Abstract: A nonperturbative procedure of solving the time-dependent Schrödinger equation, called the multi-projection approach or phase dynamics of quantum mechanics, is derived and illustrated. In addition to introducing a method with that time-dependent systems become solvable (under the assumption that corresponding time-independent systems are solvable), the new approach unveils several misconcepts related to the usual wavefunction expansion and the standard perturbation theory.

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1 Introduction

The standard methodology of quantum dynamics, initially due to Dirac\ref{1}, assumes that a wave function can always be expanded into a series of eigenfunctions

\[ \sum C_k(t) e^{-i\omega_k t} W_k(r), \]

where \( W_k(r) \) represent a set of eigenfunctions, which is, according to the customary understanding, associated with the initial Hamiltonian \( H(t_0) \) of the quantum system. By inserting the expansion into the time-dependent Schrödinger equation and doing some mathematical manipulations, a set of coupled differential equations is obtained as

\[ \frac{ih}{\hbar} \frac{dC_k}{dt} = \sum_m C_m e^{i(\omega_k - \omega_m)t} V_{km}, \]

where \( V_{km} \) are the matrix elements of the Hamiltonian variation \( V(t) \equiv H(t) - H(t_0) \). A common concept, taught everywhere and established solidly in our mind, is that the expansion (1) is the formal solution of the wave function and the equation set (2) describes the system’s behavior as exactly as the original Schrödinger equation does. Many perturbative and nonperturbative approaches are based on or related to this concept. (As an example, see the derivation of the Liouville theorem in quantum statistics.)

Our studies, however, showed that the concept outlined above, while enjoying tremendous successes, involved serious difficulties\ref{2}\ref{3}. Some of them are the following.

- In numerical computations, the sum of squared moduli of all coefficients, \( \sum |C_k(t)|^2 \), can easily go to infinity if \( V(t) \) is not truly small. That is to say, the Dirac theory, like many other types of perturbation theories, suffers from the divergence difficulty.

- The notion associated with (1) and (2) asserts that if a quantum system, say a harmonic oscillator, is initially in the ground state, it will make transition, partially though, to a higher state at the next moment, and then to a higher higher state at the next next moment. The dynamical process will never terminate if \( H(t) - H(t_0) \) is nonzero as \( t \to \infty \) (another symptom of the divergence). In contrast with that, a general analysis of quantum mechanics states that whenever the Hamiltonian \( H(t) \) becomes steady, equal to \( H(t_0) \) or not, the system will instantly be settled in a stationary state.
According to the state-transition picture provided by the Dirac theory, a ground-state system will never lose its energy under any circumstances. This conclusion does not appear to be in harmony with other theoretical and experimental observations.

The fact that an electromagnetic field can be represented by different gauge fields implies that there are an infinite number of equation sets taking the form (2) for one definite disturbance. Direct numerical calculations show that these equation sets are not equivalent to each other.

Many questions then arise, of which some must be of fundamental interest to the community. How does the Dirac formalism, derived mathematically from the Schrödinger equation, involve so many difficulties? Why can the theory, running into problems ultimately, still offer good results under certain conditions? What on earth are these conditions? Is there a method by which the time-dependent Schrödinger equation can be solved more adequately? Among them, the last question seems to be the most essential one. If we find out a proper way of solving this fundamental equation, many questions, in particular those related to the Dirac perturbation theory, can be answered accordingly.

Although a different perturbation theory[3], which corresponded to a similar theory in classical mechanics[4] and suffered less problems, was put forward, the issue was far from cleared up and the aforementioned questions remained to be open.

In this paper we wish to report on a nonperturbative procedure of solving the time-dependent Schrödinger equation. Interestingly, the new procedure, called the multi-projection approach or phase dynamics of quantum mechanics, is still based on Dirac’s idea: solving a nonstationary quantum system with knowledge of stationary quantum systems. But, unlike its predecessor, this approach suffers from no divergence difficulty and exhibits surprising effectiveness. Any dynamical system, subject to weak or strong fields, becomes analyzable and calculable provided that the corresponding stationary states can be solved analytically or numerically.

The structure of this paper is the following. Section 2 introduces our new approach, which is based on an assumption that the time-dependent Hamiltonian can be approximated by its stepwise varying counterparts. In Sec. 3, similarities and dissimilarities of the proposed approach to the influential path-integral approach are remarked. Simple applications are presented.
Section 4, where effectiveness of the new approach is illustrated. Section 5 rederives the Dirac perturbation theory and answers several related questions. Sec. 6 gives discussions on the common concept of wavefunction expansion. Sec. 7 concludes the paper.

2 Multi-projection approach

In this section, we try to determine the wave function $\Psi(t)$ on the condition that the initial wave function $\Psi(t_0)$ and the time-dependent Hamiltonian $H(t)$ are given.

According to the basic formalism of quantum mechanics, the solution of the Schrödinger equation can be written as

$$
\Psi(t) = e^{-\frac{i}{\hbar} \int_{t_0}^{t} H(\tau) d\tau} \Psi(t_0).
$$

The solution above is just a formal one except in the situation where the involved Hamiltonian is independent of time. To generally evaluate it, we slice the time interval from $t_0$ to $t$ into $n$ segments, equal to each other or not, as

$$
\Delta t_1 = t_1 - t_0, \quad \ldots, \quad \Delta t_n = t - t_{n-1}.
$$

The formal solution (3) can then be rewritten as

$$
\Psi(t) = e^{-\frac{i}{\hbar} \int_{t_{n-1}}^{t} H(\tau) d\tau} \ldots e^{-\frac{i}{\hbar} \int_{t_1}^{t_1} H(\tau) d\tau} \Psi(t_0).
$$

Or, in terms of the intermediate quantum states,

$$
\Psi(t_j) = e^{-\frac{i}{\hbar} \int_{t_{j-1}}^{t_j} H(\tau) d\tau} \Psi(t_{j-1}) \quad (j = 1, 2, \ldots).
$$

Referring to Fig. 1, we explore the possibility to replace the Hamiltonian $H(t)$ with its stepwise varying approximation $\hat{H}(t)$. Without losing generality, consider a charged particle in a time-dependent electromagnetic field, whose Hamiltonian reads

$$
H(t) = \frac{1}{2m} (p - \frac{Q}{c} A)^2 + \Phi.
$$

It is rather obvious that if $\Phi \neq 0$ and $A = 0$, the replacement of $H(t)$ with $\hat{H}(t)$ is justified by the observation that the two Hamiltonians represent roughly the same physical system. For the case in that both the
scalar potential $\Phi$ and the vector potential $A$ are nonzero, the situation becomes somewhat complicated. It is well-known, the vector potential $A$ can be separated into two parts: the longitudinal field and the transverse field [5]. By realizing that the longitudinal vector field can cause trouble to our formalism (the reason for saying it will be clear), only the scalar field and transverse vector field will be treated herein. This should be allowable in view of that an appropriate gauge transformation can always make the longitudinal vector field vanish. Under these understandings, we define the Hamiltonian $\hat{H}(t)$ as

$$\hat{H}_j(t) = \frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} H(t) dt \quad \text{for} \quad (t_{j-1} < t < t_j); \quad (8)$$

or, even in a simpler way, $\hat{H}_j(t) = [H(t_{j-1}) + H(t_j)]/2$. Thus, during each of the time segments defined by $\Delta t_j$, the new Hamiltonian $\hat{H}_j$ is independent of time and the typical intermediate state in (6) becomes

$$\Psi(t_j) \approx e^{-\frac{i}{\hbar} \hat{H}_j \Delta t_j} \Psi(t_{j-1}). \quad (9)$$

The problem is reduced to a familiar one related to solving stationary systems. For each of the intermediate Hilbert space associated with $\hat{H}_j$, we have normalized intermediate energy eigenfunctions

$$\hat{H}_j W^j_k(r) = E^j_k W^j_k(r). \quad (10)$$

After the wave function $\Psi(t_{j-1})$ is known, the wave function $\Psi(t_j)$ can be expressed by

$$\Psi(t_j) = \sum C^j_k e^{-i\omega^j_k \Delta t_j} W^j_k(r), \quad (11)$$

where $\omega^j_k = E^j_k / \hbar$ and $C^j_k$ is determined by a projection

$$C^j_k = \int \Psi(t_{j-1}) W^j_k(r)^* dr. \quad (12)$$

Or, in the Dirac notation,

$$|t_j \rangle = \sum_{k_j} \langle k_j | t_{j-1} \rangle e^{-i\omega^j_k \Delta t_j} |k_j \rangle, \quad (13)$$

where $|k_j \rangle$ stands for $W^j_k$. Eq. (13) shows that for a short time interval a dynamical system and its corresponding stationary system evolve in the
same way. By repeating the step as presented above, the wave function at
the time \( t \) can be expressed as

\[
|t⟩ = ∑_{k_n} ⋯ ∑_{k_1} e^{-i ∑_j \omega_j \Delta t_j} |k_n⟩⟨k_n|k_{n-1}⟩ ⋯ ⟨k_1|t_0⟩.
\]

(14)

For convenience of discussion, we will name \( |k_n⟩⟨k_n|k_{n-1}⟩ ⋯ ⟨k_1|t_0⟩ \) as a
minute component of \( |t_0⟩ \), by which we refer to the fact that such a compo-
nent is in usual situations very small. It is easy to prove that the sum of all
the minute components, with no phase factors involved, is just equal to the
initial wavefunction.

In deriving the above formalism, it has been assumed that the wave
function \( Ψ(t < t_0) \) does not contain any phase factor in the form \( e^{if(t,r)} \)
where \( f \) depends on \( t \) and \( r \) explicitly. Generally speaking, \( e^{if(t,r)} \) is not a
uniformly continuous function with respect to \( t \) and \( r \) and great difficulties
will be encountered if we try to expand a wavefunction having such phase
factor. A related and important point is that since we have assumed that all
\( \hat{H}_j \) contain no longitudinal vector fields, all the intermediate wave functions
are also free from such phase factors.

Numerically speaking, the accuracy of this method largely depends on
the step size. If all steps in a calculation approach infinitesimal ones, the
resultant wavefunction will approach the real dynamical one.

For a dynamical system, the energy at any moment can naturally be
defined by the intermediate Hamiltonian and the intermediate wavefunction.
In the sense of taking limit, this definition is an accurate one.

Before finishing this section, we wish to point out that this approach of-
fers a different picture about quantum dynamics. The Dirac theory leads one
to imagine how a nonstationary system makes transition from one eigenstate
to another. In contrast with that, the multi-projection approach reveals
that a nonstationary system is subjected to no other change than that each
minute component of the wavefunction acquires its own dynamical phase
factor. To be consistent with the spirit of enlightening discussions about
Berry’s phase\(^6\), the obtained formalism may be called phase dynamics of
quantum mechanics.

### 3 Equivalence to the path-integral approach

In this section, the path-integral approach\(^6\)\(^8\) is briefly reviewed and then
the relation between the path-integral approach and the multi-projection
approach is remarked.
In the path-integral approach we have

\[ \Psi(x, t) = \int \langle x, t | x_0, t_0 \rangle \Psi(x_0, t_0) dx_0, \] (15)

where \( \langle x, t | x_0, t_0 \rangle \) is named as the propagator. (For simplicity, only the one-dimensional case is considered.) It is found that the propagator takes the form

\[ \langle x, t | x_0, t_0 \rangle = \int_{x_0}^{x} e^{iS[x(\tau)]/\hbar} D[x(\tau)], \] (16)

where \( D[x(\tau)] \) is the measure associated with all possible paths \( x(\tau) \) and the action \( S \) is the integral of the Lagrangian along the path

\[ S = \int_{t_0}^{t} L(\tau)d\tau = \int_{t_0}^{t} \left[ \frac{1}{2}mv^2 - V(x(\tau), \tau) \right] d\tau. \] (17)

The formalism above is quite formal unless the discrete form of it is written out. By slicing the time \((t - t_0)\) into \(N\) equal segments and denoting \((t - t_0)/N\) by \(\varepsilon\), the action integral becomes

\[ S = \sum_{j=0}^{N-1} \left[ \frac{m(x_{j+1} - x_j)^2}{2\varepsilon} - \varepsilon V(x', \varepsilon') \right] \] (18)

where \(x'\) and \(\varepsilon'\) take values within the intervals \((x_j, x_{j+1})\) and \((t_j, t_{j+1})\) respectively. Upon this, the propagator becomes

\[ \lim_{N \to \infty} A \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \frac{i}{\hbar} \sum_{j=0}^{N-1} \left[ \frac{m(x_{j+1} - x_j)^2}{2\varepsilon} - \varepsilon V(x', \varepsilon') \right] dx_1 \cdots dx_{N-1}, \] (19)

where the factor \(A\) can be determined by the limit \(\langle x, \varepsilon | x_0, 0 \rangle \to \delta(x - x_0)\).

For purposes of this paper, we wish to take a look at how the path-integral formalism yields the standard Schrödinger equation. Consider the propagator associated with one slice of time after \(t = 0\)

\[ \langle x, \varepsilon | x_0, 0 \rangle = \left( \frac{m}{2\pi\hbar\varepsilon} \right)^{1/2} \exp \left\{ \frac{i}{\hbar} \left[ \frac{m(x - x_0)^2}{2\varepsilon} - \varepsilon V(x', \varepsilon') \right] \right\}, \] (20)

which means

\[ \Psi(x, \varepsilon) = \left( \frac{m}{2\pi\hbar\varepsilon} \right)^{1/2} \times \int_{-\infty}^{\infty} \exp \left\{ \frac{i}{\hbar} \left[ \frac{m(x - x_0)^2}{2\varepsilon} - \varepsilon V(x', \varepsilon') \right] \right\} \Psi(x_0, 0) dx_0. \] (21)
By noting that the region \((x - x_0)^2 \sim 2\varepsilon \hbar \pi / m\) gives the main contribution to the integral, we obtain, after taking approximations,

\[i\hbar \frac{\Psi(x, \varepsilon) - \Psi(x, 0)}{\varepsilon} \approx \left[-\frac{\hbar^2 \partial^2}{2m \partial x^2} + V(x, \varepsilon')\right] \Psi(x, 0),\]

which is just the finite differential form of the Schrödinger equation.

The similarities between the path-integral and the multi-projection are found by the following observations. (i) Both approaches acquire clear meaning in their discrete forms. (ii) Both approaches are approximate theories in their discrete forms and convergent to the exact theory as steps of the discrete forms become infinitesimal. (iii) For a short time interval, both approaches show indifference to whether or not the quantum system is truly time-dependent. In the path-integral approach outlined in this section, if we replace the Lagrangian with its stepwise varying approximation, all the formulas, in particular (20), (21) and (22), make no essential change; as the steps go smaller and smaller, exactly the same limits will be obtained. All these imply that the equivalence of these two approaches is indeed there.

It is now worth mentioning dissimilarities. The path-integral approach formulates the time unitary transformation of wave function in the ordinary spatial space, whereas the multi-projection approach does the same job between a series of Hilbert spaces whose bases are formed by eigenfunctions of the intermediate Hamiltonians. Due to this difference, the formalism in the path-integral approach takes a form of probability integral, while the formalism in the multi-projection approach takes a form of a series of definite projections. In terms of studying practical systems, it is expected that the two provide different computational ease.

Finally, a note about gauge. It is rather necessary for both the approaches to assume no longitudinal vector field to exist.

4 Simple applications

Some concrete examples, in which effectiveness of the multi-projection approach manifests itself, are presented in this section.

Firstly, consider a case in that the potential of a harmonic oscillator is subject to a “sudden” change and the Hamiltonian reads

\[H(t) = \frac{p^2}{2m} + \frac{S(t)}{2} kx^2,\]
with

\[
S(t) = \begin{cases} 
1 & (t \leq 0) \\
\eta & (t > 0), 
\end{cases}
\]

(24)
in which \(\eta\) is a positive constant other than 1. Physical intuition says that, if \(\eta > 1\) the total energy of the system must go larger (the spatial room for the oscillator is “compressed”); if \(\eta < 1\) the total energy must go smaller.

Though the situation given above is terribly simple, it still defines a dynamical system that needs to be solved in one way or another. As indicated in the introduction, if the standard method is applied serious difficulties arise. In what follows we try to use our multi-projection approach.

Assume that the system is initially in the ground state, namely

\[
\Psi(0) = N_0(\alpha) \exp(-\alpha^2 x^2 / 2),
\]

(25)
and the wave function after \(t = 0\) takes the form

\[
\Psi(t) = \sum_n C_n e^{-i(n+\frac{1}{2})\omega' t} N_n(\alpha') H_n(\alpha' x) e^{-\frac{1}{2} \alpha'^2 x^2}.
\]

(26)
The notation in Eqs. (25) and (26) is rather standard: \(\alpha = (mk/\hbar^2)^{1/4}\), \(\alpha' = \sqrt{\eta} \alpha\), \(\omega' = \sqrt{\eta} \omega = \sqrt{\eta k/m}\) and

\[
N_n(\alpha) = \left(\frac{\alpha}{\sqrt{\pi} 2^n n!}\right)^{1/2}, \quad H_{n+1}(\xi) = 2\xi H_n - H_{n-1} \text{ (with } H_0 = 1).\]

The energy of the system at \(t = 0\) is known to be \(E_0 = 0.5\hbar \omega\). By making use of (12), we obtain, for \(\eta = 0.5^2\),

\[
C_0 = \left(\frac{8}{9}\right)^{\frac{1}{4}}, \quad C_1 = 0, \quad C_2 = -\frac{2}{3} \left(\frac{1}{72}\right)^{\frac{1}{4}}, \quad C_3 = 0 \ldots.
\]

(27)
Numerically, we have

\[
C_0 \approx 0.9710, \quad C_2 \approx -0.2289, \quad C_4 \approx 0.0661, \quad C_6 \approx -0.0201;
\]

(28)
these imply \(\sum |C_n|^2 \approx 1\) and the final energy becomes

\[
\langle E \rangle = \sum |C_n|^2 (n + 0.5)\hbar \omega' \approx 0.6246 E_0.
\]

(29)
Similarly, we obtain, for \(\eta = 0.9^2\) and \(\eta = 1.1^2\) respectively,

\[
\langle E \rangle \approx 0.9050 E_0 \quad \text{and} \quad \langle E \rangle \approx 1.1050 E_0.
\]

(30)
It is now obvious that with the multi-projection approach (i) the normalization condition for wave function, namely $\sum |C_n|^2 = 1$, is automatically satisfied; (ii) whenever the Hamiltonian becomes steady the system is settled in a stationary state; (iii) even if a quantum system is initially in a ground state, the total energy of it can decrease or increase; (iv) no gauge issues pose problems if we let the longitudinal vector potential vanish always. In other words, all the difficulties troubling the standard dynamical theory disappear completely.

The following similar example shows that the proposed method is a good mean for evaluating the phase shift of a quantum system. If the oscillator expressed by (23) is again in the ground state at $t = 0$ and the function $S(t)$ in (23) is, instead of (24),

$$S(t) = \begin{cases} 1 & (t \leq 0) \\ \eta & (0 < t < T) \\ 1 & (t \geq T), \end{cases}$$

the formulation of the above example can still be employed except that a projection at $t = T$ onto the original Hilbert space is needed. In general, the final wave function involves energy transition and phase shift. As a special case, we consider the situation in which $T = 4\pi/\omega'$. Simple calculation tells us that at $t = T$ the wave function will come back to its original value and original phase (by virtue of $e^{i2\pi} = 1$) as if the system has been completely “frozen”. But, on the other hand, one finds that if the system gets no disturbed, it will during the same time acquire the phase factor

$$\exp(-i\omega T/2) = \exp(-i2\pi/\sqrt{\eta}).$$

(32)

It is then obvious that the disturbance makes the wavefunction have the additional phase factor $\exp(i2\pi/\sqrt{\eta})$.

An inspection of the last example states that if a perturbation is capable of changing the system’s eigenfrequencies, it can in general make the wavefunction have an additional phase shift, irrespective of whether or not there is energy transition. As shown by experiments, such phase shift has physical effects and should be treated with care.

5 Rederivation of the Dirac perturbation theory

In this section we rederive the Dirac perturbation theory with help of our proposed approach. The purpose of doing that is two folds. One is to
illustrate the analytical ability of the approach; and the other is to answer questions related to the Dirac perturbation theory.

To obtain an analytical formalism comparable to the Dirac one, we presuppose that (i) the quantum system of interest is initially in the \( n \)th eigenstate of \( H_0 \), the perturbation applies at \( t = 0 \) and vanishes completely at \( t = T \); (ii) the perturbation can be approximated by a series of pulses whose values rise sharply and vanish sharply, as illustrated in Fig. 2 (the pulse-like perturbation has the same physical effects as the real perturbation); (iii) the system’s wavefunction involves no additional phase-factor shift before and after each of the pulses. With these assumptions adopted, the derivation here can be deemed as a simple application of the “sudden approximation” due to Pauli.[11]

Consider one specific pulse of the perturbation that exists between \( t' \) and \( t' + \Delta t' \). The leading term of the wave function at \( t = t' \) is \( e^{-i\omega_n t'}|n\rangle \), at \( t = T \) it becomes, by repeated use of (13),

\[
e^{-i\omega_n t'}|n\rangle \rightarrow \sum_k e^{-i\omega_n t'} e^{-i\omega_k \Delta t'} \langle k|n\rangle|k\rangle
\]

\[
\rightarrow \sum_{k,m} e^{-i\omega_n t'} e^{-i\omega_k \Delta t'} e^{-i\omega_m (T-t'-\Delta t')} \langle m|k\rangle \langle k|n\rangle|m\rangle,
\]

where \( |m\rangle \) stand for eigenfunctions defined by the Hamiltonian \( H_0 \) and \( |k\rangle \) eigenfunctions defined by the intermediate Hamiltonian within the pulse. Note that in (33) all the phase factors take on their unperturbed values before \( t' \) and after \( t' + \Delta t' \) as if no other pulses exist, which should in general be regarded as a rough approximation, as revealed in the last section. Since \( \Delta t' \) is short, the following approximation is acceptable

\[
\sum_k \langle m|k\rangle e^{-i(\omega_k - \omega_m) \Delta t'} \langle k|n\rangle \approx \langle m|1 - \frac{i}{\hbar} \left[H(t') - H_0\right] \Delta t'|n\rangle,
\]

thus, the \( m \)th coefficient of the wave function at \( t = T \) is

\[
b_m = -\frac{i}{\hbar} \Delta t' \langle m|V|n\rangle e^{-i(\omega_n - \omega_m)t'} e^{-i\omega_m T}, \quad (m \neq n)
\]

(35)

where \( V \equiv H(t') - H_0 \). Taking contributions from all pulses into account, we obtain at \( t = T \)

\[
b_m = -\frac{i}{\hbar} e^{-i\omega_m T} \int_0^T V_{mn} e^{-i(\omega_n - \omega_m)t'} dt' \quad (m \neq n).
\]

(36)
Except the phase factor $\exp(-i\omega_m T)$, the formula above is consistent with the well-known one.

Unlike the standard derivation in textbooks, the present derivation clears up several subtle and important things, of which some were addressed in the introduction. Firstly, it indicates that if the final Hamiltonian $H(t > T)$ is not the same as the initial Hamiltonian, this formalism will assume that “perturbation pulses” are always there and thus yield misleading results. Secondly, it stresses that if the perturbation, represented by the Hamiltonian variation $V(t)$, is not relatively small or if the action time of the perturbation is not relatively short, the leading term of the wavefunction, including its phase factor, will in general be disturbed significantly and the accuracy of the formalism will partly or entirely be demolished. Thirdly, it suggests that higher-order solutions directly given by the Dirac perturbation theory are not meaningful. Finally, it implies that in order for the formalism to hold, an appropriate gauge, like the one defined in Sec. 2, needs to be adopted.

6 Discussion on the wavefunction expansion

Let us return to the subject put forth at the very beginning. What have been presented in this paper, together with those in Ref. 2, actually bring out that the usual concept about the wavefunction expansion bears problematic aspects. The formulation in Sec. 2 clearly shows that after a quantum system leaves its initial state, the eigenfunctions and eigenfrequencies associated with the initial Hamiltonian become out-of-date. If we forcefully use them to express the wavefunction at the later times, the coefficients of the would-be series have to adjust themselves violently—so violently that they cannot be regarded as continuous variables in the usual sense. If the aim of a calculation is to determine these coefficients continuously, the situation will soon become out of control: the more terms are taken into account, the faster temporal scales and the larger spatial scales get involved, the bigger and more serious errors thus enter the scheme.

In other words, although one can, at any fixed moment, expand a dynamical wavefunction into a series of the initial eigenfunctions as, up to a phase factor,

$$\sum C_n(t)W_n(r) \quad \text{or} \quad \sum C_n(t)e^{-i\omega_n t}W_n(r),$$

(37)

it is not appropriate to assume that a dynamical wavefunction can be represented by an expansion that takes the form (37) and has continuously time-
varying coefficients.

The arguments that have just been presented unveil, in another perspective, that the concepts introduced by this approach—intermediate Hilbert spaces, intermediate eigenfrequencies and discrete projections between intermediate Hilbert spaces—are more fundamental than they appear to be. (Also, it justifies the path-integral approach, whose objective is to formulate a wavefunction at certain discrete moments.)

7 Summary

We have proposed a nonperturbative procedure, called the multi-projection approach, which in a way unifies treatments of different quantum systems: stationary and nonstationary, weakly-disturbed and strongly-disturbed. In view of that knowledge about stationary systems has been accumulated for long and many systems with strong fields need to be studied, such unification appears to be quite desirable.

The equivalence of the multi-projection approach to the path-integral approach also suggests that the proposed method may find its applications in a variety of quantum fields, though it is still too early to say what exactly they are.

This paper has shown that a dynamical process of quantum system is, rather strikingly, characterized by the phase-factor evolution of each minute component of the wavefunction. This “state transition” picture is quite different from the standard one provided by the Dirac perturbation theory. The standard picture, associated usually with one fixed Hilbert space and paying much less attention to disturbance of phase factors, is applicable only if some conditions are simultaneously satisfied, which in general include: the final Hamiltonian of the system is the same as the initial Hamiltonian, the perturbation is relatively small, and the action time of the perturbation is relatively short.

We have pointed out that a dynamical wavefunction cannot be represented by a wavefunction expansion that has continuously time-varying coefficients.

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References

[1] P. A. Dirac, *The principles of Quantum Mechanics*, (Oxford, 1958). Also see, for instance, L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, (Addison-Wesley, 1958) and E. Merzbacher, *Quantum Mechanics*, 2nd ed., (Interscience, 1974).

[2] C. Y. Chen, *A puzzle of quantum dynamics and Legitimacy of wave-function expansion*, to be published. In these works, the legitimacy of wave-function expansion is analyzed. In particular, it is pointed out that almost all wave-function expansions are not uniformly convergent and cannot be generally differentiated or integrated term by term.

[3] C. Y. Chen, *Perturbation Methods and Statistical Theories*, in English, (International Academic Publishers, Beijing, 1999).

[4] C. Y. Chen, Phys. Rev. E, 47, 763 (1993).

[5] See, for instance, J. D. Jackson, *Classical Electrodynamics*, p222, 2nd ed., (John Wiley and Sons, 1975).

[6] M. V. Berry, Proc. R. Soc. London, Ser. A 392, 45, (1984).

[7] R. P. Feynman and A. R. Hibbs, *Path integrals and quantum mechanics*, (McGraw-Hill, 1965).

[8] See, for instance, R. Shankar, *Principles of quantum mechanics*, (Plenum Press, 1980); also E. R. Harris, *Introduction to modern theoretical physics*, (John Wiley and Sons, 1975).

[9] See, for instance, L. I. Schiff, *Quantum Mechanics*, 3rd ed., (McGraw-Hill, 1968).

[10] See, for instance, A. Tomita and R. Y. Chiao, Phys. Rev. Lett., 57, 937 (1986).

[11] W. Pauli, *Handbuch d. Physik*, 24, part 1, 164 (1933); See also Ref. 8.
Fig. 1, A typical time-dependent Hamiltonian and its stepwise-varying approximation.

Fig. 2, A Hamiltonian variation and its pulse-like approximation.