Abstract. The aim of this paper is to study the calcium profile governed by the advection diffusion equation. The mathematical and computational modeling has provided insights to understand the calcium signalling which depends upon cytosolic calcium concentration. Here the model includes the important physiological parameters like diffusion coefficient, flow velocity etc. The mathematical model is fractionalized using Hilfer derivative and appropriate boundary conditions have been framed. The use of fractional order derivative is more advantageous than the integer order because of the non-local property of the fractional order differentiation operator i.e. the next state of the system depends not only upon its current state but also upon all of its preceding states. Analytic solution of the fractional advection diffusion equation arising in study of diffusion of cytosolic calcium in RBC is found using integral transform techniques. Since, the Hilfer derivative is generalisation of Riemann-Liouville and Caputo derivatives so, these two are also deduced as special cases. The numerical simulation has been done to observe the effects of the fractional order of the derivatives involved in the differential equation representing the model over the concentration of calcium which is function of time and distance. The concentration profile of calcium is significantly changed by the fractional order.

1. Introduction. Calcium is a universal signalling molecule involved in regulating cell cycle and fate, metabolism and structural integrity, mobility and volume. Calcium performs information processing and various vascular function like blood flow. It functions as a messenger. A prerequisite for the proper function of the

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calcium messenger system in higher organism is that the cytosolic calcium concentration in a resting cell be kept very low. The signalling of calcium in excitable and non-excitable cells depends upon presence of a relatively lower concentration of cytosolic calcium and on the gradient of concentration of calcium existing between the cytosol and the passage of the intracellular organelles (e.g., ER (the endoplasmic reticulum)). The cytosolic calcium level can increase via calcium influx from the extracellular stores across the plasma membrane and by calcium release from the intracellular stores and that help to refill these stores.

The calcium levels within the RBCs participate in regulation of biophysical properties and also the parameters like the metabolic activity redox state and cell clearance. The level of free calcium inside the human red blood cell is maintained between 30 nM and 60 nM by the calcium pump and the low permeability of the blood cell membrane while the blood plasma calcium is nearly 1.8 mM. To maintain this level of calcium it gets activate and undergoes signalling; such activation leaves remarkable impact on various processes within the cell showing calcium a key governor in RBCs.

Fractional calculus (FC) is a mathematical approach dealing with derivatives and integrals of arbitrary and complex orders. Therefore it provides a powerful tool which has been recently employed to model complex biological systems [33, 35]. The derivatives and integrals of arbitrary order are main aspects of this branch. Nowadays, many researchers and scientists have developed a number of models using this branch as a tool [3, 4, 6, 7, 9, 10, 11, 13, 20, 22, 23, 32, 36, 42, 43, 44, 45, 46]. This special branch is significantly used in various areas of science, mathematical biology, engineering and finance [24, 25, 26, 27, 28, 34]. Atangana et al. [8] improve this field by providing new class of partial differential operators. Jarad et al. [19] established a Gronwall Inequality in the frame of Atangana-Baleanu fractional integral and also presented the condition of existence and uniqueness of solution to a certain class of ordinary differential equations involving Atangana-Baleanu fractional derivative. Imran et al. [17] analyzed the magnetohydrodynamics (MHD) unsteady free convection flow of incompressible Newtonian fluid passing over an inclined plate. Asif et al. [5] analyzed the Couette flow of an incompressible Maxwell fluid using fractional ordered derivative without singular kernel. Riaz et al [31] studied the flow of both blood and magnetic particles using Caputo-Fabrizio fractional derivative model approach. Fractional order system modelling has emerged successfully in biomedical engineering field. There are many mathematical models proposed to describe the mechanism of calcium signalling. Many models has been developed for calcium transient in various cells.

Wiesner et al. [39] developed mathematical model to explain dynamics of calcium signalling in endothelial cell lying on human umbilical vein. This model provides mathematical description of calcium-mediated signal transduction in endothelial cell. Since, the mobilization of intracellular calcium from intracellular stores depends upon binding of agonist to cell surface receptor; Agarwal et al. [2] proposed a mathematical model to study this important aspect of calcium signalling. Jha et al. [18] have developed a mathematical model to study advection diffusion of calcium in astrocyte. This model incorporates advection diffusion equation, certain physiological parameter like diffusion coefficient etc and appropriate boundary condition and its analytic solution is found using Laplace transform. Agarwal et al. [1] discussed a fractional mathematical model to analyze dynamics of cytosolic calcium ion in astrocyte. Dupont and Goldbeter [14] presented several models among which
one is regarding the activation and autophosphorylation of the Ca$^{2+}$-calmodulin-dependent protein kinase II (CaMKII) by Ca$^{2+}$ and calmodulin (CaM) which can decode Ca$^{2+}$ oscillation and to control various cellular functions. A model to describe dynamics of intracellular calcium in endothelial cell developed by Winston et al.[41]. In this model inflow of extracellular calcium and that which is released from the intracellular stores both are combinely given by a single influx term described by a permeability parameter.

In this paper, we present analytic solution of the advection diffusion equation fractionalised with respect to time variable using the Hilfer derivative which is more general in nature than Caputo and Riemann-Liouville derivatives. The article is developed in following manner: In second section the definition of certain requisite operators their properties and integral transforms are discussed. In third section the time fractional mathematical model and the solution in terms of Mittag-Leffler function and Green function by using integral transform technique is discussed. In section 4 special cases of the above model are presented. The section 5 is presenting some applications by using specific function in place of general function $f(x)$ considered in aforesaid model in section 3. Lastly illustration and discussion are presented.

2. Mathematical preliminaries. In this section, certain definitions of fractional derivative operators namely, Riemann-Liouville, Caputo and their generalization Hilfer derivative are presented. The integral transforms required in forthcoming sections are also touched upon in this section.

**Definition 2.1.** Let $f$ be a piecewise continuous on the interval $(0, \infty)$ and integrable on any finite subinterval $[0, \infty)$. Then for $t > 0$ the Riemann-Liouville fractional integral [29, Eq. 2.2, p. 45] of order $\eta$ is defined as,

\[
I_0^\eta(f(t)) = RL D_{0}^{-\eta} f(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} f(\tau) d\tau, \quad t > 0, \eta > 0.
\]  

(1)

**Definition 2.2.** Let $f$ be a function of class $C$ (class of functions described in Definition 2.1). Then the Riemann-Liouville fractional derivative [29, Eq. 2.1, p. 82] of order $\eta$ can be defined as,

\[
RL D_0^\eta(f(t)) = \left( \frac{d}{dt} \right)^{n+1} \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} f(\tau) d\tau, \quad n \leq \eta < (n+1), \quad n \in \mathbb{N}.
\]  

(2)

**Definition 2.3.** Let $f$ and its $i$th order derivative ($i = 1, 2, 3, ..., n$) be continuous in the interval $(0, \infty)$ and then the fractional derivative of order $\eta$ given by Caputo [12, Eq. 5,p. 530] is defined as,

\[
C D_0^\eta(f(t)) = \begin{cases} 
\frac{1}{\Gamma(n-\eta)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\eta+1-n}} d\tau, & \text{if } n-1 < \eta \leq n, \quad n \in \mathbb{N} \\
\frac{d^n}{dt^n} f(t), & \text{if } \eta = n
\end{cases}
\]  

(3)

**Remark 1.** For having Riemann-Liouville fractional derivative continuity or differentiability of the function at the origin is not necessary. Because of certain disadvantages while solving real world problems the use of this derivative is lesser. Developing models of real world problems using Caputo derivative, allows incorporation of initial and boundary conditions.
Hilfer [16] presented a derivative, of which, the Riemann-Liouville and the Caputo derivatives are the special cases. This derivative of order $\eta$ and type $\xi$ with respect to time is given by
\[
D_{a}^{\eta, \xi} f(t) = \left( I_{a}^{\eta(1-\xi)} \right) \frac{d}{dt} \left( I_{a}^{(1-\xi)(1-\eta)} f(t) \right), \quad 0 < \xi < 1, \quad 0 \leq \eta \leq 1, \tag{4}
\]
where, $I_{a}^{\eta(1-\xi)}$ and $I_{a}^{(1-\xi)(1-\eta)}$ are the fractional integral operators of order $\eta(1-\xi)$ and $(1-\xi)(1-\eta)$ respectively in the Riemann-Liouville sense.

On substituting $\eta = 0$, (4) get reduced to the fractional derivative in Riemann-Liouville sense and for $\eta = 1$, it reduces to the Caputo fractional derivative.

**Definition 2.4.** The Laplace transform (see, e.g. [37]) of piecewise continuous function $f(t)$ of exponential order $\alpha > 0$ with respect to variable $t$ is defined as
\[
L[f(t); p] = \hat{f}(p) = \int_{0}^{\infty} e^{-pt} f(t) dt, \quad \Re(p) > \alpha, t \geq 0. \tag{5}
\]

The inverse Laplace Transform of function $\hat{f}(p)$ with respect to $t \geq 0$ is given by
\[
L^{-1}[\hat{f}(p); t] = f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} \hat{f}(p) dp, \tag{6}
\]
where $\gamma$ is the fixed real number.

The Laplace Transform of the Hilfer derivative of a function $f(t)$ in (4) is given by Tomovski [38, Eq. 1.5],
\[
L[D_{a}^{\eta, \xi} f(t)(p)] = p^{\xi} L[f(t)(p)] - p^{\eta(\xi - 1)} \left( I_{a}^{(1-\xi)(1-\eta)} f \right)(0), \quad 0 < \xi < 1, \quad 0 \leq \eta \leq 1, \tag{7}
\]
where $\left( I_{a}^{(1-\xi)(1-\eta)} f \right)(0)$ is the integral of $f$ in Riemann-Liouville sense evaluated for the limit $t \to 0$.

**Definition 2.5.** The Fourier Transform (see, e.g. [37]) of absolutely integrable function $f(t)$ with respect to $t$ is defined as
\[
F[f(t); s] = \hat{f}(s) = \int_{-\infty}^{\infty} e^{ist} f(t) dt, \quad (s \in \mathbb{R}). \tag{8}
\]

The inverse Fourier transform of the function $\hat{f}(s)$ is defined as
\[
F^{-1}[\hat{f}(s); t] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} \hat{f}(s) ds. \tag{9}
\]

A generalisation of exponential function was introduced by Mittag-Leffler [30], named as the Mittag-Leffler function $E_{a,b}$. This special function plays remarkable role in problems of physics, engineering, biology and applied sciences. In solving fractional and integral equations, the Mittag-Leffler function is an important tool.

\[
E_{a} = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(an + 1)}, \quad \Re(a) > 0, a \in \mathbb{C}, z \in \mathbb{C}. \tag{10}
\]
Wiman [40] defined the two parameter Mittag-Leffler function as
\[ E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(na + b)}, \Re(a) > 0, \Re(b) > 0, a, b \in \mathbb{C}, z \in \mathbb{C}. \] (11)

The Mittag-Leffler function is an entire function as the series mentioned above is convergent for all values of the argument \( z \in \mathbb{C} \).

3. **Mathematical modelling.** Advection-diffusion model for cytosolic calcium describes the transport process of the \( \text{Ca}^{2+} \) ions. Advection is the movement of the \( \text{Ca}^{2+} \) ions along with the intracellular fluid. The advection diffusion equation for transport of \( \text{Ca}^{2+} \) ion is derived from the law of the conservation of mass. The processes of advection and diffusion govern the flow of species in some given medium. Let the concentration \( C(x,t) \) of the species is denoted as a function of time \( t \) and position \( x \). Here, we consider one dimensional advection diffusion equation with constant cross flow velocity.

For this advection-diffusion phenomenon, the governing partial differential equation is

\[
\frac{\partial C(x,t)}{\partial t} = -u \frac{\partial C(x,t)}{\partial x} + D \frac{\partial^2 C(x,t)}{\partial x^2},
\] (12)

where \( u \) is the flow velocity and \( D \) is the diffusion constant. Many authors have already obtained analytical solution for the advection diffusion equations fractionalised with respect to time and space variables. Here, efforts are made to obtain the analytic solution of one dimensional advection-diffusion equation with constant cross flow velocity.

Here the concentration \( C = C(x,t) \) of chemical species is function of space variable \( x \) and time \( t \) is defined as

\[
C(x,t) = \begin{cases} 
0, & \text{if } -\infty < x < 0, \\
C(x,t), & \text{if } 0 < x < \infty 
\end{cases}
\] (13)

**Theorem 3.1.** Consider the following advection diffusion equation fractionalise with respect to time,

\[ D^\mu_0 C = -u \frac{\partial C}{\partial x} + D \frac{\partial^2 C}{\partial x^2}. \] (14)

The corresponding initial and boundary conditions are,

\[
I^{(1-\xi)(1-\eta)} C(x,0) = f(x), \quad 0 < \xi \leq 1, \quad 0 \leq \eta \leq 1,
\] (15)

\[ C(0,t) = 0, \quad t > 0, \] (16)

\[ \lim_{x \to -\infty} C(x,t) = 0, \quad t > 0. \] (17)

where \( u \) is the mean velocity of the \( \text{Ca}^{2+} \) in the cell, \( D \) is the diffusion constant, \( D^\xi_0 \) is the Hilfer derivative, \( I^{(1-\xi)(1-\eta)} C(x,0) \) is the fractional integral operator in Riemann-Liouville sense of order \( (1-\xi)(1-\eta) \) and \( f(x) \) is the real valued function. The solution of (20) is,

\[ C(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y) dy \] (18)
where,
\[
G(x,t) = \frac{t^{\xi+\eta(1-\xi)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi+\eta(1-\xi)}[(i\alpha s + Ds^2)t^\xi]e^{-isx}ds,
\]
(19)

is the Green’s function.

**Proof.** Transforming (14) using Fourier transform with respect to space variable \(x\).

\[
D\eta^\xi(\hat{C}(s,t)) = i\alpha s\hat{C}(s,t) + Ds^2\hat{C}(s,t),
\]
(20)

here \(\hat{C}(s,t)\) represents the Fourier transform of \(C(x,t)\).

Taking Laplace transformation of the above equation (20) with respect to time variable \(t\),

\[
p^\xi\hat{C}(s,p) - p^{\eta(1-\xi)}I_0^{(1-\xi)(1-\eta)}C(s,0) = i\alpha s\hat{C}(s,p) + Ds^2\hat{C}(s,p), \quad \Re(p) > 0,
\]
(21)

where \(\hat{g}(.)\) represents the Laplace transform of \(g(.)\).

Fourier transform of the initial condition (15) is

\[
\mathcal{F}[I_0^{(1-\xi)(1-\eta)}C(x,0)](s) = \hat{f}(s).
\]
(22)

Therefore, we have,

\[
(p^\xi - i\alpha s - Ds^2)\hat{C} = p^{\eta(1-\xi)}\hat{f}(s),
\]
(23)

\[
\hat{C} = \frac{p^{\eta(1-\xi)}\hat{f}(s)}{p^\xi - i\alpha s - Ds^2}.
\]
(24)

Now, taking inverse Laplace transform of the above equation with respect to \(p\) and using the result given by Haubold et al [15] i.e.

\[
L^{-1}\left(\frac{p^b-1}{p^a+\alpha}\right) = e^a-b E_{a,a-b+1}(-\alpha v), \quad \Re(p) > 0, \Re(a) > 0, \Re(a-b) > -1,
\]
(25)

we get,

\[
\hat{C}(s,t) = t^{\xi+\eta(1-\xi)-1}E_{\xi,\xi+\eta(1-\xi)}[(i\alpha s + Ds^2)t^\xi]\hat{f}(s).
\]
(26)

On taking inverse Fourier Transform of above equation,

\[
C(x,t) = \frac{t^{\xi+\eta(1-\xi)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi+\eta(1-\xi)}[(i\alpha s + Ds^2)t^\xi]e^{-isx}\hat{f}(s)ds,
\]

\[
= \frac{t^{\xi+\eta(1-\xi)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi+\eta(1-\xi)}[(i\alpha s + Ds^2)t^\xi]e^{-isx}\int_{-\infty}^{\infty} e^{isy}f(y)dyds,
\]

\[
= \int_{-\infty}^{\infty}\left(\frac{t^{\xi+\eta(1-\xi)-1}}{2\pi}\int_{-\infty}^{\infty} E_{\xi,\xi+\eta(1-\xi)}[(i\alpha s + Ds^2)t^\xi]e^{-is(x-y)}dy\right)f(y)dy,
\]

\[
= \int_{-\infty}^{\infty} G(x-y,t)f(y)dy,
\]
(27)

which is the solution of (14), where

\[
G(x,t) = \frac{t^{\xi+\eta(1-\xi)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi+\eta(1-\xi)}[(i\alpha s + Ds^2)t^\xi]e^{-isx}ds,
\]
(28)

is the Green’s function.
4. **Special cases.** 1. For \( \eta = 1 \) the Hilfer derivative get reduced to the fractional derivative in Caputo sense,

**Theorem 4.1.** Consider the following advection diffusion equation fractionalised with respect to time,

\[
C_D^\xi_0 C(x,t) = -u \frac{\partial C(x,t)}{\partial x} + D \frac{\partial^2 C(x,t)}{\partial x^2}, \quad x \in (-\infty, \infty), t > 0.
\] (29)

where \( C_D^\xi_0 \) is the fractional derivative in Caputo sense, \( C(x,t) \) denotes the concentration of calcium and \( f(x) \) is the real valued function. The corresponding initial and boundary conditions are,

\[
C(x,0) = f(x), \quad 0 < \xi \leq 1,
\] (30)

\[
C(0,t) = 0, \quad t > 0,
\] (31)

\[
\lim_{x \to \infty} C(x,t) = 0, \quad t > 0.
\] (32)

For the above equation with the given condition the solution is,

\[
C(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)dy
\] (33)

where,

\[
G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi}[(iux + Ds^2)t^\xi]e^{-isx}ds.
\] (34)

where \( E_{\xi}(\cdot) \) is the one parameter Mittag-Leffler function.

For \( \eta = 0 \), Hilfer derivative get reduced to the fractional derivative in Riemann-Liouville sense.

**Theorem 4.2.** Consider the following time-fractional advection diffusion equation,

\[
RL_D^\xi_0 C(x,t) = -u \frac{\partial C(x,t)}{\partial x} + D \frac{\partial^2 C(x,t)}{\partial x^2}, \quad x \in (-\infty, \infty), t > 0.
\] (35)

where \( RL_D^\xi_0 \) is the Riemann-Liouville fractional derivative, \( C(x,t) \) is the concentration of calcium and \( f(x) \) is the real valued function. The corresponding initial and boundary conditions are,

\[
I^{1-\xi}_0 C(x,0) = f(x), \quad 0 < \xi \leq 1,
\] (36)

\[
C(0,t) = 0, \quad t > 0,
\] (37)

\[
\lim_{x \to \infty} C(x,t) = 0, \quad t > 0.
\] (38)

For the above equation with the given condition the solution is,

\[
C(x,t) = \int_{-\infty}^{\infty} G(x-y,t)f(y)dy,
\] (39)

where,

\[
G(x,t) = \frac{t^{\xi-1}}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi}[(iux + Ds^2)t^\xi]e^{-isx}ds.
\] (40)
5. Applications. Here we discuss certain applications of our main theorem in section 3 for the functions \( f(x) = \delta(x) \), the Dirac-delta function, and the exponential function \( f(x) = e^{-x} \).

**Corollary 1.** Consider the following advection diffusion equation fractionalised with respect to time,

\[
D_0^{\xi,\eta}C(x,t) = -u \frac{\partial C(x,t)}{\partial x} + D \frac{\partial^2 C(x,t)}{\partial x^2}, \quad x \in (-\infty, \infty), t > 0, \tag{41}
\]

corresponding conditions are

\[
I^{(1-\xi)(1-\eta)}C(x,0) = \delta(x), \quad 0 < \xi \leq 1, \ 0 \leq \eta \leq 1, \quad \lim_{|x| \to \infty} C(x,t) = 0, \ t > 0, \tag{42}
\]

here \( \delta(x) \) is Dirac- Delta function. The concentration \( C(x,t) \) is

\[
C(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\delta(y)dy, \tag{43}
\]

where,

\[
G(x,t) = \frac{t^{\xi+\eta(1-\xi)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi+\eta(1-\xi)}[(ius + Ds^2)t^{\xi}]e^{-isx}ds. \tag{44}
\]

**Corollary 2.** Consider the following advection diffusion equation fractionalised with respect to time,

\[
D_0^{\xi,\eta}C(x,t) = -u \frac{\partial C(x,t)}{\partial x} + D \frac{\partial^2 C(x,t)}{\partial x^2}, \quad x \in (-\infty, \infty), t > 0, \tag{45}
\]

corresponding conditions are

\[
I^{(1-\xi)(1-\eta)}C(x,0) = e^{-x}, \quad 0 < \xi \leq 1, \ 0 \leq \eta \leq 1, \quad \lim_{|x| \to \infty} C(x,t) = 0, \quad t > 0. \tag{46}
\]

Then the calcium concentration \( C(x,t) \) is,

\[
C(x,t) = \int_{-\infty}^{\infty} G(x-y,t)e^{-y}dy, \tag{47}
\]

where,

\[
G(x,t) = \frac{t^{\xi+\eta(1-\xi)-1}}{2\pi} \int_{-\infty}^{\infty} E_{\xi,\xi+\eta(1-\xi)}[(ius + Ds^2)t^{\xi}]e^{-isx}ds. \tag{48}
\]

6. Illustration and discussion. For observing the effect of fractional order derivatives involved in the advection dispersion equation which is representing our model the concentration of calcium which is function of time and distance is simulated in following figure. The graphs are plotted in Matlab using diffusion coefficient \( D = 200 \mu m^2/s \) and the flow velocity \( u = 10 \mu m/s \). For the advection diffusion equation (14), we have plotted graphs for both fractional and integral values of \( \xi \) and \( \eta \) and \( f(x) = \delta(x) \), the Dirac delta function at \( t = 0 \).

In Figure 1 is showing graphically the solution of the fractional advection diffusion equation for Riemann-Liouville case which corresponds to \( \eta = 0 \) and for Caputo derivative which corresponds to \( \eta = 1 \) is shown in Figure 2; both plots are taken for \( \xi = 1, 0.95, 0.90 \). The graphical representation of the solution of (14) for values \( \eta = 0.90 \) and \( \xi = 1, 0.95, 0.90 \) as shown in Figure 3. As the Caputo derivative permits the involvement of initial and boundary condition in the formulation of the
Figure 1. Graph between $C(x, t)$ and $t$ for various values of $\xi$ for $\eta = 0$ which corresponds to the Riemann-Liouville derivative.

Figure 2. Graph between $C(x, t)$ and $t$ for various values of $\xi$ for $\eta = 1$ which corresponds to Caputo derivative.
problem so, the Caputo derivative is more advantageous over the Riemann-Liouville derivative to solve the real world problems. The assumptions about the random behavior of a single particle, possible velocities in a flow field and the length of time it may hold; originates the advection diffusion equation. The fractional advection diffusion equation can be evoked by the use of non-integer or say arbitrary order derivative with respect to time or space variables. As the fractionalised form of our model based upon power law therefore, it explains the motion of calcium with memory in time. The numerical simulation shown by Figures 1, 2 and 3 depicts that important role is played by order of derivative on the concentration profile of the calcium.

7. **Conclusion.** In this paper, the fractional model for calcium signalling is investigated by using the Hilfer derivative. The use of fractional order derivative is more advantageous than the integer order because of the non-local property of the fractional order differentiation operator i.e. the next state of the system depends not only upon its current state but also upon all of its preceding states. The solution for the model is obtained by using integral transform technique and the Riemann-Liouville and Caputo derivatives are deduced as the special case. The numerical simulation has been done which shows the effect of fractional order over the calcium profile. It is observed that the rise in concentration of calcium inside the cell is higher for the fractional model as compare to the integral one for the same time interval making the study of calcium profile with the use of fractional model is more beneficial than the integer model. Since, to maintain the low level of \(Ca^{2+}\) concentration in the cell there is a continual low-level cycling of calcium into and out of the cell and here, it is observed that for fractional model the calcium
concentration inside the cell attains the higher level earlier than for the integral one showing sooner completion of a round of cycling of calcium before we go for the investigation of integral model and making use of fractional calculus instead of classical make this observation and conclusion possible.

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