Abstract. Let $S_{\infty}$ denote the topological group of permutations of the natural numbers. A closed subgroup $G$ of $S_{\infty}$ is called oligomorphic if for each $n$, its natural action on $n$-tuples of natural numbers has only finitely many orbits. We study the complexity of the topological isomorphism relation on the oligomorphic subgroups of $S_{\infty}$ in the setting of Borel reducibility between equivalence relations on Polish spaces.

Given a closed subgroup $G$ of $S_{\infty}$, the coarse group $M(G)$ is the structure with domain the cosets of open subgroups of $G$, and a ternary relation $AB \subseteq C$. This structure derived from $G$ was introduced in [12, Section 3.3]. If $G$ has only countably many open subgroups, then $M(G)$ is a countable structure. Coarse groups form our main tool in studying such closed subgroups of $S_{\infty}$. We axiomatise them abstractly as structures with a ternary relation. For the oligomorphic groups, and also the profinite groups, we set up a Stone-type duality between the groups and the corresponding coarse groups. In particular we can recover an isomorphic copy of $G$ from its coarse group in a Borel fashion.

We use this duality to show that the isomorphism relation for oligomorphic subgroups of $S_{\infty}$ is Borel reducible to a Borel equivalence relation with all classes countable. We show that the same upper bound applies to the larger class of closed subgroups of $S_{\infty}$ that are topologically isomorphic to oligomorphic groups.

1. Introduction

Let $S_{\infty}$ denote the Polish group of permutations of the natural numbers with the usual topology of pointwise convergence. The closed subgroups of $S_{\infty}$ (also called non-Archimedean groups) form a standard Borel space. All the classes of groups we consider will be Borel sets in this space that are invariant under conjugation by elements of $S_{\infty}$. (Details will be provided in Section 1.4.)

Kechris and two of the authors [12] determined the complexity of the topological isomorphism relation on certain classes closed subgroups of $S_{\infty}$. They used the setting of Borel reducibility between equivalence relations $E$ and $F$ on Borel spaces $X$ and $Y$, respectively: $E$ is Borel reducible to $F$, etc.
written $E \leq_B F$, if there is $f : X \to Y$ such that the preimage of any Borel set in $Y$ is Borel in $X$, and $x_0E x_1 \iff f(x_0)Ff(x_1)$ for each $x_0, x_1 \in X$. See e.g. [8] for background on Borel reducibility.

In this paper, all topological groups will be separable and all isomorphisms between them will be topological (that is, both the isomorphism and its inverse are continuous). One result in Kechris et al. [12] addresses the compact subgroups of $S_\infty$; note that these are the separable profinite groups. Their result states that the isomorphism relation for compact subgroups of $S_\infty$ is Borel equivalent to the isomorphism relation between countable graphs. In particular, it is properly analytic.

A closed subgroup $G$ of $S_\infty$ is called oligomorphic (see [3]) if for each positive natural number $n$, its canonical action on $\omega^n$, the set of $n$-tuples of natural numbers, has only finitely many orbits (these will be called $n$-orbits). Note that this is not a group theoretic property: rather, it depends on the group action and hence on the embedding of the group into $S_\infty$. The oligomorphic groups are precisely the automorphism groups of $\omega$-categorical structures with domain the set of natural numbers. They are, in a sense, opposite to compact subgroups of $S_\infty$, which are characterised by the condition that for each $n$, each $n$-orbit is finite. For background on oligomorphic groups we refer the reader to [3], and also to Tsankov [18].

We show that the isomorphism relation between oligomorphic groups is far below graph isomorphism: it is Borel reducible to a Borel equivalence relation with all classes countable. This property of an equivalence relation on a Polish space is called “essentially countable”.

Closed subgroups of $S_\infty$ that are isomorphic to oligomorphic groups will be called quasi-oligomorphic. Near the end of the paper we will show that this class is Borel, and that the same upper bound on the isomorphism relation also applies to this class.

While oligomorphic and compact subgroups of $S_\infty$ are at opposite ends of the spectrum, they have a common superclass. A Polish group $G$ is called Roelcke precompact if for every neighborhood of the identity $U$, there exists a finite set $F \subseteq G$ such that $G = UFU$. In other words, the equivalence relation $\sim_U = \{(x, y) : \exists u, v \in U \, u xv = y\}$ has only finitely many equivalence classes. Roelcke precompactness of closed subgroups of $S_\infty$ is a Borel property as noted in [12]. It is well-known that every Roelcke precompact group $G$ has only countably many open subgroups. (To see this, let $U_n$ denote the pointwise stabiliser of $\{0, \ldots, n\}$ in $G$. Each open subgroup $U$ of $G$ contains a group $U_n$, and hence is a finite union of $\sim_{U_n}$ classes. So there are only countably many possibilities for $U$.)

Figure 1 summarises the Borel reductions between isomorphism relations obtained in the earlier reference [12] and the present paper. The wavy arrows indicate known Borel reductions; unreferenced arrows are trivial “identity” reductions given by the inclusion of Borel classes.

It is well known that there are uncountably many non-isomorphic oligomorphic groups. For instance, Evans and Hewitt [6, Lemma 3.1 and its proof] show that each profinite group $K$ is isomorphic to a group of the form $\Sigma/\Phi$, where $\Sigma$ is an oligomorphic group and $\Phi$ is the intersection of all its open subgroups of finite index. (In their construction, one can see
how $\Sigma$ depends on the way $K$ is presented as a subgroup of $S_\infty$. So, one does not obtain $\Sigma$ from $K$ through a Borel function that preserves isomorphism; note that this would contradict our result that isomorphism of oligomorphic groups is essentially countable.) Alternatively, there are uncountably many pairwise non-isomorphic automorphism groups of Henson digraphs [9]: if $\text{Aut}(G)$, $\text{Aut}(H)$ are isomorphic automorphism groups of Henson digraphs, then they are conjugate by [17, Example 1 in Section 3, Theorem 2.2 & Theorem 3.2], thus inducing an isomorphism $G \cong H$ or $G \cong H^{-1} = \{(x, y) \mid (y, x) \in G\}$ by ultrahomogeneity of the digraphs.

We leave open the question whether there is a lower bound for $\cong_{\text{oligomorphic}}$ that is higher than the identity on $\mathbb{R}$. This question may have a negative answer when we require in addition that the signature of the corresponding canonical structures is finite up to interdefinability (see Subsection 1.3).

1.1. The coarse group $\mathcal{M}(G)$ associated with $G$. Coarse groups were introduced by Kechris, Nies and Tent [12, Section 3.3], in order to provide an alternative proof of their main result, that there are Borel reductions of the isomorphism relation for Roelcke precompact groups, and also for totally disconnected locally compact (t.d.l.c.) groups, to the isomorphism relation between graphs with domain $\omega$.

First we recall a few preliminaries. A Polish group is isomorphic to a closed subgroup of $S_\infty$ if and only if its neutral element has a neighbourhood basis consisting of open subgroups; see e.g. [2, Thm. 1.5.1]. Note that each left coset $aU$ of an open subgroup $U$ is also a right coset of the open subgroup $aUa^{-1}$. We will use the term open coset for some coset, left or right, of an open subgroup. $G$ will usually denote a closed subgroup of $S_\infty$. The open cosets in $G$ form a left, and also right, translation invariant base for the subspace topology on $G$. We will use letters $A, B, C, D$ to denote open cosets.

The domain of the coarse group $\mathcal{M}(G)$ associated with $G$ consists of the open cosets. Instead of the binary group operation, it has a ternary relation

**Figure 1.** Borel reductions between isomorphism relations. $E_\infty$ denotes a $\leq_B$-complete countable Borel equivalence relation. GI denotes isomorphism of countable graphs, which is $\leq_B$-complete for orbit equivalence relations given by continuous actions of $S_\infty$. $\sim$ denotes $\equiv_{\text{compact}}$.

**Table 1.** Borel reductions between isomorphism relations. $E_\infty$ denotes a $\leq_B$-complete countable Borel equivalence relation. GI denotes isomorphism of countable graphs, which is $\leq_B$-complete for orbit equivalence relations given by continuous actions of $S_\infty$. $\sim$ denotes $\equiv_{\text{compact}}$.---
\[ AB \subseteq C. \] If \( xy = z \) in \( G \) then by continuity, for each \( C \ni z \) there are \( A \ni x \) and \( B \ni y \) such that \( AB \subseteq C \). So this ternary relation approximates the group operation. Kechris, Nies and Tent \cite[Section 3.3]{KNT} assigned to a Roelcke precompact group \( G \) in a canonical, Borel way an isomorphic copy of the structure \( \mathcal{M}(G) \) with domain the natural numbers, and showed that for Roelcke precompact closed subgroups \( G, H \) of \( S_\infty \), one has
\[ G \cong H \iff \mathcal{M}(G) \cong \mathcal{M}(H). \]

Since the coarse groups can be assumed to have domain \( \omega \), by standard coding techniques this implies that isomorphism of Roelcke precompact groups is Borel reducible to isomorphism of countable graphs.

In Section 2 we will axiomatise the basic properties of an abstract coarse group \( M \). We provide axioms that govern subgroups, inclusion, and allow us to define an operation \( B = A^\circ \) approximating the inverse operation in a group. We introduce the filter group \( \mathcal{F}(M) \), which consists of the filters that contain a (unique) coset of each subgroup. Our main interest is in the case that \( M \) is countable, in which case we show that \( \mathcal{F}(M) \) is a Polish totally disconnected group. If \( G \) has only countably many open subgroups (such as when \( G \) is Roelcke precompact), then we can recover \( G \) from its coarse group in the sense that \( G \cong \mathcal{F}(\mathcal{M}(G)) \).

### 1.2. Borel duality of oligomorphic groups with their coarse groups.

Let \( \mathcal{B} \) be the closure under isomorphism of the range of the operator \( \mathcal{M} \) on the class of oligomorphic groups. Theorem 3.1 will show that \( \mathcal{B} \) is Borel, and that on \( \mathcal{B} \) one can define a Borel operator \( \mathcal{G} \) that is an “inverse up to isomorphism” of \( \mathcal{M} \), in the sense that
- \( \mathcal{G}(\mathcal{M}(G)) \cong G \) for each oligomorphic \( G \), and
- \( \mathcal{M}(\mathcal{G}(M)) \cong M \) for each \( M \in \mathcal{B} \).

Using (1) it follows that
\[ M \cong N \iff \mathcal{G}(M) \cong \mathcal{G}(N). \]

So, in a Borel fashion we can “interchange” oligomorphic groups with their coarse groups, which are countable structures.

Theorem 3.1 will be stated via a notion introduced in \cite{Mozes} as classwise Borel isomorphism; also see \cite[Def. 2.1]{Kechris}. (We have slightly adapted the terminology here to avoid using “isomorphism” in two different ways in the same sentence.) Recall that a standard Borel space \cite[Section 12B]{Kechris} consists of an uncountable set \( Y \) together with the Borel sets given by a Polish topology on \( Y \). Given an uncountable Borel subset \( B \) of a Polish space, its Borel subsets induce a standard Borel space on \( B \); in particular, the set \( B \) above carries the structure of such a space.

**Definition 1.1.** Equivalence relations \( E \) and \( F \) on standard Borel spaces \( X \) and \( Y \), respectively, are called classwise Borel bireducible if there are Borel reductions \( F: X \to Y \) of \( E \) to \( F \) and \( G: Y \to X \) of \( F \) to \( E \) such that their factorings \( \hat{F}: X/E \to Y/F \) and \( \hat{G}: Y/F \to X/E \) to the quotient spaces are bijections and satisfy \( \hat{F} = \hat{G}^{-1} \).

A main motivation for this notion comes from duality theorems, such as Stone’s. By a Stone space we mean a compact, totally disconnected
topological space. Stone duality sets up a correspondence between such spaces and Boolean algebras. This restricts to a duality between separable Stone spaces and countable Boolean algebras.

While a duality theorem merely requires that the objects are given up to isomorphism, here we are interested in Borel versions. We assume that the objects are concretely given as points in standard Borel spaces. For instance, a Borel version of Stone duality is as follows: we assume that an infinite Stone space is given as the set of paths $[T]$ through a subtree $T$ of $2^\mathbb{ω}$ with infinitely many paths and without dead ends, and that the Boolean algebras have domain $\omega$. Abstractly, for an infinite Stone space $G$, one lets $\mathcal{M}(G)$ be the Boolean algebra of clopen sets in $G$; for each countably infinite Boolean algebra $M$, one lets $\mathcal{G}(M)$ be the Stone space of ultrafilters of $M$. If the objects are concretely represented, these maps become Borel: given a tree $T$ such that $G = [T]$, one can in a Borel fashion produce a listing without repetition of the clopen sets of the space $[T]$, and hence determine $\mathcal{M}(T)$. Given a Boolean algebra $M$ with domain $\omega$, one lets $\mathcal{G}(M)$ be the tree of strings $\sigma \in 2^\mathbb{ω}$ describing a nonzero conjunction $r_\sigma$ of literals (for $i < |\sigma|$ let $l_i = i$ if $\sigma(i) = 1$ and $l_i = -i$ otherwise; let $r_\sigma = \bigwedge_{i < n} l_i$). Clearly the conditions above for classwise Borel bireducibility hold.

Our Borel version of duality between classes $\mathcal{A}, \mathcal{B}$ of (possibly topological) structures requires that both $\mathcal{A}$ and $\mathcal{B}$ are isomorphism invariant Borel sets of the canonical Polish spaces for such structures. In Section 3.5 we obtain a duality between the class $\mathcal{A}$ of profinite groups and an appropriate class $\mathcal{B}$ coarse groups. The result of Kechris, Nies and Tent [12, Section 3.3] (also see Prop. 2.13) on Roelcke precompact groups mentioned above is not yet such a duality result, because one needs to show that the closure under isomorphism of the range of the operator $\mathcal{M}$ defined on such groups is Borel, and also needs to define a Borel inverse $\mathcal{G}$ up to isomorphism. This is indeed possible for the Roelcke precompact groups, and will be carried out in a forthcoming paper of Melnikov and the first author. It is not known whether the larger class of closed subgroups of $S_\infty$ that have only countable many open subgroups is Borel, so at this stage one can’t expect a Borel duality result here.

We note that a common way of formalizing abstract duality theorems is via the notion of equivalence of categories, where the operators (now functors) also turn isomorphisms into isomorphisms. A Borel version of category equivalence can be obtained for the profinite, and even the Roelcke precompact, groups, but is unknown for the oligomorphic groups. The construction in Section 3.4 of the inverse map $\mathcal{G}$ is not uniform in that sense, because one has to pick an appropriate element $W$ of the coarse group in a Borel fashion (see Section 1.3 below). This choice is not necessarily unique. If one also wanted to turn any isomorphism between coarse groups into an isomorphism of the corresponding oligomorphic groups, one would need to make choices of elements in the coarse groups that are matched by the given isomorphism, which is only possible if the elements are uniquely determined. In contrast, the weaker formalization of Borel duality given by Definition 1.1 works.
1.3. The upper bound on the complexity of isomorphism. Once Theorem 3.1 is established, we will show that isomorphism of oligomorphic groups is Borel reducible to a countable Borel equivalence relation. We apply a result of Hjorth and Kechris [10, Theorem 4.3] about Borel invariant classes $C$ of countable structures that will be explained in more detail at the beginning of Subsection 4.2. An important further ingredient (just alluded to above) is that each oligomorphic group $G$ has an open subgroup $W$ such that the left translation action of $G$ on the left cosets of $W$ is oligomorphic, and yields a topological embedding of $G$ into $S_\infty$. ($W$ is simply the intersection of the stabilisers of finitely many numbers chosen to represent the 1-orbits; see Lemma 3.3.) We thank Todor Tsankov for communicating this fact to us.

We mention here that there is an alternative way to obtain the upper bound on isomorphism of oligomorphic groups from Theorem 3.1: via bi-interpretability of $\omega$-categorical structures. To an oligomorphic group $G$ one can in a Borel way assign a structure $N_G$ with domain $\omega$ such that $G = \text{Aut}(N_G)$: the language has $k_n$ many $n$-ary relation symbols $P^n_i$, where $k_n$ for $n \geq 1$ is the number of $n$-orbits of $G$, and $P^n_i$ denotes in $N_G$ the $i$-th $n$-orbit. Coquand (unpublished), see Ahlbrandt and Ziegler [1], showed that oligomorphic groups $G$, $H$ are topologically isomorphic if and only if $N_G$ and $N_H$ are bi-interpretable in the sense of model theory (e.g. Hodges [11, Section 5.3]); also see David Evans’ 2013 notes.

One can show that bi-interpretability of $\omega$-categorical structures is a $\Sigma^0_2$ relation. Now one applies a related result of Hjorth and Kechris in the same paper [10, Theorem 3.8], by which the existence of a Borel reduction of $\cong_B$ to a $\Sigma^0_2$ equivalence relation implies that $\cong_B$ is essentially countable.

We don’t follow this pathway because the formal details would be very tedious, while after our proof of Theorem 3.1 not too much extra effort is required to satisfy the hypothesis of [10, Theorem 4.3]. For some details on the alternative approach see our Logic Blog post [4, Section 8.5].

1.4. Preliminary: the Effros space. Given a Polish space $X$, let $\mathcal{E}(X)$ denote the set of closed subsets of $X$. The Effros Borel space on $X$ is the standard Borel space consisting of $\mathcal{E}(X)$ together with the $\sigma$-algebra generated by the sets

$$C_U = \{ D \in \mathcal{E}(X) : D \cap U \neq \emptyset \},$$

for open $U \subseteq X$. For details, see e.g. [8, Definition 1.4.5].

It is not hard to see that in $\mathcal{E}(S_\infty)$, the properties of being a (closed) subgroup of $S_\infty$, and of being an oligomorphic group, are Borel. For the former see [12, Lemma 2.5]. For the latter, note that a closed subgroup $G$ is oligomorphic if and only if for each $n$, there is $k$ such that

$$\exists x_1, \ldots, x_k \in T_n \forall y \in T_n \bigvee_{1 \leq i \leq n} G \cap U_{x_i, y} \neq \emptyset,$$

where $T_n$ is the set of $n$-tuples of natural numbers without repetitions, and $U_{x, y}$ for $x, y \in T^n$ is the open set of permutations $f$ such that $f(x(r)) = y(r)$ for each $r < n$. 
2. Coarse groups

In this section we study coarse groups, defined in Section 1.1 above, and in particular isolate some of their properties by formulating axioms. This leads to an abstract axiomatisation of coarse groups of interesting classes of groups. We let \( M \) always denote a coarse group. In our applications we will only consider the case that \( M \) is countable. We introduce the filter group \( \mathcal{F}(M) \) as a step towards recovering closed subgroups of \( S_\infty \) from their approximative countable coarse groups.

2.1. Basic definitions and axioms. Throughout, \( M \) will denote a structure for the signature with a ternary relation symbol \( R \) which will describe an coarse group. We now give the axioms for such a structure, and we will always assume that \( M \) satisfies them.

For \( A, B, C \in M \), the ternary relation \( R(A, B, C) \) will more suggestively be written as “\( AB \subseteq C \)”.

We let \( U,V,W \) denote \( \ast \)subgroups in \( M \). We write \( U \sqsubseteq V \) for \( UV \subseteq V \).

Definition 2.1 (Some definable relations).

(a) \( A \in M \) is a \( \ast \)subgroup if \( AA \sqsubseteq A \). Letters \( U,V,W \) denote \( \ast \)subgroups in \( M \). We write \( U \subseteq V \) for \( UV \subseteq V \).

(b) \( A \) is a left \( \ast \)coset of a \( \ast \)subgroup \( U \) if and only if \( U \) is the largest \( \ast \)subgroup under \( \sqsubseteq \) with \( AU \sqsubseteq A \). (\( U \) is unique by definition.) We define right \( \ast \)cosets of \( U \) analogously. We write \( LC(U) \) for the set of left \( \ast \)cosets and \( RC(U) \) for the set of right \( \ast \)cosets of a \( \ast \)subgroup \( U \).

(c) We write \( A \sqsubseteq B \) for arbitrary \( A,B \) if \( AU \sqsubseteq B \) for some \( \ast \)subgroup \( U \) such that \( A \) is a left \( \ast \)coset for \( U \). (\( U \) exists, and is unique, by Axiom 0(b).)

Axiom 0. (Basic axioms)

(a) The relation \( \sqsubseteq \) on \( \ast \)subgroups is a partial order so that any two elements have a meet.

(b) Every element is a left \( \ast \)coset of some \( \ast \)subgroup and a right \( \ast \)coset of some \( \ast \)subgroup.

(c) The relation \( \sqsubseteq \) in Definition 2.1(c) is a partial order that extends \( \sqsubseteq \) on the set of \( \ast \)subgroups.

Axiom 1. (Monotonicity)

If \( B_0B_1 \subseteq C \) and \( A_i \subseteq B_i \) for \( i \leq 1 \), then \( A_0A_1 \subseteq C \).

We say that \( A,B \) in \( M \) are disjoint if \( \neg \exists C (C \sqsubseteq A \land C \sqsubseteq B) \).\(^1\)

Axiom 2. Suppose that \( U' \sqsubseteq U \) and \( A' \in LC(U') \).

(a) There is \( A \in LC(U) \) such that \( A' \sqsubseteq A \).

(b) If \( A \in LC(U) \), then \( A' \sqsubseteq A \), or \( A' \) and \( A \) are disjoint. In particular, any two distinct left \( \ast \)cosets of the same \( \ast \)subgroup are disjoint. Similar statements hold for right \( \ast \)cosets.

\(^1\)Disjointness can be expressed via the ternary relation \( \sqsubseteq \) as \( \neg \exists C \exists D (CD \sqsubseteq A \land CD \sqsubseteq B) \); this follows from Axioms 0, 1, 2 and 5.
Remark 2.2. [12, Section 3.3]. For a closed subgroup $G$ of $\mathcal{S}_\infty$, the terms introduced above have their intended meanings in the structure $\mathcal{M}(G)$, and the axioms specified so far are satisfied.

Axiom 3. Let $U$ and $V$ be $^*$subgroups and $B \in \text{LC}(V)$. Then

$$U \subseteq V \iff \text{there is } A \in \text{LC}(U) \text{ with } A \subseteq B.$$ 

A similar statement holds for right $^*$cosets.

To see that this axiom holds in $\mathcal{M}(G)$, let $B = bV$. If $U \subseteq V$, then $A = bU \subseteq bV$. Conversely, suppose that $A = aU \subseteq B$. Then $b^{-1}aU \subseteq V$. So $a^{-1}b \in V$, and hence $U \subseteq a^{-1}bV = V$.

Each $A \in M$ contains a left $^*$coset of an arbitrarily “small” $^*$subgroup $V$.

Claim 2.3. For each $A \in M$ and each $^*$subgroup $U$, there are a $^*$subgroup $V \subseteq U$ and a left $^*$coset $B$ of $V$ such that $B \subseteq A$. A similar fact holds for right $^*$cosets.

To see this, suppose $A \in \text{LC}(W)$. Let $V = W \land U$ using Axiom 0(a). By Axiom 3 there is $B \subseteq A$ in $\text{LC}(V)$, as required.

We write $S(A, B)$ for the statement that there is a $^*$subgroup $V$ such that $A \in \text{RC}(V)$, $B \in \text{LC}(V)$, and $AB \subseteq V$. It is easily checked that in $\mathcal{M}(G)$, we have $S(A, B) \iff S(B, A) \iff B = A^{-1}$. In particular, if we are given $A \in \mathcal{M}(G)$, then $B \in \mathcal{M}(G)$ is unique.

Axiom 4 (Inverses).

(a) For each $A$, there is a unique $B$ such that $S(A, B)$.

(b) $S(A, B) \iff S(B, A)$. Assuming the axiom holds in a structure $M$, instead of $S(A, B)$ we will write $B = A^\circ$.

(c) $A \mapsto A^\circ$ is an isomorphism with respect to $\subseteq$.

Note that Axiom 4 implies that $A^{\circ\circ} = A$.

2.2. Full filters, and the filter group. A subset $x$ of $M$ is called closed upwards with respect to $\subseteq$ if for all $A \in x$ and $A \subseteq B$, we have $B \in x$. It is directed downward if for all $B, C \in x$, there is some $A \in x$ with $A \subseteq B$ and $A \subseteq C$. We now define the set of full filters $\mathcal{F}(M)$. Thereafter we will define a group operation and add axioms ensuring that $\mathcal{F}(M)$ with a canonical topology forms a Polish group.

Definition 2.4 (Full filters).

A full filter $x$ on $M$ is a subset of $M$ with the following properties.

(a) It is directed downwards, and closed upwards with respect to $\subseteq$.

(b) Each $^*$subgroup $U$ in $M$ has a left $^*$coset and a right $^*$coset in $x$.

We let $\mathcal{F}(M)$ denote the set of full filters on $M$.

Letters $x, y, z$ will denote elements of $\mathcal{F}(M)$.

We note that full filters are separating sets for the equivalence relation on $M$ of being left $^*$cosets of the same $^*$subgroup, and similarly for right $^*$cosets. In particular, they are maximal filters.

Claim 2.5. Suppose that $M$ is countable. For each $A \in M$ there is a full filter $x$ such that $A \in x$. 
To see this, let \( (U_n)_{n \in \omega} \) be a listing of all *subgroups in \( M \). We construct a \( \subseteq \)-decreasing sequence \( (A_n)_{n \in \omega} \) as follows. Let \( A_0 = A \). Given \( A_n \), find \( V_n \subseteq U_n \) and a left *coset \( B_n \) of \( V_n \) with \( B_n \subseteq A_n \) by Claim 2.3. Similarly, take \( W_n \subseteq V_n \) and a right *coset \( A_{n+1} \) of \( W_n \) with \( A_{n+1} \subseteq B_n \). Then \( \{ C : \exists n \ A_n \subseteq C \} \) is a full filter on \( M \) containing \( A \).

**Definition 2.6** (Topology on the set of full filters).
We define a topology on \( F(M) \) by declaring as subbasic the open sets
\[
\hat{A} = \{ x \in F(M) : A \in x \}
\]
where \( A \in M \). These sets form a base since filters are directed.

The Baire space \( \omega \) is endowed with a topology given by the subbasic open sets \( \{ f : f(n) = i \} \) for \( n, i \in \omega \). It is totally disconnected, i.e. any connected subset contains at most one element.

**Proposition 2.7.** Suppose that \( M \) is countable. Then \( F(M) \) is a totally disconnected Polish space.

*Proof.* Since \( M \) is countable, the *subgroups and *cosets in \( M \) can be provided with an ordering of type \( \omega \). Let \( U_n \) denote the \( n \)-th *subgroup in \( M \).

We define an injection \( \Delta \) from \( F(M) \) into Baire space \( \omega \). Suppose that \( x \in F(M) \). Let \( \Delta(x)(2n) \) be the unique \( i \) such that the \( i \)-th left *coset of \( U_n \) in \( M \) is an element of \( x \). Let \( \Delta(x)(2n+1) \) be the unique \( i \) such that the \( i \)-th right *coset of \( U_n \) in \( M \) is an element of \( x \). By Axiom 2, \( \Delta \) is well-defined and injective.

We claim that \( \Delta \) is a homeomorphism to \( \text{ran}(\Delta) \). It is clear that the image of any basic open subset \( A \) of \( F(M) \) is open. Conversely, the preimage of any subbasic open subset of the Baire space is of the form \( \{ x \in F(M) \mid A \in x \} \) for some \( A \in M \).

We show that \( \text{ran}(\Delta) \) is \( G_\delta \). Identifying a full filter \( x \) with \( \Delta(x) \), upwards closure is a closed condition. The condition in Def. 2.4(b) is satisfied automatically, because \( f(2n) \) denotes a left *coset of \( U_n \), and \( f(2n+1) \) a right *coset of \( U_n \). Downwards directedness is a \( G_\delta \) condition.

Hence \( F(M) \) is homeomorphic to a \( G_\delta \) subset of the Baire space. Since every \( G_\delta \) subspace of a Polish space is again Polish, it follows that \( F(M) \) is a Polish space. Clearly it is totally disconnected since Baire space is.

For \( x \in F(M) \) we let
\[
x^{-1} = \{ A^\circ \mid A \in x \}.
\]
We claim that \( x^{-1} \) is a full filter. It is upwards closed and directed by the previous axiom. The condition Def. 2.4(b) holds since the * operation interchanges left *cosets of \( U \) with right *cosets of \( U \). Since \( A^\circ \circ = A \), we further have \( \hat{A}^{-1} = \hat{A}^\circ \).

**Definition 2.8** (Product of full filters). For full filters \( x, y \) on \( M \), we put
\[
x \cdot y = \{ C \in M \mid \exists A \in x \exists B \in y \ AB \subseteq C \}.
\]
The next axiom ensures that this operation on \( F(M) \) is defined and continuous.
Axiom 5. Suppose $A \in RC(U) \cap LC(V)$ and $B \in RC(V) \cap LC(W)$. There is $C \in RC(U) \cap LC(W)$ such that $AB \subseteq C$, and $C$ is the least $D$ with $AB \subseteq D$. We will write $A \cdot B$ for this (unique) $C$.

This holds in $\mathcal{M}(G)$: if $A = aV = Ua'$ and $B = bW = Vb'$, then $AB = aVVb' = aVb'$. Since $aVb' = Ua'b' = abW$, $C = AB$ is a right coset of $U$ and a left coset of $W$. It is clearly minimal.

Claim 2.9. $x \cdot y$ is an element of $\mathcal{F}(M)$ for all $x, y \in \mathcal{F}(M)$.

Proof. Since the relation $\subseteq$ is transitive, $x \cdot y$ is closed upwards by definition of the product.

To see that $x \cdot y$ is directed downwards, suppose that elements $C_0, C_1 \in x \cdot y$ are given. Then there are $A_0, A_1 \in x$ and $B_0, B_1 \in y$ such that $A_iB_i \subseteq C_i$ for $i = 0, 1$. Since $x$ and $y$ are directed downwards, there are $A \in x$ with $A \subseteq A_0$ and $A \subseteq A_1$, and $B \in y$ with $B \subseteq B_0$ and $B \subseteq B_1$. By monotonicity, $AB \subseteq C_i$ for $i \leq 1$. Let $V, W$ be the *subgroups in $\mathcal{M}$ such that $B \in RC(V) \cap LC(W)$.

We can assume $A \in LC(V)$ by shrinking $A, B, V, W$ while using the fact that $x$ and $y$ are full filters to maintain that $A \in x$ and $B \in y$. In more detail, take any $A \in LC(U)$. Let $V' = U \cap V$ and $A' = LC(V') \cap x$ using that $x$ is a full filter. $A, A'$ cannot be disjoint, since $A_i, A_i' \in x$ and $x$ is a filter, so $A' \subseteq A$ by Axiom 2. One similarly obtains $B' \subseteq B$ in $RC(V') \cap y$. We have $W' \subseteq W$ for the unique *subgroup $W'$ with $B' \subseteq LC(W')$, by Axiom 3. Thus $A', B', V', W'$ are as required.

By Axiom 5, there is a unique $C \in LC(W)$ with $AB \subseteq C$ and by its minimality, $C \subseteq C_i$ for $i \leq 1$. Since $AB \subseteq C$, we have $C \in x \cdot y$.

We now show that $x \cdot y$ satisfies condition (b) in Def. 2.4. Take any *subgroup $W$ in $\mathcal{M}$ and let $B \subseteq LC(W) \cap y$. Let $V$ the *subgroup such that $B \subseteq RC(V)$, and let $A \subseteq LC(V) \cap x$. By Axiom 5, there is $C \in LC(W)$ with $AB \subseteq C$. Then $C \in x \cdot y$. Right *cosets are similar. □

The next axiom can be expressed by a $\Pi_1^1$ condition in case that $M$ is countable. An equivalent first-order axiom (Axiom 11) is introduced in Subsection 4.4. We work with the simpler $\Pi_1^1$ axiom here, since it suffices for the main results.

Axiom 6. The operation $\cdot$ on $\mathcal{F}(M)$ is associative.

Remark 2.10 (Neutral element of $\mathcal{F}(M)$). Let $1_{\mathcal{F}(M)}$ denote the filter generated by the *subgroups in $\mathcal{M}$. We have $1_{\mathcal{F}(M)} \cdot x = x \cdot 1_{\mathcal{F}(M)} = x$ and $x \cdot x^{-1} = x^{-1} \cdot x = 1_{\mathcal{F}(M)}$ for all $x \in \mathcal{F}(M)$ by Axioms 4 and 6.

The next axiom ensures that $AB \subseteq C$ and $A \subseteq B$ express the expected properties in $\mathcal{F}(M)$. It holds in $\mathcal{M}(G)$ by continuity of the group operation.

Axiom 7.

(a) $AB \subseteq C$ iff there are no *cosets $D \subseteq A$, $E \subseteq B$ and $F$ such that $DE \subseteq F$ and $C, F$ are disjoint.

(b) $A \subseteq B$ iff there is no *coset $C \subseteq A$ with $B, C$ disjoint.

Recall from Definition 2.6 that $\hat{A} = \{ x \in \mathcal{F}(M) : A \in x \}$. Note that if $A, B$ are disjoint (as defined before Axiom 2) then $\hat{A} \cap \hat{B} = \emptyset$, because filters are directed downwards. Let $\hat{A} \hat{B}$ denote the setwise product.
Claim 2.11. Let $A, B, C \in M$. Let $U$ be a *subgroup.

(a) $AB \subseteq C \iff \hat{A}B \subseteq \hat{C}$.
(b) $\hat{U}$ is a subgroup of $\mathcal{F}(M)$.
(c) $A \subseteq B \iff \hat{A} \subseteq \hat{B}$.
(d) $\hat{B} = (\hat{B})^{-1}$.
(e) $A \in LC(U) \iff \hat{A}$ is a left coset of $\hat{U}$, and similar for right cosets.

Proof. (a) The forward implication is clear. For the converse implication, we assume that $AB \subseteq C$ and find full filters $x \in \hat{A}$ and $y \in \hat{B}$ with $xy \notin \hat{C}$. By Axiom 7(a), there are $D \subseteq A$, $E \subseteq B$ and $F$ with $DE \subseteq F$ and $C$, $F$ disjoint. By Claim 2.5, take full filters $x \in \hat{D}$ and $y \in \hat{E}$. Then $xy \notin \hat{F}$. Since $\hat{C} \cap \hat{F} = \emptyset$ we conclude that $xy \notin \hat{C}$.

(c) is similar to (a) using Axiom 7(b), and (d) is easily verified.

(e) forward implication: take any $x \in \hat{A}$. We show that $x\hat{U} = \hat{A}$.

For $x\hat{U} \subseteq \hat{A}$, let $y \in \hat{U}$. Since $A$ is a left *coset of $U$, we have $AU \subseteq A$. So $x \cdot y \in \hat{AU} \subseteq \hat{A}$ by (a).

For $A \subseteq x\hat{U}$, let $y \in A$. To show that $y \in x\hat{U}$, or equivalently $x^{-1}y \in \hat{U}$, note that we have $x^{-1}y \in \hat{A}^{-1}A = \hat{A} \subseteq \hat{U}$ by (d) and (a).

(e) backward implication: suppose $\hat{A} = x\hat{V}$. There is $B \in \hat{x}$ such that $B \in LC(V)$. By the forward implication, $\hat{B}$ is a left coset of $\hat{V}$. Also $x \in \hat{A} \cap \hat{B}$, so $A, B$ are not disjoint. Since $A, B \in LC(V)$ this implies $A = B$ by Axiom 2. Right cosets are dealt with symmetrically. \qed

We can’t prove on the basis of the present axioms that $(\mathcal{F}(M), \cdot)$ is isomorphic to a closed subgroup of $S_\infty$; this will be achieved in Section 3.4 in the oligomorphic case, and in Section 3.5 in the profinite case. At the current point we can show the following.

Proposition 2.12.
Suppose that $M$ is countable. Then $(\mathcal{F}(M), \cdot)$ is a Polish group.

Proof. In view of Prop. 2.7, it suffices to show that the operation $x, y \mapsto x \cdot y^{-1}$ on $\mathcal{F}(M)$ is continuous. I.e. it suffices to show that for all $x, y$ and every $D \in M$ with $x \cdot y^{-1} \in \hat{D}$, there are $A, B \in M$ with $A \in x$, $B \in y$ such that $u \cdot v^{-1} \in \hat{D}$ holds for all $u \in \hat{A}$ and $v \in \hat{B}$.

To see this, suppose that $D \subseteq x \cdot y^{-1}$ is a left coset of $U$. We choose a left coset $B \in y$ of $U$. By Axiom 0, $B$ is a right coset of some $V$. We choose a left coset $A \in x$ of $U$. By Axiom 5, there is a left coset $C$ of $V$ with $AB^0 \subseteq C$ and hence $C \subseteq x \cdot y^{-1}$. Since $x \cdot y^{-1}$ is a full filter, we have $C = D$. By Claim 2.11, $\hat{A}B^0 = \hat{A}(\hat{B})^{-1} \subseteq \hat{C} = \hat{D}$. \qed

We call $\mathcal{F}(M)$ the filter group of $M$. Note that $\mathcal{F}(M)$ is defined abstractly as a Polish group, rather than as a permutation group. We show that closed subgroups of $S_\infty$ with countably many open subgroups can be recovered in a canonical way as the filter group of their coarse group.

Proposition 2.13 (cf. [12], after Claim 3.6).
Suppose that $G$ is a closed subgroup of $S_\infty$ such that $M(G)$ is countable. There is a natural group homeomorphism

$$\Phi : G \cong \mathcal{F}(M(G))$$
given by $g \mapsto \{A : A \ni g\}$,
with inverse $\Xi$ given by $x \mapsto g$ where $\bigcap x = \{g\}$.

This is essentially contained in [12] (note that $L_g = R_g = \{A: A \ni g\}$ in the notation there).

**Sketch of proof.** Let $x \in F(\mathcal{M}(G))$. We show that $\bigcap x$ is non-empty.

Let $U_n$ be the open subgroup of $G$ consisting of the permutations that fix $0, \ldots, n$. Since $x$ is a full filter, there are permutations $r_n, s_n \in G$ such that $r_n U_n \in x$ and $U_n s_n \in x$. Let $g(n) = r_n(n)$ and $g^*(n) = s_n^{-1}(n)$. As in [12] one shows that $g^* = g^{-1}$ using that $x$ is a filter. So $g$ is a permutation, and then clearly $g \in \cap x$ since $G$ is closed.

On the other hand, since the open cosets form a base, $\bigcap x$ has at most one element. So the map $\Xi$ is defined.

Let $g \in G$ and let $x \in F(\mathcal{M}(G))$. Trivially $\Xi(\Phi(g)) = g$. It is also trivial that $x \subseteq y = \Phi(\Xi(x))$. Since $y$ is a full filter this implies $x = y$.

One shows that $\Phi$ preserves the group operations as in [12, after Claim 3.6].

Finally $\Phi^{-1}(A) = A$ by definition, so $\Phi$ is a homeomorphism. □

**Remark 2.14.** Note that by this argument, the group $\text{Aut}(G)$ of topological automorphisms of $G$ is naturally isomorphic to $\text{Aut}(\mathcal{M}(G))$. Hence $\text{Aut}(G)$ can itself be seen as a closed subgroup of $S_\infty$.

**Remark 2.15.** The following illustrates the structure of a coarse group $M$ and will be useful when we discuss coarse groups of profinite groups in Subsection 2.5. Recall that a subgroup of a group is normal if its left cosets coincide with the right cosets. Thus, we say that a *subgroup $U$ of $M$ is normal if $LC(U) = RC(U)$.

We verify that for normal $U$, the set $LC(U)$ carries a canonical group structure. (If $M = \mathcal{M}(G)$, then this group is simply $G/U$.) Note that by Axiom 5, if $A, B \in LC(U)$ then there is a unique $C \in LC(U)$ such that $AB \subseteq C$. (Recall that we write $A \cdot B = C$.) Also, if $A \in LC(U)$ then by definition $A^0 \in RC(U) = LC(U)$. By associativity and the definition of the operation $\circ$, $(LC(U), \circ)$ is a group.

**Remark 2.16.** We sketch an alternative approach to coarse groups which may be useful e.g. in future applications to totally disconnected locally compact (t.d.l.c.) groups. On the set of open cosets, or the compact open cosets in the t.d.l.c. case, we take as primitives the following notions: the inclusion partial order $\subseteq$, *subgroups, the two operations mapping a *coset $A$ to the *subgroups $U, V$ such that $A \in RC(U)$ and $A \in LC(V)$, and the product $A \cdot B$ of cosets as in Axiom 5, namely, under the assumption that $A \in LC(V)$ and $B \in RC(V)$ for the same *subgroup $V$.

Recall that a groupoid is a small category in which every morphism has an inverse. An inductive groupoid is a groupoid that also has the structure of a meet semilattice, and satisfies some natural axioms positing that the two structures are compatible [14, Section 4.1]. In this alternative setting, a coarse group has the structure of an inductive groupoid, where $A$ as above is seen as a morphism $U \to V$.

The two approaches to coarse groups are equivalent. Each coarse group as developed above can be first-order defined in an inductive groupoid satisfying appropriate additional axioms, and vice versa. It is clear that an
inductive groupoid can be defined in the coarse group. Conversely, suppose we are given an inductive groupoid $L$. If $A \in LC(U)$ and $B \in RC(V)$, letting $W = U \wedge V$, we can define

$$AB \subseteq C :\Leftrightarrow \exists A' \subseteq A \exists B' \subseteq B [A' \in LC(W) \wedge B' \in RC(W) \wedge A' \cdot B' \subseteq C].$$

The axioms above up to Axiom 5, but with the exception of Axiom 2, turn into the usual axioms for inductive groupoids OG1-OG3 as in [14, Section 4.1]. Conversely, formulating appropriate additional axioms for $L$, one can obtain all the axioms we provide. For some details see [5, Part 1].

**Example 2.17.** As an instructive example of an inductive groupoid we consider the oligomorphic group $G = \text{Aut}(\mathbb{Q}, <)$. The open subgroups of $G$ are the stabilizers of finite sets. If $U, V$ are stabilizers of sets of the same finite cardinality, there is a unique morphism $A: U \rightarrow V$ in the sense above, corresponding to the order-preserving bijection between the two sets. The inductive groupoid for $\text{Aut}((\mathbb{Q}, <))$ is canonically isomorphic to the groupoid of finite order-preserving maps on $\mathbb{Q}$, with the partial order being reverse extension.

A filter $x$ (in either setting) corresponds to an arbitrary order-preserving map $\psi$ on $\mathbb{Q}$. The filter $x$ contains a right coset of each open subgroup if and only if $\psi$ is total, and a left coset of each open subgroup if and only if $\psi$ is onto. So the set of full filters corresponds to $\text{Aut}(\mathbb{Q})$ as expected. (Incidentally, this example shows that in Definition 2.4(b) we need both sides, and that not every maximal filter is full.)

**2.3. The action of $\mathcal{F}(M)$ on $LC(V)$.** Suppose that $V$ is a *subgroup of $M$, and as before let $LC(V) \subseteq M$ denote its set of left *cosets. We define an action

$$(2) \quad \gamma_V: \mathcal{F}(M) \actson LC(V)$$

by letting

$$(3) \quad x \cdot A = B \text{ iff } \exists S \in x [SA \subseteq B].$$

Note that such a $B$ is unique because $x$ is a filter. For $M = M(G)$, by Prop. 2.13, $\gamma_V$ is simply the natural left action of $G$ on the left cosets of the open subgroup $V$.

We verify that $\gamma_V$ is defined: For each full filter $x$, for every *subgroup $V$ and each left $A \in LC(V)$ there is $B \in LC(V)$ such that $x \cdot A = B$. Suppose that $A \in RC(U)$. Since $x$ is a full filter, it contains some $S \in LC(U)$. Then $B := S \cdot A \in LC(V)$ by Axiom 5. Hence $x \cdot A = B$.

**Claim 2.18.** For each full filter $x$, for each $C \in M$, we have $x \hat{C} = x \cdot \hat{C}$.

**Proof.** Suppose $C \in LC(V)$. By Claim 2.11(e), both $x \hat{C}$ and $x \cdot \hat{C}$ are left cosets of $\hat{V}$. So it suffices to show that $x \hat{C} \subseteq x \cdot \hat{C}$. Let $D = x \cdot C \in LC(V)$. By definition there is $S \in x$ such that $SC \subseteq D$. If $z \in x \hat{C}$, then by definition there is $y$ such that $C \in y$ and $z = x \cdot y$. So $D \in z$ as required. \qed

**Claim 2.19.** For every *subgroup $V$, $\gamma_V: \mathcal{F}(M) \actson LC(V)$ is a group action.
Proof. Clearly \(1 \cdot A = A\) for each \(A \in LC(V)\) (see Remark 2.10).

For each \(C, x, y\) by associativity of the filter product we have
\[
x(yC) = x\{y \cdot z: z \in C\} = \{(x \cdot y) \cdot z: z \in C\} = (x \cdot y)C.
\]
Also, by Claim 2.18,
\[
x(yC) = x(y \cdot C) = x(y \cdot C) \quad \text{and} \quad (x \cdot y)C = (x \cdot y) \cdot C.
\]
So \(x \cdot (y \cdot C) = (x\cdot y) \cdot C\). Then by Claim 2.11(c) \(x \cdot (y \cdot C) = (x \cdot y) \cdot C\) as required. \(\square\)

Claim 2.19 is equivalent to the statement that \((x, A) \rightarrow x \cdot A\) is an action of \(\mathcal{F}(M)\) on \(M\). For, on the basis of the axioms so far, \(M\) is partitioned into the sets \(LC(V)\) for \(*\)-subgroups \(V\), the orbits of this action. We have provided the current formulation of the claim mainly for notational convenience.

Remark 2.20.
Recall that we now regard an abstract coset structure \(M\) as having domain \(\omega\). So if \(LC(V) \subseteq \omega\) is infinite we can identify its elements \(A_0, A_1, \ldots\) with the natural numbers, and the action \(\gamma_V\) can be viewed as an action on \(\omega\).

2.4. For Roelcke precompact \(\mathcal{F}(M)\), each open coset has the form \(\hat{A}\).
In this section we provide an important tool. Introducing the new Axiom 8 for \(M\), we show that if the Polish group \(\mathcal{F}(M)\) is Roelcke precompact (see the introduction), then each open subgroup of \(\mathcal{F}(M)\) is named by a \(*\)-subgroup in \(M\). This will be needed in Section 3 to verify that \(M(\mathcal{G}(M)) \cong M\) for each \(M \in \mathcal{B}\), where \(\mathcal{B}\) is as in Subsection 1.2 and \(\mathcal{G}(M)\) is a realization of \(\mathcal{F}(M)\) as a permutation group. We will also apply this tool to characterise the coarse groups of profinite groups in Subsection 2.5. In this case, we could actually use a simpler version of Axiom 8 that only involves left cosets of a fixed subgroup, rather than double cosets.

Recall our letter conventions: letters \(A\) to \(F\) and their variants denote elements of \(M\) (called \(*\)-cosets), and letters \(U, V, W\) denote \(*\)-subgroups. Also recall from Definition 2.6 that \(\hat{A} = \{x \in \mathcal{F}(M): A \in x\}\). As always \(M\) is a structure with domain \(\omega\) in the language with one ternary relation symbol, and we generally assume that \(M\) satisfies the (still growing) list of axioms.

Note that by Claim 2.11 in Section 3 that the map \(A \mapsto \hat{A}\) is a 1-1 map from elements of \(M\) to open cosets of \(\mathcal{F}(M)\). After adding Axiom 8, we will show in Lemma 2.25 that this map is onto, assuming that \(\mathcal{F}(M)\) is Roelcke precompact: each open coset in \(\mathcal{F}(M)\) is of the form \(\hat{A}\) for some \(A \in M\).

We begin with an auxiliary claim.

Claim 2.21. For any left coset \(x \hat{V}\) in \(\mathcal{F}(M)\), there is a left \(*\)-coset \(A\) of \(V\) in \(M\) such that \(x \hat{V} = \hat{A}\).

Proof. Since \(x\) is a full filter, there is some left \(*\)-coset \(A\) of \(V\) in \(x\). We claim that \(x \hat{V} = \hat{A}\). We have \(x \hat{V} \subseteq \hat{A} \hat{V} = \hat{V}\), since \(A \in x\) and \(\hat{A}\) is a left coset of \(\hat{V}\) by Claim 2.11. To see that \(\hat{A} \subseteq x \hat{V}\), let \(y \in \hat{A}\). Since \(x, y \in \hat{A}\), \(x^{-1}y \in \hat{A}^{-1} \hat{A} = \hat{A}^{-1} \hat{A} \subseteq \hat{V}\) by Claim 2.11. Thus \(y \in x \hat{V}\). \(\square\)

Consider any open subgroup \(U\) of \(\mathcal{F}(M)\). Since \(U\) is open and \(1_{\mathcal{F}(M)} \in U\), there is an \(A \in M\) with \(1_{\mathcal{F}(M)} \in \hat{A}\) and \(\hat{A} \subseteq U\). Now \(A\) is equal to
a *subgroup $V$ in $M$, since $1_{\mathcal{F}(M)}$ contains only *subgroups by Axiom 2 and directedness of full filters. By Roelcke precompactness of $\mathcal{F}(M)$, the subgroup $U$ is a union of finitely many double cosets of the form $\hat{V}x\hat{V}$. Then, by the foregoing claim, $U = \bigcup_{i<n} \hat{V}\hat{A}_i$ for some left *cosets $A_i$ of $V$ in $M$.

To reach our goal, the main point is to show that each open subgroup in $\mathcal{F}(M)$ is of the form $\hat{U}$ for some *subgroup $U$ in $M$. It now suffices to introduce an axiom ensuring that a finite union of double cosets of $\hat{V}$ that is closed under products and inverses equals $\hat{U}$ for some subgroup $U$ in $M$. First we need to establish three claims; each one asserts that a certain semantic condition in $\mathcal{F}(M)$ is first-order definable in $M$.

**Claim 2.22** (Formula $\phi$). There is a first-order formula $\phi$ such that

$$M \models \phi(A, B, C) \iff \hat{A}\hat{B} \cap \hat{C} = \emptyset.$$ 

**Proof.** $\phi(A, B, C)$ expresses that

there are no $D \subseteq A$ and $E \subseteq B$ with $DE \subseteq C$.

For the implication from left to right, by contraposition suppose that $\hat{A}\hat{B} \cap \hat{C} \neq \emptyset$. Let $x \in \hat{A}$ and $y \in \hat{B}$ with $xy \in \hat{C}$. Since the group operation on $\mathcal{F}(M)$ is continuous by Prop. 2.12 and $\hat{C}$ is open, there are basic open subsets $\hat{D} \subseteq \hat{A}$ and $\hat{E} \subseteq \hat{B}$ with $\hat{D}\hat{E} \subseteq \hat{C}$. Then $DE \subseteq C$ by Claim 2.11.

For the implication from right to left, by contraposition suppose that $D \subseteq A$, $E \subseteq B$ and $DE \subseteq C$. Then every $x \in \hat{D}\hat{E}$ is an element of $\hat{A}\hat{B} \cap \hat{C}$. □

In the following, we repeatedly use that double cosets of $\mathcal{F}(M)$ of the form $\hat{V}\hat{A}$, where $A$ is a left *coset of $V$, are clopen. This follows from the hypothesis that $\mathcal{F}(M)$ is Roelcke precompact.

**Claim 2.23** (Formulas $\psi_n$). For each $n \geq 1$, there is a first-order formula $\psi_n$ such that

$$M \models \psi_n(A_0, \ldots, A_{n-1}, B, V) \iff \hat{B} \subseteq \bigcup_{i<n} \hat{V}\hat{A}_i,$$

for all $B$ and all left *cosets $A_0, \ldots, A_{n-1}$ of $V$.

**Proof.** $\psi_n(A_0, \ldots, A_{n-1}, B, V)$ expresses that

there is no $C \subseteq B$ such that for all $i < n$, $\phi(V, A_i, C)$.

First suppose that $\hat{B} \not\subseteq \bigcup_{i<n} \hat{V}\hat{A}_i$. Since $\bigcup_{i<n} \hat{V}\hat{A}_i$ is clopen, there is some $C$ with $\hat{C} \subseteq \hat{B}$ and $\bigwedge_{i<n} \hat{V}\hat{A}_i \cap \hat{C} = \emptyset$. Then $C \subseteq B$ by Claim 2.11, and $\bigwedge_{i<n} \phi(V, A_i, C)$ by Claim 2.22.

Conversely, assume that there is some $C \subseteq B$ with $\bigwedge_{i<n} \phi(V, A_i, C)$.

Then $\hat{C} \subseteq \hat{B}$ by Claim 2.11 and $\bigwedge_{i<n} \hat{V}\hat{A}_i \cap \hat{C} = \emptyset$ by Claim 2.22. Hence $\hat{B} \not\subseteq \bigcup_{i<n} \hat{V}\hat{A}_i$. □

**Claim 2.24** (Formulas $\theta_n$). For each $n \geq 1$, there is a first-order formula $\theta$ such that

$$M \models \theta_n(A_0, \ldots, A_{n-1}, V) \iff \bigcup_{i<n} \hat{V}\hat{A}_i$$

is a subgroup of $\mathcal{F}(M)$, for all left *cosets $A_0, \ldots, A_{n-1}$ of $V$. 

Proof. Note that $\hat{V} \hat{A}_j \hat{V} \hat{A}_l = \hat{V} \hat{A}_j \hat{A}_l$ for all $j, l < n$.

We first express that $\bigcup_{i<n} \hat{V} \hat{A}_i$ is closed under products. We will show that for all $j, l < n$, the statement $\hat{V} \hat{A}_j \hat{A}_l \not\subseteq \bigcup_{i<n} \hat{V} \hat{A}_i$ is equivalent to the following first-order formula $\rho_n(A_0, \ldots, A_{n-1}, V)$ in $M$: there are $B \subseteq V$, $C \subseteq A_j$, $D \subseteq A_l$ and $E, F$ with $BC \subseteq E$, $ED \subseteq F$ and $\bigwedge_{i<n} \phi(V, A_i, F)$.

Suppose first that $\rho(A_0, \ldots, A_{n-1}, V)$ holds in $M$ via $D, E$ and $F$. Take any $x \in \hat{B}$, $y \in \hat{C}$ and $z \in \hat{D}$. Then $x \cdot y \cdot z \in \hat{V} \hat{A}_j \hat{A}_l$ by Claim 2.11 and by hypothesis $x \cdot y \cdot z \in \hat{F}$. Since $M \models \bigwedge_{i<n} \phi(V, A_i, T)$, we have $x \cdot y \cdot z \not\in \bigcup_{i<n} \hat{V} \hat{A}_i$ by Claim 2.22.

Suppose conversely that $\hat{V} \hat{A}_j \hat{A}_l \not\subseteq \bigcup_{i<n} \hat{V} \hat{A}_i$ and take some $x \in \hat{V}$, $y \in \hat{A}_j$ and $z \in \hat{A}_l$ with $x \cdot y \cdot z \not\in \bigcup_{i<n} \hat{V} \hat{A}_i$. Since $\bigcup_{i<n} \hat{V} \hat{A}_i$ is clopen, there is $F$ disjoint from $\bigcup_{i<n} \hat{V} \hat{A}_i$ with $x \cdot y \cdot z \in \hat{F}$. By continuity in Prop. 2.12, there is $\hat{E} \subseteq \hat{V} \hat{A}_j$ such that $x \cdot y \in \hat{E}$, and $\hat{D} \subseteq \hat{A}_l$ such that $z \in \hat{D}$ and $\hat{E} \hat{D} \subseteq \hat{F}$. Again by continuity, there is $B \subseteq V$ such that $x \in \hat{B}$, and $C \subseteq A_j$ such that $y \in \hat{D}$ and $\hat{B} \hat{D} \subseteq E$. Now $\rho_n(A_0, \ldots, A_{n-1}, V)$ holds via $D, E$ and $F$ by Claims 2.11 and 2.22.

Similarly, one can express that $\bigcup_{i<n} \hat{V} \hat{A}_i$ is closed under inverses using the $\diamond$ operation in Axiom 4. We leave this case to the reader. \qed

We are now ready to express the next axiom about an $L$-structure $M$. It is the conjunction of an infinite set of first-order sentences. Note that its conclusion is equivalent to $\bigcup_{i<n} \hat{V} \hat{A}_i = \hat{U}$.

**Axiom 8.** Let $n \geq 1$. Let $A_i \in LC(V)$ for all $i < n$. If $\theta_n(A_0, \ldots, A_{n-1}, V)$ holds, then there is a *subgroup $U$ such that $\bigwedge_{i<n} [VA_i \subseteq U]$ and $\psi_n(A_0, \ldots, A_{n-1}, U, V)$.

**Lemma 2.25.** Suppose that the Polish group $\mathcal{F}(M)$ is Roelcke precompact.

(a) Every open subgroup $U$ of $\mathcal{F}(M)$ equals $\hat{U}$ for some *subgroup $U$ in $M$.

(b) Every open coset in $\mathcal{F}(M)$ equals $\hat{A}$ for some $A$ in $M$.

**Proof.**

(a) As remarked above, there are a *subgroup $V$ and left *cosets $A_0, \ldots, A_{n-1}$ of $V$ such that $U = \bigcup_{i<n} \hat{V} \hat{A}_i$. Axiom 8 yields a *subgroup $U$ in $M$ such that $U = \hat{U}$.

(b) now follows from Claim 2.21. \qed

2.5. **The profinite case.** Recall from the introduction that all our topological groups are separable, and that a topological group is profinite if and only if it is isomorphic to a compact subgroup of $S_\infty$. We show that the coarse groups of profinite groups can be characterised with only one further axiom, stated as part of Proposition 2.26. By the form this axiom takes, and given that the foregoing axioms either determine arithmetical classes or can be replaced by axioms which do so, this shows that the class of such coarse groups (with domain $\omega$) is arithmetical.

In the following recall that $U, V$ range over *subgroups. In Remark 2.15 we discussed normal *subgroups $V$, defined by the condition $LC(V) = RC(V)$. 


Proposition 2.26. Suppose $M$ satisfies the Axioms through to 8 introduced so far: Then $F(M)$ is compact $\iff M$ satisfies the condition
\[ \forall U \exists V \subseteq U \ [LC(V) = RC(V)] \land \forall U \ [LC(U) \text{ is finite}]. \]

Proof. $\Rightarrow$: Recall that $F(M)$ is totally disconnected. So, if $F(M)$ is compact, then each open subgroup of $F(M)$ contains a normal open subgroup. Since $F(M)$ is Roelcke precompact, Claim 2.11 and Lemma 2.25 imply the corresponding statement for $M$.

$\Leftarrow$: By a construction similar to the one in Claim 2.5, let $\langle N_k \rangle_{k \in \omega}$ be a descending chain of normal *subgroups such that $\forall U \exists k [N_k \subseteq U]$. Let $G_k$ be the group induced by $M$ on $LC(N_k)$ as in Remark 2.15. We define an onto map $p_k: G_{k+1} \to G_k$ as follows: given $A \in LC(N_{k+1})$, using Axiom 2 let $p_k(A) = B$ where $A \subseteq B \in LC(N_k)$. Each $p_k$ is a homomorphism by Axioms 1 and 4.

Let $G$ be the inverse limit: $G = \text{proj lim}_k(G_k, p_k)$. Thus
\[ G = (\{ f \in \prod_k G_k : \forall k f(k) = p_k(f(k+1)) \}, \), \]
which is closed and hence compact group subgroup of the Cartesian product of the $G_k$. We claim that $G \cong (F(M), \cdot)$ via the map $\Phi$ that sends $f \in G$ to the filter in $F(M)$ generated by the *cosets $f(k)$, namely
\[ \Phi(f) = \{ C \in M : \exists k f(k) \subseteq C \}. \]

It is clear that $\Phi$ is a monomorphism. For continuity of $\Phi$ at 1, let $U$ be an open subgroup of $F(M)$. By Lemma 2.25 there is a *subgroup $U \subseteq M$ such that $\hat{U} = U$. Choose $k$ such that $N_k \subseteq U$. We can view $U$ as a subgroup of $G_k$, and so $\Phi^{-1}(U) = \{ f \in G : f(k) \subseteq U \}$ is open in $G$.

To show $\Phi$ is onto, given a full filter $x \in F(M)$, for each $k$ there is $f(k) = B_k \in LC(N_k)$ such that $B_k \subseteq x$. Then $f \in G$, and clearly $\Phi(f) = x$.

This shows that $F(M)$ is compact as a continuous image of the compact space $G$. \qed

We will return to the topic of compact subgroups of $S_\infty$ in Subsection 3.5.

3. Isomorphism of oligomorphic groups, and countable models

The main result of this section establishes that an oligomorphic group $G$ can in a Borel way be interchanged with a structure with domain $\omega$, namely its corresponding coarse group $M(G)$.

Theorem 3.1. Isomorphism of oligomorphic subgroups of $S_\infty$ is classwise Borel bireducible with the isomorphism relation on an invariant Borel set of countable structures in a finite signature.

3.1. Review of the result of Kechris, Nies and Tent. Before proving the theorem, we need to review in some more detail the map $M$ defined in Kechris et al. [12, Section 3.3]. This map shows that isomorphism of Roelcke precompact groups is Borel reducible to isomorphism on the set of $L$-structures with domain $\omega$, for the language $L$ with one ternary relation symbol $R$. These structures form a Polish space $X_L = \mathcal{P}(\omega \times \omega \times \omega)$, the sets of triples of natural numbers. [12] provides a Borel map $M$ from the set of Roelcke precompact closed subgroups of $S_\infty$ to structures in $X_L$. For such
a group $G$, the set $\mathcal{N}_G$ of all open subgroups of $G$ is countable; we think of the domain of the structure $\mathcal{M}(G)$ as consisting of the cosets of subgroups in $\mathcal{N}_G$ (this structure is denoted by $\mathcal{M}_G$ in [12]). Then, by a result of Lusin-Novikov in the version of [13, 18.10], one can in a Borel way find a bijection between these cosets and $\omega$.

We note that this approach also works for Borel classes of groups where $\mathcal{N}_G$ is merely a countable neighbourhood basis of $1$ consisting of open subgroups such that $\mathcal{N}_G$ is isomorphism invariant; for instance, $\mathcal{N}_G$ could consist of the the compact open subgroups in a locally compact subgroup $G$ of $S_\infty$.

3.2. Plan of the proof. We will introduce a Borel inverse up to isomorphism of the map $\mathcal{M}$, restricted to oligomorphic groups. In more detail, let $\mathcal{B}$ be the closure under isomorphism of the range of $\mathcal{M}$ on the class of oligomorphic groups. We will show that $\mathcal{B}$ is Borel, and define a Borel map $\mathcal{G}$ from $\mathcal{B}$ to the class of oligomorphic closed subgroups of $S_\infty$ such that for each oligomorphic closed subgroup $G$ of $S_\infty$, and each structure $M$ in $\mathcal{B}$, we have

\begin{equation}
\mathcal{G}(\mathcal{M}(G)) \cong G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M.
\end{equation}

We will have $M \cong N \Leftrightarrow \mathcal{G}(M) \cong \mathcal{G}(N)$ for all $M, N \in \mathcal{B}$, as required for the proof of Theorem 3.1. The implication $M \cong N \Rightarrow \mathcal{G}(M) \cong \mathcal{G}(N)$ will follow from the definition of the map $\mathcal{G}$. The reverse implication will follow from (1) and (4).

The group $\mathcal{G}(M)$ is obtained from $\mathcal{F}(M)$ by specifying in a Borel way an embedding as an oligomorphic closed subgroup of $S_\infty$. To carry this out, we will add further “axioms” that hold for all the structures of the form $\mathcal{M}(G)$, where $G$ is oligomorphic. As before they can be expressed by monadic $\Pi_1$ sentences or $L_{\omega_1,\omega}$ sentences in the signature with one ternary relation symbol. A class $\mathcal{C}$ of $L$-structures will be defined as the set of structures satisfying all these axioms. Then $\mathcal{C}$ is $\Pi_1$. For an $L$-structure $M$ in $\mathcal{C}$ we will be able to recover an oligomorphic group $\mathcal{G}(M)$ via a Borel map in such a way that (4) holds. This implies that $\mathcal{C}$ equals $\mathcal{B}$, the closure of ran($\mathcal{M}$) under isomorphism (which is analytic), so $\mathcal{B}$ is Borel.

3.3. Ensuring that $\mathcal{F}(M)$ is isomorphic to an oligomorphic closed subgroup of $S_\infty$. Given a Polish group $G$ with a faithful action $\gamma : G \times \omega \to \omega$, we obtain a monomorphism $\Theta_\gamma : G \to S_\infty$ given by $\Theta_\gamma(g)(k) = \gamma(g,k)$. A Polish group action is continuous if and only if it is separately continuous. In the case of an action on $\omega$ (with the discrete topology), the latter condition means that for each $k, n \in \omega$, the set $\{g : \gamma(g,k) = n\}$ is open. So $\gamma$ is continuous if and only if $\Theta_\gamma$ is continuous.

**Definition 3.2.** We say that a faithful action $\gamma : G \times \omega \to \omega$ is strongly continuous if the embedding $\Theta_\gamma$ is topological.

Equivalently, the action is continuous and for each neighbourhood $U$ of $1_G$, the set $\Theta_\gamma(U)$ is open in $\Theta_\gamma(G)$, namely, there is $n$ such that $\forall k < n \gamma(g,k) = k$ implies $g \in U$. Strong continuity implies that $G$ is topologically isomorphic to a closed subgroup of $S_\infty$. Clearly, not every continuous action is strongly continuous; for instance let $G$ be the discrete group of permutations of finite support and take the natural action of $G$ on $\omega$. 
We will introduce axioms that ensure that $F(M)$ has an action on $\omega$ that is

(a) faithful, (b) oligomorphic, and (c) strongly continuous.

By the following lemma, each oligomorphic group $G$ has an open subgroup $W$ so that the natural action of $G$ on the set $LC(W) = G/W$ of left cosets of $W$ has these three properties. In the general setting of a coarse group structure $M$ we ensure the existence of a subgroup with these properties by a further axiom.

**Lemma 3.3.** Let $G$ be an oligomorphic closed subgroup of $S_\infty$. There is an open subgroup $W$ such that the left translation action $\gamma: G \rightrightarrows LC(W)$ is faithful and oligomorphic. Furthermore, for any listing without repetition $\langle A_i \rangle_{i \in \omega}$ of the cosets of $W$, when viewing $\gamma$ as an action on $\omega$ via this listing, this action is strongly continuous.

**Proof.** Let $x_1, \ldots, x_k \in \omega$ represent the 1-orbits of $G$. Let $W$ be the pointwise stabiliser of $\{x_1, \ldots, x_k\}$. If $g \in G - \{1\}$ then there are $p \in G$ and $i \leq k$ such that $g \cdot (p \cdot x_i) \neq p \cdot x_i$. So $p^{-1}qp \notin W$, and hence $g \cdot pW \neq pW$. In particular, the action is faithful, and hence $LC(W)$ is infinite.

Choose $a_i \in S_\infty$ such that $A_i = a_iW$. To show that $\Theta_\gamma$ is continuous, given $n$, let $U = \bigcap_{i<n} a_iW a_i^{-1}$, and note that $U$ is an open subgroup of $G$. Then $g \in U$ implies $\Theta_\gamma(g)(i) = i$ for $i < n$.

To show that $\Theta_\gamma^{-1}$ is continuous, given $n$, for each $i < n$ choose $p(i) \in \omega$ such that $i = a_{p(i)} x_r$ for some $r$. If $\Theta_\gamma(g)$ fixes all the numbers $p(i)$ then $\gamma(g, i) = i$ for each $i < n$.

Since $G$ is oligomorphic, it is Roelcke precompact. Then, since the action of $G$ on $LC(W)$ is strongly continuous and has finitely many 1-orbits, by Tsankov [18, Thm 2.4] this action is oligomorphic.

**Remark 3.4.** Given a $^*\text{subgroup} V$, we discuss how to express that $\gamma_V$ has properties (a), (b) and (c) above via either $\Pi_1^1$ formulas or $L_{\omega_1, \omega}$ formulas, in the signature $L$.

(a) We can say that $\gamma_V$ is faithful by expressing the following by a $\Pi_1^1$ formula: for all $x \neq 1$, there are disjoint left $^*\text{cosets} A, B$ of $V$ such that $x \cdot A = B$. Note that this makes $LC(V)$ infinite.

(b) To say that $\gamma_V$ is oligomorphic using a formula in $L_{\omega_1, \omega}$, we can require that for all $k \geq 1$, there is some $n \geq 1$ and there are $k$-tuples $\vec{C}^{0}, \ldots, \vec{C}^{n-1}$ of left $^*\text{cosets}$ of $V$ with the following property. For each $k$-tuple $\vec{B}$ of left $^*\text{cosets}$ of $V$, there is some $i < n$ and some $S$ such that for all $j < k$, we have $SB_j \subseteq C^i_j$. To show that this condition implies that $\gamma_V$ is oligomorphic, choose any $x$ such that $S \in x$. Then $x \cdot B_j = C^i_j$ for each $j$.

If $M$ satisfies the given condition, we say for short that $M$ is formally oligomorphic.

(c) We can express that $\gamma_V$ is strongly continuous by an $L_{\omega_1, \omega}$ formula. Write $\Theta_V$ for $\Theta_{\gamma_V}$. First note that $\Theta_V$ is automatically continuous at 1 (and hence continuous): a basic neighbourhood of 1 in $S_\infty$ has the form $\{p: \forall i \leq n \{p(i) = i\}\}$. For a full filter $x$, we have $xA_i = A_i$ if and only if $x \in \tilde{S}$ for some $S$ such that $SA_i \subseteq A_i$. So by the definition of the
We will reformulate the axiom in together with the following claim ensures that

\[ \forall x \exists k \exists B_1, \ldots, B_k \in LC(V) \forall x \left[ \bigwedge_i x \cdot B_i = B_i \rightarrow U \in x \right]. \]

To avoid the universal second-order quantifier \( \forall x \), we will instead use

\[ \forall U \exists k \exists B_1, \ldots, B_k \in LC(V) \forall S \left[ \bigwedge_i [SB_i \subseteq B_i] \rightarrow S \subseteq U \right]. \]

**Claim 3.5.** Let \( k \in \mathbb{N} \). Given \( V, U, B_1, \ldots, B_k \in M \), we have

\[ \forall x \left[ \bigwedge_i x \cdot B_i = B_i \rightarrow U \in x \right] \Leftrightarrow \forall S \left[ \bigwedge_i [SB_i \subseteq B_i] \rightarrow S \subseteq U \right]. \]

Thus for each \( V \), (5) is equivalent to (6).

**Proof.** We make use of Claim 2.11(c). For the implication \( \Rightarrow \) suppose that \( \bigwedge_i [SB_i \subseteq B_i] \). Let \( x \) be a full filter such that \( S \in x \). Then \( x \cdot B_i = B_i \) for each \( i \leq k \), and hence \( U \in x \). So \( \hat{S} \subseteq \hat{U} \) and hence \( S \subseteq U \).

For the implication \( \Leftarrow \) suppose that \( \bigwedge_i [x \cdot B_i = B_i] \). The by downward directness of full filters, there is \( S \in x \) such that \( \bigwedge_i [SB_i \subseteq B_i] \). So \( S \subseteq U \), whence \( U \in x \) by upward closure of full filters.

**Axiom 9.** There is a *subgroup \( W \) in \( M \) such that \( \gamma_W \) is faithful, formally oligomorphic, and strongly continuous.

Later on in Section 3.4, we will argue that we can determine such a \( W \) via a Borel function applied to \( M \). Then we will define the required oligomorphic group \( G(M) \cong \mathcal{F}(M) \) as the range of \( \Theta_W \). The first statement in (4) will then follow from Prop. 2.13. In Section 4.2 we will reformulate the axiom in order to avoid the universal second-order quantifier we are using to express faithfulness.

Lemma 3.3 together with the following claim ensures that \( \mathcal{M}(G) \) satisfies Axiom 9.

**Claim 3.6.** If \( M = \mathcal{M}(G) \) and \( V \) is a *subgroup in \( M \) such that \( \gamma_V \) is oligomorphic, then \( \gamma_V \) is formally oligomorphic.

**Proof.** Since \( \gamma_V \) is oligomorphic, we have some \( x \in \mathcal{F}(M) \) such that for all \( j < k \), \( x \cdot B_j = C^i_j \) in the notation above. It is easy to see that the action \( \mathcal{F}(M) \cap \mathcal{F}(M)/V \) induced by the group operation on \( \mathcal{F}(M) \) satisfies \( x \cdot \hat{B}_j = \hat{C}^i_j \). By continuity of the group operation in Prop. 2.12, there is some \( S \) such that \( \hat{S} \) contains \( x \) and for all \( j < k \), we have \( \hat{S} \hat{B}_j \subseteq \hat{C}^i_j \) and hence \( SB_j \subseteq C^i_j \) by Claim 2.11.

3.4. **Turning the filter group into a closed subgroup of \( S_\infty \).** We now define the Borel map \( \mathcal{G} \). Let \( \mathcal{C} \) be the set of \( L \)-structures \( M \) with domain \( \omega \) that satisfy the axioms stated above. Note that \( \mathcal{C} \) is \( \Pi_1^1 \) because all axioms can be expressed in \( \Pi_1^1 \) form or in \( L_{\omega_1,\omega} \) form. Also, \( \mathcal{C} \) contains the closure under isomorphism of the range of the map \( \mathcal{M} \), denoted \( \mathcal{B} \) in Section 3.2 above.
As mentioned above, the relation \( \{(M,W) : M \in C \land W \in M \} \) is a *subgroup in \( M \) satisfying the properties in Axiom 9\) is \( \Pi_1^1 \). By \( \Pi_1^1 \)-uniformization (Addison/Kondo, see e.g. [15, Theorem 4E.4]) there is a function \( f : C \to \omega \) with \( \Pi_1^1 \) graph that sends each \( M \in C \) to some \( W \in M \) of this kind. Recall that the embedding \( \Theta_V \), for certain *subgroups \( V \) in \( M \), was defined in (c) before Axiom 9. We define \( \mathcal{G}(M) \) as the range of \( \Theta_W \) where \( W = f(M) \). In other words, \( \mathcal{G}(M) \) is the closed subgroup of \( S_\infty \) determined by the action of \( \mathcal{F}(M) \) on \( LC(W) \). Here we use the canonical increasing bijection between \( \omega \) and \( LC(W) \) (an infinite subset of \( \omega \)) to view the action on \( LC(W) \) as an action on \( \omega \), as specified in Remark 2.20.

We are now ready to establish (4), restated here for convenience:

**Proposition 3.7.** For each oligomorphic group \( G \) and each structure \( M \in C \), we have

\[
\mathcal{G}(\mathcal{M}(G)) \cong G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M.
\]

**Proof.** As already mentioned, the first statement follows from Prop. 2.13. Given \( A \in M \), we view \( \hat{A} \) now as an open coset of \( \mathcal{G}(M) \), rather than from the filter group \( \mathcal{F}(M) \).

By Axiom 9, there is a *subgroup \( W \) in \( M \) such that \( \gamma_W \) is faithful, oligomorphic and yields a topological embedding into \( S_\infty \). Since \( \mathcal{F}(M) \) is a Polish group by Claim 2.12, the range of \( \gamma_V \) is an oligomorphic closed subgroup of \( S_\infty \). Hence \( \mathcal{F}(M) \) is Roelcke precompact by [18, Theorem 2.4].

Then, by Lemma 2.25, the map \( A \to \hat{A} \) is a bijection between \( M \) and \( \mathcal{M}(\mathcal{G}(M)) \). By Claim 2.11 it is an isomorphism. Thus we obtain the second statement. \( \square \)

Note that we actually show for each \( A \in M \) that \( \mathcal{(M(G(M)), \hat{A}) \cong (M, A)} \). This will be used below.

Proposition 3.7 implies that \( \mathcal{B} = \mathcal{C} \). Since \( \mathcal{B} \) is the closure under isomorphism of the range of a Borel measurable map defined on a Borel domain, it is analytic. Since \( \mathcal{B} \) is also coanalytic, it is Borel. Since the domain of \( f \) is Borel, the graph of \( f \) is analytic because \( f(x) \neq n \) if \( \exists \) \( m \neq n \) \( f(x) = m \). So the graph of \( f \) is Borel.

Note that \( \mathcal{G}(M) \) is an element of the Effros Borel space of \( S_\infty \) (see Section 1.4). In the following, \( \sigma \) will denote an injective map on initial segments of the integers, that is, a tuple of integers without repetitions. Let \( [\sigma] \) denote the set of permutations extending \( \sigma \):

\[
[\sigma] = \{ f \in S_\infty : \sigma \prec f \}
\]

(this is often denoted \( N_\sigma \) in the literature). The sets \( [\sigma] \) form a basis for the topology of pointwise convergence of \( S_\infty \).

**Claim 3.8.** The map \( M \mapsto \mathcal{G}(M) \), for \( M \in \mathcal{C} \), is Borel.

**Proof.** Let \( [\sigma] \) be an arbitrary basic open subset of \( S_\infty \). It is sufficient to show that \( \{ M \mid \mathcal{G}(M) \cap [\sigma] \neq \emptyset \} \) is Borel. From \( M \) we obtain \( W = f(M) \) in a Borel way, and then the list \( A_0, A_1, \ldots \) of left *cosets of \( W \) in ascending order. We have

\[
\mathcal{G}(M) \cap [\sigma] \neq \emptyset \iff \exists S \in M \ \forall i, j | \sigma(i) = j \to SA_i \subseteq A_j \],
\]
by the definition of the action $\gamma_W$ in (2).

This completes the proof of Theorem 3.1.

3.5. **Borel duality with coarse groups in the profinite case.** Let $\mathcal{C}_{\text{pro}}$ be the class of coarse groups satisfying Axioms through to 8 (but not necessarily 9), as well as the condition in Proposition 2.26 which is equivalent to compactness of $F(M)$. Here we show that $\mathcal{C}_{\text{pro}}$ is Borel, and that the compact subgroups of $S_\infty$ are classwise Borel bireducible to $\mathcal{C}_{\text{pro}}$, in analogy with the Borel version of Stone duality discussed above.

Given $M \in \mathcal{C}_{\text{pro}}$, let $\Theta : F(M) \to S_\infty$ be the map induced by the action of $F(M)$ on $M$ given by Claim 2.19 and the discussion thereafter. This action clearly is faithful, so $\Theta$ is an embedding. As in Remark 3.4(c), $\Theta$ is automatically continuous. Then, since $F(M)$ is compact, $\Theta$ is a topological embedding. Let $G_{\text{pro}}(M)$ be its range, a closed subgroup of $S_\infty$.

It is easy to establish the analog of Proposition 3.7 using Lemma 2.25. This implies that $\mathcal{C}_{\text{pro}}$ is Borel (alternatively this holds by the first-order reformulation of Axiom 6 given in Subsection 4.4). We obtain:

**Proposition 3.9.** The Borel operators $M$ and $G_{\text{pro}}$ establish classwise Borel bireducibility between the compact subgroups of $S_\infty$, and the coarse groups satisfying the Axioms 0 to 8 and the condition in Proposition 2.26.

4. **Complexity of the isomorphism relation between oligomorphic groups**

4.1. **Conjugacy.** We begin with an easy result: conjugacy of oligomorphic groups is smooth, that is, Borel reducible to the identity on $\mathbb{R}$.

For a closed subgroup $G$ of $S_\infty$, let $\mathcal{E}_G$ denote the orbit equivalence structure with domain $\omega$. For each $n$, the signature of this structure has a $2n$-ary relation symbol, denoting the orbit equivalence relation for the action of $G$ on $^n\omega$.

The following fact holds in general.

**Fact 4.1.** Let $G$ and $H$ be closed subgroups of $S_\infty$. Let $\alpha \in S_\infty$. Then $G, H$ are conjugate via $\alpha$ $\iff$ $\mathcal{E}_G \cong \mathcal{E}_H$ via $\alpha$.

**Proof.** $\Rightarrow$: This is immediate.

$\Leftarrow$: Let $M_G$ be the canonical structure for $G$, namely there are $k_\omega \leq \omega$ many $n$-ary relation symbols, denoting the individual $n$-orbits. Let $M_H$ be the structure in the same signature where the equivalence classes of $\mathcal{E}_H$ on $^n\omega$ are named so that $\alpha$ is an isomorphism $M_G \cong M_H$. Since $G \leq \text{Aut}(M_G)$, and $G$ is closed and dense, we have $G = \text{Aut}(M_G)$; similarly, $H = \text{Aut}(M_H)$. Furthermore, $\alpha^{-1} \text{Aut}(M_H)\alpha = \text{Aut}(M_G)$.

**Proposition 4.2.** The conjugacy relation between oligomorphic groups is smooth.

**Proof.** The map $G \mapsto \mathcal{E}_G$ defined on closed subgroups of $S_\infty$ is Borel because one can in a Borel way find a countable dense subgroup of $G$, which of course has the same orbits; based on that subgroup one can directly construct $\mathcal{E}_G$. 
For countable structures $S$ in a fixed countable language, mapping $S$ to its theory $Th(S)$ is Borel. The theory can be seen as a subset of $\omega$, assuming a suitable encoding of the language.

Suppose now that $G$ and $H$ are oligomorphic closed subgroups of $S_\infty$. Note that $E_G$ is interpretable without parameters in the canonical structure $N_G$ mentioned in Section 1.3. So $N_G$ is $\omega$-categorical, and hence $E_G$ is $\omega$-categorical as well.

By Fact 4.1 and since $E_G$ and $E_H$ are $\omega$-categorical, $G,H$ are conjugate $\iff E_G \cong E_H \iff Th(E_G) = Th(E_H)$, which shows smoothness. □

4.2. Essential countability of the isomorphism relation. Recall that an equivalence relation $E$ on a Polish space is called countable if every equivalence class is countable. One says that $E$ is essentially countable if $E$ is Borel reducible to a countable Borel equivalence relation.

We show that the isomorphism relation between oligomorphic subgroups of $S_\infty$ is essentially countable. As mentioned in the introduction, we apply a result of Hjorth and Kechris [10, Theorem 4.3] about Borel invariant classes $C$ of countable structures. Given a finite signature, a subset $F$ of $L_{\omega_1,\omega}$ is called a fragment if it is closed under syntactic first-order operations such as quantification over elements, or substitution. Suppose first that we had a countable fragment $F$ such that each $M \in C$ is determined up to isomorphism among the countable structures by $Th_F(M)$, its theory in this fragment. Then $\cong_C$ is smooth, because the map $M \mapsto Th_F(M)$ is Borel.

Their result uses a weaker hypothesis to yield a weaker conclusion. In [10, Theorem 4.3, (iii) $\Rightarrow$ (i)] they prove the following. Suppose that there is a fixed fragment $F$ as follows: each $M \in C$ contains a tuple of constants $\overline{a}$ such that $(M,\overline{a})$ is determined up to isomorphism among the countable structures by $Th_F(M,\overline{a})$ (i.e, $Th_F(M,\overline{a})$ is $\aleph_0$-categorical). Then $\cong_C$ is essentially countable.

Their proof proceeds as follows. They need to obtain a countable Borel equivalence relation $E$ on a Borel space $Y$ so that $E$ is Borel above $\cong_C$. The points of the Borel space $Y$ are $F$-theories of countable models extended by finitely many constants. Two theories are $E$-equivalent if they can be realised over isomorphic models in the language of $F$. (They verify as part of their proof that $Y$ is indeed a Borel space on which $E$ is Borel.) The Borel reduction maps $M$ to $Th_F(M,\overline{a})$ where $\overline{a}$ is chosen so that $Th_F(M,\overline{a})$ is $\aleph_0$-categorical. This is possible by a result in descriptive set theory due to Lusin-Novikov: one can in a Borel way uniformise a Borel relation that relates each $x$ to only countably many elements (see e.g. [13, 18.10]).

Recall from Lemma 3.3 that each oligomorphic group $G$ has an open subgroup $W$ such that the left translation action of $G$ on the left cosets of $W$ is oligomorphic, and yields a topological embedding of $G$ into $S_\infty$. The idea in applying the Hjorth-Kechris result is now as follows. Given a structure $M$ for the signature with one ternary relation satisfying the axioms so far, require the existence of $W$ axiomatically for the action of the filter group on the (abstract) left cosets of $W$. If $F$ is the least fragment
containing all the relevant formulas used in the axioms, then it can be shown that \((M, W)\), for \(W\) as above, is determined by its theory in \(F\). Thus, \((W)\) is the tuple of constants one adds to satisfy the hypothesis of the Hjorth-Kechris result.

**Theorem 4.3.** The isomorphism relation between oligomorphic subgroups of \(S_\infty\) is essentially countable.

**Proof.** Recall that \(R\) is a ternary relation symbol. Also recall that in Section 3.2 above we denoted by \(\mathcal{B}\) the closure under isomorphism of the range of the map \(\mathcal{M}\). We showed in Section 3.4 that \(\mathcal{B}\) is Borel. So by the López-Escobar theorem there is \(\sigma \in L_{\omega_1, \omega}(R)\) such that \(M \in \mathcal{B} \iff M \models \sigma\) for each model \(M\). Let \(\cong_\sigma\) denote the isomorphism relation on \(\mathcal{B}\).

Let \(F\) be the smallest fragment of \(L_{\omega_1, \omega}(R)\) containing \(\sigma\). Note that \(F\) is countable. For a structure \(M\) and \(n\)-tuple \(\overline{a}\) in \(M\), by \(\text{Th}_F(M, \overline{a})\) one denotes \(\{\phi(x_1, \ldots, x_n) \in F : (M, \overline{a}) \models \phi\}\).

By Hjorth and Kechris [10, Theorem 4.3], the following are equivalent.

(i) \(\cong_\sigma\) is essentially countable

(ii) for each \(M \in \mathcal{B}\) there is a tuple \(\overline{a}\) in \(M\) such that \(\text{Th}_F(M, \overline{a})\) is \(\kappa_0\)-categorical.

We will verify (ii), where the tuple \(\overline{a}\) has length 1: it consists of the witness \(W\) for a stronger version of Axiom 9. The problem with our formulation of faithfulness in that axiom is that it is only \(\Pi^1_1\) and hence cannot be used in a fragment. Instead, let \(\delta(V)\) denote the following first-order formula, which implies that \(\gamma_V\) is faithful, as will be verified shortly:

\[
\forall U \forall A \in LC(U) \setminus \{U\} \exists U' \subseteq U \\
\forall A' \subseteq A, A' \in LC(U') \exists C \in LC(V) \exists D \in LC(V) \setminus \{C\} A'C \subseteq D.
\]

**Axiom 10 (Replaces Axiom 9).** There is a \(*\)subgroup \(W\) in \(M\) such that \(M \models \delta(W)\), and the action \(\gamma_W\) defined in (2) is formally oligomorphic and strongly continuous.

We claim that this condition holds in \(\mathcal{M}(G)\), for any oligomorphic closed subgroup \(G\) of \(S_\infty\). By Lemma 3.3 we may assume that the action of \(G\) on \(\omega\) has a single 1-orbit. Let \(W = G_{y_0}\), the stabilizer of 0.

Suppose we are given an open subgroup \(U\) of \(G\), and let \(A \in LC(U)\), \(A \neq U\). By definition of the subspace topology on \(G\), there is tuple \(\overline{y}\) of natural numbers such that \(U' = G_{\overline{y}}\) is contained in \(U\). Take a left coset \(A' = gU' \subseteq A\). Since \(A \neq U\) we have \(A' \neq U'\), and hence \(g(y_j) \neq y_j\) for some \(j\), say \(j = 0\). Let \(h \in G\) with \(h(0) = y_0\). By definition of \(g\) and since \(U' \leq hWh^{-1} = G_{y_0}\),

\[
A'hWh^{-1} = gU'hWh^{-1} = ghWh^{-1} \neq hWh^{-1}.
\]

Thus, where \(C = hW\), \(A'C\) is a coset of \(W\) different from \(C\), as required.

**Claim 4.4.** If \(\delta(W)\) holds, then \(\gamma_W\) is faithful.

**Proof.** Suppose that \(x \neq 1\) is a full filter of \(M\). Then there is a \(*\)subgroup \(U\) and \(A \in LC(U)\) such that \(A \in x\) and \(A \neq U\). We choose \(U'\) as in the statement \(\delta(W)\). Let \(A'\) be the unique \(*\)coset in \(LC(U')\) such that \(A' \in x\). Then \(A' \subseteq A\) by Axiom 2 and since \(x\) is a filter. Choose \(C \in LC(W)\) for this \(A'\). Then \(A'C \subseteq D \neq C\), so \(x \cdot C \neq C\) as required. 

\(\square\)
Let $F$ be a countable fragment of $L_{\omega_1,\omega}$ containing $\sigma$, $\delta$ and the other formulas needed to express Axiom 10. The following now verifies Condition (ii) in the Hjorth-Kechris theorem for this fragment.

Claim 4.5. Suppose that $M, N \in \mathcal{B}$. Let $W \in M$ be a witness to Axiom 10 for $M$. Let $Z \in N$ be a *subgroup such that $\text{Th}_F(M, W) = \text{Th}_F(N, Z)$. Then $(M, W) \cong (N, Z)$.

Proof. Note that $Z \in N$ is a witness for Axiom 10 in $N$ by definition of the fragment $F$. Let $LC_M(W)$ denote the set of left *cosets of $W$ in $M$, and similarly let $LC_N(Z)$ denote the set of left *cosets of $Z$ in $N$; both sets are identified with a set of natural numbers as explained in Remark 2.20. As in Condition (c) before Axiom 9 above, by $G_W(M)$ we denote the range of the natural embedding $\mathcal{F}(M) \to S_\infty$ given by the action $\gamma_W$ of $\mathcal{F}(M)$ on $LC_M(W)$, and similarly for $G_Z(N)$. By the proof of Prop. 3.7, we have

$$(7) \quad (\mathcal{M}(G_W(M)), \hat{W}) \cong (M, W) \text{ and } (\mathcal{M}(G_Z(N)), \hat{Z}) \cong (N, Z).$$

Thus it suffices to show that the structures on the left sides are isomorphic.

Write $G = G_W(M)$ and $H = G_Z(N)$. As in Fact 4.1, let $\mathcal{E}_G$ and $\mathcal{E}_H$ be the corresponding orbit equivalence structures for the actions of $G$ on $LC_M(W)$ and of $H$ on $LC_N(Z)$. By our hypothesis we have $(M, W) \equiv (N, Z)$ (i.e., the two structures have the same first-order theory). By the definition of the group actions $\gamma_W$ and $\gamma_Z$, the structure $(\mathcal{E}_G, W)$ is interpretable in $(M, W)$, and similarly $(\mathcal{E}_H, Z)$ is interpretable in $(N, Z)$ using the same collection of formulas. This implies that $(\mathcal{E}_G, W) \equiv (\mathcal{E}_H, Z)$.

Since $G$ is oligomorphic, the orbit equivalence structures $\mathcal{E}_G$ and $\mathcal{E}_H$ are $\aleph_0$-categorical. Hence so are $(\mathcal{E}_G, W)$ and $(\mathcal{E}_H, Z)$; let $\alpha \in S_\infty$ witness that $(\mathcal{E}_G, W) \equiv (\mathcal{E}_H, Z)$.

As in the proof of Fact 4.1, $\alpha^{-1} H \alpha = G$. Since $\alpha(W) = Z$ and $\hat{W}$ is the stabiliser of $W$ and $\hat{Z}$ is the stabiliser of $Z$, we have $\alpha^{-1} \hat{Z} \alpha = \hat{W}$. Thus the map $B \mapsto \alpha B \alpha^{-1}$, for $B$ an open coset of $G$, is an isomorphism for the left hand side structures in (7), as required. \qed

This completes the proof of Theorem 4.3. \qed

4.3. Extension of the upper bound to the class of quasi-oligomorphic groups. A closed subgroup $G$ of $S_\infty$ will be called quasi-oligomorphic if it is (topologically) isomorphic to a an oligomorphic subgroup $H$ of $S_\infty$. Note that $H$, and hence $G$, is Roelcke precompact.

Fact 4.6. The class of quasi-oligomorphic groups is Borel.

Proof. Recall from Section 3.1 that Roelcke precompactness is a Borel property of closed subgroups $G$ of $S_\infty$, and that the operator $\mathcal{M}$ is defined for all Roelcke precompact groups $G$. We claim that for such a group $G$,

$$G \text{ is quasi-oligomorphic } \iff \mathcal{M}(G) \in \mathcal{B}.$$ 

Since $\mathcal{B}$ is Borel, this will suffice to establish the fact.

For the implication "$\Rightarrow$", suppose that $G \cong H$ where $H$ is oligomorphic. Then $\mathcal{M}(G) \cong \mathcal{M}(H) \in \mathcal{B}$, so $\mathcal{M}(G) \in \mathcal{B}$ as the class $\mathcal{B}$ is closed under isomorphism.
For the implication “$\iff$”, first recall that $\mathcal{F}(\mathcal{M}(G)) \cong G$ since $G$ is Roelcke precompact (Prop. 2.13). Now suppose that $\mathcal{M}(G) \in \mathcal{B}$. Then $\mathcal{G}(\mathcal{M}(G))$ is defined and oligomorphic. Since $\mathcal{G}(\mathcal{M}(G)) \cong \mathcal{F}(\mathcal{M}(G))$, this implies that $G$ is quasi-oligomorphic.

Combining the following with Theorem 4.3 shows that the isomorphism relation on the class of quasi-oligomorphic groups is essentially countable.

**Corollary 4.7.** Isomorphism on the class of quasi-oligomorphic groups is Borel equivalent to isomorphism on oligomorphic groups.

*Proof.* If $G$ is isomorphic to an oligomorphic group $H$ then $\mathcal{M}(G) \cong \mathcal{M}(H)$ and hence $\mathcal{G}(\mathcal{M}(G)) \cong H \cong G$. Since $\mathcal{G}(\mathcal{M}(G))$ is oligomorphic, the map $G \mapsto \mathcal{G}(\mathcal{M}(G))$ provides a Borel reduction of the equivalence relation in question to isomorphism of oligomorphic groups. The converse reduction exists trivially because the two classes are Borel. □

**Remark 4.8.** In contrast, conjugacy of quasi-oligomorphic groups is Borel above isomorphism of oligomorphic groups by the proof of [12, Thm. 3.1], and therefore unlikely to be smooth.

**Remark 4.9.** We note that the centre $C(G)$ of an oligomorphic group $G$ is finite, and $G/C(G)$ is quasi-oligomorphic in a natural way. See the post [4, Section 4], which is joint work with I. Kaplan.

### 4.4. Replacing the $\Pi_1^1$ Axiom 6 by a first-order axiom.

In this subsection, we replace the $\Pi_1^1$ condition in Axiom 6 (associativity of filter product) with a first-order axiom. This axiom can be verified in case that $M = \mathcal{M}(G)$, for any closed subgroup $G$ of $S_\infty$. The other axioms are given by computable $L_{\omega_1,\omega}$ sentences of finite rank (recall that we already replaced Axiom 9 by Axiom 10 which is in such a $L_{\omega_1,\omega}$ form). So the class $C$ of coarse groups for oligomorphic groups is arithmetical. This class coincides with the class of coarse groups for quasi-oligomorphic groups.

The following replaces Axiom 6. Recall that products of appropriate pairs of elements of $M$ are defined immediately after Axiom 5.

**Axiom 11.** If $A \in RC(T) \cap LC(U), B \in RC(U) \cap LC(V)$ and $C \in RC(V) \cap LC(W)$, then $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

The products in Axiom 11 are well-defined by Axiom 5. The axiom holds in $\mathcal{M}(G)$ since $A \cdot B = AB$ whenever the product $A \cdot B$ is defined.

**Claim 4.10.** The operation $\cdot$ on $\mathcal{F}(M)$ is associative.

*Proof.* It suffices to show $(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$ for any $x, y, z \in \mathcal{F}(M)$ since full filters are maximal filters.

Let $S \subseteq (x \cdot y) \cdot z$. Find $T \subseteq x \cdot y$ and $C \subseteq z$ with $TC \subseteq S$. Since $T \subseteq x \cdot y$, there are $A \in x$ and $B \in y$ with $AB \subseteq T$.

We may assume that $A \in LC(U), B \in RC(U) \cap LC(V)$ and $C \in RC(V)$ for some *subgroups* $U, V$ by shrinking $A, B, C$ similar as in the proof of Claim 2.9. In more detail, suppose that $A \in LC(U_0)$ and $B \in RC(U)$. Take a *subgroup* $U_1 \subseteq U_0, \tilde{U}$ by Axiom 0(a). There is some $A' \in LC(U_1) \cap x$, since $x$ is a full filter, and $A' \subseteq A$ by Axiom 2. We can similarly find some $B' \subseteq B$ in $RC(U_1) \cap y$.\[\]
Next, suppose that $B' \in \text{LC}(V_0)$, $C \in \text{RC}(\tilde{V})$ and take a *subgroup $V_1 \subseteq V_0, \tilde{V}$. Find $B'' \subseteq B'$ in $\text{LC}(V_1) \cap y$ and $C' \subseteq C$ in $\text{RC}(V_1) \cap z$. Let $U_2$ be a *subgroup with $B'' \in \text{RC}(U_2)$. Since $B' \in \text{RC}(U_1)$ and $B'' \subseteq B'$, we have $U_2 \subseteq U_1$ by Axiom 3. There is some $A'' \in \text{LC}(U_2) \cap x$, since $x$ is a full filter, and $A'' \subseteq A'$ by Axiom 3. Thus $A'', B'', C'$ and $U = U_2, V = V_1$ are as required.

We are now ready to show that $S \in x \cdot (y \cdot z)$. Since $A \cdot B \subseteq T$ and $T \subseteq S$, $(A \cdot B)C \subseteq S$ by monotonicity. Thus $(A \cdot B) \cdot C \subseteq S$ holds by the definition of the product. Axiom 11 yields that $A \cdot (B \cdot C) \subseteq S$. Now $A \cdot (B \cdot C) \subseteq x \cdot (y \cdot z)$ holds by Axiom 5 and the above assumptions on $A, B, C$. So $S \in x \cdot (y \cdot z)$ as required.

As a consequence we have obtained:

**Proposition 4.11.**

The class $\mathcal{C}$ of coarse groups for oligomorphic groups is arithmetical.

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