Kalb-Ramond field induced cosmological bounce in generalized teleparallel gravity

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In this paper, we consider the Kalb-Ramond (KR) field and its effects in the context of generalized teleparallel gravity in 3+1 dimensions. Teleparallel gravity is a description of gravitation in which the torsion arising from fields with spin are accommodated naturally as field strength tensor of the non-trivial tetrads. In order to describe the coupling prescription, we address the correct generalization of the Fock-Ivanenko derivative operator (FIDO) for an n-form tensor field. By varying with respect to the non-trivial tetrads, this rank-2 field is shown to source the teleparallel equivalent of Einstein's equations. We then study the possibility of reproducing two well-known cosmological bounce scenarios, namely, symmetric bounce and matter bounce in four-dimensional space-time with FLRW metric and observe that the solution requires the KR field energy density to be localized near the bounce. This feature could, in turn, explain the lack of experimental evidence in detecting the rank-2 field in the present-day universe. Furthermore, we compute the analytical expression for modifying the teleparallel gravity and energy density of the KR field for the bounce scenarios.

I. INTRODUCTION

Einstein’s General Relativity (GR) has been widely successful in explaining the data from planetary and cosmological experiments to date. In this description of gravitation, torsion-free Riemann geometry plays a crucial role. Instead of using metric as the dynamical field, one could construct an alternative description of gravity in terms of non-trivial tetrads and its field strength, torsion. This description of gravitation is called teleparallel gravity. In this framework, Lagrangian is a function of tetrads, and the torsion tensor gives its strength. Moreover, the gravitational pull on a free-falling body can be successfully described as an effect of the torsion instead of the curvature tensor. Teleparallel gravity formalism is intriguing because, aside from gravity being explained as a gauge theory of the translation group \([1]\), it also feels logical to analyse the effects of spinorial fields in terms of tetrads.

Fields transforming under non-trivial representations of the Lorentz group have been inevitable in high-energy physics. Due to the aforementioned success of GR, though not dynamical objects, ‘torsion’ or twist in the Riemannian geometry is constructed out of tetrads, and spin connections of fermions \([2]\), and higher rank anti-symmetric Kalb-Ramond (KR) field \((B_{\mu\nu})\) \([3]\). Latter, in fact, is essential to correctly reproduce the low energy string effective action \([4]\). Furthermore, massless KR field is a characteristic of any critical string theory’s massless spectrum, which emerges when the space-time is compactified to four-dimensions \([5]\). Though necessary, from a string theoretic perspective, the KR field is not yet detected in any of the experiments\([6]\).

This paper aims to address the KR field in a teleparallel setup and show that a natural explanation for its absence in the present-day Universe also leads to bouncing cosmology. We will be using generalised teleparallel description \(F(T)\) in order to accommodate its ability to explain cosmological phenomenon like the late-time acceleration of the universe \([7–14]\). The explanation for this feature is understood as the conformal equivalence of \(F(T)\) gravity is a phantom scalar field with non-minimal coupling to a boundary term \([14–16]\).

Here, we consider a generalized teleparallel gravity setup in 3 + 1 dimensions appended by an action of the Kalb-Ramond field. With the appropriate generalization of the Fock-Ivanenko derivative operator for the KR field, we compute the equivalent of Einstein’s equations by varying the action with respect to the tetrad. On the right-hand side, this gives the equivalent energy-momentum tensor of the anti-symmetric field as the source. With the setup in place, we also study the requirement to achieve bouncing cosmology. Models with bounces \([17–19]\) provide an elegant solution to the initial singularity in the Big Bang paradigm and, in some instances, could generate a scale-invariant power-law spectrum \([20]\) as well. Even though there have been immense efforts carried out in modified gravity theories with higher-order corrections\([21–22]\) and in braneworld scenarios\([23–24]\), it is interesting to understand these phenomena in the teleparallel equivalent of General Relativity (TEGR) \([25]\). We then explicitly compute the energy spectrum of the tensor field and the appropriate teleparallel gravity model for symmetric and matter bounce scenarios. And we show that the energy and pressure densities of the tensor field are indeed localized at \(t = 0\) as expected. These localized densities act as the source for the bounce and also explain why we do not observe the effects of the rank-2 tensor field in the present-day Universe.

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The paper is categorized as follows. We start with a brief review of TEGR formalism in section (II) and introduce Kalb-Ramond fields as a source of torsion. In section (III), we explain the minimal coupling prescription and develop the Fock-Ivanenko operator for the Kalb-Ramond field. As an application to cosmology, in section (IV), we compute the energy density and pressure of KR fields in the generalized teleparallel setup that will lead to correct expansion coefficients in symmetric and matter bounce scenarios. Finally, in section (V), we summarise our results.

II. TELEPARALLEL EQUIVALENT OF GENERAL RELATIVITY

In Einstein’s General Relativity (GR), the affine connection is taken to be torsionless and satisfies the metricity condition,

$$\nabla_{\mu}g_{\nu\rho} = 0$$  \hspace{1cm} (1)

where $\nabla_{\mu}$ is the covariant derivative with the Levi-Civita connection $\Gamma_{\mu}^{\nu\rho}$ playing the role of affine connection. However, in Teleparallel Gravity (TG), the Levi-Civita affine connection is replaced by the Wietzenbock connection, which is torsion-full but curvature-less and satisfies the metricity condition Eq.(1).

Although teleparallel gravity is an alternative to General Relativity, they are conceptually distinct \cite{26}. In TEGR, the space-time metric is constructed out of the dynamical non-trivial tetrad $h^a_{\mu}$, as,

$$g_{\mu\nu} = \eta_{ab}h^a_{\mu}h^b_{\nu},$$  \hspace{1cm} (2)

where $\eta_{ab}$ is the Minkowski metric of the tangent space and the tetrad $h^a_{\mu}$ is given as \cite{28},

$$h^a_{\mu} = e^a_{\mu} + \omega^a_{b\mu}x^b + B^a_{\mu}.$$  \hspace{1cm} (3)

Here, $e^a_{\mu}$ is the trivial tetrad, $\omega^a_{b\mu}$ is the purely inertial Lorentz connection, and $B^a_{\mu}$ is the translational connection corresponding to the local Lorentz transformation and local translation, respectively on the tangent space. Note that, in this paper, we will be referring Greek indices ($\mu, \nu$) to the space-time manifold and the Latin indices (a, b) to the local Minkowski tangent space $T_xM$. We also assume the signature of $\eta_{ab}$ as diag $(-+++)$.

Now, the Weitzenbock connection $\Gamma^\rho_{\mu\nu}$ can be written in terms of the tetrad as \cite{27},

$$\Gamma^\rho_{\mu\nu} = h_a\partial_{\mu}h^a_{\nu} + h_a\partial_{\nu}h^a_{\mu} + h_a\omega^a_{b\mu}h^b_{\nu}. $$  \hspace{1cm} (4)

Since we are interested in the evolution of the universe, we stick to a particular choice of tetrad with which the constructed space-time metric is flat Friedmann-Lemaitre-Robertson-Walker (FLRW) and permits the solution $\omega^a_{b\mu} = 0$ \cite{28,32}. Given this solution, one can easily show that the Weitzenbock covariant derivative of the tetrad vanishes identically, thus satisfying the metricity condition.

$$\nabla_{\mu}h^A_{\nu} \equiv \partial_{\mu}h^A_{\nu} - \Gamma^a_{\mu\nu}h^A_{a} = 0,$$  \hspace{1cm} (5)

where $\nabla_{\mu}$ represents the covariant derivative constructed with the Weitzenbock connection.

The torsion tensor is constructed from the Weitzenbock connection as given below:

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}. $$  \hspace{1cm} (6)

Straightforwardly, using Eq.(3) and Eq.(4), it can be seen that the torsion tensor acts as the field strength of the translation (gravitational) potential $B^a_{\mu}$ for zero spin.

$$T^a_{\mu\nu} = h^b_{\nu}T^a_{\mu\nu} = \partial_{\mu}B^a_{\nu} - \partial_{\nu}B^a_{\mu}. $$  \hspace{1cm} (7)

The Weitzenbock connection in teleparallel and the Levi-Civita connections in GR are then mathematically related as

$$\Gamma^\rho_{\mu\nu} - K^\rho_{\mu\nu} \equiv \tilde{\Gamma}^\rho_{\mu\nu},$$  \hspace{1cm} (8)

where $K^\rho_{\mu\nu}$ is the contorsion tensor given by

$$K^\rho_{\mu\nu} = \frac{1}{2} (T^\rho_{\nu\mu} + T^\rho_{\mu\nu} - T^\rho_{\mu\nu}). $$  \hspace{1cm} (9)

In this paper, we use over-tilde to represent quantities calculated using the Levi-Civita connection in GR to distinguish it from teleparallel gravity. It is straightforward to show that the curvature of the Weitzenbock connection also vanishes.

$$R^\rho_{\lambda\mu\nu}(\Gamma) = 0.$$  \hspace{1cm} (10)

The dual torsion tensor is defined as

$$\delta_{\rho}^{\mu\nu} = \frac{1}{2} \left[ K^\mu_{\nu\rho} - g^\mu_{\rho\nu} T^\lambda_{\mu\lambda} + g^\mu_{\nu\rho} T^\lambda_{\nu\lambda} \right].$$  \hspace{1cm} (11)

Finally, we define a quadratic function of torsion called the torsion scalar $T$ given by,

$$T = T_{\rho\mu\nu}\delta_{\rho}^{\mu\nu} = \frac{1}{2} (T^\rho_{\nu\mu} + T^\rho_{\mu\nu} - 2T^\rho_{\mu\nu}T^\nu_{\mu\rho}). $$  \hspace{1cm} (12)

The gravitational Lagrangian using the torsion scalar can be written as,

$$L_G = -\frac{h}{16\pi G}T,$$  \hspace{1cm} (13)

where $h = \text{det}(h^a_{\mu}) = \sqrt{-g}$

Using Eq.(8) in the above action and reformulating the above Lagrangian in terms of Levi-Civita connection, we can obtain the mathematical relation between the torsion scalar $T$ in teleparallel gravity and the Ricci scalar $\tilde{R}$ in GR

$$T \equiv \tilde{R} + B,$$  \hspace{1cm} (14)

where $\tilde{R}$ is the Ricci scalar and $B = 2\nabla_{\mu}(T^\nu_{\mu\nu})$ is a total divergence term. Thus this action is equivalent to the Einstein-Hilbert action, which gives Einstein’s field equations of gravity\cite{33}. 


III. COUPLING PRESCRIPTION USING FOCK–IVANEKO DERIVATIVE OPERATOR IN THE TELEPARALLEL GEOMETRY

In Minkowski space, the dynamics of the Kalb Ramond field is described by the Lagrangian \[34]\)

\[ \mathcal{L}_{KR} = -H_{abc} H^{abc} , \]

where

\[ H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab} , \]

is the field strength of the Kalb-Ramond field \(B_{ab}\), which is a rank-2 anti-symmetric tensor field.

On varying the action with respect \(B_{ab}\), we get the field equations.

\[ \partial_a H^{abc} = 0 , \]

along with the Bianchi identity

\[ \partial_{[a} H_{bcd]} = 0 , \]

For the Lorentz gauge \(\partial_a B_{ab} = 0\), the field equation \(\text{Eq.}(17)\) becomes

\[ \partial_a \partial^a B^{ab} = 0 . \]

But in teleparallel gravity, the existence of torsion destroys the gauge invariance of the theory when the KR field is used as the source of the equation of motion of the field. If we assume the coupling prescription given by,

\[ \eta^{ab} \rightarrow g^{\mu\nu} = \eta^{ab} h^a_{\mu} h^b_{\nu} \]

\[ \partial_a \rightarrow \nabla_\mu = \partial_\mu - \Gamma_\mu \]

the KR-field strength takes the form,

\[ H_{\mu\nu\rho} = \nabla_\mu B_{\nu\rho} + \nabla_\rho B_{\mu\nu} + \nabla_\nu B_{\rho\mu} = 3\partial_{[\mu} B_{\nu\rho]} + 3T_{\mu\nu\rho} \]

The last term in \(\text{Eq.}(21)\) indicates the non-minimal coupling of torsion with the KR field in teleparallel geometry and thus \(\text{Eq.}(21)\) is not invariant under \(U(1)\) gauge transformation.

In order to keep the transformation gauge invariant, in the framework of teleparallel geometry, one needs to use the minimal coupling prescription\[35,\]

\[ \eta^{ab} \rightarrow g^{\mu\nu} = \eta^{ab} h^a_{\mu} h^b_{\nu} \]

\[ \partial_a \rightarrow \nabla_\mu = \partial_\mu - \frac{i}{2} \Omega^{ab}_{\mu} J_{ab} , \]

where \(D_\mu\) is the Fock–Ivanenko Derivative Operator (FIDO) \[35,\], which acts only on the local Lorentz indices. Here, \(\Omega^{ab}_{\mu}\) is the spin connection given by,

\[\Omega^{ab}_{\mu} = -h^a_{\rho} K^{\rho\mu} h^b_{\nu} , \]

and \(J_{ab}\) is the generator in the appropriate representation of the Lorentz group. For instance, \(J_{ab}\) acting on any \(n\)-form field could be written as,

\[ J_{ab}(B^{i_1 i_2 ... i_n}) = i(\delta^{a_1}_{i_1} \eta_{b_{c_1}} - \delta^{b_{i_1}}_{a_{c_1}}) B^{c_1 ... i_n} \]

\[ + i(\delta^{a_2}_{i_2} \eta_{b_{c_2}} - \delta^{b_{i_2}}_{a_{c_2}}) B^{c_2 ... c_{i_2}} \]

\[ + ... + i(\delta^{a_n}_{i_n} \eta_{b_{c_n}} - \delta^{b_{i_n}}_{a_{c_n}}) B^{c_1 ... c_{i_n}} . \]

It is also important to note that FIDO in teleparallel gravity is equivalent to the Levi-Civita covariant derivative in the Einstein GR in the absence of contorsion, as shown in the Appendix Eq. (A). More importantly, with this coupling prescription, torsion does not violate the gauge symmetry of Kalb Ramond theory.

Using \(\text{Eq.}(23)\) and \(\text{Eq.}(24)\), we get the Fock–Ivanenko derivative acting on \(B^{ab}\) as,

\[ \mathcal{D}_\mu B^{ab} = \partial_\mu B^{ab} - \frac{i}{2} \Omega^{cd}_{\mu} (i(\delta_{c}^{a} \eta_{b_{d}} - \delta_{b}^{a} \eta_{c_{d}})) B^{bd} \]

\[ - \frac{i}{2} \Omega^{cd}_{\mu} (i(\delta_{c}^{b} \eta_{a_{d}} - \delta_{a}^{b} \eta_{c_{d}})) B^{ad} \]

\[ = \partial_\mu B^{ab} - K^{\rho}_{\mu \nu} h^{\rho}_{\nu} h^{a}_{\rho} h^{b}_{\sigma} . \]

Any spacetime tensor \(B^{\mu\nu}\) can be transformed to a Lorentz tensor \(B^{\mu\nu}\) by,

\[ B^{\mu\nu} = h^{a}_{\mu} h^{b}_{\nu} B^{ab} . \]

Now, using \(\text{Eq.}(26)\) and making use of \(\text{Eq.}(5)\) in \(\text{Eq.}(25)\), we have the teleparallel version of the covariant derivative

\[ \mathcal{D}_\mu B^{ab} = h^{a}_{\mu} h^{b}_{\nu} \nabla_{a} B^{a \sigma} , \]

with

\[ \nabla_{a} B^{a \sigma} = \nabla_{\mu} B^{\rho \sigma} - K^{\rho}_{\lambda \mu} B^{\lambda \sigma} - K^{\lambda}_{\mu \sigma} B^{\rho \lambda} \]

\[ = \partial_{\mu} B^{\rho \sigma} + \Gamma^{\rho}_{\mu \sigma} B^{\rho \lambda} + \Gamma^{\lambda}_{\mu \sigma} B^{\rho \lambda} . \]

Thus the teleparallel version of minimal coupling prescription is given as,

\[ \partial_{\mu} \rightarrow \nabla_{\mu} = \partial_{\mu} + \Gamma_{\mu} - K_{\mu} . \]

The Fock–Ivanenko derivative \(\nabla_{\mu}\) \(\text{Eq.}(22)\) turns out to be the Weitzenbock connection in teleparallel gravity minus the contorsion tensor.

With the correct prescription ready, let us now consider the Kalb-Ramond action in the teleparallel background as follows,

\[ \mathcal{L}_{m} = -h H_{\mu \nu \rho} H^{\mu \nu \rho} , \]
where $h = \sqrt{-g}$ and $H_{\mu\nu\rho}$ is given as,
\[ H_{\mu\nu\rho} = \nabla_{\mu} B_{\nu\rho} + \nabla_{\rho} B_{\mu\nu} + \nabla_{\nu} B_{\rho\mu}, \]
(32)
which is U(1) gauge invariant. The teleparallel version of field equation is given as,
\[ \nabla^{\mu} H_{\mu\nu\rho} = 0. \]
(33)
And the teleparallel version of the Bianchi identity can be written as,
\[ \nabla_{[\mu} H_{\nu\rho\sigma]} = 0. \]
(34)
Assuming Lorentz gauge $\nabla_{\mu} B_{\mu\nu} = 0$, and using the commutation relation
\[ [\nabla_{\mu}, \nabla_{\nu}] B^{\lambda\mu} = -Q^{\lambda}_{\sigma\mu\nu} B^{\sigma\mu} - Q_{\mu\nu} B^{\lambda\mu}, \]
(35)
where
\[ Q^{\theta}_{\mu\nu\rho} = \nabla_{\mu} K^{\theta}_{\rho\nu} - K^{\theta}_{\sigma\nu} K^{\sigma}_{\rho\mu} - \nabla_{\nu} K^{\theta}_{\rho\mu} + K^{\theta}_{\sigma\mu} K^{\sigma}_{\rho\nu}, \]
(36)
we can derive the field equations in teleparallel gravity to be,
\[ \nabla^{\mu} \nabla_{\mu} B^{\nu\lambda} - Q^{\nu\lambda\sigma\mu} B_{\sigma\mu} - 2Q_{\mu [\nu B^{\lambda]}_{\mu]} = 0. \]
(37)

IV. NON-SINGULAR COSMOLOGICAL BOUNCE IN THE PRESENCE OF KALB RAMOND FIELD

To study the cosmological bouncing in $F(T)$gravity, let us consider the flat homogeneous isotropic FLRW metric,
\[ ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \]
(38)
where $a(t)$ is the scale factor, which is a function of $t$. Corresponding to this metric, the non-trivial tetrad becomes,
\[ h_{\sigma\mu} = \text{diag}(1, a(t), a(t), a(t)). \]
(39)
In this geometry, the non zero components of the Wittenbock connection Eq. (4), torsion tensor Eq. (5), contorsion tensor Eq. (9) and dual torsion tensor Eq. (11) can be derived as,
\[ T^{\mu}_{\nu\rho} = H, \]
(40)
\[ T^{\nu}_{\mu\rho} = -T^{\rho}_{\mu\nu} = -H, \]
(41)
\[ K^{0}_{\mu\nu} = -Ha(t)^2, \]
(42)
\[ K^{\mu}_{\nu\rho} = -H, \]
(43)
\[ S^{\mu}_{0\nu} = -S^{\nu}_{0\mu} = H, \]
(44)
where $H = \frac{\dot{a}(t)}{a(t)}$ is the Hubble parameter. Thus, we can compute the torsion scalar using Eq. (12) as,
\[ T = 6H^2. \]
(45)
Our objective is to find the functional form of the gravitational Lagrangian $F(T)$ that can give rise to non-singular bouncing cosmology in the presence of Kalb-Ramond fields in the FLRW geometry. To do this, let us consider the action,
\[ S = \frac{1}{2\kappa^2} \left[ \int d^4x \ h \left( F(T) + \Lambda \right) - \frac{1}{2} \int d^4x \ h H_{\mu\nu\rho} H^{\mu\nu\rho} \right], \]
(46)
where $F(T) = -T + f(T)$, $\kappa = \sqrt{8\pi G}$ and $\Lambda$ is the cosmological constant. On varying this action with respect to the tetrad $h_{\mu}^{\alpha}$ [33], we get the following equations of motion,
\[ M_{\mu}^{\nu} \equiv 2h f_{TT} \partial_{\mu} T_{\nu} - 2h f_T \partial_{\nu} T_{\mu} + h S_{\mu}^{\nu} + 2h f_T \partial_{\nu} \partial_{\mu} (h S_{\alpha}^{\nu}) + 2h f_T \partial_{\mu} \partial_{\nu} (h S_{\alpha}^{\alpha}) \]
(47)
Varying the KR action with respect to the field $B_{\mu\nu}$, gives the equation of motion as,
\[ \nabla^{\mu} H_{\mu\nu\rho} = 0. \]
(48)
The Eq. (48) for the KR field can be solved by taking the ansatz,
\[ H_{\mu\nu\rho} = \varepsilon^{\mu\nu\lambda\rho} \partial_{\rho} \phi, \]
(49)
where $\phi$ is a scalar field. Putting back Eq. (49) in Eq. (48), we get
\[ \nabla_{[\mu} H_{\nu\rho\sigma]} = 0. \]
(50)
The equation of motion of $\phi$ can now be obtained from the Bianchi identity
\[ \nabla^{\lambda} \phi = 0. \]
(51)
Substituting Eq. (49) in Eq. (51) and using the fact $\nabla^{\mu} \varepsilon_{\nu\rho\sigma\lambda} \partial^{\rho} \phi = 0$ in four dimensions, we obtain the equation of motion of $\phi$ as,
\[ \nabla_{\lambda} \partial^{\lambda} \phi = 0. \]
(52)
Using Eq. (49), the equation of motion (47) becomes
\[ M_{\mu}^{\nu} = h \kappa^2 \left( 3\varepsilon_{\mu}^{\rho} \partial^{\rho} \phi \partial_{\nu} \phi - 6\delta_{\nu}^{\nu} \partial^{\lambda} \phi \partial_{\mu} \phi \right) \]
(53)
Since we are interested in how KR field affects the time evolution of the universe, for simplicity, we consider $\phi$ as
a function of the cosmic time $t$. For non-singular bouncing cosmological models, then, the initial conditions are taken to be,

$$\phi(t_b) = 0, \quad \phi'(t_b) = 1 \quad (54)$$

where $t_b$ is the time when the bounce occurs. Now, the equations of motion Eq. (52) and Eq. (53) takes the form,

$$3H^2 - 6H^2 f_T + \frac{1}{2}(f + \Lambda) = \kappa^2 \rho, \quad (55)$$

$$3H^2 + 2H' + \frac{1}{2}(f + \Lambda) - 6H^2 f_T - 2H' f_T - 2H f_T T' = -\kappa^2 p, \quad (56)$$

$$\phi'' + 3H\phi' = 0, \quad (57)$$

where $\rho$ and $p$ are the energy density and the matter pressure of the Kalb-Ramond field in the universe, given by

$$\rho = 3\phi'^2, \quad p = 3\phi'^2 a^2. \quad (58)$$

Equations (55) and (56) can be together written as,

$$2H' - 2H^2 f_T - 2H f_T T' = -3\kappa^2 \phi'^2 (a^2 + 1). \quad (59)$$

The Eq. (57) then gives the solution of the KR field as,

$$\phi(t) = \int_1^t e\left(-\int_1^\xi 3H(\xi)d\xi\right) c_1 d\xi + c_2, \quad (60)$$

where $c_1$ and $c_2$ are constants set to satisfy the initial conditions Eq. (54).

In particular, we will be looking into two cases of non-singular bouncing cosmology, namely

A. Symmetric bounce

B. Matter bounce

**A. Symmetric bounce**

In symmetric bouncing cosmology, the scale factor is given as [37, 38],

$$a(t) = a_0 \exp\left(\frac{\alpha t^2}{t^2_*}\right), \quad (61)$$

where $a_0 = a(0) > 0$ is the minimum value attained by the scale factor, $t_* > 0$ is an arbitrary time and $\alpha > 0$ is a parameter. Fig. (1a) shows the behaviour of $a(t)$ over time, where we chose the parameter $\beta = \alpha/t^2_*$. Given the expression of the scale factor, it is straightforward to calculate the Hubble parameter and the torsion scalar as

$$H(t) = 2\beta t, \quad T(t) = 24\beta^2 t^2. \quad (62)$$
In Fig. 1b, we plot the Hubble parameter over time, where $H(t)$ varies linearly with time. The Hubble parameter’s positivity determines whether a universe is expanding or contracting. The phase when $H < 0$ for $t < 0$ is the contracting phase followed by the expansion phase where $H > 0$ for $t > 0$. Clearly, the bounce occur at $t = 0$ (which is a non-singular bounce), when $H = 0$. There is a particular time $t_0 > 0$ when the scale factor becomes unity i.e $a(t_0) = 1$. We define $t_0$ to be the present cosmological time with the present Hubble parameter $H_0 \equiv H(t_0)$. The expression for $t_0$ using Eq. (61) is given as,

$$t_0 = \sqrt{\frac{\ln a_0}{\beta}}.$$  \hspace{1cm} (63)

Since $\beta > 0$, Eq. (63) restricts the range $a_0 \in (0, 1)$. Solving the equation of motion (57) using the initial conditions Eq. (54), we get the expression of $\phi(t)$ in the symmetric bounce cosmology,

$$\phi(t) = \frac{1}{2} \sqrt{\frac{\pi}{3\beta}} \text{erf} \left( \sqrt{\frac{3\beta}{2}}t \right),$$  \hspace{1cm} (64)

where erf$(x)$ is the error function. In Fig. 2, we plotted the time evolution of $\phi(t)$. $\phi(t)$ behaves as a sigmoid function, varying monotonically, but almost saturates after a certain point. This is evident from the asymptotic behavior of $\phi(t)$,

$$\lim_{t \to \infty} \phi(t) = \frac{1}{2} \sqrt{\frac{\pi}{3\beta}}.$$  \hspace{1cm} (65)

![Fig. 2: Time evolution of the scalar field $\phi(t)$ in symmetric bounce for $\beta = 0.1$, 0.5, and 1](image)

The energy density and pressure of the KR field can be obtained using Eq. (58) as

$$\rho = 3 \exp \left( -6\beta t^2 \right),$$  \hspace{1cm} (66)

$$p = 3a_0^2 \exp \left( -4\beta t^2 \right).$$  \hspace{1cm} (67)

The evolution of energy density and matter pressure with respect to the cosmic time $t$ is plotted in Fig. 1c and Fig. 1d respectively. Both the plots show a similar behavior with bell-like profile and localisation at $t = 0$. This feature is essential for the non-singular nature of symmetric bounce. Furthermore, the evolution depends on the factor $\beta$, which determines how fast the universe expands or contracts. Using Eq. (59) in terms of $T$, we get the differential equation of $f(T)$ as,

$$2Tf_{TT} + f_T = 1 + \frac{18}{T_0} a_0 \alpha \kappa^2 \exp \left( -\frac{4T}{T_0} \right)^2$$

$$+ \frac{18}{T_0} a_0 \alpha \kappa^2 \exp \left( -\frac{6T}{T_0} \right)^2,$$  \hspace{1cm} (68)

where and $T_0 \equiv T(t_0)$ is the torsion scalar at time $t_0$ which is obtained using Eq. (62) and Eq. (63) as,

$$T_0 = -24\beta \ln a_0,$$  \hspace{1cm} (69)

and $\dot{t}$ is a dimensionless quantity given by

$$\dot{t} = \frac{t_0}{t_*}.$$  \hspace{1cm} (70)

Solving the above differential equation, we finally derive the exact functional form of $F(T)$ to be,

$$F(T) = f(T) - T$$

$$- \frac{3}{2} \alpha \kappa^2 \exp \left( \frac{6T}{T_0} \right)^2 \left[ 2 + 3a_0 \exp \left( \frac{2T}{T_0} \right) \right]$$

$$+ 9\sqrt{\pi} a_0 \sqrt{\frac{T}{T_0}} \alpha \kappa^2 \dot{t} \text{erf} \left( \sqrt{\frac{T}{T_0}} \right)$$

$$+ 3\sqrt{6\pi} a_0 \sqrt{\frac{T}{T_0}} \alpha \kappa^2 \dot{t} \text{erf} \left( \sqrt{\frac{6T}{T_0}} \dot{t} \right) + C.$$  \hspace{1cm} (71)

where $C$ is an integration constant. Moreover it is important to note that the reconstructed Lagrangian Eq. (71) is an even function, and hence is symmetric with respect to the bounce at $t = 0$.

For the gravitational Lagrangian to be able to recover vacuum solutions, $T$ has to be zero in the absence of matter [39]. This is evident from Eq. (62). Also as a consequence of Eq. (55), we assume $\Lambda = 6\kappa^2$ such that it satisfies the vacuum solution constraint $f(0) = 0$. This fixes the integration constant $C$ to be,

$$C = -\frac{3}{2} \alpha \kappa^2 \left( 3a_0 + 2 \right).$$  \hspace{1cm} (72)

Fig. 5a shows the function $F(T)$ vs. torsion scalar $T$ and Fig. 5a shows the evolution of $F(T)$ with respect to the cosmic time $t$, corresponding to the symmetric bounce scenario in the presence of Kalb-Ramond field described by Eq. (54).

**B. Matter bounce**

In matter bounce cosmology [38, 40, 41] the scale factor is given as,

$$a(t) = a_0 \left( \frac{3}{2} \rho_c t^2 + 1 \right)^{\frac{1}{3}},$$  \hspace{1cm} (73)
FIG. 3: (a) Time evolution of the scale factor $a(t)$, (b) the Hubble parameter $H(t)$, (c) Energy density of the KR field, (d) Matter pressure of the KR field in matter bounce for $\rho_c = 0.01, 0.03,$ and $0.05$ with $a_0 = 1/2$

where $a(0) = a_0$ is a positive quantity, and $0 < \rho_c << 1$ is the critical density of the universe. $\rho_c$ also determines how fast the bounce occurs [12]. Fig. (3a) shows the time evolution of the scale factor in matter bounce cosmology. The present cosmological time $t_0 > 0$ can be obtained from Eq. (73) as,

$$t_0 = \sqrt{\frac{2}{3\rho_c \left( \frac{1}{a_0^3} - 1 \right)}},$$

(74)

Thus, the range of $a_0$ is restricted to $(0, 1)$, since $\rho_c > 0$.

The expressions of the Hubble parameter and the torsion scalar in matter bounce cosmology takes the form,

$$H(t) = \frac{2\rho_c t}{3\rho_c t^2 + 1} \quad T(t) = \frac{24\rho_c^2 t^2}{(3\rho_c t^2 + 1)^3}.$$

(75)

$H(t)$ is plotted in Fig. (3b), which clearly shows that a non-singular bounce occurs at $t = 0$, with a contracting and expansion phase for $t < 0$ and $t > 0$ respectively. The torsion scalar at the cosmic time $t_0$ can be obtained by substituting Eq. (74) in Eq. (75), which is given as

$$T_0 \equiv T(t_0) = 4a_0^3 \left( 1 - a_0^3 \right) \rho_c.$$

(76)

FIG. 4: Time evolution of the scalar field $\phi(t)$ in matter bounce for $\rho_c = 0.01, 0.03,$ and $0.1, a_0 = 0.5$

The corresponding energy density and matter pressure of the KR field is obtained as,

$$\rho = \frac{12}{(3\rho_c t^2 + 2)^2},$$

(77)

$$p = \frac{6a_0^2}{2^{1/2} (3\rho_c t^2 + 2)^{3/2}}.$$

(78)

These are plotted in Fig. (3c) and Fig. (3d), which shows that the maximum of energy density and matter pressure is again at $t = 0$ as expected. The energy and pressure profiles looks similar to the symmetric bounce
scenario. Upon solving the KR-field equation \([59]\), we get the expression of the scalar field \(\phi\) corresponding to the matter bounce cosmology as,

\[
\phi(t) = \sqrt{\frac{2}{3\rho_c}} \tan^{-1} \left(\sqrt{\frac{3\rho_c}{2}}\right).
\]  

(79)

The time evolution of \(\phi(t)\) is plotted in the Fig.(4). It can be observed that the behavior of \(\phi\) is again similar to what we have seen in the case of symmetric bounce, where it behaves as a sigmoid function. The asymptotic behavior of \(\phi(t)\) in matter bounce as \(t \to \infty\) is given as,

\[
\lim_{t \to \infty} \phi(t) = \frac{\pi}{\sqrt{6\rho_c}}.
\]  

(80)

Solving the functional form of \(F(T)\) using Eq.(79) in Eq.(55) and Eq.(56), we get \(F(T)\) as a function of \(t\),

\[
F(t) = \frac{24\kappa^2}{(2 + 3t^2\rho_c)^2} - \frac{48t^4\rho_c^2}{(2 + 3t^2\rho_c)^2} \\
+ \frac{6t^2\rho_c}{(2 + 3t^2\rho_c)^2} \left[8\rho_c + \kappa^2 \left(6 + 9a_0\left(\sqrt{2(2 + 3t^2\rho_c)}\right)^2\right)\right] \\
+ \frac{6\sqrt{6\rho_c}}{(2 + 3t^2\rho_c)} \kappa^2 \tan^{-1} \left(\sqrt{\frac{3\rho_c}{2}}t\right) \\
- \frac{18a_0\kappa^2t^2\rho_c}{(2 + 3t^2\rho_c)} 2F_1 \left[\frac{1}{3}; \frac{3}{2}; \frac{1}{2}; \frac{2}{2}; \frac{3 - 3t^2\rho_c}{2}\right] - 6\kappa^2,
\]  

(81)

where \(2F_1[a, b; c; d]\) represents the hypergeometric function.

The symmetry between the contraction and the expansion phase in matter bounce requires \(F(t)\) to be an even function of \(t\). In Fig.(6a), we have plotted the cosmic time evolution of \(F(t)\), which shows its symmetric behavior with respect to the bouncing point at \(t = 0\). The inverse relation \(t(T)\) can be obtained by the inversion of \(T\) in Eq.(75),

\[
t(T) = \pm \sqrt{\frac{2}{3}} \left[\frac{2}{T} - \frac{1}{\rho_c} - \frac{2}{T} \sqrt{\frac{1}{\rho_c}}\right].
\]  

(82)

Here, we have retained the solution pair that produces the desired result, \(T = 0\) at \(t = 0\). It is important to note that the solution is valid only for \(-\sqrt{\frac{2}{3\rho_c}} \leq t \leq \sqrt{\frac{2}{3\rho_c}}\). This corresponds to a characteristic time period for each matter bounce universe corresponding to the critical density \(\rho_c\), in which \(F(t)\) is valid. As the solution Eq.(81) is an even function of \(t\), both \(\pm\) solutions in Eq.(82) provide the identical form of \(F(T)\), with \(-\) and \(+\) solutions, representing the contraction and expansion phases respectively.

Substituting Eq.(82) in Eq.(81), we get the functional form of \(F(T)\) as

\[
F(T) = \frac{1}{3h(T)^2\rho_c} \left[3\kappa^2 \left(a_0^2(T - 2\rho_ch(T))\left(3 \times 2^{2/3}T\left(\frac{\rho_c}{T}\right)^{2/3}\right) \\
- 2\rho_c(3^{1/2}2^{1/2}1 - \frac{2h(T)}{\rho_c})\right) \\
- 2\rho_cTh(T)\left(\frac{2h(T)}{T} - 1 + \frac{1}{\rho_c}\right) + 1\right] - 6\kappa^2
\]

which is also restricted to \(T \leq \rho_c\), equivalently, \(-\frac{2}{\rho_c} \leq t \leq \frac{2}{\rho_c}\). In Eq.(83), we have taken \(h(T)\) as a function of \(T\), given as

\[
h(T) = 1 - \sqrt{1 - \frac{T}{\rho_c}}.
\]  

(84)

In Fig (6a), we have plotted the function \(F(T)\) in terms of \(T\). Note that the solution Eq.(83) satisfies the vacuum solution constraint \(F(0) = 0\).

V. DISCUSSIONS AND CONCLUSION

TEGR is a successful gravity description, specifically in the presence of sources that could twist the geometry to create torsion. One such example that could source torsion is the anti-symmetric rank-2 Kalb-Ramond field. These anti-symmetric tensor fields form an integral part of heterotic string models \([13, 44]\) as massless closed string modes and of some supersymmetric models like N=2 and N=8 extended SUGRA. They have also been widely studied \([3]\) in the context of electromagnetic field coupling to the Einstein-Cartan system. Though essential, it is noteworthy that there are no experimental evidence for this field in the present day universe \([8]\).

Since the presence of torsion breaks the \(U(1)\) invariance of the gauge theory, it is important to introduce a suitable coupling prescription in teleparallel gravity. To do that successfully, we first define an equivalent of the covariant derivative called the Fock-Ivanenko Derivative Operator (FIDO) in teleparallel geometry. In \([41]\), we generalize FIDO to operate on any \(n\)-form tensor field in \(d + 1\)-dimensional space, using the equations Eq.(22), Eq.(23) and Eq.(24) and in particular on KR field. For completeness, we show the equivalence of Fock-Ivanenko derivative of the KR field in teleparallel gravity to the Einstein-Cartan’s gravity in Appendix \([A]\).

We then compute the equations of motion and show that the dynamics of tetrad fields in teleparallel geometry are governed by the KR field. To keep the discussion general, we start with a generic function \(F(T)\). And we con-
FIG. 5: The plot (a) shows $F(T)$ vs $T/T_0$ in symmetric bounce scenario for $\beta = 0.1, 0.5$, and $1$. We have chosen \(\kappa = M_{Pl}\) and \(a_0 = 1/2\). The evolution of $F(T)$ with respect to the cosmic time $t$ is plotted in (b).

FIG. 6: The plot (a) shows function $F(T)$ in terms of $T/\rho_c$ in the matter bounce scenario, for different $a_0 = 0.4, 0.6$ and $0.8$. In (a), $F(T)$ is only valid for $T/\rho_c \leq 1$, equivalently $|t| \leq \sqrt{2/a_6 \rho_c}$ in (b). In plot (b), evolution $F(T)$ in terms of the time $t$ for different $a_0 = 0.4, 0.6, 0.8$ is plotted. We have chosen $\kappa = M_{Pl}$ and the critical density $\rho_c = 7 \times 10^{-6} M_{Pl}^2$, which is determined by the amplitude of the CMB spectrum $[42]$.

| Model                  | $a(t)$                                  | $\phi(t)$                                                                 | $F(T)$                                                                 |
|------------------------|-----------------------------------------|---------------------------------------------------------------------------|------------------------------------------------------------------------|
| Symmetric bounce       | \(a_0 \exp\left(\alpha \frac{t^2}{\beta^2}\right)\) | \(\frac{1}{2} \sqrt{\frac{\pi}{3\beta}} \text{erf}\left(\sqrt{3\beta} t\right)\) | \(\frac{3}{2} \alpha \kappa^2 \exp\left(-\frac{\beta^2}{2\beta_0^2}\right) \left[2 + 3a_0 \exp\left(\frac{2\beta^2}{\beta_0^2}\right)\right] + 9\sqrt{\pi a_0} \sqrt{\frac{\beta^2}{\beta_0^2}} \alpha \kappa^2 t \text{erf}\left(\sqrt{\frac{\beta^2}{\beta_0^2}} t\right) + 3\sqrt{6\pi a_0} \sqrt{\frac{\beta^2}{\beta_0^2}} \alpha \kappa^2 t \text{erf}\left(\sqrt{\frac{2\beta^2}{\beta_0^2}} t\right) - \frac{3}{2} \alpha \kappa^2 (3a_0 + 2)\) |
| Matter bounce          | \(a_0 \left(\frac{3}{2} \rho_c t^2 + 1\right)^{3/2}\) | \(\frac{1}{3\alpha(T)^{3/2}} 3 \kappa^2 \left(a_0^2 (T - 2\rho_c h(T))\right) \left(3 \times 2^{2/3} T \left(\frac{3/2}{2}\right) - 2\rho_c h(T) \left(\frac{1}{3}\right) \right.\) | \(\frac{1}{3\alpha(T)^{3/2}} 3 \kappa^2 \left(a_0^2 (T - 2\rho_c h(T))\right) \left(3 \times 2^{2/3} T \left(\frac{3/2}{2}\right) - 2\rho_c h(T) \left(\frac{1}{3}\right) \right.\) |
sidered the effect of this setting in producing two bouncing cosmologies, namely symmetric bounce and matter bounce. The absence of initial conditions and initial singularity in cosmological evolution has been a significant advantage of bouncing cosmologies over the inflationary paradigm. In these scenarios, the big bang is replaced by a continuous phase of expanding and contracting.

Note that the scale factor for both these scenarios are sourced by the localized energy density of the KR field as shown in Fig. 1c and Fig. 3c. These plots correctly indicate the nature of energy density with time ‘t’, which indicates the null results from experiments conducted to detect such anti-symmetric tensor fields.

In the symmetric bounce, the generalised teleparallel gravity is an increasing function of T, for T ∈ (0, ∞) as shown in Fig. 5a, but for large T, F(T) behaves linearly. Whereas in matter bounce, F(T) again exhibits similar behavior, albeit the evolution is valid up to some value T < ρc, as shown in Fig. 4. The analytical results are given in Table 1. In both these scenarios, the scalar field shows a wave-profile Fig. 2, 4 and energy profile Figs. 1c, 3c of a ‘kink’ which could be interesting to study further. It is also interesting to wonder how an axion/pseudo-scalar field will behave in the teleparallel setting, given the possible parity violations.

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Appendix A: Coupling prescription using Fock–Ivanenko derivative operator in the Riemannian geometry

In the framework of Riemannian geometry, the mathematical equivalent of Fock-Ivanenko derivative of the KR field B_{ab}, using Eq. (22) and Eq. (24), is given by,

\[ \mathcal{D}_\mu B^{ab} = \partial_\mu B^{ab} - \frac{i}{2} \Omega^{cd} \left( i (\delta_c^e \eta_{bd} - \delta_d^e \eta_{cb}) \right) B^{ge} \]

\[ = \partial_\mu B^{ab} - \frac{i}{2} \Omega^{cd} \left( i (\delta_c^e \eta_{bd} - \delta_d^e \eta_{cb}) \right) B^{eg} \]

\[ = \partial_\mu B^{ab} + \frac{1}{2} \Omega^{cd} \left( i (\delta_c^e \eta_{bd} - \delta_d^e \eta_{cb}) \right) B^{ge} \]

Now, using Eq. (5) and Eq. (8) in Eq. (23), we can write the spin connection as

\[ \Omega^{ab}_\mu = h^a_p \nabla_\mu h^{bp} \]

This can be equivalently written as

\[ \partial_\mu h^a_\nu + \Omega^a_\mu h^b_\nu - \Gamma^e_\nu \eta_{be} = 0 \]

Using the definition of the Fock-Ivanenko derivative of the tetrad,

\[ \mathcal{D}_\mu h^a_\nu = \partial_\mu h^a_\nu + \Omega^a_\mu h^b_\nu \]

we can rewrite Eq. (A3) as,

\[ \mathcal{D}_\mu h^a_\nu = \Gamma^a_\nu h^a_\rho \]

Substituting Eq. (A2) and Eq. (26) in Eq. (A1) and using Eq. (A5), we get,

\[ \mathcal{D}_\mu B^{ab} = h^a_\rho h^b_\sigma \nabla_\mu B^{\rho\sigma} \]

where \( \nabla_\mu B^{\rho\sigma} \) is the Levi-Civita covariant derivative of \( B^{\rho\sigma} \).

Thus, the Fock–Ivanenko derivative of the antisymmetric Lorentz tensor \( B_{ab} \) reduces to the usual Levi-Civita covariant derivative of general relativity. In other words, we can say that the minimal coupling prescription in Riemannian geometry can be written as,

\[ \partial_a \rightarrow \mathcal{D}_a = \partial_a + \Gamma^a_\mu \equiv \nabla_\mu \]

Now let us consider the Kalb-Ramond Lagrangian in the background of Riemannian geometry.

\[ \mathcal{L}_m = -\sqrt{-g} H_{\mu\nu\rho} H^{\mu\nu\rho} \]

where the field strength \( H_{\mu\nu\rho} \) is given by,

\[ H_{\mu\nu\rho} = \nabla_\mu B_{\nu\rho} + \nabla_\nu B_{\mu\rho} + \nabla_\rho B_{\mu\nu} \]

The corresponding field equation can be written as

\[ \nabla_\mu H^{\mu\nu\rho} = 0 \]

Assuming the Lorentz gauge \( \nabla_\mu B^{\mu\nu} = 0 \), and using the commutation relation,

\[ [\nabla_\mu, \nabla_\nu] B^{\lambda\mu} = \tilde{R}^{\lambda\mu}_{\sigma\rho\nu} B^{\sigma\rho} + \tilde{R}_{\mu\nu} B^{\lambda\mu} \]

we have the field equations of KR fields in teleparallel geometry,

\[ \nabla_\mu \nabla_\nu B^{\lambda\mu} + \tilde{R}^{\nu\lambda\mu}_{\nu\mu} B_{\sigma\mu} + 2 \tilde{R}^{[\nu}_{\nu} B^{\lambda]\mu} = 0 \]
[42] Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, Matter Bounce Cosmology with the \( f(T) \) Gravity, Class. Quant. Grav. 28, 215011 (2011), arXiv:1104.4349 [astro-ph.CO].

[43] M. Kalb and P. Ramond, Classical direct interstring action, Phys. Rev. D 9, 2273 (1974).

[44] E. Cremmer and J. Scherk, Spontaneous dynamical breaking of gauge symmetry in dual models, Nucl. Phys. B 72, 117 (1974).