Improved Bounds for Uniform Hypergraphs without Property B

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Abstract

A hypergraph is said to be properly 2-colorable if there exists a 2-coloring of its vertices such that no hyperedge is monochromatic. On the other hand, a hypergraph is called non-2-colorable if there exists at least one monochromatic hyperedge in each of the possible 2-colorings of its vertex set. Let \( m(n) \) denote the minimum number of hyperedges in a non-2-colorable \( n \)-uniform hypergraph. Establishing the lower and upper bounds on \( m(n) \) is a well-studied research direction over several decades. In this paper, we present new constructions for non-2-colorable \( n \)-uniform hypergraphs. These constructions improve the upper bounds for \( m(8), m(13), m(14), m(16) \) and \( m(17) \). We also improve the lower bound for \( m(5) \).

Keywords: Property B; Uniform Hypergraphs; Hypergraph 2-coloring

1 Introduction

Hypergraphs are combinatorial structures that are generalizations of graphs. Let \( H = (V, E) \) be an \( n \)-uniform hypergraph with vertex set \( V \), with each hyperedge in \( E \) having exactly \( n \) vertices in it. A 2-coloring of \( H \) is an assignment of one of the two colors red and blue to each of the vertices in \( V \). We say a 2-coloring of \( H \) to be proper if each of its hyperedges has red as well as blue vertices. \( H \) is said to be non-2-colorable if no proper 2-coloring exists for it; otherwise, it is said to satisfy Property B. For an integer \( n \geq 1 \), let \( m(n) \) denote the minimum number of hyperedges present in a non-2-colorable \( n \)-uniform hypergraph.

Establishing an upper bound on \( m(n) \) is a well-explored combinatorial problem. Erdős [6] gave a non-constructive proof to establish the currently best-known upper bound \( m(n) = O(n^22^n) \). However, there is no known construction for a non-2-colorable \( n \)-uniform hypergraph that matches this upper bound. Abbott and Moser [2] constructed a non-2-colorable \( n \)-uniform hypergraph with \( O((\sqrt{7} + o(1))^n) \) hyperedges. Recently, Gebauer [8] improved this result by constructing a non-2-colorable \( n \)-uniform hypergraph with \( O(2^{(1+o(1))n}) \) hyperedges. Even though this is the best construction known for a non-2-colorable \( n \)-uniform hypergraph for large \( n \), it is still asymptotically far from the above-mentioned non-constructive upper bound given by Erdős.

Finding upper bounds for small values of \( n \) is also a well-studied problem and several constructions have been given for establishing these. For example, it can be easily seen that \( m(1) \leq 1 \), \( m(2) \leq 3 \) (the corresponding 2-uniform hypergraph is the triangle graph) and \( m(3) \leq 7 \) (the corresponding 3-uniform hypergraph is known as the Fano plane [10], denoted by \( H_f \) in this paper).

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The previously-mentioned construction of Abbott and Moser shows that $m(4) \leq 27$, $m(6) \leq 147$ and $m(8) \leq 2187$. Moreover, their construction also gives non-trivial upper bounds on $m(n)$ for $n = 9, 10, 12, 14, 15$ and 16. For $n \geq 3$, Abbott and Hanson \(^1\) gave a construction using a non-2-colorable $(n - 2)$-uniform hypergraph to show that $m(n) \leq n \cdot m(n - 2) + 2^{n-1} + 2^{n-2}((n-1) \mod 2)$. Using the best-known upper bounds on $m(n - 2)$, this recurrence relation establishes non-trivial upper bounds as well as improve such bounds on $m(n)$ for a few small values of $n$. For example, it shows that $m(4) \leq 24$, $m(5) \leq 51$ and $m(7) \leq 421$. Seymour \(^{14}\) further improved the upper bound on $m(4)$ to $m(4) \leq 23$ by constructing a non-2-colorable 4-uniform hypergraph with 23 hyperedges. In this paper, we denote this hypergraph by $H_s$. For even integers $n \geq 4$, Toft \(^{15}\) generalized this construction using a non-2-colorable $(n - 2)$-uniform hypergraph to improve Abbott and Hanson’s result to $m(n) \leq n \cdot m(n - 2) + 2^{n-1} + \binom{n}{2}/2$. In particular, this led to establishing an upper bound $m(8) \leq 1339$. For a given integer $n \geq 3$ and a non-2-colorable $(n - 2)$-uniform hypergraph $A$, we refer to Abbott-Hanson’s construction for odd $n$ and Toft’s construction for even $n$ as Abbott-Hanson-Toft construction and denote the number of hyperedges in such a hypergraph as $m_A(n)$. It can be easily observed that $m(n) \leq m_A(n)$ for any non-2-colorable $(n - 2)$-uniform hypergraph $A$. In fact, we have already seen that the above-mentioned upper bounds $m(4) \leq 23$, $m(5) \leq 51$, $m(7) \leq 421$ and $m(8) \leq 1339$ are obtained by Abbott-Hanson-Toft constructions using the best-known constructions for non-2-colorable 2, 3, 5 and 6-uniform hypergraphs, respectively. Recently, a construction given by Mathews et al. \(^{11}\) improved the upper bound on $m(8)$ to $m(8) \leq 1269$. In addition, they modified the Abbott-Hanson-Toft construction to improve the upper bounds on $m(n)$ for $n = 11, 13$ and 17. The currently best-known upper bounds on $m(n)$ for $n \leq 17$ are given in Table \(^{1}\).

In the other direction, Erdős \(^6\) showed the lower bound on $m(n)$ to be $m(n) = \Omega(2^n)$, which was later improved by Beck \(^3\) to $m(n) = \Omega(n^{1/3-o(1)}2^n)$. The currently best-known lower bound $m(n) = \Omega(\sqrt{\ln n}2^n)$ was given by Radhakrishnan and Srinivasan \(^{13}\). Recently, a simpler proof for the same result has been given by Cherkashin and Kozik \(^4\). Note that there is a significant

| $n$ | $m(n)$ | Corresponding construction/recurrence relation |
|-----|--------|---------------------------------------------|
| 1   | $m(1) = 1$ | Single Vertex |
| 2   | $m(2) = 3$ | Triangle Graph |
| 3   | $m(3) = 7$ | Fano Plane \(^{10}\), \(^{12}\), \(^{14}\) |
| 4   | $m(4) = 23$ | \[^{12}\], \[^{14}\] |
| 5   | $m(5) \leq 51$ | $m(5) \leq 2^4 + 5m(3)$ |
| 6   | $m(6) \leq 147$ | $m(6) \leq m(2)m(3)^2$ |
| 7   | $m(7) \leq 421$ | $m(7) \leq 2^6 + 7m(5)$ |
| 8   | $m(8) \leq 1269$ | \[^{11}\] |
| 9   | $m(9) \leq 2401$ | $m(9) \leq m(3)^4$ |
| 10  | $m(10) \leq 7803$ | $m(10) \leq m(2)m(5)^2$ |
| 11  | $m(11) \leq 25449$ | $m(11) \leq 15 \cdot 2^8 + 9m(9)$ |
| 12  | $m(12) \leq 55223$ | $m(12) \leq m(3)^4m(4)$ |
| 13  | $m(13) \leq 297347$ | $m(13) \leq 17 \cdot 2^{10} + 11m(11)$ |
| 14  | $m(14) \leq 531723$ | $m(14) \leq m(2)m(7)^2$ |
| 15  | $m(15) \leq 857157$ | $m(15) \leq m(3)^5m(5)$ |
| 16  | $m(16) \leq 4831083$ | $m(16) \leq m(2)m(8)^2$ |
| 17  | $m(17) \leq 13201419$ | $m(17) \leq 21 \cdot 2^{14} + 15m(15)$ |

Table 1: Best-known upper bounds on $m(n)$ for small values of $n$
asymptotic gap between the currently best-known lower and upper bounds on \( m(n) \). Even for small values of \( n \), we are only aware of a few lower bounds for \( m(n) \) that match the corresponding upper bounds. It can be easily seen that \( m(1) \geq 1, m(2) \geq 3 \) and \( m(3) \geq 7 \) and therefore \( m(1) = 1, m(2) = 3 \) and \( m(3) = 7 \). Recently, Östergård [12] showed that \( m(4) \geq 23 \) and established \( m(4) = 23 \) as a result. The exact values of \( m(n) \) are not yet known for \( n \geq 5 \), even though it can be easily observed that \( m(n+1) \geq m(n) \) for any \( n \geq 1 \).

1.1 Our Contributions

In this paper, we give constructions that improve the best-known upper bounds on \( m(8), m(13), m(14), m(16) \) and \( m(17) \). We also establish a non-trivial lower bound on \( m(5) \).

In Section 3, we give a construction that gives the following recurrence relation. In particular, it improves the upper bound on \( m(13) \).

**Result 1.** Consider an integer \( k \geq 1 \). For an odd \( n > 2k \), \( m(n) \leq \binom{n+k-1}{k} m(n-2k) + \binom{n+k-1}{k-1} 2^{n-1} \).

For an even \( n > 2k \), \( m(n) \leq \binom{n+k-1}{k} m(n-2k) + \binom{n+k-1}{k-1} (2^{n-1} + \binom{n}{n/2})/2 \).

This construction also gives a non-2-colorable \( n \)-uniform hypergraph with \( O(3.76^{n}) \) hyperedges. Even though we note that it gives a better constructive upper bound \( m(n) = O(3.76^{n}) \) than the trivial bound \( m(n) \leq \binom{2n-1}{n} = \Theta(4^n/\sqrt{n}) \), it is asymptotically worse than the previously mentioned constructive upper bounds \( m(n) = O((\sqrt{7} + o(1))^n) \) [2] and \( m(n) = O(2^{(1+o(1))n}) \) [8].

In Section 4, we provide another construction that improves the upper bounds on \( m(8), m(13), m(14), m(16) \) and \( m(17) \).

**Result 2.** Consider an integer \( k \) satisfying \( 0 < k < n \). Let \( w = \lfloor n/k \rfloor \), \( x = n \mod k \), \( y = \lfloor k/x \rfloor \) and \( z = k \mod x \).

(a) If \( x > 0 \) and \( z > 0 \), \( m(n) \leq w \cdot m(n-k)m(k) + y \cdot m(k)w m(x) + \binom{x+z-1}{x} m(n-k)m(x)^y + \binom{x+z-1}{x} m(k)^w \).

(b) If \( x > 0 \) and \( z = 0 \), \( m(n) \leq w \cdot m(n-k)m(k) + y \cdot m(k)^w m(x) + m(n-k)m(x)^y \).

(c) If \( x = 0 \), \( m(n) \leq w \cdot m(n-k)m(k) + m(k)^w \).

In Section 4 we give a construction to prove the following result that further improves the upper bounds on \( m(13) \) and \( m(16) \).

**Result 3.** Consider an integer \( k \geq 2 \) and a non-2-colorable \((k-1)\)-uniform hypergraph \( H_{1c} \). Then, \( m(3k+1) \leq (m(k-1) + 2^{k-1})m(k+1)^2 + 2m_{H_{1c}}(k+1)m(k)^2 + 4m(k+1)m(k)^2 \).

In Section 6 we improve the currently best-known lower bound \( m(5) \geq 28 \).

**Result 4.** \( m(5) \geq 29 \).

2 Previous Results

2.1 Abbott-Moser Construction

Abbott and Moser [2] gave the construction for a non-2-colorable \( n \)-uniform hypergraph \( H = (V,E) \) by exploiting the known constructions of non-2-colorable \( a \)-uniform and \( b \)-uniform hypergraphs for
any composite \( n = ab \) for two integers \( a \geq 1, b \geq 1 \). Let \( H_a = (V_a, E_a) \) and \( H_b = (V_b, E_b) \) be non-2-colorable \( a \)-uniform and \( b \)-uniform hypergraphs, respectively. \( H \) is constructed using \( |V_a| \) identical copies of \( H_a \) by replacing each vertex of \( H_a \) with a copy of \( H_b \). Let us denote the copies of \( H_b \) as \( H_{b_1} = (V_{b_1}, E_{b_1}), H_{b_2} = (V_{b_2}, E_{b_2}), \ldots, H_{b_{|V_a|}} = (V_{b_{|V_a|}}, E_{b_{|V_a|}}) \). The vertex set of \( H \) is \( V = V_{b_1} \cup V_{b_2} \cup \cdots \cup V_{b_{|V_a|}} \). The hyperedge set of \( H \) is constructed as follows. For each hyperedge \( \{v_1, \ldots, v_n\} \) in \( E_a \), the following collection of hyperedges \( \{e_1 \cup \cdots \cup e_m : e_1 \in E_{b_{a_1}}, \ldots, e_m \in E_{b_{a_m}} \} \) is added to \( E \). The resulting hypergraph \( H \) is \( n \)-uniform and it is evident from the construction that it has \( |E_a||E_b|^n \) hyperedges. Abbott and Moser \[2\] showed that \( H \) is non-2-colorable, thereby proving the following result.

**Lemma 1.** For any composite \( n = ab \) for integers \( a, b \geq 1 \), \( m(n) \leq (m(a)m(b))^a \).

This construction gives the best-known upper bounds for some small values of \( n \). For example, it shows that \( m(6) \leq 147, m(9) \leq 2401, m(10) \leq 7803, m(12) \leq 55223, m(14) \leq 531723, m(15) \leq 857157 \) and \( m(16) \leq 4831083 \).

### 2.2 Abbott-Hanson-Toft Construction

As mentioned in the introduction, Abbott-Hanson’s construction \[1\] for odd \( n \) along with Toft’s construction \[15\] for even \( n \) is referred to as Abbott-Hanson-Toft construction. For a given \( n \geq 3 \), this construction is built using a non-2-colorable \((n-2)\)-uniform hypergraph, which we call as the core hypergraph and denote by \( H_c = (V_c, E_c) \). Let its hyperedge set be \( E_c = \{e_1, e_2, \ldots, e_{m_c}\} \).

Let \( A \) and \( B \) be two disjoint sets of vertices where \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_n\} \), each disjoint with \( V_c \). For a given \( K \subset \{1, 2, \ldots, n\} \), we define \( A_K = \bigcup_{i \in K} \{a_i\} \), \( B_K = \bigcup_{i \in K} \{b_i\} \), \( \overline{A}_K = A \setminus A_K \) and \( \overline{B}_K = B \setminus B_K \)

The construction of the non-2-colorable \( n \)-uniform hypergraph \( H = (V, E) \) is as follows. The vertex set is \( V = V_c \cup A \cup B \) and the hyperedge set \( E \) consists of the following hyperedges:

(i) \( e_i \cup \{a_j\} \cup \{b_j\} \) for every pair \( i, j \) satisfying \( 1 \leq i \leq m_c \) and \( 1 \leq j \leq n \)

(ii) \( A_K \cup \overline{B}_K \) for each \( K \) such that \( |K| \) is odd and \( 1 \leq |K| \leq \lfloor n/2 \rfloor \)

(iii) \( \overline{A}_K \cup B_K \) for each \( K \) such that \( |K| \) is even and \( 2 \leq |K| \leq \lfloor n/2 \rfloor \)

(iv) \( A \)

It is easy to observe that the number of hyperedges in \( H \) is \( 2^{n-1} + nm_c \) for odd \( n \) and \( 2^{n-1} + nm_c + \binom{n}{2}/2 \) for even \( n \). Abbott-Hanson \[1\] and Toft \[15\] proved that \( H \) is non-2-colorable, and the construction gives the upper bound on \( m(n) \) as follows.

**Lemma 2.**

\[
m(n) \leq \begin{cases} 
2^{n-1} + n \cdot m(n-2) & \text{if } n \text{ is odd} \\
2^{n-1} + n \cdot m(n-2) + \binom{n}{2}/2 & \text{if } n \text{ is even}
\end{cases}
\]

Lemma 2 gives the best-known upper bounds on \( m(n) \) for \( n = 5 \) and \( 7 \) as \( m(5) \leq 51 \) and \( m(7) \leq 421 \), respectively.

*Note that the notations used in a sub-section are not related to the notations used in other sub-sections, unless specified otherwise.*
2.3 Mathews-Panda-Shannigrahi Construction

The following construction of a non-2-colorable $n$-uniform hypergraph for $n \geq 3$ is an improvement over the Abbott-Hanson-Toft construction mentioned above. Similar to the Abbott-Hanson-Toft construction, this construction also utilizes a non-2-colorable $(n-2)$-uniform hypergraph $H_c = (V_c, E_c)$ that is called the core hypergraph in Section 2.2. Let $E_c = \{e_1, e_2, \ldots, e_{m_c}\}$. In addition, this construction uses two disjoint vertex sets $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, each disjoint from $V_c$. Let $B^1$ denote the ordered set $B^1 = (b_1, b_2, \ldots, b_n)$, where the ordering is defined as $b_1 < b_2 < \ldots < b_n$. For any $1 \leq p \leq n$, let $B^p$ denote the ordered set where $b_1$ and $b_p$ are swapped in the ordering. Let the ordered set $B^p = (b_p, b_2, \ldots, b_{p-1}, b_1, b_{p+1}, \ldots, b_n)$ be denoted by $(w_1^p, w_2^p, \ldots, w_n^p)$, where the ordering is given as $w_1^p < w_2^p < \ldots < w_n^p$. For $K \subset \{1, 2, \ldots, n\}$, let $A_K = \bigcup_{i \in K} \{a_i\}$, $\overline{A}_K = A \setminus A_K$, $B^p_K = \bigcup_{i \in K} \{w_i^p\}$ and $\overline{B}^p_K = B \setminus B^p_K$.

The construction of the non-2-colorable $n$-uniform hypergraph $H = (V, E)$ is defined as follows. The vertex set is $V = V_c \cup A \cup B$ and the hyperedge set $E$ consists of following hyperedges:

(i) $e_i \cup \{a_j\} \cup \{b_j\}$ for every pair $i, j$ satisfying $1 \leq i \leq m_c$ and $2 \leq j \leq n$

(ii) $A_K \cup \overline{B}^p_K$ for each $p$ satisfying $1 \leq p \leq n$, and each $K$ such that $|K|$ is odd and $1 \leq |K| \leq \lfloor n/2 \rfloor$

(iii) $\overline{A}_K \cup B^p_K$ for each $p$ satisfying $1 \leq p \leq n$, and each $K$ such that $|K|$ is even and $2 \leq |K| \leq \lfloor n/2 \rfloor$

(iv) $A$

It can be seen that the number of hyperedges in $H$ is at most $(n + 1)2^{n-2} + (n - 1)m_c$ when $n$ is odd and $(n + 1)2^{n-2} + (\binom{n}{n/2}/2 + (n - 1)(m_c + (\binom{n-2}{n-2}/2))$ when $n$ is even. Mathews et al. [11] showed that $H$ is non-2-colorable, which gives the following result.

**Lemma 3.**

$$m(n) \leq \begin{cases} (n + 1)2^{n-2} + (n - 1) \cdot m(n - 2) & \text{if } n \text{ is odd} \\ (n + 1)2^{n-2} + (\binom{n}{n/2}/2 + (n - 1)(m(n - 2) + (\binom{n-2}{n-2}/2)) & \text{if } n \text{ is even} \end{cases}$$

This result improved the upper bounds on $m(13)$ and $m(17)$ to $m(13) \leq 357892$ and $m(17) \leq 14304336$, respectively. However, Mathews et al. modified the above construction in the same paper to provide another construction that gives the following result.

**Lemma 4.**

$$m(n) \leq \begin{cases} (n + 4)2^{n-3} + (n - 2) \cdot m(n - 2) & \text{if } n \text{ is odd} \\ (n + 4)2^{n-3} + (n - 2) \cdot m(n - 2) + n(\binom{n-2}{n-2}/2 + (\binom{n}{n/2}/2) & \text{if } n \text{ is even} \end{cases}$$

This construction improved the upper bound on $m(11)$ to $m(11) \leq 25449$ and further improved the above-mentioned upper bounds on $m(13)$ and $m(17)$ to $m(13) \leq 297347$ and $m(17) \leq 13201419$, respectively.

3 Generalized Abbott-Hanson-Toft Construction

For any $k \geq 1$, we construct a non-2-colorable $n$-uniform hypergraph $H = (V, E)$ for an integer $n$ satisfying $n > 2k$. This construction uses a non-2-colorable $(n - 2k)$-uniform hypergraph $H_c = (V_c, E_c)$ with $E_c = \{e_1, e_2, \ldots, e_{m_c}\}$. Consider two disjoint sets of vertices $A = \{a_1, a_2, \ldots, a_{n+k-1}\}$ and $B = \{b_1, b_2, \ldots, b_{n+k-1}\}$, each disjoint with $V_c$. Let us define $I_i$ to be the collection of all
Case 3

We define \( A_I = \bigcup_{i \in I} \{a_i\}, B_I = \bigcup_{i \in I} \{b_i\}, A_{K_I} = \bigcup_{i \in K_I} \{a_i\}, B_{K_I} = \bigcup_{i \in K_I} \{b_i\}, \mathcal{A}_{K_I} = A \setminus (A_{K_I} \cup A_I) \) and \( \mathcal{B}_{K_I} = B \setminus (B_{K_I} \cup B_I) \).

The non-2-colorable \( n \)-uniform hypergraph \( H = (V, E) \) is constructed with the vertex set \( V = V_c \cup A \cup B \). The hyperedge set \( E \) consists of the following hyperedges:

(i) \( e_i \cup A_I \cup B_I \) for each \( I \in \mathcal{I}_k \) and all \( i \) satisfying \( 1 \leq i \leq m_c \)

(ii) \( A_{K_I} \cup \mathcal{B}_{K_I} \) for each \( K_I \) such that \( |K_I| \) is odd and \( 1 \leq |K_I| \leq \lfloor n/2 \rfloor \), for each \( I \in \mathcal{I}_k-1 \)

(iii) \( \mathcal{A}_{K_I} \cup B_{K_I} \) for each \( K_I \) such that \( |K_I| \) is even and \( 0 \leq |K_I| \leq \lfloor n/2 \rfloor \), for each \( I \in \mathcal{I}_k-1 \)

The number of hyperedges in \( H \) is \( (n+k-1)m_c + (n+k-1)2^{n-1} \) when \( n \) is odd and \( (n+k-1)m_c + (n+k-1)(2^{n-1} + (n/2)/2) \) when \( n \) is even.

For a 2-coloring of \( H \), we call \( \{a_i, b_i\} \) to be a matching pair if both the vertices are colored by the same color. For a given \( I \in \mathcal{I}_k-1 \), we define \( A_{blue_I} \) to be the set blue vertices in \( A \setminus A_I \), and \( B_{blue_I} = \{b_i : a_i \in A_{blue_I}\} \). Let \( \mathcal{A}_{blue_I} = A \setminus (A_I \cup A_{blue_I}) \) and \( \mathcal{B}_{blue_I} = \{b_i : a_i \in \mathcal{A}_{blue_I}\} \).

**Lemma 5.** Consider any \( I \in \mathcal{I}_k-1 \). If there is no matching pair of vertices between \( A \setminus A_I \) and \( B \setminus B_I \) in a 2-coloring \( \chi \) of hypergraph \( H \), then there exists at least one monochromatic hyperedge in the coloring \( \chi \).

**Proof.** Assume for the sake of contradiction that \( \chi \) is a proper 2-coloring of hypergraph \( H \) with no matching pair of vertices between \( A \setminus A_I \) and \( B \setminus B_I \). We arrive at a contradiction in each of the cases below.

**Case 1.** \( 1 \leq |A_{blue_I}| \leq \lfloor n/2 \rfloor \)

If \( |A_{blue_I}| \) is odd, the hyperedge \( A_{blue_I} \cup \mathcal{B}_{blue_I} \) is monochromatic in blue.

If \( |A_{blue_I}| \) is even, the hyperedge \( \mathcal{A}_{blue_I} \cup B_{blue_I} \) is monochromatic in red.

**Case 2.** \( \lfloor n/2 \rfloor < |A_{blue_I}| < n \)

If \( n - |A_{blue_I}| \) is odd, the hyperedge \( \mathcal{A}_{blue_I} \cup B_{blue_I} \) is monochromatic in red.

If \( n - |A_{blue_I}| \) is even, the hyperedge \( A_{blue_I} \cup \mathcal{B}_{blue_I} \) is monochromatic in blue.

**Case 3.** \( |A_{blue_I}| = 0 \) or \( |A_{blue_I}| = n \)

If \( |A_{blue_I}| = 0 \), \( A \setminus A_I \) is monochromatic in red. If \( |A_{blue_I}| = n \), \( A \setminus A_I \) is monochromatic in blue.

**Proof of Result.** Let us assume for the sake of contradiction that there exists a proper 2-coloring \( \chi \) for hypergraph \( H \). We know that the core hypergraph \( H_c \) is a non-2-colorable \( (n - 2k) \)-uniform hypergraph and thus has a monochromatic hyperedge in the coloring \( \chi \). Without loss of generality, assume \( H_c \) to be monochromatic in red. The hyperedges added in Step (i) ensure that no more than \( (k - 1) \) matching pairs of red vertices exist in \( \chi \). This implies that there exists an \( I \in \mathcal{I}_k-1 \) such that there is no matching pair of red vertices from \( A' = A \setminus A_I \) and \( B' = B \setminus B_I \). As a result, it follows from Lemma 5 that there exists at least one matching pair of blue vertices from \( A' \) and \( B' \). Let \( \{a_p, b_p\} \) be such a matching pair of blue vertices, where \( a_p \in A' \) and \( b_p \in B' \). This leads to a contradiction in each of the following cases.
Case 1. \(1 \leq \left| A_{\text{blue}} \right| \leq \left\lceil \frac{n}{2} \right\rceil\)

If \(\left| A_{\text{blue}} \right|\) is odd, the hyperedge \(A_{\text{blue}} \cup \overline{B}_{\text{blue}}\) is monochromatic in blue.

If \(\left| A_{\text{blue}} \right|\) is even, the hyperedge \(\overline{B}_{\text{blue}} \cup \{b_p\} \cup A_{\text{blue}} \setminus \{a_p\}\) is monochromatic in blue.

Case 2. \(\left\lceil \frac{n}{2} \right\rceil < \left| A_{\text{blue}} \right| < n\)

If \(n - \left| A_{\text{blue}} \right|\) is odd, \(\overline{B}_{\text{blue}} \cup \{b_p\}\) is even. Therefore, the hyperedge \(\overline{B}_{\text{blue}} \cup \{b_p\} \cup A_{\text{blue}} \setminus \{a_p\}\) is monochromatic in blue.

If \(n - \left| A_{\text{blue}} \right|\) is even, the hyperedge \(A_{\text{blue}} \cup \overline{B}_{\text{blue}}\) is monochromatic in blue.

Case 3. \(\left| A_{\text{blue}} \right| = n\)

If \(\left| A_{\text{blue}} \right| = n\), \(A'\) is monochromatic in blue.

This completes the proof that \(H\) is non-2-colorable. Therefore, we arrive at the following result.

\[
m(n) \leq \begin{cases} 
\binom{n + k - 1}{k} m(n - 2k) + \binom{n + k - 1}{k-1} 2^{n-1} & \text{if } n \text{ is odd} \\
\binom{n + k - 1}{k} m(n - 2k) + \binom{n + k - 1}{k-1} \left(2^{n-1} + \left(\frac{n}{\left\lfloor n/2 \right\rfloor}\right)/2\right) & \text{if } n \text{ is even}
\end{cases}
\]

We set \(k = 2\) in Result 1 and use \(m(9) \leq 2401\) from Table 1 to get an improvement of the upper bound on \(m(13)\) to \(m(13) \leq \binom{14}{2} m(9) + \binom{14}{1} 2^{12} = 275835\).

3.0.1 Optimization of \(m(n)\)

From the construction above, we obtain the following for any integer \(n\) greater than a large constant \(n_0 > 0\).

\[
m(n) \leq \binom{n + k - 1}{k} m(n - 2k) + \binom{n + k - 1}{k-1} 2^{n-1} + \left(\frac{n}{\left\lfloor n/2 \right\rfloor}\right)/2 \left(\frac{n}{\left\lfloor n/2 \right\rfloor}\right)/2
\]

Let \(k = np\), where \(\frac{1}{n} \leq p < 0.5\). Therefore,

\[
m(n) \leq \binom{n + np - 1}{np} m(n - 2np) + \binom{n + np - 1}{np-1} 2^{n} \leq \binom{n + np}{np} m(n - 2np) + \binom{n + np}{np} 2^{n}.
\]
Let us denote \( z \) and set \( n \) such that \( n(1 - 2p)^i \) is an integer for all \( i \) in the range \( 1 \leq i \leq s \), we obtain the following.

\[
m(n) < \left(\frac{e(n + np)}{np}\right)^{np} \left( m(n - 2np) + 2^n \right)
\]

\[
< \left(\frac{e(1 + p)}{p}\right)^{np} \left[ 2^n + \left(\frac{e(1 + p)}{p}\right)^{(n-2np)p} \left( m(n(1 - 2p)^2) + 2^{n-2np} \right) \right]
\]

\[
:\vdots
\]

\[
< m(n(1 - 2p)^s) \left(\frac{e(1 + p)}{p}\right)^{\sum_{i=0}^{s-1} np(1-2p)^i}
\]

\[
+ \sum_{i=0}^{s-1} 2^{np(1-2p)^i} \left(\frac{e(1 + p)}{p}\right)^{\sum_{j=0}^{i} np(1-2p)^j}
\]

\[
= m(n(1 - 2p)^s) \left(\frac{e(1 + p)}{p}\right)^{\frac{n}{2}(1-(1-2p)^s)}
\]

\[
+ \sum_{i=0}^{s-1} 2^{np(1-2p)^i} \left(\frac{e(1 + p)}{p}\right)^{\frac{n}{2}(1-(1-2p)^{(i+1)})}
\]

\[
< m(n(1 - 2p)^s) \left(\frac{e(1 + p)}{p}\right)^{\frac{n}{2}}
\]

\[
+ \left(\frac{e(1 + p)}{p}\right)^{\frac{n}{2}} \sum_{i=0}^{s-1} \left(\frac{2}{\left(\frac{e(1+p)}{p}\right)^{(1-2p)/2}}\right)^{(1-2p)^i}
\]

For any integer \( c > 0 \), we observe that there exists a constant \( c' > \frac{\ln n - \ln c}{2\ln n} \) such that \( n(1 - 2p)^s < c \) for \( s \geq c'n \ln n \). Using \( s = c'n \ln n \) in the above equation, we have

\[
m(n) \leq \left(\frac{e(1 + p)}{p}\right)^{\frac{n}{2}} \left[ m(c) + \sum_{i=0}^{c'n \ln n - 1} \left(\frac{2}{\left(\frac{e(1+p)}{p}\right)^{(1-2p)/2}}\right)^{(1-2p)^i} \right].
\]

We observe that \( 2 \left(\frac{e(1+p)}{p}\right)^{(1-2p)/2} \) increases if \( p \) increases and its value is less than 1 for \( 0 < p \leq 0.2381 \). Using \( p = 0.2381 \) in the above equation, we obtain

\[
m(n) < \left(\frac{e(1 + p)}{p}\right)^{\frac{n}{2}} \left[ m(c) + c'n \ln n \right]
\]

\[
= O(3.7596^n \cdot n \ln n)
\]

\[
= O(3.76^n).
\]

### 4 Multi-Core Construction

Consider an integer \( k \) satisfying \( 0 < k < n \). We define \( w = [n/k], x = n \mod k, y = [k/x] \) and \( z = k \mod x \). A multi-core construction makes use of a non-2-colorable \((n-k)\)-uniform hypergraph \( H_c = (V_c, E_c) \), a total of \( w \) identical non-2-colorable \( k \)-uniform hypergraphs \( H_1 = (V_1, E_1), \ldots, H_w = (V_w, E_w) \) and a total of \( y \) identical non-2-colorable \( x \)-uniform hypergraphs \( H'_1 = (V'_1, E'_1), \ldots, H'_y = (V'_y, E'_y) \). The vertex sets of the hypergraphs \( H_c, H_1, \ldots, H_w, H'_1, \ldots, H'_y \) are pairwise disjoint. Let us denote \( E_c = \{e_1, e_2, \ldots, e_{m_c}\}, E_1 = \{e_{1_1}, e_{1_2}, \ldots, e_{m_{1k}}\}, \ldots, E_w = \{e_{1w}, e_{2w}, \ldots, e_{m_{wk}}\}, E'_1 = \ldots \).
Consider a vertex set $A = \{a_1, a_2, \ldots, a_{x+z-1}\}$, disjoint with each of $V_c, V_1, \ldots, V_w, V'_1, \ldots, V'_y$. We define $A_p$ as the collection of all $p$-element subsets of the vertex set $A$. Let $\mathcal{E} = \{j_1 \cup j_2 \cup \cdots \cup j_w : (j_1, j_2, \ldots, j_w) \in E_1 \times E_2 \times \cdots \times E_w\}$ and $\mathcal{E}' = \{j'_1 \cup j'_2 \cup \cdots \cup j'_y : (j'_1, j'_2, \ldots, j'_y) \in E'_1 \times E'_2 \times \cdots \times E'_y\}$.

We define the construction of a non-2-colorable $n$-uniform hypergraph $H = (V, E)$ as follows. The vertex set is $V = V_c \cup A \cup V'_1 \cup \cdots \cup V'_w \cup V'_0$. The construction of the hyperedges belonging to $E$ depends on the values of $x$ and $z$ as follows.

Case 1. For $x > 0$ and $z > 0$, $E$ contains the following hyperedges.

(i) $e_i \cup e'_{j}$ for every triple $i, j, l$ satisfying $1 \leq i \leq m_c, 1 \leq j \leq m_k$ and $1 \leq l \leq w$

(ii) $e'_i \cup e$ for every triple $i, j, e$ satisfying $1 \leq i \leq m_x, 1 \leq j \leq y$ and $e \in \mathcal{E}$

(iii) $e_i \cup e' \cup S$ for every triple $i, e, S$ satisfying $1 \leq i \leq m_c, e' \in \mathcal{E}'$ and $S \in A_z$

(iv) $e \cup S$ for every pair $e, S$ satisfying $e \in \mathcal{E}$ and $S \in A_x$

Case 2. For $x > 0$ and $z = 0$, $E$ contains the following hyperedges.

(i) $e_i \cup e'_{j}$ for every triple $i, j, l$ satisfying $1 \leq i \leq m_c, 1 \leq j \leq m_k$ and $1 \leq l \leq w$

(ii) $e'_i \cup e$ for every triple $i, j, e$ satisfying $1 \leq i \leq m_x, 1 \leq j \leq y$ and $e \in \mathcal{E}$

(iii) $e_i \cup e'$ for every pair $i, e'$ satisfying $1 \leq i \leq m_c$ and $e' \in \mathcal{E}'$

Case 3. For $x = 0$, $E$ contains the following hyperedges.

(i) $e_i \cup e'_{j}$ for every triple $i, j, l$ satisfying $1 \leq i \leq m_c, 1 \leq j \leq m_k$ and $1 \leq l \leq w$

(ii) $e$ for each $e \in \mathcal{E}$

The number of hyperedges in $H$ is given by

$$|E| = \begin{cases} 
wm_c m_k + ym_x (m_k)^w + \left(\frac{x+z-1}{x}ight)m_c (m_x)^y + \left(\frac{x+z-1}{x}ight)(m_k)^w & \text{if } x > 0, z > 0 \\
wm_c m_k + ym_x (m_k)^w + m_c (m_x)^y & \text{if } x > 0, z = 0 \\
wm_c m_k + (m_k)^w & \text{if } x = 0
\end{cases}$$

Proof of Result 2. For the sake of contradiction, let us assume that $\chi$ is a proper 2-coloring of $H$. Without loss of generality, let the hypergraph $H_c$ be monochromatic in red in the coloring $\chi$. The hyperedges formed in Step (i) in each of the cases ensure that the hypergraphs $H_j$ are monochromatic in blue for each $j \in \{1, \ldots, w\}$.

Case 1. If $x > 0$ and $z > 0$, the hyperedges formed in Step (ii) ensure that the hypergraphs $H'_l$ are monochromatic in red for each $l \in \{1, 2, \ldots, y\}$. It can be noted from the hyperedges generated in Step (iii) that there are at most $z - 1$ red vertices in the set $A$. This implies that $A$ has at least $x$ blue vertices. The hyperedges formed in Step (iv) ensure that there are at most $x - 1$ blue vertices in $A$. Thus, we have a contradiction.

Case 2. If $x > 0$ and $z = 0$, the hyperedges formed in Step (ii) ensure that the hypergraphs $H'_l$ are monochromatic in red for each $l \in \{1, 2, \ldots, y\}$. It can be easily noted that the hyperedges generated in Step (iii) include a red monochromatic hyperedge. Thus, we have a contradiction.
Case 3. If $x = 0$, it immediately follows that we have a blue monochromatic hyperedge in the hyperedges generated by Step (ii) of the construction. This leads to a contradiction.

Thus, we have the following result on $m(n)$.

If $x > 0$ and $z > 0$,

$$m(n) \leq w \cdot m(n - k)m(k) + y \cdot m(k)^w m(x) + (x + z - 1)m(n - k)m(x)^y + (x + z - 1)m(k)^w.$$ 

If $x > 0$ and $z = 0$,

$$m(n) \leq w \cdot m(n - k)m(k) + y \cdot m(k)^w m(x) + (n - k)m(x)^y.$$ 

If $x = 0$,

$$m(n) \leq w \cdot m(n - k)m(k) + m(k)^w.$$ 

These recurrence relations give improvements on $m(n)$ for $n = 8, 13, 14, 16$ and 17 as follows:

- For $n = 8$ and $k = 5$, we have $m(8) \leq m(3)m(5) + m(5)m(3) + \binom{4}{2}m(3)m(3) + \binom{4}{3}m(5) \leq 1212$ by using $m(3) = 7$ and $m(5) \leq 51$ from Table [1]

- For $n = 13$ and $k = 5$, we obtain $m(13) \leq 2m(8)m(5) + m(5)^2m(3) + \binom{4}{2}m(8)m(3) + \binom{4}{3}m(5)^2 \leq 203139$ by using $m(3) = 7$ and $m(5) \leq 51$ from Table [1] and $m(8) \leq 1212$ obtained above.

- For $n = 14$ and $k = 5$, the recurrence relation gives $m(14) \leq 2m(9)m(5) + m(5)^2m(4) + \binom{4}{1}m(9)m(4) + \binom{4}{3}m(5)^2 \leq 528218$ by using $m(4) = 23$, $m(5) \leq 51$ and $m(9) \leq 2401$ from Table [1]

- For $n = 16$ and $k = 7$, we have $m(16) \leq 2m(9)m(7) + 3m(7)^2m(2) + \binom{7}{2}m(9)m(2)^3 + \binom{7}{3}m(7)^2 \leq 3923706$ by using $m(2) = 3$, $m(7) \leq 421$ and $m(9) \leq 2401$ from Table [1]

- Finally, for $n = 17$ and $k = 7$, we obtain $m(17) \leq 2m(10)m(7) + 2m(7)^2m(3) + \binom{7}{3}m(10)m(3)^2 + \binom{7}{3}m(7)^2 \leq 10375782$ by using $m(3) = 7$, $m(7) \leq 421$ and $m(10) \leq 7803$ from Table [1]

- It can also be noted that this construction matches the currently best-known upper bounds on $m(6)$ and $m(10)$ for $k = 3$ and $k = 5$, respectively.

5 Block Construction

For an integer $k > 0$, we describe the construction of a collection $\mathcal{H}$ of non-2-colorable $n$-uniform hypergraphs. Any hypergraph $H = (V, E)$ belonging to this collection is constructed using a non-2-colorable $(n - 2k)$-uniform hypergraph denoted by $H_c = (V_c, E_c)$ and two disjoint collections of hypergraphs $\mathcal{A}$ and $\mathcal{B}$. Let $E_c = \{e_1, e_2, \ldots, e_m\}$. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_s\}$ and $\mathcal{B} = \{H'_1, H'_2, \ldots, H'_t\}$ be the collection of hypergraphs such that each of $H_i = (V_i, E_i)$ and $H'_i = (V'_i, E'_i)$ is an identical copy of a non-2-colorable $k_i$-uniform hypergraph satisfying $k_i \geq k$ and $\sum_{i=1}^{m} k_i \geq n$. Note that the sets $V_c, V_1, V_2, \ldots, V_t, V'_1, V'_2, \ldots, V'_t$ are pairwise disjoint.

Let $P = \{i_1, i_2, \ldots, i_p\} \subseteq \{1, 2, \ldots, t\}$ such that $1 \leq i_1 < i_2 < \ldots < i_p \leq t$. Using the Cartesian products $C_P = E_{i_1} \times E_{i_2} \times \cdots \times E_{i_p}$ and $C'_P = E'_{i_1} \times E'_{i_2} \times \cdots \times E'_{i_p}$, let us define the collection of hyperedges $\mathcal{A}_P$ and $\mathcal{B}_P$ as $\mathcal{A}_P = \{j_1 \cup j_2 \cup \cdots \cup j_p : (j_1, j_2, \ldots, j_p) \in C_P\}$ and $\mathcal{B}_P = \{j'_1 \cup j'_2 \cup \cdots \cup j'_p : (j'_1, j'_2, \ldots, j'_p) \in C'_P\}$, respectively. Also, let $\mathcal{P} = \{1, 2, \ldots, t\} \setminus P$.

The hypergraph $H$ has the vertex set $V = V_c \cup V_1 \cup \cdots \cup V_t \cup V'_1 \cup \cdots \cup V'_t$ and the hyperedge set $E$ is generated from the following hyperedges, each containing at least $n$ vertices.
(i) For each \( j \) satisfying \( 1 \leq j \leq t \), \( e_i \cup e_{H_j} \cup e_{H'_j} \) for every triple \( i, e_{H_j}, e_{H'_j} \) satisfying \( 1 \leq i \leq m \), \( e_{H_j} \in E_j \) and \( e_{H'_j} \in E'_j \)

(ii) For each \( P \subset \{1, 2, \ldots, t\} \) such that \( |P| \) is odd and \( 1 \leq |P| \leq \lfloor t/2 \rfloor \), \( e_H \cup e_{H'} \) for every pair \( e_H, e_{H'} \) satisfying \( e_H \in A_P \) and \( e_{H'} \in B_P \)

(iii) For each \( P \subset \{1, 2, \ldots, t\} \) such that \( |P| \) is even and \( 0 \leq |P| \leq \lfloor t/2 \rfloor \), \( e_H \cup e_{H'} \) for every pair \( e_H, e_{H'} \) satisfying \( e_H \in A_P \) and \( e_{H'} \in B_P \)

We select an arbitrary set of \( n \) vertices from each of the hyperedges generated above to form the hyperedge set \( E \). In case a hyperedge is included more than once in \( E \) by this process, we keep only one of those to ensure that \( E \) is not a multi-set. Let us count the number of hyperedges added to the hyperedge set \( E \). Step (i) adds at most \( |E_c| \sum_{i=1}^{t} |E_i||E'_i| = \sum_{i=1}^{t} |E_i|^2 |E_c| \) hyperedges, whereas Steps (ii) and (iii) together add at most \( \prod_{i=1}^{t} |E_i|(1 + \binom{1}{t} + \cdots + \binom{t}{t/2}) \) hyperedges. Note that \( |E| \leq \sum_{i=1}^{t} |E_i|^2 |E_c| + 2^{t-1} \prod_{i=1}^{t} |E_i| \) when \( t \) is odd, and \( |E| \leq \sum_{i=1}^{t} |E_i|^2 |E_c| + (2^{t-1} + \binom{t}{t/2}/2) \prod_{i=1}^{t} |E_i| \) when \( t \) is even. In the following lemma, we prove that \( H \) is non-2-colorable by showing that any proper 2-coloring of \( H \) can be used to obtain a proper 2-coloring of any \( t \)-uniform hypergraph constructed by Abbott-Hanson-Toft construction.

**Lemma 6.** \( H \) is non-2-colorable.

**Proof.** Consider any \( t \)-uniform hypergraph \( H_{AHT} = (V_{AHT}, E_{AHT}) \) constructed by Abbott-Hanson-Toft construction using a non-2-colorable \( (t - 2) \)-uniform core hypergraph and two disjoint vertex sets \( \{p_1, \ldots, p_t\} \) and \( \{q_1, \ldots, q_t\} \). Assuming for the sake of contradiction that a proper 2-coloring exists for \( H \), we give a proper 2-coloring for \( H_{AHT} \) as follows.

- Color all vertices of the non-2-colorable \( (t - 2) \)-uniform core hypergraph of \( H_{AHT} \) with the color of the monochromatic hyperedge of \( H_c \) used in the construction of \( H \).

- Color each vertex \( p_i \) with the color of the monochromatic hyperedge of \( H_i \) used in the construction of \( H \).

- Similarly, color each vertex \( q_i \) with the color of the monochromatic hyperedge of \( H'_i \) used in the construction of \( H \).

Since \( H_{AHT} \) is non-2-colorable, we have a contradiction. As a result, we have the following recurrence relation.

\[
m(n) \leq \begin{cases} m(n - 2k) \sum_{i=1}^{t} m(k_i)^2 + 2^{t-1} \prod_{i=1}^{t} m(k_i) & \text{if } t \text{ is odd} \\ m(n - 2k) \sum_{i=1}^{t} m(k_i)^2 + (2^{t-1} + \binom{t}{t/2}/2) \prod_{i=1}^{t} m(k_i) & \text{if } t \text{ is even} \end{cases}
\]

Consider the special case when \( n = 3k + 1 \). Setting the values of \( t \) and \( k_i \)'s as \( t = 3 \), \( k_1 = k + 1 \) and \( k_2 = k_3 = k \) in this special case, we obtain the following recurrence relation.

\[
m(3k + 1) \leq m(k + 1)^3 + 6m(k)^2 m(k + 1)
\]

(1)

We give an improvement of this result below.
Modified Block Construction

Let us first repeat the detailed description for the special case mentioned above, i.e., the construction of a non-2-colorable \((3k+1)\)-uniform hypergraph \(H = (V, E)\) belonging to \(\mathcal{H}\). We construct \(H\) using a non-2-colorable \((k+1)\)-uniform hypergraph \(H_c = (V_c, E_c)\) along with non-2-colorable \((k+1)\)-uniform hypergraphs \(H_1 = (V_1, E_1)\) and \(H_2 = (V_2, E_2)\) and non-2-colorable \(k\)-uniform hypergraphs \(H_3 = (V_3, E_3), H_3' = (V_3', E_3')\). Note that each \(H_i'\) is an identical copy of \(H_i\) for \(1 \leq i \leq 3\).

For the modified construction described below, we set \(H_1\) as the Abbott-Hanson-Toft construction that uses a non-2-colorable \((k-1)\)-uniform core hypergraph \(H_{1c} = (V_{1c}, E_{1c})\) and disjoint vertex sets \(A = \{a_1, a_2, \ldots, a_{k+1}\}, B = \{b_1, b_2, \ldots, b_{k+1}\}\). Note that \(H_1'\) is not necessarily identical to \(H_1\) in this modified block construction, whereas each \(H_i'\) is an identical copy of \(H_i\) for \(2 \leq i \leq 3\).

Using the notations introduced above, the vertex set of \(H\) is \(V = V_c \cup V_{1c} \cup A \cup B \cup V_2' \cup V_2 \cup V_3' \cup V_3\). The hyperedge set \(E\) is generated from the following hyperedges.

1. \(e_{H_c} \cup e_{H_1} \cup e_{H_1'}\) for every triple \(e_{H_c}, e_{H_1}, e_{H_1'}\) satisfying \(e_{H_c} \in E_c, e_{H_1} \in E_1\) and \(e_{H_1'} \in E_1'\)
2. \(e_{H_c} \cup e_{H_2} \cup e_{H_2'}\) for every triple \(e_{H_c}, e_{H_2}, e_{H_2'}\) satisfying \(e_{H_c} \in E_c, e_{H_2} \in E_2\) and \(e_{H_2'} \in E_2'\)
3. \(e_{H_c} \cup e_{H_3} \cup e_{H_3'}\) for every triple \(e_{H_c}, e_{H_3}, e_{H_3'}\) satisfying \(e_{H_c} \in E_c, e_{H_3} \in E_3\) and \(e_{H_3'} \in E_3'\)
4. \(e_{H_1} \cup e_{H_1'}\) for every pair \(e_{H_1}, e_{H_1'}\) satisfying \(e_{H_1} \in E_1\) and \(e_{H_1'} \in \{j_2' \cup j_3' : (j_2', j_3') \in E_2' \times E_3'\}\)
5. \(e_{H_2} \cup e_{H_2'}\) for every pair \(e_{H_2}, e_{H_2'}\) satisfying \(e_{H_2} \in E_2\) and \(e_{H_2'} \in \{j_1' \cup j_3' : (j_1', j_3') \in E_1' \times E_3'\}\)
6. \(e_{H_3} \cup e_{H_3'}\) for every pair \(e_{H_3}, e_{H_3'}\) satisfying \(e_{H_3} \in E_3\) and \(e_{H_3'} \in \{j_1' \cup j_2' : (j_1', j_2') \in E_1' \times E_2'\}\)
7. All elements of the set \(\{j_1 \cup j_2 \cup j_3 : (j_1, j_2, j_3) \in E_1 \times E_2 \times E_3\}\)

Note that each of the hyperedges formed in Steps (b) to (g) has \(3k+1\) vertices. However, the hyperedges formed in Step (a) have \(3k+3\) vertices in each of them. We can remove any two vertices from each of these hyperedges to obtain the following recurrence relation. Recall that \(m_{H_{1c}}(k+1)\) denotes the number of hyperedges in the non-2-colorable \((k+1)\)-uniform hypergraph constructed by Abbott-Hanson-Toft construction that uses \(H_{1c}\) as its core hypergraph.

\[
m(3k+1) \leq m_{H_{1c}}(k+1)m(k+1)^2 + 2m_{H_{1c}}(k+1)m(k)^2 + 4m(k+1)m(k)^2 \quad (2)
\]

Whenever \(m(k+1) < m_{H_{1c}}(k+1)\), it is evident that the upper bound on \(m(3k+1)\) that this recurrence relation gives is worse than the one given by Eq. 1. However, we observe that we can improve Eq. 2 by carefully selecting the two vertices to be removed from each hyperedge formed in Step (a). Recall that each of these hyperedges is a union of three hyperedges \(e_{H_c} \in E_c, e_{H_1} \in E_1\) and \(e_{H_1'} \in E_1'\). In the following paragraph, we describe a process to create a set of \(k-1\) vertices from each hyperedge in the \((k+1)\)-uniform hypergraph \(H_1 = (V_1, E_1)\). For each hyperedge \(e_{H_c} \cup e_{H_1} \cup e_{H_1'}\) formed in Step (a), we use this process to remove two vertices from \(e_{H_1}\).

Given a hyperedge \(h \in E_1\), we create a set \(h'\) containing \(k-1\) vertices as follows.

Case 1. If \(h\) is created by Step (i) of Abbott-Hanson-Toft construction, i.e., if \(h\) is of the form \(e \cup \{a_i\} \cup \{b_i\}\) for some \(e \in E_{1c}, a_i \in A\) and \(b_i \in B\), we define \(h' = e\). In other words, we remove \(a_i\) and \(b_i\) from \(h\) to create \(h'\).

Case 2. If \(h\) is created in Step (ii) of Abbott-Hanson-Toft construction, i.e., if \(h\) is of the form \(A_K \cup \overline{B}_K\) for some \(K \subset \{1, \ldots, k+1\}\) such that \(|K|\) is odd and \(1 \leq |K| \leq \lfloor (k+1)/2 \rfloor\), we define \(h' = A_K \cup \overline{B}_K \setminus \{a_k, a_{k+1}, b_k, b_{k+1}\}\).
Case 3. If \( h \) is created in Step (iii) of Abbott-Hanson-Toft construction, i.e., if \( h \) is of the form \( \overline{A}_K \cup B_K \) for some \( K \subset \{1, \ldots, k + 1\} \) such that \( |K| \) is even and \( 2 \leq |K| \leq [(k + 1)/2] \), we define \( h' = \overline{A}_K \cup B_K \setminus \{a_k, a_{k+1}, b_k, b_{k+1}\} \).

Case 4. If \( h \) is formed in Step (iv) of Abbott-Hanson-Toft construction, i.e., if \( h = A \), we define \( h' = A \setminus \{a_k, a_{k+1}\} \).

This completes the construction of the \((3k + 1)\)-uniform hypergraph \( H \).

**Proof of Result 3.** We improve the recurrence relation given in Eq. 2 as a result of selecting \( k - 1 \) vertices from each \( h' \in E_1 \), as described above. Since this process generates multiple copies of some \((k - 1)\)-element vertex sets, the number of distinct hyperedges formed in Step (a) in the construction of \( H \) is reduced. Let us determine the cardinality of the set \( \{h' : h' \text{ is generated from some } h \in E_1\} \).

It is easy to observe that the number of distinct \( h' \)'s formed in Case 1 is \( |E_1| \). On the other hand, the total number of distinct \( h' \)'s formed in Cases 2, 3 and 4 is at most \( 2^{k-1} \). It follows from the fact that there are \( 2^{k-1} \) subsets of \( \Delta \setminus \{a_k, a_{k+1}\} \) and each \( h' \) formed in one of the Cases 2, 3 and 4 is a union of the sets \( \bigcup_{i \in P} \{a_i\} \) and \( \bigcup_{i \in \{1, \ldots, k-1\} \setminus P} \{b_i\} \) for some \( P \subseteq \{1, \ldots, k-1\} \).

Since we have shown in Lemma 5 that \( H \) is non-2-colorable, we have the following improvement over Eq. 2:

\[
m(3k + 1) \leq (m(k - 1) + 2^{k-1})m(k + 1)^2
+ 2m_{H_{1c}}(k + 1)m(k)^2 + 4m(k + 1)m(k)^2
\]

This result improves the upper bounds on \( m(n) \) for \( n = 13 \) and \( 16 \) as follows.

- For \( n = 13 \), we have \( k = 4 \). Note that \( m_{H_{1c}}(5) = 51 \), when the Fano plane \[10\] \( H_f \) having 7 hyperedges is used as the core hypergraph \( H_{1c} \). Therefore, we obtain \( m(13) \leq (m(3) + 2^3)m(5)^2 + 2m_{H_{1c}}(5)m(4)^2 + 4m(5)m(4)^2 \leq 200889 \) by using \( m(3) = 7 \), \( m(4) = 23 \) and \( m(5) \leq 51 \) from Table 1.

- For \( n = 16 \), we have \( k = 5 \). Note that \( m_{H_{1c}}(6) = 180 \), when the non-2-colorable 4-uniform hypergraph \( H_s \) with 23 hyperedges is used as the core hypergraph \( H_{1c} \). Therefore, we obtain \( m(16) \leq (m(4) + 2^4)m(6)^2 + 2m_{H_{1c}}(6)m(5)^2 + 4m(6)m(5)^2 \leq 3308499 \) by using \( m(4) = 23 \), \( m(5) \leq 51 \) and \( m(6) \leq 147 \) from Table 1.

### 6 Improved Lower Bound for \( m(5) \)

For the sake of completeness, we begin this section with a proof of the result given by Goldberg and Russell [9] for the lower bounds on \( m(n) \) for small values of \( n \). This result uses Lemma 7 and Lemma 8 in its proof. Let \( m_l(n) \) be the minimum number of hyperedges in a non-2-colorable \( n \)-uniform hypergraph with \( l \) vertices.

**Lemma 7.** [7] \( m_{2n-1}(n) = m_{2n}(n) = \binom{2n}{n-1} \).

**Lemma 8.** [8] *(Schönheim bound)* Consider positive integers \( l \geq n \geq t \geq 1 \) and \( \lambda \geq 1 \). Any \( n \)-uniform hypergraph with \( l \) vertices such that every \( t \)-subset of its vertices is contained in at least \( \lambda \) hyperedges has at least \( \left\lfloor \frac{l}{n} \right\rfloor \left\lfloor \frac{l-1}{n-1} \right\rfloor \cdots \left\lfloor \frac{\lambda(l-t+1)}{n-t+1} \right\rfloor \cdots \) hyperedges.
Lemma 9. If $n \geq 4$, then $m(n) \geq \min_{x > 2n, x \in \mathbb{N}} \left\{ \max \left\{ \left[ \frac{x}{n} \right], \left[ \frac{x - 1}{n - 1} \right] \right\} \right\}$.

Proof. Let us consider an $n$-uniform hypergraph $H = (V, E)$ such that the number of hyperedges satisfies $|E| < \min_{x > 2n, x \in \mathbb{N}} \left\{ \max \left\{ \left[ \frac{x}{n} \right], \left[ \frac{x - 1}{n - 1} \right] \right\} \right\}$. We call a 2-coloring of the hypergraph to be balanced if the coloring has $|V|/2$ red vertices and $|V|/2$ blue vertices. It can be noted that the possible number of ways to give a balanced coloring for $H$ is $\binom{|V|}{|V|/2}$ and not all of these are proper 2-colorings. Let us define $f(x) = \left[ \frac{x}{n} \right]$, $g(x) = \left[ \frac{x - 1}{n - 1} \right]$ and $r = \min_{x > 2n, x \in \mathbb{N}} \{ f(x), g(x) \}$. Let this minimum value $r$ be obtained by $x = v_{opt}$. When $x > 2n$, observe that $f(x)$ is non-increasing and $g(x)$ is non-decreasing with increasing $x \in \mathbb{N}$. Moreover, we also observe that $\binom{2n - 1}{n} \geq f(2n + 1) \geq g(2n + 1)$ for $n \geq 4$.

Case 1. If $n \leq |V| \leq 2n - 2$, any balanced coloring of its vertex set is a proper 2-coloring of $H$.

Case 2. If $|V| = 2n - 1$ or $|V| = 2n$, it follows from Lemma 7 that $m_{2n-1}(n) = m_{2n}(n) = \binom{2n-1}{n}$.

Since $|E| < r \leq \binom{2n-1}{n}$, $H$ is properly 2-colorable.

Case 3. If $2n + 1 \leq |V| \leq v_{opt}$, consider a balanced coloring of $H$. We say that such a coloring is blocked by a hyperedge if it is monochromatic in the coloring. Note that a red monochromatic hyperedge blocks $\binom{|V|}{2n}$ and a blue monochromatic hyperedge blocks $\binom{|V|}{2n}$ such colorings. In order to ensure that none of these balanced colorings is a proper 2-coloring of $H$, we need at least $f(|V|)$ hyperedges. Since $|E| < r \leq f(|V|)$ for $2n + 1 \leq |V| \leq v_{opt}$, at least one of the balanced colorings of $H$ is a proper 2-coloring of it.

Case 4. If $|V| > v_{opt}$, assume the induction hypothesis that any $n$-uniform hypergraph with $|V| - 1$ vertices and $|E|$ hyperedges is properly 2-colorable. The base case $|V| = v_{opt}$ is proved in Case 3. If there exists a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of $H$, consider a new hypergraph $H' = (V', E')$ constructed by merging $v_i$ and $v_j$ into a new vertex $v$. Since $H'$ is $n$-uniform with $|V'| = |V| - 1$ and $|E'| = |E|$, we know from the induction hypothesis that $H'$ is properly 2-colorable. This coloring of $H'$ can be extended to a proper 2-coloring of $H$ by assigning the color of $v$ to $v_i$ and $v_j$. Since $|E| < r$ and it follows from Lemma 8 that the minimum number of hyperedges required to ensure that each pair of vertices is contained in at least one hyperedge is $g(|V|) \geq r$, we are guaranteed to have a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of $H$.

Lemma 9 implies that $m(5) \geq 28$, which is obtained when $x = 23$. We improve this to $m(5) \geq 29$ using the following lemma. The first three cases of the proof for this improved lower bound are the same as the ones used in the proof above. We use Lemma 10 to prove Case 4 of the proof.

Lemma 10. Consider a positive integer $\gamma$ and a fraction $p \in [0, 1]$. Any $n$-uniform hypergraph $H = (V, E)$ satisfying $|\{\{e_1, e_2\} : e_1, e_2 \in E, |e_1 \cap e_2| = 1\}| \leq \gamma$ is properly 2-colorable if $2^{-n+1}(1 - p)n|E| + 4\gamma (2^{-2n+1} p \int_0^1 (1 - (xp)^2)^{n-1} dx) < 1$.

Proof of Result 4. Let us consider a 5-uniform hypergraph $H = (V, E)$ with at most 28 hyperedges. We show that it is properly 2-colorable.
Case 1. If $5 \leq |V| \leq 8$, any balanced coloring of its vertex set is a proper 2-coloring of $H$.

Case 2. If $|V| = 9$ or $|V| = 10$, it follows from Lemma 7 that $m_9(5) = m_{10}(5) = 126$. Since $|E| \leq 28$, $H$ has a proper 2-coloring.

Case 3. If $11 \leq |V| \leq 22$, consider a balanced coloring of $H$. We observe that a red monochromatic hyperedge blocks $\binom{|V| - 5}{\lfloor |V|/2 \rfloor - 5}$ and a blue monochromatic hyperedge blocks $\binom{|V| - 5}{\lfloor |V|/2 \rfloor - 5}$ such colorings. In order to ensure that none of these balanced colorings is a proper 2-coloring of $H$, we need at least

\[
\left(\frac{|V|}{\lfloor |V|/2 \rfloor - 5}\right)\left(\frac{|V|}{\lfloor |V|/2 \rfloor - 5} + \frac{|V| - 5}{\lfloor |V|/2 \rfloor - 5}\right)
\]

hyperedges. Since $11 \leq |V| \leq 22$, it implies that we need at least 29 hyperedges to ensure that no balanced coloring of $H$ is a proper 2-coloring.

Case 4. If $|V| = 23$ and there exists a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of $H$, we construct a new hypergraph $H' = (V', E')$ by merging vertices $v_i$ and $v_j$ into a new vertex $v$. We observe that $H'$ is 5-uniform with 22 vertices and $|E|$ hyperedges. It follows from Case 3 that $H'$ is properly 2-colorable. This coloring of $H'$ can be extended to a proper 2-coloring of $H$ by assigning the color of $v$ to $v_i$ and $v_j$. If $|E| \leq 27$, note that Lemma 8 ensures that there exists a pair of vertices not contained together in any hyperedge of $H$. Therefore, we would complete the proof by assuming that $|E| = 28$ and every pair of vertices is contained in at least one hyperedge of $H$. For such a hypergraph, we show that the cardinality of the set $\{e_1, e_2\} : e_1, e_2 \in E, |e_1 \cap e_2| = 1$ is at most 335. Setting $p = 0.3, \gamma = 335, n = 5$ and $|E| = 28$ in Lemma 10, we observe that $H$ is properly 2-colorable since $2^{-n+1}(1-p)^n|E| + 4\gamma \cdot 2^{-2n+1}p \int_0^1 (1-(xp)^2)^{n-1}dx < 1$.

In order to show that the cardinality of the set $\{e_1, e_2\} : e_1, e_2 \in E, |e_1 \cap e_2| = 1$ is at most 335, we consider the degree sequence of $H$. Note that the degree of a vertex is defined as the number of hyperedges it is contained in and the degree sequence of a hypergraph is the ordering of the degrees of its vertices in a non-increasing order. Consider an arbitrary vertex $u$ of $H$. Observe that there are 22 distinct vertex pairs involving $u$ and any hyperedge containing $u$ has 4 such pairs in it. Therefore, the degree of $u$ is at least 6 and there exists another vertex $u'$ such that $\{u, u'\}$ is contained in at least two different hyperedges of $H$. Since the sum of the degrees of the vertices of $H$ is 140, the only possible degree sequences of $H$ are $(8,6,\ldots,6)$ and $(7,7,6,\ldots,6)$. For the first sequence, the cardinality of the set $\{e_1, e_2\} : e_1, e_2 \in E, |e_1 \cap e_2| = 1$ is upper bounded by $(\binom{6}{0} - 1) \cdot 22 + (\binom{6}{2} - 1) = 335$. For the second sequence, it is upper bounded by $(\binom{6}{0} - 1) \cdot 21 + (\binom{6}{2} - 1) \cdot 2 = 334$.

Case 5. If $|V| \geq 24$, assume the induction hypothesis that any 5-uniform hypergraph with $|V| - 1$ vertices and $|E|$ hyperedges is properly 2-colorable. The base case $|V| = 23$ is proved in Case 4. If there exists a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of $H$, consider a new hypergraph $H' = (V', E')$ constructed by merging $v_i$ and $v_j$ into a new vertex $v$. Since $H'$ is 5-uniform with $|V'| = |V| - 1$ and $|E'| = |E|$, we know from the induction hypothesis that $H'$ is properly 2-colorable. This coloring of $H'$ can be extended to a proper 2-coloring of $H$ by assigning the color of $v$ to $v_i$ and $v_j$. Since it follows from Lemma 8 that the minimum number of hyperedges required to ensure that each pair of vertices is contained in at least one hyperedge is $\left\lceil \frac{|V|}{5} \left\lceil \frac{|V| - 1}{4} \right\rceil \right\rceil \geq 29$, we are guaranteed to have a pair of vertices $\{v_i, v_j\}$ not contained together in any hyperedge of $H$. 

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Table 2: Improved upper bounds on $m(n)$ for small values of $n$

| $n$ | $m(n)$ | Corresponding construction/recurrence relation |
|-----|--------|-----------------------------------------------|
| 1   | $m(1) = 1$ | Single Vertex |
| 2   | $m(2) = 3$ | Triangle Graph |
| 3   | $m(3) = 7$ | Fano Plane [10] |
| 4   | $m(4) = 23$ | |
| 5   | $m(5) \leq 51$ | $m(5) \leq 2^4 + 5m(3)$ |
| 6   | $m(6) \leq 147$ | $m(6) \leq m(2)m(3)^2$ |
| 7   | $m(7) \leq 421$ | $m(7) \leq 2^6 + 7m(5)$ |
| 8   | $m(8) \leq 1212$ | $m(8) \leq 2m(3)m(5) + \left(\frac{4}{3}\right)m(3)m(3) + \left(\frac{5}{3}\right)m(5)$ |
| 9   | $m(9) \leq 2401$ | $m(9) \leq m(3)^4$ |
| 10  | $m(10) \leq 7803$ | $m(10) \leq m(2)m(5)^2$ |
| 11  | $m(11) \leq 25449$ | $m(11) \leq 15 \cdot 2^9 + 9m(9)$ |
| 12  | $m(12) \leq 55223$ | $m(12) \leq m(3)^4m(4)$ |
| 13  | $m(13) \leq 200889$ | $m(13) \leq (m(3) + 2^3)m(5)^2 + 2m_{H_{\gamma}}(5)m(4)^2 + 4m(5)m(4)^2$ |
| 14  | $m(14) \leq 528218$ | $m(14) \leq 2m(9)m(5) + m(5)^2m(4) + \left(\frac{4}{3}\right)m(9)m(4) + \left(\frac{5}{3}\right)m(5)^2$ |
| 15  | $m(15) \leq 857157$ | $m(15) \leq m(3)^5m(5)$ |
| 16  | $m(16) \leq 3308499$ | $m(16) \leq (m(4) + 2^4)m(6)^2 + 2m_{H_{\gamma}}(6)m(5)^2 + 4m(6)m(5)^2$ |
| 17  | $m(17) \leq 10375782$ | $m(17) \leq 2m(10)m(7) + 2m(7)^2m(3) + \left(\frac{5}{3}\right)m(10)m(3)^2 + \left(\frac{5}{3}\right)m(7)^2$ |

7 Conclusion

In this paper, we establish the lower bound $m(5) \geq 29$ which is still far from the best-known upper bound $m(5) \leq 51$. We also establish improved upper bounds for $m(8)$, $m(13)$, $m(14)$, $m(16)$ and $m(17)$. In Table 2, we highlight these improved bounds on $m(n)$ for $n \leq 17$. It would be interesting to determine the exact values of $m(n)$ for $n \geq 5$.

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