SOBOLEV DIFFERENTIABILITY PROPERTIES OF LOGARITHMIC MODULUS OF REAL ANALYTIC FUNCTIONS

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Abstract. Let $f$ be the germ of a real analytic function at the origin in $\mathbb{R}^n$ for $n \geq 2$, and suppose the codimension of the zero set of $f$ at $0$ is at least 2. We show that $\log |f|$ is $W^{1,1}_{\text{loc}}$ near $0$. In particular, this implies the differential inequality $|\nabla f| \leq V|f|$ holds with $V \in L^1_{\text{loc}}$. As an application, we derive an inequality relating the Lojasiewicz exponent and singularity exponent for such functions.

Contents

1. Introduction 1
   1.1. Statement of the main results 1
   1.2. Connection with unique continuation problem 2
   1.3. An application to local invariants 3
2. Preliminaries 3
   2.1. o-minimality and tame geometry 4
   2.2. Removable singularity 6
3. Proof of the Main Theorem 8
Appendix A. Harvey-Polking’s lemma 15
References 17

1. INTRODUCTION

1.1. Statement of the main results. The main goal of this paper is to prove the following results on the log singularity for real analytic functions.

Theorem 1.1. Let $f$ be the germ of a real analytic function at the origin in $\mathbb{R}^n$, $n \geq 2$. Suppose the codimension of the zero set of $f$ at $0$ (denoted $\text{codim}_0(Z_f)$) is at least 2. Then there exists a small neighborhood $U$ of $0$ such that $\log |f| \in W^{1,1}(U)$.

For definitions of dimension and codimension of zero set of a real analytic function, see Definition 2.4 below.

We can also formulate a global version of Theorem 1.1 as follows.

Theorem 1.2. Let $f$ be a real analytic function in a neighborhood of a bounded open subset $U$ of $\mathbb{R}^n$, $n \geq 2$. Suppose the codimension of the zero set of $f$ in $U$ is at least 2. Then $\log |f| \in W^{1,1}(U)$.

We remark that the problem is purely local, and Theorem 1.2 follows from Theorem 1.1 by a simple local-to-global argument.

We say that $g$ is in $W^{1,1}_{\text{loc}}(0)$ (or $L^p_{\text{loc}}(0)$) at $0$ if there exists a neighborhood $U$ of $0$ such that $g \in W^{1,1}(U)$ (or $L^p(U)$), and we say that $g$ has an isolated zero at $0$ if there exists a neighborhood

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of $0$ such that $Z_f \cap U = \{0\}$, where $Z_f$ denotes the zero set of $f$. As a special case to Theorem 1.1, we conclude that $\log |f| \in W^{1,1}_{\text{loc}}(0)$ if $f$ has an isolated zero at the origin.

The function $\nabla \log |f|$ is not integrable in each radial direction. Indeed we show in Proposition 3.6 that the only continuous function $f : (-1,1) \to \mathbb{R}$ in $W^{1,p}(-1,1)$ satisfying $f(0) = 0$ and $|f'(x)| \leq V|f(x)|$ for $V \in L^1(-1,1)$ is the zero function.

Theorem 1.1 (and accordingly Theorem 1.2) is sharp up to both the integrability exponent and the codimension of the zero set. Indeed, take the function $f(x,y) = xy$ defined on $\mathbb{R}^2$, where the zero set is the union of $x$ and $y$ axis and has codimension 1. Let $U$ be any neighborhood of the origin, we have

$$\int_U |\nabla \log |f||^p \,dV = \int_U \left| \frac{\nabla f}{f} \right|^p \,dV \approx \int_U \left| \frac{x}{xy} \right|^p \,dV + \int_U \left| \frac{y}{xy} \right|^p \,dV = \int_U \frac{1}{|y|^p} \,dV + \frac{1}{|x|^p} \,dV$$

which is finite if and only if $p < 1$. On the other hand, for any $\varepsilon > 0$ and $2 \leq n - d \leq n$, there exists a polynomial $f$ with $\text{codim}_0 Z_f = n - d$, and $|\nabla f| \notin L^{1+\varepsilon}_{\text{loc}}(0)$; see Example 3.7.

Given any germ of a real analytic function $f$ at the origin with $f(0) = 0$ and $f$ not identically 0, we can show that $\log |f| \in L^p_{\text{loc}}(0)$ for any $0 < p < \infty$. Therefore, the main point is to prove the derivative estimate: $|\nabla \log |f|| = \frac{|\nabla f|}{f} \in L^1_{\text{loc}}$. In particular, the $L^1$ integrability reflects the cancellation between the zeros of the function and its gradient near the singular points where $\nabla f = 0$. It should be noted that in many special cases one can do much better. Take the simple example $u = |x|^{2k}$ in $\mathbb{R}^n$, with $2 \leq 2k \leq n$, then $|\nabla u(x)| \approx |x|^{2k-1}$, and $|\nabla \log |u(x)|| \approx \frac{|\nabla u(x)|}{|u(x)|} \approx \frac{1}{|x|} \in L^p_{\text{loc}}$, for any $p < n$. In fact, we prove the following result which provides a sharp upper bound for the integrability exponent.

**Theorem 1.3.** Let $f$ be the germ of a real analytic function at the origin in $\mathbb{R}^n$, $n \geq 2$, with $f(0) = 0$. Suppose $\text{codim}_0 Z_f = n - d \geq 1$ (i.e. $f$ is not identically 0). Then for each (sufficiently small) neighborhood $U$ of $0$,

$$\int_U \left| \frac{\nabla f}{f} \right|^{n-d} \,dV(x) = \infty.$$

Furthermore, the integrability exponent $n - d$ is sharp in the sense that there exists a real analytic function $f$ such that $\text{codim}_0 Z_f = n - d$ and $f \in L^p_{\text{loc}}(0)$, for any $p < n - d$ (See Example 3.7.)

The proof of Theorem 1.1 relies on the “finiteness” of analytic functions and their zero sets (which are analytic sets). A particularly useful fact is that locally analytic sets can be decomposed into finitely many connected analytic manifolds. We shall generalize this phenomenon and adopt the viewpoint that analytic sets are (locally) definable in some o-minimal structure, and accordingly they are “nice” and have limited complexity, or “tame” in their geometry and topology.

### 1.2. Connection with unique continuation problem

Our original motivation behind Theorem 1.1 comes from the study of the unique continuation properties of the differential inequality

$$|\Delta u| \leq A |u| + B |\nabla u|, \quad A \in L^p_{\text{loc}}(0), \ B \in L^q_{\text{loc}}(0), \ 0 < p, q < \infty.$$  \hspace{1cm} (1.1)

We say that the differential inequality (1.1) satisfies the unique continuation property (UCP) at 0, if every solution that vanishes in a neighborhood of a point 0 vanishes identically. The problem asks for the optimal (minimal) values for $p$ and $q$ such that (1.1) satisfies the UCP. We refer the reader to [Wol95] for an excellent exposition on this problem.

We now consider the inverse problem to the above unique continuation problem. Suppose $f$ is a function satisfying certain analytic properties, does it satisfy some kind of differential inequalities? Theorem 1.1 gives a result of this type: the differential inequality $|\nabla f| \leq V|f|$, where $V \in L^1_{\text{loc}}(0)$ holds for any real analytic function $f$ with $\text{codim}_0 Z_f \geq 2$. 

1.3. **An application to local invariants.** First, we recall the Lojasiewicz gradient inequality, which states that given \( f \) the germ of a real analytic function at 0 such that \( f(0) = 0 \), there exists some \( \beta \in (0,1) \) and a small neighborhood \( \mathcal{V} \) of 0 such that
\[
|\nabla f(x)| \geq c_\beta |f(x)|^\beta, \quad \text{for all } x \in \mathcal{V}.
\]

Here we may assume \( \nabla f(0) = 0 \) since otherwise the inequality is trivial. The *Lojasiewicz exponent of \( f \) at 0*, denoted by \( \beta_0 \), is defined to be the infimum of all \( \beta \) satisfying (1.2).

For any real analytic function \( f \) with \( f(0) = 0 \), we define the *singularity exponent* of \( f \) at 0, denoted by \( \alpha_0 \), as the supremum of all \( \alpha > 0 \) such that there exists a small neighborhood \( \mathcal{V} \) of 0 with \( \int_\mathcal{V} |f|^{-\alpha} < \infty \). By Theorem 1.1, and inequality (1.2), if \( \text{codim}_0 Z_f \geq 2 \), then there exists a neighborhood \( \mathcal{U} \) of the origin such that
\[
\int_\mathcal{U} |\nabla f| f \geq c_\beta \int_\mathcal{U} \frac{1}{|f|^{1-\beta}}
\]
for any \( \beta > \beta_0 \). This implies that \( (1-\beta) < \alpha_0 \). Letting \( \beta \to \beta_0 \), we get \( 1 - \beta_0 \leq \alpha_0 \), or
\[
\alpha_0 + \beta_0 \geq 1.
\]

We summarize the result in the following corollary:

**Corollary 1.4.** Let \( f \) be the germ of a real analytic function at the origin in \( \mathbb{R}^n \) with \( f(0) = 0 \). Suppose \( \text{codim}_0 Z_f \geq 2 \). Let \( \alpha_0 \) and \( \beta_0 \) be the singularity exponent and the Lojasiewicz exponent of \( f \) at 0, respectively. Then
\[
\alpha_0 + \beta_0 \geq 1.
\]

We point out that all of our results fail rather spectacularly if \( f \) is only assumed to be \( C^\infty \). Take \( u_\varepsilon = e^{-|x|^{-\varepsilon}}, \varepsilon > 0 \). Then \( \log u_\varepsilon = -|x|^{-\varepsilon} \). It follows that for any \( p > 0 \), there exists some large number \( \varepsilon(p) \) such that \( \log u_\varepsilon \notin L^p_{\text{loc}}(0) \) whenever \( \varepsilon > \varepsilon(p) \).

We use \( x \lesssim y \) to mean that \( x \leq Cy \) where \( C \) is a constant independent of \( x,y \), and we write \( x \approx y \) if \( x \lesssim y \) and \( y \lesssim x \). For an open subset \( \Omega \) of \( \mathbb{R}^n \), we denote by \( C^\infty_c(\Omega) \) the space of \( C^\infty \) function with compact support in \( \Omega \).

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## 2. Preliminaries

In this section we review some definitions and concepts used in later proofs.

Let \( \mathcal{M} \) be a connected \( n \)-dimensional analytic manifold and \( \mathcal{U} \) be an open subset of \( \mathcal{M} \). We denote by \( \mathcal{O}_{\mathcal{U}} \) the ring of real analytic functions from \( \mathcal{U} \) to \( \mathbb{R} \). If \( p \in \mathcal{M} \), we denote by \( \mathcal{O}_{\mathcal{M},p} \) the set of real analytic germs at \( p \). We denote the set of polynomials in \( \mathbb{R}^n \) by \( \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n] \).

**Definition 2.1.** Let \( A \subset \mathcal{O}_\mathcal{M} \). We define the *vanishing locus of \( A \)*, \( V(A) \), to be the set of points:
\[
V(A) := \{ x \in \mathcal{M} : f(x) = 0, \forall f \in A \}.
\]

**Definition 2.2.** An *algebraic subset* of \( \mathbb{R}^n \) is a set of the form \( V(A) \), where \( A \subset \mathbb{R}[x] \). A subset \( X \subset \mathcal{M} \) is a (real) analytic subset of \( \mathcal{M} \) if \( X \) is closed in \( \mathcal{M} \) and, for all \( x \in X \), there exists an open neighborhood \( \mathcal{W} \) of \( x \) in \( \mathcal{M} \) and a finite collection \( f_1, \ldots, f_j \in \mathcal{O}_\mathcal{W} \) such that \( \mathcal{W} \cap X = V(f_1, \ldots, f_j) \).

**Definition 2.3.** Let \( X \) be an analytic subset of \( \mathcal{M} \). A point \( p \in X \) is called smooth, of dimension \( d \), if there exists an open neighborhood \( \mathcal{W} \) of \( p \) in \( \mathcal{M} \) such that \( \mathcal{W} \cap X \) is an analytic sub-manifold of \( \mathcal{W} \) of dimension \( d \). In other words, there exists \( f_{d+1}, \ldots, f_n \in \mathcal{O}_\mathcal{W} \) such that \( \mathcal{W} \cap X = V(f_{d+1}, \ldots, f_n) \) and \( \nabla f_{d+1}(x), \ldots, \nabla f_n(x) \) are linearly independent at each \( x \in \mathcal{W} \).

We denote the set of smooth points of \( X \) by \( \bar{X} \), and the set of smooth points of dimension \( d \) by \( X^{(d)} \).
Definition 2.4. The dimension (over \( \mathbb{R} \)), \( \dim X \) of an analytic set \( X \subseteq M \) is the largest \( m \) such that \( X^{(m)} \) is non-empty. The dimension of \( X \) at a point \( p \in X \), denoted by \( \dim_p X \), is the largest \( d \) such that \( p \) is in the closure of \( X^{(d)} \). We say that \( X \) is pure-dimensional if the dimension of \( X \) at each point \( p \in X \) is independent of \( p \). The codimension \( \text{codim}(X) \) of an analytic set \( X \subseteq M \) is defined as \( n - d \), where \( d = \dim(X) \). The codimension of \( X \) at a point \( p \in X \) is defined as \( n - d_p \), where \( d_p := \dim_p X \).

Analytic sets can be partitioned into smooth sets, as the following result (see [BM88]) shows.

Proposition 2.5 (Stratification of analytic sets). Let \( X \) be an analytic subset of \( M \). Then there exists a collection \( \{ \mathcal{A}_\alpha \}_{\alpha} \) of subsets \( M \) such that

- \( X \) is the disjoint union of the \( \mathcal{A}_\alpha \);
- Each \( \mathcal{A}_\alpha \) is an analytic submanifold of \( M \);
- (“Condition of the frontier”) If \( \mathcal{A}_\alpha \cap \overline{\mathcal{A}_\beta} \neq \emptyset \), then \( \mathcal{A}_\alpha \subseteq \overline{\mathcal{A}_\beta} \) and \( \dim \mathcal{A}_\alpha < \dim \mathcal{A}_\beta \);
- \( \{ \mathcal{A}_\alpha \}_{\alpha} \) is locally finite.

We can also define the dimension of an analytic set \( X \) by \( \dim X = \max_{\alpha} \dim \mathcal{A}_k \). The definition is independent of the stratification: \( \dim X = d \) if and only if \( X \) contains an open set homeomorphic to an open ball in \( \mathbb{R}^d \), but not an open set homeomorphic to an open ball in \( \mathbb{R}^n \), \( n > d \). It is also clear that the definition agrees with that of Hausdorff dimension.

One result we will be using is the Lojasiewicz distance inequality for analytic functions:

Proposition 2.6. Let \( g \) be a real analytic function in a neighborhood \( U \) of the origin. Then for any compact set \( K \) in \( U \), there exist constants \( c > 0, \alpha > 0 \) which depend only on \( g \), such that

\[
|g(x)| \geq c \text{dist}(x, Z_g)^\alpha, \quad x \in K.
\]

Analytic sets are in some sense “finite” objects. The following proposition gives an important property of this nature.

Proposition 2.7 ([BM88, Remark 7.3]). Let \( X \) be an analytic subset of \( M \). Then the family of connected components of \( X \) is locally finite.

As an immediate consequence, we obtain the following

Proposition 2.8. Let \( f : (a, b) \to \mathbb{R} \) is analytic, where \( (a, b) \) is a bounded interval. For every \( (a', b') \) such that \( a' > a \) and \( b' < b \), there exists \( N < \infty \), and \( a' = a_0 < a_1 < \cdots < a_N = b' \) such that \( f \) is either constant, or strictly monotone on each subinterval \( (a_i, a_{i+1}) \).

These properties of analytic sets can be better understood when considered under the framework of o-minimal structures, a concept we introduce in the next section.

2.1. o-minimality and tame geometry. For most of the definitions and results in this section we refer the reader to the book [vdD98].

Definition 2.9. A structure on \( \mathbb{R} \) is a sequence \( S = \{ S_n \}_{n \in \mathbb{N}} \) such that for each \( n \):

1. \( S_n \) is a Boolean algebra of subsets of \( \mathbb{R}^n \), that is, \( S \) is a collection of subsets of \( \mathbb{R}^n \), \( \emptyset \in S \), and if \( A, B \in S \), then \( A \cup B \in S \) and \( \mathbb{R}^n \setminus A \in S \);
2. \( A \in S \implies A \times \mathbb{R} \in S_{n+1} \) and \( \mathbb{R} \times A \in S_{n+1} \);
3. The diagonals \( \Delta_{ij} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = x_j \} \in S_n \), for \( 1 \leq i < j \leq n \);
4. \( A \in S_{n+1} \implies \pi(A) \in S_n \), where \( \pi : \mathbb{R}^{n+1} \to \mathbb{R}^n \) is the usual projection map;
5. \( \{(x, y) \in \mathbb{R}^2 : x < y \} \in S_2 \).
6. \( S_2 \) contains the graphs of addition and multiplication.

\( S \) is called o-minimal if it satisfies the following additional axiom:

The sets in \( S_1 \) consists of exactly the finite unions of intervals and points.
Fix an o-minimal structure $\mathcal{S}$. Let $A \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}^n$. We say $A$ is $\mathcal{S}$-definable, or simply definable when the underlying structure $\mathcal{S}$ is clear, if $A \in \mathcal{S}_m$; we say the map $f$ is definable if its graph $\Gamma(f) \subset \mathbb{R}^{m+n}$ is definable. If $f$ is definable, then the domain $A$ of $f$ and its image $f(A)$ are also definable.

A structure is usually constructed as follows. Consider a family of functions, denoted by $F$, and take the smallest structure containing the graphs of all the functions in $F$. When the family consists of all constant functions, i.e. $f = c$, $c \in \mathbb{R}$, and also the graphs of addition and multiplication in $\mathbb{R}^3$, we obtain the family of semialgebraic sets, which is an o-minimal structure. On the other hand, if $F$ ranges over all restricted analytic functions, we obtain the family of so-called globally subanalytic sets, denoted as $\mathcal{S}(\mathbb{R}_{an})$. Here we call $f : \mathbb{R}^n \to \mathbb{R}$ a restricted analytic function, if there is an open subset $U$ containing $[-1,1]^n$ in $\mathbb{R}^n$, and an analytic function $g : U \to \mathbb{R}$ such that

$$f(x) = \begin{cases} g(x), & \text{for } x \in [-1,1]^n \\ \text{constant}, & \text{otherwise.} \end{cases}$$

The following result due to Gabrielov is important for our application.

**Proposition 2.10** ([Gab68]). $\mathcal{S}(\mathbb{R}_{an})$ is an o-minimal structure.

From now on we fix some o-minimal structure and talk about definable (or tame) sets and maps respect to this structure. Proposition 2.8 generalizes to definable functions of o-minimal structure.

**Proposition 2.11** ([vdD98, Theorem 1.2, Chapter 3]). Let $f : (a,b) \to \mathbb{R}$ be a definable function on the interval. Then there are points $a_0 = a < a_1 < \cdots < a_N = b$ in $(a,b)$ such that on each subinterval $(a_j,a_{j+1})$, the function is either constant, or strictly monotone.

For our proof we need a parameter version of the above result.

**Proposition 2.12** ([BGZZK21, Propostion 2.8]). Let $S$ be an o-minimal structure and let $h : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a definable function. For $x \in \mathbb{R}^m$, let $N(x)$ denote the number of times the function $h(x,\cdot) : \mathbb{R} \to \mathbb{R}$ changes monotonicity. Then $\sup_{x \in \mathbb{R}^m} N(x) < \infty$.

**Definition 2.13.** We call a set belonging to an o-minimal structure a tame set.

First we need the concept of cells, which are non-empty tame sets of particularly simple form. They are defined inductively as follows:

1. the cells in $\mathbb{R}$ are just the points $\{r\}$, and the intervals $(a,b)$;
2. Let $C \subset \mathbb{R}^n$ be a cell; if $f,g : C \to \mathbb{R}$ are definable continuous functions such that $f < g$ on $C$, then

$$(f,g) := \{(x,r) \in C \times \mathbb{R} : f(x) < r < g(x)\}$$

is a cell in $\mathbb{R}^{n+1}$. Moreover, given a definable continuous function $f : C \to \mathbb{R}$, the graph of $f$, and the sets

$$(-\infty,f) := \{(x,r) \in C \times \mathbb{R} : r < f(x)\}, \quad (f,\infty) := \{(x,r) \in C \times \mathbb{R} : f(x) < r\}$$

are cells in $\mathbb{R}^{n+1}$; finally $C \times \mathbb{R} \subset \mathbb{R}^{n+1}$ is a cell.

It turns out every tame set can be decomposed into cells.

**Proposition 2.14** (Cell Decomposition, [vdD98, Theorem 2.11, Chapter 3]). Every tame set $A \subset \mathbb{R}^m$ has a finite partition $A = C_1 \cup \cdots \cup C_\ell$ into cells $C_i$. If $f : A \to \mathbb{R}^n$ is a definable map, this partition of $A$ can be chosen such that all restrictions $f|_{C_i}$ are continuous.

A $C^k$-cell in $\mathbb{R}^m$ is defined to be a cell which is also a $C^k$ submanifold of its ambient Cartesian space.
Proposition 2.15 (Smooth Cell Decomposition, [vdD99, p. 131]). Let $k$ be a positive integer. Every tame set $A \subset \mathbb{R}^m$ admits a finite partition $A = C_1 \cup \cdots \cup C_l$ into $C^k$-cells. If $f : A \to \mathbb{R}$ is a definable map, then the partition can be chosen such that $f|_{C_i}$ is $C^k$.

The dimension of a cell is defined inductively in an obvious way (For precise definition the reader may refer to [vdD98, p. 50]), accordingly we define the dimension of a tame set as follows:

**Definition 2.16.** The dimension of a non-empty tame set $A \subset \mathbb{R}^m$ is defined by
\[
\dim A := \max \{ \dim(C_i) : \cup_i C_i \text{ is a cell decomposition of } A \},
\]
and $\dim(\emptyset) = -\infty$. The dimension of $A$ at a point $p \in A$ is defined by
\[
\dim_p A := \max \{ \dim(C_i), p \in \overline{C_i} \text{ and } \cup_i C_i \text{ is a cell decomposition of } A. \}
\]

It can be shown that the above definition does not depend on the choice of the partition, and it also agrees with Definition 2.4 for analytic sets, if we consider the latter as a definable set in the $\alpha$-minimal structure $\mathbb{R}_\text{an}$.

Now given a cell decomposition $A = C_1 \cup \cdots \cup C_k$, it is easy to see that $\pi(A) = \pi(C_1) \cup \cdots \cup \pi(C_k)$ is a cell decomposition of $\pi(A)$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the natural projection. Hence we immediately get the following result:

**Proposition 2.17.** Let $A$ be a definable set and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the natural projection map. Then $\dim A \geq \dim \pi(A)$.

Let $X \subset \mathbb{R}^n$ be a relatively compact subset. For any $\varepsilon > 0$, we denote by $M(\varepsilon, X)$ the minimal number of closed balls of radius $\varepsilon$ covering $X$.

**Proposition 2.18 ([YC04, Corollary 5.7]).** Let $A \subset \mathbb{R}^n$ be a tame set of dimension $\ell < n$. Then for any ball $B^m_r \subset \mathbb{R}^n$,
\[
M(\varepsilon, A \cap B^m_r) \leq C(n) \left( \left( \frac{r}{\varepsilon} \right)^\ell + 1 \right).
\]

**Corollary 2.19.** Let $A \subset \mathbb{R}^n$ be a tame set of dimension $\ell < n$. Denote by $N_\varepsilon(A)$ the $\varepsilon$ neighborhood of $A$. Then
\[
|N_\varepsilon(A)| \lesssim \varepsilon^{n-\ell}.
\]
where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$.

**Proof.** Let $\Lambda = \{ B(x_i, \varepsilon) \}_{i=1}^M$ be the set of balls covering $A$, where $M \leq C(n) \left( \left( \frac{\varepsilon}{\varepsilon} \right)^\ell + 1 \right)$. Then $N_\varepsilon(A)$ is contained in $\cup_{i=1}^M B(x_i, 2\varepsilon)$, and thus $|N_\varepsilon(A)| \lesssim \varepsilon^n (\varepsilon^{-\ell} + 1) \lesssim \varepsilon^{n-\ell}$. \qed

2.2. Removable singularity. In our proof of Theorem 1.1, we first show that $|\nabla \log |f|| = \frac{|\nabla f|}{|f|} \in L^1_{\text{loc}}(0)$. However, as both $\log |f|$ and $\nabla \log |f|$ are defined off the zero set of $f$, we must show that the derivative $\nabla \log |f|$ in fact exists in the sense of distribution while crossing the singular set $Z_f$, a phenomenon known as removable singularity. We now state such problem in a slightly more general form. Let $P(x, \Omega)$ be a linear partial differential operator on an open set $\Omega \subset \mathbb{R}^n$, and let $A$ be a closed subset of $\Omega$. Given a class of distributions on $\mathcal{F}(\Omega)$, we say (following Harvey-Polking [HP70]) the set $A$ is removable for $\mathcal{F}(\Omega)$ if each $f \in \mathcal{F}(\Omega)$ that satisfies $P(x, \Omega)f = 0$ in $\Omega \setminus A$ also satisfies $P(x, \Omega)f = 0$ in $\Omega$. Questions can then be asked as to what conditions on the coefficients of $P(x, \Omega)$ and on the set $A$ will ensure that $A$ is removable for $\mathcal{F}(\Omega)$. In the remainder of this section, we will prove two results of this type.

We let $d(x, E)$ to denote the Euclidean distance from the point $x$ to the set $E \subset \mathbb{R}^n$. We define the $\varepsilon$-neighborhood of $E$ by $E_\varepsilon := \{ x \in \mathbb{R}^n : d(x, E) < \varepsilon \}$. The unit ball in $\mathbb{R}^n$ is denoted by $B$, and the punctured ball $B \setminus \{ 0 \}$ by $B_\ast$. We begin with a result for isolated singularity.
Proposition 2.20. Let $\alpha$ be a multi-index with $|\alpha| = m$, for $1 \leq m < n$. Suppose $f \in L^{\frac{n}{n-\beta}}(B)$, $g \in L^1(B)$, and $D^\alpha f = g$, \((D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n})\) holds in the sense of distributions in $B_\ast$. Then $D^\alpha f = g$ holds in the sense of distributions in $B$.

Proof. Define a smooth function $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfying
\[
\varphi(x) = \begin{cases} 
0 & \text{if } |x| < 1; \\
1 & \text{if } |x| > 2.
\end{cases}
\]
Let $\varphi_n(x) := \varphi(nx)$. Then $\varphi_n$ satisfies
\[
\varphi_n(x) = \begin{cases} 
0 & \text{if } |x| < \frac{1}{n}; \\
1 & \text{if } |x| > \frac{2}{n}.
\end{cases}
\]
Let $\psi \in C_c^\infty(B)$, and set $\psi_n := \psi \varphi_n$. Since $\psi_n \in C_c^\infty(B_\ast)$, by assumption we have
\[
\int_B f(D^\alpha \psi_n) = (-1)^m \int_B g \psi_n.
\]
Since $\psi_n$ converges to $\psi$ point-wise in $B$ and $g \in L^1(B)$, by the Dominated Convergence theorem the integral on right-hand side converges to
\[
(-1)^m \int_B g \psi.
\]
For the integral on the left-hand side we have
\[
\int_B f(D^\alpha \psi_n) = \int_B f D^\alpha(\psi \varphi_n)
= \int_B f(D^\alpha \psi) \varphi_n + \sum_{|\beta| \geq 1} \int_B f(D^{\alpha-\beta} \psi)(D^\beta \varphi_n).
\]
Since $f \in L^{\frac{n}{n-\beta}}(B) \subset L^1(B)$, we have
\[
\int_B f(D^\alpha \psi) \varphi_n \xrightarrow{\epsilon \to 0} \int_B f(D^\alpha \psi).
\]
By the definition of $\varphi_n$, we see that $D^\beta \varphi_n$ is compactly supported in $\{x : |x| < \frac{2}{n}\}$, and
\[
|D^\beta \varphi_n(x)| = n^{|\beta|} |\partial_\beta \varphi(nx)| \lesssim n^{|\beta|}.
\]
Denote by $B(x,r)$ the ball centered at $x$ with radius $r$. By Hölder’s inequality,
\[
\int_B |f(D^{\alpha-\beta} \psi)(D^\beta \varphi_n)| \leq C \left( \int_{|x| < \frac{2}{n}} |f|_{\frac{n}{n-|\beta|}} \right)^{\frac{n-|\beta|}{n}} \left( \int_{|x| < \frac{2}{n}} |\partial_\beta \varphi_n|_{\frac{\beta}{n}} \right)^{\frac{|\beta|}{n}}
\leq C \|f\|_{L^{\frac{n}{n-|\beta|}}(B(0,\frac{2}{n}))} \left( \frac{2}{n} \right)^n n^{|\beta| \frac{|\beta|}{n}}
\leq C \|f\|_{L^{\frac{n}{n-|\beta|}}(B(0,\frac{2}{n}))}.
\]
Since $|\beta| = |\alpha| = m$, we have $\|f\|_{L^{\frac{n}{n-|\beta|}}(B(0,\frac{2}{n}))} \leq \|f\|_{L^{\frac{n}{n-|\alpha|}}(B(0,\frac{2}{n}))}$ which converges to 0 as $n \to \infty$.
Putting the results together and letting $n \to \infty$ in (2.1) and (2.2) we get
\[
\int_B f(D^\alpha \psi) = (-1)^m \int_B g \psi, \quad \psi \in C_c^\infty(B).
\]
Hence $D^nf = g$ in the sense of distribution in $\mathbb{B}$.\qed

Next, we prove a removable singularity result for arbitrary set of Hausdorff dimension at most \( n - 2 \).

**Proposition 2.21.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and let \( A \subset \Omega \) be a compact set of Hausdorff dimension at most \( n - 2 \). Suppose \( f \in C^\infty(\Omega \setminus A) \cap L^p_{\text{loc}}(\Omega) \) for some \( p > 2 \), and also \( \nabla f \in L^1_{\text{loc}}(\Omega) \). Then \( \nabla f \) is the derivative of \( f \) in \( \Omega \) in the sense of distributions.

**Proof.** By Harvey-Polking’s lemma (Lemma A.3), there exists a family of cut-off functions \( \{ \tilde{x}_\varepsilon \}_{\varepsilon > 0} \in C^\infty_c(\Omega) \) such that \( 0 \leq \tilde{x}_\varepsilon \leq 1 \), \( \tilde{x}_\varepsilon \equiv 1 \) in a neighborhood of \( A \), and \( \text{supp} \tilde{x}_\varepsilon \subset A \). Furthermore, the following estimate holds

\[
|\nabla \tilde{x}_\varepsilon|_{L^{p'}(\Omega)} \leq C_{\alpha,n}\varepsilon^{l-1} (\Lambda_{n-lp'}(A) + \varepsilon)^\frac{p}{p'} , \quad n - lp' > 0.
\]

Define \( \chi_\varepsilon := 1 - \tilde{x}_\varepsilon \), so that \( \chi_\varepsilon \equiv 0 \) near \( A \) and \( \chi_\varepsilon \equiv 1 \) in \( \Omega \setminus A \). Moreover, \( \chi_\varepsilon \) converges to 1 point-wise in \( \Omega \setminus A \) as \( \varepsilon \to 0 \), and \( \chi_\varepsilon \) satisfies estimate (2.3). For \( \phi \in C^\infty_c(\Omega) \), we can integrate by parts on the domain \( \Omega \cap \{ \chi_\varepsilon \geq 0 \} \) to get

\[
\int_\Omega (\nabla f) \chi_\varepsilon \phi = -\int_\Omega f (\nabla \chi_\varepsilon) \phi = -\int_\Omega f (\nabla \chi_\varepsilon) \phi - \int_\Omega f \chi_\varepsilon \nabla \phi.
\]

Since \( \nabla f \in L^1_{\text{loc}}(\Omega) \), by the Dominated Convergence theorem, the integral on the left converges to \( \int_\Omega (\nabla f) \phi \) as \( \varepsilon \to 0 \). Similarly, the second integral on the right converges to \( \int_\Omega f \nabla \phi \). By Hölder’s inequality with \( \frac{1}{p} + \frac{1}{p'} = 1 \), we get

\[
\int_\Omega |f(\nabla \chi_\varepsilon) \phi| \lesssim \| f \|_{L^p(\Omega)} \| \nabla \chi_\varepsilon \|_{L^{p'}(\Omega)} \leq C_{\alpha,n}\varepsilon^{l-1} (\Lambda_{n-lp'}(A) + \varepsilon)^\frac{p}{p'} \| f \|_{L^p(\Omega)}.
\]

Since \( p' < 2 \) (\( p > 2 \)), we can choose \( l > 1 \) such that \( lp' < 2 \), so \( n - lp' > n - 2 \). By the assumption \( \dim A \leq n - 2 \), we have \( \Lambda_{n-lp'}(A) = 0 \), and therefore

\[
\int_{\Omega \setminus A} |f(\nabla \chi_\varepsilon) \phi| \leq C_{\alpha,n}\varepsilon^{l-1+\frac{2}{p'}} \| f \|_{L^p(\Omega)}.
\]

In particular the integral converges to 0 as \( \varepsilon \to 0 \). Taking the limit in (2.4) as \( \varepsilon \to 0 \), we get

\[
\int_\Omega (\nabla f) \phi = -\int_\Omega f (\nabla \phi), \quad \phi \in C^\infty_c(\Omega).
\]

In other words, \( \nabla f \) is the derivative of \( f \) in \( \Omega \) in the sense of distributions.\qed

3. PROOF OF THE MAIN THEOREM

In this section we prove the main result of the paper, Theorem 1.1.

**Proposition 3.1.** Let \( f \) be the germ of a real analytic function at \( 0 \) in \( \mathbb{R}^n \) and suppose \( f(0) = 0 \). Then \( \log |f| \in L^p_{\text{loc}}(0) \), for any \( 0 < p < \infty \).

**Proof.** By Weierstrass Preparation theorem for real analytic functions ([KP02, Theorem 6.1.3]), one can find a real analytic coordinate \( x = (x_1, \ldots, x_n) \) defined in some cube with length \( \delta \) centered at \( 0 \) such that

\[
f(x) = u(x) (x^n + a_1(x')x_{n-1}^{m-1} + \cdots + a_{n-1}(x') x_n + a_n(x')) , \quad x' = (x_1, \ldots, x_{n-1}).
\]

Here \( m \) is the vanishing order of \( f \) in the \( x_n \) direction, \( u(0) \neq 0 \) and \( a_j \) are real analytic functions in \( n - 1 \) variables with \( a_j(0') = 0 \). We can rewrite (3.1) as

\[
f(x) = u(x) \prod_{j=1}^m (x_n - \xi_j(x')) , \quad x \in [-\delta, \delta]^n,
\]
where the roots $\xi_j(x')$ are either real or complex numbers. Assuming that $|f| < 1$ in $[-\delta, \delta]^n$, we have
\[
|\log |f(x)|| = \log \frac{1}{|f(x)|} = \log \frac{1}{|u(x)|} + \sum_{j=1}^{m} \log \frac{1}{|x_n - \xi_j(x')|}, \quad x \in [-\delta, \delta]^n.
\]
Then
\[
|\log |f(x)||^p \leq C_p \left( |\log \frac{1}{|u(x)|} + \sum_{j=1}^{m} \log \frac{1}{|x_n - \xi_j(x')|} |^p \right), \quad x \in [-\delta, \delta]^n.
\]
Hence
\[
\int_{[-\delta, \delta]^n} |\log |f(x)||^p \, dx \lesssim \int_{[-\delta, \delta]^n} |\log \frac{1}{|u|}|^p \, dx + \int_{[-\delta, \delta]^n} \left| \sum_{j=1}^{m} \log \frac{1}{|x_n - \xi_j(x')|} \right|^p \, dx.
\]
The first integral is bounded since $u$ does not vanish near $0$. For the second integral, we apply Fubini’s theorem to get
\[
\int_{[-\delta, \delta]^n} \left| \sum_{j=1}^{m} \log \frac{1}{|x_n - \xi_j(x')|} \right|^p \, dx = \int_{[-\delta, \delta]^n-1} \left( \int_{[-\delta, \delta]} \left| \sum_{j=1}^{m} \log \frac{1}{|x_n - \xi_j(x')|} \right|^p \, dx_n \right) \, dx' 
\lesssim C_p \int_{[-\delta, \delta]^n-1} \left( \sum_{j=1}^{m} \int_{[-\delta, \delta]} \left| \log \frac{1}{|x_n - \xi_j(x')|} \right|^p \, dx_n \right) \, dx'.
\]
The inner integral is bounded by (up to a constant independent of $x'$),
\[
\int_{[-\delta, \delta]} \left( \log \frac{1}{|x_n|} \right)^p \, dx_n.
\]
Since $\log \frac{1}{|x_n|} \leq C \varepsilon |x_n|^{-\varepsilon}$, for any positive $\varepsilon > 0$, we see that for each $0 < p < \infty$, the above integral is bounded by some constant $C_p$ which is uniform in $x'$. This shows that $\log |f|$ is locally $L^p$ for any $0 < p < \infty$. $\square$

In what follows we fix the o-minimal structure $\mathbb{R}_{an}$. Given the germ of a real analytic function $f$ at the origin, we may assume without loss of generality that $f$ is real analytic function in an open set $U$ containing the cube $[-1, 1]^n$. We define the restricted function
\[
(f) = \begin{cases} f(x), & \text{for } x \in [-1, 1]^n; \\ 1, & \text{otherwise} \end{cases}
\]
(3.2)

The following holds

Lemma 3.2. $Z_{(f)}$ is definable in $\mathbb{R}_{an}$.

Proof. We can write $Z_{(f)}$ as the intersection of the graph of $\overline{f}$ and $z_{n+1} = 0$ in $\mathbb{R}^{n+1}$, both of which are definable in $\mathbb{R}_{an}$. $\square$

Proposition 3.3. Let $f$ be the germ of a real analytic function at the origin in $\mathbb{R}^n$, $n \geq 2$. Suppose $\text{codim}_0 Z_f \geq 2$. Then there exists a small neighborhood $U$ of $0$ such that
\[
\int_U |\nabla \log |f(x)|| \, dV(x) < C(f, n),
\]
where the constant $C(f, n)$ depends only on $f$ and the dimension $n$. 
Proof. In view of the above remark (\((3.2)\)), it suffices to show that
\[
\int_{[-1,1]^n} |\partial_1 \log |f|| \, dx \leq C(f, n)
\]
Fix \(x' = (x_2, \ldots, x_n)\) and write \(\partial_1 \log |f(x_1, x')| = \frac{\partial_1 |f(x_1, x')|}{|f(x_1, x')|} := g_{x'}(x_1)\). By Proposition 2.12, \(g_{x'}\) changes sign on \([-1, 1]\) by a finite number of times \(M\) which is uniform in \(x' \in [-1, 1]^{n-1}\). Now for each fixed \(x'\), we break up the interval \([-1, 1]\) into at most \(M\) subintervals \([a_i(x'), b_i(x')]\), on each of which \(g_{x'}\) has the same sign. Then
\[
\int_{[-1,1]^n} |\partial_1 \log |f|| \, dx = \int_{[-1,1]^{n-1}} \int_0^1 |\partial_1 \log |f|| \, dx_1 \, dx' = \sum_{i=1}^M \int_{[-1,1]^{n-1}} \int_{a_i(x')}^{b_i(x')} |\partial_1 \log |f|| \, dx_1 \, dx'
\]
\[
= \sum_{i=1}^M \int_{[-1,1]^{n-1}} \left| \int_{a_i(x')}^{b_i(x')} \partial_1 \log |f| \, dx \right| \, dx'
\]
\[
\leq \sum_{i=1}^M \int_{[-1,1]^{n-1}} |\log |f(b_i(x'), x')|| + |\log |f(a_i(x'), x')|| \, dx'
\]
The integrand is bounded by a constant multiple of \(|\log \inf_{[-1,1]} |f(\cdot, x')|| + |\log \sup_{[-1,1]} |f(\cdot, x')||
We can assume that \(\sup_{x \in [-1,1]} |f(x)| \leq 1\), so
\[
(3.3) \quad \int_{[-1,1]^n} |\partial_1 \log |f|| \, dx \lesssim 1 + \int_{[-1,1]^{n-1}} |\log \inf_{[-1,1]} |f(\cdot, x')| \, dx'.
\]
We now break up \([-1,1]^{n-1}\) into dyadic regions of the form
\[
E_j := \{x' \in [-1,1]^{n-1} : \inf_{[-1,1]} |f(\cdot, x')| \in (2^{-j-1}, 2^{-j})\}, \quad j = 0, 1, 2, \ldots .
\]
We claim that \(E_0 = \{x' \in [-1,1]^{n-1} : \inf_{[-1,1]} |f(\cdot, x')| = 0\}\) has measure 0 with respect to the Lebesgue measure of \(\mathbb{R}^{n-1}\). Indeed, \(E_0\) is the image of the zero set \(Z_f\) under the projection map \(\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}\). Hence by Proposition 2.17, we have
\[
\dim(E_0) \leq \dim(Z_f) \leq n - 2,
\]
which implies the claim. Now, if \(a' \in E_j\), by definition there exists some \(a\) such that \(|f(a, a')| \in [2^{-j-1}, 2^{-j}]\). Let \(a_*\) be the point in \(Z_f\) such that \(\text{dist}(a, Z_f) = |a - a_*|\). Denote by \(a'_*\) the last \(n-1\) coordinates of \(a_*\). Then
\[
\text{dist}((a, a'), Z_f) = |a - a_*| \geq |a' - a'_*| \geq \text{dist}(a', E_0)
\]
where we used the fact that \(a'_* \in E_0\). By the Lojasiewicz inequality applied to \(f\) (Proposition 2.6), there exists some \(\alpha > 0\) such that
\[
2^{-j} \geq |f(a, a')| \geq c_f \text{dist}((a, a'), Z_f)^\alpha \geq c_f \text{dist}(a', E_0)^\alpha.
\]
Hence \(\text{dist}(a', E_0) \leq C_f(2^{-j})^{\frac{1}{\alpha}}\) for all \(a' \in E_j\), which implies that \(E_j\) lies in a \(C_f(2^{-j})^{\frac{1}{\alpha}}\) neighborhood of \(E_0\). By Corollary 2.19, \(|E_j| \lesssim (2^{-j})^{\frac{n-2}{\alpha}}\). It then follows from (3.3) that
\[
\int_{[-1,1]^n} |\partial_1 \log |f|| \, dx \lesssim 1 + \sum_{j=0}^\infty |E_j| \lesssim 1 + \sum_{j=0}^\infty (2^{-j})^{\frac{n-2}{\alpha}} |\log 2^{-j}| < \infty.
\]
\qed
Proof of Theorem 1.1 and Theorem 1.2.

Combining Proposition 3.1, Proposition 3.3 and Proposition 2.21, we obtain Theorem 1.1. To prove Theorem 1.2, note that the assumption implies that the dimension of $Z_f$ at each point is at least $2$. We now cover $U$ by finitely many balls. Furthermore, we can shrink the balls so that

1. For any given ball $B(p, r)$ with center $p$ and $f(p) \neq 0$, we have that $B(p, r) \cap Z_f = \emptyset$.
2. For any given ball $B(p, r)$ with center $p$ and $f(p) = 0$, $\log |f| \in W^{1,1}(B(p, r))$.

where (2) holds because of Theorem 1.1. The conclusion then follows easily.

In order to prove Theorem 1.3, we first show that the integral blows up when the exponent is the dimension of space.

**Theorem 3.4.** Let $f$ be the germ of a real analytic function at the origin in $\mathbb{R}^n$ with $f(0) = 0$. Then for every (sufficiently small) neighborhood $U$ of 0

$$
\int_U \left| \frac{\nabla f}{f} \right|^n dV(x) = \infty.
$$

*Proof.* We prove by contradiction. Without loss of generality we can assume $f$ is real analytic in $U = B_\varepsilon(0)$, a ball of radius $\varepsilon$ centered at $0$. Suppose

$$(3.4) \quad \int_U \left| \frac{\nabla f}{f} \right|^n dV(x) < \infty.$$ 

Define

$$A(x) = \begin{cases} \left| \frac{\nabla f(x)}{f(x)} \right|, & f(x) \neq 0, \\ 0, & f(x) = 0. \end{cases}$$

Then $A \in L^n(B_\varepsilon(0))$. Using spherical coordinate $x = \rho \omega, \omega \in S^{n-1}$, we have

$$(3.5) \quad \int_{B_\varepsilon(0)} A^n(x) dV(x) = \int_{S^{n-1}} \int_0^\varepsilon A^n(\rho \omega) \rho^{n-1} d\rho d\sigma(\omega),$$

where we denote by $d\sigma(\omega)$ the area element on the $n-1$ dimensional unit sphere $S^{n-1} \subset \mathbb{R}^n$. By (3.4), the integral in (3.5) is finite. Hence Fubini’s theorem implies that, for almost all $\omega_0 \in S^{n-1},$

$$\int_0^\varepsilon A^n(\rho \omega_0) \rho^{n-1} d\rho < \infty,$$

or

$$(3.6) \quad A(\rho \omega_0) \rho^{n-1} \in L^n(0, \varepsilon).$$

Fix $\omega_0 \in S^{n-1}$ such that (3.6) holds and also $f(\rho \omega_0)$ is not identically $0$ for $\rho \in (0, \varepsilon)$. Define $\varphi(\rho) := f(\rho \omega_0)$, with $\varphi(\rho_0) = 0$. By choosing $\varepsilon$ small, we have $\varphi'(\rho) \neq 0$ if $\rho \in (0, \varepsilon) \setminus \{0\}$ ($\varphi$ is a one variable real analytic function and thus have isolated zero at 0.) Taking derivative,

$$|\varphi'(\rho)| = |\nabla f(\rho \omega_0) \cdot \omega_0| \leq |\nabla f(\rho \omega_0)|.$$

By definition of $A$, we have for $\rho \neq 0$,

$$|\varphi'(\rho)| \leq |\nabla f(\rho \omega_0)|$$

$$= A(\rho \omega_0) |f(\rho \omega_0)|$$

$$= A(\rho \omega_0) |\varphi(\rho)|$$

$$= A(\rho \omega_0) \rho^{\frac{n-1}{n}} |\varphi(\rho)| \rho^{\frac{1-n}{n}}.$$ 

Hence

$$\rho^{\frac{n-1}{n}} \frac{|\varphi'(\rho)|}{|\varphi(\rho)|} \leq A(\rho \omega_0) \rho^{-\frac{1}{n}}, \quad \rho \neq 0.$$
Since \( \varphi(\rho) \) is real analytic and is not identically 0, it has finite order vanishing, i.e. there exists \( N \) such that

\[
\varphi(\rho) = \rho^N + O(\rho^{N+1}),
\]

\[
\varphi'(\rho) = N\rho^{N-1} + O(\rho^N).
\]

and \( \frac{\varphi'(\rho)}{\varphi(\rho)} \approx \frac{1}{\rho} \) for \( \rho \) small. It follows that

\[
A(\varphi_0)\rho^{\frac{n-1}{n}} \geq \rho^{\frac{n-1}{n}} \left| \frac{\varphi'(\rho)}{\varphi(\rho)} \right| \approx \rho^{\frac{n-1}{n}} \rho^{-1} = \rho^{-\frac{1}{n}} \notin L^n(0, \varepsilon),
\]

which contradicts with (3.6).

\[\square\]

We are now ready to prove Theorem 1.3, which gives a refinement of the above result based on information about codimension of the zero set.

**Theorem 3.5.** Let \( f \) be the germ of a real analytic function at the origin in \( \mathbb{R}^n \) with \( f(0) = 0 \). Suppose \( \text{codim}_0(Z_f) = n - d \). Then for every (sufficiently small) neighborhood \( U \) of 0,

\[
\int_U \left| \frac{\nabla f}{f} \right|^{n-d} = \infty.
\]

**Proof.** By definition, for any \( \varepsilon > 0 \) there exists a smooth point \( x_0 \) of \( Z_f \) in \( U \) such that \( \dim_{x_0}(Z_f) = d \). Then one can find a coordinate system \( (y', y'') \), with \( y' = (y_1, \ldots, y_d), \ y'' = (y_{d+1}, \ldots, y_n) \), in a small neighborhood \( W \) of \( x_0 \) such that \( W \subset U \) and

\[
Z_f \cap W = \{y_{d+1} = \cdots = y_n = 0\}.
\]

Here, \( y = (y_1, \ldots, y_d) \) is a local coordinate chart of \( Z_f \) in \( W \). Assume for the moment that

\[
\int_W \left| \frac{\nabla f}{f} \right|^{n-d} < \infty.
\]

Let \( f_{y'}(y'') = f(y', y'') \) be the restriction of \( f \) onto the \( y' \)-plane. For each fixed \( y' \), the function \( f_{y'} \) satisfies \( f_{y'}(0) = 0 \). Thus by Fubini’s theorem,

\[
\infty > \int_W \left| \frac{\nabla f_{y'}(y'')}{f_{y'}(y'')} \right|^{n-d} dy'' = \int \int \left| \frac{\nabla f}{f} \right|^{n-d} dy'' dy' \geq \int \int \left| \frac{\nabla f_{y'}(y'')}{f_{y'}(y'')} \right|^{n-d} dy'' dy'.
\]

Now the integral on the right-hand side is infinity by Theorem 3.4, which is contradiction. Therefore we conclude that \( \int_W \left| \frac{\nabla f}{f} \right|^{n-d} = \infty \) and

\[
\int_U \left| \frac{\nabla f}{f} \right|^{n-d} > \int_W \left| \frac{\nabla f}{f} \right|^{n-d} = \infty.
\]

\[\square\]

For functions of one variable, the differential inequality \( |\nabla f| \leq V|f| \) is rigid, as shown by the following result:

**Proposition 3.6.** Let \( f \) be a real-valued continuous function on \((-1, 1)\). Suppose \( f(0) = 0, f \in W^{1,p}(-1, 1), p \geq 1, \) and \( |\nabla f| \leq V|f| \) for \( V \in L^1(-1, 1) \). Then \( f \equiv 0 \) on \((-1, 1)\).

**Proof.** By [Bre11, Theorem 8.2],

\[
f(x) = f(0) + \int_0^x f'(t) dt = \int_0^x f'(t) dt.
\]
By replacing $V$ with $V + 1$, we can assume that $V \geq 1$ on $(-1, 1)$. It follows that for $\varepsilon > 0$,

$$
\int_{0}^{\varepsilon} V(x)|f(x)| \, dx \leq \int_{0}^{\varepsilon} V(x) \left( \int_{0}^{x} |f'(t)| \, dt \right) \, dx
$$

$$
\leq \left( \int_{0}^{\varepsilon} V(x) \, dx \right) \left( \int_{0}^{\varepsilon} V(t)|f(t)| \, dt \right)
$$

If $f$ is not identically 0 near the origin, then $\int_{0}^{\varepsilon} V(x)|f(x)| \neq 0$. Hence

$$
1 \leq \int_{0}^{\varepsilon} V(x) \, dx.
$$

Since $V \in L^1(-1, 1)$, the right-hand side goes to 0 as $\varepsilon \to 0$, which is a contradiction. Hence $f$ must be identically 0. \qed

The following example shows that the integrability exponents in all of our results are sharp.

**Example 3.7.** Let $f$ be a function on $\mathbb{R}^n$, $n \geq 2$, of the form

$$
f(x) = x_1^{2r_1} + x_2^{2r_2} + \cdots + x_k^{2r_k}, \quad k \geq 2,
$$

where $r_k$ are integers satisfying $1 \leq r_1 \leq r_2 \cdots \leq r_k$. Then for any bounded neighborhood $U$ of 0, the following holds

$$
\int_{U} \left| \nabla f \right|^{\gamma} < \infty
$$

if and only if

$$
\gamma < 1 + \frac{r_1}{r_2} + \cdots + \frac{r_1}{r_k}.
$$

If $k = 1$, i.e. $f(x) = x_1^r$, then (3.7) holds if and only if $\gamma < 1$.

**Proof.** We have, up to a nonzero constant,

$$
\left| \frac{\nabla f(x)}{f(x)} \right| = \frac{\sqrt{|x_1|^{2(2r_1-1)} + \cdots + |x_n|^{2(2r_k-1)}}}{|x_1|^{2r_1} + \cdots + |x_n|^{2r_k}}.
$$

Set $u_1 = |x_1|^{r_1}, \ldots, u_k = |x_k|^{r_k}$. Then $|x_1| = u_1^{\frac{1}{r_1}}, \ldots, |x_k| = u_k^{\frac{1}{r_k}}$, and

$$
\left| \frac{\nabla f(x)}{f(x)} \right| = \sqrt{\frac{\sum_{i=1}^{k} u_i^{4-\frac{2}{r_i}}}{u_1^{2r_1} + \cdots + u_k^{2r_k}}}.
$$

Writing $y = (y', y'')$ where $y' = (y_1, \ldots, y_k)$ and $y'' = (y_{k+1}, \ldots, y_n)$, the integral becomes

$$
\int_{U} \left| \frac{\nabla f(x)}{f(x)} \right|^{\gamma} \, dx = \int_{U'} \int_{U''} \left( \frac{y_1^{4-\frac{2}{r_1}} + \cdots + y_k^{4-\frac{2}{r_k}}}{y_1^{2r_1} + \cdots + y_k^{2r_k}} \right)^{\gamma} \frac{y_1^{\frac{1}{r_1}-1} \ldots y_k^{\frac{1}{r_k}-1}}{r_1 \cdots r_k} \, dy' \, dy''.
$$
where we denote by \( U'' \) the projection of \( U \) onto the \( y'' \) space and \( U'_{y''} := \{ y' \in \mathbb{R}^n : (y', y'') \in U \} \). Using spherical coordinate \( y = \rho \omega \) with \( \rho \in (0, \infty) \) and \( \omega = (\omega_1, \ldots, \omega_k) \in S^{k-1} \), we get

\[
\left( \frac{4 - \frac{2}{r_1} + \cdots + 4 - \frac{2}{r_k}}{y_1^2 + \cdots + y_k^2} \right)^{\gamma} = \left( \frac{(\rho \omega_1)^{4 - \frac{2}{r_1}} + \cdots + (\rho \omega_k)^{4 - \frac{2}{r_k}}}{\rho^{2\gamma}} \right)^{\frac{\gamma}{2}}
\]

\[
= \left( \frac{\rho^{4 - \frac{2}{r_1}} \left( \omega_1^4 - \frac{2}{r_1} + \rho^{\alpha_2} \omega_2^4 - \frac{2}{r_2} \cdots + \rho^{\alpha_k} \omega_k^4 - \frac{2}{r_k} \right)}{\rho^{2\gamma}} \right)^{\frac{\gamma}{2}}
\]

\[
= \rho^{\frac{\gamma}{r_1}} \left( \omega_1^{4 - \frac{2}{r_1} + \rho^{\alpha_2} \omega_2^{4 - \frac{2}{r_2}} \cdots + \rho^{\alpha_k} \omega_k^{4 - \frac{2}{r_k}} \right)^{\frac{\gamma}{2}},
\]

where we set

\[ \alpha_2 = \frac{2}{r_1} - \frac{2}{r_2}, \ldots, \alpha_k = \frac{2}{r_1} - \frac{2}{r_k}. \]

Hence up to a positive constant the inner integral in (3.8) is bounded by

\[
(3.9) \quad \int_{\rho=0}^{1} \int_{S^{k-1}} \frac{\left( \omega_1^{4 - \frac{2}{r_1} + \rho^{\alpha_2} \omega_2^{4 - \frac{2}{r_2}} \cdots + \rho^{\alpha_k} \omega_k^{4 - \frac{2}{r_k}} \right)}{\omega_1^{1 - \frac{1}{r_1}} \cdots \omega_k^{1 - \frac{1}{r_k}}} \, d\sigma(\omega) \left( \rho^{\frac{\gamma}{r_1}} \rho^{1 - \frac{1}{r_1}} \cdots \rho^{k - 1 - \frac{1}{r_k}} \right) \, d\rho,
\]

where \( d\sigma(\omega) \) denotes the area element on \( S^{k-1} \). The integral over the sphere is bounded up to a constant by

\[
(3.10) \quad \int_{S^{k-1}} \frac{\omega_1^{2\gamma - \frac{\gamma}{r_1} + \rho^{\alpha_2} \omega_2^{2\gamma - \frac{\gamma}{r_2}} \cdots + \rho^{\alpha_k} \omega_k^{2\gamma - \frac{\gamma}{r_k}}}}{\omega_1^{1 - \frac{1}{r_1}} \cdots \omega_k^{1 - \frac{1}{r_k}}} \, d\sigma(\omega).
\]

By using the standard parametrization

\[
(\omega_1, \ldots, \omega_k) = (\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \ldots, \left( \prod_{i=1}^{k-2} \sin \theta_i \right) \cos \theta_{k-1}, \prod_{i=1}^{k-1} \sin \theta_i),
\]

where \( \theta_1, \ldots, \theta_{k-2} \in [0, \pi] \) and \( \theta_{k-1} \in [0, 2\pi] \), and the formula for surface area element

\[ d\sigma(\nu) = (\sin^{k-2} \theta_1)(\sin^{k-3} \theta_2) \cdots \sin \theta_{k-1} \, d\theta_1 \cdots d\theta_{k-1}, \]

we see that the integral (3.10) is bounded for any \( \gamma > 0 \). Hence the integral (3.9) is bounded if and only if

\[
-1 < -\frac{\gamma}{r_1} + \frac{1}{r_1} - 1 + \cdots + \frac{1}{r_k} - 1 + k - 1
\]

\[
= -\frac{\gamma}{r_1} + \frac{1}{r_1} + \cdots + \frac{1}{r_k} - 1,
\]

or

\[ \gamma < 1 + \frac{r_1}{r_2} + \cdots + \frac{r_1}{r_k}. \quad \square \]

Remark 3.8. By choosing suitable values for \( r_1, \ldots, r_k \), we see that for every \( \varepsilon > 0 \), there exists a polynomial \( f \) such that \( f \notin L^{1+\varepsilon}(0) \). This shows that \( L^1 \) integrability of \( |\nabla \log |f| | \) is the best possible in Theorem 1.1. On the other hand, for \( g(x) = x_1^2 + x_2^2 + \cdots + x_k^2 \), by the above computation we have \( |\nabla g|/g \in L^p_{\text{loc}}(0) \), for any \( p < k \) with \( k = \text{codim}_{0}(Z_f) \), but \( g \notin L^k_{\text{loc}}(0) \). This proves the optimality of the exponent in Theorem 1.3.
Appendix A. Harvey-Polking’s lemma

All the proofs in this section can be found [HP70] and we provide the details here for readers’ convenience.

For a compact set $K$ in $\mathbb{R}^n$, we let $\Lambda_r(K)$ denote the $r$ dimensional Hausdorff measure of $K$.

**Lemma A.1.** Let $\{Q_i\}_{1 \leq i \leq N}$ be a finite disjoint collection of dyadic cubes of length $s_i$. For each $i$, there is a function $\varphi_i \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp} \varphi_i \subset \frac{3}{2} Q_i$ such that $\sum \varphi_i(x) = 1$ for all $x \in \bigcup_{i=1}^{N} Q_i$. Furthermore, for each multi-index $\alpha$, there is a constant $C_\alpha$ depending only on $\alpha$, for which

$$|D^\alpha \varphi_i(x)| \leq C_{\alpha,n} s_i^{-|\alpha|}$$

for all $x$ and $1 \leq i \leq N$.

**Remark A.2.** Note that in the above lemma the constant $C_\alpha$ does not depend on the choice of the collection of cubes $\{Q_i\}$.

**Proof.** Assume $s_1 \geq s_2 \geq \cdots \geq s_N$. Choose a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\psi(x) \equiv \begin{cases} 1 & \text{if } |x_i| \leq 1 \text{ for } 1 \leq i \leq n, \\ 0 & \text{if } |x_i| \geq \frac{3}{2} \text{ for some } i. \end{cases}$$

Let

$$\psi_k(x) = \psi \left( \frac{x - x_k}{s_k/2} \right),$$

where $x_k$ is the center of the cube $Q_k$. Then we have $\psi_k \equiv 1$ on $Q_k$, and $\text{supp} \psi_k \subset \frac{3}{2} Q_k$. Define for $1 \leq k \leq N$,

$$\varphi_1 = \psi_1,$$

$$\varphi_2 = \psi_2(1 - \psi_1)$$

$$\vdots$$

$$\varphi_{k+1} = \psi_{k+1} \prod_{j=1}^{k} (1 - \psi_j).$$

We show that $\{\varphi_i\}$ is the desired partition of unity. It is clear that $\varphi_i \in C_c^\infty(\mathbb{R}^n)$ and $\text{supp} \varphi_i \subset \text{supp} \psi_i \subset \frac{3}{2} Q_i$.

An easy inductive proof shows that

$$\sum_{j=1}^{k} \varphi_j = 1 - \prod_{j=1}^{k} (1 - \psi_j),$$

for $k = 1, \ldots, N$. Hence $\sum_{j=1}^{N} \varphi_j(x) = 1$ if $x \in \bigcup_{j=1}^{N} Q_j$.

It remains to show that the derivatives of $\varphi_i$ satisfies estimate (A.1). Let

$$\theta_k := \sum_{j=1}^{k} \varphi_j = 1 - \prod_{j=1}^{k} (1 - \psi_j).$$

Since $s_k \geq s_{k+1}$, it suffices to prove the estimate for $\theta_k$. We shall use the following notation. For integers $\nu_1, \ldots, \nu_r$ where $1 \leq \nu_i \leq k$, define

$$g_{\nu_1, \ldots, \nu_r} = \begin{cases} 0, & \text{if } \nu_i = \nu_j \text{ for some } i \neq j, \\ \prod_{i \neq \nu_i \ldots, \nu_r} (1 - \psi_j), & \text{if all } \nu_i \text{ are distinct.} \end{cases}$$
Then there are constants $C_{\beta_1, \ldots, \beta_r}$ depending only on the multi-index subscripts, such that
\[
D^\alpha \theta_k = C_{\beta_1, \ldots, \beta_r} \left( \sum_{\nu_1, \ldots, \nu_r = 1}^{k} g_{\nu_1, \ldots, \nu_r} (D^{\beta_1} \psi_{\nu_1}) \cdots (D^{\beta_r} \psi_{\nu_r}) \right).
\]
where the sum is over all sets of multi-indices \{\beta_1, \ldots, \beta_r\} for which $|\beta_i| \geq 1$ and $\beta_1 + \cdots + \beta_r = \alpha$. Therefore
\[
|D^\alpha \theta_k(x)| \leq \sum_{\beta_1, \ldots, \beta_r} C_{\beta_1, \ldots, \beta_r} \left( \sum_{\nu_1 = 1}^{k} |D^{\beta_1} \psi_{\nu_1}(x)| \right) \cdots \left( \sum_{\nu_r = 1}^{k} |D^{\beta_r} \psi_{\nu_r}| \right).
\]
Consider a typical sum $\sum_{\nu=1}^{k} |D^{\beta} \psi_{\nu}(x)|$. For any given $x$, $D^{\beta} \psi_{\nu} = 0$ unless $x \in \frac{3}{2} Q_{\nu}$. Hence to get the estimate for the sum we need an estimate on how many cubes $Q_{\nu}$ are there such that $\frac{3}{2} Q_{\nu}$ contains $x$. For a given size of cube with side-length $s_{\nu} = s_k 2^\nu$ (since $\nu \leq k$ and we assumed that $s_{\nu} \geq s_k$), there can be at most $2^d$ dyadic cubes $Q_{\nu}$ of length $s_{\nu}$ so that $x \in \frac{3}{2} Q_{\nu}$. Furthermore we have
\[
|D^{\beta} \psi_{\nu}(x)| \leq C_{\beta} s_{\nu}^{-|\beta|}, \quad x \in \frac{3}{2} Q_{\nu}.
\]
It follows that
\[
\sum_{\nu=1}^{k} |D^{\beta} \psi_{\nu}(x)| \leq 2^d C_{\beta} \sum_{p=0}^{\infty} (s_{\nu})^{-|\beta|} = 2^d C_{\beta} \sum_{p=0}^{\infty} (2^p s_k)^{-|\beta|} \leq C_{\beta, d} s_k^{-|\beta|}.
\]
Therefore from (A.3) we have
\[
|D^\alpha \theta_k(x)| \leq \sum_{\beta_1, \ldots, \beta_r} C_{\beta_1, \ldots, \beta_r} \left( C_{\beta_1, d} s_k^{-|\beta_1|} \right) \cdots \left( C_{\beta_r, d} s_k^{-|\beta_r|} \right) \leq C_{\alpha, d} s_k^{-|\alpha|}.
\]

**Lemma A.3** (Harvey-Polking Lemma). Let $K$ be a compact subset of $\mathbb{R}^n$. Let $l$ and $p'$ be some positive number such that $n - lp' > 0$. For each $\varepsilon > 0$, there exists some $\chi_{\varepsilon} \in C^\infty_c(\mathbb{R}^n)$ with $\chi_{\varepsilon} \equiv 1$ in a neighborhood of $K$ and $\text{supp} \chi_{\varepsilon} \subset K_{\varepsilon}$. Furthermore, for $|\alpha| < l$,
\[
|D^\alpha \chi_{\varepsilon}|_{l, p'} \leq C_{\alpha, n} \varepsilon^{-|\alpha|} \left( \Lambda_{n-lp'}(K) + \varepsilon \right)^{p'}.
\]

**Proof.** For each $\varepsilon > 0$, choose a covering of $K$ by a finite collection $\{Q_k\}_{k=1}^{N}$ of dyadic cubes of length $s_k \leq \varepsilon$, with
\[
\bigcup_{k=1}^{N} \frac{3}{2} Q_k \subset K_{\varepsilon};
\]
\[
\sum_{k=1}^{N} s_k^{n-lp'} \leq \Lambda_{n-lp'}(K) + \varepsilon,
\]
where we used the definition of Hausdorff measure. We may assume $s_1 \geq s_2 \geq \cdots \geq s_N$. Let $\{\varphi_k\}$ be the partition of unity for $\{Q_k\}$ constructed in Lemma A.1 and define
\[
\chi_{\varepsilon} := \sum_{k=1}^{N} \varphi_k.
\]
Since each $\varphi_k$ is supported on $\frac{3}{2} Q_k$, we have
\[
\text{supp} \chi_{\varepsilon} \subset \bigcup_{k=1}^{N} \frac{3}{2} Q_k \subset K_{\varepsilon}.
\]
Also, 
\[ \chi_\varepsilon(x) \equiv 1 \text{ on } \cup Q_k, \]
so clearly \( \chi_\varepsilon \equiv 1 \) in a neighborhood of \( K \). We show that \( \chi_\varepsilon \) satisfies estimate (A.5). We modify the cubes to make them disjoint. More specifically we set
\[ T_k = \frac{3}{2} Q_k - \bigcup_{k < j \leq N} \frac{3}{2} Q_j, \quad 1 \leq k \leq N. \]
Then \( \{T_k\} \) is a disjoint collection of sets with
\[ \bigcup_{k=1}^N T_k = \bigcup_{k=1}^N \frac{3}{2} Q_k, \quad T_k \subset \frac{3}{2} Q_k \quad \text{for all } k. \]
and in particular for all \( x \in T_k \), \( \varphi_j(x) = 0 \) if \( j > k \). Therefore we have
\[ \chi_\varepsilon(x) = \sum_{j=1}^N \varphi_j(x) = \sum_{j=1}^k \varphi_j(x) = \theta_k(x), \quad x \in T_k, \]
where \( \theta_k \) is given by (A.2). By estimate (A.4), we have
\[ |D^\alpha \chi_\varepsilon(x)| = |D^\alpha \theta_k(x)| \leq C_{\alpha,n} s_k^{-|\alpha|}, \quad x \in T_k, \]
where \( C_{\alpha,n} \) is a constant independent of \( \varepsilon \) or \( k \). We now estimate
\[ \|D^\alpha \chi_\varepsilon\|_{p'} = \sum_{k=1}^N \int_{T_k} |D^\alpha \chi_\varepsilon(x)|^{p'} dx, \tag{A.8} \]
where we used that \( \text{supp } \chi_\varepsilon \subseteq \bigcup_{k=1}^N \frac{3}{2} Q_k = \bigcup_{k=1}^N T_k \). Since each \( T_k \) is contained in \( \frac{3}{2} Q_k \), its volume is bounded by \( C_n s_k^n \). Combining (A.7) (A.8) and (A.6), we get
\[ \|D^\alpha \chi_\varepsilon\|_{p'} \leq C_{\alpha,n} \sum_{k=1}^N s_k^{-|\alpha| p' + n} \]
\[ \leq C_{\alpha,n} \varepsilon^{(l-|\alpha|) p'} \sum_{k=1}^N s_k^{n-l p'} \]
\[ \leq C_{\alpha,n} \varepsilon^{(l-|\alpha|) p'} (\Lambda_n-l p'(K) + \varepsilon), \quad l > \alpha. \]
Taking \((\frac{1}{p'})^\text{th}\) power we obtain estimate (A.5). \(\Box\)

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