^2 (1 + m_\theta^2 + m_\tau^2).$$
Since $m_\theta + m_\tau = n - 1$, this is bounded below by
\[
\frac{1}{n^2}(1 + \frac{1}{2}(n - 1)^2)
\]
which is less than $1/2$ (when $n \geq 3$). In this case we get no useful information about the diagonal entries of $M(t)$. We note that this lower bound is tight for conference graphs, where $m_\theta = m_\tau = (n - 1)/2$.

If $X$ is the graph of an $OA(k, n)$ and $k = \gamma n$ with $0 < \gamma < 1$, then from the expression at the start of the proof of Lemma 11.1 we see that the diagonal entries of $\hat{M}$ tend to
\[
\gamma^2 + (1 - \gamma)^2 = 1 - 2\gamma + 2\gamma^2 = \frac{1}{2} + \frac{1}{2}(1 - 2\gamma)^2
\]
If $\gamma < 1/2$, then the diagonal entries of $M(t)$ are bounded away from zero. Graphs with the above parameters exist if $n$ is a prime power.

### 13 Questions?

We do not know whether, for conference graphs, the diagonal entries of $M(t)$ can be bounded away from zero.

It might also be interesting to extend our results to walk-regular graphs (which can be characterized by the condition that the diagonals of the spectral idempotents are constant).

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