THE DIRICHLET PROBLEM IN THE PLANE WITH SEMIANALYTIC RAW DATA, QUASIANALYTICITY AND O-MINIMAL STRUCTURES

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Abstract
We investigate the Dirichlet solution for a semianalytic continuous function on the boundary of a semianalytic bounded domain in the plane. We show that the germ of the Dirichlet solution at a boundary point with angle greater than 0 lies in a certain quasianalytic class used by Ilyashenko in his work on Hilbert’s 16th problem. With this result we can prove that the Dirichlet solution is definable in an o-minimal structure if the angle at a singular boundary point of the domain is an irrational multiple of $\pi$.

Introduction
Traditional and excellent settings for ‘tame geometry’ on the reals are given by the category of semialgebraic sets and functions and by the category of subanalytic sets and functions. The sets considered may have singularities but behave still ‘tame’, i.e. various finiteness properties hold, see Bierstone-Milman [4], Bochnak et al. [5], Denef-Van den Dries [8], Lojasiewicz [32] and Shiota [35]. These categories are excellent for geometrical questions but as often observed they are insufficient for problems from analysis. For example, the solution of the differential equation $y' = \frac{y}{x^2}$ on $\mathbb{R}_{>0}$, given by $x \mapsto e^{-\frac{1}{x}}$, is not subanalytic anymore. Therefore a natural aim was a better understanding

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of the solutions of first order ordinary differential equations or more general of Pfaffian equations with polynomial or analytic raw data, and a lot of research activities was done in this direction.

It was shown that sets defined by the solutions of Pfaffian equations, so-called semi- and sub-Pfaffian sets, show a ‘tame’ behaviour, see for example Cano et al. [6], Gabrielov [18], Gabrielov et al. [19] and Lion-Rolin [30]. Also a more axiomatic understanding was obtained. This axiomatic setting is given by the framework of o-minimal structures. They generalize the category of semialgebraic sets and functions and are defined by finiteness properties. They are considered as “an excellent framework for developing tame topology, or topologie modérée, as outlined in Grothendieck’s prophetic “Esquisse d’un Programme” of 1984” (see the preface of Van den Dries [10], which provides a very good source for the definition and the basic properties of o-minimal structures).

The basic example for an o-minimal structure is given by the semialgebraic sets and functions; these are the sets which are definable from the real field $\mathbb{R}$ by addition, multiplication and the order. The subanalytic category fits not exactly in this concept (compare with Van den Dries [9]), but the globally subanalytic sets, i.e. the sets which are subanalytic in the ambient projective space, form an o-minimal structure, denoted by $\mathbb{R}_{\text{an}}$ (see Van den Dries-Miller [13]). A breakthrough was achieved by Wilkie, who showed in [40], using Khovanskii theory for Pfaffian systems (see [29]), that the real exponential field $\mathbb{R}_{\text{exp}}$, i.e. the field of reals augmented with the global exponential function $\exp: \mathbb{R} \to \mathbb{R}_{>0}$, is an o-minimal structure. Subsequently Van den Dries-Miller [12] and Van den Dries et al. [11] proved that the structure $\mathbb{R}_{\text{an,exp}}$ is o-minimal. For general Pfaffian functions o-minimality was again obtained by Wilkie [41]. This result was extended by Karpinski-Macintyre [28] and finally stated by Speissegger [36] in its most generality: the Pfaffian closure of an o-minimal structure on the real field is again o-minimal.

So first order differential equations or more general Pfaffian equations in the subanalytic context resp. in the context of o-minimal structures are well understood. As an application integration of a one variable function in an o-minimal structure can be handled (for integration with parameters this is the case so far only for subanalytic maps by the results of Lion-Rolin [31] and
Our goal is to attack higher order partial differential equations in the subanalytic resp. o-minimal setting. Compared to ordinary differential equations there are distinct classes of equations and boundary value problems, each with its own theory. A very important class of PDE’s is given by the elliptic ones and one of its outstanding representative is the Laplace equation. We consider the Dirichlet problem, i.e. the Laplace equation with boundary value problem of the first kind: let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $h \in C(\partial \Omega)$ be a continuous function on the boundary. Then the Dirichlet problem for $h$ is the following: is there a function $u$ continuous on $\Omega$ and twice differentiable in $\Omega$ such that

$$\Delta u = 0 \quad \text{in} \quad \Omega,$$

$$u = h \quad \text{on} \quad \partial \Omega.$$ 

Here $\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplace operator. Functions fulfilling the first equality are called harmonic in $\Omega$. They are actually real analytic on $\Omega$. If the answer is yes, i.e. if such a $u$ exists, we call it the Dirichlet solution for $h$. If the answer is yes for all continuous boundary functions the domain $\Omega$ is called regular. The punctured open ball for example is irregular (see Helms [20, p.168]). Simply connected domains in the plane are regular. In [25] we gave in the case that $\Omega$ is subanalytic a necessary and sufficient condition for $\Omega$ to be regular. For irregular domains there is a more general solution for the Dirichlet problem, the so-called Perron-Wiener-Brelot solution (see Armitage-Gardiner [2, Chapter 6] and [20, Chapter 8]).

We are interested in the case that $\Omega$ is a subanalytic domain and that also the boundary function $h$ is subanalytic. The natural questions are now the following. What can be said about the Dirichlet solution? Is it definable in an o-minimal structure? We consider the case that $\Omega$ is a domain in the plane (then $\Omega$ and $h$ are semianalytic; see [4, Theorem 6.13]). By [25] $\Omega$ is regular if it has no isolated boundary points. If this is not the case the Perron-Wiener-Brelot solution for a continuous boundary function coincides with the Dirichlet solution for this function after adding the finitely many isolated boundary points to $\Omega$ (compare with [20, p.168]). So from now on we may assume that $\Omega$ has no isolated boundary points. Under the additional
assumption that the boundary is analytically smooth it was shown in [24] that the Dirichlet solution is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. This result is obtained by reducing the problem to the unit ball. There the Dirichlet solutions are given by the Poisson integral (see [2, Chapter 1.3]) and we can apply the results about integration of subanalytic functions (see [7] and [31]).

The challenging part are domains with singularities. The starting point to attack singularities are asymptotic expansions. Given a simply connected domain $D$ in $\mathbb{R}^2$ which has an analytic corner at $0 \in \partial D$ (i.e. the boundary at $0$ is given by two regular analytic curves which intersect in an angle $\langle D$ greater than $0$) and a continuous boundary function $h$ which is given by power series on these analytic curves, Wasow showed in [39] that the Dirichlet solution for $h$ is the real part of a holomorphic function $f$ on $D$ which has an asymptotic development at $0$ of the following kind:

\[
(\dagger) \quad f(z) \sim \sum_{n=0}^{\infty} a_n P_n(\log z) z^{\alpha_n} \text{ as } z \to 0 \text{ on } D,
\]

i.e. for each $N \in \mathbb{N}_0$ we have

\[
f(z) - \sum_{n=0}^{N} a_n P_n(\log z) z^{\alpha_n} = o(z^{\alpha_n}) \text{ as } z \to 0 \text{ on } D,
\]

where $\alpha_n \in \mathbb{R}_{\geq 0}$ with $\alpha_n \not\to \infty$, $P_n \in \mathbb{C}[z]$ monic and $a_n \in \mathbb{C}^*$. Moreover, if $\langle D/\pi \in \mathbb{R} \setminus \mathbb{Q}$ we have that $P_n = 1$ for all $n \in \mathbb{N}_0$. (Note that $P_0 = 1$ for any angle).

To use this asymptotic development we want to have a quasianalytic property; we want to realize these maps in a class of functions with an asymptotic development as in $(\dagger)$ such that the functions in this class are determined by the asymptotic expansion. Such quasianalyticity properties are key tools in generating o-minimal structures (see [27], Van den Dries-Speissegger [14, 15] and Rolin et al. [34]; see also Badalayan [3] for quasianalytic classes of this kind).

Exactly the same kind of asymptotic development occurs at the transition map of a real analytic vector field on $\mathbb{R}^2$ at a hyperbolic singularity (see Ilyashenko [22]). Poincaré return maps are compositions of finitely many transition maps...
and are an important tool to the qualitative understanding of the trajectories and orbits of a polynomial or analytic vector field on the plane. Following Dulac’s approach (see [16]), Ilyashenko uses asymptotic properties of the Poincaré maps to prove Dulac’s problem (the weak form of (the second part) of Hilbert’s 16th problem): a polynomial vector field on the plane has finitely many limit cycles (see Ilyashenko [23] for an overview of the history of Hilbert 16, part 2). One of the first steps in Ilyashenko’s proof is to show that the transition map at a hyperbolic singularity is in a certain quasianalytic class. Formulating his result on the Riemann surface of the logarithm (compare with the introduction of [27] and with [27, Proposition 2.8]) he proves that the considered transition maps have a holomorphic extension to certain subsets of the Riemann surface of the logarithm, so-called standard quadratic domains (see [22, §0.3] and [27]; see also Section 1 below), such that the asymptotic development holds there. Quasianalyticity follows then by a Phragmén-Lindelöf argument (see [22, §3.1]). The extension of the transition map at a hyperbolic singularity and its asymptotic development is obtained by transforming the vector field by a real analytic change of coordinates into a suitable form. Then the complexification of the resulting vector field is considered and certain curves are lifted to the Riemann surfaces of its complex phase curves (see [22, §0.3] and Ilyashenko [21, Section 3]). The extension process can be rediscovered as a discrete dynamical system given by the iteration of a local biholomorphic map which fixes the origin and whose first derivative at the origin has absolute value 1 (see [21, Proposition 3]).

We can link the Dirichlet problem with semianalytic raw data and Ilyashenko’s quasianalytic class:

THEOREM A

Let \( \Omega \subset \mathbb{R}^2 \) with \( 0 \in \partial \Omega \) be a bounded semianalytic domain without isolated boundary points. Let \( h \) be a semianalytic and continuous function on the boundary and let \( u \) be the Dirichlet solution for \( h \). If the angle of \( \Omega \) at 0 is greater than 0, then \( u \) is the real part of a holomorphic function which is in the quasianalytic class of Ilyashenko described above.
We will give a precise definition of 'the angle' of $\Omega$ at a boundary point in Section 2 below.

The semianalytic boundary curves of the given domain and the semianalytic boundary function are locally given by Puiseux series (see [9, p.192]). We obtain Theorem A by extending the result of Wasow to Puiseux series and by a geometric argument: we repeat reflections of the Dirichlet solution at two real analytic curves to go all the way up and down the Riemann surface of the logarithm to get the extension to a standard quadratic domain. It is crucial that the boundary function is given by Puiseux series to be able to apply the reflection process. In the literature pairs of germs of real analytic curves resp. groups generated by two non-commuting antiholomorphic involutions (reflections at real analytic curves correspond to antiholomorphic involutions) are studied in the context of the classification of germs of biholomorphic maps fixing the origin which is investigated by the Ecalle-Voronin theory (see Ecalle [17], Voronin [37, 38] and Ahern-Gong [1]). Local biholomorphic maps which fix the origin and whose first derivatives at the origin have absolute value 1 occur also at the reflection process: the description of the repeated reflections of the Dirichlet solution involves iteration, inversion and conjugation of such maps (resp. their lifting to the Riemann surface of the logarithm). Also summation of Puiseux series is involved. We carefully estimate the functions obtained by the reflection process to get Theorem A.

Transition maps at a hyperbolic singularity exhibit a similar dichotomy of the asymptotic development as indicated in (†), depending on whether the hyperbolic singularity is resonant or non-resonant, i.e. whether the ratio of the two eigenvalues of the linear part of the vector field at the given hyperbolic singularity is rational or irrational, see [16] and [27]. In [27] it is shown that transition maps at non-resonant hyperbolic singularities are definable in a common o-minimal structure, denoted by $\mathbb{R}_Q$. This structure $\mathbb{R}_Q$ is generated by the functions (restricted to the positive line) in Ilyashenko’s quasianalytic class that have no log-terms in their asymptotic expansion.

With this result and Theorem A, taking into account the nonsingular boundary points (compare with [24]), we can prove
THEOREM B

Let $\Omega \subset \mathbb{R}^2$ be a bounded semianalytic domain without isolated boundary points and let $h$ be a semianalytic and continuous function on the boundary. Suppose that the following condition holds: if $x$ is a singular boundary point of $\Omega$ then the angle of the boundary at $x$ divided by $\pi$ is irrational. Then the Dirichlet solution for $u$ (i.e. its graph considered as a subset of $\mathbb{R}^3$) is definable in the o-minimal structure $\mathbb{R}_{\mathbb{Q},\exp}$.

As an application we obtain that the Green function of a bounded semianalytic domain fulfilling the assumptions of Theorem B, is definable in the o-minimal structure $\mathbb{R}_{\mathbb{Q},\exp}$. If the considered domain is semilinear the assumption on the angles can be dropped.

In [26] it is shown that the Riemann map from a simply connected bounded and semianalytic domain in the plane with the same assumptions on the angles as above to the unit ball is definable in $\mathbb{R}_{\mathbb{Q}}$. There it is also the key step to realize the function in question in the quasianalytic class of Ilyashenko. But the main ingredient, the reflection procedure, differs heavily. There is some overlap in the definitions, but in order to keep this paper reasonably self-contained we include all necessary definitions here.

The paper is organized as follows: Section 1 is about the Riemann surface of the logarithm and the classes of functions which we use later. In particular Ilyashenko’s quasianalytic class is introduced. In Section 2 we define the notion of an angle for semianalytic domains in a rigorous way and we present the concept of a domain with analytic corner. In Section 3 we prove Theorem A and Theorem B and give applications.

Notation

By $\mathbb{N}$ we denote the set of natural numbers and by $\mathbb{N}_0$ the set of nonnegative integers. Let $a \in \mathbb{R}^n$ and $r > 0$. We set $B(a, r) := \{ z \in \mathbb{R}^n \mid |z - a| < r \}$, $\overline{B}(a, r) := \{ z \in \mathbb{R}^n \mid |z - a| \leq r \}$ and $\dot{B}(a, r) := B(a, r) \setminus \{a\}$. Here $| \cdot |$ denotes the euclidean norm in $\mathbb{R}^n$. A domain is an open, nonempty and connected set (in a topological space). Given an open set $U$ of a Riemann surface we denote by $\mathcal{O}(U)$ the $\mathbb{C}$-algebra of holomorphic functions on $U$ with values in
By $\mathcal{O}_0$ we denote the $\mathbb{C}$-algebra of germs of holomorphic functions in open
neighbourhoods of $0 \in \mathbb{C}$. Note that $\mathcal{O}_0 = \mathbb{C}\{z\}$. We identify $\mathbb{C}$ with $\mathbb{R}^2$.

1. The Riemann surface of the logarithm

We establish the setting on the Riemann surface of the logarithm. We introduce the quasianalytic class of Ilyashenko, generalizing the setting from [27, Section 2 & 5] (we apply the results and methods of [27] to obtain Theorem B in Section 3). We define several other classes of functions we will use in the proof of Theorem A.

**Definition 1.1** (compare with [27, pp.12-13])

We define the Riemann surface of the logarithm $L$ in polar coordinates by $L := \mathbb{R}_{>0} \times \mathbb{R}$. Then $L$ is a Riemann surface with the isomorphic holomorphic projection map $\log: L \rightarrow \mathbb{C}$, $(r, \varphi) \mapsto \log r + i\varphi$. For $z = (r, \varphi) \in L$ we define the absolute value $|z| := r$ and the argument $\arg z := \varphi$. For $r > 0$ we set $B_L(r) := \{z \in L \mid |z| < r\}$. We identify $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ with $\mathbb{R}_{>0} \times ]-\pi, \pi[ \subset L$ via polar coordinates. Let $\alpha \geq 0$. We define the power function $z^\alpha$ as $z^\alpha: L \rightarrow \mathbb{C}$, $z = (r, \varphi) \mapsto \exp(\alpha \log z)$.

**Remark 1.2**

The logarithm on $L$ extends the principal branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. The power functions on $L$ extend the power functions on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

**Definition 1.3** (compare with [27, p.25])

a) Let $\rho \geq 0$. We define the holomorphic map $p^\rho: L \rightarrow L$, $(r, \varphi) \mapsto (r^\rho, \rho \varphi)$.

b) We define the holomorphic map $m: L^2 \rightarrow L$, $((r_1, \varphi_1), (r_2, \varphi_2)) \mapsto (r_1 r_2, \varphi_1 + \varphi_2)$.

**Remark 1.4**

a) Let $r > 0$ and let $f: B(0, r) \rightarrow \mathbb{C}$ be holomorphic. Then the function $f_L: B_L(r) \rightarrow \mathbb{C}$, $z \mapsto f(z^1)$, is holomorphic (compare with Definition 1.1).
b) Let $r > 0$ and let $g = \sum_{n=0}^{\infty} a_n t^n$ be a convergent Puiseux series on $]0, r[$ (with complex coefficients). Then $g_L: B_L(r) \to \mathbb{C}, z \mapsto f_L(p^\frac{1}{d}(z))$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(B(0, r^d))$, is holomorphic. Let $z \in B_L(r)$.

Then $g_L(z) = \sum_{n=0}^{\infty} a_n z^n$ as an absolutely convergent sum, i.e. $\sum_{n=0}^{\infty} |a_n| |z|^n < \infty$ (where $z^n$ as defined in Definition 1.2). So we do often not distinguish between $g$ and $g_L$. Viewed as a function on $L$ we say that $g(z) = \sum_{n=0}^{\infty} a_n z^n$ is a Puiseux series convergent on $B_L(r) \subset L$. If $d = 1$ (i.e. we are in case a)) we say that $g$ is a power series convergent on $B_L(r)$.

**Proposition 1.5**

Let $r > 0$ and let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be a Puiseux series convergent on $B_L(r) \subset L$.

a) Let $t > r$ and assume that there is some $G \in \mathcal{O}(B_L(t))$ such that $G = g$ on $B_L(r)$. Then $\sum_{n=0}^{\infty} a_n z^n$ is convergent on $B_L(t)$.

b) Let $c > 0$ such that $|g(z)| \leq c$ for $|z| < r$. Then for every $N \in \mathbb{N}$ and every $|z| < r$, we obtain

$$|g(z) - \sum_{n=0}^{N} a_n z^n| \leq c \left( \frac{|z|}{r} \right)^{\frac{N+1}{d}} \left( 1 - \left( \frac{|z|}{r} \right)^{\frac{1}{d}} \right).$$

**Proof**

a) We set $h: B_L(t^\frac{1}{d}) \to \mathbb{C}, z \mapsto G(p^d(z))$. Then $h \in \mathcal{O}(B_L(t^\frac{1}{d}))$ and $h(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < r^\frac{1}{d}$. Let $a := (1, 2\pi)$. Note that $h(m(a, z)) = h(z)$ for all $|z| < r^\frac{1}{d}$ and hence for all $|z| < t^\frac{1}{d}$. Hence $\tilde{h}: B(0, t^\frac{1}{d}) \to \mathbb{C}, z \mapsto h(|z| e^{i \text{Arg} z})$ (with $\text{Arg} z \in [-\pi, \pi]$) the standard argument of a complex number $z \in \mathbb{C}^*$, is well defined and holomorphic. We have $\tilde{h}(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < r^\frac{1}{d}$ and hence for $|z| < t^\frac{1}{d}$ by Cauchy’s Theorem.
By Remark 1.4 we see that \( G(z) = \tilde{h}_L(p^{1/2}(z)) \) is a Puiseux series which is convergent on \( B_L(t) \).

b) Let \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) be the corresponding Puiseux series on \( B(0, r) \), i.e. \( g = h_L \). Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(B(0, r^{1/2})) \). From the Cauchy estimates we obtain

\[
|f(z) - \sum_{n=0}^{N} a_n z^N| \leq c \left( \frac{|z|}{r^{1/2}} \right)^{N+1} \left( \frac{1}{1-\frac{|z|}{r}} \right)
\]

and the claim follows, since \( g = f_L \circ p^{1/2} \).

\[\Box\]

**Definition 1.6**

Let \( g = \sum_{n=0}^{\infty} a_n z^n \) be a Puiseux series on \( L \). We call \( d \in \mathbb{N} \) a denominator of \( g \). Note that \( d \) is not unique.

**Definition 1.7**

a) Let \( r > 0 \). We denote the set of all holomorphic functions \( B_L(r) \to L \) by \( \mathcal{OL}(B_L(r)) \). We define an equivalence relation \( \equiv \) on \( \bigcup_{r>0} \mathcal{OL}(B_L(r)) \) as follows: \( f_1 \equiv f_2 \) iff there is some \( r > 0 \) such that \( f_1, f_2 \in \mathcal{OL}(B_L(r)) \) and \( f_1|_{B_L(r)} = f_2|_{B_L(r)} \). We let \( \mathcal{OL}_0 \) be the set of all \( \equiv \) - equivalence classes.

b) We define \( \mathcal{OL}_0' \subset \mathcal{OL}_0 \) to be the set of all \( \varphi \in \mathcal{OL}_0 \) such that there is some \( r > 0 \), some \( h \in \mathcal{O}(B(0, r)) \) with \( h(0) = 0 \) and \( |h(z)| \leq \frac{1}{2} \) for \( |z| < r \), some \( a \in L \) and some \( k \in \mathbb{N}_0 \) such that \( \varphi(z) = m(a, m(p^k(z), 1 + h_L(z))) \) for \( z \in B_L(r) \). Note that \( 1 + h_L(z) \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \subset L \) for \( |z| < r \). We write \( r(\varphi) \), \( h(\varphi) \), \( a(\varphi) \) and \( k(\varphi) \) for the data above. We set \( \mathcal{OL}_0^\wedge := \{ \varphi \in \mathcal{OL}_0' \mid k(\varphi) \geq 1 \} \) and \( \mathcal{OL}_0^\circ := \{ \varphi \in \mathcal{OL}_0' \mid k(\varphi) = 1 \} \). We define \( s: \mathcal{OL}_0' \to \mathcal{O}_0 \), \( s(\varphi)(z) := a(\varphi)^{1/k(\varphi)}(1 + h(\varphi)(z)) \).
PROPOSITION 1.8

a) Let $\varphi \in \mathcal{OL}_0'$ and $\psi \in \mathcal{OL}_0^\land$. Then $\varphi \circ \psi \in \mathcal{OL}_0'$.

b) $\mathcal{OL}_0^\land$ is a group under composition.

Proof

a) We see immediately that $a(\varphi \circ \psi) = m(a(\varphi), p^k(a(\psi)))$, that $k(\varphi \circ \psi) = k(\varphi)k(\psi)$ and that

$$h(\varphi \circ \psi) = (1 + h(\psi))^k(1 + h(\varphi) \circ s(\psi)) - 1.$$

b) Let $\varphi \in \mathcal{OL}_0^\land$. We show that there is $\psi \in \mathcal{OL}_0^\land$ with $\varphi \circ \psi = \text{id}_L$ and are done by a). We consider $f := s(\varphi)$. Then $f(0) = 0$ and $f'(0) = a(\varphi)^1 \neq 0$. Hence $f$ is invertible at 0 and $f^{-1}$ has the form $f^{-1}(z) = f'(0)^{-1}z(1 + \tilde{h}(z))$ with $\tilde{h}(0) = 0$. Let $b \in L$ with $m(a(\varphi), b) = (1, 0)$. Then $\psi(z) := m(b, m(z, 1 + \tilde{h}_L(z)))$ fulfills the requirements.

PROPOSITION 1.9

Let $\varphi, \psi \in \mathcal{OL}_0^\land$.

a) $|\varphi(z)| \leq 2|a(\varphi)||z|$ and $|\arg(\varphi(z)) - \arg a(\varphi)| \leq |\arg z| + \frac{\pi}{2}$ for $|z| < r(\varphi)$.

b) We can choose $r(\varphi \circ \psi) = \frac{1}{10} \frac{\min(r(\varphi), r(\psi))}{\max\{1, |a(\varphi)|, |a(\psi)|\}}$.

Proof

a) We have $|h(\varphi)(z)| < 1$ for $z \in B(0, r(\varphi))$. Hence $|\varphi(z)| = |s(\varphi)(z)| = |a(\varphi)| |z| |1 + h(\varphi)(z)| \leq 2|a(\varphi)||z|$.

We have $|\arg(1 + h(\varphi)_L(z))| \leq \frac{\pi}{2}$ for $|z| < r(\varphi)$. This gives the second part of a).

b) Let $r := r(\varphi)$, $s := r(\psi)$ and $t := \frac{1}{10} \frac{\min(r, s)}{\max\{1, |a(\varphi)|, |a(\psi)|\}}$. By a) we have that $s(\psi)(z) \in B(0, r)$ for $z \in B(0, t)$. Applying the maximum principle we obtain that $|h(\varphi)(z)| \leq \frac{|z|}{r}$ for $|z| < r$ and $|h(\psi)(z)| \leq \frac{|z|}{s}$ for $|z| < s$.

With Proposition 1.8 we see that $h(\varphi \circ \psi) = h(\psi) + h(\varphi) \circ s(\psi) + h(\psi) \cdot (h(\varphi) \circ s(\psi))$ and obtain the claim by the above estimates and a).
PROPOSITION 1.10
Let \( g \) be a Puiseux series convergent on \( B_L(r) \subset L \) with denominator \( d \). Let \( \varphi \in \mathcal{O}L_0^\wedge \). Then \( g \circ \varphi \) is a Puiseux series with denominator \( d \) convergent on \( B_L(s) \subset L \) with \( s := \min \left\{ r(\varphi), \left( \frac{r}{2|a(\varphi)|} \right)^{k(\varphi)} \right\} \).

Proof
As in Proposition 1.9 a) we see that \( |\varphi(z)| \leq 2|a(\varphi)| |z|^{k(\varphi)} \) for \( |z| < r(\varphi) \). Hence \( \varphi(z) \in B_L(r) \) for \( |z| < s \) and therefore \( g \circ \varphi \in \mathcal{O}L(B_L(s)) \). By binomial expansion we obtain that \( g \circ \varphi \) is a Puiseux series convergent on \( B_L(t) \) for some \( t \leq s \). We get the claim by Proposition 1.5 a).

Definition 1.11 (compare with [27, p.6])
Let \( z \) be an indeterminate. A generalized log-power series in \( z \) is a formal expression \( g(z) = \sum_{\alpha \in \mathbb{R}_{\geq 0}} a_\alpha P_\alpha(\log z) z^\alpha \) with \( a_\alpha \in \mathbb{C} \) and \( P_\alpha \in \mathbb{C}[z] \setminus \{0\} \) monic such that \( P_0 = 1 \) and such that the support of \( g \) defined as \( \text{supp}(g) := \{ \alpha \in \mathbb{R}_{\geq 0} \mid a_\alpha \neq 0 \} \) fulfils the following condition: for all \( R > 0 \) the set \( \text{supp}(g) \cap [0, R] \) is finite. We write \( \mathbb{C}[[z^\ast]]^{\omega}_{\log} \) for the set of generalized log-power series. These series are added and multiplied by considering them as generalized power series with logarithmic polynomials as coefficients. For \( g \in \mathbb{C}[[z^\ast]]^{\omega}_{\log} \) we set \( \nu(g) := \min \text{supp}(g) \). By \( \mathbb{C}[[z^\ast]]^{\omega} \) we denote the subset of \( \mathbb{C}[[z^\ast]]^{\omega}_{\log} \) consisting of all \( g \in \mathbb{C}[[z^\ast]]^{\omega}_{\log} \) with \( P_\alpha = 1 \) for all \( \alpha \in \mathbb{R}_{\geq 0} \). By \( \mathbb{C}[[z^\ast]]^{\omega,\text{fin}}_{\log} \) (resp. \( \mathbb{C}[[z^\ast]]^{\omega,\text{fin}} \)) we denote the set of all \( g \in \mathbb{C}[[z^\ast]]^{\omega}_{\log} \) (resp. \( \mathbb{C}[[z^\ast]]^{\omega} \)) with finite support.

Convention. From now on we omit the superscript \( \omega \).

Remark 1.12

a) The set \( \mathbb{C}[[z^\ast]]_{\log} \) is a \( \mathbb{C} \)-algebra with \( \mathbb{C}[[z^\ast]] \) as subalgebra.

b) Interpreting \( \log z \) and \( z^\alpha \) with Definition 1.1 we obtain that \( g \in \mathcal{O}(L) \) for \( g \in \mathbb{C}[[z^\ast]]_{\log}^{\text{fin}} \).
Definition 1.13 (compare with [27, Definition 2.3])
A domain $W \subset L$ of the Riemann surface of the logarithm is a standard quadratic domain if there are constants $c, C > 0$ such that

$$W = \left\{ (r, \varphi) \in L \mid 0 < r < c \exp(-C\sqrt{|\varphi|}) \right\}.$$

A domain is called a quadratic domain if it contains a standard quadratic domain.

Definition 1.14
Let $U \subset L$ be a quadratic domain, let $f \in \mathcal{O}(U)$ and let $g = \sum_{\alpha \geq 0} a_{\alpha} P_{\alpha}(\log z)z^\alpha \in \mathbb{C}[[z^*]]_{\log}$. We say that $f$ has asymptotic expansion $g$ on $U$ and write $f \sim_U g$, if for each $R > 0$ there is a quadratic domain $U_R \subset U$ such that

$$f(z) - \sum_{\alpha \leq R} a_{\alpha} P_{\alpha}(\log z)z^\alpha = o(|z|^R) \quad \text{as} \quad |z| \to 0 \quad \text{on} \quad U_R.$$

We write $Tf := g$. By $\mathcal{Q}^{\log}(U)$ we denote the set of all $f \in \mathcal{O}(U)$ with an asymptotic expansion in $\mathbb{C}[[z^*]]_{\log}$. By $\mathcal{Q}(U)$ we denote the subset of all $f \in \mathcal{Q}^{\log}(U)$ with $Tf \in \mathbb{C}[[z^*]]$.

Remark 1.15
If $f \in \mathcal{Q}^{\log}(U)$ for some quadratic domain $U$ then there is exactly one $g \in \mathbb{C}[[z^*]]_{\log}$ with $f \sim_U g$, i.e. $Tf$ is well defined.

Definition 1.16
We define an equivalence relation $\equiv$ on $\bigcup_{U \subset L \text{quad.}} \mathcal{Q}^{\log}(U)$ as follows: $f_1 \equiv f_2$ if and only if there is a quadratic domain $V \subset L$ such that $f_1|_V = f_2|_V$. We let $\mathcal{Q}^{\log}$ be the set of all $\equiv$ - equivalence classes. In the same way we obtain the class $\mathcal{Q}$. Note that $\mathcal{Q} = \mathcal{Q}^1$ in the notation of [27, Definition 5.1, Remarks 5.2 & Definition 5.4].

Remark 1.17
a) We will not distinguish between $f \in \bigcup_{U \subset L \text{quad.}} \mathcal{Q}^{\log}(U)$ and its equivalence class in $\mathcal{Q}^{\log}$, which we also denote by $f$. Thus $\mathcal{Q}^{\log}(U) \subset \mathcal{Q}^{\log}$ given a quadratic domain $U \subset L$. 
b) In the same way we define $\mathcal{Q} \subset \mathcal{Q}^{\log}$. We have $\mathcal{Q}(U) \subset \mathcal{Q}$ for $U \subset \mathfrak{L}$ a quadratic domain.

c) Given a quadratic domain $U \subset \mathfrak{L}$ the set $\mathcal{Q}^{\log}(U)$ is a $\mathbb{C}$-algebra with $\mathcal{Q}(U)$ as a subalgebra. Also, $\mathcal{Q}^{\log}$ is an algebra with $\mathcal{Q}$ as a subalgebra.

d) Given a quadratic domain $U \subset \mathfrak{L}$ the well defined maps $T: \mathcal{Q}^{\log}(U) \to \mathbb{C}[[z^*]]_{\log}$, $f \mapsto Tf$, and $T: \mathcal{Q}(U) \to \mathbb{C}[[z^*]]$, $f \mapsto Tf$, are homomorphisms of $\mathbb{C}$-algebras. Also the induced maps $T: \mathcal{Q}^{\log} \to \mathbb{C}[[z^*]]_{\log}$, $f \mapsto Tf$, and $T: \mathcal{Q} \to \mathbb{C}[[z^*]]$, $f \mapsto Tf$, are homomorphisms of $\mathbb{C}$-algebras.

**PROPOSITION 1.18**

*Let $U \subset \mathfrak{L}$ be a quadratic domain. The homomorphism $T: \mathcal{Q}^{\log}(U) \to \mathbb{C}[[z^*]]_{\log}$ is injective. Therefore the homomorphism $T: \mathcal{Q}^{\log} \to \mathbb{C}[[z^*]]_{\log}$ is injective.*

**Proof**

See Ilyashenko [22, Theorem 2 p.23] and [27, Proposition 2.8].

**PROPOSITION 1.19**

a) Let $g(z) = \sum_{n=0}^{\infty} a_n z^{\frac{n}{p}}$ be a Puiseux series convergent on $B_{\mathfrak{L}}(r) \subset \mathfrak{L}$. Then $g \in \mathcal{Q}(B_{\mathfrak{L}}(r))$ with $Tg = \sum_{n=0}^{\infty} a_n z^{\frac{n}{p}}$.

b) Let $f \in \mathcal{Q}^{\log}$ and let $\rho > 0$. Then $f \circ \rho \in \mathcal{Q}^{\log}$. If $f \in \mathcal{Q}$ then $f \circ \rho \in \mathcal{Q}$.

c) Let $f \in \mathcal{Q}^{\log}$ and let $\psi \in \mathcal{O}_{\mathfrak{L}}$. Then $f \circ \psi \in \mathcal{Q}^{\log}$. If $f \in \mathcal{Q}$ then $f \circ \psi \in \mathcal{Q}$.
Proof

a) is a consequence of Proposition 1.5 b).

b) Let $c, C > 0$ such that $f \in Q^{\log}(U)$ where $U := \{(r, \varphi) \in \mathbb{L} | 0 < r < c \exp(-C\sqrt{\vert \varphi \vert})\}$. Let $d := c^2$, $D := \frac{C}{\sqrt{2}}$ and 

$$W := \left\{(r, \varphi) \in \mathbb{L} | 0 < r < d \exp(-D\sqrt{\vert \varphi \vert})\right\}.$$  

Then we see that $p^\rho(W) \subset U$ and therefore $f \circ \varphi \in \mathcal{O}(W)$. The claim follows from the following observation: let $\alpha \geq 0$ and $m \in \mathbb{N}_0$. Then  

$$(z^\alpha (\log z)^m) \circ p^\rho = \rho^m z^{\alpha \rho} (\log z)^m.$$  

c) Let $k := k(\psi), b := p^\frac{1}{2}(a(\psi))$ and $\tilde{h}(z) := \sqrt{1 + h(\psi)(z)} - 1$. Then $\chi := m(b, m(z, 1 + \tilde{h}(z)) \in \mathcal{OL}_0^*$ and $\psi = p^k \circ \chi$. Hence we can assume by b) that $\psi \in \mathcal{OL}_0^*$.

Let $c, C > 0$ such that $f \in Q^{\log}(U)$ where  

$$U := \left\{(r, \varphi) \in \mathbb{L} | 0 < r < c \exp(-C\sqrt{\vert \varphi \vert})\right\}.$$  

Let $r_0 := r(\psi)$ and $z = (r, \varphi) \in B_L(r_0)$. Then $|\psi(z)| \leq 2|a(\psi)|r$ and $|\arg \psi(z)| \leq |\arg a(\psi)| + |\varphi| + \frac{\pi}{2}$ by Proposition 1.9 a). Let 

$$d := \min \left\{\frac{c}{2|a(\psi)|} \exp\left(-C\sqrt{\vert \arg a(\psi)\vert + \frac{\pi}{2}}\right), r_0\right\}$$  

and $W := \left\{(r, \varphi) \in \mathbb{L} | 0 < r < d \exp(-C\sqrt{\vert \varphi \vert})\right\} \subset B_L(r_0)$. Then $\psi(W) \subset U$ and therefore $f \circ \psi \in \mathcal{O}(W)$. Let $Tf := \sum_{\alpha \geq 0} a_\alpha P_\alpha (\log z) z^\alpha$.

Let $R > 0$. Then there is a quadratic domain $U_R$ such that  

$$f(z) - \sum_{\alpha \leq R} a_\alpha P_\alpha (\log z) z^\alpha = o(|z|^R) \quad \text{as} \quad |z| \rightarrow 0 \quad \text{on} \quad U_R.$$  

As above we find some quadratic domain $W_R \subset B_L(r_0)$ such that $\psi(W_R) \subset U_R$. With Proposition 1.9 a) we obtain  

$$f \circ \psi(z) - \left(\sum_{\alpha \leq R} a_\alpha P_\alpha (\log z) z^\alpha\right) \circ \psi = o(|z|^R) \quad \text{as} \quad |z| \rightarrow 0 \quad \text{on} \quad W_R.$$
Hence the claim follows from the next lemma. The second part of it will be used in section 3 below.

**Lemma 1.20**

Let $\alpha > 0$ and let $m \in \mathbb{N}_0$. Let $\varphi \in O\mathcal{L}^\wedge_0$. Then there are power series $g_1, \ldots, g_m$ convergent on $B_\mathcal{L}(r(\varphi))$ such that $(z^\alpha(\log z)^m) \circ \varphi = z^{k(\varphi)\alpha} \sum_{\ell=0}^m g_\ell(z)(\log z)^\ell$.

If $\varphi \in O\mathcal{L}^*_0$ with $|a(\varphi)| = 1$ then $|g_\ell(z)| \leq 2^{m+\alpha}(|\arg a(\varphi)| + 3)^m$ for every $\ell = 1, \ldots, m$.

**Proof**

Let $a := a(\varphi)$, $k := k(\varphi)$ and $h := h(\varphi)$. By the binomial formula we obtain

$$(z^\alpha(\log z)^m) \circ \varphi = a^\alpha z^{k\alpha}(1+h(z))^{\alpha} \sum_{\ell=0}^m \binom{m}{\ell} (\log m(a, 1+h_\mathcal{L}(z)))^{m-\ell}(\log p^k(z))^\ell.$$ 

Since $\log p^k(z) = k \log z$ we see that

$$(*) \quad g_\ell := k^\ell \binom{m}{\ell} a^\alpha (1+h(z))^{\alpha} (\log m(a, 1+h_\mathcal{L}(z)))^{m-\ell}$$

fulfills the claim.

Assume that $\varphi \in O\mathcal{L}^*_0$ with $|a(\varphi)| = |a| = 1$. We have $k = 1$ and $|a^\alpha| = 1$. Since $|h_\mathcal{L}(z)| \leq \frac{1}{2}$ for $z \in B_\mathcal{L}(r(\varphi))$ we obtain $|(1+h_\mathcal{L}(z))^\alpha| \leq \left(\frac{3}{2}\right)^\alpha \leq 2^\alpha$. Since $\binom{m}{\ell} \leq 2^m$ for all $\ell$ the non-logarithmic terms in $(*)$ are bounded from above by $2^{m+\alpha}$. We have

$$\log m(a, 1+h_\mathcal{L}(z)) =$$

$$= \log a + \log(1+h(z))$$

$$= i \arg a + \log |1+h(z)| + i \arg (1+h(z)).$$

Since $\frac{1}{2} \leq |1+h(z)| \leq \frac{3}{2}$ for $z \in B_\mathcal{L}(r(\varphi))$ we see that $|\log |1+h(z)|| \leq \log 2 \leq 1$ for $z \in B_\mathcal{L}(r(\varphi))$. Since $|\arg(1+h(z))| \leq \frac{\pi}{2}$ we conclude finally that $|\log m(a, 1+k(z))| \leq \arg a + 1 + \frac{\pi}{2} \leq \arg a + 3$. This gives the claim. $\square$
2. Angles and domains with an analytic corner

We introduce the notion of an angle. This allows us to formulate Theorem A and B in a precise way.

The proof of Theorem A will be reduced in Section 3 to domains with an analytic corner as considered by Wasow (see [39]). We give the definition. Sets definable in o-minimal structures are subsets of cartesian products of the reals. So for Theorem B we have to stay on the reals. For Theorem A we have to go out on the Riemann surface of the logarithm. We have to distinguish the ambient spaces carefully and we describe the lifting process. Finally we extend the result of Wasow on the existence of asymptotic expansion (see [39]) to the case where the boundary is given by Puiseux series.

Remark 2.1
Let $\Omega$ be a bounded and subanalytic domain in $\mathbb{R}^n$. Let $x \in \partial \Omega := \overline{\Omega} \setminus \Omega$. Then the germ of $\Omega$ at $x$ has finitely many connected components. More precisely we have the following: there is $k \in \mathbb{N}_0$ such that for all neighbourhoods $V$ of $x$ exactly $k$ of the components of the set $\Omega \cap V$ have $x$ as a boundary point.

Remark 2.2
Let $\Omega \subset \mathbb{R}^2$ be a bounded and semianalytic domain without isolated boundary points. Let $x \in \partial \Omega$ and let $C$ be a connected component of the germ of $\Omega$ at $x$. Then the germ of the boundary of $\partial C$ at $x$ is given by (the germs of) one or two semianalytic curves (see [4, Theorem 6.13]). So the interior angle of $C$ at $x$, denoted by $\angle_x C$, is well defined; it takes value in $[0, 2\pi]$. If the germ of $\Omega$ at $x$ is connected we write $\angle_x \Omega$.

Definition 2.3
Let $\Omega \subset \mathbb{R}^2$ be a bounded semianalytic domain without isolated boundary points.

a) A point $x \in \partial \Omega$ is a singular boundary point if $\partial \Omega$ is not a real analytic manifold at $x$.

b) We set $\text{Sing}(\partial \Omega) := \{ x \in \partial \Omega \mid x \text{ is a singular boundary point of } \partial \Omega \}$. 
c) Let $x \in \partial \Omega$. We set $\langle \Omega, x \rangle := \{ \angle_x C \mid C$ is a component of the germ of $\Omega$ at $x$ and $x \in \text{Sing}(\partial C) \}$.

**Remark 2.4**

Let $\Omega \subset \mathbb{R}^2$ be a bounded semianalytic domain without isolated boundary points.

a) $\text{Sing}(\partial \Omega)$ is finite by analytic cell decomposition (see [13, pp.508-509]).

b) Let $x \in \partial \Omega$. Then $\langle \Omega, x \rangle = \emptyset$ if and only if $x \not\in \text{Sing}(\partial C)$ for all components $C$ of the germ of $\Omega$ at $x$. This is especially the case if $x \not\in \text{Sing}(\partial \Omega)$.

**Definition 2.5** (compare with the introduction of [39])

a) We say that a domain $D \subset \mathbb{C}$ with $0 \in \partial D$ has an analytic corner (at 0) if the boundary of $D$ at 0 is given by two analytic curves which are regular at 0 and if $D$ has an interior angle greater than 0 at 0. More precisely, the following holds.

There are holomorphic functions $\psi, \chi \in \mathcal{O}(B(0, 1))$ with $\psi(0) = \chi(0) = 0$ and $\psi'(0) \chi'(0) \neq 0$ such that for $\Gamma := \psi([0, 1])$ and $\Gamma := \chi([0, 1])$ the following holds:

i) $\partial D \cap B(0, r) = (\Gamma \cup \Gamma') \cap B(0, r)$ for some $r > 0$.

ii) The interior angle $\angle D \in [0, 2\pi]$ of $\partial D$ at 0 is greater than 0.

Note that possibly $\Gamma_1 = \Gamma_2$ if $\angle D = 2\pi$. Otherwise we may assume that $\Gamma_1 \cap \Gamma_2 = \{0\}$. We call $\psi$ and $\chi$ $D$-describing functions. Note that $D \cap B(0, r)$ is semianalytic for all sufficiently small $r > 0$.

b) Let $h : \partial D \to \mathbb{R}$ be a continuous boundary function. We call $h$ a corner function (at 0) if the following holds: There is some $\varepsilon$ with $0 < \varepsilon \leq 1$ and there are real Puiseux series $g_0, g_1 : ]0, \varepsilon[ \to \mathbb{R}$ such that $h \circ \psi(t) = g_0(t)$, $h \circ \chi(t) = g_1(t)$ for $0 \leq t < \varepsilon$ (note that $h_0 := g_0 \circ \psi^{-1}$ and $h_1 := g_1 \circ \chi^{-1}$ are Puiseux series convergent on $B_L(s)$ (compare with Proposition 1.10) and that $h = h_0$ on $\Gamma \cap B(s)$ and $h = h_1$ on $\Gamma' \cap B(s)$ for some $s > 0$).
**Definition 2.6**

a) We set \( e := \exp \circ \log : \mathbb{L} \to \mathbb{C}^*, (r, \varphi) \mapsto re^{i\varphi} \). Via the identification of \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) with \( \mathbb{R}_{>0} \times ]-\pi, \pi[ \) (see Definition 1.1) we see that \( e|_{\mathbb{C} \setminus \mathbb{R}_{\leq 0}} = \text{id}_{\mathbb{C} \setminus \mathbb{R}_{\leq 0}} \).

b) Let \( A \subset \mathbb{C}^* \) be a set. We say that \( A \) can be embedded in \( \mathbb{L} \) if there is a set \( B \subset \mathbb{L} \) such that \( e|_B \) is injective and \( e(B) = A \). We call \( B \) an embedding of \( A \) in \( \mathbb{L} \).

c) Let \( A \subset \mathbb{C}^* \) be embeddable in \( \mathbb{L} \) and let \( f : A \to \mathbb{C} \) be a function. Let \( B \) be an embedding of \( A \). We set \( f_B := f \circ e : B \to \mathbb{C} \).

**Remark 2.7**

a) Let \( U \subset \mathbb{C}^* \) be an embeddable domain. Then an embedding of \( U \) is a domain in \( \mathbb{L} \).

b) Let \( \Omega \) be a semianalytic domain such that 0 is a non-isolated boundary point of \( \Omega \). Then every component of the germ of \( \Omega \) at 0 has a semianalytic representative which is embeddable in \( \mathbb{L} \).

c) Let \( A \subset \mathbb{C}^* \) be embeddable in \( \mathbb{L} \) and let \( f : A \to \mathbb{C} \) be a function. Let \( B_1, B_2 \) be embeddings of \( A \). Then there is some \( k \in \mathbb{Z} \) such that \( B_2 = \{ m(a, z) \mid z \in B_1 \} \) and \( f_{B_2}(z) = f_{B_1}(m(a^{-1}, z)) \) where \( a := (1, 2k\pi) \) and \( a^{-1} := (1, -2k\pi) \).

**Definition 2.8**

Let \( U \subset \mathbb{C}^* \) be an embeddable domain and let \( f : U \to \mathbb{C} \) be a function. Let \( W \subset \mathbb{L} \) be a domain and \( F : W \to \mathbb{C} \) be a function. We say that \( F \) extends \( f \) if there is an embedding \( V \) of \( U \) in \( \mathbb{L} \) with \( V \subset W \) such that \( F \) extends \( f_W \).

**Theorem 2.9**

Let \( D \) be a bounded and simply connected domain with an analytic corner at 0. Let \( h : \partial D \to \mathbb{R} \) be a continuous function which is a corner function at 0. Let
Let $u$ be the Dirichlet solution for $h$. Let $r > 0$ such that $D \cap B(0, r)$ is embeddable and let $D'$ be an embedding of $D \cap B(0, r)$. Then there is some $f = f(D') \in \mathcal{O}(D') \cap C(D')$ with $\text{Re} f = u_{D'}$ and some $g = g(D') = \sum_{\alpha \geq 0} a_\alpha P_\alpha(\log z) z^\alpha \in \mathbb{C}[z^\ast]_{\log}$ such that for $R > 0$

$$f(z) - \sum_{\alpha \leq R} a_\alpha P_\alpha(\log z) z^\alpha = o(|z|^R) \quad \text{as} \quad |z| \to 0 \quad \text{on} \quad \overline{D'}.$$ 

If $\angle D/\pi \in \mathbb{R} \setminus \mathbb{Q}$ then $g \in \mathbb{C}[z^\ast]$.

**Proof**

If the boundary functions $g_0$ and $g_1$ are power series we work exactly in the setting of [39] and we obtain the statement by [39, Theorem 3 & Theorem 4].

Looking carefully at the proofs of these theorems we can generalize the results to Puiseux series: as in [32] we restrict ourselves to the case where $\alpha := \angle D/\pi$ is irrational (compare with [39, p.55]) and use the notation introduced there.

Up to the end of [39, p.53] the arguments are literally the same. But in our case the function $\phi_2(s)$ is in general a convergent real Puiseux series $\phi_2(s) = \phi(s) = \sum_{n=n_0}^{\infty} a_ns^\alpha$ where $n_0 > 0$ and $a_{n_0} \neq 0$. By (4.7) of [39] we have to consider $\frac{\partial \mu(\xi, 0)}{\partial \xi}$ which is given by (4.13) of [39] as

$$\frac{\partial \mu(\xi, 0)}{\partial \xi} = \phi'(s)/ds$$

By (4.12) of [39] we have $s = \psi(\xi) = \xi^\alpha K^{***}(\xi)$ where $K^{***}(\xi) \sim \sum_{k, \ell \geq 0} a_{k\ell} \xi^{k+\ell\alpha}$ on the real line and where $a_{00} \neq 0$. Hence by binomial expansion

$$\phi'(\psi(\xi)) \sim \sum_{k, \ell \geq 0} b_{k\ell} \xi^{k+\ell\alpha}$$

where $b_{00} \neq 0$. Since $\frac{ds}{d\xi} \sim \xi^{\alpha-1} \sum_{k, \ell \geq 0} c_{k\ell} \xi^{k+\ell\alpha}$ where $c_{00} \neq 0$ (compare with (4.15) of [39]) we see that

$$\frac{\partial \mu(\xi, 0)}{\partial \xi} \sim \sum_{k, \ell \geq 0} d_{k\ell} \xi^{k+\ell\alpha}.$$ 

Note that $\frac{n_0 \alpha}{\alpha - 1} > -1$ and that all exponents are irrational. Therefore Lemma 1, 2 & 3 of [39] can be applied to formula (4.7) of [39] and we can finish the proof by the arguments of [39, pp.54-55].
3. Proofs of Theorem A and B

We prove Theorem A by reducing the problems to the case of a domain with an analytic corner at 0. Theorem B is then deduced from Theorem A by applying the results and methods of [27].

Definition 3.1
Let $\Omega \subset \mathbb{R}^2$ be a bounded and semianalytic domain without isolated boundary points such that $0 \in \partial \Omega$. We call $C \subset \Omega$ a corner component of $\Omega$ at 0 if $C$ is a semianalytic, simply connected and embeddable representative of a connected component of the germ of $\Omega$ at 0 such that the germ of $\partial C$ at 0 consists of one or two semianalytic curves.

Theorem A gets now the following precise form:

THEOREM 3.2
Let $\Omega \subset \mathbb{R}^2$ be a bounded and semianalytic domain without isolated boundary points with $0 \in \partial \Omega$. Let $h: \partial \Omega \to \mathbb{R}$ be a continuous and semianalytic boundary function and let $u$ be the Dirichlet solution for $h$. Let $C$ be a corner component of $\Omega$ at 0. If $\angle_0 C > 0$ then there is a quadratic domain $U \subset \mathbb{L}$ and an $f \in Q^{\log}(U)$ such that $\operatorname{Re} f$ extends $u|_{C}$. If $\angle_0 C/\pi \in \mathbb{R} \setminus \mathbb{Q}$ then $f \in Q(U)$.

Note that Theorem 3.2 does not depend on the chosen embedding by Definition 1.13, Definition 1.14 and Remark 2.7 c). Theorem 3.2 can be deduced from the following:

THEOREM 3.3
Let $D$ be a bounded, simply connected and embeddable domain with an analytic corner at 0. Let $h: \partial D \to \mathbb{R}$ be a continuous function which is a corner function at 0. Let $u$ be the Dirichlet solution for $h$. Then there is a quadratic domain $U \subset \mathbb{L}$ and an $f \in Q^{\log}(U)$ such that $\operatorname{Re} f$ extends $u$. If $\angle D/\pi \in \mathbb{R} \setminus \mathbb{Q}$ then $f \in Q(U)$.

Proof of Theorem 3.2 from Theorem 3.3
We can assume that $\partial C \cap \partial \Omega$ consists of two semianalytic branches $\Gamma$ and $\Gamma'$. Moreover, we may assume that after some rotation there is a convergent Puiseux series $g: [0, \delta[ \to \mathbb{R}$ such that $\Gamma = \{(t, g(t)) \mid 0 \leq t \leq \gamma\}$ for some
$0 < \gamma < \delta$. This can be achieved by analytic cell decomposition (see [13, pp.508-509]) and the fact that bounded semianalytic functions in one variable are given by Puiseux series (see [9, p.192]). There is some $d \in \mathbb{N}$ and some convergent real power series $f: [-\delta^d, \delta^d] \to \mathbb{R}$ such that $g(t) = f(t^\gamma)$. Hence $\Gamma = \{(t^d, f(t)) | 0 \leq t \leq \gamma^d\}$. We consider $\psi: B(0, \delta^d) \to \mathbb{C}$, $z \mapsto z^d + if(z)$. Then $\psi \in \mathcal{O}(B(0, \delta^d))$ and $\Gamma = \psi([0, \gamma^d])$. Arguing similarly for $\Gamma'$ we find (after back-rotation and some dilation) holomorphic functions $\psi, \chi \in \mathcal{O}(B(0, 1))$ with $\psi(0) = \chi(0)$ such that $\Gamma = \psi([0, 1])$ and $\Gamma' = \chi([0, 1'])$. Let $m, n \in \mathbb{N}$ such that $\psi(z) = z^m \hat{\psi}(z)$, $\chi(z) = z^n \hat{\chi}(z)$ with $\hat{\psi}, \hat{\chi} \in \mathcal{O}(B(0, 1))$ and $\hat{\psi}(0) \hat{\chi}(0) \neq 0$. We can replace $\chi(z)$ by $\chi(z^m)$ and can therefore assume that $m$ divides $n$.

We choose an embedding of $C$ in $L$ which we denote again by $C$. By the above we have functions from $\mathcal{O}L_0^N$, denoted again by $\psi$ and $\chi$, which are defined on $B_L(1)$ such that $k(\psi) = m$, $k(\chi) = n$ and $\partial C \cap B_L(r) = \Gamma \cup \Gamma'$ for some $r > 0$ where $\Gamma := \psi([0, 1])$ and $\Gamma' := \chi([0, 1'])$. We apply finitely many elementary manipulations from Section 1 to $u$ and $C$. We obtain functions $u_i$ and domains $C_i$, $i = 1, 2, 3$, such that $C_3$ is an embedding of a domain with an analytic corner. This enables us to apply Theorem 3.3. The domains $C_i$ allow a similar description as $C$. We denote the data describing $C_i$ by $\Gamma_i$, $\Gamma'_i$, $\psi_i$, $\chi_i$.

1) We consider $u_1: C_1 \to \mathbb{R}$, $z \mapsto u(p^{k(\psi)}(z))$, where $C_1 := p^{1/k(\psi)}(C)$. Since $k(\psi)$ divides $k(\chi)$ we see that $C_1$ has similar properties as $C$ but additionally $k(\psi_1) = 1$.

2) We may choose a priori $C$ and $r(\psi_1^{-1})$ such that $\psi_1^{-1}$ is injective on $B_L(r(\psi_1^{-1}))$ and $C_1 \subset B_L(r(\psi_1^{-1}))$ (compare with Definition 1.7 b) and Proposition 1.8 b))

We consider $u_2: C_2 \to \mathbb{R}$, $z \mapsto u_1(\psi_1(z))$, where $C_2 := \psi_1^{-1}(C_1)$. Then $C_2$ has similar properties as $C_1$ but additionally $\Gamma_2 \subset \mathbb{R}_{>0} \times \{\varphi\}$ for some $\varphi \in \mathbb{R}$.

3) We consider $u_3: C_3 \to \mathbb{R}$, $z \mapsto u_2(p^{k(x_2)}(z))$, where $C_3 := p^{1/k(x_2)}(C_2)$. Then additionally $k(\chi_3) = 1$.

By construction $e|_{C_3}$ is injective and $e(C_3)$ is a bounded, simply connected and embeddable domain with an analytic corner at 0. Moreover $u_e := u_3 \circ (e|_{C_3})^{-1}$
is a harmonic function on \( e(C_3) \) which has a continuous extension to \( \overline{e(C_3)} \). With \( h_e \) we denote the extension to the boundary of \( e(C_3) \). With Proposition 1.8 and Proposition 1.10 we see that \( h_e \) is a corner function at 0.

By Theorem 3.3 there is a quadratic domain \( U \) and a \( g \in \mathcal{Q}^{\log}(U) \) such that \( \text{Re} g \) extends \( u_3 \). If \( \angle_{e(C_3)}/\pi \in \mathbb{R} \setminus \mathbb{Q} \) then \( g \in \mathcal{Q}(U) \). Since \( \angle_{e(C_3)} = (k(\psi)k(\chi_2))^{-1} \angle_0 C \) we get that \( g \in \mathcal{Q}(U) \) if \( \angle_0 C/\pi \in \mathbb{R} \setminus \mathbb{Q} \). By construction \( u = u_3 \circ \mathbf{p}^{\chi(\chi_2)} \circ \psi^{-1} \circ \mathbf{p}^\gamma \).

By Proposition 1.19 we obtain that \( f := g \circ \mathbf{p}^{\chi(\chi_2)} \circ \psi^{-1} \circ \mathbf{p}^\gamma \in \mathcal{Q}^{\log} \) and \( f \in \mathcal{Q} \) if \( \angle_0 C/\pi \in \mathbb{R} \setminus \mathbb{Q} \). This \( f \) fulfills the requirements. \( \square \)

It remains to prove Theorem 3.3. We prove it by doing reflections at analytic curves infinitely often. We use the fact that the given boundary function is defined at 0 by convergent Puiseux series which extend to the Riemann surface of the logarithm (see Remark 1.4). To motivate the technical statements of the upcoming proofs we give the following example for the reflection principle involved.

**Example 3.4**

Let \( R > 0 \) and let \( \chi : B(0, R) \to \mathbb{C} \) be an injective holomorphic function with \( \chi(0) = 0 \), \( \chi'(0) > 0 \) and \( \Gamma := \chi([0, R[) \subset \mathbb{C}_+ := \{ z \in \mathbb{C} \mid \text{Re} z > 0 \} \) (note that \( \Gamma \) is tangent to \( \mathbb{R}_{>0} \) at 0). Let \( R' > 0 \) such that \( \chi(B(0, R)) \supset B(0, R') \). Then \( B(0, R') \setminus \Gamma \cap \mathbb{C}_+ \) has two components, let \( V \) be one of them.

Let \( f : V \to \mathbb{C} \) be a holomorphic function which has a continuous extension to \( \Gamma \cap B(0, R') \) such that the following holds: there is a convergent Puiseux series \( h : B(0, R') \setminus \mathbb{R}_{\leq 0} \to \mathbb{R} \) such that \( \text{Re} f = h \) on \( \Gamma \cap B(0, R') \). Then there is some \( R'' \) with \( 0 < R'' < R' \) such that \( f \) has a holomorphic extension to \( B(0, R'') \) given by

\[
   z \mapsto \begin{cases} 
   f(z) & z \in V, \\
   (f - h) \circ \chi \circ \overline{\chi^{-1}(z)} + h(z) & z \notin V 
   \end{cases}
\]

(here \( \overline{z} \) denotes the complex conjugate of a complex number \( z \)). This can be seen in the following way: \( f_1 := f - h \in \mathcal{O}(V) \) has a continuous extension to \( \Gamma \cap B(0, R') \) with \( \text{Re} f_1 = 0 \) on \( \Gamma \cap B(0, R') \). Then \( f_2 := f_1 \circ \chi \in \mathcal{O}(W) \) with
$W := \chi^{-1}(V)$ has a continuous extension to $I := \chi^{-1}(\Gamma \cap B(0, R')) \subset \mathbb{R}$ with vanishing real part there. Therefore $f_3 : W \cup I \cup W^* \to \mathbb{C}$ defined by

$$z \mapsto \begin{cases} f_2(z) & z \in W \cup I, \\ -f_2(\overline{z}) & z \in W^*, \end{cases}$$

where $W^* := \{ \overline{z} \mid z \in W \}$ is a holomorphic extension of $f_2$. Then there is some $R''$ with $0 < R'' \leq R'$ such that $f_4 := f_3 \circ \chi^{-1} \in \mathcal{O}(B(0, R''))$ extends $f_2$. So $f_5 := f_4 + h \in \mathcal{O}(B(0, R''))$ extends $f$.

Remark 3.5
We define the conjugate $\tau : L \to L$ by $\tau(r, \varphi) := (r, -\varphi)$. We obtain immediately the following: let $g$ be a Puiseux series convergent on $B_L(r)$ with denominator $d$. Then $g \circ \tau$ is a Puiseux series convergent on $B_L(r)$ with denominator $d$. Let $\varphi \in \mathcal{O}_L$. Then $\psi := \tau \circ \varphi \circ \tau \in \mathcal{O}_L$ with $r(\psi) = r(\varphi)$, $k(\psi) = k(\varphi)$ and $|a(\psi)| = |a(\varphi)|$.

The proof of Theorem 3.3 relies on the iteration of the reflection process from Example 3.4. Its description involves inversion, conjugation and iteration of certain holomorphic functions and summation of Puiseux series (see (†) in the proof below). We separate the proof into two steps. In the first step we show the extension of the Dirichlet solution to a quadratic domain (compare with [21, Proposition 3] and the subsequent remarks there). In the second step we show the extension of the asymptotic development. We use the classes of functions from Section 1 to establish the dynamical system (†) and we use the estimates for these classes from Section 1 to control it to get the desired properties.

Proof of Theorem 3.3
We show the claim in two steps:

Step 1: There is a quadratic domain $U \subset L$ and an $f \in \mathcal{O}(U)$ such that $\text{Re} f$ extends $u$.

Proof of Step 1:
We choose an embedding of $D$ which we also denote by $D$. We also write $u$ for
\( u_D \) and \( h \) for \( u|_{\partial D} \). Let \( \psi, \chi \) be \( D \)-describing (see Definition 2.5). Considering \( \psi(z/|\psi'(0)|) \) and \( \chi(z/|\chi'(0)|) \) we find \( r, s > 0 \) such that the following holds:

\( \alpha \) There are functions from \( O_L^0 \), denoted again by \( \psi \) and \( \chi \), which are defined on \( B_L(r) \) and fulfil \( |a(\psi)| = |a(\chi)| = 1 \), such that \( \partial D \cap B_L(s) = \Gamma \cup \Gamma' \) with \( \Gamma := \psi([0, \varepsilon]) \) and \( \Gamma' := \chi([0, \varepsilon]) \) for some \( 0 < \varepsilon \leq r \). Moreover, we can assume that \( \psi^{-1} \) and \( \chi^{-1} \) are defined on \( B_L(r) \) and that \( \psi, \chi, \psi^{-1}, \chi^{-1} \) are injective on \( B_L(r) \).

\( \beta \) There are Puiseux series \( h_0, h_1 \) convergent on \( B_L(s) \) such that \( h = h_0 \) on \( \Gamma \) and \( h = h_1 \) on \( \Gamma' \).

We may assume that \( s \leq r \) and that \arg(a(\psi)) < \arg(a(\chi)) \). Recursively we find for \( k \geq 1 \) constants \( r_k \geq s_k > 0 \), a domain \( D_k \subset \mathbb{D} \), a function \( f_k \in O(D_k) \) and a function \( \varphi_k \in O_L^0 \) with \( |a(\varphi_k)| = 1 \), \( r_k = r(\varphi_k) = r(\varphi_k^{-1}) \) and a Puiseux series \( h_k \) convergent on \( B_L(s_k) \) with the following properties:

a) \( D_k \supset D_{k-1} \) and \( f_k \) extends \( f_{k-1} \) holomorphically for \( k \geq 2 \).

b) \( \partial D_k \cap B_L(s_k) \) consists of two boundary curves \( \Gamma_k \) and \( \Gamma'_k \) where \( \Gamma_k := \psi([0, \mu_k]) \subset \Gamma \) and \( \Gamma'_k := \varphi_k([0, \varepsilon_k]) \) for some \( 0 < \mu_k, \varepsilon_k \leq r_k \),

c) \( \text{Re } f_k = h_k \) on \( \Gamma_k \)

as follows:

\( k = 1: \) \( r_1 := r, s_1 := s, \varphi_1 := \chi, D_1 := D, f_1 := f \) from Theorem 2.9, and \( h_1 \) from \( \beta \) above.

\( k \to k + 1: \) We set \( r_{k+1} := \frac{r_k}{100}, s_{k+1} := \frac{s_k}{100}, D_{k+1} := (D_k \cup D'_k)^0 \) with

\[
D'_k := \varphi_k(\tau(\varphi_k^{-1}(D_k \cap B \left( \frac{s_k}{4} \right)))).
\]

(\( \dagger \)) \( f_{k+1}: D_{k+1} \to \mathbb{C}, \)

\[
z \mapsto \begin{cases} f_k(z) & \text{if } z \in D_k, \\ -((f_k - h_k) \circ \varphi_k \circ \tau \circ \varphi_k^{-1}(z)) + h_k(z) & \text{if } z \in D'_k, \end{cases}
\]

\( \varphi_{k+1} := \varphi_k \circ \tau \circ \varphi_k^{-1} \circ \psi \circ \tau \) and \( h_{k+1} := -(h_0 - h_k) \circ \varphi_k \circ \tau \circ \varphi_k^{-1} + h_k. \)
Note that these data are well defined:
By Proposition 1.9 a) $D'_k$ is well defined and $f_{k+1}$ exists on $D_{k+1}$. Note that the angle of $D_{k+1}$ at 0 is twice the angle of $D_k$ at 0. The function $f_{k+1}$ originates from a reflection process at $\Gamma_k$ and extends $f_k$ holomorphically (compare with Example 3.4). By Proposition 1.9 b) and Remark 3.5 we can choose $r(\varphi_{k+1}) = \frac{r_{k+1}}{\operatorname{tan}}$. By Proposition 1.9 a), Proposition 1.10 and Remark 3.5 $h_{k+1}$ is a Puiseux series convergent on $B_L\left(\frac{\psi}{k}\right)$. By construction and by Proposition 1.9 a) we see that a), b) and c) hold.

We extend $f$ holomorphically to $\bigcup_k D_k$ by setting $f|_{D_k} := f_k$. We see that $s_k = s/100^{k-1}$ and $\arg(a(\varphi_k)) - \arg(a(\psi)) = 2^{k-1}(<D)$. By the definition of $\Gamma_k$ and Proposition 1.9 a) we get that

$$\Gamma_k \subset \left\{ (r, \varphi) \in L \mid \left| \varphi - \arg a(\varphi_k) \right| \leq \frac{\pi}{2} \right\}.$$

Hence we see by b) that $f$ is holomorphic on

$$\left\{ (r, \varphi) \in L \mid 0 < r < s/100^{k-1} \text{ and } 0 < \varphi - \alpha < 2^{k-1}(<D) - \frac{\pi}{2} \right\}$$

for all sufficiently large $k \geq 1$ where $\alpha := \arg(a(\psi))$. For $\varphi > \alpha$ let $k(\varphi) \in \mathbb{N}$ be such that $2^{k-1}(<D) - \frac{\pi}{2} \leq \varphi - \alpha \leq 2^k(<D) - \frac{\pi}{2}$. Then there is some $C > 0$ such that $k(\varphi) \leq C \log(\varphi - \alpha)$ for all sufficiently large $\varphi > 0$. Hence we find some $K > 1$ such that $f$ is holomorphic on

$$\left\{ (r, \varphi) \in L \mid \varphi > \alpha \text{ and } 0 < r < K^{-\log_+ (\varphi - \alpha)} \right\}$$

where $\log_+ x := \max\{1, \log x\}$ for $x > 0$.

Repeating this process in the negative direction we see that $f$ is holomorphic on some quadratic domain $U$ since $\log(\varphi - \alpha) \leq \sqrt{\varphi}$ for all sufficiently large $\varphi$.

**Step 2**: We show that $f \in Q_{\log}(U)$. Let $g := \sum_{\alpha \geq 0} a_{\alpha} P_\alpha(\log z)z^\alpha \in \mathbb{C}[[z^*]]_{\log}$ from Theorem 2.9. Given $R > 0$ we show that there is a quadratic domain $U_R$ such that

$$f(z) - \sum_{\alpha \leq R} a_{\alpha} P_\alpha(\log z)z^\alpha = o(|z|^R) \text{ as } |z| \to 0 \text{ on } U_R.$$
Hence $Tf = g$ and we see with Theorem 2.9 that $f \in Q(U)$ if $\vartheta D/\pi \in \mathbb{R} \setminus \mathbb{Q}$.

Proof of Step 2:
We work in the setting of Step 1. We can assume that $h_0$ and $h_1$ have a common denominator $d \in \mathbb{N}$ and that there is a $c > 0$ such that $|h_i(z)| \leq c$ for $|z| < s$, $i = 0, 1$. By Proposition 1.9 a), Proposition 1.10 and Remark 3.5 we see that $h_k$ is a Puiseux series on $B_L\left(\frac{s_k-1}{4}\right)$ with denominator $d$ for all $k \geq 2$. Moreover, by induction we see that $|h_k(z)| \leq \frac{3}{4}k - 1c$ for $|z| < \frac{s_k-1}{4}$ and all $k \geq 2$.

We fix $R > 0$. Let $\gamma := \sum_{\alpha \leq R} a_{\alpha} P_{\alpha} (\log z) z^\alpha \in \mathbb{C}\llbracket z^* \rrbracket_{\log}$. We choose $R' > R$ with the following properties:

(i) $R' < \min\{\alpha \in \text{supp}(g) \mid \alpha > R\}$,
(ii) $R' < \left\lfloor \frac{Rd}{d} + 1 \right\rfloor$,
(iii) $R' < [R - \alpha] + \alpha + 1$ for all $\alpha \in \text{supp}(\gamma)$.

Here $[x]$ denotes the largest integer $n \leq x$. Let $K$ be the number of elements of $\text{supp}(\gamma)$, $L := \text{sup}\{|a| : a \text{ is a coefficient of } a_{\alpha} P_{\alpha}, \alpha \in \text{supp}(\gamma)\}$ and $M := \max_{\alpha \in \text{supp}(\gamma)} \deg P_{\alpha}$.

We may choose $r$ in Step 1 such that the following holds:

(iv) $|z^{\alpha + N} (\log z)^M| \leq |z|^{R'}$ on $B_L(r)$ for all $N > [R - \alpha]$ and all $\alpha \in \text{supp}(\gamma)$.

By Theorem 2.9 and condition (i) we find $C > 1$ such that $|f(z) - \gamma(z)| \leq C|z|^{R'}$ on $\overline{D}$.

Claim 1: Given $k \geq 1$ there is some $C_k > 0$ such that

\[ |f_k(z) - \gamma(z)| \leq C_k |z|^{R'} \]

on $\overline{D_k} \cap B_L(s_k)$.

From the proof of Claim 1 we will obtain below estimates for the $C_k$'s.

Proof of Claim 1 by induction on $k$:
The case $k = 1$ is settled by the above.

$k - 1 \rightarrow k$: By construction we have that $f_k|_{D_k-1 \cap B_L(s_k)} = f_{k-1}|_{D_{k-1} \cap B_L(s_k)}$. For the inductive step we have to consider $f_k|_{D_k' \cap B_L(s_k)}$. From (†) in Step 1 we see that

\[ f_k|_{D_k' \cap B_L(s_k)} = v_{k,1} + v_{k,2} + v_{k,3} \]
with \( v_{k,1} := -f_{k-1} \circ \varphi_{k-1} \circ \tau \circ \varphi_{k-1}^{-1}, v_{k,2} := h_{k-1} \circ \varphi_{k-1} \circ \tau \circ \varphi_{k-1}^{-1} \) and \( v_{k,3} := h_{k-1} \). To prove Claim 1 we show the following

**Claim 2:** Let \( 1 \leq i \leq 3 \). Assuming that Claim 1 holds for \( k-1 \) we show that there is some \( \gamma_{k,i} \in \mathbb{C}[[z^*]]_{\log} \) with \( \text{supp}(\gamma_{k,i}) \subseteq [0, R] \) and some \( C_{k,i} > 0 \) such that

\[
|v_{k,i} - \gamma_{k,i}| \leq C_{k,i}|z|^R
\]
on \( B_L(s_k) \).

Claim 1 follows from Claim 2: since \( f_k = f_{k-1} \) on \( \Gamma_{k-1} \) we see by applying the inductive hypothesis and by Claim 2 that \( \gamma = \gamma_{k,1} + \gamma_{k,2} + \gamma_{k,3} \) and obtain that Claim 1 holds for \( k \) with \( C_k := C_{k,1} + C_{k,2} + C_{k,3} \). (1)

**Proof of Claim 2:**

To prove Claim 2 we distinguish the three cases \( i = 1, 2, 3 \).

\( i = 1 \): Let \( \delta_{k-1} := f_{k-1} - \gamma \). Then \( |\delta_{k-1}(z)| \leq C_{k-1}|z|^R \) for \( z \in D_{k-1} \cap B_L(s_{k-1}) \) by the assumption of Claim 2. Let \( \eta_k := -\delta_{k-1} \circ \varphi_{k-1} \circ \tau \circ \varphi_{k-1}^{-1} \). Then \( v_{k,1} = -\gamma \circ \varphi_{k-1} \circ \tau \circ \varphi_{k-1}^{-1} + \eta_k \) and \( |\eta_k(z)| \leq C_{k-1}4^R |z|^R \) on \( D_k \cap B_L(s_k) \) by Proposition 1.9 a). Let \( w_{k,1} := \gamma \circ \varphi_{k-1}^{-1}, w_{k,2} := -w_{k,1} \circ \tau \) and \( w_{k,3} := w_{k,2} \circ \varphi_{k-1}^{-1} \). To prove Claim 2 in the case \( i = 1 \) we show the following

**Claim 3:** Let \( 1 \leq j \leq 3 \). We show that there is some \( \lambda_{k,j} \in \mathbb{C}[[z^*]]_{\log} \) with \( \text{supp}(\lambda_{k,j}) \subseteq [0, R] \) and some \( E_{k,j} > 0 \) such that

\[
|w_{k,j} - \lambda_{k,j}| \leq E_{k,j}|z|^R
\]
on \( B_L(s_k) \).

The case \( i = 1 \) from Claim 2 follows from Claim 3: assuming Claim 3, we can choose \( \gamma_{k,1} := \lambda_{k,3} \) and \( C_{k,1} := E_{k,3} + C_{k-1}4^R \). (2)

**Proof of Claim 3:**

To prove Claim 3 we distinguish the three cases \( j = 1, 2, 3 \).

\( j = 1 \): Let \( \alpha \in \text{supp}(\gamma) \) and \( m \leq M \). By Lemma 1.20 we find power series \( p_1, \ldots, p_m \) convergent on \( B_L(r_{k-1}) \) such that

\[
(z^\alpha (\log z)^m) \circ \varphi_{k-1} = z^\alpha \left( \sum_{\ell=0}^m p_\ell(z)(\log z)^\ell \right)
\]
with $|p_\ell(z)| \leq 2^{m+\alpha}(| \arg a(\varphi_{k-1})| + 3)^m$. Hence

$$v_{k,1} = \gamma \circ \varphi_{k-1} = \sum_{\alpha \in \text{supp}(\gamma)} \alpha \left( \sum_{\ell=0}^M q_{\alpha,\ell}(z) (\log |z|)^\ell \right)$$

with power series $q_{\alpha,\ell}$ convergent on $B_L(r_{k-1})$ and $|q_{\alpha,\ell}| \leq d_k$ on $B_L(r_{k-1})$ where $d_k := KL2^{M+R}(|\arg a(\varphi_{k-1})| + 3)^M$. Let $T_{\alpha,\ell}$ be the truncated power series expansion of $q_{\alpha,\ell}$ up to order $[R-\alpha]$. By Proposition 1.5 b) we get $q_{\alpha,\ell}(z) - T_{\alpha,\ell}(z) \leq \frac{2d_k}{r_{k-1}} |z|^{[R-\alpha]+1}$ on $B_L(\frac{r_{k-1}}{2})$. By condition (iii) and (iv) we see that there is $\lambda_{k,1} = \sum_{\alpha \leq R} b_\alpha Q_\alpha (\log |z|) z^\alpha \in \mathbb{C}[[z^*]]^{\text{fin}}$ such that on $B_L(\frac{r_{k-1}}{2})$

$$|w_{k,1}(z) - \lambda_{k,1}(z)| \leq E_{k,1}|z|^{R'}$$

where $E_{k,1} := \frac{2d_k}{r_{k-1}}$. Moreover, let $K'$ be the number of elements of $\text{supp}(\lambda_{k,1})$, $L' := \sup\{|a| : a$ is a coefficient of $b_\alpha Q_\alpha$, $\alpha \in \text{supp}(\lambda_{k,1})\}$ and $M' := \max_{\alpha \in \text{supp}(\lambda_{k,1})} \deg Q_\alpha$. Then $K' \leq K + R$, $L' \leq \frac{d_k}{r_{k-1}} L$ by Cauchy estimates and $M' \leq M$.

$j = 2$: We set $\lambda_{k,2} := -\lambda_{k,1}$ and $E_{k,2} := E_{k,1}$.

By Remark 3.5 we see that on $B_L(\frac{r_{k-1}}{2})$

$$(*) \quad |w_{k,2}(z) - \lambda_{k,2}(z)| \leq E_{k,2}|z|^{R'}.$$ 

$j = 3$: As in the case $j = 1$ we see that there is $\lambda_{k,3} \in \mathbb{C}[[z^*]]^{\text{fin}}$ with $\text{supp}(\lambda_{k,3}) \subseteq [0, R]$ such that

$$(**) \quad |\lambda_{k,2} \circ \varphi_k^{-1}(z) - \lambda_{k,3}(z)| \leq \frac{2d_k'}{r_{k-1}} |z|^{R'}$$

on $B_L(\frac{r_{k-1}}{2})$ with $d_k' := K'L'2^{M+R}||\arg a(\varphi_{k-1})|+3|^M$ (note that $\arg a(\varphi_{k-1}) = -\arg a(\varphi_{k-1})$). Hence we obtain by applying Proposition 1.9 a) to $(*)$ and by $(**)$ that on $B_L(\frac{r_{k-1}}{4})$

$$|w_{k,3}(z) - \lambda_{k,3}(z)| \leq E_{k,3}|z|^{R'}$$

where

$$E_{k,3} := \frac{2d_k'}{r_{k-1}} + 2^{R'} E_{k,2}. \quad (5)$$
So Claim 3 and therefore the case \( i = 1 \) of Claim 2 is proven since \( s_k < \frac{r_{k-1}}{4} \).

We continue with the case \( i = 2 \) of Claim 2.

\( i = 2 \): We see with Proposition 1.9 a), Proposition 1.10 and Remark 3.5 that \( v_{k,2} \) is a Puiseux series on \( B_L \left( \frac{s_{k-1}}{4} \right) \) with denominator \( d \). Moreover, \(|v_{k,2}| \leq 3^{k-1}c \) on \( B_L \left( \frac{s_{k-1}}{4} \right) \) (compare with the beginning of the proof of Step 2). Hence by Proposition 1.5 b) and condition (ii) we find \( \gamma_{k,2} \) as described such that
\[
|v_{k,2}(z) - \gamma_{k,2}(z)| \leq C_{k,2}|z|^R \quad \text{on} \quad B_L(s_k)
\]
with
\[
C_{k,2} := 3^{k-1}d' \left( \frac{4}{s_{k-1}} \right)^{\frac{R+1}{d}}
\]
and \( d' := \frac{1}{1 - \left( \frac{100}{3} \right)^2} \).

\( i = 3 \): As in the case \( i = 2 \) we can find with Proposition 1.5 b) \( \gamma_{k,3} \) and \( C_{k,3} \) as described. Moreover, we can choose \( C_{k,3} := C_{k,2} \). (7)

Hence Claim 2 and therefore also Claim 1 is proven. We revisit now the construction of the constants \( C_k \) of Claim 1. By (1) we have \( C_k = C_{k,1} + C_{k,2} + C_{k,3} \) with \( C_{k,i} \) from Claim 2. By (2) we have \( C_{k,1} = E_{k,3} + C_{k-1}4^R \) with \( E_{k,3} \) from Claim 3. The constant \( E_{k,3} \) is computed via (3), (4) and (5).

Since \( r_{k-1} = \frac{r}{100^{k-2}} \) and \( |\arg(a(\varphi_{k-1}) - \arg a(\psi))| = 2^{k-1}(\leq D) \) we see that there is some \( B > 1 \) such that \( E_{k,3} \leq B^k \) for all \( k \geq 2 \). Since \( s_{k-1} = \frac{s}{100^{k-2}} \) we see by (6) that after enlarging \( B \) if necessary \( C_{k,2} \leq B^k \) for all \( k \geq 2 \). Since \( C_{k,2} = C_{k,3} \) by (7) we finally find by (1) and (2) some \( A > 1 \) such that \( C_k \leq A^k \) for all \( k \). Hence Claim 1 gives that \(|f(z) - \gamma(z)| \leq A^k|z|^R \) on \( D_k \cap B_L(s_k) \) and all \( k \in \mathbb{N} \). Moreover, we may assume that \( A^k \leq s_k \) for all \( k \in \mathbb{N} \). We choose \( R < S < R' \) with \( R' - S \leq 1 \). We set \( t_k := A^{-\frac{R}{R-S}} \) and obtain on \( D_k \cap B_L(t_k) \)
\[
|f(z) - \gamma(z)| \leq A^k|z|^R - S|z|^S \leq |z|^S.
\]

Using a similar argument as at the end of the proof of Step 1 we find some \( K > 1 \) such that \(|f(z) - g_0(z)| \leq |z|^S \) on the set
\[
\{(r, \varphi) \in L \mid \varphi > \alpha \quad \text{and} \quad 0 < r < K^{-(\log_+(\varphi - \alpha))}\}
\]
where \( \alpha := \arg(a(\psi)) \) and \( \log_+ x := \max\{1, \log x\} \) for \( x > 0 \). Repeating these arguments in the negative direction we see that \( f(z) - \gamma(z) = o(|z|^R) \) on some admissible domain \( U_R \subset U \) since \( S > R \) and \( \log(\varphi - \alpha) \leq \sqrt{\varphi} \) eventually. \( \square \)
We need the following final ingredients for the proof of Theorem B.

**Definition 3.6** (compare with [27, Definition 3.4 & Definition 4.3])
Let \( \lambda \in \mathbb{H} \setminus \{0\} \) where \( \mathbb{H} \) denotes the upper half plane. We have \( B(|\lambda|,|\lambda|) \subseteq L \) via the identification of \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) with \( \mathbb{R}_{>0} \times -\pi, \pi \subset L \). Let \( \lambda = |\lambda| e^{i\alpha} \) with \( 0 \leq \alpha \leq \pi \). We identify \( B(\lambda,|\lambda|) \) with \( \{(r,\varphi) \in L \mid (r,\varphi - a) \in B(|\lambda|,|\lambda|)\} \).

We set \( t_\lambda: B(0,|\lambda|) \to B(\lambda,|\lambda|), z \mapsto \lambda + z \), and for \( \rho > 0 \) we define \( r^{\rho,\lambda}: L \times B(0,|\lambda|) \to L^2, (z_1,z_2) \mapsto (z_1,w_2) \), where \( w_2 := m(p^\rho(z_1), t_\lambda(z_2)) \).

**Remark 3.7**
Let \( U \subseteq L^2 \) be a 2-quadratic domain (compare with [27, Definition 2.4]) and let \( f \in Q^2_\rho(U) \) (compare with [27, Definition 5.1]). Let \( \lambda \in \mathbb{H} \setminus \{0\} \). As in [27, Proposition 4.4 & Proposition 5.15] we find some 1-quadratic domain \( V \subseteq L \times B_L(\lambda) \) such that \( r^{1,\lambda}(V) \subset U \) and the function \( r^{1,\lambda}f := f \circ r^{1,\lambda} \in Q^2_\psi(V) \).

**PROPOSITION 3.8**
Let \( U \subseteq L \) be a quadratic domain and let \( f \in Q(U) \). Let \( V \subseteq U \) be a domain such that \( e \) is injective on \( V \). Then there is some \( r > 0 \) such that \( f \circ (e|_V)^{-1}|_{B(0,r)} \) is definable in \( \mathbb{R}_Q \).

**Proof**
By considering finite coverings we can assume that \( V := \{(r,\varphi) \in U \mid \varphi_0 < \varphi < \varphi_0 + \pi\} \) for some \( \varphi_0 \in \mathbb{R} \). Let \( b := (1,\varphi_0) \in L \) and \( \psi := m(b,p^1(z)) \in OL^\circ \). Then \( f \circ \psi \in Q(W) \) for some quadratic domain \( W \subseteq L \) by Proposition 1.19 c). Let \( V' := \{(r,\varphi) \in W \mid 0 < \varphi < \pi\} \). Then \( f \circ (e|_V)^{-1}|_{B(0,r)} = f \circ \psi \circ (e|_{V'})^{-1}|_{B(0,r)} \) for all sufficiently small \( r > 0 \); we can therefore assume that \( \varphi_0 = 0 \). Hence via the identification of \( \mathbb{C} \setminus \mathbb{R}_{\leq 0} \) with \( \{(r,\varphi) \in L \mid -\pi < \varphi < \pi\} \), it suffices to show that \( f|_{\mathbb{H} \cap B(0,r)} \) is definable in \( \mathbb{R}_Q \) for some \( r > 0 \).

We define \( \Phi: U \times U \to \mathbb{C}, (z_1,z_2) \mapsto f(z_2) \). Let \( a \in [0,\pi] \). We consider \( g_a := r^{1,\lambda_a} \Phi \) with \( \lambda_a := e^{ia} \). By Remark 3.7 we have that \( g_a \in Q^2_\psi \). We set \( G_a := g_a(z_1,h_a(z_2)) \) with \( h_a(z) := e^{iz+ia} - e^{ia} \). Then \( G_a \in Q^2_1 \) by [27, Proposition 5.10]. Hence there is some \( r_a > 0 \) and some quadratic domain \( U_a \) such that \( G_a \in Q^2_1(U_a \times B(0,r_a)) \). We can assume that \( U_a = \{(r,\varphi) \in L \mid 0 < r < c_a \exp(-C_a \sqrt{|\varphi|})\} \) with some positive constants \( c_a, C_a \). We
define \( G_a^*: U_a \times B(0, r_a) \to \mathbb{C}, (z_1, z_2) \mapsto \overline{G_a(\tau(z_1), \overline{z_2})} \). Note that \( G_a^* \in Q^2 \) (compare with the proof of [27, Proposition 7.3]). We set \( RG_a := \frac{1}{2}(G_a + G_a^*) \) and \( JG_a := \frac{1}{2i}(G_a - G_a^*) \). Then \( RG_a \) and \( JG_a \) are defined on \( I_a := [0, \varepsilon_a] \times [-\varepsilon_a, \varepsilon_a] \), and \( RG_a|_{I_a} \) and \( JG_a|_{I_a} \) are definable in \( RQ \).

For \((r, \varphi) \in I_a\) we have \( RG_a(r, \varphi) = \text{Re}(re^{i(\varphi+a)}) = \text{Re}(r \cos(\varphi + a), r \sin(\varphi + a)) \) and \( JG_a(r, \varphi) = \text{Im}(re^{i(\varphi+a)}) = \text{Im}(r \cos(\varphi + a), r \sin(\varphi + a)) \). Since the polar coordinates are definable in \( RQ \) we find by a compactness argument (note that \( a \in [0, \pi] \)) some \( r > 0 \), such that \( f|_{\mathbb{H} \cap B(0, r)} \) is definable in \( RQ \). □

We obtain Theorem B:

**THEOREM 3.9**

Let \( \Omega \subset \mathbb{R}^2 \) be a semianalytic and bounded domain without isolated boundary points. Suppose that \( \angle(\Omega, x) \subset \pi(\mathbb{R} \setminus Q) \) for all \( x \in \text{Sing}(\partial \Omega) \). Let \( h: \partial \Omega \to \mathbb{R} \) be a continuous and semianalytic function on the boundary and let \( u \) be the Dirichlet solution for \( h \). Then \( u \) is definable in the o-minimal structure \( R_{Q,\exp} \).

**Proof**

Let \( x \in \overline{\Omega} \). We need to show that there is a neighbourhood \( V_x \) of \( x \) such that \( u|_{V_x \cap \Omega} \) is definable in \( R_{Q,\exp} \).

Case 1: \( x \in \Omega \). Let \( r > 0 \) with \( B(x, r) \subset \Omega \). Then \( u|_{B(x, r)} \) is harmonic and therefore real analytic (see [2, Theorem 1.8.5]). Hence \( u|_{B(x, \frac{r}{2})} \) is definable in \( R_{an} \) which is a reduct of \( R_{Q,\exp} \).

Case 2: \( x \in \partial \Omega \)

Case 2.1: \( x \notin \text{Sing}(\partial \Omega) \). Then \( \partial \Omega \) is a real analytic manifold at \( x \). Let \( r > 0 \) such that \( \Omega' := \Omega \cap B(x, r) \) is simply connected. Let \( \tilde{h} := u|_{\partial \Omega'} \). By the Riemann Mapping Theorem there is a biholomorphic map \( \varphi: \Omega' \to B(0, 1) \). By Caratheodory’s theory of prime ends (see Pommerenke [33, Chapter 2, p.18]) and by the curve selection lemma (see [10, p.94]) \( \varphi \) has a continuous extension to \( x \). Then by the Schwarz reflection principle \( \varphi \) has a holomorphic extension to a neighbourhood of \( x \). Let \( \tilde{h} := \tilde{h} \circ \varphi \) and \( y := \varphi(x) \). Then there is some \( s > 0 \) such that \( \tilde{h}|_{B(y, s)} \) is semianalytic. Let \( \tilde{u} \) be the Dirichlet solution
for $\hat{h}$ on $B(0,1)$. Then $\hat{u}$ is given by the Poisson integral (see [20, Theorem 1.8]):

$$\hat{u}(\xi) = \frac{1}{2\pi} \int_{\partial B(0,1)} \frac{1-|\xi|^2}{|\eta-\xi|^2} \hat{h}(\eta) d\sigma(\eta).$$

We have therefore

$$\hat{u}(\xi) = \frac{1}{2\pi} \int_{\partial B(0,1) \setminus B(y,s)} \frac{1-|\xi|^2}{|\eta-\xi|^2} \hat{h}(\eta) d\sigma(\eta) + \frac{1}{2\pi} \int_{\partial B(0,1) \cap B(y,s)} \frac{1-|\xi|^2}{|\eta-\xi|^2} \hat{h}(\eta) d\sigma(\eta).$$

Applying the Laplace operator to the first integral we see by switching differentiation and integration that the first summand is harmonic in $B(y,s)$ and hence real analytic in $B(y,s)$; the second one is definable in the o-minimal structure $\mathbb{R}_{\text{an}}$, $\exp$ by [7, Théorème 1’. Hence $\hat{u}|_{B(y,s)}$ is definable in $\mathbb{R}_{\text{an},\exp}$. Let $0 < t < r$ be such that $\varphi$ has a holomorphic extension to $B(x,2t)$ and $\varphi(B(x,t)) \subset B\left(y,\frac{s}{2}\right)$. Then $u|_{B(x,t)} = \hat{u} \circ \varphi|_{B(x,t)}$ is definable in $\mathbb{R}_{\text{an},\exp}$ which is a reduct of $\mathbb{R}_{\mathbb{Q},\exp}$.

Case 2.2: $x \in \text{Sing}(\partial \Omega)$. We may assume that $x = 0$. There is a semianalytic neighbourhood $V$ of 0 such that $V \cap \Omega$ is the disjoint union of finitely many corner components of $\Omega$ at 0. Let $C$ be such a corner component. Then $\angle_C \in \pi(\mathbb{R} \setminus \mathbb{Q})$ by assumption. By Theorem 3.2 there is a quadratic domain $U \subset L$ and an $f \in \mathbb{Q}(U)$ such that $\text{Re} f$ extends $u|_C$. With Proposition 3.8 we get that $u|_C$ is definable in $\mathbb{R}_{\mathbb{Q}}$ and hence in $\mathbb{R}_{\mathbb{Q},\exp}$. This shows the claim. □

As an application we obtain the definability of the Green function.

**Corollary 3.12**

Let $\Omega \subset \mathbb{R}^2$ be a semianalytic and bounded domain without isolated boundary points. Suppose that $\angle(\Omega, x) \subset \pi(\mathbb{R} \setminus \mathbb{Q})$ for all $x \in \text{Sing}(\partial \Omega)$. Let $y \in P$. Then the Green function of $\Omega$ with pole $y$ is definable in $\mathbb{R}_{\mathbb{Q},\exp}$.

**Proof**

Let $y \in P$ and $K_y : \mathbb{R}^2 \setminus \{y\} \to \mathbb{R}$, $x \mapsto \log \left|\frac{1}{x-y}\right|$, be the Poisson kernel with pole $y$. Then the Green function of $\Omega$ with pole $y$, denoted by $G^\Omega_y$, is given
by \( C^\Omega_y = K_y - u \), where \( u \) is the Dirichlet solution for \( K_y|_{\partial \Omega} \) (see [2, Chapter 4.1]). Since \( K_y|_{\partial \Omega} \) is semianalytic we have by Theorem 3.9 that \( u \) is definable in \( \mathbb{R}_{\mathbb{Q}, \exp} \). Since \( K_y \) is definable in \( \mathbb{R}_{\exp} \) we obtain the claim.

In the case of semilinear domains we can overcome the restriction on the angles:

**COROLLARY 3.13**

Let \( \Omega \subset \mathbb{R}^2 \) be a semilinear domain without isolated boundary points. Let \( h: \partial \Omega \to \mathbb{R} \) be a continuous and semianalytic function on the boundary and let \( u \) be the Dirichlet solution for \( h \). Then \( u \) is definable in the o-minimal structure \( \mathbb{R}_{\mathbb{Q}, \exp} \).

**Proof**

Let \( x \in \text{Sing}(\partial \Omega) \) and let \( C \) be a semilinear representative of a component of the germ of \( \Omega \) at \( x \) such that the germ of \( C \) at \( x \) is connected. If \( \angle_x C/\pi \in \mathbb{Q} \) we see with [24, Corollary 4] that \( u|_C \) is definable in the o-minimal structure \( \mathbb{R}_{\text{an}, \exp} \). With the proof of Theorem 3.9 we obtain the claim.

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