COHOMOLOGY OF TRIVIAL EXTENSIONS OF FROBENIUS ALGEBRAS

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ABSTRACT. We obtain a decomposition for the Hochschild cochain complex of a split algebra and we study some properties of the cohomology of each term of this decomposition. Then, we consider the case of trivial extensions, specially of Frobenius algebras. In particular, we determine completely the cohomology of the trivial extension of a finite dimensional Hopf algebra. Finally, as an application, we obtain a result about the Hochschild cohomology of Frobenius algebras.

INTRODUCTION

Let $k$ be a field, $A$ a $k$-algebra and $M$ an $A$-bimodule. The split algebra $E = A \ltimes M$, of $A$ with $M$, is the direct sum $A \oplus M$ with the associative algebra structure given by

$$(a + m)(a' + m') = aa' + am' + ma'.$$

The (co)homology of split algebras, and of several particular types of split algebras, such as the triangular matrix algebras and the trivial extensions, has been considered in several papers. See for instance [C], [C-M-R-S], [G-G1], [G-M-S], [H1], [M-P] and [Mi-P]. This study is motivated in part by the relations between the degree one Hochschild cohomology and the representation theory of a finite dimensional algebra [H2], [M-P], [S], the relations between the second and third Hochschild cohomology groups and the theory of deformations of algebras [G], [G-S], and the following question of Happel [H1]: if an algebra has only a finite number of non nulls Hochschild cohomology groups, is the algebra of finite homological dimension? Moreover, the Hochschild cohomology groups are interesting invariants of an algebra in itself.

In this work we continue the study of the Hochschild cohomology of split algebras, computing the Hochschild cohomology of a trivial extension of a Frobenius algebra $A$ in terms of the Hochschild cohomology of $A$.

Let $E = A \ltimes M$ a split algebra. This paper is organized as follows: in Section 1 we show that the canonical Hochschild complex of $E$ with coefficients in $E$ has a canonical decomposition as a direct sum of subcomplexes $X^*_p$, such that $X^*_0$ is...
the Hochschild cochain complex of $A$ with coefficients in $M$ and $X^p_n = 0$ for all $n < p - 1$. Hence,

$$
\text{HH}^n(E) = \text{H}^n(A, M) \oplus \bigoplus_{p=1}^{n+1} \text{H}^n(X^*_p).
$$

Moreover, we prove that, under suitable hypothesis, for all $p \geq 1$ there is a long exact sequence relating the cohomology of $X^*_p$, the Ext groups $\text{Ext}^*_A(M^{\otimes A p-1}, A)$ and the Ext groups $\text{Ext}^*_A(M^{\otimes A p}, M)$.

In Section 2 we consider the cohomology of the trivial extension $TA$ of an algebra $A$, which is the split algebra obtained taking $M = DA$, where $DA$ is the dual vector space of $A$, endowed with the usual $A$-bimodule structure. For these algebras we compute $\text{H}^n(X^*_1)$, for all $n \geq 0$, and $\text{H}^{p-1}(X^*_p)$, for each $p \geq 2$.

The results of Sections 1 and 2 are close to the ones obtained in [C-M-R-S].

In Section 3 we study trivial extensions of Frobenius algebras. The main result is Theorem 3.10, where we compute the Hochschild cohomology of the trivial extension of a finite order Frobenius $k$-algebra (see Definition 3.7), when the characteristic of $k$ does not divide the order of $A$. In particular, this result applies to the trivial extension of a finite dimensional Hopf algebra. Finally, as an application, we obtain a result about the Hochschild cohomology of a finite order Frobenius $k$-algebra.

1. A decomposition of the cohomology of an split algebra

Let $k$ be a field, $A$ a $k$-algebra and $M$ an $A$-bimodule. Let $E = A \times M$ the split algebra of $A$ with $M$ As it is well known, the Hochschild cohomology $\text{HH}^*(E) = \text{H}(E, E)$ is the homology of the cochain complex

$$
0 \to E \xrightarrow{b^1} \text{Hom}_k(E, E) \xrightarrow{b^2} \text{Hom}_k(E^{\otimes 2}, E) \xrightarrow{b^3} \text{Hom}_k(E^{\otimes 3}, E) \xrightarrow{b^4} \ldots,
$$

where $E^{\otimes n}$ is the $n$-fold tensor product of $E$, $b^1(x)(y) = yx - xy$ and for $n > 1$,

$$
\begin{align*}
    b^n(f)(x_1 \otimes \cdots \otimes x_n) &= x_1 f(x_2 \otimes \cdots \otimes x_n) \\
    &+ \sum_{i=1}^{n-1} (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n) \\
    &+ (-1)^n f(x_1 \otimes \cdots \otimes x_{n-1}) x_n.
\end{align*}
$$

For $0 \leq p \leq n$, let $B^p_n \subseteq E^{\otimes n}$ be the vector subspace spanned by $n$-tensors $x_1 \otimes \cdots \otimes x_n$ such that exactly $p$ of the $x_i$’s belong to $M$, while the other $x_i$’s belong to $A$ (note that $B^0_0 = k$). To unify expressions we make the convention that $B^p_0 = 0$, for $p < 0$ or $n < p$. For each $p \geq 0$ we let $X^*_p$ denote the subcomplex of $(\text{Hom}_k(E^{\otimes *}, E), b^*)$ defined by:

$$
X^*_p := \text{Hom}_k(B^p_{p-1}, A) \oplus \text{Hom}_k(B^p_p, M).
$$

It is immediate that $(\text{Hom}_k(E^{\otimes *}, E), b^*) = \bigoplus_{p=0}^\infty X^*_p$. Hence, we have proved the following result:
**Theorem 1.1.** It is holds that

\[
\text{HH}^n(E) = \bigoplus_{p=0}^{\infty} H^n(X^*_{(p)}) = H^n(A,M) \oplus \bigoplus_{p=1}^{n+1} H^n(X^*_{(p)}).
\]

We also have the following result:

**Theorem 1.2.** If \(\text{Tor}^A_i(M,M^{\otimes A}) = 0\) for \(i > 0\) and \(0 < j < p\), then there is a long exact sequence

\[
0 \rightarrow H^{p-1}(X^*_{(p)}) \rightarrow \text{Ext}^0_A(M^{\otimes A}_{p-1}, A) \rightarrow \text{Ext}^0_A(M^{\otimes A}, M) \rightarrow \\
H^p(X^*_{(p)}) \rightarrow \text{Ext}^1_A(M^{\otimes A}_{p-1}, A) \rightarrow \text{Ext}^1_A(M^{\otimes A}, M) \rightarrow \ldots,
\]

where, as usual, \(M^{\otimes A} = A\).

To prove this theorem, we need to study the cochain complexes \(X^*_{(p)}\), for \(p \geq 1\). Let \(\pi_A : E \rightarrow A\) and \(\pi_M : E \rightarrow M\) be the maps \(\pi_A(a + m) = a\) and \(\pi_M(a + m) = m\), respectively. Note that \(X^*_{(p)}\) is the total complex of the double complex

\[
\begin{array}{ccc}
\vdots & \vdots & \\
\text{Hom}_k(B^p_{p-1}, A) & \delta^{1,p}_{p-1} & \text{Hom}_k(B^p_{p-1}, M) \\
\downarrow b^0_{p} & \downarrow b^1_{p} & \downarrow b^2_{p} \\
\text{Hom}_k(B^{p+1}_{p}, A) & \delta^{1,p+1} & \text{Hom}_k(B^{p+1}_{p}, M)
\end{array}
\]

where

\[
b^0_{p}(f)(x_1 \otimes \cdots \otimes x_n) = \pi_A(x_1)f(x_2 \otimes \cdots \otimes x_n) \\
+ \sum_{i=1}^{n-1} (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n) \\
+ (-1)^n f(x_1 \otimes \cdots \otimes x_{n-1}) \pi_A(x_n),
\]

\[
b^1_{p}(g)(x_1 \otimes \cdots \otimes x_{n+1}) = x_1 g(x_2 \otimes \cdots \otimes x_{n+1}) \\
+ \sum_{i=1}^{n} (-1)^i g(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \\
+ (-1)^{n+1} g(x_1 \otimes \cdots \otimes x_n) x_{n+1},
\]

\[
\delta^{1,n}_{p}(f)(x_1 \otimes \cdots \otimes x_n) = \pi_M(x_1)f(x_2 \otimes \cdots \otimes x_n) \\
+ (-1)^n f(x_1 \otimes \cdots \otimes x_{n-1}) \pi_M(x_n),
\]
for \( f \in \text{Hom}_k(B_{p-1}^{n-1}, A) \) and \( g \in \text{Hom}_k(B_p^n, M) \).

It is immediate that the cohomology \( H^*(X_{(p)}^{0,*}) \) of the 0-column of \( X_{(1)}^{*,*} \) is the Hochschild cohomology \( HH^*(A) \). Moreover, we have the following result:

**Theorem 1.3.** The following assertions hold

1) If \( \text{Tor}_j^A(M, M^{\otimes A}) = 0 \) for \( i > 0 \) and \( 0 < j < p - 1 \), then

\[
H^n(X_{(p)}^{0,*}) = \text{Ext}^{n-p+1}_{A,*} (M^{\otimes A}_{p-1}, A) \quad \text{for all } n \geq p - 1,
\]

2) If \( \text{Tor}_j^A(M, M^{\otimes A}) = 0 \) for \( i > 0 \) and \( 0 < j < p \), then

\[
H^n(X_{(p)}^{1,*}) = \text{Ext}^{n-p+1}_{A,*} (M^{\otimes A}_p, M) \quad \text{for all } n \geq p - 1.
\]

**Proof.** We prove the second assertion. The first one follows similarly. In the proof of Theorem 2.5 of [C-M-R-S] was showed that, under our hypothesis, the complex

\[
M^{\otimes A}_p \xleftarrow{\mu^p} A \otimes B_p^p \otimes A \xleftarrow{b_1^p} A \otimes B_p^{p+1} \otimes A \xleftarrow{b_2^p} A \otimes B_p^{p+2} \otimes A \xleftarrow{b_3^p} \cdots,
\]

where \( \mu^p(a \otimes x_1 \otimes \cdots \otimes x_p \otimes a') = ax_1 \otimes_A x_2 \otimes_A \cdots \otimes_A x_p-1 \otimes_A x_p a' \) and

\[
b_n^p(x_0 \otimes \cdots \otimes x_{n+p+1}) = \pi_A(x_0x_1) \otimes x_2 \otimes \cdots \otimes x_{n+p+1} + \sum_{i=1}^{n+p-1} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+p+1} + (-1)^{n+p} x_0 \otimes \cdots \otimes x_{n+p-1} \otimes \pi_A(x_{n+p}x_{n+p+1}),
\]

is a projective resolution of \( M^{\otimes A}_p \) as an \( A \)-bimodule. The assertion follows from this fact, using that \( X_{(p)}^{1,*} \simeq \text{Hom}_A((A \otimes B_p^{p+1} \otimes A, b_{p+1}^p), M) \). \( \square \)

**Proof of Theorem 1.2.** It follows from the long exact sequence of homology of the short exact sequence

\[
0 \to X_{(p)}^{1,*-1} \to X_{(p)}^{*} \to X_{(p)}^{0,*} \to 0,
\]

using Theorem 1.3. \( \square \)

**Remark 1.4.** The vector spaces \( B_p^n \) were considered in [C-M-R-S] in order to organize the canonical Hochschild cochain complex of \( E \) as a double complex. Using the decomposition \( \text{Hom}_k(E^{\otimes *, E}), b^* = \bigoplus_{p=0}^{\infty} X_{(p)}^{*,*} \) obtained above Theorem 1.1, can be easily shown that the spectral sequence introduced at the beginning of Section 3 of [C-M-R-S] satisfies \( E^2 = E^3 = \cdots = E^\infty \). However, the term \( E^2 \) is hard to compute in general.

### 2. Trivial extensions

Given an \( A \)-bimodule \( M \), we let \( DM \) denote \( \text{Hom}_k(M, k) \) endowed with the usual \( A \)-bimodule structure.
**Definition 2.1.** Let $A$ be a $k$-algebra. The trivial extension $TA$ of $A$ is the split algebra $A \ltimes DA$.

**Theorem 2.2.** For each trivial extension $TA$, it is hold that

$$H^0(X^*_1) = HH^0(A) \quad \text{and} \quad H^n(X^*_1) = HH^n(A) \oplus \text{Ext}_{A^e}^{n-1}(DA, DA) \ \forall n \geq 1.$$  

**Proof.** Let $\delta_1^{1,*} : (\text{Hom}_k(B_0^*, A), -b_1^{0,+1}) \to (\text{Hom}_k(B_1^{+1}, DA), b_1^{1,+1})$ be the map defined by

$$\delta_1^{1,n-1}(f)(x_1 \otimes \cdots \otimes x_n) = \pi_{DA}(x_1)f(x_2 \otimes \cdots \otimes x_n)$$

$$\quad + (-1)^n f(x_1 \otimes \cdots \otimes x_{n-1})\pi_{DA}(x_n).$$

By Theorem 1.3, $H^n(X^*_1) = HH^n(A)$ and $H^{n-1}(X^*_1) = \text{Ext}_{A^e}^{n-1}(DA, DA)$. Thus, we must prove that $H^n(X^*_1) = H^n(X^0_1) \oplus H^{n-1}(X^1_1)$. Since $X^*_1$ is the mapping cone of $\delta_1^{1,*}$, to made out this task it suffices to check that $\delta_1^{1,*}$ is null homotopic. Let $\sigma_* : \text{Hom}_k(B_0^*, A) \to \text{Hom}_k(B_1^+, DA)$ be the family of maps defined by

$$\sigma_n(f)(x_{1,n})(a) = (-1)^{n+1} x_j(f(x_{j+1,n} \otimes a \otimes x_{1,j-1})) \quad \text{if} \ x_j \in DA,$$

where, to abbreviate, we write $x_{h,l} = x_h \otimes \cdots \otimes x_l$, for $h < l$. We assert that $\sigma_*$ is an homotopy from $\delta_1^{1,*}$ to 0. We have,

$$b_1^{1,n} (\sigma_n(f))(x_{1,n+1}) = \pi_A(x_1) (\sigma_n(f)(x_{2,n+1})) + (-1)^{n+1} (\sigma_n(f)(x_{1,n}))\pi_A(x_{n+1})$$

$$+ \sum_{i=1}^{n} (-1)^i \sigma_n(f)(x_{1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+1}).$$

Hence, if $x_1 \in DA$, then

$$b_1^{1,n} (\sigma_n(f))(x_{1,n+1})(x_{n+2}) = (-1)^{n+2} x_1(x_2 f(x_{3,n+2}))$$

$$+ \sum_{i=2}^{n+1} (-1)^{n+i+1} x_1(f(x_{2,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+2}),$$

if $x_j \in DA$ for $1 < j \leq n$, then

$$b_1^{1,n} (\sigma_n(f))(x_{1,n+1})(x_0) = (-1)^{(j-1)n+j} x_j(f(x_{j+1,n+1} \otimes x_{0,j-2})x_{j-1})$$

$$+ \sum_{i=0}^{j-2} (-1)^{(j-1)n+i+1} x_j(f(x_{j+1,n+1} \otimes x_{0,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,j-1}))$$

$$+ (-1)^{jn+j} x_j(x_{j+1} f(x_{j+2,n+1} \otimes x_{0,j-1}))$$

$$+ \sum_{i=j+1}^{n} (-1)^{n+i+1} x_j(f(x_{j+1,i-1} \otimes x_i x_{i+1} \otimes x_{i+2,n+1} \otimes x_{0,j-1}))$$

$$+ (-1)^{jn+n} x_j(f(x_{j+1,n} \otimes x_{n+1} x_0 \otimes x_{1,j-1})).$$
and if $x_{n+1} \in DA$, then
\[
b_{1}^{n}(\sigma_{n}(f))(x_{1,n+1})(x_{0}) = \sum_{i=0}^{n-1}(-1)^{n+i+1}x_{n+1}(f(x_{i-1} \otimes x_{i}x_{i+1} \otimes x_{i+2,n})) \\
- x_{n+1}(f(x_{0,n-1})x_{n}).
\]

On the other hand, if $x_{1} \in DA$, then
\[
\sigma_{n+1}(-b_{1}^{0,n}(f))(x_{1,n+1})(x_{n+2}) = (-1)^{n+1}x_{1}(b_{1}^{0,n}(f)(x_{2,n+2})) \\
= (-1)^{n+1}x_{1}(x_{2}f(x_{3,n+2})) + x_{1}(f(x_{2,n+1})x_{n+2}) \\
+ \sum_{i=2}^{n+1}(-1)^{n+i}x_{1}(f(x_{2,i-1} \otimes x_{i}x_{i+1} \otimes x_{i+2,n+2})),
\]

if $x_{j} \in DA$ for $1 < j \leq n$, then
\[
\sigma_{n+1}(-b_{1}^{0,n}(f))(x_{1,n+1})(x_{0}) = (-1)^{(n+1)+1}x_{1}(b_{1}^{0,n}(f)(x_{j+1,n+1} \otimes x_{0,j-1} \otimes x_{0,j-2} \otimes \cdots \otimes x_{0,j})) \\
= (-1)^{(n+1)+1}x_{j}(f(x_{j+1,n+1} \otimes x_{0,j-1} \otimes x_{0,j-2} \otimes \cdots \otimes x_{0,j})) \\
+ \sum_{i=j+1}^{n}(-1)^{(n+1)+i-j}x_{j}(f(x_{j+1,i-1} \otimes x_{i}x_{i+1} \otimes x_{i+2,n+1} \otimes x_{0,j-1} \otimes x_{0,j-2} \otimes \cdots \otimes x_{0,j})) \\
+ (-1)^{(n+1)+n-j+1}x_{j}(f(x_{j+1,n} \otimes x_{n+1,n} \otimes x_{0,j-1} \otimes x_{0,j-2} \otimes \cdots \otimes x_{0,j})) \\
+ \sum_{i=0}^{j-2}(-1)^{(n+1)+i+n-j}x_{j}(f(x_{j+1,i+1} \otimes x_{i}x_{i+1} \otimes x_{i+2,j-1} \otimes x_{j+1,n} \otimes x_{0,j-1} \otimes x_{0,j-2} \otimes \cdots \otimes x_{0,j})) \\
+ (-1)^{(n+1)+n+1}x_{j}(f(x_{j+1,n+1} \otimes x_{0,j-2} \otimes x_{j+1,n} \otimes x_{0,j-1} \otimes x_{0,j-2} \otimes \cdots \otimes x_{0,j})),
\]

and if $x_{n+1} \in DA$, then
\[
\sigma_{n+1}(-b_{1}^{0,n}(f))(x_{1,n+1})(x_{0}) = (-1)^{n+1}x_{n+1}(b_{1}^{0,n}(f)(x_{0,n})) \\
= (-1)^{n+1}x_{n+1}(x_{0}f(x_{1,n})) + x_{n+1}(f(x_{0,n-1})x_{n}) \\
+ \sum_{i=0}^{n-1}(-1)^{n+i}x_{n+1}(f(x_{0,i-1} \otimes x_{i}x_{i+1} \otimes x_{i+2,n})).
\]

The assertion follows immediately from these equalities. □

As usual, for each $A$-bimodule $M$ we write $M \otimes A^{+} = M/[A,M]$, where $[A,M]$ is the vector subspace of $M$ generated by \{$(am-ma) : a \in A, m \in M$\}. The map $\theta : \text{Hom}_{A^{+}}((DA)^{\otimes A^{p-1}},A) \to \text{Hom}_{k}((DA)^{\otimes A^{p}} \otimes A^{+},k)$ defined by
\[
\theta(f)(\psi_{1} \otimes \cdots \otimes \psi_{p}) = \psi_{p}(f(\psi_{1} \otimes \cdots \otimes \psi_{p-1}))
\]
is injective. For each $p \geq 2$, we let $\text{Cyc}_{A}^{p}(DA)$ denote the set of $k$-linear maps $g : (DA)^{\otimes A^{p}} \to k$ verifying $g(\psi_{1} \otimes \cdots \otimes \psi_{p}) = (-1)^{p-1}g(\psi_{2} \otimes \cdots \otimes \psi_{p} \otimes \psi_{1})$. Note that $\text{Cyc}_{A}^{p}(DA) \subseteq \text{Hom}_{k}((DA)^{\otimes A^{p}} \otimes A^{+},k)$.

The following lemma and its proof is inspired in the proof of Theorem 5.5 of [C-M-R-S].
Lemma 2.3. For each $p \geq 2$, it holds that
\[H^{p-1}(X^*_p) \simeq \theta(\text{Hom}_{A^e}((DA)^{\otimes A_p^{-1}}, A)) \cap \text{Cyc}^p_A(DA)\].

Moreover, if $A$ is a finite dimensional $k$-algebra, then $\theta(\text{Hom}_{A^e}((DA)^{\otimes A_p^{-1}}, A)) \cap \text{Cyc}^p_A(DA) = \text{Cyc}^p_A(DA)$.

Proof. Let us compute $H^{p-1}(X^*_p)$. It is easy to see that $H^{p-1}(X^*_p)$ is the kernel of
\[\tilde{\delta}^{1,p-1}_p: \text{Hom}_{A^e}((DA)^{\otimes A_p^{-1}}, A) \to \text{Hom}_{A^e}((DA)^{\otimes A_p}, DA),\]
where $\tilde{\delta}^{1,p-1}_p$ is the map defined by $\tilde{\delta}^{1,p-1}_p(f)(\psi_1 \cdots \psi_p) = \psi_1 f(\psi_2 \cdots \psi_p) + (-1)^p f(\psi_1 \cdots \psi_p \psi_{p-1}).$ Consider the isomorphism
\[\vartheta: \text{Hom}_{A^e}((DA)^{\otimes A_p}, DA) \to \text{Hom}_k((DA)^{\otimes A_p}, k),\]
given by $\vartheta(f)(\psi_1 \cdots \psi_p) = f(\psi_1 \cdots \psi_p)(1)$. Let $g$ be in the the image of $\vartheta$. We have
\[(\vartheta \circ \tilde{\delta}^{1,p-1}_p \circ \theta^{-1}(g))(\psi_1 \cdots \psi_p) = \psi_1(\theta^{-1}(g)(\psi_2 \cdots \psi_p)) + (-1)^p \psi_p(\theta^{-1}(g)(\psi_1 \cdots \psi_{p-1})) + g(\psi_2 \cdots \psi_p \psi_1) + (-1)^p g(\psi_1 \cdots \psi_p).
\]
Hence $H^{p-1}(X^*_p) \simeq \theta(\text{Hom}_{A^e}((DA)^{\otimes A_p^{-1}}, A)) \cap \text{Cyc}^p_A(DA)$, as desired. To finish the proof it is sufficient to note that if $\dim A < \infty$, then $\text{Hom}_k((DA)^{\otimes A_p}, k) \simeq \text{Hom}_{A^e}((DA)^{\otimes A_p^{-1}}, DA) \simeq \text{Hom}_{A^e}((DA)^{\otimes A_p^{-1}}, A)$, which implies that $\vartheta$ is an isomorphism. \(\square\)

The following result improves Theorem 5.7 of [C-M-R-S].

Corollary 2.4. For all algebra $A$ and each $n \geq 1$, we have
\[
\text{HH}^n(TA) = \text{HH}^n(A) \oplus \text{HH}_n(A)^* \oplus \text{Ext}^{n-1}_A(DA, DA) \oplus \theta(\text{Hom}_{A^e}((DA)^{\otimes A_n^{-1}}, A)) \cap \text{Cyc}^{n+1}_A(DA) \oplus \bigoplus_{p=2}^n H^n(X^*_p).
\]

Proof. It follows from Theorem 1.1, Theorem 2.2, Lemma 2.3 and the fact that $\text{HH}^n(A, DA) = \text{HH}_n(A)^*$. \(\square\)

Corollary 2.5. Let $A$ be a finite dimensional $k$-algebra. For each $n \geq 1$, we have
\[
\text{HH}^n(TA) = \text{HH}^n(A) \oplus \text{HH}_n(A)^* \oplus \text{Ext}^{n-1}_A(DA, DA) \oplus \text{Cyc}^{n+1}_A(DA) \oplus \bigoplus_{p=2}^n H^n(X^*_p).
\]

Proof. It follows from Corollary 2.4 and Lemma 2.3. \(\square\)
Lemma 2.6. Let \((DDA)^A = \{ \varphi \in DDA : a\varphi = \varphi a \text{ for all } a \in A \}\). It is hold that 
\[
\text{Ext}^0_{A^+}(DA, DA) = (DDA)^A.
\]
Proof. We have
\[
\text{Ext}^0_{A^+}(DA, DA) = \text{Hom}_{A^+}(DA, DA) \cong \text{Hom}_k(DA \otimes A, k) \\
\cong \text{Hom}_k(A \otimes A, DA, k) \cong \text{Hom}_{A^+}(A, DDA) = (DDA)^A.
\]
□

The following two results were obtained in [C-M-R-S]. They use the notation \(\text{Alt}_A(DA)\) instead of \(\text{Cyc}_A^2(DA)\).

Theorem 2.7. For each trivial extension \(TA\), it is hold that:
1) \(\text{HH}^0(TA) = \text{HH}^0(A) \oplus \text{HH}_0(A)^*\),
2) \(\text{HH}^1(TA) = \text{HH}^1(A) \oplus \text{HH}_1(A)^* \oplus (DDA)^A \oplus \theta(\text{Hom}_{A^+}(DA, A)) \cap \text{Cyc}_A^2(DA)\).

Proof. 1) By Theorems 1.1 and 2.2, and the fact that \(H^n(A, DA) = \text{HH}_n(A)^*\), we have
\[
\text{HH}^0(TA) = H^0(A, DA) \oplus H^0(X_{11}) = \text{HH}_0(A)^* \oplus \text{HH}^0(A)
\]
2) It follows immediately from Corollary 2.4 and Lemma 2.6. □

Corollary 2.8. Let \(TA\) be a trivial extension of a finite dimensional \(k\)-algebra. Then,
1) \(\text{HH}^0(TA) = \text{HH}^0(A) \oplus \text{HH}_0(A)^*\),
2) \(\text{HH}^1(TA) = \text{HH}^1(A) \oplus \text{HH}_1(A)^* \oplus A^A \oplus \text{Cyc}_A^2(DA)\).

Proof. It follows immediately from Theorem 2.7, Lemma 2.3 and the fact that \(DAA \cong A\). □

3. Trivial extensions of Frobenius algebras

We recall that a finite dimensional \(k\)-algebra is Frobenius if there exists a linear form \(\varphi : A \to k\) such that the map \(A \to DA\), defined by \(x \mapsto x\varphi\) is a left \(A\)-module isomorphism. This linear form \(\varphi : A \to k\) is called a Frobenius homomorphism. It is well known that this is equivalent to say that the map \(x \mapsto \varphi x\), from \(A\) to \(DA\), is an isomorphism of right \(A\)-modules. From this follows easily that there exists an automorphism \(\rho \in A\), called the Nakayama automorphism of \(A\) with respect to \(\varphi\), such that \(x\varphi = \varphi \rho(x)\), for all \(x \in A\). It is easy to check that a linear form \(\widetilde{\varphi} : A \to k\) is another Frobenius homomorphism if and only if there exists \(x \in A\) invertible, such that \(\widetilde{\varphi} = x\varphi\). Also it is easy to check that the Nakayama automorphism of \(A\) with respect to \(\widetilde{\varphi}\) is the map given by \(a \mapsto \rho(x)^{-1}\rho(a)\rho(x)\).

Given algebra maps \(f\) and \(g\), we let \(A_{ij}^f\) denote \(A\) endowed with the \(A\)-bimodule structure given by \(a \cdot x \cdot b := f(a)gx(b)\). To simplify notations we write \(A_f\) instead of \(A_{ij}^f\) and \(A^g\) instead of \(A_{id}^g\). We have the \(A\)-bimodule isomorphism \(\Theta : (DA)^{\otimes A} \to A_{i\rho}\), given by \(\Theta(\varphi x_1 \otimes A \cdots \otimes A \varphi x_p) = \rho^{p-1}(x_1)p^{p-2}(x_2) \cdots \rho(x_{p-1})x_p\). Let
\[
A_{i\rho} \leftarrow A \otimes A_{i\rho} \leftarrow A_{i\rho} \otimes A_{i\rho} \leftarrow A_{i\rho} \otimes A_{i\rho} \leftarrow A_{i\rho} \otimes A_{i\rho} \leftarrow \cdots
\]
be the bar resolution of \(A_{i\rho}\).
Proposition 3.1. Let \((A \otimes B_p^* \otimes A, b'_p)\) be as in the proof of Theorem 1.3. The following facts hold:

1) There is a chain map \(\Theta^p_{i+p}: (A \otimes B_p^* \otimes A, b'_p) \to (A^{\otimes s+1} \otimes A_p, b'_s)\), given by

\[
\Theta^p_{i+p}(x_0, x_{n+p+1}) = \begin{cases} x_0, \Theta(x_{n+1} \otimes A \cdots \otimes A x_{n+p}) x_{n+p+1} & \text{if } x_1, \ldots, x_n \in A, \\ 0 & \text{in other case}, \end{cases}
\]

where \(x_0, x_{n+p+1} = x_0 \otimes \cdots \otimes x_{n+p+1}\) and \(x_0 = x_0 \otimes \cdots \otimes x_n\).

2) There is a chain map \(\Psi^p_{i+p}: (A^{\otimes s+1} \otimes A_p, b'_s) \to (A \otimes B_p^* \otimes A, b'_p)\), given by

\[
\Psi^p_{i+p}(x_0 \otimes \cdots \otimes x_{n+1}) = \sum_{0 \leq i_1 \leq \cdots \leq i_p \leq n} (-1)^{0 \leq i_1 < \cdots < i_i \leq n} x_0 \otimes \cdots \otimes x_i, \otimes \rho(x_i) \otimes \varphi \otimes \rho(x_{i+1}) \otimes \cdots \otimes \rho(x_{n+1}) \otimes \varphi \otimes \cdots \\
\otimes \rho^{p-1}(x_{i+1}) \otimes \cdots \otimes \rho^{p-1}(x_{n+p}) \otimes \varphi \otimes \rho^{p}(x_{n+p+1}) \otimes \cdots \otimes \rho^{p}(x_{n+1}) \otimes x_{n+1}.
\]

3) \(\mu \circ \Theta^p = \Theta \circ \mu^p\) and \(\Theta \circ \mu^p \circ \Theta^p = \mu\), where \(\mu^p: A \otimes B_p^* \otimes A \to (DA)^{\otimes A_p}\) is the map introduced in the proof of Theorem 1.3.

Proof. We left items 1) and 3) to the reader. Let us see 2). For \(0 \leq i_1 < \cdots < i_p \leq n\) we write

\[
T_{i_1, \ldots, i_p} = (-1)^{i_1 + \cdots + i_p + p} x_0 \otimes \cdots \otimes x_{i_1} \otimes \varphi \otimes \rho(x_{i_1+1}) \otimes \cdots \otimes \rho(x_{i_2}) \otimes \varphi \otimes \cdots \\
\otimes \varphi \otimes \rho^{p-1}(x_{i_{p-1}+1}) \otimes \cdots \otimes \rho^{p-1}(x_{i_p}) \otimes \varphi \otimes \rho^{p}(x_{i_p+1}) \otimes \cdots \otimes \rho^{p}(x_{n+1}) \otimes x_{n+1}.
\]

The term of \(b'_n \circ \Psi^p_{i+p}(x_0 \otimes \cdots \otimes x_{n+1})\) obtained multiplying \(\rho^{j-1}(x_{i_j})\) by \(\varphi\) in \(T_{i_1, \ldots, i_p}\) (with \(i_j > 0\)) cancels with the term of \(b'_n \circ \Psi^p_{i+p}(x_0 \otimes \cdots \otimes x_{n+1})\) obtained multiplying \(\varphi\) by \(\rho^j(x_{i_j})\) in \(T_{i_1, \ldots, i_{j-1}, i_j-1, i_{j+1}, \ldots, i_p}\). Using this fact it is easy to see that \(b'_n \circ \Psi^p_{i+p}(x_0 \otimes \cdots \otimes x_{n+1}) = \Psi^p_{i+p} \circ b'_n(x_0 \otimes \cdots \otimes x_{n+1})\).

\[\square\]

Theorem 3.2. Let \(p \geq 1\). Assume that \(A\) is a Frobenius algebra. Then, for the trivial extension \(TA\), the double complex \(X^{(p),*}_{(p)}\), introduced below Theorem 1.2, have the same homology groups that the complex

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
\delta^0_{p+2} & \delta^1_{p+1} & \delta^2_{p+2} \\
\Hom_k(A^{\otimes 2}, A) & \to & \Hom_k(A^{\otimes 2}, A) \\
\delta^0_p & \delta^1_{p+1} & \delta^2_p \\
\Hom_k(A, A) & \to & \Hom_k(A, A) \\
\delta^0_p & \delta^1_p & \delta^2_p \\
\Hom_k(k, A) & \to & \Hom_k(k, A) \\
row p - 1 & \to & \Hom_k(k, A) \\
\uparrow & & \uparrow \\
\text{column } 0 & & \text{column } 0
\end{array}
\]
Proof. Let $\Theta_p^{0,*}$ and $\Psi_p^{1,*}$ be as in Proposition 3.1. From Theorem 1.3 and the fact that the bar resolutions of $A_{p^{p-1}}$ and $A_{p^p}$ are $A^p$-projective resolutions, it follows that the maps

$$\Theta_p^{0,*} = \text{Hom}_{A^p}(\Theta_p^{p-1}, A) \quad \text{and} \quad \Psi_p^{1,*} = \text{Hom}_{A^p}(\Psi_p^{*}, DA)$$

are quasi-isomorphism from $\text{Hom}_{A^p}((A^{o^{p-2}} \otimes A_{p^{p-1}}^e, b'_e), A)$ to $X_{(p)}^{0,*}$ and from $X_{(p)}^{1,*}$ to $\text{Hom}_{A^p}((A^{o^{p-2}} \otimes A_{p^p}^e, b'_e), DA)$ respectively. Let

$$\Xi^{0,*}_{(p)} : Y^{0,*}_{(p)} \rightarrow \text{Hom}_{A^p}((A^{o^{p-2}} \otimes A_{p^{p-1}}^e, b'_e), A)$$

and

$$\Upsilon^{1,*}_{(p)} : Y^{1,*}_{(p)} \rightarrow \text{Hom}_{A^p}((A^{o^{p-2}} \otimes A_{p^p}^e, b'_e), DA),$$

be the chain complex isomorphisms defined by

$$\Xi^{0,n}_{(p)}(f)(a_0 \otimes \cdots \otimes a_{n-p+2}) = a_0 f(a_1 \otimes \cdots \otimes a_{n-p+1}) a_{n-p+2}$$

and

$$\Upsilon^{1,n}_{(p)}(f)(a_0 \otimes \cdots \otimes a_{n-p+2}) = a_0 \varphi(f(a_1 \otimes \cdots \otimes a_{n-p+1})) a_{n-p+2}.$$
**Proposition 3.4.** Assume that $A$ is a Frobenius algebra. Then, for the trivial extension $TA$, it is hold that 

$$H^n(Y_{(1)}^*) = H^n(X_{(1)}^*) = \begin{cases} HH^n(A) & \text{if } n = 0, \\ HH^n(A) \oplus HH^{n-1}(A) & \text{if } n > 0. \end{cases}$$

**Proof.** By Theorems 3.2, 1.3 and the proof of Theorem 2.2, we get

$$H^n(Y_{(1)}^*) = H^n(X_{(1)}^*) = H^n(X_{(1)}^0) \oplus H^{n-1}(X_{(1)}^1) = HH^n(A) \oplus H^{n-1}(X_{(1)}^{1,*}).$$

Moreover, by the proof of Theorem 3.2, we know that $Y_{(1)}^{1,*} \simeq X_{(1)}^{1,*}$. Now, the result follows from Remark 3.3. \hfill \Box

**Corollary 3.5.** Assume that $A$ is a Frobenius algebra. Then, for all $n \geq 1$, we have

$$HH^n(TA) = HH_n(A)^* \oplus HH^n(A) \oplus HH^{n-1}(A) \oplus \text{Cyc}_{A}^{n+1}(DA) \oplus \bigoplus_{p=2}^{n} H^n(Y_{(p)}^*).$$

**Proof.** It follows immediately from Theorem 1.1, Lemma 2.3, Proposition 3.4, Theorem 3.2 and the fact that $H^n(A, DA) = HH_n(A)^*$. \hfill \Box

**Remark 3.6.** Let $A$ be a Frobenius algebra. By Lemma 2.3 and Theorem 3.2,

$$\text{Cyc}_{A}^n(DA) = H^n(X_{(n+1)}^*) = H^n(Y_{(n+1)}^*),$$

$$= \{ x \in A : \rho(x) = (-1)^n x \text{ and } ax = x\rho^n(a) \text{ for all } a \in A \}. $$

**Definition 3.7.** Let $A$ be a Frobenius $k$-algebra, $\varphi: A \to k$ a Frobenius homomorphism and $\rho: A \to A$ the Nakayama automorphism with respect to $\varphi$. We say that $A$ has order $m \in \mathbb{N}$ and we write $\text{ord}_A = m$ if $\rho^m = id_A$ and $\rho^r \neq id_A$ for all $r < m$. In this case we also write

$$e_A = \begin{cases} \text{ord}_A & \text{if } \text{ord}_A \text{ is even}, \\ \text{ord}_A & \text{if } \text{ord}_A \text{ is odd}. \end{cases}$$

If $\rho^m \neq id_A$ for all $m \in \mathbb{N}$, then we say that $A$ has infinite order. By the discussion at the beginning of this section these definitions are independent of $\varphi$.

From now on $A$ is a Frobenius algebra of finite order.

**Theorem 3.8.** For each $n \geq 1$ we write $n = qe_A + s$, with $0 \leq s < e_A$. Then,

$$HH^n(TA) = HH_n(A)^* \oplus \bigoplus_{i=0}^{q} (HH^{n-ie_A}(A) \oplus HH^{n-ie_A-1}(A))$$

$$\oplus \bigoplus_{j=2}^{e_A} \bigoplus_{i=0}^{q-1} H^{n-ie_A}(Y_{(j)}^*) \oplus \bigoplus_{j=2}^{s+1} H^{n-1e_A}(Y_{(j)}^*),$$

where $HH^{-1}(A) = 0$.

**Proof.** Since $\rho^{e_A} = id$ and $e_A$ is even, we have $Y_{(p)}^{r,*} = Y_{(p-e_A)}^{r,*}$ for all $p \geq e_A + 1$. Hence, if $1 \leq r \leq e_A$ is such that $p \equiv r \pmod{e_A}$, then $H^n(Y_{(p)}^*) = H^{n-p+r}(Y_{(r)}^*)$.
for all $p \geq 1$. By Theorem 1.1, the fact that $H^n(A, DA) = HH_n(A)^*$, Theorem 3.2 and the above equality, we have

$$HH^p(TA) = HH_n(A)^* \oplus \bigoplus_{p=1}^{n+1} H^n(Y^*_p)$$

$$= HH_n(A)^* \oplus \bigoplus_{j=1}^{s+1} \bigoplus_{i=0}^{q} H^{n-ite}(Y^*_j) \oplus \bigoplus_{j=1}^{s-1} \bigoplus_{i=0}^{q-1} H^{n-ite}(Y^*_j)$$

$$= HH_n(A)^* \oplus \bigoplus_{j=1}^{s+1} \bigoplus_{i=0}^{q} H^{n-ite}(Y^*_j) \oplus \bigoplus_{j=1}^{s} H^{n-ite}(Y^*_j).$$

The assertion follows immediately from Proposition 3.4. \square

Given an $\mathbb{A}^e$-projective resolution $(X_\ast, \partial_\ast)$ of $A$, let $X'_n = X_n$ endowed with the $A$-bimodule action $a \cdot x \cdot b := \rho(a)x\rho(b)$. Then $(X'_\ast, \partial_\ast)$ is an $\mathbb{A}^e$-resolution of $A^e$. Since $p: A \to A^e$ is an $A$-bimodule map, there is a chain map $\tilde{\rho}_p: (X_\ast, \partial_\ast) \to (X'_\ast, \partial_\ast)$, unique up to chain homotopy equivalence, lifting $\rho$. Let $p \geq 1$. It is easy to check that the map

$$f \mapsto \rho^{-1} \circ f \circ \tilde{\rho}_n \quad n \geq 0 \text{ and } f \in \text{Hom}_{\mathbb{A}^e}(X_n, A^{p_{-1}}),$$

is an automorphism of $\text{Hom}_{\mathbb{A}^e}((X_\ast, \partial_\ast), A^{p_{-1}})$. Taking cohomology we obtain an automorphism $\tilde{\rho}_n^\ast$ of $H^\ast(A, A^{p_{-1}})$, which do not depend of the choose resolution. Clearly $(\tilde{\rho}_n^\ast)^{ord_A}$ is the identity map. Assume that the characteristic of $k$ does not divide $ord_A$ and that $k$ has a primitive $ord_A$-th root of unity $w$. Since $X_1^{ord_A} - 1$ has distinct roots $w^i$ $(0 \leq i < ord_A)$, the cohomology $H^n(A, A^{p_{-1}})$ decomposes as the direct sum

$$H^n(A, A^{p_{-1}}) = \bigoplus_{i=0}^{ord_A - 1} H^n(A, A^{p_{-1}}),$$

where $H^n(A, A^{p_{-1}}) = \{ x \in H^n(A, A^{p_{-1}}) : \tilde{\rho}_n^\ast(x) = w^{-i}x \}$.

**Proposition 3.9.** Under the above hypothesis, we have that for each $n \geq p - 1$,

$$H^n(Y^*_p) = \begin{cases} H^{n-p+1,0}(A) \oplus H^{n-p,0}(A) & \text{if } p \text{ is odd,} \\ H^{n-p+1,0}(A) \oplus H^{n-p,0}(A) & \text{if } \text{char}(k) = 2 \text{ and } p \text{ is even,} \\ H^{n-p+1,\frac{p}{2}}(A) \oplus H^{n-p,\frac{p}{2}}(A) & \text{if } \text{char}(k) \neq 2 \text{ and } p, ord_A \text{ are even,} \\ 0 & \text{in other case,} \end{cases}$$

where $HH_{(p)}^{-1,0}(A) = HH_{(p)}^{-1,\frac{p}{2}}(A) = 0$.

**Proof.** Since the minimal polynomial $X_1^{ord_A} - 1$ of $\rho: A \to A$, has distinct roots $w^i$ $(0 \leq i < ord_A)$, the algebra $A$ becomes a $\frac{2}{ord_A}$-graded algebra

$$A = A_0 \oplus \cdots \oplus A_{ord_A - 1}, \quad \text{where } A_n = \{ a \in A : \rho(a) = w^na \}.$$
For each $0 \leq l < \text{ord}_A$, let $Y_{(p),l}^{*,*}$ be the subcomplex of $Y_{(p)}^{*,*}$ defined by

$$Y_{(p),l}^{*,*} = \bigoplus_{B_{t,n}} \text{Hom}(A_{u_1} \otimes \cdots \otimes A_{u_{n-p+1}}, A_v),$$

where $B_{t,n} = \{(u_1, \ldots, u_{n-p+1}, v) \text{ such that } v - u_1 - \cdots - u_{n-p+1} \equiv l \pmod{\text{ord}_A}\}$. It is clear that $Y_{(p)}^{*,*} = \bigoplus_{l=0}^{\text{ord}_A} Y_{(p),l}^{*,*}$. Let $f \in Y_{(p),l}^{0,n}$. A direct computation shows that

$$\tilde{\delta}_p^{1,n}(f)(x_1 \otimes \cdots \otimes x_{n-p+1}) = (-1)^n+1(1 + (-1)^p w^{-l})f(x_1 \otimes \cdots \otimes x_{n-p+1}).$$

Hence the horizontal boundary maps of $Y_{(p),l}^{*,*}$ are isomorphisms if $w^l \neq (-1)^{p-1}$, and they are zero maps if $w^l = (-1)^{p-1}$. So,

$$H^n(Y_{(p),l}^{*,*}) = \begin{cases} 0 & \text{if } w^l \neq (-1)^{p-1}, \\ H^n(Y_{(p),l}^{0,*}) \oplus H^{n-1}(Y_{(p),l}^{0,*}) & \text{if } w^l = (-1)^{p-1}. \end{cases}$$

Then,

$$H^n(Y_{(p)}^{*,*}) = \begin{cases} H^n(Y_{(p),0}^{0,*}) \oplus H^{n-1}(Y_{(p),0}^{0,*}) & \text{if } p \text{ is odd}, \\ H^n(Y_{(p),0}^{0,*}) \oplus H^{n-1}(Y_{(p),0}^{0,*}) & \text{if } \text{char}(k) = 2 \text{ and } p \text{ is even}, \\ H^n(Y_{(p),\frac{\text{ord}_A}{2}}^{0,*}) \oplus H^{n-1}(Y_{(p),\frac{\text{ord}_A}{2}}^{0,*}) & \text{if } \text{char}(k) \neq 2 \text{ and } p, \text{ord}_A \text{ are even}, \\ 0 & \text{in other case}. \end{cases}$$

The result follows easily from this fact. □

**Theorem 3.10.** Let $A$ be a finite order Frobenius $k$-algebra. Assume that the characteristic of $k$ does not divide $\text{ord}_A$ and that $k$ has a primitive $\text{ord}_A$-th root of unity $w$. For each $n \geq 1$ we write $n = qe_A + s$, with $0 \leq s < e_A$. Then,

1) if $\text{char}(k) \neq 2$ and $\text{ord}_A$ is odd,

$$\text{HH}^n(TA) = \text{HH}_n(A)^* \oplus \bigoplus_{i=0}^{q} \left( \text{HH}^{n-ire_A}(A) \oplus \text{HH}^{n-ire_A-1}(A) \right)$$

$$\oplus \bigoplus_{j=2}^{\text{ord}_A-2} \bigoplus_{i=0}^{q-1} \left( \text{HH}^{n-ire_A-j+1,0}(A) \oplus \text{HH}^{n-ire_A-j,0}(A) \right)$$

$$\oplus \bigoplus_{j=2}^{s+1} \left( \text{HH}^{n-ql_A-j+1,0}(A) \oplus \text{HH}^{n-ql_A-j,0}(A) \right),$$

where $\text{HH}^{-1}(A) = H_{(s+1)}^{-1,0}(A) = 0$. 


2) if char(k) ≠ 2 and ord_A is even,

\[ HH^n(TA) = HH_n(A)^* \oplus \bigoplus_{i=0}^{q} (HH^{n-iA}(A) \oplus HH^{n-iA-1}(A)) \]

\[ \oplus \bigoplus_{j=2}^{s+1} \left( H^{n-iA-j+1,0}_{(j)}(A) \oplus H^{n-iA-j,0}_{(j)}(A) \right) \]

\[ \oplus \bigoplus_{j=2}^{s+1} \left( H^{n-2A-j+1,0}_{(j)}(A) \oplus H^{n-2A-j,0}_{(j)}(A) \right) \]

where \( HH^{-1}(A) = H^{1,0}_{(s+1)}(A) = H^{1,0}_{(s+1)}(A) = 0. \)

3) if the characteristic of k is 2, then

\[ HH^n(TA) = HH_n(A)^* \oplus \bigoplus_{i=0}^{q} (HH^{n-iA}(A) \oplus HH^{n-iA-1}(A)) \]

\[ \oplus \bigoplus_{j=2}^{s+1} \left( H^{n-iA-j+1,0}_{(j)}(A) \oplus H^{n-iA-j,0}_{(j)}(A) \right) \]

where \( HH^{-1}(A) = H^{1,0}_{(s+1)}(A) = 0. \)

Proof. This follows immediately from Theorem 3.8 and Proposition 3.9. □

Remark 3.11. If k does not have a primitive ord_A-th root of unity, we can apply the above theorem, using that \( \overline{k} \otimes HH^*(T(A)) = HH^\overline{k}(T(\overline{k} \otimes A)) \), where \( \overline{k} \) is a suitable extension of k and HH^\overline{k}(T(\overline{k} \otimes A)) denotes the Hochschild cohomology of \( T(\overline{k} \otimes A) = \overline{k} \otimes T(A) \) as a \( \overline{k} \)-algebra.

Remark 3.12. As it is well known, every finite dimensional Hopf algebra A is Frobenius, being a Frobenius homomorphism any right integral \( \varphi \in A^* \setminus \{0\} \). Moreover, by [S, Proposition 3.6], the compositional inverse of the Nakayama map \( \rho \) with respect to \( \varphi \), is given by

\[ \rho^{-1}(h) = \alpha(h_{(1)})S^2(h_{(2)}), \]

where \( \alpha \in A^* \) is the modular element of \( A^* \) and \( S^2 \) is the compositional inverse of \( S \) (note that the automorphism of Nakayama considered in [S] is the compositional
inverse of the considered by us). Using this formula and that \( \alpha \circ S^2 = \alpha \) it is easy to check that \( \rho(h) = \alpha(S(h(1)))S^2(h(2)) \), and more generally, that
\[
\rho^l(h) = \alpha^l(S(h(1)))S^2(h(2)),
\]
where \( \alpha^l \) denotes the \( l \)-fold convolution product of \( \alpha \). Since \( \alpha \) has finite order respect to the convolution product and, by the Radford formula for \( S^4 \) (see [S, Theorem 3.8]), the antipode \( S \) has finite order respect to the composition, \( A \) has finite order. So, Theorem 3.10 applies to finite dimensional Hopf algebras.

**Theorem 3.13.** Let \( A \) be a finite dimensional Hopf algebra. Assume that \( A \) and \( A^* \) are unimodular. It is hold that

1) if \( \text{char}(k) \neq 2 \) and \( \text{ord}_A = 1 \),
\[
\text{HH}^n(TA) = \text{HH}_n(A)^* \oplus \bigoplus_{i=0}^{n} \text{HH}^i(A),
\]
2) if the characteristic of \( k \) is 2 and \( \text{ord}_A = 1 \), then
\[
\text{HH}^n(TA) = \text{HH}_n(A)^* \oplus \bigoplus_{i=0}^{n} \text{HH}^i(A) \oplus \bigoplus_{i=0}^{n-1} \text{HH}^i(A),
\]
3) if \( \text{char}(k) \neq 2 \) and \( \text{ord}_A = 2 \),
\[
\text{HH}^n(TA) = \text{HH}_n(A)^* \oplus \bigoplus_{i=0}^{n} \text{HH}^i(A) \oplus \bigoplus_{i=0}^{n-1} \text{HH}^i(A).
\]

**Proof.** By [S, Corollary 3.20], if \( A \) and \( A^* \) are unimodular, then \( \alpha \) is the counity \( \epsilon \) of \( A \) and \( S^4 = \text{id} \). Hence, in this case, \( \rho = S^2 \) and \( \rho^2 = \text{id} \). The result follows immediately from this fact and Theorem 3.10. \( \square \)

**Corollary 3.14.** Let \( A \) be a finite dimensional Hopf algebra. If \( A \) is semisimple and \( \text{char}(k) = 0 \), then
\[
\text{HH}^n(TA) = \begin{cases} \text{HH}_0(A)^* \oplus \text{HH}^0(A) & \text{if } n = 0, \\ \text{HH}^0(A) & \text{if } n > 0. \end{cases}
\]

**Proof.** Since for each semisimple finite dimensional Hopf algebra over a characteristic zero field \( A \) it is hold that \( A \) and \( A^* \) are unimodular Hopf algebra, the antipode \( S \) of \( A \) is involutive, the result follows immediately from item 1) of Theorem 3.13 and the fact that and \( A \) is separable. \( \square \)

**Theorem 3.15.** Let \( A \) be a finite order Frobenius algebra. Assume that the characteristic of \( k \) does not divide \( \text{ord}_A \) and that \( k \) has a primitive \( \text{ord}_A \)-th root of unity \( w \). Then \( \text{HH}^n(A) = \text{HH}^{n,0}_{(1)}(A) \) for all \( n \geq 0 \).

**Proof.** By Propositions 3.4 and 3.9
\[
H^n(Y^*_{(1)}) = \begin{cases} \text{HH}^n(A) & \text{if } n = 0, \\ \text{HH}^n(A) \oplus \text{HH}^{n-1}(A) & \text{if } n > 0. \end{cases}
\]
\[
= \begin{cases} \text{HH}^{n,0}_{(1)}(A) & \text{if } n = 0, \\ \text{HH}^{n,0}_{(1)}(A) \oplus \text{HH}^{n-1,0}_{(1)}(A) & \text{if } n > 0. \end{cases}
\]
From this follows easily that \( \text{HH}^n(A) = \text{HH}^{n,0}_{(1)}(A) \) for all \( n \geq 0 \), as desired. \( \square \)
Example 3.16. Let $k$ a field and $N$ a natural number. Assume that $k$ has a primitive $N$-th root of unity $w$. Let $A$ be the Taft algebra of order $N$. That is, $A$ is the algebra generated over $k$ by two elements $g$ and $x$ subject to the relations $g^N = 1$, $x^N = 0$ and $xg = wxg$. The Taft algebra $A$ is a Hopf algebra with comultiplication $\Delta$, counity $\epsilon$ and antipode $S$ given by

$$
\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g,
$$

$$
\epsilon(g) = 1, \quad \epsilon(x) = 0,
$$

$$
S(g) = g^{-1}, \quad S(x) = -xg^{-1}.
$$

Using that $t = \sum_{j=0}^{N-1} w^j g^j x^{N-1}$ is a right integral of $A$, it is easy to see that the modular element $\alpha \in A^*$ verifies $\alpha(g) = w^{-1}$ and $\alpha(x) = 0$. By Remark 3.14 the Nakayama map $\rho: A \rightarrow A$ is given by $\rho(g) = wg$ and $\rho(x) = w^{-1}x$. Hence, $A = A_0 \oplus \cdots \oplus A_{N-1}$, where

$$
A_i = \{ a \in A : \rho(a) = w^{-i}a \}
$$

$$
= \langle x^i, x^{i+1}g, \ldots, x^{N-1}g^{N-i-1}, g^{N-i}, xg^{N-i+1}, \ldots, x^{i-1}g^{N-1} \rangle.
$$

Let $C_N = \{1, t, \ldots, t^{N-1} \}$ be the cyclic of order $N$. It is easy to see that $C_N$ acts on $A_0$ via $t \cdot x^j g^i = w^{-j}x^j g^i$ and $A$ is isomorphic to the skew product of $A_0 \# C_N$. By Theorem 3.15 we have

$$
HH^n(A) = HH^n_{(1)}(A) = H^n(Y_{(1),0}^{0,*}),
$$

where $Y_{(1),0}^{0,*}$ is the complex introduced in the proof of Theorem 3.9. Let us consider the action of $C_N$ on $HH^q(A_0)$ is induced by the action of $C_N$ on $\text{Hom}_k(A_0^0, A_0)$ given by

$$
t \cdot \varphi(x^{i_1}g^{i_1} \otimes \cdots \otimes x^{i_q}g^{i_q}) = g^{N-1} \varphi(t \cdot x^{i_1}g^{i_1} \otimes \cdots \otimes t \cdot x^{i_q}g^{i_q})g.
$$

By [G-G2, 3.2.7] there is a converging spectral sequence

$$
E^2_{pq} = H^p(C_N, HH^q(A_0)) \Rightarrow H^{p+q}(Y_{(1),0}^{0,*}).
$$

Now, $N$ is invertible in $k$, since $k$ has a primitive $N$-th root of unity. Thus, the above spectral sequence collapses and

$$
HH^n(A) = H^0(C_N, HH^n(A_0)) = HH^n(A_0)^{CN}.
$$

References

[C] C. Cibils, Tensor Hochschild homology and cohomology, “Interaction between Ring Theory and Representations of Algebras.” Lecture Notes in Applied Mathematics 210 (2000), Dekker, New York, 35–51.

[C-M-R-S] C. Cibils, E. Marcos, M. J. Redondo and A. Solotar, The Cohomology of split algebras and of trivial extensions, Preprint (2001).

[G] M. Gerstenhaber, On the deformations of rings and algebras, Ann. of Math. 79 (1964), 59–103.
M. Gerstenhaber and S. D. Schack, *Relative Hochschild cohomology, rigid algebras and the Bockstein*, Journal of Pure and Applied Algebra 43 (1986), 53–74.

E. Green, E. Marcos and N. Snashall, *The Hochschild cohomology ring of a one point extension*, Preprint (2001).

J.A. Guccione and J. J. Guccione, *Cohomology of triangular matrix algebras*, Preprint 2001.

J.A. Guccione and J.J. Guccione, *Hochschild (co)homology of a Hopf crossed products*, To appear in K-theory.

D. Happel, *Hochschild cohomology of finite dimensional algebras*, “Séminaire M.-P. Malliavin, Paris, 1987-1988” Lecture Notes in Mathematics 1404 (1989), Springer-Verlag, Berlin Heidelberg New York, 108–112.

D. Happel, *Hochschild cohomology of Auslander algebras*, Topics in algebra, Part 1 (1988), Warsaw, 303–310; Banach Center Publ. 26, Part 1, PWN (1990), Warsaw.

R. Martínez-Villa and J. A. de la Peña, *The universal cover of a quiver with relations*, Journal of Pure and Applied Algebra 30 (1983), 277–292.

S. Michelena and M. I. Platzeck, *Hochschild cohomology of triangular matrix algebras*, Journal of Algebra 233 (2000), 502–525.

H. J. Schneider, *Lectures on Hopf Algebras* (1994).

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