Abstract

We consider a multi-armed bandit problem motivated by situations where only the extreme values, as opposed to expected values in the classical bandit setting, are of interest. We propose distribution free algorithms using robust statistics and characterize the statistical properties. We show that the provided algorithms achieve vanishing extremal regret under weaker conditions than existing algorithms. Performance of the algorithms is demonstrated for the finite-sample setting using numerical experiments. The results show superior performance of the proposed algorithms compared to the well known algorithms.
1 Introduction

Multi-armed bandit (MAB) is a sequential decision making framework that formalizes the explore & exploit trade-off under uncertainty. Here, the goal is to devise active sampling algorithms to identify sources generating the cumulative expected payoff (Lai and Robbins, 1985; Bubeck and Cesa-Bianchi, 2012; Bubeck et al., 2013; Slivkins, 2019; Lattimore and Szepesvári, 2020). In this paper, we focus on a special class of MAB’s called the Extreme Bandits. Extreme Bands or Max-K Bandits (Streeter and Smith, 2006a,b) are motivated by situations where only the extreme values (Steinwart et al., 2005; Cicirello and Smith, 2005), as opposed to total expected rewards in the classical bandit setting, are of interest. In Nishihara et al. (2016), it is shown that subtleties arise in the extreme bandit setting that are absent in the standard bandit setting. Using a more general regret definition, they show that no policy can be guaranteed to perform asymptotically as well as an oracle that plays the single best arm over a given duration. Thus this subclass merits independent attention owing to peculiarities observed only in the extreme bandit setting.

Applications: Real-world situations where non-parametric extreme-bandits algorithms are naturally useful have been described in literature. For example, the randomized search situations discussed in Cicirello and Smith (2005) can lead to either light-tailed or heavy-tailed reward distributions (though the paper itself utilizes a parametric approximation, the GEV distribution). The many different anomaly detection situations discussed in Carpentier and Valko (2014) and the references therein naturally include a large variety of reward distributions, some with light tails, and some with heavy tail. In solving NP-hard combinatorial optimization problems using stochastic search heuristics (Cicirello and Smith, 2005; Streeter and Smith, 2006a), where the current reward is the best solution found so far, the goal of future restarts is to find a solution that is better than the current best found.

Extreme bandit setting is also applicable in many real-world problems in diverse fields such as telecommunications, epidemiology, molecular biology, astronomy, quality control, where the objective is to detect sources that behave normally most of the time, but sometimes experience a burst of extreme events (Glaz et al., 2009); although in a limited bandit feedback setting. A complicated real-world situation is described in Apostolidis et al. (2004), where it might be possible to decide on a specific distributional model for specific situations, and then design an extreme-bandit algorithm for that model. In long tail online marketing (Skiera et al., 2010), for example, the marketer seeks to identify those markets that generate the largest sales traffic on individualized/niche products located in the long tail. These applications are naturally framed in the extreme bandit setting, which deals with sequentially choosing the distribution from a collection to sample in order to maximize the single best reward.

Related Work: There are numerous algorithms in the literature for solving extreme bandits, and these can be broadly divided into three categories: Parametric algorithms, where the distributions of the rewards are assumed to belong to specific distributions, for example Gumbel or Fréchet (Cicirello and Smith, 2005; Streeter and Smith, 2006b). Semi-parametric algorithms, where weaker semi-parametric assumptions on the distributions of the rewards are assumed, for example second order Pareto family (Carpentier and Valko, 2014; Ahab et al., 2017) or a known lower bound on the tail distribution (David and Shimkin, 2016). The above parametric/ semi-parametric settings leads to the natural questions of robustness of the algorithm with respect to inevitable deviations from the model. A distribution-free algorithm that is shown to be efficient in variety of situations, including both light-tailed distributions and heavy-tailed distributions, may avoid such issues. Another motivation behind designing extreme-bandit algorithms that do not make any parametric assumptions on the distributions of the reward is similar to the motivation behind classical exploration-exploitation algorithms, such as the Upper Confidence Bound (UCB)-type of algorithms (Lai and Robbins, 1985), for the usual average reward bandits. Even though the analysis of such algorithms often requires assumptions on the reward distributions (for example, sub-Gaussianity), there is nothing inherently parametric in the algorithm, which realizes its objective whether or not the rewards have, say, approximately, normal or beta distributions. Streeter and Smith (2006a) provide a distribution free algorithm for extreme bandits that works well for bounded rewards from any distribution that satisfies certain tail properties. However, no analysis of the algorithm is provided.
Main Contributions: We provide a *distribution-free* extreme bandit algorithm and analyze its statistical properties. It is a novel index based algorithm, where the index is constructed in a non-parametric way by considering maximum elements of carefully designed sub-sets of observed data and then computing the median of these extreme values. Instead of the optimism principle in ExtremeHunter (Carpentier and Valko, 2014), we use a particularly constructed randomization that allows one to explore arms whose index is not currently the highest. We also establish the asymptotic correctness of the algorithm. We further provide a mollified algorithm, having the same asymptotic properties, however, which is also effective in identifying best arms distinguished only by the scaling coefficients. Finally, we establish vanishing extremal regret in the strong sense (see (3)) for exponential-like and polynomial-like distributions under *weaker* assumptions than the state-of-the-art algorithms. This implies that there is no asymptotic regret of not knowing the best arm ahead of time.

2 Extreme Bandit Setting

Let $K = \{1, 2, \cdots, K\}$ denote the arms of the multi-armed bandit, where each arm $k \in K$ is associated with a reward distribution $F_k$ having a *finite* mean. Informally, at each step one “pulls an arm” and obtains an independent observation from the distribution corresponding to that arm. Let $X_{i}^{(k)}$, $t \in \mathbb{Z}^+$ be i.i.d random variables from the distribution $F_k$ for $k \in K$. Let $I_t \in K$ be the arm pulled at time $t$ to receive a reward $X_{I_t}^{(k)}$. Denote the maximum reward obtained by time $t$ as $M_t = \max_{n \leq t} X_{n}^{(I_n)}$. Define a filtration $(\mathcal{F}_0, \mathcal{F}_1, \cdots)$, where $\mathcal{F}_0$ is the trivial $\sigma$-field and $\mathcal{F}_t = \sigma(I_1, X_1^{(I_1)}, \cdots I_t, X_t^{(I_t)})$. For a time horizon $t$ a bandit strategy $\pi_t = (I_1, I_2, \cdots, I_t)$, $t \in \mathbb{Z}^+$, where each $I_n$ is $\mathcal{F}_{n-1}$ measurable is a legitimate strategy. Let $\Pi_t$ denote the collection of all legitimate strategies. The *goal* in an extreme bandit setting is to find $\pi_t \in \Pi_t$ such that

$$V_t(\pi_t) := \mathbb{E}_{\pi_t} M_t$$

is as large as possible. In general, the optimal policy may depend on $t$. An *oracle* who knows the distributions $F_k$, $k \in K$, would have chosen a strategy

$$\pi^*_t \in \arg \max_{\pi_t \in \Pi_t} V_t(\pi_t).$$

In contrast, the classical multi-armed bandit problem aims to solve the problem similar to maximizing (1) but with $\mathcal{M}_t$ replaced by $R_t = \sum_{j=1}^{t} X_j^{(I_j)}$ and the regret of a policy in the classical sense is defined as

$$R_t(\pi_t) = t \max_{i \in K} \mathbb{E}[X_i] - \mathbb{E}_{\pi_t}[R_t].$$

It is well known that there exist multiple policies with a regret of the order $\log t$, that is with a *vanishing average regret*, in the sense that $R_t(\pi_t)/t \rightarrow 0$.

In case of extreme bands, regret of any policy $\pi_t \in \Pi_t$ can be obtained by comparing $V_t(\pi_t)$ and $V_t(\pi^*_t)$. We will consider the situation usually studied in the literature on extreme bandits, where the existence of an asymptotically dominating arm is assumed. An *asymptotically dominating* arm $i^*$ is defined as:

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}[\max_{j=1,2,\cdots,n} X_j^{(i^*)}]}{\mathbb{E}[\max_{j=1,2,\cdots,n} X_j^{(i)}]} > 1$$

for each $i \neq i^*$, and we will try to detect and pull this arm most of the time.

Vanishing Extremal Regret. The following notions of regret are considered for performance evaluation of the algorithm. Suppose $i^*$ is the asymptotically dominating arm.

1. Vanishing extremal regret in a *weak* sense:

$$\frac{\mathbb{E}[\max_{n \leq t} X_n^{(I_n)}]}{\mathbb{E}[\max_{n \leq t} X_n^{(i^*)}]} \rightarrow 1, \text{ as } t \rightarrow \infty.$$  (2)
2. Vanishing extremal regret in a \textit{strong} sense:

\[ \mathbb{E}[\max_{n \leq t} X_n^{(t)}] - \mathbb{E}_x[\max_{n \leq t} X_n^{(I_n)}] \to 0, \text{ as } t \to \infty. \] (3)

Vanishing extremal regret is considered in Cicirello and Smith (2005); Carpentier and Valko (2014); Achab et al. (2017), with the aim of designing algorithms that detect an arm having the heaviest tail. This notion of regret is trivially achieved for distributions with bounded support for any policy that chooses each distribution infinitely often. It provides a meaningful notion of regret with non-trivial policies for distributions with unbounded support (Nishihara et al., 2016). So we assume that the distributions \( F_k, k \in \mathcal{K} \) have unbounded support with the only restriction of finite mean.

3 \ Max-Median Algorithm for Extreme Bandits

In this section, we provide a \textit{distribution-free} algorithm/ policy for extreme bandits. Without any room for confusion, we use policy and algorithm interchangeably. The algorithm named \textit{Max-Median}, is index based, whereby the index can be computed in \( O(KT \log T) \) time.

\begin{algorithm}
1: \( t \)–run-time index. \( K \)–number of arms. \( \varepsilon_t \)–decreasing step-size s.t \( \sum \varepsilon_t = \infty. \)
2: \( I_t \in \mathcal{K} \)–arm chosen at \( t \). \( T \)–play horizon. \( N_k(t) \)–number of \( k^{th} \) arm pulls up to \( t \).
3: \( m(t) = \min_{k \in \mathcal{K}} N_k(t) \)–minimum no. of pulls.
4: \( \mathcal{O}_{k,t}(\zeta) \in \ell^{th} \) order statistic associated with the rewards from arm \( k \).
5: Initialize: Pull each arm once
6: for \( t = K+1: T \) do
7: \quad for \( k = 1: K \) do
8: \quad \quad \( W_k(t) := \mathcal{O}_{k,t-1}(\left\lfloor \frac{N_k(t-1)}{m(t-1)} \right\rfloor) \)
9: \quad end for
10: \( I_t = \left\{ \arg \max_{k \in \mathcal{K}} W_k(t), \text{ w.p. } 1 - \varepsilon_t \right\} \)
11: end for

\textit{Discussion of Algorithm 1}: Let \( N_k(t) \) denote the number of times arm \( k \) is chosen up to time \( t \) with \( \sum_k N_k(t) = t \). Let

\[ m(t) = \min_{k \in \mathcal{K}} N_k(t) \] (4)

denote the minimum number of times any arm is pulled.

\textbf{Theorem 3.1.} Let \( \varepsilon_t \) denote the decreasing step size such that \( \sum \varepsilon_t = \infty. \) For any \( \xi > K(K-1), m(t) \) is w.p.1 lower bounded for \( t \) large enough by \( m(t) \geq \frac{1}{\xi} \sum_{d=1}^{t} \varepsilon_d. \)

Theorem 3.1 establishes that for \( \xi > K(K-1), \) the event \( \left\{ m(t) \geq (1/\xi) \sum_{d=1}^{t} \varepsilon_d \right\} \) has probability 1. This means that \( m(t) \) will be greater than or equal to \( (1/\xi) \sum_{d=1}^{t} \varepsilon_d \) for all \( t \) large enough, but the point from which this becomes true is still random, so for each fixed (large) \( t \) the event \( \left\{ m(t) \geq (1/\xi) \sum_{d=1}^{t} \varepsilon_d \right\} \) is a high probability event.

\textit{Randomization}: The decreasing step size \( \varepsilon_t \) provides an avenue for exploration and hence plays a role in the rate of convergence of the extremal regret.

\textbf{Index}: Let \( H_k(t) \) be the set of times arm \( k \) is pulled by time \( t \). Consider the following collection \( \mathcal{S}_k(t) = \{ A : A \subset H_k(t), |A| = m(t) \}. \) It is clear that the cardinality \( |\mathcal{S}_k(t)| = \left( \frac{N_k(t)}{m(t)} \right) \). Define \( \tilde{X}_{k,j,t} = \)
max_{i \in A_j} X_i^{(k)}, j = 1, 2, \cdots, \left(\frac{N_k(t)}{m(t)}\right)
by enumerating the sets in \mathcal{S}_k(t). We now introduce the following
index:
\[
\hat{W}_k(t) = \text{Median} \left\{ \hat{X}_{k,j,t-1}, j = 1, 2, \cdots, \left(\frac{N_k(t-1)}{m(t-1)}\right) \right\},
\]
the median of all the maximum rewards on each subset of \(m(t-1)\) rewards from arm \(k\) observed before time \(t\). These maxima are available for each arm; they can be compared since they are taken over sets of rewards of the same cardinality for each arm, and the median is a robust estimator of the size of these maxima, so it makes sense to compare these medians. We argue that this index is close to the index used in Algorithm 1. Let \(\mathcal{O}_{k,t}(\zeta)\) denote the \(\zeta\)th order statistic (Pickands III, 1975; Balakrishnan and Cohen, 2014) of the \(N_k(t)\) rewards from arm \(k\) observed by time \(t\); its computation involves sorting, hence the complexity of \(O(t \log t)\). We have the following result.

**Theorem 3.2.** Let \(C > 0\) be such that \(1 - x \geq e^{-Cx}\) for \(0 \leq x \leq \frac{1}{2}\), and let \(\tau = \frac{2\log(3/2)}{2C}\). For every arm \(k \in \mathcal{K}\), we have
\[
\mathcal{O}_{k,t-1}\left(\left\lfloor \frac{2N_k(t-1)}{m(t-1)} \right\rfloor\right) \leq \hat{W}_k(t) \leq \mathcal{O}_{k,t-1}\left(\left\lceil \frac{\tau N_k(t-1)}{m(t-1)} \right\rceil\right),
\]
where \([x] = \min_{n \in \mathbb{Z}^+}\{n \geq x\}\), and the upper bound holds if \(N_k(t-1)\) is large and \(m(t-1) \leq N_k(t-1)/2\).

Theorem 3.2 is established using elementary combinatorics (Brualdi, 1977). Theorem 3.2 suggests an index that is similar to the index in (5) but computationally much simpler and easy to implement. This is summarized as follows.

**Corollary 3.3.** The index for the Max-Median algorithm given as
\[
W_k(t) := \mathcal{O}_{k,t-1}\left(\left\lceil \frac{\tau N_k(t-1)}{m(t-1)} \right\rceil\right)
\]
is analogous to (5) but computationally simpler.

**Implementation Summary:** Starting with \(W_k(1) = 0, \forall k \in \mathcal{K}\), play all arms once. Pick the arm with the highest reward with probability \(1 - \varepsilon_1\). For each \(t > 2\), maintain the number of times each of the \(K\) arms is played, and also the minimum number. Sort the rewards on each arm online and select the order statistic corresponding to the index \(W_k(t)\) in (6). With probability \(1 - \varepsilon_t\) pick the arm with the largest order statistic, while with probability \(\varepsilon_t\) explore a random arm.

## 4 A Tale of Two Distributions

Typical distributions considered in the extreme bandits literature have “exponential-like” tails such as the Gumbel Generalized Extreme Valued (GEV) distribution or the exponential distribution (Cicirello and Smith, 2005; Streeter and Smith, 2006b), or “sub-exponential/ heavy tails” like the Frechét GEV distribution or the Pareto distribution (Hall and Welsh, 1984; Carpentier and Valko, 2014; Achab et al., 2017). Although the GEV distributions are the limiting distribution of the maxima of i.i.d random variables (Fisher and Tippett, 1928; De Haan and Ferreira, 2007) and hence considered in the extreme bandits literature, these distributions are often not even an approximately accurate model of the payoff distributions encountered in practice (Streeter and Smith, 2006a). So we consider rewards with more general exponential-like tails (Ryzhov and Powell, 2011) and polynomial-like tails (Carpentier and Valko, 2014; Achab et al., 2017).

We first establish the consistency of the index and the vanishing extremal regret for both exponential-like tails and polynomial-like tails. Then we provide a mollified index algorithm based on the Max-Median idea that identifies best arms distinguished only by their scaling coefficients.
4.1 Exponential-like Arms

We will show that Algorithm 1 achieves vanishing extremal regret in the strong sense (3) for exponential-like arms. The assumption of exponential-like arms means that

\[ \bar{F}_k(x) = 1 - F_k(x) \sim a_k e^{-\lambda_k x}, \quad k \in K \]  \hfill (7)

for some \( a_k > 0 \) and \( \lambda_k > 0 \). It is easy\(^2\) to see that for each \( k \)

\[ \mathbb{E}[\max_{n \leq t} X_n^{(k)}] \sim \lambda_k^{-1} \log t, \quad t \to \infty. \]  \hfill (8)

The best arm \( i^* \in K \) is identified by \( 0 < \lambda_{i^*} < \min_{k \neq i^*} \lambda_k \). It follows from (8) that for \( k \neq i^* \),

\[ \lim_{t \to \infty} \frac{\mathbb{E}[\max_{n \leq t} X_n^{(i^*)}]}{\mathbb{E}[\max_{n \leq t} X_n^{(k)}]} > 1. \]  \hfill (9)

4.1.1 Index Consistency

**Theorem 4.1.** Assume that for all \( \delta > 0 \),

\[ \sum_{t=1}^{\infty} \exp \left\{ - \left( \sum_{n=1}^{t} \epsilon_n \right)^\delta \right\} < \infty. \]

Let \( i^* \) denote the best arm as in (9). For the Max-Median policy (Algorithm 1), the following holds:

\[ \mathbb{P}(W_{i^*}(t) > W_k(t) \text{ for all } k \neq i^* \text{ and all } t \text{ large enough}) = 1. \]

In other words, w.p.1 the best arm will have the largest index eventually. This is crucial to establish vanishing regret in case of both exponential and polynomial arms. This result essentially guarantees the asymptotic correctness of Algorithm 1.

**Theorem 4.2.** Let \( i^* \) denote the best arm as in (9). For the Max-Median policy (Algorithm 1), the following holds:

\[ \lim_{t \to \infty} \left( V_t(\pi) - \mathbb{E}[\max_{n \leq t} X_n^{(i^*)}] \right) = 0, \]

where \( V_t(\pi) \) is as in (1).

According to Theorem 4.2, when the distributions of the rewards are exponential-like, Algorithm 1 achieves vanishing extremal regret in the strong sense (3). In other words, there is no asymptotic regret of not knowing the best arm ahead of time.

4.2 Polynomial-like Arms

In this section, we show that Algorithm 1 achieves vanishing extremal regret in the weak sense (2) for polynomial-like arms. Under additional assumptions which are weaker than the state-of-the-art algorithms, vanishing extremal regret in the strong sense (3) is achieved for polynomial-like arms as well. The assumption of polynomial-like arms means that

\[ \bar{F}_k(x) = 1 - F_k(x) \sim a_k x^{-\lambda_k}, \quad k \in K \]

\(^2\)An even stronger statement for the expectation (8) that involves the coefficients \( a_k \) as well is provided in the appendix section.
for some \(a_k > 0\) and \(\lambda_k > 1\). It is easy to see that

\[
\mathbb{E}[^{\max_{n \leq t} X_n^{(k)}}] \sim a_k^{1/\lambda_k} \Gamma(1 - 1/\lambda_k) t^{1/\lambda_k}, \quad t \to \infty. \tag{10}
\]

Here \(\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx\) denotes the Gamma function. The best arm \(i^* \in \mathcal{K}\) is identified by \(0 < \lambda_{i^*} < \min_{k \neq i^*} \lambda_k\). It follows from (10) that for \(k \neq i^*\),

\[
\lim_{t \to \infty} \frac{\mathbb{E}[\max_{n \leq t} X_n^{(i^*)}]}{\mathbb{E}[\max_{n \leq t} X_n^{(k)}]} = \infty. \tag{11}
\]

### 4.2.1 Index Consistency

We note that one can switch from exponential-like arms to polynomial-like arms by exponentiating the former, and switch back by taking the logarithm of the latter. Since the statement of Theorem 4.1 is invariant under monotone transformation of the rewards, the theorem holds for polynomial-like arms as well.

**Theorem 4.3.** Let \(i^*\) denote the best arm as in (11). For the Max-Median policy (Algorithm 1), the following holds:

\[
\lim_{t \to \infty} \frac{V_t(\pi)}{\mathbb{E}[\max_{n \leq t} X_n^{(i^*)}]} = 1.
\]

According to Theorem 4.3, when the distributions of the rewards are polynomial-like, Algorithm 1 achieves vanishing extremal regret in the weak sense (2). We now compare the performance of Algorithm 1 with ExtremeHUNTER (Carpentier and Valko, 2014) & ExtremeETC (Achab et al., 2017), which are specifically designed for the second order Pareto family defined by

\[
|\hat{F}_k(x) - a_k x^{-\lambda_k}| \leq c_k x^{-\lambda_k (1+\beta_k)}, \quad k \in \mathcal{K},
\]

where \(\beta_1, \beta_2, \ldots, \beta_K\) and \(c_1, c_2, \ldots, c_K\) are positive constants. ExtremeHUNTER/ ExtremeETC achieves (3) under the following condition (Carpentier and Valko, 2014; Achab et al., 2017):

\[
\min(\beta_1, \beta_2, \ldots, \beta_K) > 1/\lambda_{i^*}. \tag{12}
\]

We prove that that Algorithm 1 achieves vanishing extremal regret in the strong sense (3) under a weaker assumption \(\beta_{i^*} > 1/\lambda_{i^*}\).

**Theorem 4.4.** Suppose \(\beta_{i^*} > 1/\lambda_{i^*}\). For \(\alpha \in (1/\lambda_{i^*}, 1)\), let the exploration probabilities be chosen as \(\epsilon_t = (1 + t)^{-\alpha}\). For the Max-Median policy (Algorithm 1), the following holds:

\[
\lim_{t \to \infty} \left( V_t(\pi) - \mathbb{E}[\max_{n \leq t} X_n^{(i^*)}] \right) = 0
\]

where \(V_t(\pi)\) is as in (1).

Theorem 4.4 says that under additional assumption \(\beta_{i^*} > 1/\lambda_{i^*}\), which is clearly weaker than (12), Algorithm 1 achieves vanishing extremal regret in the strong sense (3). In other words, there is no asymptotic regret of not knowing the best arm ahead of time.

**Remark:** ExtremeHUNTER (Carpentier and Valko, 2014) is an extreme bandit algorithm that is designed with semi-parametric assumptions on the distributions of the rewards. Specifically, assuming that the rewards are realized according to a second-order Pareto family, one uses an asymptotic approximation of the expectation of the maximum of these random variables. This approximation, along with plug-in estimates of the parameters appearing in the approximation, is used to compute an index. The estimates are computed optimistically, to account for uncertainty. The policy is not randomized, and the arm with the largest index is pulled. In contrast, we do not assume that the reward distributions belong to any specific (semi)-parametric family. The index in Max-Median is constructed in a non-parametric way by considering maximum elements of carefully designed sub-sets of observed data and then computing the median of these extreme values. Instead of the optimism principle in ExtremeHUNTER, we use a particularly constructed randomization that allows one to explore arms whose index is not currently the highest.
5 Mollified Max-Median Algorithm

The Max-Median algorithm (Algorithm 1) has been showed to be effective for both exponential-like and polynomial-like arms when the best arm \( i^* \) satisfies \( 0 < \lambda_{i^*} < \min_{k \neq i^*} \lambda_k \). In this section, we propose a mollified Max-Median algorithm that can distinguish effectively between several arms with the same optimal value of \( \lambda_{i^*} \) but different values of the scaling coefficient \( a_k \). That is, we consider the situation

\[
0 < \lambda_{i_1} = \cdots = \lambda_{i_j} < \min_{k \neq \{i_1, i_2, \ldots, i_j\}} \lambda_k, \quad \text{and} \quad a_{i_1^*} > \max_{l=2, \ldots, j} a_{i_l^*}, \tag{13}
\]

applicable to both exponential-like arms and polynomial-like arms.

**Algorithm 2** mollified Max-Median: m-MM(\( \varepsilon_t, h(\cdot), \{X_h^{(ln)}\} \) for \( j \in \mathcal{K} \) and \( n \leq t \))

1. \( t \)—run-time index. \( K \)—number of arms.
2. \( \varepsilon_t \)—decreasing step-size s.t \( \sum \varepsilon_t = \infty \).
3. \( i_t \in \mathcal{K} \)—arm chosen at \( t \). \( T \)—play horizon.
4. \( N_k(t) \)—number of \( k^{th} \) arm pulls upto \( t \).
5. \( m(t) = \min_{k \in \mathcal{K}} N_k(t) \)—minimum no. of pulls.
6. \( h(m(t)) \)—index mollifier.
7. \( O_{k,t}(\zeta) \)—\( \zeta^{th} \) order statistic associated with the rewards from arm \( k \).
8. **Initialize**: Pull each arm once
9. for \( t = K+1: T \) do
10. for \( k = 1: K \) do
11. \( W_k(t) := O_{k,t-1}\left(\left\lceil \frac{N_k(t-1)}{h(m(t-1))} \right\rceil \right) \)
12. end for
13. \( I_t = \left\{ \arg\max_{k \in \mathcal{K}} W_k(t), \quad \text{w.p } 1 - \varepsilon_t \right\} \text{ for } i \in \mathcal{K} \text{ w.p } \frac{\varepsilon_t}{K} \)
14. end for

Discussion of Algorithm 2. The mollifier essentially provides a rationale to select a moderately higher order statistic for the index of each arm. The implementation is similar to Algorithm 1, except the minor modification in the index calculation. The time complexity is again \( O(KT \log T) \).

**Definition 5.1 (Mollifier).** A mollifier is any increasing function \( h : (0, \infty) \rightarrow (0, \infty) \) such that \( h(x) \rightarrow \infty \) as \( x \rightarrow \infty \) and \( h(x) = o(x/\log x) \).

Here the notation \( f_1(x) = o(f_2(x)) \) means that \( \lim_{x \rightarrow \infty} f_1(x)/f_2(x) = 0 \). Theorem 5.1 guarantees the asymptotic correctness of the mollified algorithm (Algorithm 2).

**Theorem 5.1.** Assume that there is \( \kappa > 0 \) such that

\[
\sum_{n=1}^{\infty} \left( \sum_{d=1}^{n} \varepsilon_d \right)^{-\kappa} < \infty. \tag{14}
\]

Let \( i^* = i_1^* \) denote the best arm as in (13), and let \( h \) be a mollifier. Under Algorithm 2, the following holds for either exponential-like or polynomial-like arms.

\[
P(\tilde{W}_{i^*}(t) > \tilde{W}_k(t) \text{ for all } k \neq i^* \text{ and all } t \text{ large enough}) = 1.
\]

**Remark:** Results similar to Theorem 4.2, Theorem 4.3, and Theorem 4.4 can be established for the mollified Max-Median algorithm (Algorithm 2) using arguments similar to those used for Algorithm 1. In words, Algorithm 2 achieves vanishing extremal regret in the strong sense (3) in case of exponential-like and polynomial-like arms.
6 Numerical Results

We know from Section 4 that Algorithm 1 and Algorithm 2 achieve vanishing extremal regret. So the focus of this section is to evaluate finite sample performance. In this section, we empirically evaluate Algorithm 1 & 2 on synthetic data.

Performance Evaluation Discussion

1. We employ two measures for evaluating the empirical finite sample performance: (I) Extremal regret as in (3) in a non-asymptotic sense; (II) Percentage of best arm pulls. The motivation for having another performance evaluation criterion stems from the fact that extremal regret is defined in an asymptotic sense, and we shall see that smaller extremal regret over a finite horizon need not reflect optimal play. Percentage of best arm pulls is a natural candidate for evaluation as the goal in extreme bandits can be seen as one of extreme value source identification.

2. We consider 3 types of reward distributions: polynomial arms for motivating heavy tailed data (Bubeck et al., 2013), exponential arms for motivating exponential tailed data (Ryzhov and Powell, 2011; Korda et al., 2013), and Gaussian arms for motivating real valued data (Lattimore, 2016). These distributions are sufficiently diverse to cover the commonly encountered reward distributions in bandit applications.

3. There are classical bandit algorithms like Robust-UCB (Bubeck et al., 2013) that are designed for bandits with heavy tails. In Achab et al. (2017), it is demonstrated that, even though the objectives are completely different, Robust-UCB performs comparably to ExtremeHUNTER in terms of regret under stronger assumptions. Additionally, we compare the performance against non-heavy tailed distributions as well. So we only focus on comparison of the Max-Median algorithm against other extreme bandit algorithms.

4. Time complexity: The time complexity of the implementation of the three algorithms is as follows: Max-Median (Algorithm 1) has $O(KT \log T)$, ExtremeHUNTER (Carpentier and Valko, 2014) designed for second order Pareto family has $O(T^2)$, and ThresholdAscent (Streeter and Smith, 2006a), which is distribution free, has $O(KT)$. It is noted that a faster version ExtremeETC (Achab et al., 2017), which has similar performance as ExtremeHUNTER, has $O(\log^6 T)$.

Experimental Setup. In all simulations below, the hyper-parameters of ExtremeHUNTER are chosen as in Carpentier and Valko (2014), and the hyper-parameters of ThresholdAscent are chosen as in Streeter and Smith (2006a) with manual tuning to obtain the best performance for a given distribution. The only choice parameter in Algorithm 1 is the step size that controls the exploration. All algorithms are evaluated over 5000 plays or arm pulls and values averaged over 500 trajectories, i.e., the expectation in (3) is over 500 trajectories. The number of arms $K$ is chosen to be different for different distributions, as it is known that the algorithms’ performance relative each to other is also affected by the number of bandit arms (Kuleshov and Precup, 2014).

1. Polynomial Arms-Case 1: We consider a $K = 5$ armed extreme bandit with polynomial arms having distinct distributional parameters $\lambda_k = [2.1, 1.3, 1.1, 1.9]$ and same coefficients. The distribution $\tilde{F}_k(x) \sim x^{-\lambda_k}$ is motivated by the numerical experiment in Carpentier and Valko (2014), and is considered for fair comparison. The step size that controls exploration in Algorithm 1 is chosen as $\varepsilon_t = \frac{1}{(t+1)}$. The performance of the algorithms is illustrated in Figure 1.

2. Polynomial Arms-Case 2: Next, we consider a related situation using a $K = 7$ armed extreme bandit with polynomial arms $\tilde{F}_k(x) \sim a_k x^{-\lambda_k}$ having similar distributional parameters for the best arm, where $\lambda_k = [2.5, 2.8, 4.3, 1.4, 1.4, 1.9]$ with $a_5 = 1.1, a_6 = 1.01$ and $a_j \neq (5, 6) = 1$. The step size that controls exploration in Algorithm 2 is again chosen as $\varepsilon_t = \frac{1}{(t+1)}$ with the mollifier $h(x) = \frac{\sqrt{x}}{\log x}$. This is equivalent to choosing a moderately higher-order statistic. The performance of the algorithms is illustrated in Figure 1.
3. **Exponential Arms**: Having considered a heavy tail setting with polynomial arms, we now consider an exponential tail setting with $F_k(x) \sim e^{-\lambda_k}$. In this case, a $K = 10$ armed exponential extreme bandit with $\lambda_k = [2.1, 2.4, 1.9, 1.3, 1.1, 2.9, 1.5, 2.2, 2.6, 1.4]$ is considered. The step size in Algorithm 1 is chosen as $\epsilon_t = \frac{1}{t+1}$. The performance of the algorithms is illustrated in Figure 2.

4. **Gaussian Arms**: Motivated by applications having real valued extreme value source identification, we consider a Gaussian setting with $K = 20$ arms. For the purpose of illustrating the tail identification, we consider same means with different variances for the different arms, that is $f_k(x) \sim \mathcal{N}(\mu_k, \sigma_k)$, $\mu_k = 1 \forall k \in K$ and$^3$. The step size in Algorithm 1 is chosen as $\epsilon_t = \frac{1}{t+1}$. The performance of the algorithms is illustrated in Figure 3.

$^3\sigma_k = [1.64, 2.29, 1.79, 2.67, 1.70, 1.36, 1.90, 2.19, 0.89, 0.12, 1.65, 1.19, 1.88, 0.89, 3.35, 1.5, 2.22, 3.03, 1.08, 0.48]
Figure 3: Finite sample performance for Gaussian Arms. ExtremeHUNTER is unable to deal with Gaussian setting as the best arm is pulled less than 3% of the time. Max-Median is again the best performing algorithm in terms of percentage of best arm pulled. The dominating arm corresponds to the one with the largest variance.

Figure 4: Finite sample performance for $K = 100$ Polynomial and Exponential Arms.

Next, we illustrate the performance of Algorithm 2 (Mollified Max-Median) in case of a large number of polynomial and exponential arms. The performance of Algorithm 2 in case of polynomial arms and exponential arms is shown in Figure 4. The mollifier is chosen as $h(x) = \frac{\sqrt{x}}{\log x}$. The step size in Algorithm 2 is chosen as $\varepsilon_t = \frac{1}{(t+1)}$. The coefficients were chosen equal to 1 for all arms, and the lambda values were randomly generated using a power law distribution. It should be noted that for large number of arms, Algorithm 2 is preferred over Algorithm 1.

Key Observations

1. Max-Median (Algorithm 1) has comparable extremal regret performance with ExtremeHUNTER, while performing the best amongst the chosen comparative algorithms in terms of percentage of best arm pulls, irrespective of the reward distribution.

2. ThresholdAscent performs poorly in case of all chosen distributions in terms of extremal regret (3), while performing reasonably well in terms of percentage of best arm pulls. Even though Extreme-
HUNTER performs well in terms of the regret, it performs poorly in all cases in terms of the percentage of best arm pulls.

3. From the empirical results, we infer that, for extreme bandits, extremal regret (3) is not a good measure of performance over finite-horizon settings. The extremal regret being small does not reflect the fact the dominating arms are pulled most of the time.

4. The good performance of Algorithm 1 in finite sample settings for all chosen distributions, in terms of the percentage of best arm pulls, motivates the use of Algorithm 1 for exploration in case of classical bandits (Audibert et al., 2010; Jamieson et al., 2014), and in combinatorial bandit problems (Ontanón, 2013; Nuara et al., 2018) for heavy-tail distributions.

7 Conclusion

We provided a general purpose algorithm for extreme bandits that has $O(KT \log T)$ time complexity. The index based algorithm is fashioned using combinatorics and robust statistics, where we established that the index corresponding to the best arm will have the largest value asymptotically. We also provided a mollified algorithm to select the best arm, when only the distribution coefficients are distinct. Using numerical experiments, we demonstrated the superior finite-sample performance of the algorithm against the popular algorithms. Finally, to provide a comparison with the existing semi-parametric algorithms, we established vanishing extremal regret for the Max-Median algorithm for distributions having “exponential-like tails” and “polynomial-like tails”- the most common class of distributions considered in the literature on extreme bandits- and demonstrated vanishing extremal regret under weaker conditions. It is likely that our algorithm is efficient in other situations as well, however, that is in consideration for future work.

The Max-Median algorithm uses forced randomization; and this has both advantages and drawbacks, and the analysis is sometimes more transparent in the randomized case. It is not quite clear how to construct a non-randomized procedure (such as utilizing optimism-in-face-of uncertainty) for the extremes without making some distributional assumptions. We are exploring this issue for future work.
Appendix

A Preliminaries

We list here several properties of exponential-like and polynomial-like distributions. These properties are repeatedly used in the proofs of the theorems.

A.1 Exponential-like Arms

Suppose the reward distribution of an arm is exponential-like:

\[ \bar{F}(x) = 1 - F(x) \sim ae^{-\lambda x} \]

for some \( a > 0 \) and \( \lambda > 0 \). Then a sample from this distribution satisfies

\[ \mathbb{E}[\max_{n \leq t} X_n] \sim \lambda^{-1} \log t, \ t \to \infty. \]

Moreover,

\[ \lim_{t \to \infty} \mathbb{E}\left[ \max_{n \leq t} X_n - \lambda^{-1} \log t \right] = \lambda^{-1} \log a - \lambda^{-1} \int_0^\infty e^{-x} \log x \, dx. \] (15)

Proof of (15). It is, clearly, enough to prove (15) in the case \( \lambda = 1 \). Let \( 0 < \varepsilon < 1 \), and choose \( M > 0 \) so large that both \( ae^{-M} \leq 1 \) and

\[ a^{-1}e^x P(X_1 > x) \in [1 - \varepsilon, 1 + \varepsilon] \]

for all \( x \geq M \). We have

\[ E\left[ \max_{j=1,\ldots,n} X_j \right] = \int_0^\infty \left( 1 - (1 - P(X_1 > x))^n \right) dx \]

\[ = \int_0^M \left( 1 - (1 - P(X_1 > x))^n \right) dx + \int_M^\infty \left( 1 - (1 - P(X_1 > x))^n \right) dx. \]

It is clear that

\[ \lim_{n \to \infty} \int_0^M \left( 1 - (1 - P(X_1 > x))^n \right) dx = M. \] (16)

Furthermore,

\[ \int_M^\infty \left( 1 - (1 - a(1 - \varepsilon)e^{-x})^n \right) dx \leq \int_M^\infty \left( 1 - (1 - P(X_1 > x))^n \right) dx \]

\[ \leq \int_M^\infty \left( 1 - a(1 + \varepsilon)e^{-x} \right)^n dx. \]

Write

\[ \int_M^\infty \left( 1 - (1 - ae^{-x})^n \right) dx = \int_0^{ane^{-M}} \left( 1 - (1 - w/n)^n \right) \frac{dw}{w} \]

\[ = \int_0^{ane^{-M}} \left( e^w - (1 - w/n)^n \right) \frac{dw}{w} + \int_0^{ane^{-M}} \left( 1 - e^{-w} \right) \frac{dw}{w}. \]

Changing the order of integration,

\[ \int_0^{ane^{-M}} \left( 1 - e^{-w} \right) \frac{dw}{w} = (\log a + \log n - M) \int_0^{ane^{-M}} e^{-t} dt - \int_0^{ane^{-M}} e^{-t} \log t \, dt \]

\[ = \log a + \log n - M - \int_0^\infty e^{-t} \log t \, dt + o(1). \]

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Since we that
\[ 0 \leq \int_0^{ane^{-M}} (e^{-w} - (1 - w/n)^n) \frac{dw}{w} \leq \int_0^1 (e^{-w} - (1 - w/n)^n) \frac{dw}{w} \to 0 \]
as \( n \to \infty \), (15) follows.

Let \( X_{(j:n)} \) be the \( j \)th largest order statistic from the sample \( X_1, \ldots, X_n \). Let \( m_n \to \infty \), \( m_n/n \to 0 \) as \( n \to \infty \). Then for every \( b > 0 \)
\[ \limsup_{n \to \infty} \frac{m_n}{n} \log P \left( \left| X_{(n/m_n):n} - \lambda^{-1} \log a - \lambda^{-1} \log m_n \right| > b \right) < 0. \tag{17} \]

**Proof of (17).** Once again we may assume that \( \lambda = 1 \). We have
\[ P \left( X_{(n/m_n):n} > \log a + \log m_n + b \right) = P \left( B_n \geq [n/m_n] \right), \]
where \( B_n \) has the Binomial distribution with \( n \) trials and probability for success \( p_n \sim e^{-b/m_n} \). By the exponential Markov inequality, for any \( \gamma > 0 \),
\[ P \left( B_n \geq [n/m_n] \right) \leq \left[ e^{-\gamma[n/m_n]/(1 + p_n(e^\gamma - 1))} \right]^n. \]
Choosing \( \gamma = b \), we obtain
\[ \frac{m_n}{n} \log P \left( X_{(n/m_n):n} > \log a + \log m_n + b \right) \leq b \frac{m_n}{n} - (b - m_np_n(e^b - 1)) \to -(b - 1 + e^{-b}) < 0. \]

\[ \square \]

A.2 Polynomial-like Arms
Suppose the reward distribution of an arm is polynomial-like:
\[ \bar{F}(x) = 1 - F(x) \sim ax^{-\lambda} \]
for some \( a > 0 \) and \( \lambda > 1 \). Then a sample from this distribution satisfies
\[ \mathbb{E}[\max_{n \leq t} X_n] \approx a^{1/\lambda} \Gamma(1 - 1/\lambda)t^{1/\lambda}, \quad t \to \infty. \tag{18} \]
Here \( \Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx \) for \( z > 0 \) denotes the gamma function.

B Proofs of Main Results

**Proof of Theorem 3.1.** From Algorithm 1, there is a sequence of independent \( \{0, 1\} \)-valued random variables \( J_t, \quad t = 1, 2, \ldots \) with \( P(J_t = 1) = \frac{c}{x^{1/\lambda}} \), having the following property. For \( t = 1, 2, \ldots \), let
\[ \mathcal{I}_t = \{ k \in K : N_k(t) = m(t) \}, \]
where \( \mathcal{I}_t \) is a random nonempty set. Then one of the arms in \( \mathcal{I}_{t-1} \) is pulled at time \( t \) if \( J_t = 1 \). Every time one of the arms with the smallest number of pulls is pulled, either \( m(t) \) goes up by 1, or the cardinality of the set \( \mathcal{I}_{t-1} \) is decreased by 1. Since that cardinality cannot exceed \( K \), we see that, if one of the arms
with the smallest number of pulls is pulled \( K \) times in a row, then the smallest number of times an arm is pulled goes up at least by 1. Therefore,

\[
m(t) \geq \left\lfloor \frac{1}{K} \sum_{d=1}^{t} J_d \right\rfloor \geq \frac{1}{K} \sum_{d=1}^{t} J_d - 1, \quad t = 1, 2, \ldots.
\]

(19)

Denote \( S_t = \sum_{d=1}^{t} J_d, \quad t = 1, 2, \ldots \). Note that

\[
E(S_t) = \frac{1}{K-1} \sum_{d=1}^{t} \varepsilon_d \to \infty
\]
as \( t \to \infty \) as the step-size sequence is not summable. Further,

\[
E(S_t - E(S_t))^2 = \sum_{d=1}^{t} \frac{\varepsilon_d}{K-1} \left( 1 - \frac{\varepsilon_d}{K-1} \right) \leq \frac{1}{K-1} \sum_{d=1}^{t} \varepsilon_d = E(S_t).
\]

We claim that the strong law of large numbers

\[
\frac{K-1}{\sum_{d=1}^{n} \varepsilon_d} S_n \to 1 \text{ with probability } 1
\]

(20)
holds. To see that, denote \( s_n = \sum_{d=1}^{n} \varepsilon_d, \quad n = 1, 2, \ldots \) and define

\[
m_\ell = \min \{ n = 1, 2, \ldots: s_n \geq \ell^2 \}, \quad \ell = 1, 2, \ldots.
\]

Note that

\[
ES_{m_\ell} = \frac{s_{m_\ell}}{K-1} \geq \frac{\ell^2}{K-1}, \quad \text{Var}(S_{m_\ell}) \leq \frac{s_{m_\ell}}{K-1} \leq \frac{\ell^2 + 1}{K-1}.
\]

By the Chebyshev inequality, for any \( \delta > 0 \),

\[
P \left( \left| \frac{K-1}{s_{m_\ell}} S_{m_\ell} - 1 \right| > \delta \right) \leq \frac{(K-1)(\ell^2 + 1)}{\delta^2 \ell^4}.
\]

Since this expression is summable in \( \ell \), we conclude by first Borel-Cantelli lemma that (20) holds along the subsequence \( (m_\ell) \). Next, for \( n > m_1 \) let \( K_n \) be such that

\[
m_{K_n} < n \leq m_{K_n}.
\]

Then

\[
S_{m_{K_n-1}} < S_n \leq S_{m_{K_n}}, \quad s_{m_{K_n-1}} < s_n \leq s_{m_{K_n}},
\]

so

\[
\frac{s_{m_{K_n-1}}}{s_{m_{K_n-1}}} \leq \frac{s_{m_{K_n-1}}}{s_{m_{K_n}}} \leq \frac{s_n}{s_n} \leq \frac{s_{m_{K_n}}}{s_{m_{K_n}}} \leq \frac{s_{m_{K_n}}}{s_{m_{K_n}}} \leq \frac{s_{m_{K_n}}}{s_{m_{K_n}}},
\]

Since

\[
\frac{s_{m_{K_n}}}{s_{m_{K_n-1}}} \leq \frac{K_n^2 + 1}{(K_k - 1)^2} \to 1
\]
as \( n \to \infty \), the convergence in (20) holds along all positive integers. It follows from (19) and (20) that for every \( \xi > K(K-1) \) for all \( t \) large enough each arm will be pulled at least \( \frac{1}{\xi} \sum_{d=1}^{t} \varepsilon_d \) times. \( \square \)
Proof of Theorem 3.2. For the sake of clarity and exposition, let \( n = N_k(t - 1), m = m(t - 1), x_1, \ldots, x_n \) the rewards from arm \( k \) and \( x_{(1:n)} \geq x_{(2:n)} \geq \ldots \geq x_{(n:n)} \) are the same rewards from the largest to the smallest. It is clear that \( \hat{W}_k(t) \) is one of these ordered rewards. For a set \( A \subseteq \{1, \ldots, n\} \) of cardinality \( m \) we have

\[
\max_{j \in A} x_j = x_{(L(A):n)},
\]

where

\[
L(A) = \min\{j = 1, \ldots, n : \text{there is } j' \in A \text{ such that } x_{j'} = x_{(j:n)}\}.
\]

We break the ties and make one-to-one correspondence between an order statistic and the corresponding observation in an arbitrary way. Note that there are exactly \( \binom{n - i}{m - 1} \) sets \( A \) with \( L(A) = i \).

Therefore,

\[
\hat{W}_k(t) = \mathcal{O}_{k,t-1}(l), \quad \text{where } l = \min \left\{ d \geq 1 : \sum_{i=1}^{d} \binom{n - i}{m - 1} \geq \frac{1}{2} \binom{n}{m} \right\}.
\]

We have by elementary combinatorics,

\[
\sum_{i=1}^{d} \binom{n - i}{m - 1} = \binom{n}{m} - \binom{n - d}{m}.
\]

Therefore, we can write

\[
l = \min \left\{ d \geq 1 : \binom{n - d}{m} \leq \frac{1}{2} \binom{n}{m} \right\}.
\]

Furthermore,

\[
\frac{n-d}{n} = \frac{(n-d)(n-d-1) \cdots (n-d-m+1)}{n(n-1) \cdots (n-m+1)} \leq \left( \frac{n-d}{n} \right)^m,
\]

implying that \( l \leq \left\lfloor n \left(1 - 2^{-1/m}\right) \right\rfloor \). Since \( 1 - 2^{-1/m} \leq (\log 2)/m \), we can see that \( l \ll \left\lceil \frac{2n}{m} \right\rceil \), and so

\[
\mathcal{O}_{k,t-1}\left(\left\lceil \frac{2N_k(t-1)}{m(t-1)} \right\rceil \right) \leq \hat{W}_k(t).
\]

For an upper bound, notice that for large \( n \) we have, for some \( \rho_n \downarrow 0 \) (that may change from appearance to appearance), by Stirling’s formula, uniformly in \( d \) in a bounded range,

\[
\frac{(n-d)}{n} = \frac{(n-d)!}{(n-m-d)!} \cdot \frac{1}{n!} \geq (1 - \rho_n) \frac{(n-m)^{n-m}}{(n-m-d)^{n-m-d}} \frac{(n-d)^{n-d}}{n^n} \frac{\sqrt{n-m}}{\sqrt{n-m-d}} \frac{\sqrt{n-d}}{\sqrt{n}}
\]

\[
\geq (1 - \rho_n) \left(1 - \frac{m}{n-d}\right)^d \geq (1 - \rho_n) \exp \left(-dC \frac{m}{n-d}\right)
\]

for \( C > 0 \) such that \( 1 - x \geq e^{-Cx} \) for \( 0 \leq x \leq 1/2 \). Therefore, for large \( n \),

\[
lC \frac{m}{n-l} \geq \log(3/2),
\]

and since \( m \leq n/2 \),

\[
l \geq \tau := \frac{2\log(3/2)}{2C}
\]

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This gives us the upper bound
\[ \hat{W}_k(t) \leq O_k(t) \left( \left\lfloor \frac{\tau N_k(t-1)}{m(t-1)} \right\rfloor \right), \]
and the result follows.

**Proof of Theorem 4.3.** Clearly, it is enough to prove the lower bound
\[
\liminf_{t \to \infty} \frac{V_t(\pi)}{\mathbb{E}\left[ \max_{n \leq t} X_n^{(i^*)} \right]} \geq 1. \tag{21}
\]
For \( g \geq 1 \) consider the event,
\[ A_g = \left\{ W_{i^*}(t) \leq W_k(t) \text{ for some } t > g \text{ and } k \in K/\{i^*\} \right\}. \]
By Theorem 4.1 we know that \( \mathbb{P}(A_g) \to 0 \) as \( g \to \infty \). For \( t > g \) we have
\[
V_t(\pi) \geq \mathbb{E}\left[ \max_{g < n \leq t} X_n^{(i^*)} \mathbb{1}(B_n \cap A_g^c) \right] \\
\geq \mathbb{E}\left[ \max_{g < n \leq t} X_n^{(i^*)} \mathbb{1}(B_n) \right] - \mathbb{E}\left[ \max_{g < n \leq t} X_n^{(i^*)} \mathbb{1}(B_n \cap A_g^c) \right] \\
\geq \mathbb{E}\left[ \max_{g < n \leq t} X_n^{(i^*)} \mathbb{1}(B_n) \right] - \mathbb{E}\left[ \max_{g < n \leq t} X_n^{(i^*)} \mathbb{1}(A_g) \right],
\]
where \( B_n \) is the event that the arm pulled at time \( n \) is the arm with the highest index. Letting \( K_{g+1,t} \) be the number of times between \( g+1 \) and \( t \) that the arm with the highest index is not pulled, we have
\[
\mathbb{E}\left[ \max_{g < n \leq t} X_n^{(i^*)} \mathbb{1}(B_n) \right] = \mathbb{E}\left[ \max_{j=1,...,t-g-K_{g+1,t}} X_j^{(i^*)} \right],
\]
while by (18),
\[
\lim_{t \to \infty} \frac{\mathbb{E}\left[ \max_{j=1,...,t-g-K_{g+1,t}} X_j^{(i^*)} \right]}{\mathbb{E}\left[ \max_{j=1,...,t} X_j^{(i^*)} \right]} = 1. \tag{22}
\]
To ensure (22), one needs to control the size of \( K_{g+1,t} \). Such control is provided by the fact that the sequence \( (\epsilon_t) \) converges to 0. To see this, for every \( 0 < \delta < 1 \)
\[
P\left( K_{g+1,t} > \delta t \right) \leq \frac{EK_{g+1,t} / \delta t}{\delta t} \leq \sum_{j=1}^{t} \frac{\epsilon_j / K}{\delta t} \to 0,
\]
as \( t \to \infty \). Therefore, for any such \( \delta \),
\[
E_j=1,...,t-g-K_{g+1,t} X_j^{(i^*)} \geq E_j=1,...,t-g-K_{g+1,t} X_j^{(i^*)} \mathbb{1}(K_{g+1,t} \leq \delta t) \\
\geq P(K_{g+1,t} \leq \delta t) \max_{j=1,...,(1-\delta)t-g} X_j^{(i^*)} \sim a^{1/\lambda_i} \Gamma(1 - 1/\lambda_i) (1 - \delta)^{1/\lambda_i},
\]
as \( t \to \infty \) by (18). Therefore,
\[
\liminf_{t \to \infty} \frac{E \max_{j=1,...,t-g-K_{g+1,t}} X_j^{(i^*)}}{E \max_{j=1,...,t} X_j^{(i^*)}} \geq (1 - \delta)^{1/\lambda_i}.
\]
Since this is true for all \( 0 < \delta < 1 \), we obtain
\[
\lim_{t \to \infty} \frac{E \max_{j=1,...,t-g-K_{g+1,t}} X_j^{(i^*)}}{E \max_{j=1,...,t} X_j^{(i^*)}} \geq 1.
\]
Furthermore,

\[
\frac{E \max_{j=1, \ldots, t-g-K_{s+1,t}} X_j^{(i^*)}}{E \max_{j=1, \ldots, t} X_j^{(i^*)}} \leq 1.
\]

This ensures that \( K_{s+1,t} \) behaves nicely. Therefore, (21) will follow once we show that

\[
\lim_{g \to \infty} \limsup_{t \to \infty} t^{-1/\lambda_{i^*}} E \left[ \max_{n \leq t} X_n^{(i^*)} \mathbf{1} (A_g) \right] = 0.
\]

(23)

To this end, choose \( 1 < \theta < \lambda_{i^*} \), and note that

\[
E \left[ \max_{n \leq t} X_n^{(i^*)} \mathbf{1} (A_m) \right] \leq \left\{ E \left[ \max_{n \leq t} \left( X_n^{(i^*)} \right)^{\theta} \right] \right\}^{1/\theta} \left( P(A_g) \right)^{(\theta-1)/\theta}.
\]

Replacing \( \lambda_{i^*} \) by \( \lambda_{i^*}/\theta > 1 \), we have by (18),

\[
E \left[ \max_{n \leq t} \left( X_n^{(i^*)} \right)^{\theta} \right] \leq s(\theta, \lambda_{i^*}) t^{-\theta/\lambda_{i^*}}
\]

for some \( s(\theta, \lambda_{i^*}) \) finite positive constant depending only on \( \theta \) and \( \lambda_{i^*} \). Therefore,

\[
E \left[ \max_{n \leq t} X_n^{(i^*)} \mathbf{1} (A_g) \right] \leq s(\theta, \lambda_{i^*})^{1/\theta} t^{1-1/\lambda_{i^*}} \left( P(A_g) \right)^{(\theta-1)/\theta}.
\]

Since \( P(A_g) \to 0 \) as \( g \to \infty \), (23) follows.

\[\square\]

**Proof of Theorem 4.4.** Denote

\[ M_* = \sup \{ t \geq 1 : W_{i^*}(t) \leq W_k(t) \text{ for some } k \neq i^* \}. \]

It follows from Theorem 4.1 (which holds for polynomial-like arms) that \( M_* < \infty \) a.s. For \( m \leq n \leq t \), let \( K_{m,n} \) denote the number of times between \( m \) and \( n \) that the arm with the highest index is not pulled. We have

\[
V_n(\pi) \geq E \left[ \max_{j=1, \ldots, n-M_*-K_{M_*+1,n}} X_j^{(i^*)} \right].
\]

Choose \( 1 - \alpha < \theta < 1 - 1/\lambda_{i^*} \). We have

\[
V_n(\pi) \geq E \left[ \max_{j=1, \ldots, n-\lfloor n^\theta \rfloor-K_{\lfloor n^\theta \rfloor+1,n}} X_j^{(i^*)} \mathbf{1} (M_* \leq n^\theta) \right]
\]

\[
= E \left[ \max_{j=1, \ldots, n-\lfloor n^\theta \rfloor-K_{\lfloor n^\theta \rfloor+1,n}} X_j^{(i^*)} \right] - E \left[ \max_{j=1, \ldots, n-\lfloor n^\theta \rfloor-K_{\lfloor n^\theta \rfloor+1,n}} X_j^{(i^*)} \mathbf{1} (M_* > n^\theta) \right].
\]

Note that

\[
E \left[ \max_{j=1, \ldots, n-\lfloor n^\theta \rfloor-K_{\lfloor n^\theta \rfloor+1,n}} X_j^{(i^*)} \right] \geq E \left[ \max_{j=1, \ldots, n-2\lfloor n^\theta \rfloor} X_j^{(i^*)} \mathbf{1} (K_{1,n} \leq n^\theta) \right]
\]

\[
\geq E \left[ \max_{j=1, \ldots, n-2\lfloor n^\theta \rfloor} X_j^{(i^*)} \right] - E \left[ \max_{j=1, \ldots, n-2\lfloor n^\theta \rfloor} X_j^{(i^*)} \mathbf{1} (K_{1,n} > n^\theta) \right].
\]

Using the fact that \( \beta_{i^*} > 1/\lambda_{i^*} \), we have by Theorem 1 in Carpenter and Valko (2014), for large \( n \),

\[
E \left[ \max_{j=1, \ldots, n-2\lfloor n^\theta \rfloor} X_j^{(i^*)} \right]
\]

\[
\geq (n-n^\theta a_{i^*})^{1/\lambda_{i^*}} \Gamma(1-1/\lambda_{i^*}) + o(1)
\]

\[
\geq (na_{i^*})^{1/\lambda_{i^*}} \Gamma(1-1/\lambda_{i^*}) - a_{i^*}^{1/\lambda_{i^*}} \Gamma(1-1/\lambda_{i^*})(n^{1/\lambda_{i^*}} - (n-n^\theta)^{1/\lambda_{i^*}}) + o(1)
\]

\[
= (na_{i^*})^{1/\lambda_{i^*}} \Gamma(1-1/\lambda_{i^*}) + o(1)
\]

\[
E \left[ \max_{j=1, \ldots, n} X_j^{(i^*)} \right] + o(1).
\]

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where the penultimate step is due to the choice of $\theta$. Therefore, result will follow once we prove that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \max_{j=1,\ldots,n} X_j^{(\tau')} \mathbf{1}(M > n^\theta) \right] = \lim_{n \to \infty} \mathbb{E} \left[ \max_{j=1,\ldots,n} X_j^{(\tau')} \mathbf{1}(K_{1,n} > n^\theta) \right] = 0. \tag{24}
\]

Clearly,
\[
\mathbb{E} \left[ \max_{j=1,\ldots,n} X_j^{(\tau')} \mathbf{1}(K_{1,n} > n^\theta) \right] = \mathbb{E} \left[ \max_{j=1,\ldots,n} X_j^{(\tau')} \right] \mathbb{P}(K_{1,n} > n^\theta).
\]

Since $\theta > 1 - \alpha$, an elementary exponential Markov inequality shows that $\mathbb{P}(K_{1,n} > n^\theta)$ is exponentially small. Using (18), we obtain one of the two statements in (24). Similarly,
\[
\mathbb{E} \left[ \max_{j=1,\ldots,n} X_j^{(\tau')} \mathbf{1}(M > n^\theta) \right] = \mathbb{E} \left[ \max_{j=1,\ldots,n} X_j^{(\tau')} \right] \mathbb{P}(M > n^\theta).
\tag{25}
\]

Next we estimate the probability in the right hand side above. Note that this probability does not change if we apply the same monotone increasing function to all rewards. Taking the logarithm of the rewards makes the reward distribution exponential-like, i.e., satisfy (7). Let $0 < \delta < (\min_{i \neq i^*} \lambda_i - \lambda_{i^*})/\lambda_{i^*}$. We have
\[
\mathbb{P}(M > n^\theta) \leq \mathbb{P}(W_{i^*}(d) \leq (1 + \delta)^{-1/2}\lambda_{i^*}^{-1} \log m(d - 1) \text{ for some } d > n^\theta)
\]
\[
+ \sum_{\delta < \lambda_{i^*}} \mathbb{P}(W_i(d) \geq (1 + \delta)^{-1/2}\lambda_i^{-1} \log m(d - 1) \text{ for some } d > n^\theta).
\tag{26}
\]

Next, for $a > (K-1)K$,
\[
\mathbb{P}(W_{i^*}(d) \leq (1 + \delta)^{-1/2}\lambda_{i^*}^{-1} \log m(d - 1) \text{ for some } d > n^\theta)
\]
\[
\leq \sum_{d > n^\theta} \mathbb{P} \left( m(d - 1) \leq \frac{1}{a} \sum_{l=1}^{d-1} \epsilon_l \right)
\]
\[
+ \sum_{d > n^\theta} \mathbb{P} \left( m(d - 1) > \frac{1}{a} \sum_{l=1}^{d-1} \epsilon_l, \ W_{i^*}(d) \leq (1 + \delta)^{-1/2}\lambda_{i^*}^{-1} \log m(d - 1) \right).
\]

Since $m(d - 1) \geq K_{1,d-1}/(K-1)$, another application of the exponential Markov inequality shows that $\mathbb{P} \left( m(d - 1) \leq \sum_{l=1}^{d-1} \epsilon_l/a \right)$ decreases exponentially fast, hence the sum
\[
\sum_{d > n^\theta} \mathbb{P} \left( m(d - 1) \leq \frac{1}{a} \sum_{l=1}^{d-1} \epsilon_l \right)
\]
is an exponentially fast decreasing function of $n$. Furthermore,
\[
\mathbb{P} \left( m(d - 1) > \frac{1}{a} \sum_{l=1}^{d-1} \epsilon_l, \ W_{i^*}(d) \leq (1 + \delta)^{-1/2}\lambda_{i^*}^{-1} \log m(d - 1) \right)
\]
\[
\leq \sum_{\sum_{l=1}^{d-1} \epsilon_l/n < j_1 \leq d} \mathbb{P} \left( [j_2/j_1]^{\text{th}} \text{ order statistic in } X_1^{(\tau')}, \ldots, X_{j_2}^{(\tau')} \leq (1 + \delta)^{-1/2}\lambda_{i^*}^{-1} \log j_1 \right).
\]

The latter sum is a sum of binomial probabilities and the exponential markov inequality shows that it also decays exponentially fast with $d$, hence the sum
\[
\sum_{d > n^\theta} \mathbb{P} \left( m(d - 1) > \frac{1}{a} \sum_{l=1}^{d-1} \epsilon_l, \ W_{i^*}(d) \leq (1 + \delta)^{-1/2}\lambda_{i^*}^{-1} \log m(d - 1) \right)
\]
decays exponentially fast with $n$. It follows that

$$\lim_{n \to \infty} \left\{ \mathbb{E} \left[ \max_{j=1, \ldots, n} X_j^{i^*} \right] \mathbb{P}(W_i(d) \leq (1 + \delta)^{-1/2} \lambda_i^{-1} \log m(d) \text{ for some } d > n^\theta) \right\} = 0. \quad (27)$$

In an analogous way we can show that for any $i \neq i^*$,

$$\lim_{n \to \infty} \left\{ \mathbb{E} \left[ \max_{j=1, \ldots, n} X_j^{i^*} \right] \mathbb{P}(W_i(d) \geq (1 + \delta)^{1/2} \lambda_i^{-1} \log m(d) \text{ for some } d > n^\theta) \right\} = 0, \quad (28)$$

and the remaining statement in (24) follows from (25), (26), (27) and (28).

**Remark:** One can prove Theorem 4.2 using the same arguments as in Theorem 4.4, thereby establishing the vanishing extremal regret in the strong sense in case of exponential-like arms.

**Proof of Theorem 5.1.** Note that one can switch from exponential-like arms to polynomial-like arms by exponentiation of the former, and switch back by taking the logarithm of the latter. The result is invariant under monotone transformation of the rewards, and the theorem holds for both exponential-like and polynomial-like arms. We establish the result for exponential-like arms below.

Let $a > (K - 1)K$, $b > 0$ and $A > \kappa + 2$. Denote $\nu_{d-1} = \sum_{i=1}^{d-1} \epsilon_i/a$. Using (17), we have for $k \neq i^*$, for some $c > 0$

$$\mathbb{P} \left( m(d-1) \geq \frac{1}{a} \sum_{i=1}^{d-1} \epsilon_i \tilde{W}_k(d) \geq \frac{1}{\lambda_k} \log a_k + \frac{1}{\lambda_k} \log h(m(d-1)) + b \right)$$

$$\leq \frac{1}{c} \sum_{m \geq \nu_{d-1}} \sum_{j=m}^{d-1} \exp(-cj/h(m)) \leq \frac{\exp(-cm/h(m))}{1 - \exp(-c/h(m))}$$

$$\leq \frac{1}{c} \sum_{m \geq \nu_{d-1}} h(m) \exp(-cm/h(m)) \leq \frac{1}{c} \sum_{m \geq \nu_{d-1}} h(m) \exp(-A \log m) \quad (\text{because } h(x) = o(x/\log x))$$

$$\leq \frac{1}{c} \sum_{m \geq \nu_{d-1}} m^{A-1} = O \left( \left( \sum_{i=1}^{d-1} \epsilon_i \right)^{-A-1} \right) \quad (29)$$

for all $d$ large enough. Using the condition on probabilities ($\epsilon_i$) and the first Borel-Cantelli lemma; see e.g., Durrett (2019), we see that for any $b > 0$

$$\mathbb{P} \left( m(d-1) \geq \frac{1}{a} \sum_{i=1}^{d-1} \epsilon_i \tilde{W}_k(d) \geq \frac{1}{\lambda_k} \log a_k + \lambda_k \log h(m(d-1)) + b \right)$$

$$= 0 \quad \text{for infinitely many } d$$

Since $a > K - 1$, by Theorem 3.1, for any $b > 0$

$$\mathbb{P} \left( \tilde{W}_k(d) \geq \frac{1}{\lambda_k} \log a_k + \frac{1}{\lambda_k} \log h(m(d-1)) + b \right) = 0$$

An analogous argument shows that for any $b > 0$

$$\mathbb{P} \left( \tilde{W}_k(d) \leq \frac{1}{\lambda_k} \log a_k + \frac{1}{\lambda_k} \log h(m(d-1)) - b \right) = 0,$$

and the result follows. \qed
Proof of Theorem 4.1. The proof closely follows the proof of Theorem 5.1, and is established below for exponential-like rewards. Because exponential-like and polynomial-like arms are related via a monotone transformation, it is enough to prove the statement for exponential-like arms.

First, we establish that for any $a > (K - 1)K$ and $\delta > 0$,

$$P\left(m(n) \geq \frac{1}{a} \sum_{d=1}^{n} \varepsilon_d, W_i(n) \geq (1 + \delta)\lambda_i^{-1} \log m(n) \text{ for infinitely many } n\right) = 0.$$

The above probability can be bounded as follows

$$P\left(m(n) \geq \frac{1}{a} \sum_{d=1}^{n} \varepsilon_d, W_i(n) \geq (1 + \delta)\lambda_i^{-1} \log m(n)\right) \leq \sum_{m \geq \frac{1}{a} \sum_{d=1}^{n} \varepsilon_d} \sum_{k=m}^{n} P\left(O_{i,k} \left(\left\lfloor \frac{k}{m} \right\rfloor\right) \geq (1 + \delta)\lambda_i^{-1} \log m\right).$$

The double summation on the right-hand side is the probability that a Binomial random variable with $k$ trials and the probability for success $a_i m^{-\left(1+\delta\right)}$, takes a value at least $k/m$. Using the exponential Markov inequality for the Binomial random variable $X \sim B(k, p)$, for any $\theta > 0$,

$$P(X > m) \leq e^{-\theta m} \left(e^{\theta p} + 1 - p\right)^k,$$

with $\theta = (1 + \delta) \log m$ gives us the upper bound of

$$\exp\left\{-k^\delta/2\right\}.$$

That is we have for $0 < A < \infty$,

$$P\left(m(n) \geq \frac{1}{a} \sum_{d=1}^{n} \varepsilon_d, W_i(n) \geq (1 + \delta)\lambda_i^{-1} \log m(n)\right) \leq \exp\left\{-A \left(\sum_{d=1}^{n} \varepsilon_d\right)^\delta\right\}.$$

Now using Borel-Cantelli lemma, we have that for any sub-optimal arm $i$ and any $0 < \delta < 1$, w.p.1,

$$W_i(n) \leq (1 + \delta)\lambda_i^{-1} \log m(n)$$

for all $n$ large enough. This is because the event

$$\left\{m(n) \geq \frac{1}{a} \sum_{d=1}^{n} \varepsilon_d \text{ for infinitely many } n\right\}$$

has probability 1. Since the optimal $\lambda_{i^*}$ is strictly smaller than the next best $\lambda_i$, we can find $0 < \delta < 1$ so that

$$(1 - \delta)\lambda_{i^*}^{-1} > (1 + \delta)\lambda_i^{-1},$$

for all sub-optimal $i$.

Next, we establish that for any $a > (K - 1)K$

$$P\left(m(n) \geq \frac{1}{a} \sum_{d=1}^{n} \varepsilon_d, W_i(n) < (1 - \delta)\lambda_i^{-1} \log m(n) \text{ for infinitely many } n\right) = 0.$$

The above probability can similarly be upper bounded by

$$\sum_{m \geq \frac{1}{a} \sum_{d=1}^{n} \varepsilon_d} \sum_{k=m}^{n} P\left(O_{i,k} \left(\left\lfloor \frac{k}{m} \right\rfloor\right) \leq (1 - \delta)\lambda_i^{-1} \log m\right).$$

This probability is the probability that a Binomial random variable with $k$ trials and the probability for success $a_i m^{-\left(1-\delta\right)}$, takes a value smaller than $k/m$. Again using the exponential Markov inequality for the Binomial random variable $X \sim B(k, p)$: for any $\theta > 0$,

$$P(X < m) \leq e^{\theta m} \left(e^{\theta p} + 1 - p\right)^k.$$

With $\theta = (1 - \delta) \log m$, we similarly obtain for all $0 < \delta < 1$,

$$W_i(n) \geq (1 - \delta)\lambda_i^{-1} \log m(n)$$

for all $n$ large enough w.p.1, and the result holds. □
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