SELF-DUAL AND LOGARITHMIC REPRESENTATIONS OF THE
TWISTED HEISENBERG–VIRASORO ALGEBRA AT LEVEL ZERO

DRAŽEN ADAMOVIĆ AND GORDAN RADOBOLJA

Abstract. This paper is a continuation of [2]. We present certain new applications and
generalizations of the free field realization of the twisted Heisenberg–Virasoro algebra \( \mathcal{H} \) at
level zero. We find explicit formulas for singular vectors in certain Verma modules. A free
field realization of self-dual modules for \( \mathcal{H} \) is presented by combining a bosonic construction
of Whittaker modules from [7] with a construction of logarithmic modules for vertex algebras.
As an application, we prove that there exists a non-split self-extension of irreducible self-
dual module which is a logarithmic module of rank two. We construct a large family of
logarithmic modules containing different types of highest weight modules as subquotients.
We believe that these logarithmic modules are related with projective covers of irreducible
modules in a suitable category of \( \mathcal{H} \)-modules.

1. Introduction

The twisted Heisenberg–Virasoro Lie algebra \( \mathcal{H} \) is an important example of a Lie algebra
whose associated vertex algebra has many applications in the representation theory and
conformal field theory. If the level of the corresponding Heisenberg vertex subalgebra is
non-zero, the Heisenberg–Virasoro vertex algebra is isomorphic to the tensor product of
Heisenberg vertex algebra and the (universal or simple) Virasoro vertex algebra (cf. [1],[14]).
The study of the twisted Heisenberg–Virasoro algebra at level zero was initiated by Y. Billig
[12] motivated by applications to the toroidal Lie algebras. New results on the representation
theory were obtained in recent papers [2], [19], [24].

In this paper we continue our study of free field realization of the twisted Heisenberg–
Virasoro algebra from [2]. Our main motivation is to present free field realization of self-dual
modules and certain Verma modules which we were unable to construct using methods from
[2].

Let \( \mathcal{H} \) be the twisted Heisenberg–Virasoro algebra at level zero. Let \( V^\mathcal{H}(h, h_I) \) (resp.
\( L^\mathcal{H}(h, h_I) \)) denote the Verma module (resp. the irreducible highest weight module) with
highest weight \((c_L, c_I, c_{L,I}, h, h_I)\) such that \( c_I = 0, c_{L,I} \neq 0 \) is fixed but arbitrary, and with
highest weight vector \( v_{h,h_I} \). Then \( V^\mathcal{H}(h, h_I) \) is reducible if and only if \( h_I/c_{L,I} - 1 \in \mathbb{Z} \setminus \{0\} \)
(cf. [12]).

A free field realization of irreducible highest weight modules for the twisted Heisenberg–
Virasoro algebra at level zero was presented by the authors in [2]. Using free field realization
we calculated the fusion rules for non-generic irreducible modules i.e., for those irreducible
modules which are not isomorphic to a Verma \( \mathcal{H} \)-module. In our approach, the screening

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operator $Q$ introduced in [2, Section 2] has played an important role. In particular, the singular vector in $V^{H}(h, h_I)$ in the case
\[ h_I/c_{L,I} - 1 = p \in \mathbb{Z}_{>0} \]
is expressed as $Qv_{h+p,h_I} = S_p(c) v_{h,h_I}$ where $S_p(c)$ is $p^{th}$–Schur polynomial in variables $(c(-1), c(-2), \cdots)$ defined by (4.16) where $c = -I/c_{L,I}$:
\[ S_p(c) = S_p(c(-1), \ldots, c(-p)) = S_p \left( \frac{-1}{c_{L,I}} I(-1), \ldots, \frac{-1}{c_{L,I}} I(-p) \right) \]
(see (2.7) and [2, Theorem 4.3]).

In the present paper we found a similar approach for the singular vector in the case
\[ h_I/c_{L,I} - 1 = -p \ (p \in \mathbb{Z}_{>0}) \]
(1.1)
The formula is
\[ \sum_{i=1}^{p} (L(-i)S_{p-i}(-c)) v_{h,h_I} + \left( h + \frac{c_L - 2}{24}(p - 1) \right) S_p(-c)v_{h,h_I} \]
\[ - \frac{c_L - 26}{24} \left( \sum_{i=1}^{p} (i - 1)c(-i)S_{p-i}(-c) \right) v_{h,h_I}. \]
(1.2)

Verma modules $V^{H}(h, h_I)$ in the case (1.1) are explicitly constructed in Section 5 by using certain deformation studied previously in [4]. Singular vector (1.2) admits a nice interpretation as an element $e^{\frac{c}{2}L^2 + \frac{c}{2}r + \frac{c}{2}c}$ of the group algebra in the explicit lattice realization (cf. Theorem 5.3).

It turned out that the methods from [2] do not provide a free field realization of self-dual modules $L^{H}(h, h_I)$ such that $h \neq \frac{c_{L} - 2}{24}, h_I = c_{L,I}$. On the other hand, the same paper contains certain results on vanishing of the fusion rules in the category of modules which contains self-dual module $L^{H}(h, h_I)$ as above (see [2, Theorem 5.4(3), Remark 6.5]). In order to understand these fusion rules properties from the vertex–algebraic point of view, one needs to find a bosonic realization of self-dual modules. We shall see that the realization includes both the concepts of Whittaker and logarithmic modules for vertex algebras. We shall prove that the Whittaker module $\Pi_{\lambda}$ for the vertex algebra $\Pi(0)$ introduced in [7], after certain logarithmic deformation (cf. Theorem 5.2) becomes a highly reducible $\mathcal{H}$–module $\tilde{\Pi}_{\lambda}$ which contains $L^{H}(h, h_I)$ as a submodule. Since $\tilde{\Pi}_{\lambda}$ is not a module for the Heisenberg vertex algebra $M(1)$, it is clear that such module could not have appeared in the fusion rules analysis made in [2].

In Section 2 we recall from [2] a free field realization of $\mathcal{H}$, the definition of vertex algebra $\Pi(0)$ and its modules $\Pi(p, r)$. We study an extension of the Heisenberg–Virasoro vertex algebra $\widetilde{\Pi}(0) \subset \Pi(0)$ and present a structure of $\widetilde{\Pi}(0)$–modules $\Pi(p, r)$ in Section 3. By using certain relation in $\Pi(0)$–modules we recover formula (1.2) in Section 4. Then we consider a deformed action of $\mathcal{H}$ on these modules and obtain a family of modules $\widetilde{\Pi}(p, r)$ [cf. Theorem 5.2 Theorem 5.5 with the following properties:

- $\widetilde{\Pi}(p, r)$ is a logarithmic $\mathcal{H}$–module with the following action of the element $\tilde{L}(0)$ of the Virasoro algebra:
\[\mathbb{C}[\tilde{L}(0)]v\text{ is finite-dimensional for every } v \in \Pi(p, r), p \geq 1,\]
\[\mathbb{C}[\tilde{L}(0)]v\text{ is infinite-dimensional for every } v \in \Pi(p, r), p \leq 0.\]

- For \(p \geq 1\), \(\tilde{\Pi}(p, r)\) admits a filtration
  \[\tilde{\Pi}(p, r) \cong \bigcup_{m \geq 0} Z(m),\]
  such that \(Z(m)\) is a logarithmic \(\mathcal{H}\)-module of rank \(m + 1\) and \(Z(m)/Z(m-1)\) is a weight \(\mathcal{H}\)-module. [cf. Theorem 5.4]

In Section 6 we consider a deformed action on Whittaker \(\Pi(0)\)-modules and obtain a realization of self-dual \(\mathcal{H}\)-modules \(L^\mathcal{H}(h, c_{L,I})\), \(h \neq c_{L} - \frac{2}{24}\) [cf. Theorem 6.3]

Finally, we find new applications of our results on the vertex algebra associated to the \(W(2,2)\)-algebra. We present a non-local bosonic formula (7.26) for the screening operator introduced by the authors in [3]. We hope that our new expression for screening operator can be applied to construction of a quantum group which would play the role of Kazhdan–Lusztig dual of the vertex algebra \(W(2,2)\).

We also discuss a connection of our approach with a realization of the BMS\(_3\)-algebra obtained in [11] in the case \(c_L = 26\).

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2. Results from [2]

In this section we recall from [2] the free field realization of the twisted Heisenberg–Virasoro algebra, the construction of vertex algebra \(\Pi(0)\) and their modules \(\Pi(p, r)\). The definition of the lattice is slightly changed, but the action of the generators of the Heisenberg–Virasoro algebra is the same as in [2]. Main results of Section 2 of [2] are stated in Proposition 2.1.

We also present new, explicit formulas for sub–singular vectors introduced in [2, Proposition 2.7].

Recall that the twisted Heisenberg–Virasoro algebra is an infinite–dimensional complex Lie algebra \(\mathcal{H}\) with basis

\[\{L(n), I(n) : n \in \mathbb{Z}\} \cup \{C_L, C_{LI}, C_I\}\]

and commutation relation:

\[(2.3) \quad [L(n), L(m)] = (n - m)L(n + m) + \delta_{n,-m}\frac{n^3 - n}{12}C_L,\]
\[(2.4) \quad [L(n), I(m)] = -mI(n + m) - \delta_{n,-m}(n^2 + n)C_{LI},\]
\[(2.5) \quad [I(n), I(m)] = n\delta_{n,-m}C_I,\]
\[(2.6) \quad [\mathcal{H}, C_L] = [\mathcal{H}, C_{LI}] = [\mathcal{H}, C_I] = 0.\]

Let \(V^\mathcal{H}(c_L, c_I, c_{L,I}, h, h_I)\) denote the Verma module with highest weight \((c_L, c_I, c_{L,I}, h, h_I)\), and \(L^\mathcal{H}(c_L, c_I, c_{L,I}, h, h_I)\) its irreducible quotient (cf. [12]). In this paper we consider the case
\( c_I = 0 \). For simplicity we shall denote the Verma module \( V^\mathcal{H}(c_L,0,c_L,I,h,h_I) \) with \( V^\mathcal{H}(h,h_I) \) and its irreducible quotient with \( L^\mathcal{H}(h,h_I) \).

Define the following hyperbolic lattice \( H_{yp} = \mathbb{Z}x + \mathbb{Z}y \) such that
\[
\langle x,x \rangle = \langle y,y \rangle = 0, \quad \langle x,y \rangle = 1.
\]

Let \( \mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} H_{yp} \) and extend the form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h} \). We can consider \( \mathfrak{h} \) as an abelian Lie algebra. Let \( \hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}K \) be the affinization of \( \mathfrak{h} \). Let \( \gamma \in \mathfrak{h} \) and consider \( \hat{\mathfrak{h}} \)-module
\[
M(1, \gamma) := U(\hat{\mathfrak{h}}) \otimes U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}K) \mathbb{C}
\]
where \( \mathbb{C}[t] \otimes \mathfrak{h} \) acts trivially on \( \mathbb{C} \), \( \hat{\mathfrak{h}} \) acts as \( \langle \delta, \gamma \rangle \) for \( \delta \in \mathfrak{h} \) and \( K \) acts as 1. We shall denote the highest weight vector in \( M(1, \gamma) \) by \( e^\gamma \).

We shall write \( M(1) \) for \( M(1,0) \). For \( h \in \mathfrak{h} \) and \( n \in \mathbb{Z} \) we write \( h(n) \) for \( t^n \otimes h \). Set \( h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1} \). Then \( M(1) \) is a vertex algebra which is generated by the fields \( h(z), h \in \mathfrak{h} \) (cf. [20]). Moreover, \( M(1, \gamma) \) for \( \gamma \in \mathfrak{h} \), are irreducible \( M(1) \)-modules.

Let \( V_{H_{yp}} = M(1) \otimes \mathbb{C}[H_{yp}] \) be the vertex algebra associated to the lattice \( H_{yp} \), where \( \mathbb{C}[H_{yp}] \) is the group algebra of \( H_{yp} \).

In this paper we shall consider the lattice \( L = \mathbb{Z}c + \mathbb{Z}d \) such that \( c = x \) and \( d = 2y \). Then \( V_L \) is a vertex subalgebra of \( V_{H_{yp}} \).

Define the Heisenberg and the Virasoro vector:
\[
I = -c_L,I c(-1) \quad \text{(2.7)}
\]
\[
\omega = \frac{1}{2} c(-1) d(-1) + \frac{c_L - 2}{24} c(-2) - \frac{1}{2} d(-2) \quad \text{(2.8)}
\]

Then the components of the fields
\[
I(z) = Y(I, z) = \sum_{n \in \mathbb{Z}} I(n) z^{-n-1}, \quad L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}
\]
satisfy the commutation relations for the twisted Heisenberg–Virasoro Lie algebra \( \mathcal{H} \) and \( I \) and \( \omega \) generate the simple Heisenberg–Virasoro vertex algebra \( L^\mathcal{H}(c_L, 0, c_L, I, 0, 0) \) which we shall denote by \( L^\mathcal{H}(c_L, c_L, I) \).

We have:
\[
[L(n), c(m)] = -mc(n + m) + (m^2 - m) \delta_{m+n,0} \quad \text{(2.9)}
\]
\[
[L(n), d(m)] = -md(n + m) - \frac{c_L - 2}{12} (m^2 - m) \delta_{m+n,0} \quad \text{(2.10)}
\]

Let \( u = e^c \). Then by [2] Lemma 2.4
\[
Q = \text{Res}_z Y(u, z) = u_0.
\]
is a screening operator. This means that
\[
[Q, I(n)] = [Q, L(n)] = 0 \quad \forall n \in \mathbb{Z},
\]
i.e., \( Q \) commutes with the action of the Heisenberg-Virasoro algebra.

Recall that screening operators provide an important tool for construction of singular vectors in free-field realizations (cf. [26]).

One can show that
\[
\Pi(0) = M(1) \otimes \mathbb{C}[Zc]
\]
is a simple vertex algebra (cf. [13]). Let \( Y(\cdot, z) \) be the associated vertex operator. The vertex operator \( Y(u, z) \) and the screening operator \( Q \) are well-defined on every \( \Pi(0) \)-module.

Let

\[
d^1 = d + \frac{cL - 26}{12}c, \quad d^2 = d - \frac{cL - 26}{12}c.
\]

Consider the following irreducible \( \Pi(0) \)-modules

\[
\Pi(p, r) := \Pi(0) \cdot e^{\frac{p-1}{2}d^2 + \frac{p-r}{c}} \quad \text{where} \quad p \in \mathbb{Z}, r \in \mathbb{C}.
\]

(The irreducibility of \( \Pi(p, r) \) was also proved in [13].)

Let

\[
h_{p,r} = (1 - p^2)\frac{cL - 26}{24} + 1 - p + (1 - r)\frac{p}{2}.
\]

(2.11) \[ W_{p,r} = U(\mathcal{H}) \cdot e^{\frac{p-1}{2}d^2 + \frac{p-r}{c}} \subseteq \Pi(p, r). \]

Set \( v_{p,r} = e^{\frac{p-1}{2}d^2 + \frac{p-r}{c}}. \) Note that \( h_{p,r+2} = h_{p,r} - p. \)

The following result is proved in Propositions 2.5 and 2.7 of [2].

**Proposition 2.1.** We have:

1. \( W_{p,r} \cong V^H(h_{p,r}, (1 - p)c_{L,1}) \) if and only if \( p \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}. \)
2. \( W_{p,r} \cong L^H(h_{p,r}, (1 - p)c_{L,1}) \) if \( p \in \mathbb{Z}_{\geq 1}. \)
3. For every \( r \in \mathbb{C}, W_{0,r} \cong L^H(\frac{\sqrt{-1}}{24}, c_{L,1}). \)

The following modules were not constructed in [2]:

- Reducible Verma modules \( V^H(h_{p,r}, (1 - p)c_{L,1}) \) in the case \( p \in \mathbb{Z}_{\geq 1}. \)
- Self-dual modules \( L^H(h, c_{L,1}) \) for \( h \neq \frac{\sqrt{-1}}{24}. \)

We shall present the construction of these modules in Sections 5 and 6.

We shall also need the following result.

**Lemma 2.2.** Assume that \( p \in \mathbb{Z}_{>0}. \) As an \( \mathcal{H} \)-module, \( \Pi(p, r) \) is generated by a family of \( \mathcal{H} \)-singular vectors \( \{v_{p,r-2\ell} \mid \ell \in \mathbb{Z}\} \) and a family of \( \mathcal{H} \)-cosingular vectors \( \{v_{p,r-2\ell}^{(m)} \mid \ell, m \in \mathbb{Z}, m \geq 1\}, \)
where

\[
v_{p,r-2\ell}^{(m)} = (-1)^m \frac{d(-p)^m}{2^m m!} v_{p,r-2\ell}
\]

Proof. By [2] Proposition 2.7] we have that \( \Pi(p, r) \) is generated as \( \mathcal{H} \)-module by a family of singular vectors \( \{v_{p,r-2\ell} \mid \ell \in \mathbb{Z}\} \) and by a family of cosingular vectors \( \{w_{p,r-2\ell}^{(m)} \mid \ell, m \in \mathbb{Z}, m \geq 1\} \) satisfying

(2.12) \[ Q^m w_{p,r-2\ell}^{(m)} = v_{p,r-2(\ell + m)}. \]

Let us prove that for cosingular vectors \( w_{p,r-2\ell}^{(m)} \) we may choose \( v_{p,r-2\ell}^{(m)} \).

Recall the commutator relations in \( \Pi(0) \):

\[
[d(-k), e^c_j] = 2e^c_{j-k}, \quad [e^c_j, c(-k)] = 0, \quad [e^c_0, e^c_j] = 0.
\]

By induction on \( m \) we see that \( e^c_{k} v_{p,r-2\ell}^{(m)} \) is spanned by \( S_{(k+i-1)p}(c)v_{p,r-2(\ell + 1)}^{(m-i)} \), \( i = k - 1, \ldots, m \) for each \( k \in \mathbb{Z}_{\geq 0}. \) Therefore we have

(2.13) \[ Q^j v_{p,r-2\ell}^{(m)} = \begin{cases} v_{p,r-2(\ell + j)}^{(m-j)} \mod \text{Ker} \Pi(p, r) Q^m, & j < m, \\ v_{p,r-2(\ell + m)}, & j = m. \end{cases} \]
The proof follows. \(\square\)

See Figure [\ref{fig:structure}] in the Appendix [A] for structure of \(\Pi(0)\)-module \(\Pi(p,r)\).

Remark 2.3. Note that \(\Pi(p,r) = \Pi(p,s)\) if and only if \(r-s \in 2\mathbb{Z}\). Moreover, \(\Pi(1,1) = \Pi(0)\) and \(U(\mathcal{H})v_{1,1} \cong L^\mathcal{H}(c_L,c_{L,I})\).

3. An extension of the vertex algebra \(L^\mathcal{H}(c_L,c_{L,I})\)

An extension of the Heisenberg–Virasoro vertex algebra Now we study modules \(\Pi(p,r)\) in more details. There is a vertex subalgebra of \(\Pi(0)\) which can be treated as an extension of the vertex algebra \(L^\mathcal{H}(c_L,c_{L,I})\): \(\Pi(0) = \ker_{\Pi(0)} Q\).

In this section we obtain filtrations of \(\Pi(p,r)\) by \(\Pi(0)\)-modules such that the subquotients are irreducible over \(\Pi(0)\).

Proposition 3.1. Let \(\Pi(0)^{(n)} = \ker_{\Pi(0)} Q^{n+1}\). Then we have

1. \(\Pi(0)\)

\(\Pi(0)\) is a vertex subalgebra of \(\Pi(0)\) which is as \(L^\mathcal{H}(c_L,c_{L,I})\)-module isomorphic to \(\Pi(0) = \bigoplus_{n \in \mathbb{Z}} W_{1,1-2n} \cong \bigoplus_{n \in \mathbb{Z}} L^\mathcal{H}(n,0)\).

2. For every \(n \in \mathbb{Z}_{\geq 0}\), \(\Pi(0)^{(n)}\) and \(\Pi(0)^{(n+1)}/\Pi(0)^{(n)}\) are \(\Pi(0)\)-modules. Moreover we have \(\Pi(0) = \bigcup_{n \geq 0} \Pi(0)^{(n)}\), \(\Pi(0)^{(n)} \cdot \Pi(0)^{(m)} \subset \Pi(0)^{(n+m)}\).

Proof. Since \(Q\) is a screening operator, \(\Pi(0) = \ker_{\Pi(0)} Q\) is a vertex subalgebra of \(\Pi(0)\). By using [2 Proposition 2.7] we get

\[\Pi(0) = \bigoplus_{n \in \mathbb{Z}} L^\mathcal{H}(c_L,c_{L,I}).e^{nc} = \bigoplus_{n \in \mathbb{Z}} W_{1,1-2n} \cong \bigoplus_{n \in \mathbb{Z}} L^\mathcal{H}(n,0)\.

The proof of assertion (2) is clear. \(\square\)

Condition (2) from Proposition 3.1 shows that \(\mathcal{H}\)-modules (and \(\Pi(0)\)-modules) \(\Pi(0)^{(n)}\) give a \(\mathbb{Z}_{\geq 0}\) filtration on the vertex algebra \(\Pi(0)\). In the same way we can construct a filtration on certain \(\Pi(0)\)-modules.

Theorem 3.2. Assume that \(p \in \mathbb{Z}_{>0}\). Let \(\Pi(p,r)^{(m)} = \ker_{\Pi(p,r)} Q^{m+1}\). Then we have

1. \(\Pi(p,r)^{(m)} \cong \bigcup_{m \geq 0} \Pi(p,r)^{(m)}\), \(\Pi(0)^{(n)} \cdot \Pi(p,r)^{(m)} \subset \Pi(p,r)^{(n+m)}\).

2. For every \(m \in \mathbb{Z}_{\geq 0}\) \(\Pi(p,r)^{(m)}\) is \(\Pi(0)\)-module and \(\Pi(p,r)^{(m)} \subset \Pi(p,r)^{(m+1)}\).

3. \(\Pi(p,r)^{(m+1)}/\Pi(p,r)^{(m)}\) is an irreducible \(\Pi(0)\)-module which is as \(\mathcal{H}\)-module isomorphic to

\[\bigoplus_{n \in \mathbb{Z}} W_{p,r-2n} \cong \bigoplus_{n \in \mathbb{Z}} L^\mathcal{H}(h_{p,r} + np, (1-p)c_{L,I})\.

(3.14)
Proof. The proof of assertions (1) and (2) is clear. The decomposition (3.14) essentially follows from [2, Proposition 2.7]. Let us prove the irreducibility result in (3). It suffices to prove that

$W_{1,1-2n} \cdot W_{p,r-2\ell} = W_{p,1-2(n+\ell)}$.

Recall that

$W_{1,1-2n} = U(\mathcal{H}) \cdot e^{nc}, \quad W_{p,r-2\ell} = U(\mathcal{H}) \cdot u_{p,r-2\ell} \mod \Pi(p, r)^{(m)}$,

where $u_{p,r-2\ell}$ is an $\mathcal{H}$–cosingular vector $v_{p,r-2\ell}^{(m+1)}$ such that $Q^{m+1}u_{p,r-2\ell} = v_{p,r-2\ell}$.

Since $e_{k_0}^{nc}v_{p,r-2\ell} = v_{p,r-2(\ell+n)}$ for $k_0 = -n(p - 1) - 1$ we get that

$Q^{m+1}e_{k_0}^{nc}u_{p,r-2\ell} = v_{p,r-2(\ell+n)} \neq 0$.

So $e_{k_0}^{nc}u_{p,r-2\ell} + \Pi(p, r)^{(m)}$ generates $W_{p,1-2(n+\ell)}$ in $\Pi(p, r)^{(m+1)} / \Pi(p, r)^{(m)}$. The proof follows.

Figure 1 in Appendix A represents a portion of module $\Pi(p, r)$ with action of $Q$ and $e^{L}_{-p}$ on (sub)singular generators obtained in Lemma 2.2. Quotient module $\Pi(p, r)^{(m+1)} / \Pi(p, r)^{(m)}$ is a direct sum of ”slices”, each generated by $v_{p,r-2\ell}^{(m+1)} + \Pi(p, r)^{(m)}$ and isomorphic to $W_{p,1-2(\ell+m+1)}$.

Let us now consider modules $\Pi(-p, r)$. Recall that $h_{-p,r+2} = h_{-p,r} + p$.

**Theorem 3.3.**

(1) As an $\mathcal{H}$–module $\Pi(0, r)$ is isomorphic to

$$\bigoplus_{n \in \mathbb{Z}} W_{0,r} \cong \bigoplus_{n \in \mathbb{Z}} L^{\mathcal{H}} \left( \frac{c_L - 2}{24} , c_{L,I} \right).$$

(2) Let $p \in \mathbb{Z}_{>0}$. Consider $Q$ as $\Pi(0)$–endomorphism of $\Pi(-p, r)$, and let $\Pi(-p, r)^{(m)} = \text{Im } Q^m$. Then for $m \in \mathbb{Z}_{>0}$ we have

(a) $\Pi(-p, r) \cong \Pi(-p, r)^{(m)}$ as $\Pi(0)$–modules,

(b) $\Pi(-p, r)^{(m)} / \Pi(-p, r)^{(m+1)}$ is an irreducible $\Pi(0)$–module which is as an $\mathcal{H}$–module isomorphic to

$$\bigoplus_{\ell \in \mathbb{Z}} L^{\mathcal{H}}(h_{-p,r} + \ell p, (1 + p)c_{L,I}).$$

**Proof.** It was shown in [2] that $\Pi(p, r) \cong \bigoplus_{\ell \in \mathbb{Z}} W_{p,r+2\ell}$ as $\mathcal{H}$–module when $p \notin \mathbb{Z}_{>0}$. Decomposition in (1) then follows from Proposition 2.1 (3). Since

$$Q v_{-p,r} = S_p(c) v_{-p,r-2}$$

we see that $Q(W_{-p,r+2\ell}) \subset W_{-p,r+2(\ell-1)}$ and since $Q$ commutes with the action of $\mathcal{H}$ we have $\text{Ker } Q = 0$. Therefore $\Pi(-p, r) \cong \text{Im } Q = \Pi(-p, r)^{(1)}$, so claim (a) follows by iteration.

Let us prove assertion (b). It suffices to prove the claim for $m = 0$. General statement then follows from (a). Recall that $W_{-p,r+2\ell} \cong V^{\mathcal{H}}(h_{-p,r+2\ell}, (1 + p)c_{L,I})$ and notice that $Q v_{-p,r+2(\ell+1)} = S_p(c) v_{-p,r+2\ell}$ is an $\mathcal{H}$–singular vector in $W_{-p,r+2\ell}$ which generates the maximal submodule. Now $U(\mathcal{H}) \cdot (v_{-p,r+2\ell} + \text{Im } Q) \cong L^{\mathcal{H}}(h_{-p,r+2\ell}, (1 + p)c_{L,I})$. This proves the decomposition in (b). Proof of irreducibility in (b) is similar to the proof of Theorem 5.2. □
4. Relations in $\Pi(0)$–modules

In this section we shall apply a relation in the vertex algebra $\Pi(0)$ on its modules and recover an explicit formula for a singular vector in the Verma module $V^H(h,(1-p)c_{L,I})$, for $p \geq 1$. We shall use this formula in Section 5 when we construct a free field realization of these Verma modules.

Recall first that the Schur polynomials $S_r(x_1, x_2, \ldots)$ in variables $x_1, x_2, \ldots$ are defined by

\[(4.16) \quad \exp \left( \sum_{n=1}^{\infty} \frac{x_n}{n} y^n \right) = \sum_{r=1}^{\infty} S_r(x_1, x_2, \ldots) y^r \]

Then for any $\gamma \in L$ we set $S_r(\gamma) := S_r(\gamma(-1), \gamma(-2), \ldots)$.

Let

\[ e^{-c}(z) = Y(e^{-c}, z) = \sum_{i \in \mathbb{Z}} e_i^{-c} z^{-i-1}. \]

By direct calculation we get

\[(4.17) \quad L(-2)e^{-c} = \frac{cL - 26}{24} c(-2)e^{-c} - \frac{1}{2} L(-1)(d(-1)e^{-c}). \]

Let

\[ s = (L(-2) - \frac{cL - 26}{24} c(-2))e^{-c}. \]

Then we get

Lemma 4.1. On every $\Pi(0)$–module we have

\[ Q = s_0 = \text{Res}_z Y(s, z) \]

\[ = \sum_{i=0}^{\infty} \left( (L(-2-i)e_i^{-c} + e_i^{-c} L(-1+i)) \right) \]

\[ - \frac{cL - 26}{24} \sum_{i \in \mathbb{Z}} (i+1)e_i^{-c} \]

\[ = 0. \]

Proof. The assertion follows from (4.17) and the fact that

\[ (L(-1)u)_0 = 0 \]

in every vertex operator algebra. \qed

Now we shall see some consequences of the relation $Q = 0$ for irreducible $\mathcal{H}$–modules $L^H(h, h_I)$ such that $h_I = (1-p)c_{L,I}$, $p \in \mathbb{Z}_{>0}$, which are realized as $W_{p,r}$, for $r \in \mathbb{C}$. 

We have

\[
0 = Q e^{\frac{p-1}{2} d^2 + \frac{1-r}{2} c}
\]

\[
= \left( \sum_{i=0}^{\infty} (L(-2-i)e_i^{-c}) + e_{-1}^c L(-1) + h_{p,r} e_{-2}^c \right) e^{\frac{p-1}{2} d^2 + \frac{1-r}{2} c}
\]

\[
- \frac{c_L - 26}{24} \left( \sum_{i \in \mathbb{Z}} (i+1)c(-i-2)e_i^{-c} \right) e^{\frac{p-1}{2} d^2 + \frac{1-r}{2} c}
\]

\[
= \left( \sum_{i=0}^{p} (L(-i)e_{i+2}^{-c}) + e_{-1}^c L(-1) + h_{p,r} e_{-2}^c \right) e^{\frac{p-1}{2} d^2 + \frac{1-r}{2} c}
\]

\[
- \frac{c_L - 26}{24} \left( \sum_{i=-2}^{p-2} (i+1)c(-i-2)e_i^{-c} \right) e^{\frac{p-1}{2} d^2 + \frac{1-r}{2} c}
\]

\[
(4.18)
\]

\[
= \left( \sum_{i=1}^{p} L(-i)S_{p-i}(-c) + \left( h_{p,r} - 1 + \frac{c_L - 26}{24} (p-1) \right) S_{p}(-c) \right)
\]

\[
- \frac{c_L - 26}{24} \left( \sum_{i=1}^{p} (i-1)c(-i)S_{p-i}(-c) \right) e^{\frac{p-1}{2} d^2 + \frac{1-(c+2)}{2} c}
\]

In this way we have proved:

**Proposition 4.2.** Let \( p \in \mathbb{Z}_{>0} \). In \( W_{p,r+2} \) we have:

\[
0 = \left( \sum_{i=1}^{p} L(-i)S_{p-i}(-c) \right) + \left( h_{p,r} - 1 + \frac{c_L - 26}{24} (p-1) \right) S_{p}(-c) +
\]

\[
- \frac{c_L - 26}{24} \left( \sum_{i=1}^{p} (i-1)c(-i)S_{p-i}(-c) \right) v_{p,r+2}
\]

In particular, let \( h_I = (1-p)c_{L,I} \), \( h = h_{p,r+2} \). Then the singular vector of level \( p \) in \( V^H(h, h_I) \) is \( \Phi_p(L,c) \cdot v_{h,h_I} \) where

\[
\Phi_p(L,c) := \sum_{i=1}^{p} (L(-i)S_{p-i}(-c)) + S_{p}(-c) \left( L(0) + \frac{c_L - 26}{24} (p-1) \right)
\]

\[
- \frac{c_L - 26}{24} \left( \sum_{i=1}^{p} (i-1)c(-i)S_{p-i}(-c) \right)
\]
5. Deformed action of $\mathcal{H}$ on weight $\Pi(0)$–modules and realization of Verma modules

As we noticed in Section 2, the free field realization from [2] does not provide realization of Verma modules $V^H(h, (1 - p)c_L, i)$ and their singular vectors in the case $p \geq 1$. In order to understand these Verma modules, we shall use certain deformation of free field realization from [2]. We shall use the construction from [4] to deform the action of the twisted Heisenberg–Virasoro algebra on $\Pi(0)$–modules (see also [8], [17]). Let

$$\Delta(u, z) = z^u_0 \exp\left(\sum_{n=1}^{\infty} \frac{u_n}{-n}(-z)^{-n}\right).$$

First we recall a definition of logarithmic modules. More information about structure theory of logarithmic modules can be found in literature on logarithmic vertex operator algebras (see [6], [15], [18], [22], [23] and reference therein).

**Definition 5.1.**

1. A module $(M, Y_M)$ for the conformal vertex algebra with conformal vector $\omega$ is a logarithmic module of rank $m \in \mathbb{Z}_{\geq 1}$ if

$$(L(0) - L_{ss}(0))^m = 0, \quad (L(0) - L_{ss}(0))^{m-1} \neq 0,$$

where $L_{ss}(0)$ is the semisimple part of $L(0)$.

2. If for every $m \in \mathbb{Z}_{\geq 1}$ $(L(0) - L_{ss}(0))^m \neq 0$ on $M$, we say that $(M, Y_M)$ is a logarithmic module of infinite rank.

**Theorem 5.2.** For every $\Pi(0)$–module $(M, Y_M(\cdot, z))$,

$$(\tilde{M}, \tilde{Y}_M(\cdot, z)) := (M, Y_M(\Delta(u, z), z))$$

is a $\Pi(0) := \text{Ker}_{\Pi(0)} Q$–module. The action of Heisenberg–Virasoro algebra is

$$\tilde{I}(z) = \sum_{n \in \mathbb{Z}} \tilde{I}(n)z^{-n-1} = \tilde{Y}_M(I, z) = I(z)$$

$$\tilde{L}(z) = \sum_{n \in \mathbb{Z}} \tilde{L}(n)z^{-n-2} = \tilde{Y}_M(\omega, z) = L(z) + z^{-1}Y_M(u, z).$$

In particular,

$$(5.19) \quad \tilde{I}(n) = I(n), \quad \tilde{L}(n) = L(n) + u_n.$$

and $L(0) - L_{ss}(0) = u_0 = Q$.

Recall the definition of $\Pi(0)$–modules

$$\Pi(p, r) := \Pi(0) \cdot e^{\frac{p-1}{2}d^2 + \frac{br}{c}} \text{ where } p \in \mathbb{Z}, r \in \mathbb{C}.$$  

Then $\Pi(p, r)$ are logarithmic $\mathcal{H}$–modules which are uniquely determined by the action (5.19).

We shall also consider the cyclic submodules:

$$\tilde{W}_{p, r} = \mathcal{U}(\mathcal{H}) \cdot e^{\frac{p-1}{2}d^2 + \frac{br}{c}} \subset \Pi(p, r).$$
5.1. Case $h_L = (1-p)c_{L,I}$. We saw that for the undeformed action of $\mathcal{H}$ studied in [2], vector $v_{p,r} = e^{\frac{p-1}{2}d^2 + \frac{1}{2}c}$, for $p \geq 1$, generates the irreducible highest weight module $W_{p,r}$. But we shall see below that $\tilde{W}_{p,r}$ is isomorphic to a Verma module.

Theorem 5.3. Assume that $p \in \mathbb{Z}_{>0}$, $r \in \mathbb{C}$. We have

1. $\tilde{W}_{p,r} \cong V^H(h_{p,r}, (1-p)c_{L,I})$.
2. Singular vectors in $\tilde{W}_{p,r} \cong V^H(h_{p,r}, (1-p)c_{L,I})$ are

$$\text{Sing} = \{v_{p,r-2n} \mid n \geq 0\}.$$

Proof. It is clear that $v_{p,r-2n} = e^{\frac{p-1}{2}d^2 + \frac{1}{2}c + nc}$ is a singular vector for any $n \geq 0$. We only need to prove that

$$v_{p,r-2n} \in \tilde{W}_{p,r}.$$

Assume that (5.20) holds for $n \in \mathbb{Z}_{>0}$. Since $e^c v_{p,r-2n} = 0$ for every $k \geq 1 - p$ we have

$$\tilde{L}(k) = L(k) \quad \text{on} \quad \mathbb{C}[c(-1), c(-2), \ldots].$$

But since $e^c v_{p,r-2n} = v_{p,r-2(n+1)}$ we have

$$\tilde{L}(-p)v_{p,r-2n} = L(-p)v_{p,r-2n} + v_{p,r-2(n+1)}.$$

By using the expression for singular vector in $V^H(h_{p,r}, (1-p)c_{L,I})$ from Proposition 4.2 we get for $h = h_{p,r-2n}$

$$\Phi_p(\tilde{L}, c) \cdot v_{p,r-2n} = \sum_{i=1}^{p} \left( \tilde{L}(-i)S_{p-i}(-c) \right) + S_p(-c) \left( h + \frac{c_L - 2}{24}(p - 1) \right) - \frac{c_L - 26}{24} \left( \sum_{i=1}^{p} (i - 1)c(-i)S_{p-i}(-c) \right) v_{p,r-2n}$$

$$= \Phi_p(L, c) \cdot v_{p,r-2n} + v_{p,r-2(n+1)}$$

$$= v_{p,r-2(n+1)}.$$

(Above we used the fact that $v_{p,r-2n}$ generates the irreducible $\mathcal{H}$–module $L^H(h_{p,r}, (1-p)c_{L,I})$ for the undeformed action, so $\Phi_p(L, c) \cdot v_{p,r-2n} = 0$.)

Thus we get that $v_{p,r-2(n+1)} = e^{\frac{p-1}{2}d^2 + \frac{1}{2}c + (n+1)c}$ belongs to $\tilde{W}_{p,r}$. The claim now follows by induction. □

Finally, we obtain a deformed version of Theorem 3.2.

Theorem 5.4. Let $Z^{(m)} = \text{Ker} \Pi(p,r) Q^{m+1}$. Then we have

1. $\Pi(p,r) \cong \bigcup_{m \geq 0} Z^{(m)}$, $\Pi(0)^{(n)} \cdot Z^{(m)} \subset Z^{(n+m)}$.
2. For every $m \in \mathbb{Z}_{>0}$, $Z^{(m)}$ is a logarithmic $\Pi(0)^{(n)}$–module of rank $m+1$ with respect to $L(0)$.
3. $Z^{(m)}/Z^{(m-1)}$ is a weight $\Pi(0)^{(n)}$–module which is as $\mathcal{H}$–module isomorphic to

$$\bigcup_{n \in \mathbb{Z}} \tilde{W}_{p,r-2n}.$$
\textbf{Proof.} Assertion (1) is clear. Using relation (2.13) in the proof of Lemma 2.2 we see that \( v^{(m)}_{p,r-2\ell} \in Z^{(m)} \setminus Z^{(m-1)} \). Since \( \widetilde{L}(0) - \widetilde{L}_{ss}(0) = Q \), we have that \( Z^{(m)} \) is a logarithmic module of \( L(0) \)-nilpotent rank \( m + 1 \) so (2) holds. Assertion (3) results from following facts:

(a) As an \( H \)-module \( \Pi(p,r) \) is generated by set of vectors
\[
\{v_{p,r-2\ell} \mid \ell \in \mathbb{Z}\} \cup \{v^{(m)}_{p,r-2\ell} \mid m, \ell \in \mathbb{Z}, m \geq 1\}.
\]
(b) \( Z^{(m)}/Z^{(m-1)} \) is a weight \( H \)-module (i.e., non-logarithmic) generated by vectors \( \{v^{(m)}_{p,r-2\ell} + Z^{(m-1)} \mid \ell \in \mathbb{Z}\} \).
(c) \( v^{(m)}_{p,r-2\ell} + Z^{(m-1)} \) generates the Verma module \( \widetilde{W}_{p,r-2\ell} \).

Since \( Q \) and \( e_{-j} \) commute and by using (2.12) we get
\[
Q^n e_{-j}^{-1} v^{(m)}_{p,r-2\ell} = e_{-j}^{n} v^{(m)}_{p,r-2(\ell+m)} = S_{j-p(c)} v^{(m)}_{p,r-2(\ell+m)}
\]
so \( e_{-j}^{-1} v^{(m)}_{p,r-2\ell} \in Z^{(m-1)} \) for \( j < p \). Therefore
\[
\widetilde{L}(-j)v^{(m)}_{p,r-2\ell} = \begin{cases} L(-j)v^{(m)}_{p,r-2\ell} \mod Z^{(m-1)}, & j < p, \\ L(-p)v^{(m)}_{p,r-2\ell} + v^{(m)}_{p,r-2(\ell+1)} \mod Z^{(m-1)}, & j = p. \end{cases}
\]
(5.22)

The proof of claims (a) and (b) easily follows from Theorem 3.2 and (5.22). Let us prove claim (c).

We have proved in (5.22) that \( v^{(m)}_{p,r-2\ell} + Z^{(m-1)} \) is a highest weight vector with highest weight \( (h_{p,r-2\ell}, (1-p)c_{L,I}) \). Now, repeating the same arguments as in the proof of Theorem 5.3 we get
\[
\Phi_p(L, c) \cdot v^{(m)}_{p,r-2\ell} = \Phi_p(L, c) \cdot v^{(m)}_{p,r-2\ell} + v^{(m)}_{p,r-2(\ell+1)} \mod Z^{(m-1)}
\]
This implies that \( v^{(m)}_{p,r-2\ell} + Z^{(m-1)} \) generates the Verma module \( \widetilde{W}_{p,r-2\ell} \) which contains all Verma modules \( \widetilde{W}_{p,r-2(\ell+j)}, j \in \mathbb{Z}_{\geq 1} \). This completes the proof. (See also Figure 3 in Appendix A where one can follow steps in the proof.) \( \Box \)

\textbf{5.2. Case} \( h_I = c_{L,I} \). Note that \( \Pi(0,r) \) is an \( H \)-module on which \( I(0) \) acts as multiplication by \( c_{L,I} \).

In particular, \( \widetilde{W}_{0,r} \) is a \( \mathbb{Z}_{\geq 0} \)-graded logarithmic \( H \)-module whose lowest component is
\[
\widetilde{W}_{0,r}(0) := \text{span}_C\{v_{0,r-2\ell} \mid \ell \in \mathbb{Z}_{\geq 0}\}.
\]
Moreover, since
\[
\widetilde{L}(0) - \frac{c_{L} - 2}{24} = Q, \quad Q^n v_{0,r} = v_{0,r-2n}
\]
we conclude that \( \widetilde{W}_{0,r} \) is a \( \mathbb{Z}_{\geq 0} \)-graded logarithmic module of infinite rank. See Figure 4 in Appendix A.
5.3. Case $h_I = (1+p)L_I$. We saw that $\Pi(-p, r)$ contains a descending chain of submodules $\Pi(-p, r)^{(m)}$ isomorphic to $\Pi(-p, r)$ (Theorem 5.3).

**Theorem 5.5.** Let $p \in \mathbb{Z}_{>0}$. Then $\widehat{\Pi}(-p, r)$ is a logarithmic $\Pi(0)$–module such that

$$\dim \mathbb{C}[L(0)]v = \infty \quad \text{for every } v \in \Pi(-p, r).$$

**Quotient** $\widehat{\Pi}(-p, r)/\Pi(-p, r)^{(1)}$ is a weight module such that

$$L(n)v_{-p, r} = S_{p, n}(c)v_{-p, r-2} \quad (1 \leq n \leq p) \quad \text{and} \quad L(n)v_{-p, r} = 0 \quad (n > p).$$

**Proof.** Let $S = \{v_{-p, r-2\ell} \mid \ell \in \mathbb{Z}\}$. Let $\langle S \rangle$ be the $\mathcal{H}$–submodule generated by the set $S$.

We shall first prove that $\Pi(-p, r) = \langle S \rangle$. Since, as a vector space $\Pi(-p, r) \cong \Pi(0) \cong \bigoplus_{\ell \in \mathbb{Z}} W_{-p, r-2\ell}$, it suffices to show that $W_{-p, r-2\ell} \subset \langle S \rangle$ for each $\ell$.

Take an arbitrary basis vector

$$u = c(-p_1) \cdots c(-p_s) L(-q_1) \cdots L(-q_m) v_{-p, r-2\ell}$$

of $W_{-p, r-2\ell}$, where $\ell \in \mathbb{Z}$, $p_1, \ldots, p_s, q_1, \ldots, q_m \geq 1$. Then by applying formula for $\widehat{L(n)}$ and relation $[e_m, L(n)] = me_{m+n}$ we get

$$c(-p_1) \cdots c(-p_s) L(-q_1) \cdots L(-q_m) v_{-p, r-2\ell} = c(-p_1) \cdots c(-p_s) L(-q_1) \cdots L(-q_m) v_{-p, r-2\ell} + w$$

where $w$ is a linear combination of vectors

$$c(-t_1) \cdots c(-t_{s'}) L(-u_1) \cdots L(-u_{m'}) v_{-p, r-2\ell}$$

such that $\ell' \in \mathbb{Z}$, $m' < m$. The assertion now follows by induction on $m$.

Furthermore, we have

$$(\widehat{L(0)} - L(0))^{n} v_{-p, r} = Q^{n} v_{-p, r} = (S_p(c))^{n} v_{-p, r-2n}.$$

Since $Q$ commutes with action of $\mathcal{H}$, we proved the first claim.

Taking a quotient by $\Pi(-p, r)^{(1)} = \text{Im} Q$ results in a weight module (i.e., non logarithmic module) on which $L(0) \equiv L(0)$. \hfill $\square$

See Figure 5 in Appendix A.

**Remark 5.6.** As far as we know, modules $\Pi(-p, r)/\Pi(-p, r)^{(1)}$, and their cyclic submodules generated by images of $v_{-p, r-2\ell}$ are weight $\mathcal{H}$–modules which haven’t been analysed in the literature.

6. Realization of self-dual modules via Whittaker $\Pi(0)$–modules

In Section 5 we slightly refined the free field realization from [2], but these results still don’t give a realization of all irreducible self-dual modules. In order to construct all self-dual modules we shall apply the deformation from Section 5 on the Whittaker $\Pi(0)$–module $\Pi_\lambda$ which was constructed in [7, Section 11] and used for a realization of Whittaker $A_1^{(1)}$–modules at the critical level. As a by-product we shall see that self-dual modules for $\mathcal{H}$ have non-trivial self-extensions which are logarithmic modules.
6.1. **Whittaker** $\Pi(0)$–module $\Pi_\lambda$. We shall recall the construction of a Whittaker $\Pi(0)$–module $\Pi_\lambda$ from [7, Section 11]. Let $u = e^c$, $u^{-1} = e^{-c}$.

**Theorem 6.1.** [7] Assume that $\lambda \neq 0$. There is a $\Pi(0)$–module $\Pi_\lambda$ generated by the cyclic vector $w_\lambda$ such that $c(0) = -\text{Id}$ on $\Pi_\lambda$ and

$$u_0 w_\lambda = \lambda w_\lambda, \quad u_1^{-1} w_\lambda = \frac{1}{\lambda} w_\lambda, \quad u_n w_\lambda = u_n^{-1} w_\lambda = 0 (n \geq 1).$$

As a vector space

$$\Pi_\lambda \cong \mathbb{C}[d(-n), c(-n - 1) | n \geq 0] = \mathbb{C}[d(0)] \otimes M(1).$$

$\Pi_\lambda$ is $\mathbb{Z}_{\geq 0}$–graded

$$\Pi_\lambda = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Pi_\lambda(n)$$

and lowest component is isomorphic to $\mathbb{C}[d(0)]$.

Recall also (cf. [7]) that the lowest component $\Pi_\lambda(0)$ is an irreducible Whittaker module for the associative algebra $\mathcal{A}$ defined by generators

$$d(0), e^{nc} (n \in \mathbb{Z})$$

and relations

$$[d(0), e^{nc}] = 2ne^{nc}, \quad e^{nc}e^{mc} = e^{(n+m)c} (n, m \in \mathbb{Z}).$$

6.2. **Realization of self-dual modules.** Now we can apply Theorem 5.2 on the Whittaker $\Pi(0)$–module $\Pi_\lambda$. We get $\mathcal{H}$–module $\tilde{\Pi}_\lambda$, which is as a vector space isomorphic to $\Pi_\lambda$ and the (deformed) action of $\mathcal{H}$ is as follows:

$$\text{(6.23)} \quad \tilde{I}(0) \equiv -c_{L,I}c(0) \equiv c_{L,I}\text{Id} \quad \text{on} \quad \tilde{\Pi}_\lambda,$$

and on the lowest component $\tilde{\Pi}_\lambda(0)$ we have

$$\tilde{L}(0) \equiv \frac{1}{2} d(0)(c(0) + 1) - \frac{c_{L}}{24} c(0) + u_0 \quad \text{on} \quad \tilde{\Pi}_\lambda(0)$$

$$\text{(6.24)} \quad \equiv \frac{c_{L} - 2}{24} \text{Id} + u_0 \quad \text{on} \quad \Pi_\lambda(0).$$

This implies:

$$\text{(6.25)} \quad \tilde{I}(0) w_\lambda = c_{L,I} w_\lambda, \quad \tilde{L}(0) w_\lambda = \left(\frac{c_{L} - 2}{24} + \lambda\right) w_\lambda.$$

Define also the following (logarithmic) cyclic module:

$$\tilde{\Pi}_\lambda^{(n)} = U(\mathcal{H}) \cdot d(0)^n w_\lambda.$$

**Lemma 6.2.** We have:

$$\tilde{\Pi}_\lambda^{(n+1)} \supset \tilde{\Pi}_\lambda^{(n)}, \quad \tilde{\Pi}_\lambda = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \tilde{\Pi}_\lambda^{(n)}.$$

**Proof.** By using (6.24) one can easily see that for $0 \leq m \leq n$ there is a polynomial $P(x)$ such that

$$P(\tilde{L}(0))d(0)^n w_\lambda = d(0)^m w_\lambda.$$

This proves that $\tilde{\Pi}_\lambda^{(n+1)} \supset \tilde{\Pi}_\lambda^{(n)}$ for $n \in \mathbb{Z}_{\geq 0}$. 
Take an arbitrary basis vector
\[ u = c(-p_1) \cdots c(-p_s)d(-q_1) \cdots d(-q_r)d(0)^\ell w_\lambda \]
of \( \tilde{\Pi}_\lambda \), where \( \ell \in \mathbb{Z}_{\geq 0}, p_1, \ldots, p_s, q_1, \ldots, q_r \geq 1 \). Then
\[ c(-p_1) \cdots c(-p_s)L(-q_1) \cdots L(-q_r)d(0)^\ell w_\lambda = A(-p_1) \cdots c(-p_s)d(-q_1) \cdots d(-q_r)d(0)^\ell w_\lambda + w \]
where \( A \neq 0 \) and \( w \) is a linear combination of vectors
\[ c(-t_1) \cdots c(-t_s)d(-u_1) \cdots d(-u_r)d(0)^\ell w_\lambda \]
such that \( \ell' \in \mathbb{Z}_{\geq 0}, r' < r \) or \( r = r' \) and \( u_1 + \cdots + u_{r'} < q_1 + \cdots + q_r \). The assertion now follows by induction. \( \square \)

**Theorem 6.3.** For every \( \lambda \in \mathbb{C}, \lambda \neq 0 \) we have:

1. \( \tilde{\Pi}_\lambda \) is a logarithmic \( \mathcal{H} \)-module of infinite rank with respect to \( \widetilde{L}(0) \).
2. \( \tilde{\Pi}_\lambda^{(0)} \) is an irreducible self-dual \( \mathcal{H} \)-module with highest weight
\[ (h, h_I) = \left( \frac{c_L - 2}{24} + \lambda, c_L, I \right). \]
3. \( \mathcal{H} \)-module \( \tilde{\Pi}_\lambda \) admits the \( \mathbb{Z}_{\geq 0} \)-filtration:
\[ \tilde{\Pi}_\lambda = \cup_{n \in \mathbb{Z}_{\geq 0}} \tilde{\Pi}_\lambda^{(n)} \]
such that
\[ \tilde{\Pi}_\lambda^{(0)} = L^\mathcal{H}(h, h_I), \quad \tilde{\Pi}_\lambda^{(n+1)} / \tilde{\Pi}_\lambda^{(n)} \cong L^\mathcal{H}(h, h_I). \]
Every \( \tilde{\Pi}_\lambda^{(n)} \) is a logarithmic \( \mathcal{H} \)-module of rank \( n \) with respect to \( \widetilde{L}(0) \).

**Proof.** (1) follows from the fact that on the top component \( \tilde{\Pi}_\lambda(0) \) we have
\[ Q = u_0 = \widetilde{L}(0) - \frac{c_L - 2}{24}. \]
(2) Using \([6.25]\) and the fact that the Verma module with highest weight \( (h, h_I) = \left( \frac{c_L - 2}{24} + \lambda, c_L, I \right) \) is irreducible we get
\[ L^\mathcal{H}(h, h_I) = U(\mathcal{H}) \cdot w_\lambda = \tilde{\Pi}_\lambda^{(0)}. \]
(3) First we notice that for \( m \geq 0 \) we have
\[ \widetilde{L}(m)d(0)^{n+1}w_\lambda = h\delta_{m,0}d(0)^{n+1}w_\lambda \mod \tilde{\Pi}_\lambda^{(n)}. \]
Therefore we have isomorphism
\[ L^\mathcal{H}(h, h_I) \to \tilde{\Pi}_\lambda^{(n+1)} / \tilde{\Pi}_\lambda^{(n)}. \]
The proof now follows from Lemma \([6.22]\). \( \square \)

See Figure \([6]\) in Appendix \([A]\) for reference.

We list two interesting consequences of previous theorem.

**Corollary 6.4.** Logarithmic \( \mathcal{H} \)-module \( \tilde{\Pi}_\lambda^{(1)} \) is a non-split self-extension of irreducible self-dual module \( L^\mathcal{H}(h, h_I) \):
\[ 0 \to L^\mathcal{H}(h, h_I) \to \tilde{\Pi}_\lambda^{(1)} \to L^\mathcal{H}(h, h_I) \to 0. \]

Note that the vertex algebra \( \widetilde{\Pi}(0) \) is not \( \mathbb{Z}_{\geq 0} \)-graded since for every \( n \in \mathbb{Z}, \ e^{nc} \) has weight \( n \). Irreducible \( \widetilde{\Pi}(0) \)-modules from Theorem \([5.4]\) (3) are not \( \mathbb{Z}_{\geq 0} \)-graded. But, quite surprisingly, the vertex algebra \( \widetilde{\Pi}(0) \) admits a large family of \( \mathbb{Z}_{\geq 0} \)-graded modules which are self-dual. We also construct a family of intertwining operators which haven’t appeared in \([2]\).
Corollary 6.5. We have:
(1) \( L^\mathcal{H}(h, c_{L,I}) \) is an irreducible \( \mathbb{Z}_{\geq 0} \)-graded \( \Pi(0) \)-module.
(2) For every \( n \in \mathbb{Z} \) there is a non-trivial intertwining operator of type

\[
\begin{pmatrix}
L^\mathcal{H}(h, c_{L,I}) \\
L^\mathcal{H}(n, 0) & L^\mathcal{H}(h, c_{L,I})
\end{pmatrix}.
\]

7. Some applications to the \( W(2,2) \)-algebra

In \cite{2} we introduced a free field realization of the \( W(2,2) \)-algebra as a subalgebra of the Heisenberg Virasoro algebra.

Recall that \( W(2,2) \) is realized as a subalgebra of \( L^\mathcal{H}(c_{L}, c_{L,I}) \) generated by \( L(z) \) and

\[
W(z) = c_{L,I}^2 \overline{W}(z)
\]

where

\[
\overline{W}(z) = \sum_{n \in \mathbb{Z}} \overline{W}(n) z^{-n-2} = Y(c(-1)^2 - 2c(-2), z) = c(z)^2 - 2\partial c(z).
\]

In the paper \cite{3} we discussed a free field realization of highest weight \( W(2,2) \)-modules. We constructed in \cite[Section 4]{3} \( W(2,2) \)-homomorphism \( S_1 : L^\mathcal{H}(c_{L}, c_{L,I}) \to L^\mathcal{H}(1,0) \) such that \( \text{Ker}_{L^\mathcal{H}(c_{L}, c_{L,I})} S_1 \) is the simple vertex algebra \( L^{W(2,2)}(c_{L}, -24c_{L,I}^2) \). In this paper we shall present a bosonic, non-local expression for the screening operator \( S_1 \).

The vertex algebra \( W(2,2) \) has appeared in physics literature as the Galilean Virasoro algebra (\cite{25}, \cite{9}, \cite{10}) and as BMS\(_3\) algebra (\cite{11}). We noticed a free field realization of the \( W(2,2) \)-algebra in terms of the \( \beta \gamma \) systems in \cite{11}. We shall see how this realization relates to our approach.

7.1. A bosonic formula for the second screening operator and \( W(2,2) \)-algebra.

Our approach is motivated by the realization of screening operators in LCFT from \cite{5} and \cite{6}. Recall the definition of modules \( W_{p,r} \) from \cite{2,11}. For \( r \in \mathbb{Z} \) we define:

\[
S = -\text{Res}_z \text{Res}_{z_1} \left( \log(1 - \frac{z}{z_1}) e^c(z) d^1(z_1) - \log(1 - \frac{z}{z_1}) d^1(z_1) e^c(z) \right)
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{j} (d^1(-j) e^c_j - e^c_{-j} d^1(j)) : W_{p,r} \to W_{p,r-2}.
\]

Lemma 7.1. We have:

\[
[L(m), S] = d^1(m) e^c_0 - e^c_m d^1(0) + 2\delta_{m,0} e^c_0,
\]

\[
[c(m), S] = 2e^c_m - 2\delta_{m,0} e^c_0, [W(m), S] = 0.
\]

Proof. In the proof we use the following formulas

\[
[L(n), e^c_m] = -me^c_{n+m},
\]

\[
[L(n), d^1(m)] = -md^1(n + m) - 2(m^2 - m)\delta_{m+n,0}.
\]
We have:

\[ [L(n), S] = \sum_{j=1}^{\infty} \frac{-j}{j} \left( d^1 (-j) e^c_{j+n} + e^c_{-j+n} d^1 (j) \right) \]

\[ + \sum_{j=1}^{\infty} \frac{j}{j} \left( d^1 (-j + n) e^c_{j+n} + e^c_{-j} d^1 (j + n) \right) \]

\[ -2 \sum_{j=1}^{\infty} \frac{j^2 + j}{j} \delta_{-j+n,0} e^c_{j+n} + 2 \sum_{j=1}^{\infty} \frac{j^2 - j}{j} \delta_{j+n,0} e^c_{-j} \]

\[ = -\sum_{j=1}^{\infty} \left( d^1 (-j) e^c_{j+n} + e^c_{-j+n} d^1 (j) \right) + \sum_{j=1}^{\infty} \left( d^1 (-j + n) e^c_{j} + e^c_{-j} d^1 (j + n) \right) \]

\[ -2 \sum_{j=1}^{\infty} (j + 1) \delta_{-j+n,0} e^c_{j} + 2 \sum_{j=1}^{\infty} (j - 1) \delta_{j+n,0} e^c_{-j} \]

\[ = -2(n + 1) e^c_n + 2\delta_{n,0} e^c_0 + d^1(0) e^c_n + \cdots + d^1(n - 1) e^c_1 \]

\[ - (e^c_{n-1} d^1(1) + \cdots + e^c_1 d^1(n - 1)) - \delta_{n,0} d^1(n) \]

\[ = -d^1(n) e^c_0 + e^c_n d^1(0) + 2\delta_{n,0} e^c_0. \]

Relation \([c(m), S] = 2(1 - \delta_{m,0}) e^c_m\) follows directly from the definition of the operator \(S\). Next we have

\[ [\mathbf{W}(n), S] = \left( \sum_{k \in \mathbb{Z}} [c(k)c(n - k), S] \right) + 2n [c(n - 1), S] \]

\[ = -\sum_{k \in \mathbb{Z}} (c(k) e^c_{n-k} + c(n-k) e^c_k) - 2ne^c_{n-1} \]

\[ = -4(D e^c)_n - 4ne^c_{n-1} = 0 \]

The proof follows. \(\square\)

Now, we will see that in the case \(r = 1\) our operator \(S\) is a multiple of the screening operator \(S_1\) from \([3]\):

**Corollary 7.2.** Let \(r = 1\), and consider \(S : W_{p,1} \to W_{p,-1}\). Then \(S\) commutes with the action of the \(W(2,2)\)-algebra:

\[ [S, W(n)] = [S, L(n)] = 0 \quad (n \in \mathbb{Z}). \]

Moreover, \(S\) is a \(W(2,2)\)-homomorphism which is proportional to \(S_1\).

**Proof.** In the case \(r = 1\) we have that \(d^1(0)\) and \(Q = e^c_0\) act trivially on \(W_{p,1}\), and therefore Lemma \([7.1]\) implies that

\[ [S, W(n)] = [S, L(n)] = 0 \quad (n \in \mathbb{Z}). \]

It is clear that the \(W(2,2)\)-homomorphism \(S_1 : W_{p,1} \to W_{p,-1}\) from \([3\text{ Section 4}]\) is uniquely determined by the properties

\[ [S_1, L(n)] = 0, \quad [S_1, I(n)] = -e^c_n, \]

which gives \([S_1, c(n)] = \frac{1}{c_{\mathbb{L},I}} e^c_n\). Now Lemma \([7.1]\) gives that \(S = 2c_{L,I} S_1\). \(\square\)
7.2. On the Banerjee, Jatkar, Mukhi, Neogi’s free field realization of the BMS$_3$–algebra. Recently, Banerjee, Jatkar, Mukhi and Neogi in [11] have discovered a new free field realization of the $W(2,2)$–algebra for central charge $c_L = 26$. The vertex algebra $L^{W(2,2)}(26,c_W)$ is realized inside of the $\beta\gamma$ system. Since the $\beta\gamma$–system can be embedded into the vertex algebra $\Pi(0)$, one may try to extend this realization in order to obtain an arbitrary central charge $c_L$. Quite surprisingly, even in the case of the larger vertex algebra $\Pi(0)$, one gets the $W(2,2)$–structure only for $c_L = 26$.

Recall the definition of following Virasoro vector of central charge $c_L \in \mathbb{C}$.

\[ \omega = \frac{1}{2}c(-1)d(-1) + \frac{c_L - 2}{24}c(-2) - \frac{1}{2}d(-2). \] (7.29)

We shall now deform this vector in a different way:

**Lemma 7.3.** For every $\mu \in \mathbb{C}$

\[ \tilde{\omega} = \omega + \mu e^{\frac{c}{4}}1 = \omega + \frac{\mu}{6}D^3e^{-c}. \]

is a Virasoro vector of central charge $c_L$.

The proof of lemma follows from a more general statement (which is also noticed in [11]):

**Claim:** Assume that $(V,Y,1,\omega)$ is a VOA of central charge $c$, and $v$ is a primary, commutative vector of conformal weight $-1$. Then $\tilde{\omega} = \omega + \frac{1}{6}D^3v$ is a Virasoro vector of central charge $c$.

The following result is obtained in [11, Section 2]. We include a proof of this result from which one can see that such construction works only for $c_L = 26$.

**Proposition 7.4.** [11]. The vertex algebra $L^{W(2,2)}(c_L,c_W)$ for $c_L = 26$ is isomorphic to a vertex subalgebra of $\Pi(0)$ generated by

\[ \tilde{\omega} = \frac{1}{2}c(-1)d(-1) + \frac{c_L - 2}{24}c(-2) - \frac{1}{2}d(-2) + \frac{\mu}{6}D^3e^{-c} \]

\[ w = (d(-1) + \frac{c_L - 14}{12}c(-1))e^{c} \]

where

\[ \mu = \frac{-c_W}{4}. \]

**Proof.** By direct calculation we get

\[ w_0w = \frac{c_L - 26}{3}c(-1)e^{2c} = \frac{c_L - 26}{6}De^{2c} \] (7.30)

\[ w_1w = \frac{c_L - 26}{3}c^{2c} \] (7.31)

\[ w_nw = 0 \quad (n \geq 2). \] (7.32)

By using formulas

\[ [L(n),c(m)] = -mc(n + m) + (m^2 - m)\delta_{m+n,0} \] (7.33)

\[ [L(n),d(m)] = -md(n + m) - \frac{c_L - 2}{12}(m^2 - m)\delta_{m+n,0} \] (7.34)
we get that
\[
L(1)w = \left( [L(1), d(-1)] + \frac{c_L - 14}{12} [L(1), c(-1)] \right) e^c
\]
\[
= \left( 2 - \frac{c_L - 2}{6} + \frac{c_L - 14}{6} \right) e^c
\]
\[
= 0,
\]
which easily implies that
\[
L(n)w = 2\delta_{n,0}w \quad (n \geq 0).
\] (7.35)

Since
\[
\tilde{L}(2)w = -\mu e_0^cw = -2\mu = \frac{c_W}{2},
\]
we get
\[
\tilde{L}(n)w = 2\delta_{n,0}w + \frac{c_W}{2} \delta_{n,2}w \quad (n \geq 0).
\] (7.36)

Claim now follows from (7.30)-(7.32), (7.36) and Lemma 7.3. □

Remark 7.5. The Weyl vertex algebra (also called the \(\beta\gamma\) system in the physical literature) can be realized as a subalgebra of the vertex algebra \(\Pi(0)\) as follows:
\[
\beta = (c(-1) + d(-1)) e^c, \quad \gamma = - \frac{1}{2} e^{-c}.
\] (7.37)

Then the components of the fields \(Y(\beta, z) = \sum n \beta(n) z^{-n-1}, Y(\gamma, z) = \sum n \gamma(n) z^{-n}\) satisfy the commutation relation for the Weyl algebra
\[
[\beta(n), \beta(m)] = [\gamma(n), \gamma(m)] = 0, \quad [\beta(n), \gamma(m)] = \delta_{n+m,0}.
\]

The vertex algebra \(\Pi(0)\) can be treated as a certain localization of the Weyl vertex algebra (for details see [16], [21], [7]).

By using (7.37) we see that for \(c_L = 26\) \(L^{W(2,2)}(c_L, c_W)\) is realized as a subalgebra of the Weyl vertex algebra, which corresponds to the realization in [11].

Remark 7.6. It would be interesting to investigate the structure of \(W(2,2)\)–modules \(\Pi(p,r)\) with this new action. Since
\[
\hat{\Pi}(n) = L(n) - \frac{1}{6} (n+1)n(n-1)\mu e_{n-2}^c,
\]
we have that \(\Pi(p, r)\) and \(\Pi_\lambda\) are weight \(L^{W(2,2)}(c_L, c_W)\)–modules on which \(\hat{\Pi}(0) = L(0)\). In our forthcoming papers we plan to investigate the appearance of these modules in the fusion rules analysis at \(c_L = 26\).

**Appendix A. Figures**

Here we present some visualizations of modules \(\Pi(p, r)\) and \(\hat{\Pi}(p, r)\).
\[ v^{(1)}_{p,r} + 2v^{(2)}_{p,r} - 2v^{(3)}_{p,r} \]

\[ S_p(c)v_{-p,r} - (S_p(c))^2v_{-p,r-2} \]

\[ \Phi_p(\bar{L}, c) \]

\[ \bar{L}(0) \]

\[ V^H(h_{p,r} + p, h_I) \]

\[ \bar{W}_{p,r-2} \]

\[ \bar{\Pi}(0) - \text{module } \bar{\Pi}(p, r), p \in \mathbb{Z}_{>0} \]

\[ \bar{\Pi}(0) - \text{module } \bar{\Pi}(-p, r), p \in \mathbb{Z}_{>0} \]

\[ \text{Deformed action of } \mathcal{H} \text{ on } \bar{\Pi}(p, r), p > 0. \text{ Dotted area represents a cyclic submodule of } \bar{\Pi}(p, r)/(\bar{\Pi}(p, r)^{(0)} \text{ generated by } v^{(1)}_{p,r} \text{ which is isomorphic to } \bar{W}_{p,r-2} \cong V^H(h_{p,r} + p, h_I). \text{ Arrows represent } \Phi_p(\bar{L}, c) \text{ (descending), and } \bar{L}(0) \text{ (horizontal).} \]
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Figure 4. $\Pi(0)$–module $\tilde{\Pi}(0, r)$. Dotted area represents a portion of a deformed $\mathcal{H}$–module $\tilde{W}_{0, r}$.

Figure 5. Deformed action of $\mathcal{H}$ on $\tilde{\Pi}(\lambda - p, r)$, $p \in \mathbb{Z}_{>0}$

Figure 6. A deformed Whittaker module $\tilde{\Pi}_\lambda$. Dotted area represents $\tilde{\Pi}^{(0)}_\lambda$ which is isomorphic to a self-dual $\mathcal{H}$–module $L^H(\frac{c_{\lambda I} - 2}{24} + \lambda, c_{L,I})$.

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Current address: Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia
E-mail address: adamovic@math.hr

Current address: Faculty of Science, University of Split, Rudera Boškovića 33, 21 000 Split, Croatia
E-mail address: gordan@pmfst.hr