Quantum Field Theory Treatment of Neutrino Oscillations in Vacuum and in Matter

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(March 25, 2022)

Abstract

We study neutrino oscillations in vacuum and in matter using field theoretical methods and wave-packets. In particular, we calculate the neutrino propagator in the presence of matter with constant density for the case of two flavors, obtaining the resonance formula. In the extreme relativistic limit, the result of the usual quantum mechanical treatment is recovered with interesting, but small modifications.

I. INTRODUCTION

Evidence for neutrinos having small, but finite, masses is accumulating fast. The main paradigm for this is the idea, first proposed by Pontecorvo [1], that if the neutrino flavors were a superposition of massive eigenstates, then the neutrinos could oscillate between these different flavors. There are many experiments looking for neutrino oscillations: solar neutrino measurements, atmospheric neutrino measurements, reactor experiments, and accelerator experiments. Several of them have claimed evidence for neutrino oscillations. The final breakthrough came in June 1998 when the Super-Kamiokande collaboration reported strong evidence for neutrino oscillations from the atmospheric neutrino UP-DOWN asymmetry. The measurements on the depletion of atmospheric muon neutrino flux fit well to a two flavor $\nu_\mu \leftrightarrow \nu_\tau$ oscillation model. This summer the SNO collaboration [2] showed that the solar neutrino deficit, pioneered by Davies [3], was due to conversion of electron neutrinos to mu- and tau neutrinos. Recently, also the KamLAND experiment [4] shows evidence for neutrino oscillations.

In the standard quantum mechanical treatment of neutrino oscillations, the mass eigenstates are assumed to be relativistic and to have the same momentum, and thus, different energies. The familiar quantum mechanical model describing the flavor mixing process has

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several conceptual difficulties compared to quantum field theory models, see Refs. [5–9]. For example, energy momentum conservation in the production and detection processes that forces neutrinos to be in a mass eigenstate is incompatible with neutrino oscillations. The neutrino oscillation probability is independent of the details concerning the production and detection processes only in the extremely relativistic limit. Hence, in the case that some of the mass eigenstates cannot be considered to be extremely relativistic, one has to use quantum field theory. Of course, the quantum field theory expression must reproduce the quantum mechanics oscillation probability in the ultra-relativistic limit. A very detailed review regarding different aspects and questions of neutrino oscillations in quantum field theory can be found in Ref. [5]. Another interesting question which is important in both a quantum mechanical or quantum field theory treatment is if there exists a Fock space for the flavor eigenstates, since there exists a Fock space for the mass eigenstates. This question has been discussed by Giunti et al. [10]. Fuji et al. [11] give arguments that a Fock space of flavor neutrinos does not exist.

When neutrinos propagate through matter their behavior may be affected significantly, as was pointed out by Wolfenstein [12] in 1978 and emphasized by Mikheev and Smirnov [13] as being important for neutrino oscillations. This is due to the fact that in the presence of matter the effective mass induced by the forward scattering of neutrinos by the background changes the flavor oscillating parameters. A quantum mechanical treatment of neutrinos interacting with matter can be found in almost all books treating neutrinos. Peltoniemi and Sipiläinen [14] have studied neutrino propagation in matter using a wave packet approach. Cardall and Chung [15] treats neutrino oscillations in a static uniform background by quantum field theory and show that they recover the quantum mechanical oscillation amplitude in the relativistic limit. Also Fuji et al. [11] treats neutrino oscillations in a static matter background using Bogoliubov transformations.

In the present paper, we give a short presentation of the use of field theory methods to describe neutrino oscillations in vacuum and in matter. For neutrino oscillations in matter we carry out the calculations for the case of two flavors in detail, which displays the main features of our approach. Our aim is in particular to calculate explicitly the singularity structure of the neutrino Green’s function in the presence of matter to study the resonance formula.

In Sec. II, we review the basic formalism to be used. Section III contains an elaboration of the use of wave-packets for flavor states. We then calculate the oscillation amplitude for the case of vacuum oscillations in Sec. IV, and then, treat explicitly the case of two flavors in the presence of matter with constant density in Sec. V.

II. BASIC FORMALISM

The equation of motion for the neutrino field $\nu_j$, with mass $m_j$, can be written in the following form

$$ (i\slashed{D} - m_j)\nu_j(x) = \chi_j(x), \quad (1) $$

where we have introduced the source term $\chi_j(x)$, which can be obtained from the complete Lagrangian density of the standard electroweak theory in the unitary gauge. The interaction part of primary interest to us is given by
\[
\mathcal{L}_{\text{int}}(x) = \sum_{\beta=e,\mu,\tau} \left[ -\frac{g}{\sqrt{2}} \bar{\nu}_\beta(x) \gamma^\alpha \psi_{\beta L}(x) W_\alpha(x) - \frac{g}{\sqrt{2}} \bar{\nu}_{\beta L}(x) \gamma^\alpha \nu_{\beta L}(x) W'^\dagger_\alpha(x) + \frac{g}{2 \cos \theta_W} \bar{\psi}_{\beta L}(x) \gamma^\alpha \nu_{\beta L}(x) Z_\alpha(x) \right] - \sum_{j=1}^3 \frac{1}{v} m_{\nu_j} \bar{\nu}_j(x) \nu_j(x) \sigma,
\]

where \( g \) is a dimensionless coupling constant and \( W_\alpha(x) \) and \( Z_\alpha(x) \) are the fields that describes the \( W \) and \( Z \) bosons with spin 1. The last part contains the Higgs field \( \sigma \) and the electroweak vacuum expectation value \( v \). We observe that the interaction part contains the auxiliary flavor fields \( \nu_\beta \). These fields are expressed in terms of the mass eigenfields \( \nu_j \) by the relations

\[
\nu_\beta(x) = \sum_j U_{\beta j} \nu_j(x) \quad \text{and} \quad (\bar{\nu}_\beta(x) = \sum_j U^*_{\beta j} \bar{\nu}_j(x)),
\]

where \( U \) is unitary leptonic mixing matrix to be specified below. Originally the \( U \)-matrix is defined only for the left handed fields. For simplicity we will assume that the same transformation also applies for the right handed fields. Inserting the expressions for \( \nu_\beta \) and \( \bar{\nu}_\beta \) expressed in terms of the mass eigenfields into the interaction Lagrangian, one obtains

\[
\mathcal{L}_{\text{int}}(x) = \sum_{\beta=e,\mu,\tau} \sum_{j=1}^3 \left[ -\frac{g}{\sqrt{2}} \bar{U}^*_{\beta j} \bar{\nu}_j L(x) \gamma^\alpha \psi_{\beta L}(x) W_\alpha(x) + \frac{g}{2 \cos \theta_W} \bar{\psi}_{\beta L}(x) \gamma^\alpha U_{\beta j} \nu_j L(x) W'^\dagger_\alpha(x) \right] - \frac{g}{2 \cos \theta_W} \sum_{k=1}^3 \bar{\nu}_{k L}(x) \gamma^\alpha \nu_{k L}(x) Z_\alpha(x).
\]

We have omitted the part of the interaction Lagrangian containing the \( \sigma \) field in the Higgs’ mechanism. This Lagrangian gives

\[
\chi_j(x) = \sum_{\beta=e,\mu,\tau} \left[ \frac{g}{\sqrt{2}} U^*_{\beta j} \gamma^\alpha \psi_{\beta L}(x) W_\alpha(x) + \frac{g}{2 \cos \theta_W} \gamma^\alpha \nu_{j L}(x) Z_\alpha(x) \right],
\]

describing the fundamental processes that go on in the source and the detector. The first two terms in Eq. 3 define the flavor combinations through the intermediate boson interaction containing the \( U \) matrix components. This mixture in the interaction describes that the weak bosons create a coherent superposition of neutrino mass eigenfields at the vertex. At energy scales where the neutrino rest masses are small compared to their total energy, this can give rise to neutrino oscillations. We also observe that the different neutrino fields \( \nu_j(x) \) are coupled to each other via the neutral current interaction. At the energy scale we are going to consider, the amplitudes describing the detection processes, e.g. in the SNO detector, only appear with the second order terms in the coupling constants. These processes comprise the elastic scattering of neutrinos on electrons, the inverse beta decay and the deuteron break up. To describe these processes we use the effective second order Lagrangian, obtained by neglecting the momentum of the intermediate bosons. This means that we insert the weak hadronic currents coupled to the neutrinos with the Fermi coupling constant \( G_F \), and also use the analogue of this for the purely leptonic scattering in the Cherenkov detectors. Finally, we also use this effective Lagrangian to describe neutrino interactions with matter, neglecting recoil effects. The effective Lagrangian for neutrinos interacting with matter is
\[ \mathcal{L}(x)_{\text{matter}} = -\sqrt{2}G_F \sum_{j=1}^{3} U_{ej}^* \bar{\nu}_{jL}(x) \gamma^0 \nu_{jL}(x) U_{ej} N_e(x) + \frac{1}{\sqrt{2}} G_F \sum_{j=1}^{3} \bar{\nu}_{jL}(x) \gamma^0 \nu_{jL}(x) N_n(x), \] (5)

where \( N_e(x) \) and \( N_n(x) \) are the electron and neutron density operators, respectively. In our further discussion, we will only consider the case when these densities are constant and take the neutrino fields to be the renormalized effective fields.

### III. THE FLAVOR STATES IN A WAVE PACKET APPROACH

To zeroth order in the coupling constant \( g \), the uncoupled Dirac equations for neutrinos \((i\partial - m_j)\nu_j(x) = 0\), has the following free field solutions \([16]\):

\[ \nu_j(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_j(p)}} \sum_s \left( a^s_j(p) u^s_j(p) e^{-ip\cdot x} + b^{s\dagger}_j(p) v^s_j(p) e^{ip\cdot x} \right), \] (6)

where \( u_j \) and \( v_j \) are particle and antiparticle solutions to the Dirac equation. The conjugate field \( \bar{\nu}_j(x) \) has an analogous expansion. This expression for the neutrino mass field in terms of the operators \( a^s_j(p) \) and \( b^{s\dagger}_j(p) \) can be inverted for the operators in terms of the fields in the usual way. In our view, only the mass eigenstate fields are those that build up the Fock space structure. The auxiliary “flavor fields” are defined as mixtures by the transformations given above, and are introduced for convenience. These flavor fields, of course, enter formally into the Lagrangian, as they represent the mixture produced by the weak interactions. However, they cannot be considered to represent asymptotic fields that are carrying irreducible representations of the Poincaré or Lorentz groups.

For our purposes we want to formulate the field theory slightly differently. We introduce the notation

\[ a^s_j(p, t) = a^s_j(p) e^{-iE_j(p)t}, \] (7)

where \( E_j(p) = \sqrt{p^2 + m_j^2} \). The expansion can then be written

\[ \nu_j(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_j(p)}} \sum_s \left( a^s_j(p, t) u^s_j(p) e^{ip\cdot x} + b^{s\dagger}_j(p, t) v^s_j(p) e^{-ip\cdot x} \right). \] (8)

The time dependent operators \( a^{s\dagger}_j(p, t) \) can be projected out of the neutrino fields as

\[ a^{s\dagger}_j(p, t) = \frac{1}{\sqrt{2E_j(p)}} u^s_j(p) \int d^3x e^{ip\cdot x} \nu^s_j(t, x). \] (9)

These operators and their Hermitean conjugates create and annihilate particles of definite three-momentum \( p \), at time \( t \).

A plane wave basis neutrino state \( |\nu_j; r, p, t\rangle \) with mass \( m_j \), momentum \( p \) and spin \( r \) at time \( t \) is obtained by acting with \( a^{s\dagger}_j(p, t) \) on the vacuum state \( |0\rangle \), defined such
that \( \langle 0|0 \rangle = 1 \). In order for \( |\nu_j;r,p,t\rangle \) to be covariantly normalized at equal times, i.e.,
\[
\langle \nu_j;s,q,t|\nu_j;r,p,t\rangle = 2E_j(p)(2\pi)^3\delta^{(3)}(p - q)\delta^{rs},
\]
the definition for \( |\nu_j;r,p,t\rangle \) is
\[
|\nu_j;r,p,t\rangle \equiv \sqrt{2E_j(p)}a_j^r(p,t)|0\rangle. \tag{10}
\]

The auxiliary neutrino flavor fields \( \nu_\alpha \), where \( \alpha = e, \mu, \tau \), that we will use to construct our localized states, are expressed in the mass eigenfields \( \nu_j \), where \( j = 1, 2, 3 \), by \( \nu_\alpha(x) = \sum_j U_{\alpha j}\nu_j(x) \). The flavor fields are constructed with the mass eigenfields having the same three-momentum, \( p \). The \( U_{\alpha j} \)'s are the elements of the unitary mixing matrix \( U \), which for three generations is given in standard form by
\[
U = \begin{pmatrix}
C_2C_3 & S_3C_2 & S_2e^{-i\delta} \\
-S_3C_1 - S_1S_2C_3e^{i\delta} & C_1C_3 - S_1S_2S_3e^{i\delta} & S_1C_2 \\
S_1S_3 - S_2C_1C_3e^{i\delta} & -S_1C_3 - S_2S_3C_1e^{i\delta} & C_1C_2
\end{pmatrix} \tag{11}
\]
with \( C_i \equiv \cos \theta_i \) and \( S_i \equiv \sin \theta_i \), \( i = 1, 2, 3 \). When the CP-violating phase \( \delta = 0 \), the matrix elements are real and we have \( U^*_{\alpha j} = U_{\alpha j} \) for \( \alpha = e, \mu, \tau \) and \( j = 1, 2, 3 \). In what follows, we put \( \delta = 0 \).

Let us define \( |\nu_\alpha;r,p,t\rangle \equiv \sum_j U^*_{\alpha j}|\nu_j;r,p,t\rangle \). Since we want to have an expression for \( |\nu_\alpha;r,p,t\rangle \) in terms of \( \nu^\dagger_j(t) \), we obtain
\[
|\nu_\alpha;r,p,t\rangle = \sum_j U^*_{\alpha j}\sqrt{2E_j(p)}a_j^r(p,t)|0\rangle = \sum_j U^*_{\alpha j}u_j^r(p) \int d^3x e^{ip\cdot x}\nu^\dagger_j(x)|0\rangle. \tag{12}
\]

The state \( |\nu_\alpha;r,p,t\rangle \) has a definite momentum \( p \), which means that \( \Delta p = 0 \). According to Heisenberg’s uncertainty relation, \( \Delta p \Delta x \geq \hbar/2 \), the state has therefore an infinitely wide spread in space. It also does not have a well-defined energy, since \( P^0 \) operating on the state does not satisfy \( E_\alpha = \sqrt{p^2 + m^2_\alpha} \). This is of course one of the causes of the flavor oscillations in this mode of description.

In order to have a Hilbert space state that is localized around the momentum \( p \) at position \( X \) and time \( T \), one has to use a wave-packet with some distribution function \( F_\alpha(X,P,p) = F_\alpha(p,p)e^{-|X-P|} \). We can then write such a state \( |F_\alpha(r,P,X,T)\rangle \), that is localized around \( P \) at the space time point \( X,T \) with spin projection \( r \), as
\[
|F_\alpha(r,P,X,T)\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{F_\alpha(X,P,p)}{\sqrt{2E_\alpha(p)}}|\nu_\alpha;r,p,T\rangle. \tag{13}
\]

In what follows we will use the subscripts \( s \) and \( d \) to denote source and detector, respectively. Inserting our expression for \( |\nu_\alpha;r,p,T\rangle \), we finally have
\[
|F_\alpha(r,P,\ell,X,\ell)\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{F_\alpha(X,\ell,P,\ell)}{\sqrt{2E_\alpha(p)}} \sum_j U^*_{\alpha j}u_j^r(p) \int d^3xe^{ip\cdot x}\nu^\dagger_j(\ell,x)|0\rangle \tag{14}
\]
for \( \ell = s,d \). This expression can be taken as the definition of our localized neutrino flavor states. In actual applications, it is convenient to work with Gaussian wave packets, with a mean spread \( \sigma \) around \( P \). The corresponding wave function in the plane wave basis is
\[
F_\alpha(\ell,P) = \left(\frac{2\pi}{\sigma^2_\ell}\right)^2 e^{-\frac{(P-x)^2}{4\sigma^2_\ell}} \quad \ell = s,d, \tag{15}
\]
where for simplicity will take the same $\sigma_\ell$ for all flavors. The states in Eq. (14) have correct normalization for equal times if we make the interpretation $E_\alpha(p) = \sum_j U_{\alpha j}^* E_j(p)$, i.e., $E_\alpha$ is the expectation value of the Hamiltonian in the state $|\nu_\alpha; r, p, T\rangle$.

IV. NEUTRINO OSCILLATIONS IN VACUUM

The transition amplitude $A_{\alpha\beta}^{rs} \equiv A_{\alpha\beta}^{rs}(X_s, X_d, T_s, T_d, P_s, P_d)$ in the rest frame of the detector and the source for a neutrino of flavor $\alpha$, momentum around $P_s$, and spin $r$ to be produced at the source and to be detected as a flavor $\beta$, with momentum around $P_d$, and spin $s$ is given by

$$A_{\alpha\beta}^{rs} = \langle F_\beta(s; P_d, X_d, T_d) | F_\alpha(r; P_s, X_s, T_s) \rangle. \quad (16)$$

Using the expression for $|F_\alpha(r; P, X, T)\rangle$ and its conjugate, we have

$$A_{\alpha\beta}^{rs} = \int \frac{d^3q}{(2\pi)^3} \frac{F_\beta^*(X_d, P_d, q)}{\sqrt{2E_\beta(q)}} \sum_j U_{\beta j}^* u_j^\dagger(q) \int d^3 ye^{-iqy}$$

$$\times \int \frac{d^3p}{(2\pi)^3} \frac{F_\alpha(X_s, P_s, p)}{\sqrt{2E_\alpha(p)}} \sum_k U_{\alpha k}^* u_k^\dagger(p) \int d^3xe^{ipx}$$

$$\times \langle 0|\nu_j(T_d, y)\nu_k^\dagger(T_s, x)|0 \rangle \delta_{jk}. \quad (17)$$

The expression $\langle 0|\nu_j(y)\nu_k^\dagger(x)|0 \rangle$ is the propagation amplitude for a mass eigenstate $k$ being created at $x = (T_s, x)$, propagating to $y = (T_d, y)$, where a mass eigenstate $j$ is being annihilated. In vacuum, $j$ has to be equal to $k$. On the other hand, when neutrinos propagate through matter it is possible that $j \neq k$. Since $T_d > T_s$ i.e. $y^0 > x^0$, we can insert the time-ordering operator $T$ obtaining

$$\langle 0|\nu_j(y)\nu_k^\dagger(x)|0 \rangle \rightarrow \langle 0|T(\nu_j(y)\nu_k^\dagger(x))|0 \rangle \delta_{jk}. \quad (18)$$

The last expression is simply

$$\langle 0|T(\nu_j(y)\nu_k^\dagger(x))|0 \rangle = \langle 0|T(\nu_j(y)\nu_j^\dagger(x))|0 \rangle \gamma^0 = S_j(y - x) \gamma^0, \quad (19)$$

where $S_j$ denotes the Feynman fermion propagator defined by

$$S_j(y - x) = i \int \frac{d^4k}{(2\pi)^4} \frac{(k + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik(y - x)}. \quad (20)$$

The integration in the complex $k^0$-plane has to be taken along the whole real axis. We can then rewrite the expression for the amplitude as follows

$$A_{\alpha\beta}^{rs} = \sum_j U_{\beta j}^* U_{\alpha j} \int \frac{d^3q}{(2\pi)^3} \frac{F_\beta^*(X_d, P_d, q)}{\sqrt{2E_\beta(q)}} \frac{F_\alpha(X_s, P_s, p)}{\sqrt{2E_\alpha(p)}}$$

$$\times \int d^3ye^{-iqy} \int d^3xe^{ipx} \int \frac{d^4k}{(2\pi)^4} e^{-i[k^0(T_d - T_s) - k^\dagger(y - x)]}$$

$$\times \frac{u_j^\dagger(q)(k + m_j)\gamma^0 u_k^\dagger(p)}{k^2 - m_j^2 + i\epsilon}. \quad (21)$$
We next calculate the integrals in the middle in the order \( x, y, \) and \( k \). This gives

\[
i \int \frac{dk^0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \int d^3x e^{-i\mathbf{q}\cdot\mathbf{x}} \int d^3y e^{i\mathbf{y}\cdot\mathbf{x}} \frac{(k + m_j)}{k^2 - m_j^2 + i\epsilon} e^{-ik(y-x)} = i(2\pi)^3\delta^{(3)}(p - q) \int \frac{dk^0}{2\pi} e^{-i(T_d - T_s)k^0} \frac{(k^0\gamma^0 - \mathbf{p} \cdot \gamma + m_j)}{(k^0)^2 - \mathbf{p}^2 - m_j^2 + i\epsilon}.
\]

(22)

Closing the contour in the lower \( k^0 \) half-plane and calculating the residues of the positive energy poles, it follows that

\[
A_{\alpha\beta}^{rs} = \sum_j U_{\beta j}^{rs} U_{\alpha j}^{s*} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \frac{F_{\beta}^*(\mathbf{X}_d, \mathbf{P}_d, \mathbf{q}) F_{\alpha}(\mathbf{X}_s, \mathbf{P}_s, \mathbf{p})}{\sqrt{2E_{\beta}(\mathbf{q})} \sqrt{2E_{\alpha}(\mathbf{p})}} 
\times (2\pi)^3\delta^{(3)}(p - q)e^{-i(T_d - T_s)E_j(\mathbf{p})} u^s_j(p) \frac{\mathbf{p} + m_j}{2E_j(\mathbf{p})} \frac{\gamma_i u^r_j(p)}{2E_j(\mathbf{p})}.
\]

(23)

We finally obtain

\[
A_{\alpha\beta}^{rs} = \sum_j U_{\beta j}^{rs} U_{\alpha j}^{s*} \int \frac{d^3p}{(2\pi)^3} \frac{F_{\beta}^*(\mathbf{X}_d, \mathbf{P}_d, \mathbf{p}) F_{\alpha}(\mathbf{X}_s, \mathbf{P}_s, \mathbf{p})}{\sqrt{2E_{\beta}(\mathbf{p})} \sqrt{2E_{\alpha}(\mathbf{p})}} \frac{u^s_j(p)}{2E_j(\mathbf{p})} \frac{\mathbf{p} + m_j}{2E_j(\mathbf{p})} \frac{\gamma_i u^r_j(p)}{2E_j(\mathbf{p})} e^{-iE_j(\mathbf{p})(T_d - T_s)}.
\]

(24)

The part \( u^s_j(p) \frac{\mathbf{p} + m_j}{2E_j(\mathbf{p})} \gamma_i u^r_j(p) \) can be simplified by using that \( (\mathbf{p} - m_j)u^r_j(p) = 0 \). This gives

\[
u^s_j(p) \frac{\mathbf{p} + m_j}{2E_j(\mathbf{p})} \gamma_i u^r_j(p) = (2E_j(\mathbf{p}))^2 \delta^{rs}.
\]

(25)

We finally arrive at the following rather simple expression for the transition amplitude

\[
A_{\alpha\beta}^{rs} = \int \frac{d^3p}{(2\pi)^3} \frac{F_{\beta}^*(\mathbf{X}_d, \mathbf{P}_d, \mathbf{p}) F_{\alpha}(\mathbf{X}_s, \mathbf{P}_s, \mathbf{p})}{\sqrt{2E_{\beta}(\mathbf{p})} \sqrt{2E_{\alpha}(\mathbf{p})}} M_{\alpha\beta}^{rs},
\]

(26)

where

\[
M_{\alpha\beta}^{rs} = \delta^{rs} \sum_j U_{\beta j}^{rs} U_{\alpha j}^{s*} 2E_j(\mathbf{p}) e^{-iE_j(\mathbf{p})(T_d - T_s)}.
\]

(27)

The amplitude \( A_{\alpha\beta}^{rs} \) in Eq. (26) will be our standard reference amplitude, when we later calculate the amplitude for neutrino oscillations in the presence of matter. Using Gaussian wave packets, it is possible to show that we regain the usual neutrino oscillation formula in the ultra-relativistic limit. In particular, it is useful to consider the time average of the amplitude, as we do not know when the neutrinos are emitted. We will not here elaborate further on this point, which is treated in Ref. [10], but continue with the case when neutrinos interact with matter.
V. NEUTRINOS IN MATTER: THE RESONANCE FORMULA

A. General discussion

We will next proceed to derive the neutrino oscillation probability for neutrinos propagating in matter with constant density. Although a perturbative treatment of neutrino oscillations in matter would give the influence of matter for density factors that are small compared to the energy differences $\Delta m^2/E$ between the neutrinos, the behavior of the neutrinos are quite different for densities that are of the same order as $\Delta m^2/E$. This case is known as the MSW effect and was pointed out by Wolfenstein, Mikehyev, and Smirnov as being potentially important. The quantum mechanical case can be found in many textbooks. To treat this effect with quantum field theory we have to calculate the Green's function for the neutrinos in matter. The equation of motion derived from $\mathcal{L}_{\text{eff}}(x)$ is in component form

$$(i\partial - m_j)\nu_j(x) + \sum_k A_{jk}\nu_k(x) = \chi_j(x), \quad (28)$$

where $\chi_j(x)$ is the source. Here the term responsible for the interaction with matter is given by contributions from neutral (N) and charge changing (C) currents in the static limit as

$$A^\mu(x) = -\delta^{\mu0}(A_N 1 + A_C K), \quad (29)$$

where $K_{jk} = U_{ej}^\dagger U_{ek}$ is the projector onto the electron neutrino flavor space. The terms $A_N$ and $A_C$ have the values $A_N = -\sqrt{2}G_F N_n/2$, and $A_C = \sqrt{2}G_F N_e$ expressed in terms of the number density of neutrons or electrons, respectively. For antineutrinos there is a change of sign in these terms.

By transforming to momentum space and assuming that the neutrinos all have the same three-momentum $p$, the Dirac equation in the presence of matter can be written in momentum space as

$$(\gamma^0 - p \cdot \gamma 1 - M - (A_N 1 + A_C K)\gamma^0)u(p) = 0. \quad (30)$$

In this section, we will use the symbols $u_M$ for the solutions to the Dirac equation in the presence of matter.

The Eq. (30) can be written as

$$Hu_M(p) = (-p \cdot \gamma \gamma^0 + M\gamma^0 + A_N 1 + A_C K)u_M(p) \quad (31)$$

To obtain the full set of roots one has, in the energy representation, to enlarge the flavor space with the two-dimensional energy space, by using $u_\pm = \frac{1}{2}(1 \pm \gamma^0)u$ for each flavor $f$. This gives a $2f-$dimensional representation of the Hamiltonian, that will have $2f$ roots, i.e. $f$ roots of both signs. These roots are calculated for $f = 2$ later.

It is then possible to diagonalize the Hamiltonian $H$, which is symmetric in this $2f-$dimensional representation, with a unitary transformation $U_M$ which is analogous to that one given e.g. in Ref. [17] for the case of an ordinary quantum mechanical description. The Eq. (31) then becomes
where $u_M$ is related to $u$ in the mass representation by $u_M = U_M^* u$. Since the flavor states $u_f$ are related to the matter states by $u_f = U_M^* u_M$, where $U$ is naturally extended to the same dimension, the flavor states can be expanded in the mass representation also in case of neutrinos propagating through matter (of constant density). For the antineutrinos a corresponding equation holds with different sign on the matter terms, and the same transformation $U_M$ will diagonalize the Hamiltonian also for the antineutrinos.

The Feynman propagator, as a Green’s function, is invariant under unitary transformations, and can therefore alternatively be expressed in the mass representation.

To obtain the transition amplitude we now utilize the same approach as in the previous section. The neutrino field is expanded in the solutions to Eq. (30) and the spatial momentum plane waves. The corresponding annihilation operator satisfies the equation

$$a_s^j(p, t) = a^s_j(p) e^{-iE_j t},$$

where $E_j$ now is the positive eigenvalue of $H$ and the two negative eigenvalues go with the adjoint creation operators. For the antiparticle operators these change their roles and the sign of the matter terms also change.

The amplitude of interest is in this case given by the expression

$$A^{rs}_{\alpha \beta \text{ matter}} = \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \frac{F^*_\beta(X_d, P_d, q) F^\alpha(X_s, P_s, p)}{\sqrt{2E_\beta(q)} \sqrt{2E_\alpha(p)}} M^{rs}_{\alpha \beta \text{ matter}}(q, p);$$

where

$$M^{rs}_{\alpha \beta \text{ matter}}(q, p) = \sum_{jk} U_{M \beta j} U^*_{M \alpha k} M^{rs}_{Mj \beta k}(q, p)$$

and the expression $R^{rs}_{Mj \beta k}(x)$ is given by

$$R^{rs}_{Mj \beta k}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \bar{u}_M^s(q) i\gamma^0 \langle 0 | T(\nu_j(y) \nu_j^\dagger(x)) | 0 \rangle \gamma^0 u_{Mk}^r(p).$$

Using the transformations given above and the fact that the time-ordered Green’s function can be expressed in any basis, and thus can be replaced by the Feynman propagator in the mass representation, we arrive at the expression

$$M^{rs}_{\alpha \beta \text{ matter}}(q, p) = \sum_{jk} U_{\beta j} U^*_{\alpha k} \int d^3x \int d^3y e^{i(p-x) \cdot x} e^{-i(q-y) \cdot y} R^{rs}_{jk}(y-x),$$

where

$$R^{rs}_{jk}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \bar{u}_M^s(q) i\gamma^0 S^{rs}_{jk}(A, k) \gamma^0 u_{Mk}^r(p).$$
The expression $M_{\alpha\beta}^{rs}$ in Eq. (37) will be our main task to calculate.

We first want to calculate the Feynman propagator in the presence of matter. What we seek is thus the inverse $S(A, k)$ of the operator

$$S^{-1}(A, k) = G\gamma^0 - k \cdot \gamma I - M,$$

where $G = k^0 I - A_N I - A_C K$ which operates in the tensor product space of the four dimensional Dirac space with the three dimensional mass space in the mass representation.

The neutral currents give a constant contribution to the energy that can be absorbed in its definition until the end of the calculation and will therefore be neglected from now on. The charge current contribution, on the other hand, gives a more complicated contribution and will be dealt with explicitly. For convenience we will use the notation $A \equiv A_C$ below. The calculation can then readily be done and one finds

$$S(A, k) = \left[ G\gamma^0 - k \cdot \gamma I + M + (G\gamma^0 - k \cdot \gamma I + M)\gamma^0 N^{-1} C \right] D^{-1},$$

where $N = G^2 - k^2 I - M^2$, $C = -A [M, K]$, and $D = N - CN^{-1} C$. This means that we are interested in the poles of $D^{-1}$ with respect to $k^0$. We will perform the analysis for the case of two flavors below, leaving the three-flavor case for the future.

**B. The case of two flavors**

Here we will confine ourselves to calculate the amplitude explicitly only for two flavors, since we can then easily compare our result with the quantum mechanical expression for the resonance formula.

For two flavors we thus want to calculate the appearance amplitude

$$A_{\mu\nu}^{rs \text{ matter}} = \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \frac{F_{\mu}(X_d, P_d, q) F_{\nu}(X_s, P_s, p)}{\sqrt{2E_{\mu}(q)} \sqrt{2E_{\nu}(p)}} M_{Me\mu}^{rs}(q, p).$$

Carrying out the integrations as before we end up with the expression

$$A_{\mu\nu}^{rs \text{ matter}} = \int \frac{d^3 p}{(2\pi)^3} \frac{F_{\mu}(X_d, P_d, p) F_{\nu}(X_s, P_s, p)}{\sqrt{2E_{\mu}(p)} \sqrt{2E_{\nu}(p)}}$$

$$\times i \int \frac{dk^0}{2\pi} e^{-ik^0 (T_d - T_s)} \bar{u}_\mu(p) \gamma^0 S(A, (k^0, p)) \gamma^0 u_\nu(p).$$

We are thus interested in the expression

$$M_{Me\mu}^{rs} = i \int \frac{dk^0}{2\pi} e^{-ik^0 T s} \bar{u}_\mu(p) \gamma^0 S(A, (k^0, p)) \gamma^0 u_\nu(p),$$

where $u_\ell = \sum_{j=1}^2 U_{\ell j} u_j$, $\ell = e, \mu$, and $T = T_d - T_s$.

In the case of vacuum propagation, we have $M_{Me\mu}^{rs} = \delta^{rs} \sum_{j} U_{e j}^* U_{\mu j} 2E_j(p) e^{-iE_j T}$, which in the high energy limit goes to $\delta^{rs} e^{-iET} 2E \sin 2\theta \sin [T(E_2 - E_1)/2]$, with $2E = E_1 + E_2$. In what follows, we will frequently omit the spin indices.
Let us first calculate the matrix \( D = N - CN^{-1}C \) for the two flavors \( e \) and \( \mu \). The calculation is done in the mass eigenstate basis. In this case, we have

\[
G = k^0 \mathbb{I}_2 - AK. \tag{44}
\]

Remember here that \( k^0 \) is really \( k^0 - A_N \). Then,

\[
C = -A[M,K] = A(m_2 - m_1) \begin{pmatrix} 0 & K_{12} \\ -K_{21} & 0 \end{pmatrix}, \tag{45}
\]

where \( K_{21} = K_{12} = \sin \theta \cos \theta \). Thus,

\[
C = \kappa \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{46}
\]

with \( \kappa = A(m_2 - m_1)K_{12} \). We can then readily calculate the expression \( CN^{-1}C \). The result is

\[
CN^{-1}C = -\frac{\kappa^2}{\det N} N^T = -\frac{\kappa^2}{\det N} N, \tag{47}
\]

since \( N \) is symmetric. Thus,

\[
D = N + \frac{\kappa^2}{\det N} N = \left(1 + \frac{\kappa^2}{\det N}\right) N \tag{48}
\]

and

\[
D^{-1} = \frac{1}{\det N + \kappa^2} \begin{pmatrix} N_{22} & -N_{12} \\ -N_{12} & N_{11} \end{pmatrix}. \tag{49}
\]

In calculating the expression \( \bar{u}\gamma^0 S(A, (k^0, p))\gamma^0 u \), we can use the fact that the spinors \( \bar{u} \) fulfill the relations

\[
\bar{u}(p)((E - AK)\gamma^0 - p \cdot \gamma \mathbb{I} - M) = 0. \tag{50}
\]

This leads to two different terms

\[
S^{(1)} = (k^0 + E - 2AK)D^{-1} \tag{51}
\]

and

\[
S^{(2)} = (k^0 + E - 2AK)N^{-1}CD^{-1} \tag{52}
\]

with

\[
\bar{u}_i\gamma^0 S(A, (k^0, p))_{ij}\gamma^0 u_j = u^{(1)}_i S^{(1)}_{ij} u_j + \bar{u}_i S^{(2)}_{ij} u_j. \tag{53}
\]

We are looking for the singularities of \( D^{-1} \) and \( N^{-1}CD^{-1} \). The singularities of \( D^{-1} \) are given by the roots with respect to \( k^0 \) of the equation

\[
N_{11}N_{22} - N_{12}^2 + \kappa^2 = 0. \tag{54}
\]

Now, \( N \) can be calculated to be

\[
N = \begin{pmatrix} (k^0)^2 - p^2 - m_1^2 + (A^2 - 2k^0 A)K_{11} & (A^2 - 2k^0 A)K_{12} \\ (A^2 - 2k^0 A)K_{12} & (k^0)^2 - p^2 - m_2^2 + (A^2 - 2k^0 A)K_{22} \end{pmatrix}. \tag{55}
\]
The equation (54) for \( k^0 \) is then
\[
((k^0)^2 - p^2 - m_1^2 + (A^2 - 2k^0 A)K_{11})((k^0)^2 - p^2 - m_2^2 + (A^2 - 2k^0 A)K_{22}) - (A^2 - 2k^0 A)^2K_{12}^2 + \kappa^2 = 0. \tag{56}
\]

The other singularities are determined by the expression \( N^{-1}CD^{-1} \). This can easily be calculated to be
\[
N^{-1}CD^{-1} = \frac{\kappa}{\det N + \kappa^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{57}
\]

Obviously, the singularities are again at the same location as the roots of Eq. (56).

To solve Eq. (56) we first simplify the equation using the properties \( K_{11}K_{22} - K_{12}^2 = 0 \) and \( K_{11} + K_{22} = 1 \), since \( K \) is a projector. We also use \( E_i^2 = p^2 + m_i^2 \) for \( i = 1, 2 \). Then
\[
((k^0)^2 - E_1^2)((k^0)^2 - E_2^2) + A(A - 2k^0)((k^0)^2 - K_{11}E_2^2 - K_{22}E_1^2) + \kappa^2 = 0 \tag{58}
\]

Finally, we also have \( \kappa^2 = A^2(\Delta m)^2K_{11}K_{22} = A^2(\Delta m)^2 \frac{1}{4} \sin^2 2\theta \). The term \( K_{11}E_2^2 + K_{22}E_1^2 \) can be simplified to be \( \frac{1}{2}(E_1^2 + E_2^2) + \frac{1}{2}(m_2^2 - m_1^2) \cos 2\theta \). If we, for typographical reasons, put \( k^0 = x \) the equation can be written
\[
x^4 - 2Ax^3 + (A^2 - E_1^2 - E_2^2)x^2 + A(E_1^2 + E_2^2 + (m_2^2 - m_1^2) \cos 2\theta)x \\
- \frac{1}{2}A^2(E_1^2 + E_2^2 + (m_2^2 - m_1^2) \cos 2\theta) + \frac{1}{4}A^2(\Delta m)^2 \sin^2 2\theta \\
+ E_1^2 E_2^2 = 0. \tag{59}
\]

We further introduce new variables \( E = (E_1 + E_2)/2 \) and \( \Delta m^2 = m_2^2 - m_1^2 \). Then, the equation above can be written
\[
x^4 - 2Ax^3 + \left(A^2 - 2E^2 - 2 \left(\frac{\Delta m^2}{4E}\right)^2\right)x^2 + A \left(2E^2 + 2 \left(\frac{\Delta m^2}{4E}\right)^2 + \Delta m^2 \cos 2\theta\right)x \\
- \frac{1}{2}A^2 \left(2E^2 + 2 \left(\frac{\Delta m^2}{4E}\right)^2 + \Delta m^2 \cos 2\theta\right) + A^2 \left(\frac{\Delta m^2}{2(m_1 + m_2)}\right)^2 \sin^2 2\theta \\
+ E^4 - 2E^2 \left(\frac{\Delta m^2}{4E}\right)^2 + \left(\frac{\Delta m^2}{4E}\right)^4 = 0, \tag{60}
\]
which can be solved exactly. Let us call the roots of this equation \( \lambda_i, \ i = 1, \ldots, 4 \).

These roots are of course exactly the same as those we get by solving the secular equation for the Hamiltonian given in Eq. (31), when we enlarge the two-dimensional flavor space with the two-dimensional \( u_\pm \) space related to diagonalizing \( \gamma^0 \), as was mentioned in Sec. V A.

Since we want to calculate the amplitude \( M_{e\mu}^{s} \) we are, in principle, interested in the expression
\[
\bar{u}_\mu \gamma^0 S^0_{\gamma^0} u_e = -sc\bar{u}_1 \gamma^0 S_{11} \gamma^0 u_1 - s^2 \bar{u}_1 \gamma^0 S_{12} \gamma^0 u_2 + c^2 \bar{u}_2 \gamma^0 S_{21} \gamma^0 u_1 + cs\bar{u}_2 \gamma^0 S_{22} \gamma^0 u_2 \\
= sc(\bar{u}_2 \gamma^0 S_{22} \gamma^0 u_2 - \bar{u}_1 \gamma^0 S_{11} \gamma^0 u_1) + (c^2 \bar{u}_2 \gamma^0 S_{21} \gamma^0 u_1 - s^2 \bar{u}_1 \gamma^0 S_{12} \gamma^0 u_2), \tag{61}
\]
where \( s \equiv \sin \theta \) and \( c \equiv \cos \theta \).
The amplitude can be written in the form

\[ M_{\epsilon \mu} = \bar{u}_\mu^s \gamma^0 S^0 u_e^r = \delta^{rs} \frac{1}{\det N + k^2} \left( (k^0)^3 a_3 + (k^0)^2 a_2 + k^0 a_1 + a_0 \right), \tag{62} \]

where the coefficients \( a_i \) have to be calculated. We will discuss this point later and first calculate the amplitude.

**C. Calculation of the amplitude**

The roots \( \lambda_i, i = 1, \ldots, 4 \) of Eq. (60) are all real, and we will proceed with the calculation of the amplitude. We will also assume that the roots are indexed so that in the limit \( A \to 0 \) the roots take on the values \( \lambda_1 = E_1, \lambda_2 = E_2, \lambda_3 = -E_1, \text{ and } \lambda_4 = -E_2 \). We can then close the contour in the lower half-plane and calculate the residues of the amplitude avoiding the negative roots. The amplitude is then (minus) the sum of the residues of the poles at \( \lambda_1 \) and \( \lambda_2 \).

The sum of the residues of the amplitude \( M_{\epsilon \mu} \) is then

\[ M_{\epsilon \mu} = -i \frac{\lambda_3^3 a_3 + \lambda_2^2 a_2 + \lambda_1 a_1 + a_0}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} e^{-i T \lambda_1} - i \frac{\lambda_2^3 a_3 + \lambda_3^2 a_2 + \lambda_2 a_1 + a_0}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} e^{-i T \lambda_2}. \tag{63} \]

This expression is similar to that of the matrix element of the spectral resolution of the evolution operator, only that here there is an influence from the negative energy terms through the denominators.

We first introduce the difference of the roots as \( \Delta = \lambda_2 - \lambda_1 \). Next, we observe that we can translate all the roots by \( A/2 \) so that \( x = y + A/2 \) to make the sum of the new roots \( y_i, i = 1, \ldots, 4 \) satisfy \( y_1 + y_2 + y_3 + y_4 = 0 \). This transforms away the \( x^3 \)-term in the equation. Also the denominators in the amplitude are unaffected by this translation. The numerators will obtain terms proportional to \( A \) or \( A^2 \), which we can safely neglect.

In calculating the coefficients \( a_i, i = 0, \ldots, 3 \) we will neglect small terms in the numerator. Our main interest is to locate the resonance region where the denominators are small. We therefore first put the matter parameter \( A \) to zero in the wave functions as well as in the terms \( S^{(i)}, i = 1, 2 \). We further use the approximations \( u_i^s u_j^r \approx \delta^{rs} (E_i + E_j) \) for all \( i, j \), and \( \bar{u}_i^s u_j^r \approx \delta^{rs} (m_i + m_j) \). For the diagonal elements the equality is strict.

In the limit of neglecting small terms in the numerator, we obtain the compact formulae:

\[ a_3 \approx \sin 2\theta \Delta m^2 \frac{1}{2E}, \tag{64} \]
\[ a_2 \approx \sin 2\theta \Delta m^2, \tag{65} \]
\[ a_1 \approx \sin 2\theta \Delta m^2 \frac{E}{2}, \tag{66} \]

and

\[ a_0 = 0. \tag{67} \]

The large contributions to the \( a_i \)'s given above all come from the \( S^{(1)} \) term.
We now realize that introducing the small shift in the energy by the terms $A_N$, as we did in the beginning, amounts only to add $A_N$ to $A$ in the roots of Eq. (60). This is still a very small term and does not change anything essential in the above approximation, especially not in the denominators, that contain differences of the roots. Apart from appearing in the small terms neglected, the contribution from $A_N$ appears in the overall phase factor as an addition to $A$. However, this will not affect the probability.

Finally, we approximate the sums and products of roots in terms of $y_i$ with their limiting values as $A = 0$ (and $A_N = 0$). This leads to an expression for the amplitude of the form

$$M_{e\mu} \approx -i \frac{1}{4E\Delta} e^{-iT(y_1+y_2-A)/2} \left\{ -i \left(a_32E^2 + 2Ea_2 + 2a_1 + \frac{2a_0}{E}\right) \sin(T\Delta/2) 
+ \left(a_3\Delta m^2 + \frac{a_2\Delta m^2}{2E} - \frac{a_0\Delta m^2}{2E^2}\right) \cos(T\Delta/2) \right\}. \quad (68)$$

Inserting the values for the coefficients $a_i$ from Eqs. (64) – (67), we obtain the formula

$$M_{e\mu}^r \approx -i\delta^{rs} e^{-i(E-A/2)T} 2E \sin 2\theta M \left( \frac{\Delta m^2}{4E^2} \cos(T\Delta/2) - i \sin(T\Delta/2) \right), \quad (69)$$

where

$$\sin 2\theta M = \frac{\Delta m^2 \sin 2\theta}{2E\Delta}. \quad (70)$$

In the limit $A = 0$, we have $\Delta = \Delta m^2/2E$. Neglecting the term of order $\Delta m^2/E^2$, we obtain

$$M_{e\mu}(A = 0) = e^{-iET} 2E \sin 2\theta \sin(T\Delta m^2/4E), \quad (71)$$

which is the previously derived result without matter effects and coincides with the quantum mechanical expression.

We can simplify this result in Eq. (69) further by observing that $\Delta = (y_2^2 - y_1^2)/2E$ within the approximation scheme we are envisaging. We will use a perturbative approach to obtain approximate analytic expressions for the roots $y_i$. First, we omit the linear term in Eq. (60) after translating to $x = y + A/2$ and solve the resulting equation in $y$ exactly. We then use dimensional analysis to make the Ansatz

$$y_1(\epsilon) = \sqrt{\frac{1}{4}A^2 + B^2 - \sqrt{A^2B^2 - A^2S^2 + 4a^2 - A\Delta m^2 \cos 2\theta \epsilon}}, \quad (72)$$

$$y_2(\epsilon) = \sqrt{\frac{1}{4}A^2 + B^2 + \sqrt{A^2B^2 - A^2S^2 + 4a^2 - A\Delta m^2 \cos 2\theta \epsilon}}, \quad (73)$$

where $B^2 = E^2 + (\Delta m^2)^2/4E$ and $a = \Delta m^2/4$. Inserting this into Eq. (60) gives the equations

$$\epsilon = y_i(\epsilon), \ i = 1, 2. \quad (74)$$

Solving to lowest order gives $\epsilon_1 = \sqrt{B^2 - 2a} = E_1$ and $\epsilon_2 = \sqrt{B^2 + 2a} = E_2$. The difference of the roots is then
\[ \Delta \approx \frac{1}{2E} \left( \sqrt{A^2B^2 - A^2S^2 + 4a^2 - A\Delta m^2 \cos 2\theta \sqrt{B^2 + 2a}} \right. \\
+ \sqrt{A^2B^2 - A^2S^2 + 4a^2 - A\Delta m^2 \cos 2\theta \sqrt{B^2 - 2a}} \right). \]  

(75)

This gives

\[ \sin 2\theta_M = \frac{\Delta m^2 \sin 2\theta}{2E\Delta} = \frac{2\sin 2\theta}{\tilde{\Delta}}, \]  

(76)

where

\[ \tilde{\Delta} = \sqrt{(AB/2a)^2 - (AS/2a)^2 + 1 - (A/a) \cos 2\theta \sqrt{B^2 + 2a}} \]

\[ + \sqrt{(AB/2a)^2 - (AS/2a)^2 + 1 - (A/a) \cos 2\theta \sqrt{B^2 - 2a}}. \]  

(77)

The minimum of \( \tilde{\Delta} \) is given up to small terms by

\[ \cos 2\theta = \frac{A}{4a} (\sqrt{B^2 - 2a} + \sqrt{B^2 + 2a}) = \frac{2EA}{\Delta m^2}. \]  

(78)

This minimum location coincides with the quantum mechanical location. However, the minimum value is not exactly the same as in the quantum mechanical case due to small extra terms. One of these terms is dependent upon the sum of the neutrino masses rather than their mass squared differences.

The expression for \( \tilde{\Delta} \) can be rewritten as

\[ \tilde{\Delta} = \sqrt{(A\sqrt{B^2 + 2a/2a} - \cos 2\theta)^2 + \sin^2 2\theta (1 - A^2/(m_1 + m_2)^2) - A^2/2a} \]

\[ + \sqrt{(A\sqrt{B^2 - 2a/2a} - \cos 2\theta)^2 + \sin^2 2\theta (1 - A^2/(m_1 + m_2)^2) + A^2/2a}. \]  

(79)

This is the resonance formula. The minimum at the resonance peak is dependent on the neutrino mass in the very small terms:

\[ \tilde{\Delta}_{\text{peak}} = 2\sin 2\theta \left( 1 - \frac{1}{2} \frac{A^2}{(m_1 + m_2)^2} + \ldots \right). \]  

(80)

With realistic values of \( A \approx 10^{-12} \text{ eV} \), \( a \approx 10^{-5} \text{ eV}^2 \), and \( B \) in the MeV range or higher, consistent with the experimental mass differences and the possible matter densities, the corrections are of the order of \( 10^{-14} \), and the value of \( \tilde{\Delta} \) at the minimum is given by \( \tilde{\Delta} = 2\sqrt{\sin^2 2\theta (1 + O(A^2/(m_1 + m_2)^2) + O(A^2/\Delta m^2)} \approx 2\sin 2\theta \) as in the quantum mechanical case. This indicates that the other corrections would be still smaller. The full expression for \( \Delta \) can be calculated using the formulas above without approximations, but will be quite difficult to handle and is not of much use. Our expression Eq. (69) contains the field theoretic roots to the equation. These roots are dependent not only on the mass squared differences, but also weakly dependent on the sum of the neutrino masses. This comes from the term \( \kappa \) in the equation for the singularities. This term appears from the nontrivial coupling of the two mass eigenstates in the calculation of the inverse propagator.
VI. SUMMARY AND CONCLUSIONS

In summary, we have discussed field theory methods to treat neutrino oscillation phenomena with wave packets for neutrino flavor states in vacuum and in matter with constant density. We have carried out the calculation of the oscillation amplitude with matter effects for the case of two flavors explicitly. The resulting formula coincides with the quantum mechanical expression using plane waves, up to very small terms. These terms depend on the neutrino masses, but are unfortunately too small to be observable at the present time. To the best of our knowledge this is the first detailed calculation of the neutrino oscillation formula using field theory and integrating out the poles in the Green’s function of the neutrinos in the presence of matter. Earlier works, e.g. in Ref. [11,15] have used various approximations at an earlier stage and thereby not being able to isolate the poles exactly.

Our calculation is done in what is essentially the rest frame of the detector and source, using wave packets, in which the neutrinos have the same three-momenta. It should be possible, though more complicated, to relax this condition, and let also the three-momenta in the superposition vary within the limits allowed by the wave packet parameters.

For three flavors the corresponding equation for the poles will be of sixth order, and can therefore not in general be solved exactly. We nevertheless believe that approximate analytic solutions to the equation can be found that can be used to study the singularity structure, in a way analogous to the two flavor case treated here.

ACKNOWLEDGMENT

This work was carried out with financial support from the Swedish Research Council, contract 621-2001-1978 and Göran Gustafssons Stiftelse. We gratefully acknowledge useful discussions with S. Bilenky, T. Ohlsson, and W. Winter.
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