Ensemble Kalman Filter with perturbed observations in weather forecasting and data assimilation

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Abstract

Data assimilation in Mathematics provides algorithms for widespread applications in various fields, especially in meteorology. It is of practical use to deal with a large amount of information in several dimensions in the complex system that is hard to estimate. Weather forecasting is one of the applications, where the prediction of meteorological data are corrected given the observations.

Numerous approaches are contained in data assimilation. One specific sequential method is the Kalman Filter. The core is to estimate unknown information with the new data that is measured and the prior data that is predicted. The main benefit of the Kalman Filter lies in the more accurate prediction than that with single measurements.

As a matter of fact, there are different improved methods in the Kalman Filter. In this project, the Ensemble Kalman Filter with perturbed observations is considered. It is achieved by Monte Carlo simulation. In this method, the ensemble is involved in the calculation instead of the state vectors. In addition, the measurement with perturbation is viewed as the suitable observation. These changes compared with the Linear Kalman Filter make it more advantageous in that applications are not restricted in linear systems any more and less time is taken when the data are calculated by computers. The more flexibility as well as the less pressure of calculation enable the specific method to be frequently utilised than other methods.

This project seeks to develop the Ensemble Kalman Filter with perturbed observations gradually. With the Mathematical preliminaries including the introduction of dynamical systems, the Linear Kalman Filter is built. Meanwhile, the prediction and analysis processes are derived. After that, we use the analogy thoughts to lead in the non-linear Ensemble Kalman Filter with perturbed observation and relevant formulae are imitated and deducted. Lastly, a classic Lorenz 63 model is illustrated by MATLAB. In the example, we experiment on the number of ensemble members and seek to investigate the relationships between the error of variance and the number of ensemble members. We reach the conclusion that on a limited scale the larger number of ensemble members indicates the smaller error of prediction.

Keywords: Ensemble Kalman Filter, Perturbed observation, Linear Kalman Filter, Lorenz 63 model, Dynamical System
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1 Introduction

In the Mathematical field, Data Assimilation can be utilised to make predictions in the daily routine. Actually, the specific tools in Data Assimilation save information in a dynamical model and operate the model combining the observations and the previous prediction. In this case, large quantities of information from different dimensions can be analysed so that we will not feel at loss how to integrate scattered information.

One specific topic in the Data Assimilation field is the Kalman Filter. Kalman Filter should be the classic tool widely used in both Mathematical and Computer Science world. It is aimed at predicting while making corrections based on latest observations. One of the reasons why it is welcomed is that Kalman Filter can solve problems even if the system in the model is unknown.

1.1 Numerical Weather Prediction

One application of Data Assimilation is Numerical Weather Prediction (NWP). Since 1950, the USA has started the forecast using models operated on computers. As time went by, more powerful computers arose and improvements of algorithms were made. Scientists are able to simulate the more genuine situation with more accurate results.

NWP is actually an initial value problem. With the first guess of weather situation, such as pressure, temperature, humidity, wind speed and so on, we develop approaches of Data Assimilation and forecast what is the next state based on observations. The processes are analysed in different ways according to different situations.

There are two main types of methods of Data Assimilation: variational methods and sequential methods. We only focus on the latter one in the thesis and study one specific type of it.

1.2 Goals of the Project

In the thesis, our main goal is to explore the theory of Ensemble Kalman Filter step by step, which is one general approach in the sequential methods. We will reach several goals as follows.

- the derivation of Ensemble Kalman Filter with perturbed observations
- the example of Ensemble Kalman Filter with perturbed observations
- the influence of different number of ensemble members

1.3 Outline of the Project

In the Section 2, we introduce Mathematical preliminaries to pave the way for the Kalman Filter.

In the Section 3, the Linear Kalman Filter theory is derived gradually.
In the Section 4, the non-linear Ensemble Kalman Filter with perturbed observation is formulated with the foreshadowing of the linear system.

In the Section 5, an example of Ensemble Kalman Filter with perturbed observation is given and we study the different outcomes from different parameters.

In the Section 6, we draw the conclusion and discuss the prospect of improvements.

2 Mathematical preliminaries

In this section, some prerequisite knowledge is necessary to be put forward and proved. These knowledge will be used throughout the whole thesis.

2.1 Probability Theory

This subsection is related with some definitions of the Probability Theory. Firstly, we introduce the conditional expectation given that we know the conditional probability.

Definition 2.1. If $X$ and $Y$ are two discrete random variables, then the conditional expectation of $X$ given $Y = y$ (Ross, 1998) is defined by

$$
E[X|Y = y] = E[X|y] = \sum_x xP(X = x|Y = y)
$$

where $y$ can take values such that $P(Y = y) > 0$.

Then, we introduce the Bayes’ theorem, which is an important theorem used for proof in the Linear Kalman Filter section.

Theorem 2.1. Bayes’ theorem (Ross, 2004) Let $E$ and $F$ be two events in the sample space. Then

$$
P(F|E) = \frac{P(EF)}{P(E)} = \frac{P(E|F)P(F)}{P(E)}.
$$

2.2 Matrix Theory

This subsection introduces some calculations of matrices.

Theorem 2.2. Woodbury Matrix Identity/Matrix Inversion Lemma (Hager, 1989) Let $A$, $U$, $C$, $V$ be matrices with suitable dimensions, namely $A \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times k}$ and $V \in \mathbb{R}^{k \times k}$. If $A$ and $C$ are invertible, then $A + UCV$ are invertible and

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}. \quad (2.2.1)$$

Next, we introduce some useful matrix calculations, which can be used when we aim at differentiating matrices in order to obtain the extreme values.
Properties: If matrices $A, B \in \mathbb{R}^{n \times n}$ and $B$ is a symmetric matrix, then the derivatives of $AB$, $A^TB$ and $ABA^T$ with respect to $A$ (Petersen [2012]) is defined as follows:

\[
\frac{\partial \text{Tr}(AB)}{\partial A} = B^T \quad \frac{\partial \text{Tr}(A^TB)}{\partial A} = \frac{\partial \text{Tr}(BA^T)}{\partial A} = B \quad \frac{\partial \text{Tr}(ABA^T)}{\partial A} = 2AB
\]

2.3 Dynamical Systems

In this section, some basic knowledge in regard to the dynamical systems will be introduced in advance because the Kalman Filter itself is dynamic.

To begin with, let us give the definition of a dynamical system with iterations.

**Definition 2.2.** A basic discrete-time dynamical system with iterated maps (Law, 2015) is defined by

\[
\begin{align*}
    v_{j+1} &= \Psi(v_j) \\
    v_0 &= u
\end{align*}
\]

where $\Psi$ is the operator that maps $v \in \mathbb{R}^n$ at the former time to the next time

\[
\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

\[
v_j \mapsto v_{j+1}.
\]

Next, we give an example to make the definition clear.

**Example 2.1.** Let $\Psi(v) = \lambda(v) + a$, where $v$ is one-dimensional and $\lambda$, $a$ are constants.

Then by definition, we obtain

\[
\begin{align*}
    v_{j+1} &= \lambda v_j + a \\
    v_0 &= u
\end{align*}
\]

If $\lambda \neq 1$, we obtain

\[
\begin{align*}
    v_j &= \lambda v_{j-1} + a \\
    &= \lambda(\lambda v_{j-2} + a) + a \\
    &= \lambda^2v_{j-2} + \lambda a + a \\
    &= \ldots \\
    &= \lambda^j v_0 + (a + \lambda a + \lambda^2 a + \cdots + \lambda^{j-1} a) \\
    &= \lambda^j u + \frac{a(1 - \lambda^j)}{1 - \lambda}.
\end{align*}
\]

Furthermore, if $|\lambda| < 1$, then $\lim_{j \to \infty} v_j = \frac{a}{1-\lambda}$, i.e. the limit is a fixed point.

However, in real cases, the dynamical systems are not so ideal. Sometimes some error such as the white noise will be brought to the dynamical systems, so we should define the controlled system.

**Definition 2.3.** A controlled discrete-time dynamical system (Law, 2015) is defined by

\[
\begin{align*}
    v_{j+1} &= \Psi(v_j) + \xi_j
\end{align*}
\]

where $\{\xi_j\}_{j=0}^{\infty}$ is the controller added to the system.

**Remark:** In some cases, $\xi_j$ has the form of $\xi_j = K_j(v_j - h_j(v_j))$, given another sequence $\{y_j\}_{j \in \mathbb{N}}$ and functions $K_j$, $h_j$. We aim at making $|K_j(v_j - h_j(v_j))|$ as small as possible.
In the scenario, \( \xi_j = K_j(y_j - h_j(v_j)) \), where \( K_j \) is an important intermediate variable.

As usual, we give an example of the controlled dynamical system.

**Example 2.2.** We choose \( \Psi(v) = \lambda v \) and \( h_j(v) = v \) in the special form, where \( \lambda \) is a constant. In addition, we assume \( y_j \) is obtained from the estimated value \( \hat{v}_j \) by letting \( y_j = \hat{v}_j \), where \( \hat{v}_{j+1} = \lambda \hat{v}_j \). We define the error between the true value and the estimated one as \( e_j = v_j - \hat{v}_j \). The target is to test whether the estimated value approaches to the true value in the long run.

**Conclusion:** If \( |\lambda - K_j| < 1 \), \( \forall j \), then the estimated value of \( v_j \) will approach to the true value.

**Proof:** From the conditions above, we use \( v_{j+1} = \lambda v_j + K_j(y_j - v_j) \) and \( \hat{v}_{j+1} = \lambda \hat{v}_j \) to do calculation. Then we obtain

\[
e_{j+1} = v_{j+1} - \hat{v}_{j+1} = \lambda v_j + K_j(y_j - v_j) - \lambda \hat{v}_j = \lambda(v_j - \hat{v}_j) + K_j(\hat{v}_j - v_j) = (\lambda - K_j)e_j.
\]

Continue the above step \( j + 1 \) times, then we will have

\[
e_{j+1} = (\lambda - K_j)e_j = (\lambda - K_j)(\lambda - K_{j-1})e_{j-1} = \cdots = (\lambda - K_j)(\lambda - K_{j-1})\cdots(\lambda - K_0)e_0.
\]

Take the maximum value of \( \lambda - K_j \) denoted as \( M \), then \( e_{j+1} < M^{j+1}e_0 \).

Thus, if \( |\lambda - K_j| < 1 \), we have \( \lim_{j \to \infty} v_j - \hat{v}_j = 0 \). \( \square \)

**Remark:** Example 2.2. is classic and heuristic to pave the way for sequential methods in that we replace the true value with the estimated one.

### 3 Linear Kalman Filter

With all the foreshadowing, we start the Kalman Filter theory. There are two main types of Kalman Filter in the linear system: the discrete case and the continuous case. We only focus on the discrete case.

In the discrete case, there are also numerous types of Kalman Filter. The thesis is centred on the Ensemble Kalman Filter with perturbed observation, so the target is to find out the processes of it.

To begin with, we have to build the Linear Kalman Filter in advance, which is a basic Kalman Filter to pave the way for the non-linear system.
3.1 Derivation of Linear Kalman Filter

Based on the introduction of Dynamical Systems in the Subsection 2.3, we obtain the following linear equations (Law, 2015)

\[
\begin{align*}
    v_{j+1} &= \Psi(v_j) + \xi_j \\
    y_j &= h_j(v_j) + \eta_j.
\end{align*}
\] (3.1.1)

Explanation: We make some interpretations of the system (3.1.1) as follows. In the system, in every time node \( j \), \( v_j \) is no longer a one-dimensional number. Instead, it is a vector containing information of several dimensions. We assume \( v_j \in \mathbb{R}^n \) and call it the state vector, or the signal. Actually, it is the hidden state on the grounds that it cannot be seen directly.

In comparison, \( y_j \in \mathbb{R}^m \) is the observation vector because it can be measured by humans.

Note that \( \xi_j \in \mathbb{R}^n \) and \( \eta_j \in \mathbb{R}^m \) are two types of error. For one thing, \( \xi_j \) belongs to white noise, which corresponds with Gaussian distribution. For another, \( \eta_j \) belongs to the measurement error since every measurement by human eyes or machines will cause error. In both cases, we assume that they corresponds with Gaussian distribution with zero mean because we wish there must be error but they cancel each other. To be more precise, we have \( \xi_j \sim \mathcal{N}(0, \Sigma_j) \) and \( \eta_j \sim \mathcal{N}(0, \Gamma_j) \). Moreover, they are independent of the observations and state vectors because the change of them will not affect the state vectors and the observations. In other words, there are no influences between the errors and the state vectors, the errors and the observations respectively.

Logically, \( \Psi \in \mathbb{R}^{n \times n} \) is the transition matrix and \( h \in \mathbb{R}^{m \times n} \) is observation operator. Both of them are matrices that map \( v_j \) to \( v_{j+1} \) and \( v_j \) to \( y_j \) respectively.

Next, we denote some symbols in order for the convenience of calculation in the followings. The mean and estimated mean of signals are denoted as

\[
\begin{align*}
    m_j &= \mathbb{E}[v_j|Y_j] = \mathbb{E}[v_j|y_1, y_2, \cdots, y_j] \\
    \hat{m}_{j+1} &= \mathbb{E}[v_{j+1}|Y_j] = \mathbb{E}[v_{j+1}|y_1, y_2, \cdots, y_j],
\end{align*}
\] (3.1.2)

where the conditional expectation is defined in the Subsection 2.1.

The covariance and estimated covariance of signals are denoted as

\[
\begin{align*}
    C_j &= \text{cov}[v_j|Y_j] = \text{cov}[v_j|y_1, y_2, \cdots, y_j] \\
    \hat{C}_{j+1} &= \text{cov}[v_{j+1}|Y_j] = \text{cov}[v_{j+1}|y_1, y_2, \cdots, y_j],
\end{align*}
\] (3.1.3)

where \( C_j \in \mathbb{R}^{n \times n} \) is the covariance matrix, which can be written as

\[
C_j = \mathbb{E}[(v_j - m_j)(v_j - m_j)^T|Y_j].
\]

Similarly, we know

\[
\hat{C}_j = \mathbb{E}[(v_j - \hat{m}_j)(v_j - \hat{m}_j)^T|Y_j].
\]

Finally, the initial condition also corresponds with Gaussian distribution \( v_0 \sim \mathcal{N}(m_0, C_0) \).
To make it clear, we should put symbols into a table.

Table 1: Symbols of Kalman Filter

| Symbol | Meaning                  |
|--------|--------------------------|
| \( v_j \) | state vector            |
| \( y_j \) | observation              |
| \( \Psi \) | transition matrix        |
| \( h \) | observation operator     |
| \( \xi_j \) | \( N(0, \Sigma_j) \) |
| \( \eta_j \) | \( N(0, \Gamma_j) \) |
| \( v_0 \) | \( N(m_0, C_0) \) |
| \( m_j \) | \( \mathbb{E}[v_j|Y_j] \) |
| \( \hat{m}_j \) | \( \mathbb{E}[v_j|Y_{j-1}] \) |
| \( C_j \) | \( \text{cov}[v_j|Y_j] \) |
| \( \hat{C}_j \) | \( \text{cov}[v_j|Y_{j-1}] \) |

With all the prerequisites, we formally lead in Linear Kalman Filter. In total, there are two steps in the system: the prediction model and the analysis step. \cite{Law2015}

### 3.1.1 the First Step: Prediction

The first step of Linear Kalman Filter is the prediction step. It updates the distribution of \( v_j|Y_j \) to the next time node \( j + 1 \), i.e. from \( v_j|Y_j \) to \( v_{j+1}|Y_j \), where the transition matrix \( \Phi \) and the observation operator \( h \) are linear maps as the name implies.

To be more specific, we let \( \Phi(v) = Mv \) and \( h(v) = H_j v \) in this case. The symbols are easy to be understood and recognised as linear maps.

Then, from the Equation (3.1.1) we can find that

\[
\hat{m}_{j+1} = Mm_j + \xi_j. \tag{3.1.4}
\]

To find out the detailed prediction, we do calculation by taking the expectation value of \( v_{j+1} \) given the previous observations \( y_1 \) to \( y_j \) as follows

\[
\hat{m}_{j+1} = \mathbb{E}[v_{j+1}|Y_j] \\
= \mathbb{E}[Mv_j|Y_j] + \mathbb{E}[\xi_j|Y_j] \\
= M\mathbb{E}[v_j|Y_j] + \mathbb{E}[\xi_j|Y_j]. \tag{3.1.5}
\]

Since by the assumption in the Explanation of the Subsection 3.1, \( \xi_j \) is independent of the observations \( Y_j \), then \( \mathbb{E}[\xi_j|Y_j] = 0 \). Hence, we obtain

\[
\hat{m}_{j+1} = Mm_j. \tag{3.1.6}
\]

After we find the mean prediction, it is natural for us to consider the covariance prediction, so next we calculate \( \hat{C}_{j+1} \).

By the definition of covariance \cite{Ross1998}, we know the formula of the covariance is

\[
\text{cov}[v_{j+1}|Y_j] = \mathbb{E}[(v_{j+1} - \hat{m}_{j+1})(v_{j+1} - \hat{m}_{j+1})^T|Y_j].
\]
We separate the terms of the formula and substitute the Equations (3.1.4) and (3.1.6) into it, we find out that

\[ \hat{C}_{j+1} = \mathbb{E}[(v_{j+1} - \hat{m}_{j+1})(v_{j+1} - \hat{m}_{j+1})^T|Y_j] \]
\[ = \mathbb{E}[(Mv_j + \xi_j - \hat{m}_{j+1})(Mv_j + \xi_j - \hat{m}_{j+1})^T|Y_j] \]
\[ = \mathbb{E}[(Mv_j + \xi_j - Mm_j)(Mv_j + \xi_j - Mm_j)^T|Y_j] \]

Note that during the calculation, since \( \xi_j \) is independent of \( v_j \) and \( m_j \), which is based on the assumption in the Explanation of the Subsection 3.1, we remove cross terms of \( \xi_j v_j \) and \( \xi_j m_j \) directly. Then we obtain

\[ \hat{C}_{j+1} = \mathbb{E}[(Mv_j + \xi_j - Mm_j)(Mv_j + \xi_j - Mm_j)^T|Y_j] \]
\[ = M \mathbb{E}[(v_j - m_j)(v_j - m_j)^T|Y_j] + \mathbb{E}[\xi_j^2|Y_j] \]
\[ = MC_j M^T + \Sigma_j \] (3.1.7)

With Equations (3.1.6) and (3.1.7), we finish the prediction step.

3.1.2 the Second Step: Analysis

The second step in the Linear Kalman Filter is the analysis step. In this step, we analyse the prediction we make in the first step based on the new observation and update \( v_{j+1}|Y_j \) at this time node, i.e. from \( v_{j+1}|Y_j \) to \( v_{j+1}|Y_{j+1} \).

The aim of this step is to make correction of the prediction in the previous step based on the new observations. After this step, we will obtain the more precise output. (Law, 2015)

The symbol \( v_{j+1}|Y_{j+1} \) mentioned above means in this step, it is no longer the prediction, but the data with much less error. Hence, we aim at obtaining the mean and covariance of \( v_{j+1}|Y_{j+1} \) from \( v_{j+1}|Y_j \).

To make the symbols simplified, we subtract the subscripts by one.

We start by figuring out the relationship between \( v_j|Y_j \) and \( v_j|Y_{j-1} \) as the aim of analysis step indicates.

To be more specific, we need to split \( Y_j \) into \( Y_{j-1} \) and \( y_j \) since we aim to obtain \( \mathbb{P}[v_j|Y_{j-1}] \), which is shown below.

\[ \mathbb{P}(v_j|Y_j) = \frac{\mathbb{P}(v_j \cap Y_j)}{\mathbb{P}(Y_j)} \]
\[ = \frac{\mathbb{P}(v_j \cap y_j \cap Y_{j-1})}{\mathbb{P}(Y_j)} \]
\[ = \frac{\mathbb{P}(v_j \cap y_j|Y_{j-1})}{\mathbb{P}(y_j|Y_{j-1})} \]

Then, we use the Bayes’ theorem in the Theorem 2.1 of the Subsection 2.1 (Ross, 1998) to
From the Equation (3.1.12), we are able to calculate \( \text{2.2 of the Subsection 2.2.} \) (Hager, 1989) to calculate the inverse matrices. The theorem is introduced in the Theorem 3.1.10. We find out that in the both of the equations, there are inverse matrices. Thus, to simplify make the forms more practical.

The Equations (3.1.12) and (3.1.13) are the final outcomes. However, we refine them to the Formula (3.1.9), we obtain the probability density function of Gaussian distribution (Ross, 1998) and substitute it into the Formula (3.1.9) and obtain

\[
\mathbb{P}(v_j|Y_j) \propto e^{-\frac{1}{2}(y_j-H_jv_j)^T \Gamma_j^{-1}(y_j-H_jv_j)}. \tag{3.1.10}
\]

Similarly, we find out that the left hand side of the Formula (3.1.9) corresponds with Gaussian distribution as well, i.e. \( v_j|Y_j \sim N(m_j, C_j) \). Again, we substitute it into the Formula (3.1.9) and obtain

\[
\mathbb{P}(v_j|Y_j) \propto e^{-\frac{1}{2}(v_j-m_j)^T C_j^{-1}(v_j-m_j)}. \tag{3.1.11}
\]

Combining the right hand side of the Formulae (3.1.10) and (3.1.11), we equal the coefficients of \( v_j v_j^T \) (quadradic terms) and obtain

\[
C_j^{-1} = \tilde{C}_j^{-1} + H_j^T \Gamma_j^{-1} H_j. \tag{3.1.12}
\]

Simultaneously, comparing the coefficients of \( v_j \) (linear terms) in the the right hand side of the Formulae (3.1.10) and (3.1.11) and equalling them, we obtain

\[
C_j^{-1} m_j = \tilde{C}_j^{-1} \hat{m}_j + H_j^T \Gamma_j^{-1} y_j. \tag{3.1.13}
\]

The Equations (3.1.12) and (3.1.13) are the final outcomes. However, we refine them to make the forms more practical.

We find out that in the both of the equations, there are inverse matrices. Thus, to simplify them we should introduce a useful theorem which is called Woodbury Matrix Identity (Hager 1989) to calculate the inverse matrices. The theorem is introduced in the Theorem 2.2 of the Subsection 2.2.

From the Equation (3.1.12), we are able to calculate \( C_j \) by the theorem and simplify it as
follows

\[ C_j = (\hat{C}_j^{-1} + H_j^T \Gamma_j^{-1} H_j)^{-1} \]
\[ = \hat{C}_j - \hat{C}_j H_j^T (\Gamma_j + H_j \hat{C}_j H_j^T)^{-1} H_j \hat{C}_j \]
\[ = \left( I - \hat{C}_j H_j^T (\Gamma_j + H_j \hat{C}_j H_j^T)^{-1} H_j \right) \hat{C}_j \]
\[ = (I - K_j H_j) \hat{C}_j, \quad (3.1.14) \]

where \( K_j = \hat{C}_j H_j^T (\Gamma_j + H_j \hat{C}_j H_j^T)^{-1} \).

In order to obtain \( m_j \), we observe the Equation (3.1.13) and know we should remove \( C_j \hat{C}_j^{-1} \) and \( C_j H_j^T \Gamma_j^{-1} \).

Note that from the Equation (3.1.14) we obtain

\[ C_j \hat{C}_j^{-1} = I - K_j H_j. \quad (3.1.15) \]

Then the only problem is to solve the second term of the right hand side Equation (3.1.13).

The term \( C_j H_j^T \Gamma_j^{-1} \) can appear in the equation after the transposition of Equation (3.1.12), so we simplify it and obtain the followings.

\[ C_j (\hat{C}_j^{-1} + H_j^T \Gamma_j^{-1} H_j) = I \]
\[ \implies C_j H_j^T \Gamma_j^{-1} H_j = I - C_j \hat{C}_j^{-1} \]
\[ \implies C_j H_j^T \Gamma_j^{-1} H_j = K_j H_j \quad (3.1.16) \]

Note that the last procedure of the Equation (3.1.16) comes from the Equation (3.1.15).

From the Equation (3.1.16), we obtain \( C_j H_j^T \Gamma_j^{-1} = K_j \). Therefore, we substitute it and the Equation (3.1.15) into the Equation (3.1.13). Finally we obtain

\[ m_j = (I - K_j H_j) \hat{m}_j + K_j y_j \]
\[ = \hat{m}_j + K_j (y_j - H_j \hat{m}_j) \quad (3.1.17) \]

where \( K_j = \hat{C}_j H_j^T (\Gamma_j + H_j \hat{C}_j H_j^T)^{-1} \).

**Remark:** In the above calculation, we find out that there are two outcomes of \( K_j \) in Equations (3.1.14) and (3.1.16). The reason why we can use both of them to obtain \( m_j \) during the calculation is that one solution to the Equation (3.1.16) is \( K_j = C_j H_j^T \Gamma_j^{-1} \). ([Law, 2015]) However, when we practice it in the daily routine, we only use the form \( K_j = \hat{C}_j H_j^T (\Gamma_j + H_j \hat{C}_j H_j^T)^{-1} \).

As a matter of fact, we call \( K_j \) the Kalman gain. We put forward another method to obtain it, which will not have a confusing form.

### 3.2 Another Perspective: A Different Derivation of Kalman Gain

In this subsection, in order to avoid ambiguity of the Kalman gain in the Subsection 3.1, which is stated in the remark of Subsection 3.1, we aim to obtain the Kalman gain in
Remark: In fact, when we enter into the process of analysis, the form of equations updating (3.2.1) is the assumption.

Lemma: J square errors between the true values and the predicted values mentioned above. For simplicity, we denote the sum of variances as J, which is the expectation value of the square errors between the true values and the predicted values mentioned above. Thus, we will have the following equations

\[
\begin{align*}
y_j &= H_j v_j + \eta_j \\
v_j &= \hat{v}_j + K_j (y_j - H_j \hat{v}_j)
\end{align*}
\]  

(3.2.1)

where \(K_j \in \mathbb{R}^{n \times m}\) is the Kalman gain matrix and the second equation of the Equation (3.2.1) is the assumption.

**Remark:** In fact, when we enter into the process of analysis, the form of equations updating \(v_j\) from the prediction must be the second equation of the Equation (3.2.1) corresponding with the controlled dynamical model in the Definition 2.3. Precisely, the model is based on the recursive least squares estimation. (Simon 2006)

Thus, we will have the following equations

\[
\begin{align*}
m_j &= \hat{m}_j + K_j (y_j - H_j \hat{m}_j) \\
C_j &= \hat{C}_j + K_j (y_j - H_j \hat{C}_j).
\end{align*}
\]  

(3.2.2)

For simplicity, we denote the sum of variances as \(J_j\), which is the expectation value of the square errors between the true values and the predicted values mentioned above.

**Lemma:** \(J_j = Tr(C_j)\), where \(Tr(C_j)\) is the trace of \(C_j\).

**Proof:** From the definition, we have

\[
J_j = \mathbb{E}[(v_1 - \hat{v}_1)(v_1 - \hat{v}_1)^T] + \mathbb{E}[(v_2 - \hat{v}_2)(v_2 - \hat{v}_2)^T] + \cdots + \mathbb{E}[(v_j - \hat{v}_j)(v_j - \hat{v}_j)^T]
\]

We write \((v_s - \hat{v}_s)(v_t - \hat{v}_t)^T\) as \(\epsilon_s \epsilon_t^T\). Then \(J_j = \mathbb{E}[\epsilon_1 \epsilon_1^T + \epsilon_2 \epsilon_2^T + \cdots + \epsilon_j \epsilon_j^T]\).

Since \(C_j\) is the covariance matrix of \(v_s\) and \(v_t\), where \(1 \leq s, t \leq j\), i.e. \(C_j\) contains all the covariances between each other, then we write \(C_j\) out as

\[
C_j = \\
| \mathbb{E} [\epsilon_1 \epsilon_1^T] & \mathbb{E} [\epsilon_1 \epsilon_2^T] & \cdots & \mathbb{E} [\epsilon_1 \epsilon_j^T] \\
\mathbb{E} [\epsilon_2 \epsilon_1^T] & \mathbb{E} [\epsilon_2 \epsilon_2^T] & \cdots & \mathbb{E} [\epsilon_2 \epsilon_j^T] \\
\vdots & \vdots & \ddots & \vdots \\
\mathbb{E} [\epsilon_j \epsilon_1^T] & \mathbb{E} [\epsilon_j \epsilon_2^T] & \cdots & \mathbb{E} [\epsilon_j \epsilon_j^T] |
\]

It is obvious that \(Tr(C_j) = J_j\). □

Thus, to find out \(K_j\), we only need to find what \(Tr(C_j)\) is and evaluate the extreme value of \(Tr(C_j)\).

We have to simplify \(C_j\) and do further calculations. Otherwise, with the current form, we can do nothing.

**Lemma:** \(C_j = \mathbb{E} \left[ (I - K_j H_j) (v_j - \hat{m}_j) - K_j \eta_j \right] [ (I - K_j H_j) (v_j - \hat{m}_j) - K_j \eta_j ]^T \).
Proof: By the definition from the Equation (3.1.3), \(C_j = \mathbb{E} \left[ (v_j - m_j) (v_j - m_j)^T \right]\), so we only need to show

\[ v_j - m_j = (I - K_j H_j)(v_j - \hat{m}_j) - K_j \eta_j. \]  

(3.2.3)

By expanding the right hand-side of the Equation (3.2.3), we obtain

\[ (I - K_j H_j)(v_j - \hat{m}_j) - K_j \eta_j = v_j - \hat{m}_j - K_j H_j v_j + K_j H_j \hat{m}_j - K_j (y_j - H_j v_j) \]

\[ = v_j - [\hat{m}_j - K_j (H_j \hat{m}_j - y_j)]. \]  

(3.2.4)

By substituting the Equation (3.2.2) into the Equation (3.2.4), we obtain

\[ (I - K_j H_j)(v_j - \hat{m}_j) - K_j \eta_j = v_j - m_j \]  

(3.2.5)

which is the final result. ∎

We are aimed at obtaining the trace of \(C_j\). After we observe the second lemma, which is another form of \(C_j\), we find out that it contains part of the form of \(\hat{C}_j\) because the multiplication of \((v_j - \hat{m}_j)(v_j - \hat{m}_j)^T\) will generate \(\hat{C}_j\). Thus we aim at substituting \(\hat{C}_j\) into it.

To be more specific, we again expand \(C_j\) partly in the conclusion without separating \(v_j - \hat{m}_j\). We obtain

\[ C_j = \mathbb{E} \left[ (v_j - \hat{m}_j) - K_j H_j (v_j - \hat{m}_j) - K_j \eta_j \right] \cdot \]

\[ [ (v_j - \hat{m}_j)^T - (v_j - \hat{m}_j)^T H_j^T K_j^T - \eta_j^T K_j^T] \]  

(3.2.6)

Since \(\hat{C}_j = \mathbb{E} \left[ (v_j - \hat{m}_j)((v_j - \hat{m}_j)^T)\right]\), we substitute \(\hat{C}_j\) into the Equation (3.2.6) and obtain

\[ C_j = \hat{C}_j - K_j H_j \hat{C}_j - \hat{C}_j H_j^T K_j^T + K_j H_j \hat{C}_j H_j^T K_j^T + K_j \Gamma_j K_j^T \]

\[ = \hat{C}_j - K_j H_j \hat{C}_j - \hat{C}_j H_j^T K_j^T + K_j (H_j \hat{C}_j H_j^T + \Gamma_j) K_j^T. \]  

(3.2.7)

Remark: \(\eta_j\) is independent of \(v_j, \hat{m}_j\) and \(\mathbb{E}[\eta_j \eta_j^T] = \Gamma_j\), so we can remove the relevant terms directly when doing calculations.

After we finish simplifying \(C_j\), from the Equation (3.2.7) we can rewrite \(J_j\) as

\[ J_j = Tr(\hat{C}_j) - Tr(K_j H_j \hat{C}_j) - Tr(\hat{C}_j H_j^T K_j^T) + Tr \left( K_j (H_j \hat{C}_j H_j^T + \Gamma_j) K_j^T \right). \]  

(3.2.8)

From the useful properties in the Subsection 2.2, we take the partial derivative of \(J_j\) and assign zero to it, which means \(J_j\) will be minimised in this case.

Finally we obtain

\[- (H_j \hat{C}_j)^T - \hat{C}_j H_j^T + 2K_j (H_j \hat{C}_j H_j^T + \Gamma_j) = 0.\]

Because \(\hat{C}_j\) is symmetric, we know that \((H_j \hat{C}_j)^T = \hat{C}_j H_j^T\). After combining terms with \(K_j\), we obtain the result

\[ K_j = \hat{C}_j H_j^T (H_j \hat{C}_j H_j^T + \Gamma_j)^{-1}. \]  

(3.2.9)

Summary: We conclude the processes of Linear Kalman Filter here. The Linear Kalman
Filter is composed of two steps: the prediction step and the analysis step. The flow chart of the processes is shown in Figure 1.

![Flow Chart of Linear Kalman Filter Execution](image)

Figure 1: Execution of Linear Kalman Filter

4 Ensemble Kalman Filter with perturbed observation

The Linear Kalman Filter is only related to the linear system. Almost under no circumstance does linear system always exist in the nature, so the non-linear system should be introduced. One specific filter is the Ensemble Kalman Filter (EnKF). Ensemble Kalman Filter is originated from Linear Kalman Filter. In the Ensemble Kalman Filter, an ensemble which is a collection of state vectors, is involved in making estimation and approximating...
state distribution. \cite{Mandel2009}. In fact, the effect of covariance matrix is replaced by the sample covariance matrix composed of the ensemble to avoid tremendous calculation. There are a large number of forms of the Ensemble Kalman Filter. Stochastic Ensemble Kalman Filter, which is also called Ensemble Kalman Filter with perturbed observation is one of them. In this specific Ensemble Kalman Filter, an perturbed term is added to the original observation. \cite{Katzfuss2016}

To begin with, we consider the discrete dynamical system

\[
\begin{align*}
  v_{j+1} &= \Psi(v_j) + \xi_j \\
  y_j &= H_j v_j + \eta_j,
\end{align*}
\]

where $\xi_j \sim N(0, \Sigma_j)$ and $\eta_j \sim N(0, \Gamma_j)$. The symbols in the Equation (4.0.1) have the same meaning as those in the Table 1 except $\Psi$. In this case, it represents the non-linear transition matrix.

With the basic model equations, we need to introduce the core variable: the ensemble.

**Definition 4.1.** A sample $v_j^{(n)}$, where $n = 1, 2, \cdots, N$ from the distribution $N(m_j, C_j)$ at $t = j$ is called an ensemble. \cite{Katzfuss2016} Each element of $v_j^{(n)}$ is called an ensemble member. In this case, there are $N$ ensemble members in total.

**Remark:** The ensemble members are independently and identically distributed at the initial time, which approximates the state distribution, and the mean and covariance of this new filter and the original Kalman Filter are the same at the initial time. \cite{Katzfuss2016}

Like in all the other variables, we will have the ensemble mean and covariance.

**Definition 4.2.** If there are $N$ ensemble members, then the ensemble mean and the predicted ensemble mean are defined as

\[
\begin{align*}
  m_j &= \frac{1}{N} \sum_{n=1}^{N} v_j^{(n)} \\
  \hat{m}_j &= \frac{1}{N} \sum_{n=1}^{N} v_j^{(n)}.
\end{align*}
\]

The ensemble covariance and the predicted ensemble covariance are defined as

\[
\begin{align*}
  C_j &= \frac{1}{N-1} \sum_{n=1}^{N} \left( v_j^{(n)} - m_j \right) \left( v_j^{(n)} - m_j \right)^T \\
  \hat{C}_j &= \frac{1}{N-1} \sum_{n=1}^{N} \left( \hat{v}_j^{(n)} - \hat{m}_j \right) \left( \hat{v}_j^{(n)} - \hat{m}_j \right)^T.
\end{align*}
\]

**Remark:** $\hat{m}_j$ and $\hat{C}_j$ mean the predicted ensemble mean and covariance, just like $\hat{m}_{j+1} = \mathbb{E}[v_{j+1}|Y_j]$ in the previous cases. In the following processes, we use the ensemble mean and covariance to replace the ordinary mean and covariance matrices.

Similar to the Linear Kalman Filter, there are two steps of Ensemble Kalman Filter with perturbed observation: the prediction step and the analysis step. We will make use of the constructed linear model and mimic the processes of it.
4.1 the First Step: Prediction

In the step, we let the ensemble $v_j^{(n)}$ be involved in the calculation with non-linear transition matrix $\Psi$. By substituting $v_j^{(n)}$ into the Equation (4.0.1), we obtain

$$\hat{v}_{j+1}^{(n)} = \Psi(v_j^{(n)}) + \xi_j$$

(4.1.1)

Remark: Compared with the linear model, the prediction process update the initial ensemble state, or the analysed ensemble state at the previous time node to the next time node (with hat symbol). (Houtekamer, 2005) This works for every $v_j^{(n)}$.

Next comes the mean and covariance prediction from the previous time node. As a matter of fact, the process has just appeared in the equations (4.0.2) and (4.0.3)

$$\begin{align*}
\hat{m}_{j+1} &= \frac{1}{N} \sum_{n=1}^{N} v_j^{(n)} \\
\hat{C}_{j+1} &= \frac{1}{N-1} \sum_{n=1}^{N} (v_j^{(n)} - \hat{m}_{j+1}) (v_j^{(n)} - \hat{m}_{j+1})^T
\end{align*}$$

(4.1.2)

where $v_j^{(n)}$ comes from the Equation (4.1.1) when calculating $\hat{m}_{j+1}$, and $\hat{m}_{j+1}$ comes from the just calculated $\hat{m}_{j+1}$ in the first equation of the Equation (4.1.2) when calculating $\hat{C}_{j+1}$.

Remark: Note that there is difference between $\hat{C}_j$ in this subsection and that in the Linear Kalman Filter ($\hat{C}_j = \text{cov}[v_j|Y_{j-1}]$). In the previous model, $\hat{C}_j$ is the covariance matrix of state vectors given the past observations. However, in this non-linear model, $\hat{C}_j$ is the unbiased covariance of the predicted ensembles.

4.2 the Second Step: Analysis

The analysis step, like in the Linear Kalman Filter, requires us to update the ensemble, the mean and the covariance from those in the prediction process.

To begin with, the Kalman gain $K_j$ in the ensemble Kalman Filter should be distinguished from that in the Linear Kalman Filter, which is denoted as $K_j^*$. While the Kalman gain in the Linear Kalman Filter is $K_j = \hat{C}_j H_j^T (H_j \hat{C}_j H_j^T + \Gamma_j)^{-1}$, in the Ensemble Kalman Filter, the Kalman gain is $K_j^* = \hat{C}_j H_j^T (H_j \hat{C}_j H_j^T + \Gamma_j)^{-1}$, where $\hat{C}_j$ is the predicted ensemble covariance matrix.

Remark: Note that mentioned at the end of the Subsection 4.1, the same symbol $\hat{C}_j$ in two different models mean two different meanings, one for signals and one for the ensemble.

Next, we follow the same footsteps of Linear Kalman Filter to calculate the mean update. For the convenience of calculation, we subtract the subscripts by one just like the same way before in the linear case.

After we refer to the recursive model in the Equation (3.2.1), we follow the similar pattern and obtain

$$v_j^{(n)} = \hat{v}_j^{(n)} + K_j^* (y_j - H_j \hat{v}_j^{(n)})$$

(4.2.1)
Remark: The Equation (4.2.1) is not the true analysis of ensemble in the perturbed case, which will be mentioned later, but the normal one in the ordinary Ensemble Kalman Filter. (Houtekamer 2005)

From the Equation (4.0.2), we have

\[ m_j = \frac{1}{N} \sum_{n=1}^{N} v_j^{(n)} \]

\[ = \frac{1}{N} \sum_{n=1}^{N} v_j^{(n)} + \frac{1}{N} \sum_{n=1}^{N} K_j^* y_j - \frac{1}{N} \sum_{n=1}^{N} K_j^* H_j \hat{v}_j^{(n)} \]

\[ = \hat{m}_j + K_j^* (y_j - H_j \hat{m}_j) \quad (4.2.2) \]

Subsequently, we calculate the analysis for covariance.

Conclusion: \( C_j = (I - K_j^* H_j) \hat{C}_j (I - K_j^* H_j)^T \).

Proof: The proof is composed of two steps. The approach of the proof is to change the term into the familiar definition of covariance.

First we prove

\[ v_j^{(m)} - m_j = (I - K_j^* H_j)(v_j^{(m)} - \hat{m}_j) - K_j^* \eta_j. \quad (4.2.3) \]

From the Equation (4.0.1) and the definition of Ensemble Kalman Filter, we obtain

\[ y_j = H_j v_j^{(m)} + \eta_j. \quad (4.2.4) \]

Then we expand the right hand side of the Equation (4.2.3) and substitute the Equation (4.2.4) into it, we obtain

\[ (I - K_j^* H_j)(v_j^{(m)} - \hat{m}_j) - K_j^* \eta_j = v_j^{(m)} - [\hat{m}_j + K_j^* (y_j - H_j \hat{m}_j)]. \quad (4.2.5) \]

Lastly, we substitute the Equation (4.2.2) into the Equation (4.2.5) and obtain the result directly.

In the second step, we prove

\[ (I - K_j^* H_j)(v_j^{(m)} - \hat{m}_j) - K_j^* \eta_j = (I - K_j^* H_j)(v_j^{(n)} - \hat{m}_j). \quad (4.2.6) \]

We expand both sides of the Equation (4.2.6) and substitute the Equation (4.2.4) into it. After cancelling the same terms at both side, it is equivalent to show \( v_j^{(n)} = \hat{v}_j^{(n)} + K_j^* (y_j - H_j \hat{v}_j^{(n)}). \) It is just the fact shown in the Equation (4.2.1). Therefore, we prove the conclusion. \( \square \)

Finally, we have to analyse the ensemble \( v_j^{(n)}. \) The perturbed observation means that we should replace the observation \( y_j \) with something added to it, so we utilise \( y_j + \eta_j^{(n)} \) to adjust the analysis process (Burgers 1998) and finally obtain

\[ v_j^{(n)} = \hat{v}_j^{(n)} + K_j^* (y_j + \eta_j^{(n)} - H_j \hat{v}_j^{(n)}). \quad (4.2.7) \]

Remark: The reason why we should lead in the perturbation \( \eta_j^{(n)} \sim N(0, \Gamma_j) \) is that the
average difference between the ensemble mean and the ensemble prediction in this case represents the true value of it. ([Bowler] 2013)

**Summary:** The Ensemble Kalman Filter with perturbed observation is composed of two steps: the prediction step and the analysis step.

During the prediction step, we have

\[
\begin{align*}
\hat{v}_{j+1}^{(n)} &= \Psi(v_j^{(n)}) + \xi_j^{(n)} \\
\hat{m}_{j+1} &= \frac{1}{N} \sum_{n=1}^{N} \hat{v}_{j+1}^{(n)} \\
\hat{C}_{j+1} &= \frac{1}{N-1} \sum_{n=1}^{N} \left( \hat{v}_{j+1}^{(n)} - \hat{m}_{j+1} \right) \left( \hat{v}_{j+1}^{(n)} - \hat{m}_{j+1} \right)^T.
\end{align*}
\]

During the analysis step, we have

\[
\begin{align*}
m_j &= \hat{m}_j + K_j^* (y_j - H_j \hat{m}_j) \\
C_j &= (I - K_j^* H_j) \hat{C}_j (I - K_j^* H_j)^T \\
v_j^{(n)} &= \hat{v}_j^{(n)} + K_j^* (y_j - y_j^{(n)} - H_j \hat{v}_j^{(n)}),
\end{align*}
\]

where \( K_j^* = \hat{C}_j H_j^T (H_j \hat{C}_j H_j^T + \Gamma_j)^{-1} \).

Similar to the Linear Kalman Filter, a flow chart is shown below to depict the processes.

![Flow chart](image)

**Figure 2:** Execution of Ensemble Kalman Filter with perturbed observation

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5 An Example and Experiments

There are widespread applications of the Ensemble Kalman Filter with perturbed observation in Data Assimilation. Almost all the multi-state systems can benefit from the algo-
rithms by reducing the amount of calculation as well as improving the accuracy of prediction. Taking the atmospheric prediction into consideration, studying the Ensemble Kalman Filter allows us to explore the complex earth’s atmosphere. Also, the Equation (5.0.1) in the huge system in another aspect is feasible in small occasions like hot liquids in the container. In this scenario, $\sigma$ represents the ratio of fluid viscosity to thermal conductivity, $r$ stands for the difference in temperature between the top and bottom of the system and $b$ is on behalf of the ratio of container width to container height. (Daron, 2015)

After introducing the theory part of the Kalman Filter, we will give an example: Lorenz 63 model. In fact, Edward Lorenz derived the model in 1963. (Lorenz, 1963)

The model is a chaotic system which is autonomous and composed of a series of differential equations. (Davies, 2018) The equations are as follows

$$\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - xz - y \\
\dot{z} &= xy - bz,
\end{align*}$$

(5.0.1)

where the dot notation denotes differentiation with respect to time $t$, i.e. $\dot{x} = \frac{dx}{dt}$ and the system is determined by the parameters $\sigma, r$ and $b$.

By default, we choose $\sigma = 10$, $r = 28$ and $b = \frac{8}{3}$. (Lorenz, 1963) According to the Ensemble Kalman Filter with perturbed observations, we have to set an initial point and a prediction to start the simulation. To be more specific, we set the starting point $(-10, -10, 20)$.

It is necessary to first show the figure of trajectory in advance. Hence, we use a function in MATLAB called ‘ode45’. (MATLAB, 2019) It will output the numerical solution of the Equation (5.0.1). Therefore, we are able to plot the figure from the data points. The code is shown in the Appendix A.2.

Figure 3: Trajectory of Lorenz 63 Model

Remark: As is shown in Figure 3, we find out that the butterfly-wings-shaped curve never reaches the equilibrium point $(0, 0, 0)$. The movement from one side to another never ends but with certain regularity.

Next, we carry out the experiment. Since we do not know the exact initial point in the...
reality, we can guess the point. Here, we use the guess $(-11, -12, 10)$.

We will generate ensemble members, obtain the mean and covariance of them, calculate the Kalman gain and finally obtain the analysis mean and covariance in the following steps according to the knowledge we put forward in the Section 4.

The first step is to obtain the ensemble members. We use a trick where some noise is added to the prediction point to obtain them.

Then, we will successfully obtain the mean of the prediction ensemble from the function 'mean' in MATLAB.\(^{[MATLAB\,2019]}\) Afterwards, the covariance matrix of the ensemble prediction is calculated using the mean prediction.

To carry out the analysis process in the second step, we ought to obtain the Kalman gain matrix beforehand. We again use 'ode45' to get the true value of state by using the true starting point. Then, we add some noise to the true value to simulate the error. Here, the observation operator $H$ is the identity matrix.

After that, the perturbation added to the observation is the noise $\Gamma_j$, so we still use the same method to generate the noise but with different variances. Finally, we obtain perturbed observation.

The last calculation in the second step is simply substituting the Kalman gain and perturbed observation into the Equations (4.2.9).

We push the steps above several times using a loop by adding time nodes until the end time arrives. Therefore, finally we are able to obtain the absolute error in the ensemble mean in three dimensions by calculating the difference between the true states and the prediction states, analysis states respectively.

After the conception, we are confused with the influence of the number of ensemble members. We aim at comparing the error caused by the different number of ensemble.

After we take a look at an example from one thesis 'What Is the Ensemble Kalman Filter and How Well Does it Work?'\(^{[Gillijns\,2006]}\), we guess within a scope, the larger number of ensemble can plummet the error.

Thus, we use a loop to run the scripts three times with number of ensemble members $N = 20, 50, 100$ respectively. The code is shown in Appendix A.4.\(^{[Hakim\,2012]}\)

The outcome of the codes are shown below.
When $N = 20$, the figure of the absolute error in the ensemble mean for each dimension is as follows.

![Figure 4: Absolute error in the ensemble mean for each dimension of 20 ensemble members](image)

When $N = 50$, the output is as follows.

![Figure 5: Absolute error in the ensemble mean for each dimension of 50 ensemble members](image)
When $N = 100$, the output is as follows.

Figure 6: Absolute error in the ensemble mean for each dimension of 100 ensemble members

It is shown in all three cases that the difference between prediction and analysis ensemble mean is very small. However, the error difference and the fluctuation is the smallest when the number of ensemble members is $N = 100$.

Then we compare the mean difference calculated by $\frac{1}{j} \sum_{i=1}^{j} \|v_{i, true} - m_i\|_2$ between true values and analysis mean in one graph, where $j$ is the time node when the experiment is carrying on. The code is shown in Appendix A.3. The reason why we divide the difference over the time node $j$ is because we aim at checking the relative difference, which implies the situation of the changes in the error of analysis mean and the true value, i.e. if the error becomes smooth and steady, then the error is becoming smaller as time goes by.

Figure 7: Comparison of the mean difference
It turns out that when $N = 100$, the mean difference is the smallest after 100 steps.

6 Conclusion and Prospect

**Conclusion:** Ensemble Kalman Filter with perturbed observations is composed of two steps: the prediction step and the analysis step. In the practical uses, we utilise the analysis ensemble mean to simulate the current situation.

The number of ensemble members can influence the precision. To be more specific, within a scale, the larger number of ensemble members will decrease the mean error represented by the 2-norm of difference between the true value and the analysis mean since the larger number will indicate a better result of analysis covariance output. (see in Figure 7)

Precisely, the ensemble covariance matrix is calculated by the ensemble members together. However, if the number of ensemble members is small, the Gaussian property of the ensemble members will disappear. On the other hand, providing that the number of ensemble members is large enough, then they will approach to the Gaussian distribution and tend to be identical and independent, which correspond with the property of Kalman Filter, i.e. they will have less error. (Mandel, 2009)

**Prospect:** Since the larger number of ensemble members will cause the huge calculation, we usually consider a limited number of ensemble members. However, the limited number will have a negative impact on the prediction covariance and may cause the divergence of the error. (Wu, 2017) Thus, there should be methods to improve the ill-conditioned covariance such as using the inflation. Additive inflation or multiplicative inflation are two common ways to adjust the covariance.

Last but not least, in the future we are going to verify the root mean square error between the ensemble mean prediction and the analysis together with the ensemble spread which is often defined as the deviation of ensemble prediction from the mean of ensemble members. (Zhu, 2008) In this scenario, we can explore the filter uncertainty more clearly and obtain a more persuasive conclusion.
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A Codes for Lorenz 63 model

We run each part one time.

A.1 Function: Lorenz63.m

```matlab
function derivative = Lorenz63(t,v)
    derivative(1)=10*(v(2)-v(1));
    derivative(2)=28*v(1)-v(1)*v(3)-v(2);
    derivative(3)=v(1)*v(2)-8/3*v(3);
    derivative=derivative';
end
```

A.2 Script: SEnKF1.m

```matlab
v0 = [−10,−10,20];
v0p = [−11,−12,10];
H = eye(3);
dt = 0.1;
N = [20 50 100];
steps = 100;
result = [];
[t,v] = ode45(@Lorenz63,[0 10],v0);
x = v(:,1);
y = v(:,2);
z = v(:,3);
plot3(x,y,z)
title(‘Trajectory of Lorenz 63’)
xlabel(‘x’)
ylabel(‘y’)
zlabel(‘z’)
```

A.3 Script: SEnKF2.m

```matlab
v0 = [−10,−10,20];
v0p = [−11,−12,10];
H = eye(3);
dt = 0.1;
N = [20 50 100];
steps = 100;
result = [];
v0p = v0p';
for kkk = 1:length(N)
    number_of_ensemble = N(kkk)
    [intermediate1,ensemble] = meshgrid(ones(1,number_of_ensemble),v0p);
```
for i = 1:steps
    ensemble = ensemble + 1e-1*randn(3,number_of_ensemble);
    vpm = mean(ensemble,2);
    Cp = 0;
    for j = 1:number_of_ensemble
        Cp = (ensemble(:,j) - vpm)*(ensemble(:,j) - vpm)’ + Cp;
    end
    Cp = Cp/(number_of_ensemble-1);
end
K = Cp*inv(Cp+eye(3)*1e-2);
y=v0’+diag(diag(randn(3,1))*sqrtm(0.01));
va = zeros(3,number_of_ensemble);
for k = 1:number_of_ensemble
    intermediate2 = ensemble(:,k);
    va(:,k) = intermediate2 + K*(y-intermediate2);
end
vam = vpm + K*(y-H*vpm);
Ca = (eye(3)-K*H)*Cp*(eye(3)-K*H)’;

[intermediate3 real] = ode45(@Lorenz63,[0 dt],v0’);
real = real(end,:’);
diff = norm((real-vam),2);
result(i) = diff;
v0 =real’;
ensemble = va;
end
final(kkk,:) = result;
end
defaulttn = final;
for mmm = 1:3
    intermediaten = 0;
    for nnn = 1:100
        intermediaten = defaultn(mmm,nnn)+ intermediaten;
        final(mmm,nnn) = intermediaten/nnn;
    end
end

for mmm = 1:length(N)
    plot([1:100],final(mmm,:))
    hold on
end
legend(’20_Ensemble’, ’50_Ensemble’, ’100_Ensemble’)
title(’Mean_difference_between_truth_and_analysis_mean’)
xlabel('Steps')
ylabel('Difference')

A.4 Script: SEnKF3.m

```matlab
randn('state',0);
tavg = 0;
q = 0.001;
R = eye(3)*1e-2;
nassim = 100;
ntimes = 0.1;
climo = 0;
Nens = 20;
% 20, 50, 100 respectively
inflate = 1.1;
H = eye(3);
ind = 1;
tau = 0;

x = [-10,-10,20];
[tisms states] = ode45(@Lorenz63,[0 10],x);
xt = states(end,:);
[tmp, Xa] = meshgrid(ones(1,Nens),xt);
pert = 1e-1*randn(3,Nens);
Xa = Xa + pert;
hist_ver = [];
hist_xbm = [];
hist_xam = [];
Xa_save = [];
Xaf = [];
for k=1:nassim
    Xb = Xa;
    [tims states] = ode45(@Lorenz63,[0 ntimes],xt);
    xt = states(end,:);
    Y = H*xt + diag(diag(randn(3,1)))*sqrtm(R);
    ver = xt;
    Nobs = size(Y,1);
    for n = 1:1:Nens
        [tims states] = ode45(@Lorenz63,[0 ntimes],Xb(:,n));
        Xb(:,n) = states(end,:);
    end
    xbm = mean(Xb,2);
    % ensemble mean
    [tmp Xbm] = meshgrid(ones(Nens,1),xbm);
    Xp = Xb - Xbm;
    Xbp = Xp;
```

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\[ B = \left( X_{bp}X_{bp}' \right) / (N_{ens} - 1) + q \cdot \text{eye}(3); \]

\[ X_a = \text{perturbed}\_\text{obs}(X_b, B, Y, R); \]
\[ xam = \text{mean}(X_a, 2); \]
\[ [\text{tmp } X_{am}] = \text{meshgrid}(\text{ones}(N_{ens}, 1), xam); \]
\[ X_{ap} = X_a - X_{am}; \]
\[ A = \left( X_{ap}X_{ap}' \right) / (N_{ens} - 1); \]

\[ xbe(:, k) = xbm - \text{ver}; \]
\[ xae(:, k) = xam - \text{ver}; \]
\[ xye(:, k) = Y - \text{ver}; \]
\[ xaev(:, k) = \text{diag}(A); \]
\[ xbev(:, k) = \text{diag}(B); \]
end

\[ xbe\_var = \text{var}(xbe, 0, 2); \]
\[ xae\_var = \text{var}(xae, 0, 2); \]
\[ \text{lab}(1) = ' x '; \text{lab}(2) = ' y '; \text{lab}(3) = ' z '; \]
figure(1); clf
for \( k = 1:1:3 \)
    subplot(3, 1, k)
    plot(abs(xbe(k, :)), 'b-'); hold on
    plot(abs(xae(k, :)), 'r-')
    legend('Prediction', 'Analysis')
    c = ['prediction\_mean:' num2str(mean(abs(xbe(k,2:end))),3)']
    +/-'num2str(std(abs(xbe(k,2:end))),3)'; mean analysis:
    'num2str(mean(abs(xae(k,2:end))),3)'; +/-'num2str(std(abs(xae(k,2:end))),3)];
    ylim = get(gca,'ylim');
    yoff = ylim(2)*1.1;
    ylabel(lab(k),'fontweight','bold','fontsize',12)
end

if \( k == 1 \)
    h = title('absolute\_error\_in\_mean');
    set(h,'position',[99,12,1],'fontweight','bold','fontsize',12)
elseif \( k == 3 \)
    xlabel('Steps','fontweight','bold','fontsize',12)
end
end

figure(2); clf
plot(xbev(1,3:end), 'b-'); hold on
plot(xaev(1,3:end), 'r-')
c = ['prediction'] mean: num2str(mean(xbev(3:end)),3) ' +/- \sigma' 
num2str(std(xbev(3:end)),3) 'mean analysis: ' 
num2str(mean(xaev(3:end)),3) ' +/- \sigma' num2str(std(xaev(3:end)),3)); 
yl = get(gca,'ylim'); 
yoff = yl(1) - yl(2)*.12; 

xlabel('Steps','fontweight','bold','fontsize',12); 
ylabel('Error Variance','fontweight','bold','fontsize',12); 
title('Ensemble Kalman Filter Error Variance','fontweight','bold','fontsize',12); 
set(gca,'linewidth',2,'fontweight','bold','fontsize',10); 
legend('Prediction','analysis');