JORDANIAN QUANTUM (SUPER)ALGEBRAS $U_h(g)$ VIA CONTRACTION METHOD AND MAPPING: REVIEW

B. ABDESSELAM$^a$, R. CHAKRABARTI$^b$, A. YANALLAH$^c$ and M.B ZAHAF$^d$

$^a$Laboratoire de Physique Quantique de la Matière et Modélisations Mathématiques (LPQ3M), Centre Universitaire de Mascara, 29000-Mascara, Algérie

$^b$Laboratoire de Physique Théorique d’Oran, Université d’Oran Es-Sénia, 31100-Oran, Algérie

$^c$Department of Theoretical Physics, University of Madras, Guindy Campus, Madras 600025, India

$^d$Université de Mascara, 29000-Mascara, Algérie

Abstract

Recently, a class of transformations of $R_q$-matrices was introduced such that the $q \rightarrow 1$ limit gives explicit nonstandard $R_h$-matrices. The transformation matrix is singular as $q \rightarrow 1$. For the transformed matrix, the singularities, however, cancel yielding a well-defined construction. We have shown that our method can be implemented systematically on $R_q$-matrices of all dimensions of $U_q(sl(N))$, $U_q(osp(1|2))$ and $U_q(sl(2|1))$. Explicit constructions are presented for $U_q(sl(2))$, $U_q(sl(3))$, $U_q(osp(1|2))$ and $U_q(sl(2|1))$, while choosing $R_q$ for (fund. rep.) $\otimes$ (arbitrary irrep.). Our method yields nonstandard deformations along with a nonlinear map of the $h$-Borel subalgebra on the corresponding classical Borel subalgebra, which can be easily extended to the whole algebra. Following this approach, we construct explicitly here the jordanian quantum (super)algebras (nonstandard version) $U_h(sl(2))$, $U_h(sl(3))$, $U_h(osp(1|2))$ and $U_h(sl(2|1))$. These Hopf (super)algebras are equipped with a remarkably simpler coalgebraic structure. Generalizing our results on $U_h(sl(3))$, we give the higher dimensional Jordanian (super)algebras $U_h(sl(N))$ for all $N$. The universal $R_h$-matrices are also given.

Contents

1 Introduction and motivation .......................... 1

2 $U_h(sl(N))$: Map, Hopf Algebra, Irreps. and $R_h$-matrix .................. 2

2.1 The $R_h$-matrices of the Nonstandard $U_h(sl(2))$ Algebra .................. 2

2.2 The $R_h$-matrices of the Nonstandard $U_h(sl(3))$ Algebra .................. 5

2.3 $U_h(sl(N))$: Generalization .......................... 8

3 Nonstandard quantizations of $osp(1|2)$ superalgebra ................. 9

3.1 Super-Jordanian $U_h(osp(1|2))$ algebra via contraction process ........ 10

3.2 Jordanian $U_h(osp(1|2))$ algebra: a nonlinear realization ................. 14

4 The Nonstandard superalgebra Enveloping $U_h(sl(2|1))$ ................. 18

5 Conclusion ........................................... 23

1 Introduction and motivation

It is well known that any enveloping (super)algebra $U(g)$ of an Lie (super)algebra $g$ has many quantizations: the first one called the Drinfeld-Jimbo deformation or the standard quantum deformation [1, 2] is quasitriangular, whereas the others one called the Jordanian deformations or the nonstandard quantum deformations [3] are triangular ($R_{21}R = 1$). A typical example of the Jordanian quantum algebras was first introduced by Ohn [4]. In general, nonstandard quantum algebras are obtained by applying twist [5] to the corresponding Lie algebras. A twisting that produces an algebra isomorphism to the Ohn algebra [4] is found in [6, 7].

Recently, the twisting procedure was extensively employed to study a wide variety of Jordanian deformed algebras, such as $U_h(sl(N))$ algebra [8-12], symplectic algebra $U_h(sp(N))$ [13], orthogonal algebra $U_h(so(N))$ [14-17] and orthosymplectic superalgebra $U_h(osp(1|2))$ [18-20]. It follows from these studies that:
• The nonstandard quantum algebras have undeformed commutation relations;
• The Jordanian deformation appears only in the coalgebraic structure;
• the coproduct and the antipode maps have very complicated forms in comparison with the Drinfeld-Jimbo and the Ohn deformations.

So far Jordanian quantum algebra $\mathcal{U}_h(sl(N))$ has been explicitly written, with a simple coalgebra but with deformed commutation relations, only for $N = 2$ [4]. This amounts to a choice of an appropriate basis, in which the commutation relations are deformed but the corresponding coalgebraic structure remains simple. The main object here is to construct some nonstandard versions of an enveloping Lie algebra which have deformed commutation relations; but are endowed with a relatively simpler coalgebraic structure compared to those in the previous studies [8-20].

Following this approach, we have proposed, recently [21-25], a new technic which makes possible the construction of a nonstandard version $\mathcal{U}_h(g)$ of an enveloping Lie (super)algebra $\mathcal{U}(g)$ by a suitable contraction, from the standard ones $\mathcal{U}_q(g)$. Our scheme consist to obtain the $\mathcal{R}_h$-matrix, for all dimensions of a (super)Jordanian quantum (super)algebra $\mathcal{U}_h(g)$ from the $\mathcal{R}_q$-matrix associated to the standard quantum (super)algebra $\mathcal{U}_q(g)$ through a specific transformation $G$ (singular in the $q \rightarrow 1$), as follows:

$$\mathcal{R}_h = \lim_{q \rightarrow 1} \left[ G^{-1} \otimes G^{-1} \right] \mathcal{R}_q [G \otimes G],$$

where, for example, $G = E_q \left( \frac{\hbar c_n}{q^n} \right)$ for $\mathcal{U}_q(sl(N))$ ($E_{1N}$ is the longer positive root generator of the Lie algebra $sl(N)$), $G = E_q \left( \frac{\hbar c^2}{q^{2n-1}} \right)$ for $\mathcal{U}_q(osp(1|2))$ ($c$ is the fermionic positive simple root of the Lie algebra $osp(2|1)$) and $G = E_q \left( \frac{\hbar c}{q^{2n-1}} \right)$ for $\mathcal{U}(sl(2|1))$ ($e_1$ is the bosonic positive simple root generator of the Lie algebra $sl(2|1)$). The deformed exponential map $E_q$ is defined by

$$E_q(\eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{[n]_q!}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q \times [n-1]_q!, \quad [0]_q! = 1. \quad (2)$$

This procedure yields nonstandard deformation along with a nonlinear map of the $\hbar$-Borel subalgebra on the corresponding classical Borel subalgebra, which can be easily extended to the whole algebra. Following here this strategy, we have construct the Jordanian quantum (super)algebras $\mathcal{U}_h(sl(2)), \mathcal{U}_h(sl(3)), \mathcal{U}_h(osp(1|2))$ and $\mathcal{U}_h(sl(2|1))$, wherein we use the contraction procedure and the map mentioned above. The nonstandard versions obtained here have deformed commutation relations, but the coalgebraic part is more simple and in compact form.

The manuscript is organized as follows: In section 2, the jordanian quantum algebra $\mathcal{U}_h(sl(2))$ and $\mathcal{U}_h(sl(3))$ are constructed. A nonlinear map between $\mathcal{U}_h(sl(2))$ and $\mathcal{U}(sl(2))$ (resp. $\mathcal{U}_h(sl(3))$ and $\mathcal{U}(sl(3))$) is then established. Generalizing our results on the $\mathcal{U}(sl(3))$ algebra, the higher dimensional algebras $\mathcal{U}_h(sl(N))$, $N > 4$, are also introduced via a nonlinear map and proved to be a Hopf algebra endowed with a triangular $\mathcal{R}_h$-matrix. The (super)jordanian quantum versions $\mathcal{U}_h(sl(2|1))$ and $\mathcal{U}_h(osp(1|2))$ are presented respectively in section 3 and 4. Finally, we conclude in section 5.

2 $\mathcal{U}_h(sl(N))$: Map, Hopf Algebra, Irreps. and $\mathcal{R}_h$-matrix

For our purpose, the deformation parameter $\hbar$ is an arbitrary complex number. For simplicity, we start with $\mathcal{U}_q(sl(2))$ algebra.

2.1 The $\mathcal{R}_h$-matrices of the Nonstandard $\mathcal{U}_h(sl(2))$ Algebra

The generating elements $(H, X, Y)$ of the algebra $\mathcal{U}_h(sl(2))$ obey the following commutation rules [4]

$$[H, X] = \frac{2\sinh \hbar X}{\hbar}, \quad [H, Y] = -Y (\cosh \hbar X) - (\cosh \hbar X) Y, \quad [X, Y] = H. \quad (3)$$
The non-commutative coproduct structure of \( \mathcal{U}_h(sl(2)) \) read [4]
\[
\Delta_h (X) = X \otimes 1 + 1 \otimes X, \quad \Delta_h (Y) = Y \otimes e^{hX} + e^{-hX} \otimes Y, \quad \Delta_h (H) = H \otimes e^{hX} + e^{-hX} \otimes H. \tag{4}
\]
The Borel subalgebra generated by the elements \( (H, X) \) is equipped with a Hopf structure. The universal \( \mathcal{R} \)-matrix for \( \mathcal{U}_h(sl(2)) \) may be cast in the form [26]
\[
\mathcal{R}_h = \exp \left( -hX \otimes e^{hX} H \right) \exp \left( h e^{hX} H \otimes X \right) \tag{5}
\]
which obviously coincides with the universal \( \mathcal{R} \)-matrix of the borel subalgebra. The fundamental \( (j = \frac{1}{2}) \) representation of the algebra (3) remains undeformed [4]:
\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}
\]
Using the fundamental representation (6) in the first sector of the tensor product of the operators in the expression (5) of the universal \( \mathcal{R}_h \)-matrix, we obtain the \( \mathcal{R}_h \)-matrix in the \( \left( \frac{1}{2} \otimes j \right) \) representation:
\[
\mathcal{R}_h = \begin{pmatrix} e^{hX} & -hH + \frac{h}{2} \left( e^{hX} - e^{-hX} \right) \\ 0 & q^{\frac{1}{2}} \left( 1 - q^{-2} \right) \mathcal{J}_- \end{pmatrix}. \tag{7}
\]
Absent from upper triangular form (7) and indeed from the universal \( \mathcal{R}_h \)-matrix (5) is the generator \( Y \) completing the \( \mathcal{U}_h(sl(2)) \) algebra. We will show how (7) can be obtained, directly and for arbitrary \( j \), from the corresponding \( R_q \)-matrix for \( \left( \frac{1}{2} \otimes j \right) \) representation given by (see, for example [27])
\[
R_q = \begin{pmatrix} q^{\mathcal{J}_0/2} & q^{\frac{1}{2}} \left( 1 - q^{-2} \right) \mathcal{J}_- \\ 0 & q^{-\mathcal{J}_0/2} \end{pmatrix}. \tag{8}
\]
Here the generators of \( \mathcal{U}_q(sl(2)) \) are denoted by \( \left( q^{\pm \mathcal{J}_0}, \mathcal{J}_{\pm} \right) \) satisfying the standard relations
\[
q^{\mathcal{J}_0} \mathcal{J}_{\pm} = \mathcal{J}_{\pm} q^{\mathcal{J}_0\pm 2}, \quad [\mathcal{J}_+, \mathcal{J}_-] = \frac{q^{\mathcal{J}_0} - q^{-\mathcal{J}_0}}{q - q^{-1}} \equiv [\mathcal{J}_0]. \tag{9}
\]
Hereafter we consider generic \( q \), excluding roots of unity.

For the purpose of transforming \( R_q \)-matrix in (8), we now consider a \( q \)-deformed exponential operator:
\[
E_q (\eta \mathcal{J}_+) = \sum_{n=0}^{\infty} \frac{(\eta \mathcal{J}_+)^n}{[n]!}. \quad \text{We choose, for an arbitrary finite constant } h \text{ the parameter } \eta \text{ as } \eta = \frac{h}{q-1}. \quad \text{We emphasize that through the deformed exponential } E_q(x) \text{ defined in (2) has non convenient simple expression for its inverse (comparable to the standard } q \text{-exponential [27] satisfying } \exp_q (x)^{-1} = \exp_{q^{-1}} (-x)) \text{, this is precisely what is needed for obtaining non-singular limiting forms for the } R \text{-matrix elements for arbitrary representations and other interesting properties. For any given value of } j, \text{ the series (2) may be terminated after setting } \mathcal{J}_+^{2j+1} = 0; \text{ but, we proceed quite generally as follows. Defining }
\]
\[
T_{(\alpha)} = E_q^{-1} (\eta \mathcal{J}_+) E_q (q^{\alpha} \eta \mathcal{J}_+) \tag{10}
\]
with \( T_{(0)} = 1 \), we obtain
\[
E_q^{-1} (\eta \mathcal{J}_+) q^{\alpha \mathcal{J}_0/2} E_q (q^{\alpha} \eta \mathcal{J}_+) = T_{(\alpha)} q^{\alpha \mathcal{J}_0/2}. \tag{11}
\]
For transforming \( R_q \)-matrix in (8), the operators \( T_{\pm} \) are of particular importance. We will be concerned with simple rational values of \( \alpha \). For later use, we note the identity
\[
E_q^{-1} (\eta \mathcal{J}_+) q^{\alpha \mathcal{J}_0/2} E_q (q^{\alpha} \eta \mathcal{J}_+) = \left( E_q^{-1} (\eta \mathcal{J}_+) q^{\alpha \mathcal{J}_0/2} E_q (q^{\alpha} \eta \mathcal{J}_+) \right) \left( E_q^{-1} (\eta \mathcal{J}_+) q^{\beta \mathcal{J}_0/2} E_q (q^{\beta} \eta \mathcal{J}_+) \right), \quad \text{which in notation (11) reads}
\]
\[
T_{(\alpha + \beta)} q^{(\alpha + \beta) \mathcal{J}_0/2} = \left( T_{(\alpha)} q^{\alpha \mathcal{J}_0/2} \right) \left( T_{(\beta)} q^{\beta \mathcal{J}_0/2} \right). \tag{12}
\]
Moreover using the identity \[ [\mathcal{J}_-^0, \mathcal{J}_-] = \frac{\ln q}{q-1} \left( q^{\mathcal{J}_0/2} \mathcal{J}_-^{n-1} q^{-\mathcal{J}_0/2} - q^{-\mathcal{J}_0/2} \mathcal{J}_-^{n-1} q^{\mathcal{J}_0/2} \right), \]
the following commutator is obtained \[ [E_q (\eta \mathcal{J}_+), \mathcal{J}_-] = \frac{\ln q}{q-1} \left( E_q (\eta \mathcal{J}_+) q^{\mathcal{J}_0} - E_q (q^{-1} \mathcal{J}_+) q^{-\mathcal{J}_0} \right), \]
which, in turn, leads to \[ E_q (\eta \mathcal{J}_+) = -\frac{n}{q-q^{-1}} \left( T_{(1)} q^{\mathcal{J}_0} - T_{(-1)} q^{-\mathcal{J}_0} \right) + \mathcal{J}_-. \] Evaluating term by term, the \( q \to 1 \) limits of \( T_{(\pm)} \) are found to be finite and of the form
\[
\lim_{q \to 1} T_{(\pm)} = T_{(\pm)} = \sum_{n=0}^{\infty} c_n^{(\pm)} (hJ_+)^n, \tag{13}\]
where \( (J_0, J_\pm) \) are the generators of the classical \( sl(2) \) algebra \( ([J_0, J_\pm] = \pm 2J_\pm, [J_+, J_-] = J_0) \). The first few coefficients \( \{c_n^{(\pm)} \mid n \geq 0 \} \) in (13) read \( c_0^{(\pm)} = 1, c_1^{(\pm)} = \pm 1, c_2^{(\pm)} = 1/2, c_3^{(\pm)} = 0, c_4^{(\pm)} = -1/8, c_5^{(\pm)} = 0 \). If the limits are indeed finite, then from (12) it is evident that
\[
T_{(\alpha)} = \left( T_{(1)} \right)^\alpha, \tag{14}\]
where \( T_{(\alpha)} = \lim_{q \to 1} T_{(\alpha)} \). Henceforth we write \( T_{(1)} = T \). The result obtained here suggests the following derivation of the closed form of \( T \). To this end, we left and right multiply the second commutation relation in (9) by \( E_q^{-1} (\eta \mathcal{J}_+) \) and \( E_q (\eta \mathcal{J}_+) \), respectively \( E_q^{-1} (\eta \mathcal{J}_+) \left( q^{\mathcal{J}_0} - q^{-\mathcal{J}_0} \right) \) \( E_q (\eta \mathcal{J}_+) = (q - q^{-1}) E_q^{-1} (\eta \mathcal{J}_+) [J_+, J_-] \) \( E_q (\eta \mathcal{J}_+) \). Using (10), (11) and (12), we obtain
\[
T_{(2)} q^{\mathcal{J}_0} - T_{(-2)} q^{-\mathcal{J}_0} = h(q+1) \left( T_{(1)} J_+ q^{\mathcal{J}_0} + q^{-2} T_{(-1)} J_+ q^{-\mathcal{J}_0} \right) + \left( q^{\mathcal{J}_0} - q^{-\mathcal{J}_0} \right). \tag{15}\]
Using (14), we now obtain the following equation for \( T \) as \( q \to 1 \): \( T^2 - T^{-2} = (T + T^{-1}) (2hJ_+) \), which after a factorization yields
\[
T - T^{-1} = 2hJ_. \tag{16}\]
The quadratic relation (16) in \( T \) is now solved
\[
T^{\pm 1} = \pm hJ_+ + \sqrt{1 + h^2 J_+^2}. \tag{17}\]
This our crucial result.

With all these results now in hand we go back to \( R_q \) in (8). We choose the transformation matrix as \( G = g_\frac{1}{2} \otimes g \), where \( g = E_q (\eta J_+) \) and \( g_\frac{1}{2} \equiv g \bigg|_{j=\frac{1}{2}} \). We obtain
\[
R_h = \lim_{q \to 1} \left( G^{-1} R_q G \right) = \begin{pmatrix} T & -\frac{h}{2} (T + T^{-1}) J_0 + \frac{h}{2} (T - T^{-1}) \\ 0 & T^{-1} \end{pmatrix}, \tag{18}\]
where we have defined
\[
e^{\pm hX} = T^{\pm 1} = \pm hJ_+ + \left( 1 + h^2 J_+^2 \right)^{\frac{1}{2}}, \quad H = \frac{1}{2} (T + T^{-1}) J_0 = \left( 1 + h^2 J_+^2 \right) J_0. \tag{19}\]
It is easy to verify that
\[
[H, T^{\pm 1}] = T^{\pm 2} - 1 \Rightarrow [H, X] = 2 \frac{\sinh hX}{h}. \tag{20}\]
Comparing (7) with (18), we see that the contraction scheme, which comprises our transformation and limiting procedure has furnished the \( R_h \)-matrix along with a nonlinear map of the subalgebra of \( \mathcal{U}_h(sl(2)) \) generated by \( (H, X) \) on the classical one generated by \( (J_0, J_+) \). Indeed, defining
\[
Y = J_- \frac{h^2}{4} J_+ \left( J_0^2 - 1 \right) \tag{21} \]
we complete the \( \mathcal{U}_h(sl(2)) \) algebra and show that
\[
[H, Y] = -Y (\cosh hX) - (\cosh hX) Y, \quad [X, Y] = H. \tag{22} \]
The expressions (19) and (21) may be looked as a particular realization of the \( \mathcal{U}_h(sl(2)) \) generators \( (H, X, Y) \). We have therefore, developed an invertible nonlinear map of \( (H, X, Y) \) on classical generators \( (J_0, J_+, J_-) \).
2.2 The $R_h$-matrices of the Nonstandard $\mathcal{U}_h(sl(3))$ Algebra

The major interest of our method is that can be generalized for obtaining the nonstandard $R_h$-matrices of algebras of higher dimensions as contraction limits of the corresponding $R_q$-matrices. Here we treat the $sl(3)$ algebra. Choosing the Chevalley generators corresponding to the simple roots of the $\mathcal{U}_q(sl(3))$ algebra as $k_1^+ = q^{\pm h_1}, k_2^+ = q^{\pm h_2}, k_3^+ = q^{\pm (h_1 + h_2)}, \ e_1, \ e_2, \ e_3 = \hat{e}_1 \hat{e}_2 - q^{-1} \hat{e}_2 \hat{e}_1, \ f_1, \ f_2$ and $f_3 = \hat{f}_2 \hat{f}_1 - q \hat{f}_1 \hat{f}_2$. The Hopf structure of the $\mathcal{U}_q(sl(3))$ algebra is defined by [27]

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [\hat{e}_i, \hat{f}_j] = \delta_{ij} [h_i],
\]

\[
\hat{e}_1 e_3 = q \hat{e}_3 \hat{e}_1, \quad \hat{e}_2 e_3 = q^{-1} \hat{e}_3 \hat{e}_2, \quad \hat{f}_1 \hat{f}_3 = q \hat{f}_3 \hat{f}_1, \quad f_2 \hat{f}_3 = q^{-1} f_3 f_2, \quad \Delta_q(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \Delta_q(\hat{e}_i) = \hat{e}_i \otimes q^{h_i/2} + q^{-h_i/2} \otimes \hat{e}_i, \quad \Delta_q(\hat{f}_i) = \hat{f}_i \otimes q^{h_i/2} + q^{-h_i/2} \otimes \hat{f}_i,
\]

where $[x] = \frac{q^n - q^{-n}}{q - q^{-1}}$. The Cartan matrix for the $sl(3)$ algebra reads $a = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \end{pmatrix}$. The universal matrix of the $\mathcal{U}_h(sl(3))$ algebra is given by

\[
R_q = q^{\sum_{i,j} (a^{-1})_{ij} h_i \otimes h_j} \exp_{q-2} \left( \lambda e_2 q^{h_2} \otimes q^{-h_2/2} \hat{f}_2 \right) \exp_{q-2} \left( \lambda e_3 q^{h_3} \otimes q^{-h_3/2} \hat{f}_3 \right),
\]

where $\lambda = q - q^{-1}$, $\exp_q(x) = \sum_{n=0}^{\infty} x^n / \{n\}_q!$, $\{n\}_q = \{n\} \{n - 1\}_q!$, $\{0\}_q = 1$ and $\{n\}_q = \frac{1 - q^n}{1 - q}$. We denote the classical generators ($q = 1$) of the $sl(3)$ algebra by $h_1, h_2, h_3 = h_1 + h_2, e_1, e_2, e_3 = e_1 e_2 - e_2 e_1, f_1, f_2$ and $f_3 = f_2 f_1 - f_1 f_2$.

Let us turn now to the nonstandard $R_h$-matrices. For brevity and simplicity we limit here our self to (fundamental irrep.) $\otimes$ (arbitrary irrep.). Recall that for $\mathcal{U}_q(sl(3))$ algebra the $R_q$-matrix in the representation (fund.) $\otimes$ (arb.) reads [27]:

\[
R_q = \left( \pi_{\text{fund.}} \otimes \pi_{\text{arb.}} \right) R_q = \begin{pmatrix} q^{\frac{2}{3}(2h_1 + h_2)} & q^{\frac{1}{3}(2h_1 + h_2)} & q^{\frac{1}{3}(2h_1 + h_2)} \\ 0 & q^{-\frac{1}{3}(h_1 - h_2)} & q^{-\frac{1}{3}(h_1 - h_2)} \\ 0 & 0 & q^{-\frac{1}{3}(h_1 + 2h_2)} \end{pmatrix},
\]

where

\[
\Lambda_{12} = q^{-1/2}(q - q^{-1})q^{-h_1/2} \hat{f}_1, \quad \Lambda_{13} = q^{-1/2}(q - q^{-1})f_3 q^{-\frac{1}{3}(h_1 + h_2)}, \quad \Lambda_{23} = q^{-1/2}(q - q^{-1})q^{-h_2/2} \hat{f}_2.
\]

We have shown in [21] that the nonstandard $R_h$-matrix (in the representation (fund.) $\otimes$ (arb.)) arise from the $R_q$-matrix as follows:

\[
R_h = \lim_{q \rightarrow 1} \left[ E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \otimes E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \right]^{-1} R_q \left[ E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \otimes E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \right]^{-1}
\]

\[
= \lim_{q \rightarrow 1} \begin{pmatrix} E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 & -E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \\ 0 & E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 \\ 0 & 0 & E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \end{pmatrix} R_q \begin{pmatrix} E_q^{-1} \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 & \hbar E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \\ 0 & E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) & 0 \\ 0 & 0 & E_q \left( \frac{\hbar \hat{e}_3}{q - 1} \right) \end{pmatrix}
\]

\[
= \begin{pmatrix} T & 2hT^{-1/2}e_2 & -\frac{h}{3}(T + T^{-1})(h_1 + h_2) + \frac{h}{3}(T - T^{-1}) \\ 0 & I & -2hT^{1/2}e_1 \\ 0 & 0 & T^{-1} \end{pmatrix},
\]

where

\[
T = h\hat{e}_3 + \sqrt{1 + h^2 e_3^2}, \quad T^{-1} = -h\hat{e}_3 + \sqrt{1 + h^2 e_3^2}.
\]

The following properties are pointed out:
1. The corner elements of (27) have exactly the same structure as in the $R_h$-matrix of $U_h(\mathfrak{sl}(2))$. This implies that the classical generators $e_3$, $h_3 = h_1 + h_2$ and $f_3$ of $U(\mathfrak{sl}(3))$ are deformed (for the nonstandard quantization: $U(\mathfrak{sl}(3)) \longrightarrow U_h(\mathfrak{sl}(3))$) as follows [21]:

$$T^{\pm 1} = \pm \h e_3 + \sqrt{1 + \h^2 e_3^2}, \quad H_3 = \sqrt{1 + \h^2 e_3^2} h_3, \quad F_3 = f_3 - \frac{\h^2}{4} e_3 (h_3^2 - 1),$$

and satisfy evidently the commutation relations [4]

$$TT^{-1} = T^{-1} T = 1, \quad [H_3, T^{\pm 1}] = T^{\pm 2} - 1,$$

$$[T^{\pm 1}, F_3] = \pm \h \left( H_3 T^{\pm 1} + T^{\pm 1} H_3 \right), \quad [H_3, F_3] = -\frac{1}{2} \left( T F_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1} \right).$$

With the following definition (see Ref. [4]) $E_3 = \h^{-1} \ln T = \h^{-1} \text{arcsinh} \ h e_3$, it follows that the elements $H_3$, $E_3$, and $F_3$ satisfy the relations of the $U_h(\mathfrak{sl}(2))$ algebra [4]

$$[H_3, E_3] = 2 \sinh \frac{h E_3}{h}, \quad [H_3, F_3] = -F_3 \left( \cosh h E_3 \right) - \left( \cosh h E_3 \right) F_3, \quad [E_3, F_3] = H_3,$$

where it is obvious that as $h \longrightarrow 0$, we have $(H_3, E_3, F_3) \longrightarrow (h_3, e_3, f_3)$. It is now evident from (31) that $U_h(\mathfrak{sl}(2)) \subset U_h(\mathfrak{sl}(3))$.

2. The expression (27) of the $R_h$-matrix indicate that the simple root generators $e_1$ and $e_2$ are deformed as follows:

$$E_1 = \sqrt{h e_3 + \sqrt{1 + h^2 e_3^2}} e_1 = T^{1/2} e_1, \quad E_2 = \sqrt{h e_3 + \sqrt{1 + h^2 e_3^2}} e_2 = T^{1/2} e_2.$$  

To complete our algebra $U_h(\mathfrak{sl}(3))$, let us introduce the following $h$-deformed generators:

$$F_1 = \sqrt{-h e_3 + \sqrt{1 + h^2 e_3^2}} f_1 + \frac{h}{2} \sqrt{h e_3 + \sqrt{1 + h^2 e_3^2}} h e_2 h_3 = T^{-1/2} \left( f_1 + \frac{h}{2} e_2 T h_3 \right),$$

$$F_2 = \sqrt{-h e_3 + \sqrt{1 + h^2 e_3^2}} f_2 - \frac{h}{2} \sqrt{h e_3 + \sqrt{1 + h^2 e_3^2}} h e_1 h_3 = T^{-1/2} \left( f_2 - \frac{h}{2} e_1 T h_3 \right),$$

$$H_1 = \left( -h e_3 + \sqrt{1 + h^2 e_3^2} \right) \left( \sqrt{1 + h^2 e_3^2} h_1 + \frac{h}{2} e_3 (h_1 - h_2) \right) = h_1 - \frac{h}{2} e_3 T^{-1} h_3,$$

$$H_2 = \left( -h e_3 + \sqrt{1 + h^2 e_3^2} \right) \left( \sqrt{1 + h^2 e_3^2} h_2 - \frac{h}{2} e_3 (h_1 - h_2) \right) = h_2 - \frac{h}{2} e_3 T^{-1} h_3.$$  

The expressions (29), (32) and (33) constitute a realization of the Jordanian algebra $U_h(\mathfrak{sl}(3))$ with the classical generators via a nonlinear map. This immediately yields the irreducible representations (irreps.) of $U_h(\mathfrak{sl}(3))$ in an explicit and simple manner.

**Proposition 1** The Jordanian algebra $U_h(\mathfrak{sl}(3))$ is then an associative algebra over $\mathbb{C}$ generated by $H_1$, $H_2$, $H_3$, $E_1$, $E_2$, $T$, $T^{-1}$, $F_1$, $F_2$ and $F_3$, satisfying the commutation relations [22]

$$[H_1, H_2] = 0, \quad [H_1, T^{-1} H_3] = [H_2, T^{-1} H_3] = 0,$$

$$[H_1, E_1] = 2 E_1, \quad [H_2, E_2] = 2 E_2, \quad [H_1, E_2] = -E_2, \quad [H_2, E_1] = -E_1,$$

$$[T^{-1} H_3, E_1] = E_1, \quad [T^{-1} H_3, E_2] = E_2, \quad [H_1, F_1] = -2 F_1 + h E_2 T^{-1} H_3,$$

$$[H_2, F_2] = -2 F_2 - h E_1 T^{-1} H_3, \quad [H_1, F_2] = F_2 - h E_1 T^{-1} H_3, \quad [H_2, F_1] = F_1 + h E_2 T^{-1} H_3,$$

$$[T H_3, F_1] = -T^2 F_1, \quad [T H_3, F_2] = -T^2 F_2,$$

$$[T^{-1} E_1, F_1] = \frac{1}{2} (T + T^{-1}) H_1 + \frac{1}{2} (T - T^{-1}) H_2, \quad [T^{-1} E_2, F_2] = \frac{1}{2} (T + T^{-1}) H_2 + \frac{1}{2} (T - T^{-1}) H_1,$$

$$[T^{-1} E_1, F_2] = 0, \quad [T^{-1} E_2, F_1] = 0, \quad [E_1, E_2] = \frac{1}{2 \h} (T^2 - 1),$$

$$[E_1, F_1] = \frac{1}{2} (T + T^{-1}) E_1 + \frac{1}{2} (T - T^{-1}) E_2, \quad [E_1, F_2] = \frac{1}{2} (T + T^{-1}) E_2 + \frac{1}{2} (T - T^{-1}) E_1,$$

$$[E_2, F_1] = 0, \quad [E_2, F_2] = 0, \quad [T^{-1} E_1, T^{-1} E_2] = [T^{-1} E_2, T^{-1} E_1] = 0.$$
\[ [TF_2, TF_1] = T \left( F_3 - \frac{\hbar}{2} H_3 T H_3 - \frac{\hbar}{8} (T - T^{-1}) \right) \]

\[ [TH_1, T^{\pm 1}] = \frac{1}{2} (T^{\pm 2} - 1), \quad [TH_2, T^{\pm 1}] = \frac{1}{2} (T^{\pm 2} - 1), \]

\[ [H_1, F_3] = -\frac{T^{-1}}{4} (TF_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1}) - \frac{1}{4} T^{-1} H_3^2 - \frac{1}{4} H_3 T^{-1} H_3, \]

\[ [H_2, F_3] = -\frac{T^{-1}}{4} (TF_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1}) - \frac{1}{4} T^{-1} H_3^2 - \frac{1}{4} H_3 T^{-1} H_3, \]

\[ [E_1, T] = [E_1, T^{-1}] = [E_2, T] = [E_2, T^{-1}] = 0, \]

\[ [F_1, T] = h T E_2, \quad [F_1, T^{-1}] = -h T^{-1} E_2, \quad [F_2, T] = -h T E_1, \quad [F_2, T^{-1}] = h T^{-1} E_1, \]

\[ [E_1, F_3] = -\frac{1}{2} (TF_2 + F_2 T), \quad [E_2, F_3] = \frac{1}{2} (TF_1 + F_1 T). \]

\[ [F_1, F_3] = h T F_1 - h E_2 F_3 + \frac{\hbar^2}{4} T E_2, \quad [F_2, F_3] = h T F_2 + h E_1 F_3 - \frac{\hbar^2}{4} T E_1. \] (34)

We have stated here the final results only. To obtain the expressions of $H_1$ and $H_2$, we have proceed as follows: By analogy with (29), we have first started with the ansatz $\sqrt{1 + \hbar^2 e_3^2 h_1}$ and $\sqrt{1 + \hbar^2 e_3^2 h_2}$. It is easy to see that $\sqrt{1 + \hbar^2 e_3^2 h_1}$ and $\sqrt{1 + \hbar^2 e_3^2 h_2}$, we obtain $[\sqrt{1 + \hbar^2 e_3^2 h_1} + \frac{\hbar}{2} e_3 (h_1 - h_2), F_3] = -\frac{1}{4} \left( TF_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1} \right) + \frac{\hbar^2}{4} \left( e_3 (h_1 - h_2) H_3 + H_3 e_3 (h_1 - h_2) \right)$ and $\sqrt{1 + \hbar^2 e_3^2 h_2}$, we have proceed as

\[ \sqrt{1 + \hbar^2 e_3^2 h_2} \]

\[ \frac{\hbar}{2} e_3 (h_1 - h_2), F_3] = -\frac{1}{4} \left( TF_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1} \right) + \frac{\hbar^2}{4} \left( e_3 (h_1 - h_2) H_3 + H_3 e_3 (h_1 - h_2) \right). \]

Then, if we add to $\sqrt{1 + \hbar^2 e_3^2 h_1}$ and we deduct from $\sqrt{1 + \hbar^2 e_3^2 h_2}$ the term $\frac{\hbar}{2} e_3 (h_1 - h_2)$, we obtain $[\sqrt{1 + \hbar^2 e_3^2 h_1} + \frac{\hbar}{2} e_3 (h_1 - h_2), F_3] = -\frac{1}{4} \left( TF_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1} \right) + \frac{\hbar^2}{4} T (h_1 - h_2) H_3 + \frac{\hbar^2}{4} H_3 T (h_1 - h_2)$ and $\sqrt{1 + \hbar^2 e_3^2 h_2} - \frac{\hbar}{2} e_3 (h_1 - h_2), F_3] = -\frac{1}{4} \left( TF_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1} \right) - \frac{\hbar^2}{4} T (h_1 - h_2) H_3 - \frac{\hbar^2}{4} H_3 T (h_1 - h_2)$. These commutation relations suggest to take $H_1 \sim \left( \sqrt{1 + \hbar^2 e_3^2 h_1} + \frac{\hbar}{2} e_3 (h_1 - h_2) \right)$ and $H_2 \sim \left( \sqrt{1 + \hbar^2 e_3^2 h_2} - \frac{\hbar}{2} e_3 (h_1 - h_2) \right)$.

Finally, to preserve the Cartan subalgebra, we are obliged to multiply $\left( \sqrt{1 + \hbar^2 e_3^2 h_1} + \frac{\hbar}{2} e_3 (h_1 - h_2) \right)$ and $\left( \sqrt{1 + \hbar^2 e_3^2 h_2} - \frac{\hbar}{2} e_3 (h_1 - h_2) \right)$ respectively by $T^{-1}$, i.e. to take $H_1 = T^{-1} \left( \sqrt{1 + \hbar^2 e_3^2 h_1} + \frac{\hbar}{2} e_3 (h_1 - h_2) \right) = h_1 - \frac{\hbar}{2} e_3 T^{-1} h_3$ and $H_2 = T^{-1} \left( \sqrt{1 + \hbar^2 e_3^2 h_2} - \frac{\hbar}{2} e_3 (h_1 - h_2) \right) = h_2 - \frac{\hbar}{2} e_3 T^{-1} h_3$. The expressions of $F_1$ and $F_2$ are obtained in similar way. The expressions (29), (32) and (33) may be looked now as a particular realization of the $\mathcal{U}_h (sl(3))$ generators. Others maps can be also considered.

In terms of the Chevalley generators (simple roots) $\{ E_1, E_2, F_1, F_2, H_1, H_2 \}$, the algebra $\mathcal{U}_h (sl(3))$ is defined as follows [22]:

\[ T = \left( 1 + 2 \hbar [E_1, E_2] \right)^{1/2}, \quad T^{-1} = \left( 1 + 2 \hbar [E_1, E_2] \right)^{-1/2}, \]

\[ [H_1, H_2] = 0, \quad [H_1, E_1] = 2E_1, \quad [H_2, E_2] = 2E_2, \quad [H_1, E_2] = -E_2, \]

\[ [H_2, E_1] = -E_1, \quad [H_1, F_1] = -2F_1 + \hbar E_2 (H_1 + H_2), \quad [H_2, F_2] = -2F_2 - \hbar E_1 (H_1 + H_2), \]

\[ [H_1, F_2] = F_2 - \hbar E_1 (H_1 + H_2), \quad [H_2, F_1] = F_1 + \hbar E_2 (H_1 + H_2), \]

\[ [T^{-1} E_1, F_1] = \frac{1}{2} (T + T^{-1}) H_1 + \frac{1}{2} (T - T^{-1}) H_2, \quad [T^{-1} E_2, F_2] = \frac{1}{2} (T + T^{-1}) H_2 + \frac{1}{2} (T - T^{-1}) H_1, \]

\[ [T^{-1} E_1, F_2] = [T^{-1} E_2, F_1] = 0, \]

\[ E_1^2 E_2 - 2E_1 E_2 F_1 + E_2^2 E_1 = 0, \quad E_2^2 E_1 - 2E_2 E_1 E_2 + E_1 E_2^2 = 0, \]

\[ (TF_1)^2 T F_2 - 2 T F_1 T F_2 T F_1 + T F_2 (TF_1)^2 = 0, \quad (TF_2)^2 T F_1 - 2 T F_2 T F_1 T F_2 + T F_1 (TF_2)^2 = 0, \] (35)
or, briefly

\[
[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j + T^{-1}[F_j, T](H_1 + H_2),
\]

\[
[T^{-1} E_i, F_j] = \delta_{ij} \left( T^{-1} H_i + \frac{1}{2} (T - T^{-1})(H_1 + H_2) \right),
\]

\[
(ad E_i)^{1-a_{ij}} (E_j) = (ad TF_i)^{1-a_{ij}} (TF_j) = 0, \quad i \neq j,
\]

(36)

where \((a_{ij})_{i,j=1,2}\) is the Cartan matrix of \(sl(3)\). Let us turn now to the coalgebra structure:

**Proposition 2** The Jordanian quantum algebra \(\mathcal{U}_h(sl(3))\) admits a Hopf structure with coproducts, antipodes and counits determined by \([22]\):

\[
\Delta(E_1) = E_1 \otimes 1 + T \otimes E_1, \quad \Delta(E_2) = E_2 \otimes 1 + T \otimes E_2, \quad \Delta(T^\pm 1) = T^\pm 1 \otimes T^\pm 1,
\]

\[
\Delta(F_1) = F_1 \otimes 1 + T^{-1} \otimes F_1 + hH_3 \otimes E_2, \quad \Delta(F_2) = F_2 \otimes 1 + T^{-1} \otimes F_2 - hH_3 \otimes E_1,
\]

\[
\Delta(F_3) = F_3 \otimes T + T^{-1} \otimes F_3, \quad \Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1 - \frac{1}{2} (1 - T^{-2}) \otimes T^{-1} H_3,
\]

\[
\Delta(H_2) = H_2 \otimes 1 + 1 \otimes H_2 - \frac{1}{2} (1 - T^{-2}) \otimes T^{-1} H_3, \quad \Delta(H_3) = H_3 \otimes T + T^{-1} \otimes H_3,
\]

\[
S(E_1) = -T^{-1} E_1, \quad S(E_2) = -T^{-1} E_2, \quad S(T) = T^{-1}, \quad S(T^{-1}) = T,
\]

\[
S(F_1) = -TF_1 + hTH_3 T^{-1} E_2, \quad S(F_2) = -TF_2 - hTH_3 T^{-1} E_1,
\]

\[
S(F_3) = -TF_3 T^{-1}, \quad S(H_1) = -H_1 - \frac{1}{2} (T - T^{-1}) H_3,
\]

\[
S(H_2) = -H_2 - \frac{1}{2} (T - T^{-1}) H_3, \quad S(H_3) = -TH_3 T^{-1},
\]

\[
\epsilon(a) = 0, \quad \forall a \in \left\{ H_1, H_2, H_3, E_1, E_2, F_1, F_2, F_3 \right\},
\]

\[
\epsilon(T) = \epsilon(T^{-1}) = 1.
\]

(37)

All the Hopf algebra axioms can be verified by direct calculations. Let us remark that our coproducts have simpler forms compared to those maps in \([8-20]\). This is one benefit of our procedure. Pertinent to the algebraic structures of our Hopf algebra described in \((30), (34) \text{ and } (37)\), here we obtain its universal \(R\)-matrix in the following form \([22]\):

\[
R_h = \exp \left( -hE_3 \otimes TH_3 \right) \exp \left( 2hTH_3 \otimes E_3 \right).
\]

(38)

The above universal matrix satisfies the required properties for the full Hopf structure discussed earlier. We note that the element, generated by \(E_3\) and \(H_3\) coincides with the universal \(R_h\)-matrix of the subalgebra involving the generators corresponding to the highest root, and may be connected to the results obtained by the contraction process by a suitable twist operator that can be derived as a series expansion in \(h\). The nonstandard algebras \(\mathcal{U}_h(sl(4))\) and \(\mathcal{U}_h(sl(5))\) can be derived in a similar way (see refs. \([22]\)).

### 2.3 \(\mathcal{U}_h(sl(N))\): Generalization

From these above studies, it is easy to see that:

**Proposition 3** The Jordanian quantization deform \(\mathcal{U}(sl(N))\)'s Chevalley generators as follows \([22]\):

\[
T = h[e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}]]] + \sqrt{1 + h^2([e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}]]])^2},
\]

\[
T^{-1} = -h[e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}]]] + \sqrt{1 + h^2([e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}]]])^2},
\]

\[
E_i = T^{(\delta_{ii} + \delta_{i,N-1})/2} e_i, \quad i = 1, \cdots, N - 1,
\]

\[8\]
\[ F_i = T^{-\left(\delta_{i1}+\delta_{i,N-1}\right)/2} \left( f_i + \frac{\hbar}{2} T [f_i, [e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}]]]] (h_1 + \cdots + h_{N-1}) \right) \]
\[ H_i = h_i - \frac{\left(\delta_{i1}+\delta_{i,N-1}\right)h}{2} [e_1, [e_2, \cdots, [e_{N-2}, e_{N-1}]]) T^{-1} (h_1 + \cdots + h_{N-1}) \]

and they satisfy the commutation relations
\[
[\begin{array}{ll}
H_i, H_j & = 0, \\
H_i, E_j & = a_{ij} E_j, \\
H_i, F_j & = -a_{ij} F_j + \left(\delta_{i1}+\delta_{i,N-1}\right) T^{-1} [F_j, T] (H_1 + \cdots + H_{N-1}), \\
T^{-\left(\delta_{i1}+\delta_{i,N-1}\right)} E_i, F_j & = \delta_{ij} \left( T^{-\left(\delta_{i1}+\delta_{i,N-1}\right)} H_i + \frac{\left(\delta_{i1}+\delta_{i,N-1}\right)}{2} (T - T^{-1})(H_1 + \cdots + H_{N-1}) \right), \\
[E_i, E_j] & = 0, \quad |i-j| > 1, \\
T^{\left(\delta_{i1}+\delta_{i,N-1}\right)} F_i, T^{\left(\delta_{i1}+\delta_{i,N-1}\right)} F_j & = 0, \quad |i-j| > 1, \\
(ad E_i)^{-1} a_{ij} (E_j) & = (ad T^{\left(\delta_{i1}+\delta_{i,N-1}\right)} F_i)^{-1} a_{ij} (T^{\left(\delta_{i1}+\delta_{i,N-1}\right)} F_j) = 0, \quad (i \neq j),
\end{array}\]

where \((a_{ij})_{i,j=1,\ldots,N}\) is the Cartan matrix of \(sl(N)\), i.e. \(a_{ii} = 2\), \(a_{i,i\pm1} = -1\) and \(a_{ij} = 0\) for \(|i-j| > 1\).

The algebra \(40\) is called the Jordanian quantum algebra \(U_\hbar(sl(N))\). The expressions \(39\) may be regarded as a particular nonlinear realisation of the \(U_\hbar(sl(N))\) generators. The Jordanian algebra \(U_\hbar(sl(N))\) \(40\) admits the following coalgebra structure \([22]\):
\[
\begin{align*}
\Delta(E_i) & = E_i \otimes 1 + T^{\left(\delta_{i1}+\delta_{i,N-1}\right)} \otimes E_i, \\
\Delta(F_i) & = F_i \otimes 1 + T^{-\left(\delta_{i1}+\delta_{i,N-1}\right)} \otimes F_i + T (H_1 + \cdots + H_{N-1}) \otimes T^{-1} [F_i, T], \\
\Delta(H_i) & = H_i \otimes 1 + 1 \otimes H_i - \frac{\left(\delta_{i1}+\delta_{i,N-1}\right)}{2} (1 - T^{-2}) \otimes (H_1 + \cdots + H_{N-1}), \\
S(E_i) & = -T^{-\left(\delta_{i1}+\delta_{i,N-1}\right)} E_i, \\
S(F_i) & = -T^{\left(\delta_{i1}+\delta_{i,N-1}\right)} F_i + T^2 (H_1 + \cdots + H_{N-1}) T^{-2} [F_i, T], \\
S(H_i) & = -H_i + \frac{\left(\delta_{i1}+\delta_{i,N-1}\right)}{2} (1 - T^2) (H_1 + \cdots + H_{N-1}), \\
\epsilon(E_i) & = \epsilon(F_i) = \epsilon(H_i) = 0.
\end{align*}
\]

Following \(38\), we obtain the universal \(R_\hbar\)-matrix of an arbitrary \(U_\hbar(sl(N))\) algebra in the following general form \([22]\):
\[
R_\hbar = \exp \left( -\hbar E_{1N} \otimes TH_{1N} \right) \exp \left( 2\hbar T H_{1N} \otimes E_{1N} \right).
\]

where \(H_{1N} = T (H_1 + \cdots + H_{N-1})\) and \(E_{1N} = \hbar \ln T\). The above universal \(R_\hbar\)-matrix of the full algebra Hopf algebra is obtained from the generators associated with the highest root; and its coincides with the universal \(R_\hbar\)-matrix of the Hopf subalgebra cite associated with the highest root. It is interesting to note that the nonlinear map \(40\) equips the \(\hbar\)-deformed generators \((E_i, F_i, H_i)\) with an additional induced co-commutative coproduct. Similarly, the may be also equipped with an induced co-commutative coproduct. Similarly, the undeformed generators \((e_i, f_i, h_i)\), via the inverse map, may be viewed as elements of the \(U_\hbar(sl(N))\) algebra; and, thus, may be endowed with an induced noncommutative coproduct.

3 Nonstandard quantizations of \(osp(1|2)\) superalgebra

It has been recently demonstrated \([28]\) that three distinct bialgebra structures exist on the classical \(osp(1|2)\) superalgebra, and all of them are coboundary. The classical Lie superalgebra \(osp(1|2)\) has three even \((h, b_{\pm})\) and two odd \((e, f)\) generators, which obey the commutation relations
\[
\begin{align*}
[h, e] & = e, \\
[h, f] & = -f, \\
[e, f] & = -h, \\
[h, b_{\pm}] & = \pm 2b_{\pm}, \\
b_{\pm}, b_{-} & = h, \\
b_{+}, f & = e, \\
b_{-}, e & = f, \\
b_{+} & = e^2, \\
b_{-} & = -f^2.
\end{align*}
\]
The generators $\{h, b_\pm\}$ form a subalgebra $sl(2) \subset osp(1|2)$. The inequivalent classical $r$-matrices defined on $osp(1|2)$ superalgebra have been listed in Ref. [28] as

$$r_1 = h \wedge b_+,$$
$$r_2 = h \wedge b_- - e \wedge e,$$
$$r_3 = t(h \wedge b_+ + h \wedge b_- - e \wedge e - f \wedge f).$$

The standard quasi-triangular $q$-deformation of the $osp(1|2)$ superalgebra considered in Refs. [29, 30] corresponds to $r_3$. The parameter $t$ in $r_3$ becomes irrelevant in quantization as it can be absorbed into the deformation parameter. The $r_1$ matrix is comprised of the elements of the $sl(2)$ subalgebra. This allows quantization of the $osp(1|2)$ superalgebra using the inclusion $sl(2) \subset osp(1|2)$. This is done [31] by applying Drinfeld twist for the $sl(2)$ subalgebra to the full $osp(1|2)$ superalgebra. The Hopf algebra $U_h(osp(1|2))$ obtained thereby is triangular. The quantization of the $r_1$ matrix has been obtained in Ref. [31] in terms of the classical basis set with undeformed commutation relations but with coproduct structures deformed in a complicated manner.

Recently the classical matrix $r_2$ has been quantized [32] using nonlinear basis elements. The corresponding quantized algebra $U_h(osp(1|2))$ is known [32, 33] to satisfy the triangularity condition. An important issue observed [34] in this context is that the quantum $R_h$ matrix of the $U_h(osp(1|2))$ algebra in the fundamental representation may be obtained via a contraction mechanism from the corresponding $R_q$ matrix of the standard $U_q(osp(1|2))$ algebra in the $q \to 1$ limit. A generalization of this contraction procedure for arbitrary representations, though clearly desirable as it will allow us to systematically obtain various quantities of interest of the $U_h(osp(1|2))$ algebra from the corresponding quantities of the $q$-deformed $U_q(osp(1|2))$ algebra, has not been achieved so far.

In another problem considered also here, we express the other nonstandard Hopf algebra $U_h(osp(1|2))$ corresponding to the classical $r_1$ matrix in a nonlinear basis. Here we follow the approach in Ref. [21], where the Jordanian $U_h(sl(2))$ algebra has been introduced in terms of a nonlinear basis set, while retaining the coproduct structure of these basis elements simple. A consequence of our choice of nonlinear basis elements is that, Ohn’s $U_h(sl(2))$ algebra [4] explicitly arises as a Hopf subalgebra of our $U_h(osp(1|2))$ algebra. This feature is not directly evident in the construction given in Ref. [31]. Moreover, while the algebraic relations of our $U_h(osp(1|2))$ algebra are deformed, the coproduct structures are considerably simple. Our approach may be of consequence in building physical models of many-body systems employing coalgebra symmetry [35]. Invertible nonlinear maps, and the twist operators pertaining to these maps, exist connecting the deformed and the undeformed basis sets. We will present here a class of invertible maps interrelating the Hopf algebra $U_h(osp(1|2))$, based on quantization of the $r_1$ matrix, with its classical analog $U(osp(1|2))$. The twist operators vis-à-vis the above maps will be discussed in the sequel. It is shown that a particular map called ‘minimal twist map’ implements the simplest twist given directly by the factorized form of the universal $R_h$ matrix of the $U_h(osp(1|2))$ algebra. For a ‘non-minimal’ map the twist has an additional factor. We evaluate this twist operator as a series in the deformation parameter.

Using the oft-used nomenclature we will refer to the algebra obtained by quantizing the $r_2$ matrix as the super-Jordanian $U_h(osp(1|2))$ algebra, whereas the algebra generated via quantization of $r_1$ matrix will be noted as Jordanian $U_h(osp(1|2))$ algebra.

### 3.1 Super-Jordanian $U_h(osp(1|2))$ algebra via contraction process

The Hopf structure of the super-Jordanian $U_h(osp(1|2))$ algebra using nonlinear basis elements was obtained [32] previously. For comparing with our subsequent results we list it, after a slightly altered normalization, as

$$[H, E] = \frac{1}{2} (T + T^{-1}) E,$$
$$[H, F] = -\frac{1}{4} (T + T^{-1}) F - \frac{1}{4} F (T + T^{-1}),$$
$$\{E, F\} = -H,$$
$$[H, T^{\pm 1}] = T^{\pm 2} - 1,$$
$$[H, Y] = -\frac{1}{2} (T + T^{-1}) Y - \frac{1}{2} Y (T + T^{-1}) - \frac{h}{4} E (T - T^{-1}) F - \frac{h}{4} F (T - T^{-1}) E,$$
$$[T^{\pm 1}, Y] = \pm \frac{h}{2} (T^{\pm 1} H + H T^{\pm 1}),$$
$$E^2 = \frac{T - T^{-1}}{2h},$$
$$F^2 = -Y,$$
\[ [T^{\pm 1}, F] = \pm \hbar T^{\pm 1} E, \quad [Y, E] = \frac{1}{4} (T + T^{-1}) F + \frac{1}{4} F (T + T^{-1}), \]

\[ \Delta(H) = H \otimes T^{-1} + T \otimes H + \hbar E T^{1/2} \otimes ET^{-1/2}, \quad \Delta(E) = E \otimes T^{-1/2} + T^{1/2} \otimes E, \]

\[ \Delta(F) = F \otimes T^{-1/2} + T^{1/2} \otimes F, \quad \Delta(T^{\pm 1}) = T^{\pm 1} \otimes T^{\mp 1}, \]

\[ \Delta(Y) = Y \otimes T^{-1} + T \otimes Y + \frac{\hbar}{2} E T^{1/2} \otimes T^{-1/2} F + \frac{\hbar}{2} T^{1/2} F \otimes ET^{-1/2}, \]

\[ \varepsilon(H) = \varepsilon(E) = \varepsilon(F) = \varepsilon(Y) = 0, \quad \varepsilon(T^{\pm 1}) = 1, \]

\[ S(H) = -H - \hbar E^2, \quad S(E) = -E, \quad S(F) = -F + \frac{\hbar}{2} E, \]

\[ S(T^{\pm 1}) = T^{\mp 1}, \quad S(Y) = -Y + \frac{\hbar}{2} H + \frac{\hbar^2}{4} E^2, \]  

(45)

where \( \hbar \) is the deformation parameter. The \( \mathcal{U}_q(osp(1|2)) \) algebra has only one Borel subalgebra generated by the elements \((H, E, T^{\pm 1})\). Kulish observed [34] that the \( R_\hbar \) matrix in the fundamental representation of the super-Jordanian \( \mathcal{U}_\hbar(osp(1|2)) \) algebra may be obtained via a transformation, singular in the \( q \to 1 \) limit, from the corresponding \( R_q \) matrix in the fundamental representation of the standard \( q \)-deformed \( \mathcal{U}_q(osp(1|2)) \) algebra.

Our task here is to generalize the above contraction procedure for arbitrary representations. As an application of our contraction scheme, we construct the \( L_\hbar \) operator corresponding to the Borel subalgebra of the super-Jordanian \( \mathcal{U}_\hbar(osp(1|2)) \) algebra from the corresponding \( L_q \) operator of the standard \( q \)-deformed \( \mathcal{U}_q(osp(1|2)) \) algebra. To this end, we first quote some well-known [29, 30] results on the \( \mathcal{U}_q(osp(1|2)) \) algebra. The \( \mathcal{U}_q(osp(1|2)) \) algebra is generated by three elements \((\hat{h}, \hat{e}, \hat{f})\) obeying the Hopf structure

\[ [\hat{h}, \hat{e}] = \hat{e}, \quad [\hat{h}, \hat{f}] = -\hat{f}, \quad \{\hat{e}, \hat{f}\} = -[\hbar]_q, \]

\[ \Delta(\hat{h}) = \hat{h} \otimes 1 + 1 \otimes \hat{h}, \quad \Delta(\hat{e}) = \hat{e} \otimes q^{-\hat{h}/2} + q^{\hat{h}/2} \otimes \hat{e}, \quad \Delta(\hat{f}) = \hat{f} \otimes q^{-\hat{h}/2} + q^{\hat{h}/2} \otimes \hat{f}, \]

\[ \varepsilon(\hat{h}) = \varepsilon(\hat{e}) = \varepsilon(\hat{f}) = 0, \quad S(\hat{h}) = -\hat{h}, \quad S(\hat{e}) = -q^{-\hat{h}/2} \hat{e}, \quad S(\hat{f}) = -q^{\hat{h}/2} \hat{f}, \]

(46)

where \([x]_q = (q^x - q^{-x})/(q - q^{-1})\). To facilitate our later application, we choose the \((4j + 1)\) dimensional irreducible representation of the \( \mathcal{U}_q(osp(1|2)) \) algebra in an asymmetrical manner as follows:

\[ \hat{h} \mid j m \rangle = 2m \mid j m \rangle, \quad \hat{e} \mid j m \rangle = \mid j m + 1/2 \rangle, \quad \hat{e} \mid j j \rangle = 0, \]

\[ \hat{f} \mid j m \rangle = -\mid j + m \rangle_\hbar \mid j - m + 1/2 \rangle_\hbar \mid j m - 1/2 \rangle \quad \text{for } j - m \text{ integer}, \]

\[ = \mid j + m \rangle_\hbar \mid j - m + 1/2 \rangle_\hbar \mid j m - 1/2 \rangle \quad \text{for } j - m \text{ half-integer}, \]

(47)

where \([x]_\hbar = (q^x - (1)^{2x} q^{-x})/(q^{1/2} + q^{-1/2})\).

Following the strategy adopted earlier for constructing the Jordanian deformation of the \( sl(N) \) algebra, we give here the general recipe for obtaining the quantum \( R^{1/2;j}_\hbar \) matrix of an arbitrary representation of the \( \mathcal{U}_\hbar(osp(1|2)) \) algebra. Explicit demonstration is given for the \( 1/2 \otimes j \) representation. The relevant \( R^{1/2;j}_\hbar \) matrix may be directly interpreted as the \( L \) operator corresponding to the Borel subalgebra of the \( \mathcal{U}_\hbar(osp(1|2)) \) algebra. Our construction may, obviously, be generalized for an arbitrary \( j_1 \otimes j_2 \) representation. The primary ingredient for our method is the \( R^{1/2;j}_q \) matrix [29] of the \( \mathcal{U}_q(osp(1|2)) \) algebra in the \( 1/2 \otimes j \) representation.

A suitable similarity transformation is performed on this \( R^{1/2;j}_q \) matrix. The transforming matrix is singular in the \( q \to 1 \) limit. For the transformed matrix, the singularities, however, systematically cancel yielding a well-defined construction. The transformed object, in the \( q \to 1 \) limit, directly furnishes the \( R^{1/2;j}_\hbar \) matrix for the super-Jordanian \( \mathcal{U}_\hbar(osp(1|2)) \) algebra. Interpreting, as mentioned above, the \( R^{1/2;j}_\hbar \) matrix obtained here as the \( L \) operator corresponding to the Borel subalgebra of the \( \mathcal{U}_\hbar(osp(1|2)) \) algebra, we use the standard FRT procedure [36] to construct the full Hopf structure of the said Borel subalgebra. The \( R^{1/2;j}_q \) matrix of the
tensored $1/2 \otimes j$ representation of the $U_q(osp(1|2))$ algebra reads \cite{29}

$$R_q^{\frac{1}{2}j} = \begin{pmatrix} q^\hbar \cdot \omega q^{-\frac{1}{2}} \hat{f} & -\omega (1 + q^{-1}) \hat{f}^2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} q^{-\frac{1}{2}},$$

(48)

where $\omega = q - q^{-1}$. We now introduce a transforming matrix $M$, singular in the $q \rightarrow 1$ limit, as

$$M = E_{q^2}(\eta e^2),$$

(49)

where $\eta = \frac{h}{q - 1}$. For any finite value of $j$ the series (49) may be terminated after setting $\hat{e}^{\delta j + 1} = 0$. As the transforming operator $M$ in (49) depends only on the generator $\hat{e}$, our subsequent results assume simplest form for the asymmetric choice of the representation (47). Our contraction scheme, however, remains valid independent of the choice of representation. The $R_q^{\frac{1}{2}j}$ matrix of the $U_q(osp(1|2))$ algebra may now be subjected to a similarity transformation followed by a limiting process

$$\tilde{R}_q^{\frac{1}{2}j} \equiv \lim_{q \rightarrow 1} \left[ \left( M_{j_1}^{-1} \otimes M_{j_2}^{-1} \right) R_q^{\frac{1}{2}j} \left( M_{j_1} \otimes M_{j_2} \right) \right].$$

(50)

In the followings we will present explicit results for the operator $\tilde{R}_q^{\frac{1}{2}j}$. In performing the similarity transformation (50) we may choose any suitable operator ordering. Specifically, starting from left we maintain the order $\hat{e} < \hat{h} < \hat{f}$. In our calculation a class of operators $\tilde{T}_q(\alpha) = (E_{q^2}(\eta e^2))^{-1} E_{q^2}(q^{2\alpha}\eta e^2)$ satisfying $\tilde{T}_q(\alpha + \beta) = \tilde{T}_q(\alpha) \tilde{T}_q(\beta)$, play an important role. To evaluate $q \rightarrow 1$ limiting value of the operator $\tilde{T}_q(\alpha)$, we use the identity $E_{q^2}(q^2 \eta e^2) - E_{q^2}(q^{-2} \eta e^2) = (q^2 - q^{-2}) e^2 E_{q^2}(\eta e^2) \implies \tilde{T}_q(1) - \tilde{T}_q(-1) = (q^2 - q^{-2}) e^2$. Evaluating term by term, the limiting values of $\tilde{T}_q(\pm 1)$ are found to be finite; and, for these finite operators yields $\tilde{T}_q(\pm 1) = (\tilde{T}_q(\pm 1))$, where $\tilde{T}_q(\alpha) = \lim_{q \rightarrow 1} \tilde{T}_q(\alpha)$. Writing $\tilde{T}_q(\pm 1) = \tilde{T}^\pm 1$ henceforth, we immediately observe that in the $q \rightarrow 1$ limit, the above identity assumes the form

$$\tilde{T} - \tilde{T}^{-1} = 2he^2 \implies \tilde{T}^{\pm 1} = \pm he^2 + \sqrt{1 + h^2e^4}.$$ 

(51)

This is our crucial result. Two other operator identities playing key roles are listed below:

$$\hat{f} \hat{e}^{2n} = \hat{e}^{2n} \hat{f} - \frac{1}{q + 1} \left\{ n \right\} q^2 \hat{e}^{2n-1} \hat{f} - \frac{1}{q + 1} \left\{ n \right\} q^{-2} \hat{e}^{2n-1} \hat{f},$$

(52)

$$\hat{f}^2 \hat{e}^{2n} = \hat{e}^{2n} \hat{f}^2 + q^{-1} \left\{ \frac{1}{q^2} - \frac{1}{q+1} \left\{ \frac{n}{2} \right\} q^2 \hat{e}^{2(n-1)} \hat{f}^2 \\ \frac{1}{q+1} \left\{ \frac{1}{q} - \frac{1}{q+1} \left\{ \frac{n}{2} \right\} q^2 \hat{e}^{2(n-1)} \hat{f} \right\} \right\} - \frac{q}{q+1} \left\{ \frac{1}{q} - \frac{1}{q+1} \left\{ \frac{n}{2} \right\} q^2 \hat{e}^{2(n-1)} \hat{f} \right\} - \frac{q}{q+1} \left\{ \frac{1}{q} - \frac{1}{q+1} \left\{ \frac{n}{2} \right\} q^2 \hat{e}^{2(n-1)} \hat{f} \right\}$$

(53)

where $\{ x \}_q = 1 - \frac{q}{1 - q}$, $\{ n \}_q! = \{ n \}_q \{ n-1 \}_q \cdots \{ 1 \}_q \left\{ \frac{n}{m} \right\}_q = \frac{\{ n \}_q!}{\{ n-m \}_q! \{ m \}_q!}$ and $\hat{e}^{\pm 1} = q^{\pm \hbar}$. Using the above identities systematically and passing to the limit $q \rightarrow 1$, it may be shown that in our construction of the operator $\tilde{R}_q^{\frac{1}{2}j}$ via (50), all singularities cancel yielding in a well-defined answer

$$\tilde{R}_q^{\frac{1}{2}j} = \begin{pmatrix} \tilde{T} & h\tilde{T}^{\frac{1}{2}}e & -h\tilde{H} + \frac{1}{2} (\tilde{T} - \tilde{T}^{-1}) \\ 0 & 1 & -h\tilde{T}^{\frac{1}{2}}e \\ 0 & 0 & \tilde{T}^{-1} \end{pmatrix},$$

(54)
where $\tilde{H} = \frac{1}{2}(\tilde{T} + \tilde{T}^{-1})$ or $h = \sqrt{1 + \hbar^2 e^2}$. One way of interpreting (54) is to consider it a recipe for obtaining the finite dimensional $R_h$ matrices of the $U_h(osp(1|2))$ algebra. For instance, using the classical $j = 1$ representation, obtained from (47) in the $q \to 1$ limit, we obtain the $R_h^{1/2;1} = R_h^{1/2;1}$ matrix as follows:

$$
R_h^{1/2;1} = \begin{pmatrix}
1 & 0 & h & 0 & \frac{h^2}{2} & 0 & h & 0 & \frac{h^2}{2} & 0 & h^3 \\
0 & 1 & 0 & h & 0 & \frac{h^2}{2} & 0 & -2h & 0 & \frac{h^2}{2} & 0 \\
0 & 0 & 1 & 0 & h & 0 & 0 & 0 & 0 & 0 & \frac{h^2}{2} \\
0 & 0 & 0 & 1 & 0 & h & 0 & 0 & 0 & h & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

The above matrix (54) may also be interpreted as the $L$ operator of the $U_h(osp(1|2))$ algebra. To this end, we first use the following invertible map of the quantum $U_h(osp(1|2))$ algebra (45) on the classical algebra (43):

$$
E = e, \quad H = \tilde{H}, \quad F = f + \frac{h}{4} (\frac{\tilde{T}}{\tilde{T} + 1}) e - \frac{h}{2} (\frac{\tilde{T} - 1}{\tilde{T} + 1}) eh, \quad T = \tilde{T}, \quad Y = F^2.
$$

The map (56) satisfies the algebraic relations (45); and the corresponding twist operator may also be determined. Using the map (56) the operator (54) may be recast in terms of the deformed generators of the super-Jordanian $U_h(osp(1|2))$ algebra as

$$
L \equiv \tilde{R}_h^{\pm j} = \begin{pmatrix}
T & \frac{1}{\sqrt{T}} E & -h \frac{h}{4} \left(\frac{\tilde{T}}{\tilde{T} + 1}\right) e - \frac{h}{2} \left(\frac{\tilde{T} - 1}{\tilde{T} + 1}\right) eh & T^{-1} \\
0 & 1 & -h \frac{h}{4} \left(\frac{\tilde{T}}{\tilde{T} + 1}\right) e - \frac{h}{2} \left(\frac{\tilde{T} - 1}{\tilde{T} + 1}\right) eh & T^{-1} \\
0 & 0 & T^{-1} \\
\end{pmatrix}
$$

The above $L$ operator of the $U_h(osp(1|2))$ algebra has not been obtained before. It allows immediate construction of the full Hopf structure of the Borel subalgebra of the $U_h(osp(1|2))$ algebra via the standard FRT formalism [36]. The algebraic relations for the generators $(H, E, T \pm 1)$ of the Borel subalgebra is given by

$$
R_h^{1/2;1} L_1 L_2 = L_2 L_1 R_h^{1/2;1},
$$

where $\mathbb{Z}_2$ graded tensor product has been used in defining the operators: $L_1 = L \otimes 1, L_2 = 1 \otimes L$. The coalgebraic properties of the said Borel subalgebra may be succinctly expressed as

$$
\Delta(L) = L \otimes L, \quad \varepsilon(L) = 1, \quad S(L) = L^{-1}.
$$
where $L^{-1}$ is given by

$$L^{-1} = \begin{pmatrix}
T^{-1} & -\hbar T^{-1} E & \hbar H + \frac{\hbar}{2} (T - T^{-1}) \\
0 & 1 & \hbar T^{-1} E \\
0 & 0 & T
\end{pmatrix}.$$

(60)

This completes our construction of the Hopf structure of the Borel subalgebra of the super-Jordanian $U_h(osp(1|2))$ algebra, obtained by deforming the $r_2$ matrix, by employing the contraction scheme described earlier. Our results fully coincide with the Hopf structure given in (45). This validates our contraction scheme elaborated earlier. Our recipe (50) for obtaining the $R_{ij}^{kl}$ matrix for a $j_1 \otimes j_2$ representation of the super-Jordanian $U_h(osp(1|2))$ algebra may be continued arbitrarily. The matrices such as $R_{ij}^{kl}$ may be interpreted as higher dimensional $L$ operators [37] obeying duality relations with relevant $T$ matrices.

### 3.2 Jordanian $U_h(osp(1|2))$ algebra: a nonlinear realization

The classical $r_1$ matrix has been quantized earlier [31] using the inclusion $sl(2) \subset osp(1|2)$. These authors have expressed the resultant triangular deformed $osp(1|2)$ superalgebra in terms of the classical basis set. On the other hand Ohn [4] employed a nonlinear basis set to formulate the deformed Jordanian $U_h(sl(2))$ algebra. Consequently, Ohn’s $U_h(sl(2))$ algebra [4] do not directly appear as a Hopf subalgebra of the deformed $osp(1|2)$ superalgebra considered in Ref. [31]. Moreover, if the algebraic relations are described in terms of the undeformed classical basis set, the coproduct structures tend to be complicated in nature.

In the following, we present the quantized Hopf structure corresponding to the classical $r_1$ matrix in terms of nonlinear basis elements. Our algebra explicitly includes Ohn’s $U_h(sl(2))$ algebra as a Hopf subalgebra. The coproduct structure we obtain is considerably simple. To distinguish the Jordanian deformed $osp(1|2)$ superalgebra, its generators and the deformation parameter considered here from the corresponding objects displayed above, we express them in boldfaced notations. The Jordanian $U_h(osp(1|2))$ algebra corresponding to the $r_1$ matrix is generated by the elements $(H, E, F, X, Y)$. Their classical analogs are $(h, e, f, b_\pm)$ respectively. The elements $T^{\pm 1} = \exp(\pm \hbar X)$ are also introduced. The deformation parameter is denoted by $\hbar$. The Hopf structure of the $U_h(osp(1|2))$ algebra is obtained by maintaining the following properties: (i) In the classical limit the quantum coproduct map conforms to the classical cocommutator. (ii) The coproduct map is a homomorphism of the algebra, and it satisfies the coassociativity constraint. (iii) Generator $X$ is the only primitive element. The commutation relations of the $U_h(osp(1|2))$ algebra reads [23]

\[
\begin{align*}
[H, E] &= \frac{1}{2}(T + T^{-1})E, \\
[H, F] &= -\frac{1}{4}(T + T^{-1})F - \frac{1}{4}F(T + T^{-1}) - \frac{\hbar}{8}((T - T^{-1})H + H(T - T^{-1}))E \\
&\quad - \frac{\hbar}{8}E((T - T^{-1})H + H(T - T^{-1))), \\
\{E, F\} &= -\frac{1}{4}(T + T^{-1})H - \frac{1}{4}H(T + T^{-1}), \\
[H, Y] &= \frac{1}{2}(T + T^{-1})Y - \frac{1}{2}Y(T + T^{-1}), \\
[T^{\pm 1}, Y] &= \pm \frac{\hbar}{2}(T^{\pm 1}H + HT^{\pm 1}), \\
E^2 &= \frac{1}{2\hbar}(T - T^{-1}), \\
[Y, E] &= F, \\
[T^{\pm 1}, F] &= \pm \frac{\hbar}{2}(T^{\pm 2} + 1)E, \\
F^2 &= -Y + \frac{\hbar}{8}(T - T^{-1})H^2 + \frac{\hbar}{4}(T - T^{-1})EY + \frac{3\hbar}{16}(T^2 - T^{-2})H + \frac{\hbar}{4}(T - T^{-1}) \\
&\quad + \frac{9\hbar}{128}(T - T^{-1})^3, \\
[F, Y] &= \frac{\hbar}{4}(T - T^{-1})F + \frac{\hbar}{2}(T - T^{-1})EY - \frac{\hbar^2}{4}EH^2 - \frac{3\hbar^2}{8}(T + T^{-1})EH - \frac{\hbar^2}{2}E.
\end{align*}
\]
and the corresponding coalgebraic structure is given by [23]

\[
\begin{align*}
\Delta(H) &= H \otimes T + T^{-1} \otimes H, \\
\Delta(F) &= F \otimes T^{1/2} + T^{-1/2} \otimes F + \frac{\hbar}{4} T^{-1} E \otimes \left( T^{-1/2} H + HT^{-1/2} \right) - \frac{\hbar}{4} \left( T^{1/2} H + HT^{1/2} \right) \otimes TE, \\
\Delta(T^{\pm 1}) &= T^{\pm 1} \otimes T^{\pm 1}, \\
\Delta(Y) &= Y \otimes T + T^{-1} \otimes Y, \\
\varepsilon(H) &= \varepsilon(E) = \varepsilon(F) = \varepsilon(Y) = 0, \\
S(H) &= -H + 2\hbar E^2, \\
S(E) &= -E, \\
S(F) &= -F - \frac{\hbar}{2} \left( T + T^{-1} \right) E, \\
S(T^{\pm 1}) &= T^{\mp 1}, \\
\end{align*}
\]

(62)

All the Hopf algebra axioms can be verified by direct calculation. The universal \( R_h \) matrix of the Jordanian \( U_h(osp(1|2)) \) algebra is of the factorized form [26]:

\[
R_h = G_{21}^{-1} G, \quad G = \exp \left( \hbar H T \otimes X \right),
\]

(63)

which coincides with the universal \( \mathcal{R}_h \) matrix of the \( U_h(sl(2)) \) subalgebra [26] involving the highest weight root vector.

Before discussing the general structure of a class of invertible maps of the \( U_h(osp(1|2)) \) algebra on the classical \( \mathcal{U}(osp(1|2)) \) algebra, we notice that the comultiplication map of a set of three operators

\[
T^{-1/2} E, \quad TH, \quad T^{1/2} F + \frac{\hbar}{8} T^{1/2} (T - T^{-1}) E - \frac{\hbar}{2} T^{1/2} EH,
\]

(64)

when acted by the twist operator corresponding to the factorized form of the universal \( R_h \) matrix given in (63), reduce to the classical cocommutative coproduct:

\[
G \Delta(X) G^{-1} = X \otimes I + I \otimes X,
\]

(65)

where \( X \) is an element of the set (64). From the commutation rules (61) it becomes evident that the operators in the set (64) satisfy the classical algebra generated by \( (e, h, f) \) respectively. These operators constitute an important special case of a class of maps discussed below.

To construct a class of maps interrelating the Jordanian \( U_h(osp(1|2)) \) algebra, obtained via quantization of the \( r_1 \)-matrix, and its classical analog \( \mathcal{U}(osp(1|2)) \) we proceed via an ansatz as follows:

\[
E = \varphi_1(b_+) e, \quad H = \varphi_2(b_+) h, \quad F = \varphi_3(b_+) f + u_1(b_+) e + u_2(b_+) eh,
\]

(66)

where the ‘mapping functions’ \( (\varphi_1, \varphi_2, \varphi_3; u_1, u_2) \) depend only on the classical generator \( b_+ \). In the classical limit \( \hbar \to 0 \) the above functions satisfy on the property: \( (\varphi_1, \varphi_2, \varphi_3; u_1, u_2) \to (1, 1, 1, 0, 0) \). The operators \( T^{\pm 1} \) may now be expressed as

\[
T^{\pm 1} = \pm \hbar b_+ (\varphi_1(b_+))^2 + \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^2}. \]

(67)

Substituting the ansatz (66) in the defining relations (61) for the \( U_h(osp(1|2)) \) algebra we, for a given function \( \varphi_1 \), obtain a set of six nonlinear equations for four unknown functions:

\[
\begin{align*}
(\varphi_1(b_+) + 2b_+ \varphi_1'(b_+)) \varphi_2(b_+) - \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \varphi_1(b_+) &= 0, \\
2b_+ \varphi_2(b_+ \varphi_3(b_+) - \varphi_2(b_+)) \varphi_3(b_+) + \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \varphi_3(b_+) &= 0, \\
2b_+ \varphi_2(b_+) u_1'(b_+) + \left( \varphi_2(b_+) + \sqrt{1 + \hbar^2 b_+^2 (\varphi_2(b_+))^4} \right) u_1(b_+) \\
+ \hbar^2 b_+ \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^2} (\varphi_1(b_+))^3 &= 0,
\end{align*}
\]
The classical (∆₀) cocommutative and the quantum (∆) non-cocommutative coproducts are related by the twisting element as

\[ G \Delta \circ m^{-1}(\phi) G^{-1} = (m^{-1} \otimes m^{-1}) \circ \Delta_0(\phi) \quad \forall \phi \in U(osp(1|2)), \]  

where the twisting element \( G \in U_h(osp(1|2)) \) \( \otimes 2 \) satisfies the cocycle condition

\[ (G \otimes 1) ((\Delta \otimes id)G) = (I \otimes G) ((id \otimes \Delta)G). \]
Similarly the classical ($S_0$) and the quantum ($S$) antipode maps are related as follows:

$$g S \circ m^{-1}(\phi) g^{-1} = m^{-1} \circ S_0(\phi), \quad g \in U_h(osp(1|2)).$$  \hfill (75)

The transforming operator $g$ for the antipode map may be expressed in terms of the twist operator $G$ as

$$g = \mu \circ (id \otimes S)G,$$  \hfill (76)

where $\mu$ is the multiplication map.

The first example belonging to the class of invertible maps discussed earlier plays a key role in the present construction. With the choice

$$\varphi_1(b_+) = (1 - 2hb_+)^{-1/4}, \quad \psi_1(T) = T^{-1/2},$$  \hfill (77)

we obtain, after implementing (66, 69, 70) and (72), the following direct map

$$E = (1 - 2hb_+)^{-1/4} e, \quad H = \sqrt{(1 - 2hb_+)} h, \quad T^\pm_1 = (1 - 2hb_+)^{\pm 1/2},$$

$$F = (1 - 2hb_+)^{1/4} f - \frac{h^2}{4} b_+(1 - 2hb_+)^{-3/4} e + \frac{h}{2} (1 - 2hb_+)^{1/4} eh$$  \hfill (78)

and its inverse

$$e = T^{-1/2}E, \quad h = TH, \quad f = T^{1/2}F + \frac{h}{8} T^{1/2}(T - T^{-1})E - \frac{h}{2} T^{1/2}EH.$$  \hfill (79)

It may be observed from (65) that the operator $G$ corresponding to the factorized form of the universal $R_h$ matrix given in (63) plays the role of the twist operator $G$ for the map (78) and its inverse. In this sense we refer to it as the 'minimal twist map'. The operator $g$ transforming the antipode map may be, à la (75), explicitly evaluated in a closed form:

$$g = \exp \left( -\frac{1}{2} TH(1 - T^{-2}) \right).$$  \hfill (80)

Combining (80) with the property (76) we now immediately obtain a disentanglement relation, which, if expressed in terms of classical generators, reads as follows:

$$\mu \left[ \exp \left( \frac{1}{2} h \otimes \ln(1 - 2hb_+) \right) \right] = \exp(-hhb_+).$$  \hfill (81)

In the above relation $h$ may be treated as an arbitrary parameter. To our knowledge the above disentanglement formula involving the classical $sl(2)$ generators was not observed before.

Another map, where the Cartan element $H$ of the Jordanian $U_h(osp(1|2))$ algebra remains diagonal, is given by 'mapping functions'

$$\varphi_1(b_+) = \frac{1}{\sqrt{1 - \frac{h^2b_+^2}{4}}}, \quad \psi_1(T) = \text{sech}(\frac{hX}{2}).$$  \hfill (82)

Proceeding as in the previous example, we now map the $U_h(osp(1|2))$ algebra on the classical $U(osp(1|2))$ algebra as

$$E = \frac{1}{\sqrt{1 - \frac{h^2b_+^2}{4}}} e, \quad H = h, \quad T^\pm_1 = \frac{1 \pm \frac{hb_+}{2}}{1 \mp \frac{hb_+}{2}},$$

$$F = \sqrt{1 - \frac{h^2b_+^2}{4}} f - \frac{h^2b_+}{4} \left( \frac{1 - \frac{h^2b_+^2}{4}}{4} \right)^{1/2} e - \frac{h^2b_+}{4} \sqrt{1 - \frac{h^2b_+^2}{4}} eh.$$  \hfill (83)
The inverse map now reads

$$e = \text{sech}\left(\frac{hX}{2}\right)E, \quad h = H, \quad f = \cosh\left(\frac{hX}{2}\right)F + \frac{h}{4}\sinh(hX)\cosh\left(\frac{hX}{2}\right)E + \frac{h}{2}\sinh\left(\frac{hX}{2}\right)EH. \quad (84)$$

The twist operator $G$ for the map (83), unlike the previous example of closed-form expression for the ‘minimal twist map’ given in (65), may be determined only in a series:

$$G = I \otimes I + \frac{h}{2}r + \frac{h^2}{8}(r^2 + H \otimes X^2 + X^2 \otimes H) + O(h^3), \quad (85)$$

where $r = H \otimes X - X \otimes H$. The corresponding transforming operator for the antipode map is determined from (75):

$$g = - hX + \frac{1}{2}h^2X^2 + O(h^3). \quad (86)$$

Other suitable maps belonging to the class described before may be obtained by just choosing sufficiently well-behaved, but otherwise arbitrary, ‘mapping functions’ $\varphi (b_+)$ and its inverse $\psi (T)$. Our formalism given in (69) and (72) readily produces solutions for the other ‘mapping functions’ listed in (66) and (70). These maps may be used to immediately generate representations of the Jordanian $U_h(osp(1|2))$ algebra.

4 The Nonstandard superalgebra Enveloping $U_h(sl(2|1))$

The main object of this section is to extend our construction to $U(sl(2|1))$ enveloping superalgebras. Let us just recall the more important points concerning $sl(2|1)$: The superalgebra $sl(1|2)$ is generated by six generators \{h_1, h_2, h_3, e_1, f_1, e_2, f_2, e_3, f_3\} and the commutation relations

$$\begin{align*}
[h_1, h_2] &= 0, \quad (a) \\
[h_1, h_3] &= 0, \\
[h_1, e_1] &= 2e_1, \\
[h_1, e_2] &= -2f_1, \\
[h_1, e_3] &= -e_3, \\
[h_1, f_1] &= f_1, \\
[h_1, f_2] &= -f_2, \\
[h_1, f_3] &= -f_3, \\
[h_2, e_1] &= -e_1, \\
[h_2, e_2] &= e_2, \\
[h_2, e_3] &= e_3, \\
[h_2, f_1] &= f_1, \\
[h_2, f_2] &= f_2, \\
[h_2, f_3] &= f_3, \\
[h_3, e_1] &= e_1, \\
[h_3, e_2] &= -e_2, \\
[h_3, e_3] &= -e_3, \\
[h_3, f_1] &= -f_1, \\
[h_3, f_2] &= f_2, \\
[h_3, f_3] &= 0, \\
[e_1, f_1] &= h_1, \\
[e_2, f_2] &= h_2, \\
[e_3, f_3] &= h_3, \\
[e_1, f_2] &= [e_2, f_1] = 0, \\
[e_1, e_2] &= e_3, \\
[e_1, e_3] &= e_2, \\
[e_2, e_3] &= e_1, \\
[e_3, f_2] &= f_3, \\
[f_1, e_3] &= e_2, \\
[f_2, e_3] &= f_1, \\
[f_3, e_1] &= f_2, \\
[f_3, e_2] &= f_1, \\
[f_3, e_3] &= e_1, \\
[f_3, f_1] &= 0, \\
[f_3, f_2] &= 0, \\
[f_3, f_3] &= 0, \\
&= 0, \\
&= 0, \\
&= 0, \\
&= 0, \\
&= 0, \\
&= 0. \\
&= 0. \\
&= 0. \\
&= 0. \\
&= 0.
\end{align*} \quad (87)$$

where the commutator $[,]$ is understood as the $\mathbb{Z}_2$-graded one: $[a, b] = ab - (-)^{\text{deg}(a)\text{deg}(b)}ba$. The generators $h_1, h_2, h_3, e_1$ and $f_1$ are even ($\text{deg}(h_1) = \text{deg}(h_2) = \text{deg}(h_3) = \text{deg}(e_1) = \text{deg}(f_1) = 0$), while $e_2, f_2, e_3$ and $f_3$ are odd, ($\text{deg}(e_2) = \text{deg}(f_2) = \text{deg}(e_3) = \text{deg}(f_3) = 1$). As a Hopf superalgebra, the universal enveloping $U(sl(2|1))$ of $sl(2|1)$ is generated just by six elements: it is sufficient to start from $\{h_1, h_2, e_1, f_1, e_2, f_3\}$ restricted by relations (a), (b), (c), (d), (e), (f), (g) only, and

$$e_1^2 e_2 - 2e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad f_1^2 f_2 - 2f_1 f_2 f_1 + f_2 f_1^2 = 0. \quad (88)$$

The two last equations are called the Serre relations. Let us just mention that there is a $\mathbb{C}$-algebra automorphism $\phi$ of $U(sl(2|1))$ such that

$$\phi (e_1) = e_1, \quad \phi (f_1) = f_1, \quad \phi (h_1) = h_1, \quad \phi (e_2) = f_3, \quad \phi (f_2) = -e_3, \quad \phi (h_2) = -h_3, \quad \phi (e_3) = -f_2, \quad \phi (f_3) = e_2, \quad \phi (h_3) = -h_2. \quad (89)$$

18
The quasitriangular quantum Hopf superalgebra \( \mathcal{U}_q(sl(2|1)) \) \((q\) is an arbitrary complex number\), by analogy with \( \mathcal{U}(sl(2|1)) \), is generated by six elements \( \{\hat{h}_1, \hat{h}_2, \hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2\} \) under the relations

\[
\begin{align*}
[\hat{h}_1, \hat{h}_2] &= 0, \\
[\hat{h}_1, \hat{e}_1] &= 2\hat{e}_1, \\
[\hat{h}_1, \hat{e}_2] &= -\hat{e}_2, \\
[\hat{h}_2, \hat{e}_1] &= -\hat{e}_1, \\
[\hat{h}_2, \hat{e}_2] &= 0, \\
[\hat{e}_1, \hat{f}_2] &= [\hat{e}_2, \hat{f}_1] = 0, \\
\hat{e}_2^2 - (q + q^{-1})\hat{e}_1\hat{e}_2 + 2\hat{e}_1\hat{e}_1^2 &= 0, \\
\hat{e}_2^2 - (q + q^{-1})\hat{e}_1\hat{e}_2 + \hat{e}_2\hat{e}_1^2 &= 0, \\
\hat{f}_2^2 - (q + q^{-1})\hat{f}_1\hat{f}_2 + \hat{f}_1\hat{f}_1\hat{f}_2 &= 0,
\end{align*}
\]

where \( \deg(\hat{e}_2) = \deg(\hat{f}_2) = 1 \) and \( \deg(\hat{h}_1) = \deg(\hat{h}_2) = \deg(\hat{e}_1) = \deg(\hat{f}_1) = 0 \). It is simple to note that \( \mathcal{U}_q(sl(2)) \subset \mathcal{U}_q(sl(2|1)) \). The coproducts are defined by

\[
\begin{align*}
\Delta(\hat{h}_1) &= \hat{h}_1 \otimes 1 + 1 \otimes \hat{h}_1, \\
\Delta(\hat{e}_1) &= \hat{e}_1 \otimes q^{\hat{h}_1/2} + q^{-\hat{h}_1/2} \otimes \hat{e}_1, \\
\Delta(\hat{f}_1) &= \hat{f}_1 \otimes q^{\hat{h}_1/2} + q^{-\hat{h}_1/2} \otimes \hat{f}_1, \\
\Delta(\hat{h}_2) &= \hat{h}_2 \otimes 1 + 1 \otimes \hat{h}_2, \\
\Delta(\hat{e}_2) &= \hat{e}_2 \otimes q^{\hat{h}_2/2} + q^{-\hat{h}_2/2} \otimes \hat{e}_2, \\
\Delta(\hat{f}_2) &= \hat{f}_2 \otimes q^{\hat{h}_2/2} + q^{-\hat{h}_2/2} \otimes \hat{f}_2.
\end{align*}
\]

The universal \( \mathcal{R} \)-matrix is given in ref. [27]. Note that the definition of the Hopf superalgebra differs from that of the usual Hopf algebra by the supercommutativity of the tensor product, i.e. \((a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)}(ac \otimes bd)\). For later use, we note that the fundamental representation of the algebra \((7)\) is spanned by

\[
\begin{align*}
\hat{h}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{e}_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{f}_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\hat{h}_2 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \hat{e}_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \hat{f}_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Following the technique developed above, the \( \mathcal{R}_h \)-matrix of the super-Jordanian quantum superalgebra \( \mathcal{U}_h(sl(2|1)) \), for arbitrary representations in the two tensor product sectors, can be also obtained from the \( \mathcal{R}_q \)-matrix associated with the Drinfeld-Jimbo quantum superalgebra \( \mathcal{U}_q(sl(2|1)) \) through a specific contraction. For simplicity and brevity, let us start with (fundamental irrep.) \( \otimes \) (fundamental irrep.). The \( \mathcal{R}_q \)-matrix of \( \mathcal{U}_q(sl(2|1)) \) superalgebra in the \( (\text{fund.}) \otimes (\text{fund.}) \) representation reads

\[
R_q = \begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & q^{-1} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & q^{-1} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & q \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & q^{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-2}
\end{pmatrix}.
\]
The non-standard $R_h$-matrix in the $\text{(fund.)} \otimes \text{(fund.)}$ representation is obtained, from (13), in the following manner:

$$ R_h = \lim_{q \rightarrow 1} \left[ E_q^{-1} \left( \frac{h \hat{e}_1}{q-1} \right)_{\text{fund.}} \otimes E_q^{-1} \left( \frac{h \hat{e}_1}{q-1} \right)_{\text{fund.}} \right] R_q \left[ E_q \left( \frac{h \hat{e}_1}{q-1} \right)_{\text{fund.}} \otimes E_q \left( \frac{h \hat{e}_1}{q-1} \right)_{\text{fund.}} \right] $$

$$ = \begin{pmatrix}
1 & h & 0 & -h & h^2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & h & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -h & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad (94) $$

Similarly, using a Maple program*, we obtain, for $\text{(fundamental irrep.)} \otimes \text{(arbitrary irrep.)}$, the following expression:

$$ R_h = \begin{pmatrix}
T & -hH_1 + \frac{h}{2} (T - T^{-1}) & 0 \\
0 & T^{-1} & 0 \\
0 & 0 & I
\end{pmatrix}, \quad (95) $$

where

$$ T^{\pm 1} = \pm h e_1 + \sqrt{1 + h^2 e_1^2}, \quad H_1 = \frac{1}{2} \left( T + T^{-1} \right) h_1 = \sqrt{1 + h^2 e_1^2} h_1. \quad (96) $$

It is easy to verify that

$$ TT^{-1} = T^{-1} T = 1, \quad [H_1, T^{\pm 1}] = T^{\pm 2} - 1. \quad (97) $$

We note that the contraction scheme, which comprises our transformation and limiting procedure has furnished the $R_h$-matrix along with a nonlinear map of the $h$-Borel subalgebra on its classical counterpart. Following ref. [21], let us introduce the generator

$$ F_1 = f_1 - \frac{h^2}{4} e_1 \left( h_1^2 - 1 \right). \quad (98) $$

We then show that

$$ [T^{\pm 1}, F_1] = \frac{h}{2} \left( H_1 T^{\pm 1} + T^{\pm 1} H_1 \right), \quad [H_1, F_1] = -\frac{1}{2} \left( T F_1 + F_1 T + T^{-1} F_1 + F_1 T^{-1} \right). \quad (99) $$

The coproducts $\Delta$, the counit $\epsilon$ and the antipode $S$ of $\{H, T, T^{-1}, Y\}$ read [4]

$$ \Delta(H_1) = H_1 \otimes T + T^{-1} \otimes H_1, \quad \Delta(T^{\pm 1}) = T^{\pm 1} \otimes T^{\pm 1}, \quad \Delta(F_1) = F_1 \otimes T + T^{-1} \otimes F_1, $$

$$ S(H_1) = -T H_1 T^{-1}, \quad S(T^{\pm 1}) = T^{\mp 1}, \quad S(F_1) = -T F_1 T^{-1}, $$

$$ \epsilon(H_1) = \epsilon(F_1) = 0, \quad \epsilon(T^{\pm 1}) = 1. \quad (100) $$

*Our program was performed for $\text{(fund.)} \otimes \text{(fund.)}, \text{(fund.)}, \text{etc.}$. 

20
This implies that the Ohn’s structure follows from the bosonic generators \( \{ h_1, e_1, f_1 \} \). The algebraic properties (99) exhibits clearly the embedding of \( U_h(sl(2)) \) in \( U_h(sl(2|1)) \). At this level it is natural to ask the following question. How to complete the \( U_h(sl(2|1)) \) superalgebra? To complete now the \( U_h(sl(2|1)) \) superalgebra, we introduce the following \( h \)-deformed fermionic root generator:

\[
\begin{align*}
H_2 &= h_2 - \frac{h^2}{2}e_1^2 h_1, \\
E_2 &= e_2 - \frac{h^2}{4}e_1 e_3 (2h_1 + 1), \\
F_2 &= f_2, \\
H_3 &= h_3 + \frac{h^2}{2}e_1^2 h_1, \\
E_3 &= e_3, \\
F_3 &= f_3 + \frac{h^2}{4}e_1 f_2 (2h_1 + 1)
\end{align*}
\]  

(101)

Let us note that there exist a \( \mathbb{C} \)-algebra automorphism of \( U_h(sl(2|1)) \) such that

\[
\begin{align*}
\Phi (T^{\pm 1}) &= T^{\pm 1}, \\
\Phi (F_1) &= F_1, \\
\Phi (H_1) &= H_1, \\
\Phi (E_2) &= F_3, \\
\Phi (F_2) &= -E_3, \\
\Phi (H_2) &= -H_3, \\
\Phi (E_3) &= -f_2, \\
\Phi (F_3) &= E_2, \\
\Phi (H_3) &= -H_2.
\end{align*}
\]

(102)

(For \( h = 0 \), this automorphism reduces to the classical one described by (89). The expressions (96), (97), (98) and (101) define a realisation of the super-jordanian subalgebra \( U_h(sl(2|1)) \) with the classical generators via a nonlinear map. Other invertible maps relating the super-jordanian and the classical generators may also be considered. Our construction leads to the following results (We have quoted here only the final results): The nonstandard (super-jordanian) enveloping superalgebra \( U_h(sl(2|1)) \) is an associative superalgebra over \( \mathbb{C} \) generated by \( \{ H_1, T, T^{-1}, F_1, H_2, E_2, F_2, H_3, E_3, F_3 \} \), along with (99), the commutation relations [24]

\[
\begin{align*}
[H_1, H_2] &= -\frac{1}{4} (T - T^{-1}) H_1, \\
[H_1, H_3] &= \frac{1}{4} (T - T^{-1}) H_1, \\
[H_2, H_3] &= 0, \\
[H_1, E_2] &= -\frac{1}{2} E_2 (T + T^{-1}) - \frac{h}{4} (H_1 (T - T^{-1}) + (T - T^{-1}) H_1) E_3, \\
[H_1, F_2] &= \frac{1}{2} (T + T^{-1}) F_2, \\
[H_1, E_3] &= \frac{1}{2} (T + T^{-1}) E_3, \\
[H_2, F_3] &= \frac{1}{2} F_3 (T + T^{-1}) + \frac{h}{4} (H_1 (T - T^{-1}) + (T - T^{-1}) H_1) F_2, \\
[H_2, T] &= -\frac{1}{4} (T^3 - T^{-1}), \\
[H_2, F_1] &= \frac{1}{4} (T + T^{-1})^2 F_1 - \frac{h}{4} (T - T^{-1}) H_1^2 - \frac{h}{4} (T^2 - T^{-2}) H_1 - \frac{h}{16} (T^2 - T^{-2}) (T + T^{-1}), \\
[H_2, E_2] &= \frac{h}{16} (T - T^{-1}) (T^2 - T^{-2}) E_3 + \frac{1}{8} (T - T^{-1})^2 E_2, \\
[H_2, E_3] &= -\frac{1}{8} (T^2 + 6 + T^{-2}) E_3, \\
[H_2, F_3] &= \frac{1}{8} (T^2 + 6 + T^{-2}) F_3 - \frac{h}{16} (T^2 - T^{-2}) (T + T^{-1}) F_2, \\
[H_3, T] &= \frac{1}{4} (T^3 - T^{-1}), \\
[H_3, E_1] &= \frac{1}{4} (T + T^{-1})^2 F_1 + \frac{h}{4} (T - T^{-1}) H_1^2 + \frac{h}{4} (T^2 - T^{-2}) H_1 + \frac{h}{16} (T^2 - T^{-2}) (T + T^{-1}), \\
[H_3, E_2] &= -\frac{1}{8} (T^2 + 6 + T^{-2}) E_2 - \frac{h}{16} (T^2 - T^{-2}) (T + T^{-1}) E_3, \\
[H_3, F_2] &= \frac{1}{8} (T^2 + 6 + T^{-2}) F_2, \\
[H_3, E_3] &= \frac{1}{8} (T - T^{-1})^2 E_3, \\
[H_3, F_3] &= \frac{h}{16} (T - T^{-1}) (T^2 - T^{-2}) F_2 - \frac{1}{8} (T - T^{-1})^2 F_3, \\
[E_2, F_2] &= H_2 - \frac{1}{16} (T - T^{-1})^2 - \frac{h}{4} (T - T^{-1}) E_3 F_2,
\end{align*}
\]  

21
where

\[ [E_3, F_3] = H_3 + \frac{1}{16} (T - T^{-1})^2 + \frac{h}{4} (T - T^{-1}) F_2 E_3, \quad [T^{\pm 1}, F_2] = 0, \]

\[ [E_2, F_1] = \frac{h}{4} (T - T^{-1}) E_2 + \frac{h}{2} (T - T^{-1}) E_3 F_1 - \frac{h^2}{4} E_3 H_1^2 - \frac{3h^2}{8} (T + T^{-1}) E_3 H_1 - \frac{h^2}{2} E_3 - \frac{15h^2}{64} (T - T^{-1})^2 E_3, \]

\[ E_2^2 = \frac{h}{4} (T - T^{-1}) E_3 E_2, \quad F_2^2 = E_3^2 = 0, \quad F_3^2 = -\frac{h}{4} (T - T^{-1}) F_2 F_3, \]

\[ [T^{\pm 1}, E_3] = 0. \]

The element (105) coincides with the pure \( U_h(sl(2|1)) \) universal \( \mathcal{R}_h \)-matrix of \( U_h(sl(2|1)) \) admits a Hopf structure with coproducts, antipodes and counits determined by (100) and [24]

\[
\begin{align*}
\Delta (E_2) &= E_2 \otimes T^{1/2} + T^{-1/2} \otimes E_2 + \frac{h}{4} T^{-1} E_3 \otimes \left( T^{-1/2} H_1 + H_1 T^{-1/2} \right) - \frac{h}{4} \left( T^{1/2} H_1 + H_1 T^{1/2} \right) \otimes T E_3, \\
\Delta (F_2) &= F_2 \otimes T^{-1/2} + T^{1/2} \otimes F_2, \\
\Delta (F_3) &= F_3 \otimes T^{1/2} + T^{-1/2} \otimes F_3 - \frac{h}{4} T^{-1} F_2 \otimes \left( T^{-1/2} H_1 + H_1 T^{-1/2} \right) + \frac{h}{4} \left( T^{1/2} H_1 + H_1 T^{1/2} \right) \otimes T F_2, \\
\Delta (H_2) &= H_2 \otimes 1 + 1 \otimes H_2 + \frac{1}{4} T H_1 \otimes \left( 1 - T^{-2} \right) + \frac{1}{4} \left( 1 - T^{-2} \right) \otimes T^{-1} H_1, \\
\Delta (H_3) &= H_2 \otimes 1 + 1 \otimes H_2 - \frac{1}{4} T H_1 \otimes \left( 1 - T^{-2} \right) - \frac{1}{4} \left( 1 - T^{-2} \right) \otimes T^{-1} H_1, \\
S (E_2) &= -E_2 - \frac{h}{2} \left( T + T^{-1} \right) E_3, \\
S (F_2) &= -F_2, \\
S (F_3) &= -F_3 + \frac{h}{2} \left( T + T^{-1} \right) E_3, \\
S (H_2) &= -H_2 + \frac{1}{2} \left( T^{-2} - 1 \right), \\
S (H_3) &= -H_3 + \frac{1}{2} \left( T^{-2} - 1 \right), \\
\epsilon (H_2) &= \epsilon (H_3) = \epsilon (E_2) = \epsilon (F_2) = \epsilon (F_3) = 0.
\end{align*}
\]

All the Hopf superalgebra axioms can be verified by direct calculations. The universal \( \mathcal{R}_h \)-matrix of \( U_h(sl(2|1)) \) has the following form [24]:

\[
\mathcal{R}_h = \exp \left( -h X_1 \otimes T H_1 \right) \exp \left( h T H_1 \otimes X_1 \right),
\]

where \( X_1 = h^{-1} \ln T \). The element (105) coincides with the pure \( U_h(sl(2)) \) universal \( \mathcal{R}_h \)-matrix [26].
5 Conclusion

In general, a class of nonlinear invertible maps exists relating the Jordanian quantum algebras and their classical analogues. Here we have used a particular maps realizing Jordanian $U_h(sl(2))$, $U_h(sl(3))$, $U_h(sl(N))$, $U_h(osp(2|1))$ and $U_h(sl(2|1))$. As a result a result of choice of the basis, via the maps described earlier, the algebraic commutations relations are deformed. On benifict of our procedure is that our expressions of the coalgebraic structure are considerably simpler than those found elsewhere [8-20].

Comment: Talk given by MB. ZAHAF to the Seventh Constantine High Energy Physics School (Theoretical Physics ans Cosmology), 3-7 April 2004, Constantine (Algeria).

References

[1] Drinfeld V G 1986 Quantum groups Proc. Int. Congress of Mathematicians (Berkeley, CA) vol 1 (New York: Academic) p 798.
[2] Jimbo M 1985 Lett. Math. Phys. 10 63.
[3] Demidov E E, Manin Yu I, Mukhin EE and Zhdanovich D Z 1990 Prog. Theor. Phys. Suppl. 102 203
[4] Ohn Ch 1992 Lett. Math. Phys. 25 85.
[5] Drinfeld V G (1990) Leningrad Math. J. 1 1419.
[6] Ogievetsky O V 1993 Suppl. Rendiconti Cir. Math. Palermo, serie II 37 4569.
[7] Gerstenhaber M, Giaquinto A and Schack S D 1993 Israel MMath. Conf. Proc. 7 45.
[8] Kulish P P, Lyakhovsky V D and Mudrov A I 1999 J. Math. Phys. 240 4569.
[9] Lyakhovsky V D and Del Olmo M A 1998 Peripheric extended twists Preprints math.QA/9903065
[10] Lyakhovsky V D and Del Olmo M A 1999 Extended and Reshetikhin twist for sl(3) Preprints math.QA/9811153
[11] Lyakhovsky V D and Samsonov M E 2001 Elementary parabolic twist Preprints math.QA/0107034
[12] Lyakhovsky V D, Mirolubov A M and and Del Olmo M A 2000 Quantum Jordanian twist Preprints math. QA/0010198
[13] Ananikian D N, Kulish P P and Lyakhovsky V D 2000 Chains of twists for symplectic Lie algebras Preprint math.QA/0010312.
[14] Kulish P P, Lyakhovsky V D and Del Olmo M A 1999 Chains of twist for classical Lie algebras Preprint math.QA/9908601
[15] Kulish P P and Lyakhovsky V D 2000 Jordanian twists on deformed carrier subspaces Preprint math.QA/0007182
[16] Kulish P P, Lyakhovsky V D and Stolin A 2000 Full chains of twists for orthogonal algebras Preprint math.QA/0007182
[17] Lyakhovsky V D, Stolin A and Kulish P P, 2000 Chains of Frobenius subalgebras of so(M) and the corresponding twists Preprint math.QA/0010147
[18] Celeghini E and Kulish P P 1998 *J. Phys. A: Math. Gen.* **31** L79.

[19] Kulish P P *Super-Jordanian deformation of the orthosymplectic Lie superalgebras* preprint math.QA/9806104.

[20] Aizawa N, Chakrabarti R and Segar J, *Mod. Phys. Lett.* **A18** (2003) 885.

[21] Abdesselam B, Chakrabarti A and Chakrabarti R 1998 *Mod. Phys. Lett.* **A10** 779.

[22] Abdesselam B, Chakrabarti A and Chakrabarti R 2002 *J. Phys. A: Math. Gen.* **35** 3231.

[23] Abdesselam B, Chakrabarti R, Hazzab A and Yanallah A, *On nonstandard quantizations of osp(1|2) superalgebra via contraction and mapping*, preprint math.QA/0309414.

[24] Abdesselam B, Chakrabarti R, Yanallah A and Zahaf M B, *A Nonlinear realisation of the super(Jordanian) quantum (super)algebra $\mathcal{U}_h(sl(2|1))$: Part I*, preprint to be submitted.

[25] Abdesselam B, Chakrabarti R, Yanallah A and Zahaf M B, *A Nonlinear realisation of the super(Jordanian) quantum (super)algebra $\mathcal{U}_h(sl(N|1))$: Part II*, preprint to be submitted.

[26] Ballesteros A and Herranz F J (1996), *J. Phys. A* **29** L311.

[27] Majid S *Foundations of quantum group Theory*, (Cambridge Univ. Press, 1995).

[28] Juszczak C and Sobczyk JT, *J. Math. Phys.* **39** (1998) 4982.

[29] Kulish PP and Reshetikhin N.Yu, *Lett. Math. Phys.* **18** (1989) 143.

[30] Saleur H, *Nucl. Phys.* **B336** (1990) 363.

[31] Celeghini E and Kulish PP, *J. Phys. A: Math. Gen.* **31** (1998) L79.

[32] Aizawa N, Chakrabarti R and Segar J, *Mod. Phys. Lett.* **A18** (2003) 885.

[33] Borowiec A, Lukierski J and Tolstoy VN, *Mod. Phys. Lett.* **A18** (2003) 1157.

[34] Kulish PP, *Super-Jordanian deformation of the orthosymplectic Lie superalgebras*, math.QA/9806104.

[35] Ballesteros A and Herranz FJ, *J. Phys. A: Math. Gen.* **32** (1999) 8841.

[36] Reshetikhin N. Yu, Takhtajan LA and Faddeev LD, *Leningrad Math. J.* **1** (1990) 193.

[37] Van der Jeugt J and Jagannathan R, *Czech J. Phys.* **46** (1996) 269.