Regularized boundary element/finite element coupling for a nonlinear interface problem with nonmonotone set-valued transmission conditions

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Dedicated to Professor W.L. Wendland on the occasion of his 85th birthday

Abstract

For the first time, a nonlinear interface problem on an unbounded domain with nonmonotone set-valued transmission conditions is analyzed. The investigated problem involves a nonlinear monotone partial differential equation in the interior domain and the Laplacian in the exterior domain. Such a scalar interface problem models nonmonotone frictional contact of elastic infinite media. The variational formulation of the interface problem leads to a hemivariational inequality, which lives on the unbounded domain, and so cannot be treated numerically in a direct way. By boundary integral methods the problem is transformed and a novel hemivariational inequality (HVI) is obtained that lives on the interior domain and on the coupling boundary, only. Thus for discretization the coupling of finite elements and boundary elements is the method of choice. In addition smoothing techniques of nondifferentiable optimization are adapted and the nonsmooth part in the HVI is regularized. Thus we reduce the original variational problem to a finite dimensional problem that can be solved by standard optimization tools. We establish not only convergence results for the total approximation procedure, but also an asymptotic error estimate for the regularized HVI.

Keywords: Hemivariational inequality, monotone operator, nonmonotone set-valued transmission conditions, infinite media, Clarke generalized differentiation, smoothing technique, finite elements, boundary elements, error estimate.

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1 Introduction

This paper presents a novel boundary element/finite element coupling and regularization procedure for the solution of a nonlinear scalar interface problem on an unbounded domain with nonmonotone set-valued transmission conditions that models nonlinear contact problems with nonmonotone friction in infinite elastic media. Such contact problems arise in various fields of science and technology; let us mention geophysics, see e.g. [44], soil mechanics and civil engineering of underground structures, see e.g. [47].

In this paper we employ various mathematical techniques for the solution of the interface problem that consists of a nonlinear monotone partial differential equation in a bounded domain and the Laplace operator in the exterior domain coupled by set-valued nonmonotone transmission conditions. Using Clarke generalized differentiation [9] we describe this coupled boundary value problem as a nonlinear hemivariational inequality (HVI). Further by singular boundary integral methods [31] we reduce this problem to a HVI that lives on the bounded domain and the coupling boundary, only. Thus for discretization we can develop a BEM/FEM coupling method. In addition we adapt from [42] smoothing techniques of nondifferentiable optimization and regularize the nonsmooth part in the HVI to arrive at a finite dimensional problem that can be solved by standard optimization tools. We do not only provide convergence results for the total approximation procedure, but also show an error estimate for the regularized HVI that gives the same convergence order as the error estimate of BEM/FEM coupling for the monotone Signorini contact problem in [7].

The coupling of finite element and boundary element methods combining the best of both "worlds" [53] provides nowadays a very effective tool for the numerical solution of boundary value problems in physics and engineering. Indeed, the boundary element method is better suited to problems in which the domain extends to infinity but is usually confined to regions in which the governing equations are linear and homogeneous. On the other hand, the finite element method is restricted to problems in bounded domains but is applicable to problems in which the material properties are not necessarily homogeneous and nonlinearity may occur. This method was originally proposed by engineers (see e.g. Zienkiewicz et al. [53]). The mathematical analysis goes back to Brezzi, Johnson, and Nédélec [5, 4, 32] and was extended by Wendland [48, 49]. The symmetric coupling of finite element and boundary element methods is due to Costabel [10, 12]. For the coupling of finite element and boundary element methods for various nonlinear interface problems we point out the papers [13, 46, 14, 15, 35, 8, 17, 1, 18] and for a comprehensive exposition of finite element and boundary coupling refer to [26, Chapter 12].

The theory of hemivariational inequalities has been introduced and studied since 1980s by Panagiotopoulos [43], as a generalization of variational inequalities with an aim to model many problems coming from mechanics when the energy functionals are nonconvex, but locally Lipschitz, so the Clarke generalized differentiation calculus [9] can used, see [37, 21, 20]. For more recent monographs on hemivariational inequalities with application to contact problems we refer to
In parallel with the mathematical analysis of hemivariational inequalities the interest in efficient and reliable numerical methods for their solution constantly increases. The classical book on the finite element method for hemivariational inequalities is the monograph of Haslinger et al. [29]. More recent work on numerical solution of HVIs modeling contact problems with nonmonotone friction, adhesion, cohesive cracks, and delamination for elastic bodies on bounded domains is contained in the papers [2, 33, 30, 34, 38, 28, 27]; see also [50] for variationally consistent discretization schemes and numerical algorithms for contact problems.

The plan of the paper is as follows. The next section 2 collects some basic notions of Clarke’s generalized differential calculus that are needed for the analysis of the non-monotone boundary conditions. Then we describe the interface problem under study. We settle the issues of existence and uniqueness in a general functional analytic framework. Section 3 provides a first equivalent weak variational formulation of the interface problem in terms of a hemivariational inequality (HVI). Since this HVI lives on the unbounded domain $\Omega \times \Omega^c$ (as the original problem), this hemivariational formulation cannot be treated directly and therefore provides only an intermediate step in the solution procedure. Section 4 employs boundary integral analysis to transform the interface problem to a HVI that lives on the interior bounded domain and the interface boundary, only, and so is amenable to numerical treatment. Section 5 uses regularization techniques of nonsmooth optimization and derives a regularized version of the HVI that becomes a variational equality. Section 6 turns to numerical analysis of the regularized HVI, employs the Galerkin boundary element/finite element method, and presents an asymptotic error estimate. The final section 7 summarizes our findings, gives some concluding remarks, and sketches some directions of further research.

2 Some preliminaries and the interface problem

Let us first recall the central notions of Clarke’s generalized differential calculus [9], before we pose our interface problem. Let $X$ be a (real) Banach space, let $f : X \to \mathbb{R}$ be a locally Lipschitz function. Then

$$f^0(x; z) := \lim_{y \to x, t \downarrow 0} \sup_{y + tz} \frac{f(y + tz) - f(y)}{t} \quad x, z \in X,$$

is called the generalized directional derivative of $f$ in the direction $z$. Note that the function $z \in X \mapsto f^0(x; z)$ is finite, positively homogeneous, and sublinear, hence convex and continuous; further, the function $(x, z) \mapsto f^0(x; z)$ is upper semicontinuous. The generalized gradient of the function $f$ at $x$, denoted by (simply) $\partial f(x)$, is the unique nonempty weak* compact convex subset of the dual space $X^*$, whose support function is $f^0(x; .)$. Thus

$$\xi \in \partial f(x) \iff f^0(x; z) \geq \langle \xi, z \rangle, \forall z \in X,$$

$$f^0(x; z) = \max\{\langle \xi, z \rangle : \xi \in \partial f(x)\}, \forall z \in X.$$
When $X$ is finite dimensional, then, according to Rademacher’s theorem, $f$ is differentiable almost everywhere, and the generalized gradient of $f$ at a point $x \in \mathbb{R}^n$ can be characterized by

$$\partial f(x) = \text{co} \{ \xi \in \mathbb{R}^n : \xi = \lim_{k \to \infty} \nabla f(x_k), \text{ } x_k \to x, \text{ } f \text{ is differentiable at } x_k \},$$

where "co" denotes the convex hull.

In this paper we treat the following interface problem. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with Lipschitz boundary $\Gamma$. To describe mixed transmission conditions, we assume that the boundary $\Gamma$ splits into two non-empty, open disjoint parts $\Gamma_s$ and $\Gamma_t$ such that $\Gamma = \Gamma_s \cup \Gamma_t$. Let $n$ denote the unit normal on $\Gamma$ defined almost everywhere pointing from $\Omega$ into $\Omega^c := \mathbb{R}^d \setminus \overline{\Omega}$.

In the interior part $\Omega$, we consider the nonlinear partial differential equation

$$\text{div} \left( p(|\nabla u|) \cdot \nabla u \right) + f_0 = 0 \quad \text{in } \Omega,$$

(1)

where $p : [0, \infty) \to [0, \infty)$ is a continuous function with $t \cdot p(t)$ being monotonously increasing with $t$.

In the exterior part $\Omega^c$, we consider the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega^c$$

(2)

with the radiation condition at infinity ($|x| \to \infty$)

$$u(x) = O(|x|^{-1}) \quad \text{if } d = 3,$$

(3)

$$u(x) = a + \frac{b}{2\pi} \log |x| + o(1) \quad \text{if } d = 2,$$

(4)

where $a, b$ are real constants (constant for any $u$ but varying with $u$). Let us write $u_1 := u|_{\Gamma_t}$ and $u_2 := u|_{\Gamma_s}$, then the tractions on the coupling boundary $\Gamma$ are given by the traces of $p(|\nabla u_1|) \frac{\partial u_1}{\partial n}$ and $- \frac{\partial u_2}{\partial n}$, respectively.

We prescribe classical transmission conditions on $\Gamma_t$,

$$u_1|_{\Gamma_t} = u_2|_{\Gamma_t} + u_0|_{\Gamma_t} \quad \text{and} \quad p(|\nabla u_1|) \frac{\partial u_1}{\partial n} \big|_{\Gamma_t} = \frac{\partial u_2}{\partial n} \big|_{\Gamma_t} + t_0|_{\Gamma_t},$$

(5)

and on $\Gamma_s$, analogously for the tractions,

$$p(|\nabla u_1|) \frac{\partial u_1}{\partial n} \big|_{\Gamma_s} = \frac{\partial u_2}{\partial n} \big|_{\Gamma_s} + t_0|_{\Gamma_s}$$

(6)

and the generally nonmonotone, set-valued transmission condition,

$$-p(|\nabla u_1|) \frac{\partial u_1}{\partial n} \big|_{\Gamma_s} \in \partial j(\cdot, -(u_2 + u_0 - u_1)|_{\Gamma_s}).$$

(7)

Here we consider a function $j : \Gamma_s \times \mathbb{R} \to \mathbb{R}$ such that $j(\cdot, \xi) : \Gamma_s \to \mathbb{R}$ is measurable on $\Gamma_s$ for all $\xi \in \mathbb{R}$ and $j(s, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz on $\mathbb{R}$.
for almost all (a.a.) \( s \in \Gamma_s \). We write \( \partial_j(s, \xi) := \partial j(s, \cdot)(\xi) \) for the generalized gradient of \( j(s, \cdot) \) at the point \( \xi \) in the sense of Clarke. Moreover, \( j^0(s, \cdot; \cdot) \) stands for the generalized directional derivative of \( j(s, \cdot) \).

Further, we require the following growth condition on the so-called superpotential \( j \): There exist positive constants \( c_3 \) and \( c_4 \) such that for a.a. \( s \in \Gamma_s \), all \( \xi \in \mathbb{R} \) and for all \( \eta \in \partial_j(s, \xi) \) the following inequalities hold

\[
(H(j)) \left\{ \begin{array}{l}
(i) \quad |\eta| \leq c_3(1 + |\xi|), \\
(ii) \quad \eta(-\xi) \leq c_4|\xi|.
\end{array} \right.
\]

Given data \( f_0 \in L^2(\Omega) \), \( u_0 \in H^{1/2}(\Gamma) \), and \( t_0 \in L^2(\Gamma) \) together with

\[
\int_{\Omega} f_0 \, dx + \int_{\Gamma} t_0 \, ds = 0, \text{ if } d = 2
\]

we are looking for \( u_1 \in H^1(\Omega) \) and \( u_2 \in H^1_{loc}(\Omega^c) \) satisfying (1)–(7) in a weak form.

To conclude this section, we now discuss the existence and uniqueness of a weak solution to this interface problem in functional analytic terms. To this end, let \( X := L^2(\Gamma_s) \) and introduce the real-valued locally Lipschitz functional

\[
J(y) := \int_{\Gamma_s} j(s, y(s)) \, ds,
\]

\( y \in X \).

Then by Lebesgue’s theorem of majorized convergence,

\[
J^0(y; z) = \int_{\Gamma_s} j^0(s, y(s); z(s)) \, ds,
\]

\( (y, z) \in X \times X \).

As we shall see in the subsequent sections, the weak formulation of the problem (1)–(7) leads, in an abstract setting, to a hemivariational inequality (HVI) with a nonlinear operator \( A \) and the nonsmooth functional \( J \), namely, we are looking for some \( \hat{v} \in C \) such that

\[
A(\hat{v})(v - \hat{v}) + J^0(\gamma\hat{v}; \gamma v - \gamma \hat{v}) \geq \lambda(v - \hat{v}) \quad \forall v \in C.
\]

Here \( C \neq \emptyset \) is a closed convex subset of a Banach space \( E \), the nonlinear operator \( A : E \to E^* \) is a monotone operator, and \( \gamma := \gamma_{E \to X} \) denotes the linear continuous trace operator, and the linear form \( \lambda \) belongs to the dual \( E^* \). Similar to [7], the operator \( A \) consists of a nonlinear monotone differential operator (as made precise below) that results from the PDE (1) in the bounded domain \( \Omega \) and the positive definite Poincaré–Steklov operator on the boundary \( \Gamma \) of \( \Omega \) that stems from the exterior problem (2)–(4) and can be represented by the boundary integral operators of potential theory. Thus it results that the operator \( A \) is strongly monotone with some monotonicity constant \( c_A > 0 \) and Lipschitz continuous on bounded sets. On the other hand, by the compactness of the trace map \( \gamma \), the real-valued upper semicontinuous bifunction

\[
\psi(v, w) := J^0(\gamma v; \gamma w - \gamma v), \forall (v, w) \in E \times E
\]
can be seen to be pseudo-monotone, see \cite[Lemma 1]{41}, \cite[Lemma 4.1]{24}. The latter result also shows a linear growth of \(\psi(\cdot, 0)\). This and the uniform monotonicity of \(A\) imply coercivity. Therefore by the theory of pseudo-monotone VIs \cite[Theorem 3]{22}, \cite[52]{52}, see \cite[for the application to HVIs, we have solvability of (9).

Further suppose that the generalized directional derivative \(J^0\) satisfies the one-sided Lipschitz condition: There exists \(c_J > 0\) such that
\[
J^0(y_1; y_2 - y_1) + J^0(y_2; y_1 - y_2) \leq c_J\|y_1 - y_2\|_X^2 \quad \forall y_1, y_2 \in X. \tag{10}
\]

Then the smallness condition
\[
c_J\|\gamma\|_{E \to X}^2 < c_A \tag{11}
\]
implies unique solvability of (9), see e.g. \cite[Theorem 5.1]{39} and \cite[Theorem 83]{45}.

For a deeper study that relates the conditions (10) and (11) to the jumps of \(\partial j\) we can refer to \cite{40}.

It is noteworthy that under the smallness condition (11) together with (10), fixed point arguments [6] or the theory of set-valued pseudo-monotone operators [45] are not needed, but simpler monotonicity arguments are sufficient to conclude unique solvability. Thus the compactness of the trace map \(\gamma\) is not needed either. In fact, (9) can be framed as a \textit{monotone equilibrium problem} in the sense of Blum-Oettli \cite{3}. and so the fundamental existence result \cite[Theorem 1]{3} directly applies in view of the following

\textbf{Proposition 1} Suppose (10) and (11). Then the bifunction \(\varphi : C \times C \to \mathbb{R}\) defined by
\[
\varphi(v, w) := A(v)(w - v) + J^0(\gamma v; \gamma w - \gamma v) - \lambda(w - v)
\]
has the following properties:
\(\varphi(v, v) = 0 \forall v \in C\);
\(\varphi(v, w) + \varphi(w, v) \leq 0 \forall v, w \in C\) (monotonicity);
\(\varphi(v, \cdot)\) is convex and lower semicontinuous \(\forall v \in C\);
the function \(t \in [0, 1] \mapsto \varphi(t w + (1 - t)v, w)\) is upper semicontinuous at \(t = 0\) for all \(v, w \in C\) (hemicontinuity).

Moreover, \(\varphi(v, w) + \varphi(w, v) < 0 \forall v \neq w \in C\).

\textbf{Proof.} Obviously \(\varphi\) vanishes on the diagonal and is convex and lower semicontinuous with respect to the second variable. To show (strict) monotonicity, estimate
\[
\varphi(v, w) + \varphi(w, v) = (A(v) - A(w))(w - v) + J^0(\gamma v; \gamma w - \gamma v) + J^0(\gamma w; \gamma v - \gamma w) \\
\leq -c_A\|v - w\|_E^2 + c_J\|\gamma v - \gamma w\|_X^2 \\
\leq -(c_A - c_J\|\gamma\|_{E \to X}^2)\|v - w\|_E^2.
\]
To show hemicontinuity, it is enough to consider the bifunction \((y, z) \in X \times X \mapsto J^0(y; z - y)\). Then for \((y, z) \in X \times X\) fixed, \(t \in [0, 1]\) one has
\[
J^0(y + t(z - y); z - (y + t(z - y))) = (1 - t)J^0(y + t(z - y); z - y)
\]
and thus hemicontinuity follows from upper semicontinuity of \(J^0\),
\[
\limsup_{t \downarrow 0} J^0(y + t(z - y); z - y) \leq J^0(y; z - y).
\]

3 An intermediate HVI formulation of the interface problem

In this section we provide a first equivalent weak variational formulation of the interface problem \((1) - (7)\) in terms of a hemivariational inequality (HVI). Since this HVI lives on the unbounded domain \(\Omega \times \Omega^c\) (as the original problem), this hemivariational formulation cannot be numerically treated directly and therefore provides only an intermediate step in the solution procedure.

For the bounded Lipschitz domain \(\Omega\) we use the standard Sobolev space \(H^s(\Omega)\) and the Sobolev spaces on the bounded boundary \(\Gamma\) (see e.g. [26, Chapter 3,4]),
\[
H^s(\Gamma) = \begin{cases} 
\{u |_{\Gamma} : u \in H^{s+1/2}(\mathbb{R}^d)\} & (s > 0), \\
L^2(\Gamma) & (s = 0), \\
(H^{-s}(\Gamma))^* & (s < 0).
\end{cases}
\]

Further we need for the unbounded domain \(\Omega^c = \mathbb{R}^d \setminus \overline{\Omega}\) the Frechet space (see e.g. [31, Section 4.1, (4.1.43)])
\[
H^s_{\text{loc}}(\Omega^c) = \{u \in D^s(\Omega^c) : \chi u \in H^s(\mathbb{R}^d) \forall \chi \in C_0^\infty(\Omega^c)\}.
\]

Due to the trace theorem \(u_2 |_{\Gamma} \in H^{1/2}(\Gamma)\) whenever \(u_2 \in H^1_{\text{loc}}(\Omega^c)\). Then, we define \(\Phi : H^1(\Omega) \times H^1_{\text{loc}}(\Omega^c) \to \mathbb{R} \cup \{\infty\}\) by
\[
\Phi(u_1, u_2) := \int_{\Omega} g(|\nabla u_1|) \, dx + \frac{1}{2} \int_{\Omega^c} |\nabla u_2|^2 \, dx - L(u_1, u_2 |_{\Gamma}). \tag{12}
\]

Here \(L : H^1(\Omega) \times H^{1/2}(\Gamma) \to \mathbb{R}\) is the linear functional
\[
L(u, v) := \int_{\Omega} f_0 \cdot u \, dx + \int_{\Gamma} t_0 \cdot v \, ds \tag{13}
\]
with prescribed \(f_0 \in L^2(\Omega), t_0 \in L^2(\Gamma)\). In (12) the function \(g\) is given by \(p\) (see (1)) through
\[
g : [0, \infty) \to [0, \infty), t \mapsto g(t) = \int_0^t s \cdot p(s) \, ds,
\]
where we assume that \( p \) is \( C^1 \), \( 0 \leq p(t) \leq p_0 < \infty \), and \( t \cdot p(t) \) is monotone increasing with \( t \). Then, \( 0 \leq g(t) \leq \frac{1}{2}p_0 \cdot t^2 \) and hence,

\[
G(u) := \int_{\Omega} g(\|\nabla u\|) \, dx
\]

is finite for any \( u \in H^1(\Omega) \) and strictly convex. The Frechet derivative of \( G \),

\[
DG(u; v) = \int_{\Omega} p(|\nabla u|)(\nabla u)^T \cdot \nabla v \, dx \quad \forall u, v \in H^1(\Omega)
\]

is strongly monotone in \( H^1(\Omega) \) with respect to the semi-norm \( | \cdot |_{H^1(\Omega)} = \| \nabla \cdot \|_{L^2(\Omega)} \), i.e., there exists a constant \( c_G > 0 \) such that

\[
ce_G |u - v|_{H^1(\Omega)}^2 \leq DG(u; u - v) - DG(v; u - v) \quad \forall u, v \in H^1(\Omega) \quad (14)
\]

Analogous to [7, 35] we first define

\[
\mathcal{L}_0 := \{ v \in H^1_{loc}(\Omega^c) : \Delta v = 0 \text{ in } H^{-1}(\Omega^c) \}
\]

for \( d = 2 \exists a, b \in \mathbb{R} \) such that \( v \) satisfies (4)), and then the affine, hence convex set of admissible functions

\[
C := \{ (u_1, u_2) \in H^1(\Omega) \times H^1_{loc}(\Omega^c) : u_1|_{\Gamma_s} = u_2|_{\Gamma_s} + u_0|_{\Gamma_s}, \text{ and } u_2 \in \mathcal{L}_0 \}.
\]

According to [7, Remark 4], \( C \) is closed in \( H^1(\Omega) \times H^1_{loc}(\Omega^c) \). Further, we have

\[
D\Phi((\hat{u}_1, \hat{u}_2); (u_1, u_2)) = DG(\hat{u}_1; u_1) + \int_{\Omega^c} \nabla \hat{u}_2 \cdot \nabla u_2 \, dx - \int_{\Omega} f_0 \cdot u_1 \, dx - \int_{\Gamma_s} t_0 \cdot u_2|_{\Gamma_s} \, ds.
\]

Now we are in the position to pose the hemivariational inequality problem \( (P_\Phi) \):

Find \((\hat{u}_1, \hat{u}_2) \in C\) such that for all \((u_1, u_2) \in C\) we have

\[
D\Phi((\hat{u}_1, \hat{u}_2); (u_1 - \hat{u}_1, u_2 - \hat{u}_2)) + J^0(\gamma \hat{u}_1; \gamma (u_1 - u_2 + \hat{u}_2 - \hat{u}_1)) \geq 0. \quad (15)
\]

**Theorem 1**  Problem \((P_\Phi)\) is equivalent to (1) - (7) in the sense of distributions.

**Proof.** First, taking into account the definition of the generalized gradient, we note that

\[
- \int_{\Gamma_s} p(|\nabla \hat{u}_1|) \frac{\partial \hat{u}_1}{\partial n} \psi \, ds \leq \int_{\Gamma_s} J^0(s, (\hat{u}_1 - \hat{u}_2 - u_0)|_{\Gamma_s}; \psi(s)) \, ds, \quad \forall \psi \in C^\infty(\Gamma_s)
\]

(16)

is the integral formulation of the nonmonotone boundary inclusion (7).

Let \((\hat{u}_1, \hat{u}_2) \in C\) solve (15). To show that \((\hat{u}_1, \hat{u}_2)\) solves (1) - (7) in the sense of distributions, first choose \( \eta \in C^\infty_0(\mathbb{R}^d) \) such that \((u_1, u_2) := (\hat{u}_1 + \eta|_{\Omega_s}, \hat{u}_2 + \eta|_{\Omega^c}) \in C\). Setting \((u_1, u_2)\) in (15) and integration by parts, implies

\[
0 \leq - \int_{\Omega} \left( f + \text{div} \ p(|\nabla \hat{u}_1|) \nabla \hat{u}_1 \right) \cdot \eta \, dx - \int_{\Omega} \Delta \hat{u}_2 \cdot \eta \, dx + \int_{\Gamma_s} \left( p(|\nabla \hat{u}_1|) \frac{\partial \hat{u}_1}{\partial n} - \frac{\partial \hat{u}_2}{\partial n} - t_0 \right) \cdot \eta \, ds.
\]
Note that the last term in (15) disappears. Moreover, $n$ pointing into $\Gamma_0$ yields the negative sign of $\frac{\partial w_2}{\partial n}$. Varying $\pm \eta \in C_0^\infty(\Omega)$ and $\pm \eta \in C_0^\infty(\Omega^c)$, shows that (1) and (2) hold in the sense of distributions. Hence,

$$0 \leq \int_{\Gamma} \left( p(|\nabla \hat{u}_1|) \frac{\partial \hat{u}_1}{\partial n} - \frac{\partial \hat{u}_2}{\partial n} + t_0 \right) \cdot \eta \, ds.$$ 

Since $\eta$ is arbitrary on $\Gamma$, we obtain

$$p(|\nabla \hat{u}_1|) \frac{\partial \hat{u}_1}{\partial n} = \frac{\partial \hat{u}_2}{\partial n} + t_0 \quad \text{a.e. on } \Gamma. \quad (17)$$

This proves the second relation in (5) and (6).

Next, let $\eta_1, \eta_2 \in C_0^\infty(\mathbb{R}^d)$ and consider $(u_1, u_2) := (\hat{u}_1 + \eta_1|_{\Gamma}, \hat{u}_2 + \eta_2|_{\Gamma^c}) \in C$ in an analogous way to obtain

$$0 \leq \int_{\Gamma} p(|\nabla \hat{u}_1|) \frac{\partial \hat{u}_1}{\partial n} \eta_1 - \left( \frac{\partial \hat{u}_2}{\partial n} + t_0 \right) \eta_2 \, ds + \int_{\Gamma_s} j^0(\cdot, (\hat{u}_1 - \hat{u}_2 - u_0)|_{\Gamma_s}, (\eta_1 - \eta_2)|_{\Gamma_s}) \, ds,$$

which implies by (17)

$$- \int_{\Gamma} p(|\nabla \hat{u}_1|) \frac{\partial \hat{u}_1}{\partial n} (\eta_1 - \eta_2) \, ds \leq \int_{\Gamma_s} j^0(\cdot, (\hat{u}_1 - \hat{u}_2 - u_0)|_{\Gamma_s}, (\eta_1 - \eta_2)|_{\Gamma_s}) \, ds.$$ 

Finally, we define $\psi := \eta_1 - \eta_2$. Taking $\eta_1 = \eta_2$ on $\Gamma$, we have $\psi = 0$ on $\Gamma$, but $\psi$ is arbitrary on $\Gamma_s$, what gives (16).

Vice versa we show that (15) follows from (1) - (7). Let $(u_1, u_2)$ solve (1) - (7). Due to (2) and (4), $(u_1, u_2) \in C$. Multiplying (1) and (2) with differences $w_1 := v_1 - u_1$, $w_2 := v_2 - u_2$, respectively, where one chooses $(v_1, v_2) \in C$ arbitrarily, and integrating by parts yields

$$DG(u_1; w_1) = \int_{\Gamma} p(|\nabla u_1|) \frac{\partial u_1}{\partial n} w_1 \, ds = \int_{\Omega} f_0 \cdot w_1 \, dx, \quad (18)$$

$$\int_{\Omega^c} \nabla u_2 \cdot \nabla w_2 \, dx + \int_{\Gamma} \frac{\partial u_2}{\partial n} w_2 \, ds = 0. \quad (19)$$

Combining (18), (19), and $p(|\nabla u_1|) \frac{\partial u_1}{\partial n} - \frac{\partial w_2}{\partial n} = t_0$ on $\Gamma$, we obtain

$$D \Phi((u_1, u_2); (w_1, w_2)) = \int_{\Gamma} p(|\nabla u_1|) \frac{\partial u_1}{\partial n} (w_1 - w_2) \, ds,$$

where the latter integral vanishes on $\Gamma_s$ by definition of $C$. Hence, by (7) (see in particular the integral formulation (16)) we conclude that for all $(v_1, v_2) \in C$, $w_1 = v_1 - u_1$, $w_2 = v_2 - u_2$,

$$D \Phi((u_1, u_2); (w_1, w_2)) = \int_{\Gamma_s} j^0(\cdot, (u_1 - u_2 - u_0)|_{\Gamma_s}, (w_1 - w_2)|_{\Gamma_s}) \, ds$$

$$= \int_{\Gamma_s} p(|\nabla u_1|) \frac{\partial u_1}{\partial n} (w_1 - w_2) \, ds$$

$$+ \int_{\Gamma_s} j^0(\cdot, (u_1 - u_2 - u_0)|_{\Gamma_s}, (w_1 - w_2)|_{\Gamma_s}) \, ds$$

$$\geq 0,$$
what shows that \((u_1, u_2) \in C\) solves (15). □

4 The boundary/domain HVI formulation of the interface problem

In this section we first briefly recall from \([11, 31, 26]\) some boundary integral operator theory associated to the Laplace equation and then employ it to rewrite the exterior problem (2) - (4) as a boundary variational inequality on \(\Gamma\). As a result we arrive at an equivalent hemivariational formulation of the original interface problem (1) - (7) that lives on \(\Omega \times \Gamma\) and consists of a weak formulation of the nonlinear differential operator in the bounded domain \(\Omega\), the Poincare-Steklov operator on the bounded boundary \(\Gamma\), and a nonsmooth functional on the boundary part \(\Gamma_s\).

Given \(u_2 \in L_0\), its Cauchy data on \(\Gamma\) are

\[
\left( u_2|_{\Gamma}, \frac{\partial u_2}{\partial n} \right|_{\Gamma} \right) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma).
\]

With the fundamental solution for the Laplacian,

\[
F(x, y) = -\frac{1}{2\pi} \log |x - y| \quad \text{if} \quad d = 2
\]
\[
F(x, y) = \frac{1}{4\pi} |x - y|^{-1} \quad \text{if} \quad d = 3,
\]

define for \(\phi \in C^\infty(\Gamma)\) the single layer potential \(V\) and the double layer potential \(K\) by

\[
V\phi(x') := 2 \int_{\Gamma} \phi(x) \cdot F(x', x) \, ds_x, \quad x' \in \Omega \cup \Omega^c,
\]
\[
K\phi(x') := 2 \int_{\Gamma} \phi(x) \cdot \frac{\partial}{\partial n_x} F(x', x) \, ds_x, \quad x' \in \Omega \cup \Omega^c.
\]

Then using Green’s formula (see e.g. \([26, \text{Lemma 4.1.1}]\)) and the radiation condition (4), one obtains the following representation formula, see \([31, (1.4.5)],[26, (12.28)]\).

**Lemma 1** For \(u_2 \in L_2\) with Cauchy data \((v, \psi)\) there holds

\[
u_2(x') = \frac{1}{2}(Kv(x') - V\psi(x')) + a, \quad x' \in \Omega^c,
\]

where \(a\) is the constant appearing in (4) for \(d = 2\) (and \(a = 0\) in (20) if \(d = 3\)).

Note that (20) determines \(u_2\) in \(\Omega^c\) as far as one knows its Cauchy data on \(\Gamma\).
Next define the single layer boundary integral operator $V$, the double layer boundary integral operator $K$, its formal adjoint $K'$, and the hypersingular integral operator $W$ for $\phi \in C^\infty(\Gamma)$ as follows:

\[ V\phi(\xi) := 2 \int_\Gamma \phi(x) \cdot F(\xi, x) \, ds_x, \quad \xi \in \Gamma \]
\[ K\phi(\xi) := 2 \int_\Gamma \phi(x) \cdot \frac{\partial}{\partial n_x} F(\xi, x) \, ds_x, \quad \xi \in \Gamma \]
\[ K'\phi(\xi) := 2 \int_\Gamma \phi(x) \cdot \frac{\partial}{\partial n_\xi} F(\xi, x) \, ds_x, \quad \xi \in \Gamma \]
\[ W\phi(\xi) := -\frac{\partial}{\partial n_\xi} K\phi(\xi), \quad \xi \in \Gamma. \]

Due to [11], the linear operators

\[ V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \]
\[ K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \]
\[ K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \]
\[ W : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \]

are continuous. Moreover, the operator $V$ is symmetric and positive definite and $W$ is symmetric and positive semi-definite provided that the capacity of $\Gamma$ is smaller than 1 which can be always arranged by appropriate scaling, see e.g. [26] for details. Then, $\text{Ker} W = \mathbb{R}$ (i.e. consists of the constant functions).

Further the Poincaré–Steklov operator $S$ for the exterior problem (see [7, Lemma 3.5], [26, (12.50), Lemma 12.2.4, Lemma 12.2.18]),

\[ S := \frac{1}{2} \left( W + (K' - I)V^{-1}(K - I) \right) \tag{21} \]

maps $H^{1/2}(\Gamma)$ continuously into its dual space $H^{-1/2}(\Gamma)$ and the bilinear form $\langle Su, u \rangle$ is positive definite (provided that the capacity of $\Gamma$ is smaller than 1), i.e., there exists a constant $c_\mathcal{S} > 0$ such that

\[ \langle Su, u \rangle \geq c_\mathcal{S} \| u \|^2_{H^{1/2}(\Gamma)}, \quad \forall u \in H^{1/2}(\Gamma), \tag{22} \]

where $\langle \cdot, \cdot \rangle$ extends the $L^2$ duality on $\Gamma$.

Let $E := H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)$ with $\tilde{H}^{1/2}(\Gamma_s) := \{ w \in H^{1/2}(\Gamma) : \text{supp} w \subseteq \overline{T}_s \}$. Define $\mathcal{A} : E \to E^*$ for all $(u, v), (u', v') \in E$ by

\[ \mathcal{A}(u, v)(u', v') = \mathcal{A}(u, v; u', v') := DG(u, u') + \langle S(u|_{\Gamma} - v), u'|_{\Gamma} - v' \rangle, \]

and the linear functional $\lambda \in E^*$ for all $(u, v) \in E$ by

\[ \lambda(u, v) := L(u, u|_{\Gamma} - v) + \langle Su_0, u|_{\Gamma} - v \rangle = \int_{\Omega} f_0 \cdot u \, dx + \langle t_0 + Su_0, u|_{\Gamma} - v \rangle. \]
Moreover, in case \( d = 2 \), similarly to [7], we introduce an additional linear 
constraint and consider the affine closed subspace 
\[ D := \{(u,v) \in E : \langle S1, u|\Gamma - v - u_0 \rangle = 0 \text{ if } d=2 \}. \]

Then, the hemivariational inequality problem \((P_A)\) reads: Find \((\hat{u}, \hat{v}) \in D\) 

such that for all \((u,v) \in D\) 
\[
A(\hat{u}, \hat{v}; u - \hat{u}, v - \hat{v}) + \int_{\Gamma} j^0(\cdot, \hat{v}; v - \hat{v}) \, ds \geq \lambda (u - \hat{u}, v - \hat{v}) 
\] 
(23)

Now we show the equivalence of problems \((P_b)\) and \((P_A)\).

**Theorem 2** (i) Let \((u_1, u_2) \in C\) solve \((P_b)\). Then, \((u, v) \in D\) solves \((P_A)\), 
where \(u := u_1 - c\) for \(c \in \mathbb{R}\) if \(d = 2\), \(c = 0\) if \(d = 3\) and 
\(v := (u_1 - u_2)|\Gamma - u_0\).

(ii) Let \((u, v) \in D\) solve \((P_A)\). Take \(a \in \mathbb{R}\) arbitrarily if \(d = 2\), whereas \(a = 0\) if \(d = 3\). Define \(u_1 := u\) and \(u_2\) by the representation formula (20) with 
\((u_1|\Gamma - v - u_0, -S(u_1|\Gamma - v - u_0))\) replacing \((v, \psi)\), i.e.
\[
u_2 := \frac{1}{2} (K(u|\Gamma - v - u_0) + V(S(u_1|\Gamma - v - u_0))) + a.
\]

Then, \((u_1, u_2) \in C\) solves \((P_b)\).

**Proof.**

Analogously to [7], but now with \((u, v)\) and \((u_1, u_2)\) defined in (i), (ii), respectively, 
it can be shown that the sets \(C\) and \(D\) correspond to each other.

Now let \((u_1, u_2) \in C\) solve \((P_b)\). To show that \((u, v)\) defined in (i) solves \((P_A)\) 
we simply compare the different terms in \((P_b)\) and \((P_A)\).

To this end we observe that due to the properties of the Poincaré–Steklov operator, 
\[
\frac{\partial u_2}{\partial n} = -S(u_1|\Gamma - a),
\]

where \(a\) is the constant in (4) if \(d = 2\), and \(a = 0\) if \(d = 3\). For \(\nabla u_2 \in L^2(\Omega^c),\)
\(w_2 \in H_{loc}^1(\Omega^c)\) with \(\nabla w_2 \in L^2(\Omega^c)\) there holds by Green’s formula [7, Lemma 3.4] 
observing the orientation of the normal, 
\[
\int_{\Omega^c} \nabla u_2 \cdot \nabla w_2 \, dx = -\int_{\Gamma} \frac{\partial u_2}{\partial n} w_2 \, ds.
\]

Hence 
\[
\int_{\Omega^c} \nabla u_2 \cdot \nabla w_2 \, dx = \langle S(u_2|\Gamma - a), w_2|\Gamma \rangle \tag{24}
\]

Let similarly to (i), \((u_D, v_D) \in D\) correspond to an arbitrary \((u_C, v_C) \in C\).

Therefore, we have 
\[(u_D - u)|\Gamma + (v - v_D) = (v_C - u_2)|\Gamma,
\]

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thus by the linear constraint in $D$ if $d = 2$

$$\langle S(1), (v_C - u_2) \rangle$$

$$= \langle S(1), (u_D - u) \rangle\Gamma - (v_D - v)$$

$$= \langle S(1), u_D \rangle - \langle S(1), v_D - u_0 \rangle = 0,$$

and moreover

$$u|_\Gamma = v + u_0.$$

Thus by (24),

$$\langle S(u|_\Gamma - v), (u_D - u) \rangle\Gamma + v_D - v$$

$$= \langle S(u|_\Gamma - v), (v_C - u_2) \rangle\Gamma + \langle S(u_0 + a - c), (v_C - u_2) \rangle\Gamma$$

$$= \int_{\Omega} \nabla u_2 \cdot \nabla(v_C - u_2) \, dx + \langle Su_0, (v_C - u_2) \rangle\Gamma.$$

Further, there holds

$$DG(u, u_D - u) = DG(u_1, u_C - u_1).$$

Next,

$$\lambda(u_D - u, v_D - v)$$

$$= L(u_D - u, (u_D - u)\Gamma - (v_D - v)) + \langle Su_0, (u_D - u) \rangle\Gamma - (v_D - v))$$

$$= L(u_C - u_1, (v_C - u_2)\Gamma + \langle Su_0, (v_C - u_2) \rangle\Gamma$$

and finally,

$$\int_{\Gamma^s} j^0(\cdot, v, v_D - v) \, ds = \int_{\Gamma^s} j^0(\cdot, (u_1 - u_2)\Gamma - u_0; (u_2 - v_C)\Gamma - (u_1 - u_C)\Gamma \, ds.$$

Combining all relations above we obtain

$$A(u; v; u_D - u, v_D - v) - \lambda(u_D - u, v_D - v) + \int_{\Gamma^s} j^0(\cdot, v, v_D - v) \, ds$$

$$= DG(u_1, u_C - u_1) + \int_{\Gamma^s} \nabla u_2 \cdot \nabla(v_C - u_2) \, dx$$

$$+ \langle Su_0, (v_C - u_2) \rangle\Gamma - L(u_C - u_1, (v_C - u_2)\Gamma - \langle Su_0, (v_C - u_2)\rangle\Gamma$$

$$+ \int_{\Gamma^s} j^0(\cdot, (u_1 - u_2)\Gamma - u_0; (u_2 - u_1 - v_C + u_C)\Gamma \, ds$$

$$= D\Phi((u_1, u_2); (u_C - u_1, v_C - u_2))$$

$$+ \int_{\Gamma^s} j^0(\cdot, (u_1 - u_2)\Gamma - u_0; (u_2 - u_1 - v_C + u_C)\Gamma \, ds \geq 0.$$

Therefore, $(u, v)$ is a solution of the problem $(P_A)$. The converse implication (ii) can be shown in the same way. □

Thanks to the uniform monotonicity of the nonlinear operator $DG$ in $H^1(\Omega)$ with respect to the semi-norm $|||H^1(\Omega) = ||\nabla||_{L^2(\Omega)}$ and the positive definiteness of the Poincaré–Steklov operator $S$ the following uniform monotonicity property can be derived, see [7, Lemma 4.1].
Lemma 2 There exists a constant $c_0 > 0$ such that for all $v, v' \in \tilde{H}^{1/2}(\Gamma_s)$ and all $u, u' \in H^1(\Omega)$ there holds
\[
c_0 \cdot \|(u-u', v-v')\|_{H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)}^2 \\
\leq DG(u; u-u') - DG(u'; u-u') \\
+ \langle S(u|_{\Gamma} - v - u'|_{\Gamma} + v')u, u_{\Gamma} - v - u'|_{\Gamma} + v' \rangle.
\]
Moreover, following the arguments in [41] and using the compact embedding $\tilde{H}^{1/2}(\Gamma_s) \subset L^2(\Gamma_s)$ it can be easily seen that the functional $\varphi : \tilde{H}^{1/2}(\Gamma_s) \times \tilde{H}^{1/2}(\Gamma_s) \to \mathbb{R}$ defined by
\[
\varphi(v, \tilde{v}) := \int_{\Gamma_s} j^0(\cdot, v; \tilde{v} - v) \, ds
\]
is pseudo-monotone and upper-semicontinuous. Thus the concrete hemivariational inequality problem $(P_A)$ is covered by the general theory exhibited in section 2; existence and under the smallness condition uniqueness hold for $(P_A)$.

5 A regularized boundary/domain HVI formulation of the interface problem

In this section, we first recall from [42, 41] a smoothing approximation for a class of nonsmooth functions that can be expressed by means of the plus function $p(x) = x^+ = \max\{x, 0\}$. Then we apply this smoothing approximation to nonsmooth locally Lipschitz functions $j(s, \cdot)$ and thus arrive at a regularized formulation of the hemivariational inequality problem $(P_A)$, for which we can provide existence and uniqueness results. Finally in this section, a convergence result for the developed regularization procedure is presented.

The general idea of a smoothing approximation is to use convolution. However in general, convolution is not easily applicable in practice, but for a special class of functions that can be expressed by means of the plus function, it can be explicitly computed. To see this, consider the example $f(x) = \max\{g_1(x), g_2(x)\}$, where $g_1, g_2$ are smooth functions. Then,
\[
f(x) = g_1(x) + p[g_2(x) - g_1(x)]. \tag{25}
\]
With the notations
\[
\mathbb{R}_+ = \{\varepsilon \in \mathbb{R} : \varepsilon \geq 0\}, \quad \mathbb{R}_{++} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}
\]
we define the smoothing approximation $\hat{p} : \mathbb{R}_{++} \times \mathbb{R}$ of $p$ via convolution by
\[
\hat{p}(\varepsilon, x) = \int_{\mathbb{R}} p(x - \varepsilon t) \rho(t) \, dt.
\]
Here, $\varepsilon > 0$ is a small regularization parameter and $\rho : \mathbb{R} \to \mathbb{R}_+$ is a probability density function such that
\[
\kappa = \int_{\mathbb{R}} |t| \rho(t) \, dt < \infty.
\]
Now we replace $p(x)$ by its approximation $\hat{p}(\varepsilon, x)$ and obtain $\hat{f} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$\hat{f}(\varepsilon, x) = g_1(x) + \hat{p}(\varepsilon, g_2(x) - g_1(x)),$$

(26)

as a smoothing function of $f$ in (25).

Using, for example, the Zang probability density function

$$\rho(t) = \begin{cases} 1 & \text{if } -\frac{3}{2} \leq t \leq \frac{3}{2} \\ 0 & \text{otherwise,} \end{cases}$$

leads to

$$\hat{p}(\varepsilon, x) = \int_{\mathbb{R}} p(x - t\varepsilon) \rho(t) \, dt = \begin{cases} 0 & \text{if } x < -\frac{3}{2} \\ \frac{1}{2\varepsilon}(x + \frac{3}{2})^2 & \text{if } -\frac{3}{2} \leq x \leq \frac{3}{2} \\ \frac{1}{2}g_2(x) + g_1(x) + \frac{3}{2} & \text{if } x > \frac{3}{2} \end{cases}$$

(27)

and hence,

$$\hat{f}(\varepsilon, x) := \begin{cases} g_1(x) & \text{if (i) holds} \\ \frac{1}{2\varepsilon}[g_2(x) - g_1(x)]^2 + \frac{1}{2}[g_2(x) + g_1(x)] + \frac{3}{2} & \text{if (ii) holds} \\ g_2(x) & \text{if (iii) holds;} \end{cases}$$

where the cases (i), (ii), (iii) are defined, respectively, by

(i) $g_2(x) - g_1(x) \leq -\frac{3}{2}$

(ii) $-\frac{3}{2} \leq g_2(x) - g_1(x) \leq \frac{3}{2}$

(iii) $g_2(x) - g_1(x) \geq \frac{3}{2}$

For other choices of probability density functions and other examples of smoothing functions we refer to [41, 42] and the references therein.

The representation formula (25) extends to the maximum function $f : \mathbb{R} \to \mathbb{R}$ of $m$ smooth functions $g_1, \ldots, g_m$, i.e.

$$f(x) = \max\{g_1(x), g_2(x), \ldots, g_m(x)\} = g_1(x) + P[g_2(x) - g_1(x) + \ldots + P[g_m(x) - g_{m-1}(x)]] .$$

The smoothing function $\hat{f} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is then given by

$$\hat{f}(\varepsilon, x) = g_1(x) + P(\varepsilon, g_2(x) - g_1(x) + \ldots + P(\varepsilon, g_m(x) - g_{m-1}(x))) .$$

(28)

More general nested max-min functions can be treated by an appropriate representation using the plus function, what is exploited in the numerical treatment of related nonsmooth variational problems, albeit in bounded domains, see [25]. Note that this smoothing procedure via the plus function needs only one regularization parameter $\varepsilon$.

The major properties of the function $\hat{f}(\cdot, \cdot)$ in (28) are listed in the following lemma:
Lemma 3 \([42, 41]\)

(i) For any \(\varepsilon > 0\) and for all \(x \in \mathbb{R}\),

\[
|\hat{f}(\varepsilon, x) - f(x)| \leq (m - 1)\kappa \varepsilon.
\]

(ii) The function \(\hat{f}\) is continuously differentiable on \(\mathbb{R}_{++} \times \mathbb{R}\) and for any \(x \in \mathbb{R}\) and \(\varepsilon > 0\) there exist \(\Lambda_i \in [0, 1]\) such that

\[
\sum_{i=1}^{m} \Lambda_i \varepsilon = 1
\]

and

\[
\frac{\partial \hat{f}(\varepsilon, x)}{\partial x} = \hat{f}_x(x, \varepsilon) = \sum_{i=1}^{m} \Lambda_i g'_i(x).
\]

Moreover,

\[
\limsup_{z \to x, \varepsilon \to 0^+} \hat{f}_x(\varepsilon, x) \subseteq \partial f(x).
\]

Now analogously let the superpotential \(j : \Gamma_s \times \mathbb{R} \rightarrow \mathbb{R}\) be a maximum function of smooth functions, i.e.

\[
j(s, x) = \max\{g_1(s, x), g_2(s, x), \ldots, g_m(s, x)\}
\]

and obtain the smoothing function \(\hat{j} : \Gamma_s \times \mathbb{R}_{++} \times \mathbb{R} \rightarrow \mathbb{R}\) given by

\[
\hat{j}(s, \varepsilon, x) = g_1(s, x) + P(\varepsilon, g_2(s, x) - g_1(s, x)) + \ldots + P(\varepsilon, g_m(s, x) - g_{m-1}(s, x)).
\]

Lemma 4 It holds for all \(x, z, \xi \in \mathbb{R}\), almost all (a.a.) \(s \in \Gamma_s\) that

\[
|\hat{j}_x(s, \varepsilon, x) z| \leq c(1 + |x|)|z|
\]

\[
\hat{j}_x(s, \varepsilon, x) \cdot (-x) \leq d|x|
\]

\[
\limsup_{z \to x, \varepsilon \to 0^+} \hat{j}_x(s, \varepsilon, z) \xi \leq j^0(s, x; \xi).
\]
Next we introduce \( J_\varepsilon : H^{1/2}(\Gamma_s) \to \mathbb{R} \) by
\[
J_\varepsilon(v) = \int_{\Gamma_s} \tilde{j}(s, \varepsilon, v(s)) \, ds.
\]
Since \( \tilde{j}(s, \varepsilon, \cdot) \) is continuously differentiable, the functional \( J_\varepsilon \) is Gâteaux differentiable with continuous Gâteaux derivative \( DJ_\varepsilon : H^{1/2}(\Gamma_s) \to (H^{1/2}(\Gamma_s))^* \),
\[
\langle DJ_\varepsilon(v), \bar{v} \rangle = \int_{\Gamma_s} \tilde{j}_x(s, \varepsilon, v(s)) \, \bar{v}(s) \, ds.
\]

The regularized problem \( (P_\varepsilon) \) of \( (\tilde{P}_A) \) reads now: Find \((\hat{u}_\varepsilon, \hat{v}_\varepsilon) \in D \subset E = H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)\) such that
\[
\mathcal{A}(\hat{u}_\varepsilon, \hat{v}_\varepsilon)(u - \hat{u}_\varepsilon, v - \hat{v}_\varepsilon) + \langle DJ_\varepsilon(\hat{v}_\varepsilon), v - \hat{v}_\varepsilon \rangle = \lambda(u - \hat{u}_\varepsilon, v - \hat{v}_\varepsilon) \quad \forall (u, v) \in D. \quad (37)
\]

Note that \( D \) is an affine subset of \( E = H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \). So in virtue of differentiability of \( J_\varepsilon \), \( (P_\varepsilon) \) simplifies to the variational equality \( (37) \). Assume there exists a constant \( 0 \leq \alpha < c_S \), where \( c_S \) is defined in \((22)\), such that for a.a. \( s \in \Gamma_s \) and for all \( x_1, x_2 \in \mathbb{R} \), that
\[
(\tilde{j}_x(s, \varepsilon, x_1) - \tilde{j}_x(s, \varepsilon, x_2))(x_1 - x_2) \geq -\alpha|x_1 - x_2|^2, \quad (38)
\]
then by the discussion in section 2 the regularized problem \( (P_\varepsilon) \) has a unique solution \( u_\varepsilon = (u_\varepsilon, v_\varepsilon) \).

Moreover replacing the coercive bilinear form \( a \) in \([42]\), by the strongly monotone form, see Lemma 2,
\[
[(u, v), (u', v')] \in E \times E \mapsto
DG(u; u - u') - DG(u'; u - u')
+ \langle S(u|_\Gamma - v - u'|_\Gamma + v'), u_\Gamma - v - u'|_\Gamma + v' \rangle,
\]
the approximation result \([42, \text{Theorem 5.1}]\) for the regularization procedure extends to the present situation. Thus we can summarize our findings as follows.

**Theorem 3** The regularized problem \((P_\varepsilon)\) has at least one solution \( u_\varepsilon \in D \subset E = H^1(\Omega) \times \tilde{H}^{1/2}(\Gamma_s) \). The family \( \{u_\varepsilon\} \) is uniformly bounded in \( E \). Moreover, there exists a subnet of \( \{u_\varepsilon\} \) which converges strongly in \( E \) to a solution \( u \) of the hemivariational inequality problem \((P_A)\). Under the smallness condition \( 0 \leq \alpha < c_S \) along with the condition \((38)\) strong convergence of the entire regularization procedure \( \{u_\varepsilon\} \) follows.

## 6 Numerical approximation

In this section we treat the numerical approximation for problem \((P_\varepsilon)\) by a Galerkin projection using finite elements in \( \Omega \) and boundary elements on \( \Gamma \) leading to the coupling of FEM and BEM. Here we focus to the case \( d = 2 \) and
assume the interior domain $\Omega$ to be polygonal. For the more general case of a domain with a piecewise $C^{1,1}$ boundary we can refer to the construction of an intermediate polygonal approximation with finite element/boundary element coupling and the associated error analysis presented in [15, Sections 3.3 - 3.5] for a class of nonlinear problems.

Let $(H^1_h \times \tilde{H}^{1/2}_h \times H^{-1/2}_h)_{h \in I}$ be a family of finite dimensional subspaces of $H^1(\Omega) \times H^{1/2}(\Gamma_s) \times H^{-1/2}(\Gamma)$ where $I \subseteq (0, \infty)$ with $0 \in I$. Let us specify the finite dimensional ansatz spaces $H^1_h$, $\tilde{H}^{1/2}_h$ and $H^{-1/2}_h$ as follows. With the mesh parameter $h > 0$, we consider a nested regular quasi-uniform family $(\mathcal{T}_h)_h$ of meshes of $\Omega$ consisting of triangles, say, with a diameter between $c_1 \cdot h$ and $c_2 \cdot h$ ($0 < c_1 < c_2$), and denote by $H^1_h$ the space of the continuous and piecewise linear trial functions associated to the triangulation $\mathcal{T}_h$, that is,

$$H^1_h := \{u_h \in C(\Omega) : u_h | T \in P_1(T) \quad \forall T \in \mathcal{T}_h \} \subset H^1(\Omega).$$

The family $(\mathcal{T}_h)_h$ of meshes of $\Omega$ leads to the mesh family $(\mathcal{E}_h)_h$ of edges on the boundary $\Gamma$ so that we may introduce $H^{1/2}_h$ and $H^{-1/2}_h$ as the space of all continuous and piecewise linear functions, respectively, as the space of all piecewise constant functions on $\Gamma$, associated to the partition $\mathcal{E}_h$. Further, we assume that the partition $\mathcal{E}_h$ of the boundary leads to a partition $\mathcal{E}_h$ of $\Gamma_s$, so that $\tilde{H}^{1/2}_h$ becomes the subspace of those trial functions in $H^{1/2}_h$ that are supported on $\Gamma_s$. Thus we have

$$H^{-1/2}_h := \{w_h \in L^2(\Gamma) : w_h | E \in P_0(E) \quad \forall E \in \mathcal{E}_h \} \subset H^{-1/2}(\Gamma)$$

and

$$\tilde{H}^{1/2}_h := \{v_h \in C(\Gamma_s) : v_h | E \in P_1(E) \quad \forall E \in \mathcal{E}_h \} \subset \tilde{H}^{1/2}(\Gamma_s).$$

Hence, we may simply define $E_h := H^1_h \times \tilde{H}^{1/2}_h$.

For $h \in I$ let $i_h : H^1_h \hookrightarrow H^1(\Omega)$, $j_{h} : H^{1/2}_h \hookrightarrow H^{1/2}(\Gamma)$, and $k_{h} : H^{-1/2}_h \hookrightarrow H^{-1/2}(\Gamma)$ denote the canonical imbeddings with their duals $i^*_h$, $j^*_h$, and $k^*_h$. A straightforward discretization of the problem $(P_2)$ would lead to replace $S$ by $j_{h}^*Sj_{h}$. Unfortunately, the numerical calculation of $j_{h}^*Sj_{h}$ requires the numerical computation of $V^{-1}$. Since $V^{-1}$ is, in general, not known explicitly we have to approximate $V^{-1}$ and therefore introduce the subspace $H^{-1/2}_h$ to approximate the tractions on the interface. Here we follow the Costabel symmetric BEM/FEM coupling procedure and approximate $S$ by

$$S_h := \frac{1}{2} [j_h^*Wj_h + j_h^*(I - K')k_{h}(k_h^*Vk_{h})^{-1}k_h^*(I - K)j_{h}].$$

The computation of $S_h$ requires the numerical solution of a linear system with a symmetric, positive definite matrix $V_{h} := (k_h^*Vk_{h})$. In general $S_h \neq j_h^*Sj_{h}$, since $S_h$ is the Schur complement of a discretized matrix, what only needs the knowledge of $(k_h^*Vk_{h})^{-1}$, while $j_{h}^*Sj_{h}$ is the discretized Schur complement of operators, what needs the knowledge of $V^{-1}$. Here we recall two essential properties of $S_h$. 

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Lemma 5 [26, Lemma 12.2.9] There exist a constant $c_0 > 0$ and $h_0 \in I$ such that for any $h \in I$ with $h < h_0$ and any $v_h \in H_h^{1/2}$ there holds

$$\langle S_h v_h, v_h \rangle \geq c_0 \| v_h \|^2_{H^{1/2}(\Gamma)}. \quad (39)$$

Lemma 6 [7, Lemma 5.2],[13, (3.15)] There exists a constant $c > 0$ such that for all $h \in I$ and for all $v_h \in H_h^{1/2}$ there holds

$$\| S_h v_h - j_h^h S j_h v_h \|_{H^{-1/2}(\Gamma)} \leq c \text{ dist } H^{-1/2}(\Gamma) \left( V^{-1}(I - K) j_h v_h; H_h^{-1/2} \right).$$

Let $\mathcal{D}_h$ be another partition of $\Gamma_s$ consisting of elements $K_i$ joining the midpoints $P_{i-1/2}$. $P_{i+1/2}$ of the edges $E \in \mathcal{E}_h$ lying on $\Gamma_s$ sharing $P_i$ as a common point. If $P_i$ is a vertex of $\partial \Omega$ then $K_i$ is the half of the edge. On $\mathcal{D}_h$ we introduce the space $\mathcal{Y}_h$ of all piecewise constant functions by

$$\mathcal{Y}_h = \{ \mu_h \in L^\infty(\Gamma_s) : \mu_h|_K \in \mathbb{P}_0(K) \ \forall K \in \mathcal{D}_h \}$$

and define the piecewise constant Lagrange interpolation operator $L_h : \tilde{H}_h^{1/2} \rightarrow \mathcal{Y}_h$ by

$$L_h(w_h)(x) = \sum_i w_h(P_i) \chi_{\text{int } E, K_i}(x),$$

where $\chi_{\text{int } E, K_i}$ is the characteristic function of the interior of $K_i$ in $\Gamma_s$. Then we employ the approximation

$$\langle DJ_{\varepsilon,h}(v_h), w_h \rangle = \int_{\Gamma_s} j_x(s, \varepsilon, L_h(v)(s)) L_h(w)(s) \, ds. \quad (40)$$

Further we note that there holds with some positive constant $c$ not dependent on $h$, for any $v_h \in H_h^{1/2}$

$$|L_h v_h|_{L^2(\Gamma)} \leq c \| v_h \|_{L^2(\Gamma)} \quad (41)$$

$$\| L_h v_h - v_h \|_{L^2(\Gamma)} \leq c h^{1/2} \| v_h \|_{H^{1/2}(\Gamma)}, \quad (42)$$

what follows by real interpolation from standard estimates, see e.g. [19]. To simplify matters, we assume that the datum $u_0$ in the transmission condition (5) belongs to $H_h^{1/2}$ for all $0 < h < h_0$ and so $D_h := E_h \cap D$.

Now we can state the discrete version $(P_{\varepsilon,h})$ associated to the regularized problem $(P_{\varepsilon})$: Find $(\hat{u}_{\varepsilon,h}, \hat{v}_{\varepsilon,h}) \in \mathcal{D}_h$ such that

$$\mathcal{A}_h(\hat{u}_{\varepsilon,h}, \hat{v}_{\varepsilon,h}; u_h - \hat{u}_{\varepsilon,h}, v_h - \hat{v}_{\varepsilon,h}) + \langle DJ_{\varepsilon,h}(\hat{v}_{\varepsilon,h}), v_h - \hat{v}_{\varepsilon,h} \rangle = \lambda_h(u_h - \hat{u}_{\varepsilon,h}, v_h - \hat{v}_{\varepsilon,h}) \ \forall (u_h, v_h) \in D_h, \quad (43)$$

where $\mathcal{A}_h : H_h^1 \times \tilde{H}_h^{1/2} \rightarrow (H_h^1 \times \tilde{H}_h^{1/2})^*$ is defined via

$$\mathcal{A}_h(u_h, v_h)(r_h, s_h) = \mathcal{A}_h(u_h, v_h; r_h, s_h) := DG(u_h; r_h) + \langle S_h(u_h|_\Gamma - v_h), r_h|_\Gamma - s_h \rangle$$
and \( \lambda_h : H^1_h \times \widetilde{H}^{1/2}_h \to \mathbb{R} \) by
\[
\lambda_h(u_h, v_h) := \int_{\Omega} f_0 \cdot u_h \, dx + \langle t_0, u_h |_{\Gamma} - v_h \rangle \\
+ \langle i_h \gamma^* (W + (I - K')) k_h (k_h^* V k_h)^{-1} k_h^* (I - K) u_0, u_h |_{\Gamma} - v_h \rangle.
\]
The solvability of \((P_{\epsilon,h})\) follows from the existence result in [24, Theorem 3.1] for general pseudo-monotone bifunctions. Similar to Theorem 3 we can guarantee uniqueness of solutions \( \bar{u}_{\epsilon,h} := (\bar{u}_{\epsilon,h}, \bar{v}_{\epsilon,h}) \) to \((P_{\epsilon,h})\). Moreover, since DG is strongly monotone in \( H^1(\Omega) \) with respect to the semi-norm \( | \cdot |_{H^1(\Omega)} = \| \nabla \cdot \|_{L^2(\Omega)} \), see (14), and \( S_h \) is uniformly coercive in \( H^{1/2}(\Gamma) \), see Lemma 5, (39), the uniform boundedness of the family \((\bar{u}_{\epsilon,h})_h\) can be derived. To sum up, we have the following result.

**Theorem 4** Under the smallness condition \( 0 \leq \alpha < \epsilon_S \) along with the condition (38), for \( h < h_0 \) \((h_0 \text{ given in Lemma 5})\) the problem \((P_{\epsilon,h})\) has exactly one solution. Moreover, the family \((\bar{u}_{\epsilon,h})_h\) is uniformly bounded in \( E \).

Finally we can prove the following a priori error estimate.

**Theorem 5** For \((\bar{u}_{\epsilon}, \bar{v}_\epsilon) \in D\), the solution of problem \((P_{\epsilon})\), and \((\hat{u}_{\epsilon,h}, \hat{v}_{\epsilon,h}) \in D_h\), the solution of problem \((P_{\epsilon,h})\), there holds for \( h < h_0 \) and \( \alpha > 0 \) sufficiently small (in (38)) with a constant \( C > 0 \), which is independent of \( h \), but depends on \( \alpha \),
\[
C \|(\hat{u}_\epsilon - \hat{u}_{\epsilon,h}, \hat{v}_\epsilon - \hat{v}_{\epsilon,h})\|^2_{H^1(\Omega) \times \widetilde{H}^{1/2}(\Gamma)} \\
\leq \inf_{(u_h, v_h) \in D_h} \left\{ \| \bar{u}_\epsilon - u_h \|^2_{H^1(\Omega)} + \| \bar{v}_\epsilon - v_h \|^2_{H^{1/2}(\Gamma)} \\
+ \| \bar{v}_\epsilon - v_h \|^2_{L^2(\Gamma)} + \| \bar{u}_\epsilon - L_h v_h \|^2_{H^{-1/2}(\Gamma)} \right\} \\
+ \text{dist}_{H^{1/2}(\Gamma)} (V^{-1} (I - K) (\hat{u}_\epsilon + \hat{v}_\epsilon - u_0), H^{-1/2}_h)^2 + h^{1/2}.
\]

**Proof.** First, similar to the proof of Theorem 5 in [35], in virtue of Lemma 5, (39), further by (43) and (37), there holds for \( 0 < h < h_0 \) modulo a positive constant, independent of \( h \), for all \((u_h, v_h) \in D_h\)
\[
\| u_h |_{\Gamma} - \hat{u}_{\epsilon,h} |_{\Gamma} + v_h - \hat{v}_{\epsilon,h} \|^2_{H^{1/2}(\Gamma)} \\
\leq \langle S_h (u_h |_{\Gamma} - \hat{u}_{\epsilon,h} |_{\Gamma} + v_h - \hat{v}_{\epsilon,h}), u_h |_{\Gamma} - \hat{u}_{\epsilon,h} |_{\Gamma} + v_h - \hat{v}_{\epsilon,h} \rangle \\
= \langle S_h (u_h |_{\Gamma} - \hat{u}_{\epsilon,h} |_{\Gamma} + v_h - \hat{v}_{\epsilon,h}), u_h |_{\Gamma} - \hat{u}_{\epsilon,h} |_{\Gamma} + v_h - \hat{v}_{\epsilon,h} \rangle \\
+ \langle A_h (\hat{u}_{\epsilon,h}, \hat{v}_{\epsilon,h}) - \lambda_h (u_h - \hat{u}_{\epsilon,h}, v_h - \hat{v}_{\epsilon,h}) \\
+ \langle D \bar{u}_{\epsilon,h} \rangle - \lambda (\hat{u}_{\epsilon,h} - \hat{u}_{\epsilon,h} - \hat{v}_{\epsilon,h} \rangle \\
+ \langle DJ_{\epsilon,h}(\hat{v}_{\epsilon,h}), v_h - \hat{v}_{\epsilon,h} \rangle + \langle DJ_{\epsilon,h}(\hat{v}_{\epsilon,h}), \hat{v}_{\epsilon,h} - \hat{v}_{\epsilon,h} \rangle.
\]

To conclude the a priori error estimate (44), inspecting the proof of Theorem 5 in [35], we only need to estimate the last both terms in (45). For this purpose
Next we analyze the three summands in (46) separately. For the first, we obtain by Lemma 4, (34),

$$\int_{\Gamma_s} \hat{f}_x (L_h \hat{v}_{e,h}) (L_h v_h - \hat{v}_\epsilon) \, ds \lesssim \|(1 + |L_h \hat{v}_{e,h}|)\|L^2(\Gamma_s)\|L_h v_h - \hat{v}_\epsilon\|L^2(\Gamma_s),$$

and further, using the estimate (41),

$$\|(1 + |L_h \hat{v}_{e,h}|)\|L^2(\Gamma_s) \lesssim (1 + \|\hat{v}_{e,h}\|L^2(\Gamma_s)),$$

what is uniformly bounded by Theorem 4. For the second, using the estimate (42),

$$\|\hat{v}_{e,h} - L_h \hat{v}_{e,h}\|L^2(\Gamma_s) \lesssim h^{1/2} \|\hat{v}_{e,h}\|H^{1/2}(\Gamma_s),$$

where \(\|\hat{v}_{e,h}\|H^{1/2}(\Gamma_s)\) is uniformly bounded by Theorem 4. For the third term we use the assumption (38) and estimate

$$\int_{\Gamma_s} \left( \hat{f}_x (L_h \hat{v}_{e,h}) - \hat{f}_x (\hat{v}_\epsilon) \right) (\hat{v}_\epsilon - L_h \hat{v}_{e,h}) \, ds \leq \alpha \|\hat{v}_\epsilon - L_h \hat{v}_{e,h}\|L^2(\Gamma_s),$$

then by triangle inequality and further by (42) and Theorem 4,

$$\|\hat{v}_\epsilon - L_h \hat{v}_{e,h}\|L^2(\Gamma_s)^2 \leq 2 \|\hat{v}_\epsilon - L_h \hat{v}_{e,h}\|L^2(\Gamma_s)^2 + 2 \|\hat{v}_\epsilon - \hat{v}_{e,h}\|L^2(\Gamma_s)^2 \lesssim (h + \|\hat{v}_\epsilon - \hat{v}_{e,h}\|H^{1/2}(\Gamma_s)).$$

Finally, we combine the above estimates together with (45) and (46) and thus can extend the error estimate of Theorem 5 in [35] to arrive at (44) for small enough \(\alpha\).

In order to obtain an explicit convergence order we have to assume some regularity assumptions for the solution of the regularized problem \((\hat{u}_e, \hat{v}_\epsilon)\). Further, to simplify matters, let \(u_0 = 0\).

**Corollary 1** Let \((\hat{u}_e, \hat{v}_\epsilon) \in D \cap (H^2(\Omega) \times H^{3/2}(\Gamma_s))\) be the solution of \((P_e)\) and \((\hat{u}_{e,h}, \hat{v}_{e,h}) \in D_h\) be the solution of \((P_{e,h})\). Then, there exists a constant \(c > 0\) depending on \(\alpha, \Omega, \|\hat{u}_e\|H^2(\Omega), \|\hat{v}_\epsilon\|H^{3/2}(\Gamma),\) and the smallest angle in \(T \in \mathcal{T}_h\), such that for \(\alpha\) sufficiently small and \(0 < h < h_0\),

$$\|\hat{u}_e - \hat{u}_{e,h}, \hat{v}_\epsilon - \hat{v}_{e,h}\|H^1(\Omega) \times H^{1/2}(\Gamma_s) \leq ch^{1/4}.$$
Proof. We estimate the right hand side in the error estimate in Theorem 5. We choose \( u_h = \pi_h \hat{u}_\epsilon \) and \( v_h = \pi_h \hat{v}_\epsilon \), where \( \pi_h \) denotes piecewise linear interpolation. We use classical estimates in the spaces \( L^2, H^1 \), moreover

\[
\| \hat{v}_\epsilon - \pi_h \hat{v}_\epsilon \|_{H^{1/2}(\Gamma)} \lesssim h \| \hat{v}_\epsilon \|_{H^{3/2}(\Gamma)}. \tag{47}
\]

Thus by triangle inequality and by (42),

\[
\| L_h v_h - \hat{v}_\epsilon \|_{L^2(\Gamma)} \leq \| L_h v_h - v_h \|_{L^2(\Gamma)} + \| v_h - \hat{v}_\epsilon \|_{L^2(\Gamma)} \\
\lesssim h^{1/2} \| v_h \|_{H^{1/2}(\Gamma)} + \| v_h - \hat{v}_\epsilon \|_{H^{1/2}(\Gamma)} \\
\lesssim h^{1/2} (h \| \hat{v}_\epsilon \|_{H^{3/2}(\Gamma)} + \| \hat{v}_\epsilon \|_{H^{1/2}(\Gamma)}) + h \| \hat{v}_\epsilon \|_{H^{3/2}(\Gamma)}.
\]

Finally, for the polygon \( \Gamma \) we have \( V^{-1}(1 - K)(u|_{\Gamma} + v) \in H^{3/2}(\Gamma) \) and hence

\[
\text{dist}_{H^{-1/2}(\Gamma)} \left( V^{-1}(1 - K)(\hat{u}_\epsilon|_{\Gamma} + \hat{v}_\epsilon); H^{-1/2}_h \right) \lesssim h.
\]

Thus taking into account all the estimates above one concludes the assertion of the corollary. \( \square \)

We remark that Theorem 5 extends the error estimates in [7, Theorem 4] and [35, Theorem 5] to the nonlinear interface problem with a given non-monotone boundary condition, including the \( h \)-approximation of the regularized functional \( D_\epsilon \) by \( D_{\epsilon,h} \). In particular, Corollary 1 gives the same convergence rate \( O(h^{1/4}) \) as in [39].

7 Conclusions and an Outlook

This paper has shown how various techniques from different fields of mathematical analysis and numerical analysis can be combined to a solution procedure for a nonlinear interface problem that models nonmonotone frictional contact of elastic infinite media.

Here we have considered the coercive situation. A more delicate problem arises with a loss of coercivity, when only unilateral friction conditions, but no classical transmission conditions in the form of equalities are prescribed on the coupling boundary. For such semicoercive/noncoercive problems, albeit on a bounded domain, we can refer to [25].

The error analysis in this paper has focused to the classical \( h \)-method of BEM/FEM discretization, for the more versatile \( hp \)-BEM and \( hp \)-FEM applied to unilateral contact problems, albeit in bounded domains, we refer to [40] and to [23], respectively.

Another direction of research is the coupling of the BEM with other discretizations methods for the interior problem, like discontinuous Galerkin methods and mixed finite element methods; see e.g. [16] for the numerical analysis of a transmission problem with Signorini contact.
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