MULTIVARIATE NORMAL APPROXIMATION FOR TRACES OF ORTHOGONAL AND SYMPLECTIC MATRICES

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Abstract. We show that the distance in total variation between \((\text{Tr } U, \frac{1}{\sqrt{2}} \text{Tr } U^2, \cdots, \frac{1}{\sqrt{m}} \text{Tr } U^m)\)
and a real Gaussian vector, where \(U\) is a Haar distributed orthogonal or symplectic matrix of size \(2n\) or \(2n + 1\), is bounded by \(\Gamma(2 \frac{n}{m} + 1)^{-\frac{1}{2}}\) times a correction. The correction term is explicit and
holds for all \(n \geq m^4\), for \(m\) sufficiently large. For \(n \geq m^3\) we obtain the bound \((\frac{n}{m})^{-c \sqrt{m}}\) with an explicit constant \(c\). Our method of proof is based on an identity of Toeplitz+Hankel determinants
due to Basor and Ehrhardt, see [2], which is also used to compute the joint moments of the traces.

Nous montrons que la distance en variation totale entre \((\text{Tr } U, \frac{1}{\sqrt{2}} \text{Tr } U^2, \cdots, \frac{1}{\sqrt{m}} \text{Tr } U^m)\) et un vecteur gaussien réel, où \(U\) est une matrice orthogonale ou symplectique distribuée selon la mesure
de Haar et de taille \(2n\) ou \(2n + 1\), est bornée par \(\Gamma(2 \frac{n}{m} + 1)^{-\frac{1}{2}}\) fois une correction. Cette correction a une formule explicite qui vaut pour tout \(n \geq m^4\), pour \(m\) suffisamment grand. Lorsque \(n \geq m^3\) nous obtenons la borne \((\frac{n}{m})^{-c \sqrt{m}}\) où \(c\) est une constante explicite. Notre méthode de démonstration repose sur une identité de déterminants du type Toeplitz+Hankel due à Basor et Ehrhardt, voir [2],
qui est aussi utilisée pour calculer les moments joints des traces.

0. Introduction

Let \(U\) be an element of the orthogonal, unitary or symplectic groups distributed according to normal-
ized Haar measure. In [7] Diaconis and Shahshahani showed that the joint moments of \(\text{Tr } U, \text{Tr } U^2, \cdots, \text{Tr } U^m\) up to a certain order are equal to those of \(m\) independent (complex) Gaussians if the
matrices are orthogonal or symplectic (unitary). An immediate consequence is the convergence of the vector
\((\text{Tr } U, \text{Tr } U^2, \cdots, \text{Tr } U^m)\) to a Gaussian vector as the size of the matrix \(n\) goes to infinity, and
a natural question is its rate of convergence, since the moments are equal to high order. A first answer to
this question was given by Stein in [19] who obtained a super-polynomial rate of convergence in total
variation for a single trace in the case of the orthogonal group. Later the second author of this paper
considered linear combinations of the above traces of powers of matrices and showed in [12] that the
total variation between those and a Gaussian is bounded by \(C_1 n^{-\delta_1 n}\) for the unitary case and\(C_2 e^{-\delta_2 n}\) for the orthogonal/symplectic case, for some non-explicit constants \(C_1, C_2\) and \(\delta_1, \delta_2\). This result was recently extended to the multivariate case by the second author and Lambert in [13] where they also allowed the highest power \(m\) to increase with the size of the matrix \(n\) and kept track of the constants.
They proved that the rate of convergence, again in total variation, is bounded by \(\Gamma(\frac{n}{m} + 1)^{-1}\) times a correction term, provided \(m\) grows slower than approximately \(\sqrt{n}\). This paper is devoted to the same problem but for orthogonal and symplectic matrices. It improves considerably the result in [12]. We show that if \(n \geq m^4\) and \(m\) is sufficiently large (see Corollary 0.3), the total variation is bounded by

\[
16m^\frac{3}{2} \sqrt{\Omega_m (24mN \log N)} \frac{(e^{3/2} (\log m + 1))^N}{\sqrt{N} \sqrt{\Gamma(2N + 1)}}
\]

where \(N = n/m\) and \(\Omega_m = \frac{m^2}{\Gamma(\frac{m}{2} + 1)}\) is the volume of the unit \(m\)-ball. If \(m\) is smaller than
what’s assumed in the corollary, or if we only assume \(n \geq m^3\), we lose the factor \(N^{-N}\) from the

Date: September 7, 2022.

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Gamma function which is replaced by $N^{-c_1 N}$ or $N^{-c_2 \sqrt{N}}$ for some constants $c_1$ and $c_2$ (see Corollary 5.4 and 5.5). We also mention that the case of a single power (i.e. $\text{Tr} U^k$, $k \geq 1$) for any of the above mentioned groups allows more precise estimates and is considered in a forthcoming paper [4]. Another related problem is the rate of convergence in Wasserstein distance which was studied by Dbler and Stolz in [5] for the multivariate case and the unitary, special orthogonal, and unitary symplectic groups.

In the following we let $U$ denote a random matrix drawn from either the orthogonal or symplectic group under normalized Haar measure. We consider the vector $X = (X_1, X_2, \ldots, X_m)$, where $X_k = \frac{1}{\sqrt{m}}(\text{Tr} U^k - \mathbb{E}[\text{Tr} U^k])$ and we denote by $F_{n,m}^{a,b}$ its characteristic function, where $m$ is the length of the vector, $n$ determines the size of the matrix, and $a, b$ specify which group the matrix belongs to. The values of $a$ and $b$ appear in the joint eigenvalue density of the matrices, which is given by

$$
\rho_{n}^{a,b}(x) = \frac{1}{Z_{n}^{a,b}} \prod_{1 \leq j \leq n} (1 - x_j)^a (1 + x_j)^b \prod_{1 \leq j < k \leq n} (x_j - x_k)^2
$$

on $[-1,1]^n$, where $Z_{n}^{a,b}$ is the normalization constant, $Z_{n}^{a,b} = (\pi^n n!) / 2^{n^2 - (1-a-b)n+1} \cdot (a,b < 0)$. If we set $(a, b) = (1/2, 1/2)$ we obtain the eigenvalue density of $Sp(2n)$ and $O(2n)^-$, and if $(a, b) = (-1/2, -1/2)$, $(a, b) = (-1/2, 1/2)$ and $(a, b) = (1/2, -1/2)$ we get the density of $O(2n)^+$, $O(2n+1)^-$ and $O(2n+1)^+$ respectively. For $O(2n)^-$, the probability density is actually $\rho_{n-1/2,1/2}^{1/2}$, i.e. $n$ is replaced by $n-1$. To keep notation simple we will sometimes replace $a, b$ with their respective sign. Observe that the density is supported on $[-1,1]^n$ but the eigenvalues of the random matrices all lie on the unit circle. The reason is that all eigenvalues except for 1 and -1 occur in conjugate pairs, so (0.1) is obtained by making the change of variables $x_j = \cos(\theta_j)$ in the following eigenangle densities supported on $(0, \pi)$:

$$
\rho_{n}^{-}(-\theta) = \frac{2(n-1)^2}{n! \pi^n} \prod_{1 \leq j < k \leq n} (\cos \theta_j - \cos \theta_k)^2
$$

$$
\rho_{n}^{+}(-\theta) = \frac{2n^2}{n! \pi^n} \prod_{1 \leq j \leq n} \sin^2 \theta_j \prod_{1 \leq j < k \leq n} (\cos \theta_j - \cos \theta_k)^2
$$

$$
\rho_{n}^{-}(-\theta) = \frac{2n^2}{n! \pi^n} \prod_{1 \leq j \leq n} \cos^2 \frac{\theta_j}{2} \prod_{1 \leq j < k \leq n} (\cos \theta_j - \cos \theta_k)^2
$$

$$
\rho_{n}^{+}(-\theta) = \frac{2n^2}{n! \pi^n} \prod_{1 \leq j \leq n} \sin^2 \frac{\theta_j}{2} \prod_{1 \leq j < k \leq n} (\cos \theta_j - \cos \theta_k)^2.
$$

These identities are due to H. Weyl and often called the Weyl integration formula, see [15]. Note also that there are deterministic eigenvalues: $O(2n+1)^+$ must have an eigenvalue at 1, $O(2n+1)^-$ at $-1$ and $O(2n)^-$ at both 1 and $-1$, for their determinant to have the correct value; this and the fact that the eigenvalues come in conjugate pairs explain why there are only $n$ variables in the eigenvalue density of $O(2n+1)\pm$ and $n-1$ in that of $O(2n)^-$. These deterministic eigenvalues do not appear in the above joint eigenvalue densities but they also do not affect the random vector $X$ since it is centered. Thus in the following, we will only consider random eigenvalues and write $\text{Tr} U$ for their sum, disregarding the possible eigenvalues at $\pm 1$.

We will use both (0.1) and (0.2) for the eigenvalue densities so to differentiate them we will denote by $\mathbb{E}_{n}^{a,b}$ the expected value with respect to (0.1) and by $\mathbb{E}_{G(n)}$ the expected value with respect to (0.2), where $G(n)$ denotes either $O(2n)^+$, $O(2n)^-$, $O(2n+1)^+$, $O(2n+1)^-$ or $Sp(2n)$. The characteristic
function \(F_{n,m}^{a,b}\) is therefore equal to

\[
F_{n,m}^{a,b}(\xi) = \mathbb{E}_{G(n)} \left[ \exp \left( i \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} (\text{Tr} U^k - \mathbb{E}_{G(n)}[\text{Tr} U^k]) \right) \right]
\]

\[
= \mathbb{E}_{n}^{a,b} \left[ \prod_{1 \leq j \leq n} \exp \left( i \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} \left( 2T_k(x_j) - \frac{\mathbb{E}_{G(n)}[\text{Tr} U^k]}{n} \right) \right) \right]
\]

where \(\xi = (\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^m\) and \(T_k\) is the \(k\)th Chebyshev polynomial. The mean value \(\mathbb{E}_{G(n)}[\text{Tr} U^k]\), both including and excluding the deterministic eigenvalues, is given in Proposition 1.3 and the following remark. We also introduce the functions

\[
f(x) = \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} \left( 2T_k(x) - \frac{\mathbb{E}_{G(n)}[\text{Tr} U^k]}{n} \right), \quad x \in [-1, 1]
\]

and

\[
g(\theta) = f(\cos(\theta)) = \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} \left( 2 \cos k\theta - \frac{\mathbb{E}_{G(n)}[\text{Tr} U^k]}{n} \right), \quad \theta \in [0, \pi)
\]

and write \(\text{Tr} g(U) = \sum_{1 \leq j \leq n} g(\theta_j)\). Now we have

\[
F_{n,m}^{a,b}(\xi) = \mathbb{E}_{G(n)}[e^{i\text{Tr} g(U)}] = \mathbb{E}_{n}^{a,b} \left[ \prod_{1 \leq j \leq n} e^{if(x_j)} \right].
\]

Finally we let \(p_{n,m}^{a,b}\) be the probability density of the random vector \(X\) and \(\Psi_m(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{m/2}}\) that of a standard normal vector. Hence, \(F_{n,m}^{a,b}(\xi)\) is the Fourier transform of \(p_{n,m}^{a,b}\) evaluated at \(\xi\), where our convention for the Fourier transform is chosen so that it coincides with the characteristic function,

\[
\hat{\phi}(\xi) = \int_{\mathbb{R}^m} e^{i\xi \cdot \omega} \phi(\omega) d\omega, \quad \xi \in \mathbb{R}^m.
\]

We also denote by \(\Delta_{n,m}^{(2)}\) the \(L_2\) distance between \(p_{n,m}^{a,b}\) and \(\Psi_m\) and by \(\Delta_{n,m}^{(1)}\) the \(L_1\) distance. We obtain the following bounds on \(\Delta_{n,m}^{(2)}\).

**Theorem 0.1.** Assume \(n \geq m^3\), \(m \geq 3\). For any pair \((a, b) = (\pm 1/2, \pm 1/2),\)

\[
\Delta_{n,m}^{(2)} \leq \sqrt{\Omega_m N}^{m/2} \left[ \frac{16}{15} e^{3/24} (e^{9/8} + 1) m^{3/2} \left( \frac{m}{2} \right)^{m} (\frac{m}{2})^{-1} \left( 1 - \frac{1}{c_1(m)} \right)^{m} \exp \left( - \frac{(1 - c_1(m))^2 N^2}{4c_2(m)(m + 1)^{3/2}(\log m + 1)} \right) \right]
\]

\[
+ \sqrt{3(2)e^{4m^2}} \left( \frac{\sqrt{c_3(m)m}}{2\pi n} \right) \left( \frac{m}{2} \right)^{m} \exp \left( - \frac{(1 - c_1(m))^2 N^2}{4c_2(m)(m + 1)^{3/2}(\log m + 1)} \right)
\]

\[
= m^{\frac{m}{2}} \exp \left( - \frac{(1 - c_1(m))^2 N^2}{4c_2(m)(m + 1)^{3/2}(\log m + 1)} \right) + \frac{1}{(2\pi m + 1)} e^{-\frac{N^2}{2\pi n}}
\]

where \(N = n/m\) and \(\Omega_m = \pi^{m/2}/(\Gamma(m/2) + 1)\) is the volume of the unit \(m\)-ball. The functions \(c_1(m), c_2(m)\) and \(c_3(m)\) are bounded and defined in (3.37), (3.38) and (4.44).

As a consequence we are able to derive a bound on \(\Delta_{n,m}^{(1)}\), whence on the total variation. Indeed recall that the total variation between two random variables \(U\) and \(V\) taking values in \(\mathbb{R}^m\), with p.d.f. \(p_U\) and \(p_V\) satisfies

\[
\|U - V\|_{TV} = \sup_{A \in \mathcal{A}(\mathbb{R}^m)} \left| \int_A p_U(x) dx - \int_A p_V(x) dx \right| \leq \|p_U - p_V\|_{L^1}.
\]

**Theorem 0.2.** Assume \(m \geq 4\). For any pair \((a, b) = (\pm 1/2, \pm 1/2),\)

\[
\Delta_{n,m}^{(1)} \leq 2(48 \log \Delta_{n,m}^{(2)} - 1)^{\frac{m}{2}} \Delta_{n,m}^{(2)}
\]
for $n \geq m^4$, provided $\Delta^{(2)}_{n,m} \leq 3m(12em)^{-}\frac{3}{2}$ and

$$\Delta^{(1)}_{n,m} \leq 2(80 \log \Delta^{(2)}_{n,m} - 1)^{\frac{3}{2}} \Delta^{(2)}_{n,m}$$

for $n \geq m^3$, provided $\Delta^{(2)}_{n,m} \leq 2.5m(20em)^{-\frac{m}{4}}$.

We can simplify these results by considering special cases of $m$ and $n$. For example,

**Corollary 0.3.** If $m, n$ satisfy the conditions in one column of the following table

| $n \geq m^4$ | $m^5$ | $m^6$ | $m^7$ | $m^8$ | $m^9$ | $m^{10}$ |
|-------------|-------|-------|-------|-------|-------|---------|
| $m \geq$    | $10^{19}$ | 1140 | 34    | 11    | 6     | 5        | 4        |

then,

$$\Delta^{(2)}_{n,m} \leq 8m^3 \sqrt{\Omega_m} \left(\frac{m}{2}\right)^{\frac{3}{2}} \frac{(e^{3/2} \log m + 1)^N}{\sqrt{N} \sqrt{\Gamma(2N + 1)}}$$

and

$$\Delta^{(1)}_{n,m} \leq 16m^3 \sqrt{\Omega_m (24n \log N)} \frac{(e^{3/2} \log m + 1)^N}{\sqrt{N} \sqrt{\Gamma(2N + 1)}}$$

where $N = n/m$ and $\Omega_m = \pi^{\frac{m}{2}} / \Gamma(\frac{m}{2} + 1)$ is the volume of the unit $m$-ball.

See also corollaries 5.4 and 5.5 for other conditions on $m$ and $n$. In Remark 5.1 we explain the origin of the conditions $n \geq m^4$ and $n \geq m^3$. We do not know how fast $m$ is allowed to grow relatively to $n$ to obtain a fast rate of convergence. In [7] it is suggested that there is some analogy between the present problem and the rate of convergence of the vector $(C_1, \ldots, C_m)$ containing the number of cycles of length $j, C_j$, of a uniform random permutation as its size $n$ grows to infinity. In [1], Arratia and Tavaré proved that $(C_1, \ldots, C_m)$ converges to a $m$-tuple of Poisson random variables if and only if $m/n \to 0$, and if $m/n \to 0$ then the convergence is superexponentially fast relative to $n/m$. If we believe in the analogy between the two problems we could conjecture that the fast rate of convergence for $m$-tuples of traces holds if $m/n$ goes to zero, but not if $m$ is of order $n$.

The paper is organised as follows: first we present some general facts about integrals over the orthogonal/symplectic groups: an analogue of Heine’s identity and a result of Basor and Ehrhardt [2] expressing Toeplitz+Hankel determinants using Fredholm determinants. Combined they give a new proof of the moment identities of Diaconis and Shahshahani [7]. They also give our estimates on the characteristic function for what we call the small regime of $\xi$, which we give in the second section. The intermediate and large regimes are treated in the third and fourth sections by making a certain change of variables in the integral expression of the characteristic function, a method introduced in [11]. We also reuse [2] and the results of [3]. In the last section we gather all our estimates to bound $\Delta^{(2)}_{n,m}$, the $L_2$ distance between $p^{a,b}_{n,m}$ and $\Psi_m$, via Plancherel’s theorem. We then use [2] one more time to obtain tail probabilities for $X$ which give us the final bound on the total variation $\Delta^{(1)}_{n,m}$.

1. Preliminaries

Heine’s identity expresses Toeplitz matrices as integrals over the unitary group. The next lemma is its orthogonal/symplectic analogue. It has been used previously in e.g. [12] and [14].
Lemma 1.1. For any complex function $\psi$ on $[-1,1]$, 
\[
\mathbb{E}_n^+[\prod_{j=1}^n \psi(x_j)] = \det(\hat{\phi}_{i-j} + \hat{\phi}_{i+j+1})_{0 \leq i,j \leq n-1}
\]
\[
\mathbb{E}_n^-[\prod_{j=1}^n \psi(x_j)] = \det(\hat{\phi}_{i-j} - \hat{\phi}_{i+j+1})_{0 \leq i,j \leq n-1}
\]
\[
\mathbb{E}_n^+[\prod_{j=1}^n \psi(x_j)] = \det(\hat{\phi}_{i-j} - \hat{\phi}_{i+j+2})_{0 \leq i,j \leq n-1}
\]
\[
\mathbb{E}_n^-[-\prod_{j=1}^n \psi(x_j)] = \det(\hat{\phi}_{i-j} + \hat{\phi}_{i+j})_{0 \leq i,j \leq n-1}
\]

where $\hat{\phi}_n$ is the $n$th Fourier coefficient of $\psi \circ \cos$.

Proof. The last product in the eigenvalue density (0.1) is equal to the Vandermonde determinant squared, therefore
\[
\mathbb{E}_n^{a,b}[-\prod_{j=1}^n \psi(x_j)] = \frac{1}{Z^{a,b}_n} \int_{[-1,1]^n} \prod_{j=1}^n \psi(x_j)(1-x_j)^a(1+x_j)^b \det(x_i^{j-1})_{1 \leq i,j \leq n} dx.
\]

We can perform column operations inside the determinants and obtain
\[
C \int_{[-1,1]^n} \prod_{j=1}^n \psi(x_j)(1-x_j)^a(1+x_j)^b \det(p_j^{a,b}(x_i))_{1 \leq i,j \leq n} dx
\]
for some constant $C$, where $\{p_j^{a,b}\}_{j=0}^n$ is any family of linearly independent polynomials such that $p_j$ has degree $j$. By the Andrèief identity,
\[
E_n^{a,b}[-\prod_{j=1}^n \psi(x_j)] = C \cdot N! \det(\alpha_{ij})_{0 \leq i,j \leq n-1}
\]
where
\[
\alpha_{i,j} = \int_{-1}^1 p_i^{a,b}(x) p_j^{a,b}(x) \psi(x)(1-x)^a(1+x)^b dx.
\]

We recover the identities by choosing the polynomials to be normalised Jacobi polynomials, i.e.
\[
p_j^-(\cos \theta) = \sqrt{\frac{2}{\pi}} \cos j \theta, \quad j \geq 1; \quad p_0^- \cos \theta = \frac{1}{\sqrt{\pi}}
\]
\[
p_j^+(\cos \theta) = \sqrt{\frac{2}{\pi}} \frac{\sin(j+1)\theta}{\sin \theta}
\]
\[
p_j^\circ \cos \theta = \frac{1}{\sqrt{\pi}} \frac{\sin(2j+1)\theta/2}{\cos \theta/2}; \quad p_j^- \cos \theta = \frac{1}{\sqrt{\pi}} \frac{\cos(2j+1)\theta/2}{\cos \theta/2}
\]
which are orthogonal with respect to $(1-x)^a(1+x)^b$. We see that $C = 1/N!$ by letting $\psi = 1$. □

The Toeplitz+Hankel determinants above have a Fredholm determinant expansion, found by Basor and Ehrhardt in [2], which we present in the next proposition. These identities are similar to the Borodin-Okounkov-Case-Geronimo formula that hold for Toeplitz determinants and will be the starting point of our analysis of the characteristic function, which ultimately will give us the bound on the total variation from a Gaussian vector.

In the next proposition we consider functions in the Besov class $B_1^1$, i.e. functions $\omega$ on the unit circle which satisfy
\[
\|\omega\|_{B_1^1} := \int_{-\pi}^\pi \frac{1}{y^2} \int_{-\pi}^\pi |\omega(e^{ix+iy}) + \omega(e^{ix-iy}) - 2\omega(e^{ix})| dx dy < \infty.
\]
If \( \omega \in B_1^1 \) we let \( \omega_+ \) denote its projection on \( B_1^1 \), the subspace of \( B_1^1 \) for which \( \omega_k = 0 \) for \( k < 0 \), and we write \( \hat{\omega}(e^{i\theta}) := \omega(e^{-i\theta}) \).

**Proposition 1.2.** [2] Denote by \( Q_n \) the projection operator acting on \( l_2(\mathbb{N}) \) that sets the first \( n \) coefficients to zero, and let \( H(c) \) be the Hankel operator with symbol \( c \in L^{\infty}(\mathbb{T}) \), i.e. the bounded linear operator on \( l_2(\mathbb{N}) \) with matrix representation \( H(c) = (c_{j+k+1})_{j,k=0}^{\infty} \), where \( c_k \) is the \( k \)th Fourier coefficient of \( c \). Assume that \( b_+ \in B_1^1 \), and set \( b = b_+ + b_+ \), \( a = e^b \) (so \( a_+ = e^{b_+} \)). Then,

\[
\det(\hat{a}_{i-j} + \hat{a}_{i+j+1})_{0 \leq i,j \leq n-1} = \\
\exp \left( n \log a_0 + \sum_{k=0}^{\infty} [\log a]_{2k+1} + \frac{1}{2} \sum_{k=1}^{\infty} k[\log a]_k^2 \right) \det(1 + Q_n H(a_+^{-1}\tilde{a}_+)Q_n) \\
\det(\hat{a}_{i-j} - \hat{a}_{i+j+1})_{0 \leq i,j \leq n-1} = \\
\exp \left( n \log a_0 - \sum_{k=0}^{\infty} [\log a]_{2k+1} + \frac{1}{2} \sum_{k=1}^{\infty} k[\log a]_k^2 \right) \det(1 - Q_n H(a_+^{-1}\tilde{a}_+)Q_n) \\
\det(\hat{a}_{i-j} - \hat{a}_{i+j+2})_{0 \leq i,j \leq n-1} = \\
\exp \left( n \log a_0 - \sum_{k=1}^{\infty} [\log a]_{2k} + \frac{1}{2} \sum_{k=1}^{\infty} k[\log a]_k^2 \right) \det(1 - Q_n H(t^{-1}a_+^{-1}\tilde{a}_+)Q_n) \\
\det(\hat{a}_{i-j} + \hat{a}_{i+j})_{0 \leq i,j \leq n-1} = \\
\exp \left( n \log a_0 + \sum_{k=1}^{\infty} [\log a]_{2k} + \frac{1}{2} \sum_{k=1}^{\infty} k[\log a]_k^2 \right) \det(1 + Q_n H(ta_+^{-1}\tilde{a}_+)Q_n)
\]

Here \( [\log a]_k \) stands for the \( k \)th Fourier coefficient of \( \log a \). The Fredholm determinants are well-defined because each Hankel operator is trace-class.

As a first consequence of the above proposition we can re-derive the exact formulas of Diaconis and Shahshahani in [7] for the joint moments of \( \text{Tr} U \), \( \text{Tr} U^2 \), \ldots, \( \text{Tr} U^k \).

**Proposition 1.3.** The moments of the traces of \( \text{Tr} U \), \( \text{Tr} U^2 \), \ldots, \( \text{Tr} U^k \) are given by

\[
\mathbb{E}_{O(2n)} [\prod_{j=1}^{k} \text{Tr}(M^j)^{m_j}] = \prod_{j=1}^{k} \mathbb{E}[(\sqrt{2} Z_j + \eta_j)^{m_j}], \quad \sum_{j=1}^{k} jm_j \leq 2n - 1 \\
\mathbb{E}_{O(2n)} [\prod_{j=1}^{k} \text{Tr}(M^j)^{m_j}] = \prod_{j=1}^{k} \mathbb{E}[(\sqrt{2} Z_j - \eta_j)^{m_j}], \quad \sum_{j=1}^{k} jm_j \leq 2n - 1 \\
\mathbb{E}_{O(2n+1)} [\prod_{j=1}^{k} \text{Tr}(M^j)^{m_j}] = \prod_{j=1}^{k} \mathbb{E}[(\sqrt{2} Z_j - (1-\eta_j))^{m_j}], \quad \sum_{j=1}^{k} jm_j \leq 2n \\
\mathbb{E}_{O(2n+1)} [\prod_{j=1}^{k} \text{Tr}(M^j)^{m_j}] = \prod_{j=1}^{k} \mathbb{E}[(\sqrt{2} Z_j + (1-\eta_j))^{m_j}], \quad \sum_{j=1}^{k} jm_j \leq 2n \\
\mathbb{E}_{S_p(2n)} [\prod_{j=1}^{k} \text{Tr}(M^j)^{m_j}] = \prod_{j=1}^{k} \mathbb{E}[(\sqrt{2} Z_j - \eta_j)^{m_j}], \quad \sum_{j=1}^{k} jm_j \leq 2n + 1
\]

where the \( Z_j \) are independent standard normal variables and \( \eta_j = \frac{1+(-1)^j}{2} \).

**Remark 1.4.** As we noted in the introduction, the joint eigenvalue densities do not take into account the deterministic eigenvalues of \( O(2n)^{-} \) and \( O(2n+1)^{\pm} \). Raising these to the power of \( j \) and adding them to \( \text{Tr} M^j \) above shows that \( \mathbb{E}_{G(n)}[\text{Tr} M^j] \), if including both random and deterministic eigenvalues, is actually \( \eta_j \) for all orthogonal matrices.
Remark 1.5. These moments were first computed by Diaconis and Shahshahani in [7] in the case where $U$ belongs to the unitary or orthogonal or symplectic group (for half the range in the last two cases, i.e. for $\sum_{j=1}^{k} jm_j < n/2$ where $n$ is the size of the matrix) using the representation theory of these groups. Later Hughes and Rudnick obtained the identities for $SO(n)$ and $Sp(n)$ by a combinatorial argument involving the cumulants of linear eigenvalue statistics, see [8]. The unitary case was treated in [12] (appendix) by using an identity for Toeplitz determinants, a method of proof which is similar to our proof below. See also [16] and [21] for the moments of $U(n)$, $O(n)$, $SO(n)$ and $Sp(n)$.

Proof. A partition $\lambda$ is a sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ which are called the parts of the partition. The sum of all the parts is finite and gives the weight $|\lambda|$. We will write $\lambda \vdash n$ to say that $\lambda$ is a partition of $n$, i.e. $|\lambda| = n$. The number of parts of $\lambda$ equal to $i$ is called the multiplicity of $i$ in $\lambda$ and is denoted by $m_i$, so $\lambda = (1^{m_1}2^{m_2} \ldots k^{m_k})$. We will use the identity

$$ (1.7) \exp(\sum_{n=1}^{\infty} \frac{g(n)}{n} t^n) = \sum_{\lambda} t^{\lambda} z_{\lambda}^{-1} g(\lambda) $$

where the sum is over all partitions. Here $g$ is a complex function on $\mathbb{N}$, $g(\lambda) = g(\lambda_1)g(\lambda_2) \ldots$ with $g(0) = 1$, and $z_{\lambda} = \prod_{i \geq 1} m_i^{\lambda_i}$ where $m_i$ is the multiplicity of $i$ in $\lambda$. This is a case of the ”Exponential formula” (see 5.1.9 in [20]). We now prove the proposition for $O(2n^+)$, the other cases being similar. Define

$$ f(e^{i\theta}) = \prod_{i=1}^{m} \frac{1}{1 - a_i e^{i\theta}} \frac{1}{1 - a_i e^{-i\theta}} = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(a) e^{ik\theta}}{k} (e^{ik\theta} + e^{-ik\theta})\right), \quad \theta \in [0, \pi), $$

where $a_i, t \in \mathbb{D}$, and $p_k(a) = \sum_{i=1}^{m} a_i^k$. Then

$$ \prod_{j=1}^{n} f(e^{i\theta_j}) = \exp\left(\sum_{k=1}^{\infty} \frac{p_k(a) p_k(e^{i\theta_j}, e^{-i\theta_j})}{k} t^k\right) $$

where $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n})$, $e^{-i\theta} = (e^{-i\theta_1}, \ldots, e^{-i\theta_n})$, and $p_k(e^{i\theta_j}, e^{-i\theta_j}) = \sum_{j=1}^{n} (e^{ik\theta_j} + e^{-ik\theta_j})$. So by (1.7),

$$ \prod_{j=1}^{n} f(e^{i\theta_j}) = \sum_{\lambda} t^{\lambda} z_{\lambda}^{-1} p_{\lambda}(a) p_{\lambda}(e^{i\theta_j}, e^{-i\theta_j}) $$

which gives

$$ (1.8) \mathbb{E}_{O(2n^+)} \left[ \prod_{j=1}^{n} f(e^{i\theta_j}) \right] = \sum_{\lambda} t^{\lambda} z_{\lambda}^{-1} p_{\lambda}(a) \mathbb{E}_{O(2n^+)} [p_{\lambda}(e^{i\theta_j}, e^{-i\theta_j})]. $$

Observe that if $\lambda = (1^{m_1}2^{m_2} \ldots k^{m_k})$, then

$$ \mathbb{E}_{O(2n^+)} [p_{\lambda}(e^{i\theta}, e^{-i\theta})] = \mathbb{E}_{O(2n^+)^{+}} \left[ \prod_{j=1}^{k} \text{Tr}(M^{j})^{m_j} \right]. $$

On the other hand, since $f(e^{i\theta}) = f(e^{-i\theta})$, Proposition 1.2 applies. Indeed, if we define

$$ h(x) = \exp\left(\sum_{k=1}^{\infty} \frac{2p_k(a) e^{ik\theta}}{k} T_k(x)\right), \quad x \in [-1, 1], $$

where $T_k$ is the $k$th Chebyshev polynomial, we see that

$$ \mathbb{E}_{O(2n^+)} \left[ \prod_{j=1}^{n} f(e^{i\theta_j}) \right] = \mathbb{E}_{O(n^+)}^{-} \left[ \prod_{j=1}^{n} h(x_j) \right], \quad x_j = \cos \theta_j. $$
to which we can apply Lemma 1.1 and Proposition 1.2 (log(h \circ \cos) = \log(f) \in B^1_1 + \because a_i, t \in \mathbb{D}) and obtain

\[
E_{O(2n)^+} \left[ \prod_{j=1}^{n} f(e^{i\theta_j}) \right] = \exp \left( \sum_{k=1}^{\infty} \log g_{2k} + \frac{1}{2} \sum_{k=1}^{\infty} k \log g_{2k}^2 \right) \det(1 + Q_n \cdot H(e^{i\theta} g_{+1}^{-1} g_{+}) Q_n) \tag{1.9}
\]

where \( g : [0, \pi) \to \mathbb{C}, g = h \circ \cos \). We would like the Fredholm determinant to be 1, which happens if \( m \) is sufficiently small since then \( H(e^{i\theta} g_{+1}^{-1} g_{+}) \) is annihilated by \( Q_n \). We have that

\[
g_{+1}^{-1}(\theta)g_{+}(\theta) = \exp \left( \sum_{k=1}^{m} \frac{p_k(a) t^k}{k} (-e^{ik\theta} + e^{-ik\theta}) \right) = \prod_{i=1}^{m} \frac{1 - a_i t e^{i\theta}}{1 - a_i t e^{-i\theta}}
\]

which can be expanded into

\[
g_{+1}^{-1}(\theta)g_{+}(\theta) = \sum_{0 \leq j \leq m} (-te^{i\theta})^j e_j(a) \sum_{0 \leq k} (te^{-i\theta})^k h_k(a) = \sum_{0 \leq j \leq m} \sum_{0 \leq k \leq j} (-1)^j t^{2j-k} e_j(a) h_{j-k}(a) e^{ik\theta}
\]

where \( e_k \) is the \( k \)th elementary symmetric polynomial and \( h_k \) is the \( k \)th complete symmetric polynomial. Hence the \((x,y)\) element of \( H(e^{i\theta} g_{+1}^{-1} g_{+}) \) is different from zero if and only if \( x + y \leq m \) (recall that \( H(c) = (c_{j+k+1})_{j,k=0}^{\infty} \)) so \( Q_n H(e^{i\theta} g_{+1}^{-1} g_{+}) Q_n \) is zero for \( 2n > m \). In this case (1.9) becomes

\[
E_{O(2n)^+} \left[ \prod_{j=1}^{n} f(e^{i\theta_j}) \right] = \exp \left( \sum_{k=1}^{\infty} \frac{p_{2k}(a)}{2k} t^{2k} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{p_k(a)^2}{k} t^{2k} \right).
\]

Now, by definition of \( \eta_k \) and because \( E[\exp(\xi Z_k)] = \exp(\xi^2/2) \),

\[
E_{O(2n)^+} \left[ \prod_{j=1}^{n} f(e^{i\theta_j}) \right] = \exp \left[ \sum_{k=1}^{\infty} \left( \frac{p_k(a) t^k}{k} \eta_k + \frac{k}{2} \left( \frac{p_k(a) t^k}{k} \right)^2 \right) \right]
\]

\[
= \left[ \exp \left( \sum_{k=1}^{\infty} \left( \frac{p_k(a) t^k}{k} (\eta_k + \sqrt{k} Z_k) \right) \right) \right].
\]

Using (1.7) one more time and applying the dominated convergence theorem gives

\[
E_{O(2n)^+} \left[ \prod_{j=1}^{n} f(e^{i\theta_j}) \right] = \sum_{\lambda} t^{\lambda} \gamma_{\lambda}^{-1} p_{\lambda}(a) \mathbb{E} \left[ \prod_{i=1}^{\lambda} (\lambda_i Z_{\lambda_i} + \eta_{\lambda_i}) \right].
\]

By comparing with (1.8) and matching coefficients of \( t^j \), we see that

\[
\sum_{\lambda \vdash j} \gamma_{\lambda}^{-1} p_{\lambda}(a) E_{O(2n)^+}[p_{\lambda}(e^{i\theta}, e^{-i\theta})] = \sum_{\lambda \vdash j} \gamma_{\lambda}^{-1} p_{\lambda}(a) \mathbb{E} \left[ \prod_{i=1}^{\lambda} (\lambda_i Z_{\lambda_i} + \eta_{\lambda_i}) \right], \quad j \geq 1.
\]

The power sums \( p_{\lambda}(a), \lambda \vdash j \), form a basis of the ring of symmetric polynomials of degree \( j \) in \( j \) variables. Therefore, if \( |\lambda| = m \leq 2n - 1 \) i.e. if \( \sum_{j=1}^{k} jm_j \leq 2n - 1 \),

\[
E_{O(2n)^+} \left[ \prod_{j=1}^{k} \text{Tr}(M^j)^{m_j} \right] = E_{O(2n)^+}[p_{\lambda}(e^{i\theta}, e^{-i\theta})] = \mathbb{E} \left[ \prod_{i=1}^{\lambda} (\lambda_i Z_{\lambda_i} + \eta_{\lambda_i}) \right].
\]

Thus, by the Cramér-Wold device followed by the method of moments for normal distributions, we see that the random vector \( X = (X_1, X_2, \ldots X_m) \) with \( X_j = \frac{1}{\sqrt{j}} \text{Tr}(U^j) - E_{G(n)}[\text{Tr}(U^j)] \) and \( m \) fixed converges in distribution to \( (Z_1, Z_2, \ldots Z_m) \) where the \( Z_j \) are independent standard normal variables. Also, \( E_{G(n)}[\text{Tr}(U^j)] = \pm \eta_j \) or \( E_{G(n)}[\text{Tr}(U^j)] = \pm (1 - \eta_j) \), whether the size of the matrix is even or odd. This can also be seen by considering the characteristic function of \( X \), as we do in the next corollary. First recall that

\[
f(x) = \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} \left( 2T_k(x) - \frac{E_{G(n)}[\text{Tr}(U^k)]}{n} \right), \quad x \in [-1, 1]
\]
and that
\[ g(\theta) = f(\cos(\theta)) = \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} (2 \cos k\theta - \frac{\mathbb{E}_{\mathcal{G}(n)}[\text{Tr} U^k]}{n}), \quad \theta \in [0, \pi). \]

**Corollary 1.6.** Let \( F_{n,m}^{a,b} \) be the characteristic function of \( X = (X_1, X_2, \ldots, X_m) \), \( X_j = \frac{1}{\sqrt{n}}(\text{Tr} U^j - \mathbb{E}_{\mathcal{G}(n)}[\text{Tr} U^j]) \) and \( U \) a Haar distributed orthogonal or symplectic matrix. Then, for \( \xi \in \mathbb{R}^m \),

\[
\begin{align*}
F_{n,m}^{-+}(\xi) &= e^{-\frac{1}{2}\|\xi\|^2} \det(1 + H(e^{-i2n\theta} e^{2\text{Im} g_+})) \\
F_{n,m}^{+-}(\xi) &= e^{-\frac{1}{2}\|\xi\|^2} \det(1 - H(e^{-i2n\theta} e^{2\text{Im} g_+})) \\
F_{n,m}^{-+}(\xi) &= e^{-\frac{1}{2}\|\xi\|^2} \det(1 - H(e^{-i(2n+1)\theta} e^{2\text{Im} g_+})) \\
F_{n,m}^{--}(\xi) &= e^{-\frac{1}{2}\|\xi\|^2} \det(1 + H(e^{-i(2n-1)\theta} e^{2\text{Im} g_+}))
\end{align*}
\]

where \( H \) denotes the Hankel matrix i.e. \( H(f) = (f_{j+k+1})_{j,k=0}^{\infty} \), and \( f_k \) is the \( k \)th Fourier coefficient of \( f \).

**Remark 1.7.** The fact that the Hankel operators are trace class and therefore the Fredholm determinants well-defined is part of our Proposition 1.2 which is from [2]. In our case it also follows from the identity
\[
H(ab) = T(a)H(b) + H(a)T(b)
\]
which holds for \( a, b \in L_\infty(\mathbb{T}) \), and which gives
\[
H(a_+^{-1} a_+) = T(a_+^{-1})H(a_+^{-1})
\]
since \( H(a_+) = 0 \). Now, \( a_+^{-1} = e^{-ig_+} \) and \( g \in B_1^1 \), so \( a_+^{-1} \in B_1^1 \) which is equivalent to \( H(a_+^{-1}) \) and \( H(a_+^{-1}) \) being trace class (see [17]). Applying the inequality
\[
||ABC||_{J_1} \leq ||A||_{J_1} ||B||_{J_1} ||C||
\]
(Proposition 4.2 in [10]) to \( A = T(a_+) \), \( B = H(a_+^{-1}) \), \( C = I \), and using that the operator norm of a Toeplitz operator is given by the \( L_\infty \)-norm of its symbol proves the statement.

**Proof.** Again, consider the first case for simplicity; the proof for the remaining ones is essentially the same. By definition of the characteristic function and Lemma 1.1
\[
F_{n,m}^{-+}(\xi) = \mathbb{E}_{\mathcal{G}(n)}^{-+} \left[ \prod_{1 \leq j \leq n} e^{if(x_j)} \right] = \det((e^{ig})_{i-j} + (e^{ig})_{i+j+1})_{0 \leq i,j \leq n-1}.
\]
The function \( g \) satisfies \( \hat{g} = g \) and, being a trigonometric polynomial, belongs to \( B_1^1 \). Thus the conditions of Proposition 1.2 are met and we obtain
\[
F_{n,m}^{-+}(\xi) = \exp \left( -n \sum_{k=0}^{m} \frac{\xi_k \mathbb{E}_{\mathcal{O}(2n+1)}[\text{Tr} U^k]}{\sqrt{k}} \right) + \left[ \prod_{k=0}^{(m-1)/2} \frac{\xi_{2k+1}}{\sqrt{2k+1}} \right] \left[ \sum_{k=1}^{m} \frac{\xi_k}{\sqrt{k}} \right] \det(1 + H(e^{-i2n\theta} e^{2\text{Im} g_+}))
\]
where we used that \( \mathbb{E}_{\mathcal{O}(2n+1)}[\text{Tr} U^j] = 1 - \eta_j \).

The Fredholm determinants in Corollary 1.6 converge to one as \( n \) tends to infinity. This can be seen for example from
\[
|1 - \det(1 + K)| \leq e^{\|K\|_{J_1} - 1} \leq \|K\|_{J_1} e^{\|K\|_{J_1}}
\]
(Theorem 3.3 in [10]), where \( K \) is any trace-class operator and \( J_1 \) is the trace norm, and the fact that \( \lim_{n \to 0} \|Q_n K Q_n\|_{J_1} = 0 \). This in turn follows from \( \lim_{n \to 0} \|K - (1 - Q_n) K (1 - Q_n)\|_{J_1} = 0 \) (Proposition 4.2 in [10]) and
\[
Q_n K Q_n = Q_n (K - (1 - Q_n) K (1 - Q_n)) Q_n
\]
which implies

\begin{equation}
\lim_{n \to 0} \|Q_n K Q_n\|_{L^1} \leq \lim_{n \to 0} \|K - (1 - Q_n) K (1 - Q_n)\|_{L^1} = 0
\end{equation}

where we used that \(\|Q_n\| = 1\) and (1.10).

Hence the characteristic function \(F_{n,m}^{a,b}\) converges to that of a standard normal vector, and the speed of this convergence depends on how fast the Fredholm determinants approach one. To measure the rate of convergence we first consider the \(L_2\) distance between the probability density of \(X\) and that of a normal random vector, which by Plancherel’s theorem amounts to study the \(L_2\) distance between their characteristic functions. To this end we divide the real line into three parts, or regimes, where we use different techniques to obtain as good estimates as possible. We start with the first regime which is studied using Corollary 1.6 and which will give us the dominant error term.

2. Gaussian approximation

Our goal is to prove

**Proposition 2.1.** Let \(N = n/m\) and set

\begin{equation}
A_1 = \frac{n}{2m \sqrt{\log m + 1}}.
\end{equation}

If \(N \geq m\) then, for any pair \((a, b) = (±1/2, ±1/2)\),

\begin{align}
\left( \int_{\|\xi\| < A_1} |F_{n,m}^{a,b}(\xi) - e^{-\|\xi\|^2/2} d\xi \right)^{1/2} \\
\leq \frac{16}{15} e^{13/24} (e^{9/8} + 1) m^{3/2} \sqrt{\Omega_m} \frac{(m/2)}{N} \pi \left( e^{3/2} (\log m + 1) \right)^N
\end{align}

where \(\Omega_m\) is the volume of the unit \(m\)-ball.

We start by giving two lemmas that we will need later. The first one is essentially Lemma 4.1 in [13]; we include the proof here for completeness.

**Lemma 2.2.** Fix \(m \in \mathbb{N}\) and \(\xi \in \mathbb{R}^m\). Let \(\rho = \sqrt{\log m + 1}\|\xi\|\). If \(k \geq 2m\rho\), then

\[ |(e^{2i \text{Im} g_+})_k| \leq 2e^\rho \frac{\rho^{[k/m]}}{[k/m]!}. \]

**Proof.** Let \(M \geq 1\) and define

\[ \phi_M(\omega) = \sum_{j=0}^{M} \frac{\omega^j}{j!}, \quad \omega \in \mathbb{C}. \]

Then, if \(Mm < k\),

\[ \int_{-\pi}^{\pi} \phi_M(-ig_+(\theta)) e^{ig_+(\theta) - ik\theta} \frac{d\theta}{2\pi} = 0 \]

which gives

\[ |(e^{2i \text{Im} g_+})_k| = \left| \int_{-\pi}^{\pi} e^{-i(g_+(\theta) - g_+(\theta) - ik\theta)} \frac{d\theta}{2\pi} \right| = \left| \int_{-\pi}^{\pi} (e^{-ig_+(\theta)} - \phi_M(-ig_+(\theta))) e^{ig_+(\theta) - ik\theta} \frac{d\theta}{2\pi} \right| \leq \int_{-\pi}^{\pi} |e^{-ig_+(\theta)} - \phi_M(-ig_+(\theta))| e^{1 \text{Im} g_+(\theta)} \frac{d\theta}{2\pi}. \]

The inequality \((k + j)! \geq k!(k + 1)!\) for \(k, j \in \mathbb{N}\) gives

\[ |\omega - \phi_M(\omega)| \leq \sum_{k \geq M+1} \frac{\omega^k}{k!} \leq \frac{|\omega|^{M+1}}{(M+1)!} \sum_{k \geq 0} \frac{|\omega|^k}{(M+2)^k} \leq 2 \frac{|\omega|^{M+1}}{(M+1)!}. \]
if $|\omega| \leq M/2 + 1$. Combined with
$$
\|g_+\|_\infty \leq \sum_{k=1}^{m} \frac{\|\xi_k\|}{\sqrt{k}} \leq \sqrt{1 + \log m}\|\xi\| = \rho
$$
which follows from the Cauchy-Schwarz inequality, we obtain
$$
\|(e^{2Img_+})_k\| \leq 2e^{\rho} \rho^{M+1} \frac{1}{(M + 1)!}
$$
for $k > Mm$ and $M + 2 \geq 2\rho$. These inequalities are satisfied if $M + 1 = \lfloor k/m \rfloor$ and $k \geq 2m\rho$, which proves the statement.

**Lemma 2.3.** Let $y > 0$. The function $h(x) = \frac{y^x}{\Gamma(x+1)}$ is decreasing on $[y, \infty)$.

**Proof.** The derivative of $h$ is
$$
h'(x) = \frac{y^x (\log y - \psi(x + 1))}{\Gamma(x + 1)},
$$
where $\psi$ is the Digamma function, i.e. the logarithmic derivative of $\Gamma(x + 1)$, which has the following integral representation (5.9.12 [6])
$$
\psi(x + 1) = \int_0^\infty e^{-t} - \frac{e^{-tx}}{t} - 1 dx.
$$
Now observe that
$$
\int_0^\infty \frac{e^{-t} - e^{-tx}}{t} dt = \lim_{R \to \infty} \lim_{\epsilon \to 0} \left( \int_\epsilon^R e^{-t} dt - \int_{e^R}^{\infty} e^{-t} dt \right) = \lim_{R \to \infty} \lim_{\epsilon \to 0} \left( \int_\epsilon^R e^{-t} dt - \int_{e^R}^{\infty} e^{-t} dt \right) = \log x.
$$
Thus
$$
\psi(x + 1) = \int_0^\infty \frac{e^{-t} - e^{-tx}}{t} dt > \log x
$$
since $e^t - 1 > t$. This shows that $h'(x) < 0$ on $[y, \infty)$, assuming $y > 0$. \hfill \Box

Proposition 2.1 is based on the following bound on the distance between the characteristic function $F_{n,m}^{a,b}$ of our random vector $X$ and that of a standard Gaussian vector.

**Proposition 2.4.** Let $N = n/m, \rho = \sqrt{\log m + \frac{1}{2}}\|\xi\|$. If $N \geq \max(2\rho, m)$ then
$$
|F_{n,m}^{a,b}(\xi) - e^{-\|\xi\|^2/2}| \leq \frac{32e^{1/2}(e^{9/8} + 1) me^{\rho^2N}e^{-\|\xi\|^2/2}}{15} \frac{1}{\Gamma(2N + 1)}.
$$

**Proof.** The proof is based on the following inequality
$$
|1 - \det(I + K)| \leq e^{Y_{n,m}(\xi)}(e^{(Y_{n,m}(\xi)+1)^2/2} + 1)Y_{n,m}(\xi)
$$
which holds for any trace-class operator $K$ and where $Y_{n,m}(\xi) = \max\{Tr K, \|K\|_{J_2}\}$, $\|\cdot\|_{J_2}$ being the Hilbert-Schmidt norm. It is obtained as follows: the generalised determinant $\det_2$ can be defined as $\det_2(I + K) = e^{-Tr(K)} \det(I + H)$. Hence, by definition,
$$
|1 - \det(I + K)| = |1 - e^{Tr K} \det_2(I + K)| \leq |e^{Tr K}| |1 - \det(1 + K)| + |1 - e^{Tr K}|
\leq |e^{Tr K}||1 - \det(1 + K)| + |Tr K||e^{Tr K}|
$$
Now we use that
$$
|1 - \det_2(1 + K)| \leq \|K\|_{J_2} e^{\frac{1}{2}(1 + \|K\|_{J_2})^2}
$$
which is part of Theorem 9.2. in [18]. Observe that we could use (1.11) instead of (2.16) but the latter gives us a slightly better bound.
We will apply (2.16) to the Fredholm determinants in Corollary (1.6). First consider the case 
\((a, b) = (-1/2, -1/2)\), i.e. \(K = Q_\alpha H(e^{it} e^{2\imath m g_+}) Q_n\). By Lemma 2.2, if \(n \geq m\rho\),

\[
(2.17) \quad \|Q_n H(e^{it} e^{2\imath m g_+}) Q_n\|_{\mathbb{H}_2} \leq \sum_{k \geq 2n} (k - 2n + 1)|e^{2\imath m g_+}|_k^2 \leq 4e^{2\rho} \sum_{k \geq 0} (k + 1) \frac{\rho^{[\frac{k}{m}+2N]} |\frac{k}{m} + 2N|^2}{|\frac{k}{m} + 2N|^2}.
\]

If we let \(k = jm + r\), \(0 \leq r < m\), then

\[
\sum_{k \geq 0} (k + 1) \frac{\rho^{[\frac{k}{m}+2N]} |\frac{k}{m} + 2N|^2}{|\frac{k}{m} + 2N|^2} = \sum_{j=0}^{\infty} \sum_{r=0}^{m-1} (jm + r + 1) \frac{\rho^{[j+\frac{r}{m}+2N]} |j + \frac{r}{m} + 2N|^2}{|j + \frac{r}{m} + 2N|^2} \leq m \sum_{j=0}^{\infty} \sum_{r=0}^{m-1} (j + 1) \frac{\rho^{[j+\frac{r}{m}+2N]} |j + \frac{r}{m} + 2N|^2}{|j + \frac{r}{m} + 2N|^2}.
\]

Now, \(\left\lfloor j + \frac{r}{m} + 2N \right\rfloor \geq 2N \geq \rho\), so by Lemma 2.3

\[
\frac{\rho^{[j+\frac{r}{m}+2N]} |j + \frac{r}{m} + 2N|^2}{|j + \frac{r}{m} + 2N|^2} \leq \frac{\rho^{[j+2N]}}{\Gamma(j+2N+1)^2},
\]

which combined with the inequality \(\Gamma(j + x + 1) \geq (x + 1)^j \Gamma(x + 1), j \in \mathbb{N}\) (which follows from the recurrence relation \(\Gamma(x + 1) = \Gamma(x) x\)) gives

\[
\|Q_n H(e^{it} e^{2\imath m g_+}) Q_n\|_{\mathbb{H}_2} \leq 4me^{2\rho} \sum_{j=0}^{\infty} \sum_{r=0}^{m-1} (j + 1) \frac{\rho^{[j+2N]}}{(2N+1)^2 \Gamma(2N+1)^2} \leq \frac{4m^2 e^{2\rho} \rho^{4N}}{\Gamma(2N+1)^2} \sum_{j=0}^{\infty} (j + 1) \left(\frac{\rho}{2N+1}\right)^{2j}.
\]

We set \(c_\ast = \rho/(2N)\) and obtain

\[
(2.18) \quad \|Q_n H(e^{it} e^{2\imath m g_+}) Q_n\|_{\mathbb{H}_2} \leq \frac{4m^2 e^{2\rho} \rho^{4N}}{\Gamma(2N+1)^2} \left(1 - \frac{c_\ast^2}{2}\right)^2.
\]

Similarly, by Lemma 2.2, if \(n \geq m\rho\),

\[
|\text{Tr} Q_n H(e^{it} e^{2\imath m g_+}) Q_n| \leq \sum_{k \geq n} |(e^{2\imath m g_+})_{2k}| \leq 2e^{\rho} \sum_{k \geq 0} \frac{\rho^{[2k/m+2N]}}{\Gamma(2k/m + 2N + 1)} \leq 2e^{\rho} \sum_{k \geq 0} \frac{\rho^{[2k/m+2N]}}{\Gamma(2k/m + 2N + 1)} = 2me^{\rho} \sum_{j=0}^{\infty} \frac{\rho^{2j+2N}}{\Gamma(2j + 2N + 1)} \leq 2me^{\rho} \frac{\rho^{2N}}{1 - c_\ast^2 \Gamma(2N+1)}.
\]

where for the last inequality we used that \(\Gamma(j + x + 1) \geq (x + 1)^j \Gamma(x + 1)\). Hence

\[
\Upsilon_{n,m}(\xi) \leq \frac{2me^{\rho} \rho^{2N}}{1 - c_\ast^2 \Gamma(2N+1)}.
\]

A well-known estimate for the Gamma function is

\[
\sqrt{2\pi x} x^{-1/2} e^{-x} < \Gamma(x) < \sqrt{2\pi x} e^{1/(12x)} x^{-1/2} e^{-x}, \quad x > 0,
\]

which can be found in [6] (Equation 5.6.1). Equivalently, by multiplying the above inequalities with \(x\) and using the functional equation \(\Gamma(x + 1) = x\Gamma(x)\), we obtain

\[
\sqrt{2\pi x} x^{1/2} e^{-x} < \Gamma(x + 1) < \sqrt{2\pi x} e^{1/(12x)} x^{1/2} e^{-x}, \quad x > 0,
\]

and therefore

\[
\Upsilon_{n,m}(\xi) \leq \frac{me^{\rho}}{\sqrt{\pi N (1 - c_\ast^2)}} \left(\frac{e^{\rho}}{2N}\right)^{2N} = \frac{m}{\sqrt{\pi N (1 - c_\ast^2)}} (c_\ast e^{1+c_\ast})^{2N}.
\]
If we choose \( c_* \leq 1/4 \) (so that \( c_* e^{1+c_*} < 1 \)) and let \( N \geq m \) we see that

\[
\Upsilon_{n,m}(\xi) \leq \frac{16}{15^3} \sqrt[N]{(\frac{\xi}{4})^{2N}}.
\]

The maximum is attained when \( N = (\log 256 - 5)^{-1} \), at which it is strictly less than 1/2. Therefore, (2.16) gives

\[
|F_{n,m}^- (\xi) - e^{-||\xi||^2/2}| \leq e^{1/2}(e^{9/8} + 1) \Upsilon_{n,m}(\xi) \leq \frac{32e^{1/2}(e^{9/8} + 1) me^{2\rho^2 N} e^{-||\xi||^2/2}}{15} \Gamma(2N + 1)
\]

if \( N \geq \max(2\rho, m) \).

It is easy to extend this result to the other three cases, i.e. when \( (a,b) = (1/2, 1/2) \) or \( (a,b) = \pm(1/2, -1/2) \). The operator \( k \) in the Fredholm determinants appearing in Corollary (1.6) is then equal to either \( k = -Q_nH(e^{-i\theta} e^{2im\theta^2})Q_n \) or \( K = \pm Q_nH(e^{2im\theta^2})Q_n \). Consider this latter case. By Lemma 2.2, if \( n \geq m\rho - 1/2 \), and in particular if \( n \geq m\rho \),

\[
\|Q_nH(e^{2im\theta^2})Q_n\|_{L^2}^2 = \sum_{k \geq 2n} (k - 2n + 1)[(e^{2im\theta^2})_{k+1}^2] \leq \frac{4e^{2\rho}}{15} \sum_{k \geq 0} (k + 1) \frac{\rho^{2k/m + 2N + 1/m}}{12}.
\]

But by Lemma 2.3 the right-hand side is less than the upper bound in (2.17). Thus (2.18) still applies, and a similar argument shows that so does (2.19), which all together lead to (2.22). The case \( K = -Q_nH(e^{-i\theta} e^{2im\theta^2})Q_n \) is treated similarly.

\[\square\]

Proof of Proposition 2.1. By Proposition 2.4 we have

\[
\left( \int_{||\xi|| < \Lambda_1} |F_{n,m}^{a,b}(\xi) - e^{-||\xi||^2/2}| d\xi \right)^{1/2} \leq \frac{32e^{1/2}(e^{9/8} + 1)}{15} \frac{me^{N/2}}{\Gamma(2N + 1)} \left( \int_{||\xi|| < \Lambda_1} e^{-||\xi||^2/2} d\xi \right)^{1/2} \leq \frac{32e^{1/2}(e^{9/8} + 1)}{15} \frac{me^{N/2}}{\Gamma(2N + 1)} \left( \int_{\mathbb{R}^m} ||\xi||^4 e^{-||\xi||^2/2} d\xi \right)^{1/2}.
\]

A change of variables to spherical coordinates shows that

\[
\int_{\mathbb{R}^m} ||\xi||^4 e^{-||\xi||^2/2} d\xi = \frac{m\Omega_m}{2} \Gamma(2N + m/2)
\]

where \( \Omega_m \) is the volume of the unit \( m \)-ball. Stirling’s approximation (Inequalities (2.20) and (2.21)) gives

\[
\sqrt{\frac{\Gamma(2N + m/2)}{\Gamma(2N + 1)}} \leq \left( \frac{\sqrt{\pi} e^{1/12} (2N + m/2)^{2N + m/2} e^{-2N - m}}{\sqrt{2\pi} (2N)^{2N + 1/2} e^{-2N}} \right)^{1/2} \leq e^{1/24} (m/2)^{m/2} e^{N}
\]

where we used the inequality \( 1 + x \leq e^x, x \in \mathbb{R} \) twice. This finishes the proof of the proposition. \[\square\]

3. Intermediate regime

We will now estimate \( F_{n,m}^{a,b}(\xi) \) directly from its integral expression, with probability density given by (0.2). Recall that

\[
g(\theta) = f(\cos(\theta)) = \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} \left( 2\cos k\theta - \frac{\xi_k U_{G(n)}[\text{Tr} U^k]}{n} \right), \quad \theta \in [0, \pi).
\]

We will need the following inequality that appears in [13].

Lemma 3.1. If \( y \in [-1, 1] \), \( y \neq 0 \) and \( x \in \mathbb{R} \), then

\[
1 + \left( \frac{\sinh(x)}{y} \right)^2 \leq \exp \left[ \frac{\left( \frac{x}{y} \right)^2}{2} \right].
\]
Proof. It suffices to prove the inequality for \( x \geq 0 \). Set \( \kappa(x) = (1 + \sinh^2(x)/y^2)e^{-x^2/y^2} \) and differentiate:

\[
\kappa'(x) = \frac{2e^{-x^2/y^2}}{y^2} \left( \sinh(x) \cosh(x) - x \left( 1 + \frac{\sinh^2(x)}{y^2} \right) \right)
\leq \frac{2e^{-x^2/y^2}}{y^2} (\sinh(x) \cosh(x) - x(1 + \sinh^2(x)))
= \frac{2e^{-x^2/y^2}}{y^2} \cosh(x)(\sinh(x) - x \cosh(x)) \leq 0.
\]

Thus \( \kappa(x) \leq \kappa(0) = 1 \) if \( x \geq 0 \) which proves the result for \( x \in \mathbb{R} \). \( \square \)

In the next proposition, only part (a) is needed for the intermediate regime but we include part (b) which is used in the last section and has a similar proof. We denote by \( h \) the Hilbert transform of \( g \), i.e. \( h(\theta) = \sum_{k = 1}^{m} 2 \frac{\delta_{k}}{\pi} \sin k \theta \), and \( \nu \in \mathbb{R} \) is a parameter that we will optimize over later. The technique of proof, based on the change of variable \( t = x - i\frac{\nu}{\pi} h(x) \), goes back to [11].

**Proposition 3.2.**  
(a) Let \( \xi \in \mathbb{R}^m \) and \( \nu \in \mathbb{R} \) be such that

\[
0 < 2 \frac{\nu}{n} \sqrt{\log m + 1} \|\xi\| < \frac{c_0}{m}
\]

for any constant \( c_0 > 0 \). If \( a = b = -1/2 \), then

\[
|F_{n,m}^{a,b}(\xi)| \leq \exp \left( 2 - \frac{1}{n} \right) \nu^2 m(m + 1) \|\xi\|^2 + \frac{2e^n \nu^2}{n} (m + 1)^2 (\log m + 1) \|\xi\|^3
\times E_{O(2n)} + \prod_{1 \leq j \leq n} \exp \left( \frac{\nu}{n} h(\theta_j) g'(\theta_j) \right).
\]

In the three other cases,

\[
|F_{n,m}^{a,b}(\xi)| \leq \exp \left( 2 + \frac{1}{n} \right) \nu^2 m(m + 1) \|\xi\|^2 + \frac{2e^n \nu^2}{n} (m + 1)^2 (\log m + 1) \|\xi\|^3
\times E_{G(n)} \prod_{1 \leq j \leq n} \exp \left( \frac{\nu}{n} h(\theta_j) g'(\theta_j) \right).
\]

(b) Let \( \xi \in \mathbb{R}^m \) and set

\[
\nu = \frac{\eta \sqrt{n}}{(m + 1)^2 \|\xi\|}, \quad \lambda = \nu \left( 1 - \frac{\nu^2 e^{\sqrt{2/\eta n}}}{n^2 3 \sqrt{3}} (m + 1)^4 \|\xi\|^2 \right)
\]

for any constant \( \eta \in (0, 1] \). If \( a = b = -1/2 \), then

\[
|F_{n,m}^{a,b}(\xi)| \leq e^{\eta \sqrt{n} (n + \frac{3}{2})} E_{O(2n)} + \prod_{1 \leq j \leq n} e^{-\lambda g'(\theta_j)^2}.
\]

In the three other cases,

\[
|F_{n,m}^{a,b}(\xi)| \leq e^{\eta \nu^2 (n + \frac{3}{2})} E_{G(n)} \prod_{1 \leq j \leq n} e^{-\lambda g'(\theta_j)^2}.
\]

**Proof.** We start with the case \( a = b = -1/2 \). Then

\[
F_{n,m}^{a,b}(\xi) = E_{O(2n)} + \left[ e^{i \text{Tr} \sigma(U)} \right]
\]

\[
= \frac{\gamma(n-1)^2}{n \pi^n} \int_{[0, \pi]^n} \prod_{1 \leq j \leq n} e^{i \theta_j} \prod_{1 \leq j < k \leq n} 4 \sin^2 \left( \frac{\theta_j - \theta_k}{2} \right) \sin^2 \left( \frac{\theta_j + \theta_k}{2} \right) \prod_{1 \leq j \leq n} d\theta_j.
\]

Let \( \gamma(t) = t - i\nu h(t)/n, t \in [0, \pi] \) where \( \nu \in \mathbb{R} \) and \( h \) is continuous and satisfies \( h(0) = h(\pi) = 0 \). Since the integrand above has an analytic continuation
in $\mathbb{C}^n$ we can deform the contour into the curve $\gamma\{[0,\pi]\}^n$ and then make a change of variables to get back the original contour. We obtain
\[
F_{n,m}^{a,b}(\xi) = \frac{2^{(n-1)^2}}{n!\pi^n} \int_{[0,\pi]^n} \prod_{1 \leq j \leq n} e^{ig(\theta_j - i\nu h(\theta_j))} \prod_{1 \leq j < k \leq n} 4\sin^2 \left( \frac{\theta_j - \theta_k - i\nu (h(\theta_j) - h(\theta_k))}{2} \right) \prod_{1 \leq j \leq n} \left( 1 - i\nu h'(\theta_j) \right) d\theta_j.
\]
Taking the absolute value gives the upper bound
\[
|F_{n,m}^{a,b}(\xi)| \leq \frac{2^{(n-1)^2}}{n!\pi^n} \int_{[0,\pi]^n} \prod_{1 \leq j \leq n} |e^{g(\theta_j - i\nu h(\theta_j))}| \prod_{1 \leq j < k \leq n} 4\sin^2 \left( \frac{\theta_j - \theta_k}{2} \right) \sin^2 \left( \frac{\theta_j + \theta_k}{2} \right) e^{\nu^2 \rho^2/(H(\theta_j,\theta_k)) + H(\theta_j,-\theta_k)} \prod_{1 \leq j \leq n} e^{\nu^2 \rho^2/(H(\theta_j,\theta_k))} d\theta
\]
where
\[
H(x, y) = \left( \frac{h(x) - h(y)}{2\sin(\frac{x-y}{2})} \right)^2.
\]
Here we used that $|1 + ia| \leq e^{a^2/2}$ and that
\[
\left| \sin^2 \left( \frac{\theta_j + \theta_k}{2} - i\nu (h(\theta_j) \pm h(\theta_k)) \right) \right| = \sin^2 \left( \frac{\theta_j + \theta_k}{2} \right) + \sinh^2 \left( \frac{\nu}{2n} (h(\theta_j) \pm h(\theta_k)) \right) = \sin^2 \left( \frac{\theta_j + \theta_k}{2} \right) + \sinh^2 \left( \frac{\nu}{2n} (h(\theta_j) - h(\theta_k)) \right) \leq \sin^2 \left( \frac{\theta_j + \theta_k}{2} \right) e^{\nu^2 \rho^2/(H(\theta_j,\theta_k))}
\]
where the inequality follows from Lemma 3.1. We now specialize the proof to part (a), i.e., we assume that (3.23) holds and that $h(\theta) = \sum_{k=1}^m 2\xi_k \sin k\theta/\sqrt{k}$. To bound $H$ observe that
\[
e^{ikx - e^{iky}} = \sum_{l=1}^k e^{i(l-l/2)x} e^{i(k-l+1/2)y}
\]
which gives
\[
H(x, y) = \left( \sum_{1 \leq k \leq m} \frac{\xi_k}{\sqrt{k}} \sum_{l=1}^k \left( e^{i(l-l/2)x} e^{i(k-l+1/2)y} + e^{-i(l-l/2)x} e^{-i(k-l+1/2)y} \right) \right)^2.
\]
Thus,
\[
|H(x, y)| \leq \left( \sum_{1 \leq k \leq m} 2\sqrt{k} |\xi_k| \right)^2 \leq 2m(m+1) ||\xi||^2
\]
by the Cauchy-Schwarz inequality. Next, by a Taylor expansion of $g$,
\[
\left| g(\theta - i\nu h(\theta)) - g(\theta) + i\nu h(\theta)^2 g'(\theta) \right| \leq \frac{\nu^2 h^2(\theta)}{2n^2} \sup_{t \in [0,1]} |g''(\theta - it h(\theta))| \leq \frac{\nu^2 h^2(\theta)}{2n^2} \left( \sup_{|t| \leq 2\sqrt{m+1} ||\xi||/n} |g''(\theta + it)| \right)
\]
since $||h||_\infty \leq 2\sqrt{\log m + 1} ||\xi||$ by the Cauchy-Schwarz inequality. Our hypothesis (3.23) on the parameter $\nu$ gives
\[
\sup_{|t| \leq 2\sqrt{\log m + 1} ||\xi||/n} |g''(\theta + it)| < \sup_{|t| < c_0/m} \sum_{k=1}^m 2k^{3/2} |\xi_k| e^{k|t|} < e^{c_0(m+1)^2 ||\xi||}.
\]
Now, since $g$ is real-valued,
\[
|\exp(ig(\theta_j - \frac{i}{n} h(\theta_j)))| = |\exp(ig(\theta_j - \frac{i}{n} h(\theta_j)) - ig(\theta_j) - \frac{i}{n} h(\theta_j)g'(\theta_j))\exp(\frac{i}{n} h(\theta_j)g'(\theta_j))|.
\]
Therefore,
\[
|\exp(ig(\theta_j - \frac{i}{n} h(\theta_j)))| \leq \exp\left(\frac{c_0 \nu^2}{2n^2} ||h||_\infty^2 (m + 1)^2 ||\xi||\right) \exp\left(\frac{i}{n} h(\theta_j)g'(\theta_j)\right)
\]
\[
\leq \exp\left(\frac{2c_0 \nu^2}{n^2} (\log m + 1)(m + 1)^2 ||\xi||^3\right) \exp\left(\frac{i}{n} h(\theta_j)g'(\theta_j)\right).
\]
Finally, the Cauchy-Schwarz inequality tells us that $||h'||_{\infty} \leq \sqrt{2m(m + 1)}||\xi||$. All these estimates inserted in (3.25) yield the desired upper bound.

Next consider part (b), i.e. suppose that (3.24) holds and set $h = g'$. To bound $H$ we can reuse (3.26):
\[
H(x, y) = \left( - \sum_{1 \leq k \leq m} \sqrt{k} \xi_k \sum_{l=1}^k e^{i(l-1/2)x} e^{i(k-l-1/2)y} + e^{-i(l-1/2)x} - e^{-i(k-l-1/2)y}\right)^2
\]
so by the Cauchy-Schwarz inequality,
\[
|H(x, y)| \leq \left( \sum_{1 \leq k \leq m} 2k^{3/2}||\xi_k||^2 \right)^2 \leq (m + 1)^4 ||\xi||^2.
\]
This time we Taylor expand $g$ up to second order,
\[
- \Im\left( g(\theta + i\frac{\nu}{n} g'(\theta)) - g(\theta) - \frac{\nu}{n} g'(\theta)^2 + \frac{\nu^2}{2n^2} g''(\theta)\right) \leq \frac{\nu^3 g''(\theta)^3}{6n^3} \sup_{t \in [0,1]} |g^{(3)}(\theta + i\nu g'(\theta)t/n)|
\]
\[
\leq \frac{\nu^3 g''(\theta)^3}{6n^3} \sup_{|t| \leq \nu \sqrt{2m(m+1)}||\xi||/n} |g^{(3)}(\theta + it)|
\]
where we used that $||g''||_{\infty} \leq \sqrt{2m(m + 1)}||\xi||$ from the Cauchy-Schwarz inequality. Inserting the definition of $\nu$ which was set in (3.24) gives
\[
\sup_{|x| \leq \nu \sqrt{2m(m+1)}||\xi||/n} |g^{(3)}(\theta + ix)| < \sup_{|x| < \frac{\nu \sqrt{2m(m+1)}||\xi||}{n}} \sum_{k=1}^m 2k^{3/2}||\xi_k||e^{k|x|} < \frac{\sqrt{2}e^{\sqrt{2/n} \nu}}{\sqrt{3}} (m + 1)^3 ||\xi||.
\]
Therefore (recall that $g$, $g'$ and $g''$ are real-valued),
\[
|\exp(ig(\theta + i\frac{\nu}{n} g'(\theta)))| = |\exp(ig(\theta + i\frac{\nu}{n} g'(\theta)) - ig(\theta) + \frac{\nu}{n} g'(\theta)^2 + i\frac{\nu^2}{2n^2} g''(\theta))\exp(-\frac{\nu}{n} g'(\theta)^2)|
\]
\[
\leq \exp\left(\frac{\nu^3 \sqrt{2} e^{\sqrt{2/n} \nu}}{6n^3} (m + 1)^3 ||\xi|| g'(\theta)^3 - \frac{\nu}{n} g'(\theta)^2\right)
\]
\[
\leq \exp\left(\frac{\nu^3 e^{\sqrt{2/n} \nu}}{3\sqrt{3}} (m + 1)^4 ||\xi||^2 g''(\theta)^2 - \frac{\nu}{n} g'(\theta)^2\right)
\]
\[
= \exp\left(-\lambda g'(\theta)^2\right)
\]
where we used our previous bound on $||g''||_{\infty}$. Finally, the Cauchy-Schwarz inequality gives $||g''||_{\infty} \leq (m + 1)^2 ||\xi||$. If we combine all our estimates, then (3.25) becomes
\[
|F_{n,m}^{a,b}(\xi)| \leq e^{\nu^2 (1 - \frac{1}{n})/(m+1)^4 ||\xi||^2 E_{O(2n)}[\prod_{1 \leq j \leq n} e^{-\lambda g'(\theta_j)^2}]}
\]
Inserting the definition of $\nu$ finishes the proof.
Now consider the three other cases. The proposition follows by the same approach as for the first case except that now we also need to control the additional factors that appear in the probability densities (0.2), namely
\[ \prod_{1 \leq j \leq n} \sin^2(\theta_j), \prod_{1 \leq j \leq n} \cos^2\left(\frac{\theta_j}{2}\right), \prod_{1 \leq j \leq n} \sin^2\left(\frac{\theta_j}{2}\right). \]
After the same change of variables as above, the first of these products can be bounded using Lemma 3.1:
\[ \left| \frac{\sin^2(\theta_j - ivh(\theta_j)/n)}{\sin^2(\theta_j)} \right| = 1 + \sin^2\left(\frac{v\theta_j}{n}\right) \leq \exp\left(\frac{v\theta_j}{n}\right)^2 \]
for part (a) and
\[ \left| \frac{\sin^2(\theta_j + ivg(\theta_j)/n)}{\sin^2(\theta_j)} \right| = 1 + \sin^2\left(\frac{v\theta_j}{n}\right) \leq \exp\left(\frac{v\theta_j}{n}\right)^2 \]
for part (b). Now, using that \(|\sin(k\theta)/\sin(\theta)| = \sum_{j=0}^{k-1} e^{i(k-1-2j)\theta}| \leq k\), we see that
\[ \left( \frac{h(\theta_j)}{\sin(\theta_j)} \right)^2 \leq \left( 2 \sum_{k=1}^{m} \sqrt{k} |\xi_k| \right)^2 \leq 2m(m+1)||\xi||^2 \]
for part (a) and
\[ \left( \frac{g'(\theta_j)}{\sin(\theta_j)} \right)^2 \leq \left( 2 \sum_{k=1}^{m} k^{3/2} |\xi_k| \right)^2 \leq (m+1)^4 ||\xi||^2 \]
for part (b). These last four lines of inequalities explain the additional terms that appear in the bounds of $F_{n,m}^{++}$. A similar argument gives the same bounds for the last two cases. \(\square\)

We will now use the Basor-Ehrhardt identities a second time to bound the expectation values appearing in part (a) of the previous proposition.

**Proposition 3.3.** Let $h$ be the Hilbert transform of $g$. Then, for $\xi \in \mathbb{R}^m$,
\[ \mathbb{E}_{G(n)} \left[ \exp \left( \frac{\nu}{n} \sum_{j=1}^{n} h(\theta_j)g'(\theta_j) \right) \right] \leq \exp \left( -2\nu ||\xi||^2 + 4\nu^2 m^2 (1 + \log m) ||\xi||^4 \right) \det(1 + Q_n K_{G(n)} Q_n) \]
where $K_{O(2n+1)^-} = H(a_{+1}^{-1} a_{+})$, $K_{O(2n+1)^+} = -H(a_{+1}^{-1} a_{+})$, $K_{Sp(2n)} = -H(e^{-i\theta}a_{+}^{-1} a_{+})$, $K_{O(2n)^+} = H(e^{i\theta}a_{+}^{-1} a_{+})$, and $a_{+1}^{-1} a_{+}(e^{i\theta}) = \exp \left( -2i \nu \sum_{k,j=1}^{m} k \xi_k \xi_j (\sin(k+j)\theta - \sin|k-j|\theta) \right)$.

**Proof.** Take $G = O(2n+1)^-$ for simplicity; the proof of the result for the other groups uses the same arguments. According to Lemma 1.1 and Proposition 1.2,
\[ \mathbb{E}_{O(2n+1)^-} \left[ \exp \left( \frac{\nu}{n} \sum_{j=1}^{n} h(\theta_j)g'(\theta_j) \right) \right] = \exp \left( n[\log a]_0 + \sum_{k=0}^{\infty} [\log a]_{2k+1} + \frac{1}{2} \sum_{k=1}^{\infty} k[\log a]_k^2 \right) \det(1 + Q_n H(a_{+}^{-1} a_{+}) Q_n) \]
where
\[ \begin{align*}
\text{(3.27)} a(e^{i\theta}) &= \exp \left( \frac{\nu}{n} h(\theta)g'(\theta) \right) = \exp \left( -4 \nu \sum_{k,j=1}^{m} k \xi_k \xi_j \sin k\theta \sin j\theta \right) \\
\text{and therefore} \\
a_{+}^{-1} a_{+}(e^{i\theta}) &= \exp \left( -2i \nu \sum_{k,j=1}^{m} k \xi_k \xi_j (\sin(k+j)\theta - \sin|k-j|\theta) \right). 
\end{align*} \]
First observe that by orthogonality of the sine function,
\[
[\log a]_0 = \frac{\nu}{n} \int_0^{2\pi} g'(\theta) h(\theta) \frac{d\theta}{2\pi} = -2 \frac{\nu}{n} \|\xi\|^2.
\]
Next we see that the second term in the exponential, \(\sum_{k=0}^{\infty} [\log a]_{2k+1}\), is equal to zero. Indeed, we have
\[
\log a(e^{i\theta}) = \frac{\nu}{n} \sum_{k,j=1}^m \sqrt{\frac{k}{j}} \xi_k \xi_j (e^{i(k+j)\theta} - e^{i(k-j)\theta} - e^{-i(k-j)\theta} + e^{-i(k+j)\theta})
\]
so the sum of all Fourier coefficients of positive odd order will cancel out. Finally, to bound the last term in the exponential, we will use that for any real function \(u\) on the unit circle that satisfies \(\sum_{k=1}^\infty k|\hat{u}_k|^2 < \infty\), and with Hilbert transform \(H(u) := -i \sum_{j \in \mathbb{Z}} \text{sgn}(j) \hat{u}_j e^{iy}\theta\), the following holds
\[
\sum_{k=1}^\infty k|\hat{u}_k|^2 = -\frac{1}{2} \int_0^{2\pi} u'(\theta) H(u(\theta)) \frac{d\theta}{2\pi} \leq \frac{1}{2} \|u\|_{L^2} \|u'\|_{L^2}.
\]
Thus, in our case,
\[
\frac{1}{2} \sum_{k=1}^\infty k[\log a]^2_k = \frac{\nu^2}{2n^2} \sum_{k=1}^\infty g''(\theta) \frac{d\theta}{2\pi} \leq \frac{\nu^2}{4n^2} \|g''(\theta)\|_{L^2} (\|g''(\theta)\|_{L^2} + \|g'(\theta)\|_{L^2})
\]
\[
\leq \frac{\nu^2}{4n^2} \|h\|_{\infty} \|g''(\theta)\|_{L^2} (\|g''(\theta)\|_{L^2} + \|h\|_{\infty} \|g'(\theta)\|_{L^2})
\]
\[
< \sqrt{\frac{\nu^2}{n^2}} m \sqrt{1 + \log m} (\sqrt{2m^{1/2} \sqrt{1 + \log m} + m \sqrt{1}}) \|\xi\|^4
\]
\[
< \frac{\nu^2}{n^2} m^2 (1 + \log m) \|\xi\|^4
\]
where the third inequality follows from applying the Cauchy-Schwarz inequality to bound the \(L^\infty\) norms. This finishes the proof in the case of \(O(2n+1)^-\).

**Remark 3.4.** The main heuristic behind letting \(h\) be the Hilbert transform of \(g\) in the change of variables, besides the similarity of this problem to [11], is that
\[
\frac{1}{4} \int_0^{2\pi} g'(\theta) h(\theta) \frac{d\theta}{2\pi} = \frac{1}{2} \sum_{k=1} \kappa |\hat{g}_k|^2 = -\frac{1}{2} \|\xi\|^2
\]
and the right-hand side is the limiting exponent of \(F^a_{n,b}\) (see Corollary 1.6 and the following comments).

It remains to estimate the Fredholm determinants of Proposition 3.3.

**Proposition 3.5.** Let \(m \geq 2, m \leq n^{1/3}, \xi \in \mathbb{R}^m\). With \(K_{G(n)}\) as in the previous proposition we have that
\[
\det(1 + Q_n K_{G(n)} Q_n) \leq \exp \left( \nu \frac{\sqrt{\log m + 1} + 1}{m^{1/3}} \left( \frac{m^{-1/3}}{m-1} \left( \frac{m^{-4/3}}{\sqrt{6}} (1 + m^{-1})^{5/3} + 1 \right) \|\xi\|^2 \right. \right.
\]
\[
\left. \left. + \nu^2 \frac{(m+1)^{8/3}}{n^2} \left( \sqrt{\log m + 1} + 1 \right)^2 \|\xi\|^4 \right) \right)
\]
provided \(\nu\) satisfies
\[
\frac{\nu}{n} (m+1)(1 + \sqrt{\log m + 1}) \|\xi\|^2 < \frac{2n-1}{2(m+1)^{5/3}}.
\]

**Proof.** Recall that for any trace-class operator \(K\), one can define the regularized determinant \(\det_{\nu}^2\) by
\[
\det_{\nu}^2(I + K) = e^{-\text{Tr}^2 K} \det(I + K),
\]
and prove that it satisfies the following inequality
\[
\left| \det_{\nu}^2(I + K) \right| \leq e^{\frac{\nu}{2} \|K\|_{L^2}},
\]

18
which is actually valid for the larger class of Hilbert-Schmidt operators (Theorem 9.2 in [18]). By Proposition 1.2 all our $K_{G(n)}$ are trace-class, thus

\begin{equation}
|\det(1 + Q_n K_{G(n)} Q_n)| \leq e^{\text{Tr} Q_n K_{G(n)} Q_n + \frac{1}{2} \|Q_n K_{G(n)} Q_n\|_2^2}.
\end{equation}

Since $K_{G(n)}$ is a certain type of Hankel operator with symbol $a_+^{-1}a_+$, we need bounds on the Fourier coefficients of $a_+^{-1}a_+$. We have

\[ a_+^{-1}a_+(e^{i\theta}) = \exp \left( -2i\frac{\nu}{n} \sum_{k,j=1}^{m} \sqrt{k_j \xi_k \xi_j} \sin(k+j)\theta - \sin(k-j)\theta \right). \]

Integrating by parts its $l$th Fourier coefficient twice and taking the absolute value gives

\begin{equation}
|\{a_+^{-1}a_+\}| \leq \frac{4^{\nu/n^2}}{m} \sum_{k,j=1}^{m} \sqrt{k_j \xi_k \xi_j}((k+j)^2 + (2\log m + 1)^2)\|\xi\|^2
\end{equation}

which we can bound using the Cauchy-Schwarz inequality:

\begin{equation}
|\{a_+^{-1}a_+\}| \leq \frac{4^{\nu/n^2}}{m} (m + 1)^3((\sqrt{\log m + 1} + 1)\|\xi\|^2 + \frac{2n-1}{2(m+1)^{5/3}}\|\xi\|^4).
\end{equation}

Thus, by (3.28),

\[ |\{a_+^{-1}a_+\}| \leq \frac{4^{\nu/n^2}}{m} (m + 1)^3((\sqrt{\log m + 1} + 1)\left( \frac{1}{\sqrt{6}} + \frac{2n-1}{2(m+1)^{5/3}} \right))\|\xi\|^2 := B_l. \]

Now, by definition of the kernels $K_{G(n)}$:

\[ \text{Tr} Q_n K_{O(2n+1)} Q_n = \sum_{j=1}^{\infty} \{a_+^{-1}a_+\}_{2j+1}, \quad \text{Tr} Q_n K_{O(2n+1)} Q_n = \sum_{j=1}^{\infty} \{a_+^{-1}a_+\}_{2j}, \]

so because $B_j$ is decreasing it suffices to estimate

\begin{equation}
\sum_{j=1}^{\infty} |B_j| \leq \frac{\nu}{n} (m + 1)^3((\sqrt{\log m + 1} + 1)\left( \frac{1}{\sqrt{6}} + \frac{2n-1}{2(m+1)^{5/3}} \right))\|\xi\|^2
\end{equation}

\begin{equation}
\leq \nu \left( \sqrt{\log m + 1} + 1 \left( \frac{m + 1}{\sqrt{6}(m+1)^{5/3}} \right) \right)\|\xi\|^2
\end{equation}

where we used that $m \leq n^{1/3}$.

Similarly,

\[ \|Q_n K_{O(2n+1)} Q_n\|_2^2 = \sum_{j=1}^{\infty} (j + 2n + 1)\{a_+^{-1}a_+\}_{2j+1}^2, \quad \|Q_n K_{O(2n+1)} Q_n\|_2^2 = \sum_{j=1}^{\infty} (j + 2n + 1)\{a_+^{-1}a_+\}_{2j}^2, \]

so we can restrict our attention to

\[ \sum_{j=1}^{\infty} (j + 2n + 1)B_j^2 \leq \frac{4\nu^2}{n^2(2n-1)^2}(m + 1)^6((\sqrt{\log m + 1} + 1)^2\left( \frac{1}{\sqrt{6}} + \frac{2n-1}{2(m+1)^{5/3}} \right)^2\|\xi\|^4
\end{equation}

where we used the bound

\[ \sum_{j=2}^{\infty} \frac{j - 2n + 1}{j^4} \leq \int_{2n-1}^{\infty} \frac{dx}{x^4} - (2n - 1)\int_{2n}^{\infty} \frac{dx}{x^4} = \frac{1}{2(2n-1)^2} - \frac{2n-1}{3(2n)^3} = \frac{1}{2n-1}^2 \frac{1}{4(2n-1)} \]

\[ \leq \frac{1}{2n-1}^2 \frac{1}{4(2n-1)} = \frac{1}{4(2n-1)^2} \]

\[ \leq \frac{1}{4(2n-1)^2} \]

\[ \leq \frac{1}{4(2n-1)^2} \]

\[ \leq \frac{1}{4(2n-1)^2} \]

\[ \leq \frac{1}{4(2n-1)^2} \]
since \( n \geq 8 \). The assumptions \( m \leq n^{1/3} \) and \( m \geq 2 \) give

\[
\|Q_nK_{G(n)}Q_n\|_2^2 \leq \frac{\nu^2(\log m + 1 + \frac{1}{n})^2}{n^2} \left( \frac{\sqrt{2}(m + 1)^3}{\sqrt{3}(2m^3 - 1)} + (m + 1)^{4/3} \right)^2 \|\xi\|^4
\]

\[
\leq \frac{\nu^2(m + 1)^{8/3}}{n^2} \left( \frac{\sqrt{2}(m + 1 + 1)^2}{(\log m + 1 + 1)^2} + 1 \right)^2 \|\xi\|^4
\]

\[
< 2\frac{\nu^2(m + 1)^{8/3}}{n^2} \left( \log m + 1 + 1 \right)^2 \|\xi\|^4.
\]

These bounds inserted in (3.29) give the desired inequality.

\( \square \)

Combining Propositions 3.2, 3.3 and 3.5 we see that if \( \nu \) satisfies (3.23) and (3.28), and if \( m \geq 2 \), \( m \leq n^{1/3} \), then

\[
|F_{n,m}^{a,b}(\xi)| \leq \exp(-\alpha \nu + \delta \nu^2)
\]

where

\[
\alpha = 2\|\xi\|^2 - \frac{\sqrt{\log m + 1 + 1}}{m^{5/3}} \left( \frac{1 + m^{-1})^{4/3}}{1 - m^{-3}} \left( \frac{m^{-4/3}}{\sqrt{6}} (1 + m^{-1})^{5/3} + 1 \right) \right) \|\xi\|^2
\]

and

\[
\delta = 4\frac{m^2}{n^2}(1 + \log m)\|\xi\|^4 + \left( 2 + \frac{1}{n} \right) m(m + 1)\|\xi\|^2 + \frac{2e^{\frac{c_2}{n}}}{n}(m + 1)^2(\log m + 1)\|\xi\|^3
\]

\[
+ \frac{(m + 1)^{8/3}}{n^2} \left( \log m + 1 + 1 \right)^2\|\xi\|^4,
\]

for all pairs \((a, b) = (\pm 1/2, \pm 1/2)\) and any \( \xi \in \mathbb{R}^m \). Optimizing over \( \nu \) yields \( \nu = \alpha/2\delta \) which is positive for \( m \geq 2 \). We obtain

**Proposition 3.6.** \( \text{Let } m \leq n^{1/3}, \text{ } m \geq 2. \text{ Then, for any pair } (a, b) = (\pm 1/2, \pm 1/2) \text{ and any } \xi \in \mathbb{R}^m, \)

\[
|F_{n,m}^{a,b}(\xi)| \leq \exp\left( -\frac{(1 - c_1(m))^2}{c_2(m)} \min(n^2, \|\xi\|^2) \right)
\]

where

\[
c_1(m) = \frac{\sqrt{\log m + 1 + 1}}{2m^{5/3}} \left( \frac{1 + m^{-1})^{4/3}}{1 - m^{-3}} \left( \frac{m^{-4/3}}{\sqrt{6}} (1 + m^{-1})^{5/3} \right) \right)
\]

and

\[
c_2(m) = \left( 4m^2(\log m + 1 + 2e^{\frac{4}{m^2}}(m + 1)^2(\log m + 1) + \left( 2 + \frac{1}{m^2} \right) m(m + 1) + (m + 1)^{8/3}(\sqrt{\log m + 1 + 1})^2 \right)
\]

\[
(m + 1)^{-8/3}(\log m + 1)^{-1}
\]

Consequently, for any \( \Lambda_2 \geq \Lambda_1 \) with \( \Lambda_1 \) given by (2.13),

\[
\int_{\Lambda_1 \leq \xi \leq \Lambda_2} |F_{n,m}^{a,b}(\xi)|^2 d\xi
\]

\[
\leq \Omega_m n^m \exp\left( -\frac{(1 - c_1(m))^2n^2}{2c_2(m)(m + 1)^{8/3}(\log m + 1)^2} \right) + \Omega_m A_2^m \exp\left( -\frac{2(1 - c_1(m))^2n^2}{c_2(m)(m + 1)^{8/3}(\log m + 1)} \right)
\]

**Proof.** Before inserting \( \nu = \alpha/2\delta \) in (3.35) we need to check that it satisfies (3.23) and (3.28). For the first inequality, we have

\[
\frac{\alpha}{2\delta} < \|\xi\|^2 < \frac{n}{2e^{c_0}(m + 1)^2(\log m + 1)}\|\xi\|
\]
which gives
\[ \frac{2}{n} \left( m \sqrt{\log (m+1)} \right) < \frac{1}{e^{c_0}(m+1) \sqrt{\log (m+1)}} \leq \frac{1}{c_0 \sqrt{\log (2+1)}} \]
if \( m \geq 2 \), and this will be less than \( c_0 \) if we simply choose \( c_0 = 1/3 \). For the second inequality we can use that
\[ \frac{\alpha}{2\delta} < \frac{\|\xi\|^2}{\delta} < \frac{n^2}{(m+1)^{8/3}(\sqrt{\log (m+1)}+1)^2\|\xi\|^2} \]
and therefore
\[ \frac{\nu}{n} (m+1)(1+\sqrt{\log (m+1)})\|\xi\|^2 < \frac{n}{(m+1)^{5/3}(\sqrt{\log (m+1)}+1)} \]
\[ \leq \frac{1}{2} \frac{2n-1}{(m+1)^{5/3}}. \]
for all \( m \geq 2 \). Hence (3.35) becomes
\[ \left| F_{a,b}^{n,m}(\xi) \right| \leq \exp \left( -\frac{\alpha^2}{4\delta} \right) = \exp \left( -\frac{(1-c_1(m))^2\|\xi\|^2}{\delta/\|\xi\|^2} \right). \]
for all \( m \geq 2 \). Now replace either \( \|\xi\|/n \) or \( n/\|\xi\| \) by one depending on whether \( \|\xi\| \leq n \) or \( n \leq \|\xi\| \),
\[ \left| F_{a,b}^{n,m}(\xi) \right| \leq \exp \left( -(1-c_1(m))^2(n^2,\|\xi\|^2) \times \left( 4m^2(1+\log m)+2e^{3}(m+1)^2(\log m + 1) + \left( 2 + \frac{1}{n} \right) m(m+1) + (m+1)^{8/3}(\sqrt{\log (m+1)}+1)^2 \right)^{-1} \right). \]
The denominator in the exponential is bounded by
\[ c_2(m)(m+1)^{8/3}(\log m + 1) \]
for \( n \geq m^3 \). This gives (3.36). We can now bound the \( L_2 \)-norm by writing
\[ \int_{\Lambda_1 \leq \|\xi\| \leq \Lambda_2} \left| F_{a,b}^{n,m}(\xi) \right|^2 d\xi = \int_{\Lambda_1 \leq \|\xi\| \leq n} \left| F_{a,b}^{n,m}(\xi) \right|^2 d\xi + \int_{n \leq \|\xi\| \leq \Lambda_2} \left| F_{a,b}^{n,m}(\xi) \right|^2 d\xi \]
\[ \leq \Omega_m n^m \exp \left( -2(1-c_1(m))^2\Lambda_1^2 \right) + \Omega_m \Lambda_2^m \exp \left( -2(1-c_1(m))^2n^2 \right) \]
which is (3.39) if we replace \( \Lambda_1 \) by its definition, \( \Lambda_1 = n/(2m\sqrt{\log (m+1)}). \)

4. LARGE REGIME

For the last regime we need a bound on the characteristic function \( F_{a,b}^{n,m} \) that decays with \( \xi \) since we eventually integrate it over all \( \xi \in \mathbb{R}^m \). Our method relies on the change of variables of the previous section, i.e. part (b) of Proposition 3.2 is our starting point, but instead of using the Basor-Ehrhardt formulas we apply the following lemma.

**Lemma 4.1.** For any pair \((a,b) = (\pm \frac{1}{2}, \pm \frac{1}{2})\), the joint eigenvalue probability density satisfies
\[ \sup_{\theta \in [0,\pi]} \left| \rho_n^{a,b}(\theta) \right| \leq \frac{(2e/\pi)^n}{\sqrt{2\pi n}}. \]
Proof. One can show (proof of Proposition 3.7 in [15] or Exercise 5.5.4 in [9])
\[
\begin{align*}
\rho_n^-(\theta) &= \frac{2^n}{n!\pi^n}(\det[\cos(k-1)\theta])_{1\leq j,k\leq n}^2 \\
\rho_n^+(\theta) &= \frac{2^n}{n!\pi^n}(\det[\sin k\theta])_{1\leq j,k\leq n}^2 \\
\rho_n^0(\theta) &= \frac{2^n}{n!\pi^n}(\det[\cos(k-1/2)\theta])_{1\leq j,k\leq n}^2 \\
\rho_n^r(\theta) &= \frac{2^n}{n!\pi^n}(\det[\sin(k-1/2)\theta])_{1\leq j,k\leq n}^2
\end{align*}
\]

The result follows by applying Hadamard’s formula to each determinant and Stirling’s approximation
(Inequality (2.21)) to \(n!\). □

This shows that the expected values appearing in part (b) of Proposition 3.2 satisfy
\[
E_{G(n)}[\prod_{1\leq j\leq n} e^{-\lambda g'(\theta))^2}] \leq \left(\frac{2e^n}{\sqrt{2\pi n}} \left(\frac{1}{\pi} \int_0^\pi e^{-\lambda g'(x)^2} dx\right)^n\right).
\]

To evaluate the integral on the right-hand side we will need the following result, obtained in [3].

Lemma 4.2. Let \(p_m\) be a trigonometric polynomial given by
\[
p_m(\theta) = \frac{a_0}{2} + \sum_{k=1}^m (a_k \cos k\theta + b_k \sin k\theta),
\]
where \(a_k, b_k\) are real. Define \(G(t) = \frac{1}{2\pi} \mu\{e^{i\theta} \in T, |p_m(\theta)| \leq t\}\), where \(\mu\) denotes Lebesgue measure on the unit circle \(T\). Then,
\[
G(t) \leq 2e \left(\frac{t}{\sqrt{2\|p_m\|^2}}\right)^\frac{1}{2m}.
\]

Proposition 4.3. For any pair \((a, b) = (\pm \frac{1}{2}, \pm \frac{1}{2})\) and any \(\Lambda_2 > 0\) we have that
\[
\int_{\Lambda_2 \leq \|\xi\|} |\mathcal{F}_{\mu}^{a,b}(\xi)|^2 d\xi \leq \frac{(2e)^{4n}}{2\pi n} (c_3(m) \sqrt{4\pi m}^2)^2 \frac{\Lambda_2^{n-N/2}}{N/2 - m}
\]
provided \(m \geq 3, n \geq m^3\), and where
\[
c_3(m) = \frac{e^{\frac{1}{2}(1 + \frac{1}{m^2})} (1 + m^{-1})^2}{\sqrt{2(1 - \frac{e^{1/(2m^2)}}{24\sqrt{3m^2}})}}.
\]

Proof. Define \(G(t)\) as in Lemma 4.2, with \(p_m(\theta) = g'(\theta)\). Then
\[
\frac{1}{\pi} \int_0^\pi e^{-\lambda g'(\theta)^2} d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{-\lambda g'(\theta)^2} d\theta = \int_0^\infty e^{-tG\left(\sqrt{\frac{t}{\lambda}}\right)} dt.
\]
Estimate (4.42) becomes
\[
G\left(\sqrt{\frac{t}{\lambda}}\right) \leq 2e \left(\frac{t}{2\lambda \|g'\|^2_2}\right)^{1/(4m)}
\]
which inserted in (4.45) gives
\[
\frac{1}{\pi} \int_0^\pi e^{-\lambda g'(\theta)^2} d\theta \leq 2e \left(\frac{1}{2\lambda \|g'\|^2_2}\right)^{1/4m} \int_0^\infty e^{-t} t^{1/(4m)} dt
\]
\[
= 2e \left(\frac{1}{2\lambda \|g'\|^2_2}\right)^{1/4m} \Gamma\left(1 + \frac{1}{4m}\right) < 2e \left(\frac{1}{2\lambda \|g'\|^2_2}\right)^{1/(4m)}.
\]
Recall that
\[ g'(\theta) = -2 \sum_{k=1}^{m} k \xi_k \sin k \theta. \]

Hence
\[ \|g'\|^2 = 2 \sum_{k=1}^{m} k^2 \xi_k^2 \geq 2 \sum_{k=1}^{m} \xi_k^2 = 2 \|\xi\|^2 , \]

and by (4.41),
\[ \mathbb{E}_{G(n)} \left[ \prod_{1 \leq j \leq n} e^{-\lambda g'(\theta_j)^2} \right] \leq \frac{(2e)^n}{\sqrt{2\pi n}} \left( \frac{1}{n} \int_{0}^{\pi} e^{-\lambda g'(z)^2} dz \right)^n \leq \frac{(2e)^{2n}}{\sqrt{2\pi n}} \left( \frac{1}{4\lambda \|\xi\|^2} \right)^{n/4m} . \]

It now follows from Proposition 3.2, part (b), that
\[ |F_{n,m}^{a,b}(\xi)| \leq e^{n^2(n+\frac{1}{2})} \left( \frac{(2e)^{2n}}{\sqrt{2\pi n}} \right)^{n/4m} \left( \frac{1}{4\lambda \|\xi\|^2} \right)^{n/4m} . \]

Inserting the definition of \( \lambda \) gives
\[ |F_{n,m}^{a,b}(\xi)| \leq \frac{(2e)^{2n}}{\sqrt{2\pi n}} \left( \frac{e^{\frac{1}{2}(1+\frac{1}{n})}}{\sqrt{2(1 - \frac{1}{e^{1/(2\sqrt{3}m)^2}})}} \right)^{nm} \left( \frac{\xi}{\|\xi\|} \right)^{2} . \]

Thus (recall that \( N = n/m \))
\[ (4.46) \int_{\Lambda_2 \leq \|\xi\|} |F_{n,m}^{a,b}(\xi)|^2 d\xi \leq \frac{(2e)^{2n}}{\sqrt{2\pi n}} \left( \frac{e^{\frac{1}{2}(1+\frac{1}{n})}}{\sqrt{2(1 - \frac{1}{e^{1/(2\sqrt{3}m)^2}})}} \right)^{nm} \int_{\Lambda_2 \leq \|\xi\|} \|\xi\|^{-\frac{7}{2}} d\xi . \]

A change of variables to spherical coordinates gives
\[ \int_{\Lambda_2 \leq \|\xi\|} \|\xi\|^{-\frac{7}{2}} d\xi = S_{m-1} \int_{\Lambda_2} r^{m-1-\frac{7}{2}} dr = S_{m-1} \frac{\Lambda_2^{m-\frac{5}{2}}}{\frac{5}{2} - m} , \]

where \( S_{m-1} \) is the surface area of the \( m - 1 \)-dimensional unit sphere, and where we used that \( m < \frac{N}{2} \) which follows from \( m \geq 3 \) and \( n \geq m^3 \). Observing that \( S_{m-1} = m \Omega_m \), using that \( n \geq m^3 \) and inserting the definition of \( c_3(m) \) in (4.46) prove the statement. \( \square \)

5. PROOF OF THE MAIN THEOREM

In this section we first combine all our estimates to prove the bound on the \( L_2 \) distance between our random vector and a standard normal one, given in Theorem 0.1. The total variation bound from Theorem 0.2 then follows from a result on tail probabilities. We conclude with three corollaries which give simple bounds for some special cases of \( m \) and \( n \).

Proof of Theorem 0.1. By Plancherel’s theorem, since \( p_{n,m}^{a,b} - \Psi_m \in L^2(\mathbb{R}^m) \),
\[ \| p_{n,m}^{a,b} - \Psi_m \|_2 = \| F_{n,m}^{a,b} - e^{-\|\cdot\|^2/2} \|_2 . \]

By Minkowski’s inequality,
\[ \| F_{n,m}^{a,b} - e^{-\|\cdot\|^2/2} \|_2 \leq \| F_{n,m}^{a,b} - e^{-\|\cdot\|^2/2} \|_2 1 \{ \|\xi\| \leq \Lambda_1 \} \|_2 + \| F_{n,m}^{a,b} \|_2 1 \{ \Lambda_1 \leq \|\xi\| \} \|_2 + \| e^{-\|\cdot\|^2/2} \|_2 1 \{ \Lambda_1 \leq \|\xi\| \} \|_2 \]

[23]
where \( \Lambda_1 \) is given by (2.13). By Proposition 2.1, 3.6 and 4.3,
\[
(5.47) \quad \| F^{a,b}_{n,m} - e^{-\|\cdot\|^2/2} \|_2 \leq \frac{16}{15} 13/24 \epsilon^{3/8} + 1 \frac{m^{3/2} \sqrt{\Omega_n} (m/2)^{3/2} (\log m + 1)^N}{\sqrt{N}} \theta (e^{3/2} (\log m + 1))^N / (2N + 1)
\]
\[
\| F^{a,b}_{n,m} \{ \Lambda_1 \| \leq \| \| \| \|_2 \|_2 \} \|_2 \leq \Omega_{m,n} m^{3/2} \exp \left( \frac{-(1 - c_1(m))^2 \epsilon^2}{2 \epsilon_2(m) m^2 (m + 1)^8/3 (\log m + 1)^2} \right) + \Omega_{m,n} \Lambda_{1}^{m-N/2} \left( \frac{2 \epsilon_2(m) m^2 (m + 1)^8/3 (\log m + 1)^2}{\epsilon} \right)
\]
so it remains to choose \( \Lambda_2 \) to estimate \( \| e^{-\|\cdot\|^2/2} \{ \Lambda_1 \| \leq \| \| \|_2 \} \|_2 \). For the first task, set
\[
\epsilon = \exp \left( \frac{-2(1 - c_1(m))^2 \epsilon^2}{c_2(m) (m + 1)^{8/3} (\log m + 1)} \right), \quad C = \frac{2 \epsilon_2^{4n}}{2 \pi n} \frac{c_3(m) \sqrt{\Omega_n m^2} (m + m)^{3/2}}{(m + 1)^{8/3} (\log m + 1)^2}
\]
The minimum of \( \Omega_{m,n} \Lambda_2^m (\epsilon + C \Lambda_2^{-N/2}) \) is attained when \( \Lambda_2^{N/2} = \frac{C}{\epsilon} (\frac{N}{2m} - 1) \) and equals
\[
\Omega_{m,n} \Lambda_2^{N/2} \frac{N}{N - 2m} \frac{N}{2m} (\frac{N}{2m} - 1)^{1/e^2} \epsilon^{1 - e^2}.
\]
Inserting the values of \( \epsilon \) and \( C \), taking the square root and recalling that \( m \geq 3, n \geq m^3 \) yields
\[
(5.48) \quad \| F^{a,b}_{n,m} \{ \Lambda_1 \| \leq \| \| \|_2 \} \|_2 \leq \sqrt{3 \Omega_n N^{3/2}} \left( 2 \epsilon \right)^{4n} (2 \pi n)^{1/N} \exp \left( - \frac{(1 - c_1(m))^2 \epsilon^2}{3 \epsilon_2(m) m^2 (m + 1)^8/3 (\log m + 1)} \right) + \sqrt{\Omega_n N^{3/2}} \exp \left( - \frac{(1 - c_1(m))^2 \epsilon^2}{4 \epsilon_2(m) m^2 (m + 1)^8/3 (\log m + 1)^2} \right)
\]
For the second task we make a change of variables to spherical coordinates
\[
\| e^{-\|\cdot\|^2/2} \{ \Lambda_1 \| \leq \| \| \|_2 \} \|_2 \leq \int_{\|\cdot\| > \Lambda_1} e^{-\|\cdot\|^2} d\xi = S_{m-1} \int_{\Lambda_1}^{\infty} r^{m-1} e^{-r^2} dr = \frac{S_{m-1}}{2} \int_{\Lambda_1^2}^{\infty} r^{-m/2} e^{-r} dr.
\]
Repeated integration by parts (or 8.8.10 in [6]) gives
\[
\int_{\Lambda_1^2}^{\infty} r^{-m/2} e^{-r} dr = e^{-\Lambda_1^2} \Gamma (\frac{m}{2}) \sum_{j=0}^{m-1} \frac{A_1^{2(m/2 - 1) - j}}{\Gamma (\frac{m}{2} - j)}
\]
if \( m \) is even, and
\[
\int_{\Lambda_1^2}^{\infty} r^{-m/2} e^{-r} dr = \frac{\Gamma (\frac{m}{2})}{\Gamma (\frac{m}{2} - 1)} \Gamma (-1/2, \Lambda_1^2) + e^{-\Lambda_1^2} \Gamma (\frac{m}{2}) \sum_{j=0}^{m-1} \frac{A_1^{2(m/2 - 1) - j}}{\Gamma (\frac{m}{2} - j)}.
\]
if \( m \) is odd. Here \( \Gamma (a, z) \) is the incomplete Gamma function. Observe that \( \Gamma (-1/2) < 0 \) so for any \( m \in \mathbb{N} \),
\[
\int_{\Lambda_1^2}^{\infty} r^{-m/2} e^{-r} dr \leq e^{-\Lambda_1^2} \Gamma (\frac{m}{2}) \sum_{j=0}^{m-1} \frac{A_1^{2(m/2 - 1) - j}}{\Gamma (\frac{m}{2} - j)}
\]
\[
\leq e^{-\Lambda_1^2} \sum_{j=0}^{m-1} \frac{A_1^{2(m/2 - 1) - j}}{(m/2 - 1)^j}
\]
\[
< e^{-\Lambda_1^2} \frac{A_1^m}{\Lambda_1^2 - m/2 + 1}
\]
where we used that for any \( j, m \in \mathbb{N} \), \( \Gamma (\frac{m}{2} - j) \leq \Gamma (\frac{m}{2} - j) (\frac{m}{2} - 1)^j \) (which follows from the recurrence relation \( \Gamma (z) = z \Gamma (z - 1) \)) and that \( \Lambda_1^2 > m/2 - 1 \) (by definition (2.13) of \( \Lambda_1 \) and because \( n \geq m^3 \)).
Thus, moreover, see Theorem II.3.3 and exercise II.21 in [10]. Two partial integrations give
\[ E \leq \frac{m}{N^2(1 - 2(\log m + 1)(m - 2)/m^4)} \left( \frac{N}{2\sqrt{\log m + 1}} \right)^m e^{-\frac{N}{m\log^{1/2}m}}, \]
\[ \leq \frac{2(\log m + 1)m\Omega_m}{m^N N^m} \frac{m}{N^2(1 - 2(\log m + 1)(m - 2)/m^4)} \left( \frac{N}{2\sqrt{\log m + 1}} \right)^m e^{-\frac{N}{m\log^{1/2}m}}. \]

Proof. We prove the case \((n > m)\) and if \((n \geq m)\), then the bound from the other regimes, (5.47). This holds if the smallest exponent in (5.48), \(n^2/(m^2(m + 1)^{3/2}(\log m + 1)^2)\) grows faster than \(N\log N\), i.e. if \(n > m^{3+2/3+\epsilon}\) for some \(\epsilon > 0\). We assumed \(n \geq m^4\) for simplicity. The bound in (5.48) still decays with \(n\), whence gives us some rate of convergence, as long as \(n > m^{2+1/3}\log m\); we chose to assume \(n \geq m^3\) for simplicity and because it improves our bounds from section 3.

Theorem 0.1 will give us the bound on the total variation when combined with the following result on tail probabilities.

Lemma 5.2. Assume \(L > \frac{\sqrt{\ln n}}{\sqrt{n-1}}\) and \(m \geq 4\). Let \(\square_L = [-L/2, L/2]^m\). Then, if \(n \geq m^4\),
\[ P_n^{(a,b)} [X \notin \square_L] \leq 2me^{-\frac{L^2}{\pi}}, \]
and if \(n \geq m^3\),
\[ P_n^{(a,b)} [X \notin \square_L] \leq 2me^{-\frac{L^2}{\pi}}. \]

Proof. We prove the case \((a, b) = (-1/2, -1/2)\). By Lemma 1.1, for any \(\lambda > 0\),
\[ E_{O(2n)^+}[e^{\lambda(T(U^k - E)U^k)}] = e^{-\lambda E_{O(2n)^+} \det(\hat{g}_{-k} + \hat{g}_{k})} \leq \exp(\sqrt{2}n \lambda^2), \]
where \(g(\theta) = e^{2\lambda \cos(\theta)}\). Therefore, the assumptions of Proposition 1.2 are met and we obtain (recall that \(E_{O(2n)^+} U^k = \eta_k\))
\[ E_{O(2n)^+}[e^{\lambda(T(U^k - E)U^k)}] = e^{\lambda^2} \det(1 + Q_n H(e^{i\theta} e^{-2\lambda \sin k\theta})Q_n). \]
To bound the Fredholm determinant we that for any trace class operator \(K\) given by the infinite matrix \((K_{ij})_{i,j=1}^\infty\),
\[ \det(1 + K) \leq e^{\|K\|_1} \leq e^{\sum_{i \geq 1}(\sum_{j \geq 1}(|K_{ij}|)^2)^{1/2}} \]
see Theorem II.3.3 and exercise II.21 in [10]. Two partial integrations give
\[ |(e^{-2\lambda \sin k\theta})_{i,j}| \leq \frac{1}{1^2}(4\lambda^2 + 2\lambda)k^2. \]
Moreover,
\[ \sum_{i \geq n} \left( \sum_{j \geq n} \frac{1}{(i+j)^4} \right)^{1/2} \leq \int_{n-1}^\infty \left( \int_{n-1}^\infty \frac{dx}{(x+y)^4} \right)^{1/2} dy = \frac{\sqrt{2}}{\sqrt{3(n-1)}}. \]
Thus,
\[ \det(1 + Q_n H(e^{i\theta} e^{-2\lambda \sin k\theta})Q_n) \leq \exp \left( \sum_{i \geq n} \left( \sum_{j \geq n} \frac{1}{(i+j)^2} \right)^{1/2} \right) \]
\[ \leq \exp \left( \frac{2\sqrt{2}}{\sqrt{3(n-1)}} (2\lambda^2 + \lambda)k^2 \right). \]
Now, by Markov’s inequality,
\[ P_n^{-\frac{1}{\sqrt{n}}} |\text{Tr} U^k - \mathbb{E} \text{Tr} U^k| \geq L \leq e^{-\lambda \sqrt{L}(\mathbb{E}_{O(2n)} + [e^{\lambda(\text{Tr} U^k - \mathbb{E} \text{Tr} U^k)}] + \mathbb{E}_{O(2n)} + [e^{-\lambda(\text{Tr} U^k - \mathbb{E} \text{Tr} U^k)}])}. \]
Inserting the above estimates gives
\[ P_n^{-\frac{1}{\sqrt{n}}} |\text{Tr} U^k - \mathbb{E} \text{Tr} U^k| \geq L \leq 2 \exp \left( -\lambda \left( \sqrt{L} - \frac{2\sqrt{2k^2}}{\sqrt{3(n-1)}} \right) \right) + \lambda^2 \left( \frac{k^2}{2} + \frac{4\sqrt{2k^2}}{\sqrt{3(n-1)}} \right). \]
so by choosing
\[ \lambda = \left( \frac{\sqrt{L}}{\sqrt{3(n-1)}} \right) \left( \frac{2\sqrt{2k^2}}{\sqrt{3(n-1)}} \right) \]
we obtain, for \( L > \frac{2\sqrt{6m^{3/2}}}{\sqrt{n-1}}, 1 \leq k \leq m, m \geq 4 \) and \( n \geq m^4 \),
\[ P_n^{-\frac{1}{\sqrt{n}}} |\text{Tr} U^k - \mathbb{E} \text{Tr} U^k| \geq L \leq 2 \exp \left( -\left( \frac{1}{2} \right)^2 \frac{L^2}{2 + \frac{16\sqrt{2m}}{\sqrt{3(n-1)}}} \right) \]
\[ < 2 \exp \left( -\frac{L^2}{12} \right). \]
If \( n \geq m^3 \) the last upper bound is replaced by \( 2e^{-\frac{L^2}{8}\pi} \). The claim now follows by taking the union bound:
\[ P_{n,m}[X \notin \square_L] \leq \sum_{k=1}^{m} P_n^{-\frac{1}{\sqrt{n}}} |\text{Tr} U^k - \mathbb{E} \text{Tr} U^k| \geq \frac{L}{2} \]
which is less than \( 2me^{-\frac{L^2}{8}\pi} \) for \( n \geq m^4 \) and \( 2me^{-\frac{L^2}{8}\pi} \) for \( n \geq m^3 \).

**Proof of Theorem 0.2.** We treat the case \( n \geq m^4 \), the other is analogous. First observe that
\[ 2 \int_{L/2}^{\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = 2 \sqrt{\frac{2}{\pi}} e^{-L^2/8} \int_0^{\infty} e^{-x^2/2-Lx/2} dx \leq \frac{2}{\sqrt{2\pi}} e^{-L^2/8} \int_0^{\infty} e^{-Lx^2} dx = \frac{4}{\sqrt{2\pi}L} e^{-L^2/8} \]
whence
\[ (5.49) \quad \int_{R^m \setminus \square_L} \frac{e^{-\|x\|^2/2}}{\sqrt{2\pi}} \pi dx = \left( \frac{4}{\sqrt{2\pi}L} e^{-L^2/8} \right)^m < e^{-mL^2/8} \]
if e.g. \( L \geq \sqrt{3} \). Now, by definition of \( \Delta_{n,m}^{(1)} \), and using the Cauchy-Schwarz inequality,
\[ \Delta_{n,m}^{(1)} = \left( \int_{\square_L} + \int_{R^m \setminus \square_L} \right) |p_{n,m}^{a,b}(x) - e^{-\|x\|^2/2} \sqrt{2\pi} | dx \]
\[ \leq L^{m/2} \Delta_{n,m}^{(2)} + \int_{R^m \setminus \square_L} p_{n,m}^{a,b}(x) dx + \int_{R^m \setminus \square_L} e^{-\|x\|^2/2} \sqrt{2\pi} \pi dx. \]
So by (5.49) and Lemma 5.2, assuming \( L > \frac{4\sqrt{6m^{3/2}}}{\sqrt{n-1}}, \)
\[ (5.50) \quad \Delta_{n,m}^{(1)} < L^{m/2} \Delta_{n,m}^{(2)} + 2me^{-L^2/48} + e^{-mL^2/8} < L^{m/2} \Delta_{n,m}^{(2)} + 3me^{-L^2/48}. \]
Regard the last upper bound as a function of \( L \) and consider its critical point. It satisfies
\[ (5.51) \quad e^{-L^2/48} = 4L \Delta_{n,m}^{(1)} \]
and
\[ L \leq \sqrt{48 \log \Delta_{n,m}^{(2)}}^{-1} \]
if \( m \geq 4 \) (and assuming \( L \geq 1 \)), which gives
\[ \Delta_{n,m}^{(1)} \leq (48 \log \Delta_{n,m}^{(2)} - 1) \frac{2}{2} \Delta_{n,m}^{(2)} (1 + \frac{12m}{L^2}) \leq 2(48 \log \Delta_{n,m}^{(2)} - 1) \frac{2}{2} \Delta_{n,m}^{(2)} \]
if $L \geq 2\sqrt{3}m$. But this condition follows immediately from our assumption on $\Delta_{n,m}^{(2)}$: from (5.51) we see that

$$\Delta_{n,m}^{(2)} = \frac{L^2 - \sqrt{3}m}{4} e^{-L^2/48},$$

so as a function of $L$, $\Delta_{n,m}^{(2)}$ is decreasing and therefore bounded from below by $3m(2\sqrt{3}em)^{-\frac{n}{2}}$ if $L \leq 2\sqrt{3}m$. Finally observe that if $m \geq 4$, then

$$\frac{4\sqrt{6}m^{3/2}}{\sqrt{n-1}} < 2\sqrt{3}m$$

which proves that our assumption on $L$ was correct. □

We now present some special cases for which the upper bounds in Theorem 0.1 and 0.2 simplify. The numerical constants are obtained with Wolfram Mathematica. First recall Corollary 0.3:

**Corollary 5.3.** If $m$, $n$ satisfy the conditions in one column of the following table

| $n \geq$ | $m^4$ | $m^5$ | $m^6$ | $m^7$ | $m^8$ | $m^9$ | $m^{10}$ |
|---|---|---|---|---|---|---|---|
| $m \geq$ | $10^{19}$ | 1140 | 34 | 11 | 6 | 5 | 4 |

then,

$$\Delta_{n,m}^{(2)} \leq 8m^2 \sqrt{\Omega_m} \left( \frac{m}{2} \right)^{1/2} \frac{(e^{3/2}(\log m + 1))^N}{\sqrt{N}\sqrt{\Gamma(2N+1)}}$$

and

$$\Delta_{n,m}^{(1)} \leq 16m^2 \sqrt{\Omega_m}(24n \log N)^{1/2} \frac{(e^{3/2}(\log m + 1))^N}{\sqrt{N}\sqrt{\Gamma(2N+1)}}.$$ 

**Proof.** We compare the last three terms in (0.5) with the first (which asymptotically is dominant) i.e. with

(5.52) $$\left( \frac{m}{2} \right)^{\frac{1}{2}} \frac{m^{3/2} \left(e^{3/2}(\log m + 1)\right)^N}{N^{\frac{3}{4} + 2} \sqrt{\Gamma(2N+1)}}.$$

We start with the largest (asymptotically). We seek to estimate

$$\left( \frac{m}{2} \right)^{1/2} \frac{m^{3/2} \left(e^{3/2}(\log m + 1)\right)^N}{N^{3/4 + 2} \sqrt{\Gamma(2N+1)}}.$$ 

Stirling’s Inequality (2.21) and some rearranging give the upper bound

$$\frac{1}{m^{3/2}} \exp \left( \left( \frac{1}{N} \log \left( \frac{2N}{e^{3/2}(\log m + 1)} \right) + \frac{m}{2N^2} \log(2m) + \frac{1}{4N^2} \log(4\pi e) \right) - \frac{(1 - c_1(m))^2 N^2}{4c_2(m)(m+1)^{8/3}(\log m+1)^2} \right).$$

We see directly that the exponent becomes negative for sufficiently large $m$ if $N \geq m^3$, and we check using Wolfram Mathematica that the requirements on $n$ and $m$ are those given in the table, in which case we obtain the simple upper bound $m^{-3/2}$. The other two terms are treated similarly: we divide them by (5.52), use Stirling’s approximation, rearrange them as with the previous term and check that the sign of the exponent for the ranges of $m$ and $n$ in the table is always negative. We obtain the upper bound $m^{-3/2}(\sqrt{3} + 1)$. Finally we check that (assuming simply $m \geq 4$)

$$\frac{16}{15} e^{13/24} (e^{9/8} + 1) + \frac{(\sqrt{3} + 2)}{m^{5/2}} < 8.$$
The second inequality is a consequence of Theorem 0.2. To apply it we first need to check that the assumption is satisfied but that is straightforward: the fact that \( \Omega_m = \pi^{m/2}/\Gamma(\frac{m}{2} + 1) \) and Stirling’s inequality (2.21) give

\[
8m^2 \sqrt{\Omega_m}\left(\frac{m}{2}\right)^m \frac{(e^{3/2}(\log m + 1))^N}{\sqrt{N} \sqrt{\Gamma(2N + 1)}} \leq \frac{8m^{5/4}}{\sqrt{2\pi N^2}} e^{\frac{m}{2} (\log m + 1)} N^N
\]

and the right-hand side is less than \( 3m(2\sqrt{3m})^{-\frac{m}{2}} \) if \( N \geq m^4 \) and \( m \geq 4 \). We obtain

\[
\Delta_{n,m}^{(1)} \leq 2(4\log \Delta_{n,m}^{(2)} - 1)m^{4/4} \Delta_{n,m}^{(2)}
\]

\[
\leq 16m^2 \sqrt{\Omega_m} \left( \left( 48 \log \left( 8m^2 \sqrt{\Omega_m} \left( \frac{m}{2}\right)^m \frac{(e^{3/2}(\log m + 1))^N}{\sqrt{N} \sqrt{\Gamma(2N + 1)}} \right) \right)^{-1} \right)^{m/4} \frac{(e^{3/2}(\log m + 1))^N}{\sqrt{N} \sqrt{\Gamma(2N + 1)}}
\]

since \( x \mapsto (\log x^{-1})^{m/4} x \) is non-decreasing for \( x \in [0, e^{-m/4}] \). To complete the proof we use

\[
8m^2 \sqrt{\Omega_m}\left(\frac{m}{2}\right)^m \frac{(e^{3/2}(\log m + 1))^N}{N^{m+1} \sqrt{\Gamma(2N + 1)}} \geq N^{-N}
\]

which follows again from the fact that \( \Omega_m = \pi^{m/2}/\Gamma(\frac{m}{2} + 1) \) and Stirling’s inequality (2.21). \( \square \)

If \( m \) is not sufficiently large for the assumptions of the above corollary to hold we can instead use the following.

**Corollary 5.4.** If \( n \geq m^4, m \geq 7 \), then,

\[
\Delta_{n,m}^{(2)} \leq \sqrt{\Omega_m}(N - \epsilon N)^m \exp\left(- \frac{C(m)N^2}{(m + 1)^{8/3}(\log m + 1)^2} \right)
\]

and if \( n \geq m^4, m \geq 26 \),

\[
\Delta_{n,m}^{(1)} \leq \sqrt{\Omega_m} \left( 48C(m) \right)^m \frac{\left( \frac{m}{2}\right)^m \left( e^{3/2}(\log m + 1) \right)^N}{\left( \frac{m}{2}\right)^m \sqrt{\Gamma(2N + 1)}} \exp\left(- \frac{C(m)N^2}{(m + 1)^{8/3}(\log m + 1)^2} \right)
\]

where \( \epsilon < 10^{-82} \) and \( C(m) = \frac{1-c_1(m)}{4c_2(m)} \) satisfies

\[
\begin{array}{c|ccccccccccc}
\hline
m \geq & 7 & 8 & 9 & 10 & 20 & 30 & 40 & 50 & 100 & 500 & 1000 \\
C(m) \geq & 0.052 & 0.056 & 0.059 & 0.062 & 0.077 & 0.085 & 0.091 & 0.095 & 0.106 & 0.125 & 0.131 \\
\hline
\end{array}
\]

Note that \( x \mapsto x^{\frac{m}{2}} \exp\left(- \frac{xN^2}{(m + 1)^{8/3}(\log m + 1)} \right) \), \( x > 0.077 \), is decreasing for \( n \geq m^4, m \geq 27 \).

**Proof.** This time we compare each term in (0.5) to

(5.53)

\[
\exp\left(- \frac{(1 - c_1(m))^2 N^2}{4c_2(m)(m + 1)^{8/3}(\log m + 1)^2} \right)
\]

First we divide the second term in (0.5) by (5.53) and check that it is bounded by a small constant, more precisely by \( 5 \cdot 10^{-83} \), for all \( n \geq m^4, m \geq 7 \) (for \( m \leq 6 \) it is larger than \( 10^{17} \)). This also holds for the last term in (0.5) divided by (5.53), which is smaller than \( 2^{-1022} < 3 \cdot 10^{-308} \). Next we consider

\[
16 \frac{13}{15} (e^{9/8} + 1) \left( \frac{m}{2} \right)^m \frac{m^{3/2} (e^{3/2}(\log m + 1))^{N}}{\sqrt{2\pi N}} \frac{(\frac{m}{2})^{m}}{\sqrt{\Gamma(2N + 1)}}
\]

\[
\leq 16 \frac{13}{15} (e^{9/8} + 1) \frac{m^{3/2} (m/2)^{m}}{\sqrt{2\pi N}} \frac{\left( e^{3/2}(\log m + 1) \right)^N}{\sqrt{2\pi N}^{N/2 + \frac{3}{4}}} \frac{N^{N/2 + \frac{3}{4}}}{2N}
\]
by Stirling’s inequality. The upper bound divided by (5.53) is decreasing for all $N \geq m^3$, $m \geq 4$. If $m \geq 7$, it is also bounded by $2^{-1022}$. This explains how the first inequality was obtained. For the total variation we use again Stirling’s inequality and obtain
\[
\Delta_{n,m}^{(2)} \leq \frac{(2\pi e m^{-1})^{2\pi}}{(\pi m)^{2\pi}} (m^{2\pi} + \epsilon) N^{2\pi} \exp \left( - \frac{C(m) N^2}{(m + 1)^{8/3}(\log m + 1)^2} \right)
\]
which is less than $3m(2\sqrt{3m})^{-\frac{2\pi}{3}}$ if $N \geq m^3$, $m \geq 26$. Hence Theorem 0.2 has its condition satisfied and gives
\[
\Delta_{n,m}^{(1)} \leq 2(48 \log \Delta_{n,m}^{(2)})^{\frac{2\pi}{3}} \Delta_{n,m}^{(2)}
\]
\[
\leq 2 \sqrt{\Omega_m} (m^{2\pi} + \epsilon) N^{2\pi} \left( \frac{48C(m)}{(m + 1)^{8/3}(\log m + 1)^2} \right)^{\frac{2\pi}{3}} \exp \left( - \frac{C(m) N^2}{(m + 1)^{8/3}(\log m + 1)^2} \right)
\]
since $x \mapsto (\log x)^{1/4} x$ is non-decreasing for $x \in [0, e^{-m/4}]$ and our upper bound for the $L_2$-norm is greater than exp $\left( - \frac{C(m) N^2}{(m + 1)^{8/3}(\log m + 1)^2} \right)$.

Finally, if we only assume that $n \geq m^3$, then via computations similar to those in the last corollary we obtain

**Corollary 5.5.** If $n \geq m^3$, $m \geq 68$,
\[
\Delta_{n,m}^{(2)} \leq \sqrt{\Omega_m} (m^{\frac{2\pi}{3}} + 0.2) N^{\frac{2\pi}{3}} \exp \left( - \frac{C(m) N^2}{(m + 1)^{8/3}(\log m + 1)^2} \right)
\]
and if $n \geq m^3$, $m \geq 10^{18}$,
\[
\Delta_{n,m}^{(1)} \leq \sqrt{\Omega_m} \left( \frac{80C(m)}{(m + 1)^{2m/3}(\log m + 1)^{2\pi}} \right)^{\frac{2\pi}{3}} (m^{2\pi} + 0.2) N^m \exp \left( - \frac{C(m) N^2}{(m + 1)^{8/3}(\log m + 1)^2} \right)
\]
where $C(m) = \frac{(1-c_1(m)^2)}{4c_2(m)}$ is as in the previous corollary.

**Remark 5.6.** Using these last two corollaries one can check (again with Wolfram Mathematica) that
\[
\Delta_{n,m}^{(1)} \leq N^{-0.3N}
\]
if $n \geq m^4$, $m \geq 1000$, and
\[
\Delta_{n,m}^{(1)} \leq N^{-0.8N}\sqrt{N}
\]
if $n \geq m^3$, $m \geq 10^{19}$.

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