Local Central Limit Theorem for Two-Body Potentials at Sufficiently High Temperatures

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Abstract

Dobrushin and Tirozzi [11] showed that, for a Gibbs measure with the finite-range potential, the Local Central Limit Theorem is implied by the Integral Central Limit Theorem. Campanino, Capocaccia, and Tirozzi [4] extended this result for a family of Gibbs measures for long-range pair potentials satisfying certain conditions. We are able to show for a family of Gibbs measures for long-range pair potentials not satisfying the conditions given in [4], that at sufficiently high temperatures, if the Integral Central Limit Theorem holds for a given sequence of Gibbs measures, then the Local Central Limit Theorem also holds for the same sequence. We also extend [4] when the state space is general, provided that it is equipped with a finite measure.

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1 Introduction

The Central Limit Theorem is one of the fundamental results of Probability Theory. It states that under certain conditions, the properly scaled sum of independent and identically distributed (i.i.d) random variables converges, as the sum increases to infinity to a Normal

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random variable. However, one aspect that is missing from the Central Limit Theorem is, for instance, the rate of convergence to the limiting Normal distribution. One way to generalize the Central Limit Theorem is the celebrated Berry-Esseen Theorem [2] [13] that provides a more qualitative statement, namely providing the rate at which the convergence takes place.

The Central Limit Theorem was studied for random fields generated by models coming from Statistical Mechanics (see [3] [6] [7] [8] [20] [21] [22] [24] [26] [27] [28] [29] [30] [33] [34] for various techniques of proof). The fact that the Central Limit Theorem holds for models satisfying the FKG inequality and that have a finite susceptibility was proved in [32], extending the work in [31] that was considering only monotonic functionals of the random variables. Künsch studied the model in more generality in [23], and he provided applications of the Central Limit Theorem and the second derivative of the pressure. That work provides an example on which it is known that the Central Limit Theorem holds in more generality than the case of discrete-valued spins considered in [4]. Moreover, Central Limit Theorems for the Ising ferromagnet in two or more dimensions are obtained in [18] [19] [25] [35].

Local Central Limit Theorems are also a way to provide a more refined result than the Central Limit Theorem. Their importance in Statistical Mechanics was noticed in the study of a family of models (see [4] [5] [9] [11]) in which the authors prove that assuming the Central Limit Theorem holds for a random field defined on \( \mathbb{Z}^d \), then the Local Central Limit Theorem will hold as well. In particular, in [11] the authors proved it for the short-range potentials, while [4] showed it for a family of long-range potentials. Local Central Limit Theorems are important in Statistical Mechanics since, with their help, one can deduce the equivalence of the Canonical Ensemble and the Grand-Canonical Ensemble for spin systems and for particle systems (see [11] [5] for more details).

In this paper, we consider a sufficiently high temperature regime, and we prove that for Gibbs fields with spins taking values in a measurable set \( E \), equipped with a measure \( \lambda \), with the condition \( \lambda(E) < \infty \), for which the Central Limit Theorem is satisfied, then the Local Central Limit Theorem also holds. Our result complements [4] where some families of absolutely summable long-range potentials that fail the condition in [4] still satisfy the result at sufficiently high temperatures. We also extend [4] where only discrete value spins are considered.

Our proof is similar to the proof in [4]. It uses the study of the characteristic functions of the random field for small and big values of the parameter. The analysis of these cases is done in two separate lemmas in Section 4 that use different techniques. They both rely on the analysis of the cluster expansion done in Section 3.

As an application of our main result, we consider the one-dimensional long-range Ising models with polynomially decaying interaction \( J_{xy} = |x - y|^{-2+\alpha} \), with \( 0 \leq \alpha < 1 \). Because of this particular long-range interaction, these models undergo a phase transition at low temperatures, and the conditions in [4] fail for them. As a second application of our main result, we have obtained that for the models considered in [23] for which the Central Limit Theorem holds, we have that the Local Central Limit Theorem holds as well at sufficiently high temperatures.

The paper is divided into the following sections. Section 1 is the Introduction. Section 2 introduces the models, the Local Central Limit Theorem, our main result, and some applications. In Section 3, we perform the cluster expansion corresponding to our model and obtain the absolute convergence of the corresponding series. In Section 4, we prove our main result.
2 The Models

2.1 Definitions and Notations

We consider the lattice set \( S = \mathbb{Z}^d \). The state space \((E, \mathcal{E}, \lambda)\) is a measurable space equipped with a finite measure \( \lambda \). Let \( \Omega = E^\mathbb{Z}^d \) and \((\Omega, \mathcal{F})\) be the configuration space, where \( \mathcal{F} \) is the \( \sigma \)-algebra generated by the cylinder sets. We use the notation \( \Lambda \subset \mathbb{Z}^d \) to denote that \( \Lambda \) is a finite set on \( \mathbb{Z}^d \). For \( \Lambda \subset \mathbb{Z}^d \), let \( \mathcal{F}_\Lambda \) be the smallest \( \sigma \)-algebra on \( \Omega \) containing \( \{ \sigma_\Lambda \in A \} \) over all \( \Delta \subset \Lambda \), \( A \in \mathcal{E}_\Delta \).

For a fixed \( x, y \in \mathbb{Z}^d \) with \( x \neq y \), the \( \mathcal{F}\{x,y\}\)-measurable function \( \Phi\{x,y\} : \Omega \rightarrow \mathbb{R} \) is called potential. We say that a potential is absolutely summable if

\[
\| \Phi \| := \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d, y \neq x} \| \Phi\{x,y\} \| < \infty, \tag{2.1}
\]

where \( \| \cdot \| \) denotes the sup-norm. We also assume that the potentials are translation-invariant.

Define the Hamiltonian on a finite set \( \Lambda \) with boundary condition \( \omega \in \Omega \) by

\[
H^\omega_\Lambda(\sigma) = \sum_{x, y \in \Lambda, x \neq y} \Phi\{x,y\}(\sigma) + \sum_{x \in \Lambda, y \notin \Lambda} \Phi\{x,y\}(\sigma_\Lambda \omega_\Lambda^c),
\]

where the configuration \( ((\sigma_\Lambda \omega_\Lambda^c)_x)_{x \in \mathbb{Z}^d} \) means

\[
(\sigma_\Lambda \omega_\Lambda^c)_x = \begin{cases} 
\sigma_x, & x \in \Lambda \\
\omega_x, & x \notin \Lambda 
\end{cases}
\]

For each \( A \in \mathcal{F}_\Lambda \) and \( \omega \in \Omega \), we define the finite volume Gibbs measure with boundary condition \( \omega \) and inverse temperature \( \beta > 0 \) by

\[
\mu^\omega_{\Lambda, \beta}(\sigma) = \frac{e^{-\beta H^\omega_\Lambda(\sigma)}}{Z^\omega_{\Lambda, \beta}},
\]

where \( Z^\omega_{\Lambda, \beta} \) is the partition function given by

\[
Z^\omega_{\Lambda, \beta} = \int_{E^\Lambda} e^{-\beta H^\omega_\Lambda(\sigma)} \prod_{x \in \Lambda} \lambda(d\sigma_x).
\]

2.2 Local Central Limit Theorem

Consider \( f : \Omega \rightarrow \mathbb{Z} \) be a \( \mathcal{F}_0 \)-measurable function, where 0 is the origin of the lattice \( \mathbb{Z}^d \). We assume that \( f \) is not almost-surely constant, i.e., for every \( m \in \mathbb{R} \), we have \( \lambda(f^{-1}(m)) < \lambda(E) \). For each \( x \in \mathbb{Z}^d \), define \( \theta_x : \Omega \rightarrow \Omega \) be the shift operator \( (\theta_x \sigma)_y = \sigma_{x+y} \), for all \( y \in \mathbb{Z}^d \).

For a finite cube \( \Lambda_k \subset \mathbb{Z}^d \) given by \( \Lambda_k = [-k, k]^d \), define, for a fixed \( \sigma \in \Omega \),

\[
S_k(f) = \sum_{x \in \Lambda_k} f \circ \theta_x(\sigma) \quad \text{and} \quad \bar{S}_k(f) = \frac{S_k(f) - \mu^\omega_{\Lambda_k, \beta}(S_k(f))}{\sqrt{D_k}}, \tag{2.2}
\]
where $D_k = D_k(f) = \mu_{\Lambda_k, \beta}(S_k(f)) - \mu_{\Lambda_k, \beta}(S_k(f))^2$ denotes the variance of $S_k(f)$. For simplicity, we will use the notations $S_k = S_k(f)$ and $\bar{S}_k = \bar{S}_k(f)$.

For a sequence of increasing cubes $(\Lambda_k)_{k \geq 1}$ in $\mathbb{Z}^d$, and for a sequence of boundary conditions $(\omega_k)_{k \geq 1}$, we say that the \textit{f-integral central limit theorem} holds for the sequence $(\mu_{\Lambda_k, \beta}^k)_{k \geq 1}$ if the following three conditions are satisfied:

(i) $\lim_{k \to \infty} D_k / |\Lambda_k| = \alpha$,

(ii) $\alpha > 0$,

(iii) For every $\tau \in \mathbb{R}$,

$$\lim_{k \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau} e^{-z^2/2} \, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau} e^{-z^2/2} \, dz. \quad (2.3)$$

We say that the \textit{f-local central limit theorem} holds for the sequence of Gibbs measures $(\mu_{\Lambda_k, \beta}^k)_{k \geq 1}$ if the conditions (i) and (ii) are satisfied, and

$$\lim_{k \to \infty} \sup_{p \in \mathbb{Z}} \sqrt{D_k} \mu_{\Lambda_k, \beta}^k(S_k = p) - \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{(p - \mu_{\Lambda_k, \beta}^k(S_k))^2}{2D_k} \right) = 0. \quad (2.4)$$

Our main result is the following.

**Theorem 1.** Suppose that $\Phi$ is a translation invariant and absolutely summable potential, $f : \Omega \to \mathbb{Z}$ is a bounded, non-almost-surely constant, $\mathcal{F}_0$-measurable function. Assume one of the following conditions,

(1) the potential $\Phi$ satisfies

$$\sum_{x \in \mathbb{Z}^d \setminus \{0\}} \|\Phi_{\{x,0\}}\|^{1/2} < \infty. \quad (2.5)$$

(2) the inverse temperature $\beta$ is sufficiently small.

If the \textit{f-integral central limit theorem} holds for a given sequence of Gibbs measures $(\mu_{\Lambda_k, \beta}^k)_{k \geq 1}$, then the \textit{f-local central limit theorem} holds for the same sequence of Gibbs measures.

The item (1) from Theorem 1 is an extension of the result in [4], in the sense that we extend the result for a general two-body potentials and a general state space provided with a finite measure. Note that there are absolutely summable potentials for which the condition \textcircled{2.5} fails, for instance, the long-range Ising model (see Section 2.3). We are able to show that the result in [4] is still true when we assume sufficiently high temperatures.

**Remark:** For the many-body interaction potential case, we believe that the absolute summability condition should be replaced with the condition in the respective norm for the convergence of the corresponding cluster expansion (see [16, 20]). We expect that Theorem 2 holds true at sufficiently high temperatures for the many-body interaction potential.

We are not going to show the proof when the item (1) of Theorem 1 is assumed, since the proof follows exactly the same as in [4]. Thus, the next sections contain the proof of the main result when the item (2) of Theorem 1 is assumed.
2.3 Applications

Before the proof of Theorem 1, let us present some applications.

2.3.1 Long-Range Ising Model

Let $\Omega = \{−1, 1\}^\mathbb{Z}$ be the set of configurations $\sigma = (\sigma_x)_{x \in \mathbb{Z}}$ on $\mathbb{Z}$. The Hamiltonian in a finite set $\Lambda$ with boundary condition $\omega$ is given by

$$H_\Lambda^\omega(\sigma) = -\sum_{\{x, y\} \subset \Lambda \atop x \neq y} J(|x - y|)\sigma_x \sigma_y - \sum_{x \in \Lambda \atop y \notin \Lambda} J(|x - y|)\sigma_x \omega_y,$$

(2.6)

where the coupling constants $J_{x,y} = J(|x - y|)$, with $x \neq y$, are defined by

$$J(|x - y|) = \begin{cases} J & \text{if } |x - y| = 1 \\ |x - y|^{-2+\alpha} & \text{if } |x - y| > 1 \end{cases}$$

(2.7)

where $J(1) = J > 0$ and $0 \leq \alpha < 1$. It is known that these models undergo to a phase transition at low temperatures [12, 17], they satisfy FKG inequality and have finite susceptibility at high temperatures [1].

As a first main application we obtain the Local Central Limit Theorem in the case of the long-range Ising model defined above. The fact that the Central Limit Theorem holds in the case of the long-range Ising model follows from [32]. In [32], it is proved that for models satisfying FKG inequality and that have finite susceptibility, the Central Limit Theorem holds for not necessarily monotonic functions of the random variables. This result extends the work in [31] where the functions of the random variables are assumed to be monotonic.

The Local Central Limit Theorem for these models is obtained as an application of our main result for $E = \{−1, 1\}$ and $f(\sigma) = \sigma_0$.

2.3.2 Gibbs Fields at Sufficiently High Temperatures

Under the Dobrushin’s Uniqueness Condition [10], the work in [23] proves that the Central Limit Theorem holds true for Gibbs fields with spins taking values in a compact metric space. In addition, in [23] the assumption of the finite range of the potential previously considered in [11] is discarded.

Thus, in our case as a second application, using our results we obtain that, at sufficiently high temperatures, the Local Central Limit Theorem holds true for the models considered in [23]. These models extend the work in [4] that treats only the case of discrete values for the spins, as well as the work in [11], where a Local Central Limit Theorem is obtained under the assumption on the finite-range of the potential.

3 Cluster Expansion

For all $x, y \in \mathbb{Z}^d$, since the potential $\Phi_{\{x,y\}}$ is $\mathcal{F}_{\{x,y\}}$-measurable, by abuse of notation, we will start writing $\Phi_{\{x,y\}}(\sigma_\Lambda) = \Phi_{\{x,y\}}(\sigma_x, \sigma_y)$. 
For a fixed $\Lambda \in \mathbb{Z}^d$ and $x \in \Lambda$, define

\[ h^\omega_x(\sigma_x) = h^\omega_{x,\Lambda}(\sigma_x) := \sum_{y \notin \Lambda} \Phi_{\{x,y\}}(\sigma_x, \omega_y) \]

and denote the set of all sets with two-elements in $\Lambda$ by $\mathcal{P}_2(\Lambda) = \{ A \subseteq \Lambda : |A| = 2 \}$.

For each $x \in \Lambda$ and $\omega \in \Omega$, define the probability density function $p^\omega_x : E \to \mathbb{R}$ by

\[ p^\omega_x(\sigma_x) = p^\omega_{x,\beta,\Lambda}(\sigma_x) := \frac{\exp(-\beta h^\omega_x(\sigma_x))}{\int_E \exp(-\beta h^\omega_x(\sigma_x)) \lambda(d\sigma_x)}, \]

and denote $\mathbb{E}_x$ be the expected value with respect to $p^\omega_x$.

Let $\mathcal{P}_{1,2}$ be a family of non-empty subsets $b \in \mathbb{Z}^d$, consisting of at most two points. A polymer $R$ is a set $\{b_1, \ldots, b_p\}$ of distinct elements $b_i \in \mathcal{P}_{1,2}$ that is connected in the following sense: for any $b_i, b_m \in R$, there exist $b_{k_1}, \ldots, b_{k_q} \in R$ such that $b_{k_1} = b_i$, $b_{k_q} = b_m$ and $b_{k_j} \cap b_{k_{j+1}} \neq \emptyset$. Let $\mathcal{R}$ be the set of all polymers and, if $R \in \mathcal{R}$, denote by $\bar{R}$ the subset of $\mathbb{Z}^d$ given by $\bar{R} = \bigcup_{b \in R} b$. For $t \in \mathbb{R}$, define the activity function $\zeta^t_\beta : \mathcal{R} \to \mathbb{C}$ as

\[ \zeta^t_\beta(R) := \int_{\mathcal{E}^\bar{R}} \prod_{x \in \bar{R}} p^\omega_x(\sigma_x) \prod_{b \in R} \xi_{\beta,b}(\sigma) \prod_{x \in \bar{R}} \lambda(d\sigma_x), \]

where we introduce the notations

\[ \xi_{\beta,\{x\}}(\sigma) = \zeta^t_\beta(\sigma) := \exp \left( \frac{it(f \circ \theta_x(\sigma) - \mathbb{E}_x(f \circ \theta_x))}{\sqrt{D_k}} \right) - 1, \]

\[ \xi_{\beta,\{x,y\}}(\sigma) := \exp(-\beta \Phi_{\{x,y\}}(\sigma_x, \sigma_y)) - 1. \]

**Theorem 2.** The partition function $Z^\omega_{\Lambda,\beta}$ can be written as

\[ Z^\omega_{\Lambda,\beta} = \left( \prod_{x \in \Lambda} \int_E e^{-\beta h^\omega_x(\sigma_x)} \lambda(d\sigma_x) \right) \Xi^{\omega}_{\Lambda,\beta}, \]

where

\[ \Xi^{\omega}_{\Lambda,\beta} = \Xi^{\omega}_{\Lambda,\beta}(\zeta^t_\beta) := 1 + \sum_{n=1}^{\infty} \sum_{\{R_1, \ldots, R_n\} \in \mathcal{R}_{n}} \prod_{i=1}^{n} \zeta^t_\beta(R_i). \]

Moreover,

\[ \Xi^{\omega}_{\Lambda,\beta} = \exp \left( \sum_{n=1}^{\infty} \sum_{\{R_1, \ldots, R_n\} \in \mathcal{R}_{n} \cap \Lambda} \phi^T(R_1, \ldots, R_n) \prod_{i=1}^{n} \zeta^t_\beta(R_i) \right), \]

with

\[ \phi^T(R_1, \ldots, R_n) = \frac{1}{n!} \sum_{G \in \mathcal{G}_{n}(R_1, \ldots, R_n)} (-1)^{e(G)}, \]

where the sum is over all the spanning connected graphs $G$ with $n$ vertices $\{1, \ldots, n\}$, and edges $\{i,j\}$ corresponding to pairs $\{R_i, R_j\}$ such that $\bar{R}_i \cap \bar{R}_j \neq \emptyset$, and $e(G)$ is the number of edges in $G$. 


For all $\delta > 0$, if either $t = 0$ or $0 < |t| < \delta \sqrt{D_k}$, the sum in (3.12) is absolutely convergent. Moreover, for a fixed $\varepsilon > 0$, there exists $\alpha = \alpha(\delta, \beta) > 0$ that converges to 0 when $\beta \to 0$ such that, for every polymer $R_0$,

\[
1 + \sum_{n=1}^{\infty} \sum_{R_1 \subseteq \Lambda} \ldots \sum_{R_n \subseteq \Lambda} |\phi^T(R_0, R_1, \ldots, R_n)| \prod_{i=1}^{n} |\zeta^i_\beta(R_i)| \leq e^{(1+\varepsilon)|\bar{R}_0|} \tag{3.13}
\]

and

\[
\sum_{n=1}^{\infty} \sum_{R_1 \subseteq \Lambda} \ldots \sum_{R_n \subseteq \Lambda} |\phi^T(R_1, \ldots, R_n)| \prod_{i=1}^{n} |\zeta^i_\beta(R_i)| \leq \alpha |\Lambda|. \tag{3.14}
\]

**Proof.** The proof to obtain the expansion (3.10) and the expansion (3.12) will be omitted, since they are similar to the arguments in [14] and [16] (Chapter 5), respectively.

For $t = 0$, the proof that the sum in (3.12) is absolutely convergent follows as the same argument as in [16] (Chapter 6). Assume $0 < |t| < \delta \sqrt{D_k}$ for some $\delta > 0$ (we choose a suitable $\delta$ along the proof).

Note that for every $x \in \mathbb{Z}^d$, since $|f \circ \theta_x(\sigma)| \leq \|f\|$, there exists $\delta_1 > 0$ such that, for every $0 < \delta < \delta_1$,

\[
|\xi_{\beta,\{x\}}(\sigma)| = \sqrt{2} \left|1 - \cos \left( \frac{t}{\sqrt{D_k}} (f \circ \theta_x(\sigma) - \mathbb{E}_x(f \circ \theta_x)) \right) \right| \leq \delta \|f\|. \tag{3.15}
\]

For a fixed polymer $R$ with $\bar{R} \subseteq \Lambda$, applying Equation (3.15),

\[
|\zeta^i_\beta(R)| \leq (\delta \|f\|)|\bar{R}| \int_{E^\bar{R}} \prod_{x \in \bar{R}} p^w_x(\sigma_x) \prod_{\{x,y\} \in R} |\xi_{\beta,\{x,y\}}(\sigma_x, \sigma_y)| \prod_{x \in \bar{R}} \lambda(d\sigma_x). \tag{3.16}
\]

For $\varepsilon > 0$, let

\[
\alpha := [(1 + \delta \|f\|)e^{2+\varepsilon}]^2 \sup_{x \in \mathbb{Z}^d} \sum_{y \neq x} |e^{-\beta \Phi(x,y)} - 1|. \tag{3.17}
\]

Since the potential $\Phi$ is absolutely summable, there exists $\beta_0 > 0$ such that for all $\beta < \beta_0$, we have $\alpha < 1$. For a polymer $R$, define $\gamma^1_R \subset R$ be the set of all elements of $R$ with cardinality $i$. Note that $\gamma^2_R = \bar{R}$ and $\gamma^1_R \subset \gamma^2_R$. Define, for connected $\gamma \subset \mathcal{P}_2(\Lambda)$,

\[
\hat{\zeta}_\beta(\gamma) = \int_{E^\gamma} \prod_{x \in \gamma} p^w_x(\sigma_x) \prod_{\{x,y\} \in R} |\xi_{\beta,\{x,y\}}(\sigma_x, \sigma_y)| \prod_{x \in \bar{R}} \lambda(d\sigma_x). \tag{3.18}
\]

For a fixed polymer $R_0$,

\[
\sum_{R : R \cap \bar{R}_0 \neq \emptyset} |\zeta^i_\beta(R)| e^{(1+\varepsilon)|\bar{R}|} \leq \sum_{R : R \cap \bar{R}_0 \neq \emptyset} (\delta \|f\|)^{|\bar{R}| \gamma^2} |\hat{\zeta}_\beta(\gamma^2_R)e^{(1+\varepsilon)|\bar{R}|}\|
\]

\[
= \sum_{\gamma^2, \hat{\gamma}^2 \cap \bar{R}_0 \neq \emptyset} \hat{\zeta}_\beta(\gamma^2) e^{(1+\varepsilon)|\gamma^2|} \sum_{\gamma^1, \gamma \subseteq \gamma^2} (\delta \|f\|)^{|\gamma^2|} \|
\]

\[
= \sum_{\gamma^2, \hat{\gamma}^2 \cap \bar{R}_0 \neq \emptyset} [(1 + \delta \|f\|)e^{1+\varepsilon}]^2 |\hat{\zeta}_\beta(\gamma^2)|. \tag{3.19}
\]
By the same argument as in [16] (Chapter 6), for $\delta > 0$ sufficiently small,
\begin{equation}
\max_{z \in \Lambda} \sum_{\gamma^2, z \in \mathbb{Z}^2} [(1 + \delta \| f \|) e^{1+\epsilon}]^\delta \gamma \leq \alpha. \tag{3.20}
\end{equation}

Using Equations (3.19) and (3.20),
\begin{equation}
\sup_{x \in \mathbb{Z}^d} \sum_{R \in R} |\gamma(1 + \delta \| f \|) e^{1+\epsilon} R \leq \alpha. \tag{3.21}
\end{equation}

the proof finishes by applying the result in [14].

\section{Proof of the main result}

\textbf{Proposition 1.} Let $f : \Omega \rightarrow \mathbb{Z}$ be a bounded, non-almost-surely constant, $\mathcal{F}_0$-measurable function. There exists a positive constant $d$ such that, for every $\Lambda \in \mathbb{Z}^d$ and $x \in \Lambda$,
\begin{equation}
\mathbb{E}_x ((f \circ \theta_x(\sigma) - \mathbb{E}_x (f \circ \theta_x(\sigma)))^2) \geq d \tag{4.22}
\end{equation}

and, for every $0 < \delta < \pi$, there exists a positive constant $c$ such that
\begin{equation}
|\mathbb{E}_x (e^{itf \circ \theta_x(\sigma)})| < e^{-c}, \tag{4.23}
\end{equation}

for all $\delta \leq |t| \leq \pi$ uniformly with respect to $\omega$.

\textbf{Proof.} Both (4.22) and (4.23) follow from the inequality
\begin{equation}
\frac{e^{-2\| \Phi \|}}{\lambda(E)} \leq p_{\mathbb{Z}^d}(\sigma) \leq \frac{e^{2\| \Phi \|}}{\lambda(E)},
\end{equation}

where $\| \Phi \|$ is defined in (2.1). The inequality (4.22) is concluded using Markov inequality. The estimate (4.23) is obtained from Lemma 1 in [11].

As in [4], to obtain the main result, we use the following estimate
\begin{equation}
\sup_{p \in \mathbb{Z}} \left| \sqrt{D_k} \mu_{\Lambda_k, \beta}(S_k = p) - \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(p - \mu_{\Lambda_k, \beta}(S_k))^2}{2D_k} \right) \right| \tag{4.24}
\end{equation}

\begin{align*}
\leq \int_{-B}^B |\mu_{\Lambda_k, \beta}(\exp (itS_k)) - \exp (-t^2/2)| \ dt \\
+ \int_{|t| \geq B} \exp (-t^2/2) \ dt + \int_{B \leq |t| \leq \delta \sqrt{D_k}} |\mu_{\Lambda_k, \beta}(\exp (itS_k))| \ dt \\
+ \int_{\delta \sqrt{D_k} \leq |t| \leq \pi \sqrt{D_k}} |\mu_{\Lambda_k, \beta}(\exp (itS_k))| \ dt. \tag{4.25}
\end{align*}

The convergence of the first integral is a consequence of the Central Limit Theorem, while the convergence of the second integral is a consequence of the tail of a Gaussian random variable. The convergence of the third and fourth integrals are following from Lemma 1 and 2 below. The proofs are analogous to Lemma 2.2 and 2.3 in [4], respectively. We highlight some steps in which we use the condition of sufficiently high temperatures.
Lemma 1. Suppose that $\Phi$ is an absolutely summable potential, $f : \Omega \to \mathbb{Z}$ is a bounded, non-almost-surely constant, $\mathcal{F}_0$-measurable function. There exists a positive constant $D$, not depending on $\omega_k$ and $\Lambda_k$, such that, if $\delta > 0$ and $\beta > 0$ are small enough and $|t| \leq \delta \sqrt{D_k}$, then:

$$
\left| \mu_{\Lambda_k, \beta}^{\omega_k} \left( \exp \left( it\tilde{S}_k \right) \right) \right| \leq \exp \left( -t^2 D \frac{\Lambda_k}{D_k} \right), \quad (4.26)
$$

Proof. By Taylor Remainder Theorem, there exists $0 < \theta < \delta \sqrt{D_k}$ such that

$$
\left| \mu_{\Lambda_k, \beta}^{\omega_k} \left( \exp \left( \frac{it}{D_k} \sum_{x \in \Lambda_k} \left( f \circ \theta_x (\sigma) - \mathbb{E}_{x} (f \circ \theta_x) \right) \right) \right) \right| \leq \exp \left( \frac{t^2}{2} \sum_{R, |R| = 1} \text{Re} \left| \frac{d^2}{dt^2} \zeta_\beta^l (R) \right| \right) \leq \exp \left( \frac{t^2}{2} \sum_{n, R_1, \ldots, R_n \in \Lambda_k} |\varphi^T (R_1, \ldots, R_n) \left| \left| \frac{d^2}{dt^2} \prod_{i=1}^n \zeta_\beta^l (R_i) \right| \right| \right), \quad (4.27)
$$

where $\sum^*$ means $\sum_{i=1}^n |\tilde{R}_i| \geq 2$. Differentiating the activity functions and applying Proposition [1] Equation [4.22], we get the following inequalities

$$
\text{Re} \left| \frac{d^2}{dt^2} \zeta_\beta^l (R) \right| \leq -\frac{1}{D_k} \left( d - \frac{\delta^2 \|f\|^4}{8} \right) \quad \text{and} \quad \left| \frac{d^2}{dt^2} \prod_{i=1}^n \zeta_\beta^l (R_i) \right| \leq \frac{4\|f\|^2}{D_k} \prod_{i=1}^n \zeta_\beta^0 (R_i).
$$

The proof finishes taking $\delta$ sufficiently small, applying cluster expansion in [4.28], and taking $\beta$ sufficiently small. \qed

Lemma 2. Suppose that $\Phi$ is an absolutely summable potential, $f : \Omega \to \mathbb{Z}$ is a bounded, non-almost-surely constant, $\mathcal{F}_0$-measurable function. For every $\delta > 0$, there exists a positive constant $C$, not depending on $\omega_k$ and $\Lambda_k$, such that, if $\delta \sqrt{D_k} \leq |t| \leq \pi \sqrt{D_k}$, and $\beta > 0$ is small enough, then

$$
\left| \mu_{\Lambda_k, \beta}^{\omega_k} \left( \exp \left( it\tilde{S}_k \right) \right) \right| \leq \exp \left( -C |\Lambda_k| \right). \quad (4.29)
$$

Proof. For a fixed positive real constant $c$, define the real activity function

$$
\eta^0_\beta (R) = e^{c|\tilde{R}|^2} \zeta^0_\beta (R).
$$

The cluster expansion convergence holds for this modified version of the activity function following closely the technique in [16] (Chapter 6). Indeed the proof is similar with the modification

$$
\alpha = e^{2(2+\epsilon+c)} \sup_{z \in \mathbb{Z}^d} \sum_{y \neq z} e^{-\beta \Phi(y,z)} - 1.
$$

9
Note that $\alpha < 1$ for sufficiently small $\beta$. Since

$$\sup_{x \in \mathbb{Z}^d} \max_{R \in \mathbb{R}} \sum_{x \in R} |\eta^\beta_\beta(R)| e^{(1+\varepsilon)|\tilde{R}|} \leq \alpha,$$  \hspace{1cm} (4.30)

applying the result in [14], for a fixed polymer $R_0$,

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(R_1, \ldots, R_n)} |\phi^T(R_0, R_1, \ldots, R_n)| \prod_{i=1}^{n} |\eta^\beta_i(R_i)| \leq e^{(1+\varepsilon)|\tilde{R}_0|}.$$ \hspace{1cm} (4.31)

By Proposition 1 using Equation (4.23), for every $\delta \sqrt{D_k} \leq |t| \leq \pi \sqrt{D_k}$, we have

$$|\mu^\omega_{\Lambda_k, \beta}(\exp(it\bar{S}_k))| \leq e^{-c|\Lambda_k|} \frac{\Xi_{\Lambda_k, \beta}(\eta_k)}{\Xi_{\Lambda_k, \beta}(\eta_0)}.$$ \hspace{1cm} (4.32)

Using Equations (4.30) and (4.31),

$$\frac{\Xi_{\Lambda_k, \beta}(\eta_k)}{\Xi_{\Lambda_k, \beta}(\eta_0)} \leq e^{\alpha|\Lambda_k|},$$ \hspace{1cm} (4.33)

then Equation (4.29) holds when $\beta$ is sufficiently small.

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\section*{References}

[1] M. Aizenman, J. Chayes, L. Chayes, C. Newman. Discontinuity of the Magnetization in the One-Dimensional $1/|x-y|^2$ Percolation, Ising and Potts Models. J. Stat. Phys. 50(1): 1–40, 1988.

[2] A. Berry. The accuracy of the Gaussian approximation to the sum of independent variates. Transactions of the American Mathematical Society 49(1): 122–136, 1941.

[3] E. Bolthausen. On the central limit theorem for stationary mixing random fields. The Annals of Probability: 1047–1050, 1982.

[4] M. Campanino, D. Capocaccia, and B. Tirozzi. The local central limit theorem for a Gibbs random field. Comm. Math. Phys. 70(2): 125–132, 1979.

[5] M. Campanino, G. Del Grosso, and B. Tirozzi. Local limit theorem for Gibbs random fields of particles and unbounded spins. Journal of Mathematical Physics 20(8): 1752–1758, 1979.

[6] T. Cox, and G. Grimmett. Central limit theorems for percolation models. Journal of Statistical Physics 25(2): 237–251, 1981.
[7] T. Cox, and G. Grimmett. Central limit theorems for associated random variables and the percolation model. *The Annals of Probability*: 514–528, 1984.

[8] J. De Coninck. Gaussian fluctuations for the magnetization of Lee-Yang ferromagnets at zero external field. *Journal of Statistical Physics* 47(3): 397–407, 1987.

[9] G. Del Grosso. On the local central limit theorem for Gibbs processes. *Comm. Math. Phys.* 37(2): 141–160, 1974.

[10] R.L. Dobrushin. The description of the random field by its conditional distributions and its regularity conditions. *Theor. Probab. Appl.* 13(2): 197–224, 1968.

[11] R.L. Dobrushin, and B. Tirozzi. The central limit theorem and the problem of equivalence of ensembles. *Comm. Math. Phys.* 54(2): 173–192, 1977.

[12] F.J. Dyson. Existence of a Phase Transition in a One-Dimensional Ising Ferromagnet. *Comm. Math. Phys.* 12: 91–107, 1969.

[13] C.-G. Esseen. On the Liapunoff limit of error in the theory of probability. *Arkiv for Matematik, Astronomi och Fysik A*: 1–19, 1942.

[14] R. Fernández, A. Procacci. Cluster expansion for abstract polymer models. New bounds from an old approach. *Comm. Math. Phys.* 274(1): 123–140, 2007.

[15] C.M. Fortuin, P.W. Kasteleyn, J. Ginibre. Correlation Inequalities on some Partially Ordered Sets. *Comm. Math. Phys.* 22: 89–103, 1971.

[16] S. Friedli, Y. Velenik. Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction. Cambridge University Press, Cambridge, 2017.

[17] J. Fröhlich, T. Spencer. The Phase Transition in the One-Dimensional Ising Model with $1/r^2$ interaction energy. *Comm. Math. Phys.* 84: 87–101, 1982.

[18] G. Gallavotti, G. Jona-Lasinio. Limit theorems for multidimensional Markov processes. *Comm. Math. Phys.* 41(3): 301–307, 1975.

[19] G. Gallavotti, A. Martin-Löf. Block-spin distributions for short-range attractive Ising models. *Il Nuovo Cimento B* (1971-1996) 25(1): 425–441, 1975.

[20] H.-O. Georgii. Canonical and grand canonical Gibbs states for continuum systems. *Comm. Math. Phys.* 48(1): 31–51, 1976.

[21] G. Hegerfeldt, C. Nappi. Mixing properties in lattice systems. *Comm. Math. Phys.* 53(1): 1–7, 1977.

[22] D. Iagolnitzer, B. Souillard. Lee-Yang theory and normal fluctuations. *Physical Review B* 19(3): 1515, 1979.

[23] H. Kiünisch. Decay of Correlations under Dobrushin’s Uniqueness Condition and its Applications. *Commun. Math. Phys.* 84: 207–222, 1982.

[24] V. Malyshev. A central limit theorem for Gibbsian random fields. *Doklady Akademii Nauk* Russian Academy of Sciences, 224(1), 1975.
[25] A. Martin-Löf. Mixing properties, differentiability of the free energy and the central limit theorem for a pure phase in the Ising model at low temperature. *Commun. Math. Phys.* 32(1): 75–92, 1973.

[26] B. Nakhapetyan. The central limit theorem for random fields with mixing conditions. *Advances in Probability* 6: 531–548, 1980.

[27] C. Neaderhouser. Limit theorems for multiply indexed mixing random variables, with application to Gibbs random fields. *The Annals of Probability* 6(2): 207–215, 1978.

[28] C. Neaderhouser. Some limit theorems for random fields. *Commun. Math. Phys.* 61(3): 293–305, 1978.

[29] C. Neaderhouser. Convergence of block spins defined by a random field. *Journal of Statistical Physics* 22(6): 673–684, 1980.

[30] C. Neaderhouser. An almost sure invariance principle for partial sums associated with a random field. *Stochastic Processes and their Applications* 11(1): 1–10, 1981.

[31] C.M. Newman. Normal fluctuations and the FKG inequalities. *Comm. Math. Phys.* 74(2): 119–128, 1980.

[32] C.M. Newman. A general central limit theorem for FKG systems. *Comm. Math. Phys.* 91(1): 75–80, 1983.

[33] C.M. Newman, A. Wright. An invariance principle for certain dependent sequences. *The Annals of Probability* 9(4): 671–675, 1981.

[34] C.M. Newman, A. Wright. Associated random variables and martingale inequalities. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 59(3): 361–371, 1982.

[35] D. Pickard. Asymptotic inference for an Ising lattice. *Journal of Applied Probability* 13(3): 486–497, 1976.