TILTING MODULES FOR CYCLOTONOMIC SCHUR ALGEBRAS

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ABSTRACT. This paper investigates the tilting modules of the cyclotomic $q$–Schur algebras, the Young modules of the Ariki–Koike algebras, and the interconnections between them. The main tools used to understand the tilting modules are contragredient duality, and the Specht filtrations and dual Specht filtrations of certain permutation modules. Surprisingly, Weyl filtrations — which are in general more powerful than Specht filtrations — play only a secondary role.

1. INTRODUCTION

In [7] Dipper, James and the author introduced the cyclotomic $q$–Schur algebras as another tool for studying the representations of the Ariki–Koike algebras. These algebras have a rich and beautiful combinatorial representation theory which closely resembles that of the $q$–Schur algebras. In particular, the cyclotomic $q$–Schur algebras are quasi–hereditary and so they have a theory of tilting modules by Ringel’s theorem [20]. The purpose of this paper is to describe these tilting modules.

A special case of the cyclotomic $q$–Schur algebras are the $q$–Schur algebras of Dipper and James [5]. The $q$–Schur algebra $S(d, n)$ can be realized as the endomorphism algebra $\End_{H(S_n)}(V \otimes n)$, where $V$ is the natural module for the quantum group $U_q(gl_d)$ and $H(S_n)$ is the Iwahori–Hecke algebra of the symmetric group. Donkin [9,10] showed that when $d \geq n$ the tilting modules for $S(d, n)$ are the indecomposable direct summands of the exterior powers $\bigwedge \lambda V = \bigwedge \lambda^d V \otimes \cdots \otimes \bigwedge \lambda V$, where $\lambda$ is a partition of $n$.

By definition a cyclotomic $q$–Schur algebra is the $H$–module endomorphism algebra of a certain module $M(\Lambda) = \bigoplus_{\Lambda \in \Lambda} M(\lambda)$ of the Ariki–Koike algebra $H$. We do not know how to describe $M(\Lambda)$ as a tensor product; however, in essence all of Donkin’s results generalize to the cyclotomic setting — even though the statements and proofs do not. Instead of exterior powers we consider certain hom–spaces $E(\alpha) = \Hom_H(M(\lambda), N(\alpha))$, where $N(\alpha)$ is something like an induced “sign representation” for the Ariki–Koike algebra. In order to understand these modules, we work mainly with Specht filtrations and dual Specht filtrations of the modules $M(\lambda)$ and $N(\alpha)$. Using these filtrations we are able to show that $M(\lambda)$ and $N(\alpha)$ are both self–dual $H$–modules; this implies that $E(\alpha)$ is self–dual. Further, we can “lift” these filtrations to show that $E(\alpha)$ has a Weyl filtration. Combined, these two results to show that the tilting modules of the cyclotomic $q$–Schur algebras are the indecomposable direct summands of the $E(\alpha)$.

We remark that this description of the tilting modules is valid only under some mild restrictions on the poset of multipartitions $\Lambda$ and on the defining parameters $Q_1, \ldots, Q_r$ for the cyclotomic $q$–Schur algebra; for the precise statement see Theorem 6.17. Our first restriction is that $\Lambda$ must contain all multipartitions of $n$; this is the analogue of the condition $d \geq n$ in Donkin’s theorem. The second restriction is that $Q_s \neq 0$ for any $s$; this is necessary in order to show that $E(\alpha)$ is self–dual. The final restriction is that $Q_1, \ldots, Q_r$...
must all be distinct; this is needed to force the rank of $E(\alpha)$ to be independent of the ground ring and the choice of parameters.

In more detail the contents of the paper are as follows. Section 2 recalls the notation and basic results from the representation theory of the cyclotomic $q$–Schur algebras and the Ariki–Koike algebras. Section 3 investigates the Young modules of the Ariki–Koike algebras; these are the indecomposable direct summands of the modules $M(\lambda)$ mentioned above. The Young module enjoy all of the properties of the Young modules of the symmetric groups introduced by James [14]; they are also closely related to the tilting modules of the cyclotomic $q$–Schur algebras. The fourth section of the paper deals with a duality operation on the category of $\mathcal{H}$–modules. In the case of the symmetric groups this duality corresponds to tensoring with the sign representation; applying this duality to $M(\lambda)$ produces the module $N(\lambda)$. Section 5 studies contragredient duality for the Ariki–Koike algebra; the main result here is that $M(\lambda)$ and $N(\lambda)$ are both self–dual $\mathcal{H}$–modules. Finally, building on the previous results, section 6 classifies the tilting modules as the indecomposable direct summands of the $E(\alpha)$ and section 7 describes the Ringel duals of the cyclotomic $q$–Schur algebras.

2. CYCLOTOMIC SCHUR ALGEBRAS

This section is a summary of the definitions and results that we will need from the representation theory of the cyclotomic $q$–Schur algebras and the Ariki–Koike algebras. The reader is referred to [7] for more details.

Fix positive integers $r$ and $n$ and let $\mathcal{S}_n$ be the symmetric group of degree $n$. Let $R$ be a commutative ring with 1 and let $q, Q_1, \ldots, Q_r$ be elements of $R$ such that $q$ is invertible. The Ariki–Koike algebra $\mathcal{H} = \mathcal{H}_{r,n}$ is the associative unital $R$–algebra with generators $T_0, T_1, \ldots, T_{n-1}$ and relations

\[(T_0 - Q_1) \cdots (T_0 - Q_r) = 0,\]
\[(T_i - q)(T_i + q^{-1}) = 0, \quad \text{for } 1 \leq i \leq n - 1,\]
\[T_0T_iT_0T_1 = T_1T_0T_iT_0,\]
\[T_{i+1}T_{i+1} = T_{i+1}T_{i+1}, \quad \text{for } 1 \leq i \leq n - 2,\]
\[T_iT_j = T_jT_i, \quad \text{for } 0 \leq i < j - 1 \leq n - 2.\]

The second relation is often written as $(T_i - \nu)(T_i + 1) = 0$, for $1 \leq i < n$. This presentation may be turned into the one above by renormalizing $T_i$ as $v^{-\frac{1}{2}}T_i$ and setting $q = v^2$. To do this it is necessary that $v$ have a square root in $R$; however, every field is a splitting field for $\mathcal{H}$ (because $\mathcal{H}$ is cellular), so we can adjoin a square root of $v$ without changing the representation theory of $\mathcal{H}$. We use the presentation above because it renormalizes the natural inner product on $\mathcal{H}$ and so makes many formulas nicer later on. We convert the formulas that we need from the literature without mention.

For $i = 1, \ldots, n - 1$ let $s_i$ be the transposition $(i, i + 1)$ in $\mathcal{S}_n$; then \{ $s_1, \ldots, s_{n-1}$ \} generate $\mathcal{S}_n$. If $w \in \mathcal{S}_n$, then $w = s_{i_1} \cdots s_{i_k}$ for some $i_j$; if $k$ is minimal then we say that this expression for $w$ is reduced and that $w$ has length $\ell(w) = k$. In this case we set $T_w = T_{i_1} \cdots T_{i_k}$; then $T_w$ is independent of the choice of reduced expression. We also let $L_k = T_{k-1}T_kT_1T_{k-1}$ for $k = 1, 2, \ldots, n$. These elements give a basis of $\mathcal{H}$.

2.1 (Ariki–Koike [3, Theorem 3.10]) The Ariki–Koike algebra $\mathcal{H}$ is free as an $R$–module with basis \{ $L_1^{a_1} \cdots L_n^{a_n}T_w$ | $w \in \mathcal{S}_n$ and $0 \leq a_i < r$ for $1 \leq i \leq n$ \}.

Recall that a composition of $n$ is sequence $\sigma = (\sigma_1, \sigma_2, \ldots)$ of non–negative integers such that $|\sigma| = \sum_i \sigma_i = n$; $\sigma$ is a partition if in addition $\sigma_1 \geq \sigma_2 \geq \cdots$. If $\sigma_i = 0$ for all $i > k$ then we write $\sigma = (\sigma_1, \ldots, \sigma_k)$. 

\[\begin{align*}
(T_0 - Q_1) \cdots (T_0 - Q_r) &= 0, \\
(T_i - q)(T_i + q^{-1}) &= 0, \quad \text{for } 1 \leq i \leq n - 1, \\
T_0T_iT_0T_1 &= T_1T_0T_iT_0, \\
T_{i+1}T_{i+1} &= T_{i+1}T_{i+1}, \quad \text{for } 1 \leq i \leq n - 2, \\
T_iT_j &= T_jT_i, \quad \text{for } 0 \leq i < j - 1 \leq n - 2.
\end{align*}\]
A multicomposition of $n$ is an $r$–tuple $\lambda = (\lambda^{(1)},\ldots,\lambda^{(r)})$ of compositions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. A multicomposition $\lambda$ is a multipartition if each $\lambda^{(i)}$ is a partition. If $\lambda$ is a multipartition of $n$ then we write $\lambda \vdash n$. The diagram $[\lambda]$ of the multicomposition $\lambda$ is the set $[\lambda] = \{(i,j,s) \mid 1 \leq \lambda^{(s)}_{ij} \leq i \text{ and } 1 \leq s \leq r\}$.

The set of multicompositions of $n$ is partially ordered by dominance; that is, if $\lambda$ and $\mu$ are two multicompositions then $\lambda$ dominates $\mu$, and we write $\lambda \succeq \mu$, if

$$\sum_{c=1}^{s-1} |\lambda^{(c)}| + \sum_{j=1}^{i} \lambda^{(s)}_{ij} \geq \sum_{c=1}^{s-1} |\mu^{(c)}| + \sum_{j=1}^{i} \mu^{(s)}_{ij}$$

for $1 \leq s \leq r$ and for all $i \geq 1$. If $\lambda \succeq \mu$ and $\lambda \neq \mu$ then we write $\lambda \succ \mu$.

If $\lambda$ is a multicomposition let $\mathcal{S}_\lambda = \mathcal{S}_{\lambda^{(1)}} \times \cdots \times \mathcal{S}_{\lambda^{(r)}}$ be the corresponding Young subgroup of $\mathcal{S}_n$. Set

$$x_\lambda = \sum_{w \in \mathcal{S}_\lambda} q^{\ell(w)}T_w \quad \text{and} \quad u_\lambda^+ = \prod_{s=2}^{r} \prod_{k=1}^{a_s} (L_k - Q_s),$$

where $a_s = |\lambda^{(1)}| + \cdots + |\lambda^{(s-1)}|$ for $2 \leq s \leq r$. Set $m_\lambda = x_\lambda u_\lambda^+ = u_\lambda^+ x_\lambda$ and define $M(\lambda)$ to be the right ideal $M(\lambda) = m_\lambda \mathcal{H}$ of $\mathcal{H}$.

If $\lambda = (\lambda^{(1)},\ldots,\lambda^{(r)})$ is a multipartition then a standard $\lambda$–tableau is an $r$–tuple $t = (t^{(1)},\ldots,t^{(r)})$ of standard tableau which, collectively, contain the integers $1,2,\ldots,n$ and such that $t^{(c)}$ is a standard $\lambda^{(c)}$–tableau, for $1 \leq c \leq r$. Let $T(\lambda)$ be the set of standard $\lambda$–tableau.

Let $t^\lambda$ be the standard $\lambda$–tableau with the numbers $1,2,\ldots,n$ entered in order from left to right along its rows. If $t$ is any standard $\lambda$–tableau let $d(t) \in \mathcal{S}_n$ be the unique permutation such that $t = t^\lambda d(t)$. Finally, let $*:\mathcal{H} \rightarrow \mathcal{H}$ be the anti–isomorphism given by $T_i^* = T_i^\lambda$, for $i = 0,1,\ldots,n-1$, and set $m_{st} = T_{d(t)} d(s)^\lambda T_{d(t)}$.

2.2 (Dipper–James–Mathas [F, Theorem 3.26]) The Ariki–Koike algebra $\mathcal{H}$ is free as an $R$–module with (cellular) basis $\{ m_{st} \mid s,t \in T(\lambda) \text{ for some } \lambda \vdash n \}$.

Here, and below, whenever we write expressions involving a pair of tableaux (such as $m_{st}$ or $\varphi_{ST}$), we implicitly assume that the two tableaux are of the same shape.

The basis $\{ m_{st} \}$ is the standard basis of $\mathcal{H}$. For each multipartition $\lambda$ let $\mathcal{H}(\lambda)$ be the $R$–submodule of $\mathcal{H}$ with basis $\{ m_{uv} \mid u,v \in T(\mu) \text{ for some } \mu \succ \lambda \}$; then $\mathcal{H}(\lambda)$ is a two–sided ideal of $\mathcal{H}$.

Let $S(\lambda)$ be the Specht module (or cell module) corresponding to the multipartition $\lambda$; that is, $S(\lambda) \cong (m_\lambda + \mathcal{H}(\lambda)),\mathcal{H}$, a submodule of $\mathcal{H}/\mathcal{H}(\lambda)$. For each $t \in T(\lambda)$ let $m_t = m_{1^\lambda t} + \mathcal{H}(\lambda)$; then $S(\lambda)$ is free as an $R$–module with basis $\{ m_t \mid t \in T(\lambda) \}$. Further, there is an associative symmetric bilinear form on $S(\lambda)$ which is determined by

$$\langle m_s,m_t \rangle_{\mathcal{H}(\lambda)} = (m_{s1^\lambda t}m_{1^\lambda t}) \mod \mathcal{H}(\lambda)$$

for all $s,t \in T(\lambda)$. The radical $\text{rad} S(\lambda)$ of this form is again an $\mathcal{H}$–module, so $\text{D}(\lambda) = S(\lambda)/\text{rad} S(\lambda)$ is an $\mathcal{H}$–module. When $R$ is a field, $\text{D}(\lambda)$ is either 0 or absolutely irreducible and all simple $\mathcal{H}$–modules arise uniquely in this way.

We can now give the definition of the cyclotomic $q$–Schur algebras. A set of multipartitions of $n$ is saturated if $\Lambda$ is finite and whenever $\lambda$ is a multipartition such that $\lambda \succeq \mu$ for some $\mu \in \Lambda$ then $\lambda \in \Lambda$. If $\Lambda$ is a saturated set of multicompositions let $\Lambda^+$ be the set of multipartitions in $\Lambda$. 

2.3. Definition. Suppose that $\Lambda$ is a saturated set of multipartitions of $n$. The cyclotomic $q$–Schur algebra with weight poset $\Lambda$ is the endomorphism algebra

$$\mathcal{S}(\Lambda) = \text{End}_{\mathcal{H}} (M(\Lambda)), \quad \text{where } M(\Lambda) = \bigoplus_{\lambda \in \Lambda} M(\lambda).$$

As we now describe, $\mathcal{S}(\Lambda)$ has a basis indexed by pairs of semistandard tableau.

A $\lambda$–tableau of type $\mu$ is a map $T : [\lambda] \rightarrow \{(i, s) \mid i \geq 1 \text{ and } 1 \leq s \leq r\}$ such that $\mu_i(s) = \# \{ x \in [\lambda] \mid T(x) = (i, s) \}$ for all $i \geq 1$ and $1 \leq s \leq r$. We think of a $T$ as being an $r$–tuple $T = (T^{(1)}, \ldots, T^{(r)})$, where $T^{(s)}$ is the $\lambda^{(s)}$–tableau with $T^{(s)}(i, j) = T(i, j, s)$ for all $(i, j, s) \in [\lambda]$. In this way we identify the standard tableaux above with the tableaux of type $(0, \ldots, 0, \lambda^t)$. If $T$ is a tableau of type $\mu$ then we write $\text{Type}(T) = \mu$.

Given two pairs $(i, s)$ and $(j, t)$ write $(i, s) \preceq (j, t)$ if either $s < t$, or $s = t$ and $i \leq j$.

2.4. Definition. A tableau $T$ is (row) semistandard if, for $1 \leq t \leq r$, the entries in $T^{(t)}$ are

1. weakly increasing along the rows (with respect to $\preceq$);
2. strictly increasing down columns; and,
3. $(i, s)$ appears in $T^{(t)}$ only if $s \geq t$.

Let $T^\nu_\mu(\lambda)$ be the set of semistandard $\lambda$–tableaux of type $\mu$ and let $T^\nu_\lambda(\Lambda) = \bigcup_{\mu \in \Lambda} T^\nu_\mu(\lambda)$ and $T^\nu(\Lambda^+) = \bigcup_{\lambda \in \Lambda^+} T^\nu_\mu(\lambda)$.

Usually, we will refer to row semistandard tableaux simply as semistandard tableaux. Later we will meet column semistandard tableaux (these are the conjugates of row semistandard tableaux).

Notice that if $T^\nu_\mu(\lambda)$ is non–empty then $\lambda \succeq \mu$. This observation will be used many times below.

Suppose that $t$ is a standard $\lambda$–tableau and let $\mu$ be a multicomposition. Let $\mu(t)$ be the tableau obtained from $t$ by replacing each entry $j$ with $(i, k)$ if $j$ appears in row $i$ of $t^\nu$. The tableau $\mu(t)$ is a $\lambda$–tableau of type $\mu$; it is not necessarily semistandard.

If $S$ and $T$ are semistandard $\lambda$–tableaux of type $\mu$ and $\nu$, respectively, and if $t$ is a standard $\lambda$–tableau let

$$m_{ST} = \sum_{s \in T^{(t)}(\lambda)} q^{|d(s)|} m_{st} \quad \text{and} \quad m_{ST} = \sum_{s, t \in T^{(t)}(\lambda)} q^{|d(s)| + |d(t)|} m_{st}.$$

Then we have the following two results.

2.5 \[ Theorem 4.14 \] Suppose that $\mu$ is a multicomposition of $n$. Then $M(\mu)$ is free as an $R$–module with basis $\{ m_{ST} \mid S \in T^\mu_\mu(\lambda), t \in \mathcal{T}(\lambda) \text{ for some } \lambda \vdash \mu \}$.

For $S$ and $T$ as above define $\varphi_{ST} \in \mathcal{S}(\Lambda)$ by $\varphi_{ST}(m_{\alpha h}) = \delta_{\alpha \mu} m_{ST} h$, for all $h \in \mathcal{H}$ and all $\alpha \in \Lambda$. (Here $\delta_{\alpha \nu}$ is the Kronecker delta; so, $\delta_{\alpha \nu} = 1$ if $\alpha = \nu$ and it is zero otherwise.) Then $\varphi_{ST}$ belongs to $\mathcal{S}(\Lambda)$; moreover, these elements give us a basis of $\mathcal{S}(\Lambda)$.

2.6 \[ Theorem 6.6 \] The cyclotomic $q$–Schur algebra $\mathcal{S}(\Lambda)$ is free as an $R$–module with cellular basis $\{ \varphi_{ST} \mid S, T \in T^\mu_\lambda(\lambda) \text{ for some } \lambda \in \Lambda^+ \}$.

The basis $\{ \varphi_{ST} \}$ is called the semistandard basis of $\mathcal{S}(\Lambda)$. Because this basis is cellular the map $* : \mathcal{S}(\Lambda) \rightarrow \mathcal{S}(\Lambda)$ which is determined by $\varphi_{ST}^* = \varphi_{TS}$ is an anti–isomorphism of $\mathcal{S}(\Lambda)$. This involution is closely related to the * involution on $\mathcal{H}$; explicitly, if $\varphi : M(\nu) \rightarrow M(\mu)$ is an $\mathcal{H}$–module homomorphism then $\varphi^* : M(\mu) \rightarrow M(\nu)$ is the homomorphism given by $\varphi^*(m_{\mu h}) = (\varphi(\nu))^* h$, for all $h \in \mathcal{H}$. 
In order to understand how \( \mathcal{S}(\Lambda) \) acts on its representations we need to explain how the multiplication in \( \mathcal{S}(\Lambda) \) is determined by the multiplication in \( \mathcal{H} \). Suppose that \( S, T, U \) and \( V \) are semistandard tableaux with \( \mu = \text{Type}(S) \), \( \alpha = \text{Type}(U) \) and \( \nu = \text{Type}(V) \). Then \( m_{UV} = m_\alpha h_0^{\alpha} \), for some \( h_0^{\alpha} \in \mathcal{H} \), and there exist scalars \( r_{XY} \in R \) such that

\[
m_{ST} h_0^{\nu} = \sum_{X \in T_\alpha^{\mu}(\Lambda^+)} \sum_{Y \in T_\nu(Y^\lambda)} r_{XY} m_{XY}
\]

by \([6, \text{Cor. 5.17}]\). Now, \( \varphi_{ST} \varphi_{UV}(m_\nu h) = \varphi_{ST}(m_{UV})h = \varphi_{ST}(m_\alpha h_0^{\alpha} h) = m_{ST} h_0^{\nu} h \), for all \( h \in \mathcal{H} \); so \([6, \text{Cor. 5.17}]\) determines the product \( \varphi_{ST} \varphi_{UV} \) in \( \mathcal{S}(\Lambda) \). Explicitly, we have

\[
\varphi_{ST} \varphi_{UV} = \sum_{X \in T_\alpha^{\mu}(\Lambda^+)} \sum_{Y \in T_\nu(Y^\lambda)} r_{XY} \varphi_{XY},
\]

where the \( r_{XY} \) are given by \([6, \text{Cor. 5.17}]\). Note that \( \varphi_{ST} \varphi_{UV} = 0 \) if \( \text{Type}(T) \neq \text{Type}(U) \). In addition, because \( \{ \varphi_{ST} \} \) is a cellular basis, if \( r_{XY} \neq 0 \) then \( \text{Shape}(X) \supseteq \text{Shape}(S) \), with equality only if \( X = S \). Moreover, if \( X = S \) then \( r_{XY} = r_{SY} \) depends only on \( T, U \) and \( V \); in particular, \( r_{SY} \) does not depend on \( S \). These details can be found in \([6]\); for a complete treatment of the theory of cellular algebras see \([3, 7]\).

For each multipartition \( \lambda \in \Lambda^+ \) there is a right \( \mathcal{S}(\Lambda) \)-module \( \Delta(\lambda) \), called a Weyl module. The Weyl module \( \Delta(\mu) \) is the submodule of \( \text{Hom}_{\mathcal{H}}(\mathcal{H}(\lambda), S(\lambda)) \) with basis the set of maps \( \{ \varphi_T \mid T \in T_\mu^{\mu}(\lambda), \mu \in \Lambda \} \), where \( \varphi_T(m_\mu h) = \delta_{\mu \nu} \sum_t m_t h \) and the sum is over those standard \( \lambda \)-tableaux \( t \) such that \( \mu(t) = T \). If \( T \) is a semistandard \( \lambda \)-tableau and \( \varphi_{UV} \) is a semistandard basis element then the action of \( \mathcal{S}(\Lambda) \) on \( \Delta(\lambda) \) is determined by

\[
\varphi_{ST} \varphi_{UV} = \sum_{Y \in T_\nu(Y^\lambda)} r_{XY} \varphi_Y,
\]

where \( r_{XY} \) is determined by \([6, \text{Cor. 5.17}]\) and \( \nu = \text{Type}(V) \). (As remarked above, \( r_{XY} \) is independent of \( S \).)

As with the Specht modules there is an inner product on \( \Delta(\lambda) \) which is determined by

\[
\langle \varphi_S, \varphi_T \rangle_{\varphi_\lambda} \equiv \varphi_{ST} \varphi_{TT} \equiv \text{mod } S^\lambda,
\]

where \( S^\lambda \) is the \( R \)-submodule of \( \mathcal{S}(\Lambda) \) with basis the set of maps \( \varphi_{UV} \) where \( U \) and \( V \) are semistandard \( \mu \)-tableaux with \( \mu \supset \lambda \). The quotient module \( L(\lambda) = \Delta(\lambda)/\text{rad } \Delta(\lambda) \) is absolutely irreducible and \( \{ L(\lambda) \mid \lambda \in \Lambda^+ \} \) is a complete set of non-isomorphic irreducible \( \mathcal{S}(\Lambda) \)-modules.

Recall that \( \omega = (\omega(1), \ldots, \omega(r)) \) is the multipartition with \( \omega(r) = (1^n) \) and \( \omega(s) = (0) \) for \( 1 \leq s < r \). From the definitions, \( m_\omega = 1 \) and \( \varphi_\omega = \varphi_{\omega(T)} \) is the identity map on \( \mathcal{H} \); so, \( \mathcal{H} = M(\omega) \). In particular, \( \varphi_\omega \) is an idempotent in \( \mathcal{S}(\Lambda) \) and it is easy to see that \( \mathcal{H} \cong \varphi_\omega \mathcal{S}(\Lambda) \varphi_\omega \) whenever \( \omega \in \Lambda \).

For an algebra \( A \) let \( A-\text{mod} \) be the category of finite dimensional right \( A \)-modules. As noted in \([15]\), standard arguments show that there is a functor

\[
F_\omega : \mathcal{S}(\Lambda)-\text{mod} \longrightarrow \mathcal{H}-\text{mod} : M \longrightarrow M \varphi_\omega
\]

which has the following properties.

**2.10 (The cyclotomic Schur functor \([15]\))** Suppose that \( R \) is a field and that \( \omega \in \Lambda \). Let \( \lambda \in \Lambda^+ \). Then, as right \( \mathcal{H} \)-modules,

(i) \( F_\omega(\Delta(\lambda)) \cong S(\lambda) \);
Furthermore, if $D(\mu) \neq 0$ then $[\Delta(\lambda) : L(\mu)] = [S(\lambda) : D(\mu)]$.

3. Cyclotomic Young modules

For each multicomposition $\mu$ of $n$ let $\varphi_\mu$ be the identity map on $M(\mu)$. This section describes the indecomposable summands of $M(\mu)$. We approach this question by considering the right $\mathcal{S}(\Lambda)$–modules $M(\mu) = \text{Hom}_{\mathcal{S}}(M(\Lambda), M(\mu))$.

All of these results in this section about the modules $M(\mu)$ apply without restriction on $\Lambda$; however, whenever we apply the Schur functor we implicitly assume that $\omega \in \Lambda$.

3.1. Proposition. Suppose that $\mu \in \Lambda$. Then the following hold.

(i) $M(\mu)$ is free as an $R$–module with basis

$$\{ \varphi_{ST} \mid S \in \mathcal{T}_\mu^\circ(\lambda), T \in \mathcal{T}_\nu^\circ(\lambda) \text{ for some } \nu \in \Lambda \text{ and } \lambda \in \Lambda^+ \}.$$

(ii) $M(\mu) \cong \varphi_\mu \mathcal{S}(\Lambda)$ as right $\mathcal{S}(\Lambda)$–modules; in particular, $M(\mu)$ is projective.

(iii) As $\mathcal{H}$–modules, $M(\mu) \cong F_\omega(M(\mu))$.

Proof. Part (i) is just a restatement of (2.7). For (ii), note that if $S \in \mathcal{T}_\mu^\circ(\lambda)$, for some $\lambda \in \Lambda^+$ and $\alpha \in \Lambda$, then $\varphi_\mu \varphi_{ST} = \delta_{ST} \varphi_{ST}$ for all $T \in \mathcal{T}_\alpha^\circ(\lambda)$. Hence, $\varphi_\mu \mathcal{S}(\Lambda) = M(\mu)$ by part (i). As $\varphi_\mu$ is an idempotent this also shows that $M(\mu)$ is a projective $\mathcal{S}(\Lambda)$–module. Finally, by part (i) again, the $\mathcal{H}$–module $F_\omega(M(\mu)) = \text{Hom}_{\mathcal{S}}(\mathcal{H}, M(\mu))$ is free as an $R$–module with basis $\{ \varphi_{ST} \mid S \in \mathcal{T}_\mu^\circ(\lambda), T \in \mathcal{T}(\lambda) \text{ for some } \lambda \in \Lambda^+ \}$. Hence, by (2.5), $F_\omega(M(\mu)) \cong M(\mu)$, where the isomorphism is given by the $R$–linear map determined by $\varphi_{ST} \mapsto \omega_{ST} = \varphi_{ST}(\mu)$ for all $S \in \mathcal{T}_\mu^\circ(\lambda)$ and $T \in \mathcal{T}(\lambda)$. \hfill $\Box$

An $\mathcal{S}(\Lambda)$–module $X$ has a Weyl filtration if it has an $\mathcal{S}(\Lambda)$–module filtration

$$X = X_1 \supset \cdots \supset X_k \supset X_{k+1} = 0$$

such that $X_i/X_{i+1} \cong \Delta(\lambda_i)$ for some multipartition $\lambda_i \in \Lambda^+$, for $1 \leq i \leq k$. Since each Weyl module $\Delta(\lambda)$ has simple head $L(\lambda)$ and $[\text{rad} \Delta(\lambda) : L(\mu)] \neq 0$ only if $\lambda \supset \mu$, the equivalence classes of the Weyl modules are a basis of the Grothendieck group of $\mathcal{S}(\Lambda)$; consequently, when $R$ is a field the filtration multiplicities

$$[X : \Delta(\lambda)] = \# \left\{ 1 \leq i \leq k \mid X_i/X_{i+1} \cong \Delta(\lambda) \right\}$$

are independent of the choice of filtration. Finally, note that if $X$ has a Weyl filtration as above then $F_\omega(X) = F_\omega(X_1) \supset \cdots \supset F_\omega(X_k) \supset F_\omega(X_{k+1}) = 0$ is a Specht filtration of $F_\omega(X)$ by (2.11); that is, $F_\omega(X_i)/F_\omega(X_{i+1}) \cong F_\omega(X_i/X_{i+1}) \cong S(\lambda_i)$, for $1 \leq i \leq k$.

3.2. Lemma. Suppose that $\mu \in \Lambda$. Then $M(\mu)$ has a Weyl filtration

$$M(\mu) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

and there exist multipartitions $\lambda_1, \ldots, \lambda_k$ such that $M_i/M_{i+1} \cong \Delta(\lambda_i)$, for $i = 1, \ldots, k$.

Moreover, if $\lambda_i \supset \lambda_j$ then $i > j$ and $\# \left\{ 1 \leq i \leq k \mid \lambda_i = \lambda \right\} = \# \mathcal{T}_\mu^\circ(\lambda)$ for each multipartition $\lambda$. Hence, $[M(\mu) : \Delta(\lambda)] = \# \mathcal{T}_\mu^\circ(\lambda)$ when $R$ is a field.

Proof. By Proposition 3.1, $\{ \varphi_{ST} \mid S \in \mathcal{T}_\mu^\circ(\lambda), T \in \mathcal{T}_\nu^\circ(\lambda) \text{ for some } \nu \in \Lambda \text{ and } \lambda \in \Lambda^+ \}$ is a basis of $M(\mu)$. Let $\{ S_1, \ldots, S_k \}$ be the set of semistandard tableaux of type $\mu$ ordered so that $i > j$ whenever $\text{Shape}(S_i) \supset \text{Shape}(S_j)$. Let $\lambda_i = \text{Shape}(S_i)$, for $1 \leq i \leq k$. Notice that $\lambda_i \supset \lambda_j$, for all $i$, since $\mathcal{T}_\mu^\circ(\lambda_i) \neq \emptyset$.

Fix an integer $i$, with $1 \leq i \leq k$, and let $M_i$ be the $R$–submodule of $M(\mu)$ with basis

$$\{ \varphi_{S_j T} \mid \lambda_j \supset \lambda_i \text{ and } T \in \mathcal{T}_\lambda^\circ(\lambda_j) \}.$$
Then $M_i$ is an $\mathcal{S}(\Lambda)$–module by the remarks after (2.8). Further, there is an isomorphism of $\mathcal{S}(\Lambda)$–modules $\Delta(\Lambda_i) \cong M_i/M_{i+1}$ given by $\varphi_T \mapsto \varphi_{S,T} + M_{i+1}$, for $T \in T_{\Lambda_i}(\Lambda_i)$, because $\Delta^{\lambda_i} \cap M_i \subseteq M_{i+1}$.

The simple $\mathcal{S}(\Lambda)$–modules $L(\lambda)$ are indexed by the multipartitions $\lambda \in \Lambda^+$. Fix a set \{ $P(\lambda)$ | $\lambda \in \Lambda^+$ \} of principal indecomposable $\mathcal{S}(\Lambda)$–modules where $P(\lambda)$ is the projective cover of $L(\lambda)$.

3.3. Proposition. Suppose that $R$ is a field and let $\mu$ be a multipartition of $n$. Then

$$M(\mu) \cong P(\mu) \oplus \bigoplus_{\lambda \succ \mu} c_{\lambda \mu} P(\lambda)$$

for some non–negative integers $c_{\lambda \mu}$.

Proof. By Proposition 3.7, $M(\mu)$ is a projective $\mathcal{S}(\Lambda)$–module; therefore, there exist non–negative integers $c_{\lambda \mu}$ such that $M(\mu) \cong \bigoplus_{\lambda \succ \mu} c_{\lambda \mu} P(\lambda)$. Now, by [13, Theorem 3.7] (or, more explicitly, [17, Lemma 2.19]), each $P(\lambda)$ has a Weyl filtration in which $\Delta(\lambda)$ appears with multiplicity 1. On the other hand, Lemma 3.2 $M(\mu)$ has a Weyl filtration in which the Weyl module $\Delta(\lambda)$ is a subquotient only if $T_{\lambda}(\Lambda)$ is non–empty; that is, if $\lambda \succ \mu$. Hence, $c_{\lambda \mu} \neq 0$ only if $\lambda \succ \mu$.

It remains to show that $c_{\lambda \mu} = 1$. First, observe that by Lemma 3.2 $\Delta(\mu)$ is a top composition factor of $M(\mu)$; consequently, $L(\mu)$ is also top composition factor of $M(\mu)$. On the other hand, $L(\mu)$ is a top composition factor of $P(\lambda)$ if and only if $\lambda = \mu$: hence, $P(\mu)$ is a direct summand of $M(\mu)$ and $c_{\lambda \mu} \geq 1$. Therefore, by Lemma 3.3,

$$1 = [M(\mu) : \Delta(\mu)] = \sum_{\lambda} c_{\lambda \mu} \sum_{\lambda} c_{\lambda \mu} [P(\lambda) : \Delta(\mu)] \geq c_{\lambda \mu} \geq 1.$$

We must have equality throughout; so $c_{\lambda \mu} = 1$ and the Proposition follows.

Suppose that $S \in T^\mu_{\nu}(\lambda)$ and $T \in T^\nu_{\mu}(\lambda)$. Then, since $M(\mu) = \operatorname{Hom}_\mathcal{S}(M(\Lambda), M(\mu))$, we can define an $\mathcal{S}(\Lambda)$–module homomorphism $\Phi_{ST} : M(\nu) \rightarrow M(\mu)$ by $\Phi_{ST}(f) = \varphi_{ST}f$ for all $f \in M(\nu)$. In fact, as $S$ and $T$ run over $T^\mu_{\nu}(\lambda)$ and $T^\nu_{\mu}(\lambda)$, respectively, these maps give a basis of $\operatorname{Hom}_\mathcal{S}(M(\nu), M(\mu))$.

3.4. Lemma. Suppose that $\mu \in \Lambda$. Then $\operatorname{Hom}_\mathcal{S}(\Lambda)(M(\nu), M(\mu))$ is free as an $R$–module with basis \{ $\Phi_{ST} \mid S \in T^\mu_{\nu}(\lambda)$, $\in T^\nu_{\mu}(\lambda)$ for some $\lambda \vdash n$ \}.

Proof. By definition each of the maps $\Phi_{ST}$ belongs to $\operatorname{Hom}_\mathcal{S}(\Lambda)(M(\nu), M(\mu))$ and they are certainly linearly independent. It remains to check that these homomorphisms span $\operatorname{Hom}_\mathcal{S}(\Lambda)(M(\nu), M(\mu))$. Now, if $f \in \operatorname{Hom}_\mathcal{S}(\Lambda)(M(\nu), M(\mu))$ then there exist $a_{ST} \in R$ such that $f(\varphi_{\nu}) = \sum_{ST} a_{ST} \varphi_{ST}$ by Proposition 3.1. Hence, $f = \sum a_{ST} \Phi_{ST}$ and the Lemma is proved.

For each multipartition $\lambda$ let $Y(\lambda) = F_\omega(P(\lambda))$. If $\mu \in \Lambda$ is a multipartition then $M(\mu) \cong F_\omega(M(\mu))$ by Proposition 3.1; therefore, by Proposition 3.3,

$$M(\mu) \cong Y(\mu) \oplus \bigoplus_{\lambda \succ \mu} c_{\lambda \mu} Y(\lambda).$$

As remarked above, $P(\lambda)$ has a Weyl filtration. Therefore, $Y(\lambda)$ has a Specht filtration; in particular, $Y(\lambda) \neq 0$. Following James [14], we call $Y(\lambda)$ a Young module of $\mathcal{S}$.

3.6. Theorem. Suppose that $R$ is a field and let $\mu$ be a multipartition of $n$. Then the following hold.
(i) Each $Y(\mu)$ is an indecomposable $\mathcal{H}$–module;
(ii) If $\lambda$ is another multipartition of $n$ then $Y(\lambda) \cong Y(\mu)$ if and only if $\lambda = \mu$; and,
(iii) The Young module $Y(\mu)$ has a Specht filtration

\[ Y(\mu) = Y_1 \supseteq \cdots \supseteq Y_k \supseteq Y_{k+1} = 0 \]

with $Y_i/Y_{i+1} \cong S^\lambda$, for some multipartitions $\lambda_1, \ldots, \lambda_k$.
(iv) The number of $\lambda_i$ equal to $\lambda$ is the decomposition multiplicity $[\Delta(\lambda) : L(\mu)]$.

**Proof.** First note that (2.6) and Lemma 3.4 show that

\[ \text{Hom}_{S(\Lambda)}(M(\mu), M(\lambda)) \cong \text{Hom}_{\mathcal{H}}(M(\lambda), M(\mu)) \]

as $R$–modules; explicitly, the isomorphism is given by $\Phi_{ST} \mapsto F_\omega(\Phi_{ST}) = \varphi_{ST}$. Therefore, $F_\omega$ induces an injective map $\text{Hom}_{S(\Lambda)}(P(\mu), P(\lambda)) \to \text{Hom}_{\mathcal{H}}(Y(\lambda), Y(\mu))$. Now $Y(\lambda)$ is a direct summand of $M(\mu)$ so any map from $Y(\lambda)$ to $Y(\mu)$ can be extended to a map in $\text{Hom}_{\mathcal{H}}(M(\lambda), M(\mu))$; hence, $\text{Hom}_{S(\Lambda)}(P(\mu), P(\lambda))$ and $\text{Hom}_{\mathcal{H}}(Y(\lambda), Y(\mu))$ are isomorphic $R$–modules.

In the special case where $\lambda = \nu$ the last paragraph says that $\text{End}_{S(\Lambda)}(P(\mu))$ and $\text{End}_{\mathcal{H}}(Y(\mu))$ are isomorphic rings. This proves (i) as $\text{End}_{S(\Lambda)}(P(\mu))$ is a local ring because $P(\mu)$ is indecomposable. Similarly, part (ii) follows because if $Y(\lambda) \cong Y(\mu)$ then $\text{Hom}_{\mathcal{H}}(Y(\lambda), Y(\mu))$ contains an isomorphism and this lifts to give an isomorphism $P(\mu) \cong P(\lambda)$, so $\lambda = \mu$.

We now prove (iii). Recall from the proof of Proposition 3.3 that $P(\mu)$ has a Weyl filtration $P(\mu) = P_1 \supseteq \cdots \supseteq P_k \supseteq P_{k+1} = 0$. Moreover, for each multipartition $\lambda$,

\[ \# \{ 1 \leq i \leq k \mid P_i/P_{i+1} \cong \Delta(\lambda) \} = [P(\mu) : \Delta(\lambda)] = [\Delta(\lambda) : L(\mu)] \]

where the last equality follows from [13 Lemma 2.19] (the cellular algebra analogue of the Brauer–Nesbitt cde–triangle). Setting $Y_i = F_\omega(P_i)$, and using (2.10), gives a filtration of $Y(\mu)$ with the required properties. Notice that $F_\omega(\Delta(\lambda)) \cong S(\lambda)$ for all $\lambda$; therefore, even though $F_\omega(L(\lambda)) = 0$ when $D(\lambda) = 0$ the multiplicities in the Specht filtration of $Y(\mu)$ are preserved.

In part (iii) we can do slightly better because the arguments of [13, 17] show that $P(\mu)$ can be filtered so that each of the quotients is isomorphic to a direct sum of $[\Delta(\lambda) : L(\mu)]$ copies of the Weyl module $\Delta(\lambda)$.

Let $(K, O, R)$ be a modular system (with parameters). That is, $O \subset K$ is a discrete valuation ring with residue field $R$ and we choose parameters $\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_r$ in $O$ so that the Ariki–Koike algebra $\mathcal{H}_K$ over $K$ with parameters $\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_r \in \text{semisimple and } \pi(\hat{q}) = q$ and $\pi(\hat{Q}_s) = Q_s$, for $1 \leq s \leq r$, where $\pi: O \to R$ is the canonical projection map. Let $\mathcal{H}_O$ be the Ariki–Koike algebra with parameters $\hat{q}, \hat{Q}_1, \ldots, \hat{Q}_r \in O$; then $\mathcal{H}_K \cong \mathcal{H}_O \otimes_K K$ and $\mathcal{H} = \mathcal{H}_R \cong \mathcal{H}_O \otimes_O R$.

Let $Y$ be an $\mathcal{H}_R$–module with an $O$–lattice; that is, an $O$–free $\mathcal{H}_O$–module $Y_O$ such that $Y \cong Y_O \otimes_O R$. Suppose that $Y_O$ has a Specht filtration

\[ Y_O = Y_{O,1} \supseteq \cdots \supseteq Y_{O,k} \supseteq 0 \]

and set $Y_i = Y_{O,i} \otimes_O R$, for all $i$. Then $Y \supseteq Y_1 \supseteq \cdots \supseteq Y_k \supseteq 0$ is a Specht filtration of $Y$. In this case for any multipartition $\lambda$ we define

\[ [Y : S(\lambda)] = \dim_K \text{Hom}_{\mathcal{H}_K}(Y_O \otimes_K K, S(\lambda)_K). \]

Then $[Y : S(\lambda)]$ is independent of the choice of lattice $Y_O$ and the choice of filtration ($Y$ is a modular reduction of $Y_K = Y_O \otimes_K K$ and $Y_K$ is independent of these choices being semisimple).
As Theorem 3.6(iii) holds for all rings we can rephrase Theorem 3.6(iv) as follows.

**3.7. Corollary.** Suppose that \( R \) is a field and let \( \lambda \) and \( \mu \) be multipartitions of \( n \). Then

\[
[Y(\mu) : S(\lambda)] = [\Delta(\lambda) : L(\mu)].
\]

Note that we cannot just define \([Y(\mu) : S(\lambda)]\) to be equal to the number of subquotients which are isomorphic to \( S(\lambda) \) in a Specht filtration of \( Y(\mu) \) because it can happen that \( S(\lambda) \cong S(\nu) \) even though \( \lambda \neq \nu \). This is why we have to introduce a modular system. If \( \mu = (\mu^{(1)}, \ldots, \mu^{(r)}) \) is a multicomposition of \( n \) let \( \vec{\mu} = (\vec{\mu}^{(1)}, \ldots, \vec{\mu}^{(r)}) \) be the unique multipartition of \( n \) such that \( \vec{\mu}^{(i)} \) is the partition obtained from \( \mu^{(i)} \) by reordering its parts. The following result is needed in [8].

**3.8. Corollary.** Suppose that \( R \) is a field and let \( \mu \) be a multicomposition of \( n \). Then

\[
M(\mu) \cong Y(\vec{\mu}) \oplus \bigoplus_{\lambda \succ \vec{\mu}} c_{\lambda\vec{\mu}} Y(\lambda)
\]

where the integers \( c_{\lambda\vec{\mu}} \) are as in Proposition 3.3.

**Proof.** If \( \mu \) is a multipartition then this is just a restatement of (3.5), so suppose that \( \mu \) is not a multipartition. Then \( \mathfrak{S}_\mu \) and \( \mathfrak{S}_{\vec{\mu}} \) are conjugate subgroups of \( \mathfrak{S}_n \); therefore we can find a permutation \( d \in \mathfrak{S}_n \) such that \( \mathfrak{S}_\mu = d^{-1}\mathfrak{S}_{\vec{\mu}}d \) and \( t^d \) and \( t^d \) are both row standard (see, for example, [17, Lemma 3.10]). For this \( d \) we have \( T_d m_\mu = m_{\vec{\mu}} T_d \) (by [2] 2.1(iv)); consequently, \( M(\mu) \cong T_d^{-1} M(\vec{\mu}) \cong M(\vec{\mu}) \) as right \( \mathcal{H} \)-modules. The general case now follows from (3.5).

4. Twisted cyclotomic Schur algebras

The Ariki–Koike algebra \( \mathcal{H} = \mathcal{H}_{r,n} \) has (at most) \( 2r \) one dimensional characters; namely, the \( R \)-linear maps \( \chi_{s,\alpha} : \mathcal{H}_{r,n} \rightarrow R \), for \( 1 \leq s \leq r \) and \( \alpha \in \{q, -q^{-1}\} \), which are determined by \( \chi_{s,\alpha}(T_i) = Q_s \) and \( \chi_{s,\alpha}(T_i) = \alpha \), for \( 1 \leq i \leq n \). The character \( \chi_{s,\alpha} \) is afforded by the Specht module \( S^{\lambda, s, \alpha} \) where

\[
\chi^{(t)}_{s,\alpha} = \begin{cases} 
(n), & \text{if } s = t \text{ and } \alpha = q, \\
(1^n), & \text{if } s = t \text{ and } \alpha = -q^{-1}, \\
(0), & \text{otherwise}.
\end{cases}
\]

Clearly, \( S^{\lambda, s, \alpha} \cong S^{\lambda, r, \alpha} \) if and only if \((Q_s, \alpha) = (Q_t, \beta)\). When \( \mathcal{H} \) is semisimple all of these representations are pairwise non–isomorphic.

Given any \( \mathcal{H} \)-module \( M \) we can use the character \( \chi_{s,\alpha} \) to twist the \( \mathcal{H} \)-action to give a new \( \mathcal{H} \)-module \( S_{s,\alpha} \) on which \( h \in \mathcal{H} \) acts as \( \chi_{s,\alpha}(h) \). By considering characters, in the semisimple case the effect of this operation on the Specht modules amounts to a cyclic permutation of the components of the corresponding multipartitions, and taking conjugates when \( \alpha = -q^{-1} \). In contrast, when \( \mathcal{H} \) is not semisimple the twisted Specht module \( S^{\lambda, s, \alpha}_s \) is not necessarily isomorphic to another Specht module.

This section investigates what happens when we twist modules by the ‘sign representation’ \( \chi_{r, -q^{-1}} \) of \( \mathcal{H} \). These twisted modules play a key role in understanding the tilting modules of the cyclotomic Schur algebras.

Let \( \mathbb{Z} = \mathbb{Z}[q, q^{-1}, Q_1, \ldots, Q_r] \), where \( q, \hat{Q}_1, \ldots, \hat{Q}_r \) are indeterminates over \( \mathbb{Z} \), and let \( \mathcal{H}_{\mathbb{Z}} \) be the Ariki–Koike algebra over \( \mathbb{Z} \) with parameters \( q, \hat{Q}_1, \ldots, \hat{Q}_r \). The relations of \( \mathcal{H} \) imply that \( \mathcal{H} \) has a \( \mathbb{Z} \)-algebra involution ‘ which is determined by

\[
T_i = T_i, \quad \hat{q} = -q^{-1}, \quad \text{and} \quad \hat{Q}_s = \hat{Q}_{r-s+1}.
\]
for $0 \leq i < n$ and $1 \leq s \leq r$. Then $L'_k = L_k$ and $T'_w = T_w$, for all $1 \leq k \leq n$ and $w \in \mathfrak{S}_n$. We emphasize that the involution $'$ is only defined generically (i.e. over $\mathbb{Z}$) and that $\mathcal{H}$ does not have a corresponding involution when $R$ is not a free $\mathbb{Z}$–module under specialization. Nevertheless, specialization arguments will allow us to transport the effects of $'$ into $\mathcal{H}_R$.

Suppose that $\lambda$ is a multicomposition and define

$$y_\lambda = \sum_{w \in \mathcal{G}_\lambda} (-\hat{q})^{-\ell(w)}T_w$$

and

$$u_\lambda = \prod_{s=1}^{r-1} \prod_{k=1}^{b_s} (L_k - \hat{Q}_s),$$

where $b_s = |\lambda^{(s+1)}| + \cdots + |\lambda^{(r)}|$ for $2 \leq k \leq r$. Then $y_\lambda = (x_\lambda)'$ and $u_\lambda = (u_\lambda')'$. In particular, it follows that $y_\lambda u_\lambda^{-1} = u_\lambda^{-1} y_\lambda$. Set $n_\lambda = y_\lambda u_\lambda^{-1}$ and, if $s$ and $t$ are standard $\lambda$–tableaux, define $n_{st} = T_{d(s)} d_t T_{d(t)}$; then $n_{st} = (n_{st})'$. Therefore, because $'$ is a $\mathbb{Z}$–algebra involution, $\{n_{st}\}$ is a cellular basis of $\mathcal{H}_Z$ by (2.2).

Returning to the general case, any ring $R$ with a choice of parameters $q, Q_1, \ldots, Q_r$ is naturally a $\mathbb{Z}$–module under specialization: that is, $\hat{q}$ acts on $R$ as multiplication by $q$, and $\hat{Q}_s$ acts as multiplication by $Q_s$, for $1 \leq s \leq r$. Moreover, because $\mathcal{H}$ is $R$–free this induces an isomorphism of $R$–algebras $\mathcal{H}_R \cong \mathcal{H}_Z \otimes_R \mathbb{Z}$ via $T_i \mapsto T_i \otimes 1_R$, for $0 \leq i < n$. We say that $\mathcal{H}_R$ is a specialization of $\mathcal{H}_Z$.

Hereafter, we drop the distinction between $q$ and $\hat{q}$, and $\hat{Q}_s$ and $Q_s$, and we identify the algebras $\mathcal{H} = \mathcal{H}_R$ and $\mathcal{H}_Z \otimes_R \mathbb{Z}$ via the isomorphism $T_i \mapsto T_i \otimes 1_R$ above. Thus, we have elements $y_\lambda, u_\lambda$ and $n_{st}$ in $\mathcal{H}$ and by (2.3), and the specialization argument above, we have the following.

4.1 (Du–Rui [11, 2.7]) The Ariki–Koike algebra $\mathcal{H}$ is free as an $R$–module with cellular basis $\{n_{st} \mid s, t \in T'(\lambda) \text{ for some } \lambda \vdash n \}$.

Since $\{n_{st}\}$ is a cellular basis it gives us a second collection of cell modules for $\mathcal{H}$; namely, for each multipartition $\lambda$ define the dual Specht module $S'(\lambda)$ to be the right $\mathcal{H}$–module $(n_\lambda + \mathcal{H}'(\lambda)) \mathcal{H}$, where $\mathcal{H}'(\lambda) = (\mathcal{H}(\lambda))'$ is the two–sided ideal of $\mathcal{H}$ with basis $n_{ab}$ with $\text{Shape}(u) = \text{Shape}(v) \triangleright \lambda$. Then $S'(\lambda)$ is $R$–free with basis $\{n_t \mid t \in T'(\lambda)\}$, where $t_t = n_{t_1} + \mathcal{H}'(\lambda)$. This terminology is justified in Corollary 5.7 below which shows that $S'(\lambda)$ is isomorphic to the contragredient dual of $S(\lambda')$, where $\lambda'$ is the multipartition conjugate to $\lambda$. We remark when $\mathcal{H}$ is semisimple a straightforward calculation using characters shows that $S'(\lambda) \cong S(\lambda')_{r,-q}^{-1}$.

Let $D'(\lambda) = S'(\lambda)/\text{rad} S'(\lambda)$, where $\text{rad} S'(\lambda)$ is the radical of the bilinear form on $S'(\lambda)$; the form is defined in terms of the structure constants of the cellular basis $\{n_{st}\}$. Once again, the theory of cellular algebras says that the non–zero $D'(\lambda)$ are a complete set of pairwise non–isomorphic irreducible $\mathcal{H}$–modules.

For any multicomposition $\mu$ let $N(\mu) = n_{\mu,\mathcal{H}}$. If $S$ is a semistandard $\lambda$–tableau of type $\mu$ and $t$ is a standard $\lambda$–tableau define

$$n_{St} = \sum_{s \in T'(\lambda), \mu(s) = S} (-q)^{-\ell(s)} n_{st}.$$

From the definitions, $n_{St} = n_{\mu,t}$ in $\mathcal{H}_Z$; therefore, (2.5) and the usual specialization argument show that the following holds.

4.2. Corollary. Suppose that $\mu$ is a multicomposition of $n$. Then $N(\mu)$ is free as an $R$–module with basis $\{n_{St} \mid S \in T_n'(\lambda), t \in T'(\lambda) \text{ for some } \lambda \vdash n \}$. 
Just as in [7, Cor. 4.15], this implies that \( N(\mu) \) has a dual Specht filtration in which the number of subquotients equal to \( S'(\lambda) \) is \( \#T^n(\lambda) \). This filtration can also be obtained by specializing the corresponding Specht filtration of the \( \mathcal{H} \)-module \( M(\mu) \).

Mirroring Definition 2.3, if \( \Lambda \) is a saturated set of multicompositions \( \Lambda \) define the twisted cyclotomic \( q \)-Schur algebra to be the endomorphism algebra

\[
\mathcal{S}'(\Lambda) = \text{End}_{\mathcal{H}}(N(\Lambda)), \quad \text{where} \quad N(\Lambda) = \bigoplus_{\mu \in \Lambda} N(\mu).
\]

If \( S \in T'_{\mu}(\lambda) \) and \( T \in T'_{\nu}(\lambda) \) are semistandard tableaux let

\[
n_{ST} = \sum_{s,t \in T_{\mu}(\lambda)} (-q)^{-\ell(s) - \ell(t)} n_{st},
\]

Now define the homomorphism \( \varphi_{ST} \in \mathcal{S}'(\Lambda) \) by \( \varphi_{ST}(n_{\alpha}h) = \delta_{\alpha\nu} n_{ST}h \), for all \( h \in \mathcal{H} \) and all \( \alpha \in \Lambda \). Then \( \varphi_{ST} \) belongs to \( \mathcal{S}'(\Lambda) \).

Write \( \mathcal{S}'(\Lambda) \) for the twisted cyclotomic Schur algebra over \( \mathcal{S} \). Similarly, we write \( M(\mu) \), \( N(\lambda) \), \ldots whenever we have a free \( R \)-module whose rank is independent of \( R \), \( q \) and \( Q_1, \ldots, Q_r \).

4.3. Proposition. Suppose that \( \Lambda \) is a saturated set of multicompositions.

(i) The twisted cyclotomic \( q \)-Schur algebra \( \mathcal{S}'(\Lambda) \) is free as an \( R \)-module with cellular basis

\[
\{ \varphi_{ST} \mid S \in T'_{\mu}(\lambda), T \in T'_{\nu}(\lambda) \text{ for some } \mu, \nu \in \Lambda \text{ and some } \lambda \in \Lambda^+ \}.
\]

Consequently, \( \mathcal{S}'(\Lambda) \cong \mathcal{S}'(\Lambda) \otimes_{\mathcal{S}} R \).

(ii) The twisted cyclotomic Schur algebra \( \mathcal{S}'(\Lambda) \) is quasi-hereditary.

(iii) The \( R \)-algebras \( \mathcal{S}'(\Lambda) \) and \( \mathcal{S}(\Lambda) \) are canonically isomorphic.

Proof. Using Corollary 4.3, an easy modification of the argument of [7, Prop. 6.3] shows that \( \{ n_{ST} \mid S \in T'_{\mu}(\lambda), T \in T'_{\nu}(\lambda) \text{ for some } \lambda \vdash n \} \) is a basis of \( N(\nu) \cap N(\mu) \). Part (i) now follows exactly as in the proof of [2, Sec. 4]; see [4, Theorem 6.6]. In particular, notice that because \( \mathcal{S}'(\Lambda) \cong \mathcal{S}'(\Lambda) \otimes_{\mathcal{S}} R \) we can now use specialization arguments.

Part (ii) follows from (i) using the argument of [2, Cor. 2.18]; alternatively, it may be deduced by the specialization of a hereditary chain of the algebra \( \mathcal{S}(\Lambda) \).

Finally, (iii) follows because when \( R = \mathcal{S} \),

\[
\text{Hom}_{\mathcal{S}}(M(\nu), M(\mu)) = \text{Hom}_{\mathcal{S}}(M(\nu), M(\mu)) = \text{Hom}_{\mathcal{S}}(N(\nu), N(\mu));
\]

explicitly, the isomorphism is given by \( \varphi_{ST} \mapsto \varphi'_{ST} \), for semistandard tableaux \( S \) and \( T \). As \( \mathcal{S}(\Lambda) \cong \mathcal{S}(\Lambda) \otimes R \) and \( \mathcal{S}'(\Lambda) \cong \mathcal{S}'(\Lambda) \otimes R \) this implies the general case. \( \square \)

Let \( \Delta'(\lambda) \) and \( L'(\lambda) \), respectively, be the Weyl modules and simple modules of \( \mathcal{S}'(\Lambda) \); these are defined in exactly the same way as the corresponding modules for \( \mathcal{S}(\Lambda) \). As in (2.10), if \( \omega \in \Lambda \) then there is a functor

\[
\mathcal{S}'(\Lambda) \rightarrow \mathcal{H} \rightarrow \mathcal{S}; N \mapsto N \varphi'_{\omega},
\]

where \( \varphi'_{\omega} \) is the identity map on \( \mathcal{H} \). Because \( \varphi'_{\omega} = \varphi_{\omega} \) we abuse notation and again denote this functor by \( F_{\omega} \). As in (2.10), we have \( F_{\omega}(\Delta'(\lambda)) \cong S'(\lambda), F_{\omega}(L'(\lambda)) \cong D'(\lambda) \) and

\[
[\Delta'(\lambda) : L'(\mu)] = [S'(\lambda) : D'(\mu)] \text{ whenever } D'(\mu) \neq 0.
\]

In latter sections we will be particularly interested in the analogues of the Young modules in this setup. For each \( \lambda \in \Lambda^+ \) let \( P(\lambda) \) be the projective cover of \( L'(\lambda) \). Suppose that \( \mu \in \Lambda^+ \) and let \( N(\mu) = \varphi_{\mu}' \mathcal{S}'(\Lambda) \), where \( \varphi_{\mu}' \) is the identity map on \( N(\mu) \). Then \( \varphi_{\mu}' \) is an
idempotent so \( N(\mu) \) is a projective \( \mathcal{S}(\Lambda) \)-module. Therefore, there exist non-negative integers \( c_{\lambda \mu} \geq 0 \) such that

\[
N(\mu) \cong P'(\mu) \oplus \bigoplus_{\lambda > \mu} c_{\lambda \mu} P'(\lambda).
\]

In fact, because \( N(\mu)_z = M(\mu)'_z \) it follows by a specialization argument that the integers \( c_{\lambda \mu} \) are the same as those appearing in Proposition \ref{prop:summer}. We call \( Y'(\mu) = F_w(P'(\lambda)) \) a twisted Young module.

Define the filtration multiplicities \( [Y'(\lambda) : S'(\mu)] \) exactly as in section 3.

4.4. Proposition. Suppose that \( R \) is a field and let \( \lambda \) and \( \mu \) be multipartitions of \( n \). Then

(i) \( N(\mu) \cong Y'(\mu) \oplus \bigoplus_{\lambda > \mu} Y'(\nu)^{\lambda \mu} \) where the integers \( c_{\lambda \mu} \) are the same as those appearing in Proposition \ref{prop:summer}.

(ii) \( Y'(\mu) \) is indecomposable;

(iii) \( Y'(\lambda) \cong Y'(\mu) \) if and only if \( \lambda = \mu \);

(iv) the Young module \( Y'(\mu) \) has a dual Specht filtration in which the number of subquotients equal to \( S'(\lambda) \) is \( |\Delta'(\lambda) : L'(\mu)| \); and,

(v) \( Y'(\mu) : S'(\lambda) = |\Delta'(\lambda) : L'(\mu)| \).

Proof. This can be proved in exactly the same way as in Theorem \ref{thm:summer}, alternatively, one can use a specialization argument.

We remark that the set of Young modules \( \{ Y(\lambda) \mid \lambda \vdash n \} \) and the set of twisted Young modules \( \{ Y'(\lambda) \mid \lambda \vdash n \} \) do not usually coincide; however, we always have that

\[
\{ Y(\lambda) \mid D(\lambda) \neq 0 \} = \{ Y'(\lambda) \mid D'(\lambda) \neq 0 \},
\]

because these modules are the indecomposable direct summands of \( \mathcal{H} \). T see this use Corollary \ref{cor:summer} to show that if \( \lambda \) is a multipartition and \( D(\lambda) \neq 0 \) then \( Y(\lambda) \) is the projective cover of \( D(\lambda) \) and, similarly, that \( Y'(\mu) \) is the projective cover of \( D'(\mu) \). It follows that \( D(\lambda) \cong D'(\mu) \) if and only if \( Y(\lambda) \cong Y'(\mu) \). By the results of \ref{thm:summer} and \ref{thm:summer}, the correspondence between these two different labellings of the simple \( \mathcal{H} \)-modules is given by a generalization of Kleshchev’s version of the Mullineux map (that is, in terms of paths in the associated crystal graphs).

5. Contragredient duality

We now investigate contragredient duality for the category of \( \mathcal{H} \)-modules; this will give us the connection between the Specht modules with the dual Specht modules constructed in the previous section. The aim of the section is really to construct a dual Specht filtration of \( M(\lambda) \); in essence, this is the main tool that we need in order to understand the tilting modules of the cyclotomic Schur algebras.

Recall that \( * \) is the unique anti-isomorphism of \( \mathcal{H} \) such that \( T_i^* = T_i \) for \( 0 \leq i < n \). Given a right \( \mathcal{H} \)-module \( M \) define its contragredient dual \( M^\circ \) to be the dual module \( \text{Hom}_R(M, R) \) equipped with the right \( \mathcal{H} \)-action \( (\varphi h)(m) = \varphi(mh^*) \) for all \( \varphi \in M^\circ \), \( h \in \mathcal{H} \) and \( m \in M \). A module \( M \) is self-dual if \( M \cong M^\circ \). By standard arguments, \( M \) is self-dual if and only if \( M \) possesses a non-degenerate associative bilinear form (the form \( \langle \cdot , \cdot \rangle \) is associative if \( \langle xh, y \rangle = \langle x, yh^* \rangle \) for all \( x, y \in M \) and \( h \in \mathcal{H} \)).

If \( M \) is a submodule of \( \mathcal{H} \) the reader should be careful not to confuse the dual module \( M^\circ \) with \( M^* = \{ m^* \mid m \in M \} \).

Constructing dual bases inside \( \mathcal{H} \) is, in general, quite hard. We are going to do it by comparing the two bases \( \{ m_{\alpha} \} \) and \( \{ n_{\alpha} \} \) of \( \mathcal{H} \). First we need to introduce some notation for conjugate multipartitions and tableaux.
Recall that the conjugate of a composition \( \sigma \) is the partition \( \sigma' = (\sigma'_1, \sigma'_2, \ldots) \) where \( \sigma'_i \) is the number of nodes in column \( i \) of the diagram of \( \sigma \). If \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) is a multipartition then the conjugate of \( \lambda \) is the multipartition \( \lambda' = ((\lambda^{(r)})', \ldots, (\lambda^{(1)})') \). Observe that if \( \lambda \geq \mu \) then \( \mu' \geq \lambda' \) (and conversely, if \( \lambda \) and \( \mu \) are multipartitions).

If \( T = (T^{(1)}, \ldots, T^{(r)}) \) is a \( \lambda \)-tableau of type \( \mu \) then the conjugate of \( T \) is the \( \lambda' \)-tableau \( T' = (T'^{(1)}, \ldots, T'^{(r)}) \) where \( T'^{(i,j,s)} = T(j,i,r-s+1) \) for all \( (i,j,s) \in [\lambda'] \); that is, \( T'^{(s)} \) is the tableau obtained by interchanging the rows and columns of \( T^{(r-s+1)} \). Notice that \( \text{Type}(T') = \text{Type}(T) \). Further, if \( t \) is a tableau of type \( \omega \) then \( t \) is standard if and only if \( t' \) is standard.

For each multipartition \( \lambda \) let \( t_\lambda = (t^{\lambda})' \); thus, \( t_\lambda \) is the standard \( \lambda \)-tableau with the numbers \( 1, 2, \ldots, n \) entered in order first down the columns of \( T^{\lambda} \) and then the columns of \( T^{\lambda - 1} \) and so on. Observe that if \( t \) is a standard \( \lambda \)-tableau then \( t' \geq t \geq t_\lambda \). We set \( w_\lambda = d(t_\lambda) \). The following Lemma is well–known; it can be proved by induction on \( t \).

5.1. Lemma. Suppose that \( \lambda \) is a multipartition of \( n \) and that \( t \) is a standard \( \lambda \)-tableau. Then \( d(t)d(t')^{-1} = w_\lambda \) and \( \ell(w_\lambda) = \ell(d(t)) + \ell(d(t')) \).

We also extend the dominance order to pairs of tableaux in the usual way.

If \( t \) is a standard tableau and \( k \) an integer with \( 1 \leq k \leq n \) then the residue of \( k \) in \( t \) is defined to be \( \text{res}_t(k) = q^{2d(i)}Q_s \) if \( k \) appears in row \( i \) and column \( j \) of component \( s \) of \( t \). Residues are important because of the following result.

5.2 (James–Mathas \cite{JamesMathas1996}) Suppose that \( s \) and \( t \) are standard \( \lambda \)-tableau and that \( 1 \leq k \leq n \). Then there exist \( r_{uv} \in \mathbb{R} \) such that

\[
m_{st}L_k = \text{res}_t(k)m_{st} + \sum_{(u,v)\geq(s,t)} r_{uv}m_{uv}.
\]

Let \( K = \mathbb{Q}(q, Q_1, \ldots, Q_r) \) and, following \cite{JamesMathas1996}, define \( F_t \in \mathcal{H}_K \) by

\[
F_t = \prod_{k=1}^{n} \prod_{c \in \mathcal{R}(k)} \frac{L_k - c}{\text{res}_t(k) - c}.
\]

where \( \mathcal{R}(k) = \{ q^{2d}Q_s \mid 1 \leq s \leq r \text{ and } |d| < k \text{ and } d \neq 0 \} \). Finally, given two standard \( \lambda \)-tableaux \( s \) and \( t \) set \( f_{st} = F_s^m_{st}F_t \) and \( g_{st} = F_s^m_{st}F_t \).

From the definitions, \( (\text{res}_s(k))' = \text{res}_t(k) \) in \( \mathbb{Z} \), for all tableau \( t \) and all \( k \). This implies that \( F_t' = F_t \) and hence that \( g_{st} = f_{st}^\prime \) in \( \mathcal{H}_K \); see \cite{JamesMathas1996}.

Using (5.2) we obtain the following.

5.3 (Mathas \cite{Mathas1996}) Suppose that \( \mathcal{H} = \mathcal{H}_K \).

(i) \( m_{st} = f_{st} + \sum_{a,b} r_{ab}f_{ab} \) for some \( f_{ab} \in \mathbb{R} \) with \( r_{ab} \neq 0 \) only if \( (a,b) \triangleright (s,t) \).

(ii) \( s_{st} = g_{st} + \sum_{a,b} r_{ab}g_{ab} \) for some \( f_{ab} \in \mathbb{R} \) with \( r_{ab} \neq 0 \) only if \( (a,b) \triangleright (s,t) \).

(iii) Suppose that \( s, t, u \) and \( v \) are standard tableaux. Then \( f_{st}u_{sv} = 0 \) unless \( t = u' \).

By (i) and (ii), the sets \( \{ f_{st} \} \) and \( \{ g_{st} \} \) are both bases of \( \mathcal{H}_K \). In fact, by \cite{JamesMathas1996}, both bases are self–orthogonal with respect to the bilinear form \( \langle \cdot, \cdot \rangle \) which we introduce below.

5.4. Lemma. Suppose that \( s \) and \( t \) are standard \( \lambda \)-tableaux and that \( v \) and \( u \) are standard \( \mu \)-tableaux such that \( m_{st}m_{vu} \neq 0 \). Then \( v' \geq t \).

Proof. Now, \( m_{st}m_{vu} \neq 0 \) in \( \mathcal{H} \) only if \( m_{st}m_{vu} \neq 0 \) in \( \mathcal{H}_Z \) since \( \mathcal{H} \cong \mathcal{H}_Z \otimes Z \mathbb{R} \) and specialization maps the standard basis of \( \mathcal{H}_Z \) to the standard basis of \( \mathcal{H} \) and, similarly, for the \( m_{vu} \) basis elements. Hence, by embedding \( \mathcal{H}_Z \) into \( \mathcal{H}_K \) in the natural way, we may
assume that \( \mathcal{H} = \mathcal{H}_K \). By parts (i) and (ii) of \( 5.3 \) there exists scalars \( r_{ab}, r_{cd} \in R \) such that

\[
0 \neq m_{st}m_{uv} = \left( f_{s1} + \sum_{(a,b) \triangleright (s,t)} r_{ab} f_{ab} \right) \left( g_{uv} + \sum_{(c,d) \triangleright (v,u)} r_{cd} f_{cd} \right).
\]

Therefore, there exist \((a, b) \geq (s, t)\) and \((c, d) \geq (v, u)\) such that \( f_{ab}g_{cd} \neq 0\); so, \( c' = b \) by \( 5.3 \)(iii). Consequently, \( v' \geq b \geq t \) as required.

It is possible to give a direct proof of Lemma 5.4 without using the two orthogonal bases \( \{ f_{st} \} \) and \( \{ g_{st} \} \) of \( \mathcal{H} \) (cf. [19, Lemma 4.11]); however, the proof above is both easier and nicer because it avoids long calculations with the relations in \( \mathcal{H} \). (The proof of \( 5.3 \) is straightforward and also avoids such calculations.)

Recall from \( 2.1 \) that \( \{ L^{a_1}_1 \ldots L^{a_n}_n T_w \mid 0 \leq a_i < r \text{ and } w \in \mathfrak{S}_n \} \) is a basis of \( \mathcal{H} \). Define \( \tau: \mathcal{H} \rightarrow R \) to be the \( R \)-linear map determined by

\[
\tau(L^{a_1}_1 \ldots L^{a_n}_n T_w) = \begin{cases} 1, & \text{if } a_1 = \cdots = a_n = 0 \text{ and } w = 1, \\ 0, & \text{otherwise.} \end{cases}
\]

This map was introduced by Bremke and Malle [4] who showed that \( \tau \) is a trace form; that is, \( \tau(ab) = \tau(ba) \) for all \( a, b \in \mathcal{H} \). (The definition above is slightly different from Bremke and Malle’s; it is shown in [14] that the two definitions coincide.) Combining the definition with the fact that \( \tau \) is a trace form shows that \( \tau(h^*) = \tau(h) \), for all \( h \in \mathcal{H} \).

Define a bilinear form \( \langle , \rangle: \mathcal{H} \times \mathcal{H} \rightarrow R \) on \( \mathcal{H} \) by \( \langle h_1, h_2 \rangle = \tau(h_1 h_2^*) \) for all \( h_1, h_2 \in \mathcal{H} \). Then \( \langle , \rangle \) is an associative bilinear form on \( \mathcal{H} \); further, \( \langle , \rangle \) is symmetric because \( \tau \) is a trace form.

For each multipartition \( \lambda \) set

\[
Q_{\lambda} = (-1)^{n(r-1)} \prod_{s=1}^{r} Q_s^{n-|\lambda^{(s)}|}.
\]

Then \( Q_{\lambda} \in R \) and \( Q_{\lambda} \) is a unit if and only if \( Q_s \) is a unit whenever \( |\lambda^{(s)}| < n \).

Many of the results which follow rely upon the following result.

**5.5. Theorem.** Suppose that \( (s, t) \) is a pair of \( \lambda \)-tableau and that \( (u, v) \) are \( \mu \)-tableaux. Then

\[
\langle m_{st}, n_{uv} \rangle = \begin{cases} Q_{\lambda}, & \text{if } (u', v') = (s, t), \\ 0, & \text{if } (u', v') \not\geq (s, t). \end{cases}
\]

**Proof.** Suppose first that \( \langle m_{st}, n_{uv} \rangle \neq 0 \). Now \( \langle m_{st}, n_{uv} \rangle = \tau(m_{st}n_{uv}) \), so \( m_{st}n_{uv} \neq 0 \); hence, \( v' \geq t \) by Lemma 5.4. Now \( \tau \) is a trace form and \( \tau(h) = \tau(h^*) \), for all \( h \in \mathcal{H} \); so, applying these two facts, we have \( \tau(m_{st}n_{uv}) = \tau(n_{uv}m_{st}) = \tau(m_{ts}n_{ut}) \); hence, \( m_{ts}n_{ut} \neq 0 \) and \( u' \geq s \) by Lemma 5.4. Therefore, if \( \langle m_{st}, n_{uv} \rangle \neq 0 \) then \( (u', v') \geq (s, t) \).

Now assume that \( (u', v') = (s, t) \). Then \( T_{u'x}' = T_{d(v)} T_{d(v)}^* = T_{d(s')} T_{d(s)}^* \) by Lemma 5.1.

Therefore, once again using the fact that \( \tau \) is a trace form,

\[
\langle m_{st}, n_{s't'} \rangle = \tau(m_{st}n_{s't'}) = \tau(T_{d(s')} m_{st} T_{d(v)} T_{d(v)}^* n_{s't'}) = \tau(T_{d(s')} T_{d(s)}^* m_{st} T_{w'x}' n_{s't'}). \]

Finally, \( \tau(T_{w'x}' m_{s't} T_{w'x} n_{s't'}) = Q_{\lambda} \) by [18, Prop. 5.12], so we’re done.

As a first consequence we obtain a new proof that \( \mathcal{H} \) is a symmetric algebra.
5.6. Corollary (Malle–Mathas [16]). Suppose that $q, Q_1, \ldots, Q_r$ are invertible elements of $R$. Then $(\, , \,)$ is a non–degenerate associative symmetric bilinear form on $\mathcal{H}$. Therefore, $\mathcal{H}$ is a symmetric algebra; in particular, it is self–dual

At first sight, this proof of Corollary 5.6 is considerably easier than the original proof in [16]; however, all of the work is hidden in the calculation of $\tau(T_{w \lambda}^n, m_\lambda T_{w \lambda} \cdot n_\lambda)$ from [18] and this is quite involved. The payoff for this extra effort is Theorem 5.5 which shows that the two bases $\{m_\lambda\}$ and $\{n_\lambda\}$ are almost orthogonal; this fact will be used many times in what follows.

Let $\lambda$ be a multipartition. We next show that $S'(\lambda) \cong S(\lambda')^\circ$, and so justify the term dual Specht module. Recall that $N(\lambda) = n_\lambda \mathcal{H}$ and that $S'(\lambda)$ is a quotient of $N(\lambda)$.

5.7. Corollary. Suppose that $q, Q_1, \ldots, Q_r$ are invertible elements of $R$ and let $\lambda$ be a multipartition of $\nu$. Then $S'(\lambda) \cong S(\lambda')^\circ$.

Proof. Now, $S(\lambda')$ is a submodule of $\mathcal{H} / \mathcal{H}(\lambda')$ and $S'(\lambda)$ is a submodule of $\mathcal{H} / \mathcal{H}(\lambda')$. By Theorem 5.5(ii) the modules $\mathcal{H}(\lambda')$ and $\mathcal{H}'(\lambda)$ are orthogonal with respect to the form $(\, , \,)$, as are $M(\lambda)$ and $\mathcal{H}'(\lambda)$, and $N(\lambda)$ and $\mathcal{H}(\lambda')$. Therefore, $(\, , \,)$ induces an associative bilinear form $(\, , \,)_{S(\lambda)} : S(\lambda') \times S'(\lambda) \to R$ given by

$$(a + \mathcal{H}(\lambda), b + \mathcal{H}'(\lambda))_{S(\lambda)} = (a, b) = \tau(ab^*)$$

In particular, if $s \in T'(\lambda')$ and $t \in T(\lambda)$ then

$$(m_s, n_t)_{S(\lambda)} = \begin{cases} Q_\lambda, & \text{if } t' = s, \\ 0, & \text{unless } t' \geq s, \end{cases}$$

by Theorem 5.5. Hence, $(\, , \,)_{S(\lambda)}$ is non–degenerate and $S'(\lambda) \cong S(\lambda')^\circ$ as required.

Recall from (2.3) that $M(\lambda)$ is free as an $R$–module with basis

$\{ m_{st} \mid s \in T^0_s(\lambda) \text{ and } t \in T(\lambda) \text{ for } \lambda \vdash \nu \}$.

It was shown in [17] Cor. 4.15 that this basis gives rise to a Specht filtration of $M(\mu)$. Similarly, the basis of Corollary 4.2 produces a dual Specht filtration of $N(\lambda)$. We next produce another basis of $M(\mu)$ which exhibits a dual Specht filtration of $M(\mu)$ and, similarly, a basis of $N(\mu)$ which exhibits a Specht filtration of $N(\mu)$. As a byproduct we will also obtain a non–degenerate associative bilinear form on each of these modules and hence see that they are both self–dual.

A $\lambda$–tableau $T$ is column semistandard if $T'$ is semistandard. If $\mu \in \Lambda$ and $\lambda \in \Lambda^+$ let

$T^0_\mu(\lambda) = \{ T \mid T' \in T^0_\mu(\lambda) \}$

be the set of column semistandard $\lambda$–tableaux of type $\mu$. Observe that if $T^0_\mu(\lambda) \neq \emptyset$ then $T^0_\mu(\lambda') \neq \emptyset$, so $\lambda' \succeq \mu$; equivalently, $\mu' \succeq \lambda$. We also set $T^0_\mu(\lambda^+) = \bigcup_{\lambda \in \Lambda^+} T^0_\mu(\lambda)$.

As a final piece of notation, if $v$ is any tableau and $1 \leq k \leq n$ then write $\text{comp}_v(k) = s$ if $k$ appears in component $s$ of $v$.

5.8. Lemma. Suppose that $\mu$ is a multicomposition and that $m_\mu n_{uv} \neq 0$ or $m_{\mu'} n_{uv} \neq 0$ for some standard tableaux $u$ and $v$. Then $\mu(u)$ is column semistandard.

Proof. As in the proof of Lemma 5.4 we may assume that $\mathcal{H} = \mathcal{H}_2$. We consider only the case where $m_\mu n_{uv} \neq 0$; the other case can be proved by applying the involution $\prime$.

Before we begin the proof proper we remark that it is well–known, and easy enough to check, that if $s_i \in \mathcal{S}_\mu$ then $m_\mu T_i = q m_\mu$; similarly, if $s_j \in \mathcal{S}_\lambda$ then $T_j n_\lambda = -q^{-1} n_\lambda$. 


First, because \( u' \) is standard the entries of \( \mu(u') \) are weakly increasing along rows. Suppose that the entries of \( \mu(u') \) are not strictly increasing down columns. Then we can find integers \( i < j \) such that \( i \) and \( j \) are in the same row of \( t' \) and the same column of \( u' \). The entries in \( t' \) are consecutive so this means that there exists an integer \( i \) such that \( i \) and \( i + 1 \) are in the same row of \( t' \) and the same row of \( u \). Therefore,

\[
qm\mu n_{u'} = (m\mu T_i)n_{u'} = m\mu(T_in_{u'}) = -q^{-1}m\mu n_{u'}.
\]

Consequently, \( m\mu n_{u'} = 0 \) since \( Z^\perp \) is \( Z \)-free.

It remains to show that \( \mu(u') \) satisfies condition (iii) of Definition 2.4. If \( \mu \) is a multipartition then \( m\mu = m\mu u' \); so \( m\mu u' n_{u'} \neq 0 \) and \( u' \triangleright t' \) by Lemma 5.4. Looking at the definitions, we see that \( \mu(u') \) satisfies condition Definition 2.4(iii) because \( u' \triangleright t' \).

Hence, \( u \) is column semistandard as claimed.

If \( \mu \) is a multicomposition (and not a multipartition) let \( \vec{\mu} = (\vec{\mu}^{(1)}, \ldots, \vec{\mu}^{(r)}) \) be the multpartition obtained by ordering the parts in each component \( \mu^{(s)} \) of \( \mu \). Then we can find a permutation \( d \) of minimal length in \( \mathcal{S}_|\mu| = \mathcal{S}_{|\mu^{(1)}|} \times \cdots \times \mathcal{S}_{|\mu^{(r)}|} \) such that \( d\mathcal{S}_\mu = \mathcal{S}_{\vec{\mu}d} \). Then \( T_d m\mu = m\vec{\mu} d \). Now, \( m\mu n_{u'} \) is non-zero so \( T_d m\mu n_{u'} = m\vec{\mu} d \) is also non-zero. Therefore, there exist tableaux \( a \) and \( b \) such that \( n_{ab} \) is a non-zero summand of \( T_d m\mu n_{u'} \), \( m\vec{\mu} n_{u'} \neq 0 \). By the last paragraph \( \vec{\mu}(a) \) satisfies Definition 2.4(iii). This implies that \( \mu(u) \) also satisfies Definition 2.4(iii) because \( \text{comp}_u(k) = \text{comp}_{\vec{\mu}}(k) \), for \( 1 \leq k \leq n \), by [7, Prop. 3.18] since \( d \in \mathcal{S}_{|\mu|} \). Hence, \( \mu(u) \) is column semistandard.

If \( S \) is a \( \lambda \)-tableau of type \( \mu \) let \( \hat{S} \) be the unique standard tableau such that \( \mu(\hat{S}) = S \) and \( \hat{S} \triangleright S \) whenever \( S \) is a standard \( \lambda \)-tableau with \( \mu(S) = S \). The tableau \( \hat{S} \) is denoted \( \text{first}(S) \) in [15]. The permutation \( d(\hat{S}) \) is a distinguished \( (\mathcal{S}_\lambda, \mathcal{S}_\mu) \)-double coset representative; that is, it is the unique element of minimal length in \( \mathcal{S}_\lambda d(\hat{S}) \mathcal{S}_\mu \). We emphasize that \( \hat{S} \) is a standard tableau.

5.9. Proposition. Suppose that \( \mu \) is a multicomposition of \( n \). Then \( M(\mu) \) is free as an \( R \)-module with basis \( \{ m\mu n_{s\lambda} \mid S \in T_\mu^\perp(\lambda) \text{ and } t \in T^\perp(\lambda) \text{ for some } \lambda \vdash n \} \) and \( N(\mu) \) is free as an \( R \)-module with basis \( \{ n\mu m_{s\lambda} \mid S \in T_\mu^\perp(\lambda) \text{ and } t \in T^\perp(\lambda) \text{ for some } \lambda \vdash n \} \).

Proof. We only prove for the claim for \( M(\mu) \); the second statement can be proved by a similar argument, or by specialization.

By [5.3](iii) \( \{ n_{s\lambda} \} \) is a basis of \( \mathcal{H} \), so \( M(\mu) \) is spanned by the elements \( m\mu n_{s\lambda} \), where \( s \) and \( t \) range over all pairs of standard tableaux of the same shape. Furthermore, if \( m\mu n_{s\lambda} \neq 0 \) then \( \mu(s) \) is column semistandard by Lemma 5.8. Hence, \( M(\mu) \) is spanned by the elements \( m\mu n_{s\lambda} \) with \( \mu(s) \) column semistandard. Now, if \( d(s) \) and \( d(u) \) are in the same \( (\mathcal{S}_\lambda, \mathcal{S}_\mu) \)-double coset then \( m\mu T_d(s)n_{\lambda} = \pm q^a m\mu T_d(u)n_{\lambda} \) for some integer \( a \); see the remarks at the start of the proof of Lemma 5.3. By definition \( d(\hat{S}) \) is the unique element of minimal length in its double coset; therefore, the elements in the statement of the Lemma span \( M(\mu) \). However, now we are done because \( M(\mu) \) is \( R \)-free and the number of elements in our spanning set is exactly the rank of \( M(\mu) \) by (2.5).

Combining Lemma 5.8 and the Proposition we have.

5.10. Corollary. Suppose that \( \mu \) is a multicomposition and that \( s \) and \( t \) are standard tableaux. Then \( m\mu n_{s\lambda} \neq 0 \) if and only if \( \mu(s) \) is column semistandard. Similarly, \( n\mu m_{s\lambda} \neq 0 \) if and only if \( \mu(s) \) is column semistandard.

Using Proposition 5.9, the argument of Lemma 3.2 produces the following result.
5.11. Corollary. Suppose that \( \mu \) is a multicomposition of \( n \). Then there exist filtrations

\[
M(\mu) = M_1 \supset \cdots \supset M_k \supset M_{k+1} = 0 \quad \text{and} \quad N(\mu) = N_1 \supset \cdots \supset N_k \supset N_{k+1} = 0
\]

of \( M(\mu) \) and \( N(\mu) \), respectively, and multipartitions \( \lambda_1, \ldots, \lambda_k \), such that \( \mu' \geq \lambda_i \), \( M_i/M_{i+1} \cong S'(\lambda_i) \) and \( N_i/N_{i+1} \cong S(\lambda_i) \), for \( 1 \leq i \leq k \). Moreover, if \( \lambda \) is any multicomposition of \( n \) then \( \{ 1 \leq i \leq k \mid \lambda_i = \lambda \} = \#T^\omega(\lambda) \).

5.12. Remark. As an \( R \)-module, \( N_i \) is the submodule of \( N(\mu) \) with basis the set of elements \( n_\mu m_{S\lambda} \) with \( \text{Shape}(S) \geq \lambda_i \), for \( 1 \leq i \leq k \). In particular, \( S(\mu') \cong N_k \) is spanned by \( \{ n_\mu m_{t_{\mu',i}} \mid t \in T(\mu') \} \); note that \( \mu(t_{\mu'}) \) is the unique column standard \( \mu' \)-tableau of type \( \mu \). Therefore, \( S(\mu') \cong n_\mu T^\omega_{\mu',\mu}. \mathcal{H} \); this is a result of Du and Rui [1]. Similarly, \( S'(\mu') \cong m_\mu T^\omega_{\mu',\mu}. \mathcal{H} \).

Because \( S(\lambda') \cong S(\lambda')^\omega \), the Specht filtrations of \( M(\mu) \) given by [2, Cor. 4.15] and the last result suggest that \( M(\lambda) \) is self–dual. A similar remark applies to \( N(\lambda) \). When \( r = 1 \) it is clear that both of these modules are self–dual because they are induced representations from parabolic subalgebras.

We need a non–degenerate associative bilinear form. Let \( \langle \ , \ \rangle_\mu \) be the bilinear map on \( M(\mu) \) determined by

\[
\langle m_{S\lambda}, m_\mu n_{U\rho} \rangle_\mu = \langle m_{S\lambda}, n_{U\rho} \rangle,
\]

where \( m_{S\lambda} \) and \( m_\mu n_{U\rho} \) run over the bases of \( (2.3) \) and Proposition \( 5.9 \), respectively.

If \( S \) is a semistandard tableau let \( S_{(i,s)} \) be the subtableau of \( S \) consisting of those entries \( (j,t) \) with \( (j,t) \preceq (i,s) \) (see Definition \( 2.4 \)). We extend the dominance order to the set of semistandard tableaux by defining \( S \succeq T \) if \( \text{Shape}(S_{(i,s)}) \supseteq \text{Shape}(T_{(i,s)}) \) for all \( (i,s) \). This definition coincides with our previous definition of dominance when \( S \) is a standard tableau (recall that we are identifying standard tableaux and semistandard tableaux of type \( \omega \)).

5.13. Proposition. Suppose that \( Q_1, \ldots, Q_r \) are invertible elements of \( R \) and that \( \mu \) is a multicomposition. Then \( \langle \ , \ \rangle_\mu \) is a non–degenerate associative bilinear form on \( M(\lambda) \). In particular, \( M(\mu) \) is self–dual. Similarly, \( N(\mu) \) is self–dual.

Proof. We prove the Proposition only for \( M(\mu) \); the result for \( N(\mu) \) can be obtained using a similar argument or by specialization.

Suppose that \( S \in T^\omega(\mu) \), \( t \in T(\lambda) \), \( U \in T^\omega(\rho) \) and \( v \in T(\rho) \) for some multipartitions \( \lambda \) and \( \rho \). Applying the definitions we find that

\[
\langle m_{S\lambda}, m_\mu n_{U\rho} \rangle_\mu = \langle m_{S\lambda}, n_{U\rho} \rangle = \sum_{s \in T(\mu)} \langle m_{s\lambda}, n_{U\rho} \rangle.
\]

Therefore, by Theorem \( 5.3 \), \( \langle m_{S\lambda}, m_\mu n_{U\rho} \rangle_\mu = 0 \) unless there is a standard tableau \( s \) such that \( (U', v') \succeq (s, t) \) and \( \mu(s) = S \); hence, \( (U', v') \succeq (S, t) \). (Here \( U' = (U)' \) and not \((U)'\); in general these tableaux are different.)

Next suppose that \( (U', v') = (S, t) \). Then \( \hat{U} = (S')^\omega \succeq s' \) whenever \( s' \) is a standard tableau with \( \mu(s') = S' \); therefore, \( s \succeq \hat{U}' \) whenever \( s \) is a standard tableau such that \( \mu(s) = S \). Therefore, if \( (U, v) = (S', t') \) then

\[
\langle m_{S\lambda}, n_{U\rho} \rangle_\mu = \sum_{s \in T(\mu)} \langle m_{s\lambda}, n_{U\rho} \rangle = \langle m_{\hat{S}\lambda}, n_{\hat{U}\rho} \rangle = Q_{\lambda}
\]

by Theorem \( 5.3 \).
Combining the last two paragraphs shows that the matrix \( \langle m_\mathcal{S} t, m_\mathcal{S} \mu n_\mathcal{H} \rangle \) is invertible and, hence, that the form \( \langle \ , \ \rangle \) on \( M(\mu) \) is non–degenerate.

The harder part is to prove that \( \langle \ , \ \rangle \) is associative. Now, the form \( \langle \ , \ \rangle \) on \( \mathcal{H} \) is associative, so if \( h \in \mathcal{H} \) then

\[
\langle m_\mathcal{S} t h, m_\mathcal{S} \mu n_\mathcal{H} \rangle = \langle m_\mathcal{S} h, n_\mathcal{H} \rangle = \langle m_\mathcal{S} , n_\mathcal{H} \rangle = \sum \limits_{a,b} r_{ab} \langle m_\mathcal{S} , n_{ab} \rangle,
\]

where \( n_{ab} h^* = \sum a,b r_{ab} n_{ab} \) for some \( r_{ab} \in R \). Write \( m_\mathcal{S} = m_\mathcal{S} h^*_\mathcal{S} \), for some \( h^*_\mathcal{S} \in \mathcal{H} \). Then

\[
\langle m_\mathcal{S} , n_{ab} \rangle = \tau (n_{ba} m_\mathcal{S} ) = \tau (n_{ba} m_\mathcal{S} h^*_\mathcal{S} ) = \tau (h^*_\mathcal{S} m_\mathcal{S} n_{ab} ),
\]

where the second equality uses the fact that \( \tau \) is a trace form and the last equality follows because \( \tau (a) = \tau (a^*) \) for all \( a \in \mathcal{H} \). Therefore, if \( \langle m_\mathcal{S} , n_{ab} \rangle \neq 0 \) then \( \mu(a) \) is column semistandard by Lemma 5.8. Let \( A = \mu(a) \) and recall that \( m_\mathcal{S} n_{ab} = \pm q^i m_\mathcal{S} n_{ab} \) for some integer \( i \) (which depends on \( a \)); write \( m_\mathcal{S} n_{ab} = \lambda_a m_\mathcal{S} n_{ab} \) and set \( r_{ab} = \sum a \lambda_a r_{ab} \), where the sum runs over those standard tableaux with \( \mu(a) = A \). Then we have shown that

\[
\langle m_\mathcal{S} t h, m_\mathcal{S} \mu n_\mathcal{H} \rangle = \sum \limits_{b \in T(\lambda)} r_{ab} \langle m_\mathcal{S} t , n_{ab} \rangle.
\]

On the other hand, by Lemma 5.8 again,

\[
\langle m_\mathcal{S} , m_\mathcal{S} h^*_\mathcal{S} \rangle = \sum \limits_{a,b} r_{ab} \langle m_\mathcal{S} , m_\mathcal{S} n_{ab} \rangle = \sum \limits_{a,b \in T(\lambda)} r_{ab} \langle m_\mathcal{S} , m_\mathcal{S} n_{ab} \rangle = \sum \limits_{b \in T(\lambda)} r_{ab} \langle m_\mathcal{S} , n_{ab} \rangle.
\]

Hence, \( \langle m_\mathcal{S} t h, m_\mathcal{S} \mu n_\mathcal{H} \rangle = \langle m_\mathcal{S} , m_\mathcal{S} h^*_\mathcal{S} \rangle \), so the form is associative as claimed. \( \square \)

5.14. Corollary. Suppose that \( R \) is a field and let \( \lambda \) be a multipartition of \( n \). Then both the Young module \( Y(\lambda) \) and the twisted Young module \( Y^*(\lambda) \) are self–dual.

Proof. By the Proposition, \( M(\lambda) \) and \( N(\lambda) \) are both self–dual. Hence, the result follows by induction on \( \lambda \) using Theorem 5.6 and Proposition 5.4, respectively. \( \square \)

6. The cyclotomic tilting modules

Let \( (A, X^+) \) be a quasi–hereditary algebra, where \( X^+ \) is the poset of weights for \( A \); see, for example, [10, Appendix]. For each \( \lambda \in X^+ \) there is a standard module \( \Delta(\lambda) \) with simple head \( L(\lambda) \) and a costandard module with simple socle \( L(\lambda) \). An \( A \)–module \( M \) has a \( \Delta \)–filtration if it has a filtration in which every subquotient isomorphic to a standard module; similarly, \( M \) has a \( \nabla \)–filtration if every subquotient is isomorphic to a costandard module. An \( A \)–module \( T \) is a tilting module if it has both a \( \Delta \)–filtration and a \( \nabla \)–filtration.

6.1 (Ringel [20]) Suppose that \( R \) is a field and that \( (A, X^+) \) is a quasi–hereditary algebra. Then, for each \( \lambda \in X^+ \), there is a unique indecomposable tilting module \( T(\lambda) \) such that

\[
[T(\lambda) : \Delta(\lambda)] = 1 \quad \text{and} \quad [T(\lambda) : \Delta(\mu)] \neq 0 \text{ only if } \lambda \geq \mu.
\]

Moreover, if \( T \) is any tilting module then

\[
T \cong \bigoplus_{\lambda \in X^+} T(\lambda)^{T\lambda}.
\]
for some non-negative integers $t_\lambda$.

The $T(\lambda)$ are the partial tilting modules of $A$. A full tilting module for $A$ is any tilting module which contains every $T(\lambda)$, for $\lambda \in \Lambda^+$, as a direct summand.

By [7, Cor. 6.18] the cyclotomic Schur algebras are quasi-hereditary algebras with weight poset $\Lambda^+$. The standard modules of $\mathcal{S}(\Lambda)$ are the Weyl modules and the costandard modules are their contragredient duals. In this section we will describe the partial tilting modules of $\mathcal{S}(\Lambda)$ when $\omega \in \Lambda$ and the parameters $Q_1, \ldots, Q_r$ are distinct and non-zero.

First consider the case $r = 1$. Suppose that $d \geq 1$ and let $\Lambda_{d,n}$ be the set of compositions of $n$ into at most $d$ parts and let $B$ be a free $R$–module of rank $d$. Then $\mathcal{H} (\Sigma_n)$ acts on $V^\otimes n$ (by $q$–analogues of place permutations) and $V^\otimes n \cong M(\Lambda)$; see [8]. The Dipper–James [5] $q$–Schur algebra $\mathcal{S}(d,n)$ is the cyclotomic $q$–Schur algebra $\mathcal{S}(\Lambda_{d,n})$; by the above remarks $\mathcal{S}(d,n) \cong \text{End}_{\mathcal{H} (\Sigma_n)} (V^\otimes n)$. Donkin [4] has shown that when $d \geq n$ the tilting modules for $\mathcal{S}(d,n)$ are the indecomposable direct summands of the exterior powers $\wedge V = \wedge^1 V \otimes \cdots \otimes \wedge^d V$. In the cyclotomic case we do not have a description of $M(\Lambda) = \bigoplus_\mu M(\mu)$ as a tensor product; nevertheless, we do have the following analogue of the exterior powers.

6.2. Definition. For each multicomposition $\alpha$ let $E(\alpha) = \text{Hom}_\mathcal{H} (M(\Lambda), N(\alpha))$.

By definition, $E(\alpha)$ is a right $\mathcal{S}(\Lambda)$–module. Recall that $\bar{\alpha}$ is the multipartition obtained by reordering the parts of $\alpha^{(s)}$ for each $s$. By the argument of Corollary $3.3$, $N(\alpha) \cong N(\bar{\alpha})$; therefore, $E(\alpha) \cong E(\bar{\alpha})$. Hence, there is no loss in assuming that $\alpha$ is a multipartition. The $E(\alpha)$ are very similar to the modules $M(\mu) = \text{Hom}_\mathcal{H} (M(\mu), M(\mu))$ of Proposition $3.1$. The $E(\alpha)$ play the role of exterior powers and the $M(\mu)$ the symmetric powers.

We will show that $E(\alpha)$ has a Weyl filtration and that it is self-dual; hence, it also has a dual Weyl filtration. This will enable us to show that the tilting modules of $\mathcal{S}(\Lambda)$ are the indecomposable summands of the $E(\lambda)$ as $\lambda$ runs over the multipartitions in $\Lambda^+$.

The next result is a first step towards producing a basis for $E(\alpha)$. By general principles, if $f \in N(\alpha) \cap M(\mu)^*$ then left multiplication by $f$ is an $\mathcal{H}$–module homomorphism from $M(\mu)$ into $N(\alpha)$; in fact, every element of $E(\alpha)$ arises in this way.

6.3. Lemma. Suppose that $\alpha$ and $\mu$ are multicompositions of $n$. Then there is an isomorphism of $R$–modules $\text{Hom}_\mathcal{H} (M(\mu), N(\alpha)) \cong N(\alpha) \cap M(\mu)^*$ given by $\theta \mapsto \theta(m_\mu)$. In particular, if $\theta \in \text{Hom}_\mathcal{H} (M(\mu), N(\alpha))$ and $h_\theta = \theta(m_\mu)$ then $h_\theta \in N(\alpha) \cap M(\mu)^*$ and $\theta(m) = h_\theta m$ for all $m \in M(\mu)$.

Proof. This follows from Theorem 5.16 and Lemma 5.2 of [2]. (Theorem 5.16 says that the double annihilator, $\{ h \in \mathcal{H} \mid hs = 0 \text{ whenever } m_\mu s = 0 \}$, of $m_\mu$ is $\mathcal{H} m_\mu$; Lemma 5.2 observes that this property of the double annihilator implies the Lemma.)

Therefore, to give a basis of $E(\alpha)$ it is enough to find a basis of $N(\alpha) \cap M(\mu)^*$. To do this we need to make the following assumption.

6.4. Standing assumption. For the rest of this paper assume that $Q_1, \ldots, Q_r$ are distinct.

The only place where we explicitly use Assumption 6.4 is in the proof of the following theorem; however, almost everything which follows relies on this result. Unless otherwise stated, this assumption will remain in force for the rest of the paper.

6.5. Theorem. Suppose that $Q_1, \ldots, Q_r$ are all distinct and let $\alpha$ and $\mu$ be multicompositions of $n$. Then $N(\alpha) \cap M(\mu)^*$ is free as an $R$–module with basis $\{ n_\alpha m_{ST} \mid S \in T_\mu^\alpha(\lambda), T \in T_\lambda^\mu(\lambda) \text{ for some } \lambda \vdash n \}$. 


Proof. First note that \( N(\alpha) \cap M(\mu)^* \) is free because it is a submodule of a free module. Next, by (2.5), if

\[
\text{where the sum is over all pairs } N
\]

Definition 2.4 but not condition (iii) — these are “almost” semistandard tableaux.

By Proposition 5.9, if \( x \in N(\alpha) \cap M(\mu)^* \) then \( x = \sum U_\alpha n_{\alpha} m_{\alpha} \), for some \( U_\alpha \in R \), where the sum is over all pairs \( U_\alpha \) of tableaux with \( U_\alpha \) semistandard of type \( \alpha \) and \( v \) standard. Now, if \( (i, i + 1) \in \mathcal{S}_\mu \) then \( m_{i} T_i = q \); hence, \( x T_i = q x \) and, as in [[19] (4.19)] (compare [7], Lemma 4.11), it follows that if \( r_{U \alpha} \neq 0 \) then \( i \) and \( i + 1 \) are not in the same column of \( u \) and that \( r_{ST} = r_{SV} \) where \( t = v(i, i + 1) \); that is, \( V = U(v) \) satisfies conditions (i) and (ii) of Definition 2.4. Therefore, \( x = \sum r_{UV} n_{\alpha} m_{UV} \) where the sum is over pairs \( (U, V) \) with \( U \in \mathcal{T}_\alpha^\mu(\Lambda^+) \) and \( V \in \mathcal{T}_\alpha^\mu(\Lambda^+) \). By the first paragraph, \( n_{\alpha} m_{ST} \in N(\alpha) \cap M(\mu)^* \) when \( T \) is semistandard, so we may assume that \( r_{UV} = 0 \) unless \( V \in \mathcal{T}_\alpha^\mu(\Lambda^+) \). Thus, we are reduced to showing that if we have an element \( x \in N(\alpha) \cap M(\mu)^* \) which can be written in the form

\[
x = \sum_{U \in \mathcal{T}_\alpha^\mu(\Lambda^+)} \sum_{V \in \mathcal{T}_\alpha^\mu(\Lambda^+)} r_{UV} n_{\alpha} m_{UV},
\]

for some \( r_{UV} \in R \), then \( x = 0 \). By way of contradiction, suppose that \( x \neq 0 \).

Fix \((S, T)\) with \( r_{ST} \neq 0 \) such that \( r_{UV} = 0 \) whenever \((S, T) \triangleright (U, V)\) for \( U \in \mathcal{T}_\alpha^\mu(\Lambda^+) \) and \( V \in \mathcal{T}_\alpha^\mu(\Lambda^+) \). Let \( \tilde{T} \) be the unique standard tableau such that \( \mu(\tilde{T}) = T \) and \( t \triangleright \tilde{T} \) whenever \( \mu(t) = T \); the tableau \( \tilde{T} \) is denoted \( \text{last}(T) \) in [13]. Let \( i \) be the smallest positive integer such that \( c = \text{comp}_\mu(i) \triangleright \text{comp}_\mu(i) \); such an \( i \) exists because \( T \) does not satisfy Definition 2.4(iii). If \( j < i \) then \( \text{comp}_\mu(j) \triangleright \text{comp}_\mu(i) \) by the minimality of \( i \) and the fact that \( \text{comp}_\mu(j) \triangleright \text{comp}_\mu(i) \); in particular, this implies that \( i \) must appear in the first row and first column of \( \tilde{T}\). Following [7] define

\[
y_i = T_{i-1} \ldots T_i \prod_{s=1}^{\text{comp}_\mu(i)} (L_i - Q_s).
\]

Then \( y_i \neq 0 \) since \( \text{comp}_\mu(i) \triangleright c \leq r \). Moreover, \( m_{\mu} y_i = 0 \) by [7] Lemma 5.8] (when translating into the notation of [7] note that \( y_i = \text{comp}_\mu(i) \)). Let \( t = \tilde{T}_{s_i-1} \ldots s_i \); then \( \text{res}_\mu(1) = \text{res}_\mu(i) = Q_s \), and \( \ell(d(t)) = \ell(d(\tilde{T})) + i - 1 \); so, \( m_{ST} T_{s_i-1} \ldots T_i = m_{ST} \).

Furthermore, by the cancellation property of the Bruhat–Chevalley order, \( v \triangleright \tilde{T} \) if and only if \( vs_{i-1} \ldots s_i \triangleright t \) since \( \ell(d(t)) = \ell(d(\tilde{T})) + i - 1 \); see, for example, [17], Cor. 3.9.
Therefore, using (5.2) for the third equality, we have

\[ xy_i = \left( r_{ST} n_{\alpha} m_{\mathcal{S}T} + \sum_{u,v} r_{uv} n_{\alpha} m_{uv} \right) y_i \]

\[ = \left( r_{ST} n_{\alpha} m_{\mathcal{S}t} + \sum_{u,v} r_{uv}^* n_{\alpha} m_{uv} \right) \prod_{s=1}^{\text{comp}_{\alpha}(i)} (L_1 - Q_s) \]

\[ = r_{ST} \prod_{s=1}^{\text{comp}_{\alpha}(i)} (Q_s - Q) n_{\alpha} m_{\mathcal{S}t} + \sum_{u,v} r_{uv}'' n_{\alpha} m_{uv} y_i \]

for some \( r_{uv}^*, r_{uv}'' \in R \). By Assumption 6.4 the coefficient of \( n_{\alpha} m_{\mathcal{S}t} \) in \( xy_i \) is non-zero because \( c > \text{comp}_{\alpha}(i) \) — and \( r_{ST} \neq 0 \), note also that \( n_{\alpha} m_{\mathcal{S}t} \neq 0 \) by Corollary 5.10. Therefore, \( xy_i \neq 0 \). However, \( x \in N(\alpha) \cap M(\mu)^* \) and, as remarked above, \( m_{\mu} y_i = 0 \), so this contradicts the assumption that \( x \neq 0 \). Consequently, \( x = 0 \) and the Theorem follows.

6.6. Remark. If \( Q_s = Q_1 \), where \( s \neq t \), then the rank of \( n_{\alpha} \mathcal{H} \cap \mathcal{H} m_{\mu} \) can be larger than that predicted by Theorem 6.5. For example, suppose that \( Q_1 = Q_2 \), when \( r = n = 2 \), and take \( \alpha = \mu = ((1),(1)) \). Then \( n_{\alpha} = m_{\mu} = L_1 - Q_2 \) and, by (2.6),

\[ n_{\alpha} \mathcal{H} \cap \mathcal{H} m_{\mu} = M(\mu) \cap M(\mu)^* \]

is the free \( R \)-module with basis \( \{ m_{ST} \mid S, T \in T_{\mu}(\Lambda^+) \} \); this is an \( R \)-module of rank 3. In contrast, if \( Q_1 \neq Q_2 \) then by Theorem 5.3

\[ n_{\alpha} \mathcal{H} \cap \mathcal{H} m_{\mu} = Rn_{\alpha} T_1 m_{\mu} = R(L_1 - Q_1) T_1 (L_1 - Q_2); \]

this time the intersection has rank 1. (By direct a calculation, \( (L_1 - Q_1) T_1 (L_1 - Q_2) \) is an element of \( M(\mu) \cap M(\mu)^* \) if and only if \( Q_1 = Q_2 \).

Shoji has shown that if the parameters \( Q_1, \ldots, Q_r \) are distinct then \( M(\mu) \) is an induced module (more accurately, he has shown that there exists an induced module which has the same image as \( M(\mu) \) in the Grothendieck group of \( \mathcal{H} \)). It should be possible to prove analogue of Frobenius reciprocity for the modules \( M(\mu) \) and \( N(\alpha) \) using Shoji’ work; this would give a better explanation as to why the rank of \( \text{Hom}_{\mathcal{H}}(M(\mu), N(\alpha)) \) is independent of the choice of parameters \( Q_1, \ldots, Q_r \) in the presence of Assumption 6.4.

Now we reap some consequences of Theorem 6.5. We emphasize that even though we do not explicitly state Assumption 5.4, it remains in force for all of these results.

6.7. Corollary. Suppose that \( \alpha \) and \( \mu \) are multicompositions of \( n \). Then

\[ N(\alpha) \cap M(\mu)^* = n_{\alpha} \mathcal{H} m_{\mu}. \]

Proof. Certainly, \( n_{\alpha} \mathcal{H} m_{\mu} \subseteq N(\alpha) \cap M(\mu)^* \). Conversely, by Theorem 5.5 a basis of \( N(\alpha) \cap M(\mu)^* \) is given by the elements \( n_{\alpha} m_{\mathcal{S}T} \), where \( S \in T_{\alpha}(\Lambda^+) \) and \( T \in T_{\mu}(\Lambda^+) \). As \( n_{\alpha} m_{\mathcal{S}T} \in n_{\alpha} \mathcal{H} m_{\mu} \), by (2.5), we also have the opposite inclusion.

Notice that Theorem 6.5 and Corollary 6.7 imply that if \( \mu \) is a multipartition then

\[ n_{\mu} \mathcal{H} \cap \mathcal{H} m_{\mu'} = n_{\mu} \mathcal{H} m_{\mu'} = Rn_{\mu} m_{\mathcal{T}_{\mu}(\Lambda^+)} = Rn_{\mu} T_{\mathcal{S}(\Lambda^+)} m_{\mu'}. \]
Du and Rui [11] have shown that \( n_\mu \mathcal{H} m_{\mu'} = R n_\mu T_{w_\mu} m_{\mu'} \) (without assuming that the parameters \( Q_s \) are distinct). This is interesting because the element \( n_\mu T_{w_\mu} m_{\mu'} \) generates the Specht module \( S(\mu') \); see Remark 5.12.

6.8. Corollary. Suppose that \( \text{Hom}_{\mathcal{H}}(M(\mu), N(\alpha)) \neq 0 \) for some multicompositions \( \alpha \) and \( \mu \). Then \( \alpha' \trianglerighteq \mu \).

Proof. By Lemma 6.3, \( N(\alpha) \cap M(\mu)^* \neq 0 \) since \( \text{Hom}_{\mathcal{H}}(M(\mu), N(\alpha)) \neq 0 \). Therefore, we can find a multipartition \( \lambda \) and tableaux \( S \in T^\alpha(\lambda) \) and \( T \in T^\mu(\lambda) \) such that \( n_\alpha m_{ST} \) is a non–zero element of \( \{ \alpha \} \cap M(\mu)^* \). Hence, \( \alpha' \trianglerighteq \lambda \trianglerighteq \mu \), so \( \alpha' \trianglerighteq \mu \) as required. \( \square \)

Similarly, if \( \text{Hom}_{\mathcal{H}}(N(\alpha), M(\mu)) \neq 0 \) then \( \mu' \trianglerighteq \alpha \).

6.9. Corollary. Suppose that \( \alpha \) and \( \mu \) are multicompositions of \( n \). Then
\[
\{ n_\mu m_\mu \mid S \in T^\alpha(\lambda), T \in T^\mu(\lambda) \text{ for some } \lambda \vdash n \}
\]
is a basis of \( N(\alpha) \cap M(\mu)^* \).

Proof. By Theorem 6.5, the \( R \)-module \( N(\alpha) \cap M(\mu)^* \) is stable under specialization (or, if you prefer, base change); therefore, it is enough to consider the case \( \mathcal{H} = \mathcal{H}_Z \). Applying the involutions ‘ and * to the basis of \( N(\mu) \cap M(\alpha)^* \) given by Theorem 5.3 yields the result. Alternatively, this can be proved by modifying the argument of Theorem 6.5. \( \square \)

These results allow us to give two bases for \( E(\alpha) \).

6.10. Definition. Suppose that \( \lambda \) is a multipartition of \( n \) and that \( \alpha \) and \( \mu \) are two multicompositions of \( n \). For tableaux \( S \in T^\alpha(\lambda), T \in T^\mu(\lambda), A \in T^\mu(\lambda) \) and \( B \in T^\alpha(\lambda) \) let \( \theta_{ST} \) and \( \theta'_{ST} \) be the homomorphisms in \( E(\alpha) \) determined by
\[
\theta_{ST}(m_\nu h) = \delta_{\nu \mu} n_\alpha m_{ST} h \quad \text{and} \quad \theta'_{AB}(m_\nu h) = \delta_{\nu \mu} n_{AB} m_\mu h,
\]
for all \( h \in \mathcal{H} \) and all \( \nu \in \Lambda \).

Lemma 6.5 together with Theorem 6.5 and Corollary 6.9 respectively, show that these maps are elements of \( E(\alpha) \). Indeed, these results show that each of the corresponding sets of such maps is a basis of \( E(\alpha) \). More precisely, we have the following.

6.11. Proposition. Suppose that \( Q_1, \ldots, Q_r \) are all distinct and let \( \alpha \) be a multipartition of \( n \). Then \( E(\alpha) \) is free as an \( R \)-module with bases \( \mathcal{E} \) and \( \mathcal{E}' \) where
\[
\mathcal{E} = \{ \theta_{ST} \mid S \in T^\alpha(\lambda) \text{ and } T \in T^\mu(\lambda) \text{ for some } \lambda \vdash n \}
\]
and \( \mathcal{E}' = \{ \theta'_{AB} \mid A \in T^\mu(\lambda) \text{ and } B \in T^\alpha(\lambda) \text{ for some } \lambda \vdash n \} \).

Now that we have the required notation it is a good time to note that \( E(\alpha) \) is cyclic.

6.12. Corollary. Suppose that \( \omega \in \Lambda \) and that \( \alpha \) is a multicomposition. Then \( E(\alpha) \) is a cyclic \( \mathcal{H}(\Lambda) \)-module; more precisely, \( E(\alpha) = \theta_{T_{\omega}T_{--}} \mathcal{H}(\Lambda) \) where \( T^\alpha_\omega = \alpha(t^-) \).

Proof. The map \( \theta_{T_{\omega}T_{--}} \) is the extension to \( E(\alpha) \) of the homomorphism \( \mathcal{H} \rightarrow N(\alpha) \) given by \( \theta_{T_{\omega}T_{--}}(h) = n_\alpha m_{T_{\omega}T_{--}} h = n_\alpha h \), for all \( h \in \mathcal{H} \). Suppose that in \( S \in T^\alpha(\lambda) \) and \( T \in T^\mu(\lambda) \) for some multipartition \( \lambda \). Then \( \theta_{ST} = \theta_{T_{\omega}T_{--}} \varphi_{ST} \) (both maps send \( m_\mu \) to \( n_\alpha m_{ST} \)), so \( E(\alpha) \) is cyclic as claimed. \( \square \)

As an application of Proposition 6.11 we now show that \( E(\alpha) \) has a Weyl filtration.
6.13. Theorem. Suppose that \( \omega \in \Lambda \) and that \( Q_1, \ldots, Q_r \) are all distinct. Then there exist multipartitions \( \lambda_1, \ldots, \lambda_k \) in \( \Lambda \) and an \( \mathcal{S}(\Lambda) \)-module filtration
\[
E(\alpha) = E_1 \supset \cdots \supset E_k \supset E_{k+1} = 0
\]
such that \( E_i/E_{i+1} \cong \Delta(\lambda_i) \) and \( \alpha' \succeq \lambda_i \), for \( i = 1, \ldots, k \). Moreover, if \( \lambda \) is any multipartition in \( \Lambda^+ \) then \( \# \{1 \leq i \leq k \mid \lambda_i = \lambda \} = \# T_\alpha(\lambda) \).

Proof. For the most part this is a familiar argument; however, towards the end there is a small twist so we give the details. Let \( S_1, \ldots, S_k \) be the complete set of column semistandard tableaux of type \( \alpha \) ordered so that \( i > j \) whenever \( \text{Shape}(S_i) \cong \text{Shape}(S_j) \). For each \( i \) let let \( \lambda_i = \text{Shape}(S_i) \); then \( \alpha' \succeq \lambda_i \) since \( T_\alpha(\lambda_i) \neq \emptyset \).

For \( i = 1, \ldots, k \) let \( E_i \) be the \( R \)-submodule of \( E(\alpha) \) spanned by the elements \( \theta_{S_i}T \) with \( j \geq i \) and \( T \in T_\alpha(\lambda_j) \). To prove the Theorem it is enough to show that, \( E_i \) is a submodule of \( E(\alpha) \) and that \( E_i/E_{i+1} \cong \Delta(\lambda_i) \) for each \( i \).

Suppose that \( i \geq 1 \) and let \( T \in T_\mu(\lambda_i) \), \( U \in T_\nu(\rho) \) and \( V \in T_\sigma(\rho) \), for some multicompositions \( \mu, \nu, \sigma \in \Lambda \) and \( \rho \in \Lambda^+ \). Consider the product \( \theta_{S_i}T \varphi_{UV} \). By definition, \( \theta_{S_i}T \varphi_{UV} = 0 \) unless \( \mu = \text{Type}(T) = \text{Type}(U) = \sigma \); so suppose that \( \sigma = \mu \). In order to write \( \theta_{S_i}T \varphi_{UV} \) as a linear combination of the basis elements of \( E(\alpha) \) it suffices to consider \( (\theta_{S_i}T \varphi_{UV})(m_\nu) \). Write \( m_\nu = m_\nu h^\mu_{UV} \), for some \( h^\mu_{UV} \in \mathcal{S} \). Then we have
\[
(\theta_{S_i}T \varphi_{UV})(m_\nu) = \theta_{S_i}T(m_\nu) = \theta_{S_i}T(m_\nu) h^\mu_{UV} = n_\alpha m_{S_i} h^\mu_{UV}
\]
\[
= n_\alpha \left( \sum_{\gamma \in T_\nu(\lambda_i)} r_{S_i \gamma} m_{S_i \gamma} \right) \mod H^\lambda;
\]
where the second line follows from \( 2.9 \). Hence, Theorem 6.3 implies that
\[
\theta_{S_i}T \varphi_{UV} \equiv \sum_{\gamma \in T_\nu(\lambda_i)} r_{S_i \gamma} \theta_{S_i \gamma} \mod E_{i+1}.
\]
All of our claims now follow.

\[\square\]

6.14. Remark. By Theorem 6.13 the module \( \Delta(\lambda) \) is a Weyl module composition factor of \( E(\alpha) \) whenever \( T_\alpha(\lambda) \neq \emptyset \); thus, \( \alpha' \succeq \lambda \). Moreover, since \( T_\alpha(\omega) \) is always non–empty this means that \( \Delta(\omega) \) is always a composition factor of \( E(\alpha) \); consequently, the assumption that \( \omega \in \Lambda \) is necessary in Theorem 6.13. If \( \omega \notin \Lambda \) then \( E(\alpha) \) is still an \( \mathcal{S}(\Lambda) \)-module (for all multicompositions \( \alpha \)); however, we are not able to give a Weyl filtration of \( E(\alpha) \) in this case.

6.15. Corollary. Suppose that \( R \) is a field, \( \omega \in \Lambda \) and that \( \lambda \) and \( \mu \) are multipartitions of \( n \). Then \( [E(\lambda) \Delta(\lambda')] = 1 \) and \( [E(\lambda) \Delta(\mu)] \neq 0 \) only if \( \lambda' \succeq \mu \).

Proof. By the Theorem, \( [E(\lambda) \Delta(\lambda')] = 1 \) and \( [E(\lambda) \Delta(\mu)] \neq 0 \) only if \( \lambda' \succeq \mu \). Finally, \( [E(\lambda) \Delta(\lambda')] = 1 \) because \( \lambda(\gamma') = (\lambda^\gamma)' \) is the unique column semistandard \( \lambda' \)-tableau of type \( \lambda \) (just as \( T^\lambda = \lambda(t^\lambda) \) is the unique semistandard \( \lambda \)-tableau of type \( \lambda \)).

\[\square\]

As in the previous section, if \( E \) is an \( \mathcal{S}(\Lambda) \)-module then its contragredient dual \( E^\circ \) is the dual space \( \text{Hom}_R(E, R) \) equipped with the contragredient action: \( (f \varphi)(x) = f(x \varphi^*) \), for \( f \in E^\circ, \varphi \in \mathcal{S}(\Lambda) \) and \( x \in E \). Again, \( E \) is self–dual if \( E \cong E^\circ \).

If \( E \) is an \( \mathcal{S}(\Lambda) \)-module and \( \mu \in \Lambda \) let \( E_\mu = E \varphi_\mu \) be the \( \mu \)-weight space of \( E \).
6.16. Theorem. Suppose that $Q_1, \ldots, Q_r$ are distinct invertible elements of $R$ and let $\alpha$ be a multicomposition. Then $E(\alpha)$ is self-dual.

Proof. Define a bilinear map $\{ \ , \}_\alpha : E(\alpha) \times E(\alpha) \to R$ by
\[
\{ \theta_{ST}, \theta'_{AB} \}_\alpha = \begin{cases} \langle m_{ST}, n_{AB} \rangle, & \text{if } \text{Type}(T) = \text{Type}(B), \\ 0, & \text{otherwise.} \end{cases}
\]
where $\theta_{ST}$ and $\theta'_{AB}$ run over the two bases $\mathcal{E}$ and $\mathcal{E}'$ of $E(\alpha)$ from Proposition 6.11. By definition, the different weight spaces $E_\mu$, $\mu \in \Lambda$, of $E(\alpha)$ are orthogonal with respect to $\{ \ , \}_\alpha$. Suppose then that $\text{Type}(T) = \text{Type}(B)$ and let $\lambda = \text{Shape}(S)$. Then, as in the proof of Proposition 5.13,
\[
\{ \theta_{ST}, \theta'_{AB} \}_\alpha = \begin{cases} Q\lambda, & \text{if } (A', B') = (S, T), \\ 0, & \text{if } (A', B') \neq (S, T), \end{cases}
\]
where $\lambda = \text{Shape}(S)$. Hence, $\{ \ , \}_\alpha$ is a non-degenerate bilinear form on $E(\alpha)$.

Once again, the harder task is to prove that $\{ \ , \}_\alpha$ is associative. Choose tableaux $S, T, A$ and $B$ as above. Suppose that $\varphi \in \text{Hom}_\mathcal{H}(M(\nu), M(\mu))$ for some $\nu, \mu \in \Lambda$. Then $\theta_{ST} \varphi \in E_\nu$ and $\theta_{AB} \varphi^* \in E_\mu$; therefore, if $\mu \neq \nu$ then
\[
\{ \theta_{ST} \varphi, \theta_{AB} \}_\alpha = 0 = \{ \theta_{ST}, \theta_{AB} \varphi^* \}_\alpha.
\]
because different weight spaces are orthogonal with respect to $\{ \ , \}_\alpha$. Suppose then that $\mu = \nu$. By considering weight spaces we may also assume that $\text{Type}(T) = \mu = \text{Type}(B)$. Write $\varphi(m_\mu) = m_\mu h$ for some $h \in \mathcal{H}$; then $\theta_{ST} \varphi(m_\mu h) = n_\alpha m_{ST} h$, so $\theta_{ST} \varphi$ is determined by $n_\alpha m_{ST}$. Similarly, since $\varphi(m_\mu) \in m_\mu \mathcal{H} \cap \mathcal{H} m_\mu$ we can also write $\varphi(m_\mu) = h m_\mu$. Therefore, $\varphi^*(m_\mu) = m_\mu \tilde{h}^* = (\tilde{h} m_\mu)^* = (m_\mu h)^* = h^* m_\mu$; consequently, $(\theta'_{AB} \varphi^*)(m_\mu) = \theta'_{AB}(m_\mu \tilde{h}^*) = n_{AB} m_\mu \tilde{h}^* = n_{AB} h^* m_\mu$. The bilinear form $\{ \ , \}$ on $\mathcal{H}$ is associative so, as in the proof of Proposition 5.13 we have
\[
\{ \theta_{ST} \varphi, \theta_{AB} \}_\alpha = \langle m_{ST} h, n_{AB} \rangle = \langle m_{ST}, n_{AB} h^* \rangle = \{ \theta_{ST}, \theta_{AB} \varphi^* \}_\alpha.
\]
Therefore, $\{ \ , \}_\alpha$ is associative and the proof is complete.

Combining the last two results we obtain our main theorem.

6.17. Theorem. Suppose that $R$ is a field, $\omega \in \Lambda$ and that $Q_1, \ldots, Q_r$ are distinct non-zero elements of $R$.

(i) If $\lambda \in \Lambda^+$ then
\[
E(\lambda) \cong T(\lambda') \oplus \bigoplus_{\lambda' \succcurlyeq \mu} T(\mu)^{e_{\lambda \mu}}
\]
for some non-negative integers $e_{\lambda \mu}$.

(ii) The tilting modules of $\mathcal{F}(\Lambda)$ are the indecomposable direct summands of the modules $\{ E(\lambda) \mid \lambda \in \Lambda^+ \}$.

Proof. By Theorem 6.13, $E(\lambda)$ has a $\Delta$–filtration; therefore, $E(\lambda)$ also has a $\nabla$–filtration since $E(\lambda)$ is self–dual by Theorem 6.11. Hence, $E(\lambda)$ is a tilting module. Furthermore, by Corollary 6.13, $[E(\lambda) : \Delta(\lambda')] = 1$ and if $[E(\lambda) : \Delta(\mu)] > 0$ then $\lambda' \succeq \mu$. Therefore, by Ringel’s theorem 6.13, there exist non–negative integers $e_{\lambda \mu}$ such that
\[
E(\lambda) \cong T(\lambda') \oplus \bigoplus_{\lambda' \succcurlyeq \mu} T(\mu)^{e_{\lambda \mu}}.
\]
This proves (i). Part (ii) now follows by induction on the dominance order using (6.11).
7. RINGEL DUALITY

We now turn our attention to the Ringel dual of $\mathcal{S}(\Lambda)$. By definition, the Ringel dual of $\mathcal{S}(\Lambda)$ is the algebra $\text{End}_{\mathcal{S}(\Lambda)}(T)$, where $T$ is any full tilting module for $\mathcal{S}(\Lambda)$; thus, the Ringel dual is determined only up to Morita equivalence. By Theorem 6.17 the module

$$E(\Lambda) = \bigoplus_{\alpha \in \Lambda} E(\alpha) = \bigoplus_{\alpha, \mu \in \Lambda} \text{Hom}_{\mathcal{H}}(M(\mu), N(\alpha))$$

is a full tilting module for $\mathcal{S}(\Lambda)$ when $\omega \in \Lambda$ and the parameters $Q_1, \ldots, Q_r$ are distinct and non-zero.

Let $\alpha$ and $\beta$ be two multicompositions. Then for any $S \in T_\alpha(\lambda)$ and $T \in T_\alpha(\beta)$ there is an $\mathcal{H}$–module homomorphism $\varphi'_{ST} : N(\alpha) \rightarrow N(\beta)$; this induces an $\mathcal{S}(\Lambda)$–module homomorphism $\Phi_{ST} : E(\alpha) \rightarrow E(\beta)$ given by $\Phi_{ST}(\theta) = \varphi'_{ST} \theta$, for $\theta \in E(\alpha)$. In fact, we will show that these give all of the $\mathcal{S}(\Lambda)$–module homomorphisms from $E(\alpha)$ to $E(\beta)$.

The next two results do not require that the parameters $Q_1, \ldots, Q_r$ be distinct.

7.1. Proposition. Suppose that $\omega \in \Lambda$ and let $\alpha$ and $\beta$ be multicompositions of $n$. Then $\text{Hom}_{\mathcal{S}(\Lambda)}(E(\alpha), E(\beta))$ is free as an $R$–module with basis

$$\{ \Phi_{ST} \mid S \in T_{\beta}(\lambda), T \in T_{\alpha}(\lambda) \text{ for some } \lambda \in \Lambda^+ \}.$$ 

Proof. As indicated above the maps $\Phi_{ST}$ belong to $\text{Hom}_{\mathcal{S}(\Lambda)}(E(\alpha), E(\beta))$. Moreover, they are linearly independent because the $\varphi'_{ST}$ are a basis of $\text{Hom}_{\mathcal{H}}(N(\alpha), N(\beta))$ by Proposition 4.3 (i). Thus, it remains to see that these maps span $\text{Hom}_{\mathcal{S}(\Lambda)}(E(\alpha), E(\beta))$.

Suppose that $\Phi \in \text{Hom}_{\mathcal{S}(\Lambda)}(E(\alpha), E(\beta))$. If $\theta \in \text{Hom}_{\mathcal{H}}(M(\mu), N(\alpha))$ then $\Phi(\theta)$ belongs to $\text{Hom}_{\mathcal{H}}(M(\mu), N(\beta)) \cong N(\alpha)$; that is, $\Phi$ maps weight spaces to weight spaces. Now, $E(\alpha)$ is generated by $\theta_{T_n T^{-n}}$ by Corollary 5.12, so, $\Phi$ is determined by $\Phi(\theta_{T_n T^{-n}})$. Moreover, $\Phi(\theta_{T_n T^{-n}}) \in \text{Hom}_{\mathcal{H}}(\mathcal{H}, N(\beta)) \cong N(\beta)$ since $\Phi$ maps weight spaces to weight spaces. Therefore, $\Phi(\theta_{T_n T^{-n}}) \in \text{Hom}_{\mathcal{H}}(N(\alpha), N(\beta))$ since $\Phi$ is an $\mathcal{S}(\Lambda)$–module homomorphism and $\mathcal{H} \cong \text{Hom}_{\mathcal{H}}(\mathcal{H})$ (where we identify $h \in \mathcal{H}$ with left multiplication by $h$). Hence, we can write $\Phi(\theta_{T_n T^{-n}}) = \sum_{S, T} r_{ST} \varphi'_{ST}$ for some $r_{ST} \in R$ by Proposition 4.3 (i). Therefore, $\Phi = \sum_{S, T} r_{ST} \Phi_{ST}$, completing the proof.

If $\Lambda$ is a saturated set of multicompositions let

$$E(\Lambda) = \bigoplus_{\alpha \in \Lambda} E(\alpha) = \text{Hom}_{\mathcal{H}}(M(\Lambda), N(\Lambda)) = \bigoplus_{\mu, \alpha \in \Lambda} \text{Hom}_{\mathcal{H}}(M(\mu), N(\alpha)).$$

Then $E(\Lambda)$ is an $(\mathcal{S}(\Lambda), \mathcal{S}(\Lambda))$–bimodule. Moreover, it has the following double centralizer property.

If $\Lambda$ is an algebra let $A^\text{op}$ be the opposite algebra in which the order of multiplication is reversed.

7.2. Corollary. Suppose that $\omega \in \Lambda$. Then there are canonical isomorphisms of $R$–algebras

$$\text{End}_{\mathcal{S}(\Lambda)}(E(\Lambda)) \cong \mathcal{S}(\Lambda)^{\text{op}} \quad \text{and} \quad \text{End}_{\mathcal{S}(\Lambda)}(E(\Lambda)) \cong \mathcal{S}(\Lambda).$$

Proof. The first isomorphism, $\Phi_{ST} \mapsto \varphi'_{ST}$, is immediate from (the proof of) Proposition 7.1. The second isomorphism follows by symmetry.

As a special case we have a description of the Ringel dual of $\mathcal{S}(\Lambda)$.
7.3. Corollary. Suppose that $R$ is a field, $\omega \in \Lambda$ and that $Q_1, \ldots, Q_r$ are distinct invertible elements of $R$. Then the Ringel dual of $\mathcal{H}(\Lambda)$ is isomorphic to $\mathcal{H}(\Lambda)^\varphi$.

Finally, we want to determine the $\nabla$–filtration multiplicities in the tilting modules. We actually don’t need to do any work here because the general theory of tilting modules for quasi–hereditary algebras tells us that $[T(\lambda) : \nabla(\mu)] = [\Delta(\mu'), L(\lambda')]$ (see, for example, [10, Appendix]); however, we want to show how this result can be derived using Young modules and Specht filtrations.

Recall that $F_\omega : \mathcal{H}(\Lambda)\text{-mod} \to \mathcal{H}(\Lambda)\text{-mod}; M \mapsto M_{\varphi, \omega}$ is the Schur functor.

7.4. Proposition. Suppose that $R$ is a field, $\omega \in \Lambda$ and $\lambda$ be a multipartition. Then $F_\omega\bigl(T(\lambda')\bigr) \cong Y'(\lambda)$ as $\mathcal{H}$–modules.

Proof. Applying the definitions $F_\omega(E(\lambda)) \cong \text{Hom}_{\mathcal{H}}(\mathcal{H}, N(\lambda)) \cong N(\lambda)$, where the last isomorphism comes from Lemma 6.3 (or directly). Hence, the Schur functor $F_\omega$ induces an injective map from $\text{End}_{\mathcal{H}(\Lambda)}(E(\lambda), E(\beta))$ to $\text{End}_{\mathcal{H}(\Lambda)}(N(\lambda), N(\beta))$; by Proposition 7.3(i) and Proposition 7.4 this is an isomorphism. Consequently, if an indecomposable tilting module $T(\lambda')$ is a direct summand of $E(\lambda)$ then $F_\omega(T(\lambda'))$ is an indecomposable direct summand of $N(\lambda)$. Therefore, by Proposition 4.4, $F_\omega(T(\lambda')) \cong Y'(\mu)$ for some multipartition $\mu$. Now, $E(\lambda) \cong T(\lambda') \oplus \bigoplus_{\nu \vdash \mu} T(\mu)^{\varphi, \nu}$ by Theorem 6.17(i) and $F_\omega(E(\lambda)) \cong N(\lambda) \cong Y'(\lambda) \oplus \bigoplus_{\nu \vdash \mu} Y'(\nu)^{\varphi, \nu}$ by Proposition 4.4(i). Hence, the result follows by induction on the dominance ordering.

7.5. Corollary. Suppose that $R$ is a field, $\omega \in \Lambda$ and that $Q_1, \ldots, Q_r$ are distinct non–zero elements of $R$. Let $\lambda$ and $\mu$ be multipartitions of $n$. Then $[T(\lambda') : \nabla(\mu')] = [\Delta(\mu) : L(\lambda)]$.

Proof. Now $F_\omega(\Delta(\nu)) \cong S(\nu)$ by (2.10)(i); therefore, as $F_\omega$ projects onto the $\omega$–weight space, $F_\omega(\nabla(\nu)) = F_\omega(\Delta(\nu)\varphi) \cong S(\nu)^\varphi$. Consequently, we have

$$[T(\lambda') : \nabla(\mu')] = [F_\omega(T(\lambda')) : F_\omega(\nabla(\mu'))],$$

$$= [Y'(\lambda) : S(\mu')^\varphi] \quad \text{by Proposition 7.4},$$

$$= [Y'(\lambda) : S'(\mu)], \quad \text{by Corollary 5.7},$$

$$= [\Delta(\mu) : L'(\lambda)], \quad \text{by Proposition 4.4(iv)},$$

$$= [\Delta(\mu) : L(\lambda)],$$

where the last equality follows because the isomorphism $\mathcal{H}(\Lambda) \cong \mathcal{H}(\Lambda)$ of Proposition 4.3(iii) identifies the Weyl modules $\Delta(\mu)$ and $\Delta'(\mu)$, and the simple modules $L(\lambda)$ and $L'(\lambda)$.

To conclude, we remark that in the case of the $q$–Schur algebras (i.e. when $r = 1$) our proof of Corollary 7.5 looks quite different to Donkin’s [8]; however, in spirit the two arguments are the same in that they both rely on a duality between the symmetric and exterior powers and on the isomorphism of Proposition 4.3(iii).

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