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Hermiticity of the Dirac Hamiltonian in gravitational field

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Abstract. The hermiticity properties of the Dirac Hamiltonian are discussed, both on the quantum mechanical and on the quantum field level. In a first step, it is shown that the Hamiltonian generating the time evolution of the Dirac wave function in relativistic quantum mechanics is not hermitian with respect to the covariantly defined inner product. A hermitian Hamiltonian is then defined and is shown to be directly related to the canonical field energy. In a second step, we use a manifestly covariant form of canonical Hamiltonian field theory in curved spacetime and show that, for the Dirac field, the canonical field momentum does not coincide with the generators of spacetime translations. Moreover, it is shown that the modification of the Dirac Lagrangian by a surface term leads to a momentum transfer between the Dirac field and the gravitational background field, resulting in a theory that is free of constraints, but not manifestly hermitian.

1. Hermiticity of the Dirac Hamiltonian

We start with the Dirac Lagrangian in a gravitational background (see [1])

\[ L = \frac{i}{2}(\bar{\psi}\gamma^t D_t \psi - D_t \bar{\psi}\gamma^t \psi - m \bar{\psi} \psi), \]  

and write the Dirac equation in Schrödinger form

\[ H \psi = i \partial_t \psi, \]  

where

\[ H = \frac{1}{g^{tt}}(\gamma^t m - i \gamma^t \gamma^\mu \nabla_\mu) - \frac{1}{4} \Gamma^{ab} \sigma_{ab}. \]  

All notations and conventions are identical to those used in [2]. For articles dealing with the Dirac Hamiltonian in curved spacetime see, e.g., [3, 4, 5, 6, 7]. Although it has been shown that \( H \) is not generally hermitian [3, 4], this is not widely known, since in most relevant cases, the gravitational fields are static, and \( H \) is hermitian. In this article, we will analyze in detail the hermiticity properties of \( H \).

Note that the Dirac equations lead to the following conservation law

\[ \partial_t (\bar{\psi}\gamma^t \psi) = 0. \]
where $e = \det e_i^a = \sqrt{-g}$. In order to make possible a constant in time normalization of the wave function, we therefore define the following inner product

$$\langle \psi_1, \psi_2 \rangle = \int \sqrt{-g} \bar{\psi}_1^\dagger (\gamma^0 \gamma^i) \psi_2 \, d^3x,$$

or, in an explicitly covariant form

$$\langle \psi_1, \psi_2 \rangle = \int \sqrt{-g} \bar{\psi}_1 \gamma^i \psi_2 \, d\sigma_i.$$

In order to analyze the hermiticity of $H$, we start from equation (4) and write

$$0 = \int \left[ \sqrt{-g} \left[ \partial_t \bar{\psi}_1^\dagger \gamma^0 \gamma^i \psi + \psi^\dagger \gamma^0 \gamma^i \partial_t \psi \right] + \left[ (\partial_t \sqrt{-g}) \bar{\psi}_1^\dagger \gamma^0 \gamma^i \psi + \sqrt{-g} \psi^\dagger \gamma^0 (\partial_t \gamma^i) \psi \right] \right] d^3x,$$

then use $\gamma^i_\alpha = e^i_\alpha \gamma^\alpha$, as well as the Dirac equation $H \psi = i \partial_t \psi$ and its conjugate $\psi^\dagger H^\dagger = -i \partial_t \psi^\dagger$. After some simple manipulations, we finally find

$$0 = i \left[ (H \psi, \psi) - (\psi, H \psi) \right] + (\psi, \left[ \partial_t \ln \sqrt{-g} g^{tt} \right] \psi),$$

which shows that $H$ is, in general, not hermitian. In order to write the second term in this simple form, we have assumed that $e^i_\alpha = 0$ for $\alpha = 1, 2, 3$, which is always possible by a suitable Lorentz rotation (since $e^i_\alpha$ is timelike). Moreover, we can define a hermitian operator

$$\tilde{H} = H + \frac{i}{2} \partial_t \ln \sqrt{-gg^{tt}},$$

satisfying

$$\langle \tilde{H} \psi, \psi \rangle = (\psi, \tilde{H} \psi).$$

In a similar way, we derive the relation $0 = i \left[ (p_\mu \psi, \psi) - (\psi, p_\mu \psi) \right] + (\psi, \left[ \partial_\mu \ln \sqrt{-g} g^{tt} \right] \psi)$, and define the hermitian momentum operators $\tilde{p}_\mu = p_\mu + \frac{i}{2} \partial_\mu \ln \sqrt{-gg^{tt}}$.

Summarizing, we have the four relations

$$0 = i \left[ (p_i \psi, \psi) - (\psi, p_i \psi) \right] + (\psi, \left[ \partial_i \ln \sqrt{-g} g^{tt} \right] \psi),$$

showing the non-hermiticity of the four momentum operator $i \partial_i$, as well as the hermitian four momentum operator defined as

$$\tilde{p}_i = p_i + \frac{i}{2} \partial_i \ln \sqrt{-gg^{tt}}$$

and satisfying

$$\langle \tilde{p}_i \psi, \psi \rangle = (\psi, \tilde{p}_i \psi).$$
2. Field energy and field momentum

Until now, the Hamiltonian \( H \) (and more generally, the momentum \( p_i \)) has been introduced by hand, in an attempt to render the Hamiltonian \( H \) (and the momentum \( p_i \)) hermitian. It turns out, however, that there is a straightforward, canonical, way to arrive at \( p_i \). Indeed, the expectation value of \( p_i \) is simply the canonical field momentum of the Dirac field.

Consider the canonical stress-energy tensor of the Dirac field, 
\[ t^i_k = \frac{\partial L}{\partial (\frac{\partial \gamma^i}{\partial t})} \partial_k \psi + \partial_k \bar{\psi} \frac{\partial L}{\partial (\partial_t \gamma^i)} - \delta^i_k L, \] (13)
as well as the corresponding field momentum 
\[ P_k = \int \sqrt{-g} t^i_k \ d\sigma_i = \int \sqrt{-g} t_k^i \ d^3x. \] (14)
A detailed discussion of the stress-energy tensor and on the conservation of the (total) momentum in curved spacetime can be found in [8].

It is well known that in flat spacetime the explicit expression for the field energy \( H = P_t \) is given by the expression 
\[ H = \int \bar{\psi} H \psi \ d^3x = \langle \psi, H \psi \rangle. \] (15)
Therefore, in flat space, the field Hamiltonian \( \mathcal{H} \) (which plays, in quantum field theory, the role of the generator of time translations), is simply given by the expectation value of the Dirac Hamiltonian \( H \), i.e., of the generator of the time translations of the quantum mechanical wave function.

From \( t^i_t = (i/2)(\bar{\psi} \gamma^i \partial_t \psi - \partial_t \psi \gamma^i \bar{\psi}) \), partially integrating using (4) and then the Dirac equation, or, alternatively, first the Dirac equation and then equation (7), it is not hard to show that in the general case, we find 
\[ \mathcal{H} = \int \sqrt{-g} \bar{\psi}^\dagger (\gamma^0 \gamma^t) \bar{H} \psi \ d^3x = \langle \psi, \bar{H} \psi \rangle, \] (16)
with \( \bar{H} \) from (8). Similar relations hold for the spatial components of \( P_i \), and altogether, we have 
\[ P_i = \int \sqrt{-g} \bar{\psi}^\dagger (\gamma^0 \gamma^i) \bar{p}_i \psi \ d^3x = \langle \psi, \bar{p}_i \psi \rangle. \] (17)
This result not only provides us with an interpretation of \( \bar{p}_i \), namely as the operator whose expectation values correspond to the field momentum, but in addition, it justifies, a posteriori, the introduction of this arbitrarily defined operator.

3. Second quantization

A generally, manifestly covariant formalism for classical and quantum Hamiltonian field theory on a curved background has been described in detail in [9]. The main feature is that an explicit (3+1) splitting of spacetime is avoided by replacing the conventional volume integrals by integrals over (an unspecified) spacetime hypersurface \( \sigma \) with (normalized) normal vector \( n_i \). The momentum canonically conjugated to a field \( \varphi \), i.e., \( \pi = \partial \mathcal{L}/\partial \dot{\varphi} \) is replaced by the definition \( \pi^m = \partial \mathcal{L}/\partial \varphi_m \), and the physical component of \( \pi^m \) corresponds to \( \pi = \pi^m n_m \). (Note that we are dealing with spacetime densities, since \( \mathcal{L} = \sqrt{-g}L \).) In a next step, a Poisson bracket is constructed and quantization is performed invoking a correspondence principle between the elementary Poisson bracket relations of fields and momenta and the (anti)commutator of the corresponding quantum operators. We refer to our article [9] for details, and will confine ourselves here to the presentation of the specific aspects arising in the case of the Dirac field.
3.1. Hermitian quantization

While the above procedure turns out to be straightforward in the case of a scalar fields, difficulties arise in the Dirac case. First, the Poisson bracket has to be adapted to the case of spinor fields (we need a symmetric construction, see [9]). The larger problem concerns the correct choice of the canonical variables. Formally, we have in the Lagrangian (1) two independent fields \( \psi \) and \( \tilde{\psi} \), related by hermitian conjugation, giving us the two corresponding momentum variables \( \pi^i = \frac{1}{2} \sqrt{-g} \gamma^i \tilde{\psi} \) and \( \tilde{\pi}^i = -\frac{1}{2} \sqrt{-g} \gamma^i \psi \) (recall that only the component in \( n_i \) direction is the physical canonical momentum). Obviously, the variables \( (\psi, \tilde{\psi}, \pi, \tilde{\pi}) \) are not independent, and we have two constraints in the theory, which are easily shown to be second class. In order to get a consistent theory with such constraints, it is necessary to modify the classical Poisson bracket (in order to exclude non-physical degrees of freedom) before passing over to the second quantization (see Dirac [10] for details). Fortunately, since the quantization of the special relativistic Dirac theory is well known, we can directly inspire ourselves from the corresponding theory and put it into a generally covariant form.

The consistent way is to use \( (\psi, \pi) \) as canonically conjugate variables, with \( \pi^i = i \sqrt{-g} \tilde{\psi} \gamma^i \) (the change by a factor two is a result of the Dirac procedure, i.e., the elimination of second class constraints, see [10]). The Poisson bracket can be defined as (see [9] for details)

\[
\{A, B\}_\sigma = \int_\sigma \left( \frac{\delta A}{\delta \psi^M(z)} \frac{\delta B}{\delta \pi^m_M(z)} + \frac{\delta A}{\delta \pi^m_M(z)} \frac{\delta B}{\delta \psi^M(z)} \right) \, d\sigma^m(z),
\]

(18)

where we use capital letters \( K, L, M, N \ldots \) to denote spinor indices\(^1\).

The following Poisson brackets are easily derived (for \( x, y \) on \( \sigma \))

\[
\{\psi^N(x), \psi^M(y)\}_\sigma = 0, \quad \{\pi^j_N(x), \pi^k_M(y)\}_\sigma = 0, \quad \{\psi^N(x), \pi^j_M(y)\}_\sigma = \delta^N_M \delta^j_i (x - y).
\]

(19)

Those relations are identical to those used (in the quantum theory) by Schwinger ([11]). Note that the asymmetry between \( \psi \) and \( \tilde{\psi} \) is only apparent. The corresponding relation for \( \tilde{\pi} \) and \( \pi = \gamma^0 \pi^1 \) is easily found by hermitian conjugation of (19). In other words, we could equally well start with the canonical pair \( (\tilde{\psi}, \tilde{\pi}) \) and construct the Poisson bracket accordingly.

Let us note that \( \pi^i = i \sqrt{-g} \tilde{\psi} \gamma^i \) can be inverted to \( \tilde{\psi} = -i (\sqrt{-g})^{-1} \pi^i n_i \gamma^k n_k \), where the relation \( \gamma^i n_i \gamma^k n_k = 1 \) has been used.

We now consider second quantization. We apply the correspondence principle

\[
\{A, B\}_\sigma \rightarrow \frac{1}{i} \{A, B\} = \frac{1}{i} (AB + BA),
\]

(20)

which leads us to the canonical anticommutation relations

\[
i\{\psi^N(x), \psi^M(y)\} = 0, \quad i\{\pi^j_N(x), \pi^k_M(y)\} = 0, \quad i\{\psi^N(x), \pi^j_M(y)\} = -\delta^N_M \delta^j_i (x - y).
\]

(21)

For the field momentum field (14), we find

\[
\mathcal{P}_k = \int_\sigma \frac{i}{2} \sqrt{-g} \tilde{\psi} \gamma^i \psi_k \, d\sigma_i - \int_\sigma \frac{i}{2} \sqrt{-g} \tilde{\psi} \gamma^i \psi_k \, d\sigma_i,
\]

(22)

where we have used the fact that \( \mathcal{L} = 0 \) on shell. Next, we perform a partial integration of the second term, use the expression for \( \tilde{\psi} \) in terms of \( \pi^i \), as well as the fact that

\(^1\) Upper indices are used for spinors transforming like \( \psi \) under Lorentz gauge transformations, \( \psi \rightarrow A \psi \), while lower indices are used for quantities transforming with the inverse, like \( \psi \rightarrow \psi A^{-1} \).
\[ \int_{\sigma} (\sqrt{-g} \bar{\psi} \gamma^i \psi) d\sigma_i = 0, \] as is shown by using the general relation \( \int_{\sigma} f_i d\sigma_k = \int_{\sigma} f_k d\sigma_i \) (see [11]) as well as the relation (4). The result is

\[ P_k = \int_{\sigma} \pi^i \psi, k d\sigma_i + \frac{1}{2} \int_{\sigma} \frac{1}{\sqrt{-g}} \pi^k n_k \gamma^i n_l (\sqrt{-g} \gamma^i), k \psi d\sigma_i. \tag{23} \]

Performing another partial integration, we can also write

\[ P_k = -\int_{\sigma} \pi^i, k \psi d\sigma_i + \frac{1}{2} \int_{\sigma} \frac{1}{\sqrt{-g}} \pi^k n_k \gamma^i n_l (\sqrt{-g} \gamma^i), k \psi d\sigma_i. \tag{24} \]

The following commutators are now easily derived

\[ [P_k, \psi] = -i \pi^i, k - \frac{i}{2} \frac{1}{\sqrt{-g}} \gamma^m, n_m (\sqrt{-g} \gamma^i), k n_l \psi \tag{25} \]

\[ [P_k, \pi^l] = -i \pi^i, k n_l + \frac{i}{2} \pi^k n_k (\gamma^m n_m)(\sqrt{-g} \gamma^i), k n_l n_l. \tag{26} \]

Quite generally, for an operator \( O \) that corresponds to the expectation value of a Dirac space operator \( O, \) i.e., \( O = (\psi, O \bar{\psi}) = -i \int_{\sigma} \pi^i O \psi d\sigma_i \) (see (6)), we obtain

\[ [O, \psi] = -O \psi. \tag{27} \]

In particular, for \( P_k, \) we should thus have, according to (17)

\[ [P_k, \psi] = -\tilde{p}_k \psi, \tag{28} \]

where \( \tilde{p}_k \) is the hermitian momentum operator. Indeed, for the hypersurface\(^2\) \( t = t_0 \) and assuming a gauge \( e^i_\alpha = 0, \alpha = 1, 2, 3, \) we find from (25)

\[ [P_k, \psi] = -i \left( \partial_k + \frac{1}{2} (\partial_k \ln \sqrt{-g} g^{it}) \right) \psi, \tag{29} \]

which is in perfect agreement with (11).

We conclude that \( P_k \) is not the generator of spacetime translations. This is in direct correspondence with our result of the previous section, namely that \( P_k \) is not the expectation value of the operator \( p_k = i\partial_k, \) but rather of the hermitian operator \( \tilde{p}_k. \) Also, from (27) it is clear what operator corresponds to the generators of translations: It is the expectation value of \( p_k. \) I.e., if we define \( P_k^{(1)} = (\psi, p_k \psi), \) then we have \( [P_k^{(1)}, \psi] = -p_k \psi = -i \partial_k \psi. \) There is only one problem: the operator \( p_k \) is not hermitian. Therefore, there is a second operator \( P_k^{(2)} = (p_k \psi, \psi), \) which could equally well be used to define the expectation value of \( p_k. \) How can we decide which of those operators, if any, corresponds to the physical field momentum? The reason for those ambiguities will become clear in the next subsection.

Finally, let us remark that the results are not an artifact of the quantization process. The same results can be obtained in the classical theory, see [9]. The reason is also obvious: It is the result of the fact that, due to the constraints in the theory, we were forced to deviate slightly from the canonical procedure. Thus, ultimately, the problem originates from the fact that the Lagrangian theory contains two independent field variables \( \psi \) and \( \tilde{\psi}, \) but the corresponding Hamiltonian theory is constructed only from one canonically conjugate pair of variables, \( \psi \) and \( \pi^i. \) This is not avoidable (anything else leads to inconsistencies), but it is also not unproblematic and leads to problems such as the above, concerning the interpretation of the field momentum.

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\(^2\) Note that for \( t = t_0, \) we have \( n_i = (1/\sqrt{g^{tt}}, 0, 0, 0), \) in view of the normalization \( n_i n_k g^{ik} = 1. \)
3.2. Non-hermitian quantization

Consider the Dirac Lagrangian (1) in the following form

\[ L = \sqrt{-g} \left[ \frac{i}{2} (\bar{\psi} \gamma^i \partial_i \psi - \partial_i \bar{\psi} \gamma^i \psi) \right] + L_{\text{int}}, \tag{30} \]

where \( L_{\text{int}} \) contains the interaction terms between the spin connection and the spinor fields, as well as the mass term. This can equivalently be written in the form

\[ L = \sqrt{-g} \, i \bar{\psi} \gamma^i \partial_i \psi + \frac{i}{2} \bar{\psi} (\gamma^i \sqrt{-g} \gamma_i \psi) - (\sqrt{-g} \, \bar{\psi} \gamma^i \psi)_i + L_{\text{int}}, \tag{31} \]

Omitting the surface term, we find the following Lagrangian

\[ L^{(1)} = \sqrt{-g} \, i \bar{\psi} \gamma^i \partial_i \psi + \frac{i}{2} \bar{\psi} (\gamma^i \sqrt{-g} \gamma_i \psi) + L_{\text{int}}, \tag{32} \]

which leads upon variation with respect to \( \psi \) and \( \bar{\psi} \) to the same equations as (30). Note that the role of the field variable \( \psi \) has essentially been reduced to that of a Lagrange multiplier. Further, we recall that \( L \) is on-shell zero and that \( (\sqrt{-g} \, \bar{\psi} \gamma^i \psi)_i \) is also zero (charge conservation, see, e.g., [2]). As a result, \( L^{(1)} \) too is on-shell zero. The canonical stress-energy tensor

\[ \sqrt{-g} \, t^{(1)}_{k} = \frac{\partial L^{(1)}}{\partial (\partial_i \psi)} \partial_k \psi + \partial_k \bar{\psi} \frac{\partial L^{(1)}}{\partial (\bar{\partial}_i \psi)} - \delta^{i}_{k} L^{(1)} \tag{33} \]

therefore reduces to

\[ \sqrt{-g} \, t^{(1)}_{k} = \sqrt{-g} \, \bar{\psi} \gamma^i \psi_k, \tag{34} \]

and the corresponding field momentum is of the form

\[ p^{(1)}_{k} = i \int_{\sigma} \sqrt{-g} \, \bar{\psi} \gamma^i \psi_k \, d\sigma_i, \tag{35} \]

which differs from (23) by the second term in the latter. The fact that the omission of a surface term in \( L \) leads to a modification of the stress-energy tensor (a so-called relocalization) is well known. That it leads to a modification of the integrated momentum is a little bit more surprising. It is, however, quite natural. The reason can be traced back to the fact that a strict separation between the energy and momentum of the dynamical fields (in our case, the Dirac field) on one hand and the non-dynamical background fields (in our case, gravity) is devoid of physical sense. In other words, it is rather a matter of convention which amount of energy (momentum) is attributed to one or the other part, only the total energy (momentum) being of physical relevance. (This is not a particularity of gravity. See [9] for a different example.)

Thus, although usually, surface terms lead to a relocalization of the stress-energy, but leave the integrated momentum vector unchanged, in the presence of background fields, this is not true anymore. Apart from a relocalization, a momentum transfer between dynamical and background fields is induced by surface terms. Both relocalizations and momentum transfers, however, should not be physically relevant. It is quite a matter of convention whether we attribute, e.g., the potential energy of an electron in the Coulomb field of a proton either to the electron or to the electromagnetic field.
Having reduced \( \bar{\psi} \) to a Lagrange multiplier, we consider \( \psi \) as the only true field variable in \( \mathcal{L}^{(1)} \) and define
\[
\pi^i = \frac{\partial \mathcal{L}^{(1)}}{\partial \dot{\psi}^i} = i \sqrt{-g} \ \bar{\psi} \gamma^i \, ,
\] (36)
and postulate again the anticommutation relations
\[
i \{ \psi^N(x), \psi^M(y) \} = 0, \quad i \{ \pi^N_j(x), \pi^k_j(y) \} = 0, \quad i \{ \psi^N(x), \pi^j_M(y) \} = -\delta^N_M \delta^j_i (x - y). \] (37)
According to (35), we have \( \mathcal{P}^{(1)}_k = \int_\sigma \pi^i \psi, k d\sigma_i \), and the following commutators are straightforwardly evaluated
\[
[\mathcal{P}^{(1)}_k, \psi] = -i \psi, k = -p_k \psi \quad (38)
\]
\[
[\mathcal{P}^{(1)}_k, \pi^l] = -i \pi^l, k n_i n^l \, .
\] (39)
which are now in complete correspondence to the bosonic case (see [9]) and \( \mathcal{P}^{(1)}_k \) can be interpreted as generator of spacetime translations. Consistently, we also have that \( \mathcal{P}^{(1)}_k = (\psi, p_k \psi) \).

It is needless to say that instead of \( \bar{\psi} \), we can also eliminate \( \psi \) as dynamical variable and write
\[
\mathcal{L}^{(2)} = -\sqrt{-g} \ i \partial_i \bar{\psi} \gamma^i \psi - \frac{i}{2} \bar{\psi} (\gamma^i \sqrt{-g}), i \psi + \mathcal{L}_{\text{int}},
\] (40)
where the same surface term with the opposite sign has been omitted this time. The canonical momentum is now given by \( \bar{\pi}^i = -i \sqrt{-g} \ \gamma^i \psi \), and assuming \( i \{ \bar{\psi}^N(x), \bar{\pi}^l_M(y) \} = +\delta^N_M \delta^l_i (x - y) \), we find
\[
[\mathcal{P}^{(2)}_k, \bar{\psi}] = -i \bar{\psi}, k, \quad [\mathcal{P}^{(2)}_k, \bar{\pi}^l] = -i \bar{\pi}^l, k n_i n^l \, ,
\] (41)
where we have \( \mathcal{P}^{(2)}_k = (p_k \psi, \psi) \).

### 3.3. Discussion

We have given two approaches to the canonical quantization for the Dirac field in curved spacetime. The first way is based on the manifestly hermitian Lagrangian and preserves the symmetry between the fields \( \psi \) and \( \bar{\psi} \). The field momentum \( \mathcal{P}_k \) is equal to the expectation value of the hermitian momentum \( \bar{p}_k \), but does not generate spacetime translations on the quantum fields. The second way starts from a non-hermitian Lagrangian, where one of the field variables \( \psi \) or \( \bar{\psi} \) has been eliminated by adding a suitable surface term to the Lagrangian. In this approach, the field momentum is equal to the expectation value of the non-hermitian momentum operator \( p_k = i \partial_k \) and corresponds to the generator of the spacetime translations.

On a classical level, we have traced back the differences between both approaches to a momentum transfer from the Dirac field to the gravitational background, induced by the surface term in the Lagrangian. Since the assignment of the potential (or interaction) energy to one or another of the interacting fields (Dirac and gravitational) is rather a matter of convention, both approaches should be physically equivalent. In order to establish this equivalence on the quantum level, further investigations are necessary. We suspect that both approaches can be

\footnote{For consistency with (37), we have to use the opposite sign, such that one relation results from hermitian conjugation of the other.}
related by a change of representation in operator space, involving a non-unitary transformation, very similar to the Dirac space transformation that we presented in [2] to relate the hermitian momentum $\hat{p}_k$ to the non-hermitian translation operator $p_k = i\phi_k$. An illustration of how this could work has been indicated in [9].

Finally, it is interesting to remark that in flat spacetime, both approaches are equivalent, and the generator of translations in either approach is automatically given in terms of the field momentum. It is actually quite common, in standard textbooks, to use the non-hermitian version of the Dirac Lagrangian and to omit the discussion related to the second class constraints completely. Our analysis shows that the situation in curved spacetime needs to be treated with more care.

4. Conclusions

It has been shown that the quantum mechanical Dirac Hamiltonian, as well as the momentum operator, are not hermitian in the presence of gravitational background fields. Hermitian operators have been defined and they turned out to correspond directly to the canonical field momentum of the theory.

Further, two ways of quantization have been presented. We first performed a manifestly hermitian quantization, where it turned out that the field momentum does not correspond to the generator of spacetime translations, but rather to the generator of modified translations, given by the previously introduced hermitian momentum operator. An alternative, not manifestly hermitian quantization was achieved by modifying the Lagrangian by a surface term. In this approach, the field momentum corresponds directly to the generator of translations, which are, however, given in terms of non-hermitian Dirac operators. The change of the field momentum induced by the surface term was interpreted on a classical level as momentum transfer between the Dirac field and the gravitational background field. Both approaches should be physically equivalent. In order to show this on the quantum level, further investigations are necessary. It is expected that the addition of a surface term to the Lagrangian results in a change of the representation, such that both approaches can be related by a non-unitary transformation in operator space.

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