Graphs with Conflict-Free Connection Number Two

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Abstract
An edge-colored graph \(G\) is *conflict-free connected* if every two of its vertices are connected by a path, which contains a color used on exactly one of its edges. The *conflict-free connection number* of a connected graph \(G\), denoted by \(cfc(G)\), is the smallest number of colors needed in order to make \(G\) conflict-free connected. For a graph \(G\), let \(C(G)\) be the subgraph of \(G\) induced by its set of cut-edges. In this paper, we first show that, if \(G\) is a connected non-complete graph \(G\) of order \(n \geq 9\) with \(C(G)\) being a linear forest and with the minimum degree \(\delta(G) \geq \max\{3, \frac{n-4}{5}\}\), then \(cfc(G) = 2\). The bound on the minimum degree is best possible. Next, we prove that, if \(G\) is a connected non-complete graph of order \(n \geq 33\) with \(C(G)\) being a linear forest and with \(d(x) + d(y) \geq \frac{2n-9}{5}\) for each pair of two nonadjacent vertices \(x, y\) of \(V(G)\), then \(cfc(G) = 2\). Both bounds, on the order \(n\) and the degree sum, are tight. Moreover, we prove several results concerning relations between degree conditions on \(G\) and the number of cut edges in \(G\).

Keywords  Edge-coloring · Conflict-free connection number · Degree condition

Mathematics Subject Classification  05C15 · 05C40 · 05C07

1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [3] for graph theoretical notation and terminology not described here. Let \(G\) be a graph. We use...
$V(G)$, $E(G)$, $n(G)$, $m(G)$, and $\delta(G)$ to denote the vertex-set, edge-set, number of vertices, number of edges, and minimum degree of $G$, respectively. For $v \in V(G)$, let $N(v)$ denote the neighborhood of $v$ in $G$, $\text{deg}(x)$ denote the degree of $v$ in $G$.

Let $G$ be a nontrivial connected graph with an associated edge-coloring $c : E(G) \to \{1, 2, \ldots, t\}$, $t \in \mathbb{N}$, where adjacent edges may have the same color. If adjacent edges of $G$ are assigned different colors by $c$, then $c$ is a proper (edge-)coloring. For a graph $G$, the minimum number of colors needed in a proper coloring of $G$ is referred to as the edge-chromatic number of $G$ and denoted by $\chi'(G)$. A path of an edge-colored graph $G$ is said to be a rainbow path if no two edges on the path have the same color. The graph $G$ is called rainbow connected if every pair of distinct vertices of $G$ is connected by a rainbow path in $G$. An edge-coloring of a connected graph is a rainbow connection coloring if it makes the graph rainbow connected. This concept of rainbow connection of graphs was introduced by Chartrand et al. [7] in 2008. For a connected graph $G$, the rainbow connection number $rc(G)$ of $G$ is defined as the smallest number of colors that are needed in order to make $G$ rainbow connected. Readers interested in this topic are referred to [17–19] for a survey.

Inspired by the rainbow connection coloring and the proper coloring in graphs, Andrews et al. [1] and Borozan et al. [4] independently introduced the concept of a proper connection coloring. Let $G$ be a nontrivial connected graph with an edge-coloring. A path in $G$ is called a proper path if no two adjacent edges of the path receive the same color. An edge-coloring $c$ of a connected graph $G$ is a proper connection coloring if every pair of distinct vertices of $G$ is connected by a proper path in $G$. And if $k$ colors are used, then $c$ is called a proper connection $k$-coloring. An edge-colored graph $G$ is proper connected if any two vertices of $G$ are connected by a proper path. For a connected graph $G$, the minimum number of colors that are needed in order to make $G$ proper connected is called the proper connection number of $G$, denoted by $pc(G)$. Let $G$ be a nontrivial connected graph of order $n$ and size $m$ (number of edges). Then we have $1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m$. For more details, we refer to [2,13–15] and a dynamic survey [16].

Our research was motivated by the following three results.

**Theorem 1** [5] If $G$ is a 2-connected graph of order $n = n(G)$ and minimum degree $\delta(G) > \max\{2, \frac{n+8}{20}\}$, then $pc(G) \leq 2$.

**Theorem 2** [5] For every integer $d \geq 3$, there exists a 2-connected graph of order $n = 42d$ such that $pc(G) \geq 3$.

**Theorem 3** [14] Let $G$ be a connected noncomplete graph of order $n \geq 5$. If $G \notin \{G_1, G_2\}$ and $\delta(G) \geq \frac{n}{2}$, then $pc(G) = 2$, where $G_1$ and $G_2$ are two exceptional graphs on 7 and 8 vertices.

A coloring of the vertices of a hypergraph $H$ is called conflicted-free if each hyperedge $E$ of $H$ has a vertex of unique color that is not repeated in $E$. The smallest number of colors required for such a coloring is called the conflicted-free chromatic number of $H$. This parameter was first introduced by Even et al. [12] in a geometric setting, in connection with frequency assignment problems for cellular networks. One can find many results on the conflict-free coloring, see [9,10,20].

\[ \text{Springer} \]
Recently, Czap et al. [8] introduced the concept of a conflict-free connection of graphs. An edge-colored graph \(G\) is called conflict-free connected if each pair of distinct vertices is connected by a path which contains at least one color used on exactly one of its edges. This path is called a conflict-free path, and this coloring is called a conflict-free connection coloring of \(G\). The conflict-free connection number of a connected graph \(G\), denoted by \(cfc(G)\), is the smallest number of colors needed to color the edges of \(G\) so that \(G\) is conflict-free connected. In [8], they showed that it is easy to compute the conflict-free connection number for 2-connected graphs and very difficult for other connected graphs, including trees.

This paper is organized as follows. In Sect. 2, we list some fundamental results on the conflict-free connection of graphs. In Sects. 3 and 4, we prove our main results.

2 Preliminaries

At the very beginning, we state some fundamental results on the conflict-free connection of graphs, which will be used in the sequel.

Lemma 4 [8] If \(P_n\) is a path on \(n\) edges, then \(cfc(P) = ⌈log_2(n+1)⌉\).

Let \(C(G)\) be the subgraph of \(G\) induced on the set of cut-edges of \(G\). The following lemmas respectively provide a necessary condition and a sufficient condition for graphs \(G\) with \(cfc(G) = 2\).

Recall that a linear forest is a forest where each of its components is a path.

Lemma 5 [8] If \(cfc(G) = 2\) for a connected graph \(G\), then \(C(G)\) is a linear forest whose each component has at most three edges.

Lemma 6 [8] If \(G\) is a connected graph, and \(C(G)\) is a linear forest in which each component is of order 2, then \(cfc(G) = 2\).

The following lemma, which can be seen as a corollary of Lemma 6 for \(C(G)\) being empty, is of extra interest. A rigorous proof can be found in [11].

Lemma 7 [8,11] If \(G\) is a 2-edge-connected non-complete graph, then \(cfc(G) = 2\).

A block of a graph \(G\) is a maximal connected subgraph of \(G\) that has no cut-vertex. If \(G\) is connected and has no cut-vertex, then \(G\) is a block. An edge is a block if and only if it is a cut-edge, this block is called trivial. Therefore, any nontrivial block is 2-connected.

Lemma 8 [8] Let \(G\) be a connected graph. Then from each nontrivial block of \(G\), an edge can be chosen so that the set of all such chosen edges forms a matching.

Let \(C(G)\) be a linear forest consisting of \(k (k \geq 0)\) components \(Q_1, Q_2, \ldots, Q_k\) with \(n_i = |V(Q_i)|\) such that \(2 \leq n_1 \leq n_2 \leq \cdots \leq n_k\). We now present a stronger result than Lemma 6, which will be important to show our main results.

Theorem 9 If \(G\) is a connected non-complete graph with \(C(G)\) being a linear forest with \(2 = n_1 = n_2 = \cdots = n_{k-1} \leq n_k \leq 4\) or \(C(G)\) being edgeless, then \(cfc(G) = 2\).
Proof If $C(G)$ is edgeless then the theorem is true by Lemma 7. If $C(G)$ a linear forest with at least one edge, then $G$ is a non-complete graph and therefore $cfc(G) \geq 2$. It remains to verify the converse. Note that one can choose from each nontrivial block an edge so that all the chosen edges create a matching set $S$ by Lemma 8. We define an edge-coloring of $G$ as follows. First, we color all edges from $S$ with color 2, and the edges in $E(G) \setminus (S \cup Q_k)$ with color 1. Next, we only need to color the edges of $Q_k$. If $n_k = 2$, then color the unique edge of $Q_k$ with color 1. If $n_k = 3$, then color two edges of $Q_k$ with colors 1 and 2. Suppose $n_k = 4$. It follows that $Q_k$ is a path of order 4, say $w_1w_2w_3w_4$. We color the two edges $w_1w_2$ and $w_3w_4$ with color 1, and $w_2w_3$ with color 2. It is easy to check that this coloring is a conflict-free connection coloring of $G$. Thus, we have $cfc(G) \leq 2$, and hence $cfc(G) = 2$. 

Remark 1 The following example points out that Theorem 9 is optimal in sense of the number of components of the linear forest $C(G)$ of a graph $G$ with more than two vertices. For $t \geq 3$, let $S_n$ be the graph with $n = 5t$ vertices, consisting of the path $P_6 = v_0v_1v_2v_3v_4v_5$ with four complete graphs $K_i^t$ of order $t$ sharing the vertex $v_i$ for $i \in \{0, 1, 4, 5\}$ and one more $K_t$ sharing the edge $v_2v_3$ with $P_6$. Observe that $\delta(S_n) = t - 1 = \frac{n-5}{5}$, and $C(S_n)$ is a linear forest with two components of order 3, paths $v_0v_1v_2$ and $v_3v_4v_5$. In any conflict-free connection coloring of $S_n$ with two colors the edges $v_0v_1$ and $v_1v_2$ (resp. $v_3v_4$ and $v_4v_5$) receive different colors. But then any $v_0$-$v_5$ path has a conflict. This means that $cfc(S_n) \geq 3$.

3 Degree Conditions and the Number of Cut-edges

Theorem 10 Let $G$ be a connected graph of order $n \geq k^2$, $k \geq 3$. If $\delta(G) \geq \frac{n-k+1}{k}$, then $G$ has at most $k - 2$ cut edges.

Proof Assume for the sake of contradiction that $G$ has at least $k - 1$ cut edges. Let $B$ be a set of $k - 1$ cut edges of $G$. Then the graph $G \setminus B$ has exactly $k$ components $G_1, \ldots, G_k$. Consider the following two cases.

Case 1 For every $j \in [k]$ there is a vertex $v_j \in V(G_j)$ such that $N(v_j) \subseteq V(G_j)$. Then every component $G_j$ has at least $\frac{n-k+1}{k} + 1$ vertices and we have,

$$n = |V(G)| = \sum_{j=1}^{k} |V(G_j)| \geq k \cdot \left( \frac{n-k+1}{k} + 1 \right) = n + 1,$$

a contradiction.

Case 2 There exists some $i \in [k]$ such that $N(v) \not\subseteq V(G_i)$ for every vertex $v \in V(G_i)$. Then $a = |V(G_i)| \leq k - 1$ and every vertex $v \in V(G_i)$ is incident with a cut edge from $B$.

Let $m_i$ denote the degree sum of all the vertices of $V(G_i)$ within $G[V(G_i) \cup B]$. Then we have,

$$\frac{n-k+1}{k} \cdot a \leq m_i \leq a \cdot (a - 1) + k - 1.$$
This, together with the bounds on $a$, provides
\[
0 \leq a \cdot \left( a - 1 - \frac{n - k + 1}{k} \right) + k - 1 \leq (k - 1) \cdot \left( k - 2 - \frac{n - k + 1}{k} \right) + k - 1.
\]
This leads to $n \leq k^2 - 1$, a contradiction.

The next theorem shows that the bound on the minimum degree in Theorem 3.1 cannot be lowered.

**Theorem 11** For every $k \geq 3$ and $t \geq 3$ there exists a connected $n$-vertex graph $H_n$ with $n = k \cdot t$, $\delta(H_n) = \frac{n-k}{k}$, and $k - 1$ cut edges.

**Proof** The graph $H_n$ consists of a path $P_k$ on $k$ vertices $v_1, v_2, \ldots, v_k$ together with $k$ complete graphs $K_i$ of order $t$ sharing the vertex $v_i$ for $1 \leq i \leq k$.

The following theorem shows that the bound $k^2$ on the number $n$ of vertices in Theorem 3.1 is best possible.

**Theorem 12** For every $k \geq 3$ there exists a graph $R_n$ on $n = k^2 - 1$ vertices with $\delta(R_n) = \frac{n-k+1}{k}$ and $k - 1$ cut edges.

**Proof** The graph $R_n$ is a connected graph consisting of a central block $B_0$, isomorphic to the complete graph $K_{k-1}$, $k-1$ blocks $B_1, \ldots, B_{k-1}$, that are complete graphs on $k$ vertices, and a matching $M$ of $k - 1$ cut edges. This matches the vertices of $B_0$ with the remaining blocks.

**Theorem 13** Let $G$ be a connected graph of order $n \geq \max\{k^2 + k, \frac{k}{2} \cdot k(k-2) + k^2 - 5k + 3\}$, $k \geq 5$.

If $\deg(x) + \deg(y) \geq \frac{2n - 2k + 1}{k}$ for any two non-adjacent vertices $x$ and $y$ of $G$, then $G$ has at most $k - 2$ cut edges.

**Proof** Assume for the sake of contradiction that $G$ has at least $k - 1$ cut edges. Let $B$ be a set of $k - 1$ cut edges of $G$. Then the graph $G \setminus B$ has exactly $k$ components $G_1, \ldots, G_k$. Consider the following two cases.

**Case 1** For every $j \in [k]$ there is a vertex $v_j \in V(G_j)$ such that $N(v_j) \subseteq V(G_j)$.

**Case 1.1** Let $k$ be even. Then
\[
n = |V(G)| = \sum_{j=1}^{k} |V(G_j) \cup V(G_{k-j+1})| \geq \frac{k}{2} \cdot \left( \frac{2n - 2k + 1}{k} + 2 \right) = n + \frac{1}{2},
\]
a contradiction.
Case 1.2 Let $k$ be odd. Then, without loss of generality, we can suppose that $|V(G_k)| \geq \frac{n-k+1}{k} + 1$. Therefore,

$$n = |V(G_k)| + \sum_{j=1}^{\frac{k-1}{2}} |V(G_j) \cup V(G_{k-j})| \geq \frac{n-k+1}{k} + 1$$

$$+ \frac{k-1}{2} \cdot \left( \frac{2n-2k+1}{k} + 2 \right)$$

$$= n + \frac{k+1}{2k},$$

a contradiction.

Case 2 There exists some $i \in [k]$ such that $N(v) \not\subseteq V(G_i)$ for every vertex $v \in V(G_i)$.

Case 2.1 There exists only one $i \in [k]$ such that all vertices $v \in V(G_i)$ have $N(v) \not\subseteq V(G_i)$. Observe that $|V(G_i)| = a \leq k - 1$. Notice that every vertex $v \in V(G_i)$ is incident with at least one edge from $B$, and there is a vertex $y \in V(G_i)$ with $\deg(y) \leq a - 1 + \frac{k-1}{a}$. For any component $G_j, j \neq i \in [k]$, there is,

$$|V(G_j)| \geq \left\lceil \frac{2n-2k+1}{k} \right\rceil - \deg(y) + 1 \geq \left\lceil \frac{2n-2k+1}{k} \right\rceil - a + 1 - \frac{k-1}{a} + 1.$$

This means that the number of vertices in $G$ is,

$$n = |V(G)| \geq (k-1) \cdot \left( \left\lceil \frac{2n-2k+1}{k} \right\rceil - a + 1 - \frac{k-1}{a} + 1 \right) + a$$

$$\geq (k-1) \cdot \left( \frac{2n-2k+1}{k} - a + 1 - \frac{k-1}{a} + 1 \right) + a.$$

After some manipulations we get,

$$n \leq \frac{k(k-1)}{k-2} \left( a \cdot \frac{k-2}{k-1} + \frac{k-1}{a} - \frac{1}{k} \right).$$

This, together with the bounds on $a$, provides,

$$n \leq \frac{k(k-1)}{k-2} \left( 1 \cdot \frac{k-2}{k-1} + \frac{k-1}{k} - \frac{1}{k} \right).$$

The inequality yields,

$$n \leq k^2 + k + \frac{1}{k-2}.$$
Next we check whether \( n = k^2 + k \) satisfies the original inequality,

\[
n = |V(G)| \geq (k - 1) \cdot \left( \left\lfloor \frac{2n - 2k + 1}{k} \right\rfloor - a + 1 - \frac{k - 1}{a} + 1 \right) + a.
\]

After some manipulations we get,

\[
k^2 + k \geq k^2 + 2k - 2,
\]

which is impossible. Then we have,

\[
n \leq k^2 + k - 1,
\]
a contradiction.

**Case 2.2** There exists more than one \( i \in [k] \) such that all vertices \( v \in V(G_i) \) have \( N(v) \not\subseteq V(G_i) \). Assume that there exists a pair of non-adjacent vertices \( u, w \) with \( u \in V(G_{i_1}) \) and \( w \in V(G_{i_2}) \). It is possible that \( i_1 = i_2 \). Notice that every vertex in such a component is incident with an edge from \( B \), and the two vertices \( u \) and \( w \) are incident with at most one edge from \( B \) in common, then \( \deg(u) + \deg(w) - 1 \leq k - 1 \).

It implies \( n \leq \frac{k^2 + 2k - 1}{2} \), a contradiction. Now we get that every vertex in such components is adjacent to the remaining vertices of such components. Hence all possible configurations have been excluded except for two adjacent singletons \( \{u\}, \{w\} \) as the only such two components \( V_{i_1}, V_{i_2} \). As \( \deg(u) + \deg(w) - 1 \leq k - 1 \), w.l.o.g., we assume that \( \deg(u) \leq \lfloor \frac{k}{2} \rfloor \). For any component \( G_j, j \neq i_1 \) or \( i_2 \), then

\[
|V(G_j)| \geq \frac{2n - 2k + 1}{k} - \deg(u) + 1 \geq \frac{2n - 2k + 1}{k} - \left\lfloor \frac{k}{2} \right\rfloor + 1.
\]

This means that the number of vertices in \( G \) is

\[
n = |V(G)| \geq (k - 2) \cdot \left( \frac{2n - 2k + 1}{k} - \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) + 2.
\]

After some manipulations we get

\[
n \leq \left\lfloor \frac{k}{2} \right\rfloor \cdot \frac{k(k - 2) + k^2 - 5k + 2}{k - 4},
\]
a contradiction. \( \Box \)

**Remark 2** Observe that the graph \( H_n \) of Theorem 3.2 is a good example showing that the bound on the sum of degrees in Theorem 3.4 is tight.

The next theorem shows that the bound on \( n \) cannot be lower than \( k^2 + k \).

**Theorem 14** For every \( k \geq 5 \) there exists a graph \( D_n \) on \( n = k^2 + k - 1 \) vertices with \( k - 1 \) cut edges and \( \deg(x) + \deg(y) \geq \frac{2n - 2k + 1}{k} \) for any two non-adjacent vertices \( x \) and \( y \).
Proof Let \( D_n \) be a graph consisting of a vertex \( v_0, k - 1 \) blocks \( B_1, \ldots, B_{k-1} \), that are complete graphs on \( k + 2 \) vertices, and a set \( M \) of \( k - 1 \) cut edges joining the vertex of \( v_0 \) with the \( k - 1 \) blocks \( B_1, \ldots, B_{k-1} \). Observe that \( D_n \) is a connected graph on \( k^2 + k - 1 \) vertices such that \( \deg(x) + \deg(y) \geq 2k \geq \frac{2n-2k+1}{k} \) for any two non-adjacent vertices \( x \) and \( y \).

\[ \square \]

4 Degree Conditions for \( cfc(G) = 2 \)

**Theorem 15** Let \( G \) be a connected non-complete graph of order \( n \geq 25 \). If \( C(G) \) induces a linear forest and \( \delta(G) \geq \frac{n-4}{5} \), then \( cfc(G) = 2 \).

**Proof** Observe that, by Theorem 10, the subgraph \( C(G) \) of any connected graph \( G \) with \( \delta(G) \geq \frac{n-4}{5} \) contains at most three cut edges. As \( C(G) \) is a linear forest, we conclude that \( cfc(G) = 2 \) by Theorem 9. \( \square \)

**Remark 3** The graph \( S_n \) defined in the end of Sect. 2 provides a good example showing the tightness of the minimum degree in Theorem 15.

Next, we discuss the minimum degree condition for small graphs to have conflict-free connection number 2.

**Theorem 16** Let \( G \) be a connected non-complete graph of order \( n, 9 \leq n \leq 24 \). If \( C(G) \) induces a linear forest and \( \delta(G) \geq \max\{3, \frac{n-4}{5} \} \), then \( cfc(G) = 2 \).

**Proof** We may assume that \( C(G) \neq \emptyset \) by Lemma 7. Let \( C(G) \) consist of \( k \) components \( Q_1, Q_2, \ldots, Q_k \) with \( n_i = |V(Q_i)| \) such that \( 2 \leq n_1 \leq n_2 \leq \cdots \leq n_k \). We may also assume that \( 3 \leq n_{k-1} \leq n_k \leq 4 \) by Lemma 5 and Theorem 9. Then \( G \setminus (E(Q_{k-1}) \cup E(Q_k)) \) has at least five components \( C_1, C_2, C_3, C_4, C_5 \). Since \( \delta(G) \geq 3 \), it follows that \( |V(C_i)| > 3 \) for \( 1 \leq i \leq 5 \). Notice that at most two vertices in \( C_i \) can be contained in \( Q_{k-1} \cup Q_k \), then for each \( C_i \) there exists a vertex \( u_i \) such that \( N(u_i) \subseteq V(C_i) \) for \( 1 \leq i \leq 5 \). Thus, \( |V(G)| \geq \sum_{i=1}^{5} |V(C_i)| \geq \sum_{i=1}^{5} (d(u_i) + 1) \geq 5(\frac{n-4}{5} + 1) = n + 1 > n \), a contradiction, which completes the proof. \( \square \)

**Remark 4** The following examples show that the minimum degree condition in Theorem 16 is best possible. Let \( H_i \) be a complete graph of order three for \( 1 \leq i \leq 2 \), and take a vertex \( v_i \) of \( H_i \) for \( 1 \leq i \leq 2 \). Let \( H \) be a graph obtained from \( H_1, H_2 \) by connecting \( v_1 \) and \( v_2 \) with a path of order \( t \) for \( t \geq 5 \). Note that \( \delta(H) = 2 \), but \( cfc(H) \geq 3 \). Another graph class is given as follows. Let \( G_i \) be a complete graph of order \( \frac{n}{5} \), and take a vertex \( w_i \) of \( G_i \) for \( 1 \leq i \leq 5 \). Let \( G \) be a graph obtained from \( G_1, G_2, G_3, G_4, G_5 \) by joining \( w_i \) and \( w_{i+1} \) with an edge for \( 1 \leq i \leq 4 \). Notice that \( \delta(G) = \frac{n-5}{5} \), but \( cfc(G) \geq 3 \).

**Theorem 17** Let \( G \) be a connected noncomplete graph of order \( n \) with \( 4 \leq n \leq 8 \). If \( C(G) \) induces a linear forest and \( \delta(G) \geq 2 \), then \( cfc(G) = 2 \).

**Proof** If \( |E(C(G))| \leq 3 \), then the proof follows from Theorem 9. Otherwise the subgraph \( G \setminus E(C(G)) \) has at least five components. Since \( \delta(G) \geq 2 \), at least two components of it have at least three vertices. Thus \( |V(G)| \geq 3 \times 2 + 3 = 9 > 8 \), a contradiction.
Remark 5 The following example shows that the minimum degree condition in Theorem 17 is best possible. Let $G$ be a path of order $t$ with $t \geq 5$. It is easy to see that $\delta(G) = 1$, but $cfc(G) = \lceil \log_2 t \rceil \geq 3$ by Lemma 4.

If we do not require that $C(G)$ is a linear forest in above theorems, then we can get the following theorem.

Theorem 18 Let $G$ be a connected non-complete graph of order $n \geq 16$. If $\delta(G) \geq \frac{n - 3}{4}$, then $cfc(G) = 2$.

Proof Observe that Theorem 10 shows that $C(G)$ of any connected graph $G$ with $\delta(G) \geq \frac{n - 3}{4}$ has at most two edges. This, when applying Theorem 9, immediately gives our theorem.

Remark 6 The following example shows that the minimum degree condition in Theorem 18 is best possible. Let $H_i$ be a complete graph of order $\frac{n}{4}$ for $1 \leq i \leq 4$, and take a vertex $v_i$ of $H_i$ for $1 \leq i \leq 4$. Let $H$ be a graph obtained from $H_1, H_2, H_3, H_4$ by adding the edges $v_1v_2, v_1v_3, v_1v_4$. Note that $\delta(H) = \frac{n - 4}{4}$, but $cfc(H) \geq 3$. On the other hand, the condition $n \geq 16$ in Theorem 18 is also best possible. Let $G_1, G_2, G_3, G_4$ be complete graphs of order 1, 4, 5, 5, respectively, and take a vertex $w_i$ of $G_i$ for $1 \leq i \leq 4$. Let $G$ be a graph obtained from $G_1, G_2, G_3, G_4$ by adding the edges $w_1w_2, w_1w_3, w_1w_4$. Note that $\delta(G) \geq \frac{n - 3}{4}$, but $cfc(G) \geq 3$. Also the graph $R_4$ from Theorem 12 shows the sharpness of the bound of $n$.

Theorem 19 Let $G$ be a connected non-complete graph of order $n \geq 33$. If $C(G)$ is a linear forest, and $\deg(x) + \deg(y) \geq \frac{2n - 9}{3}$ for each pair of two non-adjacent vertices $x$ and $y$ of $V(G)$, then $cfc(G) = 2$.

Proof From Theorem 13 we deduce that the subgraph $C(G)$ of $G$ has at most three edges. Now the proof follows from Theorem 9.

Remark 7 An example of the graph $S_n$, introduced in Remark 1, shows that the degree sum condition in Theorem 19 is best possible. On the other hand, the condition $n \geq 33$ in Theorem 19 is also best possible. Let $G_i$ be a complete graph of order $\frac{n - 2}{3}$ for $1 \leq i \leq 3$ and $n \leq 32$, and $G_4 = v_1u_1u_2v_2v_3$ be a path of order 5. Let $G$ be a graph obtained from $G_1, G_2, G_3, G_4$ by identifying a vertex of $G_i$ to the vertex $v_i$ for $1 \leq i \leq 3$. Note that the resulting graph $G$ satisfies that $\deg(x) + \deg(y) \geq \frac{2n - 9}{3}$ for each pair of two non-adjacent vertices $x$ and $y$ of $V(G)$ and $cfc(G) \geq 3$.

Conclusion In this paper we have shown several sufficient results for graphs $G$ to have conflict-free connection number $cfc(G) = 2$. These are conditions for the minimum degree of $G$ and the minimum degree sum for each pair of nonadjacent vertices in $G$. Moreover, we have proved several results concerning relations between degree conditions on $G$ and the number of cut edges in $G$.

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