Deformations of special Legendrian submanifolds in Sasaki-Einstein manifolds

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Abstract. In this paper, we provide a deformation theory of submanifolds which are characterized by differential forms and show that the moduli space of special Legendrian submanifolds in Sasaki-Einstein manifolds is the intersection of two deformation spaces which are smooth. We also prove that any special Legendrian submanifold admits smooth deformations given by harmonic 1-forms. We can generalize these result to Sasaki manifolds whose metric cones are almost Calabi-Yau manifolds with a certain weight.

1 Introduction

Calabi-Yau manifolds can be studied in many categories, complex geometry, Kähler geometry, algebraic geometry, etc. From the view point of calibrated geometry which is invented by Harvey and Lawson [5], any Calabi-Yau structure induces a calibration given by the real part of a holomorphic volume form. Recall that a calibration is a closed form with comass 1 and a calibrated submanifold is defined by a submanifold in which the calibration equals to a volume form. Special Lagrangian submanifolds in Calabi-Yau manifolds are calibrated submanifolds. McLean provided the deformation theory for calibrated submanifolds [10]. In particular, he showed that the moduli space of compact special Lagrangian submanifolds is smooth and the tangent space is given harmonic 1-forms. An important point of his proof was that any special Lagrangian submanifold is characterized as a submanifold where the imaginary part of the holomorphic volume form and the symplectic form vanish.

In this paper, we consider deformations of submanifolds characterized by differential forms which are not necessarily closed. We assume that \((M, g)\) is a smooth compact Riemannian manifold. We call \(X\) a submanifold in \(M\) if there exists an embedding \(\iota : X \hookrightarrow M\). Let \(X\) be a compact connected submanifold in \(M\). We consider a normal deformation of a submanifold \(X\), that is, an embedding \(f : X \hookrightarrow M\) with a family \(\{f_t\}_{t \in [0, 1]}\) of embeddings \(f_t : X \hookrightarrow M\) such that \(f_0 = \iota\), \(f_1 = f\) and \(\frac{df}{dt} f_t \in \Gamma(NX_t)\) where \(NX_t\) is the normal bundle of \(X_t = f_t(X)\) in \((M, g)\). We call the set of normal deformations of a submanifold \(X\) the moduli space of deformations.

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of $X$. Now we choose a system $\Phi = (\varphi_1, \ldots, \varphi_m) \in \otimes_{i=1}^{m} \Lambda_{i}^{k_i}$ of smooth differential forms on $M$ such that the pull-back $\iota^{*}\Phi$ of $\Phi$ by $\iota : X \to M$ vanishes:

$$\iota^{*}\Phi = (\iota^{*}\varphi_1, \ldots, \iota^{*}\varphi_m) = (0, \ldots, 0).$$

We call a deformation $f$ of $X$ with the family $\{f_t\}_{t \in [0,1]}$ such that $f_t^{*}\Phi = (0, \ldots, 0)$ a $\Phi$-deformation of $X$ and denote by $M_{X}(\Phi)$ the moduli space of $\Phi$-deformations of $X$. We can regard $X$ as an element of $M_{X}(\Phi)$ and denote it by $0_X$. We take a positive integer $s$ and a real number $\alpha$ with $0 < \alpha < 1$ and consider a deformation of $X$ of $C^{s, \alpha}$-class, that is, a $C^{s, \alpha}$-embedding $f : X \to M$ with a family $\{f_t\}_{t \in [0,1]}$ of $C^{s, \alpha}$-embeddings $f_t : X \to M$ such that $f_0 = \iota$, $f_1 = f$, $\frac{df}{dt}f_t \in \Gamma(NX_t)$ and $f_t^{*}\Phi = 0$. Let $M_{X}^{s, \alpha}(\Phi)$ denote the moduli space of $\Phi$-deformations of $X$ of $C^{s, \alpha}$-class.

A Sasaki-Einstein manifold is a $(2n+1)$-dimensional Riemannian manifold $(M, g)$ whose metric cone $(C(M), \overline{g}) = (\mathbb{R}_{>0} \times M, dr^2 + r^2g)$ is a Ricci-flat Kähler manifold where $r$ is the coordinate of $\mathbb{R}_{>0}$. We assume that $M$ is simply connected. Then the cone $C(M)$ is a complex $(n+1)$-dimensional Calabi-Yau manifold which admits a holomorphic $(n+1)$-form $\Omega$ and a Kähler form $\omega$ on $C(M)$ satisfying the Monge-Ampère equation $\Omega \wedge \overline{\Omega} = c_{n+1}\omega^{n+1}$ for a constant $c_{n+1}$. An $n$-dimensional submanifold $X$ in a Sasaki-Einstein manifold $(M, g)$ is a special Legendrian submanifold if the cone $C(X)$ is a special Lagrangian submanifold in $C(M)$. We characterize a Sasaki-Einstein manifold as a Riemannian manifold with a contact 1-form $\eta$ and a complex valued $n$-form $\psi$ such that $(\psi, \frac{1}{2}d\eta)$ is an almost transverse Calabi-Yau structure with $d\psi = (n+1)\sqrt{-1}\eta \wedge \psi$ (See Section 3.2 for the definition of almost transverse Calabi-Yau structures). Then $X$ is a special Legendrian submanifold if and only if $\iota^{*}\psi^{im} = 0$ and $\iota^{*}\eta = 0$ where $\iota$ is the inclusion $\iota : X \to M$ and $\psi^{im}$ is the imaginary part of $\psi$.

Let $X$ be a special Legendrian submanifold and $M_{X}$ the moduli space of special Legendrian deformations of $X$. Then $M_{X}$ is equal to the moduli space $M_{X}(\psi^{im}, \eta)$ of $(\psi^{im}, \eta)$-deformations of $X$. The infinitesimal deformation space of $X$ is given by the eigenspace $\text{Ker}(\Delta_0 - 2(n+1))$ of the Laplace operator $\Delta_0$ on $\Lambda_{X}^{1}$ with the eigenvalue $2(n+1)$. If $M_{X}$ is a smooth manifold, then the infinitesimal deformation space is the tangent space of $M_{X}$ at $0_X$. However, the obstruction for $M_{X}$ to be smooth does not vanish in general. Let $\omega^{T}$ be the 2-form $\frac{1}{2}d\eta$ on $M$. We denote by $\mathcal{N}_{X}$ and $\mathcal{L}_{X}$ the moduli spaces $M_{X}(\psi^{im}, \omega^{T})$ and $M_{X}(\eta)$, respectively. We fix an integer $s \geq 3$ and a real number $\alpha$ with $0 < \alpha < 1$ and denote by $\mathcal{N}_{X}^{s, \alpha}$ and $\mathcal{L}_{X}^{s, \alpha}$ the moduli space $M_{X}^{s, \alpha}(\psi^{im}, \omega^{T})$ and $M_{X}^{s, \alpha}(\eta)$, respectively.

**Theorem 1.1.** The moduli space $M_{X}$ is the intersection $\mathcal{N}_{X} \cap \mathcal{L}_{X}$ where $\mathcal{N}_{X}^{s, \alpha}$ and $\mathcal{L}_{X}^{s, \alpha}$ are smooth.

A deformation $f$ of $X$ is called a transverse $\Phi$-deformation if $f$ is a $\Phi$-deformation of $X$ with the family $\{f_t\}_{t \in [0,1]}$ such that $\frac{df}{dt}f_t \in \Gamma(NX^{T}_t)$ where $NX^{T}_t = NX_t \cap \text{Ker}(\eta)|_{X_t}$. We denote by $\mathcal{N}_{X}^{T}$ the moduli space of transverse $(\psi^{im}, \omega^{T})$-deformations of $X$.

**Theorem 1.2.** The moduli space $\mathcal{N}_{X}^{T}$ is smooth at $0_X$ and the tangent space $T_{0_X}\mathcal{N}_{X}^{T}$ is isomorphic to $H^{1}(X)$.
Let \((M, g)\) be a Sasaki manifold with an almost transverse Calabi-Yau structure \((\psi, \frac{1}{2}d\eta)\) such that \(d\psi = \kappa \sqrt{1-\eta} \wedge \psi\) for a real constant \(\kappa\). In the case \(\kappa = n + 1\), \((M, g)\) is a Sasaki-Einstein manifold. The metric cone \((C(M), \tilde{g})\) is an almost Calabi-Yau manifold which admits a holomorphic \((n + 1)\)-form \(\Omega\) and a Kähler form \(\omega\) satisfying \(\Omega \wedge \overline{\Omega} = r^{2(n+1)} c_{n+1} \omega^{n+1}\) on \(C(M)\). A special Lagrangian submanifold in \(C(M)\) is defined by an \((n + 1)\)-dimensional submanifold where \(\Omega\) and \(\omega\) vanish.

We define a special Legendrian submanifold in \(M\) as an \(n\)-dimensional submanifold \(X\) such that \(C(X)\) is a special Lagrangian submanifold in \(C(M)\). Then \(X\) is a special Legendrian submanifold if and only if \(\iota^* \psi\) and \(\iota^* \eta\) vanish, where \(\iota\) is the inclusion \(\iota : X \hookrightarrow M\).

Let \(X\) be a special Legendrian submanifold and \(\mathcal{M}_X\) the moduli space of special Legendrian deformations of \(X\). Then we can show the similar results of Theorem 1.1 and Theorem 1.2 (see Theorem 5.17 and Theorem 5.18). We say that a special Legendrian submanifold \(X\) is rigid if any special Legendrian deformation of \(X\) is induced by the group \(\text{Aut}(\eta, \psi)\) of diffeomorphisms of \(M\) preserving \(\eta\) and \(\psi\).

**Theorem 1.3.** The infinitesimal deformation space of \(X\) is isomorphic to the space \(\text{Ker}(\Delta_0 - 2\kappa)\). If \(\kappa = 0\), then \(X\) is rigid and \(\mathcal{M}_X\) is a 1-dimensional manifold. If \(\kappa < 0\), then \(X\) does not have non-trivial deformations and \(\mathcal{M}_X = \{0_X\}\).

A special Legendrian submanifold \(X\) is rigid if and only if the special Lagrangian cone \(C(X)\) is rigid, that is, any deformation of \(C(X)\) is induced by the action of the group \(\text{Aut}(\Omega, \omega, r)\) of diffeomorphisms of \(C(M)\) preserving \(\Omega\), \(\omega\) and \(r\). A typical example of Sasaki-Einstein manifolds is the odd-dimensional unit sphere \(S^{2n+1}\) with the standard metric, then the metric cone is the complex space \(\mathbb{C}^{n+1}\setminus\{0\}\). Joyce introduced the rigidity of special Lagrangian cones in \(\mathbb{C}^{n+1}\setminus\{0\}\) [9]. There exist some rigid special Lagrangian cones in \(\mathbb{C}^{n+1}\setminus\{0\}\) and the corresponding rigid special Legendrian submanifolds in \(S^{2n+1}\) [6, 8, 15]. Special Legendrian submanifolds have also the aspect of minimal Legendrian submanifolds. We call that a minimal Legendrian submanifold is rigid if any minimal Legendrian deformation of \(X\) is induced by the group \(\text{Aut}(\eta, g)\) of diffeomorphisms of \(M\) preserving \(\eta\) and \(g\). Hence there exist two rigidity conditions for special Legendrian submanifolds. We show that these conditions are equivalent:

**Theorem 1.4.** Let \(X\) be a special Legendrian submanifold. If \(\kappa > 0\), then \(X\) is rigid as a special Legendrian submanifold if and only if it is so as a minimal Legendrian submanifold.

This paper is organized as follows. In Section 2, we provide a \(\Phi\)-deformation theory of submanifolds and see some examples of \(\Phi\)-deformations. In Section 3, we introduce almost transverse Calabi-Yau structures and prove that any Sasaki-Einstein manifold is characterized by such a structure. In Section 4, we show Theorem 1.1 and Theorem 1.2. In the last section, we introduce special Legendrian submanifolds in Sasaki manifolds with almost transverse Calabi-Yau structures and prove Theorem 1.3 and a generalization of Theorem 1.1 and Theorem 1.2. We study minimal Legendrian deformations of special Legendrian submanifolds and show Theorem 1.4.
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2 Deformations of submanifolds

In this section, we assume that \((M, g)\) is a smooth Riemannian manifold. We provide a criterion for the smoothness of moduli spaces of submanifolds and apply it to some examples.

2.1 Smoothness of moduli spaces of \(\Phi\)-deformations

Let \(X\) be a compact submanifold in \(M\) with an embedding \(\iota : X \hookrightarrow M\). We choose a system \(\Phi = (\varphi_1, \ldots, \varphi_m) \in \bigoplus_{i=1}^m \wedge^k M\) of smooth differential forms on \(M\) such that the pull-back \(\iota^* \Phi\) of \(\Phi\) by \(\iota : X \to M\) vanishes:

\[\iota^* \Phi = (\iota^* \varphi_1, \ldots, \iota^* \varphi_m) = (0, \ldots, 0)\]

We call a deformation \(f\) of \(X\) with the family \(\{f_t\}_{t \in [0,1]}\) such that \(f_t^* \Phi = 0\) a \(\Phi\)-deformation of \(X\) and denote by \(\mathcal{M}_X(\Phi)\) the moduli space of \(\Phi\)-deformations of \(X\). It follows from the tubular neighbourhood theorem that there exists a neighbourhood of \(X\) in \(M\) which is identified with a neighbourhood \(U\) of the zero section of \(NX\) by the exponential map. We define \(V_1\) and \(V_2\) as the vector spaces

\[V_1 = C^\infty(NX),\]
\[V_2 = C^\infty(\bigoplus_{i=1}^m \wedge^{k_i})\]

where \(\wedge^{k_i}\) is the space of \(k_i\)-differential forms on \(X\). We denote by \(U\) the set \(\{v \in V_1 | v_x \in U, x \in X\}\):

\[U = \{v \in V_1 | v_x \in U, x \in X\}. \tag{1}\]

The exponential map induces the embedding \(\exp_v : X \hookrightarrow M\) for each \(v \in U\). Then we define the map \(F : U \to V_2\) by

\[F(v) = \exp_v^* \Phi = (\exp_v^* \varphi_1, \ldots, \exp_v^* \varphi_m)\]

for any \(v \in U\). We can consider \(X\) as an element of \(\mathcal{M}_X(\Phi)\) since \(\iota : X \to M\) can be the trivial deformation of \(X\). Hence we denote by \(0_X\) the element \(X\) of \(\mathcal{M}_X(\Phi)\). If the inverse image \(F^{-1}(0)\) of the origin of \(V_2\) is smooth at \(0_X\), then \(F^{-1}(0)\) is identified with a neighbourhood of \(0_X\) in \(\mathcal{M}_X(\Phi)\) in the \(C^1\) sense. Let \(D_1\) be the linearization of \(F\) at 0:

\[D_1 = d_0F : V_1 \to V_2.\]

Then the infinitesimal deformation space of \(X\) is given by \(\ker D_1\). Let \(D_1^*\) be the formal adjoint operator of \(D_1\).

**Lemma 2.1.** The equation \(D_1^* \circ F(v) = 0\) is a partial differential equation whose order is at most two, and quasi-linear if the order is two.

**Proof.** We assume that the dimension of \(X\) is \(n\) and the rank of \(NX\) is \(\ell\). Let \(U_X\) be an open set in \(X\) and \((x_1, \ldots, x_n)\) a coordinate of \(U_X\). Taking a local frame \(\{v_1, \ldots, v_\ell\}\) of \(NX\) over \(U_X\), then we have a trivialization \(U_X \times \mathbb{R}^\ell\) of \(NX|_{U_X}\) by
the correspondence of \((x, y_1, \ldots, y_\ell) \in U_x \times \mathbb{R}^\ell\) to \(\sum_{j=1}^{\ell} y_j v_j(x) \in NX|_{U_x}\). By the exponential map, the tubular neighbourhood \(U \cap NX|_{U_x}\) of \(NX|_{U_x}\) is isomorphic to an open set \(U_M\) in \(M\), and \((U_M, x_1, \ldots, x_n, y_1, \ldots, y_\ell)\) is a local coordinate of \(M\). Let \(v\) be an element of the set \(U\) of \([1, 2, \ldots]\). If \(v\) is expressed by \(v = \sum_{j=1}^{\ell} f_j v_j\) on \(U_M\), then the moduli space theorem implies that there exist a neighbourhood of \(D_1\) such that \(P \circ F(v) = 0\) is a partial differential equation whose order is at most one. Hence the order of the equation \(D_1^* \circ F(v) = 0\) is at most two. The first order term of \(F(v) = 0\) may contain

\[
(\frac{\partial f_1}{\partial x_1})^{m_{11}} (\frac{\partial f_1}{\partial x_2})^{m_{12}} \cdots (\frac{\partial f_1}{\partial x_n})^{m_{1n}} (\frac{\partial f_2}{\partial x_1})^{m_{21}} \cdots (\frac{\partial f_2}{\partial x_n})^{m_{2n}} (\frac{\partial f_3}{\partial x_1})^{m_{31}} \cdots (\frac{\partial f_3}{\partial x_n})^{m_{3n}} \cdots (\frac{\partial f_\ell}{\partial x_1})^{m_{\ell1}} \cdots (\frac{\partial f_\ell}{\partial x_n})^{m_{\ell n}}
\]

for integers \(m_{11}, m_{12}, \ldots, m_{\ell n}\) with \(m_{ij} = 0 \text{ or } 1\) such that \(0 \leq m_{1i} + \cdots + m_{\ell i} \leq 1\) for each \(i\). If the order of \(D_1^* \circ F(v) = 0\) is two, then the second order term is generated by the partial derivative of \(\Phi\), and hence \(D_1^* \circ F(v) = 0\) is quasi-linear.

We fix an integer \(s \geq 3\) and a real number \(\alpha\) with \(0 < \alpha < 1\). Then we set the Banach spaces

\[
V_1^{s, \alpha} = C^{s, \alpha}(NX), \\
V_2^{s-1, \alpha} = C^{s-1, \alpha}(\mathbb{R}^{m_{\ell n}}_1)
\]

with respect to the Hölder norm \(\|\cdot\|_{C^{s, \alpha}}\) induced by the Riemannian metric \(\iota^* g\) on \(X\). We define \(U_1^{s, \alpha}\) as the set \(\{v \in V_1^{s, \alpha} | v_x \in U, \ x \in X\}\). Then we can extend the map \(F\) to the smooth map \(F^{s, \alpha} : U_1^{s, \alpha} \to V_2^{s-1, \alpha}\). We define \(D_1^{s, \alpha}\) as the linearization \(d_0 F^{s, \alpha}\) of \(F^{s, \alpha}\). Let \(M_X^{s, \alpha}(\Phi)\) denote the moduli space of \(\Phi\)-deformations of \(C^{s, \alpha}\)-class.

**Proposition 2.2.** Suppose that there exist a vector space \(V_3\) of smooth sections of a vector bundle \(E\) on \(X\) and a differential operator \(D_2\) with the differential complex

\[
0 \to V_1 \overset{D_1}{\to} V_2 \overset{D_2}{\to} V_3 \to 0.
\]

Let \(D_2^*\) be a formal adjoint operator of \(D_2\). If \(P_2 = D_1 \circ D_1^* + D_2 \circ D_2^*\) is elliptic and \(\text{Im}(F) \subset \text{Im}(D_1)\), then the moduli space \(M_X^{s, \alpha}(\Phi)\) is smooth at \(0_X\) and the tangent space \(T_{0_X} M_X^{s, \alpha}(\Phi)\) is given by \(\text{Ker} D_1^{s, \alpha}\). Moreover, if \(P_1 = D_1^* \circ D_1\) is also elliptic, then the moduli space \(M_X(\Phi)\) is smooth at \(0_X\) and the tangent space \(T_{0_X} M_X(\Phi)\) is given by \(\text{Ker}(D_1)\).

**Proof.** If \(P_2\) is elliptic, then the map \(D_1^{s, \alpha} : V_1^{s, \alpha} \to V_2^{s-1, \alpha}\) has a closed image \(\text{Im}(D_1^{s, \alpha})\) and there exists a right inverse of the map \(D_1^{s, \alpha} : V_1^{s, \alpha} \to \text{Im}(D_1^{s, \alpha})\). It follows from the assumption \(\text{Im}(F) \subset \text{Im}(D_1)\) that \(\text{Im}(F^{s, \alpha}) \subset \text{Im}(D_1^{s, \alpha})\). Then we obtain the smooth map \(F^{s, \alpha} : U_1^{s, \alpha} \to \text{Im}(D_1^{s, \alpha})\) such that the derivative \(d_0 F : V_1^{s, \alpha} \to \text{Im}(D_1^{s, \alpha})\) has the right inverse \(D_1^* \circ G\) where \(G\) is Green’s operator of \(P_2\). The implicit function theorem implies that there exist a neighbourhood \(W\) of \(0\) in \(\text{Ker}(D_1^{s, \alpha})\) and a smooth map \(\varphi : W \to W^\perp\) where \(W^\perp\) is the orthogonal complement of \(W\) in \(V_1^{s, \alpha}\) such that \(\varphi(0) = 0\) and \(F(u, \varphi(u)) = 0\). Hence, \((F^{s, \alpha})^{-1}(0)\) has a manifold structure.
at $0_X$ whose tangent space is $\text{Ker}(D_1^{s,\alpha})$. Moreover, if $P_1$ is elliptic, then $\text{Ker}(D_1^{s,\alpha})$ coincides with the finite dimensional vector space $\text{Ker}(D_1)$ by the elliptic regularity method. We will show that the map $\varphi$ provides a manifold structure of $F^{-1}(0)$ at $0_X$. If we take the Taylor expansion

$$F(v) = D_1(v) + R(v)$$

of $F$ at $0$ for $v \in V_1^{s,\alpha}$ where $R(v)$ is the higher term with respect to $v$, then $D_1^s \circ F(v)$ is given by

$$D_1^s \circ F(v) = P_1(v) + D_1^s(R(v))$$

and $\frac{D_1^s(R(v))}{\|v\|} \to 0$ as $v \to 0$ in $V_1^{s,\alpha}$. Hence $D_1^s \circ F(v) = 0$ is a second order elliptic partial differential equation for any sufficiently small $v \in V_1^{s,\alpha}$. It follows form Lemma 2.1 that $D_1^s \circ F(v) = 0$ is quasi-linear elliptic, and the solution $v$ is $C^\infty$-class by Morrey’s elliptic regularity results [12]. Now an element $(u, \varphi(u))$ of $V_1^{s,\alpha}$ for $u \in W$ is a solution of $D_1^s \circ F(v) = 0$. Hence there exists a sufficiently small $\varepsilon > 0$ such that $\varphi(u)$ is $C^\infty$-class for any $u \in W$ with $\|u\|_{C^{s,\alpha}} < \varepsilon$ since $\frac{D_1^s(R(u, \varphi(u)))}{\|u\|} \to 0$ as $u \to 0$ in $W$. Thus the map $\varphi$ provides a manifold structure of $F^{-1}(0)$ at $0_X$ whose tangent space is $\text{Ker}(D_1)$, and we finish the proof. \hfill $\Box$

Remark 2.3. If we consider any $\Phi$-deformation which is not necessarily a normal deformation, that is, an embedding $f : X \hookrightarrow M$ with a family $\{f_t\}_{t \in [0, 1]}$ of embeddings such that $f_0 = \iota$, $f_1 = f$ and $f_t^* \Phi = 0$, and denote by $\tilde{M}_X(\Phi)$ a set of such $\Phi$-deformations of $X$. Then the identity component $\text{Diff}_0(X)$ of the diffeomorphism group of $X$ acts on $\tilde{M}_X(\Phi)$. We take a sufficiently small number $\varepsilon$ as in the proof of Proposition 2.2 and define $S_\varepsilon$ by the set $\{u \in \text{Ker}(D_1) \mid \|u\|_{C^{s,\alpha}} < \varepsilon\}$. Then we can consider $S_\varepsilon$ as a slice for the action of $\text{Diff}_0(X)$ on the space $\tilde{M}_X(\Phi)$.

### 2.2 Examples of $\Phi$-deformations

In this section, we see some examples of $\Phi$-deformations and the moduli spaces.

#### 2.2.1 Special Lagrangian submanifolds in Calabi-Yau manifolds

We assume that $(M, \Omega, \omega)$ is a Calabi-Yau manifold of dimension $2n$ where $(\Omega, \omega)$ is a Calabi-Yau structure on $M$, that is, $\Omega$ is a holomorphic $n$-form and $\omega$ is a Kähler form satisfying the equation $\Omega \wedge \overline{\Omega} = c_n \omega^n$ for $c_n = \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}(\frac{2}{\sqrt{-1}})^n$. We call $X$ a special Lagrangian submanifold in $M$ if $X$ is the calibrated submanifold with respect to the real part $\Omega^\text{Re}$ of $\Omega$. It is well known that $X$ is a special Lagrangian submanifold if and only if $X$ is an $n$-dimensional submanifold such that $\iota^* \Omega^\text{Im} = \iota^* \omega = 0$ where $\iota$ is the inclusion $\iota : X \hookrightarrow M$. Hence special Lagrangian deformations are $(\Omega^\text{Im}, \omega)$-deformations and the moduli space $\mathcal{M}_X$ of special Lagrangian deformations is $\mathcal{M}_X(\Omega^\text{Im}, \omega)$. The following result is provided by McLean:

**Proposition 2.4.** (Theorem 3.6 [10]) Let $X$ be a compact special Lagrangian submanifold in $M$. Then $\mathcal{M}_X$ is a smooth manifold of dimension $\dim(H^1(X))$.  

Proof. We take the set $U$ as in (1) and define the map $F: U \to \Lambda^0 \oplus \Lambda^2$ by

$$F(v) = (\exp_v \Omega^{\text{Im}}, \exp_v \omega)$$

for $v \in U$ where $\ast$ means the Hodge star operator with respect to the metric $\ast g$ on $X$. Then we can regard $F^{-1}(0)$ as a set of special Lagrangian submanifolds in $M$ which is near $X$ in $C^1$ topology. To see the infinitesimal deformation of $X$, we consider the linearization $d_0 F$ of $F$ at the origin $0 \in U$. It follows that

$$d_0 F(v) = (\ast t^* L_{\bar{v}} \Omega^{\text{Im}}, \ast t^* L_{\bar{v}} \omega)$$

$$= (d \ast t^* (i_{\bar{v}} \Omega^{\text{Im}}), d \ast (i_{\bar{v}} \omega))$$

$$= (d \ast t^* (i_{\bar{v}} \omega), d \ast (i_{\bar{v}} \omega))$$

for $v \in U$ where $\bar{v}$ is an extension of $v$ to $M$. In the last equation, we use that $\ast (i_{\bar{v}} \Omega^{\text{Im}}) = - \ast (i_{\bar{v}} \omega)$. We remark that the differential form $\ast (i_{\bar{v}} \omega) = i_{\bar{v}} \omega$ on $X$ is independent of any extension of $v$. Under the identification $NX \simeq \Lambda^1$ given by $v \mapsto i_v \omega$, we identify $d_0 F$ with the map $D_1: \Lambda^1 \to \Lambda^0 \oplus \Lambda^2$ defined by

$$D_1(\alpha) = (d^* \alpha, d\alpha)$$

for $\alpha \in \Lambda^1$. Then it turns out that

$$\text{Ker}(D_1) = \{ \alpha \in \Lambda^1 \mid d^* \alpha = d\alpha = 0 \} = \mathcal{H}^1(X)$$

where $\mathcal{H}^k(X)$ means the set of harmonic $k$-forms on $X$. We provide the differential complex

$$0 \to \Lambda^1 \xrightarrow{D_1} \Lambda^0 \oplus \Lambda^2 \xrightarrow{D_2} \Lambda^3 \to 0$$

where the operator $D_2$ is given by

$$D_2(f, \beta) = d\beta$$

for $(f, \beta) \in \Lambda^0 \oplus \Lambda^2$. Since the dual operators $D_1^*$ and $D_2^*$ are given by $D_1^*(f, \beta) = df + d^* \beta$ and $D_2^*(\gamma) = (0, d^* \gamma)$, we have

$$P_1(\alpha) = \Delta_1 \alpha,$$

$$P_2(f, \beta) = (\Delta_0 f, \Delta_2 \beta)$$

where $\Delta_i$ is the ordinary Laplace operator on $\Lambda^i$ for $i = 0, 1$ and 2. Thus $P_1$ and $P_2$ are elliptic. The image $\text{Im}(F)$ of the map $F$ is included in $d^* \Lambda^1 \oplus d\Lambda^1$ since $[f_i^* \Omega^{\text{Im}}] = [\ast \Omega^{\text{Im}}] = 0$ and $[f_i^* \omega] = [\ast \omega] = 0$. It is clear that $d^* \Lambda^1 \oplus d\Lambda^1$ is perpendicular to $\text{Ker} P_2 = \mathcal{H}^0(X) \oplus \mathcal{H}^2(X)$ and $\text{Im}(D_2^*) = \{0\} \oplus d^* \Lambda^3$. Therefore $\text{Im}(F)$ is also perpendicular to $\text{Ker} P_2 \oplus \text{Im}(D_2^*)$ and we obtain $\text{Im}(F) \subset \text{Im}(D_1)$ by the Hodge decomposition $\Lambda^0 \oplus \Lambda^2 = \text{Ker} P_2 \oplus \text{Im}(D_1) \oplus \text{Im}(D_2^*)$. It follows from Proposition 2.7 that $\mathcal{M}_X$ is smooth at $0_X$. We can show that $\mathcal{M}_X$ is smooth at any point by repeating the argument for each special Lagrangian submanifold. Hence we finish the proof. 

$\blacksquare$
2.2.2 Coassociative submanifolds in $G_2$ manifolds

We assume that $(M, g, \varphi)$ is a $G_2$ manifold where $\varphi$ is an associative 3-form on $M$. We call $X$ a coassociative submanifold in $M$ if $X$ is calibrated submanifold with respect to the Hodge dual $\ast \varphi$ of $\varphi$ where $\ast$ is the Hodge star operator with respect to the metric $g$ on $M$. An $n$-dimensional submanifold $\iota : X \hookrightarrow M$ is a coassociative submanifold if and only if $\iota^* \varphi = 0$. Hence coassociative deformations are $\varphi$-deformations and the moduli space $M_X$ of coassociative deformations is $M_X(\varphi)$.

**Proposition 2.5.** (Theorem 4.5. [10]) Let $X$ be a compact coassociative submanifold in $M$. Then $M_X$ is a smooth manifold of dimension $\dim(H^2(X))$.

**Proof.** We take the set $U$ as in (1) and define the map $F : U \to \wedge^3$ by

$$F(v) = \exp^*_v \varphi$$

for $v \in U$, then we can regard $F^{-1}(0)$ as a set of coassociative submanifolds in $M$ which is near to $X$. To see the first order deformation of $X$, we consider the linearization $d_0 F$ of $F$ at the origin $0 \in U$. It follows that for $v \in U$

$$d_0 F(v) = \iota^* L_v \varphi = d_0^*(i_v \varphi)$$

where $\tilde{v}$ is an extension of $v$ to $M$. Let $\wedge^2_-$ be the set of anti-self dual 2-forms on $X$. Under the identification $N_X \simeq \wedge^2_-$ given by $v \mapsto i_v \varphi$, we can consider $d_0 F$ as the map $D_1 : \wedge^2_- \to \wedge^3$ given by

$$D_1(\alpha) = d\alpha$$

for $\alpha \in \wedge^2_-$. Then it turns out that

$$\ker(D_1) = \{\alpha \in \wedge^2_- \mid d\alpha = 0\} = H^2_-(X)$$

where $H^2_-(X)$ is the set of harmonic anti-self dual 2-forms on $X$. Now we provide a complex as follows

$$0 \to \wedge^2_- \xrightarrow{D_1} \wedge^3 \xrightarrow{D_2} \wedge^4 \to 0 \quad (\natural)$$

where the operator $D_2$ is given by

$$D_2(\beta) = d\beta$$

for $\beta \in \wedge^3$. It is easy to see that

$$P_1(\alpha) = d^* d\alpha, \quad P_2(\beta) = (d(d^*)_+ + d^* d)\beta$$

where $(d^*)_+$ is the composition $p_+ \circ d^*$ of $d^*$ and the projection $p : \wedge^2 \to \wedge^2_-$. The complex $[\natural]$ is isomorphic to the elliptic complex

$$0 \to \wedge^2_- \xrightarrow{d^*} \wedge^1 \xrightarrow{d^*} \wedge^0 \to 0$$

by the Hodge star operator with respect to the metric $\iota^* g$ on $X$. Hence the operators $P_1$ and $P_2$ are elliptic. It follows that the image $\text{Im}(F)$ is included in $\text{Im}(D_1) = d\wedge^2_-$ from $f^*_i \varphi \in d\wedge^2$ and $d\wedge^2 = d\wedge^2_-$. Proposition $[\natural\natural]$ implies that $M_X$ is smooth at $0_X$. We can show that $M_X$ is smooth at any element by repeating the argument for each coassociative submanifold. Hence we finish the proof. \hfill \Box
2.2.3 Special Legendrian submanifolds in contact Calabi-Yau manifolds

Let $M$ be a $(2n+1)$-dimensional manifold. A pair $(\psi, \eta)$ of two differential forms is called a contact Calabi-Yau structure on $M$ if $\eta$ is a contact 1-form and $\psi$ is a $d$-closed complex valued $n$-form on $M$ such that $(\psi, \frac{1}{2}d\eta)$ is an almost transverse Calabi-Yau structure with respect to the Reeb foliation (we refer to Section 3.2 for the definition of almost transverse Calabi-Yau structures). On a contact Calabi-Yau manifold $(M, \psi, \eta)$, a calibrated submanifold $X$ with respect to the calibration $\psi^\text{Re}$ is called a special Legendrian submanifold. Then $X$ is a special Legendrian submanifold if and only if $\iota^*\psi^\text{Im} = \iota^*\eta = 0$. Hence special Legendrian deformations are $(\psi^\text{Im}, \eta)$-deformations and the moduli space $\mathcal{M}_X$ of special Legendrian deformations is $\mathcal{M}_X(\psi^\text{Im}, \eta)$. Tomassini and Vezzoni showed the following result:

**Proposition 2.6. (Theorem 4.5. [19])** Let $X$ be a compact special Legendrian submanifold in $M$. Then $\mathcal{M}_X$ is a smooth manifold of dimension $\dim(H^0(X))$.

**Proof.** We remark the moduli space $\mathcal{M}_X$ is $\mathcal{M}_X(\psi^\text{Im}, \eta, \frac{1}{2}d\eta)$ since $\iota^*\eta = 0$ is equal to $\iota^*\eta = \iota^*(\frac{1}{2}d\eta) = 0$. We take the set $U$ as in [11] and define the map $F : U \to \wedge^0 \oplus \wedge^1 \oplus \wedge^2$ by

$$F(v) = (\ast v^* \psi^\text{Im}, \exp^v \eta, \exp^v \frac{1}{2}d\eta)$$

for $v \in U$, then we can regard $F^{-1}(0)$ as a set of special Legendrian submanifolds in $M$ which is near $X$. It follows that for $v \in U$

$$d_0 F (v) = (\ast \iota^* L_{\tilde{v}}\psi^\text{Im}, \iota^* L_{\tilde{v}} \eta, \frac{1}{2} \iota^* L_{\tilde{v}} d\eta)$$

$$= (\ast dt^* (i_{\tilde{v}} \psi^\text{Im}), dt^* (i_{\tilde{v}} \eta + i_{\tilde{v}} d\eta), \frac{1}{2} dt^* (i_{\tilde{v}} d\eta))$$

$$= \left( \frac{1}{2} dt^* (i_{\tilde{v}} d\eta), dt^* (i_{\tilde{v}} \eta) + \iota^* (i_{\tilde{v}} d\eta), \frac{1}{2} dt^* (i_{\tilde{v}} d\eta) \right)$$

where $\tilde{v}$ is an extension of $v$ to $M$. In the last equation, we use that $\iota^* (i_{\tilde{v}} \psi^\text{Im}) = -\frac{1}{2} \iota^* (i_{\tilde{v}} d\eta)$. There exists the identification $NX \simeq \wedge^0 \oplus \wedge^1$ given by $v \mapsto (i_v \eta, \frac{1}{2} i_v d\eta)$. Under the above identification, we can consider $d_0 F$ as the map $D_1 : \wedge^0 \oplus \wedge^1 \to \wedge^0 \oplus \wedge^1 \oplus \wedge^2$ given by

$$D_1 (f, \alpha) = (d^* \alpha, df + 2\alpha, d\alpha)$$

for $(f, \alpha) \in \wedge^0 \oplus \wedge^1$. Then it turns out that

$$\text{Ker}(D_1) = \{(f, \alpha) \in \wedge^0 \oplus \wedge^1 \mid d^* \alpha = df + 2\alpha = 0\}$$

$$= \{(f, -\frac{1}{2} df) \in \wedge^0 \oplus \wedge^1 \mid \Delta_0 f = 0\}$$

$$= \{(f, 0) \in \wedge^0 \oplus \wedge^1 \mid f \in H^0(X)\} \simeq H^0(X).$$

Now we provide a complex as follows

$$0 \to \wedge^0 \oplus \wedge^1 \xrightarrow{D_1} \wedge^0 \oplus \wedge^1 \oplus \wedge^2 \xrightarrow{D_2} \wedge^2 \oplus \wedge^3 \to 0$$
where the operator $D_2$ is given by
\[
D_2(f, \alpha, \beta) = (d\alpha - 2\beta, d\beta)
\]
for $(f, \alpha, \beta) \in \wedge^0 \oplus \wedge^1 \oplus \wedge^2$. Since $D_1(f, \alpha, \beta) = (d^*\alpha, df + 2\alpha + d^*\beta)$ and $D_2(\beta, \gamma) = (0, d^*\beta, -2\beta + d^*\gamma)$, we have
\[
P_1(f, \alpha) = (\Delta_0 f + 2d^*\alpha, (\Delta_1 + 4)\alpha + 2df),
\]
\[
P_2(f, \alpha, \beta) = (\Delta_0 f + 2d^*\alpha, (\Delta_1 + 4)\alpha + 2df, (\Delta_2 + 4)\beta).
\]
Hence $P_1$ and $P_2$ are elliptic. The image $\text{Im}(F)$ of the map $F$ is included in
\[
\{(d^*h, \alpha, \frac{1}{2}d\alpha) \in \wedge^0 \oplus \wedge^1 \oplus \wedge^2 \mid h \in \wedge^0, \alpha \in \wedge^1\}
\]
which is perpendicular to the kernel
\[
\text{Ker } D_1^* = \{(f, -\frac{1}{2}d^*\beta, \beta) \in \wedge^0 \oplus \wedge^1 \oplus \wedge^2 \mid f \in H^0(X)\}
\]
of the operator $D_1^*$. It follows from $\text{Ker } D_1^* = \text{Ker } P_2 \oplus \text{Im}(D_2^*)$ that $\text{Im}(F) \perp \text{Ker } P_2 \oplus \text{Im}(D_2^*)$. Hence we obtain $\text{Im}(F) \subseteq \text{Im}(D_1)$ by the Hodge decomposition $\wedge^0 \oplus \wedge^2 = \text{Ker } P_2 \oplus \text{Im}(D_1) \oplus \text{Im}(D_2^*)$. It follows from Proposition 2.2 that $\mathcal{M}_X$ is smooth at $0_X$. We can show that $\mathcal{M}_X$ is smooth by repeating the argument for any special Legendrian submanifold. Hence we finish the proof. \qed

2.2.4 Legendrian submanifolds in contact manifolds

Let $(M, \eta)$ be a $(2n + 1)$-dimensional contact manifold with a contact 1-form $\eta$. A Legendrian submanifold is defined by a submanifold $\iota : X \hookrightarrow M$ such that $\iota^*\eta = 0$. Hence the space $\mathcal{M}_X(\eta)$ is the moduli space $\mathcal{M}_X$ of Legendrian deformations of $X$. The following result is well known as a consequence of the Darboux-Weinstein’s neighborhood theorem for Legendrian submanifolds in contact geometry.

**Proposition 2.7.** Let $X$ be a compact Legendrian submanifold in $M$. Then $\mathcal{M}_X^{s,\alpha}$ is a smooth manifold. The tangent space $T_0 \mathcal{M}_X^{s,\alpha}$ is isomorphic to the graph $\{(f, df) \in C^{s,\alpha}(\wedge^0 \oplus \wedge^1)\}$ of the exterior derivative $d$.

**Proof.** We remark that $\mathcal{M}_X = \mathcal{M}_X(\eta, \frac{1}{2}d\eta)$ since $\iota^*\iota^*\eta = 0$ is equal to $\iota^*d\eta = \iota^*\frac{1}{2}d\eta = 0$.

We take the set $U$ as in (1) and define the map $F : U \rightarrow \wedge^1 \oplus \wedge^2$ by
\[
F(v) = (\exp^*_v \eta, \frac{1}{2} \exp^*_v d\eta)
\]
for $v \in U$. It follows that
\[
d_0 F(v) = (d\iota^*(i_v \eta) + \iota^*(i_\tilde{v} d\eta), \frac{1}{2} d\iota^*(i_\tilde{v} d\eta))
\]
for $v \in U$ where $\tilde{v}$ is an extension of $v$ to $M$. Under the identification $\mathcal{N}X \approx \wedge^0 \oplus \wedge^1$ given by $v \mapsto (i_v \eta, \frac{1}{2} i_v d\eta)$, we identify $d_0 F$ with the map $D_1 : \wedge^0 \oplus \wedge^1 \rightarrow \wedge^1 \oplus \wedge^2$ defined by
\[
D_1(f, \alpha) = (df + 2\alpha, d\alpha)
\]
for $(f, \alpha) \in \wedge^0 \oplus \wedge^1$. Then it turns out that
\[
\text{Ker}(D_1) = \{(f, \alpha) \in \wedge^0 \oplus \wedge^1 \mid df + 2\alpha = 0\}
= \{(f, -\frac{1}{2}df) \in \wedge^0 \oplus \wedge^1\}.
\]
Now we provide a complex as follows
\[
0 \to \wedge^0 \oplus \wedge^1 \xrightarrow{D_1} \wedge^1 \oplus \wedge^2 \xrightarrow{D_2} \wedge^2 \oplus \wedge^3 \to 0
\]
where the operator $D_2$ is given by
\[
D_2(\alpha, \beta) = (d\alpha - 2\beta, d\beta)
\]
for $(\alpha, \beta) \in \wedge^1 \oplus \wedge^2$. It is easy to see that
\[
P_1(f, \alpha) = (\Delta_0 f + 2d^* \alpha, (dd^* + 2)\alpha + df),
\]
\[
P_2(\alpha, \beta) = ((\Delta_0 + 4)\alpha, (\Delta_2 + 4)\beta).
\]
Hence $P_2$ is the elliptic operator with $\text{Ker} P_2 = \{0\} \oplus \{0\}$. The space $\text{Im}(F)$ is perpendicular to $\text{Im}(D^*_2)$ since $\text{Im}(F) \subset \text{Ker } (D_2)$. Hence we obtain $\text{Im}(F) \subset \text{Im}(D_1)$ by the Hodge decomposition $\wedge^0 \oplus \wedge^1 = \text{Im}(D_1) \oplus \text{Im}(D^*_2)$. Proposition 2.2 implies that $\mathcal{M}_X^{s, \alpha}$ is smooth at $0_X$ with the tangent space $\text{Ker}(D^*_1 \alpha)$. We can show that $\mathcal{M}_X^{s, \alpha}$ is smooth by repeating the argument for any Legendrian submanifold. Hence we finish the proof.

3 Sasaki-Einstein manifolds

In this section, we assume that $(M, g)$ is a smooth Riemannian manifold of dimension $(2n + 1)$.

3.1 Transverse differential forms

Let $\mathcal{F}$ be a foliation on $M$ of codimension $2n$ and $F$ the vector bundle induced by the foliation $\mathcal{F}$. A differential form $\varphi$ on $M$ is called transverse if
\[
i_v \varphi = 0
\]
for any $v \in \Gamma(F)$. We denote by $\wedge^k_T$ the sheaf of transverse $k$-forms on the foliated manifold $(M, \mathcal{F})$. A transverse differential $k$-form can be considered as the section of $\wedge^k Q^*$ where $Q$ is the quotient vector space $TM/F$. A differential form $\varphi$ on $M$ is called basic if
\[
i_v \varphi = 0, \quad L_v \varphi = 0
\]
for any $v \in \Gamma(F)$. Let $\wedge^k_B$ be the sheaf of basic $k$-forms on the foliated manifold $(M, \mathcal{F})$. It is easy to see that for a basic form $\varphi$ the derivative $d\varphi$ is also basic. Thus the exterior derivative $d$ induces the operator
\[
d_B = d|_{\wedge^k_B} : \wedge^k_B \to \wedge^{k+1}_B
\]
by the restriction. The corresponding complex \((\wedge^*_B, d_B)\) associates the cohomology group \(H^*_B(M)\) which is called the basic de Rham cohomology group. In general, the derivative \(d\varphi\) of a transverse form \(\varphi\) is not necessarily transverse. In fact, a transverse form \(\varphi\) is basic if \(d\varphi\) is transverse. On the space \(\wedge^k_T\), there exists an orthogonal decomposition \(d\wedge^k_T = \wedge^{k+1}T + \wedge^k_T \wedge \wedge^*\) with respect to the metric \(g\). Let \(\pi_T\) denote the first projection from \(\wedge^{k+1}_T \oplus \wedge^k_T \wedge \wedge^*\) to \(\wedge^{k+1}_T\). We define a map

\[
d_T : \wedge^k_T \to \wedge^{k+1}_T
\]

by the composition \(\pi_T \circ d|_{\wedge^k_T}\) of \(\pi_T\) and the restriction \(d|_{\wedge^k_T}\) of \(d\) to \(\wedge^k_T\). Then \(d_T\varphi = d_B\varphi\) for a basic form \(\varphi\).

If there exists a complex structure \(J\) of \(Q\), then we have a decomposition

\[
\wedge^k_T = \oplus_{p+q=k} \wedge^{p,q}_T
\]

where \(\wedge^{p,q}_T\) is the set of transverse \((p, q)\)-forms on \((M, F)\). Moreover, if \(J\) is a transverse complex structure on \((M, F)\) (see the next section for the definition), then it gives rise to a decomposition \(\wedge^k_B \otimes \mathbb{C} = \oplus_{p+q=k} \wedge^{p,q}_B\) and operators

\[
\partial_B : \wedge^{p,q}_B \to \wedge^{p+1,q}_B,
\quad
\overline{\partial}_B : \wedge^{p,q}_B \to \wedge^{p,q+1}_B
\]

in the same manner as complex geometry. We denote by \(H^{p,q}_B(M)\) the cohomology of the complex \((\wedge^{p,q}_B, \overline{\partial}_B)\) which is called the basic Dolbeault cohomology group. On the space \(\wedge^{p,q}_T\), we have an orthogonal decomposition \(d\wedge^{p,q}_T = \wedge^{p+1,q}_T + \wedge^{p,q+1}_T + \wedge^{p,q}_T \wedge \wedge^*\) since \(\wedge^{p,q}_T\) is locally generated by basic forms. We denote by \(\pi_T^{1,0}\) the first projection from \(\wedge^{p+1,q}_T + \wedge^{p,q+1}_T + \wedge^{p,q}_T \wedge \wedge^*\) to \(\wedge^{p+1,q}_T\). Then we define a map

\[
\partial_T : \wedge^{p,q}_T \to \wedge^{p+1,q}_T
\]

by the composition \(\pi_T^{1,0} \circ d|_{\wedge^{p,q}_T}\). In the same manner, we can define \(\overline{\partial}_T : \wedge^{p,q}_T \to \wedge^{p,q+1}_T\). Then \(d_T = \partial_T + \overline{\partial}_T\) and \(d_T\varphi = \partial_B\varphi\) for a basic form \(\varphi\).

### 3.2 Transverse complex structures

Let \(\mathcal{F}\) be a foliation of codimension \(2n\) on \(M\). Then there exists a system \(\{U_i, f_i, \gamma_{ij}\}\) consisting of an open covering \(\{U_i\}_{i \in \Lambda}\) of \(M\), submersions \(f_i : U_i \to \mathbb{C}^n\) and diffeomorphisms \(\gamma_{ij} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)\) for \(U_i \cap U_j \neq \emptyset\) satisfying \(f_j = \gamma_{ij} \circ f_i\) such that any leaf of \(\mathcal{F}\) is given by each fiber of \(f_i\). We denote by \(M^T\) the transverse manifold \(\sqcup_i f_i(U_i)\). The foliation \(\mathcal{F}\) is a transverse holomorphic foliation (resp. a transverse Kähler foliation) if there exist a system \(\{U_i, f_i, \gamma_{ij}\}\) and a complex structure \(J_i\) (resp. Kähler structure \((g_i, J_i)\)) on each \(f_i(U_i)\) such that \(\gamma_{ij}\) is bi-holomorphic (resp. preserving the Kähler structure). Thus any transverse holomorphic foliation \(\mathcal{F}\) induces a complex structure \(J^T = \{J_i\}_{i \in \Lambda}\) on \(M^T\).

In order to characterize transverse structures on \((M, \mathcal{F})\), we consider the quotient bundle \(Q = TM/F\) where \(F\) is the line bundle associated by the foliation \(\mathcal{F}\). We define an action of \(\Gamma(F)\) to any section \(u\) of \(Q\) as follows:

\[
L_v u = \pi(L_v \tilde{u})
\]
for \(v \in \Gamma(F)\) where \(\pi\) is the quotient map \(TM \to Q\) and \(\widetilde{u} \in \Gamma(TM)\) is a lift of \(u\) by \(\pi\). We remark that \(L_v u\) is independent of the choice of the lift \(\widetilde{u} \in \Gamma(TM)\) of \(u\). The section \(u\) of \(\Gamma(Q)\) is called basic if \(L_v u = 0\) for any \(v \in \Gamma(F)\). We denote by \(\Gamma_B(Q)\) the set of basic sections of \(\Gamma(Q)\). The vector field \(\widetilde{u}\) on \(M\) is called foliated if \(L_v u \in \Gamma(F)\) for any \(v \in \Gamma(F)\). Let \(\Gamma_F(TM)\) denote the set of foliated vector fields on \(M\). Then there exists the exact sequence

\[
0 \to \Gamma(F) \xrightarrow{\iota} \Gamma_F(TM) \xrightarrow{\pi} \Gamma_B(Q) \to 0
\]

where \(\iota\) is the natural inclusion. In fact, any basic section \(u\) of \(\Gamma(Q)\) has a lift \(\widetilde{u}\) by \(\pi\) which is a foliated vector field. We can also define an action of \(\Gamma(F)\) to any section \(J \in \Gamma(\text{End}(Q))\) as follows:

\[
(L_v J)(u) = L_v(J(u)) - J(L_v u)
\]

for \(v \in \Gamma(F)\) and \(u \in \Gamma(Q)\). A section \(J \in \Gamma(\text{End}(Q))\) is called basic if \(L_v J = 0\) for any \(v \in \Gamma(F)\). If \(J\) is a complex structure of \(Q\), i.e. \(J^2 = -\text{id}_Q\), and basic as a section of \(\text{End}(Q)\), then a tensor \(N_J \in \Gamma(\otimes^2 Q^* \otimes Q)\) can be defined by

\[
N_J(u, w) = [\widetilde{J}u, \widetilde{J}w]_Q - [\widetilde{u}, \widetilde{w}]_Q - J[\widetilde{u}, \widetilde{J}w]_Q - J[\widetilde{J}u, \widetilde{w}]_Q
\]

for \(u, w \in \Gamma(Q)\), where \([u, w]_Q\) denotes \(\pi[\widetilde{u}, \widetilde{w}]\) for each lift \(\widetilde{u}\) and \(\widetilde{w}\). We call that \(J\) is integrable if \(N_J = 0\).

**Definition 3.1.** A complex structure \(J\) of \(Q\) is a transverse complex structure on 
\((M, \mathcal{F})\) if \(J\) is basic and integrable.

Any transverse holomorphic foliation \(\mathcal{F}\) induces a transverse complex structure.

**Lemma 3.2.** If \(\mathcal{F}\) is a transverse holomorphic foliation, then there exists a transverse complex structure \(J_{\mathcal{F}}\) on \((M, \mathcal{F})\).

**Proof.** We can extend a vector field \(u^T\) on \(M^T\) to a foliated vector field \(u_i\) on each \(U_i\) such that \(df_i(u_i) = u^T\) since \(U_i\) is diffeomorphic to \(f_i(U_i) \times V_i\) where \(V_i\) is an open subset of \(\mathbb{R}\). Then \(\{\pi(u_i)\}_i\) defines the section of \(Q\) and it is basic. Hence we obtain the map \(\sigma : \Gamma(TM^T) \to \Gamma_B(Q)\) by \(\sigma(u^T) = \{\pi(u_i)\}_i\). On the other hand, for any \(u \in \Gamma_B(Q)\) the family \(\{df_i(u)\}_i\) defines the vector field over \(M^T\) where \(df_i(u)\) is defined by \(df_i(\widetilde{u})\) for a lift \(\widetilde{u}\) of \(u\). We define the map \(df : \Gamma_B(Q) \to \Gamma(TM^T)\) as \(df(u) = \{df_i(u)\}_i\). Then \(df\) is the inverse map of \(\sigma\) and hence \(\Gamma_B(Q)\) is isomorphic to \(\Gamma(TM^T)\). For the complex structure \(J^T\) on \(M^T\), we can define a section \(J_{\mathcal{F}}\) of \(\text{End}(Q)\) as

\[
J_{\mathcal{F}}(u) = \sigma(J^T(df(u)))
\]

for \(u \in \Gamma_B(Q)\). This section \(J_{\mathcal{F}}\) is well-defined since any section of \(Q\) is locally generated by basic sections. Then \(J_{\mathcal{F}}\) is a complex structure of \(Q\) and basic since \(J_{\mathcal{F}}(u) = \sigma(J^T u^T)\) is basic for any \(u \in \Gamma_B(Q)\). The tensor \(N_{J_{\mathcal{F}}}\) satisfies that

\[
N_{J_{\mathcal{F}}}(u, w) = \sigma(N_{J_{\mathcal{F}}}(df(u), df(v))) = 0
\]

for \(u, w \in \Gamma_B(Q)\). It implies that \(N_{J_{\mathcal{F}}} = 0\). Hence a transversely holomorphic foliation \(\mathcal{F}\) induces the transverse complex structure \(J_{\mathcal{F}}\) on \((M, \mathcal{F})\). \(\square\)
The following result is Newlander-Nirenberg’s theorem for a transverse complex structure on a foliated manifold:

**Proposition 3.3.** Let $J$ be a complex structure of $Q$. Then the following conditions are equivalent.

(i) $J$ is a transverse complex structure on $(M, F)$.

(ii) $F$ is a transversely holomorphic foliation with $J_F = J$.

(iii) $J$ is basic and satisfies $d(\wedge_B^{1,0}) \subset \wedge_B^{2,0} \oplus \wedge_B^{1,1}$.

(iv) $J$ is basic and satisfies $d(\wedge_B^{0,1}) \subset \wedge_B^{1,1} \oplus \wedge_B^{0,2}$.

**Proof.** A basic complex structure $J$ of $Q$ corresponds to an almost complex structure on $M^T$, and so $\wedge_B^{p,q}$ is isomorphic to $\wedge_M^{p,q}$. Hence the conditions (ii), (iii) and (iv) are equivalent by Newlander-Nirenberg’s theorem for the complex manifold $(M^T, J^T)$. We already checked that the condition (ii) implies (i). Hence it suffices to show that the condition (i) implies (ii). If $J$ is a transverse complex structure on $(M, F)$, then the section $J(\sigma(u^T))$ of $Q$ is basic for any $u^T \in \Gamma(TM^T)$. Hence we can define the section $J^T$ of $\text{End}(TM^T)$ as

$$J^T(u^T) = df(J(\sigma(u^T)))$$

for $u^T \in \Gamma(TM^T)$. The section $J^T$ is an almost complex structure on $M^T$. Let $N_{J^T}$ be the Nijenhuis tensor of $J^T$. Then we obtain

$$N_{J^T}(u^T, v^T) = df(N_J(\sigma(u^T), \sigma(v^T))) = 0$$

for any $u^T, v^T \in \Gamma(TM^T)$. Hence $J^T$ is the complex structure on $M^T$. Then the foliation $F$ is transversely holomorphic and $J_F = J$ by the definition of $J_F$. It completes the proof.

**Definition 3.4.** A nowhere vanishing complex $n$-form $\psi \in \wedge^n \otimes \mathbb{C}$ is called an almost transverse $\text{SL}_n(\mathbb{C})$ structure on $(M, F)$ if $\psi$ is transverse and

$$Q \otimes \mathbb{C} = \text{Ker}_\mathbb{C} \psi/F \oplus \overline{\text{Ker}_\mathbb{C} \psi/F}$$

where $\text{Ker}_\mathbb{C} \psi/F$ is the space $\{u \in Q \otimes \mathbb{C} \mid i_u \psi = 0\}$.

An almost transverse $\text{SL}_n(\mathbb{C})$ structure $\psi$ induces a complex structure $J_\psi$ of $Q$ as follows

$$J_\psi(u) = \begin{cases} -\sqrt{-1}u & \text{for } u \in \text{Ker}_\mathbb{C} \psi/F, \\ \sqrt{-1}u & \text{for } u \in \overline{\text{Ker}_\mathbb{C} \psi/F}. \end{cases}$$

Then $Q^{0,1} = \text{Ker}_\mathbb{C} \psi/F$ and $Q^{1,0} = \overline{\text{Ker}_\mathbb{C} \psi/F}$. Therefore $\psi$ is a transverse $(n,0)$-form on $(M, F)$. Note that the section $J_\psi \in \text{End}(Q)$ is not necessarily integrable. However we have

**Proposition 3.5.** If $\psi$ satisfies $d \psi = A \wedge \psi$ for a complex valued 1-form $A$, then $J_\psi$ is basic and integrable.
Proof. At first, we show that $J_{\psi}$ is basic, that is, $L_v J_{\psi} = 0$ for $v \in \Gamma(F)$. Let $u$ be a section of $\text{Ker}_C \psi/F$. Then $L_v u$ is also the section of $\text{Ker}_C \psi/F$ since

$$\psi(L_v u) = L_v(\psi(u)) - (L_v \psi)(u)$$

$$= -(i_v d\psi)(u)$$

$$= -i_v(A \wedge \psi)(u)$$

$$= -(i_v A) \psi(u) + (A \iota_v \psi)(u)$$

$$= 0.$$

It yields that

$$(L_v J_{\psi}) u = L_v(J_{\psi}(u)) - J_{\psi}(L_v u) = L_v(-\sqrt{-1} u) - (-\sqrt{-1} L_v u) = 0$$

for any $u \in \Gamma(\text{Ker}_C \psi/F)$. In the same manner, we can prove $(L_v J_{\psi}) u = 0$ for any $u \in \Gamma(\text{Ker}_C \psi/F)$. Thus $L_v J_{\psi} = 0$ for any $v \in \Gamma(F)$, and hence $J_{\psi}$ is basic.

Secondary, we see that $J_{\psi}$ is integrable. It suffices to show that $d\wedge^{1,0}_B \subset \wedge^{1,1}_B \oplus \wedge^{2,0}_B$ by Proposition 3.3. Let $\alpha$ be an element of $\wedge^{1,0}_B$. Then $\alpha \wedge \psi = 0$ since $\psi$ is the transverse $(n,0)$-form. Then the derivative $d\alpha$ does not have the basic $(0,2)$-part. In fact, $d\alpha \wedge \psi = d(\alpha \wedge \psi) + \alpha \wedge d\psi = \alpha \wedge A \wedge \psi = 0$ since $\alpha \wedge \psi = 0$. Therefore $d\wedge^{1,0}_B \subset \wedge^{1,1}_B \oplus \wedge^{2,0}_B$. Hence $J_{\psi}$ is integrable, and we finish the proof.

If a transverse 2-form $\omega^T$ satisfies $(\omega^T)^n \neq 0$, then we call the form $\omega^T$ an *almost transverse symplectic structure* on $(M, F)$. We can consider the form $\omega^T$ as a tensor of $\wedge^2 Q^*$. 

**Definition 3.6.** Let $\psi$ be an almost transverse $\text{SL}(n, \mathbb{C})$ structure and $\omega^T$ an almost transverse symplectic structure on $(M, F)$. The pair $(\psi^T, \omega^T)$ is called an *almost transverse Calabi-Yau structure* on $(M, F)$ if following equations hold

$$\psi \wedge \omega^T = -\overline{\psi} \wedge \omega^T = 0,$$

$$\psi \wedge \psi = c_n (\omega^T)^n,$$

$$g^T = \omega^T(\cdot, J_{\psi} \cdot)$$

where $c_n = \frac{1}{n!} (-1)^{\frac{n(n+1)}{2}} \frac{2}{\sqrt{-1})^n}^n$.

If an almost transverse symplectic structure $\omega^T$ is $d$-closed, then $\omega^T$ is called a *transverse symplectic structure* on $(M, F)$. A pair $(\omega^T, J)$ is called a *transverse Kähler structure* on $(M, F)$ if the 2-tensor $\omega^T(\cdot, J\cdot)$ is positive on $Q$ and $\omega^T(J\cdot, J\cdot) = \omega^T(\cdot, \cdot)$ holds. Then we define the 2-tensor $g^T$ by $g^T(\cdot, \cdot) = \omega^T(\cdot, J\cdot)$ and call it a *transverse Kähler metric* on $(M, F)$. In [11], we introduced a transverse Calabi-Yau structure on $(M, F)$ by almost transverse Calabi-Yau structure $(\psi, \omega^T)$ such that $\psi$ and $\omega^T$ are $d$-closed.

### 3.3 Sasaki structures

We will give a brief review of some elementary results in Sasakian geometry. For much of this material, we refer to [1] and [16].
Definition 3.7. A Riemannian manifold \((M, g)\) is a Sasaki manifold if the metric cone \((C(M), \tilde{g}) = (\mathbb{R}_+ \times M, dr^2 + r^2 g)\) is Kähler for a complex structure \(I\).

We identify the manifold \(M\) with the hypersurface \(\{r = 1\}\) of \(C(M)\). Let \(I\) and \(\omega\) denote the complex structure and the Kähler form on the Kähler manifold \((C(M), \tilde{g})\), respectively. The vector field \(r \frac{\partial}{\partial r}\) is called the Euler vector field on \(C(M)\).

We define a vector field \(\xi\) and a 1-form \(\eta\) on \(C(M)\) by

\[
\xi = I(r \frac{\partial}{\partial r}), \quad \eta(X) = \frac{1}{r^2} \tilde{g}(\xi, X)
\]

for any vector field \(X\) on \(C(M)\). The vector field \(\xi\) is a Killing vector field, i.e. \(L_\xi \tilde{g} = 0\), and \(\xi + \sqrt{-1} I \xi = \xi - \sqrt{-1} r \frac{\partial}{\partial r}\) is a holomorphic vector field on \(C(M)\). It follows from \(L_\xi \eta = IL_r \frac{\partial}{\partial r} \eta = 0\) that

\[
\eta(\xi) = 1, \quad i_\xi d\eta = 0 \tag{3}
\]

where \(i_\xi\) means the interior product. The form \(\eta\) is expressed as

\[
\eta = d^c \log r = \sqrt{-1} (\bar{\partial} - \partial) \log r
\]

where \(d^c\) is the composition \(-I \circ d\) of the exterior derivative \(d\) and the action of the complex structure \(-I\) on differential forms. We define an action \(\lambda\) of \(\mathbb{R}_+\) on \(C(M)\) by

\[
\lambda_a(r, x) = (ar, x)
\]

for \(a \in \mathbb{R}_+\) and \((r, x) \in \mathbb{R}_+ \times M = C(M)\). If we put \(a = e^t\) for \(t \in \mathbb{R}\), then it follows from \(L_r \frac{\partial}{\partial r} \eta = \frac{d}{dt} \lambda_{e^t} \big|_{t=0}\) that \(\{\lambda_{e^t}\}_{t \in \mathbb{R}}\) is one parameter group of transformations such that \(r \frac{\partial}{\partial r}\) is the infinitesimal transformation. Then the Kähler form \(\omega\) satisfies \(\lambda^*_a \omega = a^2 \omega\) for \(a \in \mathbb{R}_+\) and

\[
L_r \frac{\partial}{\partial r} \omega = 2 \omega.
\]

It implies that

\[
\omega = \frac{1}{2} d(r^2 \eta) = \frac{\sqrt{-1}}{2} \partial \bar{\partial} r^2.
\]

Hence \(\frac{1}{2} r^2\) is a Kähler potential on \(C(M)\).

The 1-form \(\eta\) induces the restriction \(\eta|_M\) on \(M \subset C(M)\). Since \(L_r \frac{\partial}{\partial r} \eta = 0\), the form \(\eta\) is the extension of \(\eta|_M\) to \(C(M)\). The vector field \(\xi\) is tangent to the hypersurface \(\{r = c\}\) for each positive constant \(c\). In particular, \(\xi\) is considered as the vector field on \(M\) and satisfies \(g(\xi, \xi) = 1\) and \(L_\xi g = 0\). Hence we shall not distinguish between \((\eta, \xi)\) on \(C(M)\) and the restriction \((\eta|_M, \xi|_M)\) on \(M\). Then the form \(\eta\) is a contact 1-form on \(M\):

\[
\eta(\xi) = 1, \quad i_\xi d\eta = 0 \tag{4}
\]

since \(\omega\) is non-degenerate. The equation \([3]\) implies that
on $M$. For a contact form $\eta$, a vector field $\xi$ on $M$ satisfying the equation (4) is unique, and called the Reeb vector field. We define the contact subbundle $D \subset TM$ by $D = \ker \eta$. Then the tangent bundle $TM$ has the orthogonal decomposition
\[ TM = D \oplus \langle \xi \rangle \]
where $\langle \xi \rangle$ is the line bundle generated by $\xi$. We define a section $\Psi$ of $\text{End}(TM)$ by setting $\Psi|_D = I|_D$ and $\Psi|_{\langle \xi \rangle} = 0$. One can see that
\[ \Psi^2 = -\text{id} + \xi \otimes \eta, \]
\[ d\eta(\Psi X, \Psi Y) = d\eta(X, Y) \]
for any $X, Y \in TM$. Then the Riemannian metric $g$ satisfies
\[ g(X, Y) = \frac{1}{2}d\eta(X, \Psi Y) + \eta(X)\eta(Y) \]
for any $X, Y \in TM$. We denote by $\omega^T$ the 2-form $\frac{1}{2}d\eta$ on $M$. Then $\omega^T$ is a symplectic structure on $D$ which is compatible with $\Psi$.

We say a data $(\xi, \eta, \Psi, g)$ a contact metric structure on $M$ if for a contact form $\eta$ and a Reeb vector field $\xi$, a section $\Psi$ of $\text{End}(TM)$ and a Riemannian metric $g$ satisfy the equations (5), (6) and (7). Moreover, a contact metric structure $(\xi, \eta, \Psi, g)$ is called a $K$-contact structure on $M$ if $\xi$ is a Killing vector field with respect to $g$. Then $\Psi$ defines an almost CR structure $(D, \Psi|_D)$ on $M$. As we saw above, any Sasaki manifold $(M, g)$ has a K-contact structure $(\xi, \eta, \Psi, g)$ with the integrable CR structure $(D, \Psi|_D = I|_D)$ on $M$. Conversely, if we have such a structure $(\xi, \eta, \Psi, g)$ on $M$, then $(\xi, \frac{1}{2}d(r^2)\eta)$ is a Kähler structure on the cone $C(M)$, hence $(M, g)$ is a Sasaki manifold. We call a K-contact structure $(\xi, \eta, \Psi, g)$ with the integrable CR structure $(D, \Psi|_D)$ a Sasaki structure on $M$.

### 3.4 The Reeb foliation

Let $(\xi, \eta, \Psi, g)$ be a Sasaki structure on $M$. Then the Reeb vector field $\xi$ generates a foliation $\mathcal{F}_\xi$ of codimension $2n$ on $M$. The foliation $\mathcal{F}_\xi$ is called the Reeb foliation. We can consider $\Psi$ as a section of $\text{End}(Q)$ since $\Psi|_{\langle \xi \rangle} = 0$. Then $\Psi$ is a transverse complex structure on $(M, \mathcal{F}_\xi)$ by the integrability of the CR structure $\Psi|_D$. Let $\omega^T$ be the 2-form $\frac{1}{2}d\eta$ on $M$. Then the pair $(\Psi, \omega^T)$ is a transverse Kähler structure with the transverse Kähler metric $g^T(\cdot, \cdot) = \omega^T(\cdot, \cdot)$ on $(M, \mathcal{F}_\xi)$.

We define $\text{Ric}^T$ as the Ricci tensor of $g^T$ which is called the transverse Ricci tensor. The transverse Ricci form $\rho^T$ is defined by $\rho^T(\cdot, \cdot) = \text{Ric}^T(\cdot, \cdot)$. The form $\rho^T$ is a basic $d$-closed $(1, 1)$-form on $(M, \mathcal{F}_\xi)$ and defines a $(1, 1)$-basic Dolbeault cohomology class $[\rho^T] \in H^1_{\overline{\partial}}(M)$ as in the Kähler case. The basic class $[\frac{1}{2\pi}\rho^T]$ in $H^1_{\overline{\partial}}(M)$ is called the basic first Chern class on $(M, \mathcal{F}_\xi)$ and is denoted by $c^B_1(M)$ (for short, we write it $c^B_1$). We say the basic first Chern class is positive (resp. negative) if $c^B_1$ (resp. $-c^B_1$) is represented by a transverse Kähler form. This condition is expressed by $c^B_1 > 0$ (resp. $c^B_1 < 0$). We say that $(g^T, \omega^T)$ is a transverse Kähler-Einstein
structure with Einstein constant \( \kappa \) if \((g^T, \omega^T)\) is the transverse Kähler structure satisfying \( \text{Ric}^T = \kappa g^T \) which is equivalent to \( \rho^T = \kappa \omega^T \). If \( M \) admits such a structure, then \( 2\pi c^B_1 = \kappa [\omega^T] \), so the basic first Chern class has to be positive, zero or negative according to the sign of \( \kappa \).

On the cone \( C(M) \), a foliation \( \mathcal{F}_{\langle \xi, r \partial/\partial r \rangle} \) is induced by the vector bundle \( \langle \xi, r \partial/\partial r \rangle \) generated by \( \xi \) and \( r \partial/\partial r \). Let \( \tilde{\phi} \) be a basic form on \((C(M), \mathcal{F}_{\langle \xi, r \partial/\partial r \rangle})\). Then the restriction \( \tilde{\phi}|_M \) of \( \tilde{\phi} \) to \( M \) is also basic on \((M, \mathcal{F}_\xi)\). Conversely, for any basic form \( \phi \) on \((M, \mathcal{F}_\xi)\), the trivial extension \( \tilde{\phi} \) of \( \phi \) to \( C(M) = \mathbb{R}_{>0} \times M \) is a basic form on \((C(M), \mathcal{F}_{\langle \xi, r \partial/\partial r \rangle})\). In this paper, we identify a basic form \( \phi \) on \((M, \mathcal{F}_\xi)\) with the extension \( \tilde{\phi} \) on \((C(M), \mathcal{F}_{\langle \xi, r \partial/\partial r \rangle})\).

### 3.5 Sasaki-Einstein structures and almost transverse Calabi-Yau structures

In this section, we assume that \( M \) is a compact manifold. We provide the definition of Sasaki-Einstein manifolds.

**Definition 3.8.** A Sasaki manifold \((M, g)\) is Sasaki-Einstein if the metric \( g \) is Einstein.

Let \((\xi, \eta, \Psi, g)\) be a Sasaki structure on \( M \). Then the Ricci tensor \( \text{Ric} \) of \( g \) has following relations:

\[
\text{Ric}(u, \xi) = 2n\eta(u), \quad u \in TM
\]
\[
\text{Ric}(u, v) = \text{Ric}^T(u, v) - 2g(u, v), \quad u, v \in D
\]

Thus the Einstein constant of a Sasaki-Einstein metric \( g \) has to be \( 2n \), that is, \( \text{Ric} = 2ng \). It follows from the above equations that the Einstein condition \( \text{Ric} = 2ng \) is equal to \( \text{Ric}^T = 2(n + 1)g^T \). Moreover, the cone metric \( \overline{g} \) is Ricci-flat on \( C(M) \) if and only if \( g \) is Einstein with the Einstein constant \( 2n \) on \( M \) (we refer to Lemma 11.1.5 in [1]). Hence we can characterize the Sasaki-Einstein condition as follows

**Proposition 3.9.** Let \((M, g)\) be a Sasaki manifold of dimension \( 2n + 1 \). Then the following conditions are equivalent.

(i) \((M, g)\) is a Sasaki-Einstein manifold.

(ii) \((C(M), \overline{g})\) is Ricci-flat, that is, \( \text{Ric}_{\overline{g}} = 0 \).

(iii) \( g^T \) is transverse Kähler-Einstein with \( \text{Ric}^T = 2(n + 1)g^T \). \( \square \)

We remark that Sasaki-Einstein manifolds have finite fundamental groups from Mayer’s theorem. From now on, we assume that \( M \) is simply connected.

**Definition 3.10.** A pair \((\Omega, \omega)\) \( \in \wedge^{n+1} \otimes \mathbb{C} \oplus \wedge^2 \) is called a weighted Calabi-Yau structure on \( C(M) \) if \( \Omega \) is a holomorphic section of \( K_{C(M)} \) and \( \omega \) is a Kähler form satisfying the equation

\[
\Omega \wedge \overline{\Omega} = c_{n+1} \omega^{n+1}
\]
Deformations of special Legendrian submanifolds

\[ L_{r \frac{\partial}{\partial r}} \Omega = (n + 1)\Omega, \]
\[ L_{r \frac{\partial}{\partial r}} \omega = 2\omega. \]

If there exists a weighted Calabi-Yau structure \((\Omega, \omega)\) on \(C(M)\), then it is unique up to change \(\Omega \rightarrow e^{\sqrt{-1} \theta} \Omega\) of a phase \(\theta \in \mathbb{R}\).

**Lemma 3.11.** A Riemannian metric \(g\) on \(M\) is Sasaki-Einstein if and only if there exists a weighted Calabi-Yau structure \((\Omega, \omega)\) on \(C(M)\) such that \(g\) is the Kähler metric. □

We characterize a Sasaki-Einstein manifold by a pair of two differential forms on the manifold.

**Proposition 3.12.** The Riemannian manifold \((M, g)\) is a Sasakian-Einstein manifold if and only if there exist a contact 1-form \(\eta\) and a complex valued \(n\)-form \(\psi\) such that \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structure on \((M, \mathcal{F})\), where \(\mathcal{F}\) is the Reeb foliation induced by \(\eta\), with

\[ d\psi = (n + 1)\sqrt{-1} \eta \wedge \psi. \]

**Proof.** If \((M, g)\) is a Sasakian-Einstein manifold, then there exist a weighted Calabi-Yau structure \((\Omega, \omega)\) on \(C(M)\) with the Kähler metric \(\overline{\mathcal{F}}\). Then the Kähler form \(\omega\) is given by

\[ \omega = d\left(\frac{1}{2} r^2 \eta\right) = r dr \wedge \eta + \frac{r^2}{2} d\eta. \]

We define \(\psi'\) as the \(n\)-form

\[ \psi' = i_{r \frac{\partial}{\partial r}} \Omega \]

on \(C(M)\). Then \(\psi'\) is a transversely \((n, 0)\)-form on \((C(M), \mathcal{F}(\xi, r \frac{\partial}{\partial r}))\) such that

\[ \Omega = \left(\frac{dr}{r} + \sqrt{-1} \eta\right) \wedge \psi' \]

since \(\psi' = i_{v} \Omega + i_{\overline{\psi}} \Omega = i_{v} \Omega\) for the holomorphic vector field \(v = \frac{1}{2}(r \frac{\partial}{\partial r} - \sqrt{-1} \xi)\). The equation (9) implies that

\[ d\psi' = L_{r \frac{\partial}{\partial r}} \Omega = (n + 1)\Omega = (n + 1)\left(\frac{dr}{r} + \sqrt{-1} \eta\right) \wedge \psi'. \]

It is straightforward to

\[ \Omega \wedge \overline{\Omega} = 2(-1)^n \sqrt{-1} r^{-1} dr \wedge \eta \wedge \psi' \wedge \overline{\psi'}, \]
\[ \omega^{n+1} = (n + 1) r dr \wedge \eta \wedge \left(\frac{1}{2} r^2 d\eta\right)^n. \]

Hence it follows from the equation (8) that

\[ \psi' \wedge \overline{\psi'} = c_n r^{2(n+1)} \left(\frac{1}{2} d\eta\right)^n. \]

**Lemma 3.11.** A Riemannian metric \(g\) on \(M\) is Sasaki-Einstein if and only if there exists a weighted Calabi-Yau structure \((\Omega, \omega)\) on \(C(M)\) such that \(g\) is the Kähler metric. □
Moreover, we obtain
\[
\psi' \wedge d\eta = -2r^{-2}\sqrt{-1}i_\xi(\Omega \wedge \omega) = 0 \tag{12}
\]
since \(\Omega \wedge \omega = \frac{r^2}{2}(\frac{dr}{r} + \sqrt{-1}\eta) \wedge d\eta \wedge \psi'\). Let \(\psi\) denote the pull-back of \(\psi'\) to \(M\) by the inclusion \(i : M \to C(M) : \psi = i^*\psi'\).

Then \(\psi\) is a transversely \((n, 0)\)-form on \((M, F_\xi)\) such that \(d\psi = (n + 1)\sqrt{-1}\eta \wedge \psi\) by taking the pull-back of the equation \((10)\) by \(i\). Moreover, it follows from the equations \((11)\) and \((12)\) that \(\psi \wedge d\eta = \overline{\psi} \wedge d\eta = 0\) and \(\psi \wedge \overline{\psi} = c_n(\frac{1}{2}d\eta)^n\). Hence \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structures.

Conversely, we assume that there exist a contact form \(\eta\) and a complex valued \(n\)-form \(\psi\) such that \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structure. Then the contact form \(\eta\) induces the Reeb vector field \(\xi\) and the Reeb foliation \(F_\xi\). It follows from Proposition 3.5 that \(\psi\) defines the transverse complex structure \(J_\psi\) on \((M, F_\xi)\). Then \((\eta, \xi, J_\psi, g)\) is a Sasaki structure on \(M\) and the metric cone \((C(M), \overline{\eta})\) has the Kähler form
\[
\omega = d(\frac{1}{2}r^2\eta).
\]
It is easy to see that \(L_{r\frac{\partial}{\partial r}}\omega = 2\omega\). We define \(\psi'\) as the \(n\)-form
\[
\psi' = r^{n+1}\psi
\]
on \(C(M)\), where we consider \(\psi\) as an \(n\)-form on \(C(M)\) by the trivial extension. Then \(\psi'\) is a transversely \((n, 0)\)-form on \((C(M), F_\xi, g)\) such that \(i_\psi\psi' = 0\) for the holomorphic vector field \(v = \frac{1}{2}(r\frac{\partial}{\partial \eta} - \sqrt{-1}\xi)\). We define \(\Omega\) as the \((n + 1)\)-form
\[
\Omega = (\frac{dr}{r} + \sqrt{-1}\eta) \wedge \psi' \tag{13}
\]
on \(C(M)\). Then \(\Omega\) is a holomorphic \((n + 1)\)-form satisfying
\[
L_{r\frac{\partial}{\partial r}}\Omega = di_{r\frac{\partial}{\partial r}}\Omega = d\psi' = (n + 1)\Omega
\]
since \(i_{r\frac{\partial}{\partial \eta}}\Omega = \psi'\) and \(d\psi' = (n + 1)(\frac{dr}{r} + \sqrt{-1}\eta) \wedge \psi'\). The equation \((13)\) implies that
\[
\Omega \wedge \overline{\Omega} = 2(-1)^n \sqrt{-1} r^{-1} dr \wedge \eta \wedge \psi' \wedge \overline{\psi'} = 2(-1)^n \sqrt{-1} r^{2n+1} dr \wedge \eta \wedge \psi \wedge \overline{\psi} = 2(-1)^n \sqrt{-1} c_n r^{2n+1} dr \wedge \eta \wedge (\frac{1}{2}d\eta)^n = (n + 1)c_{n+1} r dr \wedge \eta \wedge (\frac{1}{2}r^2d\eta)^n = c_{n+1}\omega^{n+1}.
\]
Hence \((\Omega, \omega)\) is a weighted Calabi-Yau structure on \(C(M)\), and we finish the proof. \(\square\)
4 Deformations of special Legendrian submanifolds

In this section, we assume that \((M, g)\) is a simply connected and compact Sasaki-Einstein manifold of dimension \((2n + 1)\) and \((\xi, \eta, \Psi, g)\) is the Sasaki-Einstein structure on \(M\). Let \((C(M), \overline{\mathcal{F}})\) be the metric cone of \((M, g)\). We fix an almost transverse Calabi-Yau structure \((\psi, \frac{1}{2}d\eta)\) and the corresponding weighted Calabi-Yau structure \((\Omega, \omega)\) on \(C(M)\) as in Proposition B.12.

4.1 Special Legendrian submanifolds

The holomorphic \((n + 1)\)-form \(\Omega\) induces the calibration \(\Omega^\Re\) on \(C(M)\) and special Lagrangian submanifolds are defined by calibrated submanifolds. We consider such submanifolds of cone type. For any submanifold \(X\) in \(M\), the cone \(C(X) = \mathbb{R}_{>0} \times X\) is a submanifold in \(C(M)\). We identify \(X\) with the hypersurface \(\{1\} \times X\) in \(C(X)\). Then \(X\) can be considered as the link \(C(X) \cap M\).

**Definition 4.1.** A submanifold \(X\) is a special Legendrian submanifold in \(M\) if the cone \(C(X)\) is a special Lagrangian submanifold in \(C(M)\).

Any special Legendrian submanifold \(X\) is a minimal submanifold in \(M\), that is, the mean curvature vector field \(H\) of \(X\) vanishes. In fact, the mean curvature vector field \(\tilde{H}\) of the cone \(C(X)\) satisfies that \(\tilde{H}(r, x) = \frac{1}{r^2}H_x\) at \((r, x) \in \mathbb{R}_{>0} \times X = C(X)\). Hence \(H = 0\) since a special Lagrangian cone \(C(X)\) is minimal.

We also denote by \(\eta\) the extension of the contact form \(\eta\) to \(C(M)\). We provide a characterization of special Lagrangian cones in \(C(M)\).

**Proposition 4.2.** Let \(\tilde{X}\) be an \((n + 1)\)-dimensional closed submanifold in \(C(M)\) with the inclusion \(\tilde{i} : \tilde{X} \hookrightarrow C(M)\). The submanifold \(\tilde{X}\) is a special Lagrangian cone if and only if \(\tilde{i}^*\Omega^\Im = \tilde{i}^*\eta = 0\).

**Proof.** We remark that a special Lagrangian submanifold in \(C(M)\) is characterized by an \((n+1)\)-dimensional submanifold \(\tilde{X}\) in \(C(M)\) such that \(\tilde{i}^*\Omega^\Im = 0\) and \(\tilde{i}^*\omega = 0\). If \(\tilde{X}\) is a special Lagrangian cone, then the vector field \(r \frac{\partial}{\partial r}\) is tangent to \(\tilde{X}\). The vector fields \(\xi\) and \(r \frac{\partial}{\partial r}\) span a symplectic subspace of \(T_pC(M)\) with respect to \(\omega_p\) at the each point \(p \in C(M)\). We can obtain \(\tilde{i}^*\eta = 0\) since \(\eta = i_{r \frac{\partial}{\partial r}}\omega\) and \(\tilde{i}^*\omega = 0\).

Conversely, if an \((n+1)\)-dimensional submanifold \(\tilde{X}\) satisfies \(\tilde{i}^*\Omega^\Im = \tilde{i}^*\eta = 0\), then \(\tilde{X}\) is a special Lagrangian submanifold since \(\tilde{i}^*\omega = \frac{1}{2}d(r^2\tilde{i}^*\eta) = 0\). In order to see that \(\tilde{X}\) is a cone, we consider the set

\[
I_p = \{a \in \mathbb{R}_{>0} \mid \lambda_ap \in \tilde{X}\}
\]

for each \(p \in \tilde{X}\). Then \(I_p\) is a closed subset of \(\mathbb{R}_{>0}\) since \(\tilde{X}\) is closed. On the other hand, the vector field \(r \frac{\partial}{\partial r}\) has to be tangent to \(\tilde{X}\) since \(\tilde{X}\) is Lagrangian and \(\tilde{i}^*\eta = 0\). The vector field \(r \frac{\partial}{\partial r}\) is the infinitesimal transformation of the action \(\lambda\). Therefore \(I_p\) is open, and so \(I_p = \mathbb{R}_{>0}\) for each point \(p \in \tilde{X}\). Hence \(\tilde{X}\) is a cone, and it completes the proof. \(\square\)
We denote by $\psi^{\text{Re}}$ and $\psi^{\text{Im}}$ the real part and the imaginary part of $\psi$, respectively. Then we have a characterization of special Legendrian submanifolds:

**Proposition 4.3.** An $n$-dimensional submanifold $X$ in $M$ is a special Legendrian submanifold if and only if $i^*\psi^{\text{Im}} = i^*\eta = 0$.

**Proof.** Let $(\Omega,\omega)$ be the Calabi-Yau structure on $C(M)$. As in the proof of Proposition 3.12, $\Omega$ is given by

$$\Omega = \left(\frac{dr}{r} + \sqrt{-1}\eta\right) \wedge r^{n+1}\psi$$

on $C(M)$. It implies that $\Omega^{\text{Im}} = r^n dr \wedge \psi^{\text{Im}} + r^{n+1}\eta \wedge \psi^{\text{Re}}$. Thus the equation $i^*\psi^{\text{Im}} = i^*\eta = 0$ is equivalent to $i^*\Omega^{\text{Im}} = i^*\eta = 0$ where $i$ is the inclusion $i : C(X) \to C(M)$. Hence the condition $i^*\psi^{\text{Im}} = i^*\eta = 0$ holds if and only if $X$ is a special Legendrian submanifold by Proposition 4.2 and the definition of special Legendrian submanifolds.

For a real constant $\theta$, the $n$-form $e^{\sqrt{-1}\theta}\Omega$ induces a calibration $(e^{\sqrt{-1}\theta}\Omega)^{\text{Re}}$ on $(C(M),g)$. Then calibrated submanifolds are called $\theta$-special Lagrangian submanifolds for a phase $\theta$.

**Definition 4.4.** A submanifold $X$ in $M$ is $\theta$-special Legendrian if the cone $C(X)$ is a $\theta$-special Lagrangian submanifold in $C(M)$ for a phase $\theta$.

A $\theta$-special Legendrian submanifold is a special Legendrian submanifold in the Sasaki-Einstein manifold with the almost transverse Calabi-Yau structure $(e^{\sqrt{-1}\theta}\psi, \frac{1}{2}\eta)$.

Hence any $\theta$-special Legendrian submanifold is minimal. Moreover the converse is true, that is,

**Proposition 4.5.** A connected oriented submanifold $X$ in $M$ is minimal Legendrian if and only if $X$ is $\theta$-special Legendrian for a phase $\theta$.

**Proof.** We only show that a minimal Legendrian submanifold is $\theta$-special Legendrian for a phase $\theta$. Let $X$ be a connected oriented Legendrian submanifold in $M$ and $H$ the mean curvature vector of $X$. Then there exists a $\mathbb{R}/\mathbb{Z}$-valued function $\theta$ on $X$ such that $*i^*\psi = e^{-\sqrt{-1}\theta}$ where $*$ is the Hodge operator with respect to the metric $i^*g$ on $X$. If we regard $\theta$ as the function on $C(X)$ by the trivial extension, then it follows from $\Omega = \left(\frac{dr}{r} + \sqrt{-1}\eta\right) \wedge r^n\psi$ that $*i^*\tilde{\Omega} = e^{-\sqrt{-1}\theta}$ where $\tilde{i}$ is the inclusion $\tilde{i} : C(X) \hookrightarrow C(M)$ and $*$ is the Hodge operator with respect to the metric $\tilde{i}^*g$ on $C(X)$. Hence $d\theta = i^*(\tilde{H}\psi)$ for the mean curvature vector $\tilde{H}$ of $C(X)$ (Lemma 2.1). It follows from $d\theta(\frac{\partial}{\partial \theta}) = 0$ that $\tilde{H}$ has no component of $\langle \xi \rangle_{\tilde{g}}$. Since the transverse part of $\omega$ is $i_H^\perp d\eta$ and $\tilde{H} = \frac{1}{r}H$, the equation $i^*(i_H^\perp d\eta) = i^*(i_H^\perp \omega^T) = i^*(i_H^\perp \omega^T)$ holds where $\omega^T$ is the transverse 2-form $\frac{1}{2}\eta$ on $(M,\mathcal{F}_\xi)$. Thus we obtain

$$d\theta = i^*(i_H^\perp \omega^T)$$

on $M$. Hence $\theta$ is constant if $X$ is minimal. Then $X$ is special Legendrian with respect to the almost transverse Calabi-Yau structure $(e^{\sqrt{-1}\theta}\psi, \frac{1}{2}\eta)$. Therefore $X$ is the $\theta$-special Legendrian submanifold for the phase $\theta$. 

\qed
4.2 Infinitesimal deformations of special Legendrian submanifolds

Let $X$ be a compact special Legendrian submanifold in $M$. We denote by $\mathcal{M}_X$ the moduli space of special Legendrian deformations of $X$. The following result is shown by Futaki, Hattori and Yamamoto by considering deformations of the special Lagrangian cone in $C(M)$ [3]. We provide another proof from the point of view of $\Phi$-deformations.

**Proposition 4.6.** The infinitesimal deformation space of $X$ is isomorphic to the space $\text{Ker}(\Delta_0 - 2(n+1))$ of $2(n+1)$-eigenfunctions of the Laplace operator $\Delta_0$.

**Proof.** Proposition 4.3 implies that a special Legendrian deformation is a $(\psi^{\text{Im}}, \eta)$-deformation and $\mathcal{M}_X$ is the moduli space $\mathcal{M}_X(\psi^{\text{Im}}, \eta)$. We take the set $U$ as in (1) and define the map $F: U \to \wedge^0 \oplus \wedge^1$ by

$$F(v) = (\star \exp_v^{\psi^{\text{Im}}}, \exp_v\eta)$$

for $v \in U$, then we can regard $F^{-1}(0)$ as a set of special Legendrian submanifolds in $M$ which is near $X$ in $C^1$ sense. The linearization $d_0 F$ of $F$ at $0 \in U$ is given by

$$d_0 F(v) = (\ast \iota^* L_{\bar{v}} \psi^{\text{Im}}, \iota^* L_{\bar{v}} \eta)$$

$$= (\ast d\iota^* (i_{\bar{v}} \psi^{\text{Im}}) + (n+1) \iota^* (i_{\bar{v}} \eta \psi^{\text{Re}}), d\iota^* (i_{\bar{v}} \eta) + \iota^* (i_{\bar{v}} d\eta))$$

$$= \left( \frac{1}{2} d\iota^* (i_{\bar{v}} d\eta) + (n+1) \iota^* (i_{\bar{v}} \eta), d\iota^* (i_{\bar{v}} \eta) + \iota^* (i_{\bar{v}} d\eta) \right)$$

for $v \in U$ where $\bar{v}$ is an extension of $v$ to $M$. In the last equation, we use that $\iota^* (i_{\bar{v}} \psi^{\text{Im}}) = -\frac{1}{2} \iota^* (i_{\bar{v}} d\eta)$. Under the identification

$$\mathcal{N} X \simeq \wedge^0 \oplus \wedge^1$$

given by $v \mapsto (i_v \eta, \frac{1}{2} i_v d\eta)$, we identify $d_0 F$ with the map $d_0 F: \wedge^0 \oplus \wedge^1 \to \wedge^0 \oplus \wedge^1$ defined by

$$d_0 F(f, \alpha) = (d^* \alpha + (n+1)f, df + 2\alpha)$$

for $(f, \alpha) \in \wedge^0 \oplus \wedge^1$. Then it turns out that

$$\text{Ker}(d_0 F) = \{(f, \alpha) \in \wedge^0 \oplus \wedge^1 \mid d^* \alpha + (n+1)f = 0, 2\alpha + df = 0\}$$

$$= \{(f, -\frac{1}{2} df) \in \wedge^0 \oplus \wedge^1 \mid (\Delta_0 - 2(n+1))f = 0\}$$

which is isomorphic to $\text{Ker}(\Delta_0 - 2(n+1))$, and hence it completes the proof.  

The obstruction space of special Legendrian deformations is the cokernel $\text{Coker}(d_0 F)$ as in the proof of Proposition 4.6. However, the space $\text{Coker}(d_0 F)$ is isomorphic to $\text{Ker}(d_0 F)$ since $d_0 F: \wedge^0 \oplus \wedge^1 \to \wedge^0 \oplus \wedge^1$ is self dual. Hence the obstruction space does not vanish whenever $X$ has non-trivial deformations. Thus there may exist some obstruction of special Legendrian deformations.
4.3 The intersection of two deformation spaces

Let $\omega^T$ be the 2-form $\frac{1}{n}d\eta$ on $M$. We remark that $X$ is a special Legendrian submanifold if and only if $\imath^*\psi^3 = \imath^*\eta = \imath^*\omega^T = 0$. Let $X$ be a compact special Legendrian submanifold in $M$. We consider $(\psi^\text{Im}, \omega^T)$-deformations of $X$ and denote by $\mathcal{N}_X$ the moduli spaces $\mathcal{M}_X(\psi^\text{Im}, \omega^T)$. We fix an integer $s \geq 3$ and a real number $\alpha$ with $0 < \alpha < 1$ and set $\mathcal{N}_X^{s, \alpha}$ as the moduli space $\mathcal{M}_X^{s, \alpha}(\psi^\text{Im}, \omega^T)$ of $(\psi^\text{Im}, \omega^T)$-deformations of $C^{s, \alpha}$-class. Then we have

**Proposition 4.7.** The moduli space $\mathcal{N}_X^{s, \alpha}$ is smooth at $0_X$. The tangent space $T_{0_X}\mathcal{N}_X^{s, \alpha}$ is isomorphic to $\{(-\frac{1}{n+1}d^*\alpha, \alpha) \in C^{s, \alpha}(\Lambda^0 \oplus \Lambda^1) \mid d\alpha = 0\}$.

**Proof.** We take the set $U$ as in (1) and define the map $G : U \to \Lambda^0 \oplus \Lambda^2$ by

$$G(v) = (\ast \exp_v^* \psi^\text{Im}, \exp_v^* \omega^T)$$

for $v \in U$. It follows that for $v \in U$

$$d_0 G(v) = (\ast (dt^*(i_v \psi^\text{Im}) + (n+1) t^*(i_v \eta \psi^{\text{Re}}) - (n+1) t^*(\eta \wedge i_v \psi^{\text{Re}}), dt^*(i_v \omega^T))$$

$$= (d^* t^*(i_v \omega^T) + (n+1) t^*(i_v \eta), dt^*(i_v \omega^T))$$

where $\tilde{v}$ is an extension of $v$ to $M$. In the last equation, we use that $t^*(i_v \psi^\text{Im}) = -\frac{1}{2} \ast t^*(i_v d\eta)$ and $t^* \psi^{\text{Re}} = \text{vol}(X)$. Under the identification $NX \simeq \Lambda^0 \oplus \Lambda^1$ given by $v \mapsto (i_v \eta, i_v \omega^T)$, we identify $d_0 G$ with the map $D_1 : \Lambda^0 \oplus \Lambda^1 \to \Lambda^0 \oplus \Lambda^2$ defined by

$$D_1(f, \alpha) = (d^* \alpha + (n+1)f, d\alpha)$$

for $(f, \alpha) \in \Lambda^0 \oplus \Lambda^1$. Then it turns out that

$$\text{Ker}(D_1) = \{(f, \alpha) \in \Lambda^0 \oplus \Lambda^1 \mid d^* \alpha + (n+1)f = 0, d\alpha = 0\}$$

$$= \{(-\frac{1}{n+1}d^* \alpha, \alpha) \in \Lambda^0 \oplus \Lambda^1 \mid d\alpha = 0\}.$$

Now we provide a complex as follows

$$0 \to \Lambda^0 \oplus \Lambda^1 \xrightarrow{D_1} \Lambda^0 \oplus \Lambda^2 \xrightarrow{D_2} \Lambda^3 \to 0$$

where the operator $D_2$ is given by

$$D_2(f, \beta) = d\beta$$

for $(f, \beta) \in \Lambda^0 \oplus \Lambda^2$. It is easy to see that

$$P_1(f, \alpha) = ((n+1)^2 f + (n+1)d^* \alpha, \Delta_1 \alpha + (n+1)d\alpha),$$

$$P_2(f, \beta) = ((\Delta_0 + (n+1)^2) f, \Delta_2 \beta).$$

Hence $P_3$ is the elliptic operator. It follows from $\text{Im}(G) \subset \Lambda^0 \oplus d\Lambda^1$ that $\text{Im}(G)$ is perpendicular to $\text{Ker} P_3(= \{0\} \oplus \mathcal{H}^2(X))$ and $\text{Im}(D_2^*)$. Hence we obtain $\text{Im}(G) \subset \text{Im}(D_1)$ by the Hodge decomposition $\Lambda^0 \oplus \Lambda^2 = \text{Ker} P_2 \oplus \text{Im}(D_1) \oplus \text{Im}(D_2^*)$. Proposition 2.2 implies that $\mathcal{N}_X^{s, \alpha}$ is smooth at $0_X$ with the tangent space $\text{Ker}(D_1^{s, \alpha})$. Hence we finish the proof.
Let \( \mathcal{L}_X \) be the moduli space of Legendrian deformations of \( X \) of \( C^\infty \)-class and \( \mathcal{L}^{s,\alpha}_X \) that of \( C^{s,\alpha} \)-class. Then we have the following

**Theorem 4.8.** The moduli space \( M_X \) is the intersection \( \mathcal{N}_X \cap \mathcal{L}_X \) where \( \mathcal{N}^{s,\alpha}_X \) and \( \mathcal{L}^{s,\alpha}_X \) are smooth.

**Proof.** The moduli space \( M_X \) is the intersection \( \mathcal{N}_X \cap \mathcal{L}_X \) since \( \mathcal{M}_X(\psi^{\text{Im}},\eta) = \mathcal{M}_X(\psi^{\text{Im}},\eta,\omega^T) = \mathcal{M}_X(\psi^{\text{Im}},\omega^T) \cap \mathcal{M}_X(\eta) \). The space \( \mathcal{L}^{s,\alpha}_X \) is smooth by Proposition 2.7. It follows from Proposition 4.7 that \( \mathcal{N}^{s,\alpha}_X \) is smooth at any point of \( \mathcal{N}_X \cap \mathcal{L}_X \). Hence it completes the proof. \( \square \)

### 4.4 Transverse deformations of special Legendrian submanifolds

We consider a \( \Phi \)-deformation \( f \) of a submanifold \( X \) in \( M \) with the family \( \{f_t\}_{t \in [0,1]} \) such that \( \frac{d}{dt}f_t \in \Gamma(NX^T_t) \) where \( NX^T_t = NX_t \cap \ker(\eta)|_{X_t} \) and call such a deformation \( \{f_t\}_{t \in I} \) a transverse \( \Phi \)-deformation of \( X \). We define \( M^T_X(\Phi) \) as the moduli space of transverse \( \Phi \)-deformations of \( X \). We assume that \( X \) is a compact special Legendrian submanifold in \( M \). Let \( \mathcal{N}^T_X \) denote the moduli space \( M^T_X(\psi^{\text{Im}},\omega^T) \) of transverse \( (\psi^{\text{Im}},\omega^T) \)-deformations of \( X \). Then we obtain

**Theorem 4.9.** The moduli space \( \mathcal{N}^T_X \) is smooth at \( 0_X \) and the tangent space \( T_{0_X} \mathcal{N}^T_X \) is isomorphic to \( H^1(X) \).

**Proof.** Let \( U^T \) be a sufficiently small neighbourhood of the zero section in \( NX^T \) and \( U^T \) the set \( \{v \in C^\infty(NX^T) \mid v_x \in U^T, x \in X \} \). We define the map \( G^T : U^T \to \wedge^0 \oplus \wedge^2 \) by

\[
G^T(v) = (\ast \exp_v^\ast \psi^{\text{Im}}, \exp_v^\ast \omega^T)
\]

for \( v \in U^T \). Then we have

\[
d_0G^T(v) = (\ast (dt^* (i_{\tilde{v}} \psi^{\text{Im}}) + (n + 1)t^* (i_{\tilde{v}} \eta \psi^{\text{Re}})) - (n + 1)t^* (\eta \wedge i_{\tilde{v}} \psi^{\text{Re}})), dt^* (i_{\tilde{v}} \omega^T))
\]

\[
= (d^* t^* (i_{\tilde{v}} \omega^T), dt^* (i_{\tilde{v}} \omega^T))
\]

for \( v \in U^T \) where \( \tilde{v} \) is an extension of \( v \) to \( M \). In the last equation, we use that \( t^* (i_{\tilde{v}} \psi^{\text{Im}}) = -t^* (i_{\tilde{v}} \omega^T) \) and \( t^* (i_{\tilde{v}} \eta) = i_{\tilde{v}} \eta = 0 \). Under the identification

\[
NX^T \cong \wedge^1
\]

given by \( v \mapsto i_v \omega^T \), we identify \( d_0G^T \) with the map \( D_1 : \wedge^1 \to \wedge^0 \oplus \wedge^2 \) defined by

\[
D_1(\alpha) = (d^* \alpha, d\alpha)
\]

for \( \alpha \in \wedge^1 \). Then it turns out that

\[
\ker(D_1) = \{ \alpha \in \wedge^1 \mid d^* \alpha = d\alpha = 0 \} = \mathcal{H}^1(X).
\]

Now we provide a complex as follows

\[
0 \to \wedge^1 \xrightarrow{D_1} \wedge^0 \oplus \wedge^2 \xrightarrow{D_2} \wedge^3 \to 0
\]
where the operator $D_2$ is given by

$$D_2(f, \beta) = d\beta$$

for $(f, \beta) \in \wedge^0 \oplus \wedge^2$. It is easy to see that

$$P_1(\alpha) = \Delta_1 \alpha,$$

$$P_2(f, \beta) = (\Delta_0 f, \Delta_2 \beta).$$

Hence $P_1$ and $P_2$ are elliptic. It follows from $\text{Im}(G^T) \subset d^* \wedge^1 \oplus d\wedge^1$ that $\text{Im}(G^T)$ is perpendicular to $\text{Ker} P_2(= \mathcal{H}^0(X) \oplus \mathcal{H}^2(X))$ and $\text{Im}(D_2^*)$. Hence we obtain $\text{Im}(G^T) \subset \text{Im}(D_1)$ by the Hodge decomposition $\wedge^0 \oplus \wedge^2 = \text{Ker}(D_1) \oplus \text{Im}(D_2^*)$. Proposition 2.2 implies that $\mathcal{N}_X^T$ is smooth at $0_X$ with the tangent space $\text{Ker}(D_1) = \mathcal{H}^1(X)$, and hence it completes the proof.

\section{Further results}

In this section, we assume that $(M, g)$ is a simply connected and compact Riemannian manifold of dimension $(2n + 1)$.

\subsection{Sasaki manifolds with almost transverse Calabi-Yau structures}

The metric cone of a Sasaki manifold $(M, g)$ with an almost transverse Calabi-Yau structure is not a Calabi-Yau manifold unless $(M, g)$ is Sasaki-Einstein. A Kähler manifold with a non-vanishing holomorphic volume form is called an \textit{almost Calabi-Yau manifold}. We refer to \cite{7} for an almost Calabi-Yau manifolds. We provide a generalization of Proposition 3.12 as follows

\begin{proposition}
The Riemannian manifold $(M, g)$ has a contact form $\eta$ and a complex valued $n$-form $\psi$ such that $(\psi, \frac{1}{2}d\eta)$ is an almost transverse Calabi-Yau structure on $(M, \mathcal{F})$, where $\mathcal{F}$ is the Reeb foliation induced by $\eta$, with $d\psi = \kappa \sqrt{-1} \eta \wedge \psi$ for a real constant $\kappa$ if and only if the metric cone $(C(M), \overline{g})$ is an almost Calabi-Yau manifold with a non-vanishing holomorphic section $\Omega$ of $K_{C(M)}$ and a Kähler form $\omega$ such that

$$\Omega \wedge \overline{\Omega} = r^{2(\kappa - n - 1)} c_{n+1} \omega^{n+1} \quad (15)$$

and

$$L_r \frac{\partial}{\partial r} \Omega = \kappa \Omega,$$

$$L_r \frac{\partial}{\partial r} \omega = 2 \omega.$$

\end{proposition}

\begin{proof}
We assume that there exists an almost transverse Calabi-Yau structure $(\psi, \frac{1}{2}d\eta)$ with $d\psi = \kappa \sqrt{-1} \eta \wedge \psi$. Then the Kähler form $\omega$ is given by $\omega = d(\frac{1}{2} r^2 \eta)$ and satisfies $L_r \frac{\partial}{\partial r} \omega = 2 \omega$. We define $\psi'$ as the $n$-form

$$\psi' = r^\kappa \psi$$

and
on $C(M)$ and $\Omega$ as the $(n+1)$-form
\[ \Omega = \left( \frac{dr}{r} + \sqrt{-1} \eta \right) \wedge \psi' \]
on $C(M)$. Then $\Omega$ is the non-vanishing holomorphic $(n+1)$-form satisfying $L_{\frac{\partial}{\partial \theta}} \Omega = d_i \frac{\partial}{\partial \theta} \Omega = d\psi' = \kappa \Omega$. The equation (13) implies that
\[ \Omega \wedge \overline{\Omega} = 2(-1)^n \sqrt{-1} r^{-1} dr \wedge \eta \wedge \psi' \wedge \overline{\psi'} \]
Hence
\[ \psi' \wedge \overline{\psi'} = c_n r^{2k} \left( \frac{1}{2} d\eta \right)^n. \]
Remark 5.2. In Proposition \ref{prop:5.1}, the holomorphic \((n + 1)\)-form \(\Omega\) induces the calibration \(\Omega^{Re} \) with respect to the metric \(r^{-\kappa} \mathcal{F}\) where \(\kappa' = 2(\frac{1}{n+1} - 1)\). However, \(r^{-\kappa} \mathcal{F}\) is not a Kähler metric unless \(\kappa' = 0\) which is equal to \(\kappa = n + 1\).

Let \((\eta, \xi, \Psi, g)\) be a Sasaki structure on \(M\). Then we can deform \((\eta, \xi, \Psi, g)\) to another Sasaki structure \((\eta_a, \xi_a, \Psi_a, g_a)\) given by

\[
g_a = ag + (a^2 - a)\eta \otimes \eta, \quad \eta_a = a\eta, \quad \xi_a = \frac{1}{a}\xi, \quad \Psi_a = \Psi
\]

for \(a > 0\). These deformations are called the \textit{D-homothety transformations} \cite{17}. The D-homothety transformation induces the rescaling \(ag^T\) of the transverse metric \(g^T\).

Indeed, we have \(g_a^T = ag^T\).

Proposition 5.3. Let \(\kappa\) be a non-negative constant. Then the Riemannian manifold 
\((M, g)\) has a contact form \(\eta\) and a complex valued \(n\)-form \(\psi\) such that \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structure on 
\((M, \mathcal{F})\), where \(\mathcal{F}\) is the Reeb foliation induced by \(\eta\), with \(d\psi = \kappa\sqrt{-1} \eta \wedge \psi\) if and only if the metric \(g\) is Sasakian such that \(g^T\) is a transverse Kähler-Einstein metric with the Einstein constant \(2\kappa\).

Proof. The case of \(\kappa = 0\) follows from the transverse Yau’s Theorem \cite{2}. Hence it suffices to show the case of \(\kappa > 0\). If \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structure such that \(d\psi = \kappa\sqrt{-1} \eta \wedge \psi\). We take \(a = \frac{\kappa}{n+1}\) and define \(\psi_a\) and \(\eta_a\) as \(a^\frac{2}{\kappa}\psi\) and \(a\eta\), respectively. Then \((\psi_a, \frac{1}{2}d\eta_a)\) is an almost transverse Calabi-Yau structure with respect to the metric \(g_a = ag\) such that \(d\psi_a = (a + 1)^{\frac{2}{\kappa}} \eta_a \wedge \psi_a\).

It follows from Proposition \ref{prop:3.12} that \(g_a\) is a Sasaki-Einstein metric. Moreover, Proposition \ref{prop:3.12} implies that \(\text{Ric}_{g_a} = 2(n + 1)g_a^T\). Then the transverse metric \(g^T\) of \(g\) is transverse Kähler-Einstein with the Einstein constant \(2\kappa\) since the Ricci tensor is invariant of the rescaling of the metric and \(\text{Ric}_{g} = \text{Ric}_{g_a} = 2(n + 1)g_a^T = 2\kappa g^T\).

Conversely, we assume that \(g\) is a Sasaki metric with \(Ric^T = 2\kappa g^T\). Then the D-homothety transformation \((\eta_a, \xi_a, \Psi_a, g_a)\) is a Sasaki-Einstein structure for \(a = \frac{\kappa}{n+1}\) since \(\text{Ric}_{g_a} = \text{Ric}_{g^T} = 2\kappa g^T = 2(n + 1)g_a^T\). It follows from Proposition \ref{prop:3.12} that there exists an almost transverse \(\text{SL}_n(\mathbb{C})\) structure \(\psi_0\) on \((M, \mathcal{F}_\xi)\) such that \(d\psi_0 = (n + 1)^{\frac{2}{\kappa}} \eta_a \wedge \psi_0\). We define \(\psi\) as \(a^{-\frac{2}{\kappa}}\psi_0\). Then \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structure on \((M, \mathcal{F}_\xi)\) such that \(d\psi = \kappa\sqrt{-1} \eta \wedge \psi\), and hence we finish the proof.

\[\blacksquare\]

Remark 5.4. In the proof of Proposition \ref{prop:5.3}, the pair \((\psi, \frac{1}{2}d\eta)\) induces the almost transverse Calabi-Yau structure \((\psi_a, \frac{1}{2}d\eta_a)\) on \(M\). On the cone \(C(M)\), the Calabi-Yau structure \((\Omega_a, \omega_a)\) is defined by \(\Omega_a = (\frac{2\alpha}{r_a} + \sqrt{-1}\eta_a) \wedge r_a^{n+1}\psi_a\) and \(\omega_a = \frac{1}{2}d(r_a^2 \eta_a)\) where \(r_a\) is the function \(r^a\). Let \((\Omega, \omega)\) be the almost Calabi-Yau structure corresponding to \((\psi, \frac{1}{2}d\eta)\). Then the relations \(\Omega_a = a^{\frac{2}{\kappa} + 1}\Omega\) and \(\omega_a = \frac{2}{\kappa}d(r_a^2 \eta)\) hold.

A Sasaki manifold \((M, g)\) is called an \(\eta\)-\textit{Sasaki-Einstein manifold} if there exists a constant \(\lambda\) such that \(\text{Ric}_g = \lambda g + (2n - 2 - \lambda)\eta \otimes \eta\). The condition of \(\eta\)-Sasaki-Einstein for the constant \(\lambda\) is equivalent that \(g^T\) is a transverse Kähler-Einstein metric with the Einstein constant \(\lambda + 2\). Hence it follows from Proposition \ref{prop:5.3} that \((M, g)\) has an almost transverse Calabi-Yau structure \((\psi, \frac{1}{2}d\eta)\) with \(d\psi = \kappa\sqrt{-1} \eta \wedge \psi\) for \(\kappa \geq 0\) if and only if the metric \(g\) is \(\eta\)-Sasaki-Einstein for the constant \(\lambda = 2\kappa - 2\).
5.2 The automorphism group \( \text{Aut}(\eta, \psi) \)

Let \((M, g)\) be a Sasaki manifold with a Sasaki structure \((\eta, \xi, \Psi, g)\). We assume that there exists a complex valued \(n\)-form \(\psi\) such that \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structure on \((M, \mathcal{F}_\xi)\) with \(d\psi = \kappa \sqrt{-1} \eta \wedge \psi\) for a real constant \(\kappa\). In this section, we consider the group

\[
\text{Aut}(\eta, \psi) = \{ f \in \text{Diff}(M) \mid f^*\eta = \eta, \ f^*\psi = \psi \}
\]

of automorphisms preserving \((\eta, \psi)\). We also define \(\text{Aut}(\eta, [\psi])\) as the group of diffeomorphisms preserving \(\eta\) and the conformal class \([\psi]\):

\[
\text{Aut}(\eta, [\psi]) = \{ f \in \text{Diff}(M) \mid f^*\eta = \eta, \ f^*\psi = \psi, \ h \in \wedge^0 \otimes \mathbb{C} \}.
\]

Then we have

**Lemma 5.5.** \(\text{Aut}(\eta, [\psi]) = \{ f \in \text{Diff}(M) \mid f^*\eta = \eta, \ f^*\psi = e^{\sqrt{-1}\theta}\psi, \ \theta \in H^0(M) \}\)

**Proof.** If \(f^*\psi = \psi\) for a function \(h\), then \(d(f^*\psi) = dh \wedge \psi + hd\psi = dh \wedge \psi + \sqrt{-1} h\eta \wedge \psi\) and \(d(f^*\psi) = f^*(d\psi) = \eta \wedge f^*\psi\). Hence \(\overline{\partial}_T h = d\xi h = 0\) and so \(h\) is constant. Moreover, the norm \(\|h\|\) of \(h\) is 1 by taking the pull-back of \(\psi \wedge \overline{\psi} = c_n(\frac{1}{2}d\eta)^n\) by \(f\). Hence \(h = e^{\sqrt{-1}\theta}\) for a real constant \(\theta\).

From now on, we assume that \(M\) is connected. Then we can consider \(\text{Aut}(\eta, [\psi])\) as the group of phase changes \((\eta, \psi) \to (\eta, e^{\sqrt{-1}\theta}\psi)\) for \(\theta \in \mathbb{R}\). Let \(\text{Aut}(\eta, \xi, \Psi, g)\) denote the group of automorphisms preserving \((\eta, \xi, \Psi, g)\). Then we obtain

**Proposition 5.6.** \(\text{Aut}(\eta, \xi, \Psi, g) = \text{Aut}(\eta, [\psi])\)

**Proof.** We remark that \(\text{Aut}(\eta, \xi, \Psi, g) = \text{Aut}(\eta, \Psi)\). In fact, if \(f^*\eta = \eta\) and \(f_\ast \circ \Psi = \Psi \circ f_\ast\), then \(f_\ast \xi = \xi\) and \(f^* g_D = g_D\). It implies that \(f^* g = g\) since the metric \(g\) is given by \(g = g_D + \eta \otimes \eta\). Hence we will prove \(\text{Aut}(\eta, \Psi) = \text{Aut}(\eta, [\psi])\). If \(f^*\eta = \eta\) and \(f_\ast \circ \Psi = \Psi \circ f_\ast\), then \(f\) preserves the vector bundle \(D = \text{Ker} \eta\). Moreover \(f\) also preserves \(D^{0,1}\) and \(D^{1,0}\) since \(D^{0,1}\) and \(D^{1,0}\) are eigenvalue spaces of \(\Psi\). Hence \(f\) maps any transverse \((n,0)\)-form to a transverse \((n,0)\)-form by the pull back \(f^*\). In particular, \(f^*\psi = \psi\) for a function \(h\). Hence \(f \in \text{Aut}(\eta, [\psi])\). Conversely, if \(f^*\eta = \eta\) and \(f^*\psi = \psi\), then \(f\) preserves the vector bundle \(D = \text{Ker} \eta\) and \(\text{Ker} \psi\). Hence \(f\) also preserves \(D^{0,1}\) and \(D^{1,0}\). It implies that \(f_\ast \circ \Psi = \Psi \circ f_\ast\) by the definition of \(\Psi\), and hence we finish the proof.

It immediately follows that the group \(\text{Aut}(\eta, \psi)\) is the subgroup of \(\text{Aut}(\eta, \xi, \Psi, g)\). We define \(\text{aut}(\eta, \psi)\) by

\[
\text{aut}(\eta, \psi) = \{ v \in TM \mid L_v \eta = 0, \ L_v \psi = 0 \}.
\]

Let \(\text{aut}(\eta, \xi, \Psi, g)\) denote the Lie algebra of \(\text{Aut}(\eta, \xi, \Psi, g)\). Then we obtain the following relation between \(\text{aut}(\eta, \psi)\) and \(\text{aut}(\eta, \xi, \Psi, g)\).
Proposition 5.7. If $\kappa \neq 0$, then there exists the decomposition

$$\text{aut}(\eta, \psi) \oplus \langle \xi \rangle_R = \text{aut}(\eta, \xi, \Psi, g)$$

where $\langle \xi \rangle_R$ is the $\mathbb{R}$-vector space generated by $\xi$.

Proof. Proposition 5.6 implies that $\text{aut}(\eta, \xi, \Psi, g) = \{ v \in TM \mid L_v \eta = 0, L_v \psi = \sqrt{-1}c\eta, c \in \mathbb{R} \}$. The Reeb vector field $\xi$ satisfies $L_\xi \eta = 0$ and $L_\xi \psi = \sqrt{-1}\kappa \eta$. Hence the vector space $\langle \xi \rangle_R$ is a subspace of $\text{aut}(\eta, \xi, \Psi, g)$ and has the trivial intersection with $\text{aut}(\eta, \psi)$. If we take an element $v \in \text{aut}(\eta, \xi, \Psi, g)$, then there exists a real constant $c$ such that $L_v \psi = \sqrt{-1}c\eta$. The vector field $v - \frac{\kappa}{c} \xi$ is the element of $\text{aut}(\eta, \psi)$ since $L_v - \frac{\kappa}{c} \xi = 0$. Thus $v$ is in $\text{aut}(\eta, \psi) \oplus \langle \xi \rangle_R$. Hence $\text{aut}(\eta, \xi, \Psi, g)$ coincides with $\text{aut}(\eta, \psi) \oplus \langle \xi \rangle_R$, and we finish the proof.

We have the identification

$$TM \simeq \bigwedge^0 \oplus \bigwedge^1_T$$

(19)

given by $v \mapsto (i_v \eta, i_v \omega^T)$ where $\omega^T = \frac{1}{2}d\eta$.

Proposition 5.8. Under the identification (19), the Lie algebra $\text{aut}(\eta, \psi)$ is given by $\{ (f, -\frac{1}{2}df) \in \bigwedge^0_B \oplus \bigwedge^1_B \mid \Delta_B f = 4\kappa f \}$ which is isomorphic to the eigenspace $\text{Ker}(\Delta_B - 4\kappa)$ of the basic Laplacian $\Delta_B$ on $\bigwedge_B^0$.

Proof. We introduce two operators $\ast_T$ and $\ast_C$. We define an operator

$$\ast_T : \bigwedge^p_T \rightarrow \bigwedge^{2n-p}_T$$

by the formula

$$\ast \alpha = (\ast_T \alpha) \wedge \eta$$

for $\alpha \in \bigwedge^p_T$ where $\ast$ is an ordinary Hodge star operator with respect to the Riemannian metric $g$ on $M$. We can consider $\ast_T$ as an operator $\bigwedge^p_T \otimes \mathbb{C} \rightarrow \bigwedge^{2n-p}_T \otimes \mathbb{C}$ by the linearly extension. Let $\text{vol}_T$ denote the transverse volume form with respect to the metric $g^T$. Then $i_v \text{vol}_T = \ast_T v^\sharp$ for any $v \in \Gamma(Q)$ where $v^\sharp$ is the transverse 1-form defined by $v^\sharp(w) = g^T(v, w)$ for $w \in \Gamma(TM)$. The equations $\text{vol}_T = (\omega^T)^n$ and $(Jv)^\sharp = i_v(\omega^T)^n = \ast_T i_v \omega^T$. On the other hand, it follows from $i_J v \psi = \sqrt{-1}i_v \psi$ that $i_J v (\psi \wedge \overline{\psi}) = \sqrt{-1}(i_v \psi \wedge \overline{\psi} - (-1)^n \psi \wedge i_v \psi)$. It implies that

$$\sqrt{-1}c_n^{-1}(i_v \psi \wedge \overline{\psi} - (-1)^n \psi \wedge i_v \psi) = \ast_T i_v \omega^T. \quad (20)$$

We remark that the equation (20) holds for any element $v$ of $TM$. We define $\overline{\ast}_T$ as $\overline{\ast}_T(\alpha) = \overline{\ast_T \alpha}$ for any $\alpha \in \bigwedge^p_T \otimes \mathbb{C}$. It induces the map $\overline{\ast}_T : \bigwedge^{p,q}_T \rightarrow \bigwedge^{n-p,n-q}_T$. By taking the $(n-1, n)$-part of the equation (20), we obtain

$$\sqrt{-1}c_n^{-1} i_v \psi \wedge \overline{\psi} = \overline{\ast}_T(i_v \omega^T)^{1,0} \quad (21)$$

for any $v \in TM$. 

for any $v \in TM$. 

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Deformations of special Legendrian submanifolds

We introduce an operator

\[ *_C : \bigwedge^p_T \rightarrow \bigwedge^{n-p}_T \]

given by the formula

\[ \bar{*}_T \alpha = \sqrt{-1}c_n^{-1}(*_C \alpha) \wedge \bar{\psi} \] \hspace{1cm} (22)

for \( \alpha \in \bigwedge^p_T \). By taking the exterior derivative of the equation (22), we obtain that

\[ d\bar{*}_T \alpha = \sqrt{-1}c_n^{-1}(d - \sqrt{-1}\kappa\eta \wedge)(*_C \alpha) \wedge \bar{\psi} \] \hspace{1cm} (23)

for \( \alpha \in \bigwedge^p_T \). If \( \alpha \) is basic, then the left hand side of the equation (23) is the basic \((n - p + 1, n)\)-form \( \partial_B \bar{*}_B \alpha \in \bigwedge^{n-p+1}_B \). Hence we obtain that

\[ (d_\xi - \sqrt{-1}\kappa\eta \wedge) *_C \alpha = 0, \]

\[ \partial_T (*_C \alpha) \wedge \bar{\psi} = -\sqrt{-1}c_n \partial_B \bar{*}_B \alpha \] \hspace{1cm} (24)

where \( \alpha \in \bigwedge^p_B \).

We start to compute \( L_v \psi = 0 \) and \( L_v \eta = 0 \). By using the operator \( *_C \), the equation (21) is written by \( i_v \psi = *_C (i_v \omega^T)^{1,0} \) for any \( v \in TM \). It turns out that

\[ L_v \eta = di_v \eta + i_v d\eta = df + 2\theta \]

\[ L_v \psi = di_v \psi + i_v d\psi = (d - \sqrt{-1}\kappa\eta \wedge)i_v \psi + \sqrt{-1}i_v \eta \psi \]

\[ = (d - \sqrt{-1}\kappa\eta \wedge) *_C \theta^{1,0} \]

\[ = (\partial_T *_C \theta^{1,0} + \sqrt{-1}\kappa f \psi) + \bar{\partial}_T *_C \theta^{1,0} + (d_\xi - \sqrt{-1}\kappa\eta \wedge) *_C \theta^{1,0} \]

where \( f = i_v \eta \) and \( \theta = i_v \omega^T \).

If \( v \) satisfies \( L_v \psi = 0 \) and \( L_v \eta = 0 \), then \( f \) and \( \theta \) are basic since \( \theta = -\frac{1}{2} df \) is transverse. It follows from the equation (25) that \( (d_\xi - \sqrt{-1}\kappa\eta \wedge) *_C \theta^{1,0} = 0 \). We have

\[ (\partial_T *_C \theta^{1,0} + \sqrt{-1}\kappa f \psi) \wedge \bar{\psi} = \partial_T *_C \theta^{1,0} \wedge \bar{\psi} + \sqrt{-1}\kappa f \psi \wedge \bar{\psi} \]

\[ = -\sqrt{-1}c_n \partial_B \bar{*}_B \theta^{1,0} + \sqrt{-1}c_n \kappa f \bar{*}_B \]

\[ = \sqrt{-1}c_n \bar{*}_B (\bar{\partial}_B \theta^{1,0} + \kappa f). \]

It implies that \( \partial_T *_C \theta^{1,0} + \sqrt{-1}\kappa f \psi = *_C (\bar{\partial}_B \theta^{1,0} + \kappa f) \) by the definition of \( *_C \). Hence we obtain

\[ L_v \psi = -\frac{1}{2} *_C (\bar{\partial}_B \partial_B f - 2\kappa f) - \frac{1}{2} \bar{\partial}_T *_C \partial_B f \]

since \( \theta^{1,0} = -\frac{1}{2} \partial_B f \). Thus \( v \) satisfies \( L_v \eta = 0 \) and \( L_v \psi = 0 \) if and only if the corresponding \((f, \theta)\) satisfies

\[ \theta = -\frac{1}{2} df \]

\[ \Box_B f = 2\kappa f \]

\[ \bar{\partial}_T *_C \partial_B f = 0 \] \hspace{1cm} (26)

\[ (27) \]
where $\Box_B$ is the basic complex Laplace operator $\partial_B^* \partial_B + \partial_B \partial_B^* : \wedge_B^{p,q} \to \wedge_B^{p,q}$.

In order to see that the equation (27) is induced by (26), automatically, we identify $\wedge_B^{1,0}$ with $\wedge_B^{1,n} \otimes K_D^{-1}$ where $K_D$ is the basic canonical bundle $\wedge_B^{0,0}$. Let $\omega$ be the basic canonical connection of $K_D$ and $\nabla_\omega$ the covariant derivative on $K_D^{-1}$. The basic $2n$-form $(\omega^T)^n$ defines the metric of $K_D^{-1}$. Then we consider the Laplace operator $\Box_\omega : \wedge_B^{p,q} \otimes K_D^{-1} \to \wedge_B^{p,q} \otimes K_D^{-1}$ given by $\Box_\omega = \nabla_\omega^{1,0}(\nabla_\omega^{1,0})^* + (\nabla_\omega^{1,0})^* \nabla_\omega^{1,0}$ where $\nabla_\omega^{1,0}$ means the basic $(1,0)$-part of $\nabla_\omega$ and $(\nabla_\omega^{1,0})^*$ is the dual operator of $\nabla_\omega^{1,0}$. Then we see that

$$\Box_\omega = \Box_B$$

since $\nabla_\omega^{1,0}$ coincides with the operator $\partial_B$ under the identification $\wedge_B^{1,0} \simeq \wedge_B^{1,n} \otimes K_D^{-1}$.

As the same manner, we define the operator $\Box_\omega : \wedge_B^{p,q} \otimes K_D^{-1} \to \wedge_B^{p,q} \otimes K_D^{-1}$ given by $\Box_\omega = \nabla_\omega^{0,1}(\nabla_\omega^{0,1})^* + (\nabla_\omega^{0,1})^* \nabla_\omega^{0,1}$ where $\nabla_\omega^{0,1}$ means the $(0,1)$-part of $\nabla_\omega$. Then we obtain

$$\Box_\omega = \Box^2 + 2\kappa$$

by the Kodaira-Akizuki-Nakano identity on the basic vector bundle $\wedge_B^{p,q} \otimes K_D^{-1}$. If we assume $\Box_B f = 2\kappa f$, then $\Box_B \partial_B f = 2\kappa \partial_B f$. By considering $\partial_B f$ as a section of $\wedge_B^{1,n} \otimes K_D^{-1}$, we have

$$\Box_\omega \partial_B f = (\Box_\omega - 2\kappa) \partial_B f = (\Box_B - 2\kappa) \partial_B f = 0.$$  

It implies that

$$(\nabla_\omega^{0,1})^* \partial_B f = 0. \quad (28)$$

Let $U_\alpha$ be a local coordinate of $M$ and $\psi_\alpha$ a basic and transversely holomorphic local frame of $K_D$ over $U_\alpha$. Hence $\nabla_\omega^{0,1} \psi_\alpha = 0$. Then there exists a function $h_\alpha$ such that $\psi = e^{h_\alpha} \psi_\alpha$ and $\bar{T} h_\alpha = 0$. By considering $\partial_B f$ as $\partial_B f \wedge \overline{\psi}_\alpha \overline{\psi}_\alpha^{-1} = \partial_B f \wedge \overline{\psi} \wedge e^{-h_\alpha} \psi_\alpha^{-1}$, we obtain

$$\nabla_\omega^{0,1} \overline{T} \bar{\partial} B f = \nabla_\omega^{0,1} \overline{T} (\partial_B f \wedge \overline{\psi} \wedge e^{-h_\alpha} \psi_\alpha^{-1})$$

$$= \overline{T} \overline{T} \overline{T} (\partial_B f \wedge \overline{\psi}) \wedge e^{-h_\alpha} \psi_\alpha^{-1} + \overline{T} (\partial_B f \wedge \overline{\psi}) \wedge \nabla_\omega^{0,1} (e^{-h_\alpha} \psi_\alpha^{-1})$$

$$= \sqrt{-1} 2^n c_n^{-1} (\overline{T} \overline{T} \overline{T} \partial_B f) \wedge e^{-h_\alpha} \psi_\alpha^{-1}$$

where the last equation is induced by $\star_C \alpha = -\sqrt{-1} 2^{n} c_n \overline{T} (\alpha \wedge \overline{\psi})$ for $\alpha \in \wedge_T^{0,0}$. Thus the equation (28) implies $\overline{T} \overline{T} \overline{T} \partial_B f = 0$. Hence (26) implies (27).

Under the identification (19), an element $v \in \text{aut}(\eta, \psi)$ corresponds to $(f, \theta)$ such that $\theta = -\frac{1}{2} df$ and $\Box_B f = 2\kappa f$:

$$\text{aut}(\eta, \psi) = \{(f, \theta) \in \wedge^0 \oplus \wedge^1_B \mid \theta = -\frac{1}{2} df, \Box_B f = 2\kappa f\}$$

$$= \{(f, -\frac{1}{2} df) \in \wedge^0_B \oplus \wedge^1_B \mid \Delta_B f = 4\kappa f\},$$

and hence it completes the proof.

\[ \square \]

**Remark 5.9.** Futaki, Ono and Wang introduced a Hamiltonian holomorphic vector field on Sasaki manifolds in order to consider an obstruction to the existence of
transverse Kähler-Einstein metric \([4]\). They showed that the complex vector space of normalized Hamiltonian holomorphic vector fields is isomorphic to \(\text{Ker}(\Box_B - 2(n+1))\) on a Sasaki-Einstein manifold \(M\). Hence \(\text{aut}(\eta, \psi)\) is isomorphic to the real part of the space of normalized Hamiltonian holomorphic vector fields on \(M\).

Proposition 5.7 and Proposition 5.8 yield the following

**Corollary 5.10.**

(i) If \(\kappa > 0\), then \(\text{aut}(\eta, \psi) \oplus \langle \xi \rangle_\mathbb{R} = \text{aut}(\eta, \xi, \Psi, g)\).

(ii) If \(\kappa < 0\), then \(\text{aut}(\eta, \psi) = \{0\}\) and \(\text{aut}(\eta, \xi, \Psi, g) = \langle \xi \rangle_\mathbb{R}\).

(iii) If \(\kappa = 0\), then \(\text{aut}(\eta, \psi) = \langle \xi \rangle_\mathbb{R}\).

**Remark 5.11.** In the case \(\kappa \leq 0\), it is known that \(\text{aut}(\eta, \xi, \Psi, g) = \langle \xi \rangle_\mathbb{R}\) in Theorem 8.1.14 \([1]\). Hence we can see that \(\text{aut}(\eta, \psi) = \langle \xi \rangle_\mathbb{R}\) if \(\kappa = 0\).

5.3 Special Legendrian submanifolds in Sasaki manifolds with almost transverse Calabi-Yau structures

Let \((M, g)\) be a Sasaki manifold with a Sasaki structure \((\xi, \eta, \Psi, g)\). We assume that there exists a complex valued \(n\)-form \(\psi\) such that \((\psi, \frac{1}{2}d\eta)\) is an almost transverse Calabi-Yau structure on \((M, \mathcal{F}_\xi)\) with \(d\psi = \sqrt{-1} \kappa \eta \wedge \psi\) for a real number \(\kappa\). Then \((\psi, \frac{1}{2}d\eta)\) induces an almost Calabi-Yau structure \((\Omega, \omega)\) on the metric cone \((C(M), \overline{g})\) as in Proposition 5.1. An \((n + 1)\)-dimensional submanifold \(\tilde{X}\) in \(C(M)\) is called a special Lagrangian submanifold if \(\tilde{i}^* \Omega^{\text{Re}} = \tilde{i}^* \omega = 0\) where \(\tilde{i}\) is the embedding \(\tilde{i} : \tilde{X} \hookrightarrow C(M)\). We consider such submanifolds of cone type.

**Definition 5.12.** A submanifold \(X\) in \(M\) is special Legendrian if the cone \(C(X)\) is a special Lagrangian submanifold in \(C(M)\).

Then we obtain

**Proposition 5.13.** Any special Legendrian submanifold is minimal.

**Proof.** Let \(X\) be a special Legendrian submanifold. Then the special Lagrangian cone \(C(X)\) is a calibrated submanifold with respect to the metric \(r^{\kappa'} \overline{g}\) where \(\kappa' = 2(\frac{n+1}{n+2})-1\). In fact, the holomorphic \((n + 1)\)-form \(\Omega\) induces the calibration \(\Omega^{\text{Re}}\) with respect to the metric \(r^{\kappa'} \overline{g}\). Then special Lagrangian submanifolds are given by calibrated submanifolds. Therefore \(C(X)\) is minimal with respect to \(r^{\kappa'} \overline{g}\) and the mean curvature vector field \(\tilde{H}'\) vanishes. Let \(\tilde{H}\) be the mean curvature vector field of \(C(X)\) with respect to \(\overline{g}\). Then \(\tilde{H}\) also vanishes since \(\tilde{H}' = r^{-\kappa'} \tilde{H}\). Hence \(C(X)\) is minimal with respect to \(\overline{g}\), and \(X\) is a minimal submanifold with respect to \(g\). \(\square\)

We provide a characterization of special Legendrian submanifolds:

**Proposition 5.14.** An \(n\)-dimensional submanifold \(X\) in \(M\) is a special Legendrian submanifold if and only if \(\tilde{i}^* \psi^{\text{Im}} = \tilde{i}^* \eta = 0\).
Proof. By repeating the argument of the proof of Proposition\textsuperscript{[4,2]} we can show that the embedding \( i : C(X) \hookrightarrow C(M) \) is special Lagrangian if and only if \( \iota^* \Omega^\text{Re} = \iota^* \eta = 0 \). The condition \( \iota^* \Omega^\text{Re} = \iota^* \eta = 0 \) is equivalent to \( \iota^* \psi^\text{Im} = \iota^* \eta = 0 \) since \( \Omega = (\frac{d\iota}{\iota} + \sqrt{-1} \eta) \wedge r^\kappa \psi \). Hence \( X \) is a special Legendrian submanifold if and only if \( \iota^* \psi^\text{Im} = \iota^* \eta = 0 \).

Let \( X \) be a compact connected special Legendrian submanifold and \( \mathcal{M}_X \) the moduli space of special Legendrian deformations of \( X \).

**Theorem 5.15.** The infinitesimal deformation space of \( X \) is isomorphic to the eigenspace \( \text{Ker}(\Delta_0 - 2\kappa) \) of \( \Delta_0 \) with eigenvalue \( 2\kappa \). If \( \kappa = 0 \), then \( X \) is rigid and \( \mathcal{M}_X \) is a 1-dimensional manifold. If \( \kappa < 0 \), then \( X \) does not have non-trivial deformations and \( \mathcal{M}_X = \{0_X\} \).

**Proof.** Proposition\textsuperscript{[5,14]} implies that \( \mathcal{M}_X \) is the moduli space \( \mathcal{M}_X(\psi^\text{Im}, \eta) \) of \( (\psi^\text{Im}, \eta) \)-deformations of \( X \). We take the set \( U \) as in \textsuperscript{(1)} and define the map \( F : U \to \wedge^0 \oplus \wedge^1 \) by

\[
F(v) = (\ast \exp_v^* \psi^\text{Im}, \exp_v^* \eta)
\]

for \( v \in U \). We can regard \( F^{-1}(0) \) as a set of special Legendrian submanifolds in \( M \) which is close to \( X \) in \( C^1 \) sense. Then we have

\[
d_0 F(v) = (\ast \iota^* L_\psi^\text{Im}, \iota^* L_\eta)
\]

\[
= (\ast d\iota^* (i_\tilde{v}^2 \psi^\text{Im}) + \kappa \iota^* (i_\tilde{v} \eta \psi^\text{Re}), d\iota^* (i_\tilde{v} \eta) + \iota^* (i_\tilde{v} d\eta))
\]

\[
= (\frac{1}{2} d\iota^* (i_\tilde{v} d\eta) + \kappa \iota^* (i_\tilde{v} \eta), d\iota^* (i_\tilde{v} \eta) + \iota^* (i_\tilde{v} d\eta))
\]

for \( v \in U \) where \( \tilde{v} \) is an extension of \( v \) to \( M \). In the last equation, we use that \( \iota^* (i_\tilde{v} \psi^\text{Im}) = -\frac{1}{2} \ast \iota^* (i_\tilde{v} d\eta) \). Under the identification

\[
NX \cong \wedge^0 \oplus \wedge^1
\]

given by \( v \mapsto (i_v \eta, \frac{1}{2} i_v d\eta) \), we can consider \( d_0 F \) as the map \( d_0 F : \wedge^0 \oplus \wedge^1 \to \wedge^0 \oplus \wedge^1 \) defined by

\[
d_0 F(f, \alpha) = (d^* \alpha + \kappa f, d\alpha + 2\alpha)
\]

for \( (f, \alpha) \in \wedge^0 \oplus \wedge^1 \). Then it turns out that

\[
\text{Ker}(d_0 F) = \{(f, \alpha) \in \wedge^0 \oplus \wedge^1 \mid d^* \alpha + \kappa f = 0, 2\alpha + df = 0\}
\]

\[
= \{(f, -\frac{1}{2} df) \in \wedge^0 \oplus \wedge^1 \mid (\Delta_0 - 2\kappa) f = 0\}
\]

and hence the infinitesimal deformation space of \( X \) is isomorphic to the space \( \text{Ker}(\Delta_0 - 2\kappa) \). In the case of \( \kappa = 0 \), we already proved that the moduli space \( \mathcal{M}_X \) is a 1-dimensional smooth manifold in Proposition\textsuperscript{[2,2,3]}. Then the 1-dimensional deformation space is given by \( \langle \xi \rangle_\text{R} = \text{aut}(\eta, \psi) \). Thus \( X \) is rigid. If \( \kappa < 0 \), then \( \text{Ker}(\Delta_0 - 2\kappa) = \{0\} \) and \( X \) has only the trivial deformation. Hence we finish the proof. \( \square \)
Let $\omega^T$ be the transverse 2-form $\frac{1}{2}d\eta$ on $M$. We denote by $\mathcal{N}_X$ the moduli space $\mathcal{M}_X(\psi^{\text{Im}}, \omega^T)$ of $(\psi^{\text{Im}}, \omega^T)$-deformations of $X$. We fix an integer $s \geq 3$ and a real number $\alpha$ with $0 < \alpha < 1$ and set $\mathcal{N}_X^{s,\alpha}$ as the moduli space $\mathcal{M}_X^{s,\alpha}(\psi^{\text{Im}}, \omega^T)$ of $(\psi^{\text{Im}}, \omega^T)$-deformations of $C^{s,\alpha}$-class.

**Proposition 5.16.** The moduli space $\mathcal{N}_X^{s,\alpha}$ is smooth at $0_X$. If $\kappa \neq 0$, then the tangent space $T_{0_X} \mathcal{N}_X^{s,\alpha}$ is isomorphic to $\{(-\frac{1}{\kappa}d^*\alpha, \alpha) \in C^{s,\alpha}(\Lambda^0 \oplus \Lambda^1) \mid d\alpha = 0\}$. If $\kappa = 0$, then $T_{0_X} \mathcal{N}_X^{s,\alpha}$ is isomorphic to $C^{s,\alpha}(\Lambda^0) \oplus H^1(X)$.

**Proof.** We take the set $U$ as in (1) and define the map $G : U \to \Lambda^0 \oplus \Lambda^2$ by

$$G(v) = (\ast \exp_v^{\text{Im}}, \exp_v^T)$$

for $v \in U$. It follows that

$$d_0 G(v) = (\ast (dv(i\bar{v}\psi^{\text{Im}}) + (n + 1)\iota(i\bar{v}\eta\psi^{\text{Re}}) - \kappa \iota(\eta \wedge i\bar{v}\psi^{\text{Re}}), dv(i\bar{v}\omega^T))$$

for $v \in U$ where $\bar{v}$ is an extension of $v$ to $M$. In the last equation, we use that $\iota^*(i\bar{v}\psi^{\text{Im}}) = -\frac{1}{2}\iota^*(i\bar{v}d\eta)$ and $\iota^*\psi^{\text{Re}} = \text{vol}(X)$. Under the identification $NX \cong \Lambda^0 \oplus \Lambda^1$ given by $v \mapsto (i\bar{v}\eta, i\bar{v}\omega^T)$, we identify $d_0 G$ with the map $D_1 : \Lambda^0 \oplus \Lambda^1 \to \Lambda^0 \oplus \Lambda^2$ defined by

$$D_1(f, \alpha) = (d^*\alpha + \kappa f, d\alpha)$$

for $(f, \alpha) \in \Lambda^0 \oplus \Lambda^1$. Now we provide a complex as follows

$$0 \to \Lambda^0 \oplus \Lambda^1 \overset{D_1}{\to} \Lambda^0 \oplus \Lambda^2 \overset{D_2}{\to} \Lambda^3 \to 0$$

where the operator $D_2$ is given by

$$D_2(f, \beta) = d\beta$$

for $(f, \beta) \in \Lambda^0 \oplus \Lambda^2$. It is easy to see that

$$P_1(f, \alpha) = (\kappa^2 f + \kappa d^*\alpha, \Delta_1 \alpha + \kappa df),$$

$$P_2(f, \beta) = ((\Delta_0 + \kappa^2) f, \Delta_2 \beta).$$

Hence $P_2$ is the elliptic operator. It follows from $\text{Im}(G) \subset \Lambda^0 \oplus d\Lambda^1$ that $\text{Im}(G)$ is perpendicular to $\text{Ker} P_2 = \{0\} \oplus \mathcal{H}^2(X))$ and $\text{Im}(D_2^*)$. Hence we obtain $\text{Im}(G) \subset \text{Im}(D_1)$ by the Hodge decomposition $\Lambda^0 \oplus \Lambda^2 = \text{Ker} P_2 \oplus \text{Im}(D_1) \oplus \text{Im}(D_2^*)$. Proposition 2.2 implies that $\mathcal{N}_X^{s,\alpha}$ is smooth at $0_X$ with the tangent space $\text{Ker}(D_1^{s,\alpha})$. If $\kappa \neq 0$, then it turns out that

$$\text{Ker}(D_1) = \{(f, \alpha) \in \Lambda^0 \oplus \Lambda^1 \mid d^*\alpha + \kappa f = 0, d\alpha = 0\}$$

$$\text{Ker}(D_1) = \{(f, \alpha) \in \Lambda^0 \oplus \Lambda^1 \mid d^*\alpha = d\alpha = 0\} = \Lambda^0 \oplus \mathcal{H}^1(X).$$

If $\kappa = 0$, then

$$\text{Ker}(D_1) = \{(f, \alpha) \in \Lambda^0 \oplus \Lambda^1 \mid \Delta_1 \alpha = 0\} = \Lambda^0 \oplus \mathcal{H}^1(X).$$

Hence we finish the proof. \qed
Let $\mathcal{L}_X$ be the moduli space of Legendrian deformations of $X$ of $C^\infty$-class and $\mathcal{L}_X^{s,\alpha}$ that of $C^{s,\alpha}$-class. The following is a generalization of Theorem 4.8.

**Theorem 5.17.** The moduli space $\mathcal{M}_X$ is the intersection $\mathcal{N}_X \cap \mathcal{L}_X$ where $\mathcal{N}_X^{s,\alpha}$ and $\mathcal{L}_X^{s,\alpha}$ are smooth.

**Proof.** The moduli space $\mathcal{M}_X$ is the intersection $\mathcal{N}_X \cap \mathcal{L}_X$ since $\mathcal{M}_X(\psi^{\text{Im}}, \eta, \omega^T) = \mathcal{M}_X(\psi^{\text{Im}}, \omega^T) \cap \mathcal{M}_X(\eta)$. The space $\mathcal{L}_X^{s,\alpha}$ is smooth by Proposition 2.7. It follows from Proposition 5.16 that $\mathcal{N}_X^{s,\alpha}$ is smooth at $\mathcal{N}_X \cap \mathcal{L}_X$. Hence it completes the proof.

Let $\mathcal{N}_X^T$ denote the moduli space $\mathcal{M}_X^T(\psi^{\text{Im}}, \omega^T)$ of transverse $(\psi^{\text{Im}}, \omega^T)$-deformations of $X$ of $C^\infty$-class. Then we obtain

**Theorem 5.18.** The moduli space $\mathcal{N}_X^T$ is smooth at $0_X$ and the tangent space $T_{0_X} \mathcal{N}_X^T$ is isomorphic to $H^1(X)$.

**Proof.** Let $U^T$ be a sufficiently small neighbourhood of the zero section in $NX^T$ and $U^T$ the set $\{v \in C^\infty(NX^T) \mid v_x \in U^T, x \in X\}$. We define the map $G^T : U^T \to \wedge^0 \oplus \wedge^2$ by

$$G^T(v) = (\ast \exp^* v^{\text{Im}}, \exp^* v^T)$$

for $v \in U^T$. Then we have

$$d_0 G^T(v) = (\ast (d \iota^* (i_{\tilde{v}} \psi^{\text{Im}}) + \kappa \iota^* (i_{\tilde{v}} \eta \psi^\text{Re}) - \kappa \iota^* (\eta \wedge i_{\tilde{v}} \psi^\text{Re})), \; d \iota^* (i_{\tilde{v}} \omega^T))$$

for $v \in U^T$ where $\tilde{v}$ is an extension of $v$ to $M$. In the last equation, we use that $\iota^* (i_{\tilde{v}} \psi^{\text{Im}}) = - \ast \iota^* (i_{\tilde{v}} \omega^T)$ and $\iota^* (i_{\tilde{v}} \eta) = i_v \eta = 0$. Under the identification $NX^T \simeq \wedge^1$ given by $v \mapsto i_v \omega^T$, we can consider $d_0 G^T$ as the map $D_1 : \wedge^1 \to \wedge^0 \oplus \wedge^2$ given by

$$D_1(\alpha) = (d \alpha, d\alpha)$$

for $\alpha \in \wedge^1$. Then it turns out that

$$\text{Ker}(D_1) = \{\alpha \in \wedge^1 \mid d \alpha = d\alpha = 0\} = \mathcal{H}^1(X).$$

Now we provide a complex as follows

$$0 \to \wedge^1 \xrightarrow{D_1} \wedge^0 \oplus \wedge^2 \xrightarrow{D_2} \wedge^3 \to 0$$

where the operator $D_2$ is given by

$$D_2(f, \beta) = d\beta$$

for $(f, \beta) \in \wedge^0 \oplus \wedge^2$. It is easy to see that

$$P_1(\alpha) = \Delta_1 \alpha,$$

$$P_2(f, \beta) = \Delta_0 f, \Delta_2 \beta.$$
Hence $P_1$ and $P_2$ are elliptic. It follows from $\text{Im}(G^T) \subset d^*\wedge^1 \oplus d\wedge^1$ that $\text{Im}(G^T)$ is perpendicular to $\text{Ker} P_2(= \mathcal{H}^0(X) \oplus \mathcal{H}^2(X))$ and $\text{Im}(D^n_2)$. Hence we obtain $\text{Im}(G^T) \subset \text{Im}(D_1)$ by the Hodge decomposition $\wedge^0 \oplus \wedge^2 = \text{Ker} P_2 \oplus \text{Im}(D_1) \oplus \text{Im}(D^n_2)$. Proposition 2.2 implies that $\mathcal{N}_X^T$ is smooth at $0_X$ with the tangent space $\text{Ker}(D_1) = \mathcal{H}^1(X)$, and hence it completes the proof. □

5.4 Minimal Legendrian deformations of special Legendrian submanifolds

We assume that there exists an almost transverse Calabi-Yau structure $(\psi, \frac{1}{\kappa}d\eta)$ on $M$ with $d\psi = \sqrt{-1}\kappa\eta \wedge \psi$ for a real number $\kappa$. Let $(\Omega, \omega)$ be the corresponding almost Calabi-Yau structure on $C(M)$. For a real constant $\theta$, the phase change $\Omega \rightarrow e^{\sqrt{-1}\theta}\Omega$ induces the almost Calabi-Yau structure $(e^{\sqrt{-1}\theta}\Omega, \omega)$ on $C(M)$. An $(n+1)$-dimensional submanifold $\tilde{X}$ in $C(M)$ is called a $\theta$-special Lagrangian submanifold for a phase $\theta$ if $i^*(e^{\sqrt{-1}\theta}\Omega)\text{Im} = i^*\omega = 0$. Hence any $\theta$-special Lagrangian submanifold is a special Lagrangian submanifold with respect to the almost Calabi-Yau structure $(e^{\sqrt{-1}\theta}\Omega, \omega)$.

Definition 5.19. A submanifold $X$ in $M$ is $\theta$-special Legendrian if the cone $C(X)$ is a $\theta$-special Lagrangian submanifold in $C(M)$ for a phase $\theta$.

Any $\theta$-special Legendrian submanifold is a special Legendrian submanifold in the Sasaki manifold $(M, g)$ with the almost transverse Calabi-Yau structure $(e^{\sqrt{-1}\theta}\psi, \frac{1}{\kappa}d\eta)$. Hence any $\theta$-special Legendrian submanifold is minimal. Moreover the converse is true in the case $\kappa > 0$. In order to see it, we consider the relation between a mean curvature vector field and a phase of $(\psi, \frac{1}{\kappa}d\eta)$. From now on, we assume that $\kappa > 0$. Let $X$ be a connected oriented Legendrian submanifold in $M$ and $H$ the mean curvature vector field of $X$ with respect to $g$. Then there exists a $\mathbb{R}/\mathbb{Z}$-valued function $\theta$ on $X$ such that $\ast i^*\psi = e^{-\sqrt{-1}\theta}$ where $\ast$ is the Hodge operator with respect to the metric $i^*g$ on $X$.

Lemma 5.20. Let $\omega^T$ be a $2$-form $\frac{1}{2}d\eta$. Then $d\theta = i^*(i_H\omega^T)$.

Proof. We take $a = \frac{\kappa}{n+1}$ and define $\psi_a$ and $\eta_a$ as $a^\frac{2}{\kappa}\psi$ and $a\eta$, respectively. Then $(\psi_a, \frac{1}{2}d\eta_a)$ is an almost transverse Calabi-Yau structure with respect to the metric $g_a = ag$ such that $d\psi_a = (n+1)\sqrt{-1}\eta_a \wedge \psi_a$ and $\ast g_a i^*\psi_a = e^{-\sqrt{-1}\theta}$ where $\ast g_a$ is the Hodge operator with respect to $i^*g_a$ on $X$. Let $H_a$ be the mean curvature vector field of $X$ with respect to $g_a$. Then it follows from the equation (14) in Proposition 3.12 that $d\theta = i^*(i_{H_a}\omega_a^T)$ where $\omega_a^T = \frac{1}{2}d\eta_a$. Hence the equation $H_a = \frac{1}{a}H$ implies that $d\theta = i^*(i_{\frac{1}{a}i_H}\omega^T) = i^*(i_H\omega^T)$. □

Proposition 5.21. Let $X$ be a connected oriented $n$-dimensional submanifold in $M$. Then the following conditions are equivalent.

(i) $X$ is minimal Legendrian.

(ii) $X$ is $\theta$-special Legendrian for a phase $\theta$. 

(iii) $d \ast \iota^* \psi^{\text{Im}} = \iota^* \eta = 0$. \hfill \Box

**Proof.** By Proposition 5.13, (ii) implies (i). We may assume that $X$ is Legendrian and there exists a $\mathbb{R}/\mathbb{Z}$-valued function $\theta$ on $X$ such that $\ast \iota^* \psi = e^{-\sqrt{-1} \theta}$. If the condition (i) holds, then it follows from Lemma 5.20 that $\theta$ is constant and $d \ast \iota^* \psi^{\text{Im}} = d(\ast \iota^* \psi)^{\text{Im}} = 0$. Hence (iii) holds. If we assume the condition (iii), then the imaginary part of $e^{-\sqrt{-1} \theta} = \ast \iota^* \psi$ is constant, and $e^{-\sqrt{-1} \theta}$ is also constant. Hence $\ast \iota^*(e^{\sqrt{-1} \theta} \psi) = e^{\sqrt{-1} \theta} \ast \iota^* \psi = 1$ and $\iota^*(e^{\sqrt{-1} \theta} \psi)^{\text{Im}} = 0$. It yields that $X$ is a special Legendrian submanifold in the Sasaki manifold with the almost transverse Calabi-Yau structure $(e^{\sqrt{-1} \theta} \psi, \tfrac{1}{2} \iota \eta)$. Therefore the condition (ii) holds. Hence we finish the proof. \hfill \Box

Let $X$ be a compact connected special Legendrian submanifold in $M$. We consider minimal Legendrian deformations of $X$.

**Proposition 5.22.** Then the infinitesimal deformation space of $X$ as a minimal submanifold is isomorphic to $\text{Ker}(\Delta_0 - 2\kappa) \oplus \mathbb{R}$.

**Proof.** We take the set $U$ as in (1) and define the map $F : U \to \Lambda^1 \oplus \Lambda^1$ by

$$F(v) = (d \ast \exp_v^* \psi^{\text{Im}}, \exp_v^* \eta)$$

for $v \in U$. Then Proposition 5.21 implies that $F^{-1}(0)$ is regarded as a set of minimal Legendrian submanifolds in $M$ which is close to $X$. It follows that

$$d_0F(v) = (d \ast \iota^* L \tilde{v}^\psi^{\text{Im}}, \iota^* L \tilde{v} \eta) = (d(\ast \iota^* (i \tilde{v} \eta \psi^\text{Re})), d \iota^* (i \tilde{v} \eta) + \iota^* (i \tilde{v} \iota \eta)) = (d(d \ast \iota^* (i \tilde{v} \iota \eta)) + \kappa \iota^* (i \tilde{v} \eta)), d \iota^* (i \tilde{v} \eta) + \iota^* (i \tilde{v} \iota \eta))$$

for $v \in U$ where $\tilde{v}$ is an extension of $v$ to $M$. Under the identification $N X \simeq \Lambda^0 \oplus \Lambda^1$ given by $v \mapsto (i \tilde{v} \eta, \tfrac{1}{2} i \tilde{v} \iota \eta)$, we identify $d_0F$ with the map $d_0F : \Lambda^0 \oplus \Lambda^1 \to \Lambda^1 \oplus \Lambda^1$ defined by

$$d_0F(f, \alpha) = (d \ast \iota \alpha + \kappa f, df + 2\alpha)$$

for $(f, \alpha) \in \Lambda^0 \oplus \Lambda^1$. Then it turns out that

$$\text{Ker}(d_0F) = \{ (f, \alpha) \in \Lambda^0 \oplus \Lambda^1 \mid d(d \ast \iota \alpha + \kappa f) = 0, 2\alpha + df = 0 \} = \{ (f, -\tfrac{1}{2} df) \in \Lambda^0 \oplus \Lambda^1 \mid d(\Delta_0 - 2\kappa)f = 0 \}.$$ 

Since $\text{Ker}(d(\Delta_0 - 2\kappa)) = \text{Ker}(\Delta_0 - 2\kappa) \oplus \text{Ker}(d|_{\Lambda^0})$ for $\kappa \neq 0$, the space $\text{Ker}(d_0F)$ is isomorphic to the space $\text{Ker}(\Delta_0 - 2\kappa) \oplus \mathbb{R}$. Hence it completes the proof. \hfill \Box

Proposition 5.22 was shown by Ohnita from the point of view of minimal Legendrian submanifolds in $\eta$-Sasaki-Einstein manifolds [14]. In the same paper, he also provided some examples of rigid minimal Legendrian submanifolds in the standard 7-sphere $S^7$ and the real Steifel manifold $V_2(\mathbb{R}^5)$ and a non-rigid minimal Legendrian submanifold in $S^7$ (he said that any deformation of $X$ was trivial instead that $X$...
was rigid). We can consider another rigidity condition for special Legendrian submanifolds. We recall that a special Legendrian submanifold $X$ is rigid if any special Legendrian deformation of $X$ is induced by the automorphism group $\text{Aut}(\eta, \psi)$ of diffeomorphisms of $M$ preserving $\eta$ and $\psi$. Any element $f$ of $\text{Aut}(\eta, \psi)$ induces the diffeomorphism $\tilde{f}$ of $C(M)$ by $\tilde{f}(r, x) = f(x)$ for $(r, x) \in \mathbb{R}_{>0} \times M = C(M)$. Then $\tilde{f}$ is the element of the group $\text{Aut}(\Omega, \omega, r)$ of diffeomorphisms of $C(M)$ preserving $\Omega$, $\omega$ and $r$. Conversely, $\text{Aut}(\Omega, \omega, r)$ induces $\text{Aut}(\eta, \psi)$ by the restriction to the hypersurface $M = \{r = 1\}$. Under the correspondence, a special Legendrian submanifold $X$ is rigid if and only if any deformations of the special Lagrangian cone $C(X)$ is induced by the group $\text{Aut}(\Omega, \omega, r)$. Hence there exist two rigidity conditions for special Legendrian submanifolds. We show that these conditions are equivalent:

**Theorem 5.23.** Let $X$ be a compact connected special Legendrian submanifold. Then $X$ is rigid as a special Legendrian submanifold if and only if it is so as a minimal Legendrian submanifold.

**Proof.** It follows from Proposition 5.15 and Proposition 5.22 that the infinitesimal deformation space of $X$ as a minimal Legendrian submanifold is the sum of that of $X$ as a special Legendrian submanifold and the 1-dimensional vector space generated by the Reeb vector field $\xi$ on $X$. Since the group $\text{Aut}(\eta, g)$ is the automorphism group $\text{Aut}(\eta, \xi, \Psi, g)$ of the Sasaki manifold (cf. Proposition 8.1.1 [1]), Proposition 5.7 implies that $\text{aut}(\eta, g) = \text{aut}(\eta, \psi) \oplus \langle \xi \rangle_{\mathbb{R}}$. Hence $X$ is rigid as a minimal Legendrian submanifold if and only if $X$ is also rigid as a special Legendrian submanifold. It completes the proof.

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