EFFECTIVE IMPEDANCE OVER ORDERED FIELDS

ANNA MURANOVA

ABSTRACT. In this paper we study properties of effective impedance of finite electrical networks and calculate the effective impedance of a finite ladder network over an ordered field. Moreover, we consider two particular examples of infinite ladder networks (Feynman’s network or LC-network and CL-network, both with zero on infinity) as networks over the ordered Levi-Civita field \( \mathcal{R} \). We show that effective impedances of finite LC-networks converge to the limit in order topology of \( \mathcal{R} \), but the effective impedances of finite CL-networks do not converge in the same topology.

**Keywords:** weighted graphs, electrical network, ladder network, effective impedance, Laplace operator, ordered field, non-Archimedean field, Levi-Civita field.

**Mathematics Subject Classification 2010:** 05C22, 34B45, 05C25, 39A12, 12J15.

1. INTRODUCTION

It is known that electrical networks with resistances are related to weighted graphs (see e.g. [4], [10], [13]). Moreover, it is shown in [8] and [10], that effective resistance for finite networks satisfies the basic physical properties (e.g. parallel and series laws). In [4] and [8] the notion of effective resistance for infinite network is introduced. An effective resistance is tightly related to random walk and Dirichlet problem on graphs, which are described in many papers and books (e. g. [1], [7], [14],[13]). In [11] a finite electrical network with alternating current and passive elements is considered as a generalization of electrical network with resistances. It is shown there, that such a network is related to weighted graphs over non-Archimedean ordered field of rational functions \( \mathbb{R}(\lambda) \). The generalization of effective resistance for this case is called effective impedance. The inverse of effective impedance is called effective admittance. The most known in physics infinite network with passive elements is Feynman’s ladder network (LC-network, see [6]). In [15] the effective impedances of LC-network and CL-network are considered as limits of complex-valued effective impedances of corresponding sequences of finite networks.

The present paper consists of two parts. In the first part we describe some properties of electrical network over an ordered field. The main result of this part is Theorem 4 which gives the mathematical description of well-known in physics star-mesh transform. The mathematical conceptions of parallel and series laws, as well as \( \Delta - Y \) and \( Y - \Delta \) transform, follow from Theorem 4 as corollaries.

In the second part of the present paper we discuss the question whether one can generalize the notion of effective resistance for infinite networks with zero on infinity (see e.g. [8]) for the case of non-Archimedean weighted graphs. The main

This research was supported by IRTG 2235 Bielefeld-Seoul “Searching for the regular in the irregular: Analysis of singular and random systems”.

1
Theorem of this section is Theorem [13]. It shows, that a sequence of effective admittances of finite networks, exhausted a given infinite network, decreases. Unfortunately, it does not give a convergence over non-Archimedean field. As examples, we consider $LC$-network and $CL$-network (with zero on infinity) as electrical networks over ordered Levi-Civita field $\mathcal{R}$, which contains a subfield isomorphic to $\mathbb{R}(\lambda)$ (see [2], [9]). Firstly, we present the general calculation of admittance of a finite ladder network (see Figure 7) over an ordered field. Then the closeness of the Levi-Civita field in order topology ([2], [12]) gives us an opportunity to arise the question whether effective admittance of infinite network could be defined as limit of effective admittances of corresponding finite electrical networks in this case. We show, that in case of the $LC$-network the sequence of effective admittances of finite networks converge in ordered topology of the Levi-Civita field (Theorem [20]). Moreover, we show, that admittances of finite $CL$-network do not converge in the same topology (Example [23]). This shows, that in general it is not possible to generalize the notion of effective resistance for infinite networks for the case of non-Archimedean weights.

2. Properties of Effective Impedance of the Finite Network over Ordered Field

Definition 1. A network over an ordered field $(K, \succ)$ is a structure \[ \Gamma = (V, \rho, a_0, B), \]

where
- $(V, E)$ is a locally finite connected graph ($|V| \geq 2$).
- $\rho : E \to K$ is a positive function called admittance,
- $a_0 \in V$ is a fixed vertex,
- $B = \{a_1, \ldots, a_{|B|}\} \subseteq V \setminus \{a_0\}$ is a fixed non-empty subset of vertices.

Let us denote by $B_0 = B \cup \{a_0\}$ the set of boundary vertices and by $z = \frac{1}{\rho}$ the positive function of impedance.

The network is called finite if $|V| < \infty$. Otherwise, it is called infinite.

Note that we can consider $\rho$ as a function from $V \times V$ to $K$ by setting $\rho_{xy} = 0$, if $xy$ is not an edge. Then the weight $\rho_{xy}$ gives rise to a function on vertices as follows:

\[ \rho(x) = \sum_y \rho_{xy}, \]

where the notation $\sum_x$ means $\sum_{y \in V}$. Then $\rho(x)$ is called the weight of a vertex $x$. We have $0 < \rho(x) < \infty$ for any vertex $x$ of a network.

Let us consider the following Dirichlet problem on the given finite network $\Gamma = (V, \rho, a_0, B)$:

\[
\begin{cases}
\Delta_\rho v(x) = 0 & \text{on } V \setminus B_0, \\
v(x) = 0 & \text{on } B, \\
v(a_0) = 1,
\end{cases}
\]

where $\Delta_\rho v(x) = \sum_y (v(y) - v(x))\rho_{xy}$. 

The physical meaning of Dirichlet problem is the following: if we take \( K = \mathbb{R}(\lambda) \) and admittance of each edge in the form
\[
\rho_{xy} = \frac{\lambda}{L_{xy} \lambda^2 + R_{xy} \lambda + D_{xy}}, \quad L_{xy}, R_{xy}, D_{xy} \geq 0 \text{ and } L_{xy} + R_{xy} + D_{xy} \neq 0,
\]
then the real voltage at the vertex \( x \) at time \( t \) will be equal to \( \Re(v(x)e^{i\omega t}) \), assuming that we keep potential \( 1 \) at the vertex \( a_0 \), ground all the vertices from \( B \) and apply alternating current of frequency \( \omega = -i\lambda \) to the network (see (11)).

Note that if \( |V| = n \), then the Dirichlet problem (2) is a \( n \times n \) system of linear equations over the field \( K \). It can be also written in a matrix form (note, that here we already have substituted \( v(a_0) = 1, v(a_i) = 0, i = 1, |B| \) in the first \( (n - |B| - 1) \) equations):
\[
\begin{align*}
A \hat{v} &= b, \\
v(a_0) &= 1, \\
v(a_i) &= 0, i = 1, |B|,
\end{align*}
\]
where \( k = |B|, A \) is a symmetric matrix \( (A = A^T) \), \( \hat{v}, b \) are vector-columns of length \( (n - k - 1) \):
\[
A = \begin{bmatrix}
\sum_{x \sim x_1} \rho_{x_1 x_1} & -\rho_{x_1 x_2} & \cdots & -\rho_{x_1 x_{n-1}} \\
-\rho_{x_2 x_1} & \sum_{x \sim x_2} \rho_{x_2 x_2} & \cdots & -\rho_{x_2 x_{n-1}} \\
\cdots & \cdots & \cdots & \cdots \\
-\rho_{x_{n-1} x_1} & -\rho_{x_{n-1} x_2} & \cdots & \sum_{x \sim x_{n-1}} \rho_{x_{n-1} x_{n-1}}
\end{bmatrix},
\]
\[
b = (\rho_{a_0 x_1}, \rho_{a_0 x_2}, \ldots, \rho_{a_0 x_{n-1}})^T,
\]
\[
\hat{v} = (v(x_1), v(x_2), \ldots, v(x_{n-k-1}))^T.
\]

In (11) it is proved, that the Dirichlet problem (2) has a unique solution for any finite network over an ordered field.

**Definition 2.** We define effective admittance of the finite network \( \Gamma \) as
\[
Z_{\text{eff}}(\Gamma) = \frac{1}{\sum_{x,x \sim a_0} (1 - v(x)) \rho_{xa_0}},
\]
where \( v \) is the solution of the Dirichlet problem (2).

The effective admittance is defined by
\[
\mathcal{P}_{\text{eff}}(\Gamma) = \sum_{x,x \sim a_0} (1 - v(x)) \rho_{xa_0}.
\]

**Lemma 3.** For the solution \( v \) of the Dirichlet problem (2) we have
\[
(4) \quad \mathcal{P}_{\text{eff}}(\Gamma) = \sum_{i=1}^{|B|} \sum_{x:x \sim a_i} v(x)\rho_{xa_i} = \sum_{i=1}^{|B|} \Delta P v(a_i) = -\Delta P v(a_0) = \frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} v)^2 \rho_{xy},
\]
where \( \nabla_{xy} v = v(y) - v(x) \).

The proof of this result follows the same outline as the proof of the similar result in (11).

**Theorem 4.** (Star-mesh transform) Let \( \Gamma = (V, \rho, a_0, B) \) be a finite network, \( |V| = n, B_0 = B \cup \{a_0\}, \) and \( x_1, \ldots, x_m \in V, 3 \leq m \leq n, \) are such that
1. \( x_1 \notin B_0. \)
If one removes the vertex $x_1$, edges $(x_1, x_i), i = 2, m$ and change the admittances of the edges $(x_i, x_j), i, j = 2, m, i \neq j$ as follows:

$$\rho'_{x_i x_j} = \rho_{x_i x_j} + \frac{\rho_{x_1 x_i} \rho_{x_1 x_j}}{\rho(x_1)},$$

not changing the other admittances, then for the new network the solution of the Dirichlet problem (2) for all the vertices will be the same as the solution of the Dirichlet problem (2) on the original network at corresponding vertices.

**Proof.** Let us consider the Dirichlet problem for the network $\Gamma$ in a matrix form (3). Obviously, it is enough to solve the matrix equation $A \hat{v} = b$. Without loss of generality we can assume that $x_1, \ldots, x_l \not\in B_0$, where $l = m - \left|\{x_1, \ldots, x_m\} \cap B_0\right|$.

Writing equations for $x_1, \ldots, x_l$ as the first ones and denoting $k = |B|$, we have

$$A = \begin{bmatrix}
\rho(x_1) & -\rho_{x_1 x_2} & \ldots & -\rho_{x_1 x_l} & 0 & \ldots & 0 \\
-\rho_{x_1 x_2} & \rho(x_2) & \ldots & -\rho_{x_2 x_l} & -\rho_{x_2 x_{m+1}} & \ldots & -\rho_{x_2 x_{n-k-1}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
-\rho_{x_1 x_l} & -\rho_{x_2 x_l} & \rho(x_l) & -\rho_{x_l x_{m+1}} & -\rho_{x_l x_{m+1}} & \ldots & -\rho_{x_l x_{n-k-1}} \\
0 & -\rho_{x_2 x_{m+1}} & \ldots & -\rho_{x_{m+1} x_{m+1}} & \rho(x_{m+1}) & \ldots & -\rho_{x_{m+1} x_{n-k-1}} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & -\rho_{x_2 x_{n-k-1}} & \ldots & -\rho_{x_{m+1} x_{n-k-1}} & -\rho_{x_{m+1} x_{n-k-1}} & \ldots & \rho(x_{n-k-1})
\end{bmatrix},$$

since $y \not\sim x_1$ for all $y \in V \setminus \{x_2, \ldots, x_m\}$, and

$$b = (\rho_{a_0 x_1}, \rho_{a_0 x_2}, \ldots, \rho_{a_0 x_l}, \rho_{a_0 x_{m+1}}, \ldots, \rho_{a_0 x_{n-k-1}})^T.$$

Now it is easy to calculate, that the star-mesh transform is just applying of Gaussian elimination method for the first row. Indeed, applying Gaussian elimination...
method for the first row of the augmented matrix $\tilde{A} = [A|b]$ we obtain:

$$
\begin{bmatrix}
1 & -\frac{\rho_{x_1}}{\rho(x_1)} & \cdots & -\frac{\rho_{x_i}}{\rho(x_1)} & 0 \cdots 0 & \frac{\rho_{a_0x_1}}{\rho(x_1)} \\
0 & \rho^*(x_2) & -\rho_{x_1} & -\rho_{x_2, x_{i+1}} & \cdots & -\rho_{x_2, x_{n-1}} & \frac{\rho_{a_0x_2}}{\rho(x_1)} \\
-\rho_{x_3} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & -\rho_{x_{2n-2}} & \rho^*(x_{i+1}) & -\rho_{x_{i+1}, x_{i+2}} & \cdots & -\rho_{x_{2n-2}, x_{2n-1}} & \frac{\rho_{a_0x_{2n-2}}}{\rho(x_1)} \\
0 & 0 & -\rho_{x_{2n-1}} & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & -\rho_{x_{2n-1}} & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
$$

since $\rho(x_1) \neq 0$, where

$$
\rho^*(x_i) = \rho(x_i) - \frac{\rho^2_{x_i}}{\rho(x_i)} \quad \text{and} \quad \rho^*_{a_0x_i} = \rho_{a_0x_i} + \frac{\rho_{a_0x_i, x_i}}{\rho(x_i)} \quad \text{for all} \ i = 2, \ldots, m.
$$

Note, that for all $i = 2, \ldots, m$

$$
\rho'(x_i) = \rho(x_i) - \rho_{x_1} - \sum_{j=2}^{m} \rho_{x_i} + \sum_{j=2}^{m} \rho'_{x_{ij}}
$$

$$
= \rho(x_i) - \rho_{x_1} - \sum_{j=2}^{m} \rho_{x_i} + \sum_{j=2}^{m} \left( \rho_{x_i} + \frac{\rho_{x_1, x_i}}{\rho(x_1)} \right)
$$

$$
= \rho(x_i) - \rho_{x_1} + \sum_{j=2}^{m} \frac{\rho_{x_1, x_i}}{\rho(x_1)}
$$

$$
= \rho(x_i) - \rho_{x_1} + \frac{\rho_{x_1, x_i}}{\rho(x_1)} \sum_{j=2}^{m} \rho_{x_{1j}}
$$

$$
= \rho(x_i) - \rho_{x_1} + \frac{\rho_{x_1, x_i}}{\rho(x_1)} \sum_{j=2}^{m} \rho_{x_{1j}}
$$

$$
= \rho(x_i) - \rho_{x_1} + \frac{\rho_{x_1, x_i}}{\rho(x_1)} \rho(x_1) - \frac{\rho^2_{x_1}}{\rho(x_1)}
$$

$$
= \rho(x_i) - \frac{\rho^2_{x_1}}{\rho(x_1)} = \rho^*(x_i)
$$

and

$$
\rho^*_{a_0x_i} = \rho_{a_0x_i} + \frac{\rho_{a_0x_i, x_i}}{\rho(x_1)}
$$

$$
= \begin{cases} 
\rho_{a_0x_i}, & \text{if } a_0 \in \{x_2, \ldots, x_m\} \\
\rho_{a_0x_i}, & \text{otherwise, since } \rho_{a_0x_1} = 0.
\end{cases}
$$

Hence, $\tilde{A} = [A|b]$
Therefore, we can eliminate the variable $v(x_1)$ from the Dirichlet problem, changing admittances as in the statement of the theorem.

**Corollary 5.** Under the star-mesh transform of the network the effective impedance and effective admittance do not change.

**Proof.** In the proof we will use the notations from the proof of the Theorem 4.

The case $\{x_1, \ldots, x_m\} \cap B_0 = \emptyset$ is trivial. The cases, when $\{x_1, \ldots, x_m\} \cap B = \emptyset$ or $\{x_1, \ldots, x_m\} \cap \{a_0\} = \emptyset$ are obvious, due to (4).

Otherwise, we can assume, without loss of generality, that

$$x_{m-j} = a_j, j = 0, |\{x_1, \ldots, x_m\} \cap B_0|,$$

in particular, $x_m = a_0$. Then, if we denote the new network by $\Gamma'$ we have by (4)

$$\mathcal{P}_{\text{eff}}(\Gamma) = - \Delta_p v(a_0) = \sum_{x \neq a_0} (1 - v(x))\rho_{x,a_0}$$

$$= (1 - v(x_1))\rho_{x_1,a_0} + \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_i,a_0} + \sum_{x \in \{x_1, \ldots, x_m\}} (1 - v(x))\rho_{x,a_0}$$

$$= \mathcal{P}_{\text{eff}}(\Gamma') - \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_i,a_0}' + (1 - v(x_1))\rho_{x_1,a_0} + \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_i,a_0}$$

$$= \mathcal{P}_{\text{eff}}(\Gamma') - \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_i,a_0} \rho_{x_1,x_i} \rho(x_1) + (1 - v(x_1))\rho_{x_1,a_0}$$

$$= \mathcal{P}_{\text{eff}}(\Gamma') - \rho_{x_1,a_0} \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_1,x_i} \rho(x_1) + (1 - v(x_1))\rho_{x_1,a_0}$$

$$= \mathcal{P}_{\text{eff}}(\Gamma') - \rho_{x_1,a_0} \sum_{i=2}^{m-1} (1 - v(x_i))\rho_{x_1,x_i} \rho(x_1) + (1 - v(x_1))\rho_{x_1,a_0}$$

$$= \mathcal{P}_{\text{eff}}(\Gamma') - \rho_{x_1,a_0} \sum_{i=2}^{m-1} v(x_i) \rho_{x_1,x_i} \rho(x_1) + (1 - v(x_1))\rho_{x_1,a_0}$$

$$= \mathcal{P}_{\text{eff}}(\Gamma') - \rho_{x_1,a_0} \sum_{i=2}^{m-1} v(x_i) \rho_{x_1,x_i} \rho(x_1) + \rho_{x_1,a_0}$$

since

$$\rho(x_1) = \sum_{i=2}^{m} \rho_{x_1,x_i} \text{ and } v(x_1) = \sum_{i=2}^{m} v(x_i) \rho_{x_1,x_i} \rho(x_1) + \rho_{x_1,a_0} = \sum_{i=2}^{m-1} v(x_i) \rho_{x_1,x_i} \rho(x_1) + \rho_{x_1,a_0},$$

(see the first line of $\bar{A}$ and note that $v(x_j) = 0$ for all $j = j+1, m-1$ and $a_0 = x_m$).

Series law and $Y - \Delta$ transform are just particular cases of star-mesh transform. Since multigraphs are not allowed in this paper, we will use a modification of parallel law and call it parallel-series law.

**Corollary 6.** (Series law) Let $\Gamma = (V, \rho, a_0, B)$ be a finite network, $B_0 = B \cup \{a_0\}$.

Let $a, b, c \in V$ are such, that
(1) $b \notin B_0$, 
(2) $a \not\sim c$, $a \sim b$, $b \sim c$, 
(3) $b \not\sim x$ for all $x \notin \{a, c\}$.

If one removes the vertex $b$, edges $(a, b), (b, c)$ and add the edge $(a, c)$ with the admittance
\[
\rho'_{ac} = \frac{\rho_{ab}\rho_{bc}}{\rho_{ab} + \rho_{bc}},
\]
not changing other admittances, then for the new network the solution of the Dirichlet problem \((2)\) for all the vertices will be the same as the solution of the Dirichlet problem \((2)\) on the original network at corresponding vertices. The effective impedance (admittance) of the new network coincides with the effective impedance (admittance) of the original one.

**Remark 7.** The corresponding equation for impedances is then
\[
z'_{ac} = z_{ab} + z_{bc},
\]
which corresponds to the well-known physical series law.

**Figure 2. Series law**

**Proof.** Apply Theorem 4 and Corollary 5 ($x_1 = b$) for the case $m = 3$ and $\rho_{ac} = 0$. □

**Corollary 8. (Parallel-series law)** Let $\Gamma = (V, \rho, a_0, B)$ be a finite network, $B_0 = B \cup \{a_0\}$.

Let $a, b, c \in V$ are such that
\[
\begin{align*}
(1) & \quad b \notin B_0, \\
(2) & \quad a \sim b, b \sim c, a \sim c, \\
(3) & \quad b \not\sim x \text{ for all } x \notin \{a, c\}.
\end{align*}
\]

Then if one removes the vertex $b$, edges $(a, b), (b, c)$ and add the edge $(a, c)$ with the admittance
\[
\rho'_{ac} = \frac{\rho_{ab}\rho_{bc}}{\rho_{ab} + \rho_{bc}} + \rho_{ac},
\]
not changing other admittances, then for the new network the solution of the Dirichlet problem \((2)\) for all the vertices will be the same as the solution of the Dirichlet problem \((2)\) on the original network for corresponding vertices. The effective impedance (admittance) of the new network coincides with the effective impedance (admittance) of the original one.

**Remark 9.** The corresponding equation for impedances is then
\[
\frac{1}{z'_{ac}} = \frac{1}{z_{ab} + z_{bc}} + \frac{1}{z_{ac}}
\]
which corresponds to the application of the physical series law and then the physical parallel law.
Theorem 10. ($Y - \Delta$ transform) Let $\Gamma = (V, \rho, a_0, B)$ be a finite network, $B_0 = B \cup \{a_0\}$. Let $a, b, c, d \in V$ are such, that

1. $d \notin B_0$,
2. $d \sim a, d \sim b, d \sim c$,
3. $d \not\sim x$ for all $x \notin \{a, b, c\}$.

If one removes the vertex $d$, edges $(d, a), (d, b), (d, c)$ and set

\[
\rho'_ab = \frac{\rho_{da}\rho_{db}}{\rho_{da} + \rho_{db} + \rho_{dc}} + \rho_{ab},
\]

\[
\rho'_bc = \frac{\rho_{db}\rho_{dc}}{\rho_{da} + \rho_{db} + \rho_{dc}} + \rho_{bc},
\]

\[
\rho'_ac = \frac{\rho_{dc}\rho_{da}}{\rho_{da} + \rho_{db} + \rho_{dc}} + \rho_{ac},
\]

not changing other admittances, then for the new network the solution of the Dirichlet problem (2) for all the vertices will be the same as the solution of the Dirichlet problem (2) on the original network for the corresponding vertices. The effective impedance (admittance) of the new network coincides with the effective impedance (admittance) of the original one.

Remark 11. The corresponding equalities for the impedances are

\[
z'_ab = \frac{z_{dc}}{z_{da}z_{db} + z_{db}z_{dc} + z_{da}z_{dc}} + \frac{1}{z_{ab}},
\]

\[
z'_bc = \frac{z_{dc}}{z_{da}z_{db} + z_{db}z_{dc} + z_{da}z_{dc}} + \frac{1}{z_{bc}},
\]

\[
z'_ac = \frac{z_{dc}}{z_{da}z_{db} + z_{db}z_{dc} + z_{da}z_{dc}} + \frac{1}{z_{ac}}.
\]

From the physical point of view, if $\rho_{ab}, \rho_{bc}, \rho_{ac}$ are all equal to zero, then it is just $Y - \Delta$ transform, otherwise, it is $Y - \Delta$ transform and the parallel law.
**Figure 4.** $Y - \Delta$ transform

**Proof.** Theorem 4 and Corollary 5 ($x_1 = d$) for the case $m = 4$. □

The $Y - \Delta$ transform is invertible. In general, it is not the case for star-mesh transform.

**Theorem 12.** ($\Delta - Y$ transform) Let $\Gamma' = (V', \rho', a_0, B)$ be a finite network and let $a, b, c \in V$ are such, that $a \sim b, b \sim c,$ and $a \sim c$. If one add a vertex $d$ and edges $(d, a), (d, b), (d, c)$ setting

\[
\rho'_{da} = \frac{\rho'_{ac} \rho'_{bc} + \rho'_{ac} \rho'_{ab} + \rho'_{ab} \rho'_{bc}}{\rho'_{bc}},
\]

\[
\rho'_{db} = \frac{\rho'_{ac} \rho'_{bc} + \rho'_{ac} \rho'_{ab} + \rho'_{ab} \rho'_{bc}}{\rho'_{ab}},
\]

\[
\rho'_{dc} = \frac{\rho'_{ac} \rho'_{bc} + \rho'_{ac} \rho'_{ab} + \rho'_{ab} \rho'_{bc}}{\rho'_{ac}},
\]

and remove the edges $(a, b), (b, c), (a, c)$ not changing other admittances, then for the new network

\[\Gamma = (V \cup \{d\}, \rho, a_0, B),\]

the solution of the Dirichlet problem (4) for any vertex in $V$ will be the same as the solution of the Dirichlet problem (2) on the original network for the corresponding vertex. Moreover, the effective impedance and effective admittance do not change under this transform.

**Remark 13.** The corresponding equalities for the impedances are

\[
z_{da} = \frac{z'_{ab} z'_{ac}}{z'_{ab} + z'_{ac}},
\]

\[
z_{db} = \frac{z'_{ab} z'_{bc}}{z'_{ab} + z'_{bc}},
\]

\[
z_{dc} = \frac{z'_{bc} z'_{ac}}{z'_{bc} + z'_{ac}}.
\]
Proof. To prove the theorem it is enough to express $\rho_{da}$, $\rho_{db}$ and $\rho_{dc}$ from (7), assuming $\rho_{ab} = 0$, $\rho_{bc} = 0$, and $\rho_{ac} = 0$. Summing up the inverses of all three equations one obtains

$$\frac{1}{\rho_{ab}'} + \frac{1}{\rho_{bc}'} + \frac{1}{\rho_{ac}'} = \frac{(\rho_{ab} + \rho_{bc} + \rho_{ac})^2}{\rho_{da}\rho_{db}\rho_{dc}}$$

Since both sides are strictly positive, the last equation is equivalent to

$$\frac{\rho_{ab}'\rho_{bc}'\rho_{ac}'}{\rho_{ab}'\rho_{bc}' + \rho_{bc}'\rho_{ac}' + \rho_{ab}'\rho_{ac}'} = \frac{(\rho_{bc} + \rho_{ac} + \rho_{ac})^2}{\rho_{da}\rho_{db}\rho_{dc}}$$

Multiplying the both sides of the last equation by $\frac{1}{\rho_{ab}'\rho_{ac}'} = \frac{(\rho_{ab} + \rho_{bc} + \rho_{ac})^2}{\rho_{da}\rho_{db}\rho_{dc}}$, which follows from (7), we get

$$\frac{\rho_{bc}'}{\rho_{ab}'\rho_{bc}' + \rho_{bc}'\rho_{ac}' + \rho_{ab}'\rho_{ac}'} = \frac{1}{\rho_{da}}.$$ 

Then the equation for $\rho_{da}$ follows. To obtain the equations for $\rho_{db}$ and $\rho_{dc}$ one should multiply (9) by $\frac{1}{\rho_{ab}'\rho_{bc}'}$ and $\frac{1}{\rho_{ac}'\rho_{bc}'}$ respectively.

The fact that effective impedance and effective admittance do not change follows from Theorem [10].

Remark 14. All described in this section transforms preserve the positivity of admittances and impedances on the edges.

3. Effective impedance of infinite networks over an ordered field

3.1. Infinite networks with zero potential on infinity. Let $\Gamma = (V, \rho, a_0, B)$ be an infinite network over an ordered field $(K, \succ)$. Let us consider the sequence of finite graphs $(V_n, \rho|_{V_n})$, where $V_n = \{x \in V \mid \text{dist}(a_0, x) \leq n\}, n \in \mathbb{N}$.

We denote by

$$\partial V_n = \{x \in V \mid \text{dist}(a_0, x) = n\}$$

the boundary of the graph $(V_n, \rho|_{V_n})$. Note that $V_{n+1} = \partial V_{n+1} \cup V_n$.

Let us denote $B_n = B \cap V_n$. Then

$$\Gamma_n = (V_n, \rho|_{V_n}, a_0, B_n \cup \partial V_n), n \in \mathbb{N}$$
is a *sequence of finite networks exhausted the infinite network* $\Gamma$.

This is an analogue to the approach to infinite networks in \cite{3}.

Let us consider the Dirichlet problem \((2)\) on each $\Gamma_n$:

\[
\begin{cases}
\sum_{y \in \partial V_n} (v^{(n)}(y) - v^{(n)}(x)) \rho_{xy} = 0 & \text{on } V_n \setminus (\partial V_n \cup B_n \cup \{a_0\}), \\
v^{(n)}(x) = 0 & \text{on } \partial V_n \cup B_n, \\
v^{(n)}(a_0) = 1,
\end{cases}
\]

(10)

**Theorem 15.**

\[(11)\quad P_{\text{eff}}(\Gamma_{n+1}) \preceq P_{\text{eff}}(\Gamma_n).\]

**Proof.** By Dirichlet/Thomson principle \((11)\) we have

\[(12)\quad P_{\text{eff}}(\Gamma_{n+1}) \preceq \frac{1}{2} \sum_{x,y \in V_{n+1}} (\nabla_{xy} f)^2 \rho_{xy}\]

for any $f : V_{n+1} \to K$ such that $f(a_0) = 1$, $f|_{\partial V_{n+1} \cup B_{n+1}} \equiv 0$.

Since $(V_{n+1} \setminus \partial V_{n+1}) = V_n$ and $B_{n+1} \cap V_n = B_n$, the inequality \((12)\) is true for

\[
f(x) = \begin{cases} v^{(n)}(x), & \text{if } x \in V_n, \\ 0, & \text{if } x \in \partial V_{n+1}, \end{cases}
\]

where $v^{(n)}$ is the solution of \((10)\) for $\Gamma_n$. Then

\[
\frac{1}{2} \sum_{x,y \in V_{n+1}} (\nabla_{xy} f)^2 \rho_{xy} = \frac{1}{2} \sum_{x,y \in V_n} (\nabla_{xy} f)^2 \rho_{xy} + \frac{1}{2} \sum_{x,y \in \partial V_{n+1}} (\nabla_{xy} f)^2 \rho_{xy} = P_{\text{eff}}(\Gamma_n) + 0.
\]

The last equality, together with \((12)\), gives us \((11)\).

\[\square]\]

**Remark 16.** Even in a Cauchy complete non-Archimedean ordered field inequalities \((11)\) for all $n \in \mathbb{N}$ do not imply, that the sequence $\{P_{\text{eff}}(\Gamma_n)\}_{n=1}^{\infty}$ converges. Obviously, if the sequence of effective admittances of finite networks converges, then the corresponding sequence of the effective impedances also has a limit (finite or infinite).

**Definition 17.** If for given infinite network $\Gamma$ the limit of effective admittances (impedances) of exhausted finite networks exists in $K$, we call it *effective admittance (impedance) of the network $\Gamma$ with zero potential at infinity* and denote it by $P_{\text{eff}}(\Gamma) \langle Z_{\text{eff}}(\Gamma) \rangle$.

### 3.2. Examples: ladder networks over Levi-Civita field.

In this subsection we will investigate the behavior of the sequence $\{P_{\text{eff}}(\Gamma_{n\beta})\}_{n=1}^{\infty}$ of effective admittances of finite networks exhausted the ladder network $(\alpha\beta)$-network at the Figure \cite{6}($\alpha, \beta \in K, \alpha, \beta > 0$). More precisely, $\alpha\beta$-network is a network $\Gamma^{\alpha\beta} = \{V, \rho, a_0, B\}$, where

- $V = \{a_0, a_1, a_2, \ldots, x_1, x_2, \ldots\}$,
- $\rho_{a_0x_i} = \alpha$, $\rho_{x_ix_{i+1}} = \alpha$, $\rho_{a_1x_i} = \beta$, $i \in \mathbb{N}$, and $\rho_{xy} = 0$, otherwise,
- $B = \{a_0, a_1, a_2, \ldots\}$.

This network is similar to Feynman’s ladder network and $CL$-network (see \cite{6, 15}), but has zero potential at infinity. Therefore, for any ordered field $K$ the Theorem \cite{15} is true for this network. We will show (Theorem \cite{20} and Example \cite{23}
that whether \( \{ \mathcal{P}_{eff}(\Gamma_n) \}_{n=1}^\infty \) converges in Cauchy completeness of \( K \) depends on \( \alpha \) and \( \beta \).

\[
\begin{align*}
\alpha v(a_0) + \beta v(a_1) + \alpha v(x_2) - (2\alpha + \beta)v(x_1) &= 0, \\
\alpha v(x_{i-1}) + \beta v(a_i) + \alpha v(x_{i+1}) - (2\alpha + \beta)v(x_i) &= 0 \text{ for } i = 2, n-1, \\
v(x_n) &= 0, \\
v(a_i) &= 0 \text{ for } i = 1, n-1, \\
v(a_0) &= 1.
\end{align*}
\]

Using the second line in (13) we obtain the following recurrence relation for \( v(x_i), i = 2, n-1 \)

\[
v(x_{i+1}) - \left( 2 + \frac{\beta}{\alpha} \right) v(x_i) + v(x_{i-1}) = 0
\]

since \( v(a_i) = 0 \) for \( i = 2, n-1 \). The characteristic polynomial of (14) is

\[
\psi^2 - \left( 2 + \frac{\beta}{\alpha} \right) \psi + 1 = 0.
\]

Its roots are

\[
\psi_{1,2} = 1 + \frac{\beta}{2\alpha} \pm \xi,
\]

where

\[
\xi = \sqrt{\frac{\beta}{\alpha} + \left( \frac{\beta}{2\alpha} \right)^2}.
\]
Note that $\xi$ should not necessarily belong to $K$. It is known that any ordered field possesses a real-closed (or maximal ordered) extension $\overline{K}$. Then in $\overline{K}$ exists exactly one positive square root of $\frac{\beta}{\alpha} + \left(\frac{\beta}{2\alpha}\right)^2$ (13). Therefore, we fix the extension $\overline{K}$, denote the positive square root by $\xi$, and make all the further calculations in $\overline{K}$.

The solution of the recurrence equation (14) is

$$v(x_i) = c_1 \psi_1^i + c_2 \psi_2^i,$$

where $c_1, c_2 \in \overline{K}$ are arbitrary constants.

We use first and third equations in (13) as boundary conditions for this recurrence equation. Substituting (17) in the boundary conditions we obtain the following equations for the constants:

$$\begin{cases} 
1 + c_1 \psi_1^2 + c_2 \psi_2^2 - \left(2 + \frac{\beta}{\alpha}\right) (c_1 \psi_1 + c_2 \psi_2) = 0, \\
c_1 \psi_1 + c_2 \psi_2 = 0,
\end{cases}$$

which, by (15) is equivalent to

$$\begin{cases} 
c_1 + c_2 = 1, \\
c_1 \psi_1 + c_2 \psi_2 = 0.
\end{cases}$$

Therefore,

$$\begin{cases} 
c_1 = \frac{1}{1 - \psi_1^2}, \\
c_2 = \frac{\psi_1^2}{1 - \psi_1^2},
\end{cases}$$

since $\psi_1 \psi_2 = 1$ by (15).

Now we can calculate the effective admittance of $\Gamma_n$:

$$\rho_{eff}(\Gamma_n) = \alpha \left(1 - v(x_1)\right) = \alpha \left(1 - c_1 \psi_1 - c_2 \psi_2\right) = \frac{\alpha \left(\left(1 + \frac{\beta}{\alpha} + \frac{\psi_1}{1 - \psi_1^2}\right)^{2n-1} + 1\right) \left(\frac{\beta}{\alpha} + \frac{\psi_1}{1 - \psi_1^2}\right)}{\left(1 + \frac{\beta}{\alpha} + \frac{\psi_1}{1 - \psi_1^2}\right)^{2n} - 1}.$$
3.2.2. Infinite ladder networks over Levi-Civita field $\mathcal{R}$. We will consider two examples of $\alpha\beta$-network over Levi-Civita field $\mathcal{R}$. Firstly, let us describe the Levi-Civita field $\mathcal{R}$. We take the definition of $\mathcal{R}$ and theorems about its properties from [2] and [12].

**Definition 18.** A subset $M$ of the rational numbers $\mathbb{Q}$ is called left-finite if for every $r \in \mathbb{Q}$ there are only finitely elements of $M$ that are smaller than $r$.

Then the Levi-Civita field is the set of all real valued functions on $\mathbb{Q}$ with left-finite support with the following operations:

- **addition** is defined component-wise
  $$(\alpha + \beta)(q) = \alpha(q) + \beta(q),$$

- **multiplication** is defined as follows
  $$(\alpha \cdot \beta)(q) = \sum_{q_a, q_b \in \mathbb{Q}, \ q_a + q_b = q} \alpha(q_a) \cdot \beta(q_b).$$

It is proved in [2], that $\mathcal{R}$ is an ordered field with a set of positive elements

$$\mathcal{R}^+ = \{ \alpha \in \mathcal{R} \mid \alpha(\min\{ q \in \mathbb{Q} \mid \alpha(q) \neq 0 \}) > 0 \}.$$

We denote by $\tau$ the following element in $\mathcal{R}$:

$$\tau(q) = \begin{cases} 1, & \text{if } q = 1, \\ 0, & \text{otherwise}, \end{cases}$$

which plays role of infinitesimal in Levi-Civita field. Therefore, the Levi-Civita field is non-Archimedean.

By [2] we can write any $\alpha \in \mathcal{R}$ as

$$\alpha = \sum_{i=1}^{\infty} \alpha(q_i) \tau^i,$$

since $\alpha_n = \sum_{i=1}^{n} \alpha(q_i) \tau^i$ converges strongly to the limit $\alpha$ in the order topology.

The set of all polynomials over real numbers $\mathbb{R}[\tau] = \{ a_n \tau^n + \ldots + a_1 \tau + a_0 \mid a_i \in \mathbb{R}, n \in \mathbb{N} \}$ is a subring of Levi-Civita field $\mathcal{R}$ due to (21). Therefore, since $\mathcal{R}$ is a field, the field of rational functions over real numbers

$$\mathbb{R}(\tau) = \left\{ \frac{\sum_{i=k}^{n} a_i \tau^i}{\sum_{l=1}^{m} b_l \tau^l} \mid a_i, b_l \in \mathbb{R}, n, m, k, l \in \mathbb{N}_0 \right\}$$

is isomorphic to a subfield of $\mathcal{R}$.

**Example 19.** Let us find the element in $\mathcal{R}$ which corresponds to the rational function $\frac{1}{\tau^4 + 3 \tau^2}$, i.e. we should find the sequences $\{q_i\} \in \mathbb{Q}$ and $\{\alpha(q_i)\} \in \mathbb{R}$ such that

$$(3 - 4\tau + \tau^2) \left( \sum_{q_i} \alpha(q_i) \tau^{q_i} \right) = 1.$$ 

Comparing the coefficients at powers of $\tau$ at right hand side and left hand side, starting from the lowest power, one obtains

$$q_1 = 0, \alpha(q_1) = \frac{1}{3},$$
\[ q_2 = 1, \alpha(q_2) = \frac{4}{9}, \]

and the recurrence relation
\[ q_i = q_{i-1} + 1, 3\alpha(q_i) - 4\alpha(q_{i-1}) + \alpha(q_{i-2}) = 0 \text{ for } i > 2. \]

Therefore, solving the recurrence relation for \( \alpha(q_i) \) we obtain
\[ \alpha(q_i) = -\frac{1}{2^i \cdot 3^i} + \frac{1}{2} \]

and
\[ \frac{1}{3 - 4\tau + \tau^2} = \sum_{i \in \mathbb{N}} \left( -\frac{1}{2^i \cdot 3^i} + \frac{1}{2} \right) \tau^{i-1}. \]

Note, that the corresponding order in the field \( \mathbb{R}(\tau) \) is the following:
\[ a_k \tau^k + a_{k+1} \tau^{k+1} + \cdots + a_n \tau^n \approx 0 \text{ if } \frac{a_k}{b_l} > 0. \]

Therefore, we can consider Levi-Civita field \( \mathcal{R} \) as an ordered extension of the ordered field \( \mathbb{R}(\tau) \) with the positiveness defined as (23). To consider \( \mathcal{R} \) as an ordered extension of the ordered field \( \mathbb{R}(\tau) \) with the positiveness defined in [11] we make a substitution
\[ \tau = \frac{1}{\lambda}. \]

Consequently, we can consider electrical networks over field of rational numbers [11], as networks over Levi-Civita field and investigate the behavior of the sequence of effective admittances of finite electrical networks. By [2] the Levi-Civita field is Cauchy complete in order topology and real-closed.

From the physical point of view we have the following impedances of passive elements
\[ \begin{align*}
& (L\omega)^{-1} = L^{-1}\tau, L > 0 \text{ for the coil}, \\
& (C\omega) = C\tau^{-1}, C > 0 \text{ for the capacitor}, \\
& R > 0 \text{ for the resistor},
\end{align*} \]

where \( \omega \) is a frequency of the alternating current (see [11]).

Let us consider the Feynman’s infinite ladder LC-network, assuming that it has zero potential at infinity. It is an \( \alpha\beta \)-network with \( \alpha = L^{-1}\tau, \beta = C\tau^{-1} \), where \( L, C > 0, \alpha, \beta \in \mathcal{R} \).

**Theorem 20.** For the Feynman’s ladder LC-network \( (\alpha = L^{-1}\tau, \beta = C\tau^{-1}, \text{where } L, C > 0) \) with zero potential at infinity
\[ \mathcal{P}_{\text{eff}}(\Gamma_n) \to \frac{\beta}{2\alpha + \xi} \text{ as } n \to \infty \]

in the order topology of Levi-Civita field \( \mathcal{R} \), where \( \Gamma_n \) is the sequence of the exhausted finite networks

**Remark 21.** For the Feynman’s ladder LC-network
\[ \frac{\beta}{2\alpha + \xi} = \left( \frac{C}{2\tau} - \frac{\tau}{L} \sqrt{\frac{CL}{\tau^2} + \left( \frac{CL}{2\tau^2} \right)^2} \right) \]

and the motivation for this quantity was Feynman’s impedance for infinite ladder LC-network (see [6, 22-13]).
Proof. Firstly, we should write $\xi$ as an element of Levi-Civita field, i.e. as power series (21).

$$
\xi = \sqrt{\frac{CL}{2\tau^2} + \left(\frac{CL}{2\tau^2}\right)^2} = \frac{CL}{2\tau^2} \tau^2 + 1 = \frac{CL}{2\tau^2} \sum_{k=0}^{\infty} \left(\frac{1}{k}\right) \left(\frac{4\tau^2}{CL}\right)^k
$$

$$
= \frac{CL}{2\tau^2} \cdot 1 + \frac{CL}{2\tau^2} \cdot 1 \cdot \frac{4\tau^2}{CL} - \frac{1}{8} \cdot \frac{CL}{2\tau^2} \left(\frac{4\tau^2}{CL}\right)^2 + o(\tau^2)
$$

$$
= \frac{CL}{2} \tau^2 + 1 - \tau^2 + o(\tau^2).
$$

Note that here and further $o(\tau^m)$, where $m \in \mathbb{Z}$ means $\sum_{k=m+1}^{\infty} a_k \tau^k, a_k \in \mathbb{R}$.

Let us calculate the difference $\mathcal{P}_{\text{eff}}(\Gamma_n) - \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}$.

$$
\mathcal{P}_{\text{eff}}(\Gamma_n) - \frac{\beta}{\frac{2\alpha}{\alpha} + \xi} = \alpha \left(\frac{\beta}{\frac{2\alpha}{\alpha} + \xi} + \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n-1} + 1\right) - \beta \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n} - \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}
$$

The nominator $A$ of the last expression is

$$
A = \alpha \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n-1} + 1 \left(\frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^2 - \beta \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n} - \alpha(D + 1) \left(\frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^2 - \beta \left(\frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^2 - \beta \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n-1}.
$$

where $D = \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n-1}$.

Since

$$
\left(\frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^2 = \frac{\beta^2}{4\alpha^2} + \frac{\beta}{\alpha} \xi + \xi^2 = \frac{\beta^2}{2\alpha^2} + \frac{\beta}{\alpha} \xi = \frac{\beta}{\alpha} \left(\frac{\beta}{\frac{2\alpha}{\alpha} + 1 + \xi}\right)
$$

and

$$
\xi^2 = \frac{\beta}{\alpha} + \frac{\beta^2}{4\alpha^2},
$$

we have

$$
A = \alpha(D + 1) \frac{\beta}{\alpha} \left(\frac{\beta}{\frac{2\alpha}{\alpha} + 1 + \xi}\right) - \beta \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n-1} - \beta \left(1 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right)^{2n-1}
$$

$$
= 2\alpha \left(\xi^2 + \frac{\beta}{\frac{2\alpha}{\alpha} + \xi}\right) = 2\alpha \xi \left(\xi + \frac{\beta}{\frac{2\alpha}{\alpha}}\right).
$$
In this case,proof.

Since \( \frac{2\alpha \xi}{(1 + \frac{\beta}{2\alpha} + \xi)^{2n} - 1} \).

The right hand side of the last expression is, obviously,positive in \( (\mathbb{R}, \succ) \), therefore

\[
\mathcal{P}_{\text{eff}}(\Gamma_n) - \frac{\beta}{2\alpha + \xi} = \frac{2\alpha \xi}{(1 + \frac{\beta}{2\alpha} + \xi)^{2n} - 1}.
\]

\[
= \frac{2\alpha \xi}{L \left( (1 + CL\tau + \xi)^{2n} - 1 \right)}
\]

\[
= \frac{CL\tau^{-1} + 2\tau - 2\tau^3 + o(\tau^3)}{L \left( (CL\tau^{-2} + 2 - \frac{1}{CL}\tau^2 + o(\tau^2))^{2n} - 1 \right)}
\]

\[
= \frac{CL\tau^{-1} + 2\tau - 2\tau^3 + o(\tau^3)}{(CL\tau^{-2})^{2n} + o(\tau^{-4n-2})}
\]

\[
= (C\tau^{-1} + o(\tau^{-1})) \left( \frac{1}{(CL)^{2n} \tau^{4n} + o(\tau^{4n+2})} \right)
\]

\[
= \frac{C}{(CL)^{2n} \tau^{4n-1} + o(\tau^{4n-1})} \to 0,
\]

when \( n \to \infty. \)

**Remark 22.** From the proof one can see that (25) is true for \( \alpha \beta \)-network whenever for any \( \gamma \in \mathbb{R} \) exists \( N_0 \in \mathbb{N} \) such that \( n > N_0 \) implies \( \left( \frac{\beta}{\alpha} \right)^n \succ \gamma \).

**Example 23.** For the CL-network \( (\alpha = C\tau^{-1}, \beta = L^{-1}\tau, L, C > 0) \) effective admittances of the exhausted finite networks do not converge in the Levi-Civita field \( \mathbb{R} \).

**Proof.** In this case \( \xi = \frac{\tau}{\sqrt{CL}} + \left( \frac{\tau}{2\sqrt{CL}} \right)^3 + o(\tau^3) \). Let us prove, that \( \{ \mathcal{P}(\Gamma_n) \}_{n=1}^{\infty} \) is not a Cauchy sequence in \( \mathbb{R} \). Indeed

\[
\mathcal{P}(\Gamma_{n+1}) - \mathcal{P}(\Gamma_n) = \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \left( 1 + \frac{\beta}{2\alpha} + \xi \right)^{2n+1} - 1 \right) - \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \left( 1 + \frac{\beta}{2\alpha} + \xi \right)^{2n} - 1 \right)
\]

\[
= \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \left( 1 + \frac{\beta}{2\alpha} + \xi \right)^{2n+1} + 1 \right) - \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \left( 1 + \frac{\beta}{2\alpha} + \xi \right)^{2n} + 1 \right)
\]

\[
= \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \left( 1 + \frac{\beta}{2\alpha} + \xi \right)^{2n+2} - 1 \right) - \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \left( 1 + \frac{\beta}{2\alpha} + \xi \right)^{2n} - 1 \right)
\]

Since

\[
\psi_1 = 1 + \frac{\beta}{2\alpha} + \xi = 1 + \frac{\tau}{\sqrt{CL}} + o(\tau^3),
\]
we can rewrite
\[
\mathcal{P}(\Gamma_{n+1}) - \mathcal{P}(\Gamma_n) = \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \frac{\psi_1^{2n+1} + 1}{\psi_1^{2n+2} - 1} - \frac{\psi_1^{2n-1} + 1}{\psi_1^{2n} - 1} \right)
\]
\[
= C \alpha \left( \frac{\beta}{2\alpha} + \xi \right) \left( \frac{\psi_1^{2n+1} + 1}{\psi_1^{2n+2} - 1} - \frac{\psi_1^{2n-1} + 1}{\psi_1^{2n} - 1} \right)
\]
Substituting
\[
(\psi_1^{2n+1} + 1)(\psi_1^{2n} - 1) - (\psi_1^{2n-1} + 1)(\psi_1^{2n+2} - 1)
\]
\[
= \left( 2 + \frac{\tau}{\sqrt{CL}(2n+1) + o(\tau^1)} \right) \left( \frac{\tau}{\sqrt{CL}} (2n) + o(\tau^1) \right)
\]
\[
- \left( 2 + \frac{\tau}{\sqrt{CL}(2n-1) + o(\tau^1)} \right) \left( \frac{\tau}{\sqrt{CL}} (2n + 2) + o(\tau^1) \right)
\]
\[
= - 4 \frac{\tau}{\sqrt{CL}} + o(\tau^1)
\]
and
\[
(\psi_1^{2n+2} - 1)(\psi_1^{2n} - 1) = \left( \frac{\tau}{\sqrt{CL}} (2n + 2) + o(\tau^1) \right) \left( \frac{\tau}{\sqrt{CL}} (2n) + o(\tau^1) \right)
\]
\[
= \frac{\tau^2}{CL}(4n^2 + 4n) + o(\tau^2)
\]
we obtain
\[
\mathcal{P}(\Gamma_{n+1}) - \mathcal{P}(\Gamma_n)
\]
\[
= \left( \frac{C}{\sqrt{CL}} + o(\tau^0) \right) \left( -4 \frac{\tau}{\sqrt{CL}} + o(\tau^1) \right) \left( CL \frac{\tau^2}{4n^2 + 4n} + o(\tau^{-2}) \right)
\]
\[
= -\frac{4C}{n^2 + n} \tau^{-1} + o(\tau^{-1}) > \tau \text{ for any } n \in \mathbb{N} \text{ and for any } L, C > 0.
\]
Therefore, \( \{ \mathcal{P}(\Gamma_n) \}_{n=1}^\infty \) is not a Cauchy sequence in \( \mathcal{P} \). \( \square \)

Therefore, the following question:
Under what conditions the effective admittance of infinite network over non-Archimedean field could be defined?
remains open. Note, that Remark 22 gives some sufficient condition for \( \alpha \beta \)-network.

ACKNOWLEDGEMENT

The author thanks her scientific advisor, Professor Alexander Grigor’yan, for fruitful discussions on the topic.
REFERENCES

[1] Martin T. Barlow. *Random Walks and Heat Kernels on Graphs*. London Mathematical Society, Lecture Note Series: 438. Cambridge University Press, 2017.

[2] Martin Berz, Christian Bischof, George Corliss, Andreas Griewank. *Computational Differentiation: Techniques, Applications, and Tools. Chapter 2: Calculus and Numerics on Levi-Civita Fields*. eds., SIAM, 1996.

[3] N. Bourbaki. *Elements of Mathematica. Algebra II, Chapters 4-7*. Springer-Verlag Berlin Heidelberg, 2003.

[4] P.G. Doyle, J.L. Snell. *Random walks and electric networks*. Carus Mathematical Monographs 22, Mathematical Association of America. Washington, DC, 1984.

[5] Richard P. Feynman, Robert B. Leighton, Matthew Sands. *The Feynman lectures on physics, Volume 1: Mainly mechanics, radiation, and heat*. Addison-Wesley publishing company. Reading, Massachusetts, Fourth printing – 1966.

[6] Richard P. Feynman, Robert B. Leighton, Matthew Sands. *The Feynman lectures on physics, Volume 2: Mainly Electromagnetism and Matter*. Addison-Wesley publishing company. Reading, Massachusetts, Fourth printing – 1966.

[7] A. Grigor’yan. *Introduction to Analysis on Graphs*. AMS University Lecture Series, Volume: 71. Providence, Rhode Island, 2018.

[8] G. Grimmett. *Probability on Graphs: Random Processes on Graphs and Lattices*. Cambridge University Press. New York, 2010.

[9] J. F. Hall, T. D. Todorov. *Ordered Fields, the Purge of Infinitesimals from Mathematics and the Rigoroussness of Infinitesimal Calculus*. Bulgarian Journal of Physics, 2015. Vol. 42, n.2, 99–127.

[10] David A. Levin, Yuval Peres, Elizabeth L. Wilmer. *Markov Chains and Mixing Times*. AMS University Lecture Series. Providence, Rhode Island, 2009.

[11] Anna Muranova. *On the notion of effective impedance*. arXiv e-prints, page [arXiv:1905.02047], May 2019.

[12] K. Shamseddine. *New Elements of Analysis on the Levi-Civita Field*. PhD thesis, Michigan State University, East Lansing, Michigan, USA, 1999.

[13] Paolo M. Soardi. *Potential Theory on Infinite Networks*. Springer-Verlag, Berlin Heidelberg, 1994.

[14] Wolfgang Woess. *Random Walks on Infinite Graphs and Groups*. Cambridge Tracts in Mathematics: 138. Cambridge University Press, 2000.

[15] Sung Hyun Yoon. *Ladder-type circuits revisited*. European journal of physics, 2007. Vol. 22, n. 22, 277–288.

Anna Muranova: IRTG 2235, University Bielefeld, Postfach 10 01 31, 33501 Bielefeld, Germany

E-mail address: anna.muranova@gmail.com