FULL EXCEPTIONAL COLLECTIONS AND STABILITY CONDITIONS FOR DYNKIN QUIVERS

TAKUMI OTANI

Abstract. For a stability condition $\sigma$ on a triangulated category, Dimitrov–Katzarkov introduced the notion of a $\sigma$-exceptional collection. In this paper, we study full $\sigma$-exceptional collections in the derived category of an acyclic quiver. In particular, we prove that any stability condition $\sigma$ on the derived category of a Dynkin quiver admits a full $\sigma$-exceptional collection.

1. Introduction

The notion of a stability condition on a triangulated category was introduced by Bridgeland. He showed in [B1] that the space of stability conditions $\text{Stab}(\mathcal{D})$ on a triangulated category $\mathcal{D}$ has a structure of a complex manifold. It is expected that the space of stability conditions $\text{Stab}(\mathcal{D})$ is related to various “deformation theories”. In order to study the space of stability conditions $\text{Stab}(\mathcal{D})$, Macrì constructed a stability condition associated with a full exceptional collection. In several examples, such stability conditions are useful to study the homotopy type of the space of stability conditions (cf. [DK3, DK4, L, M1, M2]). Motivated by Macrì’s work, Dimitrov–Katzarkov introduced the notion of a $\sigma$-exceptional collection for a stability condition $\sigma$ on a triangulated category [DK1]. Roughly speaking, a $\sigma$-exceptional collection is an Ext-exceptional collection consisting of $\sigma$-stable objects such that their phases are in an interval of length one (see Definition 3.3).

In this paper, we study full Ext-exceptional collections in the bounded derived category $\mathcal{D}^b(Q)$ of finitely generated right $\mathbb{C}Q$-modules of an acyclic quiver $Q$ satisfying certain conditions. More precisely, we consider acyclic quivers with the two conditions:

(A1) For each $i, j = 1, \ldots, \mu$, the number of arrows from $i$ to $j$ is less than one.

(A2) Let $i, k, l = 1, \ldots, \mu$ such that $k < i < l$. If there are arrows from $k$ to $i$ and from $i$ to $l$, then there are no arrows from $k$ to $l$. 

Date: October 26, 2022.
Dynkin quivers and extended Dynkin quivers except $A^{(1)}_{1,1}$ and $A^{(1)}_{1,2}$ are contained in the class of acyclic quivers satisfying (A1) and (A2). In cases $A^{(1)}_{1,1}$ and $A^{(1)}_{1,2}$, full Ext-exceptional collections are already studied by Macrì [M1] and Dimitrov–Katzarkov [DK1], respectively. The following theorem is the main result of this paper.

**Theorem 1.1** (Theorem 3.4). Let $Q$ be an acyclic quiver satisfying the conditions (A1) and (A2), and $\mathcal{A}$ a heart of a bounded $t$-structure on $\mathcal{D}^b(Q)$. Assume that the heart $\mathcal{A}$ is obtained from the standard heart by iterated simple tilts. Then, there exists a full Ext-exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ such that $\mathcal{A} = \langle \mathcal{E} \rangle_\text{ex}$ and $\text{Sim} \mathcal{A} = \{E_1, \ldots, E_\mu\}$.

To prove the theorem, we use the result by King–Qiu [KQ]. They studied the relationship of simple tiltings of hearts in $\mathcal{D}^b(Q)$ of an acyclic quiver $Q$. Based on simple tiltings, we construct a full $\sigma$-exceptional collection consisting of simple objects in the heart. As a corollary of Theorem 1.1 for a stability condition $\sigma$ whose heart is obtained from the standard heart by iterated simple tilts, there exists a full $\sigma$-exceptional collection (Corollary 3.5).

For a Dynkin quiver $\vec{\Delta}$, it was shown by [KV] (cf. [Q]) that any heart of a bounded $t$-structure on $\mathcal{D}^b(\vec{\Delta})$ can be obtained from the standard heart by iterated simple tilts. As a direct consequence, we obtain the following theorem.

**Theorem 1.2** (Corollary 3.6). Let $\vec{\Delta}$ be one of the Dynkin quivers. For each stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}^b(\vec{\Delta})$, there exists a full $\sigma$-exceptional collection.

This theorem gives an affirmative answer to a conjecture by Dimitrov–Katzarkov [DK2, Conjecture 7.1]. As a corollary of Theorem 1.2 the space of stability conditions $\text{Stab}(\mathcal{D}^b(\vec{\Delta}))$ can be described by full Ext-exceptional collections (Corollary 3.7).

For the affine $A^{(1)}_{1,1}$ quiver and the affine $A^{(1)}_{1,2}$ quiver, it is known that for each stability condition $\sigma$ there exists a full $\sigma$-exceptional collection [M1, DK1]. Based on the results and Theorem 1.2 in the case of extended Dynkin quivers the following conjecture is expected.

**Conjecture 1.3** (Conjecture 3.10). Let $Q$ be one of the extended Dynkin quivers. For each stability condition $\sigma$ on $\mathcal{D}^b(Q)$, there exists $s \in \mathbb{C}$ such that the heart $\mathcal{P}((\text{Re}(s), \text{Re}(s)+1))$ can be obtained from the standard heart by iterated simple tilts. In particular, any stability condition $\sigma$ on $\mathcal{D}^b(Q)$ admits a full $\sigma$-exceptional collection.

This paper is organized as follows. In Section 2 we recall some notations of triangulated category and collect some facts related to stability conditions. In Section 3,
After recalling the notion of $\sigma$-exceptional collection for a stability condition $\sigma$ on a triangulated category, we state our main results. Section 4 is devoted to proving Theorem 3.4.

Acknowledgements. I am deeply grateful to my supervisor Professor Atsushi Takahashi for careful reading of this manuscript and valuable comments, and Professor Akishi Ikeda for discussions about stability conditions and constant encouragements. I would like to thank Yu Qiu for valuable suggestions and comments. I would also like to thank Yuichiro Goto, Kohei Kikuta and Yuuki Shiraishi for their helpful discussions. The author is supported by JST SPRING, Grant Number JPMJSP2138.

2. Preliminaries

Throughout this paper, for a finite dimensional $\mathbb{C}$-algebra $A$ (resp. a quiver $Q$), the bounded derived category of finitely generated right $A$-modules (resp. $\mathbb{C}Q$-modules) is denoted by $\mathcal{D}^b(A) := \mathcal{D}^b \text{mod}(A)$ (resp. $\mathcal{D}^b(Q) := \mathcal{D}^b \text{mod}(\mathbb{C}Q)$). For convenience, we always assume that for an acyclic quiver $Q$ there are no arrows from the vertex $i$ to the other vertex $j$ when $i > j$.

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category and $[1]$ the shift functor. For convenience, for $E, F \in \mathcal{D}$ we put

$$\text{Hom}_\mathcal{D}^\bullet(E, F) := \bigoplus_{p \in \mathbb{Z}} \text{Hom}_\mathcal{D}^p(E, F)[-p], \quad \text{Hom}_\mathcal{D}^p(E, F) := \text{Hom}_\mathcal{D}(E, F[p]).$$

A $\mathbb{C}$-linear triangulated category $\mathcal{D}$ is said to be of finite type if the dimension of the $\mathbb{C}$-vector space $\bigoplus_{p \in \mathbb{Z}} \text{Hom}_\mathcal{D}^p(E, F)$ is finite for all $E, F \in \mathcal{D}$.

A full subcategory $\mathcal{A}$ of a triangulated category $\mathcal{D}$ will be called extension closed if whenever $E \to F \to G$ is a triangle in $\mathcal{D}$ with $E \in \mathcal{A}$ and $G \in \mathcal{A}$, then $F \in \mathcal{A}$. The extension closed subcategory $\langle S \rangle_{\text{ex}}$ of $\mathcal{D}$ generated by a full subcategory $S \subset \mathcal{D}$ is the smallest extension closed full subcategory of $\mathcal{D}$ containing $S$.

2.1. Exceptional collections and mutations. In this section, we recall related notions of an exceptional collection.

Definition 2.1. Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category of finite type.

(i) An object $E \in \mathcal{D}$ is called exceptional if $\text{Hom}_\mathcal{D}(E, E) \cong \mathbb{C}$ and $\text{Hom}_\mathcal{D}^p(E, E) \cong 0$ when $p \neq 0$.

(ii) An ordered set $\mathcal{E} = (E_1, \ldots, E_\mu)$ consisting of exceptional objects $E_1, \ldots, E_\mu$ is called exceptional collection if $\text{Hom}_\mathcal{D}^p(E_i, E_j) \cong 0$ for all $p \in \mathbb{Z}$ and $i > j$. 
(iii) An exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ is called $\text{Ext}$ if
\[
\text{Hom}_D^p(E_i, E_j) \cong 0, \quad i \neq j, \quad p \leq 0.
\]

(iv) An exceptional collection $\mathcal{E}$ is called $\text{full}$ if the smallest full triangulated subcategory of $\mathcal{D}$ containing all elements in $\mathcal{E}$ is equivalent to $\mathcal{D}$ as a triangulated category.

(v) An exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ is called $\text{monochromatic}$ if for any $i, j = 1, \ldots, \mu$ such that $\text{Hom}_D^\bullet(E_i, E_j) \not\cong 0$, the $\mathbb{Z}$-graded $\mathbb{C}$-vector space $\text{Hom}_D^\bullet(E_i, E_j)$ is concentrated in a single non-negative degree.

Let $\mathcal{E} = (E_1, \ldots, E_\mu)$ be a monochromatic full $\text{Ext}$-exceptional collection in $\mathcal{D}$. For each $i, j = 1, \ldots, \mu$ such that $\text{Hom}_D^\bullet(E_i, E_j) \not\cong 0$, the degree in which $\text{Hom}_D^\bullet(E_i, E_j)$ is concentrated will be denoted by $\text{deg} \text{Hom}_D^\bullet(E_i, E_j) \in \mathbb{Z}_{\geq 0}$.

**Remark 2.2.** We do not consider the degree if $\text{Hom}_D^\bullet(E_i, E_j)$ vanishes.

For an exceptional object $E \in \mathcal{D}$, the smallest triangulated category containing $E$ can be identified with the bounded derived category $\mathcal{D}^b(\mathbb{C})$ of the $\mathbb{C}$-algebra $\mathbb{C}$. Hence, the tensor functor $- \otimes E : \mathcal{D}^b(\mathbb{C}) \rightarrow \mathcal{D}$ is naturally defined as follows:

\[ V^\bullet \otimes E := \bigoplus_{p \in \mathbb{Z}} V^p \otimes E[-p], \quad V^p \otimes E := E \oplus \dim \mathbb{C} V^p, \]

where $V^\bullet = \bigoplus_{p \in \mathbb{Z}} V^p[-p] \in \mathcal{D}^b(\mathbb{C})$.

**Definition 2.3.** Let $(E, F)$ be an exceptional collection. Define two objects $\mathcal{R}_F E$ and $\mathcal{L}_E F$ by the following exact triangles respectively:

\[ \mathcal{R}_F E \rightarrow E \xrightarrow{\text{ev}^*} \text{Hom}_D^\bullet(E, F)^* \otimes F, \]
\[ \text{Hom}_D^\bullet(E, F) \otimes E \xrightarrow{\text{ev}} F \rightarrow \mathcal{L}_E F, \]

where $(-)^*$ denotes the duality $\text{Hom}_C(-, \mathbb{C})$. The object $\mathcal{R}_F E$ (resp. $\mathcal{L}_E F$) is called the right mutation of $E$ through $F$ (resp. left mutation of $F$ through $E$). Then, $(F, \mathcal{R}_F E)$ and $(\mathcal{L}_E F, E)$ form new exceptional collections.

**Remark 2.4.** Our definition of mutations differs from the usual one (cf. [BP, Section 1]). In our notation, the usual left and right mutations are given by $\mathcal{R}_F E[1]$ and $\mathcal{L}_E F[-1]$, respectively.

The Artin’s braid group $\text{Br}_\mu$ on $\mu$-stands is a group presented by the following generators and relations:
Generators: \( \{ b_i \mid i = 1, \ldots, \mu - 1 \} \)

Relations: \( b_ib_j = b_jb_i \) for \( |i - j| \geq 2 \), \( b_ib_{i+1} = b_{i+1}b_i \) for \( i = 1, \ldots, \mu - 2 \).

Consider the group \( \text{Br}_\mu \rtimes \mathbb{Z}^\mu \), the semi-direct product of the braid group \( \text{Br}_\mu \) and the abelian group \( \mathbb{Z}^\mu \), defined by the group homomorphism \( \text{Br}_\mu \rightarrow \mathfrak{S}_\mu \rightarrow \text{Aut}_\mathbb{Z}\mathbb{Z}^\mu \), where the first homomorphism is \( b_i \mapsto (i, i+1) \) and the second one is induced by the natural actions of the symmetric group \( \mathfrak{S}_\mu \) on \( \mathbb{Z}^\mu \).

**Proposition 2.5** (cf. [BP, Proposition 2.1]). The group \( \text{Br}_\mu \rtimes \mathbb{Z}^\mu \) acts on the set of isomorphism classes of full exceptional collections in \( D \) by mutations and transformations:

\[
\begin{align*}
    b_i \cdot (E_1, \ldots, E_\mu) &:= (E_1, \ldots, E_{i-1}, E_{i+1}, \mathcal{R}_{E_{i+1}}E_i, E_{i+2}, \ldots, E_\mu), \\
    b_i^{-1} \cdot (E_1, \ldots, E_\mu) &:= (E_1, \ldots, E_{i-1}, \mathcal{L}_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots, E_\mu), \\
    e_i \cdot (E_1, \ldots, E_\mu) &:= (E_1, \ldots, E_{i-1}, E_i[1], E_{i+1}, \ldots, E_\mu),
\end{align*}
\]

where \( e_i \) is the \( i \)-th generator of \( \mathbb{Z}^\mu \).

**Remark 2.6.** For any full exceptional collection \( (E_1, \ldots, E_\mu) \), since \( D \) is of finite type one can choose integers \( p_1, \ldots, p_\mu \in \mathbb{Z} \) so that the shifted full exceptional collection \( (E_1[p_1], \ldots, E_\mu[p_\mu]) \) is Ext.

### 2.2. Hearts of bounded \( t \)-structures

In this section, we collect some facts about a heart of a bounded \( t \)-structure. Let \( D \) be a \( \mathbb{C} \)-linear triangulated category of finite type. Recall that a \( t \)-structure on \( D \) is a full subcategory \( \mathcal{F} \subset D \), satisfying \( \mathcal{F}[1] \subset \mathcal{F} \), such that for every object \( E \in D \), there is a triangle \( F \rightarrow E \rightarrow G \) in \( D \) with \( F \in \mathcal{F} \) and \( G \in \mathcal{F}^\perp \), where \( \mathcal{F}^\perp \) is a full subcategory given by

\[
\mathcal{F}^\perp := \{ G \in D \mid \text{Hom}_D(F, G) = 0, \ F \in \mathcal{F} \}.
\]

A \( t \)-structure \( \mathcal{P} \) is said to be **bounded** if

\[
D = \bigcup_{i,j \in \mathbb{Z}} \mathcal{F}[i] \cap \mathcal{F}^\perp[j].
\]

A heart \( \mathcal{A} \) of a \( t \)-structure \( \mathcal{F} \) is the full subcategory \( \mathcal{A} := \mathcal{F} \cap \mathcal{F}^\perp[1] \). It was proved in [BBD] that a heart \( \mathcal{A} \) of a \( t \)-structure is an abelian category, with the short exact sequences in \( \mathcal{A} \) being precisely the triangles in \( D \) all of whose vertices are objects of \( \mathcal{A} \). For simplicity, we use the term heart to mean the heart of a bounded \( t \)-structure.

Let \( \mathcal{A} \) be a heart in \( D \) and \( S \in \mathcal{A} \) a simple object. Define full subcategories

\[
{\perp} S := \{ E \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(E, S) = 0 \}, \quad \mathcal{S}^\perp := \{ E \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(S, E) = 0 \}.
\]
Then, one can consider two full subcategories

\[ \mathcal{A}_S^\sharp := \langle S[1], \perp S \rangle_{\text{ex}}, \quad \mathcal{A}_S^\flat := \langle S\perp, S[-1] \rangle_{\text{ex}}. \]

It is known that the subcategories \( \mathcal{A}_S^\sharp \) and \( \mathcal{A}_S^\flat \) are hearts in \( \mathcal{D} \). The heart \( \mathcal{A}_S^\sharp \) (resp. \( \mathcal{A}_S^\flat \)) is called the forward simple tilt (resp. backward simple tilt) of \( \mathcal{A} \) by \( S \). Denote by \( \text{Sim} \mathcal{A} \) a complete set of simple modules in a heart \( \mathcal{A} \).

**Proposition 2.7** ([KQ, Proposition 5.4]). Let \( S \) be a simple object in \( \mathcal{A} \) satisfying \( \text{Ext}^1(S, S) \cong 0 \). Assume that \( \#\text{Sim} \mathcal{A} \) is finite and \( \mathcal{A} = \langle \text{Sim} \mathcal{A} \rangle_{\text{ex}} \). Then, we have

\[
\text{Sim} \mathcal{A}_S^\sharp = \{ S[1] \} \cup \{ \psi_S^\sharp(T) \mid T \in \text{Sim} \mathcal{A}, \; T \not\cong S \},
\]

\[
\text{Sim} \mathcal{A}_S^\flat = \{ S[-1] \} \cup \{ \psi_S^\flat(T) \mid T \in \text{Sim} \mathcal{A}, \; T \not\cong S \},
\]

where \( \psi_S^\sharp(T) \) and \( \psi_S^\flat(T) \) are defined by the following exact triangles:

\[
\psi_S^\sharp(T) \longrightarrow T \longrightarrow \text{Hom}^1(T, S)^* \otimes S[1], \quad (2.1a)
\]

\[
\text{Hom}^1(S, T) \otimes S[-1] \longrightarrow T \longrightarrow \psi_S^\flat(T). \quad (2.1b)
\]

and \( \mathcal{A}_S^\sharp = \langle \text{Sim} \mathcal{A}_S^\sharp \rangle_{\text{ex}} \) and \( \mathcal{A}_S^\flat = \langle \text{Sim} \mathcal{A}_S^\flat \rangle_{\text{ex}}. \)

For an acyclic quiver, King–Qiu also showed the following

**Proposition 2.8** ([KQ, Proposition 6.4]). Let \( Q \) be an acyclic quiver and \( \mathcal{A} \) a heart in \( \mathcal{D}^b(Q) \). For any distinct simple objects \( S \) and \( T \) in the heart \( \mathcal{A} \), \( \text{Hom}^\bullet_{\mathcal{D}^b(Q)}(S, T) \) and \( \text{Hom}^\bullet_{\mathcal{D}^b(Q)}(T, S) \) are concentrated in a single positive degree. Moreover, we have \( \text{Hom}^\bullet_{\mathcal{D}^b(Q)}(S, T) \cong 0 \) or \( \text{Hom}^\bullet_{\mathcal{D}^b(Q)}(T, S) \cong 0. \)

In the Dynkin case, the following result is known. It plays an important role in this paper.

**Proposition 2.9** ([KV] (cf. [Q, Appendix A])). Any heart in \( \mathcal{D}^b(\Delta) \) can be obtained from the standard heart \( \text{mod}(\mathbb{C}\Delta) \) by iterated simple tilts.

**2.3. Stability conditions.** In this section, we recall the notion of a stability condition on a triangulated category. Let \( \mathcal{D} \) be a \( \mathbb{C} \)-linear triangulated category of finite type. Denote by \( K_0(\mathcal{D}) \) the Grothendieck group of the triangulated category \( \mathcal{D} \). We will denote by \( \mu \in \mathbb{Z}_{\geq 0} \) the rank of the Grothendieck group \( K_0(\mathcal{D}) \).

**Definition 2.10** ([B1, Definition 1.1]). A stability condition on \( \mathcal{D} \) consists of a group homomorphism \( Z : K_0(\mathcal{D}) \longrightarrow \mathbb{C} \), which is called central charge, and the family of full additive subcategories \( \mathcal{P} = \{ \mathcal{P}(\phi) \}_{\phi \in \mathbb{R}} \), called slicing, satisfying the following axioms:
(i) if $E \in \mathcal{P}(\phi)$ then $Z(E) = m(E) \exp(\sqrt{-1}\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$.
(ii) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
(iii) if $\phi_1 > \phi_2$ and $A_i \in \mathcal{P}(\phi_i)$ then $\text{Hom}_\mathcal{D}(A_1, A_2) = 0$.
(iv) for each nonzero object $E \in \mathcal{D}$ there exists a finite sequence of real numbers
$$
\phi_1 > \phi_2 > \cdots > \phi_n
$$
and a collection of triangles

\[
0 = F_0 \to F_1 \to F_2 \to \cdots \to F_{n-1} \to F_n = E
\]

with $A_i \in \mathcal{P}(\phi_i)$ for all $i = 1, \ldots, n$.
(v) (support property) There exists a constant $C > 0$ such that for all nonzero object $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$, we have
$$
\|E\| < C|Z(E)|
$$
where $\| \cdot \|$ denotes a norm on $K_0(\mathcal{D}) \otimes \mathbb{R}$.

A nonzero object $E \in \mathcal{P}(\phi)$ is called $\sigma$-semistable of phase $\phi$, and a simple object of $E \in \mathcal{P}(\phi)$ is called $\sigma$-stable. For any interval $I \subset \mathbb{R}$, we put $\mathcal{P}(I) := \langle \mathcal{P}(\phi) \mid \phi \in I \rangle_{\text{ex}}$. Then, the full subcategory $\mathcal{P}((0, 1])$ is a heart in $\mathcal{D}$, hence an abelian category. Conversely, a stability condition can be described as a heart and a certain function on the heart. First, we recall the notion of stability function. We denote by $K_0(\mathcal{A})$ the Grothendieck group of an abelian category $\mathcal{A}$.

**Definition 2.11** ([B1, Definition 2.1]). Let $\mathcal{A}$ be a heart in $\mathcal{D}$. A **stability function** on $\mathcal{A}$ is a group homomorphism $Z: K_0(\mathcal{A}) \to \mathbb{C}$ such that for all nonzero object $E \in \mathcal{A}$ the complex number $Z(E)$ lies in the semiclosed upper half plane $\mathbb{H}_- := \{ r e^{\sqrt{-1}\pi\phi} \in \mathbb{C} \mid r > 0, 0 < \phi \leq 1 \}$.

Given a stability function $Z: K_0(\mathcal{A}) \to \mathbb{C}$, the **phase** of a nonzero object $E \in \mathcal{A}$ is defined to be the real number $\phi(E) := (1/\pi)\text{arg}Z(E) \in (0, 1]$. A nonzero object $E \in \mathcal{A}$ is semistable (resp. stable) if we have $\phi(A) \leq \phi(E)$ (resp. $\phi(A) < \phi(E)$) for all nonzero subobjects $A \subset E$. We say that a stability function $Z: K_0(\mathcal{A}) \to \mathbb{C}$ satisfies the **Harder–Narasimhan property** if each nonzero object $E \in \mathcal{A}$ admits a filtration

$$
0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = E
$$
such that $F_i/F_{i-1}$ is semistable for $i = 1, \ldots, n$ with $\phi(F_1/F_0) > \phi(F_2/F_1) > \cdots > \phi(F_n/F_{n-1})$. We also say that a stability function $Z: K_0(\mathcal{A}) \rightarrow \mathbb{C}$ satisfies the support property if there exists a constant $C > 0$ such that for all semistable objects $E \in \mathcal{A}$, we have $\|E\| < C|Z(E)|$.

**Proposition 2.12** ([B1, Proposition 5.3]). To give a stability condition on a triangulated category $\mathcal{D}$ is equivalent to giving a bounded $t$-structure on $\mathcal{D}$ and a stability function on its heart with the Harder-Narasimhan property and the support property. □

Denote by $\text{Stab}(\mathcal{D})$ the space of stability conditions on $\mathcal{D}$.

**Proposition 2.13** ([B1, Theorem 1.2]). There exists a natural topology on $\text{Stab}(\mathcal{D})$ such that the forgetful morphism

$$Z: \text{Stab}(\mathcal{D}) \rightarrow \text{Hom}_Z(K_0(\mathcal{D}), \mathbb{C}), \quad (Z, \mathcal{P}) \mapsto Z$$

is a local homeomorphism. In particular, the space $\text{Stab}(\mathcal{D})$ has a structure of complex manifolds of dimension $\mu$. □

There are natural group actions on $\text{Stab}(\mathcal{D})$ commuting with each other. The first one is a $\mathbb{C}$-action

$$s \cdot (Z, \mathcal{P}) = (e^{-\pi \sqrt{-1}s} \cdot Z, \mathcal{P}_{\text{Re}(s)}), \quad s \in \mathbb{C},$$

where $\mathcal{P}_{\text{Re}(s)}(\phi) := \mathcal{P}(\phi + \text{Re}(s))$. The other action is given by the group of autoequivalences $\text{Aut}(\mathcal{D})$

$$\Phi \cdot (Z, \mathcal{P}) = (Z \circ F^{-1}, \Phi(\mathcal{P})), \quad \Phi \in \text{Aut}(\mathcal{D}).$$

For a heart $\mathcal{A}$ in $\mathcal{D}$, we can consider the subset $U(\mathcal{A}) \subset \text{Stab}(\mathcal{D})$ defined by

$$U(\mathcal{A}) := \{(Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}) \mid \mathcal{P}((0, 1]) = \mathcal{A}\}.$$

A heart $\mathcal{A}$ is said to be of finite length if $\mathcal{A}$ is artinian and noetherian. When the heart $\mathcal{A}$ is of finite length and has finitely many simple objects, any stability function $Z: K_0(\mathcal{A}) \rightarrow \mathbb{C}$ satisfies the Harder–Narasimhan property. Moreover, due to [B1, Proposition B.4], a stability function $Z: K_0(\mathcal{A}) \rightarrow \mathbb{C}$ on a heart $\mathcal{A}$ of finite length with finitely many simple objects satisfies the support property. Therefore, we have the following

**Proposition 2.14** ([B2, Lemma 5.2]). Let $\mathcal{A}$ be a heart of finite length in $\mathcal{D}$ with $\text{Sim} \mathcal{A} = \{S_1, \ldots, S_\mu\}$. Then, we have an isomorphism

$$U(\mathcal{A}) \xrightarrow{\cong} \mathbb{H}^\mu_-, \quad (Z, \mathcal{P}) \mapsto (Z(S_1), \ldots, Z(S_\mu)).$$

□
3. Main results

In this section, we study a relation between exceptional collections and stability conditions. The following proposition proved by Macrì associates full exceptional collections to stability conditions.

**Proposition 3.1** ([M1 Lemma 3.14 and Lemma 3.16]). Let \( \mathcal{E} = (E_1, \ldots, E_\mu) \) be a full Ext-exceptional collection in \( \mathcal{D} \). Then, the extension closed subcategory \( \langle \mathcal{E} \rangle_{\text{ex}} \) is a heart of finite length in \( \mathcal{D} \) and \( \text{Sim} \langle \mathcal{E} \rangle_{\text{ex}} = \{E_1, \ldots, E_\mu\} \). □

If \( \mathcal{E} \) is a full Ext-exceptional collection, then we can construct a stability condition \( \sigma = (Z, P) \) such that \( P((0,1]) = \langle \mathcal{E} \rangle_{\text{ex}} \) by Proposition 2.14. Due to Remark 2.6, we have the following

**Corollary 3.2** ([M1 Section 3.3]). For a full exceptional collection \( \mathcal{E} = (E_1, \ldots, E_\mu) \) in \( \mathcal{D} \), there exists a stability condition \( \sigma \) on \( \mathcal{D} \) such that \( E_i \) is \( \sigma \)-stable for each \( i = 1, \ldots, \mu \). □

Motivated by Macrì’s work, Dimitrov–Katzarkov introduced the notion of a \( \sigma \)-exceptional collection for a stability condition \( \sigma \).

**Definition 3.3** ([DK1 Definition 3.17]). Let \( \sigma = (Z, P) \in \text{Stab}(\mathcal{D}) \) be a stability condition on \( \mathcal{D} \). An exceptional collection \( \mathcal{E} = (E_1, \ldots, E_\mu) \) in \( \mathcal{D} \) is called \( \sigma \)-exceptional collection if the following three properties hold:

(i) For each \( i = 1, \ldots, \mu \), the object \( E_i \) is \( \sigma \)-semistable.
(ii) \( \mathcal{E} \) is an Ext-exceptional collection.
(iii) There exists a real number \( r \in \mathbb{R} \) such that \( r < \phi(E_i) \leq r + 1 \) for \( i = 1, \ldots, \mu \).

By definition, a full Ext-exceptional collection \( \mathcal{E} \) is a full \( \sigma \)-exceptional collection of a stability condition \( \sigma \) given in Proposition 3.1.

We shall prepare to state the main results. For an acyclic quiver \( Q \), we introduce the following two conditions:

(A1) For each \( i, j = 1, \ldots, \mu \), the number of arrows from \( i \) to \( j \) is less than one.
(A2) Let \( i, k, l = 1, \ldots, \mu \) such that \( k < i < l \). If there are arrows from \( k \) to \( i \) and from \( i \) to \( l \), then there are no arrows from \( k \) to \( l \).

Dynkin quivers and extended Dynkin quivers except \( A_{i_1}^{(1)} \) and \( A_{i_2}^{(1)} \) satisfy the above two conditions (A1) and (A2). The following theorem is the main result of this paper.
**Theorem 3.4.** Let $Q$ be an acyclic quiver satisfying the conditions (A1) and (A2), and $A$ a heart in $\mathcal{D}^b(Q)$. Assume that the heart $A$ is obtained from the standard heart by iterated simple tilts. Then, there exists a monochromatic full Ext-exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ such that $A = \langle \mathcal{E} \rangle_{\text{ex}}$ and $\text{Sim} \ A = \{E_1, \ldots, E_\mu\}$.

We prove Theorem 3.4 in Section 4. As a consequence of this theorem, we obtain the following result.

**Corollary 3.5.** Let $Q$ be an acyclic quiver satisfying the conditions (A1) and (A2), and $\sigma = (Z, \mathcal{P})$ a stability condition on $\mathcal{D}^b(Q)$. Assume that there exists $s \in \mathbb{C}$ such that the heart $\mathcal{P}_{\text{Re}(s)}((0,1])$ of the stability condition $s \cdot \sigma$ is obtained from the standard heart by iterated simple tilts. Then, there exists a monochromatic full $\sigma$-exceptional collection.

**Proof.** The statement follows from Theorem 3.4. \hfill \Box

In particular, when $Q$ is a Dynkin quiver $\vec{\Delta}$, we have the following

**Corollary 3.6.** Let $\vec{\Delta}$ be one of the Dynkin quivers. For each stability condition $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{D}^b(\vec{\Delta}))$, there exists a monochromatic full $\sigma$-exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$. Moreover, we can choose $\mathcal{E} = (E_1, \ldots, E_\mu)$ so that $\mathcal{P}((0,1]) = \langle \mathcal{E} \rangle_{\text{ex}}$ and $\text{Sim} \mathcal{P}((0,1]) = \{E_1, \ldots, E_\mu\}$.

**Proof.** Due to Proposition 2.9, any heart in $\mathcal{D}^b(\vec{\Delta})$ can be obtained from the standard heart by iterated simple tilts. Therefore, one can obtain the statement by Theorem 3.5. \hfill \Box

The corollary solves the conjecture given by Dimitrov–Katzarkov ([DK2, Conjecture 7.1]). As a corollary, any stability condition can be described by a full $\sigma$-exceptional collection and a stability function. Namely, we have the following

**Corollary 3.7** (cf. [W, Theorem 2.12]). We have

$$\text{Stab}(\mathcal{D}^b(\vec{\Delta})) \cong \bigcup \{U(\langle \mathcal{E} \rangle_{\text{ex}}) \mid \mathcal{E} \text{ is a monochromatic full Ext-exceptional collection}\}.$$  

**Proof.** The statement follows from Proposition 3.1 and Corollary 3.6. \hfill \Box

It is expected that the space of stability conditions $\text{Stab}(\mathcal{D}^b(\vec{\Delta}))$ can be identified with the universal unfolding (deformation) space of a polynomial $f: \mathbb{C}^3 \rightarrow \mathbb{C}$ of type ADE (cf. [E2, BQS, HKK, T]). We hope that Corollary 3.7 is useful to solve the problem.

In several cases of extended Dynkin quivers, the existence of full $\sigma$-exceptional collections is known.
Proposition 3.8 ([MT] Section 4] for $Q = A_{1,1}^{(1)}$, [DK1] Theorem 1.1] for $Q = A_{1,2}^{(1)}$). Let $Q$ be the affine $A_{1,1}^{(1)}$ quiver or the affine $A_{1,2}^{(1)}$ quiver. For each stability condition $\sigma$ on $\mathcal{D}^b(Q)$, there exists a full $\sigma$-exceptional collection. \hfill \Box

Remark 3.9. Okada classified hearts in $\mathcal{D}^b(A_{1,1}^{(1)})$ on which we can impose stability function with the Harder–Narasimhan property [O, Corollary 3.4]. Based on this result, one can check Conjecture 3.10 directly.

Motivated by Corollary 3.6 and Proposition 3.8, the following conjecture is expected.

Conjecture 3.10. Let $Q$ be one of the extended Dynkin quivers. For each stability condition $\sigma$ on $\mathcal{D}^b(Q)$, there exists $s \in \mathbb{C}$ such that the heart $\mathcal{P}((\text{Re}(s), \text{Re}(s) + 1))$ can be obtained from the standard heart by iterated simple tilts. In particular, any stability condition $\sigma$ on $\mathcal{D}^b(Q)$ admits a full $\sigma$-exceptional collection.

Remark 3.11. When $Q$ is the affine $A_{p,q}^{(1)}$ quiver, Haiden–Katzarkov–Kontsevich proved that $\text{Stab}(\mathcal{D}^b(Q))$ is identified with the moduli space of framed exponential type quadratic differentials on annulus with marked points and a grading [HKK, Theorem 6.2]. This result implies that any stability condition has a heart of finite length with finitely many simple objects up to the $\mathbb{C}$-action. Hence, in order to prove Conjecture 3.10 for the affine $A_{p,q}^{(1)}$ quiver, it is sufficient to show that any heart of finite length with finitely many simple objects can be obtained from the standard heart by iterated simple tilts.

4. Proof of Theorem 3.4

The sketch of the proof is as follows. Firstly, we show the statement in the case of the standard heart $\mathcal{A} = \text{mod}(\mathbb{C}Q)$ (Proposition 4.1). Secondly, we show the general case. We prove that a monochromatic full Ext-exceptional collection in a heart induces a new one in the simple tilted heart (Proposition 4.2).

For a monochromatic full Ext-exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$, it will be convenient to write $p_{i,j} = \deg \text{Hom}^*_D(E_i, E_j)$ if $\text{Hom}^*_D(E_i, E_j) \neq 0$. In order to show the existence of a full $\sigma$-exceptional collection for a given stability condition $\sigma$ on $\mathcal{D}$, we introduce the following two conditions (E1) and (E2) for monochromatic full Ext-exceptional collections:

(E1) For each $i, j = 1, \ldots, \mu$, we have $\dim_{\mathbb{C}} \bigoplus_{p \in \mathbb{Z}} \text{Hom}_D^p(E_i, E_j) \leq 1$. 
There exists a monochromatic full Ext-exceptional collection $E$.

Proposition 4.1. Let $Q$ be an acyclic quiver satisfying the condition (A1) and (A2). There exists a monochromatic full Ext-exceptional collection $\mathcal{E} = (E_1, \ldots, E_{\mu})$ in $\mathcal{D}^{\mathbb{C}}(Q)$ such that $\text{Sim} \mod(\mathbb{C}Q) = \{E_1, \ldots, E_{\mu}\}$ and $\mathcal{E}$ satisfies the conditions (E1) and (E2).

Proof. The simple $\mathbb{C}Q$-module $S_i$ corresponding to the vertex $i$ is an exceptional object in $\mathcal{D}$. It is known that the exceptional collection $\mathcal{E} := (S_1, \ldots, S_{\mu})$ consisting of simple $\mathbb{C}Q$-modules in the standard heart $\mod(\mathbb{C}Q)$ is full. Moreover, since $\dim \Ext^1_{\mathbb{C}Q}(S_i, S_j)$ is equal to the number of arrows from the vertex $i$ to $j$, the conditions (A1) and (A2) imply (E1) and (E2), respectively. □

Let $\mathcal{E} = (E_1, \ldots, E_{\mu})$ a monochromatic full Ext-exceptional collection in $\mathcal{D}$. Fix $i = 1, \ldots, \mu$. We can assume, by reordering $\mathcal{E}$ if necessary, that if $\Hom^\bullet_\mathcal{D}(E_j, E_i) \not\cong 0$ and $\Hom^\bullet_\mathcal{D}(E_j, E_i) \not\cong 0$ for $j < j' < i$ then we have $p_{j',i} = 1$. Indeed, if $\Hom^1_\mathcal{D}(E_j, E_i) \not\cong 0$, $\Hom^\bullet_\mathcal{D}(E_j, E_i) \not\cong 0$ and $p_{j',i} \geq 2$, then the condition (E2) implies $\Hom^\bullet_\mathcal{D}(E_j, E_j) \cong 0$. Define two indices $i^\sharp$ and $i^\flat$ by
\[
i^\sharp := \min\{j \in \{1, \ldots, i\} \mid \Hom^\bullet_\mathcal{D}(E_j, E_i) \not\cong 0, p_{j,i} \leq 1\},
\]
\[
i^\flat := \max\{j \in \{i, \ldots, \mu\} \mid \Hom^\bullet_\mathcal{D}(E_i, E_j) \not\cong 0, p_{i,j} \leq 1\}.
\]
We also define the ordered sets $\mathcal{E}_i^{\sharp}$ and $\mathcal{E}_i^{\flat}$ by
\[
\mathcal{E}_i^{\sharp} := (E_1, \ldots, E_{i-1}, E_i[1], \psi_{E_i}(E_{i+1}), \ldots, \psi_{E_i}(E_{i-1}), E_{i+1}, \ldots, E_{\mu}),
\]
\[
\mathcal{E}_i^{\flat} := (E_1, \ldots, E_{i-1}, \psi_{E_i}(E_{i+1}), \ldots, \psi_{E_i}(E_{i-1}), E_i[1], E_{i+1}, \ldots, E_{\mu}),
\]
where $\psi_{E_i}(E_j)$ and $\psi_{E_i}(E_j)$ are objects defined in (2.1).

Proposition 4.2. Let $Q$ be an acyclic quiver and $A$ a heart in $\mathcal{D} = \mathcal{D}^{\mathbb{C}}(Q)$. Suppose that there exists a monochromatic full Ext-exceptional collection $\mathcal{E} = (E_1, \ldots, E_{\mu})$ such that $A = \langle \mathcal{E} \rangle_{\text{ex}}$, $\text{Sim} A = \{E_1, \ldots, E_{\mu}\}$ and $\mathcal{E}$ satisfies the conditions (E1) and (E2). Then, we have the following:

(i) For each $i = 1, \ldots, \mu$, the ordered set $\mathcal{E}_i^{\sharp}$ is a monochromatic full Ext-exceptional collection and satisfies the conditions (E1) and (E2). Moreover, we have $A_{E_i}^{\sharp} = \langle \mathcal{E}_i^{\sharp} \rangle_{\text{ex}}$ and
\[
\text{Sim} A_{E_i}^{\sharp} = \{E_1, \ldots, E_{i-1}, E_i[1], \psi_{E_i}(E_{i+1}), \ldots, \psi_{E_i}(E_{i-1}), E_{i+1}, \ldots, E_{\mu}\}. \tag{4.1a}
\]
(ii) For each \( i = 1, \ldots, \mu \), the ordered set \( \mathcal{E}_i^* \) is a monochromatic full Ext-exceptional collection and satisfies the conditions \((E1)\) and \((E2)\). Moreover, we have \( \mathcal{A}_{\mathcal{E}_i}^* = \langle \mathcal{E}_i^* \rangle_{\text{ex}} \) and

\[
\text{Sim} \mathcal{A}_{\mathcal{E}_i}^* = \{ E_1, \ldots, E_{i-1}, \psi_{E_i}^* (E_{i+1}), \ldots, \psi_{E_i}^* (E_\mu), E_i[-1], E_{\mu+1}, \ldots, E_\mu \}. \tag{4.1b}
\]

**Proof.** We only prove the first statement (i). However, a very similar proof works for the second statement (ii). By assumption, for each \( j = i^*, \ldots, i - 1 \) we have

\[
\text{Hom}_D^\bullet (E_j, E_i)^* \otimes E_i \cong (\text{Hom}_D^1 (E_j, E_i)[-1])^* \otimes E_i \cong \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[-1].
\]

This identification yields an isomorphism \( \psi_{E_i}^* (E_j) \cong \mathcal{R}_{E_i} E_j \) for \( j = i^*, \ldots, i - 1 \). Proposition 2.5 implies that the ordered set \( \mathcal{E}_i^* \) forms a full exceptional collection.

In order to see that the full exceptional collection \( \mathcal{E}_i^* \) is monochromatic and Ext, we split the proof into four lemmas:

**Lemma 4.3.** Let \( k = 1, \ldots, i^* - 1 \) and \( j = i^*, \ldots, i - 1 \).

(i) If \( \text{Hom}_D^1 (E_j, E_i) \not\cong 0 \) or \( \text{Hom}_D^\bullet (E_k, E_i) \not\cong 0 \), then we have

\[
\text{Hom}_D^\bullet (E_k, \psi_{E_i}^* (E_j)) \cong \text{Hom}_D^\bullet (E_k, E_j). \tag{4.2a}
\]

(ii) If \( \text{Hom}_D^1 (E_j, E_i) \not\cong 0, \) \( \text{Hom}_D^\bullet (E_k, E_i) \not\cong 0 \) and \( \text{Hom}_D^\bullet (E_k, E_j) \cong 0 \), then we have

\[
\text{Hom}_D^p (E_k, \psi_{E_i}^* (E_j)) = \begin{cases} \text{Hom}_D^1 (E_j, E_i)^* \otimes \text{Hom}_D^p (E_k, E_i), & p = p_{k,i}, \\ 0, & p \neq p_{k,i}. \end{cases} \tag{4.2b}
\]

(iii) If \( \text{Hom}_D^1 (E_j, E_i) \not\cong 0, \) \( \text{Hom}_D^\bullet (E_k, E_i) \not\cong 0 \) and \( \text{Hom}_D^\bullet (E_k, E_j) \not\cong 0 \), then we have

\[
\text{Hom}_D^\bullet (E_k, \psi_{E_i}^* (E_j)) \cong 0. \tag{4.2c}
\]

**Proof.** If \( \text{Hom}_D^1 (E_j, E_i) \not\cong 0 \), the exact triangle \( \psi_{E_i}^* (E_j) \to E_j \to \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[1] \) implies \( \psi_{E_i}^* (E_j) \cong E_j \). Hence, we have \( (4.2a) \).

We assume \( \text{Hom}_D^1 (E_j, E_i) \not\cong 0 \). By the exact triangle \( \psi_{E_i}^* (E_j) \to E_j \to \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[1] \), we have the long exact sequence

\[
\cdots \to \text{Hom}_D^{p-1} (E_k, E_j) \xrightarrow{\delta_{p-1}} \text{Hom}_D^{p-1} (E_k, \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[1]) \to \\
\to \text{Hom}_D^p (E_k, \psi_{E_i}^* (E_j)) \to \text{Hom}_D^p (E_k, E_j) \xrightarrow{\delta_p} \text{Hom}_D^p (E_k, \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[1]) \to \\
\to \text{Hom}_D^{p+1} (E_k, \psi_{E_i}^* (E_j)) \to \text{Hom}_D^{p+1} (E_k, E_j) \xrightarrow{\delta_{p+1}} \cdots,
\]

where \( \delta_p : \text{Hom}_D^p (E_k, E_j) \to \text{Hom}_D^p (E_k, \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[1]) \) denotes the \( \mathbb{C} \)-linear map induced by the morphism \( E_j \to \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[1] \). Note that we have

\[
\text{Hom}_D^p (E_k, \text{Hom}_D^1 (E_j, E_i)^* \otimes E_i[1]) \cong \text{Hom}_D^1 (E_j, E_i)^* \otimes \text{Hom}_D^{p+1} (E_k, E_i).
\]
Hence, if we have $\text{Hom}_D^\bullet(E_k, E_i) \cong 0$, the long exact sequence implies the isomorphism (4.2a).

We assume $\text{Hom}_D^\bullet(E_k, E_i) \not\cong 0$. If $\text{Hom}_D^\bullet(E_k, E_j) \cong 0$, then the long exact sequence implies (4.2b).

Next, we consider the last case $\text{Hom}_D^\bullet(E_k, E_j) \not\cong 0$. Since the condition (E2) implies $p_{k,j} = p_{k,i} - 1$, the $\mathbb{C}$-linear map $\delta_p$ is nontrivial for $p = p_{k,j}$. By the condition (E1), the $\mathbb{C}$-linear map $\delta_p$ can be identified with the map $\delta_p: \mathbb{C} \to \mathbb{C}$. If $\delta_p$ is the zero map, we have $\text{Hom}_D^p(E_k, \psi_{E_i}^j(E_j)) \cong \mathbb{C}$ and $\text{Hom}_D^{p+1}(E_k, \psi_{E_i}^j(E_j)) \cong \mathbb{C}$. However, it contradicts Proposition 2.8. Therefore, the $\mathbb{C}$-linear map $\delta_p: \mathbb{C} \to \mathbb{C}$ is an isomorphism and we obtain (4.2c).

Lemma 4.4. Let $j = i^*, \ldots, i - 1$. We have

$$
\text{Hom}_D^p(E_i[1], \psi_{E_i}^j(E_j)) \cong \begin{cases} 
\text{Hom}_D^1(E_j, E_i)^*, & p = 1 \\
0, & p \neq 1.
\end{cases}
$$

Proof. Since $\psi_{E_i}^j(E_j) \cong \mathcal{R}_{E_i}E_j$, the statement is known. However, we give a proof here.

By the exact triangle $\psi_{E_i}^j(E_j) \to E_j \to \text{Hom}_D(E_j, E_i)^* \otimes E_i[1]$, we have the long exact sequence

$$
\cdots \to \text{Hom}_D^{p-1}(E_i[1], \text{Hom}_D^1(E_j, E_i)^* \otimes E_i[1]) \to \text{Hom}_D^p(E_i[1], \psi_{E_i}^j(E_j)) \to \text{Hom}_D^p(E_i[1], E_j) \to \text{Hom}_D^p(E_i[1], \text{Hom}_D(E_j, E_i)^* \otimes E_i[1]) \to \text{Hom}_D^{p+1}(E_i[1], \psi_{E_i}^j(E_j)) \to \cdots.
$$

Since $\text{Hom}_D^\bullet(E_i, E_j) \cong 0$, for each $p \in \mathbb{Z}$ we have

$$
\text{Hom}_D^p(E_i[1], \psi_{E_i}^j(E_j)) \cong \text{Hom}_D^{p-1}(E_i[1], \text{Hom}_D^1(E_j, E_i)^* \otimes E_i[1])
\cong \text{Hom}_D^{p-1}(E_i, \text{Hom}_D^1(E_j, E_i)^* \otimes E_i)
\cong \text{Hom}_D^1(E_j, E_i)^* \otimes \text{Hom}_D^{p-1}(E_i, E_i).
$$

We obtain the statement since $\text{Hom}_D^\bullet(E_i, E_i) \cong \mathbb{C}$.

Lemma 4.5. Let $j, j' = i^*, \ldots, i - 1$ such that $j < j'$. We have

$$
\text{Hom}_D^\bullet(\psi_{E_i}^j(E_j), \psi_{E_i}^{j'}(E_{j'})) \cong \text{Hom}_D^\bullet(E_j, E_{j'}).$$

Proof. Since $\psi_{E_i}^j(E_j) \cong \mathcal{R}_{E_i}E_j$, the statement is also known. However, we give a proof here.
By the exact triangle \( \psi_{E_i}^*(E_j') \to E_j \to \text{Hom}^1_D(E_j', E_i)^* \otimes E_i[1] \), we have the long exact sequence

\[
\cdots \to \text{Hom}^{p-1}_D(\psi_{E_i}^*(E_j), \text{Hom}^1_D(E_j', E_i)^* \otimes E_i[1]) \to \\
\to \text{Hom}^p_D(\psi_{E_i}^*(E_j), \psi_{E_i}^*(E_j')) \to \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_j') \to \text{Hom}^p_D(\psi_{E_i}^*(E_j), \text{Hom}^1_D(E_j', E_i)^* \otimes E_i[1]) \to \\
\to \text{Hom}^{p+1}_D(\psi_{E_i}^*(E_j), \psi_{E_i}^*(E_j')) \to \cdots.
\]

The identification \( \psi_{E_i}^*(E_j) \cong \mathcal{R}_{E_i}E_j \) implies that \( \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_i) \cong 0 \). Hence, we have

\[
\text{Hom}^p_D(\psi_{E_i}^*(E_j), E_i) \cong \text{Hom}^1_D(E_j', E_i)^* \otimes \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_i[1]) \cong 0.
\]

Therefore, there exists an isomorphism \( \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_i) \cong \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_j') \).

Next, we see \( \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_j') \cong \text{Hom}^p_D(E_j, E_j') \). By \( \text{Hom}^p_D(E_i, E_j') \cong 0 \), the long exact sequence

\[
\cdots \to \text{Hom}^{p-1}_D(\psi_{E_i}^*(E_j), E_j') \to \\
\to \text{Hom}^p_D(\text{Hom}^1_D(E_j', E_i)^* \otimes E_i[1], E_j') \to \text{Hom}^p_D(E_j, E_j') \to \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_j') \to \\
\to \text{Hom}^{p+1}_D(\text{Hom}^1_D(E_j, E_i)^* \otimes E_i[1], E_j') \to \cdots.
\]

yields an isomorphism \( \text{Hom}^p_D(\psi_{E_i}^*(E_j), E_j') \cong \text{Hom}^p_D(E_j, E_j') \).

\[\square\]

**Lemma 4.6.** Let \( j = i, \ldots, i - 1 \) and \( l = i + 1, \ldots, \mu \).

(i) If \( \text{Hom}^1_D(E_j, E_i) \cong 0 \) or \( \text{Hom}^p_D(E_i, E_i) \cong 0 \), then we have

\[
\text{Hom}^p_D(\psi_{E_i}^*(E_j), E_i) \cong \text{Hom}^p_D(E_j, E_i). \quad (4.3a)
\]

(ii) If \( \text{Hom}^1_D(E_j, E_i) \not\cong 0 \), \( \text{Hom}^p_D(E_i, E_i) \not\cong 0 \) and \( \text{Hom}^p_D(E_j, E_i) \cong 0 \), then we have

\[
\text{Hom}^p_D(\psi_{E_i}^*(E_j), E_i) \cong \begin{cases} 
\text{Hom}^1_D(E_j, E_i) \otimes \text{Hom}^p_D(E_i, E_i), & p = p_{i,l}, \\
0, & p \neq p_{i,l}.
\end{cases} \quad (4.3b)
\]

(iii) If \( \text{Hom}^1_D(E_j, E_i) \not\cong 0 \), \( \text{Hom}^p_D(E_i, E_i) \not\cong 0 \) and \( \text{Hom}^p_D(E_j, E_i) \not\cong 0 \), then we have

\[
\text{Hom}^p_D(\psi_{E_i}^*(E_j), E_i) \cong 0. \quad (4.3c)
\]

**Proof.** If \( \text{Hom}^1_D(E_j, E_i) \cong 0 \), the exact triangle \( \psi_{E_i}^*(E_j) \to E_j \to \text{Hom}^1_D(E_j, E_i)^* \otimes E_i[1] \) implies \( \psi_{E_i}^*(E_j) \cong E_j \). Hence, we have (4.3a).
We assume $\text{Hom}_D^p(E_j, E_i) \not\equiv 0$. By the exact triangle $\psi_{E_i}^*(E_j) \to E_j \to \text{Hom}_D^p(E_j, E_i)^* \otimes E_i[1]$, we have the long exact sequence

$$\cdots \to \text{Hom}_D^{p-1}(E_j, E_i) \to \text{Hom}_D^{p-1}(\psi_{E_i}^*(E_j), E_i) \to \text{Hom}_D^p(\psi_{E_i}^*(E_j), E_i) \to \text{Hom}_D^p(\psi_{E_i}^*(E_j), E_i) \to \text{Hom}_D^{p+1}(E_j, E_i) \to \cdots,$$

where $\epsilon_p: \text{Hom}_D^p(\psi_{E_i}^*(E_j), E_i) \to \text{Hom}_D^p(E_j, E_i)$ denotes the $\mathbb{C}$-linear map induced by the morphism $E_j \to \text{Hom}_D^1(E_j, E_i)^* \otimes E_i[1]$. Note that we have

$$\text{Hom}_D^p(\text{Hom}_D^1(E_j, E_i)^* \otimes E_i[1], E_i) \cong \text{Hom}_D^1(E_j, E_i) \otimes \text{Hom}_D^{p-1}(E_i, E_i).$$

Hence, if we have $\text{Hom}_D^p(E_i, E_i) \cong 0$, the long exact sequence implies the isomorphism (4.3a).

We assume $\text{Hom}_D^p(E_i, E_i) \not\equiv 0$. If $\text{Hom}_D^p(E_i, E_i) \cong 0$, then the long exact sequence implies (4.3b).

Next, we consider the last case $\text{Hom}_D^p(E_i, E_i) \not\equiv 0$. Since the condition (E2) implies $p_{j,l} = p_{k,l} + 1$, the $\mathbb{C}$-linear map $\epsilon_p$ is nontrivial for $p = p_{j,l}$. By the condition (E1), the $\mathbb{C}$-linear map $\epsilon_p$ can be identified with the map $\epsilon_p: \mathbb{C} \to \mathbb{C}$. We assume that $\epsilon_p$ is the zero map. Then, by the long exact sequence, we have $\text{Hom}_D^p(\psi_{E_i}^*(E_j), E_i) \cong \mathbb{C}$ and $\text{Hom}_D^{p-1}(\psi_{E_i}^*(E_j), E_i) \cong \mathbb{C}$. However, this fact contradicts Proposition 2.8. Therefore, the $\mathbb{C}$-linear map $\epsilon_p: \mathbb{C} \to \mathbb{C}$ is an isomorphism and we obtain (4.3c). □

Since the full exceptional collection $\mathcal{E}$ is Ext, we have

$$\text{Hom}_D^p(E_i, E_i[1]) \cong \begin{cases} \text{Hom}_D^p(E_k, E_i), & p = p_{k,i} - 1, \\ 0, & p \neq p_{k,i} - 1, \end{cases}$$

for $k = 1, \ldots, i^2 - 1$ and

$$\text{Hom}_D^p(E_i[1], E_i) \cong \begin{cases} \text{Hom}_D^p(E_i, E_i), & p = p_{l,i} + 1, \\ 0, & p \neq p_{l,i} + 1, \end{cases}$$

for $l = i + 1, \ldots, \mu$. Note $p_{k,i} \geq 2$ by definition. Therefore, the full exceptional collection $\mathcal{E}_i^\sharp$ is monochromatic and Ext.

Next, we see that the full exceptional collection $\mathcal{E}_i^\sharp$ satisfies the conditions (E1) and (E2). Lemma 4.4, 4.3, 4.6 and 4.5 imply that $\mathcal{E}_i^\sharp$ satisfies the condition (E1).

**Lemma 4.7.** The monochromatic full Ext-exceptional collection $\mathcal{E}_i^\sharp$ satisfies the condition (E2).
Full Exceptional Collections and Stability Conditions for Dynkin Quivers

Proof. For simplicity, we put $E^x_i = (E^x_i, \ldots, E^x_i)$ and $p^x_{a,b} = \deg \text{Hom}^\bullet_D(E^x_i, E^x_b)$. By Lemma 4.4, 4.3, 4.6 and 4.5 we have $\text{Hom}^\bullet_D(E^x_i, E^x_i) \neq 0$, $\text{Hom}^\bullet_D(E^x_b, E^x_i) \neq 0$ and $\text{Hom}^\bullet_D(E^x_b, E^x_b) \neq 0$ for $a < b < c$ if and only if the tuple $(E^x_a, E^x_b, E^x_c)$ is one of the following:

(i) $(E^x_a, E^x_b, E^x_c) = (E_a, E_i[1], \psi_{E_i}(E_{c-1}))$ satisfying

$$\text{Hom}^\bullet_D(E_a, E_i) \neq 0, \quad \text{Hom}^\bullet_D(E_{c-1}, E_i) \neq 0, \quad \text{Hom}^\bullet_D(E_a, E_{c-1}) \cong 0.$$ 

(ii) $(E^x_a, E^x_b, E^x_c) = (E_i[1], \psi_{E_i}(E_{b-1}), E_c)$ satisfying

$$\text{Hom}^\bullet_D(E_i, E_{b-1}) \neq 0, \quad \text{Hom}^\bullet_D(E_i, E_c) \neq 0, \quad \text{Hom}^\bullet_D(E_{b-1}, E_c) \cong 0.$$ 

(iii) $(E^x_a, E^x_b, E^x_c) = (E_a, E_i[1], E_c)$ satisfying

$$\text{Hom}^\bullet_D(E_a, E_i) \neq 0, \quad \text{Hom}^\bullet_D(E_i, E_c) \neq 0, \quad \text{Hom}^\bullet_D(E_a, E_c) \neq 0.$$ 

(iv) $(E^x_a, E^x_b, E^x_c) = (E_a, E_b, E_c)$ satisfying

$$\text{Hom}^\bullet_D(E_a, E_b) \neq 0, \quad \text{Hom}^\bullet_D(E_b, E_c) \neq 0, \quad \text{Hom}^\bullet_D(E_a, E_c) \neq 0.$$ 

In the first case (i), by Lemma 4.3 we have

$$p^x_{a,b} = \deg \text{Hom}^\bullet_D(E_a, E_i[1]) = p_{a,i} - 1,$$

$$p^x_{b,c} = \deg \text{Hom}^\bullet_D(E_i[1], \psi_{E_i}(E_{c-1})) = 1,$$

$$p^x_{a,c} = \deg \text{Hom}^\bullet_D(E_a, \psi_{E_i}(E_{c-1})) = p_{a,i}.$$ 

In the second case (ii), by Lemma 4.6 we have

$$p^x_{a,b} = \deg \text{Hom}^\bullet_D(E_i[1], \psi_{E_i}(E_{b-1})) = 1,$$

$$p^x_{b,c} = \deg \text{Hom}^\bullet_D(\psi_{E_i}(E_{b-1}), E_c) = p_{i,c},$$

$$p^x_{a,c} = \deg \text{Hom}^\bullet_D(E_i[1], E_c) = p_{i,c} + 1.$$ 

In the third case (iii), we have

$$p^x_{a,b} = \deg \text{Hom}^\bullet_D(E_a, E_i[1]) = p_{a,i} - 1,$$

$$p^x_{b,c} = \deg \text{Hom}^\bullet_D(E_i[1], E_c) = p_{i,c} + 1,$$

$$p^x_{a,c} = \deg \text{Hom}^\bullet_D(E_a, E_c) = p_{a,c}.$$ 

In the last case (iv), we have

$$p^x_{a,b} = \deg \text{Hom}^\bullet_D(E_a, E_b) = p_{a,b},$$

$$p^x_{b,c} = \deg \text{Hom}^\bullet_D(E_b, E_c) = p_{b,c},$$

$$p^x_{a,c} = \deg \text{Hom}^\bullet_D(E_a, E_c) = p_{a,c}.$$
Therefore, we obtain $p_{a,b}^\sharp + p_{b,c}^\sharp = p_{a,c}^\sharp$ by direct calculations.

The last statements $\mathcal{A}_{E_i}^\sharp \cong \langle \mathcal{E}_i \rangle_{\text{ex}}$ and (4.1a) follows from Proposition 2.7. We complete the proof of Proposition 4.2.

By Proposition 4.1 and 4.2 we obtain the statement of Theorem 3.4.

References

[BM] A. Bayer, E. Macrì, The space of stability conditions on the local projective plane, Duke Math. J. 160 (2011), no. 2, 263–322.

[BP] A. Bondal and A. Polishchuk, Homological Properties of Associative Algebras: The Method of Helices, Izv. RAN. Ser. Mat., 1993, Volume 57, Issue 2, 3-50 (Mi izv877).

[B1] T. Bridgeland, Stability conditions on triangulated categories, Ann. of Math. (2), 166 (2) : 317-345, 2007.

[B2] T. Bridgeland, Spaces of stability conditions, Algebraic geometry-Seattle 2005. Part 1, 1–21, Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI, 2009.

[BQS] T. Bridgeland, Y. Qiu and T. Sutherland, Stability conditions and $A_2$-quiver, Advances in Mathematics, Volume 365, 13 May 2020, 107049.

[BBD] A. Beilinson, J. Bernstein and P. Deligne, Faisceaux Pervers, Astérisque 100, Soc. Math de France, Paris (1983).

[DK1] G. Dimitrov and L. Katzarkov, Non-semistable exceptional objects in hereditary categories, Int. Math. Res. Not. IMRN 2016, no. 20, 6293–6377.

[DK2] G. Dimitrov and L. Katzarkov, Non-semistable exceptional objects in hereditary categories: some remarks and conjectures, Stacks and categories in geometry, topology, and algebra, 263–287, Contemp. Math., 643, Amer. Math. Soc., Providence, RI, 2015.

[DK3] G. Dimitrov and L. Katzarkov, Bridgeland stability conditions on the acyclic triangular quiver, Adv. Math. 288 (2016), 825–886.

[DK4] G. Dimitrov and L. Katzarkov, Bridgeland stability conditions on wild Kronecker quivers, Adv. Math. 352 (2019), 27–55.

[HKK] F. Haiden, L. Katzarkov, M. Kontsevich, Flat surfaces and stability structures, Publ. Math. Inst. Hautes Études Sci. 126 (2017), 247–318.

[KV] B. Keller and D. Vossieck, Aisles in derived categories, Bull. Soc. Math. Belg. 40 (1988), 239-253.

[KQ] A. King and Y. Qiu, Exchange graphs and Ext quivers, Adv. Math. 285 (2015), 1106–1154.

[L] C. Li, The space of stability conditions on the projective plane, Selecta Math. (N.S.) 23 (2017), no. 4, 2927–2945.

[M1] E. Macrì, Stability conditions on curves, Math. Res. Lett. 14 (2007), no. 4, 657–672.

[M2] E. Macrì, Some examples of spaces of stability conditions on derived categories, arXiv:math/0411613.
[O] S. Okada, *Stability manifold of $\mathbb{P}^1$*, J. Algebraic Geom. **15** (2006), no. 3, 487–505.

[Q] Y. Qiu, *Stability conditions and quantum dilogarithm identities for Dynkin quivers*, Adv. Math. **269** (2015), 220–264.

[T] A. Takahashi, *Matrix Factorizations and Representations of Quivers I*, arXiv:math/0506347.

[W] J. Woolf, *Stability conditions, torsion theories and tilting*, J. Lond. Math. Soc. (2) **82** (2010), no. 3, 663–682.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan

*Email address: u930458f@ecs.osaka-u.ac.jp*