THE HOCHSCHILD COHOMOLOGY OF THE ENVELOPING ALGEBRA OF A LIE–RINEHART PAIR

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Abstract. Let \((S, L)\) be a Lie–Rinehart pair such that \(L\) is \(S\)-projective and let \(U\) be its universal enveloping algebra. The purpose of this paper is to present a spectral sequence which converges to the Hochschild cohomology of \(U\) and whose second page involves the Lie–Rinehart cohomology of the pair and the Hochschild cohomology of \(S\) with values on \(U\).

Introduction

Let us fix a ground field \(k\). A Lie–Rinehart pair \((S, L)\) consists of a commutative algebra \(S\) and a Lie algebra \(L\) with an \(S\)-module structure that acts on \(S\) by derivations and which satisfies certain compatibility conditions. An important example is the pair \((S, \text{Der} S)\) with second component the Lie algebra of derivations of \(S\). Let \(U\) be the universal enveloping algebra of \((S, L)\). If \(M\) is a \(U\)-module, the Lie–Rinehart cohomology of the pair with values on \(M\) was defined by G. Rinehart in [Rin63] as \(H^\bullet(L|S, M) = \text{Ext}^\bullet_U(S, M)\). This generalizes the usual Lie algebra cohomology of \(L\) by taking into account its interaction with \(S\); see the article [Hue90] by J. Huebschmann for this.

We are interested in computing the Hochschild cohomology \(HH^\bullet(U)\) of the enveloping algebra \(U\). Our main result is the construction of a spectral sequence converging to it.

Theorem. If \(L\) is \(S\)-projective then there is a first-quadrant spectral sequence \(E_2\) converging to \(HH^\bullet(U)\) with second page

\[ E_2^{p,q} = H^p(L|S, H^q(S, U)) \].

In particular, there is an \(U\)-module structure on \(H^\bullet(S, U)\), the Hochschild cohomology of \(S\) with values on \(U\). We construct it using an \(U^e\)-injective resolution of \(U\) and, later, provide an alternative realization in terms of an \(S^e\)-projective resolution of \(S\), which is needed for explicit computations.

In many examples, some of which we mention throughout the article, the spectral sequence in the Theorem degenerates in the second page and therefore it allows us to...
obtain $H H^\bullet(U)$ as a graded vector space. We do not know, and do not expect, that it degenerates for every Lie–Rinehart pair.

In Section 1 we recall the definition of Lie–Rinehart pairs, their universal enveloping algebras and their cohomology theory. In Sections 2 and 3, which are the most important, we present the spectral sequence and describe the Lie module structure on $H^\bullet(S, U)$; with this in hand we work out a minimal example. In Section 4 we show how the method works in the case of the Lie–Rinehart pair that arises from a central arrangement of lines and, finally, in Section 5 we adapt a result by M. Suárez-Álvarez from [SÁ07] to obtain some information of the differential of the second page.

We will denote the tensor product over the base field $k$ simply by $\otimes$ or, sometimes, by $|$. Unless it is otherwise specified, all vector spaces and algebras will be over $k$. Given an associative algebra $A$, the enveloping algebra $A^e$ is the vector space $A \otimes A$ endowed with the product $\cdot$ defined by $a_1 \otimes a_2 \cdot b_1 \otimes b_2 = a_1 b_1 \otimes b_2 a_2$, so that the category of $A^e$-modules is equivalent to that of $A$-bimodules. The Hochschild cohomology of $A$ with values on an $A^e$-module $M$ is defined as $\text{Ext}^\bullet_{A^e}(A, M)$ and will be denoted by $H^\bullet(A, M)$ or, if $M = A$, by $H H^\bullet(A)$. The book [Wei94] by C. Weibel may serve as general reference on this subject.

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1. Lie–Rinehart pairs

In his thesis, and in [Rin63], G. Rinehart defines and studies homological properties of what we now call a Lie–Rinehart pair. A Lie–Rinehart pair $(S, L)$ consist of a commutative $k$-algebra $S$ and a $k$-Lie algebra $L$ such that $L$ acts on $S$ by $k$-linear derivations, $L$ is an $S$-module and

$$(sa)(t) = s(\alpha(t)), \quad [\alpha, s\beta] = s[\alpha, \beta] + \alpha(s)\beta$$

for $s, t$ in $S$ and $\alpha$ and $\beta$ in $L$. Given such a pair, a Lie–Rinehart module—or $(S, L)$-module—is a vector space $M$ that is at the same time an $S$-module and an $L$-Lie module in such a way that

$$(sa)(m) = s(\alpha(m)), \quad \alpha(sm) = s\alpha(m) + \alpha(s)m$$

for $s \in S$, $\alpha \in L$ and $m \in M$. A first important example is given by $M = S$, with the obvious actions of $S$ and of $L$.

Example 1.1. If $g$ is a Lie algebra, the pair $(k, g)$ is a Lie–Rinehart pair and any $g$-Lie module is a $(k, g)$-module.

Example 1.2. Any commutative algebra $S$ together with a Lie subalgebra $L$ of the algebra of derivations $\text{Der} S$ that is at the same time an $S$-module forms a Lie–Rinehart pair. In particular, for $S = k[x]$ we can take the full algebra of derivations $L = \text{Der} S$, which
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is freely generated as an \( S \)-module by the derivation \( \partial : f \mapsto f' \). The Weyl algebra \( A_1 = \frac{k[x,\partial]}{(\partial x - x\partial - 1)} \) is an \( (S,L) \)-module with actions induced by left multiplications.

**Example 1.3.** Given a finite dimensional manifold \( M \), we obtain a Lie–Rinehart pair setting \( S = C^\infty(M) \), the algebra of smooth functions, and \( L = \mathfrak{X}(M) \), the Lie-algebra of vector fields on \( M \). This is a special case of Example 1.2. Let \( E \to M \) be a smooth vector bundle on \( M \) and \( \Gamma \) be the space of smooth sections of \( E \): an \( (S,L) \)-module structure on \( \Gamma \) compatible with the usual \( S \)-module structure can be identified with a linear connection on \( E \to M \) with zero curvature.

**Example 1.4.** Another instance of the Example 1.2 arises from hyperplane arrangements. A hyperplane arrangement \( \mathcal{A} \) in a finite dimensional vector space \( V \) is a finite set of hyperplanes. The Lie algebra of derivations of the arrangement is

\[
\text{Der} \, \mathcal{A} := \{ \theta \in \text{Der}_k(S) : \alpha \text{ divides } \theta(\alpha) \text{ if } \ker \alpha \in \mathcal{A} \}.
\]

It is straightforward to check that the algebra of coordinates functions \( S = k[x_1, \ldots, x_l] \) of \( V \) and \( L = \text{Der} \, \mathcal{A} \) form a Lie–Rinehart pair —we refer to the book [OT92] by P. Orlik and H. Terao for a general reference on this subject.

As shown in [Hue90], there is an associative algebra \( U = U(S,L) \), the *universal enveloping algebra of a pair* \( (S,L) \), endowed with a morphism of algebras \( i : S \to U \) and a morphism of Lie algebras \( j : L \to U \) that satisfy, for \( s \in S \) and \( \alpha \in L \),

\[
i(s)j(\alpha) = j(s\alpha), \quad j(\alpha)i(s) - i(s)j(\alpha) = i(\alpha(s))
\]

and universal with these properties. The point of this construction is that the category of \( U \)-modules is isomorphic to that of \( (S,L) \)-modules. As a particular example, we see that \( S \) is an \( U \)-module.

**Example 1.5.** If \( g \) is Lie algebra, the universal enveloping algebra of the pair \( (k, g) \) is simply the usual enveloping algebra of \( g \).

**Example 1.6.** If \( S = k[x_1, \ldots, x_n] \), then full Lie algebra of derivations \( L = \text{Der} \, S \) is freely generated as an \( S \)-module by the \( n \) derivations \( y_i = \frac{\partial}{\partial x_i} : S \to S \) with \( 1 \leq i \leq n \). The enveloping algebra, in this case, admits the presentation

\[
k(x_1, y_i : 1 \leq i \leq n) / (y_i x_j - x_j y_i - \delta_{ij}),
\]

so it isomorphic to the algebra of differential operators \( \text{Diff}(S) = A_n \), the \( n \)th Weyl algebra.

**Example 1.7.** In the situation of Example 1.3, the enveloping algebra of the Lie–Rinehart pair \( (C^\infty(M), \mathfrak{X}(M)) \) is isomorphic to the algebra of globally defined differential operators on the manifold —we refer for this to the first section of [Hue90].
Example 1.8. A hyperplane arrangement $\mathcal{A}$ on a vector space $V$ is free, by definition, if $\text{Der} \mathcal{A}$ is a free $S$-module. In that case, as remarked by L. Narváez Macarro in [NM08], the enveloping algebra of the pair $(S, \text{Der} \mathcal{A})$ is isomorphic to the algebra of differential operators tangent to the arrangement $\text{Diff} \mathcal{A}$, that is, the associative algebra generated inside the algebra $\text{End}_k(S)$ of linear endomorphisms of the vector space $S$ by $\text{Der} \mathcal{A}$ and the set of maps given by left multiplication by elements of $S$. As seen by F. J. Calderón-Moreno in [CM99] or by M. Suárez-Álvarez in [SÁ18], it coincides with the algebra of differential operators on $S$ which preserve the ideal $QS$ of $S$ and all its powers.

We now recall another definition and a proposition from [Rin63]. Let $(S, L)$ be a Lie–Rinehart pair, let $U$ its enveloping algebra and let $M$ an $U$-module. The Lie–Rinehart cohomology of the pair with values on $M$ is defined as

$$H^\bullet(L|S, M) := \text{Ext}^\bullet_U(S, M).$$

In many important situations, some of which will be illustrated in the examples below, $L$ is a projective $S$-module, and in this case there is a well-known complex that computes the Lie–Rinehart cohomology.

**Proposition 1.9.** Suppose that $L$ is $S$-projective and let $\Lambda^\bullet_S L$ denote the exterior algebra of $L$ over $S$. The complex of $U$-modules $U \otimes_S \Lambda^\bullet_S L$ with differentials

$$d_r (u \otimes \theta_1 \wedge \cdots \wedge \theta_r) = \sum_{i=1}^r (-1)^{i+1} u \theta_i \otimes \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_r$$

$$+ \sum_{1 \leq i < j \leq r} (-1)^{i+j} u [\theta_i, \theta_j] \otimes \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \theta_r$$

whenever $\theta_1, \ldots, \theta_r \in L$, $u \in U$ and $r \geq 1$, is an $U$-projective resolution of $S$ with augmentation $\varepsilon : U \otimes_S S \ni u \otimes s \mapsto u \cdot s \in S$. In particular, the complex $\text{hom}_S(\Lambda^\bullet_S L, M)$ with Chevalley–Eilenberg differentials computes $H^\bullet(L|S, M)$. \hfill $\square$

**Example 1.10.** For the pair $(k, g)$ with $g$ a Lie algebra, $M$ is simply a $g$-Lie module and the complex $\text{hom}_k(\Lambda^\bullet_S L, M)$ is the standard complex that computes the Lie algebra cohomology $H^\bullet(g, M)$, as in §9 of the article [CE48] by C. Chevalley and S. Eilenberg.

**Example 1.11.** If $M$ is a differential manifold and $S = C^\infty(M)$, then $L = \mathfrak{X}(M)$ is finitely generated and projective over $S$—see the book by J. Nestruev [Nes03, Proposition 11.32]. The complex $\text{hom}_S(\Lambda^\bullet_S L, S)$ is precisely the de Rham complex $\Omega^\bullet(M)$ of differential forms and therefore the cohomology $H^\bullet(L|S, S)$ coincides with the de Rham cohomology of $M$.

**Example 1.12.** For the pair $(S, L)$ associated to a free hyperplane arrangement $\mathcal{A}$, the complex is $\text{hom}_S(\Lambda^\bullet_S L, S)$ is the complex of logarithmic forms $\Omega^\bullet(\mathcal{A})$, and its cohomology is isomorphic to the Orlik–Solomon algebra of $\mathcal{A}$—here we refer to the article [WY97] by J. Wiens and S. Yuzvinsky. When $k = \mathbb{C}$, this algebra is, in turn, isomorphic to the cohomology of the complement of the arrangement.
2. The spectral sequence

Let \((S, L)\) be a Lie–Rinehart pair and let \(U\) be its enveloping algebra. In this section we construct a spectral sequence that converges to the Hochschild cohomology of \(U\). In order to do so we follow the ideas and tools developed by Th. Lambre and P. Le Meur in [LLM17]. In particular, we recall from that paper the construction of a pair of adjoint functors. If \(M\) is a \(U\)-bimodule, the \(S\)-invariant subspace of \(M\) is

\[ M^S := \{ m \in M : sm = ms \text{ for all } s \in S \} . \]

This is the maximal symmetric \(S\)-subbimodule of \(M\) and it is an \(U\)-module if we define

\[ \alpha \cdot m := \alpha m - m \alpha \]

for \(\alpha \in L\) and \(m \in M^S\). The map \(\text{hom}_{Se}(S, M) \ni f \mapsto f(1) \in M^S\) is bijective and induces on its domain an \(U\)-module structure such that

\[ (\alpha \cdot \varphi)(s) = \alpha \varphi(s) - \varphi(s)\alpha - \varphi(\alpha(s)) , \quad (t \cdot \varphi)(s) = t \varphi(s) \quad (2) \]

when \(\alpha \in L, \varphi \in \text{hom}_{Se}(S, M)\) and \(s, t \in S\). What is more, the assignment

\[ G : U \text{Mod}_U \ni M \mapsto \text{hom}_{Se}(S, M) \in U \text{Mod} \]

is functorial.

Let, on the other hand, \(N\) be a left \(U\)-module. Again, the inclusion of \(S\) in \(U\) endows \(U\) with a structure of left \(S\)-module; since \(S\) is commutative, \(N\) can also be regarded as a right \(S\)-module. We can turn it into a right \(U\)-module setting, for \(u \in U\), \(n \in N\) and \(\alpha \in L\)

\[ (u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha(n) . \]

This construction extends to morphisms and thus defines a functor \(F : U \text{Mod}_U \to U \text{Mod}_U\) with \(F(N) = U \otimes_S N\). With these two functors in hand, we can state the very useful Proposition 3.4.1 of [LLM17].

**Proposition 2.1.** The functor \(F\) is left adjoint to \(G\).

Once we have established the following lemma we will be ready to construct the spectral sequence we are after.

**Lemma 2.2.** Assume that \(L\) is a projective \(S\)-module and let \(U \to I^\bullet\) be an injective resolution of \(U\) as an \(U^e\)-module.

(i) The cohomology of the complex \(\text{hom}_{Se}(S, I^\bullet)\) is \(H^\bullet(S, U)\).

(ii) The \(U\)-module structure on \(\text{hom}_{Se}(S, I^\bullet)\) defined in (2) induces an \(U\)-module structure on \(H^\bullet(S, U)\).

**Proof.** The PBW-theorem in [Rin63, §3] ensures that \(U\) is a projective \(S\)-module: using Proposition IX.2.3 of the book [CE56] by H. Cartan and S. Eilenberg, we obtain that \(U^e\) is \(S^e\)-projective. Given an injective \(U^e\)-module \(I\), the functor \(\text{hom}_{Se}(-, I)\) is naturally isomorphic to \(\text{hom}_{U^e}(U^e \otimes_S -, I)\), which is the composition of the exact functors
\[ \text{The Hochschild cohomology of } U(S, L) \]

and \( U^e \otimes_S - \), and therefore \( I \) is an injective \( S^e \)-module. As a consequence of this, \( U \to I^\bullet \) is actually a resolution of \( U \) by \( S^e \)-injective modules, so the cohomology of \( \text{hom}_{S^e}(S, I^\bullet) \) is \( \text{Ext}_{S^e}(S, U) \).

In order to prove the assertion of \((ii)\), it is enough to see that the differential of the complex \( \text{hom}_{S^e}(S, I^\bullet) \) is a morphism of \( U \)-modules, and this follows from the functoriality of \( G = \text{hom}_{S^e}(S, -) \).

\[ \square \]

Theorem 2.3. Assume \( L \) is \( S \)-projective and let \( N \) be a left \( U \)-module. There is a first-quadrant spectral sequence \( E_\bullet \) converging to \( \text{Ext}_{U^e}^\bullet(F(N), U) \) with second page
\[ E_2^{p,q} = \text{Ext}_{U^e}^p(N, H^q(S, U)). \]

Proof. Let \( Q_\bullet \to N \) be an \( U \)-projective resolution of \( N \) and let \( U \to I^\bullet \) be an \( U^e \)-injective resolution. Consider the double complex
\[ X^{\bullet, \bullet} = \text{hom}_{U^e}(Q_\bullet, G(I^\bullet)) \]
and denote its total complex by \( Z^\bullet \). There are two spectral sequences for this double complex: we will use the first one to compute \( H^\bullet(Z) \) and the second one will be the one we are looking for. From the filtration on \( Z^\bullet \) with
\[ \tilde{E}_q Z^p = \bigoplus_{r+s=p, s \geq q} X^{r,s} \]
we obtain a first spectral sequence converging to \( H(Z^\bullet) \). Its zeroth page \( \tilde{E}_0 \) has
\[ \tilde{E}_0^{p,q} = \text{hom}_{U^e}(Q_p, G(I^q)) \]
and the differential comes from the one on \( Q_\bullet \). We claim that for each \( s \geq 0 \), the functor \( \text{hom}_{U^e}(-, G(I^s)) \) is exact. Indeed, by the adjunction of Proposition 2.1 it is naturally isomorphic to \( \text{hom}_{U^e}(F(-), I^s) \), which is the composition of the functors \( F = U \otimes_S (-) \) and \( \text{hom}_{U^e}(-, I^s) \) and these are exact because \( U \) is left projective over \( S \) and \( I^s \) is \( U^e \)-injective. The first page \( \tilde{E}_1 \) of the spectral sequence is therefore given by
\[ \tilde{E}_1^{p,q} = \begin{cases} \text{hom}_{U^e}(N, G(I^q)) \cong \text{hom}_{U^e}(F(N), I^q) & \text{if } p = 0; \\ 0 & \text{if } p \neq 0 \end{cases} \]
and its differential is induced by that of \( I^\bullet \). Now, as the complex \( \text{hom}_{U^e}(F(N), I^\bullet) \) computes \( \text{Ext}_{U^e}^\bullet(F(N), U) \) using injectives, we obtain that the second page has
\[ \tilde{E}_2^{p,q} = \begin{cases} \text{Ext}_{U^e}^p(F(N), U) & \text{if } p = 0; \\ 0 & \text{if } p \neq 0. \end{cases} \]
This spectral sequence degenerates in the second page and in this way we see that \( H^\bullet(Z) \) is isomorphic to \( \text{Ext}_{U^e}^\bullet(F(N), U) \).
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The other filtration on $Z^\bullet$ is given by

$$F^p Z^q = \bigoplus_{r+s=q, r \geq p} X^{r,s}$$

and determines a spectral sequence $E_\bullet$ that also converges to $H(Z^\bullet)$. Its differential on $E_0$ is induced by the one on $I^\bullet$; as $Q_p$ is $U$-projective for each $p \geq 0$, the cohomology of $\text{hom}_U(Q_p, G(I^\bullet))$ is given in its $q$th place precisely by $E_1^{p,q} = \text{hom}_U(Q_p, H^q(S, U))$—recall that, according to Lemma 2.2, the cohomology of $G(I^\bullet)$ is $H^\bullet(S, U)$. Since the differentials in $E_1$ are induced by those of $Q^\bullet$, for each $q \geq 0$ the cohomology of the row $E_1^{\bullet,q}$ is $E_2^{p,q} = \text{Ext}_U^p(N, H^q(S, U))$. The spectral sequence $E_\bullet$ is therefore the one we were looking for. □

Specializing Theorem 2.3 to the case $N = S$ we obtain the following corollary, which is in fact the result we are mainly interested in.

**Corollary 2.4.** If $L$ is $S$-projective then there is a first-quadrant spectral sequence $E_\bullet$ converging to $HH^\bullet(U)$ with second page

$$E_2^{p,q} = H^p(L|S, H^q(S, U)).$$

We finish this section with some examples illustrating what happens in the two extreme situations.

**Example 2.5.** Suppose first that $L = 0$. The enveloping algebra $U$ is just $S$ and $\Lambda_S L = S$, so the resolution $U \otimes \Lambda_S L$ of $S$ is simply $Q_\bullet = U \otimes_S S$. The double complex $X^{\bullet,\bullet}$ is therefore $\text{hom}_S(S, \text{hom}_S(S, I^\bullet))$, which is isomorphic to $\text{hom}_S(S, I^\bullet)$ and the cohomology of $Z^\bullet$ is $HH^\bullet(S)$, the Hochschild cohomology of $S$.

**Example 2.6.** If $S = k$ and $L = \mathfrak{g}$ is a Lie algebra then $H^\bullet(S, U) = \text{Ext}^\bullet_k(k, U)$ is just $U$, the second page of our spectral sequence is $H^\bullet(\mathfrak{g}, U)$ and we recover from Corollary 2.4 the well-known fact that the Hochschild cohomology of the enveloping algebra of a Lie algebra equals its Lie cohomology with values on $U$ with the adjoint action, as in [CE56, XIII.5.1].

3. **The Lie module structure on $H^\bullet(S, U)$**

Let $(S, L)$ be Lie–Rinehart pair and let $U$ be its enveloping algebra. As we have already seen, $U$ can be regarded as an $S^e$-module with the action defined by $(s|t) \cdot u = stu$ for $s$ and $t$ in $S$ and $u$ in $U$. The Hochschild cohomology of $S$ with values on $U$, denoted as before by $H^\bullet(S, U)$, has an $U$-module structure—described in Lemma 2.2—that arises when we compute this cohomology from an injective resolution of $U$ as a module over $U^e$. The computation of this structure in particular examples is therefore rather inconvenient: indeed, we rarely compute Hochschild cohomology using injective resolutions.

The action of $U$ on $H^\bullet(S, U)$ is determined by actions of $S$ and of $L$ that satisfy the identities in (1). Let $M$ be a $U$-bimodule. In this section we construct an $L$-module
structure on $H^\bullet(S, M)$ using this time an $S^e$-projective resolution of $S$ and we show that when $M = U$, it coincides with the action of $L$ on $H^\bullet(S, U)$ that we already had. This will allow us to compute the latter in practice.

Let $\varepsilon : P_\bullet \to S$ be an $S^e$-projective resolution. Given a $U$-bimodule $M$, we will define for each $\alpha \in L$ a linear endomorphism $\alpha^\bullet$ of the complex $\text{Hom}_{S^e}(P_\bullet, M)$ which induces on its cohomology $H^\bullet(S, U)$ a Lie algebra action of $L$. In order to do so, we will adapt with minor changes the considerations in the article [SÁ17] by M. Suárez-Álvarez. There, there is a construction, for an algebra $A$, a derivation $\delta : A \to A$ and a so called $\delta$-operator $f : N \to N$, of a canonical morphism of graded vector spaces $\nabla_f : \text{Ext}^\bullet_A(N, N) \to \text{Ext}^\bullet_A(N, N)$ which, suitably specialized, gives a way to compute part of the Gerstenhaber bracket in the Hochschild cohomology of an associative algebra. The adaptation of this result to our situation is not obvious. Let us take $A = S^e$. Each $\alpha \in L$ determines a derivation of $A$; as opposed to the situation in [SÁ17], what we need here is a graded automorphism of $\text{Ext}^\bullet_A(S, M)$ and not of $\text{Ext}^\bullet_A(N, N)$. The observation that allows us to solve the problem is that there is a canonical action of $L$ on $U^e$ by derivations that restricts to the action of $L$ on $S$.

3.1. The construction of the action. Let $A$ be an algebra and let $\delta : A \to A$ a derivation. Given an $A$-module $N$, we say that a linear map $f : N \to N$ is a $\delta$-operator if for every $a \in A$ and $n \in N$ we have

$$f(an) = \delta(a)n + af(n).$$

If, moreover, $\varepsilon : P_\bullet \to N$ is an $A$-projective resolution of $N$, a $\delta$-lifting of $f$ to $P_\bullet$ is a family of $\delta$-operators $f_\bullet = (f_i : P_i \to P_{i+1}, i \geq 0)$ such that the diagram

$$\begin{array}{ccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & N \\
& & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\
& & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & N
\end{array}$$

commutes. The following proposition, extracted from [SÁ17, §1.4], ensures $\delta$-liftings exist and are in some sense unique.

**Proposition 3.1.** Let $N$ be a left $A$-module, let $f : N \to N$ be a $\delta$-operator and let $\varepsilon : P_\bullet \to N$ be a projective resolution.

(i) There exists a $\delta$-lifting $f_\bullet$ of $f$ to $P_\bullet$.

(ii) If $f_\bullet$ and $f'_\bullet$ are two $\delta$-liftings of $f$ to $P_\bullet$ then $f_\bullet$ and $f'_\bullet$ are homotopic by an $A$-linear homotopy.  \(\square\)

We return to our setting with a Lie–Rinehart pair $(S, L)$. Let $\alpha \in L$. As $L$ acts on $S$ by derivations, we can regard $\alpha$ as a derivation of $S$. It is easy to verify that the unique linear map $\alpha^e : S^e \to S^e$ such that

$$\alpha^e(s \mid t) = \alpha(s) \mid t + s \mid \alpha(t)$$

satisfies $\alpha^e|_S = \alpha$. 

\[\alpha^\bullet : \text{Ext}^\bullet(S^e, S^e) \to \text{Ext}^\bullet(S^e, S^e)\]
is a derivation of $S^e$. Viewing, as usual, $S$ as an $S^e$-module via $(s|t) \cdot f := sf$, the map $\alpha$ becomes an $\alpha^e$-operator: indeed, if $s|t \in S^e$ and $f \in S$ we have

$$\alpha((s|t)f) = \alpha(s)ft + s\alpha(f)t + sf\alpha(t) = \alpha^e(s|t)f + (s|t)\alpha(f).$$

**Example 3.2.** The standard bar resolution $B_* \to S$ is an $S^e$-projective resolution that has $B_i = S^{\otimes i+2}$ — we refer for this to [CE56, §IX.6]. Given $\alpha \in L$, there is a canonical $\alpha^e$-lifting $\alpha_*$ to $B_*$: if $i \geq 0$, the linear map $\alpha_i : B_i \to B_i$ such that

$$\alpha_i(s_0|s_1| \cdots |s_i|s_{i+1}) = \sum_{j=1}^r s_0|s_1| \cdots |\alpha(s_j)| \cdots |s_i|s_{i+1}$$

is an $\alpha^e$-operator and it is not difficult to see that $\alpha_* = (\alpha_i : i \geq 0)$ is a lifting of $\alpha$. This particular way of choosing liftings gives us a function $L \ni \alpha \mapsto \alpha_* \in \text{End}_S(P_*)$ which is, as a small calculation shows, a morphism of Lie algebras.

Fix $\alpha \in L$, an $S^e$-projective resolution $P_* \to S$ and a $U$-bimodule $M$. Let us choose one among all $\alpha^e$-liftings of $\alpha : S \to S$ to $P_*$ provided by Proposition 3.1 and call it $\alpha_*$. Given $i \geq 0$ and $\phi \in \text{hom}_{S^e}(P_i, M)$, we define $\alpha^e_i(\phi) : P_i \to M$ by

$$\alpha_i^e(\phi)(p) = [\alpha, \phi(p)] - \phi(\alpha_i(p)) \quad \text{for } p \in P_i. \quad (3)$$

**Proposition 3.3.** For each $i \geq 0$, the rule (3) defines a function

$$\alpha_i^e : \text{hom}_{S^e}(P_i, M) \to \text{hom}_{S^e}(P_i, M).$$

The collection of maps $\alpha_i^e = (\alpha_i^e)_{i \geq 0}$ is an endomorphism of the complex of vector spaces $\text{hom}_{S^e}(P_*, M)$.

**Proof.** For the first claim, we show that $\alpha_i^e(\phi)$ is a morphism of $S^e$-modules: given $p \in P_i$ and $s|t \in S^e$ we have

$$\alpha_i^e(\phi)((s|t)p) = [\alpha, s\phi(p)t] - \phi(\alpha_i((s|t)p))$$

$$= \alpha(s)\phi(p)t + s[\alpha, \phi(p)]t + s\phi(p)\alpha(t) - \phi(\alpha^e(s|t)p + (s|t)\alpha_i(p))$$

$$= s[\alpha, \phi(p)]t - (s|t)\phi(\alpha_i(p)).$$

For the second one, we must see that the map $\alpha_i^e$ commutes with the differential of $\text{hom}_{S^e}(P_*, M)$. Given $i \geq 0$ and $\phi$ in $\text{hom}_{S^e}(P_i, M)$, we have

$$d^e(\alpha_i^e(\phi))(p) = \alpha_i^e(\phi)(d(p)) = [\alpha, \phi(d(p))] - \phi(\alpha_i(d(p)))$$

and, as $\alpha_*$ is a morphism of complexes, this is equal to $\alpha_i^{e+1}(d^e\phi)$. \hfill $\square$

Proposition 3.3 implies that $\alpha_i^e$ descends to cohomology and therefore induces a graded endomorphism $\nabla_i^* \alpha$ of $H^*(S, U)$. In order to construct $\nabla_i^* \alpha$, we have chosen an $\alpha^e$-lifting $\alpha_*$: the next lemma shows that $\nabla_i^* \alpha$ is independent of that choice and, moreover, of the choice of the projective resolution $\varepsilon : P \to S$. 

Lemma 3.4. Fix $\alpha \in L$ and an $U$-bimodule $M$. Let $\varepsilon : P_\bullet \to S$ and $\varepsilon' : P'_\bullet \to S$ be two $S^e$-projective resolutions of $S$, let $\alpha_\bullet$ and $\alpha'_\bullet$ be $\alpha^e$-liftings of $\alpha$ to $P_\bullet$ and to $P'_\bullet$ respectively and, finally, let $\alpha^e_\bullet$ and $\alpha'^e_\bullet$ be defined as in Proposition 3.3. If $g : P'_\bullet \to P_\bullet$ is a morphism of complexes lifting the identity of $S$, the diagram

\[
\begin{array}{ccc}
\hom_{S^e}(P_\bullet, M) & \xrightarrow{\alpha^e_\bullet} & \hom_{S^e}(P_\bullet, M) \\
\downarrow{g^e_\bullet} & & \downarrow{g^e_\bullet} \\
\hom_{S^e}(P'_\bullet, M) & \xrightarrow{\alpha'^e_\bullet} & \hom_{S^e}(P'_\bullet, M)
\end{array}
\]

commutes up to homotopy.

Proof. The morphism of complexes of vector spaces $h_\bullet : g_\bullet \alpha'_\bullet - \alpha_\bullet g_\bullet : P'_\bullet \to P_\bullet$ is $S^e$-linear: indeed, if $i \geq 0$, $a \in S^e$ and $q \in P'_i$ we have

\[
h_i(aq) = g_i(a^e(a)q + a^e_i(q)) - \alpha_i(aq(q))
\]

\[
= a^e(a)g_i(q) + ag_i(a^e_i(q)) - \alpha^e(a)g_i(q) - a\alpha_i(g_i(q))
\]

\[
= ah_i(q).
\]

The map $h^e_\bullet : \hom_{S^e}(P_\bullet, M) \to \hom_{S^e}(P'_\bullet, M)$ induced by $h_\bullet$ is homotopic to zero because $h_\bullet$ is a lifting of the zero map in $S$ to the projective resolution $P_\bullet$. Let us show that $h^e_\bullet$ is the failure in the commutativity of the diagram. We have, for $i \geq 0$ and $\phi \in \hom_{S^e}(P_i, M)$,

\[
\left(\alpha^e_i g_i^e - g_i^e \alpha^e_i\right)(\phi) = \alpha^e_i(\phi \circ g_i) - g_i^e(\alpha^e_i(\phi)) = \alpha^e_i(\phi \circ g_i) - (\alpha^e_i(\phi)) \circ g_i,
\]

and evaluating this last expression on $q \in P'_i$ we find that $\left(\alpha^e_i g_i^e - g_i^e \alpha^e_i\right)(\phi)(q)$ is equal to

\[
[\alpha, \phi(g_i(q))] - \phi(g_i(\alpha_i^e(q))) - [\alpha, \phi(g_i(q))] + \phi(\alpha_i(g_i(q)))
\]

\[
= \phi(\alpha_i(g_i(q))) - \phi(g_i(\alpha_i^e(q)))
\]

\[
= (g_i^e \alpha_i^e - \alpha_i^e g_i^e)(\phi)(q).
\]

We see from this that $\alpha^e_i g_i^e - g_i^e \alpha^e_i = h_i^e$, which is, as we wanted, homotopic to zero. \qed

This lemma corresponds to the Lemma 1.6 of [SÁ17]; in our case, the key step was the cancellation that happened when we evaluated $\left(\alpha^e_i g_i^e - g_i^e \alpha^e_i\right)(\phi)$ on an element of $P'_i$. Now, with the help of Lemma 3.4, we see that each $\alpha \in L$ defines a canonical graded endomorphism of $H^\bullet(S, M)$.

Theorem 3.5. Let $M$ be an $U$-bimodule and let $\alpha \in L$. There is a morphism of graded vector spaces

\[
\nabla_\alpha^\bullet : H^\bullet(S, M) \to H^\bullet(S, M)
\]
such that for each \( S^e \)-projective resolution \( \varepsilon : P_\bullet \rightarrow S \) and each \( \alpha^e \)-lifting \( \alpha_\bullet \) of \( \alpha \) to \( P_\bullet \)
the diagram

\[
\begin{array}{ccc}
H(\text{hom}_{S^e}(P_\bullet, M)) & \xrightarrow{\nabla^e_\alpha} & H(\text{hom}_{S^e}(P_\bullet, M)) \\
\downarrow{}_{\cong} & & \downarrow{}_{\cong} \\
H^\bullet(S, M) & \xrightarrow{\nabla^e_\alpha} & H^\bullet(S, M)
\end{array}
\]

commutes.

**Proof.** Choosing an \( S^e \)-projective resolution \( \varepsilon : P_\bullet \rightarrow S \) and an \( \alpha^e \)-lifting of \( \alpha : S \rightarrow S \) to \( P_\bullet \), Proposition 3.3 gives us an endomorphism of complexes \( \alpha^e_\bullet \) on \( \text{hom}_{S^e}(P_\bullet, M) \): as the cohomology of this complex is \( H^\bullet(S, M) \), this induces a graded endomorphism \( \nabla^e_\alpha \) of \( H^\bullet(S, M) \). The square (4) defines an unique graded endomorphism \( \nabla^e_\alpha \) of \( H^\bullet(S, M) \); as an immediate consequence of Lemma 3.4, this endomorphism is independent of the choices of \( \varepsilon \) and of the \( \alpha^e \)-lifting. \( \square \)

**Example 3.6.** It is easy to describe the endomorphism \( \nabla^0_\alpha \) of \( H^0(S, U) \) for a given \( \alpha \in L \).
Let us choose a resolution \( P_\bullet \) of \( S \) with \( P_0 = S^e \) and augmentation \( \varepsilon : S^e \rightarrow S \) defined by \( \varepsilon(s|t) = st \). As \( \alpha^e \) is a \( \alpha^e \)-operator and \( \varepsilon \circ \alpha^e = \alpha \circ \varepsilon \), we may choose an \( \alpha^e \)-lifting with \( \alpha_0 = \alpha^e \). According to the rule (3) just before Proposition 3.3 we have

\[
\alpha_0^e(\phi(1|1)) = [\alpha, \phi(1|1)] \quad \text{for all } \phi \in \text{hom}_{S^e}(P_0, M).
\]

Identifying, as usual, each \( \phi \in \text{hom}_{S^e}(S^e, U) \) with \( \phi(1|1) \in U \), we can view \( H^0(S, U) \) as a subspace of \( U \) and the equality (5) tells us that

\[
\nabla^0_\alpha(u) = [\alpha, u]
\]

for all \( u \in H^0(S, U) \).

Theorem 3.5 defines an assignment \( \nabla : \alpha \mapsto \nabla^e_\alpha \); we will now show that it actually gives rise to a Lie action of \( L \) on \( H^\bullet(S, M) \), that is, that the identity \( \nabla_{[\alpha, \beta]} = [\nabla_\alpha, \nabla_\beta] \) holds.

Given \( \alpha \) and \( \beta \) in \( L \) and \( \varepsilon : P \rightarrow S \) an \( S^e \)-projective resolution, let \( \alpha_\bullet \) and \( \beta_\bullet \) be \( \alpha^e \)- and \( \beta^e \)-liftings of \( \alpha \) and of \( \beta \) to \( P_\bullet \). Call \( \gamma = [\alpha, \beta] \in L \): a straightforward calculation shows that \( \gamma^e = \alpha^e \circ \beta^e - \beta^e \circ \alpha^e \).

**Lemma 3.7.** In the setting of last paragraph, let \( M \) be an \( U \)-bimodule.

(i) The morphism of complexes \( \gamma_\bullet := \alpha_\bullet \circ \beta_\bullet - \beta_\bullet \circ \alpha_\bullet \) is a \( \gamma^e \)-lifting of \( \gamma : S \rightarrow S \).

(ii) Let \( \gamma^e_i \) be the endomorphism of \( \text{hom}_{S^e}(P_\bullet, M) \) induced by \( \gamma_\bullet \) as in Proposition 3.3.

We have \( \gamma^e_i = \alpha^e_i \circ \beta^e_i - \beta^e_i \circ \alpha^e_i \).

**Proof.** For each \( i \geq 0 \), the map \( \gamma_i \) is a \( \gamma^e \)-operator: given \( p \in P_i \) and \( a \in S^e \) we have

\[
(\alpha_i \circ \beta_i)(ap) = \alpha_i(\beta^e(a)p + a\beta_i(p)) \\
= \alpha^e(\beta^e(a)p + \beta^e(a)\alpha_i(p) + \alpha^e(a)\beta_i(p) + a\alpha_i\beta_i(p)
\]
and therefore \( \gamma_i(ap) = [\alpha^e, \beta^e](a)p + \gamma_i(p) \). As the a morphism of complexes \( \gamma_\bullet \) lifts \( \gamma \) because \( L \) acts as a Lie algebra on \( S \), we have proven the first statement.

In order to see the second one, we observe that for \( \phi \in \text{hom}_{S^e}(P_i, M) \) and \( p \in P_i \) we have

\[
\gamma_i^e(\phi)(p) = [\alpha, \beta, \phi(p)] - \phi(\alpha_i(\beta_i(p)) - \beta_i(\alpha_i(p)))
\]
and, on the other hand,

\[
a_i^e(\beta_i^e(\phi))(p) = [\alpha, (\beta_i^e(\phi))(p)] - (\beta_i^e(\phi))(\alpha_i(p))
= [\alpha, [\beta, \phi(p)] - [\alpha, \phi(\beta_i(p))] - [\beta, \phi(\alpha_i(p))] + \phi(\beta_i(\alpha_i(p))).
\]

These two expressions, together with the Jacobi identity, allow us to conclude that

\[
a_i^e(\beta_i^e(\phi))(p) - \beta_i^e(\alpha_i^e(\phi))(p) = \gamma_i^e(\phi)(p),
\]
which is just what we wanted. \( \square \)

**Proposition 3.8.** The assignment

\[
\nabla : L \ni \alpha \mapsto \nabla_\alpha^\bullet \in \text{End}_k(H^\bullet(S, M))
\]

is a morphism of Lie algebras.

**Proof.** Let \( \alpha, \beta \in L \) and call \( \gamma = [\alpha, \beta] \). Let \( \alpha_\bullet, \beta_\bullet \) and \( \gamma_\bullet \) be \( \alpha^e, \beta^e \) and \( \gamma^e \)-liftings, respectively. Observe that it not necessarily the case that \( \gamma_\bullet \) is the commutator of \( \alpha_\bullet \) and \( \beta_\bullet \). Let \( \alpha_\bullet^e, \beta_\bullet^e \) and \( \gamma_\bullet^e \) be the endomorphisms of \( \text{hom}_{S^e}(P_\bullet, M) \) defined as in Proposition 3.3 and consider the endomorphism \( \theta_\bullet \) of \( \text{hom}_{S^e}(P_\bullet, M) \) with

\[
\theta_i(\phi)(p) = [\gamma, \phi(p)] - \phi(\alpha_i \circ \beta_i(p) - \beta_i \circ \alpha_i(p)),
\]
where \( i \geq 0, \phi \in \text{hom}_{S^e}(P_i, M) \) and \( p \in P_i \). As we have seen in the first part of Lemma 3.7, the commutator \( [\alpha_\bullet, \beta_\bullet] \) is a \( \gamma^e \)-lifting of \( \gamma \) and therefore Lemma 3.4 tells us that the diagram

\[
\begin{array}{ccc}
\text{hom}_{S^e}(P_\bullet, M) & \xrightarrow{\gamma_\bullet^e} & \text{hom}_{S^e}(P_\bullet, M) \\
\downarrow & & \downarrow \\
\text{hom}_{S^e}(P_\bullet, M) & \xrightarrow{\theta_\bullet} & \text{hom}_{S^e}(P_\bullet, M)
\end{array}
\]

commutes up to homotopy. Now, according to the second part of Lemma 3.7 we have that \( \theta_i = \alpha_i^e \circ \beta_i^e - \beta_i^e \circ \alpha_i^e \) and therefore \( \theta_\bullet \) and \( \gamma_\bullet^e \) induce the same endomorphism on cohomology, that is,

\[
\nabla_\bullet^\gamma = H([\alpha_\bullet^e, \beta_\bullet^e]).
\]
Finally, using the linearity of the functor \( H \) we can conclude that \( \nabla_\bullet^\gamma = [\nabla_\alpha^\bullet, \nabla_\beta^\bullet] \). \( \square \)
3.2. Comparing the two actions of $L$. In Lemma 2.2 we constructed an $U$-module structure on $H^\bullet(S, U)$ using an $U^e$-injective resolution of $U$. As we have seen in Section 1, this is equivalent to having $S$- and $L$-module structures that satisfy the identities in (1). We will now show that this $L$-module structure coincides with the one defined in Subsection 3.1.1, using an $S^e$-projective resolution of $S$.

**Theorem 3.9.** Suppose $L$ is $S$-projective. The $L$-module structure on $H^\bullet(S, U)$ defined in Lemma 2.2 using injectives is equal to the one defined in Theorem 3.5 using projectives.

**Proof.** To begin with, we fix an $U^e$-injective resolution $\eta : U \to I^\bullet$, an $S^e$-projective resolution $\varepsilon : P_\bullet \to S$ and $\alpha \in L$. In Proposition 3.3, we constructed endomorphisms of complexes $\alpha^*_i$ of $\text{hom}_{S^e}(P_\bullet, U)$ and of $\text{hom}_{S^e}(P_\bullet, I^j)$ for each $j \geq 0$ —we denote them the same way—which induce the map $\nabla_{\alpha}$ on their cohomologies $H^\bullet(S, U)$ and $H^\bullet(S, I^j)$. We claim that the map

$$\eta_* : \text{hom}_{S^e}(P_\bullet, U) \ni \phi \mapsto \eta \circ \phi \in \text{hom}_{S^e}(P_\bullet, I^\bullet)$$

satisfies

$$\eta_*(\alpha^*_i(\phi)) = \alpha^*_i(\eta_*(\phi)) \quad (6)$$

for each $i \geq 0$ and $\phi \in \text{hom}_{S^e}(P_\bullet, U)$. Indeed, we have

$$\eta_*(\alpha^*_i(\phi))(p) = \eta(\alpha^*_i(\phi))(p) = \eta(\alpha_i(\phi)) - \eta(\phi(\alpha_i(p)))$$

and this is equal to $\alpha^*_i(\eta_*(\phi))$ because $\eta$ is a morphism of $U$-bimodules.

Let, on the other hand,

$$\varepsilon^* : \text{hom}_{S^e}(S, I^\bullet) \ni \varphi \mapsto \varphi \circ \varepsilon \in \text{hom}_{S^e}(P_\bullet, I^\bullet).$$

For each $\alpha \in L$ and $\varphi \in \text{hom}_{S^e}(S, I^\bullet)$ we have

$$\varepsilon^*(\alpha \cdot \varphi)(p) = \alpha \cdot \varepsilon(\varphi)(p) = [\alpha, \varphi(\varepsilon(p))] - \varphi(\alpha(\varepsilon(p))) \quad (7)$$

because, given $p \in P_0$,

$$\varepsilon^*(\alpha \cdot \varphi)(p) = \alpha \cdot \varepsilon(\varphi)(p) = [\alpha, \varphi(\varepsilon(p))] - \varphi(\alpha(\varepsilon(p)))$$

and, since $\alpha \circ \varepsilon = \varepsilon \circ \alpha_0$, this is $\alpha^*_0(\varepsilon^*(\varphi))(p)$.

As the morphisms of complexes $\varepsilon^*$ and $\eta_*$ are quasi-isomorphisms, the fact that they are equivariant with respect to the actions of $\alpha$ —as shown by (6) and (7)— allows us to conclude that the two actions of $L$ on $H^\bullet(S, U)$ coincide. 

We end this section showing how the results above work in a minimal example.

**Example 3.10.** As another instance of Example 1.2 we take $S = k[x]$, we fix a nonzero $h \in S$ and we consider the Lie algebra $L$ which, as an $S$-submodule of $\text{Der} S$, is freely generated by $y = h \frac{d}{dx}$. The enveloping algebra $U$ of the pair $(S, L)$ is isomorphic to the algebra $A_h$ with presentation

$$\frac{k\langle x, y \rangle}{(yx - xy - h)}$$
which we will identify with $U$. This algebra has been thoroughly studied by G. Benkart, S. Lopes and M. Ondrus in the series of articles that start with [BLO15a]; we observe that setting $h = 1$ we obtain the Weyl algebra that already appeared in Example 1.6. The augmented Koszul complex

$$
0 \longrightarrow S^e \xrightarrow{\delta_1} S^e \xrightarrow{\varepsilon} S
$$

with $\delta_1(s|t) = sx|t - s|xt$ and $\varepsilon(s|t) = st$ is an $S^e$-projective resolution of $S$ and therefore the Hochschild cohomology $H^\bullet(S, U)$ is the cohomology of the complex $\delta : U \to U$ with differential $\delta(u) = [x, u]$. After a small calculation we see that $H^0(S, U) = \ker \delta = \mathbb{k}[x]$ and $H^1(S, U) = \text{coker} \delta = A/hA$. As $A/hA$ is the quotient of the free noncommutative algebra in $x$ and $y$ by the relations $xy - yx = h$, and $h = 0$, we may identify $H^1(S, U)$ with $\mathbb{k}[x]$. At this point we make use of our description of the action of $U$ on $H^\bullet(S, U)$ as in Theorem 3.5. It is enough to see the action of $y$. We use Example 3.6 to see that $y$ acts on $H^0(S, U) = S$ in the obvious way. To describe its action on $H^1(S, U)$ we need a lifting $y_r$; we obtain one defining $y_0(s|t) = hs'|1 + 1|ht$ and $y_1(s|t) = hs'|1 + 1|ht' + s\Delta(h)t$, where $\Delta : S \to S$ is the unique derivation of $S$ such that $\Delta(x) = 1/1$. Since the diagram

$$
\begin{array}{ccc}
S^e & \xrightarrow{\delta_1} & S^e \\
\downarrow{y_1} & & \downarrow{y_0} \\
S^e & \xrightarrow{\delta_1} & S^e
\end{array}
$$

commutes and $y_0$ and $y_1$ are $y^e$-operators, the action of $y$ on $H^1(S, U)$ can be obtained with (3). We now compute $H^\bullet(L|S, H^i(S, U))$. Using the complex in Proposition 1.9 to compute Lie–Rinehart cohomology of $S$, we see that for each $i \in \mathbb{Z}$ this is the cohomology of the complex

$$H^i(S, U) \xrightarrow{\nabla^i_y} H^i(S, U).$$

For $i = 0$, this amounts to the cohomology of $S \xrightarrow{h} S$; the kernel of this map is $\mathbb{k}$ and its image, $hS$. Consider now the case $i = 1$ and recall that we have identified $H^1(S, U)$ with $\mathbb{k}[y]/(y^2)$; if $f \in \mathbb{k}[x]$, let us write $\overline{f}$ its class in this quotient. Given $u \in H^1(S, U)$, there are $f_0, \ldots, f_r \in \mathbb{k}[x]$ such that $u = \sum_{i=0}^r \overline{f_i} y^i$ and

$$\nabla^1_y(u) = \sum_{i=0}^r \overline{H^1_{y_i}} y^i.$$

This expression is explicit enough to compute the kernel and cokernel of $\nabla^1_y$, and this calculation, along with the help of Corollary 2.4, gives us the following description of
the Hochschild cohomology of \( A_h \):

\[
\text{HH}^i(A_h) \cong \begin{cases} 
\mathbb{k} & \text{if } i = 0; \\
S/(h) \oplus \bigoplus_{i \geq 0} \frac{S}{\gcd(h,h')} y^i & \text{if } i = 1; \\
\bigoplus_{i \geq 0} S \frac{y^i}{(h,h')} & \text{if } i = 2; \\
0 & \text{otherwise.}
\end{cases}
\]

This result had already been obtained by M. Valle in [Val17] and, partially, in [BLO15b]. With our approach, nevertheless, we have isolated the most complicated steps to different calculations and, as a consequence of that, this computation is significantly shorter.

4. A PARTICULAR EXAMPLE

In this section we describe the example that motivated us to construct the spectral sequence of Corollary 2.4: it is the algebra of differential operators \( \text{Diff} \mathcal{A} \) tangent to a central arrangements of lines \( \mathcal{A} \), whose Hochschild cohomology was studied by the author and M. Suárez-Álvarez in [KSÁ18].

Fix an integer \( r \geq 1 \) and let \( \mathcal{A} \) be a central line arrangement in the vector space \( \mathbb{k}^2 \) with \( r+2 \) lines. Let us write \( S = \mathbb{k}[x,y] \). We may suppose, up to a change of coordinates, that the line \( x = 0 \) belongs to \( \mathcal{A} \), so that we can write the defining polynomial of the arrangement as \( xF \), where \( F \in S \) is a square-free polynomial and \( x \not| F \).

There is a Lie–Rinehart pair \( (S,L) \) associated to the arrangement \( \mathcal{A} \), as we have seen in Example 1.4, with Lie algebra \( L = \text{Der} \mathcal{A} \). The very useful criterion of H. Saito [OT92, 4.19] allows us to see that the two derivations

\[
E = x\partial_x + y\partial_y, \quad D = F\partial_y
\]

form an \( S \)-basis of \( \text{Der} \mathcal{A} \). As we have already discussed in Example 1.8, the enveloping algebra \( U \) of the pair \( (S,L) \) is precisely the algebra of differential operators \( \text{Diff} \mathcal{A} \) tangent to the arrangement \( \mathcal{A} \) that we referred to as the motivating example at the beginning of this section. In [KSÁ18] we have seen that \( U \) can be regarded as the quotient of the free \( \mathbb{k} \)-algebra in variables \( x, y, D \) and \( E \) by the relations

\[
[y,x] = 0, \\
[D,x] = 0, \\
[E,x] = x, \\
[D,y] = F, \\
[E,y] = y, \\
[E,D] = rD.
\]

For \( r \geq 3 \), the Hochschild cohomology of \( U \) was computed in [KSÁ18] from an \( U^e \)-projective resolution of \( U \) after lengthy calculations. For \( r = 1 \) and \( r = 2 \) those calculations are even more tedious and rather inconvenient. With the method developed in this article we are able to obtain \( \text{HH}^*(U) \) as a vector space for every \( r \geq 1 \).

To compute the second page of the spectral sequence \( E_\bullet \) of Corollary 2.4 we use the Koszul resolution \( P_\bullet \) of \( S \), which is an \( S^e \)-projective resolution of length 2, and
compute the cohomology of $\text{hom}_{S_{\ast}}(P_{\ast}, S)$ to obtain $H^\ast(S, U)$. We then use the complex of Proposition 1.9, which also has length 2, to obtain, for each $0 \leq q \leq 2$, the Lie–Rinehart cohomology of the pair $(S, L)$ with values on $H^q(S, U)$. Since each of the complexes we used has length 2, the second page has $E^{p, q}_2 = 0$ for every $p, q \geq 3$.

It is at this point that the case $r \geq 3$ is different to the case $r = 1, 2$. As depicted in Figure 1, for $r \geq 3$ we have

$$E^{p, q}_2 = 0 \quad \text{if } p \geq 3 \text{ and } q \geq 2,$$

and, moreover, $E^{0, 1}_2 = 0$. As the differential that corresponds to the second page has bidegree $(2, -1)$, the spectral sequence degenerates at $E_2$, thus giving us a description of $HH^\ast(U)$. A problem with this description is that it is not obvious how to compute the Gerstenhaber algebra structure on $HH^\ast(U)$: in [KSÅ18], we computed cohomology by exhibiting explicit cocycles that allowed us to compute cup products and Gerstenhaber brackets. Here, we still do not know the relation between our spectral sequence and the multiplicative structure of $HH^\ast(U)$. Another consequence of the lack of explicitness of this procedure is that it is difficult to describe the formal deformations of $U$ even though we do know $HH^2(U)$.

For $r$ equal to 1 or 2 the dimensions of the components of the second page of the spectral sequence are tabulated in Figure 2. As opposed to the case in which $r \geq 3$, it is not evident that the spectral sequence degenerates at the second page: the differential $d^{0, 2}_2 : E^{0, 2}_2 \to E^{2, 1}_2$ could be non zero. Computing $HH^3(U)$ from the $U^c$-projective resolution of $U$ in [KSÅ18, 2.4] we were able to check that, in fact, $d^{0, 2}_2$ is zero, thus allowing us to obtain the dimensions of $HH^\ast(U)$ as a graded vector space. The end result is that the Hilbert series of $HH^\ast(U)$ is

$$h_{HH^\ast(U)}(t) = \begin{cases} 1 + (r + 2)t + (2r + 4)t^2 + (r + 3)t^3, & \text{if } r \geq 3; \\ 1 + (r + 2)t + (2r + 3)t^2 + (r + 2)t^3, & \text{if } r = 1, 2. \end{cases}$$

This shows that the case in which $r$ is 1 or 2 is genuinely different to that in which $r \geq 3$. 

---

**Figure 1.** Dimensions of $E_2$ for $r \geq 3$  
**Figure 2.** Dimensions of $E_2$ for $r = 1, 2$
The HoCHschild CoHOMOLOGY of $U(S, L)$

5. The differential of the second page

Let us end this article with a straightforward adaptation of the ideas in the article [SÁ07] on the change-of-rings spectral sequence to give a description of the differential of the second page of ours.

Let $(S, L)$ be a Lie–Rinehart pair such that $L$ is $S$-projective and let $U$ be its universal enveloping algebra. Let $d \geq 0$ and let $M_1$ and $M_2$ be two $U$-modules. Given $\zeta \in \text{Ext}_A^d(M_2, M_1)$, we can think of $\zeta$ as a $d$-extension of $M_2$ by $M_1$ in the sense of Yoneda; that is, an exact sequence of $U$-modules of length $d + 1$ of the form

$$
\zeta : 0 \to M_1 \to \cdots \to M_2 \to 0.
$$

If now $p \geq 0$ and $\eta$ is a $p$-extension of a left module $M_3$ by $M_2$, the Yoneda product $\zeta \circ \eta$, which is obtained by the splicing of $\zeta$ and $\eta$, is a $(p + d)$ extension of $M_3$ by $M_1$. We refer to MacLane’s book [Mac67, §III.5] for details.

**Theorem 5.1.** For each $q \geq 0$ there exists $\zeta_q \in \text{Ext}_U^q(H^q(S, U), H^{q-1}(S, U))$ such that the differential of the second page in the spectral sequence of Corollary 2.4

$$
d_{2}^{p,q} : H^p(L|S, H^q(S, U)) \to H^{p+2}(L|S, H^{q-1}(S, U))
$$

is given by $d_{2}^{p,q}(\eta) = \zeta_q \circ \eta$ for all $\eta \in H^p(L|S, H^q(S, U))$. \qed

It is to prove this that we chose to state Theorem 2.3 in a more general setting that Corollary 2.4: we need the extra freedom with respect to the first argument in order to use the argument of [SÁ07].

We have not yet been able to find an example where $d_2$ is not zero. Nevertheless, we believe that is only because we have been working with pairs $(S, L)$ where $S$ has small projective dimension as an $S^e$-module and small projective dimension as an $U$-module. In the general case, the sequence $\{\zeta_q : q \geq 1\}$ should encode important information of the pair.

In future work, we will be adressing the Hochschild cohomology of the algebra of differential operators tangent to a braid arrangement of hyperplanes via the spectral sequence: we expect it will not degenerate in the second page.

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