ADJOINT IDEALS ALONG CLOSED SUBVARIETIES
OF HIGHER CODIMENSION

SHUNSUKE TAKAGI

Abstract. In this paper, we introduce a notion of adjoint ideal sheaves along closed subvarieties of higher codimension and study its local properties using characteristic $p$ methods. When $X$ is a normal Gorenstein closed subvariety of a smooth complex variety $A$, we formulate a restriction property of the adjoint ideal sheaf $\text{adj}_X(A)$ of $A$ along $X$ involving the l.c.i. ideal sheaf $\mathcal{D}_X$ of $X$. The proof relies on a modification of generalized test ideals of Hara and Yoshida [11].

Introduction

The adjoint ideal sheaf along a divisor $D$ on a complex variety $V$ is a modification of the multiplier ideal sheaf associated to $D$, and it encodes much information on the singularities of $D$. It recently turned out that it is a powerful tool in birational geometry and has several applications, such as the study of singularities of ample divisors of low degree on abelian varieties by Ein-Lazarsfeld [4] and Debarre-Hacon [3], inversion of adjunction on log canonicity proved by Kawakita [15], and the boundedness of pluricanonical maps of varieties of general type proved by Hacon-McKernan [8] and Takayama [23]. In this paper, we introduce a notion of adjoint ideal sheaves along closed subvarieties of higher codimension and study its local properties using characteristic $p$ methods. We hope that our adjoint ideal sheaves lead to further applications.

Let $A$ be a smooth complex variety and $Y = \sum_{i=1}^m t_i Y_i$ be a formal combination, where the $t_i$ are positive real numbers and the $Y_i$ are proper closed subschemes of $A$. Let $X$ be a reduced closed subscheme of pure codimension $c$ of $A$ such that no components of $X$ are contained in the support of any $Y_i$. Suppose that $\pi : \tilde{A} \to A$ is a log resolution of $(A, X + Y)$ and $E := \sum_{j=1}^s E_j$ is smooth, where $E_1, \ldots, E_s$ are all the irreducible divisors on $\tilde{A}$ “dominating” a component of $X$. If $K_{\tilde{A}/A}$ is the relative canonical divisor of $\pi$, then we define the adjoint ideal sheaf $\text{adj}_X(A, Y)$ associated to the pair $(A, Y)$ along $X$ by

$$\text{adj}_X(A, Y) := \pi_* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) - \lfloor \pi^{-1}(Y) \rfloor + E),$$

where $\pi^{-1}(X)$ and $\pi^{-1}(Y) := \sum_{i=1}^m t_i \pi^{-1}(Y_i)$ are the scheme theoretic inverse images of $X$ and $Y$, respectively (see Definition 1.6 for the precise definition of the adjoint ideal sheaf $\text{adj}_X(A, Y)$). We say that $(A, Y)$ is purely log terminal (plt, for short) along $X$ if $\text{adj}_X(A, Y) = \mathcal{O}_A$. When $X$ is a divisor, our definitions coincide with the definitions of usual plt pairs and adjoint ideal sheaves. In order to study local

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properties of our adjoint ideal sheaves using characteristic $p$ methods, we consider a modification of generalized test ideals of Hara and Yoshida.

Let $(R, m)$ be a Noetherian local domain of characteristic $p > 0$ and $a^q = \prod_{i=1}^m a_i^{t_i}$ be a formal combination, where the $a_i \subseteq R$ are nonzero ideals and the $t_i$ are positive real numbers. Hara-Yoshida [11] introduced notions of tight closure for the pair $(R,a^q)$, called $a^q$-tight closure, and the corresponding test ideal $\tau(a^q)$. They then proved that the multiplier ideal sheaf coincides, after reduction to characteristic $p \gg 0$, with their generalized test ideal. In this paper, we define a notion of tight closure for a triple $(R, I, a^q)$, called $(I,a^q)$-tight closure, where $I \subseteq R$ is an unmixed ideal of height $c$ such that the $a_i$ are not contained in any minimal prime ideal of $I$: the $(I,a^q)$-tight closure $J^{q(I,a^q)}$ of an ideal $J \subseteq R$ is the ideal consisting of all elements $x \in R$ for which there exists $\gamma \in R$ not in any minimal prime of $I$ such that

$$\gamma J^{c(q-1)} a_1^{[t_1q]} \cdots a_m^{[t_mq]} a^q \subseteq J^q$$

for all large $q = p^e$, where $J^q$ is the ideal generated by the $q^{th}$ powers of all elements of $J$. If $N \subseteq M$ are $R$-modules, then the $(I, a^q)$-tight closure $N^{q(I,a^q)}_M$ of $N$ in $M$ is defined similarly (see Definition 2.2 for the detail). We then define the generalized test ideal $\tau_I(a^q)$ along $I$ to be the annihilator ideal of the $(I, a^q)$-tight closure $0^{q(I,a^q)}_{E_R(R/m)}$ of the zero submodule in the injective hull $E_R(R/m)$ of the residue field of $R$. When $I = R$, $(I, a^q)$-tight closure coincides with $a^q$-tight closure and the generalized test ideal $\tau_I(a^q)$ along $I$ is nothing but $\tau(a^q)$. We conjecture that the ideal $\tau_I(a^q)$ corresponds to the adjoint ideal sheaf $\operatorname{adj}_X(A, Y)$, and we obtain some partial results (Theorems 2.7 and 2.9). We use them to prove a restriction formula of our adjoint ideal sheaves.

Kawakita [16] and Ein-Mustata [5] introduced an ideal sheaf, called the l.c.i. defect ideal sheaf, which measures how far a variety is from being locally a complete intersection. They then proved a comparison of minimal log discrepancies of a variety $X$ and its ambient space $A$ with a boundary corresponding to the l.c.i. defect ideal sheaf $\mathcal{D}_X$ of $X$. Their result inspires us to formulate a restriction property of the adjoint ideal sheaf $\operatorname{adj}_X(A, Y)$ involving the l.c.i. defect ideal sheaf $\mathcal{D}_X$ of $X$.

**Theorem 3.1.** Let $A$ be a smooth complex variety and $Y = \sum_{i=1}^m t_i Y_i$ be a formal combination, where the $t_i$ are positive real numbers and the $Y_i$ are proper closed subschemes of $A$. If $X$ is a normal Gorenstein closed subvariety of codimension $c$ of $A$ which is not contained in the support of any $Y_i$, then

$$\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X) = \operatorname{adj}_X(A, Y)\mathcal{O}_X,$$

where $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$ is the multiplier ideal sheaf associated to the pair $(X, V(\mathcal{D}_X) + Y|_X)$ (see Definition 1.2 for the definition of multiplier ideal sheaves).

By making use of the partial correspondence between the adjoint ideal sheaf $\operatorname{adj}_X(A, Y)$ and the generalized test ideal $\tau_I(a^q)$, Theorem 3.1 can be reduced to a purely algebraic problem on some ideals of a ring of characteristic $p > 0$. We then solve the problem using the linkage theory of Peskine and Szpiro [19].

As a corollary of Theorem 3.1, we obtain a characterization of being plt along a Gorenstein closed subvariety in terms of Frobenius splitting (Corollary 3.1).
1. Multiplier ideals and Adjoint ideals

In this section, we first recall the definitions of multiplier ideal sheaves and adjoint ideal sheaves along divisors (our main references are \([17]\) and \([18]\)), and then we introduce a notion of adjoint ideal sheaves along closed subvarieties of higher codimension.

Let \(X\) be a \(d\)-dimensional \(\mathbb{Q}\)-Gorenstein normal variety over a field \(k\) of characteristic zero and \(Y = \sum_{i=1}^{m} t_i Y_i\) be a formal combination, where the \(t_i\) are real numbers and the \(Y_i\) are proper closed subschemes of \(X\). Since \(X\) is normal, we have a Weil divisor \(K_X\) on \(X\), uniquely determined up to linear equivalence, such that \(\mathcal{O}_X(K_X) \cong i_*\Omega^d_{X, \text{reg}}\) where \(i : X_{\text{reg}} \hookrightarrow X\) is the inclusion of the nonsingular locus. Moreover, since \(X\) is \(\mathbb{Q}\)-Gorenstein, there exists a positive integer \(r\) such that \(rK_X\) is a Cartier divisor.

Let \(E\) be a divisor over \(X\), that is, \(E\) is an irreducible divisor on some normal variety \(X'\) with a birational morphism \(f : X' \to X\). We identify two divisors over \(X\) if they correspond to the same valuation of the function field \(k(X)\). The center of \(E\) is the closure of \(f(E)\) in \(X\), denoted by \(c_X(E)\). If \(Z\) is a closed subscheme of \(X\), then we define \(\text{ord}_E(Z)\) as follows: we may assume that the scheme theoretic inverse image \(f^{-1}(Z)\) is a divisor. Then \(\text{ord}_E(Z)\) is the coefficient of \(E\) in \(f^{-1}(Z)\). We put \(\text{ord}_E(Y) := \sum_{i=1}^{m} t_i \text{ord}_E(Y_i)\) and define \(\text{ord}_E(K_{-X})\) as the coefficient of \(E\) in the relative canonical divisor \(K_{X'/X}\) of \(f\). Recall that \(K_{X'/X}\) is the unique \(\mathbb{Q}\)-divisor supported on the exceptional locus of \(f\) such that \(rK_{X'/X}\) is linearly equivalent to \(rK_X - f^*(rK_X)\). Then the log discrepancy \(a(E; X, Y)\) of \((X, Y)\) with respect to \(E\) is

\[
a(E; X, Y) := \text{ord}_E(K_{-X}) - \text{ord}_E(Y) + 1.
\]

If \(W\) is a closed subset of \(X\), then the minimal log discrepancy \(\text{mld}(W; X, Y)\) of \((X, Y)\) along \(W\) is defined by

\[
\text{mld}(W; X, Y) := \inf\{a(E; X, Y) \mid E\text{ is a divisor over } X, c_X(E) \subseteq W\}.
\]

**Definition 1.1.** Let the notation be the same as above.

(i) We say that the pair \((X, Y)\) is Kawamata log terminal (klt, for short) if \(\text{mld}(X; X, Y) > 0\). Since a resolution of singularities is obtained by blowing up subvarieties in the singular locus, this condition is equivalent to saying that \(\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^{m} Y_i; X, Y) > 0\), where \(X_{\text{sing}}\) is the singular locus of \(X\).

(ii) Let \(D\) be a reduced Cartier divisor on \(X\) such that no components of \(D\) are contained in the support of any \(Y_i\). Then we say that \((X, Y)\) is purely log terminal (plt, for short) along \(D\) if \(a(E; X, D + Y) > 0\) for all divisors \(E\) over \(X\) dominating no components of \(D\).

Suppose that \((X, Y)\) is a pair as above. A log resolution of the pair \((X, Y)\) is a proper birational morphism \(\pi : \tilde{X} \to X\) with \(\tilde{X}\) nonsingular such that all the scheme theoretic inverse images \(\pi^{-1}(Y_i)\) are divisors and in addition \(\bigcup_{i=1}^{m} \text{Supp} \, \pi^{-1}(Y_i) \cup \text{Exc}(\pi)\) is a simple normal crossing divisor. The existence of log resolutions is guaranteed by Hironaka’s desingularization theorem \([13]\).

**Definition 1.2** (\([18]\) Definition 9.3.60). Let the notation be the same as above.
(i) Fix a log resolution $\pi : \tilde{X} \to X$ of $(X, Y)$. The multiplier ideal sheaf $\mathcal{J}(X, Y)$ associated to the pair $(X, Y)$ is

$$\mathcal{J}(X, Y) = \pi_*\mathcal{O}_{\tilde{X}}([K_{\tilde{X}}/\tilde{X} - \sum_{i=1}^{m} t_i \pi^{-1}(Y_i)]) \subseteq \mathcal{O}_X.$$ 

(ii) Let $D$ be a reduced Cartier divisor on $X$ such that no components of $D$ are contained in the support of any $Y_i$. Fix a log resolution $\pi : \tilde{X} \to X$ of $(X, D + Y)$ so that the strict transform $\pi_*^{-1}D$ of $D$ is nonsingular (but possibly disconnected). Then the adjoint ideal sheaf $\text{adj}_D(X, Y)$ associated to the pair $(X, Y)$ along $D$ is

$$\text{adj}_D(X, Y) = \pi_*\mathcal{O}_{\tilde{X}}([K_{\tilde{X}}/\tilde{X} - \sum_{i=1}^{m} t_i \pi^{-1}(Y_i) - \pi^*D + \pi_*^{-1}D]) \subseteq \mathcal{O}_X.$$ 

We denote this ideal sheaf simply by $\text{adj}_D(X)$ when $Y = 0$.

**Remark 1.3.**

1. (cf. [18, Theorem 9.2.18]) $\mathcal{J}(X, Y)$ and $\text{adj}(X, Y)$ are independent of the choice of the log resolution $\pi$ used to define them (see also Lemma 1.7 (1)).

2. The pair $(X, Y)$ is klt (resp. plt along $D$) if and only if $\mathcal{J}(X, Y) = \mathcal{O}_X$ (resp. $\text{adj}_D(X, Y) = \mathcal{O}_X$).

3. ([18, Example 9.3.49]) Suppose that $X$ is an affine variety and $I$ is a nonzero ideal of $\mathcal{O}_X$. Choose a general element $f$ in $I$ so that $\text{div}_X(f)$ is reduced and no components of $\text{div}_X(f)$ are contained in the support of any $Y_i$. Then

$$\mathcal{J}(X, V(I) + Y) = \text{adj}_{\text{div}_X(f)}(X, Y)$$

(see also Claim 1 in the proof of Theorem 3.1).

An analogue of local vanishing theorem [18, Theorem 9.4.1] holds for the adjoint ideal sheaf $\text{adj}_D(X, Y)$ along a divisor $D$.

**Proposition 1.4.** Let the notation be the same as in Definition 1.2 (ii). Then for all $i > 0$,

$$R^i\pi_*\mathcal{O}_{\tilde{X}}([K_{\tilde{X}}/\tilde{X} - \pi^{-1}(Y) - \pi^*D + \pi_*^{-1}D]) = 0.$$ 

**Proof.** Set $B := [K_{\tilde{X}}/\tilde{X} - \pi^{-1}(Y) - \pi^*D]$ and $\tilde{D} := \pi_*^{-1}D$. Let $\nu : D' \to D$ be the normalization of $D$, $\mu : \tilde{D} \to D'$ be the induced morphism and $\pi_D : \tilde{D} \to D$ be the composite morphism. Then there exists an effective $\mathbb{Q}$-divisor $\text{Diff}_{D'}(0)$ on $D'$, called the different of the zero divisor on $D'$ (see [20, §3] for details), such that $K_{D'} + \text{Diff}_{D'}(0)$ is $\mathbb{Q}$-Cartier and $K_{D'} + \text{Diff}_{D'}(0) = \nu^*((K_{X} + D)|_{D})$. Now we have the following exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(B) \to \mathcal{O}_{\tilde{X}}(B + \tilde{D}) \to \mathcal{O}_{\tilde{D}}([K_{\tilde{D}} - \mu^*(K_{D'} + \text{Diff}_{D'}(0)) - \pi_D^{-1}(Y|_D)]) \to 0.$$ 

It follows from Kawamata-Viehweg vanishing theorem that

$$R^i\pi_*\mathcal{O}_{\tilde{X}}(B) = R^i\pi_*\mathcal{O}_{\tilde{D}}([K_{\tilde{D}} - \mu^*(K_{D'} + \text{Diff}_{D'}(0)) - \pi_D^{-1}(Y|_D)]) = 0$$

for all $i > 0$. Thus, we have $R^i\pi_*\mathcal{O}_{\tilde{X}}(B + \tilde{D}) = 0$ for all $i > 0$. 

$\Box$
Example 1.5. Let $X = \mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$ be the two-dimensional affine space and let $D = (x^3 + y^5 = 0) \subseteq X$. Then $\text{adj}_D(X) = (x^2, xy, y^3)$, whereas $\mathcal{J}(X, D) = (x^3 + y^5)$.

When the ambient variety is smooth, we can generalize the notion of adjoint ideal sheaves to the higher codimension case.

Let $A$ be a nonsingular variety over a field $k$ of characteristic zero and $Y = \sum_{i=1}^m t_i Y_i$ be a formal combination, where the $t_i$ are positive real numbers and the $Y_i$ are proper closed subschemes of $A$. Let $X$ be a reduced closed subscheme of pure codimension $c$ of $A$ such that no components of $X$ are contained in the support of any $Y_i$. Let $f : A' := \text{Bl}_{X} A \to A$ be the blowing-up of $A$ along $X$ and $E_1, \ldots, E_s$ be all the components of the exceptional divisor of $f$ dominating an irreducible component of $X$. Fix a log resolution $g : \tilde{A} \to A'$ of $(A', f^{-1}(X) + f^{-1}(Y))$ such that $\sum_{j=1}^s g_j^{-1}E_j$ is nonsingular (but possibly disconnected), and put $\pi := f \circ g : \tilde{A} \to A$.

Definition 1.6. In the above situation, the adjoint ideal sheaf $\text{adj}_X(A, Y)$ associated to the pair $(A, Y)$ along $X$ is

$$\text{adj}_X(A, Y) = \pi_* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - \sum_{i=1}^m [t_i \pi^{-1}(Y_i)] - c \pi^{-1}(X) + \sum_{j=1}^s g_j^{-1}E_j) \subseteq \mathcal{O}_A.$$ 

We denote this ideal sheaf simply by $\text{adj}_X(A)$ when $Y = 0$. We say that $(A, Y)$ (resp. $A$) is purely log terminal (plt, for short) along $X$ if $\text{adj}_X(A, Y) = \mathcal{O}_A$ (resp. $\text{adj}_X(A) = \mathcal{O}_A$). When $X$ is a divisor, these definitions coincide with those given in Definition 1.1(ii) and Definition 1.2(ii).

Lemma 1.7. Let the notation be as in Definition 1.6.

1. The adjoint ideal sheaf $\text{adj}_X(A, Y)$ is independent of the choice of the log resolution used to define it.
2. $(A, Y)$ is plt along $X$ if and only if

$$\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i; A, cX + Y) > 0,$$

where $X_{\text{sing}}$ is the singular locus of $X$. More generally, the adjoint ideal sheaf $\text{adj}_X(A, Y)$ is an ideal sheaf of $X$ whose sections over an open subset $U$ are those $\varphi \in \mathcal{O}_X(U)$ such that for every divisor $E$ over $X$ whose center intersects $U$ and is contained in $X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i$,

$$\text{ord}_E(\varphi) + a(E; A, cX + Y) > 0.$$

Proof. Let $f : A' \to A$ be the blowing-up of $A$ along $X$ and $E_1, \ldots, E_s$ be all the components of the exceptional divisor of $f$ dominating an irreducible component of $X$. Put $E = E_1 + \cdots + E_s$.

1. The proof is essentially the same as that of [18, Theorem 9.2.18]. We consider a sequence of morphisms $V \xrightarrow{\nu} \tilde{A} \xrightarrow{\pi} A$, where $\pi$ is a log resolution of $(A, X + Y)$ such that the strict transform $\tilde{E}$ of $E$ is nonsingular and $\nu$ is a log resolution of $(\tilde{A}, \pi^{-1}(X) + \pi^{-1}(Y))$. 

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Claim. Let \( D \) be a reduced disconnected divisor on \( \tilde{A} \) with simple normal crossing support and \( B \) be an \( \mathbb{R} \)-divisor on \( \tilde{A} \) with simple normal crossing support which has no common components with \( D \). Suppose that \( \mu : W \to \tilde{A} \) is a log resolution of \( D + B \). Then
\[
\mu_* \mathcal{O}_W(K_{W/\tilde{A}} - [\mu^*(D + B)] + \mu_*^{-1}D) = \mathcal{O}_{\tilde{A}}(-[B]).
\]

Proof of Claim. It follows from the projection formula that if the assertion holds for a given \( \mathbb{R} \)-divisor \( B \), then it holds also for \( B + B' \) whenever \( B' \) is an integral divisor on \( \tilde{A} \). Therefore, we may assume that \([B] = 0\). Setting \( \Delta = D + B \), we have
\[
K_W + \mu_*^{-1}\Delta = \mu^*(K_{\tilde{A}} + \Delta) + \sum_F (a(F; \tilde{A}, \Delta) - 1)F,
\]
where \( F \) runs through all \( \mu \)-exceptional prime divisors on \( W \). Then
\[
K_{W/\tilde{A}} - [\mu^*\Delta] + \mu_*^{-1}D = [K_{W/\tilde{A}} - \mu^*\Delta + \mu_*^{-1}\Delta] = \sum_F [a(F; \tilde{A}, \Delta) - 1]F,
\]
because \([B] = 0\). Since \( D \) is disconnected, by [17, Corollary 2.31 (3)], one has \( a(F; \tilde{A}, \Delta) > 0 \) for all \( \mu \)-exceptional prime divisors \( F \). This completes the proof. \( \square \)

By the above claim, we know that
\[
\nu_* \mathcal{O}_V(K_{V/\tilde{A}} - [\nu^{-1}\pi^{-1}(Y)] - \nu^*\tilde{E} + \nu_*^{-1}\tilde{E}) = \mathcal{O}_{\tilde{A}}(-[\pi^{-1}(Y)]).
\]
Then, setting \( h := \pi \circ \nu \), one finds using the projection formula:
\[
h_* \mathcal{O}_V(K_{V/\tilde{A}} - [h^{-1}(Y)] - c h^{-1}(X) + \nu_*^{-1}\tilde{E})
\]
\[= \pi_* \nu_* \left( \nu^* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E}) \otimes \mathcal{O}_V(K_{V/\tilde{A}} - [\nu^{-1}\pi^{-1}(Y)] - \nu^*\tilde{E} + \nu_*^{-1}\tilde{E}) \right)
\]
\[= \pi_* \left( \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E}) \otimes \nu_* \mathcal{O}_V(K_{V/\tilde{A}} - [\nu^{-1}\pi^{-1}(Y)] - \nu^*\tilde{E} + \nu_*^{-1}\tilde{E}) \right)
\]
\[= \pi_* \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - [\pi^{-1}(Y)] - c \pi^{-1}(X) + \tilde{E}).
\]
In other words, we obtain the same adjoint ideal sheaf working from \( h \) as working from \( \pi \). Since any two resolutions can be dominated by a third, the assertion follows.

(2) Since no components of \( X \) are contained in the support of any \( Y_i \), \( Y \) does not contribute to \( a(E_i; A, cX + Y) \). By [17, Lemma 2.29], one has \( a(E_i; A, cX + Y) = a(E_i; A, cX) = 0 \) for all \( i = 1, \ldots, s \). We have already seen in (1) that the adjoint ideal sheaf is independent of the choice of the log resolution used to define it. Since
\[
K_{\tilde{A}/A} - [\pi^{-1}(Y)] - c \pi^{-1}(X) + g_*^{-1}E = \sum_{F, \text{divisor on } \tilde{A}} [a(F; A, cX + Y) - 1]F + g_*^{-1}E
\]
\[= \sum_{F \neq g_*^{-1}E_i} [a(F; A, cX + Y) - 1]F
\]
for every log resolution \( g : \tilde{A} \to A' \) of \((A', f^{-1}(X) + f^{-1}(Y))\) where \( \pi = f \circ g : \tilde{A} \to A \), the pair \((A, Y)\) is plt along \( X \) if and only if \( a(F; A, cX + Y) > 0 \) for every divisor \( F \) over \( A \) dominating no components of \( E \). This implies that if \((A, Y)\) is plt along \( X \), then \( \text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m Y_i; A, cX + Y) > 0 \). The converse follows from the fact that
a log resolution of \((A', f^{-1}(X) + f^{-1}(Y))\) is obtained by blowing up subvarieties in \(f^{-1}(X_{\text{sing}} \cup \bigcup_{i=1}^{m} Y_i)\). We can prove the general case similarly.

**Example 1.8.** (1) Suppose that \(X\) is locally a complete intersection variety, and consider the blowing-up \(f : A' = \text{Bl}_X A \to A\) of \(A\) along \(X\). Then the exceptional divisor \(E := f^{-1}(X)\) is a projective bundle over \(X\). In particular, \(E\) is normal and locally a complete intersection, and therefore, so is \(A'\). By Hironaka’s embedded resolution of singularities \([13]\), there exists a log resolution \(g : A \to A'\) of \((A', E)\) which is an isomorphism over the complement of a proper closed subset of \(E\). Let \(\wt{E}\) be the strict transform of \(E\) on \(A\), and put \(\pi := f \circ g : A \to A\). Since

\[
K_{\wt{A}/A} - c\pi^{-1}(X) + \wt{E} = K_{\wt{A}/A'} - g^*E + \wt{E},
\]

\(A\) is plt along \(X\) if and only if \(A'\) is plt along \(E\). Since \(A' \setminus E\) is nonsingular, a result of Kollár \([17]\, \text{Theorem 5.50}\) says that \(A'\) is plt along \(E\) if and only if \(E\) is klt. This means that \(X\) is klt, because \(E\) is locally a product of \(X\) and an affine space. It therefore follows that \(A\) is plt along \(X\) if and only if \(X\) is klt. That is, \(\text{adj}_X(A)\) defines the non-klt locus of \(X\).

(2) Let \(X = \frac{1}{3}(1, 1, 1)\) be the quotient of \(\mathbb{C}^3 = \text{Spec} \mathbb{C}[x_1, x_2, x_3]\) by the action of \(\mathbb{Z}/3\mathbb{Z}\) given by \(x_i \mapsto \xi x_i\), where \(\xi\) is a primitive cubic root of unity. \(X\) can be embedded into \(A := \mathbb{C}^{10}\), and we will compute the ideal sheaf \(\text{adj}_X(A)\). Let \(\pi_1 : A_1 \to A\) be the blowing-up of \(A\) at the origin with exceptional divisor \(E_1\) (we use the same letter for its strict transform). Then the weak transform \(X_1\) of \(X\) is nonsingular. Next, let \(\pi_2 : A_2 \to A_1\) be the blowing-up of \(A_1\) along \(X_1\) with exceptional divisor \(E_2\). Setting \(\pi := \pi_1 \circ \pi_2 : A_2 \to A\), we have \(K_{A_2/A} = K_{A_2/A_1} + \pi_2^*K_{A_1/A} = 9E_1 + 6E_2\) and \(\pi^{-1}(X) = 2E_1 + E_2\). Thus,

\[
\text{adj}_X(A) = \pi_*\mathcal{O}_{A_2}(K_{A_2/A} - 7\pi^{-1}(X) + E_2) = \pi_*\mathcal{O}_{A_2}(-5E_1) = m^5_{A,0},
\]

where \(m_{A,0} \subseteq \mathcal{O}_A\) is the maximal ideal sheaf of the origin.

(3) Let \(X\) be the quotient of \((x^2 + y^3 + z^6 = 0) \subset \mathbb{C}^3\) by the action of \(\mathbb{Z}/5\mathbb{Z}\) given by \(x \mapsto \xi^3 x, y \mapsto \xi^2 y\) and \(z \mapsto \xi z\), where \(\xi\) is a primitive quintic root of unity. Then \(X\) can be embedded into \(A := \mathbb{C}^5\), and by an argument similar to that of (2), we have \(\text{adj}_X(A) = m^2_{A,0}\), where \(m_{A,0} \subseteq \mathcal{O}_A\) is the maximal ideal sheaf of the origin.

## 2. A Modification of Generalized Test Ideals

In this section, we consider a modification of generalized test ideals of Hara and Yoshida \([11]\), which conjecturally corresponds to our adjoint ideal sheaf.

Throughout this paper, all rings are Noetherian commutative rings with identity. For an integral domain \(R\) and an unmixed ideal \(I\) of \(R\), we denote by \(R^e\) the set of elements of \(R\) that are not in any minimal prime ideal of \(I\).

Let \(R\) be an integral domain of characteristic \(p > 0\). For an ideal \(J\) of \(R\) and a power \(q\) of \(p\), we denote by \(J^{[q]}\) the ideal of \(R\) generated by the \(q\)th powers of all elements of \(J\). Let \(F : R \to R\) be the Frobenius map, that is, the ring homomorphism sending \(x\) to \(x^p\). The ring \(R\) viewed as an \(R\)-module via the \(e\)-times iterated Frobenius map \(F^e : R \to R\) is denoted by \(\cdot^e\). Since \(R\) is reduced, \(F^e : R \to \cdot^e R\) is identified with the natural inclusion map \(R \hookrightarrow R^{1/p^e}\). We say that \(R\) is \(F\)-finite if
\(1\)R (or \(R^{1/p}\)) is a finitely generated \(R\)-module. For example, any algebra essentially of finite type over a perfect field is \(F\)-finite.

**Definition 2.1** ([Ta1, Definition 3.1]). Let \(R\) be an \(F\)-finite domain of characteristic \(p > 0\) and \(\mathfrak{a}^i = \prod_{i=1}^{m} a_i^{t_i}\) be a formal combination, where the \(a_i\) are nonzero ideals of \(R\) and the \(t_i\) are positive real numbers.

(i) The pair \((R, \mathfrak{a}^i)\) is said to be strongly \(F\)-regular if for every \(\gamma \in R^p\), there exist \(q = p^e\) and \(\delta \in a_1^{[t_1 q]} \cdots a_m^{[t_m q]}\) such that \((\gamma \delta)^{1/q} R \hookrightarrow R^{1/q}\) splits as an \(R\)-module homomorphism.

(ii) Let \(I \subseteq R\) be an unmixed ideal of height \(c\) and suppose that \(a_i \cap R^{c_i} \neq \emptyset\) for all \(i = 1, \ldots, m\). Then the pair \((R, \mathfrak{a}^I)\) is said to be purely \(F\)-regular along \(I\) if for every \(\gamma \in R^{c_i}\) there exist \(q = p^e\) and \(\delta \in I^{c(q-1)} a_1^{[t_1 q]} \cdots a_m^{[t_m q]}\) such that \((\gamma \delta)^{1/q} \to R^{1/q}\) splits as an \(R\)-module homomorphism. We say that \(R\) is purely \(F\)-regular along \(I\) if so is the pair \((R, \mathfrak{a}^I)\).

Let \(R\) be an integral domain of characteristic \(p > 0\) and \(M\) be an \(R\)-module. For each \(q = p^e\), we denote \(\mathbb{F}^e(M) = \mathbb{F}^e(M) := R \otimes_R M\) and regard it as an \(R\)-module by the action of \(R = eR\) from the left. Then we have the \(e\)-times iterated Frobenius map \(F^e_M: M \to \mathbb{F}^e(M)\) induced on \(M\). The image of an element \(z \in M\) via this map is denoted by \(z^q := F^e_M(z) \in \mathbb{F}^e(M)\). For an \(R\)-submodule \(N\) of \(M\), we denote by \(N^{[q]}_M\) the image of the induced map \(\mathbb{F}^e(N) \to \mathbb{F}^e(M)\). If \(I\) is an ideal of \(R\), then \(I^{[q]}_R = I^{[q]}\).

Now we introduce a new generalization of tight closure and the corresponding test ideal.

**Definition 2.2.** Let \(R\) be an excellent domain of characteristic \(p > 0\) and \(I \subseteq R\) be an unmixed ideal of height \(c\). Let \(\mathfrak{a}^I = \prod_{i=1}^{m} a_i^{t_i}\) be a formal combination, where the \(a_i\) are ideals of \(R\) such that \(a_i \cap R^{c_i} \neq \emptyset\) and the \(t_i\) are positive real numbers.

(i) If \(N \subseteq M\) are \(R\)-modules, then the \((I, \mathfrak{a}^I)\)-tight closure \(N^e(I, \mathfrak{a}^I)\) of \(N\) in \(M\) is defined to be the submodule of \(M\) consisting of all elements \(z \in M\) for which there exists \(\gamma \in R^{c_I}\) such that

\[
\gamma I^{c(q-1)} a_1^{[t_1 q]} \cdots a_m^{[t_m q]} z^q \subseteq N^{[q]}_M
\]

for all large \(q = p^e\).

(ii) Let \(E = \bigoplus_m E(R/m)\) be the direct sum, taken over all maximal ideals \(m\) of \(R\), of the injective hulls of the residue fields \(R/m\). The generalized test ideal \(\tau_I(R, \mathfrak{a}^I)\) associated to the pair \((R, \mathfrak{a}^I)\) along \(I\) is

\[
\tau_I(R, \mathfrak{a}^I) = \text{Ann}_R(0^e(I, \mathfrak{a}^I)) \subseteq R.
\]

We denote this ideal simply by \(\tau_I(R)\) when \(a_i = R\) for all \(i = 1, \ldots, m\).

**Remark 2.3.** When \(R\) is a normal domain and \(I = xR\) is a principal ideal, \((I, \mathfrak{a}^I)\)-tight closure coincides with divisorial (\(\text{div}(x), \mathfrak{a}^I\))-tight closure introduced in [22].

**Definition-Lemma 2.4.** Let \(R\) be an excellent domain of characteristic \(p > 0\) and \(I\) be an unmixed ideal of height \(c\). Let \(E = \bigoplus_m E(R/m)\) be the direct sum, taken over all maximal ideals \(m\) of \(R\), of the injective hulls of the residue fields \(R/m\). Fix an element \(\gamma \in R^{c_I}\). We say that \(\gamma\) is an \((I, \ast)\)-test element for \(E\) if
for all \( a^k = \prod_{i=1}^m a_i^t_i \), where the \( a_i \) are ideals of \( R \) such that \( a_i \cap R^{\geq I} \neq \emptyset \) and the \( t_i \) are positive real numbers, one has \( \gamma f^{(q-1)} a_1^{[t_1 q]} \cdots a_m^{[t_m q]} z^q = 0 \) in \( E^c(E) \) for every \( z \in 0^c_E(\delta a) \) and for every \( q = p^e \). If \( R \) is F-finite and the localized ring \( R_\gamma \) is purely F-regular along \( IR_\gamma \), then some power \( \gamma^N \) of \( \gamma \) is an \((I,*)\)-test element for \( E \).

**Proof.** It follows from an argument similar to [14] (see also the proofs of [11] Theorem 1.7 and [22] Corollary 3.10 (2)). \( \square \)

**Proposition 2.5.** Let \((R, m)\) be an F-finite local domain of characteristic \( p > 0 \) and \( I \) be an unmixed ideal of height \( c \). Let \( a^k = \prod_{i=1}^m a_i^t_i \) be a formal combination, where the \( a_i \) are ideals of \( R \) such that \( a_i \cap R^{\geq I} \neq \emptyset \) and the \( t_i \) are positive real numbers.

1. Let \( W \) be a multiplicatively closed subset of \( R \), and \( a^k_W \) and \( I_W \) be the images of \( a^k \) and \( I \) in \( R_W \), respectively. Then
   \[ \tilde{\tau}_{I_W}(R_W, a^k_W) = \tilde{\tau}_I(R, a^k)R_W. \]
2. Let \( \hat{R} \) be the \( m \)-adic completion of \( R \), and \( \hat{a}^k \) and \( \hat{I} \) be the images of \( a^k \) and \( I \) in \( \hat{R} \), respectively. Then
   \[ \tilde{\tau}_I(\hat{R}, \hat{a}^k) = \tilde{\tau}_I(R, a^k)\hat{R}. \]
3. \((R, a^k)\) is purely F-regular along \( I \) if and only if \( \tilde{\tau}_I(R, a^k) = R \).
4. If \( R \) is an F-finite regular local ring and \( \gamma \in R^{\geq I} \) is an \((I,*)\)-test element for the injective hull \( E_R(R/\mathfrak{m}) \) of the residue field \( R/\mathfrak{m} \), then \( \tilde{\tau}_I(R, a^k) \) is the unique smallest ideal \( J \) of \( R \) with respect to inclusion, such that
   \[ \gamma I^{c(q-1)} a_1^{[t_1 q]} \cdots a_m^{[t_m q]} \subseteq J^{[q]} \]
   for all (large) \( q = p^e \).

**Proof.** (1) and (2) follow from arguments similar to the proofs of [10] Propositions 3.1 and 3.2, respectively. (3) follows from an argument similar to the proof of [9] Proposition 2.1 (see also [21] Corollary 3.5) and (4) does from an argument similar to the proof of [2] Proposition 2.22. \( \square \)

**Example 2.6.** Let \((R, m)\) be an F-finite regular local ring of characteristic \( p > 0 \) and \( I = (f_1, \ldots, f_c) \subseteq R \) be an unmixed ideal generated by a regular sequence \( f_1, \ldots, f_c \). Let \( \gamma \in R^{\geq I} \) be an element such that the localized ring \( R_\gamma/IR_\gamma \) is regular, and take a sufficiently large integer \( N \). By Definition-Proposition 2.4 and Proposition 2.5

(4), \( R \) is purely F-regular along \( I \) if and only if there exists \( q = p^e \) such that \( \gamma^N I^{c(q-1)} \not\subseteq \mathfrak{m}^{[q]} \). Since \( (f_1 \cdots f_c)^{q-1} \in I^{c(q-1)} \subseteq (f_1 \cdots f_c)^{q-1}R + I^{[q]} = (I^{[q]} : I) \), this is equivalent to saying that there exists \( q = p^e \) such that \( \gamma^N (I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]} \).

It therefore follows from [7] Theorem 2.1 that \( R \) is purely F-regular along \( I \) if and only if \( R/I \) is strongly F-regular. That is, the generalized test ideal \( \tilde{\tau}_I(R) \) along \( I \) defines the non-strongly-F-regular locus of the ring \( R/I \).

**Theorem 2.7.** Let \((R, m)\) be a \( d \)-dimensional F-finite regular local ring of characteristic \( p > 0 \) and \( I \subseteq R \) be an unmixed ideal of height \( c \). Let \( a^k = \prod_{i=1}^m a_i^t_i \) be a formal combination, where the \( a_i \) are ideals of \( R \) such that \( a_i \cap R^{\geq I} \neq \emptyset \) and the \( t_i \) are positive real numbers. Set \( A = \text{Spec} \ R, X = V(I) \subseteq A \) and \( Y_i = V(a_i) \subseteq A \). Let \( f : A' \to A \) be the blowing-up of \( A \) along \( X \) and \( E_1, \ldots, E_s \) be all the components
of the exceptional divisor of \( f \) dominating an irreducible component of \( X \). Suppose that \( \pi : \tilde{A} \to A \) is a proper birational morphism from a normal scheme \( \tilde{A} \) such that the scheme theoretic inverse images \( \pi^{-1}(X) \) and \( \pi^{-1}(Y) \) are Cartier divisors, and denote by \( \tilde{E} \) the strict transform of \( E := E_1 + \cdots + E_S \) on \( \tilde{A} \). Then one has an inclusion

\[
\tau_I(R, a^I) \subseteq H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - \sum_{i=1}^{m} [t_i \pi^{-1}(Y_i)] - c \pi^{-1}(X) + \tilde{E})).
\]

**Proof.** The proof follows from essentially the same argument as that of [11, Proposition 3.8] (see also the proof of [21, Proposition 3.8] for a different strategy). For simplicity, we assume that \( \tilde{A} \) is a Cohen-Macaulay scheme. Denote the closed fiber of \( \pi \) by \( Z := \pi^{-1}(m) \) and set \( \pi^{-1}(Y) := \sum_{i=1}^{m} t_i \pi^{-1}(Y_i) \). Let

\[
\delta : H^d_m(R) \to H^d_Z(\mathcal{O}_{\tilde{A}}([\pi^{-1}(Y)] + c \pi^{-1}(X) - \tilde{E}))
\]

be the edge map \( H^d_m(R) \to H^d_Z(\mathcal{O}_{\tilde{A}}) \) of the spectral sequence \( H^m_m(R^i \pi_* \mathcal{O}_{\tilde{A}}) \Rightarrow H^{i+j}_Z(\mathcal{O}_{\tilde{A}}) \) followed by the natural map

\[
H^d_Z(\mathcal{O}_{\tilde{A}}) \to H^d_Z(\mathcal{O}_{\tilde{A}}([\pi^{-1}(Y)] + c \pi^{-1}(X) - \tilde{E})).
\]

By the local duality theorem (see [12, V, §6]), one has

\[
\text{Ann}_R(\text{Ker} \, \delta) = H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - [\pi^{-1}(Y)] - c \pi^{-1}(X) + \tilde{E})).
\]

It therefore suffices to show that \( \text{Ker} \, \delta \subseteq 0^e(I, \phi) \). Take an element \( \gamma \in R^{q,I} \) such that \( R_{R_{\gamma}}/IR_{\gamma} \) is regular. Since \( H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E})) = R_{\gamma} \), for sufficiently large integers \( N \gg 0 \), one has

\[
\gamma^N a_i^{[t_1]} \cdots a_m^{[t_m]} \subseteq a_i^{[t_1]} \cdots a_m^{[t_m]} H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - c \pi^{-1}(X) + \tilde{E}))
\]

\[
\subseteq H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}(K_{\tilde{A}/A} - [\pi^{-1}(Y)] - c \pi^{-1}(X) + \tilde{E}))
\]

\[
= \text{Ann}_R(\text{Ker} \, \delta).
\]

By Definition-Lemma 2.4, this inclusion tells us that there exists an \((I, \ast)-test\) element \( \gamma' \in \text{Ann}_R(\text{Ker} \, \delta) \cap R^{q,I} \) for \( H^d_m(R) \), because \( a_i \cap R^{0,I} \neq \emptyset \) for all \( i = 1, \ldots, m \). For every \( q = p^e \) and for every

\[
\alpha \in I^{(q-1)} a_i^{[t_1]} \cdots a_m^{[t_m]} \subseteq H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}([-q \pi^{-1}(Y)] - c(q-1) \pi^{-1}(X))),
\]

we have the following commutative diagram with exact rows:

\[
\begin{array}{c}
0 \to \text{Ker} \, \delta \to H^d_m(R) \xrightarrow{\delta} H^d_Z(\mathcal{O}_{\tilde{A}}([\pi^{-1}(Y)] + c \pi^{-1}(X) - \tilde{E})) \to 0 \\
0 \to \text{Ker} \, \delta \to H^d_m(R) \xrightarrow{\delta} H^d_Z(\mathcal{O}_{\tilde{A}}([\pi^{-1}(Y)] + c \pi^{-1}(X) - \tilde{E})) \to 0
\end{array}
\]

Then \( \alpha F^e(\text{Ker} \, \delta) \subseteq \text{Ker} \, \delta \). By the choice of the element \( \gamma' \), we can conclude that \( \gamma' I^{(q-1)} a_i^{[t_1]} \cdots a_m^{[t_m]} F^e(\text{Ker} \, \delta) = 0 \) for all \( q = p^e \), that is, \( \text{Ker} \, \delta \subseteq 0^{(I, \phi)}_{H^d_m(R)} \). \( \square \)

We conjecture that the generalized test ideal \( \tau_I(R, a^I) \) along \( I \) corresponds to the adjoint ideal sheaf \( \text{adj}_X(A, Y) \).
Conjecture 2.8. Let \((R, \mathfrak{m})\) be a regular local ring essentially of finite type over a perfect field of prime characteristic \(p\), and let \(I \subseteq R\) be a nonzero unmixed ideal. Let \(\mathfrak{a}^I = \prod_{i=1}^m a_i^{t_i}\) be a formal combination, where the \(a_i\) are ideals of \(R\) such that \(a_i \cap R^{>0} \neq \emptyset\) and the \(t_i\) are positive real numbers. Set \(A := \text{Spec } R, X := V(I)\) and \(Y := \sum_{i=1}^m t_i V(a_i)\). Assume in addition that \((R, I, \mathfrak{a})\) is reduced from characteristic zero to characteristic \(p \gg 0\), together with a log resolution \(\pi : \tilde{A} \to A\) of \((A, X + Y)\) used to define the adjoint ideal sheaf \(\text{adj}_X(A, Y)\) as in Definition 1.6. Then
\[
\text{adj}_X(A, Y) = \tau_I(R, \mathfrak{a}^I)
\]

Conjecture 2.8 is true if \(X\) is a divisor on \(A\).

Theorem 2.9 ([22 Theorem 5.3]). Let \((R, \mathfrak{m})\) be a \(\mathbb{Q}\)-Gorenstein normal local ring essentially of finite type over a perfect field of prime characteristic \(p\), and let \(f\) be a nonzero element of \(R\). Let \(\mathfrak{a}^I = \prod_{i=1}^m a_i^{t_i}\) be a formal combination, where the \(a_i\) are ideals of \(R\) such that \(a_i \cap R^{>0, f} \neq \emptyset\) and the \(t_i\) are positive real numbers. Set \(X = \text{Spec } R, D := \text{div}_X(f)\) and \(Y := \sum_{i=1}^m t_i V(a_i)\). Assume in addition that \((R, f, \mathfrak{a})\) is reduced from characteristic zero to characteristic \(p \gg 0\), together with a log resolution \(\pi : \tilde{X} \to X\) of \((X, D + Y)\) used to define the adjoint ideal sheaf \(\text{adj}_D(X, Y)\) as in Definition 1.2. Then
\[
\text{adj}_D(X, Y) = \tau_{fR}(R, \mathfrak{a}^I).
\]

3. Restriction formula of adjoint ideals

In this section, we formulate a restriction property of the adjoint ideal sheaf \(\text{adj}_X(A, Y)\) involving the l.c.i. defect ideal sheaf \(D_X\) of \(X\).

Let \(A\) be a nonsingular variety over an algebraically closed field \(k\) of characteristic zero and \(X\) be a normal Gorenstein closed subvariety of codimension \(c\) of \(A\). Kawakita [16] then defined the l.c.i. defect ideal sheaf \(D_X\) of \(X\) as follows. Since the construction is local, we may consider the germ at a closed point \(x \in X\). We take generically a closed subscheme \(Z\) of \(A\) which contains \(X\) and is locally a complete intersection of codimension \(c\). By Bertini’s theorem, \(Z\) is the scheme-theoretic union of \(X\) and another variety \(C^Z\) of codimension \(c\). Since \(X\) is Gorenstein, the closed subscheme \(D^Z := C^Z|_X\) of \(X\) is a Cartier divisor (see [24, Lemma 1]). Then the l.c.i. defect ideal sheaf \(D_X\) of \(X\) is defined by
\[
D_X := \sum_{Z \subseteq A} O_X(-D^Z),
\]
where \(Z\) runs through all the general locally complete intersection closed subschemes of codimension \(c\) which contain \(X\). Note that the support of \(D_X\) coincides with the non-locally complete intersection locus of \(X\). The reader is referred to [16, Section 2] and [5, Section 9.2] for further properties of l.c.i. defect sheaves.

Theorem 3.1. Let \(A\) be a nonsingular variety over an algebraically closed field \(k\) of characteristic zero and \(Y = \sum_{i=1}^m t_i Y_i\) be a formal combination, where the \(t_i\) are positive real numbers and the \(Y_i\) are proper closed subschemes of \(A\). If \(X\) is a normal Gorenstein closed subvariety of codimension \(c\) of \(A\) which is not contained in the support of any \(Y_i\), then
\[
\mathcal{J}(X, V(D_X) + Y|_X) = \text{adj}_X(A, Y)O_X,
\]
where $\mathcal{D}_X$ is the l.c.i. defect ideal sheaf of $X$.

**Proof.** Since the question is local, we consider the germ at a closed point $x \in X \cap \bigcap_{i=1}^n Y_i \subset A$. Let $\mathcal{I}_X \subseteq \mathcal{O}_A$ be the defining ideal sheaf of $X$ in $A$.

Fix a regular function $\varphi \in \text{adj}_X(A, Y) \setminus \mathcal{I}_X$. It then follows from Lemma [17, 2] that $\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m (X \cap Y_i); A, cX + Y - \text{div}_A(\varphi)) > 0$. Applying [[5] Remark 8.5] (see also the proof of [16 Theorem 1.1]), we have

$$\text{mld}(X_{\text{sing}} \cup \bigcup_{i=1}^m (X \cap Y_i); X, V(\mathcal{D}_X) + Y|_X - \text{div}_X(\varphi)) > 0,$$

where $\varphi$ is the image of $\varphi$ in $\mathcal{O}_X$. This means that $(X, V(\mathcal{D}_X) + Y|_X - \text{div}_X(\varphi))$ is klt, which is equivalent to saying that $\varphi$ is in $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$. Thus, we conclude that

$$\text{adj}_X(A, Y)\mathcal{O}_X \subseteq \mathcal{J}(X, V(\mathcal{D}_X) + Y|_X).$$

Next we will prove the converse inclusion. Take generically a closed subscheme $Z$ of $A$ which contains $X$ and is locally a complete intersection of codimension $c$, so $Z$ is the scheme-theoretic union of $X$ and another variety $C^Z$, and $D^Z := C^Z|_X$ is a Cartier divisor on $X$.

**Claim 1.** By a general choice of $Z$, one has

$$\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X) = \text{adj}_{D^Z}(X, Y|_X).$$

**Proof of Claim 1 (Kawakita).** Since $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X) \supseteq \text{adj}_{D^Z}(X, Y|_X)$ is clear from the definition of the ideal sheaf $\mathcal{D}_X$, we will prove the converse inclusion.

Fix a regular function $\psi \in \mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$. By the definition of the ideal sheaf $\mathcal{D}_X$, there exist closed subschemes $W_i, \ldots, W_n$ of $A$ which contain $X$ and are locally complete intersections of codimension $c$ such that $\mathcal{D}_X = \sum_{j=1}^n \mathcal{O}_X(-D^{W_j})$.

Take a log resolution $\mu : \tilde{X} \rightarrow X$ of $(X, D^{W_1} + \cdots + D^{W_n} + Y|_X)$, and let $\{E_i\}_{i \in I}$ be a collection of all divisors on $\tilde{X}$ which are supported on $\text{Exc}(\mu) \cup \bigcup_{i=1}^m \text{Supp} \mu^{-1}(Y_i|_X) \cup \bigcup_{j=1}^n \mu^{-1}D^{W_j}$. Since $\psi$ is in $\mathcal{J}(X, V(\mathcal{D}_X) + Y|_X)$, we have

$$\text{ord}_{E_i}(\psi) + \max_{1 \leq j \leq n} a(E_i; X, D^{W_j} + Y|_X) > 0$$

for all $i \in I$. Let $I_{W_j} = (f^{(j)}_1, \ldots, f^{(j)}_n) \subseteq \mathcal{O}_A$ be the defining ideal sheaf of $W_j$ in $A$, and set $I_W := (t_1f^{(1)}_1 + \cdots + t_nf^{(n)}_1, \ldots, t_1f^{(1)}_n + \cdots + t_nf^{(n)}_n) \subseteq \mathcal{O}_A[t_1, \ldots, t_n]$. Let $W \subseteq A \times T$ be the corresponding closed subscheme, where $T := \text{Spec} k[t_1, \ldots, t_n]$. Then $W$ is the scheme-theoretic union of $X \times T$ and another variety $\mathcal{C}$. Note that $\mathcal{C}|_{\mathcal{X} \times T}$ is an irreducible Cartier divisor over a generic point of $T$. One can choose a generator $h \in \mathcal{O}_X \otimes k(t_1, \ldots, t_n)$ of the principal ideal sheaf $\mathcal{O}_{\mathcal{X} \times T}(-\mathcal{C}|_{\mathcal{X} \times T})$ over a generic point of $T$ so that the restriction of $h$ to the fiber over $0, 0, 0, \ldots, 0 \in T$ is a generator of $\mathcal{O}_X(-D^{W_j})$. Thinking of the resolution $\mu \times \text{id}_T : \tilde{X} \times T \rightarrow X \times T$ induced by $\mu$, we have

$$a(\mathcal{E}_i; X \times T, \text{div}(h) + Y|_X \times T) \geq a(E_i; X, D^{W_j} + Y|_X)$$

for all $i \in I$ and all $j = 1, \ldots, n$, where $\mathcal{E}_i := E_i \times T \subset \tilde{X} \times T$. 
On the other hand, the restriction of \( h \) to the fiber over a general point \((t_1, \ldots, t_n) \in T\) is a generator of \( O_X(-D^{W_{1 \cdots t_n}})\) where \( W_{1 \cdots t_n} := \mathcal{W}|_{X \times (t_1, \ldots, t_n)}\), and \( \mu \) is a log resolution of \((X, W_{1 \cdots t_n} + Y|_X)\). Thus, for all \( i \in I \),

\[
\text{ord}_{E_i}(\psi) + a(E_i; X, D^{W_{1 \cdots t_n}} + Y|_X) = \text{ord}_{E_i}(\psi) + a(E_i; X \times T, \text{div}(h) + Y|_X \times T) \\
\geq \text{ord}_{E_i}(\psi) + \max_{1 \leq j \leq n} a(E_i; X, D^{W_j} + Y|_X) \\
> 0.
\]

This implies that \( \psi \) is in \( \text{adj}_{D^{W_{1 \cdots t_n}}}(X, Y|_X) \).

From now on, we may assume that \( A = \text{Spec } S \) and \( X = \text{Spec } R \), where \((S, n)\) is a regular local ring essentially of finite type over a field of characteristic zero and \( R = S/I \) is a Gorenstein normal quotient of \( S \). Let \( a_i \) be the ideal of \( S \) corresponding to \( Y_i \) for every \( i = 1, \ldots, n \) and denote \( a^\omega = \prod a_i^{\omega_i} \). Let \( f_1, \ldots, f_c \) be the regular sequence in \( S \) corresponding to \( Z \) and \( f \in S \) be an element whose image \( \bar{f} \) is a generator of the principal ideal \(((f_1, \ldots, f_c) : I) + I)/I \) of \( R \). Thanks to Claim 1, it is enough to prove that

\[
(1) \quad \text{adj}_{\text{div}(\bar{f})}(X, Y|_X) \subseteq \text{adj}_X(A, Y)O_X.
\]

Now we reduce the entire setup as above to characteristic \( p \gg 0 \) and switch the notation to denote things after reduction modulo \( p \). In order to prove (1), by virtue of Theorems 2.7 and 2.9 it suffices to show that

\[
(2) \quad \tilde{\tau}_{f_R}(R, a_R^\omega) \subseteq \tilde{\tau}_I(S, a_I^\omega)R.
\]

Since \( S \) is F-finite, by Lemma 2.8 forming generalized test ideals commutes with completion. Hence, we may assume that \( S \) is complete. Let \( E_S = E_S(S/n) \) and \( E_R = E_R(R/nR) \) be the injective hulls of the residue fields of \( S \) and \( R \), respectively. We can view \( E_R \) as a submodule of \( E_S \) via the isomorphism \( E_R \cong (0 : I)_{E_S} \subset E_S \). Then by Matlis duality, (2) is equivalent to saying that

\[
(3) \quad 0^{e(f_R, a_R^\omega)}_{E_R} \subseteq 0^{e(I, a_I^\omega)}_{E_S} \cap E_R.
\]

Let \( z \in 0^{e(I, a^\omega_I)}_{E_S} \cap E_R \). Let \( F_S^e : E_S \rightarrow \mathbb{F}_S(E_S) \cong E_S \) and \( F_R^e : E_R \rightarrow \mathbb{F}_R(E_R) \cong E_R \) be the \( e \)-times iterated Frobenius maps induced on \( E_S \) and \( E_R \), respectively. Since \( R = S/I \) is normal, one can choose an element \( \gamma \in S^{e, I} \) such that the image \( \bar{\gamma} \) of \( \gamma \) is not contained in any minimal prime of \( f_R \) and the localized ring \( R_{\bar{\gamma}} \) is regular. By Definition-Lemma 2.3, some power \( \gamma^N \) of \( \gamma \) is an \((I, \ast)\)-test element for \( E_S \), and then

\[
\gamma^N F_S^e(z) = 0 \text{ for all } q = p^e.
\]

On the other hand, \( I^q F_S^e(z) = 0 \) for all \( q = p^e \), because \( z \in E_R \cong (0 : I)_{E_S} \).

Claim 2. For all \( q = p^e \), one has

\[
f^{q-1}(I^q : I) \subseteq I^{c(q-1)} + I^q.
\]

Proof of Claim 2. This claim is an easy consequence of the linkage theory of Peskine and Szpiro [19]. We consider a simultaneous minimal free resolution of a natural
diagram between $S/(f_1, \ldots, f_c), S/I, S(f_1^q, \ldots, f_c^q)$ and $S/I^q$.

Here note that the ideals $I, I^q, (f_1, \ldots, f_c)$ and $(f_1^q, \ldots, f_c^q)$ are all Gorenstein of height $c$. Looking at the last step of the above diagram, by [24] Lemma 1, we obtain the equality $J^q(I^q : I) = f(f_1 \cdots f_c)^q - 1$ in $S/I^q$. Since $f$ is a regular element of $S/I$, by the flatness of the Frobenius map, $f$ is also a regular element of $S/I^q$. Therefore, $f^q - 1(I^q : I) = (f_1 \cdots f_c)^q - 1$ in $S/I^q$, which gives the assertion. 

\[\begin{array}{c}
\text{Example 3.3.} \\
(1) Let \( X = \frac{1}{3} \langle 1, 1, 1 \rangle \) be the quotient of $\mathbb{C}^3 = \text{Spec} \mathbb{C}[x_1, x_2, x_3]$ by the action of $\mathbb{Z}/3\mathbb{Z}$ given by $x_i \mapsto \xi x_i$, where $\xi$ is a primitive cubic root of unity. Denote by $m_{X,0} \subset O_X$ (resp. $m_{C^3,0} \subset O_{C^3}$) the maximal ideal sheaf of the origin. By [16] Example 2.3, the integral closure of the l.c.i. defect ideal sheaf $\mathcal{D}_X$ of $X$ is $m_{X,0}^5$. Therefore,
\[ J(X, V(\mathcal{D}_X)) = J(X, m_{X,0}^5) = J(\mathbb{C}^3, m_{C^3,0}^{15}) \cap O_X \]
\[ = m_{C^3,0}^{13} \cap O_X \]
\[ = m_{X,0}^5. \]
\end{array}\]
On the other hand, $X$ can be embedded into $A := \mathbb{C}^5$. Then we have already seen in Example 1.8 (2) that $\text{adj}_X(A) = m_{A,0}^5$, where $m_{A,0} \subset \mathcal{O}_A$ is the maximal ideal sheaf of the origin. Thus, $\mathcal{J}(X, V(D_X)) = \text{adj}_X(A)\mathcal{O}_X = m_{X,0}^5$.

(2) Let $X$ be the quotient of $(x^2 + y^5 + z^6 = 0) \subset \mathbb{C}^3$ by the action of $\mathbb{Z}/5\mathbb{Z}$ given by $x \mapsto \xi^3 x$, $y \mapsto \xi^2 y$ and $z \mapsto \xi z$, where $\xi$ is a primitive quintic root of unity. Then $X$ can be embedded into $A := \mathbb{C}^5$. Since $X$ is a Gorenstein closed subvariety of codimension three of $A$, by [16] Example 2.4 (which is an application of the structure theorem for Gorenstein ideals of codimension three in [1]), the l.c.i. defect ideal sheaf $\mathcal{D}_X$ of $X$ is the maximal ideal sheaf $m_{X,0}$ of the origin. Let $\pi : \tilde{X} \to X$ be the blowing-up of $X$ at the origin with exceptional divisor $E$. Then $\pi$ is a log resolution of $X$ and $K_{\tilde{X}/X} = -E$. Therefore,

$$\mathcal{J}(X, V(D_X)) = \pi_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}/X} - E) = \pi_*\mathcal{O}_{\tilde{X}}(-2E) = m_{X,0}^2.$$ On the other hand, we have already seen in Example 1.8 (3) that $\text{adj}_X(A) = m_{A,0}^2$, where $m_{A,0} \subset \mathcal{O}_A$ is the maximal ideal sheaf of the origin. Thus, $\mathcal{J}(X, V(D_X)) = \text{adj}_X(A)\mathcal{O}_X = m_{A,0}^2$.

We conclude this section by stating a corollary of Theorem 3.1.

Let $S$ be an algebra essentially of finite type over a field $k$ of characteristic zero, and let $I \subseteq S$ be an unmixed ideal of height $c$. Let $\mathfrak{a}^c = \prod_{i=1}^c \mathfrak{a}_i^{t_i}$ be a formal combination, where the $\mathfrak{a}_i$ are ideals of $S$ such that $\mathfrak{a}_i \cap S^{c\cdot I} \neq \emptyset$ and the $t_i$ are positive real numbers. We say that $(S, \mathfrak{a}^c)$ is of purely $F$-regular type along $I$ if there exist a finitely generated $\mathbb{Z}$-subalgebra $A$ of $k$ and an algebra $S_A$ essentially of finite type over $A$ satisfying the following conditions:

(i) $S_A$ is flat over $A$. In addition, $S_A \otimes_A k \cong S$, $I_A S = I$ and $\mathfrak{a}_i S = \mathfrak{a}_i$, where $I_A := I \cap S_A \subseteq S_A$ and $\mathfrak{a}_i$ is the restriction of $\mathfrak{a}$ to $S_A$.

(ii) $(S_\kappa, \mathfrak{a}_\kappa^c)$ is purely $F$-regular along $I_\kappa$ for every closed point $s$ in a dense open subset of $\text{Spec} A$, where $\kappa = \kappa(s)$ denotes the residue field of $s \in \text{Spec} A$, $S_\kappa = S_A \otimes_A \kappa(s)$, $I_\kappa := I_A S_\kappa$, and $\mathfrak{a}_\kappa$ is the image of $\mathfrak{a}_A$ in $S_\kappa$.

**Corollary 3.4.** In the above situation, suppose in addition that $k$ is an algebraically closed field, $S$ is a regular domain and $S/I$ is a Gorenstein quotient of $S$. Then $(\text{Spec} S, \sum_{i=1}^m t_i V(\mathfrak{a}_i))$ is plt along $V(I)$ if and only if $(S, \mathfrak{a}^c)$ is of purely $F$-regular type along $I$.

**Proof.** Since the statement is local, we may assume that $S$ is a regular local ring. The “if” part immediately follows from Proposition 2.5 (3) and Theorem 2.7. We will prove the “only if” part.

Suppose that $(\text{Spec} S, \sum_{i=1}^m t_i V(\mathfrak{a}_i))$ is plt along $V(I)$. Then $R := S/I$ is a normal local ring. Let $f_1, \ldots, f_c$ be a general regular sequence in $S$ and $f \in S$ be an element whose image $\overline{f}$ is a generator of the principal ideal $((f_1, \ldots, f_c) : I) + I$ of $R$. Then, by Theorem 3.1 and Claim 1 in the proof of Theorem 3.1 the pair $(\text{Spec} R, \sum_{i=1}^m t_i V(\mathfrak{a}_i R))$ is plt along $\text{div}(\overline{f})$. Thanks to Theorem 2.9 this implies that $(R, \mathfrak{a}^c R)$ is of purely $F$-regular type along $fR$. Applying the argument used to prove the inclusion (2) in the proof of Theorem 3.1, we know that $(S, \mathfrak{a}^c)$ is of purely $F$-regular type along $I$. □
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Department of Mathematics, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka, 812-8581 Japan

E-mail address: stakagi@math.kyushu-u.ac.jp