ON SECOND NOETHER’S THEOREM AND GAUGE SYMMETRIES IN MECHANICS

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Abstract. We review the geometric formulation of the second Noether’s theorem in time-dependent mechanics. The commutation relations between the dynamics on the final constraint manifold and the infinitesimal generator of a symmetry are studied. We show an algorithm for determining a gauge symmetry which is closely related to the process of stabilization of constraints, both in Lagrangian and Hamiltonian formalisms. The connections between both formalisms are established by means of the time-evolution operator.

1. Introduction and notation

The aim of this paper is to review, from a geometric point of view, the second Noether’s theorem in the framework of time-dependent mechanics as developed in [6], in order to analyse the search for the generating functions of infinitesimal gauge symmetries, using the tools of modern differential geometry. This theorem was stated in the context of field theories [22], but we feel convenient to analyse first the results of the theorem in the simpler case of time-dependent mechanics, before proceeding to the more general case. The second Noether’s theorem can be stated in the traditional formulation as follows:

Theorem 1. If the action integral

$$S = \int L(t, q^a, v^a) \, dt$$

is invariant under an infinitesimal transformation involving an arbitrary function and its derivatives up to order \(N\), then there exists an identity relation between the Lagrangian equations \(\delta L = 0\) and their derivatives. More precisely, if the action is invariant under the infinitesimal transformation

$$\delta q^a = \sum_{k=0}^{N} \varepsilon^{(k)} X_k^a(t, q^a, v^a), \quad \varepsilon(t) \text{ arbitrary function}, \quad \varepsilon^{(k)} = \frac{d^k \varepsilon}{dt^k},$$

then there is an identity, called Noether identity:

$$\sum_{k=0}^{N} (-1)^k \frac{d^k}{dt^k} (X_k^a \delta L) = 0.$$  

It is an easy exercise to prove the result of this theorem writing down the variation of the action under such an infinitesimal variation, performing an appropriate integration by parts, and using that the function \(\varepsilon\) is arbitrary. However, we want to understand it from a more intrinsic point of view.

The geometric setting of the Lagrangian formalism for field theories uses the theory of jet bundles [24]. One starts with a fibre bundle \(\pi: E \to R\) and for any pair of integer numbers \(k, l\) with \(k > l \geq 0\) defines the natural projections \(\pi_{k,l}: J^k \pi \to J^l \pi\), where \(J^0 \pi = E\) and denote \(\pi_k: J^k \pi \to R\) the composition \(\pi_k = \pi \circ \pi_{k,0}\). The coordinate in \(R\) will be denoted \((t)\), and we will refer to it as the time, and in \(E\) by \((t, q^a)\). Similarly, coordinates in \(J^k \pi\) will be denoted

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In the particular case of \( k = 1 \), \( J^1\pi \), usually called evolution space, its coordinates will be denoted in the more usual way \((t, q^\alpha, v^\alpha)\) and for \( k = 2 \), \( J^2\pi \), \((t, q^\alpha, v^\alpha, a^\alpha)\). When we choose a trivialization \( E \cong \mathbb{R} \times Q \) then the jet bundles also trivialize \( J^k\pi \cong \mathbb{R} \times T^kQ \), but, in general, there is no preferred trivialization.

The dynamics is defined as follows: Given a \( \pi_1 \)-semibasic 1-form \( L = L \, dt \), we define the Poincaré-Cartan forms, \( \Theta_L = dL \circ S + L \, dt \in \Lambda^1(J^1\pi) \), with \( S \) being the vertical endomorphism, and \( \Omega_L = -d\Theta_L \in \Lambda^2(J^2\pi) \). The dynamics is then described by the integral curves of a vector field \( \Gamma_L \) such that

\[
i_{\Gamma_L} \Omega_L = 0, \quad i_{\Gamma_L} dt = 1, \quad \text{and} \quad S(\Gamma_L) = 0,
\]

the last two conditions being the so-called SODE conditions. Alternatively, a SODE vector field \( \Gamma \) can be identified with a section \( \gamma : J^1\pi \to J^2\pi \). An integral curve of \( \Gamma \) is a local section \( \sigma : \mathbb{R} \to E \) of \( \pi \) such that \( j^2\sigma = \gamma \circ j^1\sigma \).

When the Lagrangian is singular \([18]\), the dynamical vector field \( \Gamma \) only exists on the primary constraint manifold \( S^1_L \). Tangency requirements on such vector may lead to new constraints and, consequently, to new constraint submanifolds, until \( \Gamma_L \) is eventually tangent to the so-called final constraint manifold \( S^f_L \). In such a way, solutions to the dynamics only exist at points of \( S^f_L \).

2. Jet fields and vector fields along maps

The necessity of considering a more general concept than that of vector field, but vector fields along a map, in order to deal in a proper way with the symmetry of the dynamics was proved in \([3]\). Similarly, the usual concept of Noether symmetry, an infinitesimal point transformation leaving the action invariant up to possible border terms, can also be extended in this more precise geometric language in terms of the Lagrangian density function \( L \), and not only jet prolongations of projectable vector fields will be considered, but the starting object will also be a vector field along the projection \( \pi_{1,0} : J^1\pi \to E \). We first establish the concept of prolongation of such an object and introduce vector fields along the natural projections \( \pi_{k+1,k} : J^{k+1}\pi \to J^k\pi \) built from \( X \in \mathfrak{X}(\pi_{1,0}) \). What follows is a summary of what can be found, written in a slightly different way, in \([11]\) and \([17]\).

**Definition 2.** If \( X \) is a \( \pi \)-projectable vector field along \( \pi_{1,0} \), we can define, for any integer number \( k \), \( X^{(k)} \in \mathfrak{X}(\pi_{k+1,k}) \) as follows:

\[
X_{j^{k+1}\phi}^{(k)}f = \left. \frac{d}{ds} \right|_{s=0} f \left( j^k_{\phi, s} (j^{k+1}_{\phi, t}) (\pi_{k,0} \circ \Psi_s \circ j^k_{\phi, t} \circ \varphi_s) \right), \quad f \in C^\infty(J^k\pi), \tag{3}
\]

where \( \Psi_s \) is the flow of any auxiliary vector field \( \dot{X} \in \mathfrak{X}(J^1\pi) \) such that \( \pi_{1,0} \circ \dot{X} = X \) and \( \varphi_s \) is the flow induced by \( X \) in the base manifold \( M \).

One can easily check that the vector field \( X^{(k)} \in \mathfrak{X}(\pi_{k+1,k}) \) along \( \pi_{k+1,k} \) is well defined and \( T\pi_{k,k-1} \circ X^{(k)} = X^{(k-1)} \circ \pi_{k+1,k} \). Then, as done in \([19]\), we consider the \( \pi_{k+1,k} \)-derivations \( i_X^{(k)} \) and \( d_X^{(k)} \) such that \( d_X^{(k)} = i_X^{(k)} \circ d - d \circ i_X^{(k)} \).

There is also a canonical jet field \( h^{(k)} : J^{k+1}\pi \to J^k\pi \) along \( \pi_{k+1,k} \) defined by

\[
h^{(k)}(j^{k+1}\phi) = j^k_{\phi, t} (j^{k+1}_{\phi, t}), \tag{4}
\]

and the associated derivation \( d_{h^{(k)}} : \Omega^*(J^k\pi) \to \Omega^*(J^{k+1}\pi) \) along \( \pi_{k+1,k} \) given by \( d_{h^{(k)}} = i_{h^{(k)}} \circ d - d \circ i_{h^{(k)}} \), corresponds to the classical operation of total time differential of a differential form. For instance, acting on a function \( f \in C^\infty(J^k\pi) \),

\[
d_{h^{(k)}} (f) = d_{\tau^{(k)}} (f) \, dt = \left[ \frac{\partial f}{\partial t} + \sum_{r=0}^{k} q^\alpha_{(r+1)} \frac{\partial f}{\partial q^\alpha_{(r)}} \right] \, dt. \tag{5}
\]
Other properties involving these derivations can be found in [1] and [17].

For $k = 1$, with the help of the map $h^{(1)}$, a SODE section $γ$ defines a holonomic jet field $γ: J^1 J^1 \pi \rightarrow J^1 J^1 π_1$ by composition $γ = h^{(1)} \circ γ$. Jet fields obtained in this way are called holonomic jet fields. In particular, the solutions to the Euler-Lagrange equations are the integral curves of the holonomic jet fields $γ_L: J^1 J^1 π \rightarrow J^1 J^1 π_1$ such that its horizontal projector $h_{γL}$ satisfies $i_{h_{γL}} Ω_L = 0.

3. Noether symmetries

Once the main geometrical ingredients have been introduced, we can express the infinitesimal variation of the Lagrangian density function $L$ with respect a vector field $X ∈ X(π_{1,0})$ in terms of the Poincaré–Cartan 1-form $Θ_L$ and the 2-form of Euler-Lagrange $δL = d_{h(1)} Θ_L + π^*_{2,1} dL \in Ω^2(J^2 π)$ (which locally takes the form $δL = δL_α dq^α ∧ dt$, where the $δL_α = \frac{∂L}{∂q^α} - d_{T(1)} (\frac{∂L}{∂q^α})$ give rise to the Euler-Lagrange equations), as indicated in the following proposition:

**Proposition 3.** Given a $π$-projectable $X ∈ X(π_{1,0})$, then

$$d_{X(1)} L = i_{X_{ev} π_{2,1}} δL^γ + d_{h(1)} (i_X Θ^γ_L),$$

where $X_{ev} = (X^α - v^α F) \partial/∂q^α$ is the evolutionary vector field of $X = F ∂/∂q^α + X^α \frac{δL}{δq^α}$ (see [1] for an intrinsic definition) and $δL^γ \in Ω^2(π_{2,0})$ and $Θ^γ_L \in Ω^1(π_{1,0})$ are the forms along $π_{2,0}$ and $π_{1,0}$ associated with the semibasic forms $δL$ and $Θ_L$ respectively (see [17]).

The concept of Noether symmetry can be introduced as follows:

**Definition 4 [17].** We say that a $π$-projectable $X ∈ X(π_{1,0})$ is a Noether symmetry if there exists a function $F ∈ C^∞(J^1 π)$ such that

$$d_{X(1)} L = d_{h(1)} F,$$

or in an equivalent way, if there is a function $G ∈ C^∞(J^1 π)$, such that

$$i_{X_{ev} π_{2,1}} δL^γ = d_{h(1)} G.$$

Observe that if $X ∈ X(π_{1,0})$ is a Noether symmetry, then $G = F - i_X Θ^γ_L$ is a conserved quantity. This is the content of the first Noether’s theorem. As a consequence of the proposition, we can restrict the study of symmetries $X ∈ X(π_{1,0})$ to those that are $π$-vertical, since we are only interested in the evolutionary part of $X$, which is $π$-vertical. We will indicate this fact by $X ∈ X^V(π_{1,0})$.

It is worth noticing that if $X ∈ X^V(π_{1,0})$ is an exact Noether symmetry, i.e. $i_{X_{ev} π_{2,1}} δL^γ = d_{h(1)} G$, then $G$ is $J^1 π_L$-projectable; in other words, $G = J^L π_H (G_H)$ for some $G_H ∈ C^∞(J^1 π^*)$ (see [11]).

When the Lagrangian is singular, we slightly modify this definition in order to take into account the relevant rôle played by the submanifold $S^L$. Two forms $ω$ and $η$ are said to be weakly equivalent, and we will write $ω ≈ η$, when their pull-backs coincide on the final constraint manifold. In the same way, they are said to be strongly equivalent if both the forms and their differentials are weakly equivalent. In this case, we will write $ω ≡ η$. So, given a $π$-vertical $X ∈ X(π_{1,0})$, it is said to be either (1) an exact Noether symmetry if $i_X δL^γ ≡ d_{h(1)} G$, (2) a weak Noether symmetry if $i_X δL^γ ≈ d_{h(1)} G$, or (3) a strong Noether symmetry when $i_X δL^γ ≡ d_{h(1)} G$. These distinctions will be useful later.
4. Hamiltonian formalism

In contrast with the autonomous case, in time-dependent Hamiltonian Mechanics the Hamiltonian is not a function but a section $h$ of a certain bundle. Given a bundle $\pi: E \to \mathbb{R}$ we consider the affine-dual bundle $\text{Aff}(J^1\pi, \mathbb{R})$, which is canonically isomorphic to $T^*E$, and also the vector bundle $p: J^1\pi^* \equiv \text{Ver}(\pi)^* \to E$ dual to the vertical bundle. We have an affine bundle fibration $\mu: T^*E \to J^1\pi^*$ and a Hamiltonian is a section $h$ of the projection $\mu$. The associated vector bundle is $p_1^*(T^*\mathbb{R}) \to J^1\pi^*$, where $p_1$ is the projection $p_1: J^1\pi^* \to \mathbb{R}$. Since we have a canonical 1-form $dt$ on $\mathbb{R}$, the bundle $p_1^*(T^*\mathbb{R})$ is canonically isomorphic to $J^1\pi^* \times \mathbb{R} \to J^1\pi^*$, the isomorphism being $(\beta, a dt) \to (\beta, a)$. Therefore, a section of the associated vector bundle will be considered as a function on $J^1\pi^*$.

Given a Hamiltonian section $h \in \text{Sec}(\mu)$, the pullback by $h$ of the canonical symplectic form $\Omega$ on $T^*E$ defines a 2-form $\Omega_h = h^*\Omega$ on $J^1\pi^*$. The associated Hamiltonian vector fields are the solutions $\Gamma_h$ to the equations

$$i_{\Gamma_h}\Omega_h = 0 \quad \text{and} \quad i_{\Gamma_h}dt = 1, \quad (9)$$

and the solution to the Hamilton equations are the integral curves of such vector fields.

The relation with the Lagrangian formalism is as follows (see [1] for details). From the Lagrangian $L$ we can define two maps, usually called the Legendre transformation $\mathcal{F}_L: J^1\pi \to J^1\pi^*$ and the extended Legendre transformations $\hat{\mathcal{F}}_L: J^1\pi \to T^*E$, related by $\mu \circ \hat{\mathcal{F}}_L = \mathcal{F}_L$. In local coordinates $(t, q^\alpha, u, p_\alpha)$ in $T^*E$ and $(t, q^\alpha, p_\alpha)$ in $J^1\pi^*$ adapted to the fibration $\mu$ we have that

$$\hat{\mathcal{F}}_L(t, q^\alpha, v^\alpha) = \left(t, q^\alpha, -E_L, \frac{\partial L}{\partial v^\alpha}\right) \quad \text{and} \quad \mathcal{F}_L(t, q^\alpha, v^\alpha) = \left(t, q^\alpha, \frac{\partial L}{\partial v^\alpha}\right).$$

In this expression $E_L$ is the energy function $E_L = v^\alpha \frac{\partial L}{\partial v^\alpha} - L$ defined by $L$.

When the Lagrangian is hyper-regular we have that $\mathcal{F}_L$ is invertible and a unique section $h$ of $\mu$ is determined by the equation $\hat{\mathcal{F}}_L = h \circ \mathcal{F}_L$. In the singular case, any section $h$ satisfying the above relation will be called a Hamiltonian for $L$, and therefore, if it exists, $h$ needs only to be defined on the primary constraint manifold $S^*_H = \mathcal{F}_L(J^1\pi)$.

Throughout this paper we will assume that the Lagrangian $L$ is such that a Hamiltonian section $h$ exists. Two Hamiltonian sections $h$ and $h'$ differ by a section of the associated vector bundle $p_1^*(T^*\mathbb{R})$, which can be identified with a function $\phi$ on $J^1\pi^*$. From $\hat{\mathcal{F}}_L = h \circ \mathcal{F}_L$ and $\mathcal{F}_L = h' \circ \mathcal{F}_L$ with $h' = h + \phi$, we get $\phi \circ \mathcal{F}_L = 0$. In other words, the difference $\phi$ between two Hamiltonian sections is a primary constraint. Thus if $\{\phi_{\alpha}\}$ is a complete set of functionally independent primary constraints then we can write a general Hamiltonian section for $L$ in the form

$$h = h_c + \lambda^I \phi_I \quad (10)$$

where $h_c$ is a particular one. As in the Lagrangian case, tangency requirements will lead to new constraints until we eventually find a solution $\Gamma_h$ tangent to the final constraint manifold $S^*_H$.

In the Hamiltonian formalism for autonomous mechanics, there are the concepts of first and second class functions (see for example [4]). In order to extend these concepts to the time-dependent case, we will introduce a Poisson bracket in $J^1\pi^*$. As we will see later, symmetries which are Hamiltonian vector fields of some functions will be well linked to symmetries in the Lagrangian formalism.

Let $\Omega$ be the canonical symplectic form on $T^*E$ and denote by $\{\cdot, \cdot\}_{T^*E}$ the associated Poisson bracket. Given two functions $f, g \in C^\infty(J^1\pi^*)$, it is easy to see that $\{\mu^*f, \mu^*g\}_{T^*E}$ is a $\mu$-projectable function, so that following [12], we can define
a unique Poisson structure \( \{ \cdot, \cdot \}_{J^1, \pi^*} \) on \( J^1\pi^* \) such that \( \mu \) is a Poisson map. In other words, the Poisson bracket on \( J^1\pi^* \) is defined by
\[
\mu^*\{f, g\}_{J^1, \pi^*} = \{\mu^*f, \mu^*g\}_{\pi^*E}.
\] (11)

In this way, we can associate to every function \( g \in C^\infty(J^1\pi^*) \) its Hamiltonian vector field
\[
Y_g = \{ \cdot, g \}_{J^1, \pi^*} = \frac{\partial g}{\partial p_\alpha} \partial q^\alpha - \frac{\partial g}{\partial q^\alpha} \partial p_\alpha.
\]

Given a section \( h \in \text{Sec}(\mu) \), for every \( \alpha \in T^*E \), we have that the linear forms \( \alpha \) and \( h(\mu(\alpha)) \) both project to \( \mu(\alpha) \) so that they differ in an element of the vector bundle \( \alpha - h(\mu(\alpha)) = h^*(\alpha) dt \). In this way, we have associated an affine function \( h^* \in C^\infty(T^*E) \) to the section \( h \in \text{Sec}(\mu) \). Moreover, the canonical Poisson bracket of two functions of this type is projectable to a function on \( J^1\pi^* \). The Hamiltonian vector field \( X_{h^*} = \{ \cdot, h^* \}_{J^1, \pi^*} \in \mathcal{X}(T^*E) \) associated to \( h^* \) is \( \mu \)-projectable and projects to the vector field \( X_h \in \mathcal{X}(J^1\pi^*) \), determined by
\[
\mu^*(\mathcal{L}_{X_h} f) = \{\mu^*f, h^*\}_{T^*E}.
\] (12)

It is easy to see that if \( h' = h + g \) then \( h'^* = h^* + \mu^*g \) and thus \( X_{h'} = X_h + Y_g \). Also notice that \( i_{X_h} \Omega_h = 0 \) on the primary constraint manifold, but in general \( X_h \) is not tangent to such manifold.

In local coordinates, the section \( h \) is of the form \( h(t,x^i,p_i) = (t,x^i,-H(t,x,p),p_i) \), the function \( h^* \) is \( h^*(t,x,u,p) = u + H(t,x,p) \) and the Hamiltonian vector field \( X_h \) is
\[
X_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_\alpha} \partial q^\alpha - \frac{\partial H}{\partial q^\alpha} \partial p_\alpha.
\] (13)

Therefore, the dynamic evolution of a function may be expressed in terms of a Poisson bracket, in the same way as in time-independent mechanics. In local coordinates
\[
\dot{f} = X_h f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial H}{\partial q^\alpha},
\] (14)

which is of the classical form \( \dot{f} = \frac{\partial f}{\partial t} + \{f, H\}_{J^1, \pi^*} \). Notice however that the vector field \( \frac{\partial}{\partial t} \) and the function \( H \) are only locally defined on \( J^1\pi^* \).

Remark 5. The above properties can also be alternatively understood as follows. We have a Lie algebroid structure over the affine bundle \( \mu_1 : J^1\mu \rightarrow J^1\pi^* \). (See [20] for the details.) The anchor is the affine map \( \rho : J^1\mu \rightarrow T^J^1\pi^* \) defined by \( \rho(j^1h) = X_h(\beta) \), for every section \( h \) of \( \mu \) and every \( \beta \in J^1\pi^* \) and the associated linear map \( \tilde{\rho} : T^*J^1\pi^* \rightarrow T^J^1\pi^* \) is determined by \( \rho(dg) = Y_g \), that is by the Poisson tensor on \( J^1\pi^* \). The action of a section \( j^1h \) of \( J^1\mu \) on a section \( \phi \) of the associated vector bundle is by Lie derivative with respect to the vector field \( X_h \).

Definition 6. A function \( g \in C^\infty(J^1\pi^*) \) is said to be first class function if, for every (final) constraint function \( \phi \), we have \( \{g, \phi\}_{J^1, \pi^*} \approx 0 \). A section \( \eta \in \text{Sec}(\mu) \) is said to be a first class section if \( \mathcal{L}_{X_\eta} \phi \approx 0 \) for every (final) constraint function \( \phi \). A section of \( \mu \) or a function of \( J^1\pi^* \) which is not of first class is said to be second class.

Therefore, we can take a complete set of independent primary constraints \( \{\phi_I\} \) and partition it \( \{\phi_I\} = \{\phi_\alpha, \phi_a\} \) into first class constraints \( \{\phi_\alpha\} \) and second class constraints \( \{\phi_a\} \). When we apply the constraint algorithm, demanding that all the constraints have to be preserved by \( Y_h \), we can determine in Eq. (10) the multipliers \( \lambda^\alpha \) which correspond to second class constraints. On the contrary, those multipliers \( \lambda^a \) corresponding to primary first class constraints will remain indeterminate, being a sign of the non-uniqueness of the solution to the dynamics.
Once we have fixed the multipliers $\lambda^\alpha$ corresponding to second class primary constraints and we have got a Hamiltonian section $h_0 = h_c + \lambda^\alpha \phi_\alpha$, we can only add first class primary constraints. Therefore any Hamiltonian for $L$ is of the form $h = h_0 + \lambda^\alpha \phi_\alpha$ where $\{\phi_\alpha\}$ is a complete set of functionally independent first class primary constraints.

**Definition 7.** Given a section $h \in \text{Sec}(\mu)$ such that $X_h$ is tangent to the primary constraint manifold $S^3_H$, a function $g \in C^\infty(J^1\pi^*)$ is said to generate a Hamiltonian (Noether) symmetry if $\{\mu^* g, h^*\}_{T^*E} \equiv 0$, where $S^2_H = \mu^{-1}(S^3_H)$.

The reason why we introduce the concept of a Hamiltonian symmetry by means of the vanishing of $\{\mu^* g, h^*\}_{T^*E}$ on $S^2_H$ will be clear in the following sections. The idea is that this kind of symmetries are in correspondence with Noether symmetries in the Lagrangian framework.

5. **The Time Evolution Operator**

In addition to the Legendre transformation, there is a second geometric object, the time evolution operator, that connects the Lagrangian and Hamiltonian formalisms.

**Theorem 8.** Given a Lagrangian $L \in C^\infty(J^1\pi)$, there exists a unique vector field $\hat{K}$ along $\mathcal{F}_L$, called the extended time evolution operator, such that

1. $T\pi_E \circ \hat{K} = i_1$, where $i_1 : J^1\pi \to TE$ is the map $i_1(j^1) = \sigma(t)$, and
2. $\iota_K \Omega = 0$, where $\Omega$ is the canonical symplectic form on $T^*E$.

In local coordinates,

$$\hat{K} = \left( \frac{\partial}{\partial t} \circ \hat{T}_L \right) + v^j \left( \frac{\partial}{\partial q^\alpha} \circ \hat{T}_L \right) + \frac{\partial L}{\partial u} \left( \frac{\partial}{\partial u} \circ \hat{T}_L \right) + \frac{\partial L}{\partial p_\alpha} \left( \frac{\partial}{\partial p_\alpha} \circ \hat{T}_L \right). \quad (15)$$

By composition with the differential of the projection $\mu$ we get a vector field $K = T\mu \circ \hat{K}$ along $\mathcal{F}_L$, the (restricted) time evolution operator, whose coordinate expression is

$$K = \left( \frac{\partial}{\partial t} \circ \mathcal{F}_L \right) + v^j \left( \frac{\partial}{\partial q^\alpha} \circ \mathcal{F}_L \right) + \frac{\partial L}{\partial u} \left( \frac{\partial}{\partial u} \circ \mathcal{F}_L \right) + \frac{\partial L}{\partial p_\alpha} \left( \frac{\partial}{\partial p_\alpha} \circ \mathcal{F}_L \right). \quad (16)$$

**Remark 9.** [Alternative construction of $\hat{K}$] Let $L \in C^\infty(J^1\pi)$ be a Lagrangian and consider the associated homogeneous Lagrangian $\hat{L} \in C^\infty(TE)$ given in coordinates by $\hat{L}(x^0, x^i, w^0, w^i) = w^0 L(x^0, x^i, w^i/w^0)$. We consider the autonomous time evolution operator $K_L$ defined by $\hat{L}$, that is $K_L = \chi \circ d\hat{L}$ where $\chi$ is the canonical isomorphism $\chi : T^*(TE) \to T(T^*E)$. By composition with the canonical inclusion $i_1 : J^1\pi \to TE$ we get the vector field $\hat{K} = K_L \circ \mathcal{F}_L$. The above properties are easy to prove from this definition.

The integral curves of $K$ are the solutions to the Euler-Lagrange equations for $L$. This is clear from the following relation between the time evolution operator and the total time derivative.

**Proposition 10.** Let $G \in C^\infty(J^1\pi^*)$ and denote by $X \in \mathcal{X}^V(\pi_{1,0})$ the vector field given by $X(v) = Tp \left( Y_G(\mathcal{F}_L(v)) \right)$, for $v \in J^1\pi$. Then

$$d_{h_{\mu^*}}(\mathcal{F}_L^* G) = \pi^*_{2,1}(K \cdot G) \pi^*_{2}(dt) - i_X \delta L^\vee. \quad (17)$$

Notice that the action of $X$ on functions $f \in C^\infty(E)$ is given by

$$X(f) = \mathcal{F}_L^* \{ p^* f, G \}_{J^1\pi}, \quad (18)$$

and locally $X = \mathcal{F}_L^* \left( \frac{\partial G}{\partial p_\alpha} \right) \frac{\partial}{\partial q^\alpha}$. 

Proposition 11. For any admissible dynamic vector field $\Gamma_L$ defined on the primary constraint manifold $S_1^L$, we have

$$\Gamma_L(T^*_L G) \approx K \cdot G,$$

for every function $G \in \mathcal{C}^\infty(J^1 \pi^*)$.

In particular, since the pullback by $\mathcal{T}_L$ of a Hamiltonian constraint is a Lagrangian constraint we get:

Corollary 12. If $\phi$ is a Hamiltonian constraint then $K \cdot \phi$ is a Lagrangian constraint.

In the light of the above proposition and taking into account our definition of the different types of Noether symmetry, we will say that a function $G \in \mathcal{C}^\infty(J^1 \pi^*)$ generates (1) an exact Noether symmetry if $K \cdot G = 0$, (2) a weak Noether symmetry if $K \cdot G \approx 0$, and (3) a strong Noether symmetry if $K \cdot G \equiv 0$.

Finally, we state another proposition that relates the action of $K$ on a function $G \in \mathcal{C}^\infty(J^1 \pi^*)$ to its dynamic evolution with respect to the part of the Hamiltonian $h_0$ determined at the first stage, and to some other terms involving primary constraints, on one hand those being first class and, on the other, those being second class. The steps taken in this direction, in the time-independent case, have served as a guideline.

Proposition 13. Let $\{\phi_I\} = \{\phi_\alpha, \phi_\beta\}$ be a complete set of independent primary constraints with $\{\phi_\alpha\}$ first class and $\{\phi_\beta\}$ second class (at primary level). Then, for every function $G \in \mathcal{C}^\infty(J^1 \pi^*)$,

$$K \cdot G = \bar{T}_L^* \{\mu^G, h_0^\alpha\}_{T^*E} + \chi_\alpha M^{\alpha\beta} \mathcal{T}_L^* \{G, \phi_\beta\}_{J^1 \pi^*} + \nu^\rho \mathcal{T}_L^* \{G, \phi_\alpha\}_{J^1 \pi^*},$$

where $M^{\alpha\beta}$ is the inverse matrix of $\mathcal{T}_L^* \{\phi_\alpha, \phi_\beta\}_{J^1 \pi^*}$, $\chi_\alpha = K \cdot \phi_\alpha$ are primary Lagrangian constraints and $\nu^\rho \in \mathcal{C}^\infty(J^1 \pi)$ are non-projectable.

This proposition will be used later to establish the connection between gauge symmetries in the Hamiltonian formalism and the Lagrangian one. However, it has an immediate but important corollary, extending the results of [23].

Corollary 14. At primary level, a Hamiltonian constraint $\phi$ is a first class constraint if and only if $K \cdot \phi$ is a projectable Lagrangian constraint.

Remark 15. We can recast proposition and some steps in its proof, in the following way. Given a Hamiltonian section $h$ for $L$ and a complete set of primary constraints $\{\phi_I\}$ there exist functions $\nu^I$ such that

$$K = (X_h \circ \mathcal{T}_L) + \nu^I(Y_{\Phi_I} \circ \mathcal{T}_L).$$

If $\Gamma_I \in \mathfrak{X}(J^1 \pi)$ are the vector fields $\Gamma_I = (T p \circ Y_{\Phi_I} \circ \mathcal{T}_L)^\nu$ (which generate the kernel of $T \mathcal{T}_L$), then the functions $\nu^I$ satisfy $\mathcal{L}_{\Gamma^I} \nu^J = \delta^I_J$, and hence they are non $\mathcal{T}_L$-projectable.

If we partition $\{\Phi_I\} = \{\Phi_0, \Phi_\alpha\}$ into first class constraints $\{\phi_\alpha\}$ and second class constraints $\{\phi_\alpha\}$, then tangency considerations imply that the multipliers $\nu^\rho$ corresponding to the second class constraint (at primary level) are determined by $\nu^\rho = M^{\rho\beta}[K \cdot \phi_\beta - \mathcal{T}_L^*(X_h \phi_\beta)]$. Moreover, by modifying the Hamiltonian section $h \rightarrow h - \lambda^\alpha \Phi_\alpha$ with $\lambda^\rho = M^{\rho\beta} \mathcal{T}_L^*(X_h \phi_\beta)$ then we have the simpler form $\nu^\rho = M^{\rho\beta}(K \cdot \phi_\beta)$. 

In this section we are going to investigate the Lie brackets of the dynamical vector fields (in both the Lagrangian and the Hamiltonian formalisms) and the vector field that generates the symmetry. We will assume that we already performed the constraint algorithm and we have arrived to a consistent solution.

In both, the Lagrangian and the Hamiltonian formalisms, this algorithm determines a final constraint submanifold and the set of solutions is an affine family of vector fields, that is, are sections of an affine subbundle of the tangent bundle. It follows that the concept of symmetry that we have to use is that of a symmetry of an affine subbundle.

In all generality, we consider a manifold $N$ and an affine subbundle $A \subset T N$ modeled on the vector bundle $\mathcal{V} \subset T N$. By an infinitesimal symmetry of $A$ we mean a vector field $Y \in \mathfrak{X}(N)$ such that $[Y, \Gamma] \in \text{Sec}(\mathcal{V})$ for every section $\Gamma \in \text{Sec}(A)$. If we fix a section $\Gamma_0 \in \text{Sec}(A)$ and a basis $\{\Gamma_\mu\}$ of sections of the underlying vector subbundle $\mathcal{V}$, then a section of $A$ can be written in the form $\Gamma = \Gamma_0 + \alpha^\mu \Gamma_\mu$. A vector field $X$ is a symmetry of $A$ if and only if for every set of functions $\alpha^\mu$ there exist functions $\beta^\mu$ such that $[X, \Gamma_0 + \alpha^\mu \Gamma_\mu] = \beta^\mu \Gamma_\mu$.

In the cases we are interested in, the manifold $N$ is a submanifold of a given manifold $M$. In such cases, instead of vector fields on $N$ we take vector fields on $M$ which are tangent to $N$. With this in mind, by an infinitesimal symmetry of $A \subset T N \subset T M$ we mean a vector field $X \in \mathfrak{X}(M)$ tangent to $N$ and such that $[Y, \Gamma]|_N \in \text{Sec}(\mathcal{V})$ for every vector field $\Gamma \in \mathfrak{X}(M)$ such that $\Gamma|_N \in \text{Sec}(A)$.

More concretely, in the cases we have in mind, the manifold $N$ is the final constraint manifold and the subbundle $A$ is the set of all vectors tangent to curves solution to the dynamics on such final constraint manifold. In the Lagrangian formalism, the vector fields solution to the dynamics on the final constraint manifold are of the form $\Gamma = \Gamma_0 + \alpha^\mu \Gamma_\mu$, where $\Gamma_0$ is a given SODE and $\Gamma_\mu$ are vertical vector fields on $J^1 \pi$. A dynamical symmetry is a vector field $X \in \mathfrak{X}(J^1 \pi)$ tangent to the final constraint manifold $S_L^J$ such that $[X, \Gamma] \approx \beta^\mu \Gamma_\mu$ for some functions $\beta^\mu \in \mathcal{C}^\infty(J^1 \pi)$. On the Hamiltonian counterpart, the vector fields solution to the dynamics are of the form $\Gamma = X_{h_0} + \alpha^\mu X_{\phi_\mu}$, where $h_0$ is a given solution and $\{\phi_\mu\}$ is a complete set of primary first class constraints (at the final level). A symmetry of the dynamics is a vector field $X \in \mathfrak{X}(J^1 \pi^*)$ such that $[X, \Gamma] \approx \beta^\mu X_{\phi_\mu}$ for some functions $\beta^\mu \in \mathcal{C}^\infty(J^1 \pi^*)$.

From now on, by a first class (respectively, second class) function we mean a function which is first class (respectively, second class) with respect to the final set of constraints, i.e. with respect to the final constraint manifold manifold. Also, in what follows in this paper we will denote by $\tilde{S}_H^k \subset T^* E$ the submanifold $\tilde{S}_H^k = \mu^{-1}(S_H^k)$.

We consider first the Lagrangian formalism. Since the symmetry $X \in \mathfrak{X}(\pi_{1,0})$ is a vector field along $\pi_{1,0}$ and any solution $\Gamma_L \in \mathfrak{X}(J^1 \pi)$, is a vector field on $J^1 \pi$, we need to prolong $X$ to a vector field defined on $J^1 \pi$, as well. As we previously said, a SODE can be identified with a section $\gamma: J^1 \pi \to J^2 \pi$ of $\pi_{2,1}$. If $L$ is singular, we may have several sections $\gamma$ solution to the dynamical equation defined on the primary constraint submanifold $S_L^J$.

Given a vector field $U \in \mathfrak{X}(p)$ along the projection $p$, we consider the vector field $X = U \circ \Psi_L \in \mathfrak{X}(\pi_{1,0})$ along the projection $\pi_{1,0}$. For any SODE section $\gamma$ we may consider the vector field $X^{(1)} \circ \gamma \in \mathfrak{X}(J^1 \pi)$. In particular, for solutions $\gamma$ of the dynamics, from proposition we have:
Proposition 16. The vector field $X^{(1)} \circ \gamma$ does not depend on the particular choice of the section $\gamma$ solution to the Lagrangian dynamical equations on $S^1_\mu$. In coordinates, if $U = U^\alpha(t, q^\alpha, p_\alpha) \frac{\partial}{\partial q^\alpha}$ then

$$X^{(1)} \circ \gamma = \mathcal{F}^*_L(U^\alpha) \frac{\partial}{\partial q^\alpha} + (K \cdot U^\alpha) \frac{\partial}{\partial p_\alpha}.$$ 

In particular, given a function $G \in C^\infty(J^1 \pi^*)$ generating a (exact, strong or weak) Noether symmetry, we consider $\gamma$. Noether symmetry, we consider $U_G = Tp \circ Y_G$, where $Y_G$ is the Hamiltonian vector field defined by $G$, and we apply the above procedure, arriving to a vector field $Z_G = (Tp \circ Y_G \circ \mathcal{F}_L)^{(1)} \circ \gamma$. In local coordinates, we find

$$Z_G = \mathcal{F}^*_L(\{q^\alpha, G\}_{J^1 \pi^*}) \frac{\partial}{\partial q^\alpha} + (K \cdot \{q^\alpha, G\}_{J^1 \pi^*}) \frac{\partial}{\partial p_\alpha}.$$ (22)

In [16] the construction of this vector field was carried out in a different but equivalent way. Now, we can establish

Theorem 17. Consider the following different conditions for a function $G$ on $J^1 \pi^*$:

(i) $K \cdot G \equiv 0$,  
(ii) $K \cdot G \approx 0$,  
(iii) $K \cdot G = 0$.

Then,

(1) The vector field $X \in \mathfrak{X}^V(\pi_{1,0})$ is a Noether symmetry with conserved quantity $G_L$ if and only if the function $G$ such that $\mathcal{F}^*_L G = G_L$ satisfies condition (ii).

(2) Condition (ii) holds if and only if $G$ generates a Hamiltonian (Noether) symmetry.

(3) If condition (iii) holds then $\mathcal{F}^*_L G = Y_G$.

(4) If condition (ii) holds then $\mathcal{F}^*_L G \approx Y_G$.

(5) If $G$ is first class and condition (i) holds then $Z_G$ is a dynamic symmetry, i.e. $[Z_G, \Gamma] \approx \beta^\mu \Gamma_\mu$, for every final solution $\Gamma$.

Sketch of the proof.

(1) $(\Leftarrow)$ The primary Lagrangian constraints may be written as $\chi L dt = i_{Z_G} \delta L$, so that there exist $r_l \in C^\infty(J^1 \pi)$ such that $K \cdot G = r_l \chi L$. It follows from Eq. (17) that $X = Z_G - r_l Z_{\phi_l} \in \mathfrak{X}^V(\pi_{1,0})$ is a Noether symmetry.

$\Rightarrow$ From equations (8) and (17) we deduce that if $X \in \mathfrak{X}^V(\pi_{1,0})$ is a Noether symmetry then $\pi_{1,0}^{-1}(K \cdot G) dt = i_{Z_G - r_l} \chi L \delta L'$. Since the left hand side of this relation does not depend on accelerations $i_{X - Z_G} \delta L$ is a primary Lagrangian constraint.

(2) $(\Rightarrow)$ See the characterization given by the second item of the theorem below. The result follows by taking into account Eq. (20) and that the pullback by $\mathcal{F}_L$ of a secondary Hamiltonian constraint is a primary Lagrangian constraint.

$\Rightarrow$ Consider a complete set of independent primary first class constraint $\{\phi_a\}$ and the corresponding vector fields $\Gamma_a \in \ker(T \mathcal{F}_L)$. Writing condition (ii) in the form $K \cdot G = r_l \chi L$ and applying $\Gamma_a$ to it we obtain $\mathcal{F}^*_L \{G, \phi_a\} = (\Gamma_a r_l) \chi L$, where we have taken into account Eq. (21). Since the left hand side is $\mathcal{F}_L$-projectable so does the right hand side. But $\chi L$ are primary Lagrangian constraints, so that only $\mathcal{F}_L$-projectable primary constraints can appear. In other words, only the pull-back by $\mathcal{F}_L$ of secondary Hamiltonian constraints can appear (see [11]), and thus $\{G, \phi_a\} \approx 0$. The result follows by taking into account Eq. (20), i.e. $\mathcal{F}^*_L \{\mu^* G, h_0\} \approx 0$, which implies $\{\mu^* G, h_0\} \approx 0$, and item (2) of theorem [18].
(4) We just have to show that $T\mathcal{F}_L \circ Z_G$ and $Y_G \circ \mathcal{F}_L$ agree on $S^I_H$ when acting on the variables $p_i$. For every $v \in S^I_L \subset J^1 \pi$,
\[
T\mathcal{F}_L(Z_G(v)) p_i = Z_G(v) \left( \frac{\partial L}{\partial \dot{v}} \right) = \left( \frac{\partial}{\partial \dot{v}} (K \cdot G) - \mathcal{F}_L^* \left( \frac{\partial G}{\partial \dot{q}} \right) \right)(v)
\approx_{S^I_L} (\mathcal{F}_L^* \{ p_i, G \})(v) = \{ p_i, G \}(\mathcal{F}_L(v)) = Y_G(\mathcal{F}_L(v)) p_i.
\]

(3) The proof is similar to that of (4) and will be omitted.

(5) The proof is based on the relation $Z_G(K \cdot F) \equiv K \cdot \{ F, G \} + Z_F(K \cdot G)$, for $G, F \in \mathcal{C}^\infty(J^1 \pi^*)$. If $K \cdot G \equiv 0$, $Z_G(K \cdot F) \approx K \cdot (Y_G F)$, and if $G$ is first class, $Z_G$ is tangent to $S^I_L$. If $\{ \psi_I \}$ is a complete set of Hamiltonian constraints then $\{ K \cdot \psi_I \}$ is a complete set of Lagrangian constraints. Then
\[
Z_G(K \cdot \psi_I) \approx K \cdot (Y_G(\psi_I)) \approx K \cdot (\text{Ham. constraints}) \approx 0.
\]

On the other hand, if $F \in \mathcal{C}^\infty(J^1 \pi^*)$, using proposition 11,
\[
Z_G(\Gamma_L(\mathcal{F}_L^* F)) \approx Z_G(K \cdot F) \approx K \cdot (Y_G F) \approx \Gamma_L(Z_G(\mathcal{F}_L^* F)),
\]
so the Lie bracket $[Z_G, \Gamma_L]$ is zero acting on $\mathcal{F}_L$-projectable functions on the final constraint manifold. Thus $[Z_G, \Gamma_L]$ is tangent to $S^I_H$ and it is in ker$(T\mathcal{F}_L)$, which concludes the proof. □

Sometimes it is useful to treat separately that part of the Hamiltonian section $h$ that is determined $h_0$ and the primary first class term.

**Theorem 18.** Let $G \in \mathcal{C}^\infty(J^1 \pi^*)$ be a first class function and let pfcc means a primary constraint that is first class with respect to $S^H_H$, and pfcc$^1$ another one that is first class with respect to $S^1_H$. Then,

1. if for every Hamiltonian section $h$ we have that $\{ \mu^*G, h^* \}_T \cong$ pfcc or, equivalently, $\{ \mu^*G, h^* \}_T \cong$ pfcc and $\{ \mu^*G, \text{pfcc} \}_T \cong$ pfcc, with $h_0$ such that $X_{h_0}$ is tangent to $S^I_H$, then $Y_G$ is a dynamical symmetry.
2. $\{ \mu^*G, h^* \}_T \cong 0$ and $\{ \mu^*G, \text{pfcc}^1 \}_T \cong 0$, with $h_0$ such that $X_{h_0}$ is tangent to $S^1_H$ if and only if $G$ generates a Hamiltonian (Noether) symmetry.

*Proof.*

1. Since the difference between any section $h$ defined on $S^I_H$ and $h_0$ (with all the multipliers associated to primary second class constraints being determined) is a primary first class constraint (with respect to $S^I_H$), the two conditions in the statement are equivalent to $\{ \mu^*G, h^* \}_T \cong$ pfcc. On the other hand $\{ Y_G, \Gamma_h \}_\Omega = d(\{ \mu^*G, h^* \}_T) \approx d(\text{pfcc})$ and $\Omega$ is symplectic, from where we deduce that $[Y_G, \Gamma_h] \cong \beta^a Y_{\phi_a}$, where $\{ \phi_a \}$ is a set of primary first class constraints. In other words, $Y_G$ is a dynamic symmetry. Finally, using Eq. 20, we get $K \cdot G \equiv 0$.

2. On the primary constraint manifold the difference between $h$ and $h_0$ (with all the multipliers associated to primary second class constraints with respect to $S^H_H$ determined) is a primary first class constraint (with respect to $S^1_H$). Thus, $G$
generates a Hamiltonian (Noether) symmetry, \( \{ \mu^*G, h^* \}_{T^*E} \approx 0 \), if and only if
\[
\{ \mu^*G, h_0^* \}_{T^*E} \approx 0 \quad \text{and} \quad \{ \mu^*G, \text{pfcc}^1 \}_{T^*E} \approx 0.
\]

\[\Box\]

7. Gauge symmetries

7.1. Lagrangian case. In this section we will analyse the existence of gauge symmetries, that is, of Noether symmetries \( X \in \mathcal{X}^V(\pi_{1,0}) \) depending on an arbitrary function \( \varepsilon(t) \) and its derivatives up to some order \( N \), i.e. \( X = \sum_{k=0}^{N} \varepsilon^{(k)} X_k \), where \( X_k \) are vertical vector fields along \( \pi_{1,0} \) and we have written \( \varepsilon^{(k)} \) for the \( k \)-th derivative of the arbitrary function \( \varepsilon(t) \). The associated conserved quantity \( G_L \) also depends on \( \varepsilon \) and its derivatives \( G_L = \sum_{k=0}^{N} \varepsilon^{(k)} G_{k}^L \), with \( G_{k}^L \in C^\infty(J^1\pi) \).

Let us suppose first that \( X \) is an exact Noether symmetry. Following [6], from \( i_X \delta L^\vee = d_{h_0}^G G \) and taking into account that \( \varepsilon \) is arbitrary, we deduce that
\[
\begin{align*}
G_{k}^L &= 0 \\
d_{h_0}^G G_{k}^L + \pi_{2,1}^* (G_{k-1}^L) dt - i_{X_k} \delta L^\vee = 0 \\
d_{h_0}^G G_0^L - i_{X_k} \delta L^\vee = 0.
\end{align*}
\]
for \( k = 1, \ldots, N \). Cariñena et al. [6] use the above relations relations (24) to draw an algorithm for determining such symmetries. The idea is as follows. From \( G_{N}^L = 0 \) and the second equation for \( k = N \), we get that \( i_{X_N} \delta L^\vee = \pi_{2,1}^* G_{N-1}^L \) is the pullback of a function in \( J^1\pi \). Thus, we must choose \( X_N \) such that its vertical lift is in the kernel of \( T\mathcal{F}_L \) (and hence in the \( \pi_{1,0} \)-vertical part of the \( \ker \Omega_L \)) in order to annihilate the terms of \( i_{X_N} \delta L^\vee \) which depends on the accelerations, and it follows that \( G_{N-1}^L \) is a primary first class constraint. In this way we recover the well known fact that only singular Lagrangians may have gauge symmetries. The algorithm proceeds by choosing at every step a vector field \( X_k \in \mathcal{X}^V(\pi_{1,0}) \) such that \( d_{h_0}^G G_{k}^L - i_{X_k} \delta L \) is a \( \pi_{2,1} \)-projectable function, that we take as \( G_{k-1}^L \), and it finishes when we can choose \( X_0 \) such that the difference \( d_{h_0}^G G_{k}^L - i_{X_k} \delta L \) vanishes, that is \( X_0 \) is a Noether symmetry of \( L \).

But it may happen that, at some stage, there is no vector field \( X_k \) such that the difference \( d_{h_0}^G G_{k}^L - i_{X_k} \delta L^\vee \) is a \( \pi_{2,1} \)-projectable function. Then the algorithm fails to determine \( X \) and we must restart it with a different choice of the initial vector field \( X_N \) until we finish off all the primary constraints (in a preselected complete set of independent constraints). Thus, this algorithm has a clear problem: we do not know when it might finish successfully and, we do not know a priori which Lagrangian primary constraints serve as input for our algorithm.

We can reinterpret Eq. (25) from the dynamic point of view. It reproduces the process of stabilization of a primary \( \mathcal{F}_L \)-projectable constraint, \( G_{N-1}^L \). To see this more clearly, we remark that the conserved quantity \( G_L \) for a Noether symmetry is always \( \mathcal{F}_L \)-projectable, \( G_L = \mathcal{F}_L^* (G) \). If we compose the expression \( d_{h_0}^G G_{k}^L + \pi_{2,1}^* (G_{k-1}^L) dt - i_{X_k} \delta L^\vee = 0 \) with any admissible dynamic section \( \gamma : S^1_\gamma \subset J^1\pi \rightarrow J^2\pi \), it is reexpressed as \( \Gamma_L (\mathcal{F}_L^* (G_k)) = \mathcal{F}_L^* (G_{k-1}) \), so we take as \( G_{k-1}^L \) the dynamic evolution of the previous constraint \( G_{k}^L \). If at some stage, \( G_{k-1}^L \) becomes non \( \mathcal{F}_L \)-projectable, the algorithm stops. If we assume that constraints obtained from the dynamic evolution of previous ones serve us to characterize the next constraint manifold in the constraint algorithm, that is to say, if no ineffective constraints appear, a constraint that never turns into non \( \mathcal{F}_L \)-projectable is related to the indeterminacy of the dynamics. If \( \Gamma_L = \Gamma_0 + \alpha^* \Gamma_\mu \) is the dynamic vector field (as we denoted throughout this paper), since \( \Gamma_\mu \in \ker T\mathcal{F}_L \), if \( \chi_{V_0} = \mathcal{F}_L^* (\phi_0) \) is a projectable constraint, imposing its dynamic evolution to be zero leads to
Thus, until at some stage \( G \) the terms involving constraints (at secondary level) have been removed, and so on. Eventually, one can easily verify that their symmetries satisfy the formalism of having a gauge symmetry. Regarding Theorem (17) item (1) and (2) and the second item of Theorem (18), considering a Hamiltonian symmetry of the type \( G = \sum_{k=0}^{N} \varepsilon(k)G_k \in \mathcal{C}^\infty(J^1\pi^*) \), where \( \varepsilon(k) \) depends only on time, implies

\[
\mu^*G_N \approx 0 \quad \text{on} \quad S_H^2 \\
\mu^*G_{k-1} + \{\mu^*G_k, h_0^*\}_{T^*E} \approx 0 \quad \text{on} \quad S_H^2 \\
\{\mu^*G_0, h_0^*\}_{T^*E} \approx 0
\]

for \( k = 1, \ldots, N \). These relations suggest us a way to find the gauge Hamiltonian (Noether) symmetries of the Hamiltonian system by means of the following algorithm. As input, we choose for \( G_N \) a constraint (with respect to \( S_H^2 \)). Then, we calculate its dynamic evolution \( \{\mu^*G_N, h_0^*\}_{T^*E} \) which we take as \( G_{N-1} \) once all the terms involving constraints (at secondary level) have been removed, and so on. Thus, until at some stage \( G_k \) is nothing but a function vanishing on \( S_H^2 \). If at some stage \( G_k \) Eq. (28) is not satisfied, the algorithm can not go on and we have to try with a different \( G_N \).

**Proposition 19.** The operator \( \hat{K} \) transforms Eqs. (27) and (28) into Eq. (26).

A different starting point to study gauge symmetries in the Hamiltonian formalism is adopted by Gomis et. al. in [13] (in the context of time-independent mechanics). They look for a dynamic gauge symmetries and, therefore, for a \( G = \sum_{k=0}^{N} \varepsilon(k)G_k \in \mathcal{C}^\infty(J^1\pi^*) \) satisfying the first item of Theorem (18). This leads them to equations equivalent to Eqs. (27) and (28) where the weak equalities \( \approx \) on \( S_H^2 \) must be replaced by \( \equiv \) \text{pfcc}. Essentially, under some regularity assumptions, i.e., that the rank of the Legendre map and the matrix of Poisson brackets of constraints is constant, and that no ineffective constraints appear, they show that is possible to build dynamic gauge symmetries, where the functions \( G_k \) are first class constraints plus a quadratic term in the rest of constraints. One can attempt to rewrite their reasonings just replacing the usual Poisson bracket of time-independent mechanics by \( \{,\}_{T^*E} \), and everything they hold carries on being valid. Nevertheless, we emphasize that the symmetries hence obtained are dynamic and not Noether. Actually, one can easily verify that their symmetries satisfy \( K \cdot G \equiv 0 \), so they are...
not, in principle, Noether (item (2) Theorem 17). Thus, as far as we know, the problem of the existence of gauge Noether symmetries is still unsolved.

8. Conclusions

We have reviewed the concept of Noether symmetry and we have clarified the relationship between Hamiltonian constraints and Lagrangian constraints in time-dependent mechanics, extending the results given in [24] for the autonomous case. In particular, by making use of the properties of the time evolution operator $K$, we have shown that Hamiltonian first class constraints with respect to $S^2_H$ are transformed into $F_L$-projectable Lagrangian constraints, and second class constraints into non-projectable ones.

We also studied Noether symmetries that come from a function defined on the dual affine bundle, imposing conditions on them to guarantee, in some sense, in which cases the symmetry commutes with the dynamic vector field. Essentially, it was done in [16], but here we use, in the Lagrangian case, the geometric content of vector fields along $\pi_{k+1,k}$ and, in the Hamiltonian case, the new tools of Hamiltonian dynamics in the cotangent bundle $T^*E$.

Finally, we have studied Noether symmetries depending on arbitrary functions of the time and its derivatives. We saw that this kind of symmetries, in the Lagrangian case, are due to the fact that there are primary $F_L$-projectable constraints that, when imposing they have to be preserved by the dynamic vector field, never turn into non-projectable ones, being in this way a sign of the indeterminacy of $\Gamma_L$. In addition, in the Hamiltonian case, we saw we can generate Noether symmetries by means of constraints whose dynamic evolution is always first class with respect to primary first class at first level when regarded on $S^2_H$. And that, under some regularity conditions, we pointed out a way to build dynamical symmetries that also depends on a arbitrary function and its derivatives, following [13]. Anyway, the existence of (Noether) gauge symmetries from a singular Lagrangian is a problem that it is not completely solved yet.

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