ON THE PRYM MAP OF GALOIS COVERINGS

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Abstract. In this paper we consider the Prym variety \( P(\overline{C}/C) \) associated to a Galois coverings of curves \( f : \overline{C} \to C \) branched at \( r \) points. We discuss some properties and equivalent definitions and then consider the Prym map \( \mathcal{P} = \mathcal{P}(G, g, r) : R(G, g, r) \to A_{p,\delta} \) with \( \delta \) the type of the polarization. For Galois coverings whose Galois group is abelian and metabelian (non-abelian) we show that the differential of this map at certain points is injective. We also consider the Abel-Prym map \( u : \overline{C} \to P(\overline{C}/C) \) and prove some results for its injectivity. In particular we show that in contrast to the classical and cyclic case, the behavior of this map here is more complicated. The theories of abelian and metabelian Galois coverings play a substantial role in our analysis and have been used extensively throughout the paper.

1. Introduction

To a given finite covering \( f : \overline{C} \to C \) between non-singular projective algebraic curves (or Riemann surfaces) one can associate a so-called Prym variety, a polarized abelian variety: \( f \) induces a norm map

\[
Nm_f : \text{Pic}^0(\overline{C}) \to \text{Pic}^0(C)
\]

\[
\sum a_ip_i \mapsto \sum a_if(p_i)
\]

The Prym variety associated to \( f \) is then defined as \( P(f) = P(\overline{C}/C) = (\ker Nm_f)^0 \), i.e., the connected component of the kernel of \( Nm_f \) containing the identity. Identifying \( \text{Pic}^0 \) with the Jacobian, one sees that the canonical (principal) polarization of \( \text{Jac}(\overline{C}) \) restricts to a polarization on \( P(f) \).

Classically, \( f \) is a double covering which is étale or branched at exactly two points. In these cases, \( P(f) \) is known to be principally polarized. In fact, these are the only cases in which the

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polarization on $P(f)$ is principal. However, the type $\delta$ of the polarization on $P(f)$ depends on the topological structure of the covering map $f$, see [2].

Let $G$ be a finite group with $n = |G|$. Consider the following moduli stack $R(G, g, r)$: Objects are couples $((C, x_1, \ldots, x_r), f : \tilde{C} \to C)$ such that

1. $(C, x_1, \ldots, x_r)$ is a smooth $r$-pointed curve of genus $g$ i.e., a point in $M_g(r)$.
2. The group $G$ acts on the smooth curve $\tilde{C}$ and the map $f : \tilde{C} \to \tilde{C}/G = C$ is the quotient map branched along the reduced divisor $D = \sum x_i$.

Note that since our problem is insensitive to level structures, we may actually consider $R(G, g, r)$ as a coarse moduli space. As a result, we omit any assumptions on the automorphism group of the base curve $C$ whose non-triviality can be remedied either by considering the moduli stack or by imposing level structures. In section 2, we will describe this moduli space intrinsically in terms of the curve $C$ using the theory of abelian and metabelian coverings that will be explained in that section.

Let $p$ be the dimension of the corresponding Prym variety and let $A_{p, \delta}$ denote the moduli space of abelian varieties with polarization of type $\delta$ over $C$. The above constructions behave well also in the families of curves and hence we obtain a morphism

$$\mathcal{P} = \mathcal{P}(G, g, r) : R(G, g, r) \to A_{p, \delta}.$$

We call the map $\mathcal{P}$ the Prym map of type $(G, g, r)$. One of our objectives in this paper is to study this map. The Prym map is even in the classical case known to be non-injective which implies that one needs to study other closely related aspects, namely the generic injectivity.

We therefore study the differential $d\mathcal{P}$ and examine at which points this differential is injective. This will be done in section 3 in which we also use the results of [4] by Lange and Ortega which partly motivated us to generalize its results to more general Galois coverings. Indeed the key points of the proof of Proposition 3.4 are direct generalizations of the results of [4] to the broader class of Galois covers. In section 3, we first describe the Prym varieties of Galois covers and give some equivalent definitions and prove some properties of this abelian variety.

There is also the so-called Abel-Prym map which is induced from the classical Abel-Jacobi map. When the Galois group of the covering is cyclic, it is shown in [4] that the Abel-Prym map is non-injective at the ramification points of the covering provided that the curve $\tilde{C}$ is not
hyperelliptic. Here we generalize this under more restrictive conditions in particular that the curve $\tilde{C}$ is not $g_{2n}^1$. We provide an example to show that these conditions are indeed necessary, otherwise the corresponding statement will be false even for very simple covers.

2. Abelian and metabelian Galois coverings

2.1. Galois covers of curves. Let us summarize some general facts about Galois coverings of curves. Let $\tilde{C}, C$ be complex smooth projective algebraic curves (equivalently Riemann surfaces) and let $f: \tilde{C} \to C$ be a Galois covering of degree $n$. By this we mean precisely that there exists a finite group $G$ with $|G| = n$ together with a faithful action of $G$ on $\tilde{C}$ such that $f$ realizes $C$ as the quotient of $\tilde{C}$ by $G$. Consider the ramification and branch divisors $R, D$ of $f$. Note that $R$ consists precisely of the points in $\tilde{C}$ with non-trivial stabilizers under the action of $G$. The deck transformation group $\text{Deck}(\tilde{C}/C)$, i.e., the group of those automorphisms of $\tilde{C}$ that are compatible with $f$ is isomorphic to the Galois group $G$ and acts transitively on each fiber $f^{-1}(x)$. If $y \in \tilde{C}$ is a ramification point with ramification index $e$, then so are all points in the fiber $f^{-1}(f(y))$. Moreover, the stabilizers of these points in $\text{Deck}(\tilde{C}/C) \cong G$ are conjugate cyclic subgroups of $\text{Deck}(\tilde{C}/C)$ (or by using the isomorphism $\text{Deck}(\tilde{C}/C) \cong G$, subgroups of $G$) of order $e$, see [9], Proposition 3.2.10. In particular the stabilizer of a point in $\tilde{C}$ is trivial, if and only if that point is not a ramification point. The stabilizer $H_y$ of a point $y \in \tilde{C}$ is also referred to as the inertia subgroup of $y$.

2.2. Abelian Galois covers. In this section we describe the building data of an abelian cover and the construction of the cover using these data and the relations among them. Notations come mostly from [9] in which such Galois coverings of algebraic varieties have been extensively studied. Let $f: \tilde{C} \to C$ be a $G$-Galois cover with $G$ finite abelian branched above the points $x_1 + \cdots + x_r$. Since the group $G$ is abelian, the inertia group above a branch point $x_i$ is independent of the chosen ramification point and we denote it by $H_i$. Then $f_* \mathcal{O}_{\tilde{C}} = \bigoplus_{\chi \in G^*} L_{\chi}^{-1}$, where each $L_{\chi}$ is an invertible sheaf on $C$ on which $G$ acts by character $\chi$. So in particular, the invariant summand $L_1$ is isomorphic to $\mathcal{O}_C$. The algebra structure on $f_* \mathcal{O}_{\tilde{C}}$ is given by the ($\mathcal{O}_C$-linear) multiplication rule $m_{\chi, \chi'}: L_{\chi}^{-1} \otimes L_{\chi'}^{-1} \to L_{\chi \chi'}^{-1}$ and compatible with the action of $G$. The choice of a primitive $n$-th root of unity $\xi$ amounts to giving a map $\{1, \ldots, r\} \to G$, the image $h_i$ of $i$ under which is the generator of the inertia group $H_i$ that is sent to $\xi^{n/n_i}$ by $\chi_i$. The line
bundles $L_\chi$ and divisors $x_i$ each labelled with an element $h_i$ as described above are called the building data of the cover. These data are to satisfy the so-called fundamental relations and determine the cover $f : \tilde{C} \to C$ up to deck automorphisms. Let us write these relations down.

For $i = 1, \ldots, r$ and $\chi \in G^*$, let $a^i_\chi$ be the smallest positive integer such that $\chi(h_i) = \xi^{a^i_\chi/n_i}$. For any two characters $\chi, \chi'$, $0 \leq a^i_\chi + a^i_{\chi'} < 2n_i$, so

$$
\epsilon^i_{\chi, \chi'} \equiv \begin{cases} 
1 & a^i_\chi + a^i_{\chi'} \geq n_i \\
0 & a^i_\chi + a^i_{\chi'} < n_i 
\end{cases}
$$

and we set $D_{\chi, \chi'} = \sum_{i=1}^{r} \epsilon^i_{\chi, \chi'} x_i$. By the fundamental relations of the cover we mean the following isomorphisms:

(2.0.1) \[ \mu_{\chi, \chi'} : L_\chi + L_{\chi'} \sim L_{\chi \chi'} \otimes \mathcal{O}_C(D_{\chi, \chi'}). \]

In particular, if $\chi' = \chi^{-1}$, then

(2.0.2) \[ L_\chi + L_{\chi^{-1}} \equiv D_{\chi, \chi^{-1}}, \]

and $D_{\chi, \chi^{-1}}$ the sum of the components $x_i$, where $\chi(h_i) \neq 1$. The cover $f : \tilde{C} \to C$ can be recovered from the fundamental relations (2.0.1) by first defining the curve $\tilde{C}$ inside the vector bundle $\mathcal{L} = \oplus_{\chi \neq 1} L_\chi$ by the equations

(2.0.3) \[ z_\chi z_{\chi'} = (\prod_i s_i^{\epsilon^i_{\chi, \chi'}}) z_{\chi \chi'}, \]

where $z_\chi$ is the fiber coordinate of the bundle $L_\chi$ which can also be viewed as the tautological section of pull-back of the bundle $L_\chi$ to $\mathcal{L}$ and $s_i \in H^0(C, \mathcal{O}_C(x_i))$ is the (pull-back to $\mathcal{L}$ of the) defining equation for $D_i$ for $i = 1, \ldots, r$. This is naturally a flat $C$-scheme. Conversely, for every choice of the sections $s_i$, equations (2.0.3) define a flat scheme $\tilde{C}$ over $C$ which is smooth if and only if each $D$ is reduced. We therefore have the following fundamental theorem proven in [6].

**Theorem 2.1.** Let $G$ be a finite abelian group. Let $\tilde{C}, C$ be smooth projective algebraic curves and let $f : \tilde{C} \to C$ be an abelian cover with Galois group $G$. With the notations as above, the following set of linear equivalences hold.

(2.1.1) \[ \mu_{\chi, \chi'} : L_\chi + L_{\chi'} \sim L_{\chi \chi'} + D_{\chi, \chi'} \quad \forall \chi, \chi' \in G^*. \]
Conversely, given a set of data \( \{ L_\chi \}_{\chi \in G^*} \) consisting respectively of invertible sheaves and reduced effective divisors on \( C \) satisfying the relation (2.1.1) determines an abelian cover. When \( C \) is furthermore complete, this abelian cover is unique.

Now let the finite abelian group \( G \) have the decomposition \( G = \langle \sigma_1 \rangle \oplus \langle \sigma_2 \rangle \oplus \cdots \oplus \langle \sigma_h \rangle \). Let us denote by \( G^* \) the character group of \( G \), called also the dual abelian group. For \( g \in G \), we denote by \( g^* \) the corresponding character of \( G \). We have naturally \( G^* = \langle \sigma_1^* \rangle \oplus \langle \sigma_2^* \rangle \oplus \cdots \oplus \langle \sigma_h^* \rangle \).

These data satisfy

\[
(2.1.2) \quad L_i^{n_i} \cong O_C(\sum_{j=1}^{n} \lambda_{ij} x_j).
\]

Denote \( d_i = \sum_{j=1}^{n} \lambda_{ij} \). The collection \( \{ (L_i)_{1 \leq i \leq h}, \{ D_i \}_{1 \leq i \leq h} \} \) is called a reduced building data for the abelian cover \( f : \tilde{C} \to C \), see [6], Definition 2.3.

**Theorem 2.2.** Let \( C \) be a projective non-singular curve. The reduced building data

\[
(2.2.1) \quad \left( \{ L_i \}_{1 \leq i \leq h}, \{ D_i \}_{1 \leq i \leq h} \right)
\]

determine the abelian cover \( f : \tilde{C} \to C \) uniquely up to isomorphisms of \( G \)-covers.

**2.3. Eigensheaves of the group action.** Let us describe the eigensheaves \( L^{-1}_\chi \) in the decomposition \( f_* \mathcal{O}_C = \underset{\chi \in G^*}{\oplus} L^{-1}_\chi \). As before, \( D \) denotes the branch divisor of \( f \) with irreducible components \( D_i \). We have already remarked that the scheme \( V \) can be constructed inside the (total space of the) vector bundle \( \mathcal{L} = \underset{\chi \in G^* \setminus \{ 1 \}}{\oplus} L^{-1}_\chi \) by the equations (2.0.3) in terms of the tautological section \( z_\chi \) of pull-back of the bundle \( L_\chi \) to \( \mathcal{L} \) and the defining equation \( s_i \in H^0(C, \mathcal{O}_C(x_i)) \) for \( x_i \). One can embed \( C \) in \( \mathcal{L} \) by the zero section of \( \mathcal{L} \to C \). Let the branch divisor \( D \) be reduced.

As a closed subscheme, \( C \) is given inside \( \mathcal{L} \) by the equations \( z_\chi = 0 \). Let \( R = f^{-1}(D) \). Let \( p : \mathcal{L} \to C \) be the bundle projection (we will use the same notation for \( \mathcal{L} \) and its total space).

Suppose \( H_j = \langle h_j \rangle, j = 1, \ldots, s \) are the (non-trivial) inertia groups of \( f \) and let \( R_j \) be the divisorial part of the reduced ramification divisor \( R_{\text{red}} \) consisting of all those points that have \( H_j \) as their stabilizer. Suppose the ramification index of \( R_j \) is \( e_j \). Consider the group of characters \( G^* \) and suppose \( \chi_j = h_j^*, j = 1, \ldots, s \) are the characters corresponding to the \( h_j \). Let \( L_j = L_{\chi_j} \) be the line bundles associated to the \( \chi_j \) for \( j = 1, \ldots, s \). It is clear that \( R_{\text{red}} = R_1 + \cdots + R_s \).
and that $R_j$ has its support in $L_j$. Note that the equations imply that the $\mathcal{C}$ has at most singularities over the singular points of the branch divisor $D$.

**Proposition 2.3.** The moduli stack $R(G,g,r)$ is isomorphic to the moduli stack of the following objects

$$((C,x_1,\ldots,x_r),\{L_\chi\}_\chi\in G^*,\{\mu_{\chi,\chi'}\}_{X,\chi,\chi'\in G^*}).$$

Where $(C,x_1,\ldots,x_r)$ is a smooth $r$-pointed curve, the $L_\chi$ are line bundle on $C$ for every $\chi \in G^*$ and the $m_{\chi,\chi'}$ are isomorphisms as in . The morphisms are morphisms of $r$-pointed curves that induce maps of line bundles compatible with the isomorphisms $\mu_{\chi,\chi'}$.

Proposition 2.3 has the following consequence: Consider the forgetful map

(2.3.1) $R(G,g,r) \to M_g(r),$

(2.3.2) $((C,x_1,\ldots,x_r),\{L_\chi\}_\chi\in G^*,\{\mu_{\chi,\chi'}\}_{X,\chi,\chi'\in G^*}) \mapsto (C,x_1,\ldots,x_r)$

Proposition 2.3 has the following consequence: Consider the forgetful map

(2.3.3) $R(G,g,r) \to M_g(r),$

(2.3.4) $((C,x_1,\ldots,x_r),\{L_\chi\}_\chi\in G^*,\{\mu_{\chi,\chi'}\}_{X,\chi,\chi'\in G^*}) \mapsto (C,x_1,\ldots,x_r)$

This map exhibits $R(G,g,r)$ as a principal homogenous space (or torsor) with the structure group $\text{Hom}(G^*,\text{Pic}^0(C))$ at a given point $((C,x_1,\ldots,x_r),\{L_\chi\}_\chi\in G^*,\{m_{\chi,\chi'}\}_{X,\chi,\chi'\in G^*}) \in R(G,g,r)$.

2.4. **Metabelian Galois covers.** In this subsection $G$ denotes a finite metabelian group. For a detailed treatment of metabelian covers we refer to [5] whose results and notations will be used throughout the paper. First of all, we have the following. First of all, we have the following.

**Definition 2.4.** A finite group $G$ is called metabelian (resp. metacyclic) if it sits in the following short exact sequence of groups.

(2.4.1) $0 \to A \to G \to N \to 0,$

where $A$ and $N$ are abelian (resp. cyclic). In other words a group is metabelian (resp. metacyclic) if it has an a normal subgroup $A$ such that both $A$ and $N = G/A$ are abelian (resp. cyclic).
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Notice that the above definition is equivalent to saying that metabelian groups are precisely the solvable groups of derived length at most 2.

It is straightforward to see that Definition 2.4 implies that $G$ is metabelian if and only if $G$ has the following presentation

$$\langle \sigma_1, \ldots, \sigma_s, \tau_1, \ldots, \tau_l \mid \sigma_i \sigma_j = \sigma_j \sigma_i, \tau_i \tau_j = \tau_j \tau_i, \sigma_i^{m_i} = 1, \sigma_i \tau_j = \tau_j \sigma_i^{r_{ij}} \rangle$$

Here $A = \langle \sigma_1, \ldots, \sigma_s \rangle$ and $N = \langle \tau_1, \ldots, \tau_l \rangle$, and $\tau_j$ denotes the image of $\tau_j$ in $N = G/A$.

Now let $G$ be a finite metabelian group as above, $X$ a smooth algebraic curve (equivalently a Riemann surface) over $\mathbb{C}$ with $G \subset \text{Aut}(X)$ and $Y$ a smooth algebraic curve such that $X/G = Y$ and the cover $\pi : X \rightarrow Y$ is Galois, i.e., $\pi$ is the quotient map $X \rightarrow X/G$. The factorization 2.4.1 gives rise to a factorization, $\pi : X \xrightarrow{p} Z \xrightarrow{q} Y$, where $p, q$ are the corresponding intermediate abelian Galois covers, i.e., $p : X \rightarrow Z = X/A$ is an abelian Galois covering with Galois group $A$ and $q : Z \rightarrow Y = Z/N$ is an abelian Galois covering with Galois group $N$. Therefore to study the Galois covering $\pi : X \rightarrow Y$, it is helpful to study these intermediate abelian coverings. In [5], the metabelian Galois coverings of algebraic varieties have been analyzed using the theory of abelian Galois coverings that we explained in Section 2 developed in particular in [6]. Explicitly, by Theorem 2.1, the cover $p : X \rightarrow Z$ is determined by the existence of invertible sheaves $(\mathcal{F}_\chi)_{\chi \in A^*}$, and reduced effective divisors $(D_i)$ without common components on $Z$ such that 2.1.1 holds.

Note the multiplication map $m_{\chi \chi'} : \mathcal{F}_\chi \otimes \mathcal{F}_{\chi'} \rightarrow \mathcal{F}_{\chi \chi'}$. Before stating one of the main structure theorems of [5], let us introduce the following notation: Suppose $\chi$ is an irreducible character of the abelian group $A = \langle \sigma_1, \ldots, \sigma_s \rangle$. Let $\tau_j \in N$. Since $A$ is a normal subgroup of $G$, $\tau_j^{-1} \sigma_u \tau_j \in A$ for every $u = 1, \ldots, s$. We define a new character $\chi_j^{(1)}$ of $A$ by $\chi_j^{(1)}(\sigma_u) = \chi(\tau_j^{-1} \sigma_u \tau_j)$ for every $u = 1, \ldots, s$. Since $\chi$ is an irreducible character, $\chi_j^{(1)}$ is also irreducible. In particular for each $\gamma \in \mathbb{N}$ one can define a character $\chi_j^{(\gamma)}$ of $A$ by setting $\chi_j^{(\gamma)}(\sigma_u) = \chi(\tau_j^{-\gamma} \sigma_u \tau_j^\gamma)$. By presentation 2.4.2, it is clear that $\chi_j^{(\gamma)} = \chi$. Now, we are ready to state our structure theorem for metabelian Galois covers.

**Theorem 2.5.** (Structure theorem for metabelian covers) A metabelian Galois cover $\pi : X \rightarrow Y$ is determined by the following data:
Proposition 2.6. The moduli stack $R(G,g,r)$ is isomorphic to the moduli stack of the following objects

$((C,x_1,\ldots,x_r),\{L_\eta\}_{\eta \in N^*},\{\phi_{R,\eta}\}_{\eta \in N^*},\{\mathcal{F}_\chi\}_{\chi \in A^*},\{\mu_{\chi,\chi'}\}_{\chi,\chi' \in A^*})$.

Where $(C,x_1,\ldots,x_r)$ is a smooth $r$-pointed curve, the $L_\eta$ are line bundle on $C$ for every $\eta \in N^*$ and the $\phi_{R,\eta}$ are isomorphisms as in (2.1.1). The $\mathcal{F}_\chi$ are line bundle on $C$ for every $\chi \in A^*$ and the $\mu_{\chi,\chi'}$ are isomorphisms. The morphisms are morphisms of $r$-pointed curves that induce maps of line bundles compatible with the isomorphisms $\mu_{\chi,\chi'}$.

3. Prym varieties and the Prym map

3.1. Generalities. In general, one can associate a Prym variety to a subtorus of an abelian variety $A$. We follow [7]. Let $A$ be a principally polarized abelian variety over $\mathbb{C}$ and let $X \overset{i}{\to} A$
be a subtorus. We denote by \( \lambda : A \to \hat{A} \) the principal polarization of \( A \). The Prym variety of \( X \) in \( A \) is defined as \( P = P(A,\lambda, X) = \lambda^{-1}(\ker \tilde{i}) \).

Now let \( f : \tilde{C} \to C \) be a covering map between smooth algebraic curves as in introduction.

Let us denote the Jacobians of the curves \( \tilde{C} \) and \( C \) respectively by \( \tilde{J} \) and \( J \). Note that by definition, if \( R \) is a Riemann surface,

\[
J(R) = \text{Jac}(R) = H^0(R, \omega_R)^*/H_1(R, \mathbb{Z}).
\]

Since the finite group \( G \) acts on \( \tilde{C} \) it also acts on the space of differential 1-forms \( H^0(\tilde{C}, \omega_{\tilde{C}}) \) and \( H_1(\tilde{C}, \mathbb{Z}) \) and hence on the Jacobian \( \tilde{J} \). In particular, we denote by \( \tilde{J}^G \) the subgroup of fixed points of \( \tilde{J} \) under the action of \( G \). The following theorem is proven in [7] (respectively, Theorem 2.5 and Proposition 3.1).

**Theorem 3.1.**

1. \( P = P(\tilde{C}/C) \) is the Prym variety of the abelian subvariety \( f^*J \) of the principally polarized abelian variety \( \tilde{J} \), i.e., \( P = P(\tilde{C}/C) = P(\tilde{J}, \tilde{\lambda}, f^*J) \).
2. \( f^*J = (\tilde{J}^G)^0 \).
3. The map \( f \) induces an isogeny \( J \times P(\tilde{C}/C) \sim \tilde{J} \)

We note that the isogeny mentioned in Theorem 3.1 is given by

\[
\phi : J \times P(\tilde{C}/C) \to \tilde{J}
\]

\[
\phi(c, \tilde{c}) = f^*c + \tilde{c}
\]

By the above mentioned \( G \)-action on \( H^0(\tilde{C}, \omega_{\tilde{C}}) \) and \( H_1(\tilde{C}, \mathbb{Z}) \), we set:

\[
H^0(\tilde{C}, \omega_{\tilde{C}})^+ = H^0(\tilde{C}, \omega_{\tilde{C}})^G(\cong H^0(C, \omega_C)) \text{ and } H^0(\tilde{C}, \omega_{\tilde{C}})^- = H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(\tilde{C}, \omega_{\tilde{C}})^+ = \bigoplus_{\chi \in \text{Irr}(G) \setminus \{1\}} H^0(\tilde{C}, \omega_{\tilde{C}})^\chi
\]

Notice that \( H^0(\tilde{C}, \omega_{\tilde{C}}) = H^0(\tilde{C}, \omega_{\tilde{C}})^+ \oplus H^0(\tilde{C}, \omega_{\tilde{C}})^- \).

The following lemma is then an immediate consequence of Theorem 3.1 above.

**Lemma 3.2.** Let \( f : \tilde{C} \to C \) be a Galois covering, then

\[
P(\tilde{C}/C) = H^0(\tilde{C}, \omega_{\tilde{C}})^-/H_1(\tilde{C}, \mathbb{Z})^-
\]

For a Galois covering \( f : \tilde{C} \to C \) of a curve \( C \) of genus \( g \) as above, one can compute the genus \( \tilde{g} = g(\tilde{C}) \) by the Riemann-Hurwitz formula. Using the isogeny \( f^*J \times P(\tilde{C}/C) \sim \tilde{J} \) we see
that the dimension of the Prym variety \( P(\tilde{C}/C) = P(f) \) is equal to \( p = \tilde{g} - g \). The canonical principal polarization on \( \tilde{J} \) restricts to a polarization of type \( \delta = (1, \ldots, 1, n, \ldots, n) \) where 1 occurs \( p - (g - 1) \) times and \( n \) occurs \( g - 1 \) times if \( r = 0 \) and 1 occurs \( p - g \) times and \( n \) occurs \( g \) times otherwise.

The canonical quotient map \( H^0(\tilde{C}, \omega_{\tilde{C}}) \to H^0(\tilde{C}, \omega_{\tilde{C}})/H^0(\tilde{C}, \omega_{\tilde{C}})^+ \) induces a map \( \Phi: \tilde{J} \to P(\tilde{C}/C) \). In fact this map induces the isogeny \( \tilde{J} \sim J \times P(\tilde{C}/C) \) mentioned in Theorem 3.1. Let \( \tilde{u}: \tilde{C} \to \tilde{J} \) be the Abel-Jacobi map for the curve \( \tilde{C} \). We have the following commutative diagram.

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{u} & P(\tilde{C}/C) \\
\downarrow{\tilde{u}} & & \downarrow{\Phi} \\
\tilde{J} & & \\
\end{array}
\]

In analogy with the Abel-Jacobi map, we call the map \( u: \tilde{C} \to P(\tilde{C}/C) \) the Abel-Prym map of the covering \( f \). We summarize some of the consequences of the above discussions in Lemma 3.3.

**Lemma 3.3.** Let \( A = V/\Lambda \) be a complex abelian variety. Denote by \( \Omega_{A,0} \) (resp. \( T_{A,0} \)) the cotangent space (resp. tangent space) of \( A \) at the origin and by \( T_A \) be the tangent bundle of the abelian variety \( A \). The rest of the notations be as in Theorem 3.1. Then it holds

1. \( \Omega_{A,0} \cong V^* \) (and hence \( T_{A,0} \cong V \)). In particular for the abelian variety \( P := P(\tilde{C}/C) \) we have \( \Omega_{P,0} \cong H^0(\tilde{C}, \omega_{\tilde{C}})^- \) and \( T_{P,0} = H^1(\tilde{C}, \Omega_{\tilde{C}})^- \).

2. \( T_A \cong T_{A,0} \otimes O_A \cong V \otimes O_A \) and \( H^1(A, T_A) \cong H^1(A, O_A)^{\otimes 2} \cong V^{\otimes 2} \). In particular, \( T_{\tilde{J}} \cong T_{\tilde{J},0} \otimes O_{\tilde{J}} \) and \( H^1(\tilde{J}, T_{\tilde{J}}) \cong H^1(\tilde{C}, \Omega_{\tilde{C}})^{\otimes 2} \) and furthermore \( H^1(P, T_P) \cong H^1(\tilde{C}, \Omega_{\tilde{C}})^{-\otimes 2} \) and \( H^1(\tilde{C}, u^* T_P) \cong H^1(\tilde{C}, \Omega_{\tilde{C}})^{- \otimes H^1(\tilde{C}, O_{\tilde{C}})}. \)

**Proof.** The first assertion of (1) is proved in [2] and the second one follows from this together with [3, 2.1] and the Serre duality.

The first assertion of (2), namely the triviality of the tangent bundle of an abelian variety is also well-known, see [2], §1.4 and the isomorphism of \( H^1(A, T_A) \) follows from this and the Serre duality again. The rest of the isomorphisms in (2) for the cohomology of the tangent bundle of the Jacobian and the Prym variety follow from the first part of (2). \( \square \)

Let

\[
(3.3.1) \quad \mathcal{P} = \mathcal{P}(G, g, r): R(G, g, r) \to A_{p, \delta}
\]
be the Prym map (of type $(G, g, r)$) associated to the above families as in introduction. In the sequel, we would like to compute the (co)differential of the map $P$ at a given point $((C, x_1, \ldots, x_r), f : \tilde{C} \to C) \in R(G, g, r)$. Therefore, we first explain the general set-up for this problem and in later sections do the computations in some special cases. Using the Proposition 2.6, one sees that the forgetful map $2.3.3$ is a principal homogeneous space over $M_g(r)$. The tangent space to $R(G, g, r)$ at a point $((C, x_1, \ldots, x_r), f : \tilde{C} \to C)$ is isomorphic to $H^1(C, \mathcal{T}_C(-D))$, where $\mathcal{T}_C$ denotes the tangent bundle of the curve $C$, see §3.4.3 in particular Example 3.4.14. Note that there is an isomorphism $H^1(C, \mathcal{T}_C(-D))^\psi \cong H^0(C, \omega_C^\otimes 2(D))$. The cotangent space to $A_p, \delta$ at the point $P := P(\tilde{C}/C)$ is isomorphic to $S^2(H^0(\Omega^1_C)) \cong S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-)$. By 3.1.2, we have

\[(3.3.2)\]

$$S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-) \cong \bigoplus_{\chi \in \text{Irr}(G) \setminus \{1\}} S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^{\chi}) \oplus \bigoplus_{\chi \in \text{Irr}(G) \setminus \{1\}} H^0(\tilde{C}, \omega_{\tilde{C}})^{\chi} \otimes H^0(\tilde{C}, \omega_{\tilde{C}})^{\eta},$$

where $\text{Irr}(G)$ denotes the set of irreducible representations of $G$. On the other hand, the action of the group $G$ on $H^0(\tilde{C}, \omega_{\tilde{C}})$ induces a natural $G$-action on the space $S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-)$. Let $\psi \in \text{Irr}(G)$ be an irreducible character of $G$ with the corresponding representation $\rho_{\psi}$. The eigenspace $S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-)^{\psi}$ of $S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-)$ corresponding to $\psi$ is

\[(3.3.3)\]

$$S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-)^{\psi} = \bigoplus_{\chi \in \text{Irr}(G) \setminus \{1\}} S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^{\chi}) \oplus \bigoplus_{\chi \in \text{Irr}(G) \setminus \{1\}} H^0(\tilde{C}, \omega_{\tilde{C}})^{\chi} \otimes H^0(\tilde{C}, \omega_{\tilde{C}})^{\eta}.$$

One obtains the following commutative diagram

\[(3.3.4)\]

$$\begin{array}{c}
S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-) \xrightarrow{p_1} H^0(\tilde{C}, \omega_{\tilde{C}})^{-} \\
\oplus H^0(\tilde{C}, \omega_{\tilde{C}})^{\chi} \otimes H^0(\tilde{C}, \omega_{\tilde{C}})^{-} \xrightarrow{\mu} H^0(\tilde{C}, \omega_{\tilde{C}})^{G},
\end{array}$$

in which the top horizontal arrow is given by

$$S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-) \to S^2(H^0(\tilde{C}, \omega_{\tilde{C}})^-)^{\psi} \xrightarrow{\text{pr}_2} \bigoplus_{\chi \in \text{Irr}(G) \setminus \{1\}} H^0(\tilde{C}, \omega_{\tilde{C}})^{\chi} \otimes H^0(\tilde{C}, \omega_{\tilde{C}})^{\eta} \xrightarrow{m} H^0(\tilde{C}, \omega_{\tilde{C}})^{\psi}$$

where $\text{pr}_2$ is the projection to the second sum in (3.3.3) and $m$ is the multiplication of differential forms, vertical arrows are natural projections in (3.3.3) to the $G$-invariant subspaces and the bottom horizontal arrow is again the multiplication of differential forms.
The Prym map. Let \( R(G, g, r) \) be as in introduction with \(|G| = n\). By the definition of the Prym variety \( P \), the tangent space \( T_P \) of \( P \) at the origin is

\[
T_P = (T_{J(\tilde{C})})^- = (H^0(\tilde{C}, \omega_{\tilde{C}}^-))^* = H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^-.
\]

The equation 3.3.5 implies that the cotangent space \( T_P^* \) of \( P \) at the origin is

\[
T_P^* = H^0(\tilde{C}, \omega_{\tilde{C}}^-).
\]

As explained earlier, the tangent space of \( R(G, g, r) \) at a point \( ((C, x_1, \ldots, x_r), f : \tilde{C} \to C) \) is \( H^1(C, \mathcal{T}_C(-B)) \cong H^0(C, \omega_C^2(B)) \), see [8], §3.4.3, especially Example 3.4.14. The isomorphism is the Serre duality. We have

**Proposition 3.4.** The codifferential \( dP \) of the Prym map can be identified with the canonical map

\[
\varphi : S^2(H^0(\tilde{C}, \omega_{\tilde{C}}^-)) \to H^0(C, \omega_C^2(B)),
\]

which is the composition \( \mu \circ p \) of 3.3.4.

In order to prove the above result, we will need the following lemma, which is [4], Lemma 4.2.

**Lemma 3.5.** Let \( X \) be a smooth projective curve and \( A = V/\Lambda \) an abelian variety. If \( u : X \to A \) is a non-constant morphism whose image generated \( A \), then the dual of the differential

\[
H^1(du) : H^1(X, \mathcal{T}_X) \to H^1(X, u^*\mathcal{T}_A)
\]

coincides with the multiplication of sections

\[
V^* \otimes H^0(X, \omega_X) \to H^0(X, \omega_X^2)
\]

Recall from 3.2.2 that the Abel-Prym map \( u : \tilde{C} \to P \) factors through Abel-Jacobi map \( u : J(\tilde{C}) \to P \). Moreover, we have an isogeny \( t : P \to \hat{P} \) induced by the polarization of \( P \). This induces an isomorphism \( T_P \cong T_{\hat{P}} \). Therefore we can identify \( T_P \) with \( H^1(P, \mathcal{O}_P) \) and by ?? with \( H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^- \). Moreover, by the fact that the tangent bundle of the abelian variety \( J(\tilde{C}) \) is trivial and its fiber over 0 is \( T_{\tilde{C}} \) one obtains \( T_{\tilde{C}} \cong T_{\tilde{C}} \otimes \mathcal{O}_{J(\tilde{C})} \) and this induces an isomorphism \( H^1(J(\tilde{C}), \mathcal{T}_{J(\tilde{C})}) = H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^{\otimes 2} \). Furthermore, the isomorphism \( T_P \cong T_{\hat{P}} \otimes \mathcal{O}_P \) gives the identifications \( H^1(P, \mathcal{T}_P) = H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^- \otimes H^1(P, \mathcal{O}_P) = H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^{-\otimes 2} \) and \( H^1(\tilde{C}, u^*\mathcal{T}_P) = \)
Lemma 3.2. In view of the above equalities, one obtains

\[(3.5.1) \quad H^1(du): H^1(\tilde{C}, O_{\tilde{C}}) \xrightarrow{H^1(d\tilde{u})} H^1(J(\tilde{C}), T_J(\tilde{C})) \xrightarrow{\varphi^* \circ p} H^1(P, T_P) \xrightarrow{u^*} H^1(\tilde{C}, u^* T_P)\]

where \(p^*\) is the projection to the subspace \(H^1(\tilde{C}, O_{\tilde{C}})^-\) (as in [3.2.2]).

**Proof.** (of Proposition 3.4) Let \(((C, x_1, \ldots, x_r), f : \tilde{C} \to C)\) be a point of \(R(G, g, r)\). The tangent space of \(A_{p, \delta}\) at the point \(P\) is equal to \(S^2 T_P = S^2 H^1(\tilde{C}, O_{\tilde{C}})^-\). The product \(H^1(\tilde{C}, T_{\tilde{C}}) \times H^0(\tilde{C}, \omega_{\tilde{C}}) \to H^1(\tilde{C}, O_{\tilde{C}})\) respects the group action and hence induces \(H^1(\tilde{C}, T_{\tilde{C}})^+ \times H^0(\tilde{C}, \omega_{\tilde{C}})^- \to H^1(\tilde{C}, O_{\tilde{C}})^-\). The induced map \(H^1(\tilde{C}, T_{\tilde{C}})^+ \to H^0(\tilde{C}, \omega_{\tilde{C}})^- \otimes H^1(\tilde{C}, O_{\tilde{C}})^-\) is symmetric so that we get a map \(H^1(\tilde{C}, T_{\tilde{C}})^+ \to S^2 H^1(\tilde{C}, O_{\tilde{C}})^-\). This is the differential of the Prym map \(\mathcal{P} : R(G, g, r) \to A_{p, \delta}\) at the point \(((C, x_1, \ldots, x_r), f : \tilde{C} \to C)\). Now \((3.5.1)\) implies that this map can be considered as a map \(H^1(\tilde{C}, T_{\tilde{C}})^+ \to H^1(\tilde{C}, u^* T_P)\) whose image lies in \(S^2 H^1(\tilde{C}, O_{\tilde{C}})^-\). Note the identities shown above. Lemma 3.3 then shows that the differential \(d\mathcal{P}\) at the point \(((C, x_1, \ldots, x_r), f : \tilde{C} \to C)\) is the multiplication map \(S^2 H^0(\tilde{C}, \omega_{\tilde{C}})^-) \to H^0(C, \omega_C \otimes \omega_{\tilde{C}}^2)\). \(\square\)

### 3.2. Prym varieties of abelian covers

In this section explain the constructions in section 3.1 for an abelian group \(G\) based on the constructions of section 2.2. So let \(f : \tilde{C} \to C\) be a \(G\)-Galos cover of \(C\), with \(G\) a finite abelian group. We have

\[(3.5.2) \quad H^0(\tilde{C}, \omega_{\tilde{C}}) = H^0(C, f_* \omega_{\tilde{C}}) = H^0(C, \oplus (\omega_C \otimes L_{\chi^{-1}})) = \oplus_{\chi \in G^*} H^0(C, \omega_C \otimes L_{\chi^{-1}})\]

where the second equality is due to the equality \((f_* \omega_{\tilde{C}})^\chi = \omega_C \otimes L_{\chi^{-1}}\) for abelian covers, see [4]. In view of the above equalities, one obtains

\[(3.5.3) \quad H^0(\tilde{C}, \omega_{\tilde{C}})^+ = H^0(C, \omega_C)\]

\[(3.5.4) \quad H^0(\tilde{C}, \omega_{\tilde{C}})^- = \oplus_{\chi \in G^* \setminus \{1\}} H^0(\tilde{C}, \omega_{\tilde{C}})^{\chi} = \oplus_{\chi \in G^* \setminus \{1\}} H^0(C, \omega_C \otimes L_{\chi^{-1}})\]

So \(P = P(\tilde{C}/C) = \oplus_{\chi \in G^* \setminus \{1\}} H^0(C, \omega_C \otimes L_{\chi^{-1}})/ \oplus_{\chi \in G^* \setminus \{1\}} H_1(\tilde{C}, Z)^{\chi}\) by virtue of (3.5.3) and Lemma 3.2.
In particular, in the abelian case the multiplication map $\mu$ in the diagram \[\mu : \oplus_{x \in G} H^0(C, \omega_C \otimes L_x) \otimes H^0(C, \omega_C \otimes L_{x^{-1}}) \to H^0(C, \omega_C^{\otimes 2}(D_{x, x^{-1}}))\] takes the following form.

\[\mu \,: \, \oplus_{x \in G} \, H^0(C, \omega_C \otimes L_x) \otimes H^0(C, \omega_C \otimes L_{x^{-1}}) \to H^0(C, \omega_C^{\otimes 2}(D_{x, x^{-1}}))\]

The Abel-Prym map. Let $G$ be a finite abelian group such that $G = \langle \sigma_1 \rangle \oplus \langle \sigma_2 \rangle \cdots \oplus \langle \sigma_h \rangle$. In other words $\sigma_1, \ldots, \sigma_h$ are independent generators of $G$. Let $P = P(\tilde{C}/C)$ be the Prym variety associated to the $G$-Galois cover $f : \tilde{C} \to C$ with the Abel-Prym map $u : \tilde{C} \to P$. In this case the map $\Phi$ in 3.2. is the map $(1 - \sigma_1) \cdots (1 - \sigma_h)$ and so $P = \text{Im}((1 - \sigma_1) \cdots (1 - \sigma_h))$. Suppose furthermore that $\text{ord}(\sigma_i) > 2$ for every $i$.

**Proposition 3.6.** Suppose the curve $\tilde{C}$ is not a $g^1_{2h}$. Then $u(p) = u(q)$ if and only if $p$ and $q$ are ramification points of $f$. In particular, if in addition $f$ is unramified, then $u$ is injective.

**Proof.** Let $c \in \tilde{C}$ be an arbitrary base point giving the Abel-Jacobi map $\tilde{u} : \tilde{C} \to \tilde{J}, p \mapsto p - c$. By definition $u(p) = u(q)$ if and only if

\[ (1 - \sigma_1) \cdots (1 - \sigma_h)(p - c) \sim (1 - \sigma_1) \cdots (1 - \sigma_h)(q - c) \]

Or equivalently $(1 - \sigma_1) \cdots (1 - \sigma_h)(p) \sim (1 - \sigma_1) \cdots (1 - \sigma_h)(q)$. After expanding and rearranging this, so that both sides are effective divisors, the assumption that $\tilde{C}$ is not a $g^1_{2h}$ implies that $p$ (resp. $q$) satisfies an equation of the form $\sigma_{1, i}^r \sigma_{2, i}^z \cdots \sigma_{h, i}^{r_i}(p) = p$ with $r_j = 1$ or 2. Since $\text{ord}(\sigma_1) > 2$ and $\sigma_i$ are independent generators by assumption, it follows that the above equation is non-trivial and hence $p$ (resp. $q$) is a ramification point. \(\square\)

The following example shows that if the assumptions of Proposition 3.6 are not satisfied, then its result will no longer be valid even for fairly simple covers.

**Example 3.7.** Suppose $f : \tilde{C} \to C$ is a Galois cover with the Galois group $G = \langle \sigma_1 \rangle \oplus \langle \sigma_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (so $\text{ord}(\sigma_1) = 2$). Suppose $p \in \tilde{C}$ is not a ramification point. Viewing $\sigma_1, \sigma_2$ as deck automorphisms of the curve $\tilde{C}$, set $q = \sigma_1 \sigma_2(p)$. Then $q$ is also not a ramification point and $(1 - \sigma_1)(1 - \sigma_2)(p) = (1 - \sigma_1)(1 - \sigma_2)(q)$ as divisors so that $u(p) = u(q)$. 

Injectivity of the differential. By Proposition 3.4 the injectivity of \( dP \) at a point \(((C, x_1, \ldots, x_r), f : \tilde{C} \to C)\) is equivalent to the surjectivity of the multiplication map

\[
\mu : H^0(C, \omega_C \otimes L_\chi) \otimes H^0(C, \omega_C \otimes L_{\chi^{-1}}) \to H^0(C, \omega_C \otimes^2 (D_{\chi, \chi^{-1}})),
\]

for every character \( \chi \in G^* \), where \( D_{\chi, \chi^{-1}} \) is the divisor in 2.1.2.

We first consider the case of étale coverings \((r = 0)\). In this case we abbreviate \( R(G, g, 0) \) by \( R(G, g) \) which is the moduli space of unramified \( G \)-Galois coverings of curves of genus \( g \).

**Proposition 3.8.** Let \( C \) be a smooth projective curve and \( G \) a finite abelian group with \( |G| = n \). In the following cases the differential \( dP \) of the Prym map \( P : R(G, g) \to A_{p, \delta} \) at a given point \(((C, x_1, \ldots, x_r), f : \tilde{C} \to C)\) is injective.

1. If \( n \) is an even number and \( \text{Cliff}(C) \geq 3 \).
2. If \( \text{Cliff}(C) \geq 2n - 1 \).

**Proof.** Take a reduced building data as in 2.1.2. Since by assumption, the covering is unramified, these relations are of the form \( L_i \cong O_C \). So \( \deg L_i = 0 \), and the Riemann-Roch together with [4], Corollary 2.3 implies that both \( \omega \otimes L_i \) and \( \omega \otimes L_i^{-1} \) are very ample. Finally, [3], Theorem 1 shows that

\[
\mu : H^0(C, \omega \otimes L_i) \otimes H^0(C, \omega \otimes L_i^{-1}) \to H^0(C, \omega_C \otimes^2 (D_{\chi, \chi^{-1}})),
\]

is surjective and this gives desired result.

**Proposition 3.9.** For \( n \geq 2 \) and \( g \geq 7 \) the Prym map \( P : R(G, g) \to A_{p, \delta} \) is generically finite.

**Proof.** Given a point \(((C, f : \tilde{C} \to C)) \in R(G, g)\), it suffices to show that the differential \( dP \) is injective at \((C, f : \tilde{C} \to C)\). This is equivalent to surjectivity of \( \mu : H^0(C, \omega_C \otimes L_\chi) \otimes H^0(C, \omega_C \otimes L_{\chi^{-1}}) \to H^0(C, \omega_C \otimes^2 (D_{\chi, \chi^{-1}})) \). Now take an \( L_i \) from a reduced building data \( 2.2.1 \). By [3], Theorem 1, this is satisfied if \( \omega_C \otimes L_i \) is ample which follows from [4], Lemma 5.4 and that the general curve of genus \( g \geq 7 \) satisfies \( \text{Cliff}(C) \geq 3 \).

Now we treat the case of ramified Galois covers. Recall from Theorem 2.2 the reduced building data of the cover. Let \( n_i \) and \( d_i \) be as in 2.2.1 Then we have
Proposition 3.10. For \( g \geq 2 \) assume that there exists \( n_i \) which is even and \( d_i \geq 6 \) or \( n_i \) is odd and \( d_i \geq 7 \). Then the differential \( dP \) of the Prym map \( P : R(G,g,r) \to A_{p,\delta} \) at the point \(((C,x_1,\ldots,x_r),f:\tilde{\mathcal{C}} \to C)\) is injective.

Proof. By our approach based on Proposition 3.4, we will show that our hypotheses imply the surjectivity of \( \mu : H^0(C,\omega_C \otimes L_1^{n_1/2}) \otimes H^0(C,\omega_C \otimes L_1^{n_1-1/2}) \to H^0(C,\omega_C^2(D_1)) \), where \( L_1 \) belongs to a reduced building data. By [3], Theorem 1, this is the case if both bundles \( \omega_C \otimes L_1^{n_1/2} \) and \( \omega_C \otimes L_1^{n_1-1/2} \) are very ample. As \( \deg(\omega_C) = 2g - 2 \), we only need to verify that \( \deg(L_1^{n_1/2}) \geq 3 \) and \( \deg(L_1^{n_1-1/2}) \geq 3 \). But \( \deg(L_1^{n_1/2}) \geq \lfloor \frac{d_1}{2} \rfloor \geq 3 \). Proof of \( \deg(L_1^{n_1-1/2}) \geq 3 \) is analogous.

One can formulate a condition without using the structure of a reduced building data. Let \( d_\chi = \deg L_\chi \).

Proposition 3.11. For \( g \geq 2 \) assume that there exists a character \( \chi \in G^* \) such that \( d_\chi \geq 3 \) and \( d_\chi - 1 \geq 3 \) or that \( L_\chi \) and \( L_\chi - 1 \) have non-zero global sections and \( d_\chi + d_\chi - 1 \geq 5 \). Then the differential \( dP \) of the Prym map \( P : R(G,g,r) \to A_{p,\delta} \) at the point \(((C,x_1,\ldots,x_r),f:\tilde{\mathcal{C}} \to C)\) is injective.

Proof. The multiplication map takes the form \( \mu : H^0(C,\omega_C \otimes L_\chi) \otimes H^0(C,\omega_C \otimes L_\chi - 1) \to H^0(C,\omega_C^2(D_{\chi,\chi - 1})) \) and the first condition implies that both \( \omega_C \otimes L_\chi \) and \( \omega_C \otimes L_\chi - 1 \) are very ample. Now use [3], Theorem 1. The second condition implies that \( L_\chi \) and \( L_\chi - 1 \) are globally generated and \( \deg(\omega_C \otimes L_\chi) + \deg(\omega_C \otimes L_\chi - 1) \geq 4g + 1 \). Again [3] implies that the multiplication map is surjective.

3.3. Prym map of metabelian covers. In this subsection we investigate the Prym map for metabelian Galois covers \( f : \tilde{\mathcal{C}} \to C \). Recall the formula [2.5.1] \( \pi_* \omega_X = \oplus (\omega_Y \otimes L' \otimes U_\chi) \), where \( U_\chi = q_* \mathcal{F}_X \) (\( q \) is the finite map in the associated factorization \( \tilde{f} : \tilde{\mathcal{C}} \to C_1 \to C \) as at the beginning of [2.3].

Here the conditions are much more complicated because we should deal with vector bundles instead of line bundles.
Let $\omega_C \otimes L'$ have a non-zero global section. Furthermore suppose there exists a $U_\chi$ such that both $U_\chi$ and $U_{\chi^{-1}}$ are generated by global sections on $C$ and that

\[(3.11.1) \quad h^0(\omega_C^2 \otimes L'^{\otimes 2} \otimes U_\chi \otimes U_{\chi^{-1}}) \leq t(h^0(\omega_C \otimes L' \otimes U_\chi) + h^0(\omega_C \otimes L' \otimes U_{\chi^{-1}})) - t^2.\]

Where $t$ is the degree of the map $q$.

**Proposition 3.12.** Let $C$ satisfy the above conditions, then the differential $dP$ is injective at the point $[f : \tilde{C} \to C]$.

**Proof.** Our assumptions imply that the invertible sheaf $\omega_C \otimes L'$ is generated by global sections. Since both $U_\chi$ and $U_{\chi^{-1}}$ are also globally generated by assumption, [1], Theorem 2.1 implies that the image of the multiplication map

$$H^0(\omega_C \otimes L' \otimes U_\chi) \otimes H^0(\omega_C \otimes L' \otimes U_{\chi^{-1}}) \to H^0(\omega_C^2 \otimes L'^{\otimes 2} \otimes U_\chi \otimes U_{\chi^{-1}})$$

has dimension $\geq t(h^0(\omega_C \otimes L' \otimes U_\chi) + h^0(\omega_C \otimes L' \otimes U_{\chi^{-1}})) - t^2$. Condition (3.11.1) implies that the multiplication is surjective and therefore the differential $dP$ of the Prym map is injective at $[f : \tilde{C} \to C]$. \qed

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