A LOOP GROUP METHOD FOR DEMOULIN SURFACES IN THE 3-DIMENSIONAL REAL PROJECTIVE SPACE

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ABSTRACT. For a surface in the 3-dimensional real projective space, we define a Gauss map, which is a quadric in $\mathbb{R}^4$ and called the first-order Gauss map. It will be shown that the surface is a Demoulin surface if and only if the first-order Gauss map is conformal, and the surface is a projective minimal coincidence surface or a Demoulin surface if and only if the first-order Gauss map is harmonic. Moreover for a Demoulin surface, it will be shown that the first-order Gauss map can be obtained by the natural projection of the Lorentz primitive map into a 6-symmetric space. We also characterize Demoulin surfaces via a family of flat connections on the trivial bundle $\mathbb{D} \times \text{SL}_4\mathbb{R}$ over a simply connected domain $\mathbb{D}$ in the Euclidean 2-plane.

INTRODUCTION

Curves and surfaces in the 3-dimensional real projective space $\mathbb{P}^3$ were the central theme of differential geometry in 19th century. Especially, various transformations for a surface in $\mathbb{P}^3$ were introduced by Darboux, Demoulin, Titzeica, Godeaux, Rozet, Wilczynski, etc., and their properties were extensively studied. The most prominent features of the theory of transformations were the Laplace sequence and the line/sphere congruences of a surface. It is well known that Toda equations which discovered in the theory of integrable systems in 1970s had been already known as the periodic Laplace sequence, and the classical Darboux and Bäcklund transformations were defined by sphere congruences and tangential line congruences, respectively.

The Demoulin surface is characterized by the coincidence of general four Demoulin transformations of a surface, which are given by the envelopes of Lie quadrics. On the one hand, the projective minimal surface is defined by a critical point of the projective area functional. It is known that Demoulin surfaces give a special class of projective minimal surfaces. Moreover, using Plücker embedding from $\mathbb{P}^3$ to $\mathbb{P}^5$, Godeaux introduced an analogue of the Laplace sequence, the so-called Godeaux sequence of a surface in $\mathbb{P}^5$. Then the surface is a Demoulin surface if and only if the Godeaux sequence is six periodic. For the modern treatment of the subject, we refer the readers to [12].

By using modern theory of integrable systems and differential geometry of harmonic maps, projective minimal surfaces and Demoulin surfaces were investigated in [8, 9, 3]. More precisely in [3], through Plücker embedding from $\mathbb{P}^3$ to $\mathbb{P}^5$ projective minimal surfaces were
characterized by Lorentz harmonicity of the conformal Gauss map, which takes values in a certain indefinite Grassmannian. In [9], Demoulin surfaces were characterized by a certain Toda equation and the Bäcklund transformation of a Demoulin surface was constructed. Moreover, many classes of surfaces characterized by geometric properties were related to various integrable systems in [8].

In this paper, we study Demoulin surfaces via a loop group method. We first define a Gauss map for a surface in \( \mathbb{P}^3 \), which is a quadric in \( \mathbb{R}^4 \), and called the \textit{first-order Gauss map}. The first-order Gauss map has the first-order contact to the surface. It will be shown that the first-order Gauss map is conformal if and only if the surface is a Demoulin surface, see Proposition 2.1.

Then the Lorentz harmonicity of the first-order Gauss map is studied. It will be shown that the first-order Gauss map is Lorentz harmonic if and only if the surface is a Demoulin or a projective minimal coincidence surface, see Theorem 2.2. We note that coincidence surfaces are simple examples of a class of surfaces which have nontrivial projective deformations, the so-called \textit{projective applicable surfaces}. Since the target space of the first-order Gauss map is a symmetric space, the Lorentz harmonic map is also characterized by a family of flat connections on the trivial bundle \( \mathbb{D} \times \text{SL}_4 \mathbb{R} \).

Combining the results in Proposition 2.1 and Theorem 2.2, we see that the first-order Gauss map is conformal Lorentz harmonic if and only if the surface is a Demoulin surface, see Corollary 2.4. Finally it will be shown that the Gauss map of a Demoulin surface can be obtained by the natural projection of the Lorentz primitive map into a 6-symmetric space, see Theorem 2.5.

In Appendix A, we review results of Thomsen [13], that is, a surface is projective minimal if and only if the conformal Gauss map is conformal Lorentz harmonic. In [3, Theorem 7], by using Plücker embedding from \( \mathbb{P}^3 \) to \( \mathbb{P}^5 \), the conformal Gauss map can be considered as the map into a certain indefinite Grassmannian in \( \mathbb{R}^6 \) and the Thomsen’s theorem was proved. Theorem A.2 is another reformulation of it.

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1. Preliminaries

1.1. Surfaces in \( \mathbb{P}^3 \) and the Wilczynski frames. The \textit{canonical system} of a surface \( S \) in the 3-dimensional real projective space \( \mathbb{P}^3 \) is given as follows. [14], [12, Section 2.2]:

\[
(1.1) \quad f_{xx} = bf_y + pf, \quad f_{yy} = cf_x + qf,
\]

where \( f \) is a lift of \( S \) in \( \mathbb{R}^4 \setminus \{0\} \), \( b, c, p \) and \( q \) are functions of real variables \( x \) and \( y \), and the subscripts \( x \) and \( y \) denote the partial derivative with respect to \( x \) and \( y \), respectively. Let \( f = (f^0, f^1, f^2, f^3)^t \in \mathbb{R}^4 \setminus \{0\} \) and assume that \( f^0 \neq 0 \). Then the surface \( S \) is given by

\[
S_{xx} = bS_y - 2(\log f^0)_x S_x \quad \text{and} \quad S_{yy} = cS_x - 2(\log f^0)_y S_y.
\]
This implies that \( x \) and \( y \) are asymptotic coordinates on \( S \). Thus the coordinates \((x, y)\) induce the Lorentz structure on the surface \( S \). It is known that \( 8bc \, dx \, dy \) is an absolute invariant symmetric quadratic form, which is called the projective metric and \( 8bc \) is called the Fubini-Pick invariant of a surface \( S \). It is also known that the conformal class of \( b \, dx^3 + c \, dy^3 \) is an absolute invariant cubic form. It is known that a surface whose Fubini-Pick invariant \( 8bc = 0 \) is ruled, thus we assume that \( bc \neq 0 \). Then the Wilczynski frame is defined as follows:

\[
F = (f, f_1, f_2, \eta),
\]

where

\[
f_1 = f_x - \frac{c_x}{2c} f, \quad f_2 = f_y - \frac{b_y}{2b} f, \quad \eta = f_{xy} - \frac{c_x}{2c} f_y - \frac{b_y}{2b} f_x + \left( \frac{b_y c_x}{4bc} - \frac{bc}{2} \right) f.
\]

Then a straightforward computation shows that the Wilczynski frame \( F \) satisfies the following equations:

\[
F_x = FU \quad \text{and} \quad F_y = FV,
\]

where

\[
U = \begin{pmatrix}
\frac{c_x}{2c} & P & k & bQ \\
1 & -\frac{b_y}{2bc} & 0 & k \\
0 & b & \frac{c_x}{2c} & P \\
0 & 0 & 1 & -\frac{c_x}{2c}
\end{pmatrix}, \quad V = \begin{pmatrix}
\frac{b_y}{2b} & \ell & Q & cP \\
0 & \frac{b_y}{2b} & c & Q \\
1 & 0 & -\frac{b_y}{2b} & \ell \\
0 & 1 & 0 & -\frac{b_y}{2b}
\end{pmatrix}.
\]

Here we introduced functions \( k, \ell, P \) and \( Q \) of two variables \( x \) and \( y \) as follows:

\[
k = \frac{bc - (\log b)_{xy}}{2}, \quad \ell = \frac{bc - (\log c)_{xy}}{2},
\]

\[
P = p + \frac{b_y}{2} \frac{c_{xx}}{2c} + \frac{c_x^2}{4c^2}, \quad Q = q + \frac{c_x}{2} - \frac{b_y}{2b} + \frac{b_y^2}{4b^2}.
\]

The compatibility conditions of (1.2) are

\[
Q_x = k_y + k \frac{b_y}{b}, \quad P_y = \ell_x + \ell \frac{c_x}{c},
\]

\[
bQ_y + 2b_y Q = cP_x + 2c_x P.
\]

These equations are nothing but the projective Gauss-Codazzi equations of a surface \( S \). Since the traces of \( U \) and \( V \) are zero, the Wilczynski frame \( F \) takes values in \( SL_4\mathbb{R} \) up to initial condition. From now on, we assume that the Wilczynski frame \( F \) takes values in \( SL_4\mathbb{R} \).

Remark 1.1. Instead of real coordinates \((x, y)\), one can use the complex coordinates \((z, \bar{z})\) with \( z = x + iy \). Then the induced conformal structure is Riemannian and the following discussion is parallel to the case of real coordinates. However, for simplicity, we consider only the case of real coordinates.
1.2. **Projective minimal surfaces and Demoulin surfaces.** It is known that the projective minimal surface is defined by a critical point of the projective area functional:

\[ \int bc \, dx \, dy, \]

where the functions \( b \) and \( c \) are defined in (1.1). Then the projective minimality can be computed as in [13]:

\[ bQ_y + 2b_y Q = 0 \quad \text{and} \quad cP_x + 2c_x P = 0, \]

where the functions \( P \) and \( Q \) are defined in (1.5).

The Demoulin surface is defined by the coincidence of general four Demoulin transformations of a surface, which are given by the envelopes of Lie quadrics. It is known that Demoulin surfaces are characterized by the functions \( P \) and \( Q \) in (1.5), see [12, Definition 2.8]:

\[ P = Q = 0. \]

**Remark 1.2.** From the equations in (1.8) and (1.9), it is easy to see that Demoulin surfaces are projective minimal surfaces.

1.3. **(Lorentz) Harmonic and (Lorentz) primitive maps into \((k\)-symmetric spaces.**

It is known that the loop group method can be applied to harmonic maps from surfaces into symmetric spaces, see [4, 6]. Let \( M \) and \( N \) be a Riemann (or Lorentz) surface and a semisimple symmetric space, respectively and \( \varphi \) a map from \( M \) into \( N \). We denote the symmetric space \( N \) as quotient \( G/K \) with semisimple Lie group \( G \) and closed subgroup \( K \) of \( G \) such that \( (G_\sigma)_o \subseteq K \subset G_\sigma \), where \( (G_\sigma)_o \) is the identity component of the fixed point group \( G_\sigma \) of the involution \( \sigma \) of the symmetric space \( N \). Let \( \Phi \) be the frame of \( \varphi \) taking values in \( G \) and \( \alpha = \Phi^{-1} d\Phi \) the Maurer-Cartan form. According to the eigenspace decomposition of \( g \) with respect to the derivative of \( \sigma \), that is \( g = \mathfrak{k} \oplus \mathfrak{p} \), we define \( \alpha^\lambda \) as follows:

\[ \alpha^\lambda = \alpha_\mathfrak{k} + \lambda^{-1} \alpha_\mathfrak{p}' + \lambda \alpha_\mathfrak{p}'' , \quad \lambda \in \mathbb{C}^\times, \]

where \( \alpha_\mathfrak{k} \) and \( \alpha_\mathfrak{p} \) denote the \( \mathfrak{k} \)- and \( \mathfrak{p} \)-parts, and \( \mathfrak{t} \) and \( \mathfrak{u} \) denote the \((1,0)\)- and \((0,1)\)-parts, respectively.

**Remark 1.3.** For a Riemann surface \( M \) with conformal coordinates \( z = x + iy \), the \((1,0)\)- and \((0,1)\)-parts denote \( dz \) and \( d\bar{z} \) parts, respectively, and for a Lorentz surface \( M \) with null coordinates \( (x, y) \), the \((1,0)\)- and \((0,1)\)-parts denote \( dx \) and \( dy \) parts, respectively.

The following theorem is a fundamental fact about (Lorentz) harmonic maps from surfaces into symmetric spaces, see [4, 6].

**Theorem 1.4.** Let \( M \) be a Riemann (or Lorentz) surface and \( N \) a semisimple symmetric space. A map \( \varphi : M \to N \) is a (Lorentz) harmonic map if and only if \( d + \alpha^\lambda \) is a family of flat connections.

If the target manifold \( N \) is a semisimple \( k \)-symmetric space \((k > 2)\), then there does not exist a loop group formulation for general (Lorentz) harmonic maps from a surface into \( N \) as the above. Instead, we restrict our attention to a rather special kind of (Lorentz) harmonic maps, the (Lorentz) primitive maps in a \( k \)-symmetric space, so that the loop group formulation can be applied.
Definition 1. Let \( \varphi \) be a map from a Riemann or Lorentz surface \( M \) into a semisimple \( k \)-symmetric space \( N = G/K \) with the order \( k \) automorphism \( \sigma \) \((k > 2)\) and \( \alpha = \Phi^{-1}d\Phi \) the Maurer-Cartan form of the frame \( \Phi \) of \( \varphi \). Moreover, let \( g = g_0 \oplus g_1 \oplus g_2 \oplus \cdots \oplus g_{k-1} \) be the eigenspace decomposition of \( g \) and \( g^C = g_0^C \oplus g_1^C \oplus g_2^C \oplus \cdots \oplus g_{k-1}^C \) the eigenspace decomposition of the complexification of \( g \) according to the derivative of \( \sigma \) and define \( g_{i+kn} = g_i^C \) for \( n \in \mathbb{Z} \). For the case of Riemann surface \( M \), \( \varphi \) is called the primitive map if

\[
(1.10)\quad \alpha' \text{ takes values in } g_0^C \oplus g_{-1}^C,
\]

where \( t \) is the \((1,0)\)-part with respect to the conformal structure on the Riemann surface \( M \). For the case of Lorentz surface \( M \), \( \varphi \) is called the Lorentz primitive map if

\[
(1.11)\quad \alpha' \text{ takes values in } g_0 \oplus g_{-1}, \quad \text{and } \alpha'' \text{ takes values in } g_0 \oplus g_1,
\]

where \( t \) and \( t' \) are the \((1,0)\)- and \((0,1)\)-parts with respect to the conformal structure on the Lorentz surface \( M \), respectively.

The following is a basic fact about (Lorentz) primitive maps, see [1].

Proposition 1.5.

1. A (Lorentz) primitive map into a semisimple \( k \)-symmetric space \( N \) \((k > 2)\) is (Lorentz) equiharmonic, that is, it is (Lorentz) harmonic with respect to any invariant metric on \( N \).
2. Let \( \varphi \) be a (Lorentz) primitive map into a semisimple \( k \)-symmetric space \( N = G/K \), \((k > 2)\), and \( \pi : N \to G/H \) with \( K \subset H \) the homogeneous projection. Then \( \pi \circ \varphi \) is (Lorentz) equiharmonic.

Let \( \varphi \) be a primitive map into a semisimple \( k \)-symmetric space \( N \) \((k > 2)\) and \( \Phi \) the corresponding frame. Moreover, let \( \alpha \) be the Maurer-Cartan form of \( \Phi \), \( \alpha = \Phi^{-1}d\Phi \). Define \( \alpha^\lambda \) as follows:

\[
\alpha^\lambda = \alpha_0 + \lambda^{-1}\alpha'_{-1} + \lambda\alpha''_1, \quad \lambda \in \mathbb{C}^x,
\]

where \( \alpha_j \) is the \( j \)-th eigenspace of the derivative of \( \sigma \), \((j = -1, 0, 1)\). The following is a well known fact, see for example, [1].

Theorem 1.6. Let \( M \) be a Riemann (or Lorentz) surface and \( N \) a semisimple \( k \)-symmetric space \((k > 2)\). If a map \( \varphi : M \to N \) is a (Lorentz) primitive map then \( d + \alpha^\lambda \) is a family of flat connections.

2. Projective minimal coincidence surfaces and Demoulin surfaces

2.1. The first-order Gauss map. Let us use the following notation:

\[
\text{diag}(a, b, c, d) = \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix},
\]

\[
\text{offdiag}(a, b, c, d) = \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}.
\]

Let \( S \) be a surface in \( \mathbb{P}^3 \) and \( F \) the corresponding Wilczynski frame defined in (1.2). We first define a map \( g_1 \) by

\[
g_1 = FJ_1 F^t,
\]
where $J_1 = \text{offdiag}(1,1,1,1)$. It is easy to see that $g_1$ maps to the space of symmetric matrices with determinant one and signature $(2,2)$, which we denote by $Q$. The special linear group $\text{SL}_4\mathbb{R}$ transitively acts on this space by $gPq^t \in Q$ with $g \in \text{SL}_4\mathbb{R}$ and $P \in Q$. Then the point stabilizer at $J_1 \in Q$ is given by $K_1 = \{X \in \text{SL}_4\mathbb{R} \mid XJ_1X^t = J_1\}$, which is isomorphic to the special orthogonal group with signature $(2,2)$, which is denoted by $\text{SO}_{2,2}$. Thus $Q$ is isomorphic to the symmetric space $\text{SL}_4\mathbb{R}/\text{SO}_{2,2}$:

$$g_1 : M \rightarrow Q \cong \text{SL}_4\mathbb{R}/K_1 = \text{SL}_4\mathbb{R}/\text{SO}_{2,2}.$$ 

This map $g_1$ is known to be a quadric which has the first order contact to the surface. Note that $g_1$ does not have the second order contact, see [10, Section 22]. We call $g_1$ the first-order Gauss map for a surface $S$ in $\mathbb{P}^3$. We now characterize the Demoulin surface by the first-order Gauss map.

**Proposition 2.1.** The first-order Gauss map $g_1$ is conformal if and only if the surface $S$ is a Demoulin surface.

**Proof.** We first introduce the inner product on the tangent space of $Q$ as follows:

$$\langle X, Y \rangle_p = \text{Tr}(p^{-1}Xp^{-1}Y), \quad X, Y \in T_pQ,$$

where $p$ is a symmetric matrix of determinant one with signature $(2,2)$. This inner product is invariant under the action of $g \in \text{SL}_4\mathbb{R}$, since

$$\langle gXg^t, gYg^t \rangle_{gpq} = \text{Tr}((gpq^{-1})^{-1}gXg^t(gpq^{-1})^{-1}gYg^t) = \langle X, Y \rangle_p.$$ 

A direct computation shows that

$$g_{1x} = 2F \left( \begin{array}{cccc} bQ & k & P & 0 \\ k & 0 & 0 & 1 \\ P & 0 & b & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) F^t, \quad g_{1y} = 2F \left( \begin{array}{cccc} cP & Q & \ell & 0 \\ Q & c & 0 & 0 \\ \ell & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right) F^t.$$ 

Thus

$$\langle g_{1x}, g_{1x} \rangle = 16P, \quad \langle g_{1y}, g_{1y} \rangle = 16Q \quad \text{and} \quad \langle g_{1x}, g_{1y} \rangle = \langle g_{1y}, g_{1x} \rangle = 8(k + \ell) + 4bc.$$ 

Since the coordinates $(x,y)$ are null for the conformal structure induced by $S$, the first-order Gauss map $g_1$ is conformal if and only if $P = Q = 0$. 

**2.2. Projective minimal coincidence surfaces and Demoulin surfaces.** Let $\tau_1$ be the outer involution on $\text{SL}_4\mathbb{R}$ associated to $Q$ in (2.1) defined by $\tau_1(X) = J_1X^{-1}J_1$, $X \in \text{SL}_4\mathbb{R}$ and $J_1 = \text{offdiag}(1,1,1,1)$. Abuse of notation, we denote the differential of $\tau_1$ by the same symbol $\tau_1$ which is an outer involution on $\text{sl}_4\mathbb{R}$:

$$\tau_1(X) = -J_1X^tJ_1, \quad X \in \text{sl}_4\mathbb{R}.$$ 

Let us consider the eigenspace decomposition of $\mathfrak{g} = \text{sl}_4\mathbb{R}$ with respect to $\tau_1$, that is, $\mathfrak{g} = \mathfrak{e}_1 \oplus \mathfrak{p}_1$, where $\mathfrak{e}_1$ is the 0th-eigenspace and $\mathfrak{p}_1$ is the 1st-eigenspace as follows:

$$\mathfrak{e}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & 0 & -a_{13} \\ a_{31} & 0 & -a_{22} & -a_{12} \\ 0 & -a_{31} & -a_{21} & -a_{11} \end{pmatrix} \mid a_{ij} \in \mathbb{R} \right\}, \quad \mathfrak{p}_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{11} & a_{23} & a_{13} \\ a_{31} & a_{32} & -a_{11} & a_{12} \\ a_{41} & a_{43} & a_{21} & a_{11} \end{pmatrix} \mid a_{ij} \in \mathbb{R} \right\}.$$ 

According to this decomposition $g = \xi_1 \oplus p_1$, the Maurer-Cartan form $\alpha = F^{-1}dF = Udx + Vdy$ can be decomposed into

$$\alpha = \alpha_{t_1} + \alpha_{p_1} = U_{t_1}dx + V_{t_1}dy + U_{p_1}dx + V_{p_1}dy,$$

where $U = U_{t_1} + U_{p_1}$ and $V = V_{t_1} + V_{p_1}$. Let us insert the parameter $\lambda \in \mathbb{R}^\times$ into $U$ and $V$ as follows:

$$U^\lambda = U_{t_1} + \lambda^{-1}U_{p_1} \quad \text{and} \quad V^\lambda = V_{t_1} + \lambda V_{p_1}.$$ 

Then a family of 1-forms $\alpha_\lambda$ is defined as follows:

$$(2.3) \quad \alpha^\lambda = \alpha_{t_1} + \lambda^{-1}\alpha_{p_1} + \lambda\alpha''_{p_1} = U^\lambda dx + V^\lambda dy.$$ 

In fact the matrices $U^\lambda$ and $V^\lambda$ are explicitly given as follows:

$$(2.4) \quad U^\lambda = \begin{pmatrix} \frac{c_x}{2} & \lambda^{-1}P & \lambda^{-1}k & \lambda^{-1}bQ \\ \lambda^{-1} - \frac{c_x}{2c} & 0 & \lambda^{-1}k \\ 0 & \lambda^{-1}b & \frac{c_x}{2} & \lambda^{-1}P \\ 0 & 0 & \lambda^{-1} & -\frac{c_x}{2c} \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} \frac{b_y}{2b} & \lambda\ell & \lambda Q & \lambda cP \\ 0 & \frac{b_y}{2b} & \lambda c & \lambda Q \\ \lambda & 0 & -\frac{b_y}{2b} & \lambda\ell \\ 0 & \lambda & 0 & -\frac{b_y}{2b} \end{pmatrix}.$$ 

The following is the main theorem in this paper.

**Theorem 2.2.** Let $S$ be a surface in $\mathbb{P}^3$ and $g_1$ the first-order Gauss map defined in (2.1). Moreover, let $\alpha^\lambda (\lambda \in \mathbb{R}^\times)$ be a family of 1-forms defined in (2.3). Then the following are mutually equivalent:

1. The surface $S$ is a Demoulin surface or a projective minimal coincidence surface.
2. The first-order Gauss map $g_1$ is a Lorentz harmonic map into $Q$.
3. $d + \alpha^\lambda$ is a family of flat connections on $\mathbb{D} \times \text{SL}_4\mathbb{R}$.

**Proof.** Let us compute the flatness conditions of $d + \alpha^\lambda$, that is, the Maurer-Cartan equation

$$d\alpha^\lambda + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0.$$ 

A straightforward computation shows that these are equivalent to

$$Q_x = P_y = 0, \quad k_y + k\frac{b_y}{b} = 0, \quad \ell_x + \ell\frac{c_x}{c} = 0,$$

$$bQ_y + 2b_yQ = 0, \quad cP_x + 2c_xP = 0.$$ 

The surfaces with $P = Q = 0$ satisfies the above equations and they are Demoulin surfaces by (1.9). Assume that $P \neq 0$ (The case of $Q \neq 0$ is similar). From the first equation and the last equation, $P$ and $(\log c)_x$ depend only on $x$. Moreover from the equation $\ell_x + \ell(\log c)_x = 0$ and the definition $\ell$ in (1.4), $(\log b)_x = -2(\log c)_x$ and thus $(\log b)_x$ depends only on $x$. Thus $(\log b/c)_{xy} = 0$, which means that it is an isothermally asymptotic surface. Using a scaling transformation and a change of coordinates, we can assume that $b = c$. Then $\ell = k$ and the equations $\ell_x + \ell(\log c)_x = 0$ and $k_y + k(\log b)_y = 0$ imply that $b(=c)$ is constant. Thus $P$ and $Q$ are constant, and from (1.5) $p \neq 0$ and $q$ are constant. Therefore, the canonical system is given by

$$f_{xx} = f_y + pf, \quad f_{yy} = f_x + qf.$$ 

A surface satisfying the above equation is the special case of the coincidence surface, [12, Example 2.19]. In fact, it is easy to see that the surface is a projective minimal coincidence surface. Thus the equivalence of (1) and (3) follows.
The equivalence of (2) and (3) follows from Theorem 1.4 since the family of 1-forms $\alpha^\lambda$ is given by the involution $\tau_1$ and it defines the symmetric space $Q = SL_4\mathbb{R}/K_1$. \hfill \Box

**Remark 2.3.** Let $F^\lambda$ be a family of frames such that $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$. It is easy to see from the forms of $U^\lambda$ and $V^\lambda$ in (2.3) that $F^\lambda$ is not the Wilczynski frame of a Demoulin surface or projective minimal coincidence surface except $\lambda = 1$. However conjugating $F^\lambda$ by $DF^\lambda D^{-1}$ with $D = \text{diag}(1, \lambda, \lambda^{-1}, 1)$, the frames $DF^\lambda D^{-1}$ give a family of Wilczynski frames for Demoulin surfaces or projective minimal coincidence surfaces. The corresponding Demoulin surfaces or projective minimal coincidence surfaces have the same projective metric $8bc\,dx\,dy$ but the different conformal classes of cubic forms $\lambda^{-3}bdx^3 + \lambda^3 cdy^3$. Moreover, the functions $P$ and $Q$ change as $\lambda^{-2}P$ and $\lambda^2 Q$, respectively.

**Corollary 2.4.** Retaining the assumptions in Theorem 2.2, the following are equivalent:

1. The surface $S$ is a Demoulin surface.
2. The first-order Gauss map $g_1$ is a conformal Lorentz harmonic map into $Q$.

**Proof.** From Proposition 2.11 it is easy to see that the first-order Gauss map is conformal if and only if it satisfies that $P = Q = 0$, that is, the surface is a Demoulin surface. Moreover, from Theorem 2.2 the Gauss map of the Demoulin surface is Lorentz harmonic. \hfill \Box

Let $S$ be a Demoulin surface or projective minimal coincidence surface and $F^\lambda$ a family of frames such that $(F^\lambda)^{-1}dF^\lambda = \alpha^\lambda$. Then $F^\lambda$ will be called the *extended Wilczynski frame* for a Demoulin surface or projective minimal coincidence surface.

We now show that the extended Wilczynski frame for a Demoulin surface has an additional order three cyclic symmetry. Let $\sigma$ be an order three automorphism on the complexification of $SL_4\mathbb{R}$ as follows:

$$\sigma X = \text{Ad}(E)X, \quad X \in SL_4\mathbb{C},$$

where $E = \text{diag}(1, \epsilon^2, \epsilon, 1)$ with $\epsilon = e^{2\pi i/3}$. Then it is easy to see that $F(\lambda)(:= F^\lambda)$ satisfies the symmetry $\sigma F(\lambda) = F(\epsilon \lambda)$, since $U(\lambda)(:= U^\lambda)$ and $V(\lambda)(:= V^\lambda)$ satisfy the same symmetry. It is also easy to see that $\tau_1$ and $\sigma$ commute, and $\kappa = \tau_1 \circ \sigma$ defines an order six automorphism. Thus, the extended Wilczynski frame $F(\lambda)$ satisfies the symmetry

$$\kappa F(\lambda) = F(-\epsilon \lambda).$$

Note that $-\epsilon$ is the 6th root of unity. From the above argument, it is easy to see that the extended Wilczynski frame $F(\lambda)$ for a Demoulin surface is an element of the twisted loop group of $SL_4\mathbb{R}$:

$$\text{ASL}_4\mathbb{R}_\kappa = \{ g : \mathbb{R}^\times \to SL_4\mathbb{R} \mid \kappa g(\lambda) = g(-\epsilon \lambda) \}.$$

**Theorem 2.5.** The first-order Gauss map of a Demoulin surface, which is conformal Lorentz harmonic in $Q = SL_4\mathbb{R}/K_1$, can be obtained by the natural projection of a Lorentz primitive map into the 6-symmetric space $SL_4\mathbb{C}/K$ with $K = \{ \text{diag}(k_1, k_2, k_2^{-1}, k_1^{-1}) \mid k_1, k_2 \in \mathbb{C}^\times \}$.

**Proof.** The 0th-eigenspace and $\pm 1$st-eigenspaces of the derivative of the order six automorphism $\kappa = \tau_1 \circ \sigma$ are described as follows:

$$g_0 = \{ \text{diag}(a_{11}, a_{22}, -a_{22}, -a_{11}) \mid a_{ij} \in \mathbb{C} \},$$
and
\[ g_{-1} = \left\{ \begin{pmatrix} 0 & 0 & a_{13} & 0 \\ a_{21} & 0 & 0 & a_{13} \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{21} & 0 \end{pmatrix} \middle| a_{ij} \in \mathbb{C} \right\}, \quad g_1 = \left\{ \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ a_{31} & 0 & 0 & a_{12} \\ 0 & a_{31} & 0 & 0 \end{pmatrix} \middle| a_{ij} \in \mathbb{C} \right\}. \]

From the matrices \( U^\lambda \) and \( V^\lambda \) in (2.4) with \( P = Q = 0 \), we see that the conditions in (1.11) of a Lorentz primitive map are satisfied. The stabilizer of \( \text{SL}_g \) is
\[ K = \{ \text{diag}(k_1, k_2, k_2^{-1}, k_1^{-1}) \mid k_1, k_2 \in \mathbb{C}^\times \}. \]

Therefore there is a Lorentz primitive map \( g = FJF^t \) with \( J = EJ_1 \) into the 6-symmetric space \( \text{SL}_4 \mathbb{C}/K \) such that \( \pi \circ g = g_1 \), where \( \pi \) is the natural projection \( \pi : \text{SL}_4 \mathbb{C}/K \to \text{SL}_4 \mathbb{C}/K_1^\mathbb{C} \). We note that the projection \( \pi \) of a general Lorentz primitive map into \( \text{SL}_4 \mathbb{C}/K \) is a harmonic map into \( \text{SL}_4 \mathbb{C}/K_1^\mathbb{C} \) not \( \text{SL}_4 \mathbb{R}/K_1 \). However, the Lorentz primitive map \( g \) induced from the first-order Gauss map \( g_1 \) has an additional real structure. The eigenspaces \( g_0 \) and \( g_{\pm 1} \) can be decomposed into the real and the imaginary parts, that is
\[ g_0 = g_0^{\text{Re}} \oplus g_0^{\text{Im}} \quad \text{and} \quad g_{\pm 1} = g_{\pm 1}^{\text{Re}} \oplus g_{\pm 1}^{\text{Im}}, \]

where \( g_0^{\text{Re}}, g_{\pm 1}^{\text{Re}} \) and \( g_0^{\text{Im}}, g_{\pm 1}^{\text{Im}} \) consist of real and imaginary entries, respectively. Since the first order Gauss map takes values in \( \mathcal{Q} = \text{SL}_4 \mathbb{R}/K_1 \), the \((1, 0)\)- and \((0, 1)\)-parts \( \alpha' \) and \( \alpha'' \) for the Maurer-Cartan form of the map \( g_1 \) take values in \( g_0^{\text{Re}} + g_{-1}^{\text{Re}} \) and \( g_0^{\text{Re}} + g_1^{\text{Re}} \), respectively. Therefore, the projection \( \pi \) combined with \( g \) gives a Lorentz harmonic map into \( \text{SL}_4 \mathbb{R}/K_1 \).

**Remark 2.6.** Since we obtained the Lorentz primitive map into the 6-symmetric space for a Denuill surface, the generalized Weierstrass type representation as in [2] can be established.

## Appendix A. Projective minimal surfaces and the conformal Gauss maps

### A.1. The conformal Gauss map

We define a map \( g_2 \) by
\[ g_2 = FJ_2F^t, \]
where \( J_2 = \text{offdiag}(1, -1, -1, 1) \). Similar to \( g_1 \), it is easy to see that \( g_2 \) maps into the space of quadrics with signature \((2, 2)\) which is isomorphic to the \( \mathcal{Q} \). In fact the special linear group \( \text{SL}_4 \mathbb{R} \) transitively acts on this space by \( gPg^t \) with \( g \in \text{SL}_4 \mathbb{R} \) and \( P \in \mathcal{Q} \). Then the point stabilizer at \( J_2 \in \mathcal{Q} \) is given by \( K_2 = \{ X \in \text{SL}_4 \mathbb{R} \mid XJ_2X^t = J_2 \} \), which is also isomorphic to \( \text{SO}_{2,2} \):
\[ g_2 : M \to \mathcal{Q} \cong \text{SL}_4 \mathbb{R}/K_2 = \text{SL}_4 \mathbb{R}/\text{SO}_{2,2}. \]

This map \( g_2 \) is known to be a Lie quadric which has the second order contact to the surface, see [10, Section 18]. We call \( g_2 \) the **conformal Gauss map** for a surface \( S \) in \( \mathbb{P}^3 \), see [13, 3]. In [11], the conformal Gauss map \( g_2 \) was called the projective Gauss map.

**Proposition A.1 (Theorem 3 in [3]).** The conformal Gauss map \( g_2 \) is conformal map.

**Proof.** We introduce the inner product on the tangent space of \( \mathcal{Q} \) as in the proof of Proposition 2.1. Then a direct computation shows that
\[ g_{2x} = 2F \text{diag}(bQ, 0, -b, 0)F^t \quad \text{and} \quad g_{2y} = 2F \text{diag}(cP, -c, 0, 0)F^t. \]
Thus
\[ \langle g_{2x}, g_{2x} \rangle = \langle g_{2y}, g_{2y} \rangle = 0 \quad \text{and} \quad \langle g_{2x}, g_{2y} \rangle = \langle g_{2y}, g_{2x} \rangle = 4bc \neq 0. \]
Since the coordinates \((x, y)\) are null for the conformal structure induced by \(S\), the conformal Gauss map \(g_2\) is conformal.

A.2. Projective minimal surfaces and the conformal Gauss maps. Let \(\tau_2\) be the outer involution on \(\text{SL}_4\mathbb{R}\) associated to the symmetric space \(Q\) in (A.1) defined by \(\tau_2(X) = J_2 X^t J_2, \ X \in \text{SL}_4\mathbb{R}\) and \(J_2 = \text{offdiag}(1, -1, -1, 1)\). Abuse of notation, we denote the differential of \(\tau_2\) by the same symbol \(\tau_2\) which is an outer involution on \(\text{sl}_4\mathbb{R}\):

\[ \tau_2(X) = -J_2 X^t J_2, \ X \in \text{sl}_4\mathbb{R}. \]  

Let us consider the eigenspace decomposition of \(g = \text{sl}_4\mathbb{R}\) with respect to \(\tau_2\), that is, \(g = \mathfrak{e}_2 \oplus p_2\), where \(\mathfrak{e}_2\) is the 0th-eigenspace and \(p_2\) is the 1st-eigenspace as follows:

\[ \mathfrak{e}_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & 0 & a_{13} \\ 0 & a_{31} & a_{32} & a_{12} \\ 0 & a_{31} & a_{21} & -a_{11} \end{pmatrix} \middle| a_{ij} \in \mathbb{R} \right\}, \quad p_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{11} & a_{23} & -a_{13} \\ a_{31} & a_{32} & -a_{11} & -a_{12} \\ a_{41} & -a_{31} & -a_{21} & a_{11} \end{pmatrix} \middle| a_{ij} \in \mathbb{R} \right\}. \]

According to this decomposition \(g = \mathfrak{e}_2 \oplus p_2\), the Maurer-Cartan form \(\alpha = F^{-1} dF = U dx + V dy\) can be decomposed into

\[ \alpha = \alpha_{\mathfrak{e}_2} + \alpha_{p_2} = U_{\mathfrak{e}_2} dx + V_{\mathfrak{e}_2} dy + U_{p_2} dx + V_{p_2} dy, \]

where \(U = U_{\mathfrak{e}_2} + U_{p_2}\) and \(V = V_{\mathfrak{e}_2} + V_{p_2}\). Let us insert the parameter \(\lambda \in \mathbb{R}^\times\) into \(U\) and \(V\) as follows:

\[ U^\lambda = U_{\mathfrak{e}_2} + \lambda^{-1} U_{p_2} \quad \text{and} \quad V^\lambda = V_{\mathfrak{e}_2} + \lambda V_{p_2}. \]

Then a family of 1-forms \(\alpha^\lambda\) is defined as follows:

\[ \alpha^\lambda = \alpha_{\mathfrak{e}_2} + \lambda^{-1} \alpha_{p_2}' + \lambda \alpha_{p_2}'' = U^\lambda dx + V^\lambda dy. \]

In fact the matrices \(U^\lambda\) and \(V^\lambda\) are explicitly given as follows:

\[ U^\lambda = \begin{pmatrix} \frac{c_0}{2c} & P & k & \lambda^{-1} b Q \\ 1 & -\frac{c_0}{2c} & 0 & k \\ 0 & \lambda^{-1} b & \frac{c_0}{2c} & P \\ 0 & 0 & 1 & -\frac{c_0}{2c} \end{pmatrix}, \quad V^\lambda = \begin{pmatrix} \frac{b y}{2b} & \ell & Q & \lambda c P \\ 0 & \frac{b y}{2b} & \lambda c & Q \\ 1 & 0 & -\frac{b y}{2b} & \ell \\ 0 & 1 & 0 & -\frac{b y}{2b} \end{pmatrix}. \]

Then the projective minimal surface can be characterized by the Lorentz harmonicity of the conformal Gauss map \([13], [3] \text{ Theorem 7}], \) and by a family of flat connections.

Theorem A.2 ([13], Theorem 7 in [3]). Let \(S\) be a surface in \(\mathbb{P}^3\) and \(g_2\) the conformal Gauss map defined in (A.1). Moreover, let \(\alpha^\lambda (\lambda \in \mathbb{R}^\times)\) be a family of 1-forms defined in (A.3). Then the following are mutually equivalent:

1. The surface \(S\) is a projective minimal surface.
2. The conformal Gauss map \(g_2\) is a conformal Lorentz harmonic map into \(Q\).
3. \(d + \alpha^\lambda\) is a family of flat connections on \(\mathbb{D} \times \text{SL}_4\mathbb{R}\).
Proof. Let us compute the flatness conditions of \( d + \alpha^\lambda \), that is, the Maurer-Cartan equation \( d\alpha + \frac{1}{2}[\alpha^\lambda \wedge \alpha^\lambda] = 0 \). It is easy to see that except the (1, 4)-entry, the Maurer-Cartan equation is equivalent to (1.6). Moreover, the \( \lambda^{-1} \)-term and the \( \lambda \)-term of the (1, 4)-entry are equivalent to that the first equation and the second equation in (1.8), respectively. Thus the equivalence of (1) and (3) follows.

The equivalence of (2) and (3) follows from Theorem 1.3, since the family of 1-forms \( \alpha^\lambda \) is given by the involution \( \tau_2 \) and it defines the symmetric space \( Q = \text{SL}_4\mathbb{R}/K_2 \).

The above theorem implies that if \( S \) is a projective minimal surface, then there exists a family of projective minimal surface \( S^\lambda (\lambda \in \mathbb{R}^\times) \) such that \( S^\lambda|_{\lambda=1} = S \). Projective minimal surfaces of the family have the same projective metric \( 8bc \, dx \, dy \) but the different conformal classes of cubic forms \( \lambda^{-1}b \, dx^3 + \lambda c \, dy^3 \). Thus the family of the Maurer-Cartan form \( \alpha^\lambda \) defines a family of Wilczynski frames \( F^\lambda \) such that \( (F^\lambda)^{-1}dF^\lambda = \alpha^\lambda \). It is easy to see that \( F^\lambda \) is an element of the twisted loop group of \( \text{SL}_4\mathbb{R} \):

\[
\Lambda_{\text{SL}_4\mathbb{R}}\tau_2 = \{ g : \mathbb{R}^\times \to \text{SL}_4\mathbb{R} \mid \tau_2 g(\lambda) = g(-\lambda) \}.
\]

This family of Wilczynski frames \( F^\lambda \) will be called the extended Wilczynski frame for a projective minimal surface.

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