On the $R$-matrix realization of Yangians and their representations

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Abstract

We study the Yangians $Y(a)$ associated with the simple Lie algebras $a$ of type $B$, $C$ or $D$. The algebra $Y(a)$ can be regarded as a quotient of the extended Yangian $X(a)$ whose defining relations are written in an $R$-matrix form. In this paper we are concerned with the algebraic structure and representations of the algebra $X(a)$. We prove an analog of the Poincaré–Birkhoff–Witt theorem for $X(a)$ and show that the Yangian $Y(a)$ can be realized as a subalgebra of $X(a)$. Furthermore, we give an independent proof of the classification theorem for the finite-dimensional irreducible representations of $X(a)$ which implies the corresponding theorem of Drinfeld for the Yangians $Y(a)$. We also give explicit constructions for all fundamental representation of the Yangians.

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1 Introduction

For any simple Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ the corresponding Yangian $Y(\mathfrak{a})$ is a canonical deformation of the universal enveloping algebra $U(\mathfrak{a}[x])$, $\mathfrak{a}[x] = \mathfrak{a} \otimes \mathbb{C}[x]$ in the class of Hopf algebras; see Drinfeld [10, 11, 12]. In accordance to Drinfeld, each Yangian $Y(\mathfrak{a})$ has at least three different presentations; see also Chari and Pressley [7, Chapter 12]. In this paper we are concerned with the one commonly known as the RTT-presentation and which preceded the other two historically. It goes back to the work of the St.-Petersburg school on the inverse scattering method; see e.g. Takhtajan and Faddeev [24], Kulish and Sklyanin [15], Tarasov [25, 26], Reshetikhin, Takhtajan and Faddeev [23]. In the case of $A$ type, i.e., $\mathfrak{a} = \mathfrak{sl}_N$, the RTT-presentation of the corresponding Yangian turns out to be particularly useful in the applications of the $R$-matrix techniques to the classical Lie algebras; see e.g. the review paper [17] and references therein. Moreover, this presentation is most convenient for the study of various subalgebras of the $A$ type Yangian which play an important role in the applications to the quantum spin chain models; see e.g. Arnaudon et al. [2, 3, 4], Molev and Ragoucy [19].

In a recent paper by Arnaudon et al. [1], the RTT-presentation of the Yangian associated with the $B$, $C$ or $D$ type Lie algebra $\mathfrak{a}$ was studied. The Yangian $Y(\mathfrak{a})$ was presented as a quotient of a quadratic algebra whose defining relations are written in the form of an RTT-relation. Below we denote this algebra by $X(\mathfrak{a})$ and call it the extended Yangian. The paper [1] contains an explicit construction of a formal series $z(u)$ whose coefficients belong to the center of $X(\mathfrak{a})$. As shown in [1], the quotient of $X(\mathfrak{a})$ by the relations $z(u) = 1$ is isomorphic to $Y(\mathfrak{a})$. In the orthogonal case $\mathfrak{a} = \mathfrak{o}_N$ ($B$ and $D$ types) this reproduces an earlier result of Drinfeld [12].

Our aim in this paper is to describe the algebraic structure of the extended Yangian $X(\mathfrak{a})$ for each orthogonal and symplectic Lie algebra $\mathfrak{a} = \mathfrak{o}_N$ and $\mathfrak{a} = \mathfrak{sp}_{2n}$ and classify its finite-dimensional irreducible representations. First, we prove an analog of the Poincaré–Birkhoff–Witt theorem for the algebra $X(\mathfrak{a})$. Then, following the approach of Molev, Nazarov and Olshanski [18], we define the Yangian $Y(\mathfrak{a})$ as a subalgebra of $X(\mathfrak{a})$. In [18], the $A$ type Yangian $Y(\mathfrak{sl}_N)$ is defined as a subalgebra of the Yangian $Y(\mathfrak{gl}_N)$ for the general linear Lie algebra $\mathfrak{gl}_N$ so that the algebra $X(\mathfrak{a})$ can be regarded as an analog of $Y(\mathfrak{gl}_N)$ for the $B$, $C$ and $D$ types. Furthermore, we show that the coefficients of the series $z(u)$ are algebraically independent and generate the center of $X(\mathfrak{a})$. This implies that the finite-dimensional irreducible representations of the algebras $X(\mathfrak{a})$ and $Y(\mathfrak{a})$ are essentially the same. These representations of the Yangian $Y(\mathfrak{a})$ were classified by Drinfeld [12]; see also Chari and Pressley [7, Chapter 12]. However, this classification is given in terms of a different presentation (new realization) of $Y(\mathfrak{a})$. At present, no explicit isomorphism between the new realization...
of the orthogonal or symplectic Yangian \( Y(\mathfrak{a}) \) and its \( RTT \)-presentation is known. (A detailed construction of such an isomorphism in the case of \( Y(\mathfrak{sl}_N) \) is recently given by Brundan and Kleshchev [5].) Therefore, the classification results of [12] do not imply an immediate description of the finite-dimensional irreducible representations of the extended Yangian \( X(\mathfrak{a}) \).

We develop an independent approach to the representation theory for the algebras \( X(\mathfrak{a}) \). We define Verma modules \( M(\lambda(u)) \) over \( X(\mathfrak{a}) \) in a standard way, where \( \lambda(u) \) is a tuple of formal series which we call the highest weight. We show that every finite-dimensional irreducible representation of \( X(\mathfrak{a}) \) is isomorphic to the unique irreducible quotient \( L(\lambda(u)) \) of \( M(\lambda(u)) \). We classify the finite-dimensional irreducible representations of \( X(\mathfrak{a}) \) by producing necessary and sufficient conditions on the highest weight \( \lambda(u) \) for the module \( L(\lambda(u)) \) to be finite-dimensional; see Theorem 5.16. Reformulating these conditions for representations of the subalgebra \( Y(\mathfrak{a}) \) of \( X(\mathfrak{a}) \) we thus obtain another proof of Drinfeld’s theorem [12] for the case of the classical Lie algebras \( \mathfrak{a} = \mathfrak{o}_N \) and \( \mathfrak{sp}_{2n} \).

As a first step, we consider the low-rank cases and construct explicit isomorphisms \( Y(\mathfrak{sp}_2) \cong Y(\mathfrak{sl}_2) \), \( Y(\mathfrak{o}_3) \cong Y(\mathfrak{sl}_2) \) and \( Y(\mathfrak{o}_4) \cong Y(\mathfrak{sl}_2) \otimes Y(\mathfrak{sl}_2) \). The former is quite immediate while the remaining two require appropriate versions of the fusion procedure for \( R \)-matrices. The representations are then described by using the known results for the Yangian \( Y(\mathfrak{sl}_2) \) which are due to Tarasov [25, 26]. For the sake of completeness, we reproduce a proof of those results which is a simpler version of the one contained in [10]. Using the above isomorphisms, we also give explicit formulas for the evaluation homomorphisms from \( X(\mathfrak{a}) \) to the universal enveloping algebra \( U(\mathfrak{a}) \) for each \( \mathfrak{a} = \mathfrak{sp}_2, \mathfrak{o}_3 \) and \( \mathfrak{o}_4 \).

In order to establish the necessary conditions for \( L(\lambda(u)) \) to be finite-dimensional, we use an induction argument which allows us to get the conditions for the rank \( n \) Lie algebra \( \mathfrak{a} \) from those of rank \( n - 1 \). The sufficient conditions on \( \lambda(u) \) are established by producing finite-dimensional modules having \( \lambda(u) \) as a highest weight. We do this first for the so-called fundamental modules and then employ the Hopf algebra structure on \( X(\mathfrak{a}) \). In particular, this proves that every finite-dimensional irreducible representations of \( X(\mathfrak{a}) \) is isomorphic to a subquotient of a tensor product of the corresponding fundamental modules. We also give an explicit construction of all fundamental modules of \( X(\mathfrak{a}) \) basically following the approach of Chari and Pressley [6] but avoiding the use of their results on the singularities of the \( R \)-matrices. For the applications of the fundamental Yangian modules to the affine Toda field theories see Chari and Pressley [8].

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2 Definitions and preliminaries

We let $a$ denote the simple complex Lie algebra of type $B_n$, $C_n$, or $D_n$. That is,

$$ a = o_{2n+1}, \ sp_{2n}, \ or \ o_{2n}, $$

(2.1)

respectively. Whenever possible, we consider the three cases simultaneously, unless otherwise stated. The Lie algebra $a$ can be regarded as a subalgebra of the general linear Lie algebra $gl_N$, where $N = 2n + 1$ or $N = 2n$, respectively. It will be convenient to enumerate the rows and columns of $N \times N$ matrices by the indices $-n, \ldots, -1, 1, \ldots, n$, if $N = 2n$, and by the indices $-n, \ldots, -1, 0, 1, \ldots, n$, if $N = 2n + 1$. For $-n \leq i, j \leq n$ set

$$ F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i} $$

(2.2)

where the $E_{ij}$ are the elements of the standard basis of $gl_N$ and

$$ \theta_{ij} = \begin{cases} 1 & \text{in the orthogonal case,} \\ \text{sgn} \ i \cdot \text{sgn} \ j & \text{in the symplectic case.} \end{cases} $$

(2.3)

The elements $F_{ij}$ span the Lie algebra $a$ and satisfy the relations

$$ F_{ij} + \theta_{ij} F_{-j,-i} = 0 $$

(2.4)

for any $-n \leq i, j \leq n$, and

$$ [F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} - \delta_{i,-l} \theta_{ij} F_{-l,-i} + \delta_{l,-j} \theta_{ij} F_{k,-i}. $$

(2.5)

For any $n$-tuple of complex numbers $\mu = (\mu_1, \ldots, \mu_n)$ we shall denote by $V(\mu)$ the irreducible representation of the Lie algebra $a$ with the highest weight $\mu$. That is, $V(\mu)$ is generated by a nonzero vector $\xi$ such that

$$ F_{ij} \xi = 0 \quad \text{for} \quad -n \leq i < j \leq n, \quad \text{and} $$

$$ F_{ii} \xi = \mu_i \xi \quad \text{for} \quad 1 \leq i \leq n. $$

The representation $V(\mu)$ is finite-dimensional if and only if

$$ \mu_i - \mu_{i+1} \in \mathbb{Z}_+ \quad \text{for} \quad i = 1, \ldots, n-1 $$

and

$$ -\mu_1 - \mu_2 \in \mathbb{Z}_+ \quad \text{if} \quad a = o_{2n}, $$

$$ -\mu_1 \in \mathbb{Z}_+ \quad \text{if} \quad a = sp_{2n}, $$

$$ -2 \mu_1 \in \mathbb{Z}_+ \quad \text{if} \quad a = o_{2n+1}. $$
Consider the endomorphism algebra \( \text{End} \mathbb{C}^N \) and let \( e_{ij} \in \text{End} \mathbb{C}^N \) be the standard matrix units (we use lower case letters to distinguish the elements of \( \text{End} \mathbb{C}^N \) from the basis elements of \( \mathfrak{gl}_N \); the latter will also be regarded as generators of the universal enveloping algebra \( U(\mathfrak{g}(N)) \)). We denote by \( F \) the \( N \times N \) matrix whose \( ij \)-th entry is \( F_{ij} \). We shall also regard \( F \) as the element

\[
F = \sum_{i,j=-n}^{n} e_{ij} \otimes F_{ij} \in \text{End} \mathbb{C}^N \otimes U(\mathfrak{a}). \tag{2.6}
\]

We shall use the transposition \( t : \text{End} \mathbb{C}^N \to \text{End} \mathbb{C}^N \) which is a linear map defined on the basis elements by the rule

\[
(e_{ij})^t = \theta_{ij} e_{-j,-i}, \tag{2.7}
\]

and the standard transposition defined by

\[
(e_{ij})' = e_{ji}. \tag{2.8}
\]

The permutation operator \( P \) is an element of \( \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N \) given by

\[
P = \sum_{i,j=-n}^{n} e_{ij} \otimes e_{ji}. \tag{2.9}
\]

We let \( Q \) denote the transposed operator \( Q = P^{t_1} = P^{t_2} \) with respect to the first or second copy of \( \text{End} \mathbb{C}^N \),

\[
Q = \sum_{i,j=-n}^{n} \theta_{ij} e_{ij} \otimes e_{-i,-j}. \tag{2.10}
\]

Whenever the double sign \( \pm \) or \( \mp \) occurs, the upper sign corresponds to the orthogonal case while the lower sign corresponds to the symplectic case. Note that the operators \( P \) and \( Q \) satisfy the relations

\[
P^2 = 1, \quad PQ = QP = \pm Q, \quad Q^2 = N Q. \tag{2.11}
\]

Set

\[
\kappa = N/2 \mp 1. \tag{2.12}
\]

The \( R \)-matrix \( R(u) \) is a rational function in a complex parameter \( u \) with values in \( \text{End} \mathbb{C}^N \otimes \text{End} \mathbb{C}^N \) defined by

\[
R(u) = 1 - \frac{P}{u} + \frac{Q}{u - \kappa}. \tag{2.13}
\]
It is well known that $R(u)$ satisfies the Yang–Baxter equation

$$R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u), \quad (2.14)$$

see [14], [27]. Here both sides take values in $\text{End} \, \mathbb{C}^N \otimes \text{End} \, \mathbb{C}^N \otimes \text{End} \, \mathbb{C}^N$ and the subscripts indicate the copies of $\text{End} \, \mathbb{C}^N$ so that $R_{12}(u) = R(u) \otimes 1$ etc.

Following the general approach of [11] and [23], we define the extended Yangian $X(a)$ as an associative algebra with generators $t_{ij}^{(r)}$, where $-n \leq i, j \leq n$ and $r = 1, 2, \ldots$ (the zero value of $i$ and $j$ is skipped if $N = 2n$), satisfying certain quadratic relations. In order to write them down, introduce the formal series

$$t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r} \in X(a)[[u^{-1}]], \quad t_{ij}^{(0)} = \delta_{ij}, \quad (2.15)$$

and set

$$T(u) = \sum_{i,j=-n}^{n} e_{ij} \otimes t_{ij}(u) \in \text{End} \, \mathbb{C}^N \otimes X(a)[[u^{-1}]]. \quad (2.16)$$

Consider the algebra $\text{End} \, \mathbb{C}^N \otimes \text{End} \, \mathbb{C}^N \otimes X(a)[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by

$$T_1(u) = \sum_{i,j=-n}^{n} e_{ij} \otimes 1 \otimes t_{ij}(u), \quad T_2(u) = \sum_{i,j=-n}^{n} 1 \otimes e_{ij} \otimes t_{ij}(u). \quad (2.17)$$

The defining relations for the algebra $X(a)$ have the form of an RTT-relation:

$$R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v). \quad (2.18)$$

Equivalently, in terms of the series (2.15) they can be written as

$$[t_{ij}(u), t_{kl}(v)] = \frac{1}{u - v} \left( t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u) \right)$$

$$- \frac{1}{u - v - \kappa} \left( \delta_{k,-i} \sum_{p=-n}^{n} \theta_{ip} t_{pj}(u) t_{-p,l}(v) - \delta_{l,-j} \sum_{p=-n}^{n} \theta_{jp} t_{k,-p}(v) t_{ip}(u) \right). \quad (2.19)$$

**Remark 2.1.** The above definition of $X(a)$ can be extended to the cases $a = o_1$ and $o_2$. However, both algebras $X(o_1)$ and $X(o_2)$ are commutative. In addition, in $X(o_2)$ we have $t_{-1,1}(u) = t_{1,-1}(u) = 0$. In what follows, we only deal with the orthogonal Lie algebras $o_N$ for $N \geq 3$. $\Box$

Consider an arbitrary formal series $f(u)$ of the form

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]. \quad (2.20)$$
Also, let $a \in \mathbb{C}$ be a constant and let $B$ be a matrix with entries in $\mathbb{C}$ such that $BB^t = 1$. It is easily derived from the defining relations for the algebra $X(a)$ that each of the mappings

$$\mu_f : T(u) \mapsto f(u) T(u),$$
$$\tau_a : T(u) \mapsto T(u - a),$$
$$T(u) \mapsto B T(u) B^t$$

defines an automorphism of $X(a)$. Furthermore, each of the mappings

$$T(u) \mapsto T(-u),$$
$$T(u) \mapsto T^t(u),$$
$$T(u) \mapsto T^{-1}(u),$$

defines an anti-automorphism of $X(a)$; cf. [18, Section 1]. This is easily verified with the use of the following property of the $R$-matrix implied by (2.11):

$$R(u) R(-u) = 1 - \frac{1}{u^2},$$

and the fact that $R(u)$ is stable under the composition of the transpositions in the first and the second copies of End $\mathbb{C}^N$.

The extended Yangian $X(a)$ is a Hopf algebra with the coproduct

$$\Delta : t_{ij}(u) \mapsto \sum_{a=-n}^{n} t_{ia}(u) \otimes t_{aj}(u),$$

the antipode

$$S : T(u) \mapsto T^{-1}(u),$$

and the counit

$$\epsilon : T(u) \mapsto 1,$$

cf. [23], [18] Section 1.

Multiplying both sides of (2.18) by $u - v - \kappa$, taking $u = v + \kappa$ and replacing $v$ by $u$ we get

$$Q T_1(u + \kappa) T_2(u) = T_2(u) T_1(u + \kappa) Q.$$  

Since $Q/N$ is a projection operator in $\mathbb{C}^N \otimes \mathbb{C}^N$ with a one-dimensional image, the expression on each side of (2.25) must be equal to $Q$ times a series $z(u)$ with coefficients in $X(a)$. Since $Q T_1(u) = Q T_2^t(u)$ and $T_1(u) Q = T_2^t(u) Q$, we have

$$T^t(u + \kappa) T(u) = T(u) T^t(u + \kappa) = z(u) 1,$$  

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where
\[ z(u) = 1 + z_1 u^{-1} + z_2 u^{-2} + \ldots, \quad z_i \in X(\mathfrak{a}). \quad (2.27) \]
Taking the \( kl \)-th entries in (2.26) we get the formulas
\[ \sum_{i=-n}^{n} \theta_{ki} t_{i,-k}(u + \kappa) t_{i}(u) = \sum_{i=-n}^{n} \theta_{li} t_{ki}(u) t_{i,-l}(u + \kappa) = \delta_{kl} z(u). \quad (2.28) \]
It was shown in [1] that all the coefficients \( z_i \) are central in \( X(\mathfrak{a}) \), and \( z(u) \) has the property
\[ \Delta : z(u) \mapsto z(u) \otimes z(u). \quad (2.29) \]
By the Hopf algebra axioms, this implies that the image of \( z(u) \) under the antipode \( S \) is found by
\[ S : z(u) \mapsto z(u)^{-1}. \quad (2.30) \]
By (2.26), we have
\[ S : T(u) \mapsto z(u)^{-1} T(u + \kappa). \]
Hence, since the transposition is involutive, we conclude that the square of the antipode is the automorphism of \( X(\mathfrak{a}) \) given by
\[ S^2 : T(u) \mapsto \frac{z(u)}{z(u + \kappa)} T(u + 2\kappa); \quad (2.31) \]
cf. [18, Section 1].

We define the Yangian \( Y(\mathfrak{a}) \) associated with the Lie algebra \( \mathfrak{a} \) as the subalgebra of \( X(\mathfrak{a}) \) which consists of the elements stable under all the automorphisms of the form (2.21). It will follow from [1] and the results below that this definition is consistent with the one given by Drinfeld [10]; cf. [18, Section 1].

3 Poincaré–Birkhoff–Witt theorem and the center of the extended Yangian

Let us denote by \( ZX(\mathfrak{a}) \) the subalgebra of \( X(\mathfrak{a}) \) generated by all the coefficients \( z_i \) of the series \( z(u) \); see (2.27).

**Theorem 3.1.** We have the tensor product decomposition
\[ X(\mathfrak{a}) = ZX(\mathfrak{a}) \otimes Y(\mathfrak{a}). \quad (3.1) \]

\[ ^1 \text{Note that the } R \text{-matrix considered in } [1] \text{ coincides with our } R(-u). \]
Proof. We follow the argument of [18, Section 2.16]. There exists a unique series $y(u)$ of the form

$$y(u) = 1 + y_1 u^{-1} + y_2 u^{-2} + \cdots, \quad y_i \in ZX(a)$$

such that $y(u)y(u + \kappa) = z(u)$. In order to see this, it suffices to write this relation in terms of the coefficients,

$$z_k = 2y_k + A_k(y_1, \ldots, y_{k-1}), \quad k \geq 1,$$

where $A_k$ is a quadratic polynomial in $k-1$ variables. By (2.26), the image of the series $z(u)$ under the automorphism (2.21) is $f(u)f(u+\kappa)z(u)$. Hence, the automorphism (2.21) takes $y(u)$ to $f(u)y(u)$. This implies that the series $\tau_{ij}(u)$ defined by

$$\tau_{ij}(u) = y(u)^{-1}t_{ij}(u), \quad i, j = -n, \ldots, n,$$

are stable under all automorphisms (2.21). Write

$$\tau_{ij}(u) = \delta_{ij} + \tau_{ij}^{(1)} u^{-1} + \tau_{ij}^{(2)} u^{-2} + \cdots.$$ 

So, the coefficients $\tau_{ij}^{(r)}$ of $\tau_{ij}(u)$ belong to the subalgebra $Y(a)$. Now the decomposition $X(a) = ZX(a) \cdot Y(a)$ follows from the relation $t_{ij}(u) = y(u)\tau_{ij}(u)$.

It remains to demonstrate that the elements $z_i$ are algebraically independent over $Y(a)$. Due to (3.2), it suffices to do this for the elements $y_i$. Suppose on the contrary, that for some positive integer $n$ there exists a nonzero polynomial $B$ in $n$ variables with the coefficients in $Y(a)$ such that

$$B(y_1, \ldots, y_n) = 0.$$ 

Take the minimal $n$ with this property. The coefficients of $B$ are stable under any automorphism (2.21). Hence, applying the automorphism (2.21) with $f(u) = 1+au^{-n}$ and $a \in \mathbb{C}$ to the equality (3.4) we get

$$B(y_1, \ldots, y_n + a) = 0$$

for any $a \in \mathbb{C}$. This means that the polynomial $B$ does not depend on its $n$-th variable, which contradicts the choice of $n$. \qed

Corollary 3.2. The Yangian $Y(a)$ is isomorphic to the quotient of $X(a)$ by the ideal generated by the elements $z_1, z_2, \ldots$, i.e.,

$$Y(a) \cong X(a)/(z(u) = 1).$$
Equivalently, $Y(a)$ is generated by the elements $\tau_{ij}^{(r)}$, where $-n \leq i, j \leq n$ and $r = 1, 2, \ldots$ subject only to the relations

\[
[\tau_{ij}(u), \tau_{kl}(v)] = \frac{1}{u-v} \left( \tau_{kj}(u) \tau_{il}(v) - \tau_{kj}(v) \tau_{il}(u) \right)
- \frac{1}{u-v-\kappa} \left( \delta_{k,-i} \sum_{p=-n}^{n} \theta_{ip} \tau_{pj}(u) \tau_{-p,l}(v) - \delta_{l,-j} \sum_{p=-n}^{n} \theta_{jp} \tau_{k,-p}(v) \tau_{ip}(u) \right)
\]

and

\[
\sum_{i=-n}^{n} \theta_{ki} \tau_{i,-k}(u + \kappa) \tau_{il}(u) = \delta_{kl}.
\] (3.6)

**Proof.** Let $I$ be the ideal of $X(a)$ introduced in the statement of the corollary. Then Theorem 3.1 implies that $X(a) = I \oplus Y(a)$ proving the first statement.

Now, the coefficients $\tau_{ij}^{(r)}$ of the series $\tau_{ij}(u)$ with $i, j = -n, \ldots, n$ generate the subalgebra $Y(a)$. Indeed, it follows from the proof of Theorem 3.1 that any element $x \in X(a)$ can be uniquely written as a polynomial $B$ in $y_1, y_2, \ldots$ such that the coefficients of $B$ are elements of the subalgebra of $X(a)$ generated by the elements $\tau_{ij}^{(r)}$. On the other hand, if $x$ belongs to the subalgebra $Y(a)$ then $B$ cannot depend on the elements $y_i$ because $x$ is stable under all automorphisms (2.21). Hence, $x$ belongs to the subalgebra of $X(a)$ generated by the $\tau_{ij}^{(r)}$.

Finally, recall that the coefficients $y_i$ of the series $y(u)$ are central in $X(a)$. Hence, we derive from (3.3) that the relation (2.19) will hold if the series $t_{ij}(u)$ are respectively replaced by $\tau_{ij}(u)$ which gives (3.5). Furthermore, (3.6) follows from (2.28). Conversely, (3.5) and (3.6) are defining relations for $Y(a)$ because they are respectively equivalent to (2.19) and the relation $z(u) = 1$. \qed

**Proposition 3.3.** The subalgebra $Y(a)$ of $X(a)$ is a Hopf algebra whose coproduct, antipode and counit are obtained by restricting those from $X(a)$.

**Proof.** The relation (2.29) implies that

\[
\Delta : y(u) \mapsto y(u) \otimes y(u).
\] (3.7)

Therefore the image of $Y(a)$ under the coproduct on $X(a)$ is contained in $Y(a) \otimes Y(a)$. By (2.30), the image of $y(u)$ under the antipode $S$ is $y(u)^{-1}$. Hence,

\[
S : y(u)^{-1} T(u) \mapsto y(u) T^{-1}(u).
\]

Any automorphism (2.21) leaves the product $y(u) T^{-1}(u)$ invariant and so the subalgebra $Y(a)$ of $X(a)$ is stable under $S$. \qed
Introduce an ascending filtration on the extended Yangian $\mathcal{X}(\mathfrak{a})$ by setting

$$\deg t_{kl}^{(r)} = r - 1$$

(3.8)

for any $k, l \in \{-n, \ldots, n\}$. Denote by $\bar{t}_{kl}^{(r)}$ and $\bar{z}_r$ the images of the elements $t_{kl}^{(r)}$ and $z_r$, respectively, in the $(r-1)$-th component of the associated graded algebra $\text{gr} \mathcal{X}(\mathfrak{a})$. Then (2.28) gives the relations

$$\bar{t}_{kl}^{(r)} + \theta_{kl} \bar{t}_{l,-k}^{(r)} = \delta_{kl} \bar{z}_r.$$  

(3.9)

Furthermore, (3.3) implies that the degree of each element $\tau_{kl}^{(r)}$ does not exceed $r-1$ and its image $\bar{\tau}_{kl}^{(r)}$ in the $(r-1)$-th component of $\text{gr} \mathcal{X}(\mathfrak{a})$ is given by

$$\bar{\tau}_{kl}^{(r)} = \frac{1}{2} (\bar{t}_{kl}^{(r)} - \theta_{kl} \bar{t}_{l,-k}^{(r)}).$$

(3.10)

The ascending filtration on the Yangian $\mathcal{Y}(\mathfrak{a})$ is induced by the one on $\mathcal{X}(\mathfrak{a})$. We denote by $\text{gr} \mathcal{Y}(\mathfrak{a})$ the associated graded algebra.

**Proposition 3.4.** The mapping

$$F_{ij} x^{r-1} \mapsto \bar{\tau}_{ij}^{(r)}$$

(3.11)

defines an algebra homomorphism $\psi : U(\mathfrak{a}[x]) \to \text{gr} \mathcal{Y}(\mathfrak{a})$.

**Proof.** By (3.10),

$$\bar{t}_{kl}^{(r)} + \theta_{kl} \bar{t}_{l,-k}^{(r)} = 0$$

for any $-n \leq k, l \leq n$ and $r \geq 1$. Furthermore, using the expansion

$$\frac{1}{u-v} = u^{-1} + u^{-2} v + \cdots,$$

take the coefficients at $u^{-r} v^{-s}$ on both sides of the relation (3.5). Keeping the highest degree terms, we come to

$$[\bar{\tau}_{ij}^{(r)}, \bar{\tau}_{kl}^{(s)}] = \delta_{kj} \bar{\tau}_{il}^{(r+s-1)} - \delta_{il} \bar{\tau}_{kj}^{(r+s-1)} - \delta_{k,-i} \theta_{ij} \bar{\tau}_{l,-j}^{(r+s-1)} + \delta_{l,-j} \theta_{ij} \bar{\tau}_{k,-i}^{(r+s-1)}.$$

It remains to compare these relations with (2.4) and (2.5). \hfill \square

Since the graded algebra $\text{gr} \mathcal{Y}(\mathfrak{a})$ is generated by the elements $\bar{\tau}_{ij}^{(r)}$, the homomorphism $\psi$ defined in Proposition 3.4 is obviously surjective. Our aim now is to show that $\psi$ is an algebra isomorphism (see Theorem 3.6 below). We shall follow the approach of Nazarov’s paper [21, Section 2], where a similar result was established for the Yangian of the queer Lie superalgebra.
Let $\rho$ be the vector representation of the Lie algebra $\mathfrak{a}$ on the vector space $\mathbb{C}^N$. So,

$$\rho: F_{ij} \mapsto e_{ij} - \theta_{ij} e_{-j,-i}.$$ 

For any $c \in \mathbb{C}$ consider the corresponding evaluation representation $\rho_c$ of the polynomial current Lie algebra $\mathfrak{a}[x]$ given by

$$\rho_c: F_{ij}^s \mapsto c^s \rho(F_{ij}), \quad s \geq 0.$$ 

For any $c_1, \ldots, c_l \in \mathbb{C}$ consider the tensor product of the evaluation representations of $\mathfrak{a}[x]$,

$$\rho_{c_1,\ldots,c_l} = \rho_{c_1} \otimes \cdots \otimes \rho_{c_l}.$$ 

**Lemma 3.5.** Let the parameters $c_1, \ldots, c_l$ and integer $l \geq 0$ vary. Then the intersection in $U(\mathfrak{a}[x])$ of the kernels of all representations $\rho_{c_1,\ldots,c_l}$ is trivial.

**Proof.** Choose a basis $Y_1, \ldots, Y_M$ of $\mathfrak{a}$, where $M = \dim \mathfrak{a}$, and set $y_i = \rho(Y_i)$. Let $A$ be a nonzero element of $U(\mathfrak{a}[x])$. Choose a total ordering on the set of basis elements $Y_i x^s$ of $\mathfrak{a}[x]$ and write $A$ as a linear combination of ordered monomials in the basis elements. Let $m$ be the maximal length of monomials which occur in $A$. For each monomial

$$(Y_{a_1} x^{s_1}) \cdots (Y_{a_m} x^{s_m}) \in U(\mathfrak{a}[x]) \quad (3.12)$$ 

occurring in $A$ consider the corresponding symmetrized elements

$$\sum_{q \in S_m} (Y_{a_1 q(1)} x^{s_{q(1)}}) \otimes \cdots \otimes (Y_{a_m q(m)} x^{s_{q(m)}}) \in (\mathfrak{a}[x])^\otimes m. \quad (3.13)$$ 

Regarding $U(\mathfrak{a}[x])$ as the quotient of the tensor algebra of $\mathfrak{a}[x]$ we derive that the elements $(3.13)$ are linearly independent. Identifying the vector spaces

$$(\mathfrak{a}[x])^\otimes m = \mathfrak{a}^\otimes m[x_1, \ldots, x_m],$$

we can regard the sum $(3.13)$ as a polynomial function in $m$ independent variables $x_1, \ldots, x_m$ with values in the vector space $\mathfrak{a}^\otimes m$,

$$\sum_{q \in S_m} x_1^{s_{q(1)}} \cdots x_m^{s_{q(m)}} Y_{a_1 q(1)} \otimes \cdots \otimes Y_{a_m q(m)}. \quad (3.14)$$

Note that

$$\rho_{c_1,\ldots,c_l}: Y_a x^s \mapsto \sum_{k=1}^l c^s_k y_a^{[k]}, \quad y_a^{[k]} = y_a^{\otimes (k-1)} \otimes y_a \otimes y_a^{\otimes (l-k)}.$$
Hence, the image of the monomial (3.12) under the representation $\rho_{c_1,\ldots,c_m}$ is given by

$$\sum_{k_1,\ldots,k_m=1}^m c_{k_1}^{s_1} \cdots c_{k_m}^{s_m} y_{a_1}^{[k_1]} \cdots y_{a_m}^{[k_m]} \in \text{End} \left( \mathbb{C}^N \right)^{\otimes m}. \quad (3.15)$$

Let us complete the set of matrices $y_1,\ldots,y_M$ to a basis $y_1,\ldots,y_N$ of $\text{End} \mathbb{C}^N$ in such a way that the identity matrix $1 \in \text{End} \mathbb{C}^N$ occurs as a basis vector $y_i$ for some $i \in \{ M + 1,\ldots,N^2 \}$. Denote by $V_m$ the subspace in $(\text{End} \mathbb{C}^N)^{\otimes m}$ spanned by the basis elements $y_{i_1} \otimes \cdots \otimes y_{i_m}$ where at least one of the tensor factors is 1. Observe that the image under the representation $\rho_{c_1,\ldots,c_m}$ of any monomial of length $< m$ occurring in $A$ is contained in $V_m$. Furthermore, modulo elements belonging to $V_m$, the sum (3.15) can be written as

$$\sum_{q \in \mathfrak{S}_m} c_1^{s_1(q)} \cdots c_m^{s_m(q)} y_{a_1(q)} \otimes \cdots \otimes y_{a_m(q)}. \quad (3.16)$$

This sum is the value of (3.14) under the specialization $x_i = c_i$ and replacement of $Y_i$ with $y_i = \rho(Y_i)$ for all $i = 1,\ldots,m$. However, since $\rho$ is faithful and the elements (3.13) are linearly independent, there exist values of the parameters $c_1,\ldots,c_m$ such that the corresponding sums (3.16) are linearly independent modulo the subspace $V_m$ which completes the proof.

We are now in a position to prove the following.

**Theorem 3.6.** The mapping $\psi : U(a[x]) \to \text{gr} Y(a)$ defined in (3.11) is an algebra isomorphism.

**Proof.** Due to Proposition 3.4, we only need to show that the kernel of $\psi$ is trivial. Let $C$ be a nonzero element of $U(a[x])$. We shall show that $\psi(C) \neq 0$. The universal enveloping algebra $U(a[x])$ has a grading defined on the generators by declaring the degree of $F_{ij}x^s$ to be equal to $s$. Then $\psi$ is obviously a homomorphism of graded algebras. Hence, we may assume that $C$ is homogeneous of degree, say, $d$. Write

$$C = \sum C_{i_1 j_1,\ldots,i_m j_m}^{r_1,\ldots,r_m} (F_{i_1 j_1} x^{r_1 - 1}) \cdots (F_{i_m j_m} x^{r_m - 1}), \quad (3.17)$$

summed over the indices $i_a, j_a, r_a$ such that $r_1 + \cdots + r_m = d + m$.

Consider the element $C' \in Y(a)$ given by the formula

$$C' = \sum C_{i_1 j_1,\ldots,i_m j_m}^{r_1,\ldots,r_m} \tau_{i_1 j_1}^{(r_1)} \cdots \tau_{i_m j_m}^{(r_m)},$$

where the summation is taken over the same set of indices as in (3.17) with the same coefficients. Then the image of $C'$ in the $d$-th component of the graded algebra $\text{gr} Y(a)$ coincides with $\psi(C)$. So, it suffices to show that $\text{deg} C' = d$. 

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Applying the standard transposition (2.8) to the third copy of $\text{End } C^N$ in the Yang–Baxter equation (2.14) and using (2.23) we come to the relation
\[ R_{12}(u - v) R'_{13}(-u) R'_{23}(-v) = R'_{23}(-v) R'_{13}(-u) R_{12}(u - v), \]
where
\[ R'(u) = 1 - \frac{P'}{u} + \frac{Q'}{u - \kappa} \]
with the transposition applied to the first (or second) copy of $\text{End } C^N$. Hence, by the defining relations (2.18) of the algebra $X(\mathfrak{a})$ we conclude that the mapping $T(u) \mapsto R'(-u)$ defines a representation of $X(\mathfrak{a})$ in the space $C^N$. Taking its composition with the automorphism (2.22) we obtain for any $c \in C^N$ the representation $\sigma_c : T(u) \mapsto R'(-u + c)$. Equivalently, in terms of the generating series (2.15) we have
\[ \sigma_c : t_{ij}(u) \mapsto \delta_{ij} + e_{ij} (u - c)^{-1} - \theta_{ij} e_{-j,-i} (u + \kappa - c)^{-1}. \]
Since the transpositions (2.7) and (2.8) commute, using (2.26) and the relations
\[ (Q')^2 = 1, \quad P'Q' = Q'P' = \pm P', \quad (P')^2 = N P', \]
we derive that the image of $z(u)$ under $\sigma_c$ is given by
\[ \sigma_c : z(u) \mapsto 1 - \frac{1}{(u - c + \kappa)^2}. \]
There exists a unique series $f_c(u) \in 1 + u^{-1}C[[u^{-1}]]$ such that
\[ f_c(u) f_c(u + \kappa) = \frac{(u - c + \kappa)^2}{(u - c + \kappa)^2 - 1}. \]
Then $\sigma_c : y(u)^{-1} \mapsto f_c(u)$ so that due to (3.19), for the image of the series $\tau_{ij}(u)$ under $\sigma_c$ we have
\[ \sigma_c : \tau_{ij}(u) \mapsto f_c(u) \left( \delta_{ij} + e_{ij} (u - c)^{-1} - \theta_{ij} e_{-j,-i} (u + \kappa - c)^{-1} \right). \]
Observe that the coefficient of the series $f_c(u)$ at $u^{-k}$ is a polynomial in $c$ of degree $\leq k - 1$. Therefore, taking the coefficient at $u^{-r}$ in (3.20) we find that the image of $\tau_{ij}^{(r)}$ under $\sigma_c$ is a polynomial in $c$ of degree $\leq r - 1$ with coefficients in $\text{End } C^N$. Moreover, the coefficient of this polynomial at $c^{-1}$ coincides with $\rho(F_{ij})$.

Using Proposition 3.3 we can construct a representation of $Y(\mathfrak{a})$ in the space $(C^N)^{\otimes 1}$ by
\[ \sigma_{c_1, \ldots, c_l} = \sigma_{c_1} \otimes \cdots \otimes \sigma_{c_l}, \quad c_i \in C. \]
The image of the element $C'$ under $\sigma_{c_1, \ldots, c_l}$ is a polynomial in $c_1, \ldots, c_l$ of degree $\leq d$. Moreover, the homogeneous component of degree $d$ of this polynomial coincides with $D = \rho_{c_1, \ldots, c_l}(C)$. By Lemma 3.5 there exist values of the parameters $c_1, \ldots, c_l$ such that $D \neq 0$. This implies that the element $C'$ has degree $d$ and so $\psi(C') \neq 0$. □
The following is an analog of the Poincaré–Birkhoff–Witt theorem for the algebra $Y(a)$. It is immediate from Theorem 3.6.

**Corollary 3.7.** Given any total ordering on the set of generators $\tau_{ij}^{(r)}$ with

$$i + j > 0, \quad r \geq 1,$$

in the orthogonal case,

and

$$i + j \geq 0, \quad r \geq 1,$$

in the symplectic case,

the ordered monomials in the generators form a basis of $Y(a)$. \qed

**Remark 3.8.** The algebra $Y(a)$ admits another filtration defined by setting the degree of the generator $\tau_{ij}^{(r)}$ to be equal to $r$. It follows from Corollary 3.2 that the associated graded algebra $\tilde{\text{gr}} Y(a)$ is commutative. Let $\tilde{\tau}_{ij}^{(r)}$ denote the image of $\tau_{ij}^{(r)}$ in the $r$-th component of $\tilde{\text{gr}} Y(a)$. By Corollary 3.7, the graded algebra $\tilde{\text{gr}} Y(a)$ is isomorphic to the algebra of polynomials in the variables $\tilde{\tau}_{ij}^{(r)}$, where the indices $i, j, r$ are subject to the same conditions as in Corollary 3.7. \qed

Recall that $ZX(a)$ is the subalgebra of $X(a)$ generated by the coefficients $z_i$ of the series $z(u)$.

**Corollary 3.9.**

(i) The center of the algebra $Y(a)$ is trivial.

(ii) The center of the algebra $X(a)$ coincides with $ZX(a)$.

(iii) The coefficients $z_1, z_2, \ldots$ of the series $z(u)$ are algebraically independent over $\mathbb{C}$, so that the subalgebra $ZX(a)$ of $X(a)$ is isomorphic to the algebra of polynomials in countably many variables.

**Proof.** It is well known that the center of the universal enveloping algebra $U(a[x])$ is trivial; see e.g. [18, Proposition 2.12]. So (i) and (ii) follow from Theorem 3.6. It is implied by the proof of Theorem 3.6 that the elements $y_1, y_2, \ldots$ of the series $y(u)$ are algebraically independent over $\mathbb{C} \subset Y(a)$. Hence so are the elements $z_i, i \geq 1$. \qed

We shall also need the following version of the Poincaré–Birkhoff–Witt theorem for the algebra $X(a)$.

**Corollary 3.10.** Given any total ordering on the set of elements $t_{ij}^{(r)}$ and $z_r$ with

$$i + j > 0, \quad r \geq 1,$$

in the orthogonal case,

and

$$i + j \geq 0, \quad r \geq 1,$$

in the symplectic case,

the ordered monomials in these elements form a basis of $X(a)$. 

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Proof. By Theorems 3.1, 3.6 and Corollary 3.9(iii), the graded algebra \( \text{gr} X(\mathfrak{a}) \) is isomorphic to the tensor product of the universal enveloping algebra \( U(\mathfrak{a}[x]) \) and the algebra of polynomials \( \mathbb{C}[\zeta, \zeta_2, \ldots] \) in indeterminates \( \zeta \). An isomorphism is given by

\[
\bar{t}_{ij}^{(r)} \mapsto F_{ij} x^{r-1} + \frac{1}{2} \delta_{ij} \zeta_r,
\]
so that \( \zeta_r \) is the image of \( \bar{z}_r \); see (3.9). This implies the statement. \( \square \)

**Proposition 3.11.** The assignment

\[
F_{ij} \mapsto \tau_{ij}^{(1)}
\]

(3.21) defines an embedding \( U(\mathfrak{a}) \hookrightarrow Y(\mathfrak{a}) \), while the assignment

\[
F_{ij} \mapsto \frac{1}{2} \left( t_{ij}^{(1)} - \theta_{ij} t_{-j,-i}^{(1)} \right)
\]

(3.22) defines an embedding \( U(\mathfrak{a}) \hookrightarrow X(\mathfrak{a}) \).

Proof. The defining relations (3.5) and (3.6) of \( Y(\mathfrak{a}) \) imply that the map (3.21) is a homomorphism. Its injectivity follows from Corollary 3.7. Furthermore, by (3.3) we have \( \tau_{ij}^{(1)} = t_{ij}^{(1)} - \delta_{ij} y_1 \). It remains to observe that \( 2y_1 = z_1 = t_{ii}^{(1)} + t_{-i,-i}^{(1)} \) for any \( i \) and \( t_{ij}^{(1)} = -\theta_{ij} t_{-j,-i}^{(1)} \) for \( i \neq j \) by (3.22). \( \square \)

4 Isomorphisms for low rank Yangians

Recall that the Yangian \( Y(\mathfrak{gl}_N) \) for the general linear Lie algebra \( \mathfrak{gl}_N \) is defined as a unital associative algebra with countably many generators \( T_{ij}^{(1)}, T_{ij}^{(2)}, \ldots \) where \( 1 \leq i, j \leq N \), and the defining relations

\[
[T_{ij}^{(r+1)}, T_{kl}^{(s)}] - [T_{ij}^{(r)}, T_{kl}^{(s+1)}] = T_{kj}^{(r)} T_{il}^{(s)} - T_{kj}^{(s)} T_{il}^{(r)},
\]

(4.1)

where \( r, s \geq 0 \) and \( T_{ij}^{(0)} = \delta_{ij} \). Equivalently, these relations can be written as

\[
[T_{ij}^{(r)}, T_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \left( T_{kj}^{(a-1)} T_{il}^{(r+s-a)} - T_{kj}^{(r+s-a)} T_{il}^{(a-1)} \right).
\]

(4.2)

Introducing the generating series,

\[
T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \cdots \in Y(\mathfrak{gl}_N)[[u^{-1}]],
\]

we can also write (4.1) in the form

\[
(u - v) [T_{ij}(u), T_{kl}(v)] = T_{kj}(u) T_{il}(v) - T_{kj}(v) T_{il}(u).
\]

(4.3)
Equivalently, using the notation of Section 2 and introducing the matrices

$$R^\circ(u) = 1 - P u^{-1}$$

(4.4)

and

$$T^\circ(u) = \sum_{i,j=1}^N e_{ij} \otimes T_{ij}(u) \in \text{End} \mathbb{C}^N \otimes Y(\mathfrak{gl}_N)[[u^{-1}]],$$

(4.5)

we can present the defining relations in the form of an RTT-relation

$$R^\circ(u - v) T^\circ_1(u) T^\circ_2(v) = T^\circ_2(v) T^\circ_1(u) R^\circ(u - v);$$

(4.6)

cf. (2.18). We use the superscript "$\circ$" here to distinguish the objects related to $Y(\mathfrak{gl}_N)$ from those related to the algebra $X(\mathfrak{a})$.

The Yangian $Y(\mathfrak{gl}_N)$ is a Hopf algebra with the coproduct

$$\Delta : T_{ij}(u) \mapsto \sum_{k=1}^N T_{ik}(u) \otimes T_{kj}(u).$$

(4.7)

An ascending filtration on $Y(\mathfrak{gl}_N)$ can be defined by setting

$$\deg T^{(r)}_{ij} = r - 1.$$

(4.8)

Let $\overline{T}^{(r)}_{ij}$ denote the image of the generator $T^{(r)}_{ij}$ in the $(r - 1)$-th component of the associated graded algebra $\text{gr} Y(\mathfrak{gl}_N)$. We have an algebra isomorphism

$$U(\mathfrak{gl}_N[x]) \rightarrow \text{gr} Y(\mathfrak{gl}_N), \quad E_{ij} x^{r-1} \mapsto \overline{T}^{(r)}_{ij}.$$ (4.9)

The assignment

$$\text{ev} : T_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1}$$

(4.10)

defines a surjective homomorphism $Y(\mathfrak{gl}_N) \rightarrow U(\mathfrak{gl}_N)$. Moreover, the assignment $E_{ij} \mapsto T^{(1)}_{ij}$ defines an embedding $U(\mathfrak{gl}_N) \hookrightarrow Y(\mathfrak{gl}_N)$.

For any series $g(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ consider the automorphism of $Y(\mathfrak{gl}_N)$ defined by

$$T_{ij}(u) \mapsto g(u) T_{ij}(u).$$

(4.11)

The Yangian for $\mathfrak{sl}_N$ is the subalgebra $Y(\mathfrak{sl}_N)$ of $Y(\mathfrak{gl}_N)$ which consists of the elements stable under all automorphisms (4.11).

The algebra $Y(\mathfrak{gl}_N)$ is isomorphic to the tensor product of its subalgebras

$$Y(\mathfrak{gl}_N) = ZY(\mathfrak{gl}_N) \otimes Y(\mathfrak{sl}_N),$$

(4.12)
where $Z_{Y(gl_N)}$ denotes the center of the algebra $Y(gl_N)$. The subalgebra $Z_{Y(gl_N)}$ is generated by the coefficients of the series $D(u)$ called the quantum determinant. In the case $N = 2$ it takes the form

$$D(u) = T_{11}(u)T_{22}(u - 1) - T_{21}(u)T_{12}(u - 1).$$

Define the series $d(u)$ with coefficients in $Z_{Y(gl_2)}$ by the relation $d(u)d(u - 1) = D(u)$. Then all the coefficients of the series $T_{ij}(u) = d(u)^{-1}T_{ij}(u)$ belong to the subalgebra $Y(sl_2)$. The series $T_{ij}(u)$ satisfy the relations

$$(u - v)[T_{ij}(u), T_{kl}(v)] = T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)$$

and

$$T_{11}(u)T_{22}(u - 1) - T_{21}(u)T_{12}(u - 1) = 1$$

which are defining relations for the algebra $Y(sl_2)$. In other words, the Yangian $Y(sl_2)$ is isomorphic to the quotient of $Y(gl_2)$ by the ideal generated by all the coefficients of $D(u)$.

For more details on the algebraic structure of the Yangians $Y(gl_N)$ and $Y(sl_N)$ see e.g. [18], [5].

4.1 Extended Yangian $X(sp_2)$

Observe that if $N = 2$ then in the symplectic case the operators $P$ and $Q$ satisfy $P + Q = 1$; see (2.9) and (2.10). Therefore, for the corresponding $R$-matrix (2.13) we have

$$R(u) = \frac{u - 1}{u - 2} \left(1 - \frac{2P}{u}\right) = \frac{u - 1}{u - 2} \cdot R^o(u/2).$$

This implies the following isomorphism where we adopt the convention of Section 2 for numbering the rows and columns of $2 \times 2$ matrices by the indices $\{-1, 1\}$.

**Proposition 4.1.** The mapping

$$t_{ij}(u) \mapsto T_{ij}(u/2), \quad i, j \in \{-1, 1\}$$

defines an isomorphism $\phi : X(sp_2) \to Y(gl_2)$.

**Proof.** This is immediate from the defining relations (2.18) and (4.6). \qed

**Corollary 4.2.** The restriction of the isomorphism (4.15) to the subalgebra $Y(sp_2)$ of $X(sp_2)$ induces an isomorphism $Y(sp_2) \to Y(sl_2)$. 

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Proof. Recall that the subalgebra $Y(\mathfrak{sp}_2)$ consists of the elements stable under all automorphisms of $X(\mathfrak{sp}_2)$ of the form (2.21). However, given a series $f(u)$ in $u^{-1}$ with complex coefficients, the mapping (4.15) takes $f(u)$ to $f(u) T_{ij}(u/2)$. So, we have the relation $\phi \circ \mu_f = \mu_g \circ \phi$, and hence $\mu_f \circ \phi^{-1} = \phi^{-1} \circ \mu_g$, where $g(u)$ is the series in $u^{-1}$ defined by $g(u) = f(2u)$. Thus, the image of $Y(\mathfrak{sp}_2)$ under the isomorphism $\phi$ coincides with the subalgebra $Y(\mathfrak{sl}_2)$ of $Y(\mathfrak{gl}_2)$, yielding the desired isomorphism.

Corollary 4.3. The mapping

$$\text{ev} : T(u) \mapsto 1 + F u^{-1}$$ (4.16)

defines a surjective homomorphism $X(\mathfrak{sp}_2) \to U(\mathfrak{sp}_2)$.

Proof. The composition of the evaluation homomorphism (4.10) and the natural projection $\mathfrak{gl}_N \to \mathfrak{sl}_N$ yields a homomorphism $Y(\mathfrak{gl}_N) \to U(\mathfrak{sl}_N)$. For $N = 2$ it takes the form

$$T_{-1,-1}(u) \mapsto 1 + (E_{-1,-1} - E_{1,1}) (2u)^{-1}, \quad T_{-1,1}(u) \mapsto E_{-1,1} u^{-1},$$

$$T_{1,1}(u) \mapsto 1 + (E_{1,1} - E_{-1,-1}) (2u)^{-1}, \quad T_{1,-1}(u) \mapsto E_{1,-1} u^{-1}.$$  

Applying the isomorphism of Proposition 4.1 and using the generators $F_{ij}$ of $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$ we get a homomorphism $X(\mathfrak{sp}_2) \to U(\mathfrak{sp}_2)$ given by

$$\text{ev} : t_{ij}(u) \mapsto \delta_{ij} + F_{ij} u^{-1}, \quad i, j \in \{-1, 1\}.$$  

Obviously, it is surjective.  

4.2 Extended Yangian $X(\mathfrak{o}_3)$

We shall now use a more standard notation for the generators of the Yangian $Y(\mathfrak{gl}_2)$, where the indices $i, j$ in the defining relations (4.1) and (4.3) run over the set $\{1, 2\}$. Consider the vector space $\mathbb{C}^2$ with its canonical basis $e_1, e_2$ and denote by $V$ the three-dimensional subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ spanned by the vectors

$$v_{-1} = e_1 \otimes e_1, \quad v_0 = \frac{1}{\sqrt{2}} (e_1 \otimes e_2 + e_2 \otimes e_1), \quad v_1 = -e_2 \otimes e_2.$$  

We identify $V$ with $\mathbb{C}^3$ regarding $v_{-1}, v_0, v_1$ as its canonical basis. In particular, the operators $P_V$ and $Q_V$ in $V \otimes V$ will be given by the respective formulas (2.9) and (2.10) so that, for instance, $P_V(v_0 \otimes v_1) = v_1 \otimes v_0$. Similarly, we regard the generator
matrix $T(u) = (t_{ij}(u))$ as an element of $\text{End } V \otimes X(\mathfrak{o}_3)[[u^{-1}]]$. Note that the operator $(1 + P)/2$ is a projection of $(\mathbb{C}^2)^{\otimes 2}$ to the subspace $V$. Due to (4.6), we have

$$
\frac{1+P}{2} \cdot T_i^\circ(2u) T_2^\circ(2u+1) = T_2^\circ(2u+1) T_1^\circ(2u) \cdot \frac{1+P}{2},
$$

because $R^\circ(-1) = 1 + P$. Therefore, we may regard each side of this relation as an element of $\text{End } V \otimes Y(\mathfrak{gl}_2)[[u^{-1}]]$.

**Proposition 4.4.** The mapping

$$
T(u) \mapsto \frac{1+P}{2} \cdot T_i^\circ(2u) T_2^\circ(2u+1)
$$

defines an isomorphism $\phi : X(\mathfrak{o}_3) \to Y(\mathfrak{gl}_2)$. More explicitly, the images of the generators under the isomorphism are given by the formulas

- $t_{-1,-1}(u) \mapsto T_{11}(2u) T_{11}(2u+1)$
- $t_{-1,0}(u) \mapsto \frac{1}{\sqrt{2}} \left( T_{11}(2u) T_{12}(2u+1) + T_{12}(2u) T_{11}(2u+1) \right)$
- $t_{-1,1}(u) \mapsto -T_{12}(2u) T_{12}(2u+1)$
- $t_{0,-1}(u) \mapsto \frac{1}{\sqrt{2}} \left( T_{11}(2u) T_{21}(2u+1) + T_{21}(2u) T_{11}(2u+1) \right)$
- $t_{0,0}(u) \mapsto T_{11}(2u) T_{22}(2u+1) + T_{21}(2u) T_{12}(2u+1)$
- $t_{0,1}(u) \mapsto -\frac{1}{\sqrt{2}} \left( T_{12}(2u) T_{22}(2u+1) + T_{22}(2u) T_{12}(2u+1) \right)$
- $t_{1,-1}(u) \mapsto -T_{21}(2u) T_{21}(2u+1)$
- $t_{1,0}(u) \mapsto -\frac{1}{\sqrt{2}} \left( T_{21}(2u) T_{22}(2u+1) + T_{22}(2u) T_{21}(2u+1) \right)$
- $t_{1,1}(u) \mapsto T_{22}(2u) T_{22}(2u+1)$.

**Proof.** We start by showing that the mapping defines an algebra homomorphism. We use a version of the well known fusion procedure for $R$-matrices; see e.g. [2] and references therein.

Consider the tensor product space $(\mathbb{C}^2)^{\otimes 4}$. As in (2.14), we use subscripts of the $R$-matrix or the permutation operator $P \in \text{End } (\mathbb{C}^2)^{\otimes 2}$ to indicate the copies of $\mathbb{C}^2$ where the operator acts. In the following we consider $V \otimes V$ as a natural subspace of $(\mathbb{C}^2)^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 2}$. Obviously, the operator

$$
\frac{1+P_{12}}{2} \cdot \frac{1+P_{34}}{2} = \frac{1}{4} \cdot R_{12}^\circ(-1) R_{34}^\circ(-1)
$$

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is a projection of \((\mathbb{C}^2)^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 2}\) to the subspace \(V \otimes V\). Let us set
\[
R_V(u) = \frac{1 + P_{12}}{2} \cdot \frac{1 + P_{34}}{2} \cdot R_{14}^\circ (2u - 1) R_{13}^\circ (2u) R_{24}^\circ (2u) R_{23}^\circ (2u + 1). \tag{4.18}
\]
Since the \(R\)-matrix \(R^\circ (u)\) satisfies the Yang–Baxter equation (2.14), we have the following equivalent expression for \(R_V(u)\),
\[
R_V(u) = R_{23}^\circ (2u + 1) R_{13}^\circ (2u) R_{24}^\circ (2u) R_{14}^\circ (2u - 1) \cdot \frac{1 + P_{12}}{2} \cdot \frac{1 + P_{34}}{2}. \tag{4.19}
\]
Clearly, the subspace \(V \otimes V\) is stable under the operator \(R_V(u)\).

**Lemma 4.5.** We have the equality of operators in \(V \otimes V\),
\[
R_V(u) = \frac{2u - 1}{2u + 1} \cdot \left(1 - \frac{P_V}{u} + \frac{Q_V}{u - 1/2}\right). \tag{4.20}
\]

**Proof.** Using the formulas of the kind
\[
(1 + P_{12}) P_{14} P_{24} = (1 + P_{12}) P_{14}
\]
and
\[
(1 + P_{12}) (1 + P_{34}) P_{14} P_{23} = (1 + P_{12}) (1 + P_{34}) P_{13} P_{24},
\]
it is easy to get a simplified expression for the operator \(R_V(u)\),
\[
R_V(u) = \frac{1 + P_{12}}{2} \cdot \frac{1 + P_{34}}{2} \cdot \left(1 - \frac{P_{14} + P_{24} + P_{13} + P_{23}}{2u + 1} + \frac{P_{13} P_{24}}{u (2u + 1)}\right).
\]
The restriction of \(R_V(u)\) to the subspace \(V \otimes V\) is given by
\[
1 - \frac{P_{14} + P_{24} + P_{13} + P_{23}}{2u + 1} + \frac{P_{13} P_{24}}{u (2u + 1)} \tag{4.21}
\]
so that the proof of the lemma is completed by the application of (4.21) to all basis vectors \(v_i \otimes v_j\) of \(V \otimes V\). For instance, we have
\[
R_V(u)(v_{-1} \otimes v_{-1}) = \left(1 - \frac{P_{14} + P_{24} + P_{13} + P_{23}}{2u + 1} + \frac{P_{13} P_{24}}{u (2u + 1)}\right)(e_1 \otimes e_1 \otimes e_1 \otimes e_1)
\]
\[
= \frac{(u - 1)(2u - 1)}{u (2u + 1)} \cdot e_1 \otimes e_1 \otimes e_1 \otimes e_1.
\]
Clearly, the application of the operator on the right hand side of (4.20) to the vector \(v_{-1} \otimes v_{-1}\) gives the same result. The remaining cases are verified by the same calculation. \(\square\)
By the lemma, the element $R_V(u)$ coincides with the $R$-matrix $\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ for $a = o_3$, up to a scalar factor. So, in order to verify that the mapping $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ defines a homomorphism $X(o_3) \rightarrow Y(gl_2)$ we need to show that the relation

$$R_V(u - v) T_V(u) T_{V'}(v) = T_{V'}(v) T_V(u) R_V(u - v)$$

(4.22)

remains valid when $T(u)$ is replaced by its image. Here we use primed indices to indicate the copies of the space $V$ in the tensor product $V \otimes V$. We reserve unprimed indices for the copies of $\mathbb{C}^2$ in the tensor product $(\mathbb{C}^2)^{\otimes 4}$. The left hand side of (4.22) reads

$$\frac{1 + P_{12}}{2} \cdot \frac{1 + P_{34}}{2} \cdot R^o_{14} (2u - 2v - 1) R^o_{13} (2u - 2v) R^o_{24} (2u - 2v) R^o_{23} (2u - 2v + 1)$$

$$\times \frac{1 + P_{12}}{2} \cdot T^o_1 (2u) T^o_2 (2u + 1) \cdot \frac{1 + P_{34}}{2} \cdot T^o_3 (2v) T^o_4 (2v + 1).$$

Writing the product of $R$-matrices in the equivalent form (4.19), we simplify this expression to

$$\frac{1 + P_{12}}{2} \cdot \frac{1 + P_{34}}{2} \cdot R^o_{14} (2u - 2v - 1) R^o_{13} (2u - 2v) R^o_{24} (2u - 2v) R^o_{23} (2u - 2v + 1)$$

$$\times T^o_1 (2u) T^o_2 (2u + 1) T^o_3 (2v) T^o_4 (2v + 1).$$

Now apply the $RTT$-relation (4.6) repeatedly to bring this expression to the form

$$\frac{1 + P_{12}}{2} \cdot \frac{1 + P_{34}}{2} \cdot T^o_3 (2v) T^o_4 (2v + 1) T^o_1 (2u) T^o_2 (2u + 1)$$

$$\times R^o_{14} (2u - 2v - 1) R^o_{13} (2u - 2v) R^o_{24} (2u - 2v) R^o_{23} (2u - 2v + 1).$$

Finally, since $R^o_{12}(-1)/2$ is a projection, we derive from (4.6) that

$$\frac{1 + P_{12}}{2} \cdot T^o_1 (2u) T^o_2 (2u + 1) = \frac{1 + P_{12}}{2} \cdot T^o_1 (2u) T^o_2 (2u + 1) \cdot \frac{1 + P_{12}}{2}.$$

Using the same property of $R^o_{34}(-1)/2$ we obtain that the resulting expression coincides with the right hand side of (4.22), where $T(u)$ is replaced with its image in accordance with (4.17).

The explicit images of the generators of $X(o_3)$ are found by taking the matrix elements in (4.17). Indeed, the application of $T(u)$ to the basis vector $v_{-1}$ of $V$ gives

$$T(u)(v_{-1}) = t_{-1,-1}(u) v_{-1} + t_{0,-1}(u) v_0 + t_{1,-1}(u) v_1,$$

while

$$\frac{1 + P_{12}}{2} \cdot T^o_1 (2u) T^o_2 (2u + 1)(v_{-1}) = \frac{1 + P_{12}}{2} \cdot T^o_1 (2u) T^o_2 (2u + 1)(e_1 \otimes e_1)$$

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\[
\begin{align*}
&= \frac{1}{2} \sum_{a,b=1}^{2} T_{a1}(2u) T_{b1}(2u + 1) (e_a \otimes e_b + e_b \otimes e_a) = T_{11}(2u) T_{11}(2u + 1) v_{-1} \\
&\quad + \frac{1}{\sqrt{2}} \left( T_{11}(2u) T_{21}(2u + 1) + T_{21}(2u) T_{11}(2u + 1) \right) v_0 - T_{21}(2u) T_{21}(2u + 1) v_1.
\end{align*}
\]

This agrees with the formulas for the images of the series \( t_{a, -1}(u) \) for \( a = -1, 0, 1 \) given in the statement. The remaining formulas are verified in the same way. Note also that the image of the series \( t_{0,0}(u) \) can be equivalently written as

\[
T_{12}(2u) T_{21}(2u + 1) + T_{22}(2u) T_{11}(2u + 1)
\]
due to the defining relation in the Yangian \( \text{Y} (\mathfrak{g} \mathfrak{l}_2) \).

In order to complete the proof of the proposition we now verify that the homomorphism \( X(\mathfrak{o}_3) \to \text{Y} (\mathfrak{g} \mathfrak{l}_2) \) given by (4.17) is bijective. Taking the coefficient at \( u^{-r} \) in \( t_{-1, -1}(u) \) we find that for any \( r \geq 1 \)

\[
t_{-1, -1}^{(r)} \mapsto 2^{-r+1} T_{11}^{(r)} + A_{r-1} (T_{11}^{(1)}, \ldots, T_{11}^{(r-1)}),
\]

where \( A_{r-1} \) stands for a quadratic polynomial in the generators \( T_{11}^{(1)}, \ldots, T_{11}^{(r-1)} \). The obvious induction on \( r \) shows that each generator \( T_{11}^{(r)} \) with \( r \geq 1 \) belongs to the image of the homomorphism. Similarly, taking the image of \( t_{1,1}^{(r)} \) we find that each generator \( T_{22}^{(r)} \) with \( r \geq 1 \) also belongs to the image. Then taking the images of \( t_{-1,0}^{(r)} \) and \( t_{0,-1}^{(r)} \) we derive the same property of the generators \( T_{12}^{(r)} \) and \( T_{21}^{(r)} \) with \( r \geq 1 \). This proves that the homomorphism is surjective.

Finally, observe that the homomorphism preserves the respective filtrations on \( X(\mathfrak{o}_3) \) and \( \text{Y} (\mathfrak{g} \mathfrak{l}_2) \). Hence, we have a homomorphism of the associated graded algebras \( \text{gr} X(\mathfrak{o}_3) \to \text{gr} \text{Y} (\mathfrak{g} \mathfrak{l}_2) \). It suffices to show that this homomorphism is injective. Identifying \( \text{gr} \text{Y} (\mathfrak{g} \mathfrak{l}_2) \) with the universal enveloping algebra \( U(\mathfrak{g} \mathfrak{l}_2[x]) \) via the isomorphism (4.9), we get

\[
\bar{t}_{0,1}^{(r)} \mapsto -\frac{1}{\sqrt{2}} E_{12} (x/2)^{r-1}, \quad \bar{t}_{1,0}^{(r)} \mapsto -\frac{1}{\sqrt{2}} E_{21} (x/2)^{r-1}, \quad \bar{t}_{1,1}^{(r)} \mapsto E_{22} (x/2)^{r-1}
\]

and

\[
\bar{z}_r \mapsto (E_{11} + E_{22}) (x/2)^{r-1}.
\]

Therefore, the injectivity of the homomorphism follows from Corollary 3.10. \( \square \)

**Corollary 4.6.** The restriction of the isomorphism \( \phi : X(\mathfrak{o}_3) \to \text{Y} (\mathfrak{g} \mathfrak{l}_2) \) to the subalgebra \( \text{Y} (\mathfrak{o}_3) \) induces an isomorphism \( \text{Y} (\mathfrak{o}_3) \to \text{Y} (\mathfrak{sl}_2) \).
Proof. Recall that the subalgebra \( Y(\mathfrak{o}_3) \) consists of the elements stable under all automorphisms of \( X(\mathfrak{o}_3) \) of the form (2.21). For any series \( f(u) \) of the form (2.20) there exists a unique series
\[
g(u) = 1 + g_1 u^{-1} + g_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]
\]
such that \( f(u) = g(2u) g(2u + 1) \). By Proposition 4.4, we have the relation \( \phi \circ \mu_f = \mu_g \circ \phi \), and hence \( \mu_f \circ \phi^{-1} = \phi^{-1} \circ \mu_g \). This implies that the image of \( Y(\mathfrak{o}_3) \) under the isomorphism \( \phi \) coincides with the subalgebra \( Y(\mathfrak{sl}_2) \) of \( Y(\mathfrak{gl}_2) \) thus yielding the desired isomorphism.

Let us denote by \( c \) the Casimir element for the Lie algebra \( \mathfrak{o}_3 \),
\[
c = \frac{1}{2} (F_{11}^2 - F_{11}) + F_{10} F_{01}.
\]
In the following we use notation (2.6).

Corollary 4.7. The mapping
\[
ev : T(u) \mapsto 1 + \frac{F}{u} + \frac{F^2 - c}{u(2u - 1)} \quad (4.23)
\]
defines a surjective homomorphism \( X(\mathfrak{o}_3) \to U(\mathfrak{o}_3) \).

Proof. Writing the homomorphism \( Y(\mathfrak{gl}_2) \to U(\mathfrak{sl}_2) \) used in the proof of Corollary 4.3 in the current notation we get
\[
T_{11}(u) \mapsto 1 + (E_{11} - E_{22}) (2u)^{-1}, \quad T_{12}(u) \mapsto E_{12} u^{-1},
\]
\[
T_{22}(u) \mapsto 1 + (E_{22} - E_{11}) (2u)^{-1}, \quad T_{21}(u) \mapsto E_{21} u^{-1}.
\]
Composing this with the isomorphism \( \mathfrak{sl}_2 \to \mathfrak{o}_3 \) given by
\[
E_{11} - E_{22} \mapsto 2F_{-1,-1}, \quad E_{12} \mapsto \sqrt{2} F_{-1,0}, \quad E_{21} \mapsto \sqrt{2} F_{0,-1},
\]
we get another homomorphism \( Y(\mathfrak{gl}_2) \to U(\mathfrak{o}_3) \) such that
\[
T_{11}(u) \mapsto 1 + F_{-1,-1} u^{-1}, \quad T_{12}(u) \mapsto \sqrt{2} F_{-1,0} u^{-1},
\]
\[
T_{22}(u) \mapsto 1 + F_{1,1} u^{-1}, \quad T_{21}(u) \mapsto \sqrt{2} F_{0,-1} u^{-1}.
\]
Finally, compose the isomorphism of Proposition 4.4 with the shift automorphism \( t_{ij}(u) \mapsto t_{ij}(u - 1/2) \) of \( X(\mathfrak{o}_3) \) and use the above formulas to get a homomorphism \( X(\mathfrak{o}_3) \to U(\mathfrak{o}_3) \). It remains to verify that the resulting formulas for the images of
$t_{ij}(u)$ agree with (4.23). This can be done by an easy straightforward calculation. For instance, for the image of $t_{0,0}(u)$ we calculate

$$t_{0,0}(u) \mapsto T_{11}(2u - 1) T_{22}(2u) + T_{21}(2u - 1) T_{12}(2u)$$

$$\mapsto \left(1 + \frac{F_{-1,-1}}{2u - 1}\right) \left(1 + \frac{F_{1,1}}{2u}\right) + 2 \cdot \frac{F_{0,-1}}{2u - 1} \cdot \frac{F_{-1,0}}{2u}.$$  \hfill (4.24)

On the other hand, formula (4.23) gives

$$t_{0,0}(u) \mapsto 1 + \frac{2F_{0,-1} F_{-1,0} + 2F_{0,1} F_{1,0} - F_{11}^2 + F_{11} - 2 F_{10} F_{01}}{2u (2u - 1)}$$

$$= 1 + \frac{2F_{0,-1} F_{-1,0} - F_{11}^2 - F_{11}}{2u (2u - 1)}.$$  

Clearly, this agrees with (4.24). All the remaining cases are verified by a similar and even shorter calculation. Obviously, the homomorphism (4.23) is surjective.

\[\square\]

### 4.3 Extended Yangian $X(\mathfrak{o}_4)$

We shall need the tensor product algebra $Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2)$. In order to distinguish the two copies of the algebra $Y(\mathfrak{gl}_2)$, we denote the corresponding generator series respectively by $T_{ij}(u)$ and $T'_{ij}(u)$ for the first and second copies, where $i, j \in \{1, 2\}$. We also identify $T_{ij}(u) \otimes 1$ with $T_{ij}(u)$ and $1 \otimes T'_{ij}(u)$ with $T'_{ij}(u)$. As before, we combine the series $T_{ij}(u)$ and $T'_{ij}(u)$ into the matrices $T^{\circ}(u)$ and $T'^{\circ}(u)$, respectively.

The algebra $Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2)$ is naturally equipped with an ascending filtration, where the degrees of the elements on each copy of $Y(\mathfrak{gl}_2)$ are defined by (4.8).

Consider the vector space $\mathbb{C}^2$ with its canonical basis $e_1, e_2$ and set $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. We identify $V$ with $\mathbb{C}^4$ regarding the vectors

$$v_{-2} = e_1 \otimes e_1, \quad v_{-1} = e_1 \otimes e_2, \quad v_1 = e_2 \otimes e_1, \quad v_2 = -e_2 \otimes e_2$$

as the canonical basis of $V$. Then $T^{\circ}_1(u) T'^{\circ}_2(u)$ may be regarded as an element of $\text{End} \, V \otimes (Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2))[[u^{-1}]]$. The operators $P_V$ and $Q_V$ in $V \otimes V$ are given by the respective formulas (2.29) and (2.10). We shall regard the matrix $T(u) = (t_{ij}(u))$ as an element of $\text{End} \, V \otimes X(\mathfrak{o}_4)[[u^{-1}]]$.

**Proposition 4.8.** The mapping

$$T(u) \mapsto T^{\circ}_1(u) T'^{\circ}_2(u),$$  \hfill (4.25)

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defines an embedding $\psi : X(0_4) \hookrightarrow Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2)$. More explicitly, the images of the generators under the embedding are given by the formulas

$$
\begin{align*}
  t_{-2,-2}(u) & \mapsto T_{11}(u) T'_{11}(u), \\
  t_{-2,1}(u) & \mapsto T_{12}(u) T'_{11}(u), \\
  t_{-1,-2}(u) & \mapsto T_{11}(u) T_{21}(u), \\
  t_{-1,1}(u) & \mapsto T_{12}(u) T'_{21}(u), \\
  t_{1,-2}(u) & \mapsto T_{21}(u) T'_{11}(u), \\
  t_{1,1}(u) & \mapsto T_{22}(u) T'_{11}(u), \\
  t_{2,-2}(u) & \mapsto -T_{21}(u) T'_{21}(u), \\
  t_{2,1}(u) & \mapsto -T_{22}(u) T'_{21}(u),
\end{align*}
$$

Proof. We start by showing that the mapping defines an algebra homomorphism. Identifying $V \otimes V$ with the tensor product space $(\mathbb{C}^2)^{\otimes 4}$, we set

$$
R_V(u) = R^{\circ}_V(u) R^{\circ}_{24}(u). \tag{4.26}
$$

Lemma 4.9. We have the equality of operators in $V \otimes V$,

$$
R_V(u) = \frac{u-1}{u} \cdot \left(1 - \frac{P_V}{u} + \frac{Q_V}{u-1}\right). \tag{4.27}
$$

Proof. We have

$$
R_V(u) = \left(1 - \frac{P_{13}}{u}\right) \left(1 - \frac{P_{24}}{u}\right) = \frac{u-1}{u} \left(1 - \frac{P_{13}P_{24}}{u} + \frac{(1-P_{13})(1-P_{24})}{u-1}\right).
$$

It remains to note that $P_V = P_{13}P_{24}$ and $Q_V = (1-P_{13})(1-P_{24})$. This is verified by the application of the operators to all basis vectors $v_i \otimes v_j$ of $V \otimes V$. For instance, by the definition of $Q_V$,

$$
Q_V(v_{-2} \otimes v_2) = v_{-2} \otimes v_2 + v_{-1} \otimes v_1 + v_1 \otimes v_{-1} + v_2 \otimes v_{-2},
$$

while

$$
(1-P_{13})(1-P_{24})(v_{-2} \otimes v_2) = (1-P_{13})(1-P_{24})(-e_1 \otimes e_1 \otimes e_2 \otimes e_2)
$$

$$
= -e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_2 \otimes e_1 \otimes e_1,
$$

which clearly coincides with $Q_V(v_{-2} \otimes v_2)$. The remaining relations are verified in the same way.\[\square\]
By the lemma, the element $R_Y(u)$ coincides with the $R$-matrix \[2.13\] for $a = \mathfrak{o}_4$, up to a scalar factor. So, in order to verify that the mapping \[4.28\] defines a homomorphism $X(\mathfrak{o}_4) \to Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2)$ we need to show that the relation

$$R_Y(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_Y(u - v)$$

remains valid when $T(u)$ is replaced by its image. The primed indices are used here to indicate the copies of the space $V$ in the tensor product $V \otimes V$. The left hand side of \[4.28\] reads

$$R_{i_1}^o(u - v) R_{i_2}^o(u - v) T_1^o(u) T_2^o(u) T_3^o(v) T_4^o(v).$$

Applying the $RTT$-relation \[4.6\] twice, we bring this expression to the form

$$T_3^o(v) T_4^o(v) T_1^o(u) T_2^o(u) R_{i_3}^o(u - v) R_{i_2}^o(u - v)$$

which coincides with the right hand side of \[4.28\], where $T(u)$ is replaced with its image in accordance with \[4.25\].

The explicit images of the generators of $X(\mathfrak{o}_4)$ are found by taking the matrix elements in \[4.25\]. Indeed, the application of $T(u)$ to the basis vector $v_{-2}$ of $V$ gives

$$T(u)(v_{-2}) = t_{-2,-2}(u) v_{-2} + t_{-1,-2}(u) v_{-1} + t_{1,-2}(u) v_1 + t_{2,-2}(u) v_2,$$

while

$$T_1^o(u) T_2^o(u)(v_{-2}) = T_1^o(u) T_2^o(u)(e_1 \otimes e_1) = \sum_{a,b=1}^{2} T_{a1}(u) T_{b1}^t(u) (e_a \otimes e_b)$$

$$= T_{11}(u) T_{11}^t(u) v_{-2} + T_{1l}(u) T_{1l}^t(u) v_{-1} + T_{2l}(u) T_{2l}^t(u) v_{-1} + T_{21}(u) T_{21}^t(u) v_1 - T_{21}(u) T_{21}^t(u) v_2.$$  

This agrees with the formulas for the images of the series $t_{a,-2}(u)$ for $a = -2, -1, 1, 2$ given in the statement. The remaining formulas are verified in the same way.

In order to demonstrate that the homomorphism $\psi$ is injective, observe that it preserves the respective filtrations on $X(\mathfrak{o}_4)$ and $Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2)$. Hence, we have a homomorphism of the associated graded algebras

$$\text{gr } X(\mathfrak{o}_4) \to \text{gr } \left(Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2)\right).$$

Identifying the graded algebra $\text{gr } Y(\mathfrak{gl}_2)$ with $U(\mathfrak{gl}_2[x])$ via the isomorphism \[4.9\], we get a homomorphism

$$\text{gr } X(\mathfrak{o}_4) \to U(\mathfrak{gl}_2[x]) \otimes U(\mathfrak{gl}_2[y])$$

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so that
\[
\tilde{t}_{-1,2}^{(r)} \mapsto -E_{12} x^{r-1}, \quad \tilde{t}_{1,2}^{(r)} \mapsto -E_{12} y^{r-1}, \quad \tilde{t}_{1,1}^{(r)} \mapsto E_{22} x^{r-1} + E_{11} y^{r-1},
\]
\[
\tilde{t}_{2,-1}^{(r)} \mapsto -E_{21} x^{r-1}, \quad \tilde{t}_{2,1}^{(r)} \mapsto -E_{21} y^{r-1}, \quad \tilde{t}_{2,2}^{(r)} \mapsto E_{22} x^{r-1} + E_{22} y^{r-1},
\]
and
\[
z_r \mapsto (E_{11} + E_{22}) x^{r} + (E_{11} + E_{22}) y^{r}.
\]
Therefore, the injectivity of \(\psi\) follows from Corollary 3.10. \(\square\)

Due to the presentation of the Yangian \(Y(\mathfrak{sl}_2)\) provided by (4.3) and (4.4) we have a natural projection \(Y(\mathfrak{gl}_2) \to Y(\mathfrak{sl}_2)\) defined by the mapping \(T_{ij}(u) \mapsto T_{ij}(u)\). Applying this projection to the first or second copy of \(Y(\mathfrak{gl}_2)\) in the tensor product algebra \(Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2)\) and taking its composition with the embedding \(\psi\) we get homomorphisms
\[
\chi^{(1)} : X(\mathfrak{o}_4) \to Y(\mathfrak{sl}_2) \otimes Y(\mathfrak{gl}_2), \quad \chi^{(2)} : X(\mathfrak{o}_4) \to Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{sl}_2).
\]

**Corollary 4.10.** The homomorphisms \(\chi^{(1)}\) and \(\chi^{(2)}\) are bijective.

**Proof.** We only consider \(\chi^{(1)}\), the proof for \(\chi^{(2)}\) is similar. By the formulas of Proposition 4.8 we have
\[
\chi^{(1)} : t_{-2,-2}(u) t_{1,1}(u - 1) - t_{1,-2}(u) t_{-2,1}(u - 1) \mapsto \left( T_{11}(u) T_{22}(u - 1) - T_{21}(u) T_{12}(u - 1) \right) T_{11}'(u) T_{11}'(u - 1) = T_{11}'(u) T_{11}'(u - 1).
\]
Therefore, all the coefficients of the series \(T_{11}'(u)\) belong to the image of \(\chi^{(1)}\). Hence, so do the coefficients of \(T_{ij}(u)\) with \(i, j \in \{1, 2\}\). This implies that \(\chi^{(1)}\) is surjective. To verify the injectivity of \(\chi^{(1)}\) we use the same argument as in the proof of Proposition 4.8. Namely, \(\chi^{(1)}\) induces a homomorphism of the associated graded algebras
\[
\text{gr} X(\mathfrak{o}_4) \to U(\mathfrak{sl}_2[x]) \otimes U(\mathfrak{gl}_2[y])
\]
and the argument is completed by the application of Corollary 3.10. \(\square\)

**Corollary 4.11.** The restriction of each isomorphism \(\chi^{(1)}\) and \(\chi^{(2)}\) to the subalgebra \(Y(\mathfrak{o}_4)\) induces an isomorphism \(Y(\mathfrak{o}_4) \to Y(\mathfrak{sl}_2) \otimes Y(\mathfrak{sl}_2)\).

**Proof.** Again, we only consider the isomorphism \(\chi^{(1)}\). The subalgebra \(Y(\mathfrak{o}_4)\) consists of the elements stable under all automorphisms of \(X(\mathfrak{o}_4)\) of the form (2.21). For any formal series \(f(u)\) of the form (2.20) consider the automorphism \(\tilde{\mu}_f\) of the algebra \(Y(\mathfrak{sl}_2) \otimes Y(\mathfrak{gl}_2)\) defined by
\[
\tilde{\mu}_f : T_{ij}(u) \mapsto T_{ij}(u), \quad T_{ij}'(u) \mapsto f(u) T_{ij}'(u).
\]

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By the definition of $\chi^{(1)}$, we have the relation $\chi^{(1)} \circ \mu_f = \tilde{\mu}_f \circ \chi^{(1)}$. This implies that if $y \in Y(o_4)$ then $\chi^{(1)}(y)$ is stable under the automorphisms $\tilde{\mu}_f$ for all series $f(u)$. Hence, the image of the subalgebra $Y(o_4)$ of $X(o_4)$ under the isomorphism $\chi^{(1)}$ coincides with the subalgebra $Y(sl_2) \otimes Y(sl_2)$ of $Y(sl_2) \otimes Y(gl_2)$ thus providing the desired isomorphism.

Let us denote by $c$ the following Casimir element for the Lie algebra $o_4$,

$$c = \frac{1}{2}(F_{11}^2 + F_{22}^2) - F_{22} + F_{21}F_{12} + F_{2,-1}F_{-1,2}.$$

In the following we use notation (2.6).

**Corollary 4.12.** The mapping

$$ev : T(u) \mapsto 1 + \frac{F}{u} + \frac{F^2 - F - c}{2u^2}$$

(4.29)

defines a surjective homomorphism $X(o_4) \to U(o_4)$.

**Proof.** Consider the isomorphism $sl_2 \oplus sl_2 \to o_4$ given by

$$E_{11} - E_{22} \mapsto -F_{11} - F_{22}, \quad E_{12} \mapsto F_{-2,1}, \quad E_{21} \mapsto F_{1,-2},$$

and

$$E'_{11} - E'_{22} \mapsto -F_{11} - F_{22}, \quad E'_{12} \mapsto F_{-2,-1}, \quad E'_{21} \mapsto F_{-1,-2},$$

where the primes indicate the basis elements of the second copy of $sl_2$. Applying the homomorphism $Y(gl_2) \to U(sl_2)$ used in the proof of Corollary 4.7 we get a homomorphism $Y(gl_2) \otimes Y(gl_2) \to U(o_4)$ such that

$$T_{11}(u) \mapsto 1 - \frac{F_{11} + F_{22}}{2} u^{-1}, \quad T_{12}(u) \mapsto F_{-2,1} u^{-1},$$

$$T_{22}(u) \mapsto 1 + \frac{F_{11} + F_{22}}{2} u^{-1}, \quad T_{21}(u) \mapsto F_{1,-2} u^{-1}$$

and

$$T'_{11}(u) \mapsto 1 + \frac{F_{11} - F_{22}}{2} u^{-1}, \quad T'_{12}(u) \mapsto F_{-2,-1} u^{-1},$$

$$T'_{22}(u) \mapsto 1 - \frac{F_{11} - F_{22}}{2} u^{-1}, \quad T'_{21}(u) \mapsto F_{-1,-2} u^{-1}.$$

Using the isomorphism of Proposition 4.8 we get a homomorphism $X(o_4) \to U(o_4)$. It remains to verify that the resulting formulas for the images of $t_{ij}(u)$ agree with

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This can be done by an easy straightforward calculation. For instance, for the image of $t_{-2,-2}(u)$ we calculate

$$
t_{-2,-2}(u) \mapsto T_{11}(u) T'_{11}(u) \mapsto \left(1 - \frac{F_{11} + F_{22}}{2} u^{-1}\right) \left(1 + \frac{F_{11} - F_{22}}{2} u^{-1}\right) = 1 + F_{-2,-2} u^{-1} + \frac{F_{2,-2}^2 - F_{-1,-1}^2}{4} u^{-2}.
$$

(4.30)

On the other hand, formula (4.29) gives

$$
t_{-2,-2}(u) \mapsto 1 + F_{-2,-2} u^{-1} + \frac{F_{2,-2}^2 + F_{-2,-1} F_{-1,-2} + F_{-2,1} F_{1,-2} - F_{-2,-2} - c}{2u^2}
$$

which agrees with (4.30). All the remaining cases are verified by a similar calculation. Obviously, the homomorphism (4.29) is surjective.

**Remark 4.13.** The respective compositions of the evaluation homomorphisms provided by Corollaries 4.3, 4.7 and 4.12 with the shift automorphism $\tau_a$ given by (2.22) yields the homomorphisms $ev_a = ev \circ \tau_a$ with the evaluation parameter $a$.

5 Representations of the extended Yangians

Here we introduce the highest weight representations for the extended Yangians $X(\mathfrak{a})$, where as before, $\mathfrak{a} = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{o}_{2n}$. We show by a standard argument that finite-dimensional irreducible representations of $X(\mathfrak{a})$ are highest weight representations. Then we give necessary and sufficient conditions for the irreducible highest weight representations to be finite-dimensional. In particular, we obtain an alternative proof of Drinfeld’s classification theorem for the finite-dimensional irreducible representations of the Yangians $Y(\mathfrak{a})$.

5.1 Highest weight representations

A representation $V$ of the algebra $X(\mathfrak{a})$ is called a highest weight representation if there exists a nonzero vector $\xi \in V$ such that $V$ is generated by $\xi$, for $-n \leq i < j \leq n$, and

$$
t_{ij}(u) \xi = 0 \quad \text{for} \quad -n \leq i < j \leq n, \quad \text{and}
$$

$$
t_{ii}(u) \xi = \lambda_i(u) \xi \quad \text{for} \quad -n \leq i \leq n,
$$

(5.1)

for some formal series

$$
\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \cdots, \quad \lambda_i^{(r)} \in \mathbb{C},
$$

(5.2)
where the value \( i = 0 \) only occurs in the case \( a = \mathfrak{o}_{2n+1} \). The vector \( \xi \) is called the highest vector of \( V \) and the tuple \( \lambda(u) = (\lambda_{-n}(u), \ldots, \lambda_n(u)) \) of the formal series is the highest weight of \( V \).

Let us identify the elements \( F_{ij} \in \mathfrak{a} \) with their images in \( X(\mathfrak{a}) \) under the embedding (3.22). The defining relations (2.19) imply

\[
[t_{ij}^{(1)}, t_{kl}(u)] = \delta_{kj} t_{il}(u) - \delta_{il} t_{kj}(u) - \delta_{k,-i} \theta_{ij} t_{-j,l}(u) + \delta_{l,-j} \theta_{ij} t_{k,-i}(u).
\]

Also, due to (2.28) we have

\[
t_{ij}^{(1)} + \theta_{ij} t_{-j,-i}^{(1)} = \delta_{ij} z_1.
\]

Therefore, \( F_{ij} = t_{ij}^{(1)} - \delta_{ij} z_1/2 \). Since \( z_1 \) is central in \( X(\mathfrak{a}) \), this gives

\[
[F_{ij}, t_{kl}(u)] = \delta_{kj} t_{il}(u) - \delta_{il} t_{kj}(u) - \delta_{k,-i} \theta_{ij} t_{-j,l}(u) + \delta_{l,-j} \theta_{ij} t_{k,-i}(u). \tag{5.3}
\]

Take the linear span of the elements \( F_{11}, \ldots, F_{nn} \) as the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{a} \) and consider the standard triangular decomposition of \( \mathfrak{a} \). Then the nonzero elements \( F_{ij} \) with \( i < j \) are the positive root vectors. The corresponding positive roots are

\[-\varepsilon_i - \varepsilon_j, \quad \varepsilon_i - \varepsilon_j \quad \text{with} \quad 1 \leq i < j \leq n\]

for \( a = \mathfrak{o}_{2n} \),

\[-2 \varepsilon_i \quad \text{with} \quad 1 \leq i \leq n \quad \text{and} \quad -\varepsilon_i - \varepsilon_j, \quad \varepsilon_i - \varepsilon_j \quad \text{with} \quad 1 \leq i < j \leq n\]

for \( a = \mathfrak{sp}_{2n} \), and

\[-\varepsilon_i \quad \text{with} \quad 1 \leq i \leq n \quad \text{and} \quad -\varepsilon_i - \varepsilon_j, \quad \varepsilon_i - \varepsilon_j \quad \text{with} \quad 1 \leq i < j \leq n\]

for \( a = \mathfrak{o}_{2n+1} \), where \( \varepsilon_i \) denotes the element of \( \mathfrak{h}^* \) defined by \( \varepsilon_i(F_{jj}) = \delta_{ij} \). The standard partial ordering on the set of weights of any \( \mathfrak{a} \)-module is now defined as follows. If \( \alpha \) and \( \beta \) are two weights, then \( \alpha \) precedes \( \beta \) if \( \beta - \alpha \) is a \( \mathbb{Z}_+ \)-linear combination of the positive roots.

**Theorem 5.1.** Every finite-dimensional irreducible representation \( V \) of the algebra \( X(\mathfrak{a}) \) is a highest weight representation. Moreover, \( V \) contains a unique, up to a constant factor, highest vector.

**Proof.** Introduce the subspace \( V^0 \) of \( V \) by

\[
V^0 = \{ \eta \in V \mid t_{ij}(u) \eta = 0, \quad -n \leq i < j \leq n \}. \tag{5.4}
\]
We show first that $V^0$ is nonzero. Consider the set of weights of $V$, where $V$ is regarded as the $a$-module defined via the embedding \((3.22)\). This set is finite and hence contains a maximal weight $\nu$ with respect to the partial ordering on the set of weights of $V$. The corresponding weight vector $\eta$ belongs to $V^0$. Indeed, if $i < j$ then by \((5.3)\) the weight of $t_{ij}(u) \eta$ has the form $\nu + \alpha$ for a positive root $\alpha$. By the maximality of $\nu$, we have $t_{ij}(u) \eta = 0$.

Next, we show that all the operators $t_{kk}(u)$ preserve the subspace $V^0$. Consider first the case $a = o_{2n+1}$. In the following argument we write $\equiv$ for an equality of operators in $V^0$. Due to \((5.3)\), it suffices to show that for any $i$ and $k$ we have

$$t_{i,i+1}(u) t_{kk}(v) \equiv 0. \tag{5.5}$$

Suppose first that $i < k$. Then \((5.5)\) is immediate from \((2.19)\) except for the cases $i = -k$ and $i = -k - 1$. In the former case, we have $k > 0$ and so \((2.19)\) gives

$$t_{-k,-k+1}(u) t_{kk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{p=k}^{n} t_{-p,-k+1}(u) t_{pk}(v), \tag{5.6}$$

while for each $p \geq k$,

$$t_{-p,-k+1}(u) t_{pk}(v) \equiv -\frac{1}{u-v-\kappa} \sum_{q=k}^{n} t_{-q,-k+1}(u) t_{qk}(v).$$

Hence, $t_{-p,-k+1}(u) t_{pk}(v) \equiv t_{-k,-k+1}(u) t_{kk}(v)$. So, \((5.6)\) implies

$$\left(1 + \frac{n-k+1}{u-v-\kappa}\right) t_{-k,-k+1}(u) t_{kk}(v) \equiv 0$$

and thus, $t_{-k,-k+1}(u) t_{kk}(v) \equiv 0$ verifying \((5.5)\).

Similarly, in the case $i = -k - 1$ we have $k \geq 0$ and so

$$t_{-k-1,-k}(u) t_{kk}(v) \equiv \frac{1}{u-v-\kappa} \sum_{p=k+1}^{n} t_{kp}(v) t_{-k-1,-p}(u). \tag{5.7}$$

For each $p \geq k + 1$ we have

$$t_{kp}(v) t_{-k-1,-p}(u) \equiv -[t_{-k-1,-p}(u), t_{kp}(v)] \equiv -\frac{1}{u-v-\kappa} \sum_{q=k+1}^{n} t_{kq}(v) t_{-k-1,-q}(u).$$

Therefore, $t_{kp}(v) t_{-k-1,-p}(u) \equiv -t_{-k-1,-k}(u) t_{kk}(v)$. So, \((5.7)\) gives

$$\left(1 + \frac{n-k}{u-v-\kappa}\right) t_{-k-1,-k}(u) t_{kk}(v) \equiv 0.$$
implying (5.5) in the case under consideration.

Suppose now that $i \geq k$. We can write

$$t_{i,i+1}(u) t_{kk}(v) \equiv -[t_{kk}(v), t_{i,i+1}(u)].$$

Now (5.5) is immediate from (2.19) except for the cases $i = -k$ and $i = -k - 1$. In the former case, we have $k \leq 0$ and so (2.19) gives

$$t_{-k,-k+1}(u) t_{kk}(v) \equiv -\frac{1}{v-u-\kappa} \sum_{p=-k+1}^{n} t_{-p,k}(v) t_{p,-k+1}(u),$$

while for each $p \geq -k + 1$,

$$t_{-p,k}(v) t_{p,-k+1}(u) \equiv -\frac{1}{v-u-\kappa} \sum_{q=-k+1}^{n} t_{-q,k}(v) t_{q,-k+1}(u).$$

Hence, $t_{-p,k}(v) t_{p,-k+1}(u) \equiv -t_{-k,-k+1}(u) t_{kk}(v)$. So, (5.8) implies

$$\left(1 + \frac{n+k}{v-u-\kappa}\right) t_{-k,-k+1}(u) t_{kk}(v) \equiv 0$$

verifying (5.5).

Finally, let $i = -k - 1$. Then $k < 0$ and

$$t_{-k-1,-k}(u) t_{kk}(v) \equiv -[t_{kk}(v), t_{-k-1,-k}(u)] \equiv -\frac{1}{v-u-\kappa} \sum_{p=-k}^{n} t_{-k-1,p}(u) t_{k,-p}(v).$$

For each $p \geq -k$ we have

$$t_{-k-1,p}(u) t_{k,-p}(v) \equiv -[t_{k,-p}(v), t_{-k-1,p}(u)] \equiv -\frac{1}{v-u-\kappa} \sum_{q=-k}^{n} t_{-k-1,q}(u) t_{k,-q}(v).$$

Therefore, $t_{-k-1,p}(u) t_{k,-p}(v) \equiv t_{-k-1,-k}(u) t_{kk}(v)$. So, (5.9) gives

$$\left(1 + \frac{n+k+1}{v-u-\kappa}\right) t_{-k-1,-k}(u) t_{kk}(v) \equiv 0$$

completing the proof of (5.5).

For the Lie algebras $\mathfrak{a} = \mathfrak{sp}_{2n}$ and $\mathfrak{o}_{2n}$ the argument is essentially the same as in the previous case. If $\mathfrak{a} = \mathfrak{sp}_{2n}$, then due to (5.3), it suffices to show that (5.5) holds for $i \in \{-n, \ldots, -2, 1, \ldots, n-1\}$ and all $k$, together with the relation

$$t_{-1,1}(u) t_{kk}(v) \equiv 0.$$

(5.10)
This relation is immediate from (2.19) for \( k > 1 \) and \( k < -1 \); for the latter we apply (2.19) to the commutator \([t_{kk}(v), t_{-1,1}(u)]\). If \( k = 1 \) or \( k = -1 \) then the claim is verified by a calculation similar to the cases (5.6) and (5.8), respectively.

If \( a = o_{2n} \), then it is sufficient to verify (5.5) for \( i \in \{-n, \ldots, -2, 1, \ldots, n-1\} \) and all \( k \), together with the relations (5.10) and \( t_{-1,2}(u) t_{kk}(v) \equiv 0 \). The calculation is again a repetition of the one for \( a = o_{2n+1} \).

Now we verify that all the operators \( t_i^{(r)} \) on the space \( V^0 \) with \( i \in \{-n, \ldots, n\} \) and \( r \geq 1 \) comprise a commutative family. First of all, by (2.19) we have \([t_{ii}(u), t_{jj}(v)] = 0\) for any \( i \neq 0 \). Furthermore, for any \( i < j \) such that \( i + j \neq 0 \) we have

\[
(u - v) [t_{ii}(u), t_{jj}(v)] = t_{ji}(u) t_{ij}(v) - t_{ji}(v) t_{ij}(u)
\]

and so, \([t_{ii}(u), t_{jj}(v)] \equiv 0\) as operators on \( V^0 \). Next, for any \( 0 \leq i \leq j \) set

\[
A_{ij} = t_{-j-i}(u) t_{ji}(v) - t_{ji}(v) t_{-j-i}(u),
\]

where the value \( i = 0 \) only occurs in the case \( a = o_{2n+1} \). By (2.19), we get

\[
A_{00} \equiv \frac{1}{u - v} A_{00} - \frac{1}{u - v - \kappa} \sum_{j=0}^{n} A_{0j}, \tag{5.11}
\]

and for any \( i > 0 \)

\[
A_{ii} \equiv -\frac{1}{u - v - \kappa} \sum_{j=i}^{n} A_{ij}, \tag{5.12}
\]

as operators on \( V^0 \), while for \( 0 \leq i < j \) we have

\[
A_{ij} \equiv -\frac{1}{u - v - \kappa} \sum_{k=i}^{n} A_{ik} - \frac{1}{u - v - \kappa} \sum_{l=j}^{n} A_{jl}.
\]

This implies

\[
A_{ij} \equiv A_{ii} - A_{jj}
\]

for \( 0 < i < j \), and

\[
A_{0j} = \frac{u - v - 1}{u - v} A_{00} - A_{jj}
\]

for \( j > 0 \). Hence, (5.12) gives

\[
\left( 1 + \frac{n - i + 1}{u - v - \kappa} \right) A_{ii} - \frac{1}{u - v - \kappa} \sum_{j=i+1}^{n} A_{jj} \equiv 0,
\]

thus proving that \( A_{ii} = [t_{-i-i}(u), t_{ii}(v)] \equiv 0 \) for all \( i > 0 \) by an obvious induction. Moreover, in the case \( a = o_{2n+1} \), we derive from (5.11) that \( A_{00} = [t_{00}(u), t_{00}(v)] \equiv 0 \).
Since the operators $t_{ij}^{(r)}$ on $V^0$ are pairwise commuting, they have a simultaneous eigenvector $\xi \in V^0$. Then $\xi$ satisfies the conditions (5.1). Moreover, since $V$ is irreducible, the submodule $X(\mathfrak{a}) \xi$ must coincide with $V$ so that $V$ is a highest weight module over $X(\mathfrak{a})$. In particular, $\xi$ is an $\mathfrak{a}$-weight vector with a certain weight $\nu$.

Finally, since the central elements $z_a$ act on $V$ as scalar operators, Corollary 3.10 implies that the vector space $V$ is spanned by the elements 
$$t_{j_1 i_1}^{(r_1)} \ldots t_{j_m i_m}^{(r_m)} \xi, \quad m \geq 0,$$
with $j_a > i_a$ and $r_a \geq 1$. Hence, by (5.3) the $\mathfrak{a}$-weight space $V_\nu$ is one-dimensional and spanned by the vector $\xi$. Moreover, if $\rho$ is a weight of $V$ and $\rho \neq \nu$ then $\rho$ strictly precedes $\nu$. This proves that the highest vector $\xi$ of $V$ is determined uniquely, up to a constant factor. \hfill \Box

Given any tuple $\lambda(u) = (\lambda_{-n}(u), \ldots, \lambda_n(u))$ of formal series of the form (5.2), we define the Verma module $M(\lambda(u))$ as the quotient of $X(\mathfrak{a})$ by the left ideal generated by all the coefficients of the series $t_{ij}(u)$ with $-n \leq i < j \leq n$, and $t_{ii}(u) - \lambda_i(u)$ for $i = -n, \ldots, n$. As we shall see below, the Verma module $M(\lambda(u))$ can be trivial for some $\lambda(u)$. In the non-trivial case, the Verma module $M(\lambda(u))$ is a highest weight representation of $X(\mathfrak{a})$ with the highest weight $\lambda(u)$ and the highest vector $1_\lambda$ which is the canonical image of the element $1 \in X(\mathfrak{a})$. Moreover, any highest weight representation of $X(\mathfrak{a})$ with the highest weight $\lambda(u)$ is isomorphic to a quotient of $M(\lambda(u))$.

Regarding $M(\lambda(u))$ as an $\mathfrak{a}$-module, we obtain the weight space decomposition
$$M(\lambda(u)) = \bigoplus_{\nu} M(\lambda(u))_\nu,$$
summed over all $\mathfrak{a}$-weights $\nu = (\nu_1, \ldots, \nu_n)$ of $M(\lambda(u))$, where
$$M(\lambda(u))_\nu = \{ \eta \in M(\lambda(u)) \mid F_{ii} \eta = \nu_i \eta, \quad i = 1, \ldots, n \}.$$ 
By (5.3), the set of weights of $M(\lambda(u))$ coincides with that of the $\mathfrak{a}$-Verma module with the highest weight $\lambda^{(1)} = (\lambda_1^{(1)}, \ldots, \lambda_n^{(1)})$. This set consists of all weights of the form $\lambda^{(1)} - \omega$, where $\omega$ is a $\mathbb{Z}_+$-linear combination of the positive roots.

One easily shows that any submodule $K$ of a non-trivial Verma module $M(\lambda(u))$ admits the weight space decomposition
$$K = \bigoplus_{\nu} K_{\nu}, \quad K_{\nu} = K \cap M(\lambda(u))_\nu.$$ 
This implies that the sum of all proper submodules is the unique maximal proper submodule of $M(\lambda(u))$. The irreducible highest weight representation $L(\lambda(u))$ of $X(\mathfrak{a})$ with the highest weight $\lambda(u)$ is defined as the quotient of the Verma module $M(\lambda(u))$ by the unique maximal proper submodule.
**Proposition 5.2.** Let $V$ be a highest weight representation of $X(\mathfrak{a})$ with the highest weight $\lambda(u) = (\lambda_{-n}(u), \ldots, \lambda_n(u))$ with some series (5.2). Then the coefficients of the series $z(u)$ act on $V$ as scalar operators determined by $z(u)|_V = \lambda_{-n}(u+\kappa) \lambda_n(u)$.

**Proof.** Let $\xi$ be the highest vector of $V$. Then $V = X(\mathfrak{a}) \xi$ so that $z(u)$ acts on $V$ as a scalar function determined by its action on $\xi$. However, taking $k = l = n$ in (2.28) we get

$$z(u) = \sum_{i=-n}^{n} \theta_{ni} t_{-i,-n}(u+\kappa) t_{in}(u).$$

Therefore, $z(u) \xi = \lambda_{-n}(u+\kappa) \lambda_n(u) \xi$. 

### 5.2 Representations of low rank Yangians

Using the results on representations of the Yangian $Y(\mathfrak{gl}_2)$ (Tarasov [25, 26]; see also [7, Chapter 12], [16]), and the isomorphisms constructed in Section 4, we describe here the finite-dimensional irreducible representations of the extended Yangians $X(\mathfrak{o}_3)$, $X(\mathfrak{sp}_2)$ and $X(\mathfrak{a}_4)$. For the sake of completeness, we also reproduce a simplified version of Tarasov’s classification theorem for the representations of $Y(\mathfrak{gl}_2)$; cf. [16].

We shall use the notation for the generators of $Y(\mathfrak{gl}_2)$ introduced in Section 4. A representation $L$ of the Yangian $Y(\mathfrak{gl}_2)$ is called a highest weight representation if there exists a nonzero vector $\zeta \in L$ such that $L$ is generated by $\zeta$ and the following relations hold

$$T_{12}(u) \zeta = 0 \quad \text{and} \quad T_{ii}(u) \zeta = \mu_i(u) \zeta \quad \text{for} \quad i = 1, 2.$$  

(5.14)  

(5.15)

for some formal series

$$\mu_i(u) = 1 + \mu_i^{(1)} u^{-1} + \mu_i^{(2)} u^{-2} + \ldots, \quad \mu_i^{(r)} \in \mathbb{C}.$$  

(5.16)

The vector $\zeta$ is called the highest vector of $L$, and the pair $\mu(u) = (\mu_1(u), \mu_2(u))$ is the highest weight of $L$. A standard argument, similar to the one used in Section 5.1 (see e.g. [16]), shows that every finite-dimensional irreducible representation of $Y(\mathfrak{gl}_2)$ is a highest weight representation. Given any pair of series $\mu(u) = (\mu_1(u), \mu_2(u))$, the corresponding Verma module $M(\mu(u))$ for $Y(\mathfrak{gl}_2)$ is the quotient of $Y(\mathfrak{gl}_2)$ by the left ideal generated by all the coefficients of the series $T_{12}(u)$ and $T_{ii}(u) - \mu_i(u)$ for $i = 1, 2$. When the components of $\mu(u)$ satisfy the condition $\mu_1(u) \mu_2(u-1) = 1$ then $M(\mu(u))$ may also be regarded as a module over the Yangian $Y(\mathfrak{gl}_2)$.

The $Y(\mathfrak{gl}_2)$-module $M(\mu(u))$ has a unique irreducible quotient $L(\mu(u))$. Thus, any finite-dimensional irreducible representation of $Y(\mathfrak{gl}_2)$ is isomorphic to $L(\mu(u))$ for a
pair $\mu(u) = (\mu_1(u), \mu_2(u))$. It remains to describe the highest weights $\mu(u)$ which correspond to finite-dimensional modules $L(\mu(u))$. This is given by the following theorem due to Tarasov [25, 26] in Drinfeld’s version [12].

**Theorem 5.3.** The irreducible highest weight representation $L(\mu(u))$ of $Y(\mathfrak{gl}_2)$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in $u$ such that

$$\frac{\mu_1(u)}{\mu_2(u)} = \frac{P(u+1)}{P(u)}.$$  \hspace{1cm} (5.17)

In this case, $P(u)$ is unique.

**Proof.** We shall need the following lemma.

**Lemma 5.4.** If $\dim L(\mu(u)) < \infty$ then there exists a formal series

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \ldots , \quad f_r \in \mathbb{C},$$

such that $f(u)\mu_1(u)$ and $f(u)\mu_2(u)$ are polynomials in $u^{-1}$.

**Proof.** By twisting the action of $Y(\mathfrak{gl}_2)$ on $L(\mu(u))$ by the automorphism (4.11) with $g(u) = \mu_2(u)^{-1}$, we obtain a module over $Y(\mathfrak{gl}_2)$ which is isomorphic to the irreducible highest weight representation $L(\nu(u), 1)$ with $\nu(u) = \mu_1(u)/\mu_2(u)$. So, we may assume without loss of generality that the highest weight of $L(\mu(u))$ has the form $\mu(u) = (\nu(u), 1)$. Let $\zeta$ denote the highest vector of the Verma module $M(\nu(u), 1)$. Since $\dim L(\nu(u), 1) < \infty$, the vectors $T^{(i)}_{21} \zeta \in M(\nu(u), 1)$ with $i \geq 1$ are linearly dependent modulo the maximal proper submodule $K$ of $M(\nu(u), 1)$. Hence, $M(\nu(u), 1)$ contains a nonzero vector $\xi \in K$ of the form

$$\xi = \sum_{i=1}^{m} c_i T^{(i)}_{21} \zeta, \quad c_i \in \mathbb{C}.$$  

Here $m$ is a positive integer and we may assume that $c_m \neq 0$. Then we have $T^{(r)}_{12} \xi = 0$ for all $r \geq 1$ because otherwise the highest vector $\zeta$ would belong to $K$. Write

$$\nu(u) = 1 + \nu^{(1)} u^{-1} + \nu^{(2)} u^{-2} + \ldots , \quad \nu^{(i)} \in \mathbb{C}.$$  

By the defining relations (4.12), in $M(\nu(u), 1)$ we have

$$T^{(r)}_{12} T^{(i)}_{21} \zeta = \sum_{a=1}^{\min(r,i)} \left( T^{(a-1)}_{22} T^{(r+i-a)}_{11} - T^{(r+i-a)}_{22} T^{(a-1)}_{11} \right) \zeta = \nu^{(r+i-1)} \zeta.$$  

Hence, for all $r \geq 1$ we have the relations

$$\sum_{i=1}^{m} c_i \nu^{(r+i-1)} = 0.$$  

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They imply
\[ \nu(u)(c_1 + c_2 u + \cdots + c_m u^{m-1}) = (b_1 + b_2 u + \cdots + b_m u^{m-1}) \]
for some coefficients \( b_i \in \mathbb{C} \) with \( b_m = c_m \). Thus, taking now
\[ f(u) = c_m^{-1} \sum_{i=1}^{m} c_i u^{-m+i} \]
we conclude that both \( f(u) \nu(u) \) and \( f(u) \) are polynomials in \( u^{-1} \).

Thus, taking the composition of the representation of \( Y(\mathfrak{gl}_2) \) on \( L(\mu(u)) \) with an appropriate automorphism of the form (4.11), we can get another highest weight representation of \( Y(\mathfrak{gl}_2) \) where both components of the highest weight are polynomials in \( u^{-1} \).

For any \( \alpha, \beta \in \mathbb{C} \) consider the irreducible highest weight representation \( L(\alpha, \beta) \) of the Lie algebra \( \mathfrak{gl}_2 \) and equip it with a \( Y(\mathfrak{gl}_2) \)-module structure via the evaluation homomorphism (4.10). Let \( \zeta \) denote the highest vector of \( L(\alpha, \beta) \). Then
\[ E_{11} \zeta = \alpha \zeta, \quad E_{22} \zeta = \beta \zeta, \quad E_{12} \zeta = 0. \]
Moreover, if \( \alpha - \beta \in \mathbb{Z}_+ \) then the vectors \( (E_{21})^r \zeta \) with \( r = 0, 1, \ldots, \alpha - \beta \) form a basis of \( L(\alpha, \beta) \) so that \( \dim L(\alpha, \beta) = \alpha - \beta + 1 \). If \( \alpha - \beta \notin \mathbb{Z}_+ \) then a basis of \( L(\alpha, \beta) \) is formed by the vectors \( (E_{21})^r \zeta \), where \( r \) runs over all nonnegative integers.

Now let \( \mu_1(u) \) and \( \mu_2(u) \) be polynomials in \( u^{-1} \) of degree not more than \( k \). Write the decompositions
\[ \mu_1(u) = (1 + \alpha_1 u^{-1}) \cdots (1 + \alpha_k u^{-1}), \]
\[ \mu_2(u) = (1 + \beta_1 u^{-1}) \cdots (1 + \beta_k u^{-1}), \] (5.18)
where the constants \( \alpha_i \) and \( \beta_i \) are complex numbers (some of them are zero if the degree of the corresponding polynomial is strictly less than \( k \)).

For any \( Y(\mathfrak{gl}_2) \)-modules \( L_1 \) and \( L_2 \), their tensor product \( L_1 \otimes L_2 \) is equipped with a \( Y(\mathfrak{gl}_2) \)-module structure defined by the coproduct (4.7).

Lemma 5.5. Re-number the parameters \( \alpha_i \) and \( \beta_i \) if necessary, so that for every index \( i = 1, \ldots, k-1 \) the following condition holds: if the multiset \( \{ \alpha_p - \beta_q \mid i \leq p, q \leq k \} \) contains nonnegative integers, then \( \alpha_i - \beta_i \) is minimal amongst them. Then the representation \( L(\mu_1(u), \mu_2(u)) \) of \( Y(\mathfrak{gl}_2) \) is isomorphic to the tensor product module
\[ L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_k, \beta_k). \] (5.19)
Proof. Let us denote the module \((5.19)\) by \(L\) and let \(\zeta_i\) be the highest vector of \(L(\alpha_i, \beta_i)\) for \(i = 1, \ldots, k\). Using the definition of the coproduct on \(Y(\mathfrak{gl}_2)\) we derive that the cyclic span \(Y(\mathfrak{gl}_2)\zeta\) of the vector \(\zeta = \zeta_1 \otimes \cdots \otimes \zeta_k\) is a highest weight module with the highest weight \((\mu_1(u), \mu_2(u))\). Therefore, the proposition will follow if we prove that the module \(L\) is irreducible.

We claim that any vector \(\xi \in L\) satisfying \(T_{12}(u)\xi = 0\) is proportional to \(\zeta\). We shall prove this claim by induction on \(k\). We assume that \(k \geq 2\). Write any such vector \(\xi\), which is assumed to be nonzero, in the form

\[
\xi = \sum_{r=0}^{p} (E_{21})^r \zeta_1 \otimes \xi_r \quad \text{where} \quad \xi_r \in L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_k, \beta_k)
\]

and \(p\) is some non-negative integer. Moreover, if \(\alpha_1 - \beta_1 \in \mathbb{Z}_+\) then we may and will assume that \(p \leq \alpha_1 - \beta_1\). We also assume that \(\xi_p \neq 0\). Applying \(T_{12}(u)\) to \(\xi\), with the use of \((5.17)\) we get

\[
\sum_{r=0}^{p} \left( T_{11}(u)(E_{21})^r \zeta_1 \otimes T_{12}(u)\xi_r + T_{12}(u)(E_{21})^r \zeta_1 \otimes T_{22}(u)\xi_r \right) = 0. \quad (5.20)
\]

Using the definition of the Yangian action on \(L(\alpha_1, \beta_1)\) and commutation relations in \(\mathfrak{gl}_2\), we obtain

\[
T_{11}(u)(E_{21})^r \zeta_1 = (1 + E_{11}u^{-1})(E_{21})^r \zeta_1 = (1 + (\alpha_1 - r)u^{-1})(E_{21})^r \zeta_1,
\]

and

\[
T_{12}(u)(E_{21})^r \zeta_1 = u^{-1}E_{12}(E_{21})^r \zeta_1 = u^{-1}r(\alpha_1 - \beta_1 - r + 1)(E_{21})^{r-1} \zeta_1.
\]

Hence, taking the coefficient at \((E_{21})^p \zeta_1\) in \((5.20)\) gives

\[
(1 + (\alpha_1 - p)u^{-1}) T_{12}(u) \xi_p = 0,
\]

implying the relation \(T_{12}(u) \xi_p = 0\). By the induction hypothesis, applied to the \(Y(\mathfrak{gl}_2)\)-module \(L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_k, \beta_k)\), the vector \(\xi_p\) must be proportional to \(\zeta_2 \otimes \cdots \otimes \zeta_k\). Therefore, using \((4.7)\) we get

\[
T_{22}(u) \xi_p = (1 + \beta_2 u^{-1}) \cdots (1 + \beta_k u^{-1}) \xi_p. \quad (5.21)
\]

In order to complete the proof of the claim it now suffices to show that \(p\) must be equal to zero. Suppose by way of contradiction that \(p \geq 1\). Then taking the coefficient at \((E_{21})^{p-1} \zeta_1\) in \((5.20)\) we derive

\[
(1 + (\alpha_1 - p + 1)u^{-1}) T_{12}(u) \xi_{p-1} + u^{-1} p(\alpha_1 - \beta_1 - p + 1) T_{22}(u) \xi_p = 0.
\]
Hence, multiplying by \( u^k \) and taking into account (5.21) we get
\[
(u + \alpha_1 - p + 1)u^{k-1}T_{12}(u)\xi_{p-1} + p(\alpha_1 - \beta_1 - p + 1)(u + \beta_2)\ldots(u + \beta_k)\xi_p = 0.
\]
Now observe that the vector \( u^{k-1}T_{12}(u)\xi_{p-1} \) depends on \( u \) polynomially. This follows by an easy induction with the use of (4.7). So, taking the value \( u = -\alpha_1 + p - 1 \) we obtain the relation
\[
p(\alpha_1 - \beta_1 - p + 1)(\alpha_1 - \beta_2 - p + 1)\ldots(\alpha_1 - \beta_k - p + 1) = 0.
\]
But this is impossible due to the conditions on the parameters \( \alpha_i \) and \( \beta_i \). Thus, \( p \) must be zero and the claim follows.

Suppose now that \( M \) is a nonzero submodule of \( L \). Then \( M \) must contain a nonzero vector \( \xi \) such that \( T_{12}(u)\xi = 0 \). Indeed, this follows from the fact that the set of \( \mathfrak{gl}_2 \)-weights of \( L \) has an upper boundary. The above argument thus shows that \( M \) contains the vector \( \zeta \). It remains to prove that the cyclic span \( K = Y(\mathfrak{gl}_2)\zeta \) coincides with \( L \).

Denote by \( \varkappa \) the anti-automorphism of the algebra \( Y(\mathfrak{gl}_2) \), defined by
\[
\varkappa : t_{ij}(u) \mapsto t_{3-i,3-j}(-u).
\]
Consider the vector space \( L^* \) dual to \( L \). That is, \( L^* \) is spanned by all linear maps \( \sigma : L \to \mathbb{C} \) satisfying the condition that the linear span of the vectors \( \eta \in L \) such that \( \sigma(\eta) \neq 0 \), is finite-dimensional. Equip \( L^* \) with a \( Y(\mathfrak{gl}_2) \)-module structure by setting
\[
(y\sigma)(\eta) = \sigma(\varkappa(y)\eta) \quad \text{for} \quad y \in Y(\mathfrak{gl}_2) \quad \text{and} \quad \sigma \in L^*, \quad \eta \in L.
\]
It is easy to see that the dual module \( L(\alpha, \beta)^* \) to the evaluation module \( L(\alpha, \beta) \) is isomorphic to \( L(-\beta, -\alpha) \). Moreover, the \( Y(\mathfrak{gl}_2) \)-module \( L^* \) is isomorphic to the tensor product module
\[
L(-\beta_1, -\alpha_1) \otimes \ldots \otimes L(-\beta_k, -\alpha_k).
\]
This is deduced from the fact that the anti-automorphism \( \varkappa \) commutes with the coproduct \( \Delta \), where \( \varkappa \) is extended to \( Y(\mathfrak{gl}_2) \otimes Y(\mathfrak{gl}_2) \) by \( \varkappa(x \otimes y) = \varkappa(x) \otimes \varkappa(y) \) for \( x, y \in Y(\mathfrak{gl}_2) \). Furthermore, the highest vector \( \zeta_i^* \) of the module \( L(-\beta_i, -\alpha_i) \) can be identified with the element of \( L(\alpha_i, \beta_i)^* \) such that \( \zeta_i^*(\zeta_i) = 1 \) and \( \zeta_i^*(\eta_i) = 0 \) for all weight vectors \( \eta_i \in L(\alpha_i, \beta_i) \) whose weights are different from \( (\alpha_i, \beta_i) \).

Suppose now that the submodule \( K \) of \( L \) is proper and consider its annihilator
\[
\text{Ann} K = \{ \xi^* \in L^* \mid \xi^*(\eta) = 0 \quad \text{for all} \quad \eta \in K \}.
\]
Then Ann $K$ is a nonzero submodule of $L^*$, which does not contain the vector $\zeta_1^* \otimes \ldots \otimes \zeta_k^*$. However, this contradicts the claim verified in the first part of the proof, because the condition on the parameters $\alpha_i$ and $\beta_i$ remain satisfied after we replace each $\alpha_i$ by $-\beta_i$ and each $\beta_i$ by $-\alpha_i$.

By this lemma, all differences $\alpha_i - \beta_i$ must be nonnegative integers because the representation $L(\lambda_1(u), \lambda_2(u))$ is finite-dimensional. Then the polynomial

$$P(u) = \prod_{i=1}^{k} (u + \beta_i)(u + \beta_i + 1) \ldots (u + \alpha_i - 1) \quad (5.23)$$

obviously satisfies (5.17).

Conversely, suppose (5.17) holds for a polynomial $P(u) = (u + \gamma_1) \ldots (u + \gamma_p)$. Set

$$\nu_1(u) = (1 + (\gamma_1 + 1)u^{-1}) \ldots (1 + (\gamma_p + 1)u^{-1}),$$
$$\nu_2(u) = (1 + \gamma_1 u^{-1}) \ldots (1 + \gamma_p u^{-1}),$$

and consider the tensor product module

$$L = L(\gamma_1 + 1, \gamma_1) \otimes L(\gamma_2 + 1, \gamma_2) \otimes \ldots \otimes L(\gamma_p + 1, \gamma_p)$$

of $Y(\mathfrak{gl}_2)$. Obviously, this module is finite-dimensional. The cyclic $Y(\mathfrak{gl}_2)$-span of the tensor product of the highest vectors of $L(\gamma_i + 1, \gamma_i)$ is a highest weight module with the highest weight $(\nu_1(u), \nu_2(u))$. Since this submodule is finite-dimensional, then so is its irreducible quotient $L(\nu_1(u), \nu_2(u))$. Since

$$\frac{\nu_1(u)}{\nu_2(u)} = \frac{\mu_1(u)}{\mu_2(u)},$$

there exists an automorphism of $Y(\mathfrak{gl}_2)$ of the form (4.11) such that its composition with the representation $L(\nu_1(u), \nu_2(u))$ is isomorphic to $L(\mu_1(u), \mu_2(u))$. Thus, the latter is also finite-dimensional.

Finally, suppose that $Q(u)$ is another monic polynomial in $u$ and

$$\frac{P(u + 1)}{P(u)} = \frac{Q(u + 1)}{Q(u)}.$$ 

This means that the ratio $P(u)/Q(u)$ is periodic in $u$ which is only possible for $P(u) = Q(u)$.

The polynomial $P(u)$ is called the Drinfeld polynomial of the representation $L(\mu(u))$.

We now apply Theorem 5.3 to the low rank extended Yangians.
Corollary 5.6. The Verma module \( M(\lambda(u)) \) over \( \mathfrak{sp}_2 \) is non-trivial for any highest weight \( \lambda(u) = (\lambda_{-1}(u), \lambda_1(u)) \). Moreover, the \( \mathfrak{sp}_2 \)-module \( L(\lambda(u)) \) is finite-dimensional if and only if there exists a monic polynomial \( P(u) \) in \( u \) such that

\[
\frac{\lambda_{-1}(u)}{\lambda_1(u)} = \frac{P(u+2)}{P(u)}.
\] (5.24)

In this case, \( P(u) \) is unique.

Proof. This is immediate from Proposition 4.11 and Theorem 5.3.

The evaluation homomorphism provided by Corollary 4.3 allows one to regard any irreducible \( \mathfrak{sp}_2 \)-module \( V(\mu) \) as an \( X(\mathfrak{sp}_2) \)-module. The corresponding evaluation module is immediately identified with an irreducible highest weight module.

Proposition 5.7. The evaluation module \( V(\mu) \) over \( X(\mathfrak{sp}_2) \) is isomorphic to \( L(\lambda(u)) \) with

\[
\lambda_{-1}(u) = 1 - \mu_1 u^{-1} \quad \text{and} \quad \lambda_1(u) = 1 + \mu_1 u^{-1}.
\]

Corollary 5.8. The Verma module \( M(\lambda(u)) \) over \( \mathfrak{o}_3 \) is non-trivial if and only if the highest weight \( \lambda(u) = (\lambda_{-1}(u), \lambda_0(u), \lambda_1(u)) \) satisfies the condition

\[
\lambda_{-1}(u - 1/2) \lambda_1(u) = \lambda_0(u - 1/2) \lambda_0(u).
\] (5.25)

Moreover, if this condition holds then the \( \mathfrak{o}_3 \)-module \( L(\lambda(u)) \) is finite-dimensional if and only if there exists a monic polynomial \( P(u) \) in \( u \) such that

\[
\frac{\lambda_0(u)}{\lambda_1(u)} = \frac{P(u+1/2)}{P(u)}.
\] (5.26)

In this case, \( P(u) \) is unique.

Proof. Let the Verma module \( M(\lambda(u)) \) be non-trivial. By Proposition 4.4, we may regard \( M(\lambda(u)) \) as a \( Y(\mathfrak{gl}_2) \)-module. In particular, we have

\[
T_{11}(2u) T_{11}(2u+1) \lambda = \lambda_{-1}(u) \lambda,
\]

where \( \lambda \) is the highest vector of \( M(\lambda(u)) \). This implies that \( \lambda \) is an eigenvector for \( T_{11}(u) \), that is, \( T_{11}(u) \lambda = \mu_1(u) \lambda \) for a certain series \( \mu_1(u) \). Moreover, this series satisfies

\[
\mu_1(2u) \mu_1(2u+1) = \lambda_{-1}(u).
\] (5.27)

Similarly, \( T_{22}(u) \lambda = \mu_2(u) \lambda \) for a series \( \mu_2(u) \) satisfying

\[
\mu_2(2u) \mu_2(2u+1) = \lambda_1(u).
\] (5.28)
Furthermore, by the defining relations \(4.3\) we have

\[
T_{12}(2u) T_{22}(2u + 1) + T_{22}(2u) T_{12}(2u + 1) = 2 T_{12}(2u + 1) T_{22}(2u).
\]

Since \(t_{0,1}(u) 1_\lambda = 0\) we derive that \(T_{12}(u) 1_\lambda = 0\). Hence, using the action of \(t_{0,0}(u)\) on \(1_\lambda\) we also get

\[
\mu_1(2u) \mu_2(2u + 1) = \lambda_0(u).
\]

This gives the condition \(5.29\).

Conversely, if the condition \(5.29\) holds for a highest weight \(\lambda(u)\) then there exist series \(\mu_1(u)\) and \(\mu_2(u)\) satisfying \(5.27\), \(5.28\) and \(5.29\). Consider the Verma module \(M(\mu_1(u), \mu_2(u))\) over \(Y(\mathfrak{gl}_2)\). Using the formulas of Proposition \(4.4\), we find that the highest vector \(1_\mu \in M(\mu_1(u), \mu_2(u))\) satisfies the conditions \(5.1\) for the action of \(X(\mathfrak{so}_3)\).

The argument of the first part of the proof shows that, regarded as a \(Y(\mathfrak{gl}_2)\)-module, the module \(L(\lambda(u))\) is isomorphic to \(L(\mu_1(u), \mu_2(u))\) with \(\mu_1(u)\) and \(\mu_2(u)\) satisfying \(5.27\), \(5.28\) and \(5.29\). So writing the relation of Theorem \(5.3\) in terms of the series \(\lambda_i(u)\), we get the desired condition. \(\square\)

The evaluation homomorphism provided by Corollary \(4.7\) allows one to regard any irreducible \(\mathfrak{so}_3\)-module \(V(\mu)\) as an \(X(\mathfrak{so}_3)\)-module.

**Proposition 5.9.** The evaluation module \(V(\mu)\) over \(X(\mathfrak{so}_3)\) is isomorphic to \(L(\lambda(u))\) with

\[
\lambda_{-1}(u) = \frac{(2u - \mu_1)(2u - \mu_1 - 1)}{2u(2u - 1)}, \\
\lambda_0(u) = \frac{(2u + \mu_1)(2u - \mu_1 - 1)}{2u(2u - 1)}, \\
\lambda_1(u) = \frac{(2u + \mu_1)(2u + \mu_1 - 1)}{2u(2u - 1)}.
\]

**Proof.** This is immediate from Corollary \(4.7\) as the Casimir element \(c\) acts on \(V(\mu)\) as multiplication by the scalar \((\mu_1^2 - \mu_1)/2\). \(\square\)

**Corollary 5.10.** The Verma module \(M(\lambda(u))\) over \(X(\mathfrak{so}_3)\) is non-trivial if and only if the highest weight \(\lambda(u) = (\lambda_{-2}(u), \lambda_{-1}(u), \lambda_1(u), \lambda_2(u))\) satisfies the condition

\[
\lambda_{-2}(u) \lambda_2(u) = \lambda_{-1}(u) \lambda_1(u).
\]

Moreover, if this condition holds then the \(X(\mathfrak{so}_3)\)-module \(L(\lambda(u))\) is finite-dimensional if and only if there exist monic polynomials \(P(u)\) and \(Q(u)\) in \(u\) such that

\[
\frac{\lambda_{-1}(u)}{\lambda_2(u)} = \frac{P(u + 1)}{P(u)} \quad \text{and} \quad \frac{\lambda_1(u)}{\lambda_2(u)} = \frac{Q(u + 1)}{Q(u)}.
\]

In this case, \(P(u)\) and \(Q(u)\) are determined uniquely.
Proof. Suppose that the Verma module $M(\lambda(u))$ over $X(\mathfrak{o}_4)$ is non-trivial. Using the isomorphism $\chi^{(1)}$ provided by Corollary 5.10 we shall regard $M(\lambda(u))$ as a module over the algebra $Y(\mathfrak{sl}_2) \otimes Y(\mathfrak{gl}_2)$. As was seen in the proof of Corollary 5.10

$$\chi^{(1)} : t_{-2,-2}(u) t_{1,1}(u-1) - t_{-1,-2}(u) t_{-2,1}(u-1) \mapsto T'_{11}(u) T'_{22}(u-1).$$

This implies that $1_{\lambda}$ is an eigenvector for $T'_{11}(u)$, that is, $T'_{11}(u) 1_{\lambda} = \mu'_1(u) 1_{\lambda}$ for a certain series $\mu'_1(u)$. Similarly, $T'_{22}(u) 1_{\lambda} = \mu'_2(u) 1_{\lambda}$ for a series $\mu'_2(u)$. Then, by the formulas of Proposition 4.18 we also have

$$T_{11}(u) 1_{\lambda} = \mu_1(u) 1_{\lambda} \quad \text{and} \quad T_{22}(u) 1_{\lambda} = \mu_2(u) 1_{\lambda}$$

for some series $\mu_1(u)$ and $\mu_2(u)$. Moreover, we have the relations

$$\lambda_{-2}(u) = \mu_1(u) \mu'_1(u), \quad \lambda_{-1}(u) = \mu_1(u) \mu'_2(u), \quad \lambda_1(u) = \mu_2(u) \mu'_1(u), \quad \lambda_2(u) = \mu_2(u) \mu'_2(u),$$

which imply (5.30). Conversely, if (5.30) holds for some series $\lambda_i(u)$, then there exist series $\mu_i(u)$ and $\mu'_i(u)$ satisfying (5.32) together with the condition $\mu_1(u) \mu_2(u-1) = 1$. Consider the $Y(\mathfrak{sl}_2) \otimes Y(\mathfrak{gl}_2)$-module $M(\mu_1(u), \mu_2(u)) \otimes M(\mu'_1(u), \mu'_2(u))$. The vector $1_{\mu} \otimes 1_{\mu'}$ satisfies the conditions (5.1) for the action of the series $t_{ij}(u)$ thus proving that the $X(\mathfrak{o}_4)$-module $M(\lambda(u))$ is non-trivial.

Finally, the argument of the first part of the proof shows that, regarded as a $Y(\mathfrak{sl}_2) \otimes Y(\mathfrak{gl}_2)$-module, the module $L(\lambda(u))$ is isomorphic to $L(\mu_1(u), \mu_2(u)) \otimes L(\mu'_1(u), \mu'_2(u))$ with the $\mu_i(u)$ and $\mu'_i(u)$ satisfying (5.32). By Theorem 5.13 the module $L(\mu_1(u), \mu_2(u)) \otimes L(\mu'_1(u), \mu'_2(u))$ is finite-dimensional if and only if there exist monic polynomials $P(u)$ and $Q(u)$ in $u$ such that

$$\frac{\mu_1(u)}{\mu_2(u)} = \frac{P(u+1)}{P(u)} \quad \text{and} \quad \frac{\mu'_1(u)}{\mu'_2(u)} = \frac{Q(u+1)}{Q(u)}.$$

Writing these formulas in terms of the $\lambda_i(u)$ we get the desired conditions. \qed

The evaluation homomorphism provided by Corollary 4.12 allows one to regard any irreducible $\mathfrak{o}_4$-module $V(\mu)$ as an $X(\mathfrak{o}_4)$-module.

**Proposition 5.11.** The evaluation module $V(\mu)$ over $X(\mathfrak{o}_4)$ is isomorphic to $L(\lambda(u))$ with

$$\lambda_{-2}(u) = \frac{(2u - \mu_1 - \mu_2)(2u + \mu_1 - \mu_2)}{4u^2},$$

$$\lambda_{-1}(u) = \frac{(2u - \mu_1 - \mu_2)(2u - \mu_1 + \mu_2)}{4u^2},$$

$$\lambda_1(u) = \frac{(2u + \mu_1 - \mu_2)(2u + \mu_1 + \mu_2)}{4u^2},$$

$$\lambda_2(u) = \frac{(2u - \mu_1 + \mu_2)(2u + \mu_1 + \mu_2)}{4u^2}.$$
Proof. This follows from Corollary 4.12 as the Casimir element \( c \) acts on \( V(\mu) \) as multiplication by the scalar \( (\mu_1^2 + \mu_2^2)/2 - \mu_2 \).

Remark 5.12. More general evaluation modules \( V(\mu)_a \) with \( a \in \mathbb{C} \) over \( X(a) \) for \( a = \mathfrak{sp}_2, \mathfrak{o}_3 \) and \( \mathfrak{o}_4 \) can be obtained by using the respective evaluation homomorphisms \( \text{ev}_a : X(a) \to U(a) \) instead of \( \text{ev} \); see Remark 4.13. Then \( V(\mu)_a \) will be isomorphic to the irreducible highest weight module \( L(\lambda(u)) \), where the components \( \lambda_i(u) \) are found from the formulas of Propositions 5.7, 5.9 or 5.11 by replacing \( u \) with \( u - a \).

### 5.3 Classification theorems

Our goal here is to prove classification theorems for the finite-dimensional irreducible representations of the extended Yangians \( X(a) \) for \( a = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \) and \( \mathfrak{o}_{2n} \). The corresponding theorem for the Yangian \( Y(\mathfrak{sl}_N) \) implies that every finite-dimensional irreducible representation of \( Y(\mathfrak{sl}_N) \) is isomorphic to a subquotient of the tensor product of the fundamental representations [12], [7, Chapter 12]. We shall use the following version of the well-known construction of the fundamental representations of \( Y(\mathfrak{gl}_N) \). They are obtained by restriction from the corresponding representation of \( Y(\mathfrak{gl}_N) \) which is obtained by a simple particular case of the fusion procedure; see e.g. [9], [20]. The vector space \( \mathbb{C}^N \) carries an irreducible representation of \( Y(\mathfrak{gl}_N) \) with the action of the generators given by

\[
T_{ij}(u) \mapsto \delta_{ij} + e_{ij} u^{-1}, \quad i, j \in \{1, \ldots, N\},
\]

where the \( e_{ij} \) denote the standard matrix units. So,

\[
T_{ij}(u) e_k = \delta_{ij} e_k + \delta_{jk} e_i u^{-1},
\]

where \( e_1, \ldots, e_N \) denote the canonical basis of \( \mathbb{C}^N \). Since for any \( b \in \mathbb{C} \) the mapping \( T_{ij}(u) \mapsto T_{ij}(u - b) \) defines an automorphism of \( Y(\mathfrak{gl}_N) \), using the coproduct (4.7), we can equip the tensor product \( (\mathbb{C}^N)^{\otimes m} \) with the action of \( Y(\mathfrak{gl}_N) \) by the rule

\[
T_{ij}(u) (e_{i_1} \otimes \cdots \otimes e_{i_m}) = \sum_{a_1, \ldots, a_{m-1} = 1}^{N} T_{ia_1}(u) e_{i_1} \otimes T_{a_1a_2}(u + 1) e_{i_2} \otimes \cdots \otimes T_{a_{m-1}j}(u + m - 1) e_{i_m}. \tag{5.33}
\]

For any \( 1 \leq m < N \) set

\[
\xi_m = \sum_{\sigma \in \mathfrak{S}_m} \text{sgn} \sigma \cdot e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(m)} \in (\mathbb{C}^N)^{\otimes m}.
\]
Then \( \xi_m \) has the properties

\[
T_{ij}(u) \xi_m = 0 \quad \text{for all } 1 \leq i < j \leq N \tag{5.34}
\]

and

\[
T_{ii}(u) \xi_m = \begin{cases} 
\frac{u + m}{u + m - 1} \xi_m & \text{if } 1 \leq i \leq m, \\
\xi_m & \text{if } m + 1 \leq i \leq N.
\end{cases}
\]

Thus, the vector \( \xi_m \) generates a highest weight module over \( Y(\mathfrak{g}_N) \) whose irreducible quotient is isomorphic to a fundamental module; see [7, Chapter 12], [16].

Consider the extended Yangian \( X(\mathfrak{a}') \) for the subalgebra \( \mathfrak{a}' \) of \( \mathfrak{a} \) of rank \( n - 1 \). That is,

\[
\mathfrak{a}' = \mathfrak{o}_{2n-1}, \mathfrak{sp}_{2n-2}, \mathfrak{o}_{2n-2} \quad \text{respectively for } \mathfrak{a} = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}.
\]

Note that \( X(\mathfrak{a}') \) is not a natural subalgebra of \( X(\mathfrak{a}) \). Let \( V \) be an \( X(\mathfrak{a}) \)-module. Set

\[
V^+ = \{ \eta \in V \mid t_{k,n}(u) \eta = 0 \quad \text{for } k < n \quad \text{and} \quad t_{-n,k}(u) \eta = 0 \quad \text{for } k > -n \}.
\]

**Lemma 5.13.** The subspace \( V^+ \) is stable under all operators \( t_{ij}(u) \) with the condition \(-n + 1 \leq i, j \leq n - 1\). Moreover, these operators form a representation of the algebra \( X(\mathfrak{a}') \) on \( V^+ \), where each operator \( t_{ij}(u) \) is the image of the generator series of \( X(\mathfrak{a}') \) with the same name.

**Proof.** For any \( \eta \in V^+ \) we have the following relations modulo elements of \( V^+ \) which are implied by (2.19): if \( k < n \) and \(-n + 1 \leq i, j \leq n - 1\) then

\[
t_{kn}(v) t_{ij}(u) \eta \equiv -[t_{ij}(u), t_{kn}(v)] \eta \equiv \frac{\delta_{k,-i}}{u - v - \kappa} \theta_{l,-n} t_{-n,j}(u) t_{nn}(v) \eta.
\]

However, applying again (2.19), we find that

\[
t_{-n,j}(u) t_{nn}(v) \eta \equiv -\frac{1}{u - v - \kappa} t_{-n,j}(u) t_{nn}(v) \eta.
\]

Therefore, \( t_{-n,j}(u) t_{nn}(v) \eta \equiv 0 \) implying \( t_{kn}(v) t_{ij}(u) \eta \equiv 0 \). A similar calculation shows that for any \( k > -n \) and \(-n + 1 \leq i, j \leq n - 1\) we also have \( t_{-n,k}(v) t_{ij}(u) \eta \equiv 0 \) proving the first part of the lemma.

In order to prove the second part, suppose that the indices \( i, j, k, l \) satisfy the condition \(-n + 1 \leq i, j, k, l \leq n - 1\). Then by (2.19) for any \( \eta \in V^+ \) we have

\[
[t_{ij}(u), t_{kl}(v)] \eta = \frac{1}{u - v} \left( t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u) \right) \eta \\
- \frac{1}{u - v - \kappa} \left( \delta_{k,-i} \sum_{p=-n}^{n} \theta_{ip} t_{pj}(u) t_{-p,l}(v) - \delta_{l,-j} \sum_{p=-n}^{n} \theta_{jp} t_{k,-p}(v) t_{ip}(u) \right) \eta.
\]

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Writing the right hand side modulo $V^+$, we get

$$\frac{1}{u-v}\left(t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u)\right) \eta$$

$$- \frac{1}{u-v-\kappa}\left(\delta_{k,-i} \sum_{p=-n+1}^{n-1} \theta_{ip} t_{pj}(u) t_{-p,l}(v) - \delta_{l,-j} \sum_{p=-n+1}^{n-1} \theta_{jp} t_{k,-p}(v) t_{ip}(u)\right) \eta$$

$$- \frac{1}{u-v-\kappa}\left(\delta_{k,-i} \theta_{i,-n} t_{n,l}(u) t_{n,l}(v) - \delta_{l,-j} \theta_{j,-n} t_{k,n}(v) t_{i,-n}(u)\right) \eta.$$

Applying again (2.19), we obtain

$$t_{-n,j}(u) t_{n,l}(v) \eta \equiv -\frac{1}{u-v-\kappa} \sum_{p=-n+1}^{n-1} \theta_{-n,p} t_{pj}(u) t_{-p,l}(v) \eta$$

$$- \frac{1}{u-v-\kappa}\left(t_{-n,j}(u) t_{n,l}(v) - \delta_{l,-j} \theta_{j,-n} t_{n,n}(v) t_{-n,n}(u)\right) \eta.$$

Hence,

$$t_{-n,j}(u) t_{n,l}(v) \eta \equiv -\frac{1}{u-v-\kappa + 1} \sum_{p=-n+1}^{n-1} \theta_{-n,p} t_{pj}(u) t_{-p,l}(v) \eta$$

$$+ \frac{1}{u-v-\kappa + 1} \delta_{l,-j} \theta_{j,-n} t_{n,n}(v) t_{-n,n}(u) \eta.$$

Similarly, $t_{k,n}(v) t_{i,-n}(u) \eta \equiv -[t_{i,-n}(u), t_{k,n}(v)] \eta$ and

$$[t_{i,-n}(u), t_{k,n}(v)] \eta \equiv -\frac{1}{u-v-\kappa} \delta_{k,-i} \theta_{i,-n} t_{-n,-n}(u) t_{nn}(v) \eta$$

$$+ \frac{1}{u-v-\kappa}\left(\sum_{p=-n+1}^{n-1} \theta_{-n,p} t_{k,-p}(v) t_{ip}(u) + t_{k,n}(v) t_{i,-n}(u)\right) \eta.$$

which gives

$$t_{k,n}(v) t_{i,-n}(u) \eta \equiv -\frac{1}{u-v-\kappa + 1} \delta_{k,-i} \theta_{i,-n} t_{-n,-n}(u) t_{nn}(v) \eta$$

$$- \frac{1}{u-v-\kappa + 1} \sum_{p=-n+1}^{n-1} \theta_{-n,p} t_{k,-p}(v) t_{ip}(u) \eta.$$

Combining these expressions, we come to the following relation

$$[t_{ij}(u), t_{kl}(v)] \eta \equiv \frac{1}{u-v}\left(t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u)\right) \eta$$

$$- \frac{1}{u-v-\kappa + 1}\left(\delta_{k,-i} \sum_{p=-n+1}^{n-1} \theta_{ip} t_{pj}(u) t_{-p,l}(v) - \delta_{l,-j} \sum_{p=-n+1}^{n-1} \theta_{jp} t_{k,-p}(v) t_{ip}(u)\right) \eta$$

$$+ \frac{1}{(u-v-\kappa)(u-v-\kappa + 1)} \delta_{k,-i} \delta_{l,-j} \theta_{ij} [t_{-n,-n}(u), t_{nn}(v)] \eta.$$
Finally, by (2.19),

\[ [t_{-n,-n}(u), t_{nn}(v)] \eta = -\frac{1}{u - v - \kappa} [t_{-n,-n}(u), t_{nn}(v)] \eta, \]

so that \([t_{-n,-n}(u), t_{nn}(v)] \eta = 0\). This yields the desired relations between the operators \(t_{ij}(u)\) on \(V^+\) since \(\kappa - 1 = \kappa'\) coincides with the value of the parameter \(\kappa\) for the Lie algebra \(\mathfrak{a}'\).

**Proposition 5.14.** The Verma module \(M(\lambda(u))\) over \(X(\mathfrak{a})\) is non-trivial if and only if the components of the highest weight \(\lambda(u)\) satisfy the conditions

\[ \frac{\lambda_{n+i-1}(u + \kappa - i)}{\lambda_{n+i}(u + \kappa - i)} = \frac{\lambda_{n-1}(u)}{\lambda_{n-1}(u)} \]

for \(i = 1, \ldots, n - 1\) if \(\mathfrak{a} = \mathfrak{o}_{2n}\) or \(\mathfrak{sp}_{2n}\), and for \(i = 1, \ldots, n\) if \(\mathfrak{a} = \mathfrak{o}_{2n+1}\).

**Proof.** Suppose first that \(M(\lambda(u))\) is non-trivial. We use induction on \(n\) taking Corollaries 5.6, 5.8 and 5.10 as the induction base. Let us apply \(t_{-n,-n+1}(u) t_{nn-1}(v)\) to the highest vector \(1_\lambda\) of \(M(\lambda(u))\). By (2.19) we have

\[ t_{-n,-n+1}(u) t_{nn-1}(v) 1_\lambda = -\frac{1}{u - v - \kappa} \left( t_{-n,-n+1}(u) t_{nn-1}(v) + \lambda_{n+1}(u) \lambda_{n-1}(v) - \lambda_n(u) \lambda_n(v) \right) 1_\lambda, \]

which implies

\[ (u - v - \kappa + 1) t_{-n,-n+1}(u) t_{nn-1}(v) 1_\lambda = \lambda_n(u) \lambda_n(v) 1_\lambda - \lambda_n(u) \lambda_{n-1}(v) 1_\lambda. \]

Putting \(u = v + \kappa - 1\) and replacing \(v\) by \(u\) we obtain (5.35) for \(i = 1\). Furthermore, by Lemma 5.13 the subspace \(M(\lambda(u))^+\) of \(M(\lambda(u))\) is a module over \(X(\mathfrak{a}')\). The highest vector \(1_\lambda\) belongs to \(M(\lambda(u))^+\) and generates a highest weight \(\lambda(\mathfrak{a}')\)-module with the highest weight \((\lambda_{n+1}(u), \ldots, \lambda_n(u))\). So, the remaining conditions hold by the induction hypothesis.

Conversely, suppose that \(\lambda(u)\) satisfies the conditions. Consider the left ideal \(I\) of the algebra \(X(\mathfrak{a})\) generated by the coefficients of the series \(t_{ij}(u)\) with \(i < j\) where \(i + j > 0\) or \(i + j \geq 0\) for the orthogonal or symplectic case, respectively; and by the coefficients of the series \(t_{ii}(u) - \lambda_i(u)\) for \(i = 1, \ldots, n\) and \(z(u) - \lambda_n(u + \kappa)\lambda_n(u)\). By Corollary 3.10 the quotient \(\widetilde{M}(\lambda(u)) = X(\mathfrak{a})/I\) is non-trivial. Let \(1_\lambda\) be the image of \(1 \in X(\mathfrak{a})\) in the quotient. It suffices to verify that the vector \(1_\lambda\) satisfies all the conditions (5.1). Now we use Corollary 3.10 again. Let us choose the total ordering on the elements \(t_{ij}^{(r)}\) and \(z_r\) with the conditions on the indices as in the statement of the corollary, in such a way that any element \(t_{ij}^{(r)}\) with \(i > j\) precedes any element \(t_{kk}^{(s)}\).
while the latter precedes any element of the form $t_{ij}^{(r)}$ with $i < j$. We shall regard $X(a)$ as the adjoint $a$-module with the action defined on the generators by $(5.3)$. For any pair $k < l$ and any $r \geq 1$ write the element $t_{kl}^{(r)}$ as a linear combination of the ordered monomials. The $a$-weight of each of the monomials coincides with the $a$-weight of $t_{kl}^{(r)}$. Then the relation $t_{kl}^{(r)} 1_{\lambda} = 0$ follows because the vector $1_{\lambda}$ is annihilated by any monomial occurring in the combination. The same argument shows that $1_{\lambda}$ is an eigenvector for the action of any element $t_{kl}^{(r)}$. Thus, the $X(a)$-module $\tilde{M}(\lambda(u))$ is a Verma module $M(\tilde{\lambda}(u))$. It remains to verify that its highest weight $\tilde{\lambda}(u)$ coincides with $\lambda(u)$. This holds for the components of $\tilde{\lambda}(u)$ with positive subscripts by the definition of $\tilde{M}(\lambda(u))$. Furthermore, since $z(u) 1_{\lambda} = \lambda_{-n}(u + \kappa) \lambda(u) 1_{\lambda}$, $(5.13)$ implies that $t_{-n,-n}(u) 1_{\lambda} = \lambda_{-n}(u) 1_{\lambda}$. So, $\tilde{\lambda}_{-n}(u) = \lambda_{-n}(u)$. By the first part of the proof, since the Verma module $M(\tilde{\lambda}(u))$ is non-trivial, the conditions $(5.35)$ must hold for the components of $\tilde{\lambda}(u)$. This shows that $\tilde{\lambda}(u) = \lambda(u)$, and thus $M(\lambda(u))$ is non-trivial.

Corollary 5.15. The irreducible highest weight module $L(\lambda(u))$ over $X(a)$ exists if and only if the conditions $(5.35)$ hold.

Proof. If $L(\lambda(u))$ exists then the conditions $(5.35)$ are derived by repeating the argument of the first part of the proof of Proposition $5.14$. Conversely, if the conditions hold then the Verma module $M(\lambda(u))$ is non-trivial by Proposition $5.14$. Therefore, the irreducible quotient $L(\lambda(u))$ of $M(\lambda(u))$ exists. 

We are now in a position to prove the classification theorem for finite-dimensional irreducible representations of the extended Yangian $X(a)$.

Theorem 5.16. Every finite-dimensional irreducible $X(a)$-module is isomorphic to $L(\lambda(u))$ where $\lambda(u)$ satisfies the conditions $(5.35)$ and there exist monic polynomials $P_1(u), \ldots, P_n(u)$ in $u$ such that

$\frac{\lambda_{i-1}(u)}{\lambda_i(u)} = \frac{P_i(u + 1)}{P_i(u)}$, \hspace{1cm} for $i = 2, \ldots, n$  

(5.36)

and also

$\frac{\lambda_0(u)}{\lambda_1(u)} = \frac{P_1(u + 1/2)}{P_1(u)}$, \hspace{1cm} if $a = o_{2n+1}$,

$\frac{\lambda_i(u)}{\lambda_1(u)} = \frac{P_1(u + 2)}{P_1(u)}$, \hspace{1cm} if $a = sp_{2n}$,

$\frac{\lambda_{i-1}(u)}{\lambda_2(u)} = \frac{P_i(u + 1)}{P_i(u)}$, \hspace{1cm} if $a = o_{2n}$.

Conversely, if $(5.35)$ and the above conditions on the highest weight $\lambda(u)$ are satisfied then $L(\lambda(u))$ exists and has finite dimension.

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The polynomials $P_1(u), \ldots, P_n(u)$ are called the Drinfeld polynomials corresponding to the finite-dimensional representation $L(\lambda(u))$.

**Proof.** Due to Theorem 5.1, every finite-dimensional irreducible $X(a)$-module is isomorphic to $L(\lambda(u))$ for some highest weight $\lambda(u)$. Then $\lambda(u)$ must satisfy (5.33) by Corollary 5.16 since $L(\lambda(u))$ exists. Now we argue by induction on $n$ taking Corollaries 5.6, 5.8 and 5.10 as the induction base. Observe that if $n \geq 2$ then by (2.19), the mapping

$$T_{ij}(u) \mapsto t_{i+n-2,j+n-2}(u), \quad i, j \in \{1, 2\}$$

defines a homomorphism $Y(sl_2) \to X(a)$. So, $L(\lambda(u))$ can be regarded as a $Y(sl_2)$-module. The highest vector $1_\lambda \in L(\lambda(u))$ then satisfies

$$T_{11}(u) 1_\lambda = \lambda_{n-1}(u) 1_\lambda, \quad T_{22}(u) 1_\lambda = \lambda_n(u) 1_\lambda, \quad T_{12}(u) 1_\lambda = 0.$$

Since the cyclic span $Y(sl_2) 1_\lambda$ is finite-dimensional, we derive from Theorem 5.3 that there exists a monic polynomial $P_n(u)$ such that (5.33) holds for $i = n$. Furthermore, by Lemma 5.13, the subspace $L(\lambda(u))^+$ is a module over $X(a')$. The highest vector $1_\lambda$ belongs to $L(\lambda(u))^+$ and generates a highest weight $X(a')$-module with the highest weight $(\lambda_{-n+1}(u), \ldots, \lambda_{n-1}(u))$. Since the cyclic span $X(a') 1_\lambda$ is finite-dimensional, the remaining conditions on the $\lambda_i(u)$ hold by the induction hypothesis.

Suppose now that the highest weight $\lambda(u)$ satisfies the given conditions. Then $L(\lambda(u))$ exists by Corollary 5.16. We need to show that $\dim L(\lambda(u)) < \infty$. Observe that the $n$-tuple of Drinfeld polynomials corresponding to an $X(a)$-module $L(\lambda(u))$ determines the highest weight $\lambda(u)$ up to a simultaneous multiplication of all components $\lambda_i(u)$ by a series $f(u) \in 1 + u^{-1} \mathbb{C}[[u^{-1}]]$. On the other hand, the composition of the action of $X(a)$ on $L(\lambda(u))$ with the automorphism (2.21) yields a representation of $X(a)$ isomorphic to $L(\lambda'(u))$, where the components of $\lambda'(u)$ are given by $\lambda'_i(u) = f(u) \lambda_i(u)$. Therefore, it suffices to prove that a particular module $L(\lambda(u))$ corresponding to an arbitrary $n$-tuple of Drinfeld polynomials is finite-dimensional.

We shall use the coproduct (2.24) to equip the tensor product of two $X(a)$-modules with an $X(a)$-module structure.

**Lemma 5.17.** Let $L(\lambda(u))$ and $L(\mu(u))$ be two irreducible highest weight modules over $X(a)$ with

$$\lambda(u) = (\lambda_{-n}(u), \ldots, \lambda_n(u)) \quad \text{and} \quad \mu(u) = (\mu_{-n}(u), \ldots, \mu_n(u)).$$

Then the tensor product $1_\lambda \otimes 1_\mu$ of the highest vectors of $L(\lambda(u))$ and $L(\mu(u))$ generates a highest weight submodule $V$ over $X(a)$ in $L(\lambda(u)) \otimes L(\mu(u))$ with the highest weight

$$(\lambda_{-n}(u) \mu_{-n}(u), \ldots, \lambda_n(u) \mu_n(u)). \quad (5.37)$$
Moreover, if the modules $L(\lambda(u))$ and $L(\mu(u))$ are finite-dimensional with the corresponding $n$-tuples of Drinfeld polynomials $(P_1(u), \ldots, P_n(u))$ and $(Q_1(u), \ldots, Q_n(u))$, respectively, then the $n$-tuple of Drinfeld polynomials corresponding to the irreducible quotient of $V$ is $(P_1(u)Q_1(u), \ldots, P_n(u)Q_n(u))$.

**Proof.** It follows easily from (2.24) that the vector $\xi = \lambda \otimes 1$ satisfies (5.1) with the highest weight (5.37). The second statement now follows from the relations defining the Drinfeld polynomials.

By the lemma, we only need to show that if an irreducible highest weight module $L(\lambda(u))$ corresponds to an $n$-tuple of Drinfeld polynomials of the form $P_j(u)$ for all $j \neq i$ and $P_i(u) = u - b$ for certain $i \in \{1, \ldots, n\}$ and $b \in \mathbb{C}$, then $\dim L(\lambda(u)) < \infty$.

Furthermore, the composition of the action of $X(a)$ on $L(\lambda(u))$ with an automorphism of the form (2.22) yields a representation of $X(a)$ whose $n$-tuple of Drinfeld polynomials is $P_j(u) = 1$ for all $j \neq i$ and $P_i(u) = u - a - b$. Thus, it suffices to prove the claim for all values of the index $i$ and a certain particular value of $b \in \mathbb{C}$.

Consider the representation of $X(a)$ on $\mathbb{C}^N$ defined in (3.19) with $c = 0$ so that

$$t_{ij}(u) = \delta_{ij} + e_{ij}u^{-1} - \theta_{ij}e_{-j,-i}(u + \kappa)^{-1}.$$ (5.39)

We claim that $\xi_m$ satisfies

$$t_{ij}(u) \xi_m = 0 \quad \text{for all} \quad -n \leq i < j \leq n \quad (5.39)$$

and

$$t_{ii}(u) \xi_m = \begin{cases} 
\frac{u + m}{u + m - 1} \xi_m & \text{if} \quad -n \leq i \leq -n + m - 1, \\
\xi_m & \text{if} \quad -n + m \leq i \leq n - m, \\
\frac{u + \kappa - 1}{u + \kappa} \xi_m & \text{if} \quad n - m + 1 \leq i \leq n. 
\end{cases} \quad (5.40)$$
Denote by \( P^{(m)} \) the operator in \((\mathbb{C}^N)^\otimes m\) which acts on the basis vectors by

\[
P^{(m)}(e_i \otimes \cdots \otimes e_m) = e_i \otimes \cdots \otimes e_1.
\]

We have \( P^{(m)}(\xi_m) = \alpha \xi_m \), where \( \alpha = 1 \) or \(-1\). The definition \((5.38)\) implies the following relation for the action of \( X(a) \) on \((\mathbb{C}^N)^\otimes m\),

\[
\theta_{ij} t_{-j, -i}(u) = P^{(m)} t_{ij}(-u - \kappa - m + 1) P^{(m)}.
\]

Due to \((5.3)\), in order to verify \((5.39)\) in the case \( a = \mathfrak{o}_{2n+1} \), it therefore suffices to consider the values \( j = i + 1 \) with \(-n \leq i \leq -1\). Since the expression for the vector \( \xi_m \) only involves the tensor products \( e_i \otimes \cdots \otimes e_n \) with negative subscripts \( i \), we may assume that the summation indices \( a_1, \ldots, a_{m-1} \) in \((5.38)\) are all negative. Indeed, \( t_{ia_1}(u) e_i = 0 \) unless \( a_1 < 0 \) implying \( t_{a_1a_2}(u + 1) e_i = 0 \) unless \( a_2 < 0 \) etc. However, in this case the formula \((5.38)\) takes the same form as its \( Y(\mathfrak{gl}_N) \)-counterpart \((5.38)\) if we take into account the convention on the basis vector indices. Therefore, the relations \( t_{i,i+1}(u) \xi_m = 0 \) and, hence \((5.39)\), are implied by the corresponding property \((5.34)\) of the vector \( \xi_m \) in the case of \( Y(\mathfrak{gl}_N) \). Moreover, this argument also proves \((5.40)\) for the non-positive values of \( i \). The application of \((5.41)\) completes the proof of \((5.40)\).

The same argument applies to the cases \( a = \mathfrak{sp}_{2n} \) and \( a = \mathfrak{o}_{2n} \) which also shows that \( t_{-1,1}(u) \xi_m = 0 \) together with \( t_{-1,2}(u) \xi_m = 0 \) for \( a = \mathfrak{o}_{2n} \).

Thus, in the case \( a = \mathfrak{sp}_{2n} \) for any \( m \in \{1, \ldots, n-1\} \) the vector \( \xi_m \) generates a highest weight submodule of \((\mathbb{C}^N)^\otimes m\) whose \( n \)-tuple of Drinfeld polynomials is \( P_j(u) = 1 \) for all \( j \neq m \) and \( P_m(u) = u + \kappa - 1 \), while \( \xi_n \) generates a highest weight submodule of \((\mathbb{C}^N)^\otimes n\) whose \( n \)-tuple of Drinfeld polynomials is \( P_1(u) = u + n - 1 \) and \( P_j(u) = 1 \) for \( j \neq 1 \). This completes the proof of the theorem in the symplectic case, as the irreducible highest weight modules over \( X(a) \) with such \( n \)-tuples of Drinfeld polynomials are finite-dimensional.

Similarly, the proof is also complete in the case \( a = \mathfrak{o}_{2n} \) and the values \( m \in \{1, \ldots, n-2\} \), as well as in the case \( a = \mathfrak{o}_{2n+1} \) for the values \( m \in \{1, \ldots, n-1\} \). In order to complete the proof in the remaining cases, we shall use the spinor representations of the orthogonal Lie algebras. The spinor representation \( V(-1/2, \ldots, -1/2) \) of the Lie algebra \( \mathfrak{o}_{2n+1} \) can be realized in the \( 2^n \)-dimensional space \( \Lambda_n \) of polynomials in \( n \) anti-commuting variables \( \xi_1, \ldots, \xi_n \),

\[
\Lambda_n = \text{span of } \{\xi_{i_1} \cdots \xi_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n, \; 0 \leq k \leq n\}.
\]

The generators of \( \mathfrak{o}_{2n+1} \) act on this space as the operators

\[
F_{ij} = \xi_i \partial_j - \frac{1}{2} \delta_{ij}, \quad F_{-j,i} = \partial_i \partial_j, \quad F_{j,-i} = \xi_i \xi_j, \quad F_{0,i} = \frac{1}{\sqrt{2}} \partial_i, \quad F_{i,0} = \frac{1}{\sqrt{2}} \xi_i,
\]

\((5.42)\).
where \( i, j \in \{1, \ldots, n\} \) and \( \partial_i \) is the left derivative over \( \xi_i \). The restriction of \( \Lambda_n \) to the subalgebra \( \mathfrak{o}_{2n} \subset \mathfrak{o}_{2n+1} \) (spanned by the elements \( F_{ij} \) with \( i, j \neq 0 \)) splits into the direct sum of two irreducible submodules, \( \Lambda_n = \Lambda_n^+ \oplus \Lambda_n^- \), where \( \Lambda_n^+ \) (respectively, \( \Lambda_n^- \)) is the subspace of \( \Lambda_n \) spanned by the even (respectively, odd) monomials in the generators \( \xi_i \). We have the isomorphisms

\[
\Lambda_n^+ \cong V(-1/2, \ldots, -1/2) \quad \text{and} \quad \Lambda_n^- \cong V(1/2, -1/2, \ldots, -1/2).
\]  

(5.43)

The highest weight vectors of the \( \mathfrak{o}_{2n} \)-modules \( \Lambda_n^+ \) and \( \Lambda_n^- \) are, respectively, the vectors 1 and \( \xi_1 \).

**Lemma 5.18.** Each spinor representation of \( \mathfrak{o}_N \) can be extended to a representation of the algebra \( X(\mathfrak{o}_N) \) by the rule

\[
t_{ij}(u) \mapsto \delta_{ij} + F_{ij} u^{-1}, \quad i, j \in \{-n, \ldots, n\}.
\]

Proof. The claim follows by a direct verification that the images of \( t_{ij}(u) \) satisfy the defining relations (2.19) with the use of the following identity of operators in each spinor representation:

\[
(F^2)_{ij} = \left( \frac{\kappa}{2} + \frac{1}{4} \right) \delta_{ij} + \kappa F_{ij},
\]

(5.44)

where \( F \) is defined in (2.6). Indeed, in the particular case \( i = j = n \), the identity is verified by a straightforward calculation. The general case then follows by commuting both sides of this particular identity with appropriate generators \( F_{ij} \).

The lemma implies that the spinor representation \( V(-1/2, \ldots, -1/2) \) of \( \mathfrak{o}_N \) becomes an irreducible highest weight representation of \( X(\mathfrak{o}_N) \) with the highest weight \( \lambda(u) \), where

\[
\lambda_i(u) = 1 + \frac{1}{2} u^{-1} \quad \text{for} \quad i \leq -1, \quad \lambda_i(u) = 1 - \frac{1}{2} u^{-1} \quad \text{for} \quad i \geq 1
\]

and \( \lambda_0(u) = 1 \) (the latter only occurs for \( N = 2n + 1 \)). The corresponding \( n \)-tuple of Drinfeld polynomials is \( (u - 1/2, 1, \ldots, 1) \) in both cases \( N = 2n \) and \( N = 2n + 1 \). Finally, the spinor representation \( V(1/2, -1/2, \ldots, -1/2) \) of \( \mathfrak{o}_{2n} \) becomes an irreducible highest weight representation of \( X(\mathfrak{o}_{2n}) \) with the highest weight \( \lambda(u) \), where

\[
\lambda_i(u) = 1 + \frac{1}{2} u^{-1} \quad \text{for} \quad i \leq -2 \quad \text{and} \quad i = 1,
\]

\[
\lambda_i(u) = 1 - \frac{1}{2} u^{-1} \quad \text{for} \quad i \geq 2 \quad \text{and} \quad i = -1.
\]

The corresponding \( n \)-tuple of Drinfeld polynomials is \( (1, u - 1/2, 1, \ldots, 1) \).
Theorem 5.16 allows us to get another proof of Drinfeld’s classifications theorem for the Yangian modules [12]; cf. [7, Chapter 12].

**Corollary 5.19.** Any finite-dimensional irreducible representation of the Yangian $Y(a)$ is isomorphic to the restriction of an $X(a)$-module $L(\lambda(u))$ to the subalgebra $Y(a)$, where the components of $\lambda(u)$ satisfy the conditions of Theorem 5.16. In particular, such representations of $Y(a)$ are parameterized by the tuples $(P_1(u), \ldots, P_n(u))$ of monic polynomials in $u$.

**Proof.** By Theorem 3.1, any finite-dimensional irreducible representation $V$ of $Y(a)$ can be extended to a representation of $X(a)$ where the elements of the center $ZX(a)$ act as scalar operators. By Theorem 5.16, the $X(a)$-module $V$ is isomorphic to $L(\lambda(u))$ for an appropriate highest weight $\lambda(u)$. This allows one to attach a tuple of polynomials $(P_1(u), \ldots, P_n(u))$ to the $Y(a)$-module $V$.

Conversely, given any $n$-tuple of polynomials $(P_1(u), \ldots, P_n(u))$, there exists a highest weight $\lambda(u)$ such that the conditions of Theorem 3.1 hold. Moreover, the components of $\lambda(u)$ are uniquely determined up to simultaneous multiplication by a formal series in $u^{-1}$. This implies that the corresponding $X(a)$-module $L(\lambda(u))$ is determined up to twisting by an appropriate automorphism (2.21). However, the subalgebra $Y(a)$ consists of the elements stable under all such automorphisms. This yields the desired parametrization of the representations of $Y(a)$. □

The finite-dimensional irreducible representations $L(\lambda(u))$ corresponding to the $n$-tuples of Drinfeld polynomials of the form $(1, \ldots, u-a, 1, \ldots, 1)$, where $a \in \mathbb{C}$ and $u-a$ is on the $i$-th position, are called the fundamental representations of $X(a)$ or $Y(a)$. The following corollary was established in the proof of Theorem 5.16.

**Corollary 5.20.** Every finite-dimensional irreducible representation of $Y(a)$ is isomorphic to a subquotient of a tensor product of the fundamental representations. □

### 5.4 Fundamental representations

In this section we give a more explicit description of the fundamental representations of the algebras $X(a)$ and $Y(a)$. We shall follow the general approach of the paper by Chari and Pressley [6]. However, contrary to [6], we avoid using the theorem describing the singularities of $R$-matrices.

We start with the orthogonal case $a = o_N$. The fundamental representations with the $n$-tuples of Drinfeld polynomials $(u-1/2, 1, \ldots, 1)$ and $(1, u-1/2, 1, \ldots, 1)$ (the latter for $N = 2n$ only), were constructed in the proof of Theorem 5.16.
Now let $N = 2n + 1$. The tensor square of the spinor representation $\Lambda_n$ of $\mathfrak{o}_{2n+1}$ has the following decomposition into irreducibles:

$$\Lambda_n \otimes \Lambda_n \cong \bigoplus_{p=0}^{n} V(\mu(p)),$$

(5.45)

where $\mu(p) = (0,\ldots,0,-1,\ldots,-1)$ with $p$ zeros. Note that $V(\mu(p))$ is a fundamental representation of $\mathfrak{o}_{2n+1}$ for any $1 \leq p \leq n - 1$. It corresponds to the fundamental weight $\omega_{n-p}$ in a more standard notation. The highest weight vector $v_p$ of $V(\mu(p))$ is given in an explicit form by

$$v_p = \sum (-1)^{j_1+\cdots+j_l} \xi_{i_1} \cdots \xi_{i_k} \otimes \xi_{j_1} \cdots \xi_{j_l},$$

(5.46)

summed over all partitions of the set $\{1,\ldots,p\}$ into the disjoint union of two subsets $\{i_1,\ldots,i_k\}$ and $\{j_1,\ldots,j_l\}$ so that $p = k + l$ with $k,l \geq 0$ while $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$.

By Lemma 5.18, we may regard $\Lambda_n$ as an $X(\mathfrak{o}_{2n+1})$-module. Furthermore, using the coproduct (2.24) and the automorphism (2.22), we can equip $\Lambda_n \otimes \Lambda_n$ with an $X(\mathfrak{o}_{2n+1})$-action by

$$t_{ij}(u)(\eta \otimes \zeta) = \sum_{k=-n}^{n} (\delta_{ik} + F_{ik} (u - a)^{-1}) \eta \otimes (\delta_{kj} + F_{kj} u^{-1}) \zeta,$$

(5.47)

where $\eta, \zeta \in \Lambda_n$ and $a \in \mathbb{C}$ is a fixed constant.

**Proposition 5.21.** If $a = p - 1/2$ then the vector $v_p \in \Lambda_n \otimes \Lambda_n$ has the properties

$$t_{ij}(u) v_p = 0 \quad \text{for} \quad -n \leq i < j \leq n$$

(5.48)

and

$$t_{ii}(u) v_p = \begin{cases} (u - p)(u + 1/2) v_p & \text{for} \quad 0 \leq i \leq p, \\ u (u - p + 1/2) v_p & \text{for} \quad p + 1 \leq i \leq n. \end{cases}$$

(5.49)

**Proof.** By the definition (5.47), we have

$$t_{ij}^{(1)} (\eta \otimes \zeta) = F_{ij} \eta \otimes \zeta + \eta \otimes F_{ij} \zeta$$

and

$$t_{ij}^{(r)} (\eta \otimes \zeta) = a^{r-2} \sum_{k=-n}^{n} F_{ik} \eta \otimes F_{kj} \zeta + a^{r-1} F_{ij} \eta \otimes \zeta$$

(5.50)
for $r \geq 2$. In particular,

$$t_{ij}^{(r+1)} (\eta \otimes \zeta) = a t_{ij}^{(r)} (\eta \otimes \zeta)$$

(5.51)

for any $r \geq 2$. Since $v_p$ is the highest weight vector in the $\mathfrak{o}_{2n+1}$-module $V(\mu^p)$, we have the relations $t_{ij}^{(1)} v_p = 0$ for $-n \leq i < j \leq n$ and

$$t_{ii}^{(1)} v_p = \begin{cases} 
0 & \text{for } 0 \leq i \leq p, \\
-v_p & \text{for } p + 1 \leq i \leq n.
\end{cases}$$

Now, (5.3) implies that

$$[F_{i-1,i}, t_{ii}^{(2)}] = t_{i-1,i}^{(2)}, \quad i = 1, \ldots, n.$$ 

Furthermore, taking the $(i - 1, i)$ entry in (2.26) and comparing the coefficients at $u^{-2}$ we get

$$t_{i-1,i}^{(2)} - \sum_{k=-n}^{n} t_{i-1,k}^{(1)} t_{k,i}^{(1)} + t_{i,-i+1}^{(2)} - \kappa t_{i,-i+1}^{(1)} = 0.$$

Hence, (5.48) will follow if we prove that $v_p$ is an eigenvector for all the operators $t_{ii}^{(2)}$ with $i = 1, \ldots, n$. By (5.50), we have the following equality of operators in $\Lambda_n \otimes \Lambda_n$,

$$t_{ii}^{(2)} = \sum_{k=-n}^{n} (F_{ik} \otimes 1)(F_{ki} \otimes 1 + 1 \otimes F_{ki}) - (F^2)_{ii} \otimes 1 + a F_{ii} \otimes 1.$$

Note that each element $F_{ki} \in \mathfrak{o}_{2n+1}$ acts on $\Lambda_n \otimes \Lambda_n$ as the operator

$$\Delta(F_{ki}) = F_{ki} \otimes 1 + 1 \otimes F_{ki}.$$

Due to (5.44), in the spinor representation $\Lambda_n$ we have $(F^2)_{ii} = n/2 + (n - 1/2) F_{ii}$. Moreover, we have $\Delta(F_{ki}) v_p = 0$ for $k < i$ and for $1 \leq i < k \leq p$. The latter follows from the fact that each vector $\Delta(F_{k,k-1}) v_p$ with $k \in \{2, \ldots, p\}$ is annihilated by all operators $\Delta(F_{j,j+1})$ and hence must be zero, as the $\mathfrak{o}_{2n+1}$-module $V(\mu^p)$ is irreducible. Recalling that $a = p - 1/2$ we thus get for any $i \in \{1, \ldots, p\}$,

$$t_{ii}^{(2)} v_p = \sum_{k=-n}^{n} (F_{ik} \otimes 1) \Delta(F_{ki}) v_p + (p - n) (F_{ii} \otimes 1) v_p - n/2 v_p.$$

Using the expression (5.46) for $v_p$ and the formulas (5.42) it is now easy to derive the relation $t_{ii}^{(2)} v_p = -p/2 \cdot v_p$. If $i \in \{p + 1, \ldots, n\}$ then

$$t_{ii}^{(2)} v_p = \sum_{k=i}^{n} (F_{ik} \otimes 1) \Delta(F_{ki}) v_p + (p - n) (F_{ii} \otimes 1) v_p - n/2 v_p.$$
Using again (5.46) and (5.42), we find that $\Delta(F_{ki})v_p = 0$ for $k > i$ which gives $t^{(2)}_{ii}v_p = (-p/2 + 1/2)v_p$. Thus, (5.48) is proved. For any $i > 0$ the relation (5.49) is now implied by (5.51) with $j = i$. Finally, we have $t^{(2)}_{00}v_p = -p/2 \cdot v_p$ which is verified by a similar calculation. This implies (5.49) for $i = 0$.

Due to Proposition 5.21, the cyclic span $W_p = X(o_{2n+1})v_p$ of the highest vector $v_p \in \Lambda_n \otimes \Lambda_n$ is a highest weight module over $X(o_{2n+1})$. By the following theorem, $W_p$ is irreducible. This module is finite-dimensional, and if $1 \leq p \leq n - 1$ then the corresponding $n$-tuple of Drinfeld polynomials is $(1, \ldots, u - 1/2, 1, \ldots, 1)$ with $u - 1/2$ on the $(p+1)$-th position; see Theorem 5.16. So, this yields a construction of the fundamental representations of $X(o_{2n+1})$ alternative to the one used in the proof of Theorem 5.16. The following is a version of a result of Chari and Pressley [6, Theorem 6.2] and earlier results of Ogievetsky, Reshetikhin and Wiegmann [22]. We assume that $1 \leq p \leq n - 1$ and $a = p - 1/2$.

**Theorem 5.22.** The $X(o_{2n+1})$-module $W_p$ is irreducible. Its restriction to the universal enveloping algebra $U(o_{2n+1})$ is given by

$$W_p|_{U(o_{2n+1})} \cong \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}).$$

**Proof.** By Corollary 5.10 and Proposition 5.21 the vector space $W_p$ is spanned by the elements

$$t^{(r_1)}_{j_1} \cdots t^{(r_m)}_{j_m}v_p, \quad m \geq 0,$$

with $j_a > i_a$ and $r_a \geq 1$. By (5.3), the $o_{2n+1}$-weights of $W_p$ have the form $\mu^{(p)} - \omega$, where $\omega$ is a $Z_+$-linear combination of the positive roots; see their description in the beginning of Section 5.1. However, any $Z_+$-linear combination of the positive roots has the form $k_1 \varepsilon_1 + \cdots + k_n \varepsilon_n$, where the $k_i$ are integers and the sum $k_1 + \cdots + k_n$ is a non-positive integer. Since $\mu^{(p)} - \mu^{(l)} = \varepsilon_{l+1} + \cdots + \varepsilon_p$ for $l < p$, we conclude that, as an $o_{2n+1}$-module,

$$W_p \subseteq \bigoplus_{s=p}^{n} V(\mu^{(s)}). \quad (5.52)$$

We shall now demonstrate that none of the irreducible $o_{2n+1}$-modules of the form $V(\mu^{(s)})$ with $s = p + 1, p + 3, \ldots$ can occur in the irreducible decomposition of $W_p$. We need the following lemma which holds for any value of the parameter $a$.

**Lemma 5.23.** For any $s \in \{2, \ldots, n\}$ in the $X(o_{2n+1})$-module $\Lambda_n \otimes \Lambda_n$ we have

$$t^{(2)}_{-s+1,s}v_s = (a - s + 1/2)v_{s-2}.$$
Proof. By (5.50), we have

\[ t^{(2)}_{-s+1,s} = \sum_{k=-n}^{n} (F_{-s+1,k} \otimes 1) \Delta(F_{ks}) - (F^2)_{-s+1,s} \otimes 1 + a F_{-s+1,s} \otimes 1. \]

Furthermore, (5.44) implies \((F^2)_{-s+1,s} = (n - 1/2) F_{-s+1,s}\). Moreover, in the \(\mathfrak{o}_{2n+1}\)-submodule \(V(\mu^{(s)})\) of \(\Lambda_n \otimes \Lambda_n\) we have \(\Delta(F_{ks}) v_s = 0\) for \(k \leq s\). Hence, applying (5.42) we obtain

\[ t^{(2)}_{-s+1,s} v_s = \sum_{k=s+1}^{n} (\partial_k \partial_s \otimes 1)(\xi_k \partial_s \otimes 1 + 1 \otimes \xi_k \partial_s) v_s + (a - n + 1/2) (\partial_s \partial_{s-1} \otimes 1) v_s. \]

Finally, using the formula (5.46) for \(v_s\) we come to

\[ t^{(2)}_{-s+1,s} v_s = (a - s + 1/2) (\partial_s \partial_{s-1} \otimes 1) v_s = (a - s + 1/2) v_{s-2}. \]

Now, if the irreducible module \(V(\mu^{(s)})\) with \(s = p + 2i - 1\) for some \(i \geq 1\) occurs in the irreducible decomposition of \(W_p\) then \(W_p\) would also contain \(V(\mu^{(p-1)})\) by Lemma 5.23. But this contradicts (5.52). Thus, as an \(\mathfrak{o}_{2n+1}\)-module,

\[ W_p \subseteq \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}). \tag{5.53} \]

We now need the following counterpart of Lemma 5.23.

**Lemma 5.24.** Let \(s \in \{2, \ldots, n\}\). If \(a \neq -s + 1/2\) then the projection of the vector \(t^{(2)}_{s,-s+1} v_{s-2} \in \Lambda_n \otimes \Lambda_n\) onto the component \(V(\mu^{(s)})\) in the decomposition (5.44) is nonzero.

**Proof.** Let us introduce a bilinear form on the vector space \(\Lambda_n\) by

\[ \langle \xi_{i_1} \cdots \xi_{i_k}, \xi_{j_1} \cdots \xi_{j_l} \rangle = \delta_{IJ}, \]

where \(I = \{i_1, \ldots, i_k\}\) and \(J = \{j_1, \ldots, j_l\}\) are subsets of \(\{1, \ldots, n\}\) such that \(i_1 < \cdots < i_k\) and \(j_1 < \cdots < j_l\), with \(\delta_{IJ} = 1\) if \(I = J\), and 0 otherwise. The form possesses the covariance property with respect to the action of \(\mathfrak{o}_{2n+1}\),

\[ \langle F_{ij} \eta, \zeta \rangle = \langle \eta, F_{ij} \zeta \rangle, \quad \eta, \zeta \in \Lambda_n. \]

Extend the form \(\langle \cdot, \cdot \rangle\) to a bilinear form on the tensor product space \(\Lambda_n \otimes \Lambda_n\) by

\[ \langle \eta_1 \otimes \eta_2, \xi_1 \otimes \xi_2 \rangle = \langle \eta_1, \xi_2 \rangle \langle \eta_2, \xi_1 \rangle. \]
One easily verifies that this form inherits the covariance property. In particular, the irreducible components $V(\mu^{(s)})$ in the decomposition (5.45) are pairwise orthogonal. So the lemma will follow if we prove that \( \langle t^{(2)}_{s,-s+1} v_{s-2}, v_s \rangle \neq 0 \). However, a direct calculation with the use of (5.50) shows that for any \( \eta, \zeta \in \Lambda_n \otimes \Lambda_n \) we have

\[
\langle t^{(2)}_{ij} \eta, \zeta \rangle = \langle \eta, (t^{(2)}_{ji} + a (1 \otimes F_{ji} - F_{ji} \otimes 1)) \zeta \rangle.
\]

Hence, using Lemma 5.23 and the formulas (5.42) we find that

\[
\langle t^{(2)}_{s,-s+1} v_{s-2}, v_s \rangle = \langle v_{s-2}, (t^{(2)}_{-s+1,s} + a (1 \otimes F_{-s+1,s} - F_{-s+1,s} \otimes 1)) v_s \rangle
\]

\[
= (-a - s + 1/2) \langle v_{s-2}, v_{s-2} \rangle \neq 0,
\]

completing the proof of the lemma.

If \( a = p - 1/2 \) then the condition of Lemma 5.24 is satisfied for any \( s \in \{2, \ldots, n\} \). Thus, Lemmas 5.23 and 5.24 imply that the \( X(\mathfrak{o}_{2n+1}) \)-module \( W_p \) is irreducible and its \( \mathfrak{o}_{2n+1} \)-irreducible decomposition coincides with the right hand side of (5.53).

Consider now the case \( a = \mathfrak{o}_{2n} \). As we mentioned in the previous section, the restriction of the \( \mathfrak{o}_{2n+1} \)-module \( \Lambda_n \) to the subalgebra \( \mathfrak{o}_{2n} \) splits into the direct sum of two irreducible submodules, \( \Lambda_n = \Lambda_n^+ \oplus \Lambda_n^- \), and we have the isomorphisms (5.43). We have the following tensor product decompositions of the \( \mathfrak{o}_{2n} \)-modules:

\[
\Lambda_n^+ \otimes \Lambda_n^+ \cong \bigoplus_{r=0}^{[n/2]} V(\mu^{(2r)}),
\]

(5.54)

\[
\Lambda_n^+ \otimes \Lambda_n^- \cong \bigoplus_{r=0}^{[(n-1)/2]} V(\mu^{(2r+1)}),
\]

(5.55)

where \( \mu^{(p)} = (0, \ldots, 0, -1, \ldots, -1) \) with \( p \) zeros. Note that \( V(\mu^{(p)}) \) is a fundamental representation of \( \mathfrak{o}_{2n} \) for any \( 2 \leq p \leq n - 1 \). The highest weight vector \( v_p \) of \( V(\mu^{(p)}) \) in the decompositions (5.54) and (5.55) is given by (5.46) with the following additional restrictions: both \( k \) and \( l \) are even for (5.54) with \( p = 2r \), while \( k \) is even and \( l \) is odd for (5.55) with \( p = 2r + 1 \).

By Lemma 5.18 we may regard \( \Lambda_n^+ \) and \( \Lambda_n^- \) as \( X(\mathfrak{o}_{2n}) \)-modules. As in the previous case, we equip the tensor products \( \Lambda_n^+ \otimes \Lambda_n^+ \) and \( \Lambda_n^+ \otimes \Lambda_n^- \) with an \( X(\mathfrak{o}_{2n}) \)-action by

\[
t_{ij}(u)(\eta \otimes \zeta) = \sum_{k=-n}^{n} \left( \delta_{ik} + F_{ik} (u - a)^{-1} \right) \eta \otimes \left( \delta_{kj} + F_{kj} u^{-1} \right) \zeta,
\]

(5.56)

where \( a \in \mathbb{C} \) is a fixed constant. In the following proposition we consider the cases of even and odd \( p \) simultaneously. If \( p = 2r \) then \( v_p \in \Lambda_n^+ \otimes \Lambda_n^+ \) and if \( p = 2r + 1 \) then \( v_p \in \Lambda_n^+ \otimes \Lambda_n^- \).
Proposition 5.25. If \( a = p - 1 \) then the vector \( v_p \) has the properties

\[
 t_{ij}(u) v_p = 0 \quad \text{for} \quad -n \leq i < j \leq n \tag{5.57}
\]

and

\[
 t_{ii}(u) v_p = \begin{cases} 
 \frac{(u - p + 1/2)(u + 1/2)}{u(u - p + 1)} v_p & \text{for} \quad -1 \leq i \leq p, \\
 \frac{(u - p + 1/2)(u - 1/2)}{u(u - p + 1)} v_p & \text{for} \quad p + 1 \leq i \leq n.
\end{cases} \tag{5.58}
\]

Proof. The proof is essentially the same as for Proposition 5.21 with the use of the relation (5.44). The calculation of the eigenvalues of the operators \( t^{(2)}_{ii} \) on \( v_p \) gives

\[
 t^{(2)}_{ii} v_p = \begin{cases} 
 (1/4 - p/2) v_p & \text{for} \quad -1 \leq i \leq p, \\
 (3/4 - p/2) v_p & \text{for} \quad p + 1 \leq i \leq n.
\end{cases}
\]

These imply the desired properties. \( \square \)

The cyclic span \( W_p = X(\mathfrak{o}_{2n}) v_p \) of the vector \( v_p \) is a highest weight module over \( X(\mathfrak{o}_{2n}) \). By the following theorem, \( W_p \) is irreducible. This module is finite-dimensional, and if \( 2 \leq p \leq n - 1 \) then the corresponding \( n \)-tuple of Drinfeld polynomials is \( (1, \ldots, u - 1/2, 1, \ldots, 1) \) with \( u - 1/2 \) on the \( (p + 1) \)-th position; see Theorem 5.16. So, \( W_p \) is a fundamental module over \( X(\mathfrak{o}_{2n}) \). The following is the \( \mathfrak{o}_{2n} \)-counterpart of Theorem 5.22. We assume that \( 2 \leq p \leq n - 1 \) and \( a = p - 1 \).

Theorem 5.26. The \( X(\mathfrak{o}_{2n}) \)-module \( W_p \) is irreducible. Its restriction to the universal enveloping algebra \( U(\mathfrak{o}_{2n}) \) is given by

\[
 W_p |_{U(\mathfrak{o}_{2n})} \cong \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}).
\]

Proof. Considering the \( \mathfrak{o}_{2n} \)-weights of \( W_p \) and using Corollary 3.10, we conclude that, as an \( \mathfrak{o}_{2n} \)-module,

\[
 W_p \subseteq \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}). \tag{5.59}
\]

The equality in (5.59) and irreducibility of the \( X(\mathfrak{o}_{2n}) \)-module \( W_p \) is implied by the following two lemmas which are verified in the same way as their \( \mathfrak{o}_{2n+1} \)-counterparts.

Lemma 5.27. For any \( s \in \{2, \ldots, n\} \) in the \( X(\mathfrak{o}_{2n}) \)-module \( \Lambda_+^n \otimes \Lambda_+^n \) or \( \Lambda_+^n \otimes \Lambda_+^n \) for even or odd \( s \), respectively, we have

\[
 t^{(2)}_{-s+1,s} v_s = (a - s + 1) v_{s-2}. \quad \square
\]
Lemma 5.28. Let $s \in \{2, \ldots, n\}$. If $a \neq -s + 1$ then the projection of the vector $t_{s,-s+1}^{(2)} v_{s-2}$ onto the component $V(\mu^{(s)})$ in the decomposition (5.51) or (5.55), respectively, is nonzero. □

In particular, if $a = p - 1$ then the condition of Lemma 5.28 is satisfied for any $s \in \{2, \ldots, n\}$. This completes the proof of the theorem. □

We conclude by showing that each fundamental representation of the Lie algebra $\mathfrak{sp}_{2n}$ can be extended to the algebra $X(\mathfrak{sp}_{2n})$ providing a fundamental representation of the latter. Due to Theorem 3.1 it suffices to prove the corresponding statement for the Yangian $Y(\mathfrak{sp}_{2n})$. We follow the argument of [3] adopting it to the presentation of $Y(\mathfrak{sp}_{2n})$ provided by Corollary 3.2. For any indices $k, l \in \{-n, \ldots, n\}$ introduce the elements $J_{kl} \in Y(\mathfrak{sp}_{2n})$ by

$$ J_{kl} = \tau_{kl}^{(2)} - \frac{1}{2} \sum_{i=-n}^{n} \tau_{ki}^{(1)} \tau_{il}^{(1)}. $$

We shall identify the universal enveloping algebra $U(\mathfrak{sp}_{2n})$ with a subalgebra of $Y(\mathfrak{sp}_{2n})$ via the embedding (3.21). Denote by $\mathcal{J}$ the subspace of $Y(\mathfrak{sp}_{2n})$ spanned by all elements $J_{kl}$.

Lemma 5.29. The subspace $\mathcal{J}$ is stable under the adjoint action of the Lie algebra $\mathfrak{sp}_{2n}$. Moreover, the $\mathfrak{sp}_{2n}$-module $\mathcal{J}$ is isomorphic to the adjoint representation.

Proof. We easily derive from (5.3) that

$$ [F_{ij}, J_{kl}] = \delta_{kj} J_{il} - \delta_{il} J_{kj} - \delta_{k,-i} \theta_{ij} J_{-l,-j} + \delta_{l,-j} \theta_{ij} J_{k,-i}. $$

This proves the first claim. For the proof of the second, take the coefficient at $u^{-2}$ in the relation (3.6). This gives

$$ \tau_{kl}^{(2)} + \theta_{kl} \tau_{-l,-k}^{(2)} + \kappa \tau_{kl}^{(1)} - \sum_{i=-n}^{n} \tau_{ki}^{(1)} \tau_{il}^{(1)} = 0, \quad (5.60) $$

where we have used the relation $\tau_{kl}^{(1)} + \theta_{kl} \tau_{-l,-k}^{(1)} = 0$. Replacing $k$ and $l$ respectively by $-l$ and $-k$ in (5.60), then multiplying it by $\theta_{kl}$ and adding the result to (5.60) yields $J_{kl} + \theta_{kl} J_{-l,-k} = 0$. The argument is completed by observing that $\dim \mathcal{J} = \dim \mathfrak{sp}_{2n}$ by Corollary 3.7. □

The following lemma is straightforward from the defining relations of $Y(\mathfrak{sp}_{2n})$ given in Corollary 3.2.
Lemma 5.30. The algebra $Y(\mathfrak{sp}_{2n})$ is generated by the elements $F_{kl}$ and $J_{kl}$ with $k, l \in \{-n, \ldots, n\}$. □

The fundamental representations of $\mathfrak{sp}_{2n}$ are the modules $V(\mu^{(p)})$ where the highest weights have the form $\mu^{(p)} = (0, \ldots, 0, -1, \ldots, -1)$ with $p$ zeros, for the values $p = 0, 1, \ldots, n - 1$. In a more common notation, $V(\mu^{(p)})$ corresponds to the fundamental weight $\omega_{n-p}$. Denote by $W_p(a)$ the fundamental representation of $Y(\mathfrak{sp}_{2n})$ corresponding to the $n$-tuple of Drinfeld polynomials $(1, \ldots, u-a, 1, \ldots, 1)$ with $a \in \mathbb{C}$ and $u-a$ on the $(p+1)$-th position. By Theorem 5.16, the $Y(\mathfrak{sp}_{2n})$-module $W_p(a)$ is isomorphic to the restriction of the $X(\mathfrak{sp}_{2n})$-module $L(\lambda(u))$ to the subalgebra $Y(\mathfrak{sp}_{2n})$, where the components of $\lambda(u)$ are given by

$$
\lambda_i(u) = \begin{cases} 
\frac{u-a-p}{u-a-p-1} & \text{if } -n \leq i \leq -p-1, \\
1 & \text{if } -p \leq i \leq p, \\
\frac{u-a}{u-a+1} & \text{if } p+1 \leq i \leq n
\end{cases}
$$

for $p = 1, \ldots, n - 1$, and

$$
\lambda_i(u) = \begin{cases} 
\frac{u-a+1}{u-a} & \text{if } -n \leq i \leq -1, \\
\frac{u-a+1}{u-a+2} & \text{if } 1 \leq i \leq n
\end{cases}
$$

for $p = 0$. So, $W_p(a)$ may also be regarded as an $X(\mathfrak{sp}_{2n})$-module. Recall that the universal enveloping algebra $U(\mathfrak{sp}_{2n})$ is embedded into $X(\mathfrak{sp}_{2n})$ via (3.22).

The following is essentially a reformulation of a particular case of [6, Theorem 6.1].

Theorem 5.31. The restriction of $W_p(a)$ to $U(\mathfrak{sp}_{2n})$ is isomorphic to the fundamental module $V(\mu^{(p)})$. Moreover, the action of $Y(\mathfrak{sp}_{2n})$ on $V(\mu^{(p)})$ is determined by the assignment $J_{kl} \mapsto b F_{kl}$ with $b = -(n-p+1)/2 + a$.

Proof. By Theorem 5.1, the $X(\mathfrak{sp}_{2n})$-module $W_p(a)$ contains a unique, up to a constant factor, highest vector $\xi$. By the Poincaré–Birkhoff–Witt theorem for $X(\mathfrak{sp}_{2n})$ and the relations (5.3), $\xi$ is a unique weight vector of the weight $\mu^{(p)}$ in the $\mathfrak{sp}_{2n}$-module $W_p(a)$. Furthermore, the irreducible decomposition of this module takes the form

$$
W_p(a) = V(\mu^{(p)}) \oplus \bigoplus_{\nu} c(\nu) V(\nu),
$$

(5.61)
summed over the weights $\nu$ strictly preceding $\mu^{(p)}$ with respect to the standard partial ordering on the set of $\mathfrak{sp}_{2n}$-weights, where the $c(\nu)$ are some multiplicities. Consider the $\mathfrak{sp}_{2n}$-module homomorphism 

$$\psi : J \otimes V(\mu^{(p)}) \to W_p(a) \quad (5.62)$$

defined by 

$$\psi : J_{kl} \otimes v \mapsto J_{kl} v, \quad v \in V(\mu^{(p)}).$$

By Lemma 5.29, the $\mathfrak{sp}_{2n}$-module $J$ is isomorphic to $V(\rho)$ with $\rho = (0, \ldots, 0, -2)$. It is well known that the irreducible decomposition of $V(\rho) \otimes V(\mu^{(p)})$ contains $V(\mu^{(p)})$ with multiplicity one, and does not contain any modules $V(\nu)$ with $\nu$ strictly preceding $\mu^{(p)}$; see e.g. [13]. Therefore, the homomorphism $\psi$ must be multiplication by a scalar on the component $V(\mu^{(p)})$ and zero on the other irreducible constituents of $V(\rho) \otimes V(\mu^{(p)})$. Then by Lemma 5.30, the subspace $V(\mu^{(p)})$ of $W_p(a)$ is stable under the action of $Y(\mathfrak{sp}_{2n})$ and thus $W_p(a) = V(\mu^{(p)})$ since $W_p$ is an irreducible $Y(\mathfrak{sp}_{2n})$-module. This proves the first part of the theorem and shows that the action of the elements $J_{kl}$ on $V(\mu^{(p)})$ is given by $J_{kl} \mapsto b F_{kl}$ for some $b \in \mathbb{C}$. By Lemma 5.30, this determines the action of $Y(\mathfrak{sp}_{2n})$ on $V(\mu^{(p)})$. Finally, the exact value of $b$ is found by calculating the eigenvalue of the operator $J_{nn}$ on the highest vector $\xi$ of $L(\lambda(u)) \cong W_p(a)$. This eigenvalue remains unchanged if we multiply all components of $\lambda(u)$ by the formal series $f(u) \in 1 + \mathbb{C}[[u^{-1}]]u^{-1}$ defined from the relation 

$$f(u) f(u + \kappa) \lambda_{-n}(u + \kappa) \lambda_n(u) = 1.$$ 

In the case $1 \leq p \leq n - 1$ we obtain 

$$f(u) = 1 + (n - p) u^{-2} + \cdots.$$ 

By Proposition 5.2, we have $z(u) = 1$ in the $X(\mathfrak{sp}_{2n})$-module $L(f(u)\lambda(u))$ so that the eigenvalue of $\tau_{nn}(u)$ on the highest vector of $L(f(u)\lambda(u))$ is $f(u) \lambda_n(u)$. This allows one to find the eigenvalue of $\tau_{nn}^{(2)}$ which turns out to be $(n - p)/2 - a + 1$. Since the eigenvalue of $\tau_{nn}^{(1)} = F_{nn}$ on the highest vector is $-1$, the eigenvalue of $J_{nn}$ is $(n - p + 1)/2 - a$ proving the claim for the case under consideration. In the case $p = 0$ the value of $b$ is found by the same calculation. 

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