FOURIER MULTIPLIERS FOR HARDY SPACES OF DIRICHLET SERIES

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Abstract. We obtain new results on Fourier multipliers for Dirichlet-Hardy spaces. As a consequence, we establish a Littlewood-Paley type inequality which yields a simple proof that the Dirichlet monomials form a Schauder basis for $p > 1$.

1. Introduction

The Dirichlet-Hardy spaces $H^p$ were first explicitly studied in the papers [2, 8]. (We refer to these papers for full details of the discussion in this section. See also [6] for some historical remarks.) For $p = 2$, they consist of Dirichlet series \( \sum_{n \in \mathbb{N}} a_n n^{-s} \) with square-summable coefficients, where $s = \sigma + it$ denotes the complex variable. By the Cauchy-Schwarz inequality, functions in $H^2$ converge on the half-plane $\mathbb{C}_{1/2} = \{ \sigma > 1/2 \}$. These spaces connect function space theory to analytic number theory. A striking illustration of this connection is given by the Riemann-zeta function $\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}$ that gives the reproducing kernel of $H^2$. Indeed, the function $k_w(s) := \zeta(s + \bar{w})$, for Re $w > 1/2$, has the property that $\langle f|k_w \rangle = f(w)$ for all $f \in H^2$, as may be verified by inspection.

For general $p > 0$, these spaces are defined to be the closure of Dirichlet polynomials in the norm

\[
\lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} a_n n^{-it} \right|^p dt \right)^{1/p}.
\]

(1)

This norm can be understood as the ergodic theorem on the infinite dimensional polydisk $\mathbb{T}^\infty$. To briefly explain this, we note that $\mathbb{T}^\infty$ is a compact Abelian group with the product of the normalised Lebesgue measures $d\theta_i/2\pi$ on each copy of $\mathbb{T}$ as its unique normalised Haar measure $d\theta$. It has dual group $\mathbb{Z}_\text{fin}^\infty$, i.e., sequences in $\mathbb{Z}^\infty$ with finitely many non-zero coefficients. So by standard Fourier analysis on groups, $F \in L^p(\mathbb{T}^\infty)$ has a Fourier expansion $F \sim \sum_{\nu \in \mathbb{Z}_\text{fin}^\infty} a_{\nu} z^\nu$, where $z \in \mathbb{T}^\infty$ and we use multi-index notation. The central observation, essentially

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called Kroenecker’s lemma, is that the path \( \phi : t \mapsto (2^{-it}, \ldots, p_i^{-it}, \ldots) \), where \( p_i \) is the \( i \)'th prime number, is ergodic in \( \mathbb{T}^\infty = \{ z = (z_1, \ldots) : z_i \in \mathbb{T} \} \). The ergodic theorem now says exactly that for continuous functions

\[
\lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} |F \circ \phi(t)|^p \, dt \right)^{1/p} = \|F\|_{L^p(\mathbb{T}^\infty)}
\]  

(2)

For \( F \) with spectral support only in the narrow cone \( \mathbb{N}_{\text{fin}}^\infty \), one checks that \( F \circ \phi \) is a Dirichlet series and that the right-hand side of this formula is exactly (1), provided we identify \( a_\nu \) with \( a_n \) when \( n = p_1^{\nu_1} p_2^{\nu_2} \cdots \). (Note that the same argument can be made using only the Stone-Weierstrass theorem, see [11]) We define the subspace \( H^p(\mathbb{T}^\infty) \) to consist of exactly these functions. By the uniqueness of prime number factorization, the map from \( H^p(\mathbb{T}^\infty) \) to \( \mathcal{H}^p \) given by \( F \mapsto F \circ \phi \) has an inverse, which is called the Bohr lift in honor of H. Bohr.

The structure of the paper is as follows. In Section 2 we use a technique of Fefferman to study certain Fourier multipliers on the spaces \( L^p(\mathbb{T}^\infty) \). These results are used in Section 3 to obtain a Littlewood-Paley inequality for the spaces \( \mathcal{H}^p \): for \( f = \sum_{n \in \mathbb{N}} a_n n^{-s} \) in \( \mathcal{H}^p \) with \( p > 1 \) and \( c > 1 \), we have

\[
\| f \|_{\mathcal{H}^p} \simeq |a_0| + \left\| \left( \sum_{k \geq 0} \left| \sum_{n \in \mathbb{N}_{\text{fin}}^{\infty}} a_n n^{-s} \right|^2 \right)^{1/2} \right\|_{\mathcal{H}^p}
\]  

(3)

As an application, we observe that the functions \( \{n^{-s}\}_{n \in \mathbb{N}} \) constitute a Schauder basis for the spaces \( \mathcal{H}^p \) for \( p > 1 \).

2. Fourier Multipliers

To state and prove our theorem on Fourier multipliers, we first introduce some notation, and review some necessary background. Throughout the section, \( p \geq 1 \).

A measurable function \( m : \mathbb{R} \to \mathbb{C} \) is called a Fourier multiplier on \( L^p(\mathbb{R}) \) if the operator \( f \mapsto \mathcal{F}^{-1}(m(\xi) \hat{f}(\xi)) \) is bounded on \( L^p(\mathbb{R}) \), where \( \mathcal{F} \) denotes the Fourier transform. On the torus \( \mathbb{T} \), a function \( m : \mathbb{Z} \to \mathbb{C} \) is called a multiplier if the map defined by the relation \( e^{int} \mapsto m(n)e^{int} \) extends to a bounded operator on \( L^p(\mathbb{T}) \). Finally, a function \( m : \mathbb{Z}_{\text{fin}}^\infty \to \mathbb{C} \) is called a multiplier if the operator

\[
\sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} a_\nu e^{i\nu \cdot} \mapsto \sum_{\nu \in \mathbb{Z}_{\text{fin}}^\infty} m(\nu)a_\nu e^{i\nu \cdot}
\]

is bounded on \( L^p(\mathbb{T}^\infty) \). Here we use the notation \( z = e^{i\theta} \) for a point in \( \mathbb{T}^\infty \). We denote the respective operator norms by \( \|m\|_{\mathcal{M}_p(X)} \), where \( X = \mathbb{R}, \mathbb{T} \) or \( \mathbb{T}^\infty \) as appropriate. We refer to the operator of multiplication by \( m \) by \( T_m \).

It is well-known that results for multipliers on \( \mathbb{T} \) may be deduced from those on the real line by the method of transference. More specifically, let \( m : \mathbb{R} \to \mathbb{C} \) be a
regulated function, i.e.,
\[ m(\xi) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} m(\xi + t)dt, \quad \forall \xi \in \mathbb{R}. \]

The basic result on transference, due to de Leeuw [4] (see [5, Section 3.6.2] for proofs), states that if a regulated function \( m \) is a multiplier on \( \mathbb{R} \), then \( m \) restricted to \( \mathbb{Z} \) is a multiplier on the torus. A converse statement also holds. In fact,
\[ \|m\|_{M_p(\mathbb{R})} = \sup_{\gamma > 0} \|m(\gamma \cdot)\|_{M_p(\mathbb{T})}. \quad (4) \]

Our argument relies in a crucial way on this formula.

To formulate our result, we introduce some additional notation. For \( \nu \in \mathbb{Z}_\infty \), one associates a unique rational number:
\[ r: \nu \mapsto - \to r_\nu \to p_1^{\nu_1} \cdots p_k^{\nu_k}, \]
where \( p_i \) is the \( i \)’th prime number. So, given a function \( m: \mathbb{Q}_+ \to \mathbb{C} \), we obtain a function \( m \circ r: \mathbb{Z}_\infty \to \mathbb{C} \). In particular, \( m \) induces in this way a densely defined Fourier multiplier on \( L^p(\mathbb{T}_\infty) \) by
\[ T_m: \sum_{\nu \in \mathbb{Z}_\infty} a_\nu e^{i \nu \cdot \theta} \mapsto \sum_{\nu \in \mathbb{Z}_\infty} m(r_\nu) a_\nu e^{i \nu \cdot \theta}. \]

Our multiplier result is as follows:

**Theorem 1.** Let \( p \in [1, \infty) \) and \( m: \mathbb{R}_+ \to \mathbb{C} \) be a regulated function continuous at rational points. Then \( m \circ r \) is a Fourier multiplier on \( L^p(\mathbb{T}_\infty) \), where \( r(\nu) = p_1^{\nu_1} \cdots p_k^{\nu_k} \) for \( \nu \in \mathbb{Z}_\infty \), if and only if \( m \circ \exp \) is a Fourier multiplier on \( L^p(\mathbb{T}_\infty) \). Moreover,
\[ \|m \circ r\|_{M_p(\mathbb{T}_\infty)} = \|m \circ \exp\|_{M_p(\mathbb{R})}. \]

**Proof.** We split the proof of the theorem into two parts.

First, we establish that \( \|m \circ r\|_{M_p(\mathbb{T}_\infty)} \leq \|m \circ \exp\|_{M_p(\mathbb{R})} \). Fix a polynomial
\[ f = \sum_{\nu \in \mathbb{Z}_\infty} a_\nu e^{i \nu \cdot \theta}. \]

Observe that since a polynomial only depends on a finite number of variables, we may restrict our attention to \( L^p(\mathbb{T}^d) \), for some \( d \in \mathbb{N} \). As a multiplier on \( L^p(\mathbb{T}^d) \), we need only consider \( \nu \in \mathbb{Z}^d \). Explicitly, we only need to consider the multiplier
\[ \nu \mapsto m(r_\nu) = m(e^{i \nu_1 \log p_1 + \cdots + \nu_d \log p_d}), \quad \nu \in \mathbb{Z}^d, \]
acting on \( L^p(\mathbb{T}^d) \). The idea is to introduce a change of variables on \( \mathbb{T}^d \) so that as a multiplier, this function only acts on the first variable.

To do this, we need to make an approximation. For \( \delta > 0 \), choose \( Q, a_1, \ldots, a_d \in \mathbb{N} \) so that
\[ \left| \frac{a_j}{Q} - \log p_j \right| < \delta, \quad \text{for } j = 1, \ldots, d. \]
We may assume that \(a_1\) and \(a_2\) are relatively prime (indeed, by the prime number theorem, we may choose both \(a_1\) and \(a_2\) to be prime), whence there exist \(q_1, q_2 \in \mathbb{N}\) so that \(a_1q_2 - a_2q_1 = 1\). This ensures that the \(d \times d\) matrix

\[
A = \begin{pmatrix}
    a_1 & a_2 & a_3 & \cdots & a_d \\
    q_1 & q_2 & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

satisfies \(\det A = 1\). A fortiori, \(A^{-1}\) also has integer coefficients, whence \(A : \mathbb{Z}^d \to \mathbb{Z}^d\) is bijective. Especially, one checks that \(A\) induces a bijective and measure preserving diffeomorphism on \(\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d\).

We next introduce a function defined on \(\nu \in \mathbb{Z}^d\) by

\[
M(\nu) = m \left( e^{\frac{\nu_1}{q_1} + \cdots + \frac{\nu_d}{q_d}} \right).
\]

Since \(m\) was assumed to be continuous on rational numbers, it follows that for any \(\epsilon > 0\) small enough, we may choose \(\delta > 0\) sufficiently small in the above approximation, so that \([M(\nu) - m(r)] < \epsilon\) uniformly on the finite index set corresponding to the set of non-zero coefficients of the polynomial \(f\). In particular, this implies that we can make \(\|T_M f - T_m f\|_{L^p(\mathbb{T}^d)}\) arbitrarily small. In light of (4), to obtain the desired inequality, we infer that it suffices to prove

\[
\|T_M f\|_{L^p(\mathbb{T}^d)} \leq \|m \circ \exp(Q^{-1}.)\|_{M_p(\mathbb{T})}.
\] (5)

To verify (5), let us first employ the change of variables \(\theta = A^T \theta'\) to get

\[
\|T_M f\|_{L^p(\mathbb{T}^d)}^p = \int_{\mathbb{T}^d} \left| \sum_{\nu \in \mathbb{Z}^d} M(\nu) a_{\nu} e^{i\nu \cdot A^T \theta'} \right|^p d\theta' = \int_{\mathbb{T}^d} \left| \sum_{\nu \in \mathbb{Z}^d} M(\nu) a_{\nu} e^{iA \nu \cdot \theta'} \right|^p d\theta'.
\]

If we change indices by \(\nu' = A \nu\), and observe that \(M(\nu) = m(e^{\nu'/Q})\), this becomes

\[
\int_{\mathbb{T}^d} \left| \sum_{\nu' \in \mathbb{Z}^d} m(e^{\nu'/Q}) a_{A^{-1} \nu'} e^{i\nu' \cdot \theta'} \right|^p d\theta' = \int_{\mathbb{T}^{d-1}} \left| \sum_{\nu'_{d-1}} b_{\nu'_{d-1}} m(e^{\nu'_{d-1}/Q}) e^{i\nu'_{d-1} \theta'} \right|^p \frac{d\theta'_2 \cdots d\theta'_d}{2\pi},
\]

where \(b_{\nu'_{d-1}} = b_{\nu'_{d-1}}(\theta'_2, \ldots, \theta'_d)\) is constant with respect to \(\theta'_1\). This is less than or equal to

\[
\|m \circ \exp(Q^{-1}.)\|_{M_p(\mathbb{T})}^p \int_{\mathbb{T}^{d-1}} \left| \sum_{\nu'_{d-1}} b_{\nu'_{d-1}} e^{i\nu'_{d-1} \theta'} \right|^p \frac{d\theta'_2 \cdots d\theta'_d}{2\pi} = \|m \circ \exp(Q^{-1}.)\|_{M_p(\mathbb{T})} \|f\|_{L^p(\mathbb{T}^d)}^p,
\]

which exactly yields the desired inequality (5).

We turn to the second part of the proof, where we establish the inequality \(\|m \circ \exp\|_{M_p(\mathbb{R})} \leq \|m \circ r\|_{M_p(\mathbb{T}^\infty)}\). By (4), it is sufficient to show that, for every \(\gamma > 0\), we have \(\|m \circ \exp(\gamma \cdot)\|_{M_p(\mathbb{T})} \leq \|m \circ r\|_{M_p(\mathbb{T}^\infty)}\). We now fix a polynomial
in one variable. As our idea is to work the previous argument backwards using only two variables, we express the polynomial as trivially depending on a second variable:

\[ f(\theta_1', \theta_2') = \sum_{|n| \leq N} a_{(n,0)} e^{i\theta_1'}. \]

Here \( a_{(n,m)} \) is zero for all \((n, m) \notin \mathbb{N} \times \{0\} \).

As in the first part of the proof, we first fix \( \delta > 0 \) and introduce a change of variables, this time induced by the matrix

\[ B = \begin{pmatrix} b + 1 & b \\ 1 & 1 \end{pmatrix}. \]

Above, the integer \( b \) is chosen so large that there exist prime numbers \( p_j, p_k \) for which

\[ |\gamma(b + 1) - \log p_j| < \delta/N, \quad \text{and} \quad |\gamma b - \log p_k| < \delta/N. \]

This is possible, since from the prime number theorem it holds that \( \log(p_{n+1}/p_n) \to 0 \) when \( n \to \infty \). As we have \( \det B = 1 \), the matrix \( B \) induces a measure preserving diffeomorphism of \( \mathbb{T}^2 \).

Setting \( \theta = B^T \theta' \) and \((n, 0)^T = B \nu \), we get

\[ \|T_{m \exp(\gamma)} f\|^p_{L^p(\mathbb{T})} = \left( \int_{\mathbb{T}^2} \left| \sum_{|n| \leq N} m(e^{\gamma n}) a_{(n,0)} e^{i(n,0) \cdot \theta'} \right|^p \frac{d\theta_1' d\theta_2'}{2\pi} \right)^{1/p} \]

\[ = \left( \int_{\mathbb{T}^2} \left| \sum_{\nu \in \mathbb{Z}^2} m(\exp(\gamma((b + 1)\nu_1 + b\nu_2))) a_{B\nu} e^{i\nu \cdot \theta} \right|^p \frac{d\theta_1 d\theta_2}{2\pi} \right)^{1/p}. \]

As \( \nu = (n, -n)^T \), we are only summing over \( \nu \in \mathbb{Z}^2 \) for which \( |\nu| \leq 2N \). So given any \( \epsilon > 0 \), by choosing \( \delta > 0 \) small enough, we make

\[ |m \circ \exp(\gamma n) - m(p_j^{\nu_1} p_k^{\nu_2})| = |m(e^{\gamma(b_1 \nu_1 + b_2 \nu_2)}) - m(e^{\nu_1 \log p_j + \nu_2 \log p_k})| < \epsilon \]

uniformly for indices \( \nu \) so that \( a_{B\nu} \) is non-zero. This implies that we only need to establish that

\[ \int_{\mathbb{T}^2} \left| \sum_{\nu \in \mathbb{Z}^2} m(p_j^{\nu_1} p_k^{\nu_2}) a_{B\nu} e^{i\nu \cdot \theta} \right|^p \frac{d\theta_1 d\theta_2}{2\pi} \leq \|m \circ r\|_{M^p(\mathbb{T}^\infty)}^p \|f\|_{L^p(\mathbb{T})}^p. \]

But this is readily seen to hold, as the left-hand side may be interpreted as \( T_{mor} \) applied to a function \( F \) depending on the \( j \)'th and \( k \)'th copy of \( \mathbb{T} \) in \( \mathbb{T}^\infty \), and where \( \|F\|_{L^p(\mathbb{T}^\infty)} = \|f\|_{L^p(\mathbb{T})} \) holds by reversing the changes in notation and variables. \( \square \)

3. Some Consequences and open problems

In this section we deduce a Littlewood-Paley inequality from Theorem 1 and also discuss Schauder bases for the spaces \( \mathcal{H}^p \).
First, we observe how a characterisation due to Marcinkiewicz is inherited by multipliers of the form discussed in the previous section. To do this, we recall that the total variation of a complex function $f$ on the interval $(a, b)$ is given by

$$
\|f\|_{BV(a,b)} = \sup\sum_{n=1}^{N} |f(x_j) - f(x_{j-1})|,
$$

where the supremum is taken over all sequences $a = x_0 < x_1 < \cdots < x_n = b$. For fixed $\eta > 1$, we also use the notation

$$
I_k = \begin{cases}
[e^{\eta^k}, e^{\eta^{k+1}}] & k \geq 1, \\
[e^{-\eta}, e^{\eta}] & k = 0, \\
[e^{-\eta^{k+1}}, e^{-\eta^k}] & k \leq -1.
\end{cases}
$$

We now get:

**Corollary 1.** Suppose that $p \in (1, \infty)$ and $\eta > 1$, then there exists a constant $C > 0$ such that for all regulated $m : \mathbb{R}_+ \rightarrow \mathbb{C}$ that are continuous at rationals we have

$$
\|m \circ r\|_{M_p(\mathbb{T}^\infty)} \leq C \left(\|m\|_{L^\infty(0,\infty)} + \sup_{k \in \mathbb{Z}} \|m\|_{BV(I_k)}\right).
$$

**Proof.** Since $m \circ \exp$ and $m$ have the same sup-norm, and $\|m \circ \exp\|_{BV(e^{\eta^k}, e^{\eta^{k+1}})} = \|m\|_{BV(I_k)}$, the Marcinkiewicz bound follows immediately from its classical counterpart, see [5, Section 5.2.1].

We also formulate a Hörmander-Mihlin type multiplier theorem for $p = 1$, see [9, 7] (a proof is also found in [5, Theorem 5.2.7]). Recall that $m : \mathbb{R} \rightarrow \mathbb{C}$ satisfies the Hörmander-Mihlin condition if $m$ is continuous and piecewise differentiable on $\mathbb{R} \setminus \{0\}$ with

$$
\|m\|_{L^\infty(\mathbb{R})} + \sup_{x \neq 0} |xf'(x)| < \infty.
$$

If this holds, then $m \in M_p(\mathbb{R})$ for any $p \in (1, \infty)$. In addition, $m$ defines a multiplier operator that is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ with norm bounded by (7). For our purposes it is useful to observe that (7) remains invariant if $m$ is replaced by $m(\lambda \cdot)$ for any $\lambda > 0$.

**Corollary 2.** Assume that $m : (0, \infty) \rightarrow \mathbb{C}$ is continuous and piecewise differentiable. Then

$$
\|m \circ r\|_{H^1(\mathbb{T}^\infty) \rightarrow L^1(\mathbb{T}^\infty)} \leq c \left(\|m\|_{L^\infty(0,\infty)} + \sup_{t > e} |t \log(t) m'(t)|\right).
$$

**Proof.** The condition of $m$ ensures that it can be modified on $(0,e)$ so that it satisfies (7) on $\mathbb{R}$. Hence $T_{m \circ \exp} : H^1 \rightarrow L^1$ is bounded in one variable with the stated bound. The case $p = 1$ of the proof of Theorem [1] now applies without changes. One simply needs to observe that after the change of variables, the assumption $f \in H^1(\mathbb{T}^\infty)$ implies that $a_{A-1,\nu} = 0$ if $\nu_1 < 0$, whence the one-dimensional multiplier $m(e^{\nu' / Q})$ is applied only to analytic functions.
We proceed to obtain a Paley-Littlewood type of theorem for $L^p(\mathbb{T}^\infty)$ as a consequence of Corollary 1. Fix a rational number $\eta > 1$, and consider intervals $I_k$ as above. For $f = \sum_{\nu \in \mathbb{Z}_m} a_\nu e^{i\nu \theta}$ in $L^p(\mathbb{T}^\infty)$, define the square function

$$S(f) = \left( \sum_k |f_k(\theta)|^2 \right)^{1/2}$$

where

$$f_k(\theta) = \sum_{\nu: r_\nu \in I_k} a_\nu e^{i\nu \theta}.$$  

The following result is clearly the most interesting in the special case of $\mathcal{H}^p$, which we stated as formula (3) in the introduction.

**Corollary 3.** Suppose that $p \in (1, \infty)$, and that $\eta > 1$ is a rational number. Then there exist constants such that for all $f \in L^p(\mathbb{T}^\infty)$, we have

$$\|f\|_{L^p(\mathbb{T}^\infty)} \approx \|S(f)\|_{L^p(\mathbb{T}^\infty)}.$$

**Proof.** We apply a standard argument. Define $m_\epsilon = \sum_{k \in \mathbb{Z}} \epsilon_k \chi_{I_k}$ for given $\epsilon \in \{-1, 1\}^\mathbb{Z}$. By Corollary 1, there exists some $C > 0$, independent of $\epsilon$, such that $\|m_\epsilon \circ \tau\|_{M_p(\mathbb{T}^\infty)} \leq C$. Here, $m_\epsilon$ is made regulated by defining it appropriately on the endpoints of the intervals $I_k$. This has no effect on the operator $T_{m_\epsilon} r_\tau$ as the endpoints are irrational. Next, since $T_{m_\epsilon} r_\tau T_{m_\epsilon} = \text{Id}$, we obtain for any $g \in L^p(\mathbb{T}^\infty)$ that $\|T_{m_\epsilon} r_\tau g\|_{L^p(\mathbb{T}^\infty)} \approx \|g\|_{L^p(\mathbb{T}^\infty)}$. This holds uniformly in $\epsilon$. The corollary now follows by averaging over $\epsilon$ and invoking Khintchine’s inequality [5, p. 435].

This result should be compared to a Paley-Littlewood inequality obtained from martingale theory. Indeed, a function $f \in L^p(\mathbb{T}^\infty)$ may be considered as a martingale $\{f(N)\}$ with respect to the filtration induced by the increasing sequence of $\sigma$-algebras corresponding to the sequence $\{\mathbb{T}^N\}_{N \in \mathbb{N}}$. The function $f(N)$, also called the conditional expectation, is obtained from $f$ by integrating away all but the $N$ first variables (see, e.g., [3] where these are called the ‘N-te Abschnitt’).  A Paley-Littlewood inequality is now obtained as a direct corollary of the classical Burkholder’s square function inequality [3] (see also [3, Theorem 5.4.7]). Set $\Delta_N f = f_N - f_{N-1}$. Then

$$\|f\|_{L^p(\mathbb{T}^\infty)} \approx \left( \sum |\Delta_N f|^2 \right)^{1/2} \|f\|_{L^p(\mathbb{T}^\infty)}.$$  

Actually, the same argument that was used to prove Corollary 3 yields (3) without using probability theory (this observation was applied in [1]).

In the following corollary, we consider the functions $1, 2^{-s}, 3^{-s}, \ldots$. It is clear that they form an orthogonal basis in $\mathcal{H}^2$. Luckily, they also yield a natural basis in $\mathcal{H}^p$:

**Corollary 4.** Suppose $p \in (1, \infty)$. Then the functions $n^{-s}$, $n = 1, 2, \ldots$, form a Schauder basis for $\mathcal{H}^p$. 

Proof. By the density and independence of these functions, and standard Schauder basis theory, it suffices to establish that the truncations \( \sum_{n=1}^{\infty} a_n n^{-s} \mapsto \sum_{n=1}^{N} a_n n^{-s} \) are bounded on \( \mathcal{H}^p \), uniformly with respect to \( N \). Let \( \alpha \in (0, 1/2) \) be an irrational number. According to Corollary \( \[\] \) the indicator functions of the intervals \( (0, N + \alpha) \) yield uniformly bounded multipliers on \( L^p(\mathbb{T}^\infty) \). The result follows. \( \square \)

Although we have not been able to find this result stated explicitly in the literature, we indicate how it can be deduced from \([10, \text{Theorem 8.7.2}]\). This result deals with the space \( L^p(G) \), where \( G \) is a compact abelian group that has a dual \( \Gamma \) which admits an order relation \( \mathcal{P} \). I.e., \( \mathcal{P} \) is a subset of \( \Gamma \) such that \( \mathcal{P} \cup (-\mathcal{P}) = \Gamma \), and \( \mathcal{P} \cap (-\mathcal{P}) = \{0\} \).

Under any such order relation one can define \( \text{sgn}(\gamma) \in \{-1, 0, 1\} \) according to whether or not \( \gamma \) is in \( \mathcal{P} \) or is in \( \{0\} \). With this, the statement is that the Hilbert transform \( T_\mathcal{P} : \sum_{\gamma \in \Gamma} a_\gamma e_\gamma \mapsto -i \sum_{\gamma \in \Gamma} \text{sgn}(\gamma) e_\gamma \) is bounded on \( L^p(G) \), where \( e_\gamma \) is the Fourier character corresponding to \( \gamma \in \Gamma \). In particular, \( \mathcal{P} = \{\nu : \log r_\nu \leq 0\} \) is an order relation in the dual \( \mathbb{Z}^\infty_{\text{fin}} \) of \( \mathbb{T}^\infty \). Hence the corresponding Riesz projection \( R_\mathcal{P} \), where \( R_\mathcal{P} e_\gamma := \chi_{\{r_\gamma \geq 0\}} e_\gamma \), is bounded on \( L^p(\mathbb{T}^\infty) \). If \( r \to \nu(r) \) is the inverse of the map \( r \), we obtain uniformly in \( N \)

\[
\left\| \sum_{n \leq N} a_n n^{-s} \right\|_{\mathcal{H}^p} = \left\| e_{\nu(N)} R_\mathcal{P} (e_{-\nu(N)} f) \right\|
\]

for functions \( f(s) = \sum_{n \in \mathbb{N}} a_n n^{-s} \) in \( \mathcal{H}^p \). As above, it follows immediately that \( \{n^{-s}\} \) is a Schauder basis for \( \mathcal{H}^p \) when \( p > 1 \).

We end with the following open questions which may seem innocent, but they could be somewhat hard taking into account the quite intractable and mysterious nature of the spaces \( \mathcal{H}^p \) for \( p \neq 2 \) as discussed, e.g., in \([11]\).

**Question 1.** Does \( \mathcal{H}^1 \) have a Schauder basis?

**Question 2.** Does \( \mathcal{H}^p \) have an unconditional basis if \( p \in (1, \infty) \setminus \{2\} \)?

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