Gaussian expansion analysis of a matrix model with the spontaneous breakdown of rotational symmetry

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Abstract: Recently the gaussian expansion method has been applied to investigate the dynamical generation of 4d space-time in the IIB matrix model, which is a conjectured nonperturbative definition of type IIB superstring theory in 10 dimensions. Evidence for such a phenomenon, which is associated with the spontaneous breaking of the SO(10) symmetry down to SO(4), has been obtained up to the 7-th order calculations. Here we apply the same method to a simplified model, which is expected to exhibit an analogous spontaneous symmetry breaking via the same mechanism as conjectured for the IIB matrix model. The results up to the 9-th order demonstrate a clear convergence, which allows us to unambiguously identify the actual symmetry breaking pattern by comparing the free energy of possible vacua and to calculate the extent of “space-time” in each direction.

Keywords: Matrix Models, Superstring Vacua, Superstrings and Heterotic Strings.
1. Introduction

It has been long considered that matrix models may be useful as a nonperturbative formulation of string theory, and hence play an important role similar to the lattice formulation in quantum field theory. Indeed matrix models have been quite successful in formulating non-critical string theory, and after the development of the notions such as “string duality” and “D-branes”, the idea has been extended also to critical strings. The IIB matrix model [1] is one of such proposals, which is conjectured to be a nonperturbative definition of type IIB superstring theory in 10 dimensions. It is a supersymmetric matrix model, which can be formally obtained by the zero-volume limit of 10d SU(N) super Yang-Mills theory.

In this model the space-time is represented by the eigenvalue distribution of ten bosonic matrices [2]. If the distribution collapses dynamically to a four-dimensional hypersurface, which in particular requires the SO(10) symmetry of the model to be spontaneously broken, we may naturally understand the dimensionality of our space-time as a result of the nonperturbative dynamics of superstring theory. In ref. [3] the first evidence for the above scenario has been obtained by calculating the free energy of space-time with various dimensionality using the gaussian expansion method up to the 3rd order. Higher-order calculations [4, 5] as well as the tests of the method itself in simpler models [6, 7] have strengthened the conclusion considerably. Refs. [8] provide another evidence for the emergence of four-dimensional space-time based on perturbative calculations around fuzzy-sphere like solutions.

While these results are certainly encouraging, it is desirable to understand the mechanism for the spontaneous symmetry breaking (SSB) of the rotational symmetry. In refs. [9]
it has been pointed out that the phase of the fermion determinant favors lower dimensional configurations since the phase becomes stationary around such configurations. Indeed Monte Carlo simulations show that SSB does not occur in various models without such a phase factor. Unfortunately including the effects of the phase in Monte Carlo simulation is technically difficult due to the so-called sign problem, but a new method, which is tested in Random Matrix Theory, was able to produce some preliminary results, which look promising. In ref. a simple matrix model which realizes the above mechanism has been proposed. The model contains flavors of Weyl fermion in the fundamental representation of , which yield a complex fermion determinant, and the large limit is taken with the ratio being fixed. The model can be solved exactly at infinitesimally small , and the SO(4) symmetry is shown to be broken down to SO(3).

In this paper we study this model at finite by the gaussian expansion method. Since the model is much simpler than the IIB matrix model, we can perform calculations up to the 9-th order with reasonable efforts. The results demonstrate a clear convergence for , which allows us to unambiguously identify the symmetry breaking pattern and to calculate the extent of “space-time” in each direction.

In fact it turns out that the SO(4) symmetry is broken down to SO(2) at finite . However, at small we reproduce the free energy as well as the extent of “space-time” in each direction obtained in ref. , which implies that the SO(3) symmetry is realized asymptotically as approaches zero. In the large region, on the other hand, the extent of “space-time” in two directions, in which the SO(2) symmetry is realized, becomes much larger than the remaining two directions. This behavior can be understood from the viewpoint of refs. since the phase of the fermion determinant becomes stationary for two-dimensional configurations, and increasing tends to amplify the effect of the phase. Thus our results nicely demonstrate the proposed mechanism for the dynamical generation of space-time in the IIB matrix model.

The rest of this paper is organized as follows. In Section 2 we define the model and review the known results. In Section 3 we explain how to apply the gaussian expansion method to the model. In Section 4 we present our results. Section 5 is devoted to a summary and discussions. The details of our calculations are given in the Appendix.

2. The model

The model we study in this paper is defined by the partition function

\[ Z = \int \! dA \, d\Phi \, d\overline{\Phi} \, e^{-(S_b + S_f)} , \]

\[ S_b = \frac{1}{2} N \text{tr} (A_\mu)^2 , \]

\[ S_f = - N \overline{\psi}_\alpha (\Gamma_\mu)_{\alpha\beta} A_\mu \psi_\beta , \]

\[ \text{See refs. [2, 10, 11] for discussions on other possible mechanisms.} \]
where $A_\mu$ ($\mu = 1, \cdots, 4$) are $N \times N$ traceless\(^2\) hermitian matrices and $\tilde{\psi}_f^\alpha$, $\psi_f^\alpha$ ($\alpha = 1, 2$; $f = 1, \cdots, N_f$) are $N$-dimensional row and column vectors, respectively, making the system SU($N$) invariant. The integration measure for $A_\mu$ is given by

$$dA = \prod_{a=1}^{N^2-1} \prod_{\mu=1}^{4} \frac{dA^a_\mu}{\sqrt{2\pi}},$$

(2.4)

where $A^a_\mu$ is the coefficient in the expansion $A_\mu = \sum_{a=1}^{N^2-1} A^a_\mu T^a$ with respect to the SU($N$) generators $T^a$ ($a = 1, \cdots, (N^2 - 1)$) normalized as $\text{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. The integration measure for the fermions is given by

$$d\psi d\bar{\psi} = \prod_{f=1}^{N_f} \prod_{i=1}^{N} \prod_{\alpha=1}^{2} d\psi^i_f d\bar{\psi}^i_\alpha.$$  

(2.5)

The system has an SO(4) symmetry, under which $A_\mu$ transforms as a vector, and $\psi_f^\alpha$ and $\tilde{\psi}_f^\alpha$ transform as Weyl spinors. The $2 \times 2$ matrices $\Gamma_\mu$ are the gamma matrices after the Weyl projection, and an explicit form is given for instance by

$$\Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.  

(2.6)$$

The fermionic part of the model can be thought of as the zero-volume limit of the system of Weyl fermions in four dimensions interacting with a background gauge field via fundamental coupling.

We take the large $N$ limit keeping the ratio $r \equiv N_f/N$ fixed (Veneziano limit). In order to discuss the SSB of the SO(4) symmetry in that limit, we consider the “moment of inertia tensor”\(^2\) \cite{2,12}

$$T_{\mu\nu} = \frac{1}{N} \text{tr} (A_\mu A_\nu),$$

(2.7)

which is a $4 \times 4$ real symmetric tensor, and denote its eigenvalues as $\{\lambda_i; i = 1, \cdots, 4\}$ with the specified order

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4.$$  

(2.8)

If the vacuum expectation values (VEVs) $\langle \lambda_i \rangle$ ($i = 1, \cdots, 4$) do not agree in the large $N$ limit, we may conclude that the SSB occurs. Thus $\langle \lambda_i \rangle$ plays the role of an order parameter. In the present model the sum of the VEVs is given by

$$\sum_{i=1}^{4} \langle \lambda_i \rangle = \sum_{\mu=1}^{4} \frac{1}{N} \text{tr} (A_\mu)^2 = 4 \left( 1 - \frac{1}{N^2} \right) + 2r$$

(2.9)

for arbitrary $N$ and $r$ due to a “virial theorem”\(^2\) \cite{17}.

\(^2\)The tracelessness condition was not imposed in the original paper \cite{17}. While this condition does not affect the large $N$ limit of the model, it simplifies our calculation drastically; see footnote 4.
At infinitesimally small $r$ the VEVs can be obtained in the large $N$ limit as [17]
\[
\langle \lambda_1 \rangle = \langle \lambda_2 \rangle = \langle \lambda_3 \rangle = 1 + r + o(r), \\
\langle \lambda_4 \rangle = 1 - r + o(r),
\]
which means that the SO(4) symmetry is spontaneously broken down to SO(3). The SSB is associated with the formation of a condensate $\langle \bar{\psi}_f \psi_f \rangle$, which is invariant under SO(3), but not under the full SO(4) transformation.

An important feature of the model that is relevant to the SSB is that the fermion determinant $\det D$, where $D$ is a $2N \times 2N$ matrix given by $D = \Gamma_\mu A_\mu$, is complex in general. If one replaces the fermion determinant by its absolute value, the same analysis at infinitesimal $r$ leads to an SO(4) symmetric result [17]. Thus the SSB of the original model occurs precisely due to the phase of the fermion determinant.

At large $r$ the effect of the phase is amplified, and we may expect that the configurations for which the phase is stationary dominate the path integral. Analogously to the situation in the IIB matrix model [9], the phase becomes stationary for 2-dimensional configurations in the present model. Therefore we anticipate the emergence of 2-dimensional “space-time” $(\lambda_1, \lambda_2 \gg \lambda_3, \lambda_4)$ as $r$ increases.

3. The gaussian expansion method

We are going to obtain results for the model (2.1) at finite $r$ using the gaussian expansion method. The method has a long history, and the original idea appeared already around 1980 in the context of solving quantum mechanical systems [18, 19], where the expansion was shown to be convergent in some concrete examples [20]. The method proved useful also in field theories [21] in various contexts. Applications to superstring/M theories using their matrix model formulations have been advocated by Kabat and Lifschytz [22], and the subsequent series of papers [23] revealed interesting blackhole thermodynamics. Applications to simplified versions of the IIB matrix model were initiated in refs. [24]. An earlier application to random matrix models can be found in ref. [25].

Similarly to the case of the IIB matrix model [3, 4, 5], let us introduce the gaussian action
\[
S_0 = \frac{1}{2} N \sum_{\mu=1}^{4} t_\mu \text{tr} (A_\mu)^2 + N \sum_{f=1}^{N_f} \sum_{\alpha, \beta=1}^{2} A_{\alpha\beta} \bar{\psi}_f^\alpha \psi_f^\beta ,
\]
which breaks the SO(4) symmetry. The $2 \times 2$ complex matrix $A$ can be expanded in terms of gamma matrices as
\[
A = \sum_{\mu=1}^{4} u_\mu \Gamma_\mu
\]
\footnote{Note that the phase of the fermion determinant is invariant under the scale transformation $A_\mu \mapsto \alpha A_\mu$. Therefore it is only the ratio of the eigenvalues that matters for the stationarity.}
\footnote{If we did not impose the tracelessness condition on $A_\mu$, we would need to consider a linear term such as $S_{\text{lin}} = N \sum_\mu h_\mu \text{tr} A_\mu$ in the gaussian action (3.1). The gaussian expansion method can be extended to such a case [7], but the calculation will be more involved.}
using 4 complex parameters $u_\mu$. Then we consider the action

\[ S_{\text{GEM}}(t, u; \lambda) = \frac{1}{\lambda} \left[ \left\{ S_0 + \lambda(S_b - S_0) \right\} + S_f \right], \]  

which reduces to the original action for $\lambda = 1$. The gaussian expansion amounts to calculating various quantities as an expansion with respect to $\lambda$ up to some finite order and setting $\lambda = 1$ eventually. As we will discuss shortly, the free parameters $t_\mu$ and $u_\mu$ in the gaussian action $S_0$ play a crucial role in the method.

In fact the gaussian expansion can be viewed as a loop expansion with the “classical action” ($S_0 + S_f$) and the “one-loop counterterms” ($S_b - S_0$). This becomes clear upon rescaling $A_\mu$ and $\psi$ as $A_\mu \mapsto \lambda A_\mu$, $\psi_\alpha \mapsto \sqrt{\lambda} \psi_\alpha$, so that the partition function takes the form

\[ Z = \int dA d\psi d\bar{\psi} e^{-(S_{\text{cl}} + S_{\text{c.t.}})}, \]  

\[ S_{\text{cl}}(t, u) = S_0 + \sqrt{\lambda} S_f, \quad S_{\text{c.t.}}(t, u) = \lambda (S_b - S_0). \]  

In actual calculations the “one-loop counter terms” can be incorporated easily by exploiting the relation

\[ S_{\text{cl}}(t, u) + S_{\text{c.t.}}(t, u) = S_{\text{cl}}(t + \lambda(1 - t), u - \lambda u). \]  

As an example, let us consider evaluating the free energy $F = -\frac{1}{N^2} \ln Z$ by the gaussian expansion method. We first calculate the free energy $\mathcal{F}(t, u)$ for the “classical action” $S_{\text{cl}}(t, u)$ defined by

\[ \exp[-N^2 \mathcal{F}(t, u)] = \int dA d\psi d\bar{\psi} e^{-S_{\text{cl}}(t, u)}. \]  

This can be done by ordinary Feynman diagrammatic calculations, where the use of Schwinger-Dyson (SD) equations reduces the number of diagrams considerably [4]. Suppose we obtain the result up to the $K$-th order as

\[ \mathcal{F}_K(t, u) = \sum_{k=0}^{K} \tilde{F}_k(t, u) \lambda^k. \]  

We shift the arguments, and obtain the new coefficients $\tilde{F}_k(t, u)$ in the expansion

\[ \mathcal{F}_K(t + \lambda(1 - t), u - \lambda u) = \sum_{k=0}^{K} \tilde{F}_k(t, u) \lambda^k + O(\lambda^{K+1}). \]  

Then the free energy for the original model can be evaluated as

\[ F_K(t, u) = \sum_{k=0}^{K} \tilde{F}_k(t, u). \]  

The result of such calculations depends on the free parameters in the gaussian action. However, in various models [4, 6, 7] a “plateau” region, in which the result becomes almost constant, was found to develop in the parameter space as one goes to higher orders of the
expansion. Moreover it turned out that the height of the plateau agrees very accurately with the correct value obtained by some other method. Therefore it is reasonable to expect that the method works in general if one can identify a plateau in the parameter space. In old literature the free parameters were determined in such a way that the result becomes most insensitive to the change of the parameters [19], but it is really the formation of a plateau that ensures the validity of the method as has been first recognized in ref. [4].

Identification of a plateau becomes a non-trivial issue when there are many parameters in the gaussian action. The histogram prescription [6, 7], which works nicely when there are only one or two real parameters, does not seem to work when the number exceeds three. (Note that we have 4 real and 4 complex parameters in the present case.) We have also attempted a Monte Carlo simulation in the parameter space to search for a plateau but with little success. We therefore use the prescription adopted for the IIB matrix model. First we solve the “self-consistency equations”

$$\frac{\partial}{\partial t_\mu} F_K(t, u) = 0 ,$$

$$\frac{\partial}{\partial u_\mu} F_K(t, u) = 0 .$$  (3.11)

Typically we obtain many solutions as we go to higher orders. If we observe that solutions concentrate in some region of the parameter space, we consider it as an indication of the plateau formation.

Although the number of parameters is much less than that (10 real and 120 complex parameters) in the IIB matrix model, it is still difficult to obtain all the solutions of the self-consistency equations at high orders. As is done in the case of the IIB matrix model [3, 4, 5], we search for solutions assuming that some subgroup of the full rotational symmetry is preserved. Here we consider the following Ansätze. For each case the independent parameters will be 2 real and 1 complex numbers.

**SO(3) Ansatz** : We assume SO(3) symmetry in the $x_2, x_3, x_4$ directions. Then the parameters are restricted to

$$t_2 = t_3 = t_4 (\equiv \hat{t}), \quad u_2 = u_3 = u_4 = 0 .$$  (3.12)

**SO(2) Ansatz** : We assume SO(2) symmetry in the $x_3, x_4$ directions. Furthermore we impose discrete symmetry under $x_1 \to x_2, x_2 \to x_1, x_4 \to -x_4$. Then the parameters are restricted to

$$t_1 = t_2, \quad t_3 = t_4 (\equiv \hat{t}), \quad u_1 = u_2, \quad u_3 = u_4 = 0 .$$  (3.13)

4. Results

For each solution of the self-consistency equations obtained within the symmetry Ansatz, we calculate the free energy. (See appendix for the details.) The free energy we plot in what follows is actually defined by

$$f = \lim_{N \to \infty} \left\{ F - 2(1 - r) \ln N \right\} ,$$  (4.1)
where the subtraction is necessary to make the quantity finite. The exact result at infinitesimal $r$ is given by $[17]$

$$f = -2 \ln 2 + (1 - \ln 2) r + o(r) . \quad (4.2)$$

Figure 1: The free energy obtained for the SO(3) Ansatz at orders 3(left top), 5(right top), 7(left bottom) and 9(right bottom) is plotted as a function of $r$.

In Fig. 1 the free energy calculated for the SO(3) Ansatz at orders 3,5,7,9 is plotted against $r$. We find that at orders 5 and 7 there are two solutions which almost coincide with each other throughout the whole region $^5$ of $0 \leq r \leq 2$. At the 9-th order there are actually three solutions lying on top of each other, which are represented by the solid lines to be distinguished from the other solutions. We consider this as an indication of the plateau formation. Similar behavior is observed for the SO(2) Ansatz as one can see from Fig. 2. Again the three solid lines in the right bottom plot represent the solutions that we consider to be concentrating.

We pick up the three solutions which concentrate at the 9-th order for the two Ansätze and plot them in Fig. 3 for comparison. Throughout the whole range of $r$ considered, the free energy for the SO(2) Ansatz is smaller than that for the SO(3) Ansatz. Thus we

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$^5$At $r \gtrsim 2$ the solutions that concentrate at the 9-th order start to separate, and we cannot obtain reliable results in that regime. Note in this regard that the number of fermion loops gives the power of $r$ in the result of the gaussian expansion. Therefore it is reasonable that the convergence becomes slower as we go to larger $r$. 

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conclude that the true vacuum is described by the SO(2) Ansatz. On the other hand, the results for the two Ansätze asymptote to each other as $r$ approaches zero. The meaning of this behavior will be clarified shortly.

Figure 2: The free energy obtained for the SO(2) Ansatz at orders 3(left top), 5(right top), 7(left bottom) and 9(right bottom) is plotted as a function of $r$.

Figure 3: Comparison of the free energy obtained for the SO(2) and SO(3) Ansätze (solid lines and dashed lines, respectively). The solutions that concentrate at the 9-th order are extracted from Figs. 1 and 2.
Let us move on to the calculation of observables. Similarly to the free energy, we can calculate an observable as an expansion with respect to $\lambda$ using the action (3.3). In our model the observable of primary interest is the eigenvalues $\lambda_i$ of the “moment of inertia tensor” (2.7). We calculate them for the SO(3) and SO(2) Ansätze at the 9-th order as a function of the free parameters in the gaussian action, and plug in the three solutions that are seen to concentrate in the study of free energy. In fact the VEVs of $\lambda_i$ can be readily obtained by diagonalizing $c_{\mu\nu}$ defined by eq. (A.1), which is calculated anyway in the calculation of free energy. Fig. 4 shows the results. Note that we are ultimately interested in the results for the SO(2) Ansatz since it gives the smaller free energy.

For the SO(3) (SO(2)) Ansatz the lines that grow almost linearly actually represent three (two) eigenvalues, which are degenerate due to the assumed symmetry. For the SO(2) Ansatz it turns out that the third largest eigenvalue comes closer to the two degenerate largest ones as one approaches $r = 0$, thus realizing the SO(3) symmetry asymptotically. In fact the results obtained for the SO(2) Ansatz are indistinguishable from those obtained for the SO(3) Ansatz at small $r$. This is consistent with our observation in Fig. 3 that the free energy for the SO(2) Ansatz have the same asymptotic behavior for $r \to 0$ as that for the SO(3) Ansatz. Actually we find that both the free energy and the observable agree asymptotically with the exact results (4.2), (2.10) at infinitesimal $r$.

At large $r$, on the other hand, the results for the SO(2) Ansatz show a clear tendency that the two degenerate eigenvalues become much larger than the other two, which implies the emergence of a two-dimensional “space-time”. This agrees with the argument based on the phase stationarity given at the end of Section 2. Thus the gaussian expansion method enables us to obtain results at finite $r$, which naturally interpolate the SO(3) symmetric exact result at infinitesimal $r$ and the two-dimensional behavior expected at large $r$.

**Figure 4:** The four eigenvalues of the “moment of inertia tensor” (2.7) obtained for the SO(3) Ansatz (left) and the SO(2) Ansatz (right) at the 9-th order are shown as a function of $r$. Note that the largest eigenvalue has 3-fold (2-fold) degeneracy for the SO(3) Ansatz and the SO(2) Ansatz, respectively. The three types of line correspond to the three solutions of the self-consistency equations (3.11) that concentrate at the 9-th order.

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6Since $c_{\mu\nu}$ takes the form (A.17) and (A.20) respectively for the SO(3) and SO(2) Ansätze, the diagonalization is actually needed only for the SO(2) Ansatz.
5. Summary and Discussions

In this paper we have applied the gaussian expansion method to a matrix model which is expected to exhibit SSB of rotational symmetry due to the phase of the fermion determinant. The free energy calculated by the gaussian expansion method depends on the free parameters in the gaussian action, and the formation of a plateau in the parameter space is crucial for the validity of the method. For each Ansatz considered for the possible breaking pattern of the SO(4) symmetry, we obtained a clear evidence for the plateau formation. By comparing the free energy obtained for each Ansatz, we concluded that the true vacuum is described by the SO(2) Ansatz. Our results for the extent of “space-time” in each direction are consistent with the exact result for infinitesimal $r$ and also with the behavior expected at large $r$.

The mechanism for the SSB of rotational symmetry demonstrated in the present model is expected to be at work also in the IIB matrix model. However, we should also note the difference of the two models. In the present model the fermionic degrees of freedom match with the bosonic ones at $r = 1$, but there the “space-time” is not really two-dimensional, but it looks like a “rugby ball” with two directions more extended than the other two. Moreover, if we consider the ten-dimensional version of the present model, we expect that there will be eight directions more extended than the other two. In the IIB matrix model, on the other hand, the result of the gaussian expansion method shows that the ratio of the extent of space-time in four directions to that in the remaining six directions increases with the order up to the 7-th order [3, 4, 5]. This suggests that four directions are much more (possibly, infinitely more) extended than the remaining six directions. It is conceivable that supersymmetry plays an important role here. Let us recall that the effective theory for the eigenvalues of the bosonic matrices in the IIB matrix model is a weakly bound system like a branched polymer due to cancellation between the bosonic and fermionic contributions [2]. This makes the space-time easy to collapse. Note also that the convergence of the gaussian expansion is not so clear in the IIB matrix model as in the present model.

While it is certainly worth while to proceed to the 8-th or 9-th orders in the IIB matrix model, we consider that Monte Carlo simulations along the line of ref. [15] are necessary to definitely confirm the emergence of a four-dimensional space time. That approach is also expected to provide an intuitive understanding of why “4” instead of 3 or 5. Note in this regard that the present simplified model shares the technical difficulty for implementing the phase of the fermion determinant in Monte Carlo simulation. Our new results obtained in this paper therefore provide a nice testing ground for the new method for simulating the IIB matrix model.

Acknowledgments

The authors would like to thank K.N. Anagnostopoulos, T. Aoyama and H. Kawai for fruitful discussions.
A. Details of the calculation

As explained in Section 3 the main part of calculations in the gaussian expansion is actually nothing but the ordinary perturbative calculation of the free energy $\mathcal{F}(t, u)$ defined by (3.7). The Feynman rules are given in Fig. 5. The first and second lines represent the bare propagators $\langle (A_\mu)_{ij}(A_\nu)_{kl} \rangle_0$ and $\langle \psi^f \bar{\psi}^g \rangle_0$, respectively, where the symbol $\langle \cdot \rangle_0$ represents a VEV obtained with the gaussian action (3.1). The third line stands for the interaction vertex coming from $S_I$. Instead of evaluating all the diagrams contributing to $\mathcal{F}(t, u)$, we use the SD equations to reduce the number of the diagrams to be computed following the idea put forward in ref. [4].

$$
\begin{align*}
\langle (A_\mu)_{ij}(A_\nu)_{kl} \rangle & = \frac{1}{N} \left( \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \frac{1}{t^\mu} \delta_{\mu \nu}, \\
\langle \psi^f \bar{\psi}^g \rangle & = \frac{1}{N} \delta_{fg} \delta_{ij} D_{\alpha \beta}, \\
\end{align*}
$$

Figure 5: The Feynman rules for the $\lambda$-expansion of (3.7).

Note that the one-point function $\langle (A_\mu)_{ij} \rangle$ is proportional to $\delta_{ij} \langle \text{tr} A_\mu \rangle$ because of the SU($N$) symmetry and therefore it vanishes due to the tracelessness condition imposed on $A_\mu$. The full propagators (connected two-point functions) can be written in the form

$$
\begin{align*}
\langle (A_\mu)_{ij}(A_\nu)_{kl} \rangle & = \frac{1}{N} \left( \delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) c_{\mu \nu}, \\
\langle \psi^f \bar{\psi}^g \rangle & = \frac{1}{N} \delta_{fg} \delta_{ij} D_{\alpha \beta}, \\
\end{align*}
$$

where the coefficients $c_{\mu \nu}$ and $D$ are given at the leading order in $\lambda$ as

$$
c_{\mu \nu} = \frac{1}{t^\mu} \delta_{\mu \nu} + O(\lambda), \quad D = A^{-1} + O(\lambda).
$$

Corresponding to eq. (3.2), the full propagator $D$ can be parametrized as

$$
D = \sum_{\mu=1}^{4} d_{\mu} \Gamma_{\mu}.
$$
where the “conjugate” gamma matrices $\bar{\Gamma}_\mu$ are defined by

$$\bar{\Gamma}_k = \Gamma_k \quad (k = 1, 2, 3), \quad \bar{\Gamma}_4 = -\Gamma_4.$$ \hspace{1cm} (A.5)

In what follows, we restrict ourselves to the large-$N$ limit with the ratio $r = \frac{N_f}{N}$ fixed, so that we have to consider planar diagrams only, but the method itself is applicable to finite $N$ as well.

By using the SD equations for the full propagators, we can reduce the calculation of the free energy to that of two-particle-irreducible (2PI) planar diagrams. Here, by “2PI diagrams” we mean those diagrams which cannot be separated into two disconnected parts by cutting two propagators. Let us consider the 2PI planar vacuum diagrams whose internal lines are all replaced by the full propagators. The sum of such diagrams is a function of $c_{\mu\nu}$ and $d_\mu$ and shall be denoted as $N^2 G(c,d)$. For example, $N^2 G(c,d)$ up to the 5th order can be computed from the 5 diagrams in Fig. 6, and it is given explicitly as

$$G(c,d) = -\frac{1}{2} \lambda r \sum_{\mu\nu} c_{\mu\nu} \text{Tr} \left( \Gamma_\mu D\Gamma_\nu D\right)$$
$$+ \frac{1}{6} \lambda^3 r^2 \sum_{\mu\nu\lambda\rho\sigma} c_{\mu\nu} c_{\lambda\rho} c_{\kappa\sigma} \text{Tr} \left( \Gamma_\mu D\Gamma_\lambda D\Gamma_\kappa D\right) \text{Tr} \left( \Gamma_\sigma D\Gamma_\rho D\Gamma_\nu D\right)$$
$$+ \frac{1}{8} \lambda^4 r^2 \sum_{\mu\nu\lambda\rho\sigma\xi} c_{\mu\nu} c_{\lambda\rho} c_{\kappa\sigma} c_{\tau\xi} \text{Tr} \left( \Gamma_\mu D\Gamma_\lambda D\Gamma_\kappa D\Gamma_\tau D\right) \text{Tr} \left( \Gamma_\xi D\Gamma_\sigma D\Gamma_\rho D\Gamma_\nu D\right)$$
$$+ \frac{1}{10} \lambda^5 r^2 \sum_{\mu\nu\lambda\rho\sigma\xi\eta\zeta} c_{\mu\nu} c_{\lambda\rho} c_{\kappa\sigma} c_{\tau\xi} c_{\eta\zeta} \text{Tr} \left( \Gamma_\mu D\Gamma_\lambda D\Gamma_\kappa D\Gamma_\tau D\Gamma_\eta D\right)$$
$$\times \text{Tr} \left( \Gamma_\xi D\Gamma_\sigma D\Gamma_\rho D\Gamma_\nu D\right)$$
$$+ O(\lambda^6),$$

(A.6)

where “Tr” implies a trace taken with respect to the spinor indices.

The complete list of 2PI planar vacuum diagrams at orders 6, 7, 8, 9 are given in Figs. 7-13. The number of diagrams at each order is 4, 9, 24, 81, respectively.

A.1 Schwinger-Dyson equations

In this section we derive a closed set of equations, which allows us to calculate $F(t,u)$ from $G(c,d)$.

Let $B_{\mu\nu}$ be the one-particle-irreducible (1PI) part of the radiative corrections to $c_{\mu\nu}$. Then we can write $c_{\mu\nu}$ as a geometric series

$$c = \frac{1}{T} + \frac{1}{T} B \frac{1}{T} + \frac{1}{T} B \frac{1}{T} B \frac{1}{T} + \cdots = \frac{1}{T - B},$$

(A.7)
Figure 6: The 2PI planar vacuum diagrams up to the 5-th order. For simplicity the propagators for bosons and fermions are represented by single (dashed and solid, respectively) lines. The first three diagrams are contributions from the orders 1, 3, 4, and the last two from the order 5. (There is no contribution from the 2nd order.) The symmetry factor is indicated below each diagram.

Figure 7: The 2PI planar vacuum diagrams at the 6-th order.

where we regard \( c_{\mu\nu}, T_{\mu\nu} \equiv t_\mu \delta_{\mu\nu}, B_{\mu\nu} \) as \( 4 \times 4 \) matrices. Note, on the other hand, that \( B_{\mu\nu} \) can be obtained from \( G(c, d) \) by

\[
B_{\mu\nu} = 2 \frac{\partial}{\partial c_{\mu\nu}} G(c, d) .
\]  \hspace{1cm} (A.8)

Combining (A.7) and (A.8), we obtain

\[
(c^{-1})_{\mu\nu} = t_\mu \delta_{\mu\nu} - 2 \frac{\partial}{\partial c_{\mu\nu}} G(c, d) .
\]  \hspace{1cm} (A.9)

Similarly we get

\[
(D^{-1})_{\alpha\beta} = A_{\alpha\beta} + \frac{1}{r} \frac{\partial}{\partial D_{\beta\alpha}} G(c, d) ,
\]  \hspace{1cm} (A.10)

which may be written in terms of \( d_\mu \) as

\[
\frac{1}{\Delta} d_\mu = u_\mu + \frac{1}{2r} \frac{\partial}{\partial d_\mu} G(c, d) ,
\]  \hspace{1cm} (A.11)
where $\Delta = \sum_{\mu} (d_\mu)^2$.

Since the SD equations (A.9), (A.10) are closed with respect to $c_{\mu \nu}$, $D_{\alpha \beta}$, we can solve them order by order in $\lambda$ once the sum of the 2PI diagrams $G(c, d)$ is obtained. For example, the results for the first few orders are as follows.

\begin{equation}
    c_{\mu \nu} = \sum_{n=0}^{\infty} \lambda^n c_{\mu \nu}^{(n)},
\end{equation}

\begin{equation}
    c_{\mu \nu}^{(0)} = \frac{1}{t_\mu} \delta_{\mu \nu},
\end{equation}

\begin{equation}
    c_{\mu \nu}^{(1)} = -r \frac{1}{t_\mu} \frac{1}{t_\nu} \text{Tr} \left( \Gamma_\mu A^{-1} \Gamma_\nu A^{-1} \right),
\end{equation}

\begin{equation}
    c_{\mu \nu}^{(2)} = -r \sum_\rho \frac{1}{t_\mu} \frac{1}{t_\rho} \frac{1}{t_\nu} \text{Tr} \left[ \{ \Gamma_\mu A^{-1}, \Gamma_\nu A^{-1} \} \{ \Gamma_\rho A^{-1} \} \{ \Gamma_\rho A^{-1} \} \right] + r^2 \sum_\rho \frac{1}{t_\mu} \frac{1}{t_\rho} \frac{1}{t_\nu} \text{Tr} \left( \Gamma_\mu A^{-1} \Gamma_\rho A^{-1} \right) \text{Tr} \left( \Gamma_\rho A^{-1} \Gamma_\nu A^{-1} \right),
\end{equation}

\begin{equation}
    D_{\alpha \beta} = \sum_{n=0}^{\infty} \lambda^n D_{\alpha \beta}^{(n)},
\end{equation}

\begin{equation}
    D_{\alpha \beta}^{(0)} = (A^{-1})_{\alpha \beta},
\end{equation}

\begin{equation}
    D_{\alpha \beta}^{(1)} = \sum_\mu \frac{1}{t_\mu} \left[ A^{-1} \left( \Gamma_\mu A^{-1} \right)^2 \right]_{\alpha \beta}.
\end{equation}
\[ D^{(2)}_{\alpha\beta} = \sum_{\mu\nu} \frac{1}{t_\mu} \frac{1}{t_\nu} \left[ A^{-1} \Gamma_\mu A^{-1} (\Gamma_\nu A^{-1})^2 \Gamma_\mu A^{-1} \right]_{\alpha\beta} \]
\[ + \sum_{\mu\nu} \frac{1}{t_\mu} \frac{1}{t_\nu} \left[ A^{-1} (\Gamma_\mu A^{-1})^2 (\Gamma_\nu A^{-1})^2 \right]_{\alpha\beta} . \]

Let us note that the free energy \( F(t, u) \) satisfies the differential equations
\[ 2 \frac{\partial}{\partial t_\mu} F(t, u) = c_{\mu\mu} , \quad (A.14) \]
\[ -\frac{1}{2r} \frac{\partial}{\partial u_\mu} F(t, u) = d_\mu . \quad (A.15) \]

By integrating these equations term by term, we finally obtain the \( \lambda \)-expansion for \( F(t, u) \).

The integration constant, which appears only in the zeroth order contributions (i.e., the one-loop diagrams), can be fixed easily by direct computation.

The explicit form of the free energy up to the 2nd order is
\[ F(t, u) = \sum_{k=0}^{\infty} \lambda^k \bar{F}_k , \quad (A.16) \]
\[ \bar{F}_0 = 2(1 - r) \ln N - 2 \ln 2 + \frac{1}{2} \sum_\mu \ln t_\mu - r \ln(\det A) , \]
\[ \bar{F}_1 = \frac{r}{2} \sum_\mu \frac{1}{t_\mu} \text{Tr} \left[ (\Gamma_\mu A^{-1})^2 \right] , \]
\[ \bar{F}_2 = \frac{r}{2} \sum_{\mu\nu} \frac{1}{t_\mu} \frac{1}{t_\nu} \text{Tr} \left[ (\Gamma_\mu A^{-1})^2 (\Gamma_\nu A^{-1})^2 \right] - \frac{r^2}{4} \sum_{\mu\nu} \frac{1}{t_\mu} \frac{1}{t_\nu} \left( \text{Tr} \left( \Gamma_\mu A^{-1} \Gamma_\nu A^{-1} \right) \right)^2 . \]

**A.2 Exploiting the Ansätze**

As explained in Section 3 we impose some Ansatz on the possible symmetry breaking pattern in order to reduce the number of free parameters in the gaussian action. In this section we write down the SD equations for each Ansatz separately. By solving the SD equations, we obtain the free energy similarly to what we have done in section A.1.

**A.2.1 SD equations for the SO(3) Ansatz**

The full propagators take the following form.
\[ c_{\mu\nu} = \begin{pmatrix} c_{11} & \hat{c} \\ \hat{c} & \hat{c} \end{pmatrix} , \quad d_2 = d_3 = d_4 = 0 . \quad (A.17) \]

The SD equations (A.9), (A.11) are rewritten as
\[ \frac{1}{c_{11}} = t_1 - 2 \frac{\partial}{\partial c_{11}} [G(c, d)]_{SO(3)} , \]
\[ \frac{1}{\hat{c}} = \hat{t} - 2 \frac{\partial}{\partial \hat{c}} [G(c, d)]_{SO(3)} , \]
\[ \frac{1}{d_1} = u_1 + \frac{1}{2r} \frac{\partial}{\partial d_1} [G(c, d)]_{SO(3)} . \quad (A.18) \]
Here and henceforth the symbol \([ \cdot ]_{\text{SO}(n)}\) implies that the number of independent variables is already reduced by the \(\text{SO}(n)\) Ansatz. The differential equations (A.14), (A.15) become

\[
\begin{align*}
2 \frac{\partial}{\partial t_1} [F(t, u)]_{\text{SO}(3)} &= c_{11}, \\
2 \frac{\partial}{\partial \tilde{t}} [F(t, u)]_{\text{SO}(3)} &= \tilde{c}, \\
-\frac{1}{2r} \frac{\partial}{\partial u_1} [F(t, u)]_{\text{SO}(3)} &= d_1. 
\end{align*}
\]

(A.19)

**A.2.2 SD equations for the \(\text{SO}(2)\) Ansatz**

The full propagators take the following form.

\[
c_{\mu\nu} = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{11} \\ \tilde{c} & \tilde{c} \end{pmatrix}, \quad d_1 = d_2, \quad d_3 = d_4 = 0. \quad (A.20)
\]

The SD equations (A.9), (A.11) are rewritten as

\[
\begin{align*}
\frac{c_{11}}{(c_{11})^2 - (c_{12})^2} &= t_1 - \frac{\partial}{\partial c_{11}} [G(c, d)]_{\text{SO}(2)}, \\
\frac{c_{12}}{(c_{11})^2 - (c_{12})^2} &= \frac{\partial}{\partial c_{12}} [G(c, d)]_{\text{SO}(2)}, \\
\frac{1}{\tilde{c}} &= \frac{\partial}{\partial \tilde{c}} [G(c, d)]_{\text{SO}(2)}, \\
\frac{1}{d_1} &= 2u_1 + \frac{1}{2r} \frac{\partial}{\partial d_1} [G(c, d)]_{\text{SO}(2)}. 
\end{align*}
\]

(A.21)

The differential equations (A.14), (A.15) become

\[
\begin{align*}
\frac{\partial}{\partial t_1} [F(t, u)]_{\text{SO}(2)} &= c_{11}, \\
\frac{\partial}{\partial \tilde{t}} [F(t, u)]_{\text{SO}(2)} &= \tilde{c}, \\
-\frac{1}{4r} \frac{\partial}{\partial u_1} [F(t, u)]_{\text{SO}(2)} &= d_1. 
\end{align*}
\]

(A.22)

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Figure 9: The 2PI planar vacuum diagrams at the 8-th order.
Figure 10: The 2PI planar vacuum diagrams at the 9-th order (to be continued).
Figure 11: The 2PI planar vacuum diagrams at the 9-th order (to be continued).
Figure 12: The 2PI planar vacuum diagrams at the 9-th order (to be continued).
Figure 13: The 2PI planar vacuum diagrams at the 9-th order.