ORDERS OF THE CANONICAL VECTOR BUNDLES OVER CONFIGURATION SPACES OF FINITE GRAPHS

FREDERICK R. COHEN AND RUIZHI HUANG

To the memory of Professor Wen-Tsün Wu (1919-2017).

ABSTRACT. We prove that the order of the canonical vector bundle over the configuration space is 2 for a general planar graph, and is 4 for a nonplanar graph.

INTRODUCTION

Let \( \xi \) be a vector bundle. If there exists a positive integer \( n \) such that the \( n \)-fold Whitney sum \( \xi^{\otimes n} \) is trivial, then we say that \( \xi \) has finite order. In this case, the smallest such \( n \) is called the order of \( \xi \), denoted by \( o(\xi) \). Meanwhile, if we are only interested in stable bundles and stable equivalences, there is a parallel notion of stable order of \( \xi \), denoted by \( s(\xi) \). It is obvious that

\[
s(\xi) \mid o(\xi).
\]

Let \( \text{Conf}(X,n) \) denote the the space of configurations of \( n \) distinct points lying in a topological space \( X \), that is,

\[
\text{Conf}(X,n) = \{(x_1, x_2, \ldots, x_n) \in X \times \cdots \times X \mid x_i \neq x_j \text{ for } i \neq j \}.
\]

If \( X \) has at least \( n \) distinct points, then \( \text{Conf}(X,n) \) is non-empty. The symmetric group \( \Sigma_n \) on \( n \)-letters acts freely on \( \text{Conf}(X,n) \) from the left by

\[
\sigma(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}), \quad \sigma \in \Sigma_n,
\]

which induces the canonical covering

\[
(1) \quad \Sigma_n \rightarrow \text{Conf}(X,n) \rightarrow \text{Conf}(X,n)/\Sigma_n.
\]

Since \( \Sigma_n \) acts canonically on the real Euclidean space \( \mathbb{R}^n \) by permuting the coordinates from the right, there is the associated vector bundle

\[
(2) \quad \xi_{X,n} : \mathbb{R}^n \rightarrow \text{Conf}(X,n) \times_{\Sigma_n} \mathbb{R}^n \rightarrow \text{Conf}(X,n)/\Sigma_n.
\]

It is an enduring interest to determine the order and stable order of \( \xi_{X,n} \) for various \( X \). The order and stable order for \( X = \mathbb{R}^n \) have been extensively studied. Cohen-Mahowald-Milgram [5] showed that \( o(\xi_{\mathbb{R}^2,n}) = 2 \). For the higher dimensional Euclidean spaces, there are studies by Yang [20] and
Cohen-Cohen-Kuhn-Neisendorfer [3]. Beyond the Euclidean case, Cohen-Cohen-Mann-Milgram [4] showed that the order for oriented surface of genus greater or equal to one is 4. Ren [14] studied the order for real projective spaces and their Cartesian products with a Euclidean space, and further he [15] investigated the order and stable order for simply connected spheres and their disjoint unions.

In this paper, we study the order of $\xi_{X,n}$ when $X$ is a finite (connected or non-connected) graph. The configuration space of particles on finite graph is interesting in both mathematics and physics. For instance, Abrams in his thesis [1] studied discrete model of configuration space on graph. The topology of configuration space on graph was studied by Farley-Sabalka [9], Barnett-Farber [2], Farber-Hanbury [8], etc, while the physical aspect on quantum statistics on graphs was investigated by Harrison-Keating-Robbins-Sawicki [10], and Maciážek [12]. Our main theorem is as follows.

**Theorem 1.** Let $\Gamma$ be a finite graph. Then for any $n \geq 2$

1. if $\Gamma$ is homeomorphic to a point, or a closed interval, or a disjoint union of finitely many of them; or if $\Gamma$ is homeomorphic to a circle with $n$ odd, then
   \[ s(\xi_{\Gamma,n}) = o(\xi_{\Gamma,n}) = 1; \]

2. if $\Gamma$ is planar but does belong to case (1)
   \[ s(\xi_{\Gamma,n}) = o(\xi_{\Gamma,n}) = 2; \]

3. if $\Gamma$ is nonplanar
   \[ s(\xi_{\Gamma,2}) = o(\xi_{\Gamma,2}) = 4. \]

**Proof.** It is well known that $\text{Conf}(\mathbb{R}^1,n)$ is equivariantly homotopy equivalent to $\Sigma_n$, and the configuration space of a disjoint union is a disjoint union of products of configuration spaces of its components. Then the unordered configuration space $\text{Conf}(\Gamma,n)/\Sigma_n$ in case (1) is homotopy equivalent to a disjoint union of points except the circle case. Hence the bundle $\xi_{\Gamma,n}$ is trivial and the order is 1. The order and stable order for $\Gamma \cong S^1$ are computed in Proposition 2.3, while for the general planar graph they are determined in Proposition 2.5. The orders for nonplanar graph are determined first for two kinds of special graphs homeomorphic to $K_5$ or $K_{3,3}$ in Proposition 3.5, and then for the general cases in Proposition 3.6.

**Notation 2.** Since the graphs in case 1 except ones homeomorphic to circle are trivial to our problem, we would like to exclude them in the computations. Hence, we let $L$ be the set of graphs homeomorphic to a point, or a closed interval, or a disjoint union of finitely many of them, and usually suppose that $\Gamma \notin L$. 

\[ \square \]
The paper is organized as follows. In Section 1 we review the discrete model of Abrams for the configuration space of particles on graph. In Section 2 and Section 3 we compute the orders and the stable orders of the canonical bundles for planar and nonplanar graphs respectively. Section 4 is devoted to the application of Theorem 1 on the stable homotopy types of generalized divided powers.

Acknowledgements. Ruizhi Huang was supported in part by National Natural Science Foundation of China (Grant no. 11801544), and “Chen Jingrun” Future Star Program of AMSS.

1. Let us first recall some useful results about configuration space of particles on a finite graph, based on the thesis of Abrams [1] and of Maciaálek [12], and the papers [2, 10]. Let $\Gamma$ be a finite graph, or equivalently a finite 1-dimensional $CW$-complex. For any point $x \in \Gamma$, as in Section 1 of [2] we define the support of $x$ by

$$\text{supp}(x) = \begin{cases} x & \text{if } x \text{ is a vertex}, \\ e & \text{if } x \in \partial e, \text{ an edge}. \end{cases}$$

For each $n \geq 2$, the Abrams discrete model of $\text{Conf}(\Gamma, n)$ is defined to be

$$A(\Gamma, n) = \{(x_1, \ldots, x_n) \in \Gamma \times \cdots \times \Gamma \mid \text{supp}(x_i) \cap \text{supp}(x_j) = \emptyset, \text{ for all } i \neq j\}.$$ 

It is obvious that

$$A(\Gamma, n) \subseteq \text{Conf}(\Gamma, n),$$

and the canonical permutation on $\Gamma \times \cdots \times \Gamma$ induces a free action on $A(\Gamma, n)$. Abrams proved the following important theorem in his thesis.

**Theorem 1.1** (Theorem 2.1 of [1]). Let $\Gamma$ be a finite graph with at least $n$ vertices. If $\Gamma$ satisfies that

1. each path between distinct vertices of degree not equal to 2 passes through at least $n - 1$ edges,
2. and each nontrivial loop passes through at least $n + 1$ edges,

then the $\Sigma_n$-equivariant inclusion $A(\Gamma, n) \hookrightarrow \text{Conf}(\Gamma, n)$ is a homotopy equivalence. In particular,

$$A(\Gamma, n)/\Sigma_n \simeq \text{Conf}(\Gamma, n)/\Sigma_n.$$

Following [1, 9, 10] we call a graph with properties 1 and 2 sufficiently subdivided. When $n = 2$, $A(\Gamma, 2)$ coincides with the “simplicial deleted product” of Shapiro [13]. In this case, Theorem 1.1 can be strengthened.
Lemma 1.2 (Theorem 2.4 of [1]). Let $\Gamma$ be a simple graph, i.e., a finite simplicial complex of dimension one. $A(\Gamma, 2)$ is a $\mathbb{Z}/2$-equivariant strong deformation retraction of Conf($\Gamma, 2$). □

This result was originally proved by Shapiro [13] and W-T Wu [18, 19]. However, as pointed out by Barnett-Farber [2] the proof of Lemma 2.1 of [13] is incorrect. Nevertheless, Abrams gave a clear proof in his thesis [1].

2.

Recall that a graph $\Gamma$ is planar if and only if it can be embedded into the real plane $\mathbb{R}^2$; otherwise $\Gamma$ is nonplanar. Moreover, if $\Gamma$ is nonplanar, it is easy to see that it can be embedded into orientable surface of higher genus (for instance, see page 53 of [17]). Let us consider planar graphs in this section. We start with a general observation which has been used for example in [4, 15] for surfaces and spheres.

Lemma 2.1. Let $n$ and $m$ be two positive integers such that $n \leq m$. For two complexes $X$ and $Y$, suppose there exists a $\Sigma_n$-equivariant map

$$Conf(X, n) \longrightarrow Conf(Y, m),$$

where $\Sigma_n$ acts on $Conf(Y, m)$ through a group monomorphism $\Sigma_n \subseteq \Sigma_m$. Then

$$s(\xi_{X, n}) \leq s(\xi_{Y, m}).$$

Moreover, if further $m = n$ then

$$o(\xi_{X, n}) \leq o(\xi_{Y, m}).$$

Proof. Denote by $f$ the equivariant map in the assumption. $f$ induces a map

$$\tilde{f} : Conf(X, n)/\Sigma_n \rightarrow Conf(Y, m)/\Sigma_m.$$

Let $\epsilon^i$ be the trivial bundle of rank $i$. Then $\xi_{Y, m}$ is pulled back to $\xi_{X, n} \oplus \epsilon^{m-n}$ along $\tilde{f}$. By definition $\xi_{Y, m}^{\otimes s(\xi_{Y, m})}$ is stably trivial, which implies that $\xi_{X, n}^{\otimes s(\xi_{X, n})}$ is stably trivial. Hence $s(\xi_{X, n}) \leq s(\xi_{Y, m})$. When $m = n$, $o(\xi_{X, n}) \leq o(\xi_{Y, m})$ by the similar argument and the lemma is proved. □

Lemma 2.2. Let $\Gamma$ be a planar finite graph. Then

$$s(\xi_{\Gamma, n}) = o(\xi_{\Gamma, n}) = 1, \text{ or } 2.$$

Proof. Let $\Gamma \hookrightarrow \mathbb{R}^2$ be an embedding. It induces a $\Sigma_n$-equivariant embedding

$$Conf(\Gamma, n) \hookrightarrow Conf(\mathbb{R}^2, n).$$

It was proved in Theorem 1.2 of [5] that $o(\xi_{\mathbb{R}^2, n}) = 2$. Hence by Lemma 2.1 $o(\xi_{\Gamma, n})$ can only be 1 or 2, and the lemma follows. □

The circle is special among all graphs and we may treat it first.
Proposition 2.3. Let $\Gamma$ be a finite graph homeomorphic to $S^1$. Then

$$s(\xi_{\Gamma,n}) = o(\xi_{\Gamma,n}) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

Proof. It is well known that $\text{Conf}(S^1, n) \simeq \bigsqcup_{(n-1)!} S^1$, $\text{Conf}(S^1, n)/\Sigma_n \simeq S^1$, and there is the map of coverings

$$\begin{array}{ccc}
\mathbb{Z}/n & \rightarrow & S^1 \\
i & & j \\
\Sigma_n & \rightarrow & \bigsqcup_{(n-1)!} S^1 \\
& & \\
& & \rightarrow \bigsqcup_{(n-1)!} S^1 \\
\end{array}$$

where the first row is the canonical $n$-fold covering of $S^1$, the second row is homotopic to the covering of configuration space (1) of $X = S^1$, $i$ is the injection of the subgroup consisting of cycles of length $n$, and $j$ is the inclusion of any component of $\bigsqcup_{(n-1)!} S^1$. In particular, the classifying map $f$ of $\xi_{S^1,n}$ can be factored as

$$f : S^1 \rightarrow B\mathbb{Z}/n \rightarrow B\Sigma_n \rightarrow BO(n),$$

where $\tilde{f}$ represents a generator of $\pi_1(B\mathbb{Z}/n) \cong \mathbb{Z}/n$, and $\rho$ is the canonical representation into the orthogonal group $O(n)$. Equivalently, this means that the structure group of $\xi_{S^1,n}$ can be lifted to $\mathbb{Z}/n$. In particular, $\omega_1(\xi_{S^1,n}) = 0$ when $n$ is odd, and $\xi_{S^1,n}$ is trivial. This proves the proposition when $n$ is odd.

On the other hand, there is the commutative diagram of group homomorphisms

$$\begin{array}{ccc}
\Sigma_n & \rightarrow & O(n) \\
\text{sgn} & & q \\
\{\pm1\} & \cong & \mathbb{Z}/2, \\
\rho & & \\
\end{array}$$

where sgn is defined by the sign of permutation, and $q$ is the quotient of $O(n)$ by the subgroup $SO(n)$. Additionally, $(B\text{sgn})^* : H^1(B\mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H^1(B\Sigma_n; \mathbb{Z}/2)$ is an isomorphism, and $Bq$ represents the universal first Stiefel-Whitney class $\omega_1 \in H^1(BO(n); \mathbb{Z}/2)$. It follows that $(B\rho)^*(\omega_1) \neq 0$, and further when $n$ is even $(Bi \circ B\rho)^*(\omega_1) \neq 0$ as sgn $i$ is surjective. Hence, from (2.1) we see that $\omega_1(\xi_{S^1,n}) \neq 0$ when $n$ is even. Then $\xi_{S^1,2}$ is stably nontrivial and $s(\xi_{\Gamma,n}) = o(\xi_{\Gamma,n}) = 2$ by Lemma 2.2. This proves the proposition when $n$ is even. \qed
Remark 2.4. When $n = 2$, the total space of $\xi_{S^1, 2}$ is the Möbius strip, and the bundle is the projection to its equatorial circle.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{$Y$-graph $K$ and $A(K, 2)/\Sigma_2$}
\end{figure}

Recall in Introduction we denote $\mathcal{L}$ to be the set of graphs homeomorphic to a point, or a closed interval, or a disjoint union of finitely many of them.

**Proposition 2.5.** Let $\Gamma$ be a planar finite graph such that $\Gamma \not\in \mathcal{L}$ and $\Gamma \not\cong S^1$. Then

\[ s(\xi_{\Gamma, n}) = o(\xi_{\Gamma, n}) = 2, \]

for any $n \geq 2$.

**Proof.** Let us firstly consider the case when $\Gamma$ is connected. By assumption $\Gamma$ contains a vertex of degree at least 3, in other words, it contains a $Y$-subgraph $K$ (Figure 1). By Example 2.2 of [1], $\text{Conf}(K, 2)/\Sigma_2 \cong S^1$. Then as in Remark 2.4, $\xi_{K, 2}$ is the Möbius strip, and $s(\xi_{K, 2}) = o(\xi_{K, 2}) = 2$ by Lemma 2.2.

For general $n$, we may first choose any proper self-embedding $i : K \hookrightarrow K$ of $K$. With this we can define a $\Sigma_2$-equivariant map

\[ \tilde{i} : \text{Conf}(K, 2) \longrightarrow \text{Conf}(K, n) \]

by sending $(x, y)$ to $(x, y, a_1, a_2, \ldots, a_{n-2})$, where the $(n - 2)$ distinct points $a_1, a_2, \ldots, a_{n-2} \in K - i(K)$. Then there is the composition of $\Sigma_2$-equivariant maps

\[ k : \text{Conf}(K, 2) \xrightarrow{i} \text{Conf}(K, n) \xrightarrow{j} \text{Conf}(\Gamma, n), \]

where $j$ is the canonical inclusion. By Lemma 2.1 $s(\xi_{\Gamma, n}) \geq s(\xi_{K, 2}) = 2$, and the proposition follows from Lemma 2.2 for the case when $\Gamma$ is connected.

When $\Gamma$ is not connected, there is a component of $\Gamma$ containing a subgraph $T$, which is either a $Y$-graph or homeomorphic to $S^1$ as $\Gamma \not\in \mathcal{L}$. If $T$ is a $Y$-graph, then the previous discussion implies the statement of the proposition. Now suppose $T$ is homeomorphic to $S^1$. Consider the canonical $\Sigma_n$-equivariant embedding $K(T, n) \hookrightarrow K(\Gamma, n)$ when $n$ is even, while
consider the $\Sigma_{n-1}$-equivariant embedding $K(T, n - 1) \leftrightarrow K(\Gamma, n)$ sending $(x_1, \ldots, x_{n-1}, y)$ with $y$ lying in a component of $\Gamma$ different from that of $T$ when $n$ is odd. Then by Lemma 2.3 and Lemma 2.2 and the similar argument above, we see that in either case $s(\xi_{\Gamma,2n}) = o(\xi_{\Gamma,2n}) = 2$. This completes the proof of the proposition.

Remark 2.6. From the proof of Proposition 2.5, we also see that the stable order of $\xi_{\Gamma,n}$ can not be 1 for any nonplanar finite graph $\Gamma$ since it contains a proper $Y$-subgraph.

3.

In this section, we determine the order and stable order of $\xi_{\Gamma,n}$ for nonplanar graph $\Gamma$.

**Lemma 3.1.** Let $\Gamma$ be a nonplanar finite graph. Then

$$o(\xi_{\Gamma,n}) = 2, \text{ or } 4,$$

for any $n \geq 2$.

**Proof.** It was showed in [4] that the order of $\xi_{M,n}$ is 4 for any closed orientable Riemann surface $M$ of genus greater than or equal to one. Then by the fact that $\Gamma$ can be embedded into orientable surface of higher genus and Lemma 2.1, $o(\xi_{\Gamma,n})$ can only 1, 2 or 4. However, 1 is impossible by Remark 2.6.

Two famous examples of non-planar graphs are the complete graph on five vertices $K_5$ and the complete bipartite graph $K_{3,3}$. A Kuratowski graph is a subdivision of $K_5$ or $K_{3,3}$. Here, a subdivision of a graph $\Gamma$ is a graph resulting from the subdivision of edges of $\Gamma$ by introducing new vertices on them. The following criterion of nonplanar graph is classical.

**Theorem 3.2** (Kuratowski’s Theorem, 1930; [11]). A graph $\Gamma$ is nonplanar if and only if $\Gamma$ contains a Kuratowski subgraph.

**Lemma 3.3** (Abrams; Section 5.1 in [1]).

$$A(K_5, 2) \cong \#_6T^2, \quad A(K_{3,3}, 2) \cong \#_4T^2,$$

where $\#_kT^2$ is the orientable closed surface of genus $k$.

In the following, Proposition 3.4 is a special case of Proposition 3.5, but it is proved in a different and simpler way.

**Proposition 3.4.** Let $\Gamma$ be a Kuratowski graph. Then

$$s(\xi_{\Gamma,2}) = o(\xi_{\Gamma,2}) = 4.$$
Proof. First since there is the covering map
\[ \mathbb{Z}/2 \to \#_{2k} T^2 \to \#_{2k+1} P^2, \]
where \( \#_{2k+1} P^2 \) is the unorientable closed surface of genus \( 2k + 1 \), we see from Lemma 1.2 and Lemma 3.3 that
\[
\text{Conf}(K_5, 2)/\Sigma_2 \cong A(K_5, 2)/\Sigma_2 \cong \#_1 P^2 \\
\text{Conf}(K_{3,3}, 2)/\Sigma_2 \cong A(K_{3,3}, 2)/\Sigma_2 \cong \#_5 P^2.
\]
Further, notice that the determinant line bundle of \( \xi_{\Gamma, 2} \) is determined by the connecting epimorphism
\[ h : \pi_1(\text{Conf}(\Gamma, 2)/\Sigma_2) \to \Sigma_2 \cong \mathbb{Z}/2, \]
which, as the orientation character, corresponds exactly to the first Stiefel-Whitney class \( \omega_1(\#_{2k+1} P^2) \) with \( k = 3 \), or 2. Hence, in either case
\[
\omega_1(\xi_{\Gamma, 2}) = \omega_1(\#_{2k+1} P^2) \neq 0.
\]
Since \( \xi_{\Gamma, 2} \) is isomorphic to the direct sum of a line bundle and the trivial line bundle, it follows that
\[
\omega(\xi_{\Gamma, 2}^\otimes 2) = (1 + \omega_1(\xi_{\Gamma, 2}))^2 = 1 + \omega_1(\#_{2k+1} P^2) \neq 1.
\]
Hence both the order \( o(\xi_{\Gamma, 2}) \) and the stable order \( s(\xi_{\Gamma, 2}) \) can not be 2. The proposition then follows from Lemma 3.1.

Proposition 3.5. Let \( \Gamma \) be a Kuratowski graph. Then
\[
s(\xi_{\Gamma, n}) = o(\xi_{\Gamma, n}) = 4,
\]
for any \( n \geq 2 \).

Proof. The case when \( n = 2 \) was showed in Proposition 3.4. For general \( n \),
\[
(3.1) \quad H_1(\text{Conf}(\Gamma, n)/\Sigma_n; \mathbb{Z}) \cong H_1(\text{Conf}(\Gamma, 2)/\Sigma_2; \mathbb{Z}) \cong \mathbb{Z}/2k
\]
where \( k = 3 \), or 2 according to \( \Gamma \cong K_5 \) or \( K_{3,3} \) by Theorem 5 of [10] and Lemma 3.3. Moreover, by the discussion before Theorem 5 of [10], both the \( \mathbb{Z}/2 \) summands in the homology are determined by a \( Y \)-subgraph \( K \) of \( \Gamma \). As in the proof of Proposition 2.5, we can define a \( \Sigma_2 \)-equivariant embedding \( \tilde{i} : \text{Conf}(K, 2) \to \text{Conf}(K, n) \) from any proper self-embedding \( i : K \hookrightarrow K \) of \( K \), and similar to (2.3) consider the composition of \( \Sigma_2 \)-equivariant maps
\[
k : \text{Conf}(K, 2) \xrightarrow{\tilde{i}} \text{Conf}(K, n) \xrightarrow{j} \text{Conf}(\Gamma, n),
\]
where \( j \) is the canonical embedding. It follows that \( k \) induces a map
\[
\tilde{k} : \text{Conf}(K, 2)/\Sigma_2 \to \text{Conf}(\Gamma, n)/\Sigma_n.
\]
Then there is the commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z} \cong H_1(\text{Conf}(K,2)/\Sigma_2;\mathbb{Z}) & \overset{\tilde{k}}{\longrightarrow} & H_1(\text{Conf}(\Gamma,n)/\Sigma_n;\mathbb{Z}) \\
\downarrow\rho_2 & & \downarrow\rho_2 \\
\mathbb{Z}/2 \cong H_1(\text{Conf}(K,2)/\Sigma_2;\mathbb{Z}/2) & \overset{\tilde{k}_*}{\longrightarrow} & H_1(\text{Conf}(\Gamma,n)/\Sigma_n;\mathbb{Z}/2) \overset{p}{\longrightarrow} \mathbb{Z}/2,
\end{array}
\]

where both \(\rho_2\) are the mod-2 reductions, and both \(p\) are the projections onto the \(\mathbb{Z}/2\)-summands determined by the \(Y\)-subgraph \(K\). Notice that the composition of maps in the top row is the mod-2 reduction. It follows that the composition of maps in the bottom row is an isomorphism. Then since by Lemma 2.3 \(\omega_1(\xi_{K,2}) \neq 0\) is the generator of \(H^1(\text{Conf}(K,2)/\Sigma_2;\mathbb{Z}/2)\), \(\omega_1(\xi_{\Gamma,n}) \neq 0\). Moreover, by (3.1) the Bockstein homomorphism

\[
\beta = Sq^1 : H^1(\text{Conf}(\Gamma,n)/\Sigma_n;\mathbb{Z}/2) \cong \bigoplus_{2k} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow H^2(\text{Conf}(\Gamma,n)/\Sigma_n;\mathbb{Z}/2)
\]

is trivial on the \(\bigoplus_{2k} \mathbb{Z}/2\)-summand and is nontrivial on the last \(\mathbb{Z}/2\)-summand corresponding to the \(Y\)-subgraph \(K\). Therefore \(\omega_1(\xi_{\Gamma,n})\) is the generator of the last \(\mathbb{Z}/2\) and satisfies \(\omega_1^2(\xi_{\Gamma,n}) = Sq^1(\omega_1(\xi_{\Gamma,n})) \neq 0\). Hence

\[
\omega(\xi_{\Gamma,n}^{\otimes 2}) \equiv 1 + \omega_1^2(\xi_{\Gamma,n}) \mod H^{\geq 3}(\text{Conf}(\Gamma,n)/\Sigma_n;\mathbb{Z}/2)
\]

is not 1, which implies that \(\xi_{\Gamma,n}^{\otimes 2}\) is not stably trivial, and the stable order \(s(\xi_{\Gamma,2})\) can not be 2. The proposition then follows from Lemma 3.1.

The general case can then be determined by either Proposition 3.4 or Proposition 3.5.

**Proposition 3.6.** Let \(\Gamma\) be a nonplanar finite graph but not a Kuratowski graph. Then

\[
s(\xi_{\Gamma,n}) = o(\xi_{\Gamma,n}) = 4,
\]

for any \(n \geq 2\).

**Proof.** By assumption, \(\Gamma\) contains a proper Kuratowski subgraph \(K\) by Theorem 3.2. Choose \(n - 2\) distinct points \(a_1, a_2, \ldots, a_{n-2} \in \Gamma - K\). Then there is a \(\Sigma_2\)-equivariant embedding

\[
\bar{\Phi} : \text{Conf}(K,2) \hookrightarrow \text{Conf}(\Gamma,n)
\]

sending \((x,y)\) to \((x,y,a_1,a_2,\ldots,a_{n-2})\). Hence, by Lemma 2.1 and Proposition 3.4

\[
o(\xi_{\Gamma,n}) \geq s(\xi_{\Gamma,n}) \geq s(\xi_{K,2}) = 4.
\]

The proposition now follows from Lemma 3.1. \(\square\)
Given two complexes $X$ and $Z$, we can consider the so-called $n$-th generalized divided power of $Z$ associated to $X$ for any $n \geq 2$ defined by

$$D_n(X, Z) := \text{Conf}(X, n)^+ \wedge \Sigma_n Z^\wedge n,$$

where $Z^\wedge n$ is the $n$-fold self-smashed product of $Z$ and inherits a $\mathbb{Z}_n$-action from the canonical permutation on $Z^\wedge n$. Let $\Sigma^i Y$ be the $i$-fold suspension of the complex $Y$. The significance of the generalized divided powers is due to a general version of Snaith’s stable splitting [16]. Indeed, following [7] define the labelled configuration space of $X$ with labels in $Z$ by

$$\text{Conf}(X, Z) := \bigsqcup_n \text{Conf}(X, n) \times_{\Sigma_n} Z^\wedge n / \sim,$$

where the equivalence relation $\sim$ is generated by

$$\{(x_1, \ldots, x_n, z_1, \ldots, z_n) \sim (x_1, \ldots, x_{n-1}, z_1, \ldots, z_{n-1}) \text{, if } z_n = \ast\}$$

with $\ast$ the based point of $Z$. Then when $Z$ is path connected there is the stable decomposition [6, 7]

$$\Sigma^\infty \text{Conf}(X, Z) \simeq \bigvee_{n=1}^{\infty} \Sigma^\infty D_n(X, Z).$$

The following lemma is due to an unpublished manuscript of Cohen and was reproved by Ren in [15].

**Lemma 4.1** (Lemma 6.1 and Corollary 6.2 of [15]). For any positive integer $t$ and $n \geq 2$, there is a homotopy equivalence

$$\Sigma^{nto(\xi_X, n)} D_n(X, Z) \longrightarrow D_n(X, \Sigma^t \xi_X, n) Z).$$

By Lemma 4.1 and Theorem 1, we immediately obtain the following proposition, which indicates that the stable homotopy types of $D_n(X, \Sigma^t Z)$ exhibit a natural periodic behavior as $t$ varies.

**Proposition 4.2.** Let $\Gamma$ be a finite graph such that $\Gamma \notin \mathcal{L}$. Then for any complex $Z$, positive integer $t$ and $n \geq 2$

1. if $\Gamma$ is homeomorphic to a circle with $n$ odd, then

$$\Sigma^{nt} D_n(\Gamma, Z) \longrightarrow D_n(\Gamma, \Sigma^t Z);$$

2. if $\Gamma$ is homeomorphic to a circle with $n$ even; or if $\Gamma$ is planar such that $\Gamma \notin \mathcal{S}^t$

$$\Sigma^{2nt} D_n(\Gamma, Z) \longrightarrow D_n(\Gamma, \Sigma^{2t} Z);$$

3. if $\Gamma$ is nonplanar

$$\Sigma^{4nt} D_n(\Gamma, Z) \longrightarrow D_n(\Gamma, \Sigma^{4t} Z).$$
REFERENCES

[1] A. Abrams, *Configuration spaces and braid groups of graphs*, PhD thesis, UC Berkeley (2000). (document), 1, 1.1, 1, 1.2, 1, 2, 3.3

[2] K. Barnett and M. Farber, *Topology of configuration spaces of two particles on a graph, I*, Alge. Geom. Topol. 9 (2009), 593-624. (document), 1, 1

[3] F. R. Cohen, R. L. Cohen, N. J. Kuhn and J. A. Neisendorfer, *Bundles over configuration spaces*, Pacific J. Math. 104 (1983), no. 1, 47-54. (document)

[4] F. R. Cohen, R. L. Cohen, B. Mann and R. J. Milgram, *Divisors and configurations on a surface*, in Algebraic Topology (Evanston 1988), Contemp. Math. 96, American Mathematical Society, Providence (1989), 103-108. (document), 2, 3

[5] F. R. Cohen, M. E. Mahowald and R. J. Milgram, *The stable decomposition for the double loop space of a sphere*, in Algebraic and Geometric Topology (Stanford 1976), Proc. Sympos. Pure Math. 32 Part 2, American Mathematical Society, Providence (1978), 225-228. (document), 2

[6] F. R. Cohen, J. P. May and L. R. Taylor, *Splitting of certain spaces CX*, Math. Proc. Cambridge Philos. Soc. 84 (1978), no. 3, 465-496. 4

[7] F. R. Cohen and L. R. Taylor, *Computations of Gelfand-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces*, Geometric Applications of Homotopy Theory I (Proc. Conf., Evanston, III., 1977), Lecture Notes in Math. 657, Springer, Berlin, (1978), pp. 106-143. 4

[8] M. Farber, E. Hanbury, *Topology of configuration space of two particles on a graph, II*, Algebr. Geom. Topol. 10 (2010), 2203-2227. (document)

[9] D. Farley and L. Sabalka, *Discrete Morse theory and graph braid groups*, Algebr. Geom. Topol. 5 (2005), 1075-1109. (document), 1

[10] J. M. Harrison, J. P. Keating, J. M. Robbins and A. Sawicki, *n-particle quantum statistics on graphs*, Comm. Math. Phys. 330 (2014), 1293-1326. (document), 1, 1, 3

[11] K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fundam. Math. 15 (1930), 271-283. 3.2

[12] T. Maciążek, *Topology of configuration spaces for particles on graphs*, PhD thesis, Center for Theoretical Physics of the Polish Academy of Sciences, Warsaw (2018). (document), 1

[13] A. Shapiro, *Obstructions to the imbedding of a complex in a euclidean space*, Ann. Math. (2) 66 (1957), 256-269. 1, 1
[14] S. Ren, *Order of the canonical vector bundle over configuration spaces of projective spaces*, Osaka J. Math. **54** (2017), no. 4, 623-634. (document)

[15] S. Ren, *Order of the canonical vector bundle over configuration spaces of spheres*, Forum Math. **30** (5) (2018), 1265-1277. (document), 2, 4, 4.1

[16] V. P. Snaith, *A stable decomposition for $\Omega^n S^n X$*, J. London Math. Soc. **7** (1974), 577-583. 4

[17] A. T. White, *Imbedding problems in graph theory*, Chapter 6 in Graphs of Groups on Surfaces: Interactions and Models (Ed. A. T. White), North-Holland Mathematics Studies, **188**, North-Holland Publishing Co., Amsterdam (2001), pp. 49-72. 2

[18] W.-T. Wu, *On the realization of complexes in Euclidean space, III*, Sci. Sinica **8** (1959), 133-150. 1

[19] W.-T. Wu, *A theory of imbedding, immersion, and isotopy of polytopes in a euclidean space*, Science Press, Peking (1965). 1

[20] S. W. Yang, *Order of the canonical vector bundle on $C_n(k)/\Sigma_k$*, Illinois J. Math. **25** (1981), no. 1, 136-146. (document)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NY 14625, USA

Email address: cohf@math.rochester.edu

INSTITUTE OF MATHEMATICS AND SYSTEMS SCIENCES, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

Email address: huangrz@amss.ac.cn

URL: https://sites.google.com/site/hrzsea