LOW FREQUENCY RESOLVENT ESTIMATES FOR LONG RANGE
PERTURBATIONS OF THE EUCLIDEAN LAPLACIAN

JEAN-FRANÇOIS BONY AND DIETRICH HÄFNER

Abstract. Let $P$ be a long range metric perturbation of the Euclidean Laplacian on $\mathbb{R}^d$, $d \geq 3$. We prove that the following resolvent estimate holds:

$$\| \langle x \rangle^{-\alpha} (P - z)^{-1} \langle x \rangle^{-\beta} \| \lesssim 1 \quad \forall z \in \mathbb{C} \setminus \mathbb{R}, |z| < 1,$$

if $\alpha, \beta > 1/2$ and $\alpha + \beta > 2$. The above estimate is false for the Euclidean Laplacian in dimension 3 if $\alpha \leq 1/2$ or $\beta \leq 1/2$ or $\alpha + \beta < 2$.

1. Introduction

There are now many results dealing with the low frequency behavior of the resolvent of Schrödinger type operators. The methods used to obtain these results are various: one can apply the Fredholm theory to study perturbations by a potential (see e.g. [6]) or a short range metric (see e.g. [9]). The resonance theory is also useful to treat compactly supported perturbations of the flat case (see e.g. [3]). Using the general Mourre theory, one can obtain limiting absorption principles at the thresholds (see e.g. [5] or [8]). The pseudodifferential calculus of Melrose allows to describe the kernel of the resolvent at low energies for compactifiable manifolds (see e.g. [7]). Concerning the long range case, Bouclet [1] has obtained a uniform control of the resolvent for perturbations in divergence form. We refer to his article and to [4] for a quite exhaustive list of previous results for perturbations of the Euclidean Laplacian.

On $\mathbb{R}^d$ with $d \geq 3$, we consider the following operator

$$P = -b \text{div}(G\nabla b) = - \sum_{i,j=1}^{d} b(x) \frac{\partial}{\partial x_i} G_{i,j}(x) \frac{\partial}{\partial x_j} b(x),$$

where $b(x) \in C^\infty(\mathbb{R}^d)$ and $G(x) \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$ is a real symmetric $d \times d$ matrix. The $C^\infty$ hypothesis is made mostly for convenience, much weaker regularity could actually be considered. We make an ellipticity assumption:

(H1) $\exists C > 0, \forall x \in \mathbb{R}^d \quad G(x) \geq CI_d \quad \text{and} \quad b(x) \geq C,$

$I_d$ being the identity matrix. We also assume that $P$ is a long range perturbation of the Euclidean Laplacian:

(H2) $\exists \rho > 0, \forall \alpha \in \mathbb{N}^d \quad |\partial_x^\alpha (G(x) - I_d)| + |\partial_x^\alpha (b(x) - 1)| \lesssim \langle x \rangle^{-\rho - |\alpha|}.$

2000 Mathematics Subject Classification. 35P25, 47A10.

Key words and phrases. Resolvent estimates, asymptotically Euclidean manifolds.
In particular, if $b = 1$, we are concerned with an elliptic operator in divergence form $P = -\text{div}(G \nabla)$. On the other hand, if $G = (g^2 g^{ij}(x))_{i,j}$, $b = (\det g^{ij})^{1/4}$, $g = \frac{1}{b}$, then the above operator is unitarily equivalent to the Laplace–Beltrami $-\Delta_g$ on $(\mathbb{R}^d, g)$ with metric

$$g = \sum_{i,j=1}^{d} g_{i,j}(x) \, dx^i \, dx^j,$$

where $(g_{i,j})_{i,j}$ is inverse to $(g^{ij})_{i,j}$ and the unitary transform is just multiplication by $g$.

**Theorem 1.** Let $P$ be of the form (1) in $\mathbb{R}^d$ with $d \geq 3$. Assume (H1) and (H2).

i) For all $\varepsilon > 0$, we have

$$\| \langle x \rangle^{-1/2-\varepsilon} (\sqrt{P} - z)^{-1} \langle x \rangle^{-1/2-\varepsilon} \| \lesssim 1,$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

ii) For all $\varepsilon > 0$, we have

$$\| \langle x \rangle^{-1/2-\varepsilon} (P - z)^{-1} \langle x \rangle^{-1/2-\varepsilon} \| \lesssim |z|^{-1/2},$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

iii) For all $\alpha, \beta > 1/2$ with $\alpha + \beta > 2$, we have

$$\| \langle x \rangle^{-\alpha} (P - z)^{-1} \langle x \rangle^{-\beta} \| \lesssim 1,$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

**Remark 2.** i) The estimate (1) is not far from optimal. Indeed, this estimate is false for the Euclidean Laplacian $-\Delta$ in dimension 3 if $\alpha \leq 1/2$ or $\beta \leq 1/2$ or $\alpha + \beta < 2$.

ii) One can interpret (1) in the following way: one needs a $\langle x \rangle^{-1/2}$ on the left and on the right to assure that the resolvent is continuous on $L^2(\mathbb{R}^d)$ and one needs an additional $\langle x \rangle^{-1}$ (distributed, as we want, among the left and the right) to guarantee that its norm is uniform with respect to $z$.

iii) By interpolation of (3) and (1), for $\alpha, \beta > 1/2$ with $\alpha + \beta \leq 2$, one obtains estimates like (4) with $|z|^{-1+\frac{\alpha+\beta}{2}-\varepsilon}$ on the right hand side.

iv) In dimension 1, the kernel of $(-\Delta - z)^{-1}$ is given by $\frac{ie^{\sqrt{|x-y|}}}{2\sqrt{\pi}}$. In particular, this operator satisfies (3) but not (1) (for any $\alpha, \beta$). Therefore it seems that (3) is more general than (1). It could perhaps be possible to prove (3) in lower dimensions (at least, in dimension 2 and when $P$ is of divergence form $P = -\text{div}(G \nabla)$).

v) For large $z$, the estimate (3) coincides with the high energy estimate in the non-trapping case. In particular, if we suppose in addition a non trapping condition for $P$, then (2) and (3) hold uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$.

The proof of the above theorem is based on the low frequency estimates of [2]. Concerning the square root of $P$, they are used to treat the wave equation. Note that in [2] they are formulated for the Laplace–Beltrami operator $-\Delta_g$, but they obviously hold for the operators studied in the present paper. Essentially, we will show that $(2) \Rightarrow (3) \Rightarrow (1)$. 
2. Proof of the results

We begin by recalling some results of [2]. For \( \lambda \geq 1 \), we set
\[
A_\lambda = \varphi(\lambda P)A_0\varphi(\lambda P),
\]
where
\[
A_0 = \frac{1}{2}(xD + Dx), \quad D(A_0) = \{ u \in L^2(\mathbb{R}^d); \ A_0u \in L^2(\mathbb{R}^d) \},
\]
is the generator of dilations and \( \varphi \in C_0^\infty([0, +\infty[; [0, +\infty[) \) satisfies \( \varphi(x) > 1 \) on some open bounded interval \( I = [1 - \varepsilon, 1 + \varepsilon], \ 0 < \varepsilon < 1 \) sufficiently small. As usual, we define the multi-commutators \( \text{ad}^j_A B \) inductively by \( \text{ad}^0_A B = B \) and \( \text{ad}^{j+1}_A B = [A, \text{ad}^j_A B] \). We recall [2, Proposition 3.1]:

**Proposition 3.** i) We have \( (\lambda P)^{1/2} \in C^2(A_\lambda) \). The commutators \( \text{ad}^j_{A_\lambda} (\lambda P)^{1/2}, \ j = 1, 2, \) can be extended to bounded operators and we have, uniformly in \( \lambda \geq 1 \),
\[
\| [A_\lambda, (\lambda P)^{1/2}] \| \lesssim 1, \\
\| \text{ad}^2_{A_\lambda} (\lambda P)^{1/2} \| \lesssim \begin{cases} 1 & \rho > 1, \\
\lambda^\delta & \rho \leq 1,
\end{cases}
\]
where \( \delta > 0 \) can be chosen arbitrary small.

ii) For \( \lambda \) large enough, we have the following Mourre estimate:
\[
\mathbb{I}_I(\lambda P) [i(\lambda P)^{1/2}, A_\lambda] \mathbb{I}_I(\lambda P) \geq \frac{\sqrt{\text{inf} I}}{2} \mathbb{I}_I(\lambda P).
\]

iii) For \( 0 \leq \mu \leq 1 \) and \( \psi \in C_0^\infty([0, +\infty[) \), we have
\[
\| \langle A_\lambda \rangle^\mu \psi(\lambda P)\langle x \rangle^{-\mu} \| \lesssim \lambda^{-\mu/2 + \delta},
\]
for all \( \delta > 0 \).

We will also need [2, Lemma B.12]:

**Lemma 4.** Let \( \chi \in C_0^\infty(\mathbb{R}) \) and \( \beta, \gamma \geq 0 \) with \( \gamma + \beta/2 \leq d/4 \). Then, for all \( \delta > 0 \), we have
\[
\| \langle x \rangle^\beta \chi(\lambda P)u \| \lesssim \lambda^{-\gamma + \delta} \| \langle x \rangle^{\beta + 2\gamma} u \|
\]
uniformly in \( \lambda \geq 1 \).

By Mourre theory (see Theorem 2.2 and Remark 2.3 of [2] for example) and Proposition 3, we obtain the following limiting absorption principle:
\[
(5) \sup_{\text{Re } z \in I, \text{Im } z \neq 0} \| \langle A_\lambda \rangle^{-1/2 - \varepsilon}((\lambda P)^{1/2} - z)^{-1} \langle A_\lambda \rangle^{-1/2 - \varepsilon} \| \lesssim \lambda^\delta,
\]
for all \( \varepsilon, \delta > 0 \). This entails the following

**Lemma 5.** For \( \Psi \in C_0^\infty([0, +\infty[) \) and \( \varepsilon > 0 \), we have
\[
(6) \| \langle x \rangle^{-1/2 - \varepsilon} \Psi(\lambda P)(\sqrt{P} - \lambda^{-1/2} z)^{-1} \langle x \rangle^{-1/2 - \varepsilon} \| \lesssim 1,
\]
\[
(7) \| \langle x \rangle^{-1/2 - \varepsilon} \Psi(\lambda P)(P - \lambda^{-1} z^2)^{-1} \langle x \rangle^{-1/2 - \varepsilon} \| \lesssim \frac{\sqrt{\lambda}}{|z|},
\]
uniformly in \( \lambda \geq 1 \) and \( z \in \mathbb{C} \setminus \mathbb{R} \) with \( \text{Re } z \in I \).
Proof. Let $\tilde{\Psi} \in C_0^\infty(]0, +\infty[)$ be such that $\Psi \tilde{\Psi} = \Psi$.

To prove the first identity, we write
\[
\|\langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P)(P^{1/2} - \lambda^{-1/2} z)^{-1}\langle x \rangle^{-1/2-\varepsilon}\| \\
\lesssim \|\langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P)(A_{\lambda})^{1/2+\varepsilon}\| \|\langle A_{\lambda} \rangle^{-1/2-\varepsilon} (P^{1/2} - \lambda^{-1/2} z)^{-1}\langle A_{\lambda} \rangle^{-1/2-\varepsilon}\| \\
\times \|\langle A_{\lambda} \rangle^{1/2+\varepsilon} \tilde{\Psi}(\lambda P)(P^{1/2} - \lambda^{-1/2} z)^{-1}\langle x \rangle^{-1/2-\varepsilon}\| \\
\lesssim \lambda^{-\frac{1}{2} - \frac{\varepsilon}{2} + \delta} \lambda^{\frac{1}{2} + \delta} \lambda^{-\frac{1}{2} - \frac{\varepsilon}{2} + \delta} \lesssim 1.
\]

Here we have used Proposition 3 iii), Lemma 4 as well as the fact that $\delta$ can be chosen arbitrary small.

To obtain (7), it is sufficient to write
\[
\|\langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P)(P - \lambda^{-1} z^2)^{-1}\langle x \rangle^{-1/2-\varepsilon}\| \\
\lesssim \lambda^{1/2} \|\langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P)((\lambda P)^{1/2} + z)^{-1}\langle x \rangle^{1/2+\varepsilon/2}\| \\
\times \|\langle x \rangle^{-1/2-\varepsilon/2} \tilde{\Psi}(\lambda P)(P^{1/2} - \lambda^{-1/2} z)^{-1}\langle x \rangle^{-1/2-\varepsilon}\| \\
\lesssim \frac{1}{\sqrt{\varepsilon + 1}}.
\]

Here we have used (9) and Lemma 4. It is clear from the proof of Lemma 4 in [2] that we can apply it to $\Psi(\lambda P)((\lambda P)^{1/2} + z)^{-1}$ and that we gain $\frac{1}{\sqrt{\varepsilon + 1}}$. Indeed, as an almost analytic extension, we can just take the almost analytic extension of $\Psi$ multiplied by the analytic function $\frac{1}{\sqrt{\varepsilon + 1}}$.

Proof of Theorem 7 We only show the third part of the theorem, the proof of the other parts is analogous. Also it is clearly sufficient to replace $z$ by $\lambda^{-1} z^2$ with $\text{Re } \tilde{z} = 1 \in I$ and $\lambda \geq 1$ (for instance, $\lambda = (\text{Re } \sqrt{z})^{-2}$ and $\tilde{z} = \sqrt{z}/(\text{Re } \sqrt{z})$). Let $\varphi, \tilde{\varphi} \in C_0^\infty([\frac{1}{3}, 3])$ and $f \in C^\infty(\mathbb{R})$ be such that $\tilde{\varphi} = 1$ on the support of $\varphi$, $f(x) = 0$ for $x < 2$ and
\[
f(x) + \sum_{\mu = 2^a, n \geq 0} \varphi(\mu x) = 1,
\]
for all $x > 0$. Since 0 is not an eigenvalue of $P$, we can write
\[
\langle x \rangle^{-a}(P - z)^{-1}\langle x \rangle^{-\beta} = \langle x \rangle^{-a} f(P)(P - z)^{-1}\langle x \rangle^{-\beta} + \sum_{\mu = 2^a, n \geq 0} \langle x \rangle^{-a} \varphi(\mu P)(P - \lambda^{-1} z^2)^{-1}\langle x \rangle^{-\beta}.
\]

Of course, since $|z| < 1$, the functional calculus gives
\[
\|\langle x \rangle^{-a} f(P)(P - z)^{-1}\langle x \rangle^{-\beta}\| \lesssim 1.
\]

Let $\bar{\alpha} = \min(\alpha, \frac{d}{4})$ and $\bar{\beta} = \min(\beta, \frac{d}{4})$. Note that $\bar{\alpha} + \bar{\beta} > 2$ since $d \geq 3$. Let $\Psi \in C_0^\infty([0, +\infty[)$ be such that $\Psi = 1$ near $[\frac{1}{17}, 12]$. Then, for $\frac{d}{4} \leq \lambda \leq 4\mu$, we have
\[
\|\langle x \rangle^{-\bar{\alpha}} \varphi(\mu P)(P - \lambda^{-1} z^2)^{-1}\langle x \rangle^{-\bar{\beta}}\| \\
\lesssim \|\langle x \rangle^{-\bar{\alpha}} \varphi(\mu P)(\lambda P)^{1/2+\varepsilon}\| \|\langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P)(P - \lambda^{-1} z^2)^{-1}\langle x \rangle^{-1/2-\varepsilon}\| \|\langle x \rangle^{1/2+\varepsilon} \tilde{\varphi}(\mu P)(\lambda P)^{1/2-\varepsilon}\| \\
\lesssim \lambda^{\frac{1}{2} + \frac{\varepsilon}{2} + 2\delta} \lambda^{-\frac{1}{2} - \frac{\varepsilon}{2} + 3\delta} \lesssim \lambda^{1+2\delta - \frac{\bar{\alpha} + \bar{\beta}}{2}} \lesssim 1,
\]
for all $\varepsilon, \delta > 0$ small enough. Here we have used (7) and two times Lemma 4. On the other hand, for $\lambda \notin \left[ \frac{4}{\varepsilon}, 4\mu \right]$, the functional calculus and Lemma 3 yield

$$\left\| (\alpha - \Delta)^{-1}(x) \right\| \leq \frac{\varepsilon}{\varepsilon - 1} \left( \left\| (\alpha - \Delta)^{-1}(x) \right\|_1 \right),$$

for all $\varepsilon > 0$. Splitting the sum into two, we get

$$\sum_{4\mu < \lambda} |\alpha - \lambda|^{-1} \mu^{-\frac{\alpha + \beta}{2} + \varepsilon} \leq \sum_{4\mu < \lambda} \mu \mu^{-\frac{\alpha + \beta}{2} + \varepsilon} \leq 1,$$

$$\sum_{\mu > 4\lambda} |\alpha - \lambda|^{-1} \mu^{-\frac{\alpha + \beta}{2} + \varepsilon} \leq \sum_{\mu > 4\lambda} \lambda \mu^{-\frac{\alpha + \beta}{2} + \varepsilon} \leq 1.$$

This finishes the proof of the theorem. \qed

**Proof of Remark 2.** Let us recall that the kernel of the resolvent of the flat Laplacian in $\mathbb{R}^3$ at $z = 0$ is given by

$$K(x, y, 0) = \frac{1}{4\pi |x - y|}.$$

Assume that $\langle x \rangle^{-\alpha}(-\Delta)^{-1}(x)^{-\beta}$ is bounded on $L^2(\mathbb{R}^3)$. Applying to $\chi \in C^\infty_0(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$, we find

$$\langle x \rangle^{-\alpha}(-\Delta)^{-1}(x)^{-\beta} \chi (x) = \int \frac{1}{4\pi |x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \chi (y) dy \gtrsim \langle x \rangle^{-\alpha - 1},$$

for $|x| \gg 1$. But $\langle x \rangle^{-1-\alpha} \in L^2(\mathbb{R}^3)$ if and only if $\alpha > 1/2$. The condition $\beta > 1/2$ is checked in the same way. We now apply the resolvent to $f(x) = \langle x \rangle^{-3/2 - \varepsilon} \in L^2(\mathbb{R}^3)$ and find

$$\langle x \rangle^{-\alpha}(-\Delta)^{-1}(x)^{-\beta} f(x) = \int \frac{1}{4\pi |x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \langle y \rangle^{-3/2 - \varepsilon} dy \gtrsim \langle x \rangle^{-\alpha - 1} \int \frac{1}{4\pi |x - y|} \langle x \rangle^{-\alpha - 3/2 - \varepsilon} dy \gtrsim \langle x \rangle^{-\alpha - 1}.$$
[8] S. Richard, *Some improvements in the method of the weakly conjugate operator*, Lett. Math. Phys. 76 (2006), no. 1, 27–36.

[9] X.-P. Wang, *Asymptotic expansion in time of the Schrödinger group on conical manifolds*, Ann. Inst. Fourier 56 (2006), no. 6, 1903–1945.

E-mail address: bony@math.u-bordeaux1.fr
E-mail address: hafner@math.u-bordeaux1.fr

Institut de Mathématiques de Bordeaux
UMR 5251 du CNRS
Université de Bordeaux I
351 cours de la Libération
33 405 Talence cedex
France