Research Article

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A functionally-analytic method for modelling axial-symmetric flows of ideal fluid

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Abstract: We consider axial-symmetric stationary flows of the ideal incompressible fluid as an important case of potential solenoid fields. We use an integral expression of the Stokes flow function via the corresponding complex analytic function for solving a boundary value problem with respect to a steady streamline of the ideal incompressible fluid along an axial-symmetric body. We describe the solvability of the problem in terms of the singularities of the mentioned complex analytic function. The obtained results are illustrated by concrete examples of modelling of steady axial-symmetric flows.

Keywords: axial-symmetric potential flows, Stokes flow function, analytic function, streamline

MSC: 35J75, 35J25

1 Introduction

It is well-known that a three-dimensional potential solenoid field symmetric with respect to the axis Ox with the Cartesian coordinates \(x, y, z\) is described in a meridian plane \(xOr\), where \(r := \sqrt{y^2 + z^2}\), in terms of the axial-symmetric potential \(\varphi\) and the Stokes flow function \(\psi\), satisfying the following system of equations:

\[
\begin{align*}
\frac{r}{\partial x} \frac{\partial \varphi(x, r)}{\partial x} - \frac{\partial \psi(x, r)}{\partial r} &= 0, \\
\frac{r}{\partial r} \frac{\partial \varphi(x, r)}{\partial r} - \frac{\partial \psi(x, r)}{\partial x} &= 0.
\end{align*}
\]

(1)

While the function \(\varphi(x, \sqrt{y^2 + z^2})\) is a spatial harmonic function in the variables \(x, y, z\), the same is not true for the function \(\psi(x, \sqrt{y^2 + z^2})\).

Under the condition that there exist continuous second-order partial derivatives of the function \(\psi(x, r)\), system (1) implies the following equation for the Stokes flow function:

\[
\frac{r}{\partial x} \frac{\partial^2 \psi(x, r)}{\partial x^2} + \frac{\partial^2 \psi(x, r)}{\partial r^2} - \frac{\partial \psi(x, r)}{\partial r} = 0,
\]

(2)

that is degenerate on the axis \(Ox\), as well as equations (1).

An important case of potential solenoid field is the velocity field of stationary flow of the ideal incompressible fluid. Moreover, an axial-symmetric flow is one of the most widespread type of spatial flows. For instance, such flows are axial-symmetric flows along fuselage of aeroplanes, missiles and dirigible balloons, cumulative charges, the movement of fluids and gases in channels with round profiles, etc. (cf., e.g., Lavrentyev and Shabat [1, 2], Lavrentyev [3], Batchelor [4], Vallander [5], Loitsyanskii [6]).

In view of the degeneration of equations (1) on the axis \(Ox\), there is considerably less theory developed for solutions of system (1) than for solutions of the classical Cauchy–Riemann system, i.e., complex analytic functions (see Lavrentyev and Shabat [1, p. 18]).

In [7], for every complex analytic function we constructed a special monogenic function taking values in an infinite-dimensional topological vector space with commutative multiplication. Such a monogenic func-

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tion is a generalization of the principal extension of a complex analytic function into a commutative Banach algebra (cf., e.g., [8, p. 165]). For simply connected domains symmetric with respect to the axis $Ox$, we established a relation between the mentioned monogenic functions and the solutions of system (1). Such a relation generalizes a similar relation established in the papers [9, 10] for domains convex in the direction of the axis $Or$. As a result, we established a relation between the solutions of system (1) in the form of integral expressions via complex analytic functions and principal extensions of these functions into the mentioned topological vector space. In such a way we developed a method for explicit construction of axial-symmetric potentials and Stokes flow functions in an arbitrary simply connected domain symmetric with respect to the axis $Ox$ by means of components of the mentioned principal extensions.

In this paper, we develop a constructive functionally-analytic method for modelling steady streamlines of the ideal incompressible fluid along axial-symmetric bodies. Such a method is essentially based on using integral expressions of the Stokes flow functions via complex analytic functions.

2 An integral expression for the Stokes flow function and a boundary value problem with respect to a steady streamline of the ideal incompressible fluid along an axial-symmetric body

Keldysh [11] describes some correct statements of boundary value problems for an elliptic equation with a degeneration on a straight line. This shows that there are essential differences with boundary value problems for elliptic equations without degeneration. Some special methods for researching boundary value problems for elliptic equations degenerating along the line are developed by Gilbert [12], Mikhailov and Radzhabov [13], Rutkauskas [14, 15].

One of the ways for solving boundary value problems for axial-symmetric potential solenoid fields is based on integral expressions of axial-symmetric potentials via analytic functions of a complex variable (cf. Whittaker and Watson [16], Bateman [17], Henrici [18], Mackie [19], Krivenkov [20], Radzhabov [21], Polozhii [22], Polozhii and Uilitko [23], Kapshiyi [24], Aleksandrov and Soloviev [25]).

Similarly, in the papers [10, 26], to solve boundary value problems for the Stokes flow functions in both bounded and unbounded domains, we use an integral expression for solutions of equation (2). Below, we shall consider a boundary value problem for the Stokes flow function in unbounded domains only.

2.1 An integral expression for the Stokes flow function

Let $D$ be an unbounded domain in the meridian plane $xOr$ and let the boundary $\partial D$ of the domain $D$ be a closed Jordan rectifiable curve symmetric with respect to the axis $Ox$. Let $D_2 := \{ z = x + ir : (x, r) \in D \}$ be the congruent domain in the complex plane $\mathbb{C}$. Its closure and boundary are denoted by $\overline{D}_2$ and $\partial D_2$, respectively. Let the boundary $\partial D_2$ cross the real axis at the points $b_1$ and $b_2$. We assume that $b_1 < b_2$.

In what follows, $z := x + ir$. For every $z \in D_2$ with $\text{Im} z \neq 0$ let us fix an arbitrary Jordan rectifiable curve $\Gamma_{z2}$ in $D_2$ that connects the points $z$ and $\overline{z}$. We suppose additionally that the curve $\Gamma_{z2}$ crosses the real axis $\mathbb{R}$ on the interval $(-\infty, b_1)$.

For $z \in D_2$ with $\text{Im} z \neq 0$ let $\sqrt{(t-z)(t-\overline{z})}$ be the continuous branch of the analytic function $G(t) = \sqrt{(t-z)(t-\overline{z})}$ outside of the cut along $\Gamma_{z2}$ for which $G(b_2) > 0$.

For every $z \in D_2$ with $\text{Im} z = 0$, we define by continuity $\sqrt{(t-z)(t-\overline{z})} := t - z$ for $z < b_1$, and $\sqrt{(t-z)(t-\overline{z})} := -(t - z)$ for $z > b_2$.

Let $C^2(D)$ be the class of functions $\psi(x, r)$ having continuous second-order partial derivatives in the domain $D$.

For a function $\psi \in C^2(D)$ satisfying the following conditions:
\begin{itemize}
\item $\psi(x, r)$ satisfies equation (2) in $D$ and the additional assumption

$$
\psi(x, 0) \equiv 0 \quad \forall (x, 0) \in D ;
$$

(3)

\item the function $\psi(x, r)$ is vanishing at infinity;

\item the function $\psi(x, r)$ is even with respect to the variable $r$;
\end{itemize}

it is proved in the papers [10, 26] that there exists a unique analytic function $F : D_z \rightarrow \mathbb{C}$ vanishing at infinity and satisfying the condition

$$
F(\bar{z}) = \overline{F(z)} \quad \forall z \in D_z
$$

(4)

and such that the equality

$$
\psi(x, r) = -\frac{1}{2\pi i} \oint_{\Gamma} \frac{F(t) (t-x)}{\sqrt{(t-z)(t-\bar{z})}} \, dt
$$

(5)

is fulfilled for all $(x, r) \in D$, where $\gamma$ is an arbitrary closed Jordan rectifiable curve in $D_z$ which surrounds $\Gamma_{zz}$. Moreover, the function $F$ has a zero at least of the second order at infinity.

Note that the function (5) satisfies equation (2) in $D$ for every function $F$ analytic in $D_z$, but this function takes real values if and only if condition (4) is satisfied. Thus, all solutions of equation (2) in $D$ with a physical interpretation are inherently represented by the integral expression (5), where the corresponding complex analytic function $F$ has the stipulated properties.

In the case when a complex analytic function $F$ satisfies condition (4) and has a zero at least of the second order at infinity, we shall call $F$ the creative function for the Stokes flow function (5).

Consider a sequence of closed Jordan rectifiable curves $\gamma_n \subset D_z$ that tend to the curve $\partial D_z$. It means that the sequence of open sets $D^n$ bounded by $\gamma_n$ and $\partial D_z$ has the following properties: $D^1 \supset D^2 \supset \cdots \supset D^n$ and $\bigcap_{n=1}^{\infty} D^n = \emptyset$.

Let us remember that a complex analytic function $F$ given in $D_z$ belongs to the Smirnov class $E_p$ in the domain $D_z$ for $p \geq 1$ (cf., e.g., [27]) if there is a sequence of closed Jordan rectifiable curves $\gamma_n$ that tends to the curve $\partial D_z$ and $\int_{\gamma_n} |F(z)|^p \, |dz| \leq c$, where the constant $c$ does not depend on $n$.

Let us note that if the creative function $F$ belongs to the Smirnov class $E_1$ in the domain $D_z$, then the formula (5) can be transformed to the form

$$
\psi(x, r) = -\frac{1}{2\pi i} \int_{\partial D_z} \frac{F(t) (t-x)}{\sqrt{(t-z)(t-\bar{z})}} \, dt
$$

(6)

for all $(x, r) \in D$, where $F(t)$ are the angular boundary values of the function $F$ which, as it is known (see, e.g., [27]), exist at almost all points $t \in \partial D_z$, and the direction of the circuit of $\partial D_z$ with the domain $D_z$ to the left is taken to be the positive direction.

### 2.2 A boundary value problem with respect to a steady streamline of the ideal incompressible fluid along an axial-symmetric body

Consider an outer boundary value problem having important applications in the hydrodynamics of potential flows. It is the following problem with respect to a steady streamline of the ideal incompressible fluid along an axial-symmetric body: to find a function $\psi_1 \in C^2(D)$ which satisfies equation (2) in $D$ and the condition

$$
\psi_1(x, r) = 0 \quad \forall (x, r) \in \partial D \cup \{(x, r) \in D : r = 0\}
$$

(7)

and have the following asymptotic

$$
\psi_1(x, r) = \frac{1}{2} v_\infty r^2 + o(1), \quad x^2 + r^2 \rightarrow \infty, \quad v_\infty > 0.
$$

(8)

For the model of steady flow of the ideal incompressible fluid condition (7) means that the boundary $\partial D$ and the axis $Ox$ are lines of flow. In the asymptotic (8), $v_\infty$ is the velocity of an unbounded flow at infinity.
We note that explicit solutions of such a problem are known in certain particular cases of steady streamline along some concrete axial-symmetric bodies (see, e.g., Lavrentyev and Shabat [1], Batchelor [4], Vallander [5] Loitsyanskii [6], Weinstein [28], Mel’nichenko and Pik [29–31], Mel’nichenko and Plaksa [10]).

2.3 Integral equation for a boundary value problem with respect to a steady streamline of the ideal incompressible fluid along an axial-symmetric body

For every \( z \in \partial D_2 \setminus \mathbb{R} \), by \( \Gamma_{z2} \) we denote that Jordan subarc of the boundary \( \partial D_2 \) with the end points \( z \) and \( \bar{z} \) which contains the point \( b_1 \). For \( z \in \partial D_2 \setminus \mathbb{R} \) let \( \sqrt{(t - z)(t - \bar{z})} \) be the continuous branch of the analytic function \( G(t) = \sqrt{(t - z)(t - \bar{z})} \) outside of the cut along \( \Gamma_{z2} \) for which \( G(b_2) > 0 \).

Since the Stokes flow function
\[
\psi(x, r) = \psi_1(x, r) - \frac{1}{2} v_\infty r^2
\]
is vanishing at infinity, we can apply the integral expression (6) for solving the boundary value problem with respect to a steady streamline of the ideal incompressible fluid along an axial-symmetric body.

To solve this problem in such a way, we obtain the integral equation
\[
\frac{1}{\pi i} \int_{\partial D_1} \frac{F(t - x)}{\sqrt{(t - z)(t - \bar{z})}} \, dt = v_\infty r^2, \quad (x, r) \in \partial D : r \neq 0,
\]
where it is necessary to find the function \( F \) creative for the function (9). Here values of the function \( \sqrt{(t - z)(t - \bar{z})} \) for \( t \in \Gamma_{z2} \) are taken on the right side of the cut \( \Gamma_{z2} \).

In the papers [10, 26] we developed a method for a transition of equation (10) to the Cauchy singular integral equation on the real axis. In a case important for applications where \( \partial D_2 \) belongs to a class being wider than the class of Lyapunov curves, the mentioned singular integral equation is reduced to the Fredholm integral equation of the second kind. Moreover, it is established in [10, 26] that in this case there exists the unique function \( F \) which satisfies equation (10) and is creative for the function (9).

In addition, for a boundary value problem about a streamline of the ideal incompressible fluid along an axial-symmetric body, in the papers [7, 10] we obtained criteria for solvability by the means of distributions of sources and dipoles on the axis of symmetry and constructed certain concrete solutions using multipoles together with dipoles distributed on the axis. Let us note that the singularities of the function \( F \) creative for the function (9) are located on the real axis only in situations considered in [7, 10].

Our immediate purpose is to generalize the mentioned results in [7, 10] with a natural physical interpretation for cases of more general locations of singularities of the creative function \( F \).

2.4 Some examples using pseudosources and pseudosinks

It was shown in the papers [7, 10] that sources, dipoles and other multipoles located at points of the axis \( Ox \) are simulated by means of creative functions having poles in these points.

In the papers of Mel’nichenko and Pik [29, 31] it was proved that the Stokes flow functions generated by sources or dipoles distributed in the three-dimensional space along a circle which has the point \( O \) as center and is perpendicular to the axis \( Ox \) are expressed by means of elliptic integrals.

Let us show that it is possible to obtain streamline pictures using more simple models simulated by means of creative functions with poles located outside of the axis \( Ox \). Elliptic integrals are not used for expressions of the Stokes flow functions in these cases.

Consider axial-symmetric flows simulated by means of creative functions having simple poles outside of the axis \( Ox \). Inasmuch as a creative function has a zero at least of the second order at infinity and satisfies condition (4), consider the creative function
\[
F(t) = \frac{q}{t + a - ib} - \frac{q}{t - a - ib} + \frac{q}{t + a + ib} - \frac{q}{t - a + ib}, \quad a, b, q > 0,
\]
having two pairs of simple poles for which the "intensities" differ only by signs: either \( q \) or \(-q\). The Stokes flow function corresponding to this creative function is

\[
\psi(x, r) = -2q \text{Re} \left( \frac{x + a - ib}{\sqrt{-(a + ib - z)(a + ib - z)}} - \frac{x - a - ib}{\sqrt{a + ib - z}(a + ib - z)} \right).
\] (11)

The lines of flow are given by the equations \( \psi(x, r) = \text{const} \) and are represented on Figure 1. It is evident that when \( b \to 0 \), the considered configuration of singularities turns into a "source-sink" pair located on the axis \( Ox \).

We shall call the singularities of the function (11) a **system of pseudosources and pseudosinks**. To use such a system for modelling a streamline picture, it is necessary to take into account the fact that the function (11) has discontinuities lines, more precisely, the segment connecting the points \(-a + ib\) and \(-a - ib\), as well as the segment connecting the points \( a + ib \) and \( a - ib \).

Consider an interaction between a flow of the ideal incompressible fluid oncoming with velocity \( v_\infty > 0 \) and the mentioned system of pseudosources and pseudosinks. If the "intensity" \( q \) is small, we have no streamline picture. An increase of the "intensity" \( q \) results in formation of a closed contour \( \Gamma \) with equation

\[
\frac{v_\infty}{2} + \frac{\psi(x, r)}{r^2} = 0,
\] (12)

for which the mentioned segments are located inside the domain bounded by \( \Gamma \).

Taking into account equation (11), we obtain the following asymptotic relations:

\[
\frac{\psi(x, r)}{r^2} = -q \text{Re} \left( \frac{1}{(x - a - ib)^2} - \frac{1}{(x + a - ib)^2} \right) + o(1) = -q \frac{4ax(x^4 - 2(a^2 + b^2)x^2 + (a^2 + b^2)(a^2 - 3b^2))}{((x - a)^2 + b^2)((x + a)^2 + b^2)} + o(1),
\]

as \( r \to 0 \).

The singularities of the function (11) are located inside the domain bounded by \( \Gamma \) if the equality (12) is fulfilled at some points \((x, 0)\) with \( x < -\sqrt{a^2 + b^2 + 2b\sqrt{a^2 + b^2}} \) and \( x > \sqrt{a^2 + b^2 + 2b\sqrt{a^2 + b^2}} \) that is realized if and only if

\[
q \geq \frac{v_\infty}{8a} \min_{x > \sqrt{a^2 + b^2 + 2b\sqrt{a^2 + b^2}}} \frac{(x - a)^2 + b^2}{x(x^2 - 2(a^2 + b^2)x^2 + (a^2 + b^2)(a^2 - 3b^2))} =: q_0.
\]

Now, we can assert:

**a)** if \( q \geq q_0 \), then the function

\[
\psi_1(x, r) = \frac{v_\infty r^2}{2} - 2q \text{Re} \left( \frac{x + a - ib}{\sqrt{-(a + ib - z)(a + ib - z)}} - \frac{x - a - ib}{\sqrt{(a + ib - z)(a + ib - z)}} \right)
\] (13)

is the solution of the problem with respect to a steady streamline for some domain \( D \).
b) if \( q < q_0 \), then there is no domain \( D \), for which the function (13) would be the solution of the problem with respect to a steady streamline.

One can see an example of streamline picture in the case \( q = q_0 \) on Figure 2. Let us note that in this case the “oval” solid of rotation with cavities (i.e., the set \( \mathbb{R}^2 \setminus D \)) is not convex in the direction of the axis \( Ox \). In the papers [7, 10] we use a quadrupole together with dipoles located on the axis \( Ox \) to obtain the picture of streamline along a “pear”, that is also not convex in the direction of the axis \( Ox \).

2.5 Expressions of solutions via distributions of pseudosources and pseudosinks

Let us consider boundary value problems for which the solvability is described in terms of the singularities of the function \( F \) creative for the function (9).

Suppose that there exists a set of cuts in the domain \( C \setminus \overline{D} \). This set includes a segment \( I^0 := [a_1, a_2] \), where \( b_1 < a_1 < a_2 < b_2 \). This set also includes a finite number \( m \) of Jordan rectifiable arcs \( I^k \) with initial point \( a_{1,k} \) and endpoint \( a_{2,k} \). Moreover, all arcs \( I^k \) are located in the domain \( \{ z \in \mathbb{C} \setminus \overline{D} : \Im z > 0 \} \) and different arcs \( I^k \) have no common points. Finally, this set includes the arcs \( I^k \) symmetric to \( I^k \) with respect to the real axis.

Now, consider a function \( F \) analytic in the domain \( D_0^2 := C \setminus I^0 \setminus (\bigcup_{k=1}^m (I^k \cup \overline{I^k})) \). We say that a function \( F \) belongs to the Smirnov class \( E_p \) in the domain \( D_0^2 \) for \( p \geq 1 \) if there are sequences of closed Jordan rectifiable curves \( \gamma^0_n, \gamma^k_n \) that tend to \( I^0, I^k, \overline{I^k} \), respectively, and

\[
\left( \int_{\gamma^k_n} |F(z)|^p \, |dz| \right) \leq c,
\]

where the constant \( c \) does not depend on \( n \).

For a function \( F(z) \) belonging to the Smirnov class \( E_1 \) in the domain \( D_0^2 \), we denote by \( F^+(t) \) and \( F^-(t) \) its angular boundary values on the set \( I^0 \cup \bigcup_{k=1}^m (I^k \cup \overline{I^k}) \) when \( z \to t \) to the left and right of this set, respectively. It is clear that \( F^+(t) \) and \( F^-(t) \) exist at almost all points \( t \in I^0 \cup \bigcup_{k=1}^m (I^k \cup \overline{I^k}) \).

**Theorem 1.** Suppose that the creative function \( F \) which is a solution of equation (10) has the form

\[
F(z) = \sum_{j=1}^n \frac{q_j}{z - x_j} + \sum_{k=1}^m \sum_{j=1}^{n_k} \left( \frac{q_{k,j}}{z - z_{k,j}} + \frac{q_{k,j}}{z - \overline{z}_{k,j}} \right) + F_1(z),
\]

where \( x_j \in I^0, z_{k,j} \in I^k, q_j \) and \( q_{k,j} \) are real numbers, the equalities

\[
\sum_{j=1}^n q_j = 0, \quad \sum_{k=1}^m n_k q_{k,j} = 0, \quad k = 1, 2, \ldots, m,
\]

are fulfilled, and the function \( F_1 \) can be extended to a function belonging to the Smirnov class \( E_1 \) in the domain \( D_0^2 \). Then the solution of the problem with respect to a steady streamline is given by the formula

\[
\psi_1(x, r) = \frac{\nu \omega t^2}{2} - \sum_{j=1}^n \frac{q_j}{2} \left( \frac{x - x_j}{(x - x_j)^2 + r^2} \right) - 2 \sum_{k=1}^m \sum_{j=1}^{n_k} q_{k,j} \Re \frac{x - z_{k,j}}{\sqrt{(z_{k,j} - z)(\overline{z}_{k,j} - \overline{z})}}
\]

\[
+ \int_{a_1}^{a_2} \int_{a_1}^{a_2} \frac{q(t)(t - x)}{\sqrt{(t - x)^2 + r^2}} \, dt \, 2 \sum_{k=1}^m \Re \int_{a_1}^{a_2} \frac{q(t)(t - x)}{\sqrt{(t - z)(t - \overline{z})}} \, dt \quad \forall (x, r) \in D,
\]

where

\[
q(t) := -\frac{1}{2m!} \left( F_1(t) - F_1(t) \right).
\]

Moreover,

\[
\left( \int_{I^0} + \sum_{k=1}^m \left( \int_{I^k} + \int_{\overline{I^k}} \right) \right) q(t) \, dt = 0.
\]
Proof. The solution of the problem with respect to a steady streamline is given by the formula

\[ \psi_1(x, r) = \frac{v_\infty r^2}{2} - \frac{1}{2\pi i} \int_{\partial D_1} F(t) \frac{t-x}{(t-z)(t-\bar{z})} \, dt, \]

where the creative function \( F \) which is a solution of equation (10) has the form (15).

Therefore, using Cauchy’s Theorem and Cauchy’s Integral Formula, one can rewrite the last equality in the form

\[ \psi_1(x, r) = \frac{v_\infty r^2}{2} - \sum_{j=1}^{n} \frac{q_j(x-x_j)}{\sqrt{(x_j-z)(x_j-\bar{z})}} - \sum_{k=1}^{m} \sum_{j=1}^{n_k} \frac{q_{k,j}(x-z_{k,j})}{\sqrt{(z_{k,j}-z)(z_{k,j}-\bar{z})}} - \frac{1}{2\pi i} \left( \int_{\gamma^0_n} + \sum_{k=1}^{m} \left( \int_{\gamma^k_n} + \int_{\gamma^k_{n-1}} \right) \right) F_1(t) \frac{t-x}{(t-z)(t-\bar{z})} \, dt, \]

where the curves \( \gamma^0_n, \gamma^k_n, \gamma^k_{n-1} \) are the same as in the inequality of type (14) for the function \( F_1 \), and all curves of integration have the same orientation with an unbounded domain, which contains \( D_2 \) as a subset, to the left.

Now, letting \( n \to \infty \) and taking into account a condition of type (4) for the function \( F_1 \), we obtain equality (17). Equality (19) follows from Cauchy’s Theorem for the function \( F_1 \) vanishing at infinity. \( \square \)

Formula (17) gives an expression for the solution of the problem with respect to a steady streamline of the ideal incompressible fluid along an axial-symmetric body \( C \setminus D_2 \) via pseudosources and pseudosinks distributed inside this body on the set \( \Gamma_0 \cup (\bigcup_{k=1}^{m} \Gamma_0^k \cup \Gamma_k) \) if the assumptions of Theorem 1 are satisfied.

The following statement converse in a certain sense to Theorem 1 is true:

**Theorem 2.** Suppose that all curves \( \Gamma_k \) satisfy the condition

\[ \sup_{z \in \Gamma_k} \mes \{ t \in \Gamma_k : |t-z| \leq \epsilon \} = O(\epsilon), \quad \epsilon \to 0, \]

(20)

where \( \mes \) denotes the linear Lebesgue measure on \( \Gamma_k \). Suppose also that the solution of the problem with respect to a steady streamline is given by formula (17), where \( q \) is summable on the set \( \Gamma^0 \cup (\bigcup_{k=1}^{m} \Gamma_0^k \cup \Gamma_k) \) to the \( p \)-th power, \( p > 1 \), and the equalities (16) and (19) are fulfilled. Then the solution \( F \) of equation (10) has the form (15), where the function \( F_1 \) can be extended to a function belonging to the Smirnov class \( E_p \) in the domain \( D^0_2 \). Moreover, in this case

\[ F_1(z) = -\left( \int_{\Gamma^0} + \sum_{k=1}^{m} \left( \int_{\Gamma^k_0} + \int_{\Gamma^k_{n-1}} \right) \right) \frac{q(t)}{t-z} \, dt \quad \forall z \in D^0_2. \]

(21)

Proof. Due to David’s Theorem 1 in [32], under condition (20) the function (21) belongs to the Smirnov class \( E_p \) in the domain \( D^0_2 \) and satisfies the boundary condition (18) at almost all points \( t \in \Gamma^0 \cup (\bigcup_{k=1}^{m} \Gamma_0^k \cup \Gamma_k) \). Now, it is easy to conclude that the creative function \( F \) which is a solution of equation (10) has the form (15). \( \square \)

Theorems 1 and 2 generalize Theorems 8 and 9, respectively, in [7], where the case \( \Gamma_k = \emptyset \) for all \( k \) was considered.

### 2.6 Expressions of solutions via distributions of pseudodipoles

Consider the Stokes flow function

\[ \psi(x, r) = 2p r^2 \Re \frac{1}{((a + ib - z)(a + ib - \bar{z}))^{3/2}} \]

(22)

corresponding to the creative function

\[ F(t) = \frac{p}{(t-a - ib)^2} + \frac{p}{(t-a + ib)^2}, \quad a, p \in \mathbb{R}, \quad b > 0, \]
which has poles of order 2 at the points $a + ib$.

It is evident that when $b \to 0$, the considered configuration of singularities turns into a dipole located on the axis $Ox$. Therefore, we shall call singularities of the function (22) pseudodipoles.

Now, using Cauchy’s Theorem and Cauchy’s Integral Formula, we prove the following statement in the same way as Theorem 1.

**Theorem 3.** Suppose that the creative function $F$ which is a solution of equation (10) has the form

$$F(z) = \sum_{j=1}^{n} \frac{p_j}{(z-x_j)^2} + \sum_{k=1}^{m} \sum_{j=1}^{n_k} \left( \frac{p_{k,j}}{(z-z_{k,j})^2} + \frac{p_{k,j}}{(z-\bar{z}_{k,j})^2} \right) + F_2(z),$$

(23)

where $x_j \in \Gamma^0$, $z_{k,j} \in \Gamma^k$, $p_j$ and $p_{k,j}$ are real numbers, and a primitive function $\mathcal{F}_2$ for the function $F_2$ can be extended to a function belonging to the Smirnov class $E_1$ in the domain $D^0_\varepsilon$. Then the solution of the problem with respect to a steady streamline is given by the formula

$$\psi_1(x, r) = \frac{\nu_0 r^2}{2} + \sum_{j=1}^{n} \int_{a_j}^{a_j} \frac{p(t)}{(t-x_j)^2 + r^2} \frac{r^2}{(t-x_j)^2 + r^2} dt - 2r^2 \sum_{k=1}^{m} \int_{\Gamma} \frac{p(t)}{(t-z)(t-\bar{z})} \frac{1}{(t-z)(t-\bar{z})} dt \quad \forall (x, r) \in D,$$

(24)

where

$$p(t) := -\frac{1}{2\pi i} (\mathcal{F}_2(t) - \mathcal{F}_2(t)).$$

The following statement which is converse in a certain sense to Theorem 3 is easy proved in the same way as Theorem 2.

**Theorem 4.** Suppose that all curves $\Gamma^k$ satisfy condition (20). Suppose also that the solution of the problem with respect to a steady streamline is given by the formula (24), where $p$ is summable on the set $\Gamma^0 \cup \bigcup_{k=1}^{m} (\Gamma^k \cup \bar{\Gamma}^k)$ to the $p$-th power, $p > 1$. Then the solution $F$ of equation (10) has the form (23), where the function $F_2$ has a primitive function $\mathcal{F}_2$ that can be extended to a function belonging to the Smirnov class $E_p$ in the domain $D^0_\varepsilon$. Moreover, in this case

$$\mathcal{F}_2(z) = -\left( \int_{\Gamma^0} \sum_{k=1}^{m} \left( \int_{\Gamma^k} \int_{\Gamma^\bar{k}} \right) \frac{p(t)}{t-z} \frac{p(t)}{t-\bar{z}} dt \right) \quad \forall z \in D^0_\varepsilon.$$

(25)

Theorems 3 and 4 generalize Theorems 11 and 12, respectively, in [7], where the case $\Gamma^k = \emptyset$ for all $k$ was considered.

Let us note that for any constant $q$ and $z_1, z_2 \in \Gamma^k$, $k = 0, 1, \ldots, m$, the following equality holds:

$$-q \frac{x-z_1}{\sqrt{(z_1-z)(z_1-\bar{z})}} + q \frac{x-z_2}{\sqrt{(z_2-z)(z_2-\bar{z})}} = -\frac{q}{(x-z)(x-\bar{z})} \frac{x-z_1}{(x-z)(x-\bar{z})} \frac{x-z_2}{(x-z)(x-\bar{z})} \quad \forall (x, r) \in D,$$

where $\sim z_1z_2$ is the subarc of $\Gamma^k$ with initial point $z_1$ and endpoint $z_2$.

Therefore, it is easy to conclude that under conditions (16) and (19), pseudosources and pseudosinks located or distributed on the set $\Gamma^0 \cup \bigcup_{k=1}^{m} (\Gamma^k \cup \bar{\Gamma}^k)$ can be replaced by pseudodipoles distributed on this set, and function (15) has the properties of the function $F_2$ from equality (23). Consequently, every solution of the problem with respect to a steady streamline of the form (17) can also be expressed by formula (24), where $p_j \equiv 0$ and $p_{k,j} \equiv 0$.

At the same time, it becomes obvious that among domains $D$ for which the solution of the problem with respect to a steady streamline is expressed by formula (24), there are domains for which the function $\psi_1$ cannot be expressed as in (17). Thus, the class of domains for which the solution to the problem with respect to a steady streamline is given by formula (24) is wider than the class of domains for which the solution of the mentioned problem is given by formula (17); cf. [7, 10], where partial cases of the mentioned formulas are considered.
2.7 Using pseudomultipoles for a construction of solutions of the problem with respect to a steady streamline

We call pseudomultipoles singularities of the Stokes flow function corresponding to a creative function with poles located outside of the axis Ox.

For instance, lines of flow $\psi(x, r) = \text{const}$ for the Stokes flow function (22) of pseudodipoles in the case of $a = 0$ are represented on the left picture on Figure 3.

![Figure 3: Pseudodipoles and pseudoquadrupoles.](image)

On the right picture on Figure 3 one can see lines of flow for the Stokes flow function

$$\psi(x, r) = 3m r^2 \text{Re} \frac{x - ib}{((ib - z)(ib - \bar{z}))^{5/2}}$$

corresponding to the creative function

$$F(t) = \frac{m}{(t - ib)} + \frac{m}{(t + ib)}, \quad m \in \mathbb{R}, \ b > 0,$$

which has poles of order 3 at the points $\pm ib$. When $b \to 0$, this configuration of singularities turns into a quadrupole located on the axis Ox. Therefore, we call such singularities pseudoquadrupoles.

Similar pictures can be constructed for pseudomultipoles of any order.

Let us consider an interaction between a flow of the ideal incompressible fluid oncoming with velocity $v_\infty > 0$ and pseudodipoles with the Stokes flow function (22) in the case of $a = 0$.

For small "intensities" $p$ we have no streamline picture. An increase of the "intensity" $p$ brings at first a formation of streamline picture of toroidal solid. The equation of its boundary $\Gamma$ is of the form $\psi_1(x, r) = c$, where

$$\psi_1(x, r) = \frac{v_\infty r^2}{2} - 2p r^2 \text{Re} \frac{1}{((ib - z)(ib - \bar{z}))^{3/2}}.$$  \hspace{1cm} (26)

The inequality $c < v_\infty b^2 / 2$ is a necessary condition for such a streamline picture because the singularity $(0, b)$ of the function $\psi_1(x, r)$ should be located inside the domain bounded by a connected part of $\Gamma$ in the half plane $r > 0$. For each fixed $c < v_\infty b^2 / 2$, an increase of the "intensity" $p$ brings a formation of the
mentioned contour $\Gamma$ when (cf. interaction between a flow and a “dipole and quadrupole” pair in [7])

$$p = \max_r \min_x \left( \frac{V_{\infty}}{2} - \frac{C}{r^2} \right) / \left( 2 \operatorname{Re} \frac{1}{(ib - z)(ib - \bar{z})^{3/2}} \right),$$

where the minimum with respect to $x$ and the maximum with respect to $r$ are taken in a neighbourhood of a prospective node, in which lines of oncoming flow and pseudodipoles are closed. For instance, on Figure 4, where one can see the streamline along a toroidal solid with the lines of flow

$$\psi_1(x, r) \equiv r^2 - 15 r^2 \operatorname{Re} \frac{1}{((i - z)(i - \bar{z}))^{3/2}} = c,$$

one of such nodes is located nearby the point $(1.5; 1.2)$.

**Figure 4:** The streamline along a toroidal solid.

**Figure 5:** The streamline along a toroidal solid without a hole.

Further, an increase of the "intensity" $p$ results finally in formation of a toroidal solid without a hole (see Figure 5). In this case the function (26) is the solution of the problem with respect to a steady streamline for a certain unbounded domain $D$ with boundary $\partial D$ that has the equation

$$\frac{V_{\infty}}{2} - 2p \operatorname{Re} \frac{1}{((i - z)(i - \bar{z}))^{3/2}} = 0.$$
Moreover, the last equality is fulfilled at some points \((x, 0)\) with \(x < 0\) and \(x > 0\), i.e., at these points we have

\[
\frac{v_\infty}{2} - 2p \Re \left( \frac{1}{(ib - x)^{3/2}} \right) = 0,
\]

from which we find

\[
p = \frac{v_\infty}{4} : \Re \left( \frac{1}{(ib - x)^{3/2}} \right) = \frac{v_\infty}{4} \left( \frac{x^2 + b^2}{x(x^2 - 3b^2)} \right).
\]

Thus, we can conclude that the function (26) is the solution of the problem with respect to a steady streamline for a certain domain \(D\) if and only if

\[
p \geq \frac{v_\infty}{4} \min_{x > \sqrt{3}b} \left( \frac{x^2 + b^2}{x(x^2 - 3b^2)} \right) = p_0.
\]

Let us note that in the case of \(p = p_0\) a toroidal solid without a hole (i.e., the set \(\mathbb{R}^2 \setminus D\)) is not convex in the direction of the axis \(Or\) (see Figure 5).

To conclude consider another example. The streamline along a “boletus” is represented on Figure 6. In this case, the lines of flow are given by the equations

\[
\psi_1(x, r) \equiv r^2 - 1, 2r^2 \Re \left( \frac{1}{(i - z)(i - \bar{z})^{3/2}} \right) - \frac{2, 6r^2}{((x - 0, 2)^2 + r^2)^{3/2}}
\]

\[
- \frac{0, 1r^2}{((x + 2)^2 + r^2)^{3/2}} - \frac{0, 06r^2}{((x + 1, 5)^2 + r^2)^{3/2}} = \text{const}.
\]

The solution \(\psi_1\) is obtained by means of three dipoles located at the points \((-2; 0), (-1, 5; 0), (0, 2; 0)\) and pseudodipoles located at the points \((0; \pm 1)\).

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