Facilitation and Internalization Optimal Strategy in a Multilateral Trading Context

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Abstract. This paper studies four trading algorithms of a professional trader at a multilateral trading facility, either internalizing or regular, observing a realistic two-sided limit order book whose dynamics are driven by the order book events. We shall show that the price switching algorithms provide lower and upper bounds of the mixed trading algorithms. The optimal price switching strategy exists and is expressed in terms of the value function. A parallelizable algorithm to numerically compute the value function and optimal price switching strategy for the discretized state process is provided.

Keywords and Phrases: Limit order book, algorithmic trading, stochastic impulse and optimal control, parallel computation, Markets in Financial Instruments Directive.

1 Introduction

1.1 Overview

Market microstructure is an interdisciplinary field involving economics, finance, probability and optimization, statistics, and even psychology, which studies the order-driven price formation processes in markets like those of stocks, futures and foreign exchanges. Due to the complexity of the phenomena, the research works on market microstructure usually focus on individual aspects of the problem. Interesting questions studied so far include econometrics of the order books and of the market maker’s inventory levels, optimal market making, a buyer or seller’s optimal order execution, and limiting behaviors of the queuing system of limit orders and bid and ask prices.

The study of market microstructure dates back to at least four decades ago, and persists up till present time. It is hard to enumerate all the literature on this field. The books O’Hara (1997) and Hasbrouck (2007) provide an overview of quantitative analysis of market microstructure. One significant development in recent years is the prevalence of electronic trading platforms as an alternative to markets where prices are determined via a market maker’s auction and the traders’ bidding; the other is the popularity of applying stochastic control to solving optimal execution and optimal market making problems. Readers are welcome to Lehalle and Laruelle (2013) for latest updates in the field of market microstructure and algorithmic trading.

Stochastic control provides the theory and methodologies to find actions that optimize an
objective, while the actions can influence the evolution of some random processes to which
the objective is associated. It naturally facilitates the study of financial markets where par-

cipants, the assembly of whose activities contribute to the price evolution, seek to maximize
profits and minimize losses. The application of stochastic control to optimizing activities in
an order book traces back to early works like Ho and Stoll (1981).

There have been many frameworks to study trading and order execution in limit order
books. Among them are the equilibrium models surveyed in Parlour and Seppi (2008), the
model with stochastic bid and ask prices and deterministic order book shape as in Alfonsi,
Schied and coauthors (2009, 2010, 2012) and in Predoiu, Shaikhet and Shreve (2011), the
model with stochastic mid price and deterministic or stochastic spread as in Avellaneda and
Stoikov (2008) and Guilbaud and Pham (2013), the Almgren-Chriss model used by many in
the industry, as in Almgren (2003), Almgren and Chriss (2000), Bouchard, Dang and Lehalle
(2011) and Gatheral and Schied (2011), and maximizing the utility by choosing an optimal
posting distance that determines the intensity of the execution process as in Guéant, Lehalle
and Fernandez-Tapia (2012a, 2012b) and Laruelle, Lehalle and Pagès (2011).

1.2 This paper

An optimal trading scheme is obviously a function of the trading constraints of the trading
agent, translated into its reward function. To express the adequate trading function, a spe-
cific market model is often needed. Up to now, two main agent types have been investigated
- directional traders and market makers.

* Directional traders: such agents already took the decision to buy or sell and the amount
of shares to buy or sell before the trading phase. Typically institutional investors like pension
funds are of this kind. The control of associated trading schemes is usually the local trading
rate, and more rarely a price (Guéant, Lehalle and Fernandez-Tapia, 2012a, 2012b). The
associated market models include classical price diffusion and a market impact component.

* Market makers: on the opposite, such agents make decisions in real time, 100% based
on the state of the order books; they are simultaneously buyers and sellers, mostly providing
liquidity to other traders. The part of high frequency traders often seen as the “new middle-
men” are of this kind (Baron, Brogaard and Kirilenko, 2012; Jovanovic and Menkveld, 2011;
Menkveld, 2013). Their control is the buying and selling prices, at which they send limit
orders around the best bid and best ask immediate prices. The associated market models
usually embed trading flows abstracted by a point process, without any market impact com-
ponent, since the nature of the market impact of limit orders has not been explored by now.

This paper models a third kind of agents: the risk taking intermediaries. The “system-
atic internalizers” defined by the European regulation are of this kind. Any investment
bank having the capabilities

(1) to internalize some of its flow against a price improvement for his external or internal
clients;
(2) to get rid of its potential inventory imbalance, like in a dark pool, in a dedicated
trading pool at the instantaneous mid price.
The controls will be the number of shares bought and sold up-to-date at every price level in
the displayed order book and in the dedicated trading pool. Hence the modeled order book
dynamics will have to embed full order book depth.

Our study will be presented as follows. Section 2 introduces the event-driven order book
dynamics. Sections 3 and 4 formulate the stochastic control problems faced by the optimal
trader and prove their well-posedness. Section 5 compares the best expected profits of a
regular trader and a systemic internalizer, either can use mixed strategies or price switching
strategies. Section 6 solves the optimal price switching problem by providing a representation of the optimal strategy, a discrete-time numerical algorithm and implementation in a Binomial model. Finally, we suggest a way to calculate a “fair” internalization premium.

The contribution of this study is multi-fold.

(1) The agent that conducts the trading activities is a risk taking intermediary. Such agents make up a significant proportion of the market participants in terms of the capital amount, but there has not yet been much research into their optimal trading strategies.

(2) One recent development in market microstructure is the event-driven limit order book models, by Rama Cont and co-authors and by Hasbrouck and Saar (2010). Especially when the trader reacts at a super speed (called “high frequency trading”), this kind of models captures the real observations, because the Central Limit Theorem that proves a diffusion-like stock price no longer applies. This paper is the first one that derives optimal trading strategies in a variation of their cutting-edge models.

(3) The optimal trading strategy will balance between the speed and cost of trading, by active orders in the book and passive orders at the mid price in the dark pool. There is another kind of strategies more passive than hidden orders, which is orders queuing up at the best available prices. Interested readers are invited to Huang, Lehalle and Rosenbaum (2013) for an empirical analysis and Lachapelle, Lasry, Lehalle and Lions (2013) for a mean field game modeling of an agent’s optimal queuing.

(4) The optimal price switching problem we shall solve belongs to the classical type of impulse control and optimal control, but its state process is non-standard, more complicated than a textbook SDE driven by Brownian motions and Poisson random measures.

(5) A parallelizable algorithm is provided for numerically computing the value function and the optimal price switching strategy for a discretized state process. The computational complexity of a stochastic control problem using backward induction should have been well known on a serial computer, while to the author’s best knowledge this paper is the first one to document the complexity on a parallel computer.

(6) The results in this paper give insights into trading activities within the Markets in Financial Instruments Directive framework, by different types of traders using different types of trading strategies.

2 The two-sided order book dynamics

As usual in optimal trading (Alfonsi and Schied, 2010, 2012; Bertsimas and Lo, 1998; Bouchard, Dang and Lehalle, 2011; Guéant, Lehalle and Fernandez-Tapia, 2012a, 2012b), the market dynamics are modeled on their own and do not specifically react to the optimal trader’s actions. We will not model any explicit market impact, following usual frameworks allowing the optimal trader to post limit orders (Avellaneda and Stoikov, 2008) as opposite to framework for optimal trading with aggressive orders (Obizhaeva and Wang, 2013) or at a larger time scale than the orderbook one (Almgren and Chriss, 2000). Our optimal trader is a “systematic internalizer” in the MiFID (Markets in Financial Instruments Directive) sense: as an intermediary or a dedicated market maker, he can capture market order flow provided that he pays a premium (i.e. he improves the price) of the liquidity taker. Our optimal trader will such implement a trading scheme close to a market making one: he will capture aggressive flows at the bid and ask, thus earn the bid-ask spread minus twice the premium he provides. As usual, he will face a market risk increasing with his inventory (Ho and Stoll, 2008). In our specific case, the optimal trader will operate a Dark Pool (Cebiroglu, Hautsch and Horst, 2013) or a similar trading platform, where he will try to unwind its inventory at the mid-price. High frequency market makers, like Knight Capital Group, operate such dark pools (“knight link” in this specific case).
2.1 Illustration of the order book

This subsection and the next will introduce the order book dynamics formed by the aggregate activities of all the market participants, when the optimal trader does not act.

In preparation, let us present a few terminologies that appear frequently in discussions about a limit order book. For every stock in the market, there are several types of orders, the most commonly used types being the limit order and the market order. A market buy (sell) order only specifies the number of shares and is executed immediately at the lowest ask (highest bid) price available in the market. A limit buy (sell) order specifies the number of shares and the highest (lowest) price at which the trader is willing to buy (sell). According to the rules of best price first and FIFO (short for “first in first out”) at the same price level, limit orders are executed when there are matching sell (buy) orders at their specified prices.

The records of all limit orders waiting to be executed are maintained. The set of the records is called a limit order book. A limit order book is a “reservoir” of limit orders. It records the number of shares, the price and the time of order arrival or cancelation for every limit order. Once a limit order is submitted, if it is not executed immediately, then this order is “stored” in the limit order book until being “released” and disappearing from the book for one of the three reasons – execution, cancelation, or expiration. The total of limit orders at each price level is called one limit. The lowest ask price (highest bid price) in the book is called the ask price (bid price) for short. The difference between the ask price and the bid price is called the spread. The distance between two adjacent price levels at which limit orders can be submitted is called the tick size.

Fig. 2.1 illustrates a snapshot of a typical limit order book at some time $t$. The vertical axis represents the different price levels in the book, where $P^a(t)$ is the current ask price, $P^b(t)$ is the current bid price and $\delta$ is the tick size. The horizontal axis represents the volume, in other words the number of shares, of limit orders at each price level. The sell side of the book is shown in gray and the buy side in dark. For example, the volume of limit sell orders at the ask price is denoted as $Q^a(t)$, which equals the length of the gray horizontal line at the price level $P^a(t)$; the volume of limit buy orders at the price level $P^b(t) - 2\delta$ is denoted as $Q^b(t)$, which equals the length of the dark horizontal line at that level. The spread is defined as $P^a(t) - P^b(t)$. Without loss of generality, the tick size is set as $\delta = 1$.

All the limit sell (buy) orders at and higher (lower) than the best ask (bid) price are displayed to the market participants. The volumes of limit orders beyond the best ask and best bid prices are constants. In the notations illustrated in Fig. 2.1, this means that $Q^a(t) = Q^a_1(t) = \cdots = \Delta^a$ and $Q^b(t) = Q^b_1(t) = \cdots = \Delta^b$, for all $0 \leq t \leq T$, where $\Delta^a$ and $\Delta^b$ are two positive constants. The number $Q^a$ of limit sell orders at the ask price and the number $Q^b$ of limit buy orders at the bid price are two stochastic processes. When the spread $P^a - P^b$ is more than one tick, limit sell orders can arrive one tick below the ask price $P^a$, and limit buy orders can arrive one tick above the bid price $P^b$. The best ask (bid) price remains constant, until either all the sell (buy) orders at the current price get depleted or new limit sell (buy) orders arrive at the price one tick lower (higher). If the number of all the sell (buy) orders at the best ask (bid) price reaches zero, then the best ask (bid) price increases (decreases) by one tick, i.e.

$$P^a(t) = P^a(t-) + 1 \text{ or } P^b(t) = P^b(t-) - 1,$$

and the volume at the new ask (bid) price is given by

$$Q^a(t) = \Delta^a \text{ or } Q^b(t) = \Delta^b. \quad (2.1)$$

If limit sell (buy) orders arrive at time $t$ at one tick below the ask price $P^a(t-) \ (\text{above the bid price } P^b(t-))$, the ask (bid) price decreases (increases) by one tick, i.e.

$$P^a(t) = P^a(t-) - 1 \text{ or } P^b(t) = P^b(t-) + 1,$$
and each arrival contains $\Delta^a (\Delta^b)$ shares, i.e. the expression (2.1) holds; the number of limit sell (buy) orders at the old ask price $P^a(t-)$ (the old bid price $P^b(t-)$) remains $Q^a(t-)$ ($Q^b(t-)$) at time $t$ and resets to $\Delta^a (\Delta^b)$ at time $t+$. We could make $Q^a_1(t), Q^a_2(t), \ldots$ and $Q^b_1(t), Q^b_2(t), \ldots$ Markov processes with independent increments. The assumption that they are constants will significantly reduce the dimensionality of the control problem while making decisions based on the major driving forces of the order book dynamics.

Besides all the displayed orders that form the limit order book in Fig. 2.1, there is a “dark pool” mechanism within the spread. Simultaneously, the trader has the opportunity to place one mid-price pegged order in the spread: these orders are posted at the price $(P^a(t) + P^b(t))/2$ at any time $t$. He has to choose if it is a buy order or sell order, since he cannot simultaneously post a buy and a sell order at the same price. Other market participants sending orders of the opposite side and having access to the dark pool will consume $\Delta^a (\Delta^b)$ shares of his order.

2.2 Mathematical formulation of the order book dynamics

This subsection will formulate rigorously the dynamics of the limit order book over a deterministic finite time horizon $[0, T]$.

Any change to the limit order book, either in the bid and ask prices, or in the available shares at each price level, is caused by one of the four types of events – limit order arrival, limit order cancelation or expiration, limit order execution, and market order arrival and immediate execution. The two sources of movements are the changes in the volumes at the limit order cancelation or expiration, limit order execution, and market order arrival and immediate execution. The randomness in the order book dynamics is modeled by the following ingredients.

- (1) Probability space $(\Omega, F, \mathbb{P})$.
- (2) Positive constants $\sigma^a$ and $\sigma^b$.
- (3) Independent standard Brownian motions $\sigma^a W^a$ and $\sigma^b W^b$, representing the evolution of $Q^a$ and $Q^b$ when there is no price change.
- (4) Known measurable functions $\theta^a, \theta^b, \lambda^a$ and $\lambda^b : \mathbb{N} \to [0, \infty)$, satisfying $\theta^a(1) = \theta^b(1) = 0$.
- (5) Inhomogeneous Poisson processes $N^a$ and $N^b$, with intensities $\theta^a (P^a(t-) - P^b(t-))$ and $\theta^b (P^a(t-) - P^b(t-))$ at time $t$. When the spread is larger than one tick, limit sell and buy orders are posted according to $N^a$ and $N^b$ at a small price improvement - the best bid plus one tick for a buy order and the best ask minus one tick for a sell order.
- (6) Inhomogeneous Poisson processes $H^a$ and $H^b$, with intensities $\lambda^a (P^a(t-) - P^b(t-))$ and $\lambda^b (P^a(t-) - P^b(t-))$ at time $t$. The trader’s buy and sell orders posted in the dark pool are filled at the mid price according to the liquidity events $H^a$ and $H^b$.
- (7) Conditioning on the spread, the next arrival times of $N^a$, $N^b$, $H^a$ and $H^b$ are independent of each other and independent of the future increment of $W^a$ and $W^b$.
- (8) The filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$, generated by the processes $W^a, W^b, N^a, N^b, H^a$ and $H^b$.

To prove the well-posedness of Problem 4.1 and thus Problem 5.1, the intensities of the order arrival processes within the spread are assumed uniformly bounded.

**Assumption 2.1** The intensity functions $\theta^a, \theta^b, \lambda^a$ and $\lambda^b$ of the inhomogeneous Poisson processes $N^a, N^b, H^a$ and $H^b$ satisfy

$$
\theta^{ia} := \sup_{p \in \mathbb{R}} \{\theta^i(p)\} < \infty \text{ and } \lambda^{ia} := \sup_{p \in \mathbb{R}} \{\lambda^i(p)\} < \infty, \quad i = a, b.
$$

(2.2)

The event-driven limit order book model and the study for an optimal trading algorithm based on it are proposed in Section 4.2 by Lehalle (2013). Consistent with existing works,
the dynamics indeed capture the main features of a limit order book. Empirical studies (Cont, Stoikov and Talreja, 2010, and Hasbrouck and Saar, 2010) observe that inhomogeneous Poisson processes are proper to model the order arrivals and cancelations at different prices, and that the orders in the neighborhoods closest to the bid and ask prices being the most influential to the stock price dynamics. An explanation for the latter observation is that the limit orders whose execution prices are far away from the bid and ask prices are more likely to be placed by speculators to profit from sudden dramatic price changes. Hence, if tracking only the volumes at the bid and ask prices, it makes a reasonable approximation to the real limit order books. The model we use is inspired by Rama Cont and co-authors. Cont, Kukanov and Stoikov (2014) proposed an order flow imbalance model to describe the stylized features of an order book, where the number of shares at each price level beyond the best prices is constant and limit order arrivals and cancelations occur only at the best bid and ask prices. Further, Cont and de Larrard (2011, 2013) have shown that a two-dimensional Brownian motion is a reasonable model for the dynamics of the volumes at the best bid and ask prices. Further, Cont and de Larrard (2011, 2013) have shown that a two-dimensional Brownian motion is a reasonable model for the dynamics of the volumes at the best bid and ask prices. Further, Cont and de Larrard (2011, 2013) have shown that a two-dimensional Brownian motion is a reasonable model for the dynamics of the volumes at the best bid and ask prices.

The number of times over $[0, t]$ that all the orders at the current ask and bid prices are depleted is

$$L^i(t) = \sum_{0 \leq s \leq t} 1_{\{Q^i(s-) \leq 0\}}, \quad 0 \leq t \leq T, \quad i = a, b.$$  \hspace{1cm} (2.3)

At every time the volume at the ask (bid) price is depleted, meaning that $L^i(t) - L^i(t-) = 1$, the $\Delta^a$ (respectively $\Delta^b$) shares at the higher (lower) price level are exposed and the ask (bid) price increases (decreases) by one tick. At every arrival of limit sell (buy) orders within the spread, meaning that $N^i(t) - N^i(t-) = 1$, the new limit at the lower (higher) price level contains $\Delta^a$ (respectively $\Delta^b$) shares and the ask (bid) price decreases (increases) by one tick. At any other time, the volumes move according to the Brownian motions and the prices remain constants.

Following the above reasoning, the dynamics of the order book can be described by the four-dimensional process $(Q^a, Q^b, P^a, P^b)$. The volumes $Q^a$ and $Q^b$ move according to

$$Q^i(t) = Q^i_0 + \sigma^i W^i(t) + \int_0^t (\Delta^i - Q^i(t-)) d (L^i + N^i)(t), \quad 0 \leq t \leq T, \quad i = a, b.$$  \hspace{1cm} (2.4)

The prices move according to

$$P^a(t) = P^a(0) + L^a(t) - N^a(t), \quad \text{and} \quad P^b(t) = P^b(0) - L^b(t) + N^b(t), \quad 0 \leq t \leq T.$$  \hspace{1cm} (2.5)

The process $(Q^a, Q^b, P^a, P^b)$ defined in (2.2) and (2.2) is Markovian.

Fig. 2.2 and Fig. 2.3 plot a simulated path of the two-sided order book dynamics (2.2) and (2.2). Fig. 2.2 shows the ask (top gray line) and bid (bottom dark line) prices. Each time of price change due to order depletion is assigned a 10% probability that it is an execution (indicated by circles) and a 90% probability that it is a cancelation. Fig. 2.3 shows the volumes at the ask (value of the top gray line) and bid (absolute value of the bottom dark line) prices respectively in the positive and negative axis. The parameters are $T = 600, P^a(0) = 20, P^b(0) = 15, Q^a(0) = Q^b(0) = \Delta^a = \Delta^b = 5, \sigma^a = \sigma^b = 10$ and $\theta^a(P^a(t) - P^b(t)) = \theta^b(P^a(t) - P^b(t)) = 0.5(P^a(t) - P^b(t))$.

### 2.3 Execution in the dark pool

This subsection will formulate rigorously the optimal trader’s activities inside the dark pool.

The trader’s decision on whether to accept an upcoming liquidity event in the dark pool is
would only buy below the price $\bar{h}$; a

hidden order strategy

a value of $\Delta$ trading hidden orders are

the execution price $P_a(t)$ for short.

In Definition 2.1, the right-continuity of $h^a$ and $h^b$ guarantees that, for each scenario $\omega \in \Omega$, the hidden orders are revised finitely many times over the time horizon $[0,T]$. The value $h^a(t) = 1$ ($h^b(t) = 1$) means that the trader places a hidden limit buy (sell) order of $\Delta^a$ ($\Delta^b$) shares with the execution the price $(P^a(t) + P^b(t))/2$ (respectively $(P^a(t) + P^b(t))/2$); the value $h^a(t) = 0$ ($h^b(t) = 0$) means that he does not place the hidden order. The processes $h^a$ and $h^b$ being zero on the sets in (2.1) and (2.1) requires that the trader’s hidden orders would only buy below the price $\bar{p}^a$ and sell above the price $\bar{p}^b$.

Suppose an liquidity sell (buy) event occurs at time $t$, meaning that $H^a(t) - H^a(t-) = 1$ (respectively $H^b(t) - H^b(t-) = 1$). If the trader placed a hidden limit buy (sell) order right before time $t$ with the execution price $(P^a(t) + P^b(t))/2$ (respectively $(P^a(t) + P^b(t))/2$), then he successfully buys $\Delta^a$ shares (sells $\Delta^b$ shares) at time $t$ and pays (receives) a cash amount of $\Delta^a (P^a(t) + P^b(t))/2$ (respectively $\Delta^b (P^a(t) + P^b(t))/2$). Using a generic hidden order strategy $h = (h^a, h^b) \in \mathcal{H}$, the trader’s stock inventory and cash amount from trading hidden orders are

$$I^h(t) = \Delta^a \int_0^t h^a(s-)dH^a(s) - \Delta^b \int_0^t h^b(s-)dH^b(s);$$

$$C^h(t) = -\Delta^a \int_0^t h^a(s-) ((P^a(t) + P^b(t))/2) dH^a(s)$$
$$+ \Delta^b \int_0^t h^b(s-) ((P^a(t) + P^b(t))/2) dH^b(s), 0 \leq t \leq T. \quad (2.8)$$

3 The optimal trading problem

Suppose the collective activities of the market participants form the order book dynamics are described in the previous section. The optimal trader will trade on top of this aggregated dynamics. He places a combination of active and hidden orders. The active orders will immediately “internalize” incoming market orders. His hidden orders in the dark pool may or may not be executed at the next moment, but once they are executed the trader receives a price half the spread lower or higher than the current ask or bid price. The flexibility to choose between active and hidden orders enables finding an optimal balance between taking the decision to internalize orders providing them a price improvement, and the naturally associated adverse selection he is exposed to via his inventory.

Depending on his informational advantage, the trader is identified as either regular or internalizing. Most traders in today’s markets, including all the traders in Europe, are regular.
They observe and only observe the current records in the order book. An internalizing trader has the priority of observing incoming orders and acting upon them immediately before the orders are displayed to other market participants. It offers an additional choice to buy or sell at a slightly inferior price so that there is no impact on the current best price.

3.1 Actively filling displayed orders

An active trading strategy places orders that will be fully and immediately executed at the best prices available, minus a “price improvement” offered to his counterpart if necessary.

The total shares that the trader has bought and sold up till time \( t \) by actively internalizing the market orders consist of those filled at the best available price, denoted by the increasing processes \( \{ Z^a(t) \}_{0 \leq t \leq T} \) and \( \{ Z^b(t) \}_{0 \leq t \leq T} \), and of those filled at the old prices plus/minus a price improvement \( \epsilon > 0 \) when new limit orders arrive within the spread, denoted by the processes \( \{ \beta^a(t) \}_{0 \leq t \leq T} \) and \( \{ \beta^b(t) \}_{0 \leq t \leq T} \) as the proportion of shares filled at the old price with respect to the total number of existing shares at the old price right before the transaction. The active trading strategy \( Z = (Z^a, Z^b, \beta^a, \beta^b) \) is called

- continuously re-balanced, if \( Z^a \) and \( Z^b \) are continuous in the time \( t \);
- discretely re-balanced, if \( Z^a \) and \( Z^b \) are pure jump processes;
- mixed, if \( Z^a \) and \( Z^b \) are mixtures of the above two.

This subsection formulates mixed active trading strategies (“mixed trading strategies” for short) respectively for an internalizing trader and a regular trader.

**Definition 3.1 (mixed trading strategy)**

(1) The set of admissible mixed trading strategies \( \mathcal{Z}^{\text{int}} \) of an internalizing trader is the collection of all trading strategies \( Z = (Z^a, Z^b, \beta^a, \beta^b) \) satisfying the following two criteria.

(1.1) The \( \mathcal{F} \)-adapted càdlàg processes \( \{ Z^a(t) \}_{0 \leq t \leq T} \) and \( \{ Z^b(t) \}_{0 \leq t \leq T} \) are non-negative and non-decreasing over the time interval \([0, T]\). The \( \mathcal{F} \)-adapted processes \( \{ \beta^a(t) \}_{0 \leq t \leq T} \) and \( \{ \beta^b(t) \}_{0 \leq t \leq T} \) take values within the interval \([0, 1]\).

(1.2) For two given positive integers \( p^b < \bar{p}^a \), the process \( Z^a \) is flat on

\[ \{(t, \omega) \in [0, T] \times \Omega | P^a(t) \geq \bar{p}^a \}, \]

and \( Z^b \) is flat on

\[ \{(t, \omega) \in [0, T] \times \Omega | P^b(t) \leq \underline{p}^b \}; \]

the process \( \beta^a \) is non-zero only on

\[ \{(t, \omega) \in [0, T] \times \Omega | P^a(t) < \bar{p}^a, N^a(t) - N^a(t-) = 1 \text{ and } Z^a(t) - Z^a(t-) < \Delta^a \}, \]

and the process \( \beta^b \) is non-zero only on

\[ \{(t, \omega) \in [0, T] \times \Omega | P^b(t) > \underline{p}^b, N^b(t) - N^b(t-) = 1 \text{ and } Z^b(t) - Z^b(t-) < \Delta^b \}. \]

(2) The set of admissible mixed trading strategies \( \mathcal{Z}^{\text{reg}} \) of a regular trader is defined as

\[ \mathcal{Z}^{\text{reg}} := \{ Z = (Z^a, Z^b, \beta^a, \beta^b) \in \mathcal{Z}^{\text{int}} | \beta^a(t) \equiv \beta^b(t) \equiv 0, \text{ for all } t \in [0, T] \}. \]

Criterion (1.1) in Definition 3.1 defines an internalizing trader’s mixed trading strategy. The bounded prices in Criterion (1.2) requires that the trader only buy below the price \( \bar{p}^a \) and sell above the price \( \underline{p}^b \). In practice, when the prices goes outside of their normal range, no trading reflects a psyche of not to take up risk or push the prices further towards the extreme; technically, it will be used to prove the well-posedness of Problem 4.1 or in other words that the value function is finite. The other requirement about \( \beta^a \) and \( \beta^b \) in
Criterion (1.2) means that it is only necessary to consider filling a fractional column at the old price when new limit orders arrive within the spread. Allowing for optimizing over this fraction enables a path-wise replication of the effect of every mixed trading strategy by a price switching strategy, as will be shown in section 5. A regular trader’s mixed trading strategy defined in Criterion (2) is the same as that of an internalizing trader, except that at the time of order arrival within the spread, the regular trader can no longer fill the orders at the old price, which is formulated as \( \beta^a \equiv \beta^b \equiv 0 \).

The rest of the current subsection will write, in the case of an internalizing trader, the order book dynamics and the trader’s stock inventory and cash amount in compact formulae. They can be verified by enumerating all the situations that could trigger a change in the bid or ask price. To adjust to the case of a regular trader is only a matter of setting \( \beta^a \equiv \beta^b \equiv 0 \).

The order book dynamics from equations (2.2), (2.2) and (2.2), is now controlled by the trader using an admissible mixed trading strategy \( Z \). The number of times over \([0, t]\) that all the orders at the current ask and bid prices are depleted can be expressed as

\[
L^i(t) = \sum_{0 \leq s \leq t} 1_{\{Q^i(s-)-(Z^i(s)-Z^i(s-)) \leq 0\}}, \quad i = a, b.
\]

For \( i = a, b \), the changes

\[
\mu^i(t) = P^a(t) - P^a(t-) \quad \text{and} \quad \mu^b(t) = - (P^b(t) - P^b(t-))
\]

in the controlled ask and bid price processes at time \( t \) can be computed by

\[
\mu^i(t) = \begin{cases} 
-1, & \text{if } N^i(t) - N^i(t-) = 1 \text{ and } Z^i(t) - Z^i(t-) < \Delta^i; \\
0, & \text{if } N^i(t) - N^i(t-) = 1 \text{ and } \Delta^i \leq Z^i(t) - Z^i(t-) < Q^i(t-) + \Delta^i, \\
or if \( t = T \) and \( Z^i(T) - Z^i(T-) < Q^i(T-) \); \\
\lfloor (Z^i(t) - Z^i(t-) - Q^i(t-)) / \Delta^i \rfloor + 1 - (N^i(t) - N^i(t-)), & \text{else}.
\end{cases}
\]

The controlled volumes \( Q^a \) and \( Q^b \) at the ask and bid prices move according to

\[
Q^i(t) = Q^i_0 + \sigma^i W^i(t) - Z^i(t) + \Delta^i \int_0^t (\mu^i(s))^+ dL^i(s) + \int_0^t 1_{\{\mu^i(s) \leq 0\}} \left( \Delta^i - (\mu^i(s))^+ - Q^i(s-) \right) dN^i(s), \quad 0 \leq t \leq T, \quad i = a, b.
\]

The controlled ask and bid prices \( P^a \) and \( P^b \) move according to

\[
P^a(t) = P^a(0) + \int_0^t (\mu^a(s))^+ dL^a(s) - \int_0^t (\mu^a(s))^- dN^a(s);
\]

\[
P^b(t) = P^b(0) - \int_0^t (\mu^b(s))^+ dL^b(s) + \int_0^t (\mu^b(s))^- dN^b(s), \quad 0 \leq t \leq T. \tag{3.1}
\]

At every time \( t \in [0, T] \), a mixed trading strategy \( Z \) gives the trader an inventory of stock shares

\[
I^Z(t) = Z^a(t) - Z^b(t) + \int_0^t \beta^a(s)Q^a(s-)dN^a(s) - \int_0^t \beta^b(s)Q^b(s-)dN^b(s). \tag{3.2}
\]

Let \( \epsilon \) be the premium per share that the trader pays his counterpart, for internalizing limit orders at the old price upon the arrival of incoming orders at the new better price. Using the mixed trading strategy \( Z \), the total cash amount that the trader pays the seller (receives
from the buyer) of the stock denoted as $C^Z_a$ (respectively $C^Z_b$) can be calculated by

$$
C^Z_a(t) = \int_0^t P^a(s-)dZ^a(s) + \int_0^t \beta^a(s)Q^a(s-) (P^a(s-) + \epsilon) dN^a(s)
+ \int_0^t \left( \frac{1}{2} \Delta^a d(P^a(s))^2 + \left( \frac{1}{2} \Delta^a - Q^a(s) \right) dP^a(s) - \Delta^a dN^a(s) \right),
$$

(3.3)

and

$$
C^Z_b(t) = \int_0^t P^b(s-)dZ^b(s) + \int_0^t \beta^b(s)Q^b(s-) (P^b(s-) - \epsilon) dN^b(s)
- \int_0^t \left( \frac{1}{2} \Delta^b d(P^b(s))^2 + \left( \frac{1}{2} \Delta^b - Q^b(s) \right) dP^b(s) - \Delta^b dN^b(s) \right).
$$

(3.4)

In the equation (3.1), the two integrals count the total number of shares bought and sold at the old prices when limit orders arrive within the spread. In each of the equations (3.1) and (3.1), the first integral on the right hand side of the identity is the amount of cash paid and received if all the orders placed at the best available prices were executed at the bid and ask prices observed right before each transaction, as they are when the prices do not change and there is no arrival within the spread. The second integral counts the cash amount paid and received from filling the limit orders at the old prices when new limit orders arrive within the spread. The third integral collects the additional cost paid for trading deeper into the book and the savings from filling the arriving limit orders within the spread.

The trader’s cumulative cash amount on his account at time $t$ from the mixed trading strategy $Z$ is

$$
C^Z(t) = -C^Z_a(t) + C^Z_b(t), \quad 0 \leq t \leq T.
$$

(3.5)

### 3.2 The goal of trading

The trader’s total stock inventory and cash amount are sums of those from trading hidden and displayed orders, being

$$
I^{h,Z}(t) = I_0 + I^h(t) + I^Z(t) \quad \text{and} \quad C^{h,Z}(t) = C_0 + C^h(t) + C^Z(t), \quad 0 \leq t \leq T,
$$

(3.6)

where the real numbers $I_0$ and $C_0$ are the initial stock inventory and the initial cash amount, and the processes $I^h$, $C^h$, $I^Z$ and $C^Z$ are defined in the equations (3.3), (3.1) and (3.1). When there is no ambiguity on which trading strategies are used, the superscripts in the stock inventory $I^{h,Z}$ and the cash amount $C^{h,Z}$ are omitted.

Let $r^I \in (-\infty, \infty)$ and $r^C \in (0, \infty)$ be two real numbers and

$$
F : \mathbb{R} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}; \quad (z, p^a, p^b) \mapsto F(z, p^a, p^b),
$$

be a measurable function with quadratic growth.

**Assumption 3.1** There exists a constant $r^F > 0$, such that for all $z \in \mathbb{R}$ and all $p^b \leq p^a$, we have

$$
|F(z, p^a, p^b)| \leq r^F \left( (z)^2 + (p^a)^2 + (p^b)^2 + 1 \right)
$$

and

$$
|F(z_1, p^a, p^b) - F(z_2, p^a, p^b)| \leq r^F \left( |p^a| + |p^b| + 1 \right) (|z_1| + |z_2| + 1) |z_1 - z_2|.
$$

The trading activities are measured by the reward

$$
\xi (I(T), C(T)) = r^C C(T) + r^I F \left( I(T), P^a(T), P^b(T) \right).
$$

(3.7)

The trader’s objective of maximizing the reward in expectation is formulated as a stochastic control problem.
Problem 3.1 (1) An internalizing trader looks for an optimal trading strategy
\[ Z^* = (Z^{a*}, Z^{b*}, \beta^{a*}, \beta^{b*}) \in \mathcal{Z}^{\text{int}} \] and \( h^* = (h^{a*}, h^{b*}) \in \mathcal{H} \) to achieve the maximum expected reward
\[ V^{\text{mix, int}}(\epsilon) := \sup_{(Z^*, h^{a*}, h^{b*}) \in \mathcal{Z}^{\text{int}}, (h^{a*}, h^{b*}) \in \mathcal{H}} \mathbb{E} \left[ \xi(I^{h,Z}(T), C^{h,Z}(T)) \right]. \quad (3.8) \]

The reward \( V^{\text{mix, int}}(\epsilon) \) is a function of the premium \( \epsilon \), because the cash amount \( C^{h,Z}(T) \) is a function of \( \epsilon \).

(2) A regular trader looks for an optimal trading strategy \( Z^* = (Z^{a*}, Z^{b*}, 0, 0) \in \mathcal{Z}^{\text{reg}} \) and \( h^* = (h^{a*}, h^{b*}) \in \mathcal{H} \) to achieve the maximum expected reward
\[ V^{\text{mix, reg}} := \sup_{(Z^*, h^{a*}, h^{b*}) \in \mathcal{Z}^{\text{reg}}, (h^{a*}, h^{b*}) \in \mathcal{H}} \mathbb{E} \left[ \xi(I^{h,Z}(T), C^{h,Z}(T)) \right]. \]

Control problems with reward functions of the form (3.2) have a five dimensional state process \( (Q^b, Q^a, I, P^a, P^b) \). Problem 3.1 is still solvable for reward functions that are not linear in \( C(T) \), in which case the state process will have the cash amount \( C \) as a sixth dimension. Assumption 3.1 is a technical assumption under which Problem 4.1 and thus Problem 3.1 are well-posed.

Several common situations where the reward criteria satisfy Assumption 3.1 are linear combination of the cash and inventory, liquidating or filling a certain number of stock shares, and holding cash only at the terminal time, corresponding to the following forms of \( \xi(I(T), C(T)) \) defined in (3.2):

(1) \( \xi(I(T), C(T)) = r^C C(T) + r^I I(T) \);

(2) \( \xi(I(T), C(T)) = r^C C(T) + r^I |I(T) - z_0| \) or \( \xi(I(T), C(T)) = r^C C(T) + r^I (I(T) - z_0)^2 \);

(3) \( \xi(I(T), C(T)) = r^C C(T) + r^I (1_{\{I(T) > 0\}}(P^b(T) - U^b) + 1_{\{I(T) < 0\}}(P^a(T) + U^a)) I(T). \)

The criterion (3.2)(1) means that the trader has a utility function linear in cash and inventory. In (3.2)(2), the coefficient \( r^I \) is negative, and the constant \( z_0 \) is the number of shares that the trader would like to hold at the terminal time \( T \). In (3.2)(3), the coefficient \( r^I \) is positive and \( U^a \) and \( U^b \) are two positive integers; if the terminal inventory \( I(T) \) is positive, the trader sells all his stocks at the price \( (P^b(T) - U^b) \) per share; if \( I(T) \) is negative, he pays the price \( (P^a(T) + U^a) \) for each share.

Remark 3.1 In application, a trader in a hedge fund or a proprietary trading firm is entitled to both buying and selling during any trading period, hence \( Z^a, Z^b, h^a \) and \( h^b \) can be all positive. For a trader in a brokerage agency, if he trades during the time \([0, T]\) to fill a buy (sell) order for the customer, a simplest way to comply with the regulations is not to sell (buy) throughout the same time period, hence, when using the results in this paper, he should set \( Z^b(t) \equiv 0, \beta^b(t) \equiv 0 \) and \( h^b(t) \equiv 0 \) (respectively \( Z^a(t) \equiv 0, \beta^a(t) \equiv 0 \) and \( h^a(t) \equiv 0 \)) for all \( 0 \leq t \leq T \). Especially, a regular trader sets \( \beta^a \equiv \beta^b \equiv 0 \). The brokerage agency’s admissible set of equivalent price switching strategies is a modification of Definition 4.1 by setting \( u^a_n \equiv -(N^a(S_n) - N^a(S_{n-1})) \) and \( h^a(t) \equiv 0 \) (respectively \( u^b_n \equiv -(N^b(S_n) - N^b(S_{n-1})) \) and \( h^b(t) \equiv 0 \)). Optimizing over the buying (selling) strategies only is a special case of the algorithm in subsection 4.2.

4 The optimal switching problem

This section presents the problem of how the trader could optimally switch the bid and ask prices and shows the well-posedness of the optimal switching problem. Compared to the optimal trading Problem 4.1, price switching considers a smaller and simpler set of active
trading strategies, which are discretely re-balanced strategies in the form of the times to trade and the number of limits to fill at each time.

4.1 Switching prices

Let \([a]\) and \(\{u\}\) respectively denote the integer part and the fractional part of a real number \(u\). The identity \(u = [u] + \{u\}\) holds. Switching the prices means that the trader chooses a sequence of times \(\{S_n\}_{n=1}^{\infty}\) and two sequences of positive numbers \(\{u_n^a\}_{n=1}^{\infty}\) and \(\{u_n^b\}_{n=1}^{\infty}\), such that the ask and bid prices are pushed from \(P^a(S_{n-1})\) to \(P^a(S_n) = P^a(S_{n-1}) + [u_n^a]\) and from \(P^b(S_{n-1})\) to \(P^b(S_n) = P^b(S_{n-1}) - [u_n^b]\). The ask and bid prices stay constants over every time interval \((S_{n-1}, S_n)\).

In the limit order book where the trader and the rest of the market participants all act, a price change could occur due to two possible reasons.

(1) Limit orders at the ask or bid price is depleted by the noise trader, or limit orders beyond the best prices. If he trades at some time \(S_n \in (S_{n-1}, T_n)\), then he has to fill all the shares at either the ask price or the bid price to trigger a price change.

After the \((n-1)\)th price switching at the time \(S_{n-1}\), if he waits until the time \(T_n\), the trader may choose to fill some shares at time \(T_n\); even if he does not trade, the rest of the market will switch the prices at time \(T_n\) any way, in which case the time of the \(n\)th price switching is set as \(S_n = T_n\). Requiring that the trader has to “trade” at time \(T_n\), though possibly zero share, makes sure that the prices are constants between two switching times and gives a neater expression of the order book dynamics.

Furthermore, formulating with the help of \(T_n\)’s reduces the price switching problem from seven-dimensional path-dependent to five-dimensional Markovian. The actual controlled state process is

\[
\{(Q^a(t^-), Q^b(t^-), I^a(t^-) + I^b(t^-), P^a(t^-), P^b(t^-), N^a(t), N^b(t))\}_{0 \leq t \leq T}.
\]

Since what matters in \((N^a(t), N^b(t))\) is their exponentially distributed arrival times, the trader only need to monitor whether or not there is an arrival of limit sell or buy orders within the spread. Viewing the controlled arrival times \(T_n\)’s as a sequence of exit times when decision marking has to take place, the state process is simplified to \(\{(Q^a(t^-), Q^b(t^-), I^a(t^-) + I^b(t^-), P^a(t^-), P^b(t^-))\}_{0 \leq t \leq T}\).

The admissible price switching strategies are switching controls defined below.

**Definition 4.1** (switching control) The admissible set of switching controls of an internalizing trader and that of a regular trader are denoted respectively as \(\mathcal{S}\) and \(\mathcal{S}^{reg}\). Let the letter \(j\) represent either the superscripts “int” or the superscript “reg”. The admissible set \(\mathcal{S}\) consists of switching controls \(\alpha := \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty}\) satisfying, for every \(n = 1, 2, \ldots\), the three criteria below, with the convention that \(S_0 = u_0^a = u_0^b = 0\).

(1) The switching time \(S_n\) is an \(\mathcal{F}\)-stopping time such that \(0 \leq S_1 < \cdots < S_{n-1} < S_n < \cdots\). If \(S_{n-1} < T\), then \(S_n = T + 1\); if \(S_{n-1} < T\), then \(S_n \in (S_{n-1}, T_n]\), where \(T_n\) defined in (4.1).
is the next time of price change if the trader does not trade.

(2) For the same positive integers $\bar{p}^a$ and $\bar{p}^b$ as in Definition 4.1, we have $(u^a_n, u^b_n) \in \mathcal{W}(S_n, P^a(S_{n-1}), P^b(S_{n-1}))$, where

$$\mathcal{W}(t, p^a, p^b) := \begin{cases} 
\{0, 1, \ldots, (\bar{p}^a - p^a)^+\} \times \{0, 1, \ldots, (p^b - \underline{p})^+\} \setminus \{(0, 0)\}, & \text{if } t < T_n, \text{ or } Q^i(t-) = 0, \ i = a \text{ or } b; \\
\{0, 1, \ldots, (\bar{p}^a - p^a)^+\} \times \{0, 1, \ldots, (p^b - \underline{p})^+\}, & \text{if } N^a(t) - N^a(t-) = 1; \\
\{0, 1, \ldots, (\bar{p}^a - p^a)^+\} \times D^i(p^b - \underline{p}) \cup \{2, \ldots, (p^b - \underline{p})^+\}, & \text{if } N^b(t) - N^b(t-) = 1; \\
\{0, (\bar{p}^a - p^a)^+\} \times \{0, (p^b - \underline{p})^+\}, & \text{if } t = T; \\
\{(0, 0)\}, & \text{if } t = T + 1.
\end{cases}$$

In expression (4.1), the sets $D^j(\bar{p}^a - p^a)$ and $D^j(p^b - \underline{p})$ of real numbers are defined as

$$D^{int}(x) := \begin{cases} 
[-1, 1], & \text{if } x \geq 1; \\
[-1, 0], & \text{if } x = 0; \\
[-1], & \text{if } x \leq -1
\end{cases}$$

for an internalizing trader, and as

$$D^{reg}(x) := \begin{cases} 
[-1, 0, 1], & \text{if } x \geq 1; \\
[-1, 0], & \text{if } x = 0; \\
[-1], & \text{if } x \leq -1
\end{cases}$$

for a regular trader. For $j = \text{int}, \text{reg}$ and $t \in [0, T]$, the set $\mathcal{A}^j_{t}$ is defined the same as $\mathcal{A}^j$ except that $S_0 = t$. Also, we denote by

$$\mathcal{A}^j_{t,1} := \{(S_1, u^a_1, u^b_1) | \alpha = \{(S_n, u^a_n, u^b_n)\}_{n=0}^\infty \in \mathcal{A}^j_t\}$$

the set of the first elements of all switching controls in $\mathcal{A}^j_{t,1}$.

In Definition 4.1, Criterion (1) specifies the trading times $S_n$. Criterion (2) specifies the possible numbers of limits to buy and sell at each transaction. Setting $S_n = T + 1$ means the trader no longer trades, hence there has to be $u^a_n = u^b_n = 0$.

The admissible switching control sets of an internalizing trader and a regular trader are different only up to the sets $D^{int}$ and $D^{reg}$ defined in (4.1) and (4.4). Throughout this paper, when a claim is valid for a trader regardless of whether he is internalizing or regular, the admissible switching control sets will be generically denoted as $\mathcal{A}$, $\mathcal{A}^i_t$ and $\mathcal{A}^j_{t,1}$.

For $n = 1, 2, \ldots$, at every time $S_n$ a switching control $\alpha \in \mathcal{A}$ causes the ask price to increase from $P^a(S_{n-1})$ to $P^a(S_n) = P^a(S_{n-1}) + [u^a_n]$ and the bid price to decrease from $P^b(S_{n-1})$ to $P^b(S_n) = P^b(S_{n-1}) - [u^b_n]$. By enumerating all the situations that could trigger a price change, the order book dynamics and the changes in the stock inventory and cash amount can be summarized in compact formulae.

The order book dynamics can be written in terms of the switching control

$$\alpha = \{(S_n, u^a_n, u^b_n)\}_{n=0}^\infty \in \mathcal{A}$$

the controlled prices move according to

$$P^a(t) = P^a(0) + \sum_{n:S_n \leq t} [u^a_n], \text{ and } P^b(t) = P^b(0) - \sum_{n:S_n \leq t} [u^b_n], \ 0 \leq t \leq T; \quad (4.5)$$
the controlled volumes move according to

\[
Q^i(t) = \left\{ \begin{array}{l}
Q^i(0) + \sigma^i W^i(t) \\
+ \sum_{n:S_n \leq t} \left( 1_{\{u_n^i \neq 0\}} (\Delta^i - Q^i(S_n^-)) - 1_{\{u_n^i = 0\}} \{u_n^i\} Q^i(S_n^-) \right), \; 0 \leq t < T; \\
+ \sum_{n:S_n = T} (1 - \{u_n^i\}) (1_{\{u_n^i \geq 1\}} \Delta^i + 1_{\{u_n^i = 0\}} Q^i(T^-)) , \; t = T, \; i = a, b.
\end{array} \right.
\]

Defining the functions \( g^a \) and \( g^b : \Omega \times [0, T] \times [0, \infty) \times \mathbb{Z} \rightarrow \mathbb{R} \) as

\[
g^i(t, q, u) = \left\{ \begin{array}{l}
1_{\{u \geq 1\}} \left( q + (u - 1) \Delta^i \right) + (N^i(t) - N^i(t-)) \left( 1_{\{u \geq 0\}} \Delta^i + 1_{\{u \leq 0\}} \{u\} q \right), \; t < T; \\
1_{\{u \geq 1\}} \left( q + (u - 1) \Delta^i \right) + 1_{\{u = 0\}} u \cdot q , \; t = T, \; i = a, b,
\end{array} \right.
\]

then \( g^a (S_n, Q^a(S_n^-), u_n^a) \) and \( g^b (S_n, Q^b(S_n^-), u_n^b) \) are respectively the number of shares that the trader buys and sells at the time \( S_n \). The quantity

\[
g_n (S_n, Q^a(S_n^-), Q^b(S_n^-), u_n^a, u_n^b) = g^a (S_n, Q^a(S_n^-), u_n^a) - g^b (S_n, Q^b(S_n^-), u_n^b)
\]

is the change in the trader’s stock inventory from transactions at the time \( S_n \).

Let \( \epsilon \) be the premium that the trader pays his counterpart for filling each share at the old price upon the arrival of limit orders at the new price. Defining the functions \( f^a \) and \( f^b : \Omega \times [0, T] \times [0, \infty) \times \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{R} \) as

\[
f^a(t, q, p, u) = \left\{ \begin{array}{l}
1_{\{u \geq 1\}} \left( p \left( q + (u - 1) \Delta^a \right) + \frac{1}{2} u (u - 1) \Delta^a \right) \\
+ (N^a(t) - N^a(t-)) \left( 1_{\{u \geq 0\}} (p - 1) \Delta^a + 1_{\{u \leq 0\}} (p + \epsilon) \{u\} q \right) , \; t < T; \\
1_{\{u \geq 1\}} \left( p \left( q + (u - 1) \Delta^a \right) + \frac{1}{2} [u] [u-1] + [u] \cdot \{u\} \cdot \Delta^a \right) \\
+ 1_{\{u = 0\}} p \cdot u \cdot q , \; t = T,
\end{array} \right.
\]

and

\[
f^b(t, q, p, u) = \left\{ \begin{array}{l}
1_{\{u \geq 1\}} \left( p \left( q + (u - 1) \Delta^b \right) + \frac{1}{2} u (u - 1) \Delta^b \right) \\
+ (N^b(t) - N^b(t-)) \left( 1_{\{u \geq 0\}} (p + 1) \Delta^b + 1_{\{u \leq 0\}} (p - \epsilon) \{u\} q \right) , \; t < T; \\
1_{\{u \geq 1\}} \left( p \left( q + (u - 1) \Delta^b \right) + \frac{1}{2} [u] [u-1] + [u] \cdot \{u\} \cdot \Delta^b \right) \\
+ 1_{\{u = 0\}} p \cdot u \cdot q \; t = T,
\end{array} \right.
\]

then \( f^a (S_n, Q^a(S_n^-), P^a(S_n^-), u_n^a) \) and \( f^b (S_n, Q^b(S_n^-), P^b(S_n^-), u_n^b) \) are respectively the total cash amount that the trader pays the seller (receives from the buyer) of the stock for transactions at the time \( S_n \). The quantity

\[
f_\alpha (S_n, Q^a(S_n^-), Q^b(S_n^-), P^a(S_n^-), P^b(S_n^-), u_n^a, u_n^b) := - f^a (S_n, Q^a(S_n^-), P^a(S_n^-), u_n^a, u_n^b) + f^b (S_n, Q^b(S_n^-), P^b(S_n^-), u_n^b)
\]

is the change in the trader’s cash amount on his account from transactions at the time \( S_n \).

Actively trading according to a generic switching control \( \alpha \in \mathcal{A} \) defined by Definition

\[\text{14}\]
the trader’s **inventory** and **cash** amount from the displayed orders are respectively

\[
I^\alpha(t) = \sum_{n:S_n \leq t} g_\alpha \left( S_n, Q^n(S_n^-), Q^b(S_n^-), u^n_n, u^b_n \right) \quad \text{and} \quad C^\alpha(t) = \sum_{n:S_n \leq t} f_\alpha \left( S_n, Q^n(S_n^-), Q^b(S_n^-), P^n(S_n^-), P^b(S_n^-), u^n_n, u^b_n \right), \quad \text{for } 0 \leq t \leq T.
\]

Taking into account the hidden orders, the trader’s total terminal stock **inventory** and **cash** amount are respectively

\[
I(T)^{h,\alpha} = I_0 + I^b(T) + I^\alpha(T) \quad \text{and} \quad C^{h,\alpha}(T) = C_0 + C^h(T) + C^\alpha(T),
\]

where the quantities \(I^b(T)\) and \(C^h(T)\) are defined in the equations (2.3). When there is no ambiguity on which trading strategies are used, the superscripts in \(I^{h,\alpha}(T)\) and \(C^{h,\alpha}(T)\) are omitted.

Since each transaction using active orders causes a change in the prices, the trader’s trading activities are a matter of choosing when and to what level to “switch” the ask and bid prices. The profit or cost from switching at time \(S_n\) is the change in his cash amount expressed as the quantity in (4.11). This is why we give the name “price switching” to the set of discretely re-balances trading strategies introduced in this section. Optimizing the trading algorithm over all the price switching strategies is a problem of switching control with impact on the state process.

**Problem 4.1** (1) An internalizing trader looks for an admissible switching control \(\alpha^* \in \mathcal{A}^{\text{int}}\) and an optimal hidden order strategy \(h^* \in \mathcal{H}\) that achieve the supremum

\[
V^{\text{swt, int}}(\varepsilon) := \sup_{\alpha \in \mathcal{A}^{\text{int}}, h \in \mathcal{H}} \mathbb{E}[\xi(I(T)^{h,\alpha}, C^{h,\alpha}(T))].
\]

Same as in Problem 3.7, the reward \(V^{\text{swt, int}}(\varepsilon)\) is a function of the premium \(\varepsilon\).

(2) A regular trader looks for an admissible switching control \(\alpha^* \in \mathcal{A}^{\text{reg}}\) and an optimal hidden order strategy \(h^* \in \mathcal{H}\) that achieve the supremum

\[
V^{\text{swt, reg}} := \sup_{\alpha \in \mathcal{A}^{\text{reg}}, h \in \mathcal{H}} \mathbb{E}[\xi(I(T)^{h,\alpha}, C^{h,\alpha}(T))].
\]

Generically for either an internalizing trader or a regular trader, Problem 4.1 requires finding an admissible switching control \(\alpha^* \in \mathcal{A}\) and an optimal hidden order strategy \(h^* \in \mathcal{H}\) that achieve the supremum

\[
\sup_{\alpha \in \mathcal{A}, h \in \mathcal{H}} \mathbb{E}[\xi(I(T)^{h,\alpha}, C^{h,\alpha}(T))]
\]

\[
= \sup_{\alpha \in \mathcal{A}, h \in \mathcal{H}} \mathbb{E} \left[ r^C \sum_{n=1}^{\infty} f_\alpha \left( S_n, Q^n(S_n^-), Q^b(S_n^-), P^n(S_n^-), P^b(S_n^-), u^n_n, u^b_n \right) + r^C C^h(T) + r^T F(I^{h,\alpha}(T), P^a(T), P^b(T)) \right].
\]

The **value process** \(V\) of (4.11) is defined as

\[
V(t) := \sup_{\alpha \in \mathcal{A}, h \in \mathcal{H}} \mathbb{E} [\xi(I(T), C(T))] \mathcal{F}(t) - r^C C(t).
\]

Then the best expected reward (4.1) can be written as

\[
\sup_{\alpha \in \mathcal{A}, h \in \mathcal{H}} \mathbb{E}[\xi(I(T), C(T))] = r^C C_0 + V(0).
\]
4.2 Well-posedness of the problem

Before looking for an optimal trading strategy that achieves the supremum in (4.1), it is necessary to verify that the optimal switching problem is well-defined. In other words, is the value process \( V \) in (4.1) finite? This subsection will prove that the answer is yes. Later in Proposition 5.1 we shall see that the optimal price switching problems provide lower and upper bounds of the value functions of the optimal trading problem, hence the optimal trading problem is consequently also well-posed. By Theorem 5.1(6)(8) and Proposition 5.1 we shall see that

\[
V_{\text{swt,reg}} \leq V_{\text{swt,int}}(\epsilon), \quad V_{\text{mix,int}}(\epsilon) \quad \text{and} \quad V_{\text{mix,reg}} \leq V_{\text{swt,int}}(0).
\]

It suffices to prove the well-posedness of the regular trader’s price switching problem and the internalizing trader’s price switching problem with zero premium. Then the well-posedness of Problem 3.1 and Problem 4.1 all follows.

Throughout this subsection, the premium \( \epsilon \) equals zero.

The well-posedness is stated as Theorem 4.1 at the end of this subsection. To prepare for the proof of the theorem, Lemma 4.1 and Lemma 4.2 will respectively provide uniform \( L^2 \) or \( L^1 \) bounds of the prices \( (P^a(T), P^b(T)) \), the stock inventory \( I^{a,\alpha}(T) \) and the cash amount \( C^{a,\alpha}(T) \) for all admissible switching controls. Because the reward criterion \( \xi \) defined in (2.2) is a function of \( (P^a(T), P^b(T)), I^{a,\alpha}(T) \) and \( C^{b,\alpha}(T) \), the two Lemmas lead to Theorem 4.1. In addition, an interpretation worth noting of Lemma 4.1 is that the trader’s trading activities will not push the prices towards explosion.

**Lemma 4.1** There exists a constant \( c_1 > 0 \), such that for any admissible switching control \( \alpha \in \mathcal{A} \) defined by Definition 4.1 the controlled ask and bid prices at any time \( t \in [0, T] \) has the \( L^2 \) bounds

\[
\mathbb{E} \left[ (P^a(t))^2 \right] \leq (P^a(0))^2 + (\bar{p}^a)^2 + c_1 T \left( (\theta^a)^2 + 1 \right) T + \theta^a + 1;
\]

\[
\mathbb{E} \left[ (P^b(t))^2 \right] \leq (P^b(0))^2 + (\bar{p}^b)^2 + c_1 T \left( (\theta^b)^2 + 1 \right) T + \theta^b + 1,
\]

where the constants \( \theta^a \) and \( \theta^b \) defined in (4.1) are the maximum arrival rates of limit orders within the spread.

**Proof.** The total number of downward (upward) movements in the ask (bid) price equals the number of times \( N^i \) that limit sell (buy) orders arrive within the spread. The total number of upward (downward) movements in the ask (bid) price equals the number of limits that the trader has filled plus the number of times when the volume at the ask (bid) price is depleted by the rest of the market. The total number of depletions at the ask (bid) price by the rest of the market participants does not exceed a renewal process \( R^i \) with independent inter-arrival times identically distributed as the leverage hitting time

\[
\inf \{ 0 \leq t \leq T \mid \Delta^i + \sigma^i W^i(t) \leq 0 \} \wedge T, \quad \text{for } i = a, b.
\]

The highest ask (lowest bid) price happens when no limit sell (buy) orders arrive within the spread and the trader fills all the limit sell (buy) order below the price \( \bar{p}^a \) (above the price \( \bar{p}^b \)). The lowest ask (highest bid) price happens when there is no depletion by the trader or the noise trader, and limit sell (buy) orders arrive according to the largest possible intensity. The prices have the upper and lower bounds

\[
P^a(0) - N^a(T) \leq P^a(0) - N^a(t) \leq P^a(t) \leq \bar{p}^a \lor P^a(0) + R^a(t) \leq \bar{p}^a \lor P^a(0) + R^a(T);
\]

\[
\underline{p}^b \land P^b(0) - R^b(t) \leq \underline{p}^b \land P^b(0) - R^b(t) \leq P^b(t) \leq P^b(0) + N^b(t) \leq P^b(0) + N^b(T).
\]

Using the inequalities in (4.2) and the bounds for variances of stationary renewal processes derived in [13] Daley (1978), we get the inequalities in (4.1). \( \square \)
Lemma 4.2 There exist positive constants $c_2$ and $c_3$, such that for any admissible switching control $\alpha \in \mathcal{A}$ defined by Definition 4.1 and any hidden order strategy $h \in \mathcal{H}$ defined by Definition 4.2, the trader’s total cash amount and stock inventory from the strategies $\alpha$ and $h$ have the bounds

$$\mathbb{E}[|C^\alpha(T) + C^h(T)|] \leq c_2 \left((Q^a(0))^2 + (Q^b(0))^2 + (P^a(0))^2 + (P^b(0))^2 + T^2 + 1\right)$$

and

$$\mathbb{E}[|I_0 + I^b(T) + I^a(T)|^2] \leq c_3 \left((Q^a(0))^4 + (Q^b(0))^4 + (I_0)^2 + (P^a(0))^4 + (P^b(0))^4 + T^4 + 1\right).$$

Proof. Throughout the time interval $[0, T]$, the ask price $P^a$ (bid price $P^b$) moves downward $N^a(T)$ times (upward $N^b(T)$ times). At least one limit at a time, the trader could take at most all the initially existing displayed limit sell (buy) orders and all the displayed limit sell (buy) orders ever arriving within the spread, as long as the ask (bid) price is below $\bar{p}^a$ (above $\bar{p}^b$). Then the total number of limits that the trader would ever buy (sell) until time $T$ does not exceed $(\bar{p}^a - P^a(0))^+ + N^a(T)$ (respectively $(P^b(0) - \bar{p}^b)^+ + N^b(T)$). At any time $t \in [0, T]$, each limit of limit sell (buy) orders contains no more than $\Delta^a + Q^a(0) + 2\sigma^a \sup_{0 \leq t \leq T} |W^a(t)|$ (respectively $\Delta^b + Q^b(0) + 2\sigma^b \sup_{0 \leq t \leq T} |W^b(t)|$) shares. Each time buying (selling), it is impossible to take more than the number of all the currently existing limit sell (buy) orders below (above) the price $\bar{p}^a$ (respectively $\bar{p}^b$). We may bound the stock inventory and cash amount from displayed orders by

$$|I^a(T)| \leq ((\bar{p}^a - P^a(0))^+ + N^a(T)) \left(\Delta^a + Q^a(0) + 2\sigma^a \sup_{0 \leq t \leq T} |W^a(t)|\right) + \left((P^b(0) - \bar{p}^b)^+ + N^b(T)\right) \left(\Delta^b + Q^b(0) + 2\sigma^b \sup_{0 \leq t \leq T} |W^b(t)|\right);$$

$$|C^a(T)| \leq (\bar{p}^a + \epsilon) \left((\bar{p}^a - P^a(0))^+ + N^a(T)\right) \left(\Delta^a + Q^a(0) + 2\sigma^a \sup_{0 \leq t \leq T} |W^a(t)|\right) + (\bar{p}^b + \epsilon) \left((P^b(0) - \bar{p}^b)^+ + N^b(T)\right) \left(\Delta^b + Q^b(0) + 2\sigma^b \sup_{0 \leq t \leq T} |W^b(t)|\right).$$

Considering the hidden orders, the trader could receive at most $\Delta^a H^a(T)$ shares of hidden limit sell orders and $\Delta^b H^b(T)$ shares of hidden limit buy orders. The greatest possible price for each share does not exceed the bounds in $[4.2]$. The stock inventory and cash amount from hidden orders can be bounded by

$$|I^h(T)| \leq \Delta^a H^a(T) + \Delta^b H^b(T);$$

$$|C^h(T)| \leq \Delta^a H^a(T) \left(P^a(0) + \bar{p}^a + N^a(T) + R^a(T) + 1\right) + \Delta^b H^b(T) \left(P^b(0) + \bar{p}^b + N^b(T) + R^b(T) + 1\right).$$

By the inequalities $[4.2]$ and $[4.2]$, by Lemma $4.1$, and by applying Burkholder-Gundy-Davis inequality to $\sup_{0 \leq t \leq T} |W^a(t)|$ and $\sup_{0 \leq t \leq T} |W^b(t)|$, the claim in this lemma can be justified. $\Box$

Theorem 4.1 There exists a constant $c_4 > 0$, such that for any switching control $\alpha \in \mathcal{A}$ and any hidden order strategy $h \in \mathcal{H}$, we have

$$\mathbb{E} \left[|\xi(C(T), I(T)) - r^C \xi C_0|\right] \leq c_4 \left((Q^a(0))^4 + (Q^b(0))^4 + (I_0)^2 + (P^a(0))^4 + (P^b(0))^4 + T^4 + 1\right) < \infty.$$

Furthermore, for all $t \in [0, T]$, the value process $V(t)$ defined in $[4.1]$ has the growth rate

$$|V(t)| \leq c_4 \left((Q^a(t))^4 + (Q^b(t))^4 + (I(t))^2 + (P^a(t))^4 + (P^b(t))^4 + (T - t)^4 + 1\right).$$
Proof. By Assumption 3.1 (1), equation (4.2) and equation (4.1), we know that
\[ |ξ(C(T), I(T)) - r^C C_0| \leq |r^C| \cdot |C^h(T) + C^α(T)| + |r^I| \cdot |r^F(z + I^h(T) + I^α(T))|^2 + 1. \] (4.22)
Substituting the inequalities (4.2) and (4.2) into inequality (4.2) gives the inequality (4.1).
Because of the Markov property of the order book dynamics, the inequality in (4.1) can be derived in the same way as (4.1).

Theorem 4.1 implies that the value process $V$ defined in (4.1) indeed exists and is finite.

5 Relation between trading and price switching

This section will state in Theorem 5.1 the relation between Problem 3.1 and Problem 4.1. Deriving from Theorem 5.1, the result in Proposition 5.1 tells that the two price switching algorithms provide lower and upper bounds of value functions of the two mixed trading algorithms. Especially, when the premium $ε$ equals zero, the internalizing trader’s optimal mixed strategy can be achieved among the set of price switching strategies.

Remark 5.1 The results in this section will have three implications.
(1) They help prove well-posedness of all the control problems, as discussed at the beginning of Subsection 4.2.
(2) When the upper and lower bounds in Proposition 5.1 do not differ much, e.g. in Fig. 6.4, the implementable lower bound switching strategy is nearly optimal for the optimal trading problem. Corollary 5.1 will show that the latter is much harder to compute.
(3) The MiFID framework defines different types of traders and strategies. Theorem 5.1 compares their best expected profits.

Definition 5.1 (step trading strategy) The admissible set of step trading strategy of an internalizing trader and that of a regular trader are denoted respectively as $S^\text{int}$ and $S^\text{reg}$.
Let the letter $j$ represent either “int” or “reg”. Let $α = \{(S_n, a_n, b_n)\}_{n=0}^{∞} \in S^j$ be an arbitrary admissible switching control. The processes $Z^α_n$ and $Z^b_n$, being the total shares that the trader has bought and sold according to the switching control $α$, are computed from
\[
Z^α_n(t) = \sum_{n:S_n \leq t} g^i(S_n, Q^i(S_n^-), a_n^i) 1_{\{u_n^i \geq 0\}}, \quad 0 \leq t \leq T, \quad i = a, b, \tag{5.1}
\]
where the mappings $g^a$ and $g^b$ are defined in (4.1). When limit orders arrive within the spread, the proportion of shares that the trader fills at the old price $P^i(S_n^-)$ is computed from
\[
β^i_α(t) = \begin{cases}
\{u_n^i\} 1_{\{u_n^i \in [-1,0]\}}, & \text{if } t = S_n \text{ and } N^i(S_n) - N^i(S_n^-) = 1; \\
0, & \text{otherwise; } 0 \leq t \leq T, \quad i = a, b. \tag{5.2}
\end{cases}
\]
The set $S^j$ of admissible step trading strategies is defined as the collection of all the trading strategies $Z_α = (Z^α_n, Z^b_n, β^a_α, β^b_α)$ satisfying (5.1) and (5.1) for some switching control $α \in S^j$.
Namely, $S^\text{int} = \{Z_α | α \in S^\text{int}\}$ and $S^\text{reg} = \{Z_α | α \in S^\text{reg}\}$.

Seen from Definition 4.1 and Definition 5.1, each step trading strategy of an internalizing or regular trader is his price switching strategy denoted in terms of the total numbers of shares bought and sold, so they are the same active trading strategy under different names. The two definitions further imply that
\[
S^\text{reg} = \{Z_α = (Z^α_n, Z^b_n, β^a_α, β^b_α) \in S^\text{int}| β^a_α(t) \equiv β^b_α(t) \equiv 0, \text{ for all } t \in [0, T]\}. \tag{5.3}
\]

Notation 5.1 (1) An internalizing trader’s best expected reward over step trading strategies is denoted as
\[
V^{\text{stp, int}}(ε) := \sup_{Z_α \in S^\text{int}, h \in H} E[ξ(I^h Z_α(T), C^h Z_α(T))].
\]
A regular trader’s best expected reward over step trading strategies is denoted as
\[ V_{\text{stp,reg}} := \sup_{Z_\alpha \in \mathcal{S}_{\text{reg}}, h \in \mathcal{H}} \mathbb{E} \left[ \xi(I_h^h, Z_\alpha(T), C_h^h, Z_\alpha(T)) \right]. \]

Theorem 5.1 The value functions of the optimal trading problem and the optimal switching problem have the relations

1. \[ V_{\text{swt,reg}} = V_{\text{stp,reg}}; \]
2. \[ V_{\text{swt,int}}(\epsilon) = V_{\text{stp,int}}(\epsilon), \text{ for all } \epsilon \geq 0; \]
3. \[ V_{\text{mix,reg}} \leq V_{\text{mix,int}}(\epsilon), \text{ for all } \epsilon \geq 0; \]
4. \[ V_{\text{mix,int}}(\epsilon) \leq V_{\text{swt,int}}(\epsilon), \text{ for all } \epsilon \geq 0; \]
5. \[ V_{\text{mix,reg}} \leq V_{\text{mix,int}}(\epsilon), \text{ for all } \epsilon \geq 0; \]
6. \[ V_{\text{mix,int}}(\epsilon) \leq V_{\text{swt,int}}(\epsilon), \text{ for all } \epsilon \geq 0; \]
7. \[ V_{\text{mix,int}}(0) = V_{\text{stp,int}}(0); \]
8. Viewing \( \epsilon \) as the variable, the two functions \( V_{\text{swt,int}}(\epsilon) \) and \( V_{\text{mix,int}}(\epsilon) \) are decreasing in \( \epsilon \).

Theorem 5.1 can be proved by the results from the next subsections 5.1 and 5.2. An outline of the proof is provided here.

Outline of Proof of Theorem 5.1

(1) and (2) By their definitions, step trading strategies (Definition 5.1) and price switching strategies (Definition 4.1) have one-to-one correspondence between each other, because the two sets in fact consist of the same active trading strategies denoted in different terms. A price switching strategy denotes the times of transactions and the numbers of limits to buy and sell at each time; a step strategy denotes the total numbers of shares bought and sold up-to-date. This explains the first and last identities in Theorem 5.1.

(3), (4), (5) and (6) The three inequalities come from the inclusions in Lemma 5.1.

(7) This identity comes from Lemma 5.2 Lemma 5.1(1) and Proposition 5.2. The main idea of the proof is to construct, in Lemma 5.2, a step trading strategy \( Z_\alpha \in \mathcal{S}_{\text{int}} \) that path-wisely replicates the stock inventory and the cash amount produced by the mixed trading strategy \( Z \in \mathcal{Z}_{\text{int}} \). By Lemma 5.1(1), every step trading strategy is a mixed trading strategy. Furthermore, as will be shown by Proposition 5.2, an internalizing trader’s two active trading strategies \( Z \) and \( Z_\alpha \) path-wisely result in the same bid ask prices, replacing the former with the latter does not change the stock inventory and cash amount produced by a passive strategy on the hidden orders.

(8) The quantities \( C_h^h, Z \) defined according to (3.1), (3.1), (3.1) and (3.2), and \( C_h^h, \alpha \) defined according to (4.1)-(6), viewed as functions in \( \epsilon \), are decreasing. The reward criterion \( \xi \) defined in (3.2) is increasing in the cash amount, because the coefficient \( r_C \) is positive. Hence \( V_{\text{mix,int}}(\epsilon) \) defined in (3.1) and \( V_{\text{swt,int}}(\epsilon) \) defined in (4.1) are decreasing in \( \epsilon \).

\[ \square \]

Proposition 5.1 The value functions of the price switching problem provide lower and upper bounds for the value functions of the optimal trading problem. (1) If the optimal trader is regular, then
\[ V_{\text{swt,reg}} \leq V_{\text{mix,reg}} \leq V_{\text{swt,int}}(0). \] (5.3)

(2) If the optimal trader is an internalizer, then
\[ V_{\text{mix,int}}(0) = V_{\text{swt,int}}(0) \] (5.4)

and
\[ V_{\text{swt,int}}(\epsilon) \leq V_{\text{mix,int}}(\epsilon) \leq V_{\text{swt,int}}(0), \text{ for all } \epsilon \geq 0. \] (5.5)

Proof. (1) The first inequality in (5.1) comes from Theorem 5.1(1)(3). The second inequality in (5.1) comes from Theorem 5.1(2)(5)(7).

(2) By Theorem 5.1(2)(7), the identity (5.1) holds. The first inequality in (5.1) comes from Theorem 5.1(2)(4). To prove the second inequality in (5.1), by Theorem 5.1(8) it holds that \( V_{\text{mix,int}}(\epsilon) \leq V_{\text{mix,int}}(0) \), the right hand side of which equals \( V_{\text{swt,int}}(0) \) by (5.1). \[ \square \]
5.1 Analysis of active strategies

It can be verified that the processes $Z_{\alpha}^b$, $Z_{\alpha}^b$, $\beta_{\alpha}^a$ and $\beta_{\alpha}^b$ defined in (4.1) and (5.1) satisfy Definition 3.1 so every step trading strategy $Z_{\alpha} = (Z_{\alpha}^a, Z_{\alpha}^b, \beta_{\alpha}^a, \beta_{\alpha}^b) \in \mathcal{F}$ is a mixed trading strategy in the admissible set $\mathcal{F}$, for $j = \text{int, reg}$. However, the contrary is not true, because a mixed trading strategy can be continuous over some time interval, but a step trading strategy is a pure jump process. By Definition 3.1(2), a regular trader’s admissible set of mixed trading strategies is the subset of an internalizing trader’s mixed trading strategies that do not fill orders at the old price at the time of order arrival within the spread. By their definitions in (4.1) and (4.1) of Definition 4.1 there is the set inclusion $D^{\text{reg}} \subsetneq D^{\text{int}}$. It follows that a regular trader’s price switching strategy is a proper subset of an internalizing trader’s price switching strategy.

Lemma 5.1 The admissible sets $\mathcal{F}^{\text{int}}$, $\mathcal{F}^{\text{reg}}$, $\mathcal{F}^{\text{int}}$, $\mathcal{F}^{\text{reg}}$, $\mathcal{F}^{\text{int}}$ and $\mathcal{F}^{\text{reg}}$ of trading strategies defined in Definition 4.1 and Definition 5.1 have the inclusion relations

$(1) \mathcal{F}^{\text{reg}} \subsetneq \mathcal{F}^{\text{reg}}$; $(2) \mathcal{F}^{\text{int}} \subsetneq \mathcal{F}^{\text{int}}$; $(3) \mathcal{F}^{\text{reg}} \subsetneq \mathcal{F}^{\text{int}}$; $(4) \mathcal{F}^{\text{reg}} \subsetneq \mathcal{F}^{\text{int}}$.

When the premium $\epsilon$ equals zero, the set of his admissible step trading strategies performs equally well as an internalizing trader’s set of admissible mixed trading strategies, though the former is a much smaller subset of the latter as stated in Lemma 5.1(1). Whatever stock inventory and cash amount a mixed trading strategy can produce at the terminal time, an internalizing trader can always find a step trading strategy that path-wisely does the same. Hence an internalizing trader’s best expected reward can be achieved over a smaller and simpler set of admissible trading strategies. This is the role of Lemma 5.2.

Lemma 5.2 Suppose the premium $\epsilon$ equals zero. For any admissible mixed trading strategy

$Z = (Z_{\alpha}^a, Z_{\alpha}^b, \beta_{\alpha}^a, \beta_{\alpha}^b) \in \mathcal{F}^{\text{int}}$, there exists an admissible switching control $\alpha = \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty} \in \mathcal{A}$ such that the step trading strategy $Z_{\alpha} = (Z_{\alpha}^a, Z_{\alpha}^b, \beta_{\alpha}^a, \beta_{\alpha}^b) \in \mathcal{F}^{\text{int} \setminus \alpha}$ defined by (5.1) and (5.1) for this $\alpha$ almost surely satisfies

$I^Z(T) = I^{Z_{\alpha}}(T)$ and $C^Z(T) = C^{Z_{\alpha}}(T).

(5.6)

Outline of Proof. It suffices to construct a specific $\alpha' = \{(S'_n, u'_n, u'_n)\}_{n=0}^{\infty} \in \mathcal{A}$ such that the step trading strategy $Z_{\alpha'} = (Z_{\alpha'}^a, Z_{\alpha'}^b, \beta_{\alpha'}^a, \beta_{\alpha'}^b)$ defined in (5.1) and (5.1) for this $\alpha'$ satisfies the identities in (5.2). Since the actual construction takes three pages, the detailed proof is omitted from the paper.

5.2 Effect on an internalizing trader’s hidden orders

Given an arbitrary admissible mixed trading strategy $Z$ from the internalizing trader, let the step trading strategy $Z_{\alpha'}$, for some $\alpha' = \{(S'_n, u'_n, u'_n)\}_{n=0}^{\infty} \in \mathcal{A}$, be the one constructed in Lemma 5.2 to replicate the terminal stock inventory and cash amount. Let $P_a^Z$ and $P_b^Z$ denote the price processes (3.1) controlled by the mixed trading strategy $Z$, and $P_a^{\alpha'}$ and $P_b^{\alpha'}$ denote the price processes (4.1) controlled by the switching control $\alpha'$. The construction is such that the two strategies also produce the same times and amounts of price change, meaning that

$P_a^Z(t) = P_a^{\alpha'}(t)$ and $P_b^Z(t) = P_b^{\alpha'}(t)$, for all $(t, \omega) \in [0, T) \times \Omega.

(5.7)

Let us recall that the intensities of the liquidity event processes $H^a$ and $H^b$ are functions of the spread only. The equations (2.8) and (5.2) further imply that, both the inventory $I^b$ and cash amount $C^b$ from an arbitrary hidden order strategy $h = (h^a, h^b) \in \mathcal{H}$ remain the same regardless of whether the mixed trading strategy $Z_{\alpha'}$ or the step trading strategy $Z_{\alpha'}$ is used. The analysis in this paragraph has verified a reinforcement of Lemma 5.2 stated as the proposition below.
Proposition 5.2 Suppose the premium $\epsilon$ equals zero. Let $h = (h^a, h^b) \in \mathcal{H}$ be an arbitrary hidden order strategy. For any admissible mixed trading strategy $Z = (Z^a, Z^b, \beta^a, \beta^b) \in \mathcal{X}^{int}$, there exists an admissible switching control $\alpha = \{(S_n, u_n^a, u_n^b)\}_{n=0}^{\infty} \in \mathcal{X}^{int}$ such that the step trading strategy $Z_\alpha = (Z_{\alpha}^a, Z_{\alpha}^b, \beta_{\alpha}^a, \beta_{\alpha}^b) \in \mathcal{X}^{int}$ defined in \eqref{5.1} and \eqref{5.1} for this $\alpha$ almost surely satisfies

$$I^{h,Z}(T) = I^{h,Z_\alpha}(T) \text{ and } C^{h,Z}(T) = C^{h,Z_\alpha}(T),$$

where $I^{h,Z}(T)$, $I^{h,Z_\alpha}(T)$, $C^{h,Z}(T)$ and $C^{h,Z_\alpha}(T)$ are defined in \eqref{3.2} and \eqref{3.1}.

6 Solving the optimal switching problem

This section will provide the characterization of an optimal trading strategy and derive a trading algorithm for the optimal switching Problem \ref{4.1}. The solution is valid regardless of whether the trader is internalizing or regular, hence the admissible set of switching controls is generically denoted as $\mathcal{A}$. Before getting down to the solution, a few notations are introduced.

The two-dimensional quantities representing both sides of the order book are denoted as $Q(t) = (Q^a(t), Q^b(t))$, $P(t) = (P^a(t), P^b(t))$, $q = (q^a, q^b)$, $p = (p^a, p^b)$, $u = (u^a, u^b)$ and $h = (h^a, h^b)$ for short. As will be shown in Theorem \ref{6.1} the decision making would only need to observe the state processes $\{(N^a, N^b)\}_{0 \leq t \leq T}$ and $\{(Q(t), I^a(t-)+I^b(t), P(t-))\}_{0 \leq t \leq T}$, which generate a smaller filtration than $\mathcal{F}(t)$. The domain of the process $\{(Q(t), I^a(t-)+I^b(t), P(t-))\}_{0 \leq t \leq T}$ is denoted as

$$\mathcal{D} = [0, \infty)^2 \times \mathbb{R} \times \{(p^a, p^b) \in (P^a(0), P^b(0)) + \mathbb{N}^2 | p^a > p^b \}.$$ 

To express the change in the order book and in the inventory from the trader’s transaction, the mapping $\gamma : \Omega \times [0, T] \times \mathcal{D} \times \mathbb{N}^2 \to [0, \infty)^2 \times \mathbb{R}$ is defined as

$$\gamma(t, q, z, u) = \begin{pmatrix} 1_{\{u^a \neq 0\}} \Delta^a + 1_{\{u^a = 0\}}(1 - \{u^a\})q^a \\ 1_{\{u^b \neq 0\}} \Delta^b + 1_{\{u^b = 0\}}(1 - \{u^b\})q^b \\ z + g_a(t, q, u) \end{pmatrix}^{\text{transpose}}.$$ 

Immediately after applying the switching control $u$ at time $t$, the volumes and inventory become

$$(Q(t), I(t)) = \gamma(t, Q(t-), I^a(t-)+I^b(t), u).$$ \hfill (6.1)

The process $\{\int_0^t r(P(s-), h(s-))ds\}_{0 \leq t \leq T}$ defined as

$$r(P(t), h(t)) = -\Delta^a h^a(t) \left( (P^a(t) + P^b(t)) / 2 \right) \lambda^a(P^a(t) - P^b(t))$$

$$+ \Delta^b h^b(t) \left( (P^a(t) + P^b(t)) / 2 \right) \lambda^b(P^a(t) - P^b(t)), \quad 0 \leq t \leq T,$$

is the finite variation part in the Doob-Meyer decomposition of the semimartingale $C^h$ defined in \eqref{5.26}. By the bound in \eqref{4.12}, the local martingale part of $C^h$ is a martingale. Then the value process $V$ defined in \eqref{4.11} can be written alternatively as

$$V(t) = \sup_{\alpha \in \mathcal{A}, h \in \mathcal{H}} \mathbb{E} \left[ r^C \sum_{n=1}^{\infty} f_\alpha(S_n, Q(S_n-), P(S_n-), u_n) \right.$$

$$+ r^C \int_0^T r(P(s-), h(s-))ds + r^C F(I^{h,\alpha}(T), P(T)) \bigg| \mathcal{F}(t) \bigg], \quad 0 \leq t \leq T,$$

$$\text{\hspace{13cm}} \hfill (6.2)$$
because $E[\xi(I(T), C(T))|\mathcal{F}(t)] - rC(t)$ equals the expectation on the right hand side of the above equation. For every $h \in \mathcal{H}_{0,t}$, the process $Y(\cdot; h)$ is defined as

$$Y(t; h) = \int_0^t r(P(s^-), h(s^-))ds + V(t), \ 0 \leq t \leq T.$$ 

For every measurable function $\phi : [0, T] \times \mathcal{D} \to \mathbb{R}$, the operator $\mathcal{M}$ is defined as

$$\mathcal{M} \phi(t, q, z, p) = \max_{u \in \mathcal{U}(t, p)} \left\{ f_\alpha(t, q, p, u) + \phi(t, \gamma(t, q, z, u), P^a + [u^a], P^b - [u^b]) \right\}.$$ 

### 6.1 Optimal trading strategy

This subsection will eventually derive, in Proposition 6.1, expressions of an optimal price switching strategy in terms of the value process. The methodology is based on the principle that the value process of a control problem is a supermartingale, and becomes a martingale if and only if the control is optimal. It is called the “martingale method”, first introduced for optimal stopping problems in Snell (1952) and for stochastic control problems in Davis (1979). The pivot of the arguments is the dynamic programming principle formulated in our setting as Theorem 6.1. A reference of the dynamic programming principle is Fleming and Soner (1993). Lemma 6.1 provides the right continuity of the value process, so that it is a qualified candidate for using the Snell envelop technique to sequentially determine each optimal time of trading. Lemma 6.2 is the characterization of the optimal trading strategy from the martingality of the value process. Because Theorem 4.1 has shown that the value process is finite, the expressions in Proposition 6.1 imply the existence of an optimal trading strategy.

**Theorem 6.1 (dynamic programming principle)** Given $(Q(t^-), I^a(t^-) + I^b(t), P(t^-)) = (q, z, p)$, there exist deterministic measurable functions $v^0$, $v^a$ and $v^b : [0, T] \times \mathcal{D} \to \mathbb{R}$, and a mapping $v : \Omega \times [0, T] \times \mathcal{D} \to \mathbb{R}$, such that the value process $V$ defined by the equation (4.1) satisfies

$$V(t) = v(t, q, z, p) = \begin{cases} 
  v^0(t, q, z, p), & \text{if } N^i(t) - N^i(t^-) = 0, \ i = a \text{ and } b; \\
  v^i(t, q, z, p), & \text{if } N^i(t) - N^i(t^-) = 1, \ i = a \text{ or } b. \end{cases} \quad (6.3)$$

The value functions $v^0$, $v^a$ and $v^b$ can be computed via the dynamic programming principle

$$v^0 \left( t, Q(t^-), I^a(t^-) + I^b(t), P(t^-) \right) = \sup_{(S_1, u_1) \in \mathcal{A}_1, \ b \in \mathcal{H}_t} \mathbb{E} \left[ r^C \left( f_\alpha(S_1, Q(S_1^-), P(S_1^-), u_1) + \int_t^{S_1} r(P(t), h(s^-))ds \right) + v \left( S_1, \gamma(S_1, Q(S_1^-), I^a(S_1^-) + I^b(S_1^-)), P^a(S_1^-) + [u^a_1], P^b(S_1^-) - [u^b_1] \right) \bigg| \mathcal{F}(t) \right],$$

(6.4)

when $N^a(t) - N^a(t^-) = 0$ and $N^b(t) - N^b(t^-) = 0$, and

$$v^i \left( t, Q(t^-), I^a(t^-) + I^b(t), P(t^-) \right) = \mathcal{M} v^0 \left( t, Q(t^-), I^a(t^-) + I^b(t), P(t^-) \right) \quad (6.5)$$

when $N^i(t) - N^i(t^-) = 1, \ i = a, b$. Especially, at the terminal time $T$, the value function satisfies the terminal condition

$$v \left( T, Q(T^-), I^a(T^-) + I^b(T), P(T^-) \right) = \mathcal{M} F(I(T^-), P(T^-)).$$

**Proof.** The existence of the functions $v^0$, $v^a$ and $v^b$ comes from the Markovian structure of the processes $(Q, I, P)$, and the memoryless property of the exponential inter-arrival times.
for the orders within the spread. In our context, the proof of the dynamic programming principle is routine. To wit, take arbitrary $\alpha \in \mathcal{A}$ and $h \in \mathcal{H}$ as defined in Definitions 4.1 and 2.1, and denote for short

$$
\Sigma^1 = r^C \left( f_0(S_1, Q(S_1), P(S_1), u_1) + \int_t^{S_1} r(P(t), h(s^-))ds \right);
$$

$$
\Sigma^2 = r^C \sum_{n=2}^{\infty} \left( f_0(S_n, Q(S_n), P(S_n), u_n) + \int_{S_{n-1}}^{S_n} r(P(S_{n-1}), h(s^-))ds \right) + r^I F(I(T), P(T));
$$

$$
\Sigma^3 = V_S(S_1, Q(S_1), I^a(S_1) + I^b(S_1), u_1) + P^a(S_1) + [u^a_1], P^b(S_1) - [u^b_1]) .
$$

Then by the same reasoning that derives equation (6) and by the law of iterated expectations, we have

$$
E \left[ \xi(I(T), C(T)) \right] - r^C C(t) = E \left[ \Sigma^1 + E \left[ \Sigma^2 \mid \mathcal{F}(S_1) \right] \right] \mathcal{F}(t) .
$$

Because

$$
\Sigma^3 = \sup_{\alpha \in \mathcal{A}_1, h \in \mathcal{H}_1} E \left[ \Sigma^2 \mid \mathcal{F}(S_1) \right]
$$

by equations (6) and (6.1), we know that

$$
E \left[ \Sigma^2 \mid \mathcal{F}(S_1) \right] \leq \Sigma^3.
$$

Taking supremum over $(S_1, u_1) \in \mathcal{A}_1, \mathcal{H}_1$ on both sides of the inequality

$$
E \left[ \Sigma^1 + E \left[ \Sigma^2 \mid \mathcal{F}(S_1) \right] \right] \mathcal{F}(t) \leq E \left[ \Sigma^1 + \Sigma^3 \right] \mathcal{F}(t)
$$

and using the equations (4.1) and (6.1), we prove that $V(t)$ is less than or equal to the right hand side of (6.1).

We know from equations (4.1) and (6.1) that

$$
V(t) \geq E \left[ \Sigma^1 + E \left[ \Sigma^2 \mid \mathcal{F}(S_1) \right] \right] \mathcal{F}(t) ,
$$

for arbitrary $\alpha \in \mathcal{A}$ and $h \in \mathcal{H}$. The expressions (6.1) and (6.1) imply that

$$
V(t) \geq E \left[ \Sigma^1 + \Sigma^3 \right] \mathcal{F}(t) ,
$$

and thus $V(t)$ greater than or equal to the right hand side of (6.1). Both sides of the inequality hold, hence

$$
V(t) = \sup_{(S_1, u_1) \in \mathcal{A}_1, h \in \mathcal{H}_1} \mathbb{E} \left[ r^C \left( f_0(S_1, Q(S_1), P(S_1), u_1) + \int_t^{S_1} r(P(t), h(s^-))ds \right)
$$

$$
+ V(S_1, \gamma(S_1), I^a(S_1) + I^b(S_1), u_1), P^a(S_1) + [u^a_1], P^b(S_1) - [u^b_1]) \mid \mathcal{F}(t) \right] .
$$

(6.10)

When $N^a(t) - N^a(t^-) = 0$ and $N^b(t) - N^b(t^-) = 0$, by (6.1) there is

$$
V(t) = v^0 \left( t, Q(t^-), I^a(t^-) + I^b(t), P(t^-) \right),
$$

hence (6.1) takes the form (6.1). When $N^a(t) - N^a(t^-) = 1$ or $N^b(t) - N^b(t^-) = 1$ or $t = T$, the trader has to “trade” at time $t$, though possibly zero share, hence (6.1) takes the form (6.1) or (6.1).
Lemma 6.1 For every $0 \leq t < t + \Delta t \leq T$, suppose the trader does not trade over the time interval $[t, t + \Delta t]$. Then the processes $\{v^i(t, Q(t-), I^a(t-)+I^b(t), P(t-))\}_{0 \leq t \leq T}$, $i = 0, a, b$, are continuous in the time $t$, meaning that

$$\lim_{|t-t'| \to 0^+} |v^i(t, Q(t-), I^a(t-)+I^b(t), P(t-)) - v^i(t', Q(t'-), I^a(t'-)+I^b(t'), P(t'-))| = 0.$$ 

Proof. It suffices to prove the continuity of $v^0$, then the continuity of $v^a$ and $v^b$ follows from the expression (6.1).

Take two arbitrary times $t \leq t' \in [0, T]$, an arbitrary price switching strategy $\alpha \in \mathcal{A}_t$ and an arbitrary hidden order strategy $h \in \mathcal{H}_t$. Denote $\Delta t := t' - t$. Suppose $N^a(t) - N^a(t-) = 0$ and $N^b(t) - N^b(t-) = 0$.

(continuity in the volume $q$) For any two sets of initial values $(q, z, p)$ and $(q', z, p) \in \mathcal{D}$, the resulted state processes are respectively denoted as $(Q, I, P)$ and $(Q', I', P')$. If the trader never trades, then, taking the ask side for example, there are three possibilities of the dynamics.

1. Limit sell orders arrive in the spread before either $Q^a(\cdot) = q^a + \sigma^a(W^a(\cdot) - W^a(t))$ or $Q^a(\cdot) = q^a + \sigma^a(W^a(\cdot) - W^a(t))$ reaches zero, in which case the new ask prices $P^a$ and $P^{a'}$ equal $p^a - 1$ and the volumes $Q^a$ and $Q^{a'}$ both become $\Delta^a$ at the time of arrival.

2. Limit sell orders does not arrive in the spread before both $Q^a(\cdot)$ and $Q^{a'}(\cdot)$ reach zero. At the time of arrival the new ask prices $P^a$ and $P^{a'}$ equal $p^a + 1$ and the volumes $Q^a$ and $Q^{a'}$ still differ by $\left| q^a - q^{a'} \right|$.\vspace{4pt}

3. Limit sell orders arrive in the spread when one of $Q^a(\cdot)$ and $Q^{a'}(\cdot)$ has reached zero and the other one has not.

In cases (1) and (2), the difference in the trader’s stock inventory does not exceed $\left| q^a - q^a' \right| + \left| q^b - q^{a'} \right|$, and that in the cash amount does not exceed $(\bar{p}^a + \epsilon) \left| q^a - q^a' \right| + (\bar{p}^b + \epsilon) \left| q^b - q^{a'} \right|$. The probability that type (3) events ever happen over the entire time horizon $[0, T]$ converges to zero, as $\left| q^a - q^a' \right| \to 0$. By Assumption 3.1 (2) and from the bounds of the stock, price inventory and cash amount in Lemma 4.1 and Proposition 4.2, we know that

$$v^0(t, q, z, p) = \lim_{q' \to q} v^0(t, q', z, p).$$ \hspace{1cm} (6.11)

(continuity in the time $t$) Take any initial values $(q, z, p) \in \mathcal{D}$ and any number $\Delta t \in [0, T - t]$. Let $\mathcal{A}_t(T - \Delta t)$ denote the active trading strategies with the terminal time $T$ replaced by $T - \Delta t$. Then there is the relation

$$v^0(t + \Delta t, q, z, p) = \left( \sup_{\alpha \in \mathcal{A}_t(T - \Delta t), h \in \mathcal{H}_t} \mathbb{E} \left[ \xi(I(T - \Delta t), C(T - \Delta t)) \mathcal{F}(t) \right] - \nu C(t) \right).$$

Because the change in the order book dynamics during the time interval $[T - \Delta t, T]$ is of the order $O(\Delta t)$, the best expected reward from terminating at the time $T - \Delta t$ or at the time $T$ differs up to $O(\Delta t)$, meaning that

$$v^0(t, q, z, p) = v^0(t + \Delta t, q, z, p) + O(\Delta t).$$ \hspace{1cm} (6.12)

The continuity of the process $\{v^0(t, Q(t-), I^a(t-)+I^b(t), P(t-))\}_{0 \leq t \leq T}$ can be concluded from the identities (6.1) and (6.1) and from the properties of Brownian motions and Poisson processes that drive the state process. \hfill \square

The proofs of Lemma 6.2 and Proposition 6.1 follow the routine procedure on how to characterize the optimal control and optimal stopping time via the martingale method. Because they are very long, the proofs are not provided here. Interested readers could find the original idea in Davis (1979) and Snell (1952), and the arguments for a most similar result in Section 2.2.2 of Li (2011).
Lemma 6.2 The price switching strategy \( \alpha^* = (S_1^*, u_1^*) \in \mathcal{A}_{t,1} \) and hidden order strategy \( h^* \in \mathcal{H}_{t,S_1^*} \) achieve the supremum in (6.1), if and only if all of the four conditions below hold.

(1) \( \{Y(t;h)\}_{0 \leq t \leq T} \) is a supermartingale, for every \( h \in \mathcal{H}_{t,T_1} \);
(2) \( \{Y(t \wedge S_1^*; h^*)\}_{0 \leq t \leq T} \) is a martingale;
(3) either \( (v^0 - \mathcal{M}v^0)(S_1^*, Q(S_1^*), I^\alpha(S_1^* -), I^h(S_1^*)), P(S_1^*) = 0 \), or \( S_1^* = T_1 \);
(4) \( u_1^* = \arg \max_{u \in \mathcal{U}(S_1^*, P(t))} \left\{ r^C \left( f_\alpha(S_1^*, Q(S_1^*), P(t), u_1^*) \right) + \int_{t}^{S_1^*} r(P(t), h^*(s-))ds + v(S_1^*, \gamma(S_1^*, Q(S_1^*), I^\alpha(S_1^* -) + I^h(S_1^*), P(t), u_1^*), P^a(t) + [u_1^*, P^b(t) - [u_1^*]] \right\} \).

Proposition 6.1 There exist an optimal switching control \( \alpha^* = \{(S_n^*, u_n^*, u_n^{b_n})\}_{n=1}^{\infty} \in \mathcal{A} \) and an optimal hidden order strategy \( h^* = (h^{a_n}, h^{b_n}) \in \mathcal{H} \), which are defined in the following way. Let \( S_0^* = u_0^* = u_0^{b_0} = 0 \). For \( n = 1, 2, \cdots \), the optimal trading time \( S_n^* \) can be expressed as

\[
S_n^* = \begin{cases} 
\inf \left\{ S_{n-1}^* - t \leq T \mid (v^0 - \mathcal{M}v^0)(t, Q(t-), I^\alpha^*(t-), h^*(t-), P(t-)) = 0 \right\} \wedge T_n, \\
if S_{n-1}^* < T; \\
T + 1, if S_{n-1}^* = T.
\end{cases}
\]

If \( S_n^* = T + 1 \), then \( u_n^* = u_n^{b_n} = 0 \); otherwise

\[
u_n^* = \arg \max_{u \in \mathcal{U}(S_n^*, P(S_{n-1}^* -))} \left\{ r^C f_\alpha(S_n^*, Q(S_{n-1}^*), P(S_{n-1}^* -), u) + v(S_n^*, \gamma(S_n^*, Q(S_{n-1}^*), I^\alpha(S_n^* -) + I^h(S_n^*), P(S_{n-1}^*), u), P^a(S_{n-1}^*) + [u^a, P^b(S_{n-1}^*) - [u^b]] \right\}.
\]

Denoting as

\[
v^0 \left( t, Q(t), I^\alpha^*(t) + I^h^*(t-), P(t), h(t) \right)
\]

\[
:= \left( v^0 \left( t, Q(t), I^\alpha^*(t) + I^h^*(t-), P(t) \right) + \Delta^a \right) - v^0 \left( t, Q(t), I^\alpha^*(t) + I^h^*(t-), P(t) \right)
\]

\[
\cdot h^a(t) \lambda^a \left( P^a(t) - P^b(t) \right) + \left( v^0 \left( t, Q(t), I^\alpha^*(t) + I^h^*(t) - \Delta^b, P(t) \right) - v^0 \left( t, Q(t), I^\alpha^*(t) + I^h^*(t), P(t) \right) \right)
\]

\[
\cdot h^b(t) \lambda^b \left( P^a(t) - P^b(t) \right),
\]

for \( 0 \leq t \leq T \), the optimal hidden order strategy \( h^* \) can be expressed as

\[
h^*(t) = \arg \max_{h(t) \in [0,1]^2} \left\{ r(P(t), h(t)) + v^0 \left( t, Q(t), I^\alpha^*(t) + I^h^*(t), P(t), h(t) \right) \right\}.
\]

6.2 Numerical algorithm

This subsection will present the numerical algorithm to compute the value function and optimal trading strategy for the discretized version of the optimal price switching Problem [4,1]. We shall specify different complexities of this algorithm on a serial computer and on a GPU cluster.

The time and the state process are discretized over a grid \( \mathcal{S} \times \mathcal{X} \), where \( \mathcal{S} \) as the grid for the time \( t \in [0,T] \) is defined as

\[
\mathcal{S} = \{0 = t_0, t_1, t_2, \cdots, t_K = T\} = \{0, \Delta t, 2\Delta t, \cdots, K \Delta t = T\}
\]
and \( \mathcal{X} \) as the grid for the state process \((Q, I, P)\) is a bounded set in \( \mathcal{D} \) with \(|\mathcal{X}| < \infty\) elements. When the grid tends finer and finer, the limit
\[
\lim_{K \to \infty, |\mathcal{X}| \to \infty} \mathcal{J} \times \mathcal{X}
\]
is assumed to be a dense set in \([0, T] \times \mathcal{D}\).

The algorithm takes the three steps to be specified below. Its outputs will be the value function and the optimal trading strategy \((\bar{v}^0(t_k, x), \bar{u}^0(t_k, x), \bar{h}^0(t_k, x))\) when limit orders do not arrive within \((t_{k-1}, t_k)\), and \((\bar{v}^1(t_k, x), \bar{u}^1(t_k, x))\) when limit sell (buy) orders arrive within \((t_{k-1}, t_k)\), for all \((t_k, x) \in \mathcal{J} \times \mathcal{X}\). The pseudo codes are provided in Table 6.1.

**Step 1.** (at the terminal time) At the time \(t_K = T\), the terminal condition is \(\bar{v}(T, x) = \mathcal{M} F(z, p)\).

**Step 1.1** Compute the reward from trading at the terminal time for every trading strategy \(u \in \mathcal{U}(T, p)\) as
\[
\bar{v}(T, x; u) = \bar{v}^0(T, x; u) = \bar{v}^1(T, x; u) = f_{\alpha}(T, q, p, u) + F(z + g_{\alpha}(T, q, u), p).
\]

**Step 1.2** The maximum reward from trading at the terminal time is
\[
\bar{v}(T, x) = \bar{v}^0(T, x) = \bar{v}^1(T, x) = \max \{\bar{v}(T, x; u) | u \in \mathcal{U}(T, p)\}.
\]

The optimal trading strategy is
\[
\bar{u}^*(T, x) = \{u \in \mathcal{U}(T, p) | \text{such that } \bar{v}(T, x; u) = \bar{v}(T, x)\}.
\]

**Step 2.** (simulate the controlled state process) With the initial values
\[
X_{t_k, x}(t_k; u) := (Q_{t_k, x}(t_k; u), I_{t_k, x}(t_k; u), P_{t_k, x}(t_k; u)) = (\gamma(t_k, q, z, p, u), p^a + [u^a], p^b - [u^b]),
\]
simulate the state process
\[
X_{t_k, x}(t_{k+1}; u, h) = (Q_{t_k, x}(t_{k+1}; u), I_{t_k, x}(t_{k+1}; u, h), P_{t_k, x}(t_{k+1}; u))\]
according to
\[
\left\{
\begin{array}{l}
Q_{t_k, x}^i(t_{k+1}; u) = Q_{t_k, x}^i(t_k; u) + \sigma^i \times \text{(a Normal r.v. with mean zero and variance } \Delta t)\
Q_{t_k, x}^i(t_{k+1}; u) = Q_{t_k, x}^i(t_k; u) + \Delta^1 1_{\{Q_{t_k, x}(t_{k+1}; u) > 0\}}; \\
I_{t_k, x}(t_{k+1}; u, h) = I_{t_k, x}(t_k; u) + \Delta^a h^a \times \text{(a Poisson r.v. with intensity } \lambda^a (p^a - p^b) \Delta t); \\
P_{t_k, x}^a(t_{k+1}; u) = p^a + [u^a] + 1_{\{Q_{t_k, x}^i(t_{k+1}; u) > 0\}}; \\
P_{t_k, x}^b(t_{k+1}; u) = p^b - [u^b] - 1_{\{Q_{t_k, x}^i(t_{k+1}; u) \leq 0\}}.
\end{array}
\right.
\]

So that the state process remains within the grid \(\mathcal{X}\), the truncated value from each simulation \(X_{t_k, x}(t_{k+1})\) is obtained from
\[
\bar{X}_{t_k, x}(t_{k+1}; u, h) := \arg\min \{ |X_{t_k, x}(t_{k+1}; u, h) - y| | y \in \mathcal{X}| \}
\]
(It is possible to directly simulate the truncated values.)
Run \(M\) simulations to get \(\bar{X}_{t_k, x}(t_{k+1}; u, h)\) according to the equations (6.2), (6.2) and (6.2).
The $M$ simulated values are denoted as $\{\tilde{X}^m_{t_k,x}(t_{k+1};u,h)\}_{m=1}^M$. For $i = a, b$, simulate $M$ Poisson random variables $\{N^i_m\}_{m=1}^M$ with the intensity $\Theta(p^a - p^b)\Delta t$ to represent whether limit orders arrive within the spread during the time interval $(t_k, t_{k+1})$.

Step 3. (value function and optimal trading strategy) This step conducts the optimization procedure by the dynamic programming principle

$$v(t_k, x) = \max_{u \in U(t_k, p), h \in \{0,1\}^2} \left\{ \left( \frac{p^a + p^b}{2} h^a \lambda^b (p^a - p^b) - \frac{p^a + p^b}{2} h^b \lambda^a (p^a - p^b) \right) \Delta t + f_a(t_k, q, p, u) + \mathbb{E} [\tilde{v}(t_k, X_{t_k,x}(t_{k+1};u,h))] \right\}. \quad (6.16)$$

Step 3.1 (approximating the expectation) The conditional expectation $\mathbb{E} [\tilde{v}(t_k, X_{t_k,x}(t_{k+1}))]$ in (6.2) is approximated by computing

$$\tilde{v}(t_k, x; u, h) := \frac{1}{M} \sum_{m=1}^M \left( 1_{\{N^a_m = 0, N^b_m = 0\}} \tilde{v}^0(t_k, \tilde{X}^m_{t_k,x}(t_{k+1};u,h)) + \sum_{i=a,b} 1_{\{N^i_m = 0\}} \tilde{v}^i(t_k, \tilde{X}^m_{t_k,x}(t_{k+1};u,h)) \right).$$

Step 3.2 (value function and trading strategy when no arrival within the spread) This is the case when there is no limit order arrival within the spread throughout the time interval $(t_{k-1}, t_k)$, meaning that $N^i(t_k) - N^i(t_{k-1}) = 0$, for $i = a$ and $b$. The reward from using a generic trading strategy $u \in U(t_k, p)$ and $h \in \{0,1\}^2$ is

$$v^0(t_k, x; u, h) = \left( \frac{p^a + p^b}{2} h^a \lambda^b (p^a - p^b) - \frac{p^a + p^b}{2} h^b \lambda^a (p^a - p^b) \right) \Delta t + f_a(t_k, q, p, u) + \tilde{v}(t_k, x; u, h). \quad (6.17)$$

The optimal value from trading is

$$\tilde{v}^0(t_k, x) = \max \left\{ \tilde{v}^0(t_k, x; u, h) | u \in U(t_k, p) \text{ and } h \in \{0,1\}^2 \right\}. \quad (6.18)$$

The optimal trading strategy is

$$(\tilde{u}^0(t_k, x), \tilde{h}^0(t_k, x)) = \left\{ u \in U(t_k, p) \text{ and } h \in \{0,1\}^2 | \text{ such that } \tilde{v}^0(t_k, x; u, h) = \tilde{v}^0(t_k, x) \right\}. \quad (6.19)$$

Step 3.3 (value function and trading strategy when there is arrival within the spread) This is the case when limit orders arrive within the spread at some point during the time interval $(t_{k-1}, t_k)$, meaning that $N^i(t_k) - N^i(t_{k-1}) = 1$, for $i = a$ or $b$. The reward from using a generic trading strategy $u \in U(t_k, p)$ is

$$v^i(t_k, x; u) = f_a(t_k, q, p, u) + \tilde{v}(t_k, x; u) \quad (6.20)$$

The optimal value from trading is

$$\tilde{v}^i(t_k, x) = \max \left\{ \tilde{v}^i(t_k, x; u) | u \in U(t_k, p) \right\}. \quad (6.21)$$

The optimal trading strategy is

$$\tilde{u}^i(t_k, x) = \left\{ u \in U(t_k, p) | \text{ such that } \tilde{v}^i(t_k, x; u) = \tilde{v}^i(t_k, x) \right\}. \quad (6.22)$$

We would like to distinguish between the computational complexities of the backward induction algorithm on a CPU and on a GPU. The corollary below draws conclusion from the discretization via the dynamic programming of the price switching problem, which is of the type of combined impulse control and optimal control. Interested readers may verify if the result is the same for other methods (PDE or backward SDE, if applicable) and other control types.
Corollary 6.1 Let $|\mathcal{T}|$, $|\mathcal{X}|$ and $|\mathcal{U}|$ respectively be the mesh sizes of the discretized time grid, space grid and admissible control set, and $M$ be the number of simulation paths to estimate the conditional expectation. Using serial computation, the time complexity of the algorithm is $|\mathcal{T}| \times |\mathcal{X}| \times |\mathcal{U}| \times M$. Using parallel computation to switch as much as possible the complexity to space complexity, the space complexity of the algorithm is $|\mathcal{X}| \times |\mathcal{U}| \times M$, and the time complexity can be reduced to at most $|\mathcal{T}| \times |\mathcal{U}|$.

Proof. Table 6.1 lists the pseudo codes of the algorithm, the computational complexity at every step and whether it is parallelizable or not. The computation at every node in the state space $\mathcal{X}$ is always parallelizable, because it uses results from the previously computed time step, and does not use any other nodes at the same time step. The number $|\mathcal{U}|$ is the complexity to get the maximum expected reward among all admissible controls, hence it cannot be carried out in parallel. For example, to get the maximum among the numbers $\{a_1, a_2, \cdots, a_N\}$, one can inductively compute $b_1 := a_1$ and $b_n := \max\{b_{n-1}, a_n\}$, for $n = 2, \cdots, N$. Then $b_N = \max\{a_1, a_2, \cdots, a_N\}$. \hfill $\Box$

Remark 6.1 There are two important observations from Corollary 6.1.

1. Because all the nodes in the state space at every time step can be computed in parallel, stochastic control problems of mediumly high dimension is no longer numerically forbidding.

2. The minimum time complexity on a GPU cluster is $|\mathcal{T}| \times |\mathcal{U}|$, which is the number of time steps multiplies the size of the admissible control set. The admissible set of mixed strategies is the cube $[0, \tilde{p}^a] \times [0, \tilde{p}^b]$, while the admissible set of price switching strategies is only the integer points inside that cube, hence the price switching problem is much simpler to implement.

6.3 Implementation

This subsection implements the algorithm in a simpler Binomial model, to the best capability of the author’s PC. An interesting application of the numerical results would be to calculate a “fair” internalization premium $\epsilon^*$. From every time step $t_k$ to $t_{k+1}$, the randomness in the model is captured by six Binomial variables. Independence is assumed unless mentioned otherwise. Other features remaining the same, the modifications from the previous subsection are the following.

1. The change in the volume $Q^i$ caused by market participants other than the trader is a random variable $R(Q^i) \in \{-1, 1\}$ with probabilities $\{0.5, 0.5\}$, $i = a, b$.

2. Let the pair of random variables $(R(N^a), R(N^b)) \in \{(0, 0), (1, 0), (0, 1)\}$ indicate whether limit sell and buy orders arrive (value one) or not (value zero) at one tick below the ask price and one tick above the bid price, when the spread is greater than one tick. The three scenarios are assigned probabilities $\{1 - p_N, p_N/2, p_N/2\}$, where $p_N = 0.3 \cdot \min\{\text{spread} - 1, 1\}$.

3. Let the pair of random variables $(R(H^a), R(H^b)) \in \{(0, 0), (1, 0), (0, 1)\}$ indicate whether there is a liquidity event (value one) or not (value zero) that consumes the trader’s hidden buy and sell orders at the mid price. The three scenarios are assigned probabilities $\{0.5, 0.25, 0.25\}$.

4. In addition, internalization here means only filling $\Delta^a$ ($\Delta^b$) shares at the time $t_k$ price $P^a(t_k)$ ($P^b(t_k)$), when at time $t_{k+1}$ limit sell (buy) orders arrive at a better price $P^a(t_{k+1}) = P^a(t_k) - 1$ ($P^b(t_{k+1}) = P^b(t_k) + 1$), indicated by $R(N^a) = 1$ ($R(N^b) = 1$).

We use $\Delta^a = \Delta^b = 5$, $\tilde{p}^a = 12$ and $\tilde{p}^b = 18$. The time mesh is

\[
\{t_0, t_1, \cdots, t_K\} = \{1, 2, \cdots, 10\}.
\]

At the terminal time $t_K = 10$, the trader’s stock inventory is valued at $P^b(10) - 2$ per share if it’s positive, and $P^a(10) + 2$ per share if negative.
Because the state space is infinite, it has to be truncated somehow on the boundary. The largest grid that the author’s PC can accept is

$$(Q^a, Q^b, I, P^a, P^b) \in \mathcal{X} = \{0, 1, \ldots, 9, 10\}^2 \times \{-20, -19, \ldots, 19, 20\} \times \{12, 13, \ldots, 17, 18\}^2.$$  

The grid contains 104181 admissible points where $P^a > P^b$.

How the trader’s optimal trading strategy interacts with simulated price paths is illustrated in Fig. 6.1 (regular trader), Fig. 6.2 (systemic internalizer) and Fig. 6.3 (systemic internalizer). The initial time is $t_0 = 1$ and terminal time $T = 10$. The initial values are $Q^a(1) = Q^b(1) = 5, P^a(1) = 16, P^b(1) = 15$ and $I(1) = 0$. The trader’s activities in all three figures display an attempt to sell short and push down the price. He indeed uses combinations of different order types - active, hidden and internalizing orders. Due to the truncation on the inventory, it is however not quite informative to compute the profit. More advanced devices are needed to allow for a wider grid, especially a larger range of the inventory variable.

It is interesting to see the effect of internalization on the trader’s best expected profit. The relative difference in best expected profits between a systemic internalizer and a regular trader is defined as

$$V^\text{diff}(t_0; a^*, b^*, \epsilon) := (V^\text{int}(t_0; a^*, b^*, z, p^a, p^b; \epsilon) - V^\text{reg}(t_0; a^*, b^*, z, p^a, p^b)) / V^\text{reg}(t_0; a^*, b^*, z, p^a, p^b),$$

where $V^\text{int}(t_0; \cdot; \epsilon)$ and $V^\text{reg}(t_0; \cdot; \epsilon)$ are respectively the time-$t_0$ value functions of systemic internalizer and regular trader defined in (10), and $\epsilon$ is the internalization premium. Fig. 6.4 shows the distribution of $V^\text{diff}(t_0; q^a, q^b, z, p^a, p^b; \epsilon = 0)$, with the initial values $(q^a, q^b, z, p^a, p^b)$ ranging over the 104181 admissible points on the grid. The systemic internalizer’s best expected profit is 1%-15% higher than that of the regular trader on about 35% of the admissible points. This means that internalization, when applied in the right situations, can be on average profitable. Furthermore, it suggests a way to specify a “fair” value $\epsilon^*$ of the internalization premium, so that both parties of the transaction will gain.

**Remark 6.2** Let a weight function $w : \mathcal{X} \rightarrow (0, 1); (q^a, q^b, z, p^a, p^b) \mapsto w(q^a, q^b, z, p^a, p^b)$ represent the likelihood of each point in the state space, satisfying

$$\sum_{(q^a, q^b, z, p^a, p^b) \in \mathcal{X}} w(q^a, q^b, z, p^a, p^b) = 1.$$

For some commonly recognized reward criterion $F$ in equation (3.2), some typical duration $T$ of a trading period and some proper grid $\mathcal{X}$ of the state space, the “fair” internalization premium $\epsilon^*$ should be a strictly positive number such that the weighted average

$$\sum_{(q^a, q^b, z, p^a, p^b) \in \mathcal{X}} V^\text{diff}(t_0; q^a, q^b, z, p^a, p^b; \epsilon^*) w(q^a, q^b, z, p^a, p^b)$$

is somewhere above zero. Since internalization is an additional choice that brings a higher best expected profit, $\epsilon^*$ should be positive. Since internalization provides price improvements to his counterpart, $\epsilon^*$ should be low enough to keep it profitable for a systemic internalizer to do so.

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Table 6.1: The parallelizable algorithm in pseudo codes

| Pseudo Codes | Computational Complexity | Parallelizable? |
|--------------|--------------------------|-----------------|
| for (x in \(X\)) { for (u in \(U(T, p)\)) { Step 1.1 | \(|X| \times |U|\) | yes |
| } | + | |
| Step 1.2 | \(|X| \times |U|\) | in x |
| print \(\bar{v}(T, x)\) and \(u^*(T, x)\) to file | + | |
| for (k = K - 1, k = -, k \geq 0) { for (x in \(X\)) { for (u in \(U(t_k, p)\) and h in \(\{0, 1\}^2\)) { Step 2 | \(|X| \times |U| \times M\) | yes |
| Step 3.1 | + | |
| equation (6.2) in Step 3.2 | + | |
| equation (6.2) in Step 3.2 | \(|X| \times |U|\) | in x |
| print \(\bar{v}^0(t_k, x), u^0(t_k, x)\) and \(h^*(t_k, x)\) to file | + | |
| for (x in \(X\)) | + | |
| for (u in \(U(t_k, p)\)) { equation (6.2) in Step 3.3 | \(|X| \times |U|\) | yes |
| } | + | |
| equation (6.2) in Step 3.3 | \(|X| \times |U|\) | in x |
| equation (6.2) in Step 3.3 | + | |
| equation (6.2) in Step 3.3 | \(|X| \times |U|\) | in x |
| print \(\bar{v}^i(t_k, x)\) and \(u^i(t_k, x)\) to file | + | |
Figures

Figures from Section 2

\[ P_b(t) - 2\delta \]

\[ P_b(t) - \delta \]

\[ P_b(t) \]

\[ P_b(t) + \delta \]

\[ P_b(t) + 2\delta \]

Figure 2.1: A limit order book
Figure 2.2: Simulation of the bid and ask prices
Figure 2.3: Simulation of the volumes
Figures from Section 6

Figure 6.1: Simulated path of prices, regular trader
Figure 6.2: Simulated path of prices, systemic internalizer
Figure 6.3: Another simulated path of prices, systemic internalizer
Figure 6.4: Relative difference in best expected profits