ON TAIL DISTRIBUTIONS OF SUPREMUM AND QUADRATIC VARIATION OF LOCAL MARTINGALES

LIPTSER R. AND NOVIKOV A.

Abstract. We extend some known results relating the distribution tails of a continuous local martingale supremum and its quadratic variation to the case of locally square integrable martingales with bounded jumps. The predictable and optional quadratic variations are involved in the main result.

1. Introduction and main result

Denote by $M(M_{\text{loc}})$ and $M^2(M_{\text{loc}}^2, M_{\text{loc}}^c)$ the classes of all martingales (local martingale) and square integrable (locally square integrable, continuous local martingales) $M = (M_t)_{t \geq 0}, M_0 = 0$ (with paths in the Skorokhod space $D_{[0, \infty)}$) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ a stochastic basis with standard general conditions. Recall that any random process $X$ with paths in the Skorokhod space and defined on the above-mentioned stochastic basis belongs to the class $D$ if the family $(X_{\tau}, \tau \in \mathcal{T})$, where $\mathcal{T}$ is the set of stopping times $\tau$, is uniformly integrable.

Henceforth $\Delta M_t := M_t - M_{t-}, \langle M \rangle_t$ and $[M, M]_t$ are the jumps, predictable quadratic variation and optional quadratic variation processes of $M$ respectively.

It is well-known (see e.g. [9], [7] and references therein) that for local martingales from $M_{\text{loc}}^2$:

$$\langle M \rangle_{\infty} < \infty, \text{ a.s.} \Rightarrow \begin{cases} [M, M]_{\infty} < \infty \text{ a.s.} \\ \lim_{t \to \infty} M_t = M_{\infty} \in \mathbb{R} \text{ a.s.} \end{cases}$$

There are many other well-known relations between $M_{\infty}$ and $\langle M \rangle_{\infty}$ (e.g., Burkholder–Gundy–Davis’s inequalities, law of large numbers for martingales, etc.) which are valid for local martingales with jumps.

If $M \in M \cap D$, then $M$ satisfies the Wald equality:

$$EM_{\infty} = 0$$

which plays a fundamental role in many applications in stochastic analysis. Often, a direct verification of the uniform integrability is difficult. In this connection, we mention one result from Novikov, [10], establishing a relation between the tail distributions of $\langle M \rangle_{\infty}$ and $EM_{\infty}$. A similar result is also proved in Elworthy, Li and Yor, [2], under slightly different conditions than in [10]. Concerning the related topic dealing with a one-sided stochastic boundary, see Peškir and Shiryaev, [13], and Vondraček [15].

1991 Mathematics Subject Classification. 60G44, 60HXX, 40E05.
Theorem*. Let $M \in \mathcal{M}_{loc}^c$ and $\langle M \rangle_\infty < \infty$ a.s. If $\sup_{t>0} E e^{\varepsilon M_t} < \infty$ for some positive $\varepsilon$, then $1^1 0 \leq EM_\infty \leq EM_\infty^+ < \infty$ and

$$\lim_{\lambda \to \infty} \lambda P(\langle M \rangle_\infty^{1/2} > \lambda) = \sqrt{\frac{2}{\pi}} EM_\infty.$$

One of our goals is a generalization of Theorem* statement for local martingales with bounded jumps.

**Theorem 1.1.** Let $M \in \mathcal{M}_{loc}^2$, $\langle M \rangle_\infty < \infty$ a.s. and $M^+ \in \mathcal{D}$. Then:

(i) $\lim_{t \to \infty} M_t := M_\infty$ exists and

$$0 \leq EM_\infty \leq EM_\infty^+ < \infty;$$

(ii) $|\Delta M| \in \mathcal{D}$ and (i) provide

$$\lim_{\lambda \to \infty} \lambda P(\sup_{t \geq 0} M_t^- > \lambda) = EM_\infty;$$

(iii) $|\Delta M| \leq K$ and

$$E e^{\varepsilon M_\infty} < \infty,$$

for some positive $K$ and $\varepsilon$, provide

$$\lim_{\lambda \to \infty} \lambda P(\langle M \rangle_\infty^{1/2} > \lambda) = \lim_{\lambda \to \infty} \lambda P([M,M]_\infty^{1/2} > \lambda) = \sqrt{\frac{2}{\pi}} EM_\infty.$$

If $M^+ \in \mathcal{D}$, Theorem 1.1 gives necessary and sufficient conditions for $M \in \mathcal{D}$ expressed in terms of $\sup_{t \geq 0} M_t^-$, $\langle M \rangle_\infty$, and $[M,M]_\infty$ which are useful in some applications (see, e.g., by Jacod and Shiryaev [8]).

**Corollary 1.** Under the assumptions of Theorem 1.1 the process $M \in \mathcal{D}$ iff any of the following conditions holds:

$$\lim_{\lambda \to \infty} \lambda P(\sup_{t \geq 0} M_t^- > \lambda) = 0,$$

$$\lim_{\lambda \to \infty} \lambda P(\langle M \rangle_\infty^{1/2} > \lambda) = 0,$$

$$\lim_{\lambda \to \infty} \lambda P([M,M]_\infty^{1/2} > \lambda) = 0.$$

A few publications preceded [10] and [2] (see Azema, Gundy and Yor, [1]; for discrete time martingales, Gundy, [5], and Galtchouk and Novikov, [6]). Takaoka, [14], presented a result similar to Theorem*.

The proofs of parts (i) and (ii) of Theorem 1.1 are obvious and might even be known. The proof of (iii) exploits a combination of techniques:

“Stochastic exponential + Tauberian theorem”

which seems to have been firstly used by Novikov, [11], to obtain asymptotics of the first passage times for Brownian motion (see also [10]) and for random walks (see, Novikov [12]). Some necessary facts on the stochastic exponential are gathered in Section 2. The proofs are given in Section 3.

The uniform boundedness assumption for $\Delta M$ might be weakened by applying a standard ”truncation” technique under some additional assumptions on the tails distribution of $\Delta M$. We show in Theorem 3.1 that the uniform boundedness assumption for $\Delta M$ is avoided if the stochastic exponential possesses an evaluation in

$^1a^+ = \max(a,0), a^- = \max(-a,0)$
terms of $\langle M \rangle_\infty$. This condition is borrowed from [10] where it is effectively applied for discrete-time martingales involving in a popular gambling strategies.

2. Preliminaries

2.1. Stochastic exponential. For discontinuous martingales, the stochastic exponential has an “intricate” structure. So, we start with recalling the necessary notions and objects involving in (ii) (for more details, see e.g. [9] or [7]).

For $M \in \mathcal{M}^2_{\loc}$. $M_0 = 0$, the decomposition $M = M^c + M^d$ is well known, where $M^c, M^d \in \mathcal{M}^2_{\loc}$ and are continuous and purely discontinuous martingales respectively. Moreover, $\langle M \rangle = \langle M^c \rangle + \langle M^d \rangle$, so the assumption $\langle M \rangle_\infty < \infty$ provides $\langle M^c \rangle_\infty < \infty$, $\langle M^d \rangle_\infty < \infty$. The measure $\mu$ is associated with the jump process $\Delta M \equiv \Delta M^d$ in the sense that for any measurable set $A$ and $t > 0 \mu((0, t] \times A) = \sum I(\Delta M_s \in A)$. Denote by $\nu = \nu(dt, dz)$ its compensator. The condition $|\Delta M| \leq K$ provides the existence of a version $\nu$ such that $\nu([R^+ \times \{|z| > K\}) = 0$. This version of $\nu$ is used in the sequel.

The purely discontinuous martingale $M^d$ is defined as the Itô integral with respect to $\mu - \nu$:

$$M^d_t = \int_0^t \int_{|z| \leq K} z(\mu(ds, dz) - \nu(ds, dz)).$$

Recall also that $\int_{|z| \leq K} \nu(\{t\}, dz) = 0$ a.s. and

$$\langle M^d \rangle_t = \int_0^t \int_{|z| \leq K} z^2 \nu(ds, dz) < \infty \text{ a.s., } t > 0.$$

Hence, $\langle M^d \rangle_t < \infty$ a.s. provides

$$\int_0^\infty \int_{|z| \leq K} z^2 \nu(ds, dz) < \infty \text{ a.s. (2.1)}.$$

This fact is important for further considerations as long as we will deal with the cumulant process

$$G_t(\lambda) = \int_0^t \int_{|z| \leq K} (e^{\lambda z} - 1 - \lambda z) \nu(ds, dz), \lambda \in \mathbb{R}.$$

The boundedness of jumps and $\nu$ implies the existence of $G_t(\lambda)$ and $G_\infty(\lambda) := \lim_{t \to \infty} G_t(\lambda) < \infty$. The cumulant process $G(\lambda)$, being increasing, possesses a nonnegative jumps process

$$\Delta G_t(\lambda) := \int_{|z| \leq K} (e^{\lambda z} - 1 - \lambda z) \nu(\{|t\}, dz).$$

A random process $\mathcal{E}(\lambda)$ with

$$\mathcal{E}_t(\lambda) = \exp \left( \frac{\lambda^2}{2} \langle M^c \rangle_t + G_t(\lambda) \right) \prod_{0 < s \leq t} (1 + \Delta G_s(\lambda)) e^{-\Delta G_s(\lambda)}$$

is known as “stochastic exponential” for the martingale $M$. Note that $\mathcal{E}_t > 0$, since $\Delta G_\lambda(\lambda) \geq 0$.

A remarkable property of the stochastic exponential is that the process $\mathcal{Z}(\lambda)$,

$$\mathcal{Z}_t(\lambda) = e^{\lambda M_t - \log \mathcal{E}_t(\lambda)}$$

(2.3)
is a positive local martingale. Indeed, applying the Itô formula to \( d\zeta_t = \frac{e^\zeta_t - 1}{1 + \Delta G_t(\zeta)} (\mu - \nu)(dt, dz) \), we get

\[
d\zeta_t = \lambda \zeta_t dM_t^\nu + \int_{|z| \leq K} \zeta_{t-}(\lambda) \left( e^{\lambda z} - 1 \right) 1 + \Delta G_t(\lambda) (\mu - \nu)(dt, dz),
\]

where the right-hand side is a sum of two local martingales. As any nonnegative local martingale, \( \zeta(\lambda) \) is also a supermartingale too (see e.g. Problem 1.4.4 in Liptser and Shiryaev [9]). The latter provides the existence of

\[
\zeta_\infty(t) := \lim_{t \to \infty} \zeta_t(\lambda) \in \mathbb{R}_+ \text{ a.s.}
\]

with \( E_{\lambda}(\zeta) \leq 1 \) for any Markov time \( \tau \); hence, in particular, \( E_{\lambda} \leq 1 \).

**Proposition 2.1.** Let \( |\Delta M| \leq K \), \( (M)_\infty \text{ a.s. and condition } \| \| \) hold. Then, with \( \varepsilon \) from \( \| \| \) and any \( \lambda \in (0, \varepsilon] \),

1) Set \( E_{\lambda}(\zeta) = 1 \).

2) \( E_{\lambda}(\zeta) = \lim_{t \to \infty} E_t(\zeta) \in \mathbb{R}_+ \text{ a.s. and } E_{\lambda}(\zeta) > 0 \text{ a.s.} \)

**Proof.** 1) Let \( t_n \) be an increasing sequence of stopping times, \( \lim_n t_n = \infty \), such that \( (M_{t \wedge t_n})_{t \geq 0} \) and \( (\zeta_{t \wedge t_n}(\lambda))_{t \geq 0} \in D \) for any \( n \). Then

\[
E_{\lambda}(\zeta) = 1.
\]

In order to finish the proof, we show that \( \zeta_{t_n}(\lambda) \) is majorized by uniformly integrable martingale \( E(e^{\lambda M_{t_n}}|\mathcal{F}_{t_n}) \), what is provided by \( \| \| \), applying Jensen’s inequality:

\[
E(e^{\lambda M_{t_n}}|\mathcal{F}_{t_n}) \geq e^{\lambda E(M_{t_n}^+)} \geq e^{\lambda M_{t_n}^+} \geq \zeta_{t_n}(\lambda).
\]

Hence, \( (\zeta_{t_n}(\lambda))_{n \geq 1} \in D \).

2) Since \( \zeta_\infty(\lambda) = e^{\lambda M_{\infty}^+ - \log \varepsilon(\lambda)} \) with \( \log 0 = -\infty \), the desired property holds true provided that \( \zeta_\infty(\lambda) < \infty \text{ a.s.} \)

3. The proof of Theorem [11]

3.1. The proof of parts (i) and (ii). 1) Let \( (t_n)_{n \geq 1} \) be an increasing sequence of stopping times, \( \lim_n t_n = \infty \), such that \( (M_{t_n})_{n \geq 1} \in D \) and, therefore, \( EM_{t_n}^+ = EM_{t_n} - n \geq 1 \). Due to the assumption \( M^+ \in D \), we have \( \lim_{n \to \infty} EM_{t_n}^+ = EM_{\infty} < \infty \).

Now, applying the Fatou theorem, we find that \( EM_{\infty}^+ \geq EM_{\infty}^- \).

Hence,

\[
EM_{\infty}^+ \geq EM_{\infty}^- = EM_{\infty} \geq 0.
\]

2) Set \( S_\lambda = \inf\{t : M_t^\lambda \geq \lambda\} \) and notice that

\[
S_\lambda < \infty = \{\sup_{t \geq 0} M_t^\lambda > \lambda\}.
\]

Since \( \Delta M = 0 \) and \( |\Delta M| \in D \), the process \( \mathcal{S} \in D_{10} \) is a uniformly integrable martingale with \( EM_{S_\lambda} = 0 \).

Write

\[
0 = EM_{S_\lambda} = EM_{S_\lambda} I_{\{S_\lambda = \infty\}} + EM_{S_\lambda} I_{\{S_\lambda < \infty\}} = EM_{S_\lambda} I_{\{S_\lambda = \infty\}} + EM_{S_\lambda} I_{\{\sup_{t \geq 0} (-M_t) \geq \lambda\}} = EM_{S_\lambda} I_{\{S_\lambda = \infty\}} + EM_{S_\lambda} I_{\{S_\lambda < \infty\}} + EM_{S_\lambda} I_{\{\sup_{t \geq 0} M_t^\lambda > \lambda\}} + EM_{S_\lambda} I_{\{\sup_{t \geq 0} M_t^\lambda > \lambda\}}.
\]

Finally, \( EM_{S_\lambda} < \infty \) provides \( \lim_{\lambda \to \infty} S_\lambda = \infty \) and \( EM_{\infty} \geq 0 \).
The desired statement holds true owing to $|M_{S_\lambda} - \lambda| \leq |\Delta M_{S_\lambda}| \leq K$, that is, $|M_{S_\lambda} - \lambda|$, $\lambda > 0$ is a uniformly integrable family. □

3.2. Proof of part (iii).

3.2.1. Auxiliary lemmas.

Lemma 3.1. Under the assumptions of Theorem 1.1 (iii),
$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left( 1 - e^{-\log E_\infty(\lambda)} \right) = E M_\infty.$$  

Proof. Recall that $\lambda \leq \varepsilon$ for $\varepsilon$ involved in assumption (ii). Since by Proposition 2.1 $\tilde{\mathbb{M}}(\lambda)$ a uniformly integrable martingale, we have $E\tilde{\mathbb{M}}(\lambda) = 1$. Hence,
$$\frac{1}{\lambda} \left( 1 - e^{-\log E_\infty(\lambda)} \right) = \frac{1}{\lambda} \left( \tilde{\mathbb{M}}(\lambda) - e^{-\log E_\infty(\lambda)} \right) = \frac{1}{\lambda} \left( e^{M_\infty} - 1 \right) e^{-\log E_\infty(\lambda)}.$$  

The required statement follows from the relation
$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} e^{-\log E_\infty(\lambda)} \left( e^{M_\infty} - 1 \right) = M_\infty,$$
$$\frac{1}{\lambda} e^{-\log E_\infty(\lambda)} \left| e^{M_\infty} - 1 \right| \leq e^{\varepsilon M_\infty}$$
and the assumption $E e^{\varepsilon M_\infty} < \infty$, see (1.1). □

Lemma 3.2. Under the assumptions of Theorem 1.1 (iii),
$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left( 1 - e^{-\frac{\lambda}{2} \langle M^c \rangle_\infty} \right) = E M_\infty.$$  

Proof. Due to Lemma 3.1, suffice it to show that
$$\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left| e^{-\log E_\infty(\lambda)} - e^{-\frac{\lambda}{2} \langle M^c \rangle_\infty} \right| = 0. \quad (3.1)$$  

In order to verify (3.1), we estimate $\log E_\infty(\lambda)$ from above and below via $\frac{\lambda^2}{2} \langle M^c \rangle_\infty$. Owing to $\log E_\infty(\lambda) \leq \frac{\lambda^2}{2} \langle M^c \rangle_\infty + G_\infty(\lambda)$, we have
$$\log E_\infty(\lambda) \leq \frac{\lambda^2}{2} \langle M^c \rangle_\infty + \left[ 1 + \frac{\lambda}{3} K e^{\lambda K} \right]. \quad (3.2)$$  

Further, with
$$G^c_\infty(\lambda) = \int_0^\infty \int_{|z| \leq K} (e^{\lambda z} - 1 - \lambda z) \nu^c(dt, dz),$$
where $\nu^c(dt, dz) := \nu(dt, dz) - \nu(\{t\}, dz)$, and $\Phi(\lambda, K) = 1 - \lambda K e^{\lambda K}$, we get
$$\log E_\infty(\lambda) \geq \frac{\lambda^2}{2} \langle M^c \rangle_\infty + \Phi(\lambda, K) \int_0^\infty \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu^c(dt, dz) \quad (3.3)$$
$$+ \sum_{t > 0} \log \left( 1 + \Phi(\lambda, K) \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu(\{t\}, dz) \right).$$
We choose \( \lambda \) so small to have \( 1 - \lambda K e^{\lambda K} > 0 \) and estimate from below the “\( \sum_{t>0} \log \)” in the last line from the above inequality by applying 

\[
\log(1 + x) \geq x - \frac{1}{2} x^2, \quad x \geq 0.
\]

This gives us the bound

\[
\sum_{t>0} \log \left( 1 + \Phi(\lambda, K) \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu(|t|, dz) \right) 
\geq \Phi(\lambda, K) \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu(|t|, dz) - \frac{1}{2} \Phi^2(\lambda, K) \left( \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu(|t|, dz) \right)^2.
\]

Since \( \nu(|t|, |z| \leq K) \leq 1 \), by the Cauchy–Schwarz inequality we find that

\[
\left( \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu(|t|, dz) \right)^2 
\leq \frac{\lambda^4}{4} \int_{|z| \leq K} z^4 \nu(|t|, dz) \leq \frac{\lambda^4 K^2}{4} \int_{|z| \leq K} z^2 \nu(|t|, dz).
\]

So, finally we get

\[
\sum_{t>0} \log \left( 1 + \Phi(\lambda, K) \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu(|t|, dz) \right) 
\geq \left( \Phi(\lambda, K) - \frac{\lambda^2}{8} K^2 \Phi^2(\lambda, K) \right) \int_{|z| \leq K} \frac{\lambda^2}{2} z^2 \nu(|t|, dz) \quad (3.4)
\]

and now choose \( \lambda \) so small to have

\[
\Phi(\lambda, K) - \frac{\lambda^2}{8} K^2 \Phi^2(\lambda, K) \geq 1 - \lambda C > 0 \quad (3.5)
\]

for some constant \( C > 0 \). Combining now (3.5), (3.4) and (3.4), we may choose a generic positive constant \( C \) and sufficiently small \( \lambda \) such that \( \mathcal{E}_\infty(\lambda) \geq [1 - C\lambda] \frac{\lambda^2}{2} \langle M \rangle_\infty \). Hence and with (3.2), for some generic positive constant \( C > 0 \) and sufficiently small \( \lambda > 0 \) we have

\[
0 < [1 - C\lambda] \frac{\lambda^2}{2} \langle M \rangle_\infty \leq \log \mathcal{E}_\infty(\lambda) \leq [1 + C\lambda] \frac{\lambda^2}{2} \langle M \rangle_\infty.
\]

These inequalities provide

\[
\frac{1}{\lambda} \left| e^{-\log \mathcal{E}_\infty(\lambda)} - e^{-\frac{\lambda^2}{2} \langle M \rangle_\infty} \right| \leq C \frac{\lambda^2}{2} \langle M \rangle_\infty e^{-\frac{\lambda^2}{2} \langle M \rangle_\infty} \xrightarrow{\lambda \to 0} 0.
\]

Since \( xe^{-x} \leq e^{-1} \), the desired result holds by Lebesgue’s dominated theorem. \( \square \)

**Lemma 3.3.** Under the assumptions of Theorem 1.1 (iii),

\[
\lim_{\lambda \to \infty} \lambda P(\langle M \rangle_\infty^{1/2} > \lambda) = c \iff \lim_{\lambda \to \infty} \lambda P([M, M]_\infty^{1/2} > \lambda) = c.
\]
Proof. It suffices to establish
\[
\lim_{\lambda \to 0} \frac{P(|M, M|^{1/2}_\infty > \lambda)}{P(|M|^{1/2}_\infty > \lambda)} \leq 1,
\]
\[
\lim_{\lambda \to 0} \frac{P(|M, M|^{1/2}_\infty > \lambda)}{P(|M|^{1/2}_\infty > \lambda)} \geq 1.
\] (3.6)

Set \( L = [M, M] - \langle M \rangle \). Since \([M, M]_\infty \leq \langle M \rangle_\infty + \sup_{t \geq 0} |L_t|\), applying the elementary inequality \((c + d)^{1/2} \leq c^{1/2} + d^{1/2}\), we find that
\[
P(|M, M|^{1/2}_\infty > \lambda) \leq P(|[M, M]_\infty + \sup_{t \geq 0} |L_t|^{1/2}_\infty > \lambda)
\]
\[
\leq P(|M|^{1/2}_\infty + \sup_{t \geq 0} |L_t|^{1/2}_\infty > \lambda)
\]
\[
\leq P(|M|^{1/2}_\infty > (1 - a)\lambda) + P(\sup_{t \geq 0} |L_t| > a\lambda), \quad a \in (0, 1).
\] (3.7)

With \( \lambda_a = (1 - a)\lambda \), the resulting bound can be rewritten as:
\[
\lambda P(|M, M|^{1/2}_\infty > \lambda) \leq (1 - a)^{-1} \lambda_a P(|M|^{1/2}_\infty > \lambda_a) + \lambda P(\sup_{t \geq 0} |L_t|^{1/2}_\infty > a\lambda). \] (3.8)

So, we shall deal with the evaluation from above of \( P(\sup_{t \geq 0} |L_t|^{1/2}_\infty > a\lambda) \). A helpful tool here is the inequality: for some absolute positive constant \( C \), any stopping time \( \tau \) and \( K \) being a bound for \( |\Delta M| \),
\[
E \sup_{t \leq \tau} |L_t|^2 \leq CK^2 E \langle M \rangle_\tau.
\] (3.9)

In order to establish (3.9), we use the following facts:
- \( L \) is the purely discontinuous local martingale with
\[
|L, L|_t = \sum_{s \leq t} (\Delta L_s)^2 = \sum_{s \leq t} ((\Delta M_s)^2 - \Delta \langle M \rangle_s)^2
\]
\[
= \sum_{s \leq t} \left( \int_{|z| \leq K} z^2 (\mu(\{s\}, dz) - \nu(\{s\}, dz)) \right)^2,
\]
- \( \langle L \rangle_t = \int_0^t \int_{|z| \leq K} z^4 (\mu(ds, dz) - \nu(ds, dz))^2 \left( \int_{|z| \leq K} z^2 (\nu ds, dz) \right)^2 \),
- \( \langle L \rangle_t \leq K^2 \left( \int_0^t \int_{|z| \leq K} z^4 (\mu(ds, dz)) \right) \),
- \( K^2 \langle M \rangle - \langle L \rangle \) is the increasing process.

Now, we refer to the Burkholder-Gundy inequality (see e.g. Theorem 1.9.7 in [9]):
for any stopping time \( \tau \),
\[
E \sup_{t \leq \tau} |L_t|^2 \leq CE[L, L]_\tau.
\]

Due to the relations \( E[L, L]_\tau = E \langle L \rangle_\tau \) and \( K^2 \langle M \rangle_\tau \geq \langle L \rangle_\tau \) (recall that \( K^2 \langle M \rangle \geq \langle L \rangle \)), we have \( E \langle L \rangle_\tau \leq K^2 E \langle M \rangle_\tau \), that is, (3.3) is valid. Due to (3.4) and the fact that \( \langle M \rangle \) is a predictable process, the Lenglart-Rebolledo inequality (see, e.g., Theorem 1.9.3 in [9]) is applicable (notice that \( \sup_{t \geq 0} |L_t|^{1/2}_\infty > a\lambda \) \( \equiv \sup_{t \geq 0} |L_t| >\)
\(a^2\lambda^2\))}, so that,

\[
P\left(\sup_{t \geq 0} |L_t|^{1/2} > a\lambda\right) \leq \frac{\lambda^{5/2}}{a^4\lambda^4} + P(CR^2\langle M\rangle_\infty > \lambda^{5/2})
\]

\[
= \frac{\lambda^{5/2}}{a^4\lambda^4} + P\left(\langle M\rangle_{\infty}^{1/2} > \lambda^{5/4}/(C^{1/2}K)\right). \tag{3.10}
\]

Hence, with \(r = 1/(C^{1/2}K)\) and \(\lambda_r = r\lambda^{5/4}\),

\[
\lambda P\left(\sup_{t \leq T_r} |L_t|^{1/2} > a\lambda\right) \leq \frac{1}{a^4\lambda^{1/2}} + \frac{1}{r\lambda^{1/4}} \lambda_r P\left(\langle M\rangle_{\infty}^{1/2} > \lambda_r\right). \tag{3.11}
\]

Now, (3.8) and (3.11) provide

\[
\lambda P\left([M, M]_{\infty}^{1/2} > \lambda\right)
\]

\[
\leq (1 - a)^{-1} \lambda a P\left(\langle M\rangle_{\infty}^{1/2} > \lambda\right) + \frac{1}{a^4\lambda^{1/2}} + \frac{r}{\lambda^{1/4}} \lambda_r P\left(\langle M\rangle_{\infty}^{1/2} > \lambda_r\right).
\]

Assume that \(c > 0\). Then, we get

\[
\frac{P\left([M, M]_{\infty}^{1/2} > \lambda\right)}{P\left(\langle M\rangle_{\infty}^{1/2} > \lambda\right)} \leq \frac{(1 - a)^{-1} \lambda a P\left(\langle M\rangle_{\infty}^{1/2} > \lambda\right)}{\lambda P\left(\langle M\rangle_{\infty}^{1/2} > \lambda\right)}
\]

\[
+ \frac{1}{a^4\lambda^{1/2}} + \frac{r}{\lambda^{1/4}} \lambda_r P\left(\langle M\rangle_{\infty}^{1/2} > \lambda_r\right)
\]

\[
\longrightarrow 1 \quad \lambda \to \infty
\]

and the first part from (3.9).

Since the second part from (3.9) is established similarly, we give only a sketch of the proof. The use of

\[
P\left(\langle M\rangle_{\infty}^{1/2} > \lambda\right) \leq P\left([M, M]_{\infty}^{1/2} > (1 - a)\lambda\right) + P\left(\sup_{t \geq 0} |L_t| > a\lambda\right), \ a \in (0, 1)
\]

provides

\[
\frac{P\left([M, M]_{\infty}^{1/2} > (1 - a)\lambda\right)}{P\left(\langle M\rangle_{\infty}^{1/2} > \lambda\right)} \geq 1 - \frac{P\left(\sup_{t \geq 0} |L_t| > a\lambda\right)}{P\left(\langle M\rangle_{\infty}^{1/2} > \lambda\right)}
\]

and the result.

If \(c = 0\), we replace \(M\) by \(M + M'\), where \(M'\) is independent of \(M\) local continuous martingale with \(M'_0 = 0\) and \(\langle M'\rangle_{\infty} < \infty\) a.s. and

\[
\lim_{\lambda \to \infty} \lambda P\left(\langle M'\rangle_{\infty}^{1/2} > \lambda\right) = c' > 0.
\]

Now, taking into account the obvious relations

\[
[M + M', M + M'] = [M, M] + [M', M'] \quad \text{and} \quad \langle M + M'\rangle = \langle M\rangle + \langle M'\rangle,
\]

with \(\delta \neq 0\) we find that

\[
\lim_{\lambda \to \infty} \lambda P\left(\langle M + \delta M'\rangle_{\infty}^{1/2} > \lambda\right) = \delta^2 c' > 0.
\]

So, by using the result already proved, we have

\[
\lim_{\lambda \to \infty} \lambda P\left([M + \delta M', M + \delta M']_{\infty}^{1/2} > \lambda\right) = \delta(c')
\]

and so, by

\[
P\left([M + \delta M', M + \delta M']_{\infty}^{1/2} > \lambda\right) \geq P\left([M, M]_{\infty}^{1/2} > \lambda\right),
\]

we find that

\[
\lim_{\delta \to 0} \lambda P\left([M, M]_{\infty}^{1/2} > \lambda\right) \leq \delta c' \quad \delta \to 0.
\]

\(\square\)
3.2.2. Final part of the proof for (iii). We refer to the Tauberian theorem.

Theorem**. (Feller, [4], XIII.5, Example (c)) Let $X$ be a nonnegative random variable such that

$$
\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left( 1 - E e^{-\frac{\lambda^2}{2} X} \right) \text{ exists in } \mathbb{R},
$$

then

$$
\sqrt{\frac{2}{\pi}} \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left( 1 - E e^{-\frac{\lambda^2}{2} X} \right) = \lim_{\lambda \to \infty} \lambda P(X^{1/2} > \lambda).
$$

Now, we are in the position to finish the proof of (ii). Letting $X = \langle M \rangle_{\infty}$, we find that

$$
\sqrt{\frac{2}{\pi}} \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left( 1 - E e^{-\frac{\lambda^2}{2} \langle M \rangle_{\infty}} \right) = \lim_{\lambda \to \infty} \lambda P(\langle M \rangle^{1/2}_{\infty} > \lambda).
$$

At the same time, Lemmas 3.1 and 3.2 provide

$$
\lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left( 1 - E e^{-\frac{\lambda^2}{2} \langle M \rangle_{\infty}} \right) = \sqrt{\frac{2}{\pi}} EM_{\infty}
$$

while by Lemma 3.3

$$
\lim_{\lambda \to \infty} \lambda P(\langle M, M \rangle^{1/2}_{\infty} > \lambda) = \sqrt{\frac{2}{\pi}} EM_{\infty}.
$$

\[\Box\]

3.3. Supplement. As it was mentioned in Introduction, the condition $|\triangle M| \leq K$ might be too restrictive to be valid for serving some examples. It is known from [10] that this condition can be replaced by a weaker one and so more useful for applications. An analog of this result is given below.

Theorem 3.1. Let $M \in M_{\text{loc}}^2$, $\langle M \rangle_{\infty} < \infty$ a.s., $M^+ \in \mathcal{D}$ and (1.1) holds. Assume also that there exist nonnegative integrable random variables $\zeta_1, \zeta_2$ such that for all sufficiently small $\lambda > 0$

$$
\frac{\lambda^2}{2} \langle M \rangle_{\infty} (1 - |\lambda| \zeta_1)^+ \leq \log \mathcal{E}_{\infty}(\lambda) \leq \frac{\lambda^2}{2} \langle M \rangle_{\infty} (1 + |\lambda| \zeta_2).
$$

Then

$$
\lim_{\lambda \to \infty} \lambda P(\langle M \rangle^{1/2}_{\infty} > \lambda) = \sqrt{\frac{2}{\pi}} EM_{\infty}.
$$

Proof. Notice that only (3.1) has to be verified under (3.12).

By (3.12), we have

$$
\frac{1}{\lambda} \left| e^{-\log \mathcal{E}_{\infty}(\lambda)} - e^{-\frac{\lambda^2}{2} \langle M \rangle_{\infty}} \right| \leq \left( \zeta_2 \lor \frac{|1 - (1 - \zeta_1 \lambda)^+|}{\lambda} \right) \frac{\lambda^2}{2} \langle M \rangle_{\infty} e^{-\frac{\lambda^2}{2} \langle M \rangle_{\infty}}
$$

$$
\leq \left( \zeta_2 \lor \zeta_1 \right) \frac{\lambda^2}{2} \langle M \rangle_{\infty} e^{-\frac{\lambda^2}{2} \langle M \rangle_{\infty}}.
$$

The right-hand side of this inequality converges to zero, as $\lambda \to 0$, and is bounded by $e^{-\lambda}(\zeta_2 \lor \zeta_1)$. Hence, in order to get (3.1) suffices it to allude on the Lebesgue dominated convergence theorem. \[\Box\]

Acknowledgements. The authors gratefully acknowledge their colleagues J. Stoyanov, E. Shinjikashvili and anonymous reviewers for comments improving presentation of the material.
References

[1] Azema, J., Gundy, R.F., Yor, M.: Sur l’intégrabilité uniforme des martingales continues. Séminaire de Probabilités. XIV, LNM 784, 249-304, Springer (1980)
[2] Elworthy, K.D., Li, X.M., Yor, M.: On the tails of the supremum and the quadratic variation of strictly local martingales. Séminaire de Probabilités XXXI, Lecture Notes in Math. 1655, 113-125, Springer (1997)
[3] Ethier, S.N.: A gambling system and a Markov chain. Ann.Appl.Probab. 6, no.4, 1248-1259 (1996)
[4] Feller, W.: An Introduction to probability and its Applications. 2, 2nd ed. Wiley (1971)
[5] Gundy, R. F.: On a theorem of F. and M. Riesz and an equation of A. Wald. Indiana Univ. Math. J. 30, no. 4, 589-605
[6] Galchouk, L. and Novikov, A.: On Wald’s equation. Discrete time case. Séminaire de Probabilités. XXXI, Lecture Notes in Math., 1655, 126-135, Springer, Berlin (1997)
[7] Jacod J., Shiryaev A.N.: Limit theorems for stochastic processes. 2nd ed. Springer-Verlag, Berlin (2003)
[8] Jacod J., Shiryaev A.N.: Local martingales and the fundamental asset pricing theorems in the discrete time case. Finance and Stochastics. 2, 255-273 (1998)
[9] Liptser, R.Sh. and Shiryaev, A.N.: Theory of Martingales. Kluwer Acad. Publ. Dordrecht (1989)
[10] Novikov, A.: Martingales, Tauberian theorem and gambling. Theory Prob., Appl. 41, no. 4, 716-729 (1996)
[11] Novikov, A.A.: Martingale approximace to first passage problems of nonlinear boundaries. Proc. Steklov Inst. Math., v. 158, 130-152 (1981)
[12] Novikov, A.: On the time of crossing a one-sided nonlinear boundary by sums of independent random variables. Theory Prob., Appl. 27, no. 4, 643-656 (1982)
[13] Peškir, G. and Shiryaev, A.N.: On the Brownian first-passage time over a one-sided stochastic boundary. Theory Probab. Appl. 42 (1998), no. 3, 444-453 (1997)
[14] Takaoka, K.: Some remark on the uniform integrability of continuous martingales. Séminaire de Probabilités. XXXIII, Lecture Notes in Math., 1709, 327-333, Springer, Berlin (1999)
[15] Vondraček, Z.: Asymptotics of first passage time over a one-sided stochastic boundary. J. Theoret. Prob. 13, no.1, 171-173 (1997)

DEPT. ELECTRICAL ENGINEERING-SYSTEMS, TEL AVIV UNIVERSITY, 69978 TEL AVIV, ISRAEL
E-mail address: <liptser@eng.tau.ac.il>

DEPT. MATHEMATICAL SCIENCES, UNIVERSITY OF TECHNOLOGY SYDNEY, PO BOX, 123. BROADWAY, NSW 2007, AUSTRALIA
E-mail address: <prob@maths.uts.edu.au>