A RESIDUE THEOREM FOR RATIONAL TRIGONOMETRIC SUMS AND VERLINDE’S FORMULA

ANDRÁS SZENES

1. Introduction

The central objects of the present work are rational trigonometric sums such as

\[ \sum \frac{1}{\sin^{2} \frac{\pi m}{k} \sin^{2} \frac{\pi n}{k} \sin^{2} \frac{\pi (m+n)}{k}}, \quad m, n \in \mathbb{Z}, \, 0 < m, n < k, \, m + n \neq k, \]

where \( k \) is a fixed positive integer.

The interest in such sums was motivated by a beautiful formula of E. Verlinde for the dimension of the “space of conformal blocks” in the WZW theory \([18]\). The data for Verlinde’s formula is a simple simply-connected Lie group \( G \), non-negative integers \( g \) and \( k \), and, in the simplest case of one puncture, a dominant highest weight \( \lambda \) of \( G \) satisfying certain conditions. The result is a non-negative integer, which we denote by \( \text{Ver}_g(\lambda; k) \). The sum in (1.1) is an example of Verlinde’s expression; up to some normalization, it represents the case of \( G = SU(3), \, g = 2, \, \lambda = 0 \).

This formula turned out to have a close relationship with the topology of the moduli spaces of flat connections over Riemann surfaces. Indeed, under certain assumptions, \( \text{Ver}_g(k\lambda; k) \) is expected to agree with the Hilbert polynomial

\[ \int_{\mathfrak{M}} e^{kc_1(\mathfrak{L})} \text{Todd}(\mathfrak{M}) \]

of a certain genus-\( g \) moduli space of flat \( G \)-connections \( \mathfrak{M} \) endowed with a line bundle \( \mathfrak{L} \), both depending on the data \((G, g, \lambda)\). The discovery of this fantastic “coincidence” opened the way to computing various intersection numbers on the moduli spaces \([17]\); such computations looked utterly impossible until then \([1]\).

Looking at (1.1), one might note that it is not at all clear that the value of this finite sum is polynomial in \( k \), as would follow from the agreement with the Hilbert polynomial of a space. To uncover the topological information hidden in Verlinde’s formula, one needs to find a calculus that replaces the sum smeared in space by a compact expression which is manifestly a

---

The research was supported by NSF grant DMS-9870053 and NSA grant #6800900.
polynomial in $k$. A similar problem arose with the evaluation of rational sums such as

$$\sum \frac{1}{m^2n^2(m+n)^2}, \quad m, n, m+n \in \mathbb{Z}^{\neq 0}.$$

These sums appeared in the work of Witten on 2-dimensional gauge theory \cite{19,20}, and they again turned out to have a close relationship to the topology of the abovementioned moduli spaces. The exact evaluation of these sums was left open in \cite{20}.

A solution to these problems was suggested by the author in \cite{13}. We conjectured that there exist certain local functionals on the space of rational and rational trigonometric sums corresponding to the Lie group $G$, which would enable one to localize both types of sums, and, moreover, that these functionals would coincide in the two cases in such a way as to provide a Riemann-Roch calculus on the moduli spaces. The functional was described in \cite{13} for the case of $G = SU(n)$ explicitly. Surprisingly, it had the form of a single iterated residue. This argument and the functional, in a somewhat modified form, was later used by Jeffrey and Kirwan \cite{8} to give a computation of the Hilbert polynomial in this case.

The functional for the case of $G = SU(n)$ is very simple, but in the case of other groups a similarly symmetric formula does not seem to exist. So we approached this problem from a more general point of view – from the point of view of arbitrary hyperplane arrangements, of which the Stiefel diagram of a Lie group is a particular example. We described the abovementioned functional in \cite{14} for the case of arbitrary rational sums. The present paper gives the answer for the rational trigonometric case.

Our effort was strongly motivated by the work of Bismut and Labourie \cite{3}. They computed the Hilbert polynomial of the moduli space for an arbitrary group in terms of rational sums, but, curiously, their formula did not seem to coincide with Verlinde’s expression. One of the main goals of this work was to prove the equality of the two expressions. We achieved this in most cases, but there is still a bit of mystery left when $g = 0$. This will be duly explained in the main body of the paper.

Finally, we need to mention a circle problems closely related to computing rational trigonometric sums: the problems of partition functions and counting lattice points in polytopes. Our localization theorem is somewhat analogous to the results of Brion and Vergne on vector partition function \cite{4}. In fact, the exact relation is worth investigating further \cite{15}.

The contents of the paper are as follows: in §2 we recall the results of \cite{14} on rational sums and Bernoulli polynomials corresponding to central hyperplane arrangements; in §3 we extend these results to affine hyperplane arrangements. The main theorem, Theorem \cite{4}, which gives a local formula for rational trigonometric sums, is given in §4 and the application to the formula of Bismut and Labourie is detailed in §5.
Acknowledgments
The author is grateful to Jean-Michel Bismut and Michèl Vergne, whose support and warm hospitality made this work possible. Noam Elkies, Pavel Etingof and Victor Kac supplied some crucial advice.

2. Central hyperplane arrangements

In this section we review the results of [14] (cf. [6] for an alternative treatment).

2.1. Notation and Conventions.

Let $\mathfrak{A}$ be a central and essential hyperplane arrangement (HPA) in an $n$-dimensional complex vector space $V$, i.e. a collection of hyperplanes in $V$ such that $\cap \mathfrak{A} = \{0\}$. Denote by $R_\mathfrak{A}$ the rational functions on $V$ with poles along $\cup \mathfrak{A}$, by $M_\mathfrak{A}$ the meromorphic functions defined in a neighborhood of 0 with poles along $\cup \mathfrak{A}$, and by $U(\mathfrak{A})$ the complement $V \setminus \cup \mathfrak{A}$.

To simplify our notation, we impose a linear ordering relation $\prec$ on $\mathfrak{A}$ and assume that the hyperplanes are indexed accordingly: $(\mathfrak{A}, \prec) = (H_1, \ldots, H_N)$. We will make this ordering explicit in the notation whenever it is used in our constructions in an essential manner. If an $m$-element subset $\mathfrak{a}$ of $\mathfrak{A}$ is ordered, we will write $\mathfrak{a} \in \mathfrak{A}^m$, and think of it as a sequence of elements of $\mathfrak{A}$. If the ordering of $\mathfrak{a}$ is consistent with $\prec$, then we will write $\mathfrak{a} \subset \prec \mathfrak{A}$.

Often it will be convenient to choose a linear form $x_i$ for each hyperplane $H_i \in \mathfrak{A}$ such that $H_i = \{x_i = 0\}$. We will use the notation $\hat{\mathfrak{A}}$ for this ordered set of linear forms. Again, we will use the corresponding notations $\hat{\mathfrak{a}} \in \hat{\mathfrak{A}}^m$ and $\hat{\mathfrak{a}} \subset \prec \hat{\mathfrak{A}}$.

Finally, we will denote the $i$th element of $\mathfrak{a}$ by $H_i^{\mathfrak{a}}$, the $i$th element of $\hat{\mathfrak{a}}$ by $x_i^{\hat{\mathfrak{a}}}$ and the function $e^{2\pi \sqrt{-1} x_i^{\hat{\mathfrak{a}}}}$ by $e_i^{\hat{\mathfrak{a}}}$.

We will call a set $\{H_i\}_{i=1}^m$ of $m$ hyperplanes in $V$ independent if $\dim \cap H_i = n - m$. This is equivalent to saying that the corresponding linear forms are linearly independent.

An important convention throughout the paper is that underlining a symbol means multiplying it by $2\pi \sqrt{-1}$. For $y \in V^*$ we will write $e_y$ for the function $e^{2\pi \sqrt{-1} y}$ on $V$, which thus may also be written as $e\hat{y}$.

Finally, the notation $1, k$ will sometimes be used to denote the set of first $k$ natural numbers.

2.2. The Constant Term.

One can associate to every hyperplane $H_i \in \mathfrak{A}$ a closed holomorphic differential 1-form $\alpha_i = dx_i/x_i$ on $U(\mathfrak{A})$, where $\{x_i = 0\} = H_i$. These 1-forms are called logarithmic differential forms, as they are homogeneous in the linear form $x_i$, they do not depend on its choice. The key fact of the topology of $U(\mathfrak{A})$ is the existence of an injective ring homomorphism $q: H^*(U(\mathfrak{A}), \mathbb{C}) \to \Omega^*(U(\mathfrak{A}))$, which assigns to every cohomology class a closed holomorphic differential form corresponding to it in de Rham theory. The image $\Omega^*_\mathfrak{A} = \operatorname{im}(q)$ of the cohomology ring is generated in degree 1 by the logarithmic 1-forms $\{\alpha_i, i = 1, \ldots, N\}$. 
The map \( q \) allows us to define a generalized constant term functional \( \text{CT}^\mathfrak{A} : M_{\mathfrak{A}} \to \mathbb{C} \), which is of degree 0 with respect to the natural grading on \( R_{\mathfrak{A}} \subset M_{\mathfrak{A}} \). Given a representation \( \sum_i Z_i \otimes \beta^i \in H_n(U(\mathfrak{A}), \mathbb{C}) \otimes H^n(U(\mathfrak{A}), \mathbb{C}) \) of the invariantly defined diagonal element derived from the natural complete pairing of \( H_n(U(\mathfrak{A}), \mathbb{C}) \) and \( H^n(U(\mathfrak{A}), \mathbb{C}) \), we may form the functional

\[
(2.1) \quad \text{CT}^\mathfrak{A} : f \mapsto \sum_i \int_{Z_i} f(\beta^i).
\]

This functional is invariantly defined; it depends solely on the HPA \( \mathfrak{A} \), with no additional choices made.

If \( |\mathfrak{A}| = n = \dim V \), i.e. the HPA is simple, then this functional is equal to the constant term of the Laurent expansion of the function \( f \) near 0. In this case we will simply write CT instead of \( \text{CT}^\mathfrak{A} \). When \( |\mathfrak{A}| > n \), a more involved algebraic computational device is available; this will be detailed below.

2.3. Iterated constant term functionals. Let \( \mathfrak{A}^n_{\text{ind}} \subset \mathfrak{A}^n \) be the set of independent ordered \( n \)-tuples of hyperplanes in \( \mathfrak{A} \). Given \( \mathbf{a} \in \mathfrak{A}^n_{\text{ind}} \) and a permutation \( \tau \in S_n \), denote by \( \mathbf{a}^\tau \) the element \( (H_{\tau(1), \mathbf{a}}, \ldots, H_{\tau(n), \mathbf{a}}) \in \mathfrak{A}^n_{\text{ind}} \).

Then it follows from the description and properties of the map \( q \) that the linear space \( \Omega^\mathfrak{A}_n = q(H^n(U(\mathfrak{A}), \mathbb{C})) \) is spanned by the forms

\[
\alpha_{a} = \alpha_{1,a} \wedge \cdots \wedge \alpha_{n,a}
\]

as \( \mathbf{a} \) varies in \( \mathfrak{A}^n_{\text{ind}} \).

Every \( \mathbf{a} \in \mathfrak{A}^n_{\text{ind}} \) defines an iterated constant term functional

\[
i\text{CT} = \text{CT}_{H_{1,a}} \text{CT}_{H_{2,a}} \cdots \text{CT}_{H_{n,a}} : M_{\mathfrak{A}} \to \mathbb{C},
\]

which is obtained by sequentially applying the 1-dimensional constant term functional with respect to each of the hyperplanes in \( \mathbf{a} \), while keeping the preceding variables constant. More precisely, the symbol \( \text{CT}_{H_{n,a}} f \) means taking the 1-dimensional constant term of \( f \) along each of the lines \( \{ x_{i,a} = a_i, \ i = 1, 2, \ldots, n-1 \} \), where \( (a_1, \ldots, a_{n-1}) \) is a fixed \((n-1)\)-tuple of complex numbers. Considering \( (a_1, \ldots, a_{n-1}) \) to be coordinates on \( H_{n,\mathbf{a}} \), we can think of \( \text{CT}_{H_{n,a}} f \) as of a function on \( H_{n,\mathbf{a}} \). Then we replace \( V \) with \( H_{n,\mathbf{a}} \) and the function \( f \) with the function \( \text{CT}_{H_{n,a}} f \) on \( H_{n,\mathbf{a}} \), and we continue the process, taking \( \text{CT}_{H_{n-1,a}} \text{CT}_{H_{n,a}} f \), etc, finally arriving at a number.

We should point out that the notation \( \text{CT}_{H_{n,a}} f \) is somewhat misleading since this quantity depends on the rest of the hyperplanes as well; this is similar to the problem with the notation used for partial derivatives. The iterated constant term functional \( i\text{CT}_a \) usually depends on the order of the hyperplanes in \( a \) (cf. [14] for further details). To simplify our notation, we will often write \( \text{CT}_a \) instead of \( \text{CT}_H \) if the appropriate forms have been introduced. Finally, while the above definition of iterated constant terms might seem somewhat complicated, computationally this procedure is very simple, and is, in fact, a built-in function of software packages such
as MAPLE. It was through such computer experimentation that the author became acquainted with the concept.

**Example 2.1.** Let
\[ f = \frac{e^{x-2y}}{xy(x+y)}. \]
Then
\[ \text{iCT}_{x,y} f = -\frac{7}{6}, \quad \text{iCT}_{y,x} f = \frac{10}{3}, \quad \text{iCT}_{x,x+y} f = \frac{9}{2}, \quad \text{iCT}_{y,x+y} f = 0. \]

One can represent \( \text{CT}^A \) as a sum of iterated constant term functionals as follows. Associate to each \( a \in A^n_{\text{ind}} \) the full flag of subspaces of \( V \):
\[ \text{Flag}(a) = (\bigcap_{i=1}^n H_{i,a} = \{0\}, \ldots, H_{n-1,a} \cap H_{n,a}, H_{n,a}). \]
Denote \( \dim H^n(U(A), C) \) by \( r(A) \), and define a subset \( B \subset A^n_{\text{ind}} \) to be an orthogonal basis of \( A \) (in degree \( n \)) if
- \( |B| = r(A) \)
- for \( a, b \in B \) and \( \tau \in S_n \) the equality \( \text{Flag}(b^\tau) = \text{Flag}(a) \) implies \( b = a \).

Then keeping the notation and concepts introduced above, one obtains the following expression for the constant term.

**Proposition 2.1.** For any orthogonal basis \( B \) of \( A \) and function \( f \in R_A \) one has
\[ \text{CT}^A(f) = \sum_{a \in B} \text{iCT}_a f. \]

**Example 2.2.** Let \( A = (H_1, \ldots, H_N) \) be a HPA in dimension 2. Then
\[ \{(H_i, H_i) | i = 2, \ldots, N\} \subset A^2 \]
is an orthogonal basis.

Note that the constant term of a function depends on the hyperplane arrangement. For example, in the notation from above, \( \text{CT}^A(1) = r(A) \).

**Example 2.3.** We compute the constant term of the function \( f = \frac{e^{x-2y}}{xy(x+y)} \) with respect to the hyperplane arrangement \( A = \{x, y, x + y\} \) using two different orthogonal bases:
\[ \text{CT}^A f = \text{iCT}_{x,y} f + \text{iCT}_{x,x+y} f = \text{iCT}_{y,x} f + \text{iCT}_{y,x+y} f = \frac{10}{3}. \]

To have an efficient method of computation of the constant term, we need to find such orthogonal bases. Fortunately, they turn out to be plentiful. The construction described below is a refined version of the usual \text{nbc} (no broken circuit) bases (cf. \[ \text{[11]} \]). Here we will make an essential use of the ordering \( \prec \) introduced at the beginning of this section.
Proposition 2.2 ([14]). The set

\[ \text{NBC}(\mathfrak{A}, \prec) = \{ a \in \mathfrak{A}_{\text{ind}}^{n} | a \subseteq \mathfrak{A} \text{ and for any } H \notin a \text{ the set } \{ H \} \cup \{ G \in a | H \prec G \} \text{ is independent} \}, \]

is an orthogonal basis of \( \mathfrak{A} \).

Remark 2.1. 1. Note that we are only considering \( \text{nbc} \)-bases in degree \( n \), which is the top degree, even though they make sense in all degrees.
2. There are examples of orthogonal bases which are not \( \text{nbc} \)-bases for any ordering. [14, 5.3].
3. The orthogonal basis in Example 2.2 is an \( \text{nbc} \)-basis.

2.4. Rational sums. Let \( V \) be an \( n \)-dimensional complex vector space and \( \Gamma \subset V \) a lattice of rank \( n \). The lattice spans an \( n \)-dimensional real subspace \( V_{\mathbb{R}} \subset V \) with dual \( V_{\mathbb{R}}^{*} \subset V^{*} \). Denote the dual lattice of \( \Gamma \) by \( \Gamma^{*} = \text{Hom}(\Gamma, \mathbb{Z}) \subset V^{*} \) and the group of characters of \( \Gamma \) by \( \hat{\Gamma} = \text{Hom}(\Gamma, U(1)) \simeq V^{*}/\Gamma^{*} \). This last isomorphism is given by the correspondence \( \tau \mapsto t + \Gamma^{*} \), where \( e_{t}(v) = e^{t(v)} \) for \( v \in V \).

We will say that a central HPA \( \mathfrak{A} \) in \( V \) is compatible with \( \Gamma \), or that \( (\mathfrak{A}, \Gamma) \) is a compatible pair in \( V \), if each hyperplane \( H_{i} \) in the arrangement is the zero-set of some linear form \( x_{i} \in \Gamma^{*} \). Define \( t \in V_{\mathbb{R}}^{*} \) to be \( \Gamma^{*} \)-special with respect to \( \mathfrak{A} \) if \( t = \lambda + \sum_{i=1}^{N} \nu_{i}x_{i} \), where \( \lambda \in \Gamma^{*} \), \( \nu_{i} \in \mathbb{R} \), and at most \( n - 1 \) of the coefficients \( \{ \nu_{i} \} \) are nonzero. While this property depends on both \( \Gamma \) and \( \mathfrak{A} \), the reference to these two objects will be omitted from the notation whenever this causes no confusion. The set of nonspecial elements is a \( \Gamma^{*} \)-invariant union of open polyhedral chambers. Since being special is a \( \Gamma^{*} \)-invariant property, we have a well-defined notion of a special character \( \tau \in \hat{\Gamma} \), as well.

For an \( n \)-tuple \( \hat{y} = (y_{1}, \ldots, y_{n}) \) of linear forms introduce the notation

\[ \Box(\hat{y}) = \left\{ t \in V^{*} | t = \sum_{i=1}^{n} \nu_{i}y_{i} \text{ with } 0 < \nu_{i} < 1, i = 1, \ldots, n \right\}. \]

Assuming that \( \hat{y} \subset \Gamma^{*} \), denote by \( \text{vol}_{\Gamma}(\hat{y}) \) the \( \Gamma^{*} \)-volume of \( \Box(\hat{y}) \), which is always a positive integer. Then we have

Proposition 2.3. Let \( (\mathfrak{A}, \Gamma) \) be a compatible pair in \( V \), and fix \( a \in \mathfrak{A}_{\text{ind}}^{n} \) and a nonspecial character \( \tau \in \hat{\Gamma} \). Pick a set of representative forms \( \hat{a} \subset \Gamma^{*} \) for \( a \).
1. Then the function

\[ \text{Todd}(\Gamma, a, \tau) = \sum \left\{ e_{\hat{t}} | e_{\hat{t}}|_{\Gamma} = \tau, \hat{t} \in \Box(\hat{a}) \right\} \frac{\prod_{i=1}^{n} e_{\hat{t}, \hat{a}}}{\text{vol}_{\Gamma}(\hat{a}) e_{\hat{t}, \hat{a}} - 1} \]

in \( M_{\mathfrak{A}} \) is independent of the choice of forms \( \hat{a} \).
2. The correspondence

\[ \alpha_{a} \mapsto \text{Todd}(\Gamma, a, \tau) \alpha_{a} \]
induces a well-defined injection
\[ \iota^\Gamma: \Omega^n_A \to \Omega^n_{\text{loc}}(U(\mathfrak{A})), \]
where \( \Omega^n_A \) is the image of \( q \) in degree \( n \), and \( \Omega^n_{\text{loc}}(U(\mathfrak{A})) \) is the space of holomorphic \( n \)-forms defined on the intersection of \( U(\mathfrak{A}) \) with a neighborhood of 0.

**Remark 2.2.** The number of terms in the sum in (2.3) is \( \text{vol}_{\hat{\Gamma}}(\hat{a}) \).

This result allows us to define a deformation \( \widetilde{q}^\Gamma: H^n(U(\mathfrak{A}), \mathbb{C}) \to \Omega^n_{\text{loc}}(U(\mathfrak{A})) \) of the map \( q \) via \( \widetilde{q}^\Gamma = \iota^\Gamma \circ q \). We emphasize that, while \( q \) is defined in all degrees, the deformation \( \widetilde{q}^\Gamma \) can only be defined in degree \( n \).

**Corollary 2.4.** For a nonspecial \( \tau \in \hat{\Gamma} \), the functional
\[ \widetilde{\text{CT}}_{\tau}^A : M_A \to \mathbb{C}; \quad \widetilde{\text{CT}}_{\tau}^A (f) = \sum_i \int_{Z_i} f \tilde{q}^\Gamma_i (b^i) \]
depends only on \( A, \Gamma \) and \( \tau \). Its value may be computed via the formula
\[ \widetilde{\text{CT}}_{\tau}^A (f) = \sum_{a \in B} i \text{CT} (\text{Todd}(\Gamma, a, \tau)) f, \]
where \( B \) is an orthogonal basis of \( \mathfrak{A} \).

After these preparations we may formulate the main result:

**Theorem 2.5 ([14]).** Given a compatible pair \( (\mathfrak{A}, \Gamma) \), let \( f \in R_A \) and \( \tau = e_{\iota^\Gamma} \). Then the Fourier series
\[ B^A(t) = \sum_{\gamma \in \Gamma \cap U(\mathfrak{A})} \tau(\gamma) f(\gamma) \]
defines a \( \Gamma^* \)-invariant distribution on \( V^*_R \) in the variable \( t \), which restricts to a polynomial function on each chamber of nonspecial elements. Moreover, the formula
\[ B_f^A(t) = (-1)^n \widetilde{\text{CT}}_{\tau}^A (f) \]
holds for each nonspecial \( \tau \).

**Example 2.4.** An example of the computation of this sum is
\[ B(u, v) = \sum_{m, n, m + n \in \mathbb{Z}^d} \frac{e^{um+vn}}{m^2 n^2 (m+n)^2}, \quad m, n, m + n \in \mathbb{Z}^d \]
\[ \frac{e^{(u)_{x+y} (v)_{x+y}}}{(1-e^x)(1-e^y)} x^2 y^2 (x+y)^2 + \frac{e^{(u-v)_{x+y} (v)_{x+y}}}{(1-e^x)(1-e^{x+y})} x(x+y) \frac{e^{(u-v)_{x+y} (v)_{x+y}}}{x^2 y^2 (x+y)^2} \]
\[ = \frac{i \text{CT}_x (xy)}{(1-e^x)(1-e^y)} \left( e^{(u)_{x+y} (v)_{x+y}} + e^{(1-(u-v))_{x+y} (v)_{x+y}} \right) \frac{e^{(u)_{x+y} (v)_{x+y}}}{x^2 y^2 (x+y)^2}. \]
Here underlining means multiplication by $2\pi\sqrt{-1}$, and $\{u\}$ is the fractional part of $u$. In the course of the computation, we first rescaled all variables by $2\pi\sqrt{-1}$, then performed the change of variables

$$\{x \to -x, y \to x + y, x + y \to y\}$$

in the second iterated constant term. Note that the constant term with respect to $x$ is the same thing as the constant term with respect to $-x$. The result is a piecewise polynomial function of degree 6 in the variables $u$ and $v$, with rational coefficients. The complete answer is too long to write down, but one has, for example, $B(\frac{1}{2}, \frac{1}{3}) = -\frac{197}{39191040}$. Looking at the answer (2.9), it would seem that, say, $B(0,0)$ is not well-defined, since the fractional part function $u \to \{u\}$ has a discontinuity at 0. It is clear from the Fourier series, however, that the function $B(u,v)$ is continuous. This simply means that the piecewise polynomial functions that we obtain “miraculously” agree on the common boundaries of their respective domains. In particular, setting $u = v = 0$ in one of these functions yields $B(0,0) = -\frac{1}{30240}$, whenever the domain of definition of the function contains the origin in its closure.

Now we recall the basic steps of the proof of Theorem 2.5 given in [14]. This will be instructive for the proof of our main result, Theorem 4.2. We refer the reader to the original paper [14] for details.

**S1.** Prove the case $n = 1$; this follows from the residue theorem in the complex plane.

**S2.** Prove the case $n = N$. (Recall that $|\mathfrak{A}| = N$.) This can be treated by considering the product case and then passing to a finite extension of $\Gamma$ if necessary.

**S3.** Show that if $N > n$, then the space of rational functions $R_{\mathfrak{A}}$ is spanned by subspaces $R_{\mathfrak{A}'}$, where $\mathfrak{A}'$ is a nontrivial subset of $\mathfrak{A}$. This is a variant of a partial fraction decomposition principle in several dimensions. Much stronger statements are true, cf. [7, 5]. Below we sketch the proof of a version, which will be used later in the paper.

**Proposition 2.6.** Let $(\mathfrak{A}, \prec)$ be an ordered essential central HPA, and let $\hat{\mathfrak{A}}$ be a set of representative linear forms. Then the space of $\mathfrak{A}$-rational functions $R_{\mathfrak{A}}$ is linearly spanned by functions of the form

$$g_{x_{1,a_1}^{\alpha_1}x_{2,a_2}^{\alpha_2}\cdots x_{n,a_n}^{\alpha_n}},$$

where $a \in \text{NBC}(\mathfrak{A}, \prec);$ $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}_{\geq 0}$, and $g$ is a polynomial in the variables $\{x_j,a_j|\alpha_j = 0\}$.

**Remark 2.3.** Note that if each of the exponents $\alpha_j$, $j = 1, \ldots, n$ in (2.10) is nonzero, then the only permissible numerator $g$ is a constant, which can be set to 1. The set of such functions is linearly independent and spans a vector space $G_{\mathfrak{A}}$, which is independent of the ordering. This space was introduced by Brion and Vergne in [3].
Proof: Order all subsequences of the sequence of forms \((x_1, \ldots, x_N)\) lexicographically as follows. Given two subsequences, \((y_1, \ldots, y_m), (y'_1, \ldots, y'_l) \subseteq \mathfrak{A}\), we will write \((y'_1, \ldots, y'_l) \prec (y_1, \ldots, y_m)\) if

- \(l > m\) and \(y_{m-k} = y'_{l-k}\) for \(k = \ldots, m-1\); or
- \(y'_{l-k} \prec y_{m-k}\), where \(k = \min\{k' | y_{m-k'} \neq y'_{l-k'}\}\).

Now we present an algorithm for exhibiting a rational function \(f \in R_{\mathfrak{A}}\) as a linear combination of fractions of the form described in the Proposition. Recall that a subsequence \((y_1, \ldots, y_m) \subseteq \mathfrak{A}\) is a broken circuit if there exists a minimally linearly dependent subsequence of the form \((y_0, y_1, \ldots, y_m) \subseteq \mathfrak{A}\). In other words, a subset of an ordered set of forms is a broken circuit if and only if there exists a form, preceding all the forms in the subset, which can be uniquely expressed as a linear combination of the forms in the subset. In particular, the forms in the subset itself should be linearly independent.

Without loss of generality, we can assume that \(f\) is a single rational fraction of the form

\[
    f = \prod_{x \in \mathfrak{A}} \frac{g}{x^{\varepsilon(x)}},
\]

where \(g\) as a polynomial and \(\varepsilon : \mathfrak{A} \to \mathbb{Z}_{\geq 0}\). Assuming there are broken circuits in the denominator of \(f\), i.e. among the forms \(\{x | \varepsilon(x) \neq 0\}\), denote by \(bc(\varepsilon) = (y_1, \ldots, y_m)\) the greatest of these with respect to the lexicographic ordering introduced above. Also, let \(m_\varepsilon = \min\{\varepsilon(y_i), i = 1, \ldots, m\}\) for this broken circuit, and let \(y_0 \in \mathfrak{A}\) be any element such that \(y_0 < y_1\) and \((y_0, y_1, \ldots, y_m)\) is linearly dependent.

Then using the obvious identity

\[
    \frac{1}{y_1 \cdots y_m} = -\sum_{i=1}^{m} \frac{\lambda_i/\lambda_0}{\prod_{j \in [0, m] \setminus \{i\}} y_j} \quad \text{if} \quad \sum_{i=0}^{m} \lambda_i y_i = 0, \tag{2.11}
\]

one can express \(f\) as a linear combination of other fractions. It is easy to check that if \(\varepsilon'\) is the exponent function of one of the new fractions, then \(bc(\varepsilon') \leq bc(\varepsilon)\), and if \(bc(\varepsilon') = bc(\varepsilon)\), then \(m_{\varepsilon'} < m_\varepsilon\). We can iterate this elementary step of the algorithm, applying it to each of the new terms separately. At the end, we arrive at an expression of \(f\) as a linear combination of fractions, each of which has no broken circuits among the forms in its denominator. After appropriate simplifications between the numerators and denominators, one arrives at the statement of the Proposition.

S4. The proof of (2.8) proceeds via an inductive comparison of the two sides. The induction is carried out on \(|\mathfrak{A}| = N\), the number of hyperplanes, and starts with S2. Thus assume that \(N > n\) and that (2.8) holds for all arrangements \(\mathfrak{B}\) with \(|\mathfrak{B}| < N\). According to S3, it is sufficient to consider functions \(f\) which are generically regular along some hyperplane \(H \in \mathfrak{A}\). Introduce the notation \(\mathfrak{A} \setminus H = \mathfrak{A} \setminus \{H\}\), and \(\mathfrak{A}|_H = \{H \cap L | L \in \mathfrak{A}\}\), the
restriction of $\mathfrak{A}$ onto $H$. Then $f \in R_{\mathfrak{A}\setminus H}$, and clearly, we have
\begin{equation}
B_f^{\mathfrak{A} \setminus H \Gamma}(t) = B_f^{\mathfrak{A} \setminus H \Gamma}(t) - B_f^{\mathfrak{A} \setminus H \cap H}(t|_H),
\end{equation}
where $f|_H$ and $t|_H$ are the natural restrictions. Note that the compatibility of $\mathfrak{A}$ and $\Gamma$ guarantees that $\Gamma \cap H$ is a lattice of full rank in $H$.

Then the proof of the inductive step follows from the relation of the triple for noncommutative no-broken-circuit bases \cite[Proposition 3.9]{14}. This relation is a simple generalization of the contraction-restriction relation of $\mathcal{B}$. Assume that the ordering $\prec$ of $\mathfrak{A}$, on which so far we have not imposed any conditions, satisfies the following properties:
- $H$ is the maximal element of $\mathfrak{A}$ with respect to $\prec$, i.e. $H_N = H$;
- if $(K, L, M) \in \mathfrak{A}$ and $K \cap H = M \cap H$ then $K \cap H = L \cap H$.

These properties assure that $\prec$ induces a natural ordering $\prec|_H$ on $\mathfrak{A}|_H$. Now by associating to each $H' \in \mathfrak{A}|_H$ the minimal hyperplane $K \in \mathfrak{A}$ such that $K \cap H = H'$, we can define a section $s : \mathfrak{A}|_H \to \mathfrak{A}$ of the canonical map $\mathfrak{A} \to \mathfrak{A}|_H$. Let $s_H : \mathfrak{A}|_H |^{n-1} \to \mathfrak{A}^n$ be the map induced by $s$ on $\mathfrak{A}|_H |^{n-1}$, composed with appending $H$ at the end of the resulting sequence from $\mathfrak{A}|_H |^{n-1}$. Then the relation of the triple reads:
\begin{equation}
\text{NBC}(\mathfrak{A}, \prec) = \text{NBC}(\mathfrak{A}\setminus H, \prec) \cup s_H(\text{NBC}(\mathfrak{A}|_H, \prec|_H)).
\end{equation}

The comparison of (2.12) and (2.13) implies the inductive step. This concludes the proof of the Theorem.

Note the key idea of the proof: we can use the flexibility provided by Propositions 2.1 and 2.2 to choose a convenient ordering depending on the function $f$. This ordering then provides us with a simple expression for the right hand side of (2.8).

2.5. Bernoulli Polynomials. The function $B_f^{\mathfrak{A} \setminus H}(t)$ introduced in Theorem 2.3 is manifestly $\Gamma^\star$-invariant, thus it cannot be a polynomial unless it is constant. The Theorem states that it coincides with a polynomial when restricted to a chamber of nonspecial elements; this polynomial will vary from chamber to chamber, however. The resulting functions were termed multiple Bernoulli polynomials in \cite{14}.

Let $\mathfrak{A}, \Gamma, f$ be as in the Theorem. For a nonspecial $u \in V_\mathbb{R}^\star$ denote by $P^{\mathfrak{A}|_H}_{[u]}(t)$ the polynomial function which coincides with the restriction of $B_{f|_H}^{\mathfrak{A} \setminus H}(t)$ to the unique chamber to which $u$ belongs. Our immediate goal is to write down a version of (2.8) with the LHS replaced by $P^{\mathfrak{A}|_H}_{[u]}(t)$. To see what needs to be changed on the RHS, recall that the essential ingredient in the definition of $\tilde{\mathfrak{A}}_{\Gamma^\star}$ was the Todd function (2.3).

Set $t = u$ and rewrite the exponential sum in the numerator as
\[
\sum \{ e_{u+w} \mid w \in \Gamma^\star, \ u + w \in \Box(\tilde{\mathfrak{A}}) \}. 
\]
Then it is clear that in order to have a global polynomial such as \( P_{[u],f}(t) \), we need to replace this sum by

\[
\sum \{ e_{t+w} \mid w \in \Gamma^*, \; u + w \in \Box(\hat{a}) \}.
\]

It will be convenient to write this modified definition of the Todd function in terms of the parameter \( \mu = t - u \), the shift vector:

\[
\text{Todd}_\mu(\Gamma, a, \tau) = \sum \{ e_{\hat{t}} \mid e_{\hat{t}} = \tau, \; \hat{t} - \mu \in \Box(\hat{a}) \} \prod_{i=1}^{n} \frac{\tau_i \hat{a}}{e_i \hat{a} - 1}.
\]

(2.14)

It is easy to check that this definition gives rise to a consistent deformation \( \tilde{q}_{\mu,\tau}^\Gamma \) of \( q \) and to a deformed constant term functional \( \tilde{CT}^\Gamma_{\mu,\tau} \). Then the new variant of Theorem 2.5 reads:

**Theorem 2.7.** Let \( \mathfrak{A}, \Gamma \) be compatible, and \( f \in R_{\mathfrak{A}} \). Let \( u, t \in V^*_R \), with \( u \) nonspecial, and set \( \mu = t - u \). Then

\[
P_{[u],f}(t) = (-1)^n \tilde{CT}^\Gamma_{\mu,\tau}(f).
\]

(2.15)

An advantage of this formulation is that it allows us to evaluate the function \( B_f^\mathfrak{A} \) at special values of \( t \) as well.

**Corollary 2.8.** Assume that \( \mathfrak{A} \) and \( \Gamma \) are compatible and \( f \in R_{\mathfrak{A}} \) is such that the series (2.7) defining \( B_f^\mathfrak{A} \) is absolutely convergent. Then for an arbitrary, possibly special \( t \in V^*_R \) one has

\[
B_f^\mathfrak{A}(t) = (-1)^n \tilde{CT}^\mathfrak{A}_{\mu,\tau}(f),
\]

(2.16)

where, as usual, \( \tau = e_{\hat{t}} \mid \Gamma \), and \( \mu \) is a sufficiently small vector in \( V^*_R \) such that \( t - \mu \) is not special.

When \( \mu = 0 \), then this statement coincides with the statement in Theorem 2.5. We will give an example of a calculation where \( t \) is special in the next section.

### 3. Affine arrangements

Here we describe a generalization of the results of the previous section to affine arrangements.

An affine hyperplane is one that does not necessarily go through the origin. For each such hyperplane \( H \), there is a unique parallel hyperplane \( H^0 \) containing the origin. Similarly, for each affine linear form \( x \) there is a unique linear form \( x^0 \) such that \( x - x^0 \) is a constant.

Let \( \mathfrak{A} \) be a collection of affine hyperplanes in the \( n \)-dimensional vector space \( V \). Consistently with the notation introduced above, we have a central HPA \( \mathfrak{A}^0 \) consisting of translates of the elements of \( \mathfrak{A} \), and for each \( n \)-tuple \( \mathbf{a} \in \mathfrak{A}^n \) we have a corresponding \( \mathbf{a}^0 \in \mathfrak{A}^0^n \). We will use a similar notation for \( n \)-tuples of forms. Note that it is possible that \( |\mathfrak{A}| > |\mathfrak{A}^0| \), since we allow parallel planes.
We will call an affine HPA $\mathfrak{A}$ and a lattice $\Gamma$ compatible if $\mathfrak{A}^0$ and $\Gamma$ are. In this situation we will assume that the chosen representative forms satisfy $\hat{\mathfrak{A}}^0 \subset \Gamma^*$, and for $y \in \hat{\mathfrak{A}}$ one has $y^0 \in \hat{\mathfrak{A}}^0$.

Define $p \in V$ to be a vertex of $\mathfrak{A}$ if $\cap \mathfrak{A}_p = \{p\}$, where $\mathfrak{A}_p = \{H \in \mathfrak{A} | p \in H\}$. We assume that the set $\text{vx}(\mathfrak{A})$ of vertices of $\mathfrak{A}$ is nonempty. Note that it is not assumed that the constants $x - x^0$ are real, thus it is possible that $p \notin V^\mathbb{R}$.

The generalization now goes as follows. As each arrangement $\mathfrak{A}_p$ is essentially a central arrangement, it has its own constant term functional $\text{CT}^\mathfrak{A}_p$ taken at the point $p \in V$. Then we define the constant term of $\mathfrak{A}$ by

$$\text{CT}^{\mathfrak{A}} = \sum_{p \in \text{vx}(\mathfrak{A})} \text{CT}^{\mathfrak{A}_p}.$$  

The definition of the deformation of $\text{CT}^{\mathfrak{A}_p}$ is similar to the deformation in the central case. One needs to generalize the definition of (2,14) for an $n$-tuple $\mathbf{a}$ of affine linear forms as follows:

$$\text{Tod}_{\mathbf{a}}(\Gamma, \mathbf{a}, \tau) = \frac{\sum \{e_i | e_i \Gamma = \tau, \dot{t} - \mu \in \square(\hat{\mathfrak{a}}^0)\}}{\text{vol}(\hat{\mathfrak{a}}^0)} \prod_{i=1}^n \frac{x_i}{x_i^{\hat{\mathfrak{a}}} - 1},$$

The statements of Proposition 2.3 still hold, since the affine forms in $\hat{\mathfrak{A}}_p$ satisfy exactly the same linear relations as the ones in $\hat{\mathfrak{A}}^0$. Merging (3.1) with the results of the previous section, (2.6), (2.14), (2.16), etc., we can write down the analogous equalities in the affine case:

$$\text{CT}^{\mathfrak{A} \Gamma}_{\mu, \tau}(f) = \sum_{p \in \text{vx}(\mathfrak{A})} \text{CT}^{\mathfrak{A}_p \Gamma}_{\mu, \tau}(f),$$

$$\text{CT}^{\mathfrak{A}_p \Gamma}_{\mu, \tau}(f) = \sum_{\mathbf{a} \in \mathbf{B}_p} \text{ICT} \left( \text{Tod}_{\mathbf{a}}(\Gamma, \mathbf{a}, \tau) f \right),$$

where $\mathbf{B}_p$ is an orthogonal basis of $\mathfrak{A}_p$ consisting of affine linear forms.

Under the same conditions as in Corollary 2.8 and using (3.3) and (3.4), we have

$$B^\mathfrak{A} \Gamma_f(t) = (-1)^n \text{CT}^{\mathfrak{A} \Gamma}_{\mu, \tau}(f),$$

As we will prove a more complicated version of this statement in the trigonometric case, the proof of this equality will be omitted.

Remark 3.1. 1. Note that this statement is a common generalization of the results of [14], sketched in the previous section, and the formulae of Brion-Vergne given in [6], where, in particular, the case of a single vertex at a generic point was considered.

2. It is very important that in (2.14) the affine forms $\hat{\mathfrak{a}}$ appear in $x_i^{\hat{\mathfrak{a}}}$ instead of the corresponding linear forms, as is the case in the rest of the formula. As an exercise, the reader may check that if $\mathfrak{A} = \mathfrak{A}_p$ has a single vertex $p \notin \Gamma$, then $\text{CT}^{\mathfrak{A}_p \Gamma}_{\mu, \tau}(1) = 0$. 

We end this short section with an example.

**Example 3.1.** We compute the sum

\begin{equation}
\sum \frac{1}{mn(2m+n-1)}, \quad m, n \in \mathbb{Z} \neq 0, \quad 2m + n \neq 1.
\end{equation}

Here $\mathcal{A} = \{ x, y, 2x + y - 1 \}$, $\Gamma = \mathbb{Z}^2$ and $\tau$ is the trivial character. This series converges absolutely, albeit painfully slowly. The arrangement has 3 simple vertices, each one contributing a single constant term. Since $\tau$ is singular, we need to choose a small shift vector $\mu$. We choose $\mu = -\epsilon y - \epsilon^2 x$, where $\epsilon$ is a small positive number. If we choose $\mu$ differently, the formulas change somewhat, but the result, naturally, remains the same.

Again, rescaling the variables by $2\pi \sqrt{-1}$ and dividing the sum by $(2\pi \sqrt{-1})^3$, we arrive at the result

\begin{equation}
\begin{aligned}
&\text{CT} \frac{1}{x, y} \frac{1}{(1-e^x)(1-e^y)(2x+y-1)} + \text{CT} \frac{1}{x, 2x+y-1} \frac{1}{(1-e^x)(1-e^{2x+y})y} \\
&+ \text{CT} \frac{1}{2x+y-1} \frac{(e^y + e^{x+y})}{y} \frac{(e^y + e^{x+y})}{(1-e^y)(1-e^{2x+y})x}.
\end{aligned}
\end{equation}

As all three vertices are simple, i.e. each is contained in exactly two hyperplanes, one can replace the iterated constant terms with the ordinary constant term. After the appropriate substitutions we obtain

\begin{equation}
\text{CT} \frac{1}{(1-e^x)(1-e^y)} \left( \frac{1}{2x+y-1} + \frac{1}{y-2x+1} + \frac{e^y - e^{x+y}}{x-y+1} \right) = \frac{\pi^2 - 8}{(2\pi \sqrt{-1})^3}.
\end{equation}

Thus the value of our infinite sum (3.6) is $\pi^2 - 8$. \hfill \Box

While it is possible to compute this answer by some ad hoc method as well, the advantage of our formula is that it does not become more complicated as the powers of the linear forms in the denominator of the function $f$ increase.

### 4. The trigonometric case

#### 4.1. Toric arrangements and rational trigonometric sums.

In this section we present a periodic version of the theory, i.e. when hyperplanes are replaced by hypertori on a torus. We keep the notation of the previous section: $\mathfrak{A}$ is an essential affine HPA and $\mathfrak{A}^o$ is the associated central arrangement in a complex vector space $V$. Let $\Theta \subset V$ be a compatible rank-$n$ lattice and denote by $T$ the complexified torus $V/\Theta$. Recall that $\Theta$ defines a real subspace $V_\mathbb{R} \subset V$.

In view of the compatibility condition, each $H \in \mathfrak{A}$ defines a hypertorus $H + \Theta \subset T$; this results in an arrangement $\mathfrak{A}/\Theta$ of hypertori in $T$. It will be convenient to work with the inverse image of this arrangement under the natural map $V \to T$. This inverse image is an infinite, periodic HPA on $V$, which we denote by $\mathfrak{A}^+/\Theta$. Recall that $\mathfrak{A}^+/\Theta$ stands for the arrangement
of those hyperplanes from $A^+\Theta$ which go through the point $p \in V$. The arrangement $A^+\Theta$ has infinitely many vertices, but we are interested in them up to a translation by a vector from $\Theta$ only; define $\text{vx}(A/\Theta) = \text{vx}(A^+\Theta)/\Theta$. Then elements of $\text{vx}(A/\Theta)$ are the vertices of the toric arrangement $A/\Theta$.

Fix an ordering $\prec$ on $A^o$, choose a set $\hat{A}^o$ of representative forms which are minimal elements of $\Theta^*$, and let $\hat{A}$ be the corresponding the of affine linear forms.

Now we introduce the periodic version of $A$-rational functions. Let $\mathbb{C}[\hat{\Theta}]$ be the polynomial ring generated by the functions $e_y$, $y \in \Theta^*$, and let $\mathbb{C}_A[\hat{\Theta}]$ be the same ring, with the functions $1 - e_y$ inverted whenever $y \in \hat{A}$. We will think of these spaces as of spaces of functions on $T$ or periodic functions on $V$, interchangeably. The elements of $\mathbb{C}_A[\hat{\Theta}]$ will be called \textit{trigonometric} $(A,\Theta)$-rational functions.

Note that the constant term functional $\text{CT}_{A^+\Theta}$ restricted to $\mathbb{C}_A[\hat{\Theta}]$ is independent of the choice of the vertex $p \in \text{vx}(A^+\Theta)$ modulo $\Theta$. Then, just as in the previous section, we may define a constant term functional $\text{CT}_{A/\Theta}: \mathbb{C}_A[\hat{\Theta}] \to \mathbb{C}$ given by the finite sum

$$\text{CT}_{A/\Theta} = \sum_{p \in \text{vx}(A/\Theta)} \text{CT}_{A^+\Theta}.$$

Here we took up the somewhat sloppy convention that summation over $p \in \text{vx}(A/\Theta)$ means taking a representative $p$ from each of the $\Theta$-equivalent classes of vertices of $A^+\Theta$.

Now let $\Gamma$ be a lattice containing $\Theta$, and let $f \in \mathbb{C}_A[\hat{\Theta}]$. Fix a character $\tau \in \text{Hom}(\Gamma/\Theta, U(1))$; this may be thought of as an element of $\hat{\Gamma}$ which is trivial on $\Theta$. We are interested in rational trigonometric sums of the form

$$Z^\text{ar}/\Theta(\tau) = \sum_{\gamma \in (\Gamma \cap U(A^+\Theta))/\Theta} \tau(\gamma)f(\gamma).$$

(4.1)

The interest in such sums was sparked by a formula given by E. Verlinde for the dimension of conformal blocks of the WZW theory [18]. We will study this formula later in the paper. We would like to write down a localized formula for the sums (4.1) similar to (2.8). Recall that, originally, when we treated the rational case in §2.4, we excluded the special characters. This was partly justified because the Fourier series $B^\text{ar}_f(t)$ may very well have singularities for special values of $t$. Later we treated the case of special values in §2.5.

In the trigonometric case, which we are investigating in this section, we cannot exclude the special characters since every sum is meaningful. Here, however, there is no applicable notion of continuity, such as the one used in §2.5.

It turns out that the shift vector $\mu$, which played an auxiliary role in §2.5, in the trigonometric case becomes essential. Going back to (3.2) and (3.3), we can write down a natural definition for the deformed constant term of
the toric arrangement $\mathcal{A}/\Theta$ with a $\mu$-shift:

$$\tilde{C}T_{\mu,\tau}^\mathcal{A}/\Theta = \sum_{p \in \text{vx}(\mathcal{A}/\Theta)} \tilde{C}T_{\mu,\tau}^\mathcal{A}/\Theta p,$$

where the functional $\tilde{C}T_{\mu,\tau}^\mathcal{A}/\Theta p$ is defined in (3.4), but here we consider it to be restricted onto $C_\mathcal{A}[\tilde{\Theta}]$.

Turning to the function $f \in C_\mathcal{A}[\tilde{\Theta}]$, define

$$(4.2) \quad \Delta_f = \{ \mu \in V_\mathcal{A}^* \mid \text{for } u \in V_\mathcal{A} \text{ not a pole of } f \text{ and for all nonzero } v \in V_\mathcal{A} \lim_{s \to +\infty} e_{\mu}(\sqrt{-1}(u + sv))f(\sqrt{-1}(u + sv)) = 0 \}.$$ 

Note that this definition is independent of the HPA, and depends on the function only. We introduce the associated linear spaces

$$C_\mathcal{A}\mu[\tilde{\Theta}] = \{ f \in C_\mathcal{A}[\tilde{\Theta}] \mid \mu \in \Delta_f \}.$$ 

By definition, after putting its terms over a common denominator, any $f \in C_\mathcal{A}[\tilde{\Theta}]$ may be represented in the form

$$(4.3) \quad f = \frac{\sum_{w \in \Theta^*} \lambda_w e_w}{\prod_{y \in \mathcal{A}} (1 - e_y)^{\varepsilon(y)^\circ}},$$

where $\varepsilon : \mathcal{A} \to \mathbb{Z}^{\geq 0}$ and the complex numbers $\lambda_w$ vanish for all but finitely many $w \in \Theta^*$. Introduce the finite set $N_{f,\varepsilon} = \{ w \in \Theta^* \mid \lambda_w \neq 0 \}$ and the convex polytope $D_\varepsilon = \{ \sum_{y \in \mathcal{A}} \nu(y)\varepsilon(y)^\circ \mid 0 < \nu(y) < 1 \}$. We will use the convention that $\bar{D}_\varepsilon$ is the closure of $D_\varepsilon$ unless $\varepsilon$ is 0, i.e. when there is nothing in the denominator. Then $D_\varepsilon = \emptyset$ and $\bar{D}_\varepsilon = \{0\}$.

**Proposition 4.1.** Given a function $f \in C_\mathcal{A}[\tilde{\Theta}]$ in the form (4.3), one has $\mu \in \Delta_f$ if and only if $D_\varepsilon$ is a nonempty open subset of $V_\mathcal{A}^*$, and $\mu + w \in \bar{D}_\varepsilon$ for every $w \in N_{f,\varepsilon}$.

Note that the set $D_\varepsilon$ is nonempty and open exactly when the linear forms which actually appear in the denominator span $V_\mathcal{A}^*$. To put the statement in formulas, introduce

$$(4.4) \quad \Delta_f^0 = \bigcap_{w \in N_{f,\varepsilon}} (D_\varepsilon - w) \quad \text{and} \quad \bar{\Delta}_f = \bigcap_{w \in N_{f,\varepsilon}} (\bar{D}_\varepsilon - w).$$

Then the statement is that $\Delta_f = \text{int}(\Delta_f^0) = \text{int}(\bar{\Delta}_f)$, where $\text{int}(S)$ stands for the interior of the set $S$. In fact, if $D_\varepsilon$ is nonempty and open, then $\Delta_f = \Delta_f^0$, and the closure of $\Delta_f$ is $\bar{\Delta}_f$.

**Proof of Lemma:** The “if” part is easy, because if the conditions on $\mu$ in the Proposition hold, then one can represent each term in $e_{\mu}f$ as a product of functions of the form $e_{\nu(y)^\circ}/(1 - e_y)$, where $0 < \nu_y < 1$ and $y \in \mathcal{A}$. As long as the linear forms $\{y^\circ \mid \varepsilon(y) \neq 0\}$ do not lie in a hyperplane in $V_\mathcal{A}^*$, for
any nonzero \( v \in V_{\mathbb{R}} \) there will be at least one form \( y \) for which \( y^0(v) \neq 0 \). In this situation the condition in (4.2) clearly holds.

For the “only if” part note that if there is a nonzero \( v \in V_{\mathbb{R}} \) such that \( y^0(v) = 0 \) for each \( y \) in the denominator, then for such \( v \) the condition in (4.2) cannot hold, thus it is necessary that \( D_{\varepsilon} \) be nonempty and open. We assume this from now on.

Now, ad absurdum, suppose that there is a \( w \in N_{f,\varepsilon} \), for which \( \mu + w \notin D_{\varepsilon} \).

Since \( D_{\varepsilon} \) is convex, there is a hyperplane separating \( \mu + w \) from \( D_{\varepsilon} \). In other words, there is a \( v \in V_{\mathbb{R}} \) such that

\[
\mu(v) + w(v) \geq \sum_{y \in \hat{A}} \varepsilon(y)(y^0(v) + |y^0(v)|)/2.
\]

Clearly, when we restrict \( f \) to the line \( \sqrt{-1}(u + sv) \) for some \( u \) and let \( s \to -\infty \), then the dominant contribution in (4.3) comes from those terms of the form \( \lambda w e^w/\prod_{i=1}^{M}(1 - e^y_i) \) for which \( w(v) \) has the maximal value \( m = \max_{w \in N_{f,\varepsilon}} w(v) \). Denoting the set of such \( w \)s by \( M_{v}(f) = \{ w \in N_{f,\varepsilon} | w(v) = m \} \), we can write the dominant contribution as

\[
e^{-2\pi(m + \mu(v))s} \prod_{y \in \hat{A}} (1 - a_y e^{-2\pi y^0(v)s}) \sum_{w \in M_{v}(f)} \lambda w e^w(u),
\]

where \( a_y \) is a nonzero constant: \( a_y = e^{-2\pi y(u)} \). Note that for generic \( u \), the coefficient in front of the fraction is not 0 because \( M_{v}(f) \) is nonempty. Thus this dominant contribution does not vanish as \( s \to -\infty \) since \( m + \mu(v) \geq \sum_{y \in \hat{A}} \varepsilon(y)(y^0(v) + |y^0(v)|)/2 \). This means that \( f \) does not satisfy (4.2), and this contradicts our assumption. The proof is complete.

4.2. The main result. Now we can formulate the main result of the paper:

**Theorem 4.2.** Let \( \Theta \subset \Gamma \) be lattices compatible with an affine HPA \( \mathfrak{A} \) in an \( n \)-dimensional complex vector space \( V \). Let \( f \in C^{\mu}_{\mathfrak{A}}(\hat{\Theta}) \) for some \( \mu \in V_{\mathbb{R}}^* \).

Then for \( t \in \Theta^* \) such that \( t - \mu \) is not \( \Gamma \)-special, the identity

\[
Z_{f}^{\mathfrak{A}/\Theta}(\tau) = \widetilde{C T}_{\mu,\tau}(f)
\]

holds, where \( \tau = e_t|_{\Gamma/\Theta} \).

For computational purposes the iterated constant term form of \( \widetilde{C T}_{\mu,\tau}(f) \) is useful:

\[
Z_{f}^{\mathfrak{A}/\Theta}(\tau) = \sum_{p \in v(\mathfrak{A}/\Theta)} \sum_{a \in B_{p}} i CT(\text{Todd}_{\mu}(\Gamma, a, \tau)f),
\]

where \( B_{p} \) is an orthogonal basis of the arrangement \( \mathfrak{A}_{p}^{\Theta} \) (see the definitions at the start of this section).
Example 4.1. The simplest nontrivial example with multiple vertices is provided by the arrangement corresponding to the Lie algebra $B_2$. Set $\mathfrak{A} = \{x, y, x + y, x - y\}$, $\Theta = \mathbb{Z}^2$, $\Gamma = \Theta/k$. Introduce the shorthand

$$T(x) = (1 - e^x)(1 - e^{-x}), \quad \delta(x, y) = \frac{1}{T(x)T(y)T(x + y)T(x - y)}.$$

Let $f(x, y) = \delta(x, y)$ and $\tau = 1$. Our aim is to compute the sum

$$Z_{\mathfrak{A}/\Theta}^\mathfrak{A}/\Theta (\tau) = \sum_{\Gamma} f(/ik, /jk), \quad 0 < i \neq j < k, \ i + j \neq k.$$

The vertices of $\mathfrak{A}/\Theta$ are at $(0, 0)$ and at $(\frac{1}{2}, \frac{1}{2})$, with corresponding orthogonal bases $\{(x, y), (x, x + y), (x, x - y)\}$ and $(x + y - 1, x - y)$. Now we need to choose a $\mu \in \Delta_f$. We will take one of the simplest choices: $\mu = -\epsilon x - \epsilon^2 y$, but note that we might have picked, for example, $3x - y - \epsilon x - \epsilon^2 y$, which would have resulted in a rather different formula.

Again, we rescale by $2\pi \sqrt{-1}$ and obtain

$$i CT_{x,y} \frac{k^2 xy \delta(x, y)}{(1 - e^{kx})(1 - e^{ky})} + i CT_{x,x+y} \frac{k^2 (x + y) \delta(x, y)}{(1 - e^{kx})(1 - e^{kx+ky})}$$

$$+ i CT_{x,x-y} \frac{k^2 x(y - x) \delta(x, y)}{(1 - e^{kx})(1 - e^{ky-2kx})} + i CT_{x+y-x-y} \frac{k^2 (x + y - 1)(x - y)(1 + e^{kx}) \delta(x, y)}{2(1 - e^{kx+ky})(1 - e^{kx-kx})}.$$

After the usual change of variables this equals

$$k^2 i CT_{x,y} \frac{xy}{(1 - e^{kx})(1 - e^{ky})} \left( \delta(x, y) + \delta(x, y - x) + \delta(x, x + y) + \frac{1 + (-1)^{k} e^{\frac{kx + ky}{2}}}{2} \delta \left( \frac{x + y + 1}{2} - \frac{x - y + 1}{2} \right) \right).$$

The answer is the somewhat intimidating

$$\frac{k^8 + 60k^6 + 5523k^4 + 133377/2 + (-1)^k(525k^4 + 5250k^2 + 30975/2)}{240 \cdot 8!},$$

which, nevertheless, does reduce to $0, 1/8, 8/25, 10/9$ for $k = 3, 4, 5, 6$, respectively.

If $\Delta_f = \emptyset$, then the statement of the Theorem is vacuous. One can still use (1.5) to compute any rational trigonometric sum using the identity

$$1 = \frac{1}{1 - z} + \frac{1}{1 - z^{-1}}.$$

Indeed, the identity clearly implies

Lemma 4.3. The space $\mathbb{C}_{\mathfrak{A}}(\Theta)$ is linearly spanned by the linear spaces

$$\{\mathbb{C}_{\mathfrak{A}}^\mu(\Theta) | \mu \in V^*_\mathfrak{A} \}.$$
Thus any $f \in C[A[\hat{\Theta}]]$ with $\Delta_f = 0$ can be represented as a sum of terms, each of which has a nonempty $\Delta$, and then one can apply the theorem to each term separately. The simplest example of this is

$$
(4.9) \sum_{\omega^k=1, \omega \neq 1} 1 = CT \left[ \frac{kx}{1 - e^{kx}} \frac{1}{1 - e^{-x}} \right] + CT \left[ \frac{kxe^{kx}}{1 - e^{kx}} \frac{1}{1 - e^{x}} \right] = \frac{k - 1}{2} + \frac{k - 1}{2} = k - 1.
$$

4.3. The proof. The proof of the theorem is parallel to that of Theorem 2.5.

Step 1. For the case of $n = 1$ we may identify $V$ with $C$ and $\Theta$ with $\mathbb{Z}$. Denote the coordinate on $T \simeq \mathbb{C} \setminus \{0\}$ by $z$. The following statement is an immediate consequence of the residue theorem in $\mathbb{C}$:

Lemma 4.4. Let $F(z)$ be a rational function, $k \in \mathbb{Z}^\geq 0$ and $l \in \mathbb{Z}$. Assume that $l'$ is an integer, such that $l' \equiv l \mod k$ and $z^l F(z)/(1 - z^k)$ vanishes both at $0$ and at $\infty$. Then we have

$$
\sum_{\omega^k=1, \omega \neq 1} \omega^l F(\omega) = \sum_{p \in \text{Pole}(F)} \text{Res}_{z=p} \frac{dz}{z} \frac{dz}{z} F(z),
$$

where $\text{Pole}(F)$ is the set of poles of the function $F$.

Now the rank-1 case of (2.8) easily follows after performing the change of variables $z \to e^{z}$.

Step 2. Here we consider the case $|A^0| = n = \dim V$. Let $x_1^0, \ldots, x_n^0 \in \Theta^*$ be minimal defining linear forms for $A^0$ and let $\beta_1 x_1^0, \ldots, \beta_n x_n^0$ be the corresponding minimal elements of $\Gamma^*$. Note that all the $\beta$s are integers since $\Gamma^* \subset \Theta^*$. Define the lattices $\Theta_A$ and $\Gamma_A$ through their duals:

$$
\Theta_A^* = \left\{ \sum_{i=1}^n k_i x_i^0 \bigg| k_i \in \mathbb{Z} \right\}, \quad \Gamma_A^* = \left\{ \sum_{i=1}^n k_i \beta_i x_i^0 \bigg| k_i \in \mathbb{Z} \right\}.
$$

Then we have $\Theta \subset \Theta_A$ and $\Gamma \subset \Gamma_A$, but not necessarily $\Theta_A \subset \Gamma$.

We prove (4.5) in three steps. We fix the data of the function $f \in C_A[\hat{\Theta}]$ and $\tau \in \text{Hom}(\Gamma/\Theta, U(1))$.

One can easily see that the equality (4.5) holds for the pair of lattices $\Theta_A \subset \Gamma_A$, since in this case both sides are simply products of the 1-dimensional case proved in Step 1. To apply the theorem in this case, we need to assume that $\tau$ is trivial on $\Theta_A$.

Replacing $\Theta_A$ with $\Theta$ is painless; if $\tau$ is trivial on $\Theta_A$, then both sides of the equality are simply multiplied by $|\Theta_A/\Theta|$ because of the $\Theta_A$-periodicity of the data. Similarly, it is easy to see that if $\tau$ is nontrivial on $\Theta_A$, then both sides of (4.5) vanish because they are multiplied by the sum of the values of a nontrivial character of $\Theta_A/\Theta$. 
To pass from $\Gamma_A$ to $\Gamma$, first recall the fact that for a finite group $G$
\begin{equation}
\sum_{\chi \in R(G)} \chi(g) = \begin{cases} |G|, & \text{if } g = e, \\
0, & \text{otherwise}, \end{cases}
\end{equation}
where $R(G)$ is the set of irreducible characters and $e$ is the unit element.
Applying this to the group $\Gamma_A/\Gamma$, one can see that
\begin{equation}
Z_f^{\Gamma_A/\Gamma}(\tau) = \frac{1}{|\Gamma_A/\Gamma|} \sum_{\tau' \in \Gamma/\Gamma_A} Z_f^{\Gamma_A/\Gamma}(\tau\tau').
\end{equation}
Checking the $\tilde{\text{CT}}$ side, we see that the change in the exponential sum in the definition (2.14), when $\Gamma_A$ is replaced by $\Gamma$, reproduces the same relation:
\begin{equation}
\tilde{\text{CT}}_{\tau,\mu}(f) = \frac{1}{|\Gamma_A/\Gamma|} \sum_{\tau' \in \Gamma/\Gamma_A} \tilde{\text{CT}}_{\tau\tau',\mu}(f).
\end{equation}
Thus knowing (4.5) for $\Gamma_A$ implies the same equality for $\Gamma$.

Step 3. While the other steps are substantially analogous to the rational case, here there are significant differences. One problem is that the direct analog of the partial fractions approximation principle formulated in Proposition 2.6 does not hold, i.e. the space of functions which have their poles on essential subsets of $\mathfrak{A}/\Theta$ does not span $C_{\mathfrak{A}}[\Theta]$. Also, even if such decomposition $f = \sum g_i$ existed, we would need to make sure that we have $\mu \in \Delta_{g_i}$ for each summand $g_i$.

We start with the trigonometric analog of (2.11).

**Lemma 4.5.** Let $y_0, \ldots, y_m$ be affine linear forms on $V$, satisfying $y_0 + y_1 + \cdots + y_m = 0$, and such that the linear forms $\{y^o_i\}_{i=0}^m$ are in $V^*_R$. Let $\tilde{\mu} = \sum_{i=1}^m \alpha_i y^o_i$ with
\begin{equation}
\begin{split}
0 < \alpha_1 < \cdots < \alpha_m < 1.
\end{split}
\end{equation}
Then the identity
\begin{equation}
\sum_{i=0}^m \prod_{j=0}^{i-1} e_{y_j} = 0
\end{equation}
holds, and every term in the sum is a trigonometric rational function which has $\tilde{\mu}$ in its $\Delta^0$ (cf. (4.4)).

**Remark 4.1.** We had to allow for the possibility of $m < n$ in this Lemma, hence the somewhat awkward formulation using $\Delta^0$ instead of $\Delta$.

**Proof.** The identity maybe easily checked by multiplying through with $\prod_{i=0}^m (1 - e_{y_i})$. To check that $\tilde{\mu}$ is in the $\Delta^0$ of the $i$th term, we express $\tilde{\mu} + \sum_{j=0}^{i-1} y^o_j$ as a linear combination of the $y^o$ variables less $y^o_i$:
\begin{equation}
\tilde{\mu} + \sum_{j=0}^{i-1} y^o_j = (1 - \alpha_i) y^o_0 + \sum_{j=1}^{i-1} (1 - \alpha_i + \alpha_j) y^o_j + \sum_{j=i+1}^m (\alpha_j - \alpha_i) y^o_j.
\end{equation}
Since all the coefficients of this linear combination are between 0 and 1, the statement now follows from the definition of $\Delta^0$. \hfill \Box

Using the Lemma we can adapt the algorithm in the proof of Proposition \ref{p:2.6} to the trigonometric case, as follows. Take a $\mu$ which is not special with respect to $\mathcal{A}$ and $\Theta$, and fix a function $f \in C^\mu[\hat{\Theta}]$. Observe that for a nonspecial $\mu$, the space $C^\mu[\hat{\Theta}]$ does not change as we vary $\mu$ in a small neighborhood. In other words, we may switch $\mu$ for a nearby vector if it is necessary.

According to Proposition \ref{p:4.1}, $f$ may be represented as a sum of elements of $C^\mu[\hat{\Theta}]$ of the form

$$e_t \prod_{y \in \hat{\mathcal{A}}} (1 - e_y)^{\varepsilon(y)}, \tag{4.12}$$

where $\varepsilon : \hat{\mathcal{A}} \to \mathbb{Z}_{\geq 0}$. We may assume without loss of generality that $f$ has this form to begin with.

Following the blueprint of the partial fraction decomposition in the rational case, we again find an ordered $m$-tuple $(y_1, \ldots, y_m)$ among $\{y \mid \varepsilon(y) \neq 0\}$, such that $(y_1^0, \ldots, y_m^0)$ is the largest possible broken circuit with respect to the lexicographic ordering. Then, by definition, there is $y_0^0 \in \hat{\mathcal{A}}$ such that $y_0^0 \prec y_1^0$, and a relation $\sum_{j=0}^m \lambda_j y_j = 0$ holds, where $\lambda_j \in \mathbb{Z}$, $j \in \{0, \ldots, m\}$. Now let

$$\tilde{y}_i = \lambda_i y_i, \quad i \in \{1, \ldots, m\} \quad \text{and} \quad \tilde{y}_0 = -\tilde{y}_1 - \cdots - \tilde{y}_m.$$  

By separating the factors in the denominator corresponding to this broken circuit and applying the formula for finite geometric progressions, we can write

$$f = \frac{e_t}{\prod_{y \in \hat{\mathcal{A}}} (1 - e_y)^{\varepsilon(y)}} \prod_{j=1}^m \frac{1 + e_{y_j} + \cdots + e_{\lambda_{j-1} y_j}}{1 - e_{\tilde{y}_j}}. \tag{4.13}$$

Expanding the numerator, we obtain a representation of $f$ as a sum of terms of the form \ref{eq:4.12}, with each term containing $\mu$ in its $\Delta$. We will consider each of these terms separately.

Using Proposition \ref{p:4.1} yet again, we see that $\mu$ may be split into a sum of two contributions: $\mu = \mu' + \mu''$, with $\mu'$ in the $\Delta^0$ of the first part of \ref{eq:4.13} and $\mu''$ is in the $\Delta^0$ of the product. Now we use our freedom of varying $\mu$ in a small neighborhood to make sure that $\mu''$ is not special with respect to $(\tilde{y}_0^0, \ldots, \tilde{y}_m^0)$; this implies that the coefficients $\alpha_i$ in the representation $\mu'' = \sum_{i=1}^m \alpha_i \tilde{y}_i^0$ are all different. Indeed, if, say, we had $\alpha = \alpha_1 = \alpha_2$, then we could represent $\mu''$ as a linear combination of the variables $\tilde{y}_0, \tilde{y}_3, \tilde{y}_4, \ldots, \tilde{y}_m$ only, by subtracting $0 = \alpha \sum_{j=0}^m \tilde{y}_j$ from the original representation.

Now we renumber the variables $\tilde{y}_j$ according to the increasing order of the coefficients $\alpha_j$, and then apply Lemma \ref{p:4.3} to $(\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_m)$ and $\tilde{\mu} = \mu''$. This allows us to use \ref{eq:4.11} to eliminate the broken circuit. Again, similarly to the proof of Proposition \ref{p:2.6}, we conclude that after performing
this transformation of the representation of \( f \), the parameter

\[
\min_{1 \leq i \leq m} \left\{ \sum \varepsilon(y), \ y^o = y_i^o \right\}
\]

in the representation (4.12) decreases, and the lexicographically largest broken circuit among the forms in the denominator does not increase.

Note, however, that our manipulations have a price. As the linear forms \( \tilde{y}_j, j = 0, \ldots, m \), are not necessarily in \( \hat{A} \), the new fractions are not going to be in \( \mathbb{C}[\hat{\Theta}] \). Rather, they will be in \( \mathbb{C}[\hat{\Theta}] \) for some affine HPA \( B \) with \( B^0 = A^o \). It is clear that the new affine forms that we need to allow are translates of \( y_0^o, \ldots, y_m^o \): for \( y_j^o \), where \( j \neq 0 \), these translates have the form \( y_j^o + i/\lambda_j, i = 0, \ldots, \lambda_j - 1 \), but for \( y_0^o \) they have an additional shift.

Iterating the procedure, we arrive at the following analog of Proposition 2.6:

**Proposition 4.6.** Let \( A \) be an affine HPA in \( V \) and \( \Theta \subset V \) a compatible lattice. Fix a non-(\( A, \Theta \))-special \( \mu \in V^*_R \). Then there exists an affine arrangement \( B \) with \( B^0 = A^0 \) and representative forms \( \hat{B} \), such that each \( f \in \mathbb{C}[\hat{\Theta}] \) may be represented as linear combination of functions from \( \mathbb{C}[\hat{\Theta}] \)

\[
eq \prod_{i=1}^{n} \frac{1}{(1 - \varepsilon(x))^i}, x \in \hat{B}, \ x^o = y_i^o \]

where \( t \in \Theta^*, (y_1^o, \ldots, y_n^o) \in NBC(\hat{A}, \prec) \), and \( \varepsilon : \hat{B} \to \mathbb{Z}^* \) is such that

\[
\sum \{ \varepsilon(x) | x \in \hat{B}, \ x^o = y_i^o \} \neq 0, \text{ for all } i \in \{1, n\}.
\]

The expression (4.14) looks a bit less satisfactory than (2.10), because we had to allow various combinations of products of powers of translates of the same linear form. There is a further normalization, however, which allows one to replace (4.14) by an element of \( \mathbb{C}_A[\hat{\Theta}] \) of the form

\[
eq \prod_{i=1}^{n} \frac{1}{(1 - \varepsilon_i)^i},
\]

where \( t \in \Theta^*, (y_1^o, \ldots, y_n^o) \in NBC(\hat{A}, \prec) \) and \( \varepsilon_i > 0 \) for \( i = 1, \ldots, n \). We postpone the proof of this statement to a later publication, as this form of the expansion is not necessary for our applications here.

Returning to the proof of our theorem, just as in the rational case, a much weaker statement suffices:

**Lemma 4.7.** For compatible \( A, \Theta \) with \( |A^o| > \dim V = n \) and a nonspecial \( \mu \in V^*_R \), there is an arrangement \( B \) with \( B^o = A^o \), with the property that any \( f \in \mathbb{C}_A[\hat{\Theta}] \) may be decomposed into a linear combination of elements of \( \mathbb{C}_B[\hat{\Theta}] \), where for each element \( g \) of the decomposition, there is a hyperplane \( H_g \in A^o \), such that \( g \) is generically regular along any hyperplane parallel to \( H_g \).
Step 4. We prove the theorem by induction on $|A^c|$. For $C_\Theta^\Phi(A)$ to be nonempty we need $|A^c| \geq n$, and the case $|A^c| = n$ was treated in Step 2. Thus we will assume $|A^c| > n$.

It will be useful to formalize the argument at the end of part S4 of Theorem 2.5 in the trigonometric context. To make our presentation more transparent, we introduce a simplified notation for the data in Theorem 4.2. The Theorem states the equality of two quantities, $Z$ and $\tilde{C}T$, associated to the data of $(V, A, \Gamma, \Theta, f, \tau, \mu)$. Now fix a hyperplane $H \in A$, and recall the notation $A \setminus H$ and $A|_H$ introduced at the beginning of S4. Assume that $g$ is generically regular along $H$. Then we have the following 3 sets of data:

$$d = (V, A, \Gamma, \Theta, g, \tau, \mu),$$
$$d \setminus H = (V, A \setminus H, \Gamma, \Theta, g, \tau, \mu),$$
$$d|_H = (H, A|_H, \Gamma \cap H, \Theta \cap H, g|_H, \tau|_H, \mu|_{H^c}).$$

This requires some explanation. First, because of the compatibility of $H$ and $\Theta$, there are two cases:

- $H \cap \Theta = \emptyset$;
- $H \cap \Theta$ is a lattice of full rank in $H$.

In the first case the object $d|_H$ does not quite make sense, and we will treat it separately. The second issue is that $H$ is an affine hyperplane, thus it does not have a canonical vector space structure. This is a minor technical problem as all of our constructions are translation invariant. We leave it to the reader to check that the choice of the origin on $H$ is immaterial.

We are ready to state the trigonometric version of the contraction-deletion principle; in the statement we assume the notation just introduced.

Lemma 4.8. If the equality $Z = \tilde{C}T$ holds for two of the three sets of data: $d$, $d \setminus H$ and $d|_H$, then it also holds for the third.

Proof: Note that we always have the equality

$$Z(d) = Z(d \setminus H) - Z(d|_H),$$

and our task is to show that the same equality holds for $\tilde{C}T$. The key to this is again (2.13), the contraction-deletion relation for nbc-bases. We impose the same relation on the ordering of $A^c$ as we did in S4. In particular, the hyperplane $H^c$ is last.

We start with the first case, $H \cap \Theta = \emptyset$, discussed above. Obviously, we have $Z(d) = Z(d \setminus H)$ here. Analyzing (2.14), it is easy to see that the function $\text{Todd}_{\Phi}(\Gamma, a, \tau)$ vanishes at the common zero $p$ of the set of affine forms $a$ unless $p \in \Gamma$. Clearly, the vertices $\text{vx}(A^c \oplus \Theta)$ which lie on $H$ are not in $\Gamma$. Then, since $g$ is regular along $H$ and $p \notin \Gamma$, the iterated constant terms coming from the image of $s_H$ in (2.13) all vanish because of the vanishing of the respective Todd functions at the vertices. This shows that

$$\tilde{C}T(d) = \tilde{C}T(d \setminus H).$$
In the second case, when \( H \cap \Theta \) is a lattice of full rank in \( H \), the proof is analogous to that of the inductive statement in part S4 of the proof of Theorem 2.5. We will not repeat the reasoning here. The only additional condition to check is that \( \mu|_{H^c} \in \Delta_{g_H} \). This immediately follows from the definition (4.12).

Now we are ready to complete the proof of the inductive step. We start with data \( d = (V, \mathfrak{A}, \Gamma, \Theta, f, \tau, \mu) \) such that \( |\mathfrak{A}| = N > n \), and assume the equality \( Z = \tilde{\mathcal{C}}T \) for all cases when \( |\mathfrak{A}^0| = M < N \).

We start with applying Lemma 4.7 to our situation. Note that there is a subtlety here: the condition imposed on \( \mu \) in the Theorem that \( t - \mu \) is not \( \Gamma \)-special is less restrictive than the condition in the Lemma, i.e. that \( \mu \) is not \( \Theta \)-special. This does not cause any problems as both conditions are open, and we may perturb \( \mu \) if necessary. What this means is that there might be several, essentially different partial fraction decompositions, which lead to the proof of the same formula \( Z = \tilde{\mathcal{C}}T \) for a particular data.

Continuing with the proof, denote again the arrangement guaranteed by the Lemma by \( \mathfrak{B} \), and let \( g \) be one of the terms of the decomposition. Then, according to Lemma 4.7, we have \( g \in C^\mu_\Theta \) for some affine arrangement \( \mathfrak{C} \subset \mathfrak{B} \) with \( |\mathfrak{C}| < N \). As \( g \in C^\mu_\Theta \), according to the inductive hypothesis (4.5) holds for the data \((V, \mathfrak{C}, \Gamma, \Theta, g, \tau, \mu)\). Since \( g \) is generically regular along all the hyperplanes in \( \mathfrak{B} \setminus \mathfrak{C} \), we are in position to use Lemma 4.8. This, combined with the inductive hypothesis, allows us to conclude that the equality \( Z = \tilde{\mathcal{C}}T \) holds for the data \((V, \mathfrak{B}, \Gamma, \Theta, f, \tau, \mu)\). As this is true for all of the terms \( g \) in the decomposition of \( f \), by additivity, we have \( Z = \tilde{\mathcal{C}}T \) for \((V, \mathfrak{B}, \Gamma, \Theta, f, \tau, \mu)\). Finally, since \( f \) is generically regular along the hyperplanes in \( \mathfrak{B} \setminus \mathfrak{A} \), we can use Lemma 4.8 again to conclude the inductive hypothesis for \((V, \mathfrak{A}, \Gamma, \Theta, f, \tau, \mu)\). This completes the proof.

4.4. Other forms of the Main Theorem. Here we rewrite Theorem 4.3 in a different form. We will relate the Bernoulli polynomials to the rational trigonometric sums.

First we return to the multiple Bernoulli polynomials introduced in §2.5. It is clear from (2.13) that if we represent \( f \) as a sum of homogeneous terms, then terms of positive degree do not contribute to the Bernoulli polynomial \( P^\mathfrak{A}_f \). This allows us to extend the definition of the Bernoulli polynomial to the case of an arbitrary \( f \in M_\mathfrak{A} \).

Using this formalism and comparing Theorems 2.7 and 4.2, we arrive at the following equality:

\[
Z_f^{\mathfrak{A}/\Theta}(\tau) = \sum_{\mu \in \mathfrak{A}/\Theta} P^{\mathfrak{A}/\Theta}_f(t),
\]

where \( t - u \in \Delta_f \).
Remark 4.2. This equality may be used to derive various polynomial relations among Bernoulli numbers. For example, the left hand side might vanish for an appropriate $\mathfrak{A}$ if $|\Gamma/\Theta|$ is small.

One may simply vary $u$ inside $t - \Delta f$. When $u$ crosses a wall of special elements, the terms on the right hand side might change, but their sum will remain the same.

Another way of finding such relations is to use the freedom in choosing $t$ such that $e^t = \tau$ when restricted to $\Gamma$. Assume for simplicity that the toric arrangement $\mathfrak{A}/\Theta$ has a single vertex at 0. Then if $\mu, \mu - s \in \Delta_f$, where $\mu = t - u$ and $s \in \Theta^*$, then we obtain two essentially different expressions for $Z_{f/\Theta}^\mathfrak{A}(\tau)$:

$$P_{[u]}^\mathfrak{A} f(t) \quad \text{and} \quad P_{[u]}^\mathfrak{A} f(t + s).$$

More generally, we can say that this polynomial will be constant on the set

$$\{ t + rs | t + rs - u \in \Delta f, r \in \mathbb{Z} \}.$$

This is a somewhat surprising "periodicity property" of a linear combination of Bernoulli polynomials.

Formula (4.15) reduces the computation of rational trigonometric sums to that of Bernoulli polynomials, albeit ones corresponding to meromorphic functions. A further computational simplification is to represent $f$ locally, near each vertex $p \in \text{vx}(\mathfrak{A} + \Theta)$ as a product $\text{Pr}_p(f)R_p(f)$, where $\text{Pr}_p(f)$ is of the form $\prod_{x \in \mathfrak{A}_p} x^{\varepsilon(x)}$ with $\varepsilon(x) \in \mathbb{Z}_{\leq 0}$, and $R_p(f)$ is a holomorphic function near $p$, which does not vanish at $p$. These conditions define $\text{Pr}(f)$, which we will call the leading principal part, up to a constant only, but this will be sufficient for our purposes.

To each linear functional $t$ on $V$ one can associate a first order linear differential operator $\nabla_t$ on $V^*$: the directional derivative. Extending this correspondence multiplicatively, we can associate to every power series $h$, defined near zero on $V$, a formal differential operator $D^h_t$ of possibly infinite order. There is an obvious affine generalization of this, when one considers affine linear forms vanishing at a point $p$ and power series near $p$ with a corresponding operator $D^h_t$. Then, for a nonspecial $t$, we clearly have

$$B_{f/\Theta}^\mathfrak{A}(t) = \left[D^{R(f)} \cdot B_{\text{Pr}(f)}^\mathfrak{A}\right](t),$$

where the notation means that the differential operator acts on the function $B$ and then the resulting function is evaluated at $t$. In this notation we can also write

$$P_{[u]}^\mathfrak{A} f(t) = \left[e^{\nabla u} \cdot B_{f/\Theta}^\mathfrak{A}\right](u),$$

where $u = t - \mu$.

Now it is easy to see that the equality (4.15) may be rewritten in the following form.

---

1The author is grateful to Michèle Vergne for pointing out these applications.
Proposition 4.9. Let $\mathfrak{g}, \Gamma \supset \Theta$ be compatible, $u = t - \mu \in V^*$ nonspecial and $f \in \mathbb{C}_\mathfrak{g}[\Theta]$. Then we have

$$Z_f^{\mathfrak{g}/\Theta}(\tau) = \sum_{p \in \text{vx}(\mathfrak{g}/\Theta)} \left[ D_{p} R_{p}(f) \cdot P_{u[p]}^\Theta \right] (t) = \sum_{p \in \text{vx}(\mathfrak{g}/\Theta)} \left[ D_{p} R_{p}(f) e^{\nabla \mu} \cdot B_{p}^\Theta \right] (t - \mu).$$

This might seem like a clumsy way of presenting our formula, but, as it turns out, this is exactly what we need for our application in the next section.

5. The Verlinde formula and the work of Bismut and Labourie

5.1. Preliminaries and Verlinde’s formula. In this section we apply our results to a special case which, in fact, motivated our work. The computation is related to a natural example of the setup of the previous section, one provided by Lie theory. Let $\mathfrak{g}$ be the Lie algebra of $G$. Then in terms of our earlier notation

- $V$ is the Cartan subalgebra of $\mathfrak{g}$;
- $\Re$ is the HPA on $V$ induced by the set of roots; denote by $\hat{\Re}$ the set of positive roots;
- $\Theta \subset V$ is the coroot lattice;
- $\Gamma \subset V$ is the lattice for which $\Gamma^*$ is the lattice generated by the long roots; for a positive integer $k$ let $\Gamma[k] = \{ v \in V | kv \in \Gamma \}$.

There is an inner product $(,)$ on $V^*$ called basic, such that the long roots have square length 2. The resulting identification of $V$ and $V^*$ induces a one-to-one map between $\Theta$ and $\Gamma^*$. We will need some additional notation from Lie theory:

- Let $\rho = \frac{1}{2} \sum_{\alpha \in \Re} \alpha$,
- denote by $\theta$ the highest root; the integer $h = (\theta, \rho) + 1$ is called the dual Coxeter number,
- the two important functions of Lie theory
  $$d = \prod_{\alpha \in \hat{\Re}} \alpha \quad \text{and} \quad \delta = \prod_{\alpha \in \hat{\Re}} 2\sqrt{-1} \sin(\pi \alpha),$$
  the generalized Vandermonde determinant and the Weyl denominator, are in $R_{\Re}$ and $\mathbb{C}_\mathfrak{g}[\Theta]$, correspondingly. Note that $d^{-1}$ is the leading principal part of $\delta^{-1}$ at the origin.

It is easy to identify the vertices of our arrangement:

$$\text{vx}(\Re/\Theta) = \{ p + \Theta | (\alpha \in \hat{\Re}| \alpha(p) \in \mathbb{Z})_{\text{lin}} = V^* \} \quad \text{and} \quad \Re_p = \{ \{ v | \alpha(v) = \alpha(p) \} | \alpha \in \hat{\Re}, \alpha(p) \in \mathbb{Z} \},$$
where $\langle \rangle_{\text{lin}}$ is the linear span of a set. Visually, the vertices correspond to the intersections of $n$ or more root hyperplanes in the Stiefel diagram of the group. For example, in the case of the rank-2 group $G_2$ there are 5 vertices.

Given positive integers $k, g$ and a dominant weight $\lambda$ such that $(\theta, \lambda) \leq k$, there is a remarkable formula discovered by E. Verlinde for a nonnegative integer $\text{Ver}_g(\lambda; k)$, which stands for the dimension of a certain vector space, the space of “conformal blocks” in an appropriate conformal field theory. It takes the form

\[(5.1)\quad \text{Ver}_g(\lambda; k) = \frac{((-1)^{|\mathcal{R}|} |\Gamma/\Theta| (k + h)^n)^{g-1}}{|W|} \sum_{\gamma \in \Gamma[k+h]/\Theta} \chi_\lambda(\gamma) \delta(\gamma)^{2-2g},\]

where $\chi_\lambda$ is the character of the irreducible representation with highest weight $\lambda$ lifted to the Cartan subalgebra by the exponential map, and $W$ is the Weyl group of $G$. The number $\text{Ver}_g(\lambda; k)$ vanishes if $\lambda$ is not an integer linear combination of roots.

Before we proceed, we introduce the shorthand

- $Z_m^{\mathcal{R}[k]}(\tau)$ for the rational trigonometric sum corresponding to the function $\delta^{-m}$, the lattice $\Gamma[k]$ and some exponential weight $\tau \in \tilde{\Theta}$,
- $d_p$ for $\prod_{\alpha \in \mathcal{S}^+ \cap \Theta} (\alpha - \alpha(p))$, which is the inverse of the principal part $\text{Pr}_p(\delta^{-1})$ at a vertex $p$,
- $B_m^{\mathcal{R}[k]}[\lambda]$ for the rational sum corresponding to the arrangement $\mathcal{S}_p^{\mathcal{R}[k]}$, the lattice $\Gamma[k]$ and the rational function $d_p^{-m}$, and

Recall now the Weyl character formula,

\[\chi_\lambda = \frac{1}{\delta} \sum_{w \in W} \text{sign}(w)e_w(\lambda + \rho),\]

and the fact that the function $\delta$ is Weyl-antisymmetric. Using our notation, we may write:

\[(5.2)\quad \text{Ver}_g(\lambda; k) = ((-1)^{|\mathcal{R}|} |\Gamma/\Theta| (k + h)^n)^{g-1} Z^{\mathcal{R}[k+h]}(\lambda + \rho).\]

Now we can apply Theorem 4.2 to compute the value of $\text{Ver}_g(\lambda; k)$:

\[(5.3)\quad \text{Ver}_g(\lambda; k) = (-1)^n ((-1)^{|\mathcal{R}|} |\Gamma/\Theta| (k + h)^n)^{g-1} \sum_{p \in \text{vx}(\mathcal{R}/\Theta)} \tilde{\text{G}}_{\mu, \lambda + \rho}^{\mathcal{R}[k+h]/\Theta} (\delta^{1-2g}),\]

where $\mu \in \Delta_{\delta^{1-2g}}$. Clearly, $\Delta_{\delta^{1-2g}} = (2g - 1)\Delta_{\delta^{1}}$. Denote the convex polytope $\Delta_{\delta^{1}}$ simply by $\Delta$ from now on. The following statement easily follows from the definition of the function $\delta$ and Proposition 4.1:

**Lemma 5.1.** The set $\Delta$ is the convex hull of the Weyl orbit of the weight $\rho$. 
For the brave souls who want to use (5.3) for actual computations, here is the “reverse engineered” form:

\[
\text{Ver}_g(\lambda; k - h) = (-1)^n((-1)^{|\mathcal{R}|\Gamma/\Theta}|k^n)^{g-1} \times \\
\sum_{p \in \text{vx}(\mathcal{R}/\Theta)} \sum_{a \in \mathcal{B}_p} \frac{\prod_{i=1}^n \delta_{\tilde{e}_i, \tilde{a}}}{\text{vol}_\Gamma(\tilde{a}^\circ)} \left( \prod_{i} k_{\tilde{e}_i, \tilde{a}} \right)
\]

where \(\text{vx}(\mathcal{R}/\Theta)\) is defined above and \(\mathcal{B}_p\) is a set of ordered \(n\)-tuples of root hyperplanes from \(\mathcal{R}_p\). There are many choices for \(\mathcal{B}_p\); one possibility is given by (2.2), which depends on a linear ordering of the roots. We replaced \(k\) by \(k - h\) to make the formula more readable.

Let us point out once more the computational advantage of the formula (5.3) over the finite sum in (5.1): in the localized formula the number of terms does not depend on \(k\), while in Verlinde’s expression the number of terms increases polynomially in \(k\). The toric arrangement in Example 4.1 corresponds to the case of the group Spin(5); the lattice \(\Gamma\) in the Example is slightly different from the one appearing in the Lie data, but this difference causes only minor modifications in the calculations.

### 5.2. The Riemann-Roch numbers of the moduli spaces of flat connections

We will try to give an ultrashort introduction to this subject here. The reader is referred to [3, 12] for more details.

We start with the same data as was necessary to describe Verlinde’s formula: we need a simple, simply-connected compact Lie group \(G\), an integer \(g\) and an integer \(k\) called the level and a dominant weight \(\lambda\) in the root lattice such that \((\lambda, \theta) \leq k\). Given these, one may construct a possibly singular symplectic manifold \(M\), a moduli space of flat connections on a Riemann surface of genus \(g\). This manifold is endowed with a prequantum line bundle \(L\). The manifold and the line bundle depend on the data but we will omit this dependence from the notation. Modulo some technical difficulties, one can define an integer \(\chi(M, L)\), the Riemann-Roch number of this pair. We will denote this number by \(\chi(g, k, \lambda)\) when we want to emphasize the dependence on our parameters.

Assume that holomorphic structures compatible with the symplectic form are fixed on \(M\) and \(L\). Then one may define \(\chi(M, L)\) as the alternating sum of dimensions of the sheaf cohomology groups of the space of holomorphic sections of \(L\):

\[
\chi(M, L) = \sum_{i=0}^{\dim M} \dim H^i(M, L).
\]

It is expected that a vanishing theorem holds in this case, which means that \(\dim H^i(M, L) = 0\) if \(i > 0\). On the other hand, the space of conformal blocks, whose dimension is computed by Verlinde’s formula, may be identified with the space of sections \(H^0(M, L)\) (cf. [2]). Thus one would
conjecture that

\[ \chi(g, k, \lambda) = \text{Ver}_g(\lambda; k) \quad (5.5) \]

Bismut and Labourie give an explicit formula for the Riemann-Roch number \( \chi(g, k, \lambda) \), and they prove the identity (5.5) in various cases, notably for large values of \( k \) when \( \lambda/k \) is fixed. Our goal in this section is to show how our formalism fits with their formula and to prove (5.5) in general. Our efforts here are a significant improvement on Section 7 of [3].

Note that so far we have described a restricted case of the whole story, when the “number of punctures”, denoted by \( s \) in [3], is equal to 1. Our proof easily implies the case of \( s > 1 \) as well if \( g > 0 \). However, when \( s > 2 \), then \( g \), the genus of the Riemann surface, is allowed to be 0, and we were not able to cover this case.

We would like to end this section with a sketchy and incomplete review of the existing results regarding (5.5):

- The equality \( \dim H^0(\mathfrak{M}, \Sigma) = \text{Ver}_g(\lambda; k) \) is known in most cases (cf. [2, 12]).
- The vanishing theorem mentioned above is easy to show in some cases, and apparently follows from a recent result of Teleman [16].
- The first two points thus provide, albeit a rather rocky road to the proof of (5.5). In a recent paper [10], Meinrenken and Woodward prove this equality in complete generality, including the genus 0 case, using completely different methods.

### 5.3. The formula of Bismut and Labourie

Now we are ready to compare our residue theorem to the formula of Bismut and Labourie for \( \chi(g, k, \lambda) \). We will assume that the reader has [3] available for the comparison, since even introducing all the notation from this reference would have unreasonably lengthened our paper.

The formula for \( \chi(g, k, \lambda) \) is given in Theorems 6.16 and 6.26 of [3]. We start with listing the correspondence of the relevant notation ([3] \( \mapsto \) this paper):

- \((p \mapsto k), (c \mapsto h), (u \mapsto p), (l \mapsto |\mathfrak{M}|)\)
- \((p + c)(g-1)\dim(\mathfrak{g}(u)) + \dim(\mathfrak{g}(u)/t) \mapsto (k + h)(2g-2+s)|\mathfrak{R}_p| + n(g-1)\)
- \(\mathcal{C}/\mathfrak{R} \mapsto \Theta, R_t \mapsto \Gamma,\)
- According to [3, Proposition 1.3] \( \text{Vol}(T)^2 = |\Gamma/\Theta|.\)
- The function \( e_t(p)P_{p,m,q}(t) \) defined by formula (2.166) of [3] is equal to \( q^m|\mathfrak{R}_p| B_m^{\mathfrak{R}_p[q]}(qt) \). This follows from (4.110).

Consider the case \( s = 1 \) first. In this case, the formula (6.112) of [3] simplifies a little: one may forego the summation over the Weyl group because of the symmetry. Then the formula of Bismut and Labourie may be
rewritten in our notation as

\[(5.6) \quad \chi(g, k, \lambda) = ((-1)^{|\mathfrak{R}|} |\Gamma/\Theta|(k + h)^n)^{g-1} \sum_{p \in \text{vx}(\mathfrak{R}/\Theta)} \left[ D^{(\delta/d)_p} e^{\nabla_{\mu} \cdot B_{2g-1}} \right] (\lambda + \rho - \mu), \]

where \( \mu = \rho - \frac{h}{k} \lambda \).

This formula has exactly the same form as the one appearing in Proposition 4.9. Taking into account (5.2), we may conclude

**Proposition 5.2.** Let \( k, g > 0, s = 1 \). Then \( \chi(g, k, \lambda) = \text{Ver}_g(\lambda; k) \) as long as \( \mu = \rho - \frac{h}{k} \lambda \in \Delta \) and \( \frac{\lambda}{k} \) is not \( \Gamma \)-special.

Note that here we wrote \( \Delta \) instead of \( (2g-1)\Delta \), i.e. we set \( g = 1 \), as this case clearly implies the cases of higher genera.

Geometrically, the condition that \( \lambda/k \) is special means that the moduli space \( \mathfrak{M} \) is singular. Let us write (5.6) in the form (4.15). We obtain

\[\chi(g, k, \lambda) = ((-1)^{|\mathfrak{R}|} |\Gamma/\Theta|(k + h)^n)^{g-1} \sum_{p \in \text{vx}(\mathfrak{R}/\Theta)} P^{\mathfrak{R}_p \Gamma[k+h]} (\lambda + \rho),\]

where \( u = (k + h) \lambda k \). Now if \( \frac{\lambda}{k} \) is special, then we can use Remark 4.2 and conclude that by perturbing \( u \) here a bit, we still have a valid formula. This perturbation corresponds to computing the Riemann-Roch number on a different moduli space, which is smooth. This moduli space may be different if it moves in different directions, but Remark 4.2 tells us that we will always get the same answer (cf. [3, Theorem 6.40]). A similar trick may be used if \( \mu \) is on the boundary of \( \Delta \). The reader is referred to [3, Section 6.11] for further explanations.

The case of multiple punctures, \( s > 1 \), easily reduces to the same argument. Indeed, each term in the sum over the \( s \) copies of the Weyl group in formula (6.112) of [3] is of the form of a product of factors of the kind that we encounter in the \( s = 1, g > 0 \) case. Thus if the condition in the Proposition holds, then we have \( \mu_i \in \Delta_{\delta-1} \) for each \( i = 1, \ldots, s \) and then we can use the easy direction of Proposition 4.1 to conclude that the appropriate condition \( \mu \in \Delta \) holds in this case.

When \( s > 2 \) and \( g = 0 \), then the number of punctures exceeds the number of factors of \( \delta \) in the denominator, and this argument would require \( \mu \in \frac{1}{3} \Delta \), which does not always hold.

### 5.4. Studying the condition

Now we turn to the study of the condition in Proposition 5.2. We maintain the notation of the previous section. Recall that \( \lambda \) is a dominant weight satisfying \( (\theta, \lambda) \leq k \). We denoted the polytope \( \Delta_{\delta-1} \) simply by \( \Delta \), and according to Lemma 5.1, \( \Delta \) is the convex hull of the \( W \)-orbit of \( \rho \). Then the condition in Proposition 5.2 may be rewritten as follows.
Proposition 5.3. Let $\mathfrak{g}$ be a root system of a simple Lie algebra with a chosen dominant chamber $\mathcal{C} \subset V^*$. Then if $(\theta, \nu) < h$ and $\nu \in \mathcal{C}$, then $\rho - \nu \in \Delta$.

Proof: The statement can be proved by direct computation using the classification of simple Lie algebras. We will demonstrate the method in the cases of the Lie algebras $A_n$ and $D_n$. Note that the lattices associated to the Lie algebra do not figure in this statement; we only need to know the set of roots to formulate it.

Denote the simple roots by $\{\alpha_i\}_{i=1}^n$; thus $\mathcal{C} = \{\nu|\langle \alpha_i, \nu \rangle \geq 0, i \in \overline{1,n}\}$. Denote by $\omega_i$, $i \in \overline{1,n}$, the corresponding fundamental weights rescaled as follows: $(\alpha_i, \omega_j) = 0$ for $i \neq j$, and $(\theta, \omega_i) = h$. Then the set $\mathcal{C}^{\leq h} = \{\nu \in \mathcal{C}|(\theta, \nu) \leq h)\}$ is a simplex with vertices at $0, \omega_1, \ldots, \omega_n$. As $\Delta$ is the convex hull of the $W$-orbit of $\rho$, it is clear that to prove the statement, one needs to show that $\rho - \omega_i \in \Delta$ for $i \in \overline{1,n}$.

For the Lie algebra $A_n$ this is true in a remarkable fashion. Here we may think of $V^*$ as the subspace in the Euclidean $n + 1$-space with coordinate vectors $\delta_i, i = 0, \ldots, n$. To simplify the notation, let $v = \sum_{i=0}^n \lambda_i \delta_i = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ be a generic vector in this space. Then

- $V^* = \{v|\lambda_0 + \lambda_1 + \cdots + \lambda_n = 0\}$,
- $\mathcal{C} = \{v \in V^*|\lambda_0 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$,
- $\rho = \frac{1}{2} \sum_{i=0}^n (n - 2i) \delta_i$,
- $\theta = \delta_0 - \delta_n$, $h = n + 1$ and
- $\omega_m = m \sum_{i=0}^{n-m} \delta_i - (n + 1 - m) \sum_{i=n-m+1}^n \delta_i, i = 1, \ldots, n$.

Computing the coordinates of the vector $\rho - \omega_m$ for each $m$, one discovers that they are the same as those of $\rho$, but in different order. Since the Weyl group of $A_n$ acts by permuting the coordinates this means that the vertices of the simplex $\rho - \mathcal{C}^{\leq h}$ are a subset of the vertices of the convex polytope $\Delta$. This is, of course, a much stronger statement than what we needed, but it also shows that our computation is “sharp” in a certain sense.

This sharp statement does not hold for the other simple Lie algebras. The general method of checking the condition goes as follows. For $v \in V^*$, denote by $v^\#$ the vector in $\mathcal{C}$ which is Weyl equivalent to $v$. Since the codimension-$1$ faces of the polytope $\Delta$ lie in hyperplanes perpendicular to Weyl shifted fundamental weights, and $\Delta$ is Weyl invariant, we have

$$\Delta = \{v \in V^*|\langle v^\#, \omega_i \rangle \leq \langle \omega_i, \rho \rangle, i \in \overline{1,n}\}.$$

Hence in order to prove the Proposition for $\mathfrak{g}$, one needs to check the $n^2$ inequalities:

$$(5.7) \quad \langle (\rho - \omega_m)^{\#}, \omega_i \rangle \leq \langle \omega_i, \rho \rangle, \quad i, m \in \overline{1,n}.$$

Here is what happens in the case of $D_n$. We have

- $V^* = \{v = \sum_{i=1}^n \lambda_i \delta_i\}$ is the Euclidean $n$-space,
- $\mathcal{C} = \{v \in V^*|\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n|\},$
• The $j$th coordinate of $v^#$ is the $j$th largest number among the absolute values of the coordinates of $v$, with the possible exception of the $n$th coordinate, which has a minus sign if the number of negative coordinates of $v$ is odd.

• $\rho = \sum_{i=1}^n (n-i) \delta_i$,

• $\theta = \delta_1 + \delta_2$, $h = 2n - 2$,

• $\omega_1 = 2(n-1) \delta_1$, $\omega_m = (n-1) \sum_{i=1}^m \delta_i$ for $m \in \mathbb{Z}$, $\omega_{n-1} = \omega_{n-2} + (n-1)(\delta_{n-1} + \delta_n)$, and $\omega_n = \omega_{n-2} + (n-1)(\delta_{n-1} - \delta_n)$.

The inequalities (5.7) are easy to check now. Indeed, for example, for $m \in \mathbb{Z}$ we have

$$\rho - \omega_m = \sum_{i=1}^m (1-i) \delta_i + \sum_{i=m+1}^n (n-i) \delta_i.$$ 

In this case, we see that comparing the $i$th coordinates, we have

$$|\langle \delta_i, \rho \rangle| \geq |\langle \delta_i, \rho - \omega_m \rangle|,$$

which immediately implies the inequalities.

Note that while for $A_n$ we have $(\rho - \omega_m)^# = \rho$ for all $m \in \mathbb{Z}$, here this only holds for $m = 1, n-1, n$. The cases of the Lie algebras $B_n$ and $C_n$ are completely analogous and are left as an exercise to the reader. The exceptional Lie algebras need to be checked on a case by case basis. We will not present these tedious calculations here; rather, we hope that someone will find a conceptual Lie theoretic proof of the Proposition, which will not rely on the classification theorem.

Let us summarize our results. If we assume that $\chi(g,k,\lambda)$ is the appropriately modified definition of the Riemann-Roch number of the moduli spaces (cf. the discussion after Proposition 5.2), then Proposition 5.3 combined with Proposition 5.2 implies

**Theorem 5.4.** For arbitrary $k$ and $\lambda$, we have

$$\chi(g,k,\lambda) = \text{Ver}_g(\lambda;k)$$

whenever $g > 0$.

As we pointed out earlier, our arguments prove the same statement in the case of an arbitrary number of punctures as well. The reader is referred to [10] for a different approach to this Theorem which also covers the genus 0 case.

5.5. **Quasipolynomial behavior and the topology of moduli spaces.**

As we mentioned in the introduction, one of the reasons for searching for a localization formula for Verlinde’s function $\text{Ver}_g(\lambda,k)$ was to uncover the polynomial nature of this function. The coefficients of this polynomial, in turn, give information about the topology of the moduli spaces in terms of certain characteristic numbers. This point, first raised in [17], was elaborated upon in [18], mainly with the group $SU(n)$ in mind. As explained in the paper of Bismut and Labourie [3], the situation is more complicated.
for other groups: the Riemann-Roch Theorem needs to be replaced by the Kawasaki-Riemann-Roch Theorem for orbifolds.

As we will see, in this case instead of polynomials one deals with quasipolynomials. We say that a function on the positive integers $q(k)$ is a quasipolynomial in $k$ of order $L$ if the function $q(kL+a)$ is a polynomial in $k$ for every $a \in \mathbb{Z}$. Equivalently, a quasipolynomial of order $L$ is a linear combination of characters of the cyclic group $\mathbb{Z}/L\mathbb{Z}$ with values in polynomials.

In this concluding paragraph, we would like to explain how one may obtain information about the components of the quasipolynomial behavior. This might be useful in problems such as determining the Picard group of the moduli space (cf. [9]). Such computations also serve as a handy mnemonic for understanding the stratification of the moduli spaces worked out in [3]. We would like to emphasize that the statements detailed below may be read off from the results of [3] already. We only add a somewhat more compact and transparent formalism for the computations.

We start by noting a few simple scaling properties of the Bernoulli functions:

- For the case when the vertex $p$ is at the origin, we have
  
  \[ B^Q_m[k](kt) = k^{m|\mathfrak{g}|} B^Q_m[1](t). \]

- If the vertex is not at the origin, but we have $p \in \Gamma$, then this identity is generalized as follows:
  
  \[ B^Q_m[p][k](kt) = e_{tp(kp)k^{m|\mathfrak{g}| \mathfrak{p}|} B^Q_m[1](t). \]

- If the vertex $p$ is not in $\Gamma$, then the scaling properties are a bit more complicated. However, for any vertex $p \in \text{vx}(\mathfrak{g}+\Theta)$ we have $Np \in \text{vx}(\mathfrak{g}+\Theta)$ for all integers $N$; also, we have $Mp \in \Gamma$ for some integer $M$. In particular, $\Gamma \subset \mathfrak{g}+\Theta$.

As a next step, we amend the material of §5.2 slightly. We need to make the relation between the parameters $g, k, \lambda$ and the prequantized moduli spaces $(\mathfrak{M}, \mathfrak{L})$ more precise.

- The space $\mathfrak{M}(g,k,\lambda)$ depends on $g$ and the quotient $\lambda/k$ only. Thus we may write $\mathfrak{M}(g,\lambda/k)$.
- The $m$th power of the line bundle $\mathfrak{L}(g,k,\lambda)$, defined on $\mathfrak{M}(g,\lambda/k)$, is simply the bundle $\mathfrak{L}(g,mk,m\lambda)$
- One may, in fact, define $\mathfrak{M}(g,t)$ for any $t \in \mathbb{C}^{\leq 1}$; these spaces form a smooth family, when $t$ varies inside a chamber of nonspecial elements.

For a given set of data $(g,k,\lambda)$, we are interested in computing the orbifold Riemann-Roch number

\[ \chi(g,km,m\lambda) \int_{\mathfrak{M}} e^{m\mathfrak{L}} \text{Todd}(\mathfrak{M}) \]
as a function of $m$. Analyzing (5.6), we see that $\mu = \rho - ht$, where $t = \lambda/k$, and thus does not depend on $m$. Then the formula has the general form

$$\chi(g, mk, m\lambda) = \text{const } (mk + h)^{n(g-1)} \sum_{p \in \text{vx}(\mathcal{R}/\Theta)} D_p \cdot B_{2g-1}^{\mathcal{R}_p[mk+h]}(\text{((mk + h)t)}),$$

where $D_p$ is an operator that does not depend on $m$. Naturally, here we assumed that $t$ is not special. If $t$ is special, the we need to use the formalism of the Bernoulli polynomials $P_{[u]}$.

Comparing this formula for $\chi$ to the scaling properties of the Bernoulli functions that we listed above, we arrive at the following general conclusions.

**Proposition 5.5.** The vertices $\text{vx}(\mathcal{R}/\Theta) \subset T$ are a union of cyclic subgroups, one of which is $\Gamma/\Theta$. The orbifold Riemann-Roch number $\chi(g, mk, m\lambda)$ is a polynomial in $m$ if $kp \in \Gamma$ and $\lambda(p) \in \mathbb{Z}$ for all $p \in \text{vx}(\mathcal{R}^+/\Theta)$.

For example, in the case of the Lie group $G_2$ the set of vertices $\text{vx}(\mathcal{R}/\Theta)$ consists of 5 points: the union of a cyclic group of order 3, which is $\Gamma/\Theta$, and a two cyclic groups of order 2. This means that if we take the line bundle corresponding to the trivial character ($\lambda = 0$) at level 1 ($k = 1$), then the orbifold Euler characteristic will be a quasipolynomial of order 2. If we take $\lambda \neq 0$, then $\chi(g, mk, m\lambda)$ could end up being a quasipolynomial of order 6.
References

[1] M. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523–615.

[2] A. Beauville, Y. Laszlo, Conformal blocks and generalized theta functions, Comm. Math. Phys. 164 (1994), no. 2, 385–419.

[3] J.-M. Bismut, F. Labourie, Symplectic geometry and the Verlinde formulas. Surveys in differential geometry: differential geometry inspired by string theory, pp. 97–311, Surv. Differ. Geom., 5, Int. Press, Boston, MA, 1999

[4] M. Brion, M. Vergne, Residue formulae, vector partition functions and lattice points in rational polytopes, J. Amer. Math. Soc. 10 (1997), no. 4, 797–833.

[5] M. Brion; M. Vergne, Arrangement of hyperplanes. I. Rational functions and Jeffrey-Kirwan residue, Ann. Sci. cole Norm. Sup. (4) 32 (1999), no. 5, 715–741.

[6] M. Brion, M. Vergne, Arrangement of hyperplanes. II. The Szenes formula and Eisenstein series, Duke Math. J. 103 (2000), no. 2, 279–302.

[7] I. M. Gelfand, A. V. Zelevinski, Algebraic and combinatorial aspects of the general theory of hypergeometric functions. (Russian) Funktsional. Anal. i Prilozhen. 20 (1986), No.3, 17–34.

[8] L. Jeffrey, F. Kirwan, Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface, Ann. of Math. (2) 148 (1998), no. 1, 109–196.

[9] S. Kumar, M. S. Narasimhan, Picard group of the moduli spaces of $G$-bundles Math. Ann. 308 (1997), no. 1, 155–173.

[10] E. Meinrenken, C. Woodward, Hamiltonian loop group actions and Verlinde factorization, J. Differential Geom. 50 (1998), no. 3, 417–469.

[11] P. Orlik, H. Terao, Arrangements of hyperplanes, Springer-Verlag, Berlin, 1992

[12] C. Sorger, La formule de Verlinde, Sminaire Bourbaki, Vol. 1994/95. Astrisque No. 237 (1996), Exp. No. 794, 3, 87–114.

[13] A. Szenes, The combinatorics of the Verlinde formulas, in Vector bundles in algebraic geometry (Durham, 1993), 241–253, London Math. Soc. Lecture Note Ser., 208.

[14] A. Szenes Iterated Residues and multiple Bernoulli polynomials Internat. Math. Res. Notices 1998, no. 18, 937–956. (arXiv:hep-th/9707114).

[15] A. Szenes, M. Vergne, in preparation.

[16] C. Telemann, The quantization conjecture revisited, Ann. of Math. (2) 152 (2000), no. 1, 1–43.

[17] M. Thaddeus, Conformal field theory and the cohomology of the moduli space of stable bundles, J. Differential Geom. 35 (1992), no. 1, 131–149.

[18] E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, Nuclear Phys. B 300 (1988), no. 3, 360–376.

[19] E. Witten, On quantum gauge theories in two dimensions Comm. Math. Phys. 141 (1991), no. 1, 153–209.

[20] E. Witten, Two-dimensional gauge theories revisited, J. Geom. Phys. 9 (1992), no. 4, 303–368.

Massachusetts Institute of Technology, Department of Mathematics
E-mail address: szenes@math.mit.edu