Multi-parameter deformations of the module of symbols of differential operators

B. Agrebaoui‡ F. Ammar§ V. Ovsienko¶

Abstract

The space of symbols of differential operators on a smooth manifold (i.e., the space of symmetric contravariant tensor fields) is naturally a module over the Lie algebra of vector fields. We study, in the case of $\mathbb{R}^n$ with $n \geq 2$, multi-parameter formal deformations of this module. The space of linear differential operators on $\mathbb{R}^n$ provides an important class of such formal deformations; we show, however, that the whole space of deformations is much larger.

‡Faculté des Sciences de Sfax, BP. 802 3018 Sfax Tunisie, B.Agreba@fss.rnu.tn
§Faculté des Sciences de Sfax, BP. 802 3018 Sfax Tunisie, Faouzi.Ammar@fss.rnu.tn
¶CNRS, Luminy Case 907, F–13288 Marseille, Cedex 9, France ovsienko@cpt.univ-mrs.fr
1 Introduction

The space of linear differential operators on tensor densities over a smooth manifold is naturally a module over the Lie algebra of vector fields. This module structure has been studied in a series of recent papers (see [6, 14, 4, 15, 11, 17, 3, 5] and references therein). The module of differential operators can be viewed as a deformation of the corresponding module of symbols; the general framework of the deformation theory (see e.g. [10, 13, 20, 9, 8]), therefore, relates its study to the cohomology of the Lie algebra of vector fields, cf. [6, 15].

The first cohomology space of the Lie algebra of vector fields, classifying the infinitesimal deformations of the module of symbols has been calculated, for an arbitrary smooth manifold, in [15] (see also [3] for the details in the one-dimensional case). Of course, not for every infinitesimal deformation there exists a formal deformation containing the latter as an infinitesimal part. The obstructions are characterized in terms of Nijenhuis-Richardson products of non-trivial first cohomology classes. The main problem considered in this paper is to determine the integrability condition, i.e., a necessary and sufficient condition for an infinitesimal deformation that guarantees existence of a formal deformation. We provide such a condition in the case of $\mathbb{R}^n$, where $n \geq 2$.

Let $\mathcal{F}_\lambda$ be the space of tensor densities of degree $\lambda \in \mathbb{R}$ on $\mathbb{R}^n$. The two-parameter family of Vect($\mathbb{R}^n$)-modules, $\mathcal{D}_{\lambda,\mu}$ of linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ will provide us with an important class of examples of non-trivial deformations of the module of symbols.

The classical deformation theory traditionally deals with one-parameter deformations (cf. [10, 13, 20]). We will study multi-parameter deformations and adopt here a modern viewpoint of miniversal deformations (see [8]). Our methods are similar to those of [19, 18]. The first cohomology space of Vect($\mathbb{R}^n$) has in our case a canonical basis; we consider a commutative algebra generated by the parameters of deformation, corresponding to all non-trivial cohomology classes. This allows us to consider the most general multi-parameter infinitesimal deformation. The obstructions for integrability of an infinitesimal deformation is expressed in terms of algebraic relations between the generators.

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2 The general framework

Let us start with the notion of (multi-parameter) deformations over a commutative algebra. Our approach will be similar to those of [19, 18]; it corresponds to the notion of miniversal deformations [8] in a special case when one can choose a basis of the first cohomology space.

2.1 Polynomial deformations

Let $g$ be a Lie algebra and $(V, \rho)$ a $g$-module, where $V$ is a vector space and $\rho$ is a homomorphism $\rho : g \to \text{End}(V)$. We will consider multi-parameter formal deformations, i.e., formal series

$$\rho(t) = \rho + \sum_{1 \leq m < \infty} \varphi_m(t)$$  \hspace{1cm} (2.1)

where $t = (t_1, \ldots, t_p)$ are the parameters of deformation and each term $\varphi_m(t)$ is a linear map $\varphi_m(t) : g \to \text{End}(V) \otimes \mathbb{C}[t]$ which is a homogeneous polynomial in $t$ of degree $m$. The expression $\rho(t)$ must satisfy the homomorphism condition, that is, for every $X, Y \in g$

$$\rho(t)([X, Y]) = [\rho(t)(X), \rho(t)(Y)]$$  \hspace{1cm} (2.2)

where the bracket in the right hand side stands for the standard commutator in $\text{End}(V)$ extended to the formal series $\text{End}(V) \otimes \mathbb{C}[[t]]$.

2.2 The Maurer-Cartan equation

The standard Chevalley-Eilenberg differential (see [9]) retains, in the case of linear maps from $g$ to $\text{End}(V)$ to the following formula. Given a linear map $a : g \to \text{End}(V)$, its differential $\delta a$ is the bilinear skew-symmetric map

$$\delta a(X, Y) = a([X, Y]) - \{\rho(X), a(Y)\} + \{\rho(Y), a(X)\}$$

The standard cup-product of linear maps $a, b : g \to \text{End}(V)$ is a bilinear map $[a, b] : g \otimes g \to \text{End}(V)$ defined by

$$[a, b](X, Y) = -[a(X), b(Y)] + [a(Y), b(X)]$$  \hspace{1cm} (2.3)

It is also called the Nijenhuis-Richardson product [13].
Put $\varphi(t) = \rho(t) - \rho$, one easily checks that the condition (2.2) reads

$$\delta \varphi(t) + \frac{1}{2} [\varphi(t), \varphi(t)] = 0$$

(2.4)

This is the Maurer-Cartan equation (also called the deformation equation, cf. [13]). Although it is equivalent to (2.2), it is useful to relate the deformations (2.1) with the cohomology theory.

### 2.3 Equivalent deformations

Two deformations $\rho(t)$ and $\rho'(t)$ are called *equivalent* if there exists an inner automorphism $I(t) : \text{End}(V) \otimes \mathbb{C}[[t]] \to \text{End}(V) \otimes \mathbb{C}[[t]]$ of the form

$$I(t) = \exp \left( \sum_{1 \leq i \leq p} t_i \text{ad}A_i + \sum_{1 \leq i,j \leq p} t_i t_j \text{ad}A_{ij} + \cdots \right),$$

(2.5)

where $A_i, A_{ij}, \ldots$ are some elements of $\text{End}(V)$, satisfying the relation

$$I(t) \rho(t) = \rho'(t).$$

(2.6)

### 2.4 Infinitesimal deformations and first cohomology

The first-order term $\varphi_1(t)$ of the expression (2.1) is called an infinitesimal deformation. It is of the form

$$\varphi_1(t) = t_1 c_1 + \cdots + t_p c_p.$$  

(2.7)

It is easy to check that the equation (2.4) implies that each linear map $c_i : \mathfrak{g} \to \text{End}(V)$ is a 1-cocycle (see e.g. [4] for the details). Furthermore, if $\rho(t)$ and $\rho'(t)$ are equivalent deformations, then the corresponding cocycles in the infinitesimal deformations are cohomologous, namely $c_i = c'_i + \delta A_i$. Therefore, an infinitesimal deformation is defined, up to equivalence, by the cohomology classes $[c_1], \ldots, [c_p]$ in $H^1(\mathfrak{g}; \text{End}(V))$.

It is natural to assume the classes $[c_1], \ldots, [c_p]$ linearly independent. Moreover, we will choose a basis of $H^1(\mathfrak{g}; \text{End}(V))$ and consider the most general multi-parameter deformation.
2.5 Obstructions and commutative algebras

Given an infinitesimal deformation of a $\mathfrak{g}$-module $V$, it is called integrable if there exists a formal deformation containing it as an infinitesimal part.

Developing (2.4), one obtains a recurrent system

$$
\delta \varphi_m(t) + \frac{1}{2} \sum_{i+j=m} [[\varphi_i(t), \varphi_j(t)]] = 0 \quad (2.8)
$$

The second-order term in (2.8) is $\delta \varphi_2(t) + \frac{1}{2}[[\varphi_1(t), \varphi_1(t)]] = 0$. The cohomology class of $[[\varphi_1(t), \varphi_1(t)]]$ is, therefore, an obstruction to existence of the second-order term $\varphi_2(t)$. It is an element of $H^2(\mathfrak{g}; \text{End}(V)) \otimes \mathbb{C}[t]$, where polynomial coefficients that are homogeneous second-order polynomials in $t$. For existence of $\varphi_2(t)$ it is necessary and sufficient that these obstructions vanish. One thus obtains second-order relations for the parameters $t_1, \ldots, t_p$.

In the same way, each term of the system (2.8) is a homogeneous polynomial of order $m$ in $t$. This leads to a system of algebraic relations on the formal parameters: $R_m(t) = 0$, where $m \geq 2$. To construct multi-parameter formal deformations of the form (2.1), one has to consider a commutative associative algebra generated by $t_1, \ldots, t_p$ such that all the relations, $R_m(t) = 0$ are satisfied.

A notion of versal deformation introduced in [8] is a universal object of the category of multi-parameter deformations. Any multi-parameter deformation can be obtained from the versal deformation as a homomorphism of the corresponding commutative algebras. If one chooses a basis $[c_1], \ldots, [c_p]$ in $H^1(\mathfrak{g}; \text{End}(V))$, the versal deformation corresponds to the commutative algebra

$$
A = \mathbb{C}[t_1, \ldots, t_p]/R,
$$

where $R$ is the ideal generated by the relations, $R_m(t) = 0$.

It worth noticing that up to now we were considering only deformations with a finite number of parameters (just as in the above definitions). However, following [8], we will include into the considerations the case of graded modules with infinitely many independent parameters of deformation.
3 Deformations of $\mathbb{Z}$-graded modules

Let us consider in more details a particular case when the module $(V, \rho)$ is split into a direct sum of $\mathfrak{g}$-modules:

$$V = \bigoplus_{k \in \mathbb{Z}} V_k$$  \hspace{1cm} (3.1)

Suppose that for some values $i \in \mathbb{Z}$ there exist non-trivial cocycles $c_i$ on $\mathfrak{g}$ with values in $\text{End}(V)$ such that for all $X \in \mathfrak{g}$ one has

$$c_i(X)|_{V_k} \subset V_{k-i}.$$  \hspace{1cm} (3.2)

Assume, furthermore, that there is a deformation of the form:

$$\rho(\tau) = \rho + \sum_{i \in \mathbb{Z}} \tau_i c_i + (\tau^2)$$  \hspace{1cm} (3.3)

where $\tau_i$ are the free parameters, i.e., the parameters generate the free commutative algebra $\mathbb{C}[\tau_i]$.

The following construction is meant to use the extra degrees of freedom given by the decomposition (3.1). We will add formal parameters indexed by $k \in \mathbb{Z}$. Consider each cocycle $c^k_i : \mathfrak{g} \to \text{Hom}(V_k, V_{k-i})$ defined by the restriction:

$$c^k_i(X) := c_i(X)|_{V_k}$$  \hspace{1cm} (3.4)

as independent.

**Proposition 3.1.** There exists a formal deformation of the form

$$\rho(t) = \rho + \sum_{i,k \in \mathbb{Z}} t^{k-i}_i t^k + (t^2)$$  \hspace{1cm} (3.5)

where $t^{k}_i$ are formal parameters satisfying the relation

$$t^{k-j}_i t^{k}_j = t^{k}_i t^{k-j}_i.$$  \hspace{1cm} (3.6)

**Proof.** The original deformation (3.3) satisfies the Maurer-Cartan equation (2.4). In each order $m$ the equation (2.8) for the deformation (3.3) has a solution $\rho_m(\tau) \in \text{Hom}(\mathfrak{g}; \text{End}(V)) \otimes \mathbb{C}[\tau]$ which is a homogeneous polynomial in $\tau$ of degree $m$. Replacing in $\rho_m(\tau)|_{V_k}$ each monomial $\tau_{i_1} \cdots \tau_{i_m} \tau_{i_m}$ by $t^{k-i_2-\cdots-i_m}_i t^{k-i_m-i_{m-1}}_i t^{k}_{i_m}$, one, obviously, gets a solution $\rho_m(t)$.
4 The main results

4.1 The space of symbols

Consider the Lie algebra Vect($\mathbb{R}^n$) of smooth vector fields on $\mathbb{R}^n$ and the space $\mathcal{S}$ of smooth symmetric contravariant tensor fields on $\mathbb{R}^n$. The space $\mathcal{S}$ is naturally isomorphic to the space of functions on $T^*\mathbb{R}^n$ polynomial on fibers. Clearly, $\mathcal{S}$ has a structure of a Poisson algebra with natural graduation

$$\mathcal{S} = \bigoplus_{k=0}^{\infty} \mathcal{S}_k,$$

where $\mathcal{S}_k$ is the space of $k$-th order tensor fields.

The space $\mathcal{S}$ is a Vect($\mathbb{R}^n$)-module since Vect($\mathbb{R}^n$) $\subset$ $\mathcal{S}$. In Darboux coordinates, the action of $X \in$ Vect($\mathbb{R}^n$) on $\mathcal{S}$ is given by the Hamiltonian vector field

$$L_X = \frac{\partial X}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{\partial X}{\partial x^i} \frac{\partial}{\partial \xi_i},$$

which is nothing but the Lie derivative of tensor fields.

The aim of this paper is to study multi-parameter formal deformations of this module. We will restrict our considerations to the multi-parameter formal deformations which are differentiable, i.e., each term in the formal series \( (2.1) \) supposed to be a differential operator on $\mathcal{S}$.

4.2 Description of the infinitesimal deformations

According to the general framework, one needs an information about the space of the first cohomology of Vect($\mathbb{R}^n$) with coefficients in End($\mathcal{S}$) in order to describe the infinitesimal deformations. The module End($\mathcal{S}$) is decomposed as follows:

$$\text{End}(\mathcal{S}) = \bigoplus_{k,\ell} \text{Hom}(\mathcal{S}_k, \mathcal{S}_\ell).$$

To study the Vect($\mathbb{R}^n$)-cohomology with coefficients in End($\mathcal{S}$) it then suffice to consider the cohomology with coefficients in each module Hom($\mathcal{S}_k, \mathcal{S}_\ell$). We will, furthermore, restrict ourself to the subspace $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell) \subset \text{Hom}(\mathcal{S}_k, \mathcal{S}_\ell)$ given by differential operators from $\mathcal{S}_k$ to $\mathcal{S}_\ell$.

1 Here and below sum over repeated indices is understood.
The space of first cohomology of the Lie algebra of vector fields with coefficients in \( D(S_k, S_\ell) \) has been calculated, for an arbitrary manifold \( M \) of \( \dim M \geq 2 \), in \([15]\).

We recall here the result in the case \( M = \mathbb{R}^n \).

\[
H^1(\text{Vect}(\mathbb{R}^n); D(S_k, S_\ell)) = \begin{cases} 
\mathbb{R}, & \text{if } k - \ell = 0, \quad k - \ell = 1 \text{ and } \ell \neq 0, \quad k - \ell = 2 \\
0, & \text{otherwise} 
\end{cases} \quad (4.3)
\]

One has, therefore, infinitely many non-trivial cohomology classes generating an infinitesimal deformation of the \( \text{Vect}(\mathbb{R}^n) \)-module \( \mathcal{S} \).

Let us give the explicit formulæ for corresponding 1-cocycles.

a) For all \( k \geq 0 \) there is a 1-cocycle with values in \( D(S_k, S_k) \) that associates to \( X \in \text{Vect}(\mathbb{R}^n) \) the operator of multiplication by the function

\[
c_0(X) = \text{Div}(X) \quad (4.4)
\]

b) For all \( k \geq 2 \) there is a 1-cocycle with values in \( D(S_k, S_{k-1}) \) given by

\[
c_1(X) = \frac{\partial^2 X}{\partial x^i \partial x^j} \frac{\partial^2}{\partial \xi_i \partial \xi_j} \quad (4.5)
\]

**Remark.** More geometrically, this cocycle can be written as the Lie derivative of the (flat) connection on \( \mathbb{R}^n \), namely, \( c_1(X) = L_X(\nabla) \).

c) For all \( k \geq 2 \) there is a 1-cocycle with values in \( D(S_k, S_{k-2}) \) given by

\[
c_2(X) = \frac{\partial^3 X}{\partial x^i \partial x^j \partial x^l} \frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_l} - 3 \frac{\partial^3 X}{\partial x^i \partial x^j \partial \xi_l} \frac{\partial^2}{\partial \xi_i \partial \xi_j} \frac{\partial}{\partial x^l} \quad (4.6)
\]

**Remark.** This cocycle is related to the famous Moyal product, namely for \( P \in S_k \), \( c_2(X)(P) \) coincides with the trird-order term in the Moyal product of \( X \) and \( P \).

As in Section 3, we will use the notation

\[
c_i^k = c_i|_{S_k}, \quad i = 0, 1, 2
\]

and deal with independent cocycles \( c_0^k, c_1^k, c_2^k \).
4.3 Integrability condition

According to the results of [15] (see Section 4.2), the infinitesimal deformations of the Lie derivative (4.2) are of the form:

\[ \varphi_1(t) = L_X + \varphi_1(t)(X) \]

with

\[ \varphi_1(t) = \sum_{0 \leq k < \infty} t_0^k c_0^k + \sum_{2 \leq k < \infty} (t_1^k c_1^k + t_2^k c_2^k) \]  \hspace{1cm} (4.7)

where the symbols \( t_0^k, t_1^k, t_2^k \) stand for independent formal parameters. We, therefore, have to deal with infinitesimal deformations with infinite number of parameters.

Let us formulate the main result of this paper.

**Theorem 4.1.** The following relations

a) one series of second-order relation \( R_2^k(t) \):

\[ t_1^k t_2^{k-1} - t_1^{k-2} t_2^k = 0 \quad , \quad k \geq 4 \]  \hspace{1cm} (4.8)

b) two series of third-order relations, namely \( R_3^k(t) \):

\[ (t_0^k - t_0^{k-1}) t_1^k t_2^{k-1} = 0 \quad , \quad k \geq 3 \]  \hspace{1cm} (4.9)

and \( \widetilde{R}_3^k(t) \):

\[ (t_0^k - t_0^{k-2}) t_1^k t_2^{k-2} = 0 \quad , \quad k \geq 4 \]  \hspace{1cm} (4.10)

are necessary and sufficient for integrability of the infinitesimal deformation (4.7).

The following statement can be considered as a corollary of Theorem 4.1, but we will give its elementary proof.

**Proposition 4.2.** An infinitesimal deformation (4.4) with additional series of relations: \( t_2^k = 0 \) for all \( k \), is integrable without any condition on \( t_0^k \) and \( t_1^k \).

**Proof.** This is an immediate consequence of the fact that all the Richardson-Nijenhuis products of the two first non-trivial cohomology classes vanish:

\[ [c_0, c_0] = [c_0, c_1] = [c_1, c_1] = 0 \]  \hspace{1cm} (4.11)

and so do the obstructions.

\[ \square \]

The proof that the relations (4.8)-(4.10) are necessary is just a result of a straightforward computation; it will be given in Section 6.1. The proof that these conditions are sufficient is based on the existence of an important class of deformations corresponding to the \( \text{Vect}(\mathbb{R}^n) \)-modules of differential operators.
5 Module of differential operators

Consider the space $\mathcal{D}$ of linear differential operators on $\mathbb{R}^n$. It is isomorphic to $\mathcal{S}$ as a vector space, but its structure as a Vect($\mathbb{R}^n$)-module is quite different. In this section we interpret $\mathcal{D}$ as a deformation of the Vect($\mathbb{R}^n$)-module $\mathcal{S}$.

5.1 Lie derivative of differential operators

The composition of differential operators is defined by:

$$A \circ B = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k A}{\partial \xi_{i_1} \cdots \partial \xi_{i_k}} \frac{\partial^k B}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

(5.1)

Of course, since $A$ is a polynomial in $\xi$, there are only finite number of terms in this sum. There is a filtration of the associative algebra $\mathcal{D}$

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \cdots \subset \mathcal{D}^r \subset \cdots,$$

(5.2)

where $\mathcal{D}^r$ is the space of $r$-th order differential operators (isomorphic to $\bigoplus_{i \leq r} \mathcal{S}_i$ as a vector space). One has $\mathcal{S} = \text{gr}\mathcal{D}$ as well as an associative algebra and as a Lie algebra. The space $\mathcal{S}$ is usually called the space of symbols associated to $\mathcal{D}$.

The space $\mathcal{D}$ is a Vect($\mathbb{R}^n$)-module since Vect($\mathbb{R}^n$) is a Lie subalgebra of $\mathcal{D}$. Moreover, there is a family of embeddings Vect($\mathbb{R}^n$) $\hookrightarrow \mathcal{D}$ depending on a parameter $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) given by

$$i^\lambda : X \mapsto X + \lambda \text{Div}(X)$$

where $X \in \text{Vect}(\mathbb{R}^n)$ and $\text{Div}(X)$ is the divergence with respect to the standard volume form on $\mathbb{R}^n$. This defines a one-parameter family of Vect($\mathbb{R}^n$)-module structures on the space $\mathcal{D}$. More generally, one can define a two-parameter family of Vect($\mathbb{R}^n$)-modules on $\mathcal{D}$ by

$$\mathcal{L}_X^{\lambda,\mu}(A) = i^\mu(X) \circ A - A \circ i^\lambda(X)$$

(5.3)

These modules are denoted $\mathcal{D}_{\lambda,\mu}$.

Remark. From the geometrical viewpoint, the module $\mathcal{D}_{\lambda,\mu}$ is the space of differential operators acting on the space of tensor densities (cf. [4, 6, 15, 5]); the first-order differential operator $i^\lambda(X)$ is a Lie derivative of tensor densities of degree $\lambda$. 

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Lemma 5.1. The explicit formula of the \(\text{Vect}(\mathbb{R}^n)\)-action on \(\mathcal{D}_{\lambda,\mu}\) is

\[
\mathcal{L}^\mu_\lambda_X = L_X + (\mu - \lambda)\text{Div}(X)
\]

\[
- \sum_{k=2}^{\infty} \frac{1}{k!} \left( \frac{\partial^k X}{\partial x_{i_1} \cdots \partial x_{i_k}} \frac{\partial^k}{\partial \xi_{i_1} \cdots \partial \xi_{i_k}} + k\lambda \frac{\partial^{k-1} \text{Div}(X)}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}} \frac{\partial^{k-1}}{\partial \xi_{i_1} \cdots \partial \xi_{i_{k-1}}} \right)
\]

where \(L_X\) is as in (4.2).

Proof. This formula readily follows from (5.1).

\[\square\]

5.2 The Weyl symbols

Consider the operator \(\text{Div}\) on \(\mathcal{D}_{\lambda,\mu}\) given by

\[\text{Div} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial \xi_i}\]

that extends the divergence of vector fields to the space of symmetric contravariant tensor fields. Recall that the linear map

\[\exp (\lambda \text{Div}) : \mathcal{D} \to \mathcal{S}\]

defines the famous Weyl symbol of a differential operator (see [2]). Note that the parameter in this formula is usually interpreted in terms of the Planck constant, namely \(\lambda = \frac{ih}{2}\).

Lemma 5.2. The action (5.4) becomes after the transformation (5.6) as follows: The action \(\tilde{\mathcal{L}}^\mu_\lambda\) is of the form

\[
\tilde{\mathcal{L}}^\mu_\lambda_X = L_X + \tau_0 c_0(X) + \tau_1 c_1(X) + \tau_2 c_2(X) + \sum_{m \geq 3} L_m(X)
\]

with

\[
\tau_0 = \mu - \lambda, \quad \tau_1 = \lambda - \frac{1}{2}, \quad \tau_2 = \lambda(\lambda - 1).
\]

where \(L_m(X)\) are the terms with the degree shift \(m\), that is, for the operators from \(\mathcal{S}_k\) to \(\mathcal{S}_\ell\) with \(\ell - k = m\).

Proof. By definition, \(\tilde{\mathcal{L}}^\mu_\lambda = \exp(-\lambda \text{Div}) \circ \mathcal{L}^\mu_\lambda \circ \exp(\lambda \text{Div})\), a straightforward computation then yields (5.7) and (5.8).

\[\square\]

This new expression of the \(\text{Vect}(\mathbb{R}^n)\)-action on \(\mathcal{D}_{\lambda,\mu}\) allows us to consider this module as a deformation of \(\mathcal{S}\).
5.3 Differential operators and formal deformations

The modules $D_{\lambda,\mu}$ allow us to prove the existence of a big class of formal deformations. The idea is to consider the parameters $\tau_0, \tau_1, \tau_2$ as independent using the fact that the expressions (5.8) does not satisfy any non-trivial homogeneous relation.

**Lemma 5.3.** There exists a (formal) deformation of the form (5.7) such that the parameters $\tau_0, \tau_1, \tau_2$ are independent.

**Proof.** Let us use the existence of modules $D_{\lambda,\mu}$. Each term $L_m$ in (5.7) polynomially depends on $\tau_0, \tau_1, \tau_2$. The operator $L_X^{\lambda,\mu}$ defines a Vect($\mathbb{R}^n$)-action and, so, satisfies the homomorphism condition (2.2). A term of degree of shift $m$ in (2.2) is again a polynomial in $\tau_0, \tau_1, \tau_2$, more precisely, a sum of the terms

$$\tau_0^{m_0} \tau_1^{m_1} \tau_2^{m_2}, \quad m_1 + 2m_2 = m \quad (5.9)$$

with operator coefficients. But, all the monomials (5.9) with $\tau_0, \tau_1, \tau_2$ given by (5.8) are, obviously, linearly independent and, so, the equation (2.2) has to be satisfied independently for the operator coefficients of all monomials (5.9). These conditions are therefore independent on $\tau_0, \tau_1, \tau_2$. \qed

Applying the construction from Section 3 to obtain a formal deformation with the infinitesimal part of the form (4.7), one then obtains the following intermediate result.

**Proposition 5.4.** The following relations:

$$t_1^k t_2^k - t_1^{k-2} t_2^k = 0, \quad k \geq 4 \quad (5.10)$$

$$(t_0^k - t_0^{k-1}) t_1^k = 0, \quad k \geq 3 \quad (5.11)$$

$$(t_0^k - t_0^{k-2}) t_2^k = 0, \quad k \geq 4 \quad (5.12)$$

are sufficient for integrability of the infinitesimal deformation (4.4).

**Proof.** The conditions (5.10)-(5.12) coincide with the conditions (3.6) from Proposition 3.1 that are sufficient for integrability. \qed

**Remark.** The conditions (5.11) and (5.12) are slightly stronger than (4.9) and (4.10) respectively. So, the ideal generated by these polynomials in (5.10)-(5.12) is bigger than the one generated by $R_2(t), R_3(t)$ and $R'_3(t)$. Therefore, the formal deformation naturally related to the modules of differential operators turns out to be not the most general one. In other words, it is not a versal deformation in the sense of [8].
6 Proof of the main theorem

The proof contains two parts. First, we show by a straightforward computation that the conditions (4.8) - (4.10) are necessary. Second, we use the existence of the deformation constructed in the preceding section to prove that these conditions are, indeed, sufficient.

6.1 The origin of the integrability conditions

Let us give here the details in the case of quadratic relation (4.8).

It suffice to look for the solutions of the Maurer-Cartan equation which are homogeneous with respect to the partial derivatives in \(x\) and \(\xi\). More precisely, one has

**Lemma 6.1.** If there is a solution of the equation (2.8), then there exists one of the form:

\[
\varphi_m = \sum_{0 \leq t \leq s \leq 3m} \left( \alpha_{st}^k \frac{\partial^{s-t}X}{\partial x_{i_1} \cdots \partial x_{i_{s-t}}} \frac{\partial^t}{\partial x_{i_{s-t+1}} \cdots \partial x_{i_s}} \frac{\partial^s}{\partial \xi_{i_1} \cdots \partial \xi_{i_s}} 
+ \beta_{st}^k \frac{\partial^{s-t+1}X}{\partial x_{i_1} \cdots \partial x_{i_{s-t}} \partial \xi_{i_1}} \frac{\partial^t}{\partial x_{i_{s-t+1}} \cdots \partial x_{i_s} \partial \xi_{i_2}} \frac{\partial^s}{\partial \xi_{i_3} \cdots \partial \xi_{i_s}} 
+ \gamma_{st}^k \frac{\partial^{s-t}X}{\partial x_{i_1} \cdots \partial x_{i_{s-t}} \partial \xi_{i_{s-t}}} \frac{\partial^{t+1}}{\partial x_{i_{s-t+1}} \cdots \partial x_{i_s} \partial \xi_{i_{s-t+1}}} \frac{\partial^s}{\partial \xi_{i_3} \cdots \partial \xi_{i_s} \partial \xi_{i_{s-t+2}}} \right) 
\] (6.1)

**Proof.** The cocycles (4.4)-(4.6) are precisely of this form. The cup product (2.3) of two such linear maps is a bilinear map which is also homogeneous in \(x\) and \(\xi\). Finally, the coboundary operator \(\delta\) preserves the homogeneity in the same way. □

**Remark.** The formula (6.1) is the most general differential operator on \(S\) which is invariant with respect to the \(\text{GL}(n, \mathbb{R})\)-action on \(\mathbb{R}^n\).

The Maurer-Cartan equation (2.8) in the second order reads:

\[
\delta \varphi_2(t) |_{S_k} = -\frac{1}{2} \sum_{i,j} t_{i}^{k-j} t_{j}^{k} [c_{i}^{k-j}, c_{j}^{k}] 
\] (6.2)
Obviously, \([c_0, c_0] = [c_0, c_1] = 0\). The non-zero cup products are

\[
\frac{1}{2}[c_1, c_1](X, Y) = -2 \frac{\partial^2 X}{\partial x^2 \partial x^j} \frac{\partial^4 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} - (X \leftrightarrow Y)
\]

\[
[c_0, c_2](X, Y) = 3 \frac{\partial^3 X}{\partial x^2 \partial x^j} \frac{\partial^3 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial^2}{\partial \xi_m \partial \xi_p} - (X \leftrightarrow Y)
\]

\[
[c_1, c_2](X, Y) = -2 \frac{\partial^2 X}{\partial x^2 \partial x^j} \frac{\partial^4 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial^4}{\partial \xi_m \partial \xi_p} + 6 \frac{\partial^3 X}{\partial x^2 \partial x^j \partial \xi_k} \frac{\partial^3 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial^4}{\partial \xi_m \partial \xi_p} - 6 \frac{\partial^3 X}{\partial x^2 \partial x^j \partial \xi_k} \frac{\partial^3 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial^4}{\partial \xi_m \partial \xi_p} -(X \leftrightarrow Y)
\]

\[
\frac{1}{2}[c_2, c_2](X, Y) = -6 \frac{\partial^3 X}{\partial x^2 \partial x^j \partial \xi_k} \frac{\partial^3 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial}{\partial \xi_m \partial \xi_p} \frac{\partial^4}{\partial \xi_m \partial \xi_p} + 9 \frac{\partial^3 X}{\partial x^2 \partial x^j \partial \xi_k} \frac{\partial^3 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial}{\partial \xi_m \partial \xi_p} \frac{\partial^4}{\partial \xi_m \partial \xi_p} - 3 \frac{\partial^3 X}{\partial x^2 \partial x^j \partial \xi_k} \frac{\partial^3 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial}{\partial \xi_m \partial \xi_p} \frac{\partial^4}{\partial \xi_m \partial \xi_p} - 6 \frac{\partial^3 X}{\partial x^2 \partial x^j \partial \xi_k} \frac{\partial^3 Y}{\partial x^l \partial x^p \partial \xi_j \partial \xi_l} \frac{\partial}{\partial \xi_m \partial \xi_p} \frac{\partial^4}{\partial \xi_m \partial \xi_p} -(X \leftrightarrow Y)
\]

as well as \([c_2, c_0] = [c_0, c_2]\) and \([c_2, c_1] = [c_1, c_2]\).

The second-order term of a formal deformation is: \(\varphi_2(t)(X) = \sum t_i t_j \varphi_{ij}^k(X)\), where \(\varphi_{ij}^k(X)\) are differential operators on \(S\) of the form (6.1) homogeneous with respect to the partial derivatives in \(x\) and in \(\xi\) of degree \(i + j + 2\). Tedious but direct computation yields:

\[
\varphi_2(t)(X) = \alpha^k \frac{\partial^2 X}{\partial x^2 \partial x^j \partial \xi_i \partial \xi_l} \frac{\partial^4}{\partial \xi_m \partial \xi_p} + \beta^k \frac{\partial^3 X}{\partial x^2 \partial x^j \partial x^l \partial \xi_i \partial \xi_l} \frac{\partial^4}{\partial \xi_m \partial \xi_p} + \gamma^k \frac{\partial^4 X}{\partial x^2 \partial x^j \partial x^l \partial \xi_i \partial \xi_l} \frac{\partial^4}{\partial \xi_m \partial \xi_p}
\]

where the coefficients \(\alpha^k, \beta^k, \gamma^k\) are quadratic polynomials in \(t_i^k\) satisfying the following system.
This system has a unique solution if and only if the condition (4.8) is satisfied. This proves that this condition is necessary for existence of the second order term \( \varphi_2(t) \).

The proof in the case of (4.9) and (4.10) are analogous but one has to consider the third-order terms in (2.8).

### 6.2 The conditions of integrability are sufficient

Let us show that the conditions (4.8)-(4.10) are, indeed, sufficient.

Let us suppose that there is a condition of integrability in order \( m \), i.e., a relation \( R_m(t) = 0 \), where \( R_m(t) \) is a homogeneous polynomial of degree \( m \) in \( t_0^k, t_1^k, t_2^k \). One has to prove that the polynomial \( R_m(t) \) belongs to the ideal, \( R \), generated by the relations (4.8)-(4.10).

Proposition 5.4 insures that \( R_m(t) \) belongs to the ideal generated by the polynomials in (5.10)-(5.12). Therefore, \( R_m(t) \) is split into a sum: \( R_m(t) = R_{m,1}(t) + R_{m,2}(t) + R_{m,3}(t) \) of polynomials divisible by (5.10), (5.11) and (5.12) respectively.

The polynomial \( R_{m,1}(t) \) already belongs to \( R \).

Consider, the second term \( R_{m,2}(t) \). A direct computation (cf. Section 6.1) shows that the only second-order condition is (4.8), one then can assume \( m \geq 3 \). Then, the relation \([c_0, c_1] = 0 \) implies that each monomial in \( R_m(t) \) has to contain some parameter \( t_2^k \) as a multiple (cf. Proposition 1.2). By assumption, the polynomial \( R_{m,2}(t) \) is a multiple of \((t_0^k - t_0^{k-1}) t_1^k \) for some \( k \). But, modulo the relation (4.8), any expression of the form \((t_0^k - t_0^{k-1}) t_1^k \cdots t_2^k \) is divisible by (4.10) and so \( R_{m,2}(t) \), indeed, belongs to \( R \).

Since the Nijenius-Richardson product \([c_0, c_2] \) commutes with \( c_0 \), then \( R_{m,3}(t) \) has to contain the terms of the form \((t_0^k - t_0^{k-2}) t_2^k \cdots t_1^k \) or \((t_0^k - t_0^{k-2}) t_2^k \cdots t_2^k \). But, using the relation (4.8) one readily gets that these terms are divisible by (4.9) and (4.10) respectively and, therefore, belong to \( R \).
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