Polish $G$-spaces, the generalized model theory and complexity

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Abstract. Given Polish space $Y$ and a continuous language $L$ we study the corresponding logic $\text{Iso}(Y)$-space $Y_L$. We build a framework of generalized model theory towards analysis of Borel/algorithmic complexity of subsets of $Y^k_L \times (\text{Iso}(Y))^l$.

2010 Mathematics Subject Classification: 03E15, 03C07, 03C57
Keywords: Polish G-spaces, Continuous logic, Generalized model theory.

0 Introduction

Let $(Y, d)$ be a Polish space and $\text{Iso}(Y)$ be the corresponding isometry group endowed with the pointwise convergence topology. Then $\text{Iso}(Y)$ is a Polish group.

For any countable continuous signature $L$ the set $Y_L$ of all continuous metric $L$-structures on $(Y, d)$ can be considered as a Polish $\text{Iso}(Y)$-space. We call this action logic. It is known that the logic action is universal for Borel reducibility of orbit equivalence relations of Polish $G$-spaces with closed $G \leq \text{Iso}(Y)$ [15], [30]. Moreover by a result of J.Melleray in [10] every Polish group $G$ can be realised as the automorphism group of a continuous metric structure on an appropriate Polish space $(Y, d)$ and the structure is approximately ultrahomogeneous.

On the other hand typical notions naturally arising for logic actions can be applied in the general case of a Polish $G$-space $X$ with $G$ as above. If we consider $G$ together with a family of grey subgroups, then distinguishing an appropriate family $B$ of grey subsets of $X$ we arrive at the situation very similar to the logic space $Y^k_L$ (see [2], [12].
for non-Archimedian $G$ and \cite{30} for the general case). For example we can treat elements of $B$ as continuous formulas. Then many theorems of traditional model theory can be generalized to topological statements concerning spaces with nice topologies (i.e. defined by $B$). This approach is called the generalized model theory, \cite{3}.

The aim of this paper is to demonstrate that the tools of the generalized model theory nicely work for some other aspects of logic actions. Our basic concern is as follows. Viewing the logic space $Y_L$ as a Polish space and using the recipe of generalized model theory we distinguish some subsets of $Y_L^k \times (\text{iso}(Y))^l$ and then study Borel/algorithmic complexity of them.

We usually fix a countable dense subset $S_Y$ of $Y$ and study subsets of $Y_L$ which are invariant with respect to isometries stabilizing $S_Y$ setwise. This is the approach where a structure on $Y$ (say $M$) is considered together with its presentation over $S_Y$, i.e. together with the set

$$\text{Diag}(M, S_Y) = \{ (\phi, q) : M \models \phi < q, \text{ where } q \in [0, 1] \cap \mathbb{Q} \text{ and } \phi \text{ is a continuous sentence with parameters from } S_Y \}.$$ 

The best example of this situation is the logic space $\mathcal{U}_L$ over the bounded Urysohn metric space $\mathcal{U}$\footnote{in this paper we do not use the Urysohn space $\mathcal{U}$; we only use the ball of it of diameter 1 and denote it by $\Omega$} where distinguishing the countable counterpart $\mathcal{Q}\mathcal{U}$ of $\mathcal{U}$ (see Section 3 of \cite{30}) we study $\text{iso}(\mathcal{Q}\mathcal{U})$-invariant subsets of $\mathcal{U}_L$.

We demonstrate in Section 2 that this setting appears as a part of generalized model theory studied in \cite{3} and \cite{30}. Moreover our methods also work in the Hilbert space case and in the measure algebras case.

In Section 2.4 we consider other examples suggested by topological notions involving nice topologies. Since they correspond to natural logic constructions we view this material as a further development of generalized model theory.

In Section 3 we show that our approach gives a framework to computable members of $Y_L$ and their computable indexations. We will see that it supports standard approaches both to computable model theory and to effective metric spaces. Moreover it is suited to the general setting of computable Polish group actions presented in the recent paper \cite{41}.

In Section 4 we examine our approach in the cases of separable categoricity and ultrahomogeneity. In particular in Section 4.1 we find a Borel subset $\mathcal{S}\mathcal{C}$ of $Y_L$ which is $\text{iso}(S_Y)$-invariant and can be viewed as the set of all presentations over $S_Y$ of separably categorical structures on $Y$. Moreover in Section 4.2 we study complexity of the index set of computable members of $\mathcal{S}\mathcal{C}$.

The paper is self-contained. We address it to logicians and do not assume any special background. We believe that the ideas presented in it can be helpful in model theory, descriptive set theory and computability theory.
1 Logic space of continuous structures

Our paper belongs to a field of modern logic which can be situated between Invariant Descriptive Set Theory ([4], [27]) and Continuous Model Theory ([6], [10], [12]), see also [7], [13] - [15], [44]. The definitions below basically correspond to these sources.

1.1 Polish group actions.

A Polish group (group) is a separable, completely metrizable topological space (group). Sometimes we extend the corresponding metric to tuples by
\[ d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max(d(x_1, y_1), \ldots, d(x_n, y_n)). \]

If a Polish group \( G \) continuously acts on a Polish space \( X \), then we say that \( X \) is a Polish \( G \)-space. We say that a subset of \( X \) is invariant if it is \( G \)-invariant.

Let \((Y, d)\) be a Polish space and \(\text{Iso}(Y)\) be the corresponding isometry group endowed with the pointwise convergence topology. Then \(\text{Iso}(Y)\) is a Polish group. A compatible left-invariant metric can be obtained as follows: fix a countable dense set \( S = \{ s_i : i \in \{1, 2, \ldots\} \} \) and then define for two isometries \( \alpha \) and \( \beta \) of \( Y \)
\[ \rho_S(\alpha, \beta) = \sum_{i=1}^{\infty} 2^{-i} \min(1, d(\alpha(s_i), \beta(s_i))). \]

We will study closed subgroups of \(\text{Iso}(Y)\). We fix a dense countable set \( \Upsilon \subset \text{Iso}(Y) \).

In any closed subgroups of \(\text{Iso}(Y, d)\) we distinguish the base consisting of all sets of the form \( N_{\sigma, q} = \{ \alpha : \rho_S(\alpha, \sigma) < q \} \), \( \sigma \in \Upsilon \) and \( q \in \mathbb{Q} \). We may assume that \( \Upsilon \) is a subgroup of \(\text{Iso}(Y)\). To get this it is enough to replace \( \Upsilon \) by \( G_0 = \langle \Upsilon \rangle \).

1.2 Continuous structures.

We now fix a countable continuous signature
\[ L = \{ d, R_1, \ldots, R_k, \ldots, F_1, \ldots, F_l, \ldots \}. \]

Let us recall that a metric \( L \)-structure is a complete metric space \((M, d)\) with \( d \) bounded by 1, along with a family of uniformly continuous operations on \( M \) and a family of predicates \( R_i \), i.e. uniformly continuous maps from appropriate \( M^{k_i} \) to \([0, 1]\). It is usually assumed that \( L \) assigns to each predicate symbol \( R_i \) a continuity modulus \( \gamma_i : [0, 1] \rightarrow [0, 1] \) so that any metric structure \( M \) of the signature \( L \) satisfies the property that if \( d(x_j, x'_j) < \gamma_i(\varepsilon) \) with \( 1 \leq j \leq k_i \), then the inequality
\[ |R_i(x_1, \ldots, x_j, \ldots, x_{k_i}) - R_i(x_1, \ldots, x'_j, \ldots, x_{k_i})| < \varepsilon. \]

holds for the corresponding predicate of \( M \). It happens very often that \( \gamma_i \) coincides with \( \text{id} \). In this case we do not mention the appropriate modulus. We also fix continuity moduli for functional symbols.
Note that each countable structure can be considered as a complete metric structure with the discrete \{0, 1\}-metric.

Atomic formulas are the expressions of the form \(R_i(t_1, ..., t_r), d(t_1, t_2)\), where \(t_i\) are simply classical terms (built from functional \(L\)-symbols). We define formulas to be expressions built from 0, 1 and atomic formulas by applications of the following functions:

\[
x/2, \ x - y = \max(x - y, 0), \ \min(x, y), \ \max(x, y), \ |x - y|, \\
\neg(x) = 1 - x, \ x \dot{+} y = \min(x + y, 1), \ \sup_x \text{ and } \inf_x.
\]

**Statements** concerning metric structures are usually formulated in the form

\[\phi = 0,\]

where \(\phi\) is a formula. Sometimes statements are called **conditions**; we will use both names. A **theory** is a set of statements without free variables (here \(\sup_x\) and \(\inf_x\) play the role of quantifiers).

We often extend the set of formulas by the application of **truncated products** by positive rational numbers. This means that when \(q \cdot x\) is greater than 1, the truncated product of \(q\) and \(x\) is 1. Since the context is always clear, we preserve the same notation \(q \cdot x\). The continuous logic after this extension does not differ from the basic case.

It is worth noting that the choice of the set of connectives guarantees that for any continuous relational structure \(M\), any formula \(\phi\) is a \(\gamma\)-uniform continuous function from the appropriate power of \(M\) to \([0, 1]\), where \(\gamma(\varepsilon)\) is of the form

\[
\frac{1}{n} \cdot \min\{\gamma'(\varepsilon) : \gamma' \text{ is a continuity modulus of an } L\text{-symbol appearing in the formula}\},
\]

where the number \(n\) only depends on the complexity of \(\phi\).

This follows from the fact that when \(\phi_1\) and \(\phi_2\) have continuity moduli \(\gamma_1\) and \(\gamma_2\) respectively, then the formula \(f(\phi_1, \phi_2)\) obtained by applying a binary connective \(f\), has a continuity modulus of the form \(\min(\gamma_1(\frac{1}{2}x), \gamma_2(\frac{1}{2}x))\).

It is observed in Appendix A of [12] that instead of continuity moduli one can consider **inverse continuity moduli**. Slightly modifying that place in [12] we define it as follows.

**Definition 1.1** A continuous monotone function \(\delta : [0, 1] \to [0, 1]\) with \(\delta(0) = 0\) is an inverse continuity modulus of a map \(F(x) : X^n \to [0, 1]\) if for any \(\bar{a}, \bar{b}\) from \(X^n\),

\[
|F(\bar{a}) - F(\bar{b})| \leq \delta(d(\bar{a}, \bar{b})).
\]

The choice of the connectives above guarantees that the following statement holds (see [30]).

**Lemma 1.2** For any continuous relational structure \(M\), where each \(n\)-ary relation has \(n \cdot \text{id}\) as an inverse continuity modulus, any formula \(\phi\) admits an inverse continuity modulus which is of the form \(k \cdot \text{id}\), where \(k\) depends on the complexity of \(\phi\).
Remark 1.3 By Lemma 4.1 of [10] each \( n \)-ary functional symbol \( F \) can be replaced by the predicate \( D_F(\bar{x}, y) = d(F(\bar{x}), y) \). It is clear that the continuity moduli with respect to variables from \( \bar{x} \) are the same and \( \text{id} \) works as a continuity modulus for \( y \). Thus we may always assume that \( L \) is relational.

For a continuous structure \( M \) defined on \( (Y, d) \) let \( \text{Aut}(M) \) be the subgroup of \( \text{Iso}(Y) \) consisting of all isometries preserving the values of atomic formulas. It is easy to see that \( \text{Aut}(M) \) is a closed subgroup with respect to the topology on \( \text{Iso}(Y) \) defined above.

For every \( c_1, ..., c_n \in M \) and \( A \subseteq M \) we define the \( n \)-type \( \text{tp}(\bar{c}/A) \) over \( A \) as the set of all \( \bar{x} \)-conditions with parameters from \( A \) which are satisfied by \( \bar{c} \) in \( M \). Let \( S_n(T_A) \) be the set of all \( n \)-types over \( A \) of the expansion of the theory \( T \) by constants from \( A \). There are two natural topologies on this set. The logic topology is defined by the basis consisting of sets of types of the form \([\phi(\bar{x}) < \varepsilon]\), i.e. types containing some \( \phi(\bar{x}) \leq \varepsilon' \) with \( \varepsilon' < \varepsilon \). The logic topology is compact.

The \( d \)-topology is defined by the metric
\[
d(p, q) = \inf \{d(\bar{c}, \bar{b}) \mid \text{there is a model } M \text{ with } M \models p(\bar{c}) \land q(\bar{b}) \}.\]

By Propositions 8.7 and 8.8 of [6] the \( d \)-topology is finer than the logic topology and \( (S_n(T_A), d) \) is a complete space.

The following notion is helpful when we study some concrete examples, for example the Urysohn space. A relational continuous structure \( M \) is approximately ultrahomogeneous if for any \( n \)-tuples \((a_1, ..., a_n)\) and \((b_1, ..., b_n)\) with the same quantifier-free type (i.e. with the same values of predicates for corresponding sub tuples) and any \( \varepsilon > 0 \) there exists \( g \in \text{Aut}(M) \) such that
\[
\max \{d(g(a_j), b_j) : 1 \leq j \leq n\} \leq \varepsilon.
\]

As we already mentioned any Polish group can be chosen as the automorphism group of a continuous metric structure which is approximately ultrahomogeneous.

The bounded Urysohn space \( U \) (see Section 2.3) is ultrahomogeneous in the traditional sense: any partial isomorphism between two tuples extends to an automorphism of the structure [17]. Note that this obviously implies that \( U \) is approximately ultrahomogeneous.

We will use the continuous version of \( L_{\omega_1\omega} \) from [8] (see also [10]. We remind the reader that continuous \( L_{\omega_1\omega} \)-formulas are defined by the standard procedure applied to countable conjunctions and disjunctions (see [8]). Each continuous infinite formula depends on finitely many free variables. The main demand is the existence of continuity moduli of such formulas. It is usually assumed that a continuity modulus \( \delta_{\phi,x} \) satisfies the equality
\[
\delta_{\phi,x}(\varepsilon) = \sup \{\delta_{\phi,x}(\varepsilon') : 0 < \varepsilon' < \varepsilon\}
\]
and
\[
\delta_{\bigwedge \Phi,x}(\varepsilon) = \sup \{\delta_{\bigwedge \Phi,x}(\varepsilon') : 0 < \varepsilon' < \varepsilon\}, \text{ where } \delta_{\bigwedge \Phi,x} = \inf \{\delta_{\phi,x} : \phi \in \Phi\}.
\]
1.3 Logic action

Fix a countable continuous signature

\[ L = \{d, R_1, ..., R_k, ..., F_1, ..., F_l, ...\} \]

and a Polish space \((Y, d)\). Let \(S\) be a dense countable subset of \(Y\). Let \(\text{seq}(S) = \{\bar{s}_i : i \in \omega\}\) be the set (and an enumeration) of all finite sequences (tuples) from \(S\). Let us define the space of metric \(L\)-structures on \((Y, d)\). Using the recipe as in the case of \(\text{Iso}(Y)\) we introduce a metric on the set of \(L\)-structures as follows. Enumerate all tuples of the form \((\varepsilon, j, \bar{s})\), where \(\varepsilon \in \{0, 1\}\) and when \(\varepsilon = 0\), \(\bar{s}\) is a tuple from \(\text{seq}(S)\) of the length of the arity of \(R_j\), and for \(\varepsilon = 1\), \(\bar{s}\) is a tuple from \(\text{seq}(S)\) of the length of the arity of \(F_j\). For metric \(L\)-structures \(M\) and \(N\) let

\[
\delta_{\text{seq}(S)}(M, N) = \sum_{i=1}^{\infty} \{2^{-i}|R_j^M(\bar{s}) - R_j^N(\bar{s})|^2 : i \text{ is the number of } (\varepsilon, j, \bar{s})\}.
\]

Since the predicates and functions are uniformly continuous (with respect to moduli of \(L\)) and \(S\) is dense in \(Y\), we see that \(\delta_{\text{seq}(S)}\) is a complete metric. Moreover by an appropriate choice of rational values for \(R_j(\bar{s})\) we find a countable dense subset of metric structures on \(Y\), i.e. the space obtained is Polish. We denote it by \(Y_L\). It is clear that \(\text{Iso}(Y)\) acts on \(Y_L\) continuously. Thus we consider \(Y_L\) as an \(\text{Iso}(Y)\)-space and call it the space of the logic action on \(Y\).

Remark 1.4 It is worth noting that in this definition a structure on \(Y\) (say \(M\)) is identified with its presentation \(\text{Diag}(M, S)\), see Introduction.

It is convenient to consider the following basis of the topology of \(Y_L\). Fix a finite sublanguage \(L' \subset L\), a finite subset \(S' \subset S\), a finite tuple \(q_1, ..., q_l \in Q \cap [0, 1]\) and a rational \(\varepsilon \in [0, 1]\) with \(1 - \varepsilon < 1/2\). Consider a diagram \(D\) of \(L'\) on \(S'\) of some inequalities of the form

\[
d(F_j(\bar{s}), s') > \varepsilon, \quad d(F_j(\bar{s}), s') < 1 - \varepsilon, \quad |R_j(\bar{s}) - q_i| > \varepsilon, \quad |R_j(\bar{s}) - q_i| < 1 - \varepsilon, \quad \text{with } \bar{s} \in \text{seq}(S'), s' \in S'.
\]

(i.e. in the case of relations we consider negations of statements of the form: \(|R_j(\bar{s}) - q_i| \leq \varepsilon\), \(|R_j(\bar{s}) - q_i| \geq 1 - \varepsilon\)). The set of metric \(L\)-structures realizing \(D\) is an open set of the topology of \(Y_L\) and the family of sets of this form is a basis of this topology. Compactness theorem for continuous logic (see [12]) shows that the topology is compact. We will call it logic too.

If in Remark 1.4 one relax the conditions on formulas \(\phi\) used in \(\text{Diag}(M, S)\) (for example allowing \(\phi\) to be from some \(L_{\omega_1\omega}\)-fragments) the topology can become richer (and the basis should be corrected). Moreover by the continuous version of the Lopez-Escobar theorem ([10], [15]) every Polish group action arises as an action of some closed \(G \leq \text{Iso}(Y)\) on the space of separable continuous structures of some \(L_{\omega_1\omega}\)-sentence, [15]. This possibility will be discussed in the next section.

\(2\) resp. \(2^{-1}d(F_j^M(\bar{s}), F_j^N(\bar{s}))\) when \(\varepsilon = 1\)
2 Good and nice topologies

In Section 2.2 we give the main concepts of the generalized model theory. They are based on the notion of a grey subset introduced in [9]. The corresponding preliminaries are given in Section 2.1. In Section 2.3 we describe the most important examples of the situation. In Section 2.4 we demonstrate several applications of our approach. They concern complexity of some subsets of the logic space. In a sense this section explains the reason why the questions of complexity are considered under the framework of good/nice topologies (of Section 2.2).

2.1 Grey subsets

The notion of grey subsets was introduced in [9]. It has become very fruitful, see [10], [15], [30] and [14].

A function $\phi$ from a space $X$ to $[-\infty, +\infty]$ is upper (lower) semi-continuous if the set $\phi_{<r}$ (resp. $\phi_{>r}$) is open for all $r \in \mathbb{R}$ (here $\phi_{<r} = \{ z \in X : \phi(z) < r \}$, a cone). A grey subset of $X$, denoted $\phi \subseteq X$, is a function $X \rightarrow [0, \infty]$. It is open (closed), $\phi \subseteq_o X$ (resp. $\phi \subseteq_c X$), if it is upper (lower) semi-continuous. We also write $\phi \in \Sigma_1$ when $\phi \subseteq_o X$ and we write $\phi \in \Pi_1$ when $\phi \subseteq_c X$. We will assume below that values of a grey subset belong to $[0, 1]$.

Let us return to the situation of Section 1.3. We fix a language $L$, a countable dense subset $S$ of $Y$ and study subsets of $Y_L$. One of the basic observations is that any first-order continuous sentence $\phi(\bar{c})$, $\bar{c} \in S$, defines a grey subset of $Y_L$:

$$\phi(\bar{c}) \text{ takes } M \text{ to the value } \phi^M(\bar{c}).$$

Moreover Proposition 2.1 below says that $\phi(\bar{c})$ defines a grey subset of $Y_L$ which belongs to $\Sigma_n$ for some $n$. It is Proposition 1.1 in [30].

**Proposition 2.1** For any continuous formula $\phi(\bar{v})$ of the language $L$ there is a natural number $n$ such that for any tuple $\bar{a} \in S$ and $\varepsilon \in [0, 1]$, the subset

$$\text{Mod}(\phi, \bar{a}, < \varepsilon) = \{ M : M \models \phi(\bar{a}) < \varepsilon \}$$

(or $\text{Mod}(\phi, \bar{a}, > \varepsilon) = \{ M : M \models \phi(\bar{a}) > \varepsilon \}$)

of the space $Y_L$ of $L$-structures, belongs to $\Sigma_n$.

When $G$ is a Polish group, then a grey subset $H \subseteq G$ is called a grey subgroup if

$$H(1) = 0, \forall g \in G(H(g) = H(g^{-1})) \text{ and } \forall g, g' \in G(H(gg') \leq H(g) + H(g')).$$

This is equivalent to Definition 2.5 from [9]. It is worth noting that by Lemma 2.6 of [9] an open grey subgroup is clopen.
If $H$ is a grey subgroup, then for every $g \in G$ we define the grey coset $Hg$ and the grey conjugate $H^g$ as follows:

$$Hg(h) = H(hg^{-1})$$
$$H^g(h) = H(ghg^{-1}).$$

Observe that if $H$ is open, then $Hg$ is an open grey subset and $H^g$ is an open grey subgroup.

**Definition 2.2** Let $X$ be a continuous $G$-space. A grey subset $\phi \subseteq X$ is called invariant with respect to a grey subgroup $H \subseteq G$ if for any $g \in G$ and $x \in X$ we have $\phi(g(x)) \leq \phi(x) + H(g)$.

Since $H(g) = H(g^{-1})$, the inequality from the definition is equivalent to $\phi(x) \leq \phi(g(x)) + H(g)$.

**Remark 2.3** (see Section 2.1 of [30]). It is clear that for every continuous structure $M$ (defined on $Y$) any continuous formula $\phi(\bar{x})$ defines a clopen grey subset of $M|_{\bar{x}}$. Moreover note that when $\phi(\bar{x}, \bar{c})$ is a continuous formula with parameters $\bar{c} \in M$ and $\delta$ is a linear inverse continuous modulus for $\phi(\bar{x}, \bar{y})$ (see Definition 1.1), then $\phi$ is invariant with respect to the open grey subgroup $H_{\delta, \bar{c}} \subseteq \text{Aut}(M)$ defined by

$$H_{\delta, \bar{c}}(g) = \delta(d((c_1, \ldots, c_n), (g(c_1)), \ldots, g(c_n))),$$

where $g \in \text{Aut}(M)$,

i.e.

$$\phi(g(\bar{a}), \bar{c}) \leq \phi(\bar{a}, \bar{c}) + H_{\delta, \bar{c}}(g).$$

In the space of continuous $L$-structures $Y_L$ this remark has the follows version (see Lemma 2.2 in [30]).

**Lemma 2.4** Let $\delta$ be an inverse continuity modulus for $\phi(\bar{x})$, which is linear. The grey subset defined by $\phi(\bar{c}) \subseteq Y_L$ is invariant with respect to the grey stabiliser $H_{\delta, \bar{c}} \subseteq \text{Iso}(Y)$ defined as follows.

$$H_{\delta, \bar{c}}(g) = \delta(d((c_1, \ldots, c_n), (g(c_1)), \ldots, g(c_n))),$$

where $g \in \text{Iso}(Y)$.

### 2.2 Nice bases

In this section we consider a certain class of Polish $G$-spaces. To describe it we need the following definition.

**Definition 2.5** A family $\mathcal{U}$ of open grey subsets of a Polish space $X$ with a topology $\tau$ is called a grey basis of $\tau$ if the family $\{\phi_{<r} : \phi \in \mathcal{U}, r \in \mathbb{Q} \cap (0, 1)\}$ is a basis of $\tau$.

We now describe our typical assumptions on $G$:
• $G$ is a Polish group;

• we distinguish a countable dense subgroup $G_0 < G$ and a countable family of clopen grey subsets $\mathcal{R}$ of $G$ which is a grey basis of the topology of $G$;

• we assume that $\mathcal{R}$ consists of all $G_0$-cosets of grey subgroups from $\mathcal{R}$, i.e. for each $\rho \in \mathcal{R}$ there is a grey subgroup $H \in \mathcal{R}$ and an element $g_0 \in G_0$ so that for any $g \in G$, $\rho(g) = H(g g_0^{-1})$;

• we assume that $\mathcal{R}$ is closed under $G_0$-conjugacy, under $\text{max}$ and truncated multiplication by positive rational numbers.

Remark 2.6 In Remark 2.9 of [30] it is observed that for every Polish group $G$ there is a a countable $G_0 < G$ and a countable family of open grey subsets $\mathcal{R}$ satisfying these assumptions.

We will see below that if the space $(Y, d)$ is good enough (for example the bounded Urysohn space) and $S$ is a dense countable subset of $Y$, then the family $\mathcal{R}$ of grey subsets of $G = \text{Iso}(Y)$ can be chosen among grey cosets of the form

\[ \rho(g) = q \cdot d(\bar{b}, g(\bar{a})) , \]

where $q \in \mathbb{Q}^+$ and

$\bar{a}, \bar{b}$ are tuples from $S$ which are isometric in $Y$.

If the metric is bounded by 1 we mean the truncated multiplication by $q$ in the formula above.

When we fix $G_0$, $\mathcal{R}$ and consider a Polish $G$-space $(X, d)$ we also distinguish a countable grey basis $\mathcal{U}$ of the topology of $X$. Let $\tau$ be the corresponding topology.

The approach of generalized model theory of H. Becker from [3] suggests that along with the $d$-topology $\tau$ we shall consider some special topology on $X$ which is called nice. In the case of Polish $G$-spaces this idea has been realized in [30] with using continuous logic. Since we do not need the corresponding material in exact form we introduce the following very general definition.

Definition 2.7 Let $\mathcal{R}$ be a grey basis of $G$ consisting of cosets of open grey subgroups of $G$ which also belong to $\mathcal{R}$. Assume that the subfamily of $\mathcal{R}$ of all open grey subgroups is closed under $\text{max}$ and truncated multiplication by numbers from $\mathbb{Q}^+$.

We say that a family $\mathcal{B}$ of Borel grey subsets of the $G$-space $(X, \tau)$ is a good basis with respect to $\mathcal{R}$ if:

(i) $\mathcal{B}$ is countable and generates the topology finer than $\tau$;

(ii) for each $\phi \in \mathcal{B}$ there exists an open grey subgroup $H \in \mathcal{R}$ such that $\phi$ is $H$-invariant.

It will be usually assumed that all constant functions $q$, $q \in \mathbb{Q} \cap [0, 1]$ are in $\mathcal{B}$.

Definition 2.8 A topology $\mathbf{t}$ on $X$ is $\mathcal{R}$-good for the $G$-space $(X, \tau)$ if the following conditions are satisfied.

(a) The topology $\mathbf{t}$ is Polish, $\mathbf{t}$ is finer than $\tau$ and the $G$-action remains continuous
with respect to $t$.

(b) There exists a grey basis $\mathcal{B}$ of $t$ which is good with respect to $\mathcal{R}$.

Nice bases and nice topologies introduced in [30] are good. We remind the reader that a good basis $\mathcal{B}$ with respect to $\mathcal{R}$ is nice if the following additional properties hold:

(iii) for all $\phi_1, \phi_2 \in \mathcal{B}$, the functions $-\phi_1, \min(\phi_1, \phi_2), \max(\phi_1, \phi_2), |\phi_1 - \phi_2|, \phi_1 - \phi_2, \phi_1 + \phi_2$ belong to $\mathcal{B}$;

(iv) for all $\phi \in \mathcal{B}$ and $q \in \mathbb{Q}^+$ the truncated product $q \cdot \phi$ belongs to $\mathcal{B}$;

(v) for all $\phi \in \mathcal{B}$ and open grey subsets $\rho \in \mathcal{R}$ the Vaught transforms (see Section 2.1 in [30]) $\phi^\rho, \phi^{\Delta \rho}$ belong to $\mathcal{B}$.

In the situation of standard examples (see Section 2.3) the property that the basis is good is straightforward. It is much more difficult to prove that the basis is nice. Theorem 3.2 from [30] is an example of a result of this kind. The following theorem gives existence of nice topologies. This is Theorem 2.12 in [30]. Note that the assumptions on the grey basis $\mathcal{R}$ follow from the conditions of/before Remark 2.6.

**Theorem 2.9** Let $G$ be a Polish group and $\mathcal{R}$ be a countable grey basis satisfying the assumptions of Definition 2.7 and the following closure property:

for every grey subgroup $H \in \mathcal{R}$ and every $g \in G$ if $Hg \in \mathcal{R}$, then $H^g \in \mathcal{R}$.

Let $(X, \tau)$ be a Polish $G$-space and $\mathcal{F}$ be a countable family of Borel grey subsets of $X$ generating a topology finer than $\tau$ such that for any $\phi \in \mathcal{F}$ there is a grey subgroup $H \in \mathcal{R}$ such that $\phi$ is invariant with respect to $H$.

Then there is an $\mathcal{R}$-nice topology for $G$-space $(X, \tau)$ such that $\mathcal{F}$ consists of open grey subsets.

In Section 2.4 we describe possible applications of statements of this kind.

### 2.3 Countable approximating substructures

In this section we give basic examples of good bases and topologies on some logic spaces. The following definition is taken from [7].

**Definition 2.10** Let $(M, d)$ be a Polish metric structure with universe $M$. We say that a (classical) countable structure $N$ is a countable approximating substructure of $M$ if the following conditions are satisfied:

- The universe $N$ of $N$ is a dense countable subset of $(M, d)$.
- Any automorphism of $N$ extends to a (necessarily unique) automorphism of $M$, and $\text{Aut}(N)$ is dense in $\text{Aut}(M)$.

Let $G_0$ be a dense countable subgroup of $\text{Aut}(N)$. We may consider it as a subgroup of $\text{Aut}(M)$. 
Family $\mathcal{R}^M(G_0)$. Let $\mathcal{R}_0$ be the family of all clopen grey subgroups of $\text{Aut}(M)$ of the (truncated) form

$$H_{q,s} : g \rightarrow q \cdot d(g(s),\bar{s}), \text{ where } \bar{s} \subseteq N, \text{ and } q \in \mathbb{Q}^+.$$ 

It is clear that $\mathcal{R}_0$ is closed under conjugacy by elements of $G_0$. Consider the closure of $\mathcal{R}_0$ under the function $\max$ and define $\mathcal{R}^M(G_0)$ to be the family of all $G_0$-cosets of grey subgroups from $\max(\mathcal{R}_0)$. Then $\mathcal{R}^M(G_0)$ is countable and the family of all $(H_{q,s})_{<l}$ where $H \in \mathcal{R}_0$ and $l \in \mathbb{Q}$, generates the topology of $\text{Aut}(M,d)$. Moreover it is easy to see that $G_0$ and $\mathcal{R}^M(G_0)$ satisfy all the conditions of/before Remark 2.6 for $\mathcal{R}$ and in particular $\mathcal{R}^M(G_0)$ satisfies the conditions of Theorem 2.9 for $\mathcal{R}$.

Family $\mathcal{B}_L$. Let $L$ be a relational language of a continuous signature with inverse continuity moduli $\leq n \cdot \text{id}$ for $n$-ary relations. We will assume that $L$ extends the language of the structure $M$.

Let $\mathcal{L}$ be a countable fragment of $L_{\omega_1\omega}$, in particular $\mathcal{L}$ is closed under first-order connectives. Note that inverse continuity moduli of first-order continuous formulas (with connectives as in Introduction) can be taken linear (of the form $k \cdot \text{id}(x)$). Thus it is easy to see that every formula of $\mathcal{L}$ has linear inverse continuity moduli.

Let $\mathcal{B}_L$ be the family of all grey subsets defined on the logic space $M_L$ by continuous $\mathcal{L}$-sentences (with parameters) as follows

$$\phi(s) : M \rightarrow \phi^M(\bar{s}), \text{ where } \bar{s} \subseteq N \text{ and } \phi(\bar{x}) \in \mathcal{L}.$$ 

By linearity of inverse continuity moduli it is easy to see that for any continuous sentence $\phi(s)$ there is a number $q \in \mathbb{Q}$ (depending on the continuity modulus of $\phi$) such that the grey subset as above is $H_{q,s}$-invariant. As a result we have the following statement.

Let $\mathcal{B}_L$ be a family of grey subsets corresponding to a countable continuous fragment $\mathcal{L}$ of $L_{\omega_1\omega}$. Then the family $\mathcal{B}_L$ is a good basis with respect to $\mathcal{R}^M(G_0)$.

(A) Let us consider the following example. The Urysohn space of diameter 1 is the unique Polish metric space of diameter 1 which is universal and ultrahomogeneous. This space $\mathfrak{U}$ is considered in the continuous signature $\langle d \rangle$.

The countable counterpart of $\mathfrak{U}$ is the rational Urysohn space of diameter 1, $\mathbb{Q}\mathfrak{U}$, which is both ultrahomogeneous and universal for countable metric spaces with rational distances and diameter $\leq 1$. The space $\mathfrak{U}$ is interpreted as $M$ above and $\mathbb{Q}\mathfrak{U}$ will be our $N$. It is shown in Section 6.1 of [7] that there is an embedding of $\mathbb{Q}\mathfrak{U}$ into $\mathfrak{U}$ so that:

(i) $\mathbb{Q}\mathfrak{U}$ is an approximating substructure of $\mathfrak{U}$: it is dense in $\mathfrak{U}$; any isometry of $\mathbb{Q}\mathfrak{U}$ extends to an isometry of $\mathfrak{U}$ and $\text{Iso}(\mathbb{Q}\mathfrak{U})$ is dense in $\text{Iso}(\mathfrak{U})$;

(ii) for any $\varepsilon > 0$, any partial isometry $h$ of $\mathbb{Q}\mathfrak{U}$ with domain $\{a_1,...,a_n\}$ and any isometry $g$ of $\mathfrak{U}$ such that $d(g(a_i),h(a_i)) < \varepsilon$ for all $i$, there is an isometry $\hat{h}$ of $\mathbb{Q}\mathfrak{U}$ that extends $h$ and is such that for all $x \in \mathfrak{U}$, $d(\hat{h}(x),g(x)) < \varepsilon$. 

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Let $G_0$ be a dense countable subgroup of $\text{Iso}(\mathbb{Q})$. By (i) we may consider it as a subgroup of $\text{Iso}(\mathcal{U})$. We now define $\mathcal{R}^\mathcal{U}(G_0)$ by the recipe above. As we already know $G_0$ and $\mathcal{R}^\mathcal{U}(G_0)$ satisfy all the conditions of/before Remark 2.6 and in particular $\mathcal{R}^\mathcal{U}(G_0)$ satisfies the conditions of Theorem 2.9.

Let $L$ be a relational language of a continuous signature as above. Let $\mathcal{L}$ be a countable fragment of $L^{\omega_1\omega}$ and let $\mathcal{B}_\mathcal{L}$ be the family of all grey subsets defined by continuous $\mathcal{L}$-sentences (with parameters from $\mathbb{Q}$) as above. We already know that $\mathcal{B}_\mathcal{L}$ is a good basis. It is proved in [30] (see Theorem 3.2) that this basis is nice.

**Theorem 2.11** The family $\mathcal{B}_\mathcal{L}$ is a $\mathcal{R}^\mathcal{U}(G_0)$-nice basis.

(B) A separable Hilbert space. We follow [7] and [43]. Let us consider the complex Hilbert space $l_2(\mathbb{N})$. Let $Q$ denote the algebraic closure of $\mathbb{Q}$, and consider the countable subset $Ql_2$ of $l_2(\mathbb{N})$ of all sequences with finite support and coordinates from $Q$. It is shown in Section 6.2 of [7] (with use Section 7 of [43]), that it is an approximating substructure of $l_2(\mathbb{N})$. In particular we have another pair playing the role of $(\mathbb{M}, N)$. Since $l_2(\mathbb{N})$ is unbounded, the authors of [7] consider instead its closed unit ball, equipped with functions $x \to \alpha x$ for $|\alpha| \leq 1$ and $(x, y) \to \frac{x + y}{2}$, from which $l_2(\mathbb{N})$ can be recovered. According Remark 1.3 we will consider a relational language for this structure.

The automorphism group of the unit ball is $U(l_2(\mathbb{N}))$, the unitary group of the whole complex Hilbert space $l_2(\mathbb{N})$. The topology of pointwise convergence is the strong operator topology.

Let $G_0$ be a dense countable subgroup of $U(Ql_2)$. We may consider it as a subgroup of $U(l_2(\mathbb{N}))$. We now apply the procedure of $\mathcal{R}^\mathcal{U}(G_0)$ and $\mathcal{B}_\mathcal{L}$. As a result we obtain the family $\mathcal{R}^\mathcal{H}(G_0)$ and a grey basis defined on $U(l_2(\mathbb{N}))$.

Let $L$ be a relational language of a continuous signature extending the language of the unit ball and satisfying the assumptions above and let $\mathcal{L}$ be a countable fragment of $L^{\omega_1\omega}$. Let $\mathcal{B}_\mathcal{L}$ be the corresponding family of all grey subsets of the logic space $l_2(\mathbb{N})_L$. This is a good basis with respect to $\mathcal{R}^\mathcal{H}(G_0)$.

(C) The measure algebra on $[0, 1]$. Denote by $\lambda$ the Lebesgue measure on the unit interval $[0, 1]$. We view its automorphism group $\text{Aut}([0, 1], \lambda)$ as the automorphism group of the Polish metric structure

$$(MALG, 0, 1, \land, \lor, \neg, d),$$

where $MALG$ denotes the measure algebra on $[0, 1]$ and $d(A, B) = \lambda(A \Delta B)$ (see [Kec95]). The approximating substructure is the countable measure algebra $A$ generated by dyadic intervals. This is observed in Section 6.3 od [7]. Exactly as in the case of $\mathcal{U}$ and $l_2(\mathbb{N})$ one can define a family of open grey subgroups of $\text{Aut}([0, 1], \lambda)$, say $\mathcal{R}^\text{Aut}(G_0)$, and a good bases of the corresponding logic spaces with respect to $\mathcal{R}^\text{Aut}(G_0)$. 

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Remark 2.12 The basic continuous metric structures which appear in (A) - (C), i.e. \( \mathcal{U} \), the unit ball of \( l_2(\mathbb{N}) \) and MALG, are ultahomogeneous structures in the classical sense: any partial isomorphism between two tuples extends to an automorphism of the structure. This is in particular mentioned in Section 3.1 of [5].

2.4 The Effros Borel structure of \( \text{Iso}(\mathcal{U}) \). Applications

Given a Polish space \( Y \) let \( \mathcal{F}(Y) \) denote the set of closed subsets of \( Y \). The Effros structure on \( \mathcal{F}(Y) \) is the Borel space with respect to the \( \sigma \)-algebra generated by the sets

\[ C_U = \{ D \in \mathcal{F}(Y) : D \cap U \neq \emptyset \}, \]

for open \( U \subseteq Y \). For various \( Y \) this space serves for analysis of Borel complexity of families of closed subsets (see [33] and [44] some recent results). It is convenient to use the fact that there is a sequence of Kuratowski-Ryll-Nardzewski selectors \( s_n : \mathcal{F}(Y) \to Y, \ n \in \omega \), which are Borel functions such that for every non-empty \( F \in \mathcal{F}(Y) \) the set \( \{ s_n(F) ; n \in \omega \} \) is dense in \( F \).

Given a Polish group \( G \) and a continuous (or Borel) action of \( G \) on a Polish space \( Y \) one can consider the Borel space

\[ \mathcal{F}(Y)^m \times \mathcal{F}(G)^n. \]

In the situation when \( Y \) and \( G \) have grey bases \( \mathcal{B} \) and \( \mathcal{R} \) respectively which satisfy the conditions of Theorem 2.9 one can consider \( Y^m \) with respect to the good topology induced by \( \mathcal{B} \) (say \( \mathbf{t} \)). Then many natural Borel subsets of \( Y^m \times G^n \) can be viewed as elements of

\[ \mathcal{F}((Y, \mathbf{t}))^m \times \mathcal{F}(G)^n. \]

By Theorem 2.2 of [15] for any Polish group \( G \) and any standard Borel \( G \)-space \( X \) there is a continuous group monomorphism \( \Phi : G \to \text{Iso}(\mathcal{U}) \) and a Borel \( \Phi \)-equivariant injection \( f : X \to \mathcal{U}_L \). We only need here that the language \( L \) is countable relational with 1-Lipschitz symbols of unbounded arity. As a result all Polish groups can be considered as elements of \( \mathcal{F}(\text{Iso}(\mathcal{U})) \), all Polish spaces are elements of \( \mathcal{F}(\mathcal{U}_L) \) and Polish \( G \)-spaces are pairs from

\[ \mathcal{F}(\mathcal{U}_L) \times \mathcal{F}(\text{Iso}(\mathcal{U})). \]

Let \( \mathcal{R}^\mathcal{U}(G_0) \) and \( \mathcal{B}_\mathcal{L} \) be grey bases defined in Section 2.3 in the case of \( \text{Iso}(\mathcal{U}) \) and \( \mathcal{U} \). Let \( \mathbf{t} \) be the corresponding nice topology. The following proposition is a version of a well-known fact.

Proposition 2.13 (1) The following relations from \( \mathcal{F}(\text{Iso}(\mathcal{U})))^2, (\mathcal{F}(\mathcal{U}_L))^2, (\mathcal{F}(\mathcal{U}_L, \mathbf{t}))^2, (\mathcal{F}(\text{Iso}(\mathcal{U})))^3, \mathcal{F}(\text{Iso}(\mathcal{U})) \times \mathcal{F}(\mathcal{U}_L) \times \mathcal{F}(\mathcal{U}_L) \) and \( \mathcal{F}(\text{Iso}(\mathcal{U})) \times \mathcal{F}(\mathcal{U}_L, \mathbf{t}) \times \mathcal{F}(\mathcal{U}_L, \mathbf{t}) \) (under natural interpretations) are Borel:

\[ \{(A, B) : A \subseteq B\}, \{(A, B, C) : AB \subseteq C\}. \]
(2) The closed subgroups of \( \text{Iso}(\mathfrak{H}) \) form a Borel set \( \mathcal{U}(\text{Iso}(\mathfrak{H})) \) in \( \mathcal{F}(\text{Iso}(\mathfrak{H})) \).

(3) The Polish \( G \)-spaces form a Borel set in \( \mathcal{F}(\text{Iso}(\mathfrak{H})) \times \mathcal{F}(\mathfrak{U}_L) \times \mathcal{F}(\mathfrak{U}_L) \) and closed \( G \)-subspaces of \( (\mathfrak{U}_L, \mathfrak{t}) \) form a Borel set in \( \mathcal{F}(\text{Iso}(\mathfrak{H})) \times \mathcal{F}(\mathfrak{U}_L, \mathfrak{t}) \times \mathcal{F}(\mathfrak{U}_L, \mathfrak{t}) \).

Proof. Statement (2) is well-known: see Section 3.2 of [44]. Moreover statements (1) and (2) are variants of Lemmas 2.4 and 2.5 from [33] which were proved for \( S_{\infty} \). It is also mentioned in [33] that they hold in general. Statement (3) follows from (1) and (2). We only mention here that a Polish \( G \)-space is viewed as a triple consisting of \( G \), the subspace and the graph of the action. \( \square \)

In model theory a theory \( T_1 \) is a model companion of \( T \) if \( T_1 \) is model complete and every model of \( T \) embeds into a model of \( T_1 \) and vice versa. One of the definitions of model completeness states that any formula is equivalent to an existential one (or a universal one).

In the case of the logic space \( \mathfrak{U}_L \) theories are identified with \( \mathfrak{t} \)-closed invariant subsets. It is convenient to fix an enumeration of the sets

\[
\mathcal{B}_L(Q) = \{ (\phi)_{<r} : \phi \in \mathcal{B}_L \text{ and } r \in Q \cap [0,1] \},
\]

\[
\mathcal{B}_{oL}(Q) = \{ (\phi)_{<r} : \text{is a clopen member of } \mathcal{B}_L \text{ and } r \in Q \cap [0,1] \},
\]

where the latter one is a basis of the topology \( \tau \). The following definition is a version of Definition 1.3 from [37] and Proposition 1.4 of [30].

Definition 2.14 Let \( X_0 \) and \( X_1 \) be closed invariant subsets of \( (\mathfrak{U}_L, \mathfrak{t}) \). We say that \( X_1 \) is a companion of \( X_0 \) if \( \tau \)-closures of \( X_0 \) and \( X_1 \) coincide and any element of \( \mathcal{B}_L \) is \( \tau \)-clopen on \( X_1 \).

Theorem 2.15 The set of pairs \( (X_0, X_1) \) of \( \text{Iso}(\mathfrak{H}) \)-invariant members of \( \mathcal{F}(\mathfrak{U}_L, \mathfrak{t}) \) with the condition that \( X_1 \) is a companion of \( X_0 \) is Borel.

Proof. Applying Proposition 2.13 we consider pairs \( (X_0, X_1) \) of \( \mathfrak{t} \)-closed invariant subsets as elements of the corresponding Borel set of triples \( (\text{Iso}(\mathfrak{H}), X_0, X_1) \). Using Kuratowski-Ryll-Nardzewski selectors the condition that \( \tau \)-closures of \( X_0 \) and \( X_1 \) are the same can be written as follows:

\[
(\forall A_k \in \mathcal{B}_{oL}(Q)) \exists i \exists j \exists l(s_i(X_1) \in A_k \rightarrow s_j(X_0) \in A_k) \land (s_i(X_0) \in A_k \rightarrow s_l(X_1) \in A_k).
\]

Now note that for any two \( \mathfrak{t} \)-closed \( A \) and \( B \) the condition \( A \cap X_1 \subseteq B \cap X_1 \) is equivalent to the formula:

\[
\forall j(s_j(X_1) \in A \rightarrow s_j(X_1) \in B).
\]

In particular this condition is Borel. We can now express that any element of \( \mathcal{B}_L(Q) \) is \( \tau \)-clopen on \( X_1 \) as follows:

\[
\forall i(\forall B_l \in \mathcal{B}_L(Q))(\exists A_k \in \mathcal{B}_{oL}(Q))(\exists A_m \in \mathcal{B}_{oL}(Q))(s_i(X_1) \in B_l \rightarrow s_i(X_1) \in A_k \land (A_k \cap X_1 \subseteq B_l \cap X_1) \land (s_i(X_1) \notin B_l \rightarrow s_i(X_1) \in A_m \land (A_m \cap X_1 \subseteq X_1 \setminus B_l))).
\]

\( \square \)
Remark 2.16 The theorem above is a counterpart of the statement that identifying theories a language $L$ with closed subsets of the compact space of complete $L$-theories the binary relation to be a model companion is Borel. Although the authors have not found it in literature, it is true and possibly is folklore.

Theorem 2.15 confirms that the approach of good/nice topologies is useful. It provides a topological tool for a general property from logic (model companions).

3 Computable presentations

If $X$ is of the form $X_L$ then it makes sense to study complexity of sets of indices of computable structures of natural model-theoretic classes. In the case of first order structures this approach is traditional, see [1], [22], [23] and [28]. In Section 3.1 we give an appropriate generalization and in Section 3.2 we illustrate it in the case of $L_L$ for relational $L$.

These ideas were already presented by the authors in Section 5 of preprint [29]. We have discovered that they are closely related to the approach of the recent paper of A.G. Melnikov and A. Montalbán [41]. In fact the main concern below is to realize the situation of Sections 2.1 - 2.2 of [41] in cases (A) - (C) of Section 2.3. Having this we arrive in a field where the results of [41] work.

It is worth noting here that the approach of Section 3.1 can be applied when one considers computable presentations of Polish spaces, see [42] and [48]. Some details are given in Remark 3.2. In a sense this is the easiest case. The approach of [41] also works here.

3.1 Computable grey subsets

Consider the situation of Section 2.2. Let $G$ be a Polish group and $R$ be a distinguished countable family of clopen grey cosets which is a grey basis of $G$:

- the family $\{\rho_q : \rho \in R \text{ and } q \in \mathbb{Q}^+ \cap [0, 1]\}$ forms a basis of the topology of $G$.

We fix a countable dense subgroup $G_0 \triangleleft G$ so that:

- $R$ is closed under $G_0$-conjugacy and consists of all $G_0$-cosets of grey subgroups from $R$;
- the set of grey subgroups from $R$ is closed under $\max$ and truncated multiplication by positive rational numbers.

Let

$$R^+_{\sigma q} = \{\sigma_r : \sigma \in R \text{ and } r \in \mathbb{Q}^+ \cap [0, 1]\},$$

$$R^-_{\rho q} = \{\rho > q : \rho \in R \text{ and } q \in \mathbb{Q}^+ \cap [0, 1]\}.$$
• there is a computable 1-1-enumeration of the family $\mathcal{R}_Q = \mathcal{R}_Q^+ \cup \mathcal{R}_Q^-$ so that the relation of inclusion between members of this family is computable.

**Remark 3.1** Having this assumption we arrive at the case that $(G, \mathcal{R}_Q)$ is a **computably presented $\omega$-continuous domain**, see [19] and [20]. In fact our assumptions are slightly stronger. Moreover in A1 - A4 below we will make them much stronger.

• Let $(X, \tau)$ be a Polish $(G, G_0, \mathcal{R})$-space together with a distinguished countable $G_0$-invariant grey basis $\mathcal{U}$ (see Definition 2.5) of clopen grey subsets which is closed under $\max$ and truncated multiplying by positive rational numbers.

• Let

$$ U_Q^+ = \{ \sigma_{<r} : \sigma \in \mathcal{U} \text{ and } r \in \mathbb{Q}^+ \cap [0, 1] \} \text{ (a basis of } (X, \tau) \text{)} , $$

$$ U_Q^- = \{ \rho_{>q} : \rho \in \mathcal{U} \text{ and } q \in \mathbb{Q}^+ \cap [0, 1] \} \text{ and } U_Q = U_Q^+ \cup U_Q^- , $$

and the relation of inclusion between sets of $U_Q$ be computable (under an appropriate computable coding).

As a result $(X, U_Q)$ is a computably presented $\omega$-continuous domain.

Note that in the discrete case these circumstances are standard and in particular arise when one studies computability in $S_\infty$-spaces of logic actions.

**Remark 3.2** It is worth noting that when we have a recursively presented Polish space in the sense of the book of Moschovakis [42] (Section 3), then a basis of the form $U_Q$ as above (in fact $U_Q^+$) can be naturally defined. Indeed, let us recall that a recursive presentation of a Polish space $(X, d)$ is any sequence $S_X = \{ x_i : i \in \omega \}$ which is a dense subset of $X$ satisfying the condition that $(i, j, m, k)$-relations

$$ d(x_i, x_j) \leq \frac{m}{k+1} \text{ and } d(x_i, x_j) < \frac{m}{k+1} $$

are recursive. If in this case for all $i$ we define grey subsets $\phi_i(x) = d(x, x_i)$, then all balls $(\phi_i)_{<r}$, $r \in \mathbb{Q}$, form a basis $U_Q^+ = \{ B_i : i \in \omega \}$ of $X$ which under appropriate enumeration (together with co-balls $(\phi_i)_{>r}$) satisfies our requirements above. When $G$ is a Polish group with a left-invariant metric $d$, then for any $q_1, ..., q_k \in \mathbb{Q}$ and any tuple $h_1, ..., h_k \in G$ the grey subset $\phi_{q, h}(x) = \max_{i \leq k} (q_i \cdot d(h_i, xh_i))$ is a grey subgroup $\overline{\mathbb{Q}}$. If $G$ is a recursively presented space with respect to a dense countable subgroup $G_0$ and the multiplication is recursive, then let $V$ consist of all $\phi_{q, h}$ with $h \in G_0$ and let $R$ consist of all $G_0$-cosets of these grey subgroups. The structure (domain) $(\mathcal{R}_Q, \subset)$ is computably presented.

If $G$ isometrically acts on $X$ and $x_1, ..., x_k$ is a finite subset of the recursive presentation $S_X$ then as we already know the function $\psi_{q, x}(g) = \max_{i \leq k} (q_i \cdot d(h_i, g(x_i)))$ also defines a grey subgroup. When $G$ has a recursive multiplication and a recursive action

\footnote{apply $d(h_i, xyh_i) \leq d(h_i, xh_i) + d(xh_i, yh_i) = d(h_i, xh_i) + d(h_i, yh_i)$ together with the fact that $\max$ applied to grey subgroups gives grey subgroups again}
on $X$ (see Section 3 of [42]) so that the recursive presentation $S_X$ is $G_0$-invariant, then let $V$ consist of these subgroups and $R$ consist of the $G_0$-cosets. Then the structure $(R_Q, \subset)$ is computably presented.

As we will see below the following assumptions are satisfied in the majority of interesting cases.

**Computability assumptions.**

A1. We assume that under our 1-1-enumerations of the families $R_Q$ and $U_Q$ the sets of indices of $U^+_Q$, $R^+_Q$ and the set of rational cones

$$V^+_Q = \{ H <_r : H \text{ is a graded subgroup from } R \}$$

$$V^-_Q = \{ H >_r : H \text{ is a graded subgroup from } R \}$$

are distinguished by computable unary relations on $\omega$.

A2. We assume that under our 1-1-enumerations of the families $R_Q$ and $U_Q$ the binary relation to be in the pair $\sigma <_r$, $\sigma >_r$ for $\sigma \in R$ or $\sigma \in U$ is computable.

A3. We also assume that the following relation is computable:

$$\text{Inv}(V, U) \Leftrightarrow (V \in V^+_Q) \land (U \in U^+_Q) \land (U \text{ is } V\text{-invariant})$$

By invariantness we mean the property that $U$ is presented as $\{ x \in X : \phi(x) < r \}$, the set $V$ is presented as $\{ g \in G : H(g) < s \}$ and $\phi$ is an $H$-invariant grey subset (in particular the inequality $\phi(g(x)) < r + s$ holds for $x \in U$ and $g \in V$).

A4. We assume that there is an algorithm deciding the problem whether for a natural number $i$ and for a basic set of the form $\sigma <_r$ for $\sigma$ from $U$ or $R$ and $r \in \mathbb{Q}$, the diameter of $\sigma <_r$ is less than $2^{-i}$.

Under this setting we introduce the main notion, which is a counterpart of a computable structure. In [42] in the case of a recursively presented Polish space it is defined that a point $x \in X$ is recursive if the set $\{ s : x \in B, B \in U_Q \}$ is computable. We imitate it in the following definition.

**Definition 3.3** We say that an element $x \in X$ is computable if the relation

$$\text{Sat}_x(U) \Leftrightarrow (U \in U_Q) \land (x \in U)$$

is computable.

In the case of the logic action of $S_\infty$, when $x$ is a structure on $\omega$ and all $H$ and $\phi$ are two-valued, this notion is obviously equivalent to the notion of a computable structure.

We will denote by $\text{Sat}_x(U_Q)$ the set $\{ C : C \in U_Q \text{ and } \text{Sat}_x(C) \text{ holds} \}$.

**Remark 3.4** In [41] computable topological spaces are considered under so called formal inclusion $\ll$ (it corresponds to terms ”approximation” or ”way-below” in other sources). In Definition 2.2 of [41] it is defined for computable Polish metric spaces, but in fact this relation can be defined in more general situations. Axioms (F1) - (F4)
given in [41] after Definition 2.2 describe the field of applications of this notion. It
is always assumed in [41] that ≪ is computably enumerable. In our framework this
relation can be defined as follows:
\[ \sigma_{<r} \ll \sigma'_{<r'} \iff \exists r_1 (r < r_1 \land \sigma_{<r_1} \subseteq \sigma'_{<r'} \land \text{diam}(\sigma_{<r}) \leq \frac{1}{2} \text{diam}(\sigma'_{<r'})) \].

Then it is computably enumerable and satisfies (F1) - (F4) of [41]. In particular the
results of [41] hold in the cases of Section 3.2 below. We only add here that in [41]
computable elements are those \( x \) for which \( \text{Sat}_x(U_Q) \) is computably enumerable.

We now make few basic observations which are very helpful when one tries to
estimate the complexity of some families of computable structures. The following
lemma follows from the assumption that \( U \) is a grey basis and satisfies \( A4 \).

**Lemma 3.5** If \( x \in X \) is computable then there is a computable function \( \kappa : \omega \rightarrow U_Q^+ \) such that for all natural numbers \( n \), \( x \in \kappa(n) \) and \( \text{diam}(\kappa(n)) \leq 2^{-n} \).

We also say that

an element \( g \in G \) is **computable** if the relation \( (N \in R_Q) \land (g \in N) \) is computable.

Then there is a computable function realizing the same property as \( \kappa \) above but already in the case of the basis \( R_Q \).

In the following lemma we use standard indexations of the set of computable
functions and of the set of all finite subsets of \( \omega \).

**Lemma 3.6** The following relations belong to \( \Pi^0_2 \):

1. \( \{ e : \text{the function } \varphi_e \text{ is a characteristic function of a subset of } U_Q \} \);
2. \( \{(e,e') : \text{there is a computable element } x \in X \text{ such that the function } \varphi_e \text{ is a characteristic function of the set } \text{Sat}_x(U_Q) \text{ and the function } \varphi_{e'} \text{ realizes the corresponding function } \kappa \text{ defined in Lemma 3.5}\} \);
3. \( \{(e,e') : \text{there is an element } g \in G \text{ such that the function } \varphi_e \text{ is a characteristic function of the subset } \{N \in R_Q : g \in N\} \text{ and the function } \varphi_{e'} \text{ realizes the corresponding function } \kappa \text{ defined as in Lemma 3.5}\} \).

**Proof.** (1) Obvious. Here and below we use the fact that a function is computable
if and only if its graph is computably enumerable.

(2) Under \( A1 \) and \( A4 \) the corresponding definition can be described as follows:

\( "e \text{ is a characteristic function of a subset of } U_Q" \) \land
\( \forall n \exists d \exists n' (\varphi_{e'}(n) \in U^+_Q) \land (\varphi_{e'}(n) \neq \emptyset) \land (\varphi_{e'}(\varphi_{e'}(n)) = 1) \land (\text{diam}(\varphi_{e'}(n)) < 2^{-n})) \land
\( \forall d \exists n ("\text{every element } U' \text{ of the finite subset of } U_Q \text{ with the canonical index } d \text{ satisfies } \varphi_e(U') = 1") \leftrightarrow ("\varphi_{e'}(n) \text{ is contained in any element } U' \text{ of the finite subset of } U_Q \text{ with the canonical index } d") \).
The last part of the conjunction ensures that the intersection of any finite subfamily of \( U_Q \) of cones \( U' \) with \( \varphi_e(U') = 1 \) contains a closed cone of the form \( \phi_{\leq r} \) of sufficiently small diameter. Now the existence of the corresponding \( x \) follows by Cantor’s intersection theorem for complete spaces.

(3) is similar to (2). □

We say that \( e \) is an index of a computable element \( x \in X \) if \( \varphi_e \) is a characteristic function of \( \text{Sat}_x(U_Q) \). We now have the following straightforward proposition.

**Proposition 3.7** The set of indices of computable elements of \( X \) belongs to \( \Sigma^0_3 \).

### 3.2 Computable approximating structures

In this section we show that the computability assumptions \( A_1 - A_4 \) given in Section 3.1 are satisfied in the case of good graded bases presented in (A) - (C) of Section 2.3. In fact we only consider case (A). Cases (B),(C) are similar.

(D) **Computable presentation of the logic space over \( \mathcal{U} \).** Let \( L \) be a relational language satisfying assumptions of Section 2.3. Let us consider the space \( \mathcal{U}_L \) and the family of grey cosets \( \mathcal{R}^\mathcal{U}(G_0) \) defined in Section 2.3 (A). The latter will be interpreted as \( \mathcal{R} \) of Section 3.1.

To define the **grey basis** \( \mathcal{U} \) of Section 3.1 we use the recipe of the definition of the basis of the topology of \( Y_L \) in Section 1.3. For a finite sublanguage \( L' \subset L \), a finite subset \( S' \subset \mathbb{Q}U \) and a finite tuple \( q_1,...,q_t \in \mathbb{Q} \cap [0,1] \) consider the maximum

\[
\max_{\bar{s} \in \text{seq}(S')} |R_j(\bar{s}) - q_i|, 1 - |R_j(\bar{s}) - q_i|, \text{ with } \bar{s} \in \text{seq}(S'), s' \in S'.
\]

When \( \sigma \) is this maximum, the inequality \( \sigma < \varepsilon \) corresponds to a basic open set of the topology of \( Y_L \) as in Section 1.3.

Let \( \mathcal{L} \) be the fragment of all first order continuous formulas. Let \( \mathcal{B}_0 \) be the nice basis corresponding to \( \mathcal{L} \) (see Theorem 2.11). It is worth noting that the **grey basis** \( \mathcal{U} \) is a subfamily of the family of all grey subsets from \( \mathcal{B}_0 \). Moreover \( \mathcal{U} \) corresponds to quantifier free \( L \)-formulas.

To verify that the \( \text{Iso}(\mathcal{U}) \)-space \( \mathcal{U}_L \) and the bases \( \mathcal{R}, \mathcal{U} \) satisfy the computability conditions of Section 3.1 (in particular \( A_1 - A_4 \)), we need the following proposition.

**Proposition 3.8** The elementary theory of the structure \( \mathbb{Q}\mathcal{U} \) in the binary language of inequalities

\[ d(x, x') \leq ( \text{ or } \geq ) q, \text{ where } q \in \mathbb{Q} \cap [0,1], \]

extended by all constants from \( \mathbb{Q}U \) is decidable.

**Proof.** It is well known that \( \mathbb{Q} \) can be identified with the natural numbers so that the ordering of the rational numbers becomes a computable relation. Thus the language in the formulation can be considered as a computable one.
It is noticed in [35] that the first order structure $\mathbb{Q}U$ is universal ultrahomogeneous in the language

$$d(x, x') = q,$$

where $q \in \mathbb{Q} \cap [0, 1]$.

This obviously implies that $\mathbb{Q}U$ is a universal ultrahomogeneous first-order structure in the language of inequalities as in the statement of the proposition. So one can present $\mathbb{Q}U$ in this language as a Fraïssé limit of an effective sequence of finite structures. Enumerating elements of structures from this sequence and describing distances between them, we obtain an effective set of axioms of the form

$$d(c, c') \leq (\text{ or } \geq) q,$$

where $c, c' \in \mathbb{Q}U$ and $q \in \mathbb{Q} \cap [0, 1]$.

We also add all standard $\forall\exists$-axioms stating that the age of $\mathbb{Q}U$ is an amalgamation class. The obtained axiomatization describes a complete theory having elimination of quantifiers. □

**Corollary 3.9** The structure $\mathbb{Q}U$ under the language of binary relations

$$d(x, y) \leq (\text{ or } \geq) q,$$

has a presentation on $\omega$ so that all relations first-order definable in $\mathbb{Q}U$, are decidable.

This obviously follows from Proposition 3.8. Let us fix such a presentation.

**Coding $R_\mathbb{Q}$, cones of grey cosets.** Let

$$H_{q, \bar{s}} : g \to q \cdot d(g(\bar{s}), \bar{s}),$$

where $\bar{s} \subset \mathbb{Q}U$, and $q \in \mathbb{Q}^\dagger$.

be a grey subgroup and $g_0 \in G_0$ take $\bar{s}'$ to $\bar{s}$.

Then we can code the $q'$-cone of the grey coset

$$H_{q, \bar{s} g_0} : g \to q \cdot d(g(\bar{s}'), \bar{s}),$$

by the number of the tuple $(q, \bar{s}, \bar{s}', q', *)$, where $\bar{s}, \bar{s}'$ are identified with the corresponding tuples from $\omega$ with respect to the presentation of Corollary 3.9 and * is one of the symbols $<, \leq, >, \geq$. Note that the tuples $\bar{s}, \bar{s}'$ have the same quantifier free diagram (which is determined by a finite subdiagram). By Corollary 3.9 the set of all tuple $(q, \bar{s}, \bar{s}', q', *)$ of this form is computable and by ultrahomogeneity of the structure from this corollary they code all possible cones.

To see that the relation of inclusion between cones of this form is decidable note that

$$(q, \bar{s}, \bar{s}', q', *)$$

defines a subset of the cone of $(q_1, \bar{s}_1, \bar{s}'_1, q'_1, *)_1$ if for every tuple $\bar{s}''_1 \bar{s}_1'$ of the same quantifier free type with $\bar{s}'\bar{s}_1'$ which also satisfies the *-inequality between $q \cdot d(\bar{s}'', \bar{s})$ and $q'$, the corresponding *$_1$-inequality between $q_1 \cdot d(\bar{s}''_1, \bar{s}_1)$ and $q'_1$ holds.
Indeed by ultrahomogeneity this exactly states that if for an automorphism \( g \) the \( \ast \)-inequality between \( q \cdot d(g(s'), s) \) and \( q' \) holds, then the corresponding \( \ast_1 \)-inequality between \( q_1 \cdot d(g(s'_1), s_1) \) and \( q'_1 \) also holds. Thus to decide the inclusion problem between these cones it suffices to formulate the statement above as a formula (with parameters \( s', s, s'_1, s_1 \)) and to verify if it holds in the structure \( \mathcal{Q}_\mathcal{M} \).

**Cones of grey subgroups** (i.e. the set \( \mathcal{V}_\mathcal{Q} \)) are distinguished in the set of codes of \( \mathcal{R}_\mathcal{Q} \) by the computable subset of tuples as above with \( \bar{s} = s' \).

**Coding \( \mathcal{U}_\mathcal{Q} \).** Since we interpret elements of \( \mathcal{B}_0 \) by first order \( L \)-formulas with parameters from \( \mathcal{Q}_\mathcal{M} \) and without free variables, it is obvious that both \( \mathcal{B}_0 \) and \( \mathcal{U} \) can be coded in \( \omega \) so that the operations of connectives are defined by computable functions. Moreover \( \mathcal{U} \) is a decidable subset of \( \mathcal{B}_0 \). Thus the elements of the grey basis \( \mathcal{U} \) are coded as a computable set. Now all cones of the form \( \sigma_{<q}, \sigma_{>q}, \sigma_{\leq q}, \sigma_{\geq q} \) can be enumerated so that all natural relations between them (in particular relations from \( \mathbf{A2} \)) are computable. For example if \( S' \) is a finite subset of \( \mathcal{Q}_\mathcal{M} \) and cones \( \sigma_{<q} \) and \( \sigma'_{<q'} \) correspond to inequalities of the form

\[
|R_j(s) - q_i| < q, 1 - |R_j(s) - q_i| < q,
\]

then it can be verified effectively if the inequalities of the cone \( \sigma_{<q} \) follow from the ones of the cone \( \sigma'_{<q'} \) together with the diagram of the metric on \( S' \) and the inequalities provided by the continuity moduli. It is worth noting here that the diagram of the metric on \( S' \) (i.e. all equalities \( d(s, s') = q \) with \( s, s' \in S', q \in [0, 1] \cap \mathbb{Q} \)) is decidable by Corollary 3.9.

**Satisfying \( \mathbf{A3} \).** Let \( U \) be of the form \( \sigma_{<q} \) for \( \sigma \in \mathcal{U} \) and \( V \) be of the form \( H_{<k} \) for \( H \in \mathcal{V} \). We assume that the inequalities of \( \sigma_{<q} \) are as in the previous paragraph. Let \( S' \) be a finite subset of \( \mathcal{Q}_\mathcal{M} \) which contains all parameters which appear in the definition of \( U \) and \( H \). Since \( \mathcal{Q}_\mathcal{M} \) is ultrahomogeneous the condition \( \text{Inv}(V, U) \) is satisfied if and only if the following property holds.

If \( \gamma \) is a partial isometry of \( \mathcal{Q}_\mathcal{M} \) with domain \( S' \) and fixing the parameters appearing in \( H \), then the value of \( \sigma \) (with respect to the parameters \( S' \)) is preserved under \( \gamma \).

When this property holds for all possible interpretations of the \( L \)-symbols on \( \mathcal{M} \) it just follows from the diagram of the metric on \( S' \) and the inequalities provided by the continuity moduli. Thus by Corollary 3.9 this relation is decidable.

**Satisfying \( \mathbf{A4} \).** Let \( \sigma \) be a \textit{max}-formula of the previous paragraphs which defines an element of \( \mathcal{U} \). To compute \( \text{diam}(\sigma_{<q}) \) consider the definition of the metric \( \delta_{\text{seq}(\mathcal{Q}_\mathcal{M})} \) of the space \( \mathcal{U}_L \) with respect to \( \text{sec}(\mathcal{Q}_\mathcal{M}) \) in the beginning of Section 1.1. Find all numbers \( i \) of tuples \((j, s')\) such that \( R_j(s') \) appears in \( \sigma \). We may assume that appearance of such subformulas forces inequalities of the form \( q'_i \leq R_j(s') \leq q_i \) for rational
$0 \leq q_i' < q_i \leq 1$. Let $I$ be the (finite) subset of such $i$. Then $\text{diam}(\sigma_{<q})$ is computed by

$$\sum_{i=1}^{\infty} \{2^{-i} : i \not\in I\} + \sum_{i \in I} 2^{-i}|q_i - q_i'|.$$ 

In particular we have an algorithm for comparing it with powers $2^{-i}$.

The case of basic clopen sets of $\mathcal{R}^U$ is similar.

(E) Decidability. We have found that our arguments for the computability assumptions A1 - A4 can be applied for some other related questions and possibly the most natural ones are involved into decidability of continuous theories. This explains why we now consider this issue.

Let $\Gamma$ be a set of continuous formulas of a continuous signature $L$ with a metric. Let $\phi$ be a continuous $L$-formula.

Definition 3.10 (see [11], Section 9) The value $\sup\{\phi^M : \text{for } M \models \Gamma = 0\}$ is called the degree of truth of $\phi$ with respect to $\Gamma$. We denote this value by $\phi^\circ$.

If the language $L$ is computable, the set of all continuous $L$-formulas and the set of all $L$-conditions of the form

$$\phi \leq \frac{m}{n}, \text{ where } \frac{m}{n} \in \mathbb{Q}_+,$$

are computable.

We remind the reader that a real number $r \geq 0$ is computable if there is an algorithm which for any natural number $n$ finds a natural number $k$ such that

$$\frac{k - 1}{n} \leq r \leq \frac{k + 1}{n}.$$ 

Corollary 9.11 of [11] states that when $\Gamma$ is computably enumerable and $\Gamma = 0$ axiomatizes a complete theory, then the value $\phi^\circ$ is a recursive real which is uniformly computable from $\phi$. The latter exactly means that the corresponding complete theory is decidable. Note that in this case the value $\phi^\circ$ coincides with the value of $\phi$ in models of $\Gamma = 0$.

The following theorem shows that in the situations of examples of Section 2.3 the expansion of the structure by the countable approximating substructure has decidable continuous theory.

Theorem 3.11 The structure $(\mathfrak{U}, s)_{s \in \mathbb{Q}_+}$ of the expansion of the bounded Urysohn space by constants from $\mathbb{Q}\mathfrak{U}$ has decidable continuous theory.

The same statement holds for structures $(\mathfrak{M}, s)_{s \in \mathbb{N}}$ where $(\mathfrak{M}, d) \in \{l_2(\mathbb{N}), MALG\}$ and $\mathbb{N}$ is the corresponding countable approximating substructure see Section 2.3, (B) and (C)).
Proof. To prove the theorem we use Corollary 9.11 of [11]. We only consider the case of \((\U, s)\in \mathbb{Q}_U\). The remaining cases are similar.

Let \(T_{\mathbb{Q}_U}\) be the set of the standard axioms of \(\U\) (with rational \(\varepsilon\) and \(\delta\), see Section 5 in [17]) together with all quantifier free axioms describing distances between constants from \(\mathbb{Q}_U\). We claim that the set \(T_{\mathbb{Q}_U}\) is computable. Since the set of all standard axioms of \(\U\) is computable (see [17]), it suffices to check that the set of all axioms of the form

\[
d(c, c') = q, \quad \text{where } c, c' \in \mathbb{Q}_U \text{ and } q \in \mathbb{Q} \cap [0, 1],
\]

is computable. This follows from the fact that the elementary (not continuous) theory of the structure \(\mathbb{Q}_U\) in the language of binary relations together with all constants \(c \in \mathbb{Q}_U\) is decidable, Proposition 3.8.

Note that \(T_{\mathbb{Q}_U}\) axiomatizes the continuous theory of a single continuous structure, i.e. the corresponding continuous theory is complete. Indeed, otherwise there is a separable continuous structure \(M \models T_{\mathbb{Q}_U}\) such that for some tuple \(\bar{s} \in \mathbb{Q}_U\) the structures \((\U, \bar{s})\) and the reduct of \(M\), say \(M'\), to the signature \((d, \bar{s})\), do not satisfy the same inequalities of the form

\[
\phi(\bar{s}) \leq (\prec)q \text{ or } \phi(\bar{s}) \geq (\succ)q \text{ where } q \in \mathbb{Q} \cap [0, 1].
\]

On the other hand since \(\U\) is separably categorical (see Section 4) and ultrahomogeneous, the structures \(M'\) and \((\U, \bar{s})\) are isomorphic, contradicting the previous sentence.

By Corollaries 9.8 and 9.11 of [11] there is an algorithm which for every continuous sentence \(\phi(\bar{s})\) computes its value in \(\U\). □

Remark 3.12 It is worth noting that when we apply Proposition 3.8 we only need computability of the set of axioms of the form

\[
d(c, c') = q, \quad \text{where } c, c' \in \mathbb{Q}_U \text{ and } q \in \mathbb{Q} \cap [0, 1],
\]

This can be shown as in the proof of Proposition 3.8. Moreover the corresponding argument works in the cases of \(l_2(\mathbb{N})\) and \(MALG\).

4 Complexity of some subsets of the logic space

In this section we fix a countable continuous signature

\[
L = \{d, R_1, ..., R_k, ..., F_1, ..., F_l, ...\},
\]

a Polish space \((Y, d)\), a countable dense subset \(S_Y\) of \(Y\) and study subsets of \(Y_L\) which are invariant with respect to isometries stabilising \(S_Y\) setwise. Viewing the logic space \(Y_L\) as a Polish space one can consider Borel/algorithmic complexity of some natural subsets of \(Y_L\) of this kind. This approach differs from the one of
Section 2.4. It corresponds to considering a structure on $Y$ (say $M$) together with its **presentation over** $S_Y$, i.e. the set

$$
\text{Diag}(M, S_Y) = \{(\phi, q) : M \models \phi < q, \text{ where } q \in [0, 1] \cap \mathbb{Q} \text{ and } \phi \text{ is a continuous sentence with parameters from } S_Y\}.
$$

It is natural in the cases of examples (A) - (C) of Section 2.3 and the corresponding computable presentations as in Section 3. Moreover it corresponds to the approach of computable model theory.

We will concentrate on separable categoricity.

A theory $T$ is **separably categorical** if any two separable models of $T$ are isomorphic. A useful reformulation of this notion is given in Theorem 4.1. Since we will only use this theorem below all necessary facts concerning separable categoricity (together with the proof of Theorem 4.1) are given in Appendix.

### 4.1 Separable categoricity

We preserve all the assumptions of Section 1 on the space $(Y, d)$. For simplicity we assume that all $L$-symbols are of continuity modulus $\text{id}$. Simplifying notation we put $S = S_Y$. We reformulate separable categoricity as follows.

**Theorem 4.1** Let $M$ be a non-compact, separable, continuous, metric structure on $(Y, d)$. The structure $M$ is separably categorical if and only if for any $n$ and $\varepsilon$ there are finitely many conditions $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$, so that any $n$-tuple of $M$ satisfies one of these conditions and the following property holds:

for any $i \in I$, and any $a_1, ..., a_n \in M$ realizing $\phi_i(\bar{x}) \leq \delta_i$ and any finite set of formulas $\Delta(x_1, ..., x_n, x_{n+1})$ realized in $M$ and containing $\phi_i(\bar{x}) \leq \delta_i$, there is a tuple $b_1, ..., b_n, b_{n+1}$ realizing $\Delta$ such that $\max_{i \leq n} d(a_i, b_i) < \varepsilon$.

We now introduce a class of structures which is justified by its formulation. If we assume that all parameters appearing in it can be taken from $S$ we arrive at the following definition.

**Definition 4.2** Let $\mathcal{SC}_S$ be the set of all $L$-structures $M$ on $Y$ with the following condition:

for every $n$ and rational $\varepsilon$ there is a finite set $F$ of tuples $\bar{a}_i$ from $S$ together with conditions $\phi_i(\bar{x}) \leq \delta_i$ ($i \in I$ and all $\delta_i$ are rational) with $\phi_i^M(\bar{a}_i) \leq \delta_i$, $i \in I$, and the following properties

- any $n$-tuple $\bar{a}$ from $S$ satisfies in $M$ one of these $\phi_i(\bar{x}) \leq \delta_i$
- when $\phi_i^M(\bar{a}) \leq \delta_i$ and $\bar{c}$ is an $(n + 1)$-tuple from $S$ with $c_1, ..., c_n$ satisfying $\phi_i(\bar{x}) \leq \delta_i$ in $M$,

for any finite set $\Delta$ of $L$-formulas $\phi(\bar{y})$, $|\bar{y}| = n + 1$ with $\phi^M(\bar{c}) = 0$ there is an $(n + 1)$-tuple $\bar{b} = (b_1, ..., b_{n+1}) \in S$ so that $\max_{j \leq n} (d(a_j, b_j)) \leq \varepsilon$ and $\phi^M(\bar{b}) = 0$ for all formulas $\phi \in \Delta$.

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Note that Theorem 4.1 implies that if $M$ is a separably categorical structure on $Y$, there is a dense set $S' \subseteq Y$ so that $M$ belongs to the corresponding set of $L$-structures $\mathcal{S} \mathcal{C}_{S'}$. To see this we just extend $S$ to some countable $S'$ which satisfies the property of Definition 4.2 in which we additionally require that $a_1, \ldots, a_n \in S'$. Thus the following statement becomes interesting.

**Proposition 4.3** The subset $\mathcal{S} \mathcal{C}_S \subset Y_L$ is $\text{iso}(S)$-invariant and Borel.

**Proof.** It is clear that $\mathcal{S} \mathcal{C}_S$ is $\text{iso}(S)$-invariant. To see that $\mathcal{S} \mathcal{C}_S$ is a Borel subset of $Y_L$ it suffices to note that given rational $\varepsilon > 0$, finitely many formulas $\phi_i(\bar{x})$, $i \in I$, with $|\bar{x}| = n + 1$, and an $n$-tuple $\bar{a}$ from $S$ the set of $L$-structures $M$ on $Y$ with the property that

there is an $(n+1)$-tuple $\bar{b} \in S$ so that $\max_{j \leq n}(d(a_j, b_j)) \leq \varepsilon$ and $\phi_i^M(\bar{b}) = 0$

for all $i \in I$,

is a Borel subset of $Y_L$. The latter follows from Lemma 2.1, which in particular says that any set of $L$-structures of the form

$$\{ M : M \models \max_{j \leq n}(d(a_j, b_j) - \varepsilon), \max_{i \in I}(\phi_i(\bar{b})) = 0 \}$$

is a Borel subset of $Y_L$. $\square$

The proof above demonstrates that $\mathcal{S} \mathcal{C}_S$ is of Borel level $\omega$.

In Section 4.3 we will discuss the conjecture that any separably categorical continuous $L$-structure on $Y$ is homeomorphic to a structure from $\mathcal{S} \mathcal{C}_S$.

Since $\mathcal{S} \mathcal{C}_S$ is a subset of the standard space $Y_L$ we do need to specify grey bases $\mathcal{R}$, $\mathcal{U}$ as in Section 3.2. To be definite one can generate $\mathcal{R}$ by grey stabilizers and $\mathcal{U}$ by atomic formulas. This issue becomes important in the next section when we consider computable members of $\mathcal{S} \mathcal{C}_S$.

### 4.2 Computable members

The following proposition is an effective version of Proposition 4.3 in the case (A) of Section 2.3. We work in the effective presentation given in Section 3.2 (D).

**Proposition 4.4** Let $\mathcal{S} \mathcal{C}_{\mathcal{Q} \mathcal{U}}$ be the $\text{iso}(\mathcal{Q} \mathcal{U})$-invariant Borel subset of $\mathcal{U}_L$ defined as in Section 4.1.

Then the subset of indices of computable structures from $\mathcal{S} \mathcal{C}_{\mathcal{Q} \mathcal{U}}$ is hyperarithmetical.

**Proof.** Under the framework of Sections 3.1 and 3.2 (D) the following statement holds.

The set of all pairs $(i, j)$ where $j$ is an index of a cone from $(\mathcal{B}_0)_\mathcal{Q}$ and $i$ is an index of a computable structure from this cone, is hyperarithmetical of level $\omega$. 

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This is an effective version of Proposition 2.1. It follows from Lemma 3.6 by standard arguments. Note that as we have shown in Section 3.2 (D) all assumptions of Lemma 3.6 are satisfied under the circumstances of our proposition.

It remains to verify that the definition of $\mathcal{SC}_{U}$ defines a hyperarithmetical subset of indices of computable structures. This is straightforward (similar to the proof of Proposition 4.3).

4.3 Countable dense homogeneity

We conjecture that when $(Y, d)$ is as in the cases (A) - (C) of Section 2.3 and the dense subset $S$ is chosen as the corresponding approximating substructure, then any separably categorical structure from $Y_L$ is homeomorphic to an element of $\mathcal{SC}_S$. In this section we connect it with countable dense homogeneity.

A separable space $X$ is countable dense homogeneous (CDH) if given any two countable dense subsets $D$ and $E$ of $X$ there is a homeomorphism $f : X \to X$ such that $f(D) = E$ (see [25]). It is known that the unbounded Urysohn space and the spaces $l^2$ are CDH (and they are homeomorphic).

In [17] J. Dijkstra introduced Lipschitz CDH as follows.

**Definition 4.5** A metric space $(X, d)$ is called Lipschitz countable dense homogeneous if given $\varepsilon$ and $A, B$ countable dense subsets of $X$ there is a homeomorphism $h : X \to X$ such that $f(A) = B$ and

$$1 - \varepsilon < \frac{d(h(x), h(y))}{d(x, y)} < 1 + \varepsilon \text{ for all } x, y \in X.$$  

He has proved in [17] that every separable Banach space is Lipschitz CDH. Moreover it is shown in [36] that the unbounded Urysohn space is also Lipschitz CDH.

Regarding the property $\mathcal{SC}_S$ this notion seems very helpful. Indeed, as we have already noted Theorem 4.1 implies that if $M$ is a separably categorical structure on $Y$, there is a dense set $S' \subseteq Y$ so that $M$ belongs to the corresponding Borel set of $L$-structures $\mathcal{SC}_{S'}$. To support the conjecture of this subsection we need a homeomorphism which takes $S'$ onto $S$ and takes $M$ into $\mathcal{SC}_S$. However the latter condition is not easy to control.

4.4 Complexity of sets of approximately ultrahomogeneous structures

If in the definition of the class $\mathcal{SC}_S$ we restrict ourselves by only quantifier free formulas we arrive at a definition of a subset of $\mathcal{SC}_S$ which we denote by $\mathcal{SCU}_S$. The approach of Sections 4.1 and 4.2 works in this case too. As in Section 4.3 one can conjecture that when $(Y, d)$ is as in the cases (A) - (C) of Section 2.3 and the dense subset $S$ is chosen as the corresponding approximating substructure, then any separably categorical ultrahomogeneous structure from $Y_L$ is homeomorphic to an element of $\mathcal{SCU}_S$.  

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It is worth noting here that since any Polish group can be realized as the automorphism group of an approximately ultrahomogeneous structure it makes sense to study Polish groups by description of the corresponding classes of approximately ultrahomogeneous structures and to study the complexity of these classes. For example we do not know if the class of approximately ultrahomogeneous $L$-structures on $Y$ is a Borel subset of $Y_L$.

4.5 CLI Polish groups

In this section we give a different example of complexity of a subclass of $Y_L$ for a Polish space $(Y,d)$. It somehow corresponds to the result of M.Malicki [39] that the set of all Polish groups admitting compatible complete left-invariant metrics is coanalytic non-Borel as a subset of a standard Borel space of Polish groups.

The following statement is Lemma 9.1 of [10].

Let $G$ be the automorphism group of a continuous $L$-structure $M$ on the space $(Y,d)$. Then the group $G$ admits a compatible complete left-invariant metric (i.e. $\text{Aut}(M)$ is CLI) if and only if each $L_{\omega_1\omega}$-elementary embedding of $M$ into itself is surjective.

What is the complexity of the class of structures from this proposition? We have some remark concerning this question.

Proposition 4.6 Let $\mathcal{L}$ be a countable fragment of $L_{\omega_1\omega}$. The subset of $Y_L$ consisting of structures $M$ admitting proper $\mathcal{L}$-elementary embeddings into itself is analytic. $\text{Iso}(S)$-invariant and coanalytic.

Proof. Consider the extension of $L$ by a unary function $f$. All expansions of $L$-structures satisfying the property that $f$ is an isometry which preserves the values $\mathcal{L}$-formulas, form a Borel subset of the (Polish) space of all $L \cup \{f\}$-structures on $Y$.

If $s \in S = S_Y$ and $\varepsilon \in Q \cap [0,1]$ then the condition that $f(S)$ does not intersect the $\varepsilon$-ball of $s$ is open. Thus the set of $L \cup \{f\}$-structures with a proper embedding $f$ into itself, is Borel. The rest is easy. □

5 Appendix. Proof of Theorem 4.1

We need the following definition.

Definition 5.1 Let $A \subseteq M$. A predicate $P : M^n \to [0,1]$ is definable in $M$ over $A$ if there is a sequence $(\phi_k(x) : k \geq 1)$ of $L(A)$-formulas such that predicates interpreting $\phi_k(x)$ in $M$ converge to $P(x)$ uniformly in $M^n$.

Let $p(\bar{x})$ be a type of a theory $T$. It is called principal if for every model $M \models T$, the predicate $\text{dist}(\bar{x},p(M))$ is definable over $\emptyset$.

By Theorem 12.10 of [4] a complete theory $T$ is separably categorical if and only if for each $n > 0$, every $n$-type $p$ is principal. Another property equivalent to separable
categoricity states that for each \( n > 0 \), the metric space \( (S_n(T), d) \) is compact. In particular for every \( n \) and every \( \varepsilon \) there is a finite family of \( n \)-types \( p_1, \ldots, p_m \) so that their \( \varepsilon \)-neighbourhoods cover \( S_n(T) \).

In the classical first order logic a countable structure \( M \) is \( \omega \)-categorical if and only if \( \text{Aut}(M) \) is an \textbf{oligomorphic} permutation group, i.e. for every \( n \), \( \text{Aut}(M) \) has finitely many orbits on \( M^n \). In continuous logic we have the following modification.

\textbf{Definition 5.2} An isometric action of a group \( G \) on a metric space \( (X, d) \) is said to be \textbf{approximately oligomorphic} if for every \( n \geq 1 \) and \( \varepsilon > 0 \) there is a finite set \( F \subset X^n \) such that

\[
G \cdot F = \{ g\bar{x} : g \in G \text{ and } \bar{x} \in F \}
\]

is \( \varepsilon \)-dense in \( (X^n, d) \).

Assuming that \( G \) is the automorphism group of a non-compact separable continuous metric structure \( M \), \( G \) is approximately oligomorphic if and only if the structure \( M \) is separably categorical (C. Ward Henson, see Theorem 4.25 in [45]). It is also known that separably categorical structures are \textbf{approximately homogeneous} in the following sense: if \( n \)-tuples \( \bar{a} \) and \( \bar{c} \) have the same types (i.e. the same values \( \phi(\bar{a}) = \phi(\bar{b}) \) for all \( L \)-formulas \( \phi \)) then for every \( c_{n+1} \) and \( \varepsilon > 0 \) there is a tuple \( b_1, \ldots, b_n, b_{n+1} \) of the same type with \( \bar{c}, c_{n+1} \), so that \( d(a_i, b_i) \leq \varepsilon \) for \( i \leq n \). In fact for any \( n \)-tuples \( \bar{a} \) and \( \bar{b} \) there is an automorphism \( \alpha \) of \( M \) such that

\[
d(\alpha(\bar{c}), \bar{a}) \leq d(tp(\bar{a}), tp(\bar{c})) + \varepsilon.
\]

(i.e \( M \) is \textbf{strongly} \( \omega \)-near-homogeneous in the sense of Corollary 12.11 of [46]).

To prove Theorem 4.4 we start with the following observation.

\textbf{Lemma 5.3} Let \( M \) be a non-compact, separable, continuous, metric structure on \( (Y, d) \). The structure \( M \) is separably categorical if and only if for any \( n \) and \( \varepsilon \) there are finitely many conditions \( \phi_i(\bar{x}) \leq \delta_i \), \( i \in I \), so that any \( n \)-tuple of \( M \) satisfies one of these conditions and the following property holds:

for any \( i \in I \), any \( a_1, \ldots, a_n \in M \) realising \( \phi_i(\bar{x}) \leq \delta_i \) and any type \( p(x_1, \ldots, x_n, x_{n+1}) \) realized in \( M \) and containing \( \phi_i(\bar{x}) \leq \delta_i \), there is a tuple \( b_1, \ldots, b_n, b_{n+1} \) realizing \( p \) such that \( \max_{i \leq n} d(a_i, b_i) < \varepsilon \).

\textbf{Proof.} By Theorem 12.10 of [46] a complete theory \( T \) is separably categorical if and only if for each \( n > 0 \), every \( n \)-type is principal. An equivalent condition states that for each \( n > 0 \), the metric space \( (S_n(T), d) \) is compact. In particular for every \( n \) and every \( \varepsilon \) there is a finite family of principal \( n \)-types \( p_1, \ldots, p_m \) so that their \( \varepsilon/2 \)-neighbourhoods cover \( S_n(T) \).

Thus when \( M \) is separably categorical, given \( n \) and \( \varepsilon \), we find appropriate \( p_i \), \( i \in I \), define \( P_i(\bar{x}) = \text{dist}(\bar{x}, p_i(M)) \), the corresponding definable predicates and \( n \)-conditions \( \phi_i(\bar{x}) \leq \delta_i \) describing the corresponding \( \varepsilon/2 \)-neighbourhoods of \( p_i \). The rest follows by strong \( \omega \)-near-homogeneity.
To see the converse assume that $M$ satisfies the property from the formulation. To see that $G = Aut(M)$ is approximately oligomorphic take any $n$ and $\varepsilon$ and find finitely many conditions $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$, satisfying the property from the formulation for $n$ and $\varepsilon/4$. Choose $\bar{a}_i$ with $\phi_i(\bar{a}_i) \leq \delta_i$ and let $F = \{\bar{a}_i : i \in I\}$. To see that $G \cdot F$ is $\varepsilon$-dense we only need to show that if $\bar{a}$ satisfies $\phi_i(\bar{x}) \leq \delta_i$, then there is an automorphism which takes $\bar{a}$ to the $\varepsilon$-neighbourhood of $\bar{a}_i$. This is verified by "back-and-forth" as follows. Let $(\varepsilon_k)$ be an infinite sequence of positive real numbers whose sum is less than $\varepsilon/4$. At every step $l$ (assuming that $l \geq n$) we build a finite elementary map $\alpha_l$ and $l$-tuples $\bar{c}_l$ and $\bar{d}_l$ so that

- $\bar{c}_n = \bar{a}$ and $\bar{d}_n = \bar{a}_i$;
- for $l > n$, $\alpha_l$ takes $\bar{c}_l$ to $\bar{d}_l$;
- for $l > n + 1$, the first $l - 1$ coordinates of $\bar{c}_l$ (resp. $\bar{d}_l$) are at distance less than $\varepsilon_l$ away from the corresponding coordinates of $\bar{c}_{l-1}$ (resp. $\bar{d}_{l-1}$);
- the sets $\bigcup \{\bar{c}_l : l \in \omega\}$ and $\bigcup \{\bar{d}_l : l \in \omega\}$ are dense in $M$.

In fact we additionally arrange that for even $l$, $\bar{c}_{l+1}$ extends $\bar{c}_l$ and for odd $l$ $\bar{d}_{l+1}$ extends $\bar{d}_l$. In particular the type of $\bar{c}_{l+1}$ always extends the type of $\bar{c}_l$. At the $(n+1)$-th step we find finitely many conditions $\phi'_j(\bar{x}) \leq \delta'_j$, $j \in J$, so that any $(n+1)$-tuple of $M$ satisfies one of these conditions and for any $j \in J$, any $a'_1, ..., a'_{n+1} \in M$ realising $\phi'_j(\bar{x}) \leq \delta'_j$ and any type $p(x_1, ..., x_{n+1}, x_{n+2})$ realized in $M$ and containing $\phi'_j(\bar{x}) \leq \delta_j$, there is a tuple $b_1, ..., b_{n+1}, b_{n+2}$ realizing $p$ such that $\max_{l \leq n+1}d(a'_l, b_l) < \varepsilon_{n+1}$. Now by the choice of $i$ for any extension of $\bar{a} = \bar{c}_n$ to an $(n+1)$-tuple $\bar{c}_{n+1}$ we can find a tuple $\bar{d}_{n+1}$ realizing $tp(\bar{c}_{n+1})$ so that the first $n$ coordinates of $\bar{d}_{n+1}$ are at distance less than $\varepsilon/4$ away from the corresponding coordinates of $\bar{d}_n = \bar{a}_i$. If $n$ is even we choose such $\bar{c}_{n+1}$ and $\bar{d}_{n+1}$; if $n$ is odd we replace the roles of $\bar{c}_{n+1}$ and $\bar{d}_{n+1}$. For the next step we fix the condition $\phi'_j(\bar{x}) \leq \delta'_j$ satisfied by $\bar{c}_{n+1}$ and $\bar{d}_{n+1}$.

The $(l+1)$-th step is as follows. Assume that $l$ is even (the odd case is symmetric). Extend $\bar{c}_l$ to an appropriate $\bar{c}_{l+1}$ (aiming to density of $\bigcup \{\bar{c}_l : l \in \omega\}$). There are finitely many conditions $\phi''_k(\bar{x}) \leq \delta''_k$, $k \in K$, so that any $(l+1)$-tuple of $M$ satisfies one of these conditions and for any $k \in K$, any $a'_1, ..., a'_{l+1} \in M$ realising $\phi''_k(\bar{x}) \leq \delta''_k$ and any type $p(x_1, ..., x_{l+1}, x_{l+2})$ realized in $M$ and containing $\phi''_k(\bar{x}) \leq \delta''_k$, there is a tuple $b_1, ..., b_{l+1}, b_{l+2}$ realizing $p$ such that $\max_{l \leq l+1}d(a'_l, b_l) < \varepsilon_{l+1}$. We find the condition satisfied by $\bar{c}_{l+1}$ and a tuple $\bar{d}_{l+1}$ realizing $tp(\bar{c}_{l+1})$ so that the first $l$ coordinates of $\bar{d}_{l+1}$ are at distance less than $\varepsilon_l$ away from the corresponding coordinates of $\bar{d}_l$.

As a result for every $k$ we obtain Cauchy sequences of $k$-restrictions of $\bar{c}_k$-s and $\bar{d}_k$-s. For $k = n$ their limits are not distant from $\bar{a}$ and $\bar{a}_i$ more than $\varepsilon/2$. Moreover the limits $\lim \{\bar{c}_l\}$ and $\lim \{\bar{d}_l\}$ are dense subsets of $Y$ and realize the same type. This defines the required automorphism of $M$. □

**Proof of Theorem 4.1** It suffices to show that the condition of the formulation implies the corresponding condition of Lemma 5.3. Given $n$ and $\varepsilon$ take the family $\phi_i(\bar{x}) \leq \delta_i$, $i \in I$, satisfying the condition of the theorem for $n$ and $\varepsilon/2$. Let $p(\bar{x}, x_{n+1})$ be a type with $\phi_i(\bar{x}) \leq \delta_i$ and $a_1, ..., a_n$ be as in the formulation.
Let \((\varepsilon_k)\) be an infinite sequence of positive real numbers whose sum is less than \(\varepsilon/2\). Now apply the condition of the formulation of the theorem to \(n + 1\) and \(\varepsilon_1/2\) and find an appropriate finite family of inequalities such that one of them, say \(\psi(\bar{x}, x_{n+1})  \leq \tau\), belongs to \(p\) and for any \(c_1, \ldots, c_n, c_{n+1} \in M\) realising \(\psi(\bar{x}, x_{n+1})  \leq \tau\), and any finite subset \(\Delta \subset p\) containing \(\psi(\bar{x}, x_{n+1})  \leq \tau\) there is a tuple \(c'_1, \ldots, c'_n, c'_{n+1}\) realizing \(\Delta\), such that \(\max_{i \leq n+1} d(c_i, c'_i) < \varepsilon_1/2\). Then let \(b_1^1, \ldots, b_{n+1}^1\) be a tuple realizing \(\phi_i(\bar{x}) \leq \delta_i\) and \(\psi(\bar{x}, x_{n+1})  \leq \tau\) such that \(\max_{i \leq n} d(a_i, b_i^1) < \varepsilon/2\).

For \(n + 1\) and \(\varepsilon_2/2\) find an appropriate condition \(\psi'(\bar{x}, x_{n+1})  \leq \tau'\) from \(p\) so that any \(c_1, \ldots, c_n, c_{n+1} \in M\) realizing \(\psi'(\bar{x}, x_{n+1})  \leq \tau'\), and any finite subset \(\Delta \subset p\) containing \(\psi'(\bar{x}, x_{n+1})  \leq \tau'\) there is a tuple \(c'_1, \ldots, c'_n, c'_{n+1}\) realizing \(\Delta\), such that \(\max_{i \leq n+1} d(c_i, c'_i) < \varepsilon_2/2\). Let \(b_1^2, \ldots, b_{n+1}^2\) be a tuple realizing \(\phi_i(\bar{x}) < \delta_i\), \(\psi(\bar{x}, x_{n+1})  \leq \tau\) and \(\psi'(\bar{x}, x_{n+1})  \leq \tau'\) such that \(\max_{i \leq n+1} d(b_i^1, b_i^2) < \varepsilon_1/2\). Note that \(\max_{i \leq n} d(a_i, b_i^2) < \varepsilon/2 + \varepsilon_1/2\).

Continuing this procedure we obtain a Cauchy sequence of \((n + 1)\)-tuples so that its limit satisfies \(p\) and is not distant from \(\bar{a}\) more than \(\varepsilon\). \(\square\)

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