MODELING PSYCHOLOGICAL MEASUREMENTS WITH QUANTUM INSTRUMENTS: COMBINATION OF QUESTION ORDER EFFECT, RESPONSE REPLICABILITY EFFECT, AND QQ-EQUALITY

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ABSTRACT. We continue to analyze matching of some basic constraints on human’s decision making with quantum theory of measurement. As was found, quantum measurement theory based on the projection postulate does not match with combination of the question order effect (QOE) and the response replicability effect (RRE). This was the alarm signal for quantum-like modeling of decision making. Recently, it was shown that this objection to quantum-like modeling can be removed on the basis of the general measurement theory based on quantum instruments. In the present paper, the problem of combination QOE, RRE with the famous QQ-equality is analyzed. This equality was derived by Busemeyer and Wang and it was shown (in the joint paper with Solloway and Shiffrin) that statistical data from many social opinion polls satisfies it. Now, we construct quantum instruments satisfying QQE, RRE, and QQ-inequality. Moreover, we show that this model reproduces the statistics of the famous Clinton-Gore Poll data almost faithfully with a prior belief state independent of the question order. This model successfully removes the order effect from the data to determine the genuine distribution of the opinion to the Poll. Independently of modeling of the concrete psychological effects, the paper provides psychologist-friendly presentation of the theory of quantum instruments - the most general mathematical framework for quantum measurements. We hope that this theory will attract attention of psychologists and generate further psychological applications.

keywords: quantum-like models, decision making, social science, quantum instruments, order effect, response replicability effect, QQ-equality

1. INTRODUCTION

We start with remind that recently the top level experts in cognition, psychology, decision making, economics, finances, and social and political sciences as well as in molecular biology started to be interested in application of the quantum formalism. The majority of applications are based on quantum measurement theory (QMT) that is used to model the basic psychological effects.
However, as was shown in [5], canonical QMT based on self-adjoint operators (representing questions and tasks) and the projection postulate (representing the mental state update) confronts with combination of some psychological effects. So, QMT works separately for each of such effects, but it collapses to explain their combinations. In [5], it was shown that combination of the question order effect (QOE\(^1\)) and the response replicability effect (RRE\(^2\)) cannot be described by canonical QMT. This was the alarm signal for quantum-like modeling of decision making (cognition and psychology).

However, since the 1970s QMT was developed very much by generalizing canonical QMT used during the first years of quantum physics. Nowadays, in quantum physics, especially in quantum information theory, one uses QMT based on theory of quantum instruments\([15–18]\). So, say in quantum information theory, nobody would be surprised that in some situations canonical QMT (projection postulate + self-adjoint operators) cannot be applied.

Therefore, the right reply to the challenge presented in [5] should be based on theory of quantum instruments. The first attempt to proceed in this direction was done in paper of Basieva and Khrennikov [6]. But, they used too restricted class of quantum instruments. (At the same time, this class is standard for quantum information theory.) And their paper confirmed the impossibility statement formulated originally in [5]. Only very recently it was shown [8] that by using quantum instruments, it is possible to describe the combination of QOE and RRE. Thus, quantum-like modeling program was secured from the objection of paper [5]. We point out that the instruments under consideration are not typical for using in quantum information theory. Hence, mental measurements are different from typical quantum physical measurements.

Immediately after publication of article [8], Dzhafarov (the private communication) raised the question whether the model presented in [8] match the famous QQ-equality (QQE). The latter is one of the most important fruits of the quantum-like approach to decision making. This is a special constraint on probabilities that was derived by Busemeyer and Wang [3] in the QP-framework. So, from the CP-viewpoint there is no reasons for QQE to hold, but manipulation with quantum states leads to it. In paper [9], it was shown that statistical data from a bunch of social opinion polls satisfies QQE. We stress that this equality was derived on the basis of canonical QMT [3] and that it can

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\(^1\)QOE: dependence of the (sequential) joint probability distribution on questions’ order: \(p_{AB} \neq p_{BA}\); see [3,9] for its modeling with canonical QMT.

\(^2\)RRE [5]: Suppose that after answering the \(A\)-question with say the “yes”-answer, Alice is asked another question \(B\). She replied to it with some answer. And then she is asked \(A\) again. In the social opinion polls and other natural decision making experiments, Alice definitely repeats her original answer to \(A\), “yes”. This is \(A - B - A\) response replicability. (In the absence of \(B\)-question, we get \(A - A\) replicability). Combination of \(A - B - A\) and \(B - A - B\) replicability forms RRE.

\(^3\)Generalized quantum observables in the form of positive operator valued measures (POVMs) (which are so widely used in quantum information theory and started to be used in quantum-like modeling [11,12]) appear naturally in the framework of theory of quantum instruments (see appendix). But, we shall not use them in the present paper.
be violated in general QMT \cite{15-18}. Thus, combination of QOE and RRE with QQE is a delicate problem: one has to go beyond canonical QMT in rather special way.

The purpose of this paper is two-fold:

A). We present the solution of this problem (QOE+RRE+QQE) by constructing corresponding quantum instruments. This step is important to justify the use of the quantum-like models for decision making. We emphasize that our model is a complete model for a certain set of data satisfying the QQ-equality including the Clinton-Gore poll in the following sense. Since the Clinton-Gore poll data sets approximately satisfy QQE, there is a uniform method to make the original data to satisfy QQE with small distortion. Here the main point is that \textit{such a method can be defined independent of the model.} We show that our model completely reproduces the modified data of the Clinton-Gore poll data and this means our model reproduce the original data with the error equal to the distortion to fit the data to QQE. Moreover, we show that this model reproduces the statistics of the famous Clinton-Gore Poll data almost faithfully with a prior belief state independent of the question order. Thus, this model successfully removes the order effect from the data to determine the genuine distribution of the opinion to the Poll.

Our paper does not contain a discussion of alternative theories. We only offer quantum models as viable candidates for these results, because we are not psychologists and do not have sufficient knowledge on classical model. The reader can find such a discussion in works \cite{19-21}.

B). Irrespectively to the (QOE+RRE+QQE)-problem, the paper can be useful for psychologists as representing the most general formalism of QMT based on theory of quantum instruments. We hope that this theory will attract attention of psychologists and generate further psychological applications.

Finally, we remark that the projective type (von Neumann-Lüders \cite{14,28}) instruments, i.e., describing measurement’s feedback onto system’s state by orthogonal projections (section 2.2), are widely used in quantum-like modeling in cognition, psychology, and decision making \cite{2,3,9-12,26-27}. Although this model is very attractive due to its simplicity, one has to be careful by applying it. In \cite{5}, the role of RRE was elevated. In fact, RRE is a special exhibition of memory in combination with keeping

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4It should be pointed out that we do not try to compare the classical \cite{13} and quantum \cite{14} probability models applied for decision making. We remark that in quantum physics comparison of these models started 1920th (with the Wigner function) and it is still continued, e.g., in hot discussions on Bell’s inequality \cite{15-21}. How far can one proceed with classical probability in quantum physics? This is the very complex problem (see, e.g., \cite{22-25}) and it is too early to make the final conclusion. Nevertheless, the quantum formalism has been successfully applied to numerous theoretical and engineering problems. This formalism is powerful and successful, irrespectively to the (im)possibility to proceed with classical probability. One of its advantages is the linear structure of the state space. This reduces calculations to simple linear algebra for matrices and vectors. In fact, the linear space structure of mental spaces has been widely used in cognitive science and psychology. And the quantum-like modeling matches well such modeling with linear state spaces, with advantage of exploring the mathematical formalism and the methodology which were elaborated in quantum physics, one of the most successful branches of science.
consistency in decisions (actions). So, RRE is one of the basic characteristics of rational cognition. Quantum instruments constructed in this paper express the memory effect and decision consistency that are sufficient to generate RRE. In contrast, the projective type instruments do not. They express very strong correlations between the A-outcome and successive B-outcome. Such a behavior is irrational. So, such instruments can be used to model irrational behavior. In social opinion polls, generally people behave rationally, in spite of the order effect. The latter does not contradict to rationality.

2. QUANTUM INSTRUMENTS

We briefly present quantum instruments (see [8] for detailed non-physicist friendly presentation, see also basic papers [15–18]; see appendix for coupling with generalized observables given by POVMs.

2.1. Quantum states and observables. In quantum theory, it is postulated that every quantum system S corresponds to a complex Hilbert space \( \mathcal{H} \); denote the scalar product of two vectors by the symbol \( \langle \psi_1 | \psi_2 \rangle \). Throughout the present paper, we assume \( \mathcal{H} \) is finite dimensional. States of the quantum system \( S \) are represented by density operators acting in \( \mathcal{H} \) (positive semi-definite operators with unit trace). Denote this state space by the symbol \( S(\mathcal{H}) \).

In quantum physics (especially quantum information theory), there are widely used notations invented by Dirac: a vector belonging to \( \mathcal{H} \) is symbolically denoted as \( |\psi\rangle \); orthogonal projector on this vector is denoted as \( |\psi\rangle\langle\psi| \), it acts to the vector \( |\xi\rangle \) as \( \langle\psi|\xi\rangle|\psi\rangle \).

Any density operator \( \rho \) of rank one is of the form \( \rho = |\psi\rangle\langle\psi| \) with a unit-norm vector \( |\psi\rangle \). In this case, \( |\psi\rangle \) is called a pure state, so \( |\psi\rangle \in \mathcal{H}, \|\psi\| = \sqrt{\langle\psi|\psi\rangle} = 1 \).

Observables are represented by Hermitian operators in \( \mathcal{H} \). These are just symbolic expressions of physical observables, say the position, momentum, or energy. Each Hermitian operator \( A \) can be represented as

\[
A = \sum_x x E^A(x),
\]

where \( x \) labels the eigenvalues and \( E^A(x) \) is the spectral projection of the observable \( A \) corresponding to the eigenvalue \( x \). We note that spectral projectors sum up to the unit operator \( I \):

\[
\sum_x E^A(x) = I.
\]

We also remark that each orthogonal projector \( E \) is Hermitian and idempotent, i.e., \( E^* = E \) and \( E^2 = E \). This implies that equality (2) can be rewritten in the form:

\[
\sum_x E^A(x)[E^A(x)]^* = I.
\]
The operator \( A \) can be considered as the compact mathematical representation for probabilities of outcomes of the physical observable. These probabilities are given by the **Born rule** that states if an observable \( A \) is measured in a state \( \rho \), then the probability distribution \( \Pr\{A = x \| \rho\} \) of the outcome of the measurement is given by

\[
\Pr\{A = x \| \rho\} = \text{Tr}[E^A(x)\rho] = \text{Tr}[E^A(x)\rho E^A(x)].
\]

For a pure state \( |\psi\rangle \), this leads to the relation

\[
\Pr\{A = x \| |\psi\rangle\} = \frac{\|E^A(x)|\psi\rangle\|^2}{\|E^A(x)|\psi\rangle\|^2},
\]

as \( \text{Tr}[E^A(x)\rho] = \|E^A(x)|\psi\rangle\|^2 \).

This is the formal operational quantum model.

However, the theory should also reflect the evident fact that the same quantum observable, say energy, can be measured with a variety of quantum measuring instruments. In the quantum formalism, these instruments are characterized by back-actions of measurements to system’s states. Suppose the observable \( A \) is measured and the result \( A = x \) is obtained. The Born rule gives the probability of this output. But, what is happened with system’s state?

In particular, to determine the joint probability distribution \( \Pr\{A = x, B = y \| \rho\} \) of outcomes of successive measurements of observables \( A \) and \( B \) (first measurement of \( A \) and then measurement of \( B \)) for the input state \( \rho \), we need another postulate to determine the state after the measurement.

### 2.2. Von Neumann-Lüders instruments.

In the conventional quantum measurement theory (QMT), the **projection postulate** is posed, stating: in the measurement of an observable \( A \), the input state \( \rho \) is changed to the output state

\[
\rho_{\{A=x\}} = \frac{E^A(x)\rho E^A(x)}{\text{Tr}[E^A(x)\rho]},
\]

provided that the measurement leads to the outcome \( A = x \). If the input state is the pure state \( |\psi\rangle \), the output state is also the pure state \( |\psi\rangle_{\{A=x\}} \) such that

\[
|\psi\rangle_{\{A=x\}} = \frac{E^A(x)|\psi\rangle}{\|E^A(x)|\psi\rangle\|^2}.
\]

In this case, i.e., for the density operator \( \rho = |\psi\rangle\langle\psi| \), we have the relations

\[
\rho_{\{A=x\}} = \frac{E^A(x)\rho E^A(x)}{\text{Tr}[E^A(x)\rho]} = \frac{E^A(x)|\psi\rangle\langle\psi|E^A(x)}{\|E^A(x)|\psi\rangle\|^2} = |\psi_{\{A=x\}}\rangle\langle\psi_{\{A=x\}}|.
\]

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5For observables given by Hermitian operators with non-degenerate spectra, this postulate was suggested by von Neumann [14]. Then Lüders extended it even to observables with degenerate spectra [28]. For the latter, von Neumann used a more general rule (in the spirit of theory of quantum instruments).

6We recall that this rule was adopted by Wang and Busemeyer [3].
According to the projection postulate, if observables \( A \) and \( B \) are successively measured in this order in the initial input state \( \rho \), the joint probability distribution of their outcomes is given by

\[
Pr\{A = x, B = y \| \rho\} = Pr\{E^B(y)E^A(x)\rho E^A(x)\}.
\]

For the vector state \( |\psi\rangle \) we have

\[
Pr\{A = x, B = y \| |\psi\rangle\} = \|E^B(y)E^A(x)|\psi\rangle\|^2.
\]

Thus, in the conventional QMT the outcome probability distribution and the state change caused by the measurement are uniquely determined by the Hermitian operator \( A \). However, in modern QMT, a more flexible rule is adopted, in which the state change caused by the measurement is not uniquely determined by the Hermitian operator (symbolically representing the physical observable). There are many ways to measure the same observable.

2.3. The Davis-Lewis-Ozawa quantum instruments. It has been known that the projection postulate is too restrictive to describe all the physically realizable measurements of the observable \( A \). Davies-Lewis [15] proposed to abandon this postulate and proposed a more flexible approach to QMT.

The space \( \mathcal{H} \) of linear operators in \( \mathcal{H} \) is a linear space over the complex numbers. Moreover, \( \mathcal{L}(\mathcal{H}) \) is the complex Hilbert space with the scalar product, \( \langle A | B \rangle = \text{Tr}A^*B \). We consider linear operators acting on it; they are called superoperators. The latter terminology is used to distinguish operators acting in the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{L}(\mathcal{H}) \). Otherwise superoperators are usual linear operators. In particular, for \( \mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \), there is well defined its adjoint operator \( \mathcal{T}^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \). However, some basic notions are specific for superoperators. A superoperator \( \mathcal{T} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) is called positive if it maps the set of positive semi-definite operators into itself. A superoperator is called completely positive if its natural extension \( \mathcal{T} \otimes \mathcal{I} \) to the tensor product \( \mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \) is again a positive superoperator on \( \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \).

(This notion is rather complicated technically. Therefore we shall not discuss it in more detail.)

Consider now a general measurement on the system \( S \). The statistical properties of any measurement are characterized by

- (i) the output probability distribution \( Pr\{x = x \| \rho\} \), the probability distribution of the output \( x \) of the measurement in the input state \( \rho \):
- (ii) the quantum state reduction \( \rho \mapsto \rho_{\{x=x\}} \), the state change from the input state \( \rho \) to the output state \( \rho_{\{x=x\}} \) conditional upon the outcome \( x = x \) of the measurement.

According to Davies–Lewis [15] and one of the present authors [17], the modern quantum measurement theory postulates that any measurement of the system \( S \) is described by a mathematical structure called a quantum instrument. This is any map \( x \to \mathcal{I}(x) \), where for each real \( x \), the map \( \mathcal{I}(x) \) is a completely positive superoperator satisfying the normalization condition \( \sum_x \text{Tr}[\mathcal{I}(x)\rho] = 1 \) for any state \( \rho \).
Davies–Lewis [15] originally postulated that the superoperator $\mathcal{I}(x)$ should be positive. However, Yuen [29] pointed out that the Davies–Lewis postulate is too general to exclude physically non-realizable instrument. One of the present authors [17] introduced complete positivity to ensure that every quantum instrument is physically realizable.

Given a quantum instrument $\mathcal{I}$, the output probability distribution for the input state $\rho$ is defined by the generalized Born rule in the trace-form,

$\text{Pr}\{x = x\|\rho\} := \text{Tr}[\mathcal{I}(x)\rho]$,

and the quantum state reduction is defined by

$\rho \mapsto \rho_{\{x=x\}} := \frac{\mathcal{I}(x)\rho}{\text{Tr}[\mathcal{I}(x)\rho]}$.

The above general formulation of quantum instruments reflects variety of real measuring instruments for the same system $S$ that measure an observable $A$ accurately or with some error. A quantum instrument $\mathcal{I}_A$ is called an instrument measuring an observable $A$, or an $A$-measuring instrument, if the output probability distribution satisfies Born’s rule (4) for the $A$-measurement, i.e,

$\text{Tr}[\mathcal{I}_A(x)\rho] = \text{Tr}[E^A(x)\rho]$.

The projective $A$-measuring instrument is defined by

$\mathcal{I}_A(x)\rho := E^A(x)\rho E^A(x)$

for any state $\rho$ and real number $x$. Then, this instrument satisfies not only the Born formula (4), but also the projection postulate (6). Thus, the projection postulate is no longer the requirement for the measurement of the observable $A$ but only one type of the measurement of $A$.

2.4. Quantum order effect from the quantum instrument viewpoint. Finally, we make the probabilistic comment on quantum instruments. Typically the main attention is given to Born’s rule (13) as generating probabilities from quantum states, $\rho \mapsto \text{Pr}\{A = x\|\rho\}$. We would like to elevate the role of the state transform (12), measurement back-action. It generates quantum probability conditioning. If after measurement of observable $A$ with outcome $A = x$, one measures observable $B$, then the probability to get output $B = y$ is given by Born’s rule for state $\rho_x$ :

$\text{Pr}\{B = y\|A = x\|\rho\} = \frac{\text{Tr}[\mathcal{I}_B(y)\rho_{\{A=x\}}]}{\text{Tr}[\mathcal{I}_A(x)]} = \frac{\text{Tr}[\mathcal{I}_B(y)\mathcal{I}_A(x)\rho]}{\text{Tr}[\mathcal{I}_A(x)]}$.

However, it is not clear whether the condition of complete positivity is important for mental measurements. So, may be the original Davies–Lewis instruments are of some value for the quantum cognition project.
Now, by using quantum conditional probability we can define the sequential joint probability distribution of $A$ (first) and $B$ (last),

$$\text{Pr}\{A = x, B = y|\rho\} = \text{Pr}\{A = x|\rho\} \text{Pr}\{B = y|A = x|\rho\} = \text{Tr} [I_B(y)I_A(x)\rho].$$

It is clear that if superoperators $I_A(x)$ and $I_B(y)$ do not commute, i.e.,

$$[I_A(x), I_B(y)] = I_A(x)I_B(y) - I_B(y)I_A(x) \neq 0,$$

then generally $\text{Pr}\{A = x, B = y|\rho\} \neq \text{Pr}\{B = y, A = x|\rho\}$. This is the mathematical representation of QOE.

In the framework of the quantum instrument theory, one has to distinguish noncommutativity of Hermitian operators symbolically representing quantum observables and noncommutativity of instruments. The standard noncommutativity condition

$$[A, B] \neq 0,$$

describes incompatibility of observables, in the sense that their JPD is not well defined, JPD for their joint measurement. So, condition (18) is related to joint measurements, not to sequential measurement and not to QOE. The latter is characterized not via nonexistence of JPD, but via noncommutativity of the state updates, formalized via condition (17).

2.5. **Quantum instruments as representation of indirect measurements.** The basic model for construction of quantum instruments is based on the scheme of indirect measurements. This scheme formalizes the following situation. As was permanently emphasized by Bohr (one of the founders of quantum mechanics), the results of quantum measurements are generated in the process of interaction of a system $S$ with a measurement apparatus $M$. This apparatus consists of a complex physical device interacting with $S$ and a pointer that shows the result of measurement, say spin up or spin down. An observer can see only outputs of the pointer and he associates these outputs with the values of the observable $A$ for the system $S$. So, the observer approaches only the pointer, not the system by itself. Whether the outputs of the pointer can be associated with the “intrinsic properties” of $S$ or not is one of the main problems of quantum foundations, it is still the topic for hot foundational debates [19]-[25]. Thus, the indirect measurement scheme involves:

- the states of the systems $S$ and the apparatus $M$;
- the operator $U$ representing the interaction-dynamics for the system $S + M$;
- the meter observable $M_A$ giving outputs of the pointer of the apparatus $M$.

We shall make the following remark on the operator $U$ of the interaction-dynamics. As all operations in quantum mechanics, it is a linear operator. In quantum formalism, dynamics of the state of an isolated system is described by the Schrödinger equation and
its evolution operator is unitary. In the indirect measurement scheme, it is assume that (approximately) the compound system $S + M$ is isolated. Hence, its evolution operator $U$ is unitary.

Formally, an **indirect measurement model**, introduced in [17] as a “(general) measuring process”, is a quadruple

$$(K, \sigma, U, M_A)$$

consisting of a Hilbert space $K$, a density operator $\sigma \in S(K)$, a unitary operator $U$ on the tensor product of the state spaces of $S$ and $M$, $U : H \otimes K \to H \otimes K$, and a self-adjoint operator $M_A$ on $K$. By this measurement model, the Hilbert space $K$ describes the states of the apparatus $M$, the unitary operator $U$ describes the time-evolution of the composite system $S + M$, the density operator $\sigma$ describes the initial state of the apparatus $M$, and the Hermitian operator $M_A$ describes the meter observable of the apparatus $M$. Then, the output probability distribution $Pr\{A = x\| \rho\}$ in the system state $\rho \in S(H)$ is given by

$$(19) \quad Pr\{A = x\| \rho\} = Tr[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^*],$$

where $E^{M_A}(x)$ is the spectral projection of $M_A$ for the eigenvalue $x$.

The change of the state $\rho$ of the system $S$ caused by the measurement for the outcome $A = x$ is represented with the aid of the map $I_A(x)$ in the space of density operators defined as

$$(20) \quad I_A(x)\rho = Tr_K[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^*],$$

where $Tr_K$ is the partial trace over $K$. Then, the map $x \mapsto I_A(x)$ turn out to be a quantum instrument. Thus, the statistical properties of the measurement realized by any indirect measurement model $(K, \sigma, U, M_A)$ is described by a quantum measurement. We remark that conversely any quantum instrument can be represented via the indirect measurement model [17]. Thus, quantum instruments mathematically characterize the statistical properties of all the physically realizable quantum measurements.

Now, we point to a few details which were omitted in the above considerations. The measuring interaction between the system $S$ and the apparatus $M$ turns on at time $t_0$, the time of measurement, and turns off at time $t = t_0 + \Delta t$. We assume that the system $S$ and the apparatus $M$ do not interact each other before $t_0$ nor after $t = t_0 + \Delta t$ and that the compound system $S + M$ is isolated in the time interval $(t_0, t)$. The **probe system** $P$ is defined to be the minimal part of apparatus $M$ such that the compound system $S + P$ is isolated in the time interval $(t_0, t)$. Then the above scheme is applied to the probe system $P$, instead of the whole apparatus $M$. The rest of the apparatus $M$ performs the pointer measurement on the probe $P$. In particular, the unitary evolution operator $U$ describing the state-evolution of the system $S + P$ has the form $H = e^{-i\Delta tH}$, where $H = H_S + H_P + H_{SP}$ is Hamiltonian of $S + P$ with the terms $H_S$ and $H_P$ representing the internal dynamics in the subsystems $S$ and $P$ of the compound system and $H_{SP}$ describing the interaction between the subsystems.
Introduction of probe systems may be seen as unnecessary complication of the scheme of indirect measurements. However, it is useful if the apparatus $M$ is a very complex system that interacts (often parallely) with many systems $S_j$, $j = 1, 2, \ldots, m$. Its different probes are involved solely in interaction with the concrete systems, $P_j$ with $S_j$. And the system $S_j + P_j$ can be considered as an isolated system; in particular, from interactions with other systems $S_i$ and probes $P_i$.

The indirect measurement scheme is part of theory of open quantum systems \[30\]. Instead of a measurement apparatus $M$, we can consider the surrounding environment $E$ of the system $S$ (see \[31\]-\[33\] for applications to psychology).

3. INDIRECT MEASUREMENT OF MENTAL OBSERVABLES: UNCONSCIOUSNESS AS A SYSTEM AND CONSCIOUSNESS AS A MEASUREMENT APPARATUS

The scheme of indirect measurements presented in the previous section was created in quantum physics. Our aim is to adapt it to mental measurements. The main question is about the cognitive analogs of the system $S$ and the measurement apparatus $M$. We suggest to use the framework that was developed in article \[34\] of one of the authors of this paper for quantum-like modeling of the Helmholtz sensation-perception theory. This scheme can be extended to the general scheme of unconscious-conscious interaction in the process of decision making.

The measured system $S$ is a sensation (or generally any state of unconsciousness). Consciousness as the measurement apparatus to unconsciousness interacts with sensations to make the decisions (to generate the outcomes of measurements). Consciousness is, of course, much larger than a single probe. It is a large environment with many probes interacting with unconsciousness. There should be a rule of transformation from a sensation to a conscious decision. This is unitary transformation and measurement of the meter in the probe. The operational description neglects the neuro-physiological and electrochemical structure of interaction. Unconsciousness is a black box that is mathematically described by the state of sensation space (= the unconscious state), so that the unconscious state probabilistically determines the decision (by interaction with consciousness), then the unconscious state is changed according to the previous unconscious state and the decision made. Thus, each probe is described by its quantum instrument. Instruments are probe dependent.

For the question-measurements, the question $A$ is transferred into unconsciousness, it plays the role of a sensation, so to say a high mental level sensation. Then, interaction described by the unitary operator $U$ generates a new state of the compound system - unconsciousness-consciousness. And consciousness performs the final “pointer reading”, the measurement of the meter observable. Pointer reading can be treated as generation of a perception, a high mental level perception.

We understand that appealing to the unconsciousness-consciousness description of cognitive processing is not so common in modern psychology. However, this description matches very well with the indirect measurement scheme. We can appeal to the authority of James \[35\] (appealing to Freud \[37\] might generate the negative reaction), see also
4. CONSTRUCTIONS OF INSTRUMENTS

4.1. Observables $A$ and $B$. We consider two questions $A$ and $B$ for a client to answer “yes” ($y$) or “no” ($n$) and consider the joint probability $p(AaBb)$ for obtaining the answer $a$ ($a = y$ or $n$) for the question $A$ and the answer $b$ ($b = y$ or $n$) for the question $B$, if the question $B$ is asked after the question $A$, and the analogous joint probability $p(BbAa)$ if the question $A$ is asked after the question $B$.

We model the above joint probability distributions by the joint probability distributions of outcomes of successive measurements of 2-valued observables $A$ and $B$ in a quantum system in a given state $\rho$ in the $A$–$B$ order and in the $B$–$A$ order. According to the general QMT, the joint probability distributions are well defined by two 2-valued observables (represented by projections) $A$, $B$ in a fixed Hilbert space $H$, a quantum state $\rho$, and an $A$-measuring instrument $I_A$ and a $B$-measuring instrument $I_B$ by the relations

\begin{align}
    p(AaBb) &= \Pr\{A = a, B = b||\rho\} = \Tr[I_B(b)I_A(a)\rho], \\
    p(BbAa) &= \Pr\{B = b, A = a||\rho\} = \Tr[I_A(a)I_B(b)\rho].
\end{align}

Let $A$ and $B$ be projections on a Hilbert space $\mathcal{H}$. They correspond to questions labeled $A$ and $B$, respectively. The eigenvalue 1 means the answer “yes” to the questions, and the eigenvalue 0 means the answer “no” to the questions. We consider the case where projections $A$ and $B$ commute. Let $C^2 = \{|0\rangle, |1\rangle\}^\perp\perp$ and $C^3 = \{|0\rangle, |1\rangle, |3\rangle\}$, where $^\perp\perp$ stands for the orthogonal complement in $\mathcal{H}$, so that $S^\perp\perp$ stands for the subspace spanned by a subset $S$ of $\mathcal{H}$. We let $\mathcal{H} = C^2 \otimes C^2 \otimes C^3$. Denote by $I_1$, $I_2$, and $I_3$ the identity operators on the first, second, and the third component of $\mathcal{H}$, respectively. We let $A = |1\rangle\langle 1| \otimes I_2 \otimes I_3$, and $B = I_1 \otimes |1\rangle\langle 1| \otimes I_3$.

4.2. Instrument $I_A$ measuring $A$. We construct an instrument $I_A$ measuring $A$ as follows. The instrument $I_A$ carries out a measurement of $A$ by a measuring interaction between the object $S$ described by the state space $\mathcal{H}$ and the probe $\tilde{S}$ described by the state space $K = C^2 \otimes C^2$, which is prepared in the state $|00\rangle$ just before the measuring interaction. Denote by $I_4$ and $I_5$ the identity operators on the first and second component of $K$. The time evolution of the composite system $S + \tilde{S}$ during the measuring interaction
is described by a unitary operator \( U_A \) on \( \mathcal{H} \otimes \mathcal{K} \) satisfying

\[
\begin{align*}
(23) \quad U_A : |000\rangle |00\rangle &\mapsto |000\rangle |00\rangle, \\
(24) \quad U_A : |010\rangle |00\rangle &\mapsto |010\rangle |01\rangle, \\
(25) \quad U_A : |100\rangle |00\rangle &\mapsto |100\rangle |10\rangle, \\
(26) \quad U_A : |110\rangle |00\rangle &\mapsto |110\rangle |11\rangle, \\
(27) \quad U_A : |001\rangle |00\rangle &\mapsto |011\rangle |00\rangle, \\
(28) \quad U_A : |011\rangle |00\rangle &\mapsto |011\rangle |01\rangle, \\
(29) \quad U_A : |101\rangle |00\rangle &\mapsto |101\rangle |10\rangle, \\
(30) \quad U_A : |111\rangle |00\rangle &\mapsto |101\rangle |11\rangle, \\
(31) \quad U_A : |002\rangle |00\rangle &\mapsto |002\rangle |00\rangle, \\
(32) \quad U_A : |012\rangle |00\rangle &\mapsto |002\rangle |01\rangle, \\
(33) \quad U_A : |102\rangle |00\rangle &\mapsto |112\rangle |10\rangle, \\
(34) \quad U_A : |112\rangle |00\rangle &\mapsto |112\rangle |11\rangle,
\end{align*}
\]

and the outcome of the measurement is obtained by measuring the meter observable

\[
(35) \quad M_A = |1\rangle \langle 1| \otimes I_5
\]

of the probe \( \tilde{S} \). Note that both \( A \) and \( M_A \) have the same spectrum \( \{0, 1\} \) and they are projections. We denote by \( I_\mathcal{H} \) and \( I_\mathcal{K} \) the identity operators on \( \mathcal{H} \) and \( \mathcal{K} \), respectively, and denote by \( M_A^\perp \) the orthogonal complement of the projection \( M_A \), i.e., \( M_A^\perp = I_\mathcal{K} - M_A \).

Suppose that the probe \( P \) is prepared in the state \( |00\rangle \) just before the measuring interaction, the measuring process described by the indirect measurement model \((\mathcal{K}, |00\rangle, U_A, M_A)\) defines the instrument \( \mathcal{I}_A \) by

\[
(36) \quad \mathcal{I}_A(a)\rho = \text{Tr}_\mathcal{K}[(I_\mathcal{H} \otimes P^{M_A}(a))U_A(\rho \otimes |00\rangle \langle 00|)U_A^\dagger(I_\mathcal{H} \otimes P^{M_A}(a))],
\]

for any density operator \( \rho \) on \( \mathcal{H} \), where \( \text{Tr}_\mathcal{K} \) stands for the partial trace over \( \mathcal{K} \) and \( P^{M_A}(a) \) denotes the spectral projection of an observable \( M_A \) for \( a \), i.e., \( P^{M_A}(0) = M_A^\perp \) and \( P^{M_A}(1) = M_A \). Consequently,

\[
\begin{align*}
\mathcal{I}_A(0)\rho &= \text{Tr}_\mathcal{K}[(I_\mathcal{H} \otimes M_A^\perp)U_A(\rho \otimes |00\rangle \langle 00|)U_A^\dagger(I_\mathcal{H} \otimes M_A^\perp)], \\
\mathcal{I}_A(1)\rho &= \text{Tr}_\mathcal{K}[(I_\mathcal{H} \otimes M_A)U_A(\rho \otimes |00\rangle \langle 00|)U_A^\dagger(I_\mathcal{H} \otimes M_A)],
\end{align*}
\]

for any \( \rho \).

The instrument \( \mathcal{I}_A \) determines the probability distribution \( \text{Pr}\{a = a|\rho\} \) of the outcome \( a \) of the measurement, where \( a = 0, 1 \), and the state change \( \rho \mapsto \rho_{\{a=a\}} \) caused
by the measurement is determined by

\[
Pr\{ a = a | \rho \} = \text{Tr}[I_A(a) \rho],
\]

\[
\rho \mapsto \rho_{(a = a)} = \frac{I_A(a) \rho}{\text{Tr}[I_A(a) \rho]}.
\]

Suppose that the object state \(|\psi\rangle\) just before the measurement is arbitrarily given, i.e.,

\[
|\psi\rangle = \sum_{\alpha, \beta, \gamma} c_{\alpha, \beta, \gamma} |\alpha, \beta, \gamma\rangle.
\]

Then, by the Born formula the probability distribution of the observable \(A\) is defined as

\[
Pr\{ A = 0 | |\psi\rangle \} = \| A_0 |\psi\rangle \|^2 = \sum_{\beta, \gamma} |c_{0, \beta, \gamma}|^2,
\]

\[
Pr\{ A = 1 | |\psi\rangle \} = \| A_1 |\psi\rangle \|^2 = \sum_{\beta, \gamma} |c_{1, \beta, \gamma}|^2.
\]

By linearity of \(U_A\), it follows from Eqs. (23)–(26) that

\[
\sum_{\alpha, \beta} (c_{\alpha, \beta, 0}|\alpha, \beta, 0\rangle + c_{\alpha, \beta, 1}|\alpha, \alpha, 0\rangle + c_{\alpha, \beta, 2}|\alpha, \alpha, 2\rangle)|\alpha, \beta\rangle.
\]

Then, we have

\[
(I_H \otimes P_{M_A}(a)) U_A |\psi\rangle |00\rangle
\]

for \(a = 0, 1\). Consequently,

\[
(I_H \otimes M_A) U_A |\psi\rangle |00\rangle = \sum_{\beta} (c_{0, \beta, 0}|0, \beta, 0\rangle + c_{0, \beta, 1}|0, 1, 1\rangle + c_{0, \beta, 2}|0, 0, 2\rangle)|0, \beta\rangle,
\]

\[
(I_H \otimes M_A) U_A |\psi\rangle |00\rangle = \sum_{\beta} (c_{1, \beta, 0}|1, \beta, 0\rangle + c_{1, \beta, 1}|1, 0, 1\rangle + c_{1, \beta, 2}|1, 1, 2\rangle)|1, \beta\rangle.
\]

The probabilities of obtaining the outcomes \(a = 0\) and \(a = 1\) are given by

\[
Pr\{ a = 0 | \rho \} = Pr\{ M_A = 0 | U_A |\psi\rangle |00\rangle \} = \| (I_H \otimes M_A^\dagger) U_A |\psi\rangle |00\rangle \|^2 = \sum_{\beta, \gamma} |c_{0, \beta, \gamma}|^2,
\]

\[
Pr\{ a = 1 | \rho \} = Pr\{ M_A = 1 | U_A |\psi\rangle |00\rangle \} = \| (I_H \otimes M_A) U_A |\psi\rangle |00\rangle \|^2 = \sum_{\beta, \gamma} |c_{1, \beta, \gamma}|^2.
\]

This shows

\[
Pr\{ a = 0 | \rho \} = Pr\{ A = 0 | |\psi\rangle \},
\]

\[
Pr\{ a = 1 | \rho \} = Pr\{ A = 1 | |\psi\rangle \}.
\]
for any state $|\psi\rangle$ in $\mathcal{H}$. It follows that the instrument $\mathcal{I}_A$ measures the observable $A$, i.e.,

$$\Pr\{a = 0|\rho\} = \text{Tr}[\mathcal{I}_A(0)\rho] = \text{Tr}[A^\perp\rho],$$

$$\Pr\{a = 1|\rho\} = \text{Tr}[\mathcal{I}_A(1)\rho] = \text{Tr}[A\rho],$$

for any density operator $\rho$ on $\mathcal{H}$.

From Eqs. (36) and (42) we have

$$\mathcal{I}_A(a)|\psi\rangle\langle\psi| = \sum_\beta |c_{a,\beta,0}|^2|a, \beta, 0\rangle\langle a, \beta, 0| + |c_{a,\beta,1}|^2|a, a^\perp, 1\rangle\langle a, a^\perp, 1| + |c_{a,\beta,2}|^2|a, a, 2\rangle\langle a, a, 2|$$

$$= \sum_\beta |a, \beta, 0\rangle\langle a, \beta, 0|\psi\langle a, \beta, 0| + |a, a^\perp, 0\rangle\langle a, a^\perp, 0|\psi\langle a, \beta, 1| + |a, a, 2\rangle\langle a, a, 2|\psi\langle a, \beta, 2|$$

for $a = 0, 1$. By linearity of $\mathcal{I}_A(a)$ we conclude

$$\mathcal{I}_A(a)\rho = \sum_\beta |a, \beta, 0\rangle\langle a, \beta, 0|\rho|a, \beta, 0\rangle\langle a, \beta, 0|$$

$$+ |a, a^\perp, 0\rangle\langle a, \beta, 1|\rho|a, \beta, 1\rangle\langle a, \beta, 1| + |a, a, 2\rangle\langle a, a, 2|\rho|a, \beta, 2\rangle\langle a, a, 2|.$$  

for any density operator $\rho$ on $\mathcal{H}$ and $a = 0, 1$.

4.3. **Instrument $\mathcal{I}_B$ measuring $B$.** The instrument $\mathcal{I}_B$ is constructed with the same probe system $P$ prepared in the state $|00\rangle$. The unitary operator $U_B$ on $\mathcal{H} \otimes \mathcal{K}$, describing the time evolution of the composite system $S + \tilde{S}$ during the measuring interaction, is supposed to satisfy

$$U_B : |000\rangle|00\rangle \mapsto |000\rangle|00\rangle,$$

$$U_B : |010\rangle|00\rangle \mapsto |010\rangle|01\rangle,$$

$$U_B : |100\rangle|00\rangle \mapsto |100\rangle|10\rangle,$$

$$U_B : |110\rangle|00\rangle \mapsto |110\rangle|11\rangle,$$

$$U_B : |001\rangle|00\rangle \mapsto |101\rangle|00\rangle,$$

$$U_B : |011\rangle|00\rangle \mapsto |011\rangle|01\rangle,$$

$$U_B : |101\rangle|00\rangle \mapsto |101\rangle|10\rangle,$$

$$U_B : |111\rangle|00\rangle \mapsto |011\rangle|11\rangle,$$

$$U_B : |002\rangle|00\rangle \mapsto |002\rangle|00\rangle,$$

$$U_B : |012\rangle|00\rangle \mapsto |112\rangle|01\rangle,$$

$$U_B : |102\rangle|00\rangle \mapsto |002\rangle|10\rangle,$$

$$U_B : |112\rangle|00\rangle \mapsto |112\rangle|11\rangle,$$
and the meter observable on $\mathcal{K}$ is given by

$$M_B = I_4 \otimes |1\rangle\langle 1|.$$  

Then the instrument $I_B$ of this measuring process $(\mathcal{K}, |00\rangle, U_B, M_B)$ is defined by

$$I_B(b)\rho = \text{Tr}_\mathcal{K}[\left(I_{\mathcal{H}} \otimes P^{M_B}(b)\right)U_B(\rho \otimes |00\rangle\langle 00|)U_B^\dagger\left(I_{\mathcal{H}} \otimes P^{M_B}(b)\right)]$$

for any density operator $\rho$ on $\mathcal{H}$ and $b = 0, 1$. Then we have

$$I_B(b)\rho = \sum_\alpha |\alpha, b, 0\rangle\langle \alpha, b, 0| \rho |\alpha, b, 0\rangle\langle \alpha, b, 0| + |b^\perp, b, 1\rangle\langle b^\perp, b, 1| \rho |b^\perp, b, 1\rangle\langle b^\perp, b, 1| + |b, b, 2\rangle\langle b, b, 2| \rho |b, b, 2\rangle\langle b, b, 2|.$$

for any density operator $\rho$ on $\mathcal{H}$ and for any $b = 0, 1$. Thus, the probability distribution of the outcome $b$ of the instrument $I_B$ is given by

$$\text{Pr}\{b = 0|\rho\} = \text{Tr}[I_B(0)\rho] = \text{Tr}[B^\perp\rho],$$

$$\text{Pr}\{b = 1|\rho\} = \text{Tr}[I_B(1)\rho] = \text{Tr}[B\rho]$$

for any density operator $\rho$ on $\mathcal{H}$, and hence the instrument $I_B$ measures the observable $B$.

5. MIND STATES

In modeling the Clinton-Gore experiment, we assume that the subject’s mind state is represented by the space $\Omega = \{0, 1\}^2 \times \{0, 1, 2\}$. Here we suppose that any individual subject has one of the mind states, $\omega \in \Omega$, and any statistical ensemble of the subjects is characterized by a probability distribution on $\Omega$. Thus, the statistical data under consideration should be explained solely by one of the probability distributions on $\Omega$. The mind state $\alpha$ represents the state in which the subject will answer “yes” for the question A if $\alpha = 1$ and answer “no” otherwise. The mind state $\beta$ represents the analogous state for the question B. Thus, the mind state $(\alpha, \beta)$ represents the subject’s prior belief as long as the questions A and B are concerned. For $(\alpha, \beta, \gamma) \in \Omega$ we write $\delta_{(\alpha, \beta, \gamma)} = |\alpha, \beta, \gamma\rangle\langle \alpha, \beta, \gamma| \in \mathcal{H}$. In this way, we identify the mind state $(\alpha, \beta, \gamma) \in \Omega$ with the quantum state $|\alpha, \beta, \gamma\rangle\langle \alpha, \beta, \gamma| \in \mathcal{H}$, and any probability distribution $\mu$ of the mind states $\omega \in \Omega$ with the quantum state $\hat{\mu} = \sum_\omega \mu(\omega)\delta_\omega$. The dynamics of producing the answer to the question and preparing the mind state for the next question is supposed to be described as a process of quantum measurement, or equivalently a mathematical object called a quantum instrument, which we believe is the most flexible framework consistent with the psycho-physical parallelism.

Our previously defined two quantum instruments $I_A$ and $I_B$ describe the measurement on the mind state as follows.
Theorem 5.1. For any \((\alpha, \beta, \gamma) \in \Omega\) and \(a, b = 0, 1\), we have

\[
\begin{align*}
\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)} &= \delta_{\alpha}(a)\delta_{(\alpha,\beta,0)}, \\
\mathcal{I}_A(a)\delta_{(\alpha,\beta,1)} &= \delta_{\alpha}(a)\delta_{(\alpha,\beta,1)}, \\
\mathcal{I}_A(a)\delta_{(\alpha,\beta,2)} &= \delta_{\alpha}(a)\delta_{(\alpha,\alpha,2)}, \\
\mathcal{I}_B(b)\delta_{(\alpha,\beta,0)} &= \delta_{\beta}(b)\delta_{(\alpha,\beta,0)}, \\
\mathcal{I}_B(b)\delta_{(\alpha,\beta,1)} &= \delta_{\beta}(b)\delta_{(\beta,\beta,1)}, \\
\mathcal{I}_B(b)\delta_{(\alpha,\beta,2)} &= \delta_{\beta}(b)\delta_{(\beta,\beta,2)}.
\end{align*}
\]

Proof. The relations follow from Eq. (43) and Eq. (58) by routine computations. \(\square\)

The mind state \(\gamma\) represents the personality of the subject in such a way that if \(\gamma = 1\), the subject changes his/her mind to prepare the answer for the other question to be the opposite to the previous answer, and if \(\gamma = 2\), the subject changes his/her mind to prepare the answer for the other question to be the same as the previous answer; on the other hand, if \(\gamma = 0\), the subject’s mind is so robust that the question will not affect his/her mind.

Thus, the state change is the Bayesian update if \(\gamma\) has only the value \(\gamma = 0\), and yet if the value \(\gamma = 1\) or \(\gamma = 2\) is allowed, the statistics of the answers does not follows the Bayesian update rule.

From Eq. (43) we have

\[
\mathcal{I}_A(0)\alpha, \beta, \gamma\langle \alpha', \beta', \gamma'| = \sum_{\beta''} |0, \beta'', 0\rangle\langle 0, \beta'', 0|\alpha, \beta, \gamma\langle \alpha', \beta', \gamma'|0, \beta'', 0\rangle\langle 0, \beta'', 0| \\
+ \text{Tr}[(A^\dagger \otimes |1\rangle\langle 1|)[\alpha, \beta, \gamma] \langle \alpha', \beta', \gamma'|010\rangle\langle 010| \\
+ \text{Tr}[(A^\dagger \otimes |2\rangle\langle 2|)[\alpha, \beta, \gamma] \langle \alpha', \beta', \gamma'|002\rangle\langle 002| \\
= |0, \beta, 0\rangle\langle 0, \beta, 0|\alpha, \beta, \gamma\langle \alpha', \beta', \gamma'|0, \beta, 0\rangle\langle 0, \beta, 0| \\
+ \langle \alpha', \beta', \gamma'|0, 0\rangle I_2\otimes |1\rangle\langle 1|\alpha, \beta, \gamma\langle 010|\langle 010| \\
+ \langle \alpha', \beta', \gamma'|0, 0\rangle I_2\otimes |2\rangle\langle 2|\alpha, \beta, \gamma\langle 002|\langle 002| \\
= \delta_{\alpha}(0)\delta_{\alpha'}(0)\delta_{\beta}(\beta')\delta_{\gamma}(0)(0)\delta_{\gamma'}(0)[0, \beta, 0\rangle\langle 0, \beta, 0| \\
+ \delta_{\alpha}(0)\delta_{\alpha'}(0)\delta_{\beta}(\beta')\delta_{\gamma}(1)\delta_{\gamma'}(1)[010\rangle\langle 010| \\
+ \delta_{\alpha}(0)\delta_{\alpha'}(0)\delta_{\beta}(\beta')\delta_{\gamma}(2)\delta_{\gamma'}(2)[002\rangle\langle 002|.
\]

It follows that if \(\langle \alpha, \beta, \gamma|\alpha', \beta', \gamma'\rangle = 0\) then

\[
\mathcal{I}_A(0)\langle \alpha, \beta, \gamma|\alpha', \beta', \gamma'| = 0.
\]

Similarly, if \(\langle \alpha, \beta, \gamma|\alpha', \beta', \gamma'\rangle = 0\) then

\[
\mathcal{I}_A(a)\langle \alpha, \beta, \gamma|\alpha', \beta', \gamma'| = 0,
\]

\[
\mathcal{I}_B(b)\langle \alpha, \beta, \gamma|\alpha', \beta', \gamma'| = 0
\]
for any $a, b$. Therefore, for any density operator $\rho$ we have

$$I_A(a)\rho = \sum_{\alpha, \beta, \gamma} \mu(\alpha, \beta, \gamma) I_A(a) \delta_{(\alpha, \beta, \gamma)} = I_A(a)\rho',$$

$$I_B(b)\rho = \sum_{\alpha, \beta, \gamma} \mu(\alpha, \beta, \gamma) I_B(b) \delta_{(\alpha, \beta, \gamma)} = I_B(b)\rho',$$

where $\mu(\alpha, \beta, \gamma) = \langle \alpha, \beta, \gamma | \rho | \alpha, \beta, \gamma \rangle$, and

$$\rho' = \sum_{\alpha, \beta, \gamma} \mu(\alpha, \beta, \gamma) \delta_{(\alpha, \beta, \gamma)}.$$

6. Response Replicability Effect (RRE)

Here, we consider two properties of successive measurements of observables $A$ and $B$.

In A-A and A-B-A paradigms, the response to $A$ is repeated (with probability 1). We call this response replicability effect (RRE).

In $A-B$ vs. $B-A$ paradigm, the joint probabilities of the two responses are different on a set of states with a positive Legesgue measure. This is question order effect (QOE).

The psychological problem raised by the Clinton-Gore experiment is as follows. We are given two observables $A$ and $B$ whose joint probability distributions (JPDs) of successive measurements shows QOE and is naturally considered to satisfy RRE. We cannot represent the JPDs as the JPDs of classical random variables $A$ and $B$ defined by Kolmogorov, which does not show QOE, nor the JPDs of the outcomes of successive measurements of non-commuting quantum observables $A$ and $B$ defined by von Neumann and Lüders, which does not satisfy RRE.

We shall show that instruments $I_A$ and $I_B$ have both Response Replicability Effect (RRE) in this section, and Question Order Effect (QOE) in the next section.

**Theorem 6.1.** The instruments $I_A$ and $I_B$ have the Response Replicability Effect; namely, they satisfy the following relations.

(i) $\sum_a \text{Tr}[I_A(a)I_A(a)\rho] = 1$ for any density operator $\rho$.

(ii) $\sum_b \text{Tr}[I_B(b)I_B(b)\rho] = 1$ for any density operator $\rho$.

(iii) $\sum_{a,b} \text{Tr}[I_A(a)I_B(b)I_A(a)\rho] = 1$ for any density operator $\rho$.

(iv) $\sum_{a,b} \text{Tr}[I_B(b)I_A(a)I_B(b)\rho] = 1$ for any density operator $\rho$. 
Proof. From Eqs. (65) and (66) we assume without any loss of generality that \( \rho = \delta_{(\alpha,\beta,\gamma)} \) for some \( \alpha, \beta, \gamma \). For any \( \alpha, \beta \), we have

\[
\sum_a \text{Tr}[\mathcal{I}_A(a)\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] = \sum_a \delta_{\alpha}(a)\text{Tr}[\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] = \sum_a \delta_{\alpha}(a)\text{Tr}[\delta_{(\alpha,\beta,0)}] = \text{Tr}[\delta_{(\alpha,\beta,0)}] = 1.
\]

\[
\sum_a \text{Tr}[\mathcal{I}_A(a)\mathcal{I}_A(a)\delta_{(\alpha,\beta,1)}] = \sum_a \delta_{\alpha}(a)\text{Tr}[\mathcal{I}_A(a)\delta_{(\alpha,\beta,1)}] = \sum_a d\delta_{\alpha}(a)\text{Tr}[\delta_{(\alpha,\beta,1)}] = \text{Tr}[\delta_{(\alpha,\beta,1)}] = 1.
\]

\[
\sum_a \text{Tr}[\mathcal{I}_A(a)\mathcal{I}_A(a)\delta_{(\alpha,\beta,2)}] = \sum_a \text{Tr}[\delta_{\alpha}(a)\mathcal{I}_A(a)\delta_{(\alpha,\beta,2)}] = \sum_a \text{Tr}[\delta_{\alpha}(a)\delta_{(\alpha,\beta,2)}] = \text{Tr}[\delta_{(\alpha,\beta,2)}] = 1.
\]

Thus, relation (i) follows, and relation (ii) follows similarly. For any \( \alpha, \beta \), we have

\[
\sum_{a,b} \text{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] = \sum_{a,b} \delta_{\alpha}(a)\text{Tr}[\mathcal{I}_A(a)\mathcal{I}_B(b)\delta_{(\alpha,\beta,0)}] = \sum_{a,b} \delta_{\alpha}(a)\delta_{\beta}(b)\text{Tr}[\mathcal{I}_A(a)\delta_{(\alpha,\beta,0)}] = \sum_{a,b} \delta_{\alpha}(a)\delta_{\beta}(b)\delta_{\alpha}(a)\text{Tr}[\delta_{(\alpha,\beta,0)}] = \text{Tr}[\delta_{(\alpha,\beta,0)}] = 1.
\]
$$\sum_{a,b} \text{Tr}[I_A(a)I_B(b)I_A(a)\delta_{(\alpha,\beta,1)}] = \sum_{a,b} \delta_\alpha(a) \text{Tr}[I_A(a)I_B(b)\delta_{(\alpha,\alpha^+,1)}]$$
$$= \sum_{a,b} \delta_\alpha(a) \delta_\beta(b) \text{Tr}[I_A(a)\delta_{(\alpha,\alpha^+,1)}]$$
$$= \sum_{a,b} \delta_\alpha(a) \delta_\beta(b) \delta_\alpha(a) \text{Tr}[\delta_{(\alpha,\alpha^+,1)}]$$
$$= \text{Tr}[\delta_{(\alpha,\alpha^+,1)}]$$
$$= 1,$$

$$\sum_{a,b} \text{Tr}[I_A(a)I_B(b)I_A(a)\delta_{(\alpha,\beta,2)}] = \sum_{a,b} \delta_\alpha(a) \text{Tr}[I_A(a)I_B(b)\delta_{(\alpha,\alpha,2)}]$$
$$= \sum_{a,b} \delta_\alpha(a) \delta_\beta(b) \text{Tr}[I_A(a)\delta_{(\alpha,\alpha,2)}]$$
$$= \sum_{a,b} \delta_\alpha(a) \delta_\beta(b) \delta_\alpha(a) \text{Tr}[\delta_{(\alpha,\alpha^+,2)}]$$
$$= \text{Tr}[\delta_{(\alpha,\alpha^+,2)}]$$
$$= 1.$$

Thus, relation (iii) follows, and relation (iv) follows similarly. \(\square\)

7. QUESTION ORDER EFFECT (QOE)

Let \(\hat{\mu} = \sum_{\alpha,\beta,\gamma} \mu(\alpha, \beta, \gamma) \delta_{(\alpha,\beta,\gamma)}\), we write

\[
\begin{align*}
\text{(67)} & \quad p(AaBb) = \text{Tr}[I_B(b)I_A(a)\hat{\mu}], \\
\text{(68)} & \quad p(BbAa) = \text{Tr}[I_A(a)I_B(b)\hat{\mu}],
\end{align*}
\]

where \(a, b = 0, 1\). We will write \(Ay, An, By, Bn\) instead of \(A1, A0, B0, B1\). we have

\[
p(AaBb) \\
= \sum_{\alpha,\beta,\gamma} \mu(\alpha, \beta, \gamma) p(AaBb|\delta_{(\alpha,\beta,\gamma)}) \\
= \sum_{\alpha,\beta} \mu(\alpha, \beta, 0) \delta_\alpha(a) \delta_\beta(b) + \mu(\alpha, \beta, 1) \delta_\alpha(a) \delta_{\alpha^+}(b) + \mu(\alpha, \beta, 2) \delta_\alpha(a) \delta_\alpha(b) \\
= \mu(a, b, 0) + \delta_{\alpha^+}(b) \sum_\beta \mu(a, \beta, 1) + \delta_\alpha(b) \sum_\beta \mu(a, \beta, 2).
\]
and

\[ p(BbAa) = \sum_{\alpha, \beta, \gamma} \mu(\alpha, \beta, \gamma)p(BbAa|\delta_{(\alpha, \beta, \gamma)}) \]

\[ = \sum_{\alpha, \beta} \mu(\alpha, \beta, 0)\delta_{a}(\alpha)\delta_{b}(\beta) + \mu(\alpha, \beta, 1)\delta_{\beta}(a)\delta_{\beta}(b) + \mu(\alpha, \beta, 2)\delta_{\beta}(a)\delta_{\beta}(b) \]

\[ = \mu(a, b, 0) + \delta_{b}(a) \sum_{\alpha} \mu(\alpha, b, 1) + \delta_{b}(a) \sum_{\alpha} \mu(\alpha, b, 2). \]

Thus,

\[ p(AyBy) = \mu(1, 1, 0) + \mu(1, 1, 2) + \mu(1, 0, 2). \]
\[ p(AyBn) = \mu(1, 0, 0) + \mu(1, 0, 1) + \mu(1, 1, 1). \]
\[ p(AnBy) = \mu(0, 1, 0) + \mu(0, 1, 1) + \mu(0, 0, 1). \]
\[ p(AnBn) = \mu(0, 0, 0) + \mu(0, 0, 2) + \mu(0, 1, 2). \]
\[ p(ByAy) = \mu(1, 1, 0) + \mu(1, 1, 2) + \mu(0, 1, 2). \]
\[ p(ByAn) = \mu(0, 1, 0) + \mu(0, 1, 1) + \mu(1, 1, 1). \]
\[ p(ByAn) = \mu(0, 0, 0) + \mu(0, 0, 2) + \mu(1, 0, 2). \]

**Theorem 7.1.** The instruments \( I_A \) and \( I_B \) have the Question Order Effect; namely, we have the following statements. Let \( \rho \) be a density operator on \( \mathcal{H} \).

(i) The relation

\[ \text{Tr}[I_B(1)I_A(1)\rho] = \text{Tr}[I_A(1)I_B(1)\rho] \]

holds if and only if \( \langle 1, 0, 2|\rho|1, 0, 2 \rangle = \langle 0, 1, 2|\rho|0, 1, 2 \rangle. \)

(ii) The relation

\[ \text{Tr}[I_B(0)I_A(1)\rho] = \text{Tr}[I_A(1)I_B(0)\rho] \]

holds if and only if \( \langle 1, 1, 1|\rho|1, 1, 1 \rangle = \langle 0, 0, 1|\rho|0, 0, 1 \rangle. \)

(iii) The relation

\[ \text{Tr}[I_B(1)I_A(0)\rho] = \text{Tr}[I_A(0)I_B(1)\rho] \]

holds if and only if \( \langle 1, 1, 1|\rho|1, 1, 1 \rangle = \langle 0, 0, 1|\rho|0, 0, 1 \rangle. \)

(iv) The relation

\[ \text{Tr}[I_B(0)I_A(0)\rho] = \text{Tr}[I_A(0)I_B(0)\rho] \]

holds if and only if \( \langle 1, 0, 2|\rho|1, 0, 2 \rangle = \langle 0, 1, 2|\rho|0, 1, 2 \rangle. \)

One of Eqs. (77), (78), (79), (80) for a general density operator \( \rho \) holds only on a set of density operators \( \rho \) with Lebesgue measure 0.
Proof. Eq. (77) holds if and only if $p(AyBy) = p(ByAy)$. Thus, assertion (i) follows from Eqs. (65), (66), (69), and (73). Assertions (ii)–(iv) follow similarly. The last assertion follows from the fact that each relation holds on a submanifold of the space of density operators with co-dimension 1.  □

8. QQ-EQUALITY

If two questions, adjacent to each other, are asked in different orders, then the quantum model of measurement order makes an a priori and parameter-free prediction, named the QQ equality [2, 3, 9]:

$$q = [p(ByAy) + p(BnAn)] - [p(AyBy) + p(AnBn)]$$

$$= [p(AyBn) + p(AnBy)] - [p(ByAn) + p(BnAy)] = 0.$$  

Note that the formula in [9, P. 9435, left column] is not correct in that the first and second lines are not equal but have the opposite signs; the correct definition here is due to [3]. We have

$$p(AyBy) - p(ByAy) = \mu(1, 0, 2) - \mu(0, 1, 2),$$
$$p(AnBn) - p(BnAn) = \mu(0, 1, 2) - \mu(1, 0, 2),$$
$$p(AyBn) - p(BnAy) = \mu(1, 1, 1) - \mu(0, 0, 1),$$
$$p(AnBy) - p(ByAn) = \mu(0, 0, 1) - \mu(1, 1, 1).$$

Thus, it is easy to check that the QQ equality holds and we have

**Theorem 8.1.** The instruments $\mathcal{I}_A$ and $\mathcal{I}_B$ satisfy the QQ equality.

9. QQE-RENORMALIZATIONS

Given the joint probabilities $p(AyBy), p(AyBn), p(AnBn), p(ByAy), p(ByAn), p(BnAy)$, and $p(BnAn)$, let

$$S_1 = \frac{p(AyBy) + p(AnBn) + p(ByAy) + p(BnAn)}{2},$$
$$S_2 = \frac{p(AyBn) + p(AnBy) + p(ByAn) + p(BnAy)}{2}.$$
Their QQE-renormalizations \( \bar{p}(AyBy), \bar{p}(AyBn), \bar{p}(AnBn), \bar{p}(ByAy), \bar{p}(ByAn), \bar{p}(BnAy), \) and \( \bar{p}(BnAn) \), are defined as follows.

\[
\begin{align*}
\bar{p}(AyBy) &= S_1 \times \frac{p(AyBy)}{p(AyBy) + p(AnBn)}, \\
\bar{p}(AnBn) &= S_1 \times \frac{p(AnBn)}{p(AyBy) + p(AnBn)}, \\
\bar{p}(ByAy) &= S_1 \times \frac{p(ByAy)}{p(ByAy) + p(BnAn)}, \\
\bar{p}(BnAn) &= S_1 \times \frac{p(BnAn)}{p(ByAy) + p(BnAn)}, \\
\bar{p}(AyBn) &= S_2 \times \frac{p(AyBn)}{p(AyBn) + p(AnBy)}, \\
\bar{p}(AnBy) &= S_2 \times \frac{p(AnBy)}{p(AyBn) + p(AnBy)}, \\
\bar{p}(ByAn) &= S_2 \times \frac{p(ByAn)}{p(ByAn) + p(BnAy)}, \\
\bar{p}(BnAy) &= S_2 \times \frac{p(BnAy)}{p(ByAn) + p(BnAy)}.
\end{align*}
\]

**Theorem 9.1.** The following statements hold.

(i) The QQE-renormalizations \( \bar{p}(AaBb), \bar{p}(BaAb) \) with \( a, b = y, n \) define joint probability distributions, i.e., \( 0 \leq \bar{p}(AaBb), \bar{p}(BaAb) \leq 1 \) for all \( a, b = y, n \), \( \sum_{a,b=y,n} \bar{p}(AaBb) = \sum_{a,b=y,n} \bar{p}(BaAb) = 1 \).

(ii) The joint probability distributions \( \{ \bar{p}(AaBb), \bar{p}(BaAb) \mid a, b = y, n \} \) satisfy the QQE, i.e.,

\[
q = [\bar{p}(ByAy) + \bar{p}(BnAn)] - [\bar{p}(AyBy) + \bar{p}(AnBn)] \\
= [\bar{p}(AyBn) + \bar{p}(AnBy)] - [\bar{p}(ByAn) + \bar{p}(BnAy)] = 0.
\]

(iii) If the joint probability distributions \( p(AaBb), p(BaAb) \) with \( a, b = y, n \) satisfy the QQE then \( \bar{p}(AaBb) = p(AaBb) \) and \( \bar{p}(BaAb) = p(BaAb) \) for all \( a, b = y, n \).

(iv) The following relations hold.

\[
\begin{align*}
(81) \quad \bar{p}(AyBy) + \bar{p}(AnBn) + \bar{p}(ByAy) + \bar{p}(BnAn) &= p(AyBy) + p(AnBn) + p(ByAy) + p(BnAn), \\
(82) \quad \bar{p}(AyBn) + \bar{p}(AnBy) + \bar{p}(ByAn) + \bar{p}(BnAy) &= p(AyBn) + p(AnBy) + p(ByAn) + p(BnAy).
\end{align*}
\]
(v) The following relations hold.

\[
\begin{align*}
\bar{p}(AyBy) - p(AyBy) &= \frac{q}{2[p(AyBy) + p(AnBn)]} = \frac{q}{2S_1 - q}, \\
\bar{p}(AnBn) - p(AnBn) &= \frac{q}{2[p(AyBy) + p(AnBn)]} = \frac{q}{2S_1 - q}, \\
\bar{p}(ByAy) - p(ByAy) &= \frac{-q}{2[p(ByAy) + p(BnAn)]} = \frac{-q}{2S_1 + q}, \\
\bar{p}(BnAn) - (BnAn) &= \frac{-q}{2[p(ByAy) + p(BnAn)]} = \frac{-q}{2S_1 + q}, \\
\bar{p}(AyBn) - p(AyBn) &= \frac{-q}{2[p(AyBn) + p(AnBn)]} = \frac{-q}{2S_2 + q}, \\
\bar{p}(AnBn) - p(AnBn) &= \frac{-q}{2[p(AyBn) + p(AnBn)]} = \frac{-q}{2S_2 + q}, \\
\bar{p}(ByAn) - p(ByAn) &= \frac{q}{2[p(ByAn) + p(BnAy)]} = \frac{q}{2S_2 - q}, \\
\bar{p}(BnAy) - p(BnAy) &= \frac{q}{2[p(ByAn) + p(BnAy)]} = \frac{q}{2S_2 + q}.
\end{align*}
\]

\textbf{Proof.} The assertions can be verified straightforward calculations. \hfill \Box
10. Independence of Personality State

In the previous sections, we have not assumed that in the probability distribution \( \mu \) of the mind state, the personality state is independent from the belief state. Since the personality state should determine only the way of changing the belief state, it would be more desirable to describe it to be independent from the belief state \((\alpha, \beta)\).

Here, we assume the independence of the personality state. Thus, we suppose

\[
\mu(\alpha, \beta, \gamma) = p(\alpha, \beta)q(\gamma),
\]

\[
p(\alpha, \beta) = \sum_{\gamma} \mu(\alpha, \beta, \gamma),
\]

\[
q(\gamma) = \sum_{\alpha, \beta} \mu(\alpha, \beta, \gamma).
\]

Then, Eqs. (69)–(76) are written as follows.

\[
p(AyBy) = p(1, 1)q(0) + p(1, 1)q(2) + p(1, 0)q(2).
\]

\[
p(AyBn) = p(1, 0)q(0) + p(1, 0)q(1) + p(1, 1)q(1).
\]

\[
p(AnBy) = p(0, 1)q(0) + p(0, 1)q(1) + p(0, 0)q(1).
\]

\[
p(AnBn) = p(0, 0)q(0) + p(0, 0)q(2) + p(0, 1)q(2).
\]

\[
p(ByAy) = p(1, 1)q(0) + p(1, 1)q(2) + p(0, 1)q(2).
\]

\[
p(BnAy) = p(1, 0)q(0) + p(1, 0)q(1) + p(0, 0)q(1).
\]

\[
p(ByAn) = p(0, 1)q(0) + p(0, 1)q(1) + p(1, 1)q(1).
\]

\[
p(BnAn) = p(0, 0)q(0) + p(0, 0)q(2) + p(1, 0)q(2).
\]

Let \( p(Ay) = p(AyBy) + p(AyBn) \), etc. Then, \( p(Ay) = p(1, 1) + p(1, 0) \), etc. We have

\[
p(AyBy) = p(1, 1)q(0) + p(Ay)q(2).
\]

\[
p(AyBn) = p(1, 0)q(0) + p(Ay)q(1).
\]

\[
p(AnBy) = p(0, 1)q(0) + p(An)q(1).
\]

\[
p(AnBn) = p(0, 0)q(0) + p(An)q(2).
\]

\[
p(ByAy) = p(1, 1)q(0) + p(By)q(2).
\]

\[
p(BnAy) = p(1, 0)q(0) + p(Bn)q(1).
\]

\[
p(ByAn) = p(0, 1)q(0) + p(By)q(1).
\]

\[
p(BnAn) = p(0, 0)q(0) + p(Bn)q(2).
\]
Thus, if \( p(A_y) \neq p(B_y) \) and \( p(A_y) \neq p(B_n) \), we can determine \( q(0), q(1), q(2) \) by the experimental data:

\[
q(2) = \frac{p(A_yB_y) - p(B_yA_y)}{p(A_y) - p(B_y)} ,
\]
\[
q(1) = \frac{p(A_yB_n) - p(B_nA_y)}{p(A_y) - p(B_n)} ,
\]
\[
q(1) = \frac{p(A_nB_y) - p(B_yA_n)}{p(A_n) - p(B_y)} ,
\]
\[
q(2) = \frac{p(A_nB_n) - p(B_nA_n)}{p(A_n) - p(B_n)} .
\]

Note that since the output data \( \{p(A_aB_b), p(B_bA_a) | a, b = y, n\} \) obtained from our model satisfy the QQ equality, the above relations are consistent, so that \( q(1) \) and \( q(2) \) are uniquely determined from the experimental data and then we determine \( q(0) \) by \( q(0) + q(1) + q(2) = 1 \).

If we obtain the personality state \( q(0), q(1), q(2) \), the brief state is determined by the following relations.

\[
p(1, 1) = \frac{\bar{p}(A_yB_y) - \bar{p}(A_y)q(2)}{q(0)} ,
\]
\[
p(1, 0) = \frac{\bar{p}(A_yB_n) - \bar{p}(A_y)q(1)}{q(0)} ,
\]
\[
p(0, 1) = \frac{\bar{p}(A_nB_y) - \bar{p}(A_n)q(1)}{q(0)} ,
\]
\[
p(0, 0) = \frac{\bar{p}(A_nB_n) - \bar{p}(A_n)q(2)}{q(0)} ,
\]
\[
p(1, 1) = \frac{\bar{p}(B_yA_y) - \bar{p}(B_y)q(2)}{q(0)} ,
\]
\[
p(0, 1) = \frac{\bar{p}(B_yA_n) - \bar{p}(B_y)q(1)}{q(0)} ,
\]
\[
p(1, 0) = \frac{\bar{p}(B_nA_y) - \bar{p}(B_n)q(1)}{q(0)} ,
\]
\[
p(0, 0) = \frac{\bar{p}(B_nA_n) - \bar{p}(B_n)q(2)}{q(0)} .
\]
11. Clinton-Gore poll

Consider the following data from Clinton-Gore experiment \[3,4,9\].

\[(130)\] \( p(AyBy) = 0.4899, \)
\[(131)\] \( p(AyBn) = 0.0447, \)
\[(132)\] \( p(AnBy) = 0.1767, \)
\[(133)\] \( p(AnBn) = 0.2887, \)
\[(134)\] \( p(ByAy) = 0.5625, \)
\[(135)\] \( p(ByAn) = 0.1991, \)
\[(136)\] \( p(BnAy) = 0.0255, \)
\[(137)\] \( p(BnAn) = 0.2129. \)

We have

\[p(ByAy) - p(AyBy) = 0.5625 - 0.4899 = 0.0726,\]
\[p(AnBn) - p(BnAn) = 0.2887 - 0.2129 = 0.0758,\]
\[p(AyBn) - p(BnAy) = 0.0447 - 0.0255 = 0.0192,\]
\[p(ByAn) - p(AnBy) = 0.1991 - 0.1767 = 0.0224.\]

Thus, the QQ equality is approximately satisfied with good accuracy.

\[q = \left[p(ByAy) + p(BnAn)\right] - \left[p(AyBy) + p(AnBn)\right] = \left[p(AyBn) + p(AnBy)\right] - \left[p(ByAn) + p(BnAy)\right] = -0.0032.\]

Thus, their QQE-renormalization \(\bar{p}(Aa,Bb),\bar{p}(Bb,Aa)\) are expected to approximate the original data \(p(Aa,Bb), p(Bb,Aa)\) with good accuracy.

For the Clinton-Gore poll, we have

\[S_1 = \frac{p(AyBy) + p(AnBn) + p(ByAy) + p(BnAn)}{2} = \frac{0.4899 + 0.2887 + 0.5625 + 0.2129}{2} = 0.777,\]
\[S_2 = \frac{p(AyBn) + p(AnBy) + p(ByAn) + p(BnAy)}{2} = \frac{0.0447 + 0.1767 + 0.1991 + 0.0255}{2} = 0.223,\]
and we obtain their QQE-renormalizations as follows.

\[
\tilde{p}(AyBy) = S_1 \times \frac{p(AyBy)}{p(AyBy) + p(AnBn)} = 0.777 \times \frac{0.4899}{0.4899 + 0.2887} = 0.4889. \\
E = (0.48889/0.4899) - 1 = -0.00206.(-0.21%).
\]

\[
\tilde{p}(AnBn) = S_1 \times \frac{p(AnBn)}{p(AyBy) + p(AnBn)} = 0.777 \times \frac{0.2887}{0.4899 + 0.2887} = 0.2881. \\
E = (0.28810/0.2887) - 1 = -0.00207.(-0.21%).
\]

\[
\tilde{p}(ByAy) = S_1 \times \frac{p(ByAy)}{p(ByAy) + p(BnAn)} = 0.777 \times \frac{0.5625}{0.5625 + 0.2129} = 0.5636, \\
E = (0.56366/0.5625) - 1 = 0.00206.(+0.21%).
\]

\[
\tilde{p}(BnAn) = S_1 \times \frac{p(BnAn)}{p(ByAy) + p(BnAn)} = 0.777 \times \frac{0.2129}{0.5625 + 0.2129} = 0.2133, \\
E = (0.21333/0.2129) - 1 = 0.00201.(+0.20%).
\]

\[
\tilde{p}(AyBn) = S_2 \times \frac{p(AyBn)}{p(AyBn) + p(AnBn)} = 0.223 \times \frac{0.0447}{0.0447 + 0.1767} = 0.0450, \\
E = (0.04502/0.0447) - 1 = 0.00715.(+0.72%).
\]

\[
\tilde{p}(AnBy) = S_2 \times \frac{p(AnBn)}{p(AyBn) + p(AnBn)} = 0.223 \times \frac{0.1767}{0.0447 + 0.1767} = 0.1779, \\
E = (0.17797/0.1767) - 1 = 0.00718.(+0.72%).
\]

\[
\tilde{p}(ByAn) = S_2 \times \frac{p(ByAn)}{p(ByAn) + p(BnAy)} = 0.223 \times \frac{0.1991}{0.1991 + 0.0255} = 0.1976, \\
E = (0.19768/0.1991) - 1 = -0.00713.(-0.71%).
\]

\[
\tilde{p}(BnAy) = S_2 \times \frac{p(BnAy)}{p(ByAn) + p(BnAy)} = 0.223 \times \frac{0.0255}{0.1991 + 0.0255} = 0.0253, \\
E = (0.02531/0.0255) - 1 = -0.00745.(-0.75%).
\]

We have

\[
\tilde{p}(Ay) = \tilde{p}(AyBy) + \tilde{p}(AyBn) = 0.48889 + 0.04502 = 0.53391, \\
\tilde{p}(An) = \tilde{p}(AnBy) + \tilde{p}(AnBn) = 0.17797 + 0.28810 = 0.46607, \\
\tilde{p}(By) = \tilde{p}(ByAy) + \tilde{p}(ByAn) = 0.56366 + 0.19768 = 0.76134, \\
\tilde{p}(Bn) = \tilde{p}(BnAy) + \tilde{p}(BnAn) = 0.02531 + 0.21333 = 0.23864.
\]
Under the assumption of the independence of the personality state, we can determine $q(0), q(1), q(2)$ by the QQE-normalized data.

$$q(2) = \frac{\bar{p}(AyBy) - \bar{p}(ByAy)}{\bar{p}(Ay) - \bar{p}(By)} = \frac{0.48889 - 0.56366}{0.53391 - 0.76134} = 0.32876,$$

$$q(1) = \frac{\bar{p}(AyBn) - \bar{p}(BnAy)}{\bar{p}(Ay) - \bar{p}(Bn)} = \frac{0.04502 - 0.02531}{0.53391 - 0.23864} = 0.06675,$$

$$q(1) = \frac{\bar{p}(AnBy) - \bar{p}(ByAn)}{\bar{p}(An) - \bar{p}(By)} = \frac{0.17797 - 0.19768}{0.46607 - 0.76134} = 0.06675,$$

$$q(2) = \frac{\bar{p}(AnBn) - \bar{p}(BnAn)}{\bar{p}(An) - \bar{p}(Bn)} = \frac{0.28810 - 0.21333}{0.46607 - 0.23864} = 0.32876.$$

We determine $q(0)$ by $q(0) + q(1) + q(2) = 1$, so that we obtain

$$q(2) = 0.32876,$$

$$q(1) = 0.06675,$$

$$q(0) = 0.60449.$$

We can determine $\bar{p}(\alpha, \beta)$ for $\alpha, \beta = 0, 1$ by Eqs. (110)–(117) as follows.

$$\bar{p}(1, 1) = \frac{\bar{p}(AyBy) - \bar{p}(Ay)q(2)}{q(0)} = 0.51839,$$

$$\bar{p}(1, 0) = \frac{\bar{p}(AyBn) - \bar{p}(Ay)q(1)}{q(0)} = 0.01551,$$

$$\bar{p}(0, 1) = \frac{\bar{p}(AnBy) - \bar{p}(An)q(1)}{q(0)} = 0.24294,$$

$$\bar{p}(0, 0) = \frac{\bar{p}(AnBn) - \bar{p}(An)q(2)}{q(0)} = 0.22312,$$

$$\bar{p}(1, 1) = \frac{\bar{p}(ByAy) - \bar{p}(By)q(2)}{q(0)} = 0.51839,$$

$$\bar{p}(0, 1) = \frac{\bar{p}(ByAn) - \bar{p}(By)q(1)}{q(0)} = 0.24294,$$

$$\bar{p}(1, 0) = \frac{\bar{p}(BnAy) - \bar{p}(Bn)q(1)}{q(0)} = 0.01551,$$

$$\bar{p}(0, 0) = \frac{\bar{p}(BnAn) - \bar{p}(Bn)q(2)}{q(0)} = 0.22312.$$
Thus, we have

\begin{align}
(141) & & \bar{p}(1, 1) = 0.5183_9, \\
(142) & & \bar{p}(1, 0) = 0.0155_1, \\
(143) & & \bar{p}(0, 1) = 0.2429_4, \\
(144) & & \bar{p}(0, 0) = 0.2231_2, \\
(145) & & \end{align}

Note that the unity of the total probability is satisfied.

\[ \bar{p}(1, 1) + \bar{p}(1, 0) + \bar{p}(0, 1) + \bar{p}(0, 0) = 0.9999_6. \]

Therefore, the belief state \( \bar{p}(\alpha, \beta) \) and the personality state \( q(\gamma) \) are determined from the experimental data. Then, our quantum model with the belief state \( \bar{p}(\alpha, \beta) \) and the personality state \( q(\gamma) \) accurately reconstructs the QQE-renormalized data \( \bar{p}(Aa, Bb), \bar{p}(Bb, Aa) \) for \( a, b = y, n \) as follows.

\[
\begin{align*}
\bar{p}(AyBy) &= \bar{p}(1, 1)q(0) + [\bar{p}(1, 1) + \bar{p}(1, 0)]q(2) \\
&= 0.51839 \times 0.60449 + (0.51839 + 0.01551) \times 0.32876 = 0.4888_8, \\
\bar{p}(AyBn) &= \bar{p}(1, 0)q(0) + [\bar{p}(1, 1) + \bar{p}(1, 0)]q(1) \\
&= 0.01551 \times 0.60449 + (0.51839 + 0.01551) \times 0.06675 = 0.0450_1, \\
\bar{p}(AnBy) &= \bar{p}(0, 1)q(0) + [\bar{p}(0, 1) + \bar{p}(0, 0)]q(1) \\
&= 0.24294 \times 0.60449 + (0.24294 + 0.22312) \times 0.06675 = 0.1779_6, \\
\bar{p}(AnBn) &= \bar{p}(0, 0)q(0) + [\bar{p}(0, 1) + \bar{p}(0, 0)]q(2) \\
&= 0.22312 \times 0.60449 + (0.24294 + 0.22312) \times 0.32876 = 0.2880_9, \\
\bar{p}(BnAy) &= \bar{p}(1, 1)q(0) + [\bar{p}(0, 1) + \bar{p}(1, 1)]q(2) \\
&= 0.51839 \times 0.60449 + (0.24294 + 0.51839) \times 0.32876 = 0.5636_5, \\
\bar{p}(BnAn) &= \bar{p}(0, 1)q(0) + [\bar{p}(0, 1) + \bar{p}(1, 1)]q(1) \\
&= 0.24294 \times 0.60449 + (0.24294 + 0.51839) \times 0.06675 = 0.1976_7, \\
\bar{p}(BnAy) &= \bar{p}(1, 0)q(0) + [\bar{p}(0, 0) + \bar{p}(1, 0)]q(1) \\
&= 0.01551 \times 0.60449 + (0.22312 + 0.01551) \times 0.06675 = 0.0253_0, \\
\bar{p}(BnAn) &= \bar{p}(0, 0)q(0) + [\bar{p}(0, 0) + \bar{p}(1, 0)]q(2) \\
&= 0.22312 \times 0.60449 + (0.22312 + 0.01551) \times 0.32876 = 0.2133_2.
\end{align*}
\]

Therefore, all data of the QQR-renomalizations are accurately reproduced, and we conclude that the quantum model reproduces the statistics of the Clinton-Gore Poll data almost faithfully with a prior belief state \( \{\bar{p}(0, 0), \ldots, \bar{p}(1, 1)\} \) independent of the question order. Thus, this model successfully removes the order effect from the data to determine the genuine distribution of the opinion to the Poll.
We restrict considerations to POVMs with a discrete domain of definition $X = \{x_1, \ldots, x_N\}$. POVM is a map $x \rightarrow D^A(x)$ Here, for each $x \in X, D^A(x)$ is a positive Hermitian operator (called an effect), and the normalization condition

$$\sum_x D^A(x) = I$$

holds, where $I$ is the unit operator. This map defines an operator valued measure on the algebra of all subsets of $X$, for $O \subset X, D^A(O) = \sum_{x \in O} D^A(x)$. The condition (146) characterizes “probabilistic operator valued measures”.

POVMs represent statistics of measurements of quantum observables with the following generalization of the Born’s rule:

$$\Pr\{A = x \parallel \rho\} = \text{Tr}[\hat{D}^A(x)\rho].$$

We remark that equality (146) implies that $\sum_x \Pr\{A = x \parallel \rho\} = 1$.

POVM does not represent state update. The latter is typically determined (non-uniquely) via representation of effects in the form:

$$D^A(x) = V(x)V(x)^*,$$

where $V(x)$ is a linear operator in $H$. Hence, the normalization condition has the form $\sum_x V(x)V(x)^* = I$. From (3), we see that the map $x \rightarrow \hat{E}^A(x)$ is a special sort of POVM, the projector valued measure - PVM. The Born rule can be written similarly to (4):

$$\Pr\{A = x \parallel \rho\} = \text{Tr}[V(x)\rho V^*(x)].$$

It is assumed that the post-measurement state transformation is based on the map:

$$\rho \rightarrow \mathcal{I}_A(x)\rho = V(x)\rho V^*(x),$$

so

$$\rho \rightarrow \rho_x = \frac{\mathcal{I}_A(x)\rho}{\text{Tr}\mathcal{I}_A(x)\rho}. \tag{150}$$

Now, we remark that the map $x \rightarrow \mathcal{I}_A(x)$ given by (149) is a (very special) quantum instrument. We would like to elevate the role of the use of quantum instruments, comparing with the use of just POVMs. An instrument provides both statistics of the measurement-outputs and the rule for the state update, but POVM should always be endowed with ad hoc condition (147).

Finally, we remark that any instrument generates POVM by the rule:

$$D^A(x) = \mathcal{I}(x)^*I. \tag{151}$$

However, its state update need not have the form (149).
According to the Kraus theorem \[40\] for any $A$-measuring instrument $\mathcal{I}_A$, there exists a family $\{M_{xj}\}_{x,j}$ of operators, called the measurement operators for $\mathcal{I}_A$, in $H$ such that

$$\mathcal{I}_A(x)\rho = \sum_j M_{xj}\rho M_{xj}^\dagger$$

(152)

for any state $\rho$. Thus it determines POVM

$$D^A(x) = \mathcal{I}(x)^* I = \sum_j M_{xj}^\dagger M_{xj}.$$  

(153)

Hence, each instrument determines POVM uniquely, but not vice versa.

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