GEOMETRY OF THE AHARONOV-BOHM EFFECT

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We show that the connection responsible for any abelian or non abelian Aharonov-Bohm effect with \( n \) parallel “magnetic” flux lines in \( \mathbb{R}^3 \), lies in a trivial \( G \)-principal bundle \( P \to M \), i.e. \( P \) is isomorphic to the product \( M \times G \), where \( G \) is any path connected topological group; in particular a connected Lie group. We also show that two other bundles are involved: the universal covering space \( \tilde{M} \to M \), where path integrals are computed, and the associated bundle \( P \times_G \mathbb{C}^n \to M \), where the wave function and its covariant derivative are sections.

Key words: Aharonov-Bohm effect; fibre bundle theory; gauge invariance.

As is well known, the magnetic Aharonov-Bohm (A-B) effect\textsuperscript{1,2} is a gauge invariant, non local quantum phenomenon, with gauge group \( U(1) \), which takes place in a non simply connected space. It involves a magnetic field in a region where an electrically charged particle obeying the Schroedinger equation cannot enter, i.e. the ordinary 3-dimensional space minus the space occupied by the solenoid producing the field; in the ideal mathematical limit, the solenoid is replaced by a flux line. Locally, the particle couples to the magnetic potential \( \vec{A} \) but not to the magnetic field \( \vec{B} \); however, the effect is gauge invariant since it only depends on the flux of \( \vec{B} \) inside the solenoid.

The fibre bundle theoretic description of this kind of phenomena has proved to be very useful to obtain a more profound insight into the relation between physical processes and pure mathematics.\textsuperscript{3} In the present case, since by symmetry, the dimension along the flux line can be ignored, the problem reduces to the effect on the charged particle of an abelian connection in a \( U(1) \)-bundle with base space the plane minus a point. The wave function representing the particle is a section of an associated vector bundle. As shown in ref. 4, the bundle turns out to be trivial and then its total space is isomorphic to the product \( (\mathbb{R}^2 - \{ \text{point} \}) \times U(1) \). Since \( \mathbb{R}^2 \) is topologically equivalent to an open disk, and \( U(1) \) is the unit circle, the bundle structure is summarized by

\begin{equation}
U(1) \to T^2_{\ast} \to D^2_{\ast}
\end{equation}

where \( D^2_{\ast} \) is the open disk minus a point and \( T^2_{\ast} \) is the open solid 2-torus minus a circle.

As suggested by Wu and Yang\textsuperscript{6}, Yang-Mills fields can give rise to non abelian A-B effects. In ref. 6 the authors studied an \( SU(2) \) gauge configuration leading to an A-B effect; later, several authors studied the effect with gauge groups \( SU(3) \)\textsuperscript{7} and \( U(N) \)\textsuperscript{8}. Also, in refs. 9-14 the effect was studied in the context of gravitation theory. In all these examples, as in the magnetic case, there is a principal bundle structure

\begin{equation}
\xi : G \to P \to \pi_G \to M
\end{equation}

whose total space \( P \), where the connection giving rise to the effect lies, is however never specified. We shall restrict ourselves to connections \( \omega \) giving rise to \( n \) \(( n = 1, 2, 3, ... \)\) flux lines in \( \mathbb{R}^3 \). In this note we prove the following:

**Theorem:** Let \( G \) be a path connected topological group (for example a connected Lie group) and \( \xi \) a continuous principal \( G \)-bundle over \( \mathbb{R}^3 - \{ n \text{ parallel lines} \} \), \( n = 1, 2, 3, ... \) Then the bundle \( \xi \) is trivial, i.e. isomorphic to the product bundle.

**Proof of the theorem**
The classification of bundles over $\mathbb{R}^3 \setminus \{n \text{ parallel lines} \}$ is the same as that over $\mathbb{R}^2 \setminus \{n \text{ points} \}$ which is topologically equivalent to $D^2_\circ \setminus \{n \text{ points } \} \equiv D^2_\circ \setminus n$. Denote this set of points by $\{b_1, ..., b_n\}$. By symmetry along the dimension of the flux tubes, $D^2_\circ n$ is the space where it can be considered that the charged particles move.

Let $x_0$ be a point in $D^2_\circ n$. We construct a bouquet of $n$ loops $\gamma_1, \gamma_2, ..., \gamma_n$ through $x_0$, with the $k$-th loop surrounding the point $b_k$, $k = 1, 2, ..., n$. This space is homeomorphic to the wedge product (or reduced join) $S^1 \vee ... \vee S^1 \equiv \vee_n S^1 \equiv S^1(\{1\}) \vee ... \vee S^1(\{n\})$ of $n$ circles $15$, and the classification of bundles over $D^2_\circ n$ is the same as that over $\vee_n S^1$, namely

$$B_{D^2_\circ n}(G) = B_{\vee_n S^1}(G)$$

(3)

where $B_M(G)$ is the set of isomorphism classes of $G$-bundles over $M$ $16$.

By explicit construction we shall prove that, up to isomorphism, the unique $G$-bundle over $\vee_n S^1$ is the product bundle $\vee_n S^1 \times G$ (which is a purely topological result). With this aim, we cover the circle $S^1(\{k\})$ with two open sets $U_{k+}$ and $U_{k-}$ such that

$$U_{k+} \cap U_{k-} \simeq \{x_0, a_k\}, \quad k = 1, ..., n,$$

(4)

$$U_{i+} \cap U_{j+} \simeq U_{i-} \cap U_{j-} \simeq U_{i+} \cap U_{j-} \simeq \{x_0\}, \quad i, j = 1, ..., n, \quad i \neq j$$

(5)

where $\simeq$ denotes homotopy equivalence, and $a_k \in S^1(\{k\})$ with $a_k \neq x_0$. We then have

$$2 \times \left( \frac{2n}{2} \right) + 2n = \frac{2 \times (2n)!}{(2n-2)!2!} + 2n = 4n^2$$

(6)

transition functions

$$g_{\alpha, \beta} : U_{\alpha} \cap U_{\beta} \rightarrow G$$

(7)

with $g_{\beta, \alpha} = g_{\alpha, \beta}^{-1}$. Up to homotopy they are given by

$$g_{k+, k-} : \{x_0, a_k\} \rightarrow G, \quad x_0 \mapsto g_{0k}, \quad a_k \mapsto g_k, \quad k = 1, ..., n$$

(8)

and

$$g_{i+, j+} \circ g_{i-, j-} : \{x_0\} \rightarrow G, \quad i, j = 1, ..., n, \quad i \neq j,$$

$$x_0 \mapsto g_{ij++}, \quad x_0 \mapsto g_{ij-}, \quad x_0 \mapsto g_{ij+\cdot}$$

(9)

The $2n$ transition functions $g_{i+, i+}, g_{i-, i-}, i \in \{1, ..., n\}$, give the identity in $G$. The number of cocycle relations $g_{\beta, \alpha}g_{\alpha, \gamma} = g_{\beta, \gamma}$ on the $2n(2n-1)$ non trivial transition functions $g_{\mu, \nu}$ is

$$\left( \frac{2n}{3} \right) = \frac{n(2n-1)(2n-2)}{3}.$$
Extending continuously (as constants) $\Lambda$ and $G$ there exists a unique $U$ for $x$, therefore $\bar{H}$ for $i,j$ for $k\in \mathbb{N}_{+}$, then the map $H_i\times [0,1] \times U \rightarrow G$, $H_i(x_0,t) = c_{ij}(t)$ between $g_i$ with $c$ then the map by $g_i$ then the homotopy class of maps from $\{x_0, a_k\}$ to $G$ has only one element, namely $[g_{k,+,-}]$ i.e.

$$\{x_0, a_k\}, G_{\sim} = \{[g_{k,+,-}]_{\sim}\}. \quad (11)$$

Similarly, let $g_{ij}$ with $\mu, \nu \in \{+, -, \}$ be any of the functions in (9), and let $g'_{ij}: \{x_0\} \rightarrow G$ be given by

$$g'_{ij}(x_0) = g'_{ij}, \nu; \quad (12)$$

then the map

$$H_{ij}: \{x_0\} \times [0,1] \rightarrow G, \quad H_{ij}(x_0,t) = c_{ij}(t) \quad (13)$$

with $c_{ij}: [0,1] \rightarrow G$ a continuous path in $G$ satisfying $c_{ij}(0) = g_{ij}$ and $c_{ij}(1) = g'_{ij}$, is a homotopy between $g_{ij}$ and $g'_{ij}$ i.e.

$$\{x_0\}, G_{\sim} = \{[g_{ij}]_{\sim}\}. \quad (14)$$

It is then easy to show that, if we define (constant) functions

$$\Lambda_{k}\mu: U_{k} \rightarrow G, \quad p \rightarrow \lambda_{k}\mu, \quad k = 1, ..., n, \quad \mu = +, -, \quad (15)$$

then

$$g'_{k+,-}(x_0) = \lambda_{k} - g_{k+,-}(x_0)\lambda_{k}^{-1}, \quad (16)$$

$$g'_{k+,-}(a_k) = \lambda_{k} - g_{k+,-}(a_k)\lambda_{k}^{-1}, \quad (17)$$

for $k = 1, ..., n$, and

$$g'_{ij}(x_0) = \lambda_{ij}g_{ij}(x_0)\lambda_{ij}^{-1} \quad (18)$$

for $i, j \in \{1, ..., n\}, \ i \neq j$, $\mu, \nu \in \{+, -, \}$. In fact, for arbitrary two sets of homotopic transition functions $g_{ij, \alpha} \sim g_{ij, \beta}$, one has continuous maps $H : (U_{ij} \cap U_{\alpha}) \times [0,1] \rightarrow G$ such that $H(x, 0) = g_{ij, \alpha}(x)$ and $H(x, 1) = g'_{ij, \alpha}(x)$. Given a map $\Lambda_{\alpha} : U_{\beta} \cap U_{\alpha} \rightarrow G$, $x \rightarrow \Lambda(x) = \lambda_{\alpha}$, one defines $\bar{H}(x, 1) = H(x, 1)\Lambda_{\alpha}(x)$ and therefore $\bar{H}(x, 1) = g_{ij, \alpha}(x)\Lambda_{\alpha}(x)$. Now, one defines $\Lambda_{\beta} : U_{ij} \cap U_{\alpha} \rightarrow G$ through $\Lambda_{\beta}(x) = \bar{H}(x, 1)\Lambda_{\alpha}(x)^{-1} = g_{ij, \alpha}(x)\Lambda_{\alpha}(x)^{-1}$ i.e. $g_{ij, \alpha}(x) = \Lambda_{\beta}(x)g_{ij, \alpha}(x)\Lambda_{\alpha}(x)^{-1} \ (*)$ for all $x \in U_{ij} \cap U_{\alpha}$. In our case, for $U_{k+} \cap U_{k-} \simeq \{x_0, a_k\}$, the formulae corresponding to (*) are

$$g'_{k+,-}(x_0) = \Lambda_{k-}(x_0)g_{k+,-}(x_0)\Lambda_{k+}(x_0)^{-1}$$

and

$$g'_{k+,-}(a_k) = \Lambda_{k-}(a_k)g_{k+,-}(a_k)\Lambda_{k+}(a_k)^{-1}. \quad (19)$$

Extending continuously (as constants) $\Lambda_{k+}$ and $\Lambda_{k-}$ respectively to the open sets $U_{k+}$ and $U_{k-}$:

$$\Lambda_{k+}: U_{k+} \rightarrow G, \quad p \rightarrow \lambda_{k+}, \quad \Lambda_{k-}: U_{k-} \rightarrow G, \quad q \rightarrow \lambda_{k-} \quad (16)$$

(in particular one has $\Lambda_{k+}(x_0) = \lambda_{k+}(a_k) = \lambda_{k+}$ and $\Lambda_{k-}(x_0) = \lambda_{k-}(a_k) = \lambda_{k-}$), one obtains equations (16) and (17). Proceeding similarly for the cases $U_{ij} \cap U_{\alpha} \simeq \{x_0\}$, one gets equation (18).

Then, up to isomorphisms, and according to a general theorem for coordinate bundles, there is a unique $G$-bundle generated by the transition functions given by the equations (8) and (9), namely the product bundle. We then have

$$B_{\Lambda, S^1}(G) = \{[G \rightarrow \bigvee S^1] \times \bigvee S^1\}, \quad (19)$$
For the cases of the examples in refs. 1, 6 and 7, we have, respectively,

\[ \mathcal{B}_{D^2} (U(1)) = \{ [U(1) \to D^2_{\omega^2} \times U(1) \to D^2_{\omega^2}] \}, \quad (20) \]

\[ \mathcal{B}_{D^2} (SU(2)) = \{ [SU(2) \to D^2_{\omega^2} \times SU(2) \to D^2_{\omega^2}] \}, \quad (21) \]

and

\[ \mathcal{B}_{D^2} (SU(3)) = \{ [SU(3) \to D^2_{\omega^2} \times SU(3) \to D^2_{\omega^2}] \}. \quad (22) \]

In the gravitational case, the gauge group is \( SL(2, \mathbb{C}) \) \(^{18}\), which is the universal covering group of the connected component of the Lorentz group \( L^1 \); then, for weak gravitational fields with a distribution of \( n \) gravitomagnetic flux lines as above we would have the trivial bundle

\[ SL(2, \mathbb{C}) \to D^2_{\omega^2} \times SL(2, \mathbb{C}) \to D^2_{\omega^2} \cdot n. \quad (23) \]

We want to stress that, even if the A-B connection is flat (though not exact), one of the sufficient conditions for the automatic triviality of the A-B bundle fails: though paracompact, the bouquet \([ \cdot \cdot \cdot ]\) denotes here the equivalence class of bundles isomorphic to the product bundle. QED

It is interesting to notice that there are two other fibre bundles related to the A-B effect. The first bundle is the universal covering space \(^{19}\) of the base manifold (“laboratory” or physical space where the particles coupled to the A-B potential move) which is the \( \pi_1(M;x_0) \) (non trivial) bundle \( \xi_c : \tilde{M} \xrightarrow{\sim} M \), where \( \pi_1(M;x_0) \equiv \pi_1(M) \) if \( M \) is connected is the fundamental group of \( M \). In our case,

\[ \pi_1(\mathbb{R}^2 \setminus \{ \text{n points} \}; x_0) \cong \{ c_1, ..., c_n \} \]

is the freely generated group with \( n \) non commuting generators \( c_1, ..., c_n \). In particular, for the original abelian A-B effect with group \( U(1) \), \( \mathbb{R}^2 = RS(\text{Log}) \); the Riemann surface of the logarithm, and \( \pi_1(\mathbb{R}^2) \equiv \mathbb{Z} \).

The particle propagator in \( M, K(x'', t''; x', t') \) with \( t'' > t' \), is a sum of homotopy propagators \(^7\) multiplied by corresponding gauge factors \(^{20}\): the former are given by unrestricted path integrals computed in \( \tilde{M} \), the paths in these path integrals project onto the corresponding homotopy classes of paths in the non simply connected space \( M \); the latter are Wilson loops given by

\[ T \exp \int_{\xi(c)} \tilde{A} \cdot d\tilde{l} \]

where \( T \) denotes time order, \( \tilde{A} \) is the A-B potential, and \( c \) is a loop in \( \tilde{M} \) beginning and ending respectively at \( y_0 \) and \( y'' \) in \( \pi^{-1} \{ x'' \} \) \( \cong \pi_1(M) \), with \( y_0 \) fixed and arbitrary. Then one has the group homomorphism (many-to-one or one-to-one)

\[ y'' \xrightarrow{\psi} \Psi(y'') \xrightarrow{\varphi} T \exp \int_{\pi(c)} \tilde{A} \cdot d\tilde{l} \in G \quad (24) \]

whose image in \( G, \varphi(\pi_1(M)) \), responsible for the A-B effect, is the holonomy of the connection. \(^{21}\)

The second bundle is the associated complex vector bundle \( \xi_{\mathbb{C}^m} : \mathbb{C}^m - P_{\mathbb{C}^m} \xrightarrow{\pi_{\mathbb{C}^m}} M \ (m = 2s + 1 \) is the dimension of the spinor space and \( s \) is the spin; for scalar particles \( m = 1 \)\), where

\[ P_{\mathbb{C}^m} = (M \times G) \times_G \mathbb{C}^m = \{ [(x, g, \bar{z})] \} \}_{(x, g, \bar{z}) \in (M \times G) \times_G \mathbb{C}^m}, \]
\[ [(x, g), \vec{z}] = \{(x, gg'), g'^{-1}\vec{z}\} \quad \forall g' \in G. \]  

(25)

\(\xi_{\mathbb{C}^m}\) is trivial since \(\xi\) is trivial, and the quantum mechanical wave functions of the particles are global sections of \(\xi_{\mathbb{C}^m}\):

\[ \psi \in \Gamma(\xi_{\mathbb{C}^m}) \]

i.e. \(\psi : M \to P_{\mathbb{C}^m}\) with \(\pi_{\mathbb{C}^m} \circ \psi = \text{Id}_M\).

Notice that while the propagator is computed in \(\xi_c\), the wave function lies in \(\xi_{\mathbb{C}^m}\), with

\[ \psi(x'', t'') = \int_M dx' K(x'', t''; x', t') \psi(x', t'). \]  

(26)

If \(\omega\) is the A-B connection in \(P\), then the coupling \(\omega - \psi\) is the covariant derivative

\[ \nabla^\omega \psi = \psi_{V^\uparrow(\gamma_\psi)} \in \Gamma(\xi_{\mathbb{C}^m}), \] 

(27)

where \(V\) is a vector field in \(M\), \(V^\uparrow\) its horizontal lifting in \(P\) by \(\omega\), \(\gamma_\psi\) and \(V^\uparrow(\gamma_\psi)\) are equivariant functions from \(P\) to \(\mathbb{C}^m\) with \(\gamma_\psi(p) = \vec{z}\) where \(\psi(\pi_G(p)) = [p, \vec{z}]\), and \(\psi_{V^\uparrow(\gamma_\psi)}(x) = [p, V^\uparrow(\gamma_\psi)(p)]\) for any \(p \in \pi_G^{-1}(\{x\})\).

Locally, of course, \(\nabla^V \psi\) reproduces the usual minimal coupling between \(\vec{A}\) and \(\psi\).

Finally, though \(\varphi : \pi_1(M) \to G\) is a group homomorphism, and \(\tilde{M} \xrightarrow{f} M \times G\) given by \(f(y) = (\pi(y), 1)\) is a canonical map, there is no bundle map between \(\xi_c\) and \(\xi\): the pair of functions \((f \times \varphi, f)\) is not a principal bundle homomorphism.

In summary, the three bundles are related by the following diagram:

\[
\begin{array}{ccc}
\pi_1(M) & \xrightarrow{\varphi} & G \\
\downarrow & & \downarrow \\
\tilde{M} & \xrightarrow{f} & M \times G \\
\downarrow \pi & & \downarrow \pi_G \\
M & = & M \end{array}
\]

\[
\begin{array}{ccc}
& & \mathbb{C}^m \\
\downarrow & & \\
(M \times G) \times_G \mathbb{C}^m & \xrightarrow{\iota} & \mathbb{C}^m \\
\downarrow \pi_{\mathbb{C}^m} & \uparrow \psi & \uparrow \nabla^V \psi \\
M & = & M
\end{array}
\]

where \(\iota\) is the canonical injection of the bundle \(\xi\) into its associated bundle i.e. \(\iota(p) = [p, 0]\).

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