Proper connection number and 2-proper connection number of a graph

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Abstract

A path in an edge-colored graph is called a proper path if no two adjacent edges of the path are colored with one same color. An edge-colored graph is called \(k\)-proper connected if any two vertices of the graph are connected by \(k\) internally pairwise vertex-disjoint proper paths in the graph. The \(k\)-proper connection number of a \(k\)-connected graph \(G\), denoted by \(pc_k(G)\), is defined as the smallest number of colors that are needed in order to make \(G\) \(k\)-proper connected. For \(k = 1\), we write \(pc(G)\) other than \(pc_1(G)\), and call it the proper connection number of \(G\). In this paper, we present an upper bound for the proper connection number of a graph \(G\) in terms of the minimum degree of \(G\), and give some sufficient conditions for a graph to have 2-proper connection number two. Also, we investigate the proper connection numbers of dense graphs.

Keywords: proper path, proper connection number, 2-proper connection number, dense graph

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1 Introduction

In this paper we are concerned with simple connected finite graphs. We follow the terminology and the notation of Bondy and Murty [3]. For a graph \(G = (V, E)\) and two

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disjoint subsets $X$ and $Y$ of $V$, denote by $B_G[X,Y]$ the bipartite subgraph of $G$ with vertex set $X \cup Y$ and edge set $E(X,Y)$, where $E(X,Y)$ is the set of edges of $G$ that have one end in $X$ and the other in $Y$. A graph is called pancyclic if it contains cycles of all lengths $r$ for $3 \leq r \leq n$. The join $G_1 \vee G_2$ of two edge-disjoint graphs $G_1$ and $G_2$ is obtained by adding edges from each vertex in $G_1$ to every vertex in $G_2$.

An edge-coloring of a graph $G$ is an assignment $c$ of colors to the edges of $G$, one color to each edge of $G$. If adjacent edges of $G$ are assigned different colors by $c$, then $c$ is a proper (edge-)coloring. For a graph $G$, the minimum number of colors needed in a proper coloring of $G$ is referred to as the chromatic index of edge-chromatic number of $G$ and denoted by $\chi'(G)$. A path in an edge-colored graph with no two edges sharing the same color is called a rainbow path. An edge-colored graph $G$ is said to be rainbow connected if every pair of distinct vertices of $G$ is connected by at least one rainbow path in $G$. Such a coloring is called a rainbow coloring of the graph. For a connected graph $G$, the minimum number of colors needed in a rainbow coloring of $G$ is referred to as the rainbow connection number of $G$ and denoted by $rc(G)$. The concept of rainbow coloring was first introduced by Chartrand et al. in [6]. In recent years, the rainbow coloring has been extensively studied and a variety of nice results have been obtained, see [5, 7, 10, 11, 13] for examples. For more details we refer to a survey paper [14] and a book [15].

Inspired by the rainbow coloring and proper coloring of graphs, Andrews et al. [1] introduced the concept of proper-path coloring. Let $G$ be an edge-colored graph, where adjacent edges may be colored with the same color. A path in $G$ is called a proper path if no two adjacent edges of the path are colored with a same color. An edge-coloring $c$ is a proper-path coloring of a connected graph $G$ if every pair of distinct vertices $u, v$ of $G$ is connected by a proper $u-v$ path in $G$. A graph with a proper-path coloring is said to be proper connected. If $k$ colors are used, then $c$ is referred to as a proper-path $k$-coloring. An edge-colored graph $G$ is $k$-proper connected if any two vertices are connected by $k$ internally pairwise vertex-disjoint proper paths. For a $k$-connected graph $G$, the $k$-proper connection number of $G$, denoted by $pc_k(G)$, is defined as the smallest number of colors that are needed in order to make $G$ $k$-proper connected. Clearly, if a graph is $k$-proper connected, then it is also $k$-connected. Conversely, any $k$-connected graph has an edge-coloring that makes it $k$-proper connected; the number of colors is easily bounded by the edge-chromatic number which is well known to be at most $\Delta(G)$ or $\Delta(G) + 1$ by Vizing's Theorem [16] (where $\Delta(G)$, or simply $\Delta$, is the maximum degree of $G$). Thus $pc_k(G) \leq \Delta(G) + 1$ for any $k$-connected graph $G$. For $k = 1$, we write $pc(G)$ other than $pc_1(G)$, and call it the proper connection number of $G$.

Let $G$ be a nontrivial connected graph of order $n$ and size $m$. Then the proper
connection number of \( G \) has the following apparent bounds:

\[
1 \leq pc(G) \leq \min\{\chi'(G), rc(G)\} \leq m.
\]

Furthermore, \( pc(G) = 1 \) if and only if \( G = K_n \) and \( pc(G) = m \) if and only if \( G = K_{1,m} \) is a star of size \( m \).

The paper is organized as follows: In Section 2, we give the basic definitions and some useful lemmas. In Section 3, we study the proper connection number of bridgeless graphs, and present a tight upper bound for the proper connection number of a graph by using this result. In Section 4, we give some sufficient conditions for graphs to have 2-proper connection number two. In Section 5, we investigate the proper connection number of dense graphs.

## 2 Preliminaries

At the beginning of this section, we list some fundamental results on proper-path coloring.

**Lemma 2.1.** [1] If \( G \) is a nontrivial connected graph and \( H \) is a connected spanning subgraph of \( G \), then \( pc(G) \leq pc(H) \). In particular, \( pc(G) \leq pc(T) \) for every spanning tree \( T \) of \( G \).

**Lemma 2.2.** [1] If \( T \) is a nontrivial tree, then \( pc(T) = \chi'(T) = \Delta(T) \).

Given a colored path \( P = v_1v_2 \ldots v_sv_s \) between any two vertices \( v_1 \) and \( v_s \), we denote by \( start(P) \) the color of the first edge in the path, i.e., \( c(v_1v_2) \), and by \( end(P) \) the last color, i.e., \( c(v_{s-1}v_s) \). If \( P \) is just the edge \( v_1v_s \), then \( start(P) = end(P) = c(v_1v_s) \).

**Definition 2.1.** Let \( c \) be an edge-coloring of \( G \) that makes \( G \) proper connected. We say that \( G \) has the strong property under \( c \) if for any pair of vertices \( u, v \in V(G) \), there exist two proper paths \( P_1, P_2 \) between them (not necessarily disjoint) such that \( start(P_1) \neq start(P_2) \) and \( end(P_1) \neq end(P_2) \).

In [4], the authors studied the proper-connection numbers in 2-connected graphs. Also, they presented a result which improves the upper bound \( \Delta(G) + 1 \) of \( pc(G) \) to the best possible whenever the graph \( G \) is bipartite and 2-connected.

**Lemma 2.3.** [4] Let \( G \) be a graph. If \( G \) is bipartite and 2-connected, then \( pc(G) = 2 \) and there exists a 2-edge-coloring \( c \) of \( G \) such that \( G \) has the strong property under \( c \).
Lemma 2.4. Let $G$ be a graph. If $G$ is 2-connected, then $pc(G) \leq 3$ and there exists a 3-edge-coloring $c$ of $G$ such that $G$ has the strong property under $c$.

Lemma 2.5. Let $H = G \cup \{v_1\} \cup \{v_2\}$. If there is a proper-path $k$-coloring $c$ of $G$ such that $G$ has the strong property under $c$. Then $pc(H) \leq k$ as long as $v_1, v_2$ are not isolated vertices of $H$.

As a result of Lemma 2.5, we obtain the following corollary.

Corollary 2.6. Let $H$ be the graph that is obtained by identifying $u_i$ of $G$ to $v_i$ of a path $P_i$ for $i = 1, 2$, where $d_{P_i}(v_i) = 1$. If there is a proper-path $k$-coloring $c$ of $G$ such that $G$ has the strong property under $c$, then $pc(H) \leq k$.

Lemma 2.7. Let $C_n = v_1v_2\ldots v_nv_1$ be an $n$-vertex cycle and $G = C_n + v_{n-1}v_1$. One has that $pc_2(G) = 2$.

Proof. If $n$ is an even integer, it is obvious that $pc_2(G) \leq pc_2(C_n) = 2$. So we only need to consider the case that $n$ is odd. Let $C' = v_1v_2v_3\ldots v_{n-1}v_1$. Then we have that $C'$ is an even cycle. We color the edges $v_{2i-1}v_{2i}$ by color 1 for $i = 1, 2, \ldots, \frac{n-1}{2}$ and color the other edges by color 2. Now we show that for any $v_i, v_j$, there are two disjoint proper paths connecting them. If $i, j \neq n$, we can see that $P_1 = v_i v_{i+1} \ldots v_j$ and $P_2 = v_i v_{i-1} \ldots v_1 v_{n-1} \ldots v_j$ are two disjoint proper paths connecting $v_i$ and $v_j$. If $i = n$, we also have that $Q_1 = v_nv_1v_2\ldots v_{j-1}$ and $Q_2 = v_nv_{n-1}v_{n-2}\ldots v_j$ are two disjoint proper paths connecting $v_n$ and $v_j$. The proof is thus complete.

3 Upper bounds of proper connection number

In [4], the authors studied the proper connection number for 2-(edge)-connected graphs by (closed) ear-decomposition. Here, we reproof the result for 2-edge-connected graphs by using another method.

Theorem 3.1. If $G$ is a connected bridgeless graph with $n$ vertices, then $pc(G) \leq 3$. Furthermore, there exists a 3-edge-coloring $c$ of $G$ such that $G$ has the strong property under $c$.

Proof. We prove the result for connected bridgeless graphs by induction on the number of blocks in $G$. First, the result clearly holds when $G$ is 2-connected by Lemma 2.4. Suppose that $G$ has at least two blocks. Let $X$ be the set of vertices of an end-block of $G$, that is, $X$ contains only one cut vertex, say $x$. From Lemma 2.4, we know that
Assume that $G$ with the number of blocks one less than $G$. By the induction hypothesis, we have that $pc(H) \leq 3$ and $H$ has a 3-edge-coloring $c_2$ such that $H$ has the strong property under $c_2$. Let $c$ be the edge-coloring of $G$ such that $c(e) = c_1(e)$ for any $e \in E(G[X])$ and $c(e) = c_2(e)$ otherwise. We now show that $G$ has the strong property under the coloring $c$. It suffices to consider the pairs $u, v$ such that $u \in X \setminus \{x\}$ and $v \in V(G) \setminus X$. Let $P_1, P_2$ be two proper paths in $G[X]$ between $u$ and $x$ such that $start(P_1) \neq start(P_2)$ and $end(P_1) \neq end(P_2)$, and let $Q_1, Q_2$ be the two proper paths in $H$ between $v$ and $x$ such that $start(Q_1) \neq start(Q_2)$ and $end(Q_1) \neq end(Q_2)$. It is obvious that either $P = P_1 \cup Q_1, Q = P_2 \cup Q_2$ or $P = P_1 \cup Q_2, Q = P_2 \cup Q_1$ are two proper paths between $u$ and $v$ with the property that $start(P) \neq start(Q)$ and $end(P) \neq end(Q)$. This completes the proof. 

With a similar analysis and by Lemma 2.3, we have the following theorem.

**Theorem 3.2.** If $G$ is a bipartite connected bridgeless graph with $n$ vertices, then $pc(G) \leq 2$. Furthermore, there exists a 2-edge-coloring $c$ of $G$ such that $G$ has the strong property under $c$.

An Eulerian graph is clearly bridgeless. As a result of Theorem 3.1 we have the following corollary.

**Corollary 3.3.** For any Eulerian graph $G$, one has that $pc(G) \leq 3$. Furthermore, if $G$ is Eulerian and bipartite, one has that $pc(G) = 2$.

**Lemma 3.4.** Let $G$ be a graph and $H = G − PV(G)$, where $PV(G)$ is the set of the pendant vertices of $G$. Suppose that $pc(H) \leq 3$ and there is a proper-path 3-coloring $c$ of $H$ such that $H$ has the strong property under $c$. Then one has that $pc(G) \leq \max\{3, |PV(G)|\}$.

**Proof.** Assume that $PV(G) = \{v_1, v_2, \ldots, v_k\}$. If $k \leq 2$, we have that $pc(G) \leq 3$ by Lemma 2.3. So we consider the case that $k \geq 3$. Let $u_i$ be the neighbor of $v_i$ in $G$ for $i = 1, 2, \ldots, k$, and let $\{1, 2, 3\}$ be the color-set of $c$. We first assign color $j$ to $u_j v_j$ for $j = 4, \ldots, k$. Then we color the remaining edges $u_1 v_1, u_2 v_2, u_3 v_3$ by colors $1, 2, 3$.

If $u_1 = u_2 = u_3$, we assign color $i$ to $u_i v_i$ for $i = 1, 2, 3$. If $u_1 = u_2 \neq u_3$, let $P$ be a proper path of $G$ connecting $u_1$ and $u_3$. Then there are two different colors in $\{1, 2, 3\} \setminus \{start(P)\}$. We assign these two colors to $u_1 v_1$ and $u_2 v_2$, respectively, and choose a color that is distinct from $end(P)$ in $\{1, 2, 3\}$ for $u_3 v_3$. If $u_i \neq u_j$ for $1 \leq i \neq j \leq 3$, suppose that $P_{ij}$ is a proper path of $G$ between $u_i$ and $u_j$. We choose a
color that is distinct from $\text{start}(P_{12})$ and $\text{start}(P_{13})$ in $\{1,2,3\}$ for $u_1v_1$. Similarly, we color $u_2v_2$ by a color in $\{1,2,3\} \setminus \{\text{end}(P_{12}),\text{start}(P_{23})\}$, and color $u_3v_3$ by a color in $\{1,2,3\} \setminus \{\text{end}(P_{13}),\text{end}(P_{23})\}$.

One can see that in all these cases, $v_i$ and $v_j$ are proper connected for $1 \leq i \neq j \leq k$. Moreover, as $H$ has the strong property under edge-coloring $c$, it is obvious that $v_i$ and $u$ are proper connected for $1 \leq i \leq k$ and $u \in V(H)$. Therefore, we have that $pc(H) \leq k = |PV(G)|$. Hence, we obtain that $pc(G) \leq \max \{3, |PV(G)|\}$. 

**Lemma 3.5.** Let $G$ be a graph with a cut-edge $v_1v_2$, and $G_i$ be the connected graph obtained from $G$ by contacting the connected component containing $v_i$ of $G - v_1v_2$ to a vertex $v_i$, where $i = 1, 2$. Then $pc(G) = \max \{pc(G_1), pc(G_2)\}$

**Proof.** First, it is obvious that $pc(G) \geq \max \{pc(G_1), pc(G_2)\}$. Let $pc(G_1) = k_1$ and $pc(G_2) = k_2$. Without loss of generality, suppose $k_1 \geq k_2$. Let $c_1$ be a $k_1$-proper coloring of $G_1$ and $c_2$ be a $k_2$-proper coloring of $G_2$ such that $c_1(v_1v_2) = c_2(v_1v_2)$ and $\{c_2(e) : e \in E(G_2)\} \subseteq \{c_1(e) : e \in E(G_1)\}$. Let $c$ be the edge-coloring of $G$ such that $c(e) = c_1(e)$ for any $e \in E(G_1)$ and $c(e) = c_2(e)$ otherwise. Then $c$ is an edge-coloring of $G$ using $k_1$ colors. We will show that $c$ is a proper-path coloring of $G$. For any pair of vertices $u, v \in V(G)$, we can easily find a proper path between them if $u, v \in V(G_1)$ or $u, v \in V(G_2)$. Hence we only need to consider that $u \in V(G_1) \setminus \{v_1, v_1\}$ and $v \in V(G_2) \setminus \{v_1, v_2\}$. Since $c_1$ is a $k_1$-proper coloring of $G_1$, there is a proper path $P_1$ in $G_1$ connecting $u$ and $v_1$. Since $c_2$ is a $k_2$-proper coloring of $G_2$, there is a proper path $P_2$ in $G_2$ connecting $v$ and $v_2$. As $c_1(v_1v_2) = c_2(v_1v_2)$, then we know that $P = uP_1v_2v_1P_2v$ is a proper path connecting $u$ and $v$ in $G$. Therefore, we have that $pc(G) \leq k_1$, and the proof is thus complete.

Let $B \subseteq E$ be the set of cut-edges of a graph $G$. Let $C$ denote the set of connected components of $G' = (V; E \setminus B)$. There are two types of elements in $C$, singletons and connected bridgeless subgraphs of $G$. Let $S \subseteq C$ denote the singletons and let $D = C \setminus S$. Each element of $S$ is, therefore, a vertex, and each element of $D$ is a connected bridgeless subgraph of $G$. Contracting each element of $D$ to a vertex, we obtain a new graph $G^*$. It is easy to see that $G^*$ is a tree, and the edge set of $G^*$ is $B$. According to the above notations, we have the following theorem.

**Theorem 3.6.** If $G$ is a connected graph, then $pc(G) \leq \max \{3, \Delta(G^*)\}$.

**Proof.** For an arbitrary element $A$ of $C$, $A$ is either a singleton or a connected bridgeless subgraph of $G$. Let $C(A)$ be the set of cut-edges in $G$ that has an end-vertex in $A$. It is obvious that $|C(A)| \leq \Delta(G^*)$. We use $A_0$ to denote the subgraph of $G$ obtained from $A$
by adding all the edges of \( C(A) \) to \( A \). If \( A \) is a singleton, we have that \( pc(A_0) = |C(A)| \leq \max\{3, \Delta(G^*)\} \). Otherwise, from Theorem 3.1 we know that \( pc(A) \leq 3 \) and there is a coloring \( c \) of \( A \) such that \( A \) has the strong property under \( c \). Then by Lemma 3.4 we have that \( pc(A) \leq 3 \) and there is a coloring \( c \) of \( A \) such that \( A \) has the strong property under \( c \). Then by Lemma 3.4, we have that 

\[
pc(A_0) = \max\{pc(A) \leq 3, |C(A)| \leq \max\{3, \Delta(G^*)\}\}.
\]

Hence, by Lemma 3.5, we can obtain that 

\[
pc(G) = \max_{A \in C} pc(A_0) \leq \max\{3, \Delta(G^*)\}.
\]

Let \( rK_t \) be the disjoint union of \( r \) copies of the complete graph \( K_t \). We use \( S^t_r \) to denote the graph obtained from \( rK_t \) by adding an extra vertex \( v \) and joining \( v \) to one vertex of each \( K_t \).

**Corollary 3.7.** If \( G \) is a connected graph with \( n \) vertices and minimum degree \( \delta \geq 2 \), then

\[
pc(G) \leq \max\{3, \frac{n-1}{\delta+1}\}.
\]

Moreover, if \( \frac{n-1}{\delta+1} > 3 \), and \( n \geq \delta(\delta + 1) + 1 \), we have that

\[
pc(G) = \frac{n-1}{\delta+1}
\]

If and only if \( G \cong S^t_r \), where \( t = 1 = \delta \) and \( rt + 1 = n \).

**Proof.** Since the minimum degree of \( G \) is \( \delta \geq 2 \), we know that each leaf of \( G^* \) is obtained by contracting an element with at least \( \delta + 1 \) vertices of \( D \). Therefore, \( D \) has at most \( \frac{n-1}{\delta+1} \) such elements, and so, one can see that \( \Delta(G^*) \leq \frac{n-1}{\delta+1} \). From Theorem 3.3, we know that 

\[
pc(G) \leq \max\{3, \frac{n-1}{\delta+1}\}.
\]

If \( \frac{n-1}{\delta+1} > 3 \) and \( pc(G) = \frac{n-1}{\delta+1} \), one can see that \( G^* \) is a star with \( \Delta(G^*) = \frac{n-1}{\delta+1} \), and each leaf of \( G^* \) is obtained by contracting an element with \( \delta + 1 \) vertices of \( D \), that is, \( G \cong S^t_r \), where \( t = \delta \) and \( rt + 1 = n \). On the other hand, if \( G \cong S^t_r \), where \( t = \delta \) and \( rt + 1 = n \), we can easily check that \( pc(G) = r = \frac{n-1}{\delta+1} \).

### 4 Graphs with 2-proper connection number two

At the beginning of this section, we list an apparent sufficient condition for graphs to have proper connection number two.

**Proposition 4.1.** Let \( G \) be a simple noncomplete graph on at least three vertices in which the minimum degree is at least \( n/2 \), then \( pc(G) = 2 \).

We should mention that the condition \( \delta(G) \geq n/2 \) is quite rough. In [4], the authors gave a much better result for graphs with appreciable quantity of vertices to have proper connection number two. They proved that if \( G \) is connected noncomplete graph with \( n \geq 68 \) vertices, and \( \delta \geq n/4 \), then \( pc(G) = 2 \).

It is easy to find that the 2-proper connection number of any simple 2-connected graph is at least 2, and every complete graph on at least 4 vertices evidently has the property that \( pc_2(K_n) = 2 \). But suppose that our graph has considerably fewer edges. In
particular, we may ask how large the minimum degree of $G$ must be in order to guarantee the property that $pc_2(G) = 2$. Motivated by Proposition 4.4, we consider the condition $\delta \geq n/2$ and get the following theorem.

**Theorem 4.2.** Let $G$ be a connected graph with $n$ vertices and minimum degree $\delta$. If $\delta \geq n/2$ and $n \geq 4$, then $pc_2(G) = 2$.

**Proof.** Since $\delta \geq n/2$, we know that there exists a Hamiltonian cycle $C = v_1v_2\ldots v_n$ in $G$. If $n$ is even, one has that $pc_2(G) \leq pc_2(C_n) = 2$. Hence, we only need to consider the case that $n = 2k + 1$. Let $H = G - v_n$, one has that $d_H(v_1) \geq d_G(v_i) - 1 \geq k = |V(H)|/2$.

Hence, there exists a Hamiltonian cycle $C' = v'_1v'_2\ldots v'_{2k}$ in $H$. As $d_G(v_n) \geq k + 1$, one can see that there is an edge, say $v'_1v'_2$, such that $v_nv'_1, v_nv'_2 \in E(G)$. Hence, there is a spanning subgraph $G'$ of $G$ with $E(G') = E(C') \cup \{v_nv'_1, v_nv'_2\}$. By Lemma 2.7, we have that $pc_2(G) \leq pc_2(G') = 2$, and so the proof is complete. \qed

**Remark:** The condition $\delta \geq n/2$ is best possible. In fact, we can find graphs with minimum degree less than $n/2$ which is not 2-connected, and so we cannot calculate the 2-proper connection number. For example, let $G = K_1 \vee (2K_k)$. We know that $\delta(G) = k < |V(G)|/2$, whereas $pc_2(G)$ does not exists.

Though the condition on the minimum degree can not be improved, we may consider some weaker conditions. We need an important conclusion which can be found in [2]. We use $cir(G)$ to denote the circumference (length of a longest cycle) of $G$.

**Lemma 4.3.** [2] If $G = G(n, m)$ and $m \geq n^2/4$, then $G$ contains a cycle $C_r$ of length $r$ for each $3 \leq r \leq cir(G)$.

**Theorem 4.4.** Let $G$ be a simple graph on at least four vertices in which the degree sum of any two nonadjacent vertices is at least $n$. Then $pc_2(G) = 2$.

**Proof.** Since the degree sum of any two nonadjacent vertices of $G$ is at least $n$, we know that $G$ is Hamiltonian. Suppose that a Hamiltonian cycle of $G$ is $C = v_1v_2\ldots v_n$. If $n$ is even or $n = 5$, it is obvious that $pc_2(G) \leq 2$.

If $\delta \leq 3$, suppose, without loss of generality, that $v_n$ is the vertex which has minimum degree. There exists a $2 < j < n - 2, j \neq i$ such that $v_j$ and $v_n$ are not adjacent. We have that $d(v_j) \geq n - d(v_n) \geq n - 3$, and so we know that $\{v_{j-2}, v_{j+2}\} \cap N(v_j) \neq \emptyset$, where the subscripts are modulus $n$. Hence, we can see that $pc_2(G) \leq 2$ by Lemma 2.7. In what follows of the proof, we only consider the case that $n$ is an odd number which is larger than 7 and $\delta \geq 4$. To continue our proof, we need the following claim.

**Claim:** $G$ is pancyclic.
Proof of the Claim: Let $\overline{E}$ be the edge set of $\overline{G}$. One can see that for any $uv \in \overline{E}$, $d_G(u) + d_G(v) \geq n$. It is obvious that
\[
\sum_{uv \in \overline{E}} (d_G(u) + d_G(v)) = (n - 1) \sum_{v \in V(G)} d_G(v) = 2(n - 1)m. \tag{1}
\]

One the other hand,
\[
\sum_{uv \in E} (d_G(u) + d_G(v)) = \sum_{u \in V(G)} d_G(u)^2 \geq n(\sum_{u \in V(G)} d_G(u)/n)^2 = 4m^2/n, \tag{2}
\]
and
\[
\sum_{uv \in E} (d_G(u) + d_G(v)) \geq \binom{n}{2} - mn, \tag{3}
\]
where the equality holds in (2) if and only if $G$ is a regular graph and the equality holds in (3) if and only if $d_G(u) + d_G(v) = n$ for each pair of nonadjacent vertices $u$ and $v$.

Thus we have
\[2(n - 1)m \geq 4m^2/n + \binom{n}{2} - mn,\]
i.e.,
\[(m - n^2/4)(m - \binom{n}{2}) \leq 0.\]

Hence, we have that $m \geq n^2/4$, with equality holds if and only if $G$ is a regular graph with degree $n/2$. We know that $G$ is pancyclic from Lemma 4.3.

Assume that $C' = u_1u_2 \ldots u_{n-1}$ is a cycle of $G$ with $n - 1$ vertices and $v \notin V(C')$. Without loss of generality, assume that $\{u_i, u_i, u_j, u_k\} \subseteq N(v)$ such that $1 < i < j < k \leq n - 1$. Let $c(u_{2i-1}u_{2i}) = 1$ and $c(u_{2i+1}u_{2i}) = 2$ for $i = 1, 2, \ldots, \frac{n-1}{2}$, and let $c(vu_1) = c(u_{n-1}u_1), c(vu_i) = c(u_iu_{i+1}), c(vu_j) = c(u_{j-1}u_j), c(vu_k) = c(u_ku_{k+1})$. Now we prove that for any $x, y \in V(G)$, there are two disjoint proper paths connecting them. We only need to consider the case that $x = v, y = u_l \in V(C')$. If $1 \leq l \leq k$, then $P_1 = vu_1u_2 \ldots u_l$ and $P_2 = vu_ku_{k-1} \ldots u_l$ are two disjoint proper paths connecting them. If $k < l \leq n - 1$, then $P_1 = vu_1u_{i-1} \ldots u_{i}u_{n-1} \ldots u_l$ and $P_2 = vu_ju_{j+1} \ldots u_l$ are two disjoint proper paths connecting them. Hence, we have that $pc_2(G) \leq 2$. \hfill \qed

Remark: The condition that “the degree sum of any two nonadjacent vertices of $G$ is at least $n$” cannot be improved. For example, $C_5$ and $K_1 \vee (2K_2)$ have the property that the degree sum of any two nonadjacent vertices of is one less than their number of vertices, whereas $pc_2(C_5) = 3$ and $pc_2(K_1 \vee (2K_2)$ does not exist.
5 Proper connection number of dense graphs

In this section, we consider the following problem:

Problem 1. For every $k$ with $1 \leq k \leq n - 1$, compute and minimize the function $f(n, k)$ with the following property: for any connected graph $G$ with $n$ vertices, if $|E(G)| \geq f(n, k)$, then $pc(G) \leq k$.

In [9], this kind of question was suggested for rainbow connection number $rc(G)$, and in [12], the authors considered the case $k = 3$ and $k = 4$ for rainbow connection number $rc(G)$. We first show a lower bound for $f(n, k)$.

Proposition 5.1. $f(n, k) \geq \binom{n - k - 1}{2} + k + 2$.

Proof. We construct a graph $G_k$ as follows: Take a $K_{n - k - 1}$ and a star $S_{k+2}$. Identify the center-vertex of $S_{k+2}$ with an arbitrary vertex of $K_{n - k - 1}$. The resulting graph $G_k$ has order $n$ and size $E(G_k) = \binom{n - k - 1}{2} + k + 1$. It can be easily checked that $pc(G_k) = k + 1$. Hence, $f(n, k) \geq \binom{n - k - 1}{2} + k + 2$. □

Lemma 5.2. Let $G$ be a graph with $n$ ($n \geq 6$) vertices and at least $\binom{n - 1}{2} + 3$ edges. Then for any $u, v \in V(G)$, there is a $2$-connected bipartite spanning subgraph of $G$ with $u, v$ in the same part.

Proof. Let $\overline{G}$ be the complement of $G$. Then we have that $|E(\overline{G})| \leq n - 4$. Let $S = N(u) \cap N(v)$, we have that $|S| \geq 2$. Since otherwise, $|S| \leq 1$, then one has that for any $w \in V(G) \setminus (S \cup \{u, v\})$, either $uw \in E(\overline{G})$ or $vw \in E(\overline{G})$, and thus $|E(\overline{G})| \geq n - 3$, which contradicts the fact that $|E(\overline{G})| \leq n - 4$. Therefore, we know that $B_G[S, \{u, v\}]$ is a $2$-connected bipartite subgraph of $G$ with $u, v$ in the same part.

Suppose that $H = B_G[X, Y]$ is a $2$-connected bipartite subgraph of $G$ with $u, v$ in the same part and $H$ has as many vertices as possible. Then, if $V(G) \setminus V(H) \neq \emptyset$, one has that there exists a vertex $w \in V(G) \setminus V(H)$, such that $|N(w) \cap X| \geq 2$ or $|N(w) \cap Y| \geq 2$. Since otherwise,

$$|E(\overline{G})| \geq (n - |V(H_1)|)(|V(H_1)| - 2) \geq n - 3,$$

which contradicts the fact that $|E(\overline{G})| \leq n - 4$. Then $w$ can be added to $X$ if $|N(w) \cap X| \geq 2$ or added to $Y$ otherwise, which contradicts the maximality of $H$. So, we know that $H$ is a $2$-connected bipartite spanning subgraph of $G$ with $u, v$ in the same part, which completes the proof. □

Lemma 5.3. Every $2$-connected graph on $n$ ($n \geq 12$) vertices with at least $\binom{n - 1}{2} - 5$ edges contains a $2$-connected bipartite spanning subgraph.
Proof. The result is trivial if $G$ is complete. We will prove our result by induction on $n$ for noncomplete graphs. First, if $|V(G)| = 12$ and $|E(G)| \geq 50$, one can find a 2-connected bipartite spanning subgraph of $G$. So we suppose that the result holds for all 2-connected graphs on $n_0$ ($13 < n_0 < n$) vertices with at least $(n_0 - 1) - 5$ edges. For a 2-connected graph $G$ on $n$ vertices with $|E(G)| \geq (\binom{n}{2} - 5)$, let $v$ be a vertex with minimum degree of $G$, and let $H = G - v$. If $d(v) = 2$, then $|E(H)| \geq (\binom{n}{2}) - 7$. Let $N_G(v) = \{v_1, v_2\}$. We know that $H$ contains a 2-connected bipartite spanning subgraph with $v_1, v_2$ in the same part by Lemma 5.2. Clearly, $G$ contains a 2-connected bipartite spanning subgraph. Otherwise, $3 \leq d(v) \leq n - 2$, then $|E(H)| \geq (\binom{n - 1}{2}) - 5 - (n - 2) = (\binom{n - 1}{2} - 5)$ and $\delta(H) \geq 2$. If $H$ has a cut-vertex $u$, then each connected component of $H - u$ contains at least 2 vertices.

We have that $|E(H)| \leq (\binom{n - 3}{2} + 3 < (\binom{n - 2}{2}) - 5$, a contradiction. Hence, $H$ is 2-connected. By the induction hypothesis, we know that $H$ contains a 2-connected bipartite spanning subgraph $B_H[X, Y]$. Since $d(v) \geq 3$, at least one of $X$ and $Y$ contains at least 2 neighbors of $v$. Hence, $G$ contains a 2-connected bipartite spanning subgraph.

Theorem 5.4. Let $G$ be a connected graph of order $n \geq 14$. If $(\binom{n - 3}{2}) + 4 \leq |E(G)| \leq \binom{n}{2} - 1$, then $pc(G) = 2$.

Proof. The result clearly holds if $G$ is 3-connected. We only consider of the graphs with connectivity at most 2.

Claim 1: $\delta(G) \leq 5$.

Proof of Claim 1: Suppose to the contrary that $\delta(G) \geq 6$. If $G$ has a cut-vertex, say $x$, then each connected component of $G - x$ has at least 6 vertices. Hence, we have that $|E(G)| \leq (\binom{n - 6}{2}) + \binom{6}{2}$, which is less than $(\binom{n - 3}{2}) + 4$ when $n \geq 14$, a contradiction. If $G$ is 2-connected with a 2-vertex cut $\{x, y\}$, then each connected component of $G - x - y$ has at least 5 vertices. We have that $|E(G)| \leq (\binom{n - 5}{2}) + \binom{7}{2} - 1$, which is less than $(\binom{n - 3}{2}) + 4$ when $n \geq 14$. We can also get a contradiction. Hence, we get our conclusion $\delta(G) \leq 5$.

Let $v$ be a vertex with the minimum degree in $G$, and let $H = G - v$. Then $|V(H)| = n - 1$ and $|E(H)| \geq (\binom{n - 3}{2}) + 4 - 5 = (\binom{n - 3}{2}) - 1$.

Note that if $H$ is 3-connected, one can get that $pc(H) \leq 2$. Then by Lemma 25, one has that $pc(G) \leq 2$. So, we only consider the case that the connectivity of $H$ is at most 2.

Claim 2: $\delta(H) \leq 4$.

Proof of Claim 2: We use the same method as in the proof of Claim 1. Suppose that $\delta(H) \geq 5$. If $H$ has a cut-vertex, say $x$, then each connected component of $H - x$ has at least 5 vertices. Hence, we have that $|E(H)| \leq (\binom{n - 6}{2}) + \binom{6}{2}$, which is less than $(\binom{n - 3}{2} - 1$ when $n \geq 14$, a contradiction. If $H$ is 2-connected with a 2-vertex cut $\{x, y\}$, then
each connected component of $H - x - y$ has at least 4 vertices. Hence, we have that $|E(H)| \leq \binom{n-5}{2} + \binom{6}{2} - 1$, which is less than $\binom{n-3}{2} - 1$ when $n \geq 14$. Hence we get our conclusion that $\delta(H) \leq 4$.

Let $u$ be a vertex with the minimum degree in $H$, and let $F = H - u = G - v - u$. Then $|V(F)| = n - 2$ and $|E(F)| \geq \binom{n-3}{2} - 5 = \binom{n-2}{2} - 5$. If $F$ is 2-connected, we know that $F$ contains a bipartite 2-connected spanning subgraph by Lemma 5.3 and hence $pc(H) \leq 2$. By Lemma 2.5 we have that $pc(G) \leq 2$. So, we only need to consider the case that $F$ has cut-vertices. As in the proof of Lemma 5.3, we know that $F$ has a pendent vertex $w$, and so $\delta(G) \leq d_G(w) \leq 3$. Let $F' = F - w = G - u - v - w$, then $|E(F')| \geq \binom{n-3}{2} - 6$. From Lemma 5.2 we know that $F'$ contains a 2-connected bipartite spanning subgraph, and so $pc(F') \leq 2$. If $d_G(w) = 1$, then $u$ and $v$ are also pendent vertices in $G$. We have that $|E(G)| \leq \binom{n-3}{2} + 3$, which contradicts the fact that $|E(G)| \geq \binom{n-3}{2} + 4$. Thus, $d(w) \geq 2$. If $uv \in E(G)$, one can see that $pc(G) = 2$ by Corollary 2.6. If $uv \notin E(G)$, we have that $u$ has a neighbor in $F'$. Since otherwise, $d(u) = 1$ and $d(v) = 1$, $|E(G)| \leq \binom{n-3}{2} + 3$, a contradiction. So, we know that either $v$ has a neighbor in $F'$ or $uv \in E(G)$. By Corollary 2.6 we have that $pc(G) = 2$. The proof is thus complete. 

\[\textbf{Theorem 5.5.} \text{ Let } G \text{ be a connected graph of order } n \geq 15. \text{ If } |E(G)| \geq \binom{n-4}{2} + 5, \text{ then } pc(G) \leq 3.\]

\[\textbf{Proof.} \text{ If } G \text{ is 2-edge connected, then } pc(G) \leq 3 \text{ clearly holds from Lemma 5.1. If } \delta(G) = 1, \text{ let } H = G - v, \text{ where } v \text{ is a pendent vertex. Then, } H \text{ has } n - 1 \text{ vertices and } |E(H)| \geq \binom{n-1-3}{2} + 4. \text{ From Theorem 5.4 we know that } pc(H) = 2, \text{ and so } pc(G) \leq 3. \text{ In the following, we only consider the graphs with cut-edges and without pendent vertices. Let } e \text{ be a cut-edge of } G, \text{ and let } G_1, G_2 \text{ be the two connected components of } G - e \text{ with } |V(G_1)| \leq |V(G_2)|. \text{ If } |V(G_1)| \geq 5, \text{ we know that } E(G) \leq \binom{n-5}{2} + 11 < \binom{n-4}{2} + 5, \text{ a contradiction. So, we know that } |V(G_1)| \leq 4. \text{ Since } G \text{ has no pendent vertices, we know that } |V(G_1)| \geq 3. \text{ Hence, } G_1 \text{ has three or four vertices with at most one pendent vertex in } G_1. \text{ It can be easily checked that } pc(G_1) \leq 2. \text{ We claim that } pc(G_2) \leq 2. \text{ In fact, if } |V(G_1)| = 3, \text{ then } G_2 \text{ has } n - 3 \text{ vertices and } |E(G_2)| \geq \binom{n-4}{2} + 5 - 4 = \binom{n-3-1}{2} + 1. \text{ If } |V(G_1)| = 4, \text{ then } G_2 \text{ has } n - 4 \text{ vertices and } |E(G_2)| \geq \binom{n-4}{2} + 5 - 7 = \binom{n-4}{2} - 2. \text{ In both cases, we know that } pc(G_2) \leq 2. \text{ Consequently, we can easily get that } pc(G) \leq 3. \text{ }\Box\]
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