Contact problems of multilayer slabs interaction on an elastic foundation

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Abstract. A mathematical model and a method to assess the internal force factors in multilayer
strip – slab on elastic foundation under various static loads were developed in the paper. A detailed
review of well-known publications on assessing the stress-strain state and dynamic behavior of
various structures interacting with the base is given. A closed system of integro-differential
equations describing the strain process in multilayer slabs on an elastic foundation was derived.
The problem under consideration was reduced (using the Chebyshev polynomial) to the solution of
infinite systems of algebraic equations. The regularities of infinite systems of algebraic equations
were proved and the corresponding estimates were obtained. A number of terms of the Chebyshev
polynomial for obtaining a solution to the problem with the required accuracy was determined.
The efficiency of the method was shown on the example of test problems.

1. Introduction

At present, through the efforts of researchers, many different methods for calculating structures on a
deformable foundation are developed; the properties of these structures are described by various physical
models. These include building foundations, airfield and road surfaces, slabs, hydro-technical structures,
rails and railway sleepers, etc. In most of the methods developed, an attention is paid to the analysis of the
relationship between the contacting structural elements with the soil base [1–4].

In [5–8], various laws of strain and structural destruction of soil were considered, as well as seismic wave
propagation, and the rigid body-soil interaction. There are a number of works in which the joint work of the structure with the base is considered:
- in [9], the contact problems of the indentation of a rectangular stamp with a flat base into an elastic
rough half-space in the presence of Coulomb friction with unknown zones of cohesion and slippage
were considered;
- in [10] the statement and mathematical methods for solving problems of hydro-elasticity of three-layer
structural elements were presented;
- in [11–15], various studies were presented to assess the strength and dynamics of earth dams, both
taking/and not taking into account the base under static and dynamic effects;
- in [16] the interaction of a nonlinear system, i.e. the earth base, with the gravity support was studied. An
ideal elastoplastic model was used in the superstructure model, and the Winkler's model was adopted for
the foundation;
the behavior of a structure resting on a foundation was investigated in [17] during an earthquake. It was assumed that the surface of soil and foundation is a set of discrete non-linear elements, consisting of springs, attachment units and clearance elements;

-in [18], the problems of evaluating the critical stress and strain in a hinge-supported rectangular slab beyond the elastic limit was considered and the stability of a bent slab was evaluated.

Besides, there is a number of publications devoted to the dynamics, where the work of the structure is considered together with the foundation using artificial boundary conditions at the boundary of the foundation final area:

- in [19], the solution of the plane problem of wave propagation from a stamp located on the surface of a half-space was considered;
- in [20], the problem of axisymmetric vibrations of a flexible ring lying on a viscoelastic layered foundation was studied. The damping properties of the system were analyzed under various excitation frequencies;
- in [21] a linear problem of interaction of a Rayleigh surface wave propagating in a sandy medium with a rigid structure partially buried in soil was solved;
- in [22], vibrations, stress state and stability of road foundations under machines were considered, taking into account the sub-soil base;
- in [23], the dynamic behavior of concrete earth dams was estimated together with the foundation.

Therefore, the development of mathematical models and methods for assessing the stress-strain state of multilayer strip-slabs lying on an elastic foundation, taking into account their geometrical and physical features, is a relevant task.

In the present study, a closed system of integro-differential equations was derived to describe the strain process of multilayer slabs on an elastic foundation. The considered problems were solved by the expansion in a series of reactive pressure on the foundation by orthogonal polynomials and were reduced to the study of infinite systems of algebraic equations. Moreover, the regularity of infinite systems of algebraic equations was proved and the corresponding estimates were obtained. To show the effectiveness of the methods, a number of test problems were solved.

2. Mathematical models of the problem

![Figure 1. Design scheme of n-layered strip-slabs](image1)

![Figure 2. Design scheme of n-layered beam slabs](image2)

Consider \( n \) -layered strip-slabs lying on an elastic foundation (Fig. 1). The width and thickness of each layer of the strip are \( 2l, h_1, h_2, \ldots, h_n \), respectively. Each layer of the strip is loaded with external
loads $q_1, q_2, ..., q_n$, respectively, constant loads along the slab, arbitrary loads across it. Assume that there is an elastic filler between the strips, and the response of the filler is proportional to the difference in deflections of the connecting strips. The lower strip, tightly fit to the base, in addition to external loads and the response of the upper strip filler, is also influenced by the reactive response of the base. To simulate the strain process of a multilayer strip-slab, we take $n$-layered beam slabs cut with a width equal to one. This allows us to reduce the problem under consideration to the problem of $n$-layered beam slabs of length $2l$, thickness $h_1, h_2, ..., h_n$ and width equal to one (Fig. 2). Establishing the origin of the coordinate at the symmetrical center of the beam slabs with the abscissa on the segment $[-l; l]$, i.e. $-l \leq x \leq l$, allows considering the slab deflections $y_1, y_2, ..., y_n$ as a function of the variable $X$, i.e. $y_i = y_i(x), i = 1, 2, ..., n$. Here $y_i$ is the deflection of the $i$-th beam slab.

To simulate the strain process of $n$-layered beam slabs, a system of differential equations is written for the unknown deflections of beam slabs in the form:

$$
D_i \cdot y^{(n)}_i = q_i - k_{i-1} \cdot (y_n - y_{n-1})
$$

$$
D_2 \cdot y^{(n-1)}_1 = q_1 + k_1 \cdot (y_n - y_{n-1}) - k_2 \cdot (y_{n-1} - y_{n-2})
$$

$$
D_2 \cdot y^{(n-2)}_2 = q_2 + k_2 \cdot (y_{n-1} - y_{n-2}) - k_3 \cdot (y_{n-2} - y_{n-3})
$$

$$
\vdots \quad \vdots \quad \vdots
$$

$$
D_n \cdot y'_1 = q_n + k_n \cdot (y_2 - y_1) - p
$$

(1)

Here $D_i = \frac{E_i h_i^3}{12(1 - \nu_i^2)}$; $E_i, \nu_i$ – are the modulus of elasticity and Poisson's ratios of the slab materials; $k_i$ – are the filler stiffness coefficients; $q_i = q_i(x)$ – external loads of the $i$-th slab; $p = p(x)$ – reactive normal pressure in the base.

The equation relating the settlement of a homogeneous base $V(x)$ to the reactive pressure $p(x)$ under conditions of plane deformation according to [4] can be represented as:

$$
V(x) = \frac{2l(1 - \nu_o^2)}{\pi E_o} \int_{-l}^{l} p(s) \ln \frac{1}{|y - s|} ds
$$

(2)

Here $E_o, \nu_o$ – are the modulus of elasticity and Poisson's ratio of the base material, respectively.

Further, it is assumed that there is a two-way link between the slab surface and the base, therefore, the contact conditions can be written in the following form:

$$
y_i(x) = V(x), \quad -l \leq x \leq l
$$

(3)

Now the problem under consideration can be formulated as follows: It is necessary to determine the deflections of multilayer beam slabs satisfying the system of equations (1) and the base settlement satisfying the equation (2).

3. Solution Methods

In what follows, a dimensionless coordinate $X$ is introduced, equal to the ratio of the absolute coordinate to the half-length of the beam $l$.

The reactive pressure distribution law is sought in the form of a series using the Chebyshev polynomial [24]:
\[ p(x) = (1-x^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_n \cdot T_n(x) \]  

where \( T_n(x) \) is the Chebyshev polynomial of the first kind \([16]\); \( A_n \) are the unknown coefficients.

The reactive pressures of the foundation \( p(x) \) must satisfy the equilibrium equation of the beam slabs, i.e.:

\[ \int_{-1}^{1} p(x) d(x) = \frac{P}{l}, \quad \int_{-1}^{1} xp(x) dx = \frac{M}{l^2} \]  

where \( P, M \) are the sum of all vertical forces and the sum of their moments relative to the middle of the beam slabs, respectively.

Substituting (4) into (5) and taking into account the orthogonality of Chebyshev polynomials with respect to the weight \( (1-x^2)^{\frac{1}{2}} \), we have

\[ A_0 = \frac{P}{\pi l}, \quad A_1 = \frac{2M}{\pi l^2} \]  

Substituting (4) into (2) and using the well-known relation \([25]\)

\[ \int_{-1}^{1} \ln|x-s| \cdot (1-s^2)^{\frac{1}{2}} T_k(s) ds = \begin{cases} \pi \ln 2, & \text{at } k = 0 \\ -\frac{\pi}{k} T_k(x), & \text{at } k = 1,2,3, \ldots \end{cases} \]

for the base settlement we get:

\[ V(x) = \frac{2(1-v^2)l}{E_i} \left[ -A_0 \ln 2 + \sum_{n=1}^{\infty} A_n \cdot \frac{T_n(x)}{n} \right] \]  

If the remaining coefficients of series (4) are assumed to be zero, then this is the case of an absolutely rigid beam slab. The terms of series (4) for values of \( n = 2 \) represent a correction that distinguishes the distribution of reactive pressure for the absolutely rigid beam slabs.

For simplicity, consider a two-layer beam slab interacting with a homogeneous elastic foundation. Then the system of differential equations for beam slabs deflections (1) takes the form

\[ \frac{D_1}{l^4} \cdot y''_2 = q_2 - k_1 \cdot (y_2 - y_1) \quad \frac{D_2}{l^4} \cdot y''_2 = q_1 + k_1 \cdot (y_2 - y_1) - p \]  

The general solution of the system of differential equations (8) taking into account (4) is represented in the following form

\[ y_1 = \frac{l^4}{D_1 + D_2} \left[ \sum_{i=1}^{4} C_i x^{4-i} + f_q(x) - \sum_{n=0}^{\infty} A_n \cdot f_n(x) - \sum_{i=1}^{4} B_i \cdot u_i(\alpha x) + \psi_q(x) + \frac{1}{D_1} \sum_{n=0}^{\infty} A_n \cdot \varphi_n(x) \right] \]  

\[ y_2 = \frac{l^4}{D_1 + D_2} \left[ \sum_{i=1}^{4} C_i x^{4-i} + f_q(x) - \sum_{n=0}^{\infty} A_n \cdot f_n(x) - \sum_{i=1}^{4} B_i \cdot u_i(\alpha x) + \psi_q(x) + \frac{1}{D_1} \sum_{n=0}^{\infty} A_n \cdot \varphi_n(x) \right] \]
\[
y_2 = \frac{l^4}{D_1 + D_2} \cdot \left\{ \sum_{i=1}^{4} C_i x^{4-i} + f_q(x) - \sum_{n=0}^{\infty} A_n \cdot f_n(x) + \right. \\
+ D_1 \cdot \left[ \sum_{i=1}^{4} B_i \cdot u_i(\alpha x) + \psi_q(x) + \frac{1}{D_1} \cdot \sum_{n=0}^{\infty} A_n \varphi_n(x) \right] \right\}
\]  
(10)

where \( C_i, B_i \) — are the integration constants determined from the boundary conditions of the problem under consideration;

\[
u_1(x) = ch \cdot \cos x; \quad u_2(x) = ch \cdot \sin x + sh \cdot \cos x;
\]

\[
u_3(x) = sh \cdot \sin x; \quad u_4(x) = ch \cdot \sin x - sh \cdot \cos x
\]

\[
f_q^{IV}(x) = q_l(x) + q_2(x)
\]

\[
\psi_q(x) = \frac{1}{4\alpha^2} \int_{0}^{x} \left[ q_1(z) - q_2(z) \right] u_4 \left[ \alpha(x-z) \right] dz
\]  
(12)

\[
f_n^{IV}(x) = (1 - x^2)^{\frac{1}{2}} T_n(x)
\]  
(13)

\[
\varphi_n(x) = \frac{1}{4\alpha^2} \int_{0}^{x} u_4 \left[ \alpha(x-z) \right] (1 - z^2)^{\frac{1}{2}} T_n(z) dz
\]  
(14)

\[
f_n(x) = \frac{1}{2^4 n(n-1)(n-2)(n-3)} (1 - x^2)^{\frac{7}{2}} P_{n-4}^{(7/2, 7/2)}(x), \quad n > 3
\]  
(15)

\[
f_n^{'}(x) = -\frac{1}{2^3 n(n-1)(n-2)} (1 - x^2)^{\frac{5}{2}} P_{n-3}^{(5/2, 5/2)}(x), \quad n > 2
\]  
(16)

\[
f_n^{''}(x) = \frac{1}{2^2 n(n-1)} (1 - x^2)^{\frac{3}{2}} P_{n-2}^{(3/2, 3/2)}(x), \quad n > 1
\]  
(17)

\[
f_n^{'''}(x) = -\frac{1}{2n} (1 - x^2)^{\frac{1}{2}} P_{n-1}^{(1/2, 1/2)}(x), \quad n > 0
\]  
(18)

Here

\[P_n^{(\alpha, \beta)}(x)\] — are Jacobi polynomials [24].

Using the explicit form

\[T_n(x) = \cos(n \cdot \arccos x)\]  
(19)

of Chebyshev polynomials, it is possible to obtain an explicit form for the function \( f_n(x) \) for \( n < 4 \).

Formulas (9) and (10) determining deflections of beam slabs are of a general nature, i.e. arbitrary stiffness of slabs corresponds to arbitrary laws of external loads distribution.

With specific laws of distribution of external loads, it is possible to determine the deflections of the beam slabs satisfying the boundary conditions. In this case, the integration constants \( B_i, C_i \) are determined from the boundary conditions of the problems. Coefficients \( A_n \) in formulas (9), (10) and (7) determining the slab deflection and foundation settlement are the unknowns. The contact conditions (3) are used to determine the unknown coefficients \( A_n \). The deflection of the lower beam slabs satisfying the boundary conditions (9) and the base settlement (7) are introduced into the contact conditions (3). Further, the
obtained equality is multiplied by \((1 - x^2)^{-\frac{1}{2}} \cdot T_n(x)\) and then integrated between \(-1\) and \(1\). After integration, an infinite system of algebraic equations with infinite unknowns for unknown coefficients \(A_n\) can be obtained. This infinite system of algebraic equations is solved by the reduction method. The application of the reduction method is justified strictly mathematically.

Certain coefficients \(A_n\) are substituted in (4), (7), (9), (10) and the regularities in reactive pressure, base settlement and deflections of beam slabs are found. The deflections of the beam slabs found make it possible to determine the patterns of change in internal forces of the slabs corresponding to the changes in external loads, the filler stiffness coefficient and the base response.

**Problem 1.** To illustrate the effectiveness of the above methods, consider the problem of assessing the deflection of a two-layer beam slab loaded with uniformly distributed external load:

\[ q_1(x) = q_2(x) = q = \text{const} \]

In this case, due to the symmetry of load, series (4) includes only even polynomials

\[ p(x) = (1 - x^2)^{-\frac{1}{2}} \cdot \sum_{n=0}^{\infty} A_{2n} \cdot T_{2n}(x) \]  \((20)\)

From the equilibrium equation (5), we have

\[ A_0 = 4q \cdot \pi^{-1} \]  \((21)\)

Expression (3), describing the base settlement, takes the form

\[ V(x) = \frac{2l(1 - V_0^2)}{E_0} \left[ -A_0 \ln 2 + \sum_{n=1}^{\infty} A_{2n} \cdot \frac{T_{2n}(x)}{2n} \right] \]  \((22)\)

Expressions (11), (12) have the form

\[ f_q(x) = 2q \cdot \frac{x^4}{24} \]  \((23)\)

\[ \psi_q(x) = \frac{1}{4\alpha^3} \cdot \frac{D_1 - D_2}{D_1 \cdot D_2} \cdot q \cdot \frac{1}{\alpha} \cdot \ln \left[ 1 - u_i(\alpha x) \right] \]  \((24)\)

With formulas satisfying the boundary conditions and relative deflections of the slab

\[ \bar{y}_1(x) = y_1(x) - y_1(0), \quad \bar{y}_2(x) = y_2(x) - y_2(0) \]

we obtain the following expression to determine the deflections of the slab.

\[ \bar{y}_1 = \frac{l^4}{D_1 + D_2} \cdot \left\{ q \cdot \left[ \frac{1}{2} \cdot x^2 + \frac{x^4}{24} \right] - \frac{D_2}{D_1} \cdot \sum A_{2n} \cdot \left[ \bar{\varphi}_{2n} \cdot (u_1(\alpha x) - 1) + \bar{\varphi}_{2n} \cdot u_3(\alpha x) + \varphi_{2n}(x) + \frac{D_1}{D_2} \cdot (f_{2n}(x) - f_{2n}(0)) \right] \right\} \]  \((25)\)

\[ \bar{y}_2 = \frac{l^4}{D_1 + D_2} \cdot \left\{ q \cdot \left[ \frac{1}{2} \cdot x^2 + \frac{x^4}{24} \right] + \sum A_{2n} \cdot \left[ \bar{\varphi}_{2n} \cdot (u_1(\alpha x) - 1) + \bar{\varphi}_{2n} \cdot u_3(\alpha x) + \varphi_{2n}(x) - (f_{2n}(x) - f_{2n}(0)) \right] \right\} \]  \((26)\)

Here
\[ \varphi_{2n} = \frac{b^{-1}}{8\alpha^3} \cdot \int_{0}^{1} \left\{ 2u_1(\alpha) \cdot u_1[\alpha(1-z)] + u_4(\alpha) \cdot u_2[\alpha(1-z)] \right\} \cdot (1-z^2)^{-\frac{1}{2}} \cdot T_{2n}(z) \, dz \quad (27) \]

\[ \overline{\varphi}_{2n} = \frac{b^{-1}}{8\alpha^3} \cdot \int_{0}^{1} \left\{ 2u_3(\alpha) \cdot u_1[\alpha(1-z)] - u_2(\alpha) \cdot u_2[\alpha(1-z)] \right\} \cdot (1-z^2)^{-\frac{1}{2}} \cdot T_{2n}(z) \, dz \quad (28) \]

\[ b = u_1(\alpha) \cdot u_2(\alpha) + u_3(\alpha) \cdot u_4(\alpha) \]

Next, expressions (22) and (25) are substituted in (3), then, the obtained equalities are multiplied by

\[ (1-x^2)^{-\frac{1}{2}} \cdot T_{2k}(x), \quad k=1,2,3,... \]

and integrated between \(-1\) and 1. Taking into account the orthogonality of Chebyshev polynomials, the following infinite system of algebraic equations for unknown coefficients \(A_{2n}\) is obtained

\[ a_{2k} + \sum_{n=1}^{\infty} a_{2n,2k} \cdot A_{2n} = A_{2k} \cdot \frac{2l \cdot (1-v_0^2)}{E_0} \cdot \frac{\pi}{2k}, \quad k=1,2,3,... \quad (29) \]

where

\[ a_{2k} = \frac{l^4}{D_1 + D_2} \cdot \int_{-1}^{1} \left\{ q \cdot \left( \frac{1}{2} \cdot x^2 + x^4 \right) - \frac{D_2}{D_1} A_0 \cdot [\overline{\varphi}_0 \cdot (u_1(\alpha x) - 1) + \overline{\varphi}_0 \cdot u_3(\alpha x)] \right\} \cdot (1-x^2)^{-\frac{1}{2}} \cdot T_{2k}(x) \, dx \quad (30) \]

\[ a_{2n,2k} = -\frac{l^4}{D_1 + D_2} \cdot \frac{D_2}{D_1} \cdot \int_{-1}^{1} \left\{ \overline{\varphi}_{2n} \cdot u_1(\alpha x) - 1 + \overline{\varphi}_{2n} \cdot u_3(\alpha x) + \varphi_{2n}(x) + \frac{D_1}{D_2} \cdot [f_{2n}(x) - f_{2n}(0)] \right\} \cdot (1-x^2)^{-\frac{1}{2}} \cdot T_{2k}(x) \, dx \quad (31) \]

To eliminate the singularity of integrals (30) and (31), integrating by parts, they are reduced to the following form:

\[ a_{2k} = \frac{l^4}{D_1 + D_2} \cdot \int_{-1}^{1} \left\{ q \cdot \left( 1 + \frac{x^2}{2} \right) - A_0 \cdot \frac{D_2}{D_1} \cdot [\overline{\varphi}_0 \cdot u_1(\alpha x) + \overline{\varphi}_0 \cdot u_3(\alpha x)] \right\} \cdot (1-x^2)^{\frac{3}{2}} \cdot P_{2k-2}^{(3/2,3/2)}(x) \, dx \quad (32) \]

\[ a_{2n,2k} = -\frac{l^4}{D_1 + D_2} \cdot \frac{D_2}{D_1} \cdot \int_{-1}^{1} \left\{ \overline{\varphi}_{2n} \cdot u_1(\alpha x) - 1 + \overline{\varphi}_{2n} \cdot u_3(\alpha x) + \varphi_{2n}(x) + \frac{D_1}{D_2} \cdot f_{2n}(x) \right\} \cdot (1-x^2)^{\frac{3}{2}} \cdot P_{2k-2}^{(3/2,3/2)}(x) \, dx \quad (33) \]

where
\[ \alpha_{2k} = \frac{1}{2k \cdot (2k-1)} \cdot \frac{2^{4k} \cdot [(2k)!]^2}{(4k)!} \]  

(34)

The obtained formulas (32) and (33) are convenient and allow obtaining exact values of the integrals. Thus, the problem under consideration is reduced to the study of infinite systems of algebraic equations (29). It is known that regular infinite systems of algebraic equations have a unique bounded solution. A regular infinite system of algebraic equations can be solved by the reduction methods. Therefore, to apply the reduction method, it is necessary to prove the regularity of the system of infinite equations (29).

4. Substantiation of the solution method

To prove the regularity of infinite systems (29), coefficients \( a_{2n} \), \( a_{2n,2k} \) are used, defined by formulas (32), (33), respectively.

The free terms \( a_{2n} \) of system (29) should be bounded in modulus, i.e.

\[ |a_{2n}| < \infty, \quad k = 1, 2, 3, \ldots \]  

(35)

The absolute series consisting of coefficients \( a_{2n,2k} \) must be convergent and have finite sums \( S_{2k} \), i.e.

\[ S_{2k} = \sum_{n=1}^{\infty} |a_{2n,2k}|, \quad k = 1, 2, 3, \ldots \]  

(36)

If, under conditions (35) and (36), the following is true:

\[ \lim_{k \to \infty} S_k = S < 1 \]  

(37)

then infinite system (29) is regular, if

\[ \lim_{k \to \infty} S_k = 0 \]  

(38)

then infinite system (29) is quasiregular.

To check the fulfillment of condition (35), (37) or (38), the following inequalities are taken into account

\[ \left\| \int_a^b f_1(x) \cdot f_2(x) \, dx \right\| \leq \left\{ \int_a^b f_1^2(x) \, dx \right\}^{1/2} \cdot \left\{ \int_a^b f_2^2(x) \, dx \right\}^{1/2} \]  

(39)

\[ (1 - x^2)^\alpha \leq (1 - x^2)^\beta, \quad \alpha \geq \beta, \quad -1 \leq x \leq 1 \]  

(40)

Expressions (39) are called the Cauchy - Bunyakovsky inequality.

When checking the fulfillment of condition (36), the continuity of the integrands in (32) on the integrable intervals is taken into account. Using (39) and (40) in (32), we obtain the following estimate

\[ |a_{2k}| \leq \alpha \cdot \alpha_{2k} \cdot \left\| P_{2k-2}^{(3/2, 3/2)} \right\|^{1/2} \]  

(41)

where

\[ \alpha = \frac{L}{D_1 + D_2} \cdot \left\{ \int_{\lambda \in L} q \cdot \left( 1 + \frac{x^2}{2} \right) - A_0 \cdot \frac{D_z}{D_1} \cdot (\bar{\phi}_0 \cdot u_0^\prime(\alpha x) + \bar{\phi}_0 \cdot u_0^\prime(\alpha x) + \phi_0^\prime(x) + \frac{D_z}{D_2} \cdot \phi_0^\prime(x) \right)^2 \, dx \}^{1/2} \]  

(42)

\[ \left\| P_n^{(\alpha, \beta)} \right\| \]  - is the norm of Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) defined by the formula
Taking into account (34) in (41), we have
\[
\lim_{k \to \infty} |a_{2k}| = 0
\] (44)

The fulfillment of conditions (35) confirms the obtained limit values (44).

Now investigate the series (36). For this, consider the general term of series (36), defined by formula (33), which can be rewritten as
\[
a_{2n,2k} = \frac{l^4}{D_1 + D_2} \cdot \alpha_{2k} \cdot \left( \varphi_{1,2k} \cdot \varphi_{2n} + \varphi_{3,2k} \cdot \varphi_{2n,2k} + f_{2n,2k} \right)
\] (45)

where
\[
\varphi_{1,2k} = \int_{-1}^{1} u_1'(\alpha x) \cdot (1 - x^2)^{3/2} \cdot P_{2k-2}^{3/2, 3/2}(x) dx
\] (46)
\[
\varphi_{3,2k} = \int_{-1}^{1} u_3''(\alpha x) \cdot (1 - x^2)^{3/2} \cdot P_{2k-2}^{3/2, 3/2}(x) dx
\] (47)
\[
\varphi_{2n,2k} = \int_{-1}^{1} \varphi_2''(x) \cdot (1 - x^2)^{3/2} \cdot P_{2k-2}^{3/2, 3/2}(x) dx
\] (48)
\[
f_{2n,2k} = \int_{-1}^{1} f_2''(x) \cdot (1 - x^2)^{3/2} \cdot P_{2k-2}^{3/2, 3/2}(x) dx
\] (49)

Taking into account inequalities (39), (40), the following estimates can be obtained from (52), (53)
\[
|\varphi_{1,2k}| \leq a_1 \cdot \left| P_{2k-2}^{3/2, 3/2} \right|^{1/2}; \quad |\varphi_{3,2k}| \leq a_3 \cdot \left| P_{2k-2}^{3/2, 3/2} \right|^{1/2}
\] (50)

Where
\[
a_1 = \left\{ \int_{-1}^{1} 4\alpha^4 \cdot u_2^3(\alpha x) dx \right\}^{1/2}; \quad a_3 = \left\{ \int_{-1}^{1} 4\alpha^4 \cdot u_2^3(\alpha x) dx \right\}
\] (51)

We will estimate the coefficients \( \overline{\varphi}_{2n} \) and \( \overline{\varphi}_{2n} \), determined by formulas (27) and (28), respectively. For this, using the integration by parts, we reduce them to the following form
\[
\overline{\varphi}_{2n} = \frac{b^{-1}}{8\alpha^3} \cdot (-2\alpha^2 \cdot \alpha_{2n}) \cdot \int_{0}^{1} \{ 2u_1(\alpha) \cdot u_3[\alpha(1-z)] +
\]
\[
+ u_4(\alpha) \cdot u_4[\alpha(1-z)] \} \cdot (1 - z^2)^{3/2} \cdot P_{2k-2}^{3/2, 3/2}(x) dx
\] (52)
\[
\overline{\varphi}_{2n} = \frac{b^{-1}}{8\alpha^3} \cdot 2\alpha b \cdot \alpha_{2n} \cdot P_{2n-2}^{3/2, 3/2}(0) - \frac{b^{-1}}{8\alpha^3} \cdot 2\alpha^2 \cdot \alpha_{2n} \cdot \int_{0}^{1} \{ 2u_3(\alpha) \cdot u_3[\alpha(1-z)] -
\]
\[
- u_4(\alpha) \cdot u_4[\alpha(1-z)] \} \cdot (1 - z^2)^{3/2} \cdot P_{2k-2}^{3/2, 3/2}(x) dx
\] (53)

Taking into account inequalities (39) and (40), the following estimates can be obtained from (52), (53)
\[
|\overline{\varphi}_{2n}| \leq \overline{\varphi} \cdot \alpha_{2n} \left| P_{2k-2}^{3/2, 3/2} \right|^{1/2}, \quad |\overline{\varphi}_{2n}| \leq \overline{\varphi} \cdot \alpha_{2n} \left| P_{2k-2}^{3/2, 3/2} \right|^{1/2}
\] (54)

Where
It is necessary to estimate the coefficient $\varphi_{2n,2k}$ determined by formula (48).

For this, taking into account (14) in function $\varphi_{2n}^a(x)$, participating in the integral (48), and integrating by parts, we obtain

$$\varphi_{2n}^a = \alpha_{2n} \left[ (1-x^2)^{3/2} \cdot P_{2n-2}^{(3/2, 3/2)}(x) - u_1(\alpha x) \cdot P_{2n-2}^{(3/2, 3/2)}(0) - 4\alpha^4 \cdot \varphi_{2n}(x) \right]$$  \hspace{1cm} (57)

Substituting (57) into (48), we have

$$\varphi_{2n,2k} = \alpha_{2n} \left( J_{1,2n,2k} + J_{2,2n,2k} + J_{3,2n,2k} \right)$$  \hspace{1cm} (58)

where

$$J_{1,2n,2k} = \int_{-1}^{1} \left( 1-x^2 \right)^{3/2} \cdot P_{2n-2}^{(3/2, 3/2)}(x) \cdot (1-x^2)^{3/2} \cdot P_{2k-2}^{(3/2, 3/2)}(x) dx$$  \hspace{1cm} (59)

$$J_{2,2n,2k} = P_{2n-2}^{(3/2, 3/2)}(0) \int_{-1}^{1} u_1(\alpha x) \cdot (1-x^2)^{3/2} \cdot P_{2k-2}^{(3/2, 3/2)}(x) dx$$  \hspace{1cm} (60)

$$J_{3,2n,2k} = -4\alpha^4 \int_{-1}^{1} \varphi_{2n}(x) \cdot (1-x^2)^{3/2} \cdot P_{2k-2}^{(3/2, 3/2)}(x) dx$$  \hspace{1cm} (61)

Applying (39) and (40) in (59), we obtain the following estimates

$$\left| J_{1,2n,2k} \right| \leq 4\alpha^3 \cdot \left\| P_{2n-2}^{(3/2, 3/2)} \right\|^{1/2} \cdot \left\| P_{2k-2}^{(3/2, 3/2)} \right\|^{1/2}$$  \hspace{1cm} (62)

Applying (39) and (40) in (60) and taking into account

$$\left| P_{2n-2}^{(3/2, 3/2)}(0) \right| \leq \left\| P_{2n-2}^{(3/2, 3/2)} \right\|^{1/2}$$

we obtain the following estimate

$$\left| J_{2,2n,2k} \right| \leq \bar{\alpha}_1 \cdot \left\| P_{2n-2}^{(3/2, 3/2)} \right\|^{1/2} \cdot \left\| P_{2k-2}^{(3/2, 3/2)} \right\|^{1/2}$$  \hspace{1cm} (63)

where

$$\bar{\alpha}_1 \leq 4\alpha^3 \cdot \left\{ \int_{-1}^{1} u_1^2(\alpha x) dx \right\}^{1/2}$$  \hspace{1cm} (64)

Applying (39) and (40) in (61), we obtain

$$\left| J_{3,2n,2k} \right| \leq \left\{ \int_{-1}^{1} 16\alpha^8 \cdot \varphi_{2n}^2(x) dx \right\}^{1/2} \cdot \left\| P_{2k-2}^{(3/2, 3/2)} \right\|^{1/2}$$

or

$$\left| J_{3,2n,2k} \right| \leq 8\alpha^4 \cdot \left\| \varphi_{2n}(1) \right\| \cdot \left\| P_{2k-2}^{(3/2, 3/2)} \right\|^{1/2}$$  \hspace{1cm} (65)

where
Applying (39) and (40) in (66), we obtain
\[
\varphi_{2n}(1) = \frac{1}{4\alpha} \cdot \frac{1}{0} u_4[a(1-z)](1-x^2)^{3/2} \cdot P_{2n-2}^{3/2, 3/2}(x)dx
\] (66)

Substituting (67) into (65) we obtain the following estimates
\[
\left| J_{3,2n,2k} \right| \leq a_4 \cdot \parallel P_{2n-2}^{3/2, 3/2} \parallel^{1/2} \cdot \parallel P_{2k-2}^{3/2, 3/2} \parallel^{1/2}
\] (68)

where
\[
a_4 = 2\alpha \left\{ \int_{-1}^{1} u_4^2(\alpha x)dx \right\}^{1/2}
\] (69)

With (62), (63), (65) we obtain the following estimates in (58)
\[
\left| \varphi_{2n,2k} \right| \leq \alpha_{2n} \cdot \left( 4\alpha^3 + \bar{a}_1 + a_4 \right) \cdot \parallel P_{2n-2}^{3/2, 3/2} \parallel^{1/2} \cdot \parallel P_{2k-2}^{3/2, 3/2} \parallel^{1/2}
\] (70)

Substituting (17) into (49), then applying (39), (40), we obtain the following estimates
\[
\left| J_{2,n,2k} \right| \leq \alpha_{2n} \cdot \parallel P_{2n-2}^{3/2, 3/2} \parallel^{1/2} \cdot \parallel P_{2k-2}^{3/2, 3/2} \parallel^{1/2}
\] (71)

From equality (45), taking into account (50), (54), (70), (71), we have
\[
\left| a_{2n,2k} \right| \leq a \cdot \alpha_{2n} \cdot \parallel P_{2n-2}^{3/2, 3/2} \parallel^{1/2} \cdot \alpha_{2k} \cdot \parallel P_{2k-2}^{3/2, 3/2} \parallel^{1/2}
\] (72)

where
\[
a = \frac{l^4}{D_1 + D_2} \cdot \frac{D_2}{D_1} \left( \frac{\varphi \cdot a_1 + \varphi \cdot a_3 + \bar{a}_1 + a_4 + 1}{} \right)
\] (73)

By introducing (72) into (35), we obtain
\[
S_{2k} \leq a \cdot \alpha_{2k} \cdot \parallel P_{2k-2}^{3/2, 3/2} \parallel^{1/2} \cdot \sum_{n=1}^{\infty} \alpha_{2n} \cdot \parallel P_{2n-2}^{3/2, 3/2} \parallel^{1/2}
\] (74)

In this inequality, the numerical series
\[
\sum_{n=1}^{\infty} \alpha_{2n} \cdot \parallel P_{2n-2}^{3/2, 3/2} \parallel^{1/2}
\] (75)
is absolutely convergent. This can be verified by substituting (34) and (43) in (75). If we denote the sums of series (75) by \( S \), then inequality (74) takes the form
\[
S_{2k} \leq S \cdot a \cdot \alpha_{2k} \cdot \parallel P_{2k-2}^{3/2, 3/2} \parallel^{1/2}
\] (76)

with (34) and (43), one can show that
\[
\lim_{k \to \infty} \alpha_{2k} \cdot \parallel P_{2k-2}^{3/2, 3/2} \parallel^{1/2} = 0
\] (77)

Equality (77) shows that
\[
S_{2k} \to 0, \quad \text{as} \quad k \to \infty
\] (78)

The obtained limiting values (78), asserts that in this problem conditions (38) are satisfied. It follows from condition (38) that the infinite system of algebraic equations (29) is quasiregular and the reduction method is completely acceptable for its solution.
Problem 2. Consider a numerical example by solving system (29) and determine the bending moment of two-layer beam slabs using the described method. In this case, we use the following characteristics of the foundation and soil [26]:

\[ \nu_0 = 0.3, \quad E_0 = 5 \cdot 10^2 \text{ kgf} / \text{sm}^2 \]

And the slab characteristics

\[ l = 500 \text{ sm}, \quad h_1 = h_2 = 45 \text{ sm}, \quad \nu_1 = \nu_2 = 0.167 \]

\[ E_1 = E_2 = 1.25 \cdot 10^5 \text{ kgf} / \text{sm}^2 \]

Table 1. Results of solving algebraic equations corresponding to different values of the filler stiffness \((k)\)

| \(k(\text{kgf} / \text{sm}^3)\) | \(A_0 / q\)    | \(A_2 / q\)    | \(A_4 / q\)    | \(A_6 / q\)    |
|------------------|------------|------------|------------|------------|
| 0.25             | 1.273239   | -0.299585  | -0.003142  | 0.000273   |
| 1.00             | 1.273239   | -0.300361  | -0.003195  | 0.000270   |
| 2.5              | 1.273239   | -0.300584  | -0.003174  | 0.000267   |
| 10.0             | 1.273239   | -0.300619  | -0.003209  | 0.000269   |
| 25               | 1.273239   | -0.300971  | -0.003214  | 0.000262   |

Table 2. Results of maximum values of bending moments of slabs

| \(k(\text{kgf} / \text{sm}^3)\) | \(M_1(x)/(qL^2)\) | \(M_2(x)/(qL^2)\) |
|------------------|------------------|------------------|
| 0.25             | 0.103563         | 0.069591         |
| 1.00             | 0.098766         | 0.074678         |
| 2.5              | 0.087723         | 0.085346         |
| 10.0             | 0.087133         | 0.082961         |
| 25               | 0.086412         | 0.086095         |

Based on the reduction method, we restrict ourselves to the first four terms in series (20). Then the system of infinite equations (29) turns into a system of three equations with three unknown coefficients \(A_2, A_4, A_6\). The coefficient \(A_0\) is considered as known and its values are calculated by the formula (21). The numerical values of coefficients \(A_0, A_2, A_4, A_6\) for different values of the filler coefficient \(k\) are given in Table 1. According to Table 1, it can be said that the change in the values of filler coefficient do not significantly affect the change in the solution of the system (29), nor does it affect the change in the reactive pressure based on the formula (20). Table 2 shows the bending moment values in the section \(x = 0\) of the strip-beam at various values of the filler coefficient \(k\). Table 2 shows that the filler stiffness significantly depends on the change in bending moments of the beam-slabs. With a filler stiffness value of
$k = 0.25 \text{kgf} / \text{m}^3$, the bending moment of the upper slab is approximately 34% less than the bending moment of the lower slab. With an increase in the values of the filler stiffness, the values of bending moments in the lower slab decrease, and in the upper slab they increase, i.e. the bending moments of the slabs approach each other. The results obtained show the effectiveness of the methods in determining the internal force factors in a multilayer slab-beam.

5. Conclusions
1. A mathematical model was developed to assess the internal force factors of multilayer strip-slabs on an elastic foundation under various static loads.
2. To assess the internal force factors of multilayer strip-slabs, an analytical method to solve the problem was proposed based on orthogonal polynomials on approximations.
3. The possibility of using the proposed methods to solve the problem of assessing the internal force factors of a strip-slab under various static loads was substantiated theoretically.
4. The required number of terms in Chebyshev polynomial was established to obtain the results with satisfactory accuracy.
5. The effectiveness of the methods for solving the problem was validated by the test problems solution.
6. It was stated that an account for the filler stiffness characteristics leads to a redistribution of internal forces in the slab.

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