Quasi normal modes for a spin-3/2 field in a 4-dimensional Reissner-Nordström black hole background.

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**Abstract.** We will present a theoretical framework for studying Quasi Normal Modes (QNMs) for spin-3/2 fields in a 4-dimensional Reissner-Nordström black hole background, as a foundation for understanding higher-dimensional black holes. This work is useful in cases where we wish to study the signatures of micro higher-dimensional black holes, and how we might detect them during the experiments performed in the LHC at CERN. We shall make use of the Wentzel-Kramers-Brilloin (WKB) method to compute the QNMs for black holes perturbed by fields. We take the approximation to 6th order and compare these results to those we obtain using the improved asymptotic iterative method. Using the WKB method to 3rd order we are able to generate absorption probabilities associated to our spin-3/2 fields. Finally we compare these results to those obtained for the Schwarzschild black hole.

**1. Introduction**
Quasi Normal Modes (QNMs) are a result of perturbations of black holes and they can give an insight into the stability of the black hole [1, 2]. The black hole perturbations considered here are due to spin-3/2 fields. In this work we will be comparing Quasi Normal Frequency (QNF) results we obtained for a spin-3/2 field in a 4-dimensional Reissner-Nordström background using the Wentzel-Kramers-Brilloin (WKB) and Asymptotic Iterative Method (AIM) to those obtained for the Schwarzschild black hole. Using the equations of motion we are able to derive a radial equation which has the form

\[
\frac{d^2 \Psi}{dr^2} + \left( \omega^2 - V(r) \right) \Psi = 0. \tag{1}
\]

This Schrödinger like form is necessary in order to apply the chosen approximation methods. The QNMs are given by \( \omega \) and \( V(r) \) is the potential felt by the field near the black hole. At the boundaries of our space time we require that our wave functions behave like plane waves. To ensure this we assume our functions are of the following form:

\[
\Psi \propto e^{i \omega r_\ast} \quad \text{for} \quad r_\ast \to \infty,
\]

\[
\Psi \propto e^{-i \omega r_\ast} \quad \text{for} \quad r_\ast \to -\infty, \tag{2}
\]
where $r_*$ is the tortoise coordinate given as $dr_*/dr = 1/f(r)$, with $1/f(r)$ the function for the radial component of the metric. In order to derive our equations of motion we use the massless form of the Rarita-Schwinger equation [3, 4]:

$$\gamma^{\mu\nu\rho}D_\nu\Psi_\rho = 0,$$

where in Eq. (3) $\gamma^{\mu\nu\rho}$ is the anti-symmetric gamma tensor and is given as

$$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]} = \gamma^{\mu\nu}\gamma^{\rho} - \gamma^{\mu\rho}\gamma^{\nu} + \gamma^{\nu\rho}\gamma^{\mu},$$

where our $\gamma^{\mu}$'s denote the gamma matrices. Our metric is given as [5]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2,$$

with $f(r) = 1 - 2M/r + Q/r^2$, where the location of our event horizons are determined by $r = 1 \pm \sqrt{M^2 - Q^2}$, as such for a physical solution we must have $Q \leq M$. The case of $Q = M$ represents the extremal case for our black hole, which we will investigate and compare these results to those obtained for the Schwarzschild black hole given in Ref. [6]. We hope that by considering the extremal case we will identify any significant differences in our new results.

Solving for the equations of motion as done in Ref. [6] we achieve the following radial equation

$$-\frac{d^2}{dr_*^2}\tilde{\phi}_1 + V_1\tilde{\phi}_1 = \omega^2\tilde{\phi}_1; \quad -\frac{d^2}{dr_*^2}\tilde{\phi}_2 + V_2\tilde{\phi}_2 = \omega^2\tilde{\phi}_2,$$

where $r_*$ is the tortoise coordinate given as $d/dr_* = f d/dr$, $V_{1,2} = \pm f(r)dW/dr + W^2$, and

$$W = \frac{\sqrt{f}}{ABr} [\bar{\lambda}AB + CAB + \bar{\lambda} + C - \bar{\lambda}f],$$

with

$$C = \frac{Q}{r}; \quad A = \sqrt{f + \bar{\lambda} + \frac{Q}{r}} \quad \text{and} \quad B = \sqrt{f - \bar{\lambda} - \frac{Q}{r}}.$$

In the above expressions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are the wave functions associated to our fields, and $\bar{\lambda}$ are the eigenvalues for our spinors on a sphere given as $\bar{\lambda} = j + 1/2$ with $j = 3/2, 5/2, ...$ [6].

In Fig. 1 we have plotted the effective potential for the extremal Reissner-Nordström black hole and the Schwarzschild black hole, where the results for the uncharged Schwarzschild black hole are given in Ref. [6]. We observe that in both cases the potential function increases with an increase in the value of $j$. However there are some interesting differences between our two cases, firstly the Reissner-Nordström black hole has its event horizon closer to the singularity, implying it is more compact than the Schwarzschild black hole. Secondly, the Reissner-Nordström black hole has a greater potential maximum, for all equivalent values of $j$, so the black hole should be producing more energetic QNMs.

Once we have obtained the effective potential for our spin-$3/2$ particles we may begin applying approximation methods to determine our allowed QNMs, this is is outlined in the next section. In section 3 we present these results and use them to determine the absorption probabilities associated to our black holes in section 4. Finally we provide some concluding remarks and outline future works.
Figure 1: QNMs for both the Schwarzschild and Reissner-Norström black holes in 4 dimensions for values of $j = 3/2$ to $j = 13/2$.

**2. Approximation methods**

2.1. WKB

The WKB approximation method is a semi-classical approach to solving Schrödinger-like equations. The method was first used to study black hole perturbations due to scalar fields, and later the first order WKB method was used to study QNMs associated to black hole perturbations [1, 7]. The method was extended to 6th order by R. A. Konoplya where he showed that the QNMs can be obtained using the following equation [7]

$$\frac{i (\omega^2 - V_0)}{\sqrt{-2V''_0}} - \sum_{i=2}^{6} \Lambda_i = n + \frac{1}{2},$$

where the $\Lambda_i$ are correction terms, and can be found in Ref. [8]. $V_0$ and $V''_0$ are evaluated at the point $r_0$ which represents the radial location of our maximal value for the potential. So once we have determined the values of $V_0$ and $V''_0$ we may solve for $\omega$, which represents our complex QNMs.

2.2. Improved AIM

The improved AIM method can be used to solve second order differential equations, such as our radial equation. The original AIM used an iterative process, in which we must take a derivative in each step to obtain a solution for the second order differential equation. This is computationally time consuming as such the new method uses instead a recursion relation which does away with the derivative operators. As we have already explained this method in a previous proceedings we refer the reader to Ref. [9, 10] for a full explanation of the method.

**3. QNMs**

In Tab. 1 we observe that the results for the AIM and the WKB to 6th order agree very closely, whereas as those for the WKB 3rd order do not have a strong agreement with the other two results. This strong agreement in the 6th order WKB and the AIM suggests that our results are correct. We note that the higher modes for each value of $j$ have an increased dampening term. This means that if we were to detect QNMs from a Reissner-Nordström black we are most likely to detect the lower $n$ modes and least likely to detect the higher modes. In Fig. 2 we
Table 1: Low-lying \((n \leq l, \text{ with } l = j - 3/2)\) spin-3/2 field quasi normal mode frequencies using the WKB and the AIM methods in a 4-dimensional Reissner-Nordström black hole background.

| \(l\) | \(n\) | WKB 3rd order | WKB 6th order | AIM |
|-------|-------|---------------|---------------|-----|
| 0     | 0     | 0.5410-0.0867i| 0.5414-0.0865i| 0.5414-0.0864i|
| 1     | 0     | 0.8173-0.0874i| 0.8174-0.0874i| 0.8174-0.0873i|
| 1     | 1     | 0.8027-0.2638i| 0.8032-0.2636i| 0.8032-0.2636i|
| 2     | 0     | 1.0810-0.0878i| 1.0811-0.0878i| 1.0811-0.0877i|
| 2     | 1     | 1.0701-0.2643i| 1.0704-0.2643i| 1.0703-0.2642i|
| 2     | 2     | 1.0491-0.4435i| 1.0491-0.4435i| 1.0490-0.4435i|
| 3     | 0     | 1.3395-0.0880i| 1.3396-0.0880i| 1.3395-0.0879i|
| 3     | 1     | 1.3307-0.2646i| 1.3309-0.2646i| 1.3308-0.2645i|
| 3     | 2     | 1.3136-0.4430i| 1.3136-0.4430i| 1.3135-0.4430i|
| 3     | 3     | 1.2889-0.6239i| 1.2880-0.6245i| 1.2880-0.6245i|
| 4     | 0     | 1.5953-0.0881i| 1.5953-0.0881i| 1.5953-0.0881i|
| 4     | 1     | 1.5879-0.2648i| 1.5880-0.2648i| 1.5880-0.2648i|
| 4     | 2     | 1.5734-0.4427i| 1.5734-0.4427i| 1.5734-0.4427i|
| 4     | 3     | 1.5523-0.6225i| 1.5517-0.6228i| 1.5517-0.6228i|
| 4     | 4     | 1.5252-0.8047i| 1.5232-0.8060i| 1.5232-0.8060i|
| 5     | 0     | 1.8494-0.0882i| 1.8494-0.0882i| 1.8494-0.0882i|
| 5     | 1     | 1.8430-0.2649i| 1.8431-0.2649i| 1.8431-0.2649i|
| 5     | 2     | 1.8305-0.4425i| 1.8305-0.4425i| 1.8305-0.4425i|
| 5     | 3     | 1.8121-0.6216i| 1.8117-0.6218i| 1.8117-0.6218i|
| 5     | 4     | 1.7883-0.8026i| 1.7869-0.8034i| 1.7869-0.8034i|
| 5     | 5     | 1.7596-0.9857i| 1.7563-0.9879i| 1.7563-0.9879i|

Figure 2: QNMs for both the Schwarzschild and Reissner-Nordström black holes in 4 dimensions for values of \(l = 0\) to \(l = 5\).

have plotted a comparison between the Schwarzschild black hole and the Reissner-Nordström black hole, using the results of Ref. [6]. Comparing our results we see that our QNFs are indeed larger and therefore more energetic for the Reissner-Nordström black hole. We also see that the dampening terms for the Reissner-Nordström black hole have not increased as drastically as the QNFs.
4. Absorption probabilities

In order to relate our QNMs to cross sections we need to determine the absorption probabilities associated with the QNMs associated with the Reissner-Nordström black hole. The method and theory for determining the absorption probabilities is given in Ref. [1, 4, 6, 11]. Due to the high energy of some of our emitted fields we need to use the WKB to 3rd order approach to determine our absorption probabilities. The formula is given as

$$|A_j(\omega)|^2 = \frac{1}{1 + e^{2S(\omega)}}$$

where

$$S(\omega) = \pi k^{1/2} \left[ \frac{1}{2} z_0^2 + \left( \frac{15}{64} b_2^2 - \frac{3}{18} b_4 \right) z_0^4 \right] + \pi k^{-1/2} \left[ \frac{3}{16} b_4 - \frac{7}{64} b_6 \right]$$

$$+ \pi k^{1/2} \left[ \frac{1155 b_4^3}{2048} - \frac{315 b_2 b_4}{128} + \frac{25 b_2^2}{128} + \frac{55 b_4 - 35/64 b_6}{128} \right] z_0^6$$

$$- \pi k^{-1/2} \left[ \frac{1365 b_4^3}{2048} \right] - \frac{25 b_2 b_4}{128} + \frac{95 b_4 - 35 b_6}{128} z_0^2,$$

and

$$z = r - r_0; \quad z_0^2 \equiv -\frac{2}{P_0}; \quad k \equiv \frac{1}{2} P_0'';$$

$$b_n \equiv \left( \frac{2}{n! P_0^n} \right) \frac{d^n P}{dr^n} \bigg|_{r_0}; \quad \frac{d}{dr} = \left( 1 - \frac{2M}{r} + \frac{Q}{r^2} \right) \frac{d}{dr},$$

with $P = \omega - V$ [6]. Subscript 0’s denote evaluations at $r_0$, the radial location of the maximum of our potential. Plugging the determined values for our QNMs into this equation gives us the results as shown in Fig. 3.

![Figure 3](image)

Figure 3: Absorption probabilities for both the Schwarzschild and Reissner-Norström black holes in 4 dimensions for values of $j = 3/2$ to $j = 13/2$.

In Fig. 3 we have plotted the values for our absorption probabilities associated to the Schwarzschild and the Reissner-Nordström black hole. We notice that increase in the value of $j$ results in an increase in the minimum energy required for total absorption to occur. It is also clearly visible that total absorption occurs at higher energies for the Reissner-Nordström black hole.
5. Concluding Remarks
The allowed QNMs for our spin-3/2 particles near our extremal Reissner-Nordstrom black hole are given in Tab. 1, where we can clearly see that an increase in the value of quantum number $l$ results in an increase in the frequency of our emitted QNMs. Furthermore we see that an increase in $n$ results in an increase in the dampening term of our QNMs. These results are consistent with those obtained in Ref. [6], comparing our result to those in this paper we also note that the Reissner-Nordstrom black hole emits QNMs with a higher energy when compared to the Schwarzschild black hole.
In Fig. 3 we have plotted the absorption values of our fields for values of $j = 3/2$ to $j = 13/2$. We note that an increase in the value of $j$ results in an increase in the minimum energy required for total absorption to occur, these results are again consistent to the results we have obtained in Ref. [6]. We also note that the minimum energy required for total absorption to occur in the Reissner-Nordstrom black hole is higher than the Schwarzschild black hole.
In this paper we have considered only the 4-dimensional Reissner-Nordstrom case, in a future paper we will extend this to higher dimensional Reissner-Nordstrom black holes. Having done this we will use the same method and tools of analysis to determine the allowed QNMs for the Anti-de Sitter type black hole.

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