Low Energy Sum Rules For Pion-Pion Scattering and Threshold Parameters*

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Abstract

We derive a set of sum rules for threshold parameters of pion-pion scattering whose dispersion integrals are rapidly convergent and are dominated by S- and P-waves absorptive parts. Stringent constraints on some threshold parameters are obtained.

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1 Introduction

Pion-pion scattering is a fundamental strong interaction process that is particularly well suited for theoretical investigations. The pion is the lightest hadron and the principles of axiomatic field theory lead to a wealth of rigorous results, some of which have a direct physical relevance \cite{1}. These results are consequences of analyticity, unitarity and crossing symmetry (isospin violation effects are ignored). On another front, chiral perturbation theory provides an extension of the current algebra techniques and produces explicit representations for the low energy pion-pion scattering amplitudes \cite{2}. These amplitudes exhibit the required general properties within their domain of validity and their specific structure reflects the fact that the pion is a Goldstone boson associated with the spontaneous breaking of the axial symmetry of the massless quark limit of QCD \cite{3}.

Unfortunately, experimental information on pion-pion scattering is hard to obtain. Phase shift analyses for the S- and P-waves are available above threshold and up to 1400 MeV \cite{4} but large uncertainties prevail for the threshold parameters (scattering lengths and effective ranges) \cite{5}. This is an awkward state of affairs since these parameters play a central role in chiral perturbation theory. The situation can be improved by constructing solutions of Roy’s rigorous partial wave equations \cite{6} which are consistent with experimental information \cite{7}. However this procedure does not fix the
threshold parameters uniquely.

In this paper we present an alternative approach to the problem of low energy pion-pion scattering which does not resort to chiral perturbation theory or to the Roy equations. We derive constraints on pion-pion threshold parameters which are consequences of exact properties combined with the known low energy features of pion-pion scattering. Our tools are sum rules involving dispersion integrals that are dominated by the low energy S- and P- waves. The well known Olsson sum rules cannot be used since they are sensitive to the high energy absorptive parts. We have found three sum rules which fulfill our needs. Their dispersion integrals being dominated by low energy contributions, depend significantly on threshold parameters and this dependence cannot be ignored. This leads us to the following strategy: the S- and P- wave absorptive parts occurring in the integrands are parametrized to reproduce the main characteristics of the low energy cross-sections with the scattering lengths and effective ranges as free parameters. The parametrization we use has been proposed by Schenk. The sum rules become non-linear equations for the S- and P- wave threshold parameters and a combination of D- wave scattering lengths. We show that the solutions of these equations which are compatible with the data are confined to a rather small portion of the experimentally allowed domain. This is our main result and it establishes the relevance of our sum rules. One may hope that the expected improved data will allow a detailed check of their impli-
cations. Furthermore, the sum rules presented here could be used as a tool to estimate corrections to certain one loop predictions of chiral perturbation theory [11].

We derive our sum rules in Section 2 using a crossing symmetric decomposition of the definite isospin amplitudes into an S- and P-wave term and a higher waves contribution. Their implications are established in Section 3 by means of quadratic and linear fits of the equations for the threshold parameters. The constraints are discussed and compared with chiral perturbation theory results in Section 4.

2 Low energy Sum Rules

We explain in this section how one obtains the approximate relations between the threshold parameters and low energy S- and P-wave absorptive parts which are at the basis of our analysis. We also present exact counterparts of these relations which include the complete absorptive parts.

We exploit the quite remarkable fact established some time ago [7, 12], that there is a set of analytic amplitudes $\hat{T}$ which have the exact S- and P-wave absorptive parts, are crossing symmetric and respect the Froissart bound. These unique amplitudes are given by:
\[
\hat{T}(s, t, u) = \frac{1}{4} (s + tC_{st} + uC_{su})a_0 + \frac{1}{\pi} \int_{4}^{\infty} \frac{dx}{x(x - 4)} \cdot \left\{ \left[ \frac{s(s - 4)}{(x - s)} + \frac{t(t - 4)}{(x - t)}C_{st} + \frac{u(u - 4)}{(x - u)}C_{su} \right] \operatorname{Im}f_0(x) \right. \\
+ \left. 3 \left[ \frac{s(t - u)}{x - s} + \frac{t(s - u)}{x - t}C_{st} + \frac{u(t - s)}{x - u}C_{su} \right] \operatorname{Im}f_1(x) \right\}
\]

Our notations are standard:

\[
T = \begin{pmatrix} T^0 \\ T^1 \\ T^2 \end{pmatrix},
\quad a_0 = \begin{pmatrix} a_0^0 \\ 0 \\ a_0^2 \end{pmatrix},
\quad f_0 = \begin{pmatrix} f_0^0 \\ 0 \\ f_0^2 \end{pmatrix},
\quad f_1 = \begin{pmatrix} 0 \\ f_1^1 \\ 0 \end{pmatrix}
\]

Here \(T^I\) designates the isospin \(I\) \(s\)-channel amplitude, \(f^I_l\) is its \(l\)-th partial wave amplitude, \(a^I_0\) is an S-wave scattering length and \(C_{st}\) and \(C_{su}\) denote the crossing matrices. Our normalization of \(T^I\) is such that its \(s\)-channel partial wave expansion is

\[
T^I(s, t, 4 - s - t) = \sum_{l=0}^{\infty} (2l + 1) f^I_l(s) P_l(1 + \frac{2t}{s - 4}),
\]

where \(s\) is the square of the center of mass energy and \(t\) is the square of the momentum transfer, both in units of \(m_{\pi}^2\), \(m_{\pi}\) being the pion mass. Below the inelastic threshold the partial wave amplitudes are given in terms of their phase shifts \(\delta^I_l\):

\[
f^I_l(s) = \sqrt{\frac{s}{s - 4}} e^{i\delta^I_l(s)} \sin \delta^I_l(s),
\]
If we decompose the full amplitudes $T'$ according to

$$T = \hat{T} + \mathcal{T}$$  \hspace{1cm} (2.5)

the absorptive parts of the second term $\mathcal{T}$ turn out to be sums of partial waves absorptive parts with $l \geq 2$. It then follows that eq. (2.5) represents a unique crossing symmetric decomposition of the amplitude $T'$ into an S- and P-wave contribution $\hat{T}'$ and a higher waves term $\mathcal{T}'$. We call $\hat{T}'$ a truncated amplitude. The properties of $\mathcal{T}'$ are consequences of rigorous twice subtracted fixed-t dispersion relations. Note that $\hat{T}'$ is analytic in the three variables $s$, $t$ and $u$ with cuts $[4, \infty)$. Therefore $\mathcal{T}'$ is analytic too. Clearly, neither $\hat{T}'$ nor $\mathcal{T}'$ fulfills the unitarity condition. It should also be kept in mind that $\hat{T}'$ does not carry the complete S- and P-waves; $\mathcal{T}'$ contributes to the real parts of their amplitudes.

According to (2.5), every pion-pion threshold parameter is a sum of a truncated part coming from $\hat{T}$ and a higher waves term due to $\mathcal{T}$. Definition (2.1) implies that the truncated S-wave scattering lengths coincide with the full scattering lengths. The other truncated threshold parameters are obtained from (2.1) as combinations of integrals over S- and P-wave absorptive parts and S-wave scattering lengths. We are looking for threshold parameters, or linear combinations of such parameters, which are well approximated by their truncated part. That is to say, we have to find combinations for which the higher waves contribution is under control and can be assumed to be small. We first try to do this for $\pi^0-\pi^0$ parameters.
Let \( T(s, t, u) \equiv \frac{1}{3}(T^0(s, t, u) + 2T^2(s, t, u)) \) be the full \( \pi^0-\pi^0 \) amplitude. According to (2.1) its truncated version is:

\[
\hat{T}(s, t, u) = a + \frac{1}{\pi} \int_4^\infty \frac{dx}{x(x-4)} \left[ \frac{s(s-4)}{x-s} + \frac{t(t-4)}{x-t} + \frac{u(u-4)}{x-u} \right] \text{Im} f(x)
\]

with \( f = \frac{1}{3}(f_0^0 + 2f_0^2) \), \( a = \frac{1}{3}(a_0^0 + 2a_0^2) \). Eq. (2.6) gives for \( t = 0 \):

\[
\text{Re} \left\{ \hat{T}(s, 0, 4-s) - \hat{T}(4, 0, 0) \right\} = \frac{s-4}{s-4} \pi \int_4^\infty dx \frac{1}{x(x-4)} \frac{2x-4}{x(x-4) - s(s-4)} \text{Im} f(x)
\]

Threshold parameters specify the behaviour of a scattering amplitude as \( s \to 4 \) from above. Some care is required in taking the limit of the integral in (2.7) since it appears to diverge at first sight. We have to exploit the threshold behaviour of \( \text{Im} f \) which allows us to write:

\[
\text{Im} f(x) = \frac{1}{16\pi} \sqrt{x(x-4)} \sigma(x)
\]

with \( \sigma \) regular at \( x = 4 \) (\( \sigma \) is the S-wave \( \pi^0-\pi^0 \) total cross-section). After insertion of (2.8) into (2.7) one finds \( \sigma(x) \) can be replaced by \( (\sigma(x) - \sigma(4)) \) in the integrand if \( s > 4 \), without changing the value of the integral. This is due to the identity:

\[
P \int_4^\infty dv' \frac{1}{\sqrt{v' - 4(v' - v)}} = 0,
\]
which is true if \( v > 4 \). The limit \( s \to 4^+ \) can be taken safely once this subtraction has been performed.

The limit of the left-hand side of (2.7) is equal to \( \hat{b}/4 \), \( \hat{b} \) being the truncated \( \pi^0-\pi^0 \) S-wave effective range. We use the standard definition of scattering lengths \( a^I_l \) and effective ranges \( b^I_l \):

\[
\text{Re} f^I_l(\nu) = \nu^I_l(a^I_l + b^I_l \cdot \nu + ...) \tag{2.10}
\]

where \( \nu \) is the square of the center of mass momentum, \( \nu \equiv (s - 4)/4 \). Using \( \nu \) as the integration variable, we obtain the following sum rule

\[
\hat{b} = \frac{1}{4\pi^2} \int_0^\infty d\nu \frac{2\nu + 1}{(\nu(\nu + 1))^{3/2}}(\sigma(\nu) - \sigma(0)) \tag{2.11}
\]

A second sum rule is obtained by combining the derivative of (2.6) with respect to \( t \) at threshold with (2.11). It gives the truncated D-wave scattering length \( \hat{a}^{(2)} \) (\( a^{(2)} = \frac{1}{3}(a_0^2 + 2a_2^2) \)):

\[
\hat{a}^{(2)} = \frac{1}{60\pi^2} \int_0^\infty d\nu \frac{\nu^{1/2}}{(\nu + 1)^{5/2}}\sigma(\nu) \tag{2.12}
\]

An important point is that we have sum rules not only for the truncated \( \hat{a}^{(2)} \) and \( \hat{b} \) but also for the complete D-wave scattering length \( a^{(2)} \) and S-wave effective range \( b \) \[13\]. The latter are consequences of the exact analyticity properties of the full amplitudes \( T^I \) and crossing symmetry. Combining them with (2.11) and (2.12) one obtains the decompositions:
\[ b = \hat{b} + \bar{b} \]
\[ a_{(2)} = \hat{a}_{(2)} + \bar{a}_{(2)} \]

The higher wave contributions given by
\[
\bar{b} = \frac{1}{4\pi^2} \int_0^\infty d\nu \frac{2\nu + 1}{(\nu(\nu + 1))^{3/2}}(\sigma(\nu)),
\]
\[
\bar{a}_{(2)} = \frac{1}{60\pi^2} \int_0^\infty d\nu \left[ \frac{\nu^{1/2}}{(\nu + 1)^{5/2}} \sigma(\nu) + 8\pi \frac{2\nu + 1}{(\nu(\nu + 1))^2} \frac{\partial}{\partial t}A(\nu,0) \right],
\]

where \( A(\nu, t) \) is the absorptive part of \( T \) and \( \sigma(\nu) \) is the higher waves contribution to the total cross section: \( \sigma(\nu) = 4\pi(\nu(\nu + 1))^{-1/2}A(\nu,0) \).

The weight functions appearing in (2.11), (2.12) and (2.14) favor the low energy parts of the integrals. As the higher partial waves are small at low energies we may expect that \( \bar{a}_{(2)} \) and \( \bar{b} \) are small with respect to \( \hat{a}_{(2)} \) and \( \hat{b} \).

Indeed, one finds that the contribution of the D-wave resonance \( f_2(1270) \) to \( \bar{a}_{(2)} \) and \( \bar{b} \) is of the order of 10% of the accepted values of \( a_{(2)} \) and \( b \).

This indicates that \( \bar{a}_{(2)} \) and \( \bar{b} \) are in fact small compared to \( \hat{a}_{(2)} \) and \( \hat{b} \) but not negligibly small. Therefore an evaluation of \( \hat{b} \) and \( \hat{a}_{(2)} \) gives only a relatively crude estimate of the full parameters \( b \) and \( a_{(2)} \). As it is our ambition to derive more precise predictions, we have to find sum rules giving low energy parameters which are well approximated by truncated integrals. We must admit that our \( \pi^0-\pi^0 \) sum rules do not really meet our requirements.
In order to achieve our aims, we have to work with amplitudes having the same analyticity properties as the scattering amplitudes $T^I$ but a better asymptotic behaviour. The $t - u$ antisymmetry of $T^1(s, t, u)$ implies that

$$H(s, t, u) = \frac{T^1(s, t, u)}{t - u} \quad (2.15)$$

is such an amplitude. Furthermore, we find that the following three functions are suitable for our purposes:

$$F_2(s, t) = H(t, u, s) \mp (H(s, t, u) + H(u, s, t))$$

$$(2.16)$$

$$F_3(s, t) = H(t, u, s) + H(u, s, t)$$

Proceeding as in the $\pi^0-\pi^0$ case, one finds, for the truncated versions of $\hat{F}_\alpha$ of the $F_\alpha$’s:

$$-96 \lim_{s \to 4+} \frac{\partial}{\partial s} \hat{F}_1(s, 0) =$$

$$\left(2\hat{a}_0^0 - 5\hat{a}_0^2\right) - 18\hat{a}_1^1 + 18\hat{b}_1^1 = \quad (2.17)$$

$$\frac{1}{4\pi^2} \int_0^\infty d\nu \frac{1}{\nu^{1/2} (\nu + 1)^{5/2}} \left[\frac{3\nu + 2}{\nu} (2\sigma_0(\nu) - 5\sigma_2(\nu)) + \frac{9\nu^2 + 6\nu + 2}{\nu(\nu + 1)} \sigma_1(\nu)\right]$$

$$-96 \lim_{s \to 4+} \frac{\partial}{\partial s} \hat{F}_2(s, 0) =$$

$$3\left(2\hat{a}_0^0 - 5\hat{a}_0^2\right) - 2\left(2\hat{b}_0^0 - 5\hat{b}_0^2\right) - 18\hat{b}_1^1 = \quad (2.18)$$

$$\frac{1}{4\pi^2} \int_0^\infty d\nu \frac{1}{\nu^{1/2} (\nu + 1)^{3/2}} \left[-\frac{3\nu + 2}{\nu + 1} (2\sigma_0(\nu) - 5\sigma_2(\nu)) + 2(\nu + 1) (2\sigma_0(0) - 5\sigma_2(0)) - \frac{3\nu^2 + 6\nu + 2}{\nu(\nu + 1)} \sigma_1(\nu)\right]$$
\[ \lim_{s \to 4^+} \left( \frac{1}{\nu} \frac{\partial}{\partial t} \hat{F}_3(s, 0) \right) \]

\[ = \left( 2\hat{a}_0^3 - 5\hat{a}_0^2 \right) - 18\hat{a}_1^2 + 30 \left( 2\hat{a}_2^0 - 5\hat{a}_2^2 \right) \]

\[ = \left( -18\hat{a}_1^2 + 18\hat{b}_1^2 \right) \]

\[ = \frac{1}{4\pi^2} \int_0^\infty d\nu \frac{\nu}{(\nu(\nu + 1))^{5/2}} \left[ \nu (2\sigma_2(\nu) - 5\sigma_2(\nu)) + 3\sigma_1(\nu)(\nu - 2) \right] \]

As in (2.7), \( \sigma_1(\nu) = 4\pi(2l + 1)(\nu(\nu + 1))^{1/2} \text{Im} f_I^l(\nu), l = 0 \) for \( I = 0, 2 \) and \( l = 1 \) for \( I = 1 \).

One finds that

\[ \lim_{s \to 4^+} \frac{1}{\nu} \frac{\partial}{\partial t} H(s, 0) \]

gives a sum rule of the same type for \( a_3^1 \), the \( I = 1 \), F-wave scattering length, which we shall not use.

Again there are exact sum rules for the higher waves contributions to the combinations of threshold parameters appearing in (2.17)-(2.19). Such sum rules can be obtained by a straightforward application of the technique used for the \( \pi^0-\pi^0 \) amplitude in Ref. \[13\] to \( F_1 \) and another totally symmetric amplitude constructed in Ref. \[14\]. We display the results without going through the proofs:

\[ -18\overline{a}_1^1 + 18\overline{b}_1^1 = \]

\[ = \frac{1}{4\pi^2} \int_0^\infty d\nu \left[ \frac{1}{\nu^{1/2}(\nu + 1)^{5/2}} \left( 2\sigma_2(\nu) - 5\sigma_2(\nu) \right) \right. \]

\[ + \left. 3\left( \frac{\nu^2 + 4\nu + 2}{(\nu(\nu + 1))^{5/2}} \overline{\sigma}_1(\nu) \right) \right] \]

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\[-2 \left( \bar{b}_0^0 - \bar{b}_0^2 \right) - 18 \bar{b}_1^1 = \]
\[
\frac{1}{4\pi^2} \int_0^\infty d\nu \frac{1}{(\nu(\nu + 1))^{3/2}} \left[ -\frac{3\nu + 2}{\nu + 1} \left( 2\sigma^0(\nu) - 5\sigma^2(\nu) \right) \right.
\]
\[
\left. -3\frac{\nu^2 + 6\nu + 2}{\nu(\nu + 1)} \sigma_1(\nu) \right] \] (2.21)

\[-18\sigma_1^1 + 30 \left( 2\pi_2^0 - 5\pi_2^2 \right) = \]
\[
\frac{1}{4\pi^2} \int_0^\infty d\nu \left[ \frac{1}{\nu^{1/2}(\nu + 1)^{5/2}} \left( 2\sigma_0^0 - 5\sigma_0^2 + 3\sigma_1^1 \right) - \frac{6(2\nu + 1)}{\nu(\nu + 1))^{5/2}} \sigma_1^1 \right] \] (2.22)

In these integrals, \( \sigma'^I \) and \( \bar{A}'^I \) are the higher waves contributions to the isospin \( I \) total cross-section and absorptive part. Furthermore, the fact that \( \pi_0^0 = \pi_0^2 = 0 \) has been taken into account.

We now have weight functions decreasing more rapidly than the corresponding \( \pi^0-\pi^0 \) weight functions we had before. This results in the contributions of \( f_2(1270) \) to the integrals reduced to the order of 1% of the expected values of the combinations of complete threshold parameters. As a consequence, we can now transform eqs. (2.17)-(2.19) into reliable approximate sum rules by replacing the truncated parameters in the left-hand sides by their complete counterparts. In this manner, we arrive at a very helpful tool for the analysis of low energy pion-pion scattering, which will be established in the next Section.
It is worth noting that the existence of the truncated amplitudes defined in eq. (2.1) is due to the convergence of twice subtracted dispersion relations. Their uniqueness implies that crossing symmetry alone does not constrain the S- and P- wave absorptive parts. Furthermore, the uniqueness of $\hat{T}^I$ implies the uniqueness of the right hand sides of (2.17)-(2.19). This does not apply to the right hand sides of (2.20)-(2.22). As a matter of fact, there are various inequivalent methods leading to different expressions for the right hand sides. The fact that the values of these expressions have to be equal, leads to constraints on the higher waves absorptive parts, in contrast to those of the S- and P- waves.

3 Transforming the sum rules into equations for threshold parameters

The presently available data do not allow a reliable evaluation of the integrals appearing in our sum rules. In particular they are quite sensitive to the values of the threshold parameters. If these quantities were to be known precisely, one could use the sum rules to test their consistency with field theoretic predictions. In the present situation some threshold parameters are only poorly known and the best one can possibly do is to turn the sum rules into non-linear equations for these parameters and determine if and how
their possible values are constrained. To achieve this aim in a simple way we require an analytic parametrization of the S- and P-wave phase shifts, containing the scattering lengths and effective ranges as free parameters, and reproducing their main known features above threshold and below the $K-\bar{K}$ threshold. A parametrization has been provided by Schenk [9] along these lines, which we use with the $I = 1$ P-wave modified slightly in such a way that it depends only on $a_1^1$ and $b_1^1$. The explicit form of the parametrization we shall use is:

\[
\tan \delta_{I}^I(\nu) = \frac{\nu^{1/2}}{(\nu + 1)^{1/2}} \left\{ a_I + [b_I - a_I/\nu_0^I + (a_I)^3]\nu \right\} \frac{\nu_0^{I}}{(\nu_0^{I} - \nu)}, \quad I = 0, 2, \tag{3.1}
\]

\[
\tan \delta_{1}^1(\nu) = \frac{\nu^{3/2}}{(\nu + 1)^{1/2}} \left\{ a_1 + [b_1 - a_1/\nu_\rho]\nu \right\} \frac{\nu_\rho}{(\nu_\rho - \nu)} \tag{3.2}
\]

The S- and P-wave parameters have been relabelled: $a_I = a_I^l$, $b_I = b_I^l$, $l = 0$ if $I = 0, 2$ and $l = 1$ for $I = 1$. We take $\nu_0^0 = 8.5$, $\nu_0^2 = -5.0$ as in Ref. [9] and $\nu_\rho = 6.6$, which is the position of the $\rho(770)$ resonance. Note that these representations for the phase shifts ensure normal threshold behaviour.

Another representation of the S- and P-wave phase shifts may be obtained from numerical solutions to the Roy equations that are consistent with experimental data [7]. Nevertheless, the difference between this and the representation we use has been found to yield a difference at the level of a few percent in the present analysis when the parameters in (3.1) and (3.2) are correctly adjusted [16].

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Once the cross-sections $\sigma_I$ determined by (3.1) and (3.2) are inserted into the integrals of (2.17)-(2.19), these integrals become non-linear functions of the 6 parameters $a_I$ and $b_I$. When we evaluate these integrals numerically, we cut them off at $\nu = 11$ corresponding to a total energy of 970 MeV, contributions from higher energies being negligible.

We shall explore a restricted domain of the space spanned by these parameters:

\[
\begin{align*}
    a_0 &\in (0,1), \ a_2 \in (-0.1,0), \ a_1 \in (0,0.1) \\
    b_0 &\in (0,1), \ b_2 \in (-0.2,0), \ b_1 \in (0,0.02)
\end{align*}
\] (3.3)

The experimental data for the 5 first parameters give [5]:

\[
\begin{align*}
    a_0 &\in (0.21,0.31), \ a_2 \in (-0.040,-0.016), \ a_1 \in (0.036,0.040) \\
    b_0 &\in (0.22,0.28), \ b_2 \in (-0.090,-0.074),
\end{align*}
\] (3.4)

while no experimental information on $b_1$ is available. Note that these values are well inside the domain defined by (3.3).

Since the sum rules integrals are smooth functions of the parameters, we approximate them in the domain defined by eq. (3.3) by least square quadratic fits. The fits $I_1$, $I_2$ and $I_3$ of the integrals in (2.17), (2.18) and (2.19) have the form ($\alpha = 1,2,3$):

\[
I_{\alpha} = C_{\alpha} + \sum_{I=0}^{2}[R_{\alpha I}a_I + S_{\alpha I}b_I + T_{\alpha I}(a_I)^2 + U_{\alpha I}(b_I)^2 + V_{\alpha I}a_Ib_I] \tag{3.5}
\]
The values of the coefficients are given in Table 1. The relative standard
deviation of the fit (3.5) is less than 4\% for all the integrals and the correlation
coefficients squared are all larger than 0.99965.

With (3.5) the sum rules produce 3 equations of second degree in the S-
and P- wave parameters $a_I$ and $b_I$ and the D- wave parameter

$$A_2 \equiv 2a_2^0 - 5a_2^2$$

(3.6)

One may ask how many solutions of these equations are located in the domain
(3.3). In order to find the answer, we observe that the following sum rule

$$b_1 - \frac{5}{3} A_2 = \frac{1}{12\pi^2} \int_0^\infty \frac{3\nu + 1}{\nu(\nu + 1)^{5/2}} \sigma_1(\nu)$$

(3.7)

is a consequence of (2.17)-(2.19). Remarkably, its integral depends only on
the $I = 1$ P- wave cross section. The quadratic version of the integral is
$(I_1 - I_3)/18$. One sees that it depends only on the $I = 1$ P- wave parameters;
no fictitious S- wave dependence is introduced through our fit of the $I_\alpha$’s.

Solving the quadratic approximation of (3.7) with respect to $b_1$, we find

$$b_1 = -1.80 + 1.26a_1 \pm [3.24 - 4.51a_1 + 3.72(a_1)^2 + 6.31A_2]^{1/2}$$

(3.8)

Fits to the experimental data give $^3$

$$A_2 = 2a_2^0 - 5a_2^2 = (0.275 \pm 0.210) \cdot 10^{-2}$$

(3.9)

If we constrain $A_2$ to this range, we find that only the $+$ solution in (3.7)
belongs to the domain (3.3). Introducing this solution into (2.18) and (2.19)
one gets two equations for $a_0$, $b_0$, $a_2$, $b_2$ and $a_1$ at fixed $A_2$. It turns out
that they have only one solution compatible with (3.3) if $A_2$ is in the interval
(3.9). That is to say that if one chooses any triplet among the first 5 of these
parameters which fulfills (3.3) and if $A_2$ is fixed according to (3.9), then $b_1$ is
determined by (3.8) and only one value of the remaining pair of parameters
obeys (3.3) and is allowed by the sum rules (2.18) and (2.19).

This results leads naturally to the question of the existence of sets of
parameters which are compatible with the sum rules as well as with the
experimental data. To answer this question, we restrict the S- and P- wave
parameters to the domain (3.4) and the physically unknown $b_1$ to the interval
(0.002, 0.010) by using (3.8). The sum rule integrals behave nearly linearly
in this domain and we simplify our discussion by replacing the quadratic fits
(3.5) by linear ones. The new correlation coefficients are larger than 0.996.

The linearized version of (3.7) is

$$ b_1 = 1.795 A_2 + 0.053 a_1 - 0.001 $$  

(3.10)

Using (3.4) and (3.8) this gives

$$ b_1 = 0.006 \pm 0.004. $$  

(3.11)

The + solution in (3.7) gives the same value, the large error coming mainly
from the large uncertainty on $A_2$ in (3.9). As far as we know, we have here
the first determination of the $I = 1$ P- wave effective range based on sum
rules.
In addition to (3.7) we have the 2 independent sum rules (2.18) and (2.19). Eliminating \( b_1 \) by means of (3.10) in their linearized versions one is left with 2 linear equations relating 6 parameters. We find it convenient to express these equations in terms of \( a_0, a_1, b_0, A_0, B_0 \) and \( A_2 \) where \( A_0 \) and \( B_0 \) are corrected versions of the differences \((0.4a_0 - a_2)\) and \((0.4b_0 - b_2)\) appearing in the left hand sides of the sum rules:

\[
A_0 \equiv 0.27a_0 - a_2, \quad B_0 \equiv 0.27b_0 - b_2 \quad (3.12)
\]

The linearized sum rules (2.18) and (2.19) can now be written as:

\[
\begin{align*}
A_0 - 0.529B_0 &= -0.001a_0 + 0.004b_0 + 0.797a_1 - 0.009 \\
A_0 - 0.689B_0 - 1.692A_2 &= 0.010a_0 - 0.002b_0 - 0.009
\end{align*} \quad (3.13)
\]

The equations above and eq. (3.10) express the constraints imposed by our sum rules in a domain of threshold parameters consistent with the data. These constraints are analyzed in the next section.

### 4 Discussion of the constraints on threshold parameters

It is convenient to discuss the implications of eq. (3.13) in the \((A_0, B_0, A_2)\) space. The experimental data (3.4) define a domain \( \Delta \) for these parameters:
\[ \Delta \equiv \{ A_0 \in (0.072, 0.124), \ B_0 \in (0.134, 0.166), \]
\[ A_2 \in (0.065 \cdot 10^{-2}, 0.485 \cdot 10^{-2}) \} \]  

(4.1)

For given values of \( a_0, b_0 \) and \( a_1 \), the equations (3.13) constrain the point with co-ordinates \((A_0, B_0, A_2)\) to a straight line \(d(a_0, b_0, a_1)\). One finds that this line intersects \( \Delta \) for all values of \( a_0, b_0 \) and \( a_1 \) allowed by (3.4) along a segment \( \overline{d}(a_0, b_0, a_1) \) as shown in Fig. 1. The values of \((A_0, B_0, A_2)\) which are compatible with experimental data and the sum rules define a domain \( \overline{\Delta} \) which is the union of the segments \( \overline{d}(a_0, b_0, a_1) \) corresponding to all values of \((a_0, b_0, a_1)\) in the domain (3.4). As a consequence of our linearization, \( \overline{\Delta} \) is a convex domain; it is shown in Fig. 2; apart from eq. (3.10) it displays all the information we have derived from our approximate sum rules.

The constraints we obtain are quite spectacular. \( \overline{\Delta} \) is a very narrow prism bounded by four planes and truncated by the faces of \( \Delta \). The faces \( A_0 = 0.072 \) and \( A_0 = 0.124 \) are completely excluded and the same is nearly true for the faces \( A_2 = 0 \) and \( A_2 = 0.005 \). Whereas \( B_0 \) is unconstrained there are strong correlations on the possible values of \( A_0 \) and \( A_2 \) at given \( B_0 \).

Fig. 3 shows the projections of \( \overline{\Delta} \) onto the \((A_0, A_2)\)-, \((A_2, B_0)\)- and \((A_0, B_0)\)-planes. A very narrow strip is selected in the \((A_0, B_0)\)-plane. The central experimental values are slightly outside this strip.

If we restrict ourselves to the S-wave scattering lengths, we see that \( A_0 \)
is confined to the interval $(0.090, 0.112)$. This defines a band bounded by $|0.27a_0 - a_2 - 0.101| < 0.011$ in the $(a_0, a_2)$ plane. This band is shown in Fig. 4. A similar band shows up in many other analyses of pion-pion scattering; the one used in Ref. [5] is also shown in Fig. 4 together with the rectangle compatible with the data. Clearly, most of the correlations encoded in the shape of $\Delta$ are washed out by the projection onto this $(a_0, a_2)$ plane. Despite this, the sum rules still impose efficient constraints.

Finally we compare our results with the predictions of two versions of chiral perturbation theory (CHPT), the so called standard one (SCHPT) [2, 17, 18] and the generalized one (GCHPT) [19, 20, 21]. Table 2 gives three sets of central values of threshold parameters obtained from various one loop pion-pion scattering amplitudes [18, 20]. They define points $P$ in the $(A_0, B_0, A_2)$-space whose locations are indicated in Figs. 5a-c together with the relevant portions of our allowed domain $\Delta$. The SCHPT point $P_a$ is just at the border of this domain although its $a_0$ value is slightly below the interval allowed in (3.4). The GCHPT points $P_b$ and $P_c$ are outside $\Delta$ whereas their $(a_0, b_0, a_1)$ obey (3.4). In other words, one loop SCHPT threshold parameters can be considered consistent with our sum rules combined with Schenck’s parametrization while this is not quite true for GCHPT.

When discussing the positions of the points $P$ with respect to sections of $\Delta$ we do not take into account the CHPT values of $a_0$, $b_0$ and $a_1$. We do this in a second exercise: inserting the values of $a_0$, $b_0$, $a_1$ and $B_0$ from Table 2
into the constraints (3.13) we obtain values of \( A_0 \) and \( A_2 \) also given in Table 2. They define new points \( Q \) in Figs. 5a-c. That is to say, these points are produced by our sum rules implemented with Schenk’s parametrization (3.1)-(3.2) taken at CHPT values of \( a_0, b_0 \) and \( a_1 \). They are inside their sections of \( \Delta \), indeed as they must be, but do not coincide with the points \( P \) defined earlier. It may be inferred from the peculiar fact that one loop chiral amplitudes fulfill our sum rules identically \([11]\) that the discrepancy between the \( Q \)’s and \( P \)’s is essentially due to the difference between Schenk’s and the one loop chiral absorptive parts. The former being certainly closer to the true ones above threshold and at intermediate energies, we conclude that our results establish the necessity of non-negligible higher order corrections to one loop calculations.

Two loop computations in the framework of GCHPT are presented in Ref. \([21]\) and a sample of two loop threshold parameters is displayed in Table 2. This sample defines two points \( Q \) and \( P \) which are practically identical (Fig. 5d). This spectacular improvement must come from the two loop corrections to the absorptive parts and a larger flexibility in the choice of effective coupling constants. The circumstance that this choice is partly based on sum rules may also be playing a role. These are sum rules based on twice subtracted dispersion relations involving high energy contributions in contrast with the low energy sum rules analyzed here. For a check of sum rule (3.7), the values of \( b_1 \) as obtained via eq. (3.10) from the CHPT data
for $A_2$ and $a_1$ are given in Table 2. Whereas the sum rule predictions differ from CHPT values at one loop, the agreement is again excellent at two loop GCHPT.

Our discussion of CHPT pion-pion scattering illustrates the relevance of low energy sum rules. They reveal definitely the need of two loop corrections. However, as no two loop results obtained in the strict SCHPT framework are available at present, we cannot tell whether our tools allow a discrimination between that scheme and GCHPT.

Although our analysis is based on exact sum rules we have had to make two major approximations which are not under precise quantitative control. First, the contributions from the higher partial waves due to $T_\sim$ in the decomposition (2.3) have been neglected. Second, we have played our game using the very simple analytic parametrization (3.1)-(3.2) for the S- and P-waves. An improved parametrization will modify the shape of $\Delta$ whereas an evaluation of the size of the $T_\sim$ contributions would allow an estimation of the uncertainties coming from these contributions. This would enlarge $\Delta$. Since our domain $\Delta$ is well inside $\Delta$, we believe that our results are robust and will survive these improvements.

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Table Captions

**Table 1** Coefficients of the quadratic fits (3.5) of the sum rules integrals in eqs. (2.17)-(2.19).

**Table 2** Threshold parameters of chiral perturbation theory (CHPT). For each set of parameters the first column gives central CHPT values and the second column gives sum rules predictions based on the values found in the first column. The values of $A_0$ and $A_2$ in a first column define $P$ in Fig. 5 whereas those in the corresponding second column define Q. (a) Threshold parameters of a standard one loop amplitude carrying the coupling constants $L_1$, $L_2$ and $L_3$ given in the 4th column of Table 2 in Ref. [18], (b)-(c) threshold parameters of two generalized one loop amplitudes displayed in Table 1 of [20] corresponding to two values of the coupling constant $L_3$, (d) threshold parameters of an extended two loop amplitude: 5th line in Table 1 of [21].
Figure Captions

**Fig. 1.** The two planes in the \((A_0, B_0, A_2)\)-space defined by eq. (3.13) for the central values of the right hand sides according to (3.4). The points belonging to the intersection of these planes which are inside the domain \(\Delta\) [defined in eq. (4.1)] are physically admissible, compatible with the sum rules and the experimental central values of \((a_0^0, b_0^0, a_1^1)\).

**Fig. 2.** The domain \(\overline{\Delta}\) in the \((A_0, B_0, A_2)\)-space which is compatible with experimentally allowed values of \((a_0^0, b_0^0, a_1^1)\) as defined in (3.4). The vertices of \(\overline{\Delta}\) are marked by dots: they are at the intersection of a prism with the faces of \(\Delta\).

**Fig. 3.** The projections of \(\overline{\Delta}\) and the central experimental values onto the \((A_0, A_2)\)-, \((A_2, B_0)\)- and \((A_0, B_0)\)-planes. The dots represent the central experimental values obtained from (3.4).

**Fig. 4.** Constraints on the S-waves scattering lengths. The experimental data define a rectangle of allowed values, the “universal” strip [5] is bounded by the dashed lines \(a_0^2 = 0.4a_0^0 - 0.131 \pm 0.010\) and our band is limited by the full lines \(a_0^2 = 0.27a_0^0 - 0.101 \pm 0.011\). The constraints confine \((a_0^0, a_0^2)\) to the shaded area.

**Fig. 5.** Sections of \(\Delta\) and \(\overline{\Delta}\) at five fixed values of \(B_0\) obtained from CHPT, (a) one loop SCHPT, \(B_0 = 0.140\), (b)-(c) one loop GCHPT, \(B_0 = 0.149, 0.151\), (d) two loop GCHPT, \(B_0 = 0.146\). The significance of the points \(Q\) and \(P\) is explained in the text.
| α | $C_\alpha$ | $R_{\alpha 0}$ | $R_{\alpha 1}$ | $R_{\alpha 2}$ | $S_{\alpha 0}$ | $S_{\alpha 1}$ | $S_{\alpha 2}$ | $T_{\alpha 0}$ | $T_{\alpha 1}$ | $T_{\alpha 2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00 | 0.07 | −0.06 | 0.18 | 0.08 | 1.19 | 0.03 | 0.68 | 10.28 | −0.01 |
| 2 | 0.02 | 0.63 | 0.33 | 1.93 | −0.41 | −0.89 | −0.59 | 4.03 | −16.15 | −0.16 |
| 3 | 0.00 | 0.07 | −0.10 | 0.18 | 0.08 | 0.31 | 0.03 | 0.68 | 0.10 | −0.01 |

| α | $U_{\alpha 0}$ | $U_{\alpha 1}$ | $U_{\alpha 2}$ | $V_{\alpha 0}$ | $V_{\alpha 1}$ | $V_{\alpha 2}$ |
|---|---|---|---|---|---|---|
| 1 | 0.02 | −12.42 | −0.11 | 0.09 | 10.11 | −0.69 |
| 2 | −0.16 | −240.3 | 0.75 | −2.49 | 1.24 | 0.58 |
| 3 | 0.02 | −7.67 | −0.11 | 0.09 | −1.82 | −0.69 |

Table 1
|       | SCHPT one loop | GCHPT one loop | GCHPT two loops |
|-------|----------------|----------------|-----------------|
|       | (a)            | (b)            | (c)             | (d)             |
|       | Sum rules      | Sum rules      | Sum rules       | Sum rules       |
| $a_0^0$ | 0.20           | 0.27           | 0.28            | 0.26            |
| $b_0^0$ | 0.25           | 0.26           | 0.28            | 0.25            |
| $a_1^0$ | 0.037          | 0.039          | 0.039           | 0.0026          |
| $A_0$   | 0.095 0.095    | 0.096 0.102    | 0.097 0.099     | 0.0026 0.0038   |
| $B_0$   | 0.140           | 0.149          | 0.151           | 0.0044 0.0056   |
| $A_2$   | 0.0026 0.0038 | 0.0024 0.0036 | 0.0014 0.0028   | 0.0048 0.0054   |
| $b_1^1$ | 0.0044 0.0056 | 0.0048 0.0054 | 0.0028 0.0034   | 0.0044 0.0056   |

Table 2
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5