SENSITIVE DEPENDENCE OF GEOMETRIC GIBBS STATES

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Abstract. For quadratic-like maps, we show a phenomenon of sensitive dependence of geometric Gibbs states: There are analytic families of quadratic-like maps for which an arbitrarily small perturbation of the parameter can have a definite effect on the low-temperature geometric Gibbs states. Furthermore, this phenomenon is robust: There is an open set of analytic 2-parameter families of quadratic-like maps that exhibit sensitive dependence of geometric Gibbs states. We introduce a geometric version of the Peierls condition for contour models ensuring that the low-temperature Gibbs states are concentrated near the critical orbit.

1. Introduction

A central problem in statistical mechanics and the thermodynamic formalism, is the study of phase transitions. Here we focus on “zero-temperature” phase transitions, or the chaotic dependence of Gibbs states on the temperature parameter as it drops to zero. In good situations, as in the case of contour models satisfying the Peierls condition, Gibbs states converge to a ground state as the temperature drops to zero, see for example [Sin82, §2] or the summary in vEFS93, §B.4, and [Bré03, CGU11, Con16, Lep05] and references therein for other convergence results. There are several examples of divergence, see [BGT15, CH10, vER07], and the companion paper [CRL15b].

Here we focus on the thermodynamic formalism of smooth maps and geometric potentials. In this setting the potential is entirely determined by the map, whereas in the setting considered in BGT15, Bré03, CGU11, CH10, CRL15b, Con16, vER07, Lep05 the map is fixed, and the potential is allowed to vary independently of the map. The geometric potential arises naturally in several important problems, like in the construction of physical measures, as in the pioneering work of Sinai [Sin72], Ruelle [Rue76], and Bowen [Bow75]. The pressure of the geometric potential, as a function of the inverse temperature, is also connected to several multifractal spectra, and large deviations rate functions.

The simplest case of interest is that of circle expanding maps. A folklore result asserts that generically there is a unique ground state for the geometric potential, and that geometric Gibbs states converge to this ground state as the temperature drops to zero [CRL17a]. On the other hand, some of the divergence examples mentioned above can be adapted to the case of smooth circle expanding maps, as shown in [CRL17a]. However, these examples, as well as those in BGT15, CH10, CRL15b, vER07, are given by constructions that require infinitely many conditions, and they are of infinite codimension.

*For real analytic maps it is an open problem to show that for every real analytic circle expanding map the geometric Gibbs states converge to a ground state as the temperature drops to zero, see [CRL17a].
Our goal is to show that in the next simplest case, of (real and complex) quadratic-like maps on a single variable, the occurrence of zero-temperature phase transitions is a robust phenomenon for 2-parameter families. In fact, our main result implies that there is an open set of 2-parameter families of quadratic-like maps that exhibit a phenomenon of sensitive dependence of Gibbs states, which is similar in spirit to the sensitive dependence on initial conditions that is characteristic of chaotic dynamical systems. More precisely, for a 2-parameter family of quadratic-like maps in this open set, an arbitrarily small perturbation of the parameters can have a drastic effect on the low-temperature geometric Gibbs states. In the companion paper [CRL17b] we show a similar phenomenon at positive temperature, and in [CRL15b] we study it for classical lattice systems. The situation is however significantly simpler in [CRL15b], since the potential is independent of the system and there are no differentiability issues.

One of the main technical difficulties to study of geometric Gibbs states for quadratic-like maps is the presence of the critical point, which is a serious obstruction to uniform hyperbolicity. This leads to some complications, like the fact that there is no obvious characterization of ground states, since there are quadratic maps without a Lyapunov minimizing measure, see for example [BK98, Example 5.4], [BT06, Corollary 2], and [CRL15b, Main Theorem]. In particular, the ergodic optimization approach to study low-temperature Gibbs states, described for example in [BL13, Con16], breaks down for quadratic-like maps.

The main tool introduced in this paper is the “Geometric Peierls condition”. Roughly speaking, it ensures that the geometric Gibbs states concentrate on the critical orbit as the temperature drops to zero, provided there is a well defined Lyapunov exponent at the critical value. Combined with an erratic critical orbit, this creates the divergence of geometric Gibbs states. A somewhat similar idea was used by Hofbauer and Keller to produce an example of a quadratic map without a physical measure [HK90], see also [HK95]. However, the mechanisms are different: Hofbauer and Keller used long parabolic cascades to control almost every point with respect to the Lebesgue measure; we use a fine control of derivatives of orbits far from the critical orbit to control the mass of the geometric Gibbs states at low temperatures.

To state our results more precisely, we recall the concept of quadratic-like maps of Douady and Hubbard [DH85]. Given simply connected subsets $U$ and $V$ of $\mathbb{C}$ such that the closure of $U$ is compact and contained in $V$, a holomorphic map $f: U \to V$ is a quadratic-like map if it is proper of degree 2. Such a map has a unique point at which the derivative $Df$ vanishes; it is the critical point of $f$. The filled-in Julia set of a quadratic-like map $f: U \to V$ is

$$K(f) := \{z \in U \mid \text{for every integer } n \geq 1, f^n(z) \in U\}.$$  

The Julia set $J(f)$ of $f$ is the boundary of $K(f)$, and it coincides with the closure of the repelling periodic points of $f$.

Given a quadratic-like map $f$, denote by $\mathcal{M}_f$ the space of all probability measures on $J(f)$ that are invariant by $f$. For $\mu$ in $\mathcal{M}_f$ denote by $h_\mu(f)$ the measure-theoretic entropy of $\mu$, and for each $t$ in $\mathbb{R}$ put

$$P_f(t) := \sup \left\{ h_\mu(f) - t \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f \right\}. $$
It is the pressure of $f|_{I(f)}$ for the potential $-t \log |Df|$. A measure $\mu$ realizing the supremum above is an equilibrium state of $f|_{I(f)}$ for the potential $-t \log |Df|$ or a geometric Gibbs state.

A quadratic-like map $f: U \to V$ is real if $U$ and $V$ are invariant under complex conjugation and if $f$ commutes with complex conjugation. The critical point $c$ of such a map is real. A real quadratic-like map is essentially topologically exact if $f^2(c)$ is defined and is different from $f(c)$, if $f$ maps the interval $I(f)$ bounded by $f(c)$ and $f^2(c)$ to itself, and if $f|_{I(f)}$ is topologically exact. For such a map $f$ we consider both, the interval map $f|_{I(f)}$, and the complex map $f$ acting on its Julia set $J(f)$.

Let $f$ be a real quadratic-like map that is essentially topologically exact. Denote by $\mathcal{M}_f^\mathbb{R}$ the space of all probability measures on $I(f)$ that are invariant by $f$. For $\mu$ in $\mathcal{M}_f^\mathbb{R}$ we denote by $h_\mu(f)$ the measure-theoretic entropy of $\mu$, and for each $t$ in $\mathbb{R}$ we put

$$P^\mathbb{R}_f(t) := \sup \left\{ h_\mu(f) - t \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f^\mathbb{R} \right\}.$$ 

It is the pressure of $f|_{I(f)}$ for the potential $-t \log |Df|$. A measure $\mu$ realizing the supremum above is an equilibrium state of $f|_{I(f)}$ for the potential $-t \log |Df|$ or a geometric Gibbs state.

**Definition 1.1** (Sensitive dependence of Gibbs states). Let $\Lambda$ be a topological space, and $(f_\lambda)_{\lambda \in \Lambda}$ a continuous family of real or complex quadratic-like maps. The family $(f_\lambda)_{\lambda \in \Lambda}$ has sensitive dependence of low-temperature geometric Gibbs states, if there is a parameter $\lambda_0$ such that for every sequence of inverse temperatures $(\beta_\ell)_{\ell \in \mathbb{N}}$ satisfying $\beta_\ell \to +\infty$ as $\ell \to +\infty$, there is a parameter $\lambda$ in $\Lambda$ arbitrarily close to $\lambda_0$ such that the following property holds: For each $\ell > 0$ there is a unique equilibrium state $\rho_\ell^\mathbb{R}(\lambda)$ of $f|_{I(f_\lambda)}$ (resp. $\rho_\ell(\lambda)$ of $f|_{J(f_\lambda)}$) for the potential $-t \log |Df_\lambda|$, and the sequence of equilibrium states $(\rho_\ell^\mathbb{R}(\lambda))_{\ell \in \mathbb{N}}$ (resp. $(\rho_\ell(\lambda))_{\ell \in \mathbb{N}}$) diverges.

Our main result is stated as the Main Theorem in §3.2. The following is a simple consequence of this result, which is easier to state.

**Sensitive Dependence of Geometric Gibbs States.** There is an open sub-set $\Lambda_0$ of $\mathbb{C}$ intersecting $\mathbb{R}$, a holomorphic family of quadratic-like maps $(\widehat{f}\lambda)_{\lambda \in \Lambda_0}$, and a compact subset $\Lambda$ of $\Lambda_0 \cap \mathbb{R}$, such that the following properties hold. For every real parameter $\lambda$ in $\Lambda$ the map $\widehat{f}\lambda$ is real, and the family of real (resp. complex) maps $(\widehat{f}\lambda)_{\lambda \in \Lambda}$ has sensitive dependence of low-temperature geometric Gibbs states.

We prove that the conclusions of the Sensitive Dependence of Geometric Gibbs States hold for an open set of holomorphic 2-parameter families of quadratic-like maps, see Remark 3.4. Thus, for quadratic-like maps, the sensitive dependence of Gibbs states is a robust phenomenon for 2-parameter families.

Note that the Sensitive Dependence of Geometric Gibbs States does not say anything about the behavior of the low-temperature geometric Gibbs states of $\widehat{f}\lambda_0$. We show that the parameter $\lambda_0$ can be chosen so that the geometric Gibbs states of $\widehat{f}\lambda_0$ converge as the temperature drops to zero, and that $\lambda_0$ can be chosen so that they converge, see Remark 3.5. In the former case we show that the set of accumulation measures of the geometric Gibbs states of $\widehat{f}\lambda_0$ is a segment joining certain periodic measures, see Remark 3.6. In the latter case we show that the convergence of the
geometric Gibbs states is super-exponential, and that the large deviation principle for Gibbs states studied in [BLL13] holds with a degenerated rate function, see Remark 3.7. Our estimates also show that for every $\lambda$ in $\Lambda$, the geometric pressure of $\hat{f}_\lambda$ is super-exponentially close to its asymptote, see Remark 3.8.

For each $\lambda$ in $\Lambda$, the map $\hat{f}_\lambda$ has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense. In fact, the maps in the family $(\hat{f}_\lambda)_{\lambda \in \Lambda}$ satisfy various non-uniform hyperbolicity conditions with uniform constants. For example, the critical orbit is non-recurrent in a uniform way: There is a neighborhood of $z = 0$ that for each $\lambda$ in $\Lambda$ is disjoint of the forward orbit of the critical value of $\hat{f}_\lambda$. Furthermore, all the maps in the family $(\hat{f}_\lambda)_{\lambda \in \Lambda}$ satisfy the Collet-Eckmann condition with uniform constants: There are constants $C > 0$ and $\eta > 1$, such that for every $\lambda$ in $\Lambda$ and every integer $n \geq 1$, we have $|D\hat{f}_\lambda^n(\hat{f}_\lambda(0))| \geq C\eta^n$. Moreover, all maps in $(\hat{f}_\lambda)_{\lambda \in \Lambda}$ have uniform “goodness constants” in the sense of [BBS15, Definition 2.2], cf. Proposition 4.3. This supports the idea that the lack of expansion is not responsible for the sensitive dependence of geometric Gibbs states.

The Sensitive Dependence of Geometric Gibbs States provides the first examples of an analytic map having a “zero-temperature” phase transition. In the case of a quadratic-like map $f$, this completes the classification of phase-transitions for $t > 0$:

- **High-temperature**: A phase transition at the first zero of the geometric pressure function. Such a phase transition appears if and only if $f$ is not uniformly hyperbolic, and if it does not satisfy the Collet-Eckmann condition, see [NS98, Theorem A] or [RL12, Corollary 1.3] for the real case, and [PLS03, Main Theorem] for the complex case;
- **Low-temperature**: A phase transition occurring after the first zero the geometric pressure function. In this case $f$ cannot be uniformly hyperbolic, and it must satisfy the Collet-Eckmann condition, see [CRL13, CRL15a, CRL17b];
- **Zero-temperature**: For every $t > 0$ the geometric pressure function is real analytic at $t$, and there is a unique geometric Gibbs state for the potential $-t \log |Df|$, but these measures diverge as $t \to +\infty$. A map exhibiting such a phase transition must be uniformly hyperbolic, or satisfy the Collet-Eckmann condition.

Roughly speaking, the mechanism responsible for high-temperature phase transitions is the lack of (non-uniform) expansion. However, the lack of (non-uniform) expansion is not responsible for low and zero-temperature phase transitions. The irregular behavior of the critical orbit seems to be responsible for low and zero-temperature phase transitions. As mentioned above, in [CRL17a] we give an example of a smooth circle expanding map having a zero-temperature phase transition. However, it is an open problem if there is a uniformly hyperbolic quadratic-like map having a zero-temperature phase transition.

1.1. **Notes and references.** The family of quadratic-like maps $(\hat{f}_\lambda)_{\lambda \in \Lambda_0}$ in the theorem is given explicitly in [3.3].

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† Compare with the discussion in the introduction of [CRL13].
It follows from the proof of the Sensitive Dependence of Geometric Gibbs states that there is a definite oscillation of the Gibbs states. More precisely, there is a continuous function $\varphi : \mathbb{C} \to [0, 1]$ that only depends on $\lambda_0$, such that for every $(\beta_\ell)_{\ell \in \mathbb{N}}$ and $\lambda$ as in the definition of sensitive dependence of geometric Gibbs states, we have
\[
\limsup_{\ell \to +\infty} \int \varphi \, d\rho_{\beta_\ell}(\lambda) = 1, \quad \text{and} \quad \liminf_{\ell \to +\infty} \int \varphi \, d\rho_{\beta_\ell}(\lambda) = 0.
\]
In fact, at certain temperatures the geometric Gibbs state is super-exponentially close to a certain periodic measure, and at others temperatures they are close to a different periodic measure, see the Main Theorem in §3.2.

1.2. Organization. After some preliminaries about the quadratic family in §2 we state the Main Theorem in §3 and prove the Sensitive Dependence of Geometric Gibbs states assuming this result (§3.4). The Main Theorem is stated for “uniform families” of quadratic-like maps, which are defined in §3.1. This notion is inspired from the work of Douady and Hubbard [DHSS], and it is satisfied for a large class of holomorphic families of quadratic-like maps, see Remark 3.2.

The rest of the paper is devoted to the proof of the Main Theorem. In §4 we introduce the Geometric Peierls Condition (Definition 4.1), which roughly speaking requires the derivatives along the orbit of the critical value to outweigh the derivatives of orbits that stay far from the critical point. We also give a criterion for this condition (Proposition 4.3) whose proof occupies §§4.1, 4.2. In §4.3 we make various estimates for uniform families of maps, most of which are deduced from analogous estimates for quadratic maps in [CRL13]. In §5 we implement an inducing scheme (§5.1), analogous to that in [CRL13] for quadratic maps. For a map satisfying the Geometric Peierls Condition, we also show how to control the pressure of the induced map in terms of the derivatives of the map along the orbit of the critical value (Proposition 6.2 in §5.2).

The proof of the Main Theorem is given in §6. We first estimate in §6.1 the postcritical series in terms of certain 2 variables series that only depends on the combinatorics of the postcritical orbit (Lemma 6.1). The main estimates needed in the proof of the Main Theorem can be stated only in terms of these 2 variables series, and are relegated to Appendix A. These are given in an abstract setting that is independent of the rest of the paper. The proof of the Main Theorem is completed in §6.2.

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2. Preliminaries

We use $\mathbb{N}$ to denote the set of integers that are greater than or equal to 1, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For a Borel measure $\rho$ on $\mathbb{C}$, denote by $\text{supp}(\rho)$ its support.

For an annulus $A$ contained in $\mathbb{C}$, we use $\text{mod}(A)$ to denote the conformal modulus of $A$. 
2.1. Koebe principle. We use the following version of Koebe distortion theorem that can be found, for example, in [McM94]. Given an open subset $G$ of $\mathbb{C}$ and a map $f: G \to \mathbb{C}$ that is a biholomorphism onto its image, the distortion of $f$ on a subset $C$ of $G$ is

$$\sup_{x, y \in C} |Df(x)|/|Df(y)|.$$ 

Koebe Distortion Theorem. For each $A > 0$ there is a constant $\Delta > 1$ such that for each topological disk $\hat{W}$ contained in $\mathbb{C}$ and each compact set $K$ contained in $\hat{W}$ and such that $\hat{W} \setminus K$ is an annulus of modulus at least $A$, the following property holds: For each open topological disk $U$ contained in $\mathbb{C}$ and every biholomorphic map $f: U \to \hat{W}$, for every $x, y$ and $z$ in $f^{-1}(K)$ we have

$$\Delta^{-1}|Df(z)| \leq \frac{|f(x) - f(y)|}{|x - y|} \leq \Delta|Df(z)|.$$ 

Moreover, the distortion of $f$ on $f^{-1}(K)$ is bounded by $\Delta$.

2.2. Quadratic polynomials, Green’s functions, and Böttcher coordinates. In this subsection and the next we recall some basic facts about the dynamics of complex quadratic polynomials, see for instance [CG93] or [MI06] for references.

For $c$ in $\mathbb{C}$ we denote by $f_c$ the complex quadratic polynomial

$$f_c(z) := z^2 + c,$$

and by $K_c$ the filled Julia set of $f_c$; that is, the set of all points $z$ in $\mathbb{C}$ whose forward orbit under $f_c$ is bounded in $\mathbb{C}$. The set $K_c$ is compact and its complement is the connected set consisting of all points whose orbit converges to infinity in the Riemann sphere. Furthermore, we have $f_c^{-1}(K_c) = K_c$ and $f_c(K_c) = K_c$. The boundary $J_c$ of $K_c$ is the Julia set of $f_c$.

For a parameter $c$ in $\mathbb{C}$, the Green’s function of $K_c$ is the function $G_c: \mathbb{C} \to [0, +\infty)$ that is identically 0 on $K_c$, and that for $z$ outside $K_c$ is given by the limit,

$$G_c(z) := \lim_{n \to +\infty} \frac{1}{2^n} \log |f_c^n(z)| > 0. \quad (2.1)$$

The function $G_c$ is continuous, subharmonic, satisfies $G_c \circ f_c = 2G_c$ on $\mathbb{C}$, and it is harmonic and strictly positive outside $K_c$. On the other hand, the critical values of $G_c$ are bounded from above by $G_c(0)$, and the open set

$$U_c := \{z \in \mathbb{C} \mid G_c(z) > G_c(0)\}$$

is homeomorphic to a punctured disk. Notice that $G_c(c) = 2G_c(0)$, thus $U_c$ contains $c$ if 0 is not in $K_c$.

By Böttcher’s Theorem there is a unique conformal representation

$$\varphi_c: U_c \to \{z \in \mathbb{C} \mid |z| > \exp(G_c(0))\},$$

and this map conjugates $f_c$ to $z \mapsto z^2$. It is called the Böttcher coordinate of $f_c$ and satisfies $G_c = \log |\varphi_c|$.

2.3. External rays and equipotentials. Let $c$ be in $\mathbb{C}$. For $v > 0$ the equipotential $v$ of $f_c$ is by definition $G_c^{-1}(v)$.

A Green’s line of $G_c$ is a smooth curve on the complement of $K_c$ in $\mathbb{C}$ that is orthogonal to the equipotentials of $G_c$ and that is maximal with this property. Given $t$ in $\mathbb{R}/\mathbb{Z}$, the external ray of angle $t$ of $f_c$, denoted by $R_c(t)$, is the Green’s line of $G_c$ containing

$$\{\varphi_c^{-1}(r \exp(2\pi it)) \mid \exp(G_c(0)) < r < +\infty\}.$$
By the identity $G_c \circ f_c = 2G_c$, for each $v > 0$ and each $t$ in $\mathbb{R}/\mathbb{Z}$ the map $f_c$ maps the equipotential $v$ to the equipotential $2v$ and maps $R_c(t)$ to $R_c(2t)$. For $t$ in $\mathbb{R}/\mathbb{Z}$ the external ray $R_c(t)$ lands at a point $z$, if $G_c : R_c(t) \to (0, +\infty)$ is a bijection and if $G_c^{-1}_{|R_c(t)}(v)$ converges to $z$ as $v$ converges to 0 in $(0, +\infty)$. By the continuity of $G_c$, every landing point is in $J_c = \partial K_c$.

The Mandelbrot set $\mathcal{M}$ is the subset of $\mathbb{C}$ of those parameters $c$ for which $K_c$ is connected. The function
\[
\Phi : \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \text{cl}(\mathbb{D})
\]
\[c \mapsto \Phi(c) := \varphi_c(c)
\]
is a conformal representation, see [DHS84, VIII, Théorème 1]. For $v > 0$ the equipotential $v$ of $\mathcal{M}$ is by definition
\[
\mathcal{E}(v) := \Phi^{-1}(\{z \in \mathbb{C} \mid |z| = v\}).
\]
On the other hand, for $t$ in $\mathbb{R}/\mathbb{Z}$ the set
\[
\mathcal{R}(t) := \Phi^{-1}(\{r \exp(2\pi i t) \mid r > 1\})
\]
is called the external ray of angle $t$ of $\mathcal{M}$. We say that $\mathcal{R}(t)$ lands at a point $z$ in $\mathbb{C}$, if $\Phi^{-1}(r \exp(2\pi i t))$ converges to $z$ as $r \searrow 1$. When this happens $z$ belongs to $\partial \mathcal{M}$.

2.4. The wake 1/2. In this subsection we recall a few facts that can be found for example in [DHS84] or [Mil00].

The external rays $\mathcal{R}(1/3)$ and $\mathcal{R}(2/3)$ of $\mathcal{M}$ land at the parameter $c = -3/4$, and these are the only external rays of $\mathcal{M}$ that land at this point, see for example [Mil00, Theorem 1.2]. In particular, the complement in $\mathbb{C}$ of the set
\[
\mathcal{R}(1/3) \cup \mathcal{R}(2/3) \cup \{-3/4\}
\]
has 2 connected components; we denote by $\mathcal{W}$ the connected component containing the point $c = -2$ of $\mathcal{M}$.

For each parameter $c$ in $\mathcal{W}$ the map $f_c$ has 2 distinct fixed points; one of the them is the landing point of the external ray $R_c(0)$ and it is denoted by $\beta(c)$; the other one is denoted by $\alpha(c)$. The only external ray landing at $\beta(c)$ is $R_c(0)$, and the only external ray landing at $-\beta(c)$ is $R_c(1/2)$.

Moreover, for every parameter $c$ in $\mathcal{W}$ the only external rays of $f_c$ landing at $\alpha(c)$ are $R_c(1/3)$ and $R_c(2/3)$, see for example [Mil00, Theorem 1.2]. The complement of $R_c(1/3) \cup R_c(2/3) \cup \{\alpha(c)\}$ in $\mathbb{C}$ has 2 connected components; one containing $-\beta(c)$ and $z = c$, and the other one containing $\beta(c)$ and $z = 0$. On the other hand, the point $\alpha(c)$ has 2 preimages by $f_c$: Itself and $\alpha(c) := -\alpha(c)$. The only external rays landing at $\alpha(c)$ are $R_c(1/6)$ and $R_c(5/6)$.

2.5. Yoccoz puzzles and para-puzzle. In this subsection we recall the definitions of Yoccoz puzzle and para-puzzle. We follow [Roe00].

Definition 2.1 (Yoccoz puzzles). Fix $c$ in $\mathcal{W}$ and consider the open region $X_c := \{z \in \mathbb{C} \mid G_c(z) < 1\}$. The Yoccoz puzzle of $f_c$ is given by the following sequence of graphs $(I_{c,n})_{n=0}^{+\infty}$ defined for $n = 0$ by:
\[
I_{c,0} := \partial X_c \cup (X_c \cap \text{cl}(R_c(1/3)) \cap \text{cl}(R_c(2/3))），
\]
and for $n \geq 1$ by $I_{c,n} := f_{c^{-n}}(I_{c,0})$. The puzzle pieces of depth $n$ are the connected components of $f_{c^{-n}}(X_c) \setminus I_{c,n}$. The puzzle piece of depth $n$ containing a point $z$ is denoted by $P_{c,n}(z)$. 
Note that for a real parameter $c$, every puzzle piece intersecting the real line is invariant under complex conjugation. Since puzzle pieces are simply-connected, it follows that the intersection of such a puzzle piece with $\mathbb{R}$ is an interval.

**Definition 2.2** (Yoccoz para-puzzle). Given an integer $n \geq 0$, put

$$J_n := \{ t \in [1/3, 2/3] \mid 2^n t \ (\text{mod} \ 1) \in \{1/3, 2/3\}\},$$

let $\mathcal{X}_n$ be the intersection of $\mathcal{W}$ with the open region in the parameter plane bounded by the equipotential $E(2^{-n})$ of $\mathcal{M}$, and put

$$I_n := \partial \mathcal{X}_n \cup \left( \mathcal{X}_n \cap \bigcup_{t \in J_n} \text{cl}(R(t)) \right).$$

Then the Yoccoz para-puzzle of $\mathcal{W}$ is the sequence of graphs $(I_n)_{n=0}^{+\infty}$. The para-puzzle pieces of depth $n$ are the connected components of $\mathcal{X}_n \setminus I_n$. The para-puzzle piece of depth $n$ containing a parameter $c$ is denoted by $P_n(c)$.

Observe that there is only 1 para-puzzle piece of depth 0, and only 1 para-puzzle piece of depth 1; they are bounded by the same external rays but different equipotentials. Both of them contain $c = -2$.

Fix a parameter $c$ in $\mathcal{P}_0(-2)$. There are precisely 2 puzzle pieces of depth 0: $P_{c,0}(\beta(c))$ and $P_{c,0}(-\beta(c))$. Each of them is bounded by the equipotential 1 and by the closures of the external rays landing at $\alpha(c)$. Furthermore, the critical value $c$ of $f_c$ is contained in $P_{c,0}(-\beta(c))$ and the critical point in $P_{c,0}(\beta(c))$. It follows that the set $f_c^{-1}(P_{c,0}(\beta(c)))$ is the disjoint union of $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$, so $f_c$ maps each of the sets $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$.

Moreover, there are precisely 3 puzzle pieces of depth 1:

$$P_{c,1}(-\beta(c)), P_{c,1}(0) \text{ and } P_{c,1}(\beta(c));$$

$P_{c,1}(-\beta(c))$ is bounded by the equipotential 1/2 and by the closures of the external rays that land at $\alpha(c)$; $P_{c,1}(\beta(c))$ is bounded by the equipotential 1/2 and by the closures of the external rays that land at $\tilde{\alpha}(c)$; and $P_{c,1}(0)$ is bounded by the equipotential 1/2 and by the closures of the external rays that land at $\alpha(c)$ and at $\tilde{\alpha}(c)$. In particular, the closure of $P_{c,1}(\beta(c))$ is contained in $P_{c,0}(\beta(c))$.

It follows from this that for each integer $n \geq 1$ the map $f_c^n$ maps $P_{c,n}(-\beta(c))$ biholomorphically to $P_{c,0}(\beta(c))$.

2.6. The uniformly expanding Cantor set. For a parameter $c$ in $\mathcal{P}_3(-2)$, the maximal invariant set $\Lambda_c$ of $f_c^3$ in $P_{c,1}(0)$ plays an important rôle in the proof of the Main Theorem.

Fix $c$ in $\mathcal{P}_3(-2)$. There are precisely 2 connected components of $f_c^{-3}(P_{c,1}(0))$ contained in $P_{c,1}(0)$ that we denote by $Y_c$ and $\tilde{Y}_c$. The closures of these sets are disjoint and contained in $P_{c,1}(0)$. The sets $Y_c$ and $\tilde{Y}_c$ are distinguished by the fact that $Y_c$ contains in its boundary the common landing point of the external rays $R_c(7/24)$ and $R_c(17/24)$, denoted $\gamma(c)$, and that $\tilde{Y}_c$ contains in its boundary the common landing point of the external rays $R_c(5/24)$ and $R_c(19/24)$.

\[\dagger\] In contrast to [Roe00], we only consider para-puzzles contained in $\mathcal{W}$. 

map \( f_c^3 \) maps each of the sets \( Y_c \) and \( \tilde{Y}_c \) biholomorphically to \( P_{c,1}(0) \). Thus, if we put
\[
g_c : Y_c \cup \tilde{Y}_c \to P_{c,1}(0)
\]
\[
z \mapsto g_c(z) := f_c^3(z),
\]
then
\[
\Lambda_c = \bigcap_{n \in \mathbb{N}} g_c^{-n}(\text{cl}(P_{c,1}(0))).
\]

2.7. Parameters. In this subsection we recall the definition of a certain parameter sets in [CRL13, Proposition 3.1] that are important in what follows.

Given an integer \( n \geq 3 \), let \( \mathcal{K}_n \) be the set of all those real parameters \( c < 0 \) such that
\[
f_c(c) > f_c^2(c) > \cdots > f_c^{n-1}(c) > 0 \quad \text{and} \quad f_c^n(c) \in \Lambda_c.
\]
Note that for a parameter \( c \) in \( \mathcal{K}_n \), the critical point of \( f_c \) cannot be asymptotic to a non-repelling periodic point. This implies that all the periodic points of \( f_c \) in \( \mathbb{C} \) are hyperbolic repelling and therefore that \( K_c = J_c \), see [Mil06]. On the other hand, we have \( f_c(c) > c \) and the interval \( I_c = [c, f_c(c)] \) is invariant by \( f_c \). This implies that \( I_c \) is contained in \( J_c \) and hence that for every real number \( t \) we have \( P_c^t(t) \leq P_c(t) \). Note also that \( f_c | I_c \) is not renormalizable, so \( f_c \) is topologically exact on \( I_c \), see for example [MS03, Theorem III.4.1].

Since for \( c \) in \( \mathcal{K}_n \) the critical point of \( f_c \) is not periodic, for every integer \( k \geq 0 \) we have \( f_c^{n+3k}(c) \neq 0 \). Thus, we can define the sequence \( \iota(c) \) in \( \{0,1\}^{\mathbb{N}_0} \) for each \( k \geq 0 \) by
\[
\iota(c)_k := \begin{cases} 0 & \text{if } f_c^{n+3k}(c) \in Y_c; \\ 1 & \text{if } f_c^{n+3k}(c) \in \tilde{Y}_c. \end{cases}
\]

**Proposition 2.3.** For each integer \( n \geq 3 \), the set \( \mathcal{K}_n \) is a compact subset of \( \mathcal{P}_n(-2) \cap (-2, -3/4) \), and the function \( \iota : \mathcal{K}_n \to \{0,1\}^{\mathbb{N}_0} \) is homeomorphism. Finally, for each \( \delta > 0 \) there is \( n_0 \geq 3 \) such that for each integer \( n \geq n_0 \) the set \( \mathcal{K}_n \) is contained in the interval \((-2, -2 + \delta)\).

**Proof.** Except for the assertion that \( \iota \) is a homeomorphism, this is [CRL13, Proposition 3.1]. In this last result it is shown that \( \iota \) is a bijection, so it only remains to observe that, since for each \( c \) in \( \mathcal{P}_3(-2) \) the map \( f_c \) is uniformly expanding on \( \Lambda_c \) [CRL13, §3.3], the map \( \iota \) is continuous, and therefore a homeomorphism. \( \square \)

3. Main Theorem

In this section we state the Main Theorem, and prove the Sensitive Dependence of Geometric Gibbs States assuming this result.

The Main Theorem, stated in §3.2, is stated for “uniform families” of quadratic-like maps, which are defined in §3.1. By the work of Douady and Hubbard [DHS85], there is a large class of holomorphic families of quadratic-like maps that are uniform, see Remark 3.2. We use this to exhibit in §3.3 a concrete (real) 1-parameter family of quadratic-like maps satisfying the hypotheses of the Main Theorem. This family is used in §3.4 to prove the Sensitive Dependence of Geometric Gibbs States assuming the Main Theorem.
3.1. Uniform families of quadratic-like maps. A quadratic-like map \( f: U \to V \) is normalized, if its unique critical point is 0, and if \( D^2 f(0) = 2 \). For such a map \( f \) there is a holomorphic function \( R_f: U \to \mathbb{C} \) such that for \( w \) in \( U \) we have
\[
f(w) = f(0) + w^2 + w^3 R_f(w).
\]
Note that \( f \) is uniquely determined by its critical value \( f(0) \), and the function \( R_f \).

By the straightening theorem of Douady and Hubbard [DH85], for every quadratic-like map \( f: U \to V \) there is \( c \in \mathbb{C} \) and a quasi-conformal homeomorphism \( h: \mathbb{C} \to \mathbb{C} \) that conjugates the quadratic polynomial \( f_c \) to \( f \) on a neighborhood of \( J_c \). In the case \( f \) is real, \( c \) is real, and \( h \) can be chosen so that it commutes with the complex conjugation. In all the cases, the quasi-conformal homeomorphism \( h \) can be chosen to be holomorphic on a neighborhood of infinity, and tangent to the identity there.

Put
\[
\mathcal{X} := \{ c \in \mathbb{C} \mid G_c(c) \leq 1 \} \quad \text{and} \quad \hat{\mathcal{X}} := \{ c \in \mathbb{C} \mid G_c(c) \leq 2 \},
\]
and for \( c \) in \( \mathbb{C} \), put
\[
X_c := \{ z \in \mathbb{C} \mid G_c(z) \leq 1 \} \quad \text{and} \quad \hat{X}_c := \{ z \in \mathbb{C} \mid G_c(z) \leq 2 \}.
\]
Note that \( X_c \) is contained in the interior of \( \hat{X}_c \), and that
\[
\mathcal{X} = \{ c \in \mathbb{C} \mid c \in X_c \} \quad \text{and} \quad \hat{\mathcal{X}} = \{ c \in \mathbb{C} \mid c \in \hat{X}_c \}.
\]

**Definition 3.1** (Uniform family of quadratic-like maps). A family \( \mathcal{F} \) of normalized quadratic-like maps is uniform, if there are constants \( K \geq 1 \) and \( R > 0 \), such that for each \( f \) in \( \mathcal{F} \) there are \( c(f) \) in \( \mathcal{X} \) and a \( K \)-quasi-conformal homeomorphism \( h_f \) of \( \mathbb{C} \) satisfying the following properties.

1. The homeomorphism \( h_f \) conjugates \( f_{c(f)} \) on \( \hat{X}_{c(f)} \) to \( f \) on \( h_f(\hat{X}_{c(f)}) \). Furthermore, if \( f \) is real, then \( h_f \) commutes with the complex conjugation.
2. The set \( \hat{X}_{c(f)} \) is contained in \( B(0, R) \), and the homeomorphism \( h_f \) is holomorphic on \( \mathbb{C} \setminus \text{cl}(B(0, R)) \), and it is tangent to the identity at infinity.

Note that property 1 implies that \( h_f(0) = 0 \).

**Remark 3.2.** Although it is not needed in this paper, we remark that a family \( \mathcal{F} \) of normalized quadratic-like maps with connected Julia sets is uniform if and only if the following property holds: There is a constant \( m > 0 \) such that for each \( f: U \to V \) in \( \mathcal{F} \) there is an essential annulus in \( V \setminus U \) whose conformal modulus is at least \( m \).

Let \( \mathcal{F} \) be a uniform family of quadratic-like maps. For each \( f \) in \( \mathcal{F} \) put
\[
X_f := h_f(X_{c(f)}) \quad \text{and} \quad \hat{X}_f := h_f(\hat{X}_{c(f)}).
\]
By the definition of uniform family, the puzzle pieces of \( f_{c(f)} \) can be push-forward to \( X_f \) by \( h_f \). We call to these sets the puzzle pieces of \( f \). We say that a puzzle piece of \( f \) has depth \( n \) if it is the push-forward of a puzzle piece of \( c(f) \) with depth \( n \). The puzzle piece of depth \( n \) of \( f \) containing \( w \) is denoted \( P_{f,n}(w) \). Thus, we have
\[
P_{f,n}(w) := h_f(P_{c(f),n}(h_f^{-1}(w))).
\]
Set
\[
\beta(f) := h_f(\beta(c(f))) \quad \text{and} \quad \bar{\beta}(f) := h_f(-\beta(c(f))).
\]
For every integer \( n \geq 0 \), put
\[
\mathcal{P}_n(\mathcal{F}) := \{ f \in \mathcal{F} \mid c(f) \in \mathcal{P}_n(-2) \},
\]
and for \( n \geq 3 \), put
\[
\mathcal{K}_n(\mathcal{F}) := \{ f \in \mathcal{F} \mid c(f) \in \mathcal{K}_n \}.
\]
Moreover, for \( f \) in \( \mathcal{P}_2(\mathcal{F}) \) put
\[
Y_f := h_f(Y_{c(f)}), \quad \text{and} \quad \tilde{Y}_f := h_f(\tilde{Y}_{c(f)}),
\]
and let \( g_f : h_f(Y_{c(f)} \cup \tilde{Y}_{c(f)}) \to P_{1,1}(0) \) be defined by \( g_f := h_f^{-1} \circ g_{c(f)} \circ h_f \). Denote by \( p(f) \) and \( p^+(f) \) the unique fixed point of \( g_f \) in \( Y_f \) and \( \tilde{Y}_f \), respectively, and denote by \( p^- (f) \) the unique fixed point of \( g_f^2 \) in \( \tilde{Y}_f \) that is different from \( p^+(f) \); it is a periodic point of \( g_f \) of minimal period 2. Furthermore, denote by
\[
\mathcal{O}^+(f) := \{ f^j(p^+(f)) \mid j \in \{ 0, 1, 2 \} \} \quad \text{and} \quad \mathcal{O}^-(f) := \{ f^j(p^-(f)) \mid j \in \{ 0, 1, \ldots, 5 \} \}
\]
the orbits of \( p^+(f) \) and \( p^-(f) \) under \( f \), respectively.

For each integer \( n \geq 5 \), and each \( f \in \mathcal{K}_n(\mathcal{F}) \), put \( \iota(f) := \iota(c(f)) \), see \S2.7 and note that for every integer \( j \geq 0 \) we have
\[
\iota(f)_j := \begin{cases} 0 & \text{if } f^{n+1+3j}(0) \in Y_f; \\ 1 & \text{if } f^{n+1+3j}(0) \in \tilde{Y}_f. \end{cases}
\]

Finally, for every \( f \) in \( \mathcal{F} \) such that \( c(f) \) is real, denote by \( I(f) \) the image under \( h_f \) of the interval \([c(f), f_{c(f)}(c(f))]\). When \( c(f) \) is in \([-2, 0] \), the set \( I(f) \) is invariant by \( f \). If in addition \( f \) is real, then \( I(f) \) is a subinterval of \( \mathbb{R} \).

### 3.2. Main Theorem
For every normalized quadratic-like map \( f \), and every periodic point \( p \) of \( f \) with period \( m \) in \( \mathbb{N} \), put
\[
\chi_f(p) := \frac{1}{m} \log |Df^m(p)|.
\]

In this subsection we state the Main Theorem, which is based on the following concept.

**Definition 3.3** (Admissible family of quadratic-like maps). A uniform family of quadratic-like maps \( \mathcal{F} \) is **admissible**, if for every sufficiently large integer \( n \geq 6 \) the following properties hold.

1. If we endow \( \mathcal{F} \) with the topology of locally uniform convergence, then there is a continuous function \( s_n : \mathcal{K}_n \to \mathcal{K}_n(\mathcal{F}) \) such that \( c \circ s_n \) is the identity.
2. For every \( f \) in \( s_n(\mathcal{K}_n) \), we have
\[
\chi_f(p(f)) > \chi_f(p^+(f)) \quad \text{and} \quad \chi_f(p^+(f)) = \chi_f(p^-(f)).
\]

Endow the set \( \{+,-\} \) with the discrete topology, and \( \{+,-\}^\mathbb{N} \) with the corresponding product topology.

**Main Theorem.** For every \( R > 0 \) there is a constant \( K_0 > 1 \) such that if \( \mathcal{F} \) is an admissible uniform family of quadratic-like maps with constants \( K_0 \) and \( R \), then for every sufficiently large integer \( n \) there is a continuous subfamily \( \{ f_\xi \}_{\xi \in \{+,-\}^\mathbb{N}} \) of \( s_n(\mathcal{K}_n) \) such that the following properties hold.

1. For each \( \xi \) in \( \{+,-\}^\mathbb{N} \) the map \( f_\xi \) is essentially topologically exact. Moreover, for each \( t > 0 \) there is a unique equilibrium state \( \rho_t(\xi) \) (resp. \( \rho_t(\xi) \)) of \( f_\xi^{I(f_\xi)} \) (resp. \( f_\xi^{J(f_\xi)} \)) for the potential \(-t \log |Df_\xi|\).
2. There are constants $C_0 > 0$ and $v_0 > 0$, and a continuous function $A: \{+, -\}^N \to (0, +\infty)$, such that for every sequence $\varsigma = (\varsigma(m))_{m \in \mathbb{N}}$ in $\{+, -\}^N$, the following properties hold. Let $m$ and $\tilde{m}$ be integers such that

$$\tilde{m} \geq m \geq 1 \quad \text{and} \quad \varsigma(m) = \cdots = \varsigma(\tilde{m}),$$

and let $t$ be in $[A(\varsigma)m, A(\varsigma)\tilde{m}]$. Then the equilibrium state $\rho^\varsigma(\hat{f}_\varsigma)$ (resp. $\rho_t(\hat{f}_\varsigma)$) of $f_{\hat{2}}(f_{\hat{1}})$ (resp. $f_{\hat{1}}(f_{\hat{2}})$) is super-exponentially close to the orbit $O^{(m)}(f_{\hat{\varsigma}})$ of $p^{(m)}(f_{\hat{\varsigma}})$, in that

$$\rho^\varsigma(\hat{f}_\varsigma) \left(B \left(O^{(m)}(f_{\hat{\varsigma}}), \exp(-v_0t^2)\right)\right) \geq 1 - C_0 \exp(-v_0t^2)$$

(resp. $\rho_t(\hat{f}_\varsigma) \left(B \left(O^{(m)}(f_{\hat{\varsigma}}), \exp(-v_0t^2)\right)\right) \geq 1 - C_0 \exp(-v_0t^2)$).

Note that for each $\varsigma$ in $\{+, -\}^N$ the map $f_{\hat{\varsigma}}$ has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense.

**Remark 3.4 (Robustness).** It follows from the theory of quadratic-like maps of Douady and Hubbard [DHSS] that condition 1 of Definition 3.3 is satisfied for every holomorphic 1-parameter family of quadratic-like maps $(\hat{f}_\lambda)_{\lambda \in \Lambda_0}$ intersecting the combinatorial class of the quadratic map $f_{-2}$ transversally. That is, if there is a parameter $\lambda_0$ in $\Lambda_0$ such that

$$\hat{f}_{\lambda_0}(0) = \beta(\hat{f}_{\lambda_0}), \quad \text{and} \quad \frac{\partial}{\partial \lambda} \left(\hat{f}_{\lambda}(0) - \beta(\hat{f}_{\lambda})\right)|_{\lambda = \lambda_0} \neq 0.$$

So, condition 1 of Definition 3.3 is satisfied for an open set of holomorphic 1-parameter families of quadratic-like maps. If in addition $\chi_{\hat{f}_{\lambda_0}}(p(\hat{f}_{\lambda_0})) > \chi_{\hat{f}_{\lambda_0}}(p^+(\hat{f}_{\lambda_0}))$, then the inequality in (3.1) is also satisfied for an open set of holomorphic 1-parameter families of quadratic-like maps.

On the other hand, the equality in (3.1) imposes a restriction, but there is an open set of holomorphic 2-parameter families of quadratic-like maps that have a holomorphic 1-parameter subfamily satisfying this condition. Thus, the conclusions of the Main Theorem hold for an open set of holomorphic 2-parameter families of quadratic-like maps.

**Remark 3.5 (Sensitivity is compatible with divergence, and with convergence).** In the proof of the Main Theorem we show that for any choice of $\varsigma$ in $\{+, -\}^N$, a uniform family $\mathcal{F}$ as in the Main Lemma has sensitive dependence of low-temperature geometric Gibbs states at $f_{\hat{\varsigma}}$. If we choose $\varsigma$ that is not eventually constant, then the Main Theorem implies that the geometric Gibbs states of $f_{\hat{\varsigma}}$ diverge as the temperature drops to zero. On the other hand, if $\varsigma$ is eventually constant, then the geometric Gibbs states converge. This shows that in the Sensitive Dependence of Geometric Gibbs states the parameter $\lambda_0$ can be chosen so that the geometric Gibbs states of $f_{\hat{\lambda_0}}$ diverge as the temperature drops to zero, and that it can also be chosen so that they converge.

**Remark 3.6 (Accumulation measures).** Our estimates show that for every $\varsigma$ in $\{+, -\}^N$, and every $t > 0$ we have

$$\rho^\varsigma(\hat{f}_\varsigma) \left(B \left(O^+(f_{\hat{\varsigma}}) \cup O^-(f_{\hat{\varsigma}}), \exp(-v_0t^2)\right)\right) \geq 1 - C_0 \exp(-v_0t^2)$$

(resp. $\rho_t(\hat{f}_\varsigma) \left(B \left(O^+(f_{\hat{\varsigma}}) \cup O^-(f_{\hat{\varsigma}}), \exp(-v_0t^2)\right)\right) \geq 1 - C_0 \exp(-v_0t^2)$).
see Remark 5.2. In particular, every accumulation measure of $(\rho_t^\beta(\xi))_{t>0}$ and of $(\rho_t(\xi))_{t>0}$ is supported on $\mathcal{O}^+(f_\xi) \cup \mathcal{O}^-(f_\xi)$. Combined with the Main Theorem this implies that, if the sequence $\xi$ is not eventually constant, then the set of accumulation measures of $(\rho_t^\beta(\xi))_{t>0}$ and that of $(\rho_t(\xi))_{t>0}$, are both equal to the segment joining the invariant probability measure supported on $\mathcal{O}^+(f_\xi)$ to the invariant probability measure supported on $\mathcal{O}^-(f_\xi)$.

**Remark 3.7 (Speed of convergence to ground states).** If the sequence $\xi$ is eventually constant, then the Main Theorem implies that, as the temperature drops to zero, the geometric Gibbs states of $f_\xi$ converge super-exponentially to the periodic measure supported on $\mathcal{O}^+(f_\xi)$ or $\mathcal{O}^-(f_\xi)$. In other situations the convergence is only exponential, as in the case of the shift map and a locally constant potential [Bre03]. For the shift map and a potential admitting a unique ground state, the exponential convergence can be derived from the large deviation principle in [BLL13, §3.1.3], using the fact that the rate function is finite on a dense set. The Main Theorem shows that this large deviation principle holds, and that the corresponding rate function is everywhere equal to $+\infty$, except on $\mathcal{O}^+(f_\xi)$ or on $\mathcal{O}^-(f_\xi)$ (depending on the choice of $\xi$ where it vanishes.

**Remark 3.8 (Pressure at low temperatures).** Our estimates show that there is a constant $\gamma$ in $(0, 1)$ such that for every $\xi$ in $\{+, -\}^\mathbb{N}$, and every sufficiently large $t > 0$ we have

$$P^R_{f_\xi}(t) \sim P_{f_\xi}(t) \sim -t \frac{\chi_{\text{crit}}(f_\xi)}{2} + \frac{\log t}{3} \gamma f^{\beta}(f_\xi)$$

see [6.11] and [6.13] for precisions.

### 3.3. A concrete admissible family.

In this subsection we exhibit a concrete (real) 1-parameter family of quadratic-like maps satisfying the hypotheses of the Main Theorem. We use this family to prove the Sensitive Dependence of Geometric Gibbs States, in §3.4 below.

For each parameter $\lambda$ in $\mathcal{P}_3(-2)$, put $p^-(\lambda) := p^-(f_\lambda)$ and define the polynomial

$$P_\lambda(w) := (w^2 - \beta(\lambda))^2 \prod_{i=0}^2 [(w - f_\lambda^i(p(\lambda))) (w - f_\lambda^{i+1}(\lambda))]^2 \cdot (w - p^-(\lambda))^{\sum_{j=1}^5 (w - f_\lambda^j(p^-(\lambda)))^2}.$$

Noting that $DP_\lambda(p^-(\lambda)) \neq 0$, define

$$\omega(\lambda) := \frac{2}{p^-(\lambda)^2 DP_\lambda(p^-(\lambda))} \frac{(DF^p_\lambda(p^+(\lambda)))^2}{DF^p_\lambda(p^-(\lambda))} - 1,$$

and the polynomial

$$\hat{f}_\lambda(w) := \lambda + w^2 + w^3 \omega(\lambda) P_\lambda(w).$$

Note that each of the coefficients of $\hat{f}_\lambda$ depends holomorphically with $\lambda$ in $\mathcal{P}_3(-2)$, and that $\hat{f}_\lambda$ is real when $\lambda$ is real. Moreover, we have $\omega(-2) = 0$, so $\hat{f}_{-2}$ coincides with the quadratic polynomial $f_{-2}$.

By definition, for each $\lambda$ in $\mathcal{P}_3(-2)$ the polynomial $\hat{f}_\lambda$ coincides with $f_\lambda$ on $\pm \beta(\lambda)$ and on the orbits of $p(\lambda)$, $p^+(\lambda)$, and $p^-(\lambda)$. Moreover, the derivative of $\hat{f}_\lambda$ coincides

with the derivative of $f_\lambda$.
with that of $f_\lambda$ at every point in the orbit of $p(\lambda)$ and $p^+(\lambda)$, so
\begin{equation}
\chi_{\hat{f}_\lambda}(p(\lambda)) = \chi_{f_\lambda}(p(\lambda)) \quad \text{and} \quad \chi_{\hat{f}_\lambda}(p^+(\lambda)) = \chi_{f_\lambda}(p^+(\lambda)).
\end{equation}

On the other hand,
\[ D\hat{f}_\lambda(p^-(\lambda)) = 2p^-(\lambda) \frac{(Df_\lambda^3(p^+(\lambda)))^2}{Df_\lambda^3(p^-(\lambda))}, \]
and for each $j$ in $\{1, \ldots, 5\}$ the derivative of $\hat{f}_\lambda$ coincides with that of $f_\lambda$ at $f_\lambda^j(p^-(\lambda))$.
Thus $D\hat{f}_\lambda^j(p^-(\lambda)) = (Df_\lambda^j(p^+(\lambda)))^2$ and
\begin{equation}
\chi_{\hat{f}_\lambda}(p^-(\lambda)) = \chi_{f_\lambda}(p^+(\lambda)) = \chi_{\hat{f}_\lambda}(p^+(\lambda)).
\end{equation}

**Lemma 3.9.** Let $K_0 > 1$ be given. For each $\lambda$ in $P_3(-2)$, put $U_\lambda := \hat{f}_\lambda^{-1}(B(0, 80))$.
Then there is $r_\# > 0$ such that for every $\lambda$ in $B(-2, r_\#)$ the map $\hat{f}_\lambda : U_\lambda \to B(0, 80)$ is a normalized quadratic-like map, and the family
\[ \mathcal{F}_0 := \{ \hat{f}_\lambda : U_\lambda \to B(0, 80) \mid \lambda \in B(-2, r_\#) \} \]
is uniform with constants $K_0$ and $R = 80$, and such that for some $\delta > 0$ the map $c$ maps $\{ \hat{f}_\lambda \mid \lambda \in [-2, -2 + r_\#]\}$ homeomorphically onto $[-2 - 2 + \delta]$. Moreover, there is $n_\# \geq 1$ such that for every integer $n \geq n_\#$ there is a continuous map $\sigma_n : K_n \to [-2, -2 + r_\#]$ such that $\lambda \mapsto c(\hat{f}_{\sigma_n(\lambda)})$ is the identity on $K_n$, and such that for every $f$ in $\sigma_n(K_n)$ we have $\chi_f(p(f)) > \chi_f(p^+(f))$.

**Proof.** Since $\omega(-2) = 0$, and $\omega$ and $P_\lambda$ are holomorphic in $\lambda$, we can choose $r_1 > 0$ such that $B(-2, r_1)$ is contained in $P_3(-2)$ and such that for every $\lambda$ in $B(-2, r_1)$ the closure of the open set $U_\lambda$ is contained in $B(0, 80)$ and $\hat{f}_\lambda : U_\lambda \to B(0, 80)$ is a quadratic-like map. For each $r$ in $(0, r_1]$, consider the family of quadratic-like maps
\[ \mathcal{F}(r) := \{ \hat{f}_\lambda : U_\lambda \to B(0, 80) \}_{\lambda \in B(0, r)}. \]

Noting that for $\lambda$ close to $-2$ the set $\partial U_\lambda$ is an analytic Jordan curve that is close to $\partial U_{-2}$ in the $C^1$ topology, it follows that there is $r_2$ in $(0, r_1)$ such that the family of quadratic-like maps $\mathcal{F}(r_2)$ is analytic in the sense of [DH85, §II, 1]. Moreover, the considerations in [DH85, §II, 2] imply that there is $r_3$ in $(0, r_2)$ such that the family $\mathcal{F}(r_3)$ is uniform with constants $K_0$ and $R = 80$, and such that for every real parameter $\lambda$ in $B(0, r_3)$ the conjugacy $h_{\hat{f}_\lambda}$ commutes with the complex conjugation, and therefore $c(\hat{f}_\lambda)$ is real. For $\lambda$ in $B(-2, r_3)$ put $c(\lambda) := c(\hat{f}_\lambda)$. If $\lambda$ in $B(-2, r_3)$ is real and satisfies $\lambda > -2$, then we have $\hat{f}_\lambda(0) = \lambda > -\beta(\lambda)$. Together with $\hat{f}_\lambda(-\beta(\lambda)) = \hat{f}_\lambda(\beta(\lambda)) = \beta(\lambda)$ this implies that $c(\hat{f}_\lambda) > -2$. By CRL13 Lemma A.1] there is $\varphi > 0$ such that for every $\lambda$ in $(-2 - 2 + \varphi)$ we have by \cite{EH85} Lemma A.1]
\begin{equation}
\chi_{\hat{f}_\lambda}(p(\lambda)) = \chi_{f_\lambda}(p(\lambda)) > \chi_{f_\lambda}(p^+(\lambda)) = \chi_{\hat{f}_\lambda}(p^+(\lambda)).
\end{equation}

Thus, reducing $r_3$ if necessary, the inequality in \cite{EH85} Lemma A.1] is satisfied.

Finally, note that [DH85] Proposition 17 and Theorem 4] implies that there is $r_4$ in $(0, r_3)$ such that $c$ is locally injective at each point of $B(-2, r_4) \setminus \{-2\}$. Thus, there is $\delta > 0$ such that $c$ maps $[-2, -2 + r_4)$ bijectively onto $[-2 - 2 + \delta]$. Since by Proposition 2.3 there is $n_\# \geq n_1$ such that for every integer $n \geq n_\#$ the set $K_n$ is contained in $[-2, -2 + \delta]$, this completes the proof of the lemma with $r_\# = r_4$. □
3.4. Proof of the Sensitive Dependence of Geometric Gibbs States assuming the Main Theorem. Let $K_0$ be the constant given by the Main Theorem with $R = 80$, and let $\mathcal{F}_0$ be the family of quadratic-like maps given by Lemma 3.9 for this choice of $K_0$. This lemma and (3.3) imply that $\mathcal{F}_0$ is uniform with constants $K_0$ and 80, and that it is admissible. Fix a sufficiently large integer $n$ for which the conclusions of the Main Theorem are satisfied with $\mathcal{F} = \mathcal{F}_0$, and let $(f_\xi)_{\xi \in \{+, -\}^n}$ and $A$ be as in the statement of the Main Theorem. Given $\xi$ in $\{+, -\}^N$, denote by $\lambda(\xi)$ the unique parameter in $\Lambda_0$ such that $f_{\lambda(\xi)} = f_\xi$. By Lemma 3.3 $\lambda(\xi)$ is real. Then we prove the Sensitive Dependence of Geometric Gibbs States with $\Lambda = \{\lambda(\xi) \mid \xi \in \{+, -\}^N\}$.

Let $\mathcal{G}$ be the family of quadratic-like maps given by $0$ and $F$. This lemma and (3.3) imply that $\mathcal{G}$ is uniform for maps in a uniform family to satisfy this condition with uniform constants. We also make other uniform estimates that are used in the rest of the paper, which are mostly deduced from analogous estimates for quadratic maps in [CRL13].

4. The Geometric Peierls condition, and uniform estimates

In this section we introduce the Geometric Peierls condition, and give a criterion for maps in a uniform family to satisfy this condition with uniform constants. We also make other uniform estimates that are used in the rest of the paper, which are mostly deduced from analogous estimates for quadratic maps in [CRL13].
To state the Geometric Peierls condition, we introduce some notation. For every normalized quadratic-like map \( f \), put

\[
\chi_{\text{crit}}(f) := \lim \inf_{n \to +\infty} \frac{1}{n} \log |Df^n(f(0))|.
\]

Let \( \mathcal{F} \) be a uniform family of quadratic-like maps, \( n \geq 5 \) an integer, and \( f \) a map in \( K_n(\mathcal{F}) \). Put

\[
V_f := P_{f,n+1}(0) = f^{-1}(P_{f,n}(\tilde{\beta}(f))),
\]

and

\[
D_f = \{ w \in \mathbb{C} \setminus V_f | f^m(w) \in V_f \text{ for some } m \in \mathbb{N} \}.
\]

For \( w \) in \( D_f \) denote by \( m_f(w) \) the least integer \( m \geq 1 \) such that \( f^m(w) \in V_f \), and call it the \textit{first landing time} of \( w \) to \( V_f \). The \textit{first landing map} to \( V_f \) is the map \( L_f : D_f \to V_f \) defined by \( L_f(w) := f^m_f(w) \).

**Definition 4.1** (Geometric Peierls Condition). Given \( \kappa > 0 \) and \( \upsilon > 0 \), a quadratic-like map \( f \) in \( \mathcal{F} \) satisfies the \textit{Geometric Peierls Condition with constants} \( \kappa \) and \( \upsilon \), if for every \( z \) in \( L_f^{-1}(V_f) \) we have

\[
|DL_f(z)| \geq \kappa \exp((\chi_{\text{crit}}(f)/2 + \upsilon)m_f(z)).
\]

**Remark 4.2.** The analogy between (4.1) and the usual Peierls conditions for contour models is as follows. We use terminology in [Sin72, §II]. As usual, the one-point interaction energy corresponds to the geometric potential \(- \log |Df|\). The (orbit of the) critical point \( z = 0 \) of \( f \) plays the role of the unique ground state. In contrast to the usual Peierls condition for contour models where the ground state is assumed to be supported on a periodic configuration, it is crucial for the Main Theorem to allow the orbit of 0 to be nonperiodic. However, we do require later that the Lyapunov exponent \( \lim_{n \to +\infty} \frac{1}{n} \log |Df^n(f(0))| \) exists, so that the “\( \lim \inf \)” that defines \( \chi_{\text{crit}}(f) \) is actually a limit. Consider an initial condition \( w \) near the critical point 0 of \( f \). Following the definition of the boundary of a configuration [Sin72, Definition 2.2], we see that for the “boundary” of \( w \) with respect to 0 to be finite, it is enough to assume that for some integer \( \tau \geq 1 \) we have \( f^\tau(w) = f^\tau(0) \). The orbit of \( w \) shadows that of 0 up to a certain time \( \ell \), so that the derivatives of \( f^{\ell-1} \) at \( f(w) \) and at \( f(0) \) are comparable. After time \( \ell \), the orbit of \( w \) can be significantly different from that of 0. To simplify, assume that the point \( z := f^\ell(w) \) satisfies \( L_f(z) = 0 \), so we have \( f^{\ell+m_f(z)}(z) = f^{\ell+m_f(z)}(0) \), and therefore the boundary of \( w \) with respect to 0 is bounded from above by \( m_f(z) \). Up to a uniform distortion constant, the Hamiltonian at \( z \) relative to \( w \) is equal to

\[
- \log |Df^{\ell+m_f(z)}(w)| - \left( -\frac{\ell + m_f(z)}{2} \chi_{\text{crit}}(f) \right) \sim - \log |DL_f(z)| + \frac{m_f(z)}{2} \chi_{\text{crit}}(f).
\]

Thus condition (4.1) becomes Peierls condition as in [Sin72, §II, Definition 2.3].

The following is a criterion for the Geometric Peierls Condition. For future reference, it is stated in a slightly stronger form than what is needed for this paper.

---

\( ^\dagger \)Here we replaced \( \log |Df^{\ell+m_f(z)}(0)| = -\infty \) by \( (\ell + m_f(z))\chi_{\text{crit}}(f)/2 \), which is what appears naturally in several estimates, see for example Lemmas 4.10 and 4.3.

\( ^\dagger \)To follow the analogy, we should require the constant \( \kappa \) to be larger than the implicit distortion constant in the computation above. Later on we compensate a possible small value of \( \kappa \) by assuming that the map \( f \) is in \( K_n(\mathcal{F}) \) for a sufficiently large integer \( n \).
Proposition 4.3. For every \( v > 0 \) satisfying \( v < \frac{1}{2} \log 2 \) and every \( R > 0 \), there are constants \( K_1 > 1 \), \( n_1 \geq 6 \), and \( k_1 > 0 \), such that the following property holds. If the family of quadratic-like maps \( \mathcal{F} \) is uniform with constants \( K_1 \) and \( R \), then for every integer \( n \geq n_1 \), every element \( f \) of \( \mathcal{K}_n(\mathcal{F}) \) satisfies the Geometric Peierls Condition with constants \( k_1 \) and \( v \). Furthermore, we have

\[
\chi_f(\beta(f)) > \chi_{\text{crit}}(f) + 2v, \chi_{\text{crit}}(f) > 2v, \quad \text{and} \quad \chi_f(p(f)) < \chi_f(p^+(f)) + v/4.
\]

After some uniform geometric estimates in §4.1, the proof of Proposition 4.3 is given in §4.2. In §4.3, we make various uniform estimates.

4.1. Uniform geometric estimates. In this subsection, we use Mori's theorem on the modulus of continuity of normalized quasi-conformal maps, to obtain some preliminary estimates for quadratic-like maps in a given uniform family.

Lemma 4.4. Given \( K > 1 \) and \( R > 0 \) there is a constant \( C_1 > 1 \) such that for every uniform family \( \mathcal{F} \) of quadratic-like maps with constants \( K \) and \( R \), the following property holds for every \( f \) in \( \mathcal{F} \) and \( w \) in \( \hat{X}_f \):

\[
C_1^{-1}|w| \leq |f(w) - f(0)|^{1/2} \leq C_1|w|.
\]

Moreover, if in addition \( w \) is in \( X_f \), then

\[
C_1^{-1}|w| \leq |Df(w)| \leq C_1|w|,
\]

\[
C_1^{-1}|Df_c(f)(h_f^{-1}(w))|^K \leq |Df(w)| \leq C_1|Df_c(f)(h_f^{-1}(w))|^{1/K}.
\]

The proof of this lemma is after the following one.

Lemma 4.5. For each \( R > 0 \) there is a constant \( C_2 > 1 \) such that the following property holds. Let \( K \geq 1 \) be given, and let \( h \) be a \( K \)-quasi-conformal homeomorphism of \( \mathbb{C} \) that is holomorphic outside \( \text{cl}(B(0,R)) \) and that is tangent to the identity at infinity. Then for every \( z \) and \( z' \) in \( B(0,2R) \) we have

\[
C_2^{-K}|z - z'|^K \leq |h(z) - h(z')| \leq C_2|z - z'|^{1/K}.
\]

Proof. Replacing \( h \) by \( h - h(0) \) if necessary, assume \( h(0) = 0 \). Put \( D := h(B(0,2R)) \), let \( \varphi : D \to B(0,2R) \) be a bi-holomorphic map fixing \( z = 0 \), and note that \( \varphi \circ h|_{B(0,2R)} \) is a \( K \)-quasi-conformal homeomorphism of \( B(0,2R) \) fixing \( z = 0 \). Thus, Mori's theorem implies that for every \( z \) and \( z' \) in \( B(0,2R) \), we have

\[
(16K^22^{K-1})^{-1}|z - z'|^K \leq |\varphi \circ h(z) - \varphi \circ h(z')| \leq 16(2R)^{-4} |z - z'|^{1/K},
\]

see for example [Ah66 p. 47]. It remains to estimate the distortion of \( \varphi \) on \( D \).

Note first that the holomorphic function \( g : B(0,(R^{-1})^{-1}) \setminus \{0\} \to \mathbb{C} \) defined by \( g(\zeta) = h(\zeta^{-1}) \) extends holomorphically to \( \zeta = 0 \), and that the extension, also denoted by \( g \), satisfies \( g(0) = 0 \) and \( Dg(0) = 1 \). By Koebe's \( \frac{1}{2} \)-theorem and the version of the Koebe Distortion theorem in [CG93 Theorem 1.6], we have

\[
B(0,(8R)^{-1}) \subset g(B(0,(2R)^{-1})) \subset B(0,2R^{-1}),
\]

and therefore,

\[
B(0,R/2) \subset D \subset B(0,8R).
\]

By Schwarz' Lemma and Koebe's \( \frac{1}{2} \)-theorem we have \( \frac{1}{4} \leq |D\varphi(0)| \leq 4 \). Next we show that \( \varphi \) has a univalent extension to \( \tilde{D} := h(B(0,4R)) \). Note first that \( \varphi \)
extends continuously to cl$(D)$, since \( \partial D \) is a Jordan curve; we denote this extension also by \( \varphi \). Consider the holomorphic involution \( \iota \) of \( A := h(B(0, 4R) \setminus \text{cl}(B(0, R))) \) defined by \( \iota(w) := h(4R^2/h^{-1}(w)) \). Let \( \hat{\varphi} : \hat{D} \to \mathbb{C} \) be the function that coincides with \( \varphi \) on \( \text{cl}(D) \), and that for \( z \) in \( \hat{D} \setminus \text{cl}(D) \) is given by \( \hat{\varphi}(z) := 4R^2/\varphi(\iota(z)) \). Then \( \hat{\varphi} \) is homeomorphism from \( \hat{D} \) to \( B(0, 4R) \), and by Schwarz reflection principle it is holomorphic. By the Koebe Distortion Theorem, there is a universal constant \( \Delta > 1 \) independent of \( h \) such that for every distinct \( z \) and \( z' \) in \( B(0, 2R) \) we have

\[
(4\Delta)^{-1} \leq \Delta^{-1}|D\varphi(0)| \leq \frac{|\varphi \circ h(z) - \varphi \circ h(z')|}{|h(z) - h(z')|} \leq \Delta|D\varphi(0)| \leq 4\Delta.
\]

Together with (4.5), this proves the desired chain of inequalities with \( C_2 = 16 \cdot 2R \cdot 4\Delta \).

**Proof of Lemma 4.4** Let \( C_2 > 0 \) be the constant given by Lemma 4.5. Then

\[
\hat{D} := \sup_{f \in \mathcal{F}} \text{diam}(f(\hat{X}_f)) \leq C_2 \left( \sup_{c \in \mathcal{X}} \text{diam}(f_c(\hat{X}_c)) \right)^{\frac{1}{K}} < +\infty.
\]

Observe that, since \( \hat{X}_f \subset f(\hat{X}_f) \) and since \( \hat{X}_f \) contains \( 0 \), for every \( w \) in \( \hat{X}_f \) we have \( |w| \leq \hat{D} \).

On the other hand, since for \( c \in \hat{X} \) the sets

\[
\partial X_c := \{ z \in \mathbb{C} \mid G_c(z) = 1 \} \quad \text{and} \quad \partial \hat{X}_c := \{ z \in \mathbb{C} \mid G_c(z) = 2 \}
\]

are disjoint and depend continuously with \( c \), we have

\[
r := \inf_{c \in \hat{X}} \text{dist}(\partial X_c, \partial \hat{X}_c) > 0.
\]

In particular, for every \( c \in \hat{X} \) the set \( \hat{X}_c \) contains \( B(0, r) \). Combined with Lemma 4.5 this implies that, if we put \( \tilde{r} := (r/C_2)^K \), then for every \( f \) in \( \mathcal{F} \) and \( w \) in \( X_f \) the set \( \hat{X}_f \) contains \( B(w, \tilde{r}) \).

To prove (4.2), note that for every \( f \) in \( \mathcal{F} \) and \( w \) in \( \hat{X}_f \) we have

\[
|w^2 + w^3R_f(w)| = |f(w) - f(0)| \leq \text{diam}(f(\hat{X}_f)) \leq \hat{D},
\]

and therefore \( |w^3R_f(w)| \leq \hat{D} + \hat{D}^2 \). So, if we put \( \tilde{R} := (\hat{D} + \hat{D}^2)/r^3 \), then the maximum principle implies \( |R_f| \leq \tilde{R} \) on \( B(0, \tilde{r}) \). Letting \( \tilde{r}_0 := \min \left\{ \tilde{r}, 1/(2\tilde{R}) \right\} \), for every \( w \) in \( B(0, \tilde{r}_0) \) we have \( |wR_f(w)| \leq 1/2 \), and therefore

\[
\frac{1}{2} \leq \left| \frac{f(w) - f(0)}{w^2} \right| \leq \frac{3}{2}.
\]

Let \( w \in \hat{X}_f \setminus B(0, \tilde{r}_0) \) be given and put \( z := h_f^{-1}(w) \). Applying Lemma 4.5 with \( K = K \) twice and using \( h_f(0) = 0 \) we obtain

\[
|f(w) - f(0)| = |h_f(f_c(f_c(z))) - h_f(f_c(f_c(0)))| \geq C_2^{-K}|f_c(f_c(z)) - f_c(f_c(0))|^{K} = C_2^{-K}|z|^{2K} \geq C_2^{-K - 2K^2}|w|^{2K^2} \geq C_2^{-K - 2K^2}d_0^{2K^2}.
\]

Together with (4.6) these estimates imply (4.2) with

\[
C_1 = \max \left\{ 2, C_2^{K + 2K^2}D_0^{-2K^2}, \hat{D}^{-2K^2} \right\}.
\]
To prove (4.3) and (4.4), note first that by Schwarz’ Lemma for every \( w \) in \( B(0, \tilde{r}/2) \) we have \( |DR_f(w)| \leq 2\tilde{R}/\tilde{r} \). Then, putting \( \tilde{r}_1 := \min \{ \tilde{r}, 1/(10\tilde{R}) \} \), for every \( w \) in \( B(0, \tilde{r}_1) \) we have \( |3wR_f(w) + w^2DR_f(w)| \leq 1/2 \), so

\[
\frac{3}{2} \leq \frac{|Df(w)|}{|w|} \leq \frac{5}{2}.
\]

Let \( w \) in \( X_c \setminus B(0, \tilde{r}_1) \) be given. Using that \( B(w, \tilde{r}) \) is contained in \( \widetilde{X}_f \) and the definition of \( \tilde{D} \), Schwarz’ lemma implies \( |Df(w)|/|w| \leq (\tilde{D}/\tilde{r}_1) \). To estimate \( |Df(w)|/|w| \) from below, put

\[
z := h_f^{-1}(w), \quad r_1 := \min \{ C_2^{-K}r_1^2, r \}, \quad \text{and} \quad B(z) := B(z, r_1).
\]

By Lemma 4.5 we have \( r_1 \leq |z| \), so \( f_{c(f)} \) is injective on \( B(z) \). By Koebe’s \( \frac{1}{4} \)-theorem, the set \( f_{c(f)}(B(z)) \) contains

\[
B(f_{c(f)}(z), |Df_{c(f)}(z)|r_1/4),
\]

and therefore \( B(f_{c(f)}(z), r_1^2/2) \). By Lemma 4.6 we have

\[
h_f(B(z)) \subset B \left( w, C_2r_1^2 \right)
\]

and

\[
\hat{B}(w) := B(f(w), C_2^{-K}r_1^2/2) \subset h_f(f_{c(f)}(B(z))).
\]

Thus, if we put \( \varepsilon := C_2^{-1-K}2^{-K}r_1^2/2 \), then Schwarz’ lemma applied to \( f^{-1}|_{\hat{B}(w)} \) implies

\[
|Df(w)| \geq \varepsilon \quad \text{and} \quad |Df(w)|/|w| \geq \varepsilon/\hat{D}.
\]

This completes the proof of (4.3) with \( C_1 := \max \{ 3, \hat{D}/(\tilde{r}_1), \hat{D}/\varepsilon \} \). Combined with Lemma 4.5 this last estimate implies,

\[
(4.7) \quad \frac{\tilde{C}^{-1}}{(2C_2)^K}|Df_{c(f)}(z)|^K = \frac{\tilde{C}^{-1}}{C_2^K}|z|^K \leq \tilde{C}^{-1}|h_f(z)| = \tilde{C}^{-1}|w|
\]

\[
\leq |Df(w)| \leq \tilde{C}|w| = \tilde{C}|h_f(z)| \leq \widetilde{C}C_2|z|^{\tilde{r}_1^2/2} = \frac{\tilde{C}C_2}{2^{1/K}}|Df_{c(f)}(z)|^{\frac{\tilde{r}_1^2}{2}}.
\]

This proves (4.4) with \( C_1 = \tilde{C}(2C_2)^K \), and completes the proof of the lemma. \( \square \)

4.2. Proving the Geometric Peierls Condition. In this subsection we prove Proposition 4.3. The proof is given after the following lemmas.

**Lemma 4.6.** Given \( K > 1 \) and \( R > 0 \) there is a constant \( C_3 > 1 \) such that for every uniform family of quadratic-like maps \( \mathcal{F} \) with constants \( K \) and \( R \), every integer \( n \geq 6 \), and every \( f \) in \( K_n(\mathcal{F}) \), the following property holds. Let \( w \) be a point in \( X_f \) and \( m \geq 1 \) an integer such that \( f^m \) maps a neighborhood of \( w \) biholomorphically onto \( P_{f,1}(0) \). Then

\[
C_3^{-1}|Df^m_{c(f)}(h_f^{-1}(w))|^{\tilde{r}_1^2/2} \leq |Df^m(w)| \leq C_3|Df^m_{c(f)}(h_f^{-1}(w))|^K.
\]

**Proof.** Let \( C_2 \) be the constant given by Lemma 4.5. From the proof of Lemma 5.4, we have

\[
\Xi := \inf_{c \in \mathcal{P}_4(-2)} \text{diam}(P_{c,1}(0)) > 0.\]
We prove the lemma with \( C_3 = C_3^1 + K \Xi \frac{1}{\kappa} - K \). Let \( w \) be as in the statement of the lemma. By hypothesis, there is a neighborhood \( W \) of \( w \) which is mapped biholomorphically onto \( P_{f,1}(0) \) by \( f^m \). Take arbitrary points \( u \) and \( v \) in \( W \), and put

\[
z := h_f^{-1}(w), \quad x := h_f^{-1}(u), \quad \text{and} \quad y := h_f^{-1}(v).
\]

By Lemma 4.5 we have

\[
\frac{|f^m(u) - f^m(v)|}{|u - v|} = \frac{|f^m(h_f(x)) - f^m(h_f(y))|}{|h_f(x) - h_f(y)|}
\leq C_2^{1+K} \left| \frac{f_{c(f)}^m(x) - f_{c(f)}^m(y)}{x - y} \right|^{\frac{1}{\kappa}}
\leq C_2^{1+K} \left| \frac{f_{c(f)}^m(x) - f_{c(f)}^m(y)}{x - y} \right|^{\frac{1}{\kappa}} \left| \frac{f^m_{c(f)}(x) - f^m_{c(f)}(y)}{x - y} \right|^{K}
\leq C_2^{1+K} \text{diam}(P_{c(f),1}(0)) \left| \frac{f^m_{c(f)}(x) - f^m_{c(f)}(y)}{x - y} \right|^{K}.
\]

Since \( u \) and \( v \) are arbitrary points of \( W \), we conclude that

\[
|Df^m(w)| \leq C_2^{1+K} \text{diam}(P_{c(f),1}(0)) \left| \frac{f^m_{c(f)}(x) - f^m_{c(f)}(y)}{x - y} \right|^{K}
\leq C_2^{1+K} \Xi \left| \frac{f^m_{c(f)}(x) - f^m_{c(f)}(y)}{x - y} \right|^{K}. 
\]

The proof of the other inequality follows similar arguments.

**Lemma 4.7.** For every \( K > 1, R > 0, \) and \( \varepsilon > 0 \) there is an integer \( n_2 \geq 5 \) such that for every uniform family of quadratic-like maps \( \mathcal{F} \) with constants \( K \) and \( R \), the following property holds. For every integer \( n \geq n_2 \), and every \( f \) in \( K_n(\mathcal{F}) \), we have

\[
K^{-1}(1 - \varepsilon) \log 2 \leq \chi_{\text{crit}}(f) \leq K(1 + \varepsilon) \log 2,
\]

and for every periodic point \( p \) of \( f \) in \( h_f(\Lambda_c(f)) \), we have

\[
K^{-1}(1 - \varepsilon) \log 2 \leq \chi_f(p) \leq K(1 + \varepsilon) \log 2.
\]

**Proof.** Let \( C_3 \) be the constant given by Lemma 4.6.

Combining [CRL13, Lemma 4.2] and [CRL13, Lemma 5.3] with \( m_1 = 4 \), we conclude that there are constants \( \tilde{C}_0 > 0 \) and \( n_0 \geq 3 \) such that for each integer \( n \geq n_0 \) and each parameter \( c \) in \( K_n \), we have for every \( z \) in \( \Lambda_c \) and every integer \( m \geq 1 \),

\[
\tilde{C}_0^{-1}2^{(1+\varepsilon)m} \leq |Df^m_c(z)| \leq \tilde{C}_02^{(1+\varepsilon)m}.
\]

Note that \( f^m_c \) is in \( \Lambda_c \), so we can take \( z = f^n_c(c) \) above. Noting that for every \( f \) in \( K_n(\mathcal{F}) \) and every integer \( k \geq 1 \) the map \( f^{3k} \) maps a neighborhood of \( h_f(f^{n+1}(0)) = f^{n+1}(0) \) biholomorphically onto \( P_{f,1}(0) \) (cf., [CRL13, Lemma 5.1]), by Lemma 4.6 for every integer \( m \geq 1 \) we have

\[
C_3^{-1}\tilde{C}_0^{-1}2^{(1+\varepsilon)m} \leq |Df^m_c(f^{n+1}(0))| \leq C_3\tilde{C}_02^{K(1+\varepsilon)m},
\]

and

\[
C_3^{-1}\tilde{C}_0^{-1}2^{K(1+\varepsilon)m} \leq |Df^m_c(p)| \leq C_3\tilde{C}_02^{K(1+\varepsilon)m}.
\]

Taking logarithms, dividing by \( m \), and letting \( m \to +\infty \), we conclude the proof of the lemma. \( \square \)
Proof of Proposition 4.3. Let \( \varepsilon > 0 \) be sufficiently small so that \( \varepsilon < \frac{2}{3} \left( \frac{1}{2} - \frac{\varepsilon}{\log 2} \right) \) and \( \varepsilon < \frac{\varepsilon}{8 \log 2} \), and let \( K_1 > 1 \) be sufficiently close to 1 so that
\[
v' := \left( \frac{1 - \varepsilon}{K_1} - \frac{K_1(1 + \varepsilon)}{2} \right) \log 2 > \frac{2}{\log 2} \left( \frac{1 - \varepsilon}{K_1} \right) \log 2.
\]
Let \( n_2 \) be given by Lemma 4.7 for this value of \( \varepsilon \). In view of Proposition 2.3 and of the formula \( Df_{-2}(\beta(-2)) = 4 \), we can take \( n_2 \) larger if necessary so that for every integer \( n \geq n_2 \) and every parameter \( c \) in \( K_n \), we have \( \chi_f(\beta(f)) \geq 1 - \varepsilon \log 4 \).
Assume \( \mathcal{F} \) is uniform with constants \( R \) and \( K_1 \), and let \( C_3 \) be the constant given by Lemma 4.6 with \( K = K_1 \). Note that \( C_3 \) depends on \( R \) and \( \upsilon \) only.
By Lemma 4.7 for every integer \( n \geq n_2 \) and every \( f \) in \( K_n(\mathcal{F}) \), we have (4.8) with \( K \) replaced by \( K_1 \). On the other hand, by [CRL13, Proposition B] there are \( \hat{\kappa}_1 \) and \( \tilde{\kappa}_1 \), such that for every integer \( n \geq \tilde{\kappa}_1 \), every parameter \( c \) in \( K_n \), and every \( z \) in \( L_c^{-1}(V_c) \), we have
\[
|DL_c(z)| \geq \hat{\kappa}_1 2(1 - \varepsilon) m_c(z).
\]
Noting that for each \( f \) in \( K_n(\mathcal{F}) \) and each \( w \) in \( F_c^{-1}(V_c) \) the map \( f^{m_f(z)} \) maps a neighborhood of \( w \) biholomorphically onto \( P_f(0) \), by Lemma 4.6 we have
\[
|DL_f(w)| \geq C_3 \hat{\kappa}_1 2^{1 - \varepsilon} m_f(z).
\]
Noting that by definition of \( \upsilon' \) we have
\[
2 \frac{\upsilon'}{\upsilon} = 2^{\frac{\upsilon'}{\upsilon}} \exp(\upsilon') \geq \exp(\chi_{\text{crit}}(f) / 2 + \upsilon'),
\]
inequality (4.8) implies the first part of the lemma with
\[
n_1 = \min\{n_2, \tilde{\kappa}_1\} \quad \text{and} \quad \kappa_1 = C_3 \hat{\kappa}_1.
\]
Now we prove the last part. The second inequality follows from (4.8), and the definition of \( \upsilon' \). To prove the third inequality, note that by (4.8) and the definition of \( \upsilon' \), we have
\[
\chi_f(p(f)) - \chi_f(p^+(f)) \leq \left( \frac{K_1(1 + \varepsilon)}{2} \right) \log 2 \leq \frac{\upsilon'}{4}
\]
To prove the first inequality, note that by Lemma 4.6 and our choice of \( n_2 \), we have
\[
\chi_f(\beta(f)) \geq \frac{1}{K_1} \chi_{f_{\text{crit}}}(\beta(c(f))) \geq \frac{1 - \varepsilon}{K_1} \log 4.
\]
Combined with (4.8) and the definition of \( \upsilon' \), this implies
\[
\chi_f(\beta(f)) - \chi_{\text{crit}}(f) \geq \left( \frac{1 - \varepsilon}{K_1} - \frac{K_1(1 + \varepsilon)}{2} \right) \log 4 = 2\upsilon' \geq 2\upsilon.
\]
This we concludes the proof of the lemma.

4.3. Uniform estimates. In this subsection we prove various uniform estimates. Throughout this subsection we fix a uniform family of quadratic-like maps \( \mathcal{F} \), with constants \( K \) and \( R \).

Lemma 4.8. There is a constant \( \Delta_1 > 1 \) that only depends on \( K \) and \( R \), such that for each \( f \) in \( \mathcal{P}_2(\mathcal{F}) \) the following properties hold for each integer \( k \geq 2 \): For each point \( y \) in \( P_{f,k}(\beta(f)) \) or in \( P_{f,k}(\beta(f)) \) we have
\[
\Delta_1^{-1} |Df(\beta(f))|^k \leq |Df^k(y)| \leq \Delta_1 |Df(\beta(f))|^k.
\]
Proof. The proof follows the same lines that [CRL13] Lemma 3.6], and we only need to check that some constants are finite and others are positive. Let \( C_1 \) be the constant given by Lemma [4.4] and let \( C_2 \) be that given by Lemma [4.5]. By the proof of [CRL13] Lemma 3.6] we have
\[
\Xi_1 := \sup_{c \in P_0(-2)} \sup_{z \in P_{c,1}(\beta(c))} |Df(z)| < +\infty,
\]
\[
\Xi_2 := \inf_{c \in P_2(-2)} \inf_{z \in P_{c,1}(\beta(c))} |Df(z)| > 0,
\]
and
\[
\Xi_3 := \inf_{c \in P_2(-2)} \text{mod}(P_{c,0}(\beta(c)) \setminus \text{cl}(P_{c,1}(\beta(c)))) > 0.
\]
By Lemmas [4.4] and [4.5] we have
\[
\hat{\Xi}_1 := \sup_{f \in P_0(\mathcal{F})} \sup_{w \in P_{f,1}(\beta(f))} |Df(w)| \leq C_1 \Xi_1^\frac{1}{C_0} < +\infty,
\]
\[
\hat{\Xi}_2 := \inf_{f \in P_2(\mathcal{F})} \inf_{w \in P_{f,1}(\beta(f))} |Df(w)| \geq C_1^{-1} \Xi_2^K > 0,
\]
and since for every \( f \) in \( \mathcal{F} \) the conjugacy \( h_f \) is \( K \)-quasi-conformal
\[
\hat{\Xi}_3 := \inf_{f \in P_2(\mathcal{F})} \text{mod}(P_{f,0}(\beta(f)) \setminus \text{cl}(P_{f,1}(\beta(f)))) \geq \frac{1}{K} \Xi_3 > 0.
\]
Let \( \Delta > 1 \) be the constant given by Koebe Distortion Theorem with \( A = \hat{\Xi}_3 \). The desired inequalities follow from the fact that \( f^{k-1} \) maps each of the sets \( P_{f,k}(\beta(f)) \) and \( P_{f,k}(\beta(f)) \) biholomorphically to \( P_{f,1}(\beta(f)) \) with \( \Delta_1 = \Delta \Xi_1 \hat{\Xi}_2^{-1} \).

For a parameter \( c \) in \( P_2(-2) \) the external rays \( R_c(7/24) \) and \( R_c(17/24) \) land at the point \( \gamma(c) \) in \( P_{c,1}(0) \), see [CRL13] Section 3.3]. Let \( \hat{U}_c \) be the open disk containing \( -\beta(c) \) that is bounded by the equipotential 2 and by
\[
R_c(7/24) \cup \{ \gamma(c) \} \cup R_c(17/24).
\]
Put \( \hat{W}_c := f_c^{-1}(\hat{U}_c) \), and for every \( n \geq 3 \) and every \( f \) in \( \mathcal{K}_n(\mathcal{F}) \) put \( \hat{W}_f := h_f(\hat{W}_c(f)) \).

**Lemma 4.9** (Uniform distortion bound). There is a constant \( \Delta_2 > 1 \) that only depends on \( K \) and \( R \), such that for each integer \( n \geq 4 \), and each \( f \) in \( \mathcal{K}_n(\mathcal{F}) \) the following properties hold: For each integer \( m \geq 1 \) and each connected component \( W \) of \( f^{-m}(P_{f,1}(0)) \) on which \( f^m \) is univalent, \( f^m \) maps a neighborhood of \( W \) biholomorphically to \( \hat{W}_f \) and the distortion of this map on \( W \) is bounded by \( \Delta_2 \).

**Proof.** We follow the proof of [CRL13] Lemma 4.3]. From that proof we have that for each parameter \( c \) in \( P_4(-2) \) the set \( \hat{W}_c \) contains the closure of \( P_{c,1}(0) \) and
\[
A := \inf_{c \in P_4(-2)} \text{mod}(\hat{W}_c \setminus \text{cl}(P_{c,1}(0))) > 0.
\]
Since for every \( f \) in \( \mathcal{F} \) the conjugacy \( h_f \) is \( K \)-quasi-conformal, we have
\[
A := \inf_{f \in P_4(\mathcal{F})} \text{mod}(\hat{W}_f \setminus \text{cl}(P_{f,1}(0))) \geq \frac{A}{K} > 0.
\]
By [CRL13] Lemma 4.2], \( f_c^m \) maps a neighborhood of \( h_f^{-1}(W) \) biholomorphically to \( \hat{W}_c(f) \). By conjugacy, \( f^m \) maps a neighborhood of \( W \) biholomorphically to \( \hat{W}_f \). The conclusion follows from Koebe Distortion Theorem with \( A = \hat{A} \). \( \square \)
Lemma 4.10. There is a constant $C_4 > 1$ that only depends on $K$ and $R$, such that for each integer $n \geq 4$ and each $f$ in $K_n(F)$, the following properties hold for each integer $q \geq 1$: For each open set $W$ that is mapped biholomorphically to $P_{\ell,1}(0)$ by $f^q$, and each $x$ in $W$, we have

$$|Df(x)| \geq C_4^{-1}|Df^{q-1}(f(x))|^{-\frac{1}{2}}.$$ 

Proof. We follow the proof of [CRL13] Lemma 5.4. Let $C_1$ be the constant given by Lemma 4.4 and let $C_2$ be that given by Lemma 4.5. Let $\Delta_1 > 1$ and $\Delta_2 > 1$ be the constants given by Lemmas 4.8 and 4.9 respectively. From the proof of [CRL13] Lemma 5.4, we have

$$\Xi_1 := \inf_{c \in \mathcal{P}_4(-2)} \text{diam}(P_{c,1}(0)) > 0 \quad \text{and} \quad \Xi_2 := \sup_{c \in \mathcal{P}_4(-2)} |Df_c(\beta(c))| < +\infty,$$

and that for each $c$ in $\mathcal{P}_3(-2)$ the closure of $P_{c,1}(0)$ is contained in $\widetilde{W}_c$ and

$$\Xi_3 := \inf_{c \in \mathcal{P}_4(-2)} \text{mod}(\widetilde{W}_c \setminus \text{cl}(P_{c,1}(0))) > 0.$$ 

Since for every $f$ in $F$, the conjugacy $h_f$ is $K$-quasi-conformal, we have

$$\hat{\Xi}_3 := \inf_{f \in \mathcal{P}_4(F)} \text{mod}(\widetilde{W}_f \setminus \text{cl}(P_{f,1}(0))) \geq \frac{\Xi_3}{K} > 0,$$

and by Lemma 4.5 and inequality 4.4, we have

$$\hat{\Xi}_1 := \inf_{f \in \mathcal{P}_4(F)} \text{diam}(P_{f,1}(0)) \geq C_2^{-K} \Xi_1 > 0$$

and

$$\hat{\Xi}_2 := \sup_{f \in \mathcal{P}_4(F)} |Df(\beta(f))| \leq C_1 \Xi_2^{\frac{1}{2}} < +\infty.$$ 

Let $n \geq 4$ be a integer and $f$ in $K_{n}(F)$. Note that $f^q$ maps a neighborhood $\widetilde{W}$ of $W$ biholomorphically to $\widetilde{W}_f$ (Lemma 4.9). So, if we put $\widetilde{W}' := f(\widetilde{W})$, then $f(0)$ is not in $\widetilde{W}'$ and $f^{q-1}$ maps $W'$ biholomorphically to $\widetilde{W}_f$; in particular we have

$$\text{mod}(\widetilde{W}' \setminus \text{cl}(f(W))) = \text{mod}(\widetilde{W}_f \setminus \text{cl}(P_{f,1}(0))) \geq \hat{\Xi}_3.$$

Thus there is a constant $A_1 > 0$ independent of $n$, $f$ and $q$ such that for every $x$ in $W$, we have

$$|f(x) - f(0)| \geq \text{dist}(f(W), f(0)) \geq \text{dist}(f(W), \partial \widetilde{W}') \geq A_1 \text{diam}(f(W))$$

(cf. [LV78] Teichmüller’s module theorem, §II.1.3). Thus, if we put $A_2 := C_1^{-2}(A_1 \Delta_1^{-1} \hat{\Xi}_1)^{1/2}$, then by Lemmas 4.4 and 4.9 with $m = q - 1$ and with $W$ replaced by $f(W)$, we have

$$|Df(x)| \geq C_4^{-2} A_1^{1/2} \text{diam}(f(W))^{1/2} \geq A_2 |Df^{q-1}(f(x))|^{-1/2}.$$ 

This proves the lemma with constant $C_4 = A_2^{-1}$.

Lemma 4.11. There are constants $C_5 > 0$ and $v_1 > 0$ that only depend on $K$ and $R$, such that for every $f$ in $\mathcal{P}_3(F)$, every $\ell$ in $\mathbb{N}$, and every connected component $W$ of $g_f^{-\ell}(P_{f,1}(0))$, we have

$$\max\{\text{diam}(W), \text{diam}(f(W)), \text{diam}(f^2(W))\} \leq C_5 \exp(-v_1 \ell).$$
Proof. Let \( K \geq 1 \) and \( R > 0 \) be the constants of the family \( \mathcal{F} \). For this \( R \), let \( C_2 \) be the constant of Lemma 4.5. By [CRL15a, Lemma 2.4] there are constants \( C_0' > 0 \) and \( \nu_0' > 0 \) such that for every \( c \in \mathcal{P}_5(-2) \), every \( \ell \in \mathbb{N} \), and every connected component \( W' \) of \( g_c^{-\ell}(P_{c,1}(0)) \), we have
\[
\text{diam}(W') \leq C_0' \exp(-\nu_0' \ell).
\]
Put
\[
\Xi_0 := \inf_{c \in \mathcal{P}_5(-2)} \text{dist}(Y_c \cup \tilde{Y}_c, \partial P_{c,1}(0)) > 0,
\]
\[
\Xi_1 := \sup_{c \in \mathcal{P}_5(-2)} \text{diam}(\{ z \in \mathbb{C} \mid G_c(z) \leq 2 \}) < +\infty,
\]
and \( \hat{C}_0 := \max\{ C_0', \Xi_0^{-1} \Xi_1 C_0' \} \). Fix \( c \) in \( \mathcal{P}_5(-2) \), \( \ell \) in \( \mathbb{N} \), and a connected component \( W' \) of \( g_c^{-\ell}(P_{c,1}(0)) \). For every \( w \) in \( Y_c \cup \tilde{Y}_c \) define the holomorphic maps
\[
z \mapsto \frac{f_c(z) - f_c(w)}{z - w} \quad \text{and} \quad z \mapsto \frac{f_c^2(z) - f_c^2(w)}{z - w}
\]
on \( X_c \). Notice that for \( z \) in \( \partial P_{c,1}(0) \) both maps are bounded from above by \( \Xi_0^{-1} \Xi_1 \).

By the maximum principle for every \( z \) and \( w \) in \( Y_c \cup \tilde{Y}_c \) we have
\[
\max\{ |f_c(z) - f_c(w)|, |f_c^2(z) - f_c^2(w)| \} \leq \Xi_0^{-1} \Xi_1 |z - w|.
\]

In particular,
\[
\max\{ \text{diam}(W'), \text{diam}(f_c(W')), \text{diam}(f_c^2(W')) \} \leq \hat{C}_0 \exp(-\nu_0' \ell).
\]

From Lemma 4.5 by putting \( C_3 := C_2 \hat{C}_0 \) and \( \nu_1 := \nu_0' / K \), we conclude the proof of the lemma. \( \square \)

5. Estimating the geometric pressure function

The aim of this section is to prove Proposition 4.1 stated at the beginning of §5.2. For a uniform family, this proposition allows us to control the geometric pressure function by the itinerary of the critical point, using an inducing scheme. We achieve this aim by adapting a similar result for quadratic maps [CRL13, Proposition D]. The general scheme of this adaptation is the following. The arguments in the original proof can be grouped in 3 types. Purely combinatorial arguments depending only on the combinatorics of the Yoccoz puzzle. Geometric estimates of the sizes of the puzzle pieces. An estimate of the derivative of the first landing map to a neighborhood of the critical point and an estimate of the Lyapunov exponent of the critical value. In the adaptation, the combinatorial arguments follow directly from the conjugacy, the geometric estimates follow from the Hölder continuity of the conjugacy (cf., Lemma 4.5), and the last part use the Geometric Peierls conditions in a crucial way (Definition 4.1).

In §5.1 we introduce an inducing scheme, and we prove a result on the existence of conformal measures and equilibrium states (Proposition 5.2) that is analogous to general results in [PRL11]. In §5.2 we state Proposition 4.1 and prove a Bowen type formula, and other general properties of the geometric pressure function. Finally, the proof of Proposition 4.1 is given in §5.3.

Throughout this section we fix a uniform family of quadratic-like maps \( \mathcal{F} \), with constants \( K \) and \( R \).
5.1. Inducing scheme. In this subsection we introduce the inducing scheme to estimate the geometric pressure function for maps in $\mathcal{K}_n(\mathcal{F})$.

Let $n \geq 5$ be an integer and $f$ in $\mathcal{K}_n(\mathcal{F})$. Put

$$D_f := \{ z \in V_f \mid f^m(z) \in V_f \text{ for some } m \geq 1 \}.$$ 

For $w$ in $D_f$ put $m_f(w) := \min\{ m \in \mathbb{N} \mid f^m(w) \in V_f \}$, and call it the first return time of $w$ to $V_f$. The first return map to $V_f$ is defined by

$$F_f : D_f \to V_f$$

$$w \mapsto F_f(w) := f^{m_f(w)}(w).$$

It is easy to see that $D_f$ is a disjoint union of puzzle pieces; so each connected component of $D_f$ is a puzzle piece. Note furthermore that in each of these puzzle pieces $W$, the return time function $m_f$ is constant; denote the common value of $m_f$ on $W$ by $m_f(W)$.

Throughout the rest of this subsection we put $\hat{V}_f := P_{f,4}(0)$. The proof of the following lemma is the same as for [CRL13, Lemma 6.1]. The reason is that the combinatorics and Koebe space are preserved by the conjugacy.

Lemma 5.1 (Uniform distortion bound). There is a constant $\Delta_3 > 1$ that only depends on $K$ and $R$, such that for each integer $n \geq 5$, and each $f$ in $\mathcal{K}_n(\mathcal{F})$ the following property holds: For every connected component $W$ of $D_f$ the map $F_f|_W$ is univalent and its distortion is bounded by $\Delta_3$. Furthermore, the inverse of $F_f|_W$ admits a univalent extension to $\hat{V}_f$ taking images in $V_f$. In particular, $F_f$ is uniformly expanding with respect to the hyperbolic metric on $\hat{V}_f$.

Denote by $\mathcal{D}_f$ the collection of connected components of $D_f$ and if $c(f)$ is real denote by $\mathcal{D}_f^c$ the sub-collection of $\mathcal{D}_f$ of those sets intersecting $I(f)$. For each $W$ in $\mathcal{D}_f$ denote by $\phi_W : \hat{V}_f \to V_f$ the extension of $f|^{-1}_W$ given by Lemma 5.1. Given an integer $\ell \geq 1$ we denote by $E_{f,\ell}$ (resp. $E_{f,\ell}^c$) the set of all words of length $\ell$ in the alphabet $\mathcal{D}_f$ (resp. $\mathcal{D}_f^c$). Again by Lemma 5.1 for each integer $\ell \geq 1$ and each word $W_1 \cdots W_\ell$ in $E_{f,\ell}$ the composition

$$\phi_{W_1 \cdots W_\ell} = \phi_{W_1} \circ \cdots \circ \phi_{W_\ell}$$

is defined on $\hat{V}_f$. We also put

$$m_f(W_1 \cdots W_\ell) = m_f(W_1) + \cdots + m_f(W_\ell).$$

For $t, p$ in $\mathbb{R}$ and an integer $\ell \geq 1$ put

$$Z_{t}(t, p) := \sum_{W \in E_{f,\ell}} \exp(-m_f(W)p) \left( \sup\{|D\phi_W(z)| \mid z \in V_f \} \right)^t$$

and

$$Z_{t}^R(t, p) := \sum_{W \in E_{f,\ell}^c} \exp(-m_f(W)p) \left( \sup\{|D\phi_W(z)| \mid z \in V_f \} \right)^t.$$ 

For a fixed $t$ and $p$ in $\mathbb{R}$ the sequence

$$\left( \frac{1}{\ell} \log Z_{t}(t, p) \right)_{\ell=1}^{+\infty}\text{ resp. }\left( \frac{1}{\ell} \log Z_{t}^R(t, p) \right)_{\ell=1}^{+\infty}$$

converges to the pressure function of $F_f$ (resp. $F_f|_{D_f \cap I(f)}$) for the potential $-t \log |D F_f| - pm_f$; we denote it by $\mathcal{P}_f(t, p)$ (resp. $\mathcal{P}_f^R(t, p)$). On the set where it is finite, the
function $\mathcal{P}_f$ (resp. $\mathcal{P}^R_f$) so defined is continuous and strictly decreasing in each of its variables.

Given $t > 0$ and $p$ in $\mathbb{R}$, a finite measure $\tilde{\mu}$ on $\mathbb{C}$ that is supported on the maximal invariant set of $F|_{D_f \cap \mathbb{R}}$ (resp. $F$) is $(t, p)$-conformal for $F_f$ if for every $W$ in $\mathfrak{D}_f^R$ (resp. $\mathfrak{D}_f$), and every Borel subset $U$ of $W \cap \mathbb{R}$ (resp. $W$), we have

$$\tilde{\mu}(F_f(U)) = \exp(pm_f(W)) \int_U |DF_f|^t \, d\tilde{\mu}.$$ 

Note that in this case we have

$$\exp(-pm_f(W)) \inf_{z \in W} |DF_f(z)|^{-t} \leq \tilde{\mu}(W) \leq \exp(-pm_f(W)) \sup_{z \in W} |DF_f(z)|^{-t}.$$

**Proposition 5.2.** Let $n \geq 5$ be an integer, $f$ in $\mathcal{K}_n(\mathcal{F})$, and $t > 0$ such that

(5.2) $\mathcal{P}_f^R(t, P_f^R(t)) = 0$ (resp. $\mathcal{P}_f(t, P_f(t)) = 0$).

Then there is a $(t, P_f^R(t))$-conformal (resp. $(t, P_f(t))$-conformal) probability measure $\tilde{\mu}$ for $F_f$, and there is a probability measure $\tilde{\rho}$ that is invariant by $F_f$, absolutely continuous with respect to $\tilde{\mu}$, and whose density satisfies

(5.3) $\Delta_3^{-t} \leq \frac{d\tilde{\rho}}{d\tilde{\mu}} \leq \Delta_3^t$.

If in addition

(5.4) $\sum_{W \in \mathfrak{D}^R} m_f(W) \cdot \exp(-m_f(W)P_f^R(t)) \sup_{w \in W \cap \mathbb{R}} |DF_f(w)|^{-t}$

(resp. $\sum_{W \in \mathfrak{D}} m_f(W) \cdot \exp(-m_f(W)P_f(t)) \sup_{w \in W} |DF_f(w)|^{-t}$)

is finite, then the measure

$$\hat{\rho} := \sum_{W \in \mathfrak{D}^R} \sum_{j=0}^{m_f(W)-1} (f^j)_*(\tilde{\rho}|_{W \cap \mathbb{R}}) \quad \text{(resp.} \sum_{W \in \mathfrak{D}} \sum_{j=0}^{m_f(W)-1} (f^j)_*(\tilde{\rho}|_W)\text{)}$$

is finite and the probability measure proportional to $\hat{\rho}$ is the unique equilibrium state of $f|_{I(f)}$ (resp. $f|_{J(f)}$) for the potential $-t \log |DF|$. 

**Proof.** The proof is standard, refer to [PRL11, §4] for precisions. The existence of the conformal measure follows from the same arguments given in [PRL11, Theorem A in §4 and Proposition 4.3]. To construct an absolutely continuous invariant measure, let $\ell \geq 1$ be an integer, and let $W$ be a word in $E_{f, \ell}$. Then by Lemma 5.1.
and (5.1), for every integer $\ell' \geq 1$ we have in the complex case
\[
\tilde{\mu}(F_{\ell}^{-\ell'}(\phi_{W}(V_f))) = \sum_{W' \in E_{\ell',\ell'}} \tilde{\mu}(\phi_{W'} \circ \phi_{W}(V_f)) \geq \tilde{\mu}(\phi_{W}(V_f)) \sum_{W' \in E_{\ell',\ell'}} \exp\left(-m_f(W')P_f(t)\right) \inf_{z \in \phi_{W'}(V_f)} |D\phi_{W'}(z)|^t \\
\geq \Delta^{-t}_\ell \tilde{\mu}(\phi_{W}(V_f)) \sum_{W' \in E_{\ell',\ell'}} \tilde{\mu}(\phi_{W'}(V_f)) = \Delta^{-t}_\ell \tilde{\mu}(\phi_{W}(V_f)).
\]
A similar argument shows that $\tilde{\mu}(F_{\ell}^{-\ell'}(\phi_{W}(V_f))) \leq \Delta^{t}_\ell \tilde{\mu}(\phi_{W}(V_f))$. Analogous inequalities also hold in the real case. Since these inequalities hold for every $\ell \geq 1$, and every $W$ in $E_{\ell',\ell}$, it follows that any weak* accumulation measure of $\left(\frac{1}{r} \sum_{\ell=1}^{k} (F_{\ell})_{*} \tilde{\mu}\right)_{k=1}^{+\infty}$ is an invariant probability measure satisfying the desired properties.

To prove the last statement, note that by (5.1) and (5.3), our hypothesis (5.4) implies that
\[
\sum_{W \in S^k} m_f(W) \hat{\rho}(W) \left(\text{resp.} \sum_{W \in S} m_f(W) \hat{\rho}_i(W)\right)
\]
is finite, so the measure $\hat{\rho}$ is finite. The last statement of the proposition follows as in the proof of [CRL13, Proposition A].

### 5.2. Estimating the 2 variables pressure function.

The following is our main tool to estimate the 2 variables pressure function, in order to verify the hypotheses of Proposition 5.2.

**Proposition I.** Let $\kappa > 0$ and $\nu > 0$ be given. Then there are $n_3 \geq 5$ and $C_6 > 1$ that only depend on $K$, $R$, $\kappa$, and $\nu$, such that for every integer $n \geq n_3$, and every $f$ in $K_n(\mathcal{F})$ satisfying the Geometric Peierls Condition with constants $\kappa$ and $\nu$, the following properties hold for each $t \geq 2 \log 2/\nu$.

1. For $p$ in $[-t\chi_{\text{crit}}(f)/2, 0)$ satisfying
\[
\sum_{k=0}^{+\infty} \exp(-(n + 3k)p)|Df^{n+3k}(f(0))|^{-t/2} \geq C_6^t,
\]
we have $\mathcal{P}_f(t, p) > 0$ and $P_f^{R}(t) \geq p$. If in addition the sum above is finite, then $\mathcal{P}_f(t, p)$ is finite and $P_f^{R}(t) > p$.

2. For $p \geq -t\chi_{\text{crit}}(f)/2$ satisfying
\[
\sum_{k=0}^{+\infty} \exp(-(n + 3k)p)|Df^{n+3k}(f(0))|^{-t/2} \leq C_6^{-t},
\]
we have $\mathcal{P}_f(t, p) < 0$ and $P_f(t) \leq p$. 

3. For $p \geq -t\chi_{\text{crit}}(f)/2$ satisfying
\[
\sum_{k=0}^{+\infty} k \cdot \exp(-(n + 3k)p)|Df^{n+3k}(f(0))|^{-t/2} < +\infty,
\]
we have
\[ \sum_{W \in D_f} m_f(W) \cdot \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} < +\infty. \]

The proof of this proposition is given in §5.3. In the rest of this subsection we prove 2 results that are used in the proof of the proposition above. The first is a Bowen type formula relating \( P^R \) (resp. \( P_f \)) to the 2 variables pressure function of \( F_f \) (Proposition II). The second is a lower bound for the pressure function (Proposition III).

**Proposition II** (Bowen type formula). For every \( \kappa > 0 \), every \( \nu > 0 \), every integer \( n \geq 5 \), and every \( f \) in \( \mathcal{K}_n(\mathcal{F}) \) satisfying the Geometric Peierls Condition with constants \( \kappa \) and \( \nu \), we have for each \( t \geq 2 \log 2/\nu \),
\[ P^R(t) = \inf \{ p \mid \mathcal{P}^R_f(t, p) \leq 0 \} \text{ (resp. } P_f(t) = \inf \{ p \mid \mathcal{P}_f(t, p) \leq 0 \} \).

The proof of this proposition is at the end of the subsection. It uses several results is also used in the next subsection. The proof that the geometric pressure function is smaller or equal than the infimum is simple and depends basically on Lemma 4.1. The other inequality is much more involved. It requires the Geometric Peierls condition, and a lower bound on the pressure function that we proceed to state and prove.

**Proposition III** (Critical line). For every integer \( n \geq 5 \), and every \( f \) in \( \mathcal{K}_n(\mathcal{F}) \), we have
\[ \chi^R_{\inf} := \inf \left\{ \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f \right\} \leq \chi(\text{crit})/2. \]

In particular, for each \( t > 0 \) we have
\[ P_f(t) \geq P^R(t) \geq -t\chi(\text{crit})/2. \]

The proof of this proposition is given after the following lemma.

**Lemma 5.3.** There is a constant \( C_7 > 0 \) that only depends on \( K \) and \( R \), such that for each integer \( k \geq 0 \) and each \( f \) in \( \mathcal{K}_n(\mathcal{F}) \), the following property holds: For every integer \( k \geq 0 \) there is a connected component \( \hat{W} \) of \( D_f \) contained in \( P_f^{n+3k+2}(0) \), such that \( h_f^{-1}(\hat{W}) \) intersects \( \mathcal{R} \), and such that \( m_f(\hat{W}) = n + 3k + 3 \) and
\[ \sup_{z \in \hat{W}} |DF_f(z)| \leq C_7 |DF^{n+3k}(f(0))|^{1/2}. \]

**Proof.** We follow the proof of [CRL13, Lemma 6.3]. Let \( C_1 \) be the constant given by Lemma 4.1 and let \( C_2 \) be the constant given by Lemma 4.5. Let \( \Delta_2 > 1 \) and \( \Delta_3 > 1 \) be the constants given by Lemmas 4.9 and 5.1 respectively. From the proof of [CRL13, Lemma 6.3], we have
\[ \Xi_0 := \sup_{c \in P_4(-2)} \text{diam}(P_{c,1}(0)) < +\infty \]
and
\[ \Xi_1 := \sup_{c \in P_4(-2)} \sup_{z \in P_{c,1}(0)} |DF^2_c(z)| < +\infty. \]

By Lemma 4.5 and inequality (4.4), we have
\[ \Xi_0 := \sup_{f \in P_4(\mathcal{F})} \sup_{z \in P_{f,1}(0)} |DF^2_{f_c}(z)| < +\infty, \]
and
\[ \tilde{\Xi}_1 := \sup_{f \in \mathcal{P}_d(\mathcal{F})} \sup_{z \in P_{r, 1}(0)} |Df^2(z)| \leq C_7^2 \Xi_1^2 < +\infty. \]

Fix an integer \( n \geq 5 \), a \( f \) in \( \mathcal{K}_n(\mathcal{F}) \), and an integer \( k \geq 0 \). By the proof of [CRL13 Lemma 6.3] there is a connected component \( W \) of \( D_{c(f)} \) contained in \( P_{c(f), n+3k+2}(0) \), that intersects \( \mathbb{R} \), and such that \( m_{c(f)}(W) = n + 3k + 3 \). The set \( \hat{W} = h_f(W) \) verifies the desired properties of the lemma. To finish the proof it remains to prove the inequality.

Let \( z_{\hat{W}} \) be the unique point in \( \hat{W} \) such that \( f^{n+3k+3}(z_{\hat{W}}) = 0 \). Then \( f^{n+3k+1}(z_{\hat{W}}) \) belongs to \( P_{r, 1}(0) \), so by definition of \( \hat{\Xi}_0 \) we have
\[ |f^{n+3k+1}(z_{\hat{W}}) - f^{n+3k}(f(0))| \leq \text{diam}(P_{r, 1}(0)) \leq \hat{\Xi}_0. \]

Since \( f^n \) maps \( P_{c,n+1}(c) \) biholomorphically to \( P_{c,1}(0) \) and \( f^n(c) \in \Lambda_c \), it follows that \( f^n(c) \) maps \( P_{c,n+3k+1}(c) \) biholomorphically to \( P_{c,1}(0) \), and the same holds for \( f^{n+3k} \) and the sets \( P_{f,n+3k+1}(f(0)) \) and \( P_{f,1}(0) \); so the distortion of \( f^{n+3k} \) on \( P_{f,n+3k+1}(f(0)) \) is bounded by \( \Delta_2 \) (Lemma 4.4) and for each point \( y \) in \( P_{f,n+3k+1}(f(0)) \) we have
\[ \Delta_2^{-1}|Df^{n+3k}(f(0))| \leq |Df^{n+3k}(y)| \leq \Delta_2|Df^{n+3k}(f(0))|. \]

Together with (5.5) this implies that
\[ |f(z_{\hat{W}}) - f(0)| \leq \Delta_2 \hat{\Xi}_0 |Df^{n+3k}(f(0))|^{-1} \]
and by Lemma 4.4
\[ |Df(z_{\hat{W}})| \leq C_7^2 \Delta_2^{1/2} \hat{\Xi}_0^{1/2} |Df^{n+3k}(f(0))|^{-1/2}. \]

Combined with (5.6) with \( y = f(z_{\hat{W}}) \), this implies
\[ |Df^{n+3k+1}(z_{\hat{W}})| \leq C_7^2 \Delta_2^{3/2} \hat{\Xi}_0^{1/2} |Df^{n+3k}(f(0))|^{1/2}. \]

Putting \( C_7 := C_7^2 \Delta_2 \hat{\Xi}_0 \Delta_2^{3/2} \hat{\Xi}_0^{1/2} \), we get by Lemma 5.1
\[ \sup_{z \in \hat{W}} |DF(z)| \leq \Delta_3 |Df^{n+3k+3}(z_{\hat{W}})| \leq \Delta_3 \hat{\Xi}_1 |Df^{n+3k+1}(z_{\hat{W}})| \leq C_7 |Df^{n+3k}(f(0))|^{1/2}. \]

\[ \square \]

**Proof of Proposition III.** The proof follows from Lemma 5.3 by constructing a sequence of measures supported in periodic points whose Lyapunov exponents converge to \( \chi_{\text{crit}}(f) \). For details see the proof of [CRL13 Proposition 6.2]. \[ \square \]

**Lemma 5.4.** Let \( \kappa > 0 \) and \( \nu > 0 \) be given. Then there is \( C_8 > 1 \) that only depends on \( K \), \( R \), \( \kappa \), and \( \nu \), such that for every integer \( n \geq 5 \), and every \( f \) in \( \mathcal{K}_n(\mathcal{F}) \) satisfying the Geometric Peierls Condition with constants \( \kappa \) and \( \nu \), the following property holds: For every \( t \geq 2 \log 2/\nu \), \( p \geq -t(\chi_{\text{crit}}(f) + 2\nu)/2 \), and \( y \) in \( V_f \) we have
\[ L_{t, p}(y) := 1 + \sum_{z \in L_1^+(y)} (m_f(z) + 1) \exp(-m_f(z)p) |DL_f(z)|^{-t} \leq C_8^t. \]
Proof. Fix $\kappa > 0$ and $\nu > 0$, and put $t_0 := 2 \log 2 / \nu$. We prove the lemma with
\[
C_8 := \max\{1, \kappa^{-1}\} \left(1 - \exp\left(\log 2 - (2/3) \nu t_0\right)\right)^{-2/t_0}.
\]
Fix $n \geq 5$ and $f$ in $\mathcal{K}_n(\mathcal{F})$ satisfying the Geometric Peierls Condition with constants $\kappa$ and $\nu$.

Fix $t \geq t_0$, $p \geq -t(\chi_{\text{crit}}(f)/2 + \nu/2)$, and $y$ in $V_f$. Since $f$ satisfies the Geometric Peierls Condition, for every $z \in L_f^{-1}(y)$ we have
\[
\exp(-m_f(z)p)|DL_f(z)|^{-1} \leq \kappa^{-t} \exp(-(2/3)\nu \cdot m_f(z)t).
\]
On the other hand, for each integer $m \geq 1$ the set \{ $z \in L_f^{-1}(y) \mid m_f(z) = m$ \} is contained in $f^{-m}(y)$ and therefore it contains at most $2^m$ points. So, we have
\[
\tilde{L}_{t,p}(y) \leq \max\{1, \kappa^{-1}\}^{+\infty} \sum_{m=0}^{+\infty} (m + 1) \exp(m \log 2 - (2/3) \nu t)) \leq C_8. 
\]

\begin{lemma}
Given an integer $n \geq 5$ and $f$ in $\mathcal{K}_n(\mathcal{F})$, the following property holds for every $t > 0$ and every real number $p$: If $\mathcal{P}^f_f(t,p) > 0$ (resp. $\mathcal{P}^f_f(t,p) > 0$), then the series
\begin{equation}
\sum_{j=1}^{+\infty} \exp(-jp) \sum_{y \in f^{-j}(0)} |Df^j(y)|^{-t} \left( \text{resp.} \sum_{j=1}^{+\infty} \exp(-jp) \sum_{y \in f^{-j}(0)} |Df^j(y)|^{-t} \right)
\end{equation}
diverges. On the other hand, if for some $\kappa > 0$ and $\nu > 0$ the map $f$ satisfies the Geometric Peierls Condition with constants $\kappa$ and $\nu$, then for every
\[
t \geq 2 \log 2 / \nu \quad \text{and} \quad p \geq \mathcal{P}^f_f(t) - \frac{\nu}{2} \left( \text{resp.} \quad p \geq \mathcal{P}^f_f(t) - \frac{\nu}{2} \right)
\]
satisfying $\mathcal{P}^f_f(t,p) < 0$ (resp. $\mathcal{P}^f_f(t,p) < 0$), the series above converges.
\end{lemma}

Proof. The proof of the first part of the lemma depends on Lemma 5.1 and it follows the same lines that the first part of [CRIT].

We prove the last assertion concerning $f|_{\ell_f}$; the arguments apply without change to $f|_{\ell(f)}$. Fix some positive constants $\kappa$ and $\nu$ and let $C_8 > 1$ be given by Lemma 5.4 for the constants $\kappa$ and $\nu$. Let $f$ be a map in $\mathcal{F}$ satisfying the Geometric Peierls Condition with constants $\kappa$ and $\nu$, and let
\[
t \geq 2 \log 2 / \nu \quad \text{and} \quad p \geq \mathcal{P}^f_f(t) - \frac{\nu}{2}
\]
be such that $\mathcal{P}^f_f(t,p) < 0$. By Proposition 5.1 we have
\[
p \geq -t(\chi_{\text{crit}}(f) + \frac{\nu}{2})/2,
\]
so $t$ and $p$ satisfy the hypotheses of Lemma 5.4. Given an integer $m \geq 1$ and a point $z$ in $f^{-m}(0)$ denote by $\ell(z)$ the number of those $j \in \{0, \ldots, m - 1\}$ such that $f^{j}(z)$ is in $V_f$. In the case where $z$ is not in $V_f$, this point is in the domain of $L_f$ and we have $\ell(z) = 0$ if and only $L_f(z) = 0$. Moreover, if $z$ is not in $V_f$
and \( \ell(z) \geq 1 \), then \( L_f(z) \) is in the domain of \( F_f^{\ell(z)} \) and \( F_f^{\ell(z)}(L_f(z)) = 0 \). So, if \( z \) is not in \( V_f \) we have in all the cases,
\[
|Df^m_f(z)| = |DF_f^{\ell(z)}(L_f(z))| \cdot |DL_f(z)|.
\]
Then Lemma 5.4 and our hypothesis \( \mathcal{P}_f(t,p) < 0 \) imply that the series \( (5.7) \) is bounded from above by
\[
\bar{L}_{t,p}(0) + \sum_{\ell=1}^{+\infty} \sum_{y \in F^{-\ell}_{-f}(0)} \bar{L}_{\ell,p}(y) \exp\left(\sum_{i=0}^{\ell-1} m_f(F_f^{-i}(y)) + \sum_{i=\ell}^{+\infty} m_f(y)\right) |DF_f^{\ell}(y)|^{-t}
\leq C_8 \left(1 + \sum_{\ell=0}^{+\infty} Z_{f,\ell}(t,p)\right) < +\infty.
\]

**Proof of Proposition II**. We follow the proof of [CRL13, Proposition C].
We prove the assertion for \( f|_{\mathcal{U}(f)} \); the arguments apply without change to \( f|_{\mathcal{U}(f)} \).
Let \( \Delta_3 > 1 \) be given by Lemma 5.4. Let \( n \geq 5 \) be an integer and let \( f \) be in \( K_n(F) \).
We use that fact that for each \( t > 0 \) we have
\[
(5.8) \quad P_f(t) = \limsup_{m \to +\infty} \frac{1}{m} \log \sum_{y \in f^{-m}(0)} |Df^m(y)|^{-t},
\]
see for example [PRLS04].
Fix \( t \geq 2 \log 2/v \). We use the fact that the function \( p \mapsto \mathcal{P}_f(t,p) \) is strictly decreasing where it is finite, see 5.1. In particular, for each \( p \) satisfying \( p < p_0 := \inf\{p \mid \mathcal{P}_f(t,p) \leq 0\} \) we have \( \mathcal{P}_f(t,p) > 0 \). Lemma 5.5 implies that for such \( p \) the series \( (5.7) \) diverges and by \( (5.8) \) we have \( P_f(t) \geq p \). It follows that, \( P_f(t) \geq p_0 \).
To prove the reverse inequality, suppose by contradiction \( p_0 < P_f(t) \) and let \( p \) be in the interval \( (p_0, P_f(t)) \) satisfying \( p \geq P_f(t) - t \frac{8}{\log 2} \). Then \( \mathcal{P}_f(t,p) < 0 \) and by Lemma 5.5 the series \( (5.7) \) converges. Then \( (5.8) \) implies \( P_f(t) \leq p \) and we obtain a contradiction that completes the proof of the proposition.

**5.3. Proof of Proposition II**. The final step in the proof of Proposition II is given after the following proposition, which estimates the partition function of the induced map in terms of the derivative of the iterates of the map at its critical value.

Let \( n \geq 4 \) be an integer and \( f \) in \( K_n(F) \). Since the critical point \( z = 0 \) does not belong to \( D_f \) (cf. [CRL13, Lemma 4.2]), for each integer \( \ell \geq 1 \), each connected component of \( D_f \) intersecting \( P_f,\ell(0) \) is contained in \( P_f,\ell(0) \). Define the *level* of a connected component \( W \) of \( D_f \) as the largest integer \( k \geq 0 \) such that \( W \) is contained in \( P_f,n+3k+2(0) \). Given an integer \( k \geq 0 \) denote by \( \mathfrak{D}_{f,k} \) the collection of all connected components of \( D_f \) of level \( k \); we have \( \mathfrak{D}_f = \bigcup_{k=0}^{+\infty} \mathfrak{D}_{f,k} \), and for every \( W \) in \( \mathfrak{D}_{f,k} \) we have \( m_f(W) \geq n + 3k + 1 \).

**Proposition 5.6.** Let \( \kappa > 0 \) and \( v > 0 \) be given. Then there are \( n_4 \geq 5 \) and \( C_9 > 1 \) that only depend on \( K, R, \kappa, \text{and } v \), such that for every integer \( n \geq n_4 \), and every \( f \) in \( K_n(F) \) satisfying the Geometric Peiersl Condition with constants \( \kappa \) and \( v \), the following properties hold for each \( t \geq 2 \log 2/v \) and each integer \( k \geq 0 \):
1. For each $p < 0$, we have
\[
\sum_{W \in \mathcal{D}, k \cap \mathcal{D}^p} \exp(-m_f(W)p) \inf_{z \in W} |DF_f(z)|^{-t} > C_0^{-t} \exp(-(n + 3k)p)|Df^{n+3k}(f(0))|^{-t/2}.
\]

2. For each $p \geq -\chi_{\text{crit}}(f)/2 - \nu/3$, we have
\[
\sum_{W \in \mathcal{D}, k} \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t}
\leq \sum_{W \in \mathcal{D}, k} (m_f(W) - (n + 3k)) \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t}
\leq C_0^{-t} \exp(-(n + 3k)p)|Df^{n+3k}(f(0))|^{-t/2}.
\]

The proof of this proposition is given after the following lemma.

**Lemma 5.7.** There is $C_{10} > 1$ that only depends on $K$ and $R$, such that for each integer $n \geq 5$, each $f$ in $\mathcal{K}_n(\mathcal{F})$, each integer $k \geq 0$, and each pair of real numbers $t > 0$ and $p$, we have
\[
\sum_{W \in \mathcal{D}, k} \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t}
\leq 2C_{10}^{t} \exp(-(n + 3k + 1)p)|Df^{n+3k}(f(0))|^{-t/2}
\cdot \left(1 + \sum_{w \in L_f^{-1}(0) \text{ in } P_f(0)} \exp(-m_f(w)p)|DL_f(w)|^{-t}\right).
\]

Moreover,
\[
\sum_{W \in \mathcal{D}, k} (m_f(W) - (n + 3k)) \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t}
\leq 2C_{10}^{t} \exp(-(n + 3k + 1)p)|Df^{n+3k}(f(0))|^{-t/2}
\cdot \left(1 + \sum_{w \in L_f^{-1}(0) \text{ in } P_f(0)} (m_f(w) + 1) \exp(-m_f(w)p)|DL_f(w)|^{-t}\right).
\]

**Proof.** The proof follows the same lines that the proof of [CRL13, Lemma 7.1], and it depends on Lemmas [4.9] [4.10] and [5.1] as well as on [CRL13, Lemma 5.1]. The second inequality does not appear in [CRL13, Lemma 7.1]. It follows from the first displayed equation in the proof of [CRL13, Lemma 7.1], and from [CRL13, (7.1)]. See the proof of [CRL13, Lemma 7.1] for further details. \(\square\)

**Proof of Proposition 5.6.** The proof depends on Lemmas [3.3] [5.4] and [5.7] and on Proposition [11] and it follows the same lines that the proof of [CRL13, Lemma 7.2]. There are some differences in part 2 since the condition on $p$ is slightly different and we add a new inequality. We include the proof of part 2 here.

Let $C_8 > 0$ and $C_{10} > 0$ be given by Lemmas [5.4] and [5.7] respectively. Let $n_2$ be the integer given by Lemma [4.7] with $\varepsilon = \frac{1}{M}$. Put $t_0 := 2 \log 2/\nu$. We prove the
lemma for $n_4 = n_2$. Fix an integer $n \geq n_4$, a map $f$ in $K_n(\mathcal{F})$, $t \geq t_0$, and an integer $k \geq 0$.

To prove part 2, note that the first inequality follows from the fact that for every $W$ in $D_{f,k}$ we have $m_f(W) \geq n + 3k + 1$. To prove the second inequality, let $p \geq -t_{\chi_{\text{crit}}}(f)/2 - tv/3$ be given. By Lemma 4.7, we have

$$
\chi_{\text{crit}}(f) \leq 1.1K\log 2.
$$

Thus $-p \leq t(0.55K + \frac{p}{3\log 2})\log 2 < t(0.55K + 1/t_0)\log 2$ and therefore $2\exp(-p) < 2t(0.55K + 2/t_0)$. Combined with Lemmas 5.4 and 5.7 we obtain part 2 of the proposition with $C_9 = 2^{0.55K + 2/t_0}C_{10}C_8$.

**Proof of Proposition 7.** We follow the proof of [CRL13, Proposition D]. Let $n_4$ and $C_9$ be given by Proposition 5.6 and put $t_0 := 2\log 2/v$. To prove the proposition, fix an integer $n \geq n_4$, a $f$ in $K_n(\mathcal{F})$, and $t \geq t_0$.

To prove part 1, let $p$ be in $[-t_{\chi_{\text{crit}}}(f)/2, 0)$. By part 1 of Proposition 5.6 if the sum

$$
\sum_{k=0}^{+\infty} \exp(-(n + 3k)p)|Df^n + 3k(f(0))|^{-t/2}
$$

is greater than or equal to $C_9'$, then $\mathcal{P}_f^R(t, p) > 0$ and by Proposition IV we have $P_f^R(t) \geq p$. This proves the first part of part 1 with $C_5 = C_9$. To complete the proof of part 1, suppose (5.9) is finite and greater than or equal to $(2C_9)'$. Then there is $p' > p$ such that (5.9) with $p$ replaced by $p'$ is greater than or equal to $C_9'$. As shown above, this implies $P_f^R(t) \geq p' > p$. On the other hand, by part 2 of Proposition 5.6 if $k \geq 0$, the sum

$$
\sum_{W \in D_f} \exp(-m_f(W)p) \sup_{z \in W} |Df(z)|^{-t}
$$

is finite, so $\mathcal{P}_f^R(t, p)$ is also finite. This completes the proof of part 1 with $C_5 = 2C_9$.

To prove part 2, let $p \geq -t_{\chi_{\text{crit}}}(f)/2$ be given. By part 2 of Proposition 5.6 if (5.9) is less than or equal to $C_9^{-t}$, then $\mathcal{P}_f(t, p) < 0$ and by Proposition IV we have $P_f(t) \leq p$. This proves part 2 of the proposition with $C_5 = C_9$.

To prove part 3, let $p \geq -t_{\chi_{\text{crit}}}(f)/2$ be given and put $p' := p - t_{\mathcal{V}}$. By part 2 of Proposition 5.6 with $k = 0$, the sum

$$
\sum_{W \in D_{f,0}} \exp(-m_f(W)p) \sup_{z \in W} |Df(z)|^{-t}
$$

is finite. Let $A > 0$ be a constant such that for every pair of integers $k \geq 1$ and $m \geq 3k + 1$, we have

$$
m \leq Ak \exp(t_0v(m - 3k)/3).
$$
Applying part 2 of Proposition 5.6 with $p$ replaced by $p'$, we obtain that for each integer $k \geq 1$ we have

$$
\sum_{W \in \mathcal{D}_{f,k}} m_f(W) \cdot \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \leq \sum_{W \in \mathcal{D}_{f,k}} Ak \exp(tv (m_f(W) - 3k)/3) \exp(-m_f(W)p) \sup_{z \in W} |DF_f(z)|^{-t} \leq (Ac^t \exp(-tvn/3)) k \cdot \exp(-(n + 3k)p)|Df^{n+3k}(f(0))|^{-t/2}.
$$

Summing over $k \geq 0$ we obtain the desired assertion. \hfill \Box

## 6. Sensitive dependence of geometric Gibbs states

In this section we prove the Main Theorem. In §6.1 we estimate the postcritical series in terms of certain 2 variables series that only depends on the combinatorics of the postcritical orbit (Lemma 6.1). The main estimates needed in the proof of the Main Theorem can be stated only in terms of these 2 variables series, and are relegated to Appendix A. The proof of the Main Theorem is given in §6.2 and it is divided in 3 parts. In the first part (§6.2.1), we introduce the family of maps $(f_k)_{k \in \{+,-\}^\mathbb{N}}$, which is mostly defined through the combinatorics of the postcritical orbit. In the second part (§6.2.2), we estimate the geometric pressure, and prove the existence and uniqueness of geometric Gibbs states, as well as the existence of conformal measures. The third, and most difficult part of the proof is given in §6.2.3 where we show that on certain intervals of inverse temperatures the geometric Gibbs states are concentrated near the orbit of $p^+$, or the orbit of $p^-$.

### 6.1. The 2 variables series

Denote by $\tilde{\Sigma}$ the set of all those sequences $(\tilde{x}_j)_{j \in \mathbb{N}_0}$ in $\{0, 1^+, 1^-\}^{\mathbb{N}_0}$ such that for each $j$ in $\mathbb{N}_0$ satisfying $\tilde{x}_j = 1^+$ (resp. $\tilde{x}_j = 1^-$), we have $\tilde{x}_{j+1} \neq 1^-$ (resp. $\tilde{x}_{j+1} \neq 1^+$). A sequence $(x_j)_{j \in \mathbb{N}_0}$ in $\{0, 1\}^{\mathbb{N}_0}$ is compatible with a sequence $(\tilde{x}_j)_{j \in \mathbb{N}_0}$ in $\tilde{\Sigma}$ if for every $j$ in $\mathbb{N}_0$ such that $\tilde{x}_j = 0$ (resp. $\tilde{x}_j = 1^+$, $\tilde{x}_j = 1^- = 1^-$), we have $x_j = 0$ (resp. $x_j = 1$, $x_j \neq x_{j+1}$).

Fix a sequence $(\tilde{x}_j)_{j \in \mathbb{N}_0}$ in $\tilde{\Sigma}$. Define $N: \mathbb{N}_0 \to \mathbb{N}_0$ by $N(0) := 0$, and for $k$ in $\mathbb{N}$ by

$$
N(k) := 2\{j \in \{0, \ldots, k-1\} \mid \tilde{x}_j = 0\}.
$$

Moreover, define $B: \mathbb{N}_0 \to \mathbb{N}_0$ by $B(0) := 0$, $B(1) := 1$, and for $k \geq 2$ by

$$
B(k) := 1 + 2\{j \in \{0, \ldots, k-2\} \mid \tilde{x}_j \neq \tilde{x}_{j+1}\}.
$$

Note that for $k$ in $\mathbb{N}$ the function $B(k)$ is equal to the number of blocks of $0$’s, $1^+$’s, and $1^-$’s in the sequence $(\tilde{x}_j)_{j=0}^{k-1}$.

Throughout the rest of this subsection, fix a uniform family of quadratic-like maps $\mathcal{F}$, and let $\Delta_2 > 1$ be the constant given by Lemma 4.9. For each integer $n \geq 5$, and each $f$ in $\mathcal{K}_n(\mathcal{F})$, put

$$
\theta(f) := \left| \frac{Dg_f(p(f))}{Dg_f(p^+(f))} \right|^{1/2}.
$$

(6.1)
Note that the condition $\theta(f) > 1$ is equivalent to $\chi_f(p(f)) > \chi_f(p^+(f))$. When this holds, define

$$\xi(f) := \frac{\log \Delta_2}{2 \log \delta(f)}.$$

If in addition the itinerary $i(f)$ is compatible with $(\hat{\xi}_j)_{j \in \mathbb{N}_0}$, let $\xi \geq \xi(f)$ be given and define for each integer $k \geq 0$ and each $(\tau, \lambda)$ in $[0, +\infty) \times [0, +\infty)$,

$$\pi_{f,k}^x(\tau, \lambda) := 2^{-\lambda_k - \lambda N(k) + \xi \tau B(k)}.$$

The purpose of this subsection is to prove the following lemma.

**Lemma 6.1.** Let $\Delta_1$ be the constant given by Lemma 4.9, and let $\hat{\xi}$ be a sequence in $\hat{\Sigma}$ such that $N(k)/k \to 0$ as $k \to +\infty$, and such that the length of every maximal block containing only $1^-$'s is even. Then, for every integer $n \geq 6$ and every $f$ in $K_n(\mathcal{F})$ such that $\chi(p^-(f)) = \chi(p^+(f))$, and such that the itinerary $i(f)$ is compatible with $\hat{\xi}$, we have

$$\chi_{\text{crit}}(f) = \frac{1}{3} \log |Dg_f(p^+(f))|.$$  

If moreover

$$\chi_f(p(f)) > \chi_f(p^+(f)),$$

then the following property holds for every choice of $\xi \geq \xi(f)$. For every integer $k \geq 0$, and every $t > 0$ and $\delta > 0$, we have

$$\Delta_1 \frac{1}{2} \exp(-n\delta) \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^2 \pi_{f,k}^x \left( \frac{\log \delta(f)}{\log 2}, \frac{3\delta}{\log 2} \right) \leq \exp \left( -(n + 3k) \left( -t \frac{\chi_{\text{crit}}(f)}{2} + \delta \right) \right) |Df^{n+3k}(f(0))|^{-\frac{1}{2}}$$

$$\leq \Delta_1 \frac{1}{2} \exp(-n\delta) \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^2 \pi_{f,k}^x \left( \frac{\log \delta(f)}{\log 2}, \frac{3\delta}{\log 2} \right).$$

**Proof.** Let $\Delta_2$ be the constant given by Lemma 4.3. Put $\tilde{c} := f^{n+1}(0)$. For every $k$ in $\mathbb{N}$ and every $j$ in $\{0, 1, 2\}$, we have by the chain rule

$$Df^{3k+j}(f(0)) = Df^j((f^{3k})(\tilde{c})) \cdot Df^{3k}(\tilde{c}) \cdot Df^n(f(0))$$

$$= Df^j(g_f^k(\tilde{c})) \cdot Dg_f^k(\tilde{c}) \cdot Df^n(f(0)).$$

Since $|Df^j((g_f^k)(\tilde{c}))|$ is bounded independently of $k$ and $j$, we have

$$\chi_{\text{crit}}(f) = \liminf_{m \to +\infty} \frac{1}{m} \log |Df^m(f(0))| = \frac{1}{3} \liminf_{k \to +\infty} \frac{1}{k} \log |Dg_f^k(\tilde{c})|.$$  

On the other hand, by Lemma 4.3, the assumption $\chi(p^-(f)) = \chi(p^+(f))$, and the fact that the blocks of $1^-$'s in $\hat{\xi}$ have even length, we have that for each integer $k$ in $\mathbb{N}$,

$$\Delta_2^{B(k)} \leq \frac{|Dg_f^k(\tilde{c})|}{|Dg_f(p^+(f))|^{k-N(k)}|Dg_f(p(f))|^{N(k)} \leq \Delta_2^{B(k)}.$$

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Taking logarithm yields
\[-B(k) \log \Delta_2 + N(k) \log \frac{|Dg_f(p(f))|}{|Dg_f(p^+(f))|} \leq \log |Dg_f^k(\hat{\mathcal{C}})| - k \log |Dg_f(p^+(f))| \]
\[\leq B(k) \log \Delta_2 + N(k) \log \frac{|Dg_f(p(f))|}{|Dg_f(p^+(f))|}.\]
Since for each \(k \in \mathbb{N}\) we have \(B(k) \leq 2N(k) + 1\), using the hypothesis that \(N(k)/k \to 0\) as \(k \to +\infty\), we conclude that
\[\lim_{k \to +\infty} \frac{1}{k} \log |Dg_f^k(\hat{\mathcal{C}})| = \log |Dg_f(p^+(f))|.\]
Combined with (6.4), this completes the proof of (6.2).

In the case where \(k = 0\), the chain of inequalities (6.3) is given by Lemma 4.8. Fix \(k \in \mathbb{N}\) and \(t > 0\). Using
\[Df^{n+3k}(f(0)) = Dg_f^k(f^{n+1}(0)) \cdot Df^n(f(0))\]
and Lemmas 4.8 and 4.9, the assumption \(\chi(p^-(f)) = \chi(p^+(f))\), and the fact that the maximal blocks of \(1\)'s in \(\hat{\mathcal{C}}\) have even length, we have
\[\Delta_1^{-t} \theta(f)^{-2tN(k)} \Delta_2^{-tB(k)} \leq \frac{|Df^{n+3k}(f(0))|^{-t}}{|Dg_f(p^+(f))|^{-tk} |Df(\beta(f))|^{-tn}} \leq \Delta_1^{-t} \theta(f)^{-2tN(k)} \Delta_2^{-tB(k)} .\]
Since by (6.2) we have
\[\exp((n + 3k)t\chi_{\text{crit}}(f)) = \exp(nt\chi_{\text{crit}}(f)|Dg_f(p^+(f)|^t),\]
if we multiply each term in the chain of inequalities (6.5) by
\[\left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{tn},\]
then we get
\[\Delta_1^{-t} \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{tn} \theta(f)^{-2tN(k)} \Delta_2^{-tB(k)} \leq \exp((n + 3k)t\chi_{\text{crit}}(f))|Df^{n+3k}(f(0))|^{-t} \leq \Delta_1^{-t} \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{tn} \theta(f)^{-2tN(k)} \Delta_2^{-tB(k)} .\]
Taking square roots, and then by multiplying by \(\exp(-(n + 3k)\delta)\) in each of the terms of the chain of inequalities above, we obtain
\[\Delta_1^{-t/2} \exp(-n\delta) \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{tn}{2}} \exp(-3k\delta) \theta(f)^{-tN(k)} \Delta_2^{-tB(k)/2} \leq \exp\left(-n\left(\frac{3\chi_{\text{crit}}(f)}{2} + \delta\right)\right)|Df^{n+3k}(f(0))|^{-\frac{t}{2}} \leq \Delta_1^{-t/2} \exp(-n\delta) \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{tn}{2}} \exp(-3k\delta) \theta(f)^{-tN(k)} \Delta_2^{-tB(k)/2} .\]
Together with our choice \(\xi \geq \xi(f)\), and our definition of \(\pi_{f,k}^+\), this implies the desired chain of inequalities.
6.2. Proof of the Main Theorem. Put \( v := \frac{1}{3} \log 2 \), let \( R > 0 \) be given, and let \( K_1, \kappa_1 \), and \( n_1 \) be given by Proposition 4.3. We prove that the Main Theorem holds with \( K_0 = K_1 \). Let \( F \) be a uniform family of quadratic-like maps with constants \( K_1 \) and \( R \) that is admissible. By Proposition 4.3 for every \( n \geq n_1 \), every \( f \) in \( K_n(F_0) \) satisfies the Geometric Peierls Condition with constants \( \kappa_1 \) and \( v \), and we have

\[
\chi_f(\beta(f)) > \chi_{\text{crit}}(f) + 2v. \tag{6.6}
\]

Taking \( n_1 \) larger if necessary, assume that for every \( n \geq n_1 \) there is a continuous function \( s_n: K_n \to K_n(F) \) such that \( c \circ s_n \) is the identity, and that (3.1) holds for every \( f \) in \( s_n(K_n) \).

Let \( \Delta_1, \Delta_2, C_5, v_1 \) and \( \Delta_3 \) be the constants given by Lemmas 4.8, 4.9, 4.11 and 5.1 respectively. Moreover, let \( n_3 \) and \( C_6 \) the constants given by Proposition 3.6 and \( \kappa = \kappa_1 \), let \( n_4 \) and \( C_6 \) be given by Proposition 3.6 and let \( n_6 \geq \max\{6, n_3, n_4\} \) be sufficiently large so that

\[
\exp(n_6; v) \geq \Delta_1 C_6 (2 + \Delta_2). \tag{6.7}
\]

6.2.1. The subfamily. In this subsection we define the family \( \{f_\xi\}_{\xi \in \{+,-\}^n} \), as in the statement of the Main Theorem.

Fix an integer \( n \geq n_6 \), let \( c_k \) in \( K_n \) be such that \( \nu(c_k) \) is the constant sequence equal to 0, and put \( f_{c_k} := s_n(c_k) \). By (3.1) we have \( \theta(f_{c_k}) > 1 \), so there is \( r_{c_k} > 0 \) such that for \( c \) in \( B(c_k, r_{c_k}) \) \( \cap K_n \) the number \( \theta(s_n(c)) \) is defined, and depends continuously with \( c \). Reducing \( r_{c_k} \) if necessary, assume that for every \( c \) and \( c' \) in \( B(c_k, r_{c_k}) \) \( \cap K_n \) we have \( \theta(s_n(c)) \leq \theta(s_n(c'))^2 \). By Proposition 2.3 it follows that there is an integer \( q_{c_k} \geq 0 \) such that the set

\[
\{c \in K_n \mid \text{for every } j \in \{0, \ldots, q_{c_k}\}, \nu(c)_j = 0\}
\]

is a compact set contained in \( B(c_k, r_{c_k}) \). On the other hand, by (3.1) for each \( c \) in \( K_n \) we can define the number \( \xi(s_n(c)) \) as in (3.1). It follows that this number depends continuously with \( c \) in \( K_n \), so

\[
\xi := \sup_{c \in K_n \cap B(c_k, r_{c_k})} \xi(s_n(c)) < +\infty.
\]

Put \( \Xi := [2\xi] + 1 \) as in (A.2) and let \( q \geq q_{c_k} \) be an integer satisfying the conditions in (A.1) and such that in addition \( q + \Xi \) is even. For each real number \( s \geq 0 \), let \( I_s \) and \( J_s \) be the intervals defined in (A.1) for these choices of \( \Xi \) and \( q \). By definition, as \( s \) varies in \( \mathbb{N}_0 \) these intervals form a partition of \([1, +\infty)\). Moreover, for each integer \( s \) in \( \mathbb{N}_0 \), the end points of \( I_s \) and \( J_s \) are even.

Endow the set \( \{0, 1^+, 1^-\} \) with the discrete topology, and \( \{0, 1^+, 1^-\}^{\mathbb{N}_0} \) with the corresponding product topology. Moreover, endow the subset \( \hat{\Sigma} \) of \( \{0, 1^+, 1^-\}^{\mathbb{N}_0} \), defined in (6.1) with the induced topology.

Given \( \xi \) in \( \{+,-\}^{\mathbb{N}} \), let \( \hat{\xi}(\xi) \) be the sequence in \( \{0, 1^+, 1^-\}^{\mathbb{N}_0} \) defined by

\[
\hat{\xi}(\xi)_j := \begin{cases} 
0 & \text{if for some } s \in \mathbb{N}_0 \text{ we have } j + 1 \in I_s; \\
1^+ & \text{if } j + 1 \in J_0; \\
1^{(m)} & \text{for } j + 1 \in J_{4m-3} \cup J_{4m-2} \cup J_{4m-1} \cup J_{4m}.
\end{cases}
\]

Note that by definition \( \hat{\xi}(\xi) \) is in \( \hat{\Sigma} \) and that the first \( q \) entries of this sequence are equal to 0. Moreover, the map \( \hat{\xi}: \{+,-\}^{\mathbb{N}} \to \hat{\Sigma} \) so defined is continuous. Finally, note that the length of each maximal block of \( 1^- \)'s in \( \hat{\xi}(\xi) \) is even.
Define the family of itineraries \((\iota(\varsigma))_{\varsigma \in \{+, -\}^n}\) in \(\{0, 1\}^\mathbb{N}_0\), by

\[
\iota(\varsigma)_j = \begin{cases} 
0 & \text{if } \overline{x}(\varsigma)_j = 0; \\
1 & \text{if } \overline{x}(\varsigma)_j = 1^+; \\
0 & \text{if } \overline{x}(\varsigma)_j = 1^- \text{ and } j \text{ is even}; \\
1 & \text{if } \overline{x}(\varsigma)_j = 1^- \text{ and } j \text{ is odd}.
\end{cases}
\]

Note that \(\iota(\varsigma)\) is compatible with \(\overline{x}(\varsigma)\) in the sense of §6.1 and that the first \(q\) entries of this sequence are equal to 0. Moreover, \(\iota(\varsigma)\) depends continuously with \(\varsigma\) in \(\{+, -\}^\mathbb{N}\).

Given \(\varsigma\) in \(\{+, -\}^\mathbb{N}\), let \(c(\varsigma)\) in \(K_n\) be the unique parameter such that \(\iota(f_{c(\varsigma)}) = \iota(\varsigma)\) (Proposition 2.3), and put \(f_\varsigma := s_n(c(\varsigma))\). Note that the function \(\varsigma \mapsto f_\varsigma\) so defined is continuous. On the other hand, since for each \(\varsigma\) in \(\{+, -\}^\mathbb{N}\) the first \(q \geq q_\varsigma\) entries of \(\iota(f_\varsigma) = \iota(\varsigma)\) are equal to 0, the parameter \(c(\varsigma)\) is in \(B(\lambda_{k}, r_{k})\). So, for every \(\varsigma\) and \(\varsigma'\) in \(\{+, -\}^\mathbb{N}\) we have

\[
\theta(f_{\varsigma'}) \leq \theta(f_{\varsigma})^2. \tag{6.8}
\]

6.2.2. Pressure estimates, and the existence of equilibria. The purpose of this subsection is to prove part 1 of the Main Theorem, and at the same time to estimate for each \(\varsigma\) in \(\{+, -\}^\mathbb{N}\) the pressure functions of \(f_\varsigma\) at large values of \(n\). That for each \(\varsigma\) in \(\{+, -\}^\mathbb{N}\) the interval map \(f_\varsigma|_{I(f_\varsigma)}\) is topologically exact follows from the fact that this map is not renormalizable, see [CRL13, §3] for details. Thus, to prove part 1 of the Main Theorem we only need to prove the assertions about equilibrium states.

Let \(N : \mathbb{N}_0 \to \mathbb{N}_0\) and \(B : \mathbb{N}_0 \to \mathbb{N}_0\) be the functions defined in §A.1 for our choices of \(\Xi\) and \(q\). Clearly, \(N(k)/k \to 0\) as \(k \to +\infty\), and for each \(\varsigma\) in \(\{+, -\}^\mathbb{N}\), these functions coincide with those defined in §6.1 with \((\overline{x}_j)_{j \in \mathbb{N}_0} = \overline{x}(\varsigma)\). It follows that for each integer \(k \geq 0\), the 2 variables functions \(\pi_{k}\) defined in §6.1 for our choice of \(\xi\) in §6.2.1 are independent of \(\varsigma\); denote them by \(\pi_k\). Note that the 2 variables series defined in §A.1 \(\Pi\), and for each integer \(s \geq 0\), the series \(I_s\), and \(J_s\), satisfy

\[
\Pi = \sum_{k=0}^{+\infty} \pi_k, I_s = \sum_{k \in I_s} \pi_k, \text{ and } J_s = \sum_{k \in J_s} \pi_k.
\]

For each real number \(s \geq 0\) put \(\lambda(s) := |J_s|^{-1}\), as in §A.2.

Let \(A : \{+, -\}^\mathbb{N} \to (0, +\infty)\) be the continuous functions defined by

\[
A(\varsigma) := \frac{4 \log 2}{\log \theta(f_{\varsigma})},
\]

define \(A_{\text{sup}} := \sup_{\varsigma \in \{+, -\}^n} A(\varsigma)\),

\[
\eta_0 := \sup \left\{ \exp(\chi_{f_\varsigma}(\beta(f_\varsigma)) - \chi_{\text{crit}}(f_\varsigma)) \mid \varsigma \in \{+, -\}^\mathbb{N} \right\},
\]

and let \(t_{k} \geq 2 \log 2/\nu\) be sufficiently large so that

\[
\theta(f_{\varsigma}) \leq \frac{1}{2} A_{\text{sup}},
\]

\[
\left(\inf_{\varsigma \in \{+, -\}^n} \chi_{f_\varsigma}(p^+(f_\varsigma))\right)^{-1}, t_{k} \geq \frac{25}{2} A_{\text{sup}},
\]

\[
2 \left(\frac{1}{\nu_{\text{crit}}^2} \right)^2 t_{k} \geq 2^n \Delta_{f}^\frac{1}{2} C_{6}^\frac{5}{2} \eta_0^\frac{5}{2}, \text{ and } \frac{\log C_5}{t_{k}^{\frac{1}{4}}} \leq \frac{8}{A_{\text{sup}}}. \tag{6.9}
\]
For the rest of this subsection we fix \( \varsigma \) in \( \{+,-\}^N \), and put
\[
\begin{align*}
  f &:= f_{\varsigma},
p^+ &:= p^+(f_{\varsigma}), p^- := p^-(f_{\varsigma}), \\
P^R &= P^R_{f_{\varsigma}}, \mathcal{P}^R = \mathcal{P}^R_{f_{\varsigma}}, P := P_{f_{\varsigma}}, \text{ and } \mathcal{P} := \mathcal{P}_{f_{\varsigma}}.
\end{align*}
\]
Moreover, fix \( t \geq t_k \), and put
\[
\tau := \frac{4}{A(\varsigma)} t = \frac{\log \theta(f)}{\log 2},
\]
\[
P^+ := -i \frac{\chi_{\text{crit}}(f)}{2} + \log \frac{2}{3} \lambda(\tau - 1), \quad \text{and} \quad P^- := -i \frac{\chi_{\text{crit}}(f)}{2} + \log \frac{2}{3} \lambda(\tau).
\]
Note that by (6.2) we have \( \chi_{\text{crit}}(f) = \chi_{f}(p^+) \), and that by (6.3) we have \( \tau \geq 50 \) and \( P^- < P^+ < 0 \). Moreover, by (6.8) we have
\[
2^\tau \xi = \theta(f) \xi \leq \theta(f) t \xi = \Delta t.
\]
Combined with (3.1), Lemma 6.1 with \( \hat{x} = \hat{x}(\varsigma) \) and \( \delta = \frac{\log 2}{3} \lambda(\tau - 1) \), (6.6), (6.7), and part 1 of Lemma A.2, this implies
\[
\sum_{k=0}^{+\infty} \exp \left( -(n + 3k) P^+ \right) |Df^{n+3k}(f(0))|^{-\frac{1}{2}} \leq \Delta^{\frac{1}{2}} \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{n}{2}} \Pi^+ (\tau, \lambda(\tau - 1)).
\]
\[
\leq \left( \Delta^{\frac{1}{2}} \exp(-nu) \right)^{t} (2 + 2^\tau \xi)
\]
\[
\leq \left( \Delta^{\frac{1}{2}} \exp(-nu) (2 + \Delta_{2}) \right)^{t}
\]
\[
\leq C^{-t}.
\]
Together with part 2 of Proposition \( \mathbb{I} \) this implies
\[
P^R(t) \leq P(t) \leq P^+ \quad \text{and} \quad \mathcal{P}^R(t, P^+) \leq \mathcal{P}(t, P^+) < 0.
\]
On the other hand, by (3.1), Lemma 6.1 with \( \delta = \frac{\log 2}{3} \lambda(\tau) \leq \log 2 \), the definition of \( \eta_0 \), (6.9), and part 2 of Lemma A.2 we have
\[
\sum_{k=0}^{+\infty} \exp \left( -(n + 3k) P^- \right) |Df^{n+3k}(f(0))|^{-\frac{1}{2}} \geq \Delta^{-\frac{1}{2}} \exp \left( -n \frac{\log 2}{3} \lambda(\tau) \right) \left( \frac{\exp(\chi_{\text{crit}}(f))}{|Df(\beta(f))|} \right)^{\frac{n}{2}} \Pi^- (\tau, \lambda(\tau)).
\]
\[
\geq 2^{-n} \left( \Delta^{\frac{1}{2}} \eta_0^{\frac{1}{2}} \right)^{-t} 2^\tau \xi
\]
\[
\geq \left( 2^n \Delta^{\frac{1}{2}} \eta_0 \left( \frac{1}{\Delta^{\frac{1}{2}}} \right)^{2} \right)^{t}
\]
\[
\geq C_0^t.
\]
Then part 1 of Proposition \( \mathbb{I} \) implies
\[
P(t) \geq P^R(t) \geq P^- \quad \text{and} \quad \mathcal{P}(t, P^-) \geq \mathcal{P}^R(t, P^-) > 0.
\]
We proceed to prove the existence and uniqueness of equilibrium states. Combining (6.11) and (6.13), we have that the number \( \chi_{\text{inf}}^R(f) \) defined in the statement of Proposition III satisfies
\[
\chi_{\text{inf}}^R(f) = -\lim_{t \to +\infty} \frac{P^R(t)}{t} = \frac{\chi_{\text{crit}}(f)}{2}.
\]
Similarly,
\[
\chi_{\text{inf}}(f) := \inf \left\{ \int \log |Df| \, d\mu \mid \mu \in \mathcal{M}_f \right\} = \frac{\chi_{\text{crit}}(f)}{2}.
\]
Using (6.13) again, we conclude that for every \( t > 0 \) we have
\[
P^R(t) > -t\chi_{\text{inf}}^R(f) \quad \text{and} \quad P(t) > -t\chi_{\text{inf}}(f).
\]
The existence and uniqueness of equilibrium states follows from [PRL14, Theorem A] in the real case. In the complex case it is proved in [PRL11, Main Theorem] for rational maps, and the proof applies without changes to quadratic-like maps. This completes the proof of part 1 of the Main Theorem.

6.2.3. Temperature dependence. In this subsection we complete the proof of the Main Theorem by showing part 2. We give the proof in the complex setting; except for the obvious notational changes, it applies to the real case without modifications. We adopt the notation introduced in the previous subsections.

Fix \( t \geq t_\kappa \), and let \( m_0 \) be the integer in \( \mathbb{N} \) such that \( t \) is in \( (A(\varsigma)(m_0-1), A(\varsigma)m_0] \). Note that \( \tau \geq 50 \), and that the integer \( \tau_0 := \lceil \tau \rceil \) satisfies \( 4m_0 - 3 \leq \tau_0 \leq 4m_0 \). On the other hand, by (6.11) and (6.13) there is \( s_0 \in [\tau - 1, \tau] \) such that
\[
(6.14) \quad P(t) = -t\frac{\chi_{\text{crit}}(f)}{2} + \frac{\log 2}{3} \lambda(s_0^C).
\]
Put \( s_0 := \lceil s_0^C \rceil \) and note that \( s_0 \) is either equal to \( \tau_0 - 1 \) or \( \tau_0 \).

We first prove that the hypotheses of Proposition 5.2 are satisfied for this value of \( t \). By (6.11), Lemma 6.1 with \( \delta = \frac{\log 2}{3} \lambda(\tau) \), and part 1 of Lemma A.3 we have
\[
(6.15) \quad \sum_{k=0}^{+\infty} k \cdot \exp \left( -(n + 3k)P^- \right) |Df^{n+3k}(f(0))|^{-\frac{\delta}{2}} < +\infty.
\]
In particular, this implies that the sum in (6.12) is finite, so by part 1 of Proposition [PRL14] we have \( \mathcal{P}(t, P^-) < +\infty \). This implies that \( \mathcal{P}(t, \cdot) \) is continuous and strictly decreasing on \( [P^-, +\infty) \), so by (6.11), (6.13), and Proposition III we have the second equality in (6.2). Finally, combining (6.13), and part 3 of Proposition III we obtain that the second sum in (6.4), with \( P_f(t) \) replaced by \( P^- \), is finite. In view of (6.13), this implies that the second sum in (6.4) is finite. This completes the proof that the hypotheses of Proposition 5.2 are satisfied.

Let \( \bar{\rho} \), and \( \hat{\rho} \) be the measures given by Proposition 5.2. Moreover, put \( \mathfrak{D} := \mathfrak{D}_f \), \( F := F_f \), and for every integer \( k \geq 0 \) put \( \mathfrak{D}_k := \mathfrak{D}_{f,k} \). For each integer \( s \geq 0 \) let \( a_s \) and \( b_s \) be the left and right endpoint of \( I_s \), respectively as in [A.1]. Thus \( a_0 = 1 \) and for every integer \( s \geq 0 \) we have
\[
I_s = [a_s, b_s] \quad \text{and} \quad J_s = [b_s, a_{s+1}].
\]
Note that by part (b) of Lemma A.1 and the hypothesis \( q \geq 50(\Xi + 1) \), we have \( a_{s+1} - b_s = |J_s| \geq (s + 1)^2 \). For each integer \( \varsigma \in [\tau_0 - 3, s_0] \) put
\[
\hat{\rho}_\varsigma' := \sum_{k=b_\varsigma + \varsigma^2}^{a_{s+1} - 1} \sum_{j=n + 3b_{\varsigma} - 2}^{n + 3(k+1-\varsigma^2)} \sum_{W \in \mathcal{D}_k} (f_j)^s (\hat{\rho}|W),
\]
and put
\[
\hat{\rho}'' := \sum_{k=J_0}^{n + 3(a_{s_0} + 1 - \varsigma^2)} \sum_{j=n + 3b_{s_0} - 1}^{n + 3(k+1-\varsigma^2)} \sum_{W \in \mathcal{D}_k} (f_j)^s (\hat{\rho}|W).
\]
In part 1 we estimate the total mass of the measure
\[
\hat{\rho} := \left( \sum_{\varsigma=\tau_0-3}^{s_0} \hat{\rho}_\varsigma' \right) + \hat{\rho}''.
\]
from below, and in part 2 we show that the total mass of \( \hat{\rho} - \hat{\rho}' \) is small in comparison to that of \( \hat{\rho}' \). In part 3 we complete the proof of part 2 of the Main Theorem by showing that \( \hat{\rho}' \) is supported on a small neighborhood of the orbit of \( p^+ \) or \( p^- \).

The following series, defined in §A, are used in parts 1 and 2 below: \( \Pi^\pm, I_s^+, J_s^+, \) and \( \tilde{J}_s^\pm \). They satisfy
\[
\Pi^+ = \sum_{k=0}^{+\infty} k \cdot \pi_k^+, I_s^+ = \sum_{k \in I_s} k \cdot \pi_k^+, J_s^+ = \sum_{k \in J_s} k \cdot \pi_k^+, \quad \text{and} \quad \tilde{J}_s^\pm = \sum_{k=b_\varsigma + \varsigma^2}^{a_{s+1} - 1} (k+1-b_\varsigma-s^2) \pi_k^\pm.
\]

1. To estimate the total mass of \( \hat{\rho}' \) from below, put \( \Upsilon_1 := \Delta_3 C_9 \Delta_1^\frac{1}{2} \eta_0 2^n \), and for each \( \varsigma \in [\tau_0 - 3, s_0] \), put
\[
H_\varsigma := \{ k \in \mathbb{N}_0 \mid b_\varsigma + \varsigma^2 \leq k \leq a_\varsigma + 1 - 1 \}.
\]
By part 1 of Proposition A.1, Lemma 6.1 with \( \delta = \frac{\log 2}{3} \lambda(s^C) \leq \log 2 \), the definition of \( \eta_0 \), (5.1), (5.3), and (6.14), we have
\[
(6.16) \quad |\hat{\rho}'| \geq \sum_{\varsigma=\tau_0-3}^{s_0} |\hat{\rho}_\varsigma'| \geq \sum_{\varsigma=\tau_0-3}^{s_0} \sum_{k \in H_\varsigma} 3(k+1-b_\varsigma-\varsigma^2) \sum_{W \in \mathcal{D}_k} \tilde{\rho}(W) \geq (\Delta_3 C_9)^{-t} \sum_{\varsigma=\tau_0-3}^{s_0} \sum_{k \in H_\varsigma} 3(k+1-b_\varsigma-\varsigma^2) \exp(-(n+3k)P(t))|Df^{n+3k}(f(0))|^{-t/2} \geq \left( \Delta_3 C_9 \Delta_1^\frac{1}{2} \left( \frac{|Df(\beta(f))|}{\exp(\lambda_{\text{crit}}(f))} \right)^\frac{3}{2} \right)^{-t} \sum_{\varsigma=\tau_0-3}^{s_0} \sum_{k \in H_\varsigma} 3(k+1-b_\varsigma-\varsigma^2) \pi_k^- (\varsigma, \lambda(s^C)) \geq 3 \Upsilon_1^{-t} \sum_{\varsigma=\tau_0-3}^{s_0} \tilde{J}_\varsigma^- (\varsigma, \lambda(s^C)).
\]
2. By \(\text{(6.11)}\), part 2 of Proposition \(\text{(5.6)}\) and \(\text{(6.14)}\), we have

\[
|\hat{\rho} - \hat{\rho}'| = \sum_{k \in \mathbb{N}_0} \sum_{W \in \mathcal{D}_k} m_f(W) \hat{\rho}(W)
\]

\[
+ \sum_{\varsigma = \tau_0 - 3}^{s_0 - 1} \sum_{k \in H_{\varsigma}} \sum_{W \in \mathcal{D}_k} (m_f(W) - 3(k + 2 - b_{\varsigma} - \varsigma^2)) \hat{\rho}(W)
\]

\[
+ \sum_{k \in H_{s_0}} \sum_{W \in \mathcal{D}_k} (m_f(W) - 3(k + 4 - b_{s_0} - 2s_0^2 + |J_{s_0 - 1}|)) \hat{\rho}(W)
\]

\[
\leq (\Delta_3 C_3)^t \left[ \sum_{k \in \mathbb{N}_0} \sum_{\varsigma = \tau_0 - 3}^{s_0 - 1} \sum_{k \in H_{\varsigma}} \sum_{W \in \mathcal{D}_k} (n + 3k + 1) \exp(-(n + 3k)P(t))|Df^{n+3k}(f(0))|^{-\frac{t}{2}}
\]

\[
+ \sum_{\varsigma = \tau_0 - 3}^{s_0 - 1} \sum_{k \in H_{\varsigma}} (n + 3(b_{\varsigma} + \varsigma^2)) \exp(-(n + 3k)P(t))|Df^{n+3k}(f(0))|^{-\frac{t}{2}}
\]

\[
+ \sum_{k \in H_{s_0}} (n + 3(b_{s_0} + 2s_0^2 - |J_{s_0 - 1}|)) \exp(-(n + 3k)P(t))|Df^{n+3k}(f(0))|^{-\frac{t}{2}} \right].
\]

Thus, if we put \(Y_2 := \Delta_3 C_3 \Delta_1^t \exp(-n\upsilon)\), then by \(\text{(3.11)}\), Lemma \(\text{6.1}\) with \(\delta = \frac{\log 2}{3}\lambda(s^C)\), and part 2 of Lemma \(\text{A.3}\),

\[
|\hat{\rho} - \hat{\rho}'| \leq (n + 4)Y_2 \left[ \Pi^+(\tau, \lambda(s^C)) - \sum_{\varsigma = \tau_0 - 3}^{s_0} \hat{J}_+^\varsigma(\tau, \lambda(s^C)) - (|J_{s_0 - 1}| - s_0^2)\hat{J}_{s_0}^+(\tau, \lambda(s^C)) \right]
\]

\[
\leq (n + 4)Y_2 2^{-q\tau^2} \sum_{\varsigma = \tau_0 - 3}^{s_0} \hat{J}_-^\varsigma(\tau, \lambda(s^C)).
\]

Together with \(\text{(6.10)}\) and the definitions of \(\tau\) and \(A_{sup}\), the previous chain of inequalities implies

\[
|\hat{\rho} - \hat{\rho}'| \leq 3(n + 4) (Y_1 Y_2)^t 2^{-q\tau^2} |\hat{\rho}'| \leq 3(n + 4) (Y_1 Y_2)^t 2^{-q(\frac{1}{A_{sup}})^2} |\rho'|.
\]

Thus, if we put

\[
\upsilon_0' := \frac{1}{2^q} \left( \frac{4}{A_{sup}} \right)^2 \log 2, \quad \text{and} \quad C_0' := 3(n + 4) \exp \left( \frac{(\log(Y_1 Y_2)^2)^2}{4\upsilon_0'^2} \right),
\]

then

\[
\left(6.17\right) \quad \frac{|\hat{\rho} - \hat{\rho}'|}{|\hat{\rho}|} \leq \frac{|\hat{\rho} - \hat{\rho}'|}{|\hat{\rho}'|} \leq C_0' \exp(-\upsilon_0'^2 t^2).
\]

3. Using the inequality \(\tau \geq 50\), and the definitions of \(\tau\) and \(A_{sup}\) we have for every \(\varsigma\) in \([\tau_0 - 3, s_0]\),

\[
\varsigma^2 \geq \frac{\tau^2}{2} \geq \frac{8}{A_{sup}^2} t^2.
\]

So, if we put \(\upsilon_0'' := \upsilon_1 \frac{8}{A_{sup}^2} - \log C_3 \frac{1}{\varsigma^2} > 0\), then

\[
\left(6.18\right) \quad C_3 \exp(-\upsilon_1^2) \leq \exp(-\upsilon_0'' t^2).
\]
For $\zeta \in \{+, -\}$, denote by $\mathcal{O}^\zeta$ the forward orbit of $p^\zeta$ under $f$. Let $\zeta$ in $[\tau_0 - 3, s_0]$ be given, and put $m(\zeta) := \lfloor \zeta/4 \rfloor$, so that $4m(\zeta) - 3 \leq \zeta \leq 4m(\zeta)$. For every integer $j$ such that $j + 1$ is in $J_\zeta$ we have $\hat{\omega}(\zeta)_j = 1^{(m(\zeta))}$, so

$$
\hat{\omega}(\zeta)_j = \begin{cases} 
1 & \text{if } \zeta(m(\zeta)) = +; \\
0 & \text{if } \zeta(m(\zeta)) = - \text{ and } j \text{ is even}; \\
1 & \text{if } \zeta(m(\zeta)) = - \text{ and } j \text{ is odd}.
\end{cases}
$$

Since $b_\zeta$ is even, for every $\ell$ in $[0, a_{\zeta+1} - 1 - b_\zeta]$ the points $f^{n+1+3(b_\zeta+\ell-1)}(0)$ and $f^{3\ell}(p^{(m(\zeta))})$ are both in $Y_f$ or both in $\hat{Y}_f$. It follows that

$$
P_{f,3(a_{\zeta+1}+1-b_\zeta)+4}(f^{n+1+3(b_\zeta+1)}(0)) = P_{f,3(a_{\zeta+1}+1-b_\zeta)+4}(p^{(m(\zeta))}).
$$

Then, for each integer $j$ in $[b_\zeta - 1, a_{\zeta+1} - 2]$ we have

$$
P_{f,3(a_{\zeta+1}-1-b_\zeta)+4}(f^{n+1+3j}(0)) = P_{f,3(a_{\zeta+1}-1-b_\zeta)+4}(f^{3j}(p^{(m(\zeta))})),
$$

which implies that for each integer $k$ in $J_\zeta$ and for each integer $j$ in $[b_\zeta - 1, k - 1]$, we have

$$
P_{f,3(k-j)+1}(f^{n+1+3j}(0)) = P_{f,3(k-j)+1}(f^{3j}(p^{(m(\zeta))})).
$$

Note that by definition of $\mathcal{D}_k$, every element $W$ of $\mathcal{D}_k$ is contained in $P_{f,n+3k+2}(0)$, so, if in addition we have $k \geq b_\zeta + \varsigma^2$ and $j \leq k - \varsigma^2$, then by (6.18) and Lemma 4.11 we obtain

$$
f^{n+1+3j}(W) \cup f^{n+1+3j}(W) \cup f^{n+1+3j}(W) \subset B(\mathcal{O}^{(m(\zeta))}, \exp(-v_0''t^2)).
$$

This proves that $\hat{\rho}''$ is supported on $B(\mathcal{O}^{(m(\zeta))}, \exp(-v_0''t^2))$.

On the other hand, for each integer $k$ in $J_{s_0}$, every element $W$ of $\mathcal{D}_k$ is contained in $P_{f,n+3k+2}(0)$, and hence in $P_{f,n+3k+2}(0)$. Thus, by (6.19) with $\zeta = s_0 - 1$, (6.18), and Lemma 4.11 we have that for every integer $j$ in $[b_{s_0-1} - 1, a_{s_0} - s_0]$, we have

$$
f^{n+1+3j}(W) \cup f^{n+1+3j}(W) \cup f^{n+1+3j}(W) \subset B(\mathcal{O}^{(m(s_0-1))}, \exp(-v_0''t^2)),
$$

which proves that $\hat{\rho}''$ is supported on $B(\mathcal{O}^{(m(s_0-1))}, \exp(-v_0''t^2))$.

Assume that there are integers $m$ and $\hat{m}$ as in the statement of the Main Theorem, so that

$$
\hat{m} \geq m \geq 1, \omega(m) = \cdots = \omega(\hat{m}), \text{ and } t \in [A(\omega)m, A(\omega)\hat{m}].
$$

Then $4m \leq \tau_0 \leq 4\hat{m}$, so for every $\zeta$ in $[\tau_0 - 3, s_0]$ we have $\zeta(m(\zeta)) = \zeta(m)$. It follows from the considerations above that the measure $\hat{\rho}''$ is supported on $B(\mathcal{O}^{(m_0)}, \exp(-v_0''t^2))$.

Since the equilibrium state $\rho_0$ of $f|_{\mathcal{H}f}$ for the potential $-t \log |Df|$ is the probability measure proportional to $\hat{\rho}$, by (6.17) we have

$$
\rho_0(\mathcal{C} \setminus B(\mathcal{O}^{(m_0)}, \exp(-v_0''t^2)) \leq \frac{|\hat{\rho} - \hat{\rho}|}{|\hat{\rho}|} \leq C_0' \exp(-v_0''t^2).
$$

Under our assumption $t \geq t_\zeta$, this proves part 2 of the Main Theorem with $v_0 = \min\{v_0', v_0''\}$ and $C_0 = C_0'$. In the case where $t$ is in $(0, t_\zeta)$, it suffices to take the same value of $v_0$ and replace $C_0$ by a constant bounded from below by $\exp(v_0't_\zeta)$, if necessary. The proof of the Main Theorem is thus complete.

Remark 6.2. Without assuming the existence of $m$ and $\hat{m}$ satisfying (6.20), the measure $\hat{\rho}$ is supported on $B(\mathcal{O}^+ \cup \mathcal{O}^-, \exp(-v_0't^2))$, and the estimate above gives that for every $t > 0$ we have

$$
\rho_0(\mathcal{C} \setminus B(\mathcal{O}^+ \cup \mathcal{O}^-, \exp(-v_0't^2)) \leq C_0 \exp(-v_0't^2).
$$
Appendix A. Estimating the 2 variables series

In this appendix we make some of the main estimates in the proof of the Main Theorem, in an abstract setting that is independent of the rest of the paper.

After describing the setting and making some preliminary estimates in §6.1, the main estimates are given in §A.2 and §A.3.

A.1. Setting and preliminary estimates. Given an integer \(\Xi \geq 0\), fix \(q \geq 50(\Xi + 1)\). For each real number \(s\) in \([0, +\infty)\) define:

\[
a_s := 2^{qs^3} \quad \text{and} \quad b_s := 2^{qs^3} + q(2s + 1) + \Xi,
\]

and note that \(b_s < a_{s+1}\). Moreover, define the following intervals of \(\mathbb{R}\):

\[
I_s := [a_s, b_s) \quad \text{and} \quad J_s := [b_s, a_{s+1}).
\]

Note that \(|I_0| = q + \Xi\), and that for integer values of \(s\), the intervals \(I_s\) and \(J_s\) form a partition of \([1, +\infty)\). We use this partition in §6 to define a certain family of itineraries. For \(s\) in \([0, +\infty)\) that is not necessarily an integer, the interval \(J_s\) is used in the proof of Lemmas A.2 and A.3 in §A.2.

Define the function \(N: \mathbb{N}_0 \to \mathbb{N}_0\), by

\[
N(0) := 0 \quad \text{and} \quad N(k) := \# \{ j \in \{0, \ldots, k-1\} \mid j + 1 \in \bigcup_{s \in \mathbb{N}_0} I_s \},
\]

and the function \(B: \mathbb{N}_0 \to \mathbb{N}_0\) by \(B(0) := 0\), and for \(s\) in \(\mathbb{N}_0\) by

\[
B^{-1}(2s + 1) = I_s \quad \text{and} \quad B^{-1}(2(s + 1)) = J_s.
\]

Observe that for every \(s\) in \(\mathbb{N}_0\), we have for every \(k\) in \(\mathbb{N}\) by

\[
N(k) = \sum_{j=0}^{s} |I_j| = \sum_{j=0}^{s} (q(2j + 1) + \Xi) = q(s + 1)^2 + \Xi \cdot (s + 1)
\]

and for every \(k\) in \(I_s\)

\[
N(k) = k - (2^{qs^3} - 1) + qs^2 + \Xi s
\]

Lemma A.1. The following properties hold.

(a) For each real number \(s \geq 0\), we have \(b_s \leq a_{s+1}/2\).

(b) For each real number \(s \geq 0\), we have \(a_{s+1}/2 \leq |J_s|\).

(c) For each real number \(s \geq 1\), we have \(b_s/a_s \leq 5/4\).

Proof. Part (a) with \(s = 0\) follows from our hypothesis \(q \geq 50(\Xi + 1)\). For \(s > 0\), it follows from this and from the fact that the derivative of the function

\[
s \mapsto 2q(s+1)^3 - 1 - (2^{qs^3} + q(2s + 1) + \Xi)
\]

is strictly positive on \([0, +\infty)\). Part (b) follows easily from part (a). For part (c) notice that by our hypothesis \(q \geq 50(\Xi + 1)\) it is enough to prove that for every \(s \geq 1\) we have \(2q(s+1) \leq (1/4) \cdot 2^{qs^3}\). The case \(s = 1\) is given by our hypothesis \(q \geq 50(\Xi + 1)\). For \(s > 1\), it follows from this and from the fact that the derivative of the function

\[
s \mapsto 2^{qs^3} - 8q(s + 1)
\]

is strictly positive on \([1, +\infty)\). \(\square\)
A.2. Estimating the 2 variables series. Let $\xi > 0$ be given, put $\Xi := \lceil 2\xi \rceil + 1$, and let $q$, $N$, and $B$ be as in the previous subsection. For $s$ in $\mathbb{N}_0$ define the following 2 variables series on $[0, +\infty) \times [0, +\infty)$,

$$ I_s^\pm(\tau, \lambda) := \sum_{k \in I_s} 2^{-\lambda k - \tau B(k)}, \quad \text{and} \quad J_s^\pm(\tau, \lambda) := \sum_{k \in J_s} 2^{-\lambda k - \tau B(k)}, $$

and put

$$ \Pi^\pm(\tau, \lambda) := 1 + \sum_{s=0}^{+\infty} I_s^\pm(\tau, \lambda) + \sum_{s=0}^{+\infty} J_s^\pm(\tau, \lambda). $$

Note that by (A.1) and (A.2), for every $\lambda, \tau, \lambda > 0$

(1) $\Pi^+(\tau, \lambda(\lambda - 1)) \leq 2 + 2^\tau \xi$.

(2) $2^\tau E \leq \Pi^-(\tau, \lambda(\lambda)).$

Proof.

1. By (A.1), (A.3), our hypothesis $\tau \geq 2$, and the inequality $\Xi - 2\xi \geq 1$, for every $\lambda \geq 0$ we have

$$ \sum_{s=0}^{+\infty} I_s^+(\tau, \lambda) \leq \sum_{s=0}^{+\infty} \sum_{m=1}^{1 - \lambda(\lambda - 1)} 2^{-\tau(qs^2 + \Xi s + m) + \tau \xi (2s + 1)} \leq 2^\tau \xi \sum_{s=0}^{+\infty} 2^{-(\Xi - 2\xi)1s} \sum_{m=1}^{+\infty} 2^{-\tau m} \leq 2^\tau \xi \frac{2^{-\tau}}{1 - 2^{-\tau}} \sum_{s=0}^{+\infty} 2^{-(\Xi - 2\xi)1s} \leq 2^\tau \xi \frac{2^{-\tau}}{(1 - 2^{-\tau})^2}. $$

(1) $\Pi^+(\tau, \lambda(\lambda - 1)) \leq 2 + 2^\tau \xi$.

(2) $2^\tau E \leq \Pi^-(\tau, \lambda(\lambda)).$

To complete the proof of part 1, note that

$$ \sum_{m=1}^{+\infty} 2^{-\lambda(\lambda - 1)m} = \frac{1}{2^{\lambda(\lambda - 1)} - 1} \leq \frac{1}{\lambda(\lambda - 1) \log 2} \leq 2|J_{\tau - 1}| \leq 2 \cdot a_\tau. $$

Combined with (A.4) and the inequality $\Xi - 2\xi \geq 1$, the previous chain of inequalities implies that for every $j$ in $\mathbb{N}_0$ we have

$$ J_j^+(\tau, \lambda(\lambda - 1)) \leq 2 \cdot 2^q \tau^3 - q\tau^2(j + 1)^2 - (\Xi - 2\xi)\tau(j + 1) \leq 2 \cdot 2^q \tau^3 - q\tau^2(j + 1)^2 - \tau(j + 1). $$

We obtain for every integer $j \geq |\tau| \geq \tau - 1$

$$ J_j^+(\tau, \lambda(\lambda - 1)) \leq 2 \cdot 2^q \tau^3 - q\tau^2(j + 1)^2 - \tau(j + 1) \leq 2 \cdot 2^\tau(j + 1). $$
To estimate $J_j^+(τ, λ(τ − 1))$ for $j$ in $\{0, \ldots, |τ| − 1\}$, note that

$$\sum_{m=1}^{|J_j|} 2^{−λ(τ−1)m} \leq |J_j| \leq a_{j+1}.$$  

Combined with (A.4) and the inequality $Ξ − 2ξ ≥ 1$, this implies that for every integer $j$ in $\{0, \ldots, |τ| − 1\}$ we have

$$J_j^+(τ, λ(τ − 1)) \leq 2^{q(τ+1)−q τ(τ+1)−2(Ξ+2ξ)τ−λ(τ+1)} \leq 2^{−τ(τ+1)}.$$  

Thus,

$$\sum_{j=0}^{∞} J_j^+(τ, λ(τ − 1)) \leq 2^{−τ(τ+1)} = 2^{2−τ−τ} ≤ 2.$$  

Together with (A.5) this implies the desired inequality.

2. Put $τ_0 := [τ]$. By part (b) of Lemma (A.1) and the definition of $λ(τ)$, we have

$$λ(τ) = |J_τ|−1 ≤ \frac{2}{a_{τ+1}} ≤ \frac{2}{a_τ}.$$  

From this inequality, part (c) of Lemma (A.1) and our hypothesis $τ ≥ 2$, we obtain

$$λ(τ)(b_{τ_0} − 1) ≤ \frac{b_{τ_0}}{a_τ} ≤ 3.$$  

On the other hand, note that for every $m$ in $\{1, \ldots, |J_τ|\}$ we have $λ(τ)m ≤ 1$, so by part (b) of Lemma (A.1) we have

$$\sum_{m=1}^{|J_τ|} 2^{−λ(τ)m} ≥ \frac{1}{2}(|J_τ| − 1) ≥ \frac{1}{2^2}|J_τ| ≥ \frac{1}{2^3} 2^{q(τ+1)^3}.$$  

Suppose $τ ≥ τ_0 − 1/3$. In view of (A.4) and (A.8), the previous chain of inequalities implies

$$\frac{1}{2^6} 2^{q(τ+1)^3−qτ(τ_0+1)^3−(Ξ+2ξ)τ−(τ_0+1)} \leq \frac{1}{2^4} \left( \sum_{m=1}^{|J_τ|} 2^{−λ(τ)m} \right) 2^{−qτ(τ_0+1)^2−(Ξ+2ξ)τ−(τ_0+1)} \leq \frac{1}{2^4} \left( \sum_{m=1}^{|J_τ|} 2^{−λ(τ)m} \right) 2^{−λ(τ)(b_{τ_0} − 1)−qτ(τ_0+1)^2−(Ξ+2ξ)τ−(τ_0+1)} = J_τ^−(τ, λ(τ)).$$  

On the other hand, by our assumption $τ ≥ τ_0 − 1/3$ we have

$$(τ+1)^3−τ(τ_0+1)^2 ≥ (τ+1)^3−τ \left( τ + \frac{4}{3} \right)^2 = \frac{τ^2}{3} + \frac{11τ}{9} + 1 ≥ \frac{τ^2}{4} + \frac{τ(τ_0+1)}{12}.$$  

Combined with our hypotheses $q ≥ 50(Ξ + 1) ≥ 25(Ξ + 2ξ + 3)$ and $τ ≥ 2$, and with (A.9), this implies part 2 of the lemma when $τ ≥ τ_0 − 1/3$. 
To complete the proof, suppose \( \tau \leq \tau_0 - 1/3 \). Similarly as above we have
\[
\frac{1}{2^6} 2^{q_0^3 - q \tau \tau_0^2 - (\Xi + 2 \xi) \tau \tau_0}
\leq \frac{1}{2^4} \left( \sum_{m=1}^{J_{\tau_0-1}} 2^{-\lambda(\tau_0-1)m} \right) 2^{-q \tau \tau_0^2 - (\Xi + 2 \xi) \tau \tau_0}
\leq \frac{1}{2^4} \left( \sum_{m=1}^{\lambda(\tau_0-1)} 2^{-\lambda(\tau_0-1)m} \right) 2^{-q \tau \tau_0^2 - (\Xi + 2 \xi) \tau \tau_0}
\leq \left( \sum_{m=1}^{\lambda(\tau_0-1)} 2^{-\lambda(\tau_0-1)m} \right) 2^{-q \tau \tau_0^2 - (\Xi + 2 \xi) \tau \tau_0}
= J_{\tau_0-1}(\tau, \lambda(\tau)).
\]
On the other hand, our assumption \( \tau \leq \tau_0 - 1/3 \) implies
\[
\tau_0^3 - \tau_0^2 \geq \frac{\tau_0^2}{3} \geq \frac{\tau^2}{4} + \frac{\tau \tau_0}{12}.
\]
Combined with our hypotheses \( q \geq 50(\Xi + 1) \geq 25(\Xi + 2 \xi + 3) \) and \( \tau \geq 2 \), and with (A.10), we obtain part 2 of the lemma when \( \tau \leq \tau_0 - 1/3 \). The proof of the lemma is thus complete.

**A.3. Estimating the weighted 2 variables series.** For each \( s \) in \( \mathbb{N}_0 \), \( \tau > 0 \), and \( \lambda \geq 0 \) put
\[
\tilde{I}_s^+(\tau, \lambda) := \sum_{k \in I_s} k \cdot 2^{-\lambda k - \tau N(k) + \tau \xi B(k)},
\]
\[
\tilde{J}_s^+(\tau, \lambda) := \sum_{k \in J_s} k \cdot 2^{-\lambda k - \tau N(k) + \tau \xi B(k)},
\]
and
\[
\tilde{\Pi}^+(\tau, \lambda) := 1 + \sum_{s=0}^{+\infty} \tilde{I}_s^+(\tau, \lambda) + \sum_{s=0}^{+\infty} \tilde{J}_s^+(\tau, \lambda).
\]
Noting that by part (b) of Lemma A.1 we have \( a_{s+1} - b_s = |J_s| \geq s^2 + 1 \), define for each \( \tau > 0 \) and \( \lambda \geq 0 \),
\[
\tilde{J}_s^+(\tau, \lambda) := \sum_{k=b_s + s^2}^{a_{s+1}-1} (k + 1 - b_s - s^2) \cdot 2^{-\lambda k - \tau N(k) + \tau \xi B(k)}.
\]

**Lemma A.3.** For each \( \tau \geq 50 \), the following properties hold:

1. \( \tilde{\Pi}^+(\tau, \lambda(\tau)) < +\infty \).
2. Let \( s \) in \( [\tau - 1, \tau] \) be given, put \( s_0 := \lfloor s \rfloor \) and \( \tau_0 := \lceil \tau \rceil \), and note that \( s_0 \) is equal to either \( \tau_0 - 1 \) or \( \tau_0 \). Then
\[
\Pi^+(\tau, \lambda(s)) \leq \tilde{\Pi}^+(\tau, \lambda(s)) - \sum_{\varsigma = \tau_0 - 3}^{s_0} \tilde{J}_\varsigma^+(\tau, \lambda(s)) - (|J_{s_0-1}| - s_0^2) J_{s_0}^+(\tau, \lambda(s)) \leq 2^{-q \tau^2} \sum_{\varsigma = \tau_0 - 3}^{s_0} \tilde{J}_\varsigma^+(\tau, \lambda(s)).
\]
The proof of this lemma is given after the following one.
Sublemma A.4. Given $\tau \geq 50$ and $s$ in $[\tau - 1, \tau]$, put $\tau_0 := \lceil \tau \rceil$ and $s_0 := \lceil s \rceil$. Then the following properties hold.

1. $\tilde{J}_{s_0}^-(\tau, \lambda(s)) \geq 2^{2q(s+1)^3 - q\tau(s_0+1)(s_0+2)}$.
2. $\tilde{J}_{s_0}^-(\tau, \lambda(s)) \geq 2^{q\tau^2(\tau-4)}$.
3. For every integer $\varsigma$ in $[\tau_0 - 3, s_0 - 1]$ we have

   $$(b_\varsigma + \varsigma^2)J_\varsigma^+(\tau, \lambda(s)) \leq \frac{1}{20}2^{-q\tau^2} \cdot \tilde{J}_\varsigma^-(\tau, \lambda(s)).$$

4. $$(b_{s_0} - |J_{s_0-1}| + 2s_0^2)J_{s_0}^+(\tau, \lambda(s)) \leq \frac{1}{2}2^{-q\tau^2} \cdot \tilde{J}_{s_0}^-(\tau, \lambda(s)).$$

Proof.

1. By part (b) of Lemma A.1 and the definition of $\lambda(s)$, we have

   $$\lambda(s) = \frac{1}{|J_s|} \leq \frac{2}{a_{s+1}} \leq \frac{2}{a_{s_0}}.$$  

On the other hand,

$$\lambda(s_0) s_0^2 = \frac{s_0^2}{|J_{s_0}|} \leq \frac{s_0^2}{2q s_0^2 - 1} \leq \frac{1}{q s_0} \leq \frac{1}{100}.$$  

From these 2 inequalities and part (c) of Lemma A.1 we obtain

$$\lambda(s)(b_{s_0} + s_0^2) \leq \frac{2b_{s_0}}{a_{s_0}} + \frac{1}{100} \leq 3.$$  

By (A.1), (A.2), and (A.12), we have

$$\tilde{J}_{s_0}^-(\tau, \lambda(s)) \geq \frac{1}{2^5}2^{-q\tau(s_0+1)^2 - (\Xi + 2\xi)(\tau(s_0+1))} \sum_{m=1}^{N} m \cdot 2^{-\lambda(s)m}.$$  

Noticing that for every integer $N \geq 1$ we have

$$\sum_{m=1}^{N} m \cdot 2^{-\lambda(s)m} = \frac{2^{\lambda(s)}}{(2^{\lambda(s)} - 1)^2} \left(1 - (N + 1)2^{-\lambda(s)N} + N2^{-\lambda(s)(N+1)}\right),$$

and that the function

$$\eta \mapsto 1 - (N + 1)\eta^N + N\eta^{N+1}$$
Lemma A.1, the previous chain of inequalities implies

\[ 2 \geq (|J_{s_0}| - s_0^2 + 1)2^{-\lambda(s_0)(|J_{s_0}| - s_0^2)} + (|J_{s_0}| - s_0^2)2^{-\lambda(s_0)(|J_{s_0}| - s_0^2 + 1)} \]

Together with (A.13), the inequality is decreasing on \([0, 1]\), we have by (A.11) and the inequality \(1 - 2^{-\lambda(s_0)} \leq \lambda(s_0) \log 2\)

\[ (A.14) \sum_{m=1}^{\lfloor |J_{s_0}| - s_0^2 \rfloor} m \cdot 2^{-\lambda(s)m} \geq \frac{2\lambda(s)}{(2\lambda(s) - 1)^2} \cdot \left(1 - (|J_{s_0}| - s_0^2 + 1)2^{-\lambda(s_0)(|J_{s_0}| - s_0^2)} + (|J_{s_0}| - s_0^2)2^{-\lambda(s_0)(|J_{s_0}| - s_0^2 + 1)}\right) \]

\[ = \frac{2\lambda(s)}{(2\lambda(s) - 1)^2} \cdot \left(1 - 2\lambda(s_0)s_0^2 - 1 - 2\lambda(s_0)s_0^2 - 1 (|J_{s_0}| - s_0^2) \left(1 - 2^{-\lambda(s_0)}\right)\right) \]

\[ \geq \frac{2\lambda(s)}{(2\lambda(s) - 1)^2} \left(1 - 2\lambda(s_0)s_0^2 - 1 (1 + \log 2)\right) \]

\[ \geq \frac{1}{2^4 (2\lambda(s) - 1)^2}. \]

Note that by \(\lambda(s) \leq 1\), we have \(2\lambda(s) - 1 \leq \lambda(s)\). Thus, together with part (b) of Lemma [A.3] the previous chain of inequalities implies

\[ \sum_{m=1}^{\lfloor |J_{s_0}| - s_0^2 \rfloor} m \cdot 2^{-\lambda(s)m} \geq \frac{1}{24} \cdot |J_s|^2 \geq \frac{1}{26} \cdot 2^{2q(s+1)^3}. \]

Together with (A.13), the inequality \(\Xi \geq 2\xi\), and our hypotheses \(q \geq 50(\Xi + 1)\) and \(\tau \geq 50\), this implies

\[ \tilde{J}_{s_0}(\tau, \lambda(s)) \geq \frac{1}{24} 2^{2q(s+1)^3 - q \tau (s_0 + 1)^2 - (\Xi + 2\xi)\tau (s_0 + 1)} \geq 2^{2q(s+1)^3 - q \tau (s_0 + 1)(s_0 + 2)}. \]

This proves part 1.

2. When \(s_0 \leq \tau\) we have by our hypothesis \(\tau \geq 50\),

\[ 2(s + 1)^3 - \tau \cdot (s_0 + 1)(s_0 + 2) \geq 2\tau^3 - \tau (\tau + 1)(\tau + 2) \geq \tau^2 (\tau - 4). \]

On the other hand, in the case where \(s_0 \geq \tau\) we have by our hypothesis \(\tau \geq 50\),

\[ 2(s + 1)^3 - \tau \cdot (s_0 + 1)(s_0 + 2) \geq 2\tau^2 s_0 - \tau \cdot (s_0 + 1)(s_0 + 2) \]

\[ = \tau s_0(s_0 - 3) - 2\tau \geq \tau^2 (\tau - 3) - 2\tau \geq \tau^2 (\tau - 4). \]

In all the cases, part 2 follows from part 1.

3. Let \(\zeta\) be an integer in \([\tau_0 - 3, s_0]\) and note that by (A.1), (A.4), and the definition of \(\tilde{J}_{\zeta}\), we have

\[ (A.15) \frac{J^+(\tau, \lambda(s))}{J^- (\tau, \lambda(s))} = 2^{4\tau \xi (\zeta + 1) + \lambda(s) \xi^2} \sum_{m=1}^{\lfloor |J_{s_0}| \rfloor} 2^{-\lambda(s)m} m 2^{-\lambda(s)m}. \]

Suppose \(\zeta\) is in \([\tau_0 - 3, s_0 - 1]\). Then \(\lambda(s)|J_{s_0}| \leq 1\), so

\[ \frac{J^+(\tau, \lambda(s))}{J^- (\tau, \lambda(s))} \leq 2 \cdot 2^{4\tau \xi (\zeta + 1) + \lambda(s) \xi^2} \sum_{m=1}^{\lfloor |J_{s_0}| \rfloor} 2^{-\lambda(s)m} m 2^{-\lambda(s)m} \]

\[ \leq 2^2 \cdot 2^{4\tau \xi (\zeta + 1) + \lambda(s) \xi^2} \frac{|J_{s_0}|}{(|J_{s_0}| - \xi^2)^2}. \]
Noting that by parts (a) and (b) of Lemma A.1 we have
\[
\lambda(s)\xi^2 \leq \xi^2/|J_s| \leq 1 \quad \text{and} \quad |J_s| \leq 2(|J_s| - \xi^2),
\] by part (c) of Lemma A.1, the inequality \(\Xi \geq 2\xi\), and our hypotheses \(q \geq 50(\Xi + 1)\) and \(\tau \geq 50\) we obtain
\[
(\beta + \xi^2)\frac{J^+(\tau, \lambda(s))}{J^-(\tau, \lambda(s))} \leq 2^{6q}2^{4\tau \xi(\xi+1)}|J_s|^{-1}
\leq 2^{7+4\tau \xi(\xi+1) - q(\xi+1)^3} \leq \frac{1}{20}2^{-q\xi^2}.
\]
This proves part 3.

4. By our hypotheses \(q \geq 50(\Xi + 1)\) and \(\tau \geq 50\) we have
\[
b_{s_0} - |J_{s_0-1}| + 2s_0^2 = 2^{q(s_0-1)^3} + 2s_0^2 + 4qs_0 + 2\Xi \leq 2^{q(s_0-1)^3} + qs_0^2 \leq 2 \cdot 2^{q(s_0-1)^3}.
\]
Thus, by part (b) of Lemma A.1 (A.14), (A.15), and the inequality \(\lambda(s) \leq 1\), we have
\[
(b_{s_0} - |J_{s_0-1}| + 2s_0^2)\frac{J^+(\tau, \lambda(s))}{J^-(\tau, \lambda(s))} \leq 2^{5+2^{q(s_0-1)^3+4\tau \xi(s_0+1)+\lambda(s)s_0^2}(2\lambda(s) - 1)}
\leq 2^{5}\lambda(s) \cdot 2^{q(s_0-1)^3+4\tau \xi(\xi+1)+\lambda(s)4s_0^2}
\leq 2^{5} \cdot 2^{-q(s+1)^3+q(s_0-1)^3+4\tau \xi(s_0+1)+\lambda(s)4s_0^2}
\]
Using \(\lambda(s)s_0^2 \leq s_0^2|J_s|^{-1} \leq 1\), the inequality \(\Xi \geq 2\xi\), and our hypotheses \(q \geq 50(\Xi + 1)\) and \(\tau \geq 50\), we have
\[
(b_{s_0} - |J_{s_0-1}| + 2s_0^2)\frac{J^+(\tau, \lambda(s))}{J^-(\tau, \lambda(s))} \leq 2^{-q(s_0-1)^3+q(s_0-1)^3+q\xi(s_0+1)}
\leq 2^{-3q(s_0-1)+q\xi(s_0+1)} \leq \frac{1}{4}2^{-q\xi^2}.
\]
This completes the proof of part 4 and of the lemma.

**Proof of Lemma A.3.** 1. Note that for every \(s \geq 0\), we have \(\lambda(s) \leq 1\) and
\[
(A.16) \quad \sum_{m=1}^{+\infty} m \cdot 2^{-\lambda(s)m} = \frac{2^{\lambda(s)}}{(2^{\lambda(s)} - 1)^2} \leq \frac{2^{\lambda(s)}}{(\lambda(s) \log 2)^2} \leq 2^{3d} |J_s|^2 \leq (2^3) 2^{2q(s+1)^3}.
\]
Together with (A.1), (A.2), (A.3), and the inequality \(\Xi - 2\xi \geq 1\), for every \(j \in \mathbb{N}_0\) we have
\[
(A.17) \quad \vec{J}^+_j(\tau, \lambda(s)) + \vec{I}^+_j(\tau, \lambda(s)) \leq 2^{-\tau(q(j+1)^2+\Xi(j+1)+\tau\xi(2j+3)} \sum_{k \in J_j \cup I_j} k \cdot 2^{-\lambda(s)^k}
\leq (2^{\tau\xi+3})2^{2q(s+1)^3+q\xi(2j+3) - \tau(j+1)}.
\]
Taking \( s = \tau \), for every \( j \geq 2\tau + 1 \) we have
\[
\tilde{J}_j^+(\tau, \lambda(\tau)) + \tilde{I}_{j+1}^+(\tau, \lambda(\tau)) \leq (2^{\tau\xi+3}) 2^{-\tau(j+1)}.
\]
This implies that \( \tilde{\Pi}^+(\tau, \lambda(\tau)) \) is finite, as wanted.

2. The first inequality follows directly from the definitions. To prove the second inequality, note that by (A.1) and (A.3), and our hypotheses \( \tau \geq 50 \) and \( \xi > 0 \), we have
\[
(A.18) \quad 1 + \tilde{I}_0^+(\tau, \lambda(s)) \leq 1 + \sum_{k=1}^{+\infty} k \cdot 2^{-\tau k + \tau \xi} = 1 + 2^{\tau\xi} \frac{2^{-\tau}}{(1 - 2^{-\tau})^2} \leq 2^{\tau\xi}.
\]
On the other hand, by part (c) of Lemma A.1, (A.1), (A.3), and the inequality \( \Xi - 2\xi \geq 1 \), for every integer \( j \geq 1 \) we have
\[
\tilde{I}_j^+(\tau, \lambda(s)) \leq (2^{\tau\xi}) 2^{-\tau(q^2 + (\Xi - 2\xi)j)} \sum_{m=1}^{I_j} \left( 2^{qj^3 + m} \right) 2^{-\tau m}
\leq (2^{\tau\xi+1}) 2^{qj^3 - q\tau j^2 - (\Xi - 2\xi)\tau j} \frac{1}{1 - 2^{-\tau}}
\leq (2^{\tau\xi+2}) 2^{q(j-\tau)j^2 - \tau j}.
\]
Combined with (A.18), the inequality \( \Xi \geq 2\xi \), and our hypotheses \( q \geq 50(\Xi + 1) \) and \( \tau \geq 50 \), this implies
\[
1 + \sum_{j=0}^{\tau_0+1} \tilde{I}_j^+(\tau, \lambda(s)) \leq (2^{\tau\xi+2}) \frac{2^{q(\tau+2)^2}}{1 - 2^{-\tau}} \leq 2^{2q\tau(\tau+6)}.
\]
Together with part 2 of Sublemma A.4 and our hypothesis \( \tau \geq 50 \), this chain of inequalities implies
\[
(A.19) \quad 1 + \sum_{j=0}^{\tau_0+1} \tilde{I}_j^+(\tau, \lambda(s)) \leq \frac{1}{2q^2} 2^{-q\tau^2} \cdot \tilde{J}_{\tau_0}^+(\tau, \lambda(s)).
\]
On the other hand, by (A.1), (A.2), and our hypothesis \( \tau \geq 50 \), for every \( j \) in \( \{0, \ldots, \tau_0 - 4\} \) we have
\[
\tilde{J}_j^+(\tau, \lambda(s)) = 2^{-\tau(q(j+1)^2 + \Xi(j+1)) + 2\tau\xi(j+1)} \sum_{k \in J_j} k \cdot 2^{-\lambda(s)k}
\leq |J_j| 2^{q(j+1)^3 - q\tau(j+1)^2 - (\Xi - 2\xi)\tau(j+1)}
\leq 2^{2q(j+1)^3 - q\tau(j+1)^2 - (\Xi - 2\xi)\tau(j+1)}
\leq 2^{q(j+1)^2(2j+2-\tau)}
\leq 2^{q(\tau-2)^2(\tau-4)}
\leq 2^{q\tau^2(\tau-7)}.
\]
Together with part 2 of Sublemma A.4 and with our hypothesis \( \tau \geq 50 \), this implies
\[
(A.20) \quad \frac{\sum_{j=0}^{\tau_0-4} \tilde{J}_j^+(\tau, \lambda(s))}{J_{\tau_0}^+(\tau, \lambda(s))} \leq \tau 2^{-3q\tau^2} \leq \frac{1}{2q^2} 2^{-q\tau^2}.
\]
On the other hand, by (A.17), part 1 of Sublemma A.4 the inequality \( \Xi \geq 2\xi \) and our hypothesis \( q \geq 50(\Xi + 1) \), for every integer \( j \geq s_0 + 1 \) we have
\[
\frac{\tilde{J}_j^+ (\tau, \lambda(s)) + \tilde{I}_{j+1}^+ (\tau, \lambda(s))}{\tilde{J}_{s_0}^+ (\tau, \lambda(s))} \leq (2^{\xi+3}) \frac{2^{-q^2 \tau \cdot (j+1)^2 - (s_0 + 1)(s_0 + 2)}}{2^{-q^2 \tau \cdot (j-s_0)}} \leq (2^{\xi+3}) \frac{2^{-q^2 \tau \cdot (s_0 + 2)(j-s_0)}}{2^{-q^2 \tau \cdot (j-s_0)}} \leq (2^{\xi+3}) \frac{2^{-q^2 \tau \cdot (j-s_0)}}{1}.
\]
Summing over \( j \geq s_0 + 1 \) and using our hypotheses \( q \geq 50(\Xi + 1) \) and \( \tau \geq 50 \), we obtain
\[
\sum_{j=s_0+1}^{\infty} \left( \frac{\tilde{J}_j^+ (\tau, \lambda(s)) + \tilde{I}_{j+1}^+ (\tau, \lambda(s))}{\tilde{J}_{s_0}^+ (\tau, \lambda(s))} \right) \leq \frac{1}{2^{\xi+3}} \frac{2^{-q^2 \tau \cdot 2}}{1 - 2^{-q^2 \tau \cdot 1}} \leq \frac{1}{2^{\xi+3}} \frac{2^{-q^2 \tau \cdot 2}}{2^{-q^2 \tau \cdot 1}}.
\]
Combined with (A.19) and (A.20), this implies
\[
\tilde{\Pi}^+ (\tau, \lambda(s)) - \sum_{\varsigma=\tau_0-3}^{s_0} \tilde{J}_\varsigma^+ (\tau, \lambda(s)) \leq \frac{1}{2} 2^{-q^2 \tau \cdot 2} \cdot \tilde{J}_{s_0}^- (\tau, \lambda(s)).
\]
For each integer \( \varsigma \) in \( [\tau_0 - 3, s_0] \), we have
\[
\tilde{J}_\varsigma^+ (\tau, \lambda(s)) - \tilde{J}_\varsigma^- (\tau, \lambda(s)) \leq \sum_{k=b_\varsigma}^{b_\varsigma + 2^{-1}} k \cdot 2^{-\lambda_k - \tau \cdot N(k) + \tau \cdot B(k)} + \sum_{k=b_\varsigma}^{a_{\varsigma+1}-1} (b_\varsigma + \varsigma^2) 2^{-\lambda_k - \tau \cdot N(k) + \tau \cdot B(k)} \leq (b_\varsigma + \varsigma^2) \tilde{J}_\varsigma^+ (\tau, \lambda(s)).
\]
Together with part 3 of Sublemma A.4 this implies that for \( \varsigma \) in \( [\tau_0 - 3, s_0 - 1] \) we have
\[
\tilde{J}_\varsigma^+ (\tau, \lambda(s)) - \tilde{J}_\varsigma^- (\tau, \lambda(s)) \leq \frac{1}{20} 2^{-q^2 \tau \cdot 2} \cdot \tilde{J}_{\varsigma_0}^- (\tau, \lambda(s)).
\]
On the other hand, (A.22) with \( \varsigma = s_0 \) and part 4 of Sublemma A.4 imply
\[
\tilde{J}_s^+ (\tau, \lambda(s)) - \tilde{J}_s^- (\tau, \lambda(s)) - (|\tilde{J}_{s_0-1}^-| - s_0^2) J_{s_0}^+ (\tau, \lambda(s)) \leq \frac{1}{4} 2^{-q^2 \tau \cdot 2} \cdot \tilde{J}_{s_0}^- (\tau, \lambda(s)).
\]
Together with (A.21) and (A.23), this implies the desired inequality and completes the proof of the lemma. \( \square \)

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