A Note on Nonvanishing Properties of Drichlet \( L \)-values Mod \( \ell \) and Applications to K-groups

Abstract

Let \( \chi \) be a Dirichlet character, \( \psi_n \) the character of \( \mathbb{Z}_p^* \) with order \( p^n \). Let \( \ell \) be a prime not equal to \( p \). We note that by directly using a theorem of Sinnott, it can be proved that \( L(-k, \chi \psi_n) \) is a \( \ell \)-unit for sufficiently large \( n \). Applying this, some boundness result of \( \ell \)-part of \( 4n + 2 \)-th K-groups of rings of integers in a cyclotomic \( \mathbb{Z}_p \) extension of real abelian fields are proved.

1 Introduction

Let \( F \) be a number field. Let \( p \) be a prime number. Iwasawa theory study the ideal class groups of \( \mathbb{Z}_p \) extension of \( F \). In [3], Washington proved the \( \ell \)-part of the class numbers in cyclotomic \( \mathbb{Z}_p \) extension of \( F \) is bounded when \( F \) is an abelian number field and \( \ell \neq p \) is a prime. Since the ideal class group is isomorphic to the torsion part of the 0-th K group of \( \mathcal{O}_F \), where \( \mathcal{O}_F \) is the ring of integers of \( F \). Hence Washington’s theorem says the size of \( \ell \)-part of \( K_0(\mathcal{O}_F) \) is bounded in a cyclotomic \( \mathbb{Z}_p \) extension. By class number formula, this is essentially a mod \( \ell \) nonvanishing property of Dirichlet L-functions at \( s = 0 \).

In [4], Sinnott gave a different proof by algebraic methods. For nonvanishing properties of Dirichlet L-functions at \( s = -k \), this is proved in [7] by a refinement of Washington’s method. In this paper, we note that the nonvanishing properties of Dirichlet L-functions for \( s = -k \) can also be proved by directly using Sinnott’s theorem in [4] . Since the Lichtenbaum conjecture which relates the Dedekind zeta functions at \( s = -k \) and higher K-groups is proved for abelian number fields by Huber and Kings [9], we give some bounded results on non-\( p \) part of higher K-groups in cyclotomic \( \mathbb{Z}_p \) extensions of a real abelian number field.

This paper is organized as follows. In section 1, we state Sinnott’s theorem about rational measures on \( p \)-adic integers \( \mathbb{Z}_p \). In section 2, we use Sinnott’s theorem to show the mod \( \ell \) nonvanishing properties of \( L \)-values. In section 3, we give some applications on higher K-groups in a cyclotomic \( \mathbb{Z}_p \) extension of a real abelian number field.
2 Sinnot’s Theorem

Let \( \ell, p \) be two distinct prime numbers. A \( \overline{\mathbb{F}}_1 \) value measure on \( \mathbb{Z}_p \) is a finitely additive \( \overline{\mathbb{F}}_1 \)-valued functions on collection of compact open subsets of \( \mathbb{Z}_p \). Equivalently, a measure is a finitely additive \( \overline{\mathbb{F}}_1 \)-valued functions on the sets \( \{ c + p^n \mathbb{Z}_p | c \in \mathbb{Z}_p \} \). If \( \phi : \mathbb{Z}_p \to \overline{\mathbb{F}}_1 \) is a locally constant function, say constant on the cosets of \( p^n \mathbb{Z}_p \) in \( \mathbb{Z}_p \), then we define

\[
\int_{\mathbb{Z}_p} \phi(x) \alpha(x) = \sum_{a \mod p^n} \phi(a) \alpha(a + p^n \mathbb{Z}_p)
\]

The ring of measures. Let \( \alpha, \beta \) be two measures and \( U \) be an open compact subset of \( \mathbb{Z}_p \). Then \( \alpha + \beta \) is a measure. Define the convolution \( \alpha \ast \beta(U) = \int \int_{1U(x+y)d\alpha(x)d\beta(y)} \). Then \( \alpha \ast \beta \) is a measure and the measures is a ring respect to +, ∗.

Proposition 1. The ring of \( \overline{\mathbb{F}}_1 \)-valued measures on \( \mathbb{Z}_p \) is isomorphic to the ring of \( \overline{\mathbb{F}}_1 \)-valued functions on \( \mu_p^\infty \).

Proof. Given a measure \( \alpha \), define

\[
\hat{\alpha}(\zeta) = \int_{\mathbb{Z}_p} \zeta^x d\alpha(x)
\]

for \( \zeta \in \mu_p^\infty \).

Given a function \( f \) on \( \mu_p^\infty \), define

\[
\hat{f}(a + p^n \mathbb{Z}_p) = \frac{1}{p^n} \sum_{\zeta^{p^n}=1} f(\zeta) \zeta^{-a}.
\]

It is easy to verify these two map establish the ring isomorphism. They are Fourier transform and Fourier inversion. \( \square \)

A measure \( \alpha \) is a rational function if there is a rational function \( R(Z) \in \overline{\mathbb{F}}(Z) \) such that \( \hat{\alpha}(\zeta) = R(\zeta) \) for all but finitely many \( \zeta \in \mu_p^\infty \).

\( \alpha \) is supported on \( \mathbb{Z}_p \) if and only if \( \sum_{\epsilon^{p^n}=1} \hat{\alpha}(\epsilon \zeta) = 0 \) for every \( \zeta \in \mu_p^\infty \).

Let \( U = 1 + 2p\mathbb{Z}_p \). Let \( \psi \) be a character from \( \mathbb{Z}_p^\times \) to \( \mu_p^\infty \). Hence it is a character of \( U \) to \( \mu_p^\infty \subset \overline{\mathbb{F}}_1 \). We can view \( \psi \) as a character of \( \text{Gal}(\mathbb{Q}_\infty / \mathbb{Q}) \), where \( \mathbb{Q}_\infty \) is the cyclotomic \( \mathbb{Z}_p \) extension of \( \mathbb{Q} \). Let \( \Psi \) be the group of characters \( \mathbb{Z}_p^\times \) to \( \mu_p^\infty \subset \overline{\mathbb{F}}_1 \).

The following theorem proved by Sinnot is the main tool of this article.

Theorem 1 (Sinnot). Let \( \alpha \) be a rational function measure on \( \mathbb{Z}_p \) with values in \( \overline{\mathbb{F}}_1 \), and let \( R(Z) \in \overline{\mathbb{F}}(Z) \) be the associated rational function. Assume that \( \alpha \) is supported on \( \mathbb{Z}_p^\times \). If

\[
\Gamma_\alpha(\psi) := \int_{\mathbb{Z}_p^\times} \psi(x) d\alpha(x) = 0
\]
for infinitely many $\psi \in \Psi$, then

$$R(Z) + R(Z^{-1}) = 0.$$ 

### 3 Special values of Drichlet L-functions

Let $\chi$ be a nontrivial Drichlet character with conductor $f_\chi$. It is well-known $L(-k, \chi)$ is algebraic when $k$ is a non-positive integer. $L(-k, \chi) \neq 0$ if and only if $(-1)^k \chi(-1) = -1$, see [2]. Let $\ell$ be a prime. Fix an isomorphic $\iota$ from $\mathbb{C} \cong \mathbb{Q}_l$.

In this article, we define $\mathcal{L}(1, \chi)$.

**Proposition 2.** $L(-k, \chi) = (Z \frac{d}{dz})^k \mathcal{L}(Z)|_{Z=1}$, in particular, if $\ell \nmid f_\chi$, then $L(-k, \chi) \in \mathbb{Q}_l$.

**Proof.** Firstly, note that $\mathcal{L}(Z)$ is independent on $f$ for any multiper of $f_\chi$, since

$$\sum_{a=1}^{f_\chi} \chi(a) = \sum_{a=1}^{f_\chi} \chi(a)(1 + \left( Z \frac{d}{dz} \right)^{-1}) = \sum_{a=1}^{f_\chi} \chi(a)Z^a.$$

By [2] we know that $L(-k, \chi) = -\frac{B_{k+1}}{k+1}$, where $B_{k+1}$ is defined by the following Taylor expansion

$$\sum_{a=1}^{f_\chi} \chi(a) = \sum_{a=0}^{\infty} \frac{B_{n, \chi}}{n!}.$$

Since $\chi$ is nontrivial, we have $B_{0, \chi} = \sum_{a=1}^{f_\chi} \chi(a) = 0$. So we can write

$$\sum_{a=1}^{f_\chi} \chi(a) e^{at} = \sum_{a=0}^{\infty} \frac{B_{n+1, \chi}}{(n+1)!} e^n = \sum_{n=0}^{\infty} \frac{-L(-n, \chi)}{n!} e^n.$$

On the other hand, by setting $e^t = Z$, we have

$$\left( \frac{d}{dt} \right)^n \sum_{a=1}^{f_\chi} \chi(a) e^{at} = \sum_{a=1}^{f_\chi} \chi(a) e^{at} = -((Z \frac{d}{dZ})^n F_\chi(Z)|_{Z=1}).$$
Hence
\[ \sum_{a=1}^{f} \chi(a)e^{at}/(e^{ft} - 1) = \sum_{n=0}^{\infty} -((Z \frac{d}{dZ})^n F\chi(Z)|_{Z=1})t^n. \]

Therefore \( L(-k,\chi) = (Z \frac{d}{dZ})^k F\chi(Z)|_{Z=1}. \) Hence the rational function
\[ (Z \frac{d}{dZ})^k F\chi(Z) \]
has finite value at \( Z = 1. \) Note that the denominator of this rational function is a power of \( (1 - Zf) = [(1 - Z)(1 + Z + \cdots + Z^{f-1})], \) we know that the denominator in fact is a factor of a power of \( (1 + Z + \cdots + Z^{f-1}). \) Hence its denominator of the value at \( Z = 1 \) divides \( f. \) Therefore its values at \( Z = 1 \) is in \( \mathbb{Z}_l \) if \( \ell \nmid f. \)

Let \( \theta \) be a Dirichlet character. Let \( f\theta \) be its conductor and let \( f = 2pf\theta. \) Let
\[ R_k(Z) = (Z \frac{d}{dZ})^k \frac{\sum_{a=1,p|a} \theta(a)Z^a}{1 - Zf}. \]

Let \( \tilde{R}_k(Z) \in \overline{\mathbb{F}}(Z) \) be the rational function obtained from \( R(Z) \) by modulo its coefficients. We define the associated rational \( \mathbb{F}_l \)-valued measure \( \alpha_k \) on \( \mathbb{Z}_p \) by setting \( \hat{\alpha}_k(\zeta) = \tilde{R}_k(\zeta) \) for \( \zeta \in \mu_{p^\infty} \) for which \( \zeta \neq 1, \) and setting \( \hat{\alpha}_k(\zeta) = 0 \) for \( \zeta = 1. \) Note that \( \alpha \) is supported on \( \mathbb{Z}_p \times \) since \( \sum_{\epsilon \in \mathbb{F}_l} R(\epsilon Z) = 0. \)

**Proposition 3.** For any character \( \psi \in \Psi \) whose conductor \( p^m \) does not divide \( f, \) we have
\[ \frac{1}{2} L(-k, \theta \psi) = \int_{\mathbb{Z}_p^\times} \psi(x) d\alpha_k(x) \]

**Proof.** View \( \psi \) as a function on \( \mathbb{Z}/p^m \mathbb{Z} \) by letting \( \psi(a) = 0 \) if \( p|a. \) Then by Fourier transform, we have
\[ \psi(x) = \sum_{\zeta \in \mu_{p^m}} \tau(\psi, \zeta)\zeta^x, \]
where
\[ \tau(\psi, \zeta) = \frac{1}{p^m} \sum_{x \mod p^m} \psi(x)\zeta^{-x}. \]
\( \tau(\psi, \zeta) \) vanishes unless \( \zeta \) has order \( p^m, \) see [2].

Note that
\[ \int_{\mathbb{Z}_p^\times} \psi(x) d\alpha_k(x) = \int_{\mathbb{Z}_p^\times} \sum_{\zeta \in \mu_{p^m}} \tau(\psi, \zeta)\zeta^x d\alpha_k(x) \]
\[ = \sum_{\zeta \in \mu_{p^m}} \tau(\psi, \zeta) R_k(\zeta) = \sum_{\zeta \in \mu_{p^m} \setminus \mu_{p^{m-1}}} \tau(\psi, \zeta) R_k(\zeta) \]
Since $R_k(Z) + R_k(Z^{-1}) = (Z \frac{d}{dZ})^k \left( \frac{\sum_{a=1}^f \theta(a)Z^a}{1-Z} \right) = (Z \frac{d}{dZ})^k \left( \frac{\sum_{a=1}^f \theta(a)Z^a}{1-Zf^m} \right)$, if $\zeta$ is a primitive $p^m$-th root of unity,
\[ R(\zeta) + R(\zeta^{-1}) = (Z \frac{d}{dZ})^k \left( \frac{\sum_{a=1}^f \theta(a)\zeta^a Z^a}{1-Zf^m} \right) |_{Z=1}. \]

Now, since $\psi$ is even, we have $\tau(\psi, \zeta) = \tau(\psi, \zeta^{-1})$; hence
\[ 2 \int_{\mathbb{Z}_p^*} \psi(x)d\alpha_k(x) = \sum_{\zeta \in \mu_{p^n} \setminus \mu_{p^n-1}} (\tau(\psi, \zeta) + \tau(\psi, \zeta^{-1}))R_k(\zeta) \]
\[ = \sum_{\zeta \in \mu_{p^n} \setminus \mu_{p^n-1}} \tau(\psi, \zeta)(R_k(\zeta) + R_k(\zeta^{-1})) \]
\[ = (Z \frac{d}{dZ})^k \left( \frac{\sum_{a=1}^f \theta(a)\psi(a)Z^a}{1-Zf^m} \right) |_{Z=1} \]
\[ = L(-k, \theta\psi). \]

**Theorem 2.** Let $p, \ell$ be two different primes. $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$. Let $k \geq 0$ be an integer. Let $\theta$ be a Dirichlet character such that $\theta(-1)(-1)^k = 1$. Then there are only finitely many characters of $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ such that $\text{ord}_\ell(L(-k, \theta\psi)) > 0$. If $\ell$ does not divide the conductor of $\theta$, $L(-k, \theta\psi)$ is a $\ell$-unit for all but finitely many characters of $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$.

**Proof.** If there are infinitely many $\psi$ such that $\text{ord}_\ell(L(-k, \theta\psi)) > 0$, then by Theorem 1 we know that $\hat{R}_k(Z) + \hat{R}_k(Z^{-1}) = 0$, by Taylor expansion at $Z = 0$, we see that the coefficient of $Z$ of $\hat{R}_k(Z) + \hat{R}_k(Z^{-1})$ is 1. If $\ell$ does not divide the conductor of $\theta$, then $\frac{1}{2}L(-k, \theta\psi)$ is already in $\mathbb{Z}_l^*$ by Proposition 1.

**4 An application on $\ell$-part of K-groups in $\mathbb{Z}_p$ extensions**

We use the above theorem to give an application on K-groups in $\mathbb{Z}_p$ extension.

We have the following theorem relates the zeta value and the order of higher K-groups the ring of integers. See [6] or [1]. For general number field, this is conjectured by Lichtenbaum.

**Theorem 3.** If $F$ is totally real abelian number field, then
\[ \zeta_F(1 - 2k) = (-1)^k |\mathbb{Q}[F]:\mathbb{Q}|2^{|K_{4k-2}(\mathcal{O}_F)|} \left| \frac{K_{4k-2}(\mathcal{O}_F)}{K_{4k-1}(\mathcal{O}_F)} \right| \]
for some $\epsilon \in \mathbb{Z}$. 

5
**Theorem 4.** Let \( \ell > 2 \) and \( p \) be two distinct primes and \( F \) a real abelian number field. Let \( F_\infty/F \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \) and \( F_n \) its \( n \)-th layer. For an integer \( m \equiv 2, 3 \mod 4 \), let \( \ell^{e_m} \) be the exact power of \( \ell \) dividing \( |K_m(O_{F_n})| \). Then \( e_m \) is bounded as \( n \to \infty \). In addition, the \( 2 \)-part of the \( K_2(O_{F_n}) \) is also bounded.

Let \( q = p \) if \( p \) is an odd prime and \( q = 4 \) if \( p = 2 \). Let \( F \) be a real abelian number field with conductor \( dp^q \), where \( p \nmid r \).

Let \( F_\infty \) be the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \) and \( F_n \) be its \( n \)-th layer, hence \( \text{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z} \). Fix a prime \( \ell \neq p \), we concern about the \( \ell \)-part of \( K \) groups of \( F_n \).

When \( n \geq r \), the character group of \( \text{Gal}(F_n/Q) \) is a subgroup of the character groups of \( \text{Gal}(\mathbb{Q}(\zeta_{dp^n})/\mathbb{Q}) \cong (\mathbb{Z}/dp^n)\times \). Any Drichlet character of \( (\mathbb{Z}/dp^n)\times \) can be written in the form \( \chi \) or \( \chi\psi_m \) where \( \chi \) is a Drichlet character with \( \chi(-1) = 1 \) such that \( pq \) does not divide the conductor of \( \chi \), and \( \chi_m \) has order \( p^m \) and conductor \( qp^m \) with \( 1 \leq m \leq n \).

Let \( X \) be the Dirichlet character group of \( F \). When \( n \geq r \), \( \zeta_{F_n}(s) = \prod_{\chi \in X} \prod_{\psi \in \chi} L(s, \chi \psi) \), where \( \psi \) is a character of \( \text{Gal}(\mathbb{Q}_\infty/Q) \) which has order less than \( p^n \). Then

\[
\zeta_{F_n}(1 - 2k)/\zeta_{F_n-1}(1 - 2k) = \prod_{\chi \in X} \prod_{\psi \text{ order } p^n} L(1 - 2k, \chi \psi)
\]

By Theorem 2 and Theorem 3 we know that the \( \ell \)-part of \( |K_{4k-2}(F_n)|/|K_{4k-1}(F_n)| \) is bounded as \( n \to \infty \). It remains to show that they are bounded separately. This is easy from the following proposition quoted from [1].

**Proposition 4.** Let \( m \geq 3 \) be an odd integer. Set \( i = \frac{m+1}{2} \).

\[
K_m(O_F) \cong \begin{cases} 
Z^{r_1+r_2} \oplus Z/\omega_i(F) & n \equiv 1 \pmod{8} \\
Z^{r_2} \oplus Z/2\omega_i(F) \oplus (Z/2)^{r_1-1} & n \equiv 3 \pmod{8} \\
Z^{r_1+r_2} \oplus Z/\omega_i(F) & n \equiv 5 \pmod{8} \\
Z^{r_2} \oplus Z/\omega_i(F) & n \equiv 7 \pmod{8}
\end{cases}
\]

where \( \omega_i(F) \) is an integer defined in Section 5.3 [1]. We denote the \( \ell \)-exact power of \( \omega_i(F) \) by \( \omega_i^{(\ell)}(F) \). In [1] Section 5.3, it is proved that

\[
\omega_i^{(\ell)}(F) = \max \{ \ell^v | \text{Gal}(F(\mu_{\ell^v})/F) \text{ has exponent dividing } i \}.
\]

Thus it is clear that the \( \ell \)-part of the \( K_m(F_n) \) is bounded, since \( F_\infty \supset \cdots \supset F_n \supset \cdots \supset F_0 = F \) is cyclotomic \( \mathbb{Z}_p \) extension where \( p \neq \ell \).

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