TORUS ACTIONS ON STABLE MODULE CATEGORIES, PICARD GROUPS, AND LOCALIZING SUBCATEGORIES

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Abstract. Given an abelian $p$-group $G$ of rank $n$, we construct an action of the torus $T^n$ on the stable module $\infty$-category of $G$-representations over a field of characteristic $p$. The homotopy fixed points are given by the $\infty$-category of module spectra over the Tate construction of the torus. The relationship thus obtained arises from a Galois extension in the sense of Rognes, with Galois group given by the torus. As one application, we give a homotopy-theoretic proof of Dade’s classification of endotrivial modules for abelian $p$-groups. As another application, we give a slight variant of a key step in the Benson-Iyengar-Krause proof of the classification of localizing subcategories of the stable module category.

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1. Introduction

1.1. Stable module categories. Let $G$ be a finite group and let $k$ be a field of characteristic $p$. Our object of study is the stable module category of $G$, denoted $\text{StMod}_G$. The objects of $\text{StMod}_G$ are the $k[G]$-modules, i.e., $G$-representations over $k$. Given two objects $M, N \in \text{StMod}_G$, we have

$$\text{Hom}_{\text{StMod}_G}(M, N) \overset{\text{def}}{=} \text{Hom}_{k[G]}(M, N)/\sim,$$

where we take the quotient by the subspace of all $k[G]$-module maps $M \to N$ which factor through a projective $k[G]$-module. The stable module category $\text{StMod}_G$ has a symmetric monoidal structure given by the $k$-linear tensor product of representations.

Definition 1.1. An object $M \in \text{StMod}_G$ with $\dim_k M < \infty$ is called endotrivial if its endomorphism ring $M \otimes_k M^\vee$ is equivalent in $\text{StMod}_G$ to the unit $k$; equivalently, if $M$ belongs to the Picard group of $\text{StMod}_G$.

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The unit $k$ is obviously endotrivial. Additional examples of endotrivial modules arise from the fact that $\text{StMod}_G$ is actually a triangulated category, where the suspension of an object $M \in \text{StMod}_G$ is obtained by choosing an embedding $M \rightarrow M'$ for $M'$ free and then taking the quotient $M'/M$. The triangulated and tensor structures are appropriately compatible, so the suspensions $\Sigma^i k$ for $i \in \mathbb{Z}$ are also endotrivial.

**Theorem 1.2** (Dade [Dad78, Th. 10.1]). Suppose $G$ is an abelian $p$-group. Then every endotrivial module is equivalent in $\text{StMod}_G$ to $\Sigma^i k$ for some $i \in \mathbb{Z}$.

The Picard group of $\text{StMod}_G$ for a general finite $p$-group $G$ is an important object of study in modular representation theory, and has been completely calculated in the work of Carlson–Thevenaz [CT00, CT05, CT04]. One of our goals in this paper is to give a homotopy-theoretic proof of Dade’s theorem, using descent theory. We note that related methods of tensor-triangulated categories have been used to study Picard groups of stable module categories in [Bal10] and in particular to obtain information about Picard groups after inverting the characteristic.

### 1.2. Structured ring spectra

We now give an overview of our proof of Dade’s theorem. Suppose we were working not in the stable module category but the ordinary representation category. Then we could make the following argument to show that the Picard group is trivial: every element of the abelian $p$-group $G$ induces multiplication by a scalar in $k^\times$ on our representation. Since $k^\times$ is $p$-torsion free, every element of $G$ acts trivially and we are done.

One can start this argument in the stable module category. However, it is now only true that every element of the group acts as the identity up to homotopy (i.e., modulo projectives) on an invertible object. This is insufficient to conclude that the object is trivial. Nonetheless, we will still be able to carry out a more sophisticated version of this argument, for which the language of $\infty$-categories and higher algebra [Lur09, Lur14] will be essential. In particular, we will henceforth work with the $\infty$-categorical enhancement of $\text{StMod}_G$ (which by abuse of notation we will denote by the same symbol).

Our starting point is that $\text{StMod}_G$ is equivalent to the $\infty$-category of module spectra over the Tate construction $k^{tG}$, an $E_\infty$-ring spectrum obtained as a suitable localization of the cochain algebra $k^{kG} \simeq F(BG_+, k)$. In particular, the group of stable equivalence classes of endotrivial modules is isomorphic to the Picard group of the $E_\infty$-ring spectrum $k^{tG}$. Dade’s theorem states that the Picard group of $k^{tG}$ is generated by the suspensions of the unit if $G$ is abelian.

Picard groups of structured ring spectra form a recurring topic in stable homotopy theory, starting with the observation of Hopkins that the Picard groups of the $L_n$ and $K(n)$-localized stable homotopy categories contain significantly more than suspensions (or algebraically flat objects). Given an $E_\infty$-ring spectrum $R$, a general theme is that the Picard group will be easier to understand if the homotopy groups of $R$ are homologically simple. For example, if $\pi_*(R)$ is regular and concentrated in even degrees, the Picard group of $R$ can be determined purely algebraically, cf. [BR05], [HM15, Th. 6.4].

A general approach [GL] to the computation of Picard groups of $E_\infty$-ring spectra is to decompose the $\infty$-category of $R$-modules into $\infty$-categories of modules over $E_\infty$-rings with better-behaved homotopy groups, and apply a descent spectral sequence. We refer to [MS, HMS15] for examples of this approach.

### 1.3. Our approach

If $G$ is abelian, then the $E_\infty$-ring $k^{tG}$ has fairly complicated homotopy groups, homologically. For instance, if $p$ is odd and $G$ has rank one, then
\[
\pi_* k^{tG} \simeq E(\alpha_{-1}) \otimes_k k[\beta_{-2}^{\pm 1}],
\]
i.e., the tensor product of an exterior algebra and a Laurent polynomial algebra, and this has infinite homological dimension. When $G$ has higher rank, the positive homotopy groups of $\pi_*(k^{tG})$ are entirely square-zero.

A direct approach using descent theory to the Picard group of $k^{tG}$ is problematic because of the presence of these exterior classes; instead, we will use a sort of reverse descent, following ideas that we used in [Mat14, §9] to study the Galois group. The main step is the following result.
**Theorem 1.3.** If $G$ is an abelian $p$-group of rank $n$, then one can construct an action of the $n$-torus $\mathbb{T}^n$ on the symmetric monoidal $\infty$-category $\text{StMod}_G$ such that the homotopy fixed points $\text{StMod}^{\mathbb{T}^n}_G$ are given by the analog of the stable module $\infty$-category for the torus.

The above arises from a faithful $\mathbb{T}^n$-Galois extension in the sense of Rognes [Rog08] that runs $k^{\mathbb{T}^n} \to k^G$. The Picard group of the $E_\infty$-ring $k^{\mathbb{T}^n}$ can be calculated directly using descent theory. In particular, we can prove the analog of Dade’s theorem for $\text{StMod}^{\mathbb{T}^n}_G$ relatively easily. We will show using an obstruction-theoretic calculation (Theorem 5.3) that any invertible object in $\text{StMod}_G$ can be “descended” to the homotopy fixed points $\text{StMod}^{\mathbb{T}^n}_G \simeq \text{Mod}(k^{\mathbb{T}^n})$. Since the Picard group of the latter is cyclic, this will complete the proof. This relies on techniques with descent spectral sequences following [MS].

As a result, we will also obtain a partial local version of Dade’s theorem. Choose an identification $\text{Proj}(H^{even}(G;k))_\text{red} \simeq \mathbb{P}^{n-1}_k$. Let $U \subset \mathbb{P}^{n-1}_k$ be a Zariski open subset. We say that a morphism $M \to N$ in $\text{StMod}_G$ is an $U$-equivalence if its cofiber has cohomology supported on the complement of $U$. We can then form an associated Bousfield localization $(\text{StMod}_G)_U$ where we invert all $U$-equivalences. Then we will prove:

**Theorem 1.4.** Suppose $U \subset \mathbb{P}^{n-1}_k$ is affine and $p > 2$. Let $M \in \text{StMod}_G$ be a compact object (i.e., one represented by a finite-dimensional $\kappa[G]$-representation). Suppose that the $U$-localization of $M$ is invertible in $(\text{StMod}_G)_U$. Then the $U$-localization of $M$ is equivalent to the $U$-localization of $\Sigma^i k$ for some $i$.

### 1.4. Thick and localizing subcategories

The idea of comparing the $E_\infty$-ring $k^G$ (or $k^hG$) with an $E_\infty$-ring whose homotopy groups do not have the exterior algebra classes is not new: it is used prominently in the stratification [BIK11] of localizing subcategories of the stable module category of an $p$-group. The main result runs as follows:

**Theorem 1.5** (Benson-Iyengar-Krause [BIK11]). Let $G$ be a finite $p$-group. The localizing subcategories of $\text{Mod}(k^G)$ are in bijection with the subsets of the set of homogeneous prime ideals in $\pi_*(k^G)$. The localizing subcategories of the stable module category $\text{StMod}_G \simeq \text{Mod}(k^G)$ are in bijection with subsets that do not contain the irrelevant ideal.

Earlier work of Benson-Carlson-Rickard [BCR97] classifies the thick subcategories of the compact objects of $\text{StMod}_G$. The above result has been extended in the work of Stevenson [Ste14], for example for the singularity categories of complete intersection local rings.

Suppose $G$ is in fact elementary abelian, $G \simeq C^n_p$. Then Benson-Iyengar-Krause prove Theorem 1.5 by replacing $k^{hC^n_p}$ with a different $E_\infty$-ring $R$, given by the (derived) ring of functions on the classifying stack of the $G$-group scheme $\alpha^+_p$. The underlying $E_1$-rings of $R$ and $k^{hC^n_p}$ are equivalent, so it suffices to classify localizing subcategories of $\text{Mod}(R)$. Now the $E_\infty$-ring $R$ receives a map $R' \to R$, where $\pi_*(R')$ has only polynomial classes (cf. [BIK11, §7]). It is shown (cf. [BIK11, Th. 4.4]) how to use a classification result for localizing subcategories of $\text{Mod}(R')$ to obtain one for $\text{Mod}(R)$. The classification of localizing subcategories of $\text{Mod}(R')$ can be proved using a technique with “residue fields” which goes back to [HPS97], as $\pi_*(R')$ is regular.

In this paper, we study a $\mathbb{T}^n$-Galois extension of $E_\infty$-rings $k^{h\mathbb{T}^n} \to k^{hC^n_p}$. This gives an $E_\infty$-approximation to $k^{hC^n_p}$ that only sees the polynomial classes. On $E_1$-rings, this extension is equivalent to the one of [BIK11], but it is constructed using topology rather than graded Hopf algebras.

We will prove a general result that for a faithful $\mathbb{T}^n$-Galois extension of $E_\infty$-rings $R_1 \to R_2$, the localizing subcategories of $\text{Mod}(R_1)$ and $\text{Mod}(R_2)$ are in canonical bijective. As a result, we can reduce the classification of localizing subcategories of $\text{Mod}(k^{hC^n_p})$ to the analog in $\text{Mod}(k^{h\mathbb{T}^n})$, where one can use the method of residue fields. This gives a slightly different approach to some of the technical steps in [BIK11], which does not require modifying the $E_\infty$-structure.
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2. **Stable module categories and Tate spectra**

2.1. **Construction of the stable module ∞-category.** We start by reviewing the construction of the stable module ∞-category of a finite group. Although these results are not new, we have spelled out some of the details for the convenience of the reader.

**Construction 2.1.** Let $G$ be a finite group and let $k$ be a field of characteristic $p > 0$. Consider the group ring $k[G]$ and the symmetric monoidal category $C$ of (discrete) $k[G]$-modules, equipped with the $k$-linear tensor product.

The category $C$ admits a combinatorial model structure [Hov99, §2.2] where:

1. The fibrations are the surjections.
2. The cofibrations are the injections.
3. The weak equivalences are the stable equivalences. Given a morphism $f : V_1 \to V_2$ in $C$, one says that it is a stable equivalence if there exists a morphism $g : V_2 \to V_1$ such that the endomorphisms $g \circ f - id_{V_1}$ and $f \circ g - id_{V_2}$, of $V_1, V_2$ respectively, each factor through a projective $k[G]$-module.

One checks in addition that $C$ is a symmetric monoidal model category. Note that every object in the model category $C$ is cofibrant-fibrant. It is shown in [Hov99, §2.2] that the morphisms in homotopy category $Ho(C)$ are given by the classical stable module category, i.e.,

$$\text{Hom}_{Ho(C)}(V_1, V_2) \simeq \text{Hom}_C(V_1, V_2) / \sim,$$

where we identify two morphisms if their difference factors through a projective.

**Definition 2.2.** The stable module ∞-category $\text{StMod}_G$ is the ∞-categorical localization $C[W^{-1}]$ for $W \subset C$ the weak equivalences [Lur14, Def. 1.3.4.15]. This inherits the structure of a symmetric monoidal ∞-category [Lur14, Prop. 4.1.3.4].

The stable module ∞-category also has another description as a Bousfield localization which we review. We use [MNN15] as a reference for some of these ideas in the context of equivariant stable homotopy theory, but they are of course much older (e.g., [HPS97]).

**Construction 2.3.** Consider the symmetric monoidal ∞-category $\text{Fun}(BG, Perf(k))$ of perfect $k$-module spectra equipped with a $G$-action. We form the Ind-completion $\text{Ind}(\text{Fun}(BG, Perf(k)))$ and, inside here, the $A^{-1}$-localization (cf. [MNN15, §3]) for $A = F(G_+, k) \in CAlg(\text{Fun}(BG, Perf(k)))$, which we denote $L_{A^{-1}} \text{Ind}(\text{Fun}(BG, Perf(k)))$.

These constructions have been extensively studied in the literature for any noetherian ring, not only group rings, starting with the work of Krause [Kra05]. We refer to the work of Benson-Krause [BK08] for a construction of $\text{Ind}(\text{Fun}(BG, Perf(k)))$ via a model category of complexes of injectives. Finally, these ∞-categories have been studied by Gaitsgory in the more general setting of DG-algebras and DG-schemes [Gai13].

**Theorem 2.4.** There is an equivalence of symmetric monoidal ∞-categories between $\text{StMod}_G$ and $L_{A^{-1}} \text{Ind}(\text{Fun}(BG, Perf(k)))$.

**Proof.** The main point is the construction of the functor in this language. Let $C$ be, as before, the model category of (discrete) $k[G]$-modules. Note that $C = \text{Ind}(C')$ for $C'$ the category of finitely generated (discrete) $k[G]$-modules.
Here Perf($k$) denotes the ∞-category of perfect $k$-module spectra. Let Perf^{\circ}(k) ⊂ Perf($k$) be the full subcategory spanned by the discrete, finite-dimensional $k$-vector spaces. We have a functor
\[ C' \to \text{Fun}(BG, \text{Perf}^{\circ}(k)) \subset \text{Fun}(BG, \text{Perf}(k)) \]
which extends by Ind-completion to a symmetric monoidal functor
\[ \Phi: C \to \text{Ind}(\text{Fun}(BG, \text{Perf}(k))). \]
Here $\Phi$ commutes with filtered colimits. The composite
\[ C \xrightarrow{\Phi} \text{Ind}(\text{Fun}(BG, \text{Perf}(k))) \rightarrow L_{A^{-1}}\text{Ind}(\text{Fun}(BG, \text{Perf}(k))) \]
sends projectives to zero, so it respects weak equivalences and factors through a symmetric monoidal functor
\[ \varphi: \text{StMod}_G \simeq C[W^{-1}] \rightarrow L_{A^{-1}}\text{Ind}(\text{Fun}(BG, \text{Perf}(k))). \]
This is the desired functor.

We claim that the above functor $\varphi$ commutes with homotopy colimits. We first show that $\varphi$ is exact. Suppose given a cofiber sequence $M' \to M \to M''$ in StMod$_G$. Using the model structure on $C$, we can represent this cofiber sequence by a short exact sequence in $C$, e.g.,
\[ 0 \to M' \to M \to M'' \to 0. \]
The short exact sequence is a filtered colimit in $C$ of short exact sequences
\[ 0 \to M'_\alpha \to M_\alpha \to M''_\alpha \to 0, \]
which belong to $C'$, i.e., $M_\alpha$ is finitely generated. Observe that $\Phi$ sends each of the short exact sequences (2.3) to a cofiber sequence in Fun($BG$, Perf($k$)). Therefore, $\Phi$ (and therefore $\varphi$) sends (2.2) to a cofiber sequence too. Next, we observe that $\varphi$ respects arbitrary direct sums (since $\Phi$ does). It follows that $\varphi$ is cocontinuous.\[ \text{Lur14, Prop. 1.4.4.1}. \]

To check that $\varphi$ is an equivalence, we check that it is fully faithful on the homotopy category. Since Fun($BG$, Perf($k$)) is generated as a thick subcategory by Fun($BG$, Perf$^{\circ}(k)$), essential surjectivity will then be automatic.

Note that StMod$_G$ is stable and the objects represented by finitely generated $k[G]$-modules are easily seen to be compact (note that compactness can be checked at the level of the homotopy category, cf. [Lur14, Prop. 1.4.4.1]). Using duality, one reduces to showing that if $M \in C'$, then the natural map
\[ \text{Hom}_{\text{Ho}(C)}(1, M) \xrightarrow{\varphi} \pi_0\text{Hom}_{L_{A^{-1}}\text{Ind}(\text{Fun}(BG, \text{Perf}(k)))}(1, M). \]
is an isomorphism. However, both are known to be the Tate cohomology $\hat{H}^0(G; M)$:

1. In $\text{Ho}(C)$, this is now an easy calculation using the description of the homotopy category (2.1).
2. We need to show that in $L_{A^{-1}}\text{Ind}(\text{Fun}(BG, \text{Perf}(k)))$, maps from the unit to the $A^{-1}$-localization of $M$ are as claimed. This follows from the explicit description of $A^{-1}$-localization, cf. [MNN15, §3] and the following subsection.

We can also spell out the details directly. The $A^{-1}$-localization of $M$ in Ind(Fun($BG$, Perf($k$))) can be computed as the cofiber of a map
\[ |M \otimes (\mathbb{D}A)^{\otimes +1}| \to M, \]
(cf. [MNN15, §3]) for $\mathbb{D}$ denoting duality. Since $\text{Hom}_{\text{Ind}(\text{Fun}(BG, \text{Perf}(k)))}(1, \cdot)$ commutes with arbitrary colimits, one now calculates directly that the mapping spectrum
\[ \text{Hom}_{\text{Ind}(\text{Fun}(BG, \text{Perf}(k)))}(1, |M \otimes (\mathbb{D}A)^{\otimes +1}|) \]
is connective with $\pi_0$ given by the coinvariants $M_G$, and that the map (2.5) induces the norm map on $\pi_0$. This implies the desired description of the right-hand-side of (2.4). $\square$
2.2. Connection with Greenlees-May. In the stable module $\infty$-category $\text{StMod}_G$, endomorphisms of the unit are given by the Tate construction. We review the connection between $\text{StMod}_G$ and the Greenlees-May definition of the Tate construction [GM95b].

Fix a finite group $G$.

**Definition 2.5** ([GM95b]). Let $\text{Sp}_G$ denote the symmetric monoidal $\infty$-category of $G$-spectra. Given a $G$-spectrum $X$, the Tate construction is the $G$-spectrum $\tilde{E}G \land F(EG+, X)$. We will write $X^{iG}$ for the $G$-fixed points $(\tilde{E}G \land F(EG+, X))^G$. When the context is clear, we will also call this the Tate construction.

Let $A = F(G_+, S_0^G)$; this is a commutative algebra object in $\text{Sp}_G$. As in [MNN15, §6], we recall that $EG_+$ is the $A$-acylization (cf. [MNN15, §3]) of the unit and $\tilde{E}G$ is the $A^{-1}$-localization. In particular, $F(EG+, X)$ is the $A$-completion of $X$.

**Construction 2.6.** Let $\underline{k} \in \text{CAlg}(\text{Sp}_G)$ denote the Borel-equivariant (or cofree) form of $k$, i.e., the $G$-spectrum representing Borel-equivariant cohomology with coefficients in $k$. The symmetric monoidal $\infty$-category $\text{Perf}_{\text{Sp}_G}(\underline{k})$ of compact $\underline{k}$-modules in $\text{Sp}_G$ is identified with $\text{Fun}(BG, \text{Perf}(k))$.

We review the connection between $\text{StMod}_G$ and $\text{Mod}(k^{iG})$ of the unit and $\tilde{E}G$. As in [MNN15, §6.3]. In particular, it follows by applying $\text{Ind}$-completion that:

**Theorem 2.7.** We have an equivalence of symmetric monoidal $\infty$-categories:

\[(2.6) \quad \text{Mod}_{\text{Sp}_G}(\underline{k}) \simeq \text{Ind}(\text{Fun}(BG, \text{Perf}(k))).\]

Under this, the two objects that we have denoted $A$ correspond. Applying $A^{-1}$-localization, one finds:

**Corollary 2.8.** We have an equivalence of symmetric monoidal $\infty$-categories $\text{StMod}_G \simeq \text{Mod}_{\text{Sp}_G}(\underline{k} \land \tilde{E}G)$. The endomorphisms of the unit are given by the $E_\infty$-algebra $k^{iG} = (\underline{k} \land \tilde{E}G)^G$.

We specialize to the case where $G$ is a $p$-group. We find:

**Theorem 2.9** (cf. Keller [Kel94]). If $G$ is a $p$-group, then there is an equivalence of symmetric monoidal $\infty$-categories between $\text{StMod}_G$ and $\text{Mod}(k^{iG})$ for $k^{iG}$ the Tate construction of $G$.

**Proof.** In this case, $\text{StMod}_G$ is generated as a localizing subcategory by the unit because $\text{Fun}(BG, \text{Perf}(k))$ is generated as a thick subcategory by the permutation modules. This follows from the fact that any nontrivial $G$-representation over $k$ has a nontrivial fixed vector.

By the symmetric monoidal version of the Schwede-Shipley theorem [Lur14, Prop. 7.1.2.7], one obtains a symmetric monoidal equivalence

\[(2.7) \quad \text{Ind}(\text{Fun}(BG, \text{Perf}(k))) \simeq \text{Mod}_{\text{Sp}_G}(\underline{k}) \simeq \text{Mod}(k^{iG}),\]

where $k^{iG} \simeq F(BG+, k)$ is the $E_\infty$-ring of $k$-valued cochains on the classifying space $BG$. The result follows. \hfill $\square$

**Example 2.10.** Suppose $G$ is an abelian $p$-group of rank $n$. In this case, $\pi_*k^{iG}$ contains a polynomial algebra $k[x_1, \ldots, x_n]$ on classes $x_i$, with $|x_i| = -2$.

We will need to know that in this case, the Tate construction $k^{iG}$ can be identified with the localization of $k^{iG}$ away from the ideal $(x_1, \ldots, x_n)$ (cf. [GM95a]). Equivalently, the localizing subcategory of $\text{Mod}(k^{iG}) \simeq \text{Ind}(\text{Fun}(BG, \text{Perf}(k)))$ generated by the iterated cofiber $k^{iG}/(x_1, \ldots, x_n)$ (using the first description of this $\infty$-category) is equal to that generated by $k^{iG}/G_+$ (using the second).

Although this description is classical, we will describe the argument in the case where $G$ is elementary abelian. The $x_i$’s are the Euler classes of complex line bundles for a linearly independent set of
maps $G \to \mu_p \subset S^1$. In more detail, given a complex character $\chi$ of $G$, we form the unit sphere $S(\chi)$, the Euler sequence $S(\chi)_+ \to S^0 \to S^\chi$, and the induced cofiber sequence in $\text{Fun}(BG, \text{Perf}(k))$

$$S(\chi)_+ \wedge k \to k \to \Sigma^2 k.$$ 

In $\text{Mod}(k^{hG})$, we obtain a corresponding cofiber sequence

$$(S(\chi)_+ \wedge k)^{hG} \to k^{hG} \xrightarrow{e(\chi)} \Sigma^2 k^{hG},$$

for $e(\chi) \in \pi_2(k^{hG})$ the Euler class of the character. From this, it follows that the cofiber of $x_i$ is represented in $\text{Fun}(BG, \text{Perf}(k))$ by the smash product of $k$ and a finite $G$-complex with isotropy in a maximal proper subgroup of $G$. As a result, the smash product $k^{hG}/(x_1, \ldots, x_n)$ of the cofibers of each of the $x_i$’s has to belong to the thick subcategory generated by $G_\times$. The converse direction is easier: the thick subcategory generated by any nonzero $X \in \text{Fun}(BG, \text{Perf}(k))$ contains $k \wedge G_+$; one sees this from the projection formula and the fact that $k \wedge G_+$ belongs to the thick subcategory generated by the unit.

Every compact object in the stable module $\infty$-category can be represented by a finitely generated $k[G]$-module. A general result that includes this appears in [Miy07]. Any invertible object in $\text{StMod}_G$ is compact (as the unit is), and therefore one can identify the group of stable equivalence classes of (finite-dimensional) endotrivial modules in the sense of Definition 1.1 and the Picard group of the $E_\infty$-ring $k^{hG}$.

3. Torus actions on $\infty$-categories

3.1. Generalities. We start by reviewing some generalities about group actions.

**Definition 3.1.** Let $\mathcal{V}$ be an $\infty$-category and let $G$ be a topological group. A $G$-action on an object $x \in \mathcal{V}$ will mean a functor of $\infty$-categories $BG \to \mathcal{V}$ which maps the basepoint of $BG$ to $x$. We will call $\text{Fun}(BG, \mathcal{V})$ the $\infty$-category of objects in $\mathcal{V}$ equipped with a $G$-action. Recall that if $\mathcal{V}$ admits limits, the functor

$$\mathcal{V} \to \text{Fun}(BG, \mathcal{V}),$$

which gives an object of $\mathcal{V}$ the trivial $G$-action, admits a right adjoint of homotopy fixed points which we will denote $(\cdot)^{hG}$.

For example, we can consider a $G$-action on an $\infty$-category by taking $\mathcal{V}$ to be the $\infty$-category $\text{Cat}_\infty$ of $\infty$-categories.

**Example 3.2.** Let $\mathcal{C}$ be a 1-category. To give an $S^1$-action on $\mathcal{C}$ is equivalent to giving an automorphism of the identity functor of $\mathcal{C}$. This follows from considering $S^1 = BZ$ as a monoidal category with one object whose endomorphisms are $Z$: to give an $S^1$-action on $\mathcal{C}$ is to give a multiplication map $BZ \times \mathcal{C} \to \mathcal{C}$ satisfying natural identities.

Given an $S^1$-action on $\mathcal{C}$, the homotopy fixed points $\mathcal{C}^{hS^1}$ are the full subcategory of $\mathcal{C}$ spanned by those objects on which this automorphism is the identity. We note that actions of the *monoid* $Z_{\geq 0}$ were considered in [Dri04] under the name $Z_{\ast}$-category.

**Example 3.3.** Given a finite group $G$ and a central element $g \in G$, we obtain an $S^1$-action on the category of (discrete) $k[G]$-modules given by multiplying by $g$.

**Example 3.4.** Let $R$ be a smooth commutative algebra over a field of characteristic zero and let $r \in R$ be a nonzerodivisor. Consider the $\infty$-category $\text{Perf}(R)$ of all perfect $R$-module spectra. The subcategory $\text{Perf}(R)_{r\text{-tors}} \subset \text{Mod}(R)$ of all $r$-power torsion perfect $R$-module spectra has an $S^1$-action, which gives an $r$-power torsion perfect $R$-module spectrum $M$ the automorphism of multiplication by $1 + r$. In this case, we have an equivalence of $\infty$-categories

$$(\text{Perf}(R)_{r\text{-tors}})^{hS^1} \simeq \text{Coh}(R/r),$$

where $\text{Coh}(R/r) \subset \text{Mod}(R/r)$ is the subcategory of those $R/r$-module spectra that are perfect as $R$-modules, by a theorem of Teleman. We refer to [Pre12, §3] for a detailed treatment.
3.2. The action on the stable \(\infty\)-category. We now construct the analog of this action on the stable module \(\infty\)-category of an abelian \(p\)-group. It is possible to use Example 3.3 together with \(\infty\)-categorical localization, but it is convenient for our purposes to take a different approach.

**Construction 3.5.** Let \(A\) be an abelian \(p\)-group of rank \(n\), i.e., \(A\) can be minimally generated by \(n\) elements. Then we can find a free abelian group \(L\) of rank \(n\) and a surjection \(L \to A\), with kernel \(L' \subset L\). Since \(A\) and \(L\) are abelian, we can model \(BL\) and \(BA\) by topological abelian groups, so that \(BL \to BA\) is a morphism of topological abelian groups. We thus obtain an action of \(BL\) on \(BA\) via this homomorphism and the \(BA\)-action on itself by left translation.

We have a fiber sequence of spaces

\[
BL \to BA \to B^2L',
\]

exhibiting \(B^2L'\) as the homotopy orbits \((BA)_hBL\). Note that \(BL\) is a torus \(\mathbb{T}^n\) and \(B^2L'\) is \((\mathbb{C}P^\infty)^n\) (noncanonically). Taking cochains with \(k\)-valued coefficients, we obtain a morphism of \(E_\infty\)-rings

\[
F(B^2L'_+, k) \simeq k^{hBL'} \to k^{hA} \simeq F(BA_+, k),
\]

and \(BL\) acts on \(k^{hA}\) in the \(\infty\)-category of \(E_\infty\)-\(k^{hBL'}\)-algebras such that the natural map \(k^{hBL'} \to (k^{hA})^{hBL}\) is an equivalence.

This action has a Galois property, which enables one to determine the \(\infty\)-category of \(BL\)-homotopy fixed points. We discuss this below.

3.3. Rognes’s Galois theory and descent. Let \(G\) be a topological group with the homotopy type of a finite complex. We will need the notion of a faithful \(G\)-Galois extension of \(E_\infty\)-rings.

**Definition 3.6** (Rognes [Rog08, Def. 4.1.3]). Consider an extension \(R \to R'\) of \(E_\infty\)-rings and suppose \(G\) acts on \(R'\) in \(E_\infty\)-\(R\)-algebras. The extension is said to be a **faithful \(G\)-Galois extension** if:

1. \(R \to (R')^hG\) is an equivalence.
2. The natural map \(R' \otimes_R R' \to F(G_+, R')\) is an equivalence.
3. Given an \(R\)-module \(M\) such that \(R' \otimes_R M\) is contractible, \(M\) is itself contractible.

**Proposition 3.7.** Suppose \(R \to R'\) is a morphism of \(E_\infty\)-rings and suppose \(R'\) is given a \(G\)-action in \(E_\infty\)-\(R\)-algebras. Then \(R \to R'\) is a faithful \(G\)-Galois extension if and only if there exists an \(E_\infty\)-\(R\)-algebra \(\tilde{R}\) such that:

1. The \(E_\infty\)-\(\tilde{R}\)-algebra \(R' \otimes_R \tilde{R}\) is \(G\)-equivariantly isomorphic to \(F(G_+, \tilde{R})\) (where the latter has a \(G\)-action by translation)
2. \(\tilde{R}\) is descendable (cf. [Mat14, §3-4], Definition 6.2 below) as an \(E_\infty\)-\(R\)-algebra.

In particular, \(G\)-Galois extensions are stable under base-change.

**Proof.** Suppose there exists \(\tilde{R}\) as above, so that we need to show that \(R \to R'\) is a faithful \(G\)-Galois extension. We first need to show that \(R \to (R')^hG\) is an equivalence. We claim more strongly that if \(\{X_n\}\) is the standard tower of \(R\)-module spectra converging to \((R')^hG\) (based on the simplicial filtration of \(BG\)) then the cofiber of the natural map of towers \(\{R\} \to \{X_n\}\) is nilpotent (cf. [Mat15, Sec. 3.1]). To check this, we may make a base-change along the descendable morphism \(R \to \tilde{R}\). In fact, descent along \(R \to \tilde{R}\) allows to reduce to the case \(R' \simeq F(G_+, R)\) to begin with, and in this case the cosimplicial object computing \((R')^hG\) is split augmented over \(R\).

Next, the natural map \(\tilde{R} \otimes_R \tilde{R} \to F(G_+, \tilde{R})\) is an equivalence as that can be checked after base-change along \(R \to R'\). The final condition (of faithfulness) in Definition 3.6 can similarly be checked by descent.

Conversely, if \(R \to R'\) is a faithful \(G\)-Galois extension, then \(R'\) is a dualizable \(R\)-module [Rog08, Prop. 6.2.1] and the third condition of faithfulness then implies that \(R'\) is itself descendable [Mat14, Th. 3.36] so that the conditions above are satisfied with \(\tilde{R} = R'\). \(\square\)
We now state the Galois descent theorem. For $G$ finite, this also appears in [Mei12, GL].

**Proposition 3.8** (cf. [Mat14, §9]). Given a faithful $G$-Galois extension $R \to R'$, one obtains a $G$-action on the symmetric monoidal $\infty$-category $\text{Mod}(R')$ of $R'$-modules, and the natural functor

$\text{Mod}(R) \to \text{Mod}(R')^{hG}$

is an equivalence of symmetric monoidal $\infty$-categories.

We return to our setup.

**Proposition 3.9** (cf. [Mat14, §9]). Notation as in 3.5, the natural map $k^{hBL'} \to k^{hA}$ exhibits the target as a faithful $BL$-Galois extension of the source.

**Proof.** In fact, one checks by an explicit calculation on homotopy groups that the map $k^{hBL'} \to k^{hA}$ makes the target into a free module (with shifts) over the source, e.g., $F(BC_{p+}, k) = k^{hC_p}$ is free over $F(BS^1, k) \simeq k^{hS^1}$. In particular, it is descendable. Moreover, we have a fiber square of spaces

$$
\begin{array}{ccc}
BL \times BA & \longrightarrow & BA \\
\downarrow & & \downarrow \\
BA & \longrightarrow & B^2L'
\end{array}
$$

i.e., $BA \to B^2L'$ is a $BL$-torsor. By the convergence of the Eilenberg-Moore spectral sequence (cf. [Lur11, §1.1]), we have an equivalence of $E_\infty^{-k^{hA}}$-algebras,

$F(BL_+, k^{hA}) \simeq F((BL \times BA)_+, k) \simeq k^{hA} \otimes_{k^{hBL'}} k^{hA},$

which is equivariant for the $BL$-action, by translation on the left-hand-side and on the second factor on the right-hand-side. This implies the claim. \qed

As a result of Proposition 3.8 and Proposition 3.9, the canonical functor

$\text{Mod}(F(\mathbb{C}P^\infty_+^+, k)) \simeq \text{Mod}(k^{hBL'}) \to \text{Mod}(k^{hA})^{hBL}$

is an equivalence of symmetric monoidal $\infty$-categories. Moreover, this holds after arbitrary base-change along the source.

We will now extend this to the stable module $\infty$-category.

**Definition 3.10.** Let $T^n$ be the standard $n$-torus. Recall that $\pi_* k^{hT^n} \simeq k[x_1, \ldots, x_n]$ where, for each $i$, $|x_i| = -2$. We will let $k^{T^n}$ denote the localization of $k^{hT^n} = F(BT^n_+, k)$ away from the iterated cofiber $F(BT^n_+, k)/(x_1, \ldots, x_n)$ (cf. [GM95a]), or equivalently the $A^{-1}$-localization in $\text{Mod}(k^{hT^n})$ for $A = k^{hT^n}/(x_1, \ldots, x_n) \in \text{Alg}(\text{Mod}(k^{hT^n}))$ (cf. [MNN15, §3]). If $M$ is a finitely generated free $\mathbb{Z}$-module, we will write $k^{iBM}$ for the above construction where we identify $BM$ with an $n$-torus (in any manner).

We have a pushout square of $E_\infty$-rings

$$
\begin{array}{ccc}
k^{hBL'} & \longrightarrow & k^{iBL'} \\
\downarrow & & \downarrow \\
k^{hA} & \longrightarrow & k^{iA}
\end{array}
$$

because both the Tate constructions are obtained by the equivalent finite localization (cf. Example 2.10).

**Construction 3.11.** As a localization of $k^{hA}$, the Tate construction $k^{iA}$ inherits a $BL$-action, and $k^{iBL'} \to k^{iA}$ is a faithful $BL$-Galois extension. The action of $BL$ on the $E_\infty$-ring $k^{iA}$ induces a $BL$-action on the symmetric monoidal $\infty$-category $\text{StMod}_{k^{iA}} \simeq \text{Mod}(k^{iA})$. Note also that the $BL$-action extends to a $BA$-action (which originates with the $BA$-action on itself by left translation).
We summarize the above analysis in the following.

**Theorem 3.12** (cf. [Mat14, §9]). Let $A$ be an abelian $p$-group of rank $n$, and choose a short exact sequence $0 \to L' \to L \to A \to 0$ where $L \simeq \mathbb{Z}^n$. Then one obtains an action of the torus $BL \simeq \mathbb{T}^n$ on the symmetric monoidal, $k$-linear $\infty$-category $\mathrm{StMod}_A$ such that:

1. The homotopy fixed points $\mathrm{StMod}_A^{kBL}$ are canonically identified with the symmetric monoidal $\infty$-category $\mathrm{Mod}(k^{BL'})$.
2. The action of $BL$ extends to an action of $BA$ on $\mathrm{StMod}_A$.

4. Torus actions on spectra

In this section, we will describe some tools for working with torus actions on spectra. We note that most of the subtleties encountered here disappear after 2 is inverted.

4.1. $E_\infty$-rings and $gl_1$. Let $R$ be an $E_\infty$-ring. Recall the spectrum of units $gl_1(R)$ [May77, ABG+14]. We have a natural equivalence of connected spaces

\[(\Omega^\infty R)_{\geq 1} \simeq (\Omega^\infty gl_1(R))_{\geq 1}\]

and in particular natural isomorphisms of homotopy groups

\[\psi: \pi_i(R) \simeq \pi_i(gl_1(R))\]

for $i > 0$. However, these isomorphisms are generally not compatible with the $\pi_* (S^0)$-module structure.

**Proposition 4.1.** Let $n \in \{1, 3, 7\}$, so that there exists an element $\delta_n \in \pi_n (S^0)$ of Hopf invariant one. Then if $R$ is any $E_\infty$-ring and if $x \in \pi_n (R)$, we have

\[\delta_n \psi (x) = \psi (\delta_n x + x^2) \in \pi_{2n} (gl_1(R)).\]

**Proof.** By naturality, it suffices to prove this in the case where $R$ is the free $E_\infty$-ring on $S^n$, so that

\[R \simeq S^0 \lor S^n \lor (S^n \lor S^n)_{h \Sigma^2} \lor \ldots,\]

and $x \in \pi_n (R)$ is the tautological element. In this case, $\pi_{2n} (R) \simeq \pi_{2n} (S^0) \oplus \pi_{2n} (S^n) \oplus \mathbb{Z} / 2$. It follows that we have a “universal” formula

\[\delta_n \psi (x) = \psi (a_1 x + a_2 + a_3 x^2),\]

where $a_1 \in \pi_n (S^0), a_2 \in \pi_{2n} (S^0), a_3 \in \mathbb{Z} / 2$. It remains to determine the values of these coefficients by considering several specific examples.

First, we claim that $a_2 = 0$. This follows because we have a map of $E_\infty$-rings $R \to S^0$ that annihilates $x$, so that $\delta_n \psi (x)$ must belong to the kernel of the induced map. Second, we claim that $a_1 = \delta_n$; this follows by considering the case of the square-zero $E_\infty$-ring $S^0 \lor S^n$, so that we have a canonical equivalence of spectra $gl_1 (S^0 \lor S^n) \simeq gl_1 (S^0) \lor S^n$ by [MS, Prop. 6.5.3].

Finally, we claim that $a_3 \neq 0$. For this, we consider the $E_\infty$-ring $R = \tau_{\leq 2n} F_2 \{x_n\}$ obtained by truncating the free $E_\infty$-algebra over $F_2$ on a degree $n$ class to degrees $[0, 2n]$, so that $\pi_* (R) \simeq F_2 [x_n] / (x_n^2)$. It follows [MS, Prop. 5.2.2] that the $k$-invariant of $gl_1 (R)$ is the map $H F_2 [n] \to H F_2 [2n + 1]$, given by $S^{n + 1}$. However, this implies that the classes in degree $n$ and $2n$ of $\pi_* gl_1 (R)$ are connected by a $\delta_n$, as desired. \(\square\)
4.2. The group algebra. Let $X$ be a spectrum with an $S^1$-action. Equivalently, $X$ is a module over the topological group ring $\Sigma_+^\infty S^1$. Since $S^1$ is a commutative topological group, the group ring $\Sigma_+^\infty S^1$ has the structure of an $E_\infty$-ring, and we obtain an action of $\pi_*(\Sigma_+^\infty S^1)$ on $\pi_*(X)$.

We will need the following description of the former, which is well-known to experts.

**Proposition 4.2.** The homotopy groups of the ring spectrum $\Sigma_+^\infty S^1$ are given by the algebra $\pi_*(S^0)[y]/(y^2+\eta y)$ where $|y|=1$. More generally, we have an equivalence of algebras

$$\pi_*(\Sigma_+^\infty \mathbb{T}^n) \simeq \pi_*(S^0)[y_1, \ldots, y_n]/(y_i^2=\eta y_i), \quad |y_i|=1.$$  

**Proof.** As a spectrum, $\Sigma_+^\infty S^1 \simeq S^0 \vee S^1$, so it suffices to determine the multiplicative structure. We consider the map of $E_\infty$-ring spectra $\Sigma_+^\infty S^1 \rightarrow S^0$ obtained from the map of groups $S^1 \rightarrow 1$. Let $y \in \pi_1(\Sigma_+^\infty S^1)$ denote a generator of the kernel of $\pi_1(\Sigma_+^\infty S^1) \rightarrow \pi_1(S^0)$.

Using the universal property of $\mathfrak{gl}_1$ [ABG+14, Th. 5.2], we see that there exists a map

$$HZ[1] \rightarrow \mathfrak{gl}_1(\Sigma_+^\infty S^1),$$

whose image in homotopy is generated by $\psi(y)$. This map is adjoint to the identity map

$$\Sigma_+^\infty \mathcal{O}_\infty(HZ[1]) \simeq \Sigma_+^\infty S^1 \rightarrow \Sigma_+^\infty S^1.$$ 

In particular, we find that $\eta \psi(y)=0$ in $\pi_2(\mathfrak{gl}_1(\Sigma_+^\infty S^1))$. It follows from (4.1) that

$$\psi(\eta y) = \psi(y^2),$$

so that $y^2 = \eta y$ as $\psi$ is an isomorphism. \hfill $\square$

**Remark 4.3.** We can also see this as follows. The ring structure on $\Sigma_+^\infty S^1$ comes from the multiplication map $m : S^1 \times S^1 \rightarrow S^1$. On unreduced suspensions, we obtain a map

$$\Sigma_+^\infty m : \Sigma_+^\infty(S^1 \times S^1) \simeq S^0 \vee S^1 \vee S^1 \vee S^2 \rightarrow \Sigma_+^\infty S^1 \simeq S^0 \vee S^1.$$ 

However, under this identification the stable map $S^2 \rightarrow S^1$ that one obtains is the stabilization of the unstable Hopf fibration $S^3 \rightarrow S^2$. This determines the multiplicative structure on $\pi_*(\Sigma_+^\infty S^1)$ as claimed. We are grateful to Mike Hill for this remark.

**Remark 4.4.** Let $X$ be a spectrum with an $S^1$-action. We can describe multiplication by $y$ in the following manner on $\pi_0$. Consider the $H$-space $\Omega^\infty X$ and let $* \rightarrow \Omega^\infty X$ be a map of unpointed spaces corresponding to an element in $\pi_0 X$. The $S^1$-action on $\Omega^\infty X$ then extends this to an unpointed map $S^1 \rightarrow \Omega^\infty X$, which we can (using the $H$-space structure on $\Omega^\infty X$) turn into a pointed map $S^1 \rightarrow \Omega^\infty X$, i.e., a class in $\pi_1(X)$. This gives the map

$$\pi_0(X) \rightarrow \pi_1(X)$$

of multiplication by $y$.

It follows from this that if $X$ is a spectrum with $S^1$-action, the multiplication by $y$ on $\pi_0$ is determined by the $S^1$-action on $\Omega^\infty X$ together with its $H$-space structure. As a result, the multiplication by $y$ on $\pi_i, i > 0$ is determined by the action on $\Omega^\infty X$ to begin with.

4.3. The spectral sequence. In this subsection, we review the spectral sequence for taking homotopy fixed points of $\mathbb{T}^n$-actions. This spectral sequence is essentially the one described in [Hes96, §1.4] (where it starts from $E_1$ instead).

**Construction 4.5.** We can make $S^0$ into a $\Sigma_+^\infty \mathbb{T}^n$-module, or a spectrum with $\mathbb{T}^n$-action, by making $\mathbb{T}^n$ act trivially. Let $X$ be any spectrum with $\mathbb{T}^n$-action. Then we have an equivalence of spectra

$$X^{H\mathbb{T}^n} \simeq \text{Hom}_{\Sigma_+^\infty \mathbb{T}^n}(S^0, X),$$

and, thanks to the calculation (4.2), we obtain a spectral sequence [EKMM97, Ch. IV, Th. 4.1]

$$E_2^{s,t} = \text{Ext}^{s,t}_{\pi_*(S^0)[y_1, \ldots, y_n]/(y_i^2+\eta y_i)}(\pi_*(S^0), \pi_*(X)) \implies \pi_{t-s}(X^{H\mathbb{T}^n}).$$
Here the module structure of $\pi_*(S^0)$ is such that each $y_i$ acts trivially. The differentials run $d_r : E_r^{s,t} \to E_{r+r-1}^{s,r+t}$.

**Construction 4.6.** Suppose $\eta$ acts by zero on $\pi_*(X)$. In this case, by using natural Ext adjunctions in both variables, we can rewrite the $E_2$ page of (4.3) as

$$E_2^{s,t} \simeq \Ext_{(\pi_*(S^0)/\eta)[y_1,\ldots,y_n]/(y_i^2)}(\pi_*(S^0)/\eta, \pi_*(X)) \simeq \Ext_{\mathbb{Z}[y_1,\ldots,y_n]/(y_i^2)}(\mathbb{Z}, \pi_*(X)).$$

Here we use the inclusion $\mathbb{Z}[y_1,\ldots,y_n]/(y_i^2) \subset \pi_*(\Sigma^\infty T^n)/\eta$ and we make the $y_i$ act by zero on $\mathbb{Z}$.

We will need a condition that ensures that this spectral sequence degenerates at $E_2$. Let $A = \bigotimes_{i=1}^n \mathbb{Z}[y_i]/y_i^2$.

**Proposition 4.7.** If $M$ is an abelian group, consider $M \otimes_{\mathbb{Z}} A$ as an $A$-module by extending scalars. Then we have

$$\Ext_A^s(\mathbb{Z}, M \otimes_{\mathbb{Z}} A) = 0, \quad s > 0.$$

**Proof.** This follows because there is an isomorphism of $A$-modules

$$M \otimes_{\mathbb{Z}} A \simeq \text{Hom}_{\mathbb{Z}}(A, M),$$

and now one can use Ext adjunctions to obtain

$$\Ext_A^s(\mathbb{Z}, M \otimes_{\mathbb{Z}} A) \simeq \Ext_A^s(\mathbb{Z}, \text{Hom}_{\mathbb{Z}}(A, M)) \simeq \Ext_{\mathbb{Z}}^s(\mathbb{Z}, M) = 0, \quad s > 0.$$

**Definition 4.8.** Let $X$ be a spectrum with a $T^n$-action. Suppose $\eta$ acts by zero on $\pi_*(X)$. We will say that $\pi_*(X)$ is **relatively projective** if, as a module over $A = \mathbb{Z}[y_1, y_2, \ldots, y_n]/(y_i^2) \subset \pi_*(\Sigma^\infty T^n)/\eta$, it is a retract of a direct sum of modules that are obtained from $\mathbb{Z}$-modules by extending scalars.

**Example 4.9.** Let $Y$ be a spectrum such that $\eta$ acts by zero on $\pi_*(Y)$. Then we can form the coinduced spectrum $X = F(T^n, Y)$, which inherits a $T^n$-action from the first factor. The homotopy groups are relatively projective. In fact, as spectra with $T^n$-actions, we have an equivalence $X \simeq \Sigma^{-n}\Sigma^\infty T^n \wedge Y$ by Atiyah duality; it is clear that the latter has relatively projective homotopy groups.

**Corollary 4.10.** Suppose $X$ is a spectrum with $T^n$-action. Suppose that $\eta$ acts trivially on $\pi_*(X)$ and that $\pi_*(X)$ is relatively projective. Then the map $\pi_*(X^{kT^n}) \to \pi_*(X)$ is an injection whose image is given by $\bigcap_{i=1}^n \ker(y_i)$.

**Proof.** By Proposition 4.7 above, one sees that the spectral sequence degenerates after $E_2$ with the desired outcome.

We now consider a few different examples of the spectral sequence.

**Example 4.11.** Let $X$ be a discrete spectrum (i.e., $\pi_i X = 0$ if $i \neq 0$) with a $T^n$-action, necessarily trivial. Then one finds that the above spectral sequence is given additively at $E_2$ by $\pi_0 X \otimes \mathbb{Z}[\xi_1,\ldots,\xi_n]$ where each $\xi_i$ is in bidegree $(1,-1)$. The spectral sequence collapses at $E_2$ and one obtains the cohomology of product of copies of a projective space.

**Corollary 4.12.** Suppose $X$ is a spectrum with a $T^n$-action and suppose that $\pi_*(X)$ is concentrated in even degrees. Then $\pi_*(X^{kT^n})$ is concentrated in even degrees.

**Proof.** By induction, we may take $n = 1$ and $T^n = S^1$. In this case, we see that $\pi_*(X)$ is, as a module over $\mathbb{Z}[y]/y^2$, a sum of modules $M_{2i}$ concentrated in a single degree $2i$ (for $i \in \mathbb{Z}$). The spectral sequence (4.4) is easily seen to degenerate for reasons of degree and give the result. Alternatively, the result follows by filtering $X$ via the Postnikov filtration.

**Corollary 4.13.** Let $X$ be a spectrum concentrated in degrees 0 and 1 with an $T^n$-action; we do not assume $\eta$ acts by zero. Suppose given $x \in \pi_0 X$. Then the following are equivalent:
(1) $x$ lifts to an element of $\pi_0 X^{ht^n}$.
(2) $x$ is annihilated by the operators $y_i, i = 1, \ldots, n$.

**Proof.** This follows from the spectral sequence (4.4). In particular, if $x$ is annihilated by the operators $y_i$, it defines a class in the 0-line $\text{Hom}_{\pi_0 \Sigma^{\infty+}}(\pi_0(S^n), \pi_0(X))$. We need to show that it survives the spectral sequence, i.e., it supports no differentials. However, we observe that

$$\text{Ext}_{\pi_0 \Sigma^{\infty+}}^{s, s-1}(\pi_0(S^n), \pi_0(X)) = 0$$

for $s \geq 2$ for grading reasons, cf. Example 4.11. This should be compared to Example 3.2 via the identification between spectra concentrated in degree $[0, 1]$ and symmetric monoidal groupoids where every object is invertible. 

\[\square\]

5. **Proof of Dade’s theorem**

5.1. **Picard spectra.** Given a symmetric monoidal $\infty$-category $\mathcal{C}$, we will let $\text{pic}(\mathcal{C})$ denote the (connective) Picard spectrum of $\mathcal{C}$, cf., e.g., [MS, §2.2] for a discussion. Recall that:

1. $\pi_0 \text{pic}(\mathcal{C}) = \text{Pic}(\mathcal{C})$ is the Picard group of isomorphism classes of invertible objects in $\mathcal{C}$.
2. $\pi_1 \text{pic}(\mathcal{C})$ is the group of homotopy classes of self-equivalences of the unit $1 \in \mathcal{C}$.
3. $\pi_i \text{pic}(\mathcal{C})$ for $i \geq 2$ is the $(i-1)$st homotopy group of the endomorphism space of the unit.

Given an $E_\infty$-ring $R$, we will also write $\text{pic}(R)$ for $\text{Pic}(\text{Mod}(R))$. We have a natural equivalence of spectra $\tau_{\geq 1} \text{pic}(R) \simeq \Sigma gl_1(R)$.

Let $\text{Cat}_{\otimes}$ be the $\infty$-category of symmetric monoidal $\infty$-categories and symmetric monoidal functors between them, and let $\text{Sp}_{\geq 0}$ denote the $\infty$-category of connective spectra. We obtain a functor

$$\text{pic}: \text{Cat}_{\otimes} \to \text{Sp}_{\geq 0},$$

which commutes with limits [MS, Prop. 2.2.3].

For instance, it follows from Theorem 3.12 (with the notation from there) that we have equivalences of connective spectra

$$(5.1) \quad \text{pic}(k^{t B L'}) \simeq \tau_{\geq 0}(\text{pic}(k^{t A})^{hBL}) \simeq \tau_{\geq 0}(\text{pic}(k^{t A})^{ht^n}) \simeq \tau_{\geq 0}(\text{pic}(\text{StMod}_A)^{ht^n}).$$

To prove Dade’s theorem, we will calculate the left-hand-side of this equivalence, and show that any element of the Picard group of $\text{StMod}_A$ survives to the homotopy fixed points $\text{pic}(\text{StMod}_A)^{ht^n}$.

5.2. **The general tool.** In this subsection, we prove our main general tool for Picard groups (Theorem 5.3), which gives a criterion for when invertible modules can be descended along a faithful $T^n$-Galois extension.

**Proposition 5.1.** Let $R \to R'$ be a faithful $T^n$-Galois extension. Suppose $\pi_*(R) \to \pi_*(R')$ exhibits the target as a free module over the source and that $\eta = 0$ in $\pi_*(R)$. Then $\pi_*(R')$ is relatively projective as a module over $\mathbb{Z}[y_1, \ldots, y_n]/(y_2^2) \subset \pi_*(\Sigma^{\infty+} / \eta)$.

**Proof.** The assumption of freeness of $\pi_*(R')$ as $\pi_*(R)$-module imply that it suffices to check the claim about relative projectivity after base-change along $R \to R'$, i.e., to show that the action on $\mathbb{Z}[y_1, \ldots, y_n]/(y_2^2)$ on $\pi_*(R' \otimes_R R')$ makes the target relatively projective. However, this follows again because we have a $T^n$-equivariant equivalence $R' \otimes_R R' \simeq F(T^n, R')$. 

We will now need a general result about homotopy fixed points.

**Proposition 5.2.** Let $R \to R'$ be a faithful $T^n$-Galois extension of $E_\infty$-rings. Suppose that:

1. $\eta$ acts trivially on $\pi_*(R)$.
2. Every element in $\pi_1 R$ squares to zero.
3. $\pi_*(R')$ is a free module over $\pi_*(R)$.
Then we have:

\begin{equation}
\pi_i(\tau_{\geq 1} \mathfrak{gl}_1(R')^{ht^{n+1}}) = \begin{cases} 
\pi_i \mathfrak{gl}_1(R) & i \geq 1 \\
0 & i \leq 0 \text{ and } i \text{ even} \\
?? & i \leq 0 \text{ and } i \text{ odd}
\end{cases}
\end{equation}

If we do not assume hypothesis (2), the above conclusion (5.2) still holds after inverting 2.

Of course, the conclusion of the theorem is the vanishing of \(\pi_i(\tau_{\geq 1} \mathfrak{gl}_1(R')^{ht^n})\) for \(i \leq 0\) even, but we have stated in this manner for ease of reference.

**Proof.** First, we consider the case where \(n = 1\) so that \(R \to R'\) is an \(S^1\)-Galois extension. We claim first that \(\eta\) acts by zero on the homotopy groups of \(\tau_{\geq 1} \mathfrak{gl}_1(R)\). On \(\pi_1\), this follows from Proposition 4.1. On higher homotopy groups, this follows because we have an equivalence of spectra

\[ R_{[i,i+1]} \simeq \mathfrak{gl}_1(R_{[i,i+1]}), \quad i \geq 2 \]

by [MS, §5.1]. It follows that \(\pi_\ast(\tau_{\geq 1}(R))\) is the direct sum of relatively projective \(\mathbb{Z}[y]/y^2\)-modules and graded \(\mathbb{Z}[y]/y^2\) modules \(M\) concentrated in degree one. By Remark 4.4 and our assumptions, the same holds for \(\pi_\ast(\tau_{\geq 1} \mathfrak{gl}_1(R))\). The spectral sequence now implies that \(\pi_i(\tau_{\geq 1} \mathfrak{gl}_1(R)^{ht^{n+1}})\) is as desired.

Now consider a torus of arbitrary rank \(\mathbb{T}^n\). Choose a decomposition \(\mathbb{T}^n \simeq \mathbb{T}^{n-1} \times S^1\). By induction on \(n\), we can assume that \(\pi_i(\tau_{\geq 1} \mathfrak{gl}_1(R'))^{ht^{n-1}} = 0\) for \(i \leq 0\) even. We consider the \(S^1\)-equivariant cofiber sequence

\[ \tau_{\geq 1} \mathfrak{gl}_1(R'^{ht^{n-1}}) \to (\tau_{\geq 1} \mathfrak{gl}_1(R'))^{ht^{n-1}} \to C, \]

where by the inductive assumption the homotopy groups of \(C\) are concentrated in negative, odd degrees. Taking \(S^1\)-homotopy fixed points, we obtain a cofiber sequence of spectra

\[ (\tau_{\geq 1} \mathfrak{gl}_1(R'^{ht^{n-1}}))^{hS^1} \to (\tau_{\geq 1} \mathfrak{gl}_1(R'))^{ht^n} \to C^{hS^1}. \]

By Corollary 4.12, it follows that \(C^{hS^1}\) is concentrated in negative, odd degrees. The map \(R \to R'^{ht^{n-1}}\) is a faithful \(S^1\)-Galois extension to which the hypotheses of this result applies, and the case of rank one implies that \((\tau_{\geq 1} \mathfrak{gl}_1(R'^{ht^{n-1}}))^{hS^1}\) has vanishing \(\pi_i\) for \(i \leq 0\) even. Combining these completes the proof. When we invert 2 on \(\mathfrak{gl}_1(R)\), we note that one does not have to worry about \(\eta\) and the above goes through more simply. \(\square\)

We will now consider the following general question. Let \(R \to R'\) be a faithful \(\mathbb{T}^n\)-Galois extension and let \(\mathcal{L} \in \text{Pic}(R')\). We ask when \(\mathcal{L}\) descends to an \(R\)-module. Since there is a \(\mathbb{T}^n\)-action on the \(\infty\)-category \(\text{Mod}(R')\), each element \(a \in \pi_1(\mathbb{T}^n)\) induces a natural automorphism of the identity functor of \(\text{Mod}(R)\). In particular, each \(a\) induces a natural automorphism of \(R'\)-modules \(\mathcal{L} \simeq \mathcal{L}^a\), which is classified by an element in \(\pi_0(R'^\infty)\). A necessary condition for \(\mathcal{L}\) to descend is that this monodromy automorphism should be the identity.

**Theorem 5.3.** Let \(R \to R'\) be a faithful \(\mathbb{T}^n\)-Galois extension of \(\mathbf{E}_\infty\)-ring spectra. Suppose the hypotheses of Proposition 5.2 are satisfied. Let \(\mathcal{L} \in \text{Pic}(R')\). Then \(\mathcal{L}\) descends to an invertible \(R\)-module if and only if for every \(a \in \pi_1(\mathbb{T}^n)\), the induced monodromy automorphism \(a: \mathcal{L} \to \mathcal{L}\) is the identity.

If we assume only that the first and third hypotheses of Proposition 5.2 are satisfied, then given any such \(\mathcal{L}\), the tensor power \(\mathcal{L}^{\otimes 2^n}\) descends for \(n \gg 0\).

**Proof.** By Galois descent, we need to show that the map

\[ \text{pic}(R')^{ht^n} \to \text{pic}(R') \to \tau_{\leq 1} \text{pic}(R') \]
has the property that the class of \( \mathcal{L} \) belongs to the image in \( \pi_0 \). For this, we consider the diagram of spectra
\[
\Sigma(\tau_{\geq 1}\mathfrak{gl}_1(R'))^{hT^n} \longrightarrow (\text{pic}(R'))^{hT^n} \longrightarrow (\tau_{\leq 1}\text{pic}(R'))^{hT^n} \downarrow \\
\tau_{\leq 1}\text{pic}(R'),
\]
where the top row is a cofiber sequence of spectra. The assumption about monodromy implies that \( \mathcal{L} \) belongs to the image of the vertical map of spectra, cf. Corollary 4.13. It therefore suffices to show that the horizontal map
\[
(\text{pic}(k^{lA}))^{hT^n} \rightarrow (\tau_{\leq 1}\text{pic}(k^{lA}))^{hT^n}
\]
induces a surjection on \( \pi_0 \). This in turn follows because \( \pi_{-2}(\tau_{\geq 1}\mathfrak{gl}_1(R'))^{hT^n} = 0 \) by Proposition 5.2.

\[\square\]

5.3. Dade’s theorem and its variants. In this subsection, we complete the proof of Dade’s theorem and its version over an open subset of projective space. Our first goal is to calculate the Picard group of \( k^{T^n} \) for any \( n \).

**Proposition 5.4.** The Picard group of \( k^{T^n} \) is cyclic, generated by the suspension \( \Sigma k^{T^n} \).

**Proof.** As explained in [Mat14, §9.4], there exists an even periodic derived scheme \( \mathfrak{X} \) (with structure sheaf \( \mathcal{O}^{\text{top}} \)) such that \( \Gamma(\mathfrak{X}, \mathcal{O}^{\text{top}}) \simeq k^{T^n} \) and with the following properties:

1. The underlying ordinary scheme of \( \mathfrak{X} \) is \( \mathbb{P}^{n-1}_k \).
2. \( \tau_2\mathcal{O}^{\text{top}} \) is the sheaf \( \mathcal{O}(-1) \) on \( \mathbb{P}^{n-1}_k \).
3. There is an equivalence of symmetric monoidal \( \infty \)-categories \( \text{Mod}(k^{T^n}) \simeq \text{QCoh}(\mathfrak{X}) \). Equivalently, \( \text{QCoh}(\mathfrak{X}) \) admits the unit as a compact generator [Lur14, Prop. 7.1.2.7].

The derived scheme \( \mathfrak{X} \) is constructed by inverting each of the polynomial generators in \( k^{T^n} \) to obtain an even periodic \( \mathbb{E}^{\infty} \)-ring with \( \pi_0 \) a polynomial algebra over \( k \). One glues together these affine spaces to form a \( \mathbb{P}^{n-1}_k \). The third assertion follows from the amplexness of \( \tau_{-2}(\mathcal{O}^{\text{top}}) \), cf. [MM15, Prop. 3.2.2].

Zariski locally, \( \mathcal{O}^{\text{top}} \) is a sheaf of even periodic ring spectra with regular \( \pi_0 \), so that the Picard group is algebraic (cf. [BR05], [MS, Th. 2.4.4]). In particular, it follows that we can calculate the Picard group of \( k^{T^n} \) using the descent spectral sequence [MS, §3]. Using the cohomology of line bundles on projective space [Har77, III.5], one sees that there is no contribution to the Picard group. We note that \( H^i(\mathbb{P}^{n-1}, \mathcal{O}(-r)) = 0 \) if \( i \notin \{0, n-1\} \) for all \( r \in \mathbb{Z} \). We have \( H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}(-r)) \simeq H^{n-1}(\mathbb{P}^{n-1}, \pi_2, \mathcal{O}^{\text{top}}) \neq 0 \) for \( r \geq n \), but these will not contribute in the Picard spectral sequence. \[\square\]

**Theorem 5.5** (Dade [Dad78, Th. 10.1]). If \( A \) is an abelian \( p \)-group and \( k \) is a field of characteristic \( p \), then the Picard group of \( \text{StMod}_A \) is cyclic, generated by the suspension of the unit.

**Proof.** If \( A = C_2 \), the homotopy groups of \( k^{lA} \) are a graded field \( k[u_1^{\pm 1}] \) and every module over \( k^{lA} \) is free, so the Picard group is trivial. We will therefore assume \( A \neq C_2 \). Then, every element of \( \pi_1(k^{lA}) \) squares to zero.

First, we use Theorem 2.9 to identify \( \text{StMod}_A \) and \( \text{Mod}(k^{lA}) \), so we equivalently need to determine the Picard group of the \( \mathbb{E}^{\infty} \)-ring \( k^{lA} \). We have a faithful \( \mathbb{T}^n \)-Galois extension of \( \mathbb{E}^{\infty} \)-rings \( k^{lT^n} \rightarrow k^{lA} \). Note that \( \pi_*(k^{lA}) \) is a free module over \( \pi_*(k^{lT^n}) \); in fact, one checks this for the faithful \( \mathbb{T}^n \)-Galois extension \( k^{lT^n} \rightarrow k^{lA} \) of which this is a base-change.

Fix now an invertible \( k^{lA} \)-module \( M \). By Theorem 5.3, it descends to \( k^{lT^n} \) if and only if the monodromy elements in \( \pi_1(\mathbb{T}^n) \) act trivially on \( M \), i.e., give the identity in \( (\pi_0 k^{lA})^\times = k^\times \). Since the \( \mathbb{T}^n \)-action on \( \text{Mod}(k^{lA}) \) extends to a \( BA \)-action, the monodromy necessarily acts by \( p \)-power torsion elements. Since \( k^\times \) is \( p \)-torsion free, the monodromy is trivial. It follows that \( M \) descends
to an invertible module over $k^{\mathbb{P}_n}$. By Proposition 5.4, the Picard group of $k^{\mathbb{P}_n}$ is generated by the suspension of the unit, so we are done. \qed

We now describe how one can use the above arguments to obtain partial results about the Picard groups of localized stable module categories.

**Construction 5.6.** Let $U \subset \mathbb{P}^{n-1}_k$ be an open subset. We can then form an appropriate localization of the stable module category. Namely, we construct the even periodic derived scheme $\mathfrak{X}$ as in Proposition 5.4 and consider the open subscheme $\mathfrak{U} \subset \mathfrak{X}$ corresponding to $U$. We construct the $\mathbb{E}_\infty$-rings

$$k_U^{\mathbb{P}_n} \overset{\text{def}}{=} \Gamma(\mathfrak{U}, \mathcal{O}^{\text{top}}), \quad k_U^A \overset{\text{def}}{=} k_U^{\mathbb{P}_n} \otimes_{k^{\mathbb{P}_n}} k^A.$$

The associated module $\infty$-category $\text{Mod}(k_U^A)$ is the localization (over $U$) of the stable module $\infty$-category.

In general, we do not know how to control the Picard group of these localized stable module $\infty$-categories. However, we can obtain the following result.

**Proposition 5.7.** Given an element $\mathcal{L} \in \text{Pic}(k_U^A)$, $\mathcal{L}^{\otimes p^r}$ is a suspension of the unit for $s \gg 0$. In particular, the quotient of $\text{Pic}((\text{StMod}_A)_U)$ by the cyclic part generated by the suspensions of the unit is $p$-power torsion.

**Proof.** We first argue that the Picard group of $k_U^{\mathbb{P}_n}$ is cyclic after $p^{-1}$-localization. First, $\mathfrak{U}$ is also $0$-affine (cf. [MM15, Prop. 3.28]), i.e., one has an equivalence of symmetric monoidal $\infty$-categories $\text{Mod}(k_U^{\mathbb{P}_n}) \simeq \text{QCoh}(\mathfrak{U})$. Now the Picard group of the scheme $U$ is cyclic, generated by the twisting sheaf since the restriction map $\text{Pic}(\mathbb{P}^{n-1}_k) \to \text{Pic}(U)$ is surjective by regularity. The remaining terms in the descent spectral sequence as in Proposition 5.4 that can contribute to the Picard group of $k_U^{\mathbb{P}_n}$ (or equivalently $\mathfrak{U}$) are all $p$-power torsion and there are only finitely many of them.

The descent from $k_U^A$ to $k_U^{\mathbb{P}_n}$ can be carried out similarly. Namely, we claim that if $\mathcal{L} \in \text{Pic}(k_U^A)$, there exists $r$ such that $\mathcal{L}^{\otimes p^r}$ descends to $\text{Pic}(k_U^{\mathbb{P}_n})$. In fact, by Theorem 5.3 it suffices to show that the monodromy action on $\mathcal{L}^{\otimes p^r}$ (which gives an element of $\pi_0(k_U^A)^\times$) is trivial. Since this monodromy action is the $p^r$th power of the monodromy element for $\mathcal{L}$ itself, it suffices to take $r$ large enough that $p^r$ annihilates $A$. Note that if $p = 2$, there may be elements in $\pi_1$ which do not square to zero, but we use instead the last claim of Theorem 5.3. \qed

**Theorem 5.8.** Suppose $U \subset \mathbb{P}^{n-1}_k$ is affine and that $p > 2$. Let $\mathcal{L} \in \text{Pic}((\text{StMod}_A)_U)$. Suppose $\mathcal{L}$ is the $U$-localization of a compact object in $\text{StMod}_A$. Then $\mathcal{L}$ is the suspension of a unit.

**Proof.** Suppose $\mathcal{L} \in \text{Pic}((\text{StMod}_A)_U)$. Each generator $a \in A$ induces an $S^1$-action on $\text{StMod}_A$ given by multiplication by $a$. In particular, multiplication by $a$ on $\mathcal{L}$ gives an element in $\pi_0(k_U^A)^\times$. We claim that this element is trivial, or equivalently that $a: \mathcal{L} \to \mathcal{L}$ is homotopic to the identity. Since $\mathcal{L}$ is invertible in $\text{StMod}_A$, it suffices to show that the trace of the endomorphism $a - 1$ of $\mathcal{L}$ is equal to zero. However, we know that $\mathcal{L}$ arises as the $U$-localization of a finite-dimensional $A$-representation $V$, which is dualizable in $\text{StMod}_A$. The trace of $a - 1$ on $V$ is easily seen to be zero, so the trace of $a - 1$ on $\mathcal{L}$ is zero as well. This shows that every generator $a \in A$ acts as the identity on $\mathcal{L}$. By Theorem 5.3, this means that $\mathcal{L}$ descends to an invertible module over $k_U^{\mathbb{P}_n}$. Since $k_U^{\mathbb{P}_n}$ is regular, the Picard group is algebraic and the result follows. \qed

Given a compact object in $(\text{StMod}_A)_U$ (e.g., an invertible one), we recall that the obstruction to its being the $U$-localization of a compact object in $\text{StMod}_A$ lives in $K_0$ (cf. [TT90, 5.2.2] for the analogous result for extensions of perfect complexes over schemes).
6. Stratification via projective space

It is known by the work of Benson-Iyengar-Krause that one can “stratify” objects of the stable module category $\text{StMod}_G$ of a $p$-group $G$ via the cohomology $H^\text{even}(BG; k)$ and its homogeneous prime ideals. In particular, one can classify the localizing subcategories $[\text{BIK11}]$. More generally, one can carry this out for $\text{Mod}(k^hG)$.

The key case is when $G$ is elementary abelian. In $[\text{BIK11}]$, this is proved by showing that the classification of localizing subcategories of $\text{Mod}(k^hG)$ is equivalent to that of the $\infty$-category of modules over a ring spectrum whose homotopy groups are a polynomial ring. Here the classification is much simpler.

In this section, we use the comparison with the Tate construction for a torus to give another approach to this reduction, and thus to their results.

6.1. Generalities. Let $C$ be a presentable stable $\infty$-category.

Definition 6.1. We recall that a localizing subcategory $C' \subset C$ is a full stable subcategory closed under arbitrary colimits. We will also assume that any object of $C$ equivalent to an object in $C'$ is itself in $C'$. We will let $\text{Loc}(C)$ denote the class of all localizing subcategories of $C$.

Given a full subcategory $D \subset C$, the intersection of all localizing subcategories containing $D$ is a localizing subcategory, and is said to be the localizing subcategory generated by $D$.

Let $C, D$ be presentable stable $\infty$-categories and let $F: C \to D$ be a cocontinuous functor. Then if $M \in C$ belongs to the localizing subcategory generated by the $\{M_\alpha\}_{\alpha \in A} \subset C$, we can conclude that $F(M)$ belongs to the localizing subcategory generated by the $\{F(M_\alpha)\}_{\alpha \in A} \subset D$. This follows easily because the preimage of a localizing subcategory is a localizing subcategory. We obtain a map

$$F_*: \text{Loc}(C) \to \text{Loc}(D),$$

which sends a localizing subcategory $C' \subset C$ to the localizing subcategory of $D$ generated by $F(C')$.

In this subsection, we will show that this map is an isomorphism for a faithful Galois extension of $E_\infty$-rings with a connected Galois group. It will be convenient to use the notion of descendability $[\text{Mat14}, \S3-4]$.

Definition 6.2. A morphism of $E_\infty$-rings $A \to B$ is descendable if the thick $\otimes$-ideal that $B$ generates in $\text{Mod}(A)$ is all of $\text{Mod}(A)$.

Given an $A$-module $M$, one can form the cobar construction, which is an augmented cosimplicial object

$$M \to \left( M \otimes_A B \to M \otimes_A B \otimes_A B \to \cdots \right).$$

If $A \to B$ is descendable, then $M$ is the totalization of the above cobar construction; moreover, $M$ is a retract of the partial totalization at a finite stage. It follows, for example, that the thick $\otimes$-ideals in $\text{Mod}(A)$ that $M$ and $M \otimes_A B$ generate are equal.

Proposition 6.3. Let $R \to R'$ be a morphism of $E_\infty$-rings which is descendable. Then the map $\text{Loc}(\text{Mod}(R)) \to \text{Loc}(\text{Mod}(R'))$ obtained by base-change is injective, and has a section $\text{Loc}(\text{Mod}(R')) \to \text{Loc}(\text{Mod}(R))$ obtained by restriction of scalars.

Proof. We have a forgetful right adjoint functor $\text{Mod}(R') \to \text{Mod}(R)$ given by restriction of scalars, which is also cocontinuous. We claim that this induces a section to the map $\text{Loc}(\text{Mod}(R)) \to \text{Loc}(\text{Mod}(R'))$. Equivalently, we need to show that if $M \in \text{Mod}(R)$, then the $R$-modules $M$ and $R' \otimes_R M$ generate the same localizing subcategory of $\text{Mod}(R)$. This follows because, as remarked above, $M$ and $R' \otimes_R M$ even generate the same thick $\otimes$-ideal in $\text{Mod}(R)$. \qed
Proposition 6.4. Let $R$ be an $E_\infty$-ring and let $X$ be a connected finite complex. Then base-change $\text{Mod}(R) \to \text{Mod}(F(X_+, R))$ induces an isomorphism $\text{Loc}(\text{Mod}(R)) \simeq \text{Loc}(\text{Mod}(F(X_+, R)))$ (whose inverse is induced by the restriction of scalars functor).

Proof. Choose a basepoint $* \in X$, so that we obtain a map $s : F(X_+, R) \to R$ of $E_\infty$-rings. The morphism $\text{Loc}(\text{Mod}(R)) \to \text{Loc}(\text{Mod}(F(X_+, R)))$ admits a section since $s$ is a section of $R \to F(X_+, R)$. This section $\phi : \text{Loc}(\text{Mod}(F(X_+, R))) \to \text{Loc}(\text{Mod}(R))$ comes from the functor $\text{Mod}(F(X_+, R)) \otimes_{F(X_+, R)} R \to \text{Mod}(R)$, given by extension of scalars along $s$. However, $s : F(X_+, R) \to R$ is descendable [Mat14, Prop. 3.34], so $\phi$ is also an injection by Proposition 6.3. Therefore, $\phi$ is an inverse to the map in question. $\square$

Theorem 6.5. Let $R$ be an $E_\infty$-ring and let $G$ be a topological group with the homotopy type of a connected finite complex. Let $R \to R'$ be a faithful $G$-Galois extension. Then extension and restriction of scalars induce isomorphisms $\text{Loc}(\text{Mod}(R)) \simeq \text{Loc}(\text{Mod}(R'))$ inverse to one another.

Proof. It suffices to show that the restriction of scalars functor

$\text{Mod}(R') \to \text{Mod}(R)$

induces an injection on Loc, by Proposition 6.3.

Choose a descendable $E_\infty$-$R$-algebra $\tilde{R}$ such that $R' \otimes_R \tilde{R} \simeq F(G_+, \tilde{R})$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\text{Mod}(R') & \overset{\otimes_{R} \tilde{R}}{\longrightarrow} & \text{Mod}(R' \otimes_{R} \tilde{R}) \\
\downarrow & & \downarrow \\
\text{Mod}(R) & \overset{\otimes_{R} \tilde{R}}{\longrightarrow} & \text{Mod}(\tilde{R})
\end{array}
$$

The horizontal arrows are given by extension of scalars along $R \to \tilde{R}$ and the vertical arrows come from restriction of scalars. Applying Loc, we obtain a commutative diagram

$$
\begin{array}{ccc}
\text{Loc}(\text{Mod}(R')) & \overset{\otimes_{R} \tilde{R}}{\longleftarrow} & \text{Loc}(\text{Mod}(R' \otimes_{R} \tilde{R})) \\
\downarrow & & \downarrow \simeq \\
\text{Loc}(\text{Mod}(R)) & \overset{\otimes_{R} \tilde{R}}{\longleftarrow} & \text{Loc}(\text{Mod}(\tilde{R}))
\end{array}
$$

The left vertical arrow is an isomorphism by Proposition 6.4, and the horizontal arrows are injections by Proposition 6.3.

The diagram shows that restriction of scalars induces an injection $\text{Loc}(\text{Mod}(R')) \hookrightarrow \text{Loc}(\text{Mod}(R))$. By Proposition 6.3, it is also a surjection and is therefore an isomorphism (with inverse induced by extension of scalars) as desired. $\square$

The above should be compared with [BIK11, Th. 4.4], which is the necessary comparison tool in the Benson-Iyengar-Krause proof.

6.2. The classification. We note the stratification result for cochains on the classifying space of the torus. Here the argument uses the existence of “residue fields” as in [Ang08].

Proposition 6.6 (cf. [BIK11, Th. 5.2]). Let $k$ be a field of characteristic $p$. The localizing subcategories of $\text{Mod}(k^{MT})$ (resp. $\text{Mod}(k^{TT})$) are in bijection with the subsets of the set of homogeneous prime ideals of $\pi_*(k^{MT})$ (resp. those not containing the irrelevant ideal).
Let $A$ be an abelian $p$-group and construct the $\mathbb{T}^n$-action on $\text{StMod}_A \simeq \text{Mod}(k^{tA})$ as before, as well as the map $k^{tA} \to k^{tA}$, which is a Galois extension with Galois group isomorphic to $\mathbb{T}^n$. Combining this result with the previous subsection, 3.11, and Theorem 3.12, one obtains:

**Theorem 6.7** (Benson-Iyengar-Krause [BIK11]). Let $A$ be an abelian $p$-group of rank $n$. The functor $\text{Mod}(k^{hA}) \to \text{Mod}(k^{tA})$ induced by restriction of scalars along $k^{tA} \to k^{hA}$ induces an isomorphism on $\text{Loc}$. As a result, the localizing subcategories of $\text{Mod}(k^{hA})$ are in bijection with the subsets of the set of homogeneous prime ideals of $\pi_*(k^{tA})$.

**Theorem 6.8** (Benson-Iyengar-Krause [BIK11]). Let $A$ be an abelian $p$-group of rank $n$. The functor $\text{StMod}_A \to \text{Mod}(k^{tA})$ induced by restriction of scalars along $k^{tA} \to k^{tA}$ induces an isomorphism on $\text{Loc}$. As a result, the localizing subcategories of $\text{StMod}_A$ are in bijection with the subsets of the set $\mathbb{P}_k^{n-1}$ of homogeneous prime ideals of $\pi_*(k^{tA})$ which do not contain the irrelevant ideal.

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