Strategies for Estimating Quantum Lossy Channels

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Abstract

Due to the anisotropy of quantum lossy channels one must choose optimal bases of input states for best estimating them. In this paper, we obtain that the equal probability Schrödinger cat states are optimal for estimating a single lossy channel and they are also the optimal bases of input states for estimating composite lossy channels. On the other hand, by using the symmetric logarithmic derivative (SLD) Fisher information of output states exported from the lossy channels we obtain that if we take the equal probability Schrödinger cat states as the bases of input states the maximally entangled inputs are not optimal, however if the bases of the input states are not the equal probability Schrödinger cat states the maximally entangled input states may be optimal for the estimating composite lossy channel.

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I. INTRODUCTION

A quantum noisy channel can be expressed with a trace-preserving completely positive map: \( \varepsilon : \rho \to \rho' (\zeta) \). Here, \( \rho \) and \( \rho' (\zeta) \) are density matrices in the Hilbert space \( \mathcal{H} \), and \( \zeta = (\psi_1, \psi_2, ..., \psi_n) \in \Gamma \) are parameters characterizing the channel. So a single parameter quantum noisy channel can be expressed as

\[
\rho'(\zeta) = \varepsilon (\rho).
\]

Due to the complete positivity of the map, it can be expanded to composite quantum systems in \( \mathcal{H} \otimes \mathcal{H} \). The composite channels in the expanded systems have two forms, one is

\[
\rho'(\zeta) = \varepsilon \otimes I (\rho),
\]

called mixed noisy channel; the other is

\[
\rho''(\zeta) = \varepsilon \otimes \varepsilon (\rho),
\]

called double noisy channel. Here, \( \rho, \rho', \rho'' \) are density matrices in \( \mathcal{H} \otimes \mathcal{H} \), and \( I, \varepsilon \) are the single quantum identity and noisy channel.

For a known quantum noisy channel at least two subjects are interested. One is to determine its information capacities, the best probability to understand the input state under assumption that the action of the quantum channel is known. Although the capacities of quantum noisy channels have not been solved thoroughly much effort has been put into this topic and many results are obtained \[1\]. Another topic, the estimation of quantum noisy channel has also been attracted much attention in last years \[2\] \[3\] \[4\] \[7\] because it is also important in quantum information theory. The estimation of quantum noisy channel is to identify a quantum noisy channel as the type of the channel is known but its quality is unknown. The quality of the channel can be characterized with some parameters. Thus estimating some channel is equal to estimating its certain parameters, which may be appealed to the quantum estimation theory \[2\].

Quantum estimation theory is one about seeking the best strategy for estimating one or more parameters of a density operator of a quantum mechanical system. About how to estimate a quantum noisy channel we refer the readers to the Refs. \[2\] and recent \[3\]. In this paper we restrict our attention in two aspects of the estimation of a quantum noisy channel, lossy channel (which will be described in section II). Because of the anisotropy of the quantum lossy channels, different input states must have different effects for estimating the quantum noisy channel. So at first, we will discuss what coherent states are optimal for estimating the single quantum lossy channel and what bases of the input states are optimal for estimating the composite lossy channels? Because entanglement has been taken a kind of resource for processing quantum information, secondly, we will discuss: can the estimation of the composite channels \( \varepsilon \otimes I (\rho) \) and \( \varepsilon \otimes \varepsilon (\rho) \) be improved by using entangled input states? This paper is constructed as follows. In section II we shall set up a model of quantum lossy channel and explain why we choose coherent states to estimate these channels. In section III we shall seek for the optimal input coherent states or optimal bases of input states for estimating the single and the composite lossy channels. In section IV we shall calculate the symmetric logarithmic derivative (SLD) Fisher information of output states exported from the channels \( \varepsilon, \varepsilon \otimes I \) and \( \varepsilon \otimes \varepsilon \) and answer whether the entangled input states improve the estimation. A brief conclusion will close this paper in last section.

II. LOSSY CHANNEL AND SCHRODINGER CAT STATE

We set the lossy channel to be estimated is described by the following physical model. A quantum system, such as photons in state \( \rho \) is in a vacuum environment, the evolution of the state is a completely positive map: \( \rho' = \varepsilon (\rho) \). In this model, making use of the language of master equation we can obtain that the interaction of the in question system with its environment makes the
system evolving according to

$$\frac{\partial \rho}{\partial \tau} = \dot{\rho} + \hat{L}\rho; \dot{\rho} = \eta \sum_{i} a_i \rho a_i^\dagger,$$

$$\hat{L}\rho = -\sum_{i} \frac{\eta}{2} \left( a_i a \rho + \rho a_i^\dagger a \right), \quad (4)$$

where $\eta$ is the energy decay rate. The formal solution of Eq. (4) may be written as

$$\rho (\tau) = \exp \left( \left( \hat{J} + \hat{L} \right) \tau \right) \rho (0), \quad (5)$$

which leads to the solution for the initial single-mode state $\alpha \langle \beta |$

$$\exp \left[ \left( \hat{J} + \hat{L} \right) \tau \right] |\alpha \rangle \langle \beta | = |\beta \rangle |\alpha \rangle \exp \left( \eta \tau \right), \quad (6)$$

where $t = e^{-\frac{\text{i}}{2} \pi \eta \tau}$. Estimating this channel is equal to estimating the parameter $\eta$.

As known, in order to estimate the channel, one, for example Alice must prepare many identical initial states, input states $\rho$ and another one, for example Bob must measure the output samples exported from this channel. For enhancing the detection efficiency we use coherent states to be the input states $|\tilde{\rho}_{\alpha} \rangle$. Because we do not know what coherent state is the optimal input state for the estimation in advance, we generally set this state be the superposition of coherent states $|\alpha \rangle$ and $|\alpha^* \rangle$, namely, a Schrödinger cat state

$$|\varphi \rangle = A |\alpha \rangle + B |\alpha^* \rangle. \quad (7)$$

This state is considered one of realizable mesoscopic quantum systems [8]. Zheng [9] has shown the method for preparing this state and the measurement scheme of this state has been given in [10]. Set $|\alpha| \gg 1$ (in fact only if $|\alpha| \geq 3$, then $|\alpha \rangle - |\alpha^* \rangle \simeq 0$. Thus, $|\alpha \rangle$ and $|\alpha^* \rangle$ can be taken into a pair orthogonal bases. Setting $A \approx \sin \theta$, $B \approx \cos \theta$ we have

$$|\varphi \rangle = \sin \theta |\alpha \rangle + \cos \theta |\alpha^* \rangle. \quad (8)$$

In the time-varying bases $|a_1 t \rangle, |a_2 t \rangle$, the state $\rho = |\varphi \rangle \langle \varphi |$ passing through the lossy channel becomes

$$\rho' = \mathcal{E} (\rho) \approx \begin{pmatrix} \sin^2 \theta & \sin \theta \cos \theta e^{-2|\alpha|^2 \eta \tau} \\ \sin \theta \cos \theta e^{-2|\alpha|^2 \eta \tau} & \cos^2 \theta \end{pmatrix},$$

$$= \frac{1}{2} I + \chi \sin \theta \cos \theta \sigma_x + \left( \sin^2 \theta - \frac{1}{2} \right) \sigma_z, \quad (9)$$

where $\chi = e^{-2|\alpha|^2 \eta \tau}, \sigma_i (i = x, y, z)$ are Pauli matrices and $|\alpha \rangle - \alpha \rangle \rightarrow 0$ is set (namely, we set the time of the system evolution is very short, $\tau \rightarrow 0$, $\tau \rightarrow 1$). We call the map of Eq. (9) the single lossy channel.

III. OPTIMAL INPUT STATES AND OPTIMAL BASES OF INPUT STATES FOR ESTIMATING THE LOSSY CHANNELS

In this section we answer the first question put forward in section II, namely, what (coherent) input states are optimal for estimating the single lossy channel and what bases of the (coherent) input states are optimal for estimating the composite lossy channels? In the following we will firstly investigate the single lossy channel then the mixed lossy channel $\varepsilon \otimes I$ and the double lossy channel $\varepsilon \otimes \varepsilon$.

In the quantum estimation theory, in order to estimate a channel, at first, one must introduce a cost function. In general, delta function is chosen for this aim, namely,

$$C (\hat{\zeta}, \zeta) = -\prod_{i=1}^{m} \delta (\hat{\zeta}_i - \zeta_i), \quad (10)$$

where $\hat{\zeta}_i$ called estimators which are always a function of observing data and it describes the strategy for calculating the estimates; $\zeta_i$ are parameters to be estimated. It is given that the optimal estimation is to seek the POVM generators $d\Pi_m (\zeta)$ for which

$$\left\{ (\mathbb{Y} - W_m (\zeta)) d\Pi_m (\zeta) = 0, \mathbb{Y} - W_m \geq 0 \right\}. \quad (11)$$

Here,

$$W_m (\zeta) = Z (\zeta) \rho_m (\zeta), \quad (12)$$

where $Z (\zeta)$ is the prior probability density function (PDF), and

$$\mathbb{Y} = \sum_{m} \mathbb{Y}_m = \sum_{m} \int W_m (\zeta) d\Pi_m (\zeta), \zeta \in \Gamma. \quad (13)$$

In our problem, we shall find out a optimal input state, namely a optimal angle $\theta$ in Eq. (9), where $0 < \chi \leq 1$ is supposed. In the following subsection A, we shall investigate what state is the optimal input state for estimating the single lossy channel. In subsection B, we shall investigate that when we use two-mode entangled state estimating the composite lossy channels (include mixed lossy channel and double lossy channel), if we measure the output state separately, what bases of the input states are optimal. In subsection C, we shall look for the optimal bases of input states for estimating composite channels as we measure the output states jointly.

A. Single lossy channel

The schematic diagram for estimating the single lossy channel can be expressed as Fig.1 (above part). Because there is no prior knowledge about the angle $\theta$, we assign to it a uniform prior probability density function, namely

$$Z (\chi) = \frac{1}{2\pi} \quad (14)$$
The POVM generator is
\[
d\Pi(\chi) = \frac{1}{2\pi} (I + \chi \sigma_x \sin 2\theta - \sigma_z \cos 2\theta) d\theta, \tag{15}
\]
and
\[
W(\chi) = \frac{1}{4\pi} (I + \chi \sigma_x \sin 2\theta - \sigma_z \cos 2\theta). \tag{16}
\]
So we have
\[
\Upsilon - W(\chi) = \frac{1}{8\pi} [(1 + \chi^2) I - 2\chi \sigma_x \sin 2\theta + 2\sigma_z \cos 2\theta]. \tag{17}
\]
The eigenvalues of \(\Upsilon - W(\chi)\) are
\[
\lambda = 1 + \chi^2 \pm 2\sqrt{\cos^2 2\theta + \chi^2 \sin^2 2\theta}. \tag{18}
\]
Because \(0 < \chi \leq 1\), if and only if \(\theta = \pm \pi/4, \pm 3\pi/4\) we have
\[
\begin{cases} 
[\Upsilon - W(\chi)] d\Pi(\chi) = 0, \\
\Upsilon - W(\chi) \geq 0.
\end{cases} \tag{19}
\]
It shows that the equal probability Schrödinger cat states \(|\varphi\rangle = (|\alpha\rangle \pm |\alpha\rangle)/\sqrt{2}\) are optimum for estimating the lossy channel \(\epsilon\).

B. Composite lossy channels (separated measurements)

In the following, we shall find out the optimal bases of the input states for estimating the mixed lossy channel as we separately measure the out states exported from composite lossy channels. The schematic diagram is shown in Fig.1 (below part). If the estimated channel is a mixed channel we can use possibly entangled states as their inputs. Set the input state be \(\rho = |\Phi\rangle \langle \Phi|\), where \(|\Phi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2\). By the Schmidt decomposition, the vector \(|\Phi\rangle\) is represented as
\[
|\Phi\rangle = \sqrt{\gamma} |\varphi_1\rangle |\varphi_2\rangle + \sqrt{1 - \gamma} |\psi_1\rangle |\psi_2\rangle, \tag{20}
\]
where \(\gamma\) is a real number between 0 and 1, and \(|\varphi_1, \psi_1\rangle\) and \(|\varphi_2, \psi_2\rangle\) are orthonormal bases of \(\mathcal{H}_1 = C^2\) and \(\mathcal{H}_2 = C^2\). We generally set they are
\[
|\varphi_1\rangle = \sin \theta |\alpha\rangle + \cos \theta |\alpha\rangle - \alpha),
|\psi_1\rangle = \cos \theta |\alpha\rangle - \sin \theta |\alpha\rangle. \tag{21}
\]
From Eq. (20) we have
\[
\rho = |\Phi\rangle \langle \Phi| = \gamma |\varphi_1\rangle \langle \varphi_1| \otimes |\varphi_2\rangle \langle \varphi_2| + \sqrt{\gamma (1 - \gamma)} |\psi_1\rangle \langle \psi_1| \otimes |\varphi_2\rangle \langle \varphi_2| + (1 - \gamma) |\psi_1\rangle \langle \psi_1| \otimes |\psi_2\rangle \langle \psi_2|. \tag{22}
\]
As the state \(\rho\) pass through the mixed channel it becomes
\[
\rho' = \varepsilon \otimes I(\rho) = \gamma \varepsilon (|\varphi_1\rangle \langle \varphi_1|) \otimes I(|\varphi_2\rangle \langle \varphi_2|) + \sqrt{\gamma (1 - \gamma)} \varepsilon (|\psi_1\rangle \langle \psi_1|) \otimes I(|\varphi_2\rangle \langle \varphi_2|) + (1 - \gamma) \varepsilon (|\psi_1\rangle \langle \psi_1|) \otimes I(|\psi_2\rangle \langle \psi_2|). \tag{23}
\]
If Bob samples the data by separate measuring the out states from channel \(\varepsilon_1\) OR channel \(\varepsilon_2\), say channel \(\varepsilon_1\), means
\[
\rho_1' = \text{tr}_2 (\rho') = |\varphi_2\rangle \langle \varphi_2| + |\psi_2\rangle \langle \psi_2| = \gamma |\varphi_1\rangle \langle \varphi_1| + (1 - \gamma) |\psi_1\rangle \langle \psi_1| = \frac{1}{2} (I + k \chi \sigma_x \sin 2\theta - k \sigma_z \cos 2\theta), \tag{24}
\]
where \(k = 2\gamma - 1\). Thus,
\[
d\Pi(\chi) = \frac{1}{2\pi} (I + k \chi \sigma_x \sin 2\theta - k \sigma_z \cos 2\theta) d\theta, \tag{25}
\]
and
\[
W'(\chi) = \frac{1}{4\pi} (I + k \chi \sigma_x \sin 2\theta - k \sigma_z \cos 2\theta). \tag{26}
\]
From Eq. (23), we have
\[
\tilde{\Upsilon} = \frac{I}{4\pi} \left[1 + \frac{1}{2} k^2 (\chi^2 + 1)\right]. \tag{27}
\]
So
\[
\tilde{\Upsilon} - W'(\chi)
= \frac{1}{8\pi} [k^2 (\chi^2 + 1) - 2k \chi \sigma_x \sin 2\theta + 2k \sigma_z \cos 2\theta]. \tag{28}
\]
which has eigenvalues
\[
\tilde{\lambda} = k^2 (\chi^2 + 1) + 2k \sqrt{\cos^2 2\theta + \chi^2 \sin^2 2\theta}. \tag{29}
\]
It is clear that if and only if \(\theta = \pm \pi/4, \pm 3\pi/4\) we have
\[
\begin{cases}
[\tilde{\Upsilon} - W'(\chi)] d\Pi(\chi) = 0, \\
\tilde{\Upsilon} - W'(\chi) \geq 0.
\end{cases} \tag{30}
\]
Which shows that it is optimal that the equal probability Schrödinger cat states $|\varphi\rangle = (|\alpha\rangle + |\alpha\rangle)/\sqrt{2}$ and $|\psi\rangle = (|\alpha\rangle - |\alpha\rangle)/\sqrt{2}$ are taken as the bases of the input states for the estimation of the mixed lossy channel.

Now we investigate the double lossy channel. Similarly, we take the input state $|\varphi\rangle$ as Eq. (21) and the bases as Eq. (23). Thus, the initial state $\rho$ passing through the double lossy channel becomes

$$\rho'' = \varepsilon \otimes \varepsilon (\theta) = \gamma \varepsilon (|\varphi_1\rangle \langle \varphi_1|) \otimes \varepsilon (|\varphi_2\rangle \langle \varphi_2|) + \sqrt{\gamma (1 - \gamma)} \varepsilon (|\varphi_1\rangle \langle \varphi_2|) \otimes \varepsilon (|\varphi_2\rangle \langle \varphi_1|) + (1 - \gamma) \varepsilon (|\varphi_2\rangle \langle \varphi_2|) \otimes \varepsilon (|\varphi_2\rangle \langle \varphi_2|).$$

(31)

Because of

$$\langle \varphi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\varphi_2\rangle = 1 + \frac{1}{2} (\chi - 1) \sin^2 2\theta,$$

$$\langle \varphi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\varphi_2\rangle = -\frac{1}{2} (\chi - 1) \sin^2 2\theta,$$

$$\langle \varphi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\varphi_2\rangle = \frac{1}{2} (\chi - 1) \sin 2\theta \cos 2\theta,$$

$$\langle \varphi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\varphi_2\rangle = \frac{1}{2} (\chi - 1) \sin 2\theta \cos 2\theta,$$

(32)

and

$$\langle \psi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\psi_2\rangle = -\frac{1}{2} (\chi - 1) \sin^2 2\theta,$$

$$\langle \psi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\psi_2\rangle = 1 + \frac{1}{2} (\chi - 1) \sin^2 2\theta,$$

$$\langle \psi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\psi_2\rangle = -\frac{1}{2} (\chi - 1) \sin 2\theta \cos 2\theta,$$

$$\langle \psi_2 \varepsilon (|\varphi_2\rangle \langle \varphi_2|) |\psi_2\rangle = -\frac{1}{2} (\chi - 1) \sin 2\theta \cos 2\theta,$$

(33)

we have

$$\rho''_1 = tr_2 (\rho'') = \langle \varphi_2 | \rho'' | \varphi_2 \rangle + \langle \psi_2 | \rho'' | \psi_2 \rangle = \gamma |\varphi_1\rangle \langle \varphi_1 | + (1 - \gamma) |\psi_1\rangle \langle \psi_1 | + \frac{1}{2} (I + k \chi \sigma_x \sin 2\theta - k \sigma_z \cos 2\theta),$$

(34)

which is similar to $\rho'_1$. So we can obtain the similar results.

C. Composite lossy channels (joint measurements)

For some purposes jointly measure the output state exported from the composite channels is needed. The schematic diagram of the joint measurement is shown in Fig. 2. In this case, what are the optimal bases of the input states for estimating the composite lossy channel? In this subsection we shall answer this question.

We use the projector $\Pi = |\Psi\rangle \langle \Psi|$ to measure the output state exported from the composite channels. Here, we suppose

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\varphi\rangle |\varphi\rangle + |\psi\rangle |\psi\rangle).$$

(35)

Measuring the the output state $\rho'$ with projector $\Pi$ we can obtain its probability as

$$p' = \langle \Psi | \rho' | \Psi \rangle = \frac{1}{2} \left[ \left( \langle \varphi | \langle \varphi | + \langle \psi | \langle \psi | \right) \rho' (|\varphi\rangle \langle \varphi | + |\psi\rangle \langle \psi |) \right]$$

$$= \frac{1}{2} \left[ 1 + \sqrt{\gamma (1 - \gamma)} \left( \sin^2 \theta + 2 \gamma \sin^4 \theta + \cos^4 \theta \right) \right].$$

(36)

Similarly, measuring the output state $\rho''$ with projector $\Pi$ we can obtain its probability as

$$p'' = \langle \Psi | \rho'' | \Psi \rangle = \frac{1}{2} \left[ \left( \langle \varphi | \langle \varphi | + \langle \psi | \langle \psi | \right) \rho'' (|\varphi\rangle \langle \varphi | + |\psi\rangle \langle \psi |) \right]$$

$$= \frac{1}{2} \left[ 1 + \sqrt{\gamma (1 - \gamma)} \left( \sin^2 \theta + 2 \gamma \sin^4 \theta + \cos^4 \theta \right) \right].$$

(37)

It shows that when $\theta = \pm \pi/4$, the probabilities $p'$ and $p''$ take their maximum.

In this section we have obtained that the equal probability Schrödinger cat states are optimal for estimating the single lossy channel and they are also the optimal bases of input states for estimating the composite lossy channels.

IV. SLD FISHER INFORMATION OF THE OUTPUT STATES OF LOSSY CHANNELS

Based on the above results, in this section we shall investigate the second problem put forward in section II, namely, can the entangled input states improve the estimation of the composite lossy channels? To accomplish
this task we may use another method, to calculate the SLD Fisher information of the output states of the lossy channels. At first, we briefly review this theory. Given a one-parameter family $\rho(\varsigma)$ of density operator, an estimator for parameter $\varsigma$ is represented by a Hermitian operator $T$. It is shown that if the system is in the state $\rho_\varsigma$, then the expectation $E_\varsigma[T] := \text{Tr}\rho_\varsigma T$ of the estimator $T$ should be identical to $\varsigma$ and the estimator $T$ for the parameter $\varsigma$ satisfies the quantum Cramér-Rao inequality $V_\varsigma[T] \geq (J_\varsigma)^{-1}$, where $V_\varsigma[T] := \text{Tr}\rho_\varsigma (T - \varsigma)^2$ is the variance of estimator $T$, and $J_\varsigma := J(\rho_\varsigma) := \text{Tr}\rho_\varsigma (L_\varsigma)^2$ is the quantum SLD Fisher information with $L_\varsigma$ the symmetric logarithmic derivative. Here, the Hermitian operator $L_\varsigma$ satisfies the equation

$$
\frac{d\rho_\varsigma}{d\varsigma} = \frac{1}{2} (L_\varsigma \rho_\varsigma + \rho_\varsigma L_\varsigma).
$$

It is important to notice that the lower bound $(J_\varsigma)^{-1}$ in the quantum Cramér-Rao inequality is achievable (at least locally). In other words, the inverse of the SLD Fisher information gives the ultimate limit of estimation. So in this problem the bigger of the SLD Fisher information is, the more accurately the estimation may be made. In [2], the author has proved that the SLD Fisher information is convex so we only need to investigate the pure state inputs. In the following we will calculate the SLD Fisher information of output states for above three lossy channels.

### A. Optimal inputs cases

In the previous section we obtain that the equal probability Schrödinger cat states are the optimal input states for estimating the single lossy channel, and they are also the optimal bases of the input states for estimating the composite channels. In the following we calculate the SLD Fisher information of the output state for above three lossy channels by use of optimal input states or optimal bases of the input states.

At first, we calculate the SLD Fisher information of single lossy channel when the input state is $\rho = |\varphi\rangle \langle \varphi|$. In this problem, the parameter $\varsigma$ in above formulas is $\chi$. In the output states of lossy channels. From Eq.\(\text{(38)}\) we have

$$
\frac{d\rho_\varsigma}{d\varsigma} = \begin{pmatrix}
0 & -|\alpha|^2 \tau \\
-|\alpha|^2 \tau^\dagger & 0
\end{pmatrix}, \quad \text{(39)}
$$

and

$$
L_\varsigma = \frac{2|\alpha|^2 \chi}{1 - \chi^2} \begin{pmatrix}
\chi & -1 \\
-1 & \chi
\end{pmatrix}. \quad \text{(40)}
$$

So we can easily calculate the SLD Fisher information of output state $\rho_\varsigma^\prime$ as

$$
J_\varsigma^\prime |_{\theta = \frac{\pi}{4}} = \frac{4|\alpha|^4 \tau^2 e^{-4|\alpha|^2 \eta \tau}}{1 - e^{-4|\alpha|^2 \eta \tau}}. \quad \text{(41)}
$$

Secondly, we calculate the SLD Fisher information of the mixed lossy channel by using the input state $\varrho = |\Phi\rangle \langle \Phi|$ (see Eq.\(\text{(20)}\)). Here

$$
|\varphi_1\rangle = |\varphi_2\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle + |-\alpha\rangle),
$$

$$
|\psi_1\rangle = |\psi_2\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle - |-\alpha\rangle). \quad \text{(42)}
$$

Thus, we have

$$
\rho_\varsigma^\prime |_{\theta = \frac{\pi}{4}} = \varepsilon \otimes I (\varrho) = \frac{1}{4} \begin{pmatrix}
u & w \chi & u \chi & w \\
v & vv \chi & w \chi & v \\
u \chi & w \chi & u & w \\
w & v & w & v
\end{pmatrix}, \quad \text{(43)}
$$

where $u = 1 + 2\sqrt{\gamma (1 - \gamma)}$, $v = 1 - 2\sqrt{\gamma (1 - \gamma)}$, $w = 2\gamma - 1$. Here, the expression of Hermitian operator $L_\varsigma$ is too complex, we do not give it. Fortunately, by using Eq.\(\text{(39)}\) we can obtain a simple expression of its SLD Fisher information as

$$
J_\varsigma^\varepsilon |_{\theta = \frac{\pi}{4}} = \frac{4|\alpha|^4 \tau^2 e^{-4|\alpha|^2 \eta \tau}}{1 - e^{-4|\alpha|^2 \eta \tau}}. \quad \text{(44)}
$$

It is shown that the SLD Fisher information of mixed channel do not vary with the changing of the entanglement degree of input state, namely, it does not vary with $\gamma$’s changing. The SLD Fisher information of mixed lossy channel is accurately equal to one of one-shot single lossy channel. It means that when we take the equal probability Schrödinger cat states as the bases the entangled input states the estimation of the mixed lossy channel can not be improved.

Thirdly, we investigate the double lossy channel. By using the input state $\varrho = |\Phi\rangle \langle \Phi|$ we have the output state as

$$
\varrho_\varsigma^\prime |_{\theta = \pi/4} = \varepsilon \otimes \varepsilon (\varrho) = \begin{pmatrix}
\frac{1 + 2A}{4} & \frac{A}{4} & \frac{A}{4} & \frac{A}{4} \\
\frac{A}{4} & \frac{1 - 2A}{4} & \frac{A}{4} & \frac{A}{4} \\
\frac{A}{4} & \frac{A}{4} & \frac{x^2(1 - 2A)}{4} & \frac{1 - 2A}{4} \\
\frac{A}{4} & \frac{A}{4} & \frac{1 - 2A}{4} & \frac{1 + 2A}{4}
\end{pmatrix}, \quad \text{(45)}
$$

where, $A = \sqrt{\gamma (1 - \gamma)}$, $B = 2\gamma - 1$. By using the $\varrho_\varsigma^\prime |_{\theta = \pi/4}$ we can calculate its SLD Fisher information $J_\varsigma^\varepsilon |_{\theta = \pi/4}$. But the expression of $J_\varsigma^\varepsilon |_{\theta = \pi/4}$ is too complex and too long. Here, we do not give it, too. We plot the $J_\varsigma^\varepsilon |_{\theta = \pi/4}$ with $\gamma$ and time $\tau$ as Fig.3 where $\eta = 0.25$ and $|\alpha| = 3$. From Fig.3, we see that when we take the equal probability Schrödinger cat states as the bases of the input states, the entangled inputs can not improve the estimation of the double lossy channel, too.

Before we end this subsection we analytically investigate two kinds of specific cases. Namely, we calculate the SLD Fisher information of output state of double lossy channel when the input states are product state and maximally entangled state where optimal bases is still held.
The Fisher information of double lossy channel is
\[
\gamma F_{\chi} = \frac{16 |\alpha|^4 \tau^2 e^{-8|\alpha|^2 \eta \tau}}{1 - e^{-8|\alpha|^2 \eta \tau}}.
\]

This is the SLD Fisher information of channel $\varepsilon \otimes \varepsilon$ when the input state is the maximally entangled state. A numerical work shows that $J_{\chi}^{\varepsilon \otimes \varepsilon} |_{\gamma=1/2, \theta=\pi/4}$ is less than $J_{\chi}^{\varepsilon \otimes \varepsilon} |_{\gamma=0, \theta=\pi/4}$ for $|\alpha| > 1$.

In this subsection we obtained that when we take the equal probability Schrödinger cat states as the bases of the input states the entangled input states can not improve the estimation of composite lossy channels.

### B. Non-optimal inputs cases

In this subsection, we discuss another case. When the input states are some non-optimal states, namely, when $\theta \neq \pi/4$ in the input states, whether the entangled inputs are better than the non-entangled inputs for estimating these composite lossy channels? In the following, we only discuss the cases of $\theta = 0, \pi/2$. Thus, In this cases the input state $|\alpha\rangle$ becomes

\[
|\Phi\rangle = \sqrt{\gamma} |\alpha\rangle |\alpha\rangle + \sqrt{1 - \gamma} |\alpha\rangle |\alpha\rangle.
\]

For the mixed channel, the output state is

\[
|\Phi\rangle = \sqrt{\gamma} |\alpha\rangle |\alpha\rangle + \sqrt{1 - \gamma} |\alpha\rangle |\alpha\rangle.
\]

Thus we can calculate the SLD Fisher information of $\dot{\theta}_{\chi}^{\varepsilon \otimes \varepsilon}$ as

\[
J_{\chi}^{\varepsilon \otimes \varepsilon} |_{\gamma=1/2, \theta=\pi/2} = \frac{16 |\alpha|^4 \tau^2 e^{-8|\alpha|^2 \eta \tau}}{1 - e^{-8|\alpha|^2 \eta \tau}}.
\]

Here, we do not give out the SLD operator for its very complex. Similarly, we can obtain the output state of the double lossy channel as

\[
\dot{\theta}_{\chi}^{\varepsilon \otimes \varepsilon} |_{\theta=0, \theta=\pi/2} = \begin{bmatrix}
\gamma & 0 & 0 & \sqrt{\gamma (1 - \gamma)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\sqrt{\gamma (1 - \gamma)} & 0 & 0 & 1 - \gamma
\end{bmatrix}.
\]
Its SLD Fisher information is

\[
J_{\chi}^{\otimes I} \bigg|_{\theta=0, \theta=\pi/2} = \frac{64\gamma (\gamma - 1) \chi^8 \tau^2 |\alpha|^4}{4\gamma (1 - \gamma) \chi^4 + (8\gamma^2 - 8\gamma + 1) \chi^2 - (2\gamma - 1)^2}.
\]

(56)

Here, we also do not give the the SLD operator for its very complex. From Eqs. (55), (56) we can easily obtain that \(J_{\chi}^{\otimes I} \bigg|_{\theta=0, \theta=\pi/2}\) and \(J_{\chi}^{\otimes \epsilon} \bigg|_{\theta=0, \theta=\pi/2}\) take their maximum at \(\gamma = 1/2\) (see Figs.4 and 5), which shows that when the bases of input states are not optimal bases (equal probability Schrödinger cat states) the entangled input states may improve the estimation of the composite lossy channels.

V. CONCLUSIONS

In this paper, we have discussed two corresponding problems to estimating quantum lossy channels. Firstly, we have investigated what the optimal input states is for estimating the single lossy channel and what the best bases are of the input states for estimating the composite lossy channels. We obtain that the equal probability Schrödinger cat states are the optimal input states for estimating the single lossy channel and they are also the optimal bases of input states for estimating the composite lossy channels. Secondly, we have investigated whether the entangled input states can improve the estimation of the quantum lossy channels. By calculating the SLD Fisher information of the output states we obtained that when we take the equal probability Schrödinger states as the bases of the input states the entangled input states can not improve the estimation of the composite channel, however when we take the coherent states \(|\pm\alpha\rangle\) instead of the equal probability Schrödinger cat states as the bases, the maximally entangled input states can improve the estimation of the composite lossy channels.
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