In this paper we study how to attack under the self-similarity hypothesis a perfect fluid Bianchi I model with variable $G$ and $\Lambda$, but under the condition $\text{div} \ T \neq 0$. We arrive to the conclusion that: $G$ and $\Lambda$ are decreasing time functions (the sign of $\Lambda$ depends on the equation of state), while the exponents of the scale factor must satisfy the conditions $\sum_{i=1}^{3} \alpha_i = 1$ and $\sum_{i=1}^{3} \alpha_i^2 < 1$, $\forall \omega \in (-1, 1)$, relaxing in this way the Kasner conditions. We also show the connection between the behavior of $G$ and the Weyl tensor.

Keywords: Time varying constants; Bianchi I; Self-similarity.

1. Introduction.

Ever since Dirac first considered the possibility of a $G$ variable (see 1), there have been numerous modifications of general relativity to allow for a variable $G$, nevertheless these theories have not gained wide acceptance. However, recently (see 2-13) a modification has been proposed treating $G$ and $\Lambda$ as non-constants coupling scalars. So it is considered $G$ and $\Lambda$ as coupling scalars within the Einstein equations, $R_{ij} - \frac{1}{2}g_{ij} = GT_{ij} - \Lambda g_{ij}$, while the other symbols have their usual meaning and hence the principle of equivalence then demands that only $g_{ij}$ and not $G$ and $\Lambda$ must enter the equation of motion of particles and photons. In this way the usual conservation law, $\text{div} \ T = 0$, holds. Taking the divergence of the Einstein equations and using the Bianchi identities we obtain the an equation that controls the variation of $G$ and $\Lambda$. These are the modified field equations that allow to take into account a variable $G$ and $\Lambda$. Nevertheless this approach has some drawbacks, for example, it cannot derived from a Hamiltonian, although there are several advantages in the approach.

There are many publications devoted to study the variation of $G$ and $\Lambda$ in the framework of flat FRW symmetries (see for example 2-13) and all this works have been extended to more complicated geometries, like for example Bianchi I models, which represent the simplest generalization of the flat FRW models (see for example
Bianchi I models are important in the study of anisotropies. But in our opinion, the problem arises when one tries to solve the resulting field equations (FE). It seems that it is unavoidable to make simplifying hypotheses, or to impose ad hoc some particular behavior for some of the quantities of the model, in order to obtain an exact solution to the FE. Such simplifying hypothesis are made for mathematical reason (in order to reduce the number of unknowns) although are justified from the physical point of view. Usually such assumptions or simplifying hypotheses follow a power law, for example, the quantity $X$ follows a power law i.e. $X = X_0 t^\alpha$, where $X_0$ is an appropriate dimensional constant, $t$ is the cosmic time (for example) and $\alpha \in \mathbb{R}$ (usually $\alpha \in \mathbb{Q}$, but this other question), and depending on the nature of the quantity $X$, $\alpha$ will be positive or negative. Actually we think that although all these simplifying hypotheses are correct or at least bring us to obtain correct results, it is not necessary to do that, since they may be deduced from symmetry principles in such a way that one may justify (deduce) them from a correct mathematical principle, and usually all these approaches have physical meaning.

Therefore the main goal of this paper is to apply the well known tactic (approach) of self similarity (SS) in order to study and find exact solutions for a perfect fluid Bianchi I models with variable $G$ and $\Lambda$, but under the condition $\text{div} \, T \neq 0$, and trying to make the lowest number of assumptions or neither. We will try to show that with this approach all the usual simplifying hypotheses may be deduced from a correct mathematical principle.

The paper is divided in the following sections. Section two is devoted to outline all the ingredients as well as the field equations. In section three, we study the model under the self-similarity hypothesis. We start this section introducing briefly some ideas about self-similarity and self-similar spacetimes. Once we have found the homothetic vector field we go next to calculate the scale factors, where they obviously follow a power law solution, as well as the derived quantities from them as the Hubble parameter $H$, the deceleration parameter $q$ and the shear $\sigma$, since they only depend of the scale factors and will be the same for all the studied cases. We study four models. The first of them is the classical one i.e. where $G$ is a true constant and the cosmological constant $\Lambda$ vanish. We have preferred starting with this model in order to check how works the employed tactic. We emphasize that the obtained solution (that will be the same in all the studied cases) satisfies the condition, $\sum_{i=1}^{3} \alpha_i = 1$ and $\sum_{i=1}^{3} \alpha_i^2 < 1$, relaxing in this way the Kasner conditions ($\sum_{i=1}^{3} \alpha_i = \sum_{i=1}^{3} \alpha_i^2 = 1$, only valid for a vacuum solution), but it is only valid for $\omega = 1$, i.e. ultra-stiff matter, and where $(\alpha_i)_{i=1}^{3}$ are the exponents of the scale factors. If $\omega \neq 1$, the model collapse to the standard flat FRW one. We then study the curvature invariants as well as the Weyl tensor and its invariant, and end calculating the gravitational entropy. We show that the definitions of gravitational entropy do not work well in this kind of spacetimes, the self-similar ones, since these
definitions are dimensionless which means that this quantity remains constant along the homothetic trajectories of any self-similar spacetime. We would like to relate in same way the behavior of the Weyl tensor with the behavior of a $G$ time-varying. So after showing that the tactic works correctly, we pass to study a simple model where we consider that only vary $G$. In this case we get the same solution as in the above model with the same restriction for the equation of state i.e. the solution is only valid if $\omega = 1$. In the third of the studied model we consider only a $\Lambda$ time-varying. As we have pointed out previously the solution is the same with regard to the exponents of the sale factors i.e. they must satisfy the conditions $\sum_{i=1}^{3} \alpha_i = 1$ and $\sum_{i=1}^{3} \alpha_i^2 < 1$, but in this case this solution is only valid iff $\omega \in (-1, 1)$ i.e. $\omega \neq 1$, since if $\omega = 1$, $\Lambda = 0$. We find always that $\Lambda$ behaves as a negative decreasing time function. The last of the studied cases considers that both “constants”, $G$ and $\Lambda$ are time-varying. We find that $G$ is a decreasing time function (i.e. has a similar behavior as the Weyl tensor) and $\Lambda \approx t^{-2}$, with $\Lambda < 0$, and as in the above case this solution is only valid $\forall \omega \in (-1, 1)$. If $\omega > 1$ then $\Lambda > 0$. We also show how to regain the “classical” case where $\text{div} \, T = 0$. We end we a brief conclusions.

2. The Model.

Throughout the paper $M$ will denote the usual smooth (connected, Hausdorff, 4-dimensional) spacetime manifold with smooth Lorentz metric $g$ of signature $(-, +, +, +)$. Thus $M$ is paracompact. A comma, semi-colon and the symbol $\mathcal{L}$ denote the usual partial, covariant and Lie derivative, respectively, the covariant derivative being with respect to the Levi-Civita connection on $M$ derived from $g$. The associated Ricci and stress-energy tensors will be denoted in component form by $R_{ij}(\equiv R^c_{jcd})$ and $T_{ij}$ respectively. A diagonal Bianchi I space-time is a spatially homogeneous space-time which admits an abelian group of isometries $G_3$, acting on spacelike hypersurfaces, generated by the spacelike KVs $\xi_1 = \partial_x, \xi_2 = \partial_y, \xi_3 = \partial_z$. In synchronous co-ordinates the metric is:

$$ds^2 = -dt^2 + A_1^2(t)(dx^1)^2$$

(1)

where the metric functions $A_1(t), A_2(t), A_3(t)$ are functions of the time co-ordinate only (Greek indices take the space values 1, 2, 3 and Latin indices the space-time values 0, 1, 2, 3). In this paper we are interested only in proper diagonal Bianchi I space-times (which in the following will be referred for convenience simply as Bianchi I space-times), hence all metric functions are assumed to be different and the dimension of the group of isometries acting on the spacelike hypersurfaces is three. Therefore we consider the Bianchi type I metric as

$$ds^2 = -c^2 dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2,$$

(2)

see for example (23-29).

For a perfect fluid with energy-momentum tensor:

$$T_{ij} = (\rho + p) \, u_i u_j + pg_{ij},$$

(3)
where we are assuming an equation of state \( p = \omega \rho, (\omega = \text{const.}) \). Note that here we have preferred to assume this equation of state but as we will show in the following sections this equation may be deduced from the symmetries principles as for example the self-similar one. The 4-velocity is defined as follows

\[
u = \left(\frac{1}{c}, 0, 0, 0\right), \quad u_i u^i = -1. \tag{4}
\]

The time derivatives of \( G \) and \( \Lambda \) are related by the Bianchi identities

\[
\left(R_{ij} - \frac{1}{2} R g_{ij}\right)^{;j} = \left(\frac{8\pi G}{c^4} T_{ij} - \Lambda g_{ij}\right)^{;j}, \tag{5}
\]

in this case this equation reads:

\[
\dot{\rho} + \rho (1 + \omega) \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right) = -\frac{8\pi G}{c^4} - \frac{\dot{G}}{G} \rho. \tag{6}
\]

Therefore the resulting field equations are:

\[
\begin{align*}
\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} &= \frac{8\pi G}{c^4} \rho + \Lambda c^2, \\
\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} &= -\frac{8\pi G}{c^4} \omega \rho + \Lambda c^2, \\
\frac{\ddot{X}}{X} + \frac{\ddot{Z}}{Z} &= -\frac{8\pi G}{c^4} \omega \rho + \Lambda c^2, \\
\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} &= -\frac{8\pi G}{c^4} \omega \rho + \Lambda c^2.
\end{align*} \tag{7}
\]

\[
\dot{\rho} + \rho (1 + \omega) \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right) = -\frac{8\pi G}{c^4} - \frac{\dot{G}}{G} \rho. \tag{11}
\]

Now, we define

\[
H = \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right) = 3\frac{\dot{R}}{R} \quad \text{and} \quad R^3 = X Y Z, \quad q = \frac{d}{dt} \left(\frac{1}{H}\right) - 1, \tag{12}
\]

Since we have defined the 4-velocity by eq. \(4\) then the expansion \( \theta \) is defined as follows:

\[
\theta := u^i_i, \quad \theta = \frac{1}{c} \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right) = \frac{1}{c} H, \tag{13}
\]

and therefore the acceleration is: \( a_i = u_{i;j} u^j \), in this case \( a = 0 \), while the shear is defined as follows: \( \sigma_{ij} = \frac{1}{2} \left(u_{i;\alpha} h^\alpha_j + u_{j;\alpha} h^\alpha_i - \frac{4}{3} \theta h_{ij}\right), \)

\[
\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}, \quad \sigma^2 = \frac{1}{3c^2} \left(\left(\frac{\dot{X}}{X}\right)^2 + \left(\frac{\dot{Y}}{Y}\right)^2 + \left(\frac{\dot{Z}}{Z}\right)^2 - \frac{\dot{X}}{X} \frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} \frac{\dot{Z}}{Z} - \frac{\dot{Y}}{Y} \frac{\dot{Z}}{Z}\right) \tag{14}
\]
3. Self-similar solution.

In general relativity, the term self-similarity can be used in two ways. One is for the properties of spacetimes, the other is for the properties of matter fields. These are not equivalent in general. The self-similarity in general relativity was defined for the first time by Cahill and Taub (see 35, and for general reviews 36-44). Self-similarity is defined by the existence of a homothetic vector $V$ in the spacetime, which satisfies

$$\mathcal{L}_V g_{ij} = 2\alpha g_{ij},$$

where $g_{ij}$ is the metric tensor, $\mathcal{L}_V$ denotes Lie differentiation along $V$ and $\alpha$ is a constant. This is a special type of conformal Killing vectors. This self-similarity is called homothety. If $\alpha \neq 0$, then it can be set to be unity by a constant rescaling of $V$. If $\alpha = 0$, i.e. $\mathcal{L}_V g_{ij} = 0$, then $V$ is a Killing vector.

Homothety is a purely geometric property of spacetime so that the physical quantity does not necessarily exhibit self-similarity such as $\mathcal{L}_V Z = dZ$, where $d$ is a constant and $Z$ is, for example, the pressure, the energy density and so on. From equation (15) it follows that

$$\mathcal{L}_V R_{ijkl} = 0,$$

and hence

$$\mathcal{L}_V R_{ij} = 0, \quad \mathcal{L}_V G_{ij} = 0.$$  

(17)

A vector field $V$ that satisfies the above equations is called a curvature collineation, a Ricci collineation and a matter collineation, respectively. It is noted that such equations do not necessarily mean that $V$ is a homothetic vector. We consider the Einstein equations

$$G_{ij} = 8\pi G T_{ij},$$

(18)

where $T_{ij}$ is the energy-momentum tensor.

If the spacetime is homothetic, the energy-momentum tensor of the matter fields must satisfy

$$\mathcal{L}_V T_{ij} = 0,$$

(19)

through equations (18) and (17). For a perfect fluid case, the energy-momentum tensor takes the form of eq. (3) i.e. $T_{ij} = (p + \rho)u_i u_j + pg_{ij}$, where $p$ and $\rho$ are the pressure and the energy density, respectively. Then, equations (15) and (19) result in

$$\mathcal{L}_V u^i = -\alpha u^i, \quad \mathcal{L}_V \rho = -2\alpha \rho, \quad \mathcal{L}_V p = -2\alpha p.$$  

(20)

As shown above, for a perfect fluid, the self-similarity of the spacetime and that of the physical quantity coincide. However, this fact does not necessarily hold for more general matter fields. Thus the self-similar variables can be determined from dimensional considerations in the case of homothety. Therefore, we can conclude homothety as the general relativistic analogue of complete similarity.
From the constraints (20), we can show that if we consider the barotropic equation of state, i.e., \( p = f(\rho) \), then the equation of state must have the form \( p = \omega \rho \), where \( \omega \) is a constant. This class of equations of state contains a stiff fluid \( (\omega = 1) \) as special cases, within this theoretical framework. There are many papers devoted to study Bianchi I models (in different context) assuming the hypothesis of self-similarity (see for example 46-47) but here, we would like to try to show how taking into account this class of hypothesis one is able to find exact solutions to the field equations within the framework of the time varying constants.

The homothetic equations are given by eq. (15) so it is a straightforward task to find the homothetic vector field, where in this case is as follows:

\[
V = t \partial_t + \left( 1 - t \frac{\dot{X}}{X} \right) x \partial_x + \left( 1 - t \frac{\dot{Y}}{Y} \right) y \partial_y + \left( 1 - t \frac{\dot{Z}}{Z} \right) z \partial_z,
\]

(21)

Therefore, we have obtained the following behavior for the scale factors:

\[
X = X_0 t^{\alpha_1}, \quad Y = Y_0 t^{\alpha_2}, \quad Z = Z_0 t^{\alpha_3},
\]

(22)

with \( X_0, Y_0, Z_0 \) are integrating constants and \( (\alpha_i)_{i=1}^3 \in \mathbb{R} \). In this way we find that

\[
H = \left( \sum_{i=1}^3 \alpha_i \right) \frac{1}{t} = \frac{\alpha}{t}, \quad q = \frac{d}{dt} \left( \frac{1}{H} \right) = \frac{1}{\alpha} - 1, \quad \sigma^2 = \frac{1}{3c^2} \left( \sum_{i}^{3} \alpha_i^2 - \sum_{i \neq j}^{3} \alpha_i \alpha_j \right) \frac{1}{t^2},
\]

(23)

with \( \alpha_1 + \alpha_2 + \alpha_3 = \alpha \).

In this section we are going to study several Bianchi I models and we will show how it is possible to find exact solutions to the field equations (without the condition \( \text{div}T = 0 \)) under the hypothesis of SS.

The time derivatives of \( G \) and \( \Lambda \) are related by the Bianchi identities i.e. eq. (11) that in this case collapses to the following one:

\[
\dot{\rho} + \rho (1 + \omega) H = f(t) = -\frac{\dot{G}}{8\pi G} - \frac{\dot{G}}{G^2} \rho,
\]

(24)

where \( f(t) \) is a function that depends on time and controls the time variation of the constant \( G \) or/and \( \Lambda \). If \( G = \text{const.} \) and \( \Lambda \) vanish then \( f(t) = 0 \), so the model collapses to the standard one. This idea was pointed out by Rastall (see 55) and improved (in the theoretical framework of time varying constants) by Harko and Mak (see 56).

Therefore the resulting field equations are (7-10) together to the new one

\[
\dot{\rho} + \rho (1 + \omega) H = f(t),
\]

(25)

3.1. “Constants” constants. The classical model.

In this case we consider \( f(t) = 0 \), so this means that \( G = \text{const.} \) and \( \Lambda \) vanish and therefore we get that from eq. (25) that

\[
\dot{\rho} + \rho (1 + \omega) H = 0, \quad \Rightarrow \quad \rho = \rho_0 t^{-(\omega+1)\alpha}.
\]

(26)
From the field equations (7) we get that

$$\rho_0 = \frac{A c^2}{8 \pi G}, \quad \alpha = \frac{2}{(\omega + 1)},$$

where \(A = \alpha_1 \alpha_2 + \alpha_3 \alpha_1 + \alpha_2 \alpha_3\).

The shear has the following behavior, \(\sigma^2 \neq 0\), as it is observed \(\sigma \rightarrow 0\) as \((\alpha_i \rightarrow \alpha_j)\). As in the previous sections, we may calculate the coefficients \((\alpha_i)\) by solving the following system of equations:

\[
\begin{align*}
\alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_2 &= -A \omega, \\
\alpha_1 (\alpha_1 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_1 &= -A \omega, \\
\alpha_2 (\alpha_2 - 1) + \alpha_1 (\alpha_1 - 1) + \alpha_1 \alpha_2 &= -A \omega, \\
\alpha (\omega + 1) &= 2,
\end{align*}
\]

where \(A = \alpha_1 \alpha_2 + \alpha_3 \alpha_1 + \alpha_2 \alpha_3\), and \(\alpha = \alpha_1 + \alpha_2 + \alpha_3\).

So we have the following solutions for this system of equations:

\[
\begin{align*}
\alpha_1 &= 1 - \alpha_2 - \alpha_3, \quad \omega = 1, \\
\alpha_1 &= \alpha_2 = \alpha_3 = \frac{2}{3 (\omega + 1)},
\end{align*}
\]

as it is observed only solution (32) is interesting for us. The second solution is the usual FRW one, so it is not interesting for us (see Einstein-de Sitter (50) for \(\omega = 0\), and Harrison (51) \(\forall \omega\)). Nevertheless we have found that solution (32) verifies the conditions

\[
\alpha = \sum \alpha_i = 1, \quad \sum \alpha_i^2 < 1,
\]

but iff \(\omega = 1\), (see [47]) while other authors claim that must be satisfies the condition \(\sum \alpha_i^2 = 1\), (see [48], [49], and [52]), and in particular, in this context (see [47]).

Nevertheless we have found that this solution only verifies the first of the condition of the Kasner like solutions i.e. \(\alpha = \sum \alpha_i = 1\), while the second condition \(\sum \alpha_i^2 = 1\), it is not verified (see [48] and [49]). In this case we find that it is verified the condition \(\sum \alpha_i^2 < 1\). Therefore we have found the same behavior as the obtained one in (46). Before ending we would like to make a little comment about the Kasner like solutions. If a solution of (28) verifies the relationships \(\sum \alpha_i^2 = \sum \alpha_i = 1\), i.e. they are Kasner’s type (see [48], [49], and in particular [57]), then this means that \(A = \alpha_1 \alpha_2 + \alpha_3 \alpha_1 + \alpha_2 \alpha_3 = 0\) (i.e. the model is Ricci flat), which brings us to get the following result: \(\alpha_1 = \frac{1}{2} (1 - \alpha_3 - \sqrt{1 + 2 \alpha_3 - 3 \alpha_3^2}) < 0, \alpha_2 = \frac{1}{2} (1 - \alpha_3 + \sqrt{1 + 2 \alpha_3 - 3 \alpha_3^2}) > 0, \forall \alpha_3 \in (0, 1)\), we think that this class of solutions are unphysical and have a pathological curvature behavior as it is shown below. Furthermore, as we can see, if \(A = 0\), then from eq. (27) we get \(\rho = 0\), as it is expected for this class of solutions (vacuum solutions) so they are not interested for us. Nevertheless relaxing the condition \(\sum \alpha_i^2 = 1\), to our result i.e. \(\sum \alpha_i^2 < 1\), we are able to obtain solutions with \((\alpha_i) > 0, \forall i\), and \(\rho \neq 0\).
Therefore we have obtained the following behavior for the main quantities:

\[ H = \frac{1}{t}, \quad \Rightarrow \quad q = 0, \]  

(35)

so it is quite difficult to reconcile this model with the observational data. With regard to the energy density we find that

\[ \rho = \frac{Ac^2}{8\pi G}t^{-2}, \quad \sigma^2 = \frac{1}{3c^2} (1 + 3A) \frac{1}{t^2}, \]  

(36)

and with regard to the constants \((\alpha_i)_{i=1}^3\) we have that only obtain a BI solution iff \(\alpha_1 = 1 - \alpha_2 - \alpha_3\), (where furthermore we suppose that \(\alpha_2 \neq \alpha_3\) and that this result only is possible if the equation of state is \(\omega = 1\), i.e. ultra-stiff matter (see 46). For a review of Bianchi I solutions see for example (53).

With regard to the curvature behavior, we may see that the full contraction of the Riemann tensor (see for example (40), (41)) \(I_1 := R_{ijkl}R^{ijkl}\), and the full contraction of the Ricci tensor, \(I_2 := R_{ij}R^{ij}\), are:

\[ I_1 = \frac{K}{c^4t^4}, \quad I_2 = \frac{4(-\alpha_2 - \alpha_3 + \alpha_2^2 + \alpha_2\alpha_3 + \alpha_3^2)^2}{c^4t^4}, \]  

(37)

where \(K = K(\alpha_i) = \text{const} \neq 0\), i.e.

\[ K = \left[ 3(\alpha_2^2 + \alpha_3^2) + 2\alpha_2\alpha_3 + 9\alpha_2^2\alpha_3^2 + 3(\alpha_2^2 + \alpha_3^2) - 6(\alpha_2^2 + \alpha_3^2) + \alpha_2\alpha_3(6(\alpha_2^2 + \alpha_3^2) - 8(\alpha_2 + \alpha_3)) \right]. \]  

(38)

and the scalar curvature \(R\) is: \(R = R_i^i\).

The non-zero components of the Weyl tensor are:

\[ C_{1212} = K_1 t^{-2(1-\alpha_1)}, \quad C_{1313} = K_2 t^{-2(1-\alpha_2)}, \quad C_{1414} = K_3 t^{-2(1-\alpha_3)}, \]

\[ C_{2323} = K_4 t^{-2\alpha_3}, \quad C_{2424} = K_5 t^{-2\alpha_2}, \quad C_{3434} = K_6 t^{-2\alpha_1}, \]  

(39)

where the numerical constants \((K_i)_{i=1}^6 = K(\alpha_i) = \text{const} \neq 0\). As we can see with the obtained solution for \((\alpha_i)_{i=1}^3\) the Weyl \(\rightarrow \infty\) as \(t \rightarrow 0\), in the next models we shall show that \(G(t)\) has the same behavior as the Weyl tensor. In a forthcoming paper we study models with \(\text{Weyl} \rightarrow 0\) as \(t \rightarrow 0\) (i.e. models that verify the Weyl tensor hypothesis) and with a growing \(G\), i.e. in same way exists a relationship between both quantities.

Now taking into account a very famous result by Hall et al (see 45) we may check that \(\mathcal{L}_V C_{ijkl}^\alpha = 0\), as it is shown in (45) if a vector field \(V \in \mathfrak{X}(M)\), verifies the conditions \(\mathcal{L}_V C_{ijkl} = 0\), and \(\mathcal{L}_V T_{ij} = 0\) (as it is known, if \(V\) is a homothetic vector field, then it is also a matter collineation), then \(\mathcal{L}_V g = 2g\) i.e. it is a homothetic vector field, but in this case we have arrived to the conclusion that \(\mathcal{L}_V g = 2g \iff \mathcal{L}_V T_{ij} = 0\), and that it is also verified the relationship \(\mathcal{L}_V C_{ijkl}^\alpha = 0\).

The Weyl scalar is defined as: \(I_3 := C_{abcd} C_{abcd}\), as it is observed \(I_3\), is also defined as follows: \(I_3 = I_1 - 2I_2 + \frac{1}{4}R^2\), this definition is only valid when \(n = 4\). Therefore, \(I_3\) has the following behavior

\[ I_3 = \frac{\tilde{K}}{c^4t^4}, \]  

(40)
with \( \dot{K} \) given by \( \dot{K} = \frac{4}{9} \left[ \alpha_2^2 + \alpha_3^2 - \alpha_2 \alpha_3 + 3 \alpha_2^2 \alpha_3^2 + \alpha_4^2 + \alpha_3^4 - 2 \left( \alpha_2^2 + \alpha_3^2 \right) + \alpha_2 \alpha_3 \left( 2 \left( \alpha_2^2 + \alpha_3^2 \right) - (\alpha_2 + \alpha_3) \right) \right] \).

The non-zero components of the electric part of the Weyl tensor are:

\[ E_{22} = \tilde{K}_1 t^{-2(1-\alpha_1)}, \quad E_{33} = \tilde{K}_2 t^{-2(1-\alpha_2)}, \quad E_{44} = \tilde{K}_3 t^{-2(1-\alpha_3)}, \]

while the magnetic part of the Weyl tensor vanish, \( H_{ij} = 0 \).

The gravitational entropy is defined as follows (see 31-32):

\[ P^2 = I_3^2 - I_2^2 - \frac{1}{3} R^2 = I_1 I_2 - \frac{1}{3} R^2 - 2, \]  

finding that

\[ P^2 = \text{const.} \neq 0, \]

note that \( P^2 = I_3/I_2 \). As have been pointed out by Nicos Pelavas et al (see 33) this definition is not an acceptable candidate for gravitational entropy along the homothetic trajectories of any self-similar spacetime. Nor indeed is any “dimensionless” scalar. This implies that \( I_3/I_2 \) is constant along timelike homothetic trajectories. As a consequence, \( I_3/I_2 \) does not provide a measure of gravitational entropy along homotheticities and therefore \( I_3/I_2 \) cannot be a candidate for a measure of gravitational entropy in self-similar spacetimes.

### 3.2. G-variable.

In this case we are going to consider that only vary “constant” \( G \). This only possible if we take into account the condition \( \text{div} T \neq 0 \) and therefore \( f(t) = -\frac{G'}{G} \rho \), so eq. (25) collapses to the following one.

\[ \frac{\rho'}{\rho} + \frac{G'}{G} = - (1 + \omega) \frac{\alpha}{t}, \quad \Rightarrow \quad \rho G = t^{-(1+\omega)\alpha}, \]  

From the field equations (7) we get that

\[ G\rho = \frac{c^2}{4 \pi} \frac{A}{\alpha (1 + \omega) t^2}, \quad \alpha = \frac{2}{(\omega + 1)}. \]  

The shear has the following behavior, \( \sigma^2 \neq 0 \), as it is observed \( \sigma \rightarrow 0 \) as \( (\alpha_i \rightarrow \alpha_j) \). As in the previous sections, to calculate the coefficients \( (\alpha_i) \) we need to solve the filed equations obtaining the same system of equations as in the above case i.e. eqs. (28-31), so we get the same solution as in the above case i.e. \( \sum_{i=1}^{3} \alpha_i = 1 \), and \( \sum_{i=1}^{3} \alpha_i^2 < 1 \), and only valid if \( \omega = 1 \).

Therefore we have obtained the following behavior for the main quantities:

\[ H = \frac{1}{t}, \quad \Rightarrow \quad q = 0, \]  

and

\[ G\rho = \frac{A c^2}{8 \pi t^2}, \quad \sigma^2 = \frac{1}{3 c^2} (1 - 3A) \frac{1}{t^2}. \]
note that this result is quite similar to the obtained one in the last solution i.e. the obtained one in eq. \((36)\), but we are not able to get a separate behavior for the quantities \(G\) and \(\rho\).

### 3.3. \(\Lambda\)–variable.

In this case we consider only the variation of the cosmological constant \(\Lambda\), so eq. \((25)\) yields
\[
\dot{\rho} + \rho (1 + \omega) \left( \frac{X}{X} + \frac{Y}{Y} + \frac{Z}{Z} \right) = -\frac{\dot{\Lambda} c^4}{8\pi G},
\]
and therefore from the field equations \((7)\) we get that
\[
\Lambda' = -\frac{A}{c^2} \frac{2 \rho}{c^4} - \frac{8\pi G}{c^4} \rho',
\]
and hence
\[
\rho = \frac{c^2 A}{4\pi G} \frac{1}{(1 + \omega) \alpha t^2}.
\]

Now, we next to calculate the quantity \(\Lambda\), from eq. \((49)\) we get
\[
\Lambda = \frac{A}{c^2} \left( 1 - \frac{2}{(1 + \omega) \alpha} \right) \frac{1}{t^2},
\]
in this way it is observed that
\[
\Lambda = \Lambda_0 t^{-2}, \quad \begin{cases} 
\Lambda_0 > 0 & \iff (\omega + 1)\alpha > 2 \\
\Lambda_0 = 0 & \iff (\omega + 1)\alpha = 2 \\
\Lambda_0 < 0 & \iff (\omega + 1)\alpha < 2 
\end{cases}
\]

The shear has the following behavior, \(\sigma^2 \neq 0\), by hypothesis. As in the previous sections, we calculate the coefficients \((\alpha_i)\) from following system of equations:
\[
\begin{align*}
\alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_2 &= A \left( \frac{\alpha - 2}{\alpha} \right), \\
\alpha_1 (\alpha_1 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_1 &= A \left( \frac{\alpha - 2}{\alpha} \right), \\
\alpha_2 (\alpha_2 - 1) + \alpha_1 (\alpha_1 - 1) + \alpha_1 \alpha_2 &= A \left( \frac{\alpha - 2}{\alpha} \right),
\end{align*}
\]
where \(A = \alpha_1 \alpha_2 + \alpha_3 \alpha_1 + \alpha_2 \alpha_3\), and \(\alpha = \alpha_1 + \alpha_2 + \alpha_3\).

So we have the following solutions for this system of equations:
\[
\begin{align*}
\alpha_1 &= \alpha_2 = \alpha_3, \\
\alpha_1 &= 1 - \alpha_2 - \alpha_3,
\end{align*}
\]
as it is observed solution \((56)\) is not interesting for us, since it is unphysical (in this context). Only the second solution has physical meaning and it is valid \(\forall \omega \in (-1, 1)\).
Therefore, we have found that the solution is \( \sum_{i=1}^{3} \alpha_i = 1 \), and \( \sum_{i=1}^{3} \alpha_i^2 < 1 \), but in occasion this solution is only valid if \( \forall \omega \in (-1, 1) \).

Therefore we have obtained the following behavior for the main quantities:

\[
H = \frac{1}{t}, \quad \implies \quad q = 0,
\]

while with regard to the energy density we find that

\[
\rho = \frac{c^2 A}{4\pi G (1 + \omega) t^2},
\]

so, if \( \omega < -1 \implies \rho \) is negative (phantom cosmologies), for the rest of the values of \( \omega \), i.e. \( \omega \in (-1, 1) \), \( \rho \) is a decreasing function on time.

The cosmological “constant” behaves as follows

\[
\Lambda = \Lambda_0 t^{-2}, \quad \Lambda_0 = \begin{cases} 
\Lambda_0 = 0 & \iff \omega = 1, \\
\Lambda_0 < 0, & \forall \omega \in (-1, 1)
\end{cases}
\]

so we have found that \( \Lambda \) is a “negative decreasing function” on time. Note that \( \Lambda_0 > 0 \) if \( \omega > 1 \). As we can see this solution is quite different of the previous ones, since here we have obtained a solution type Bianchi I \( \forall \omega \in (-1, 1) \) while in the previous ones this only happens if \( \omega = 1 \). Here if \( \omega = 1 \) then we regain the first of the studied cases i.e. which one where \( \Lambda \) vanish and \( G \) behaves as a true constant.

\subsection*{3.4. \( G & \Lambda \text{—variable.} \)}

In this case we are going to consider that both “constants” \( G \) and \( \Lambda \) vary, therefore eq. (25) yields

\[
\dot{\rho} + \rho (1 + \omega) H = -\frac{\dot{\Lambda} c^4}{8\pi G} - \frac{\dot{G}}{G} \rho,
\]

so from the field equations (7) and (11) and following the same steps as in the above models we get that

\[
G \rho = \frac{c^2 A}{4\pi (\omega + 1) t^2}.
\]

as we can see it is verified the relationship \( G \rho \approx t^{-2} \), as it is expected. In fact it is impossible to separate both functions (to get the behavior of both functions independently), to do that we need to impose a condition, but precisely we are trying to avoid such way.

Now taking into account again eq. (7) we get

\[
\Lambda = \Lambda_0 t^{-2}, \quad \Lambda_0 = \frac{A}{c^2} \left( 1 - \frac{2}{(\omega + 1)\alpha} \right),
\]

in this way it is observed that

\[
\Lambda = \Lambda_0 t^{-2}, \quad \begin{cases} 
\Lambda_0 > 0 & \iff (\omega + 1)\alpha > 2 \\
\Lambda_0 = 0 & \iff (\omega + 1)\alpha = 2 \\
\Lambda_0 < 0 & \iff (\omega + 1)\alpha < 2
\end{cases}
\]
The shear behaves (see eq. (14)) as follows: \( \sigma^2 \neq 0 \), by hypothesis. In order to find the value of constants \( (\alpha_i) \), we make them verify the field eqs. so in this case we get the following system of eqs.: 

\[
\begin{align*}
\alpha_2 (\alpha_2 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_2 &= A \left( \frac{\alpha - 2}{\alpha} \right), \\
\alpha_1 (\alpha_1 - 1) + \alpha_3 (\alpha_3 - 1) + \alpha_3 \alpha_1 &= A \left( \frac{\alpha - 2}{\alpha} \right), \\
\alpha_2 (\alpha_2 - 1) + \alpha_1 (\alpha_1 - 1) + \alpha_1 \alpha_2 &= A \left( \frac{\alpha - 2}{\alpha} \right),
\end{align*}
\]

(note that this system is the same as eqs. \([53-55]\) where \( A = \alpha_1 \alpha_2 + \alpha_3 \alpha_2 + \alpha_2 \alpha_3 \), and \( \alpha = \alpha_1 + \alpha_2 + \alpha_3 \). Therefore we obtain the same solution as in the last studied case, i.e. \( \sum_{i=1}^{3} \alpha_i = 1 \), and \( \sum_{i=1}^{3} \alpha_i^2 < 1 \), with \( \omega \in (-1, 1) \), finding in this way the following behavior for the main quantities:

\[
H = \frac{1}{t}, \quad \implies \quad q = 0,
\]

and with regard to the product \( G\rho \) we get

\[
G\rho = \frac{c^2}{4\pi} \frac{A}{(\omega + 1)} t^{-2},
\]

but we cannot say anything more. The cosmological “constant” behaves as follows

\[
\Lambda = \Lambda_0 t^{-2}, \quad \Lambda_0 = \begin{cases} 
\Lambda_0 = 0 & \iff \omega = 1 \\
\Lambda_0 < 0, & \forall \omega \in (-1, 1)
\end{cases}
\]

so we have found that \( \Lambda \) is a negative decreasing function on time. As in the above case we get a positive cosmological “constant” if \( \omega > 1 \).

In order to try to find a separate behavior for the functions \( \rho \) and \( G \), we may suppose that

\[
\rho = \rho_0 t^{-a}, \quad G = G_0 t^{a-2}, \quad \implies \quad G\rho = \frac{c^2}{4\pi} \frac{A}{(\omega + 1)} t^{-2} = K t^{-2},
\]

with \( a \in \mathbb{R}^+ \), i.e. for example we may choose

\[
G = \frac{c^2}{4\pi \rho_0} \frac{A}{(\omega + 1)} t^{a-2} = \frac{K}{\rho_0} t^{a-2},
\]

therefore, it is verified the field eq. \([61]\) for all the possible values of \( a \). We may find other possibilities as for example

\[
\rho = \rho_0 t^{-2+a}, \quad G = G_0 t^{-a}
\]

with \( a \in (-\infty, 2) \). At this point we would like to stress the relationship between the behavior of the Weyl tensor and the behavior of \( G(t) \), since \( \text{Weyl} \to \infty \) as \( t \to 0 \) in the same way as \( G(t) \).
But if \( a = (\omega + 1) \) then we regain the condition \( \text{div} \, T = 0 \) as well as \( f(t) = 0 \), i.e.

\[
\text{div} \, T = \dot{\rho} + \rho (1 + \omega) H = 0 = \frac{\dot{\Lambda}}{8\pi G} - \frac{\dot{G}}{G} \rho = f(t).
\]

(74)

but this case has been already studied in (19).

4. Conclusions.

We have shown how to attack a perfect fluid Bianchi I with \( G \) and \( \Lambda \) variable under the condition \( \text{div} \, T \neq 0 \), taking account only the hypothesis of SS. Our arguments exploit the symmetry properties of homothetically self-similar spacetimes. These calculations are of physical interest since self-similar spacetimes are very widely studied (42) and are, for example, believed to play an important role in describing the asymptotic properties of more general models. In this way we have shown that it is not necessary to make any assumption “ad hoc” or to take into account any previous hypothesis or considering any hypothetical behavior for any quantity since all these hypotheses could be deduced from the symmetry principles, as for example the self-similar hypothesis SS (for other approaches see 19). As have seen to get the solution under the SS hypothesis it very simple since once we have calculate the homothetic vector field it is a trivial task to obtain the behavior of the scale factors, where the must follow a power law solution, \( X = X_0 t^{\alpha_1}, Y = Y_0 t^{\alpha_2}, Y = Y_0 t^{\alpha_3} \), in this way we only need to calculate the value of these exponents i.e. \( (\alpha_i)_{i=1}^{3} \), since the rest of the quantities will depend on these values. We have started studying the simplest case where \( G \) is a true constant and \( \Lambda \) vanish, i.e. the classical model in order to check how works the purposed method. For this model we arrive to the already known result \( \sum \alpha_i = 1 \), and \( \sum \alpha_i^2 < 1 \), stressing that this result is only valid if the equation of state verifies the relationship \( \omega = 1 \), i.e. the result is only valid for ultra-stiff matter, otherwise the model collapses to the FRW solution. We have discussed why it is not possible to get the vacuum (\( \rho = 0 \)) Kasner solution, \( \sum \alpha_i = 1 = \sum \alpha_i^2 = 1 \). We think that this class of solutions are unphysical since necessarily one of the scale factors must be a decreasing time function (maybe such class of solutions would have any interest in the study of singularities). At the same time, we have studied the curvature invariants, \( I_1 \) and \( I_2 \), i.e. the Kretschmann’s scalars, showing that the obtained solution is singular as well as the Weyl tensor and its scalar, \( I_3 \). We have performed all these calculations in order to show that there is a relationship between the behavior of the Weyl tensor and the behavior of the variable Newton constant \( G(t) \). All these considerations are valid for the rest of the studied models. We also have calculated the gravitational entropy, \( P^2 = I_3 / I_2 \), showing that this definition is not valid for self-similar spacetimes since this quantity is dimensionless and therefore remains constant. Actually we have arrived to the same conclusion as the obtained one by Nicos Pelavas et al (see 33). Furthermore if one gets the Kasner solution then \( I_2 = 0 \), i.e. the model collapses to a Ricci flat model and therefore \( P^2 = \infty \).
With regard to the second of the studied models, where only vary \( G(t) \), we have shown that it is not possible to get a separate behavior of the “constants” \( G \) and \( \rho \), obtaining \( G\rho \sim t^{-2} \), as it is expected. For this model we have obtained the same solution as in the previous case i.e. that the exponents of the scale factors must satisfy the relationships \( \sum \alpha_i = 1 \), and \( \sum \alpha_i^2 < 1 \), and only valid for \( \omega = 1 \). As we have shown in the last of the studied models in order to get a separate behavior between \( G \) and \( \rho \), it is possible to follow several ways but in this paper we are trying to do the slightest number of hypotheses and in any case to avoid to make previous assumptions on the behavior of any quantity.

In the third of the studied models, where only vary the cosmological constant \( \Lambda \), we have found that \( \Lambda \) is a negative decreasing function on time \( \forall \omega \in (-1, 1) \) and a positive time decreasing function if \( \omega > 1 \). If \( \omega = 1 \) then \( \Lambda \) vanish, so the exponents verify the same relationship as above but in this case this solution is only valid if \( \omega \in (-1, 1) \) since if \( \omega = 1 \) then we get the first of the studied models.

In the fourth model, we have considered that both constant vary. In this case we arrive to similar conclusions as in the above cases, i.e. \( \Lambda \) is a negative decreasing function on time \( \forall \omega \in (-1, 1) \), vanish if \( \omega = 1 \) and it is a positive decreasing time function if \( \omega > 1 \) while \( G\rho \sim t^{-2} \) and the exponents of the scale factors must satisfy the relationships \( \sum \alpha_i = 1 \), and \( \sum \alpha_i^2 < 1 \), valid \( \forall \omega \in (-1, 1) \). In this occasion we have made an assumption on the behavior of \( G \) in of to try to know its behavior finding in this way that is a decreasing time function on time. It is quite surprising result since in similar models with FRW symmetries this quantity is always growing. At the same time we have shown the similitude between its behavior and the behavior of the Weyl tensor. Both quantities tend to infinite as \( t \) runs to zero. We have finished howing how to regain, in a trivial way, the condition \( \text{div} T = 0 \).

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