A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture

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Abstract

Doing surgery on the 5-torus, we construct a 5-dimensional closed spin-manifold \( M \) with \( \pi_1(M) \cong \mathbb{Z}^4 \times \mathbb{Z}/3 \), so that the index invariant in the \( KO \)-theory of the reduced \( C^* \)-algebra of \( \pi_1(M) \) is zero. Then we use the theory of minimal surfaces of Schoen/Yau to show that this manifolds cannot carry a metric of positive scalar curvature. The existence of such a metric is predicted by the (unstable) Gromov-Lawson-Rosenberg conjecture.

1 Obstructions to positive scalar curvature

1.1 Definition. A manifold \( M \) which admits a metric of positive scalar curvature is called a pscm-manifold.

We start with a discussion of the index obstruction for spin manifolds to be pscm, constructed by Lichnerowicz [3], Hitchin [4] and in the following refined version due to Rosenberg [9].

1.2 Theorem. Let \( M^m \) be a closed spin-manifold, \( \pi := \pi_1(M) \). One can construct a homomorphism, called index, from the singular spin bordism \( \Omega^\text{spin}_{*}(B\pi) \) to the (real) \( KO \)-theory of the reduced real \( C^* \)-algebra of \( \pi \):

\[
\text{ind} : \Omega^\text{spin}_{*}(B\pi) \to KO_{*}(C_{\text{red}}^*\pi)
\]

Let \( u : M \to B\pi \) be the classifying map for the universal covering of \( M \). If \( M \) is pscm, then

\[
\text{ind}(u : M \to B\pi) = 0 \in KO_m(C_{\text{red}}^*)
\]
Gromov/Lawson \[3\] and Rosenberg \[8\] conjectured that the vanishing of the index should also be sufficient for existence of a metric with positive scalar curvature on \(M\) if \(m \geq 5\). This was proven by Stefan Stolz \[14\] for \(\pi = 1\), and subsequently by him and other authors also for a few other groups \[8, 5, 1, 10\].

In dimension \(\geq 5\) there is only one additional obstruction for pscm, the minimal surface method of Schoen and Yau, which we will recall now. (In dimension 4, the Seiberg-Witten theory yields additional obstructions). The first theorem is the differential geometrical backbone for the application of minimal surfaces to the pscm problem:

### 1.3 Theorem
Let \((M^m, g)\) be a manifold with positive scalar curvature, \(\dim M = m \geq 3\). If \(V\) is a smooth \(m - 1\)-dimensional submanifold of \(M\) with trivial normal bundle, and if \(V\) is a local minimum of the volume functional, then \(V\) admits a metric of positive scalar curvature.

**Proof.** Schoen/Yau: \[11, 5.1\] for \(m = 3\), \[12, proof of Theorem1\] for \(m > 3\).

The next statement from geometric measure theory implies applicability of the previous theorem if \(\dim(M) \leq 7\).

### 1.4 Theorem
\[4\, \text{chapter 8}\] and references therein, in particular \[2, 5.4.18\]
Suppose \(M^m\) is a smooth orientable closed manifold, \(\dim M = m \leq 7\). Suppose \(0 \neq x \in H_{m-1}(M, \mathbb{Z})\). Then a smooth orientable closed \(m - 1\)-dimensional submanifold \(V\) of \(M\) exists which represents \(x\) and which has minimal volume under all currents which represent \(x\). In particular, \(V\) is a local minimum of the volume functional with orientable (hence trivial) normal bundle.

This implies the following statement about homology and cohomology which was observed by Stephan Stolz. Let \(X\) be any space.

### 1.5 Definition
For \(m \geq 2\) we define
\[
H^+_m(X, \mathbb{Z}) := \{f_*[M] \in H_m(X, \mathbb{Z}); \, f : M^m \to X \text{ and } M \text{ is pscm}\}
\]

### 1.6 Corollary
Let \(X\) be any space, \(\alpha \in H^1(X, \mathbb{Z})\). Cap-product with \(\alpha\) induces a map
\[
\alpha \cap : H_m(X, \mathbb{Z}) \to H_{m-1}(X, \mathbb{Z}).
\]
If \(3 \leq m \leq 7\), then \(\alpha \cap\) maps \(H^+_m(X, \mathbb{Z})\) to \(H^+_{m-1}(X, \mathbb{Z})\).

**Proof.** If \(f : M^m \to X\) represents \(x \in H^+_m(X, \mathbb{Z})\) and \(M\) is a pscm-manifold, then by theorems \[1.3\] and \[1.4\] the class \(f^*\alpha \cap [M]\) is represented by \(N^{m-1} \hookrightarrow M\) and \(N\) is also pscm. In particular \(\alpha \cap x = f_*(f^*\alpha \cap [M])\) is represented by \(f \circ j : N \to X\), i.e. \(\alpha \cap x \in H^+_{m-1}(X, \mathbb{Z})\).
2 The Counterexample

To produce a counterexample to the unstable Gromov-Lawson-Rosenberg conjecture, we use the only other known obstruction for pscm, namely the minimal surface method explained above.

The fundamental group will be $\pi := \mathbb{Z}^4 \times \mathbb{Z}/3$. We start with the computation of the $KO$-theory of the $C^\ast\pi$-algebra. Note that the reduced $C^\ast$-algebra of the product of two groups is the (minimal) tensor product of the individual $C^\ast$-algebras [13, p. 14–15]. By [13, p. 14 and 1.5.4]

$$KO_n(C^\ast_{\text{red}}(\mathbb{Z}^4 \times \mathbb{Z}/3)) \cong \bigoplus_{i \in I} KO_{n-n_i}(C^\ast_{\text{red}}(\mathbb{Z}/3)); \quad |I| < \infty$$

For finite groups, it is well known that their $KO$-theory is a direct sum of copies of the $KO$-theories of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. In particular, it is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. Therefore, the same is true for $\pi$:

**2.1 Proposition.** $KO_\ast(C^\ast_{\text{red}}\pi)$ is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. In particular, its torsion is only 2-torsion.

We will now construct a spin manifold $M^5$ with $\pi_1(M) = \pi$, so that the class $[u : M \to B\pi] \in \Omega_5^{\text{spin}}$ is 3-torsion. Then, automatically

$$\text{ind}(u : M \to B\pi) = 0 \in KO(C^\ast_{\text{red}}\pi)$$

**2.2 Example.** Let $p : S^1 \to B\mathbb{Z}/3$ be a map so that $\pi_1(p)$ is surjective and equip $S^1$ with the spin structure induced from $D^2$. This is 3-torsion since $\Omega_1^{\text{spin}}(B\mathbb{Z}/3) \cong H_1(B\mathbb{Z}/3, \mathbb{Z}) \cong \mathbb{Z}/3$ (use the Atiyah-Hirzebruch spectral sequence). Consider the singular manifold

$$f = id \times p : S^1 \times \cdots \times S^1 \times S^1 \to S^1 \times \cdots \times S^1 \times B\mathbb{Z}/3 = B\pi$$

This is then 3-torsion in $\Omega_5^{\text{spin}}(B\pi)$. Doing surgery we can construct a bordism $F : W \to B\pi$ in $\Omega_5^{\text{spin}}(B\pi)$ from $f$ to some $u : M \to B\pi$ where $u$ is an isomorphism on $\pi_1$.

Now, $M$ is a manifold with trivial index, and we have to show that it is not pscm. Assume that the converse is true.

We study the homology and cohomology of $\pi$ first. By the Künneth theorem

$$H_1(B\pi, \mathbb{Z}) = x_1 \mathbb{Z} \oplus \cdots \oplus x_4 \mathbb{Z} \oplus y\mathbb{Z}/3$$

$$H^1(B\pi, \mathbb{Z}) = a_1 \mathbb{Z} \oplus \cdots \oplus a_4 \mathbb{Z}$$

$$0 \neq w = x_1 \times \cdots \times x_4 \times y \in H_5(B\pi, \mathbb{Z})$$

$$0 \neq z = x_4 \times y = a_1 \cap (a_2 \cap (a_3 \cap w)) \in H_2(B\pi, \mathbb{Z})$$

We use the map

$$B_* : \Omega_*^{\text{spin}}(X) \to H_*(X, \mathbb{Z}) : [f : M \to X] \mapsto f_*[M]$$
which is an edge homomorphism in the Atiyah-Hirzebruch spectral sequence. Of course, \( w = f_\ast([T^5]) = u_\ast[M] \) is the image under this transformation of the singular manifold we consider.

If \( M \) would admit a pscm, then

\[
w \in H^+_5(B\pi)
\]

Iterated application of theorem 1.6 implies that

\[
0 \neq z \in H^+_2(B\pi)
\]

But there is only one two dimensional oriented manifold with positive curvature, namely \( S^2 \). Since \( \pi_2(B\pi) = 0 \) any map \( g : S^2 \to B\pi \) is null homotopic. In particular \( g_\ast[S^2] = 0 \in H_2(B\pi, \mathbb{Z}) \), and therefore \( H^+_2(\pi) = 0 \).

This is the desired contradiction and \( M \) does not admit a metric with positive scalar curvature.

The proof relies on the existence of torsion in \( \pi \). Therefore, one may still conjecture that the unstable Gromov-Lawson-Rosenberg conjecture is true for torsion free groups.

3 Acknowledgements

To work on the counterexample was inspired by talks of Stephan Stolz where he expressed his opinion that the original GLR-conjecture is false. The author wants to thank Stephan Stolz for useful and enlightening conversations on the subject. Stolz conjectures that a weaker form, the so called stable GLR-conjecture, is true (compare [15]) and shows [16] that this conjecture follows from the Baum-Connes conjecture.

References

[1] Botvinnik, B.I., Gilkey, P., and Stolz, S.: “The Gromov-Lawson-Rosenberg conjecture for space form groups”, in preparation

[2] Federer, H.: “Geometric measure theory”, vol. 153 of Grundlehren der math. Wissenschaften, Springer (1969)

[3] Gromov, M. and Lawson, H.B.: “Positive scalar curvature and the Dirac operator on complete Riemannian manifolds”, Publ. Math. IHES 58, 83–196 (1983)

[4] Hitchin, N.: “Harmonic spinors”, Advances in Mathem. 14, 1–55 (1974)

[5] Kwasik, S. and Schultz, R.: “Positive scalar curvature and periodic fundamental groups”, Comm. Math. Helv. 65, 271–286 (1990)

[6] Lichnerowicz, A.: “Spineurs harmoniques”, C.R. Acad. Sci. Paris, sérées 1 257, 7–9, Zentralblatt 136.18401 (1963)
[7] Morgan, F.: “Geometric measure theory, a beginner’s guide”, Academic press (1988)

[8] Rosenberg, J.: “C*-algebras, positive scalar curvature, and the Novikov conjecture”, Publ. Math. IHES 58, 197–212 (1983)

[9] Rosenberg, J.: “C*-algebras, positive scalar curvature, and the Novikov conjecture III”, Topology 25, 319–336 (1987)

[10] Rosenberg, J. and Stolz, S.: “The “stable” version of the Gromov-Lawson conjecture”, Contemporary Mathematics 181, 405–418 (1995)

[11] Schoen, R. and Yau, S.-T.: “Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature”, Annals of Mathematics 110, 127–142, MR 81k:5802 (1979)

[12] Schoen, R. and Yau, S.T.: “On the structure of manifolds with positive scalar curvature”, manuscr. mathem. 28, 159–183 (1979)

[13] Schröder, H.: “K-theory for real C*-algebras and its applications”, vol. 290 of Pitman Research Notes in Mathematics, Longman Scientific & and Technical (1993)

[14] Stolz, S.: “Simply connected manifolds of positive scalar curvature”, Annals of Mathematics 136, 511–540 (1992)

[15] Stolz, S.: “Positive scalar curvature metrics—existence and classification”, preprint (1994)

[16] Stolz, S.: “The Baum-Connes conjecture implies the stable Gromov-Lawson-Rosenberg Conjecture”, Vortrag auf der Konferenz ”Geometric Groups and Bounded Topology“, Schloß Ringberg (1996)