Some inequalities associated with the Hermite–Hadamard–Fejér type for convex function

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Abstract In this paper, we extend some estimates of the right-hand side of a Hermite–Hadamard–Fejér type inequality for functions whose first derivatives’ absolute values are convex. The results presented here would provide extensions of those given in earlier works.

Keywords Hermite–Hadamard–Fejer inequality · Trapezoid inequality · Convex function · Hölder inequality.

Mathematics Subject Classification 26D07 · 26D15

Introduction

Definition 1 The function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that $f$ is concave if $(-f)$ is convex.

The following inequality is well known in the literature as the Hermite–Hadamard integral inequality (see, [2, 4]):

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$

(1.1)

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$.

In [1], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1 Let $f : I' \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I'$, $a, b \in I'$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:

$$\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx = \frac{b - a}{2} \int_0^1 (1 - 2t)f'(ta + (1 - t)b) \, dt.$$ 

(1.2)

Theorem 1 Let $f : I' \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I'$, $a, b \in I'$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{(b - a)}{8} (|f'(a)| + |f'(b)|).$$

(1.3)

Theorem 2 Let $f : I' \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I'$, $a, b \in I'$ with $a < b$, $f' \in L[a, b]$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2(2p + 1)^{1/p}} \times \left( \frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2} \right)^{(p-1)/p}.$$ 

(1.4)

The most well-known inequalities related to the integral mean of a convex function are the Hermite–Hadamard
inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities (see [5–14]). In [3], Fejér gave a weighted generalization of the inequalities (1.1) as the following:

**Theorem 3** \( f : [a, b] \to \mathbb{R} \) be a convex function, then the inequality

\[
 f\left(\frac{a+b}{2}\right) \int_a^b w(x)dx \leq \frac{1}{b-a} \int_a^b f(x)w(x)dx 
\]

\[
 \leq \frac{f(a) + f(b)}{2} \int_a^b w(x)dx 
\]

(1.5)

holds, where \( w : [a, b] \to \mathbb{R} \) is nonnegative, integrable, and symmetric about \( x = \frac{a+b}{2} \).

In [5], some inequalities of Hermite–Hadamard–Fejer type for differentiable convex mappings were proved using the following lemma.

**Lemma 2** Let \( f : I_a \to \mathbb{R} \) be a differentiable mapping on \( I = [a, b] \) with \( a < b \), and \( w : [a, b] \to [0, \infty) \) be a differentiable mapping. If \( f' \in L[a, b] \), then the following equality holds:

\[
\frac{f(a) + f(b)}{2} \int_a^b w(x)dx - \frac{1}{b-a} \int_a^b f(x)w(x)dx 
\]

\[
= \frac{(b-a)^2}{2} \int_0^1 p(t)f'(ta + (1-t)b)dt 
\]

(1.6)

for each \( t \in [0, 1] \), where

\[
p(t) = \int_t^1 w(as + (1-s)b)ds - \int_0^t w(as + (1-s)b)ds. 
\]

In this article, using functions whose derivatives’ absolute values are convex, we obtained new inequalities of Hermite–Hadamard–Fejer type. The results presented here would provide extensions of those given in earlier works.

**Main results**

We will establish some new results connected with the right-hand side of (1.5) and (1.1). Now, we prove our main theorems:

**Theorem 4** Let \( f : I_a \to \mathbb{R} \) be a differentiable mapping on \( I = [a, b] \) with \( a < b \) and let \( w : [a, b] \to \mathbb{R} \) be continuous on \( [a, b] \). If \( |f'| \) is convex on \( [a, b] \), then for all \( x \in [a, b] \), the following inequalities hold:

\[
\left( \int_a^b w(s)ds \right)^x f(b) - \left( \int_a^b w(s)ds \right)^x f(a) 
\]

\[
- \alpha \int_a^b \left( \int_a^b w(s)ds \right)^{x-1} w(t)f(t)dt 
\]

(2.1)

where \( x > 0 \) and \( ||w|| = \sup_{t \in [a, b]} |w(t)| \).

**Proof** By integration by parts, we have the following equalities:

\[
\int_a^b \left( \int_a^b w(s)ds \right)^x f(t)dt 
\]

\[
= \left( \int_a^b w(s)ds \right)^x f(b) - \left( \int_a^b w(s)ds \right)^x f(a) 
\]

\[
- \alpha \int_a^b \left( \int_a^b w(s)ds \right)^{x-1} w(t)f(t)dt. 
\]
\[
\left| \left( \int_{a}^{b} w(s) \, ds \right)^{z} f(b) - \left( \int_{a}^{x} w(s) \, ds \right)^{z} f(a) \right|
- \alpha \left( \int_{a}^{b} \left( \int_{x}^{t} w(s) \, ds \right)^{z-1} w(t) f(t) \, dt \right)
\leq \int_{a}^{b} \left( \int_{x}^{t} w(s) \, ds \right)^{z} |f'(t)| \, dt + \int_{x}^{b} \left( \int_{x}^{t} w(s) \, ds \right)^{z} |f'(t)| \, dt
\]
\[
\leq \|w\|_{[a,b],\infty}^{z} \int_{a}^{x} (t-x)^{z} |f'(t)| \, dt + \|w\|_{[x,b],\infty}^{z} \int_{x}^{b} (t-x)^{z} |f'(t)| \, dt
\]
\[
= \|w\|_{[a,b],\infty}^{z} \left[ \int_{a}^{x} (t-x)^{z} |f'(b) - t - \frac{t-a}{b-a} b | dt \right]
+ \|w\|_{[x,b],\infty}^{z} \left[ \int_{x}^{b} (t-x)^{z} |f'(b) - t + \frac{t-a}{b-a} b | dt \right]
\]
\[
\leq \|w\|_{[a,b],\infty}^{z} \left\{ \frac{|f'(b)|}{b-a} \left[ \frac{(b-x)^{z+1} (x-a)}{(x+1)(x+2)} + \frac{(x-a)^{z+2}}{(x+2)} \right] \right\}
\]
for all \( x \in [a,b] \). Hence, the proof of theorem is completed.

**Corollary 1** Under the same assumptions of Theorem 4 with \( w(s) = 1 \), then the following inequality holds:
\[
(b-x)^{z} f(b) - (a-x)^{z} f(a) - \alpha \int_{a}^{b} (t-x)^{z-1} f(t) \, dt
\]
\[
\leq \frac{1}{(b-a)} \left\{ \frac{|f'(a)|}{(x-a+1)} + \frac{(b-x)^{z+2}}{(x+1)(x+2)} \right\}
\]
for all \( x \in [a,b] \).

**Remark 1** If we take \( \alpha = 1 \) and \( x = \frac{a+b}{2} \) in (2.2), the inequality (2.2) reduces to (1.3).

**Corollary 2** (Fejer Type Inequality) Under the same assumptions of Theorem 4 with \( \alpha = 1 \), then the following inequalities hold:
\[
|f(b) - f(a)| \int_{a}^{b} w(s) \, ds + |f(a)| \int_{a}^{b} w(s) \, ds - \int_{a}^{b} w(t) f(t) \, dt
\]
\[
\leq |f'(a)| \frac{(x-a)^{z} (3b-2a-x) \|w\|_{[a,b],\infty} + \|w\|_{[x,b],\infty} (b-x)^{3}}{6(b-a)}
\]
\[
+ |f'(b)| \frac{(b-x)^{z} (x-3a+2b) \|w\|_{[a,b],\infty} + (x-a)^{3} \|w\|_{[a,b],\infty}}{6(b-a)}
\]
\[
\leq |f'(a)| \frac{(x-a)^{z} (3b-2a-x) + (b-x)^{3}}{6(b-a)} \|w\|_{[a,b],\infty}
\]
\[
+ |f'(b)| \frac{(b-x)^{z} (x-3a+2b) + (x-a)^{3}}{6(b-a)} \|w\|_{[a,b],\infty}
\]
which is proved by Tseng et al. in [8].

**Corollary 3** (Weighted Trapezoid Inequality) Let \( w : [a,b] \to \mathbb{R} \) be symmetric to \( \frac{a+b}{2} \) and \( x = \frac{a+b}{2} \) in Corollary 2. Then the following inequalities hold:
\[ \left| \frac{f(a)+f(b)}{2} \int_a^b w(s)ds - \int_a^b w(t)f(t)dt \right| \]
\[ \leq \frac{(b-a)^2}{48} \left[ 5\|w\|_{[a,b]!}\|f'\|_{[a,b]!} + 5\|w\|_{[a,b]!}\|f'\|_{[a,b]!} \right] \]
\[ + \left[ \|w\|_{[a,b]!} + 5\|w\|_{[a,b]!} \right] |f'(a)| \]
\[ \leq \frac{(b-a)^2}{8} \left| f'(a) \right| + \left| f'(b) \right| \]
which is proved by Tseng et al. in [8].

**Theorem 5** Let \( f: I' \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I' \), \( a, b \in I' \) with \( a < b \) and let \( w: [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b] \). If \( |f'|^q \) is convex on \([a, b] \), \( q > 1 \), then for all \( x \in [a, b] \), the following inequalities hold:

\[ \left| \int_a^b w(s)ds \right|^z - \left| \int_a^b w(s)ds \right|^z \]
\[ - \frac{b}{a} \left( \int_a^b w(t)f(t)dt \right) \]
\[ \leq \frac{(b-a)^2}{8} \left( \left| f'(a) \right| + \left| f'(b) \right| \right) \]
\[ \left( \int_a^b w(t)f(t)dt \right) \]
\[ \leq \frac{b-a}{6} \left( \left| f'(a) \right| + \left| f'(b) \right| \right) \]
\[ \left( \int_a^b w(t)f(t)dt \right) \]

Since \( |f'|^q \) is convex on \([a, b] \)

\[ \left| f' \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} \right) \right|^q \leq \frac{b-t}{b-a} \left| f'(a) \right|^q + \frac{t-a}{b-a} \left| f'(b) \right|^q. \]

From (2.3), it follows that

\[ \left| \int_a^b w(s)ds \right|^z - \left| \int_a^b w(s)ds \right|^z \]
\[ - \frac{b}{a} \left( \int_a^b w(t)f(t)dt \right) \]
\[ \leq \frac{(b-a)^2}{8} \left( \left| f'(a) \right| + \left| f'(b) \right| \right) \]
\[ \left( \int_a^b w(t)f(t)dt \right) \]
\[ \leq \frac{b-a}{6} \left( \left| f'(a) \right| + \left| f'(b) \right| \right) \]
\[ \left( \int_a^b w(t)f(t)dt \right) \]

where \( \alpha > 0, \frac{1}{p} + \frac{1}{q} = 1 \), and \( \|w\|_\infty = \sup_{t \in [a,b]} |w(t)| \).

**Proof** We take absolute value of (2.1). Using Holder's inequality, we find that
\[
\left( x-a \right)^{\frac{1}{2}} \|w\|_{[a,b],\infty} \left( \frac{\left( b-a \right)^2 - (b-x)^2}{2} \right)^{\frac{1}{2}} \left| f'(a) \right|^{\frac{1}{2}} + \left( x-a \right)^{\frac{1}{2}} \|f'\|_{[a,b],\infty} ^{\frac{1}{2}} \left( \frac{\left( b-a \right)^2 - (x-a)^2}{2} \right)^{\frac{1}{2}} \left| f'(b) \right|^{\frac{1}{2}} \right) \left( \frac{b-a}{(x-a)^{\frac{1}{2}}} + \frac{x-a}{2} \left| f'(b) \right|^{\frac{1}{2}} \right) \right) \frac{1}{b-a} \int_a^b f(t) \, dt \right) \right) \leq \frac{(b-a)^2}{4(p+1)^2} \left[ \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right]^{\frac{1}{q}} + \left( f(a)^q + 3|f'(b)|^q \right)^{\frac{1}{q}}.
\]

**Corollary 6** (Fejer Type Inequality) Under the same assumptions of Theorem 5 with \( z = 1 \), then the following inequalities hold:

\[
\left| f(b) \int_a^b w(s) ds + f(a) \int_a^b w(s) ds - \int_a^b w(t) f(t) dt \right| \leq \frac{(x-a)^{\frac{1}{2}}}{(b-a)^{\frac{1}{2}}} \|w\|_{[a,b],\infty} \left( \frac{\left( b-a \right)^2 - (b-x)^2}{2} \right)^{\frac{1}{2}} \left| f'(a) \right|^{\frac{1}{2}} + \left( x-a \right)^{\frac{1}{2}} \|f'\|_{[a,b],\infty} ^{\frac{1}{2}} \left( \frac{\left( b-a \right)^2 - (x-a)^2}{2} \right)^{\frac{1}{2}} \left| f'(b) \right|^{\frac{1}{2}} \right) \left( \frac{b-a}{(x-a)^{\frac{1}{2}}} + \frac{x-a}{2} \left| f'(b) \right|^{\frac{1}{2}} \right) \right) \left( \frac{b-a}{(x-a)^{\frac{1}{2}}} + \frac{x-a}{2} \left| f'(b) \right|^{\frac{1}{2}} \right) \right) \frac{1}{b-a} \int_a^b f(t) \, dt \right) \right) \leq \frac{(b-a)^2}{4(p+1)^2} \left[ \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right]^{\frac{1}{q}} + \left( f(a)^q + 3|f'(b)|^q \right)^{\frac{1}{q}}.
\]

**Corollary 7** (Weighted Trapezoid Inequality) Let \( w : [a, b] \to \mathbb{R} \) be symmetric to \( a+b \) and \( x = \frac{a+b}{2} \) in Corollary 6. Then the following inequalities hold:

\[
\left| f(0) + f(b) \int_a^b w(s) ds - \int_a^b w(t) f(t) dt \right| \leq \frac{(b-a)^2}{4(p+1)^2} \left[ \|w\|_{[a,b],\infty} \left( \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right) \right]^{\frac{1}{q}} + \left( f(a)^q + 3|f'(b)|^q \right)^{\frac{1}{q}}.
\]

**Corollary 5** Let the conditions of Corollary 4 hold. If we take \( z = 1 \) and \( x = \frac{a+b}{2} \) in (4.5), then the following inequality holds:

\[
\left( f(a) + f(b) \int_a^b w(s) ds - \int_a^b w(t) f(t) dt \right) \right) \leq \frac{(b-a)^2}{4(p+1)^2} \left[ \|w\|_{[a,b],\infty} \left( \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right) \right]^{\frac{1}{q}} + \left( f(a)^q + 3|f'(b)|^q \right)^{\frac{1}{q}}.
\]
Theorem 6 Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \), \( a, b \in I \) with \( a < b \) and let \( w : [a, b] \to \mathbb{R} \) be continuous on \([a, b] \). If \( |f'|^q \) is convex on \([a, b] \), \( q > 1 \), then for all \( x \in [a, b] \), the following inequality holds:

\[
\left| \left( \int_x^b w(s) \, ds \right)^{\frac{1}{p}} f(b) - \left( \int_x^a w(s) \, ds \right)^{\frac{1}{p}} f(a) \right| - \left( \int_x^b \int_x^t \frac{w(t)f(t)}{w(s)} \, ds \, dt \right)^{\frac{1}{p}} \leq \frac{(b-a)^{\frac{1}{p}}|w|_{\infty}^{\frac{1}{p}}}{(2p+1)^{\frac{1}{2}}} \left[ (x-a)^{pq+1} + (b-x)^{pq+1} \right]^{\frac{1}{2}} \left( |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{2}}.
\]

which this completes the proof.

Corollary 8 Under the same assumptions of Theorem 6 with \( w(s) = 1 \), then the following inequality holds:

\[
\left| (b-x)^{q}f(b) - (a-x)^{q}f(a) - x \int_a^b (t-x)^{q-1}f(t) \, dt \right| \leq \frac{(b-a)^{\frac{1}{p}}}{(2p+1)^{\frac{1}{2}}} \left[ (x-a)^{pq+1} + (b-x)^{pq+1} \right]^{\frac{1}{2}} \left( |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{2}}.
\]

(2.5)

Remark 2 Let the conditions of Corollary 8 hold. If we take \( \alpha = 1 \) and \( x = \frac{a+b}{2} \) in (2.5), then the inequality becomes the inequality (1.4).

Corollary 9 (Fejer Type Inequality) Under the same assumptions of Theorem 6 with \( \alpha = 1 \), then the following inequality holds:

\[
\left| f(b) \int_x^b \frac{w(s) \, ds + f(a) \int_x^a w(s) \, ds}{a} - f(a) \int_x^b \frac{w(t)f(t) \, dt}{a} \right| \leq \frac{(b-a)^{\frac{1}{p}}|w|_{\infty}^{\frac{1}{p}}}{(p+1)^{\frac{1}{2}}} \left[ (x-a)^{pq+1} + (b-x)^{pq+1} \right]^{\frac{1}{2}} \left( |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{2}}.
\]

Corollary 10 (Weighted Trapezoid Inequality) Let \( w : [a, b] \to \mathbb{R} \) be symmetric to \( \frac{a+b}{2} \) and \( x = \frac{a+b}{2} \) in Corollary 9. Then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} \int_a^b w(s) \, ds - \int_a^b w(t)f(t) \, dt \right| \leq \frac{(b-a)^{\frac{1}{p}}|w|_{\infty}^{\frac{1}{p}}}{(2p+1)^{\frac{1}{2}}} \left( |f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{2}}.
\]
We take absolute value of (2.1). Using Holder's inequality and the convexity of \(|f'(t)|^q\), we find that

\[
\left| \left( \int_a^b w(s)ds \right)^z f(b) - \left( \int_a^b w(s)ds \right)^z f(a) \right| \\
\geq \frac{1}{(x+1)(x+2)^2(b-a)^2} \left( (x-a)^z+1+(b-x)^z+1 \right)^z
\]

We find that

\[
\frac{1}{(x+1)(x+2)^2(b-a)^2} \left( (x-a)^z+1+(b-x)^z+1 \right)^z
\]

which this completes the proof.

**Corollary 11** Under the same assumptions of Theorem 7 with \(w(s) = 1\), then the following inequality holds:

\[
\left| (b-a)^2 f(b) - (a-x)^2 f(a) - x \int_a^b (t-x)^{z-1} f(t)dt \right| \\
\leq \frac{1}{(x+1)(x+2)^2(b-a)^2} \left( (x-a)^z+1+(b-x)^z+1 \right)^z
\]

\[
\times \left( \left( (x+1)(b-a)(x-a)^{z-1}+(b-x) \right) \left( (x-a)^{z-1}+(b-x)^{z-1} \right) \right)\left| f'(a) \right|^q \\
\times \left( (x+1)(b-a)(b-x)^{z-1}+(x-a) \right) \left( (x-a)^{z-1}+(b-x)^{z-1} \right) \left| f'(b) \right|^q
\]

\[
\leq \frac{(b-a)^2}{4} \left( \left| f'(a) \right|^q+\left| f'(b) \right|^q \right)^{\frac{3}{2}}.
\]

**Corollary 12** Let the conditions of Corollary 11 hold. If we take \(a = 1\) and \(x = \frac{a+b}{2}\) in (2.6), then the following inequality holds:

\[
\left| \frac{f(a)+f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
\leq \frac{(b-a)^2}{4} \left( \left| f'(a) \right|^q+\left| f'(b) \right|^q \right)^{\frac{3}{2}}.
\]

**Corollary 13** (Fejer Type Inequality) Under the same assumptions of Theorem 7 with \(a = 1\), then the following inequality holds:

\[
\left| \int_a^b w(s)ds \int_a^b w(t)f(t)dt - \int_a^b w(s)ds \int_a^b w(t)f(t)dt \right| \\
\leq \frac{\left| \int_a^b w(t)f(t)dt \right|}{(x+1)(x+2)^2(b-a)^2} \left( (x-a)^z+1+(b-x)^z+1 \right)^z
\]

\[
\times \left( \left( (x+1)(b-a)(x-a)^{z-1}+(b-x) \right) \left( (x-a)^{z-1}+(b-x)^{z-1} \right) \right)\left| f'(a) \right|^q \\
\times \left( (x+1)(b-a)(b-x)^{z-1}+(x-a) \right) \left( (x-a)^{z-1}+(b-x)^{z-1} \right) \left| f'(b) \right|^q
\]

\[
\leq \frac{(b-a)^2}{2} \left( \left| f'(a) \right|^q+\left| f'(b) \right|^q \right)^{\frac{3}{2}}.
\]

**Corollary 14** (Weighted Trapezoid Inequality) Let \(w : [a, b] \to \mathbb{R}\) be symmetric to \(\frac{a+b}{2}\) and \(x = \frac{a+b}{2}\) in Corollary 13. Then the following inequality holds:

\[
\left| \frac{f(a)+f(b)}{b-a} \int_a^b w(s)ds - \int_a^b w(t)f(t)dt \right| \\
\leq \frac{(b-a)^2}{4} \left( \left| \int_a^b w(t)f(t)dt \right| \left| f'(a) \right|^q+\left| f'(b) \right|^q \right)^{\frac{3}{2}}.
\]
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