NON-UNIQUENESS IN LAW OF THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS DIFFUSED VIA A FRACTIONAL LAPLACIAN WITH POWER LESS THAN ONE HALF

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Abstract. Non-uniqueness of three-dimensional Euler equations and Navier-Stokes equations forced by random noise, path-wise and more recently even in law, have been proven by various authors. We prove non-uniqueness in law of the three-dimensional Navier-Stokes equations forced by random noise and diffused via a fractional Laplacian that has power between zero and one half. The solution we construct has Hölder regularity with a small exponent rather than Sobolev regularity with a small exponent. For the power sufficiently small, the non-uniqueness in law holds at the level of Leray-Hopf regularity. In particular, in order to handle transport error, we consider phase functions convected by not only a mollified velocity field but a sum of that with a mollified Ornstein-Uhlenbeck process if noise is additive and a product of that with a mollified exponential Brownian motion if noise is multiplicative.

Keywords: convex integration; fractional Laplacian; Navier-Stokes equations; non-uniqueness; random noise.

1. Introduction

1.1. Motivation from physics and mathematics. Various ways to describe dissipation have been proposed by atmospheric scientists and geophysicists (e.g., frictional dissipation in [49]). In particular, in models such as surface quasi-geostrophic equations, diffusion in a form of a fractional Laplacian appears naturally (e.g., [11]). Specifically, \((-\Delta)^m\) for \(m \in \mathbb{R}_+\) as a Fourier operator with a symbol \(|\xi|^{2m}\) so that \((-\Delta)^m \hat{f}(\xi) = |\xi|^{2m} \hat{f}(\xi)\) for any integrable function \(f\) on \(\mathbb{R}^n\) or \(T^n = [-\pi, \pi]^n, n \in \mathbb{N}\), \(n = \{1, 2, \ldots\}\). Introduced for the first time by Lions [35, p. 263] who subsequently in [36, Equ. (6.164) on p. 97] claimed the uniqueness of its Leray-Hopf weak solution (see Definition 1.1) when \(m \geq \frac{1}{2} + \frac{\Phi}{n}\), the generalized Navier-Stokes (GNS) equations (1) that has diffusion in the form of \((-\Delta)^m\) (so that it recovers the classical NS equations when \(m = 1\)) has captured the interests of mathematicians for more than sixty years. Except logarithmic improvements in the case of smooth initial data that was initiated by Tao [53] (also [2]), Lions’ threshold of \(\frac{1}{2} + \frac{\Phi}{n}\) remains unbroken. On the other hand, non-uniqueness of Leray-Hopf weak solutions to the GNS equations (1) when \(m = 1\) was famously conjectured by Ladyzhenskaya [33] and remains open. Analogous statements may be made for the NS equations forced by random noise that have received much attention from researchers for more than half a century since the work of Novikov [47] (e.g., [13] for the GNS equations forced by random noise). In particular, failure of path-wise uniqueness of Leray-Hopf weak solution to the GNS equations forced by random noise with exponent \(m = 1\) remains open. This research direction

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concerning non-uniqueness has received special attention from the general community of stochastic partial differential equations and significant progress has been made for a certain heat equation (e.g., [41] [42] [43]); however, extending the techniques developed therein to the GNS equations that are non-linear and non-local seems to be completely out of reach. In this manuscript, we prove non-uniqueness, not only path-wise but even in law (see Definition 1.2). of the GNS equations with exponent \( m \in (0, \frac{1}{2}) \) forced by random noise \( \mathbf{g} \) at the level of spatial regularity \( C^\gamma \), \( \gamma > 0 \) sufficiently small (see Theorems 2.1-2.4). Consequences of our results include non-uniqueness in law of the GNS equations forced by random noise \( \mathbf{g} \) at the level of Leray-Hopf regularity when \( m \) is sufficiently small (see Remark 2.1). In what follows, we elaborate to make these statements precise.

1.2. Previous works. We denote \( \partial_t \triangleq \frac{\partial}{\partial t} \), velocity and pressure fields, and viscosity by \( u : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \pi : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}, \text{ and } \nu \geq 0 \), respectively. Then the GNS equations read

\[
\partial_t u + (u \cdot \nabla)u + \nabla \pi + \nu (\Delta u) = 0, \quad \nabla \cdot u = 0, \quad \text{ for } t > 0. \tag{1}
\]

The case \( m = 1, \nu > 0 \) gives the classical NS equations while \( \nu = 0 \) the Euler equations.

**Definition 1.1.** (E.g., [7] Def. 3.5 and 3.6) Suppose \( \nu > 0 \). If \( u(t, \cdot) \) is weakly divergence-free, mean-zero, satisfies (1) distributionally and \( \|u(t)\|_{L_2}^2 + 2\nu \|u\|_{L_2^\gamma}^2 \leq \|u(0)\|_{L_2^\gamma}^2 \) for any \( t \in [0, T] \), then \( u \in C_0^{\text{weak}}([0, T]; L_2^2) \cap L_2^2([0, T]; H_2^\mu) \) is a Leray-Hopf weak solution of (1). On the other hand, if \( u(t, \cdot) \) is weakly divergence-free, mean-zero, and satisfies (1) distributionally for any \( t \in [0, T] \), then \( u \in C_0^1 L_2^2 \) is a weak solution of (1).

The global existence of a Leray-Hopf weak solution to (1) in case \( m = 1 \) is classical [30] [54], while the case \( m \in (0, 1) \) can be found in [10] The. 1.1]. Next, let us consider

\[
du + [\nu (\Delta u) + \text{div}(u \otimes u) + \nabla \pi]dt = F(u)dB, \quad \nabla \cdot u = 0, \quad \text{ for } t > 0. \tag{2}
\]

**Definition 1.2.** Suppose that \( F \) is a certain operator (to be stated precisely in Section 2) and \( B \) is a Brownian motion. Then the existence of a Leray-Hopf weak solution to (2) in case \( m = 1 \), i.e., at the regularity of \( L_2^\infty L_2^2 \cap L_2^2 H_2^1 \) and the energy inequality, was proven in [25] (see [25] Def. 3.3); cf. [24] in which the existence of a weak solution to (2) in case \( m = 1 \) was proven but without the energy inequality (see [24] Def. 3.1). If for any solution \( (u, B) \) and \( (\tilde{u}, B) \) with same initial distributions, defined potentially on different filtered probability spaces, \( \mathcal{L}(u) = \mathcal{L}(\tilde{u}) \), holds, where \( \mathcal{L}(v) \) represents the law of \( v \), then uniqueness in law holds for (2). On the other hand, if for any solutions \( (u, B) \) and \( (\tilde{u}, B) \) with common initial data defined on same probability space, \( u(t) = \tilde{u}(t) \) for all \( t \) with probability one, then path-wise uniqueness holds for (2). While uniqueness in law does not imply path-wise uniqueness (see [8] Exa. 2.2]), Yamada-Watanabe theorem implies the converse. Moreover, if a solution is adapted to the canonical right continuous filtration generated by \( B \) and augmented by all the negligible sets, then it is a strong solution. By [8] The. 3.2], existence of a strong solution and uniqueness in law together imply path-wise uniqueness.

We point out that a typical proof of path-wise uniqueness, when possible, is similar to the deterministic case. For such a reason and more, a general consensus has been to devote effort to prove uniqueness in law for (2) (see [17] p. 878–879]), until the recent breakthrough developments of convex integration, which we review next.

Gromov [27] Par. 2.4 considered the \( C^1 \) isometric embedding theorem due to Nash [46] and Kuiper [32] as a primary example of homotopy-principle and developed convex integration technique. Müller and Šverak applied this technique to prove the existence of unexpected solutions to various equations in [44], and extended it to Lipschitz mappings.
in [45]. De Lellis and Székelyhidi Jr. [18] extended the technique and proved the global existence of a weak solution to $n$-dimensional ($n$D) Euler equations, $n \geq 2$, in $L^2_{x,v}$ with compact support in space-time, extending the previous works of [51, 52] in case of $n = 2$ and regularity only in $L^2_{x,v}$. Improving the convex integration technique was further motivated in effort to prove the negative direction of Onsager’s conjecture [48] (see [12, 23] for positive direction) and wealth of remarkable results flourished (e.g., [4, 19, 20, 21]) until Isett [31] provided its complete resolution. Although the convex integration technique was limited to the Euler equations up to this point, Buckmaster and Vicol [6] introduced new tool called intermittent Beltrami flows and proved the non-uniqueness of 3D NS equations. This inspired many variations: non-uniqueness of 3D GNS equations for tool called intermittent Beltrami flows and proved the non-uniqueness of 3D NS equations. and furthermore in the class of weak solutions with bounded kinetic energy, integrable vorticity that are smooth outside a fractal set of singular times with Hausdorff dimension strictly less than one [5]; non-uniqueness of 2D GNS equations for $m \in (0, 1)$ [37] (and Boussinesq system [39]).

The implications of convex integration reached the stochastic case as well: non-uniqueness path-wise of compressible Euler [5, 9] (29 for non-uniqueness in law); non-uniqueness in law of stochastic NS equations (2) with $n = 3, m = 1$ in [28], $n = 3, m \in (\frac{9}{7}, \frac{3}{2})$ in [56], $n = 2, m \in (0, 1)$ in [57] (and stochastic Boussinesq system in [58]). A natural question is whether such non-uniqueness results can be extended to the case $n = 3, m \in (0, \frac{3}{2})$. The heart of the matter in the proof is the careful adaptation of convex integration technique to the stochastic case and upon a close inspection, it turns out that the convex integration part of [56] cannot be extended to the case $m \leq \frac{1}{3}$. The previous works in the deterministic case (e.g., [5, Th. 1.5] and [38, Th. 1]) also required $m \geq 1$. This direction of research was partially explored by the authors in [10, 22] who proved the non-uniqueness of Leray-Hopf weak solution to the 3D GNS equations for $m \in (0, \frac{1}{2})$ (see [10, Th. 1.2] and [22, Th. 1.2]) while commenting without providing a full proof that appropriate modifications of their arguments can prove the non-uniqueness of weak solution for $m \in (0, \frac{1}{2})$ (see [10, Cor. 2.3], [22, p. 337]). While this raises hope that appropriately adapting the proofs within [10, 22] to the probabilistic settings of [28] can lead to the non-uniqueness in law of the 3D stochastic GNS equations (2) with $m \in (0, \frac{1}{2})$, unfortunately, major obstacles arise, which the author has not been able to resolve directly. The main iteration schemes within [10, 22] consist of an estimate of a convective derivative, and therefore a time derivative, of Reynolds stress, e.g., $\|H_0 R_{q+1} + v_{q+1} \cdot \nabla R_{q+1}\|_0$ in [10, Equ. (68)] (see also [10, Equ. (15)], [22, Eq. (5.43)]). The respective Reynolds stress $R_{q+1}$ for the stochastic GNS equations (2) in an additive noise case for example is given in [108a] that consists of $R_{m_{const}}$ defined in [107b] which in turn consists of an Ornstein-Uhlenbeck process that is only in $C^0_{\gamma}$ for $\alpha < \frac{1}{2}$ by Proposition [22] (see [57, Rem. 1.2] for similar explanation). Therefore, modifying the arguments in [10, 22] suitably to the stochastic case seems very difficult.

2. Statement of main results

Despite the obstacles aforementioned in Subsection [12], we obtain the following results; for simplicity we assume hereafter that $\nu = 1$ in (2) and denote an adjoint operator by an asterisk.

**Theorem 2.1.** Suppose that $n = 3, m \in (0, \frac{1}{2}), F \equiv 1, B$ is a GG-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\text{Tr}((-\Delta)^{m+2\sigma} GG^*) < \infty$ for some $\sigma > 0$. Then given $T > 0, K > 1$, and $\kappa \in (0, 1)$, there exist $\gamma \in (0, 1)$ and a $P$-almost surely (a.s.) strictly positive stopping time $t$ such that

$$P(\{t \geq T\}) > \kappa$$

(3)
and the following is additionally satisfied. There exists an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \(u\) that is a weak solution to (2) starting from a deterministic initial condition \(u^0\), satisfies
\[
\text{esssup}_{\omega \in \Omega} ||u(\omega)||_{C^\gamma_t} < \infty,
\]
and on the set \(\{t \geq T\}, \)
\[
||u(T)||_{L^2} > K||u^m||_{L^2} + K(T Tr(G^*))^{\frac{1}{2}},
\]
(5)

**Theorem 2.2.** Suppose that \(n = 3, m \in (0, \frac{1}{2}), F \equiv 1, B\) is a \(GG^*\)-Wiener process on \((\Omega, \mathcal{F}, \mathbb{P}),\) and \(Tr((-\Delta)^{-m+2\gamma}G^* ) < \infty\) for some \(\gamma > 0\). Then non-uniqueness in law holds for (2) on \([0, \infty)\). Moreover, for all \(T > 0\) fixed, non-uniqueness in law holds for (2) on \([0, T]\).

**Theorem 2.3.** Suppose that \(n = 3, m \in (0, \frac{1}{2}), F(u) = u,\) and \(B\) is a \(\mathbb{R}\)-valued Wiener process on \((\Omega, \mathcal{F}, \mathbb{P})\). Then given \(T > 0, K > 1,\) and \(\kappa \in (0, 1)\), there exist \(\gamma \in (0, 1)\) and a \(\mathbb{P}\)-a.s. strictly positive stopping time \(t\) such that (3) holds and the following is additionally satisfied. There exists an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \(u\) which is a weak solution to (2) starting from a deterministic initial condition \(u^m\), satisfies (4), and on the set \(\{t \geq T\}, \)
\[
||u(T)||_{L^2} > Ke^{\frac{1}{2}} ||u^m||_{L^2},
\]
(6)

**Theorem 2.4.** Suppose that \(n = 3, m \in (0, \frac{1}{2}), F(u) = u,\) and \(B\) is a \(\mathbb{R}\)-valued Wiener process on \((\Omega, \mathcal{F}, \mathbb{P})\). Then non-uniqueness in law holds for (2) on \([0, \infty)\). Moreover, for all \(T > 0\) fixed, non-uniqueness in law holds for (2) on \([0, T]\).

We emphasize that the spatial regularity \(C^\gamma_\Omega\) for \(\gamma > 0\) in (4) is higher than \(H^\gamma_t\) of the solutions constructed in previous works such as [28, 56].

**Remark 2.1.** From [56, 57] in case of Theorems 2.1, 2.2, and (175) in case of Theorems 2.3, 2.4, we see that the only condition on \(\gamma\) is that \(\gamma < \beta\); we choose not to pursue the explicit lower bound of this \(\beta \in (0, \frac{1}{2})\) because it is taken to be quite small in the proofs of Theorems 2.1 and 2.3. Nonetheless, because \(C^\gamma_\Omega \subset L^2_\Omega\), we see that there exists \(m \in (0, \frac{1}{2})\) such that the non-uniqueness in law stated in Theorems 2.2 and 2.4 hold for solutions at the level of Leray-Stampacchia regularity. Proving the non-uniqueness of Leray-Stampacchia weak solution requires additionally showing that appropriate energy inequality holds, and that seems difficult (cf. analogous situation for the deterministic Hall-magnetohydrodynamics system [15]).

**Remark 2.2.** There are two reasons to believe that extensions of Theorems 2.1, 2.2 to higher spatial regularity beyond (4) or \(m \geq \frac{1}{2}\) will require new ideas. The first reason is simply technical: in order to handle the diffusive term \((-\Delta)^\mu u\) in (2), we rely on Lemma 6.3 and its hypothesis requires that \(2m + \epsilon \leq 1\) for some \(\epsilon > 0\) and consequently \(m < \frac{1}{2}\) (see [109] and (211)). Second, the solution to the GNS equations (1) possesses scaling-invariance of \(u_\lambda(t, x) \equiv \lambda^{2m-1} u(\lambda^{2m} t, \lambda x)\) for any \(\lambda > 0\), and it follows from the definition of Hölder semi-norm that \(C^{1-2m}(\mathbb{T}^3)\) is a critical space; i.e., \(||u_\lambda(t)||_{C^{1-2m}} = ||u(\lambda^{2m} t)||_{C^{1-2m}}\) (cf. [54] Sec. 5). We note that local well-posedness of GNS equations in critical Besov spaces have been studied in [40, 54, 55]; however, in this largest critical Besov space \(B^{1-2m}_{\infty, \infty} = C^{1-2m}\) (see (1) p. 99), we were able to locate only [59] in which Yu and Zhai proved the local existence and uniqueness of solution to the deterministic GNS equations in \(B^{1-2m}_{\infty, \infty} = C^{1-2m}\) but only in case \(m \in (\frac{1}{2}, 1)\) so that the non-uniqueness result in case \(m < \frac{1}{2}\) can be seen as a complimentary result to [59] and the case \(m = \frac{1}{2}\) remains intriguingly open. We were not able to locate in the literature an extension of [59] to the case \(m \in (0, \frac{1}{2})\).
our work does not cross out such possibilities because $1 - 2m > 0$ for any $m \in (0, \frac{1}{2})$ and $\gamma$ in (4) is quite small. In fact, while $\gamma \in (0, \beta)$ from (56), the estimate of (111) in the proof of Theorem 2.7 requires $\beta < \frac{1}{3}(1 - 2m - \epsilon)$ where $\epsilon \in (0, 1 - 2m)$ and therefore $\gamma < \frac{1}{3}(1 - 2m - \epsilon) < 1 - 2m$ as expected (and identically in (215) within the proof of Theorem 2.3). In this perspective, it seems that one should not expect any better regularity than what we achieved in case $m < \frac{1}{2}$ but close to $\frac{1}{2}$. This is in sharp contrast to the solutions $u \in H'(\mathbb{T}^3)$ for $s > 0$ quite small which were previously constructed in case $m = 1$ (e.g., [6, 28]) that has potential to rise to the level of $H'(\mathbb{T}^3)$ for any $s < \frac{1}{2}$ as the relevant critical space in this case is $H^s(\mathbb{T}^3) = B_{1, \infty}^s(\mathbb{T}^3)$.

Heuristically, our proofs of Theorems 2.7, 2.4 consists of extending “upward” to the GNS equations the approach on the Euler equations in [7, Sec. 5] which applied convex integration at level of $C_{t,x}$ to give a simple proof of [21, Th. 1.1] similarly to how [10, 24] extended the work of [41] on the Euler equations. Simultaneously, we must adapt such arguments to a probabilistic setting from [28] while facing major difficulty due to a transport error within the Reynolds stress on which we will elaborate in Remarks 2.4 and 2.7. Our proof can be readily simplified to prove analogous results in the deterministic case as well, and therefore gives a new simple proof of the non-uniqueness of weak solution to the 3D GNS equations (1) when $m \in (0, \frac{1}{2})$.

Remark 2.3. As aforementioned, non-uniqueness in law of the GNS equations (2) in [56, 57] were successfully extended to the Boussinesq system [58]. An attempt at extensions of Theorems 2.7, 2.4 to the Boussinesq system was countered by a surprising but somewhat inherent difficulty. In the Boussinesq system, the equation of velocity field (2) contains $\theta e^3$ where $\theta : \mathbb{R}_+ \times \mathbb{T}^3 \rightarrow \mathbb{R}$ represents temperature and $e^3$ the standard basis of $\mathbb{R}^3$. Consequently, $R_{q+1}$ in (1058) would consist of $(\theta - \theta_q)e^3$ where $\theta_q$ is after mollification in space-time (see [58, Eq. (93), (116), and (117a)]). Although the iteration scheme in [58] required only $||R_{q+1}||_{C_{t,x}}$ (see [58, Eq. (60b)]), those in the current manuscript will require $||R_{q+1}||_{C_{t,x}}$ (see (452)). Considering that one can apply mollifier estimates to $\theta_1 - \theta_q$, we can split

\[ ||R((\theta_1 - \theta_q)e^3)||_{C_{t,x}} \leq ||R((\theta_1 - \theta_0)e^3)||_{C_{t,x}} + ||R((\theta_0 - \theta_q+1)e^3)||_{C_{t,x}} \]

(see [58, Eq. (183)]) and reduce the workload of $||R((\theta_1 - \theta_q)e^3)||_{C_{t,x}}$ to $||R((\theta_0 - \theta_q+1)e^3)||_{C_{t,x}}$ where $R$ is a divergence-inverse operator (see Lemma 6.7). Now one way to proceed is, similarly to [58, Eq. (126)], to rely on $W^{1,4}(\mathbb{T}^3) \hookrightarrow C(\mathbb{T}^3)$ and obtain

\[ ||R((\theta_0+1 - \theta_q)e^3)||_{C_{t,x}} \leq C||\theta_0+1 - \theta_q||_{C_{t,x}} \leq C||v_{q+1} - v_q||_{C_{t,x}} \int_0^\infty ||\theta||_{W^{1,4}}dr. \]  

(7)

For $\|v_{q+1} - v_q\|_{C_{t,L^2}}$ within [7, 55] offers a bound by $(2\pi)^3\|v_{q+1} - v_q\|_{C_{t,x}} \leq (2\pi)^3 M_0(\hat{\epsilon})\delta_{q+1}$ (see (42) for definitions of $M_0(\hat{\epsilon})$ and $\delta_{q+1}$); however, we need to bound (7) by a constant multiple of $M_0(\hat{\epsilon})\delta_{q+2}$ (see (45c)) and this will not be small enough because $\delta_{q+2} \ll \delta_{q+1}$.

Here, there is a room for improvement between $\|v_{q+1} - v_q\|_{C_{t,L^2}}$ of (7) and $\|v_{q+1} - v_q\|_{C_{t,x}}$ in [55]. Indeed, in [58, Eq. (133)], this issue is overcome by first splitting $\|v_{q+1} - v_q\|_{C_{t,L^2}}$ to $\|v_{q+1} - v_q\|_{C_{t,L^2}} + \|v - v_q\|_{C_{t,L^2}}$ where the second term can be handled via standard mollifier estimates while the first term by careful $L^p(\mathbb{T}^3)$-estimates. Unfortunately, the convex integration schemes within this manuscript are extensions of the approach on the Euler equations and will be completely at the level of $C(\mathbb{T}^3)$ (see [45]), in contrast to the approach on the NS equations which can be on $L^p(\mathbb{T}^3)$ for $p < \infty$ (e.g., [7, Sec. 7]).
3. Preliminaries

We denote by \([x]\) the smallest integer \(j\) such that \(j \geq x\). For vector \(v\), we denote its \(j\)-th component by \(v^{(j)}\). We write \(A \leq_{ab} B\) and \(A \geq_{ab} B\) to imply that there exists a constant \(C(a, b) \geq 0\) such that \(A \leq C(a, b)B\) and \(A = C(a, b)B\) due to (\(\cdot\)), respectively. We set \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). For \(j, m \in \mathbb{N}_0\) we denote supremum norm by

\[
\|f\|_{C_{t,x}} \triangleq \|f\|_{C^{0}_{t,x}} \triangleq \sup_{s \in [0,T]} |f(s, x)| \quad \text{and} \quad \|f\|_{C^{m}_{t,x}} \triangleq \sum_{0 \leq \beta \leq m} \|\partial_\beta f\|_{C^{0}_{t,x}}.
\] (8)

Furthermore, given \(\alpha \in (0, 1)\), we define Hölder semi-norms and norms respectively by

\[
\|f\|_{C^{\alpha}_{t,x}} = \max_{|\beta| = m} \|D_\beta f\|_{C^{0}_{t,x}}, \quad \|f\|_{C^{\alpha}_{t,x} + s} = \max_{|\beta| = m} \sup_{s, \xi, \eta \in [0,T]} \frac{|D_\beta f(s, \xi) - D_\beta f(s, \eta)|}{|\xi - \eta|^s},
\] (9a)

\[
\|f\|_{C^{\alpha}_{t,x}} \triangleq \sum_{j=0}^m \|f\|_{C^{\alpha-\frac{j}{m}}_{t,x}}, \quad \|f\|_{C^{\alpha}_{t,x} \cap s} \triangleq \|f\|_{C^{\alpha}_{t,x}} + \|f\|_{C^{\alpha}_{t,x} + s};
\] (9b)

here, \(\beta\) is a multi-index over \(\mathbb{T}^3\). Let us recall from [44] Equ. (128) that for \(r \geq s \geq 0\),

\[
\|f\|_{C^{\alpha}_{t,x}} \leq \|f\|_{C^{\alpha}_{t,x} \cap s}. \quad (10)
\]

We denote \(L^2_\sigma \triangleq \{f \in L^2(\mathbb{T}^3) : \langle \nabla \cdot f = 0, \int_{\mathbb{T}^3} f \, dx = 0\}\) to be the Leray projection operator and denote by \(\hat{\phi}\) the trace-free part of a tensor product. For any Polish space \(H\), we define \(\mathcal{B}(H)\) to be the \(\sigma\)-algebra of Borel sets in \(H\). Given any probability measure \(P\) on \(\mathcal{B}(\Omega_0)\) we denote \(\mathbb{P}\) to be the mathematical expectation with respect to \((\text{w.r.t.}~P)\). We represent an \(L^2(\mathbb{T}^3)\)-inner product of \(A\) and \(B\) and a quadratic variation of \(A\) by \(\langle A, B\rangle\) and \(\langle\langle A\rangle\rangle\), respectively. We define \(\mathcal{P}(\Omega_0)\) to be the set of all probability measures on \((\Omega_0, \mathcal{B})\) where \(\Omega_0 \triangleq C([0, \infty); H^{-3}(\mathbb{T}^3)) \cap L^\infty_{\text{loc}}([0, \infty); L^2_\sigma)\) and \(\mathcal{B}\) is the Borel \(\sigma\)-field of \(\Omega_0\). We define the canonical process \(\tilde{\xi}\) from \((\Omega_0, \mathcal{B})\) to \(H^{-3}(\mathbb{T}^3)\) by \(\tilde{\xi}(\omega) \triangleq \omega(t)\). Similarly, for \(t \geq 0\) we define \(\tilde{\Omega} \triangleq C([t, \infty); H^{-3}(\mathbb{T}^3)) \cap L^\infty_{\text{loc}}([t, \infty); L^2_\sigma)\) and the following Borel \(\sigma\)-algebras: \(\mathcal{B}_t^n \triangleq \sigma(\tilde{\xi}(s) : s \geq t)\), \(\widetilde{\mathcal{B}}^0_t \triangleq \sigma(\tilde{\xi}(s) : s \leq t)\), \(\mathcal{B}_1 \triangleq \bigcap_{t \geq 0} \mathcal{B}^0_t\). For any Hilbert space \(U\) we denote by \(L^2(U, L^2_\sigma)\) the space of all Hilbert-Schmidt operators from \(U\) to \(L^2_\sigma\) with its norm \(\|\cdot\|_{L^2(U, L^2_\sigma)}\). We require \(G : L^2_\sigma \rightarrow L^2(U, L^2_\sigma)\) to be \(\mathcal{B}(L^2_\sigma)\)-\(\mathcal{B}(L^2(U, L^2_\sigma))\)-measurable and satisfy for any \(\psi \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma\)

\[
\|G(\psi)\|_{L^2(U, L^2_\sigma)} \leq C(1 + \|\psi\|_{L^2_\sigma}) \quad \text{and} \quad \lim_{j \rightarrow \infty} \|\langle \theta \rangle^j \psi - G(\theta)^j \psi\|_U = 0
\] (11)

for some constant \(C \geq 0\) such that \(\lim_{j \rightarrow \infty} \|\theta_j - \theta\|_{L^2_\sigma} = 0\). Furthermore, we assume the existence of another Hilbert space \(U_1\) such that the embedding \(U \hookrightarrow U_1\) is Hilbert-Schmidt. We define \(\widetilde{\Omega} \triangleq C([0, \infty); H^{-3}(\mathbb{T}^3) \times U_1) \cap L^\infty_{\text{loc}}([0, \infty); L^2_\times U_1)\) and \(\mathcal{P}(\tilde{\Omega})\) as the set of all probability measures on \((\tilde{\Omega}, \tilde{\mathcal{B}})\), where \(\tilde{\mathcal{B}}\) is the Borel \(\sigma\)-algebra on \(\tilde{\Omega}\). Analogously, we also define the canonical process on \(\tilde{\Omega}\) as \((\tilde{\xi}, \theta) : \tilde{\Omega} \rightarrow H^{-3}(\mathbb{T}^3) \times U_1\) by \((\tilde{\xi}(\omega), \theta(\omega)) \triangleq \omega(t)\). We extend the previous definitions of \(\mathcal{B}_t^n, \mathcal{B}_1^n, \mathcal{B}_1^0\), and \(\tilde{\mathcal{B}}_t^n, \mathcal{B}_t^0\) to \((\tilde{\mathcal{B}}_t^n, \mathcal{B}_t^0)\) and \((\mathcal{B}_1^n, \mathcal{B}_1^0)\) for \(t \geq 0\), respectively.

The convex integration scheme we will employ in this manuscript is different from those in [10] [22] (deterministic) or [28] [56] [57] [58] (stochastic). We recall some setups from [7] [21] which were actually applied to the 3D deterministic Euler equations rather than the GNS equations. First, given \(\zeta \in \mathbb{S}^2 \cap \mathbb{Q}^3\), let \(A_\zeta \in \mathbb{S}^2\) satisfy

\[
A_\zeta \cdot \dot{\zeta} = 0 \quad \text{and} \quad A_{-\zeta} = A_\zeta.
\] (12)

We define

\[
B_\zeta \triangleq 2^{-\frac{1}{2}}(A_\zeta + i\zeta \times A_\zeta) \in \mathbb{C}^3.
\] (13)
It follows that
\[ |B_\zeta| = 1, \quad B_\zeta \cdot \zeta = 0, \quad i\zeta \times B_\zeta = B_\zeta, \quad \text{and} \quad B_{-\zeta} = \bar{B}_\zeta. \quad (14) \]

Next, for any \( \lambda \in \mathbb{Z} \) such that \( \lambda \zeta \in \mathbb{Z}^3 \) we define
\[ W_{\zeta,\lambda}(x) \triangleq W_\zeta \lambda(x) \triangleq B_\zeta e^{i\lambda \zeta \cdot x} \quad (15) \]

so that it is \( T^3 \)-periodic, divergence-free, and
\[ \nabla \times W_{\zeta}(x) = \lambda W_{\zeta}(x). \quad (16) \]

**Lemma 3.1.** ([21 Pro. 3.1], [7 Pro. 5.5]) Let \( \Lambda \) be a given finite subset of \( S^2 \cap \mathbb{Q}^3 \) such that \( -\Lambda = \Lambda \), and \( \lambda \in \mathbb{Z} \) be such that \( \lambda \Lambda \subset \mathbb{Z}^3 \). Then for any choice of coefficients \( a_\zeta \in \mathbb{C} \) such that \( \bar{a}_\zeta = a_{-\zeta} \) and \( B_\zeta \) defined by (13), the vector field
\[ W(x) \triangleq \sum_{\zeta \in \Lambda} a_\zeta B_\zeta e^{i\lambda \zeta \cdot x} \quad (17) \]
is a \( \mathbb{R} \)-valued, divergence-free Beltrami vector field such that \( \nabla \times W = \lambda W \), and thus it is a stationary solution of the Euler equations
\[ \text{div}(W \otimes W) = \nabla |W|^2 \quad (18) \]

Furthermore, the following identities hold:
\[ B_\zeta \otimes B_{-\zeta} + B_{-\zeta} \otimes B_\zeta = \text{Id} - \zeta \otimes \zeta \quad \text{and} \quad \int_{T^3} W \otimes W \, dx = \frac{1}{2} \sum_{\zeta \in \Lambda} |a_\zeta|^2 (\text{Id} - \zeta \otimes \zeta). \quad (19) \]

**Lemma 3.2.** ([21 Lem. 3.2], [7 Pro. 5.6]) There exists a sufficiently small constant \( C_* > 0 \) with the following properties. Let \( B_{C_*} \text{(Id)} \) denote the closed ball of symmetric \( 3 \times 3 \) matrices, centered at \( \text{Id} \) of radius \( C_* \). Then there exist pair-wise disjoint subsets
\[ \Lambda_\alpha \subset S^2 \cap \mathbb{Q}^3, \quad \alpha \in \{0, 1\}, \quad (20) \]
and smooth positive functions
\[ \gamma_\zeta^{(\alpha)} \in C^\omega(B_{C_*}(\text{Id})), \quad \alpha \in \{0, 1\}, \zeta \in \Lambda_\alpha, \quad (21) \]
such that for every \( \zeta \in \Lambda_\alpha \) we have \( -\zeta \in \Lambda_\alpha \) and \( \gamma_\zeta^{(\alpha)} = \gamma_{-\zeta}^{(\alpha)} \), while for every \( R \in B_{C_*}(\text{Id}) \) we have the identity
\[ R = \frac{1}{2} \sum_{\zeta \in \Lambda_\alpha} (\gamma_\zeta^{(\alpha)}(R))^2 (\text{Id} - \zeta \otimes \zeta). \quad (22) \]

It suffices to consider index sets \( \Lambda_0 \) and \( \Lambda_1 \) in Lemma [3.2] to have 12 elements (cf. [6 Rem. 3.3]). By abuse of notation, we hereafter denote \( \Lambda_j = \Lambda_j \mod 2 \) for \( j \in \mathbb{N}_0 \). For convenience, we denote a universal constant \( M \) such that for both \( j \in \{0, 1\} \)
\[ \sum_{\zeta \in \Lambda_j} \|\gamma_\zeta^{(j)}\|_{L^1(|x|^{1+j}) \cap C^0(B_{C_*}(\text{Id}))} \leq M. \quad (23) \]

We leave rest of the preliminaries in the Appendix [6].

4. PROOFS OF THEOREMS [2.1] [2.2]

Without loss of generality we assume that \( \sigma \) in the hypothesis of Theorems [2.1] [2.2] satisfy \( \sigma \in (0, 1) \).
4.1. Proof of Theorem [2.2] assuming Theorem [2.1] We fix $\gamma \in (0, 1)$ for the following definitions.

**Definition 4.1.** Let $s \geq 0$ and $\xi^m \in L^m_\omega$. Then $P \in \mathcal{P}(\Omega_0)$ is a martingale solution to [2] with initial condition $\xi^m$ at initial time $s$ if

1. $P(\{\xi(t) = \xi^m \; \forall \; t \in [0, s]\}) = 1$ and for all $l \in \mathbb{N}$

$$P(\{\xi \in \Omega_0 : \int_0^s \|G(\xi(r))\|^2_{L^2(U \cup \mathbb{L}^2 \cup L^2)} dr \leq \infty\}) = 1,$$

2. for every $\psi_i \in C^\infty(\bar{\mathbb{T}^3}) \cap L^2_\omega$ and $t \geq s$, the process

$$M^i_{t,s} \triangleq \langle \xi(t) - \xi(s), \psi_i + \int_s^t \langle \text{div}(\xi(r) \otimes \xi(r)) + (-\Delta)^m \xi(r), \psi_i \rangle dr \rangle$$

is a continuous, square-integrable $(\mathcal{B}_t)_{t \geq s}$-martingale under $P$ with $\langle \langle M^i_{t,s} \rangle \rangle = \int_s^t \|G(\xi(r))\psi_i\|^2_{L^2} dr$.

3. for any $q \in \mathbb{N}$, there exists a function $t \mapsto C_{t,q} \in \mathbb{R}_+$ such that for all $t \geq s$,

$$\mathbb{E}_t^P \left[ \sup_{r \in [s, t]} \|\xi(r)\|^2_{L^q} + \int_s^t \|\xi(r)\|^2_{H^2} dr \right] \leq C_{t,q}(1 + \|\xi^m\|^2_{L^2}).$$

The set of all such martingale solutions with the same constant $C_{t,q}$ in (26) for every $q \in \mathbb{N}$ and $t \geq s$ will be denoted by $C(s, \xi^m, C_{t,q})_{q \in \mathbb{N}, t \geq s}$.

In the current case of additive noise, if $\{\psi_i\}_{i=1}^m$ is a complete orthonormal system that consists of eigenvectors of $GG^*$, then $M^i_{t,s} \triangleq \sum_{i=1}^m M^i_{t,s} \psi_i$ becomes a $GG^*$-Wiener process starting from initial time $s$ w.r.t. the filtration $(\mathcal{B}_t)_{t \geq s}$ under $P$.

**Definition 4.2.** Let $s \geq 0$, $\xi^m \in L^m_\omega$ and $\tau : \Omega_0 \mapsto [s, \infty]$ be a stopping time of $(\mathcal{B}_t)_{t \geq s}$. Define the space of trajectories stopped at $\tau$ by

$$\Omega_{0,\tau} \triangleq \{\omega(\cdot \wedge \tau(\omega)) : \omega \in \Omega_0\} = \{\omega \in \Omega_0 : \xi(t, \omega) = \xi(t \wedge \tau(\omega), \omega) \; \forall \; t \geq 0\}.$$  

Then $P \in \mathcal{P}(\Omega_{0,\tau})$ is a martingale solution to [2] on $[s, \tau]$ with initial condition $\xi^m$ at initial time $s$ if

1. $P(\{\xi(t) = \xi^m \; \forall \; t \in [0, s]\}) = 1$ and for all $l \in \mathbb{N}$

$$P(\{\xi \in \Omega_0 : \int_{s \wedge \tau}^{s \wedge \tau} \|G(\xi(r))\|^2_{L^2(U \cup \mathbb{L}^2 \cup L^2)} dr \leq \infty\}) = 1,$$

2. for every $\psi_i \in C^\infty(\bar{\mathbb{T}^3}) \cap L^2_\omega$ and $t \geq s$, the process

$$M^i_{t,s} \triangleq \langle \xi(t \wedge \tau) - \xi^m, \psi_i + \int_{s \wedge \tau}^{t \wedge \tau} \langle \text{div}(\xi(r) \otimes \xi(r)) + (-\Delta)^m \xi(r), \psi_i \rangle dr \rangle$$

is a continuous, square-integrable $(\mathcal{B}_t)_{t \geq s}$-martingale under $P$ with $\langle \langle M^i_{t,s} \rangle \rangle = \int_{s \wedge \tau}^{t \wedge \tau} \|G(\xi(r))\psi_i\|^2_{L^2} dr$.

3. for any $q \in \mathbb{N}$, there exists a function $t \mapsto C_{t,q} \in \mathbb{R}_+$ such that for all $t \geq s$,

$$\mathbb{E}_t^P \left[ \sup_{r \in [s \wedge \tau, t \wedge \tau]} \|\xi(r)\|^2_{L^q} + \int_{s \wedge \tau}^{t \wedge \tau} \|\xi(r)\|^2_{H^2} dr \right] \leq C_{t,q}(1 + \|\xi^m\|^2_{L^2}).$$

The proof of the following proposition concerning existence and stability of martingale solutions to [2] is identical to that of [56, Pro. 4.1], which in turn follows [28, The. 3.1], because it makes use of the range of $m$ only in a few parts of its proof, which are flexible, and hence can readily be extended to our current case $m \in (0, \frac{1}{2})$. 

Proposition 4.1. For any \((s, \xi^m) \in [0, \infty) \times L^2, \) there exists \(P \in \mathcal{P}(\Omega_0)\) which is a martingale solution to \((2)\) with initial condition \(\xi^m\) at initial time \(s\) according to Definition 2. For all \(t \geq s\), let \(\mathcal{G}_t\) be the \(\sigma\)-algebra associated to \(\mathcal{F}_t\). Moreover, if there exists a family \(\{(s_t, \xi_t)\}_{t \geq s} \subset [0, \infty) \times L^2\) such that \(\lim_{t \to \infty} \|s_t - s\|_{L^2} = 0\) and \(P_t \in C(s_t, \xi_t, \mathcal{C}_{\mathcal{G}_t})\) is the martingale solution corresponding to \((s_t, \xi_t)\), then there exists a subsequence \(\{P_{t_k}\}_{k \geq 1}\) that converges weakly to some \(P \in C(s, \xi^m, \mathcal{C}_{\mathcal{G}_t})\).

Proposition 4.1 leads to the following two results from \([28]\) which apply to our case as their proofs do not rely on the specific form of the diffusive term. Let \(\mathcal{B}_\tau\) represent the \(\sigma\)-algebra associated to any given stopping time \(\tau\).

Lemma 4.2. (cf. \([28]\), Pro. 3.2) Let \(\mathcal{B}_\tau\) be a bounded stopping time of \((\mathcal{B}_\tau)_{\geq 0}\). Then for every \(\omega \in \Omega_0\), there exist \(Q_\omega \doteq \delta_\omega \otimes \tau(\omega) R_{\tau(\omega), \xi(\tau(\omega), \omega)} \in \mathcal{P}(\Omega_0)\) where \(\delta_\omega\) is a point-mass at \(\omega\) and \(R_{\tau(\omega), \xi(\tau(\omega), \omega)} \in \mathcal{P}(\Omega_0)\) is a martingale solution to \((2)\) with initial condition \(\xi(\tau(\omega), \omega)\) at initial time \(\tau(\omega)\) such that
\[
Q_\omega(\omega' \in \Omega_0 : \xi(t, \omega') = \omega(t) \forall t \in [0, \tau(\omega)]) = 1,
\]
\[
Q_\omega(A) = R_{\tau(\omega), \xi(\tau(\omega), \omega)}(A) \forall A \in \mathcal{B}(\mathbb{R}) \cap \tau(\omega),
\]
and the mapping \(\omega \mapsto Q_\omega(B)\) is \(\mathcal{B}_\tau\)-measurable for every \(B \in \mathcal{B}\).

Lemma 4.3. (cf. \([28]\), Pro. 3.4) Let \(\tau\) be a bounded stopping time of \((\mathcal{B}_\tau)_{\geq 0}\), \(\xi^m \in L^2\), and \(P\) be a martingale solution to \((2)\) on \([0, \tau]\) with initial condition \(\xi^m\) at initial time \(0\) according to Definition 4.1. Suppose that there exists a Borel set \(N \subset \Omega_0\) such that \(P(N) = 0\) and \(Q_\omega\) from Lemma 4.2 satisfies for every \(\omega \in \Omega_0 \setminus N\)
\[
Q_\omega(\omega' \in \Omega_0 : \tau(\omega') = \tau(\omega))) = 1.
\]
Then the probability measure \(P \otimes \tau, R \in \mathcal{P}(\Omega_0)\) defined by
\[
P \otimes \tau, R(\cdot) \doteq \int_{\Omega_0} Q_\omega(\cdot) P(d\omega)
\]
satisfies \(P \otimes \tau, R_{\Omega_0, \tau} = P|_{\Omega_0}\), and it is a martingale solution to \((2)\) on \([0, \infty)\) with initial condition \(\xi^m\) at initial time \(0\) according to Definition 4.1.

Now we see that if
\[
d\mathcal{z} + (-\Delta)^m \mathcal{z} dt + \nabla \pi^1 dt = dB, \quad \nabla \cdot \mathcal{z} = 0 \quad \text{for } t > 0, \quad \mathcal{z}(0, x) = 0, \quad (\text{34a})
\]
\[
\partial_t \pi + (-\Delta)^m \pi + \text{div}(\pi \otimes \mathcal{z}) + \nabla \pi^1 = 0, \quad \nabla \cdot \pi = 0 \quad \text{for } t > 0, \quad \pi(0, x) = u^m(x) \quad (\text{34b})
\]
so that \(\mathcal{z}(t) = \int_0^t \mathbb{E} e^{(-\Delta)^m(t-s)} dB(s)\), then \(u = v + \mathcal{z}\) solves \((2)\) with \(\pi = \pi^1 + \pi^2\). Let us formally fix a \(GG^+\)-Wiener process \(B\) on \((\Omega, \mathcal{F}, P)\) with \((\mathcal{F}_t)_{t \geq 0}\) as the canonical filtration of \(B\) augmented by all the \(P\)-negligible sets. We have the following results concerning regularity of \(\mathcal{z}\).

Proposition 4.4. For all \(\delta \in (0, \frac{1}{4}), T > 0, \) and \(l \in \mathbb{N}\),
\[
\mathbb{P}^P \|z\|_{C^l_r \mathcal{H}_1} + \|z\|_{C^l_r \mathcal{H}_2} < \infty
\]

Proof of Proposition 4.4. This is an immediate consequence of \([58]\), Pro. 4.4 and the hypothesis of Theorems 2.1, 12.2 that \(\text{Tr}((-\Delta)^{\frac{1}{2} m^2 + 2 \gamma} GG^+) < \infty\).
Next, for every \( \omega \in \Omega_{0} \) we define

\[
M_{t,0}^{\omega} \triangleq \omega(t) - \omega(0) + \int_{0}^{t} \mathbb{P} \text{div}(\omega(r) \otimes \omega(r)) + (-\Delta)^{m} \omega(r) dr,
\]

\[
Z^{\omega}(t) \triangleq M_{t,0}^{\omega} - \int_{0}^{t} \mathbb{P}(-\Delta)^{m} e^{-(t-r)(-\Delta)^{m}} M_{r,0}^{\omega} dr.
\]

If \( P \) is a martingale solution to (2), then the mapping \( \omega \mapsto M_{t,0}^{\omega} \) is a \( GG' \)-Wiener process under \( P \) and it follows from (36a), (36b) that

\[
Z(t) = \int_{0}^{t} \mathbb{P} e^{-(t-r)(-\Delta)^{m}} dM_{r,0}.
\]

It follows from Proposition 4.4 that for any \( \delta \in (0, \frac{1}{3}) \), \( Z \in C_{T} \mathcal{H}_{\delta}^{t} \subset C_{T}^{1+\delta} \mathcal{H}_{\delta}^{t} \) \( P \)-almost surely. For \( \omega \in \Omega_{0}, I \in \mathbb{N}, \) and \( \delta \in (0, \frac{1}{23}) \), we define

\[
\tau_{I}^{(t)}(\omega) \triangleq \inf\{t \geq 0 : C_{\delta}[Z^{\omega}(t)] \|
\mathcal{H}_{\delta}^{t} \geq (L - \frac{1}{I})^{\frac{1}{2}} \}
\]

\[
\land \inf\{t \geq 0 : C_{\delta}[Z^{\omega}](t) \| \mathcal{H}_{\delta}^{t} \geq (L - \frac{1}{I})^{\frac{1}{2}} \} \land \tau_{I} = \lim_{I \to \infty} \tau_{I},
\]

where \( C_{\delta} > 0 \) is the Sobolev constant such that \( \|f\|_{L_{\infty}} \leq C_{\delta}\|f\| \mathcal{H}_{\delta}^{t} \) for all \( f \in \mathcal{H}_{\delta}^{t} \) that is mean-zero. We note that the condition of \( \delta \in (0, \frac{1}{23}) \) is more restrictive than \( \delta \in (0, \frac{1}{12}) \) in previous works such as (28) [56], and this is needed in (231). By (28) Lem. 3.5] it follows that \( \tau_{I} \) is a stopping time of \((\mathcal{B}_{j})_{j \geq 0}\). We define for \( C_{\delta} \geq 0 \) in (38), \( \tau_{I} \land L \geq 1, \) and \( \delta \in (0, \frac{1}{23}) \),

\[
T_{\tau} \triangleq \inf\{t \geq 0 : C_{\delta}[Z(t)] \| \mathcal{H}_{\delta}^{t} \geq L^{\frac{1}{2}} \} \land \inf\{t \geq 0 : C_{\delta}[Z(t)] \| \mathcal{H}_{\delta}^{t} \geq L^{\frac{1}{2}} \} \land L,
\]

and realize that \( T_{\tau} > 0 \) and \( \lim_{L \to \infty} T_{\tau} = \infty \) \( P \)-a.s. due to Proposition 4.4 The stopping time \( \tau \) in the statement of Theorem 2.1 is actually \( T_{\tau} \) for \( L > 1 \) sufficiently large. Next, we assume Theorem 2.1 on \( (\Omega, \mathcal{F}, (\mathcal{F}_{t})_{t \geq 0}, \mathbb{P}) \) and denote the solution constructed therein by \( u \) and \( P = \mathcal{L}(u) \) the law of \( u \). Then the following propositions can be proven identically to [56] Pro. 4.5 and 4.6 as the proofs therein do not rely on the range of \( m \). We only mention that a consequence from the proof of Proposition 4.5 is that \( \tau_{I} \) from (38) satisfies

\[
\tau_{I}(u) = T_{\tau} \quad \mathbb{P} \text{-almost surely.}
\]

**Proposition 4.5.** Let \( \tau_{I} \) be defined by (38). Then \( P = \mathcal{L}(u) \) where \( u \) is constructed by Theorem 2.1 is a martingale solution to (2) on \([0, \tau_{I}]\) according to Definition 4.2.

**Proposition 4.6.** Let \( \tau_{I} \) be defined by (38) and \( P = \mathcal{L}(u) \) constructed from Theorem 2.1. Then \( P \otimes_{\tau} R \) in (33) is a martingale solution to (2) on \([0, \infty)\) according to Definition 4.1.

At this point we are ready to prove Theorem 2.2 due to its similarity to previous works [28] [56]. we leave this in the Appendix.

4.2. **Proof of Theorem 2.1 assuming Proposition 4.8.** Considering (34b), for \( q \in \mathbb{N}_{0} \) we aim to construct a solution \((v_{q}, \hat{R}_{q})\) to

\[
\partial_{t}v_{q} + (-\Delta)^{m}v_{q} + \text{div}((v_{q} + z) \otimes (v_{q} + z)) + \nabla \pi_{q} = \text{div} \hat{R}_{q}, \quad \nabla \cdot v_{q} = 0 \quad \text{for } t > 0,
\]

where \( \hat{R}_{q} \) is a trace-free symmetric matrix. For any \( a \in \mathbb{N}, \beta \in (0, \frac{1}{2}) \), and \( L > 1 \), we set

\[
\lambda_{q} \triangleq a^{\beta}, \quad \delta_{q} \triangleq \lambda_{q}^{2\beta}, \quad \text{and} \quad M_{0}(t) \triangleq L^{\delta_{q}}
\]
so that \( \delta_{q}^{2} \lambda_q < \delta_{q+1}^{2} \lambda_{q+1} \). We note that one can also set \( \lambda_q = a^{2q} \) for \( b \in \mathbb{N} \) similarly to some previous works (e.g., [6,23]); we chose \( a^{2q} \) for simplicity because choosing \( b \neq 2 \) will not improve our results. We see from (39) that for any \( \delta \in (0, \frac{1}{2}) \) and \( t \in [0, T_L] \),

\[
\|z(t)\|_{L^0_{\gamma}} \leq L^{+}, \quad \|z(t)\|_{Q^{1,0}_{\gamma}} \leq L^{+} \quad \text{and} \quad \|\|C\|_{\gamma}^{1,\gamma} \|_{L^0_{\gamma}} \leq L^{+}
\]

by definition of \( C \) from (38). Now we see that if

\[
a^{2b} > 1 + 2(2\pi)^{\frac{1}{2}},
\]

which we will formally state in (48b), then \( \sum_{1 \leq q \leq \gamma} \delta_q^{2} < \frac{1}{2(2\pi)^{\frac{1}{2}}} < \frac{1}{2} \) for any \( q \in \mathbb{N} \). We set the convention that \( \sum_{1 \leq q \leq \gamma} \delta_q^{2} = 0 \), denote by \( c_{R} > 0 \) a universal constant to be described subsequently (see (73), (81), (95)) and assume the following inductive bounds: for \( q \in \mathbb{N}_0 \) and \( t \in [0, T_L] \),

\[
\|v_{\gamma}\|_{C_{\gamma}} \leq M_0(t)^{\frac{1}{2}}(1 + \sum_{1 \leq q \leq \gamma} \delta_q^{2}) \leq 2M_0(t)^{\frac{1}{2}},
\]

\[
\|v_{\gamma}\|_{C_{\gamma}}^{1,\gamma} \leq M_0(t)^{\frac{1}{2}} \delta_q^{2},
\]

\[
\|\tilde{R}_{0}\|_{C_{\gamma}} \leq c_{R} M_0(t)^{\frac{1}{2}} \delta_q^{2}.
\]

**Proposition 4.7.** For \( L > 1 \), define

\[
v_0(t, x) = (2\pi)^{\frac{1}{2}}L^{2}e^{2Lt}(\sin(x^3) \quad 0 \quad 0)^{T}.
\]

Then together with

\[
\tilde{R}_0(t, x) = \frac{2L^{3}e^{2L}}{(2\pi)^{\frac{1}{2}}}(0 \quad 0 \quad -\cos(x^3) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) + (\mathcal{R}(-\Delta)^{m}v_0 + v_0 \partial_0 z + z \partial_0 v_0 + z \partial_0 z)(t, x),
\]

it satisfies (41) at level \( q = 0 \). Moreover, (35) at level \( q = 0 \) is satisfied provided

\[
\frac{2C_{S}}{\sqrt{2L}} + \frac{20}{(2\pi)^{\frac{1}{2}}L^{+}} + \frac{10}{L^{+}} \leq 1 - \frac{4}{(2\pi)^{\frac{1}{2}}},
\]

\[
(1 + 2(2\pi)^{\frac{1}{2}})^{\frac{1}{2}} < a^{40} \leq c_{R} L,
\]

where the first inequality of (48b) guarantees (44). Furthermore, \( v_0(0, x) \) and \( \tilde{R}_0(0, x) \) are both deterministic.

**Proof of Proposition 4.7** The facts that \( v_0 \) is incompressible, mean-zero, \( \tilde{R}_0 \) is trace-free and symmetric, (41) at level \( q = 0 \) holds with \( \pi_0 \pm -\frac{1}{2}(2v_0 \cdot z + |z|^2) \), as well as \( v_0(0, x) \) and \( \tilde{R}_0(0, x) \) both being deterministic can be readily verified (see [56 Pro. 4.7]). Concerning the three estimates of (45a)-(45c), first we can directly compute from (46)

\[
\|v_0\|_{C_{\gamma}} = (2\pi)^{\frac{1}{2}}M_0(t)^{\frac{1}{2}} \leq M_0(t)^{\frac{1}{2}}, \quad \|v_0\|_{C_{\gamma}}^{1,\gamma} = (2\pi)^{\frac{1}{2}}L^{2}e^{2Lt}2(L + 1) \leq M_0(t)^{\frac{1}{2}}L_0, \quad (49)
\]

and

\[
\|v_0(t)\|_{L^{2}} \leq \frac{M_0(t)^{\frac{1}{2}}}{\sqrt{2}}.
\]

Moreover, we can estimate

\[
\|\tilde{R}_0\|_{C_{\gamma}} \leq (2\pi)^{\frac{1}{2}}4L^{3}e^{2Lt} + \|\mathcal{R}(-\Delta)^{m}v_0\|_{C_{\gamma}} + 20\|v_0\|_{C_{\gamma}}\|\alpha\|_{C_{\gamma}} + 10\|\alpha\|_{C_{\gamma}}^{2}.
\]

(51)
Next, for \( C_S > 0 \) from (38) we can estimate by the Sobolev embeddings \( H^{1-2m}(\mathbb{T}^3) \hookrightarrow H^{\frac{1}{2}}(\mathbb{T}^3) \hookrightarrow C(\mathbb{T}^3) \) for functions that are mean-zero, and the fact that \( \Delta v_0 = -v_0 \),

\[
\|R(-\Delta)^m v_0\|_{L^1_{t,\mathbb{T}^3}} \leq C_S \|R(-\Delta)^m v_0\|_{C_t H^{\frac{1}{2}}(\mathbb{T}^3)} \overset{(38a)}{\leq} C_S 2 \|v_0\|_{C_t L^2} = C_S 2 \frac{M_0(t)^{\frac{3}{2}}}{\sqrt{2}}. \tag{52}
\]

Therefore, applying (52) to (51) gives us

\[
\|\hat{R}_0\|_{C_t} \leq \frac{M_0(t)}{L} \left[ \frac{4}{(2\pi)^{\frac{3}{2}}} + \frac{2C_S}{(2\pi)^{\frac{3}{2}}} + \frac{20}{(2\pi)^{\frac{3}{2}}} + \frac{10}{L^2} \right] \leq c_R M_0(t) \delta_1. \tag{53}
\]

\[\square\]

**Proposition 4.8.** Let \( L \) satisfy

\[L > c_R^{-1}(1 + 2(2\pi)^{\frac{3}{2}})^2\]

and (48a). Suppose that \((v_q, \hat{R}_q)\) is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process that solves (41) and satisfies (45a)–(45c). Then there exist a choice of parameters \( a \) and \( \beta \) such that (48b) is fulfilled and an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \((v_{q+1}, \hat{R}_{q+1})\) that solves (41), satisfies (45a)–(45c) at level \( q + 1 \) and for all \( t \in [0, T_L] \)

\[
\|v_{q+1} - v_q\|_{C_t L^2} \leq M_0(t) \delta_{q+1}^{\frac{3}{2}}. \tag{55}
\]

Finally, if \( v_q(0, x) \) and \( \hat{R}_q(0, x) \) are deterministic, then so are \( v_{q+1}(0, x) \) and \( \hat{R}_{q+1}(0, x) \).

Taking Proposition 4.8 for granted, we are able to prove Theorem 2.1 now.

**Proof of Theorem 2.1 assuming Proposition 4.8** Given any \( T > 0, K > 1 \), and \( \kappa \in (0, 1) \), starting from \((v_0, \hat{R}_0)\) in Proposition 4.7, Proposition 4.8 gives us \((v_q, \hat{R}_q)\) for all \( q \geq 1 \) that are \((\mathcal{F}_t)_{t \geq 0}\)-adapted and satisfy (41), (45a)–(45c), and (55), as well as \( a \) and \( \beta \) such that (48b) is fulfilled. Then for all \( t \in [0, T_L] \), \( \gamma \in (0, \beta) \), using the fact that \( 2^{q+1} \geq 2(q + 1) \) for all \( q \in \mathbb{N}_0 \),

\[
\sum_{q \geq 0} \|v_{q+1} - v_q\|_{C_t L^2} \leq \sum_{q \geq 0} \|v_{q+1} - v_q\|_{C_t L^2} \leq M_0(t) \|v_q\|_{C_t L^2} \leq M_0(t) \delta_{q+1}^{\frac{3}{2}}. \tag{55}
\]

Therefore, \((v_q)_{q=1}^\infty\) is Cauchy in \( C([0, T_L]; C^\gamma(\mathbb{T}^3)) \) and hence we can deduce a limiting solution \( \lim_{q \to \infty} v_q = v \in C([0, T_L]; C^\gamma(\mathbb{T}^3)) \). It follows that there exists a deterministic constant \( C_L > 0 \) such that

\[
\|v\|_{C_t L^2} \leq C_L. \tag{57}
\]

Because each \( v_q \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted, \( v \) is also \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Because \( \lim_{q \to \infty} \hat{R}_q = 0 \) in \( C_{Lx} \) by (45c), we see that \( v \) is a weak solution to (34b) and considering (34a) we see that \( u = v + z \) solves (2) weakly. Now for the universal constant \( c_R > 0 \) determined from the proof of Proposition 4.8 (see (73), (81), (95)), we choose \( L > 1 \) sufficiently large so that it satisfies (54), (48a), and additionally

\[
\frac{3}{2} + \frac{1}{L} < \left( \frac{1}{\sqrt{2\pi}} - \frac{1}{2} \right) e^{LT}, \tag{58a}
\]

\[L^{\gamma}(2\pi)^{\frac{3}{2}} + K(T \text{Tr}(GG^*))^{\frac{3}{2}} \leq (e^{LT} - K)\|u^m\|_{C_t L^2} + L e^{LT}. \tag{58b}\]

As \( \lim_{L \to \infty} T_L = +\infty \) P-a.s. due to (39), for the \( T > 0 \) and \( \kappa > 0 \) already fixed, increasing \( L \) larger if necessary gives us (3). Because \( z \) is \((\mathcal{F}_t)_{t \geq 0}\)-adapted, and we already verified that
v is \((F_t)_{t\geq 0}\)-adapted, we deduce that \(v\) is also \((F_t)_{t\geq 0}\)-adapted. Moreover, \((4.3)\) and \((57)\) give us \((4.3)\). Next, for all \(t \in [0, T_L]\), using the fact that \(2^{q+1} > 2(q + 1)\) for \(q \in \mathbb{N}_0\),

\[
\|v - v_0\|_{C_{t,x}} \leq M_0(t)^{\frac{1}{2}} \sum_{q \geq 0} \delta_{q+1} M_0(t)^{\frac{1}{2}} \sum_{q \geq 0} a_{2(q+1)\beta} \leq \frac{M_0(t)^{\frac{1}{2}}}{2(2\pi)^{\frac{1}{2}}}. \tag{59}
\]

This implies that

\[
\|v - v_0\|_{C_{t,x}} \leq \frac{M_0(t)^{\frac{1}{2}}}{2} \tag{60}
\]

and therefore

\[
\|v(0)\|_{C_{t,x}} + L_e^{L_T} \leq \frac{3}{2} M_0(0)^{\frac{1}{2}} + L_e^{L_T} \leq \frac{1}{\sqrt{2}} - \frac{1}{2} M_0(T)^{\frac{1}{2}} \leq \|v(0)\|_{C_{t,x}} - \|v(T) - v_0(T)\|_{C_{t,x}} \leq \|v(T)\|_{C_{t,x}}. \tag{61}
\]

Therefore, on \([T_L \geq T]\) we obtain

\[
\|u(t)\|_{C_{t,x}} \leq \|u(0)\|_{C_{t,x}} + L_e^{L_T} - \|v(T)\|_{C_{t,x}} \geq \|u^{in}\|_{C_{t,x}} + K(TT_{Tr}(GG^*)^2). \tag{62}
\]

This verifies \((5)\). Finally, because \(v(0, x)\) is deterministic by Proposition \((4.7)\) Proposition \((4.8)\) implies that \(v(0, x)\) remains deterministic; by \((34a)\) this implies that \(u^{in}\) is deterministic. \(\Box\)

### 4.3. Proof of Proposition \((4.8)\)

#### 4.3.1. Mollification

We fix \(L > 0\) that satisfies \((54)\) and \((48a)\) and see that taking \(a \in \mathbb{N}\) sufficiently large and then \(\beta \in (0, \frac{1}{2})\) sufficiently small can give us \((48b)\). Now we define

\[
L = \frac{1}{\lambda_q^{\frac{1}{2}}}. \tag{63}
\]

We let \(\{\varphi_i\}_{i>0}\) and \(\{\varphi_q\}_{q>0}\) be families of standard mollifiers with mass one on \(\mathbb{R}^3\) with compact support and \(\mathbb{R}\) with compact support on \(\mathbb{R}_+\), respectively. Then we mollify \(v_q, \tilde{R}_q\), and \(z\) in space and time to obtain

\[
v_t \equiv (v_q * \varphi_1) *_t \varphi_t, \quad \tilde{R}_t \equiv (\tilde{R}_q *_x \varphi_q) *_t \varphi_t, \quad z_t \equiv (z *_x \varphi_t) *_t \varphi_t. \tag{64}
\]

It follows from \((41)\) that \((v_t, \tilde{R}_t)\) satisfies

\[
\partial_t v_t + (-\Delta)^\beta v_t + \text{div}(v_t + z_t) \otimes (v_t + z_t) + \nabla \pi_t = \text{div}(\tilde{R}_t + R_{\text{com}1}), \quad \nabla \cdot v_t = 0 \tag{65}
\]

for \(t > 0\) where

\[
R_{\text{com}1} \equiv R_{\text{com2}} \equiv (v_t + z_t) *_x (v_t + z_t) - ((v_t + z) *_x (v_t + z)) *_x \varphi_t *_t \varphi_t. \tag{66a}
\]

\[
\pi_t \equiv (\pi_q *_x \varphi_t) *_t \varphi_t - \frac{1}{3} (v_t + z_t) - ((v_t + z)^2) *_x \varphi_t *_t \varphi_t. \tag{66b}
\]

Let us observe that because \(\beta \in (0, \frac{1}{2})\) and mollifiers have mass one, for any \(N \in \mathbb{N}\), by taking \(a \in \mathbb{N}\) sufficiently large,

\[
\|v_q - v_0\|_{C_{t,x}} \leq \frac{M_0(t)^{\frac{1}{2}}}{2} \sum_{q \geq 0} \lambda_q \leq M_0(t)^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}. \tag{67a}
\]

\[
\|v_0\|_{C_{t,x}} \leq \frac{1}{2} M_0(0)^{\frac{1}{2}} \sum_{q \geq 0} \lambda_q \leq \frac{1}{2} \sum_{q \geq 0} \lambda_q M_0(t)^{\frac{1}{2}}, \tag{67b}
\]

\[
\|v_t\|_{C_{t,x}} \leq \|v_q\|_{C_{t,x}} + M_0(t)^{\frac{1}{2}} \sum_{q \geq 0} \lambda_q. \tag{67c}
\]
4.3.2. Perturbation. Next, in order to attain acceptable estimates for transport and corrector errors subsequently, we fix \([0, T_L]\) into an interval of size \(l\), define \(\Phi_j : [0, T_L] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3\) for \(j \in \{0, \ldots, [I^{-1}T_L]\}\) a \(T^3\)-periodic solution to
\[
(\partial_t + (v_l + z_l) \cdot \nabla)\Phi_j = 0,
\]
\[
\Phi_j (j, x) = x.
\]
Let us comment in Remark 4.1 on the importance of including \(z_j\) on (68a). We now collect suitable estimates on \(\Phi_j\).

**Proposition 4.9.** For all \(j \in \{0, \ldots, [I^{-1}T_L]\}\) and \(t \in [l(j-1), l(j+1)]\) with appropriate modification in case \(j = 0\) and \([I^{-1}T_L]\),
\[
\|\nabla \Phi_j(t) - \text{Id} \|_{C^1} \lesssim \log^{\frac{3}{2}} \lambda_q M_0(t) \ll 1, \tag{68a}
\]
\[
\frac{1}{2} \leq \|\nabla \Phi_j(t, x)\| \leq 2 \forall x \in \mathbb{T}^3 \text{ and } \|\Phi_j(t)\|_{C^1} \leq 1, \tag{68b}
\]
\[
\|\partial_t \Phi_j(t)\|_{C^1} \lesssim M_0(t)^{\frac{1}{2}}, \tag{68c}
\]
\[
\|\nabla \Phi_j(t)\|_{C^0} \lesssim I^{-N+1} M_0(t) \delta_q^{\frac{1}{2}} \lambda_q \quad \forall N \in \mathbb{N}, \tag{68d}
\]
\[
\|\partial_t \nabla \Phi_j(t)\|_{C^0} \lesssim I^{-N} M_0(t)^{\frac{1}{2}} \delta_q^{\frac{1}{2}} \lambda_q \quad \forall N \in \mathbb{N}_0 \tag{68e}
\]
(cf. [7] Equ. (5.19a) and (5.19c) on p. 206). [4] Lem. 3.1).

**Proof of Proposition 4.9.** These are just direct consequences of [4] Pro. D.1. Specifically, first, (68a) follows from [7] Equ. (135)) as
\[
\|\nabla \Phi_j(t) - \text{Id}\|_{C^0} \lesssim \log^{\frac{3}{2}} \lambda_q M_0(t) \ll 1.
\]
Second, the first estimate of (68b) follows from (69a) and the second estimate of (68b) follows from [4] Equ. (132)-(133), (67c), and (43). Third, (69c) follows directly from (68a), (69a), (67c), and (43). Fourth, (69d) follows from [4] Equ. (136)) as follows:
\[
\|\nabla \Phi_j(t)\|_{C^0} \lesssim \|I^{-N} M_0(t) \delta_q^{\frac{1}{2}} \lambda_q + \frac{1}{2} I^{-N} \delta_q^{\frac{1}{2}} \lambda_q \|_{C^0} \lesssim I^{-N+1} M_0(t) \delta_q^{\frac{1}{2}} \lambda_q.
\]
Finally, we can take \(\nabla\) on (68a) and estimate for all \(N \in \mathbb{N},
\|
\partial_t \nabla \Phi_j(t)\|_{C^0} \lesssim \|\nabla (v_l + z_l)\| \cdot \|\nabla \Phi_j(t)\|_{C^0} \lesssim \|\nabla (v_l + z_l)\| \cdot \|\nabla \Phi_j(t)\|_{C^0}
\]
\[
\lesssim \lesssim \|I^{-N} M_0(t) \delta_q^{\frac{1}{2}} \lambda_q + \frac{1}{2} I^{-N} \delta_q^{\frac{1}{2}} \lambda_q \|_{C^0} \lesssim I^{-N+1} M_0(t) \delta_q^{\frac{1}{2}} \lambda_q
\]
while the case \(N = 0\) can be proven similarly and more easily. \(\square\)

Next, we introduce a non-negative bump function \(\chi\) that is supported in \((-1, 1)\) such that \(\chi|_{(-1, 1)} \equiv 1\) and shifted bump functions for \(j \in \{0, 1, \ldots, [I^{-1}T_L]\}\)
\[
\chi_j(t) \doteqdot \chi(I^{-1}t - j),
\]
which satisfy for all \(t \in [0, T_L],
\[
\sum_j \chi_j^2(t) = 1 \text{ and supp } \chi_j \subset ((j-1), (j+1));
\]
consequently, for all \(t \in [0, T_L]\), at most two cutoffs are non-trivial. Next, we recall Lemma 3.2 and introduce an amplitude function
\[
a_{ij}(t, x) \doteqdot a_{i+1, j}(t, x) \doteqdot e_q \frac{l}{c} \delta_{q+1} M_0(t)^{\frac{1}{2}} \chi_j(t) \gamma \left\{ \text{Id} - \frac{\partial_t \tilde{R}_j(t, x)}{c R \delta_{q+1} M_0(t)} \right\}.
\]
Thus, for all \((t, x) \in [0, T_L] \times \mathbb{T}^3\), by applying Young’s inequality for convolution, taking \(c_R^+ \leq C_s\) where \(C_s\) is the constant from Lemma 3.2 and relying on the fact that mollifiers have mass one, we obtain

\[
\left| \frac{\hat{R}(t, x)}{c_R^+ \delta_{q+1} M_0(t)} \right| \leq \frac{\|\hat{R}\|_{C_s}}{c_R^+ \delta_{q+1} M_0(t)} \leq \frac{\|\hat{R}\|_{C_s}}{c_R^+ \delta_{q+1} M_0(t)} \leq \frac{\|\hat{R}\|_{C_s}}{c_R^+ \delta_{q+1} M_0(t)} \leq C_s, \tag{73}
\]

and hence \(\frac{\|\hat{R}\|}{c_R^+ \delta_{q+1} M_0} \|_{C_s} \leq C_s\), from which it follows that

\[
\text{Id} - \frac{\hat{R}(t, x)}{c_R^+ \delta_{q+1} M_0(t)} \in B_{C_s}(\text{Id}). \tag{74}
\]

We also obtain the following crucial point-wise identity:

\[
\frac{1}{2} \sum_j \sum_{\xi \in \Lambda_j} a_{\xi}(t, x) (\text{Id} - \zeta \otimes \zeta) \overset{\text{(73)}}{=} c_R^+ \delta_{q+1} M_0(t) - \hat{R}(t, x). \tag{75}
\]

For convenience, let us record suitable estimates of the amplitude function \(a_{\xi}\).

**Proposition 4.10.** The amplitude function \(a_{\xi}\) in (72) satisfies the following bounds on \([0, T_L]\):

\[
\|a_{\xi}\|_{C_s C^N} \overset{(71)}{\leq} c_R^+ \delta_{q+1} M_0(t) \|\gamma_{\xi}\|_{C^{\infty}(B_{C_s}(\text{Id}))^l}^N \quad \forall N \in \mathbb{N}, \tag{76a}
\]

\[
\|a_{\xi}\|_{C_s C^N} \overset{(71)}{\leq} c_R^+ \delta_{q+1} M_0(t) \|\gamma_{\xi}\|_{C^{\infty}(B_{C_s}(\text{Id}))^{l-1}}^N \quad \forall N \in \mathbb{N}. \tag{76b}
\]

**Proof of Proposition 4.10** The first estimate (76a) in case \(N = 0\) follows immediately from (71) and (74). In case \(N \in \mathbb{N}\) we see that

\[
\|a_{\xi}\|_{C_s C^N} \overset{(71)}{\leq} c_R^+ \delta_{q+1} M_0(t) \left\| \gamma_{\xi} \left( \text{Id} - \frac{\hat{R}(s, x)}{c_R^+ \delta_{q+1} M_0(s)} \right) \right\|_{C_s C_c N} \tag{77}
\]

where we can rely on [4] Equ. (129) to deduce

\[
\left\| \gamma_{\xi} \left( \text{Id} - \frac{\hat{R}(s, x)}{c_R^+ \delta_{q+1} M_0(s)} \right) \right\|_{C_s C_c N} \overset{\text{(73)}}{\leq} \|\gamma_{\xi}\|_{C^{\infty}(B_{C_s}(\text{Id}))^l}^N c_R^+ \tag{78}
\]

so that applying (78) to (77) verifies (76a) in this case as well. For estimate (76b) we can directly differentiate (72) w.r.t. \(t\) so that relying on (45c) in case \(N = 0\) while additionally applying [4] Equ. (129) in case \(N \in \mathbb{N}\) can give us the desired results. \(\square\)

Next, we define

\[
w_{\xi}^{(p)}(t, x) \equiv w_{\xi}^{(p)}_{q+1, \lambda}(t, x) \pm a_{q+1, \lambda}(t, x) W_{c_s, \lambda^+, \lambda^+}(\Phi(t, x)) = a_{q+1, \lambda}(t, x) B_{\xi} e^{i\lambda x + \zeta \Phi(t, x)}. \tag{79}
\]

where \(a_{q+1, \lambda}, W_{c_s, \lambda^+, \lambda^+},\) and \(B_{\xi}\) are defined in (72), (15), and (13), respectively. Then we define the principal part \(w_{q+1}^{(p)}\) of a perturbation \(w_{q+1}\), to be defined in (87), as

\[
w_{q+1}^{(p)}(t, x) \equiv \sum_j \sum_{\xi \in \Lambda_j} w_{\xi}^{(p)}(t, x). \tag{80}
\]
It follows by choosing $c_R \leq (2 \sqrt{3}M)^{-1}$ and taking advantage of the fact that for any $s \in [0, t]$ fixed, there exist at most two non-trivial cutoffs that

$$
||w^{(p)}_{q+1}||_{C_r} \leq c_R R^{-\frac{1}{2}} M_0(t)^{\frac{1}{2}} \sup_{\xi \in \Lambda_j} \sum_{j} ||\gamma_{\xi}||_{C(B_{c_r}(Id))} \leq \delta^{\frac{1}{2}} q_{+1} M_0(t)^{\frac{1}{2}}.\tag{81}
$$

Next, we define the scalar phase function for $\zeta \in \Lambda_j$

$$
\phi_{\zeta}(t, x) \doteq \phi_{q+1, j, \zeta}(t, x) \doteq e^{i\lambda_{q+1} \zeta (\Phi_j(t, x) - x)}
$$

so that we can rewrite

$$
 w^{(p)}_{(r)}(t, x) \doteq a_{(r)}(t, x) B_{c} \phi_{(r)}(t, x) e^{i\lambda_{q+1} \zeta x} \doteq a_{(r)}(t, x) \phi_{(r)}(t, x) W_{(r)}(x).
$$

Due to (16) we can obtain

$$
 a_{(r)} \phi_{(r)} W_{(r)} = \lambda_{q+1}^{-1} \nabla \times (a_{(r)} \phi_{(r)} W_{(r)} - \lambda_{q+1}^{-1} \nabla (a_{(r)} \phi_{(r)}) \times W_{(r)}).
$$

Therefore, if we define

$$
 w^{(c)}_{(r)}(t, x) \doteq \lambda_{q+1}^{-1} \nabla (a_{(r)} \phi_{(r)})(t, x) \times B_{c} e^{i\lambda_{q+1} \zeta x},
$$

then

$$
 w^{(c)}_{(r)}(t, x) \doteq \lambda_{q+1}^{-1} (\nabla a_{(r)} + a_{(r)} \lambda_{q+1} \zeta \cdot (\nabla \Phi_j - Id))(t, x) \times B_{c} e^{i\lambda_{q+1} \zeta \Phi_j(t, x)}
$$

Now we can define the incompressibility corrector $w^{(c)}_{q+1}$ and then the perturbation $w_{q+1}$ as

$$
 w^{(c)}_{q+1}(t, x) = \sum \sum \sum w^{(c)}_{(r)}(t, x) \text{ and } w_{q+1} = w^{(p)}_{q+1} + w^{(c)}_{q+1}
$$

so that

$$
 w_{q+1} \doteq \sum \sum \sum w^{(p)}_{(r)} + w^{(c)}_{(r)} = \sum \sum \sum \lambda_{q+1}^{-1} \nabla \times (a_{(r)} W_{(r)} \circ \Phi_j)
$$

from which we clearly see that $w_{q+1}$ is mean-zero and divergence-free as desired. Next, for $a \in \mathbb{N}$ sufficiently large

$$
||w^{(c)}_{q+1}||_{C_r} \leq 2 \sup_j \lambda_{q+1}^{-1} \|\nabla a_{(r)}\|_{C_r} + \|a_{(r)}\|_{C_r} \sup_s \|\nabla \Phi_j(s) - Id\|_{(0,(j-1),k_{(j+1)})}(s)||_{C_r} \leq \delta^{\frac{1}{2}} q_{+1} M_0(t)^{\frac{1}{2}}.\tag{82}
$$

It follows now that for $a \in \mathbb{N}$ sufficiently large

$$
||w_{q+1}||_{C_r} \leq ||w^{(p)}_{q+1}||_{C_r} + ||w^{(c)}_{q+1}||_{C_r} \leq \frac{3 \delta^{\frac{1}{2}} q_{+1} M_0(t)^{\frac{1}{2}}}{4}.\tag{90}
$$

Thus, if we define

$$
 v_{q+1} = v_{l} + w_{q+1},
$$

then we may verify (55) as follows:

$$
||v_{q+1} - v_{l}||_{C_r} \leq \delta^{\frac{1}{2}} q_{+1} M_0(t)^{\frac{1}{2}}.\tag{92}
$$

Next, we can verify (43a) at level $q + 1$ as follows:

$$
||v_{q+1}||_{C_r} \leq ||v_{l}||_{C_r} + \frac{3}{4} \delta^{\frac{1}{2}} q_{+1} M_0(t)^{\frac{1}{2}} \leq M_0(t)^{\frac{1}{2}} (1 + \sum \delta^{\frac{1}{2}}).\tag{93}
$$
Next, in order to verify (45b) at level \( q + 1 \), we compute using the fact that for any fixed time \( s \in [0, t] \), there are at most two non-trivial cutoffs

\[
\|[\partial t w]^{(p)}_{q+1}\|_{C_s^\alpha} + \|\nabla w^{(p)}_{q+1}\|_{C_s^\alpha} \leq M_c R^{q+1} [M_0(t)]^{q+1} + \lambda_{q+1} M_0(t), 
\]

(94a)

\[
\|[\partial t w]^{(c)}_{q+1}\|_{C_s^\alpha} + \|\nabla w^{(c)}_{q+1}\|_{C_s^\alpha} \leq M_c R^{q+1} [\lambda^{-1}_{q+1} M_0(t)]^{q+1} + M_0(t) \lambda^q + \lambda_{q+1} M_0(t) \lambda^q. 
\]

(94b)

Thus, taking \( c_L \ll M^{-1} \) and \( a \in \mathbb{N} \) sufficiently large gives us

\[
\|w_{q+1}\|_{C_s^{\alpha}} \leq \frac{3}{4} \delta_{q+1} [M_0(t)]^{q+1} + \|\partial t w_{q+1}\|_{C_s^\alpha} + \|\nabla w_{q+1}\|_{C_s^\alpha} + \|\partial t w_{q+1}\|_{C_s^\alpha} + \|\nabla w_{q+1}\|_{C_s^\alpha}
\]

\[
\leq \frac{3}{4} \delta_{q+1} [M_0(t)]^{q+1} + C \lambda_{q+1} \delta_{q+1} [M_0(t)]^{q+1} + \frac{\lambda_{q+1} \delta_{q+1} [M_0(t)]^{q+1}}{2}. 
\]

We are now ready to verify (45b) at level \( q + 1 \) as follows. Because mollifiers have mass one, for \( \beta \in (0, \frac{1}{4}) \), we can take \( a \) in \( \mathbb{N} \) sufficiently large to attain due to (91), (93), and (95)

\[
\|v_{q+1}\|_{C_s^{\alpha}} \leq \frac{\lambda_{q+1} \delta_{q+1} [M_0(t)]^{q+1}}{2}. 
\]

(96)

Subsequently, we will rely on Lemma 6.2 and estimate Reynolds stress. We observe that due to (69b), if we choose \( a \in \mathbb{N} \) sufficiently large, then \( \frac{1}{4} \leq |\nabla \Phi(t, x)| \leq 2 \) for all \( t \in [l(j - 1), l(j + 1)] \) and \( x \in \mathbb{T}^3 \) so that (25) is satisfied with \( C = 2 \). Thus, as discussed on [7] p. 210, for any \( \alpha \in (0, 1) \), \( p \in \mathbb{N} \), and \( a \) that is smooth, periodic, \( a = a_1(0, j - 1, j + 1) \) such that

\[
\|a\|_{C_s^{\alpha}} \leq C_a \gamma^N, \forall N \in \mathbb{N}_0 \cap [0, p + 1], p + 1 \geq \max(\frac{1}{\alpha}, 8), 
\]

we can estimate

\[
\|R(a W_{\zeta} \otimes \Phi_j)\|_{C_s^{\alpha}} \leq \frac{C_a}{A^{j_1 - q+1}}. 
\]

(98)

Additionally, because for \( \zeta \in \Lambda_j \) and \( \zeta' \in \Lambda_{j'} \) such that \( |j - j'| \leq 1 \) and \( \zeta + \zeta' \neq 0 \), there exists \( C_{\Lambda} \in (0, 1) \) such that \( |\zeta + \zeta'| \geq C_{\Lambda} \) (cf. [6] p. 110) and [38] Equ. (91)), it follows from (69) again that for a smooth, periodic function \( a(x) \) that satisfies \( a = a_1(0, j-1, j+1) \), and (97), from [7] Equ. (53) we have an estimate of

\[
\|R(a (W_{\zeta} \otimes \Phi_j \otimes W_{\zeta'} \otimes \Phi_{j'})\|_{C_s^{\alpha}} \leq a_{\alpha} \frac{C_a}{A^{j_1 - q+1}}. 
\]

(99)

4.3.3. Reynolds stress. The following decomposition of the Reynolds stress at level \( q + 1 \) is crucial to attain the necessary estimates. First,

\[
\text{div} \hat{R}_{q+1} = - \nabla \pi_{q+1}
\]

(100)

To take advantage of mollifier estimates, we make the following arrangements by (91):

\[
- \text{div}(v_j \otimes z_j + z_j \otimes v_j + z_j \otimes z_j)
\]

(101)
Concerning in (138), we point out that within (101), the most difficult term is $\text{div}(z_l \otimes w_{q+1}) = (z_l \cdot \nabla)w_{q+1}$, which is absent in the deterministic case. First, a naive attempt of rewriting

$$\text{div}(z_l \otimes w_{q+1}) = \text{div}(z_l \otimes w_{q+1}) + \nabla\left(\frac{1}{3} z_l \cdot w_{q+1}\right)$$

and estimating on $||z_l \otimes w_{q+1}||_{C_{\alpha}}$ fails as

$$||z_l \otimes w_{q+1}||_{C_{\alpha}} \leq ||z_l||_{C_{\alpha}}||w_{q+1}||_{C_{\alpha}} \lesssim L^\delta \delta_{q+1} M(t)^{\frac{2}{3}} \approx \delta_{q+2} 2^{2q+3/3} L^\delta M(t)^{\frac{2}{3}}$$

which clearly cannot be bounded by $c_R M_0(t)\delta_{q+2}$ that is needed to attain (45a) at level $q + 1$. Second, the approach of writing $\text{div}(z_l \otimes w_{q+1}) = (z_l \cdot \nabla)w_{q+1}$ and relying on (98) also fails, because $\nabla$ is applied on $e^{i\lambda_{t+\zeta} \Phi}$, and the $\lambda_{t+\zeta}$ from its chain rule becomes too large to handle. Our new idea to overcome this difficulty is to include $(z_l \cdot \nabla)\Phi$ in (68a), and include this problematic term $(z_l \cdot \nabla)w_{q+1}$ within the transport and corrector errors in $R_{\text{corr}}$ and $R_{\text{corr}}$ to be defined respectively in (107b) and (107b), so that not only the term when $\nabla$ is applied on $e^{i\lambda_{t+\zeta} \Phi}$ in (68a) vanishes, but the term when $\nabla$ is applied on $e^{i\lambda_{t+\zeta} \Phi}$ in $(z_l \cdot \nabla)w_{q+1}$ also vanishes, as we will see in (126) and (138). Let us make this precise.

Let us write $\text{div}(z_l \otimes w_{q+1})$ in (101) as

$$\text{div}(z_l \otimes w_{q+1}) = (z_l \cdot \nabla)w_{q+1}^{(p)} + (z_l \cdot \nabla)w_{q+1}^{(c)}$$

and apply (101) to (100) to write

$$\text{div}\hat{R}_{q+1} = \nabla_\pi_{q+1}$$

Concerning $R_{\text{corr}}$ and $\pi_{\text{corr}}$ in (103), first we see that $\chi_j(t)\chi_j(t) = 0$ if $|j - j'| \geq 2$ because $\chi_j$ has support in $(l(j - 1), l(j + 1))$. Second, by Lemma 3.2 we know that $\Lambda_j \cap \Lambda_j = 0$ if $|j - j'| = 1$. Third, using an identity of

$$(A \cdot \nabla B) + (B \cdot \nabla A) = \nabla(A \cdot B) - A \times \nabla B - B \times \nabla A$$

and (16), we can compute

$$\text{div}(W_{(x)} \otimes W_{(y)} + W_{(x)} \otimes W_{(y)}) = \nabla(W_{(x)} \cdot W_{(y)}).$$

Taking into account of these observations allows us to rewrite

$$\text{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \hat{R}_k)$$
by relying on Lemma 6.3 we obtain 
\[
\text{div}(\sum_j \sum_{\xi \in \Lambda_j} w_{j+1}^{(p)}(\xi) \otimes w_{j+1}^{(p)}(\xi) + \hat{R}_t) + \sum_{j,f} \sum_{\xi \in \Lambda_j} \text{div}(w_{j+1}^{(p)}(\xi) \otimes w_{j+1}^{(p)}(\xi))
\]
Thus, (103) and (106) motivate us to define in addition to \(R_{\text{com1}}\) and \(\pi_t\) defined in (66),
\[
\begin{align*}
R_{\text{line}} & \triangleq R_{\text{linear}} \triangleq \mathcal{R}((-\Delta)^m w_{q+1} + (w_{q+1} \cdot \nabla) z_t), & (107a) \\
R_{\text{transp}} & \triangleq R_{\text{transport}} \triangleq \mathcal{R}((\partial_t + (v_t + z_t) \cdot \nabla) w_{q+1}^{(p)}), & (107b) \\
R_{\text{osc}} & \triangleq R_{\text{oscillation}} \triangleq \mathcal{R}(\sum_{j,f} \sum_{\xi \in \Lambda_j} (W(\xi) \otimes W(\xi)) - \frac{1}{2} |\nabla (a_{j\xi}(\xi) \phi_{j\xi}(\xi) \phi_{j\xi}(\xi))|, & (107c) \\
\pi_{\text{osc}} & \triangleq \pi_{\text{oscillation}} \triangleq \frac{1}{2} \sum_{j,f} \sum_{\xi \in \Lambda_j} \sum_{\xi' \neq 0} a_{j\xi}(\xi) \phi_{j\xi}(\xi) \phi_{j\xi}(\xi) (W(\xi) \cdot W(\xi)), & (107d) \\
R_{\text{Nash}} & \triangleq \mathcal{R}(w_{q+1} \cdot \nabla) v_t, & (107e) \\
R_{\text{corr}} & \triangleq R_{\text{corrector}} \triangleq \mathcal{R}(\sum_{j} \sum_{\xi \in \Lambda_j} (a_{j\xi}(\xi) \phi_{j\xi}(\xi) \phi_{j\xi}(\xi) (W(\xi) \cdot W(\xi)))), & (107f) \\
\pi_{\text{corr}} & \triangleq \pi_{\text{corrector}} \triangleq \frac{1}{3} |w_{q+1}^{(c)}|^2 + \frac{2}{3} |w_{q+1}^{(p)}| + \frac{1}{3} |w_{q+1}^{(c)}|, & (107g) \\
R_{\text{com2}} & \triangleq R_{\text{commutator2}} \triangleq \mathcal{R}(\sum_{j} \sum_{\xi \in \Lambda_j} (v_{q+1} \delta(z - z_t) + (z - z_t) \delta(v_{q+1} + z_t)(z - z_t) + (z - z_t) \delta z_t), & (107h) \\
\pi_{\text{com2}} & \triangleq \pi_{\text{commutator2}} \triangleq \frac{1}{3} (2v_{q+1} \cdot (z - z_t) + |z|^2 - |z_t|^2). & (107i)
\end{align*}
\]

We define from (103),
\[
\begin{align*}
\hat{R}_{q+1} & \triangleq R_{\text{line}} + R_{\text{transp}} + R_{\text{osc}} + R_{\text{Nash}} + R_{\text{corr}} + R_{\text{com1}} + R_{\text{com2}}, & (108a) \\
\pi_{q+1} & \triangleq \pi_t - \pi_{\text{osc}} - \pi_{\text{corr}} - \pi_{\text{com2}}. & (108b)
\end{align*}
\]
First, we work on \(R_{\text{line}}\) from (107a). As \(m \in (0, \frac{1}{2})\) by hypothesis, for any \(\epsilon \in (0, 1 - 2m)\), by relying on Lemma 6.3, we obtain
\[
\|R(-\Delta)^m w_{q+1}\|_{C^{\epsilon}} \leq \epsilon \|R w_{q+1}^{(p)}\|_{C^{\epsilon}} + \|\nabla R w_{q+1}^{(c)}\|_{C^{\epsilon}}. & (109)
\]
First, using the fact that for all \(s \in [0, t]\) fixed, there exist at most two non-trivial cutoffs
\[
\|R w_{q+1}^{(p)}(\xi)\|_{C^{\infty}_t} \leq 2 \sup_{\xi} \sum_{j} \|R(a_{j\xi}(\xi) W(\xi))\|_{C^{\infty}_t}. & (110)
\]
From (76a) we see that (97) is satisfied by “\(C_a^\prime = \delta_{q+1}^{2}M_0(t)^{2}\|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))}\) for all \(0 \leq N \leq \lceil \frac{1}{2m} \rceil \vee 8\). Therefore, by taking \(\beta < \frac{1}{4}(1-2m-\varepsilon)\) and \(a \in \mathbb{N}\) sufficiently large, continuing from (110)

\[
\|Rw_{q+1}^{(p)}\|_{C_2(C^{2m+\varepsilon})} \lesssim M_0^{\frac{1}{2}}(t)^{\frac{1}{2}} a^{2m+\varepsilon-1} c_R M_0(t) \delta_{q+2}^{2} a^{2m+\varepsilon-1} \ll c_R M_0(t) \delta_{q+2} .
\]

(111)

Next, because for all \(s \in [0, t]\) there exist at most two non-trivial cutoffs, we have

\[
\|R_w^{(c)}\|_{C,C^{2m+\varepsilon}} \lesssim 2 \sup_j \sum_{\zeta \in \Lambda_j} \|R((\nabla - \nabla\zeta) + i a_{\zeta} \zeta \cdot (\nabla \Phi_j - \text{Id})) \cdot W_\zeta(\Phi_j)\|_{C_2(C^{2m+\varepsilon})}.
\]

(112)

Now, for all \(N \in \mathbb{N}\) we can estimate

\[
|a_{\zeta}(\nabla \zeta - \text{Id})\|_{C,C^{\varepsilon}} \lesssim \delta_{q+1}^{1} M_0(t)^{2} \delta_{q+1} \|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))} t^{-N+1}
\]

(113)

and hence deduce for all \(N = 0, \ldots, \lceil \frac{1}{2m} \rceil \vee 8\), by taking \(a \in \mathbb{N}\) sufficiently large

\[
\|a_{\zeta}(\nabla \zeta - \text{Id})\|_{C,C^{\varepsilon}} \lesssim A_{q+1}^{1,2} M_0(t)^{2} \|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))} t^{-N+1}.
\]

(114)

Therefore, (111) shows that (97) holds with “\(C_a^\prime = A_{q+1}^{1,2} M_0(t)^{2} \|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))} l^{-1}\) for \(N = 0, \ldots, (\lceil \frac{1}{2m} \rceil + 1) \vee 8\), so that by (98) we can continue from (112) by taking \(\beta < \frac{1}{4}(\frac{3}{2} - 4m - 2\varepsilon)\) and taking \(a \in \mathbb{N}\) sufficiently large

\[
\|Rw_{q+1}^{(p)}\|_{C_2(C^{2m+\varepsilon})} \lesssim \sup_j \sum_{\zeta \in \Lambda_j} A_{q+1}^{1,2} M_0(t)^{2} \|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))} t^{-1}.
\]

(115)

(116)

Applying (111) and (115) to (109) gives us

\[
\|R(-\Delta)^{1/2}w_{q+1}\|_{C_2} \ll c_R M_0(t) \delta_{q+2}.
\]

Next, within \(R\) line from (107a) we first split

\[
\|R((w_{q+1} \cdot \nabla)\zeta)\|_{C_2} \lesssim \|R((w_{q+1}^{(p)} \cdot \nabla)\zeta)\|_{C_2} + \|R((w_{q+1}^{(c)} \cdot \nabla)\zeta)\|_{C_2}.
\]

(117)

First, we compute

\[
\|R((w_{q+1}^{(p)} \cdot \nabla)\zeta)\|_{C_2} \lesssim \|\sum_j a_{\zeta}(B_{\zeta} \epsilon^{\delta_{q+1}^{1/2}(\Phi_j)(t,x)} \cdot \nabla \zeta)\|_{C_2}.
\]

(118)

For any \(\varepsilon \in (\frac{3}{4}, 1)\), for all \(N = 0, \ldots, \lceil \frac{1}{2m} \rceil \vee 8 = 8\), we can estimate

\[
|a_{\zeta}(\nabla \zeta)\|_{C_2} \lesssim \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))} L^{\frac{1}{2}}.
\]

(119)

Thus, (97) holds with “\(C_a^\prime = \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))} L^{\frac{1}{2}}\) so that, as for all time \(s \in [0, t]\) fixed, there exist at most two non-trivial cutoffs, continuing from (118), choosing \(\beta < \frac{1}{4}(1-\varepsilon)\) and \(a \in \mathbb{N}\) sufficiently large,

\[
\|R((w_{q+1}^{(p)} \cdot \nabla)\zeta)\|_{C_2} \lesssim \sup_j \sum_{\zeta \in \Lambda_j} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \|\gamma_\zeta\|_{C^1(\mathbb{R},(l^4))} L^{\frac{1}{2}}.
\]
Second, we use that for all $s \in [0, t]$ there exist at most two non-trivial cutoffs to write

\[ \|R((w_{q+1}^e) \cdot \nabla) \|_{C_s} \leq 2 \sup_{j} \|R((\lambda_{q+1}^{-1} \nabla a)_{\xi} + ia_{\xi} (\nabla \Phi_j - \text{Id})) \times W_{\alpha}(\Phi_j) \cdot \nabla \|_{C_s}. \]  

(121)

For any $\epsilon \in (1, 1)$, for all $N = 0, \ldots, \lfloor \frac{1}{\epsilon} \rfloor$ and $Q = 8$, we can estimate by taking $a \in \mathbb{N}$ sufficiently large

\[ \| (\lambda_{q+1}^{-1} \nabla a_{\xi} + ia_{\xi} (\nabla \Phi_j - \text{Id})) \cdot \nabla \|_{C_s} \leq \lambda_{q+1}^{-1} \| \nabla \|_{C_s} \leq \left( \frac{1}{\epsilon} \right) \| \nabla \|_{C_s}. \]  

(122)

Therefore, (127) holds with “$C_a$” $= \delta_{q+1}^{-\frac{1}{2}} \| \nabla \|_{C_s} L^2 \| M_0(t) \|_{L^2} \| \nabla \|_{C_s}$ so that choosing $\beta < \frac{1}{8} (\frac{1}{\epsilon} - 2\epsilon)$ and $a \in \mathbb{N}$ sufficiently large, we can continue from (121) as

\[ \| R((w_{q+1}^e) \cdot \nabla) \|_{C_s} \leq \delta_{q+1}^{-\frac{1}{2}} \| M_0(t) \|_{L^2} \| \nabla \|_{C_s} \| \nabla \|_{C_s}. \]  

(123)

Applying (120) and (123) to (117) gives us

\[ \| R((w_{q+1} \cdot \nabla) \|_{C_s} \leq c_R M_0(t) \delta_{q+2}. \]  

(124)

Together with (116) and (107a), (124) allows us to conclude that

\[ \| R_{\text{line}} \|_{C_s} \leq \| R(-\nabla)^n w_{q+1} \|_{C_s} + \| R((w_{q+1} \cdot \nabla) \|_{C_s} \leq c_R M_0(t) \delta_{q+2}. \]  

(125)

Next, we look at $R_{\text{man}} = R((\partial_t + (v_i + z_j) \cdot \nabla) w_{q+1}^p)$ in (107b). We make the following key observation that the worst term when $\nabla$ falls on $W_{\alpha}(\cdot) \cdot \Phi_j$ vanishes:

\[ \nabla \nabla \sum_{j} \sum_{j} \sum_{j} \partial_t a_{\xi} (v_i + z_j) \cdot \nabla a_{\xi} W_{\alpha}(\Phi_j) + \partial_t a_{\xi} \nabla W_{\alpha}(\Phi_j) \cdot (v_i + z_j) \cdot \nabla \Phi_j. \]  

(126)

For any $\epsilon \in (1, 1)$, for all $N = 0, \ldots, \lfloor \frac{1}{\epsilon} \rfloor$ and $Q = 8$, we estimate

\[ \| (v_i + z_j) \cdot \nabla a_{\xi} \|_{C_s} \leq L^{(N+1)} (1 + \| v_i \|_{C_s} + \| \gamma \|_{C_s}) \delta_{q+1}^{-\frac{1}{2}} M_0(t) \| \nabla \|_{C_s} \leq L^{(N+1)} \]  

(127)

Hence, together with (106h), for all $N = 0, \ldots, \lfloor \frac{1}{\epsilon} \rfloor$ and $Q = 8$, we have

\[ \| \partial_t a_{\xi} + (v_i + z_j) \cdot \nabla a_{\xi} \|_{C_s} \leq L^{(N+1)} \delta_{q+1}^{-\frac{1}{2}} M_0(t) L^{(N+1)} \| \nabla \|_{C_s} \leq L^{(N+1)} \]  

(128)
Therefore, (97) is satisfied with $C_a^* = \frac{3}{q+1}C a_{(t)}^{(t-1)}\|\gamma_c\|_{C^{(B_{c}(t))}}$ so that we can take $\beta < \frac{1}{q}(\frac{1}{2} - 2\epsilon)$ and $a \in \mathbb{N}$ sufficiently large to compute by (28)

$$
\|R\|_{C^{(t)}} \lesssim \sup_{\zeta \lambda} \sum_{j} |R(\partial a_{(c)} + (v_i + z_i) \cdot \nabla a_{(c)}) W_{(c)} \circ \Phi_j|_{C^{(t)}}
$$

$$
\lesssim \delta_{q+1}^\frac{1}{2}C a_{(t)}^{(t-1)} \approx c_R D_{(t)} \delta_{q+2} a^{2(q^2 - \frac{1}{2} - 2\epsilon)} \ll c_R D_{(t)} \delta_{q+2}.
$$

Next, we work on $R_{osc}$ from (107c): by relying on the identities of

$$
\nabla(\phi_{(c)}\phi_{(c)}) = i_{\lambda_{q+1}^j}(\nabla \Phi_j - Id)\phi_{(c)}\phi_{(c)} + i_{\lambda_{q+1}^j}\zeta^j \cdot (\nabla \Phi_j - Id)\phi_{(c)}\phi_{(c)}, W_{(c)}\phi_{(c)} = W_{(c)} \circ \Phi_j
$$

for $\zeta \in \Lambda_j, \zeta^j \in \Lambda_j^j$, we can rewrite

$$
R_{osc} \lesssim \sum_{\zeta \in \Lambda_j, \zeta^j \in \Lambda_j^j} \nabla(\phi_{(c)}\phi_{(c)})[i_{\lambda_{q+1}^j}(\nabla \Phi_j - Id) + i_{\lambda_{q+1}^j}\zeta^j \cdot (\nabla \Phi_j - Id)]].
$$

(130)

Now for any $\epsilon \in (\frac{1}{2}, \frac{1}{4})$, for all $N = 0, \ldots, [\frac{1}{\epsilon}] \vee 8 = 8$, by taking $a \in \mathbb{N}$ sufficiently large we obtain

$$
\|\nabla(\phi_{(c)}\phi_{(c)})[i_{\lambda_{q+1}^j}(\nabla \Phi_j - Id) + i_{\lambda_{q+1}^j}\zeta^j \cdot (\nabla \Phi_j - Id)]\|_{C^{(t)}}
\leq \|\nabla(\phi_{(c)}\phi_{(c)})[i_{\lambda_{q+1}^j}(\nabla \Phi_j - Id) + i_{\lambda_{q+1}^j}\zeta^j \cdot (\nabla \Phi_j - Id)]\|_{C^{(t)}}
\leq \delta_{q+1}^\frac{1}{2}C a_{(t)}^{(t-1)} F^{N-1}.
$$

(131)

Therefore, (97) is satisfied with $C_a^* = \delta_{q+1}^\frac{1}{2}C a_{(t)}^{(t-1)} F^{N-1}$. Hence, we can choose $\beta < \frac{1}{q}(\frac{1}{2} - 2\epsilon)$, as well as $a \in \mathbb{N}$ sufficiently large, continue from (130), use the fact that for any $s \in [0, t]$ fixed there exist at most two non-trivial cutoffs, and compute

$$
\|R_{osc}\|_{C^{(t)}} \lesssim \sup_{\zeta \lambda} \sum_{j} \nabla(\phi_{(c)}\phi_{(c)})[i_{\lambda_{q+1}^j}(\nabla \Phi_j - Id) \times \Phi_j] \cdot \nabla v_i.
$$

(133)

Now for any $\epsilon \in (\frac{1}{2}, \frac{1}{4})$, for all $N = 0, \ldots, [\frac{1}{\epsilon}] \vee 8 = 8$ we can estimate

$$
\|\nabla v_i\|_{C^{(t)}} \lesssim \delta_{q+1}^\frac{1}{2}C a_{(t)}^{(t-1)} F^{N-1}\delta_{q+1}^\frac{1}{2}C a_{(t)}^{(t-1)} F^{N-1}.
$$

(134)

On the other hand, for all $N = 0, \ldots, [\frac{1}{\epsilon}] \vee 8 = 8$, we can estimate by taking $a \in \mathbb{N}$ sufficiently large,

$$
\|\nabla(\lambda_{q+1}^j\nabla a_{(c)} + i_{\lambda_{q+1}^j}\zeta^j \cdot (\nabla \Phi_j - Id)) \cdot \nabla v_i\|_{C^{(t)}}
\leq \|\nabla v_i\|_{C^{(t)}} \lesssim \delta_{q+1}^\frac{1}{2}C a_{(t)}^{(t-1)} F^{N-1}\delta_{q+1}^\frac{1}{2}C a_{(t)}^{(t-1)} F^{N-1}.
$$

(135)
Therefore, for all $N = 0, \ldots, \lceil \frac{1}{\epsilon} \rceil \vee 8 = 8$,
\[
\|a_{t_0} \cdot \nabla a_{t_0}\|_{C(0)}^{\epsilon} + \|a_{t_0} \cdot \nabla \Phi_{t_0} - \nabla \Phi_{t_0}\|_{C(0)}^{\epsilon} \leq \|\gamma\|_{C^{\epsilon}(t_0)}^{\epsilon} \delta_{q+1}^2 M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}}.
\] (136)

Hence, (97) is satisfied with \(C_a = \|\gamma\|_{C^{\epsilon}(t_0)}^{\epsilon} \delta_{q+1}^2 M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}}\) so that by (98), choosing \(\beta < \frac{1}{2}(1 - 2\epsilon)\) and \(a \in \mathbb{N}\) sufficiently large, continuing from (133) and taking advantage of the fact that for any \(s \in [0, t]\) fixed there exist at most two non-trivial cutoffs give
\[
\|R_{\text{Nash}}\|_{C(0)}^{\epsilon} \leq \sup_j \sum_{t \in \mathcal{A}} \|\mathcal{R}(a_{t_0} W_{t_0}) \cdot \Phi_j + (\lambda_{q+1}^{-1} \nabla a_{t_0} + i a_{t_0} \xi \cdot (\nabla \Phi_{t_0} - \Id)) \times W_{t_0}(\Phi_j) \cdot \nabla v_i\|_{C(0)}^{\epsilon}.
\] (137)

Next, we work on \(R_{\text{cont}}\) from (107). First, we again make the important observation that
\[
(\partial_t + (v_i + z_i) \cdot \nabla) u_{t_0} \leq \sum_j \sum_{t \in \mathcal{A}} \|\mathcal{R}(a_{t_0} W_{t_0}) \cdot \Phi_j + (\lambda_{q+1}^{-1} \nabla a_{t_0} + i a_{t_0} \xi \cdot (\nabla \Phi_{t_0} - \Id)) \times W_{t_0}(\Phi_j)\|_{C(0)}^{\epsilon}.
\] (138)

For any \(\epsilon \in \left(\frac{1}{8}, \frac{1}{4}\right)\), for all $N = 0, \ldots, \lceil \frac{1}{\epsilon} \rceil \vee 8 = 8$, we can estimate
\[
\|\|\partial_t + (v_i + z_i) \cdot \nabla)(\lambda_{q+1}^{-1} \nabla a_{t_0} + i a_{t_0} \xi \cdot (\nabla \Phi_{t_0} - \Id))\|_{C(0)}^{\epsilon} \leq \lambda_{q+1}^{\frac{1}{2}} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} + \|a_{t_0} \nabla \Phi_{t_0} - \Id\|_{C(0)}^{\epsilon} + \lambda_{q+1}^{-1} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon}.
\] (139)

We can estimate separately for all $N = 0, \ldots, \lceil \frac{1}{\epsilon} \rceil \vee 8 = 8$, by taking \(a \in \mathbb{N}\) sufficiently large
\[
\lambda_{q+1}^{-1} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} \leq \lambda_{q+1}^{\frac{1}{2}} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} + \|\nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} + \lambda_{q+1}^{-1} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon}.
\] (140a)

\[
\|\partial_t a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} \leq \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \|\gamma\|_{C^{\epsilon}(t_0)}^{\epsilon} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}}.
\] (140b)

\[
\|\partial_t a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} \leq \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \|\gamma\|_{C^{\epsilon}(t_0)}^{\epsilon} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}}.
\] (140c)

\[
\lambda_{q+1}^{-1} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} \leq \lambda_{q+1}^{\frac{1}{2}} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} + \|\nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} + \lambda_{q+1}^{-1} \nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon}.
\] (140d)

\[
\|\partial_t a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} \leq \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \|\gamma\|_{C^{\epsilon}(t_0)}^{\epsilon} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}}.
\] (140e)

\[
\|\nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon} \leq \|\gamma\|_{C^{\epsilon}(t_0)}^{\epsilon} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}}.
\] (140f)

Applying (140) to (139) gives us by taking \(a \in \mathbb{N}\) sufficiently large, for all $N = 0, \ldots, \lceil \frac{1}{\epsilon} \rceil \vee 8 = 8$,
\[
\|\|\partial_t + (v_i + z_i) \cdot \nabla)(\lambda_{q+1}^{-1} \nabla a_{t_0} + i a_{t_0} \xi \cdot (\nabla \Phi_{t_0} - \Id))\|_{C(0)}^{\epsilon} \leq \|\gamma\|_{C^{\epsilon}(t_0)}^{\epsilon} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} + \|\nabla a_{t_0}(\nabla \Phi_{t_0} - \Id)\|_{C(0)}^{\epsilon}.
\] (141)
This implies that (127) holds with “$C_k$” = $\|\gamma\|_{L^\infty(R_k, t\cdot Id)}\delta_t^{\frac{1}{q+1}}F_{q+1}M_0(t)$ and hence via (88), continuing from (138), taking $\beta < \frac{1}{8}(1-2\epsilon)$ and $a \in \mathbb{N}$ sufficiently large, we obtain

$$\sup_{j \in \mathbb{N}} \|\mathcal{R}(\partial_t + (v_j + z_j) \cdot \nabla)w_{q+1}(v_j)\|_{L^\infty} \leq c_R M_0(t) \delta_t^{2(q+2)2^{(q+2)^{-1}}} \leq c_R M_0(t) \delta_t^{q+2}.$$

Next, within $R_{\text{cor}}$ from (107), we can directly estimate by taking $\beta < \frac{1}{8}$ and $a \in \mathbb{N}$ sufficiently large,

$$\|w_{q+1}(v_j)\|_{L^\infty} \leq c_R M_0(t) \delta_t^{q+2} \leq c_R M_0(t) \delta_t^{q+2}.$$

Applying (142) and (143) to (107) gives us

$$\|R_{\text{cor}}\|_{L^\infty} \leq c_R M_0(t) \delta_t^{q+2}.$$

Finally, for $\beta < \frac{5}{24}$, taking $a \in \mathbb{N}$ sufficiently large we can directly estimate $R_{\text{com}}$ from (107b) as follows: as $\frac{1}{8} - 2\delta > \frac{5}{24}$, by taking $a \in \mathbb{N}$ sufficiently large we can compute

$$\|\mathcal{R}_{\text{com}}\|_{L^\infty} \leq c_R M_0(t) \delta_t^{q+2} \leq c_R M_0(t) \delta_t^{q+2}.$$

Applying (129), (132), (137), (144)-(146) to (108a) shows that (45c) at level $q+1$ holds.

At last, following similar arguments in [28] we comment on how $(v_{q+1}, \hat{R}_{q+1})$ is $(\mathcal{F}_{q+1})_{\geq 0}$-adapted and that $v_{q+1}(0, x)$ and $\hat{R}_{q+1}(0, x)$ are deterministic if $v_q(0, x)$ and $\hat{R}_q(0, x)$ are deterministic. First, $z(t)$ from (43a) is $(\mathcal{F}_{q+1})_{\geq 0}$-adapted. Due to the compact support of $\varphi_1$ in $\mathbb{R}_+$, it follows that $\tilde{z}_1$ is $(\mathcal{F}_{q+1})_{\geq 0}$-adapted. Similarly, because $(v_q, \hat{R}_q)$ are both $(\mathcal{F}_{q+1})_{\geq 0}$-adapted by hypothesis, so are $(v_j, \hat{R}_j)$. Because $M_0(t)$ from (42), $\chi_j(t)$ from (70), and $\gamma_j$ from Lemma 3,2 are deterministic, $a_6(t)$ from (72) is also $(\mathcal{F}_{q+1})_{\geq 0}$-adapted. It follows that $w_{q+1}^{(p)}(v_j)$ from (79) is $(\mathcal{F}_{q+1})_{\geq 0}$-adapted and consequently so are $w_{q+1}^{(p)}(v_j)$ from (80). Similarly, $w_{q+1}^{(p)}(v_j)$ from (85) and hence in turn $w_{q+1}^{(p)}(v_j)$ and $\hat{w}_{q+1}^{(p)}(v_j)$ are also $(\mathcal{F}_{q+1})_{\geq 0}$-adapted. Therefore, $w_{q+1}$ from (87) is $(\mathcal{F}_{q+1})_{\geq 0}$-adapted, indicating that $v_{q+1}$ from (91) is $(\mathcal{F}_{q+1})_{\geq 0}$-adapted. It follows that all of $R_{\text{line}}, R_{\text{tan}}, R_{\text{osc}}, R_{\text{Nash}}, R_{\text{cor}}, and R_{\text{com}}$ from (107), and $R_{\text{com}}$ from (108a) are $(\mathcal{F}_{q+1})_{\geq 0}$-adapted and consequently so is $R_{q+1}$ from (108a).
Similarly, due to the compact support of \( \varphi_i \) in \( \mathbb{R}^+ \), if \( v_q(0, x) \) and \( \tilde{R}_q(0, x) \) are deterministic, then so are \( v(0, x), \tilde{R}_q(0, x) \), and \( \partial_t \tilde{R}_q(0, x) \). Similarly, \( z(0, x) \) is also deterministic because \( z(t) \equiv 0 \) by (34a). Because \( M_0(t) \), \( \gamma_t \), and \( \chi_j \) are deterministic, it follows that \( a_{\gamma_t} \) and \( \partial_t a_{\gamma_t} \) from (72) are both deterministic. It follows that \( w^{(p)}(0, x) \) from (79) is deterministic; therefore, \( w^{(p)}(0, x) \) and \( \partial_t w^{(p)}(0, x) \) from (80) are deterministic. Similarly, \( w^{(c)}(0, x) \) from (85) is deterministic and hence so is \( w^{(c)}(0, x) \), as well as \( \partial_t w^{(c)}(0, x) \) from (87). This implies that \( w^{(c)}(0, x) \) is also deterministic and thus so is \( v^{(c)}(0, x) \) from (91). Finally, all of \( R_{\text{line}}(0, x), R_{\text{ran}}(0, x), R_{\text{osc}}(0, x) \), \( R_{\text{Nash}}(0, x), R_{\text{cont}}(0, x) \), and \( R_{\text{com2}}(0, x) \) from (107), and \( R_{\text{com1}}(0, x) \) are all deterministic and hence so is \( \tilde{R}_q(0, x) \) from (108a). This completes the proof of Proposition 4.3.

5. Proofs of Theorems 2.3, 2.4

5.1. Proof of Theorem 2.4 assuming Theorem 2.3

We recall \( U_1, \tilde{Q}, \) and \( \tilde{B}_t \) from Section 3 fix any \( \gamma \in (0, 1) \), and state the definition of a probabilistically weak solution:

**Definition 5.1.** Let \( s \geq 0, \xi^m \in L^2, \) and \( \theta^m \in U_1. \) Then \( P \in \mathcal{P}(\tilde{Q}) \) is a probabilistically weak solution to (2) with initial condition \((\xi^m, \theta^m)\) at initial time \( s \) if

\[
\begin{align*}
&M(1) \quad P(\{(\xi(t) = \xi^m, \theta(t) = \theta^m) & \forall t \in [0, s]\}) = 1 \text{ for all } l \in \mathbb{N} \\
&M(2) \quad \text{under } P, \theta \text{ is a cylindrical } (\tilde{B}_t)_{t \geq s}-\text{Wiener process on } U \text{ starting from initial condition } \theta^m \text{ at initial time } s \text{ and for every } q \in \mathbb{N}, \text{ there exists a function } t \mapsto C_{l,q} \in \mathbb{R}_+ \text{ such that for all } t \geq s,} \\
&M(3) \quad \text{for any } q \in \mathbb{N}, \text{ there exists a function } t \mapsto C_{l,q} \in \mathbb{R}_+ \text{ such that for all } t \geq s,} \\
&M(4) \quad \mathbb{E}^P \left[ \sup_{r \in [0,t]} \|\xi^m(r)\|_{L^2}^2 + \int_t^\infty \|\xi^m(r)\|_{H^1}^2 dr \right] \leq C_{l,q}(1 + \|\xi^m\|_{L^2}^2). \\
\end{align*}
\]

The set of all such probabilistically weak solutions with the same constant \( C_{l,q} \) in (149) for every \( q \in \mathbb{N} \) and \( t \geq s \) is denoted by \( \mathcal{W}(s, \xi^m, \theta^m, \{C_{l,q}\}_{q \in \mathbb{N}, l \geq s}) \).

**Definition 5.2.** Let \( s \geq 0, \xi^m \in L^2, \) and \( \theta^m \in U_1. \) Let \( \tau \geq s \) be a stopping time of \((\tilde{B}_t)_{t \geq s}\) and set

\[
\tilde{Q}_\tau \triangleq \{\omega(\cdot \wedge \tau, \omega) : \omega \in \tilde{Q} : (\xi, \theta)(t, \omega) = (\xi, \theta)(t \wedge \tau, \omega), \omega\}. \\
\]

Then \( P \in \mathcal{P}(\tilde{Q}_\tau) \) is a probabilistically weak solution to (2) on \([s, \tau]\) with initial condition \((\xi^m, \theta^m)\) at initial time \( s \) if

\[
\begin{align*}
&M(1) \quad P(\{(\xi(t) = \xi^m, \theta(t) = \theta^m) & \forall t \in [0, s]\}) = 1 \text{ for all } l \in \mathbb{N} \\
&M(2) \quad \text{under } P, \theta(\cdot \wedge \tau, l_1) \in U, \text{ where } \{l_1\}_{l \in \mathbb{N}} \text{ is an orthonormal basis of } U, \text{ is a continuous, square-integrable } (\tilde{B}_t)_{t \geq s}-\text{martingale with initial condition } \theta^m, l_1 \text{ at initial time } s \text{ with a quadratic variation process given by } (t \wedge \tau - s)\|l_1\|_{L^2}^2 \text{ and for every } q \in \mathbb{N}, \text{ there exists a function } t \mapsto C_{l,q} \in \mathbb{R}_+ \text{ such that for all } t \geq s,} \\
&M(3) \quad \mathbb{E}^P \left[ \sup_{r \in [0,t]} \|\xi^m(r)\|_{L^2}^2 + \int_t^\infty \|\xi^m(r)\|_{H^1}^2 dr \right] \leq C_{l,q}(1 + \|\xi^m\|_{L^2}^2). \\
\end{align*}
\]
Comparing (2) and (148) we see that \( F \) weakly to some \( \bar{\mathcal{F}} \), (28, Pro. 5.2)
Let \( L_{\bar{\mathcal{F}}} \) be a bounded \( B \)-valued Wiener process. Then for every \( \omega \in \bar{\Omega} \), there exists a probabilistically weak solution \( \bar{P} \in \mathcal{P}(\bar{\Omega}) \) to (2) with initial condition \( (\xi, 0) \) at initial time 0 according to Definition 5.1. Moreover, if there exists a family \( \{ (s_1, \xi_1, \theta_1) \}_{n \in \mathbb{N}} \subset [0, \infty) \times \mathbb{R}_+^2 \times \mathbb{R}_+ \) such that \( \lim_{n \to \infty} \| (s_1, \xi_1, \theta_1) - (s, \xi_0, \theta_0) \|_{L_\infty \times \mathbb{R}_+ \times \mathbb{R}_+} = 0 \) and \( P_t \in \mathcal{W}(s_1, \xi_1, \theta_1, (C_r, \psi \in \mathcal{B}_r \Rightarrow \mathcal{B}_r \notin \Omega)) \), then there exists a subsequence \( \{ P_{\bar{t}_n} \}_{n \in \mathbb{N}} \) that converges weakly to some \( P \in \mathcal{W}(s, \xi_0, \theta_0, (C_r, \psi \in \mathcal{B}_r \Rightarrow \mathcal{B}_r \notin \Omega)) \).

Lemma 5.2. (28, Pro. 5.2) Let \( \tau \) be a bounded \( \bar{\mathcal{F}}_\tau \)-stopping time. Then for every \( \omega \in \bar{\Omega} \), there exists \( Q_\omega \in \mathcal{P}(\bar{\Omega}) \) such that \( Q_\omega(\omega' \in \bar{\Omega} : (\xi, 0)(t, \omega') = (\xi, 0)(t, \omega) \forall t \in [0, \tau(\omega))] = 1 \), (154a)
\( Q_\omega(A) = R_{(\omega, \xi(\omega), \theta(\omega))}(A) \forall A \in \bar{\mathcal{F}}(\omega) \), (154b)
where \( R_{(\omega, \xi(\omega), \theta(\omega))}(A) \in \mathcal{P}(\bar{\Omega}) \) is a probabilistically weak solution to (2) with initial condition \( (\xi(\omega), \theta(\omega), \theta(\omega), \omega)) \) at initial time \( \tau(\omega) \).

Lemma 5.3. (28, Pro. 5.3) Let \( \tau \) be a bounded \( \bar{\mathcal{F}}_\tau \)-stopping time, \( \xi_0 \in L_\infty^2 \), and \( P \in \mathcal{P}(\bar{\Omega}) \) be a probabilistically weak solution to (2) on \( [0, \tau] \) with initial condition \( (\xi_0, 0) \) at initial time 0 according to Definition 5.2. Suppose that there exists a Borel set \( N \subset \Omega_\tau \) such that \( P(N) = 0 \) and \( \bar{Q}_\omega \) from Lemma 5.2 satisfies for every \( \omega \in \Omega_\tau \setminus N \)
\( \bar{Q}_\omega(\omega' \in \bar{\Omega} : \tau(\omega') = \tau(\omega)) = 1 \). (155)

Then the probability measure \( P \otimes R \in \mathcal{P}(\bar{\Omega}) \) defined by
\[ P \otimes R(\cdot) \equiv \int_{\bar{\Omega}} Q_\omega(\cdot) P(d\omega) \] (156)
satisfies \( P \otimes R \vert_{\Omega_\tau} = P \vert_{\Omega_\tau} \) and it is a probabilistically weak solution to (2) on \( [0, \infty) \) with initial condition \( (\xi_0, 0) \) at initial time 0.

Now we fix \( \mathbb{R} \)-valued Wiener process \( B \) on \( (\Omega, \mathcal{F}, P) \) with \( (\mathcal{F}_t)_{t \geq 0} \) as its normal filtration. For \( l \in \mathbb{N} \), \( L > 1 \), and \( \delta \in (0, \frac{1}{2L}) \) we define
\[ \tau'_L(\omega) \equiv \inf \{ t \geq 0 : |\theta(t, \omega)| > (L - \frac{1}{2})^2 \} \land \inf \{ t \geq 0 : \|\theta(t, \omega)\|_{C^\infty_{-L}} > (L - \frac{1}{2})^2 \} \land L \] (157a)
\[ \tau_L \equiv \lim_{l \to \infty} \tau'_L. \] (157b)
Comparing (2) and (148) we see that \( F(\xi(\omega)) = \xi(\omega), \theta = B \); as Brownian path is locally Hölder continuous with exponent \( \alpha \in (0, \frac{1}{2}) \), it follows by (28, Lem. 3.5) that \( \tau_L \) is a stopping time of \( (\bar{\mathcal{B}}_t)_{t \geq 0} \). For the fixed \( (\Omega, \mathcal{F}, P) \), we assume Theorem 2.3 and denote by \( u \) the solution constructed by Theorem 2.3 on \( [0, t] \) where \( t = T_L \) for \( L > 1 \) sufficiently large and
\[ T_L \equiv \inf \{ t > 0 : |B(t)| \geq L^2 \} \land \inf \{ t > 0 : \| B \|_{C^\infty_{-L}} \geq L^2 \} \land L. \] (158)
With \( P \) representing the law of \((u, B)\), the following two results also follow immediately from previous works ([28], [56], [57]) making use of the fact that
\[
\theta(t, (u, B)) = B(t) \quad \forall \ t \in [0, T_L] \ \textbf{P}-\text{almost surely.} \tag{159}
\]

**Proposition 5.4.** (cf. [28] Pro. 5.4, [56] Pro. 5.4]) Let \( \tau_L \) be defined by \((157)\). Then \( P = \mathcal{L}(u, B) \), is a probabilistically weak solution to \((2)\) on \([0, \tau_L] \) that satisfies Definition 5.2.

**Proposition 5.5.** (cf. [28] Pro. 5.5, [56] Pro. 5.5]) Let \( \tau_L \) be defined by \((157)\) and \( P = \mathcal{L}(u, b) \). Then \( P \otimes_R \mathbb{R} \) in \((156)\) is a probabilistically weak solution to \((2)\) on \([0, \infty) \) that satisfies Definition 5.1.

Similarly to Theorem 2.2 at this point we are ready to prove Theorem 2.3 due to its similarity to previous works ([28], [56]), we leave this in the Appendix.

### 5.2. Proof of Theorem 2.3 assuming Proposition 5.7

We define \( \Upsilon(t) = e^{R(t)} \) and \( v = \Upsilon^{-1}u \) for \( t \geq 0 \). It follows from Itô’s product formula that
\[
\partial_t v + \frac{1}{2} v + (-\Delta)^m v + \Upsilon \text{div}(v \otimes v) + \nabla(\Upsilon^{-1} \pi) = 0, \quad \nabla \cdot v = 0 \quad \text{for} \ t > 0. \tag{160}
\]

For every \( q \in \mathbb{N}_0 \) we aim to construct \((v_q, \hat{R}_q)\) that satisfies
\[
\partial_t v_q + \frac{1}{2} v_q + (-\Delta)^m v_q + \Upsilon \text{div}(v_q \otimes v_q) + \nabla p_q = \text{div} \hat{R}_q, \quad \nabla \cdot v_q = 0 \quad \text{for} \ t > 0. \tag{161}
\]

We define \( \lambda_q \) and \( \delta_q \) identically to the additive case in \((42)\) but define differently
\[
M_0(t) = e^{4Lq+2L} \quad \text{and} \quad m_L = \sqrt{3L} e^{L}. \tag{162}
\]

Due to \((158)\), for all \( L > 1, \delta \in (0, \frac{1}{\delta_0}) \), and \( t \in [0, T_L] \) we have
\[
|B(t)| \leq L^2 \quad \text{and} \quad ||B||_{C^2_{\tau_L}} \leq L^2 \tag{163}
\]

which implies
\[
||\Upsilon||_{C^2_{\tau_L}} + ||\Upsilon(t)|| + ||\Upsilon^{-1}(t)|| \leq e^{L^2} L^2 + \sum_{\delta_q \leq \frac{1}{2}} \leq m_L^2. \tag{164}
\]

For inductive bounds we assume that \((v_q, \hat{R}_q)\) for all \( q \in \mathbb{N}_0 \) satisfy the following on \([0, T_L] \) with another universal constant \( c_R > 0 \) to be determined subsequently (see \((193)\) and \((204)\)):
\[
||v_q||_{C_{\tau_L}} \leq m_L M_0(t) \sum_{\delta_q \leq \frac{1}{2}} \leq 2m_L M_0(t) \tag{165a}
\]
\[
||v_q||_{C_1_{\tau_L}} \leq m_L M_0(t) \delta_q \lambda_q, \tag{165b}
\]
\[
||\hat{R}_q||_{C_{\tau_L}} \leq c_R M_0(t) \delta_q \tag{165c}
\]

where again we follow the convention that \( \sum_{\delta_q \leq \frac{1}{2}} \delta_q = 0 \) and assume \((44)\), to be formally stated in \((168)\), so that \( \sum_{1 \leq q \leq \frac{1}{2}} \delta_q^2 < \frac{1}{2} \) for any \( q \in \mathbb{N} \) and hence the second inequality of \((165a)\) is justified.

**Proposition 5.6.** For \( L > 1 \), define
\[
v_0(t, x) = (2\pi)^{-\frac{1}{2}} m_L e^{2Lt+L} \left( \sin(\lambda^T x) \quad 0 \quad 0 \right)^T. \tag{166}
\]
Then together with
\[
\hat{R}_0(t, x) = \frac{m_L(2L + \frac{1}{2})e^{2Lt+L}}{(2\pi)^{\frac{1}{2}}} \begin{pmatrix}
0 & 0 & -\cos(x) \\
0 & 0 & 0 \\
-\cos(x) & 0 & 0
\end{pmatrix} + \mathcal{R}(-\Delta)^m v_0,
\] (167)
it satisfies (161) at level \( q = 0 \). Moreover, (165) at level \( q = 0 \) is satisfied provided
\[
\sqrt{3}[1 + 2(2\pi)^{2}]\leq \sqrt{3}a^{-\delta} \leq \frac{c_R e^L}{L^2(4L + 1 + C_S \sqrt{2})e^{2Lt}}
\] (168)
where the first inequality guaranties (44). Furthermore, \( v_0(0, x) \) and \( \hat{R}_0(0, x) \) are both deterministic.

**Proof of Proposition 5.6.** The facts that \( v_0 \) is incompressible, mean-zero, \( \hat{R}_0 \) is trace-free and symmetric, and (161) at level \( q = 0 \) holds with \( p_0 \equiv 0 \), as well as both \( v_0(0, x) \) and \( \hat{R}_0(0, x) \) both being deterministic can be readily verified (see [56, Pro. 5.6]). Concerning the three estimates (165a)-(165c) we compute
\[
\|v_0\|_{C_{x,s}} = (2\pi)^{-\frac{1}{2}}m_L M_0(t)^{\frac{1}{2}} \leq m_L M_0(t)^{\frac{1}{2}},
\] (169a)
\[
\|v_0\|_{C_{x,s}} = (2\pi)^{-\frac{1}{2}}(2L + 1)m_L M_0(t)^{\frac{1}{2}} \leq m_L^2 M_0(t)\delta_0^2 \lambda_0,
\] (169b)
and
\[
\|v_0(t)\|_{L^2} \leq \frac{m_L M_0(t)^{\frac{1}{2}}}{\sqrt{2}}.
\] (170)
Finally,
\[
\|\hat{R}_0\|_{C_{x,s}} \leq (2\pi)^{-\frac{1}{2}}m_L (2L + \frac{1}{2})e^{2Lt+L} + \|\mathcal{R}(-\Delta)^m v_0\|_{C_{x,s}}.
\] (171)
By the same computations in (52) of the proof of Proposition 4.7, we know \( \|\mathcal{R}(-\Delta)^m v_0\|_{C_{x,s}} \leq C_S 2\|v_0\|_{C_{x,s}}^2 \) for the same \( C_S > 0 \) from (38) because \( v_0 \) in (166) also satisfies \( \Delta v_0 = -v_0 \). Therefore, applying (170) to (171) leads us to
\[
\|\hat{R}_0\|_{C_{x,s}} \leq \frac{m_L (4L + 1)e^{2Lt+L}}{(2\pi)^{\frac{1}{2}}} + \frac{C_S 2m_L M_0(t)^{\frac{1}{2}}}{\sqrt{2}} \leq C_R M_0(t)\delta_1.
\] (172)

**Proposition 5.7.** Let \( L > 1 \) satisfy
\[
\sqrt{3}[1 + 2(2\pi)^{2}] < \frac{c_R e^L}{L^2(4L + 1 + C_S \sqrt{2})e^{2Lt}}.
\] (173)
Suppose that \((v_q, \hat{R}_q)\) is an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process that solves (161) and satisfies (165a)-(165c). Then there exist a choice of parameters \( a \) and \( \beta \) such that (163) is fulfilled and an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \((v_{q+1}, \hat{R}_{q+1})\) that solves (161), satisfies (165a)-(165c) at level \( q + 1 \), and for all \( t \in [0, T_L] \)
\[
\|v_{q+1} - v_q\|_{C_{x,s}} \leq m_L M_0(t)^{\frac{1}{2}} \delta_{q+1}^2.
\] (174)
Finally, if \( v_0(0, x) \) and \( \hat{R}_0(0, x) \) are deterministic, then so are \( v_{q+1}(0, x) \) and \( \hat{R}_{q+1}(0, x) \).

Taking Proposition 5.7 for granted, we are able to prove Theorem 2.3 now.
Proof of Theorem 2.3 assuming Proposition 5.7. Given any \( T > 0, K > 1, \) and \( \kappa \in (0, 1) \), starting from \((v_0, \tilde{R}_0)\) in Proposition 5.6, Proposition 5.7 gives us \((v_q, \tilde{R}_q)\) for all \( q \geq 1 \) that are \((\mathcal{F}_t)_{t \geq 0}\)-adapted and satisfy \((161), (165a)-(165c), \) and \((174)\), as well as \( a \) and \( \beta \) such that \((168)\) is fulfilled. Then for all \( \gamma \in (0, \beta) \), similarly to \((56)\), using the fact that \( 2^{q+1} \geq 2(q+1) \) for all \( q \in \mathbb{N}_0 \),

\[
\sum_{q \geq 0} \|v_{q+1} - v_q\|_{C_{\kappa}^1}^{10} \leq \sum_{q \geq 0} \|v_{q+1} - v_q\|_{C_{\kappa}^1}^{10} \|v_{q+1} - v_q\|_{C_{\kappa}^1}^{10} \leq m_L\|v_{q+1} - v_q\|_{C_{\kappa}^1}^{10} \leq m_L^{1+3\gamma} M_0(t)^{128} \sum_{q \geq 0} a^{2q+1} \leq m_L^{1+3\gamma} M_0(t)^{128}. \tag{175}
\]

This implies that \( \{v_q\} \) is Cauchy in \( C([0, T_L]; \mathcal{C}^\gamma) \) and hence we can deduce a limiting solution \( v = \lim_{q \to \infty} v_q \in C([0, T_L]; \mathcal{C}^\gamma) \) that is \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Because \( u = \Upsilon v = e^{\delta t}v \), due to \((163)\) we can deduce \((4)\). Because \( \lim_{q \to \infty} \|\tilde{R}_q\|_{C_{\kappa}^1} \leq \lim_{q \to \infty} c_k M_0(t) \delta_q = 0 \) due to \((165c)\), we see that \( v \) is a weak solution to \((160)\) on \([0, T_L]\). Then \( u = \Upsilon v \) is a \((\mathcal{F}_t)_{t \geq 0}\)-adapted solution to \((2)\). Moreover, similarly to \((59)\) we can show

\[
\|v - v_0\|_{C_{\kappa}^1} \leq m_L M_0(t) \|v - v_0\|_{C_{\kappa}^1} \leq m_L M_0(t)^{1/2} \leq \frac{m_L M_0(t)^{1/2}}{2}. \tag{176}
\]

Therefore,

\[
\|v - v_0\|_{C_{\kappa}^1} \leq (2\pi)^{1/2} \|v - v_0\|_{C_{\kappa}^1} \leq \frac{m_L M_0(t)^{1/2}}{2}. \tag{177}
\]

Next, we take \( L > 1 \) sufficiently large so that not only \((173)\) but

\[
\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) e^{2LT} > \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) e^{2LT} \quad \text{and} \quad L > [\ln(K e^{\frac{1}{2}})]^2 \tag{178}
\]

hold. It follows that

\[
e^{2LT} \|v(0)\|_{L^2} \leq \frac{1}{2} e^{2LT} m_L M_0(0) \|v(0)\|^2 \left(\frac{1}{2} + \frac{1}{2}\right). \tag{179}
\]

This implies that on a set \((T_L \geq T)\),

\[
\|v(T)\|_{L^2} \leq \frac{m_L M_0(0)^{1/2}}{\sqrt{2}} - \|v(T) - v_0(T)\|_{L^2} \leq \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) e^{2LT} m_L M_0(0)^{1/2} \leq e^{2LT} \|v(0)\|_{L^2}. \tag{180}
\]

This gives on the set \((T_L \geq T)\),

\[
\|u(T)\|_{L^2} \geq |e^{\delta t}v| e^{2LT} \|v(0)\|_{L^2} \geq e^{2LT} \|u^in\|_{L^2} \geq K e^{\frac{1}{2}t} \|u^in\|_{L^2}. \tag{181}
\]

which verifies \((6)\). Finally, taking \( L > 1 \) larger if necessary achieves \((3)\) due to \((158)\). We also note that \( u^m(x) = \Upsilon(0)v(0, x) = v(0, x) \) is deterministic because \( v_q(0, x) \) is deterministic for all \( q \in \mathbb{N}_0 \) due to Propositions 5.6 and 5.7. \(\square\)

5.3. Proof of Proposition 5.7.
5.3.1. Mollification. We fix \( L > 1 \) sufficiently large so that \( (173) \) holds, and then take \( a \in \mathbb{N} \) sufficiently large while \( \beta \in (0, \frac{1}{2}) \) sufficiently small so that \( (168) \) holds. Now we define \( l \) identically to \( (63) \) and mollify \( v_l \) and \( \hat{R}_q \) identically to \( (64) \) while

\[
T_l \doteq T \ast \varphi_l. \tag{182}
\]

Because \((v_l, \hat{R}_q)\) solves \( (161)\), we see that

\[
\partial_t v_l + \frac{1}{2} v_l + (-\Delta)^m v_l + T_l \text{div}(v_l \otimes v_l) + \nabla p_l = \text{div}(\hat{R}_l + R_{\text{comm}}) \tag{183}
\]

where

\[
R_{\text{comm}} \doteq R_{\text{commator}} \doteq -(T_l (v_l \delta_l v_l)) \ast \varphi_l + T_l (v_l \delta_l v_l), \tag{184a}
\]

\[
p_l \doteq (p_l \ast \varphi_l) \ast \varphi_l - \frac{1}{3} (T_l ||v_l||^2 - (T_l |v_l|^2) \ast \varphi_l). \tag{184b}
\]

Let us compute for \( N \in \mathbb{N}, \beta \in (0, \frac{1}{2}), \) and \( a \in \mathbb{N} \) sufficiently large

\[
\begin{align*}
||v_q - v||_{C^\beta_a} &\lesssim l m^a_{\frac{1}{2}} M_0(t) \delta_q T_{l} \lambda_q \ll m_L M_0(t) \delta_q T_{l} \lambda_q, \tag{185a} \\
||v||_{C^\beta_a} &\lesssim l -N m^a_{\frac{1}{2}} M_0(t) \delta_q T_{l} \lambda_q \ll l -N m_L M_0(t), \tag{185b} \\
||v||_{C^\beta_a} &\ll ||v||_{C^\beta_a} \leq m_L M_0(t) \delta_q T_{l} \lambda_q \ll l \sum_{1 \leq q \leq q} \delta_q T_{l} \lambda_q. \tag{185c}
\end{align*}
\]

5.3.2. Perturbation. Differently from \((68a) - (68b)\) we define \( \Phi_j : [0, T_L] \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \) for \( j \in \{0, \ldots, [-l T_L]\} \) a \( \mathbb{T}^3 \)-periodic solution to

\[
(\partial_t + (T_l v_l) \cdot \nabla) \Phi_j = 0, \tag{186a}
\]

\[
\Phi_j(jl, x) = x. \tag{186b}
\]

Let us comment in Remark 5.1 on the importance of multiplying \( v_l \) by \( T_l \) within \( (186a) \). We collect necessary estimates of \( \Phi_j \).

**Proposition 5.8.** For all \( j \in \{0, \ldots, [-l T_L]\} \) and \( t \in (l(j - 1), l(j + 1)) \) with appropriate modification in case \( j = 0 \) and \( [-l T_L] \),

\[
\begin{align*}
||\nabla \Phi_j(t) - \text{Id}||_{C^\beta_a} &\lesssim e^{l \frac{1}{2} m^a_{\frac{1}{2}} M_0(t) \delta_q T_{l} \lambda_q} \ll 1, \tag{187a} \\
\frac{1}{2} \leq ||\nabla \Phi_j(t, x)|| \leq 2 \forall x \in \mathbb{T}^3 \text{ and } ||\Phi_j(t)||_{C^1} \leq 1, \tag{187b} \\
||\partial_t \Phi_j(t)||_{C^\beta_a} &\ll e^{l \frac{1}{2} m_L M_0(t) \delta_q T_{l} \lambda_q}, \tag{187c} \\
||\nabla \Phi_j(t)||_{C^\beta_a} &\ll e^{l \frac{1}{2} m^a_{\frac{1}{2}} M_0(t) \delta_q T_{l} \lambda_q} l^{-N} \forall N \in \mathbb{N}, \tag{187d} \\
||\partial_t \nabla \Phi_j(t)||_{C^\beta_a} &\ll e^{l \frac{1}{2} m^a_{\frac{1}{2}} M_0(t) \delta_q T_{l} \lambda_q} l^{-N} \forall N \in \mathbb{N}_0. \tag{187e}
\end{align*}
\]

(cf. \([7\) Equ. (5.19a) and (5.19c)] and \([4\) Lem. 3.1]).

**Proof of Proposition 5.8.** The proof is similar to that of Proposition 4.9 relying on \([4\) Pro. D.1]. First, due to \([4\) Equ. (135)], \( a \in \mathbb{N} \) sufficiently large gives

\[
||\nabla \Phi_j(t) - \text{Id}||_{C^\beta_a} \ll e^{l \frac{1}{2} m^a_{\frac{1}{2}} M_0(t) \delta_q T_{l} \lambda_q} - 1 \lesssim le^{l \frac{1}{2} m^a_{\frac{1}{2}} M_0(t) \delta_q T_{l} \lambda_q} \ll 1. \tag{188}
\]
Second, the first estimate in (187b) is an immediate consequence of (187a) while the second estimate in (187b) follows from [4, Eq. (132)-(133)] and (185b). Third, (187c) follows from directly estimating on $\partial_t\Phi_j(t)$ from (186a) via (163), (185c) and (187b). Fourth, (187d) can be verified via [4, Eq. (136)] as follows:

\[
\|\nabla \Phi_j(t)\|_{C^N} \leq e^{L^2 t} \|\nabla \Phi_j(t)\|_{C^N} \leq e^{L^2 t} M_0(t) \delta \eta_q \lambda_q \leq L^N e^{L^2 t} M_0(t) \delta \eta_q.
\]

Finally, we can directly apply $\nabla$ on (186a) and estimate in case $N \in \mathbb{N}$

\[
\|\nabla \Phi_j(t)\|_{C^N} \leq e^{L^2 t} \left[ \|\nabla \Phi_j(t)\|_{C^N} + \|\nabla \Phi_j(t)\|_{C^N} \right] \leq e^{2L^2 t} M_0(t) \delta \eta_q \lambda_q \|\nabla \Phi_j(t)\|_{C^N}.
\]

While the case $N = 0$ can be achieved similarly and more simply.

Let us define $\chi$ and $\chi_j$ for $j \in \{0, 1, ..., \left[ \Gamma^{-1} T_L \right] \}$ identically to (70) in the proof of Proposition 4.8 so that (71) continues to be satisfied. On the other hand, while we continue to define $a(t)$ identically to (72) except $M_0(t)$ is defined by (162) rather than (42), we define a modified amplitude function as

\[
\tilde{a}(t, x) = \tilde{a}_{q+1}(t, x) \equiv \gamma_j^{-1} \tilde{a}_{q}(t, x) = e^{L^2 t} \frac{1}{e^{L^2 t} M_0(t)} \sum_{\gamma_j \subseteq \gamma_j(t, x)} \tilde{a}_{q}(t, x) \left( \text{Id} - \frac{\tilde{R}_j(t, x)}{e^{L^2 t} M_0(t)} \right).
\]

Convenience of defining $\tilde{a}_q$ as $\gamma_j^{-1} \tilde{a}_q$ will be clear in the derivations of (209) and (222). As we have not changed the inductive hypothesis of $\tilde{R}_j$ (cf. (165) and (165c)), the computations of (73) and (74) go through without any issue so that $\text{Id} - \frac{R_j}{e^{L^2 t} M_0(t)}$ lies in the domain of $\gamma_j(t, x)$ from (21). Moreover, we derive the following crucial point-wise identity:

\[
\gamma_j(t, x) \leq \sum_{\gamma_j \subseteq \gamma_j(t, x)} \tilde{a}_{q}(t, x) \left( \text{Id} - \frac{\tilde{R}_j(t, x)}{e^{L^2 t} M_0(t)} \right) \leq \frac{L^{N-1}}{e^{L^2 t} M_0(t)}.
\]

Next, we obtain necessary estimates for $\tilde{a}_q$:

**Proposition 5.9.** The modified amplitude function $\tilde{a}_q$ in (189) satisfies the following bounds on $[0, T_L]$:

\[
\|\tilde{a}_q\|_{C^N} \leq e^{L^2 t} \frac{1}{e^{L^2 t} M_0(t)} \|\nabla \Phi_j(t)\|_{C^N} \leq e^{L^2 t} M_0(t) \delta \eta_q \lambda_q \|\nabla \Phi_j(t)\|_{C^N}.
\]

\[
\|\tilde{a}_q\|_{C^N} \leq e^{L^2 t} \frac{1}{e^{L^2 t} M_0(t)} \|\nabla \Phi_j(t)\|_{C^N} \leq e^{L^2 t} M_0(t) \delta \eta_q \lambda_q \|\nabla \Phi_j(t)\|_{C^N}.
\]

**Proof of Proposition 5.9.** The first inequality (191a) follows from the estimate (76a) in Proposition 4.10 and the fact that $\|\nabla \Phi_j(t)\|_{C^N} \leq e^{L^2 t} M_0(t)$ due to (163). Although the definition of $M_0(t)$ in (162) is different from (42), this makes no difference in the computations of (77)–(78). Next, we can directly apply $\partial_t$ on (189) and estimate

\[
\|\tilde{a}_q\|_{C^N} \leq \|\nabla \Phi_j(t)\|_{C^N} + \|\nabla \Phi_j(t)\|_{C^N} \leq e^{L^2 t} M_0(t) \delta \eta_q \lambda_q \|\nabla \Phi_j(t)\|_{C^N}.
\]

Then we can apply (163) and (76a)–(76b) and immediately obtain the desired result (191b). 

□
Next, we recall $\bar{a}(x)$, $W_{\varepsilon,L_{q+1}}$, and $B_x$ respectively from (15a), (15), and (13), and define

$$\begin{align*}
w^{(p)}_{q+1}(t,x) &\triangleq \sum_{j \in \Lambda_j} \bar{a}(x) B_x \Phi_j(t,x) = \bar{a}(x) B_x e^{i t \varepsilon \cdot \zeta} \Phi_j(t,x),
(192a)
$$

$$\begin{align*}
w^{(p)}_{q+1}(t,x) &\triangleq \sum_{j \in \Lambda_j} \sum_{c \in \Lambda_j} \bar{a}(x) B_x e^{i t \varepsilon \cdot \zeta} \Phi_j(t,x),
(192b)
$$

we note that defining $w^{(p)}_{q+1}$ this way with $\bar{a}(x)$ instead of $a(x)$ makes sure to eliminate a difficult term in $R_{osc}$, as we will subsequently see in (209). Thus, by choosing $c_R \leq (2 \sqrt{M})^{-4}$ and using the facts that $\| \nabla_i \|_{C_t} \leq e^{i t \frac{L}{2}}$ and for any $s \in [0, t]$ fixed, there exist at most only two non-trivial cutoffs, we obtain

$$\|w^{(p)}_{q+1}\|_{C_t, s} \leq e^{i t \frac{L}{2}} L^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} M_0(t) \frac{1}{2} \sqrt{M} \leq 2^{-1} e^{i t \frac{L}{2}} \delta_{q+1}^{\frac{1}{2}} M_0(t).$$

(193)

Next, we define $\Phi \Phi_j$ identically to (82) so that

$$\begin{align*}
w^{(p)}_{q+1}(t,x) &\triangleq \sum_{j \in \Lambda_j} \bar{a}(x) B_x \Phi_j(t,x) = \bar{a}(x) B_x e^{i t \varepsilon \cdot \zeta} \Phi_j(t,x).
(194)
$$

Then

$$\bar{a}(x) \Phi_j W_{q+1} = \lambda_{q+1}^{-1} \nabla \cdot (\bar{a}(x) \Phi_j W_{q+1}) - \lambda_{q+1}^{-1} \nabla (\bar{a}(x) \Phi_j W_{q+1}) \times W_{q+1}. \n(195)
$$

Next, let us define

$$\begin{align*}
w^{(p)}_{q+1}(t,x) &\triangleq \lambda_{q+1}^{-1} \nabla \cdot (\bar{a}(x) \Phi_j W_{q+1}) - \lambda_{q+1}^{-1} \nabla (\bar{a}(x) \Phi_j W_{q+1}) \times W_{q+1},
(196)
$$

the reason to incorporate $\bar{a}(x)$ within $w^{(p)}_{q+1}$ is to make $w^{(p)}_{q+1}$ divergence-free as we will subsequently see in (195a). It follows that

$$w^{(p)}_{q+1}(t,x) = \lambda_{q+1}^{-1} (\nabla \bar{a}(x) W_{q+1} + \bar{a}(x) \nabla (\Phi_j(t,x) - 1) \times (\Phi_j(t,x) - 1)) \times B_x e^{i t \varepsilon \cdot \zeta} \Phi_j(t,x).
(197)
$$

Thus, if we define $w^{(p)}_{q+1}$ and $w^{(p)}_{q+1}$ identically to (87), then

$$w^{(p)}_{q+1} + \sum_{j \in \Lambda_j} \sum_{c \in \Lambda_j} \Phi_j \Phi_j W_{q+1} \leq w^{(p)}_{q+1} + \sum_{j \in \Lambda_j} \sum_{c \in \Lambda_j} \Phi_j \Phi_j W_{q+1}, \n(198)
$$

which shows that $w^{(p)}_{q+1}$ is mean-zero and divergence-free because $\nabla \cdot (\nabla f) = 0$ for all $f$. Next, we can estimate using the fact that for all $s \in [0, t]$ fixed, there are only at most two non-trivial cutoffs,

$$\begin{align*}
||w^{(p)}_{q+1}||_{C_t, s} &\leq 2 \sup_{j \in \Lambda_j} \lambda_{q+1}^{-1} ||\nabla \bar{a}(x)||_{C_t, s} + ||\bar{a}(x)||_{C_t, s} \sup_{s \in [0, t]} ||(\nabla \Phi_j(s) - 1) \Phi_j(s)||_{C_t, s} \\
&\leq \delta_{q+1} \epsilon^{i t \frac{L}{2}} M_0(t) \frac{1}{2} \lambda_{q+1}^{-1} \leq e^{i t \frac{L}{2}} \delta_{q+1}^{\frac{1}{2}} M_0(t).
(199)
$$

It follows that

$$\begin{align*}
||w^{(p)}_{q+1}||_{C_t, s} &\leq ||w^{(p)}_{q+1}||_{C_t, s} + ||w^{(p)}_{q+1}||_{C_t, s}
\leq \delta_{q+1}^{\frac{1}{2}} M_0(t) \frac{1}{2} \lambda_{q+1}^{-1} \leq e^{i t \frac{L}{2}} \delta_{q+1}^{\frac{1}{2}} M_0(t).\n(200)
$$

Thus, if we define the velocity field at level $q + 1$ identically to (91), then we can verify (174) as follows:

$$||v_{q+1} - v_0||_{C_t, s} \leq ||w^{(p)}_{q+1}||_{C_t, s} + ||v_{q+1} - v_0||_{C_t, s}
\leq m \lambda_{q+1} \delta_{q+1}^{\frac{1}{2}} M_0(t).$$

(201)
Additionally, we can verify (165a) as follows:

\[
\|v_{q+1}\|_{C_{s,t}} \leq \|v_t\|_{C_{s,t}} + \|w_{q+1}\|_{C_{s,t}} \quad \leq \quad m_L M_0(t) \frac{1}{2} (1 + \sum_{1 \leq \delta q+1} \delta^\frac{1}{q+1}).
\]  

(202)

Next, in order to verify (165b) at level \(q+1\), we compute similarly to (23) using the fact that for any fixed \(s \in [0, t]\), there are at most two non-trivial cutoffs

\[
\|\partial_t w_{q+1}^{(p)}\|_{C_{s,t}} + \|\nabla w_{q+1}^{(c)}\|_{C_{s,t}} \quad \leq \quad M c_K \delta^{\frac{1}{q+1}} [\varepsilon^{q+1} M_0(t) \frac{1}{4} + \lambda_{q+1}] \varepsilon^{q+1} M_0(t) m_L \frac{1}{2},
\]  

(203a)

\[
\|\partial_t w_{q+1}^{(c)}\|_{C_{s,t}} + \|\nabla w_{q+1}^{(c)}\|_{C_{s,t}} \quad \leq \quad M c_K \delta^{\frac{1}{q+1}} [\varepsilon^{q+1} \lambda_{q+1}] \varepsilon^{q+1} M_0(t) m_L \frac{1}{2} + e^{q+1} \lambda_{q+1} \varepsilon^{q+1} M_0(t) m_L \frac{1}{2}.
\]  

(203b)

with \(M\) from (24). Therefore, by taking \(c_K \ll M^4\) and \(a \in \mathbb{N}\) sufficiently large

\[
\|w_{q+1}\|_{C_{s,t}} \quad \leq \quad \frac{3}{4} m_L \delta^{\frac{1}{q+1}} M_0(t) \frac{1}{2} \quad + \quad \|\partial_t w_{q+1}^{(p)}\|_{C_{s,t}} + \|\nabla w_{q+1}^{(p)}\|_{C_{s,t}} + \|\partial_t w_{q+1}^{(c)}\|_{C_{s,t}} + \|\nabla w_{q+1}^{(c)}\|_{C_{s,t}} \quad \leq \quad \frac{\lambda_{q+1} \delta^{\frac{1}{q+1}} M_0(t) m_L}{2}.
\]  

(204)

We are now ready to verify (165b) at level \(q+1\) as follows. By Young’s inequality for convolution and the fact that mollifiers have mass one and \(\beta \in (0, \frac{1}{2})\),

\[
\|v_{q+1}\|_{C_{s,t}} \quad \leq \quad \lambda_{q+1} \delta^{\frac{1}{q+1}} M_0(t) m_L^2 \left( 2^{n-1+\beta} + \frac{1}{2} \right) \quad \leq \quad \lambda_{q+1} \delta^{\frac{1}{q+1}} M_0(t) m_L^2.
\]  

(205)

Subsequently, similarly to the proof of Proposition 4.8 we will rely on Lemma 6.2 to estimate Reynolds stress. Due to (187b) this time, by choosing \(a \in \mathbb{N}\) sufficiently large we have \(\frac{1}{2} \leq |\nabla \Phi(t, x)| \leq 2\) for all \(t \in [j-1, j+1]\) and \(x \in T^3\) so that (234) is satisfied with \(C = 2\). Therefore, (97) leads to (98)-(99) again.

5.3.3. **Reynolds stress.** First, we observe that

\[
\text{div} \hat{\mathbf{R}}_{q+1} - \nabla p_{q+1} = \text{div} (v_t \otimes v_t) - \nabla p_t + \text{div}(\hat{R}_{t} + R_{\text{com}}) = \partial_t w_{q+1}^{(p)} + \partial_t w_{q+1}^{(c)} + \frac{1}{2} w_{q+1} + (-\Delta w_{q+1} + \text{div}(v_{q+1} \otimes v_{q+1})).
\]  

(206)

We have an identity of

\[
- \partial_t \text{div}(v_t \otimes v_t) + \text{div}(v_{q+1} \otimes v_{q+1}) = \partial_t \text{div}(v_{q+1} \otimes v_{q+1}) + \text{div}(v_{q+1} \otimes v_{q+1}) - (\partial_t - \partial_t \text{div}(v_{q+1} \otimes v_{q+1})).
\]  

(207)

Applying this identity (207) in (206) leads to

\[
\text{div} \hat{\mathbf{R}}_{q+1} - \nabla p_{q+1} = (\partial_t + \text{div}(v_t \cdot \nabla)) w_{q+1}^{(p)} + \text{div}(\text{div}(\partial_t v_{q+1} \otimes v_{q+1} + \hat{R}_{t}))
\]

\[
+ \text{div}(\text{div}(\text{div}(v_{q+1} \cdot \nabla)v_{q+1} + (\partial_t + \text{div}(v_t \cdot \nabla)) w_{q+1}^{(p)} + \text{div}(v_{q+1} \otimes w_{q+1} + w_{q+1} \otimes v_{q+1}))
\]

\[
\text{div}(R_{\text{com}} + \nabla p_{q+1})
\]

\[
\text{div}(R_{\text{com}} + \nabla p_{q+1})
\]

\[
\text{div}(R_{\text{com}} + \nabla p_{q+1})
\]
\[ \text{Remark 5.1.} \text{ Similarly to Remark 4.1, we strategically multiplied } v \text{ and } \bar{v} \text{ in } R_{\text{osc}} \text{ in } R_{\text{tra}} \text{ and } R_{\text{corr}} \text{ in } R_{\text{corr}}. \text{ As we will see in (219) and (223), this leads to a crucial cancellation of the most difficult term when } \nabla \text{ is applied on } e^{\lambda_{p^-} \phi_{\bar{v}}} \text{ in which } \lambda_{p+1} \text{ from chain rule makes such terms too large to handle.} \]

Concerning \( R_{\text{osc}} \) in (208), making use of the fact that \( \gamma_{\bar{v}} = \gamma_{-\bar{v}} \) from Lemma 3.2 so that \( \bar{a}_{\bar{v}} = \bar{a}_{-\bar{v}} \) in (189).

\[ \text{div}(\nabla_{\bar{v}} w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \hat{R}) \]

(209)

and hence \( R_{\text{osc}} \) and \( p_{\text{osc}} \) are same as those in (107a) - (107d); therefore, the estimate (132) directly applies to the current case. Thus, besides \( R_{\text{osc}}, p_{\text{osc}}, \text{ and } R_{\text{com1}} \) in (184a), we define from (208)

\[ R_{\text{line}} \triangleq R_{\text{linear}} \triangleq R(1 - \frac{1}{2} \Delta^m w_{q+1}), \]  

(210a)

\[ R_{\text{tra}} \triangleq R_{\text{transport}} \triangleq R((\partial_t + \gamma_{\bar{v}}(v_l \cdot \nabla)) w_{q+1}^{(p)}), \]  

(210b)

\[ R_{\text{Nash}} \triangleq R((\gamma_{\bar{v}}(v_l \cdot \nabla)) w_{q+1}^{(p)}), \]  

(210c)

\[ R_{\text{corr}} \triangleq R_{\text{correction}} \triangleq R((\partial_t + \gamma_{\bar{v}}(v_l \cdot \nabla)) w_{q+1}^{(p)} + \gamma_{\bar{v}}(v_l \cdot \nabla)) w_{q+1}^{(p)}), \]  

(210d)

\[ p_{\text{com1}} \triangleq p_{\text{com1}} \triangleq (\gamma_{\bar{v}}(v_l \cdot \nabla)) w_{q+1}^{(p)}), \]  

(210e)

\[ R_{\text{com2}} \triangleq R_{\text{com2}} \triangleq (\gamma_{\bar{v}}(v_l \cdot \nabla)) w_{q+1}^{(p)}), \]  

(210f)

\[ p_{\text{com2}} \triangleq p_{\text{com2}} \triangleq (\gamma_{\bar{v}}(v_l \cdot \nabla)) w_{q+1}^{(p)}), \]  

(210g)

and \( p_{\text{q+1}} \triangleq p_{\text{q+1}} - p_{\text{com1}} - p_{\text{com2}} \) while \( \hat{R}_{q+1} \) identically to (108a).

Now we start to work on \( R_{\text{line}} \) from (210a). For any \( \epsilon \in (0, 1 - 2m) \) fixed, we can estimate via Lemma 6.3

\[ ||R(\Delta)^m w_{q+1}||_{C^0} \leq ||Rw_{q+1}^{(p)}||_{C^0} + ||Rw_{q+1}^{(c)}||_{C^0} \]

(211)
First, we can rely on the fact that for any \( s \in [0, t] \) there are at most two non-trivial cutoffs to obtain
\[
\|Rw_{q+1}\|_{C_tC^m} \leq 2 \sup_{\xi \in \mathcal{A}_q} \sum_j \|R(\tilde{u}_q W_{q+1}(\Phi_j))\|_{C_tC^m}.
\] (212)
By (191) we see that (97) is satisfied with \( C_a \) valid for all \( N = 0, \ldots, \lfloor \frac{1}{2m} \rfloor \) and we can choose \( \beta < \frac{1}{4}(1 - 2m - \epsilon) \) and \( a \in \mathbb{N} \) sufficiently large to deduce
\[
\|Rw_{q+1}\|_{C_tC^m} \leq c_0M_0(t)\delta_{q+2}[e^{\frac{1}{2}(q_1+2m+\epsilon-1)}] \ll c_0M_0(t)\delta_{q+2}. \] (213)
Second, we see that
\[
\|Rw_{q+1}\|_{C_tC^m} \leq 2 \sup_{\xi \in \mathcal{A}_q} \sum_j \|R(\tilde{u}_q W_{q+1}(\Phi_j))\|_{C_tC^m}.
\] (214)
where for any \( N = 0, \ldots, \lfloor \frac{1}{2m} \rfloor \) we can estimate by taking \( a \in \mathbb{N} \) sufficiently large
\[
\|\lambda_{q+1}^{-1} \nabla \tilde{u}_q + i\tilde{u}_q \zeta \cdot (\nabla \Phi_j - Id)\|_{C_tC^m} \ll \lambda_{q+1}^{-1} L^{q+1} M_0(t)^{1/2} \|\zeta\|_{C^{\lfloor \frac{1}{2m} \rfloor}(B_{R_0}(\Phi_j))} L^{-N-1} + e^{\frac{1}{2}q_1+2m+\epsilon} M_0(t)^{1/2} \|\zeta\|_{C^{\lfloor \frac{1}{2m} \rfloor}(B_{R_0}(\Phi_j))} L^{-N-1} \ll \lambda_{q+1}^{-1} \|\zeta\|_{C^{\lfloor \frac{1}{2m} \rfloor}(B_{R_0}(\Phi_j))} M_0(t)^{1/2} \lambda_{q+1}^{-1} L^{q+1} e^{\frac{1}{2}q_1+2m+\epsilon}. \] (215)
Therefore, (97) holds with \( C_a = e^{\frac{1}{2}(q_1+2m+\epsilon-1)} \lambda_{q+1}^{-1} L^{q+1} \) and hence by taking \( \beta < \frac{1}{4}(1 - 2m - \epsilon) \) and \( a \in \mathbb{N} \) sufficiently large we obtain
\[
\|Rw_{q+1}\|_{C_tC^m} \ll \sup_{\xi \in \mathcal{A}_q} \sum_j \delta_{q+1} M_0(t)^{1/2} \|\zeta\|_{C^{\lfloor \frac{1}{2m} \rfloor}(B_{R_0}(\Phi_j))} L^{-N-1} \ll c_0M_0(t)\delta_{q+2}. \] (216)
Therefore, we conclude by applying (213) and (216) to (211) that
\[
\|R(-\Delta)^m w_{q+1}\|_{C_tC^m} \ll c_0M_0(t)\delta_{q+2}. \] (217)
As \( \|Rw_{q+1}\|_{C_tC^m} \ll \|Rw_{q+1}\|_{C_tC^m} \), we can apply the same estimates in (213) and (216) to get (218)
\[
\|R_{\text{time}} w_{q+1}\|_{C_tC^m} \|R\frac{1}{2}w_{q+1} + (-\Delta)^m w_{q+1}\|_{C_tC^m} \ll c_0M_0(t)\delta_{q+2}. \] (219)
Next, in order to work on \( R_{\text{time}} \) from (210b) we make the key observation that
\[
(\partial_t + \tau_j(v_t \cdot \nabla))w_{q+1}(t, x) = (1) \sum_j \sum_{\xi \in \mathcal{A}_q} \left[ \partial_t \tilde{u}_q(t, x) + \tau_j(v_t \cdot \nabla) \tilde{u}_q(t, x) \right] W_{q+1}(\Phi_j(t, x)) \nonumber + \tilde{u}_q(t, x) \nabla W_{q+1}(\Phi_j(t, x)) \cdot \left[ \partial_t \Phi_j(t, x) + \tau_j(v_t \cdot \nabla) \Phi_j(t, x) \right] \nonumber + \tilde{u}_q(t, x) \nabla W_{q+1}(\Phi_j(t, x)) \cdot \left[ \partial_t \Phi_j(t, x) + \tau_j(v_t \cdot \nabla) \Phi_j(t, x) \right] W_{q+1}(\Phi_j(t, x)). \] (219)
For any $\epsilon \in (\frac{1}{2}, \frac{1}{4})$ and $N = 0, \ldots, \lceil \frac{1}{\epsilon} \rceil \vee 8 = 8$, 

$$\|\partial_t \bar{a}_i + \mathcal{T}_i(v_i \cdot \nabla)\bar{a}_i\|_{C^0_t} \leq e^{\frac{2}{1-\epsilon} t} \frac{c_R}{\sqrt{\epsilon}} \delta_{q+1}^N M_0(t)^{\frac{3}{5}} \|\gamma\|_{C^{0,1}(B_{r_0}(Id))} L^{-1}$$

\[ \leq e^{\frac{2}{1-\epsilon} t} \frac{c_R}{\sqrt{\epsilon}} \delta_{q+1}^N m_L M_0(t) L^{-1} \|\gamma\|_{C^{0,1}(B_{r_0}(Id))} L^{-1}. \]

Therefore, (207) is satisfied with “$C_\alpha$” = $\|\gamma\|_{C^{0,1}(B_{r_0}(Id))} \delta_{q+1}^N m_L M_0(t) L^{-1}$. Hence, by taking $\beta < \frac{1}{(2 - 2\epsilon)}$ and $a \in \mathbb{N}$ sufficiently large we obtain

$$\|R_{\text{ran}}\|_{C^0_t} \leq e^{\frac{2}{1-\epsilon} t} \delta_{q+1}^N m_L \approx c_R M_0(t) \delta_{q+1}^N [a^{2(\theta - \frac{1}{2} + 2\epsilon)}e^{\frac{2}{1-\epsilon} t} m_L] \approx c_R M_0(t) \delta_{q+1}^N.$$

Next, we work on $R_{\text{Nash}}$ in (210c) which may be written using the fact that $\bar{a}_i = \gamma_i^\alpha \gamma \bar{a}_i$ due to (158) as follows:

$$R_{\text{Nash}} \leq \sum_{j} \sum_{\xi_k} \mathcal{R}(a_i \gamma_{\alpha} \gamma \bar{a}_i \cdot \nabla v_i) + ((\lambda_{q+1}^{-1} \nabla a_i + ia_i \zeta \cdot (\nabla \Phi_j - Id)) \times W(\Phi_j) \cdot \nabla v_i).$$

Now $\|\gamma_i^\alpha \gamma_i \bar{a}_i \|_{C^0_t} \leq e^{\frac{2}{1-\epsilon} t}$ by (163) and thus considering (137), for $\epsilon \in (\frac{1}{2}, \frac{1}{4})$ and choosing $\beta < \frac{1}{(1 - 2\epsilon)}$ gives us immediately for $a \in \mathbb{N}$ sufficiently large

$$\|R_{\text{Nash}}\|_{C^0_t} \leq \sum_{j} \sum_{\xi_k} \|\mathcal{R}(a_i \gamma_{\alpha} \gamma \bar{a}_i \cdot \nabla v_i)\|_{C^0_t} \leq \sum_{j} \sum_{\xi_k} \frac{c_R M_0(t) \delta_{q+1}^N [a^{2(\theta - \frac{1}{2} + 2\epsilon)}e^{\frac{2}{1-\epsilon} t} m_L]}{X} \leq c_R M_0(t) \delta_{q+1}^N.$$

Next, we look at $R_{\text{corr}}$ from (210d). Again, we make the key observation that

$$\|\partial_t (v_i \cdot \nabla) \gamma \bar{a}_i\|_{C^0_t} \leq \sum_{j} \sum_{\xi_k} \|\mathcal{R}(a_i \gamma_{\alpha} \gamma \bar{a}_i \cdot (\nabla \Phi_j - Id)) \times W(\Phi_j)\|_{C^0_t}$$

\[ \leq \sum_{j} \sum_{\xi_k} \|\mathcal{R}(a_i \gamma_{\alpha} \gamma \bar{a}_i \cdot (\nabla \Phi_j - Id)) \times W(\Phi_j)\|_{C^0_t} \leq \sum_{j} \sum_{\xi_k} \frac{c_R M_0(t) \delta_{q+1}^N [a^{2(\theta - \frac{1}{2} + 2\epsilon)}e^{\frac{2}{1-\epsilon} t} m_L]}{X} \leq c_R M_0(t) \delta_{q+1}^N. \]

For any $\epsilon \in (\frac{1}{2}, \frac{1}{4})$ and $N = 0, \ldots, \lceil \frac{1}{\epsilon} \rceil \vee 8 = 8$, by taking $a \in \mathbb{N}$ sufficiently large we can separately estimate

$$\lambda_{q+1}^{-1} \partial_t \nabla \bar{a}_i \|_{C^0_t} \leq \lambda_{q+1}^{-1} e^{\frac{2}{1-\epsilon} t} \delta_{q+1}^N \|\gamma_i\|_{C^{0,1}(B_{r_0}(Id))} L^{-1}.$$

\[ \leq e^{\frac{2}{1-\epsilon} t} \delta_{q+1}^N M_0(t) \|\gamma_i\|_{C^{0,1}(B_{r_0}(Id))} L^{-1} \leq e^{\frac{2}{1-\epsilon} t} \delta_{q+1}^N m_L M_0(t) L^{-1}. \]

\[ \leq e^{\frac{2}{1-\epsilon} t} \delta_{q+1}^N m_L M_0(t) L^{-1}. \]
\[ \| \tilde{a}(\tilde{C}) \|_{C, C} \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \delta_{q+1} \lambda q M_0(t)^{2} m_0^2 \| \chi \|_{C^{0}(B_{c_{1}, (d))}} \delta_q \lambda_q \tilde{M}^{-N}, \quad (225c) \]

\[ \lambda_{q+1}^{-1} \| I'(v_1 \cdot \nabla) \tilde{a}(\tilde{C}) \|_{C, C} \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \lambda q \tilde{M} \lambda_0 M_0(t)^{2} F_{N-2}, \quad (225d) \]

\[ \| I'(v_1 \cdot \nabla) \tilde{a}(\tilde{C}) (\nabla \Phi_j - Id) \|_{C, C} \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \lambda q \lambda_0 M_0(t)^{2} m_0^2 \| \chi \|_{C^{0}(B_{c_{1}, (d))}} \tilde{M}^{-N}, \quad (225e) \]

\[ \| I'(v_1 \cdot \nabla) \tilde{a}(\tilde{C}) \cdot \nabla \Phi_j \|_{C, C} \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \lambda q \lambda_0 M_0(t)^{2} m_0^2 \| \chi \|_{C^{0}(B_{c_{1}, (d))}} \tilde{M}^{-N}. \quad (225f) \]

Using (225), we can estimate for all \( N = 0, \ldots, \left[ \frac{1}{\varepsilon} \right] \) and \( \delta_{q+1}^N \)

\[ \| \tilde{a}(q+1) \|_{C, C} + \| \tilde{a}(q+1) \|_{C, C} + \| \tilde{a}(q+1) \|_{C, C} \]

\[ + \lambda_{q+1}^{-1} \| I'(v_1 \cdot \nabla) \tilde{a}(\tilde{C}) \|_{C, C} + \| I'(v_1 \cdot \nabla) \tilde{a}(\tilde{C}) (\nabla \Phi_j - Id) \|_{C, C} + \| I'(v_1 \cdot \nabla) \tilde{a}(\tilde{C}) \cdot \nabla \Phi_j \|_{C, C} \]

\[ \leq \delta_{q+1}^{-1} \lambda_{q+1}^{-1} \tilde{M}^{-N} \| \tilde{a} \|_{C^{0}(B_{c_{1}, (d))}} \tilde{M}^{-N}. \quad (226) \]

Hence, (27) holds with \( C_a = \delta_{q+1}^{-1} \lambda_{q+1}^{-1} \tilde{M}^{-N} \| \tilde{a} \|_{C^{0}(B_{c_{1}, (d))}} \tilde{M}^{-N}. \quad (226) \)

Next, we can directly estimate within \( R_{cont} \) from (210d) as follows. First, we can write

\[ \| R(\tilde{a} \cdot \nabla \tilde{a}) \|_{C, C} \]

\[ \leq \sup_j \sum_{j \in \Lambda_j} \| R \|_{C, C} (\lambda_{q+1}^{-1} \tilde{a}(\tilde{C}) \cdot (\nabla \Phi_j - Id) \times W_{C_{s}}(\Phi_j)) \]

\[ \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \| \tilde{a}(q+1) \|_{C, C} \| \tilde{a}(q+1) \|_{C, C} \]

\[ \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \| \tilde{a}(q+1) \|_{C, C} \| \tilde{a}(q+1) \|_{C, C} \]

\[ \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \| \tilde{a}(q+1) \|_{C, C} \| \tilde{a}(q+1) \|_{C, C} \]

\[ \leq \varepsilon^{\frac{1}{2} + \frac{1}{2}} \| \tilde{a}(q+1) \|_{C, C} \| \tilde{a}(q+1) \|_{C, C} \]

\[ \| R_{cont} \|_{C, C} \leq c_{R} M_0(t) \delta_{q+2} \| a^{(d_{q+2}+1)} \|_{C, C} \| \tilde{a}(q+1) \|_{C, C} \]

\[ \leq c_{R} M_0(t) \delta_{q+2} \| a^{(d_{q+2}+1)} \|_{C, C} \| \tilde{a}(q+1) \|_{C, C} \]

\[ \leq c_{R} M_0(t) \delta_{q+2} \| a^{(d_{q+2}+1)} \|_{C, C} \| \tilde{a}(q+1) \|_{C, C} \]

\[ \| R_{cont} \|_{C, C} \leq c_{R} M_0(t) \delta_{q+2}. \quad (229) \]

Finally, we can estimate \( R_{cont 1} \) in (184a) and \( R_{cont 2} \) in (210f) as follows. First, we can write

\[ \| (\tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q})) * \chi_\tilde{q}, \chi_\tilde{q} = \tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q}) \]

\[ \quad + (\tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q})) * \chi_\tilde{q} = \tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q}) \]

\[ \quad + (\tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q})) * \chi_\tilde{q} = \tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q}) \]

\[ \quad + (\tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q})) * \chi_\tilde{q} = \tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q}) \]

\[ \quad + (\tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q})) * \chi_\tilde{q} = \tilde{T}(v_\tilde{q} \tilde{\Omega} v_\tilde{q}) \]

apply standard commutator estimate to it (e.g., [7] Prop. 6.5) or [14] Equ. (5)) so that we can estimate by taking \( \beta < \frac{1}{2} \) and \( \alpha \in \mathbb{N} \) sufficiently large, as well as using the fact that \( \delta \in (0, \frac{1}{2}) \)

\[ \| R_{cont 1} \|_{C, C} \leq c_{R} M_0(t) \delta_{q+2}. \quad (229) \]
We point out that in contrast to [28, 56], this is where we need \( \delta \in (0, \frac{1}{15}) \) rather than \( \delta \in (0, \frac{1}{5}) \) from previous works such as [28, p. 43] and [56, p. 30], essentially due to a new choice of \( \ell \) in (65). Second, as \( |Y(t) - Y'_i(t)| \leq l^4 \|Y\| \|e^{-\delta t}M_0(0)\|_p \|^4 \leq l^4 \|e^{-\delta t}M_0(0)\|_p \|^4 \) due to (164) and \(-\frac{2}{3} + 3\delta < -\frac{5}{8}\) because \( \delta \in (0, \frac{1}{15}) \), by taking \( \beta < \frac{5}{8} \) and \( a \in \mathbb{N} \) sufficiently large we obtain

\[
\|R_{\text{com2}}\|_{C_{\ell}} \lesssim l^4 \|e^{-\delta t}M_0(0)\|_p \|^4 \leq c_R M_0(t)\|R_{\text{com2}}\|_{C_{\ell}} \lesssim c_R M_0(t)\|R_{\text{com2}}\|_{C_{\ell}}.
\]

(232)

Applying (218), (221), (132), (225), (229), (231), (232) to (108a) verifies (165c) at level \( q + 1 \).

The verification of how \((v_{q+1}, \hat{R}_{q+1}) = (F_i)_{i \geq 20}\)-adapted and that \(v_q(0, x)\) and \(\hat{R}_{q+1}(0, x)\) are deterministic if \(v_q(0, x)\) and \(\hat{R}_q(0, x)\) are deterministic is similar to the proof of Proposition 4.8 and previous works [28, 56].

6. Appendix

6.1. Further preliminaries.

**Lemma 6.1.** ([7, Equ. (5.34)]) For any \( v \in C^\infty(\mathbb{T}^3) \) that is mean-zero, define

\[
(Rv)_{kl} \triangleq (\partial_i^2 \Delta^{-1} v^l + \partial_i^2 \Delta^{-1} v^k) - \frac{1}{2} (\partial_i v^l + \partial_i v^k) \Delta^{-1} v
\]

(233)

for \( k, l \in \{1, 2, 3\} \). Then \( Rv(x) \) is a symmetric trace-free matrix for each \( x \in \mathbb{T}^3 \) that satisfies \( \text{div}(Rv) = 0 \). When \( v \) does not satisfy \( \int_{\mathbb{T}^3} v dx = 0 \), we overload notation and denote \( Rv \triangleq R(v - \int_{\mathbb{T}^3} v dx) \). Moreover, \( R \) satisfies the classical Calderón-Zygmund and Schauder estimates:

\[
\|(-\Delta)^s R\|_{L^p_{\ell} \to L^p_{\ell}} + \|R\|_{L^p_{\ell} \to L^p_{\ell}} + \|R\|_{C^{1,1}_{\ell}} \lesssim 1 \quad \text{for all } p \in (1, \infty).
\]

The following stationary phase lemma played a crucial role in our proofs.

**Lemma 6.2.** ([7, Lem. 5.7], [16, Lem. 2.2]) Let \( \lambda \in \mathbb{Z}^3 \), \( \alpha \in (0, 1) \), and \( p \in \mathbb{N} \). Assume that \( a \in C^{p+\alpha}(\mathbb{T}^3) \) and \( \Phi \in C^{p+1+\alpha}(\mathbb{T}^3) \) are smooth functions such that the phase function \( \Phi \) obeys

\[
C^{-1} \leq |\nabla \Phi| \leq C
\]

(234)

on \( \mathbb{T}^3 \), for some constant \( C \geq 1 \). Then

\[
\|R(a(x)e^{i\lambda \cdot \Phi(x)})\|_{C^{1,1}_{\ell}} \lesssim \frac{|a|_{C^{1,1}_{\ell}} + |a|_{C^{1,1}_{\ell}}|\nabla \Phi|_{C^{p+\alpha}_{\ell}}}{\lambda^{1-\alpha}}.
\]

(235)

**Lemma 6.3.** ([50, The. 1.4], [22, The. B.1]) Let \( \gamma, \epsilon > 0 \) and \( \beta \geq 0 \) such that \( 2\gamma + \beta + \epsilon \leq 1 \), and let \( f(t) : \mathbb{T}^3 \mapsto \mathbb{R} \). If \( f \in C^{2\gamma+\beta+\epsilon}_{\ell} \), then \((-\Delta)^\gamma f \in C^\beta_{\ell} \), and there exists a constant \( C = C(\epsilon) > 0 \) such that

\[
\|(-\Delta)^\gamma f\|_{C^\beta_{\ell}} \leq C(\epsilon)\|f\|_{C^{2\gamma+\beta+\epsilon}_{\ell}}.
\]

(236)

6.2. Proof of Theorem 2.2 The proof is similar to those of previous works [28, 56]. In short, we can fix \( T > 0 \) arbitrarily, any \( k \in (0, 1) \) and \( K > 1 \) such that \( kk^2 \geq 1 \), rely on Theorem 2.1 and Proposition 4.6 to deduce the existence of \( L > 1 \) and a measure \( P \otimes_{T} R \) that is a martingale solution to (4) on \( [0, \infty) \) starting from a deterministic initial condition.
\[ \xi \text{ of Theorem 2.1 which coincides with } P = \mathcal{L}(u) \text{ over a random interval } [0, \tau_L] \text{ and satisfies} \]
\[ P \otimes_{\tau_L} R((\tau_L \geq T)) \overset{(156)-(159)}{=} \int_{\Omega_0} Q_{\omega}((\tau_L(\omega) \geq T)) P(d\omega) \overset{(159)}{=} P((\tau_L \geq T)) \overset{3}{=} \kappa. \tag{237} \]

It follows that
\[ \mathbb{E}^{\mathbb{P}^\omega, R}[[|\xi(T)|^2]^{2/3}] \overset{(33)(40)}{=} \kappa [K|\xi_{\text{in}}|^2] \geq \kappa K^2 [\langle |\xi_{\text{in}}|^2 \rangle + T \text{Tr}(GG^*)]. \tag{238} \]

On the other hand, via a standard Galerkin approximation scheme (e.g., [25, 26]), one can readily construct a probabilistically weak solution \( \Theta \) which starts from the same initial condition \( \xi_{\text{in}} \) and satisfies
\[ \mathbb{E}^{\Theta}[|\xi(T)|^2] \leq |\xi_{\text{in}}|^2 + T \text{Tr}(GG^*). \]

Because \( \kappa K^2 \geq 1 \), this implies \( P \otimes_{\tau_L} R \neq \Theta \) and hence (2) fails the uniqueness in law.

6.3. Proof of Theorem 2.4

The proof is similar to that of Theorem 2.2, we sketch it for completeness. We fix \( T > 0 \) arbitrarily, any \( \kappa \in (0, 1) \), and \( K > 1 \) such that \( \kappa K^2 \geq 1 \). The probability measure \( P \otimes_{\tau_L} R \) from Proposition 5.5 satisfies \( P \otimes_{\tau_L} R((\tau_L \geq T)) > \kappa \) due to (156)-(159) and (3), which, together with (6), implies
\[ \mathbb{E}^{P^\omega, R}[[|\xi(T)|^2]^{2/3}] \geq \kappa K^2 e^T \| \xi_{\text{in}} \|^2. \tag{239} \]

where \( \xi_{\text{in}} \) is the deterministic initial condition constructed through Theorem 2.3. On the other hand, via a standard Galerkin approximation scheme (e.g., [25, 26]), one can readily construct a probabilistically weak solution \( \Theta \) to (2) starting also from \( \xi_{\text{in}} \) such that
\[ \mathbb{E}^{\Theta}[|\xi(T)|^2] \leq e^T \| \xi_{\text{in}} \|^2. \]

This implies the lack of joint uniqueness in law for (2) and consequently the non-uniqueness in law for (2) by [28, The. C.1], which is an infinite-dimensional version of [8, The. 3.1].

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