A note on the Neumann eigenvalues of the biharmonic operator

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We study the dependence of the eigenvalues of the biharmonic operator subject to Neumann boundary conditions on the Poisson’s ratio $\sigma$. In particular, we prove that the Neumann eigenvalues are Lipschitz continuous with respect to $\sigma \in [0,1]$ and that all the Neumann eigenvalues tend to zero as $\sigma \to 1^-$. Moreover, we show that the Neumann problem defined by setting $\sigma = 1$ admits a sequence of positive eigenvalues of finite multiplicity that are not limiting points for the Neumann eigenvalues with $\sigma \in [0,1]$ as $\sigma \to 1^-$ and that coincide with the Dirichlet eigenvalues of the biharmonic operator. Copyright © 2016 John Wiley & Sons, Ltd

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ (i.e., a bounded connected open set) of class $C^{2,\alpha}$ for some $\alpha \in ]0,1[$. Let $\sigma \in [0,1]$. We consider the Neumann eigenvalue problem for the biharmonic operator, namely, the problem

\begin{align}
\Delta^2 u = \lambda u, & \quad \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = 0, & \quad \text{on } \partial\Omega, \\
\frac{\partial u}{\partial n} + (1 - \sigma) \text{div}_{\Omega} (D^2 u \cdot v)_{\partial\Omega} = 0, & \quad \text{on } \partial\Omega,
\end{align}

(1)

in the unknowns $u$ (the eigenfunction) and $\lambda$ (the eigenvalue). Here, $\nu$ is the outer unit normal to $\partial\Omega$, $\text{div}_{\Omega} F$ denotes the tangential divergence of a vector field $F$, which is defined by $\text{div}_{\Omega} F = \text{div}_{\Omega}(DF \cdot v) - (DF \cdot v)_{\partial\Omega}$, $F_{\partial\Omega}$ denotes the projection of a vector field $F$ onto the tangent space to $\partial\Omega$, and $D^2 u$ is the Hessian of $u$ (we refer to [1] for the derivation of the boundary conditions in (1)). For $N = 2$, this problem is related to the study of the transverse vibrations of a thin plate with a free edge and which occupies at rest a planar region of shape $\Omega$. The coefficient $\sigma$ represents the Poisson’s ratio of the material that the plate is made of. We refer, for example, to [2] for more details on the physical interpretation of problem (1) and on the Poisson’s ratio $\sigma$. We mention the paper [3], where the authors study the dependence of the vibrational modes of a plate subject to homogeneous boundary conditions upon the Poisson’s ratio $\sigma \in ]0,\frac{1}{2}[$, providing also a perturbation formula for the frequencies as functions of the Poisson’s coefficient.

We note that eigenvalue problems for the biharmonic operator have gained significant attention in the last decades. In particular, there are several papers concerning the dependence of the eigenvalues upon different parameters that enter the problem, such as the shape or the coefficients. We refer to the book [4] for more information on shape optimization problems for the biharmonic operator. We also refer to [5], where it is discussed the dependence of the eigenvalues of polyharmonic operators upon variation of the mass density, and to [6] where the authors consider Neumann and Steklov-type eigenvalue problems for the biharmonic operator with particular attention to shape optimization and mass concentration phenomena. We also mention [7], where the author considers the shape sensitivity problem for the eigenvalues of the biharmonic operator (in particular, also those of problem (1)) for $\sigma \in ]\frac{1}{N+1},1[$. We note that other issues have been addressed in the literature for polyharmonic operators, such as analyticity, continuity, and stability estimates for the eigenvalues with respect to the shape; we refer to [8–13] and the references therein.

We recall that problem (1) admits an infinite sequence of nonnegative eigenvalues of finite multiplicity that depend on $\sigma \in [0,1]$ and that we denote here by

$$0 = \lambda_1(\sigma) = \lambda_2(\sigma) = \cdots = \lambda_{N+1}(\sigma) < \lambda_{N+2}(\sigma) \leq \cdots \leq \lambda_j(\sigma) \leq \cdots.$$
We note that $\lambda = 0$ is an eigenvalue of (1) of multiplicity $N + 1$, and a set of linearly independent eigenfunctions associated with $\lambda = 0$ is given by $\{1, x_1, \ldots, x_N\}$.

If we set $\sigma = 1$, problem (1) reads

$$
\begin{align*}
\Delta^2 u &= \lambda u, \quad \text{in } \Omega, \\
\Delta u &= 0, \quad \text{on } \partial \Omega, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \partial \Omega.
\end{align*}
$$

We note that the differential operator associated with problem (2) is not a Fredholm operator. Indeed, all the harmonic functions in $\Omega$ are eigenfunctions corresponding to the eigenvalue $\lambda = 0$. We also note that the boundary conditions in (2) do not satisfy the so-called complementing conditions (see [14, §10] and [15] for details), which are necessary conditions for the well-posedness of a differential problem. Nevertheless, problem (2) admits a countable number of positive eigenvalues of finite multiplicity diverging to $+\infty$, which we denote here by

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots.$$ 

In this paper, we show that $\lambda_j(\sigma) \to 0$ as $\sigma \to 1^-$ for all $j \in \mathbb{N}$. Thus, the positive eigenvalues of problem (2) are not limiting points for the eigenvalues of problem (1) as $\sigma \to 1^-$. Moreover, we show that the positive eigenvalues $\lambda_j$ of problem (2) coincide with the eigenvalues of the Dirichlet problem for the biharmonic operator, namely, the problem

$$
\begin{align*}
\Delta^2 w &= \mu w, \quad \text{in } \Omega, \\
w &= 0, \quad \text{on } \partial \Omega, \\
\frac{\partial w}{\partial n} &= 0, \quad \text{on } \partial \Omega.
\end{align*}
$$

We recall that, for $N = 2$, problem (3) models the transverse vibrations of a thin plate that has a clamped edge (see, e.g., [2] for details). We also recall that the eigenvalues of (3) are positive and of finite multiplicity and form an increasing sequence diverging to $+\infty$, which we denote here by

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_j \leq \cdots.$$ 

The present paper is organized as follows: in Section 2, we characterize the eigenvalues of problems (1), (2), and (3). In Section 3, we prove that all the eigenvalues of problem (1) go to zero as $\sigma \to 1^-$, and moreover, we prove that $\lambda_j = \mu_j$ for all $j \in \mathbb{N}$. Finally, in Section 4, we consider problems (1), (2), and (3) in the case of the unit ball in $\mathbb{R}^N$ centered at zero, where it is possible to recover the results of Section 3 thanks to explicit computations.

### 2. Eigenvalues of Neumann and Dirichlet problems

We consider problems (1), (2), and (3) in their weak formulation. The weak formulation of problem (1) when $\sigma \in [0, 1]$ is

$$
\int_{\Omega} (1 - \sigma)D^2 u : D^2 \varphi + \sigma \Delta u \varphi \, dx = \lambda \int_{\Omega} u \varphi \, dx,
$$

for all $\varphi \in H^2(\Omega)$, in the unknowns $u \in H^2(\Omega)$, $\lambda \in \mathbb{R}$, where $D^2 u : D^2 \varphi = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ denotes the Frobenius product. Actually, we will recast problem (5) in $H^2(\Omega)/\mathcal{N}$, where $\mathcal{N} \subset H^2(\Omega)$ is the subspace of $H^2(\Omega)$ generated by the functions $\{1, x_1, \ldots, x_N\}$. To do so, we set

$$H^2_N(\Omega) := \left\{ u \in H^2(\Omega) : \int_{\Omega} u \, dx = \int_{\Omega} \frac{\partial u}{\partial x_i} \, dx = 0, \forall i = 1, \ldots, N \right\}.$$ 

In the sequel, we will think of the space $H^2_N(\Omega)$ as endowed with the bilinear form given by the left-hand side of (5). From the fact that $|D^2 u|^2 \geq \frac{1}{\beta} (\Delta u)^2$ for all $u \in H^2(\Omega)$ and from the Poincaré–Wirtinger inequality, it follows that such bilinear form defines on $H^2_N(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. We denote by $\pi_N$, the map from $H^2(\Omega)$ to $H^2_N(\Omega)$ defined by

$$\pi_N[u] := u - \frac{1}{|\Omega|} \int_{\Omega} u + \frac{1}{|\Omega|^2} \sum_{i=1}^N \left( \int_{\Omega} \frac{\partial u}{\partial x_i} \, dx \right) x_i,$$

for all $u \in H^2(\Omega)$. We denote by $\pi_N^p$, the map from $H^2(\Omega)/\mathcal{N}$ onto $H^2_N(\Omega)$ defined by the equality $\pi_N^p = \pi_N \circ p$, where $p$ is the canonical projection of $H^2(\Omega)$ onto $H^2(\Omega)/\mathcal{N}$. The map $\pi_N^p$ turns out to be a continuous isomorphism. Let $F(\Omega)$ be defined by

$$F(\Omega) := \{ G \in H^2(\Omega)' : G[1] = G[x] = 0, \forall i = 1, \ldots, N \}.$$ 

Then, we consider the operator $\mathcal{P}_\sigma$, as an operator from $H^2_N(\Omega)$ to $F(\Omega)$ defined by

$$\mathcal{P}_\sigma[u][\varphi] := \int_{\Omega} (1 - \sigma)D^2 u : D^2 \varphi + \sigma \Delta u \varphi \, dx, \quad \forall u \in H^2_N(\Omega), \varphi \in H^2(\Omega).$$

It turns out that $\mathcal{P}_\sigma$ is a continuous isomorphism of $H^2_N(\Omega)$ onto $F(\Omega)$. We denote by $\mathcal{J}$ the continuous embedding of $L^2(\Omega)$ into $H^2(\Omega)'$ defined by

$$\mathcal{J}[u][\varphi] := \int_{\Omega} u \varphi \, dx, \quad \forall u \in L^2(\Omega), \varphi \in H^2(\Omega).$$
Finally, we define the operator $T_\alpha$ acting on $H^2(\Omega)/\mathcal{N}$ as follows:

$$T_\alpha = (\pi_\alpha)^{(-1)} \circ J^{-1} \circ i \circ \pi_\alpha^\mathcal{N},$$

where $i$ denotes the embedding of $H^2(\Omega)$ into $L^2(\Omega)$.

**Lemma 6**

The pair $(\lambda, u)$ of the set $(\mathbb{R} \setminus \{0\}) \times (H^2(\Omega) \setminus \{0\})$ satisfies (5) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times (H^2(\Omega)/\mathcal{N} \setminus \{0\})$ satisfies the equation $\lambda^{-1} p[u] = T_\alpha p[u]$.

We have the following theorem.

**Theorem 7**

The operator $T_\alpha$ is a nonnegative compact self-adjoint operator in $H^2(\Omega)/\mathcal{N}$, whose eigenvalues coincide with the reciprocals of the positive eigenvalues of problem (5). In particular, the set of eigenvalues of problem (5) is contained in $[0, +\infty[$ and consists of a sequence increasing to $+\infty$, and each eigenvalue has finite multiplicity. Moreover, the first eigenvalue is $\lambda = 0$ and has multiplicity $N + 1$, and a set of linearly independent eigenfunctions corresponding to $\lambda = 0$ is given by $\{1, x_1, \ldots, x_N\}$.

**Proof**

It is easy to prove that the operator $T_\alpha$ is self-adjoint. The compactness of the operator $T_\alpha$ follows from the compactness of the embedding $i$. The last statement is straightforward.

In an analogous way, it is possible to show that the eigenvalues of (3) are positive and of finite multiplicity. In fact, the weak formulation of problem (3) reads: find $(u, \lambda) \in H^2(\Omega) \times \mathbb{R}$ such that $u$ solves equation $\int_\Omega \Delta u \Delta \varphi dx = \lambda \int_\Omega u \varphi dx$ for all $\varphi \in H^2(\Omega)$. We note that this is equivalent to finding $(u, \lambda) \in H^2(\Omega) \times \mathbb{R}$ such that Equation (5) holds for all $\varphi \in H^2(\Omega)$. From the Poincaré inequality, it follows that the bilinear form given by the left-hand side of (3) defines on $H^2(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. Therefore, the analogous of Theorem 7 holds; hence, the eigenvalues of problem (3) are positive and can be represented by means of an infinite sequence diverging to $+\infty$ of the form (4), and the corresponding eigenfunctions form an orthonormal basis of $H^2(\Omega)$.

Finally, we show that problem (2) admits an infinite sequence of positive eigenvalues. We have already observed that all harmonic functions in $H^2(\Omega)$ are eigenfunctions corresponding to the eigenvalue $\lambda = 0$. We start by recalling the following direct decomposition of the space $H^2(\Omega)$ (see [16, Theorem 4.7] for details):

$$H^2(\Omega) = H^2_0(\Omega) \oplus \Delta(H^2(\Omega) \cap H^2_0(\Omega)),$$

where $H^2_0(\Omega) = \{h \in H^2(\Omega) : \Delta h = 0\}$ is the space of harmonic functions in $H^2(\Omega)$.

In order to characterize the positive eigenvalues of problem (2) and to get rid of the harmonic functions that are the eigenfunctions associated with $\lambda = 0$, we will obtain a problem in $\Delta(H^2(\Omega) \cap H^2_0(\Omega))$. Thus, we consider the following weak formulation of problem (2) for $\lambda \neq 0$:

$$\int_\Omega \Delta^2 u \Delta \varphi dx = \lambda \int_\Omega \Delta u \Delta \varphi, \quad \forall u, \varphi \in H^2(\Omega) \cap H^2_0(\Omega),$$

(8)

in the unknowns $u \in H^2(\Omega) \cap H^2_0(\Omega), \lambda \in \mathbb{R}$. (In the case $\lambda = 0$, the solutions of (2) are exactly the harmonic functions in $H^2(\Omega)$.)

We note that there exists a constant $C > 0$ such that $\int_\Omega \Delta^2 u \Delta \varphi dx \leq C\|u\|_{H^2(\Omega)} \|\varphi\|_{H^2(\Omega)}$ and $\|u\|_{H^2(\Omega)} \leq C\|\Delta^2 u\|_{L^2(\Omega)}$ for all $u, \varphi \in H^2(\Omega) \cap H^2_0(\Omega)$ (the second inequality follows from standard elliptic regularity for the Dirichlet problem for the biharmonic operator and from the regularity assumptions of $\Omega$; see [15, Theorem 2.20] for details). Therefore, the bilinear form given by the left-hand side of (8) defines on $H^2(\Omega) \cap H^2_0(\Omega)$ a scalar product whose induced norm is equivalent to the standard norm of $H^2(\Omega)$. Thus, the analogue of Theorem 7 holds.

**Theorem 9**

The set of eigenvalues of problem (2) is contained in $[0, +\infty[$. The eigenspace corresponding to the eigenvalue $\lambda = 0$ has infinite dimension, and all harmonic functions in $H^2(\Omega)$ are eigenfunctions associated with $\lambda = 0$. Moreover, the set of positive eigenvalues consists of a sequence increasing to $+\infty$. Each positive eigenvalue has finite multiplicity, and the corresponding eigenfunctions form an orthonormal basis of $\Delta(H^2(\Omega) \cap H^2_0(\Omega))$.

### 3. Dependence of the Neumann eigenvalues upon the Poisson’s ratio

In the first part of this section, we consider the behavior of the eigenvalues of problem (1) as $\sigma \to 1^-$. In the second part, we show that the positive eigenvalues of problem (2) and the eigenvalues of problem (3) coincide. We start with the following theorem.

**Theorem 10**

For all $j \in \mathbb{N}$, it holds $\lim_{\sigma \to 1^-} \lambda_j(\sigma) = 0$. Moreover, the function $\lambda_j$ from $[0, 1]$ to $\mathbb{R}$, which maps $\sigma \in [0, 1]$ to $\lambda_j(\sigma)$ and extended at $\sigma = 1$ by setting $\lambda_j(1) = 0$, is Lipschitz continuous on $[0, 1]$.

**Proof**

The proof is divided into three steps. In the first step, we prove that $\lim_{\sigma \to 1^-} \lambda_j(\sigma) = 0$ for all $j \in \mathbb{N}$. In the second step, we prove that $\lambda_j(\sigma)$ is locally Lipschitz continuous on $[0, 1]$. In the third step, we prove that the function $\lambda_j(\sigma)$ extended with continuity at $\sigma = 1$ is Lipschitz continuous in a neighborhood of $\sigma = 1$. 

Copyright © 2016 John Wiley & Sons, Ltd Mathematical Methods in the Applied Sciences.
Step 1. We recall that for each \( \sigma \in [0, 1] \), we have the following formula for \( \lambda_j(\sigma) \):

\[
\lambda_j(\sigma) = \inf_{E \in \mathcal{P}(\Omega)} \sup_{u \in E} \frac{\int_\Omega (1 - \sigma) |D^2 u|^2 + \sigma (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx}.
\]

(11)

We also recall that the space \( H^2_{x}(\Omega) \) is closed in \( H^2(\Omega) \), and therefore, it is a Hilbert space, endowed with the standard scalar product of \( H^2(\Omega) \). Let \( \{u_i\}_{i=1}^\infty \) be a set of linearly independent functions in \( H^2_{x}(\Omega) \) such that \( \int_\Omega u_i u_k = \delta_{ik} \) for all \( i, k \in \mathbb{N} \). Then, from (11), we have that for all \( j \in \mathbb{N} \) it holds

\[
\lambda_j(\sigma) \leq \sup_{c_1, \ldots, c_j \in \mathbb{R}} \frac{(1 - \sigma) \int_\Omega \left( \sum_{i=1}^j c_i D^2 u_i \right)^2 \, dx}{\int_\Omega \left( \sum_{i=1}^j c_i u_i \right)^2 \, dx},
\]

where we have chosen as \( j \)-dimensional space \( E \) in (11) the space generated by \( \{u_1, \ldots, u_j\} \). Then, we have

\[
\sup_{c_1, \ldots, c_j \in \mathbb{R}} \frac{(1 - \sigma) \int_\Omega \left( \sum_{i=1}^j c_i D^2 u_i \right)^2 \, dx}{\int_\Omega \left( \sum_{i=1}^j c_i u_i \right)^2 \, dx} \leq \sup_{c_1, \ldots, c_j \in \mathbb{R}} j(1 - \sigma) \frac{\sum_{i=1}^j c_i^2 \int_\Omega |D^2 u_i|^2 \, dx}{\sum_{i=1}^j c_i^2 \int_\Omega |D^2 u|_i \, dx} \leq j(1 - \sigma) \max_{i=1, \ldots, j} \int_\Omega |D^2 u_i|^2 \, dx,
\]

(12)

and therefore,

\[
\lim_{\sigma \to 1} \lambda_j(\sigma) = 0,
\]

(13)

for all \( j \in \mathbb{N} \).

Step 2. For each \( \sigma_1, \sigma_2 \in [0, 1] \) and \( u \in H^2(\Omega) \), we have

\[
\left| \frac{\int_\Omega (1 - \sigma_1) |D^2 u|^2 + \sigma_1 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} - \frac{\int_\Omega (1 - \sigma_2) |D^2 u|^2 + \sigma_2 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} \right| \leq |\sigma_1 - \sigma_2| \frac{\int_\Omega |D^2 u|^2 + (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} \leq (1 + N)|\sigma_1 - \sigma_2| \frac{\int_\Omega |D^2 u|^2 + (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx}.
\]

(14)

From (14), it follows that

\[
\frac{\int_\Omega (1 - \sigma_1) |D^2 u|^2 + \sigma_2 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} \left( 1 - (1 + N)|\sigma_1 - \sigma_2| \frac{1}{1 - \sigma_2} \right) \leq \frac{\int_\Omega (1 - \sigma_1) |D^2 u|^2 + \sigma_1 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} \leq \frac{\int_\Omega (1 - \sigma_2) |D^2 u|^2 + \sigma_2 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} \left( 1 + (1 + N)|\sigma_1 - \sigma_2| \frac{1}{1 - \sigma_2} \right).
\]

(15)

If \( \sigma_1, \sigma_2 \) satisfy \( (1 + N)|\sigma_1 - \sigma_2| < 1 - \sigma_2 \), then taking the infimum and the supremum in (15) yields

\[
|\lambda_j(\sigma_1) - \lambda_j(\sigma_2)| \leq (1 + N) \frac{\lambda_j(\sigma_2)}{1 - \sigma_2} |\sigma_1 - \sigma_2|.
\]

By repeating the same arguments earlier, it is possible to prove that

\[
|\lambda_j(\sigma_1) - \lambda_j(\sigma_2)| \leq (1 + N) \frac{\lambda_j(\min \{\sigma_1, \sigma_2\})}{1 - \min \{\sigma_1, \sigma_2\}} |\sigma_1 - \sigma_2|,
\]

(16)

for all \( \sigma_1, \sigma_2 \) satisfying \( (1 + N)|\sigma_1 - \sigma_2| < 1 - \min \{\sigma_1, \sigma_2\} \). Then, the function \( \lambda_j(\sigma) \) is locally Lipschitz on \([0, 1]\).

We note that from (16) it follows that for all \( \epsilon \in (0, 1] \), the function \( \lambda_j(\sigma) \) is Lipschitz continuous on \([0, 1 - \epsilon]\). Moreover, from (13), it follows that the function \( \lambda_j(\sigma) \) can be extended with continuity at \( \sigma = 1 \) by setting \( \lambda_j(1) := 0 \).

Step 3. Now, we prove that the function \( \lambda_j(\sigma) \) extended with continuity at \( \sigma = 1 \) is Lipschitz on \([0, 1]\). We note that (16) does not allow to prove that \( \lambda_j(\sigma) \) is Lipschitz in a neighborhood of \( \sigma = 1 \). We need a refined estimate for \( |\lambda_j(\sigma_1) - \lambda_j(\sigma_2)| \) near \( \sigma = 1 \). Let \( \sigma_1, \sigma_2 \in \left[ \frac{1}{2}, 1 \right] \). By using the same arguments of step 2, we have that

\[
\left| \frac{\int_\Omega (1 - \sigma_1) |D^2 u|^2 + \sigma_1 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} - \frac{\int_\Omega (1 - \sigma_2) |D^2 u|^2 + \sigma_2 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} \right| \leq |\sigma_1 - \sigma_2| \frac{\int_\Omega |D^2 u|^2 + (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx} \leq |\sigma_1 - \sigma_2| \frac{\int_\Omega (1 - \sigma_1) |D^2 u|^2 + \sigma_1 (\Delta u)^2 \, dx}{\int_\Omega u^2 \, dx},
\]

for \( i = 1, 2 \). Hence, from the same arguments of step 2, we deduce that

\[
|\lambda_j(\sigma_1) - \lambda_j(\sigma_2)| \leq \frac{\lambda_j(\sigma_1)}{1 - \sigma_i} |\sigma_1 - \sigma_2|,
\]
for all $\sigma_1, \sigma_2 \in [\frac{1}{2}, 1]$ with $|\sigma_1 - \sigma_2| < 1 - \sigma_i$, for $i = 1, 2$. In particular, we note that if $\sigma_1 > \sigma_2$, then $|\sigma_1 - \sigma_2| < 1 - \sigma_2$. Therefore,

$$\left| \lambda_j(\sigma_1) - \lambda_j(\sigma_2) \right| \leq \frac{\lambda_j(\min \{\sigma_1, \sigma_2\})}{1 - \min \{\sigma_1, \sigma_2\}} |\sigma_1 - \sigma_2|,$$

(17)

for all $\sigma_1, \sigma_2 \in [\frac{1}{2}, 1]$. Moreover, from (12), it follows that there exists a constant $C_j$ that does not depend on $\sigma$, such that

$$\lambda_j(\sigma) \leq C_j(1 - \sigma),$$

(18)

for all $\sigma \in [0, 1]$. From (17) and (18), it follows that

$$\left| \lambda_j(\sigma_1) - \lambda_j(\sigma_2) \right| \leq C_j|\sigma_1 - \sigma_2|,$$

for all $\sigma_1, \sigma_2 \in [\frac{1}{2}, 1]$. Then, $\lambda_j(\sigma)$ is Lipschitz in a neighborhood of $\sigma = 1$; hence, it is Lipschitz on $[0, 1]$. This concludes the proof of the theorem.

Thus, the positive eigenvalues of problem (2) are not limiting points for the eigenvalues of problem (1) as $\sigma \to 1$.

Now, we consider problems (2) and (3). We note that, under the assumptions that $\Omega$ is of class $C^{4,\alpha}$, we have that the eigenfunctions $w$ of problem (3) are of class $C^{4,\alpha}(\Omega)$ (see [15, Theorem 2.20]). We have the following theorem.

**Theorem 19**

All the positive eigenvalues of problem (2) coincide with the eigenvalues of problem (3).

**Proof**

Let $\mu$ be an eigenvalue of problem (3) and let $w \in H^2_0(\Omega)$ be an eigenfunction associated with $\mu$. Let $\nu_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ be the unique solution of

$$\begin{align*}
\Delta \nu_0 &= w, & \text{in } \Omega, \\
\nu_0 &= 0, & \text{on } \partial\Omega.
\end{align*}$$

We set $\nu_h = \nu_0 + h$ for some harmonic function $h$. Now, we consider the following problem: find a harmonic function $h$ such that

$$\begin{align*}
\Delta^2 h &= \mu \nu_h, & \text{in } \Omega, \\
\Delta \nu_h &= 0, & \text{on } \partial\Omega, \\
\frac{\partial \Delta \nu_h}{\partial n} &= 0, & \text{on } \partial\Omega.
\end{align*}$$

Clearly, $\Delta \nu_h|_{\partial\Omega} = \frac{\partial \Delta \nu_h}{\partial n}|_{\partial\Omega} = 0$ for all harmonic functions $h$. As for the differential equation, we have $\Delta^2 (\nu_0 + h) = \mu (\nu_0 + h)$ if and only if $\Delta (\Delta \nu_0 + \Delta h) = \mu (\nu_0 + h)$, that is $\Delta \nu = \mu (\nu_0 + h)$ and therefore $h = \frac{\Delta \nu}{\mu} - \nu_0$, which is clearly harmonic and belongs to $H^2(\Omega)$. Therefore, each eigenvalue $\mu$ of problem (3) is an eigenvalue of problem (2), and a corresponding eigenfunction is given by $\nu = \frac{\Delta \nu}{\mu}$. On the other hand, suppose that $\lambda > 0$ is an eigenvalue of problem (2) and let $u \in \Delta H^2(\Omega) \cap H^2_0(\Omega))$ be a corresponding eigenfunction. Then, the function $w = \Delta u$ is in $H^2_0(\Omega)$ and solves

$$\begin{align*}
\Delta^2 w &= \lambda w, & \text{in } \Omega, \\
w &= 0, & \text{on } \partial\Omega, \\
\frac{\partial w}{\partial n} &= 0, & \text{on } \partial\Omega;
\end{align*}$$

therefore, $\lambda$ is an eigenvalue of problem (3) with corresponding eigenfunction $\Delta u$.

**4. Neumann and Dirichlet eigenvalues in the case of the unit ball**

In this section, we consider problems (1), (2), and (3) when $\Omega = B$ is the unit ball in $\mathbb{R}^N$ centered at zero. In this case, it is possible to perform explicit computations that allow to recast the eigenvalue problems (1), (2), and (3) into suitable equations of the form $F(\lambda) = 0$ and then gather information on the behavior of the eigenvalues.

It is convenient to use the standard spherical coordinates $(r, \theta) \in \mathbb{R}_+ \times \partial B$ in $\mathbb{R}^N$. We refer, for example, to [17] for more details on spherical coordinates in $\mathbb{R}^N$. We denote by $\Delta_S$ the Laplace–Beltrami operator on the unit sphere $\partial B$ of $\mathbb{R}^N$. We denote by $H_m(\theta)$ a spherical harmonic of order $m \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We recall that for all $m \in \mathbb{N}_0$, $H_m$ is a solution of the equation $-\Delta_S H_m = m(m + N - 2)H_m$.

As customary, for $m \in \mathbb{N}_0$, we denote by $f_m$ and $l_m$ the ultraspherical and modified ultraspherical Bessel functions of the first species and order $m$, respectively, which are defined by

$$f_m(z) = z^{1 - \frac{m}{2}} J_{\frac{m}{2} - 1}(z), \quad l_m(z) = z^{1 - \frac{m}{2}} I_{\frac{m}{2} - 1}(z),$$

where $J_j(z)$ and $I_j(z)$ are the Bessel and modified Bessel functions of the first species and order $j$, respectively (see [18, §9] for details).

We consider first problem (3) on $B$. For the convenience of the reader, we recall a result from [1].
Lemma 20
Given an eigenvalue $\mu$ of problem (3) on $B$, a corresponding eigenfunction $w$ is of the form $w(r, \theta) = W_m(r)H_m(\theta)$, for some $m \in \mathbb{N}_0$, where
\[
W_m(r) = \alpha j_m(\sqrt{\lambda}r) + \beta i_m(\sqrt{\lambda}r),
\] (21)
for suitable $\alpha, \beta \in \mathbb{R}$.
We establish now an implicit characterization of the eigenvalues of (3) on $B$.

Lemma 22
The eigenvalues $\mu$ of problem (3) on $B$ are given implicitly as zeroes of the equation
\[
j_m(\sqrt{\lambda})j_m'(\sqrt{\lambda}) - i_m(\sqrt{\lambda})i_m'(\sqrt{\lambda}) = 0.
\] (23)

Proof
By Lemma 20, an eigenfunction $w$ associated with an eigenvalue $\mu$ is of the form $w(r, \theta) = W_m(r)H_m(\theta)$, where $W_m(r)$ is given by (21). We recall that in spherical coordinates, the Dirichlet boundary conditions are written as
\[
w_{r_{n-1}} = \partial_r w_{r_{n-1}} = 0.
\]
By imposing boundary conditions to $w(r, \theta)$, we obtain a homogeneous system of two equations in two unknowns $\alpha$ and $\beta$, which has solutions if and only if its determinant vanishes. This yields formula (23).

Now, we consider problem (1) on $B$. For the convenience of the reader, we recall the following result from [1].

Lemma 24
Given an eigenvalue $\lambda$ of problem (1) with $\sigma \in [0, 1]$ on $B$, a corresponding eigenfunction $u$ is of the form $u(r, \theta) = U_m(r)H_m(\theta)$, for some $m \in \mathbb{N}_0$, where
\[
U_m(r) = \alpha j_m(\sqrt{\lambda}r) + \beta i_m(\sqrt{\lambda}r),
\] (25)
for $\alpha, \beta \in \mathbb{R}$.
We have the following lemma on the eigenvalues of problem (1) on $B$.

Lemma 26
The eigenvalues $\lambda$ of problem (1) with $\sigma \in [0, 1]$ on $B$ are given implicitly as zeroes of the equation
\[
\det M(\lambda, \sigma) = 0,
\] (27)
where $M(\lambda, \sigma)$ is the $2 \times 2$ matrix defined by
\[
\begin{bmatrix}
\sqrt{\lambda}j_m''(\sqrt{\lambda}) + (N - 1) \sqrt{\lambda}j_m'(\sqrt{\lambda}) & \sqrt{\lambda}j_m''(\sqrt{\lambda}) + (N - 1) \sqrt{\lambda}j_m'(\sqrt{\lambda}) \\
-m(m + N - 2)\sigma j_m(\sqrt{\lambda}) & -m(m + N - 2)\sigma j_m(\sqrt{\lambda})
\end{bmatrix}
\begin{bmatrix}
\sqrt{\lambda}j_m''(\sqrt{\lambda}) + (N - 1) \sqrt{\lambda}j_m'(\sqrt{\lambda}) & \sqrt{\lambda}j_m''(\sqrt{\lambda}) + (N - 1) \sqrt{\lambda}j_m'(\sqrt{\lambda}) \\
-m(m + N - 2)(\sigma - 3)j_m(\sqrt{\lambda}) & -m(m + N - 2)(\sigma - 3)j_m(\sqrt{\lambda})
\end{bmatrix}.
\] (28)

Proof
By Lemma 24, an eigenfunction $u$ associated with an eigenvalue $\lambda$ is of the form $u(r, \theta) = U_m(r)H_m(\theta)$, where $U_m(r)$ is given by (25). We recall that in spherical coordinates, the Neumann boundary conditions are written as
\[
\begin{align*}
\{ (1 - \sigma)\partial_r^2 u + \sigma \Delta u &\}_{r_{n-1}} = 0, \\
\partial_r(\Delta u) + (1 - \sigma)\frac{1}{r}\Delta S (\partial_r u - \frac{u}{r}) &\}_{r_{n-1}} = 0;
\end{align*}
\]
see [1] for details. By imposing boundary conditions to the function $u$, we obtain a system of two equations in two unknowns $\alpha$ and $\beta$, and the associated matrix is given by (28). Thus, the eigenvalues must solve Equation (27).

We give now an alternative proof of Theorem 19 when $\Omega = B$ is the unit ball in $\mathbb{R}^N$ centered at zero based on the explicit representations of the eigenvalues discussed in this section. We have the following theorem.

Theorem 29
Equations $\det M(\lambda, 1) = 0$ and (23) admit the same nonzero solutions.
Proof
We consider (28) with \( \sigma = 1 \). Let \( \lambda > 0 \) be a solution of \( \det M(\lambda, 1) = 0 \). We compute \( F(\lambda) = \det M(\lambda, 1) \). We have

\[
F(\lambda) = -\sqrt{\lambda}m(N + m - 2)(N + m - 1) \left( j_m(\sqrt{\lambda}) i'_m(\sqrt{\lambda}) - i_m(\sqrt{\lambda}) j'_m(\sqrt{\lambda}) \right) \\
+ \sqrt{\lambda}m(N + 1)(m + N - 2) \left( j'_m(\sqrt{\lambda}) i''_m(\sqrt{\lambda}) - i'_m(\sqrt{\lambda}) j''_m(\sqrt{\lambda}) \right) \\
- \lambda^{3/4} (N(N - 1) + m(N + m - 2)) \left( j'_m(\sqrt{\lambda}) i''_m(\sqrt{\lambda}) - i'_m(\sqrt{\lambda}) j''_m(\sqrt{\lambda}) \right) \\
+ \lambda^{3/4} (m + N - 2)(m + N - 1) \left( j'_m(\sqrt{\lambda}) i''_m(\sqrt{\lambda}) - i'_m(\sqrt{\lambda}) j''_m(\sqrt{\lambda}) \right)
\]

(30)

We set \( C^\pm_m(z) = I_{2m + 1}(z) J_{2m - 1} \pm I_{2m - 1}(z) J_{2m + 1} \). We use the well-known recurrence formulas for Bessel functions and their derivatives (see [18, 9.1.27 and 9.6.26]) to acquire

\[
j_m(z) i'_m(z) - i_m(z) j'_m(z) = z^{1-N} C^+_m(z),
\]

(31)

\[
j'_m(z) i''_m(z) - i'_m(z) j''_m(z) = z^{1-N} \left( 2z I_{2m + 1}(z) J_{2m - 1} - (N - 1) C^+_m(z) \right),
\]

(32)

\[
j''_m(z) i'''_m(z) - i''_m(z) j'''_m(z) = z^{-N} \left( z^2 C^+_m(z) + 2mz I_{2m + 1}(z) J_{2m - 1} - m(m + N - 2) C^+_m(z) \right),
\]

(33)

\[
j''_m(z) i'''_m(z) - i''_m(z) j'''_m(z) = z^{-N} \left( z^2 C^-_m(z) + 2(1 - N + m)z I_{2m + 1}(z) J_{2m - 1} + (N - 1)(m + N - 2) C^+_m(z) \right) \\
+ (N(N - 1) + m(m + N - 2)) C^-_m(z),
\]

(34)

\[
j''_m(z) i'''_m(z) - i''_m(z) j'''_m(z) = z^{-1-N} \left( -2z I_{2m + 1}(z) J_{2m - 1} + (1 - N + 2m)z^2 C^-_m(z) \\
+ 2m(1 - N + m)z I_{2m + 1}(z) J_{2m - 1} + m(m + N - 2)(N + 1) C^+_m(z) \right),
\]

(35)

\[
j''_m(z) i'''_m(z) - i''_m(z) j'''_m(z) = z^{-2-N} \left( -z^2 C^+_m(z) + 2(N - 1)z I_{2m + 1}(z) J_{2m - 1} - (N + 1)(2m + 1)z^2 C^-_m(z) \\
- 2(N - 3)(m - 1)mz I_{2m + 1}(z) J_{2m - 1} + m(m - 1)(m + N - 2)(m + N - 1) C^+_m(z) \right).
\]

(36)

Thanks to (31)–(36), expression (30) simplifies to

\[
F(\lambda) = \lambda^{3/4} \left( j_m(\sqrt{\lambda}) i'_m(\sqrt{\lambda}) - i_m(\sqrt{\lambda}) j'_m(\sqrt{\lambda}) \right).
\]

(37)

Therefore, by comparing (37) with (23), we see that the nonzero eigenvalues of problem (2) and the eigenvalues of problem (3) on the unit ball coincide.

Figure 1. Solution branches of Equation (27) with \( N = 2 \) for \((\sigma, \lambda) \in [0, 1] \times [0, 500]\). The dashing refers to the choice of \( m \) in 27. In particular: black line \((m = 0)\), gray line \((m = 1)\), dashed line from \( m = 2 \) (longest segments) to \( m = 9 \) (shortest segments).

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Remark 38

From Theorem 10, it follows that all the eigenvalues \( \lambda(\sigma) \to 0 \) as \( \sigma \to 1^- \). This means that there are infinitely many branches of solutions \( \sigma \mapsto \lambda(\sigma) \) of Equation (27) such that \( \lambda(\sigma) \to 0 \) as \( \sigma \to 1^- \). Theorem 29 shows that there are also infinitely many branches \( \sigma \mapsto \lambda(\sigma) \) such that \( \lambda(\sigma) \to \mu \) as \( \sigma \to 1^- \), for some solution \( \mu > 0 \) to Equation (23) (Figure 1).

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