Fast Marginal Likelihood Estimation of the Ridge Parameter in Ridge Regression

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Abstract

Ridge regression provides coefficient estimates via shrinkage, even when the observed design matrix contains correlated covariates, or when it is singular, as when the number of covariates exceeds the number of observations. This shrinking usually improve predictions in the linear model, compared to ordinary least-squares. However, the estimation and prediction accuracy of the ridge model depend on the choice of the ridge parameter. The current approaches to estimating the ridge parameter are based on minimizing cross-validation or information loss criteria, which are either computationally expensive, asymptotically inconsistent, and only approximate (~2 times) the marginal likelihood of the model parameters. The marginal likelihood depends on a matrix determinant, which is computationally demanding when the number of covariates is large. This paper shows that after taking a singular value decomposition of the design matrix, the marginal likelihood can be simplified into an equation involving no matrix operations such as determinants or inverses. This simplification allows for a fast estimation of the ridge parameter based on a simple optimization algorithm, which typically completes in less than one-tenth of a second, even for data sets where the number of covariates and/or sample size is very large. Also, the marginal likelihood estimate of the ridge parameter is the “Bayes empirical Bayes” posterior mode, and is preferred according to the Bayes factor, over pair-wise comparisons of all possible ridge parameter estimates. We illustrate the speed and viability of the ridge parameter estimation method through the analysis of several real data sets, involving hundreds to several thousand covariates and observations, and involving more covariates than observations.

KEYWORDS: Ridge Regression, Bayes Empirical Bayes, Marginal Likelihood Estimation, Bayes factor, High-dimensional regression.

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1 Introduction

The linear regression model is ubiquitous in statistics. For $p$ regression coefficient parameters $\beta = (\beta_1, \ldots, \beta_p)^\top$, and error variance parameter $\sigma^2$, the classical ordinary least-squares (OLS) estimator of the linear model is given by $\hat{\beta} = (X^\top X)^{-1}Xy$ and $\hat{\sigma}^2 = \frac{1}{n-p}||y - X\hat{\beta}||^2$, for observed dependent variable observations, $y = (y_1, \ldots, y_n)^\top$, and for an observed $n \times p$ design matrix of $p$ covariates, $X = (x_{ik})_{n \times p}$. Throughout, with no loss of generality, it is assumed that all variables have been zero-mean centered, and that all covariates are also rescaled to have variance 1, so that $\sum_{i=1}^n y_i = 0$, $\sum_{i=1}^n x_{ki} = 0$, and $\sum_{i=1}^n x_{ki}^2 = 0$, for $k = 1, \ldots, p$. Then the intercept parameter $\beta_0$ is ignorable, since $\hat{\beta}_0 = \frac{1}{p} \sum_{i=1}^n y_i = 0$.

The OLS estimate $\hat{\beta}$ can be poorly determined and exhibit high variance when the given design matrix $X$ contains highly-correlated covariates. The OLS estimate does not even exist when $X^\top X$ is singular, such as when covariates have extremely high correlations, or when the number of covariates exceeds the number of observations (i.e., when $p > n$).

Ridge regression (Hoerl & Kennard, 1970) addresses both of these concerns, by providing the alternative estimator $\overline{\beta}_\lambda = (X^\top X + \lambda I_p)^{-1}Xy$. Here, $\lambda > 0$ is the ridge parameter that shrinks the estimate of the coefficients $\beta$ towards zero, with the amount of shrinking an increasing function of $\lambda$. Compared to OLS, this shrinking feature enable ridge regression to provide coefficient estimates with lower variance. In turn, this usually lowers the prediction error of the linear regression model. Unlike OLS, the ridge estimate $\overline{\beta}_\lambda$ exists even when $X^\top X$ is singular, because $(X^\top X + \lambda I_p)$ is necessarily non-singular when $\lambda > 0$. Obviously when $\lambda = 0$, the estimator $\overline{\beta}_\lambda$ reduces to the OLS estimator $\hat{\beta}$.

From a Bayesian perspective (Polson & Scott 2012), there are good reasons why pure shrinkage methods like ridge regression, which cannot shrink the coefficient estimates to exactly zero, may be preferred over other popular methods. The other methods include sparse shrinkage methods that can shrink estimates to exactly zero (e.g., the LASSO; Tibshirani, 1996), and Bayesian variable selection methods which use latent indicator $(0,1)$ parameters that set coefficients to $\beta_k = 0$ with positive prior probability (e.g., Bottolo & Richardson, 2012). First, under sparse shrinking, the posterior probability is zero that a coefficient $\beta_k = 0$; under Bayesian variable selection, the posterior average of $\beta$ is surely non-zero, after averaging over the latent indicator parameters. From this perspective, point-null hypotheses of the form $\beta_k = 0$ seem unrealistic, and the posterior average of $\beta$ obtained under either of these methods may be practically indistinguishable from the posterior average of $\beta$ obtained under a pure shrinkage method. Second, a pure shrinkage method can provide computational gains over sparse shrinking and Bayesian variable selection, in terms of speed and simplicity.

In ridge regression, the quality of coefficient estimates $\overline{\beta}_\lambda$ and predictions depend crucially on the choice of the ridge parameter, $\lambda$. Hence, several methods have been proposed to estimate the ridge parameter. They include automatic plug-in estimators, and methods based on cross-validation, information criteria, and Markov chain Monte Carlo (MCMC).

Many automatic plug-in estimators were proposed (see Kibria, 2003). Virtually all of these are variations of the classical, Hoerl-Kennard-Baldwin (1975) estimator, given by $\hat{\lambda}_{HKB} = p\hat{\sigma}^2/\hat{\beta}^2\hat{\beta}$, and based on OLS estimates. For the orthonormal setting, where $X^\top X = I_p$, this estimator provides the optimal choice of $\lambda$ that minimizes the expected prediction error in the ridge regression model (Hoerl & Kennard, 1970). The estimator has
been extended to \( n > p \) settings (Cule & Delorio, 2013). In general, a plug-in estimator is attractive because it quickly obtains a ridge estimate \( \hat{\lambda} \) without iteration. However, a plug-in estimate may not be optimal for the given data set at hand.

Methods that estimate \( \lambda \) directly from the data are often considered in practice. One method is based on cross validation. This method involves first specifying a grid of trial values of \( \lambda \), and then estimating the ridge parameter by selecting the trial value that provides the smallest prediction error for ridge regression in \( K \)-fold cross validation, such as 10-fold or \( n \)-fold (Hastie et al. 2009, Ch. 3.7). However, \( K \)-fold cross validation is computationally expensive for large data sets (i.e., when \( n \) or \( p \) is large), because it requires \( K \) separate estimations of the ridge regression model, for every trial value of \( \lambda \). Also, the optimal estimate of \( \lambda \) may lie outside the grid of \( \lambda \) values that happens to be considered by the statistician.

An alternative method is to estimate the posterior distribution of the parameters \((\beta, \sigma^2, \lambda)\) of the ridge regression model via MCMC, after assigning a conjugate normal inverse-gamma prior to \( \beta, \sigma^2 | \lambda \) and a gamma prior to \( \lambda \) (Dension et al. 2002). However, this estimation procedure is also computationally expensive for large data sets. Also, it is impossible to fully confirm MCMC convergence (Cowles & Carlin, 1996), and checking the MCMC convergence of coefficient posterior estimates is time consuming when \( p \) is large.

Generalized cross-validation (GCV) (Golub, et al. 1979), which approximates \( n \)-fold cross-validation, provides a computationally-fast and hence popular method to estimate the ridge parameter \( \lambda \). The GCV criterion is defined by:

\[
\text{GCV}(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{y_i - \mathbf{x}_i^T \hat{\beta}_\lambda}{1 - \text{df}_\lambda/n} \right)^2,
\]

where \( \text{df}_\lambda \) is the effective degrees of freedom in the ridge model, given \( \lambda \). For this model, the degrees of freedom can be quickly computed by \( \text{df}_\lambda = \sum_{k=1}^{\min(n,p)} \frac{d_k^2}{d_k^2 + \lambda} \), where \( d_1^2 > \cdots > d_{\min(n,p)}^2 \) are the eigenvalues obtained from a singular value decomposition (s.v.d.) of \( \mathbf{X} \). Then the GCV(\( \lambda \)) criterion can be efficiently computed for a grid of trial values of \( \lambda \), so that the ridge parameter can be quickly estimated by selecting the trial value that attains the lowest value of this criterion. As a selector of the ridge parameter, the GCV(\( \lambda \)) criterion behaves similarly to the Akaike’s Information Criterion (AIC; Akaike, 1973) with penalty term \( 2\text{df}_\lambda \) (Yang, 2005). However, like AIC, \( K \)-fold cross-validation including GCV(\( \lambda \)) is inconsistent, i.e., it does not surely lead to the selection of the true-value of \( \lambda \) as \( n \to \infty \) (Shao, 1997). Also, the estimation of \( \lambda \) based on GCV(\( \lambda \)) (and on AIC) tends to yield an over-fitted ridge regression model (Zhang, et al. 2009).

The ridge parameter \( \lambda \) can be consistently estimated (selected) as a minimizer of the modified Bayesian information criterion, defined by:

\[
\text{BIC}(\lambda) = \log \frac{1}{n} ||\mathbf{y} - \mathbf{X}\hat{\beta}_\lambda||^2 + \frac{a_n}{n} \text{df}_\lambda \log n,
\]

where \( a_n \) is suitably chosen to diverge to infinity as a function of \( n \) (Fan & Tang, 2013). Like GCV(\( \lambda \)), the BIC(\( \lambda \)) can be quickly calculated over a range of values of \( \lambda \), to facilitate the use of a fast minimization algorithm that finds the optimal estimate of the ridge parameter.

However, the BIC(\( \lambda \)) only provides a Laplace approximation to \( -2\pi(\mathcal{D}_n | \lambda) \), where \( \pi(\mathcal{D}_n | \lambda) \) is the marginal likelihood of the data \( \mathcal{D}_n = (\mathbf{X}, \mathbf{y}) \) given \( \lambda \) (Ripley, 1996, p. 64). In
particular, the ratio $\pi(D_n | \lambda)/\pi(D_n | \lambda')$ gives a Bayes factor, describing evidence in terms of the odds of one ridge parameter $\lambda$ versus another $\lambda'$ (Kass & Raftery, 1995).

A more principled approach to estimating the ridge parameter $\lambda$ is by maximizing the marginal likelihood $\pi(D_n | \lambda)$, instead of minimizing some approximation to $-2\pi(D_n | \lambda)$, such as BIC($\lambda$) or GCV($\lambda$). However, according to standard theory for Bayesian normal linear models, the marginal density $\pi(D_n | \lambda)$ depends on the determinant $|X^TX + \lambda I_p|$. Therefore, $\pi(D_n | \lambda)$ seems to be computationally demanding and slow, especially when $p$ is large and/or when $\pi(D_n | \lambda)$ is computed over many trial values of $\lambda$ (Bottolo & Richardson, 2012, p. 586).

However, in this paper we show that after taking the singular value decomposition (s.v.d.) of $X$, the density $\pi(D_n | \lambda)$ can be equivalently expressed by a simpler equation involving only simple vector operations, and involving no matrix determinants or inverses. Then the marginal density $\pi(D_n | \lambda)$, which is a log-concave function of $\lambda$, can be rapidly evaluated over many trial values of $\lambda$. This permits the use of a fast and simple optimization algorithm that can quickly find the estimate $\hat{\lambda}$ that maximizes $\pi(D_n | \lambda)$, even for very large data sets where either $p$ or $n$ is very large.

From a frequentist statistics point of view, $\hat{\lambda}$ is a marginal maximum likelihood estimate. We will show later that if the ridge parameter $\lambda$ is assigned a uniform prior, then $\hat{\lambda}$ is also an estimate of the posterior mode under the ”Bayes empirical Bayes” framework, a fully Bayesian approach to empirical Bayes (Deely & Lindley, 1981). Moreover, the $\hat{\lambda}$ is preferred according to the Bayes factor, over pair-wise comparisons of all possible ridge parameter estimates.

After obtaining the ridge estimate $\hat{\lambda}$, the posterior mean $\overline{\beta}_\hat{\lambda}$ and the posterior covariance of $\beta$ can be directly calculated, given $\hat{\lambda}$. Then based on the posterior covariance estimate, the subset of ”significant” predictors, among the $p$ given covariates, can be identified according to the scaled neighborhood (SN) criterion (Li & Lin, 2010). Under the SN criterion, a covariate is decided to be a significant predictor, when the marginal posterior probability is less than 1/2, that its coefficient $\beta_k$ is within one (marginal) posterior standard deviation of 0. This criterion was validated under the Bayesian elastic net model (Li & Lin, 2010), which encompasses ridge regression, in a MCMC estimation setting.

Conditioning posterior inferences of $\beta$ on an estimate $\hat{\lambda}$ does not account for the extra variability that is inherent in the estimation of the ridge parameter $\lambda$. For practice involving real data, this extra variability is negligible when $n$ is sufficiently large, and may not matter much when it is simply of interest to identify significant predictors (covariates) as having SN criteria below the threshold of 1/2. Again, this paper focuses on estimation methods that provide fast, deterministic, and approximate posterior estimation of model parameters. For large data sets that frequently appear in statistical practice, and/or for data sets having important policy implications, such a deterministic estimation method may be preferred over slower and random-output MCMC posterior estimation methods that aim to fully account for the uncertainty in all ridge model parameters ($\beta, \sigma^2, \lambda$).

In the following sections we describe and illustrate the marginal maximum likelihood estimation methodology, for ridge regression. In Section 2 we review the Bayesian linear model, emphasizing ridge regression, as in classical texts on Bayesian regression (Denison, et al. 2002). In Section 3, we show how the marginal density $\pi(D_n | \lambda)$ can be simplified into
an equation involving no matrix determinants, after taking a singular value decomposition (s.v.d.) of $X$, thereby adopting a view of the linear model in an orthogonalized space. Using that simplified equation, we describe a simple and fast algorithm that finds the marginal likelihood estimate $\hat{\lambda}$ of the ridge parameter. The estimate provides a basis to directly calculate the posterior mean and covariance estimate of $(\beta, \sigma^2)$, as well as auxiliary statistics such as the SN criterion.

In Section 4, we briefly review the literature to support the idea that the ridge regression model actually has a large scope for applied statistics, because of the model’s ability to handle data with a very large number of covariates $p$ (e.g., hundreds or thousands), and possibly with $p$ possibly growing with $n$ and/or $p > n$. Specifically, ridge regression can provide a flexible and approximate Bayesian nonparametric modeling (Müller & Quintana, 2004), even for linear classification of a binary dependent variable. For the marginal maximum likelihood estimation of the ridge parameter, we will also describe the connections with the “Bayes empirical Bayes” statistical framework.

Section 5 illustrates the marginal likelihood estimation methodology through the analysis of several real data sets. The illustrations, among other things, report the algorithm’s computation times in the estimation of the ridge parameter $\lambda$, as well as the added time for the computation of other statistics. The other statistics include the posterior mean and covariance of $(\beta, \sigma^2)$, and the SN criteria. For most of these data sets, the number of covariates range from several hundred to around fifteen thousand, and some of the data sets involve more covariates than observations. Section 6 concludes with some possible extensions of the computational methodology.

## 2 Bayesian Ridge Regression

For a set of data, $D_n = (X, y)$, the Bayesian ridge linear regression model, which assigns a conjugate normal-inverse gamma (NIG) prior density to $(\beta, \sigma^2)$, is defined by:

\[
\begin{align*}
  f(y \mid X, \beta, \sigma^2) &= n_n(y \mid X\beta, \sigma^2 I_n) = \prod_{i=1}^{n} n(y_i \mid x_i^T \beta, \sigma^2) \quad (1a) \\
  \pi(\beta, \sigma^2) &= n_p(\beta \mid 0, \sigma^2 \lambda^{-1} I_p) \text{ig}(\sigma^2 \mid a, b), \quad (1b) \\
  &= \text{nig}(\beta, \sigma^2 \mid 0, \lambda^{-1} I_p, a, b) \quad (1c)
\end{align*}
\]

with $n_n(\cdot \mid \mu, \Sigma)$ and $n(\cdot \mid \mu, \sigma^2)$ the density functions of the $n$-variate normal distribution and the univariate normal distribution (resp.); with ig$(\cdot \mid a, b)$ the density function of the inverse-gamma (IG) distribution with shape $a$ and rate $b$ (and scale $1/b$); and with $I_p$ the $p$-dimensional identity matrix. Also, nig$(\cdot, \cdot \mid \cdot, \cdot, \cdot, \cdot)$ is the density function of the NIG distribution (Lindley & Smith, 1972). Throughout, as typically done in practice, we assume that $\sigma^2$ is assigned a non-informative prior, approximated by ig$(\sigma^2 \mid a = 0^+, b = 0^+)$, with $0^+ = \lim_{t \downarrow 0} t$.

Under the ridge model, the prior predictive distribution of an observable dependent response, $y$, conditionally on chosen covariates $x$, is a Student distribution with density:

\[
\text{st}(y \mid 0, b(1 + x^T \lambda^{-1} x), a) = \int \int n(y \mid x^T \beta, \sigma^2) \text{nig}(\beta, \sigma^2 \mid 0, \lambda^{-1} I_p, a, b) d\beta d\sigma^2. \quad (2)
\]
A set of data $D_n = (X, y)$ updates the NIG prior density (1c) to a posterior density $\pi(\beta, \sigma^2 | D_n)$. Given the conjugacy of the NIG prior (1a) with the normal likelihood (1a), the posterior density is also a NIG density, and is given by:

$$
\pi(\beta, \sigma^2 | D_n, \lambda) = \frac{n_n(y | X_\beta, \sigma^2 I_n)\text{nig}(\beta, \sigma^2 | 0, \lambda^{-1}I_p, a, b)}{\int \int n_n(y | X_\beta, \sigma^2 I_n)\text{nig}(\beta, \sigma^2 | 0, \lambda^{-1}I_p, a, b)d\beta d\sigma^2}
$$

(3a)

$$
= n_p(\beta | \overline{\beta}_\lambda, \sigma^2 \overline{\nabla}_\lambda)\text{ig}(\sigma^2 | \overline{\alpha}, \overline{b}_\lambda)
$$

(3b)

$$
= \text{nig}(\beta, \sigma^2 | \overline{\beta}_\lambda, \overline{\nabla}_\lambda, \overline{\alpha}, \overline{b}_\lambda),
$$

(3c)

with (data) updated parameters

$$
\overline{\nabla}_\lambda = (\lambda I_p + X^T X)^{-1},
$$

(4a)

$$
\overline{\beta}_\lambda = \overline{\nabla}_\lambda X^T y,
$$

(4b)

$$
\overline{\alpha} = a + n/2,
$$

(4c)

$$
\overline{b}_\lambda = b + (y^T y - \overline{\beta}_\lambda^T \overline{\nabla}_\lambda^{-1} \overline{\beta}_\lambda)/2,
$$

(4d)

where the second term of $\overline{b}_\lambda$ is equal to half the sum of squared residuals of the fit, using the posterior mean $\overline{\beta}_\lambda$ of $\beta$. The marginal posterior of $\beta$ is a Student distribution with mean $\overline{\beta}_\lambda$, covariance $(\overline{\beta}_\lambda / (\overline{\alpha} - 1)) \overline{\nabla}_\lambda$, and degrees of freedom $2a + n$. Also, $\sigma^2$ has an inverse-gamma $\text{ig}(\sigma^2 | \overline{\alpha}, \overline{b}_\lambda)$ marginal posterior distribution.

The posterior predictive distribution of an observable dependent response, $y$, given a chosen covariate vector $x$, is a Student distribution with density:

$$
st(y | x^T \overline{\beta}_\lambda, \overline{\alpha}(1 + x^T \overline{\nabla}_\lambda x), \overline{\alpha}) = \int \int n(y | x^T \beta, \sigma^2)\text{nig}(\beta, \sigma^2 | \overline{\beta}_\lambda, \overline{\nabla}_\lambda, \overline{\alpha}, \overline{b}_\lambda)d\beta d\sigma^2,
$$

(5)

along with mean $\mathbb{E}_{n\lambda}(Y | x) = x^T \overline{\beta}_\lambda$ and variance $\mathbb{V}_{n\lambda}(Y | x) = \{\overline{b}_\lambda(1 + x^T \overline{\nabla}_\lambda x)/\overline{\alpha} - 2\}$. The normalizing constant (denominator) of the posterior density (3a) is the marginal likelihood of the ridge regression model, conditionally on $\lambda$, and is given by:

$$
\pi(D_n | \lambda) = \frac{|\overline{\nabla}_\lambda|^{1/2} \overline{\alpha} \Gamma(\overline{\alpha})}{|\lambda^{-1}I_p|^{1/2} \overline{b}_\lambda \Gamma(a) \pi^{n/2}}.
$$

(6)

Then given the marginal maximum likelihood estimate, $\hat{\lambda}$, the marginal posterior mean and covariance of $\beta$ is directly computed by $\overline{\beta}_\lambda$ and $(\overline{\beta}_\lambda / (\overline{\alpha} - 1)) \overline{\nabla}_\lambda$, respectively. The error variance $\sigma^2$ has posterior mean $\overline{\sigma}_\lambda^2 = \overline{b}_\lambda / (\overline{\alpha} - 1)$ and posterior standard deviation $\mathbb{V}_{n\lambda}^{1/2}(\sigma^2) = [\overline{b}_\lambda / \{(\overline{\alpha} - 1)^2(\overline{\alpha} - 2)\}]^{1/2}$. Also, the usual auxiliary statistics can be calculated. They include the standardized residuals $z_i = \{y_i - \mathbb{E}_{n\lambda}(Y_i | x_i)\} / \mathbb{V}_{n\lambda}^{1/2}(\sigma^2)$, $i = 1, \ldots, n$, for the purposes of outlier detection; and R-squared $R^2 = 1 - \{||y - X\overline{\beta}_\lambda||^2 / ||y - \overline{y}||^2\}$, with $\overline{y} = \frac{1}{n} \sum_{i=1}^n y_i$.

Since the posterior probability that $H_0 : \beta_k = 0$ is always zero (for $k = 1, \ldots, p$), a common practice is to view the null hypothesis $H_0 : \beta_k = 0$ as an approximation to the null hypothesis $H_0 : \beta_k \in [-\overline{\sigma}_\lambda^{1/2} \overline{\alpha}^{1/2}, \overline{\sigma}_\lambda^{1/2} \overline{\alpha}^{1/2}]$ (Berger 1993; see Li & Lin, 2010). Then according to the SN criterion, a given covariate $X_k$ is decided to be a "significant" predictor of $Y$, when
the marginal posterior probability that $\beta_k \in [-v_{\lambda_k}^{1/2}, v_{\lambda_k}^{1/2}]$ is less than 1/2 (Li & Lin, 2010). Here, the $v_{\lambda_k}$ (for $k = 1, \ldots, p$) are the diagonal elements of the matrix $(\overline{b}_\lambda / (\overline{\alpha} - 1)) \overline{\nabla}_\lambda$. For a fitted ridge regression model, this posterior probability is directly calculated by

$$\text{St}_{2a+n}(\{v_{\lambda_k}^{1/2} \overline{\beta}_k\}/v_{\lambda_k}^{1/2}) - \text{St}_{2a+n}(\{-v_{\lambda_k}^{1/2} \overline{\beta}_k\}/v_{\lambda_k}^{1/2}),$$

with $\text{St}_{2a+n}(\cdot)$ the c.d.f. of the standard Student distribution with $2a + n$ degrees of freedom.

As shown in (6), the marginal likelihood $\pi(D_n | \lambda)$ depends on the matrix determinant $|\overline{\nabla}_\lambda|^{1/2} = |\lambda p + X'X|^{-1/2}$. This determinant is computationally expensive when $p$ is large, and when $X'X$ is non-diagonal, as is common in real applied statistical practice.

In the next section, we show that the original marginal likelihood $\pi(D_n | \lambda)$ equation can be simplified, after taking a singular value decomposition (s.v.d.) of $X$. This simplified equation form allows for the fast computation of $\pi(D_n | \lambda)$ over a set of trial values of $\lambda$. Moreover $\pi(D_n | \lambda)$ is log-concave as a function of $\lambda$. These properties permit the construction of a simple and fast algorithm for obtaining the marginal maximum likelihood estimate $\lambda$ of the ridge parameter.

### 3 Estimation of $\lambda$ Based on the S.V.D.

The singular-value decomposition (s.v.d.) of a design matrix $X$ is given by $X = UDW^T$, where $U$ and $W$ are orthogonal matrices of dimensions $n \times q$ and $p \times q$, respectively, and $q = \min(n, p)$. Also, $D = \text{diag}(d_1, \ldots, d_q)$ is the diagonal matrix of singular values $d_1 > \cdots > d_q > 0$, corresponding to eigenvalues $(d_1^2, \ldots, d_q^2)^T$, which are the diagonal elements of $Z'Z$, with $Z = UD$.

The linear model, $n_n(y \mid X\beta, \sigma^2 I_n)$, and the linear model $n_n(y \mid XW\alpha, \sigma^2 I_n)$ in canonical form, define identical normal densities. Here, $\beta = W\alpha$, with $\alpha = (\alpha_1, \ldots, \alpha_q)^T$ the regression coefficients for the orthogonalized space, with OLS estimate $\hat{\alpha} = (Z'Z)^{-1}Z'y = D^{-1}U'y$.

Recall that equations (4a)-(4d) provide the updated parameters $(\overline{\nabla}_\lambda, \overline{\beta}_\lambda, \overline{\alpha}, \overline{b}_\lambda)$ for the posterior density $\pi(\beta, \sigma^2 | D_n, \lambda)$ of (3). It is known that $\overline{\beta}_\lambda = \overline{\nabla}_\lambda X'y = W\overline{\alpha}_\lambda$, where $\alpha$ has posterior mean:

$$\overline{\alpha}_\lambda = \left(\frac{d_1^2/\lambda}{1 + d_1^2/\lambda}, \ldots, \frac{d_q^2/\lambda}{1 + d_q^2/\lambda}\right)^T,$$

with posterior covariance $\sigma^2(D^2 + \lambda I_q)$ given $\sigma^2$, and with marginal posterior $\sigma^2 \sim \text{ig}(\overline{\alpha}, \overline{\beta}, \lambda)$ (Polson & Scott, 2012, p. 307). In addition, here, using simple algebra, it can be easily shown that

$$\overline{b}_\lambda = \frac{1}{2}(y'y - \overline{\beta}_\lambda'\overline{\nabla}_\lambda^{-1}\overline{\beta}_\lambda) = \frac{1}{2}\left(y'y - \sum_{k=1}^q \overline{\alpha}_k^2 d_k^4 \frac{d_k^2}{d_k^2 + \lambda}\right), \quad (7)$$

and that the following ratio identity holds:

$$\frac{|\overline{\nabla}_\lambda|^{1/2}}{|\lambda^{-1} I_p|^{1/2}} = \frac{\lambda^{q/2}}{\prod_{k=1}^q (d_k^2 + \lambda)^{1/2}}.$$
In conclusion, this means that the marginal density $\pi(D_n | \lambda)$ of (6) can be simplified to:

$$
\pi(D_n | \lambda) = \frac{\lambda^{q/2} b^a \Gamma(\overline{a})}{\{\prod_{k=1}^{q} (d_k^2 + \lambda)\}^{1/2} \, b^a \lambda \Gamma(a) \pi^{n/2}}.
$$

(8)

The marginal density $\pi(D_n | \lambda)$, as presented in the simplified form above, is not computationally demanding, because it does not involve any matrix determinants or inverses. Specifically, when evaluating the density $\pi(D_n | \lambda)$ over a set of trial values of $\lambda$, the terms $q$, $b^a$, $\Gamma(\overline{a})$, $d_k^2$, $\pi$, and $\Gamma(a) \pi^{n/2}$, remain as fixed constants. The remaining terms $b^\lambda$ (see (7)) and $\{\prod_{k=1}^{q} (d_k^2 + \lambda)\}^{1/2}$, which do need to be evaluated for each distinct trial value of $\lambda$, involve only simple vector (not matrix) operations that can be computed quickly.

Therefore, working with the simplified density equation (8), the log marginal density can be expressed as:

$$
\log \pi(D_n | \lambda) = \frac{q}{2} \log \lambda - \frac{1}{2} \sum_{k=1}^{q} \log (d_k^2 + \lambda) + \log \Gamma(\overline{a}) - \log \Gamma(a) + a \log b - \overline{a} \log b - \frac{a}{2} \log \pi,
$$

and after retaining only the terms that depend on $\lambda$, we find that the marginal maximum likelihood estimate $\hat{\lambda}$ is the choice $\lambda \in [0, \infty)$ which maximizes:

$$
\frac{q}{2} \log \lambda - \frac{1}{2} \left\{ \sum_{k=1}^{q} \log (d_k^2 + \lambda) \right\} - \overline{a} \log \left( \frac{1}{2} \sum_{k=1}^{q} \tilde{a}_k^2 d_k^4 \right).
$$

The function $\log \pi(D_n | \lambda)$ is log-concave, as illustrated in Section 5. Given data $D_n$, the marginal maximum likelihood estimate $\hat{\lambda}$ can be quickly obtained by a two-step algorithm. The first step evaluates $\log \pi(D_n | \lambda)$ for successive values $\lambda = \frac{1}{4} k$, $k = 0, 1, 2, \ldots$, until a value $k^*$ is found such that $\log \pi(D_n | \lambda = \frac{1}{4} k^*) < \log \pi(D_n | \lambda = \frac{1}{4} (k^* - 1))$. The second step obtains the estimate $\hat{\lambda}$ as the the value $\lambda \in [\max\{0, \frac{1}{4} (k^* - 1)\}, \frac{1}{4} (k^* + 1)]$ which minimizes $- \log \pi(D_n | \lambda)$. The minimization step can be performed by using the \texttt{fminbnd()} function of MATLAB (Natick, MA).

4 On the Ridge Model and Estimation Approach

Sections 1-3 focused on the ridge regression linear model, assuming a continuous-valued dependent variable. The ridge model and the estimation approach may appear to be limited in scope. However, by drawing from the existing literature, it may be argued that this scope can be large for applied regression analysis. We list some of the arguments below.

- The ridge regression model allows for a fast least-squares estimation of model parameters, even when the number of covariates $p$ is very large (e.g., in the hundreds or thousands), and when $p > n$. This modeling approach, where the number of coefficient parameters $p$ may be very large, is similar in spirit to the Bayesian nonparametric (BNP) modeling approach, which involves the specification of models with massively many parameters, for the purposes of providing flexible and robust statistical inference.
(Müller & Quintana, 2004). Technically, a BNP model has an infinite number of parameters (Bernardo and Smith, 1994). So for the ridge regression model, if the number of covariates $p$ is chosen to be an increasing function of $n$, then the model meets this technical definition of the BNP model as $n \to \infty$. For example, a flexible linear model where $p$ grows with $n$, may assume the mean function:

$$E[Y \mid x] = x^\top \beta = \sum_{k=1}^{L} \beta_k x_k + \sum_{i=1}^{n} \beta_i B_i(x),$$

(9)

where $B_i(x)$ is a multivariate spline, such as a cubic spline $B_i(x) = ||x - x_i||^3$ with knots $x_i$, for $i = 1, \ldots, n$ (see Müller & Rios Insua, 1998; Denison et al., 2002, p. 102). Here, $\sum_{i=1}^{n} \beta_i B_i(x)$ is the linear combination of basis functions that captures departures of linearity of the underlying regression function. The covariates of this mean function (9) can be easily specified (before centering and scaling all the $p = L + n$ covariates), before subjecting $\beta$ to ridge shrinkage posterior estimation. Also, high dimensional (large $p$) shrinkage linear regression models can be characterized by Lévy processes (see Polson & Scott, 2012).

- As the prior predictive density equation (2) shows, the Bayesian ridge regression model assumes that $h(x) = x^\top \beta$ is a Student process with zero mean function $\mu(x) = E(h(x)) = 0$ and non-stationary covariance function $C(x, x') = E(h(x)h(x')) = b(1 + x^\top \lambda^{-1} x')(a - 2)^{-1}$. Given $\sigma^2$, $h(x) = x^\top \beta$ is a zero-mean Gaussian process (GP) under the "weight-space view," with covariance function $C(x, x'; V) = x^\top \sigma^2 \lambda^{-1} x'$ (Rasmussen & Williams, 2006). Since the Student distribution assigns more probability in the tails, compared to the normal distribution, the Student process provides more robust inference than the GP (Dension et al. 2002, p. 29). However, the difference between the two processes is minimal for reasonable sample sizes ($n > 100$), because the Student process tends to a GP as $\pi \to \infty$, as shown by the posterior predictive (5). Nevertheless, the GP, and the Student process, provide examples of flexible BNP models (Müller & Quintana, 2004). While the "weight-space view" may not provide a fully-flexible GP modeling approach (Rasmussen & Williams, 2006), an important limitation of the GP modeling approach is that it usually requires repeated inversions of $n \times n$ matrices, in order to compute the covariance function for different values of the covariance parameters. Such matrix inversions are computationally demanding or even prohibitive when $n$ is sufficiently large. A possible remedy for large-$n$ settings is to approximate a pure GP model by using a lower-dimensional covariance function matrix. Alternatively, as done in this paper, a flexible and more interpretable GP can always be specified, in a far more computationally-efficient manner, by adopting the weight-space view and taking the number of covariates $p$ to be very large, with the covariates possibly including spline terms.

- As mentioned in Section 1, we assume that dependent variable observations are zero-mean centered and continuous. This is done with no loss of generality, as we argue now. First, binary regression, that is regression involving a binary (two-class) dependent variable, is often of interest in statistical practice. Logistic regression provides a standard binary regression model. However, both frequentist and Bayesian estimation of the logistic regression model requires iteration, and this estimation can become
too computationally expensive for data sets that are sufficiently large. An alternative and computationally-fast (albeit less natural) approach is to code the binary class observations as $\tilde{Y}_i \in \{-1, 1\}$ (for $i = 1, \ldots, n$), and then apply the cutoff of 0 to the predictions based on a shrinkage linear regression model fit by penalized least-squares (Hastie et al., 2009, Section 16.4). In this spirit, the ridge regression model can be fit to the dependent responses $y = \tilde{Y} - \bar{Y}$ (with $\bar{Y}$ the mean of $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_n)^\top$). Then for a given covariate vector $x$, the posterior predictive probability of a positive class $Y_i(\epsilon) = 1$ can be estimated by

$$\hat{\Pr}(Y + \bar{Y} \geq 0 \mid x) = \int_0^\infty \text{st}(y + \bar{Y} \mid x^\top \beta, \bar{\sigma}_\lambda(1 + x^\top \nabla \lambda x), \pi) dy,$$

and the posterior predictive probability of class $\tilde{Y} = -1$ can be estimated by $1 - \hat{\Pr}(Y + \bar{Y} \geq 0 \mid x)$, where the Student density in (10) is from (5).

- When there are more than two classes, labeled as $c = 1, \ldots, C$, with observations $c_l$ (for $l = 1, \ldots, n_c$), it is possible to take a ”one-versus-all” approach to linear classification (Rifkin & Klatau, 2004). That is, consider the vector of $n = n_c C$ centered dependent variable observations $y = y^{(c)} - \bar{y}$, where $y^{(c)} = (y_1^{(c)}, \ldots, y_{n_c}^{(c)})^\top$, \(y_l^{(c)} = (y_l^{(1)}(c), \ldots, y_l^{(C)(c)})^\top\) and $y_l^{(c)} = 1(c_l = c) - 1(c_l \neq c)$ for $c = 1, \ldots, C$ and $l = 1, \ldots, n_c$, and with $\bar{y}$ the mean of $y^{(c)}$. Each observation vector $y_l^{(c)}$, for $l = 1, \ldots, n_c$, corresponds to a covariate matrix $X_l$ consisting of row vectors $X_l^{(c)} = (1(c_l = 1)X_l^I, \ldots, 1(c_l = C)X_l^I)$, for $c = 1, \ldots, C$ respectively. Also let $X$ be the design matrix, after centering and scaling each of the columns of $X = (X_1^I, \ldots, X_{n_c}^I)^\top$, with $X$ having column means $\mu_x$ and column standard deviations $\sigma_x$. Then we may consider a linear model $n_c(y \mid X \beta, \sigma^2 I_{n_c})$, with $\beta$ subject to ridge regression shrinkage estimation. For a given covariate vector $x$, we may estimate the posterior predictive probabilities $\hat{\Pr}(Y + \bar{Y} \geq 0 \mid z_c)$, for $c = 1, \ldots, C$, with $z_c = (x_c - \mu_x)\sigma_x^{-1}$ and $X_l^{(c)} = (1(c_l = 1)X_l^I, \ldots, 1(c_l = C)X_l^I)$. Then the optimal class prediction is given by the class $c \in \{1, \ldots, C\}$ with the highest $P_c = \hat{\Pr}(Y + \bar{Y} \geq 0 \mid z_c)$. Moreover, $P_c - \xi$, for $c = 0, 1, \ldots, C$, collectively provides a corrected histogram probability density estimate for the $C$ classes, provided that $\xi$ is chosen so that $\sum_{c=1}^C P_c - \xi = 1$ (Glad et al. 2003). And if so desired, the given histogram may be smoothed, say, by linear interpolation. Here, the linear model provides a type of ”Bayesian density regression” (e.g., see Karabatsos & Walker, 2012). Also, this histogram/density correction method provides another way to estimate class probabilities from linear classifiers (see Wu et al., 2004).

- Finally, as mentioned, the marginal likelihood estimate $\hat{\lambda}$ is the value $\lambda \in [0, \infty)$ that maximizes $\pi(D_n \mid \lambda)$. In the ”Bayes Empirical Bayes” statistical framework (Deely & Lindley, 1981), this estimate $\hat{\lambda}$ can be characterized as an estimate of the posterior mode under a uniform prior density $u(\lambda \mid 0, \lambda_{\text{max}})$, for large $\lambda_{\text{max}} > 0$. Specifically, under this framework, the posterior density of $(\beta, \sigma^2)$ for the ridge regression model is defined by:

$$\pi_u(\beta, \sigma^2 \mid D_n) = \int \text{nig}(\beta, \sigma^2 \mid \overline{\beta}, \nabla \lambda, \pi, \tilde{\beta}_\lambda) \pi_D(\lambda \mid D_n) d\lambda,$$

$$\pi(\lambda \mid D_n) \propto \pi(D_n \mid \lambda) u(\lambda \mid 0, \lambda_{\text{max}}).$$
As shown above, if the estimate $\hat{\lambda}$ is the maximizer of $\pi(D_n | \lambda)$, then it is also the mode of the posterior density function $\pi(\lambda | D_n)$.

5 Illustrations

We illustrate the marginal maximum likelihood estimation algorithm on a variety of data sets that range widely in terms of sample size ($n$) and the number of covariates ($p$). The eight data sets are described as follows:

- The Iris data set (Fisher, 1936), with sepal length as the dependent variable, and with covariates of sepal width, petal length, and petal width.

- The Teacher data set, a time series data set collected to investigate the effect of a new teacher curriculum, versus the old curriculum, on the respective teaching abilities of undergraduate teacher education students who attended one of four Chicago universities (see Karabatsos & Walker, 2014). The dependent response is a test score on math teaching ability, obtained after completing a math teaching course. The covariates include time (in years), indicator (0,1) of new curriculum, and 347 indicators (0,1) of student, which were included for the purposes of providing a robust posterior estimate of the new curriculum effect (coefficient).

- Two versions of the classic, Diabetes data set (Efron et al. 2004), each with dependent variable defined by a measure of disease progression one year after baseline. This first version of the data set, DiabetesQ, includes 65 covariates defined by 10 covariates at baseline, their squares, and their two-way interactions. The second version of the data set, DiabetesS, includes over 500 covariates, including the 65 quadratic covariates, plus 442 65-variate cubic splines with knots defined by all $n = 442$ observed values of the 65-dimensional covariate vectors (see Section 4).

- The Meaning data set, which contains the ratings of 1,194 high school students from 9 public high schools of Chicago and New York. Each student provided ratings on an 18-item survey about the meaningfulness of past reading experiences (Tatum & Karabatsos, 2013). In total, the data set contains nearly 21,000 ratings, and contains information on over 100 student, teacher, and school background covariates.

- The Blog data set, with the dependent variable being the number of comments on the blog after 24 hours. In the data set, the sample size exceeds 52,000. Also there are over 2,500 covariates, formed by taking the powers of 1, 2, and 3 of each of the 840 original covariates.

- The Lymphoma data set, containing nearly 80 observations on over 7,000 covariates, with dependent variable indicating ($1, -1$) the presence of diffuse large b-cell lymphoma versus Follicular Lymphoma (FL) morphology (Shipp et al. 2012).

- The Cancer data set (Petricoin, et al. 2002), containing observations of a binary dependent variable indicating ($1, -1$) either the presence or absence of ovarian cancer, and containing data on over 15,000 covariates.
For each data set, before fitting the ridge regression model, the observations of each of the variables were mean-centered, and each of the \( p \) covariates were also scaled to have variance 1, as mentioned in Section 1. For each of the Iris, Teacher, Diabetes, Meaning, and Blog data sets, the dependent responses were also scaled to have variance 1.

For each data set, Table 1 presents the computation time (in seconds) of the marginal maximum likelihood estimation algorithm to obtain the ridge estimate \( \hat{\lambda} \). This includes the time needed to compute the singular value decomposition (s.v.d.) of the given design matrix \( X \). As shown, for each of the 8 data sets, the algorithm calculated the ridge estimate in less than one-tenth of a second. All the computation times, reported in this section, are based on a modern 64-bit laptop computer with 2.8 GHz Intel Core i7 processor and 16 megabytes RAM. However, relatively fast computation times were also obtained on a much older 64-bit laptop with a 2.4 GHz Intel i5 processor and 4 megabytes RAM. Also, no parallel computing techniques were required.

| Data Set  | \( n \) | \( p \) | Algorithm | Total | \( \hat{\lambda} \) | \( \pi(D_n|\hat{\lambda}) \) |
|-----------|--------|--------|-----------|-------|----------------|------------------|
| Iris      | 150    | 3      | 0.002     | 0.003 | 0.17           | -61.73           |
| Teacher   | 347    | 349    | 0.003     | 0.011 | 0.0001         | -396.88          |
| DiabetesQ | 442    | 65     | 0.006     | 0.011 | 67.70          | -389.63          |
| DiabetesS | 442    | 507    | 0.056     | 0.074 | 446.05         | -375.83          |
| Meaning   | 20,994 | 113    | 0.018     | 0.023 | 250.45         | -21262.87        |
| Blog      | 52,397 | 2,520  | 0.071     | 0.185 | 410.88         | -44738.55        |
| Lymphoma  | 77     | 7,129  | 0.001     | 12.818| 0.00005        | -73.32           |
| Cancer    | 253    | 15,154 | 0.002     | 120.810| 0.00005      | 52.74            |

Table 1: For various data sets that differ by the sample size (\( n \)) and the number of covariates (\( p \)), the computation time (in seconds) for the marginal maximum likelihood estimation algorithm, and total computation time after adding the computation of other statistics.

Table 1 also presents the total computation times, after adding the computation times for other statistics. The other statistics include the posterior mean estimate \( \bar{\beta}_\lambda \); the marginal posterior covariance matrix \( (\bar{b}_\lambda/(a-1))\nabla_\lambda \); the scaled neighborhood criteria for the \( p \) covariates; the posterior mean and standard deviation of the error variance \( \sigma^2 \); log marginal likelihood at \( \lambda \); and the standardized residuals. For almost all of the data sets, the computation times were less than two-tenths of a second. The Lymphoma data set required under 13 seconds, while the Cancer data set took about 2 minutes. This is because in each case, a large-scale matrix inversion was needed to obtain the covariance matrix estimate \( \nabla_\lambda \).

For each of the eight data sets, Figure ?? presents the marginal likelihood \( \pi(D_n|\lambda) \) as a function of \( \lambda \), along with the estimate \( \hat{\lambda} \) shown by a vertical dashed line. This figure shows the concavity of \( \log \pi(D_n|\lambda) \).

For further illustration, Figure ?? presents the results of the DiabetesQ data. The left panel presents the coefficient estimates \( \bar{\beta}_\lambda \) for the 65 covariates. For each of the covariates, the right panel presents the results of the scaled neighborhood criterion. That panel shows that the 17 following covariates were significant predictors of disease progression: \textit{sex}, \textit{bmi}, \textit{map}, \textit{hdl}, \textit{tch}, \textit{ltg}, \textit{glu}, \textit{age*sex}, \textit{age*ltg}, \textit{sex*map}, \textit{bmi*map}, \textit{map*glu}, \textit{ldl*ltg}, \textit{age^2}, \textit{age^2},
sex^2, bmi^2, and glu^2. Each of these covariates correspond to a posterior probability of \( \beta_k \in \left[ -v_{1k}^{1/2}, v_{1k}^{1/2} \right] \) of less than 1/2.

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Figure 1

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Figure 2

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6 Conclusions

For ridge regression, the mainstream methods for estimating the ridge parameter are based on computationally expensive and inconsistent cross-validation methods, or on information criteria, which can only approximate (minus 2 times) the log marginal likelihood of the ridge regression model. This paper introduced a fast algorithm that quickly obtained the marginal maximum likelihood estimate of the ridge parameter. The maximum of the marginal likelihood corresponds to the posterior mode of the ridge parameter under the Bayes Empirical Bayes statistical framework, when the parameter is assigned a uniform prior. The construction of the fast estimation algorithm was facilitated by a singular value decomposition of the covariate design matrix, which gave rise to a simplified and equivalent form of the marginal likelihood equation that involves no computationally-demanding matrix inverses or determinants; and was helped by the fact that the marginal likelihood is log-concave over the range of the ridge parameter.

Then given the marginal maximum likelihood estimate of the ridge parameter, the coefficient estimates, the posterior covariance matrix, the SN criteria indicating the subset of significant predictors (covariates), and regression residuals, can all then be computed in less than a second. We illustrated the computational speed and viability of the estimation algorithm on many large data sets, involving hundreds to several thousands of observations and/or covariates. MATLAB code for the full ridge regression estimation algorithm can be obtained from the author.

When the number of covariates \( p \) is at least several thousand, the computation of the marginal posterior covariance matrix of \( \beta \) can take a noticeable amount of time on a modern computer. This is because then the computation involves taking the inverse of a very high-dimensional \( p \times p \) matrix. However, in real practice, it is usually of interest to infer only the diagonal elements of this covariance matrix, because they provide the information necessary to compute the significance values of predictors, as done via the SN criteria. In such a case, a fast algorithm can be used to extract only the diagonal elements of the inverse matrix (Lin et al., 2009).

While the current manuscript focused on ridge regression shrinkage, the same marginal maximum likelihood estimation algorithm can also be used to help guide the optimal choice of regularization parameter \( \lambda \) for popular sparse shrinkage estimation methods based in penalized least-squares. Methods include the LASSO (Tibshirani, 1996) and the Elastic Net (Zou & Hastie, 2006). This guided search idea is valid because \( \lambda \) has a similar interpretation
across the methods. For example the Elastic Net estimator is based on a convex combination of the LASSO and ridge regression. Also, this guided search can be performed without requiring computationally-expensive $K$-fold cross-validation.

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References

Akaike, H. (1973). Information theory and the an extension of the maximum likelihood principle. In B. Petrov & F. Csaki (Eds.), *Second international symposium on information theory* (p. 267-281). Budapest: Academiai Kiado.

Berger, J. (1993). *Statistical decision theory and Bayesian analysis (second edition)*. New York: Springer.

Bernardo, J., & Smith, A. (1994). *Bayesian theory*. Chichester, England: Wiley.

Bottolo, L., & Richardson, S. (2010). Evolutionary stochastic search for Bayesian model exploration. *Bayesian Analysis, 5*, 583-618.

Cowles, M., & Carlin, B. (1996). Markov chain Monte Carlo convergence diagnostics: A comparative review. *Journal of the American Statistical Association, 91*, 833-904.

Cule, E., & Iorio, M. D. (2013). Ridge regression in prediction problems: Automatic choice of the ridge parameter. *Genetic Epidemiology, 37*, 704-714.

Deely, J., & Lindley, D. (1981). Bayes empirical Bayes. *Journal of the American Statistical Association, 76*, 833-841.

Denison, D., Holmes, C., Mallick, B., & Smith, A. (2002). *Bayesian methods for nonlinear classification and regression*. New York: John Wiley and Sons.

Efron, B., Hastie, T., Johnstone, I., & Tibshirani, R. (2004). Least angle regression. *Annals of Statistics, 32*, 407-499.

Fan, Y., & Tang, C.-Y. (2013). Tuning parameter selection in high dimensional penalized likelihood. *Journal of the Royal Statistical Society: Series B, 75*, 531-552.

Fisher, R. (1936). The use of multiple measurements in taxonomic problems. *Annals of Eugenics, 7*, 179-188.

Glad, I., Hjort, N., & Ushakov, N. (2003). Correction of density estimators that are not densities. *Scandinavian Journal of Statistics, 30*, 415-427.

Golub, G., Heath, M., & Wahba, G. (1979). Generalized cross-validation as a method for choosing a good ridge parameter. *Technometrics, 21*, 215-223.
Hastie, T., Tibshirani, R., & Friedman, J. (2009). *The elements of statistical learning: Data mining, inference, and prediction* (2nd edition). New York: Springer-Verlag.

Hoerl, A., & Kennard, R. (1970). Ridge regression: Biased estimation for nonorthogonal problems. *Technometrics, 12*, 55-67.

Hoerl, A., Kennard, R., & Baldwin, K. (1975). Ridge regression: Some simulations. *Communications in Statistics: Theory and Methods, 4*, 105-123.

Karabatsos, G., & Walker, S. (2012). Adaptive-modal Bayesian nonparametric regression. *Electronic Journal of Statistics, 6*, 2038-2068.

Karabatsos, G., & Walker, S. (2014). *A Bayesian nonparametric causal model for regression discontinuity designs* (Tech. Rep. No. 1311.4482). arXiv preprint.

Kass, R., & Raftery, A. (1995). Bayes factors. *Journal of the American Statistical Association, 90*, 773-795.

Kibria, B. (2003). Performance of some new ridge regression estimators. *Communications in Statistics-Simulation and Computation, 32*, 419-435.

Li, Q., & Lin, N. (2010). The Bayesian elastic net. *Bayesian Analysis, 5*, 151-170.

Lin, L., Lu, J., Ying, L., Car, R., & Weinan, E. (2009). Fast algorithm for extracting the diagonal of the inverse matrix with application to the electronic structure analysis of metallic systems. *Communications in Mathematical Sciences, 7*, 755-777.

Lindley, D., & Smith, A. (1972). Bayes estimates for the linear model (with discussion). *Journal of the Royal Statistical Society, Series B, 34*, 1-41.

Müller, P., & Insua, D. (1998). Issues in Bayesian analysis of neural network models. *Neural Computation, 10*, 749-770.

Müller, P., & Quintana, F. (2004). Nonparametric Bayesian data analysis. *Statistical Science, 19*, 95-110.

Petricoin, E., Ardekani, A., Hitt, B., Levine, P., Fusaro, V., Steinberg, S., Mills, G., Simone, C., Fishman, D., Kohn, E., & Liotta, L. (2002). Use of proteomic patterns in serum to identify ovarian cancer. *The Lancet, 359*, 572-577.

Polson, N., & Scott, J. (2012). Local shrinkage rules, Lévy processes and regularized regression. *Journal of the Royal Statistical Society: Series B, 74*, 287-311.

Rasmussen, C., & Williams, C. (2006). *Gaussian processes for machine learning*. Cambridge, MA: The MIT Press.

Rifkin, R., & Klautau, A. (2004). In defense of one-vs-all classification. *Journal of Machine Learning Research, 5*, 101-141.
Ripley, B. (1996). *Pattern recognition and neural networks*. Cambridge: Cambridge University Press.

Shao, J. (1993). Linear model selection by cross-validation. *Journal of the American Statistical Association*, 88, 486-494.

Shipp, M., Ross, K., Tamayo, P., Weng, A., Kutok, J., Aguiar, R., Gaasenbeek, M., Angelo, M., Reich, M., Pinkus, G., Ray, T., Koval, M., Last, K., Norton, A., Lister, T., Mesirov, J., Neuberg, D., Lander, E., Aster, J., & Golub, T. (2002). Diffuse large b-cell lymphoma outcome prediction by gene-expression profiling and supervised machine learning. *Nature medicine*, 8, 68-74.

Tatum, A., & Karabatsos, G. (2013). *A survey study of teens and texts* (working manuscript) (Tech. Rep.). University of Illinois-Chicago.

Tibshirani, R. (1996). Regression shrinkage and selection via the Lasso. *Journal of the Royal Statistical Society, Series B*, 58, 267-288.

Wu, T.-F., Lin, C.-J., & Weng, R. (2004). Probability estimates for multi-class classification by pairwise coupling. *The Journal of Machine Learning Research*, 5, 975-1005.

Yang, Y. (2005). Can the strengths of AIC and BIC be shared? a conflict between model identification and regression estimation. *Biometrika*, 92, 937-950.

Zhang, Y., Li, R., & Tsai, C.-L. (2010). Regularization parameter selections via generalized information criterion. *Journal of the American Statistical Association*, 105, 312-323.

Zou, H., & Hastie, T. (2005). Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society: Series B*, 67, 301-320.
Figure Captions

Figure 1. The marginal likelihood function $\pi(D_n | \lambda)$, for each of the eight data sets. Dashed vertical line: The marginal maximum likelihood estimate $\hat{\lambda}$. (When $\hat{\lambda}$ is near zero, the dashed line is not clearly visible due to the solid vertical axis line).

Figure 2. Left panel: Coefficient estimates for the DiabetesQ data. Right panel: Corresponding significance probability values according to the scaled neighborhood (SN) criterion.
