The (revised) Szeged index and the
Wiener index of a nonbipartite graph

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Abstract

Hansen et. al. used the computer programm AutoGraphiX to study the
differences between the Szeged index $Sz(G)$ and the Wiener index $W(G)$,
and between the revised Szeged index $Sz^*(G)$ and the Wiener index for a
connected graph $G$. They conjectured that for a connected nonbipartite
graph $G$ with $n \geq 5$ vertices and girth $g \geq 5$, $Sz(G) - W(G) \geq 2n - 5$.
Moreover, the bound is best possible as shown by the graph composed
of a cycle on 5 vertices, $C_5$, and a tree $T$ on $n - 4$ vertices sharing a
single vertex. They also conjectured that for a connected nonbipartite
graph $G$ with $n \geq 4$ vertices, $Sz^*(G) - W(G) \geq \frac{n^2 + 4n - 6}{4}$. Moreover, the
bound is best possible as shown by the graph composed of a cycle on 3
vertices, $C_3$, and a tree $T$ on $n - 3$ vertices sharing a single vertex. In
this paper, we not only give confirmative proofs to these two conjectures
but also characterize those graphs that achieve the two lower bounds.

Keywords: Wiener index, Szeged index, revised Szeged index.

AMS subject classification 2010: 05C12, 05C35, 05C90, 92E10.

1 Introduction

All graphs considered in this paper are finite, undirected and simple. We refer the
readers to [2] for terminology and notation. Let $G$ be a connected graph with vertex
set $V(G)$ and edge set $E(G)$. For $u, v \in V$, $d_G(u, v)$ denotes the distance between $u$
and $v$ min $G$. The Wiener index of $G$ is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v).$$

This topological index has been extensively studied in the mathematical literature; see,
e.g., [3,7]. Let $e = uv$ be an edge of $G$, and define three sets as follows:

$$N_u(e) = \{w \in V : d_G(u,w) < d_G(v,w)\},$$
\[ N_v(e) = \{ w \in V : d_G(v, w) < d_G(u, w) \}, \]
\[ N_0(e) = \{ w \in V : d_G(u, w) = d_G(v, w) \}. \]

Thus, \( \{N_u(e), N_v(e), N_0(e)\} \) is a partition of the vertices of \( G \) respect to \( e \). The number of vertices of \( N_u(e), N_v(e) \) and \( N_0(e) \) are denoted by \( n_u(e), n_v(e) \) and \( n_0(e) \), respectively.

A long time known property of the Wiener index is the formula \([6, 14]\):

\[ W(G) = \sum_{e=uv \in E(G)} n_u(e)n_v(e), \]

which is applicable for trees. Motivated the above formula, Gutman \([4]\) introduced a graph invariant, named as the \textit{Szeged index} as an extension of the Wiener index and defined by

\[ Sz(G) = \sum_{e=uv \in E(G)} n_u(e)n_v(e). \]

Randić \([12]\) observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge, and so he conceived a modified version of the Szeged index which is named as the \textit{revised Szeged index}. The revised Szeged index of a connected graph \( G \) is defined as

\[ Sz^*(G) = \sum_{e=uv \in E(G)} \left( n_u(e) + \frac{n_0(e)}{2} \right) \left( n_v(e) + \frac{n_0(e)}{2} \right). \]

Some properties and applications of the Szeged index and the revised Szeged index have been reported in \([1, 3, 9–11, 15]\).

We have known that for a connected graph, \( Sz^*(G) \geq Sz(G) \geq W(G) \). It is easy to see that \( Sz^*(G) = Sz(G) = W(G) \) if \( G \) is a tree, which means \( m = n - 1 \). So we want to know the differences between \( Sz(G) \) and \( W(G) \), and between \( Sz^*(G) \) and \( W(G) \) for a connected graph with \( m \geq n \).

In \([8]\) Hansen et. al. used the computer programm AutoGraphiX and made the following conjectures:

**Conjecture 1.1** Let \( G \) be a connected bipartite graph with \( n \geq 4 \) vertices and \( m \geq n \) edges. Then

\[ Sz(G) - W(G) \geq 4n - 8. \]

Moreover, the bound is best possible as shown by the graph composed of a cycle on 4 vertices \( C_4 \) and a tree \( T \) on \( n - 3 \) vertices sharing a single vertex.

**Conjecture 1.2** Let \( G \) be a connected bipartite graph with \( n \geq 4 \) vertices and \( m \geq n \) edges. Then

\[ Sz^*(G) - W(G) \geq 4n - 8. \]

Moreover, the bound is best possible as shown by the graph composed of a cycle on 4 vertices \( C_4 \) and a tree \( T \) on \( n - 3 \) vertices sharing a single vertex.
Conjecture 1.3 Let $G$ be a connected graph with $n \geq 5$ vertices and girth $g \geq 5$ and with an odd cycle. Then

$$Sz(G) - W(G) \geq 2n - 5.$$ 
Moreover, the bound is best possible as shown by the graph composed of a cycle on 5 vertices $C_5$ and a tree $T$ on $n - 4$ vertices sharing a single vertex.

Conjecture 1.4 Let $G$ be a connected graph with $n \geq 4$ vertices and $m \geq n$ edges and with an odd cycle. Then

$$Sz^*(G) - W(G) \geq \frac{n^2 + 4n - 6}{4}.$$ 
Moreover, the bound is best possible as shown by the graph composed of a cycle on 3 vertices $C_3$ and a tree $T$ on $n - 3$ vertices sharing a single vertex.

In [3] we showed that both Conjecture 1.1 and 1.2 are true. In this paper, we will give confirmative proofs to Conjecture 1.3 and Conjecture 1.4. During the proof of Conjecture 1.3 we find another case which also makes the equality holds, that is the graph composed of a cycle on 5 vertices, $C_5$, and two trees with roots $v_1, v_2$ in $C_5$ satisfying $v_1v_2 \in E(C_5)$. So we get the following theorem:

Theorem 1.5 Let $G$ be a connected nonbipartite graph on $n \geq 5$ vertices and girth $g \geq 5$. Then

$$Sz(G) - W(G) \geq 2n - 5.$$ 
Equality holds if and only if $G$ is composed of a cycle $C_5$ on 5 vertices, and one tree rooted at a vertex of the cycle $C_5$, or two trees, respectively, rooted at two adjacent vertices of the cycle $C_5$.

We notice that the method used in the proof of Theorem 1.5 can also be used to prove the bipartite case, and therefore this gives another proof to Conjecture 1.1 other than that in [3].

2 Main results

We start this section with two definitions that are needed in our later proofs frequently.

**Definition 1.** Let $P$ be a shortest path between two vertices $x$ and $y$ in a graph $G$, $P'$ another path from $x$ to $y$ in $G$. We call $P'$ the second shortest path between $x$ and $y$, if $P' \neq P$, $|P'|-|P|$ is minimum, and if there are more than one path satisfying the condition, we choose $P'$ as a one with the most common vertices with $P$ in $G$.

**Definition 2.** A subgraph $H$ of a graph $G$ is called isometric if distance between any pair of vertices in $H$ is the same as their distance in $G$. 


In [13] Gutman gave another expression for the Szeged index:

$$sz(G) = \sum_{e = uv \in E(G)} n_u(e)n_v(e) = \sum_{e = uv \in E(G)} \sum_{x \leq V(G)} \mu_{x,y}(e)$$

where \(\mu_{x,y}(e)\), interpreted as contribution of the vertex pair \(x\) and \(y\) to the product \(n_u(e)n_v(e)\), is defined as follows:

$$\mu_{x,y}(e) = \begin{cases} 1, & \text{if } d_G(x, u) < d_G(x, v) \quad \text{and} \quad d_G(y, v) < d_G(y, u), \\ 0, & \text{otherwise.} \end{cases}$$

From above expressions, we know that

$$sz(G) - w(G) = \sum_{\{x, y\} \leq V(G)} \sum_{e \in E(G)} \mu_{x,y}(e) - \sum_{\{x, y\} \leq V(G)} d_G(x, y)$$

$$= \sum_{\{x, y\} \leq V(G)} \left( \sum_{e \in E(G)} \mu_{x,y}(e) - d_G(x, y) \right).$$

For convenience, let \(\pi(x, y) = \sum_{e \in E(G)} \mu_{x,y}(e) - d_G(x, y)\).

Let \(G\) be a connected graph. For every pair \(\{x, y\} \subseteq V(G)\), let \(P_1\) be the shortest path between \(x\) and \(y\). We know that for all \(e \in E(P_1)\), \(\mu_{x,y}(e) = 1\), which means that \(\pi(x, y) \geq 0\). Let \(P_2\) be the second shortest path between \(x\) and \(y\) (if there exists). Then \(P_1 \Delta P_2 = C\), where \(C\) is a cycle. Let \(P'_1 = P_1 \cap C = x'P_2y'\). If \(E(P_1) \cap E(P_2) = \emptyset\), then \(x' = x, y' = y\).

Now we have the following lemma.

**Lemma 2.1** For every pair \(\{x, y\} \subseteq V(G)\), and \(C, x', y'\) defined as above,

(1) if \(C\) is an even cycle, then \(\pi(x, y) \geq d_C(x', y') \geq 1\);

(2) if \(C\) is an odd cycle and \(d_C(x', y') \geq 2\), then \(\pi(x, y) \geq 1\).

**Proof.** Firstly, we prove that for every \(v \in V(C)\), \(d_C(x', v) = d_G(x', v)\). If \(v \in P'_1\), it is simply true; otherwise, we can find a shorter path between \(x'\) and \(y'\), and then we can find a shorter path between \(x\) and \(y\). If \(v \in P'_2\) and \(d_C(x', v) > d_G(x', v) = |E(P_3)|\), where \(P_3\) is a shortest path between \(x'\) and \(v\) in \(G\), then the path \(xP_2x'P_3vP_2y'P_2y\) between \(x\) and \(y\) is shorter than \(P_2\), a contradiction. For the same reason, we have \(d_C(y', v) = d_G(y', v)\) for all \(v \in V(C)\). Similarly, it is easy to see that a shortest path from \(x\) (or \(y\)) to the vertex \(v\) in \(P'_2\) is \(xP_2x'(yP_2y')\) together with a shortest path from \(x'(y')\) to \(v\) in \(C\). So, if an edge \(e\) in \(E(C)\) makes \(\mu_{x', y'}(e) = 1\), then we have \(\mu_{x,y}(e) = 1\).
Lemma 2.2
For every pair \(\{x, y\} \subseteq V(C)\), where \(C\) is an isometric cycle,

1. if \(C\) is an even cycle, then \(\pi(x, y) \geq d_C(x, y) \geq 1\);
2. if \(C\) is an odd cycle and \(d_C(x, y) \geq 2\), then \(\pi(x, y) \geq 1\).

Next we show that \(d_C(x_1, y_1) \geq 1\). From equations (2.1) and (2.2), we have

\[
d_C(x', x_1) = d_C(x', x_2),
\]
\[
d_C(y', y_1) = d_C(y', y_2).
\]

Let \(d_C(x_1, y_1) = \min\{d_C(x_i, y_j), i, j \in \{1, 2\}\}\). For every edge \(e\) in a shortest path between \(x_1\) and \(y_1\), we have \(\mu_{x, y}(e) = \mu_{x', y'}(e) = 1\). So, \(\sum_{e \in E(G)} \mu_{x, y}(e) \geq d_C(x, y) + d_C(x_1, y_1)\), which means that \(\pi(x, y) \geq d_C(x, y)\).

Next we show that \(d_C(x_1, y_1) \geq 1\). From equations (2.1) and (2.2), we have

\[
d_C(x', x_1) = d_C(x', y') + d_C(y', x_1) - 1,
\]
\[
d_C(y', y_1) = d_C(x', y') + d_C(x', y_1) - 1.
\]

If \(d_C(x_1, y_1) = 0\), that is \(x_1 = y_1\), then by adding the above two equations, we get
\[d_C(x', y') = 1,\]
which contradicts the assumption \(d_C(x', y') \geq 2\).

From the proof of Lemma 2.1, we also get the following lemma.

Lemma 2.2 For every pair \(\{x, y\} \subseteq V(C)\), where \(C\) is an isometric cycle,

1. if \(C\) is an even cycle, then \(\pi(x, y) \geq d_C(x, y) \geq 1\);
2. if \(C\) is an odd cycle and \(d_C(x, y) \geq 2\), then \(\pi(x, y) \geq 1\).

Now, we give a confirmative proof of Theorem 1.5.

Proof of Theorem 1.5. Let \(C = v_1v_2\cdots v_kv_1\) be a shortest odd cycle of \(G\) with length \(k\), where \(k \geq g \geq 5\). It is obvious that \(C\) is an isometric cycle. We consider the pair \(\{x, y\} \subseteq V(G)\).

Case 1. \(\{x, y\} \subseteq V(C)\).

If \(d_C(x, y) \geq 2\), then by Lemma 2.2, we have \(\pi(x, y) \geq 1\). Otherwise, \(\pi(x, y) \geq 0\). Therefore,
\[
\sum_{\{x, y\} \subseteq V(C)} \pi(x, y) \geq \binom{k}{2} - k.
\]

Case 2. \(x \in V(C), y \in V(G)\backslash V(C)\).
We will prove that for every \( y \in V(G) \setminus V(C) \), there exist two vertices \( x_1, x_2 \) in \( C \) such that \( \pi(x_1, y) \geq 1 \) and \( \pi(x_2, y) \geq 1 \).

Assume that \( v_i \) is the vertex on \( C \) such that \( d_{G}(v_i, y) = \min_{v \in V(C)} d_{G}(v, y) \), and \( P_1 \) is a shortest path between \( v_i \) and \( y \). Let \( |E(P_1)| = p_1 \). It is obvious that \( P_1 \) does not contain any vertex in \( C \).

Now we show that \( \pi(v_i+2, y) \geq 1 \). Since \( P_2 = yP_1v_{i+1}v_{i+2} \) is a path from \( y \) to \( v_{i+2} \),

\[
p_1 = d_{G}(v_i, y) \leq d_{G}(v_{i+2}, y) = p_1 + 2.
\]

**Subcase 2.1.** \( d_{G}(v_{i+2}, y) = p_1 + 2 \).

In this case, \( P_2 \) is a shortest path from \( y \) to \( v_{i+2} \). Let \( P_3 \) be a second shortest path between \( y \) and \( v_{i+2} \), \( C_1 = P_2 \triangle P_3 \), \( C_1 \cap P_2 \cap P_3 = \{x', y'\} \). By Lemma \[2.1\] \( \pi(v_{i+2}, y) \geq 1 \) except for the case that \( C_1 \) is an odd cycle and \( d_{C_1}(x', y') = 1 \). In this case, the length of \( P_3 \) is \( (p_1 + 2) + |C_1| - 2 = p_1 + |C_1| \), which is not less than \( p_1 + k \).

Consider the path \( yP_1v_{i+1}v_{i-2} \cdots v_{i+2} \). It is a path between \( y \) and \( v_{i+2} \), and its length is \( p_1 + (k - 2) < p_1 + k \), contrary to the choice of \( P_3 \).

**Subcase 2.2.** \( p_1 \leq d_{G}(v_{i+2}, y) < p_1 + 2 \).

Let \( P_2' \) be a shortest path from \( y \) to \( v_{i+2} \), and \( P_3' \) a second shortest path between \( y \) and \( v_{i+2} \). Let \( C_1' = P_2' \triangle P_3', C_1' \cap P_2' \cap P_3' = \{x', y'\} \). If \( P_3' = P_2 \), since \( g \geq 5 \) and \( |E(P_2')| \geq |E(P_1)| \), then \( d_{C_1'}(x', y') \geq 2 \), and by Lemma \[2.1\] we have \( \pi(v_{i+2}, y) \geq 1 \). If \( P_3' \neq P_2 \), by Lemma \[2.1\] \( \pi(v_{i+2}, y) \geq 1 \) except for the case that \( C_1' \) is an odd cycle and \( d_{C_1'}(x', y') = 1 \). But, this case cannot happen because the length of \( P_3' \) is \( |E(P_2')| + |C_1'| - 2 \geq p_1 + |C_1'| - 2 \geq p_1 + k - 2 \geq p_1 + 3 \), which is larger than the length of \( P_2 \), contrary to the choice of \( P_3' \).

No matter which cases happen, we always have \( \pi(v_{i+2}, y) \geq 1 \). Similarly, we have \( \pi(v_{i-2}, y) \geq 1 \). Because \( k \geq 5 \), \( v_{i-2} \) is different from \( v_{i+2} \). For all the remaining vertices in \( C \), \( \pi(v_j, y) \geq 0 \) for \( j \neq i - 2, i + 2 \). Then, for a fixed \( y \in V(G) \setminus V(C) \), we get that \( \sum_{x \in V(C)} \pi(x, y) \geq 2 \).

Therefore,

\[
\sum_{x \in V(C), y \in V(G) \setminus V(C)} \pi(x, y) \geq 2(n - k).
\]

**Case 3.** \( x, y \in V(G) \setminus V(C) \).

In this case, \( \pi(x, y) \geq 0 \).
From above cases, we have
\[
\begin{align*}
S_z(G) - W(G) &= \sum_{\{x,y\} \subseteq V(G)} \pi(x, y) \\
&= \sum_{\{x,y\} \subseteq V(G)} \pi(x, y) + \sum_{x \in V(C)} \pi(x, y) + \sum_{\{x,y\} \subseteq V(G) \setminus V(C)} \pi(x, y) \\
&\geq \binom{k}{2} - k + 2(n - k) \\
&= 2n + \frac{1}{2}k(k - 7) \\
&\geq 2n - 5.
\end{align*}
\]
for \(k \geq 5\).

From the above inequalities, we see that equality holds if and only if \(k = g = 5\), \(\pi(x, y) = 1\) for all the nonadjacent pairs \(\{x, y\}\) in \(C\), and there are exactly two vertices \(v_1, v_2\) in \(C\) such that \(\pi(v_1, y) = 1, \pi(v_2, y) = 1\) for all \(y \in V(G) \setminus V(C)\), and \(\pi(x, y) = 0\) for every pair \(\{x, y\} \subseteq V(G) \setminus V(C)\).

We first claim that if the equality holds, then \(G\) is unicyclic. Suppose that \(C\) is the set of all cycles except the shortest cycle \(C\). Let \(C'\) is a shortest cycle of \(C\), then \(C'\) is an isometric cycle. If \(C'\) is an even cycle, and there exists a pair of vertices \(\{x, y\} \subseteq V(C') \setminus V(C)\), then by Lemma 2.2, \(\pi(x, y) \geq 1\), a contradiction. So there is only one vertex \(x \in V(C') \setminus V(C)\). Let \(v_i, v_j\) be the neighbors of \(x\) in \(C'\). Then \(v_ix, xv_j\) together with a shortest path between them in \(C\) is a cycle \(C''\) different from \(C\). Since the length of \(C\) is 5, \(d(v_i, v_j) \leq 2\), and the length of \(C''\) is at most 4, contrary to the assumption that \(g \geq 5\).

If \(C'\) is an odd cycle, and there exists a pair of nonadjacent vertices \(\{x, y\} \subseteq V(C') \setminus V(C)\). Then by Lemma 2.2, \(\pi(x, y) \geq 1\), a contradiction. If there are only two adjacent vertices \(x, y\) on \(V(C') \setminus V(C)\), and let \(v_i\) be the neighbor of \(x\) in \(C\) and \(v_j\) the neighbor of \(y\) in \(C\), then \(v_ixyv_j\) together with a shortest path between them in \(C\) is a cycle \(C_1\) different from \(C\). Since the length of \(C\) is 5 and \(g \geq 5\), \(d(v_i, v_j) = 2\). Then \(C_1\) is an isometric cycle, and by Lemma 2.2, \(\mu_{v_i,v_j}(xy) = 1\), and so \(\pi(v_i, v_j) \geq 2\), a contradiction. If there is only one vertex \(x \in V(C') \setminus V(C)\), and let \(v_i, v_j\) be the neighbors of \(x\) in \(C'\), then \(v_ix, xv_j\) together with a shortest path between them in \(C\) is a cycle \(C_2\) different from \(C\). Since the length of \(C\) is 5, \(d(v_i, v_j) \leq 2\), and the length of \(C_2\) is at most 4, contrary to the assumption that \(g \geq 5\).

So, we have that \(G\) is a unicyclic graph with the only cycle \(C\) of length 5. Let \(C = v_1v_2 \cdots v_5v_1, T_i\) be the component of \(E(G) \setminus E(C)\) that contains the vertex \(v_i(1 \leq i \leq 5)\).

If there are at least three nontrivial \(T_i\)s, say \(T_i, T_j, T_k\), then there is a pair of vertices, say \(\{v_i, v_j\}\) which are not adjacent. Let \(x \in V(T_i) \setminus \{v_i\}, y \in V(T_j) \setminus \{v_j\}\). Then
\( \{x, y\} \subseteq V(G) \setminus V(C) \). Since \( d_C(v_i, v_j) = 2 \), by Lemma 2.1 \( \pi(x, y) \geq 1 \), a contradiction. Therefore, there are at most two nontrivial \( T \)s, say \( T_i, T_j \). If \( v_i, v_j \) are not adjacent, similarly we can find \( \{x, y\} \subseteq V(G) \setminus V(C) \) satisfying \( \pi(x, y) \geq 1 \), a contradiction. Thus, \( v_i, v_j \) must be adjacent. In this case, for any \( x \in V(T_i) \setminus \{v_i\} \), \( y \in V(T_j) \setminus \{v_j\} \), \( \pi(x, y) = 0 \), and for any \( x \in V(T_i) \setminus \{v_i\} \), \( \pi(x, v_{i-2}) = 1 \), \( \pi(x, v_{i+2}) = 1 \), and \( \pi(x, v_k) = 0 \) for \( k \neq i, j \). \( y \in V(T_j) \setminus \{v_j\} \) is similar to the x case. By calculation, we have \( Sz(G) - W(G) = 2n - 5 \). If there is only one nontrivial \( T \), we also can calculate that \( G \) satisfies \( Sz(G) - W(G) = 2n - 5 \).

Here we notice that by the above same way, we can give another proof to Conjecture 1.1 and get the following result:

**Theorem 2.3** Let \( G \) be a bipartite connected graph with \( n \geq 4 \) vertices and \( m \geq n \) edges. Then

\[
Sz(G) - W(G) \geq 4n - 8.
\]

Equality holds if and only if \( G \) is composed of a cycle on 4 vertices \( C_4 \) and a tree \( T \) on \( n - 3 \) vertices sharing a single vertex.

**Proof.** Let \( C \) be a shortest cycle of \( G \), and assume that \( C = v_1v_2 \cdots v_gv_1 \). Simply, \( C \) is an isometric cycle. We consider the pair \( \{x, y\} \subseteq V(G) \).

**Case 1.** \( \{x, y\} \subseteq V(C) \).

By Lemma 2.2 \( \pi(x, y) \geq d_C(x, y) \). Thus, if \( xy \) is an edge of \( G \), then \( \pi(x, y) \geq 1 \). Otherwise, \( \pi(x, y) \geq 2 \). Therefore,

\[
\sum_{\{x, y\} \subseteq V(C)} \pi(x, y) \geq g + 2 \left( \left( \binom{g}{2} - g \right) \right).
\]

**Case 2.** \( x \in V(C), y \in V(G) \setminus V(C) \).

Assume that \( v_i \) is a vertex on \( C \) such that \( d_G(v_i, y) = \min_{v \in V(C)} d_G(v, y) \), and \( P_1 \) is a shortest path between \( v_i \) and \( y \). Then \( P_1 \) does not contain any vertices on \( C \); otherwise, if \( v_j \in P_1 \), then \( d_G(v_j, y) < d_G(v_i, y) \), contrary to the choice of \( v_i \).

If there is only one path between \( y \) and \( v_i \), then \( \pi(y, v_i) = 0 \) and \( v_i \) is a cut vertex. For any other vertex \( v_j \) in \( C \), the path from \( y \) to \( v_j \) must go through \( v_i \), and thus, \( \mu_{v_i,v_j}(e) = \mu_{y,v_j}(e) \) for \( e \in E(C) \). From Lemma 2.2 we have that if \( v_iv_j \) is an edge of \( C \), then \( \pi(y, v_j) \geq 1 \). If \( d_C(v_i, v_j) \geq 2 \), then \( \pi(y, v_j) \geq 2 \). Therefore,

\[
\sum_{x \in V(C)} \pi(x, y) \geq 2 + 2(g - 3) = 2g - 4 \geq g.
\]

If there are at least two paths between \( y \) and \( v_i \), then, since \( G \) is a bipartite graph, by Lemma 2.1 \( \pi(y, v_i) \geq 1 \). And for each \( v_j \in V(C) \setminus \{v_i\} \), there are at least two paths
from \( y \) to \( v_j \), so \( \pi(y, v_j) \geq 1 \). Therefore,

\[
\sum_{x \in V(C)} \pi(x, y) \geq g.
\]

**Case 3.** \( x \in V(G) \setminus V(C), y \in V(G) \setminus V(C) \).

In this case, \( \pi(x, y) \geq 0 \).

From the above cases, we have

\[
Sz(G) - W(G) = \sum_{\{x, y\} \subseteq V(G)} \pi(x, y) + \sum_{x \in V(C)} \pi(x, y) + \sum_{\{x, y\} \subseteq V(G) \setminus V(C)} \pi(x, y)
\]

\[
\geq g + 2\left(\frac{g}{2} - g\right) + g(n - g)
\]

\[
= g(n - 2)
\]

\[
\geq 4n - 8.
\]

From the above inequalities, one can see that if equality holds, then \( g = 4 \), and \( \pi(x, y) = 1 \) for all the adjacent pairs \( \{x, y\} \subseteq V(C) \), \( \pi(x, y) = 2 \) for all the nonadjacent pairs \( \{x, y\} \subseteq V(C) \) and \( \pi(x, y) = 0 \) for every pair \( \{x, y\} \subseteq V(G) \setminus V(C) \).

Now we show that if equality holds, then \( G \) is a unicyclic graph. Suppose that \( C \) is the set of all cycles except the shortest cycle \( C \). Let \( C' \) is a shortest cycle of \( C \). Then \( C' \) is an isometric cycle. Since \( G \) is bipartite, \( C' \) is an even cycle. If there exists a pair of vertices \( \{x, y\} \subseteq V(C') \setminus V(C) \), then by Lemma 2.1 \( \pi(x, y) = 1 \), a contradiction. So there is only one vertex \( x \in V(C') \setminus V(C) \). Let \( v_i, v_j \) be the neighbors of \( x \) in \( C' \). Then \( v_i x, x v_j \) together with a shortest path between them in \( C \) is a cycle \( C'' \) different from \( C \). Since the length of \( C \) is 4, \( d(v_i, v_j) = 2 \), and the length of \( C'' \) is 4, \( \mu_{v_i,v_j}(xv_i) = \mu_{v_i,v_j}(xv_j) = 1 \). Thus, \( \pi(v_i, v_j) \geq 4 \), a contradiction. Therefore, \( G \) is unicyclic.

Let \( T_i \) be the component of \( E(G) \setminus E(C) \) that contains the vertex \( v_i \) (1 \( \leq i \leq 4 \)).

If there are at least two nontrivial \( T_i \)s, say \( T_i, T_j \), and let \( x \in V(T_i) \setminus \{v_i\}, y \in V(T_j) \setminus \{v_j\} \), then \( \{x, y\} \subseteq V(G) \setminus V(C) \), and there are at least two paths between \( x \) and \( y \). By Lemma 2.1 \( \pi(x, y) \geq 1 \), a contradiction. Therefore, there is only one nontrivial \( T_i \). In this case, we can calculate that \( G \) satisfies \( Sz(G) - W(G) = 4n - 8 \). Hence, equality holds if and only if \( G \) is the graph composed of a cycle on 4 vertices, \( C_4 \), and a tree \( T \) on \( n - 3 \) vertices sharing a single vertex.
Since for a bipartite graph, we have \( Sz^*(G) = Sz(G) \), which immediately implies Conjecture \[13\].

Next, we give a proof to Conjecture \[14\]. At first we need the following Lemmas.

**Lemma 2.4** (\[13\]) For a connected graph \( G \) with at least two vertices,
\[
Sz(G) \geq W(G),
\]
with equality if and only if each block of \( G \) is a complete graph.

**Lemma 2.5** Let \( G \) be a connected graph with \( n \geq 4 \) vertices and \( m \geq n \) edges and with an odd cycle. Then for every vertex \( u \in V(G) \), there exists an edge \( e = v_1v_2 \in E(G) \) such that \( u \in N_0(e) \), that is, \( \sum_{e \in E(G)} n_0(e) \geq n \).

**Proof.** Suppose that there is a vertex \( u \in V(G) \) such that for every \( e = xy \in E(G) \), we have \( d_G(u, x) \neq d_G(u, y) \). Let \( d = ecc(u) \), \( N^i(u) = \{ v \in V(G) | d_G(u, v) = i \}, 1 \leq i \leq d \). By the assumption, we know that there is no edge in \( N^i(u) \) for every \( i \), that is, \( N^i(u) \) is an independent set. Set \( X = \{ u \} \cup \bigcup_{1 \leq i \leq d} N^i(u) \), \( Y = \bigcup_{1 \leq i \leq d} N^i(u) \) is odd \( N^i(u) \). Then \( G = G[X, Y] \) is a bipartite graph with partite sets \( X \) and \( Y \). But, \( G \) is a connected graph with an odd cycle, a contradiction. Hence, for every vertex \( u \in V(G) \), there exists an edge \( e = v_1v_2 \in E(G) \) such that \( u \in N_0(e) \), and so we have \( \sum_{e \in E(G)} n_0(e) \geq n \).

Now we turn to solving Conjecture \[14\] and get the following result:

**Theorem 2.6** Let \( G \) be a connected nonbipartite graph with \( n \geq 4 \) vertices. Then
\[
Sz^*(G) - W(G) \geq \frac{n^2 + 4n - 6}{4}.
\]
Equality holds if and only if \( G \) is composed of a cycle on 3 vertices, \( C_3 \), and a tree \( T \) on \( n - 3 \) vertices sharing a single vertex.

**Proof.** By using \( n_u(e) + n_v(e) + n_0(e) = n \) for every \( e \in E(G) \), we have
\[
Sz^*(G) - W(G) = \sum_{e = uv \in E(G)} \left( n_u(e) + \frac{n_0(e)}{2} \right) \left( n_v(e) + \frac{n_0(e)}{2} \right) - W(G)
\]
\[
= \sum_{e = uv \in E(G)} n_u(e)n_v(e) + \sum_{e = uv \in E(G)} \left( \frac{n_0(e)}{2} (n - n_0(e)) + \frac{n_0^2(e)}{4} \right) - W(G)
\]
\[
= Sz(G) - W(G) + \sum_{e = uv \in E(G)} \left( \frac{n_0(e)}{2} n - \frac{n_0^2(e)}{4} \right)
\]
Let $n_0 = \sum_{e=uv \in E(G)} \left( \frac{n_0(e)}{2} n - \frac{n_0^2(e)}{4} \right)$. If there are two edges $e', e''$ such that $n_0(e') \geq n_0(e'')$, and put $n_0'(e') = n_0(e') + 1$, $n_0'(e'') = n_0(e'') - 1$, $n_0'(e) = n_0(e)$ for other edges, then

\[
\begin{align*}
&n_0' - n_0 \\
&= \sum_{e=uv \in E(G)} \left( \frac{n_0'(e)}{2} n - \frac{n_0'^2(e)}{4} \right) - \sum_{e=uv \in E(G)} \left( \frac{n_0(e)}{2} n - \frac{n_0^2(e)}{4} \right) \\
&= \frac{n_0(e'') - n_0(e') - 1}{2} \\
&< 0.
\end{align*}
\]

Let $C$ be a shortest odd cycle of $G$ with length $g$, and its edges be $e_1, e_2, \ldots, e_g$. Then $C$ is isometric. For every edge $e = uv \in E(C)$, there is a vertex $x \in V(C)$ such that $d_C(x, u) = d_C(x, v) = d_C(x, u) = d_C(x, v)$. Therefore, $n_0(e) \geq 1$ for every $e \in E(C)$. If there are two edges $e', e''$ such that $n_0(e') \geq n_0(e'')$, we could do the operation as above, which makes $n_0$ smaller. Thus, $n_0$ attains its minimum when $n_0(e_i) = 1$ except for $n_0(e_1), n_0(e) = 0$ for all the remaining edges. By Lemma \ref{lem2.5} \[
\sum_{e \in E(G)} n_0(e) \geq n, \text{ and so } n_0(e_1) \geq n - g + 1. \text{ Hence,}
\]

\[
n_0 \geq (g-1)\left(\frac{n}{2} \frac{1}{4} \right) + \frac{n - g + 1}{2} n - \frac{(n - g + 1)^2}{4} \geq \frac{n^2}{2} - \frac{1}{4} (2 + (n - 2)^2) = \frac{n^2 + 4n - 6}{4}.
\]

From the above inequalities, we can see that equality holds if and only if $g = 3$, $S_2(G) = W(G)$ and $n_0(e_1) = n - 2, n_0(e_2) = 1, n_0(e_3) = 1, n_0(e) = 0$ for all the remaining edges.

Now we conclude that $G$ is unicyclic. Suppose that $G$ is not unicyclic. By Lemma \ref{lem2.4} we know there is a block $H$ different from $C$ which is a complete graph of order at least three. Then, $n_0(e) \geq 1$ for every $e \in E(H)$, a contradiction.

Let $T_i$ be the component of $E(G) \setminus E(C)$ that contains the vertex $v_i (1 \leq i \leq 3)$.

If there are at least two nontrivial $T_i$s, say $T_1, T_2$, then $n_0(v_2v_3) = |V(T_1)| \geq 2, n_0(v_1v_3) = |V(T_2)| \geq 2$, a contradiction. Therefore, there is only one nontrivial $T_i$. In this case, we can calculate that $G$ satisfies $Sz^*(G) - W(G) = \frac{n^2 + 4n - 6}{4}$. Hence, equality holds if and only if $G$ is the graph composed of a cycle on 3 vertices, $C_3$, and a tree $T$ on $n - 3$ vertices sharing a single vertex.

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