FROM REGULAR TO CHAOTIC STATES
IN ATOMIC NUCLEI

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1. Introduction

An interesting aspect of nuclear dynamics is the co-existence, in atomic nuclei, of regular and chaotic states [1]. This is a vast subject and, for reasons of space, in this paper we limit ourselves to a few examples only.

In order to highlight the difference between the regular and chaotic states, it is perhaps useful to remember that the low energy states (0 ∼ 4 MeV above the ground state), termed regular, are described by a variety of models: the shell-model, the collective model and their various extensions [2–8]. By means of these models, all the properties of nuclear levels such as excitation energies, transition probabilities, magnetic and quadrupole momenta, etc. may be accurately calculated.

When the excitation energy increases, the level density rises, making the calculation of nuclear properties in terms of single levels both of no physical interest and impossible. Instead, a statistical description, called Statistical Nuclear Spectroscopy (SNS), is used, based on the division into global and local nuclear properties [9–17]. A typical example of this is the separation of the level density into a global component, the secular variation, and a local component, the fluctuations, which are well described by the random matrix ensembles [18–21].

In recent years, the study of quantum levels in classically chaotic regions has shown that they have the same fluctuation properties as those predicted by random matrix ensembles [28–30] in a large energy range; we term these states chaotic. The physical significance of statistical concepts in atomic nuclei is therefore understood through the link with chaotic motion in hamil-
tonian dynamics [22–25].

In the first part of the present work, we review the state of the art of nuclear dynamics and use a schematic shell model to show how a very simple and schematic nucleon–nucleon interaction can produce an order→chaos transition. The second part is devoted to a discussion of the wave function behaviour and decay of chaotic states using some simple models.

2. Regular states in atomic nuclei: shell and collective models

As is well known, the regular states of atomic nuclei are described, roughly speaking, by the mean field approximation of the shell model and by the oscillations about the mean field which give rise to collective excitations. Nuclear potential is essentially symmetric; nucleons move in regular orbits and, besides the energy, there are other constants of motion and other quantum numbers [6,7]. For the sake of completeness we report below some basic notions of the shell model and collective models.

In the shell model [5] the nuclear states, like the electronic states in atoms, are described in terms of the motion of nucleons in a mean–field. But, as is well known, while the nuclear field is generated by the interactions of the nucleons, the atomic field is mainly governed by the interaction of the electrons with the nucleus.

Experimental data suggests a shell structure for the atomic nucleus, i.e. the greater stability of the nuclei with 2, 8, 20, 28, 50, 82 and 126 neutrons or 2, 8, 20, 28, 50, 82 protons. These nuclei are called magic nuclei and their nucleonic numbers are termed magic numbers. The low energy levels of these
**magic nuclei** are very high, so their structure is particular stable. In the *shell model* the *magic numbers* represent the numbers of nucleons that saturate the nuclear shells. *Magic nuclei* are to nuclei what noble gases are to atoms.

The hamiltonian of the *shell model* can be written as:

\[ \hat{H}_0 = \sum_{i=1}^{A} -\frac{\hbar^2}{2m_i} \Delta_i + \hat{U}_i, \]  

(2.1)

where \( A \) is the number of nucleons, and \( \hat{U}_i \) is the mean-field. The choice of the mean-field is crucial; \( \hat{U}_i \) may be obtained by the usual methods using the many body theory. However, phenomenological approximations have traditionally been used instead:

\[ \hat{U}_i = \hat{U}_{C,i}(r) + \hat{U}_{LS,i} \vec{L}_i \cdot \vec{S}_i, \]  

(2.2)

where \( \hat{U}_{C,i}(r) \) is the central potential, i.e. with a spherical symmetry, and \( \hat{U}_{LS,i} \vec{L}_i \cdot \vec{S}_i \) is the spin–orbit interaction. It is common to use a central potential that behaves like the charge density of the nucleus (Saxon–Woods potential):

\[ \hat{U}_{C,i}(r) = \frac{-U_0}{1 + \exp \left[(r - R_0)/a\right]}, \]  

(2.3)

where \( R_0 \) is the nuclear radius, \( a \approx 0.53 \cdot 10^{-15} \text{m} \) and \( U_0 \approx (50–60) \text{MeV} \). With this potential the calculations are quite complicated, so simpler potentials are usually used.

To obtain a good agreement between the shell model results and the experimental data, it is necessary to add a residual interaction \( \hat{H}_R \) so that the total hamiltonian \( \hat{H} \) can be written:

\[ \hat{H} = \hat{H}_0 + \hat{H}_R, \]  

(2.4)
where $\hat{H}_R$ is the part of nucleon–nucleon interaction not included in $\hat{H}_0$. Using second quantization formalism [8], we can write:

$$\hat{H}_0 = \sum_{i=1}^{A} \epsilon_i \hat{a}_i^+ \hat{a}_i, \quad (2.5a)$$

$$\hat{H}_R = \sum_{ijkl}^{A} V_{ijkl} \hat{a}_i^+ \hat{a}_j^+ \hat{a}_k \hat{a}_l, \quad (2.5b)$$

where $\epsilon_i$ and $V_{ijkl}$ are the single–particle energies and the residual interactions respectively. The operators $\hat{a}_i^+$ and $\hat{a}_i$ are the creation and annihilation operators of the ith single nucleon state:

$$[\hat{a}_i, \hat{a}_j^+] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0, \quad (2.6)$$

where $[\hat{a}, \hat{b}]_+ = \hat{a} \hat{b} + \hat{b} \hat{a}$. $\hat{H}_0$ and $\hat{H}_R$ can be calculated by the Hartree–Fock equations [8], starting from the free nucleon interactions; but often, as mentioned above, phenomenological approximations are used (the values of $\epsilon_i$ are obtained by the experimental spectrum of the nuclei with only one nucleon added to a double magic core).

In order to obtain the different observables, the following procedure is generally used. As first step we solve the unperturbed equation:

$$\hat{H}_0 |\phi_\alpha> = E^{(0)}_\alpha |\phi_\alpha>, \quad (2.7)$$

where $E^{(0)}_\alpha = \sum_{i=1}^{A} \epsilon_i$. To obtain the eigenvalues and eigenstates of $\hat{H}$, the Schrödinger equation:

$$\hat{H} |\psi_n> = E_n |\psi_n>, \quad (2.8)$$

must be solved and $|\psi_n>$ expanded by a complete unperturbed base of eigenstates:

$$|\psi_n> = \sum_{\alpha} C_{n\alpha} |\phi_\alpha>, \quad (2.9)$$
In this way we obtain the secular problem:

$$\sum_{\beta} E^{(0)}_{\alpha} \delta_{\alpha\beta} + <\phi_\alpha | \hat{H}_R | \phi_\beta > C_{n\beta} = E_n C_{n\alpha}. \quad (2.10)$$

It is clear that the sums in (2.9) and (2.10) are infinite and so the secular problem is not solvable. To overcome this difficulty, it is standard procedure to cut the basis states by introducing a finite number of configurations which are sufficient to describe the first excitation states. Many nucleons are frozen in the deeper shells of the mean field potential and form an inert core; only a few nucleons partially populate the single particle shells outside the core. These are called valence–nucleons. So there are $N$ valence–nucleons, $m$ active shells and a finite number of energy levels.

Now, if we know the states $|\psi_n>$, it is possible to calculate all the observables $A$, which characterize a nuclear state. To obtain the value of $A$ in the state $|\psi_n>$, we must calculate the diagonal element $<\phi_n | \hat{A} | \phi_n >$ of the operator $\hat{A}$ associated to $A$:

$$A_{\psi_n} = <\psi_n | \hat{A} | \psi_n > = \sum_{\alpha, \beta} C_{n\alpha} C_{n\beta} <\phi_\alpha | \hat{A} | \phi_\beta >. \quad (2.11)$$

The non–diagonal elements $<\psi_f | \hat{A} | \psi_i >$ are associated with transition probabilities between the initial states $|\psi_i>$ and the final states $|\psi_f>$. If the number of valence–nucleons is high, this procedure becomes very complicated; on the other hand, the low energy spectrum of nuclei with many nucleons outside the closed shells shows a simple behaviour which changes systematically from one nucleus to another. These regularities are explained by describing the nuclear correlations with collective motions, corresponding
to variations in the shape of the nucleus. In this way, we obtain a general-
ization of the shell model: the mean field is not an isotropic static potential,
but becomes a variable field that can assume various shapes, not only with
spherical symmetry, but also, for example, ellipsoidal. For these nuclei the
collective motions can be divided into rotations and vibrations [6,7]. The first
correspond to the rotation of the nuclear orientation with shape conservation
and have low excitation energies; the second correspond to oscillations of the
nucleus around its equilibrium shape.

The nuclear surface in polar coordinates can be written:

\[ R(\theta, \phi) = R_0 \{1 + \sum_{\lambda=0}^{\infty} \sum_{\mu=\pm \lambda}^{\lambda} \alpha_{\lambda \mu} Y_{\lambda \mu}(\theta, \phi)\} = R_0 + \Delta R, \]  \hspace{1cm} (2.12)

where \( R_0 \) is the mean nuclear radius, \( Y_{\lambda \mu}(\theta, \phi) \) are the spherical harmonics
and \( \alpha_{\lambda \mu} \) are the coefficients which describe the deformation, generally time–
dependent. Small oscillations around the equilibrium shape may be described
by harmonic oscillators, so the hamiltonian of the system can be written:

\[ \hat{H}_\lambda = \frac{B_\lambda}{2} \sum_{\mu=-\lambda}^{\lambda} |\dot{\alpha}_{\lambda \mu}|^2 + \frac{C_\lambda}{2} \sum_{\mu=-\lambda}^{\lambda} |\alpha_{\lambda \mu}|^2, \]  \hspace{1cm} (2.13)

where the coefficients \( B_\lambda \) and \( C_\lambda \) are related to the frequency vibration \( \omega_\lambda \)
by:

\[ \omega_\lambda = \sqrt{\frac{B_\lambda}{C_\lambda}}. \]  \hspace{1cm} (2.14)

If we consider the low energy spectrum (0 ∼ 4 MeV), only \( \lambda = 2, 3 \) are
important, because \( \lambda = 0 \) represents a compression (or dilatation) without
shape change and \( \lambda = 1 \) represents a translation of the entire nucleus or a
dipole oscillation. Both modes are outside the energy range considered. So
each quantum excitation, called phonon, has an energy $E_\lambda = \hbar \omega_\lambda$ and spin $I = \lambda \hbar$.

For deformed nuclei, the rotation excitation energies may be written:

$$E_{I,k} = \frac{\hbar^2}{2J} [I(I + 1) - k^2] + \frac{\hbar^2}{2J_3} k^2,$$

where $I$ is the total angular momentum of the nucleus and $k$ the projection of $I$ on the symmetry axis. $J (= J_1 = J_2)$ and $J_3$ are the momenta of inertia of the nucleus referred to the principal axes. Equation (2.15) becomes simpler in the case of the $k=0$ rotational band:

$$E_I = \frac{\hbar^2}{2J} I(I + 1),$$

where the allowed values for the angular momentum $I$ are 0, 2, 4, 6, ... . For low angular momenta ($I < 8\hbar$), there is a good agreement between the energy values of the rotational model and the experimental data.

3. Chaotic states in atomic nuclei

In the low energy excitation spectrum of an atomic nucleus, the average level density $\bar{\rho}(E)$ is small and, as discussed in the previous section, one might expect to be able to describe most of the states in detail using nuclear models. However, the average level density increases very rapidly with the excitation energy $E$ (Bethe’s law):

$$\bar{\rho}(E) = \frac{C}{(E - \Delta)^\frac{3}{2}} \exp (A \sqrt{E - \Delta}),$$

where $A, C, \Delta$ are constants for a given nucleus [2]. Therefore, once the region of the neutron emission threshold is reached ($\sim 8$ MeV), the number of levels
is so high that a description of the individual levels has to be abandoned. The aim of nuclear models at this and higher excitation energies is rather to describe special states, like giant resonances and other collective states, which have a peculiar structure. But the detailed description of the sea of background states around the collective ones is fruitless. Thus, for example, observations of levels of heavy nuclei in the neutron–capture region give precise information concerning a sequence of levels from number N to number (N + n), where N is an integer of the order of $10^6$ (see fig. 1) [16]. For these densities of states a level assignment based on shell or collective models becomes a very difficult task. It is therefore reasonable to inquire whether the highly excited states may be understood from the opposite point of view, assuming as working hypothesis that the shell structure is completely washed out and that no quantum numbers other than spin and parity remain good. The outcome of such an approach is termed a statistical theory of energy levels [10–12].

The statistical theory will not predict the detailed sequence of levels in any one nucleus, but it will describe the general appearance and the degree of irregularity of the level structure that is expected to occur in the atomic nuclei, which are too complicated to be understood in detail.

We describe a complex nucleus as a ”black box”, in which a large number of particles are interacting according to unknown laws [13]. The problem is then to define, in a mathematically precise way, an ensemble in which all possible laws of interaction are equally probable. This program, initiated by Wigner and developed by many authors, has, to a large extent, been successful [13].
3.1 The random matrix theory

An appropriate means to define an ensemble of random matrices is provided by the random matrix theory [14,15]. The hamiltonian matrix $H$ is an $N \times N$ stochastic matrix (its matrix elements $h_{ij}$ are random variables) and the probability density is specified by:

$$P(H)dH = P(h_{11}, h_{12}, ..., h_{NN})dh_{11}dh_{12}...dh_{NN},$$  \hspace{1cm} (3.2)

where the probability information is:

$$I\{P(H)\} = \int dHP(H) \ln P_N(H).$$ \hspace{1cm} (3.3)

The aim is to find the function $P(H)$ that minimizes $I$. This is equivalent to assuming the least possible knowledge about $P(H)$. We impose that the $h_{ij}$ are real and, to limit the eigenvalues of $H$ to a finite range, a condition on its norm $[Tr(H^2)]^{1/2}$ is also imposed. Thence $P(H)$ should minimize $I$ subject to the constraints:

$$\int dHP(H) = 1, \int dHP(H)Tr(H^2) = C,$$ \hspace{1cm} (3.4)

which leads to:

$$P(H) = \exp \{\lambda_1 + \lambda_2 Tr(H^2)\}.$$ \hspace{1cm} (3.5)

By inserting (3.5) in (3.4), the Lagrange multipliers $\lambda_1, \lambda_2$ may be determined. This probability distribution defines the so-called Gaussian Orthogonal Ensemble (GOE), since $P(H)$ is invariant for orthogonal transformation and the elements $h_{ij}$ are independent random variables. We can observe that, without the time invariance, the $h_{ij}$ are complex numbers and we have the Gaussian Unitary Ensemble [14].
With the help of (3.5) one can easily obtain \( P(E) = P(E_1, E_2, ..., E_N) \) and so the statistic of the energy spectrum. Mehta defines "statistic" thus: “A statistic is a quantity which can be calculated from an observed sequence of levels alone, without other information and whose average value and variance are known from the theoretical model. A suitable statistic is one which is sensitive for the property to be compared or distinguished and is insensitive for other details” [15]. Various statistics may be used to show the local correlations of the energy levels; in this work we shall use \( P(s) \) and \( \Delta_3(L) \) mainly.

The first statistic measures the probability that two neighboring eigenvalues are a distance “s” apart, in the average level distance unit. For this statistic the GOE average is closely approximated by the Wigner distribution:

\[
P(s) = \frac{\pi}{2} s \exp(-\frac{\pi}{4}s^2),
\]

which gives level repulsion.

The second statistic is defined for a fixed interval \((-L/2, L/2)\), as the least–square deviation of the staircase function \( N(E) \) from the best straight line fitting it:

\[
\Delta_3(L) = \frac{1}{L} \min_{A,B} \int_{-L/2}^{L/2} [N(E) - AE - B]^2 dE,
\]

where \( N(E) \) is the number of levels between \( E \) and zero for positive energy, between \(-E\) and zero for negative energy. The \( \Delta_3(L) \) statistic provides a measure of the degree of rigidity of the spectrum: for a given interval \( L \), the smaller \( \Delta_3(L) \) is, the stronger is the rigidity, signifying the long–range
correlations between levels. For this statistic in the GOE ensemble:

\[ \Delta_3(L) = \begin{cases} \frac{L}{15}, & L \ll 1 \\ \frac{1}{\pi} \log L, & L \gg 1 \end{cases} \quad (3.8) \]

For the GUE there are similar results [14].

If the mean level density for the GOE is calculated, we obtain:

\[ \rho(E) = \begin{cases} \frac{1}{\pi \sigma^2} \sqrt{2N\sigma^2 - E}, & |E| < 2N\sigma^2 \\ 0, & |E| < 2N\sigma^2 \end{cases} \quad (3.9) \]

Although this is an unrealistic result, it can be explained by remembering that global and local behaviours are on different scales and GOE is a good model for local properties.

3.2 Comparison of GOE predictions with experimental data

Neutron resonance spectroscopy on a heavy even-even nucleus typically leads to the identification of about 150 to 170 s-wave resonances with \( J^\pi = \frac{1}{2}^+ \) located 8–10 MeV above the ground state of the compound system, with average spacings around 10 eV and average total widths around 1 eV. Proton resonance spectroscopy yields somewhat shorter sequences of levels with fixed spin and parity, with typically 60 to 80 members. For the statistical analysis, it is essential that the sequences be pure (no admixture of levels with different spin or parity) and complete (no missing levels) [17]. Only such sequences were considered by Haq, Pandey and Bohigas [18]. Scaling each sequence to the same average level spacing and lumping together all sequences one leads to the ”Nuclear Data Ensemble” (NDE), which contains 1726 level spacings.
As shown in fig. 2 the agreement between the experimental data and the 
GOE predictions is surprisingly good (in the GOE model there are no free 
parameters) [19].

It is also important to observe that this agreement in not limited to 
nuclear levels, but is valid for all quantum systems (nuclear, atomic and 
molecular) in a high energy range [1,29].

The behaviour of spectral statistics near the ground state is also of con-
siderable interest. Garrett, German, Courtney and Espino [20] (fig. 3) and 
also Shriner, Mitchell and Von Egidy [21] (fig. 4) have shown that for these 
energies nuclei do not follow the GOE results but behave like a Poisson en-
semble or with an intermediate behaviour between Poisson and GOE. In the 
Poisson ensemble:

\[ P(s) = e^{-s}, \quad \Delta_3(L) = \frac{L}{15}, \]

which gives a high probability for the occurrence of near degeneracies, no 
level repulsion and no correlation between levels.

To sum up, we have seen that the spectra of atomic nuclei show a Poisson→GOE 
transition on increasing the excitation energy (see fig. 5) over the yrast line. 
In the next section we show how this behaviour is not of a purely quantal 
nature but has a classical counterpart in the regular and chaotic hamiltonian 
systems.

4. Order and chaos in classical and quantum mechanics

In this section we introduce the dynamical systems and their stability, 
which may be studied by means of the Lyapunov exponents and metric en-
tropy. With the help of these quantities, we clarify the concept of ergodic system giving a hierarchy of chaos. Then we extend the study to hamiltonian systems, which are essential in order to study the transition order→chaos in nuclear physics.

A dynamical system [22–25] is defined by \( N \) differential equations of the first order:

\[
\frac{d}{dt} \vec{z}(t) = \vec{f}(\vec{z}(t), t),
\]

where the variables \( \vec{z} = (z_1, ..., z_N) \) are in the phase space \( \Omega \) (the euclidean space \( \mathbb{R}^N \), unless otherwise specified). These equations describe the time evolution of the variables and the system they represent.

A solution of the dynamical system is a vector function \( \vec{z}(\vec{z}_0, t) \), that satisfies (4.1) and the initial condition:

\[
\vec{z}(\vec{z}_0, 0) = \vec{z}_0,
\]

often written simply \( \vec{z}(t) \) without the initial condition dependence.

The time evolution of \( \vec{z} \in \Omega \) is obtained with the one parameter group of diffeomorphism \( g^t : \Omega \to \Omega \), so that:

\[
\frac{d}{dt} (g^t \vec{z})|_{t=0} = \vec{f}(\vec{z}, 0).
\]

The group \( g^t \) is called phase flux and the solution is called orbit. The system is called hamiltonian, if the dimension of \( \Omega \) is even and there exists a function \( H(\vec{z}, t) \) given by:

\[
\vec{f}(\vec{z}(t), t) = J \nabla H(\vec{z}, t),
\]

where:

\[
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]
is the symplectic matrix. $H(z, t)$ is the *hamiltonian*.

On the phase space $\Omega$ one usually defines a probability measure $\mu : \Omega \to \Omega$, so that $\mu(\Omega) = 1$. If we choose a subspace $A$ of $\Omega$, the system is measure preserving if:

$$\mu(g^t A) = \mu(A). \quad (4.6)$$

It is well known that hamiltonian systems preserve their measure: the Liouville measure. Dynamical systems which do not preserve their measure are called *dissipative*, and usually have a measure contraction in time evolution.

The dynamic of a system is called *regular* if the orbits are stable to infinitesimal variations of initial conditions. It is called *chaotic* if the orbits are unstable to infinitesimal variations of initial conditions [23–25]. Useful quantities to calculate this behaviour are the Lyapunov exponents, which give the stability of a single orbit, and the metric entropy, which represents a mean exponent for the entire system. A vector of the tangent space $T_{\vec{z}} \Omega$ to the phase space $\Omega$ in the position $\vec{z}$ is given by:

$$\vec{\omega}(\vec{z}) = \lim_{s \to 0} \frac{\vec{q}(s) - \vec{q}(0)}{s}, \quad (4.7)$$

where $\vec{q}(0) = \vec{z}$ and $\vec{q}(s) \in \Omega$. The tangent space vectors are the velocity vectors of the curves on $M$; there are obviously $N$ independent vectors.

Now we can define the Lyapunov exponent:

$$\lambda(z) = \lim_{t \to \infty} \frac{1}{t} \ln |\vec{\omega}(t)|, \quad (4.8)$$

where $\vec{\omega}(t)$ is a tangent vector to $\vec{z}(t)$ with the condition that $|\vec{\omega}(0)| = 1$.

It can be demonstrated that the limit given by (4.8) exists for a compact phase space, and that it is metric independent. Fixing an orbit in the $N$
dimensional phase space, there are $N$ distinct exponents $\lambda_1, \ldots, \lambda_N$, called first order Lyapunov exponents. If the orbit has positive Lyapunov exponents, it is chaotic.

To characterize globally the chaoticity of a system, we introduce the metric entropy. If $\alpha(0)$ is a partition of non–overlapping sets that completely cover the phase space $\Omega$ at the initial instant:

$$\alpha(0) = \{A_i(0) : \Sigma(E) = \cup_i A_i(0), A_i(0) \cap A_j(0) = \phi\}, \quad (4.9)$$

the partition $\alpha(0)$ can evolve in a discretized time flux:

$$\alpha(0), \alpha(1), \alpha(2), \ldots, \alpha(n). \quad (4.10)$$

Then we define $\beta(n)$ as the intersection set of all the sets at every instant:

$$\beta(n) = \{B_i(n) : B_i(n) = A_i(0) \cap A_j(1) \cap \ldots \cap A_k(n)\}. \quad (4.11)$$

In this way, the number of sets defined by $\beta(n)$ do not decrease when $n$ is increased. By introducing a probability measure $\mu$ preserved by dynamics, one can define the metric information of the partition $B(n)$:

$$I_n = -\sum_i \mu(B_i(n)) \ln \mu(B_i(n)). \quad (4.12)$$

The information $I_n$ has the following properties:

(i) $I_n = 0$ if, and only if, there exists a set $B_i(n)$ of the partition $\beta(n)$ so that $\mu(B_i(n)) = 1$;

(ii) $I_n$ assumes the maximum value when $\beta(n) = \{B_1(n), \ldots, B_N(n)\}$ and in this way $\mu(B_i(n)) = \frac{1}{N} \quad \forall i$ i.e. $I_n = \ln N$. 

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The metric entropy of the partition $\alpha(0)$ is defined as:

$$h(A_i(0), \mu) = \lim_{n \to \infty} \frac{I_n}{n}$$

(4.13)
i.e. a time average of the metric information. If all the elements of $\alpha(t)$ have a measure that decreases, on average, exponentially with $t$, then the metric entropy will be positive. A positive entropy indicates that there does not exist a finite number of measures to guess the next one.

We indicate the maximum value of $h$ on all partitions $A_i$ with $h_{KS}$:

$$h_{KS} = \max \{ h(A_i(0), \mu), \forall A_i(0) \}$$

called Kolmogorov–Sinai entropy. According to a very important theorem [26]:

$$h_{KS}(\mu) = \int_A d\mu(\vec{z}) \sum_{\lambda_i > 0} \lambda_i(\vec{z})$$

(4.15)
with $A$ subspace of $\Omega$ and $\lambda_i$ Lyapunov exponents. Therefore the Kolmogorov–Sinai entropy is a very useful tool for showing chaotic behaviour in the region $A$.

A system is called ergodic if the time average is equal to phase space average:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t dt f(g^t(\vec{z}(t))) = \int_{\Omega} d\mu(\vec{z}) f(\vec{z}).$$

(4.16)
Incidentally, as is well known, Boltzmann started from the “ergodic hypothesis” to obtain statistical mechanics of equilibrium. But ergodicity is not sufficient to reach an equilibrium state: one must consider mixing systems.

In a mixing system, every finite element of the phase space occupies for $t \to \infty$ the entire phase space $\Omega$; more precisely: $\forall A, B \subset \Omega$ with $\mu(A)$ and
\( \mu(B) \neq \phi, \)
\[
\lim_{t \to \infty} \frac{\mu(B \cap g^tA)}{\mu(B)} = \mu(A). \tag{4.17}
\]
To have quantitative information of orbit separations, we must introduce \( K\)-systems (Kolmogorov), which are mixing systems with a positive metric entropy:

\[ h_{KS} > 0. \tag{4.19} \]
Such systems are typical chaotic systems.

Among the \( K\)-systems, the most unpredictable ones are the \( B\)-systems (Bernoulli), which have the Kolmogorov–Sinai entropy equal to the entropy of every partition:

\[ h_{KS} = h(A_i(0), \mu), \quad \forall A_i(0). \tag{4.20} \]

4.1 Classical chaos

In classical mechanics, the state of a system of coordinates \( q_i \) and moments \( p_i, \ i = 1, \ldots, n \) in the \( N=2n \) dimensional phase space \( \Omega \), is specified by the hamiltonian \( H = H(\vec{p}, \vec{q}) \), with \( \vec{p} = (p_1, \ldots, p_n) \), \( \vec{q} = (q_1, \ldots, q_n) \) [22].

As is well known, the time evolution is obtained by the Hamilton equations:

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{4.21}
\]
These equations, with the position \( \vec{z} = (q_1, \ldots, q_n, p_1, \ldots, p_n) \), can be written in the more compact form (4.4).

The hamiltonian system is integrable if there are \( n \) functions defined on \( \Omega \):

\[ F_i = F_i(\vec{z}) \quad i = 1, \ldots, N \]
in involution:

\[
[F_i, F_j]_{PB} = \sum_{k=1}^{n} \frac{\partial F_i}{\partial q_k} \frac{\partial F_j}{\partial p_k} - \frac{\partial F_j}{\partial q_k} \frac{\partial F_i}{\partial p_k} = 0, \quad \forall i, j
\] (4.23)

and linearly independent. \([ , ]_{PB}\) are the Poisson Brackets.

For the conservative systems we have \(F_1 = H(\vec{z})\) and also:

\[
\frac{dF_i}{dt} = [H, F_i]_{PB} = 0.
\] (4.24)

Because there are \(n\) constants of motion, every orbit can explore only the \(n\) dimensional manifold \(\Omega_f\):

\[
\Omega_f = \{ \vec{z} : F_i(\vec{z}) = f_i, i = 1, ..., N \}
\] (4.25)

If \(\Omega_f\) is compact and connected, it is equivalent to a \(n\) dimensional torus:

\[
T^n = \{ (\theta_1, ..., \theta_n) \mod 2\pi \}
\] (4.26)

There are \(n\) irreducible and independent circuits \(\gamma_i\) on \(\Omega_f\) and there exists a canonical transformation:

\[
(\vec{p}, \vec{q}) \rightarrow (\vec{I}, \vec{\theta})
\] (4.26)

generated by the function \(S(\vec{q}, \vec{I})\), so that:

\[
I_i = \oint_{\gamma_i} d\vec{q} \cdot \vec{p}, \quad \theta_i = \frac{\partial S}{\partial I_i}
\] (4.27)

The \(I_i\) are called action variables and the \(\theta_i\) are called angle variables. The moments \(\vec{p}\) and coordinates \(\vec{q}\) are periodic functions of \(\vec{\theta}\) with period \(2\pi\):

\[
\vec{q} = \sum_{\vec{m}} \vec{q}_{\vec{m}}(\vec{I}) e^{i\vec{m} \cdot \vec{\theta}}, \quad \vec{p} = \sum_{\vec{m}} \vec{p}_{\vec{m}}(\vec{I}) e^{i\vec{m} \cdot \vec{\theta}}
\] (4.28)
where $\vec{m} = (m_1, \ldots, m_n)$ is an integer vector.

The hamiltonian depends only on action variables: $H = H(\vec{I})$, and so the new equation of motion are:

$$
\dot{\theta}_i = \frac{\partial H(\vec{I})}{\partial I_i} = \omega_i(\vec{I}), \quad \dot{I}_i = -\frac{\partial H(\vec{I})}{\partial \theta_i} = 0, \quad \text{(4.29)}
$$

so:

$$
\theta_i = \omega_i(\vec{I})t + \alpha_i, \quad \text{(4.30)}
$$

where $\alpha_i$ are constants and $\omega_i$ are the frequencies on the torus.

The orbits of an integrable system are quasi–periodic, with $n$ quasi–periods:

$$
T_i = \frac{2\pi}{\omega_i}, \quad \text{(4.31)}
$$

and the orbit is closed if there exists a period $\tau$ so that:

$$
\tilde{\theta}(\tau) = \tilde{\theta}(0) + 2\pi \vec{D}, \quad \text{(4.32)}
$$

with $\vec{D}$ integer vector. To have the closure, we must have:

$$
\vec{\omega} \cdot \vec{m}_i = 0, \quad \text{(4.33)}
$$

and the period is:

$$
\tau = \frac{2\pi}{\omega_c} = \frac{2\pi D_i}{\omega_i} = \frac{D_i}{T_i}. \quad \text{(4.34)}
$$

If we do not have $n$ independent relations among frequencies, the orbits do not close: we have the so called ”torus ergodicity”, but the motion is still regular (see fig. 6).

Adding a small perturbation $V(\vec{I}, \tilde{\theta})$ to an integrable hamiltonian $H_0(\vec{I})$, the total hamiltonian can be written:

$$
H(\vec{I}, \tilde{\theta}) = H_0(\vec{I}) + \chi V(\vec{I}, \tilde{\theta}), \quad \text{(4.35)}
$$
and, generically, the integrability is destroyed. As a consequence, parts of phase space become filled with chaotic orbits, while in other parts the toroidal surfaces of the integrable system are deformed but not destroyed; thus we have a quasi-integrable system.

With growing $\chi$, chaotic motion develops near the regions of phase space where all the $\omega_i$ are commensurate; or, more precisely, where the equation (4.33) is obeyed by integer vectors.

Conversely, tori of the integrable system on which the $\omega_i$ are incommensurate are deformed, but not destroyed immediately (KAM theorem). As $\chi$ increases, the phase space generically develops a highly complex structure, with islands of regular motion (filled with quasi-periodic orbits) interspersed in regions of chaotic motion, but containing in turn more regions of chaos. As $\chi$ grows further, the fraction of phase space filled with chaotic orbits grows until it reaches unity as the last KAM surface is destroyed. Then the motion is completely chaotic everywhere, except possibly for isolated periodic orbits [22,23].

It is very useful to plot a $2n-1$ surface of section $P \subset \Omega$, called Poincaré section (see fig. 7). As shown in figure 8, for an integrable system with two degrees of freedom, the $x = 0$ Poincaré section of a rational (resonant) torus is a finite number of points along a closed curve, while the section of an irrational (non resonant) torus is a continuous closed curve.

Adding a perturbation, the section presents closed curves (KAM tori), whose points are stable (elliptic), and also curves formed by substructures, residua of resonant tori, whose points are unstable (hyperbolic). As the perturbation parameter increases, the closed curves are distorted and reduced...
in number.

4.2 Quantum chaos

In non relativistic quantum mechanics, the time evolution of a state $|\psi>$ of a system with hamiltonian $\hat{H}$ is given by the Schrödinger equation [27]:

$$i\hbar \frac{\partial}{\partial t} |\psi> = \hat{H} |\psi>.$$  \hfill (4.36)

The unitary time–evolution operator $\hat{U}(t)$ is:

$$\hat{U}(t) = \exp \left(-\frac{i\hat{H}}{\hbar}\right),$$  \hfill (4.37)

so that $|\psi(t)> = \hat{U}(t)|\psi>$. If we choose a new state $|\psi'>$ near to $|\psi>$ we have:

$$<\psi'(t)|\psi(t)> = <\psi'|\hat{U}^+(t)\hat{U}(t)|\psi(t)> = <\psi'|\psi>.$$  \hfill (4.38)

The distance between two quantum states is always time-independent, so there is no chaos in quantum mechanics in the sense of time exponential divergence of near states. However an interesting question is: whether there are quantum–mechanical manifestations of classical chaotic motion. We shall use the term quantum chaotic system in the precise, and restricted, sense of a quantum system whose classical analogue is chaotic [24,25].

The studies of the eigenvalues of billiards with different shapes [28,29] have shown that if the system is classically integrable the spectral statistics are well modelled by the Poisson ensemble (3.10) and if the system is classically chaotic, by the GOE (3.6, 3.8) or GUE ensembles, depending on the time–reversal symmetry (fig. 9).
When the classical dynamics of a physical system is regular, the short-range properties of the corresponding quantal spectrum tend to resemble those of a spectrum of randomly distributed numbers (the Poisson spectrum). This is because regular classical motion is associated with integrability or separability of the classical equations of motion. In quantum mechanics, the separability corresponds to a number of independent conserved quantities (such as angular momentum), and each energy level can be characterised by the associated quantum numbers. Superimposing the terms arising from the various quantum numbers, a spectrum is generated like that of random numbers, at least over short intervals. When the classical dynamics of a physical system is chaotic, the system cannot be integrable and there must be fewer constants of motion than degrees of freedom. Quantum mechanically, this means that once all good quantum numbers due to obvious symmetries etc. are accounted for, the energy levels cannot simply be labelled by quantum numbers associated with certain constants of motion. The short-range properties of the energy spectrum then tend to resemble those of eigenvalue spectra of matrices with randomly chosen elements.

Berry and Tabor [30] predict level clustering for any integrable system (with the exception of the harmonic oscillator that has equal spaced levels [31]) in the asymptotic high energy regime, using the semiclassical ($\hbar \to 0$) Einstein–Brillouin–Keller (EKB) quantization of action variables [24]:

$$E_{n_1,\ldots,n_N} = H(I_1 = \hbar(n_1 + a_1/4), \ldots, I_N = \hbar(n_N + a_N/4)),$$

(4.39)

where $a_l$ is the Maslov index for the $I_l$ action variable. In one dimension: $a = 0$ for the rotator and $a = \frac{1}{2}$ for the oscillator.
In quasi-integrable and chaotic systems the EKB quantization is not applicable but we can calculate the level density $\rho(E)$ using a semiclassical formula obtained by Gutzwiller [32,33]:

$$\rho(E) = \tilde{\rho}(E) + \rho_{\text{osc}}(E), \quad (4.40)$$

where:

$$\tilde{\rho}(E) = \frac{1}{(2\pi \hbar)^N} \int d\vec{p} d\vec{q} \delta (E - H(\vec{p}, \vec{q})), \quad (4.41)$$

$$\rho_{\text{osc}}(E) = \sum_j \frac{T_j}{2f(\lambda_j)} \cos \left( \frac{S_j(E) \hbar}{\hbar} - \frac{a_j \pi}{2} \right). \quad (4.42)$$

The summation is over all the periodic orbits of the classical phase space, $T_j$ is the $j$-periodic orbit period, $f$ is a function of the $j$–periodic orbits Lyapunov exponent and $S_j(E) = \oint j d\vec{q} \cdot \vec{p}$.

With the help of (4.42) Berry [34] has calculated the semiclassical behaviour of the spectral rigidity $\Delta_3(L)$ for a chaotic time–reversal system:

$$\Delta_3(L) = \begin{cases} \frac{L}{15}, & L \ll 1 \\ \frac{1}{\pi} \log L, & 1 \ll L \ll L_{\text{max}} \\ \frac{1}{2\pi^2} \log eL_{\text{max}} + \text{oscillations}, & L \gg L_{\text{max}} \end{cases}, \quad (4.43)$$

where $L_{\text{max}} = \frac{\hbar \rho}{T_{\text{min}}}$ with $T_{\text{min}}$ the period of the shortest periodic orbits of the classical phase space. The long–range periodic orbits contribute to the universal behaviour, but there is also a non universal behaviour not predicted by GOE.

In the next section, we describe a schematic shell model to show how an order to chaos transition may occur by studying the classical and quantum properties of the model.
5. Order to chaos transition in a schematic shell model

To study the transition from regular to chaotic states in atomic nuclei, an extension of the two level Lipkin–Meshkov–Glick (LMG) model was proposed by Meredith, Koonin and Zirnbauer (MKZ) [35–37]. The two level LMG model has only one degree of freedom, i.e. the particle number in the upper level, and so its classical limit does not have chaotic behaviour. The generalization proposed by MKZ has the SU(3) symmetry and two degrees of freedom: its classical limit shows an order to chaos transition.

The SU(3) model consists of M identical particles, labelled by the index n, each of which can be in three single–particle states having energy $\epsilon_i$. The hamiltonian is:

$$\hat{H} = \sum_{i=1}^{3} \epsilon_i \hat{G}_{ii} + \sum_{ij=1}^{3} V_{ij} \hat{G}_{ij}^2,$$  \hspace{1cm} (5.1)

with $\epsilon_3 = \epsilon_1 = \epsilon$ and $\epsilon_2 = 0$, where $V_{ij}$ is the strength of the two–body interaction between states i and j:

$$V_{ij} = V_{ji} \hspace{0.5cm} \text{and} \hspace{0.5cm} V_{ii} = 0,$$  \hspace{1cm} (5.2)

with:

$$\hat{G}_{ij} = \sum_{n=1}^{M} \hat{a}_{ni}^+ \hat{a}_{nj}.$$  \hspace{1cm} (5.3)

The operator $\hat{a}_{ni}^+$ creates a particle n in state i, and $\hat{a}_{ni}$ annihilates a particle n from state i; each term of $\hat{G}_{ij}$ takes a particle, n, out of state j and into state i, and each $\hat{G}_{ii}$ counts the particles in state i.

Applying the usual fermion anti–commutation rules to the $\hat{a}_{ni}$ operators, we get:

$$[\hat{G}_{ij}, \hat{G}_{hk}]_- = \hat{G}_{ik} \delta_{jh} - \hat{G}_{jh} \delta_{ik},$$  \hspace{1cm} (5.4)
and the 9 operators $\hat{G}_{ij}$ are generators for $U(3)$. Taking into account the number conservation, $\sum_{i=1}^{3} \hat{G}_{ii} = M$, this becomes $SU(3)$.

5.1 Classical limit

The quantum 4–dimensional phase space of $SU(3)$ is $U(3)/U(2) \otimes U(1)$, where $U(2) \otimes U(1)$ is the maximal stability subgroup which leaves the ground state $|00\rangle$ invariant up to a phase factor [38].

The coherent states of $U(3)/U(2) \otimes U(1)$ are:

$$|SU(3), \hat{\Phi} > = \hat{\Phi}|00\rangle,$$

where

$$\hat{\Phi} = \exp \{ \sum_{i=2}^{3} (\nu_i \hat{G}_{i1} - \nu_i^* \hat{G}_{1i}) \},$$

with $\nu_i$ complex numbers.

The Bergmann kernel is:

$$k(\vec{z}, \vec{z}^*) = <SU(3), \hat{\Phi}|SU(3), \hat{\Phi} >= (1 + Z^+ Z)^M,$$

where $Z$ is the 3x1 matrix:

$$Z = (z_i) = (\nu_i) \frac{\tan(\nu)}{\nu},$$

with $\nu = \sqrt{\nu_2 \nu_2^* + \nu_3 \nu_3^*}$. The metric of $U(3)/U(2) \otimes U(1)$ is then given by:

$$g_{ij} = \frac{\partial^2 \ln k(\vec{z}, \vec{z}^*)}{\partial z_i \partial z_i^*},$$

and the nondegenerate closed two–form $\omega$ of $U(3)/U(2) \otimes U(1)$ is:

$$\omega = \sum_{i,j=2}^{3} g_{ij} dz_i \wedge dz_j^*,$$
with Poisson Brackets:

\[
[f, g]_{PB} = -i \sum_{i,j=2}^{3} g^{ij} \left[ \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z^*_j} - \frac{\partial f}{\partial z^*_i} \frac{\partial g}{\partial z_j} \right].
\] (5.11)

By introducing the canonical coordinates:

\[
\frac{1}{\sqrt{2}} (q_i + ip_i) = \sqrt{M} \nu_i \sin \nu,
\] (5.12)

one can show that (5.12) will be transformed into the following canonical form:

\[
[f, g]_{PB} = \sum_{i,j=2}^{3} \left[ \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_i} \right].
\] (5.13)

The semiquantal dynamics of this many–body system with fixed nucleons number \( M \) is determined by varying the effective action:

\[
S = \int \left[ \frac{1}{2} (\vec{p} \cdot d\vec{q} - \vec{q} \cdot d\vec{p}) - H(\vec{p}, \vec{q}) \right] dt.
\] (5.14)

The result of such a variation is the set of classical–like dynamical equations:

\[
\frac{dq_i}{dt} = \frac{\partial H(\vec{p}, \vec{q})}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H(\vec{p}, \vec{q})}{\partial q_i}
\] (5.15)

with the Hamiltonian functions given by:

\[
H(\vec{p}, \vec{q}) = < SU(3), \hat{H} | SU(3), \hat{\Phi} >.
\] (5.16)

Equation (5.15) is in fact equivalent to the time–dependent Hartree–Fock (TDHF) dynamical equations. It should be noted that (5.15) is not a classical limit because there is at least first–order quantum correlation in \( H(\vec{p}, \vec{q}) \).

The phase space representation of the generators \( \hat{G}_{ij} \) in the coherent–states basis is:

\[
< |\hat{G}_{11}| > = \frac{1}{2} (2M - p^2 - q^2),
\] (5.17)
< |\hat{G}_{ii}| > = \frac{1}{2}(q_i + ip_i)\sqrt{2M - p^2 - q^2},
< |\hat{G}_{ij}| > = \frac{1}{2}(q_j + ip_j)(q_i - ip_i),
< |\hat{G}_{ji}| > = < |\hat{G}_{ij}| >^*,

where | > represents |SU(3), \Phi > and (q_2^2 + q_3^2 + p_2^2 + p_3^2) \leq 2M.

The quantum correlations of the quadratic function of the generators can be computed and the results are:

$$\Delta(\hat{G}_{ij}^2) = < |\hat{G}_{ij}^2| > - < |\hat{G}_{ij}| >^2 = -\frac{1}{M} < |\hat{G}_{ij}| >^2.$$  \text{(5.18)}

In this way the mean value of the hamiltonian in the coherent states representation is:

$$H(\vec{p}, \vec{q}) = \frac{\epsilon_1}{2}[2 - (p_2^2 + p_3^2 + q_2^2 + q_3^2)] + \frac{\epsilon_2}{2}(p_2^2 + q_2^2) + \frac{\epsilon_3}{2}(p_3^2 + q_3^2) + \frac{\chi}{4}[1 - \frac{1}{M}](p_2^2 + p_3^2)^2 - (q_2^2 + q_3^2)^2 + (q_2^2 - p_2^2)(q_3^2 - p_3^2) + 4(q_2q_3p_2p_3).$$  \text{(5.19)}

It should be noted that the phase space has been scaled in (5.19) such that \(q_2^2 + q_3^2 + p_2^2 + p_3^2 \leq 2\), with \(\chi = MV/\epsilon\). The classical limit of the SU(3) model can be realized by the limit \(M \rightarrow \infty\), and, in this limit, the quantum correlation \(\Delta(\hat{G}_{ij}^2)\) goes to zero (see 5.18). So the classical hamiltonian is \((\epsilon = 1)\):

$$H_{cl} = -1 + \frac{1}{2}q_2^2(1 - \chi) + \frac{1}{2}q_3^2(2 - \chi) + \frac{1}{2}p_2^2(1 + \chi) + \frac{1}{2}p_3^2(2 + \chi) + \frac{\chi}{4}[1 - \frac{1}{M}]((q_2^2 + p_2^2)^2 - (q_2^2 + q_3^2)^2 - (q_2^2 - p_2^2)(q_3^2 - p_3^2) - 4q_2q_3p_2p_3].$$  \text{(5.20)}

with the phase space given by \(\Omega = \{(q_2, q_3, p_2, p_3) \in R^4 : (q_2^2 + q_3^2 + p_2^2 + p_3^2) \leq 2\}\). This hamiltonian represents two oscillators coupled non-linearly.
by the parameter $\chi$ (the interaction between the nucleons). The system is integrable if the interactions are neglected but the interactions break this regular behaviour and produce an order to chaos transition.

5.2 Classical calculations

Using the Hamilton equations of (5.20), in order to analyze the stability of the system, we calculated the periodic orbits of this model:

$$\dot{\vec{z}} = J \nabla H_{cl}(\vec{z}, \chi), \quad (5.21)$$

where $\vec{z} = (q_2, q_3, p_2, p_3)$, $\nabla = (\frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_3})$, $e J$ is the 4x4 symplectic matrix:

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (5.22)$$

where $I$ is the 2x2 identity matrix.

To explore the phase space $\Omega$, we chose a 4–dimensional lattice of initial conditions $\vec{z}_0 = \vec{z}(t = 0)$, $\vec{z}_0 \in \Omega$. The side of the lattice has about $10^4$ initial conditions.

The time evolution of (5.21), with the initial condition $\vec{z}_0$, is obtained using a 4$^{th}$ order Runge–Kutta method [39]. The following function can then be constructed:

$$d(\vec{z}_0, t) = d(\vec{z}_0, \vec{z}_t) = |\vec{z}_t - \vec{z}_0|, \quad (5.23)$$

where $\vec{z}_t = \vec{z}(t)$. (5.23) is the function to minimize, with minimum value equal to 0. To obtain a periodic orbit, it is not actually necessary to vary all five parameters $(\vec{z}_0, t)$; in fact we can fix $E = H(\vec{z}_0)$ or $t$, which correspond to focusing either on constant energy or constant period in the E–T plot.
Another algorithm [40] has been used to find the minimum of \( d \). Since \( d \) is the result of a complicated calculation and its derivatives are not available, this algorithm is particularly useful in our case because the derivatives of \( d \) are not required [41].

These periodic trajectories occur in one parameter families, each of which describes a continuous curve in the energy–period plane (fig. 10); just as in quantum mechanics the set of energy levels is characteristic of a given hamiltonian \( H \), so \( H_{cl} \) can be characterized by the plots \( E–T \), where \( E \) is the energy of a periodic trajectory and \( T \) the corresponding period.

If \( \vec{w} \) is a vector of the tangent space \( T\Omega_{\vec{z}} \) of the phase space manifold \( \Omega \) at \( \vec{z} \), its time evolution is given by:

\[
\dot{\vec{w}} = J \frac{\partial^2 H(\vec{z})}{\partial \vec{z}^2} \vec{w}.
\]  

(5.24)

By (5.21) and (5.24) the Lyapunov exponents can be calculated:

\[
\lambda(\vec{z}) = \lim_{t \to \infty} \frac{1}{t} \ln |\vec{w}(t)|.
\]  

(5.25)

In terms of the stability matrix \( M(0, t) \) [42], defined as:

\[
M_{ij}(0, t) = \frac{\partial z_i(t)}{\partial z_j(0)},
\]  

(5.26)

\( \lambda(\vec{z}) \) can be written:

\[
\lambda(\vec{z}) = \lim_{t \to \infty} \frac{1}{t} \ln |M(0, t)|,
\]  

(5.27)

where \( |M(0, t)| \) is a norm of the matrix \( M(0, t) \).

This matrix can be calculated by solving its equations of motion:

\[
\dot{M} = J \frac{\partial^2 H(\vec{z})}{\partial \vec{z}^2} M,
\]  

(5.28a)
with the initial conditions:

\[ M(0) = I \]  

(5.28b)

where \( I \) is the 4x4 identity matrix. The calculation of the Lyapunov exponents is related to that of the eigenvalues \( \sigma_i \) of the matrix \( M(0, T) \):

\[ \lambda_i(\vec{z}) = \frac{1}{T} \ln \sigma_i. \]  

(5.29)

Now, using the unitarity of \( M \), a periodic orbit is unstable if

\[ Tr(M) > 4 \quad \text{or} \quad Tr(M) < 0 \]  

(5.30a)

and stable if

\[ 0 < Tr(M) < 4. \]  

(5.31b)

It is interesting to study the change of stability of periodic trajectories as a function of \( \chi \). In figure 11 the ratio between the number of stable orbits and the number of total orbits with period \( T < 30 \) is plotted vs \( \chi \). As can be seen the sensitivity of the orbits to a small change of \( \chi \) is quite different for different values of \( \chi \), reflecting the transition order \( \rightarrow \) chaos as the coupling constant increases.

As mentioned in section 4.2, the calculation of the long–range behaviour (non–universal) of \( \Delta_3 \) requires the knowledge of the shortest period \( T_{\text{min}} \) of the closed trajectories. For a chaotic system with time–reversal symmetry [34]:

\[ \Delta_3^\infty = \frac{1}{\pi^2} \ln (e L_{\text{max}}) - \frac{1}{8}, \]  

(5.32a)

where:

\[ L_{\text{max}} = \frac{\hbar \rho}{T_{\text{min}}} \]  

(5.32b)
and $\bar{\rho}$ is the average level density. Obviously $\bar{\rho}$ and $T_{\text{min}}$ are functions of the coupling constant $\chi$.

Figure 12 shows $\Delta_3(L)$ for $\chi = 100$, $T_{\text{min}} = 0.04$, $\bar{\rho} = 115$. As can be seen, the breaking of universality is near $L=38$. Obviously the semi–classical estimate of the $\Delta_3(L)$ saturation does not agree with the quantal calculations for $\chi = 0.5$ and $\chi = 2$ because the equation (5.32) is valid only for a \textit{chaotic} system [43].

Incidentally, Åberg [44] gives a rough estimate of the shortest periodic orbit in a nucleus:

$$T_{\text{min}} \approx \frac{4R}{v_F} \approx \frac{4 \cdot 1.2A^{1/3}}{0.3c}, \quad \text{ (5.33)}$$

and thus $L_{\text{max}} \approx 80A^{-1/3}\bar{\rho}$. For $^{152}\text{Dy}$ at $E_x \approx 3.0$ MeV, $\bar{\rho} \approx 560$ MeV$^{-1}$, we get $L_{\text{max}} \approx 8000$, i.e. we would need a sequence of more than 8000 energy levels with the same spin and parity. This example shows that we should hardly expect to see a non–generic behaviour of long–range spectrum fluctuations due to short periodic orbits in nuclei.

\textbf{5.3 Quantum calculations}

In order to find the eigenstates and eigenvalues of the hamiltonian (5.1), we need to find a complete set of basis states. A natural basis can be written $|bc>$ meaning $b$ particles in the second level, $c$ in the third and, of course, $M-b-c$ in the first level; in this way $|00>$ is the ground state with all the particles in the lowest level [36,37]. We can write the general basis state:

$$|bc> = \sqrt{\frac{1}{b!c!} \hat{G}_{21}^b \hat{G}_{31}^c} |00>, \quad \text{ (5.34)}$$
with $\sqrt{\frac{1}{bc!}}$ the normalizing constant.

From the commutation relation (5.4) we can calculate the expectation values of $\hat{H}_M$ and thus, eigenvalues and eigenstates of $\hat{H}_M$; in this way the energy spectrum range is independent of the number of the particles:

$$
< b'c' | \frac{\hat{H}}{M} | bc > = \frac{1}{M}(-M + b + 2c)\delta_{bb'}\delta_{cc'} - \frac{\chi}{2M^2}Q_{b'c',bc},
$$

(5.35)

where:

$$
Q_{b'c',bc} = \sqrt{b(b-1)(M-b-c+1)(M-b-c+2)}\delta_{b-2,b'}\delta_{c,c'}
$$

+ $\sqrt{(b+1)(b+2)(M-b-c)(M-b-c-1)}\delta_{b+2,b'}\delta_{c,c'}$

+ $\sqrt{c(c-1)(M-b-c+1)(M-b-c+2)}\delta_{b,b'}\delta_{c-2,c'}$

+ $\sqrt{(c+1)(c+2)(M-b-c)(M-b-c-1)}\delta_{b,b'}\delta_{c+2,c'}$

+ $\sqrt{(b+1)(b+2)c(c-1)}\delta_{b+2,b'}\delta_{c-2,c'}$

+ $\sqrt{b(b-1)(c+1)(c+2)}\delta_{b-2,b'}\delta_{c+2,c'}$

and $\chi = MV/\epsilon$. The expectation value $< \hat{H}_M >$ is real and symmetric. For any given number of particles M, we can set up the complete basis state, write down the matrix elements of $< \hat{H}_M >$ and then diagonalize $< \hat{H}_M >$ to find its eigenvalues. $< \hat{H}_M >$ connects only states with $\Delta b = -2, 0, 2$ and $\Delta c = -2, 0, 2$ which makes the problem easier. We group states with $b, c$ even; $b, c$ odd; $b$ even and $c$ odd; $b$ odd and $c$ even. This means that $< \hat{H}_M >$ becomes block diagonal containing 4 blocks which can be diagonalized separately; these matrices are referred to as ee, oo, oe and eo.

To separate regular and chaotic state in a quantum system a powerful tool is the study of spectral statistics. We obtain [45] a good agreement with
GOE in the classically chaotic region and, in the classically regular region, a good agreement with Poisson statistic (see fig. 13). In this figure the continuous curve is the Brody distribution, discussed in section 6.5.

On the basis of the semiclassical torus quantization, the presence of crossings in a small $\chi$ neighbourhood may be interpreted as the signature of quasi crossing in the true system, and thus as a signature of torus destruction, when the exact levels are “split” at the crossing [46–48].

Another method to study the quantum stochasticity is the sensibility of the energy levels to variations of the perturbation parameter; the behaviour of the curvature of the energy level $E(\chi)$ in a small $\chi$ neighbourhood is an example [49].

When the parameter $\chi = 0$, then the hamiltonian represents two oscillators, and there are many degenerations, but for $\chi \neq 0$ these degenerations are broken. We calculated [45], for a large number of particles (semiclassical limit) the density of quasi crossings outside the degeneration region as function of the parameter $\chi$:

$$\rho(\chi) = \frac{\Delta N}{\Delta \chi}$$  \hspace{1cm} (5.37)

where $\Delta N$ is the number of quasi crossings in the parameter range $\Delta \chi$. To obtain $\Delta N$, we fixed three values $\chi - \Delta \chi$, $\chi$, and $\chi + \Delta \chi$ and imposed that:

$$s_i(\chi - \Delta \chi) > s_i(\chi) \hspace{1cm} (5.38a)$$

$$s_i(\chi + \Delta \chi) > s_i(\chi) \hspace{1cm} (5.38b)$$

where $s_i(\chi) = E_{i+1}(\chi) - E_i(\chi)$. The results (fig. 14) show for all the classes a maximum of quasi crossings for $\chi = 2$ in agreement with the transition to
chaos of fig. 12.

In order to study the sensitivity of the energy levels to small changes of the parameter $\chi$, we have used the statistic $\Delta^2(E)$ [50], defined as:

$$\Delta^2(E_i) = |E_i(\chi + \Delta \chi) + E_i(\chi - \Delta \chi) - 2E_i(\chi)|.$$  \hspace{1cm} (5.39)

This statistic measures the curvature of $E_i$ in a small range $\Delta \chi$. Fig. 15 shows $\Delta^2(E)$ in the $[-0.5,0.5]$ energy range for different values of $\chi$; in this case we note the formation of a peak in correspondence to the $\chi = 2$ region.
6. Wave function behaviour and EM decay of regular and chaotic states

As we have seen in the previous sections, it is now well known that in atomic nuclei there are two different states: regular and chaotic. These states can be highlighted with the aid of various statistics. The statistics we have considered before concern the eigenvalues of the system, and the most frequently used are those of level spacing; i.e. the nearest neighbour distribution, \( P(s) \), and the stiffness of the spectrum \( \Delta_3(L) \). Since these statistics display clearly different behaviour depending on the regularity or chaoticity of the nucleus, they are very useful to distinguish between the two regimes and, at the same time, characterize the transition from one region to the other.

However, it is well known that, in general, the wave functions and the transition probabilities are much more sensitive to the purity of states than the eigenvalues are. Consequently, it is interesting to undertake the study of statistics directly related to the wave functions with the aim of characterizing the ordered or chaotic behaviour of atomic nuclei.

Using the SU(3) schematic shell model for the nucleus, described in Section 5, we have studied the various statistics concerning the wave functions, such as the intensity of the momenta of the wave function \( I_m \), the information measure or entropy \( S \), and the correlation functions \( K_{n,m} \).

The momenta \( I_m \) and the information measure \( S \) are defined \[51\]:

\[
I_m = \sum_{b,c} |\psi(b,c)|^{2m},
\]  

\[(6.1)\]
\[ S = - \sum_{b,c} |\psi(b, c)|^2 \ln |\psi(b, c)|^2. \]  

(6.2)

Figure 16 shows that the lower momenta (m=2,3,4) diverge in the regular region but have a constant behaviour in the chaotic region, as predicted by [52].

The information measure \( S \) also shows different behaviours in the two regions, assuming high values in the chaotic region and low values in the regular region because wave functions are localized (see fig. 17).

Another way to separate regular from chaotic states is with the help of the transition probabilities among different states [53]. Only a few strong transitions between regular states and many weak transitions between chaotic states are expected. The transition probability is defined:

\[ P_{ij}^{(k)} = |< \psi^\tau_i | T_k | \psi^\tau_j '>|^2; \]  

(6.3)

where:

\[ T_1 = \frac{1}{2}(G_{12} - G_{21}), \quad T_2 = \frac{1}{2}(G_{13} - G_{31}), \quad T_3 = \frac{1}{2}(G_{23} - G_{32}). \]  

(6.4)

\[ |\psi^\tau_i > = \sum_{\alpha=1}^{d_\tau} x_{i,\alpha}^\tau |b, c > \]  

(6.5)

is the wave function of the j–state of class \( \tau \) with dimension \( d_\tau \) (ee, oo, eo and oe). Scaling \( P_{ij}^k \) with a local mean value \( < P_{ij}^k > \), the distributions \( \frac{P_{ij}^k}{< P_{ij}^k >} \) follow the above previsions very well (see fig. 18).

The previous results clearly show that the wave function behaviour of atomic nuclei is a very powerful tool for distinguishing regular and chaotic regimes. In particular, the study of the transition probabilities between different states is of great interest.
6.1 Decay of regular and chaotic states

Recently, new interest has been shown in the study of the decay of unstable quantum states. In this section we briefly review the literature of major interest. The section is organized as follows: first, we look at the current experimental stage to illustrate the reasons and interest for studying the decay of quantum systems. Secondly, we review the main current theoretical approaches to the problem. We begin with the study of "warm" rotational nuclei, to continue with that of unstable systems described by means of an anti-hermitian effective hamiltonian. Finally, we briefly discuss the Interacting Boson Model (IBM) that affords a wide spectrum of applications, from vibrational to rotational nuclei as well as transitional regions.

6.2 Experimental stage

The new generation of experimental instruments like GASP, EUROBALL, EUROGAM, etc [54], which provide a resolution in the order of a hundred times greater than that of the previous generation, will open a new way in nuclear spectroscopy research. Over the last few years, as discussed above, great interest has been shown in the study of the properties of the atomic nuclei from a statistical point of view. This has made it possible, with the aid of the Random Matrix Theory [14], to distinguish between regular and chaotic regimes, the onset of chaos, the transition from one regime to the other, etc.

Let us now turn our attention to the recent statistical analysis of $\gamma$-ray spectra of "warm" rotational nuclei by Garrett, Hagemann and Herskind.
In their work, two and three-fold energy correlation measurements between gamma rays were analyzed to investigate high spin states of the so-called quasi-continuum that spreads up to 5-10 MeV above the yrast line. This study had already been proposed by Guhr and Weidenmüller [56] who stressed the coexistence of collectivity and chaos in nuclei at several MeV of excitation energy; they suggested the study of stretched collective E2 decays of high spin states in deformed even-even nuclei at, or above, the yrast line. It was shown that the collective E2 transitions from states of spin $I$, located several MeV above the yrast line, populate many states (of the same parity) of $(I - 2)$ spin rather than a single one (as happens with the E2 transitions from states near the yrast line, where a well defined rotational band structure exists). Thus the transition strength is spread over an energy interval that is called rotational damping width. The analysis of the group of Herskind is indeed oriented in this direction. By performing a fluctuation analysis of the 2–D $E_{\gamma_1} x E_{\gamma_2}$ spectrum they obtained information on the number of decay routes the nuclear decay flow takes to go from the initial high spin state to the damped region (which is supposed to be in the diagonal valley, $x = y$, of the 2–D spectra). A schematic illustration of the average flow of $\gamma$-ray decay from high-spin states induced by heavy-ion compound reactions is shown in fig. 19.

The 2–D and 3–D fluctuation analysis method developed by this group [55] has been applied to the spectra from a triple coincidence experiment made by the Manchester–Daresbury–Copenhagen–Bonn collaboration at the Daresbury Tandem Accelerator. The reaction consisted in bombarding $^{124}Sn$ by a beam of $^{48}Ca$ of 215 MeV, to form the residual nuclei of $^{168}Yb$ at the
highest spin $I_{\text{max}} = 60\hbar$. From the fluctuation analysis done on $^{168}\text{Yb}$ they concluded that the rotational correlations originating from well defined rotational band structures only exist up to 1 MeV above the yrast line. Over this limit the main decay routes spread out into many branches. This may be accounted for by the rotational damping phenomenon, as Guhr and Weidenmüller [56] had previously suggested in 1989 from a theoretical calculation with a GOE hamiltonian weakly perturbed by a residual interaction. In the conclusions of this work, the authors proposed the measurement of the spreading width of E2-strength for a high-spin state well above the yrast line, to determine the possible onset of chaos in “warm” deformed nuclei. The fluctuation analysis mentioned above has in fact taken up this suggestion.

The first conclusions extracted from this new analysis of γ-ray data suggest that there is much interest to be had from the study of states near and above the yrast line, ”warm” rotational nuclei and the decay of unstable quantum states. New theoretical attention has been devoted to these phenomena.

**6.3 Rotational damping motion**

Due to the recent experiments in γ-ray detection, a renewed interest has developed in the study of the decay of chaotic states and the possibility of clarifying the mechanisms by which an excited nucleus (6–8 MeV over yrast line) undergoes a transition from a regular to a chaotic behaviour.

In this sense, the work of Matsuo, Dossing, Herskind and Frauendorf [57]
is very interesting. They investigate the properties of energy levels and rotational transitions as a function of the excitation energy, in "warm" deformed nuclei, and show that, at an excitation energy of the order of $E_x \simeq 8$ MeV, there is considerable fragmentation of the rotational E2 strength distribution when it is represented against the $\gamma$ energy (see fig. 20). This fragmentation with increasing energy is due to the mixing of the rotational bands caused by the Surface Delta Interaction (SDI):

$$V(1, 2) = 4\pi V_0 \delta(r_1 - R_0) \delta(r_2 - R_0) \cdot \sum_{\lambda, \mu} Y^*_{\lambda, \mu}(r_1) \cdot Y_{\lambda, \mu}(r_2), \quad (6.6)$$

which contains all the possible multipolarities. Actually, the high multipole components of this SDI are shown to be responsible for the mixing of rotational bands which is reflected in fluctuations typical of chaotic behaviour. The strength fluctuations of the E2 transitions, for instance, obey a Porter–Thomas distribution above a certain excitation energy, assumed to be the threshold for the realization of quantum chaos in the system (fig. 21).

In particular, the same authors have also studied high spin levels in nuclei with a cranked shell model extended to include residual two-body interactions [58]. Here again the residual interaction induces the transition from a regular to a chaotic regime, since when the rotational bands do not interact, $\gamma$-ray energies behave like random variables, i.e., they obey a Poisson distribution typical of a regular regime. On the contrary, when the residual interaction is added, at an excitation energy over 600 KeV, a gradual rotational damping is established, and, at 1.8 MeV above the yrast line, the complete damping is observed and typical GOE fluctuations of the energy levels and transition strengths are produced (fig. 22).
A similar approach is used by Åberg [59] to study the rotational damping of rapidly rotating nuclei. As in the preceding approach, a cranked Nilsson model with a schematic residual interaction is used to study the distribution of the E2-strength, and $\gamma - \gamma$ correlations. In particular the normal deformed (ND) $^{168}$Yb and the superdeformed (SD) $^{152}$Dy are studied, and the different behaviours of these nuclei with increasing energy are compared. The main difference between the approach of Åberg and that of the authors of ref. [58] is that in the former the energies of the np-nh states are given in the laboratory frame as functions of the angular momentum I. So, diagonalization is performed for each fixed I value. This is reflected in a change of the individual energies, but the conclusions regarding the different statistical properties are the same. The Åberg approach is perhaps more useful if a direct comparison with $E_\gamma - E_\gamma$ experimental spectra is to be carried out. The "classical" eigenvalue statistics are studied (i.e. $P(s)$, $\Delta_3(L)$) with special regard to the $\Delta_3(L)$ since it gives a measure of long-range correlations. In this model the transition from order to chaos is brought about by changing the strength of the two-body force ($\Delta$) which is taken as a parameter, $V_2 = \pm \Delta$, where the sign is chosen randomly to avoid coherent effects. Following the procedure of Brody [60], the $\Delta_3(L)$ spectrum is fitted with a single parameter, $q$, to study the mixing between the Poisson and GOE distribution:

$$\Delta_3(L, q) = \Delta_3^P[(1 - q)L] + \Delta_3^{GOE}(qL).$$ (6.7)

The dynamical properties are studied in this model and again a rotational damping is manifested in the fragmentation of E2 strength in many daughter states. The average standard deviation of the E2-strength function saturates

42
for sufficiently large values of the strength of the two-body residual interaction, following the same trend as $q$ parameter, which gives the mixing of a Poisson and GOE behaviour of spectrum properties (see figs. 23–24).

In addition to the previously mentioned rotational damping, Åberg also studies motional narrowing, i.e., the decrease of the width of nuclear magnetic resonances when the temperature increases, a phenomenon which may be studied in a simple two-band model. The narrowing of the strength function is accomplished by a change from a Gaussian shaped strength to a Breit-Wigner shape. This phenomenon is understood in terms of time scales: when the time scale of the fluctuations in available rotational frequencies is much less than the time the wave function takes to spread out over the basis states, the intensity spectrum is Gaussian, while in the opposite case the corresponding spectrum is of the Breit-Wigner type.

Since, due to the motional narrowing the E2 strength may become very narrow at the end, a small number of relatively strong E2 transitions may be observed, and, consequently, a "suppression of chaos". In fact the transition entropy defined as [44]:

$$H^{T,\alpha} = - \sum_{\alpha'} [M_{\alpha\alpha'}]^2 \cdot \ln([M_{\alpha\alpha'}]^2),$$

(6.8)

where:

$$M_{\alpha\alpha'}(I) = \langle \alpha', I - 2|\text{M}(E2)|\alpha, I \rangle$$

(6.9)

decreases when motional narrowing sets in.

The $\gamma - \gamma$ correlations between consecutive $\gamma$-rays are also analyzed to study the fragmentation of E2 strength. Correlations are revealed at small
excitation energies or when a small $\Delta$ is used, while at high excitation energies these correlations disappear.

6.4 The study of unstable systems using a complex effective hamiltonian

A different and more general approach to the study of unstable quantum systems is that of Sokolov and Zelevinsky [61] and more recently that of Mizutori and Zelevinsky [62], in which the statistical theory of spectra, formulated in terms of random matrix theory, is generalized to treat unstable states, i.e. those coupled to open channels. In this way the influence of the coupling with the continuum on the properties of internal states can be better understood.

In general, as we have seen up to now, the level statistics are treated as if the states were stationary but, in reality, all excited states are resonances embedded in the continuum. So if one intends to study the most general states of a system (including the unstable ones) a new approach to the problem has to be used.

When the widths of the states are small as compared with level spacings, the approximation of discrete levels is reasonable for long-lived states. However, when the widths increase and the levels overlap, the effects of the finite lifetime of those become crucial and the application of a random matrix ensemble of hermitian hamiltonians is not enough appropriate.

Since the standard gaussian ensembles are applicable to discrete stationary states, while an excited state decays via open channels, the use of a
non-hermitian effective hamiltonian $H$ is imposed to take into account the width of the states due to their finite life [61,62].

Within this model the reaction amplitudes are represented, using the general theory of resonant nuclear reactions, as sums of pole terms in the complex plane. Such poles, $\epsilon_n = E_n - (1/2)i\Gamma_n$ are the eigenvalues of the non-hermitian effective hamiltonian $\bar{H} = H - (i/2)W$ where $H$ is the hermitian part belonging to the GOE. The amplitudes of the antihermitian part, $W$, have a separable structure $W_{nm} = \Sigma_c A_n^c A_m^c$ due to the unitarity of the scattering matrix, where $A_n^c$ are the amplitudes for the decay of intrinsic states of the system, $|n\rangle (n=1,...,N)$ into different channels, $c$ ($c=1,...,k$) and that must be real according to time reversal invariance of the full hamiltonian, $\bar{H}$. Those eigenvalues correspond to the intermediate unstable states of energies $E_n$ and widths $\Gamma_n$.

Decaying systems are hence described by ensembles of random non-hermitian hamiltonians represented by N-dimensional matrices where the hermitian part of the effective hamiltonian, $H$, belongs to the GOE. The decay amplitudes $A_n^c$, on the other hand, are assumed to be Gaussian random variables completely uncorrelated with the matrix elements of $H$ and among each other.

Within this model the distribution function of complex energies, the level spacing distribution and the width distribution are studied in ref. [61,62] for weak and strong coupling to the continuum, as well as for the transitional region.

The authors of ref. [62] discussed in great detail the single channel case:
They assumed that $<A_n> = 0$, $<H_{mn}> = 0$, and:

$$<H_{mn'}H_{m'n'}> = \frac{a^2}{4N}(\delta_{mn'}\delta_{m'n'} + \delta_{mn}\delta_{m'n'}). \quad (6.10)$$

The strength of coupling to the continuum is given by the overlap parameter \(\chi = \eta/a = 2<\Gamma>/D\) where \(\eta = \sum_j <\Gamma_j>\); \(D\) is the mean level spacing.

When the overlap parameter is small (\(\chi \ll 1\)), that is, when coupling with the continuum is weak, the hermitian part \(H\) dominates, while the antihermitean part, \(W\), prevails in the opposite limit (\(\chi \gg 1\)).

Typical signatures of the transition from weak to strong coupling are observed: first, concerning the level spacing distribution, which gives the simplest characterization of spectral correlations, it is observed that for a weak coupling (where the hermitian part, \(H\), of the effective hamiltonian dominates) the distribution of levels corresponds to a Wigner distribution, and deviations from it are small. As the coupling with the continuum increases, the Wigner function ceases to be a good approximation to the GOE and a new feature is observed: the level repulsion at short distances disappears.

The authors of ref. [62] have fitted the \(P(s)\) distribution by a simple one-parameter superposition of the normalized Wigner, \(P_W\), and Gaussian, \(P_G\) distributions, that is:

$$P(s) = \alpha P_W(s) + (1 - \alpha)P_G(s) = \frac{1}{\Delta}[\frac{\alpha}{\Delta}s + (1 - \alpha)\sqrt{\frac{2}{\pi}}]\exp\left(-\frac{s^2}{2\Delta^2}\right), \quad (6.11)$$

where the parameter \(\Delta\) is determined, for a fixed \(\alpha\), by the mean level spacing:

$$D = \Delta[\sqrt{\frac{\pi}{2}}\alpha + \sqrt{\frac{2}{\pi}}(1 - \alpha)]. \quad (6.12)$$

In the fig. 25 the trend of the mixing parameter \(\alpha\) for the \(P(s)\) as a function of the coupling constant \(\chi\) is plotted, showing a pronounced minimum.
(α = 0.74) at the transition point χ = 1. Both small and large values of χ correspond to the nearest level statistics close to a Wigner distribution \( P_W \). It is well known that the GOE implies rigid spectra with small level spacing fluctuations. So, we can conclude that coupling to the decay channels softens the spacing distribution so that the spacing fluctuations increase gradually in the transition region.

As regards the distribution of complex energies, it is has been observed that, for a random matrix ensemble, the widths distribution corresponds to a Porter-Thomas one, i.e.:

\[
P^{PT}(\Gamma) = \frac{1}{\sqrt{2\pi \langle \Gamma \rangle}} \exp\left(-\frac{\langle \Gamma \rangle}{2}\right).
\] (6.13)

This, however, is no longer correct when the level spacings become very small compared to the widths, i.e., when the overlap between neighbour states is considerable. This is a consequence of the coupling to the continuum, since for weak coupling the typical widths are small compared to spacings. When, however, the coupling to the continuum is important, the widths become larger than the mean level spacing. This behaviour had already been observed for complex random matrix ensembles [63] and in chaotic dissipative systems [64].

The most interesting statistic for highlighting the phase transition is the width distribution. In fig. 26 the widths, plotted as a function of the coupling parameter χ, show a collectivization. The mean fraction \( \langle \Gamma_1 \rangle / \sum \Gamma_j \) of the total width is accumulated by the broadest state \( \Gamma_1 \).

In conclusion, we can say that within these works a standard statistical spectroscopy of discrete levels and unstable states, as well as for the transi-
tional region, has been performed using ensembles of random non-hermitian hamiltonians represented by N-dimensional matrices. They show [61,62] that the instability of states changes the statistics remarkably, removing level repulsion when distances are smaller than widths. When the coupling to the continuum increases, that is, when the matrix elements of the antihermitian part of the effective hamiltonian become comparable with the spacings of eigenvalues of the hermitian part, a transition to a new regime occurs.

Related to the study of unstable systems and in the same spirit as the preceding work, a paper by Haake et al. [65] is noteworthy. The authors study the level density of different classes of random non-hermitian matrices, $\tilde{H} = H + i\Gamma$, where the damping, $\Gamma$, is chosen quadratic in Gaussian random numbers, to describe the decay of resonances through various channels. When the notion of level spacing is extended to the Euclidian distance between complex eigenvalues of non-hermitian operators, two different behaviours of $P(s)$ statistic (as in the ordinary case of real eigenvalues with the Wigner and Poisson statistics) allow the distinction between regular and chaotic dynamics: cubic repulsion tends to be typical under conditions of global classical chaos while linear repulsion signals classical integrability. This classification also arises for dissipative systems [66].

6.5 The IBM model in the study of unstable systems

The profound understanding of the statistics of low-lying levels of nuclei and the underlying chaotic dynamics requires realistic theoretical models of the nucleus. In general, the models used have only two degrees of freedom.
However, since the quadrupole deformation plays an important role in collective nuclear dynamics, a realistic model requires at least five degrees of freedom.

The work of Whelan and Alhassid [67] have been along these lines. Using the Interacting Boson Model (IBM) [3], whose classical limit is obtained with coherent states and which provides a good description of low-lying energy levels and EM transitions of heavy nuclei, they studied, classically and quantum mechanically, the chaotic properties of low-lying collective states of atomic nuclei. The six degrees of freedom come from one monopole s-boson \((0^+)\) and five \(d_\mu (\mu = -2, ..., 2)\) bosons \((2^+)\) with which a U(6) dynamical algebra is constructed. The most general hamiltonian is then constructed with all one and two-body scalars that conserve the total number of bosons, \(N = ss^+ + \sum_\mu d^+_\mu d_\mu\).

The most useful parameterization of the IBM hamiltonian is given by:

\[
H = E_0 + c_0 n_d + c_2 Q^x \cdot Q^x + c_1 L^2, \tag{6.14}
\]

where \(n_d = d^+ \cdot \bar{d}\) is the number of d-bosons, \(L\) is the angular momentum and \(Q^x\) is the quadrupole operator:

\[
Q^x = (d^+ \times \bar{s} + s^+ \times \bar{d})(2) + \chi (d^+ \times \bar{d})(2), \tag{6.15}
\]

that depends on a parameter \(\chi\), and where \(\bar{d}_\mu = (-)^\mu d_\mu\) so that \(\bar{d}_\mu\) transforms under rotations like \(d^+_\mu\).

Its classical limit is obtained for the number of bosons going to infinity, since \(1/N\) plays the role of \(\hbar\). That classical limit is:

\[
\hbar = \epsilon_0 + \tilde{c}[\eta m_d - (1 - \eta)q^x \cdot q^x] + c_1 L^2, \tag{6.16}
\]
where:
\[
\frac{\eta}{1 - \eta} = -\frac{c_0}{Nc_2}, \quad \bar{c} = \frac{c_0}{\eta},
\]  
(6.17)
with \(0 < \eta < 1\). In an algebraic model like this, a dynamical symmetry occurs when the hamiltonian can be written as a function of the Casimir invariants \(C\) of a chain of subalgebras of the original algebra. So, the authors of ref. [67] analyze the rotational nuclei (SU(3) symmetry), vibrational nuclei (U(5) symmetry) and \(\gamma\)-unstable nuclei (O(6) symmetry):
\[
\begin{align*}
U(6) & \supset U(5) \supset O(5) \supset O(3) \text{ (vibrational nuclei)} \\
U(6) & \supset SU(3) \supset O(3) \text{ (rotational nuclei)} \\
U(6) & \supset O(6) \supset O(5) \supset O(3) \text{ (\(\gamma\)-unstable nuclei)}.
\end{align*}
\]

The authors study the character of the classical dynamics of a nucleus described by the classical limit of the quantum IBM hamiltonian. As a first result, they found that the quantal fluctuations, which are well correlated with the classical results, are independent of the number of bosons. The transitions between the different dynamical symmetries of the model (i.e., between the different dynamics of nuclei) with the variation of the parameter \(\chi\), of both classical and quantal hamiltonian, are studied.

The transition between deformed rotational nuclei (SU(3)) and spherical vibrational nuclei (U(5)) is obtained for \(\chi = -1/2\sqrt{7}\) and \(0 < \eta < 1\). With these parameters, classical and quantal measures of chaos are studied: the average maximal Lyapunov exponent \(\lambda\) and the fraction of chaotic trajectories \(\sigma\), on the one hand, and the parameter \(\omega\) of the Brody distribution of level spacing [60] on the other:
\[
P_\omega(s) = As^\omega \exp(-\alpha s^{\omega+1}).
\]  
(6.18)
The Brody distribution, as the $\Delta^L_3(L)$ mentioned above, interpolates between a Poisson distribution ($\omega = 0$) for a regular system and a Wigner one ($\omega = 1$) for a chaotic system. $\alpha$ and $A$ are chosen such that $P_\omega(s)$ is normalized to 1 and $<S> = 1$. A second quantal measure of chaos is $\nu$, which characterizes the $B(E2: I \rightarrow I)$ distribution for the levels with $I = 2^+$:

$$P_\nu(y) = \frac{1}{\Gamma(\frac{\nu}{2})} \left( \frac{\nu}{2} \right)^{\frac{\nu}{2} - 1} \exp \left( -\frac{\nu y^2}{2} \right),$$

where:

$$y = |<f|T|i>|^2,$$ (6.20)

$|i>$ and $|f>$ being, respectively, the initial and final states and $T$ the transition operator. $P(y)dy$ is the probability of having intensity $P(y)$ in the interval $dy$ around $y$. $P(y)$ reduces to a Porter-Thomas distribution for $\nu = 1$ (chaotic limit) and, as the system becomes more regular, $\nu$ decreases towards small positive values.

The maximum of chaos is obtained for $\eta = 0.5 - 0.7$. The quantal results show a strong correlation with the classical dynamics, since $\omega$ and $\nu$ are largest around $\eta = 0.5 - 0.7$. Secondly, the transition between rotational nuclei (SU(3)) and $\gamma$-unstable nuclei (O(6)) is obtained for $\eta = 0$ and $-1/2\sqrt{7} < \chi < 0$, chaos being settled for intermediate values of $\chi$. Finally, the transition between $\gamma$-unstable (O(6)) and vibrational nuclei (U(5)), for $\chi = 0$ and $0 < \eta < 1$, is always completely regular.

The most important result is, on the one hand, the discovery of a new nearly regular region that lies between a rotational (SU(3)) and vibrational (U(5)) regime of the nucleus, and that is not related to any of the known dynamical symmetries of the model. Since the fraction of chaotic trajectories
(σ) is < 0.3, that region is not completely regular. The signatures of this nearly regular region are identified by the rather sharp minimum in all measures of chaos, as seen in fig. 27. The authors suggest that this new nearly regular region may be connected with a previously unknown approximate symmetry of the model. On the other hand, a spin dependence of the degree of chaoticity is revealed. When both classical (λ,σ) and quantal (ω,ν,q) measures of chaos are plotted versus spin, an interesting dependence is found: in all measures (except λ) a weak dependence on the spin, at low and medium spins, is found. However, at high spins (I > 20ℏ) there is a rapid decrease of chaoticity and the motion becomes regular.

In conclusion, strong correlations are observed between the onset of classical and “quantum” chaos. Although the IBM model can be useful for the study of the degree of chaoticity of low-lying collective states of nuclei, at higher spin and/or energy, bosons may break into quasi-particles and it is also important to take into account the additional fermionic degrees of freedom, to obtain a realistic description [4].

In this section we have presented some of the most interesting recent works, which have attempted to explain the mechanisms of the decay of quantum states and its connection with the onset of chaos.

The theoretical efforts are also supported by the new detector generation, which, with their higher resolution, can provide a new and profound knowledge in the field of nuclear spectroscopy.
7. Conclusions and open problems

To conclude this presentation of some regular and chaotic aspects in nuclear dynamics, it is important to remember two points. The first is the good agreement between the theoretical previsions concerning the Poisson→GOE transition and the experimental data; the second is the possibility of studying the order to chaos transition using simple models.

As pointed out by many authors (see for instance ref. [1]), the static nuclear mean field is too regular (as witnessed by the existence of shell structure) to be responsible of chaotic behaviour, and chaos must be caused by residual interaction, as shown in the models discussed above. Because of the exclusion principle, the role of the latter increases with the excitation energy. This makes one wonder whether nuclear motion might be more regular in the ground–state region than, say, at neutron threshold. Indeed, at several MeV excitation energy, collective states and giant resonances acquire substantial spreading widths [1]. They lose their individuality as excitation states, and show up only as bumps in strength functions. The same kind of statement applies also to rotational bands 1 or 2 MeV above the yrast line. Much work remains to be done to elucidate the role of chaotic motion and its interplay with collectivity.

There is also the problem of showing which properties of the nucleon–nucleon interaction justify the use of random matrix models. These, according to ref. [1], are related to nonintegrability and chaos, but the matter requires further research to obtain a deeper understanding of the problem.

We have shown that chaos in nuclei is a rich field; it permeates many
aspects of nuclear structure and reactions, and gives rise to new analyses of the experimental data.
Figure Captions

Figure 1: The total neutron cross-section on $^{232}Th$ vs neutron energy (adapted from [7]).

Figure 2: The NDE experimental results compared with the theoretical predictions (adapted from [19]).

Figure 3: $P(s)$ for "cold" deformed rare–earth nuclei (adapted form [20]).

Figure 4: Spectral statistics for nuclei with $24 < A < 244$ and excitation energy of few MeV; (a) $2^+$ and $4^+$ states, even–even nuclei; (b) all other states, even–even nuclei; (c) states with non–natural parity, odd–odd nuclei; (d) states with natural parity, odd–odd nuclei (adapted from [21]).

Figure 5: A diagramatic representation of the ranges of excitation energy $E_x$ and angular momentum $I$ associated with each set of data previously discussed (adapted form [20]).

Figure 6: Closed (a) and not closed (b) orbits on the torus.

Figure 7: A Poincarè section.

Figure 8: Poincarè sections: (a) regular closed orbit, (b) regular not closed orbit, (c) chaotic orbit.

Figure 9: $P(s)$ and $\Delta_3(L)$ for billiards. Regular billiard: circular boundary (left), chaotic billiard: stadium boundary corresponding to eigenfunctions with odd–odd symmetry (right) (adapted from [29]).

Figure 10: $E–T$ plots for different values of $\chi$ and for different families: $\triangle$ initial conditions along the axis $q_1$, $\Box$ initial conditions along the axis $q_2$, $\circ$ initial conditions near the minima of the static potential: (a) $\chi = 0.5$, (b)
\( \chi = 2, \) (c) \( \chi = 100 \) (adapted from [43]).

**Figure 11:** Ratio between the number of stable periodic orbits and the number of total periodic orbits vs \( \chi \) (adapted from [45]).

**Figure 12:** Universal and long–range behaviour of \( \Delta_3(L) \) vs \( L \). The continuous curve is the GOE behaviour (adapted from [43]).

**Figure 13:** \( P(s) \) vs \( s \) for the \( eo \) class with \( M = 102 \); (a) \( \chi = 0.75 \), (b) \( \chi = 2 \), (c) \( \chi = 3 \). The continuous curve is the Brody distribution (6.12) with \( \omega = 1 - \Delta N_{st}/\Delta N_{tot} \) (adapted from [45]).

**Figure 14:** Density of quasi–crossing vs \( \chi \) for all the classes with \( M = 102 \) (adapted from [45]).

**Figure 15:** \( \Delta^2(E) \) vs energy \( E \) for different values of \( \chi \) for the \( eo \) class with \( M = 102 \) (adapted from [45]).

**Figure 16:** Momenta \( I_m \) vs energy \( E \) for the \( eo \) class with \( M = 102 \): (a) chaotic region, (b) quasi–integrable region (adapted from [51]).

**Figure 17:** Information measure \( S \) vs energy \( E \) for the \( eo \) class with \( M = 102 \): (a) chaotic region, (b) quasi–integrable region (adapted from [51]).

**Figure 18:** Transition probability \( (ee) \to (oo) \) with \( M = 50 \): (a) regular region, (b) chaotic region (adapted from [53]).

**Figure 19:** A schematic illustration of the average flow of \( \gamma \)-ray decay from high-spin states induced by heavy-ion compound reactions. The insert illustrates the spreading of the E2 transitions (adapted from [55]).

**Figure 20:** The rotational E2 strength distribution vs the energy \( E_\gamma \) for four initial levels with different excitation energies above the yrast line (adapted from [57]).
**Figure 21:** The probability distribution of the rotational E2 strengths \( s_{ij} \) for \( \gamma \)-ray energies in the interval \( 0.90 < E_\gamma < 1.05 \) MeV. The dashed curve represents the Porter-Thomas distribution. The excitation energy above the yrast line is indicated for each bin (adapted from [57]).

**Figure 22:** The rotational–strength distributions obtained for transitions from 50 levels in each bin. The excitation energy above the yrast line is indicated for each bin. The smoothed distribution function is drawn with a solid curve. The transitions shown are for \((-1,1)\) in \(^{168}\)Yb at \( \hbar \omega = 0.5 \) MeV (adapted from [58]).

**Figure 23:** Mixing parameter \( q \) vs the strength of the two-body interaction \( \Delta \), at two excitation energy intervals, \( 2.0 - 2.5 \) MeV (dashed line) and \( 3.0 - 3.5 \) MeV (solid line) in superdeformed \(^{152}\)Dy, and \( 0.5 - 1.0 \) MeV (dashed line) and \( 1.5 - 2.0 \) MeV (solid line) in normal-deformed \(^{168}\)Yb. In all cases \( I^\pi = 50^+ \) (adapted from [59]).

**Figure 24:** Average standard deviation of the E2-strength function vs \( \Delta \) for \( 50^+ \rightarrow 48^+ \) transitions emerging from the excitation energy intervals \( 2.0 - 2.5 \) MeV (dashed line) and \( 3.0 - 3.5 \) MeV (solid line) in superdeformed \(^{152}\)Dy, and \( 0.5 - 1.0 \) MeV (dashed line) and \( 1.5 - 2.0 \) MeV (solid line) in normal-deformed \(^{168}\)Yb (adapted from [59]).

**Figure 25:** Mixing parameter \( \alpha \) vs the continuum coupling constant \( \chi \) (adapted from [62]).

**Figure 26:** Mean ratio of the width of the broadest state to the total summed width of all complex eigenvalues vs the continuum coupling constant \( \chi \). Error bars correspond to the spread over 50 matrices 160x160 (adapted from [62]).
Figure 27: Classical and quantal measures of chaos vs $\nu$ ($\nu = 0$: SU(3), $\nu = 1$: U(5)). Left: $\bar{\lambda}$ and $\sigma$ for angular momentum per boson $l = 0.1$. Right: $\omega$ and $\nu$ for the levels with $I = 2^+$ (adapted from [67]).
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