In quasi-proportional auctions, each bidder receives a fraction of the allocation equal to the weight of his bid divided by the sum of the weights of all bids, where a bid’s weight is determined by a chosen weight function. We study the relationship between the weight function, bidders’ private values, and the auctioneer’s revenue in equilibrium.

It has been shown that if one bidder has a much higher private value than the others, then the revenue-optimizing weight function is nearly flat. Essentially, threatening the bidder with the highest valuation to share the allocation of the rewards maximizes the auctioneer’s revenue. We show that as bidder private values approach parity, the revenue-maximizing weight function becomes steeper, making the quasi-proportional auction more like a winner-take-all auction.

For flatter weight functions, there is known to be a unique pure-strategy Nash equilibrium. We show that this also holds for steeper weight functions, at least in the two-bidder case. We also give some analysis and intuition about why this may hold for more bidders.

1. INTRODUCTION
Quasi-proportional auctions [Buchanan et al. 1980; Kelly 1997] award each bidder a fraction of the total allocation equal to the weight of their bid divided by the sum of weights of all bids, where a bid’s weight is determined by a chosen weight function. Hence, the allocation for bidder $i$ is

$$a_i(b) = \frac{f(b_i)}{\sum_j f(b_j)},$$

where $b$ is the vector of bids and $f(\cdot)$ is the weight function. In this paper, we focus on the winners-pay quasi-proportional auctions, in which bidders pay their bid times their allocation. A well-known alternative is the all-pay auction, in which all bidders pay their full bid regardless of allocation [Buchanan et al. 1980]. The all-pay auction has been used as a model for disparate interests plying officials with gifts and favors in hopes of influencing policy.

Why use quasi-proportional auctions? It is well known that the revenue-optimal auction (for a single item, non-repeated) is the second-price auction with an optimal reserve price [Myerson 1981; Riley & Samuelson 1981]. The optimal reserve price is based on knowledge of prior distributions from which bidders draw their private values. Without this knowledge, we are in a prior-free setting [Goldberg et al. 2002; Hartline & Karlin 2007], in which it can be a challenge to set an effective reserve price [Muthukrishnan 2009]. If the auction is repeated and priors are stable over time, then the priors may be learnable [Li et al. 2010; Cole & Roughgarden 2014; Dughmi et al. 2014; Hummel & McAfee 2014]. However, in many practical scenarios, either the auction is not repeated, or the priors are not stable over time.

In the prior-free setting without reserve prices, Mirrokni et al. [2010] shows that a quasi-proportional auction has better worst-case performance than a second-price auction. In their worst case, the bidder with the highest private value has a much higher private value than the other bidders. In this case, they show that quasi-proportional auctions with functions $f(x) = x^p$ and $p \leq 1$, called Tullock auctions [Buchanan et al. 1980], can achieve $\Omega(\sqrt{\alpha})$ revenue, where $\alpha$ is the ratio of the highest private value to the next-highest private value. These results are called prior-free revenue results.
Nguyen & Vojnovic [2010] show that for the prior-free setting without reserve prices, there is an upper bound of \( o\left(\frac{v_1}{\log(v_1)}\right) \) on equilibrium revenue, where \( v_1 \) and \( v_2 \) are the two highest private values of bidders. They also give a mechanism that achieves revenue \( \Omega\left(\frac{v_1}{\log^2(\frac{v_1}{v_2} + 1)}\right) \), which is similar to the best known prior-free result with reserve prices [Lu et al. 2006].

Other than a lack of priors, some reasons why a quasi-proportional allocation might be useful include:

1. The item is always awarded. For auctions with reserve prices, the item may be withheld from all bidders. This may create a problem for the seller and is avoided here.
2. There is a shared allocation. The seller may desire a shared allocation if a zero allocation to runner-up bidders makes them unlikely to participate in future auctions, which can decrease competition and revenue in those auctions. A shared allocation awards a “second prize for second price.” (A single allocation is also possible: the auctioneer can award the item at random, with each bidder’s probability of winning the item equal to its fraction of the allocation [Muthukrishnan 2009].)

For the winners-pay quasi-proportional auction, assume bidder \( i \) has utility function

\[
u_i(b) = a_i(b)(v_i - b_i),
\]

(2)

where \( v_i \) is bidder \( i \)'s private value for a full allocation. Section 3 and 2 focus on a two-bidder auction, in which we assume \( v_1 \geq v_2 \) and define \( \alpha = \frac{v_1}{v_2} \) to express the disparity in bidders’ private values. Without loss of generality, in this paper we assume the weight function \( f(\cdot) \) has the form \( f(x) = x^p \), where the exponent \( p \) is to be specified. Similar to Mirrokni et al. [2010], this paper also studies how to select \( p \) based on \( \alpha \) to optimize the auctioneer’s equilibrium revenue.

The main contributions of this paper are summarized below:

1. We prove that for any \( p > 0 \), there is a unique pure-strategy Nash equilibrium in the two-bidder case, and bound the bids and bid ratio at this equilibrium. Note that Mirrokni et al. [2010] also shows that for \( 0 < p \leq 1 \), the auction has a unique pure-strategy Nash equilibrium. However, their proof breaks down for steeper functions (\( p > 1 \)), because the bidder utility functions are not concave over their whole domains. We also discuss why we hope there is a unique pure-strategy Nash equilibrium in cases with \( p > 1 \) and more than two bidders.
2. We bound the equilibrium revenue of the auctioneer in the two-bidder case for all \( p > 0 \). We also compare the equilibrium revenue in a quasi-proportional auction with the standard second-price auction, and show that the equilibrium revenue in a quasi-proportional auction is an increasing function of \( \alpha \).
3. We show that as \( \alpha \to 1 \), the revenue-maximizing \( p \) increases beyond one. This is complementary to the existing result that as \( \alpha \) increases, the revenue-maximizing \( p \) decreases (see Mirrokni et al. [2010]).

Finally, it is worth mentioning that since our results require at least approximate knowledge of the ratio of bidders’ private values, these results are not completely prior-free. However, they are scale-free, in the sense that we only need knowledge about the ratios of bidders’ private values (the level of competitiveness) but not the magnitudes of the private values. (Allocations based on functions \( f(x) = x^p \) are scale-free – multiplying all bids by a constant does not alter the allocation.)
2. ANALYSIS OF THE TWO PLAYER CASE

In this section, we prove that for the two player case, there is a unique pure-strategy Nash equilibrium. We also bound the bids and revenues at this equilibrium, and we prove the equilibrium revenue is strictly increasing in the private value ratio $\alpha$.

Specifically, we assume that bidder 1 has a private value $v_1 = \alpha > 1$ and bids $b_1 \in (0, \alpha]$, and bidder 2 has a private value $v_2 = 1$ and bids $b_2 \in (0, 1)$. The weight function under consideration is $f(x) = x^p$, where $p > 0$. For each bidder $i = 1, 2$, his fraction allocation function $a_i(\cdot)$ is defined as

$$a_i(b) = \frac{f(b_i)}{\sum_{j \in \{1, 2\}} f(b_j)},$$

and his utility function $u_i(\cdot)$ is defined as

$$u_i(b) = a_i(b)(v_i - b_i).$$

The revenue of the auctioneer (seller) is $r = a_1(b_1)b_1 + a_2(b_2)b_2$.

2.1. Uniqueness of the Pure-Strategy Nash Equilibrium

The following theorem proves the uniqueness of the pure-strategy Nash equilibrium. It also bounds the bids at the equilibrium.

**Theorem 2.1.** For any $\alpha > 1$ and any $p > 0$, the two-bidder case with $v_1 = \alpha$, $v_2 = 1$, and weight function $f(x) = x^p$ has a unique pure strategy Nash equilibrium $b^* = (b_1^*, b_2^*)$. Moreover, at this Nash equilibrium, (1) $0 < b_1^* < \frac{\alpha p}{1 + p}$, (2) $0 < b_2^* < \frac{p}{1 + p}$, and (3) $1 < \alpha^{\frac{1}{1-p}} < b_1^*/b_2^* < \alpha$.

**Proof.** We first show that the best responses of both players are unique. Notice that

$$\frac{\partial u_1}{\partial b_1} = \frac{b_1^{p-1}}{(b_1^p + b_2^p)^2} \left[ \alpha pb_2^p - (1 + p)b_1b_2^p - b_1^{p+1} \right],$$

and

$$\frac{\partial u_2}{\partial b_2} = \frac{b_2^{p-1}}{(b_1^p + b_2^p)^2} \left[ pb_1^p - (1 + p)b_2b_1^p - b_2^{p+1} \right].$$

Since $b_1, b_2 > 0$, the sign of $\frac{\partial u_i}{\partial b_i}$ is determined by $l(b_1; b_2) = \alpha pb_2^p - (1 + p)b_1b_2^p - b_1^{p+1}$. We further note that for any $b_2 > 0$, $l(b_1; b_2)$ is a strictly decreasing function of $b_1$. Moreover, $l(0; b_2) = \alpha pb_2^p > 0$ and $l(\alpha; b_2) = -\alpha b_2^p - b_2^{p+1} < 0$. Thus, for any $b_2 > 0$, there exists a unique best response $b_1 = BR_1(b_2) \in (0, \alpha)$ s.t. $l(BR_1(b_2); b_2) = 0$. Moreover, for $b_1 < BR_1(b_2)$, $l(b_1; b_2) > 0$, and hence $u_1$ is strictly increasing in $b_1$; for $b_1 > BR_1(b_2)$, $l(b_1; b_2) < 0$ and hence $u_1$ is strictly decreasing in $b_1$. Hence, $b_1 = BR_1(b_2) \in (0, \alpha)$ is the unique maximizer of $u_1(\cdot, b_2)$. In other words, $b_1 = BR_1(b_2)$ is the unique best response to $b_2$. Similarly, we can prove that the (unique) best response of bidder 2 to a given $b_1 > 0$ is the unique solution to the equation $pb_1^p - (1 + p)b_2b_1^p - b_2^{p+1} = 0$. Moreover, this unique best response is in interval $(0, 1)$.

Thus, a bid vector $b = (b_1, b_2) \in (0, \alpha) \times (0, 1)$ is a pure-strategy Nash equilibrium in this two-bidder case if and only if it satisfies the following equation

$$\begin{cases} \alpha pb_2^p - (1 + p)b_1b_2^p - b_1^{p+1} = 0 \\ pb_1^p - (1 + p)b_2b_1^p - b_2^{p+1} = 0 \end{cases},$$

\footnote{Without loss of generality, we assume that $\alpha > 1$. For the special case with $\alpha = 1$, it is straightforward to show the unique pure-strategy Nash equilibrium is $b^* = \left( \frac{p}{2}, \frac{p}{2} \right)$.}
which is equivalent to
\[
\begin{cases}
    b_2^{p+1} [\alpha p - (1 + p)b_1] = b_1^{p+1} \\
    b_2^{p} [(p - (1 + p)b_2] = b_2^{p+1}
\end{cases}
\]  
\tag{8}
\]

Notice that the above equation implies that at a Nash equilibrium, \(0 < b_1 < \frac{\alpha p}{1 + p}\) and \(0 < b_2 < \frac{p}{1 + p}\). It also implies that at a Nash equilibrium, \(b_1 \neq b_2\), since \(\alpha > 1\).

In the remainder of this proof, we define an auxiliary variable \(z \triangleq b_1/b_2\). Note that based on the above discussion, at a Nash equilibrium, \(z \neq 1\). By dividing the first equation in Equation (8) by the second equation in Equation (8), we have
\[
z^{2p+1} = \frac{\alpha p - (1 + p)z b_2}{p - (1 + p)b_2},
\]  
\tag{9}
\]
which implies that
\[
b_2 = \frac{p(z^{2p+1} - \alpha)}{(1 + p)z(z^{2p} - 1)} \quad \text{and} \quad b_1 = \frac{p(z^{2p+1} - \alpha)}{(1 + p)(z^{2p} - 1)},
\]  
\tag{10}
\]
Notice that the above equation implies that at a Nash equilibrium, \(z = b_1/b_2 > \alpha z^{2p+1}\). To characterize a Nash equilibrium, it is sufficient to compute \(z\). Notice that from Equation (8), we also have
\[
[p - (1 + p)b_2]^2 z^{2p} = b_2^2,
\]
due to Equation (9). Since at equilibrium we have \(b_2 < \frac{p}{1 + p}\), the above equation is equivalent to
\[
[p/b_2 - (1 + p)] z^p = 1.
\]  
\tag{11}
\]
Plugging in the first equation of Equation (10), we have
\[
\left[\frac{(1 + p)z(z^{2p} - 1)}{z^{2p+1} - \alpha} - (1 + p)\right] z^p = 1,
\]
which is equivalent to
\[
(1 + p)z^p \frac{\alpha - z}{z^{2p+1} - \alpha} = 1.
\]
Notice that the above equation implies that at a Nash equilibrium, \(z < \alpha\). Rewriting the above equation as a polynomial equation, we have that
\[
z^{2p+1} + (1 + p)z^{p+1} - (1 + p)\alpha z^p - \alpha = 0.
\]  
\tag{12}
\]
Thus, to prove this game has a unique pure-strategy Nash equilibrium, it is sufficient to prove that
— Equation (12) has a unique solution \(z^*\) in \((0, \infty)\),
— The value \(z^*\), and \(b^* = (b_1^*, b_2^*)\) derived by plugging \(z^*\) into Equation (10), satisfy all the constraints on Nash equilibria discussed above.

We first prove that Equation (12) has a unique solution \(z^*\) in \((0, \infty)\). Let
\[
g(z) \triangleq z^{2p+1} + (1 + p)z^{p+1} - (1 + p)\alpha z^p - \alpha,
\]
notice that \( g(0) = -\alpha < 0 \) and \( \lim_{z \to \infty} g(z) = \infty \). On the other hand, we have
\[
g'(z) = (2p + 1)z^{2p} + (1 + p)^2 z^p - (1 + p)\alpha z^{p-1} = z^{p-1} \left[(2p + 1)z^{p+1} + (1 + p)^2 z - (1 + p)\alpha \right].
\]
(13)

Let \( h(z) \overset{\Delta}{=} (2p + 1)z^{p+1} + (1 + p)^2 z - (1 + p)\alpha \). Then for \( z \in (0, \infty) \), the sign of \( g'(z) \) is completely determined by \( h(z) \). Notice that \( h(z) \) is strictly increasing in \( z \), \( h(0) = -(1 + p)\alpha < 0 \), and \( \lim_{z \to \infty} h(z) = \infty \). Thus, there exists a \( \tilde{z} \in (0, \infty) \), s.t. (1) \( g'(<\tilde{z}) = h(\tilde{z}) = 0 \), (2) \( \forall z \in (0, \tilde{z}), h(z) < 0 \) and \( g'(z) < 0 \), and (3) \( \forall z \in (\tilde{z}, \infty), h(z) > 0 \) and \( g'(z) > 0 \). Thus, \( g(z) \) decreases first, then increases. Since \( g(0) = -\alpha < 0 \) and \( \lim_{z \to \infty} g(z) = \infty \), there exists a unique \( z^* \in (0, \infty) \) s.t. \( g(z^*) = 0 \).

Finally, we prove that \( z^* \) and \( b^* = (b_1^*, b_2^*) \) satisfy all the constraints on Nash equilibria discussed above. Notice that since \( \alpha > 1 \)
\[
g \left(\alpha^{1/(1+2p)}\right) = (1 + p) \left[\alpha^{1+2p} - \alpha^{\frac{1+2p}{1+p}}\right] < 0
\]
and
\[
g(\alpha) = \alpha^{2p+1} - \alpha > 0.
\]
Thus, \( z^* \in \left(\alpha^{\frac{1}{1+p}}, \alpha\right) \). From Equation (10), \( z^* \in \left(\alpha^{\frac{1}{1+p}}, \alpha\right) \) implies that \( 0 < b_2^* < \frac{p}{1+p} \) and \( 0 < b_1^* < \frac{p}{1+p} \). □

2.2. Equilibrium Revenue

Let \( R^* \) denote the revenue of the auctioneer (seller) at the unique pure-strategy Nash equilibrium \( b^* = (b_1^*, b_2^*) \). In the remainder of this section, we rewrite \( R^* \) as \( R^*(\alpha, p) \) to emphasize its dependence on \( \alpha \) and \( p \). The following theorem provides upper/lower bounds on \( R^*(\alpha, p) \):

**Theorem 2.2.** For any \( \alpha > 1 \) and \( p > 0 \), we have
\[
\frac{p}{p + 2} < \alpha \frac{\frac{p+1}{1+p}}{\alpha^{\frac{p}{1+p}}} < R^*(\alpha, p) < \frac{p + \alpha^{p+1}}{1 + \alpha^p} < \frac{\alpha^p}{1 + (1 + p)\alpha^p}.
\]

**Proof.** Recall that \( b^* = (b_1^*, b_2^*) \) is the unique pure-strategy Nash equilibrium, and \( z^* = b_1^*/b_2^* \). Notice that
\[
R^*(\alpha, p) = \frac{z^p b_1^*}{1 + z^p b_1^*} + \frac{1}{1 + z^p b_2^*} = \frac{1 + z^p b_1^* b_2^*}{1 + z^p b_1^* b_2^*} = \frac{1 + z^p b_1^*}{1 + z^p} + \frac{1 + z^p b_2^*}{1 + z^p} = \frac{p}{1 + (1 + p)z^p}.
\]
(17)

where the last equality follows from Equation (11). Recall that from Theorem 2.1, \( 1 < \alpha^{\frac{1}{1+p}} \) is a strictly increasing function of \( x \) for \( x \in (0, \infty) \), and \( h_2(x) = \frac{1 + x^{p+1}}{1 + x^p} \) is a strictly increasing function of \( x \) for \( x \in [1, \infty) \). Thus we have
\[
ph_1(1)h_2(1) < ph_1(\alpha^{\frac{1}{1+p}})h_2(\alpha^{\frac{1}{1+p}}) < R^*(\alpha, p) = ph_1(z^*)h_2(z^*) < ph_1(\alpha)h_2(\alpha),
\]
which is Equation (16). □

Notice that the lower bound \( \frac{p}{p + 2} < R^*(\alpha, p) \) is an \( \alpha \)-independent lower bound. This bound is tight since for \( \alpha = 1 \), we have \( b_1^* = b_2^* = \frac{p}{p + 2} \) and \( R^*(1, p) = \frac{p}{p + 2} \). Based on Theorem 2.2, we have the following corollaries:

**Corollary 2.3.** For any \( \alpha > 1 \) and \( p > 0 \), we have \( R^*(\alpha, p) < \alpha \min\{p, 1\} \). Moreover, \( \lim_{p \to 0} R^*(\alpha, p) = 0 \).
\textbf{PROOF.} Notice that \( \frac{1+z^{p+1}}{1+z} < \frac{\alpha z^{p+1}}{1+\alpha z} = \alpha, \ \frac{1+z^{p}}{1+(1+p)z} < 1, \) and \( \frac{z^p}{1+(1+p)z} < 1, \) then from Theorem 2.2 we have \( R^\ast(\alpha, p) < \alpha \min\{p, 1\}. \) Thus

\[ 0 \leq \lim_{p \to 0} R^\ast(\alpha, p) \leq \lim_{p \to 0} \alpha \min\{p, 1\} = 0. \]

\( \square \)

\textbf{COROLLARY 2.4.} For any \( \epsilon > 0 \) and any \( p \geq \frac{2}{\epsilon} - 2, \) we have

\[ R^\ast(\alpha, p) > 1 - \epsilon \ \forall \alpha > 1. \]

Moreover, \( \lim_{p \to \infty} R^\ast(\alpha, p) \geq 1 \) for all \( \alpha > 1. \)

\textbf{PROOF.} Notice that from Theorem 2.2 \( R^\ast(\alpha, p) > \frac{p}{p+2}. \) The first part of the corollary follows from the fact that \( \frac{p}{p+2} \geq 1 - \epsilon \) if \( p \geq \frac{2}{\epsilon} - 2, \) and the second part of the corollary follows from

\[ \lim_{p \to \infty} R^\ast(\alpha, p) \geq \lim_{p \to \infty} \frac{p}{p+2} = 1. \] (18)

\( \square \)

We now briefly discuss the results of Corollary 2.3 and 2.4. Recall that for the two-bidder case with \( v_1 = \alpha > 1 \) and \( v_2 = 1, \) the equilibrium revenue of the auctioneer (seller) in the standard second-price auction (without a reserve price) is 0. Corollary 2.3 shows that for a given \( \alpha > 1, \) a quasi-proportional auction with weight function exponent \( p < \frac{1}{\alpha} \) has an equilibrium revenue strictly less than that of the second price auction. It also shows that the equilibrium revenue in a quasi-proportional auction approaches to 0 if \( \alpha \) is fixed and \( p \to 0. \) These results indicate that though the revenue-optimizing weight function is nearly flat for very large \( \alpha \) (see [Mirrokni et al. 2010]), however, a too flat weight function can significantly reduce the seller’s revenue.

In many practical cases, the private value ratio \( \alpha \) is estimated based on some training data, and there can be large estimation errors. In such cases, it is important to design an auction mechanism under which the worst-case equilibrium revenue is satisfactory. Corollary 2.4 shows that for any \( \epsilon > 0, \) we can choose a large enough \( p \) so that the worst-case equilibrium revenue is at least \( 1 - \epsilon, \) where 1 is the equilibrium revenue in the standard second-price auction. It also shows that if \( \alpha \) is fixed and \( p \to \infty, \) then the equilibrium revenue of the quasi-proportional auction is at least as good as that of the standard second-price auction.

The following theorem states that for a fixed \( p > 0 \) (i.e. a fixed quasi-proportional auction mechanism), \( R^\ast(\alpha, p) \) is a strictly increasing function of \( \alpha: \)

\textbf{THEOREM 2.5.} For a fixed \( p > 0, \) \( R^\ast(\alpha, p) \) is a strictly increasing function of \( \alpha \) for \( \alpha \in [1, \infty). \) Moreover, \( \lim_{\alpha \to \infty} R^\ast(\alpha, p) = \infty. \)

\textbf{PROOF.} From the proof of Theorem 2.2, we have

\[ R^\ast(\alpha, p) = \frac{1+z^{p+1}}{1+z} \frac{z^p}{1+(1+p)z^p}. \] (19)

Moreover, as we have discussed in the proof of Theorem 2.2 if \( p \) is fixed, then \( R^\ast \) is a strictly increasing function of \( z^* \) for \( z^* \in [1, \infty). \) Thus, to prove the first part of this theorem, we only need to show that if \( p \) is fixed, then \( z^* \) is a strictly increasing function of \( \alpha. \) Recall that from the proof of Theorem 2.1, \( z^* \) is the unique positive solution of
Thus we have
\[
\frac{dz^*}{d\alpha} = \left. \frac{\frac{(2p+1)z^{2p} + (1+p)^2z^p - (1+p)pz^{p-1}}{(1+p)z^p + 1}}{\frac{p+1)z^{2p} + (1+p)^2z^p - (1+p)pz^{p-1}}{(1+p)z^p + 1}} \right|_{z=z^*},
\]
where the first equality is based on the implicit differentiation, and the second equality is based on \((1+p)\alpha z^{p-1} = z^{2p} + (1+p)z^p - \alpha / z\). Thus, \(\frac{dz^*}{d\alpha} > 0\) and hence \(R^*\) is a strictly increasing function of \(\alpha\).

As to the second part of the theorem, recall from Theorem 2.2, we have
\[
R^*(\alpha,p) > p \frac{1 + \alpha \frac{p+1}{p+2}}{1 + \alpha \frac{p}{p+2}} \frac{1 + \alpha \frac{p+1}{p+2} \frac{p}{2}}{1 + \alpha \frac{p}{p+2} \frac{p+2}{2}} \geq \frac{1 + \alpha \frac{p+1}{p+2} \frac{p}{2}}{1 + \alpha \frac{p}{p+2} \frac{p+2}{2}} = \frac{1 + \alpha \frac{p+1}{p+2} \frac{p}{2}}{1 + \alpha \frac{p}{p+2} \frac{p+2}{2}} \rightarrow \infty.
\]
Thus \(\lim_{\alpha \to \infty} R^*(\alpha,p) = \infty\). □

3. REVENUE-OPTIMIZING WEIGHT FUNCTIONS

Figures 1 and 2 show that the revenue-optimizing value of \(p\) increases smoothly as the ratio of private values \(\alpha\) decreases. Figure 1 shows equilibrium revenue values for large competitive ratios: \(100 \leq \alpha \leq 1000\). For these values, the revenue-optimizing \(p\) values are less than one. Now look at Figure 2. It shows equilibrium revenue values for smaller competitive ratios: \(1 \leq \alpha \leq 10\). For these values, the revenue-optimizing \(p\) values are greater than one. (The two plots have different scales on their horizontal axes.) So it makes sense to use \(p > 1\) when we know \(\alpha \leq 10\).

The plots are based on a two-bidder auction, with bidders’ private values \(v_1 = 1\) and \(v_2 = \alpha\). Notice that as \(\alpha\) increases, the equilibrium revenue decreases as a fraction of the highest private value: it is about 40/1000 = 4% for \(\alpha = 1000\) and about 1.7/10 = 17% for \(\alpha = 10\). This agrees with the intuition that as competition decreases, the seller receives a lower fraction of the highest private value.

The plots were generated by using hill-climbing to evolve bids to equilibrium. For each data point plotted, initial bids are \((b_1, b_2) = (0.5, 0.5)\). At each step, we update the bid for bidder 1, then for bidder 2, as follows. First, we estimate the partial derivative of utility with respect to bid:
\[
\frac{\partial u_i}{\partial b_i} \approx \frac{(u_i(b_i + \Delta, b_{-i}) - u_i(b_i, b_{-i}))}{\Delta},
\]
where
\[
\begin{align*}
u_i(b) &= u_i(b_i, b_{-i}) = a_i(b)(v_i - b_i), \quad (22) \\
a_i(b) &= f(b_i)/\sum_{j \in \{1, 2\}} f(b_j), \quad (23)
\end{align*}
\]
and $\Delta = 0.001$. Note that as is standard in game theory, we use $b_{-i}$ to denote the bids of all bidders other than bidder $i$. If the estimated partial derivative is positive, then we add $\Delta$ to the bid; otherwise we subtract $\Delta$ from the bid. We repeat for 100,000 steps. Then we compute the revenue:

$$r = a_1(b_1) + a_2(b_2).$$ (25)

4. BIDDER UTILITY FUNCTIONS FOR $P > 1$

In this section, we show that, even with more than two bidders and $p > 1$, there is reason to hope that bidders have a unique pure-strategy Nash equilibrium. For $p > 1$, individual bidders’ utility functions are not necessarily concave over the whole domain of bids that give nonegative utility: $[0,v]$, where $v$ is the bidder’s private value. So we cannot apply results such as those by Rosen [1965] to show that there is a unique Nash equilibrium in pure strategies. However, in this section we offer some hope for conver-
gence to a pure-strategy equilibrium by showing that each bidder’s utility function, given other bidders’ bids, is concave at its maximum.

We will use \( x \) to represent a bidder’s bid. Holding the other bids fixed, the allocation is

\[
a(x) = \frac{x^p}{x^p + s},
\]

where \( s \) is the sum of weight functions of the other bidders’ bids. The bidder’s utility function is

\[
u(x) = a(x)(v - x),
\]

where \( v \) is the bidder’s private value. We want to show:

**Theorem 4.1.** For the value \( x^* \) at which \( \frac{\partial u}{\partial x} = 0 \), \( \frac{\partial^2 u}{\partial x^2} |_{x^*} < 0 \).
PROOF.

\[
\frac{\partial u}{\partial x} = \frac{p}{x}a(x)(1 - a(x)) - (p + 1)a(x)(1 - p/(p + 1)a(x)) \\
= pa(x)(1 - a(x))\left(\frac{v}{x} - 1\right) - a(x).
\]

(28)

Set \( \frac{\partial u}{\partial x} = 0 \) and solve for \( x \):

\[
x = v \frac{p(1 - a(x))}{1 + p(1 - a(x))}.
\]

(30)

Equivalently,

\[
x = v \frac{p(1 - a(x))}{1 + p(1 - a(x))}.
\]

(31)

As an aside, note that

\[
x v = 1 - \frac{1}{1 + p(1 - a(x))}.
\]

(32)

So the utility-maximizing bid as a portion of the private value increases with \( p \) but decreases with the allocation. But the allocation depends on \( p \): the allocation to the highest bidder increases with \( p \). So \( p \) mediates a tradeoff: smaller \( p \) values deny some allocation to the highest bidder, spurring them to bid higher, but greater \( p \) values foster competition among bidders, because they make the auction closer to winner-take-all.

The second derivative is:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{p}{x}a(x)(1 - a(x))[p(1 - 2a(x))(v/x - 1) - v/x - 1].
\]

(33)

Note that

\[
\frac{\partial a}{\partial x} = \frac{p}{x}a(x)(1 - a(x)).
\]

(34)

So we have:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial a}{\partial x}[p(1 - 2a(x))(\frac{v}{x} - 1) - \frac{v}{x} - 1] \\
= \frac{\partial a}{\partial x}p\left[\frac{v}{x}(1 - 2a(x) - 1/p) - (1 - 2a(x) + 1/p)\right].
\]

(35)

(36)

Insert the maximum \( \frac{x}{v} \) from Equation 31

\[
\frac{\partial^2 u}{\partial x^2} |_{x/v} = \frac{p(1 - a(x))}{1 + p(1 - a(x))} = \frac{\partial a}{\partial x}[-a(x) + \frac{1}{p}] \\
= \frac{a(x) + \frac{1}{p}}{1 - a(x) - 2].
\]

(37)

Since \( \frac{\partial a}{\partial x}, a(x) + \frac{1}{p}, \) and \( 1 - a(x) \) are all positive, this is negative. \( \square \)

Figures 3, 4, and 5 show the utility function and its first two derivatives for \( p = 1, p = 2, \) and \( p = 4 \). In each case, we set \( s \), the sum of weight functions for other bids, to 7, and the bidder’s private value to 10. In Figure 3 with \( p = 1 \), the utility function is concave (negative second derivative) over the whole bid domain \([0, 10]\). In Figures 4 and 5, the utility functions are not concave at \( x = 0 \), but the second derivatives are negative at the \( x \) values where the first derivatives are zero — the utility functions are concave at their maximum values. In Figure 3, the utility function is a smoothly rounded curve. In Figure 4, the utility function is still somewhat rounded, but there is a slight S-curve starting at zero, and the curve is less rounded and more linear to the right of the maximum. In Figure 5 these effects are more pronounced.
5. CONCLUSION & NEXT STEPS

In this paper, we have proved the existence and uniqueness of a pure-strategy Nash equilibrium for the two-bidder quasi-proportional auction, and bounded the bids, bid ratio, and the auctioneer's revenue at this equilibrium. We have also provided a numerical solution to find the revenue-optimal exponent $p$ for a known ratio of valuation $\alpha$, and shown that as $\alpha \to 1$, the revenue-optimal exponent $p$ increases beyond 1.

We became interested in quasi-proportional auctions while considering how to design a search advertising exchange. We envisioned having a just a few bidders with competitive valuations within an order of magnitude of each other. Yet one bidder might be likely to consistently monetize queries somewhat better than the others.

The shared allocations of quasi-proportional auctions can help maintain competition over time. Allocating some inventory to runners-up encourages them to participate in the auction initially if they can profitably serve the inventory they receive. Having runners-up actively serving ads on inventory should make it easier for them to ramp up if they win more inventory over time, which maintains competitive pressure on
the highest bidder, who might otherwise use the startup or restart costs of runner-up bidders as a barrier to entry or re-entry that allows the initial highest bidder to bid less over time.

Similarly, delivering some inventory to runner-up bidders allows them to learn from the inventory, to gain information about the value of the inventory and to improve that value through optimization. It could be argued that the value of the inventory would be most increased by sending it all to the highest bidder, so they could use it all for learning and optimization. However, the seller’s concern is to increase how much of the value the seller extracts through pricing, and strengthening competition among buyers can contribute to that.

One interesting direction for future work is to determine when and how to eliminate bidders from the auction in order to optimize revenue. An inherent problem with quasi-proportional auctions is that a bidder can enter a very small bid and still get something. [Nguyen & Vojnovic][2010] show that revenue can decrease as the number of bidders increases. One open question, for example, is which criteria for eliminating
bidders offer the highest revenue? Is it optimal to eliminate all but the top $k$ bidders for some $k \geq 2$? Or is it better to eliminate bidders who bid less than some fraction of the highest bid or of the runner-up bid? Another open question is how to set elimination criteria for repeated quasi-proportional auctions, to balance higher revenue from fewer bidders today against maintaining competition from more bidders tomorrow. A model to answer this question might consider the effects of allocations both on the probabilities of bidders’ participation in future auctions and on changes in their valuations through learning and optimization.

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