The energy-momentum conservation law in two-particle system for twist-deformed Galilei Hopf algebras

Marcin Daszkiewicz

Institute of Theoretical Physics
University of Wroclaw pl. Maxa Borna 9, 50 – 206 Wroclaw, Poland
e – mail : marcin@ift.uni.wroc.pl

Abstract

In this article we discuss the energy-momentum conservation principle for two-particle system in the case of canonically and Lie-algebraically twist-deformed Galilei Hopf algebra. Particularly, we provide consistent with the coproducts energy and momentum addition law as well as its symmetric with respect the exchange of particles counterpart. Besides, we show that the vanishing of total fourmomentum for two Lie-algebraically deformed kinematical models leads to the discret values of energies and momenta only in the case of the symmetrized addition rules.
1 Introduction

The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2], [3] and String Theory models [4], [5], indicating that space-time at the Planck scale should be noncommutative, i.e., it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. [6], [7]) as well as with field theoretical models (see e.g. [8], [9]), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic [10] and nonrelativistic [11] symmetries, one can distinguish three basic types of space-time noncommutativity (see also [12] for details):

1) Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [13]-[15]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu},$$

(1)

2) Lie-algebraic modification of classical space-time [15]-[18]

$$[x_\mu, x_\nu] = i\theta^\rho_{\mu\nu} x_\rho,$$

(2)

and

3) Quadratic deformation of Minkowski and Galilei spaces [15], [18]-[20]

$$[x_\mu, x_\nu] = i\theta^\rho_\mu \theta^\tau_\nu x_\rho x_\tau,$$

(3)

with coefficients $\theta_{\mu\nu}$, $\theta^\rho_{\mu\nu}$ and $\theta^\rho_\mu \theta^\tau_\nu$ being constants.

Moreover, it has been demonstrated in [12], that in the case of the so-called N-enlarged Newton-Hooke Hopf algebras $\mathcal{U}_0^{(N)}(NH_{\pm})$ the twist deformation provides the new space-time noncommutativity of the form [12]

$$[t, x_i] = 0, \ [x_i, x_j] = if_\pm \left(\frac{t}{\tau}\right) \theta_{ij}(x),$$

(4)

with time-dependent functions

$$f_+ \left(\frac{t}{\tau}\right) = f \left(\sinh \left(\frac{t}{\tau}\right), \cosh \left(\frac{t}{\tau}\right)\right), \ f_- \left(\frac{t}{\tau}\right) = f \left(\sin \left(\frac{t}{\tau}\right), \cos \left(\frac{t}{\tau}\right)\right),$$

$\theta_{ij}(x) \sim \theta_{ij} = \text{const or } \theta_{ij}(x) \sim \theta^k_{ij} x_k$ and $\tau$ denoting the time scale parameter - the cosmological constant. Besides, it should be noted that the all above quantum spaces 1),

1 $x_0 = ct$.

2 The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [13], [14]), for the quantum N-enlarged Newton-Hooke Hopf algebras.
2) and 3) can be obtained by the proper contraction limit of the commutation relations
4). It is commonly known that the above mentioned Poincare and Galilei Hopf algebras are given by the algebraic as well as by the coalgebraic sector respectively. Since the first of them defines the commutation relations for the generators, the second one introduces particularly the addition rules of two momenta and energies. Usually, such a constructed composition law remains nonsymmetric with respect the exchange of particles [21]-[25] and the different solutions of the problem have been proposed in papers [21], [22] and [25].

In this article we provide consistently with coproduct the four-momentum addition laws for two canonically and three Lie-algebraically twist-deformed Galilei Hopf algebras [15]. We show that after the proper symmetrization only two of them (for $U_{\kappa_{1}}(G)$ and $U_{\kappa_{2}}(G)$ quantum groups) are still deformed while the third one becomes classical. Further, we demonstrate that the corresponding two-particle energy-momentum conservation principle with vanishing conserved quantity $\bar{p}_{tot}$ and $E_{tot}$ leads to the discrete values of momenta and energies only in the case of the symmetrized addition rules i), ii) and iii).

The paper is organized as follows. In second Section we remaind the canonically-deformed Galilei Hopf algebras proposed in [15]. The third Section is devoted to the corresponding four-momentum conservation principle. In the fourth Section we recall the Lie-algebraically nonrelativistic Hopf structure while the proper conservation rules are provided and analyzed in fifth Section. The final remarks are discussed in the last Section.

2 Canonical twist-deformations of Galilei Hopf algebra

The two canonically deformed Galilei Hopf algebras $U_{\theta_{i}}(G)$ and $U_{\theta_{0}}(G)$ have been provided in article [15] by the proper contractions of their relativistic counterparts. They are given by the classical algebraic sector

\[
[K_{ij}, K_{kl}] = i (\delta_{il} K_{jk} - \delta_{jl} K_{ik} + \delta_{jk} K_{il} - \delta_{ik} K_{jl}) ,
\]

\[
[K_{ij}, V_{k}] = i (\delta_{jk} V_{i} - \delta_{ik} V_{j}) ,
[K_{ij}, \Pi_{\rho}] = i (\eta_{\rho j} \Pi_{i} - \eta_{\rho i} \Pi_{j}) ,
\]

\[
[V_{i}, V_{j}] = [V_{i}, \Pi_{j}] = 0 ,
[V_{i}, \Pi_{0}] = -i \Pi_{i} ,
[\Pi_{\rho}, \Pi_{\sigma}] = 0 ,
\]

3) Such a result indicates that the twisted N-enlarged Newton-Hooke Hopf algebra plays a role of the most general type of quantum group deformation at nonrelativistic level.

4) The symmetrization problem has been mainly considered in the case of $\kappa$-Poincare Hopf structure $U_{\kappa}(P)$.

5) Before symmetrization the condition $\bar{p}_{tot} = 0$ and $E_{tot} = 0$ gives the classical solution $\bar{p}_{1} = -\bar{p}_{2}$ and $E_{1} = -E_{2}$ respectively.
where \( K_{ij}, \Pi_0, \Pi_i \) and \( V_i \) can be identified with rotation, time translation, momentum and boost operators as well as by the following twisted coproducts

\[
\Delta_{\theta_{ij}}(\Pi_\mu) = \Delta_0(\Pi_\mu) \quad , \quad \Delta_{\theta_{ij}}(V_i) = \Delta_0(V_i) ,
\]

\[
\Delta_{\theta_{ij}}(K_{ij}) = \Delta_0(K_{ij}) - \theta^{kl}[(\delta_{ki}\Pi_j - \delta_{kj}\Pi_i) \otimes \Pi_l + \Pi_k \otimes (\delta_{li}\Pi_j - \delta_{lj}\Pi_i)] ,
\]

(8)

and

\[
\Delta_{\theta_{0i}}(\Pi_\mu) = \Delta_0(\Pi_\mu) ,
\]

\[
\Delta_{\theta_{0i}}(K_{ij}) = \Delta_0(K_{ij}) - \theta^{0k}\Pi_0 \wedge (\delta_{ki}\Pi_j - \delta_{kj}\Pi_i) ,
\]

\[
\Delta_{\theta_{0i}}(V_i) = \Delta_0(V_i) - \theta^{0k}\Pi_i \wedge \Pi_k ,
\]

(9)

(10)

(11)

(12)

respectively. Besides, in the case of \( \mathcal{U}_{\theta_{ij}}(G) \) Hopf structure the corresponding quantum space-time is given by

\[
[ t, x_i ] = 0 \quad , \quad [ x_i, x_j ] = i\theta_{ij} ,
\]

(13)

while for \( \mathcal{U}_{\theta_{0i}}(G) \) it looks as follows

\[
[ t, x_i ] = i\theta_{0i} \quad , \quad [ x_i, x_j ] = 0 .
\]

(14)

Of course, for deformation parameters \( \theta_{ij} \) and \( \theta_{0i} \) approaching zero the above relations become classical.

3 **Canonically deformed energy-momentum conservation law**

Let us turn to the momentum addition rules corresponding to the \( \Delta(P) \)-coproducts [8] and [10]. Due to the fact that both of them are primitive, for momenta \( \vec{p}_1 = [p_{11}, p_{12}, p_{13}] \) and \( \vec{p}_2 = [p_{21}, p_{22}, p_{23}] \) as well as for energies \( E_1 \) and \( E_2 \), we have

\[
\vec{p}_1 + A \vec{p}_2 = \vec{p}_3 \quad ; \quad A = \theta_{ij} \text{ or } A = \theta_{0i} ,
\]

\[
E_1 + A E_2 = E_3 ,
\]

(15)

(16)

with

\[
\vec{p}_3 = [p_{11} + p_{21}, p_{12} + p_{22}, p_{13} + p_{23}] ,
\]

\[
E_3 = E_1 + E_2 ,
\]

(17)

(18)
in the case of $U_{\theta_{ij}}(G)$ and $U_{\theta_{i0}}(G)$ Hopf structures. It means that for both algebras the addition law remains undeformed and the energy-momentum conservation principle for two-particle system takes the standard form

$$\bar{p}_1 + A \bar{p}_2 = \bar{p}_1 + \bar{p}_2 = \bar{p}_{\text{tot}} = \text{const.},$$  

(19)

$$E_1 + A E_2 = E_1 + E_2 = E_{\text{tot}} = \text{const.}.$$  

(20)

Of course, for $\bar{p}_{\text{tot}}$ and $E_{\text{tot}}$ equal to zero we get

$$\bar{p}_1 = -\bar{p}_2,$$  

(21)

and

$$E_1 = -E_2,$$  

(22)

respectively.

4 Twisted Lie-algebraically deformed Galilei Hopf structures

The three Lie-algebraically twist-deformed Galilei Hopf structures $U_{\kappa_1}(G), U_{\kappa_2}(G)$ and $U_{\kappa_3}(G)$ have been introduced in article [15] as well. Their algebraic sectors remain classical (see formulas (5)-(7)) while the coproducts are given by

$$\Delta_{\kappa_1}(\Pi_0) = \Delta_0(\Pi_0),$$  

(23)

$$\Delta_{\kappa_1}(\Pi_i) = \Delta_0(\Pi_i) + \sin\left(\frac{1}{\kappa_1} \Pi_{\gamma}\right) \wedge (\delta_{ki} \Pi_l - \delta_{li} \Pi_k)$$

$$+ \left[\cos\left(\frac{1}{\kappa_1} \Pi_{\gamma}\right) - 1\right] \perp (\delta_{ki} \Pi_k + \delta_{li} \Pi_l),$$  

(24)

$$\Delta_{\kappa_1}(K_{ij}) = \Delta_0(K_{ij}) + K_{kl} \wedge \frac{1}{\kappa_1} (\delta_{i\gamma} \Pi_j - \delta_{j\gamma} \Pi_i)$$

$$+ i[K_{ij}, K_{kl}] \wedge \sin\left(\frac{1}{\kappa_1} \Pi_{\gamma}\right)$$

$$+ [[K_{ij}, K_{kl}], K_{kl}] \perp \left[\cos\left(\frac{1}{\kappa_1} \Pi_{\gamma}\right) - 1\right]$$

$$+ K_{kl} \sin\left(\frac{1}{\kappa_1} \Pi_{\gamma}\right) \perp \frac{1}{\kappa_1} (\psi_{\gamma} \Pi_k - \chi_{\gamma} \Pi_l)$$

$$+ \frac{1}{\kappa_1} (\psi_{\gamma} \Pi_l + \chi_{\gamma} \Pi_k) \wedge K_{kl} \left[\cos\left(\frac{1}{\kappa_1} \Pi_{\gamma}\right) - 1\right],$$  

(25)

The indexes $k$, $l$, $\gamma$ are fixed, spatial and different.

$\gamma a \wedge b = a \otimes b - b \otimes a$, $a \perp b = a \otimes b + b \otimes a$.

$\psi_{\lambda} = \eta_{\nu\lambda} \eta_{\beta\mu} - \eta_{\mu\lambda} \eta_{\beta\nu}$, $\chi_{\lambda} = \eta_{\nu\lambda} \eta_{\alpha\mu} - \eta_{\mu\lambda} \eta_{\alpha\nu}$.
\[ \Delta_{\kappa_1}(V_i) = \Delta_0(V_i) + i [V_i, K_{kl}] \wedge \sin \left( \frac{1}{\kappa_1} \Pi \right) \]
\[ + \left[ |V_i, K_{kl} \rangle, K_{kl} \right] \perp \left[ \cos \left( \frac{1}{\kappa_1} \Pi \right) - 1 \right], \]

in case of the first quantum group

\[ \Delta_{\kappa_2}(\Pi_0) = \Delta_0(\Pi_0), \quad (27) \]
\[ \Delta_{\kappa_2}(\Pi_i) = \Delta_0(\Pi_i) + \sin \left( \frac{1}{\kappa_2} \Pi \right) \wedge (\delta_{ki} \Pi_l - \delta_{li} \Pi_k) \]
\[ + \left[ \cos \left( \frac{1}{\kappa_2} \Pi \right) - 1 \right] \perp (\delta_{ki} \Pi_k + \delta_{li} \Pi_l), \]

\[ \Delta_{\kappa_2}(K_{ij}) = \Delta_0(K_{ij}) + K_{kl} \wedge \frac{1}{\kappa_2} (\delta_{0i} \Pi_j - \delta_{0j} \Pi_i) \]
\[ + i [K_{ij}, K_{kl}] \wedge \sin \left( \frac{1}{\kappa_2} \Pi \right) \]
\[ + \left[ |K_{ij}, K_{kl} \rangle, K_{kl} \right] \perp \left[ \cos \left( \frac{1}{\kappa_2} \Pi \right) - 1 \right] \]
\[ + K_{kl} \sin \left( \frac{1}{\kappa_2} \Pi \right) \perp \left( \frac{1}{\kappa_2} (\psi_0 \Pi_k - \chi_0 \Pi_l) \right) \]
\[ + \left[ \frac{1}{\kappa_2} (\psi_0 \Pi_l + \chi_0 \Pi_k) \right] \wedge K_{kl} \left[ \cos \left( \frac{1}{\kappa_2} \Pi \right) - 1 \right], \]

\[ \Delta_{\kappa_2}(V_i) = \Delta_0(V_i) + \frac{1}{\kappa_2} K_{kl} \wedge \Pi_i + i [V_i, K_{kl}] \wedge \sin \left( \frac{1}{\kappa_2} \Pi \right) \]
\[ + \left[ |V_i, K_{kl} \rangle, K_{kl} \right] \perp \left[ \cos \left( \frac{1}{\kappa_2} \Pi \right) - 1 \right] \]
\[ + K_{kl} \sin \left( \frac{1}{\kappa_2} \Pi \right) \perp \left( \frac{1}{\kappa_2} (\delta_{ki} \Pi_l - \delta_{li} \Pi_k) \right) \]
\[ - \frac{1}{\kappa_2} (\delta_{ki} \Pi_k + \delta_{li} \Pi_l) \wedge K_{kl} \left[ \cos \left( \frac{1}{\kappa_2} \Pi \right) - 1 \right], \]

for the second Hopf algebra and

\[ \Delta_{\kappa_3}(\Pi_0) = \Delta_0(\Pi_0) + \frac{1}{\kappa_3} \Pi_l \wedge \Pi_k, \quad (31) \]
\[ \Delta_{\kappa_3}(\Pi_i) = \Delta_0(\Pi_i), \quad \Delta_{\kappa_3}(V_i) = \Delta_0(V_i), \quad (32) \]
\[ \Delta_{\kappa_3}(K_{ij}) = \Delta_0(K_{ij}) + \frac{i}{\kappa_3} [K_{ij}, V_k] \wedge \Pi_l + \frac{1}{\kappa_3} V_k \wedge (\delta_l \Pi_j - \delta_j \Pi_l) , \] (33)

for the third, \( \mathcal{U}_{\kappa_3}(G) \) Hopf structure. One can also check that that the corresponding quantum space-times look as follows

\[ [x_i, x_j] = \frac{i}{\kappa_1} \delta_{ij} (\delta_{ki} x_l - \delta_{li} x_k) + \frac{i}{\kappa_1} \delta_{ij} (\delta_{lj} x_k - \delta_{jk} x_l) , \quad [t, x_i] = 0 , \] (34)

\[ [t, x_i] = \frac{i}{\kappa_2} (\delta_{li} x_k - \delta_{lk} x_i) , \quad [x_i, x_j] = 0 , \] (35)

and

\[ [x_i, x_j] = \frac{i}{\kappa_3} t (\delta_{li} \delta_{kj} - \delta_{kj} \delta_{li}) , \quad [t, x_i] = 0 . \] (36)

respectively. Obviously, for all deformation parameters \( \kappa_1, \kappa_2 \) and \( \kappa_3 \) running to infinity the above relations become commutative.

5 Lie-algebraically deformed energy-momentum conservation law

Due to the \( \Delta(P) \)-coproducts (23), (24), (27), (28), (31) and (32) the two-particle energy-momentum addition rules for \( \vec{p}_1 = [p_{1\gamma}, p_{1k}, p_{1l}] \) and \( \vec{p}_2 = [p_{2\gamma}, p_{2k}, p_{2l}] \) as well as for energies \( E_1 \) and \( E_2 \) take the form

\[ \vec{p}_1 +_{\kappa_i} \vec{p}_2 = \vec{p}_3 ; \quad i = 1, 2, 3 , \] (37)

\[ E_1 +_{\kappa_i} E_2 = E_3 , \] (38)

with

\[ p_{3\gamma} = p_{1\gamma} + p_{2\gamma} , \] (39)

\[ p_{3k} = p_{1k} + p_{2k} + \sin \left( \frac{p_{1\gamma}}{\kappa_1} \right) p_{2l} - \sin \left( \frac{p_{2\gamma}}{\kappa_1} \right) p_{1l} + \] 
\[ + \left[ \cos \left( \frac{p_{1\gamma}}{\kappa_1} \right) - 1 \right] p_{2k} + \] 
\[ + \left[ \cos \left( \frac{p_{2\gamma}}{\kappa_1} \right) - 1 \right] p_{1k} , \] (40)

\[ p_{3l} = p_{1l} + p_{2l} - \sin \left( \frac{p_{1\gamma}}{\kappa_1} \right) p_{2k} + \sin \left( \frac{p_{2\gamma}}{\kappa_1} \right) p_{1k} + \] 
\[ + \left[ \cos \left( \frac{p_{1\gamma}}{\kappa_1} \right) - 1 \right] p_{2l} + \] 
\[ + \left[ \cos \left( \frac{p_{2\gamma}}{\kappa_1} \right) - 1 \right] p_{1l} , \] (41)

\[ E_3 = E_1 + E_2 , \] (42)
in case of the first Hopf algebra

\[ p_{3\gamma} = p_{1\gamma} + p_{2\gamma}, \quad (43) \]

\[ p_{3k} = p_{1k} + p_{2k} + \sin \left( \frac{E_1}{\kappa_2} \right) p_{2l} - \sin \left( \frac{E_2}{\kappa_2} \right) p_{1l} + \]

\[ + \left[ \cos \left( \frac{E_1}{\kappa_2} \right) - 1 \right] p_{2k} + \]

\[ + \left[ \cos \left( \frac{E_2}{\kappa_2} \right) - 1 \right] p_{1k}, \quad (44) \]

\[ p_{3l} = p_{1l} + p_{2l} - \sin \left( \frac{E_1}{\kappa_2} \right) p_{2k} + p_{1k} \sin \left( \frac{E_2}{\kappa_2} \right) + \]

\[ + \left[ \cos \left( \frac{E_1}{\kappa_2} \right) - 1 \right] p_{2l} + p_{1l} \left[ \cos \left( \frac{E_2}{\kappa_2} \right) - 1 \right], \quad (45) \]

\[ E_3 = E_1 + E_2, \quad (46) \]

for the second quantum group and

\[ p_{3\gamma} = p_{1\gamma} + p_{2\gamma}, \quad p_{3k} = p_{1k} + p_{2k}, \quad p_{3l} = p_{1l} + p_{2l}, \quad (47) \]

\[ E_3 = E_1 + E_2 + \frac{1}{\kappa_3} (p_{1l}p_{2k} - p_{1k}p_{2l}), \quad (48) \]

for the third Hopf structure. Consequently, the components \((39)-(48)\) become nonsymmetric with respect the exchange of indexes 1 and 2. In order to improve the problem we modify the above rules in the most simple and natural way as follows:\footnote{The terms of laws \((40)-(41)\) and \((44)-(45)\) including cosinus function are together symmetric with respect the exchange of indexes 1 and 2. Hence, they remain untouched by our procedure. However, the terms with sinus function are together antysymmetric. It is easy to see that their following symmetrization (for example in the case of the first Hopf algebra)

\[ \sin \left( \frac{p_{1\gamma}}{\kappa_1} \right) p_{2l} - p_{1l} \sin \left( \frac{p_{2\gamma}}{\kappa_1} \right) + p_{1l} \sin \left( \frac{p_{2\gamma}}{\kappa_1} \right) - \sin \left( \frac{p_{1\gamma}}{\kappa_1} \right) p_{2l} = 0, \]

cancels all of them. Consequently, we get the formulas \((50)-(51)\) and \((54)-(55)\) respectively. In the similar way we proceed with rules \((48)-(50)\) and \((56)-(58)\).}

\[ \text{i}) \quad p_{3\gamma} = p_{1\gamma} + p_{2\gamma}, \quad (49) \]

\[ p_{3k} = p_{1k} + p_{2k} + \left[ \cos \left( \frac{p_{1\gamma}}{\kappa_1} \right) - 1 \right] p_{2k} + p_{1k} \left[ \cos \left( \frac{p_{2\gamma}}{\kappa_1} \right) - 1 \right], \quad (50) \]

\[ p_{3l} = p_{1l} + p_{2l} + \left[ \cos \left( \frac{p_{1\gamma}}{\kappa_1} \right) - 1 \right] p_{2l} + p_{1l} \left[ \cos \left( \frac{p_{2\gamma}}{\kappa_1} \right) - 1 \right], \quad (51) \]

\[ E_3 = E_1 + E_2, \quad (52) \]
ii) \[ p_{3\gamma} = p_{1\gamma} + p_{2\gamma}, \quad (53) \]
\[ p_{3k} = p_{1k} + p_{2k} + \left[ \cos \left( \frac{E_1}{\kappa_2} \right) - 1 \right] p_{2k} + p_{1k} \left[ \cos \left( \frac{E_2}{\kappa_2} \right) - 1 \right] , \quad (54) \]
\[ p_{3l} = p_{1l} + p_{2l} + \left[ \cos \left( \frac{E_1}{\kappa_2} \right) - 1 \right] p_{2l} + p_{1l} \left[ \cos \left( \frac{E_2}{\kappa_2} \right) - 1 \right] , \quad (55) \]
\[ E_3 = E_1 + E_2, \quad (56) \]

iii) \[ p_{3\gamma} = p_{1\gamma} + p_{2\gamma}, \quad p_{3k} = p_{1k} + p_{2k}, \quad p_{3l} = p_{1l} + p_{2l}, \quad E_3 = E_1 + E_2. \]

Then, the energy-momentum conservation law for two-particle system takes the form
\[ \vec{p}_1 + \kappa_i \vec{p}_2 = \vec{p}_{\text{tot}} = \text{const.}, \quad \quad (57) \]
\[ E_1 + \kappa_i E_2 = E_{\text{tot}} = \text{const.}, \quad (58) \]
with the components of total vector \( \vec{p}_{\text{tot}} \) and with total energy \( E_{\text{tot}} \) given by equations (39)-(48) in the case of twisted coproduct \( \Delta(P) \) as well as by \( \text{i), ii) and iii) formulas for their symmetrized counterparts respectively. \)

Let us now turn to the special situation when \( \vec{p}_{\text{tot}} = 0 \) and \( E_{\text{tot}} = 0 \), i.e., when
\[ \vec{p}_1 + \kappa_i \vec{p}_2 = 0, \quad \quad (59) \]
and
\[ E_1 + \kappa_i E_2 = 0, \quad \quad (60) \]
or, equivalently
\[ p_{3\gamma} = 0 = p_{3k} = p_{3l}, \quad E_3 = 0. \quad \quad (61) \]

Then, by direct calculation one can check that the conditions (61) are satisfied by classical set of solutions (21) for all three Lie-algebraically deformed Hopf algebras \( \mathcal{U}_\kappa(G) \) for both symmetrized and nonsymmetrized addition rules, as well as by
\[ p_{1\gamma} = -p_{2\gamma} = \kappa_1 \pi \left( n - \frac{1}{2} \right); \quad n \in \mathbb{Z}, \quad \quad (62) \]
\[ E_1 = -E_2, \quad \quad (63) \]
\[ p_{1k} \text{ and } p_{1l} - \text{arbitrary real numbers}, \quad \quad (64) \]
in the case of the first quantum group \( \mathcal{U}_\kappa_1(G) \) and
\[ p_{1\gamma} = -p_{2\gamma}, \quad \quad (65) \]
\[ E_1 = -E_2 = \kappa_2 \pi \left( n - \frac{1}{2} \right); \quad n \in \mathbb{Z}, \quad \quad (66) \]
\[ p_{1k} \text{ and } p_{1l} - \text{arbitrary real numbers}, \quad \quad (67) \]
for the second Hopf structure \( \mathcal{U}_\kappa_2(G) \) only after the symmetrization. Obviously, for parameters \( \kappa_i \) approaching infinity the above formulas become commutative.
6 Final remarks

In this article we provide the addition rules for two momenta of particles in the case of three Lie-algebraically and two canonically twist-deformed Galilei Hopf algebras. The proposed prescription remains consistent with $\Delta(P)$-coproducts given by the formulas (8), (10), (23), (24), (27), (28), (31) and (32) respectively. Besides, we formulate the energy-momentum conservation principle for all considered systems. Particularly, we show that the total energy-momentum vanishing condition $\vec{p}_{\text{tot}} = 0$ and $E_{\text{tot}} = 0$ leads to the quantization of three-momentum in the case of $\mathcal{U}_{s1}(G)$ Hopf algebra as well as to the discretisation of energy values for $\mathcal{U}_{s2}(G)$ Hopf structure only in the case of the symmetrized addition rules i), ii) and iii).

It should be noted that the above considerations can be extended to the N-particle twist-deformed relativistic and nonrelativistic kinematical models as well. The works in this direction already started and are in progress.

Acknowledgments

The author would like to thank J. Lukierski for valuable discussions.

References

[1] H.S. Snyder, Phys. Rev. 72, 68 (1947)
[2] S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B 331, 39 (1994); Comm. Math. Phys. 172, 187 (1995); hep-th/0303037
[3] A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997); hep-th/9612084
[4] A. Connes, M.R. Douglas, A. Schwarz, JHEP 9802, 003 (1998); hep-th/9711162
[5] N. Seiberg and E. Witten, JHEP 9909, 032 (1999); hep-th/9908142
[6] A. Deriglazov, JHEP 0303, 021 (2003); hep-th/0211105
[7] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001); hep-th/0010175
[8] P. Kosinski, J. Lukierski, P. Maslanka, Phys. Rev. D 62, 025004 (2000); hep-th/9902037
[9] M. Chaichian, P. Prešnajder and A. Tureanu, Phys. Rev. Lett. 94, 151602 (2005); hep-th/0409096
[10] S. Zakrzewski, "Poisson Structures on the Poincare group"; q-alg/9602001
[11] Y. Brihaye, E. Kowalczyk, P. Maslanka, "Poisson-Lie structure on Galilei group"; math/0006167

[12] M. Daszkiewicz, Mod. Phys. Lett. A 27 (2012) 1250083; arXiv: 1205.0319 [hep-th]

[13] R. Oeckl, J. Math. Phys. 40, 3588 (1999)

[14] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, Phys. Lett. B 604, 98 (2004); hep-th/0408069

[15] M. Daszkiewicz, Mod. Phys. Lett. A 23, 505 (2008); arXiv: 0801.1206 [hep-th]

[16] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264, 331 (1991)

[17] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz, Phys. Lett. B 286, 57 (1992)

[18] J. Lukierski and M. Woronowicz, Phys. Lett. B 633, 116 (2006); hep-th/0508083

[19] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, Comm. Math. Phys. 150, 495 (1992)

[20] P. Aschieri, L. Castellani, A.M. Scarfone, Eur. Phys. J. C 7, 159 (1999); q-alg/9709032

[21] M. Daszkiewicz, J. Lukierski, M. Woronowicz, Mod. Phys. Lett. A 23, 653 (2008); arXiv: hep-th/0703200

[22] M. Daszkiewicz, J. Lukierski, M. Woronowicz, Phys. Rev. D 77 (2008) 105007; arXiv: 0708.1561 [hep-th]

[23] J. Kowalski-Glikman, Lect. Notes Phys. 669, 131 (2005); arXiv: hep-th/0405273

[24] J. Kowalski-Glikman, Int. Jour. Mod. Phys. A 32 (2017) 1730026; arXiv: 1711.00665 [hep-th]

[25] C.A.S. Young, R. Zegers, Nucl. Phys. B 804, 342 (2008); arXiv: 0803.2659 [hep-th]