Second-order gravitational effects of local inhomogeneities on CMB anisotropies in nonzero-$\Lambda$ flat cosmological models

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Nonlinear gravitational effects of large-scale local inhomogeneities on Cosmic Microwave Background (CMB) anisotropies are studied, based on the relativistic second-order theory of perturbations in nonzero-$\Lambda$ flat cosmological models, which has been analytically derived by the present author, and on the second-order formula of CMB anisotropies derived by Mollerach and Matarrese. In this paper we derive the components of the CMB anisotropy power spectra in the range of $l = 1 - 22$ which are caused by asymmetric local inhomogeneities on scales of 300 Mpc. Using our results it is found that there is a possibility to explain the small north-south asymmetry of CMB anisotropies which has recently been observed.

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I. INTRODUCTION

In most studies of Cosmic Microwave Background (CMB) anisotropies, the comparison between observed and theoretical quantities have so far been done, assuming the linear approximation for cosmological perturbations. It seems to be successful enough to determine the cosmological parameters [1, 2, 3]. The present state of our universe is, however, locally complicated and associated with nonlinear behavior on various scales, and so the observed quantities of CMB anisotropies may include some small effects caused by large-scale local inhomogeneities through nonlinear process. Recently it has been reported by Eriksen et al.[4, 5], Hansen et al.[6, 7, 8], Vielva et al.[9] and Park[10] that there is a non-trivial north-south asymmetry in various quantities about CMB anisotropies. These observational results may suggest the existence of the above effect.

In this paper we study these nonlinear effects of large-scale local inhomogeneities on CMB anisotropies, based on the relativistic second-order theory of cosmological perturbations, which we have recently derived[11] and on Mollerach and Matarrese’s second-order formula of CMB anisotropies[12]. In Sec. II, we show the second-order perturbations in nonzero-$\Lambda$ flat cosmological models and the corresponding CMB anisotropies. In Sec. III, we derive the expressions for the second-order power ($\Delta C_l$) of CMB anisotropies, assuming a dipole form of local inhomogeneities. In Sec. IV, we derive numerically the first-order and second-order anisotropy power spectra, and consider the condition that the local inhomogeneities can cause the observed asymmetry of the CMB anisotropy, assuming four simple model types of the radial dependence in local inhomogeneities. It is found that there is a possibility that the observed decrease of low multipoles of CMB anisotropy[13, 14] also may be explained together with the above asymmetry. The derivation of main equations in Sec. II and III are shown in Appendix. Concluding remarks follow in Section V.

II. SECOND-ORDER PERTURBATIONS AND CMB ANISOTROPIES

First we review the background spacetime and the perturbations which were derived in the previous paper. The background flat model with dust matter is expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j],$$

where the Greek and Latin letters denote 0, 1, 2, 3 and 1, 2, 3, respectively, and $\delta_{ij} (= \delta^i_j = \delta_j^i)$ are the Kronecker delta. The conformal time $\eta = x^0$ is related to the cosmic time $t$ by $dt = a(\eta)d\eta$. The matter density $\rho$ and the scale factor $a$ have the relations

$$pa^2 = 3(a'/a)^2 - \Lambda a^2, \quad \text{and} \quad pa^3 = \rho_0,$$

where a prime denotes $\partial/\partial\eta$, $\Lambda$ is the cosmological constant, $\rho_0$ is an integration constant and the units $8\pi G = c = 1$ are used.

The first-order and second-order metric perturbations $\delta_1 g_{\mu\nu}(= h_{\mu\nu})$ and $\delta_2 g_{\mu\nu}(= \ell_{\mu\nu})$, respectively, were derived explicitly by imposing the synchronous coordinate condition:

$$h_{00} = h_{0i} = 0 \quad \text{and} \quad \ell_{00} = \ell_{0i} = 0.$$
Here we show their expressions only in the growing mode:

\[
\begin{align*}
\delta T_r/T &= P(\eta) F_{i,j}, \\
\delta T_\tau/T &= P(\eta) L_i^j + P^2(\eta) M_i^j + Q(\eta) N_i^j + C_i^j, 
\end{align*}
\]  

(2.4)

where \( F \) is an arbitrary potential function of spatial coordinates \( x^1, x^2 \) and \( x^3 \), \( \delta T_r/T \), \( \delta T_\tau/T \), \( N_i^j \) and \( P(\eta) \) and \( Q(\eta) \) satisfy

\[
\begin{align*}
P'' + \frac{2a'}{a} P' - 1 &= 0, \\
Q'' + \frac{2a'}{a} Q' &= P - \frac{5}{2}(P')^2. 
\end{align*}
\]  

(2.5)

The three-dimensional covariant derivative \( |i \) are defined in the space with metric \( dl^2 = \delta_{ij} dx^i dx^j \) and their suffices are raised and lowered by use of \( \delta_{ij} \). The functions \( L_i^j \) and \( M_i^j \) are defined by

\[
\begin{align*}
L_i^j &= \frac{1}{28} \left[-3 F_i F_j - 2 F \cdot F_{i,j} + \frac{1}{2} \delta_{ij} F_{i,l} F_{l} \right], \\
M_i^j &= \frac{1}{28} \left[19 F_i F_{j,l} - 12 F_{i,j} \Delta F - 3 \delta_{ij} \left[ F_{k,l} F_{k,l} - (\Delta F)^2 \right] \right]. 
\end{align*}
\]  

(2.6)

and \( N \) is defined by

\[
\Delta N = \frac{1}{28} \left[(\Delta F)^2 - F_{k,l} F_{k,l} \right].
\]  

(2.7)

The last term \( C_i^j \) satisfies the wave equation

\[
\square C_i^j = \frac{3}{14} (P/a)^2 G_i^j + \frac{1}{7} \left[P - \frac{5}{2}(P')^2 \right] \tilde{G}_i^j,
\]  

(2.8)

where \( G_i^j \) and \( \tilde{G}_i^j \) are second-order traceless and transverse functions of spatial coordinates, and the operator \( \square \) is defined by

\[
\square \phi \equiv g^{\mu\nu} \phi_{,\mu\nu} = -a^{-2} \left( \partial^2/\partial \eta^2 + \frac{2a'}{a} \partial/\partial \eta - \Delta \right) \phi.
\]  

(2.9)

So \( C_i^j \) represents the second-order gravitational waves caused by the first-order density perturbations.

The velocity perturbations \( \delta_1 u^\mu \) and \( \delta_2 u^\nu \) vanish, i.e. \( \delta_1 u^0 = \delta_1 u^i = 0 \) and \( \delta_2 u^0 = \delta_2 u^i = 0 \), and the density perturbations are

\[
\begin{align*}
\delta_1 \rho/\rho &= \frac{1}{\rho a^2} \left( \frac{a'}{a} P' - 1 \right) \Delta F, \\
\delta_2 \rho/\rho &= \frac{1}{2\rho a^2} \left\{ \frac{1}{2} \left(1 - \frac{a'}{a} P'\right) (3 F_i F_{i,l} + 8 F \Delta F) + \frac{1}{2} P [(\Delta F)^2 + F_{k,l} F_{k,l}] \\
&+ \frac{1}{4} \left[[P'(P')^2 - \frac{2a'}{a} P'Q]\left[(\Delta F)^2 - F_{k,l} F_{k,l}\right] - \frac{1}{\rho a} PP'[4 F_{k,l} F_{k,l} + 3(\Delta F)^2] \right].
\end{align*}
\]  

(2.10)

Next let us consider the CMB temperature \( T = T^{(0)}(1 + \delta T/T) \), which \( T^{(0)} \) is the background temperature and \( \delta T/T = \delta_1 T/T + \delta_2 T/T \) is the perturbations. The present temperature \( T^{(0)} \) is related to the emitted background temperature \( T_e^{(0)} \) at the decoupling epoch \( T_e^{(0)} = (1 + z_e)T_o^{(0)} \), the temperature perturbation \( \tau \equiv (\delta T/T)_e \) at the decoupling epoch is determined by the physical state before that epoch , and the present temperature perturbations \( (\delta T/T)_p \) is related to \( (\delta T/T)_e \) by the gravitational perturbations along the light ray from the epoch to the present epoch. The light ray is described using the background wave vector \( k^\mu \equiv dx^\mu/\lambda \), where \( \lambda \) is the affine parameter, and its component is \( k^{(0)\mu} = (1, -e^i) \), and the ray is given by \( x^{(0)\mu} = [\lambda, (\lambda_0 - \lambda)e^i] \), where \( e^i \) is the directional unit vector.

The first-order temperature perturbation is

\[
\delta T/T = \tau + \frac{1}{2} \int_{\lambda_0}^{\lambda_e} d\lambda P'(\eta) F_{i,j} e^i e^j.
\]  

(2.11)
Using the relation \( dP/d\lambda = P' \) and \( dF/d\lambda = -F_i e^i \), this equation is expressed as

\[
\delta T/T = \Theta_1 + \Theta_2
\]

(2.12)

where

\[
\Theta_1 = \tau - \frac{1}{2} [(P' F_i)_o - (P' F_i)_e] e^i,
\]

\[
\Theta_2 = \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P''(\eta)F_i e^i.
\]

(2.13)

\( \Theta_1 \) and \( \Theta_2 \) represent the intrinsic and Sachs-Wolfe effects, respectively. The latter can be divided into the ordinary Sachs-Wolfe effect \( \Theta_{sac} \) and the Integral Sachs-Wolfe effect \( \Theta_{isw} \), where

\[
\Theta_{sac} = \frac{1}{2} [(P'' F)_o - (P'' F)_e],
\]

\[
\Theta_{isw} = \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P''(\eta)F.
\]

(2.14)

The second-order temperature perturbation is

\[
\delta T/T = I_1(\lambda_e) \left[ \frac{1}{2} I_1(\lambda_e) - \tau \right] - \int_{\lambda_o}^{\lambda_e} d\lambda \left( \frac{1}{2} A^{(1)'} + A^{(1)} A^{(1)'} - A^{(1)''} \right) \int_{\lambda_o}^{\lambda_e} d\bar{\lambda} A^{(1)}(\bar{\lambda}) \right) + \frac{\partial \tau}{\partial d^{(1)j}} d^{(1)j},
\]

(2.15)

where \((\eta, x^i) = (\lambda, \lambda_o - \lambda)\) in the integrands and

\[
I_1(\lambda_e) = \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P' F_{ij} e^i e^j,
\]

\[
A^{(1)} = \frac{1}{2} P F_{ij} e^i e^j,
\]

\[
A^{(2)} = \frac{1}{2} [P L_{ij} + P^2 M_{ij} + Q N_{ij} + C_l^i] e^i e^j.
\]

(2.16)

Now let us assume an appropriate form of \( F \) as follows, to represent a situation including local inhomogeneities:

\[
F(x) = F_P(x) + F_L(x),
\]

(2.17)

where the part of primordial density perturbations \( (F_P) \) and the part of local homogeneities \( (F_L) \) are expressed as

\[
F_P = \int dk \alpha(k) e^{ikx},
\]

\[
F_L = R(r) Y_l^m(\theta, \phi).
\]

(2.18)

In the former equation, \( \alpha(k) \) is a random variable, \( Y_l^m(\theta, \phi) \) is spherical harmonics, and \((r, \theta, \phi)\) is the polar coordinates, i.e. \( r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \). For the above \( F(x) \), we have the first-order density perturbation

\[
\frac{\delta \rho}{\rho} = \frac{1}{\rho a^2} \left( \frac{d}{a} P' - 1 \right) (\Delta F_P + \Delta F_L),
\]

\[
= \frac{1}{\rho a^2} \left( \frac{d}{a} P' - 1 \right) \left[ - \int dk \alpha(k) k^2 e^{ikx} + \hat{R}(r) Y_l^m(\theta, \phi) \right],
\]

(2.19)

where

\[
\hat{R}(r) = \frac{1}{r^2} \frac{d}{dr} (r^2 R_{rr}) - \frac{l(l+1)}{r^2} R
\]

(2.20)

with \( R_{rr} = dR/dr \).
The part of $\Delta F_P$ is for the ordinary primordial perturbations, so that by the averaging process, we have

$$
\langle \alpha(k)\alpha(k') \rangle = (2\pi)^{-2} P_F(k) \delta(k+k'),
$$

with

$$
P_F(k) = P_{F0} k^{-3} (k/k_0)^{n-1} T_s^2(k),
$$

where $T_s(k)$ is the matter transfer function [15] and $P_{F0}$ is the normalization constant. Then the average value of $(\delta_1 \rho/\rho)^2$ is

$$
\langle (\delta_1 \rho/\rho)^2 \rangle = \left[ \frac{1}{\rho a^2} \left( \frac{d' P'}{\rho} - 1 \right) \right]^2 (2\pi)^{-2} \int dk P_F(k) k^4 (\Delta F_L)^2
$$

$$
= \left[ \frac{1}{\rho a^2} \left( \frac{d' P'}{\rho} - 1 \right) \right]^2 (2\pi)^{-2} P_{F0} \int dk k (k/k_0)^{n-1} T_s^2(k) + (\Delta F_L)^2. \tag{2.23}
$$

The part of local inhomogeneities represents a realization of cosmic variance (in our neighborhood) in primordial perturbations with small $l$. In order to consider directly the north-south asymmetry, we assume in the following a simplest case with $l = 1$ and $m = 0$, i.e. $F_L = R(r) \cos \theta$. As the observational background of this dependence, there is an asymmetric distribution of galaxies within 200 – 300 Mpc around us. According to the studies of galaxy number counts in the Sloan Digital Sky Survey (Yasuda et al.[16]), the galactic number density in the stripes toward the Southern Galactic Cap is larger than that in the stripes toward Northern Galactic Cap. Such an asymmetry of matter distribution may extend to the region on these scales.

Then the first-order temperature perturbations are

$$
\delta T/T = \left. \Theta_P + \Theta_L \cos \theta \right|_1,
$$

where

$$
\Theta_P = -\frac{1}{2} \int dk \alpha(k) \int_{\lambda_o}^{\lambda_e} d\lambda P'(\eta)(k\mu)^2 e^{ikx},
$$

$$
\Theta_L = \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P'(\eta) R_{rr}, \tag{2.25}
$$

and $\tau$ in Eq.(2.11) was here neglected, because we pay attentions to the Sachs-Wolfe effect in the low-$l$ cases such as $l < 30$.

The former equation can be rewritten as

$$
\Theta_P = \int dk \alpha(k) \left\{ -\frac{1}{2} [(P''_o + i k (P'_o) P_{l}(\mu)] + \frac{1}{2} \sum_l (-i)^l (2l + 1) \Theta_{P(l)} P_l(\mu) \right\}, \tag{2.26}
$$

where

$$
\Theta_{P(l)} = \int_{\lambda_o}^{\lambda_e} d\lambda P''(\eta) j_l(kr) - \{ k(P'_o)(2l + 1)^{-1} [j_{l+1}(kr_e) - j_{l-1}(kr_e)] + j_l(kr_e)(P''_o) \}. \tag{2.27}
$$

In these equations, we have $\eta = \lambda$ and $r = \lambda_o - \lambda$. In the derivation of Eq.(2.27), we used the relations[17, 18]

$$
e^{ikx} = e^{ikr_{l+1}} = \sum_l (-i)^l (2l + 1) j_l(kr) P_l(\mu), \tag{2.28}
$$

and

$$
(2l + 1) \mu P_l(\mu) = (l + 1) P_{l+1}(\mu) + l P_{l-1}(\mu), \tag{2.29}
$$

where $\mu \equiv \cos \theta_k$ and $\theta_k$ is the angle between the vectors $k^i$ and $e^i$. In the above equation for $\Theta_L$, $\theta$ is the angle between $x^3$-axis and the vector $e^i$, i.e. $e^1 = \sin \theta \cos \phi, e^2 = \sin \theta \sin \phi, e^3 = \cos \theta$.

The second-order temperature perturbation consists of three components:

$$
\delta T/T = \Theta_{LP} + \Theta_{LL} + \Theta_{PP}, \tag{2.30}
$$

where

$$
\Theta_{LP} = \int dk \alpha(k) \int_{\lambda_o}^{\lambda_e} d\lambda P'(\eta) j_l(kr) - \{ k(P'_o)(2l + 1)^{-1} [j_{l+1}(kr_e) - j_{l-1}(kr_e)] + j_l(kr_e)(P''_o) \}, \tag{2.31}
$$

$$
\Theta_{LL} = \frac{1}{2} \int_{\lambda_o}^{\lambda_e} d\lambda P'(\eta) R_{rr}, \tag{2.32}
$$

$$
\Theta_{PP} = \int dk \alpha(k) \left\{ -\frac{1}{2} [(P''_o + i k (P'_o) P_{l}(\mu)] + \frac{1}{2} \sum_l (-i)^l (2l + 1) \Theta_{P(l)} P_l(\mu) \right\}. \tag{2.33}
$$
where $\Theta_{LP}$ is a component including the product of primordial perturbations and local inhomogeneities, $\Theta_{LL}$ is a component including only the product of local inhomogeneities, and $\Theta_{PP}$ is a component including only the product of primordial perturbations. In this paper we are concerned mainly studying how CMB anisotropies are influenced by local inhomogeneities, and $\Theta_{PP}$ is small enough, compared with $\Theta_P$ in the first-order. From now, therefore, we neglect $\Theta_{PP}$ and treat only $\Theta_{LL}$ and $\Theta_{LP}$.

In Eq. (2.15) for $d_2T/T$, the term with $A^{(2)}$ includes $P_i^j, M_j^l, N_i^j$ and $C_i^j$. The terms with $N_i^j$ and $C_i^j$ are small, compared with the terms with $P_i^j$ and $M_j^l$, because we have $Q_i/P_i^2 < 10^{-2}$ always and the contribution of gravitational radiation is very small. So they are neglected in the following.

For $\Theta_{LL}$, we obtain from Eq. (2.15)

$$\Theta_{LL} = \Theta_{LL}^{(0)} + \Theta_{LL}^{(2)} \cos^2 \theta,$$

where

$$\Theta_{LL}^{(0)} = \frac{1}{16} \int_{\lambda_0}^{\lambda_e} d\lambda P'(\eta)r^{-2} \left\{ R^2 + \frac{2}{7} P[13(R/r)^2 + 12R_rR/r - 6(R/r)^2] \right\},$$  

$$\Theta_{LL}^{(2)} = \frac{1}{8} \left( \int_{\lambda_0}^{\lambda_e} d\lambda P'R_r \right)^2 - \frac{1}{4} \int_{\lambda_0}^{\lambda_e} d\lambda PP'(R_{rr})^2 + \frac{1}{4} \int_{\lambda_0}^{\lambda_e} d\lambda P'' R_{rr} \int_{\lambda_0}^{\lambda_e} d\bar{\lambda} PP(\bar{\lambda})R_{rr}(\bar{\lambda})$$

$$+ \frac{1}{4} \int_{\lambda_0}^{\lambda_e} d\lambda P' \left\{ \frac{5}{4}(R_r)^2 - RR_{rr} - \frac{1}{4}(R/r)^2 \right\}$$

$$+ \frac{1}{14} P[7(R_{rr})^2 - 7(R/r)^2 - 6R_r(R_r - R/r)^2 - 6r^{-3}R(2R_r - R/r)] \right\}.$$

For $\Theta_{LP}$, we obtain from Eq.(A23) in Appendix using the relation (2.29) and performing partial integrations

$$\Theta_{LP}/\cos \theta = \frac{1}{4} \int d\kappa \left( \sum_{l_1} (-l_1^2) (2l + 1) H_{LP}^{(l_1)} P_l(\mu), \right)$$

where

$$H_{LP}^{(l_1)} = \left[ -(P^r)_{c j_1}(kr) + \int_{\lambda_0}^{\lambda_e} d\lambda P'' j_1(\bar{\lambda}) \right] \times \int_{\lambda_0}^{\lambda_e} d\bar{\lambda} P'(\bar{\lambda})R_r(\bar{\lambda})$$

$$- \int_{\lambda_0}^{\lambda_e} d\lambda j_1(\bar{\lambda}) \left\{ P'' R_{rr} + (P^r)_{c} R_{rr} + \frac{5}{2} \frac{d}{d\lambda} (P^r)_{c} \right\}$$

$$+ \frac{1}{7} \frac{d^2}{d\lambda^2} \left[ 7P' R + PP'(4R_{rr} + 9R_r/r - 9R/r^2) \right]$$

$$- \frac{3}{7} PP' k^2 (R_{rr} - R/r + R/r^2) - \frac{d^2}{d\lambda^2} \left[ P'' \int_{\lambda_0}^{\lambda_e} d\lambda P'(\bar{\lambda})R_{rr}(\bar{\lambda}) \right] \right\}$$

$$+ \int_{\lambda_0}^{\lambda_e} d\lambda P'' R_{rr} \int_{\lambda_0}^{\lambda_e} d\bar{\lambda} P'(\bar{\lambda})j_1(\bar{\lambda}) + \int_{\lambda_0}^{\lambda_e} d\lambda R_{rr} \left\{ k_{j_1}^{(1)}(kr) \right\} \left[ (P^r)_{c} P' + (P^r)_{c} P'' \right]$$

$$+ k_{j_1}^{(2)}(kr)(P^r)_{c} P',$$

and we neglected the terms with $\phi_k$ in Eq.(A23) because they vanish in the $\kappa$ integration. Since the primordial random perturbations are rotationally symmetric and the local inhomogeneity is assumed to be proportional to $\cos \theta$, it is reasonable that $\Theta_{LP}$ is represented as the product of $\cos \theta$ and rotationally symmetric perturbations (cf. Eq.(2.34)). The latter can be expanded using the Legendre polynomials.

In the above equations, we used the auxiliary functions defined by

$$j_{l_1}^{(1)}(kr) = \frac{1}{2l + 1} \left[ l j_{l-1}(kr) - (l + 1) j_{l+1}(kr) \right],$$

$$j_{l_2}^{(2)}(kr) = \frac{1}{2l + 1} \left[ \frac{2l^2 + 2l - 1}{2l - 1} (2l + 1) j_{l}(kr) - \frac{l(l - 1)}{2l + 1} j_{l-2}(kr) - \frac{(l + 1)(l + 2)}{2l + 3} j_{l+2}(kr) \right].$$
III. POWER SPECTRA OF CMB ANISOTROPIES

The CMB anisotropies in the present analysis include the contributions from primordial perturbations and local homogeneities, and only the former perturbations are regarded as statistically random quantities. In order to derive the power spectra, therefore, we take the statistical average \( \langle \rangle \) only for the contribution from the primordial perturbations, and \( \langle (\delta T/T)^2 \rangle \) is expressed as

\[
\langle (\delta T/T)^2 \rangle = \langle (\delta T/T)^2 \rangle + 2\langle T/T \delta T/T \rangle,
\]

where we took first two terms and neglected higher order terms. For the first-order anisotropies, we have

\[
\langle (\delta T/T)^2 \rangle = \langle (\Theta_P)^2 \rangle + (\Theta_L)^2 \cos^2 \theta,
\]

where \( \Theta_L \) is defined in Eq. (2.25) and

\[
(T_0)^2\langle (\Theta_P)^2 \rangle = \sum_l \frac{2l+1}{4\pi} C_l.
\]

The power spectra \( C_l \) are

\[
C_l = (T_0)^2 \int dk k^2 P_F(k) |\mathcal{H}_P^{(l)}(k)|^2,
\]

where \( T_0 \) is the present CMB temperature and

\[
\mathcal{H}_P^{(0)}(k) = -(P''_o) - (P'_o) (P''_e) - (P'_e) + \int_0^{\lambda_o} d\lambda P''_e j_0(\lambda k),
\]

\[
\mathcal{H}_P^{(1)}(k) = \frac{1}{3} [(P'_o) - (P'_e) j_1(1)(kr_e) - (P''_e) j_1(1)(kr_e)] + \int_0^{\lambda_o} d\lambda P''_e j_1(\lambda k).
\]

For \( l \geq 2 \), we have

\[
\mathcal{H}_P^{(l)}(k) = k (P'_o) - k (P''_e) j_l(1)(kr_e) + \int_0^{\lambda_o} d\lambda P''_e j_l(\lambda k).
\]

In the derivation of \( C_l \), we used Eq. (2.21) for \( \langle \alpha(k) \alpha^*(k') \rangle \), and the formulas for \( P_l(\mu) : \int_{l-1}^l P_l(\mu) P_{l'}(\mu) d\mu = 2/(2l+1), \) for \( l = l', l \neq l' \), respectively.

For the second term of Eq. (3.1), we have

\[
\langle (\delta T/T \delta T/T) \rangle = \langle \Theta_P \Theta_{LP} \rangle + \Theta_L \Theta_{LL} \cos \theta,
\]

where

\[
\Theta_{LL} \cos \theta = \Theta_{LL}^{(0)} \cos \theta + \Theta_{LL}^{(2)} \cos^2 \theta
\]

and

\[
\langle \Theta_P \Theta_{LP} \rangle / \cos \theta = \frac{1}{4}(2\pi)^{-1} \sum_l (2l+1) \int dk k^2 P_F(k) \mathcal{H}_P^{(l)} \mathcal{H}_{PL}^{(l)}.
\]

If we put \( \langle \Theta_P \Theta_{LP} \rangle \) in the form

\[
(T_0)^2 \langle \Theta_P \Theta_{LP} \rangle = \sum_l \frac{2l+1}{4\pi} \Delta C_l \cos \theta,
\]

we have

\[
\Delta C_l = \frac{1}{2}(T_0)^2 \int dk k^2 P_F(k) \mathcal{H}_P^{(l)} \mathcal{H}_{PL}^{(l)},
\]
where $\mathcal{H}_{PL}^{(l)}$ is expressed by performing the $\lambda$ differentiation in Eq.(2.35) as
\begin{equation}
\mathcal{H}_{PL}^{(l)} = \int_{\lambda_0}^{\lambda_1} d\lambda \Phi \, j_l(kr) + [-(\hat{P}^{''})_c j_l(kr) + \int_{\lambda_0}^{\lambda_1} d\lambda P^{'''}_c j_l(kr) + \int_{\lambda_0}^{\lambda_1} d\lambda P^{''}_c (\hat{P}_c) R_r(\hat{\lambda}) \\
+ \int_{\lambda_0}^{\lambda_1} d\lambda P^{''}_c R_{rr} \int_{\lambda_0}^{\lambda_1} d\lambda P^{'''}_c j_l(kr) + \int_{\lambda_0}^{\lambda_1} d\lambda R_{rr} \{k_l^{(l)}(kr) [(P^c)_c P^c + (P^c)_c P] + k^2 j_l^{(2)}(kr)(P^c)_c P\},
\end{equation}
(3.12)
where
\[ \Phi \equiv -PP^{''}R_{rrr} + [2P^{''}P + \frac{1}{2}P^c - (P^{'''})_c P] R_{rr} - \frac{1}{2}P^{''}P_r - P^{'''r} R \\
+ \frac{2}{7}((P^c)^2 + PP^c)(4R_{rr} + 9R_r/r - 9R/r^2) + \int_{\lambda_0}^{\lambda_1} d\lambda PR_{rr} \\
+ \frac{2}{7}((P^c)^2 + PP^c)(4R_{rr} + 9R_r/r - 18R/r^2 + 18R/r^3) \\
+ \frac{1}{7}PP'(4R_{rr} + 9R_{rr}/r - 27R_{rr}/r^2 + 54R_r/r^3 - 54R/r^4) \\
+ \frac{3}{7}PP'k^2(R_{rr} - R_r/r + R/r^2).
\] (3.13)

It is found by numerical calculations that the dominant contribution to $\Delta C_l$ comes from the last two terms (proportional to $PP'$) in $\Phi$, especially the terms with highest-order differentiations with respect to $r$.

The total anisotropies in Eq.(3.1) consist of the rotationally symmetric component ($\langle (\Phi)^2 \rangle$), the asymmetric component ($\langle (\Theta T) \rangle$, and the dipole and quadrupole components with $\langle (\Theta L) \rangle^2$ and $\Theta_T\Theta_{LL}$. The key point in this paper is that the north-south asymmetry affecting the CMB power spectra of $l \geq 2$ is caused by the component $\langle (\Theta_T \Theta_{LL}) \rangle$ which is proportional to $\cos \theta$.

IV. SIMPLE MODELS OF LARGE-SCALE LOCAL INHOMOGENEITIES

In order to study the gravitational influence of local inhomogeneities on CMB, we consider a simple model with north-south asymmetry, in which $F_L(x)$ is expressed as
\[ F_L(x) = R(r) \cos \theta. \] (4.1)
This corresponds to the case $l = 1$ and $m = 0$ in Eq.(2.18). For $R$, four types of functional forms are considered and compared:
\[ R = R_0 \exp[-\alpha(x - 1)^2], \quad \frac{1}{2}R_0[1 + \cos 2\pi(x - 1)], \]
\[ R_0x^2 \exp[-\alpha(x - 1)^2], \quad \text{and} \quad \frac{1}{2}R_0x^2[1 + \cos 2\pi(x - 1)] \] (4.2)
in the interval $x = [x_1, x_2]$ with $x \equiv r/r_c$, where $x_1 \equiv r_1/r_c = 0.5$ and $x_2 \equiv r_2/r_c = 1.5$. In all types we have $R = 0$ for $x > x_2$ or $x < x_1$. $R_0$ is the normalization constant and a constant $\alpha$ is chosen as 20. Here $a_0 r_c$ is assumed to be $\approx 300h^{-1}\text{Mpc}$ ($H_0 = 100h \text{ km/s/Mpc}$). These types are called in the following the Gaussian type (G), the sine type (S), the modified Gaussian type (MG), and the modified sine type (MS), respectively. In the first two types, $R$ is radially symmetric around the surface $x = 1$. In the second two types, $R$ has small asymmetry around it.

First, we derive the first-order CMB anisotropies for comparison. They are calculated using Eq.(3.4). In Table I, the behavior of $l(l+1)C_l$ is shown for $n = 0.97$. The root mean square ($A_1$) of $l(l+1)C_l$ for $l = 2 - 11$ is 0.172 for $n = 0.97$, where $A_1 \equiv \{\sum_{l=2}^{11}l(l+1)C_l^2/10^{1/2}, \xi \equiv 2\pi/|P_{T0}(T_0)|^2$ and $T_0$ is the present CMB temperature. The normalization constant $P_{T0}$ can be determined as $P_{T0} = [870(\mu K)^2/(2.7K)^2](2\pi)/A_1 = 2.1 \times 10^{-8}$, so that it may be consistent with the observed CMB anisotropy.

The first-order anisotropy $\Theta_L$ due to local inhomogeneities is derived from Eq.(2.25) in the four types and the ratios of their values to $R_0$ in the four types are shown in Table II.

Next we derive the second-order anisotropy for local inhomogeneities of the above four types, using Eq.(3.11). The behavior of $\Delta C_l$ is shown in Table I for $l = 1 \sim 22$. It is found that $\Delta C_l$ change the signs often in this interval of $l$, but their absolute values are comparable and increase slowly with the increase of $l$. The total
TABLE I: CMB anisotropy powers \(l(l+1)C_l\) and \(l(l+1)\Delta C_l\) in the case \(n = 0.97\). The latter is caused by the coupling of cosmological perturbations and local inhomogeneities of types G, S, MG and MS. Here \(\xi \equiv 2\pi/[P_{P_0}(T_0)^2]\), and \(P_{P_0}\) and \(R_0\) are the normalization factors.

| \(l(l+1)C_l\xi\) | \(10^{-3} \times 2l(l+1)\Delta C_l\xi/R_0\) |
|------------------|------------------|
| \(l\)       | G | S | MG | MS | mean |
| 1             | 4.550 | -1.07 | -2.37 | -0.99 | -1.73 | -1.54 |
| 2             | 0.184 | -0.54 | -0.61 | -0.48 | -0.55 | -0.54 |
| 3             | 0.177 | 0.50 | -0.13 | 0.68 | -0.016 | 0.27 |
| 4             | 0.170 | -0.66 | -0.86 | -0.11 | -0.22 | -0.46 |
| 5             | 0.168 | 1.88 | 0.021 | 1.58 | 0.084 | 0.89 |
| 6             | 0.166 | 1.69 | 0.58 | 2.41 | 1.65 | 1.58 |
| 7             | 0.167 | 2.26 | 1.03 | -0.031 | -0.32 | 0.74 |
| 8             | 0.165 | 2.99 | 1.26 | 2.21 | 0.51 | 1.74 |
| 9             | 0.172 | -5.32 | -3.62 | -7.52 | -3.93 | -5.10 |
| 10            | 0.173 | -1.49 | -1.08 | -3.44 | -3.36 | -2.34 |
| 11            | 0.179 | -5.04 | 1.02 | -1.69 | 3.41 | -0.58 |
| 12            | 0.176 | -4.29 | -1.79 | -3.31 | 0.43 | -2.24 |
| 13            | 0.191 | 5.37 | 3.11 | 10.26 | 2.78 | 5.38 |
| 14            | 0.191 | 4.26 | 2.73 | 6.15 | 7.43 | 5.14 |
| 15            | 0.203 | 7.27 | -9.40 | 4.51 | -9.52 | -1.78 |
| 16            | 0.197 | 2.39 | 3.36 | 0.40 | -0.84 | 1.33 |
| 17            | 0.215 | -2.04 | 9.66 | -11.56 | 3.92 | -0.23 |
| 18            | 0.220 | -4.90 | -2.12 | -4.33 | -4.31 | -3.92 |
| 19            | 0.230 | -19.76 | -8.06 | -8.44 | 1.24 | -8.76 |
| 20            | 0.231 | 2.80 | -6.55 | 3.65 | 2.13 | 0.51 |
| 21            | 0.240 | 32.68 | 13.26 | 35.11 | 12.92 | 23.49 |
| 22            | 0.260 | 2.49 | 13.18 | -0.79 | 1.44 | 4.08 |

TABLE II: CMB anisotropies caused by only local inhomogeneities of types G, S, MG and MS. \(R_0\) is the normalization factor.

| model types | G | S | MG | MS | mean |
|-------------|---|---|----|----|------|
| \(\Theta_{L0}(R_0)^2\) | -9.1 \times 10^3 | -9.1 \times 10^2 | -9.1 \times 10^2 | -8.7 \times 10^2 | -9.0 \times 10^2 |
| \(\Theta_{L2}(R_0)^2\) | 2.6 \times 10^4 | 1.6 \times 10^4 | 3.1 \times 10^4 | 2.4 \times 10^4 | 2.4 \times 10^4 |

The ratios \((A_2/A_1)\) are shown in Table III, where \(A_2\) is defined as the mean square root \((A_2)\) of \(2l(l+1)\Delta C_l\xi\) in \(l = 2 \sim 11\), that is, \(A_2 \equiv \left\{\sum_{l=2}^{11}2l(l+1)\Delta C_l\xi/10\right\}^{1/2}\). The dipole component \((l = 1)\) is separately treated from the multiple components \((l \geq 2)\). Here we remark that we have not specified the value of \(a_0r_c\) yet, and \(R_0\) is arbitrary. From Table III, we find that, as the mean value, we have

| \((A_2/A_1)/|R_0|\) | G | S | MG | MS | mean |
|-----------------|---|---|----|----|------|
| 1.6 \times 10^4 | 0.81 | 1.7 \times 10^4 | 1.2 \times 10^4 | 1.3 \times 10^4 |
their values in four types are shown in Table II. For the above values of $P_{l0}$ and $R_0$, $\Theta_L$, $\Theta_{LL0}$ and $\Theta_{LL2}$ are $\approx 3.6 \times 10^{-5}(A_2/A_1), -5.1 \times 10^{-6}(A_2/A_1)^2$ and $1.2 \times 10^{-3}(A_2/A_1)^2$, respectively.

In order to examine the significance of $R_0$ given in Eq.(4.3), we consider here the first-order density perturbation due to local inhomogeneities. It is expressed as

$$
(\delta \rho/\rho)_L = \frac{1}{\rho a^2} \left( \frac{a'}{a} P' - 1 \right) \tilde{R} \cos \theta,
$$

where

$$
\tilde{R} = (r_c)^{-2}[(x^2 R_x)/x^2 - 2R/x^2].
$$

The ratio $(\delta M/M)$ of perturbed mass $(\delta M)$ to background mass $(M)$ in the interval $x = [x_1, x_2]$ is defined by

$$
J = \int_{x_1}^{x_2} \int_0^1 (\delta \rho/\rho)_L x^2 dx d\mu/ \left\{ \frac{1}{3} [(x_2)^3 - (x_1)^3] \int_0^1 d\mu \right\}
$$

$$
= \frac{1}{2 (pa^2)_0 (r_c)^2} \left( \frac{a'}{a} P' - 1 \right) \int_0^1 [(x_2)^3 - (x_1)^3]^{-1} \int_{x_1}^{x_2} [(x^2 R_x)/x - 2R] dx.
$$

The factor $[(a'/a)P' - 1]$ can be regarded as having the value at present epoch, because the inhomogeneities are at the place of $z = 0.05 \sim 0.15$, and it is equal to $-0.456$ for the concordant background model with $\Omega_0 = 0.27$ and $A_0 = 0.73$. Moreover, $(pa^2)_0 (r_c)^2 = 3(\rho_c/c/H_0)^2$, where we have used the unit $c = 8\pi G = 1$ in this paper. Therefore we have

$$
J = -7.0(300h^{-1}\text{Mpc}/a_0 r_c)^2 \int_{x_1}^{x_2} [(x^2 R_x)/x - 2R] dx,
$$

where $x_2 - 1 = -x_1 - 1 = 0.5$ and $c/H_0 = 3000h^{-1}$ Mpc. After performing the integration in Eq.(4.7), we obtain

$$
J = (2.4, 14.0, 13.6, 10.3) R_0 (300h^{-1}\text{Mpc}/a_0 r_c)^2,
$$

for types G, S, MG and MS, respectively, and their mean value is $\bar{J} = 10.1 R_0 (300h^{-1}\text{Mpc}/a_0 r_c)^2$. For $R_0$ given in Eq.(4.3), $\bar{J}$ leads to

$$
\bar{J} = 7.6 \times 10^{-4}(A_2/A_1) (300h^{-1}\text{Mpc}/a_0 r_c)^2 \times R_0/|R_0|.
$$

On the other hand, the observed value of $(\langle (\delta M/M)^2 \rangle)^{1/2}$ is about 1 on the scale of $8h^{-1}$ Mpc and for the power spectrum $P(k) \propto k^n$, we have $\delta M/M \propto M^{-(n+3)/6} \propto r^{-(n+3)/2} = r^{-1.985}$ for $n = 0.97$. Since we are considering a local inhomogeneity included in the sphere of radius $2a_0 r_c$, its scale is regarded as $4\epsilon a_0 r_c (\approx 1200h^{-1}\text{Mpc})$ with $\epsilon \approx 1$, so that the value of $\delta M/M$ for the inhomogeneity is

$$
(\delta M/M)^{\text{power}} = (8h^{-1}/4\epsilon a_0 r_c)^{1.985} / b = 4.8 \times 10^{-5}(b^{1.985} - 1)(300h^{-1}\text{Mpc}/a_0 r_c)^{1.985},
$$

where $b$ is the biasing factor[19, 20]. If we assume $|\bar{J}| = (\delta M/M)^{\text{power}}$, we obtain from Eqs.(4.9) and (4.10)

$$
A_2/A_1 = 0.063(b^{1.985} - 1)(300h^{-1}\text{Mpc}/a_0 r_c)^{0.015}.
$$

For this value of $A_2/A_1$, we have

$$
|\Theta_L| = 2.3 \times 10^{-6}/(b^{1.985}), \Theta_{LL0} = -2.0 \times 10^{-8}/(b^{1.985})^2, \Theta_{LL2} = 4.8 \times 10^{-6}/(b^{1.985})^2 \text{ for } n = 0.97,
$$

neglecting the factor of $a_0 r_c$. If $b^{1.985} \sim 1$, these are small, compared with $\sqrt{C_1}/T_0(\sim 5 \times 10^{-5})$ and $\sqrt{C_2}/T_0(\sim 10^{-5})$. If $b^{1.985} \lesssim 0.5$, however, $\Theta_{LL2}$ is comparable with $\sqrt{C_2}/T_0$ or larger than it.

Finally let us compare our above theoretical results with the observed anisotropy spectra by Eriksen et al.[4]. Here we assume that the matter density perturbations in the Southern and Northern hemispheres are positive and negative, respectively, corresponding to the galactic number counts in SDSS[16]. Then we have $R_0 < 0$ in order that the directions $\theta = 0$ and $\pi$ may be in the Galactic North and South, respectively. In our results of calculations (from Table I), we find the trend that $\Delta C_l$ for $l = 2, 3$ and $4$ are positive for $R_0 < 0$ and $\Delta C_l$ for $l \geq 5$ change their signs in the period of $\Delta t = 3$. This trend seems to be seen in Fig.2 of Eriksen et al.’s paper[4]. It is found, moreover, that corresponding to the north-south asymmetry (proportional to $\cos \theta$) the
ratio $\Delta C_l/C_l$ seems to be $0.3 \sim 0.5$ by reading the points in Fig. 2 of their paper. From Eq.(4.11) and Eriksen et al.’s observational result, accordingly, we obtain the condition $b_\epsilon^{1.985} = 0.1 \sim 0.2$. This condition can be satisfied in the reasonable range of $b$. For instance we have $\epsilon = 0.45 \sim 0.63$ for $b = 0.5$.

Under this condition, $\Theta_{LL2}$ is comparable with $\sqrt{C_2}/T_0$, so that the measured value of $C_2$ may be disturbed by $\Theta_{LL2}$, because of their similar angular dependence. $\Theta_{LL2}$ and $\sqrt{C_2}$ depend on $P_2(\cos \theta)$ and $P_2(\cos \theta_k)$, where $\theta$ and $\theta_k$ are the angles between a directional vector $e$ and the $x^3$-axis and between $e$ and $k$, respectively (cf. Appendix), but both of them may be measured as the quadrupole components. So its measured value may have been given a value smaller than its theoretical expected value by the offset effect.

V. CONCLUDING REMARKS

In this paper we derived the asymmetry of CMB anisotropy powers (proportional to $\cos \theta$) with comparatively low $l$ using the second-order perturbation theory by assuming the local matter distribution is dipole-like. This assumption seems to be rough, but it is consistent with the observed situation that the asymmetry of CMB anisotropies disappears when they are averaged in the whole sky and the asymmetry in the distribution of galaxies is also disappears similarly in the whole sky, though it has only a small north-south asymmetry. If we add a correction term $\propto \cos^2 \theta$ to $F_L$, more realistic simulations to observed $\Delta C_l$ may be obtained, especially for $l > 20$. Moreover, to clarify the relation between the derived asymmetry and non-Gaussianity, it is necessary to investigate the multi-point correlations of anisotropies. Detailed analyses on these points are to be done in the next step.

In our result, $\Delta C_l/C_l$ depend not so on the distance to the local inhomogeneity (i.e. the value of $a_0 r_c$), but sensitively on $b_\epsilon^{(n+3)/2}$. It was found that the behavior of $\Delta C_l$ in the case $n = 0.90$ is not so different from that in the case $n = 0.97$, so the conclusion in our result does not depend on $n$.

The northern and southern hemispheres were assumed here as those in the Galactic coordinate frame. However our theory itself does not depend on any coordinate frames and can treat any local inhomogeneities which can be supposed.

When the measurements of CMB anisotropies are more accurate, the relation between the local matter distribution and the CMB asymmetry will be more realistic through the second-order perturbation theory.

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APPENDIX A: DERIVATION OF $\delta_2 T/T$

From Eq.(2.15), we obtain

$$\frac{\delta T}{T} = \frac{1}{2} [I_1(\lambda_c)]^2 - A^{(1)'} \int_{\lambda_o}^{\lambda_c} d\lambda A^{(1)}$$

$$- \int_{\lambda_o}^{\lambda_c} d\lambda \left\{ \frac{1}{2} A^{(2)'} + A^{(1)'} A^{(1)'} - A^{(1)''} \right\} \int_{\lambda_o}^{\lambda_c} d\lambda A^{(1)}(\lambda),$$

(A1)

neglecting the terms with $\tau$ at the decoupling epoch. Each term in Eq.(A1) is expressed as follows:

$$I_1(\lambda_c) = - \frac{1}{2} \int_{\lambda_o}^{\lambda_c} d\lambda \rho'(F_L,ij\epsilon^i\epsilon^j + F_P,ij\epsilon^i\epsilon^j),$$

(A2)

so that

$$[I_1(\lambda_c)]^2 = \frac{1}{4} \left( \int_{\lambda_o}^{\lambda_c} d\lambda \rho'(F_L,ij\epsilon^i\epsilon^j) \right)^2 + \frac{1}{2} \int_{\lambda_o}^{\lambda_c} d\lambda \rho'(F_L,ij\epsilon^i\epsilon^j) \int_{\lambda_o}^{\lambda_c} d\lambda \rho'(F_P,ij\epsilon^i\epsilon^j),$$

(A3)

where the term $\int_{\lambda_o}^{\lambda_c} d\lambda \rho'(F_P,ij\epsilon^i\epsilon^j)$ is neglected by the reason described in the text. Similarly we obtain

$$A^{(1)'} A^{(1)'} = \frac{1}{4} \rho' \left[ (F_L,ij\epsilon^i\epsilon^j)^2 + 2 F_L,ij\epsilon^i\epsilon^j F_P,kl\epsilon^k\epsilon^l \right],$$

(A4)
where $\lambda$ corresponds to Eq.(2.18), where $\theta_i$, $\partial\theta/\partial x$ $F_k$ together with $\lambda L_{ij} e_i = \phi_{ij}$ to $F_{k,i} e^k e^i + F_{P,i} e^i e_j$ 

\[ M_i^j e^j e^i = \frac{1}{28} \left[ 9 [19 F_{L,il} F_{L,jl} - 12 \Delta F_L F_{L,i} e^j - 3 [F_{L,k} F_{L,kl} - (\Delta F_L)^2] + 38 F_{L,il} F_{P,jl} - 12 \Delta F_L F_{P,i} e^j - 6 [F_{L,k} F_{P,kl} - \Delta F_L \Delta F_P] \right]. \]  

Using the radial coordinate $r$, we have 

\[ F_{L,j} e^j = R_r Y_l^m, \]
\[ F_{P,j} e^j = \int d\kappa (k) i k \mu e^{ik \mathbf{x}}, \]  

(A8)

To express $F_{L,j} k^j$ and $F_{P,j} k^j k^l$ in terms of polar coordinates, we introduce two unit (three-dimensional) vectors $e_i^\phi$ and $e_i^\theta$ together with $e^i$, and make an orthonormal triad vectors with components:

\[ (e^i) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \]
\[ (e^i_\phi) = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \]
\[ (e^i_\theta) = (-\sin \phi, \cos \phi, 0), \]  

(A10)

respectively, where $\theta$ is the angle between the vector $e^i$ and the $x^3$-axis. Using these vectors, we have $\partial r/\partial x^1 = e_1$, $\partial \theta/\partial x^2 = r^{-1} e_{\theta_1}$ and $\partial \phi/\partial x^3 = (r \sin \theta)^{-1} e_{\phi_1}$ for the polar coordinates $(r, \theta, \phi)$.

Moreover, the projection of $\mathbf{k}$ to the triad is expressed using another angles $\theta_k$ and $\phi_k$ as

\[ k^i (e_\theta i, e_\phi i, e_i) = k (\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, \cos \theta_k), \]  

(A11)

where $e_i = \delta_{ij} e^j$, etc., and the angle $\theta_k$ is the angle between the vectors $e^i$ and $k^i$. Then we have

\[ F_{L,j} k^j = F_{L,j} (e^1 k^1 e_i + e_\phi^1 k^1 e_{\phi_1} + e_\theta^1 k^1 e_{\theta_1}) = k [F_{L,r} \cos \theta_k + F_{L,\theta} r^{-1} \sin \theta_k \cos \phi_k + F_{L,\phi} (r \sin \theta)^{-1} \sin \theta_k \sin \phi_k], \]

(A12)

\[ F_{L,j} k^j k^l = k^j k^l [F_{L,rt} e_j e_i + (F_{L,\theta} + r F_{L,r}) r^{-2} e_{\theta_1} e_{\phi_1} + (F_{L,\phi} + r \sin^2 \theta F_{L,\theta}) + \sin \theta \cos \theta F_{L,\phi} (r \sin \theta)^{-2} e_{\phi_1} e_{\phi_1}] + 2 (F_{L,rt} r^{-1} F_{L,\theta} r^{-1} e_{\theta_1} e_{\phi_1} + 2 (F_{L,\phi} - r^{-1} F_{L,\phi} (r \sin \theta)^{-1} e_{\phi_1} e_{\phi_1} + (F_{L,\phi} - \cot \theta F_{L,\phi}) (r^2 \sin \theta)^{-1} e_{\phi_1} e_{\phi_1}). \]  

(A13)

Substituting $F_P$ and $F_L$ in Eq.(2.18) with $l = 1$ and $m = 0$ into Eq.(A3), we obtain

\[ (I_1)^2 = \frac{1}{4} \left( \int_{\lambda_0} d\lambda P' R_{rt} \right)^2 \cos^2 \theta + \frac{1}{2} \int_{\lambda_0} d\lambda P' R_{rt} \cos \theta \int_{\lambda_0} d\lambda P' i \int d\kappa (k) \mu e^{ik \mathbf{x}}, \]

(A14)

\[ A^{(1)} A^{(1)'} = \frac{1}{4} P P' \left[ \left( R_{rt} \right)^2 \cos^2 \theta - 2 i R_{rt} \cos \theta \int d\kappa (k) \mu (k) e^{ik \mathbf{x}} \right], \]  

(A15)
\[ A^{(1)'}_{\mu} \int_{\lambda_0}^{\lambda} d\lambda A^{(1)} = \frac{1}{4} P'' \left[ R_{rr} \int_{\lambda_0}^{\lambda} d\lambda P(\bar{\eta}) R_{rr} (\bar{r}) \cos^2 \theta - R_{rr} \cos \theta \right] \int_{\lambda_0}^{\lambda} d\lambda P \int d\mathbf{k} \alpha (k) (k\mu)^2 e^{ikx} \\
\quad - \int d\mathbf{k} \alpha (k) (k\mu)^2 e^{ikx} \int_{\lambda_0}^{\lambda} d\lambda P R_{rr} \cos \theta \right]. \]  

(A16)

For the terms in \( A^{(2)'} \), we have

\[ FF_{ij} e^{i\lambda} = RR_{rr} \cos^2 \theta + \int d\mathbf{k} \alpha (k) e^{ikx} [R_{rr} - (k\mu)^2 R] \cos \theta, \]

(A17)

\[ F_{ij} F_{ij} = (R_{rr})^2 \cos^2 \theta + R^2 \sin^2 \theta \cos^2 \theta + 2 \int d\mathbf{k} \alpha (k) e^{ikx} \text{ik}[R_{rr} \cos \theta \mu - r^{-1} \sin \theta \sin \theta_k \cos \phi_k], \]

(A18)

\[ F_{ij} F_{ij} = (R_{rr})^2 \cos^2 \theta + (R_{rr})^2 \sin^2 \theta \cos^2 \theta + 2 \int d\mathbf{k} \alpha (k) e^{ikx} [-R_{rr} \cos \theta (k\mu)^2 \\
\quad - R_{rr} \sin \theta \mu \sin \theta_k \cos \phi_k], \]

(A19)

\[ \Delta FF_{ij} e^{i\lambda} = (R_{rr} + 2r^{-1} R_{rr} - 2r^{-2} R) R_{rr} \cos^2 \theta \\
\quad - \int d\mathbf{k} \alpha (k) e^{ikx} [R_{rr} + 2r^{-1} R_{rr} - 2r^{-2} R] (k\mu)^2 + R_{rr} k^2 \cos \theta, \]

(A20)

\[ F_{ij} F_{ij} - (\Delta F)^2 = -4r^{-1} (R_{rr} - r^{-1} R)(R_{rr} + r^{-1} R - r^{-2} R) \cos^2 \theta + 2r^{-2} (R_{rr} - r^{-1} R)^2 \\
\quad - 2 \int d\mathbf{k} \alpha (k) k^2 e^{ikx} \{|R_{rr} + 2r^{-1} R_{rr} - 2r^{-2} R| \cos \theta \sin^2 \theta_k \\
\quad - 2 \sin \theta \sin \theta_k \cos \theta_k - (R_{rr} + 2r^{-1} R_{rr} - 2r^{-2} R) \cos \theta \right]. \]

(A21)

Using the above equations, we obtain \( \Theta_{LL} \) and \( \Theta_{LP} \) as follows:

\[ \Theta_{LL} = \frac{1}{4} \left[ \int_{\lambda_0}^{\lambda} d\lambda P' R_{rr} \cos \theta \right]^2 - \frac{1}{4} \int_{\lambda_0}^{\lambda} d\lambda P' (R_{rr} \cos \theta)^2 \\
+ \frac{1}{4} \int_{\lambda_0}^{\lambda} d\lambda P' R_{rr} R_{rr} (\bar{r}) \cos^2 \theta \\
+ \frac{1}{4} \int_{\lambda_0}^{\lambda} d\lambda P' \left\{ - [R_{rr} + 5 \frac{1}{2} (R_{rr})^2 \cos^2 \theta + \frac{1}{4} r^{-2} R^2 \sin^2 \theta \\
+ \frac{1}{14} P [19 (R_{rr})^2 \cos^2 \theta + 19 (R'/r')^2 \sin^2 \theta - 12 R_{rr} (R_{rr} + 2r^{-1} R_{rr} - 2r^{-2} R) \cos^2 \theta \\
+ 12r^{-1} (R_{rr} - r^{-1} R)(R_{rr} + r^{-1} R_{rr} - r^{-2} R) \cos^2 \theta - 6 (R_{rr} - r^{-1} R)^2] \right\} \]

(A22)

and

\[ \Theta_{LP} = -\frac{1}{4} \int_{\lambda_0}^{\lambda} d\lambda P' R_{rr} \cos \theta \times \int_{\lambda_0}^{\lambda} d\lambda P' \int d\mathbf{k} \alpha (k) (k\mu)^2 e^{ikx} \\
+ \frac{1}{2} \int_{\lambda_0}^{\lambda} d\lambda P' R_{rr} \cos \theta \int d\mathbf{k} \alpha (k) (k\mu)^2 e^{ikx} \\
- \frac{1}{4} \int_{\lambda_0}^{\lambda} d\lambda P' \left\{ R_{rr} \int_{\lambda_0}^{\lambda} d\lambda P(\bar{\eta}) e^{ikx} + e^{ikx} \int_{\lambda_0}^{\lambda} d\lambda P(\bar{\eta}) R_{rr} P(\bar{\eta}) \right\} \cos \theta \\
+ \frac{1}{4} \int_{\lambda_0}^{\lambda} d\lambda P' \left\{-5 \frac{1}{2} R_{rr} \cos \theta \int d\mathbf{k} \alpha (k) k\mu e^{ikx} - \int d\mathbf{k} \alpha (k) e^{ikx} [R_{rr} - (k\mu)^2 R] \cos \theta \\
- \frac{1}{2} \int d\mathbf{k} \alpha (k) e^{ikx} [R_{rr} \cos \theta \mu \sin \theta_k \cos \phi_k] \right\} \\
+ \frac{1}{28} \int_{\lambda_0}^{\lambda} d\lambda P' \int d\mathbf{k} \alpha (k) e^{ikx} [19 - R_{rr} \cos \theta (k\mu)^2 + R_{rr} \sin \theta \mu \sin \theta_k \cos \phi_k] \\
+ 6k^2 [R_{rr} + 2r^{-1} R_{rr} - 2r^{-2} R]^2 + R_{rr}^2 \cos \theta + 3k^2 [R_{rr} \cos \theta \mu]^2] \]
\[ \begin{align*}
&+ r^{-1}(R_r - r^{-1}R)(\cos \theta \sin^2 \theta_k - 2 \sin \theta \sin \theta_k \cos \phi_k \cos \phi_k) - (R_{rr} + 2r^{-1}R_r - 2r^{-2}R) \cos \theta) \\
&+ \frac{1}{4}(P')_e \int d\lambda \kappa(k)(k\mu)^2 e^{ikx} \int_{\lambda_0}^{\lambda_e} d\lambda P \cos \theta. \\ 
&\text{(A23)}
\end{align*} \]

By integrating Eq.(A23) partially with respect to \( \lambda \), we obtain Eq.(2.34). In the process of integrations, it is to be noticed that we use the boundary condition \((R)_e = (R)_o = 0\) and \((d^mR/dr^m)_e = (d^mR/dr^m)_o = 0\) for \(1 \leq m \leq 4\).

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