A note on the foundation of relativistic mechanics

II: Covariant hamiltonian general relativity

Carlo Rovelli
Centre de Physique Théorique, Luminy, F-13288 Marseille, EU

Physics Department, Pittsburgh University, PA-15260, USA
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I illustrate a simple hamiltonian formulation of general relativity, derived from the work of Esposito, Gionti and Stornaiolo, which is manifestly 4d generally covariant and is defined over a finite dimensional space. The spacetime coordinates drop out of the formalism, reflecting the fact that they are not related to observability. The formulation can be interpreted in terms of Toller’s reference system transformations, and provides a physical interpretation for the spinnetwork transition amplitudes computable in principle in loop quantum gravity and in the spin foam models.

I. THE PROBLEM

In the companion paper [1], I have discussed the possibility of a relativistic foundation of mechanics and I have argued that the usual notions of state and observable have to be modified in order to work well in a relativistic context. Here I apply this point of view to field theory. In the context of field theory, the relativistic notion of observable suggests to formulate the hamiltonian theory over a finite dimensional space, for two reasons. First the space of the relativistic ("partial" [2]) observables of a field theory is finite dimensional. Second, the infinite dimensional space of the initial values of the fields, which is the conventional arena for hamiltonian field theory, is based on the notion of instantaneous surface, which has little general significance in a relativistic context. The possibility of defining hamiltonian field theory over a finite dimensional space has been explored by several authors (See [3–5], and ample references therein), developing the classic works of Weil [6] and De Donder [7] on the calculus of variations in the 1930’s. In section II I briefly illustrate the main lines of this formulation using the example of a scalar field, and I discuss its relation with the relativistic notions of state and observable considered in [1]. I then apply these ideas to general relativity (GR) in Section II.

Unraveling the hamiltonian structure of GR has taken decades. The initial intricacies faced by Dirac [8] and Bergmann [9] were slowly reduced in various steps, from the work of Arnowit, Deser and Misner [10] all the way to the Ashtekar formulation [11] and its variants. Here, I discuss a far simpler hamiltonian formulation of GR, constructed over a finite dimensional configuration space and manifestly 4d generally covariant. The formulation is largely derived from the work of Esposito, Gionti and Stornaiolo in [12]. (On covariant hamiltonian formulations of GR, see also [13–18].) I discuss two interpretations of this formalism. The first uses the coordinates, while the second makes no direct reference to spacetime. The four spacetime coordinates drop out from the formalism (as the time coordinate drops out from the ADM formalism) in an appropriate sense. I find this feature particularly attractive: the general relativistic spacetime coordinates have no relation with observability and a formulation of the theory in which they do not appear was long due.

I expect this formulation of GR to generalize to any matter coupling and to any diffeomorphism invariant theory. I think that it sheds some light on the coordinate-independent physical interpretation of the theory and helps clarifying issues that appear confused in the hamiltonian formulations which are not manifestly covariant. In particular: what are “states” and “observables” in a theory without background spacetime, without external time and without an asymptotic region? I close discussing the relevance of this analysis for the problem of the interpretation of the formalism of quantum gravity. The formulation presented can be interpreted in terms of Toller’s reference system transformations [19], and provides a physical interpretation for the spinnetwork transition amplitudes which can be computed in principle in loop quantum gravity [20] and in the spin foam models [21].

II. RELATIVISTIC

HAMILTONIAN FIELD THEORY

There are several ways in which a field theory can be cast in hamiltonian form. One possibility is to take the space of the fields at fixed time as the (nonrelativistic) configuration space $Q$. This strategy badly breaks special and, in a general covariant theory, general relativistic invariance. Lorentz covariance is broken by the fact that one has to choose a Lorentz frame for the $t$ variable. I find far more disturbing the conflict with general covariance. The very foundation of general covariant physics is the idea that the notion of a simultaneity surface all over the universe is devoid of physical meaning. Seems to me that it is better not to found hamiltonian mechanics on a notion devoid of physical significance.

A second alternative is to formulate mechanics on the
space of the solutions of the equations of motion. The idea goes back to Lagrange. In the generally covariant context, a symplectic structure can be defined over this space using a spacelike surface, but one can show that the definition is surface independent and therefore it is well defined. This strategy as been explored by Witten, Ashtekar and several others \cite{18}. The structure is viable in principle, but very difficult to work with in practice. The reason is that we do not know the space of the solutions of an interacting theory. Therefore we must either work over a space that we can’t even coordinatize, or coordinatize the space with the initial data on some instantaneousy surface, and therefore, effectively, go back to the conventional fixed time formulation. Thus, this formulation has the merit of telling us that the hamiltonian formalism is actually intrinsically covariant, but it does not really indicate how to effectively deal with it in a covariant manner.

The third possibility, which I consider here, is to use a covariant finite dimensional space for formulating hamiltonian mechanics. I noticed in the companion paper \[\text{[1]}\] that in the relativistic context the double role of the phase space, as the arena of mechanics and the space of the states, is lost. The space of the states, namely the phase space $\Gamma$ is infinite dimensional in field theory, virtually by definition of field theory. But this does not imply that the arena of hamiltonian mechanics has to be infinite dimensional as well. In particular, a main result of \[\text{[1]}\] is that the natural arena for relativistic mechanics is the extended configuration space $C$ of the partial observables. Is the space of the partial observables of a field theory finite or infinite dimensional?

Consider a field theory for a field $\phi(x)$ with $k$ components, defined over spacetime $M$ with coordinates $x$, and taking values in a $k$ dimensional target space $T$ 

$$\phi : M \rightarrow T$$

$$x \mapsto \phi(x). \quad (1)$$

For instance, this could be Maxwell theory for the electric and magnetic fields $\phi = (\vec{E}, \vec{B})$, where $k = 6$. In order to make physical measurements on the field described by this theory we need $k$ measuring devices to measure the components of the field $\phi$, and 4 devices (say one clock and three devices giving us the distance from three reference objects) to determine the spacetime position $x$. Field values $\phi$ and positions $x$ are therefore the partial observables of a field theory. Therefore the operationally motivated extended configuration space for a field theory is the finite dimensional $4+k$ dimensional space

$$C = M \times T. \quad (2)$$

A correlation is a point $(x, \phi)$ in $C$. It represents a certain value $(\phi)$ of the fields at a certain spacetime point $(x)$. This is the obvious generalization of the $(t, \alpha)$ correlations of the pendulum of the example in \[\text{[1]}\].

A motion is a physically realizable ensemble of correlations. A motion is determined by a solution $\phi(x)$ of the field equations. Such a solution determines a 4-dimensional surface in the $(4+k)$ dimensional space $C$. The space of the solutions of the field equations, namely the phase space $\Gamma$, is therefore an (infinite dimensional) space of 4d surfaces in the $(4+k)$-d configuration space $C$. Each state in $\Gamma$ determines a surface in $C$.

Hamiltonian formulations of field theory defined directly on $C = M \times T$ are possible and have been studied. The main reason is that in a local field theory the equations of motion are local, and therefore what happens at a point depends only on the neighborhood of that point. There is no need, therefore, to consider full spacetime to find the hamiltonian structure of the field equations. I refer the reader to \[\text{[3,5]}\] the ample references therein, and especially the beautiful and detailed \[\text{[4]}\]. What comes below is a very simple and self-contained illustration of the formalism.

The difference with the finite dimensional case is that curves in $C$ are replaced by 4d surfaces. Thus, we need a hamiltonian formalism determining these four dimensional surfaces in $C$. At a point $p$ of $C$, a curve has a tangent, leaving in $T_pC$, the tangent space of $C$ at $p$. A 4d surface has four independent tangents $X_\mu$, or a “quadrivet” $X = \epsilon^{\mu\nu\rho\sigma} X_\mu \otimes X_\nu \otimes X_\rho \otimes X_\sigma$.

Consider a self interacting scalar field $\phi(x)$ defined on Minkowski space $M$, with interaction potential $V(\phi)$. Its field equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi + V'(\phi) = 0 \quad (3)$$

can be derived from a hamiltonian formalism as follows. To test the theory \[\text{[3]}\] we need five measuring devices: a clock reading $x^0$, three devices that give us the spatial position $\vec{x}$, and a device measuring the value of the field $\phi$. Therefore, the space $C$ of the partial observables is the five dimensional space $C = M \times T$, with coordinates $(x, \phi)$. Here $T = \mathbb{R}$ is the target space of the field: $\phi \in \mathbb{R}$. Let $\Omega$ be a 10d space with coordinates $(x, \phi, p, p')$ carrying the the Poincaré-Cartan four-form

$$\theta = p \, dt \land x + p' \, d\phi \land dx^3 \land dx^4. \quad (4)$$

Here $dt \land x \equiv dx^1 \land dx^2 \land dx^3 \land dx^4$ and $d^3 x_\mu \equiv d^4 x(\partial_\mu)$. (Geometrically, $\Omega$ is not the cotangent space of $C$; it is a subspace of the bundle of its four-forms, or a dual first jet bundle \[\text{[3]}\].) Consider the constraint

$$H = p + (\frac{1}{4} m^4 \, p_\mu + \frac{1}{2} m^2 \phi^2 + V(\phi)) = 0 \quad (5)$$

on $\Omega$. (Here $p$ is the DeDonder-Weyl hamiltonian function.) Let $\omega$ be the restriction of the five form $d\theta$ to the surface $\Sigma$ defined by $H = 0$. Then the solutions of $\omega$ are the orbits of $\omega$. An orbit of $\omega$ is an integral surface of its null quadriectors. That is, it is a 4d surface immersed in $\Sigma$ whose quadri-tangent $X$ satisfies

$$\omega(X) = 0. \quad (6)$$
For simplicity, let’s say that this surface can be coordi-

nized by \( x^\mu \), that is, it is given by \( x, \phi(x), p^\nu(x) \). Then \( \phi(x) \) is a solution of the field equa-

tions \((\mathcal{B})\). Thus, \((\mathcal{B})\) is equivalent to the field equations \((\mathcal{B})\).

A state determines a 4d surface \( (x, \phi(x)) \) in the ex-

tended configurations space \( \mathcal{C} \). It represents a set of com-

binations of measurements of partial observables that can be real-

ized in nature. The phase space \( \mathcal{F} \) is the space of these

states, and is infinite dimensional.

In the classical theory, a state determines whether or not a certain correlation \( (x, \phi) \), or a certain set of correla-
tions \((x_1, \phi_1), \ldots (x_n, \phi_n)\), can be observed. They can be ob-
erved iff the points \((x_i, \phi_i)\) lie on the 4d surface that
represents the state. Viceversa, the observation of a certain set of correlations gives us information on the
state: the surface has to pass by the observed points. In the
quantum theory, a state determines the probability amplitude of observing a correlation, or a set of correla-

tions \((\mathcal{D})\).

Notice that there is an important difference between a
finite dimensional system and a field theory. For the first, the measurement of a finite number of correlations
can determine the state. In the quantum theory, a single
correlation may determine the state. For instance, if we have seen the pendulum in the position \( \alpha \) at
time \( t \), we then know the quantum state uniquely. It follows that quantum phenomena determines uniquely the
probability amplitude \( W(\alpha', t'; \alpha, t) \) for observing a correlation \( (\alpha', t') \) after having observed a correlation \( (\alpha, t) \). Clearly

\[
W(\alpha', t'; \alpha, t) = \langle \alpha', t' | \alpha, t \rangle
\]

(7)

where \( | \alpha, t \rangle \) is the eigenstate of the Heisenberg oper-
ator \( \hat{\alpha}(t) \) with eigenvalue \( \alpha \). In field theory, on the other hand, an infinite number of measurement is required in principle to uniquely determine the state. We can measure any finite number of correlations, and still do not
know the state. Predictions in field theory are therefore always given on the basis of some a priori assumption
on the state. In quantum theory, this additional assumption, typically, is that the field is in a special state such as
the vacuum. Thus, a prediction of the quantum theory takes the
following form: if the system is in the vacuum state and we observe a certain set of correlations \((x_1, \phi_1), \ldots (x_n, \phi_n)\), what is the probability amplitude

\[
W(x_1', \phi_1', \ldots x_n', \phi_n'; x_1, \phi_1, \ldots x_n, \phi_n)
\]

(8)
of observing a certain other set \((x_1', \phi_1'), \ldots (x_n', \phi_n')\) of cor-
relations? The quantities \((\mathcal{E})\) are directly related to the
usual \( n \) point distributions of field theory. The relation is the same as the one that transforms the position basis to the energy basis for an harmonic oscillator, that is, for instance

\[
W(x', \phi'; x, \phi) = \sum_{n,m} \bar{\nu}_n(\phi') \psi_m(\phi) \langle n, x'| m, x \rangle
\]

(9)

where \( | n, x \rangle \sim (a^\dagger(x))^n | 0 \rangle \) is the state with \( n \) particles in
\( x \). The distributional character of these quantities will
be studied elsewhere.

For later comparison with GR, notice that the spacetime
component \( \mathcal{M} \) of the relativistic configuration space
\( \mathcal{C} = \mathcal{M} \times \mathcal{T} \) is essential in the description, since the
predictions of the theory regard precisely the dependence of
the partial observable \( \phi \) on the partial observables \( x^\mu \).

The simplicity, covariance and elegance of this hamil-
tonian formulation is quite remarkable. I find it par-
ticularly attractive from the conceptual point of view, because the notions of observable and state on which it
is based are operationally well founded, relativistic and
covariant. I now apply these ideas to GR.

III. COVARIANT HAMILTONIAN GR

GR can be formulated in tetrad-Palatini variables as
follows. I indicate the coordinates of the spacetime
manifold \( M \) as \( x^\mu \), where \( \mu = 0, 1, 2, 3 \). The fields are a
tetrad field \( e^I_\mu \) and a Lorentz connection field \( A^I J_\mu \) (antisym-
metric in \( I, J \)) where \( I = 0, 1, 2, 3 \) is a 4d Lorentz index, raised and
lowered with the Minkowski metric. The action is

\[
S = \int e^I_\mu e^J_\nu F^{KL}_{\mu \nu} \epsilon^{\mu \nu \rho \sigma} \epsilon_{IJKL} d^4 x
\]

(10)

where \( F^{IJ}_{\mu \nu} \) is the curvature of \( A^{I J}_\mu \). The field equations
turn out to be

\[
D_\nu (e^I_\mu e^J_\nu) \epsilon^{\mu \nu \rho \sigma} \epsilon_{IJKL} = 0.
\]

(11)

where \( D_\mu \) is the covariant derivative of the Lorentz
connection. The first equation implies that \( A^{I J}_\mu \) is the
spin connection determined by the tetrad field. Using this,
the second is equivalent to the Einstein equations. That
is, if \( (e^I_\mu(x), A^{I J}_\mu(x)) \) satisfy \((\mathcal{F})\), then the metric
tensor \( g_{\mu \nu}(x) = e_{\mu}^I(x) e_{\nu}^J(x) \) satisfies the Einstein equations.
I shall thus refer at \((\mathcal{F})\) as the Einstein equations. Below,
these equations are derived from a hamiltonian formal-
ism, derived from \((\mathcal{G})\).

A. First version

Consider the \((4+16+24)\) dimensional space \( \tilde{\mathcal{C}} \) with co-
ordinates \((x^\mu, e^I_\mu, A^{I J}_\mu)\). We have \( \tilde{\mathcal{C}} = M \times \mathcal{T} \), where \( \mathcal{T} \)
is the target space on which the tetrad-Palatini fields of
GR take value and \( M \) is the spacetime manifold on which
they are defined.

\[
e^I_\mu = e^I_\mu dx^\mu, \quad A^{I J}_\mu = A^{I J}_\mu dx^\mu
\]

(12)

are one-forms on \( \tilde{\mathcal{C}} \). For any function or form on \( \tilde{\mathcal{C}} \) with
Lorentz indices, the Lorentz covariant differential is defined
by

\[
Dv^I = dv^I + A^{I J} v^J.
\]

(13)

I now introduce the main objects of the formalism: the
form on \( \tilde{\mathcal{C}} \)
\[
\theta = \frac{1}{2} \epsilon_{IJKL} e^I \wedge e^J \wedge DA^{KL}
\] (14)

and the presymplectic form \( \omega = d\theta \). (Notice that \( dA^{KL} = d(A^I_{\mu} dx^\mu) = dA^I_{\mu} \wedge dx^\mu \), because \( A^I_{\mu} \) and \( x^\mu \) are independent coordinates on \( C \).) Now, the remarkable fact is that the presymplectic form \( \omega \) defines GR completely. In fact, its orbits, defined by

\[
\omega(X) = 0
\] (15)

where \( X \) is the quadri tangent to the orbit, are the solutions of the Einstein equations. That is: assuming for simplicity that the \( x^\mu \) coordinatize the orbit, namely that the orbit is given by \((x^\mu, e^I_{\mu}(x), A^I_{\mu}(x))\), then it follows from (14) that \((e^I_{\mu}(x), A^I_{\mu}(x))\) solve the Einstein equations. Vice versa, if \((e^I_{\mu}(x), A^I_{\mu}(x))\) solve the Einstein equations, then \((x^\mu, e^I_{\mu}(x), A^I_{\mu}(x))\) is an orbit of \( \omega \). Equation (15) is equivalent to the Einstein equations (see Appendix).

The demonstration is a straightforward calculation given in the Appendix.

**B. Second version**

The simplicity of the formulation of GR described above is quite striking. I find the following observations even more remarkable. The space \( \mathcal{C} \) contains the field variables \( A^I_{\mu} \) and \( e^I_{\mu} \) as well as the spacetime coordinates \( x^\mu \). Since the theory is coordinate invariant, we are in a situation analogous to the finite dimensional cases of the relativistic particle, or the cosmological model, described in [1]. In those examples, the unphysical lagrangian evolution parameter \( t \) dropped out of the formalism; not surprisingly, since it had nothing to do with observability. Here, similarly, we should expect the coordinates \( x^\mu \) to drop out of the formalism. In fact, it is well known that the gauge invariant quantities of GR are coordinate independent. They are obtained by solving away the coordinates from quantities constructed out of the fields. Therefore the theory should actually live on the sole field space \( T \) with coordinates \( A^I_{\mu} \) and \( e^I_{\mu} \), without reference to the spacetime coordinates \( x^\mu \). Is this possible?

The remarkable aspect of the expression (14) of the form \( \theta \) is that the differentials of the spacetime coordinates \( dx^\mu \) appear only within the one-forms \( e^I = e^I_{\mu} dx^\mu \) and \( A^I_{\mu} = A^I_{\mu} dx^\mu \). This fact indicates that the sole role of the \( x^\mu \) is to arbitrarily coordinatize the orbits in the 4d space of the fields \((e^I_{\mu}, A^I_{\mu})\). We can therefore reinterpret the formalism of the previous section dropping the spacetime part of \( \mathcal{C} \). Let \( V \) be a 4d vector space. Clearly \( V \) is not spacetime; it can be interpreted as a “space of directions”. Let \( \mathcal{C} \) be the 4d space of the one-forms \((e^I_{\mu}, A^I_{\mu})\) on \( V \). Notice that \( \mathcal{C} \) is the space \( \mathcal{C} = V^* \otimes \mathcal{P} \) of the 4d one-forms with value in the algebra \( P \) of the Poincaré group. Choosing a basis \( a_\mu \) in \( V \), the coordinates on \( \mathcal{C} \) are \((e^I_{\mu}, A^I_{\mu})\) and \( C \) can be identified with the target space \( T \) of the tetrad-Palatini fields. Consider a 4d surface immersed in \( \mathcal{C} \). The tangent space \( T_p \) to this surface at a point \( p = (e^I_{\mu}, A^I_{\mu}) \) is a 4d vector space. This space can be identified with \( V \). In particular, given an arbitrary choice of coordinates \( x^\mu \) on the surface, we identify the basis \( \partial_\mu \) of \( T_p \) with the basis \( a_\mu \) of \( V \). Therefore we have immediately the ten one-forms \((e^I_{\mu}, A^I_{\mu})\) on the tangent space \( T_p \). That is, \( e^I(\partial_\mu) = e^I_{\mu} \) and \( A^I(\partial_\mu) = A^I_{\mu} \). Consider now a form \( \alpha = \alpha_I dx^I \) on \( C \). \( \alpha \) is a one-form on \( T_p \) (valued in the one-forms over \( C \)), and we can write \( \alpha(\partial_\mu) = \alpha_I dx^I_{\mu} + \alpha_{IJ} dA^J_{\mu} \). But notice that \( \alpha \) determines also a two-form on \( T_p \) by

\[
\alpha(\partial_\mu \otimes \partial_\nu) = \alpha_I \partial_\mu e^I_{\nu} + \alpha_{IJ} \partial_\mu A^J_{\nu}.
\] (16)

It follows that \( \omega = d\theta \), with \( \theta \) given by (14), acts on multivectors in \( T_p \). In particular, it acts on the quadri tangent \( X = e_{\mu
u} = \partial_\nu \otimes \partial_\nu \otimes \partial_\nu \otimes \partial_\nu \) of \( T_p \). The vanishing (15) of \( \omega(X) \) is equivalent to the Einstein equations (see Appendix).

Therefore the theory is entirely defined on the 4d space \( \mathcal{C} \). The states are the 4d surfaces in \( \mathcal{C} \), whose tangents are in the kernel of \( d\theta \). This is all of GR. The spacetime part \( M \) of \( \mathcal{C} \sim M \times \mathcal{C} \) is eliminated from the formalism. The only residual role of the \( x^\mu \) is to arbitrarily parametrize the states, precisely as for the unphysical lagrangian parameter in the examples of [1]. Below I study the physical interpretation of \( \mathcal{C} \).

**IV. PHYSICAL INTERPRETATION OF \( \mathcal{C} \)**

**A. Classical theory: reference system transformations**

As discussed in [1], in general, coordinates of the extended configuration space \( \mathcal{C} \) are the partial observables of the theory. A point in \( \mathcal{C} \) represents a correlation between these observables, that is, a possible outcome of a simultaneous measurements of the partial observables, which give information on the state of the system, or that can be predicted by the theory. What are the partial observables and the correlations of GR captured in the space \( \mathcal{C} \)? Can we give the space of the Poincaré algebra valued 4d one-forms \( C \) a direct physical interpretation?

Assume that the measuring apparatus includes a reference system formed by physical objects defining three orthogonal axes and a clock. Following Tolfer [19], we take a transformation \( T \) of the Poincaré group (in the reference system) as the basic operation defining the theory. That is, assume that the basic operation that we can perform is to displace the reference system by a certain length in a spacial direction, or wait a certain time, or rotate it by a certain amount, or boost it at a certain velocity. The operational content of GR can be taken to be the manner in which the transformations \( T \) fit together [19]. That is, we assume that a number of these
transformation, \( \mathcal{T}_1, ..., \mathcal{T}_n \), can be performed and that it is operationally meaningful to say that two transformations \( \mathcal{T}_r \) and \( \mathcal{T}_s \) start at the same reference system, or arrive at the same reference system, or one starts where the other arrives. We identify one transformation as the measurement of a partial observable (the value of the partial observable is given by the Poincaré group element giving the magnitude of the transformation).

A set of such measurements is therefore an oriented graph \( \gamma \) where each link \( l \) carries an element \( U_l \) of the Poincaré group. Arbitrarily coordinatize the nodes of the graph with coordinates \( y \). In the classical theory, we assume that arbitrarily many and arbitrarily fine transformations can be done, so that we can take the elements of the Poincaré group as infinitesimal, namely in the algebra, and the coordinates \( y \) as smooth. An infinitesimal transformation can therefore be associated to an infinitesimal coordinate change \( dy \). As there is a 10d algebra of available transformations, there is a 10d space of infinitesimal transformations and the coordinates \( y \) are ten dimensional. However, it is an experimental fact—coded in the theory—that rotations and boosts close and realize the Lorentz group. That is, the relations between sets of physical rotations or boosts are entirely determined kinematically by the Lorentz group. The same is not true for displacements. Therefore the space of the Lorentz group \[19\]. Coordinatize the remaining 4d space \( y \) not true for displacements. Therefore the space of the fibers is operationally meaningful to say that two transformations, \( T \), can be performed, and that it is operationally meaningful to say that two transformations \( T_r \) and \( T_s \) start at the same reference system, or arrive at the same reference system, or one starts where the other arrives. We identify one transformation as the measurement of a partial observable (the value of the partial observable is given by the Poincaré group element giving the magnitude of the transformation).

In the quantum theory, quantum discreteness does not allow us to go to the continuous description. A finite set of partial measurements must therefore be represented by the graph \( \gamma \) with elements of the Poincaré group \( U_l \) associated the links \( l \). If we restrict to configuration observables we can take the \( U_l \) to be in the Lorentz group, or, in gauge fixed form, in the rotation group. In general, quantum theory gives the probability amplitude to observe a certain ensemble of partial observables, given that a certain other ensemble of observables has been observed. See Section IV of \[1\] and \[23\]. Therefore, we should expect that the predictions of a quantum theory of GR can be cast in the form of probability amplitudes \( W(\gamma', U'_l; \gamma, U_l) \).

Now, the quantities \( W(\gamma', U'_l; \gamma, U_l) \) are precisely of the form spin network to spin network transition amplitudes which can be computed, in principle, in loop quantum gravity (see \[20\], and references therein) and in the spinfoam models (see \[21\] and references therein). More precisely, we can write, in analogy with \[1\]

\[
W(\gamma', U'_l; \gamma, U_l) = \sum_{j'\neq j} \psi_{j'}(U'_l) \psi_k(U_l) \langle \gamma', j' | \gamma, j \rangle
\]

where \( j' \) (respectively \( j \) ) represents the possible labels of a spin network with graph \( \gamma' \) (respectively \( \gamma \) ), \( \psi_{j'}(U_l) \) is the spin network function on the group, and \( \langle \gamma', j' | \gamma, j \rangle \) is the (physical) spin network to spin network transition amplitude. See \[24\] for details. Therefore the hamiltonian structure illustrated here provides a conceptual framework for the interpretation of these transition amplitudes. Notice that no trace of position or time remains in these expressions.

V. CONCLUSION

The shift in perspective defended in the companion paper \[1\] is partially motivated by special relativity, but it is really forced by general relativity. The notion of initial data spacelike surface conflicts with diffeomorphism invariance. A generally covariant notion of instantaneous state, or evolution of states and observables in time, make little physical sense. In a general gravitational field we cannot assume that there exists a suitable asymptotic region, and therefore we do not have a notion of scattering amplitude and \( S \) matrix. In this context, it is not clear what we can take as states and observables of the theory, and what is the meaning of dynamics. In the paper \[1\] and in this paper I have attempted a relativistic foundation of mechanics, that could provide clean notions of states and observables, making sense in an arbitrary general relativistic situation, as well as in quantum theory.

I have argued that mechanics can be seen as the theory of the evolution in time only in the nonrelativistic
limit. In general, mechanics is a theory of relative evolution of partial observables with respect to each other. More precisely, it is a theory of correlations between partial observables. Given a state, classical mechanics determines which correlations are observable and quantum mechanics gives the probability amplitude (or probability density) for each correlation.

In this paper I have applied the ideas of [1] to field theory. I have argued that the relativistic notions of state and observable lead naturally to the formulation of field theory over a finite dimensional space. The application of this formulation to general relativity leads to a remarkably simple hamiltonian formalism, in which the physical irrelevance of the spacetime coordinates becomes manifest.

General relativity can be formulated simply as the pair (\(\mathcal{C}, d\theta\)). \(\mathcal{C}\) is the 40d space of the Poincaré valued 4d one-forms, and \(d\theta\) is given by (14). The orbits of \(d\theta\) in \(\mathcal{C}\), solutions of equation (15), are the solutions of the Einstein equations and form the elements of the phase space \(\Gamma\). This is all of general relativity.

The compactness and simplicity of this hamiltonian formalism is quite remarkable. Notice for instance that general relativity can be formulated simply as the pair (\(\mathcal{C}, d\theta\)) and is illustrated in Section IV. In the quantum domain, it leads directly to the spin-network to spin-network amplitudes computed in loop quantum gravity.

**APPENDIX**

Here we prove the claim that equations (13) is equivalent to the Einstein equations (1). Let us first write \(\omega\) explicitly. We have from (14)

\[
\theta = \frac{1}{2} \epsilon_{IJKL} e_I^\mu d\mu^J \wedge e_J^\nu dx^\nu \wedge (dA_{KL}^\sigma \wedge dx^\sigma + A_{KLM}^\sigma dx^\sigma \wedge A_{M}^L) dx^\sigma
\]

\[
= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} e_I^\rho dA_{KL}^\mu \wedge dx^\sigma
\]

It follows

\[
\omega = d\theta = \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} e_I^\rho d\mu^J \wedge dA_{KL}^\sigma \wedge dx^\sigma
\]

\[
+ \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} d(e_I^\rho e_J^\sigma A_{KL}^\mu A_{M}^L) \wedge d^4x. \tag{18}
\]

The components of \(X = \epsilon^{\mu\nu\rho\sigma} X_\mu \sigma X_\nu \rho X_\rho \sigma \sigma_\sigma \) that give a nonvanishing contribution when contracting with \(\omega\) are the ones with at least two \(\partial_\mu = \partial/\partial x^\mu\) components. These are (I leave the \(\otimes\) understood in the notation)

\[
X_\mu = \frac{\partial}{\partial x^\mu} + \partial_\mu^I e_I^\nu \frac{\partial}{\partial e_\nu^I} + \partial_\mu A_{IJ}^I \frac{\partial}{\partial A_{IJ}^I} + \ldots \] \tag{20}

From (13) and (21), we obtain

\[
\omega(X) = K_\mu^I d\mu^I + K_I^J dA_{IJ}^I + K_\mu dx^\mu, \tag{22}
\]

where

\[
K_\mu^I = \epsilon_{IJKL} \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho}^J e_\sigma^I,
\]

\[
K_I^J = \epsilon_{IJKL} D_J e_\sigma^K e_\sigma^L e^{\mu\nu\rho\sigma}. \tag{23}
\]

while \(K_\mu\) vanishes if \(K_\mu^I\) and \(K_I^J\) do. It follows immediately that \(\omega(X) = 0\) give the Einstein equations (13).

The Einstein equations are obtained even more directly in the second version of the formalism. Of the four \(\partial_\mu\), three contract the three \(e^I\) and \(A_{IJ}\) forms, giving their components, and one contracts either \(de^I_\mu\) or \(dA_{IJ}^I\), leaving simply

\[
\omega(X) = K_\mu^I d\mu^I + K_I^J dA_{IJ}^I \tag{24}
\]

where \(K_\mu^I\) and \(K_I^J\) are again given by (23).

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