AN INEQUALITY BETWEEN MULTIPOINT SESHADRI CONSTANTS

J. ROÉ AND J. ROSS

ABSTRACT. Let $X$ be a projective variety of dimension $n$ and $L$ be a nef divisor on $X$. Denote by $\epsilon_{d}(r; X, L)$ the $d$-dimensional Seshadri constant of $r$ very general points in $X$. We prove that

$$\epsilon_{d}(rs; X, L) \geq \epsilon_{d}(r; X, L) \cdot \epsilon_{d}(s; \mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(1))$$

for $r, s \geq 1$.

1. Introduction

Let $L$ be a nef divisor on a projective variety $X$ of dimension $n$. The Seshadri constant of dimension $d \leq n$ at $r$ points $p_{1}, \ldots, p_{r}$ in the smooth locus of $X$ is defined (see [14]) to be the real number

$$\epsilon_{d}(p_{1}, \ldots, p_{r}; X, L) = \inf \left\{ \left( \frac{L_{d} \cdot Z}{\sum \text{mult}_{p_{i}} Z} \right)^{1/d} : Z \subset X \text{ effective cycle of dimension } d \right\}.$$ 

By semicontinuity of multiplicities if the points are in very general position then the Seshadri constant does not depend on the actual points chosen [23]. Thus one can define

$$\epsilon_{d}(r; X, L) = \epsilon_{d}(p_{1}, \ldots, p_{r}; X, L)$$

where $p_{1}, \ldots, p_{r}$ is any collection of $r$ very general points in $X$. We shall prove the following inequality comparing Seshadri constants of very general points $X$ and those in projective space.

Theorem 1.1. Let $X$ be an $n$ dimensional projective variety, $L$ be a nef divisor on $X$ and $r, s, d \geq 1$ be integers, with $d \leq n$. Then

$$\epsilon_{d}(rs; X, L) \geq \epsilon_{d}(r; X, L) \cdot \epsilon_{d}(s; \mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(1)).$$

Remark 1.2. When $d = 1$ the Seshadri constant $\epsilon_{1}$ reduces to the usual Seshadri constant as introduced by Demailly [5], and in this case Theorem 1.1 is due to Biran [2]. When $r = 1, d = n - 1$ the Theorem is due to Roé [12].

It is well known and not hard to see that $\epsilon_{d}(r; X, L) \leq \sqrt[n]{L^{n}/r}$ for $d = 1$ and $d = n - 1$ (see remark 1.4). On the other hand, explicit values or even lower bounds are in general hard to compute. Thus Theorem 1.1 should be seen as a lower bound on $\epsilon_{d}(rs; X, L)$, which will be useful if Seshadri constants on projective space are known. As an example, let us recall the most general form of a famous conjecture by Nagata:
Conjecture 1.3 (Nagata-Biran-Szemberg, [14]). Let $X$ be an $n$ dimensional projective variety, $L$ be a nef divisor on $X$ and $1 \leq d \leq n$ be an integer. Then there is a positive integer $r_0$ such that for every $r \geq r_0$, \( \epsilon_d(r; X, L) = \sqrt[2r]{L^n/r} \).

**Remark 1.4.** Already Demailly observed that $\epsilon_1(r; X, L) \leq \epsilon_d(r; X, L)$ for all $d = 2, \ldots, n$ (which combined with the obvious equality $\epsilon_n(r; X, L) = \sqrt[2n]{L^n/r}$ gives the upper bound $\epsilon_1(r; X, L) \leq \sqrt[2n]{L^n/r}$). Standard arguments also show that $\epsilon_{n-1}(r; X, L) \leq \sqrt[2n-1]{L^n/r} = \epsilon_d(r; X, L)$ and that $\epsilon_1(r; X, L) = \sqrt[n]{L^n/r}$ implies $\epsilon_d(r; X, L) = \sqrt[2n]{L^n/r}$ for all $d = 2, \ldots, n$, so if the conjecture above holds for $d = 1$ then it holds for all $d$. In view of these facts, it is tempting to ask if the inequalities $\epsilon_{d_1}(r; X, L) \leq \epsilon_{d_2}(r; X, L)$ hold for all $d_1 \leq d_2$.

In this context Theorem 1.1 implies, for instance, that if (1) the Nagata-Biran-Szemberg conjecture is true for $d$-dimensional Seshadri constants on $\mathbb{P}^n$, and (2) $\epsilon_d(1; X, L) = \sqrt[2n]{L^n}$, then the Nagata-Biran-Szemberg conjecture is true for the $d$-dimensional Seshadri constants of $(X, L)$. See [12] for more applications along this line.

The proof uses the degeneration to the normal cone of $r$ very general points of $X$. That is, let $p_1, \ldots, p_r$ be very general points in $X$ and $\pi: \mathcal{X} \to \mathbb{A}^1$ be the blowup of $X \times \mathbb{A}^1$ at points $p_i \times \{0\}$ for $1 \leq i \leq r$. The central fibre of $\mathcal{X}$ over $0 \in \mathbb{A}^1$ is reducible, having one component which is the blowup of $X$ at $p_1, \ldots, p_r$ (with exceptional divisor $E_i \simeq \mathbb{P}^{n-1}$ over $p_i$) and $r$ exceptional components $F_i \simeq \mathbb{P}^n$, with each $E_i$ glued to $F_i$ along a hyperplane.

Regard a collection of $rs$ points in $X$ as coming from $r$ sub-collections each consisting of $s$ points. Shrinking $\mathbb{A}^1$ if necessary for each $1 \leq i \leq r$ we can find $s$ sections of $\pi$ that pass through very general points in $F_i \setminus E_i$. Then the inequality in Theorem 1.1 follows by showing that the existence of highly singular cycles at $rs$ points of $X$ implies the existence of highly singular cycles at $rs$ points of the central fiber chosen at will, which means cycles with higher multiplicities at the $s$ chosen points.

It is clear that this argument actually produces something stronger. For instance we do not have to divide the collection of $rs$ points evenly.

**Theorem 1.5.** Let $X$ be an $n$ dimensional projective variety and $L$ be a nef divisor on $X$. Suppose $1 \leq s_1 \leq s_2 \leq \cdots \leq s_r$ are integers and set $s = \sum_{i=1}^r s_i$. Then

$$\epsilon_d(s; X, L) \geq \epsilon_d(r; X, L) \cdot \epsilon_d(s_r; \mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(1)).$$

We can also consider weighted Seshadri constants as in [9], defined as follows. In addition to the nef divisor $L$ on $X$ and $r$ points $p_1, \ldots, p_r$ in the smooth locus of $X$, let a nonzero real vector $\ell = (l_1, \cdots, l_r) \in \mathbb{R}^r_+$ be given with each $l_i \geq 0$. The Seshadri constant of dimension $d \leq n$ at these points with weights $\ell$ is defined to be the real number

$$\epsilon_d(\ell_1 p_1, \ldots, l_r p_r; X, L) = \inf \left\{ \left( \frac{L^d \cdot Z}{\sum_{i=1}^r l_i \text{mult}_{p_i} Z} \right)^{1/d} : Z \subset X \text{ effective cycle of dimension } d \right\}.$$

Again by semicontinuity if the points are in very general position then the weighted Seshadri constant does not depend on the actual points chosen [23]. Thus one can define

$$\epsilon(\ell, r; X, L) = \epsilon(\ell_1 p_1, \ldots, l_r p_r; X, L)$$
where \( p_1, \ldots, p_r \) is any collection of \( r \) very general points in \( X \).

**Theorem 1.6.** Let \( X \) be an \( n \)-dimensional projective variety, let \( L \) be a nef divisor on \( X \), and let \( r, d \geq 1 \) be integers with \( d \leq n \). Suppose \( 1 \leq s_1 \leq s_2 \leq \cdots \leq s_r \) are integers and set \( s = \sum_{i=1}^{r} s_i \). For each \( i \) let \( \ell_i \) be a vector in \( \mathbb{R}^{s_i} \) and set \( \ell = (\ell_1, \ell_2, \ldots, \ell_r) \in \mathbb{R}^s_+ \). Then

\[
\epsilon_d(s; \ell; X, L) \geq \epsilon_d(r; X, L) \cdot \inf_{i=1,\ldots,r} \epsilon_d(s_i; \ell_i; \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)).
\]

**Remark 1.7.** Since the \( L \)-degree and multiplicity at a point of a \( d \)-dimensional scheme coincide with those of the associated \( d \)-dimensional cycle, the Seshadri constant can be equivalently defined as

\[
\epsilon_d(p_1, \ldots, p_r; X, L) = \inf \left\{ \left( \frac{d \cdot Z}{\sum \text{mult}_{p_i} Z} \right)^{\frac{1}{d}} : Z \subset X \text{ closed subscheme of dimension } d \right\}.
\]

For convenience, we shall work with this definition.

**Conventions:** By a variety \( X \) we mean a possibly reducible reduced scheme of finite type over an algebraically closed uncountable field \( K \).

If \( X \) is a variety then a very general point \( p \in X \) is a point that lies outside a countable collection of proper subvarieties of \( X \). We say that a collection of \( r \) points \( p_1, \ldots, p_r \) is very general if the point \((p_1, \ldots, p_r) \in X^r \) in the \( r \)-th power of \( X \) is very general. The uncountable hypotheses on the base field guarantees that a set of very general points is nonempty and dense; over a countable or finite field claims on very general points are void and \( \epsilon(r; X, L) \) is not well defined.

2. **Preliminaries**

2.1. **Semicontinuity of Seshadri constants.** Our proof of Theorem 1.1 will rely on the semicontinuity property according to which Seshadri constants can only “jump down” in families. When \( d = 1 \) this semicontinuity is well-known and comes from interpreting Seshadri constants in terms of ampleness of certain line bundles on a blowup of \( X \) [7, Thm. 5.1.1]. To deal with the general case \( d \geq 1 \), let \( p: \mathcal{X} \to B \) be a projective flat morphism, and for \( b \in B \), denote \( \mathcal{X}_b = p^{-1}(b) \). We assume that \( \mathcal{X} \) is a variety, \( B \) is a reduced and irreducible scheme, and all the fibres \( \mathcal{X}_b \) are (possibly reducible) varieties. The following result, well known in singularity theory, follows from the Hilbert-Samuel stratification [8] (in particular, from the finiteness proved in [8, Theorem 4.15]).

**Theorem 2.1.** Let \( \mathcal{Y} \subset \mathcal{X} \) be a closed subscheme, and let \( m \) be a nonnegative integer. The set of \( y \in \mathcal{Y} \) such that \( \mathcal{Y}_{p(y)} \) has multiplicity at least \( m \) at \( y \) is Zariski-closed in \( \mathcal{Y} \).

For our purposes, \( \mathcal{Y} \to B \) will be a family of subschemes of dimension \( d \) in the family of (possibly reducible) varieties \( \mathcal{X} \to B \). Since we are interested in the multiplicities at several points at a time for the computation of Seshadri constants, we need a multipoint analogue of Theorem 2.1. Denote \( \mathcal{X}^r_B := \mathcal{X} \times_B \cdots \times_B \mathcal{X} \to B \) (respectively \( \mathcal{Y}^r_B \to B \)) the family whose fiber over \( b \) is \((\mathcal{X}_b)^r \) (respectively \((\mathcal{Y}_b)^r \)). The next Corollary is an immediate consequence of (2.1).
Corollary 2.2. Let \( \mathcal{Y} \subset \mathcal{X} \) be a closed subscheme, and let \( m_1, \ldots, m_r \) be non-negative integers. The set of tuples \( (y_1, \ldots, y_r) \in (\mathcal{Y}_b)^r, \ b \in B \) such that \( \mathcal{Y}_b \) has multiplicity at least \( m_i \) at each \( y_i \) is Zariski-closed in \( \mathcal{Y}_B^r \).

Given a closed subscheme \( \mathcal{Y} \subset \mathcal{X} \) and a sequence \( \mathbf{m} = (m_1, \ldots, m_r) \), denote \( \mathcal{I}_m(\mathcal{Y}) \subset \mathcal{Y}_B^r \) the closed set given by (2.4). Fix be a relatively nef divisor \( \mathcal{L} \) on \( \mathcal{X} \), and denote \( \mathcal{L}_b = \mathcal{L}|_{\mathcal{X}_b} \). Since the definition of multipoint Seshadri constants only makes sense for distinct smooth points, we will usually restrict to the complement of the diagonals and the singularities in \( \mathcal{X}_B^r \), which is Zariski open.

Lemma 2.3. Let \( r \) be a positive integer, let \( \mathcal{X}_B^r \) be the complement of the diagonals and the singularities in \( \mathcal{X}_B^r \), and let a real number \( \epsilon > 0 \) and a nonzero real vector \( \ell = (l_1, \ldots, l_r) \) be given with each \( l_i \geq 0 \). Then the set of tuples \( (p_1, \ldots, p_r) \in \mathcal{X}_B^r \) such that

\[
\epsilon_d \left( l_1 p_1, \ldots, l_r p_r; \mathcal{X}_b, \mathcal{L}_b \right) \leq \epsilon
\]

is the union of at most countably many Zariski closed sets of \( \mathcal{X}_B^r \).

Proof. Let \( \text{Hilb}_d(\mathcal{X}/B) \) denote the relative Hilbert scheme of subschemes of \( \mathcal{X} \) of (relative) dimension \( d \). It has countably many irreducible components, which are irreducible projective schemes over \( B \) (see [9]). Let \( H \) be one of these components, and let \( \mathcal{H} \subset \mathcal{X} \times_B H \to H \) be the corresponding universal family. By the standard properties of the Hilbert scheme, the intersection with \( \mathcal{L}_b \) is constant, i.e., \( \mathcal{L}_b^d \cdot \mathcal{H}_{b,h} \) does not depend on \( b \in B \), \( h \in H \). Denote this number by \( \mathcal{L}_B^d \cdot H \).

For each choice of a sequence \( \mathbf{m} = (m_1, \ldots, m_r) \) of \( r \) nonnegative integers, we apply corollary 2.2 above to the universal family \( \mathcal{H} \to H \) and get a Zariski closed set

\[
\mathcal{I}_m(\mathcal{H}) \subset \mathcal{H}_H^r \subset (\mathcal{X} \times_B H)^r_H = \mathcal{X}_B^r \times_B H.
\]

Let \( \mathcal{P}_m(\mathcal{H}) \subset \mathcal{X}_B^r \) be the image of \( \mathcal{I}_m(\mathcal{H}) \) by projection on the first factor. Since \( H \) is projective, \( \mathcal{P}_m(\mathcal{H}) \) is Zariski closed, and \( \mathcal{P}_m(\mathcal{H}) := \mathcal{P}_m(\mathcal{H}) \cap \mathcal{X}_B^r \) is Zariski closed in \( \mathcal{X}_B^r \). By the definition of Seshadri constants, for all \( (p_1, \ldots, p_r) \in \mathcal{P}_m(\mathcal{H}) \) with \( p_i \in \mathcal{X}_b \), \( \epsilon_d(l_1 p_1, \ldots, l_r p_r; \mathcal{X}_b, \mathcal{L}_b) \leq (\mathcal{L}_B^d \cdot H/ \sum l_i m_i)^{1/d} \).

Since every \( d \)-dimensional subscheme is represented by a point in some component of the Hilbert scheme, it follows that

\[
\epsilon_d \left( p_1, \ldots, p_r; \mathcal{X}_b, \mathcal{L}_b \right) = \inf_{(p_1, \ldots, p_r) \in \mathcal{P}_m(\mathcal{H})} \left\{ \left( \frac{\mathcal{L}_B^d \cdot H}{\sum l_i m_i} \right)^{1/d} \right\}.
\]  

(2.4)

Thus, for each \( \epsilon \in \mathbb{R} \), the set of tuples \( (p_1, \ldots, p_r) \) such that \( \epsilon_d(p_1, \ldots, p_r; X, L) \leq \epsilon \) is exactly

\[
\bigcup_{\mathcal{P}_m(\mathcal{H}) \cap \mathcal{X}_B^r, \epsilon \leq \epsilon_d} \mathcal{P}_m(\mathcal{H}),
\]

hence the claim. \( \square \)

Remark 2.5. From the previous Lemma applied to \( \mathcal{X} = X, B = \text{Spec} \ K \), for very general points \( p_1, \ldots, p_r \) the Seshadri constant \( \epsilon_d(p_1, \ldots, p_r; X, L) \) and its weighted counterparts are independent of the points chosen, and thus \( \epsilon_d(r; X, L) \) is well defined. Moreover, (2.4) shows that

\[
\epsilon_d(r; X, L) = \inf_{\mathcal{P}_m(\mathcal{H}) = X^r} \left\{ \left( \frac{\mathcal{L}_B^d \cdot H}{\sum m_i} \right)^{1/d} \right\}.
\]
Remark 2.6. If \( d = 1 \), then the infimum in (2.4) can be taken over all \( m \) and all components \( \mathcal{H} \) of the Hilbert scheme of \( X \) in all dimensions (including, e.g., the isolated point corresponding to the whole variety \( X \)). Doing so, the Nakai-Moishezon for \( \mathbb{R} \)-divisors \( [3] \) implies that the infimum is attained by some \( \mathcal{H} \) and \( m \) (see the proof of [13, Proposition 4], and also [4, 1]) hence the set of values effectively taken by the Seshadri constant as points vary is either finite or countable.

Remark 2.7. For surfaces, a stronger version of Lemma (2.3) is known to hold. Namely a finite number of Zariski closed sets (hence a single one) suffices, and moreover the set of values effectively taken by the Seshadri constant is either finite or has exactly one accumulation point which is \( \sqrt{L^2/r} \). This has been proved by Oguiso [10] for \( r = 1 \) and follows from Harbourne-Roe [6] for \( r > 1 \). Unfortunately, the methods used to prove such finiteness do not seem to extend to varieties of higher dimension.

Remark 2.8. For Seshadri constants at higher dimensional centers (i.e. measuring the multiplicities at non-closed points) as defined by Paolelli in [11] similar semi-continuity results hold; in this case, the Hilbert scheme has to be used as parameter space in the place of the \( r \)th product of the variety.

3. PROOFS OF THEOREMS

3.1. Proof of Theorems 1.1 and 1.5. We start with Theorem 1.1 and consider the degeneration to the normal cone of very general points \( p_1, \ldots, p_r \), in \( X \). In detail set \( B = \mathcal{M}^1 \) and let \( \pi: \mathcal{X} \to X \times B \) be the blowup at the points \( \tilde{p}_i = (p_i, 0) \in X \times B \).

The exceptional divisor \( F \) is a disjoint union \( F = \bigcup F_i \) where \( F_i = \mathbb{P}(T_{p_i} \oplus \mathcal{C}) \simeq \mathbb{P}^n \) is the projective completion of the tangent space \( T_{p_i}X \) of \( X \) at \( p_i \). We denote by \( q_1, q_2 \) the projections from \( X \times B \) to the factors and let \( q = q_2 \circ \pi: \mathcal{X} \to B \).

Set \( \delta = \epsilon_d(r; X, L) \) and fix \( 1 \leq i \leq r \). By replacing \( B \) with an open set around \( 0 \) if necessary we can find sections \( \sigma'_{i,j} \) for \( j = 1, \ldots, s \) which pass through \( p_i \) and have very general tangent direction there. Denote by \( \sigma_{i,j} \) the section obtained from the proper transform of \( \sigma'_{i,j} \) in \( \mathcal{X} \). Then the collection \( \Sigma_i = \{ \sigma_{i,1}(0), \ldots, \sigma_{i,s}(0) \} \) is a set of \( s \) very general points in \( F_i \simeq \mathbb{P}^n \). Denote \( \sigma' := (\sigma'_{1,1}, \ldots, \sigma'_{r,s}) : B \to X^{rs} \) and \( \sigma := (\sigma_{1,1}, \ldots, \sigma_{r,s}) : B \to X^{rs}_B \).

Each component of the relative Hilbert scheme of \( X \times B \) is of the form \( H' = Z \times B \) where \( Z \) is a component of the Hilbert scheme of \( X \); for each such \( H' \) there is a unique component \( H \) of the Hilbert scheme of \( \mathcal{X} \) with \( H' \cap q_2^{-1}(B \setminus \{0\}) = H \cap q^{-1}(B \setminus \{0\}) \). We resume notations as in the previous section, with \( Z \to Z, H' \to H' \), and \( H \to H \) as universal Hilbert families on \( X, X \times B \) and \( \mathcal{X} \) respectively.

Recycling notation from above, let \( \mathcal{P}_m(Z) \) be the projection of \( \mathcal{I}_m(Z) \) to \( X^{rs} \) and consider for a while a sequence \( m = (m_1, \ldots, m_{rs}) \) of \( rs \) nonnegative integers such that \( \mathcal{P}_m(Z) = X^{rs} \). For convenience, set also \( m_{i,j} = m_{(i-1)s+j} \) for \( i = 1, \ldots, r, j = 1, \ldots, s \). Then \( \mathcal{P}_m(H') = (X \times B)^{rs}_B = X^{rs} \times B \) and \( \mathcal{P}_m(H) = X^{rs}_B \).

Consider the following pullback diagram:

\[
\begin{array}{ccc}
\mathcal{I}_m(H') \times \mathcal{P}_m(H') & \xrightarrow{\tau_2} & B \\
\downarrow{\tau_1} & & \downarrow{\sigma' \times i} \\
\mathcal{I}_m(H') & \xrightarrow{\mathcal{P}_m} & \mathcal{P}_m(H') = X^{rs} \times B 
\end{array}
\]
\(\tau_2\) is onto and proper, so (restricting to a smaller neighbourhood of 0 if needed) we may choose a section; denote by \(\zeta : B \to Z\) the composition of this section with the natural maps

\[
\mathcal{I}_m(\mathcal{H}') \times \mathcal{P}_m(\mathcal{H}') B \overset{\tau_2}{\to} \mathcal{I}_m(\mathcal{H}') \hookrightarrow \mathcal{H}'_{H'} \hookrightarrow X_{rs} H \to H = Z \times B \to Z.
\]

Let \(\mathcal{Y'} \to B\) be the family obtained from the universal family \(Z \to Z\) by base change through \(\zeta\). By construction, each fiber \(\mathcal{Y}'_b\) with \(b \neq 0\) is a subscheme of \(X\) of dimension \(d\) with a point of multiplicity \(\geq m_{i,j}\) at \(\sigma_{i,j}(b)\). Consider the strict transform \(\mathcal{Y'}\) of \(\mathcal{Y'}\) in \(X\). By flatness, \(\mathcal{Y}_0\) has dimension \(d\), and by semicontinuity of multiplicities it has a point of multiplicity \(\geq m_{i,j}\) at \(\sigma_{i,j}(0)\). Therefore for every \(i\) such that some \(m_{i,j} > 0\), \(\mathcal{Y}_0\) is a \(d\)-dimensional subscheme of \(F_i \cong \mathbb{P}^n\) of degree at least \(m_i := (\epsilon_d(s; \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))^d \sum_j m_{i,j}\). Since this degree is exactly the multiplicity of \(\mathcal{Y}_0\) at \(p_i\), it follows that

\[
L \cdot Z = L \cdot \mathcal{Y}_0' \geq \delta^d \sum_{i=1}^r m_i = \delta^d (\epsilon_d(s; \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))^d \sum_{k=1}^{rs} m_k.
\]

Since this is true whenever \(\mathcal{P}_m(Z) = X_{rs}\), in view of (2.5.4) the claimed bound on \(\epsilon_d(r; s; X, L)\) follows.

The proof of Theorem 1.5 follows easily as if \(s_1 \leq s_2 \leq \cdots \leq s_r\) then \(s = \sum_{i=1}^r s_i \leq rs_r\) and

\[
\epsilon_d(s; X, L) \geq \epsilon_d(rs_r; X, L) \geq \epsilon_d(r; X, L) \cdot \epsilon_d(s; \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \geq \epsilon_d(s; X, L).
\]

3.2. Proof of Theorem 1.6. Essentially, the proof of Theorem 1.1 works also in this more general setting. We have proved the particular case first for clarity, and give next a sketch of the changes needed for 1.6.

Just as above let \(X \to B\) be the degeneration to the normal cone of very general points \(p_1, \ldots, p_r\) in \(X\), with exceptional divisors \(\mathbb{P}^n\). Also just as above, by shrinking \(B\) if necessary, for every component \(Z\) of \(\text{Hilb}_d(X)\) and every \(\mathbf{m} = (m_1, \ldots, m_s)\) with \(\mathcal{P}_m(Z) = X^s\) there exist schemes in \(\text{Hilb}_d(X)\) with multiplicities at least

\[
m_i := (\epsilon_d(s_i; \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)))^d \sum_{j=s_{i-1}+1}^{s_i} m_j
\]

at general points \(p_i, i = 1, \ldots, r\). From this the result follows exactly as in the previous case. \(\square\)

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