Elastic torsion of cylindrically anisotropic nano/microtubes is examined by Saint-Venant approach. It is assumed that the tubes were obtained by rolling the plates of the cubic crystals with plane orientation (011). The analytical expression for the torsional stiffness of such nano/microtubes is obtained. It is found, that the torsional stiffness is dependent on the three compliance coefficients of cubic crystal: thickness parameter, chirality angle and radius of the tube. Numerical analysis of the torsional stiffness of nano/microtubes is given. It was shown that for most of crystals the dimensionless ratio of torsional stiffness to torsional stiffness at zero chiral angle of nano/microtubes slightly varies with the thickness parameter of tubes. Materials with the substantial change in the dimensionless ratio of torsional stiffnesses are found. It is shown, that the torsion of chiral nano/microtubes obtained from cubic crystal plates in the absence of tensile forces is accompanied by linear Poynting's effect. Comparative analysis of the dimensionless ratios of torsional stiffness to torsional stiffness at zero chirality angle for nano/microtubes obtained by rolling the crystal planes (001) and (011) is given. It is shown that the variability of torsional stiffness for nano/microtubes obtained by rolling the crystal planes (011) is much higher than for nano/microtubes produced by rolling the crystal planes (001). Comparative analysis of linear Poynting's effect for nano/microtubes created by rolling the crystal planes (001) and (011) is also presented.

Keywords: nanotubes, microtubes, torsion, anisotropy, Poynting's effect.
1. Introduction

Many nano- and mesomaterials were synthesized in the form of nanotubes and microtubes. Nano/microtubes were obtained, for example, by rolling thin crystalline plates into tubes [1–7]. This method makes it possible to obtain nano- and microtubes from almost all crystalline materials.

To describe mechanical properties of nano- and microtubes in the framework of continuum mechanics it is often used a model of a cylindrical shell (see, e.g., the reviews [8,9]). A model of a hollow cylindrical rod was proposed in [10–12]. This model is acceptable for describing the carbon and non-carbon nano/microtubes with curvilinear cubic, hexagonal, rhombohedral and tetragonal anisotropy. It has been shown by the example of the problem on extension the nano/microtubes obtained by rolling the crystal planes (001), that such tubes can have negative Poisson’s ratio [10–12] and manifest linear Poynting’s effect [13]. Below we will consider the problem on torsion nano/microtubes, obtained by rolling the cubic crystal planes (011), in the framework of the model of a hollow cylindrical rod.

2. From rectilinear-anisotropic cubic crystals to cylindrically anisotropic nano/microtubes

The method of rolling thin crystal plates is one of the most effective methods of nanotubes and microtubes fabrication from single crystals. The elastic properties of the produced tubes can be described in terms of the elasticity theory, if the thicknesses of their walls are significantly higher than the atomic and interatomic distances, i.e. several nanometers [14].

We assume that the crystal plate with orientation (011) is rolled into a cylindrical tube so that the axis which is perpendicular to the plane (011) passes into the radial axis of the tube. Other crystal axis which is rotated by an angle $\chi$ relative to the crystallographic axis in the plate corresponds to the longitudinal axis of the tube (see Fig. 6 in [12]). The symmetry of formed curvilinear anisotropic tube is reduced relative to the initial cubic symmetry of the crystal due to rotation to the specified chiral angle. Its symmetry corresponds to the symmetry of the monoclinic system with thirteen compliance coefficients $s'_{ij}$, $s'_{ij} = s_{ij} - s_{ii} \xi^2 - 0.5 s_{ii} \xi \chi$, $s'_{13} = s_{13} - 0.5 \Delta$, $s'_{23} = s_{23} + 1.5 s_{ii} \xi \chi$, $s'_{33} = s_{33} + 0.5 s_{ii} \chi^2$, $s'_{44} = s_{44} + 2 \Delta s_{ii} \chi$, $s'_{55} = s_{55} + 2 \Delta s_{ii} \chi$, $s'_{66} = s_{66} + \Delta s_{ii} \chi$, $s'_{16} = s'_{15} + s'_{14} s'_{13} - s'_{15} s'_{12}$, $s'_{26} = s'_{25} + s'_{24} s'_{23} - s'_{25} s'_{22}$, $s'_{36} = s'_{35} + s'_{34} s'_{33} - s'_{35} s'_{32}$, $s'_{46} = s'_{45} + s'_{44} s'_{44} - s'_{45} s'_{43}$, $s'_{56} = s'_{55} + s'_{54} s'_{54} - s'_{55} s'_{53}$, $s'_{66} = s'_{65} + s'_{64} s'_{64} - s'_{65} s'_{63}$, depending on the chiral angle $\chi$ and the initial three cubic crystal compliances $s_{ij}$ as follows

$$s'_{11} = s_{11} - 2 \Delta \sin^2 \chi \cos^2 \chi - 0.5 \Delta \sin^2 \chi, \quad s'_{22} = s_{22} - 2 \Delta \sin^2 \chi \cos^2 \chi - 0.5 \Delta \sin^2 \chi, \quad s'_{33} = s_{33} - 0.5 \Delta, \quad s'_{12} = s_{12} + 1.5 \Delta \sin^2 \chi \cos^2 \chi, \quad s'_{13} = s_{13} + 0.5 \Delta \sin^2 \chi, \quad s'_{23} = s_{23} + 0.5 \Delta \sin^2 \chi, \quad s'_{44} = s_{44} + 2 \Delta s_{ii} \chi, \quad s'_{55} = s_{55} + 2 \Delta s_{ii} \chi, \quad s'_{66} = s_{66} + \Delta s_{ii} \chi, \quad s'_{16} = s'_{15} + s'_{14} s'_{13} - s'_{15} s'_{12}, \quad s'_{26} = s'_{25} + s'_{24} s'_{23} - s'_{25} s'_{22}, \quad s'_{36} = s'_{35} + s'_{34} s'_{33} - s'_{35} s'_{32}, \quad$$

The dependences of the coefficients $s'_{ij}$ on chiral angle are periodic with period $\pi$.

Curvilinear-anisotropic elasticity of nano/microtubes thus characterized by linear equations of Hooke’s law

$$u_{zz} = s'_{zz} \sigma_{zz} + s'_{zz} \sigma_{yy} + s'_{zz} \sigma_{rr} - s'_{zz} \sigma_{zz}, \quad u_{rr} = s'_{rr} \sigma_{zz} + s'_{rr} \sigma_{yy} + s'_{rr} \sigma_{rr} - s'_{rr} \sigma_{zz}, \quad u_{yz} = s'_{yz} \sigma_{zz} + s'_{yz} \sigma_{yy} + s'_{yz} \sigma_{rr} - s'_{yzz} \sigma_{zz}, \quad u_{yz} = s'_{yz} \sigma_{zz} + s'_{yz} \sigma_{yy} + s'_{yz} \sigma_{rr} - s'_{yzz} \sigma_{zz}, \quad 2u_{zz} = s'_{zz} \sigma_{zz} + s'_{zz} \sigma_{yy} + s'_{zz} \sigma_{rr} - s'_{zz} \sigma_{zz}, \quad 2u_{rr} = s'_{rr} \sigma_{zz} + s'_{rr} \sigma_{yy} + s'_{rr} \sigma_{rr} - s'_{rr} \sigma_{zz}. \quad (2)$$

3. Torsion of nano/microtubes obtained by rolling the crystal planes (011)

We will consider the problem on torsion of curvilinear-anisotropic nano/microtubes, obtained by rolling the crystal planes (011) of a cubic crystal. Let the integral boundary conditions

$$P_z = \int \sigma_{zz} dS = 0, \quad M_z = \int \sigma_{zz} r dS \quad (3)$$

are satisfied at the ends of tubes ($P_z, M_z$ are the total tensile force and torsion moment, respectively). Tension is absent and full torque is given. Local conditions of the absence of stress on the sides of the hollow tubes, i.e. on the inner surface $r = r_0$ and outer surface $r = R_0 = r_0 \phi$, are assumed.

Later we will assume that there is an axial symmetric radial-inhomogeneous stress state $\sigma_{zz}(r), \sigma_{yy}(r), \sigma_{rr}(r), \sigma_{rr}(r), \sigma_{zz}(r), \sigma_{zz}(r)$. Then the equilibrium equations are simplified

$$\sigma_{zz}(r) = \frac{d}{dr} (r \sigma_{zz}(r)), \quad \frac{d}{dr} (r \sigma_{zz}(r)) = 0, \quad \frac{d}{dr} (r^2 \sigma_{zz}(r)) = 0. \quad (5)$$

The last two equations (5) and the zero boundary conditions on the side surfaces of the tube lead to $\sigma_{zz}(r) = \sigma_{zz}(0) = 0$. This in turn leads to a simplification of the equations of Hooke’s law, which will contain now three normal stresses $\sigma_{zz}(r), \sigma_{zz}(r), \sigma_{zz}(r)$, and one shear stress $\sigma_{zz}(r)$. The deformations are also radially inhomogeneous according to these equations. They allow to obtain unambiguous displacements if the following constraints are satisfied

$$u_{zz}(r) = \varepsilon, \quad u_{zz}(r) = \frac{d}{dr} (r \varepsilon_{zz}(r)), \quad 2u_{zz}(r) = \varepsilon \tau r. \quad (6)$$

The condition (6), together with the equilibrium equation (4) and the equations of Hooke’s law allow us to express all stresses through the one component $\sigma_{zz}(r)$ that satisfies the differential equation of the second order

$$\frac{d}{dr} \left( r \frac{d}{dr} (r \sigma_{zz}(r)) \right) = a_0 \sigma_{zz}(r) + a_1 (1 - a_0) \varepsilon + a_2 (4 - a_0) \varepsilon \tau r, \quad (7)$$

where

$$a_0 = \frac{t_{11} - t_{13}}{t_{11} - t_{12}}, \quad a_1 = \frac{t_{11} - t_{12}}{t_{11} - t_{13}}, \quad a_2 = \frac{t_{11} - 2t_{12} + 2t_{13} - t_{13}}{t_{11} t_{12} - t_{13}}, \quad t_{13} = \frac{1}{s_{16} t_{16} (t_{14} - t_{15})}, \quad t_{14} = \frac{4 t_{16} t_{13} - 2t_{15}}{t_{11} t_{12} - t_{13} \frac{1}{4}}, \quad t_{15} = \frac{t_{11} + 4t_{16} t_{13} - 2t_{15} t_{14}}{t_{11} t_{12} - t_{13}}. \quad (8)$$

The solution of this equation for the stress component $\sigma_{zz}(r)$
has the powerlike form
\[ \sigma_\varepsilon(r) = a_\varepsilon \epsilon + a_\tau \tau r_\text{t}_0 \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]  
(9)

The remaining stress components have a similar structure. Using the equilibrium equations (4), (5), Hooke's law (2), and the additional conditions (6), we find
\[ \sigma_{\text{ext}}(r) = a_\varepsilon \epsilon + 2a_\tau \tau r_\text{t}_0 \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]
\[ t_1 \sigma_{\text{ext}}(r) = [1 - a_1(t_{12} + t_{13})] \epsilon + \frac{a_2}{s_\text{e}} a_2 \left(t_{23} - a_2(2t_{12} + t_{13}) \right) r_\text{t}_0 \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]
\[ - \sum t_1 + t_3 (1 + \lambda_\varepsilon) A_i \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]
\[ s_{\text{lat}} \sigma_{\text{ext}}(r) = \frac{s_{\text{lat}}}{t_{11} + a_{1} \left(s_{23} + s_{36} - a_2 t_{12} + t_{13} \right) r_\text{t}_0 \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]
\[ + \sum s_{\text{lat}} - s_{\text{lat}} t_{13} + s_{\text{lat}} t_{13} + s_{\text{lat}} t_{13} + (1 + \lambda_\varepsilon) A_i \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]

The obtained representations of stress components contain four parameters \( \epsilon, \tau, A_i, A \). The boundary conditions of the stress absence on the inner and outer walls of the cylindrical tubes allow us to express \( A_i, A \) through \( \epsilon, \tau \) as follows
\[ A_i = a_1 \left( \rho^3 - \rho \right) - a_2 \rho^3 + A_1 \rho^3 - A_2 \rho^3 \tau r_\text{t}_0 \]
\[ A = a_1 \left( \rho^3 - \rho \right) - a_2 \rho^3 + A_1 \rho^3 - A_2 \rho^3 \tau r_\text{t}_0 \]

(11)

Here, the thickness parameter \( \rho = R_0/r_\text{t} \) is the ratio of the outer radius to the inner radius of the tube.

The integration of \( \sigma(r) \) and \( \sigma(r)r \) on the tube cross-sectional area allows one to fulfill the integral conditions (2). We ultimately obtain relationship between the parameters \( \epsilon, \tau \), and the dependence of torsional stiffness \( C = M/r \) on the tube radius, ratio of internal and external radii \( \rho = R_0/r_\text{t} \), compliances \( s_{\text{lat}} \) and chiral angle \( \chi \) (see expressions \( s_{\text{lat}}, t_{13} \) though \( s_{\text{lat}}, \chi \) in (1), (8))
\[ \varepsilon = \Gamma \tau r_\text{t}_0 \]
\[ C = \frac{2\pi r_\text{t}_0}{s_{\text{lat}}} \left[ \frac{s_{\text{lat}}}{t_{11} + a_1 \left(s_{23} + s_{36} - a_2 t_{12} + t_{13} \right) r_\text{t}_0 \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]
\[ + \sum s_{\text{lat}} - s_{\text{lat}} t_{13} + s_{\text{lat}} t_{13} + s_{\text{lat}} t_{13} + (1 + \lambda_\varepsilon) \left( \frac{r}{r_\text{t}_0} \right)^{ \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon} }, \quad \lambda_\varepsilon = -1 \pm \sqrt{\delta_\varepsilon}. \]
\[ \Omega = \frac{a_1 \left( \rho^3 - \rho \right) + a_2 \left( \rho^3 - \rho \right)}{\rho^3 - \rho}, \quad \Gamma = \frac{\Omega}{\Gamma_1}, \quad \Gamma_1 = \frac{a_1 \left( \rho^3 - \rho \right) - a_2 \left( \rho^3 - \rho \right)}{\rho^3 - \rho}, \quad \Gamma = \frac{\Omega}{\Gamma_1} - \frac{\rho^3 - \rho}{\rho^3 - \rho}, \quad \Omega = \frac{\rho^3 - \rho}{\rho^3 - \rho}, \quad \Gamma = \frac{\Omega}{\Gamma_1} - \frac{\rho^3 - \rho}{\rho^3 - \rho}. \]

At zero chiral angle the result for torsional stiffness takes very simple form
\[ C_0 = C_{t \rightarrow 0} = \frac{\pi}{2s_{\text{lat}}} \rho^4 (\rho^4 - 1) \]

The dimensionless ratio \( C/C_0 \) depends on the dimensionless combinations of coefficient compliances, dimensionless scaling parameter \( \rho \) and chiral angle \( \chi \). Numerical analysis of torsion was based on experimental data for the cubic crystals from handbook [15] was performed for auxetic nano/microtubes from Table 11 in [12]. The dimensionless ratio \( C/C_0 \) is greater than one for the majority of the nano/microtubes. However, it could be less than one for thin-walled tubes from Yb, TmSe and Tm_90Se. Examples of the dependencies of this ratio from \( \rho \) and \( \chi \) for nano/microtubes of cubic crystals Tm_90Se and Cu are shown in Fig. 1. Let us note that the anisotropy coefficient \( \Delta = s_{11} - s_{12} - 0.5s_{44} \) is negative (−14.26) for Tm_90Se and positive (+14.65) for Cu. It was found that the ratio \( C/C_0 \) increases slightly with increasing thickness parameter. The growth of \( C/C_{0} \) is most noticeable during the angular surroundings of \( \chi = 5\pi/16 \) and \( \chi = 11\pi/16 \).

Received higher connection of parameters \( \epsilon, \tau \) reflects the effect of occurrence of longitudinal strain of tubes in their torsion, Poynting's effect. The effect, experimentally observed originally by Poynting, was nonlinear [16]. Here, the connection of \( \epsilon \) with \( \tau \) is linear due to the linearity of Hooke's law. The dimensionless proportionality coefficient \( \Gamma = \gamma \) (see expressions (1), (8)) on the relative thickness of the walls of the tubes (\( \rho < 1 \)), dimensionless combinations of three compliance coefficients \( s_{11}, s_{12}, s_{44} \) and chiral angle \( \chi \). The dependence of the effect on chiral angle is odd, and in particular, disappears when \( \chi = 0 \), and when \( \chi = \pi/2 \). It can be seen from the above formulas and Fig. 2. The oscillating character of linear Poynting's effect can also be seen from this figure.

It should be noted that the differences exist between the nano/microtubes obtained by rolling the crystal planes (001) and (011). Comparisons are given for the dimensionless ratio of torsional stiffness \( C/C_0 \) and the dimensionless coefficients \( C_0 \) for nano/microtubes obtained by rolling the crystal planes (001) and (011) of cubic crystals Tm_90Se (a) and Cu (b).
of this dimensionless ratio showed that in most cases the external and internal radii, dimensionless combinations of torsional stiffness and internal radii at zero chiral angle depends on the dimensionless ratio of the torsional stiffness to the difference between the fourth power of the external radii, and internal radii at zero chiral angle. Torsional stiffness, rolling the crystal planes (011) of cubic crystals is proportional to the difference between the fourth power of the external radii, and internal radii at zero chiral angle. Table 11 [12], shows similar behavior of coefficient $\Gamma$ for the nano/microtubes from other cubic crystals, presented in Fig. 3b. The analysis carried out for nano/microtubes from other cubic crystals, presented in Table 11 [12], shows similar behavior of coefficient $\Gamma$ for the tubes fabricated by rolling the crystal planes (001) and (011), depending on the sign of the anisotropy coefficient.

3. Conclusion

Torsional stiffness of chiral nano/microtubes fabricated by rolling the crystal planes (011) of cubic crystals is proportional to the difference between the fourth power of the external and internal radii at zero chiral angle. Torsional stiffness, made dimensionless by dividing on the torsional stiffness at zero chirality depends on the dimensionless ratio of the external and internal radii, dimensionless combinations of elastic compliances and chiral angle. Numerical analysis of this dimensionless ratio showed that in most cases the dimensionless ratio of torsional stiffness slightly increases with increasing the thickness parameter of tubes at a fixed chiral angle. This ratio can increase substantially at a chiral angle close to $5\pi/16$ or $11\pi/16$. Torsion of chiral nano/microtubes of cubic crystals is accompanied by changes in their length, even in the absence of tensile forces. Such Poynting’s effect disappears in special cases of chiral angles equal to $0$ and $\pi/2$. Nano/microtubes lengthening or shortening under torsion can occur at different values of the chiral angle.

It is shown that the variability of torsional stiffness for nano/microtubes obtained by rolling the crystal planes (011), is much higher than for nano/microtubes produced by rolling the crystal planes (001). In the case thin-walled nano/microtubes from cubic crystals with a positive anisotropy coefficient, torsion in the case (001) planes can be accompanied by a stronger elongation/shortening than for the nano/microtubes in the case of planes (011). The opposite situation occurs for nano/microtubes from cubic crystals with a negative coefficient of anisotropy.

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