DZIOBEK EQUILIBRIUM CONFIGURATIONS ON A SPHERE

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ABSTRACT. We investigate the n-body problem on a sphere with a general interaction potential that depends on the mutual distances. We focus on the equilibrium configurations, especially on the Dziobek equilibrium configurations, which is an analogy of Dziobek central configurations of the classical n-body problem. We obtain a criterion and then reduce it to two sets of equations. Then we apply these equations to the curved n-body problem in $S^3$. In the end, we find the derivative of the Cayley-Menger determinant.

Key Words: curved n-body problem; Dziobek configurations; equilibrium configurations; stability; Cayley-Menger determinant.

1. INTRODUCTION

The classical n-body problem has been generalized in many ways, for example, under the potential $\sum \frac{m_im_j}{r_{ij}}$, or in higher dimensional Euclidean space. In particular, the curved n-body problem, which generalizes the classical n-body problem to surfaces of constant curvature has received lot of attentions in the last decade (cf [2, 6, 10, 13] and the references therein ).

Motivated by those work, we study the generalization of the n-body problem to unit sphere of the Euclidean space. We assume that the potential depends on the shortest geodesic distance. We also assume that the potential is attractive (repulsive) in most cases. We only specify the potential in the last section.

One major distinction between the generalization and the classical n-body problem is the existence of equilibrium configurations, due to the compactness of spheres. This paper is devoted to the study of equilibrium configurations on spheres.

In particular, we consider the equilibrium configurations formed by $N$ bodies on some $(N-2)$-dimensional sphere. We call them the Dziobek equilibrium configurations. In the classical n-body problem, Otto Dziobek [9] first introduced a set

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of equations for non collinear four-body central configurations on $\mathbb{R}^2$, an approach proved fruitful in the study of four-body central configurations (cf [1,11] and the references therein).

We obtain a criterion similar to that of Otto Dziobek for the Dziobek equilibrium configurations in the n-body problem on a sphere. If the potential is attractive (repulsive), there is an obstacle for the equilibrium configurations, namely, the particles could not lie on one hemisphere. By this property, we can further separate the criterion into two sets of equations. One set of the equations can be used to determine the manifold in the configuration space that admits equilibrium configurations, then the other set of equations can be used to determine the corresponding masses.

The paper is organized as follows. In Section 2 we discuss the basic setting of the n-body problem on a sphere and the equilibrium configurations. In Section 3 we define the Dziobek equilibrium configurations and obtain a criterion. Then we separate the criterion into two sets. In Section 4 we turn to the curved n-body problem in $S^3$. We apply the criterion to equilibrium configurations of three-four- and five-body in $S^3$ and discuss the stability of associated equilibria. We discuss the derivative of the Cayley-Menger determinant in the Appendix.

2. THE EQUILIBRIUM CONFIGURATIONS FOR MECHANICAL SYSTEM ON A SPHERE

Let $S^n$ be the unit sphere of the Euclidean space $\mathbb{R}^{n+1}$. Let us consider $N$ points of positive mass $m_i$ on $S^n$ that interacting mutually by a potential depending on the shortest geodesic distance between the points. The position vector of the $i$-th point is $q_i = (x_{i1}, ..., x_{i,n+1})^T$ with $x_{i1}^2 + ... + x_{i,n+1}^2 = 1$, $i = 1, ..., N$. Denote the configuration by $\mathbf{q} = (q_1, ..., q_N)$. The configuration space is

$$Q = (S^n)^N \setminus \{\text{collisions and configurations where the vector field is undefined}\}.$$

The mechanical system is given by the Lagrangian $L : TQ \to \mathbb{R}$

$$L(q, \dot{q}) = T_q(q) - V(q)$$

where $T$ is a Riemannian metric on the configuration space and $V(q)$ is the interaction potential. Denote the distance between two points $q_i, q_j$ by $d_{ij}$. Then

$$\cos d_{ij} = q_i \cdot q_j.$$  

Assume that the potential $V$ is

$$V(q) = \sum_{1 \leq i < j \leq N} m_i m_j G(d_{ij}),$$

where $G : (0, \pi) \to \mathbb{R}$ is some given smooth function.

**Definition 1 ([13]).** A potential $V$ as given by (2) is called attractive (repulsive) if the binary potential $G$, is such that $G'(x) > 0$ ($G'(x) < 0$) for all $x \in (0, \pi)$. 


The equilibrium motion, or simply equilibrium, is solution in the form of \(q(t) = q(0)\). The configuration \(q(0)\), called an equilibrium configuration, is a critical point of \(V\). The derivative of \(V\) is

\[
\nabla_{q_i} V(q) = \sum_{j=1, j \neq i}^{N} \frac{m_i m_j G'(d_{ij}) \nabla_{q_i} d_{ij}}{\sin d_{ij}} - \sum_{j=1, j \neq i}^{N} \frac{m_i m_j G'(d_{ij}) \nabla_{q_i} \cos^{-1} q_i \cdot q_j}{\sin d_{ij}}.
\]

By extending \(q_i \cdot q_j\) into a homogeneous function of degree zero in \(\mathbb{R}^{2(n+1)} \setminus \{0\}\), i.e., \(\frac{q_i}{\sqrt{q_i \cdot q_i}} = \frac{q_j}{\sqrt{q_j \cdot q_j}}\), we obtain

\[
\nabla_{q_i} V(q) = \sum_{j=1, j \neq i}^{N} \frac{-m_i m_j G'(d_{ij})}{\sin d_{ij}} \left[ q_j - \cos d_{ij} q_i \right].
\]

Hence, a configuration \(q \in Q\) is an equilibrium configuration if \(q\) satisfies the following system

\[
\sum_{j=1, j \neq i}^{N} \frac{m_i m_j G'(d_{ij})}{\sin d_{ij}} \left[ q_j - \cos d_{ij} q_i \right] = 0, \ i = 1, \ldots, N.
\]

**Remark 1.** The Lyapunov stability of such equilibrium is related with the second variation of \(V\). In particular, the well-known Lagrange-Dirichlet Theorem says it is stable if the configuration is an isolated minimum. The converses of this theorem is widely discussed (cf. [12, 14] and the references therein). If the potential is analytic, then it is unstable if it is not a minimum. The equilibrium configurations also lead to relative equilibria of the system [16].

**Proposition 1.** The \(i\)-th equation of system (3) holds if and only if there is a constant \(\theta_i\) such that

\[
\sum_{j=1, j \neq i}^{N} \frac{m_i m_j G'(d_{ij})}{\sin d_{ij}} q_j + \theta_i q_i = 0.
\]

**Proof.** Assume that equation (4) holds. Multiply \(q_i\) to the both sides of equation (4). Since \(q_i \cdot q_j = \cos d_{ij}\) and \(q_i \cdot q_i = 1\), we obtain \(\theta_i = -\sum_{j=1, j \neq i}^{N} \frac{m_j m_i G'(d_{ij}) \cos d_{ij}}{\sin d_{ij}}\).

Thus equation (4) is equivalent to the \(i\)-th equation of (3). \(\square\)

The following result generalizes one result of Diacu [6] for the curved n-body problem, see Section 4.

**Theorem 1.** Assume that the potential is attractive (repulsive). There is no equilibrium configuration for any positive masses in any closed hemisphere of \(\mathbb{S}^n\) (i.e. a hemisphere that contains its boundary), as long as at least one body does not lie on the boundary.
Proof. Let \( q \) be a configuration that lies in a closed hemisphere of \( S^n \) and that there is at least one body not on the boundary. Then there is some point \( v \in S^n \) such that \( v \cdot q_i \geq 0 \) for all \( i \) and at least one of them is strictly positive. Assume that \( v \cdot q_1 \) is the smallest. Then \( \nabla q_i V = 0 \) implies

\[
\sum_{j=1}^{N} \frac{m_j G''(d_{ij})}{\sin d_{ij}} [v \cdot q_j - \cos d_{ij} v \cdot q_1] = 0.
\]

Since we have assumed \( G'(d_{ij}) \) is of the same sign for all \( j \), this is a contradiction.

We end this section by several examples of equilibrium configurations for equal masses. The examples extend those constructed by Diacu in [6] for the curved \( n \)-body problem (Section [4]). Denote the standard bases of \( \mathbb{R}^{n+1} \) by \( e_1, ..., e_{n+1} \). Denote the unit sphere in \( \text{span}\{e_1, ..., e_{k+1}\} \) by \( S^k \). We assume that the configurations constructed below are not those where \( G'(d_{ij}) \) is undefined.

**Example 1** (regular simplex with equal masses). Consider a regular \( k \)-simplex. Place one unit mass at each of the vertices. The configuration obtained is an equilibrium configuration. It is enough to check that equation (1) holds for \( i = 1 \). Since \( d_{ij} = d_{12} \) for any pair of \( \{i, j\} \), we find that

\[
\sum_{i=2}^{k+1} m_i G'(d_{1i}) q_i = \frac{G'(d_{21})}{\sin d_{21}} (\sum_{i=2}^{k+1} q_i) = -\frac{G'(d_{21})}{\sin d_{21}} q_1.
\]

**Example 2** (regular polygon with equal masses). Consider a regular \( 2k + 1 \)-gon located on the unit circle of \( \text{span}\{e_1, e_2\} \). Place one unit mass at each of the vertices. By complex number notation, the position vectors are \( q_j = e^{i\phi_j}, j = 1, ..., 2k + 1 \). Let us check equation (1) for \( j = 2k + 1 \). Since \( d_{2k+1,j} = d_{2k+1,2k+1-j} \), we have

\[
\frac{m_j G'(d_{2k+1,j}) e^{i\phi_j}}{\sin d_{2k+1,j}} + \frac{m_{2k+1-j} G'(d_{2k+1,2k+1-j}) e^{i\phi(2k+1-j)}}{\sin d_{2k+1,2k+1-j}} = \frac{2 G'(d_{2k+1,j})}{\sin d_{2k+1,j}} \cos j \phi e^{i\phi(2k+1)}.
\]

Then it follows that equation (1) holds for \( i = 2k + 1 \), then for all \( i \) by symmetry.

Similarly, the regular polygon of even vertices with equal masses is also an equilibrium configuration.

**Example 3** (two regular polygons with equal masses on two complementary circles). Consider one regular \( n_1 \)-polygon located on the unit circle of \( \text{span}\{e_1, e_2\} \) and another regular \( n_2 \)-polygon located on the unit circle of \( \text{span}\{e_3, e_4\} \). Place
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Let us check that equation (4) holds for any \(1 \leq i \leq n_1 + n_2\), say \(i = 1\).

\[
\sum_{i=2}^{n_1+n_2} \frac{m_i G'(d_{i1})}{\sin^3 d_{i1}} q_i = \sum_{i=2}^{n_1} \frac{G'(d_{i1})}{\sin d_{i1}} q_i + G'(\frac{\pi}{2}) \sum_{i=n_1+1}^{n_1+n_2} q_i.
\]

The first part is collinear with \(q_1\) by the above example, and the second part is zero. Thus this configuration is one equilibrium configuration.

3. Dziobek Equilibrium Configurations

In this section, we consider equilibrium configurations where \(N\) masses span an \((N-2)\)-sphere. We obtain a criterion, then separate it into two sets of equations, the shape equations and the mass equations. In the classical n-body problem, a central configuration of \(N\) bodies that span an \((N-2)\)-dimensional affine plane are called Dziobek central configurations [9, 11]. For equilibrium configurations on sphere, equation (4) implies that the \(N\) position vectors are always dependent, so \(1 \leq \text{rank}(q_1, \ldots, q_N) \leq N - 1\).

**Definition 2.** A Dziobek configuration of \(N\) bodies on sphere is one such that \(\text{rank}(q_1, \ldots, q_N) = N - 1\).

Let \(\{q_1, \ldots, q_N\}\) be a collection of vectors in \(\mathbb{R}^{N-1}\). Assume the rank of these \(N\) vectors is \(N - 1\). Consider the \((N-1) \times N\) matrix:

\[
X = [q_1, \ldots, q_N].
\]

Since the rank of \(X\) is \(N - 1\), \(\dim \ker X = 1\). The kernel can be found as follows. Let \(X_k\) be the \((N-1) \times (N-1)\) matrix obtained from \(X\) by deleting the \(k\)-th column and let \(|X_k|\) denote its determinant.

**Lemma 1.** Let

\[
\Delta = (\Delta_1, \ldots, \Delta_N) = (|X_1|, -|X_2|, \ldots, (-1)^{k+1}|X_k|, \ldots).
\]

Then \(\Delta^T\) is the base of \(\ker X\). In other words, \(\Delta \neq 0\) and \(\Delta_1 q_1 + \cdots + \Delta_N q_N = 0\).

**Proof.** Assume that \(\Delta_N = (-1)^N|X_N| \neq 0\). Consider the linear system in \(X_N u = q_N, u = (u_1, \ldots, u_{N-1})^T\). By Cramer’s rule, we obtain \(u_k = \frac{\Delta_k}{\Delta_N}, k = 1, \ldots, N-1\). Then it follows that \(\Delta_1 q_1 + \cdots + \Delta_N q_N = 0\).

**Proposition 2.** Consider a Dziobek configuration of \(N\) bodies on \(\mathbb{S}^{N-2}\). Then the configuration is not on a hemisphere if and only if all \(\Delta_i\) are of the same sign.
Proof. We only prove that if not all $\Delta_i$ are of the same sign the Dziobek configuration lies on a hemisphere. There are two cases.

If there is some $\Delta_i = 0$, say $\Delta_1$, then rank $\{q_2, \ldots, q_N\} = N - 2$. Let $\Pi$ be the hyperplane spanned by $\{q_2, \ldots, q_N\}$ and $\vec{n}$ be the normal of $\Pi$ in $\mathbb{R}^{N-1}$ with the property $\vec{n} \cdot q_1 > 0$. Then we have

$$\vec{n} \cdot q_i \geq 0, \quad i = 1, \ldots, N,$$

which implies that the Dziobek configuration lies on a hemisphere.

If all $\Delta_i$ are nonzero, there are two consecutive elements of $\Delta$ that are of different sign, say $\Delta_1 > 0$, $\Delta_2 < 0$. Then

$$|X_1| = \det(q_2, q_3, \ldots, q_N) > 0, \quad |X_2| = \det(q_1, q_3, \ldots, q_N) > 0.$$

Let $\tilde{\Pi}$ be the $(N-2)$-dimensional hyperplane spanned by $\{q_3, \ldots, q_N\}$ and $\vec{m}$ be the normal of $\tilde{\Pi}$ in $\mathbb{R}^{N-1}$ with the property $\vec{m} \cdot q_1 > 0$. Assume that $q_2 = \lambda_1 q_1 + \sum_{i=3}^{N} \lambda_i q_i$.

Then

$$|X_1| = \det(\lambda_1 q_1 + \sum_{i=3}^{N} \lambda_i q_i, q_3, \ldots, q_N) = \lambda_1 |X_2|.$$

Then $\lambda_1 > 0$. Hence we have

$$\vec{m} \cdot q_i \geq 0, \quad i = 1, \ldots, N,$$

which implies that the Dziobek configuration lies on a hemisphere. □

Denote the quantity $\frac{G'(d_{ij})}{\sin d_{ij}}$ by $S_{ij}$. Then equation (4) becomes

$$\sum_{j \neq i} m_j S_{ij} q_j + \theta_i q_i = 0, \quad 1 \leq i \leq N.$$

Theorem 2. Assume that the potential is attractive (repulsive) and that $q = (q_1, \ldots, q_N)$ is a Dziobek configuration in $S^{N-2}$. Then the configuration $q$ is an equilibrium configuration if and only if there is a nonzero real number $p$ such that

$$m_i m_j S_{ij} = p \Delta_i \Delta_j \quad \text{for any } i \neq j.$$  \hspace{1cm} (6)

Proof. The proof of the sufficient conditions: Since $q$ is an equilibrium configuration, equation (4) holds. That is, there is some nonzero real number $p_j$ such that

$$\sum_{j \neq i} m_j S_{ij} q_j + \theta_i q_i = 0, \quad 1 \leq i \leq N.$$  \hspace{1cm} (7)

by Lemma 1. System (7) is equivalent to

$$S = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix} (\frac{\Delta_1}{m_1}, \ldots, \frac{\Delta_N}{m_N}), \quad \text{where} \quad S = \begin{bmatrix} \theta_1 & S_{12} & \cdots & S_{1N} \\ S_{21} & \theta_2 & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & \theta_N \end{bmatrix}$$
Since the left matrix \( S \) is symmetric, we see that \( p_j \frac{\Delta}{m_i} = p_i \frac{\Delta}{m_j} \), or, by Proposition 2
\[
(p_1 m_1, ..., p_N m_N) = \frac{p_1 m_1}{\Delta_1} (\Delta_1, ..., \Delta_N).
\]

Let \( M = \text{diag}\{m_1, ..., m_N\} \). We have
\[
MSM = \frac{p_1 m_1}{\Delta_1} (\Delta_1, ..., \Delta_N)^T (\Delta_1, ..., \Delta_N),
\]
which gives (6).

The proof of the sufficient conditions: Let \( (p_1 m_1, ..., p_N m_N) = p\Delta \). The system (6) implies system (7), so the condition is also sufficient.

The system (6) can be obtained in another way, see Appendix. It imply that all \( \Delta_i \) \((i \geq 1)\) are of the same sign, which agrees with Proposition 2. Eliminating the constant \( p \), we get a system of \( \frac{N(N-1)}{2} - 1 \) equations from (6). The system can be written in a form with the property that most of the equations are just constraints on the shapes, or, independent of the masses.

**Proposition 3.** Let \( A = (a_{ij}) \) be a symmetric matrix and \( b = (b_1, ..., b_n) \). Assume that \( A = b^T b \) and \( b_1 b_2 b_n \neq 0 \). Consider the system consists of the \( \frac{n(n-1)}{2} \) equations
\[
a_{ij} = b_i b_j, \ i = 1, ..., n-1, \ j = i + 1, ..., n.
\]
The system of equations is equivalent to
\[
\begin{align*}
(8) \quad a_{1n} &= b_1 b_n, \ b_k &= b_n \frac{a_{1k}}{a_{1n}}, \ k = 2, ..., n-1; \\
(9) \quad b_1 &= b_n \frac{a_{12}}{a_{2n}}, \ a_{2k} a_{1n} = a_{1k} a_{2n}, \ k = 3, ..., n-1; \\
(10) \quad a_{jn} a_{12} &= a_{1j} a_{2n}, \ a_{jk} a_{1n} = a_{1k} a_{jn}, \ j = 3, ..., n-2, \ k = j + 1, ..., n-1; \\
(11) \quad a_{n-1,n} a_{12} &= a_{1n-1} a_{2n}.
\end{align*}
\]

Proof. From the first system, we see \( a_{ij} a_{kl} = b_i b_j b_k b_l = a_{ik} a_{jl} \) holds for any 4-tuple \( \{i, j, k, l\} \) of \( \{1, ..., n\} \). Then we derive the second system from the first one.

Let us derive the first system from the second one. For convenience, put the first system in an upper triangular form
\[
E = \begin{bmatrix}
a_{12} = b_1 b_2 & a_{13} = b_1 b_3 & \cdots & a_{1n} = b_1 b_n \\
a_{23} = b_2 b_3 & a_{24} = b_2 b_4 & \cdots & a_{2n} = b_2 b_n \\
\vdots & \vdots & \ddots & \vdots \\
a_{n-1,n} = b_{n-1} b_n
\end{bmatrix}
\]

By (8), we can recover the first row of \( E \). By (9) and the first row of \( E \), we see
\[
a_{2n} = a_{12} \frac{b_n}{b_1} = b_2 b_n, \ a_{2k} = a_{1k} a_{2n} \frac{b_n}{a_{1n}} = b_2 b_n \frac{b_k}{b_n} = b_2 b_k, \ k = 3, ..., n-1.
\]
Hence the second row of $E$ is recovered. Similarly, the $j$-th row can be obtained by

$$a_{jn} = a_{1j} \frac{b_n}{b_1} = b_j b_n, a_{2k} = a_{1k} a_{jn} = \frac{b_k b_j b_n}{b_n} = b_j b_k, k = j + 1, \ldots, n - 1.$$ 

Thus, we obtain all equations of $E$. This completes the proof. \[ \square \]

Applying the above result to the system (6), where $A = S$ and $b = \sqrt{p} \Delta M^{-1}$, we get

**Theorem 3.** Assume that the potential is attractive (repulsive) and that $q = (q_1, \ldots, q_N)$ is a Dziobek configuration in $S^{N-2}$ with masses $m_1, \ldots, m_N$. Then $q$ is an equilibrium configuration if and only if the following system of equations are satisfied

$$\begin{cases}
m_2 = \frac{S_{1N} \Delta_2}{S_{12} \Delta_N} m_N, & \ldots, m_{N-1} = \frac{S_{1N} \Delta_{N-1}}{S_{1N-1} \Delta_N} m_N, m_1 = \frac{S_{2N} \Delta_1}{S_{12} \Delta_N} m_N, \\
S_{1j} S_{2N} = S_{12} S_{jN}, j = 3, \ldots, N - 1; \\
S_{jk} S_{1N} = S_{1k} S_{jN}, k = j + 1, \ldots, N - 1, j = 2, \ldots, N - 2.
\end{cases} \tag{12}$$

Note that the first $N - 1$ equations are involved with the masses, while the remaining equations are not. Let us call the first $N - 1$ equations the mass equations, and the others the shape equations.

The system (12) determines the Dziobek equilibrium configurations, including the configurations and the corresponding masses. The shape equations alone can not determine the configurations. Indeed, there are configurations that satisfies the shape equations, but the configuration lies on a hemisphere (see Remark 3 in Section 4). Thanks to Proposition 2 to build a Dziobek equilibrium configuration, we may first construct a configuration that satisfies the shape equations and the condition that all $\Delta_i$ are of the same sign (or equivalently, configurations not on a hemisphere), then determine the corresponding positive masses by the mass equations.

**Remark 2.** Recall that for Dziobek central configurations of the classical $n$-body problem [11], one can only be certain that at least two elements of $\Delta$ are nonzero. Hence, we can get a system similar to (12) there, which is only necessary but not sufficient. However, for $n = 4$, we know that all $\Delta_i$ are nonzero, so the central configuration equations can be written in a form similar to (12). The difference is that there are further restrictions on the mutual distances such that the masses are positive, see Corbera et al. [5].

4. **Example: the curved N-body problem in $S^3$**

In this section we consider the problem in $S^3$ with the gravitational interaction. The potential is defined as spherical-symmetric solutions of the Laplace equation...
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on $\mathbb{S}^3$. This is the curved n-body problem in $\mathbb{S}^3$. For more on this problem, see [2]. For any two points $q_i$ and $q_j$, the binary potential and the potential are

$$G(d_{ij}) = -m_i m_j \cot d_{ij}, \quad V(q) = \sum_{1 \leq i < j \leq N} -m_i m_j \cot d_{ij}$$

respectively. Since $G(x) = -\cot x$, $G'(x) = \frac{\sin x}{\sin^2 x} > 0$, the potential is attractive and $S_{ij} = \frac{G'(d_{ij})}{\sin d_{ij}} = \frac{1}{\sin d_{ij}}$.

Note that the potential is undefined at $d_{ij} = \pi$, so we must exclude those configurations with points diametrically opposite in the examples considered in Section 2 for instance, the regular polygons with even vertices. Moreover, there is no equilibrium configuration for two masses. Otherwise, the equation (4) implies that $d_{12}$ is 0 or $\pi$. The equilibrium configurations are also called special central configurations in the curved n-body problem in $\mathbb{S}^3$ [16].

4.1. Criteria for Dziobek Equilibrium Configurations of Three, Four and Five Bodies. By Theorem 3, we obtain the following criteria for Dziobek equilibrium configurations of 3, 4 and 5 bodies respectively. The regular 2, 3, and 4-simplex with equal masses (see Example 1) satisfies the following criteria respectively.

**Corollary 1** ($N = 3, \mathbb{S}^1$). Consider one configuration $q = (q_1, q_2, q_3)$ on $\mathbb{S}^1$. Then $q$ is a Dziobek equilibrium configuration if and only if the masses are

$$(m_1, m_2, m_3) = \left( \frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{23} \Delta_3}, \frac{\sin^3 d_{12} \Delta_2}{\sin^3 d_{13} \Delta_3}, 1 \right) m_3.$$  

**Corollary 2** ($N = 4, \mathbb{S}^2$). Consider one configuration $q = (q_1, q_2, q_3, q_4)$ on $\mathbb{S}^2$. Then $q$ is a Dziobek equilibrium configuration if and only if

$$\sin d_{12} \sin d_{34} = \sin d_{13} \sin d_{24} = \sin d_{14} \sin d_{23}$$

and the masses are

$$(m_1, m_2, m_3, m_4) = \left( \frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{24} \Delta_4}, \frac{\sin^3 d_{12} \Delta_2}{\sin^3 d_{14} \Delta_4}, \frac{\sin^3 d_{13} \Delta_3}{\sin^3 d_{14} \Delta_4}, 1 \right) m_4.$$  

**Corollary 3** ($N = 5, \mathbb{S}^3$). Consider one configuration $q = (q_1, q_2, q_3, q_4, q_5)$ in $\mathbb{S}^3$. Then $q$ is a Dziobek equilibrium configuration if and only if

$$\sin d_{13} \sin d_{25} = \sin d_{12} \sin d_{35} = \sin d_{23} \sin d_{15}, \quad \sin d_{24} \sin d_{15} = \sin d_{14} \sin d_{35},$$

$$\sin d_{14} \sin d_{25} = \sin d_{12} \sin d_{45} = \sin d_{24} \sin d_{15},$$

and the masses are

$$(m_1, \ldots, m_5) = \left( \frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{25} \Delta_5}, \frac{\sin^3 d_{12} \Delta_2}{\sin^3 d_{15} \Delta_5}, \frac{\sin^3 d_{13} \Delta_3}{\sin^3 d_{15} \Delta_5}, \frac{\sin^3 d_{14} \Delta_4}{\sin^3 d_{15} \Delta_5}, 1 \right) m_5.$$
For the case of $N = 3$, the constraint on the shape is only that the configuration is not in one semicircle, in other words, $\varphi_{i+1} - \varphi_i < \pi$ for $i = 1, 2, 3$ with $q_i = (\cos \varphi_i, \sin \varphi_i)$. Then all angles are acute, and the configuration forms an acute triangle, see Figure 1. Let $d_{12} = \alpha, d_{23} = \beta$. Then $d_{13} = 2\pi - (\alpha + \beta)$ and

$$0 < \alpha < \pi, \quad 0 < \beta < \pi, \quad \pi < \alpha + \beta < 2\pi.$$ 

Note that $\Delta_1 = |q_2, q_3| = \sin d_{23} = \sin \beta$, $\Delta_2 = -|q_1, q_3| = \sin d_{13} = |\sin(\alpha + \beta)|$, $\Delta_3 = |q_1, q_2| = \sin d_{12} = \sin \alpha$. Thus the masses satisfy

$$\frac{m_2}{\sin^2 \alpha} = \frac{m_3}{\sin^2(\alpha + \beta)}, \quad \frac{m_1}{\sin^2 \alpha} = \frac{m_3}{\sin^2 \beta}, \quad \frac{m_2}{\sin^2 \beta} = \frac{m_1}{\sin^2(\alpha + \beta)}.$$ 

The above system gives all Dziobek equilibrium configurations for three masses and it has been obtained by direct computations in [7]. The constraint of the masses is found as

$$m_1^2 m_2^2 + m_2^2 m_3^2 + m_2^2 m_3^2 - 2m_1 m_2 m_3 < 0,$$

if we assume that $\sum_{i=1}^3 m_i = 1$. It is easy to see that all such configurations are local minima of the potential $V$ restricted on $S^1$. These equilibria are stable on $S^1$ (Remark 1), see [7] and the generalization in [13].

For the case of $N = 4$, the system is not trivial and an equivalent system has been obtained by direct computation in [3]. We do not know much besides the regular tetrahedron equilibrium configuration on $S^2$ with four equal masses. Now we present a family of 4-body Dziobek equilibrium configurations which contains the regular tetrahedron. Consider a tetrahedron configuration of four masses with position vectors

$$q_1 = (1, 0, 0)^T \quad q_2 = (-c, r, 0)^T$$
$$q_3 = (-c, -\frac{1}{2}r, \frac{\sqrt{3}}{2}r)^T \quad q_4 = (-c, -\frac{1}{2}r, -\frac{\sqrt{3}}{2}r)^T$$

where $m_2 = m_3 = m_4$, and $c \in (0, 1), \ r^2 + c^2 = 1$, see Figure 2. Denote such a configuration by $q_c$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{triangle.png}
\caption{An acute triangle configuration}
\end{figure}
Proposition 4. The configuration $q_c, c \in (0, 1)$ is a Dziobek equilibrium configuration if

$$m_1 m_4 = \frac{8\sqrt{3}c}{3(1 + 3c^2)^2}.$$  

By numerical study, all such equilibrium configurations are not minima of the potential $V$ restricted on $S^2$. These equilibria are unstable on $S^2$, see Remark 1.

Proof. The tetrahedron is not on one hemisphere and the shape equations are satisfied since $d_{12} = d_{13} = d_{14}$, and $d_{23} = d_{24} = d_{34}$. The last two of the mass equations $m_i m_4 = \frac{\sin^3 d_{i1} \Delta_i}{\sin^3 d_{i1} \Delta_i}$, $i = 2, 3$ are true since $\Delta_2 = \Delta_3 = \Delta_4$. We only need to check the first mass equation.

Direct computation leads to

$$\cos d_{12} = -c, \quad \sin^3 d_{12} = r^3, \quad \cos d_{24} = c^2 - \frac{1}{2}r^2, \quad \sin^3 d_{24} = 3\sqrt{3}r^3\left(\frac{1}{4} + \frac{3}{4}c^2\right)^{\frac{3}{2}},$$

and $\Delta_4 = \frac{\sqrt{3}}{2}r^2, \Delta_1 = 3c \Delta_4$. Thus the configuration is a Dziobek equilibrium configuration if and only if

$$m_1 m_4 = \frac{\sin^3 d_{12} \Delta_1}{\sin^3 d_{24} \Delta_4} = \frac{r^3 3c \Delta_4}{3\sqrt{3}r^3\left(\frac{1}{4} + \frac{3}{4}c^2\right)^{\frac{3}{2}} \Delta_4} = \frac{8\sqrt{3}c}{3(1 + 3c^2)^2}.$$  

Remark 3. Consider the configuration with $c = -\frac{1}{3}$. Then $\sin d_{12} = \sin d_{24}$, so the shape equations are satisfied. However, the configuration is on the north hemisphere.

As $c \to 0$, we have $m_i m_4 \to 0$. This is intuitively clear. As $c \to 0$, the three masses $m_2, m_3, m_4$ tend to form an equilibrium configuration of their own on the equator. Then we may place an infinitesimal mass at $\pm(1, 0, 0)$ to form an equilibrium.
configuration of 4 bodies. The function \( f(c) = \frac{8\sqrt{3}c}{3(1+3c^2)^{3/2}}, c \in (0, 1) \), is increasing on \((0, \frac{\sqrt{3}}{6})\) and decreasing on \( (\frac{\sqrt{3}}{6}, 1) \). The maximum is \( \frac{16}{9\sqrt{3}} > 1 \), \( \lim_{c \to 0} f(c) = 0 \) and \( \lim_{c \to 1} f(c) = \frac{\sqrt{3}}{3} \).

**Corollary 4.** Consider four masses \((\bar{m}, m, m, m)\) on \( S^2 \). If \( \frac{\bar{m}}{m} \in (0, \frac{16}{9\sqrt{3}}) \), then there is at least one Dziobek equilibrium configuration. If \( \frac{\bar{m}}{m} \in (\frac{\sqrt{3}}{3}, \frac{16}{9\sqrt{3}}) \), then there are at least two Dziobek equilibrium configurations. Especially, there are at least two equilibrium configurations for four equal masses.

For the case of \( N = 5 \), the system is not trivial and an equivalent system has been obtained by direct computation in [3]. We do not know much besides the regular pentatope equilibrium configuration on \( S^3 \) with five equal masses. Nevertheless, it is easy to construct a family of 5-body Dziobek equilibrium configurations similar to the 4-body equilibrium configurations constructed above and obtain conclusions similar to Proposition 4 and Corollary 4.

### 4.2. Another Example.

Consider Dziobek equilibrium configurations of \( N \) masses with the property that \( \sum_{i=1}^{N} m_i q_i = 0 \). By Lemma 1, the vector \((m_1, \ldots, m_N)\) is a multiple of \((\Delta_1, \ldots, \Delta_N)\). Then equations of (6) implies that \( \sin d_{ij} \) is a constant for all pairs of all \( \{i, j\} \). Thus, there is some \( c \in (0, \pi) \) such that \( d_{ij} = c \), or \( \pi - c \).

If all \( d_{ij} \) equal to \( c \), thus the configuration is a regular simplex, which implies that \( \Delta_1 = \Delta_2 = \ldots = \Delta_N \) and \( m_1 = m_2 = \ldots m_N \). For instance, on \( S^1 \), this is the only possibility. However, this is not the only case if the sphere is of higher dimension. A similar phenomenon happens in [8].

For example, consider the following Dziobek configuration on \( S^2 \) with position vectors

\[
\begin{align*}
q_1 &= (a, b, 0)^T \\
q_2 &= (-c, r, 0)^T \\
q_3 &= (-c, -\frac{1}{2}r, \frac{\sqrt{3}}{2}r)^T \\
q_4 &= (-c, -\frac{1}{2}r, -\frac{\sqrt{3}}{2}r)^T
\end{align*}
\]

where \( a, b, c \in (0, 1) \), \( r^2 + c^2 = 1 \), \( a^2 + b^2 = 1 \). We show that there are values of \( a, c \) such that \( \sin d_{ij} \) is a constant for all pairs of \( \{i, j\} \). Since the configuration is not on one hemisphere, this configuration leads to a Dziobek equilibrium configuration with the property \( \sum m_i q_i = 0 \) but not a regular simplex. Indeed, we only need to solve

\[
q_1 \cdot q_2 = q_2 \cdot q_3, \quad q_1 \cdot q_2 = -q_1 \cdot q_3.
\]

In coordinates, the system is

\[ 4ac = \sqrt{(1 - a^2)(1 - c^2)}, \quad 3c^2 - 1 = 6ac. \]

The two algebraic curves defined by the equations has one intersection in \((0, 1) \times \)
Thus, there is Dziobek equilibrium configuration on $S^2$ that is not regular simplex but satisfies $\sum m_i q_i = 0$.

**Appendix:** The derivative of the Cayley-Menger determinant

For a Dziobek configuration of $n$-body in $S^{n-2}$, recall the $(n-1) \times n$ matrix $X = [q_1, \ldots, q_n]$. Since rank $X = n - 1$, the corresponding Gram matrix $X^T X$ has rank $n - 1$. Then the determinant $F = 0$. We may call the quantity $F$ the spherical Cayley-Menger determinant, [4]. For instance, for $n = 4$,

$$F = \begin{vmatrix}
1 & \cos d_{12} & \cos d_{13} & \cos d_{14} \\
\cos d_{12} & 1 & \cos d_{23} & \cos d_{24} \\
\cos d_{13} & \cos d_{23} & 1 & \cos d_{34} \\
\cos d_{14} & \cos d_{24} & \cos d_{34} & 1
\end{vmatrix}.$$

A by-product of equation (6) is the following. A Dziobek configuration on $S^{n-2}$ can be parametrized by the $C_{2,n}^2$ quantities $\{\cos d_{12}, \ldots, \cos d_{n-1,n}\}$ with the relation $F = 0$. Then any equilibrium configuration of the system (11) is the critical point of $V + \lambda F$. Then equation (6) implies $\frac{\partial F}{\partial \cos d_{ij}} = \alpha \Delta_i \Delta_j$ for some $\alpha$.

**Proposition 5.** Let $q_1, \ldots, q_n$ be a Dziobek configuration in $S^{n-2}$. Let $d_{12}, \ldots, d_{n-1,n}$ and $F$ be the corresponding mutual distances and the spherical Cayley-Menger determinant. Then we have

$$\frac{\partial F}{\partial \cos d_{ij}} = 2\Delta_i \Delta_j \text{ for any } 1 \leq i < j \leq n.$$

where $\Delta_i$ is the signed determinant defined in (5).

**Proof.** By the symmetry of $X^T X$, we have $\frac{\partial F}{\partial \cos d_{ij}} = 2F_{ij}$, with $F_{ij}$ being the $(i,j)$ cofactor of matrix $X^T X$, i.e.,

$$\frac{\partial F}{\partial \cos d_{ij}} = 2(-1)^i(-1)^j |A_{ij}|,$$
where \( A_{ij} \) is the \((i, j)\) minor of matrix \( X^T X \). Let \( X_k \) be the square matrix of order \( n - 1 \) obtained from \( X \) by deleting the \( k \)-th column. Then \( X_i^T X_j = A_{ij} \). Thus, we have \( \frac{\partial F}{\partial \cos d_{ij}} = 2\Delta_i \Delta_j \).

This derivative formula enables us to obtain equation (6) directly.

For a Dziobek configuration \( \mathbf{x} = (x_1, \ldots, x_n) \) in \( \mathbb{R}^{n-2} \), the mutual distances satisfy a relation and its derivative formula is similar to the above one. Due to the translational symmetry, the appropriate Gram matrix is \( \tilde{X}^T \tilde{X} \), with

\[
\tilde{X} = [x_2 - x_1, \ldots, x_n - x_1].
\]

It is easy to see that \(|\tilde{X}^T \tilde{X}| = 0 \). Note that the entries of \( X^T \tilde{X} \) are not in terms of the mutual distances. By using the formula \((x_i - x_1) \cdot (x_j - x_1) = \frac{1}{2} (d_{ij}^2 + d_{1j}^2 - d_{1i}^2) \) and some bordering technique, \([4]\), we can obtain another determinant

\[
\Gamma = \begin{vmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & d_{12}^2 & \cdots & d_{1n}^2 \\
1 & d_{21}^2 & 0 & \cdots & d_{2n}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & d_{n1}^2 & d_{n2}^2 & \cdots & 0
\end{vmatrix}, \quad \text{and} \quad \Gamma = (-1)^n 2^{n-1} |\tilde{X}^T \tilde{X}|.
\]

Usually, it is \( \Gamma \) instead of \(|\tilde{X}^T \tilde{X}| \) that is called the Cayley-Menger determinant. Let

\[
X = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n
\end{bmatrix}_{(n-1) \times n},
\]

and \( X_k \) be the square matrix of order \( n - 1 \) obtained from \( X \) by deleting the \( k \)-th column. Let \( \Delta_k = (-1)^{k-1} |X_k| \). For \( n = 4 \), Dziobek \([9]\) observed a formula that is equivalent to

\[
\frac{\partial \Gamma}{\partial d_{ij}^2} = -8\Delta_i \Delta_j \text{ for any } 1 \leq i < j \leq n.
\]

With the technique used to relate \( \Gamma \) and \(|\tilde{X}^T \tilde{X}| \), we have

**Proposition 6.** Let \( x_1, \ldots, x_n \) be a Dziobek configuration in \( \mathbb{R}^{n-2} \). Let \( d_{12}, \ldots, d_{n-1,n} \) be the corresponding mutual distances. Let \( \Gamma \) and \( \Delta_i \) be the determinants defined above. Then we have

\[
\frac{\partial \Gamma}{\partial d_{ij}^2} = (-2)^{n-1} \Delta_i \Delta_j \text{ for any } 1 \leq i < j \leq n.
\]

**Proof.** By the symmetry, we have \( \frac{\partial \Gamma}{\partial d_{ij}^2} = 2(-1)^i (-1)^j |B_{ij}| \) where \( B_{ij} \) is the \((i+1, j+1)\) minor of \( \Gamma \). On the other hand, note that

\[
\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 1 \\
0 & x_1 & x_2 & \cdots & x_n
\end{vmatrix} = -\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
0 & x_1 & x_2 & \cdots & x_n
\end{vmatrix}.
\]
Bordering $X_j$ in the same way without exchanging the first two row, we obtain

$$
|X^T X_j| = -
\begin{vmatrix}
0 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & x_1 \cdot x_1 & \cdots & x_1 \cdot x_i & \cdots & x_1 \cdot x_{j-1} & x_1 \cdot x_{j+1} & \cdots & x_1 \cdot x_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{i-1} \cdot x_1 & \cdots & x_{i-1} \cdot x_i & \cdots & x_{i-1} \cdot x_{j-1} & x_{i-1} \cdot x_{j+1} & \cdots & x_{i-1} \cdot x_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{i+1} \cdot x_1 & \cdots & x_{i+1} \cdot x_i & \cdots & x_{i+1} \cdot x_{j-1} & x_{i+1} \cdot x_{j+1} & \cdots & x_{i+1} \cdot x_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & x_j \cdot x_1 & \cdots & x_j \cdot x_i & \cdots & x_j \cdot x_{j-1} & x_j \cdot x_{j+1} & \cdots & x_j \cdot x_n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & x_n \cdot x_1 & \cdots & x_n \cdot x_i & \cdots & x_n \cdot x_{j-1} & x_n \cdot x_{j+1} & \cdots & x_n \cdot x_n 
\end{vmatrix}
$$

We then replace $x_i \cdot x_j$ by $\frac{1}{2}(|x_i|^2 + |x_j|^2 - d_{ij}^2)$, and eliminate all the $|x_i|^2$ by subtracting the appropriate multiple of the first row and column from the others. We obtain

$$
\Delta_i \Delta_j = (-1)^{i+j} |X^T X_j| = (-1)^{i+j} 2^{n-2} (-1)^{n-1} |B_{ij}|.
$$

Hence follows the formula $\frac{d}{dt^2} = (-2)^{n-1} \Delta_i \Delta_j$. \hfill \square

Central configuration in $\mathbb{R}^n$ of dimension $n-2$ are considered in [11]. The equations of them are derived by vectorial method there. Note that these equations follows easily from the above derivative formula.

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