On the spectra of generalized Fibonacci and Fibonacci-like operators

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January 19, 2013

Abstract

We analyze the spectra of generalized Fibonacci and Fibonacci-like operators in Banach space $l^1$. Some of the results have application in population dynamics.

1 Introduction and preliminaries

Let $l^1$ denote the Banach space of all real sequences $x = (x_1, x_2, x_3, \cdots)$ such that $\|x\|_1 = \sum |x_k| < \infty$. Let $H : l^1 \to l^1$ be a linear operator on $l^1$. The resolvent set of $H$, $\rho(H)$ is the set of all complex numbers $\lambda$ such that the operator $\lambda I - H$ has a bounded inverse, where $I : l^1 \to l^1$ is the identity operator. The set $\sigma(H) = \mathbb{C} \setminus \rho(H)$ is the spectrum of $H$. The spectrum is further subdivided into three mutually disjoint parts, the point spectrum $\sigma_p(H)$, the continuous spectrum $\sigma_c(H)$ and the residual spectrum $\sigma_r(H)$. The point spectrum is the set of all $\lambda \in \mathbb{C}$ such that $\lambda I - H$ has no inverse. As in the finite dimensional case, such $\lambda$ are also called eigenvalues and the corresponding non-zero vectors $x \in l^1$, such that $(\lambda I - H)x = 0$ are called eigenvectors. The continuous spectrum is the set of all $\lambda$ not in $\rho(H)$ or $\sigma_p(H)$ for which the range of $\lambda I - H$ is dense in $l^1$. The residual spectrum is the set of all $\lambda$ in $\sigma(H)$ which are not in $\sigma_p(H)$ or $\sigma_c(H)$. The spectral radius of $H$ is

$$r_{\sigma}(H) = \sup_{\lambda \in \sigma(H)} |\lambda|.$$ (1)

The operator $H$ has a matrix representation $\mathbf{H}$ in the standard basis $e_{ik} = \delta_{ik}$, where $\delta_{ik}$ is the Kronecker symbol.

We shall also use two standard results: first, if the operator $H$ is bounded or closed and has a matrix representation $\mathbf{H}$, then the transpose matrix $\mathbf{H}^t$ is the matrix representation of the operator $H^t : l^\infty \to l^\infty$ and (see e.g. [TaLa80], [Gol66, Corollary II.5.3] or [Hal55, Theorems 3.2 and 3.3])

$$\sigma_p(H^t) \subseteq \sigma_p(H) \cup \sigma_r(H), \quad \sigma_r(H) \subseteq \sigma_p(H^t).$$ (2)
Second, if $H$ is bounded, then (see for example [TaLa80, (3-5)])

$$r_{\sigma}(H) = \lim_{k \to \infty} \|H^k\|_1^{1/k}.$$  \hfill (3)

Our aim is to classify spectra of two classes of generalized Fibonacci and Fibonacci-like operators. For the first class of operators their spectral radii are expressed in terms of largest real positive roots of certain polynomials and the coefficients of their powers behave like generalized Fibonacci sequences, as we shall see in section 2.

The second class of operators, which also has applications in mathematical biology, is analyzed in a similar manner in section 3.

2 Generalized Fibonacci operators

Let the linear operator $F_n : l^1 \to l^1$ be defined by

$$(x_1, x_2, x_3, \cdots) \to \left( \sum_{k=n+1}^{\infty} x_k, x_1, x_2, x_3, \cdots \right), \quad n = 1, 2, 3, \ldots$$  \hfill (4)

Each $F_n$ is bounded and its matrix representation in the standard basis is

$$F_n = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix},$$  \hfill (5)

Following the analysis of the spectrum of $F_1$ by Halberg [Hal55], the spectrum of $F_n$ is classified in several steps which are summarized as follows:

1. first, by solving the equation

$$(\lambda I - F_n) x = 0, \quad x \neq 0,$$  \hfill (6)

we show that the point spectrum is

$$\sigma_p(F_n) = \{ \lambda \in \mathbb{C} : \lambda^{n+1} - \lambda^n - 1 = 0, \ |\lambda| > 1 \},$$  \hfill (7)

2. second, by solving the equation

$$(\lambda I - F_n) x = y, \quad x \neq 0,$$  \hfill (8)

we compute the inverse $(\lambda I - F_n)^{-1}$ and show that the resolvent set consists of all $\lambda$ such that $|\lambda| > 1$ which are not in $\sigma_p(F_n)$, that is,

$$\rho(F_n) = \{ \lambda \in \mathbb{C} : |\lambda| > 1, \ \lambda^{n+1} - \lambda^n - 1 \neq 0 \},$$  \hfill (9)
3. third, we analyze the transposed operator $F^t_n$ and show that

$$\sigma_p(F^t_n) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1, \lambda \neq 1 \}, \quad (10)$$

which, together with $(2)$, implies that the residual spectrum of $F_n$ is

$$\sigma_r(F_n) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1, \lambda \neq 1 \}. \quad (11)$$

4. Finally, since the spectrum of $F_n$ is closed, it also contains the point $\lambda = 1$. Since this point is neither in the point spectrum nor in the residual spectrum, it must be in the continuous spectrum, that is

$$\sigma_c(F_n) = \{ 1 \}. \quad (12)$$

We proceed with the detailed analysis of each step.

**Step 1.** The equation $(6)$ can be written as

$$0 = \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots,$$

$$x_1 = \lambda x_2,$$

$$x_2 = \lambda x_3,$$

$$\vdots$$

$$x_k = \lambda x_{k+1},$$

$$\vdots$$

Since $\lambda = 0$ implies $x = 0$, zero is not an element of $\sigma_p(F_n)$. If $\lambda \neq 0$, by applying $(13)$ recursively, we have

$$x_{k+1} = \frac{1}{\lambda} x_k = \frac{1}{\lambda^2} x_{k-1} = \frac{1}{\lambda^3} x_{k-2} = \cdots = \frac{1}{\lambda^k} x_1, \quad k \geq 1. \quad (14)$$

Thus

$$x = x_1 \left( 1 \quad \frac{1}{\lambda} \quad \frac{1}{\lambda^2} \quad \cdots \quad \frac{1}{\lambda^{k}} \quad \cdots \right)^t \quad (15)$$

and

$$\|x\|_1 = |x_1| \sum \frac{1}{|\lambda|^k}. \quad (16)$$

If $|\lambda| \leq 1$, then $\|x\|_1 = \infty$, so $x \notin l^1$. If $|\lambda| > 1$, then $\|x\|_1 = |x_1| |\lambda|/(|\lambda| - 1)$. Inserting $(14)$ into the first equality of $(13)$ gives

$$0 = \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots$$

$$= \lambda x_1 - \frac{1}{\lambda^n} x_1 - \frac{1}{\lambda^{n+1}} x_1 - \frac{1}{\lambda^{n+2}} x_1 - \cdots$$

$$= x_1 \left[ \lambda - \frac{1}{\lambda^n} \left( 1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3} + \cdots \right) \right]$$

$$= x_1 \left( \lambda - \frac{1}{\lambda^{n-1}} \right)$$

$$= x_1 \frac{\lambda^{n+1} - \lambda^n - 1}{\lambda^{n-1}(\lambda - 1)}. \quad (17)$$
Since $x_1 \neq 0$, we conclude that $\sigma_p(F_n)$ consists of those roots of the polynomial

$$p_{n+1}(\lambda) \overset{\text{def}}{=} \lambda^{n+1} - \lambda^n - 1$$

for which $|\lambda| > 1$, as stated in (17).

Since $p_{n+1}(1) = -1 < 0$ and $p_{n+1}'(\lambda) > 0$ for $\lambda \in \mathbb{R}, \lambda \geq 1$, that is, $p_{n+1}$ is strictly increasing for $\lambda > 1$, we conclude that $F_n$ has exactly one real eigenvalue larger than one. Let us denote this eigenvalue by $\lambda_{\text{max}}(F_n)$. By Ostrovsky’s theorem \cite[Theorem 1.1.4, p. 3]{Pra04}, $\lambda_{\text{max}}(F_n)$ is the unique positive root of $p_{n+1}(\lambda)$ and the absolute values of all other roots are strictly smaller. Consequently, all other eigenvalues of $F_n$ are in absolute value strictly smaller than $\lambda_{\text{max}}(F_n)$ which, in turn, implies

$$r_{\sigma}(F_n) = \lambda_{\text{max}}(F_n).$$

(18)

Figure 1 shows $\sigma_p(F_n)$ for various values of $n$.

Step 2. The equation (8) can be written as

$$y_1 = \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots,$$
$$x_2 = \frac{1}{\lambda} (x_1 + y_2),$$
$$x_3 = \frac{1}{\lambda} (x_2 + y_3),$$
$$\vdots$$
$$x_{k+1} = \frac{1}{\lambda} (x_k + y_{k+1}),$$
$$\vdots$$

(19)

By setting

$$u = \sum_{k=n+1}^{\infty} x_k, \quad v = \sum_{k=n+1}^{\infty} y_k,$$

and using (19), we have

$$u = \frac{1}{\lambda} x_n + \frac{1}{\lambda} u + \frac{1}{\lambda} v, \quad y_1 = \lambda x_1 - u.$$

After rearranging, we have

$$u = \frac{1}{\lambda - 1} (x_n + v).$$

Thus,

$$y_1 = \lambda x_1 - \frac{1}{\lambda - 1} (x_n + v).$$

(20)

These roots are the eigenvalues and the vectors $x$ defined by (15) are the corresponding eigenvectors.
By recursively applying (19), we have

\[ x_2 = \frac{1}{\lambda} (x_1 + y_2), \]
\[ x_3 = \frac{1}{\lambda} (x_2 + y_3) = \frac{1}{\lambda^2} x_1 + \frac{1}{\lambda^2} y_2 + \frac{1}{\lambda} y_3, \]
\[ \vdots \]
\[ x_{k+1} = \frac{1}{\lambda} (x_k + y_{k+1}) = \frac{1}{\lambda^k} x_1 + \frac{1}{\lambda^k} y_2 + \frac{1}{\lambda^{k-1}} y_3 + \frac{1}{\lambda^{k-2}} y_4 + \cdots + \frac{1}{\lambda} y_{k+1}, \quad (21) \]
\[ \vdots \]

Inserting \( x_n \) into (20) gives

\[ y_1 = \lambda x_1 - \frac{1}{\lambda - 1} \left( \frac{1}{\lambda^{n-1}} x_1 + \frac{1}{\lambda^{n-1}} y_2 + \frac{1}{\lambda^{n-2}} y_3 + \cdots + \frac{1}{\lambda} y_n + v \right), \]
and solving for $x_1$ gives
\[
x_1 = \frac{1}{\lambda^{n+1} - \lambda^n - 1} \left( \lambda^{n-1}(\lambda - 1) y_1 + y_2 + \lambda y_3 + \lambda^2 y_4 + \cdots + \lambda^{n-2} y_n + \lambda^{n-1} v \right).
\]
By inserting this into (21) we have
\[
x = (\lambda I - F_n)^{-1} y = \frac{1}{\lambda^{n+1} - \lambda^n - 1} (A + B) y
\]
where the matrix representations of $\lambda$ and $\lambda$ are given by
\[
A = \begin{pmatrix}
(\lambda - 1)\lambda^{n-1} & 1 & \lambda & \lambda^2 & \lambda^3 & \cdots & \lambda^{n-2} & \lambda^{n-1} & \lambda^{n-1} & \cdots \\
(\lambda - 1)\lambda^{n-2} & \frac{1}{\lambda} & 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-3} & \lambda^{n-2} & \lambda^{n-2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(\lambda - 1)\lambda & \frac{\lambda}{\lambda^n} & \frac{\lambda}{\lambda^n} & \frac{\lambda}{\lambda^n} & \cdots & \frac{\lambda}{\lambda^n} & 1 & \lambda & \lambda & \cdots \\
(\lambda - 1) & \frac{1}{\lambda^n} & \frac{1}{\lambda^n} & \frac{1}{\lambda^n} & \cdots & \frac{1}{\lambda^n} & \frac{1}{\lambda^n} & \frac{1}{\lambda^n} & \frac{1}{\lambda^n} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},
\]
and
\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{\lambda} & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{\lambda^n} & \frac{1}{\lambda^n} & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix},
\]
respectively. Obviously, for $|\lambda| > 1$ we have $\|A\|_1 < \infty$ and $\|B\|_1 < \infty$. Thus, for $|\lambda| > 1$ and $\lambda$ not being the root of $\lambda^{n+1} - \lambda^n - 1$, the operator $\lambda I - F_n$ has a bounded inverse, so the resolvent set of $F_n$ is given by $[2]$.

Step 3. The point spectrum of the transposed operator $F_n^t$ consists of all $\lambda \in \mathbb{R}$ such that
\[
(\lambda I - F_n^t) x = 0, \quad x \neq 0, \quad \|x\|_\infty < \infty.
\]
This is equivalent to
\[
x_2 = \lambda x_1, \\
x_3 = \lambda x_2 = \lambda^2 x_1, \\
\vdots \\
x_{n+1} = \lambda x_n = \lambda^n x_1, \\
x_{n+2} = \lambda x_{n+1} - x_1 = (\lambda^{n+1} - 1) x_1, \\
x_{n+3} = \lambda x_{n+2} - x_1 = (\lambda^{n+2} - \lambda - 1) x_1, \\
\vdots \\
x_k = (\lambda^{k-1} - \lambda^{k-n-2} - \lambda^{k-n-3} - \cdots - \lambda - 1) x_1, \\
\vdots
\]

$^2$Next row of $A$ is obtained by dividing the previous row by $\lambda$. 

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Therefore,
\[ x_k = \left( \lambda^{k-1} - \frac{\lambda^{k-n-1} - 1}{\lambda - 1} \right) x_1. \]

For \( |\lambda| \leq 1, \lambda \neq 1 \) we have
\[ |x_k| < \left( 1 + \frac{2}{|\lambda - 1|} \right) |x_1|, \]
which implies \( \|x\|_\infty < \infty \). For \( \lambda = 1 \) we have
\[
\begin{align*}
x_2 &= x_1, \\
x_3 &= x_1, \\
&\vdots \\
x_{n+1} &= x_1, \\
x_{n+2} &= 0, \\
x_{n+3} &= -x_1, \\
x_{n+4} &= -2x_1, \\
&\vdots \\
x_k &= -(k - n - 2) x_1,
\end{align*}
\]
so \( \|x\|_\infty = \infty \). We conclude that the point spectrum of \( F^t_n \) is given by \( [10] \). This, in turn, implies \( [11] \) and \( [12] \) as described before.

### 2.1 Relationship to generalized Fibonacci sequences

In this section we describe the relationship between operators \( F_n \) and generalized Fibonacci sequences. A generalized Fibonacci sequence \( \{f^{(n)}\} \) is defined by
\[
\begin{align*}
f_1^{(n)} &= 1, & f_2^{(n)} &= 1, & \ldots, & f_{n+1}^{(n)} &= 1, & f_k^{(n)} &= f_{k-1}^{(n)} + f_{k-n-1}^{(n)}, & k > n + 1. \tag{22}
\end{align*}
\]

For \( n = 1 \) this definition yields the classical Fibonacci sequence
\[
\begin{align*}
f_1 &= 1, & f_2 &= 1, & f_k &= f_{k-1} + f_{k-2}, & k > 2. \tag{23}
\end{align*}
\]
By induction we can prove that the $k$-th power of the matrix $F_n$ from (5) for $k > n$ has the form

$$F_n^k = \begin{pmatrix}
    f_{k-n}^{(n)} & f_{k-n+1}^{(n)} & f_{k-n+2}^{(n)} & \cdots & f_{k-1}^{(n)} & f_k^{(n)} & f_k^{(n)} & \cdots \\
    f_{k-n}^{(n)} & f_{k-n+1}^{(n)} & f_{k-n+2}^{(n)} & \cdots & f_{k-1}^{(n)} & f_k^{(n)} & f_k^{(n)} & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
    f_1^{(n)} & f_2^{(n)} & f_3^{(n)} & \cdots & f_n^{(n)} & f_{n+1}^{(n)} & f_{n+1}^{(n)} & \cdots \\
    0 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
    0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\
    1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
    0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}.$$

We conclude that

$$||F_n^k||_1 = 1 + \sum_{i=1}^{k} f_i^{(n)}. \quad (24)$$

By applying (22) to the terms in parentheses we have

$$2 \sum_{i=1}^{k} f_i^{(n)} = f_1^{(n)} + \cdots + f_n^{(n)} + (f_{n+1}^{(n)} + f_{n+2}^{(n)} + \cdots + f_{k-1}^{(n)}) + f_k^{(n)}$$

$$= f_1^{(n)} + f_2^{(n)} + \cdots + f_k^{(n)} + f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)}$$

$$= f_1^{(n)} + f_2^{(n)} + \cdots + f_n^{(n)} + (f_{n+1}^{(n)} + f_{n+2}^{(n)} + \cdots + f_k^{(n)})$$

$$+ f_k^{(n)} + f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)}$$

$$= \sum_{i=1}^{k} f_i^{(n)} - f_{n+1}^{(n)} + f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} + f_k^{(n)}.$$

From this, by applying (22) again recursively, we obtain

$$\sum_{i=1}^{k} f_i^{(n)} = f_{k-n}^{(n)} + f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} + f_k^{(n)} - 1$$

$$= f_{k-n+1}^{(n)} + \cdots + f_k^{(n)} + f_{k+1}^{(n)} - 1$$

$$= f_{k-n+2}^{(n)} + \cdots + f_k^{(n)} + f_{k+2}^{(n)} - 1$$

$$\vdots$$

$$= f_{k+n-1}^{(n)} - 1.$$
and from (18) it follows that

$$\lim_{k \to \infty} \left( f_{k+n+1}^{(n)} \right)^{1/k} = \lambda_{\max}(F_n).$$

Also, by using standard techniques in analyzing linear recurrence relations with constant coefficients, we can prove that for all $i, j$

$$\lim_{k \to \infty} \frac{[F_k]_{i,j}}{[F_k]_{i+1,j}} \equiv \lim_{m \to \infty} \frac{f_{m+1}^{(n)}}{f_m^{(n)}} = \lambda_{\max}(F_n).$$

For example, by setting $n = 1$ we have for the Fibonacci sequence (23)

$$\lim_{k \to \infty} \left( f_{k+2} \right)^{1/k} = r_\sigma(F_1) = \frac{1 + \sqrt{5}}{2},$$

$$\lim_{k \to \infty} \frac{f_{k+1}}{f_k} = \frac{1 + \sqrt{5}}{2}.$$  

3 Fibonacci-like operators

Now we would like to consider the family of linear operators $\Gamma_n : l^1 \to l^1$ defined by

$$(x_1, x_2, x_3, \cdots) \to \left( \rho \sum_{k=n+1}^\infty (k-n) x_k, x_1, x_2, x_3, \cdots \right), \quad n = 1, 2, 3, \ldots \quad (26)$$

for some real positive $\rho$. The domain of $\Gamma_n$ is

$$\text{Dom } \Gamma_n = \left\{ x \in l^1 : \sum_{k=n+1}^\infty (k-n)x_k < \infty \right\},$$

and its matrix representation in the standard basis is

$$\Gamma_n = \begin{pmatrix}
0 & 0 & \cdots & 0 & \rho & 2\rho & 3\rho & 4\rho & 5\rho & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots \\
& & & & & & & & & \ddots
\end{pmatrix} \quad (27)$$

However, the operator $\Gamma_n$ is not closed as illustrated by the following example.

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3The proof is derived using the fact that $f_l^{(n)}$ has the form $f_l^{(n)} = \alpha \lambda_{\max}(F_n) + \sum_{i=1}^n \alpha_i \lambda_i$, where $\lambda_{\max}(F_n)$ and $\lambda_i$ are the roots of the characteristic polynomial (17), and $|\lambda_{\max}(F_n)| > |\lambda_i|$. 

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**Example 1** Let us define the sequence \( \{ x^{(m)} \} \) of vectors in \( l^1 \) by

\[
x^{(m)} = \left( 0 \; \cdots \; 0 \; \frac{1}{m} \; 0 \; \cdots \right)^t.
\]

Then

\[
x^{(m)} \to (0 \; 0 \; \cdots)^t,
\]

while

\[
\Gamma_n x^{(m)} = (\rho \; 0 \; \cdots \; 0 \; \frac{1}{m} \; 0 \; \cdots)^t \to (\rho \; 0 \; 0 \; \cdots)^t.
\]

Although the point spectrum of \( \Gamma_n \) is defined and can be computed in a standard manner (see later), the resolvent set of \( \Gamma_n \) is empty, which makes the analysis of \( \Gamma_n \) less interesting. Instead, we shall consider the family of operators \( G_n : l_1 \to l_1 \) formally defined by

\[
G_n = D_n \Gamma_n D_n^{-1},
\]

where

\[
D_n = \text{diag}(1, \cdots, 1, \rho, 2\rho, 3\rho, 4\rho, \cdots).
\]

That is, for \( n \in \mathbb{N} \) the operator \( G_n \) is defined by

\[
(x_1, x_2, x_3, \cdots) \to \left( \sum_{k=n+1}^{\infty} x_k, x_1, x_2, \cdots, x_{n-1}, \rho x_n, 2 x_{n+1}, \frac{3}{2} x_{n+2}, \frac{4}{3} x_{n+3}, \frac{5}{4} x_{n+4}, \cdots \right),
\]

and its matrix representation in the standard basis is

\[
G_n = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & \rho & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 2 & 0 & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \frac{3}{2} & 0 & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \frac{4}{3} & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{5}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}.
\]

Let us define the polynomial \( q_{n+1}(\lambda) \) by

\[
q_{n+1}(\lambda) = \lambda^{n+1} - 2 \lambda^n + \lambda^{n-1} - \rho.
\]

Similarly as in section 2, the spectrum of \( G_n \) is classified in several steps which are summarized as follows:
1. first, by solving the equation

\[(\lambda I - G_n) x = 0, \quad x \neq 0,\]  

we show that the point spectrum is

\[\sigma_p(G_n) = \{\lambda \in \mathbb{C} : q_{n+1}(\lambda) = 0, \ |\lambda| > 1\}, \quad n \geq 2.\]  

(31)

2. second, by solving the equation

\[(\lambda I - G_n) x = y, \quad x \neq 0,\]  

we compute the inverse \((\lambda I - G_n)^{-1}\) and show that the resolvent set consists of all \(\lambda\) such that \(|\lambda| > 1\) which are not in \(\sigma_p(G_n)\),

\[\rho(G_n) = \{\lambda \in \mathbb{C} : |\lambda| > 1, \ \lambda \notin \sigma_p(G_n)\},\]  

(33)

3. third, we analyze the transposed operator \(G_n^t\) and show that

\[\sigma_p(G_n^t) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1, \ \lambda \neq 1\},\]  

(34)

which, together with (2), implies that the residual spectrum of \(G_n\) is

\[\sigma_r(G_n) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1, \ \lambda \neq 1\}.\]  

(35)

4. Finally, since the spectrum of \(G_n\) is closed, is also contains the point \(\lambda = 1\). Since this point is neither in the point spectrum nor in the residual spectrum, it must be in the continuous spectrum, that is

\[\sigma_c(G_n) = \{1\}.\]  

(36)

The proofs are similar to the ones from section 2 but more tedious.

**Step 1.** The equation (30) can be written as

\[0 = \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots,\]  

\[x_k = \lambda x_{k+1}, \quad k = 1, \cdots, n - 1,\]  

\[\rho x_n = \lambda x_{n+1},\]  

\[\frac{k - n + 1}{k - n} x_k = \lambda x_{k+1}, \quad k = n + 1, n + 2, \cdots.\]  

(37)

Since \(\lambda = 0\) implies \(x = 0\), zero is not an element of \(\sigma_p(G_n)\). If \(\lambda \neq 0\), by applying (37) recursively, we obtain

\[x_k = \frac{1}{\lambda^{k-1}} x_1, \quad k = 2, 3, \cdots, n,\]  

\[x_k = \rho \frac{k - n}{\lambda^{k-1}} x_1, \quad k = n + 1, n + 2, \cdots.\]  

(38)
This, in turn, implies
\[ x = x_1 \left( 1 \frac{1}{\lambda} \frac{1}{\lambda^2} \cdots \frac{1}{\lambda^n} \frac{2\rho}{\lambda^{n+1}} \frac{3\rho}{\lambda^{n+2}} \cdots \frac{(k-n)\rho}{\lambda^{k-1}} \cdots \right)^t, \]
so that
\[ \|x\|_1 = |x_1| \left( \frac{1}{\lambda^n - \frac{1}{\lambda}} + \frac{\rho}{\lambda^n} \left( 1 + \frac{2}{\lambda} + \frac{3}{\lambda^2} + \cdots \right) \right). \]
If |\lambda| \leq 1, then \( \|x\|_1 = \infty \), so \( x \notin l^1 \). If |\lambda| > 1, then, by using differentiation of the geometric series, we have
\[ \|x\|_1 = |x_1| \left( \frac{\lambda^n - 1}{\lambda^n - 1} + \frac{\rho}{\lambda^n} \left( \lambda^2 - 1 \right) \right) < \infty, \]
thus, \( x \in l^1 \). By inserting (38) into the first equality of (37) and using differentiation of the geometric series we have
\[ 0 = \lambda x_1 - \rho \left( \frac{1}{\lambda} x_1 + \frac{2}{\lambda^2} x_1 + \frac{3}{\lambda^3} x_1 + \cdots \right) = x_1 \left( \lambda - \frac{\rho}{\lambda^n} \left( 1 + \frac{2}{\lambda} + \frac{3}{\lambda^2} + \frac{4}{\lambda^3} + \cdots \right) \right) = x_1 \left( \lambda - \frac{\rho}{\lambda^n} \left( 1 - \frac{1}{\lambda} \right)^2 \right). \]
Finally, solving this equation with \( x_1 \neq 0 \) gives (31).

We shall now prove that \( \sigma_p(G_n) \) consists of \( \lambda_{\text{max}}(G_n) \), a unique simple real eigenvalue larger than one and all other eigenvalues have modulus smaller than \( \lambda_{\text{max}}(G_n) \). This also implies
\[ r_\sigma(G_n) = \lambda_{\text{max}}(G_n). \] (39)
The proof is based on the ideas from the proof of [Pra04, Theorem 1.1.4, pp. 3-4]. Indeed, if \( n = 1 \) then the roots of \( q_2(\lambda) \) are \( 1 \pm \sqrt{\rho} \) and the statement holds. For \( n \geq 2 \) we have
\[ q_{n+1}(\lambda) = \lambda^{n-1}(\lambda - 1)^2 - \rho, \] (40)
\[ q'_{n+1}(\lambda) = \lambda^{n-2}[(n + 1)\lambda^2 - 2n\lambda + (n - 1)]. \] (41)
Since \( q_{n+1}(1) = -\rho < 0 \) and \( q'_{n+1}(\lambda) > 0 \) for \( \lambda \in \mathbb{R}, \lambda > 1 \), that is, \( q_{n+1}(\lambda) \) is strictly increasing for \( \lambda > 1 \), we conclude that \( q_{n+1}(\lambda) \) has exactly one real root larger than one or, equivalently, that \( G_n \) has exactly one real eigenvalue larger than one. Let us denote this eigenvalue by \( \lambda_{\text{max}}(G_n) \). Let \( z \neq \lambda_{\text{max}}(G_n) \) be some other real or complex eigenvalue of \( G_n \) and let \( \zeta = |z| > 1 \). Since \( z \) is also the root of \( q_{n+1}(\lambda) \), the relation (40) implies
\[ z^{n-1}(z - 1)^2 = \rho, \]
which, in turn, implies
\[ |z|^{n-1}|z - 1|^2 = \rho. \] (42)
Since \( \zeta > 1 \), this implies
\[ \zeta^{n-1}(\zeta - 1)^2 \leq \rho. \]
or
\[ q_{n+1}(\zeta) = \zeta^{n-1}(\zeta - 1)^2 - \rho \leq 0. \tag{43} \]

Since \( q_{n+1}(\lambda) \) is strictly increasing for \( \lambda > 1 \), and \( q_{n+1}(\lambda_{\text{max}}(G_n)) = 0 \), we conclude that \( \zeta \leq \lambda_{\text{max}}(G_n) \) and that the equality in (43) holds only if \( \zeta = \lambda_{\text{max}}(G_n) \). But, the equality in (43) and (42) imply
\[ |z - 1| = \zeta - 1, \]
that is, \( z \in \mathbb{R} \) and \( z = \pm \zeta = \pm \lambda_{\text{max}}(G_n) \). The choice \( z = -\lambda_{\text{max}}(G_n) \) is impossible since \( q_{n+1}(-\lambda_{\text{max}}(G_n)) \neq 0 \), and the second choice contradicts the assumption \( z \neq \lambda_{\text{max}}(G_n) \). Therefore, \( \zeta < \lambda_{\text{max}}(G_n) \) as desired.

**Remark 1** Although the above analysis is sufficient for our purposes, by inspecting the polynomial \( q_{n+1}(\lambda) \) and its derivative from (40) and (41), respectively, we can establish further facts about its roots. From (41) we see that the derivative \( q'_{n+1}(\lambda) \) has exactly two real positive simple roots \( \lambda_1 = \frac{n}{n+1} \) and \( \lambda_2 = 1 \) and, if \( n > 2 \), also the root \( \lambda_0 = 0 \). If \( n > 3 \) then \( \lambda_0 \) is multiple. Let \( \rho_0 = 4\lambda^{n-1}/(n+1)^2 \). The number of real roots of \( q_{n+1}(\lambda) \) in the open interval \((0, \lambda_{\text{max}}(G_n))\) is governed by \( \rho \) as follows: if \( \rho > \rho_0 \), then there are no such roots, if \( \rho = \rho_0 \) there is exactly one root equal to \( \lambda_1 \) and if \( \rho < \rho_0 \) there are exactly two roots, one smaller and one larger than \( \lambda_1 \). Finally, if \( n \) is odd, then \( q_{n+1}(\lambda) \) also has a simple negative real root. As we have already proved, \( \lambda_{\text{max}}(G_n) \) is the root with strictly maximal absolute value.

**Remark 2** It is easy to see that the point spectrum of \( \Gamma_n \) from (26) and (27) is equal to the point spectrum of \( G_n \).

**Step 2.** The equation (32) can be written as
\[ y_1 = \lambda x_1 - x_{n+1} - x_{n+2} - x_{n+3} - \cdots, \tag{44} \]
\[ x_{k+1} = \frac{1}{\lambda} (x_k + y_{k+1}), \quad k = 1, \ldots, n-1, \]
\[ x_{n+1} = \frac{1}{\lambda} (\rho x_n + y_{n+1}), \]
\[ x_{k+1} = \frac{1}{\lambda} \left( \frac{k-n+1}{k-n} x_k + y_{k+1} \right), \quad k = n+1, n+2, \ldots. \]

By recursively applying the above three equalities, we obtain
\[ x_k = \frac{(k-n)\rho}{\lambda^{k-1}} x_1 + \frac{(k-n)\rho}{\lambda^{k-2}} y_2 + \frac{(k-n)\rho}{\lambda^{k-3}} y_3 + \cdots + \frac{(k-n)\rho}{\lambda^{k-n+1}} y_n + \frac{k-n}{1} \frac{1}{\lambda^{k-n+2}} y_{n+2} + \frac{k-n}{3} \frac{1}{\lambda^{k-n+3}} y_{n+3} + \cdots + \frac{k-n}{k-n-1} \frac{1}{\lambda^2} y_{k-1} + \frac{1}{\lambda} y_k, \quad k = n+1, n+2, \ldots. \tag{45} \]

For \(|\lambda| > 1\), by inserting (45) into (44), rearranging, and using differentiation of the geometric series, we have
\[ y_1 = \lambda x_1 - \frac{\lambda^2}{(\lambda - 1)^2} \left( \frac{\rho}{\lambda} x_1 + \frac{\rho}{\lambda^2} y_2 + \frac{\rho}{\lambda^3} y_3 + \cdots + \frac{\rho}{\lambda^2} y_n + \frac{1}{\lambda} y_{n+1} \right) - \alpha_{n+2} y_{n+2} - \alpha_{n+3} y_{n+3} - \alpha_{n+4} y_{n+4} - \cdots, \tag{46} \]
where

\[
\alpha_{n+k} = \frac{1}{\lambda} + \frac{k+1}{k} \frac{1}{\lambda^2} + \frac{k+2}{k} \frac{1}{\lambda^3} + \frac{k+3}{k} \frac{1}{\lambda^4} + \cdots, \quad k = 2, 3, 4, \ldots,
\]

and

\[
|\alpha_{n+k}| \leq \frac{1}{|\lambda|} \left( 1 + \sum_{i=1}^{\infty} \frac{k+i}{k} \frac{1}{|\lambda|^i} \right) \leq \frac{1}{|\lambda|} \left( 1 + \sum_{i=1}^{\infty} (i+1) \frac{1}{|\lambda|^i} \right) = \frac{|\lambda|}{(|\lambda| - 1)^2}. \tag{47}
\]

Solving for (146) for \( x_1 \) gives

\[
x_1 = \frac{1}{q_{n+1}(\lambda)} \left[ \lambda^{n-2}(\lambda - 1)^2 y_1 + \rho y_2 + \rho \lambda y_3 + \rho \lambda^2 y_4 + \cdots + \rho \lambda^{n-2} y_n + \lambda^{n-1} y_{n+1} + \right.
\]

\[\left. + \tilde{\alpha}_{n+2} y_{n+2} + \tilde{\alpha}_{n+3} y_{n+3} + \tilde{\alpha}_{n+4} y_{n+4} + \cdots \right], \tag{48}
\]

where

\[
\tilde{\alpha}_{n+k} = \lambda^{n-2}(\lambda - 1)^2 \alpha_{n+k},
\]

and

\[
|\tilde{\alpha}_{n+k}| \leq |\lambda|^{n-1} \frac{(|\lambda| + 1)^2}{(|\lambda| - 1)^2}. \tag{49}
\]

By inserting (48) into (44) and (45), we have

\[
x = (\lambda I - G_n)^{-1} y = \frac{1}{q_{n+1}(\lambda)} \left( A + B \right) y
\]

where the matrix representations of \( A \) and \( B \) are given by

\[
A = \begin{pmatrix}
(\lambda - 1)^2 \lambda^{-n+1} & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
(\lambda - 1)^2 \lambda^{-n+3} & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
(\lambda - 1)^2 \lambda^{-n+4} & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
(\lambda - 1)^2 \lambda^{-n+5} & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(\lambda - 1)^2 \lambda & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
(\lambda - 1)^2 \lambda & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
(\lambda - 1)^2 \lambda & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
(\lambda - 1)^2 \lambda & \rho & \rho \lambda & \rho \lambda^2 & \cdots & \rho \lambda^{n-2} & \lambda^{n-1} & \tilde{\alpha}_{n+2} & \tilde{\alpha}_{n+3} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
respectively. For $|\lambda| > 1$, by using differentiation of the geometric series, (49), and the argument used in (47), we have $\|A\|_1 \leq \infty$ and $\|B\|_1 \leq \infty$. Thus, for $|\lambda| > 1$ and $q_{n+1}(\lambda) \neq 0$, the operator $\lambda I - G_n$ has a bounded inverse and its resolvent set is given by (63).

**Step 3.** The point spectrum of the transposed operator $G^t_n$ consists of all $\lambda \in \mathbb{R}$, $|\lambda| \leq 1$, such that

$$(\lambda I - G^t_n)x = 0, \quad x \neq 0, \quad \|x\|_\infty < \infty.$$ 

This is equivalent to

$$x_k = \lambda^{k-1} x_1, \quad k = 2, \ldots, n,$$

$$x_{n+1} = \frac{\lambda^n}{\rho} x_1,$$

$$x_{n+k} = \frac{1}{k} \left( \frac{\lambda^{n+k-1}}{\rho} - \lambda^{k-2} - 2 \lambda^{k-3} - 3 \lambda^{k-4} - \cdots - (k-2) \lambda - (k-1) \right) x_1, \quad k \geq 2.$$ 

If $\lambda \neq 1$, we have

$$x_{n+k} = \frac{1}{k} \left[ \frac{\lambda^{n+k-1}}{\rho} - \left( \lambda^{k-2} + 2 \lambda^{k-3} + \cdots + (k-2) \lambda + (k-1) \right) \frac{(\lambda-1)^2}{(\lambda-1)^2} \right] x_1$$

$$= \frac{1}{k} \left( \frac{\lambda^{n+k-1}}{\rho} - \frac{\lambda^k - k \lambda + (k-1)}{(\lambda-1)^2} \right) x_1, \quad k \geq 2.$$ 

Therefore

$$|x_{n+k}| \leq \frac{1}{k} \left( \frac{1}{\rho} + \frac{1 + k + k - 1}{|\lambda - 1|^2} \right) x_1 \leq \left( \frac{1}{2 \rho} + \frac{2}{|\lambda - 1|^2} \right) x_1, \quad k \geq 2,$$

which implies $\|x\|_\infty < \infty$. For $\lambda = 1$ we have

$$x_{n+k} = \frac{1}{k} \left( \frac{1}{\rho} - 1 - 2 - 3 - \cdots - (k-2) - (k-1) \right) x_1$$

$$= \frac{1}{k} \left( \frac{1}{\rho} - \frac{(k-1)k}{2} \right) x_1, \quad k \geq 2.$$
so $\|x\|_\infty = \infty$. We conclude that the point spectrum of $G_n^t$ is given by (33). This, in turn, implies (35) and (36) as described before.

Example 2 The lesion forming plant pathogen potato late blight (phytophthora infestans) grows radially on a leaf with a constant daily rate. The latency period for a lesion to become infectious is five days, and the sporulating area is infectious for one day. In [PSV05] the epidemic spread of such pathogen is modeled with the infinite dimensional Leslie matrix of the form of $\Gamma_5$ as defined in (27). Further, the upper bound for the speed of invasion in computed via minimization of the largest eigenvalue $\lambda_{\text{max}}(\Gamma_5(s))$. From Remark 2 it follows that this eigenvalue is the largest unique positive root of $q_6(\lambda)$ from (29). Here the parameter $\rho$ has the form $\rho(s) = \text{const} \times M(s)$ where $M(s)$ is some moment-generating function (for example, $M(s) = \exp(\sigma^2 s^2/2)$ for the Gaussian kernel or $M(s) = 1/(1 - \sigma^2 s^2)$ for the Laplace kernel). Here $\Gamma_5(s)$ appears naturally due to the fact that the considered pathogen has a latency period of five days. It is interesting that $\lambda_{\text{max}}(\Gamma_5(s))$ can be computed analytically:

$$\lambda_{\text{max}}(\Gamma_5) = \frac{1}{3} + \frac{2^{1/3}}{3(2 + 27\sqrt{\rho} + \sqrt{108\sqrt{\rho} + 729\rho})^{1/3}} + \frac{(2 + 27\sqrt{\rho} + \sqrt{108\sqrt{\rho} + 729\rho})^{1/3}}{3 \cdot 2^{1/3}}.$$

The speed of invasion is bounded by

$$v^* = \min_{0<s<s^*} \frac{1}{s} \ln \left[ \lambda_{\text{max}}(\Gamma_5(s)) \right],$$

where $s^*$ is the maximum $s$ for which $M(s)$ is defined. For details about a rather complex derivation of this model we refer the reader to [PSV05].

Acknowledgment. The author wishes to thank the anonymous referee for the valuable comments and remarks, in particular for correcting an error in the previous version of the manuscript and for providing Example 1. The author also wishes to thank Jussi Behrndt from the Technische Universität Berlin for helpful discussions.

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