Cavity QED and Quantum Computation in the Weak Coupling Regime

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Abstract

In this paper we consider a model of quantum computation based on n atoms of laser–cooled and trapped linearly in a cavity and realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields.

We solve the Schrödinger equation of the model in the case of n=2 and construct the controlled NOT gate by making use of a resonance condition and rotating wave approximation associated to it. Our method is not heuristic but completely mathematical, and the significant feature is a consistent use of Rabi oscillations.

We also present an idea of the construction of three controlled NOT gates in the case of n=3 which gives the controlled–controlled NOT gate.
1 Introduction

Quantum Computation (or Computer) is a challenging task in this century for not only physicists but also mathematicians. Quantum Computation is in a usual understanding based on qubits which are based on two level systems (two energy levels or fundamental spins) of atoms, See [1] as for general theory of two level systems.

In a realistic image of Quantum Computer we need at least one hundred atoms. However, then we may meet a very severe problem called Decoherence which destroy a superposition of quantum states in the process of unitary evolution of our system. At the present it is not easy to control the decoherence. See for example [2] as an introduction.

An optical system like Cavity QED may have some advantage on this problem, therefore we consider a quantum computation based on Cavity QED. As an approximate model we realize it as the n atoms Tavis–Cummings Hamiltonian interacting with n external (laser) fields. As to the Tavis–Cummings model see [3]. To perform the quantum computation we must first of all show that our system is universal [4]. To show it we must construct the controlled NOT operator (gate) explicitly in the case of $n = 2$, [4], [5].

For that we must embed a system of two qubits in a space of wave functions of the model and solve the Schrödinger equation. In a reduced system we can construct the controlled NOT by use of some resonance condition and the rotating wave approximation associated to it. Then we need to assume that the coupling constants are small enough (the weak coupling regime in the title).

Next we want to construct the controlled–controlled NOT operator in the case of $n = 3$. For that purpose the construction of three controlled NOT gates is required because three atoms are trapped linearly in the cavity, and we present an idea toward explicit construction. If this point will be overcome our system of quantum computation may become complete.

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1 In the study of Cavity QED Quantum Computation this (important) point is missed
2 A Model Based on Cavity QED

We consider a quantum computation model based on \( n \) atoms of laser–cooled and trapped linearly in a cavity and realize it as the \( n \) atoms Tavis–Cummings Hamiltonian interacting with \( n \) external (laser) fields. This is of course an approximate theory. In a more realistic model we must add other dynamical variables such as positions of atoms and their momenta etc. However, since such a model is almost impossible to solve we consider a simple one.

Then the Hamiltonian is given by

\[
H = \omega 1_L \otimes a^\dagger a + \frac{\Delta}{2} \sum_{j=1}^{n} \sigma_j^{(3)} \otimes 1 + g \sum_{j=1}^{n} \left( \sigma_j^{(+)} \otimes a + \sigma_j^{(-)} \otimes a^\dagger \right) + \sum_{j=1}^{n} h_j \left( \sigma_j^{(+)} e^{i(\Omega_j t + \phi_j)} + \sigma_j^{(-)} e^{-i(\Omega_j t + \phi_j)} \right) \otimes 1
\]  

(1)

where \( \omega \) is the frequency of radiation field, \( \Delta \) the energy difference of two level atoms, \( a \) and \( a^\dagger \) are annihilation and creation operators of the field, and \( g \) a coupling constant, \( \Omega_j \) the frequencies of external fields which are treated as classical fields, \( h_j \) coupling constants, and \( L = 2^n \). Here \( \sigma_j^{(+)} \), \( \sigma_j^{(-)} \) and \( \sigma_j^{(3)} \) are given as

\[
\sigma_j^{(s)} = l_2 \otimes \cdots \otimes l_2 \otimes \sigma_s \otimes l_2 \otimes \cdots \otimes l_2 \ (j - \text{position}) \in M(L, \mathbb{C})
\]  

(2)

where \( s \) is +, – and 3 respectively and

\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad l_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]  

(3)

In the case of \( n = 2 \) (which is the target through this paper) see the figure 1. Here we state our scenario of quantum computation. Each external field generates a unitary element of the corresponding qubit (atom) like \( a \otimes b \) where \( a, b \in U(2) \), while an photon inserted generates an entanglement among such elements like \( \sum_j a_j \otimes b_j \). As a whole we obtain any element in \( U(4) \).

Here let us rewrite the Hamiltonian (1). If we set

\[
S_+ = \sum_{j=1}^{n} \sigma_j^{(+)} , \quad S_- = \sum_{j=1}^{n} \sigma_j^{(-)} , \quad S_3 = \frac{1}{2} \sum_{j=1}^{n} \sigma_j^{(3)},
\]  

(4)
Figure 1: The dotted line means a single photon inserted in the cavity and two curves mean external (laser) fields (which are treated as classical ones) subjected to atoms

then (1) can be written as

\[
H = \omega 1_L \otimes a^\dagger a + \Delta S_3 \otimes 1 + g \left( S_+ \otimes a + S_- \otimes a^\dagger \right) + \\
\sum_{j=1}^n h_j \left( \sigma_j^{(+) e^{i(\Omega_j t + \phi_j)}} + \sigma_j^{(-) e^{-i(\Omega_j t + \phi_j)}} \right) \otimes 1 \equiv H_0 + V(t),
\]

which is relatively clear. \(H_0\) is the Tavis–Cummings Hamiltonian and we treat it as an unperturbed one. We note that \(\{S_+, S_-, S_3\}\) satisfy the \(su(2)\)–relation

\[
[S_3, S_+] = S_+, \quad [S_3, S_-] = -S_-, \quad [S_+, S_-] = 2S_3.
\]

However, the representation \(\rho\) defined by

\[
\rho(\sigma_+) = S_+, \quad \rho(\sigma_-) = S_-, \quad \rho(\sigma_3/2) = S_3
\]

is a full representation of \(su(2)\), which is of course not irreducible.

We would like to solve the Schrödinger equation

\[
i \frac{d}{dt} U = HU = (H_0 + V) U,
\]

where \(U\) is a unitary operator. We can solve this equation by using the method of constant variation. The equation \(i \frac{d}{dt} U = H_0 U\) is solved to be

\[
U(t) = \left( e^{-it\omega S_3} \otimes e^{-it\omega N} \right) e^{-itg(S_+ \otimes a + S_- \otimes a^\dagger)} U_0
\]

where \(N = a^\dagger a\) is the number operator and \(U_0\) a constant unitary. Here we have used the resonance condition

\[
\omega = \Delta
\]
Therefore we can calculate the term
\[ \frac{d}{dt} U_0 = e^{itg(S_+ \otimes a + S_- \otimes a^1)} \left( e^{it\omega S_3} \otimes e^{it\omega N} \right) \left( e^{-it\omega S_3} \otimes e^{-it\omega N} \right) e^{-itg(S_+ \otimes a + S_- \otimes a^1)} U_0 \]  (9)
after some algebras. We would like to calculate the right hand side of (9) explicitly, which is however a very hard task due to the term \( e^{-itg(S_+ \otimes a + S_- \otimes a^1)} \). It has been done only for \( n = 1, 2 \) and 3 as far as we know, [5, 6]. The case \( n = 1 \) which is just the Jaynes–Cummings model is not interesting from the point of view of a quantum computation, so we restrict to the case \( n = 2 \) in the following.

First let us write down each term (9). From the result in [6] and some algebras we have
\[
e^{-itg(S_+ \otimes a + S_- \otimes a^1)} = \left( \frac{2N+2}{2N+3} f(N+1) + 1 - ik(N+1)a - ik(N+1)a^\dagger \frac{2}{2N+3} f(N+1)a^2 \right. \\
- ik(N)a^\dagger f(N) + 1 f(N) - ik(N)a \\
- ik(N)a^\dagger f(N) f(N) + 1 - ik(N)a \\
\left. \frac{2}{2N-1} f(N-1)a^2 \right) - ik(N-1)a^\dagger - ik(N-1)a^\dagger \frac{2N}{2N-1} f(N-1) + 1
\]  (10)
where
\[
f(N) = \frac{-1 + \cos \left( tg \sqrt{2(2N+1)} \right)}{2}, \quad k(N) = \frac{\sin \left( tg \sqrt{2(2N+1)} \right)}{\sqrt{2(2N+1)}},
\]
and
\[
\left( e^{it\omega S_3} \otimes e^{it\omega N} \right) V(t) \left( e^{-it\omega S_3} \otimes e^{-it\omega N} \right) = \left( \begin{array}{cccc} 0 & h_2 e^{i\{(\Omega_2 + \omega)t + \phi_2\}} & h_1 e^{i\{(\Omega_1 + \omega)t + \phi_1\}} & 0 \\
 h_2 e^{-i\{\Omega_2 + \omega)t + \phi_2\}} & 0 & 0 & h_1 e^{i\{(\Omega_1 + \omega)t + \phi_1\}} \\
 h_1 e^{-i\{\Omega_1 + \omega)t + \phi_1\}} & 0 & 0 & h_2 e^{i\{(\Omega_2 + \omega)t + \phi_2\}} \\
 0 & h_1 e^{-i\{\Omega_1 + \omega)t + \phi_1\}} & h_2 e^{-i\{\Omega_2 + \omega)t + \phi_2\}} & 0 \end{array} \right) \otimes 1.
\]  (11)
Therefore we can calculate the term
\[
F(t) \equiv e^{itg(S_+ \otimes a + S_- \otimes a^1)} \left( e^{it\omega S_3} \otimes e^{it\omega N} \right) V(t) \left( e^{-it\omega S_3} \otimes e^{-it\omega N} \right) e^{-itg(S_+ \otimes a + S_- \otimes a^1)} \]  (12)
from (10) and (11). However, we omit the explicit form because of being too complicated.

Next let us go to a quantum computation based on two atoms of laser–cooled and trapped linearly in a cavity.

3 Quantum Computation

Let us make a short review of two–qubits. Each element can be written as

$$\psi = a_{++}|+\rangle \otimes |+\rangle + a_{+-}|+\rangle \otimes |-\rangle + a_{-+}|-\rangle \otimes |+\rangle + a_{--}|-\rangle \otimes |-\rangle$$

with two bases $|+\rangle$ and $|-\rangle$ and $|a_{++}|^2 + |a_{+-}|^2 + |a_{-+}|^2 + |a_{--}|^2 = 1$. Here if we identify

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then $\psi$ above becomes

$$\psi = \begin{pmatrix} a_{++} \\ a_{+-} \\ a_{-+} \\ a_{--} \end{pmatrix}.$$

(13)

How do we embed two–qubits in our quantized system? It is not known at the moment, which will depend on some method of experimentalists. Therefore let us consider the simplest one like

$$|\psi(t)\rangle = \begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} \otimes |0\rangle,$$

(14)

where $|0\rangle$ is the ground state of the radiation field ($a|0\rangle = 0$). We note that in full theory
we must consider the following superpositions

\[ |\Psi(t)\rangle = \sum_{n=0}^{\infty} \left( \begin{array}{c} a_{++\,n}(t) \\ a_{+-\,n}(t) \\ a_{-+\,n}(t) \\ a_{--\,n}(t) \end{array} \right) \otimes |n\rangle \]

as a wave function, which is however too complicated to solve.

To determine a dynamics that the coefficients \(a_{++}, a_{+-}, a_{-+}, a_{--}\) will satisfy we substitute (14) into the equation

\[ id \frac{d}{dt} |\psi(t)\rangle = F(t) |\psi(t)\rangle . \]  

(15)

See Appendix for the full calculations. The equation is not satisfied under the restrictive ansatz (14). However, excited states \(|1\rangle, |2\rangle, |3\rangle\) which have no corresponding kinetic terms contain the coupling constants \(h_1\) and \(h_2\), so the equation is approximately satisfied if they are small enough (namely, in the weak coupling regime in the title).

Therefore the (full) equation is reduced to the equations of \(\{a_{++}, a_{+-}, a_{-+}, a_{--}\}\) at the ground state.

\[ i \frac{d}{dt} a_{++} = \]

\[ h_1 e^{i((\Omega_1+\omega)t+\phi_1)} \left\{ f(0) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} + \]

\[ h_2 e^{i((\Omega_2+\omega)t+\phi_2)} \left\{ 1 + f(0) + \frac{2}{3} f(1) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} a_{+-} + \]

\[ h_1 e^{i((\Omega_1+\omega)t+\phi_1)} \left\{ 1 + f(0) + \frac{2}{3} f(1) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} + \]

\[ h_2 e^{i((\Omega_2+\omega)t+\phi_2)} \left\{ f(0) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} a_{-+} , \]  

(16)

\[ i \frac{d}{dt} a_{+-} = \]

\[ h_1 e^{-i((\Omega_1+\omega)t+\phi_1)} \left\{ f(0) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} + \]

\[ h_2 e^{-i((\Omega_2+\omega)t+\phi_2)} \left\{ 1 + f(0) + \frac{2}{3} f(1) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} a_{++} + \]

\[ h_1 e^{i((\Omega_1+\omega)t+\phi_1)} \left\{ 1 + f(0) \right\} + h_2 e^{i((\Omega_2+\omega)t+\phi_2)} f(0) a_{--} , \]  

(17)
\[
\frac{d}{dt} a_{+-} = \\
\left[ h_1 e^{-i(\Omega_1 + \omega)t + \phi_1} \left\{ 1 + f(0) + \frac{2}{3} f(1) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} \right. + \\
\left. h_2 e^{-i((\Omega_2 + \omega)t + \phi_2)} \left\{ f(0) + \frac{2}{3} f(0) f(1) + k(0) k(1) \right\} \right] a_{++} + \\
\left[ h_1 e^{i((\Omega_1 + \omega)t + \phi_1)} f(0) + h_2 e^{i((\Omega_2 + \omega)t + \phi_2)} \{ 1 + f(0) \} \right] a_{--},
\]  
(18)

or in a matrix form

\[
\frac{d}{dt} \begin{pmatrix}
    a_{++}(t) \\
    a_{+-}(t) \\
    a_{-+}(t) \\
    a_{--}(t)
\end{pmatrix} = \begin{pmatrix}
    1 & 1 & 1 & 1 \\
    1 & 1 & 1 & 1 \\
    1 & 1 & 1 & 1 \\
    1 & 1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
    a_{++}(t) \\
    a_{+-}(t) \\
    a_{-+}(t) \\
    a_{--}(t)
\end{pmatrix}
\]  
(20)

where \(\#\) is the corresponding matrix element from the above equations.

We obtained the system of complete equations, which is still complicated. How do we solve it? We use some resonance condition and the rotating wave approximation associated to it. Since

\[
f(0) = \left\{ -1 + \cos \left( t g \sqrt{2} \right) \right\} / 2, \quad f(1) = \left\{ -1 + \cos \left( t g \sqrt{6} \right) \right\} / 2,
\]

\[
k(0) = \sin \left( t g \sqrt{2} \right) / \sqrt{2}, \quad k(1) = \sin \left( t g \sqrt{6} \right) / \sqrt{6},
\]

the products \(f(0)f(1)\) and \(k(0)k(1)\) contain the term \(e^{-itg(\sqrt{2}+\sqrt{6})}\) by the Euler formulas

\[
\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2, \quad \sin(\theta) = (e^{i\theta} - e^{-i\theta})/2i.
\]

Noting

\[
e^{i((\Omega_1 + \omega)t + \phi_1)} e^{-itg(\sqrt{2}+\sqrt{6})} = e^{i((\Omega_1 + \omega - (\sqrt{2}+\sqrt{6})g)t + \phi_1)},
\]

we set a new resonance condition

\[
\Omega_1 + \omega - (\sqrt{2} + \sqrt{6})g = 0.
\]  
(21)
The solution is easily obtained to be

\[
\frac{d}{dt} \begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} = -\frac{(\sqrt{3} - 1)h_1}{24} \begin{pmatrix} 0 & e^{i\phi_1} & e^{i\phi_1} & 0 \\ e^{-i\phi_1} & 0 & e^{-i\phi_1} & 0 \\ e^{-i\phi_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix}. \tag{22}
\]

Then (20) reduces to a very simple matrix equation

\[
\begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} = \exp \left\{ \frac{i(\sqrt{3} - 1)h_1 t}{24} \begin{pmatrix} 0 & e^{i\phi_1} & e^{i\phi_1} & 0 \\ e^{-i\phi_1} & 0 & e^{-i\phi_1} & 0 \\ e^{-i\phi_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} a_{++}(0) \\ a_{+-}(0) \\ a_{-+}(0) \\ a_{--}(0) \end{pmatrix}
\]

\[
= \begin{pmatrix} \cos(\alpha t) & \frac{ie^{i\phi_1}}{\sqrt{2}} \sin(\alpha t) & \frac{ie^{i\phi_1}}{\sqrt{2}} \sin(\alpha t) & 0 \\ \frac{ie^{-i\phi_1}}{\sqrt{2}} \sin(\alpha t) & \frac{1 + \cos(\alpha t)}{2} & -\frac{1 + \cos(\alpha t)}{2} & 0 \\ \frac{ie^{-i\phi_1}}{\sqrt{2}} \sin(\alpha t) & -\frac{1 + \cos(\alpha t)}{2} & \frac{1 + \cos(\alpha t)}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{++}(0) \\ a_{+-}(0) \\ a_{-+}(0) \\ a_{--}(0) \end{pmatrix} \tag{23}
\]

where we have set \( \alpha = \frac{\sqrt{6} - \sqrt{2}}{24} h_1 \). That is, we obtained the unitary operator \( U(t) \). In particular, if we choose \( t_0 \) satisfying \( \cos(\alpha t_0) = -1 \) (\( \sin(\alpha t_0) = 0 \)), then

\[
U(t_0) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}
\]

\[
= - \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}
\]. \tag{24}
At this stage we use a very skillful method. That is, we exchange two atoms in the cavity

which introduces the exchange (swap) operator

\[
P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{25}
\]

Multiplying \(U(t_0)\) by \(P\) gives

\[
PU(t_0) = -\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{26}
\]

This is just the controlled \(\sigma_z\) operator except for the overall constant \(-1\) (an overall constant can be always neglected). From this it is easy to construct the controlled NOT operator, namely

\[
C_{\text{NOT}} = (1_2 \otimes W)C_{\sigma_z}(1_2 \otimes W) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

\(^2U(t_0)\) is imprimitive in the sense of [8], so the main theorem in it says that our system is universal (namely, we can construct any element in \(U(4)\)). However, how to construct a unitary element explicitly is not given in [8].
where \( W \) is the Walsh–Hadamard operator given by

\[
W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = W^{-1}.
\] (27)

See for example [5]. As to a construction of \( W \) by making use of Rabi oscillations see [10]. Therefore our system is universal [4], [8].

A comment is in order.

(a) In the equation (20) we can set another resonance condition in place of (21) and obtain a unitary operator like \( U(t) \) in (23).

(b) In place of the ansatz (14) we can set for example

\[
|\psi(t)\rangle = \begin{pmatrix} a_{++}(t) \\ a_{+-}(t) \\ 0 \\ 0 \end{pmatrix} \otimes |0\rangle + \begin{pmatrix} 0 \\ 0 \\ a_{-+}(t) \\ a_{--}(t) \end{pmatrix} \otimes |1\rangle.
\]

Then we can trace the same line shown in this section and obtain a unitary operator under some resonance condition like (24). This is a good exercise, so we leave it to the readers.

4 Controlled-Controlled NOT Gate

Our quantum computation model is based on \( n \) atoms of laser–cooled and trapped linearly in a cavity, so we have another problem on the controlled NOT operators (of three types) when \( n = 3 \).

**Problem**: Let us consider the case of three atoms in a cavity. How can we construct C-NOT (or C-unitary) operators for any two atoms among them?

See the figure 2. These constructions are very crucial in realizing quantum logic gates, for example, the controlled–controlled NOT gate shown as a picture.
Figure 2: The Controlled NOT gates (of three types) for the three atoms in the cavity

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

The (usual) construction by making use of controlled NOT or controlled U gates is shown as a picture ([5], [4]).
where \( V \) is a unitary matrix given by

\[
V = \frac{1}{2} \begin{pmatrix}
1 + i & 1 - i \\
1 - i & 1 + i
\end{pmatrix} \implies V^2 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \sigma_1.
\]

However, we have not seen “realistic” constructions in any references, so we must give the explicit construction.

To solve this let us state our idea. First we consider the construction of controlled NOT operator between the first and second atoms, namely

\[
C\text{-NOT} \otimes 1
\]

Our strategy is as follows.

(i) We move the third atom from the cavity.

(ii) We insert a photon in the cavity as two atoms interact with it and subject laser fields to the atoms, and next exchange the two atoms, which gives the controlled NOT operator as shown in the preceding section.

(iii) We return the third atom (outside the cavity) to the former position.

See the figure 3.

If an influence of the “getting the third atom in and out” on the states space is small enough (namely, the unitary operator induced is near to the identity \( I_4 \)), then we certainly
Figure 3: The process to construct the controlled NOT gate between the first atom and second one for the three atoms in the cavity
Figure 4: The general setting for a quantum computation based on Cavity QED. The dotted line means a single photon inserted in the cavity and all curves mean external (laser) fields (which are treated as classical ones) subjected to atoms

obtain the controlled NOT gate (namely, $C_{NOT} \otimes 1_2$) that we are looking for. Similarly we can obtain the remaining two ones.

It is easy to generalize our idea to the $n$–atoms case. To perform a quantum computation we need to construct (many) controlled–controlled NOT gates or controlled–controlled unitary ones for three atoms among $n$–atoms, see [4], §7. The method is almost same, so we leave it to the readers. See the figure 4.

In principle, we can construct general quantum networks.

By the way, a quick construction of controlled–controlled NOT gates is essential in general quantum networks [4].

We have given the exact form of evolution operator for the three atoms Tavis–Cummings model [4], therefore we can in principle track the same line shown in this paper and it may be possible to get the controlled–controlled NOT or many unitary gates directly (without combining many elementary gates like the construction of controlled–controlled NOT gate above).

However, such a calculation for the three atoms case becomes very difficult (because we must treat $8 \times 8$ matrices at each step of calculations). We will attempt it in the near future.

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3We must estimate an influence of the “getting atoms (which are not our target) in and out” on the whole states space, which is however difficult in our model. For that we must add in further terms necessary to calculate it.
5 Discussion

In this paper we constructed the controlled NOT operator in the quantum computation based on Cavity QED which showed that our system is universal. We also constructed the controlled–controlled NOT operator (under some assumption). Therefore we can in principle perform a quantum computation.

We expect strongly that some experimentalists will check whether our method works good or not.

See [12] and their references for some experiments on Cavity QED (which may be related to our method).

We conclude this paper by making a comment (which is important at least to us). The Tavis–Cummings model is based on (only) two energy levels of atoms. However, an atom has in general infinitely many energy levels, so it is natural to use this possibility. We are also studying a quantum computation based on multi–level systems of atoms (a qudit theory) [11]. Therefore we would like to extend the Tavis–Cummings model based on two–levels to a model based on multi–levels. This is a very challenging task.

Acknowledgment. We wish to thank Shin’ichi Nojiri for his helpful comments and suggestions, and thank the referees for careful readings and useful suggestions. K. Fujii wish to thank Gilles Nogues for teaching him some experimental facts on Cavity QED.

Appendix

One Qubit Operators by Classical Fields

Let us make a brief review of theory without the radiation field, whose states space is only tensor product of two level systems of each atom. See the figure 5.

The Hamiltonian in this case is

\[
H = \sum_{j=1}^{n} \left\{ \frac{\Delta}{2} \sigma_j^{(3)} + \hbar \left( \sigma_j^{(+)} e^{i(\Omega_j t + \phi_j)} + \sigma_j^{(-)} e^{-i(\Omega_j t + \phi_j)} \right) \right\}
\]  

(28)
Figure 5: The $n$ atoms in the cavity without a photon (in Figure 4) from (1), where we have omitted the unit operator $1$. Then

$$H = \sum_{j=1}^{n} \left( \begin{array}{cc} \Delta/2 & h_j e^{i(\Omega_j t + \phi_j)} \\ h_j e^{-i(\Omega_j t + \phi_j)} & -\Delta/2 \end{array} \right)_j$$

$$= \sum_{j=1}^{n} \left\{ \left( \begin{array}{cc} e^{i\Omega_j t + \phi_j} \\ e^{-i\Omega_j t + \phi_j} \end{array} \right)_j \left( \begin{array}{cc} \Delta/2 & h_j \\ h_j & -\Delta/2 \end{array} \right)_j \left( \begin{array}{cc} e^{-i\Omega_j t + \phi_j} \\ e^{i\Omega_j t + \phi_j} \end{array} \right)_j \right\}$$

$$= (U_1 \otimes \cdots \otimes U_n) \sum_{j=1}^{n} \left( \begin{array}{cc} \Delta/2 & h_j \\ h_j & -\Delta/2 \end{array} \right)_j (U_1 \otimes \cdots \otimes U_n)^\dagger, \quad (29)$$

where

$$U_j = \left( \begin{array}{cc} e^{i\Omega_j t + \phi_j} \\ e^{-i\Omega_j t + \phi_j} \end{array} \right)_j \quad \text{and} \quad M_j = 1_2 \otimes \cdots \otimes 1_2 \otimes M \otimes 1_2 \otimes \cdots \otimes 1_2.$$  

The wave function defined by $i \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle$ with (28) can be written as a tensor product

$$|\Psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle, \quad (30)$$

so if we define

$$|\tilde{\Psi}\rangle \equiv (U_1 \otimes \cdots \otimes U_n)^\dagger |\Psi\rangle,$$

then it is easy to see

$$i \frac{d}{dt} |\tilde{\Psi}\rangle = \sum_{j=1}^{n} \left( \begin{array}{cc} \Delta-\Omega_j/2 & h_j \\ h_j & -\Delta-\Omega_j/2 \end{array} \right)_j |\tilde{\Psi}\rangle.$$

The solution is easy to obtain

$$|\tilde{\Psi}(t)\rangle = \otimes_{j=1}^{n} \exp \left\{ -it \left( \begin{array}{cc} \Delta-\Omega_j/2 & h_j \\ h_j & -\Delta-\Omega_j/2 \end{array} \right)_j \right\} |\tilde{\Psi}(0)\rangle.$$
Therefore, the solution that we are looking for is

\[ |\Psi(t)\rangle = (U_1 \otimes \cdots \otimes U_n) |\tilde{\Psi}(t)\rangle \]

\[ = \bigotimes_{j=1}^{n} \left( e^{i\frac{\eta_j + \phi_j}{2}} e^{-i\frac{\eta_j + \phi_j}{2}} \right) \exp \left\{ -i t \left( \frac{\Delta - \Omega_j}{h_j} \frac{h_j}{2} \right) \right\} |\Psi(0)\rangle. \]

(31)

Last we note that

\[ \exp \left\{ -i t \left( \begin{array}{cc} \frac{\theta}{2} & h \\ h & -\frac{\theta}{2} \end{array} \right) \right\} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \]

where

\[ x_{11} = \cos \left( t \sqrt{\frac{\theta^2}{4} + h^2} \right) - i \frac{\theta}{2} \sin \left( t \sqrt{\frac{\theta^2}{4} + h^2} \right), \]

\[ x_{12} = x_{21} = -i h \frac{\sin \left( t \sqrt{\frac{\theta^2}{4} + h^2} \right)}{\sqrt{\frac{\theta^2}{4} + h^2}}, \]

\[ x_{22} = \cos \left( t \sqrt{\frac{\theta^2}{4} + h^2} \right) + i \frac{\theta}{2} \sin \left( t \sqrt{\frac{\theta^2}{4} + h^2} \right). \]

We can always construct unitary operators in \( U(2) \) at each atoms by using Rabi oscillations, see for example [10].

**Explicit Form of the Equation (15)**

Let us give the explicit form to (15) for avoiding errors in the calculations. The left hand side of (15) is

\[ \left( i \frac{d}{dt} a_{++}(t) \right) \otimes |0\rangle \]

(32)

, while each of the right hand side becomes

1-component = \( h_1 e^{i(\Omega_1 + \omega)t + \phi_1} \times \)
\[ \left[ -ia_{++} \left\{ k(1) + \frac{4}{5}k(1)f(2) - \frac{4}{3}f(1)k(2) \right\} \right| 1 \right] + a_{--} \left\{ f(0) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} \right| 0 \right] + \\
h_2 e^{i(\Omega_2 + \omega)t + \phi_2} \times \\
\left[ -ia_{++} \left\{ k(1) + \frac{4}{5}k(1)f(2) - \frac{4}{3}f(1)k(2) \right\} \right| 1 \right] + a_{--} \left\{ f(0) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} \right| 0 \right], \\
(33) \\
2\text{-component} = h_1 e^{i((\Omega_1 + \omega)t + \phi_1)} \times \\
\left[ \sqrt{2}a_{++} \left\{ \frac{2}{3}f(1) + k(1)k(2) + \frac{2}{3}f(1)f(2) \right\} \right| 2 \right] - ia_{+-} \left\{ k(0) - f(0)k(1) + k(0)f(1) \right\} \right| 1 \right] \\
-ia_{-+} \left\{ k(0) - f(0)k(1) + k(0)f(1) \right\} \right| 1 \right] + a_{-} \left\{ 1 + f(0) \right\} \right| 0 \right] + \\
h_2 e^{i(\Omega_2 + \omega)t + \phi_2} \times \\
\left[ \sqrt{2}a_{++} \left\{ k(1)k(2) + \frac{2}{3}f(1)f(2) \right\} \right| 2 \right] + ia_{+-} \left\{ k(1) + f(0)k(1) - k(0)f(1) \right\} \right| 1 \right] \\
+ia_{-+} \left\{ f(0)k(1) - k(0)f(1) \right\} \right| 1 \right] \right| 0 \right] + \\
h_1 e^{-i(\Omega_1 + \omega)t + \phi_1} a_{++} \left\{ f(0) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} \right| 0 \right] + \\
h_2 e^{-i(\Omega_2 + \omega)t + \phi_2} a_{++} \left\{ 1 + f(0) + \frac{2}{3}f(1) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} \right| 0 \right], \\
(34) \\
3\text{-component} = h_1 e^{i((\Omega_1 + \omega)t + \phi_1)} \times \\
\left[ \sqrt{2}a_{++} \left\{ k(1)k(2) + \frac{2}{3}f(1)f(2) \right\} \right| 2 \right] + ia_{+-} \left\{ f(0)k(1) - k(0)f(1) \right\} \right| 1 \right] \\
+ia_{-+} \left\{ k(1) + f(0)k(1) - k(0)f(1) \right\} \right| 1 \right] \right| 0 \right] + \\
h_2 e^{i(\Omega_2 + \omega)t + \phi_2} \times \\
\left[ \sqrt{2}a_{++} \left\{ \frac{2}{3}f(1) + k(1)k(2) + \frac{2}{3}f(1)f(2) \right\} \right| 2 \right] - ia_{+-} \left\{ k(0) - f(0)k(1) + k(0)f(1) \right\} \right| 1 \right] \\
-ia_{-+} \left\{ k(0) - f(0)k(1) + k(0)f(1) \right\} \right| 1 \right] + a_{-} \left\{ 1 + f(0) \right\} \right| 0 \right] + \\
h_1 e^{-i(\Omega_1 + \omega)t + \phi_1} a_{++} \left\{ 1 + f(0) + \frac{2}{3}f(1) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} \right| 0 \right] + \\
h_2 e^{-i(\Omega_2 + \omega)t + \phi_2} a_{++} \left\{ f(0) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} \right| 0 \right], \\
(35) \\
4\text{-component} = h_1 e^{i((\Omega_1 + \omega)t + \phi_1)} \times
\[
\begin{align*}
-2\sqrt{6}a_{++} & \left\{ \frac{1}{5}k(1)f(2) - \frac{1}{3}f(1)k(2) \right\} |3\rangle + \sqrt{2}a_{+-} \left\{ \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} |2\rangle \\
+\sqrt{2}a_{+-} & \left\{ \frac{2}{3}f(1) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} |2\rangle + ia_{--}k(0)|1\rangle \\
\hbar_2 e^{i(\Omega_2 + \omega)t + \phi_2} & \times \\
\begin{align*}
-2\sqrt{6}a_{++} & \left\{ \frac{1}{5}k(1)f(2) - \frac{1}{3}f(1)k(2) \right\} |3\rangle + \sqrt{2}a_{+-} \left\{ \frac{2}{3}f(1) + \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} |2\rangle \\
+\sqrt{2}a_{+-} & \left\{ \frac{2}{3}f(0)f(1) + k(0)k(1) \right\} |2\rangle + ia_{--}k(0)|1\rangle \\
\hbar_1 e^{-i(\Omega_1 + \omega)t + \phi_1} & \times \\
\begin{align*}
ia_{++} & \left\{ k(0) - k(1) + \frac{2}{3}k(0)f(1) - 2f(0)k(1) \right\} |1\rangle + a_{+-} \left\{ 1 + f(0) \right\} |0\rangle + a_{+-}f(0)|0\rangle \\
\hbar_2 e^{-i(\Omega_2 + \omega)t + \phi_2} & \times \\
\begin{align*}
ia_{++} & \left\{ k(0) - k(1) + \frac{2}{3}k(0)f(1) - 2f(0)k(1) \right\} |1\rangle + a_{+-}f(0)|0\rangle + a_{+-} \left\{ 1 + f(0) \right\} |0\rangle
\end{align*}
\end{align*}
\end{align*}
\]

(36) after a long calculation by making use of (12).

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