Walrasian Equilibria in Markets with Small Demands

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Abstract. We study the complexity of finding a Walrasian equilibrium in markets where the agents have \(k\)-demand valuations, where \(k\) is a constant. This means that the maximum value of every agent comes from a bundle of size at most \(k\). Our results are threefold. For unit-demand agents, where the existence of a Walrasian equilibrium is guaranteed, we show that the problem is in quasi-\(\mathcal{NC}\). Put differently, we give the first parallel algorithm that finds a Walrasian equilibrium in polylogarithmic time. This comes in striking contrast to all existing algorithms that are highly sequential. For \(k = 2\), we show that it is \(\text{NP-hard}\) to decide if a Walrasian equilibrium exists even if the valuations are submodular, while for \(k = 3\) the hardness carries over to budget-additive valuations. In addition, we give a polynomial-time algorithm for markets with 2-demand single-minded valuations, or unit-demand valuations. Our last set of results consists of polynomial-time algorithms for \(k\)-demand valuations in unbalanced markets; markets where the number of items is significantly larger than the number of agents, or vice versa.

Keywords: Walrasian equilibrium · \(\text{NP hardness}\) · Parallel algorithms.

1 Introduction

One of the most significant problems in market design is finding pricing schemes that guarantee good social welfare under equilibrium. Evidently, the most compelling equilibrium notion in markets with indivisible items is a Walrasian equilibrium, henceforth WE, \cite{33}: an allocation of items to the agents and a pricing, such that every agent maximizes her utility and all items are allocated. By the First Welfare Theorem, WE has the nice property that maximizes social welfare. The existence of WE seems to heavily rely on the class of valuation functions of

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the agents. When parameterized by the valuation function class, the existence of WE is (relatively) clear due to Gul and Stracchetti [20] and Milgrom [28]: WE are guaranteed to exist only in the class of gross substitutes valuation functions.

Two of the most central and interesting problems regarding WE are:

(a) decide if a WE exists,
(b) compute a WE (if it exists).

We study the aforementioned two problems when valuation functions are parameterized by an integer \( k \) which denotes the maximum bundle size for which every player is interested. Such a class of \( k \)-demand valuation functions can be seen as an extension of the unit-demand functions, where each agent, for a given bundle \( X \) values only the most valuable \( k \)-subset of \( X \). This is a natural class of valuation functions since in real markets it could be the case that more than \( k \) items have the same value as \( k \) of them; one can hang only a certain number of paintings on their house’s walls. We investigate the complexity of the aforementioned problems when we are restricted to the intersection of the standard valuation classes and the \( k \)-demand classes. Our results contain hardness results as well as efficient algorithms.

As an example of the effect that \( k \)-demand valuation functions have on the complexity of these problems, we present unbalanced markets. In such markets, the available items are significantly more than the allocated items, or vice versa. We provide algorithms for the aforementioned problems parameterized by \( k \). In particular, in these markets when \( k \) is constant these problems are in \( \mathcal{P} \).

1.1 Walrasian Equilibria and Valuation Functions

We consider markets with a set \( N \) of \( n \) agents and a set \( M \) of \( m \) items. Every agent \( i \) has a valuation function \( v_i : 2^M \rightarrow \mathbb{R}_{\geq 0} \); for every subset, or bundle, of items \( X \subseteq M \) agent \( i \) has value \( v_i(X) \). A valuation function \( v_i \) is monotone if \( X \subseteq Y \) implies \( v_i(X) \leq v_i(Y) \), and it is normalized if \( v_i(\emptyset) = 0 \). In what follows, we assume that all the agents have monotone and normalized valuation functions.

There are many different valuation functions studied over the years and we focus on several of them.\(^5\)

- Unit-demand (UD): for agent \( i \) there exist \( m \) values \( v_{i1}, \ldots, v_{im} \) and \( v_i(X) = \max_{j \in X} v_{ij} \), for every \( X \subseteq M \).

- Additive (AD): for agent \( i \) there exist \( m \) values \( v_{i1}, \ldots, v_{im} \) and \( v_i(X) = \sum_{j \in X} v_{ij} \), for every \( X \subseteq M \).

- Budget-additive (BA): for every agent \( i \) there exist \( m + 1 \) values \( v_{i1}, \ldots, v_{im}, B_i \), such that for every \( X \subseteq M \) it is \( v_i(X) = \min \left\{ B_i, \sum_{j \in X} v_{ij} \right\} \).

\(^5\) When we refer to a valuation function as general we mean that the value for any bundle does not depend on other bundles’ values. It is clear that the set of general functions contains all other classes of functions.
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- Single-minded (SM): for agent \( i \) there exist a set \( X_i \subseteq M \) and a value \( B_i \), and \( v_i(X) = B_i \), if \( X_i \subseteq X \), and \( v_i(X) = 0 \), otherwise.

- Submodular (SUBM): for agent \( i \) and every two sets of items \( X \) and \( Y \) it holds \( v_i(X) + v_i(Y) \geq v_i(X \cup Y) + v_i(X \cap Y) \).

- Fractionally subadditive (XOS): for every agent there exist vectors \( v_{i1}, \ldots, v_{ik} \in \mathbb{R}^m \) and \( v_i(X) = \max_{j \in [k]} \sum_{l \in X} v_{ik}(l) \), for every \( X \subseteq M \).

- Subadditive (SUBA): for agent \( i \) and every two sets of items \( X \) and \( Y \) it holds \( v_i(X) + v_i(Y) \geq v_i(X \cup Y) \).

We will focus on constrained versions of the aforementioned valuation functions, where the constraint bounds the cardinality of the sets an agent has value for.

**Definition 1** \((k\text{-demand valuation})\). A valuation function \( v : 2^m \to \mathbb{R}_{\geq 0} \) is \( k\text{-demand} \) if for every bundle \( X \subseteq M \) it holds that

\[
v(X) = \max_{X' \subseteq X, |X'| \leq k} v(X').
\]

\(k\text{-demand valuations} \) naturally generalize unit-demand valuations, but, at the same time, they keep the structure of more complex valuation functions. In addition, when \( k \) is a constant, they have a succinct representation.

An allocation \( S = (S_0, S_1, \ldots, S_n) \) is a partition of \( M \) to \( n + 1 \) disjoint bundles, thus \( S_i \cap S_j = \emptyset \), where agent \( i \in [n] \) gets bundle \( S_i \). Items in \( S_0 \) are not allocated to any agent. The social welfare of allocation \( S \) is defined as \( SW(S) = \sum_{i \in [n]} v_i(S_i) \). An allocation \( S \) is optimal if it maximizes the social welfare, i.e., \( SW(S) \geq SW(S') \), for every possible allocation \( S' \). A pricing \( p = (p_1, \ldots, p_m) \) defines a price for every item, where \( p_j \geq 0 \) is the price of item \( j \). For \( X \subseteq M \), we denote \( p(X) = \sum_{j \in X} p_j \). Given an allocation \( S \) and a pricing \( p \), the utility of agent \( i \) is

\[
u_i(S, p) := v_i(S_i) - p(S_i).
\]

The demand correspondence of agent \( i \) with valuation \( v_i \) under pricing \( p \), denoted \( D(v_i, p) \), is the set of items that maximize the utility of the agent; formally \( D(v_i, p) := \{ S \subseteq M : u_i(S, p) \geq u_i(T, p) \text{ for all } T \subseteq M \} \). Any element of \( D(v_i, p) \) is called demand set of agent \( i \).

**Definition 2** \((\text{Gross substitutes (GS)})[23]\). A valuation function satisfies the gross substitutes property when for any price vectors \( p \in \mathbb{R}_m \) and \( S \in D(v, p) \), if \( p' \) is a price vector \( p \leq p' \) (meaning that for all \( l \in S \), \( p_l \leq p'_l \)), then there is a set \( S' \in D(v, p') \) such that \( S \cap \{ j; p_j = p'_j \} \subseteq S' \).

Intuitively, a valuation is gross substitute if after the increase of the prices of some items in some demand set \( S \) of an agent the agent still has a demand set \( S' \) that contains the items with unchanged prices.
It is known that \( UD \subset BA \subset SubM \), that \( AD \subset GS \subset SubM \), and finally that \( SubM \subset XOS \subset SubA \). In addition, SMi valuation functions are not contained in any of these valuation classes.

**Definition 3 (Walrasian Equilibrium).** An allocation \( S = (S_0, S_1, \ldots, S_n) \) and a pricing \( p = (p_1, \ldots, p_m) \) form a Walrasian equilibrium (WE), if the following two conditions hold.

1. For every agent \( i \) and any bundle \( X \subseteq M \) it holds that \( v_i(S_i) - p(S_i) \geq v_i(X) - p(X) \).
2. For every item \( j \in S_0 \) it holds that \( p_j = 0 \).

**Walrasian**

**Input:** A market with \( n \) agents and \( m \) items, and a valuation function for each agent.

**Task:** Decide whether the market possesses a Walrasian equilibrium, and if it does, compute one.

The **First Welfare Theorem** states that for any Walrasian equilibrium \((S, p)\), partition \( S \) corresponds to an optimal allocation \([25]\). Hence, **walrasian** can be decomposed into the following two problems.

**WinnerDetermination**

**Input:** A market with \( n \) agents and \( m \) items, and a valuation function for each agent.

**Task:** Find an optimal allocation \( S^* \) for the items.

**WalrasianPricing**

**Input:** A market with \( n \) agents and \( m \) items, a valuation function for each agent, and an optimal allocation \( S^* \).

**Task:** Find a pricing vector \( p \) such that \((S^*, p)\) is a Walrasian equilibrium, or decide that there is no Walrasian equilibrium for the instance.

This decomposition highlights that a WE exists if and only if there exists a pricing vector \( p \) that satisfies the conditions of Definition 3 for any optimal allocation \( S^* = (S^*_0, S^*_1, \ldots, S^*_n) \). In addition, one of the results of Roughgarden and Talgam-Cohen \([30]\) states that if for some class \( \mathcal{V} \) of valuation functions **WinnerDetermination** is computationally harder than finding the demand for each agent, then there exist instances in \( \mathcal{V} \) with no WE.

For \( k \)-demand valuation functions, the conditions of Definition 3 correspond to the solution of the following linear system of \( m \) variables and \( n \cdot \sum_{j=1}^{k} \binom{m}{j} + m \) equality/inequality constraints, where each constraint has at most \( 2k \) variables.

\[
\begin{align*}
  v_i(S^*_i) - p(S^*_i) & \geq v_i(X) - p(X), & \forall X \subseteq M \text{ where } |X| \leq k, \quad \forall i \in N \\
p_j & \geq 0, & \forall j \notin S^*_0 \\
p_j & = 0, & \forall j \in S^*_0.
\end{align*}
\]
Note that when \( k \) is a constant, as mentioned earlier, the above constraints are \( n \cdot \sum_{j=1}^{k} \left( \frac{m_j}{j!} \right) + m \) which is at most linear in \( n \) and polynomial in \( m \), since

\[
\sum_{j=1}^{k} \left( \frac{m_j}{j!} \right) \leq \sum_{j=1}^{k} \left( \frac{m_j}{j!} \right) \leq \sum_{j=1}^{k} \left( \frac{m_j}{j!} \right) \leq e^k \sum_{j=1}^{k} \left( \frac{m_j}{j!} \right) 
\]

A solution to linear system (1) can be found in time polynomial in \( n \) and \( m \) by formulating it as an LP with objective function set to a constant. So, for this case, the problem of deciding the existence of WE and the problem of computing one (if it exists) essentially reduce to finding an optimal allocation \( S^* \). Throughout the paper we study cases with constant \( k \), hence we exploit this computational equivalence of WinnerDetermination and Walrasian, and focus on the former.

### 1.2 Related Work

**Existence of Walrasian Equilibria.** The most general class of valuation functions for which existence of WE is guaranteed has been proved by Gul and Stracchetti [20] and Milgrom [28] to be gross substitutes. Other valuation classes (that can be seen as special market settings) outside gross substitutes that guarantee WE existence have also been discovered, including the “tree valuations” in [7], and the valuation classes of [4,8,9]. Interestingly, the former admits also a polynomial time algorithm.

Non-existence of WE has been shown for many valuation classes, mostly by constructing an ad hoc market that does not identify some particular pattern as responsible for the non-existence (e.g. [20,24,13]). Roughgarden and Talgam-Cohen in [30] reprove some of these results and show a systematic way of proving non-existence of WE for more general valuation and pricing classes via standard complexity assumptions. The latter paper shows the remarkable relation between computability of arbitrary problems and existence of equilibria in markets.

**Computation of Walrasian Equilibria.** On the computational side, in markets that do not guarantee existence of WE, the problem of deciding existence is \( \text{NP} \)-hard for all the most important valuation classes. This has been established by proving that WinnerDetermination for budget-additive valuations is \( \text{NP} \)-hard via the “knapsack” problem in [24] and via the strongly \( \text{NP} \)-hard problem “bin packing” in [30]. By the fact that a WE corresponds to an optimal allocation, it is immediate that existence of WE is at least as hard as WinnerDetermination. Since budget-additive functions are a subset of submodular functions, it seems that as soon as valuation functions are allowed to be outside gross substitutes to the next bigger well-studied superset, i.e. submodular, the problem is already \( \text{NP} \)-hard. Also, for the class of single-minded agents (which is incomparable to the rest of the classes), WinnerDetermination is \( \text{NP} \)-hard [10]. In this work we
further refine this apparent dichotomy by introducing a hierarchy of $k$-demand valuation functions, and showing how the problem’s complexity changes according to $k$.

It is worth mentioning the “tollbooth” problem on trees, defined in [21] (see also [11]), for which, even though WE existence is not guaranteed, finding one (if it exists) is in $P$.

**Relaxations/Approximations.** Due to [20] and [28], existence of WE is guaranteed only in a restrictive class of functions, namely *gross substitutes*. This fact has ignited a line of works that, in essence, question the initially defined WE as being the equilibrium that occurs in actual markets. These works consider relaxed or approximate versions of WE. Some of the most interesting results on such relaxations are the following:

- If only two-thirds of the agents are required to be utility maximizers then a relaxed *Walrasian equilibrium* exists for single-minded agents ([10,11]).
- If the seller is allowed to package the items into indivisible bundles prior to sale, not all items have to be sold, and additionally only half of the optimal social welfare is required (*Combinatorial Walrasian equilibrium*) then such an equilibrium exists for general valuation functions and can be found in polynomial time ([17]).
- If agents exhibit *endowment effect*, meaning that the agents’ valuations for a bundle they already possess is multiplied by a factor $a$, then for any $a \geq 2$ there exists an *$a$-endowed equilibrium* for the class of submodular functions ([2]). For stronger notions of endowment, endowed equilibria exist even for XOS functions, and additionally, bundling guarantees equilibria for general functions ([16]).

### 1.3 Contribution

In this work we study WE under their classic definition with no relaxation or approximation notions involved. We introduce a hierarchy of valuation functions, parallel to the already existing one. Our valuation functions are called $k$-demand and are a generalization of unit-demand with parameter $k$ that determines at most how many items from a bundle the agent cares about. By definition, it is easy to see that the class of $j$-demand is included in $(j+1)$-demand for any $j \in [n-1]$. The purpose of considering valuation functions from the intersection of some $k$-demand class and some other known class, is to refine the complexity of the WE-related problem.

Algorithms and hardness results on the existence of WE and/or the problem of computing one in the current literature show an interesting dependence on the parameter $k$ that we define here. For example, existence of WE is guaranteed in the well studied case of unit-demand valuation functions (i.e. $k = 1$), and a WE can be computed in polynomial time [15,25]. Non-existence of WE is established in [30] by proving that even *WinnerDetermination* is NP-hard and this is achieved for valuation functions according to which the players are
only interested for at most 2 items (i.e. \( k = 2 \)). Furthermore, non-existence of WE and \( \mathsf{NP} \)-hardness of \textsc{WinnerDetermination} is proven for single-minded agents via a reduction to instances where agents are interested in at most 3 items (i.e. \( k = 3 \)) \cite{10}. For each of the above cases of \( k \) we push the state of the art forward: we supplement the “easy” case \( k = 1 \) with a novel quasi-\( \mathsf{NC} \) algorithm\footnote{This is the first parallel algorithm for computing WE to the authors’ knowledge.}, and the “hard” cases with stronger \( \mathsf{NP} \)-hardness results in the sense that ours imply the existing ones. We also go beyond these cases and study the case of constant \( k \) and \( m \gg k \cdot n \) or \( n \gg \frac{m}{k} \).

Mixing the standard valuations’ hierarchy and the \( k \)-demand hierarchy results to a two-dimensional landscape of valuation classes that aims to break down the complexity of the WE-related problems. For example, a possible result could be that below some threshold of \( k \) and below some standard valuation class, deciding WE existence is in \( \mathsf{P} \). Our results however indicate that this is not the case: even for \( k = 2 \) and submodular functions \textsc{WinnerDetermination} is \( \mathsf{NP} \)-hard, and therefore deciding existence of WE is also \( \mathsf{NP} \)-hard. This is an improvement over the result of Roughgarden and Talgam-Cohen \cite{30}, where \( \mathsf{NP} \)-hardness is proven for \( k = 2 \) but general functions. Our reduction is entirely different than the one in \cite{30}, and in particular, it is from the problem “3-bounded 3-dimensional matching” to a market with \( n \) agents, \( m \) items and 2-demand submodular valuations. Furthermore, in \cite{24} \textsc{WinnerDetermination} is proven to be weakly \( \mathsf{NP} \)-hard for budget-additive functions by reducing “knapsack” to a market with 2 agents, and \( m \) items. We show that the problem is strongly \( \mathsf{NP} \)-hard for \( k \)-demand budget-additive functions even for \( k = 3 \). The case \( k = 2 \) for the latter problem remains open.

On the positive side, we show a clear dichotomy for the problem of deciding WE existence with single-minded agents. It was proven in \cite{10} that \textsc{WinnerDetermination} is \( \mathsf{NP} \)-hard, via a reduction from “exact cover by 3-sets” to a market with single-minded agents who actually used 3-demand valuations. We show that \textsc{WinnerDetermination} is solvable in polynomial time for single-minded agents with 2-demand valuations by a reduction to the maximum weight matching problem. Then, by the decomposition shown at the end of Section 1.1, one can find a WE pricing via an LP (if such a pricing exists).

As we showed, even for the smallest possible value for \( k \), the problem of deciding WE existence remains \( \mathsf{NP} \)-hard for the next greater well-studied class of valuations outside gross substitutes (submodular). This means that in order to get efficient algorithms we have to further restrict our market design to markets that retain constant demand \( k \), but with either reduced number of agents, or reduced number of items. In our last set of results we present polynomial-time algorithms for \( k \)-demand general valuations for constant \( k \) in \textit{unbalanced markets}. These markets have either significantly more agents than items or vice versa.

A very important remark is that we are interested in constant \( k \) so that our problems have succinct representation, namely polynomial in the number of agents and items, i.e. \( \Theta (n \cdot \binom{m}{k} \cdot \log V) \), where \( V \) is the maximum valuation among all bundles and among all agents. This makes our setting computationally
interesting and also removes the need for access to some value oracle or demand oracle: the former takes as input a bundle and returns its value, and the latter, for some indicated agent, takes a pricing as input and outputs the most preferable bundles for the agent. Having such oracles when \( k \) is constant is redundant since one can compute in polynomial time the value for every bundle and even though the number of bundles is \( 2^m \), they can only take \( \binom{m}{k} \) many values. Also, a demand oracle is not needed since one can search in polynomial time through the list of \( \binom{m}{k} \) sets of bundles (arranged according to the payoff they induce to the given agent) and recover the most preferable bundles. In contrast, a great line of works has studied the complexity of the WE-related problems, provided that value oracles and demand oracles are available (e.g. \([6,29,20,32,26]\)).

Other works have also considered special classes of valuations that have as parameter the cardinality of the valuable bundles (\([13]\) and \([12]\)). However these valuation functions are not identical to ours. In \([13]\) the valuation function of each agent, called \( k \)-wise dependent, is encoded in a hypergraph whose vertices are the items and each hyperedge has a positive or negative weight that determines the additional value of the bundle in case all of its adjacent vertices are a subset of the bundle. This class of valuations is incomparable to ours by definition. The model of \([12]\) is the same as that of \([13]\), as argued in the latter. Recently, Berger et al. in \([5]\) introduced a hierarchy of valuation functions similar to ours, called "\( k \)-demand" that also generalize unit-demand functions. The same definition of functions appears also in \([14]\). However, those are a special case of our \( k \)-demand functions (i.e. also additive), and in fact they are gross substitutes.

The paper is organized in sections so that each deals with a particular value or group of values for \( k \). We study unit-demand valuations in Section 2, \( 2 \)-demand valuations in Section 3, \( 3 \)-demand valuations in Section 4, and \( k \)-demand valuations for constant \( k \) and unbalanced markets in Section 5. We conclude with a discussion in Section 6.

## 2 Unit-demand Valuation Functions

The simplest case of markets is when the agents have unit-demand valuation functions. The existence of WE in this class of markets was shown in the seminal paper of Demange, Gale, and Sotomayor \([15]\) via an algorithm that resembles the tâtonnement process of Walras \([34]\). This algorithm is pseudopolynomial in general, and polynomial when the values of the agents are bounded by some polynomial. In \([25]\) an algorithm (Algorithm 1) is presented and it is shown that a modification of it finds a WE in time \( O(m^2 n + m^4 \log V) \), where \( V \) is the maximum valuation of some item between all agents.

In this section we show that Walrasian in these markets is in quasi-\( \text{NC} \). The complexity class quasi-\( \text{NC} \) is defined as quasi-\( \text{NC} = \bigcup_{k \geq 0} \) quasi-\( \text{NC}^k \), where quasi-\( \text{NC}^k \) is the class of problems having uniform circuits of quasi-polynomial size, \( n^{\log \log \log n} \), and polylogarithmic depth \( O(\log^k n) \) \([3]\). Here “uniform” means that the circuit can be generated in polylogarithmic space. Put differently, quasi-
\( \text{NC} \) contains problems that can be solved in polylogarithmic parallel time using quasi-polynomially many processors with shared memory.

In this class of markets, \text{WinnerDetermination} can be reduced to a maximum weight matching on a complete bipartite graph. On the left side of the graph there exist \( n \) nodes corresponding to the agents, on the left side there are \( m \) nodes corresponding to the items and the weight of the edge \((i, j)\) equals to the value of agent \( i \) for item \( j \). The recent breakthrough of Fenner, Gurjar, and Thierauf \cite{18} states that the maximum weight perfect matching in bipartite graphs is in quasi-\( \text{NC} \) when the edge-weights are bounded by some polynomial; later Svensson and Tarnawski \cite{31} extended this result for general graphs. Thus, if we augment the bipartite graph that corresponds to the market by adding dummy items with zero value for every agent, or dummy agents with zero value for every item, we can guarantee that it contains a perfect matching without changing any optimal allocation. Then, we can use the algorithm of \cite{18} and compute an optimal allocation in polylogarithmic time.

Given an optimal allocation, \text{WalrasianPricing} for this markets has a special structure. It is a linear feasibility problem with polynomially many inequalities and at most two variables per inequality. For this special type of feasibility systems there exists a quasi-\( \text{NC} \) algorithm \cite{27}.

**Theorem 1.** \text{Walrasian} in unit-demand markets with polynomial valuations is in quasi-\( \text{NC} \).

*Proof.* When shared memory is available, as in quasi-\( \text{NC} \), we can solve \text{WinnerDetermination} in polylogarithmic parallel time via the algorithm of \cite{18} and store it in the shared memory. Then, the processors will read the solution, build the linear system for \text{WalrasianPricing} and solve it in polylogarithmic time via the the algorithm of \cite{27} on the shared memory. Hence, the composition of the two algorithms can be done in polylogarithmic time using quasi-polynomially many processors. \( \Box \)

We observe that this is the current best possible result, since any improvement would imply better parallel algorithms for other important problems like maximum weight matching and feasibility of systems with linear inequalities. We have to state though that it is open whether both aforementioned problems are in \( \text{NC} \). On the other hand, it is known that the maximum weight problem in graphs with polynomial weights is in pseudo-deterministic \( \text{RNC} \) \cite{1,19}. Hence, a first improvement would be to place \text{WalrasianPricing} in pseudo-deterministic \( \text{RNC} \).

### 3 2-demand Valuation Functions

In this section we resolve the complexity of deciding existence of WE for 2-demand valuation functions. A version of 2-demand valuations, termed pair-demand valuations, was studied in \cite{30}, where every agent \( i \) has a value \( v_i(j, k) \) for every pair of items and the value of \( i \) for a bundle \( S \) is \( v_i(S) = \max_{j, k \in S} v_i(j, k) \). These are general valuation functions that can allow complementarities. We
strengthen the results of [30] and prove that WinnerDetermination is $\text{NP}$-hard even when the valuation functions of the agents are 2-demand submodular and every agent has positive value for at most six items.

**Theorem 2.** WinnerDetermination is strongly $\text{NP}$-hard even for 2-demand submodular functions.

**Proof.** We reduce from 3-bounded 3-dimensional matching, termed 3DM(3). The input of a 3DM(3) instance consists of three sets $X, Y, Z$, where $|X| = |Y| = |Z|$, and a set $S$ of triplets (hyperedges) $(x, y, z)$ where $x \in X, y \in Y$, and $z \in Z$. In addition, every element of $X, Y, Z$ appears in at most three triplets and every triplet shares at most one element with any other triplet. The task is to decide if there is a subset of non-intersecting triplets of $S$ of cardinality $|X|$. The problem is known to be $\text{NP}$-complete [22].

For every element $x \in X$ we create an agent and for every element in $Y \cup Z$ we create an item. Let $S_x$ denote the set of items that correspond to the $j$th triplet of $S$ that $x$ belongs to. Recall that there exist at most three such triplets. In addition, since any two triplets of $S$ share at most one element, we have that $S_x$s are disjoint. Moreover, let $S_x$ be the union of the elements from $S_x$s. Then, the valuation function of agent $x$ for a subset of items $T$ is defined as follows:

- $v_x(T) = 2$, if $T$ contains some $S_x$;
- $v_x(T) = 0$, if $|T \cap S_x| = 0$;
- $v_x(T) = 1$, if $|T \cap S_x| = 1$;
- $v_x(T) = 1.5$, if $|T \cap S_x| = 2$ and $T$ does not contain any $S_x$;
- $v_x(T) = 1.75$, if $|T \cap S_x| = 3$ and $T$ does not contain any $S_x$.

Observe that if $|T \cap S_x| > 3$, then $T$ will contain some $S_x$, hence the definition of the valuation function is complete. It is not hard to verify that $v_x$ is indeed a 2-demand submodular function.

We claim that there is an allocation with welfare $2|X|$ if and only if the 3DM(3) instance is satisfiable. Firstly, assume that indeed the 3DM(3) instance has a solution $S'$, i.e., $S'$ contains $|X|$ non-intersecting triplets in $S$. Then, if the triplet $(x, y, z)$ belongs to $S'$ we allocate the items that correspond to $y$ and $z$ to the agent that corresponds to $x$ and the agent has value 2 for the bundle. Clearly, the allocation achieves welfare $2|X|$. For the other direction, assume that there is an allocation for the items with welfare $2|X|$. This means that every agent gets utility 2 from her allocated bundle. Then, by construction, each agent $x$ alongside her allocated bundle corresponds to a triplet from $S$. Observe, that the allocation consists of non-overlapping bundles, hence we get $|X|$ non-intersecting triplets in $S$. \qed

Theorem 2 implies that Walrasian is $\text{NP}$-hard for any class of valuation functions that contains the class of 2-demand submodular valuations. Thus, it is $\text{NP}$-hard for 2-demand XOS valuations and 2-demand subadditive valuations.

**Corollary 1.** Walrasian is strongly $\text{NP}$-hard even if all the agents have 2-demand submodular valuation functions.
Closing the gap in single-minded valuations. In addition to the above hardness results we study single-minded agents with 2-demand valuations and we show that in this case Walrasian is easy, contrary to the case of 3-demand valuations where it is NP-hard [10]. To prove this, for agents that are single-minded for bundles of size 2, we reduce WinnerDetermination to a maximum weight matching problem over a graph $G$. Every item corresponds to a vertex of $G$. For every pair of items that is the most preferable by an agent we create the corresponding edge with weight the value of the agent for the items; if there are more than one agents that want the same pair of items we keep only the weight for the highest valuation. Clearly, any maximum weight matching corresponds to an optimal allocation.

Next we show how to handle instances where every agent is either unit-demand or 2-demand single-minded. Recall that a unit-demand agent might have positive value for various items, and a 2-demand single-minded agent has positive value only for a particular pair of items. To achieve this, we extend the construction described above as follows. For every unit-demand agent $j$, we add a new vertex $j$ and the edges $(i, j)$, where $j$ is a vertex that corresponds to an item, with weight equal to the agent’s value for item $j$. Again, a maximum weight matching for the constructed graph corresponds to an optimal allocation.

**Theorem 3.** Walrasian is in P for markets where every agent is unit-demand or 2-demand single-minded.

4 3-demand Valuation Functions

In this section we prove strong NP-hardness for WinnerDetermination for 3-demand budget-additive valuation functions.

**Theorem 4.** WinnerDetermination is strongly NP-hard even when all the agents have identical 3-demand budget-additive valuation functions.

**Proof.** We prove the theorem with a reduction from 3-partition. An instance of 3-partition consists of a multiset of $3n$ positive integers $a_1, a_2, \ldots, a_{3n}$ summing up to $S$. The question is whether the multiset can be partitioned into $n$ triplets such that the elements of each triplet sum up to $B = \frac{S}{n}$. So, given an instance of 3-partition we create a WinnerDetermination instance with $n$ agents and $3n$ items. All the agents have the same 3-demand budget-additive valuation: they have value $a_i$ for item $i$ and budget $B$.

The question we would like to decide is whether there exists an allocation with social welfare $n \cdot B$. It is not hard to see that if there is a solution to 3-partition, then there exists an allocation for WinnerDetermination with social welfare $n \cdot B$. On the other hand, observe that, due to the budget-additive valuations, social welfare $n \cdot B$ for the instance can be achieved only when there exists an allocation where every agent gets value $B$. In addition, since the agents have 3-demand valuation functions it means that any allocation that maximizes the social welfare, without loss of generality, allocates exactly three items to every
agent; otherwise some agent gets more than 3 items and value get wasted since, by definition of 3-demand valuation, the agent will only appreciate the 3 most valuable of the items. Hence, if there exists an allocation for the constructed instance with social welfare \( n \cdot B \), necessarily, every agent gets exactly 3 items whose values sum up to \( B \). This allocation trivially defines a solution to 3-partition.

\[ \square \]

**Corollary 2.** **Walrasian** is strongly \( \text{NP} \)-hard even if all the agents have identical 3-demand budget-additive valuation functions.

### 5 Constant-demand Valuation Functions

In this section we study markets where the agents have \( k \)-demand valuation functions, where \( k \) is constant. Our results from the previous sections imply that deciding the existence of a WE is \( \text{NP} \)-hard even when \( k = 2 \) and the valuation functions are submodular. In addition, we showed that the problem is \( \text{NP} \)-hard for \( k = 3 \) even for budget-additive valuations. This means that in order to get efficient algorithms we have to further restrict our market design in markets that retain constant demand \( k \), but with either reduced number of agents, or reduced number of items. For this reason, we study *unbalanced* markets.

A market is *unbalanced* if the number of available items is significantly larger than the number of the most valuable items to be allocated (\( m \gg k \cdot n \)), or the other way around (\( n \gg \frac{m}{k} \)).

**Theorem 5.** In markets with \( k \)-demand valuations, \( n \) agents and \( m \) items, where \( k \), \( n \) are constant, **WinnerDetermination** is in \( \text{P} \).

**Proof.** We consider the unbalanced market where the number of available items \( m \) is a lot greater than the number of items \( k \cdot n \) to be allocated. The number \( k \cdot n \) comes from the fact that in an optimum allocation, not more than \( k \cdot n \) items will be appreciated by the agents (by definition of the \( k \)-demand valuation function). Therefore allocating more than these items does not improve the sum of valuations. In this case, we can find all possible subsets of size \( k \cdot n \) of items, that is, all candidate sets of items to be allocated to the agents. Formally, we consider the set \( I := \{ L \subseteq M \mid |L| = k \cdot n \} \) that consists of all \((k \cdot n)\)-subsets of \( M \). It is \( |I| = \binom{m}{k \cdot n} \in O((m - k \cdot n)^{k \cdot n}) \), which is a polynomial in \( m \) when \( k \) and \( n \) are constant.

Observe now that, given a subset \( L \) of items with size \( k \cdot n \), one can construct a \( k \) + 1-uniform hypergraph, i.e. a hypergraph all of whose hyperedges have size \( k + 1 \), in the following way. Have its vertex set be \( L \cup N \), and for every \( k \)-subset \( L_k \) of \( L \) have a hyperedge \( L_k \cup \{ i \} \) for every \( i \in N \). Also, assign to each hyperedge a weight equal to the valuation of player \( i \) for the item bundle \( L_k \), namely \( v_i(L_k) \). On this graph one can run a brute-force algorithm to find a maximum weight \((k + 1)\)-dimensional matching in constant time, since the graph is of constant size. Then, by repeating the same routine for all \((k \cdot n)\)-subsets of \( I \) in time polynomial in \( m \), we pick the one that yields the maximum sum of
weights in the matching. The optimal allocation of items to agents corresponds to the aforementioned optimum matching. The running time of this algorithm is \( O(m^c) \) for some constant \( c \), i.e. polynomial in the input size, since the input size is \( \Omega(n \cdot \binom{m}{k} \cdot \log V) \) bits, where \( V := \max_{X \subseteq M} v_i(X) \); that is because every agent has to declare how much is her valuation for every \( k \)-subset of items. \( \square \)

**Corollary 3.** In markets with \( k \)-demand valuations, \( n \) agents and \( m \) items, where \( k, n \) are constant, WALRASIAN is in \( \mathbb{P} \).

**Theorem 6.** In markets with \( k \)-demand valuations, \( n \) agents and \( m \) items, where \( k \) is constant and \( m \in O\left(\frac{\log n}{\log \log n}\right) \), WinnerDetermination is in \( \mathbb{P} \).

**Proof.** We consider the unbalanced market where the number of available items \( m \) is a lot smaller than the number of items \( k \cdot n \) to be allocated. Here, we consider again \( k \) to be constant so that the LP based algorithm under Definition 3 for computing WE prices runs in polynomial time. Without loss of generality, we will assume that \( m/k \) is an integer, since we can always add up to \( k - 1 \) “dummy” items whose value is 0 for every agent and do not affect the valuation when inserted in some bundle. Now we will enumerate all possible partitions \( P = (P_1, P_2, \ldots, P_{m/k}) \) of \( M \) into \( m/k \) many \( k \)-subsets. In particular, we consider the set \( J := \{P \mid |P_i| = k, \forall i \in \{m/k\}\} \) that consists of all possible partitions of \( M \) into \( k \)-sized parts. It is \( |J| = \binom{m}{k, \ldots, k} = \frac{m!}{(k!)^{m/k}} \leq m! \). Since for the gamma function \( \Gamma(m + 1) := m! \) it holds that \( \Gamma^{-1}(m) \in \Theta\left(\frac{\log m}{\log \log m}\right) \), when \( m \in O\left(\frac{\log n}{\log \log n}\right) \) it is \( |J| \in O\left(\frac{\log n}{\log \log n}\right) \) for some constant \( c \).

Now, starting from some partition \( P = (P_1, P_2, \ldots, P_{m/k}) \), we can construct a bipartite graph in the following way. Consider as one part of vertices \( A := \{P_1, P_2, \ldots, P_{m/k}\} \), the other part of vertices \( B := N \), and have each vertex of \( A \) connect to every vertex of \( B \). Also, assign on each edge that connects \( P_j \) with \( i \in N \), the valuation of agent \( i \) for the item bundle \( P_j \), namely \( v_i(P_j) \). One can find a maximum weight matching on this complete bipartite graph in time polynomial in \( n \). By repeating this routine for all possible partitions inside \( J \) in time polynomial in \( n \), we pick the partition that yields the maximum sum of weights in the matching. The optimal allocation of items to agents corresponds again to the aforementioned optimum matching. The described algorithm runs in time \( O(n^d) \) for some constant \( d \), and it is polynomial in the input size, which is \( \Omega\left(n \cdot \binom{m}{k} \cdot \log V\right) \) bits as explained in the proof of Theorem 5. \( \square \)

**Corollary 4.** In markets with \( k \)-demand valuations, \( n \) agents and \( m \) items, where \( k \) is constant and \( m \in O\left(\frac{\log n}{\log \log n}\right) \), WALRASIAN is in \( \mathbb{P} \).

### 6 Discussion

In this paper we study the complexity of computing Walrasian equilibria in markets with \( k \)-demand valuations. For markets with \( k = 1 \), known as “matching
markets”, we prove that the problem is in quasi-$\mathcal{NC}$. We view this as a very interesting result since all the known algorithms for the problem are highly sequential. Can we design an $\mathcal{NC}$ algorithm for the problem via a form of a simultaneous auction? This would be remarkable since it would imply that bipartite weighted matching is in $\mathcal{NC}$. For $k = 2$ we show that WinnerDetermination is intractable even for submodular functions, and for $k = 3$ the hardness remains for an even stricter class, namely budget-additive functions. In order to completely resolve the complexity of 2-demand valuations, it remains to solve WinnerDetermination for 2-demand budget-additive valuations. Is the problem \textbf{NP}-hard, or is there a polynomial time algorithm for it? Answering this question would provide a complete dichotomy for the complexity of the problems WinnerDetermination and also Walrasian. Another very intriguing direction is to study approximate Walrasian equilibria. The recent results of Babaioff, Dobzinski, and Oren [2] and of Ezra, Feldman, and Friedler [16] propose some excellent notions of approximation. Can we get better results if we assume $k$-demand valuations?

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