Variations on the theme of Marcinkiewicz’ inequality

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1 Introduction

Let $h$ be a smooth function on $\mathbb{R}$ with a compact support, and

$$g(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{h(t)}{t-x} \, dt$$

be its Hilbert transform. Set $f = g + ih$ and introduce the function

$$u_f(z) = \int_{\mathbb{R}} H(zf(t)) \, dt, \quad H(z) = \log |1-z| + \text{Re}(z),$$

called the logarithmic determinant of genus one. It is subharmonic in $\mathbb{C}$, and its Riesz measure is $d\mu_f(\zeta) = d\nu_f(\zeta^{-1})$, where $d\nu_f$ is the distribution measure of $f$:

$$\nu_f(E) = \text{meas} \left( \{ t : f(t) \in E \} \right), \quad E \text{ is a borelian subset of } \mathbb{C},$$

and $\text{meas}(\cdot)$ stands for the Lebesgue measure on $\mathbb{R}$. Let

$$\mu_f(r) = \mu_f(\{|z| \leq r\}) = \text{meas} \left( \{|f| \geq r^{-1} \} \right)$$

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be a (conventional) counting function of $d\mu_f$, and let

$$n_f(r) = \mu_f (\{|z - ir/2| \leq r/2\}) + \mu_f (\{|z + ir/2| \leq r/2\})$$

$$= \mu_f (\{|\text{Im}(z^{-1})| \geq r^{-1}\}) = \text{meas} (\{|h| \geq r^{-1}\})$$

be its Levin-Tsuji counting function, see [9], [16], and [5, Chapter 1]. Then the classical estimates of the Hilbert transform can be easily rewritten as upper bounds of $\mu_f(r)$ by $n_f(r)$.

For example, Marcinkiewicz’ inequality (see [7, Chapter V])

$$m_f(\lambda) \leq C \left\{ \frac{1}{\lambda^2} \int_0^\lambda \text{sm}_h(s)ds + \frac{1}{\lambda} \int_\lambda^\infty m_h(s)ds \right\}, \quad 0 < \lambda < \infty, \quad (1.1)$$

where

$$m_f(\lambda) = \text{meas} (\{|f| \geq \lambda\}) = \nu_f (\{|w| \geq \lambda\}) = \mu_f(\lambda^{-1}),$$

and

$$m_h(\lambda) = \text{meas} (\{|h| \geq \lambda\}) = \nu_f (\{|\text{Im}w| \geq \lambda\}) = n_f(\lambda^{-1}),$$

reads:

$$\mu_f(r) \leq C \left\{ r \int_0^r \frac{n_f(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{n_f(t)}{t^3} \, dt \right\}, \quad 0 < r < \infty. \quad (1.2)$$

From this, one readily obtains

$$\mu_f(r) \leq Cr \int_0^\infty \frac{n_f(t)}{t^2} \, dt, \quad 0 < r < \infty, \quad (1.3)$$

which is equivalent to Kolmogorov’s weak $L^1$-type inequality: $\lambda m_f(\lambda) \leq C||h||_{L^1}$, $0 < \lambda < \infty$, and

$$\int_0^\infty \frac{\mu_f(t)}{t^{p+1}} \, dt \leq C(p) \int_0^\infty \frac{n_f(t)}{t^{p+1}} \, dt, \quad 1 < p < 2, \quad (1.4)$$

which is equivalent to M. Riesz’ inequality:

$$\|f\|_{L^p} \leq C(p)||h||_{L^p}. \quad (1.5)$$

The classical proof of inequality (1.1) is based on the interpolation technique which later served as one of the cornerstones of an abstract theory of interpolation of operators in Banach spaces [2].
A natural question arises: what is special about the subharmonic function
\( u_f(z) \) which makes inequality (1.2) valid? The answer is proposed in [12]: a
key fact ensuring this, is the positivity condition:

\[
\begin{align*}
u_f(z) &\geq 0, & z \in \mathbb{C},
\end{align*}
\]

which can be easily checked with the help of Green’s formula (see [3, Lemma 5])
or by applying of the Cauchy residue theorem (see [13]).

This leads to a heuristic principle which suggests that

- results about the distribution of the Hilbert transform can be deduced
  from inequality (1.6) by using methods of the subharmonic function
  theory.

As will be shown, inequality (1.6) is even too strong, and in many cases
it suffices to assume that \( u_f(x) \geq 0 \) on \( \mathbb{R} \), or to control the negative part
\( u_f^- = \max(0, -u_f) \) on \( \mathbb{R} \).

The principle shows a path to new results on the Hilbert transform. In
[13], its application leads to a complete description of the distribution of the
Hilbert transform of \( L^1 \)-functions and measures of finite variations. At the
same time, putting known estimates of the Hilbert transform into this setting,
we arrive at new interesting questions about the argument-distribution of the
Riesz measure in certain classes of subharmonic functions. For example, the
proofs of the inequalities of Kolmogorov and M. Riesz found in [12] give new
bounds for zeros of polynomials. Positivity condition (1.6) links our work
with the theory of uniform algebras and Jensen measures (see [4]).

In this work we put forward a new approach to the Marcinkiewicz in-
equality (1.1) (or (1.2)). The methods applied in [12], [14] are too rigid for
this. Here we use a different technique.

Here and later on, we use the following notations:
\( \phi(s) \lesssim \psi(s) \) means that there is a positive numerical constant \( C \) such that,
for each \( s > 0 \), \( \phi(s) \leq C\psi(s) \);
\( \phi(s) \lesssim_\alpha \psi(s) \) means the same as above but \( C \) may depend on a parameter \( \alpha \);
\( H(z) = \log |1 - z| + \text{Re}(z) \) is the canonical kernel of genus one;
\( \mathbb{C}_\pm \) are the upper and lower half-planes.
2 Main results

Define a subharmonic canonical integral of genus one:

\[ u(z) = \int_C H(z/\zeta) \, d\mu(\zeta), \quad (2.1) \]

where \( d\mu \) is a non-negative locally finite measure on \( \mathbb{C} \) such that

\[ \int_C \min\left(\frac{1}{|\zeta|}, \frac{1}{|\zeta|^2}\right) \, d\mu(\zeta) < \infty. \quad (2.2) \]

Let \( \mu(r) = \mu(\{|z| \leq r\}) \) be a (conventional) counting function of the measure \( d\mu \), and let \( n(r) = \mu(\{|\text{Im}(1/z)| \geq 1/r\}) = \mu(\{|z - ir/2| \leq r/2\}) + \mu(\{|z + ir/2| \leq r/2\}) \) be its Levin-Tsuji counting function [9], [16] (see also [3, Chapter 1]).

Let \( M(r,u) = \max_{|z| \leq r} u(z) \). Then by the Jensen inequality, \( \mu(r) \lesssim M(r,u) \). In the opposite direction, a standard estimate of the kernel

\[ H(z) \lesssim \frac{|z|^2}{1 + |z|}, \quad z \in \mathbb{C}, \]

yields Borel’s estimate

\[ M(r,u) \lesssim r \int_0^r \frac{\mu(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{\mu(t)}{t^3} \, dt. \quad (2.3) \]

In particular,

\[ M(r,u) = \begin{cases} 
  o(r), & r \to 0 \\
  o(r^2), & r \to \infty. 
\end{cases} \quad (2.4) \]

**Theorem 1.** Let \( u(z) \) be a canonical integral (2.1) of genus one, then

\[ M(r,u) \lesssim r \int_0^r \frac{n(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{n(t)}{t^3} \, dt + r^2 \int_r^\infty \frac{m(t,u)}{t^2} \, dt, \quad (2.5) \]

where

\[ m(r,u) = \frac{1}{2\pi} \int_0^{2\pi} u^{-1}(re^{i\theta}|\sin \theta|) \frac{d\theta}{\sin^2 \theta} \]
is the Tsuji proximity function of $u$.

If the function $u$ is non-negative in $\mathbb{C}$, then the proximity function $m(r, u)$ vanishes, and applying Jensen’s inequality we arrive at

**Corollary 1.** Let $u$ be a canonical integral of genus one which is non-negative in $\mathbb{C}$. Then

$$
\mu(r) \lesssim r \int_0^r \frac{n(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{n(t)}{t^3} \, dt.
$$

As we explained in the introduction, this result immediately yields the classical Marcinkiewicz inequality (1.1). In this case one can apriori assume that the function $f$ is bounded, so that most of the technicalities needed for the proof of Theorem 1 (see Lemmas 2 and 4 below) are redundant, and our proof of Marcinkiewicz’ inequality, being conceptually new, is comparable in length to the classical one.

There is a curious reformulation of Corollary 1. Let $\mathcal{M}$ be a measurable space endowed with a locally finite non-negative measure $dm$, let $f$ be a complex-valued measurable function on $\mathcal{M}$, and let $m_f(\lambda) = m(\{|f| \geq \lambda\})$ be the distribution function of $f$. If

$$
\int_{\mathcal{M}} \min(|f|, |f|^2) \, dm < \infty,
$$

then we define the logarithmic determinant of $f$ of genus one

$$
u_f(z) = \int_{\mathcal{M}} H(zf(t)) \, dm(t),
$$

which is subharmonic in $\mathbb{C}$, and moreover is represented by a canonical integral of genus one. Applying Corollary 1, we obtain

**Corollary 2.** If $f$ satisfies condition (2.6), and the logarithmic determinant $u_f$ is non-negative in $\mathbb{C}$, then

$$
m_f(\lambda) \lesssim \left\{ \frac{1}{\lambda^2} \int_0^\lambda \text{sm}_{\text{Im}} f(s) \, ds + \frac{1}{\lambda} \int_\lambda^\infty \text{sm}_{\text{Im}} f(s) \, ds \right\}, \quad 0 < \lambda < \infty.
$$

In particular,

$$
\|f\|_{L^p(m)} \lesssim_p \|\text{Im} f\|_{L^p(m)}, \quad 1 < p < 2,
$$
and
\[ m_f(\lambda) \lesssim \frac{||\text{Im} f||_{L^1(\lambda)}}{\lambda}, \quad 0 < \lambda < \infty. \]

Corollary 1 can also be applied to Jensen measures in \( \mathbb{C} \). A compactly supported finite measure \( \sigma \) in \( \mathbb{C} \) is called a \textit{Jensen measure} (with respect to the origin) if for an arbitrary subharmonic function \( h \) in \( \mathbb{C} \)
\[ h(0) \leq \int h d\sigma. \tag{2.7} \]

A simple argument shows that (2.7) then holds true for subharmonic functions in a domain \( G \) such that \( 0 \in G \) and \( \text{supp}(\sigma) \subset G \). For a harmonic function, the equality sign must occur in (2.7). Therefore,
\[ \sigma(\mathbb{C}) = \int 1 d\sigma = 1, \]
that is, \( \sigma \) is a probability measure, and
\[ \int \zeta^k d\sigma(\zeta) = 0, \quad k = 1, 2, \ldots. \tag{2.8} \]

Define the potential
\[ V_\sigma(z) = \int \log |1 - z\zeta| d\sigma(\zeta). \tag{2.9} \]

Then, due to (2.7) and (2.8),
\[ 0 \leq V_\sigma(z) \leq \log^+(c|z|), \quad z \in \mathbb{C}, \tag{2.10} \]
for some \( c > 0 \). The opposite is also true: if, for some \( c > 0 \), a subharmonic function \( V \) satisfies (2.10), then it is a potential of a of a Jensen measure \( \sigma \), [§4].

Due to condition (2.8), every potential \( V_\sigma \) of a Jensen measure can be represented by a canonical integral of genus one:
\[ V_\sigma(z) = \int_{\mathbb{C}} H(z/\zeta) d\mu(\zeta), \quad d\mu(\zeta) = d\sigma(1/\zeta). \]

Thus, Theorem 1 is applicable to the potential \( V_\sigma \), and we obtain
Corollary 3. Let \( \sigma \) be a Jensen measure in \( \mathbb{C} \), \( \sigma(\lambda) = \sigma(|z| \geq \lambda) \), \( \sigma_I(\lambda) = \sigma(|\text{Im} z| \geq \lambda) \). Then

\[
\sigma(\lambda) \lesssim \frac{1}{\lambda^2} \int_0^\lambda s \sigma_I(s) \, ds + \frac{1}{\lambda} \int_\lambda^\infty \sigma_I(s) \, ds.
\]

The class of Jensen measures is invariant with respect to the holomorphic mappings. More precisely, let \( G \) be a domain which contains the origin and \( \text{supp}(\sigma) \), and let \( F \) be an analytic function in \( G \), \( F(0) = 0 \). Then the push forward \( F_*\sigma \) is defined by

\[
\int \phi \, dF_*\sigma = \int \phi \circ F \, d\sigma,
\]
where \( \phi \) is an arbitrary continuous function in \( \mathbb{C} \), and \( \phi \circ F \) is a composition of \( \phi \) and \( F \). By the monotone convergence theorem this equation also holds for semicontinuous functions. The measure \( F_*\sigma \) automatically has a compact support since \( F \) is bounded on \( \text{supp}(\sigma) \). If \( h \) is subharmonic in \( \mathbb{C} \), then \( h \circ F \) is subharmonic in \( G \), and

\[
\int h \, dF_*\sigma = \int h \circ F \, d\sigma \geq h(F(0)) = h(0).
\]
Hence, \( F_*\sigma \) is a Jensen measure.

Corollary 4. Let \( \sigma \) be a Jensen measure in \( \mathbb{C} \), and let \( f = g + ih \) be an analytic function in a domain \( G \) which contains the origin and \( \text{supp}(\sigma) \), and \( f(0) = 0 \). Let

\[
m_{f,\sigma}(\lambda) = \sigma(|f| \geq \lambda), \quad m_{h,\sigma}(\lambda) = \sigma(|h| \geq \lambda).
\]

Then

\[
m_{f,\sigma}(\lambda) \lesssim \frac{1}{\lambda^2} \int_0^\lambda s m_{h,\sigma}(s) \, ds + \frac{1}{\lambda} \int_\lambda^\infty m_{h,\sigma}(s) \, ds.
\]  \hspace{1cm} (2.11)

Corollary 4 probably holds true under a weaker (and more natural) assumption \( g(0) = 0 \) rather than \( f(0) = 0 \). In that case, using Theorem 2 (see below) one can get an estimate which is slightly weaker than (2.11).

In the next result, we shall not assume that \( u(z) \) is non-negative in \( \mathbb{C} \) and instead introduce the quantity

\[
\delta(r) = n(r) + [u^-(r) + u^-(r)]
\]
which we keep under control.

We assume that the integrals
\[
\int_0^\infty \frac{\delta(t)}{t^2} \, dt \quad \text{and} \quad \int_0^\infty \frac{\delta(t)}{t^3} \left(1 + \log t\right) \, dt \tag{2.12}
\]
are convergent and define
\[
\delta^*(r) = r \int_0^r \frac{\delta(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{\delta(t)}{t^3} \left(1 + \log \frac{t}{r}\right) \, dt \tag{2.13}
\]
The function \(\delta^*(r)\) does not decrease, \(r^{-2}\delta^*(r)\) does not increase, and therefore
\[
\delta^*(r) \leq \delta^*(2r) \leq 4\delta^*(r), \quad 0 < r < \infty. \tag{2.14}
\]

**Theorem 2.** Let \(u(z)\) be an arbitrary subharmonic function in \(\mathbb{C}\) represented by a canonical integral of genus one. Then
\[
M(r, u) \lesssim r^2 \left[ \int_r^\infty \frac{\sqrt{\delta^*(t)}}{t^2} \, dt \right]^2. \tag{2.15}
\]

Observe, that the RHSs of (2.5) and (2.15) do not depend on the bound for the integral (2.2). Estimate (2.15) is slightly weaker than (2.5); however, it suffices for deriving estimates of M. Riesz and Kolmogorov, as well as of the weak \((p, \infty)\)-type estimate (see Corollary 6 below).

Fix an arbitrary \(\epsilon > 0\). Then by the Cauchy inequality
\[
\left[ \int_r^\infty \frac{\sqrt{\delta^*(t)}}{t^2} \, dt \right]^2 = \left[ \int_r^\infty \frac{(1 + \log^{1+\epsilon} \frac{t}{r}) \delta^*(t)}{t^{3/2}} \frac{dt}{t^{1/2} \sqrt{1 + \log^{1+\epsilon} \frac{t}{r}}} \right]^2
\]
\[
\lesssim_{\epsilon} \int_r^\infty \frac{\delta^*(t)}{t^3} \left(1 + \log^{1+\epsilon} \frac{t}{r}\right) \, dt
\]
\[
\lesssim_{\epsilon} \frac{1}{r} \int_0^r \frac{\delta(s)}{s^2} \, ds + \int_r^\infty \frac{\delta(s)}{s^3} \left(1 + \log^{3+\epsilon} \frac{s}{r}\right) \, ds.
\]
Thus we get

**Corollary 5.** For each \(\epsilon > 0\),
\[
M(r, u) \lesssim_{\epsilon} r \int_0^r \frac{\delta(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{\delta(t)}{t^3} \left(1 + \log^{3+\epsilon} \frac{t}{r}\right) \, dt. \tag{2.16}
\]
We do not know whether the term \( \log^{3+\varepsilon} \) is really needed on the RHS of (2.14). Apparently, our method does not allow us to omit it. Rewriting (2.16) in the form

\[
M(r, u) \lesssim \varepsilon \int_0^1 \frac{\delta(rs)}{s^2} \, ds + \int_1^\infty \frac{\delta(rs)}{s^3} \left( 1 + \log^{3+\varepsilon} s \right) \, ds,
\]

we immediately obtain

**Corollary 6.** The following inequalities hold for canonical integrals of genus one:

**M. Riesz-type estimate:**

\[
\int_0^\infty \frac{\mu(r)}{r^{p+1}} \, dr \lesssim_p \int_0^\infty \frac{M(r, u)}{r^{p+1}} \, dr \lesssim_p \int_0^\infty \frac{\delta(r)}{r^{p+1}} \, dr, \quad 1 < p < 2, \quad (2.17)
\]

**weak \((p, \infty)\)-type estimate:**

\[
\sup_{r \in (0, \infty)} \frac{\mu(r)}{r^p} \lesssim_p \sup_{r \in (0, \infty)} \frac{M(r, u)}{r^p} \lesssim_p \sup_{r \in (0, \infty)} \frac{\delta(r)}{r^p}, \quad 1 < p < 2, \quad (2.18)
\]

and **Kolmogorov-type estimate:**

\[
\sup_{r \in (0, \infty)} \frac{\mu(r)}{r} \lesssim_p \sup_{r \in (0, \infty)} \frac{M(r, u)}{r} \lesssim_p \int_0^\infty \frac{\delta(r)}{r^2} \, dr. \quad (2.19)
\]

Estimates weaker than (2.17) and (2.19) were obtained in [12] and [14] under additional restrictions which now appear to be redundant. Estimate (2.18) is apparently new.

If we assume that \( d\mu \) is supported by \( \mathbf{R} \), that is, \( u(z) \) is harmonic in \( \mathbf{C}_\pm \), then our technique gives a better result:

Let \( u(z) \) be a canonical integral of genus one of a measure \( d\mu \) supported by \( \mathbf{R} \). Then, for \( 0 < r < \infty \),

\[
M(r, u) \lesssim r \int_0^r \frac{u^-(t) + u^-(t)}{t^2} \, dt + r^2 \int_r^\infty \frac{u^-(t) + u^-(t)}{t^3} \, dt. \quad (2.20)
\]

Note, that one cannot replace \( u^-(x) \) by \( u^+(x) \) on the RHS of our estimates. For example, the function

\[
u_p(z) = r^p \cos \left( \frac{\pi}{2} - |\theta| \right), \quad 1 < p < 2,
\]
is subharmonic in $\mathbb{C}$, harmonic in the upper and lower half-planes $\mathbb{C}_\pm$, represented by the canonical integral of genus one of the measure $d\mu(x) = c_p |x|^{p-1} \, dx$ ($c_p > 0$), and non-positive on $\mathbb{R}$.

There is a corollary to Theorem 2 which is parallel to Corollary 2. Let $\mathcal{M}$ be a measurable space endowed with a locally finite non-negative measure $d\mu$, and let $f : \mathcal{M} \to \mathbb{R}^{n+1}$, $n \geq 1$, be a measurable function such that

$$\int_{\mathcal{M}} \min(||f||, ||f||^2) \, dt < \infty, \quad (2.21)$$

where $||\cdot||$ stands for the $n+1$-dimensional Euclidean norm. We start to enumerate the coordinates in $\mathbb{R}^{n+1}$ with $j = 0$, and denote by $e_0$ the vector in $\mathbb{R}^{n+1}$ with the zeroth coordinate equal one, and other coordinates vanishing.

Let $f_j(t)$ be the $j$-th coordinate function of $f(t)$, and $\hat{f}(t) = \left\{ \sum_{j=1}^n f_j^2(t) \right\}^{1/2}$. We define the logarithmic determinant

$$v_f(x) = \int_{\mathcal{M}} [\log ||e_0 - x f(t)|| + x f_0(t)] \, d\mu(t)$$

$$= \int_{\mathcal{M}} [\log \sqrt{1 - 2x f_0(t) + x^2 ||f||^2 + x f_0(t)}] \, d\mu(t), \quad x \in \mathbb{R},$$

where the integral converges due to assumption (2.21). Then, if the function $v_f(x)$ is non-negative on $\mathbb{R}$, we may estimate its distribution function $m_f(\lambda) = m(\{||f|| \geq \lambda\})$ by the distribution function $m_{\hat{f}} = m(\{\hat{f} \geq \lambda\})$ of $\hat{f}$.

For this, observe that

$$v_f(x) = \int_{\mathcal{M}} H(x f_C(t)) \, d\mu(t), \quad x \in \mathbb{R},$$

where $f_C$ is a “complex-valued surrogate” of $f$: $f_C = f_0 + i \hat{f}$. That is, $v_f$ has a subharmonic continuation from $\mathbb{R}$ to $\mathbb{C}$ by a canonical integral of genus one

$$u_{f_C}(z) = \int_{\mathcal{M}} H(z f_C(t)) \, d\mu(t), \quad z \in \mathbb{C}.$$

Next, observe that $m_f(\lambda) = m(\{f_0^2 + \hat{f}^2 \geq \lambda^2\}) = m_{f_C}(\lambda)$, and $m_f(\lambda) = m_{\text{Im} f_C}(\lambda)$ for $0 < \lambda \leq \infty$. Hence, Theorem 2 is applicable in this situation.

For simplicity, we restrict ourselves to the case when $v_f$ is non-negative on the real axis.
Corollary 7. Let $f$ satisfy condition (2.21), and let the logarithmic determinant $v_f$ be non-negative on the real axis. Then, for $0 < \lambda < \infty$ and $\epsilon > 0$,

$$m_f(\lambda) \lesssim \epsilon \frac{1}{\lambda^2} \int_0^\lambda s \left(1 + \log^{3+\epsilon} \frac{\lambda}{s}\right) m_f(s) \, ds + \frac{1}{\lambda} \int_\lambda^\infty m_f(s) \, ds .$$

In particular,

$$||f||_{L^p(m)} \lesssim ||\hat{f}||_{L^p(m)} , \quad 1 < p < 2 ,$$

and

$$m_f(\lambda) \lesssim \frac{||\hat{f}||_{L^1(m)}}{\lambda} .$$

This corollary may be of some interest in view of the results of Aleksandrov and Kargaev [1].

Our third result pertains to a more general class of subharmonic functions represented by a generalized canonical integral of genus one. It gives a Kolmogorov-type estimate which can be applied to a wider class of functions than (2.19):

Theorem 3. Let $d\mu$ be a non-negative locally finite measure on $\mathbb{C}$ such that

$$\int_{\{ |\zeta| \geq 1 \}} \frac{d\mu(\zeta)}{|\zeta|^2} < \infty ,$$

and let there exist a finite principal value integral

$$\lim_{\epsilon \to 0} \int_{\{ |\zeta| \leq \epsilon \}} \frac{d\mu(\zeta)}{\zeta} .$$

Let

$$u(z) = \lim_{\epsilon \to 0} \int_{|\zeta| > \epsilon} H(z/\zeta) \, d\mu(\zeta) ,$$

then

$$\sup_{0 < r < \infty} \frac{M(r, u)}{r} \lesssim \int_0^\infty \frac{\delta(t)}{t^2} \, dt + \limsup_{r \to 0} \frac{\mu(r)}{r} . \quad (2.22)$$

It is easy to see that if the integral (2.2) converges at the origin, then the upper limit on the RHS of (2.22) vanishes, and in this case (2.22) coincides with (2.19).
In fact, our proof yields a stronger result
\[ \int_{-\infty}^{\infty} \frac{u^+(t)}{t^2} \, dt + \limsup_{r \to \infty} \frac{M(r, u)}{r} \lesssim \int_{0}^{\infty} \frac{\delta(t)}{t^2} \, dt + \limsup_{r \to 0} \frac{\mu(r)}{r}, \] (2.23)
which gives control over the positive harmonic majorants of \( u \) in the upper and lower half-planes. Applying a known technique of functions of Cartwright class [11], [8], one can extract from (2.23) information about the asymptotic regularity of \( u \) and \( \mu \) at infinity and near the origin.

Notice, that one can reformulate Theorem 3 in the spirit of Corollaries 2 and 7. We leave this to the reader.

3 Auxiliary Lemmas

We shall need several known facts about harmonic and subharmonic functions.

Lemma 1. Let \( v \) be a subharmonic function in the angle \( S = \{ z : 0 < \arg z < \alpha \}, 0 < \alpha < 2\pi \), let
\[ \limsup_{z \to \zeta, z \in S} v^+(z) \leq \Phi(|\zeta|), \quad \zeta \in \partial S; \] (3.1)
and let
\[ \int_{0}^{\alpha} v^+(re^{i\theta}) \sin \left( \frac{\pi}{\alpha} \theta \right) \, d\theta = o(r^{\pi/\alpha}), \quad r \to \infty. \] (3.2)

Then, for \( z = re^{i\theta} \in S \),
\[ v(re^{i\theta}) \sin \left( \frac{\pi}{\alpha} \theta \right) \lesssim r^{-\pi/\alpha} \int_{0}^{r} \Phi(t)t^{\pi/\alpha-1} \, dt + r^{\pi/\alpha} \int_{r}^{\infty} \frac{\Phi(t)}{t^{\pi/\alpha+1}} \, dt. \] (3.3)

If the majorant \( \Phi(t) \) does not decrease, then the factor \( \sin(\pi\theta/\alpha) \) on the LHS of (3.3) can be omitted.

Proof: The general case is easily reduced to the special case when \( S = \mathbb{C}_+ \), so that, without loss of generality, we assume that \( \alpha = \pi \). First, we show that \( v(z) \) is majorized by the Poisson integral of \( \Phi(|t|) \), and then we estimate this integral.

Denote by \( h_R(z) \) a harmonic function in the semi-disk \( \{ \text{Im} z > 0, |z| < R \} \) with boundary values \( h_R(t) = \Phi(|t|), -R < t < R, \) and \( h_R(Re^{i\theta}) = v^+(Re^{i\theta}) \),
0 < \theta < \pi. Applying the Poisson-Nevanlinna representation in this semi-disk (see [5, Chapter 1, Theorem 2.3], [10, Section 24.3]), we obtain for \( z = re^{i\theta}, \ r < R, \)

\[
v(z) \leq h_R(z) = \int_{-R}^{R} \Phi(|t|)K_1(z,t) \, dt + \int_{0}^{\pi} v^+(Re^{i\phi})K_2(z,Re^{i\phi}) \, d\phi , \tag{3.4}
\]

where

\[
K_1(z,t) = \frac{r \sin \theta}{\pi} \left\{ \frac{1}{|z-t|^2} - \frac{R^2}{|R^2 - zt|^2} \right\} , \tag{3.5}
\]

\[
K_2(z,Re^{i\phi}) = \frac{1}{2\pi} \frac{4Rr(R^2 - r^2) \sin \phi \sin \theta}{(R^2 + r^2 - 2Rr \cos(\phi - \theta))(R^2 + r^2 - 2Rr \cos(\phi + \theta))} , \tag{3.6}
\]

By condition (3.2), the second integral on the RHS of (3.4) tends to 0 as \( R \to \infty. \) Therefore, letting \( R \to \infty \) in (3.4), we obtain

\[
v(z) \leq \frac{r \sin \theta}{\pi} \int_{-\infty}^{\infty} \frac{\Phi(|t|)}{|z-t|^2} \, dt . \tag{3.7}
\]

Making use of straightforward estimates of the Poisson kernel, we get

\[
v(z) \leq \frac{2}{\pi r \sin \theta} \int_{0}^{2r} \Phi(t) \, dt + \frac{4r}{\pi} \int_{2r}^{\infty} \frac{\Phi(t)}{t^2} \, dt ,
\]

and estimate (3.3) follows.

If the majorant \( \Phi(t) \) does not decrease, then we modify the previous argument:

\[
v(z) \leq \frac{4}{r} \int_{0}^{r/2} \Phi(t) \, dt + \frac{4r}{r} \int_{2r}^{\infty} \frac{\Phi(t)}{t^2} \, dt \lesssim \frac{1}{r} \int_{0}^{r} \Phi(t) \, dt + r \int_{r}^{\infty} \frac{\Phi(t)}{t^2} \, dt ,
\]

completing the proof. \( \square \)

The next lemma asserts that under certain conditions the Carleman integral formula [10, Lecture 24], [5, Chapter 1] holds without remainder.

**Lemma 2.** Let \( v(z) \) be a subharmonic function on \( D_R = \{ z \in \overline{C}_+ : |z| \leq R \} \) which satisfies conditions

\[
\int_{0}^{\pi} v^+(Re^{i\theta}) \sin \theta \, d\theta = o(r) , \quad r \to 0 , \tag{3.8}
\]
and
\[ \int_0^\infty \frac{\delta(t)}{t^2} \, dt < \infty. \quad (3.9) \]

Then
\[ \frac{1}{2\pi} \int_{-R}^R v(t) \left( \frac{1}{t^2} - \frac{1}{R^2} \right) \, dt + \frac{1}{\pi R} \int_0^\pi v(Re^{i\phi}) \sin \phi \, d\phi \]
\[ = \int_{D_R} \left( \frac{1}{|\zeta|^2} - \frac{1}{R^2} \right) \text{Im} \zeta \, d\mu(\zeta), \quad (3.10) \]

where the first integral on the LHS is absolutely convergent.

**Proof:** We start with the Nevanlinna representation
\[ \int_{-R}^R v^+(t) K_1(z, t) \, dt + \int_0^\pi v^+(Re^{i\phi}) K_2(z, Re^{i\phi}) \, d\phi \]
\[ = v(z) + \int_{-R}^R v^-(t) K_1(z, t) \, dt + \int_0^\pi v^-(Re^{i\phi}) K_2(z, Re^{i\phi}) \, d\phi \]
\[ + \int_{D_R} K_3(z, \zeta) \, d\mu(\zeta), \]

where the kernels \( K_1 \) and \( K_2 \) were defined by (3.5) and (3.6), and
\[ K_3(z, \zeta) = \log \left| \frac{z - \bar{\zeta}}{z - \zeta} \cdot \frac{R^2 - z\bar{\zeta}}{R^2 - z\zeta} \right|. \quad (3.11) \]

We multiply both the left and right hand sides of the Nevanlinna representation by \( r^{-1} \sin \theta \), integrate it with respect to \( \theta \) from 0 to \( \pi \) and change the integration order in all terms. We shall use the formulas:
\[ \frac{1}{r} \int_0^\pi K_1(re^{i\theta}, t) \sin \theta \, d\theta = \frac{1}{2} \left[ \min \left( \frac{1}{t^2} - \frac{1}{r^2} \right) - \frac{1}{R^2} \right], \quad (3.12) \]
\[ \frac{1}{r} \int_0^\pi K_2(re^{i\theta}, Re^{i\phi}) \sin \theta \, d\theta = \frac{1}{R} \sin \phi, \quad (3.13) \]
and
\[ \frac{1}{r} \int_0^\pi K_3(re^{i\theta}, \zeta) \sin \theta \, d\theta = \pi \text{Im} \zeta \left[ \min \left( \frac{1}{|\zeta|^2} - \frac{1}{r^2} \right) - \frac{1}{R^2} \right]. \quad (3.14) \]
Observe that the RHS of relations (3.12)-(3.14) are non-decreasing functions of $r^{-1}$. Therefore, making the limit transition $r \to 0$, and using the monotone convergence theorem and condition (3.9) of the lemma, we get

\[
\frac{1}{2} \int_{-R}^{R} v^+(t) \left( \frac{1}{t^2} - \frac{1}{R^2} \right) dt + \frac{1}{R} \int_{0}^{\pi} v^+(Re^{i\phi}) \sin \phi d\phi \\
= \frac{1}{2} \int_{-R}^{R} v^-(t) \left( \frac{1}{t^2} - \frac{1}{R^2} \right) dt + \frac{1}{R} \int_{0}^{\pi} v^-(Re^{i\phi}) \sin \phi d\phi \\
+ \pi \int_{D_R} \left( \frac{1}{|\zeta|^2} - \frac{1}{R^2} \right) \text{Im} \zeta d\mu(\zeta).
\]

The first and third integrals on the RHS are finite due to condition (3.9). This completes the proof. $\square$

**Remark.** Condition (3.8) holds true for canonical integrals of genus one defined in (2.1).

Indeed, if $u(z)$ is such an integral, then due to (2.3)

\[
\int_{0}^{2\pi} u^+(re^{i\theta}) d\theta = o(r), \quad r \to 0.
\]

Since $u(0) = 0$, this yields

\[
\int_{0}^{2\pi} |u(re^{i\theta})| d\theta = 2 \int_{0}^{2\pi} u^+(re^{i\theta}) d\theta - \int_{0}^{2\pi} u(re^{i\theta}) d\theta \\
= 2 \int_{0}^{2\pi} u^+(re^{i\theta}) d\theta = o(r), \quad r \to 0.
\]

The third lemma was proved in [14] (cf. [10, Lecture 26]). Its proof uses the Nevanlinna representation for the semi-disk.

**Lemma 3.** Let $v(z)$ be a function which is harmonic in $\mathbb{C}_+$, subharmonic in $\overline{\mathbb{C}}_+$, and satisfies conditions (3.8) and (3.9) of Lemma 2. Then, for $z = re^{i\theta} \in \mathbb{C}_+$,

\[
v(re^{i\theta}) \sin \theta \leq \frac{1}{\pi} \int_{0}^{\pi} v^-(2re^{i\varphi}) \sin \varphi d\varphi + \frac{r}{2\pi} \int_{-2\pi}^{2\pi} \frac{v^-(t)}{t^2} dt.
\]
The next lemma is a version of the Levin integral formula ([9], [5, Chapter 1]) without a remainder.

**Lemma 4.** Let $v$ be a subharmonic function in $C$ such that $v(z)$ and $v(\bar{z})$ satisfy conditions (3.8) and (3.9) of Lemma 2. Then

$$
\frac{1}{2\pi} \int_0^{2\pi} v(Re^{i\theta}\sin\theta) \frac{d\theta}{R\sin^2\theta} = \int_0^R \frac{n(t)}{t^2} dt,
$$

(3.16)

where $n(t)$ is the Levin-Tsuji counting function, and the integral on the LHS is absolutely convergent.

**Proof:** It suffices to prove that

$$
\frac{1}{2\pi} \int_0^{\pi} v(Re^{i\theta}\sin\theta) \frac{d\theta}{R\sin^2\theta} = \int_{|\Im \frac{1}{\zeta}| > \frac{1}{R}} \left[ \frac{1}{\zeta} - \frac{1}{R} \right] d\mu(\zeta).
$$

(3.17)

Then (3.16) follows by adding to (3.17) a similar formula for the integral from $\pi$ to $2\pi$.

First, we prove that the integral on the LHS of relation (3.16) is absolutely convergent. Making use of notations introduced in (3.5), (3.2) and (3.11), observe that the Nevanlinna formula implies that

$$
|v(z)| \leq \int_{-R}^R |v(t)|K_1(z,t) dt + \int_0^\pi |v(Re^{i\phi})|K_2(z,Re^{i\phi}) d\phi + \int_{D_R} K_3(z,\zeta) d\mu(\zeta).
$$

We set $z = Re^{i\theta}\sin\theta$, multiply the formula by $(R\sin^2\theta)^{-1}$, integrate it with respect to $\phi$ from 0 to $\pi$, and change the integration order in all terms. We shall use the following relations:

$$
\int_0^\pi K_1(Re^{i\theta}\sin\theta,t) \frac{d\theta}{R\sin^2\theta} = \frac{1}{t^2} - \frac{1}{R^2},
$$

and

$$
\int_0^\pi K_2(Re^{i\theta}\sin\theta,Re^{i\phi}) \frac{d\theta}{R\sin^2\theta} = \frac{2}{R} \sin \phi,
$$

and

$$
\int_0^\pi K_3(Re^{i\theta}\sin\theta,\zeta) \frac{d\theta}{R\sin^2\theta} = 2\pi \left[ \min \left( \left| \frac{1}{\zeta} \right|, \frac{1}{R} \right) - \frac{\Im \zeta}{R^2} \right].
$$

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Using these relations, we verify that
\[
\int_0^\pi |v(Re^{i\theta} \sin \theta)| \frac{d\theta}{R \sin^2 \theta} \leq \int_{-R}^R |v(t)| \left( \frac{1}{t^2} - \frac{1}{R^2} \right) dt \\
+ \frac{2}{R} \int_0^\pi |v(Re^{i\phi})| \sin \phi \, d\phi + 2\pi \int_{\partial R} \left| \frac{1}{\zeta} \right| \, d\mu(\zeta).
\]

The first integral on the RHS is finite due to Lemma 2, and the third is finite due to condition (3.9). That is, the integral on the LHS of (3.16) is absolutely convergent.

Now, we write the Nevanlinna formula in the form

\[
v(z) = \int_{-R}^R v(t) K_1(z, t) \, dt + \int_0^\pi v(Re^{i\phi}) K_2(z, Re^{i\phi}) \, d\phi - \int_{\partial R} K_3(z, \zeta) \, d\mu(\zeta).
\]

Again, we set here \(z = Re^{i\theta} \sin \theta\), multiply by \((R \sin^2 \theta)^{-1}\), integrate with respect to \(\theta\) from 0 to \(\pi\) and change the integration order in all terms. We can do this since we already know that the integrals with \(|v|\) instead of \(v\) are finite. As a result, we obtain the equation

\[
\int_0^\pi v(Re^{i\theta} \sin \theta) \frac{d\theta}{R \sin^2 \theta} = \int_{-R}^R v(t) \left( \frac{1}{t^2} - \frac{1}{R^2} \right) dt + \frac{2}{R} \int_0^\pi v(Re^{i\phi}) \sin \phi \, d\phi \\
-2\pi \int_{\partial R} \left[ \min \left( \left| \frac{1}{\zeta} \right|, \frac{1}{R} \right) - \frac{\text{Im} \zeta}{R^2} \right] \, d\mu(\zeta).
\]

Taking into account (3.10), we get

\[
\int_0^\pi v(Re^{i\phi} \sin \phi) \frac{d\phi}{R \sin^2 \phi} = 2\pi \int_{\partial R} \left[ \frac{1}{|\zeta|^2} - \frac{1}{R^2} \right] \text{Im} \zeta \, d\mu(\zeta) \\
-2\pi \int_{\partial R} \left[ \min \left( \left| \frac{1}{\zeta} \right|, \frac{1}{R} \right) - \frac{\text{Im} \zeta}{R^2} \right] \, d\mu(\zeta) \\
= 2\pi \int_{|\text{Im} \frac{1}{\zeta}| \geq \frac{1}{R}} \left[ \left| \frac{1}{\zeta} \right| - \frac{1}{R} \right] \, d\mu(\zeta).
\]

Then (3.17) follows and the proof is complete. \(\Box\)
In other words, in the assumptions of Lemma 4, the first fundamental theorem for Tsuji characteristics holds without a remainder term:

\[
\mathcal{T}(r,u) = m(r,u) + \int_0^r \frac{n(t)}{t^2} \, dt, \quad 0 < r < \infty, \tag{3.19}
\]

where

\[
\mathcal{T}(r,u) = \frac{1}{2\pi} \int_0^{2\pi} u^+(re^{i\theta} | \sin \theta)) \frac{d\theta}{\sin^2 \theta},
\]

and

\[
m(r,u) = \frac{1}{2\pi} \int_0^{2\pi} u^-(re^{i\theta} | \sin \theta)) \frac{d\theta}{\sin^2 \theta}.
\]

The last lemma was proved in a slightly different setting in [11] (see also [5, Lemma 5.2, Chapter 6]):

**Lemma 5.** Let \( u(z) \) be a subharmonic function in \( C \), and let

\[
T(r,u) = \frac{1}{2\pi} \int_0^{2\pi} u^+(re^{i\theta} \sin \theta) \frac{d\theta}{\sin^2 \theta}
\]

be its Nevanlinna characteristic function. Then, for \( 0 < R < \infty \),

\[
\int_R^\infty \frac{T(r,u)}{r^3} \, dr \leq \int_R^\infty \frac{\mathcal{T}(r,u)}{r^2} \, dr. \tag{3.20}
\]

### 4 Proof of Theorem 1

Using monotonicity of \( T(r,u) \), Lemma 5, and then Lemma 4, we obtain

\[
\frac{T(R,u)}{R^2} \leq 2 \int_R^\infty \frac{T(r,u)}{r^3} \, dr \leq 2 \int_R^\infty \frac{\mathcal{T}(r,u)}{r^2} \, dr \leq 2 \int_R^\infty \frac{dr}{r^2} \left( \int_0^r \frac{n(t)}{t^2} \, dt + m(r,u) \right)
\]

\[
= \frac{2}{R} \int_0^R \frac{n(t)}{t^2} \, dt + 2 \int_R^\infty \frac{n(t)}{t^3} \, dt + 2 \int_R^\infty \frac{m(t,u)}{t^2} \, dt.
\]

Then the inequality \( M(r,u) \leq 3T(2r,u) \) completes the proof. □
5 Proof of Theorem 2

We split the proof into several parts. Without loss of generality, we assume convergence of the integrals
\[ \int_0^\infty \frac{\delta(t)}{t^2} \log t \, dt \quad \text{and} \quad \int_0^\infty \frac{\delta(t)}{t^3} \, dt. \]

We define a measure \( \mu_1 \), supp(\( \mu_1 \)) \( \subset \bar{C}_- \), by reflecting at the real axis the part of the measure \( \mu \) which lies in the upper half-plane. Formally,
\[ \mu_1(E) = \mu(E \cap \bar{C}_-) + \mu(E^- \cap \bar{C}_-) , \]
where \( E \subset \mathbb{C} \) is a borelian set, and \( E^- = \{ z : \bar{z} \in E \} \). Then the measure \( \mu_1 \) also satisfies condition (2.1) and we denote by \( u_1(z) \) its canonical integral of genus one. Observe that \( u_1(t) = u(t) \), so that \( \delta(t, u_1) = \delta(t, u) \), \( t \in \mathbb{R} \).

5.1 Estimate of \( u_1^-(iy), y > 0 \).

We have
\[ H(iy/\zeta) = \log \left| 1 + y \operatorname{Im} \frac{1}{\zeta} - i y \operatorname{Re} \frac{1}{\zeta} \right| - y \operatorname{Im} \frac{1}{\zeta} \]
\[ \geq \log \left| 1 + y \operatorname{Im} \frac{1}{\zeta} \right| - y \operatorname{Im} \frac{1}{\zeta} . \]
Since the RHS is non-positive for \( y > 0 \) and \( \zeta \in \bar{C}_- \),
\[ u_1^-(iy) \leq - \int_{\bar{C}_-} \left[ \log \left| 1 + y \operatorname{Im} \frac{1}{\zeta} \right| - y \operatorname{Im} \frac{1}{\zeta} \right] d\mu(\zeta) \]
\[ = - \int_0^\infty \left[ \log \left( 1 + \frac{y}{t} \right) - \frac{y}{t} \right] d\mathbb{N}(t) \]
\[ = y^2 \int_0^\infty \frac{n(t)}{t^2(t + y)} \, dt \]
\[ \leq y \int_0^y \frac{n(t)}{t^2} \, dt + y^2 \int_y^\infty \frac{n(t)}{t^3} \, dt. \]
(5.1)
5.2 Estimates of $u_1^+(re^{i\theta})$, $0 < \theta < \pi$.

Using harmonicity of the function $u_1$ in the upper half-plane, we transform the lower bound for $u_1$ into the upper bound. We shall show that

$$u_1^+(re^{i\theta}) \sin \theta \lesssim \delta^*(r) \quad 0 < r < \infty, \quad 0 < \theta < \pi,$$

(5.2)

where $\delta^*(r)$ is defined by (2.13).

Consider the function $-u_1(z)$ and apply Lemma 1 to the angles $\{0 < \arg z < \pi/2\}$ and $\{\pi/2 < \arg z < \pi\}$ with

$$\Phi(r) = [u_1^-(r) + u_1^-(r)] + r \int_0^r \frac{n(t)}{t^2} dt + r^2 \int_r^\infty \frac{n(t)}{t^3} dt.$$

Condition (3.1) holds due to estimate (5.1), and condition (3.2) holds due to estimate (2.4) combined with Jensen’s inequality:

$$\int_0^\pi u_1^-(re^{i\theta}) d\theta \leq \int_0^{2\pi} u_1^+(re^{i\theta}) d\theta \leq M(r, u_1) = o(r^2), \quad r \to \infty.$$

Therefore,

$$-u_1(re^{i\theta}) \sin 2\theta \lesssim \frac{1}{r^2} \int_0^r \Phi(t) t dt + r^2 \int_r^\infty \frac{\Phi(t)}{t^3} dt \quad \lesssim \frac{1}{r^2} \int_0^r [u_1^-(t) + u_1^-(t)] t dt + r^2 \int_r^\infty \frac{u_1^-(t) + u_1^-(t)}{t^3} dt$$

$$+ r \int_0^r \frac{n(s)}{s^2} ds + r^2 \int_r^\infty \frac{n(s)}{s^3} \left(1 + \log \frac{s}{r}\right) ds \lesssim \delta^*(r).$$

(5.3)

Observe that the factor $|\sin 2\theta|$ on the LHS of (5.3) can be replaced by $\sin \theta$. This follows from inspection of the proof of Lemma 1 (since on the imaginary axis the function $-u(iy)$ has an increasing majorant). Alternatively, one may again apply Lemma 1 to a small angle around the imaginary axis, say in $\{(|\theta - \pi/2| < \pi/8\}$. That is, we have

$$-u_1(re^{i\theta}) \sin \theta \lesssim \delta^*(r).$$

(5.4)

Using Lemma 3 we obtain

$$u_1^+(re^{i\theta}) \sin \theta \lesssim \int_0^\pi u_1^-(2re^{i\phi}) \sin \phi d\phi + r \int_0^{2r} \frac{u_1^-(t) + u_1^-(t)}{t^2} dt \lesssim \delta^*(r),$$

proving estimate (5.2).
5.3 Estimate of \( u^+(re^{i\theta}), \theta \neq 0, \pi \).

Here we prove that, for an arbitrary \( \eta > 0 \),

\[
u^+(re^{i\theta}) \lesssim \frac{\delta^*(r)}{\eta \sin \theta} + \frac{m^2}{\sin^2 \theta} \int_r^\infty \frac{M(t,u)}{t^3} dt.
\] (5.5)

For this, we shall need several upper bounds for the difference

\[
D = D(z,\zeta) = H(z/\zeta) - H(z/\bar{\zeta}) = \log \left| \frac{1 - z/\zeta}{1 - z/\bar{\zeta}} \right| + \text{Re} \left[ z \left( \frac{1}{\zeta} - \frac{1}{\bar{\zeta}} \right) \right],
\]

when \( z,\zeta \in \mathbb{C}_+ \).

First,

\[
D = \log \left| \frac{z - \zeta}{z - \bar{\zeta}} \right| + 2\text{Im} z \left| \frac{1}{\zeta} \right| \leq 2|z| \left| \frac{1}{\zeta} \right|. \tag{5.6}
\]

We shall use this estimate when \( |z| \left| \frac{1}{\zeta} \right| \geq 1 \).

Next, let \( t = |z|/|\zeta|, \theta = \arg(z), \phi = \arg(\zeta) \). Then

\[
D = \frac{1}{2} \log \left[ 1 - \frac{4t \sin \theta \sin \phi}{|1 - te^{i(\theta+\phi)}|^2} \right] + 2t \sin \theta \sin \phi
\]

\[ \leq -\frac{2t \sin \theta \sin \phi}{|1 - te^{i(\theta+\phi)}|^2} + 2t \sin \theta \sin \phi
\]

\[ = 2t \sin \theta \sin \phi \frac{-2t \cos(\theta + \phi) + t^2}{|1 - te^{i(\theta+\phi)}|^2}
\]

\[ \lesssim t \sin \theta \sin \phi \frac{\max(t, t^2)}{|1 - te^{i(\theta+\phi)}|^2}. \tag{5.7}
\]

If \( t \leq 1/2 \), then

\[ |1 - te^{i(\theta+\phi)}|^2 \gg 1, \]

and we obtain

\[ D \lesssim t^2 \sin \theta \sin \phi \lesssim \eta t^2 + \eta^{-1} t^2 \sin^2 \phi = \eta \left| \frac{z}{\zeta} \right|^2 + \frac{|z|^2}{\eta} \left| \frac{1}{\zeta} \right|^2, \tag{5.8}
\]

with an arbitrary \( \eta > 0 \).

If \( t \geq 1/2 \), then

\[ |1 - te^{i(\theta+\phi)}|^2 \gg t^2 \sin^2 \theta, \]

and we obtain

\[ D \lesssim t^2 \sin \theta \sin \phi \lesssim \eta t^2 + \eta^{-1} t^2 \sin^2 \phi = \eta \left| \frac{z}{\zeta} \right|^2 + \frac{|z|^2}{\eta} \left| \frac{1}{\zeta} \right|^2, \tag{5.8}
\]

with an arbitrary \( \eta > 0 \).
so that (5.7) gives us

\[ D \lesssim t \frac{\sin \phi}{\sin \theta} \lesssim \frac{\eta}{\sin^2 \theta} + \frac{t^2}{\eta} \sin^2 \phi = \frac{\eta}{\sin^2 \theta} + \frac{|z|^2}{\eta} \left| \Im \frac{1}{\zeta} \right|^2, \quad (5.9) \]

again, with an arbitrary positive \( \eta \). We shall use the bounds (5.8) and (5.9) when \( |z| \Im \frac{1}{\zeta} \leq 1 \).

Now, for \( z \in \mathbb{C}_+ \), \( r = |z| \), we have

\[ u(z) - u_1(z) = \int_{\mathbb{C}_+} D(z, \zeta) \, d\mu(\zeta) \]

\[ \lesssim \left( \int_{|\Im \zeta| \geq \frac{1}{2}} + \int_{|\Im \zeta| \leq \frac{1}{2}, |\zeta| \geq 2r} + \int_{|\Im \zeta| \leq \frac{1}{2}, |\zeta| \leq 2r} \right) D(z, \zeta) \, d\mu(\zeta) \]

\[ \lesssim r \int_0^r \frac{\ln(t)}{t} + \frac{r^2}{\eta} \int_r^{\infty} \frac{\ln(t)}{t^2} + \frac{\eta}{\sin^2 \theta} \int_0^r \frac{d\mu(t)}{t} + \frac{\eta r^2}{\sin^2 \theta} \int_r^{\infty} \frac{d\mu(t)}{t^2} \]

\[ \lesssim \frac{\delta(r)}{\eta} + \frac{\eta r^2}{\sin^2 \theta} \int_r^{\infty} \frac{M(t, u)}{t^3} \, dt. \]

Then, using estimate (5.2) for \( u_1^+(z) \) in the upper half-plane, we obtain estimate (5.5) for \( 0 < \theta < \pi \). The same argument applies for the lower half-plane, and the proof of (5.5) is complete.

### 5.4 Integral inequality for \( M(r, u) \).

Here we prove the integral inequality

\[ M(r, u) \lesssim \sqrt{\delta^*(r)} \frac{r^2}{t^3} \int_r^{\infty} \frac{M(t, u)}{t^3} \, dt. \quad (5.10) \]

First, we improve estimate (5.3) near the real axis. Consider the function \( u(z) \) in the angles \( \{ |\arg z| \leq \pi/6 \} \) and \( \{ |\arg z - \pi| \leq \pi/6 \} \). On the boundary of these angles,

\[ u(re^{i\theta}) \lesssim \Phi(r), \quad \theta = \pm \frac{\pi}{6}, \quad \pi \pm \frac{\pi}{6}, \]

where

\[ \Phi(r) = \eta^{-1} \delta^*(r) + \eta r^2 \int_r^{\infty} \frac{M(t, u)}{t^3} \, dt. \]
Applying Lemma 1 to $u(z)$ in these angles, we obtain for $|\theta| \leq \pi/8$ and $|\pi - \theta| \leq \pi/8$,

$$u(re^{i\theta}) \lesssim r^{-3} \int_0^r \Phi(t)t^2dt + r^3 \int_r^\infty \frac{\Phi(t)}{t^4}dt \lesssim \Phi(r).$$

The second inequality follows since the function $\Phi(r)$ does not decrease, and the function $r^{-2}\Phi(r)$ does not increase.

Thus, for $0 < r < \infty$,

$$M(r, u) \lesssim \Phi(r) = \eta^{-1}\delta^*(r) + \eta r^2 \int_r^\infty \frac{M(t, u)}{t^3}dt.$$

Choosing

$$\eta = \sqrt{\delta^*(r)} : \sqrt{r^2 \int_r^\infty \frac{M(t, u)}{t^3}dt},$$

we obtain inequality (5.10).

### 5.5 Solution of the integral inequality (5.10).

We set

$$M_1(r) = \int_r^\infty \frac{M(t, u)}{t^3}dt.$$

Then

$$M(r, u) = -r^3M'_1(r),$$

and inequality (5.10) takes the form

$$-M'_1(r)r^2 \lesssim \sqrt{\delta^*(r)} M_1(r)$$

or

$$-\frac{d\sqrt{M_1(r)}}{dr} \lesssim \frac{\sqrt{\delta^*(r)}}{r^2}.$$

Integrating this inequality from $\infty$ to $r$, we obtain

$$M_1(r) \lesssim \left[ \int_r^\infty \frac{\sqrt{\delta^*(t)}}{t^2}dt \right]^2.$$
On the other hand, since $M(r, u)$ does not decrease,

$$M_1(r) \geq M(r, u) \int_r^\infty \frac{dt}{t^3} = \frac{M(r, u)}{2r^2}.$$  

Therefore,

$$M(r, u) \lesssim r^2 M_1(r) \lesssim r^2 \left[ \int_r^\infty \frac{\sqrt{\delta^*(t)}}{t^2} \, dt \right]^2,$$

completing the proof of Theorem 2. \(\blacksquare\)

## 6 Proof of Theorem 3

We divide the proof into 4 parts. Set

$$B := \limsup_{r \to 0} \frac{\mu(r)}{r},$$

$$C := \int_0^\infty \frac{\delta(t)}{t^2} \, dt.$$

Without loss of generality, we assume that both values $B$ and $C$ are finite.

First, we shall prove the theorem under the additional assumption

$$\text{supp}(\mu) \subset \overline{C_-}, \quad (6.1)$$

and till Section 6.4 we assume that the function $u(z)$ is harmonic in $C_+.$

### 6.1 The function $u(z)$ has nonnegative harmonic majorants in $C_\pm.$

Consider the function

$$U(z) := -u(z) - \frac{y}{\pi} \int_{-\infty}^\infty \frac{u^-(t)dt}{(t-x)^2 + y^2}.$$  

This function is harmonic in $C_+$ and $U(x) \leq 0$, $x \in \mathbb{R}$. Moreover, for $y > 0$, 

$$U(iy) = -u(iy) = -\lim_{\varepsilon \to 0} \int_{|\xi| \geq \varepsilon, \xi \in C_-} \log \left| 1 - \frac{iy}{\xi} \right| + \text{Re} \frac{iy}{\xi} \, d\mu(\xi)$$
\[
\begin{align*}
\leq & - \lim_{\varepsilon \to 0} \int_{|\zeta| \geq \varepsilon, \zeta \in \mathbb{C}_-} \frac{iy}{\zeta} \, d\mu(\zeta) \\
= & \ y \int_{\mathbb{C}_-} \frac{1}{\zeta} \, d\mu(\zeta) \leq Cy.
\end{align*}
\]

By the Poisson-Nevanlinna representation of harmonic functions in the semi-disk \(D_R\) (cf. Section 3), we have

\[
U(z) \leq \frac{1}{2\pi} \int_0^\pi U(Re^{i\phi}) K_2(z, Re^{i\phi}) \, d\phi
\]

\[
\leq \frac{2Rr(R + r)}{\pi(R - r)^3} \int_0^\pi u^-(Re^{i\phi}) \, d\phi
\]

\[
\leq \frac{4Rr(R + r)}{(R - r)^3} T(R, u), \tag{6.2}
\]

where \(T(R, u)\) is the Nevanlinna characteristic of \(u\).

Note that, for any \(\delta > 0\), the function \(u\) can be represented in the form

\[
u(z) = \int_{|\zeta| > \delta} \left[ \log \left| 1 - \frac{z}{\zeta} \right| + \text{Re} \left( \frac{z}{\zeta} \right) \right] \, d\mu(\zeta)
\]

\[
+ \int_{|\zeta| \leq \delta} \log \left| 1 - \frac{z}{\zeta} \right| \, d\mu(\zeta) + \text{Re} \left( z \int_{|\zeta| < \delta} \frac{d\mu(\zeta)}{\zeta} \right)
\]

\[
= : u^\delta(z) + v^\delta(z) + \text{Re} \left( z \int_{|\zeta| < \delta} \frac{d\mu(\zeta)}{\zeta} \right). \tag{6.3}
\]

The well-known Borel estimates

\[
\max_{|z| \leq r} u^\delta(z) = o(|z|^2), \quad \max_{|z| \leq r} v^\delta(z) = o(|z|), \quad z \to \infty,
\]

imply that \(T(R, u) = o(R^2), \ R \to \infty\). Therefore, by setting \(R = 2r\) in (6.2), we get

\[
U^+(z) = o(|z|^2), \quad z \to \infty, \quad \text{Im} \ z > 0.
\]

Applying the Phragmén-Lindelöf principle in the angles \(\{0 < \arg z < \pi/2\}\) and \(\{\pi/2 < \arg z < \pi\}\), we conclude that

\[
U(z) \leq Cy, \quad z = x + iy \in \mathbb{C}_+; \quad \tag{6.4}
\]

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i.e. \(-U(z) + Cy\) is a nonnegative harmonic function in \(\mathbb{C}_+\). Since \(u(z) \leq -U(z) + Cy\), the function \(u(z)\) also has a nonnegative harmonic majorant in \(\mathbb{C}_+\).

For \(z \in \mathbb{C}_-\), we write

\[
|u(z) - u(\bar{z})| = \lim_{\varepsilon \to 0} \int_{|\zeta| \geq \varepsilon, \zeta \in \mathbb{C}_-} \log \left| \frac{1 - z/\zeta}{1 - z/\bar{\zeta}} \right| + \text{Re} \left[ \frac{z}{\zeta - \bar{\zeta}} \right] d\mu(\zeta)
\]

\[\leq \int_{\mathbb{C}_-} \text{Re} \left[ z2i\text{Im} \frac{1}{\zeta} \right] d\mu(\zeta) \leq 2C|y|. \tag{6.5}\]

Because \(u(\bar{z})\) has a nonnegative harmonic majorant in \(\mathbb{C}_-\), we get the desired conclusion.

### 6.2 Estimate of \(u(z)\) near the origin.

Set

\[
I(r) := \frac{1}{r} \int_{0}^{\pi} u(re^{i\theta}) \sin \theta \, d\theta.
\]

Let us prove that

\[
\lim_{r \to 0} \sup I(r) \lesssim B. \tag{6.6}
\]

For any given \(\varepsilon > 0\), choose a positive \(\delta < \varepsilon\) such that

\[
\mu(r) < (B + \varepsilon)r, \quad \text{for} \quad 0 < r < \delta.
\]

Let us represent \(u\) by the formula (6.3) with this \(\delta\). Since

\[
u^\delta(z) = O(|z|^2), \quad z \to 0,
\]

we have

\[
\lim_{r \to 0} \sup I(r) \leq \lim_{r \to 0} \sup I^\delta(r) + \left| \int_{|\zeta| < \delta} \frac{d\mu(\zeta)}{\zeta} \right|, \tag{6.7}
\]

where

\[
I^\delta(r) = \frac{1}{r} \int_{0}^{\pi} v^\delta(re^{i\theta}) \sin \theta \, d\theta.
\]

It suffices to show that

\[
I^\delta(r) \lesssim B + \varepsilon + \int_{|\zeta| < \delta} \text{Im} \frac{1}{\zeta} d\mu(\zeta), \quad 0 < r < \delta. \tag{6.8}
\]
Indeed, if (6.8) is valid, then substituting it into (6.7), we get

\[
\limsup_{r \to 0} I(r) \lesssim B + \varepsilon + \int_{|\zeta| < \delta} \Im \frac{1}{\zeta} d\mu(\zeta) + \int_{|\zeta| < \delta} \frac{d\mu(\zeta)}{\zeta}.
\]

Taking the limit as \( \varepsilon \to 0 \) (then \( \delta \to 0 \) as well), we obtain (6.6).

To prove (6.8), we set for \( |z| = r, 0 < r < \delta \):

\[ v^\delta(z) = \int_{r < |\zeta| < \delta} + \int_{|\zeta| < r} =: v^\delta_1(z) + v^\delta_2(z), \]

and

\[ I^\delta_j(r) := \frac{1}{r} \int_0^\pi v^\delta_j(re^{i\theta}) \sin \theta \, d\theta, \quad j = 1, 2. \]

Note that

\[
\left| \int_0^\pi \log \left| 1 - \frac{re^{i\theta}}{\zeta} \right| \sin \theta \, d\theta \right| \leq 2 \int_0^{2\pi} \log^+ \left| 1 - \frac{re^{i\theta}}{\zeta} \right| \, d\theta - \int_0^{2\pi} \log \left| 1 - \frac{re^{i\theta}}{\zeta} \right| \, d\theta
\]

\[ \leq 4\pi \log \left( 1 + \frac{\delta}{|\zeta|} \right) + 2\pi \log^+ \frac{r}{|\zeta|}. \]

This estimate will allow us to change the integration order in the double integrals that arise when estimating \( I^\delta_j(r), \ j = 1, 2, \) below.

Write

\[ I^\delta_1(r) = \frac{1}{r} \int_{r < |\zeta| < \delta} d\mu(\zeta) \int_0^\pi \log \left| 1 - \frac{re^{i\theta}}{\zeta} \right| \sin \theta \, d\theta. \]

For \( r < |\zeta| \), we have

\[
\int_0^\pi \log \left| 1 - \frac{re^{i\theta}}{\zeta} \right| \sin \theta \, d\theta = -\text{Re} \sum_{k=1}^\infty \frac{r^k}{k\zeta^k} \int_0^\pi e^{ik\theta} \sin \theta \, d\theta
\]

\[ = \frac{\pi r}{2} \Im \frac{1}{\zeta} + \text{Re} \sum_{m=1}^\infty \frac{r^{2m}}{m(4m^2 - 1)} \frac{1}{\zeta^{2m}}. \]

Hence

\[ I^\delta_1(r) \leq \frac{\pi}{2} \int_{|\zeta| < \delta} \Im \frac{1}{\zeta} d\mu(\zeta) + \sum_{m=1}^\infty \frac{r^{2m-1}}{m(4m^2 - 1)} \int_{r < |\zeta| < \delta} \frac{d\mu(\zeta)}{|\zeta|^{2m}}. \]
Since
\[
\int_{r<|\zeta|<\delta} \frac{d\mu(\zeta)}{|\zeta|^{2m}} = \int_r^\delta \frac{d\mu(t)}{t^{2m}} \leq \frac{\mu(\delta)}{\delta^{2m}} + 2m \int_r^\delta \frac{\mu(t)}{t^{2m+1}} dt \leq (B + \varepsilon)\delta^{-2m+1} + 2(B + \varepsilon)r^{-2m+1} < 3(B + \varepsilon)r^{-2m+1},
\]
we get
\[
I_1^\delta(r) \leq \pi \int_{|\zeta|<\delta} \frac{1}{\zeta} d\mu(\zeta) + (B + \varepsilon) \sum_{m=1}^\infty \frac{1}{m(4m^2-1)}. \quad (6.9)
\]
Further,
\[
I_2^\delta(r) = \frac{1}{r} \int_{|\zeta|<r} d\mu(\zeta) \int_0^\pi \log \left| 1 - \frac{r e^{i\theta}}{\zeta} \right| \sin \theta d\theta.
\]
For \(|\zeta| \leq r\), we have
\[
\int_0^\pi \log \left| 1 - \frac{r e^{i\theta}}{\zeta} \right| \sin \theta d\theta = 2 \log \frac{r}{|\zeta|} + \int_0^\pi \log \left| 1 - \frac{\zeta e^{i\theta}}{r} \right| \sin \theta d\theta
\]
\[
= 2 \log \frac{r}{|\zeta|} - \text{Re} \sum_{k=1}^\infty \frac{\zeta^k}{k r^{k}} \int_0^\pi e^{i k \theta} \sin \theta d\theta
\]
\[
= 2 \log \frac{r}{|\zeta|} - \frac{\pi}{2r} \text{Im} \zeta + \text{Re} \sum_{m=1}^\infty \frac{\zeta^{2m}}{m(4m^2-1) r^{2m}}.
\]
Whence
\[
I_2^\delta(r) \leq \frac{1}{r} \int_{|\zeta|<r} \left[ 2 \log \frac{r}{|\zeta|} + \frac{\pi}{2} + \sum_{m=1}^\infty \frac{1}{m(4m^2-1)} \right] d\mu(\zeta)
\]
\[
\lesssim \frac{1}{r} \int_0^r \log \frac{r}{t} d\mu(t) + \frac{\mu(r)}{r}
\]
\[
\lesssim B + \varepsilon \quad (6.10)
\]
Since \(I^\delta = I_1^\delta + I_2^\delta\), the desired inequality \((6.8)\) follows from \((6.9)\) and \((6.10)\).

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6.3 Estimate of $u^+(z)$ on the real and imaginary axes.

Let us prove that

$$
\int_{-\infty}^{\infty} \frac{u^+(t)}{t^2} dt + \limsup_{y \to +\infty} \frac{u^+(iy)}{y} \lesssim B + C .
$$

(6.11)

Since $u$ has a nonnegative harmonic majorant in $\mathbb{C}_+$, we have

$$
\int_{-\infty}^{\infty} \left| u(t) \right| dt \lesssim \frac{1}{1 + t^2} < \infty ,
$$

and $u$ admits the Poisson representation

$$
u(re^{i\varphi}) = \frac{r \sin \varphi}{\pi} \int_{-\infty}^{\infty} \frac{u(t)}{t^2 + r^2 + 2rt \cos \varphi} + kr \sin \varphi ,
$$

(6.12)

where

$$
k = \limsup_{y \to +\infty} \frac{u(iy)}{y} \neq \infty .
$$

Note that inequality (6.4) implies

$$
u(iy) \geq -\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u^- (t) dt}{t^2 + y^2} - Cy = o(y) - Cy, \quad y \to \infty ;
$$

i.e. $k \geq -C$.

Multiplying (6.12) by $\sin \varphi$, integrating against $\varphi$ from 0 to $\pi$, and taking into account that

$$
\int_{0}^{\pi} \frac{\sin^2 \varphi \, d\varphi}{r^2 + t^2 - 2rt \cos \varphi} = \frac{\pi}{2} \min \left( \frac{1}{r^2}, \frac{1}{t^2} \right) ,
$$

we get

$$
\int_{0}^{\pi} \nu(re^{i\varphi}) \sin \varphi \, d\varphi = \frac{r}{2} \int_{-\infty}^{\infty} u(t) \min \left( \frac{1}{r^2}, \frac{1}{t^2} \right) \, dt + \frac{k\pi r}{2} .
$$

Hence

$$
\int_{-\infty}^{\infty} \left[ u^+(t) \min \left( \frac{1}{r^2}, \frac{1}{t^2} \right) \right] \, dt + k^+\pi = \int_{-\infty}^{\infty} u^-(t) \min \left( \frac{1}{r^2}, \frac{1}{t^2} \right) \, dt + k^-\pi + 2I(r) .
$$

Letting $r \to 0$, we obtain by the monotone convergence theorem

$$
\int_{-\infty}^{\infty} \frac{u^+(t)}{t^2} \, dt + k^+\pi = \int_{-\infty}^{\infty} \frac{u^-(t)}{t^2} \, dt + k^-\pi + 2 \lim_{r \to 0} I(r)
$$

(it turns out that the last limit exists). Taking into account that $k^- \leq C$ and using (6.6), we get (6.11).
6.4 Concluding steps.

From (6.4) and (6.11) we obtain

$$\limsup_{y \to -\infty} \frac{u^+(iy)}{|y|} \leq \limsup_{y \to +\infty} \frac{u^+(iy) + 2Cy}{y} \lesssim B + C.$$  

Since $u$ has nonnegative harmonic majorants in both upper and lower half-planes, the following inequality holds in the whole plane:

$$u(z) \lesssim \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{u^+(t) dt}{(t-x)^2 + y^2} + (B + C)|y|, \quad z = x + iy. \quad (6.13)$$

The assertion of Theorem 3 can be obtained from this inequality and (6.11) by applying a known argument (cf. [12], [14]). First, one applies (6.13) and (6.11) to get the upper bound for $u(z)$ in the angles $\{|\arg z\pm \pi/2| \leq \pi/4\}$, and then, using the Phragmén-Lindelöf principle, one obtains the upper bound for $u(z)$ in the complementary angles. This gives $u(z) \lesssim (B + C)|z|$, and completes the proof of estimate (2.22) for the special case (6.1).

Now, let $\mu$ be an arbitrary measure in $\mathbb{C}$ satisfying conditions of Theorem 3 and having finite value $C$. As in Section 5, we define the measure $\mu_1$, $\text{supp} \mu_1 \subset \bar{\mathbb{C}}_-$, by reflecting with respect to the real axis the part of $\mu$ which charges $\mathbb{C}_+$. Since

$$\int_{\mathbb{C}_-} \frac{1}{\zeta} d\mu_1(\zeta) \leq C < \infty,$$

the measure $\mu_1$ also satisfies the conditions of Theorem 3, and we can define the corresponding generalized canonical integral $u_1(z)$ of this measure. Then a straightforward estimate (cf. 6.5) shows that for $z \in \mathbb{C}_+$

$$u(z) \leq u_1(z) + 2|y| \int_{\mathbb{C}} \left| \frac{1}{\zeta} \right| d\mu(\zeta) \lesssim (B + C)|z|.$$  

The same estimate holds in the lower half-plane, and the general case of Theorem 3 follows. □
References

[1] A. Aleksandrov and P. Kargaev, *Hardy classes of functions that are harmonic in a half-space*, Algebra & Analysis 5 (1993), no. 2, 1–73. (Russian) English transl. in St. Petersburg Math. J. 5 (1994).

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*. Acad. Press, London, 1988.

[3] M. Essén, *Some best constants inequalities for conjugate functions*, Internat. Ser. Numer. Math. 103, Birkhäuser, Basel, 1992.

[4] T. W. Gamelin, *Uniform algebras and Jensen measures*. London Math. Soc. Lecture Note Series, Cambridge Univ. Press, 1978.

[5] A. A. Goldberg and I. V. Ostrovskii, *Value distribution of meromorphic functions*. Nauka, Moscow, 1970. (Russian)

[6] B. Khabibullin, *Sets of uniqueness in spaces of entire functions of one variable*, Math. USSR Izv. 39 (1992), 1063–1083.

[7] P. Koosis, *Introduction to $H_p$ spaces, 2nd ed.* Cambridge Univ. Press, 1998.

[8] P. Koosis, *Leçons sur le Théorème de Beurling et Malliavin*. Les Publications CRM, Montréal, 1996.

[9] B. Ya. Levin, *On functions holomorphic in a half-plane*, Travaux de l’Université d’Odessa (Math) 3 (1941), 5-14. (Russian)

[10] B. Ya. Levin, *Lectures on Entire Functions*. Transl. Math. Monographs, vol. 150, Amer. Math. Soc., Providence RI, 1996.

[11] B. Ya. Levin and I. V. Ostrovskii, *The dependence of the growth of an entire function on the distribution of the zeros of its derivatives*, Sibirsk. Mat. Zh. 1 (1960), 427–455. (in Russian). English transl. in Amer. Math. Soc. Transl. (2) 32 (1963), 323–357.

[12] V. Matsaev and M. Sodin, *Variations on the theme of M. Riesz and Kolmogorov*, Intern. Math. Res. Notices, no. 6 (1999), 287–297.
[13] V. Matsaev and M. Sodin, *Distribution of the Hilbert transforms of measures*, Geom. Funct. Anal. **10** (2000), 160–184.

[14] V. Matsaev and M. Sodin, *Compact operators with $S_p$-imaginary component and entire functions*, Proc. Israel Math. Conf., to appear.

[15] R. Nevanlinna, *Über die Eigenschaften meromorpher Funktionen in einem Winkelraum*, Acta Soc. Sci. Fenn. **50** no. 12 (1925).

[16] M. Tsuji, *On Borel’s directions of meromorphic functions of finite order*, Tôhoku Math. J. **2** (1950), 97-112.

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