Littelmann Paths  
for the Basic Representation  
of an Affine Lie Algebra

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Abstract

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Lie algebra. Let $\hat{V}(\Lambda_0)$ be the basic irreducible level-one representation of $\hat{\mathfrak{g}}$, and $\hat{V}_{\lambda'}(\Lambda_0)$ the Demazure module corresponding to the translation $-\lambda'$ in the affine Weyl group. Suppose $\lambda'$ is a sum of minuscule coweights of $\mathfrak{g}$ (which exist if $\mathfrak{g}$ is of classical type or $E_6, E_7$).

We give a new model for the crystal graphs of $\hat{V}(\Lambda_0)$ and $\hat{V}_{\lambda'}(\Lambda_0)$ which combines Littelmann’s path model and the Kyoto path model. As a corollary, we prove that $\hat{V}_{\lambda'}(\Lambda_0)$ is isomorphic as a $\mathfrak{g}$-module to a tensor product of fundamental representations of $\mathfrak{g}$.

1 Main Results

1.1 Product Theorems

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac-Moody algebra. The basic representation $\hat{V}(\Lambda_0)$ is the simplest and most important $\hat{\mathfrak{g}}$-module (see Sec. 2.1 for definitions, as well as [6 Ch. 14], [21 Ch. 10]). One of its remarkable properties is the Tensor Product Phenomenon. In many cases, the Demazure modules $\hat{V}_z(\Lambda_0) \subset \hat{V}(\Lambda_0)$ are representations of the finite-dimensional algebra $\mathfrak{g}$, and they factor into a tensor product of many small $\mathfrak{g}$-modules. Hence the full $\hat{V}(\Lambda_0)$ could be constructed by extending the $\mathfrak{g}$-structure on the semi-infinite tensor power $V \otimes V \otimes \cdots$ of a small $\mathfrak{g}$-module $V$.

The Kyoto school of Jimbo, Kashiwara et al., has established this phenomenon for $\mathfrak{g}$ of classical type (and for a large class of $\hat{\mathfrak{g}}$-modules $\hat{V}(\Lambda)$) via the theory of perfect crystals [7], [10], [11], [14], [23] a development of the earlier theory of semi-infinite paths [2]. Pappas and Rapoport [20] have given a geometric version of the phenomenon for type $A$: they construct a flat deformation...
of Schubert varieties of the affine Grassmannian into a product of finite Grassmannians.

In this paper, we extend the Tensor Product Phenomenon for $\hat{V}(\Lambda_0)$ to the non-classical types $E_6$ and $E_7$ by a uniform method which applies whenever $\mathfrak{g}$ possesses a minuscule representation, or more precisely a minuscule coweight. We shall rely on a key property of such coweights which may be taken as the definition. Let $\hat{X}$ be the extended Dynkin diagram (the diagram of $\hat{\mathfrak{g}}$).

A coweight $\varpi$ of $\mathfrak{g}$ is minuscule if and only if it is a fundamental coweight $\varpi = \varpi_i$ and there exists an automorphism $\sigma$ of $\hat{X}$ taking the node $i$ to the distinguished node $0$. Such automorphisms exist in types $A, B, C, D, E_6, E_7$.

We let $V(\lambda)$ denote the irreducible $\mathfrak{g}$-module with highest weight $\lambda$, and $V(\lambda)^*$ its dual module. Our main representation-theoretic result is:

**Theorem 1** Let $\lambda$ be an element of the coroot lattice of $\mathfrak{g}$ which is a sum:

$$\lambda = \lambda_1 + \cdots + \lambda_m,$$

where $\lambda_1, \ldots, \lambda_m$ are minuscule fundamental coweights (not necessarily distinct), with corresponding fundamental weights $\lambda_1, \ldots, \lambda_m$. Let $\hat{V}_\lambda(\Lambda_0) \subset \hat{V}(\Lambda_0)$ be the Demazure module corresponding to the anti-dominant translation $t_{-\lambda}$ in the affine Weyl group.

Then there is an isomorphism of $\mathfrak{g}$-modules:

$$\hat{V}_\lambda(\Lambda_0) \cong V(\lambda_1)^* \otimes \cdots \otimes V(\lambda_m)^*.$$

Now fix a minuscule coweight $\varpi$ and its corresponding fundamental weight $\varpi$. Let $N$ be the smallest positive integer such that $N\varpi$ lies in the coroot lattice of $\mathfrak{g}$. Then we have the following characterization of the basic irreducible $\hat{\mathfrak{g}}$-module:

**Theorem 2** The tensor power $V_N := V(\varpi)^{\otimes N}$ possesses non-zero $\mathfrak{g}$-invariant vectors. Fix such a vector $v_N$, and define the $\mathfrak{g}$-module $V^{\otimes \infty}$ as the direct limit of the sequence:

$$V_N \hookrightarrow V_N^{\otimes 2} \hookrightarrow V_N^{\otimes 3} \hookrightarrow \cdots$$

where each inclusion is defined by: $v \mapsto v_N \otimes v$.

Then $\hat{V}(\Lambda_0)$ is isomorphic as a $\mathfrak{g}$-module to $V^{\otimes \infty}$.

It would be interesting to define the action of the full algebra $\hat{\mathfrak{g}}$ on $V^{\otimes \infty}$, and thus give a uniform “path construction” of the basic representation (cf. [2]): that is, to define the raising and lowering operators $E_0, F_0$, as well as the charge operator $d$. Combinatorial definitions of the charge produce for $\mathfrak{g}$ of classical type produce generalizations of the Kostka-Foulkes polynomials (c.f. [19]).

**1.2 Crystal Theorems**

Our basic tool to prove the above results is Littelmann’s combinatorial model [15], [16], [14] for representations of Kac-Moody algebras, a vast generalization
of Young tableaux. Littelmann’s paths and path operators give a flexible construction of the crystal graphs associated to quantum $\mathfrak{g}$-modules by Kashiwara [8] and Lusztig [17] (see also [5],[4]). Roughly speaking, we prove Theorem 1 by reducing it to an identity of paths: we construct a path crystal for the affine Demazure module which is at the same time a path crystal for the tensor product. We carry this out in Sec. 2.

Theorem 2 follows as a corollary in Sec. 3. To describe the crystal graph of the semi-infinite tensor product, we pass to a semi-infinite limit of Littelmann paths which we call *skeins*. We thus recover the Kyoto path model for classical $\mathfrak{g}$, and our results are equally valid for $E_6$, $E_7$.

To be more precise, we briefly sketch Littelmann’s theory. Let us define a $\mathfrak{g}$-crystal as a set $\mathcal{B}$ with a weight function $\text{wt} : \mathcal{B} \to \oplus_{i=1}^r \mathbb{Z} \omega_i$, and partially defined crystal operators $e_1, \ldots, e_r, f_1, \ldots, f_r : \mathcal{B} \to \mathcal{B}$ satisfying:

$$\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i \quad \text{and} \quad e_i(b) = b' \iff f_i(b') = b.$$ 

Here $\omega_1, \ldots, \omega_r$ are the fundamental weights and $\alpha_1, \ldots, \alpha_r$ are the roots of $\mathfrak{g}$. A dominant element is a $b \in \mathcal{B}$ such that $e_i(b)$ is not defined for any $i$. We say that a crystal $\mathcal{B}$ is a model for a $\mathfrak{g}$-module $V$ if the formal character of $\mathcal{B}$ is equal to the character of $V$, and the dominant elements of $\mathcal{B}$ correspond to the highest-weight vectors of $V$. That is:

$$\text{char}(V) = \sum_{b \in \mathcal{B}} e^{\text{wt}(b)} \quad \text{and} \quad V \cong \bigoplus_{b \text{ dom}} V(\text{wt}(b)),$$

where the second sum is over the dominant elements of $\mathcal{B}$. Clearly, a $\mathfrak{g}$-module $V$ is determined up to isomorphism by any model $\mathcal{B}$.

We construct such $\mathfrak{g}$-crystals $\mathcal{B}$ consisting of polygonal paths in the vector space of weights, $\mathfrak{h}_R^* := \oplus_{i=1}^r \mathbb{R} \omega_i$. Specifically:

- The elements of $\mathcal{B}$ are continuous piecewise-linear mappings $\pi : [0,1] \to \mathfrak{h}_R^*$, up to reparametrization, with initial point $\pi(0) = 0$. We use the notation $\pi = (v_1 \ast v_2 \ast \cdots \ast v_k)$, where $v_1, \ldots, v_k \in \mathfrak{h}_R^*$ are vectors, to denote the polygonal path starting at 0 and moving linearly to $v_1$, then to $v_1 + v_2$, etc.

- The weight of a path is its endpoint:

$$\text{wt}(\pi) := \pi(1) = v_1 + \cdots + v_k.$$

- The crystal lowering operator $f_i$ is defined as follows (and there is a similar definition of the raising operator $e_i$). Let $\ast$ denote the natural associative operation of concatenation of paths, and let any linear map $w : \mathfrak{h}_R^* \to \mathfrak{h}_R^*$ act pointwise on paths: $w(\pi) := (w(v_1) \ast \cdots \ast w(v_k))$. We will divide a path $\pi$ into three well-defined sub-paths, $\pi = \pi_1 \ast \pi_2 \ast \pi_3$, and reflect the middle piece by the simple reflection $s_i$:

$$f_i\pi := \pi_1 \ast s_i \pi_2 \ast \pi_3.$$
The pieces $\pi_1, \pi_2, \pi_3$ are determined according to the behavior of the $i$-height function $h_i(t) = h_i^\pi(t) := \langle \pi(t), \alpha_i^\vee \rangle$, as the point $\pi(t)$ moves along the path from $\pi(0) = 0$ to $\pi(1) = \text{wt}(\pi)$. This function may attain its minimum value $h_i(t) = M$ several times. If, after the last minimum point, $h_i(t)$ never rises to the value $M+1$, then $f_i \pi$ is undefined. Otherwise, we define $\pi_2$ as the last sub-path of $\pi$ on which $M \leq h_i(t) \leq M+1$, and $\pi_1, \pi_3$ as the remaining initial and final pieces of $\pi$.

A key advantage of the path model is that the crystal operators, while complicated, are universally defined for all paths. Hence a path crystal is completely specified by giving its set of paths $B$.

Also, the dominant elements have a neat pictorial characterization, as the paths $\pi$ which never leave the fundamental Weyl chamber: that is, $h_i^\pi(t) \geq 0$ for all $t \in [0,1]$ and all $i = 1, \ldots, r$. For simplicity we restrict ourselves to integral dominant paths, meaning that all the steps are integral weights: $v_1, \ldots, v_k \in \bigoplus_{i=1}^r \mathbb{Z} \alpha_i$. (For arbitrary dominant paths, see [15].)

Littelmann’s Character Theorem [16] states that if $\pi$ is any integral dominant path with weight $\lambda$, then the set of paths $\mathcal{B}(\pi)$ generated from $\pi$ by $f_1, \ldots, f_r$ is a model for the irreducible $\mathfrak{g}$-module $V(\lambda)$, (This $\mathcal{B}(\pi)$ is also closed under $e_1, \ldots, e_r$.) Note that we can choose any integral path $\pi$ which stays within the Weyl chamber and ends at $\lambda$, and each such choice gives a different (but isomorphic) path crystal modelling $V(\lambda)$. In principle, any reasonable indexing set for a basis of $V(\lambda)$ should be in natural bijection with $\mathcal{B}(\pi)$ for some choice of $\pi$. For example, classical Young tableaux correspond to choosing the steps $v_j$ to be coordinate vectors in $\mathfrak{h}^*_R \cong \mathbb{R}^p$.

Furthermore, we have Littelmann’s Product Theorem [16]: if $\pi_1, \ldots, \pi_m$ are dominant integral paths of respective weight $\lambda_1, \ldots, \lambda_m$, then $\mathcal{B}(\pi_1) \cdot \ldots \cdot \mathcal{B}(\pi_m)$, the set of all concatenations, is a model for the tensor product $V(\lambda_1) \otimes \ldots \otimes V(\lambda_m)$.

Everything we have said also holds for the affine algebra $\widehat{\mathfrak{g}}$, provided we replace the roots $\alpha_1, \ldots, \alpha_r$ of $\mathfrak{g}$ by the roots $\alpha_0, \alpha_1, \ldots, \alpha_r$ of $\widehat{\mathfrak{g}}$; and the weights $\varpi_1, \ldots, \varpi_r$ of $\mathfrak{g}$ by the weights $\Lambda_0, \Lambda_1, \ldots, \Lambda_r$ of $\widehat{\mathfrak{g}}$. We also replace the vector space $\mathfrak{h}^*_R$ by $\mathfrak{h}^*_R := \bigoplus_{i=0}^r \mathbb{R} \Lambda_i \oplus \mathbb{R} \delta$, where $\delta$ is the non-divisible positive imaginary root of $\widehat{\mathfrak{g}}$. (Indeed, the theory works uniformly for all symmetrizable Kac-Moody algebras.) We denote path crystals for $\mathfrak{g}$ and $\widehat{\mathfrak{g}}$ by $\mathcal{B}$ and $\widehat{\mathcal{B}}$ respectively.

We can also model the affine Demazure module $\widehat{V}_\pi(\Lambda)$, where $z \in \widehat{W}$, the Weyl group of $\widehat{\mathfrak{g}}$. Indeed, if $z = s_{i_1} \cdots s_{i_m}$ is a reduced decomposition and $\pi$ is an integral dominant path of weight $\Lambda$, we define the Demazure path crystal:

$$\widehat{\mathcal{B}}_z(\pi) := \{ f_{i_1}^{k_1} \cdots f_{i_m}^{k_m} \pi \mid k_1, \ldots, k_m \geq 0 \}.$$  

Then the formal character of $\widehat{\mathcal{B}}_z(\pi)$ is equal to the character of $\widehat{V}_\pi(\Lambda)$, and $\pi$ is the unique dominant path [15]. Now suppose $z = t_{-\lambda'}$, an anti-dominant translation in $\widehat{W}$, so that $\widehat{V}_{\lambda'}(\Lambda) := \widehat{V}_\Lambda(\Lambda)$ is a $\mathfrak{g}$-submodule of $\widehat{V}(\Lambda)$; and consider $\widehat{\mathcal{B}}_{\lambda'}(\pi) := \widehat{\mathcal{B}}_z(\pi)$ as a $\mathfrak{g}$-crystal by forgetting the action of $f_0, e_0$ and
projecting the affine weight function to $h^*_\mathfrak{g}$. Then Littelmann’s Restriction Theorem \[16\] implies that the $\mathfrak{g}$-crystal $\hat{B}_{\mathcal{X}}(\pi)$ is a model for the $\mathfrak{g}$-module $\hat{V}_{\mathcal{X}}(\Lambda)$.

Now we are ready to state our main combinatorial results. For $\lambda$ a dominant weight, define its dual weight $\lambda^*$ by the dual $\mathfrak{g}$-module:

$$V(\lambda^*) = V(\lambda)^*.$$ 

**Theorem 3** Let $\lambda^*$ be as in Theorem 1. Then there exists an integral $\hat{\mathfrak{g}}$-dominant path $\pi$ with weight $\Lambda_0$ which generates the $\hat{\mathfrak{g}}$ Demazure path crystal:

$$\hat{B}_{\mathcal{X}}(\pi) = \Lambda_0 \ast B(\lambda_1^*) \ast \cdots \ast B(\lambda_m^*) \mod R\delta.$$ 

This is to be understood as an equality of sets of paths in $h^*_\mathfrak{g} \mod R\delta$.

Theorem 1 follows immediately from this. Indeed, $s_i \Lambda_0 = \Lambda_0$ for $i = 1, \ldots, r$, so $f_i(\Lambda_0 \ast \pi') = \Lambda_0 \ast f_i(\pi')$ for any path $\pi'$. Thus the crystal on the right-hand side of the above equation is isomorphic to $B(\lambda_1^*) \ast \cdots \ast B(\lambda_m^*)$, which models $V(\lambda^*) \otimes \cdots \otimes V(\lambda_r^*)$. See \[3\] for methods of enumerating the paths in this crystal (and hence computing the dimension of the corresponding representation).

Next we give the following crystal version of Theorem 2:

**Theorem 4** Let $\varpi', N$ be as in Theorem 2. Define the $N$-fold concatenation $B_N = B(\varpi^*) \ast \cdots \ast B(\varpi^*)$. Then $\Lambda_0 \ast B_N$ contains a unique $\hat{\mathfrak{g}}$-dominant path $\Lambda_0 \ast \pi_N$.

Define $\hat{B}_\infty$ as the direct limit of the sequence:

$$\Lambda_0 \ast B_N \hookrightarrow \Lambda_0 \ast B_N \ast B_N \hookrightarrow \Lambda_0 \ast B_N \ast B_N \ast B_N \hookrightarrow \cdots$$

where the inclusions are given by $\Lambda_0 \ast \pi \hookrightarrow \Lambda_0 \ast \pi \ast \pi \ast \pi$. Then $\hat{B}_\infty$ has a natural $\hat{\mathfrak{g}}$-crystal structure which is isomorphic to the $\hat{\mathfrak{g}}$-crystal of $\hat{V}(\Lambda_0)$.

In Sec. 3.2, we will use crystals of semi-infinite paths (‘skeins’) to give a natural meaning to the direct limit above.

**1.3 Example: $E_6$**

Referring to Bourbaki \[1\], we write the extended Dynkin diagram $\hat{X} = \hat{E}_6$:

```
 0
\bullet \bullet \bullet
1 2 3 4 5 6
```

The simple roots are defined inside $\mathbb{R}^6$ with standard basis $\epsilon_1, \ldots, \epsilon_6$. (Our $\epsilon_6$ is $\frac{1}{\sqrt{3}}(-\epsilon_0 - \epsilon_7 + \epsilon_8)$ in Bourbaki’s notation.) They are:

$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{1}{2} \sqrt{3} \epsilon_6, \quad \alpha_2 = \epsilon_1 + \epsilon_2,$$

$$\alpha_3 = \epsilon_2 - \epsilon_1, \quad \alpha_4 = \epsilon_3 - \epsilon_2, \quad \alpha_5 = \epsilon_4 - \epsilon_3, \quad \alpha_6 = \epsilon_5 - \epsilon_4.$$
Since $E_6$ is simply laced, the coroots and coweights may be identified with the roots and weights, with the natural pairing given by the standard dot product on $\mathbb{R}^6$.

We focus on the minuscule coweight $\varpi_1^\vee$ corresponding to the diagram automorphism $\sigma$ with $\sigma(1) = 0$ and $\sigma(0) = 6$. In this case, the corresponding fundamental representation $V(\varpi_1)$ is also minuscule, meaning that all of its weights are extremal weights $\lambda \in W(E_6) \cdot \varpi_1$. The roots $\alpha_2, \ldots, \alpha_6$ generate the root sub-system $D_5 \subset E_6$, and the reflection subgroup $W(D_5) = \text{Stab}_{W(E_6)}(\varpi_1)$ acts by permuting $\epsilon_1, \ldots, \epsilon_5$ (the subgroup $W(A_4) = S_5$) and by changing an even number of signs $\pm \epsilon_1, \ldots, \pm \epsilon_5$. We have $\dim V(\varpi_1) = |W(E_6)/W(D_5)| = 27$.

The weights are:

$$\varpi_1 = \frac{2\sqrt{3}}{3} \epsilon_6,$$

$$S_5 \cdot \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6} \epsilon_6,$$

$$S_5 \cdot \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6} \epsilon_6,$$

$$-\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5) + \frac{\sqrt{3}}{6} \epsilon_6,$$

$$\pm S_5 \cdot \epsilon_1 - \frac{\sqrt{3}}{3} \epsilon_6,$$

The lowest weight is $-\varpi_6 = -\epsilon_5 - \frac{\sqrt{3}}{2} \epsilon_6$, so that $V(\varpi_1)^* = V(\varpi_6)$ and $\varpi_1^* = \varpi_6$.

The simplest path crystal for $V(\varpi_1^*)$ is the set of 27 straight-line paths from 0 to the negatives of the above extremal weights:

$$\mathcal{B}(\varpi_1^*) = \{ (v) \mid v \in -W(E_6) \cdot \varpi_1 \}$$

We have $3 \varpi_1^\vee \in \bigoplus_{i=1}^6 \mathbb{R} \alpha_i^\vee$ the coroot lattice, so that $N = 3$ in Theorem 2, and this $N$ is also the order of the automorphism $\sigma$. The path crystal $\mathcal{B}_3 := \mathcal{B}(\varpi_1^*) \ast \mathcal{B}(\varpi_1^*) \ast \mathcal{B}(\varpi_1^*)$, the set of all 3-step walks with steps chosen from the 27 weights of $V(\varpi_1^*)$, is a model for $V(\varpi_1^*)^{\otimes 3}$. In this case there is a unique $\mathfrak{g}$-dominant path of weight 0,

$$\pi_3 := (\varpi_6) \ast (\varpi_1 - \varpi_6) \ast (-\varpi_1),$$

which corresponds to the one-dimensional space of $\mathfrak{g}$-invariant vectors in $V(\varpi_1^*)^{\otimes 3}$.

Now Theorem 3 states that the affine Demazure module $\hat{V}_{3m\varpi_1^\vee}(\Lambda_0)$ is modelled by the $\hat{\mathfrak{g}}$-path crystal:

$$\mathcal{B}_{3m} = \{ (\Lambda_0 \ast v_1 \ast \cdots \ast v_{3m}) \mid v_j \in -W(E_6) \cdot \varpi_1 \},$$

the set of all 3$m$-step walks in $\Lambda_0 \oplus \mathbb{R}^6$ starting at $\Lambda_0$, with steps chosen from the 27 weights of $V(\varpi_1^*)$. This path crystal is generated from its unique $\hat{\mathfrak{g}}$-dominant path $\Lambda_0 \ast \pi_3 \ast \cdots \ast \pi_3$. As a corollary of Theorem 4, we can realize the $\hat{\mathfrak{g}}$-crystal of the basic $\hat{\mathfrak{g}}$-module $\hat{V}(\Lambda_0)$ as the set of all infinite walks (or "skeins") of the form:

$$\pi = \Lambda_0 \ast \pi_3 \ast \cdots \ast \pi_3 \ast v_1 \ast \cdots \ast v_{3m},$$

infinite
Furthermore, the affine Weyl group is a semi-direct product of the finite Weyl group and the isomorphism naturally on the chamber $h$. The highest root of $\Delta$ as $\theta = a_1a_1 + \cdots + a_r\alpha_r$, and its co-root as $\theta^\vee = a_1^\vee a_1^\vee + \cdots + a_r^\vee a_r^\vee$. Warning: If $\mathfrak{g}$ is not simply laced, $\theta^\vee$ is not the highest root of the dual root system $\Delta^\vee$.

The Weyl group $W$ of $\mathfrak{g}$ is generated by reflections $s_1, \ldots, s_r$ defined by $s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$ for $\lambda \in \mathfrak{h}_{\mathbb{R}}^* := \oplus_{i=1}^r \mathbb{R}\alpha_i^\vee$. We have the fundamental Weyl chamber $C = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid (\lambda, \alpha_i^\vee) \geq 0, \ i = 1, \ldots, r\}$. The Weyl group also acts naturally on $\mathfrak{h}_{\mathbb{R}}$. If we choose a $W$-invariant bilinear form $(\cdot | \cdot)$ on $\mathfrak{h}_{\mathbb{R}}$, we have the isomorphism $\nu : \mathfrak{h}_{\mathbb{R}} \to \mathfrak{h}_{\mathbb{R}}^*$ defined by $\langle \nu(h), h' \rangle = (h|h')$ for $h, h' \in \mathfrak{h}_{\mathbb{R}}$. We normalize so that $\nu(\theta^\vee) = \theta$ and $\nu(\alpha_i^\vee) = \frac{\alpha_i^\vee}{\alpha_i^\vee}w_i$.

Now let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus CK \oplus C\mathbb{d}$ be the untwisted affine Lie algebra of $\mathfrak{g}$, where $K$ is a central element and $\mathbb{d} = t\frac{\partial}{\partial t}$ is a derivation. (Cf. Kac [6, Ch. 6 and 7].) Then $\widehat{\mathfrak{g}}$ has Cartan subalgebra $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus CK \oplus C\mathbb{d}$, with dual $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus C\Lambda_0 \oplus C\delta$, where $\langle \Lambda_0, h \rangle = \langle \delta, h \rangle = 0$ and $\langle \Lambda_0, K \rangle = \langle \delta, d \rangle = 1$.

The simple roots of $\widehat{\mathfrak{g}}$ are $\alpha_1, \ldots, \alpha_r$ and $\alpha_0 = \delta - \theta$; the simple coroots are $\alpha_1^\vee, \ldots, \alpha_r^\vee$ and $\alpha_0^\vee = K - \theta^\vee$. The fundamental weights are $\Lambda_0$ and $\Lambda_i = \alpha_i^\vee \Lambda_0$ for $i = 1, \ldots, r$. The affine Weyl group $\widehat{W}$ is generated by the reflections $s_0, s_1, \ldots, s_r$ acting on $\widehat{\mathfrak{h}}^*_R$. The fundamental Weyl chamber of $\widehat{\mathfrak{g}}$ is the cone $\widehat{C} = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \Lambda, \alpha_i^\vee \rangle \geq 0, \ i = 0, \ldots, r\}$ with extremal rays $\Lambda_0, \ldots, \Lambda_r$.

For $\Lambda = \lambda + \ell\Lambda_0 + m\delta$, we will ignore the charge $m = \langle \Lambda, d \rangle$ and work modulo $\mathbb{R}\delta$, even when we do not indicate this explicitly. Since $\langle \delta, \alpha_i^\vee \rangle = 0$ for $i = 0, \ldots, r$, the charge has no effect on the path operators $f_i, e_i$.

Consider the lattice $M = \nu(\oplus_{i=1}^r \mathbb{Z}\alpha_i^\vee)$ in $\mathfrak{h}_{\mathbb{R}}^*$. For any $\mu \in M$, there is an element $t_\mu \in \widehat{W}$ which acts on the weights of level $\ell$ as translation by $\ell\mu$: that is,

$$t_\mu(\lambda + \ell\Lambda_0) = \lambda + \ell\mu + \ell\Lambda_0 \pmod{\mathbb{R}\delta}.$$ 

Furthermore, the affine Weyl group is a semi-direct product of the finite Weyl group with the lattice of translations: $\widehat{W} = W \rtimes t_M$.
Consider the anti-dominant translation \( z = t_{-\lambda} \) corresponding to a dominant weight \( \lambda = \nu(\lambda') \in C \cap M \). We denote the resulting Demazure module as \( \tilde{V}_\chi(\Lambda_0) := V_\chi(\Lambda_0) \). Then the \( \widehat{\mathfrak{n}}_+ \)-module \( \tilde{V}_\chi(\Lambda_0) \) is also a \( \mathfrak{g} \)-submodule of \( V(\Lambda_0) \):

\[
\mathfrak{g} \cdot \tilde{V}_\chi(\Lambda_0) \subset \tilde{V}_\chi(\Lambda_0),
\]

and these are the only \( z \in \widehat{W} \) for which \( \tilde{V}_\chi(\Lambda_0) \) is a \( \mathfrak{g} \)-module.

### 2.2 Minuscule weights and coweights

We collect needed facts concerning minuscule weights in root systems. The statements below are well-known and easily verified from tables [11, 12], although direct proofs are also not difficult (cf. [18]).

We say a non-zero coweight \( \varpi_i' \in \mathfrak{h}^*_R \) is **minuscule for** \( \Delta \) if \( \langle \alpha, \varpi_i' \rangle = 0 \) or 1 for all positive roots \( \alpha \in \Delta_+ \). Equivalently, \( \varpi_i' = \varpi_i' \) for some \( i = 1, \ldots, r \) with \( a_i = \langle \theta, \varpi_i' \rangle = 1 \). This implies that \( a_i \) is 1 as well, so that \( \nu(\varpi_i') = \varpi_i \).

The classification of the minuscule \( \varpi_i' \) is most concisely described by listing the pairs \((X, X\setminus\{i\})\), where \( X \) is the Dynkin diagram of \( \mathfrak{g} \). We have \( \varpi_i' \) minuscule when:

\[
(X, X\setminus\{i\}) \cong (A_r, A_{r-k} \times A_{k-1}), \quad k=1, \ldots, r
\]

\[
(B_r, B_{r-1}), \quad (C_r, A_{r-1}),
\]

\[
(D_r, D_{r-1}), \quad (D_r, A_r),
\]

\[
(E_6, D_5), \quad (E_7, E_6).
\]

There are no minuscule \( \varpi_i' \) for \( X = E_8, F_4, \) or \( G_2 \).

Now define the extended Weyl group \( \widehat{W} \) as a group of linear mappings on \( \widehat{\mathfrak{h}}_R^* \) by \( \widehat{W} := W \ltimes t_L \), where \( L = \nu(\oplus_{i=1}^r \mathbb{Z} \varpi_i') \). Let

\[
\Sigma := \{ \sigma \in \widehat{W} \mid \sigma(C) = \widehat{C} \},
\]

the symmetries in \( \widehat{W} \) of the fundamental chamber of \( \widehat{\mathfrak{h}}_R^* \). The set \( \Sigma \) is a system of coset representatives for \( \widehat{W} / \widetilde{W} \), so that \( \widehat{W} = \Sigma \ltimes \widetilde{W} \). We can extend the Bruhat length function to \( \widehat{W} \) as: \( l(\sigma w) = l(w) \) for \( \sigma \in \Sigma, w \in \widetilde{W} \). Each element \( \sigma \in \Sigma \) defines an automorphism of the Dynkin diagram of \( \widehat{\mathfrak{g}} \) which we also write as \( \sigma \). For \( j = 0, \ldots, r \), we have:

\[
\sigma(\Lambda_j) = \Lambda_{\sigma(j)} \quad \text{and} \quad \sigma(\alpha_j) = \alpha_{\sigma(j)}.
\]

There is a natural correspondence between elements of \( \Sigma \) and minuscule coweights. Each \( \sigma \in \Sigma \) can be written uniquely as:

\[
\sigma = \sigma t_{-\nu(\varpi_i')} = \sigma t_{-\varpi_i},
\]

for \( \sigma \in W \) and \( \varpi_i' \) a minuscule coweight. In fact, \( \sigma = w_0 w_i \), where \( w_0 \) is the longest element of \( W \) and \( w_i \) is the longest element of the parabolic subgroup \( W_i := \text{Stab}_W(\varpi_i) \). We have \( \sigma(\alpha_j) = \alpha_{\sigma(j)} \) for \( j \neq i \), and \( \sigma(\alpha_i) = -\theta \).
We have \( \sigma(\Lambda_i) = \bar{\sigma} - w_i(\varpi_i + \Lambda_0) = \Lambda_0 \), so that
\[
\sigma(i) = 0.
\]
Also, \( \bar{\sigma}(\varpi_i) = w_0 w_i(\varpi_i) = w_0(\varpi_i) = -\varpi_i^* \), and \( \varpi_{\sigma(0)} = \varpi_i^* \).

The number \( N \) appearing in Theorems 2 and 4 is the order of \( \sigma \) in the group \( \Sigma \), and is also the order of \( \varpi_\lor \) in \( \bigoplus_{i=1}^r R \varpi_i / \bigoplus_{i=1}^r R \alpha_i^\lor \).

The definition of a minuscule weight \( \varpi \) for \( \Delta_\lor \) is dual to the above:
\[
\langle \varpi, \alpha_\lor \rangle = 0 \text{ or } 1 \text{ for all positive coroots } \alpha_\lor \in \Delta_\lor^+; \text{ or equivalently } \varpi = \varpi_i \text{ and } \langle \varpi, \theta^* \rangle = 1,
\]
where \( \theta^* \) is highest in the root system \( \Delta_\lor \). The fundamental representation \( V(\varpi) \) corresponding to a minuscule \( \varpi \) has a basis consisting of extremal weight vectors \( v_{w(\varpi)} \) for \( w \in W \). Note, however, that \( \varpi \) need not be minuscule even when the corresponding coweight \( \varpi_\lor \) is minuscule. In fact, we have:

**Lemma 5** Let \( \varpi_i^\lor \) be a minuscule coweight for \( \Delta \), and \( \varpi_i \) the corresponding weight.

(i) If \( \Delta \) is simply laced (i.e., all root vectors have the same length), then \( \varpi_i \) is minuscule for \( \Delta_\lor \).

(ii) If \( \Delta \) is not simply laced, then \( \varpi_i \) is minuscule for \( \Delta_\lor^s \), the simply-laced root system of short vectors in \( \Delta_\lor \). Furthermore, \( \alpha_i^\lor \in \Delta_\lor^s \).

**Proof.** Part (i) follows from \( \nu(\varpi_i) = (a_i/a_i^\lor) \varpi_i = \varpi_i \). Part (ii) is immediately verified for the relevant types \( B_r \) and \( C_r \). □

Consider the parabolic Bruhat order on the \( W \)-orbit \( W \cdot \varpi_i \). That is, the partial order generated by the relations: \( \tau_1 < \tau_2 \) if \( \tau_1 = \tau_2 - d\alpha_i \) for some positive root \( \alpha_i \) and some \( d > 0 \). The following result says that if \( \varpi_i^\lor \) is minuscule, then this “strong” order is identical to the “weak” order:

**Lemma 6 (Stembridge [22])** Suppose the coweight \( \varpi_i^\lor \) is minuscule. Then the Bruhat order on \( W \cdot \varpi_i \) has covering relations:
\[
\tau_1 < \tau_2 \text{ whenever } \tau_1 = \tau_2 - d\alpha_j
\]
for some simple root \( \alpha_j \) of \( W \) and some \( d > 0 \).

**Proof.** This follows from Stembridge’s formulation by noting that \( W \cdot \varpi_i \cong W/W_i \cong W \cdot \varpi_i^\lor \). □

## 2.3 Lakshmibai-Seshadri Paths

We examine in detail the path crystals of \( V(\varpi_i) \) where \( \varpi_i^\lor \) is minuscule. For any dominant weight \( \lambda \), the most canonical choice of dominant path is the straight-line path from 0 to \( \lambda \), denoted \( \pi = (\lambda) \). The corresponding path crystal \( B(\lambda) \) can be described non-recursively by the Lakshmibai-Seshadri (LS) chains [15].
These are saturated chains in the parabolic Bruhat order on $W\cdot \lambda$, weighted with certain rational numbers:

$$(\tau_1 > \cdots > \tau_m; \; 0 < a_1 \leq \cdots \leq a_{m-1} < 1)$$

with $\tau_j \in W \cdot \lambda$, $a_j \in \mathbb{Q}$, and $m \geq 1$. We require that if $\tau_{j+1} = \tau_j - d_j \alpha$ for $\alpha \in \Delta_+$, $d_j \in \mathbb{N}$, then $a_j = n_j/d_j$ for some $n_j \in \mathbb{N}$. An LS chain corresponds to the LS path defined as:

$$\pi = (a_1 \tau_1 \ast (a_2 - a_1) \tau_2 \ast (a_3 - a_2) \tau_3 \ast \cdots \ast (1 - a_{m-1}) \tau_m).$$

Notice that if $a_{j+1} = a_j$ then we may omit the step $0 \tau_j$; in this case there may be more than one LS chain producing the same LS path. However, $B(\lambda)$ is equal to the set of all distinct LS paths \([15]\).

**Proposition 7** Let $\varpi_i^\vee, \varpi_j^\vee$ be two minuscule coweights (possibly identical), and let $\bar{\sigma}_i$ be the linear mapping of $\mathfrak{h}_R^\ast$ corresponding to $\varpi_i^\vee$. For each $\pi \in B(\varpi_i)$, we have $\bar{\sigma}_i \pi \in B(\varpi_i)$. That is, the linear mapping $\bar{\sigma}_i$ permutes the paths in $B(\varpi_i)$.

**Proof.** Consider an LS chain $(\tau_1 > \cdots > \tau_m; \; 0 < a_1 \leq \cdots \leq a_{m-1} < 1)$ corresponding to a path $\pi \in B(\varpi_i)$. If $g$ is simply laced, then $\varpi_i$ is minuscule by Lemma 5(i), and the denominator of $a_j$ is:

$$d_j = \langle \tau_j, \alpha_i \rangle = \langle w \varpi_i, \alpha_i \rangle = \langle \varpi_i, w \alpha_i \rangle = 0 \text{ or } 1.$$

Since $0 < a_j = \frac{n_j}{d_j} < 1$, this means that $m = 1$ and $\pi = (\tau_1) = (w \varpi_i)$ for $w \in W$, a straight-line path of extremal weight. Since $\bar{\sigma}_i$ is an automorphism of the root system $\Delta$, it permutes the elements of a $W$-orbit, and hence $\bar{\sigma}_i \pi$ is another straight-line LS path.

If $g$ is not simply laced, the paths of $B(\varpi_i)$ are more complicated. By Lemma 6, we may assume that $\tau_{j+1} = \tau_j - d_j \alpha_{k(j)}$, where $k(j) \in \{1, \ldots, r\}$ and $\alpha_{k(j)}$ is a simple root. The turned path is:

$$\bar{\sigma}_i \pi = (a_1 \bar{\sigma}_i \tau_1 \ast (a_2 - a_1) \bar{\sigma}_i \tau_2 \ast \cdots).$$

Suppose $k(j) = l$ for some $j$. By Lemma 5(ii), $\alpha_i^\vee$ is a short root of $\Delta_i^\vee$, and so is $w \alpha_i^\vee$, so we have:

$$d_j = \langle \tau_j, \alpha_i^\vee \rangle = \langle w \varpi_i, \alpha_i^\vee \rangle = \langle \varpi_i, w \alpha_i^\vee \rangle = 0 \text{ or } 1$$

by the same Lemma. This again means that $m = 1$ and $\bar{\sigma}_i \pi$ is a straight-line LS path.

Finally, suppose $k(j) \neq l$ for all $j = 1, \ldots, m$. Then:

$$\bar{\sigma}_i \tau_{j+1} = \bar{\sigma}_i \tau_j - d_j \bar{\sigma}_i \alpha_{k(j)} = \bar{\sigma}_i \tau_j - d_j \alpha_p,$$

where $p := \sigma_i(k(j)) \in \{1, \ldots, r\}$. Hence $\bar{\sigma}_i \tau_j > s_p \bar{\sigma}_i \tau_j = \bar{\sigma}_i \tau_{j+1}$, and $\bar{\sigma}_i \pi$ is an LS path. \(\square\)

Although we do not need it here, we note that for any minuscule $\varpi_i^\vee$, the LS-paths of $B(\varpi_i)$ have at most two linear pieces (cf. \([12]\)).

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2.4 Twisted Demazure operators

For a Weyl group element with reduced decomposition \( z = s_{i_1} \cdots s_{i_m} \in \hat{W} \) and any path \( \pi \) (not necessarily dominant), we define the crystal Demazure operator:

\[
\hat{B}_z(\pi) := \{ f_{k_1}^{i_1} \cdots f_{k_m}^{i_m} \pi_\lambda \mid k_1, \ldots, k_m \geq 0 \}.
\]

We can extend \( \hat{B}_z \) to an operator taking any set of paths \( \Pi \) to a larger set of paths: \( \hat{B}_z(\Pi) := \bigcup_{\pi \in \Pi} \hat{B}_z(\pi) \). We have \( \hat{B}_y(\hat{B}_z(\Pi)) = \hat{B}_{yz}(\Pi) \) whenever \( l(yz) = l(y) + l(z) \).

Similarly, we let \( B(\Pi) \) be the set of all paths generated from \( \Pi \) by \( f_1, \ldots, f_r, e_1, \ldots, e_r \).

It will be convenient to define a Demazure module \( \hat{V}_z(\Lambda) \) for any \( z \in \hat{W} = \tilde{W} \). Now, \( \sigma \in \Sigma \) induces an automorphism of \( \hat{g} \), so for a module \( \hat{V} \) we have the twisted module \( \sigma \hat{V} \) defined by the action \( g \circ v := \sigma^{-1}(g)v \) for \( g \in \hat{g} \), \( v \in \hat{V} \). That is, \( \sigma \hat{V}(\Lambda) \cong \hat{V}(\sigma\Lambda) \), and in particular \( \sigma \hat{V}(\lambda_i) \cong \hat{V}(\Lambda_{\sigma(i)}) \). Now, for \( z = \sigma y \) with \( y \in \hat{W} \) define: \( \hat{V}_z(\Lambda) := \sigma(\hat{V}_y(\Lambda)) \subset \sigma \hat{V}(\Lambda) \), a twist of an ordinary Demazure module. Thus, \( \hat{V}_{\sigma y}(\Lambda) \cong \hat{V}_{\sigma y \sigma^{-1}}(\sigma \Lambda) \) and

\[
\hat{V}_{g \sigma}(\Lambda) \cong \hat{V}_g(\sigma \Lambda).
\]

Furthermore, \( \hat{V}_{\lambda^\vee}(\Lambda_0) := \hat{V}_z(\Lambda_0) \) for \( z = t_{-\nu(\lambda^\vee)} \) is a \( g \)-module for any dominant integral coweight \( \lambda^\vee \in \oplus_{i=1}^r N \varpi_i^\vee \).

The combinatorial counterpart of this construction is:

\[
\hat{B}_{\sigma y}(\pi) := \sigma \hat{B}_y(\pi)
\]

for \( \sigma \in \Sigma \) and \( y \in \hat{W} \). All of our statements regarding \( \hat{V}_z(\Lambda) \) and \( \hat{B}_z \) for \( z \in \hat{W} \) remain valid for \( z \in \tilde{W} \).

2.5 Proof of Theorems 1 and 3

Let \( \lambda^\vee \) be a dominant integral coweight (not necessarily in the root lattice) which can be written:

\[
\lambda^\vee = \lambda_1^\vee + \cdots + \lambda_m^\vee,
\]

where \( \lambda_j^\vee \in \{ \varpi_1, \ldots, \varpi_r \} \) are minuscule fundamental coweights (not necessarily distinct) with corresponding weights \( \lambda_j \) and dual weights \( \lambda_j^\ast = -w_0(\lambda_j) \).

Then we claim (extending Theorem 1 to the coweight lattice) that there is an isomorphism of \( g \)-modules:

\[
\hat{V}_{\lambda^\vee}(\Lambda_0) \cong V(\lambda_1^\ast) \otimes \cdots \otimes V(\lambda_m^\ast).
\]

This follows immediately from Littelmann’s Character and Restriction Theorems combined with the following extension of Theorem 3.

We will choose a certain \( \hat{g} \)-dominant path \( \pi_m \) of weight \( \Lambda_0 \) and show that:

\[
\hat{B}_{\lambda^\vee}(\pi_m) = \Lambda_0 \ast B(\lambda_1^\ast) \ast \cdots \ast B(\lambda_m^\ast).
\]
Let $\sigma_j \in \Sigma$ correspond to $\lambda^*_j$ for $j = 1, \ldots, m$. We define $\pi_m$ inductively as the last of a sequence of paths $\pi_0, \pi_1, \ldots, \pi_m$:

$$\pi_0 := \Lambda_0, \quad \pi_j := \sigma^{-1}_j(\pi_{j-1} \star \lambda^*_j).$$

We may picture $\pi_m$ as jumping up to level $\Lambda_0$, winding horizontally around the fundamental alcove $A = (\mathfrak{h}_R^s + \Lambda_0) \cap \hat{C}$, and ending at $\Lambda_0$. Indeed, note that $\text{wt}(\pi_0) = \Lambda_0$. For $j > 0$, write $\varpi := \lambda_j$ and $\Lambda := \varpi + \Lambda_0$, so that $\sigma_j(\Lambda) = \Lambda_0$ and $\sigma_j(\varpi) = -\varpi^*$. Then by induction:

$$\text{wt}(\pi_j) = \sigma^{-1}_j(\text{wt}(\pi_{j-1}) + \varpi^*) = \sigma^{-1}_j \Lambda_0 + \sigma^{-1}_j \varpi^* = \Lambda - \varpi = \Lambda_0,$$

so that each $\pi_j$ has weight $\Lambda_0$. Furthermore, since $\Lambda_0 + \lambda^*_j \in \hat{C}$ and $\sigma_j$ is an automorphism of $\hat{C}$, it is clear that each $\pi_j$ is indeed a $\hat{g}$-dominant path.

We will now prove Theorem 3 by showing that the Demazure operator $B_{\lambda^*}$ "unwinds" $\pi_m$ starting from its endpoint. To compute:

$$B_{\lambda^*}(\pi_m) = B_{\lambda^*_1} \cdots B_{\lambda^*_m}(\pi_m),$$

it suffices to prove:

**Lemma 8** For $j = m, m-1, \ldots, 1$, we have:

$$B_{\lambda^*_j} \left( \pi_j \star B(\lambda^*_i + 1) \star \cdots \star B(\lambda^*_m) \right) = \pi_{j-1} \star B(\lambda^*_i) \star B(\lambda^*_j + 1) \star \cdots \star B(\lambda^*_m).$$

**Proof.** For $j = m$, we compute:

$$B_{\lambda^*_m}(\pi_m) = \hat{B}_{w_m w_0 \sigma_m} \left( \sigma^{-1}_{m-1} (\pi_{m-1} \star \lambda^*_m) \right) = \hat{B}_{w_m w_0} (\pi_{m-1} \star \lambda^*_m) \quad \text{(I)} \quad = \hat{B}_{w_m w_0} (\pi_{m-1} \star \lambda^*_m) \quad \text{(II)} \quad = \pi_{m-1} \star \hat{B}_{w_0} (\lambda^*_m) \quad \text{(III)} \quad = \pi_{m-1} \star B(\lambda^*_m).$$

The equalities are justified as follows: (I) Here $w_m^* := w_0 w_m w_0$, the longest element in $\text{Stab}_W(\lambda_m^*) = \text{Stab}_W(-w_0 \lambda_m)$. (II) We say a path $\pi$ is $i$-neutral if $\langle \pi(t), \alpha_i^* \rangle \geq 0$ and $\langle \text{wt} \, \pi, \alpha_i^* \rangle = 0$. It is clear from the definition of $f_\lambda$ that $f_\lambda(\pi \star \pi') = \pi \star f_\lambda(\pi')$ for any path $\pi'$, and the two sides are both defined or both undefined. Now note that the $\pi_j$ are $i$-neutral for $i=1, \ldots, r$. (III) Follows from $B_{w_0 w_m^*}(\lambda^*) = B_{w_0 w_m^*} \hat{B}_{w_m^*}(\lambda^*) = \hat{B}_{w_0}(\lambda^*)$. 

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For $j < m$, letting $\mathcal{B}_{j+1} := \mathcal{B}(\lambda_{j+1}^*) \otimes \cdots \otimes \mathcal{B}(\lambda_{m}^*)$, we have:

$$
\mathcal{B}_{\lambda_j^*}(\pi_j \otimes \mathcal{B}_{j+1}) = \hat{\mathcal{B}}_{w_0^\lor \sigma_j} \left( \sigma_j^{-1}(\pi_{j-1} \otimes \lambda_j^*) \otimes \mathcal{B}_{j+1} \right)
$$

Here (I) and (II) are as above. (IV) Follows from Proposition 7. (V) Follows from [13, Prop. 12], which implies that $\lambda_j^* \otimes \mathcal{B}_{j+1}$ is isomorphic to a union of Demazure crystals $\mathcal{B}_y(\mu)$ with $y \geq w_0^\lor m$. (VI) Both sides are stable under the $f_i, e_i$ for $i = 1, \ldots, r$, and they contain the same $\mathfrak{g}$-dominant paths, hence they are identical.

This concludes the proof of the Lemma, and hence of Theorem 3. □

3 Semi-infinite Crystals

3.1 Proof of Theorem 2

Fix a minuscule coweight $\varpi$ with corresponding weight $\varpi^*$, dual weight $\varpi^*$, and automorphism $\sigma \in \Sigma$ of order $N$. The $N$-fold concatenation $\mathcal{B}(\varpi^*) \otimes \cdots \otimes \mathcal{B}(\varpi^*)$ contains the path $\pi_N := (\sigma^{-N}(\varpi^*) \otimes \cdots \otimes \sigma^{-2}(\varpi^*) \otimes \sigma^{-1}(\varpi^*))$, which is $\mathfrak{g}$-dominant with weight 0. Thus $V(\varpi^*)^{\otimes N}$ possesses a corresponding invariant vector $v_N$.

Since $t_{-N\varpi^*} \in \hat{W}$, the twisted Demazure module $V_{mN\varpi^*}(\Lambda_0)$ is an ordinary Demazure submodule of $\hat{V}(\Lambda_0)$. In fact, the basic $\hat{\mathfrak{g}}$-module, considered as a $\mathfrak{g}$-module, is a direct limit of these Demazure modules:

$$
\hat{V}(\Lambda_0) = \lim_{\rightarrow} V_{N\varpi^*} \otimes V_{2N\varpi^*} \otimes V_{3N\varpi^*} \otimes \cdots .
$$

By Theorem 1, the $\mathfrak{g}$-module on the right hand side is isomorphic to:

$$
\lim_{\rightarrow} V(\varpi^*)^{\otimes N} \xrightarrow{\phi_1} V(\varpi^*)^{\otimes 2N} \xrightarrow{\phi_2} V(\varpi^*)^{\otimes 3N} \xrightarrow{\phi_3} \cdots
$$

for some injective $\mathfrak{g}$-module morphisms $\phi_1, \phi_2, \phi_3, \ldots$. Now, if $\phi, \psi : V_1 \rightarrow V_2$ are any two injective $\mathfrak{g}$-module morphisms, complete reducibility implies that there exist automorphisms $\xi_1, \xi_2$ forming the commutative diagram:

$$
\begin{array}{ccc}
V_1 & \xrightarrow{\phi} & V_2 \\
\xi_1 \downarrow & & \xi_2 \downarrow \\
V_1 & \xrightarrow{\psi} & V_2
\end{array}
$$

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Thus, the inclusions $\psi_m : V(\pi^*) \otimes N^m \hookrightarrow V(\pi^*) \otimes N^{m+1}$, $v \mapsto v_N \otimes v$ appearing in Theorem 2 define the same $g$-module as the above maps $\phi_m$, and we conclude that $V(\Lambda_0)$ is isomorphic as a $g$-module to the infinite tensor product as claimed.

### 3.2 The Skein model

In this section we prove Theorem 4 by a crystal analog of the above argument. We introduce a notation for a path $\pi$ which emphasizes the vector steps going toward the endpoint $\Lambda = \text{wt}(\pi)$ rather than away from the starting point 0. Define

$$\pi = (v_k \cdots v_1 \vdash \Lambda) := (v' \star v_k \cdots \star v_1),$$

the path with endpoint $\Lambda$, last step $v_1$, etc, and first step $v' := \Lambda - (v_k + \cdots + v_1)$, a makeweight to assure that the steps add up to $\Lambda$.

A **skein** is an infinite list:

$$\pi = (\cdots \star v_2 \star v_1 \vdash \Lambda),$$

where $\Lambda \in \bigoplus_{i=0}^r \mathbb{Z} \Lambda_i$ and $v_j \in \mathfrak{h}_\mathbb{R}^*$ (not $\hat{\mathfrak{h}}_\mathbb{R}^*$), subject to conditions (i) and (ii) below. For $i = 0, \ldots, r$ and $k > 0$, define:

$$h_i[k] := \langle \Lambda - (v_1 + \cdots + v_k), \alpha_i^\vee \rangle.$$

We require:

(i) For each $i$ and all $k \gg 0$, we have $h_i[k] \geq 0$.

(ii) For each $i$, there are infinitely many $k$ such that $h_i[k] = 0$.

We think of the skein $\pi$ as a “projective limit” of the paths

$$\pi[k] := (\star v_k \cdots \star v_1 \vdash \Lambda) \quad \text{as} \quad k \to \infty.$$

The conditions on $\pi$ assure that only a finite number of steps of $\pi$ lie outside the fundamental chamber $\hat{C}$, and that $\pi$ touches each wall of $\hat{C}$ infinitely many times. Note that $\pi$ stays always at the level $\ell = \langle \Lambda, K \rangle$.

**Lemma 9** For a skein $\pi$ and $i = 0, \ldots, r$, one of the following is true:

(i) $f_i(\pi[k])$ is undefined for all $k \gg 0$;

(ii) there is a unique skein $\pi'$ such that $\pi'[k] = f_i(\pi[k])$ for all $k \gg 0$.

In the second case, we define $f_i \pi := \pi'$.

**Proof.** Recall that a path $\pi$ is $i$-neutral if $h_i^\pi(t) \geq 0$ for all $t$ and $h_i^\pi(1) = 0$. For a fixed $i$, divide $\pi$ into a concatenation: $\pi = (\cdots \star \pi_2 \star \pi_1 \star \pi_0 \vdash \Lambda)$, where each $\pi_j$ is an $i$-neutral finite path except for $\pi_0$, which is an arbitrary finite path. Now it is clear that if $f_i(\pi_0)$ is undefined, then (i) holds. Otherwise (ii) holds and

$$f_i \pi = (\cdots \star \pi_2 \star \pi_1 \star f_i(\pi_0) \vdash \Lambda - \alpha_i). \quad \square$$
We can immediately carry over the definitions of the path model to skeins, including that of (Demazure) path crystals. For example, we say that \( \pi \) is an integral dominant skein if \( \pi[k] \) is integral dominant for \( k \gg 0 \), and hence for all \( k \). There exist integral dominant skeins of level \( \ell = 1 \) only when \( g \) has a minuscule coweight. We cannot concatenate two skeins, but we can concatenate a skein \( \pi_1 \) and a path \( \pi_0 \): that is, \( \pi_1 \star \pi_0 := (\pi_1 \star \pi_0 \vdash \wt(\pi_1) + \wt(\pi_0)) \).

**Proposition 10** For an integral dominant skein \( \pi \) of weight \( \Lambda \), the crystal \( \hat{B}(\pi) \) is a model for \( \hat{V}(\Lambda) \), and for Demazure modules.

**Proof.** Given an integral dominant skein \( \pi \) and a Weyl group element \( z \in \tilde{W} \), we can divide \( \pi = \pi_1 \star \pi_0 \) in such a way that the Demazure operator \( \hat{B}_z \) acts on \( \pi \) by reflecting intervals in \( \pi_0 \) rather than \( \pi_1 \). This gives an isomorphism between the Demazure crystals generated by the path \( \wt(\pi_1) \star \pi_0 \) and by the skein \( \pi \):

\[
\hat{B}_z(\wt(\pi_1) \star \pi_0) \overset{\sim}{\rightarrow} \hat{B}_z(\pi_1 \star \pi_0) \overset{\sim}{\rightarrow} \hat{B}_z(\pi)
\]

\[
\wt(\pi_1) \star \pi' \mapsto \pi_1 \star \pi'.
\]

This proves the assertion about Demazure modules.

Now, given an infinite chain of Weyl group elements \( z_1 < z_2 < \cdots \), we have the morphisms of \( \hat{g} \)-crystals:

\[
\hat{B}_{z_1}(\Lambda) \leftarrow \hat{B}_{z_1}(\wt(\pi_1) \star \pi_0) \rightarrow \hat{B}_{z_1}(\pi)
\]

\[
\hat{B}_{z_1}(\Lambda) \leftarrow \hat{B}_{z_2}(\wt(\pi_1') \star \pi_0') \rightarrow \hat{B}_{z_2}(\pi)
\]

\[
\vdots
\]

\[
\hat{B}(\Lambda) \leftarrow \hat{B}(\pi)
\]

Since the \( \hat{g} \) crystals at the bottom are the unions of their Demazure crystals, they are isomorphic: \( \hat{B}(\Lambda) \cong \hat{B}(\pi) \). \( \square \)

Now consider the situation of Theorem 4. Define the skein \( \pi_\infty := (\cdots \star \pi_N \star \pi_N \vdash \Lambda_0) \), and consider the commutative diagram:

\[
\hat{B}_{N}(\Lambda_0 \star \pi_N) \rightarrow \hat{B}_{N}(\pi_\infty)
\]

\[
\hat{B}_{2N}(\Lambda_0 \star \pi_N \star \pi_N) \rightarrow \hat{B}_{2N}(\pi_\infty)
\]

\[
\vdots
\]

\[
\hat{B}(\pi_\infty)
\]

where the vertical maps on the left are those of the direct limit. Theorem 4 now follows from the above Proposition.
References

[1] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 4, 5 et 6, Hermann, Paris (1968).

[2] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, A Path space realization of the basic representation of $A_n^{(1)}$, in Infinite-Dimensional Lie Algebras and Groups, Adv. Ser. in Math. Phys. 7, World Sci. (1989), 108–123.

[3] D. Grabiner, P. Magyar, Random walks in Weyl chambers and the decomposition of tensor powers, J. Algebraic Combin. 2 (1993), 239–260.

[4] J. Hong and S-J. Kang, Introduction to Quantum Groups and Crystal Bases, Grad. Stud. in Math. 42, Amer. Math. Soc. (2002).

[5] A. Joseph, Quantum Groups and Their Primitive Ideals, Springer Verlag, New York (1994).

[6] V.G. Kac, Infinite Dimensional Lie Algebras, 3rd ed., Cambridge University Press (1990).

[7] S-J. Kang, M. Kashiwara, K.C. Misra, T. Miwa, T. Nakashima and A. Nakayashiki, Perfect crystals of quantum affine Lie algebras, Duke Math. J. 68 (1992), 499–607.

[8] M. Kashiwara, On crystal bases of $q$-analog of universal enveloping algebras, Duke Math. J. 63 (1991), 465–516.

[9] M. Kashiwara, On level-zero representation of quantized affine algebras, Duke Math. J. 112 (2002), 117–195.

[10] A. Kuniba, K. Misra, M. Okado, T. Takagi, J. Uchiyama, Demazure modules and perfect crystals, Comm. Math. Phys. 192 (1998), 555–567.

[11] A. Kuniba, K. Misra, M. Okado, T. Takagi, J. Uchiyama, Paths, Demazure crystals, and symmetric functions, J. Math. Phys. 41 (2000), 6477–6486.

[12] V. Lakshmibai and C.S. Seshadri, Standard monomial theory, Proceedings of the Hyderabad Conference on Algebraic Groups, ed. S. Ramanan, Manoj Prakashan Madras (1991), 279-323.

[13] V. Lakshmibai, P. Littelmann and P. Magyar, Standard monomial theory for Bott-Samelson varieties, Compositio Math. 130 (2002), 293–318.

[14] V. Lakshmibai, P. Littelmann, P. Magyar, Standard monomial theory and applications, in Representation Theories and Geometry, ed. A. Broer, Kluwer Academic Publishers (1998), 319–364.

[15] P. Littelmann, A Littlewood-Richardson formula for symmetrizable Kac-Moody algebras, Invent. Math. 116 (1994), 329–346.

[16] P. Littelmann, Paths and root operators in representation theory, Ann. Math. 142 (1995), 499–525.
[17] G. Lusztig, Introduction to Quantum groups, Progress in Mathematics 110, Birkhäuser Verlag (1993).

[18] I.G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Cambridge Tracts in Mathematics 157, Cambridge University Press (2003).

[19] M. Okado, A. Schilling, M. Shimozono, Crystal bases and $q$-identities, in $q$-series with applications to combinatorics, number theory, and physics, Contemp. Math. 291, Amer. Math. Soc. (2001), 29–53.

[20] G. Pappas and M. Rapoport, Local models in the ramified case I: the EL-case, J.Alg.Geom 12 (2003), 107–145.

[21] A. Pressley and G. Segal, Loop Groups, Oxford Univ. Press (1986).

[22] J. Stembridge, Minuscule elements of Weyl groups, J. Algebra 235 (2001), 722–743.