On the inversion of the Laplace transform

In Memory of Dimitris Gatzouras

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Abstract

The Laplace transform is a useful and powerful analytic tool with applications to several areas of applied mathematics, including differential equations, probability and statistics. Similarly to the inversion of the Fourier transform, inversion formulae for the Laplace transform are of central importance; such formulae are old and well-known (Fourier-Mellin or Bromwich integral, Post-Widder inversion). The present work is motivated from an elementary statistical problem, namely, the unbiased estimation of a parametric function of the scale in the basic model of a random sample from exponential distribution. The form of the uniformly minimum variance unbiased estimator of a parametric function $h(\lambda)$, as well as its variance, are obtained as series in Laguerre polynomials and the corresponding Fourier coefficients, and a particular application of this result yields a novel inversion formula for the Laplace transform.

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1 Introduction

For a function $u : (0, \infty) \to \mathbb{R}$, its Laplace transform is defined by the integral

$$\phi(\lambda) = \int_0^\infty \exp(-\lambda x)u(x)dx, \quad (1.1)$$

provided that there exists $\lambda_0 \geq 0$ such that

$$\int_0^\infty \exp(-\lambda_0 x)|u(x)|dx < \infty. \quad (1.2)$$

There is a second version of the Laplace transform, related to probability measures $\mu$ supported in (a subset of) $[0, \infty)$, namely,

$$\phi_\mu(\lambda) = \int_{[0,\infty)} \exp(-\lambda x)d\mu(x); \quad (1.3)$$
it is just a matter of notation to express \( \phi_\mu(\lambda) \) as \( E \exp(-\lambda X) \) where the nonnegative random variable \( X \) has distribution function \( F(x) = \mu([0, x]), \ x \geq 0, \) and \( E \) denotes expectation. In this setup, \( \phi_\mu(\lambda) \) is denoted as \( M_X(-\lambda) \) and it is called the moment generating function of \( X \). It is clear that formulae (1.1) and (1.2) coincide if \( X \) has a density \( u \) (w.r. to Lebesgue measure on \( [0, \infty) \)). An inversion formula for the probabilistic version (1.2) can be found in Billingsley (1995) or Schilling et al (2012), and it is based on an ingenious application of the law of large numbers. The formula can be written as \( (x > 0) \)

\[
\mu([0, x]) + \frac{1}{2} \mu([1, x]) = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{(-1)^k}{k!} \left( \frac{N}{x} \right)^k \phi_\mu^{(k)}\left( \frac{N}{x} \right). \tag{1.3}
\]

Regarding (1.1), it is known from Lerch (1903) that the transformation \( u \to \phi \) is one to one. Furthermore, there are two well-known inversion formulae for (1.1), namely, the Fourier-Mellin or Bromwich integral (see Boas (1983), Cohen (2007)),

\[
u(x) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma-iT}^{\gamma+iT} \exp(sx) \phi(s) ds, \tag{1.4}\]

where \( \gamma \geq 0 \) is greater than the real part of every pole of (the analytic extension of) \( \phi \), and the Post (1930) or Post-Widder formula (see Widder (1946), Post (1930), Cohen (2007)),

\[u(x) = \lim_{n} \frac{(-1)^n}{n!} \left( \frac{n}{x} \right)^{n+1} \phi^{(n)}\left( \frac{n}{x} \right). \tag{1.5}\]

The above inversions hold under some mild restrictions, e.g., (1.4) is satisfied for almost all \( x \in (0, \infty) \) (clearly, this is the best we can expect, but the formula in itself is complicated and, so, inconvenient for purposes of computation, as can be seen when applied to trivial exemplary cases), and (1.5) holds at continuity points of \( u \), provided that \( u \) is smooth in pieces and that the growth of \( |u| \) at infinity is at most of exponential order.

The present work is motivated from an elementary statistical inference problem which, at a first glance, seems to be unrelated to Laplace inversion. The problem is to find the minimum variance unbiased estimator of a given parametric function \( h(\lambda) \), based on a random sample \( X_1, \ldots, X_n \) from \( \text{Exp}(\lambda) \), with \( \lambda > 0 \) unknown, or, more generally, from \( \Gamma(a, \lambda) \) with \( a > 0 \) fixed and known and \( \lambda > 0 \) an unknown parameter (for the definitions see Section 2). The main result provides necessary and sufficient conditions on \( h \) so that a solution of this problem exists, and shows that the solution (whenever exists) can be presented as a series in Laguerre polynomials,

\[L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}. \tag{1.6}\]

A particular application of the main result yields a novel inversion formula for the Laplace transform; see Section 3.
2 On the best unbiased estimator of a parametric function of the scale parameter in exponential/gamma models

2.1 Preliminaries and a simple parametric inference problem

The probability density of the exponential distribution, Exp(\(\lambda\)), is given by

\[ f_\lambda(x) = \lambda \exp(-\lambda x), \quad x > 0, \]

while the Gamma distribution, \(\Gamma(a, \lambda)\), has probability density

\[ f_\lambda(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} \exp(-\lambda x), \quad x > 0, \tag{2.1} \]

where \(a > 0\) and \(\lambda > 0\) are positive constants, so that \(\text{Exp}(\lambda) \equiv \Gamma(1, \lambda)\).

From now on, we suppose that \(a > 0\) is known (given), and we assume that \(\lambda > 0\) is the (unique) unknown parameter to be estimated from the data. More generally, we wish to estimate an arbitrary parametric function \(h(\lambda)\) by using a suitable choice of an estimator

\[ T = T(X_1, \ldots, X_n), \]

where \(T\) is a real valued measurable function with domain \((0, \infty)^n\) and \(X_1, \ldots, X_n\) are iid (independent, identically distributed) random variables with density (2.1). Of course, the actual value of \(T\) (when \(X_1 = x_1, \ldots, X_n = x_n\)) must not vary with \(\lambda\), but \(T\) may depends on \(n\) or \(a\) (since both are fixed and known).

So, the problem can be formulated as follows:

**Problem 1.** Let \(h(\lambda) : (0, \infty) \to \mathbb{R}\) be a given (arbitrary) parametric function and suppose that \(X_1, \ldots, X_n\) are iid with density (2.1). Under what conditions on \(h\) is it possible to find an estimator \(T = T(X_1, \ldots, X_n)\) such that

1. \(\mathbb{E}_\lambda T = h(\lambda)\) for all \(\lambda > 0\), and
2. \(\mathbb{E}_\lambda T^2 < \infty\) for all \(\lambda > 0\)?

And, in case that such a \(T\) exists, how can we obtain the best possible estimator for \(h\)?

An estimator satisfying condition 1 is called *unbiased*; as we shall see, unbiasedness restricts the class of possible estimators in such a way that the family of permitted parametric functions \(h\) is quite narrow. Condition 2 means that \(T \in \cap_{\lambda>0} L^2(\mu_\lambda)\), where \(\mu_\lambda\) is the product probability measure of \((X_1, \ldots, X_n)\) on \([0, \infty)^n\). Then, provided \(\mathbb{E}_\lambda T = h(\lambda)\), the quantity \(\mathbb{E}_\lambda (T - h(\lambda))^2\) can be written as \(\text{Var}_\lambda T = \mathbb{E}_\lambda T^2 - h(\lambda)^2\), and it is called the *variance* of the estimator \(T\). Even if \(T\) is not unbiased, the quantity \(\mathbb{E}_\lambda (T - h(\lambda))^2\) is called MSE (mean squared error), and it is the most important measure of closeness between a point estimator \(T(X_1, \ldots, X_n)\) and a parametric function \(h(\lambda)\), traditionally used in statistics for a long time. The subscript \(\lambda\) in \(\mathbb{E}_\lambda\) and \(\text{Var}_\lambda\) denotes that the true probability measure of the \(X_i\)'s is as in (2.1).
It is clear that, if we restrict ourselves to the class of unbiased estimators, those with smaller variance are preferable. In the plausible case that we can pick an estimator $T^*$ satisfying

1. $E_\lambda T^* = h(\lambda)$ for all $\lambda > 0$,
2. $\text{Var}_\lambda T^* < \infty$ for all $\lambda > 0$, and
3. for any unbiased estimator $T$ and for all $\lambda > 0$, $\text{Var}_\lambda T^* \leq \text{Var}_\lambda T$,

it follows that this is the best we can do. Such an estimator $T^*$ is then called uniformly minimum variance unbiased estimator (UMVUE for short), and this is what we could name as best. In order to be able to obtain the UMVUE it is necessary and sufficient that the class

$$\mathcal{T}_h = \{ T : T \text{ is an unbiased estimator for } h(\lambda) \text{ with finite variance (for all } \lambda > 0) \}$$

is nonempty. This follows from one of the most fundamental result in parametric inference, adapted to the present particular case of Gamma distributions with $a$ known. Indeed, the following is true; see Lehmann and Casella (1998).

**Theorem 2.1.** (Rao-Blackwell / Lehmann-Scheffé). Let $X_1, \ldots, X_n$ be a random sample from (2.1) with $\lambda > 0$ unknown and $a > 0$ known. Let also $h : (0, \infty) \to \mathbb{R}$ be a parametric function, and suppose that $\mathcal{T}_h$ is nonempty. Set $X = X_1 + \cdots + X_n$. Then

(i) The conditional probability distribution of $(X_1, \ldots, X_n)$ given $X$ is independent of $\lambda$.
(ii) For any $T \in \mathcal{T}_h$, the (unique w.p. 1) UMVUE is given by the conditional expectation

$$T^*(X) := E \left( T(X_1, \ldots, X_n) \mid X \right).$$

(iii) Equivalently, the UMVUE of $h(\lambda)$ is the unique (w.p. 1) unbiased estimator in $\mathcal{T}_h$ which is a function of $X$, $u = u(X)$. Hence, $u(X) = E \left( T(X_1, \ldots, X_n) \mid X \right) = T^*(X)$.

**Remark 2.1.** It is well-known that the distribution of $X = X_1 + \cdots + X_n$ is $\Gamma(na, \lambda)$. In view of Theorem 2.1, and substituting $a$ for $na > 0$, Problem 1 reduces to an equivalent, much simper, reformulation, as follows.

**Problem 2.** Let $h(\lambda) : (0, \infty) \to \mathbb{R}$ be a given (arbitrary) parametric function and suppose that $X$ is a random variable having probability density (2.1), with $a > 0$ fixed and known, and $\lambda > 0$ an unknown parameter. Under what conditions on $h$ does the UMVUE $u = u(X)$ of $h(\lambda)$ exists for all $\lambda$? And, in case that it exists, how can we obtain its form?

Since, by definition, $E_\lambda \psi(X) = \int_0^\infty f_\lambda(x)\psi(x)dx$ for arbitrary measurable $\psi$, the imposed condition of a finite second moment on $u$ for all $\lambda$ implies that

$$\int_0^\infty x^{a-1} \exp(-\lambda x)u(x)^2dx < \infty.$$  (2.2)
In other words, \( u \in L^2(\lambda) \) for all \( \lambda > 0 \), where \( L^2(\lambda) \) is the Lebesgue space of functions \( u : (0, \infty) \rightarrow \mathbb{R} \) satisfying (2.2). Thus, it is reasonable to define

\[
L_0^2 := \bigcap_{\lambda > 0} L^2(\lambda).
\] (2.3)

We can rewrite the unbiasedness restriction \( E_\lambda u(X) = h(\lambda) \) as

\[
\frac{\Gamma(a)h(\lambda)}{\lambda^a} = \int_0^\infty x^{a-1} \exp(-\lambda x)u(x)dx, \quad \lambda > 0.
\] (2.4)

It is then obvious that the rhs of (2.4) defines a holomorphic function in the right half-plane \( \mathbb{C}^+ = \{ \lambda \in \mathbb{C} : \Re(\lambda) > 0 \} \) whenever \( u \in L_0^2 \). This means that the function \( \lambda \rightarrow \Gamma(a)\lambda^{-a}h(\lambda) \) is holomorphic, and hence, \( h(\lambda) \) must be holomorphic in \( \mathbb{C}^+ \). This already imposes a serious restriction to the allowable parametric functions, e.g., it is necessary that \( h \in C^\infty(0, \infty) \); in fact, the analytic extension of \( h \) should have no singularities in the right half-plane. As a simple example, for the \( C^\infty(0, \infty) \) parametric function \( h(\lambda) = \frac{1}{\lambda^2 - 2\lambda + 2} \), no unbiased estimator exists (for all \( \lambda > 0 \)), because of the poles \( 1 \pm i \) of \( h \). However, regarding Problem 2, the analyticity of \( h \) is not sufficient to provide a positive answer. To see this, it suffices to observe that for \( u \in L_0^2 \), \( \int_0^\infty x^{a-1} \exp(-\lambda x)u(x)dx \rightarrow 0 \) as \( \lambda \rightarrow +\infty \), by dominated convergence. Then, any holomorphic function \( h \) that grows faster than \( \lambda^a \) at infinity, e.g. \( h(\lambda) = \exp(\lambda) \), cannot be written as the expectation of some \( u \in L_0^2 \); see (2.4).

### 2.2 Results

We are now in a position to state the main results.

**Theorem 2.2.** Assume that \( X \) is a random variable with probability density \( f_1 \) as in (2.1), with \( \lambda > 0 \) unknown. For a given parametric function \( h(\lambda) \), its UMVUE \( u(X) \) exists in \( L_0^2 \) if and only if the following two conditions are satisfied.

1. The function \( h \) can be extended to an holomorphic function in \( \mathbb{C}^+ \), and
2. For any \( \lambda > 0 \),

\[
\sum_{n=0}^\infty \beta_n(\lambda)^2 < \infty,
\] (2.5)

where

\[
\beta_n(\lambda) = \frac{(-1)^n}{\sqrt{n!} [a]_n} \left( \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right] \right);
\] (2.6)

here, \([a]_n = \prod_{j=0}^{a-1} (a + j) = \Gamma(a + n)/\Gamma(a)\) denotes the ascending factorial (Pochhammer symbol).

**Theorem 2.3.** Let \( h \) be a parametric function satisfying (1) of Theorem 2.2. For fixed \( \lambda > 0 \) define the function

\[
H_\lambda(y) := h\left( \frac{\lambda}{1 - y} \right), \quad |y| < 1.
\] (2.7)
Then, an alternative simplified form of the constants $\beta_n(\lambda)$ in (2.6) is given by

$$\beta_n(\lambda) = \frac{(-1)^n H^{(n)}_1(0)}{\sqrt{n!} [\alpha]_n}. \quad (2.8)$$

**Theorem 2.4.** Assume that (1) and (2) of Theorem 2.2 are satisfied and fix $\lambda_0 > 0$. Then, the function $u(x)$ for which $u(X)$ is the UMVUE of $h(\lambda)$ is given by

$$u(x) = \sum_{n=0}^{\infty} \beta_n(\lambda_0) q_n(x), \quad (2.9)$$

where $\{q_n(x)\}_{n=0}^{\infty}$ is the complete orthonormal polynomial system corresponding to the weight function $f_{\lambda_0}$, with the convention that each $q_n$ is of degree $n$ and with strictly positive leading coefficient. The series converges a.e. on $(0, \infty)$ and in $L^2(\lambda)$ for every $\lambda \geq \lambda_0$, and the resulting function $u(x)$, given by (2.9), is independent of the choice of $\lambda_0$. Furthermore, for any $\lambda > 0$, the variance of the UMVUE is given by

$$\text{Var}_{\lambda} u(X) = \sum_{n=1}^{\infty} \beta_n(\lambda)^2, \quad (2.10)$$

where the constants $\beta_n(\lambda)$ are completely determined from $h(\lambda)$; see (2.6) or (2.8).

**Example 1.** We compare the expression (2.10) with the classical information inequality, namely, the famous Cramér-Rao (CR) lower bound (LB$_{CR}$). Since, as is well-known, the regularity conditions are satisfied for $f_{\lambda_0}$, the bound states that for a random sample $X_1, \ldots, X_n$ (of size $n$) from $f_{\lambda_0}$, and for any unbiased estimator $T = T(X_1, \ldots, X_n)$ of $h(\lambda)$, the inequality $\text{Var}_{\lambda} T \geq h'(\lambda)^2/(nI(\lambda)) := LB_{CR}$ is satisfied; here, $I(\lambda)$ is the Fisher information, defined as

$$I(\lambda) := \mathbb{E}_\lambda \left[ \left( \frac{\partial}{\partial \lambda} \log f_{\lambda}(X_1) \right)^2 \right] = \frac{a}{\lambda^2}.$$

Thus, the CR-bound reads as $\text{Var}_{\lambda} T \geq \lambda^2 h'(\lambda)^2/(na)$. On the other hand, the series expansion (2.10) (applied with $na$ in place of $a$; see Problems 1 and 2) yields

$$\text{Var}_{\lambda} u(X) = \sum_{m=1}^{\infty} \frac{\lambda^2}{m! [na]_m} \left( \frac{d^m}{d\lambda^m} \left[ \lambda^{m-1} h(\lambda) \right] \right)^2.$$

Since $u(X)$ is the UMVUE and thus, $\text{Var}_{\lambda} T \geq \text{Var}_{\lambda} u(X)$ for any unbiased estimator $T$, it is clear that the CR-bound is implied by the preceding series, on just keeping its first term.

### 2.3 Proofs

We first state some auxiliary lemmas.

**Lemma 2.1.** For $x > 0$, $a > 0$ and $\lambda > 0$,

$$\frac{d^n}{dx^n} [\lambda^n f_{\lambda}(x)] = \lambda \frac{d^n}{d\lambda^n} [\lambda^{n-1} f_{\lambda}(x)], \quad n = 0, 1, 2, \ldots. \quad (2.11)$$
Proof. By Leibnitz formula and (2.1) it is easily seen that both sides of (2.11) are equal to
\[
\Gamma(a + n)f_\lambda(x) \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (\lambda x)^k \Gamma(a + k).
\]

Lemma 2.2. (Rodrigues’ formula; see Afendras and Papadatos (2015)). For \( x > 0, \ a > 0 \) and \( \lambda > 0 \),
\[
\frac{d^n}{dx^n} \left[ x^n f_\lambda(x) \right] = (-1)^n \sqrt{|a|} n! n^n f_\lambda(x) q_{n,\lambda}(x), \quad n = 0, 1, 2, \ldots,
\]
where \( \{q_{n,\lambda}(x)\}_{n=0}^{\infty} \) is the complete orthonormal system with respect to \( f_\lambda \) standardized so that \( q_{n,\lambda} \) has degree \( n \) and positive leading coefficient. The polynomials \( q_{n,\lambda} \) satisfy the orthogonality condition
\[
E_\lambda \left[ q_{n,\lambda}(X)q_{m,\lambda}(X) \right] = \int_0^\infty f_\lambda(x) q_{n,\lambda}(x) q_{m,\lambda}(x) \, dx = \begin{cases} 1 & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}
\]

One important observation is that, as (2.12) and (2.11) show, we may produce the orthonormal set \( q_{n,\lambda} \) by differentiate w.r. to the parameter \( \lambda \), instead of \( x \); more precisely,
\[
q_{n,\lambda}(x) = \frac{(-1)^n}{\sqrt{n! |a| n^n}} \frac{d^n}{dx^n} \left[ x^n f_\lambda(x) \right] = \frac{(-1)^n}{\sqrt{n! |a| n^n}} \left( \lambda \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} f_\lambda(x) \right] \right).
\]
Thus, (2.13) obtains the following

Corollary 2.1. For \( x > 0, \ a > 0, \lambda > 0 \) and \( n \in \{0, 1, \ldots\} \),
\[
q_{n,\lambda}(x)f_\lambda(x) = \frac{(-1)^n}{\sqrt{n! |a| n^n}} \left( \lambda \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} f_\lambda(x) \right] \right).
\]

We now proceed to verify the results of Theorems 2.2–2.4.

Assume first that the UMVUE of \( h(\lambda) = u(X) \), and suppose that it has finite variance for all \( \lambda > 0 \). Multiplying the equation \( E_\lambda u(X) = h(\lambda) \) by \( \lambda^{n-1} \) and then taking \( n \) derivatives w.r. to \( \lambda \), we subsequently obtain
\[
h(\lambda) = \int_0^\infty f_\lambda(x) u(x) \, dx,
\]
\[
\lambda^{n-1} h(\lambda) = \int_0^\infty \lambda^{n-1} f_\lambda(x) u(x) \, dx,
\]
\[
\frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right] = \int_0^\infty \left( \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} f_\lambda(x) \right] \right) u(x) \, dx,
\]
\[
\frac{\lambda}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right] = \int_0^\infty \left( \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} f_\lambda(x) \right] \right) u(x) \, dx,
\]
\[
\frac{(-1)^n}{\sqrt{n! |a| n^n}} \left( \lambda \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right] \right) = \int_0^\infty q_{n,\lambda}(x)f_\lambda(x) u(x) \, dx; \tag{2.15}
\]
note that the differentiation can be passed under the integral sign, due to the assumed (squared) integrability of \( u \) with respect to \( f_\lambda \) for all \( \lambda > 0 \). We conclude from (2.15) that
the constants $\beta_n(\lambda)$ of (2.6) are the Fourier coefficients of $u$ with respect to the orthonormal polynomial system $\{q_{n,\lambda}\}_{n=0}^\infty$, corresponding to the weight function $f_{\lambda}$. It should be noticed that the orthonormal polynomial system corresponding to a probability measure (having finite moments of any order) is unique, apart from a possible multiplication of each polynomial by $\pm 1$. Moreover, since our system $\{q_{n,\lambda}\}_{n=0}^\infty$ is complete in $L^2(\lambda)$, see Afendras et al (2011), Parseval’s identity yields

$$E_{\lambda} u(X)^2 = \int_0^{\infty} f_{\lambda}(x)u(x)^2 dx = \sum_{n=0}^\infty \beta_n(\lambda)^2 < \infty.$$ 

Thus, assuming that $u \in L^2_0$, and since $\beta_0(\lambda) = E_{\lambda} u(X) = h(\lambda)$, we get

$$\text{Var}_{\lambda} u(X) = \sum_{n=1}^\infty \beta_n(\lambda)^2, \quad \text{for all } \lambda > 0.$$ 

Conversely, assume that $h$ is holomorphic in $C^+$ and that the series in (2.6) is finite for all $\lambda > 0$. Then we may define the function $u(x; \lambda)$ as

$$u(x; \lambda) := \sum_{n=0}^\infty \beta_n(\lambda) q_{n,\lambda}(x), \quad x > 0,$$  

(2.16)

where, by Riesz-Fisher, the series converges in $L^2(\lambda)$, that is,

$$\int_0^{\infty} \left( u_N(x; \lambda) - u(x; \lambda) \right)^2 f_{\lambda}(x) dx \to 0, \quad N \to \infty,$$

with $u_N(x; \lambda) = \sum_{n=0}^N \beta_n(\lambda) q_{n,\lambda}(x)$. It remains to show that the limiting function $u(x; \lambda)$ does not depend on $\lambda$, and that it is the unique UMVUE of $h(\lambda)$. To this end, choose a fixed $\lambda_0 > 0$ with $\lambda_0 < \lambda$ and write

$$u_N(x) = \sum_{n=0}^N \beta_n(\lambda_0) q_{n,\lambda_0}(x), \quad x > 0, \quad N = 0, 1, 2, \ldots$$  

(2.17)

From the convergence of the series (2.5) (with $\lambda = \lambda_0$) it is easily seen that $u_N(x)$ is Cauchy $L^2(\lambda_0)$, and hence, it converges (in the norm of $L^2(\lambda_0)$) to a function $u(x) \in L^2(\lambda_0)$. Moreover, is easy to check that for any $\lambda \geq \lambda_0$, we can find a constant $C_\lambda = C(\lambda, \lambda_0)$ such that $f_{\lambda}(x) \leq C_\lambda f_{\lambda_0}(x)$ for all $x > 0$. This implies that $u_N$ is also Cauchy $L^2(\lambda)$ for any fixed $\lambda \geq \lambda_0$; indeed, if $\varepsilon > 0$ is arbitrary, we can find $N(\varepsilon)$ such that $\int_0^{\infty} \left( u_{N_1}(x) - u_{N_2}(x) \right)^2 f_{\lambda_0}(x) dx < \varepsilon / C_\lambda$ for all $N_1, N_2 > N(\varepsilon)$ and, then,

$$\int_0^{\infty} \left( u_{N_1}(x) - u_{N_2}(x) \right)^2 f_{\lambda}(x) dx \leq C_\lambda \int_0^{\infty} \left( u_{N_1}(x) - u_{N_2}(x) \right)^2 f_{\lambda_0}(x) dx < \varepsilon.$$ 

The preceding argument verifies that the limiting function $u$, defined as the $L^2(\lambda_0)$-limit of the sequence in (2.17), belongs to $L^2(\lambda)$ for all $\lambda \geq \lambda_0$, in symbols, $u(x) \in \bigcap_{\lambda \geq \lambda_0} L^2(\lambda)$. 

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From the orthogonality of the polynomials \( q_{n,\lambda_0} \) \( (n \geq 1) \) and \( q_{0,\lambda_0} \equiv 1 \) we immediately see that \( E_{\lambda_0} u_N(X) = \beta_0(\lambda_0) = h(\lambda_0) \), and clearly, this is also true for \( u \), i.e., \( E_{\lambda_0} u(X) = h(\lambda_0) \). However, the situation is different when \( \lambda \neq \lambda_0 \), that is, the expectation of \( u_N(X) \) w.r. to \( f_\lambda \) varies with both \( N \) and \( \lambda \). More precisely, since \( q_{0,\lambda}(x) \equiv 1 \),

\[
E_\lambda u_N(X) = h(\lambda_0) + \sum_{n=1}^{N} \beta_n(\lambda_0) E_\lambda q_{n,\lambda_0}(X), \quad N = 1, 2, \ldots \quad \lambda > 0.
\]

On the other hand, we have shown that for \( \lambda \geq \lambda_0 \), \( E_\lambda \left( u_N(X) - u(X) \right)^2 \rightarrow 0 \), so that, by the Cauchy-Schwarz inequality,

\[
\left| E_\lambda u_N(X) - E_\lambda u(X) \right| \leq E_\lambda \left| u_N(X) - u(X) \right| \leq \left( E_\lambda \left| u_N(X) - u(X) \right|^2 \right)^{1/2} \rightarrow 0.
\]

It follows that \( E_\lambda u(X) = \lim_N E_\lambda u_N(X) \). Hence, the expectation of \( u(X) \) w.r. to \( f_\lambda \) can be obtained as the limit of the expectations of \( u_N(X) \) (w.r. to \( f_\lambda \)). Next, we see that the calculation of \( E_\lambda u_N(X) \) requires evaluation of the expectations \( E_\lambda q_{n,\lambda_0}(X) \), that is, integrals of the polynomials \( q_{n,\lambda_0}(x) \) w.r. to a different weight function \( (f_\lambda \text{ instead of } f_{\lambda_0}) \), under which these polynomials are no longer orthogonal.

In order to calculate \( E_\lambda q_{n,\lambda_0}(X) \) we proceed as follows. We have

\[
E_\lambda q_{n,\lambda_0}(X) = \int_0^\infty \frac{f_\lambda(x)}{f_{\lambda_0}(x)} f_{\lambda_0}(x) q_{n,\lambda_0}(x) \, dx
= \left( \frac{\lambda}{\lambda_0} \right)^n \int_0^\infty f_{\lambda_0}(x) \exp\left( - (\lambda - \lambda_0) x \right) q_{n,\lambda_0}(x) \, dx.
\]

The last integral can be viewed as the \( n \)-th Fourier coefficient of the bounded \( C^\infty(0, \infty) \) function \( w(x) := \exp\left( - (\lambda - \lambda_0) x \right) \), \( x > 0 \), with respect to the corresponding orthonormal polynomial system \( \{q_{n,\lambda_0}\}_{n=0}^{\infty} \) in \( L^2(\lambda_0) \). On the other hand, it is known that the same Fourier coefficients can be conveniently obtained by using the identity (see Afendras and Papadatos (2015), Afendras et al (2011))

\[
E_{\lambda_0} \left[ q_{n,\lambda_0}(X) w(X) \right] = \frac{1}{\sqrt{n!} |a|_n} \, E_{\lambda_0} \left[ X^n w^{(n)}(X) \right],
\]

provided \( E_{\lambda_0} \left[ X^n \left( w^{(n)}(X) \right)^2 \right] < \infty \). Since \( w^{(n)}(x) = (\lambda_0 - \lambda)^n \exp\left( - (\lambda - \lambda_0) x \right) \) is a bounded function of \( x \), because \( \lambda \geq \lambda_0 \), we can apply the preceding formulae to deduce

\[
E_\lambda q_{n,\lambda_0}(X) = \frac{\lambda}{\lambda_0} \left( \frac{\lambda_0 - \lambda}{\sqrt{n!} |a|_n} \right)^n E_{\lambda_0} \left[ X^n \exp\left( - (\lambda - \lambda_0) X \right) \right].
\]

A straightforward computation now yields

\[
E_{\lambda_0} \left[ X^n \exp\left( - (\lambda - \lambda_0) X \right) \right] = \frac{\lambda_0}{\Gamma(a)} \int_0^\infty x^{a-1} \exp(-\lambda x) \, dx = [a]_n \frac{\lambda_0^a}{\lambda^{a+n}}.
\]
and thus,
\[ E_{\lambda} q_{n;\lambda_0}(X) = (-1)^n \sqrt{\frac{[a]_n}{n!}} \left( 1 - \frac{\lambda_0}{\lambda} \right)^n. \]

Recalling that \( \beta_n(\lambda_0) \) is given by (2.6) with \( \lambda = \lambda_0 \), we have
\[ E_{\lambda} u_{N}(X) = \sum_{n=0}^{N} \frac{(-1)^n \lambda_0}{n!} \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right]_{\lambda=\lambda_0} \left( (-1)^n \sqrt{\frac{[a]_n}{n!}} \left( 1 - \frac{\lambda_0}{\lambda} \right)^n \right) \]
\[ = \sum_{n=0}^{N} \frac{1}{n!} \left( 1 - \frac{\lambda_0}{\lambda} \right)^n \left\{ \lambda_0 \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right] \right\}_{\lambda=\lambda_0} \].

(2.18)

Though the preceding formula appears to be quite complicated at a first glance (e.g., it seems that it is not an easy task to obtain its limiting value as \( N \to \infty \)), this is not the case. In fact, (2.18) represents a Taylor development around \( y = 0 \) of the function \( H_{\lambda_0}(y) := h\left( \frac{\lambda_0}{\lambda} \right), \ |y| < 1 \). Recall that \( h(\lambda) \) has been assumed to be holomorphic in \( \Re(\lambda) > 0 \), so that \( H_{\lambda_0}(y) \) is analytic in the open disc \( |y| < 1 \). Writing \( H^{(n)}_{\lambda_0}(y) \) for \( \frac{d^n}{dy^n} H_{\lambda_0}(y) \), we shall verify below the equality
\[ H^{(n)}_{\lambda_0}(0) = \left\{ \lambda_0 \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right] \right\}_{\lambda=\lambda_0}, \quad n = 0, 1, \ldots. \]

(2.19)

Assuming for a while that (2.19) is valid, and substituting it into (2.18), we obtain the simple formula
\[ E_{\lambda} u_{N}(X) = \sum_{n=0}^{N} \frac{H^{(n)}_{\lambda_0}(0)}{n!} \left( 1 - \frac{\lambda_0}{\lambda} \right)^n. \]

Since \( |1 - \lambda_0/\lambda| < 1 \) (for \( \lambda > \lambda_0/2 \)), we conclude from Taylor’s theorem that \( E_{\lambda} u_{N}(X) \to H_{\lambda_0}(1 - \lambda_0/\lambda) = h(\lambda) \). Thus, \( E_{\lambda} u(X) = \lim_N E_{\lambda} u_{N}(X) = h(\lambda) \), and this verifies that \( u(X) \) is the (unique) UMVUE of \( h(\lambda) \), for every \( \lambda \geq \lambda_0 \). (To see uniqueness, repeat the previous construction with \( \lambda_1 \) in place of \( \lambda_0 \). Then, as we showed, the produced estimator \( \tilde{u}(X) \) will satisfy \( E_{\lambda} \tilde{u}(X) = h(\lambda) = E_{\lambda} u(X) \) for all \( \lambda \geq \max\{\lambda_0, \lambda_1\} \), so it must be identical to \( u(X) \).) Furthermore, (2.6) shows that \( u \) has the same Fourier coefficients as the function \( u(x; \lambda) \) defined by (2.16); thus \( u(x) = u(x; \lambda) \) is independent of \( \lambda \), and Parseval’s identity yields (2.10). The orthogonal polynomials for the weight function \( f_{\lambda} \) are called generalized Laguerre (Laguerre when \( \alpha = 1 \)). The a.e. convergence of the Laguerre series expansion of a function \( u \in L^2(\lambda) \) is a well-known (Carleson-Hunt type) result that can be found in Mackenhaupt (1970); see also Uspensky (1927) and Stempak (2000).

It remains to show (2.19). Using Leibnitz formula we first calculate
\[ \lambda \frac{d^n}{d\lambda^n} \left[ \lambda^{n-1} h(\lambda) \right] = (n - 1)! \sum_{k=1}^{n} \binom{n}{k} \frac{\lambda^k h^{(k)}(\lambda)}{(k - 1)!}, \quad n = 1, 2, \ldots, \]

(2.20)

while the lhs equals to \( h(\lambda) \) for \( n = 0 \). Next, we define \( H_{\lambda}(y) = h(\lambda/(1 - y)), \ |y| < 1 \), so that \( H^{(0)}_{\lambda}(y) = H_{\lambda}(y) \) and \( H^{(0)}_{\lambda}(0) = h(\lambda) \). For \( n = 1 \), \( H^{(1)}_{\lambda}(y) = \lambda h'(\lambda/(1 - y))/(1 - y)^2 \), and
\( H'_0(0) = \lambda h'(\lambda) \) equals to the sum in the rhs of (2.20) (with \( n = 1 \)). We shall prove, using induction on \( n \), the formula (valid for \( \lambda > 0, |y| < 1 \))

\[
H^{(n)}_{\lambda}(y) = (n-1)! \sum_{k=1}^{n} \left( \frac{n}{k} \right) \frac{\lambda^k h^{(k)}(\lambda/(1-y))}{(k-1)!(1-y)^{n+k}} , \quad n = 1, 2, \ldots , \tag{2.21}
\]

which, setting \( y = 0 \), yields the rhs of (2.20); then, the substitution \( \lambda \to \lambda_0 \) verifies (2.19).

Noting that (2.21) is true for \( n = 1 \), we assume that it is true for some \( n \). Then,

\[
\begin{align*}
H^{(n+1)}_{\lambda}(y) &= \frac{d}{dy} \left\{ (n-1)! \sum_{k=1}^{n} \left( \frac{n}{k} \right) \frac{\lambda^k h^{(k)}(\lambda/(1-y))}{(k-1)!(1-y)^{n+k}} \right\} \\
&= (n-1)! \sum_{k=1}^{n} \left( \frac{n}{k} \right) \frac{\lambda^k d}{dy} \left\{ \frac{h^{(k)}(\lambda/(1-y))}{(1-y)^{n+k}} \right\} \\
&= (n-1)! \sum_{k=1}^{n} \left( \frac{n}{k} \right) \frac{\lambda^k h^{(k-1)}(\lambda/(1-y))}{(k-1)! (1-y)^{n+k}} \frac{\lambda}{(1-y)^2} \\
&\quad + (n-1)! \sum_{k=1}^{n} \left( \frac{n}{k} \right) \frac{\lambda^k h^{(k)}(\lambda/(1-y))}{(k-1)! (1-y)^{n+k+1}} (n+k) \\
&= (n-1)! \sum_{k=2}^{n+1} (k-1) \left( \frac{n}{k-1} \right) \frac{\lambda^k h^{(k)}(\lambda/(1-y))}{(k-1)! (1-y)^{n+1+k}} \\
&\quad + (n-1)! \sum_{k=1}^{n} (n+k) \left( \frac{n}{k} \right) \frac{\lambda^k h^{(k)}(\lambda/(1-y))}{(k-1)! (1-y)^{n+1+k}} \\
&= (n-1)! \sum_{k=1}^{n+1} \left\{ (k-1) \left( \frac{n}{k-1} \right) + (n+k) \left( \frac{n}{k} \right) \right\} \frac{\lambda^k h^{(k)}(\lambda/(1-y))}{(k-1)! (1-y)^{n+1+k}},
\end{align*}
\]

where the last equality follows from \( \binom{n}{k} = 0 \) for \( k = n+1 \) and \( (k-1) \binom{n}{k-1} = 0 \) for \( k = 1 \). It is now obvious that

\[
(k-1) \binom{n}{k-1} + (n+k) \binom{n}{k} = \frac{(k-1)n!}{(k-1)!(n-k+1)!} + \frac{(n+k)n!}{k!(n-k)!} \\
= \frac{\{k(k-1) + (n+k)(n+1-k)\}n!}{k!(n+1-k)!} \\
= \frac{n(n+1)}{k!(n+1-k)!} \\
= n \binom{n+1}{k}.
\]

This shows that (2.21) holds with \( n + 1 \) in place of \( n \), and concludes the inductive argument.
### 3 A novel inversion formula of the Laplace transform

The results of Section 2 apply to the particular case where \( a = 1 \), i.e., when \( X \) follows the exponential distribution with parameter \( \lambda > 0 \), \( \text{Exp}(\lambda) \), with probability density

\[
f_\lambda(x) = \lambda \exp(-\lambda x), \quad x > 0.
\]

In this case, Lemma 2.2 produces the corresponding orthonormal polynomial system, namely,

\[
q_{n,1}(x) = \sum_{k=0}^{n} (-1)^n (\frac{n}{k}) (\frac{\lambda x}{k})^k.
\]

The preceding polynomials are functions of \( \lambda x \) (this is also true for \( a \neq 1 \), since it is easily seen that \( q_{n,1}(x) = q_{n,1}(\lambda x) \)). Hence, it is convenient to define \( p_n(x) = q_{n,1}(x) \), so that \( q_{n,1}(x) = p_n(\lambda x) \). Then, the polynomial system \( \{p_n(x)\}_{n=0}^{\infty} \) is the complete orthonormal system corresponding to \( f_1 \) (i.e., \( \text{Exp}(1) \)), that is,

\[
E[p_n(X)p_m(X)] = \int_0^\infty e^{-x} p_n(x)p_m(x)dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \end{cases}
\]

where \( E \) stands for \( E_1 \). Traditionally, the polynomials \( L_n(x) = (-1)^n p_n(x) \) (with alternating leading coefficients) are called Laguerre polynomials, see (1.6), and they are also orthonormal w.r. to \( f_1(x) = e^{-x}, x > 0 \).

Consider now Problem 2 with \( a = 1 \). This reduces in finding the function

\[
u \in L^2_0 := \bigcap_{\lambda > 0} L^2((0, \infty), e^{-\lambda x})
\]

for which

\[
E_{\lambda} u(X) := \int_0^\infty \lambda \exp(-\lambda x)u(x)dx = h(\lambda), \quad \lambda > 0,
\]

provided that \( h(\lambda) \) allows such a construction. Theorem 2.2 provides a necessary and sufficient condition on \( h \), namely, \( h(\lambda) \) is holomorphic for \( \lambda \in \mathbb{C}^+ = \{\lambda \in \mathbb{C} : \Re(\lambda) > 0\} \) satisfying

\[
\sum_{n=0}^{\infty} \left( \frac{(-1)^n}{n!} \left( \lambda \frac{d^n}{d\lambda^n} \left[ \lambda^{\nu-1} h(\lambda) \right] \right) \right)^2 < \infty, \quad \lambda > 0.
\]

In view of Theorem 2.3, the preceding condition can be rewritten as

\[
\sum_{n=0}^{\infty} \left( \frac{(-1)^n H_n^{(\nu)}(0)}{n!} \right)^2 < \infty, \quad \lambda > 0,
\]

where \( H_n^{(\nu)}(y) = h(\lambda/(1 - y)), |y| < 1 \).

It is obvious that the equation \( E_{\lambda} u(X) = h(\lambda) \) can be written in terms of the Laplace transform of \( u \), (1.1), as

\[
\lambda \phi(\lambda) = \int_0^\infty \lambda \exp(-\lambda x)u(x)dx = E_{\lambda} u(X) = h(\lambda).
\]

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Hence, given the (holomorphic in \( C^+ \)) Laplace transform \( \phi \), one can check the validity of either (3.2) or (3.3) for \( h(\lambda) := \lambda \phi(\lambda) \), in order to ensure that the inverse function \( u(x) \) of \( \phi(\lambda) \) exists in \( L^2_0 \); if this is the case, then Theorem (2.4) applies and \( u \) is obtained as a Laguerre polynomial series with constants derived from the derivatives of \( \phi \).

Translating Theorems 2.2–2.4 to the Laplace-transform case, we obtain the following

**Theorem 3.1.** (A) Assume that \( \phi(\lambda) \) is an holomorphic function of \( \lambda \in C^+ \), such that

\[
\sum_{n=0}^{\infty} \left( \frac{1}{n!} \left( \lambda \frac{d^n}{d\lambda^n} [\lambda^n \phi(\lambda)] \right) \right)^2 < \infty, \quad \lambda > 0,
\]

or, equivalently,

\[
\sum_{n=0}^{\infty} \left( \frac{\Phi^{(n)}(0)}{n!} \right)^2 < \infty, \quad \lambda > 0,
\]

where

\[
\Phi_{\lambda}(y) = \frac{\lambda}{1-y} \phi\left(\frac{\lambda}{1-y}\right), \quad |y| < 1.
\]

Then, \( \phi \) is the Laplace transform of a function \( u \in L^2_0 \). Moreover, for every fixed \( \lambda_0 > 0 \), the inverse Laplace transform, \( u \), is given by

\[
u(x) = \sum_{n=0}^{\infty} \frac{\Phi^{(n)}(0)}{n!} L_n(\lambda_0 x),
\]

where the Laguerre polynomials \( L_n \) are given by (1.6). The series converges a.e. and in \( L^2(\mathbb{R}_+, e^{-\lambda x}) \) for every \( \lambda \geq \lambda_0 \), and the sum of the series does not depend on the particular choice of \( \lambda_0 \).

(B) If \( \phi \) is the Laplace transform of a function \( u \in L^2_0 \) then \( \phi \) is holomorphic in \( C^+ \) and satisfies (3.4) (equivalently, (3.5)).

Since the choice of \( \lambda_0 \) does not affect the validity of (3.7), we may set \( \lambda_0 = 1 \). Then, the function \( \Phi_{\lambda} \) in (3.6) reduces to \( \Phi_1(y) = (1-y)^{-1} \phi((1-y)^{-1}) = \Phi(y) \), say, and from (3.7) we obtain the (Taylor-like) Laplace inversion formula

\[
u(x) = \sum_{n=0}^{\infty} \frac{\Phi^{(n)}(0)}{n!} L_n(x), \quad \text{where} \quad \Phi(y) = \frac{1}{1-y} \phi\left(\frac{1}{1-y}\right),
\]

which is valid almost everywhere in \((0, \infty)\).

At this point we note that all inversion formulae of \( \phi(\lambda) \) provide approximating functions for \( u(x) \) in some sense. For instance, (3.8) says that

\[
u_N(x) := \sum_{n=0}^{N} \frac{\Phi^{(n)}(0)}{n!} L_n(x) \to u(x), \quad \text{a.e.,}
\]

while (1.4) can be written in our case as

\[
u_N(x) := \frac{1}{2\pi i} \int_{1-iN}^{1+iN} \exp(sx)\phi(s)ds \to u(x), \quad \text{a.e.,}
\]
and (1.5) reads as
\[ v_N(x) := \frac{(-1)^N}{N!} \left( \frac{N}{x} \right)^{N+1} \phi^{(N)}\left( \frac{N}{x} \right) \rightarrow u(x) \] at continuity points \( x \) of \( u(x) \).

Hence, it would be desirable to compare the degree of approximation of the preceding formulae; however, this is beyond the scope of the present work. We merely point out a possible advantage of the new inversion formula: The approximating functions \( u_N \) in (3.9) are polynomials, and the formula becomes exact for any polynomial \( u \) when \( N \geq \deg(u) \).

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