Microscopic physics of quantum self-organization of optical lattices in cavities

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Abstract. We study quantum particles at zero temperature in an optical lattice coupled to a resonant cavity mode. The cavity field substantially modifies the particle dynamics in the lattice, and for strong particle-field coupling leads to self-organization of the particles, a configuration with only every second site occupied. We study the growth of this order out of a homogeneous initial distribution for few particles. Simulations reveal that the growth dynamics crucially depends on the initial quantum many-body state of the particles and can be monitored via the cavity fluorescence. Studying the relaxation time of the ordering reveals inhibited tunnelling due to the interaction with the cavity field. However, the relaxation becomes very quick for strong coupling.

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1. Introduction

Ultracold atoms in an optical lattice formed by a far detuned laser field constitute an ideal system to study quantum phase transitions, i.e. phase transitions at zero temperature [1]. In the most prominent example first predicted theoretically [2] and confirmed experimentally [3] it was found that the particle ground state changes from a superfluid (SF) state where all atoms are delocalized to a perfectly ordered Mott insulator state for increasing lattice depth. More complex phases as supersolids, etc. were predicted if long-range interactions or mixed species set-ups [4, 5] are used, but these are harder to realize and measure experimentally. While the final states are well understood, these phase transitions require the build-up or decay of long-range correlations, the mechanism and timescale of which is not fully understood.

In a parallel development, a dynamical transition to a self-organized phase in optical lattices was found for classical particles, when the lattice is placed inside an optical resonator. It originates from interference of the resonantly enhanced light field scattered by the atoms into the cavity mode with the lattice light itself and leads to a preferred occupation of every second site [6]. For a finite temperature cloud thermal density fluctuations are amplified and lead to a runaway self-organization by feedback from the cavity field. However, for a Bose–Einstein condensate (BEC) ($T \approx 0$) the initial density is perfectly homogeneous and only quantum fluctuations which go beyond a mean-field description of the cold gas can start the self-organising process when the cavity interaction is switched on. Tunnelling results in a dynamical change of the atomic phase at $T = 0$ which gets irreversible only if cavity decay is included. In this work, we study this quantum dynamics on the microscopic level and show how it depends on the precise quantum properties of the initial atomic state beyond any mean-field density.

The paper is also intended to show the limitations of the effective Bose–Hubbard type model developed earlier [7] for atom-cavity systems. We demonstrate that in the regime of moderate coupling the low-dimensional Bose–Hubbard approach reproduces very well the results of a full Monte Carlo wavefunction (MCWF) simulation, while it breaks down in the regime of stronger coupling. Even in this regime, however, it predicts the steady state surprisingly well, whereas the relaxation time to this state is predicted wrong. We show that the relaxation time of the system exhibits a highly nontrivial behaviour. In the regime of moderate atom-cavity coupling the relaxation time is composed of the timescale of photon counts and that of tunnelling. The combination of these two timescales leads to a minimum behaviour in the relaxation time, while for stronger coupling the relaxation becomes very quick, a behaviour observed in the full simulations but not reproducible with the Bose–Hubbard approach.

We first describe our system, the model, and solution methods applied. Afterwards, we go on investigating the dynamics of a single atom in the system. Although self-organization cannot be defined in this case, we demonstrate that already here the relaxation time exhibits the same behaviour we find later for two atoms, and here it is easier to give a qualitative picture of this behaviour. We finally turn to the case of two atoms and show how the increasing coupling results in a transition from a $T = 0$ homogeneous initial condition into a self-organized configuration in steady state.

2. System, models

The proposed set-up is depicted in figure 1. It consists of a one-dimensional optical lattice within a cavity sustaining a single mode with its axis aligned orthogonally to the lattice
axis—such systems have been studied in diverse theoretical contexts before [8]–[11], and are available experimentally [12, 13]. We assume that the cavity mode function is constant along the lattice direction, still, as we will see below, it modifies significantly the dynamics of atoms in the lattice.

A standard quantum optical model for the system with a single atom moving in one-dimension along the lattice axis is obtained by adiabatic elimination of the atomic excited state(s). This is justified in the regime where the driving—in our case, the laser generating the lattice (pump)—is far detuned from the atomic transition frequency [14, 15]. The Hamiltonian for a single atom and the cavity then reads ($\hbar = 1$):

$$H = \frac{p^2}{2\mu} + V_0 \sin^2(Kx) - (\Delta_C - U_0) a^\dagger a + \text{sign}(U_0)\sqrt{U_0 V_0} \sin(Kx) (a^\dagger + a).$$

Here $x$, $p$ and $\mu$ are the atomic position and momentum operators and the mass, respectively; $a$ is the cavity field operator, $\Delta_C = \omega - \omega_C$ is the cavity detuning—$\omega$ is the laser, $\omega_C$ is the cavity frequency, and $K$ is the lattice field wavenumber. The first two terms describe the atomic motion in the lattice, which appears as a classical potential after the elimination of the atomic internal dynamics. The potential depth for a two-level atom $V_0 = \eta^2/\Delta_A$ where $\eta$ is the pump Rabi frequency and $\Delta_A = \omega - \omega_A$, $\omega_A$ is the atomic transition frequency. The third term is the free Hamiltonian of the cavity mode with the detuning shifted by $U_0 = g^2/\Delta_A$, where $g$ is the atom-cavity interaction coupling constant. The last term describes the atom-mode interaction which stems from stimulated absorption from the pump followed by stimulated emission into the cavity mode and the reverse process. Atomic spontaneous emission is strongly suppressed due to the large atom-pump detuning and therefore neglected. The cavity mode is, however, coupled to the surrounding EM modes, resulting in the decay of cavity photons (escape through

**Figure 1.** Scheme of the system consisting of a one-dimensional optical lattice with lattice axis $x$ and a cavity sustaining a single EM mode aligned orthogonally.
the mirrors). The process is described by the Liouvillean dynamics [16]

\[ \mathcal{L}\rho = \kappa \left( 2a \rho a^\dagger - [a^\dagger a, \rho] \right). \]  

(1b)

Note that model (1) is not specific to the two-level atom but generally applicable to linearly polarizable particles, atoms and molecules alike. In this case, the parameter \( U_0 \) is proportional to the susceptibility of the particles [15]. In the following, we shall hence speak about particles without any further specification.

As described theoretically [6, 17] and observed experimentally [18] the models (1a) and (1b) feature a phase transition termed self-organization for a finite temperature classical gas. As this occurs for red-detuned driving, i.e. high-field seeking particles with \( U_0, \Delta_C < 0 \) we will restrict ourselves to this case. Self-organization can be qualitatively understood as follows: the system has three steady-state configurations in a mean-field description: (i) an ‘unorganized’ configuration, where the particles are equally distributed through all lattice sites and scatter no light into the cavity due to destructive interference and (ii) two ‘organized’ configurations in which the particles occupy either only the even or only the odd sites of the lattice and scatter superradiantly into the cavity. In the latter case, the last term in the Hamiltonian (1a) deepens the lattice potential at the positions of the particles, so that they are self-trapped or ‘self-organized’.

Configurations (ii) have lower energy and entropy (lower symmetry) than configuration (i). At a given temperature the system chooses between configuration (i) and one of configurations (ii) so as to minimize the free energy. Lowering the temperature results in a phase transition when the symmetry of configuration (i) is spontaneously broken into the lower symmetry of one of configurations (ii).

The above qualitative picture is modified by the fact that the system never reaches thermal equilibrium with some external heat bath as energy is continuously flowing through the system from the pump via scattering on the particles into the cavity field and then out via the cavity loss channel. Self-organization is therefore a dynamical phase transition for which the above mentioned configurations are steady-state patterns. In steady state, the particles have a momentum distribution determined by the cavity field fluctuations, which, in most cases of physical interest, resembles very much a thermal distribution [19]. In this sense, it is justified to speak about an effective temperature of the particles and use the picture of an equilibrium phase transition as we did above.

Let us now turn to the case of zero temperature and envisage a fixed number of classical point particles at each lattice site. In contrast to above, no matter whether we are above or below the threshold, no dynamics will arise because a homogeneous gas scatters no field into the cavity due to destructive interference. If no photons are present initially, such a classical gas cannot break the symmetry and is unable to escape the initial homogeneous configuration (i).

In the following, we show that this is quite different for quantum particles at \( T = 0 \), a situation which can be prepared by loading a BEC or particles coupled out from a BEC into the cavity [20, 21]. Interestingly, in this case both for a Mott insulator and SF state (BEC) as initial condition, quantum fluctuations and the possibility of tunnelling between lattice sites immediately start self-organization and the superradiant build-up of the cavity photon number. This is combined with an intricate measurement-induced dynamics related to the information gained via the dissipation channel of the photon-loss.

\footnote{Note that self-organization is a phase transition without any direct particle–particle interaction: only an effective interaction exists generated by the single cavity mode with which all particles interact.}
Let us emphasize that there are two main differences as compared to the above-described classical self-organization: the initial temperature of the particles is zero and the particles in the lattice are confined strongly enough so that hopping between the lattice sites is due solely to tunnelling. Redistribution thus is a coherent quantum process and requires no direct inter-particle interaction.

We are using two approaches to the problem. The first one is the direct simulation of the system (1) by the MCWF method, which unravels the corresponding Master equation in terms of individual quantum trajectories. This approach takes into account the full particle and cavity dynamics, with the cavity decay accounted for by quantum jumps. We are using a new simulation framework presented in [22]. The second approach analogous to standard Bose–Hubbard models is based on a second-quantized form of the Hamiltonian (1a):

$$\int dx \, \Psi(x) \, H \, \Psi(x),$$

where the field operator $\Psi(x)$ is restricted to the lowest vibrational band of the lattice.

To obtain the smallest possible system useful for studying self-organization, we restrict the dynamics to only one lattice wavelength, that is, two lattice sites (cf figure 1) with periodic boundary condition. This is the smallest system which can seize the difference between the configurations described above and contains all the essential physics.

With two lattice sites, the lowest vibrational band constitutes a two-dimensional Hilbert space, for which the localized Wannier basis with state $|l\rangle$ localized at the left and $|r\rangle$ at the right lattice site can be used. Hence $\Psi(x) = \langle x|l \rangle b_l + \langle x|r \rangle b_r$, where $b_l$ and $b_r$ are the corresponding bosonic annihilation operators. Putting the restricted field operator back into the second-quantized Hamiltonian, we obtain the Bose–Hubbard type Hamiltonian:

$$H_{BH} = J \left( b_l^\dagger b_l + b_r^\dagger b_r \right) - (\Delta_C - NU_0) a^\dagger a + \bar{J} \left( b_l^\dagger b_l - b_r^\dagger b_r \right) \left( a^\dagger + a \right),$$

where $J \equiv \langle l| (p^2/(2\mu) + V_0 \sin^2(Kx)) |r\rangle$ and $\bar{J} / (\text{sign}(U_0) \sqrt{U_0 V_0}) \equiv \langle l| \sin(Kx) |l\rangle = -\langle r| \sin(Kx) |r\rangle$.

The dynamics of particles in cavities as described by such Hamiltonians in system configurations different from the one investigated here has been discussed in [7, 23, 24]. A very attractive feature of Hamiltonian (2) is that it is simple enough so that together with the Liouvillean (1b) the full time-dependent Master equation can be solved even for several particles. For discussions below, we will imagine this Master equation as unravelled in quantum trajectories.

### 3. Discussion

Self-organization of an initially thermal gas of classical point-like particles in the system with classical cavity field (coherent state) has been described above to set the scene for the two-particle quantum results presented in section 5, and to raise two questions concerning quantum particles in a quantum-mechanically described cavity field:

1. whether in the steady state of the quantum Master equations (1a) and (1b), there is some transition, a quantum analogue for the above-described self-organization, which is a classical dynamical phase transition [17].
2. if there is such a transition then whether there is a mechanism via which an ultracold homogeneous gas can escape the unorganized configuration and evolve into the organized...
one(s), which is not possible with a classical mean-field description of either the atoms or
the cavity field [25].

In section 5, both questions will be answered to the positive for the case of two particles in the
system restricted to two lattice sites, which is the smallest system for which the two questions
make sense. For two particles, we certainly cannot expect a ‘quantum phase transition’. Nonetheless, we do observe a very smooth transition in the structure of the quantum steady state, which is analogous to self-organization as happens in the second-order density correlation.

From the discussion in section 2, we also see that unlike in the classical case, where
the model could be solved even for several thousand particles [6, 14, 17], full quantum
mechanical solution for several particles is difficult. Indeed, to account for the dynamical
nature of the lattice, i.e. the fact that the field scattered by the particles from the pump into
the cavity modifies the potential felt by the particles (cf last term in the Hamiltonian (1a)),
several single-particle states have to be incorporated in the simulation: in the MCWFS, we
typically use 30 momentum eigenstates (for details see [22]). For \( N \) particles this gives a \( 30^N \)
dimensional Hilbert space, which is further multiplied by the cavity-field Hilbert space. With the
computational resources available nowadays this approach can be practically pursued up to two
particles. In the Bose–Hubbard approach, on the other hand, there are only two single-particle
states, but these are defined by the lattice potential solely. Hence, it remains to be demonstrated
that this approach can be applied at all to describe self-organization, a phenomenon based on
the dynamical nature of the lattice. This problem is also addressed in section 5, where we compare
our two-particle MCWFS and Bose–Hubbard results.

4. Single-particle dynamics

We first consider the dynamics of a single particle initially prepared in one of the localized states
(say, \( |r\rangle \)). Without coupling to the cavity (\( U_0 = 0 \)) the particle moves unperturbed in the lattice
via tunnelling, which, in the case of two sites corresponds to an oscillation between states \( |r\rangle \)
and \( |l\rangle \). This can be monitored via the expectation value \( \langle K x \rangle \), which, as displayed in figure 2(a)
(red line), oscillates accordingly between \( \pm \pi/2 \) (cf also figure 1).

This simple behaviour is significantly modified in the presence of even a weakly coupled
cavity, \( U_0 \neq 0 \). Now the particle scatters photons from the lattice field into the cavity mode,
depending on its state. Photons can decay according to the Liouvillean (1b) and allow us to
monitor the particle motion. The decay of a cavity photon can be modelled by a quantum jump,
which is mathematically described by the application of the cavity field operator \( a \) on the state
vector of the system. This, in turn, changes the whole particle-field wavefunction and thus gives
feedback on the particle localization.

When the coupling is weak enough, the field in the cavity will be small, and the contribution
of the last term of Hamiltonian (1a) to the potential felt by the particle (second term in the same
Hamiltonian) is negligible, so that it still makes sense to define the localized particle states
solely from the lattice potential.

We assume that these states are well localized. When a point-like particle is placed into the
lattice at position \( x \) in steady state it radiates a coherent field \( |\alpha(x)\rangle \) into the cavity, where the
amplitude is determined by the Liouvillean dynamics (1) and reads

\[
\alpha(x) = \frac{\sqrt{U_0 V_0}}{U_0 - \Delta_C - i \kappa} \sin(K x) = \frac{\sqrt{U_0 V_0}}{\kappa} \frac{1}{1 - i} \sin(K x),
\]

(3)
Figure 2. Simulated data for a single particle in the lattice-cavity system. Parameters: $V_0 = -10 \omega_{\text{rec}}$, $\kappa = 10 \omega_{\text{rec}}$, $\Delta_C - U_0 = -\kappa$, with the recoil frequency $\omega_{\text{rec}} \equiv \hbar K^2 / (2\mu)$. The colour code: red corresponds to $U_0 = 0$, green to $U_0 = -0.005\omega_{\text{rec}}$ blue to $U_0 = -0.5\omega_{\text{rec}}$ and magenta to $U_0 = -10\omega_{\text{rec}}$; with maximal cavity photon numbers amounting to 0, $2.5 \times 10^{-4}$, $2.5 \times 10^{-2}$ and 0.5, respectively. (a) Example single MCWF trajectories. On the green trajectory green arrows mark the instance of cavity decays (photon escapes): the resulting jump in the tunnelling oscillation’s phase is visible. (b) Ensemble average data—lines without points: solution of the time-dependent Master equation based on the Bose–Hubbard Hamiltonian (2). Lines with points: ensemble average of several MCWF trajectories.

where the second equality holds under the resonance condition $\Delta_C - NU_0 = -\kappa$ ($N$ is the particle number), to which we restrict ourselves in the following. This makes that the cavity field increases monotonically with increasing coupling.
Accordingly, in state $|r\rangle$ the particle will radiate an approximately coherent state $|\alpha\rangle$, while in state $|l\rangle$ a coherent state with opposite phase $|-\alpha\rangle$, where $\alpha = \alpha(x = \pi/(2K))$. It can also be expressed from the Bose–Hubbard model as $\alpha = J/(\kappa(1 - i))$.

If we assume that tunnelling is much slower than cavity field evolution, then the latter will follow adiabatically the former. Without cavity jumps the system evolves coherently and the back action of the cavity field on the particle motion is negligible by our assumption, this evolution amounts to an oscillation between states $|r,\alpha\rangle$ and $|l,-\alpha\rangle$, hence at a given time instant $t$ the overall state of the particle-cavity system reads approximately

$$|\Psi(t)\rangle = \cos(Jt) |r,\alpha\rangle + i \sin(Jt) |l,-\alpha\rangle.$$

Now imagine that at time $t$ a jump happens: immediately after the jump the state of the system reads

$$|\Psi'(t)\rangle \propto a |\Psi(t)\rangle \propto \cos(Jt) |r,\alpha\rangle - i \sin(Jt) |l,-\alpha\rangle,$n

that is, the cavity jump is reflected back on to the particle motion and results in a jump of the phase of the tunnelling oscillation.

This behaviour is verified by the simulations, an example trajectory is displayed in figure 2(a) (green line). Here, the parameters were chosen such that the maximal expectation value of the cavity photon number is only $2.5 \times 10^{-4}$—this maximum is achieved when the particle is prepared perfectly localized at a lattice site.

The jump is a stochastic event and in ensemble average the jumps of the phases on individual trajectories result in a dephasing and hence damping of the oscillation. This behaviour, as displayed in figure 2(b) (green lines) is verified by both the MCWF and the simulation of the time-dependent Master equation based on the Bose–Hubbard Hamiltonian (2). In this regime of very low cavity photon number, the correspondence between the two models is very good. Increasing the photon number results in several jumps happening in one tunnelling cycle: in ensemble average this corresponds to the over-damped regime of the tunnelling oscillation (cf figures 2(a) and (b) blue line).

The above picture of the dynamics on one Monte Carlo trajectory is not valid in the regime of stronger coupling where the photon number is higher. Here, the cavity field modifies significantly the potential felt by the particle and hence the states $|r\rangle$ and $|l\rangle$ defined solely by the lattice potential lose their significance because many other particle spatial states enter the dynamics. Cavity decays are much more frequent, and the stronger field fluctuations are reflected back on to the potential. Accordingly, as we observe in figure 2 (magenta line), the Bose–Hubbard approach being based on those two states breaks down in this regime.

The steady state of the dynamics is, quite independently of the coupling, the mixed state

$$\rho_{ss} \approx \langle r,\alpha| \langle r,\alpha| + |l,-\alpha\rangle \langle l,-\alpha|)/2.$$n

As displayed in figure 3, the relaxation time to this state, that is, the timescale of the damping of the tunnelling oscillation exhibits a nontrivial behaviour. For moderate coupling, according to the discussion above, this is composed of two timescales: the timescale of cavity photon decay, and that of tunnelling. The former being inversely proportional to the photon number gets faster with increasing coupling.

On the other hand, the latter gets slower, a behaviour verified by both the MCWF and the Bose–Hubbard approach. In the following, we give an interpretation in the framework of the Bose–Hubbard approach. Here, when initially in the state $|r,\alpha\rangle$, the system first evolves into $|l,\alpha\rangle$ via tunnelling (first term of Hamiltonian (2)), and this state is then transformed into $|l,-\alpha\rangle$ by the dissipative cavity evolution on a timescale of $\kappa^{-1}$, since in steady state the particle in state $|l\rangle$ radiates the approximately coherent field $|-\alpha\rangle$. States $|r,\alpha\rangle$ and $|l,\alpha\rangle$, however,
Figure 3. Relaxation time of the damping of the tunnelling oscillation simulated with the MCWF and Bose–Hubbard approach. Same parameters as above. The dotted hyperbole has been fit on the weak-coupling part of the data and is proportional to the cavity photon count rate. The dotted horizontal line is the timescale of tunnelling with no coupling.

Figure 4. Effective level scheme for one particle in the Bose–Hubbard approach. Tunnelling acts as an off-resonant driving while the dissipative cavity field dynamics as damping of the higher-lying states.

have different energies because of the last term in Hamiltonian (2), so that the tunnelling becomes non-resonant and hence slows down as compared to the free (no-cavity) case. Using the $\alpha$ value calculated above from the Bose–Hubbard approach, the energy difference reads $\Delta = \frac{2J^2}{\kappa} \propto |U_0|$. With increasing $U_0$ the energy difference increases, so that tunnelling gets slower. The above discussion is summarized in figure 4.
The combination of one accelerating and one slowing timescale results in the minimum behaviour of the relaxation time in figure 2 around \( U_0 = -0.05 \omega_{\text{rec}} \). Ultimately, with high enough coupling the cavity-generated potential starts to dominate the lattice potential, in which regime strong fluctuations and self-trapping lead to fast relaxation as observed in the MCWFS (red line \(|U_0| \gtrsim \omega_{\text{rec}}\)). Obviously, the Bose–Hubbard approach is unable to reproduce the behaviour in this regime.

5. Two-particle dynamics

Having understood the dissipative quantum dynamics of a single particle in our lattice–cavity system, we now turn to the case of two particles. Two particles on two lattice sites with periodic boundary conditions is the minimal system that can exhibit the difference between the configurations described above for self-organization. In the Bose–Hubbard approach where there is only one state at each lattice site the Hilbert space for the particles is spanned by only three states: \(|1, 1\rangle \equiv |0\rangle\) — the Mott insulator (MI) state, which corresponds to the homogeneous distribution or unorganized configuration, and \(|2, 0\rangle \equiv |\rangle\) and \(|0, 2\rangle \equiv |\rangle\) corresponding to the two organized configurations. \(|0\rangle\) scatters no field (and no photons) into the cavity due to destructive interference between the fields scattered by the two particles, while \(|\rangle\) and \(|\rangle\) scatter \(|-2\alpha\rangle\) and \(|2\alpha\rangle\), respectively, the factor 2 being due to constructive interference. The difference between the two configurations can be monitored via the density correlation \(\langle n_l n_r \rangle\), which is 1 for the MI state and 0 in the subspace spanned by \(|\pm\rangle\).

When the particles are initially in the MI state, then for \( U_0 = 0 \) at time \( t \) the particle state is

\[
|\Psi(t)\rangle = \cos(2Jt) |0, 0\rangle + \frac{i}{\sqrt{2}} \sin(2Jt) (-\rangle + \rangle).
\]  

(6)

With \( U_0 \neq 0 \) under the simplifying conditions we discussed above for the single-particle case we have for the joint system

\[
|\Psi(t)\rangle = \cos(2Jt) |0, 0\rangle + \frac{i}{\sqrt{2}} \sin(2Jt) (-\rangle , -2\alpha\rangle + \rangle, 2\alpha\rangle).
\]  

(7)

If a jump happens in this state (application of \( a \)), then the state immediately after the jump reads

\[
|\Psi'(t)\rangle \propto a |\Psi(t)\rangle \propto |-, -2\alpha\rangle - |+, 2\alpha\rangle.
\]  

(8)

There are two points worth noting here: firstly, the quantum jump in the photon number erases all information about the phase of the tunnelling oscillation in the particle Hilbert space. Secondly, after the escape of one single photon from the cavity tunnelling stops immediately. Indeed, in the state (8) both \(|-, -2\alpha\rangle\) and \(|+, 2\alpha\rangle\) tunnels to \(|0, 0\rangle\) (note that we assume again the cavity field following adiabatically the tunnelling), but their phases are opposite, and hence the two paths interfere destructively. A second jump at \( t' > t \), however, makes the phases match again, and puts the state \(|\Psi''(t')\rangle \propto a|\Psi'(t')\rangle \propto |-, -2\alpha\rangle + |+, 2\alpha\rangle\) back into the tunnelling cycle (7).

An example MCWF trajectory exhibiting this behaviour is plotted in figure 5(a) (green line). We observe that a quantum jump brings the system into the state (8), signalled by \(\langle n_l n_r \rangle = 0\), and it remains there until the next jump, when it starts to oscillate anew.
Figure 5. Simulated data for two particles in the lattice-cavity system with same parameters as above. The colours correspond to the same values of $U_0$, here with maximal cavity photon numbers amounting to 0, $10^{-3}$, 0.1 and 2. (a) Example single MCWF trajectories—the vertical dotted lines mark the time instants of cavity decays on the trajectory plotted in green. (b) Ensemble average data—lines without points: data stemming from the Bose–Hubbard approach; lines with points: MCWF approach.

In ensemble average, these stochastic ‘dark’ periods of the tunnelling oscillation lead to damping just as in the single-particle case. The final steady state is always a mixture

$$
\rho_{ss} = w |0, 0\rangle \langle 0, 0| + \frac{1 - w}{2} (-2\alpha) |-, -2\alpha\rangle \langle -, -2\alpha| + |+, 2\alpha\rangle \langle +, 2\alpha|.
$$

At this point, it becomes clear that any mean-field description of this system is bound to fail: a mean cavity field description would prohibit the possibility of a coherent superposition of different particle configurations radiating different fields as in (7), which is essential for the onset of the dynamics from a homogeneous zero-temperature initial condition (see also [25]).
On the other hand, a particle mean field cannot seize the difference between states (9) with different $w$, because this appears only in the density correlation.

As displayed in figure 5(b), our two approaches for simulating the damping agree well in the regime of moderate coupling. However, just as in the single-particle case, strong coupling—in the regime where the cavity-generated potential dominates the lattice one—results in extremely quick damping, which cannot be reproduced by the Bose–Hubbard approach. Here, the relaxation time exhibits the same behaviour as we have seen in the single-particle case (cf figure 6(a)): with increasing coupling it has a minimum, after which the Bose–Hubbard approach gives a monotonic increase of the relaxation time, while the MCWFS gives a peak, and for even stronger coupling very quick relaxation. For the interpretation of this behaviour the same discussion applies as above for the single-particle case.

Figure 6. (a) Relaxation time for two particles and (b) weight of the MI component in the steady-state mixture (9), $\langle n_1 n_i \rangle_{ss} = w$, as functions of the coupling. Same parameters as above.
Increasing coupling results in a decrease of the weight $w$ of the MI component in steady state, cf figure 6(b), and with strong enough coupling ($|U_0| \gtrsim 10 \omega_{\text{rec}}$ in our case) the steady state is confined solely into the self-organized subspace. This proves our initial assertion that even when the system is started from a $T = 0$ homogeneous state (here the MI state) self-organization can occur. On the same figure, we also see that this is confirmed by both approaches, only the relaxation time of the process is predicted wrongly by the Bose–Hubbard approach for strong coupling.

It is easy to see that starting the system from the SF state

$$|\text{SF}\rangle \propto |0\rangle + \frac{1}{\sqrt{2}} (|-angle + |+\rangle)$$

instead of the MI as above does not change the steady state since already the first quantum jump erases the information about the initial condition completely. The process of relaxation will, however, be different. This process can be monitored by a time resolved analysis of the intensity escaping the cavity, which is proportional to the cavity photon number: an example is displayed in figure 7. Here, we are in the over-damped regime of the tunnelling oscillation. When prepared in the MI state, the particles do not scatter initially, and the build-up of the cavity field occurs on the timescale of the self-organization process. With the SF initial state, on the other hand, some field is built up almost instantly (on the timescale of $\kappa^{-1}$), because in the SF the states $|\pm\rangle$ have finite weight; while the rest of the field is built up on the longer timescale.

Figure 7. Initial build-up of the cavity photon number with the particles prepared initially in the Mott insulator or SF state. $U_0 = -10\omega_{\text{rec}}$, other parameters are the same as above.
6. Conclusions

In summary, we have seen that coupling two particles in an optical lattice to a cavity induces an irreversible reorganization of the particles, a process which can be monitored directly in an experiment by the time-resolved analysis of the intensity escaping the cavity. We have shown that no classical or mean-field description of either the particles or the cavity field can account for the phenomenon. For strong enough coupling the process leads to a fast self-organization of the particles, a configuration in which they occupy every second site in the lattice, and scatter superradiantly into the cavity. An important conclusion of the work is that while an effective low-dimensional Bose–Hubbard type model cannot reproduce the time evolution in the strong-coupling regime as observed in more detailed simulations, it can still predict the steady-state remarkably well. This model can therefore be used in the future for a high number of particles to study possible quantum phase transitions occurring in the steady state of this dissipative system.

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