KAWAMATA-VIEHWEG VANISHING FAILS FOR LOG DEL PEZZO SURFACES IN CHAR. 3

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Abstract. We construct a log del Pezzo surface in characteristic 3 violating the Kawamata-Viehweg vanishing theorem. As a consequence we show that there exists a Kawamata log terminal threefold singularity which is not Cohen-Macaulay in characteristic 3.

1. Introduction

In characteristic zero the main technical tool used to establish the Minimal Model Program (MMP for short) is the Kawamata-Viehweg vanishing theorem. Unfortunately, vanishing theorems are known to fail in general for varieties over fields of positive characteristic and a great amount of work has been done to construct examples of pathological varieties violating them and to study their geography (see [9], [10], [21], [23]).

In this context, (log) Fano varieties over an algebraically closed field of characteristic \( p > 0 \) violating Kodaira-type vanishing theorems seem rather rare and in fact are conjectured to exist only for small primes in each dimension. As far as the author knows, only two classes of log Fano varieties violating Kawamata-Viehweg vanishing have been constructed and both are in characteristic 2 (see [18], [19] for a six-dimensional smooth example and [3] for a two-dimensional one with klt singularities) and the only positive result towards the conjecture in every dimension is [22, Theorem 1.9], where it is proven that the classical Kodaira vanishing theorem holds for the first cohomology group on a klt Fano variety \( X \) if \( p > 2 \dim X \).

In the case of surfaces, however, the situation is much better understood: it is known that Kawamata-Viehweg vanishing holds for smooth del Pezzo surfaces over an algebraically closed field (see [4, Appendix A]), for regular del Pezzo surfaces over a (possibly imperfect) field of characteristic \( p > 3 \) (see [7, Theorem 1.1]) and for log del Pezzo surfaces over fields of sufficiently large characteristic (see [5, Theorem 1.2]), although we do not know an explicit lower bound on the characteristic. In this note we generalize an example of Keel and McKernan (see [13, Section 9] and [3, Section 4]) to characteristic 3 and we show the following:

Theorem 1.1 (See Section 3). Over any algebraically closed field of characteristic 3, there exists a log del Pezzo for which the Kawamata-Viehweg vanishing theorem does not hold.

One important application of the Kawamata-Viehweg vanishing theorem for the MMP in characteristic zero is the proof that klt singularities are Cohen-Macaulay and rational (see [17, Theorem 5.22]). In characteristic \( p > 0 \), due to the failure of vanishing theorems, general cohomological properties of klt singularities are still largely unknown but, according to a local-global principle, they are expected to be strictly related to vanishing theorems for log Fano varieties.

In dimension three, the main result of [11] shows that klt threefold singularities are Cohen-Macaulay and rational in large characteristic and the main ingredient of their proof is the Kawamata-Viehweg vanishing for log del Pezzo surfaces. As for low characteristic, in [3, Theorem 1.3] the authors give an example of a klt not Cohen-Macaulay threefold in characteristic 2. Using a generalized cone construction and Theorem 1.1 we show:

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Theorem 1.2 (See Section 4). Over any algebraically closed field of characteristic 3, there exists a \( \mathbb{Q} \)-factorial klt threefold singularity which is not Cohen-Macaulay.

An open problem is whether Kawamata-Viehweg vanishing holds for log del Pezzo over fields of characteristic \( p \geq 5 \). Despite not being able to solve this problem, in the last section we present a Kodaira-type vanishing theorem for ample Cartier divisors on a log del Pezzo surfaces of characteristic \( p \geq 5 \), partially answering to a question of Cascini and Tanaka (see [4, Remark 3.2]).

Theorem 1.3 (See Theorem 5.4). Let \( X \) be a log del Pezzo surface of characteristic \( p \geq 5 \) and let \( A \) be an ample Cartier divisor. Then,

\[
H^1(X, \mathcal{O}_X(A)) = 0.
\]

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2. Preliminaries

2.1. Notation. Throughout this paper, \( k \) denotes an algebraically closed field. By variety we mean an integral scheme which is separated and of finite type over \( k \). If \( X \) is a normal variety, we denote by \( K_X \) the canonical divisor class.

Given a scheme \( X \) defined over a field \( k \) of characteristic \( p > 0 \), we denote by \( F^e : X \to X \) the absolute Frobenius morphism and for any natural number \( e > 0 \) we denote the \( e \)-th iterate of Frobenius by \( F^e \). We denote by \( W_m(k) \) the ring of Witt vectors of length \( m \).

We say that \( (X, \Delta) \) is a log pair if \( X \) is normal, \( \Delta \) is an effective \( \mathbb{Q} \)-divisor and \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. We say it is log smooth if \( X \) is smooth and \( \text{Supp}(\Delta) \) is a snc divisor. We refer to [17] and [16] for the definition of the singularities appearing in the MMP (e.g. klt, plt, lc). We say \( f : Y \to X \) is a log resolution if \( f \) is a birational morphism, \( Y \) is regular, \( \text{Exc}(f) \) has pure codimension one and the pair \((Y, \text{Supp}(f^{-1}\Delta + \text{Exc}(f)))\) is log smooth. We say that a two dimensional log pair \((X, \Delta)\) is log del Pezzo if \((X, \Delta)\) has klt singularities and 
\(- (K_X + \Delta) \) is ample.

Let \( X \) be a normal variety. Given a \( \mathbb{Q} \)-Weil divisor, we define the reflexive sheaf \( \mathcal{O}_X(D) \) which to an open subset \( U \subset X \) associates 

\[
H^0(U, \mathcal{O}_X(D)) = H^0(U, \mathcal{O}_X(\lfloor D \rfloor)) \ni \varphi \in K(X) \mid \text{div}(\varphi)|_U + D|_U \geq 0 \}
\]

If \( L \) is a reflexive sheaf of rank 1 on \( X \), there exists a Weil divisor \( D \) such that \( L \simeq \mathcal{O}_X(D) \). We denote by \( L^{[m]} \) the double dual of \( L^\otimes m \), which is isomorphic to \( \mathcal{O}_X(mD) \). By \( \text{Cl}(X) \) we denote the abelian group of \( \mathbb{Z} \)-Weil divisors modulo linear equivalence and we use the abbreviation \( \text{Cl}(X)_{\mathbb{Q}} := \text{Cl}(X) \otimes \mathbb{Q} \). We say that a variety \( X \) is \( \mathbb{Q} \)-factorial it admits an affine open covering \((U_i)_{i \in I}\) such that \( \text{Cl}(U_i)_{\mathbb{Q}} = 0 \). We denote by \( \rho(X) := \rho(X/\text{Spec } k) \) the Picard number of \( X \).

For us, an \( A_n \)-type singularity is a klt surface singularity such that its minimal resolution is a string of \( n \) smooth rational curves. Note in particular that for us an \( A_n \)-type singularity is not necessarily canonical.
2.2. Frobenius splitting. We fix an algebraically closed field $k$ of characteristic $p > 0$. For the convenience of the reader we recall the definition of $F$–splitting.

**Definition 1.** Let $X$ be a projective variety over $k$. We say that $X$ is *globally $F$-split* if for some $e > 0$ the natural map

$$\mathcal{O}_X \xrightarrow{F^e} F^e_* \mathcal{O}_X$$

splits as an homomorphism of $\mathcal{O}_X$-modules.

**Remark 1.** In the definition of $F$-splitting for a variety $X$ one can equivalently ask that for all $e > 0$ the $\mathcal{O}_X$-module homomorphism $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ splits.

Being globally $F$-split implies powerful vanishing results for the cohomology of ample divisors on $X$. We will need the following result, which is a mild generalization of [2, Theorem 1.2.9] to $\mathbb{Q}$-Cartier Weil divisors:

**Proposition 2.1.** Let $X$ be a normal Cohen-Macaulay projective variety of dimension $n$ over $k$. If $X$ is globally $F$-split, then for any ample $\mathbb{Q}$-Cartier Weil divisor $A$

$$H^i(X, \mathcal{O}_X(-A)) = 0 \text{ if } i < n.$$  

**Proof.** By Remark 1 we know that for any large $g \geq 0$ there exists a splitting:

$$\mathcal{O}_X \to F^g_* \mathcal{O}_X \to \mathcal{O}_X.$$  

Restricting to the regular locus $U$ and tensoring by $\mathcal{O}_U(-A)$ we have the following splitting:

$$\mathcal{O}_U(-A) \to F^g_* \mathcal{O}_U(-p^g A) \to \mathcal{O}_U(-A).$$

Since $X$ is a normal variety and each sheaf in the sequence is reflexive we deduce that the splitting holds on the whole $X$:

$$\mathcal{O}_X(-A) \to F^g_* \mathcal{O}_X(-p^g A) \to \mathcal{O}_X(-A).$$

Passing to cohomology, we have an injection:

$$H^i(X, \mathcal{O}_X(-A)) \hookrightarrow H^i(X, \mathcal{O}_X(-p^g A)) \text{ for all } i \geq 0.$$  

Let $m$ be the Cartier index of $A$ and let us write $m = pfh$ where $\gcd(p, h) = 1$. Then for large enough and sufficiently divisible $e$ we have that $m$ divides $p^f (p^e - 1)$ and thus

$$p^f (p^e - 1)A \text{ is Cartier.}$$  

Now consider $g = f + e$. We have

$$-p^g A = -(p^{e+f} - p^f)A - p^f A,$$

and applying Serre duality we have

$$H^i(X, \mathcal{O}_X(-p^g A)) \cong H^{n-i}(X, \mathcal{O}_X((p^{e+f} - p^f)A) \otimes \mathcal{O}_X(K_X + p^f A))^*.$$  

By choosing $e$ sufficiently large and divisible we conclude that the last cohomology group vanishes for $i < n$ by Serre vanishing criterion for ample line bundles. \hfill \Box

2.3. A cone construction for Weil $\mathbb{Q}$-Cartier divisors. In this section we recall a generalized cone construction for Weil $\mathbb{Q}$-Cartier divisors, introduced by Demazure [8]. We will use it in Section 4 to construct a klt not CM threefold singularity in characteristic 3. For this reason we show how to compute the local cohomology group at the vertex of the cone and we describe the divisor class group.

Let $T$ be a projective normal variety over a field $k$ of arbitrary characteristic and let $L$ be a Weil divisorial sheaf of rank 1 such that for some $m > 0$ the sheaf $L^{[m]}$ is an ample line bundle.

Consider the generalized cone

$$C_a(T, L) := \text{Spec}_k \sum_{m \geq 0} H^0(T, L^{[m]}).$$
The point defined by the ideal $\sum_{m \geq 1} H^0(X, L^m)$ is called the vertex of the cone and we denote it by $v$. Let us define the affine $\mathbb{A}^1$-fibration

$$Y := \text{Spec}_T \sum_{m \geq 0} L^m \xrightarrow{f} T.$$ 

The morphism $f$ comes with a natural section $s: T \to Y$ defined by the vanishing of the ideal sheaf $\sum_{m \geq 1} L^m$. We denote by $E$ the image of $s$.

We have a natural birational morphism

$$U := \text{Spec}_T \sum_{m \in \mathbb{Z}} L^m = Y \setminus E$$

is isomorphic to $C_a(T, L) \setminus v$.

We show that the local cohomology at the vertex of the cone is controlled by the cohomological behaviour of $L$:

**Proposition 2.2.** For $i \geq 2$,

$$H^i_v(C_a(T, L), \mathcal{O}_{C_a(T, L)}) \simeq \sum_{m \in \mathbb{Z}} H^{i-1}(T, L^m).$$

**Proof.** Since $C_a(T, L)$ is affine, the cohomology groups $H^i(C_a(T, L), \mathcal{O}_{C_a(T, L)})$ vanish for $i \geq 1$. Thus, by the long exact sequence in local cohomology (see [12, Chapter III, ex. 2.3]), we have

$$H^i_v(C_a(T, L), \mathcal{O}_{C_a(T, L)}) \simeq H^{i-1}(U, \mathcal{O}_U) \text{ for } i \geq 2.$$ 

Since $f$ is affine, we have

$$H^{i-1}(U, \mathcal{O}_U) = H^{i-1}(T, f_* \mathcal{O}_U) = \sum_{m \in \mathbb{Z}} H^{i-1}(T, L^m),$$

thus concluding. \qed

We recall the results of [24] to compute the divisor class group and the canonical class in our specific case. Let us fix a Weil $\mathbb{Q}$-Cartier divisor $D$ such that $L \simeq \mathcal{O}_T(D)$. We have a natural map

$$\varphi: \text{Div}(T) \to \text{Div}(C_a(T, L))$$

$$E \mapsto \sum_{m \in \mathbb{Z}} H^0(T, \mathcal{O}_T(E + mD)).$$

The map $\varphi$ descends to the divisor class groups and we have

**Proposition 2.3.** There is a short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{\theta} \text{Cl}(T) \xrightarrow{\varphi} \text{Cl}(C_a(T, L)) \to 0,$$

where $\theta(1)$ is the divisor class of $D$.

**Proof.** Apply the short exact sequence of [24, Theorem 1.6] in the special case where $D$ has integral coefficients. \qed

As for the canonical class we have the following

**Proposition 2.4.** The generalized cone $C_a(X, L)$ is $\mathbb{Q}$-Gorenstein if and only if there exists $r \in \mathbb{Q}$ such that $rK_T \sim_{\mathbb{Q}} D$.

**Proof.** We just need to understand under which conditions there exists an $m \in \mathbb{Z}$ such that the divisor $mK_{C_a(T, L)}$ is linearly equivalent to zero. By [24, Theorem 2.8] $mK_{C_a(T, L)} = \varphi(mK_T)$ and since the sequence in Proposition 2.3 is exact, we conclude that $mK_{C_a(T, L)}$ is Cartier if and only if $mK_T$ is linearly equivalent to a multiple of $D$, thus concluding. \qed
3. A LOG DEL PEZZO SURFACE VIOLATING KV VANISHING IN CHARACTERISTIC 3.

In this section and in the following we fix $k$ to be an algebraically closed field of characteristic 3. We prove Theorem 1.1 by constructing a log del Pezzo surface of Picard rank 1 not satisfying the Kawamata-Viehweg vanishing theorem.

3.1. Keel-McKernan surface in characteristic 3. In [13, Section 9], the authors construct a family of log del Pezzo surfaces in characteristic 2 violating the Bogomolov bound on the number of singular points. In [3] it was noted that their example gives various counterexamples to the Kawamata-Viehweg vanishing theorem. We adapt their construction to the characteristic 3 case.

Let us consider the smooth rational curve $C$ inside $\mathbb{P}_x^1 \times \mathbb{P}_y^1$ defined by the equation:

$$C := \{ ([x_1 : x_2], [y_1 : y_2]) \mid x_2 y_1^3 - x_1 y_2^3 = 0 \}.$$

We denote by $\pi_x: \mathbb{P}_x^1 \times \mathbb{P}_y^1 \to \mathbb{P}_x^1$ the natural projection onto the first coordinate and we say $F_p = \pi_x^{-1}(p)$ for $p \in \mathbb{P}_x^1$ is the vertical fibre over $p$.

The main property of $C$ is that the morphism $\pi_x|_C: C \to \mathbb{P}_x^1$ is the relative Frobenius morphism. Geometrically, the curve $C$ has the following “funny” property: every vertical fibre $F_p$ is a triple tangent to $C$.

Fix a closed point $p_1$ on $C$ and consider the vertical fiber $F_1$ passing through this point. Since such a fibre is a triple tangent to $C$ at the point $p_1$ we perform three successive blow-ups to separate $C$ from $F_1$. The order of the blow-ups is as follows: at each step we blow-up the intersection point of the strict transform of $F_1$ and the strict transform of $C$. After these birational modifications the strict transform of $C$ and $F_1$ (which, by abuse of notation, we denote with the same letter) and the exceptional divisors $E_1, G_1, H_1$ are in the following configuration:

![Diagram of curves and singular points](image)

where all the curves are smooth and rational with the following intersection numbers:

$$H_1^2 = -2, G_1^2 = -2, F_1^2 = -3, E_1^2 = -1, C \cdot E_1 = 1, E_1 \cdot F_1 = 1, E_1 \cdot H_1 = 1, H_1 \cdot G_1 = 1.$$

Note that the self-intersection of $C$ has dropped by 3. Performing the same operation with other two points $p_2, p_3$ on the curve $C$ we construct a birational morphism $f: S \to \mathbb{P}_x^1 \times \mathbb{P}_y^1$ where the strict transform of $C$ has become a $(−3)$-curve. Over each point $p_i$ we have the exceptional curves $H_i, G_i, E_i$ and the strict transform of the fibre $F_i$ in the same configuration as the one described above for $p_1$.

On $S$ there are the $(-3)$-curves $F_1, F_2, F_3$ and three cycles of type $A_2$ of $(-2)$-curves formed by $H_i$ and $G_i$ for $i = 1, \ldots, 3$. Thus we can contract such curves together with $C$ to construct a birational morphism $\psi: S \to T$. We denote, with a slight abuse of notation, the pushforward of a divisor $D$ via $\psi$ with the same letter $D$.

On $T$ we have the following configuration of curves and singular points:
Remark 2. The singularity at the points of type $A_2$ (resp. $A_1$) is formally isomorphic to the quotient of $A_k^2$ by the action of the group scheme $\mu_3$ with weights $(1, 2)$ (resp. $(1, 1)$).

The following proposition justifies why this surface is a generalization of Keel-McKernan construction in characteristic 3:

**Proposition 3.1.** The surface $T$ is a log del Pezzo surface of Picard rank 1.

**Proof.** It is straightforward to see that $\rho(T) = 1$. Since we contract only cycle of $(-2)$-curves and $(-3)$-curves, $T$ has klt singularities. We are only left to show that $-K_T$ is an ample divisor. By an explicit computation we have

$$\psi^* K_T = K_S + \sum_{i=1}^{3} \frac{1}{3} F_i + \frac{1}{3} C.$$  

Since $\rho(T) = 1$ it is enough to prove that the anticanonical divisor has positive intersection with an effective curve.

Let $F_p$ be the fibre of the the map $\pi_x \circ f: S \to \mathbb{P}_x^1$ over a general point $p \in \mathbb{P}_x^1$. By projection formula we have:

$$-K_T \cdot \psi_* F_p = -\psi^* K_T \cdot F_p = 2 - \frac{1}{3} C \cdot F_p = 2 - 1 > 0.$$  

$\square$

Remark 3. It is possible to perform a similar construction for characteristic $p > 3$, but the resulting surface has ample canonical divisor class.

Remark 4. In [13, Section 9] the authors prove the Bogomolov bound: a log del Pezzo surface of Picard rank one over an algebraically closed field of characteristic zero has at most six singular points. The sharp bound of four singular points was later proven by [1]. The surface $T$ shows that the Bogomolov bound cannot hold in characteristic 3. It is an open question whether the Bogomolov bound holds for large characteristic.

We show that there are no anticanonical sections on $T$:

**Proposition 3.2.** $H^0(T, \mathcal{O}_T(-K_T)) = 0$.

**Proof.** Since

$$-\psi^* K_T = -K_S - \frac{1}{3} \sum_{i=1}^{3} F_i - \frac{1}{3} C$$

we have $H^0(T, \mathcal{O}_T(-K_T)) = H^0(S, \mathcal{O}_S(-K_S - \sum_{i=1}^{3} F_i - C))$. A direct computation shows

$$-K_S - \sum_{i=1}^{3} F_i - C \sim f^*(-K_{\mathbb{P}_x^1 \times \mathbb{P}_y^1} - \sum_{i=1}^{3} F_i - C) + \sum_{i=1}^{3} (G_i + 2H_i + 3E_i).$$

Therefore

$$H^0(T, \mathcal{O}_T(-K_T)) = H^0(\mathbb{P}_x^1 \times \mathbb{P}_y^1, \mathcal{O}(-K_{\mathbb{P}_x^1 \times \mathbb{P}_y^1} - \sum_{i=1}^{3} F_i - C)) = H^0(\mathbb{P}_x^1 \times \mathbb{P}_y^1, \mathcal{O}(-2, -1)) = 0.$$  

$\square$
3.2. Failure of Kawamata-Viehweg vanishing for $T$. We show that the Kawamata-Viehweg vanishing theorem fails on the surface $T$.

We consider the following ample $\mathbb{Q}$-Cartier Weil divisor

$$A := E_2 + E_3 - E_1$$

**Theorem 3.3.** The Kawamata-Viehweg vanishing theorem fails for the Weil divisor $A$; i.e.

$$H^1(T, \mathcal{O}_T(-A)) \neq 0.$$

**Proof.** The strategy is to pull-back the divisor to the minimal resolution $S$ and compute there the cohomology groups. Let us consider the pull-back of $A$ to $S$ as a $\mathbb{Q}$-divisor:

$$-\psi^*A = E_1 + \frac{1}{3}F_1 + \frac{2}{3}H_1 + \frac{1}{3}G_1 - E_2 - \frac{2}{3}F_2 - \frac{2}{3}H_2 - \frac{2}{3}G_2 - \frac{2}{3}C - E_3 - \frac{1}{3}F_3 - \frac{2}{3}H_3 - \frac{1}{3}G_3 - \frac{1}{3}C;$$

thus

$$[-\psi^*A] = E_1 - E_2 - F_2 - H_2 - G_2 - E_3 - F_3 - H_3 - G_3 - C.$$ 

We have

$$\psi_*\mathcal{O}_S([-\psi^*A]) = \mathcal{O}_T(-A),$$

and we compute the cohomology group using the Leray spectral sequence

$$E_2^{i,j} = H^j(T, R^i\psi_*\mathcal{O}_S([-\psi^*A])) \Rightarrow H^{i+j}(S, \mathcal{O}_S([-\psi^*A])).$$

We show that $R^i\psi_*\mathcal{O}_S([-\psi^*A]) = 0$ for $i > 0$. By the Kawamata-Viehweg vanishing theorem for birational morphism between surfaces (see [16, Theorem 10.4]) we just need to check that $[-\psi^*A]$ is $\psi$-nef:

$$[-\psi^*A] \cdot C = 2,$$

$$[-\psi^*A] \cdot F_1 = 1,$$

$$[-\psi^*A] \cdot H_1 = 1,$$

$$[-\psi^*A] \cdot G_1 = 0,$$

$$[-\psi^*A] \cdot F_2 = 2,$$

$$[-\psi^*A] \cdot H_2 = 0,$$

$$[-\psi^*A] \cdot G_2 = 1 \text{ for } i = 2, 3.$$ 

Therefore the Leray spectral sequence (1) degenerates at the $E_2$ page and we have for all $i \geq 0$:

$$H^i(T, \mathcal{O}_T(-A)) \simeq H^i(S, \mathcal{O}_S([-\psi^*A])).$$

By a direct computation we have

$$K_S \cdot [-\psi^*A] = -2 \text{ and } [-\psi^*A]^2 = -6.$$ 

Therefore, by the Riemann-Roch theorem on $S$, we deduce

$$\chi(T, \mathcal{O}_T(A)) = \chi(S, \mathcal{O}_S([-\psi^*A])) = -1,$$

which implies $h^1(T, \mathcal{O}_T(-A)) \neq 0$. \qed

We now conclude that the surface $T$ gives a generalization to [5, Theorem 1.3] to characteristic 3. For the definition of liftability for log pairs we refer to [5, Definition 2.15].

**Corollary 3.4.** Over any algebraically closed field $k$ of characteristic 3 there exists a log del Pezzo surface $T$ which is not globally $F$-split and such that for any log resolution $\mu$: $S \rightarrow T$ the log smooth pair $(S, \text{Exc}(\mu))$ does not lift to $W_2(k)$.

**Proof.** The surface $T$ constructed above is not globally $F$-split by Proposition 2.1 and Theorem 3.3.

By Proposition 3.3 and Serre duality we have

$$H^1(T, \mathcal{O}_T(K_T + A)) \neq 0.$$

If the pair $(S, \text{Exc}(\mu))$ lifted to $W_2(k)$, we could apply [5, Lemma 6.1] to the $\mathbb{Z}$-divisor $D := K_T + A$, thus getting a contradiction. \qed
4. A klt not Cohen-Macaulay threefold singularity in characteristic 3

The aim of this section is to prove Theorem 1.2.

With the same notation as in Section 2.3, let us consider the generalized cone over the log del Pezzo surface $T$:

$$X := C_\alpha(T, \mathcal{O}_T(A)),$$

and denote the vertex by $v$.

As an application of Proposition 2.2 and Theorem 3.3 we deduce

**Proposition 4.1.**

$$H^2_v(X, \mathcal{O}_X) \cong \sum_{m \in \mathbb{Z}} H^1(T, \mathcal{O}_T(mA)) \neq 0.$$

Therefore $X$ is not Cohen-Macaulay.

In the remaining part of this section, we prove that $X$ has klt singularities. If we were working over a field of characteristic zero we could use Inversion of Adjunction to deduce that $X$ is klt (see [15, Paragraph 4.42]). However, since we are working in positive characteristic, we cannot apply Inversion of Adjunction and thus we have to study in more detail the singularities of $X$ to conclude it is klt. We start by studying the singularities of its partial resolution:

$$Y := \text{Spec}_T \sum_{m \geq 0} \mathcal{O}_T(mA) \xrightarrow{\pi} X.$$ 

The exceptional locus of the birational morphism $\pi$ is the prime divisor $E$, which is isomorphic to $T$. We denote by $f$ the natural affine map $f : Y \to T$. We thus have the following diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow f & & \downarrow \\
T & & 
\end{array}$$

**Proposition 4.2.** $Y$ is a $\mathbb{Q}$-factorial variety with isolated singularities and the pair $(Y, E)$ is log canonical.

**Proof.** To check that $Y$ is $\mathbb{Q}$-factorial it is sufficient to work in an analytic neighbourhood of the singular locus by [20, (24.E)]. The same is true to compute the discrepancies. Thus we can reduce to study the preimage $f^{-1}(U) \subset Y$ of an analytic neighbourhood $U$ of the singular points of $T$ because outside the preimage of those points the pair $(Y, E)$ is log smooth.

As explained in Remark 2, in $T$ there are two different types of singular points. We show the result is true for the $A_2$-type singular points and for the $A_1$-type singular points the computation is similar.

Let us consider a singular point $p \in T$, which is formally isomorphic to the quotient of $\mathbb{A}^2_{u,v}$ by the group $\mu_3$ with weight $(1, 2)$. In local coordinates,

$$k^2_{u,v} / \mu_3 = \text{Spec}_k k[u^3, v^3, uv] \cong \text{Spec}_k k[x, y, z]/(z^3 - xy),$$

and the Weil divisorial sheaf $\mathcal{O}_T(A)$ is isomorphic to the Weil divisorial ideal $D := (x, z)$. In this case we have

$$Y = \text{Spec}_{k^2 / \mu_3} \sum_{m \geq 0} \mathcal{O}_T(mD) \cong \text{Spec}_k \frac{k[x, y, z, a, b, c, d]}{(z^3 - xy, a^2 - cx, ab - cz, a^3 - dx^2, ac - dx, b^3 - dy, bc - dz)},$$

the fibration $f$ is the natural morphism

$$\text{Spec}_k \frac{k[x, y, z, a, b, c, d]}{(z^3 - xy, a^2 - cx, ab - cz, a^3 - dx^2, ac - dx, b^3 - dy, bc - dz)} \to \text{Spec}_k \frac{k[x, y, z]}{(z^3 - xy)}.$$
and the section $E$ is the subvariety defined by the ideal $(a, b, c, d)$.

A more conceptual way to understand the $\mathbb{A}^1$-fibration $Y \to T$ and its singularities is to see it locally as a quotient of the trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^2$. Let us consider the line bundle

$$\mathbb{L} := \text{Spec}_k \left( \sum_{k \geq 0} (u)^k \right) \simeq \text{Spec}_k k[u, v, s]$$

as a $\mathbb{A}^1$-fibration $Y \to T$. Let us consider the line bundle $L := \text{Spec}_k \sum_{k \geq 0} (u)^k \simeq \text{Spec}_k k[u, v, s]$ together with the section $S = (s = 0)$. We have a natural action of $\mu_3$ on $L$ of weight $(1, 2, 1)$ and we can construct the quotient $p: L \to L/\mu_3$.

A direct computation shows that the quotient pair $(L/\mu_3, p(S))$ is isomorphic to $(Y, E)$. With this description, we deduce that $Y$ is a $\mathbb{Q}$-factorial variety by [17, Lemma 5.16] and that the singularities of $Y$ are isolated.

Moreover we have shown that, near the preimage via $f$ of the singular points of $T$, the pair $(Y, E)$ is toroidal and thus by [6, Proposition 11.4.24] we conclude it has log canonical singularities.

To study discrepancies on $X$ we need the following:

**Lemma 4.3.** $X$ is $\mathbb{Q}$-factorial.

*Proof.* Tensoring by $\mathbb{Q}$ the short exact sequence in Proposition 2.3, we have the short exact sequence of $\mathbb{Q}$-vector spaces

$$0 \to \mathbb{Q} \to \text{Cl}(T)_\mathbb{Q} \to \text{Cl}(X)_\mathbb{Q} \to 0.$$ 

Since $\rho(T) = 1$ and $H^1(T, \mathcal{O}_T) = 0$, we have $\text{Cl}(T)_\mathbb{Q} \simeq \mathbb{Q}$. Therefore $\text{Cl}(X)_\mathbb{Q} = 0$, concluding the proof.

We can now prove that $X$ is a klt not Cohen-Macaulay threefold singularity:

*Proof of Theorem 1.2.* By Proposition 4.1 $X$ is not Cohen-Macaulay. We are left to check it is klt. By Lemma 4.3, $K_X$ is $\mathbb{Q}$-Cartier and therefore we can compare the canonical classes via the birational morphism $\pi: Y \to X$:

$$K_Y = \pi^* K_X + a(E)E.$$ 

Since $Y$ is a threefold and the singularities are isolated, we can apply the usual adjunction formula:

$$K_E = K_Y + E|_E = (a(E) + 1)E|_E.$$ 

We know that $-E$ is a $\pi$-ample divisor since $X$ is $\mathbb{Q}$-factorial. Since $E \simeq T$ is a log del Pezzo surface we conclude that $a(E) + 1 > 0$.

By Proposition 4.2, $Y$ is a $\mathbb{Q}$-factorial variety and the pair $(Y, E)$ has log canonical singularities along $E$. Therefore the pair $(Y, -a(E)E)$ is sub-klt, thus concluding that $X$ has klt singularities.

5. **Kodaira-type vanishing for log del Pezzo surfaces in characteristic $p$**

Throughout this section, $k$ is an algebraically closed field of characteristic $p > 0$. The aim of this section is to collect some Kodaira-type vanishing for ample line bundles on log del Pezzo surfaces for arbitrary $p > 0$ and to prove Theorem 1.3.

We start by discussing the case of log del Pezzo surfaces with at worst canonical singularities.

**Proposition 5.1.** Let $A$ be a big and nef Cartier divisor on a log del Pezzo surface $X$ such that $| - K_X| \neq \emptyset$. Then

$$H^1(X, \mathcal{O}_X(A)) = 0.$$
Proof. Since $A$ is effective, the divisor $A - K_X$ is effective and ample by hypothesis and thus by an application of Serre duality we have
\[ h^1(X, O_X(A)) = h^1(X, O_X(K_X - A)), \]
which is zero by [3, Proposition 3.3].

**Corollary 5.2.** Let $X$ be a log del Pezzo surface with at worst canonical singularities. Then for any big and nef Cartier divisor $A$ we have:
\[ H^1(X, O_X(A)) = 0 \]

Proof. By Proposition 3.2, it is sufficient to check $h^0(X, O_X(-K_X)) \neq 0$. Since the singularities are canonical, the canonical divisor is Cartier and we can thus apply the Riemann-Roch formula to show $h^0(X, O_X(-K_X)) \geq 1 + K_X^2 > 0$. □

Remark 5. Let us note however that the Kawamata-Viehweg vanishing theorem is not valid for del Pezzo surfaces with canonical singularities in characteristic 2 by [3, Theorem 3.1].

We recall a result of Kollár (see [14, Theorem II.6.2, Remark II.6.2.4 and Remark II.6.7.2]), which is a combination of Ekedahl’s purely inseparable trick and Bend and Break techniques.

**Theorem 5.3.** Let $X$ be a normal surface over $k$. Let $L$ be a big and nef Weil $\mathbb{Q}$-Cartier divisor on $X$ such that $H^1(X, L^\vee) = 0$. Assume that $X$ is covered by a family of curves $\{D_t\}$ such that $X$ is smooth along the general curve $D_t$ and such that
\[ ((p-1)L - K_X) \cdot D_t > 0. \]
Then for any point $x \in X$ there exists a rational curve $C_x$ passing through $x$ such that
\[ L \cdot C_x \leq 2 \dim X \frac{L \cdot D_t}{((p-1)L - K_X) \cdot D_t}. \]

As an application we deduce an effective vanishing for the $H^1$ of a positive line bundle on a log del Pezzo surface:

**Theorem 5.4.** Let $X$ be a log del Pezzo surface over $k$ and let $A$ be a big and nef Cartier divisor. Then

1. $H^1(X, O_X(-A)) = 0$;
2. If $p \geq 5$, then $H^1(X, O_X(A)) = 0$;
3. If $p = 3$, then $H^1(X, O_X(2A)) = 0$;
4. If $p = 2$, then $H^1(X, O_X(4A)) = 0$.

Proof. To prove (1), it is enough to show that $H^0(X, A) \neq 0$ by [3, Proposition 3.3]. So denoting by $f : Y \to X$ the minimal resolution, we have
\[ H^0(X, O_X(A)) = H^0(Y, O_Y(f^*A)). \]
Since $Y$ is a rational surface we have $h^2(Y, O_Y(f^*A)) = h^0(Y, O_Y(K_Y - f^*A)) = 0$ and therefore
\[ h^0(Y, O_Y(f^*A)) \geq 1 + \frac{1}{2}(f^*A)(f^*A - K_Y) = 1 + \frac{1}{2}(A^2 - K_X \cdot A) > 0. \]
To prove (2), let us note that if $H^1(X, O_X(A)) \neq 0$ we have $H^1(X, O_X(K_X - A)) \neq 0$ by Serre duality. Let us define a Weil $\mathbb{Q}$-Cartier ample divisor
\[ L := O_X(A - K_X). \]
Considering a covering family $\{D_t\}$ of curves for $X$ belonging to a very ample linear system we have that
\[ (p-1)(L \cdot D_t) - K_X \cdot D_t > 0. \]
Therefore we can apply Theorem 5.3 for every point \( x \in X \) we can find a curve \( C_x \) passing through \( x \) such that

\[
L \cdot C_x \leq 4\frac{L \cdot D_t}{(p-1)L \cdot D_t - K_X \cdot D_t} < \frac{4}{p-1}.
\]

Moreover, if \( x \in X \) is chosen to be generic we have

\[
L \cdot C_x = A \cdot C_x - K_X \cdot C_x > 1
\]

since \( A \) is big and nef and Cartier. Thus \( p < 5 \).

In the case where \( p = 3 \), we apply the same proof to \( L = 2A - K_X \) with the same notation to the curves \( D_t \). In this case by Theorem 5.3 we can find that for any point \( x \) there exists a rational curve \( C_x \) passing through \( x \) such that

\[
L \cdot C_x < 4 = 2
\]

However choosing \( x \) generic enough we have

\[
L \cdot C_x = 2A \cdot C_x - K_X \cdot C_x > 2,
\]

thus getting a contradiction. The proof for the case \( p = 2 \) is analogous. \( \square \)

We conclude by discussing the special case where the linear system induced by \( A \) is birational.

**Proposition 5.5.** Let \( (X, \Delta) \) be a log del Pezzo surface over \( k \). Let \( A \) be a big and nef Cartier divisor such that the linear system \( |A| \) is birational onto the image. Then

\[
H^1(X, \mathcal{O}_X(A)) = 0.
\]

**Proof.** Let \( f: Y \to X \) be the minimal resolution. We have for a certain effective boundary divisor \( \Delta_Y \):

\[
K_Y + \Delta_Y = f^*(K_X + \Delta).
\]

By the relative Kawamata-Viehweg vanishing theorem for birational morphism between surfaces (see [16, Theorem 10.4]) we deduce

\[
H^i(Y, \mathcal{O}_Y(f^*A)) = H^i(X, \mathcal{O}_X(A)).
\]

By hypothesis, there exists an irreducible section \( C \in |f^*A| \). The curve \( C \) is Gorenstein with dualizing sheaf:

\[
\omega_C = \mathcal{O}_Y(K_Y + C) \otimes \mathcal{O}_C.
\]

Consider the following short exact sequence

\[
0 \to \mathcal{O}_Y \to \mathcal{O}_Y(C) \to \mathcal{O}_C(C) \to 0.
\]

By considering the long exact sequence and using the fact that \( H^i(\mathcal{O}_Y) = 0 \) for \( i = 1, 2 \) we have

\[
H^1(Y, \mathcal{O}_Y(f^*A)) = H^1(Y, \mathcal{O}_Y(C)) \simeq H^1(C, \mathcal{O}_C(C)).
\]

Now, using Serre duality on \( C \) we have

\[
H^1(C, \mathcal{O}_C(C)) \simeq H^1(C, \omega_C \otimes \mathcal{O}_C(-K_Y|_C)) \simeq H^0(C, \mathcal{O}_C(K_Y|_C))^*.
\]

It is easy to see that \( K_Y|_C \) is an anti-ample divisor because

\[
K_Y \cdot C = (f^*(K_X + \Delta) - \Delta_Y) \cdot f^*A = (K_X + \Delta) \cdot A - \Delta_Y \cdot f^*A < 0.
\]

Therefore

\[
H^0(C, \mathcal{O}_C(K_Y|_C)) = 0,
\]

thus concluding the proof. \( \square \)
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