In this paper, research on the dynamic behaviors of the electro-optical bistable system, especially for its chaotic dynamic behavior, was carried out. The dynamic equation is resolved by the method of the diagrammatized mode. The state of the electro-optical bistable system with changing system parameter is analyzed in detail, and the concrete positions of the bifurcation points are calculated. The system can generate tangent bifurcation and period double bifurcation as the systemic parameter changes, thus the system can generate chaos through period double bifurcation; on the other hand, the system can generate intermittent chaos.

Keywords: electro-optical bistable system chaos; stability; diagrammatized mode; periodic window

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1. Introduction

Li (2006) and Szőke, Danue, Goldhar, and Kurnit, (1969) advanced the optical bistable system for the first time in 1969. Because it was very important in all optical communications, optical switch, optical detect, optical logic gate, optical information storage, optical computer and so on, the optical bistable system received broad attention consequently (Chen, Shen, Jiang, & Shi, 2007; Hua & Lu, 2000; Jin, Chen, Huang, & Jiang, 2009; Wang & Yan, 2009; Yang, Dai, & Zhang, 1994). Optical bistable system includes all optical bistable and hybrid bistable systems (Shen, 2000), and there are many works on the acousto-optical bistable system and the hybrid bistable system (Liu, Wang, Gao, & Lu, 1999; Lv, Li, & Cao, 2004; Réal & Claude, 1985; Zhang, Pan, Luo, & Gao, 2008; Zhang, Zhai, Zhang, & Liu, 2001), but there are few on the electro-optical bistable system. Gao, Narducci, Schulman, Squicciarini, and Yuan (1983) did research on the dynamic characters of the electro-optical bistable system, because this system’s structure is simple and can be realized easily in experiment, and aroused the interest of many researchers. Up to now, some researchers have made some works on system chaos with change in the input intensity levels (Hopf, Kaplan, Rose, & Sanders, 1986; Niu, Ma, & Wang, 2009; Zhang, 1997; Zhang, Li, Zheng, Jiang, & Gao, 1998), but there is no work on system chaos caused by the system parameter as we know. In this paper, the stability of the electro-optical bistable system with change in the system parameter is analyzed, and the relation between the periodic windows and intermittent chaos was discovered.

2. The electro-optical bistable system

Figure 1 shows the electro-optical bistable system. LP is the polaroid, A is an amplifier and PLZT is the nonlinear medium made by Pb-based lanthanum-doped zirconate titanate. A/D and D/A stand for the equipments that made the analogue signal to digital signal and digital to analogue, respectively. The light outputs from the He–Ne laser enter into the electro-optic modulator, which is composed of the polaroid LP and PLZT. It is converted into the electrical signal by a photoelectric detector. The signal is delayed by the delay device and feedback to PLZT nonlinear medium. The refractive index is changed nonlinearly, which causes the change in output light intensity.

The dynamic behavior of the electro-optical bistable system can be described by the following equation (Niu et al., 2009; Zhang et al., 1998):

\[
\frac{dx(t)}{dt} + x(t) = I \left\{ 1 - k \cos[x(t - \tau_d) + \theta] \right\}
\]

where \( I \) and \( x(t) \) represent the input and output intensity levels, \( I \) always represents bifurcation parameter researched in chaos control (Niu et al., 2009; Zhang et al., 1998), \( x(t) \) is proportional to the feedback voltage, \( \tau_d \) is the effective delay time in feedback loop, \( \theta \) is the extinct parameter and \( k \) is the modulation depth of the device, which represents the bifurcation parameter in this paper. Both \( \tau_d \) and the time variable \( t \) are scaled to the natural response time of the electro-optical bistable system.
The dynamic equation of the electro-optical bistable system (1) can be expressed as the dimensionless iterate equation at extended delayed:

\[ x_{n+1} = I \left[ 1 - k \cos(x_n + \theta) \right] \]

where \( n \) represents the iterate times and \( k \) is the bifurcation parameter.

3. Analysis for the stability of the electro-optical bistable system

In this paper, the parameter \( I \) and \( \theta \) are set as 2 and \( \pi \), for other values, the results are similar.

3.1. Period one state

When the system is in period one state, and let \( x_{n+1} = x_n \), Equation (2) can be expressed as the following equation, \( x^* \) is the steady-state solution:

\[ x^* = 1 + k \cos x^*. \]  

If the stability of the electro-optical bistable system is analyzed with a diagrammatized mode, Equation (3) can be switched as the following equation:

\[ y = x^*, \]

\[ y = 1 + k \cos x^*. \]  

Figure 2 shows the steady period one-state solution of equation (4), we find the points of intersection between the beeline \( y = x^* \) and the curves \( y = 1 + k \cos x^* \) are steady-state solutions. On the other hand, for Equation \( y = x^* \), if there is a very small deviation \( \epsilon_k \) at \( x^* \) for \( n = 0 \) and \( \epsilon_k \) at \( x^* \) for \( n = k \), then, if \( |dy/dx^*| > 1 \), \( \epsilon_k \to \infty \) when \( k \to \infty \), so, if \( |dy/dx^*| > 1 \), the steady-state solution is not stable; if \( |dy/dx^*| < 1 \), \( \epsilon_k \to 0 \) when \( k \to \infty \), so, the steady-state solution is stable and if \( |dy/dx^*| = 1 \), the points of \( k \) are critical points (bifurcation points) (Liu, 1994). Obviously, \( k = \pm 1/\sin x^* \) accords with the conditions. The critical points are also steady-state solutions, so \( x^* \) accords with the following equation:

\[ k = \frac{x^* - 1}{\cos x^*} = \pm \frac{1}{\sin x^*}. \]  

It can also be written as

\[ x^* \sin x^* - \sin x^* \mp \cos x^* = 0. \]

We obtain the values of \( x^* \) from Equation (6) through the method of numerical calculation, and substituting these values into Equation (5), we can obtain the values of \( k \). At the points \( k = 1.04 \), the system bifurcates from period one state to period two state.

In Figure 2, there are tangent points between \( y = x^* \) and \( y = 1 + k \cos x^* \) at different places. From Figure 3(a) one can find that \( y = x^* \) and \( y = 1 + 4.05 \cos x^* \) have a point of intersection when \( x^* \) is close to 1.5. It implies there is a steady-state solution. By numerically solving \( x^* = 1 + 4.05 \cos x^* \), we can obtain the accurate value \( x^* = 1.4576 \). When \( k = 4.05 \) and \( x^* = 1.4576 \), the slope of \( y = 1 + 4.05 \cos x^* \) equals \(-4.0241\left\{ -4.0241 \right\} > 1\),
which means that this is an unstable point. When \( x^* \) is close to \(-3\), \( y = x^* \) and \( y = 1 + 4.05 \cos x^* \) are tangential, the slope of \( y = 1 + 4.05 \cos x^* \) equals 1. It infers that this point is a critical point (bifurcation point). If \( k < 4.05 \), there is no point of intersection. It means there is no steady-state solution; if \( k > 4.05 \), the slope’s absolute value of \( y = 1 + 4.05 \cos x^* \) is less than 1. That is to say, there is a steady-state solution. As \( k < 4.05 \), a slit will be made between \( y = x^* \) and \( y = 1 + k \cos x^* \). In the slit, the difference of evolvement is very small. It is an approximate periodic movement, but with an increase in the iterative times, the value of \( x \) will walk out of the slit quickly. This will generate an analogous stochastic oscillation. It implies that the system generates intermittent chaos. Of course, the area of \( k \), which makes the system to be in intermittent chaos state, is not very broad, because the intermittent chaos needs a narrower slit between \( y = x^* \) and \( y = 1 + k \cos x^* \). If the slit becomes very wide, it does not generate intermittent chaos. Therefore, when \( k \) is close to 4.05, the system will evolve abruptly from period one state to intermittent chaos with the decline of \( k \).

3.2. Period two state

When the system is in period two state, and let \( x_{n+2} = x_n \), Equation (2) can be expressed as the following equation, \( x^* \) is the steady-state solution:

\[
x^* = 1 + k \cos(1 + k \cos x^*).
\]  

(7)

If the stability of the electro-optical bistable system is analyzed with a diagrammatized mode, Equation (7) can be switched as the following equation:

\[
y = x^*, \\
y = 1 + k \cos(1 + k \cos x^*).
\]  

(8)

Figure 4 is the steady period two-state solution diagram for the diagrammatized mode. In this diagram, the points of intersection are between the beeline \( y = x^* \) and the curves \( y = 1 + k \cos(1 + k \cos x^*) \) are steady-state solutions. At the same time, when the absolute value of \( \frac{dy}{dx^*} = k^2 \sin x^* \sin(1 + k \cos x^*) \) equals 1, the point \( k \) is the critical point (bifurcation point).

From Figure 5(a) we find that there are three points of intersection where \( x^* \) are close to \(-1.2\), 1.5 and 2.3. It implies that there are steady-state solutions whose accurate values are \(-1.188\), \(1.4408\) and \(2.2699\). The slopes of \( x^* = 1 + 3.4 \cos(1 + 3.4 \cos x^*) \) equal \(-8.2075\), \(11.36576\) and \(-8.2079\), their absolute values are all more than 1, and
the three points of intersection are all unstable points. When $x^*$ are close to 0 and 4.4, $y = x^*$ and $y = 1 + 3.4 \cos(1 + 3.4 \cos x^*)$ are tangential, the slope of $y = 1 + 3.4 \cos(1 + 3.4 \cos x^*)$ equals 1. It infers that these points are also critical points (bifurcation points). When $k < 3.4$, there is no steady-state solution; but at the side of $k > 3.4$, the slope's absolute value of $y = 1 + k \cos(1 + k \cos x^*)$ equals 1; hence, the points of $k$ are critical points (bifurcation points).

3.3. Period three state

When the system is in period three state, and let $x_{n+3} = x_n$, Equation (2) can be expressed as the following equation, $x^*$ is the steady-state solution:

$$x^* = 1 + k \cos(1 + k \cos(1 + k \cos x^*)).$$

(9)

If the stability of the electro-optical bistable system is analyzed with the diagrammatized mode, Equation (9) can be switched as the following equation:

$$y = x^*,
\quad y = 1 + k \cos(1 + k \cos(1 + k \cos x^*)).$$

(10)

From Figure 6 one can find that the points of intersection are between the beeline $y = x^*$ and the curves $y = 1 + k \cos(1 + k \cos(1 + k \cos x^*))$ are steady-state solutions. At the same time, the absolute value of $\frac{dy}{dx^*} = -k^3 \sin x^* \sin(1 + k \cos x^*) \sin(1 + k \cos(1 + k \cos x^*))$ equals 1; hence, the points of $k$ are critical points (bifurcation points).

Figure 7(a)–(d) is the steady period three-state solution diagrams for the diagrammatized mode at $k$ equalling 2.68, 3.18, 4.88 and 5.88. At the points of $k = 2.68$, 3.18, 4.88 and 5.88, the beeline $y = x^*$ and the curves $y = 1 + k \cos(1 + k \cos(1 + k \cos x^*))$ are tangential, the slope of $y = 1 + k \cos(1 + k \cos(1 + k \cos x^*))$ equals 1. It implies that these points are also critical points (bifurcation points). When $k < 2.68$, there is no point of intersection, i.e. there is no steady-state solution also; if $k > 2.68$, the slope's absolute value of $y = 1 + 2.68 \cos(1 +
2.68 \cos(1 + 2.68 \cos x^*) is less than 1, then there is a steady-state solution. When \( k < 2.68 \), a slit will be made between \( y = x^* \) and \( y = 1 + 2.68 \cos(1 + 2.68 \cos(x^*)) \), it can generate intermittent chaos. Therefore, when \( k \) is close to 2.68, the system will evolve abruptly from period three state to intermittent chaos with the decline of \( k \). The cases where \( k \) equals 3.18, 4.88 and 5.88 are alike. The system also can generate intermittent chaos at these points.

### 3.4. Period \( m \) state

We can extend this method to a random period. When the system is in period \( m \) state (\( m \) is a random positive integer), and let \( x_{n+m} = x_n \), Equation (2) can be expressed as the following equation, where \( x^* \) is the steady-state solution:

\[
x^* = 1 + k \cos(1 + k \cos(1 + k \cos(\cdots))).
\] (11)
If the stability of the electro-optical bistable system is analyzed with the diagrammatized mode, Equation (11) can be switched as the following equation:

\[
\begin{align*}
y &= x^k, \\
y &= 1 + k \cos(1 + k \cos(1 + k \cos(\cdots))).
\end{align*}
\] (12)

**4. Results of numerical calculation**

The parameters \( I \) and \( \theta \) are the same as mentioned in Section 3. Figure 8(a) shows the bifurcation diagram of the electro-optical bistable system with \( k \) changing. From Figure 8(a), one can find, as \( k \) changes the system goes into period one, period two, period four, period eight and chaotic state successively, on the other hand, there are many transparent windows (periodic windows) in the chaos area. For example, when \( k \) is close to 2.6, there is a window of period three, which agrees with the theory that period three implies chaos advanced by Li Tianyan and York in 1973 (Shen, 2000), and accords with the results analyzed by Figure 7(a).

Figure 8(b) is the largest Lyapunov exponent (\( \lambda_L \)) diagram corresponding to Figure 8(a). Figure 8(b) shows critical points (where \( \lambda_L = 0 \)) are in accordance with the bifurcation points in Figure 8(a). By comparing Figure 8(a) and 8(b), one can find, for \( \lambda_L > 0 \), the system is in chaotic states including the intermittent chaos and the chaos being generated through doubling bifurcations.

Figure 8(a) shows that the system is in period one state for \( k \in (0, 1) \), and is in period two state for \( k \in (1, 1.8) \); obviously, \( k = 1 \) is the system bifurcate point from period one to period two state. Then, the system enters into chaos through period double bifurcates if \( k > 2.2 \). For \( k \in (3.5, 4.5) \) there are a few fuscous lines in the chaotic region, and the boundary is orderly because the chaos is generated through period doubling bifurcations and intermittent chaos. The system changes from tangent bifurcation to period one state. As \( k \) increases the system goes again into period one, period two, period four, period eight and chaos state, successively. If \( k \in (5, 5.2) \), there is a fuscous area in the hyponastic chaos. It is also caused by period doubling bifurcations and intermittent chaos, and others only caused
by intermittent chaos. The system changes from tangent bifurcation to period one state again as \( k \) is close to 5.2. With \( k \) increasing the system goes again into period one, period two, period four, period eight and chaos state, successively. This phenomenon accords with the results which are analyzed by the diagrammatized mode.

Figure 9(a) is the bifurcation diagram at \( k \in (3, 4.4) \). When \( k \) is close to 3.2, there is a window of period three, which is in accordance with the results analyzed in Figure 7(b). There is a window of period two when \( k \) is close to 3.4. It is in accordance with the results which are analyzed in Figure 5(a). The system brings tangent bifurcation at \( k = 4.05 \). The point with \( k = 4.05 \) and \( x^* = 1.45 \) is the end point of the branch in period doubling bifurcation, which is in accordance with the results that are analyzed in Figure 3(a). Figure 9(b) is the bifurcation diagram at \( k \in (4.4, 5.4) \), when \( k \) is close to 4.9 there is a window of period three, which is in accordance with the results analyzed in Figure 7(c). The system brings tangent bifurcation at \( k = 5.2 \). The points that \( k = 4.05 \), \( x^* = -3.6 \), \(-2 \) and \( 1.4 \) are the end points of the branches in period doubling bifurcation, which is in accordance with the results analyzed in Figure 3(b). Figure 9(c) is the bifurcation diagram at \( k \in (5.4, 6.4) \), when \( k \) is close to 5.9 there is a window of period three, which is in accordance with the results analyzed in Figure 7(d). When \( k \) is close to 6.06, there is a window of period two, which is in accordance with the results analyzed in Figure 5(b). These phenomena are all in accordance with the results that are analyzed by the diagrammatized mode.

5. Results and discussion

In this paper, the stability of the electro-optical bistable system is theoretically analyzed by the diagrammatized mode. It is analyzed for the stability of the electro-optical bistable system with change in the system parameter. It infers that the chaos can be generated through period doubling bifurcations and intermittent chaos in this system. The results of numerical calculation show that the analysis in theory is reasonable. We discover that periodic window implies intermittent chaos from Figure 3(a)–(b), Figure 5(a)–(b) and Figure 7(a)–(d). This work offers a useful method to research periodic window.

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