EVERY SMOOTHLY BOUNDED $p$-CONVEX DOMAIN IN $\mathbb{R}^n$ ADMITS A $p$-PLURISUBHARMONIC DEFINING FUNCTION

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ABSTRACT. We show that every bounded domain $D$ in $\mathbb{R}^n$ with smooth $p$-convex boundary for $2 \leq p < n$ admits a smooth defining function $\rho$ which is $p$-plurisubharmonic on $\overline{D}$; if in addition $bD$ has no $p$-flat points then $\rho$ can be chosen strongly $p$-plurisubharmonic on $D$. If $bD$ is 2-convex then for any open connected conformal surface $M$ and conformal harmonic map $f : M \to \mathcal{T}$, either $f(M) \subset D$ or $f(M) \subset bD$. In particular, every conformal harmonic map $\mathbb{D}^n \to D$ from the punctured disc extends to a conformal harmonic map $\mathbb{D} \to D$.

1. INTRODUCTION

It is classical that a bounded convex domain with smooth boundary in $\mathbb{R}^n$ has a smooth convex defining function; see Herbig and McNeal [14] and the references therein (in particular, Weinstock [16] pp. 402–403 and Gilbarg and Trudinger [10] pp. 354–357). In this paper we prove the analogous result for $p$-convex domains for any $2 \leq p < n$. This class of domains, and the associated $p$-plurisubharmonic functions, are closely related to $p$-dimensional minimal submanifolds; in particular, to minimal surfaces when $p = 2$. They were studied by Harvey and Lawson [13]; see also [11, 12] and [1, Sect. 8.1].

Let $x = (x_1, \ldots, x_n)$ denote the coordinates on $\mathbb{R}^n$. Given a $C^2$ function $\rho : \Omega \to \mathbb{R}$ on a domain $\Omega \subset \mathbb{R}^n$, we denote by $\nabla^2 \rho(x) = \text{Hess}_\rho(x)$ the Hessian of $\rho$ at $x \in \Omega$, i.e., the quadratic form on $T_x \mathbb{R}^n = \mathbb{R}^n$ represented by the matrix $(\frac{\partial^2 \rho}{\partial x_i \partial x_j}(x))$. The trace of $\text{Hess}_\rho(x)$ equals the Laplacian of $\rho$ at $x$: $\text{tr} \text{Hess}_\rho(x) = \Delta \rho(x)$. For $2 \leq p \leq n - 1$ we denote by $G_p(\mathbb{R}^n)$ the Grassman manifold of $p$-planes in $\mathbb{R}^n$. Given $\Lambda \in G_p(\mathbb{R}^n)$, we denote by $\text{tr}_\Lambda \text{Hess}_\rho(x) \in \mathbb{R}$ the trace of the restriction of $\text{Hess}_\rho(x)$ to $\Lambda$. Our main result is the following.

**Theorem 1.1.** Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with boundary of class $C^{r,\alpha}$ for some $r \in \{2, 3, \ldots\}$ and $0 < \alpha \leq 1$, and let $p \in \{2, \ldots, n - 1\}$. If the principal interior curvatures $\nu_1(x) \leq \nu_2(x) \leq \cdots \leq \nu_{n-1}(x)$ of $bD$ at every point $x \in bD$ satisfy

\begin{equation}
\nu_1(x) + \nu_2(x) + \cdots + \nu_p(x) \geq 0,
\end{equation}

then there exists a defining function $\rho$ of class $C^{r,\alpha}$ for $D$ such that

\begin{equation}
\text{tr}_1 \text{Hess}_\rho(x) \geq 0 \quad \text{for all } x \in \overline{D} \text{ and } \Lambda \in G_p(\mathbb{R}^n).
\end{equation}

If there are no points $x \in bD$ with $\nu_i(x) = 0$ for $i = 1, \ldots, p$ (such a point is called $p$-flat), then $\rho$ can be chosen such that strict inequality holds in (1.2) for all $x \in D$ and $\Lambda \in G_p(\mathbb{R}^n)$.

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Indeed, choosing (1.3) restricted to the tangent space \( T \) to the inner normal vector \( -\nabla \rho \) admits a smooth defining function in an interior collar around \( bD \) by using an unbounded defining function is a more convenient property, yielding applications which are not possible (see also [16, pp. 402–403] and [10, pp. 354–357]).

A \( \mathcal{C}^2 \) function \( \rho \) satisfying (1.2) is said to be \( p \)-plurisubharmonic on \( \overline{D} \); it is strictly \( p \)-plurisubharmonic at \( x \) if strict inequality holds at this point. Recall that \( \rho \) is \( p \)-plurisubharmonic on a domain \( \Omega \subset \mathbb{R}^n \) is and only if the restriction \( \rho|_M \) to any minimal \( p \)-dimensional submanifold \( M \subset \Omega \) is a subharmonic function on \( M \) in the induced metric, \( \Delta_M(\rho|_M) \geq 0 \) (cf. [13] and [1, Sect. 8.1]). For affine subspaces this follows by observing that if \( v_1, \ldots, v_p \in \mathbb{R}^n \) is an orthonormal basis of \( \Lambda \in G_p(\mathbb{R}^n) \) and we set \( \bar{\rho}(u_1, \ldots, u_p) = \rho(x + \sum_{j=1}^{p} u_j v_j) \), then

\[
\text{tr}_A \text{Hess}_\rho(x) = \Delta \bar{\rho}(0).
\]

The analogous formula holds on a minimal submanifold (cf. [13, Equation (2.10)]). Denoting by \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p \) the eigenvalues of \( \text{Hess}_\rho(x) \), we have

\[
\min_{\Lambda \in G_p(\mathbb{R}^n)} \text{tr}_A \text{Hess}_\rho(x) = \lambda_1 + \lambda_2 + \cdots + \lambda_p.
\]

(See [13, Lemma 2.5 and Corollary 2.6] or Lemma 2.1 below.) A domain \( D \subset \mathbb{R}^n \) is said to be \( p \)-convex if it admits a strongly \( p \)-plurisubharmonic exhaustion function \( \rho : D \to \mathbb{R} \), i.e., such that the set \( \{ x \in D : \rho(x) \leq c \} \) is compact for every \( c \in \mathbb{R} \).

Suppose now that \( \rho \) is a \( \mathcal{C}^2 \) defining function for \( D \), i.e., \( D = \{ \rho < 0 \} \) and \( d\rho \neq 0 \) on \( bD = \{ \rho = 0 \} \). Condition (1.1) is then equivalent to the following restricted version of (1.2):

\[
(1.3) \quad \text{tr}_A \text{Hess}_\rho(x) \geq 0 \quad \text{for every point } x \in bD \quad \text{and } p \text{-plane } \Lambda \subset T_x bD.
\]

Indeed, choosing \( \rho \) such that \( |\nabla \rho| = 1 \) on \( bD \), the Hessian \( \text{Hess}_\rho(x) \) at the point \( x \in bD \) restricted to the tangent space \( T_x bD \) is the second fundamental form of \( bD \) at \( x \) with respect to the inner normal vector \( -\nabla \rho(x) \) (see [13, Remark 3.11]), and

\[
\min_{\Lambda \subset T_x bD} \text{tr}_A \text{Hess}_\rho(x) = \nu_1(x) + \nu_2(x) + \cdots + \nu_p(x).
\]

(The minimum is over all \( p \)-planes \( \Lambda \subset T_x bD \). See [13, Remark 3.12] and Lemma 2.1.) For a bounded domain \( D \subset \mathbb{R}^n \) with smooth boundary we have (see [13, Summary 3.16])

\[
D \text{ is } p\text{-convex } \iff bD \text{ is } p\text{-convex},
\]

and these are further equivalent to the condition that \( -\log \text{dist}(\cdot, bD) \) is a \( p \)-plurisubharmonic function in an interior collar around \( bD \). However, the existence of a \( p \)-plurisubharmonic defining function is a more convenient property, yielding applications which are not possible by using unbounded \( p \)-plurisubharmonic exhaustion functions.

With this terminology, Theorem 1.1 can be stated as follows.

**Theorem 1.2.** Every bounded \( p \)-convex domain \( D \subset \mathbb{R}^n \) (\( 2 \leq p < n \)) with smooth boundary admits a smooth defining function \( \rho \) which is \( p \)-plurisubharmonic on \( \overline{D} \). If in addition \( bD \) has no \( p \)-flat points then \( \rho \) can be chosen strongly \( p \)-plurisubharmonic on \( D \).

Following the terminology introduced in [8] and used in [1], 2-convex domains are called **minimally convex** and 2-plurisubharmonic functions are called **minimal plurisubharmonic** (as they pertain to minimal surfaces). Thus, a bounded minimally convex domain \( D \subset \mathbb{R}^n \),
$n \geq 3$, with smooth boundary admits a defining function which is minimal plurisubharmonic on $\overline{D}$. This has the following corollary. Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and $\mathbb{D}^* = \mathbb{D} \setminus \{ 0 \}$.

**Corollary 1.3.** Let $D$ be a bounded minimally convex domain in $\mathbb{R}^n$, $n \geq 3$, with $\mathcal{C}^r$ boundary for some $r > 2$. If $M$ is a connected open conformal surface and $f : M \to \mathbb{R}^n$ is a conformal harmonic map with $f(M) \subset \overline{D}$, then either $f(M) \subset D$ or $f(M) \subset bD$. If $K \subset M$ is a removable set for bounded harmonic functions, then every conformal harmonic map $M \setminus K \to D$ extends to a conformal harmonic map $M \to D$. In particular, every conformal harmonic map $\mathbb{D}^* \to D$ from the punctured disc extends to a conformal harmonic map $\mathbb{D} \to D$.

**Proof.** Let $\rho$ be a defining function for $D$ such that (1.2) holds for $p = 2$. The first claim follows from the maximum principle applied to the subharmonic function $\rho \circ f : M \to (-\infty, 0]$, and the second claim is an immediate consequence. The last claim follows by observing that every bounded harmonic function on $\mathbb{D}^*$ extends to a harmonic function on $\mathbb{D}$. □

**Remark 1.4.** If $bD$ fails to be minimally convex at a point $p \in bD$, then there is an embedded minimal disc $f : \mathbb{D} \to D \cup \{ p \}$ with $f(0) = p$ and $f(\mathbb{D} \setminus \{ 0 \}) \subset D$ (see [13, Lemma 3.13]). Hence, the conclusion of Corollary 1.3 fails in this case.

**Remark 1.5.** Corollary 1.3 generalizes the classical fact that two immersed minimal surfaces in $\mathbb{R}^3$ which touch at a point but locally at that point lie on one side of one another must coincide. In fact, assume that $D \subset \mathbb{R}^3$ is a bounded domain whose smooth boundary contains a domain $\Sigma$ in an immersed minimal surface $\widetilde{\Sigma} \subset \mathbb{R}^3$ (here, $\Sigma$ is embedded), and $bD$ is strongly minimally convex at each point $p \in bD \setminus \Sigma$. Then, the domain $D$ is minimally convex. Let $\rho$ be a minimal plurisubharmonic defining function for $D$ provided by Theorem 1.1. Given a connected open conformal surface $M$ and a nonconstant conformal harmonic map $f : M \to \overline{D}$, the maximum principle for the subharmonic function $\rho \circ f : M \to (-\infty, 0]$ shows that either $f(M) \subset D$ or $f(M) \subset \Sigma$.

We shall see that a suitable convexification of the signed distance function to $bD$,

$$\delta(x) = \text{dist}(x, D) - \text{dist}(x, \mathbb{R}^n \setminus D), \quad x \in \mathbb{R}^n$$  

satisfies the conclusion Theorem 1.1 in an interior collar around $bD$. Assuming that $bD$ is of Hölder class $\mathcal{C}^{r,\alpha}$ for some $r \in \{2, 3, \ldots \}$ and $0 < \alpha \leq 1$, $\delta$ is of the same class $\mathcal{C}^{r,\alpha}$ in a neighbourhood $V \subset \mathbb{R}^n$ of $bD$ (see Li and Nirenberg [15]). This gives the following corollary.

**Corollary 1.6.** Let $D$ be a bounded $p$-convex domain with $\mathcal{C}^r$ boundary in $\mathbb{R}^n$ for some $2 \leq p < n$ and real number $r > 2$. Then every domain $D_t = \{ z \in D : \delta(z) < t \}$ for $t < 0$ close to 0 is $p$-convex, and it is strongly $p$-convex if $bD$ does not contain any $p$-flat points.

On a domain $\Omega \subset \mathbb{C}^n$ and for a complex line $\Lambda \subset \mathbb{C}^n$, $\text{tr}_\Lambda \text{Hess}_\rho(x)$ is the Levi form of $\rho$ at $x \in \Omega$ on any unit vector $\xi \in \Lambda$. It follows that every minimal plurisubharmonic function on a domain in $\mathbb{C}^n$ is also plurisubharmonic in the usual sense of complex analysis, and every 2-convex domain $D$ in $\mathbb{C}^n$ is pseudoconvex; the converse fails. Assuming that $bD$ is of class $\mathcal{C}^r$ for some $r > 2$, Levi pseudoconvexity of $bD$ is characterized by nonnegativity of the Hessian of $\delta$ applied in complex tangent directions to $bD$. From the differential geometric viewpoint, Levi pseudoconvexity says that the mean sectional curvature of $bD$ in complex tangent directions is nonnegative (see (2.4) and (3.1)). This leads to an observation regarding
strong pseudoconvexity of domains $D_t = \{ z \in D : \delta(z) < t \}$ for $t < 0$ close to 0; see Theorem 3.1

2. Proof of Theorem 1.1

Let $Q$ be a real symmetric $n \times n$ matrix for $n \geq 2$ and $Q$ be the associated quadratic form $Q(x) = Qx \cdot x$, $x \in \mathbb{R}^n$. (The dot denotes the Euclidean inner product.) Let $1 \leq p < n$. Given a $p$-plane $\Lambda \in G_p(\mathbb{R}^n)$, we denote by $\text{tr}_\Lambda Q$ the trace of the restricted quadratic form $Q|_\Lambda$. The following result is \cite{13} Lemma 2.5 and Corollary 2.6; we include a simple proof.

**Lemma 2.1.** Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $Q$. For every $\Lambda \in G_p(\mathbb{R}^n)$ we have that $\text{tr}_\Lambda Q \geq \lambda_1 + \lambda_2 + \cdots + \lambda_p$, with equality if and only if $\Lambda$ admits an orthonormal basis $v_1, \ldots, v_p$ such that $Qv_i = \lambda_i v_i$ for $i = 1, \ldots, p$.

**Proof.** This is obvious for $p = 1$, and we proceed by induction on $p$. We may assume that in the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ the matrix $Q$ is diagonal with $Qe_i = \lambda_i e_i$ for $i = 1, \ldots, n$.

The subspace $\Lambda' = \Lambda \cap (\{0\} \times \mathbb{R}^{n-1})$ has dimension at least $p - 1$. Pick an orthonormal basis $v_1, v_2, \ldots, v_n$ of $\Lambda$ with $v_2, \ldots, v_n \in \Lambda'$. Then, $\text{tr}_\Lambda Q = Qv_1 \cdot v_1 + \sum_{j=2}^{p} Qv_j \cdot v_j$. Clearly, $Qv_1 \cdot v_1 \geq \lambda_1$ with equality if and only if $Qv_1 = \lambda_1 v_1$. Since $\Lambda'' := \text{span}\{v_2, \ldots, v_p\}$ is contained in the invariant subspace $\{0\} \times \mathbb{R}^{n-1}$ of $Q$ with eigenvalues $\lambda_2 \leq \cdots \leq \lambda_n$, the inductive hypothesis gives $\sum_{j=2}^{p} Qv_j \cdot v_j \geq \lambda_2 + \cdots + \lambda_p$, with equality if and only if $\Lambda''$ is spanned by eigenvectors for eigenvalues $\lambda_2, \ldots, \lambda_p$. This completes the induction step. \hfill $\square$

**Proof of Theorem 1.1.** Let $D \subset \mathbb{R}^n$ be a bounded domain with $C^{r,\alpha}$ boundary for some integer $r \geq 2$ and real number $0 < \alpha \leq 1$. Denote by $\delta$ the signed distance function \cite{4} from $bD$. By Li and Nirenberg \cite{15}, $\delta$ is of the same class $C^{r,\alpha}$ in a collar neighbourhood $V \subset \mathbb{R}^n$ of $bD$. We recall some further properties of $\delta$, referring to Bellettini \cite{3} Theorem 1.18, p. 14] and Gilbarg and Trudinger \cite{10} Section 14.6. Shrinking $V$ around $bD$ if necessary, there is a $C^r$ projection $\pi : V \to bD$ such that for any $x \in V$, $p = \pi(x) \in bD$ is the unique nearest point to $x$ on $bD$. The gradient $\nabla \delta$ has constant norm $|\nabla \delta| = 1$ on $V$, and it has constant value on the intersection of $V$ with the normal line $N_p = p + \mathbb{R} \cdot \nabla \delta(p)$ at $p \in bD$. By a translation and an orthogonal rotation we may assume that $p = 0$, $T_0 bD = \{ x_n = 0 \}$, and the standard basis vectors $e_1, e_2, \ldots, e_n = \nabla \delta(0)$ of $\mathbb{R}^n$ diagonalize the matrix $H$ of $\text{Hess}_0(\delta)$:

$$He_i = \nu_i e_i, \quad i = 1, \ldots, n - 1; \quad He_n = 0.$$

The restriction of $\text{Hess}_0(\delta)$ to the tangent space $T_0 bD = \mathbb{R}^{n-1} \times \{0\}$ is the second fundamental form of $bD$ at 0 and $\nu_1, \ldots, \nu_{n-1}$ are the principal curvatures of $bD$ at 0 from the side $-e_n = -\nabla \delta(0)$. By reordering we may assume that $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-1}$. For any point $x = (0, \ldots, 0, t) \in N_0 \cap V$ the basis $e_1, \ldots, e_n$ also diagonalizes $\text{Hess}_0(\delta(x))$, with the eigenvalues

$$(2.1) \quad \nu_i(x) = \frac{\nu_i}{1 + tv_i}, \quad i = 1, \ldots, n.$$

In particular, $\nu_n(x) = 0$. Note that for $t < 0$ (which corresponds to $x \in N_0 \cap V \cap D$) we have $\nu_i(x) \geq \nu_i$, while for $t > 0$ (i.e., $x \in N_0 \cap V \setminus D$) we have $\nu_i(x) \leq \nu_i$. In both cases equality holds if and only if $\nu_i = 0$. Furthermore, $\nu_1(x) \leq \cdots \leq \nu_{n-1}(x)$ for all $x \in N_0 \cap V$.

Assume now that $bD$ is $p$-convex at 0 in the sense that condition (1.1) holds, i.e.,

$$\nu_1 + \nu_2 + \cdots + \nu_p \geq 0.$$
By Lemma 2.1 applied with $Q = \mathrm{Hess}_\delta(0)$ restricted to $T_0bD$, this is equivalent to
\[
\mathrm{tr}_A \mathrm{Hess}_\delta(0) \geq 0 \quad \text{for every } p\text{-plane } A \subset T_0bD.
\]
Consider the family of domains $D_t = \{\delta < t\}$ for $t$ near 0. As $t$ increases, the domains $D_t$ strictly increase, and $D_0 = D$. The tangent space to $bD_t = \{\delta = t\}$ at $x = (0, \ldots, 0, t) \in bD_t$ is $\mathbb{R}^{n-1} \times \{0\}$, $e_n = \nabla \delta(x)$ is the unit outer normal vector at $x$, and the numbers $\nu_i(x)$ (2.1) for $i = 1, \ldots, n-1$ are the principal normal curvatures of $bD_t$ at $x$. From what has been said, we see that at $x = (0, \ldots, 0, t) \in bD_t$ with $t < 0$ we have that
\[
\nu_1(x) + \nu_2(x) + \cdots + \nu_p(x) \geq \nu_1 + \nu_2 + \cdots + \nu_p \geq 0
\]
with equality if and only if $\nu_1 = \nu_2 = \cdots = \nu_p = 0$, i.e., the point $0 \in bD$ is $p$-flat. Since this analysis holds at any point $p \in bD$, we conclude that for $t < 0$ close to 0 the boundary $bD_t$ is $p$-convex, and it is strongly $p$-convex if and only if $bD$ does not contain any $p$-flat points.

It remains to find a defining function for $D$, having the domains $D_t$ as sublevel sets, which is $p$-plurisubharmonic on $\overline{D}$, and is strongly $p$-plurisubharmonic on $D$ provided that $bD$ has no $p$-flat points. Consider the function $h \circ \delta : V \to \mathbb{R}$, where $h$ is a smooth convex increasing function on $\mathbb{R}$ with $h(0) = 0$, $h(0) = 1$, and $\dot{h}(0) = a > 0$ (to be determined). Then,
\[
\mathrm{Hess}_{h \circ \delta} = (\dot{h} \circ \delta) \mathrm{Hess}_\delta + (\dot{h} \circ \delta) \nabla \delta \cdot (\nabla \delta)^T.
\]
Assume that for $p \in bD$ the orthonormal vectors $v_1, \ldots, v_{n-1}, v_n$ diagonalize $\mathrm{Hess}_\delta(p)$, where $T_pD = \text{span}\{v_1, \ldots, v_{n-1}\}$ and $v_n = \nabla \delta(p)$. Then the vectors $v_1, \ldots, v_{n-1}$ lie in the kernel of the matrix $\nabla \delta(p) \cdot \nabla \delta(p)^T$, while $v_n$ is an eigenvector with eigenvalue 1. Hence, the basis $v_1, \ldots, v_n$ diagonalizes $\mathrm{Hess}_{h \circ \delta}(x)$ at every point $x = p + \delta(x)v_n \in N_p \cap V$, the eigenvalues corresponding to $v_1, \ldots, v_{n-1}$ get multiplied by the number $\dot{h}(\delta(x))$, which is close to 1 if $x$ is close to $p$, and the eigenvalue in the normal direction $v_n = \nabla \delta(x)$ is $\dot{h}(\delta(x)) \approx a$. By choosing $a$ such that $a + \lambda_1(p) > 0$ for all $p \in bD$ and shrinking the neighbourhood $V$ around $bD$, the function $h \circ \delta$ is $p$-plurisubharmonic on $\overline{D} \cap V$, and it is strongly $p$-plurisubharmonic on $D \cap V$ if $bD$ has no $p$-flat points. Choose $c < 0$ such that $\{c \leq \delta \leq 0\} \subset \overline{D} \cap V$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex increasing function such that $\phi(t) = 2c/3$ for $t \leq c$ and $\phi(t) = t$ for $t \geq c/2$. Then, $\rho = \phi \circ h \circ \delta$ is a well-defined function on $\overline{D} \cup V$ which equals $h \circ \delta$ outside $D_{c/2}$ and is $p$-plurisubharmonic on $\overline{D}$.

Assume now that $h \circ \delta$ is strongly $p$-plurisubharmonic on a collar $\{c \leq \delta < 0\}$. Then, $\rho = \phi \circ h \circ \delta$ is strongly $p$-plurisubharmonic on $\{c/2 \leq \delta < 0\} \subset D \cap V$. We take
\[
\tilde{\rho}(x) = \rho(x) + \epsilon \chi(x) \cdot |x|^2,
\]
where $\chi : \mathbb{R}^n \to [0, 1]$ is a smooth function which equals 1 on $\overline{D}_{c/3}$ and has compact support contained in $D$. The support of the differential of $\chi$ is then compact and contained in the set $\{c/2 < \delta < 0\}$ where $\rho$ is strongly $p$-plurisubharmonic. Note that $|x|^2 = \sum_{i=1}^n x_i^2$ is strongly $p$-plurisubharmonic on $\mathbb{R}^n$. Choosing $\epsilon > 0$ small enough therefore ensures that $\tilde{\rho}$ is strongly plurisubharmonic on $D$. This proves Theorem 1.1.

Let us look more closely at the case $p = 2$ of Theorem 1.1 which is of interest for the theory of 2-dimensional minimal surfaces. In particular, every 2-convex domain in $\mathbb{R}^n$, $n \geq 3$, admits plenty of proper minimal surfaces parameterized by conformal harmonic immersions from any given bordered Riemann surface (see [1] Theorems 8.3.1 and 8.3.11). In this case, condition 1.3 (which is equivalent to condition 1.1 in Theorem 1.1) can be expressed in
more intrinsic differential geometric terms, thereby exposing a connection to Levi pseudoconvexity studied in complex analysis; see the following section for the latter.

Let $D$ be a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$. Given a point $p \in bD$ and a 2-plane $\Lambda \subset T_p bD$, let $S(p, \Lambda)$ be the surface obtained by intersecting $bD$ with the affine 3-plane $p \in \Sigma_p \cong \mathbb{R}^3$ spanned by $\Lambda$ and the normal vector to $bD$ at $p$. Let $Q_p$ denote the second fundamental form of $bD$ at $p$. The principal curvatures $\lambda_1(p, \Lambda), \lambda_2(p, \Lambda)$ of $S(p, \Lambda)$ at $p$ are the eigenvalues of the restricted quadratic form $Q_p|\Lambda$. Their sum

$$H(p, \Lambda) = \lambda_1(p, \Lambda) + \lambda_2(p, \Lambda) \tag{2.2}$$

is the mean curvature of $S(p, \Lambda)$ at $p$, and their product

$$K(p, \Lambda) = \lambda_1(p, \Lambda) \cdot \lambda_2(p, \Lambda) \tag{2.3}$$

is the Gaussian curvature of $S(p, \Lambda)$ at $p$. These numbers are, respectively, the sectional mean curvature and the sectional Gaussian curvature of $bD$ at $p$ on the 2-plane $\Lambda$. While $H$ reflects the way how $bD$ sits in $\mathbb{R}^n$, $K$ is an intrinsic quantity depending only on the induced metric on $bD$. If $\rho$ is a defining function for $D$ such that $|\nabla \rho(p)| = 1$, then $Q_p$ equals the Hessian $\Hess_\rho(p)$ restricted to $T_p bD$, and hence

$$H(p, \Lambda) = \tr_\Lambda \Hess_\rho(p) = \Delta(\rho|_{p+\Lambda})(p). \tag{2.4}$$

If $H(p, \Lambda) = 0$ then $K(p, \Lambda) = -\lambda_1(p, \Lambda)^2 \leq 0$, and $K(p, \Lambda) = 0$ if and only if $Q_p|\Lambda$ vanishes. Theorem 1.1 for $p = 2$ can now be stated as follows.

**Theorem 2.2.** Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with $C^{r,\alpha}$ boundary for some $r \geq 2$ and $0 < \alpha \leq 1$. If

$$H(p, \Lambda) \geq 0 \text{ for every point } p \in bD \text{ and 2-plane } \Lambda \subset T_p bD, \tag{2.5}$$

then $D$ admits a $C^{r,\alpha}$ defining function $\rho$ satisfying $(1.2)$ with $p = 2$ (i.e., $\rho$ is minimal plurisubharmonic on $D$). If in addition there is no $(p, \Lambda)$ as above such that $H(p, \Lambda) = K(p, \Lambda) = 0$, then $\rho$ can be chosen strongly minimal plurisubharmonic on $D$.

3. A REMARK ON LEVI PSEUDOCONVEX DOMAINS

For a smoothly bounded domain $D$ in a complex Euclidean space $\mathbb{C}^n$, $n \geq 2$, the 2-convexity condition $(1.3)$ applied only to complex lines $\Lambda \subset T_p bD$ is precisely the Levi pseudoconvexity condition. Indeed, if $v \in \Lambda = C v$ is a unit vector then

$$(dd^c \rho)(v, Jv) = \Delta(\rho|_{p+\Lambda})(p) = \tr_\Lambda \Hess_\rho(p) = H(p, \Lambda), \tag{3.1}$$

where $J$ is the standard almost complex structure operator on $\mathbb{C}^n$.

It is well known that if $D$ is a bounded pseudoconvex domain in $\mathbb{C}^n$ then the function $-\log \text{dist}(\cdot, bD)$ is plurisubharmonic on $D$, and hence the domains $D_t = \{z \in D : \text{dist}(z,bD) > t\} = \{\delta < -t\}$ for $t > 0$ are pseudoconvex. This is a stronger result than we can get for minimally convex domains, and it does not require any smoothness of $bD$. However, the following observation related to Theorem 2.2 seems worthwhile recording.

**Theorem 3.1.** Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with $C^{r,\alpha}$ boundary for some $r \geq 2$ and $0 < \alpha \leq 1$, and let $\delta$ be given by $(1.4)$. If the sectional curvature $K(p, \Lambda)$
is nonzero for every point $p \in bD$ and complex line $\Lambda \subset T_p bD$ on which the Levi form \((3.2)\) vanishes, then the domains $D_t = \{ \delta < t \}$ for $t < 0$ close to 0 are strongly pseudoconvex.

**Proof.** Let $\delta$ denote the signed distance function \((1.4)\) to $bD$. Fix a point $p \in bD$ and let $v = \nabla \delta(p)$. Let $J$ denote the standard almost complex structure on $\mathbb{C}^n$, which corresponds to multiplication by $i$ on tangent complex vectors. The tangent space to $bD$ at $p$ is an orthogonal direct sum $T_p bD = T_p^c bD \oplus E$, where $\Sigma := T_p^c bD = T_p bD \cap J(T_p bD)$ is the maximal complex subspace of $T_p bD$ (a complex hyperplane) and $E$ is the real line orthogonal to $T_p^c bD$. Note that $J(E) = \mathbb{R}v$ is the normal line to $bD$ at $p$. The Levi pseudoconvexity condition is that $H(p, \Lambda) = \text{tr}_\Lambda \text{Hess}_\delta(p) \geq 0$ for every complex line $\Lambda \subset T_p^c bD$ (see \((3.1)\)).

Let $D_t = \{ \delta < t \}$ for $t < 0$ close to 0 and $x = p + tv \in bD_t$. Then, $T_x bD_t$ is the orthogonal complement to $v$, and hence $T_x bD_t = \Sigma \oplus E$, i.e., the same orthogonal decomposition of $T_x bD_t$ holds along the line $N_p = p + \mathbb{R}v$. To prove the theorem, it suffices to show that $\text{tr}_\Lambda \text{Hess}_\delta(x) \geq H(p, \Lambda)$, with equality if and only if $H(p, \Lambda) = K(p, \Lambda) = 0$. This will imply that $bD_t$ for $t < 0$ close to 0 is Levi pseudoconvex, and it is strongly Levi pseudoconvex if and only if there are no complex lines $\Lambda \subset T_p bD$, $p \in bD$, with $H(p, \Lambda) = K(p, \Lambda) = 0$.

Let $v_1, \ldots, v_{n-1} \in T_p bD$ be orthonormal eigenvectors of $\text{Hess}_\delta(p)$ with the eigenvalues $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-1}$. Fix a complex line $\Lambda \subset T_p bD$. For any orthonormal basis $\xi = \sum_{i=1}^{n-1} \xi_i v_i$ and $\eta = \sum_{i=1}^{n-1} \eta_i v_i$ for $\Lambda$ we have that

\[(3.2) \quad \text{tr}_\Lambda \text{Hess}_\delta(x) = \sum_{i=1}^{n-1} \nu_i(x)(\xi_i^2 + \eta_i^2) \geq \sum_{i=1}^{n-1} \nu_i(\xi_i^2 + \eta_i^2) = \text{tr}_\Lambda \text{Hess}_\delta(p) = H(p, \Lambda),\]

where we used $\nu_i(x) \geq \nu_i$ (see \((2.1)\)). Equality holds if and only if $\xi_i^2 + \eta_i^2 = 0$ whenever $\nu_i \neq 0$ (since for such $i$ we have $\nu_i(x) > \nu_i$). Clearly this holds if and only if the vectors $\xi$ and $\eta$ are contained in the kernel of $\text{Hess}_\delta(p)$. Since this argument holds for every orthonormal basis of $\Lambda$, we infer that equality holds in \((3.2)\) if and only if $H(p, \Lambda) = K(p, \Lambda) = 0$. \[\square\]

**Remark 3.2.** (A) Regarding plurisubharmonic defining functions, there is a major difference between the minimal case (cf. Theorem \((1.1)\)) and the complex (Levi) case. It has been known since the 1970s through examples of Diederich and Fornaess \([6]\) and Fornaess \([9]\) Example 5] that there exist bounded weakly pseudoconvex domains $D \subset \mathbb{C}^2$ with smooth real analytic boundaries which do not admit a defining function that is plurisubharmonic on $\overline{D}$, in the sense that its Levi form is nonnegative at every point of $\overline{D}$ in all complex directions. Furthermore, on a pseudoconvex domain admitting a plurisubharmonic defining function the $\overline{\partial}$–Neumann problem is exactly regular; see Boas and Straube \([4]\). This is not true on all weakly pseudoconvex domains; it fails in particular on certain worm domains according to Barrett \([2]\) and Christ \([5]\). Hence, Theorem \((2.2)\) has no analogue in the complex case. The reason is that Levi pseudoconvexity provides much less information since it only pertains to complex lines, as opposed to all 2-planes tangent to the boundary. In particular, it does not give any condition in the direction normal to the maximal complex subspace of $T_p bD$.

(B) On the other hand, it was shown by Diederich and Fornaess in \([7]\) that locally near any boundary point of a smoothly bounded weakly pseudoconvex domain in $\mathbb{C}^n$ one can make a local holomorphic change of coordinates so that the level sets of the signed distance function are strongly pseudoconvex in the interior of the new local domain. The point is that, while a biholomorphic map preserves the sign of the Levi form on complex directions (and hence of
the function $H(p, \Lambda)$ in (3.1), the sectional Gaussian curvature $K(p, \Lambda)$ (2.3) may very well change from zero to a nonzero value. If we ensure that this happens on all complex tangent directions on which the Levi form vanishes, then Theorem 3.1 applies.

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