The torsional contact of coated bodies

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Abstract. This paper advances an algorithm for the simulation of the elastic contact undergoing a constant normal force and an oscillating torsional moment, under the assumption of partial slip. The solution of this type of contact can be achieved with a numerical formulation based on influence coefficients that express the contribution of elementary rectangular patches of uniform surface tractions, normal or tangential, to the displacement field induced in the elastic half-space. Summation of these elementary contributions allows for the calculation of the displacement field for arbitrary distributions of tractions, thus allowing for an iterative problem resolution that can handle arbitrary contact geometry, various frictional regimes and consideration of coated materials. The displacement response of multi-layered bodies is calculated based on the frequency response functions, and a previously advanced technique for calculation of convolution products in the Fourier transform domain is implemented for increased accuracy. The solution of the tangential contact problem is achieved by the treatment of the arising linear system of equations with the conjugate gradient method, which provides superior computational efficiency. The interaction between the normal and the tangential contact tractions is discussed. The influence of the Young modulus ratio between the coating and the substrate on the shear tractions in the partial slip torsional contact is investigated through numerical simulations.

1. Introduction
Existing studies of the torsional contact involve the coupling of the well-known Hertz solution with the finding of the shear tractions arising in the partial slip contact subjected to a torsional moment. The latter contact scenario is covered in the literature [1-3] under strong limiting assumptions. The torsional contact solution for a sphere in partial slip was presented by Johnson [4] based on the theoretical framework developed by Mindlin [5]. This solution was later enhanced by Dintwa et al. [6] and applied to viscoelastic materials. More recent developments in the numerical study of fretting [7] addressed the problem of circumferential slip, i.e. the fretting mode III.

The contact of coated bodies was extensively studied [8-10] based on the mathematical framework developed by Burminster [11], who pioneered the study of multi-layered bodies. In case of the contact problem, the lack of analytical solutions encourages the semi-analytical treatment by employing modern numerical tools such as the fast Fourier transform [12-14]. The latter technique is particularly advantageous for the study of multi-layered materials, for which most of the results are obtained [8-11] in closed-form only in the frequency domain, whereas their space domain counterparts are lacking.
Spinu and Glovnea [15] advanced an efficient algorithm for the study of the fretting modes I and II, i.e. a constant normal force and an oscillating tangential force, involving homogeneous bodies with different elastic properties. The latter algorithm is extended in this paper to allow for a different loading program, consisting in a constant normal force and an oscillating torsional moment, thus reproducing the conditions of the fretting mode III. Moreover, the assumption of homogeneity is lifted to allow for a more general solution involving coated materials. To this end, a recently developed technique [16] for the calculation of displacements in multi-layered materials is integrated into the contact solver [15] for homogeneous bodies.

The end result is a computer simulation technique that is very efficient from the point of view of algorithmic complexity, due to the use of state-of-the-art numerical tools. The precision of the advanced method is first checked against existing analytical solutions [6] for specific contact scenarios, and new results for more general cases that lack analytical solution are then obtained through numerical simulation. The method ability to handle various contact configurations and loading program makes it a good candidate for the future study of the fretting processes.

2. Model outline

The starting point in the analysis of the torsional contact of coated bodies is the algorithm [15] developed for the slip-stick contact between dissimilar elastic materials. The latter method allows for contact simulations with loading regimes resulting in transversal or radial slip (i.e., fretting modes I and II, respectively). In the latter cases, the load consists in a constant normal force and an oscillating tangential force, with no moments transmitted through the contact. A different scenario, corresponding to a fretting mode III, i.e. circumferential slips, is treated in this paper. To this end, the loading program, depicted in figure 1, consists in a constant normal force $W$ and an oscillating torsional moment $M_z$ applied along the normal direction $z$. The time length of a single loading from zero is denoted by $\tau$, which is used as normalizer for time moments. No tangential force is applied, yet shear tractions arise on the contact area due to: (a) the mismatch between the elastic properties of the contacting bodies, and (b) the transmitted moment $M_z$. The contact schematic is depicted in figure 2, showing an elastic homogeneous spherical indenter loaded against a coated half-space. Although the indenter and the half-space substrate are assumed similar elastic, having a Young modulus $E_2$ and a Poisson’s ratio $\nu_2$, the discrepancy in elastic properties stems from the elastic properties of the coating, $E_1$ and $\nu_1$. The simulations conducted in this work assume that the ratio $\nu_1/\nu_2$ is kept constant and equal to unity, whereas the Young modulus mismatch $E_1/E_2$ is varied.

![Figure 1. The loading history.](image1)

![Figure 2. Schematic of the torsional contact.](image2)
The framework for the study of the slip-stick contact is well developed, although analytical solutions exist only under strong limiting assumptions concerning: (a) the contact geometry, limited to a few cases, and (b) the connection between the normal and the shear tractions $p, q_x$ and $q_y$. Numerical analysis may be employed to relieve these limitations and to advance the understanding of the contact processes. The connection between the contact model equations in the normal and in the tangential direction is best suggested by the equation of displacements $u_x, u_y$ and $u_z$ arising in the slip-stick contact, when the technique of influence coefficients is used:

$$
\begin{bmatrix}
  u_x \\
  u_y \\
  u_z
\end{bmatrix} = \begin{bmatrix}
  K_{xx} & K_{xy} & K_{xz} \\
  K_{yx} & K_{yy} & K_{yz} \\
  K_{zx} & K_{zy} & K_{zz}
\end{bmatrix} \otimes \begin{bmatrix}
  q_x \\
  q_y \\
  p
\end{bmatrix},
$$

(1)

where $K_{ij}, i, j = x, y, z$ is the matrix of influence coefficients expressing the contribution of contact tractions along the $j$-axis to the displacement $u_i$, and symbol $\otimes$ denotes convolution product. These influence coefficients are analytical solutions obtained by integration of the fundamental solutions derived for point forces acting on the elastic half-space boundary (i.e., the Boussinesq and Cerruti solutions). In this paper, the problem discretization entails a rectangular computational domain that is discretized into elementary rectangular patches of equal size. The closed-form solutions for the influence coefficients $K_{ij}$ are based on the solution obtained by Love [18] and Gallego et al. [7].

Contact problem discretization results in the ability to compute the convolution product in the equation (1), for arbitrary tractions distributions and contact area. It should be noted that the analytical counterpart of equation (1) in the continuous model is an integral having the Boussinesq and Cerruti solutions $G_q$ as kernels:

$$
\int G_q(x-x', y-y') q_j(x', y') dx' dy'.
$$

(2)

The main idea of the algorithm developed in [15] is to split the matrix equation (1), having the contact tractions as unknown, into two equations: the first for the normal contact direction, having the pressure $p$ as unknown, and the second for the tangential direction, with the shear tractions $q_x$ and $q_y$ as unknowns. In each case, the static force equilibrium, the surface separation equation and the boundary conditions should be added to form a system with a unique solution. The resulting two sets of equations are not independent, but the global solution can be achieved via an iterative process as suggested in [19].

The same algorithmic strategy can be applied to the contact scenario considered in this paper, except for: (a) the loading program, which entails a vanishing tangential forces and an oscillating torsional moment, and (b) the lower contacting body is a coated body for which there exist no fundamental solutions of the Boussinesq-Cerruti type. The algorithm modifications to overcome these variations are detailed in the following sections.

3. Solution of the tangential sub-problem involving torsion

The subset of equations describing the contact in the normal direction is similar to the one presented in [15,17], and can be solved in the same manner. The contact equations in the tangential direction, for the loading program depicted in figure 1, will be restated here for clarity and completion:
\[
\begin{bmatrix}
s_x(i, j) \\
s_y(i, j)
\end{bmatrix}
= \begin{bmatrix}
u_x(i, j) \\
u_y(i, j)
\end{bmatrix} - \phi_2 \begin{bmatrix}
y(i) \\
x(j)
\end{bmatrix}, \quad (i, j) \in A_C;
\]

(3)

\[
\|q(i, j)\| \leq \mu p(i, j), \|\sigma(i, j)\| = 0, \quad (i, j) \in A_g;
\]

(4)

\[
\|q(i, j)\| = \mu p(i, j), \|\sigma(i, j)\| > 0, \quad (i, j) \in A_C - A_g;
\]

(5)

\[
\sum_{(i, j) \in A_C} \left[ x(j)q_y(i, j) + y(i)q_x(i, j) \right] = M_z/\Delta,
\]

(6)

where \(i\) and \(j\) are indexes for positioning of the referred elementary cell in the computational domain, \(A_C\) and \(A_g\) the contact and the stick area, respectively (i.e., the set of elementary patches for which a contact or slip status, respectively, is assigned), \(\Delta\) is the elementary cell area, \(\phi_2\) the rigid-body angle of torsion, \(\mu\) the frictional coefficient (assumed uniform over the contact area), \(\sigma(s_1, s_2)\) the vector of the relative slip distances, and \(\|\mathbf{x}\|\) the norm of any vector \(\mathbf{x}\). Equation (3) compares geometry parameters after the deformation, by relating the initial and the final positions of various particles on the contact surface. Equations (4) and (5) impose the boundary conditions: cells within the stick region \(A_g\) have vanishing slip and the norm of the shear tractions is related to pressure by a static friction law, whereas cells in the slip region \(A_C - A_g\) have non-vanishing norm of the slip vectors, and the tractions obey a kinetic friction law. Equation (6) expresses the static moment equilibrium.

The system of equations (3) - (6) must be solved in \(q\), under the assumption that the pressure distribution \(p\) and the contact area \(A_C\) are known from the solution of the normal contact sub-problem, but otherwise can be arbitrary. By plugging equation (1) into (3), a linear system is obtained, subjected to the constraints (4) - (6). The latter system should be solved to obtain the shear tractions \(q\) in the stick zone \(A_g\). Additional difficulty arise because the stick region is also unknown, in other words, the size of the system is not known a priori. From this point of view, the situation is similar to that of the normal contact sub-problem, in which the pressure should be calculated without knowing the contact area in advance. For the latter model, an algorithm was advanced [17] that iterates simultaneously the pressure distribution and the contact area, in a single loop scheme. A similar approach is taken in this paper, by concurrent stabilisation of the stick area and of the related shear tractions, whereas the resolution of the linear system is performed with the Conjugate Gradient Method (CGM). A detailed algorithm description is provided below.

The iterative process starts with the assumption that all contact area is in stick. The initial approximation for the shear tractions should be linear in coordinates:

\[
\begin{cases}
q_x(i, j) = y(i) \cdot a; \\
q_y(i, j) = x(j) \cdot a,
\end{cases}
\]

(7)

with the slope \(a\) obtained by plugging equation (7) into (6). A set of auxiliary variables are initialized as follows: \(\theta = 0, S_0 = 1, \mathbf{d}(i, j) = \mathbf{0}, (i, j) \in A_C\). The loop consists in the following steps:

1. Compute the displacement field according to equation (1).
2. Estimate the angle of torsion \(\phi_2\), considering that:

\[
\begin{bmatrix}
u_x(i, j) \\
u_y(i, j)
\end{bmatrix} - \phi_2 \begin{bmatrix}
y(i) \\
x(j)
\end{bmatrix} = 0, \quad (i, j) \in A_g.
\]

(8)
By defining a new function $N(\phi_2)$:

$$N(\phi_2) = \left[ u_x(i,j) - \phi_2 \cdot y(i) \right]^2 + \left[ u_y(i,j) - \phi_2 \cdot x(j) \right]^2, (i,j) \in A_{St},$$

the optimal (i.e., that verifies best the system (8)) value can be derived by setting the partial derivative $\partial N/\partial \phi_2$ to zero, resulting in:

$$\phi_2 = \left\{ \sum_{(i,j) \in A_{St}} \left[ u_x(i,j) y(i) + u_y(i,j) x(j) \right] \cdot \left\{ \sum_{(i,j) \in A_{St}} \left[ (x(j))^2 + (y(i))^2 \right] \right\}^{-1} \right\}. \quad (10)$$

3. Compute the relative slip distances $s$ from equation (3) and the square sum $S$:

$$S = \sum_{(i,j) \in A_{St}} \left[ s_x^2(i,j) + s_y^2(i,j) \right]. \quad (11)$$

4. Calculate the descent direction $d$ from the CGM, by taking into consideration only data from the stick area:

$$\begin{bmatrix} d_x(i,j) \\ d_y(i,j) \end{bmatrix} = \begin{bmatrix} s_x(i,j) \\ s_y(i,j) + \frac{S}{S_0} d_x(i,j) + d_y(i,j) \end{bmatrix}, (i,j) \in A_{St};$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}^T, (i,j) \in A_c - A_{St}. \quad (12)$$

5. Store $S$ for the subsequent iteration: $S_0 = S$.

6. Compute the step length $\alpha$ along the descent direction according to the CGM:

$$\alpha = \left[ s_x \ s_y \right] \cdot \left[ d_x \ d_y \right]^T \cdot \left[ \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \otimes \begin{bmatrix} d_x \\ d_y \end{bmatrix} \right]^T \cdot \left[ \begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} \otimes \begin{bmatrix} d_x \\ d_y \end{bmatrix} \right]^{-1}. \quad (13)$$

7. Store the current solution $q$ in a different variable for convergence check: $q_0 = q$.

8. Improve the solution precision according to the CGM: $q \leftarrow q - \alpha \cdot d$.

9. Impose the boundary conditions (4) and (5) to adjust the system size (i.e., the stick area $A_{St}$).

Elementary cells $(i,j)$ having the norm of the shear tractions greater than the product between pressure and the frictional coefficient are removed from the stick region, and the associated shear tractions are forced to the upper value of frictional slip:

$$A_{St} \leftarrow A_{St} - \{(i,j) : \| q(i,j) \| > \mu p(i,j) \} ; \quad (14)$$

$$q(i,j) \leftarrow \mu p(i,j) \frac{q(i,j)}{\| q(i,j) \|}, (i,j) \in \{(i,j) : \| q(i,j) \| > \mu p(i,j) \} . \quad (15)$$

Cells with the slip vectors $s(i,j)$ not opposite to the shear tractions $q(i,j)$ are reincluded in $A_{St}$:

$$A_{St} \leftarrow A_{St} \cup \{(i,j) : q(i,j)s(i,j) > 0 \} . \quad (16)$$
10. Adjust the solution to obey the static equilibrium:

\[
\begin{bmatrix}
q_x(i, j) \\
q_y(i, j)
\end{bmatrix} = \begin{bmatrix}
q_x(i, j) + y(i) \cdot b \\
n \cdot b
\end{bmatrix}, \quad (i, j) \in A_N,
\]  

(17)

where the slope \( b \) results by plugging equation (17) into (6):

\[
b = \left\{ \frac{M_z}{\Delta} - \sum_{(i, j) \in D_R} \left[ q_x(i, j)x(j) + q_y(i, j)y(j) \right] \right\}^{-1} \left\{ \sum_{(i, j) \in A_S} \left[ (x(j))^2 + (y(j))^2 \right] \right\}^{-1}.
\]  

(18)

11. Check convergence and perform a new iteration if the precision goal \( \varepsilon \) is not attained:

\[
\sum_{(i, j) \in A_C} \left\| \mathbf{q}^{(i)} - \mathbf{q}^{(i)}_0 \right\| \leq \varepsilon.
\]  

(19)

With this algorithm, the shear tractions in the torsional contact can be computed for arbitrary contact geometry, pressure distribution or frictional coefficient. The latter was assumed uniform without loss of generality, as mapped distributions can be equally considered in equations (4) & (5).

4. Displacement computation for coated bodies

The second variation from the case considered in [15] is that the lower contacting body is a coated half-space. As opposed to the homogeneous case, no analytical solution for the fundamental solutions \( G_y \) from equation (2) is available in the time/space domain. The Fourier transforms of the functions \( G_y \) are however readily available in closed-form [8,9] in the frequency domain, and are known in the literature as the frequency response functions (FRF). The FRFs can be used to compute the convolutions arising in equation (1) as described in [16]. Only an algorithm outline is given here, with description of the main steps:

1. Establish an extended virtual domain for displacement calculation. It should be noted that an extension of the discretized region in the time/space domain with an imposed ratio translates to a decrease of the data interval in the Fourier transform domain with the same factor. The increase in resolution in the spectral domain is required to minimize the aliasing phenomenon. This step concludes with the creation of a mesh in the spectral domain.

2. Compute the discrete values of the FRF in the nodes of the newly created mesh.

3. Obtain the zero-padded digitized tractions distribution and calculate the discrete Fourier transform of the latter by means of the fast Fourier Transform.

4. Compute the element-wise product between the spectral series created in steps 2 and 3.

5. Rearrange the result in wrap-around order.

6. Transfer the result back to the time/space domain by means of the inverse fast Fourier transform.

7. Retain only the terms in the considered computational domain (as opposed to the extended virtual domain created in step 1).

This method can be used for the calculation of displacements in the coated half-space according to equation (1), and its precision was extensively verified [16].

5. Results and discussions

The novel algorithm is first validated against the existing analytical solution for the slip-stick torsional contact involving homogeneous contacting bodies. In a cylindrical coordinate system \((r, \theta, z)\), the
shear circumferential stress \( q_\theta(r) \) due to a torsional moment \( M_z \) subsequent to a normal indentation resulting in a Hertz contact radius \( a_H \), can be expressed as [6]:

\[
q_\theta(r) = \begin{cases} 
\frac{3\mu W}{2\pi a_H^3} \left[ 1 - \left( \frac{r}{a_H} \right)^2 \right]^2 \left[ \frac{\pi}{2} + k^2 D(k) F(k^*, \varphi) - K(k^*) E(k^*, \varphi) \right], & r \leq a_S; \\
\frac{3\mu W}{2\pi a_H^3} \left( 1 - \left( \frac{r}{a_H} \right)^2 \right)^2, & a_S \leq r \leq a_H, 
\end{cases}
\tag{20}
\]

where \( a_S \) is the stick radius, and

\[
k^* = a_S/a_H; \quad k = \sqrt{1 - (k^*)^2}; \quad \varphi = \arcsin \left[ \frac{1}{k^2} \sqrt{\frac{(k^*)^2 - (r/a_H)^2}{1 - (r/a_H)^2}} \right]. \tag{21}
\]

\( F(k^*, \varphi) \) and \( E(k^*, \varphi) \) are the incomplete elliptical integrals of first and of the second kind, of modulus \( k^* \) and amplitude \( \varphi \), \( K(k) \) and \( E(k) \) are the complete elliptical integrals of the first and of the second kind, and \( D(k) = (K(k) - E(k))/k^2 \). The maximum value for the torsional moment, i.e. the value for which gross-slip is imminent, results as \( M_{z_{\text{max}}} = 3\pi \mu W a_H / 16 \).

The distributions of shear tractions in the plane \( y = 0 \) for \( E_1/E_2 = 1 \) are presented in figure 3(a). The curves are anti-symmetric with respect to the \( z \)-axis, and therefore are depicted only for \( x/a_H > 0 \). The predictions of the numerical program, displayed with dashed lines, agree with the analytical distributions given by equations (20) and (21), depicted using continuous black lines. The shear tractions for specific time moments from the loading program are presented in figure 3(b). There exist a periodicity suggested by the overlapping of the curves corresponding to times \( t/\tau = 1 \) and \( t/\tau = 5 \). Extending the loading program past \( t/\tau = 5 \) results in the same states as for the first loading/unloading cycle. The influence of the elastic modulus ratio \( E_1/E_2 \) on the shear tractions is presented in figure 4.

![Figure 3. Shear tractions in the contact of homogeneous materials: (a) validation; (b) loading history.](image-url)
6. Conclusions

An existing numerical model for the simulation of slip-stick elastic contacts undergoing fretting modes I and II is enhanced in this paper to allow for circumferential slip, i.e., fretting mode III. To this end, the numerical solution of the tangential contact problem is modified to account for the new loading program, consisting in a constant normal force and an oscillating torsional moment. Moreover, the numerical model is extended to allow for the contact of coated bodies, which requires a special procedure for displacement calculation based on exiting analytical solutions in the Fourier transform domain.

The conjugate gradient method is employed to solve the tangential contact problem. The numerical contact solution is compared with the existing analytical result for the torsional contact between homogeneous and similar elastic materials, and a good agreement is found. A loading/unloading cycle is simulated and a periodicity is found in the distributions of shear tractions. Additional simulations show the influence of the elastic modulus ratio between the coating and the substrate on the distribution of shear tractions.

The conducted simulations prove the method potential for the study of fretting contact processes with a moderate computational effort, provided by the use of modern numerical tools such as the fast Fourier transform. A study of the subsurface stresses arising in the coated system under the considered load, with localisation of the global maximum of the equivalent stress, is proposed for a future work.

7. References

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