Vibration Characteristics of Elastic Metamaterial Rod

Ziyue Zheng1, Haisheng Shu1, Shuowei A1, Di Mu1, Lei Zhao1 and Yuqiang Su1

1Harbin Engineering University, College of Mechanical and Electrical Engineering, 150001, Harbin, China

shujs@hrbeu.edu.cn

Abstract. This paper investigates the free vibration and steady forced vibration of elastic metamaterial rods. Firstly, the elastic metamaterial rod is equivalent to the effective medium rod based on the effective medium theory. Then the free vibration and steady forced vibration method of the effective medium rod are established from the classical theory, and with this method we can explore the difference characteristics between the elastic metamaterial rod and the natural media rods. The results show that the effective medium rod has some strange phenomenon such as the same shape of different natural frequencies, the accumulation of natural frequencies near the lower band and the loss of natural frequency. This work is expected to help the vibration control of the elastic metamaterial for finite structure.

1. Introduction

In 2000, Liu took the lead in proposing a local resonance phononic crystal structure, and pointed out that since the scatterer has a local resonance of the mode "spring-mass", the structure can be regarded as a homogenous structure, which has frequency and negative density under the long wave limit [1]. Subsequently, a large number of scholars have designed different unit cell structures and initiated different types of resonance modes (monopoles, dipoles, quadrupoles, etc.) of the scatterers to obtain shear modulus, bulk modulus, etc., which have elastic/acoustic metamaterial structure [2-7]. Thanks to these strange characteristic different from those of natural materials, scholars applied it to low frequency vibration [8-10], negative refraction [2, 5, 11], acoustic/elastic wave hiding [12, 13], seismic wave protection [14-16, 27] and other aspects.

In recent years, due to the development of effective medium theory [2, 17-22, 29], more and more scholars have begun to use this method to explore the propagation characteristics of acoustic/elastic waves in infinite or semi-infinite elastic metamaterial [3, 16, 23-26]. The effective medium theory is considered that under the long-wave limit, when the wavelength in the matrix of the elastic metamaterial is much larger than the lattice constant, the structure can be equivalent to the homogeneous structure macroscopic which material parameter is frequency changed (equivalent density \( \rho_{\text{eff}} \), equivalent shear modulus \( \mu_{\text{eff}} \), the equivalent bulk modulus \( K_{\text{eff}} \)). Compared with the modeling of the unit cell level, this not only greatly simplifies the calculation, but also facilitates people to understand the propagation behavior of long waves in elastic metamaterials. The most classic is the interpretation of the band gap and negative refraction characteristics [5, 19]. Williams [25] and others analytically calculated the equivalent density of the multi-vibrator metamaterial beam to explain the propagation behavior of the Lamb wave. R. Zhu [26] verified the anisotropy of equivalent density by experimental method, and discussed the propagation behavior of waves in anisotropic
elastic metamaterial plates. In addition, based on the effective medium theory, our group \cite{16, 24} analyzed the Lamb wave in the elastic metamaterial layer and the propagation behavior of the SH wave in the associated half space, and predicted the possible traveling wave components. Using the effective medium theory, the above work is a good analysis of the propagation of elastic waves in an infinite or semi-infinite elastic metamaterial medium, which can be applied to occasion where the wavelength is much smaller than the propagation size, such as seismic wave protection \cite{14}.

However, it should be noted that when the concept of elastic metamaterials is introduced into practical applications, the corresponding structures are generally waveguide structures of finite size, and the main concern is the vibration characteristics of these special structures.

In this paper, we use the effective medium theory to equate an elastic metamaterial rod into an effective medium rod. Starting from the classical theory (one-dimensional wave equation), the free vibration and forced vibration analysis methods of the effective medium rod are established, and the free vibration and forced vibration characteristics of the elastic metamaterial rod are explored by this method. When investigating free vibration, due to the frequency change and the negative value of the equivalent density, the finite elastic metamaterial rod has significantly different modal characteristics from the natural medium rod, including the same mode shape at different natural frequency points, the dense distribution of natural frequencies near the band edge and the absence of natural frequencies and modes. The analysis of the forced vibration characteristics further demonstrates the effectiveness of free vibration analysis.

2. Analysis of Free Vibration of EM Rod

Under the condition of two-dimensional plane stress, this paper adopts the elastic metamaterial rod structure shown in figure 1.a, which is made up of a single cell structure as shown in figure 1.b through a 10*3 array. The material and structural parameters are seen in table 1. With the help of the "Feel and Response" method \cite{2}, the unit cell equivalent parameters are calculated as shown in figure 1.b. The equivalent Young's modulus $E_{\text{eff}}$ can be obtained by

$$E(\omega) = \frac{9KG}{3K + G}.$$  

It can be seen from figure 1.b that in 0-2500 Hz, the equivalent density varies greatly due to the excitation of the scatterer dipole mode, and a negative value occurs in the 570-1258 Hz (gray region). However, the equivalent shear modulus and the equivalent bulk modulus hardly change, so this paper will focus on the frequency change of the equivalent density. At the same time, in order to facilitate the subsequent analysis, we show the dispersion relationship of the longitudinal wave in such a rod as shown in figure 1.c, which satisfies:

$$\omega = c_L k = \sqrt{\frac{E_{\text{eff}}}{\rho_{\text{eff}}} \cdot k}.$$  

Due to the frequency variation of the equivalent density, the longitudinal wave is dispersed, and the energy band curve is decomposed into two branches, the sound branches and the optical branch which are separated by a band gap (grey area), and the band gap interval corresponds to the negative density interval \cite{19, 4}. The band structure can also be given by FEM as shown in figure 1.c. This result shows good agreement with the theoretical calculation results of equivalent medium, which confirms the validity of the effective medium theory.
Figure 1. (a) composition of unit cell and equivalent parameters (b) 1 * 3 primitive cell and its band structure (c) equivalent process.

Table 1. Unit cell material parameter.

| component | $\rho$ (kg/m$^3$) | E(Pa) | G(Pa) |
|-----------|-------------------|-------|-------|
| matrix    | 1180              | 0.435e10 | 0.159e10 |
| scatterer | 11600             | 0.408e11 | 0.149e11 |
| layer     | 1300              | 1.175e5  | 4e4   |

Table 2. Unit cell structure and parameters.

| a  | R  | r  |
|----|----|----|
| 10 | 3.5| 3  |

Next, the above effective medium rod free vibration is analyzed. It is noted that at this time the rod is no longer a Cauchy medium, and its material parameters, especially the equivalent density, have a strong frequency change, especially a negative value in the band gap, then it is necessary to establish a free vibration analysis method of effective medium rod. Following the classical theory, the effective medium rod governing equation can be written as [31]:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_x(\omega)^2} \frac{\partial^2 u}{\partial t^2} = 0 \quad 0 \leq x \leq l \quad -\infty < t < -\infty$$

(1)

Where $l$ is the length of the rod, $c_x(\omega)$ is the velocity of longitudinal wave in the rod, and the magnitude is $\sqrt{\frac{E(\omega)}{\rho(\omega)}}$, where the velocity of longitudinal wave, the elastic modulus and the density are both a
function of the circular (angle) $\omega$ frequency. We find that when $\frac{1}{c_j^2(\omega)}$ is a positive number, this variable $\omega$ has no effect on the solution of equation (1). Therefore, according to the classical theory method, let the standing wave solution be brought into equation (1), we can get:

$$\frac{X'}{X} = \frac{1}{c_j^2(\omega)} \frac{T'}{T} = -k(\omega)^2 \tag{2}$$

When considering the boundary condition of the effective medium rod as free at both ends, to put the boundary condition with zero stress at both ends into equation (2), and the non-zero solution is obtained to obtain the natural frequency distribution as follows:

$$\omega = \frac{n \pi \sqrt{E(\omega)/\rho(\omega)}}{l}, n = 1, 2, 3,... \tag{3}$$

In particular, when the equivalent density of figure 1 also exhibits a negative value in the band gap frequency, to solve the governing equation (1), it is noted that $\frac{1}{c_j^2(\omega)}$ is also a negative value.

If: $k \neq 0$, for the second-order partial differential equation, it can also be solved by the standing wave method, and its solution is:

$$u = X'T = (Ae^{-k'x} + Be^{k'x})(Ce^{j\omega t} + De^{-j\omega t}) \tag{4}$$

Where $k'$ is a non-negative real number, satisfied $k' = \frac{i}{k}$. Bring the free boundary conditions at both ends into equation (4):

$$\begin{bmatrix} -k' & k' \\ -k'e^{-i\eta} & k'e^{i\eta} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{5}$$

If we have a non-zero solution in equation (5), $k'$ is zero only. Obviously, this contradicts the assumptions, so we give up.

If: $k = 0$, then $\omega = c_0k = \frac{\pi \sqrt{E(\omega)/\rho(\omega)}}{l}$, means that there is no natural frequency in the effective medium rod. At this time, the right side of equation (2) is 0, then the solution of the differential equation is:

$$u = X'T = (A + Bx)(C + Dt) \tag{6}$$

Bring the boundary conditions of free at both ends, we can get $B = 0$ in equation (6). From $t \rightarrow \infty$, the above formula $u$ must be a finite number to have physical meaning, and get: $D = 0$. Therefore you can get:

$$u = X'T = A'C = u_0 \tag{7}$$
The free vibration analysis method of the effective medium rod is given above. Due to the existence of the variables in the governing equation (1), the equation (3) characterizing the natural frequency distribution and the corresponding modal shape is obviously different from the classical theoretical results. Also, the above also gives the analysis of free vibration when the density is negative, and this case has never been covered by classical theory. Next, we will use the above analysis method to analyze the free vibration characteristics of the effective medium rod shown in figure 1.

As shown in figure 1, \( \rho_{\text{eff}} \) varies with frequency. When approaching 570 Hz, that is, the lower band edge \( f_0^- \) of the band gap, \( \rho_{\text{eff}} \) will rise sharply with frequency; while near 1258 Hz, that is, the upper band edge \( f_1^+ \) of the band gap, \( \rho_{\text{eff}} \) will rise relatively slowly from 0; Between \( f_0^- \) and \( f_1^+ \) \( \rho_{\text{eff}} \) is negative.

First consider \( \rho_{\text{eff}} \) as positive and bring its value into equation (3). For the convenience of expression, let the left side of the equation (3) equal sign \( y_1^n \) and the right side \( y_2 \), and figure 2 gives the function image of \( y_1^n \) and \( y_2 \). The abscissa of the intersection of the two functions is the natural frequency, and the corresponding mode is \( \cos \left( \frac{n \pi x}{l} \right) \), and \( n \) represents the \( n \)th mode. For comparison, the natural frequency and modal state in the natural medium are also given, where \( E \) and \( \rho \) are taken when \( \omega \to 0 \).

![Figure 2. The function image of \( y_1^n \) and \( y_2 \).](image)

We know that there is only one intersection of \( y_2 \) and \( y_1^n \) in the natural media rod as shown in figure 2, which means that a natural frequency corresponds to a modality \( \cos \left( \frac{n \pi x}{l} \right) \). However, for the effective medium rod, due to the frequency variation of its density, \( y_2 \) splits into two upper and lower branches, resulting in an intersection of \( y_1^n \) and \( y_2 \) before and after band gap, which means that the same mode will correspond to two natural frequencies. And the two natural frequencies are the \( n \)th order before the band gap and the \( n \)th order after the band gap, which is obviously different from the natural dielectric rod. In addition, in figure 2 we also observe when the frequency approaches the lower band edge \( f_0^- \), \( y_2 \to 0 \), \( y_1^n \to \infty \), then the first intersection with this will be clustered nearby \( f_0^- \), and the intersection spacing becomes narrower and narrower. That is to say, the effective medium rods in the vicinity of \( f_0^- \) will have an infinite number of natural frequencies and their modal density will increase.

In fact, the essence of the above-mentioned peculiar phenomenon is derived from the modulation of
the scatterer dipole resonance on the longitudinal wave in the rod, and the dispersion is given in figure 1. On the one hand, it can be seen that for longitudinal wave and traveling waves of the same frequency, longitudinal wave propagation of two wavelengths is allowed in the elastic metamaterial due to modulation. Since the modality can be seen as a standing wave form and consists of a set of traveling waves propagating in the opposite direction of the same frequency, and the modal shape is determined only by the wavelength. Then, two sets of traveling wave with the same wavelength and different frequencies in the elastic metamaterial rod naturally produce the same modal shape but different modal frequencies. On the other hand, near the vicinity of \( f_0 \) (area A of figure 1), the dispersion curve tends to be a straight band. This means that the two different wavelengths of the longitudinal waves will have a very small difference in frequency, and this difference will become smaller and smaller as the frequency approaches \( f_0 \). When two sets of traveling waves with different wavelengths and almost the same frequency form a mode, these modal frequencies will naturally be close, and as the frequency approaches \( f_0 \), the modal frequency interval becomes smaller, that is, the modal density increases.

When the frequency is between \( f_0^+ \) and \( f_0^- \), \( \rho_{ef} \) is negative, it can be seen from the above analysis that there is no natural frequency in the effective medium rod, and equation (7) means that the motion state of the rod can be regarded as the rigid body is still and always at the initial displacement \( u_0 \). In fact, this solution is in line with the physical nature of negative density. From the unit cell level, when the dipole resonance of the scatterer in the rod is excited, the displacement in the rod matrix is offset to zero by the displacement in the scatterer at the band gap frequency (negative density interval), that is, at this time, there is no mode in the matrix that provides vibration.

3. Analysis of Steady Forced Vibration

Considering that the structure will always be subjected to a certain dynamic load rather than free vibration under actual conditions, we will discuss the forced vibration of the effective medium rod in steady state. Similarly, it is first necessary to establish an effective medium, a forced vibration method for a non-cauchy dielectric rod. Compared with the classical theory, \( c_0^2 \) of the governing equation (1) has frequency change and negative values, but we find that this does not affect the solution, that is, the steady solution of equation (1) can still be written as:

\[
u(x,t) = u^r + u' = (Ae^{-j\omega x} + Be^{j\omega x})e^{j\omega t}
\]

(8)

The wave vector is \( k = \omega / c_0(\omega) \). \( A, B \) are determined by the boundary conditions of the rod. Consider the boundary condition of the rod in the second section, where the left end is excited by the harmonic displacement \( u = u_0 e^{j\omega t} \), and when the right end is free, the equation (8) has:

\[
A = \frac{u_0 e^{j\omega l}}{e^{j\omega l} + e^{-j\omega l}}, \quad B = \frac{u_0 e^{-j\omega l}}{e^{j\omega l} + e^{-j\omega l}}.
\]

The displacement transfer rate from the left end to the right end of the rod \( Trans \) can be obtained by the following formula.

\[
Trans = 20 \cdot \lg \frac{u_r}{u_0}
\]

(9)

Where, \( u_r \) is the displacement amplitude at the right end of the rod can be obtained from equation (8). Figure 3a shows the effective medium rod displacement transfer rate curve in the 1 to 2500 Hz band.
Figure 3. (a) Forced vibration transfer rate curve (b) Theoretical analysis of the peak displacement field distribution (c) Simulation of the peak displacement field distribution and the corresponding unit cell displacement field magnified view, the arrow represents the displacement vector.

It is observed in the transfer rate curve of figure 3a that a large number of formants are enriched near the lower band edge and the interval is smaller and smaller; the transfer rate in the band gap frequency is very small. Using the equation (8) to calculate the displacement field at the formats in and after the band gap, it is found that the existence is similar to that in the free vibration analysis, that is, the displacement field of the nth formant before the band gap and after the band gap remain consistent. In fact, from the point of view of the modal superposition method, these steady forced vibration characteristics of the rod are actually another manifestation of its free vibration characteristics. Since the effective medium rod is concentrated at the natural frequency near the lower band edge, when the excitation is close to these concentrated natural frequencies, the format generated by the rod response is naturally concentrated; since there is not a natural frequency in the negative density interval (band gap frequency) of the effective medium rod, its motion form can be regarded as the static state of the rigid body. When the excitation frequency is in the negative density interval, the motion state of the rod is dominated by the stationary state, which leads to the decrease of the transfer rate; the band gap has the same modal shape successively. The mode is naturally excited to exhibit a similar displacement field response. In addition, it should be pointed out that due to the relationship between the steady forced vibration of the effective medium rod and the free vibration, the forced vibration characteristic results also confirm the reliability of the above free vibration analysis.

In order to further verify the effectiveness of the above-mentioned effective medium rod forced vibration analysis, we used COMSOL to model the elastic metamaterial rod at the unit cell level and perform steady state calculation. Applying a simple harmonic displacement excitation $u = u_0 e^{j\omega t}$ to
the left end of the rod which direction is \( x \) as shown in figure 1c, \( u_0 \) is the displacement amplitude, \( \omega \) is the circle (angle) frequency, the other surfaces are free. Take the average displacement of the line at the right end of the pickup rod \( u_l \), the displacement transfer rate curve can be obtained by using equation (9) as shown in figure 3a.

We can see that the transfer rate curve of the effective medium rod and the elastic metamaterial rod modeled from the unit cell level shows a good agreement at the low frequency band, especially at the band gap position. However, we still need to point out that this analysis process is based on the dynamic homogenization theory under the long-wave limit and cannot describe the unit cell motion state. For example, in figure 3c, although the corresponding one of the first formants before and after the band gap is macroscopically consistent, the displacement field on a single unit cell (microscopic) exhibits an inconsistency similar to that in the free vibration analysis. Moreover, the accuracy of this algorithm depends to a large extent on the accuracy of the effective medium theory. It can be seen from figure 3a that the transfer rate curve deteriorates after 1500 Hz, which is due to the boundary effect and high frequency factors leading to the error of the equivalent medium theory [20]. Even so, we still need to emphasize that the above method can still provide the ideal accuracy for the forced vibration of the elastic metamaterial rod at low frequencies (maximum error of peak position is 4.3%).

4. Conclusion
In this paper, we use the effective medium theory to analyze the free vibration and steady forced vibration analysis method of a finite-size effective medium rod from the governing equation of the rod (one-dimensional wave equation). And using this method, explore the vibration characteristics of the elastic metamaterial rod different from the natural medium rod. The theoretical results at the effective medium level together with the simulation results at the unit cell level show that: 1. Due to the dipole resonance for the modulation of the longitudinal waves in the rod, the nth order modal shape of the elastic metamaterial rod before and after the band gap is macroscopically uniform. In the effective medium theory, different natural frequencies correspond to the same natural mode; 2, the elastic metamaterial rod does not have a natural frequency in the band gap frequency interval (negative density interval). Its motion form is similar to a stationary rigid body; 3. The natural frequency of the metamaterial rod is concentrated at the lower band edge, and the mode density is continuously increased. In this paper, an analytical method for free vibration and steady forced vibration of an effective medium rod is established. This method can be extended to the elastic metamaterial finite structures such as plates, shells and films, and it is expected that this work will help people to understand and solve the dynamic response of elastic metamaterial finite structures from the half-wavelength scale and more fully utilize its characteristics to control the vibration displacement field and vibration.

References
[1] Liu Z, Zhang X, Mao Y, et al. 2000 Sci. Locally resonant sonic materials, 289(5485) 1734-36.
[2] Liu X N, Hu G K, Huang G L, et al. 2011 Appl. Phys. Lett. An elastic metamaterial with simultaneously negative mass density and bulk modulus, 98(25) 251907.
[3] Lai Y, Wu Y, Sheng P, et al. 2011 Nature Mater. Hybrid elastic solids, 10(8) 620.
[4] Yang Z, Mei J, Yang M, et al. 2008 Phys. Membrane-type acoustic metamaterial with negative dynamic mass, 101(20) 204301.
[5] Wu Y, Lai Y and Zhang Z Q 2011 Physic. Rev. Lett. Elastic metamaterials with simultaneously negative effective shear modulus and mass density, 107(10) 105506.
[6] Oh J W, Kwon Y E, Lee H J and Kim Y Y 2016 Sci. Rep. Elastic metamaterials for independent realization of negativity in density and stiffness, 6 23630.
[7] Li J, Chan C T 2004 Physic. Rev. E Double-negative acoustic metamaterial, 70 055602.
[8] Zhu R, Liu X N, Hu G K, et al. 2014 J. Sound Vib. A chiral elastic metamaterial beam for broadband vibration suppression, 333(10) 2759-73.
[9] Liu X N, Hu G K, Sun C T, et al. 2011 J. Sound Vib. Wave propagation characterization and design of two-dimensional elastic chiral metacomposite, 330(11) 2536-53.
[10] Xiao Y, Wen J, Yu D, et al. 2013 *J. Sound Vib.* Flexural wave propagation in beams with periodically attached vibration absorbers: Band-gap behavior and band formation mechanisms, 332(4) 867-93.

[11] Feng L, Liu X P, Lu M H, et al. 2006 *Physic. Rev. Lett.* Acoustic backward-wave negative refractions in the second band of a sonic crystal, 96(1) 014301.

[12] Farhat M, Guenneau S and Enoch S 2009 *Physic. Rev. Lett.* Ultrabroadband elastic cloaking in thin plates, 103(2) 024301.

[13] Stenger N, Wilhelm M and Wegener M 2012 *Physic. Rev. Lett.* Experiments on elastic cloaking in thin plates, 108(1) 014301.

[14] Andrea C, Philippe R, Sebastien G, et al. 2016 *Scientific Rep.* Forests as a natural seismic metamaterial: Rayleigh wave bandgaps induced by local resonances, 6 19238.

[15] Aravantinoszafiris N and Sigalas M M 2015 *J. Appl. Phys.* Large scale phononic metamaterials for seismic isolation, 118 064901.

[16] Shi X, Shu H, Zhou H, et al. 2017 *J. Appl. Phys.* SH wave propagation in joined half-spaces composed of elastic metamaterials, 122(21) 215104.

[17] Geers M G D, Kouznetsova V G, Massart T J, et al. 2009 Computational homogenization of structures and materials.

[18] Pham K, Kouznetsova V G and Geers M G D 2013 *J. Mech. Phys. Solids* Transient computational homogenization for heterogeneous materials under dynamic excitation, 61(11) 2125-46.

[19] Wu Y, Lai Y and Zhang Z Q 2007 *Phys. Rev. B Condensed Matt.* Effective medium theory for elastic metamaterials in two dimensions, 76(20) 205313

[20] Roca D, Lloberas-Valls O, Cante J, et al. 2018 *Comput. Methods Appl. Mech. Eng.* A computational multiscale homogenization framework accounting for inertial effects: Application to acoustic metamaterials modelling, 330 415-46.

[21] Huang H H, Sun C T and Huang G L 2009 *Int. J. Eng. Sci.* On the negative effective mass density in acoustic metamaterials, 47(4) 610-7.

[22] Sridhar A, Kouznetsova V G and Geers M G D 2016 *Computation. Mech.* Homogenization of locally resonant acoustic metamaterials towards an emergent enriched continuum, 57(3) 423-35.

[23] Torrent D and Sánchez-Dehesa J 2009 *Phys. Rev. Lett.* Radial wave crystals: radially periodic structures from anisotropic metamaterials for engineering acoustic or electromagnetic waves, 103(6) 064301.

[24] Shu H, Xu L, Shi X, et al. 2016 *J. Appl. Phys.* Traveling Lamb wave in elastic metamaterial layer, 120(16) 165103.

[25] Williams E G, Roux P, Rupin M, et al. 2015 *Physic. Rev. B* Theory of multiresonant metamaterials for A0 Lamb waves, 91(10) 104307.

[26] Zhu R, Liu X N, Huang G L, et al. 2012 *Physic. Rev. B* Microstructural design and experimental validation of elastic metamaterial plates with anisotropic mass density, 25(14) 144307.

[27] Krödel S, Thomé N and Daraio C 2015 *Extreme Mech. Lett.* Wide band-gap seismic metastructures, 4 111-7.

[28] Ping S 1995 *Waves Random Complex Media* Introduction to Wave Scattering, Localization and Mesoscopic Phenomena, 17(2) 235-7.

[29] Srivastava A 2015 *Int. J. Smart Nano Mater.* Elastic Metamaterials and Dynamic Homogenization: A Review, 6(1) 41-60.

[30] Shen X H, Sun C T, Barnhart M V and Huang G L 2018 *J. Vib. Acoust.* Analysis of Dynamic Behavior of the Finite Elastic Metamaterial-Based Structure with Frequency Dependent Properties, 140(3) 031012.

[31] Achenbach J D 1980 Wave Propagation in Elastic Solids, North-Holland.

**Acknowledgments**

This work was funded by the project (grant number 51875112) supported by the National Natural Science Foundation of China, the project (grant number LBH-Q15029) supported by the Postdoctoral Scientific Research Developmental Fund of Heilongjiang Province of China.