Abstract

We apply the theory of unimodular random rooted graphs to study the metric geometry of large, finite, bounded degree graphs whose diameter is proportional to their volume. We prove that for a positive proportion of the vertices of such a graph, there exists a mesoscopic scale on which the graph looks like \( \mathbb{R} \) in the sense that the rescaled ball is close to a line segment in the Gromov-Hausdorff metric.

1 Introduction

The aim of this modest note is to prove that large graphs with diameter proportional to their volume must `look like \( \mathbb{R} \)’ from the perspective of a positive proportion of their vertices, after some rescaling that may depend on the choice of vertex. We write \( d_{GH}^{\text{loc}} \) for the Gromov-Hausdorff metric, which is a measure of similarity between locally compact pointed metric spaces (see Section 2.1).

Theorem 1.1. Let \( (G_n)_{n \geq 1} = ((V_n, E_n))_{n \geq 1} \) be a sequence of finite, connected graphs with \( |V_n| \to \infty \), and suppose that there exists a constant \( C < \infty \) such that \( |V_n| \leq C \text{diam}(G_n) \) for every \( n \geq 1 \). Suppose furthermore that the set of degree distributions of the graphs \( G_n \) are uniformly integrable. Then there exists a sequence of subsets \( A_n \subseteq V_n \) with \( \liminf_{n \to \infty} |A_n|/|V_n| \geq C^{-1} \) such that

\[
\lim_{n \to \infty} \sup_{v \in A_n} \inf_{\epsilon > 0} d_{GH}^{\text{loc}}\left( (V, \epsilon d_{G_n}, v), (\mathbb{R}, d_{\mathbb{R}}, 0) \right) = 0,
\]

where we write \( d_{G_n} \) for the graph metric on \( G_n \) and \( d_{\mathbb{R}}(x, y) = |x - y| \) for the usual metric on \( \mathbb{R} \).

Our result should be compared to the (much more difficult) result of the first author, Finucane, and Tessera [2] that transitive graphs with volume proportional to their diameter converge to the circle when rescaled by their diameter; here we have much weaker hypotheses but also a much weaker result. We remark that in [2] it is shown more generally that every sequence of transitive graphs with volume at most polynomial in their diameter has a subsequence converging to a torus (equipped with an invariant Finsler metric) when rescaled by their diameters. Various further results on the scaling limits of transitive graphs satisfying polynomial growth conditions have subsequently been obtained by Tessera and Tointon [7]. It seems unlikely that polynomial growth
assumptions such as \(|V_n| = O(\text{diam}(G_n)^C)\) will imply much at all about the metric geometry of graphs without the assumption of transitivity.

Note that it is not possible to control the scale on which the rescaled graph looks like \(\mathbb{R}\). Indeed, if \(r : \mathbb{N} \rightarrow \mathbb{N}\) is any function with \(r(n)/n \rightarrow 0\) and \(r(n) \rightarrow \infty\) as \(n \rightarrow \infty\), then the graph formed by attaching \(2\lceil n/r(n) \rceil\) line segments of length \(\lceil r(n)/2 \rceil\) to a line segment of length \(n\) in a regularly spaced manner has \(\text{diam}(G_n) \sim n\) and \(|V_n| \sim 2n\) but it metrically distinguishable from \(\mathbb{R}\) on the scale \(r(n)\) from the perspective of every vertex in the sense that

\[
\lim_{n \to \infty} \inf_{v \in V} d_{\text{GH}} \left( \left( B_{G_n}(v, r(n)), r(n)^{-1}d_{G_n}, v \right), \left( [-1, 1], d_\mathbb{R}, 0 \right) \right) > 0.
\]

Note also that the hypotheses do not allow us to take the sets \(A_n\) to have \(|A_n|/|V_n| \rightarrow 1\). Indeed, consider taking an \(n \times n\) square grid and attaching to the grid three disjoint paths of length \(n^2\):

The volume of this graph is about twice its diameter, the grid has about \(1/4\) of the total vertices of the graph, and from a vertex of the grid the graph is metrically distinguishable from both \(\mathbb{R}\) and \(\mathbb{R}_+\) at every scale. A similar example shows that the dependence \(\lim \inf_{n \to \infty} |A_n|/|V_n| \geq C^{-1}\) is optimal.

We remark that the uniform integrability hypothesis of Theorem 1.1 holds in particular if there exists a constant \(M\) such that all the graphs \((G_n)_{n \geq 1}\) have degrees bounded by \(M\). We use this uniform integrability assumption to guarantee that if \(\rho_n\) is a uniform random root vertex of \(G_n\) for each \(n \geq 1\) then the sequence of random rooted graphs \((G_n, \rho_n)\) is tight with respect to the local topology \([5, \text{Proposition 2.1}]\), so that there exists a subsequence \(\sigma(n)\) such that \(G_{\sigma(n)}\) Benjamini-Schramm converges to some infinite random rooted graph \((G, \rho)\). Indeed, we will use this fact to deduce Theorem 1.1 from the following closely related result via a compactness argument. The relevant definitions are reviewed in Section 2.2.

**Theorem 1.2.** Let \((G_n)_{n \geq 1} = ((V_n, E_n))_{n \geq 1}\) be a sequence of finite, connected graphs Benjamini-Schramm converging to some infinite random rooted graph \((G, \rho) = ((V, E), \rho)\). Suppose that there exists a constant \(C < \infty\) such that \(|V_n| \leq C \text{diam}(G_n)\) for every \(n \geq 1\). Then there exists an event \(\Omega\) of probability at least \(1/C\) on which the following hold:

1. The graph \(G\) is two-ended and has linear growth almost surely.

2. The pointed metric space \((V, \varepsilon d_G, \rho)\) converges to \((\mathbb{R}, d_\mathbb{R}, 0)\) in the local Gromov-Hausdorff topology almost surely as \(\varepsilon \downarrow 0\).

Here, a connected graph is said to have **linear growth** if \(\limsup_{n \to \infty} \frac{1}{n} |B_G(v, n)| < \infty\) for some (and hence every) vertex \(v\) of \(G\). An infinite, connected, locally finite graph is said to be \(k\)-ended if deleting a finite set of vertices from \(G\) results in a maximum of \(k\) infinite connected components.

Similarly to above, note that the hypotheses do not ensure that \(G\) is two-ended or has linear growth almost surely: Consider for example taking \(G_n\) to be a path of length \(n^2\) connected to a
\(n \times n\) square grid, whose Benjamini-Schramm limit is equal either to \(\mathbb{Z}\) or \(\mathbb{Z}^2\), each with probability \(1/2\).

2 Definitions

2.1 The Gromov-Hausdorff metric

We now define the Gromov-Hausdorff metric, referring the reader to [4] for a detailed treatment of this metric and its properties. Let \(X\) and \(Y\) be pointed sets, i.e., non-empty sets with a distinguished point. A correspondence between \(X\) and \(Y\) is a set \(R \subseteq X \times Y\) such that \(\{x\} \times Y \cap R \neq \emptyset\) and \(X \times \{y\} \cap R \neq \emptyset\) for every \(x \in X\) and \(y \in Y\). If \((X, x_0)\) and \((Y, y_0)\) are pointed sets, i.e., non-empty sets each with a distinguished point, then we say that a correspondence \(R\) between \(X\) and \(Y\) is a correspondence between \((X, x_0)\) and \((Y, y_0)\) if \((x_0, y_0) \in R\).

If \((X, d_X)\) and \((Y, d_Y)\) are metric spaces and \(R\) is a correspondence between \(X\) and \(Y\), we define the distortion of \(R\) to be

\[
\text{dis} R = \sup \left\{ |d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in R \right\}.
\]

Given two pointed metric spaces \((X, d_X, x_0)\) and \((Y, d_Y, y_0)\), we define the Gromov-Hausdorff distance to be

\[
d_{GH}((X, d_X, x_0), (Y, d_Y, y_0)) = \frac{1}{2} \inf \left\{ \text{dis} R : R \text{ is a correspondence between } (X, x_0) \text{ and } (Y, y_0) \right\}.
\]

The function \(d_{GH}\) defines a metric on the space of isometry classes of compact pointed metric spaces. Similarly, the local Gromov-Hausdorff topology on (isometry classes of) locally compact pointed metric spaces is defined to be the topology induced by the metric

\[
d_{GH}^{\text{loc}}((X, d_X, x_0), (Y, d_Y, y_0)) = \sum_{r \geq 1} 2^{-r} d_{GH}\left((B_X(x_0, r), d_X, x_0), (B_Y(y_0, r), d_Y, y_0)\right)
\]

where we write \(B_X(x, r)\) for the ball of radius \(r\) around the point \(x\) in the metric space \(X = (X, d_X)\).

This topology has the property that \((X_n, d_n, x_n)\) converges to \((X, d_X, x_0)\) if and only if

\[
\lim_{n \to \infty} d_{GH}\left((B_{X_n}(x_n, r), d_n, x_n), (B_X(x_0, r), d_X, x_0)\right) = 0
\]

for every \(r \geq 0\).

2.2 Unimodular random rooted graphs

We now review the notions of Benjamini-Schramm convergence and unimodular random rooted graphs, referring the reader to [5] for more detailed treatments. A rooted graph is a connected, locally finite graph \(g\) together with a distinguished root vertex \(v\). A graph isomorphism between rooted graphs is an isomorphism of rooted graphs if it preserves the root. The space of rooted graphs is denoted \(\mathcal{G}_\ast\), and is equipped with the local topology, which is induced by the metric

\[
d_{\text{loc}}((g_1, v_1), (g_2, v_2)) = 2^{-R((g_1, v_1), (g_2, v_2))}
\]
where $R((g_1, v_1), (g_2, v_2))$ is the supremal value of $r$ for which the balls of radius $r$ around $v_1$ and $v_2$ are isomorphic as rooted graphs. The space $G_*$ is a Polish space with respect to this topology [5, Theorem 2].

Similarly, we define a rooted oriented-edge-labeled graph to be a rooted graph $(g, v)$ together with a function from the set of oriented edges of $g$ to $\{0, 1\}$. (Although our graphs are undirected, we can still think of each edge as a pair of oriented edges.) We write $G_\{0,1\}$ for the space of isomorphism classes of rooted oriented-edge-labeled graphs, which is equipped with a local topology that is defined similarly to the unlabelled case. Finally, we define the spaces of doubly-rooted graphs $G_{**}$ and doubly-rooted oriented-edge-labeled graphs $G_\{0,1\}_{**}$ similarly to above except that we now have an ordered pair of distinguished root vertices.

A probability measure $\mu$ on $G_*$ is said to be unimodular if it satisfies the mass-transport principle, which states that the identity

$$\mu \left[ \sum_{v \in V} F(G, \rho, v) \right] = \mu \left[ \sum_{v \in V} F(G, v, \rho) \right]$$

is satisfied for every measurable function $F : G_* \to [0, \infty]$, where we write $(G, \rho)$ for a random variable sampled from the measure $\mu$. Unimodular probability measures on $G_\{0,1\}$ are defined similarly. This notion was first introduced by Benjamini and Schramm in [3] and developed systematically by Aldous and Lyons [1]. A different form of the mass-transport principle was first considered by Häggström in the context of Cayley graphs [6].

The set of unimodular probability measures on $G_*$ is convex and closed under the weak topology [5, Theorem 8]. It also includes all the laws of random graphs of the form $(G, \rho)$ where $G$ is a finite connected graph and $\rho$ is a uniform random root vertex of $G$. Similar statements hold for rooted oriented-edge-labeled graphs. A unimodular random rooted graph $(G, \rho)$ is said to be the Benjamini-Schramm limit of a sequence of finite connected graphs $(G_n)_{n \geq 1}$ if the random variables $(G_n, \rho_n)$ converge in distribution to $(G, \rho)$ when we take $\rho_n$ to be a uniform random root vertex of $G_n$ for each $n \geq 1$.

A set $\Omega \subseteq G_\{0,1\}$ is said to be (re-rooting) invariant if $(g, v) \in \Omega$ if and only if $(g, u) \in \Omega$ for every vertex $u$ of $g$. It is easily seen that if $\mu$ is a unimodular probability measure on $G_\{0,1\}$ and $\Omega$ is a measurable invariant set with $\mu(\Omega) > 0$ then the conditional measure $\mu(\cdot \mid \Omega)$ is also unimodular.

### 3 Proof

**Proof of Theorem 1.2.** Let $(G_n, \rho_n)$ and $(G, \rho)$ be as in the statement of the theorem. Let $E_n^\to$ and $E^\to$ denote the sets of oriented edges of $G_n$ and $G$ respectively. We first argue that (by passing to a bigger probability space if necessary) it is possible to endow $G$ with a random oriented-edge-labelling $\gamma \in \{0, 1\}^{E^\to}$ such that the following hold:

1. $(G, \rho, \gamma)$ is a unimodular random rooted oriented-edge-labelled graph.
2. The event $\Omega = \{ \gamma(e) = 1 \text{ for some } e \in E^\to \}$ has $\mathbb{P}(\Omega) \geq C^{-1}$.

3. On the event $\Omega$, $\gamma$ is an oriented doubly-infinite geodesic of $G$ almost surely.

We will construct the edge-labelling $\gamma$ via a limiting procedure. For each $n \geq 1$, let $\gamma_n$ be an oriented geodesic of maximum length in $G_n$, which we consider as an element of $\{0, 1\}^{E_n^\to}$. The sequence of oriented-edge-labelled graphs $(G_n, \rho_n, \gamma_n)$ is tight, so that there exists a subsequence $(G_{\sigma(n)}, \rho_{\sigma(n)}, \gamma_{\sigma(n)})$ converging to some infinite unimodular random edge-labelled graph $(G, \rho, \gamma)$. The notation here is justified since forgetting the oriented-edge-labelling gives back the same law on random rooted graphs that described our original random rooted graph $(G, \rho)$. It remains to argue that this graph satisfies properties (2) and (3) above. We say that a vertex $v$ is incident to $\omega \in \{0, 1\}^{E \to}$ if there is an oriented edge with $v$ as one of its endpoints and with $\omega(e) = 1$. For (2), we clearly have that

$$\mathbb{P}(\rho_n \text{ is incident to an edge of } \gamma_n) = \frac{\text{diam}(G_n) + 1}{|V_n|} \geq C^{-1}$$

for every $n$ such that $|V_n| \geq 2$, and consequently that

$$\mathbb{P}(\gamma(e) = 1 \text{ for some } e \in E^\to) \geq \mathbb{P}(\rho \text{ is incident to an edge of } \gamma) \geq C^{-1}$$

also. For (3), we consider the set of all $(g, x, \omega) \in \mathcal{G}_{\bullet}^{\{0,1\}}$ such that $\omega$ does not contain any oriented cycles, any two vertices incident to $\omega$ are connected in $\omega$ by exactly one path, and this path is an oriented geodesic in $g$: We observe that this set is closed in $\mathcal{G}_{\bullet}^{\{0,1\}}$, and deduce that the set $\{\gamma(e) : e \in E^\to\}$ is almost surely an oriented geodesic of $G$ on the event that it is nonempty. Moreover, since $G$ is infinite, the mass-transport principle implies that $\gamma$ must be a doubly-infinite geodesic on this event: If not, there would exist one or two special vertices of $(G, \rho, \gamma)$ that lied at the endpoints of $\gamma$, and considering the mass-transport in which every vertex sends mass 1 to each of these special vertices would lead to a contradiction of the mass-transport principle. (Indeed, the expected mass out would be at most 2, while the root would receive infinite mass with positive probability by [5, Proposition 11] and therefore have infinite expected mass in.)

We now use the random oriented doubly-infinite geodesic $\gamma$ to argue that both claims of the theorem hold on the event $\Omega$. Let $\Gamma$ be the set of vertices visited by $\gamma$. Let $\mu$ be the law of $(G, \rho, \gamma)$ conditioned on the event $\Omega$, which, since $\Omega$ is an invariant event, is the law of a unimodular random rooted oriented-edge-labelled graph. Since $\gamma$ is oriented, we can put a total ordering on $\Gamma$ that encodes the order in which the vertices of $\Gamma$ are visited by $\gamma$. For each $v \in \Gamma$, we write $(\sigma^n(v))_{n \in \mathbb{Z}}$ for the vertices obtained by shifting up and down the oriented geodesic $\gamma$. For each vertex $v$ of $G$, let $g(v)$ of $\Gamma$ at minimal distance to $v$, choosing $g(v)$ to be the point that is minimal in the total order on $\Gamma$ if there are multiple points at minimal distance to $v$. The mass-transport principle implies that

$$\mu \left[ \# \{ v \in V : g(v) = \rho \} \right] = \mu \left[ \sum_{v \in V} 1(g(v) = \rho) \right] = \mu \left[ \sum_{v \in V} 1(g(\rho) = v) \right] = 1$$

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and hence that
\[ \mu \{ v \in V : g(v) = \rho \} = \mu(\rho \in \Gamma)^{-1} < \infty. \]

Let \( X_0 = g(\rho) \) and let the sequence \((X_n)_{n \in \mathbb{Z}}\) be defined by shifting up and down the oriented geodesic \( \gamma \). Let \( \tilde{\mu} \) be the law of \((G, \rho, \gamma)\) conditioned on the event that \( \rho \in \Gamma \). Then the mass-transport principle implies that the sequence of random variables \((G, X_n, \gamma)_{n \in \mathbb{Z}}\) is stationary under the measure \( \tilde{\mu} \): Indeed, if \( \mathcal{A} \subseteq G_{\bullet}^{(0,1)} \) is any measurable set and \( n \in \mathbb{Z} \) then
\[
\mu ((G, X_n, \gamma) \in \mathcal{A} \mid \rho \in \Gamma) = \mu(\rho \in \Gamma)^{-1} \mu \left[ \sum_{v \in V} 1(\rho \in \Gamma, v = \sigma^n(\rho), (G, v, \gamma) \in \mathcal{A}) \right]
\]
\[
= \mu(\rho \in \Gamma)^{-1} \mu \left[ \sum_{v \in V} 1(\rho \in \Gamma, v = \sigma^n(v), (G, \rho, \gamma) \in \mathcal{A}) \right]
\]
\[
= \mu ((G, \rho, \gamma) \in \mathcal{A} \mid \rho \in \Gamma),
\]
which establishes the desired stationarity. Setting \( V_n = \{ v \in V : g(v) = X_n \} \), it follows that \((|V_n|)_{n \in \mathbb{Z}}\) is a stationary sequence of finite mean random variables under the measure \( \tilde{\mu} \), and we deduce from the ergodic theorem and the Borel-Cantelli lemma that
\[
\limsup_{n \to \infty} \frac{1}{2n+1} \sum_{m=-n}^{n} |V_m| < \infty \quad \text{and} \quad (1)
\]
\[
\limsup_{n \to \infty} \frac{1}{2n+1} \max \{ |V_m| : -n \leq m \leq n \} = 0 \quad (2)
\]
almost surely under \( \tilde{\mu} \). On the other hand, we have that
\[
\mu((G, X_n, \gamma)_{n \in \mathbb{Z}} \in \mathcal{A}) = \mu \left[ \sum_{v \in V} 1(v = \rho, ((G, \sigma^n(v), \gamma)_{n \in \mathbb{Z}} \in \mathcal{A}) \right]
\]
\[
= \mu \left[ \sum_{v \in V} 1(g(v) = \rho, ((G, \sigma^n(\rho), \gamma)_{n \in \mathbb{Z}} \in \mathcal{A}) \right]
\]
\[
= \mu(\rho \in \Gamma) \tilde{\mu} \{ v \in V : g(v) = \rho \} 1(((G, X_n, \gamma)_{n \in \mathbb{Z}} \in \mathcal{A})
\]
for every measurable set \( \mathcal{A} \subseteq \left(G_{\bullet}^{(0,1)}\right)^{\mathbb{Z}} \), so that the laws of \((G, X_n, \gamma)_{n \in \mathbb{Z}}\) under \( \mu \) and \( \tilde{\mu} \) are absolutely continuous and hence that \((1)\) and \((2)\) also hold almost surely under \( \mu \).

We will now argue that this implies the two claims. We begin with the first. Linear growth follows obviously from \((1)\) since the ball of radius \( n \) around \( X_0 \) is contained in the set \( \bigcup_{m=-n}^{n} V_m \). Since \( G \) has linear growth it must have at most two-ends, since unimodular random graphs with more than two ends always have infinitely many ends and exponential growth a.s. To see that \( G \) is two-ended rather than one-ended, observe that if \( V_n \) and \( V_m \) are adjacent then we must have that \( \text{diam}(V_n) + \text{diam}(V_m) + 1 \geq |n - m| \). Moreover, if \( v \in V_n \) for some \( n \in \mathbb{Z} \) then the geodesic connecting \( v \) to \( X_n \) is also contained in \( X_n \), so that
\[
|V_n| \geq \text{diam}(V_n) \quad (3)
\]
for every $n \in \mathbb{Z}$ and we deduce from (2) that $\bigcup_{n \geq N} V_n$ and $\bigcup_{n \geq N} V_{-n}$ are not adjacent for $N$ sufficiently large almost surely. It follows that the union $\bigcup_{n=-N}^{N} V_n$ is a finite set separating $G$ into two disjoint sets of vertices for sufficiently large $N$ almost surely, so that $G$ is two-ended almost surely as claimed.

Finally, the fact that $(V, \varepsilon d_G(x, y), \rho)$ converges to $(\mathbb{R}, |x - y|, 0)$ in the pointed Gromov-Hausdorff topology as $\varepsilon \downarrow 0$ follows easily from (2) and (3). Indeed, these estimates imply that if $n(v)$ denotes the unique index such that $v \in V_{n(v)}$ for each $v \in V$ then

$$\limsup_{r \to \infty} \max_{u, v \in B(\rho, Ar)} \frac{1}{r} |d(u, v) - |n(u) - n(v)|| \leq \limsup_{r \to \infty} \max_{-Ar \leq n \leq Ar} \frac{2}{r} \operatorname{diam}(V_n) = 0$$

for every $A \geq 1$. This implies that the correspondence

$$\left\{(v, -A \lor r^{-1}n(v) \land A) : v \in B_G(\rho, Ar)\right\} \cup \left\{(X_{[nx]}, x) : x \in [-A, A]\right\}$$

between $(B_G(\rho, Ar), r^{-1}d_G, \rho)$ and $([-A, A], d_r, 0)$ has distortion tending to zero as $r \to \infty$ for every $A \geq 1$, which implies the claim. \hfill \Box

\textit{Remark 3.1.} With a little more work one can show that any two-ended unimodular random rooted graph has linear volume growth and converges to $\mathbb{R}$ under rescaling. This should be compared to the results of [8].

We now deduce Theorem 1.1 from Theorem 1.2; this will be an exercise in compactness.

\textit{Proof of Theorem 1.1.} For each $\varepsilon > 0$ define $\mathcal{E}_\varepsilon : G_{\bullet} \to [0, 1]$ by

$$\mathcal{E}_\varepsilon(g, v) = d_{GH}^h\left((V, \varepsilon d_{G_n}, v), (\mathbb{R}, d_\mathbb{R}, 0)\right) = \sum_{k \geq 1} 2^{-k} d_{GH}\left((B_g(v, k\varepsilon^{-1}), \varepsilon d_g, v), (-k, k), d_\mathbb{R}, 0\right).$$

Since the Gromov-Hausdorff distance between two compact, metric spaces is always bounded by their diameter (consider the correspondence $\mathcal{R} = X \times Y$), the sum defining $\mathcal{E}_\varepsilon$ converges uniformly and, since each summand is clearly continuous, we deduce that $\mathcal{E}_\varepsilon$ is continuous on $G_{\bullet}$ for each $\varepsilon > 0$. It follows in particular that the set $\{(g, v) : \inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(g, v) < x\}$ is open in $G_{\bullet}$ for every $x > 0$. By the Portmanteau theorem, $\mu \to \mu(\inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(G, \rho) < x)$ is a lower semi-continuous function with respect to the weak topology on the space of probability measures $\mathcal{P}(G_{\bullet})$ for each $x > 0$.

Let $\rho_n$ be a uniform root vertex of $G_n$ and let $\mu_n$ be the law of $(G_n, \rho_n)$. The uniform integrability assumption ensures that the set $A = \{\mu_n : n \geq 1\}$ is a compact subset of $\mathcal{P}(G_{\bullet})$ [5, Proposition 2.1]. Moreover, its closure $\overline{A}$ is contained in the set of $unimodular$ probability measures on $G_{\bullet}$. Since $G_{\bullet}$ is a Polish space, $\mathcal{P}(G_{\bullet})$ is also a Polish space. Since $|V_n| \to \infty$, it follows that the set $\overline{A} \setminus A$ coincides with the set of limits of subsequences of $(\mu_n)_{n \geq 1}$. Thus, Theorem 1.2 implies that

$$\mu\left(\inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(G, \rho) < x\right) \geq C^{-1}$$
for every $\mu \in \overline{A} \setminus A$ and $x > 0$. It therefore follows by a standard compactness argument using lower semi-continuity that

$$\liminf_{n \to \infty} \mu_n \left( \inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(G, \rho) < x \right) \geq C^{-1}$$

for every $x > 0$, which is equivalent to the claim: Indeed, if not then there exists $x > 0$ and a subsequence $\sigma(n)$ such that $\lim_{n \to \infty} \mu_{\sigma(n)} \left( \inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(G, \rho) < x \right) < C^{-1}$. Using compactness and taking a further subsequence $\tau$ we can ensure that $\lim_{n \to \infty} \mu_{\tau(n)} \left( \inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(G, \rho) < x \right) < C^{-1}$ and that $\mu_{\tau(n)}$ converges to some $\mu \in \overline{A} \setminus A$. But then lower semi-continuity gives that $C^{-1} > \lim_{n \to \infty} \mu_{\tau(n)} \left( \inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(G, \rho) < x \right) \geq \mu \left( \inf_{\varepsilon > 0} \mathcal{E}_\varepsilon(G, \rho) < x \right) \geq C^{-1}$, a contradiction.

\[\square\]

Acknowledgments

We thank Jonathan Hermon and Matthew Tointon for helpful comments on a draft.

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