STRONG DISCRETE MORSE THEORY AND SIMPLICIAL L–S CATEGORY: A DISCRETE VERSION OF THE LUSTERNIK-SCHNIRELMANN THEOREM.

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Abstract. We prove a discrete version of the Lusternik–Schnirelmann theorem for discrete Morse functions and the recently introduced simplicial Lusternik–Schnirelmann category of a simplicial complex. To accomplish this, a new notion of critical object of a discrete Morse function is presented, which generalizes the usual concept of critical simplex (in the sense of R. Forman). We show that the non-existence of such critical objects guarantees the strong homotopy equivalence (in the Barmak and Minian’s sense) between the corresponding sublevel complexes. Finally, we establish that the number of critical objects of a discrete Morse function defined on $K$ is an upper bound for the non-normalized simplicial Lusternik–Schnirelmann category of $K$.

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1. INTRODUCTION

Since its inception by Robin Forman [9], discrete Morse theory has been a powerful and versatile tool used not only in diverse fields of mathematics, but also in applications to other areas [14] as well as a computational tool [6]. Its adaptability stems in part from the fact that it is a discrete version of the beautiful and successful "smooth" Morse theory [11]. While smooth...
Morse theory touches many branches of math, one such area that has its origins in critical point theory is that of Lusternik–Schnirelmann category. The (smooth) Lusternik–Schnirelmann category or L–S category of a smooth manifold $X$, denoted $\text{cat}(X)$, was first introduced in [10], where the authors proved what is now known as the Lusternik–Schnirelmann Theorem (see also [5] for a detailed survey of the topic). A version of this result can be stated as follows:

**Theorem 1.1.** If $M$ is a compact smooth manifold and $f : M \to \mathbb{R}$ is a smooth map, then

$$\text{cat}(M) + 1 \leq \sharp(\text{crit}(f))$$

where $\text{crit}(f)$ is the set of all critical points of the function $f$.

There are many “smooth” versions of this theorem in various contexts (see for example [13]). The aim of this paper is to view Forman’s discrete Morse theory from a different perspective in order to prove a discrete version of the L–S theorem compatible with the recently defined simplicial L–S category developed by three of the authors [8]. This simplicial version of L–S category is suitable for simplicial complexes. Other attempts have been made to develop such a “discrete” L–S category. In [1], one of the authors developed a discrete version of L–S category and proved an analogous L–S theorem for discrete Morse functions. Our version of the L–S Theorem, Theorem 4.5, relates a new generalized notion of critical object of a discrete Morse function to the simplicial L–S category of [8].

In this paper we use the notion of simplicial L–S category defined in [8]. This simplicial approach of L–S category uses strong collapses in the sense of Barmak and Minian [3] as a framework for developing categorical sets. As opposed to standard collapses, strong collapses are natural to consider in the simplicial setting since they correspond to simplicial maps (see Figure 1.1).

![Figure 1.1. An elementary strong collapse from $K$ to $K - \{v\}$.](image)

This is especially contrasted with (standard) elementary collapses, which in general do not correspond to simplicial maps (see Figure 1.2). Furthermore, elementary strong collapses correspond to the deletion of the open star of a dominated vertex. Notice that in general, the deletion of a vertex is not a simplicial map.

This paper is organized as follows. Section 2 contains the necessary background and basics of simplicial complexes and collapsibility. Section 3 is
devoted to both reviewing discrete Morse theory and introducing a generalized notion of critical object in this context. Here we develop a collapsing theorem for discrete Morse functions which is analogous to the classical result of Forman (Theorem 3.3 of [9]).

Section 4 is the heart of the paper. In this section, we recall the definitions and basic properties of the simplicial L–S category and prove the simplicial L–S theorem in Theorem 4.5. The rest of the section is devoted to examples and immediate applications.

2. Fundamentals of simplicial complexes

In this section we review some of the basics of simplicial complexes (see [12] for a more detailed exposition). Let us start with the definition of simplicial complex. The more usual way to introduce this notion is to do it geometrically. First, let us introduce the basic building blocks, that is, the notion of a simplex. Given $n + 1$ points $v_0, \ldots, v_n$ in general position in an Euclidean space, the $n$-simplex $\sigma$ generated by them, $\sigma = (v_0, \ldots, v_n)$, is defined as their convex hull. A simplex $\sigma$ contains lower dimensional simplices $\tau$, denoted by $\tau \leq \sigma$, called faces just by considering the corresponding simplices generated by any subset of its vertices. A simplicial complex is a collection of simplices satisfying two conditions:

- Every face of simplex in a complex is also a simplex of the complex.
- If two simplices in a complex intersect, then the intersection is also a simplex of the complex.

Alternatively, it is possible to define a simplicial complex abstractly. It will avoid confusion when parts of the simplicial structure are removed.
Given a finite set \( [n] := \{0, 1, 2, 3, \ldots, n\} \), an abstract simplicial complex \( K \) on \( [n] \) is a collection of subsets of \( [n] \) such that:
- If \( \sigma \in K \) and \( \tau \subseteq \sigma \), then \( \tau \in K \).
- \( \{i\} \in K \) for every \( i \in [n] \).

The set \( [n] \) is called the vertex set of \( K \) and the elements \( \{i\} \) are called vertices or 0-simplices. We also may write \( V(K) \) to denote the vertex set of \( K \).

An element \( \sigma \in K \) of cardinality \( i + 1 \) is called an \( i \)-dimensional simplex or \( i \)-simplex of \( K \). The dimension of \( K \), denoted \( \text{dim}(K) \), is the maximum of the dimensions of all its simplices.

**Example 2.1.** Let \( n = 5 \) and \( V(K) := \{0, 1, 2, 3, 4, 5\} \).
Define \( K := \{\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{1, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}, \emptyset\} \).

This abstract simplicial complex may be regarded geometrically as follows:

```
       2
      / \  \
     3   5
    /     \  \
   1-----4
   |
   0
```

Further definitions are in order.

**Definition 2.2.** A subcomplex \( L \) of \( K \), denoted \( L \subseteq K \), is a subset \( L \) of \( K \) such that \( L \) is also a simplicial complex.

We use \( \sigma^{(i)} \) to denote a simplex of dimension \( i \), and we write \( \tau < \sigma^{(i)} \) to denote any subsimplex of \( \sigma \) of dimension strictly less than \( i \). If \( \sigma, \tau \in K \) with \( \tau < \sigma \), then \( \tau \) is a face of \( \sigma \) and \( \sigma \) is a coface of \( \tau \). A simplex of \( K \) that is not properly contained in any other simplex of \( K \) is called a facet of \( K \).

At this point we recall a key concept in simple-homotopy theory: the notion of simplicial collapse [4].

**Definition 2.3.** Let \( K \) be a simplicial complex and suppose that there is a pair of simplices \( \sigma^{(p)} < \tau^{(p+1)} \) in \( K \) such that \( \sigma \) is a face of \( \tau \) and \( \sigma \) has no other cofaces. Such a pair \( \{\sigma, \tau\} \) is called a free pair. Then \( K - \{\sigma, \tau\} \) is a simplicial complex called an elementary collapse of \( K \) (see Figure 1.2). The action of collapsing is denoted \( K \downarrow K - \{\sigma, \tau\} \).

More generally, \( K \) is said to collapse onto \( L \) if \( L \) can be obtained from \( K \) through a finite series of elementary collapses, denoted \( K \downarrow L \). In the case where \( L = \{v\} \) is a single vertex, we say that \( K \) is collapsible.

Now we introduce some basic subcomplexes related to a vertex. They play analogous role in the simplicial setting as the closed ball, sphere, and open ball play in the continuous approach.
Definition 2.4. Let $K$ be a simplicial complex and let $v \in K$ be a vertex.

The star of $v$ in $K$, denoted $\text{st}(v)$, is the subcomplex of simplices $\sigma \in K$ such that $\sigma \cup \{v\} \in K$.

The link of $v$ in $K$, denoted $\text{lk}(v)$, is the subcomplex of $\text{st}(v)$ of simplices which do not contain $v$.

The open star of $v$ in $K$ is $\text{st}^o(v) := \text{st}(v) - \text{lk}(v)$. Note that $\text{st}^o(v)$ is not a simplicial subcomplex.

Finally, given a simplicial complex $K$ and a vertex $v \not\in K$, the cone $vK$ is the simplicial complex whose simplices are $\{v_0, \ldots, v_n\}$ and $\{v, v_0, \ldots, v_n\}$ where $\{v_0, \ldots, v_n\}$ is any simplex of $K$.

It is important to point out that simplicial collapses are not simplicial maps in general. This suggests that it is natural to consider a special kind of collapse which is a simplicial map. The notion of strong collapse, introduced by Barmak and Minian in [2, 3], satisfies this requirement.

Definition 2.5. Let $K$ be a simplicial complex and suppose there exists a pair of vertices $v, v' \in K$ such that every maximal simplex containing $v$ also contains $v'$. Then we say that $v'$ dominates $v$ and $v$ is dominated by $v'$.

If $v$ is dominated by $v'$ then the inclusion $i : K - \{v\} \to K$ is a strong equivalence. Its homotopical inverse is the retraction $r : K \to K - \{v\}$ which is the identity on $K - \{v\}$ and such that $r(v) = v'$. This retraction is called an elementary strong collapse from $K$ to $K - \{v\}$, denoted by $K \downarrow \downarrow K - \{v\}$.

A strong collapse is a finite sequence of elementary strong collapses. The inverse of a strong collapse is called a strong expansion and two complexes $K$ and $L$ have the same strong homotopy type if there is a sequence of strong collapses and strong expansions that transform $K$ into $L$.

See Figure 1.1 to illustrate the above notions.

Equivalently, $v$ is a dominated vertex if and only if its link is a cone [2].

3. STRONG DISCRETE MORSE THEORY

We are now ready to introduce a key object of our study, that is, discrete Morse functions in the R. Forman’s sense [9]. More precisely, we are interested in a generalized notion of critical object suitable for codifying the strong homotopy type of a complex.

Definition 3.1. Let $K$ be a simplicial complex. A discrete Morse function $f : K \to \mathbb{R}$ is a function satisfying for any $p$-simplex $\sigma \in K$:

(M1) $\# \left( \{\tau^{(p+1)} > \sigma : f(\tau) \leq f(\sigma)\} \right) \leq 1$.

(M2) $\# \left( \{\nu^{(p-1)} < \sigma : f(\nu) \geq f(\sigma)\} \right) \leq 1$.

Roughly speaking, we can say that it is a weakly increasing function which satisfies the property that if $f(\sigma) = f(\tau)$, then one of both simplices is a maximal coface of the other one.

Definition 3.2. A critical simplex of $f$ is a simplex $\sigma$ satisfying:
If \( \sigma \) is a critical simplex, the number \( f(\sigma) \in \mathbb{R} \) is called a critical value. Any simplex that is not critical is called a regular simplex while any output value of the discrete Morse function which is not a critical value is a regular value. The set of critical simplices of \( f \) is denoted by \( \text{crit}(f) \).

Given any real number \( c \), the level subcomplex of \( f \) at level \( c \), \( K(c) \), is the subcomplex of \( K \) consisting of all simplices \( \tau \) with \( f(\tau) \leq c \), as well as all of their faces, that is,

\[
K(c) = \bigcup_{f(\tau) \leq c} \bigcup_{\sigma \leq \tau} \sigma.
\]

Any discrete Morse function induces a gradient vector field.

**Definition 3.3.** Let \( f \) be a discrete Morse function on \( K \). The induced gradient vector field \( V_f \) or \( V \) when the context is clear, is defined by the following set of pairs of simplices:

\[
V_f := \{(\sigma(p), \tau^{(p+1)}): \sigma < \tau, f(\sigma) \geq f(\tau)\}.
\]

Note that critical simplices are easily identified in terms of the gradient field as precisely those simplices not contained in any pair in the gradient field.

**Definition 3.4.** Let \( f: K \to \mathbb{R} \) be a discrete Morse function and \( V_f \) its induced gradient vector field. For each vertex/edge pair \((v, uv) \in V_f\), write \( \text{St}(v, u) := \text{st}(v) \cap \text{st}(u) \). Define \( m_v := \min\{f(\tau): f(\tau) > f(uv), \tau \in (K - \text{St}(v, u)) \cup (\text{crit}(f) \cap \text{St}(v, u))\} \).

**Example 3.5.** We illustrate Definition 3.4. Let \( K \) be the simplicial complex with discrete Morse function \( f \) given in Figure 3.1. Observe that taking \( v = f^{-1}(10) \) and \( u = f^{-1}(0) \), we have \((v, uv) \in V_f\) and

\[
\text{St}(v, u) = f^{-1}(\{9, 10, 11, 12, 13, 14\})
\]

Since \( \text{St}(v, u) \) in Example 3.5 does not contain any critical simplex, then determining \( m_v \) consists on finding the smallest value greater than \( f(uv) \) outside of \( \text{St}(v, u) \). In this case, \( m_v = 15 \).

**Example 3.6.** Now \( f \) will be slightly modified to create a new discrete Morse function \( g \) on \( K \) (Figure 3.2). It is clear that \((v, uv) \in V_g\), but in this case \( m_v = 11 \).

Notice that the values that take both functions \( f \) and \( g \) on some simplices coincide. It is interesting to point out that both examples have the same induced gradient vector field and consequently, contain the same critical simplices in the usual Forman sense. This justifies the need to take into
Figure 3.1. A 2-dimensional simplicial complex with a discrete Morse function.

Figure 3.2. The same 2-dimensional simplicial complex of Figure 3.1 with another discrete Morse function.

account additional information in this new approach and thus a more general concept of critical object.

**Definition 3.7.** Continuing with the notation used in Definition 3.4, define \( l_v \) as the largest regular value in \( f(\text{St}(v, u)) \) such that

- \( f(uv) \leq l_v \leq m_v \)
- every maximal regular simplex of \( K(l_v) \cap \text{St}(v, u) \) contains the vertex \( u \).

**Definition 3.8.** Under the notation of Definition 3.4, the strong collapse of \( v \) under \( f \), denoted \( S_v^f \), is given by \( S_v^f := \{ (\sigma, \tau) \in V_f : f(uv) \leq f(\tau) \leq l_v \} \) and the interval \( I(S_v^f) = [f(uv), l_v] \) is called the strong interval of \( S_v \).
The elements of the set $C(V_f) := V_f - \bigcup S_v$ are the critical pairs of $f$ while each element in $\bigcup S_v$ is a regular pair of $f$. If $(\sigma, \tau)$ is a critical pair, the value $f(\tau)$ is a critical value of $f$.

A critical object is either a critical simplex (in the standard Forman sense) or a critical pair. The set of all critical objects of $f$ is denoted by $\text{scrit}(f)$. In order to avoid confusion, we call the images of all critical objects (either in the Forman sense or from a critical pair) strong critical values.

Remark 1. It is worthwhile to mention that critical pairs are detecting when a standard collapse has been made. Notice that it may happen either due to combinatorial reasons (e.g. the non-existence of dominated vertices in the corresponding subcomplex) or to a bad choice of the values of the discrete Morse function (i.e. noise). This second option induces a bad ordering in the way the standard collapses are made inside a potential strong one. The key idea is that every elementary strong collapse should ideally be made as an uninterrupted sequence of standard collapses. So every time it is not made in this way, a new critical object appears.

Example 3.9. Define $f$ and $g$ as in Examples 3.5 and 3.6. We see that $c_f^v = 14$ while $l_g^v = 10$. Hence (by abuse of notation), $(16, 15)$ is a critical pair under $f$ while $(14, 13), (16, 15)$ and $(18, 17)$ are critical pairs under $g$. Notice that the strong intervals are $I(S_f^9) = [9, 14]$ and $I(S_g^9) = [9, 10]$, respectively.

Example 3.10. To further illustrate Definition 3.7, we shall consider two different discrete Morse functions defined on a collapsible but non-strongly collapsible triangulation of the 2-disc. Notice that both of them have only a single critical value (in the Forman sense), but very different numbers of critical objects, due to a bad election in the ordering of normal collapses of $g$.

Let $f$ be given in Figure 3.3. By abuse of notation, we refer to the simplices by their labeling under the discrete Morse function (the fact that pairs in $V_f$ are given the same label should not cause confusion). For each of the pairs $(9, 9), (6, 6), (3, 3), (2, 2), (1, 1) \in V_f$, we have the corresponding values $l_9 = 11, l_6 = 8, l_3 = 5, l_2 = 2,$ and $l_1 = 1$. The corresponding strong collapses under the indicated vertices are given by

$S_f^9 = \{(9, 9), (10, 10), (11, 11)\}$,  
$S_f^6 = \{(6, 6), (7, 7), (8, 8)\}$,  
$S_f^3 = \{(3, 3), (4, 4), (5, 5)\}$,  
$S_f^2 = \{(2, 2)\}$,  
$S_f^1 = \{(1, 1)\}$.

Thus we obtain the following strong intervals:

$I(S_f^9) = [9, 11], I(S_f^6) = [6, 8], I(S_f^3) = [3, 5], I(S_f^2) = [2, 2] = \{2\}$  
and $I(S_f^1) = [1, 1] = \{1\}$.
Hence there is a single critical pair, namely, (13, 13), so that \( \text{scrit}(f) = \{0, (13, 13)\} \).

It is interesting to point out that this discrete Morse function can be considered as optimal in the sense that it minimizes the number of critical objects, as we will see in Theorem 4.5.

Now let \( g \) be the discrete Morse function given in Figure 3.4.

Now we consider the pairs \((9, 9), (8, 8), (7, 7), (5, 5), (3, 3) \in V_g \) and obtain corresponding values \( l_9 = 10, l_8 = 8, l_7 = 7, l_5 = 6 \) and \( l_3 = 3 \). The corresponding strong collapses are then given by

\[
S^g_9 = \{(9, 9)\}, \\
S^g_8 = \{(8, 8)\}, \\
S^g_7 = \{(7, 7)\}, \\
S^g_5 = \{(5, 5), (6, 6)\} \quad \text{and} \\
S^g_3 = \{(3, 3)\}.
\]

It follows that the strong intervals are:

\[
I(S^g_9) = [9, 9] = \{9\}, \\
I(S^g_8) = [8, 8] = \{8\}, \\
I(S^g_7) = [7, 7] = \{7\}, \\
I(S^g_5) = [5, 6] \\
\text{and} \\
I(S^g_3) = [3, 3] = \{3\}.
\]

We conclude that \( \text{scrit}(g) = \{0, (15, 15), (13, 13), (14, 14), (10, 10), (12, 12), (11, 11)\} \) for a total of 7 critical objects.

The following Theorem is the analogue of Forman’s classical result Theorem 3.3 of [9] and also analogous to the classical result in the smooth setting establishing homotopy equivalence of sublevel sets by assuming non-existence of critical values.
Theorem 3.11. Let $f$ be a discrete Morse function on $K$ and suppose that $f$ has no strong critical values on $[a,b]$. Then $K(b) \searrow \nearrow K(a)$. In particular, if $I(S_v) = [f(uv), l_v]$ is a strong interval for some vertex $v \in K$, then $K(l_v) \searrow \nearrow K(f(uv))$.

Proof. Let us consider the interval $[a,b]$ such that it does not contain any strong critical value. If additionally this interval does not contain any strong interval, then we conclude that $K(l_1) = K(l_2)$. Hence, assume that $[a,b]$ contains strong intervals. We may then partition $[a,b]$ into subintervals such that each one of them contains exactly one strong interval. Let us suppose $I(S_v) = [f(uv), l_v] \subseteq [a,b]$ is the unique strong interval in $[a,b]$. Again, since $f$ does not take values in $(a, f(uv))$ or $(l_v, b)$, it follows that $K(a) = K(f(uv))$ and $K(l_v) = K(b)$. Thus it suffices to show that $K(l_v) \searrow \nearrow K(f(uv))$. By definition of $l_v$, $u$ is contained in every maximal regular simplex of $K(l_v) \cap \text{St}(v, u)$. Furthermore, since $l_v < m_v$, all the new simplices that are attached from $K(f(uv))$ to $K(l_v)$ are contained within $\text{St}(v, u)$. Hence $K(l_v) - K(f(uv) - \epsilon)$ is an open cone with apex $u$ and thus, $K(l_v) \searrow \nearrow K(f(uv))$.

\[\square\]

4. Simplicial Lusternik–Schnirelmann category

In this section, we recall the fundamental definitions and results found in [8].

Definition 4.1. Let $K, L$ be simplicial complexes. We say that two simplicial maps $f : K \to L$ and $g : L \to K$ are homotopic if there exists a sequence of simplicial maps $f_0, f_1, \ldots, f_n$ such that $f_0 = f$, $f_n = g$, and $f_i$ is homotopic to $f_{i+1}$ for each $i$.

Figure 3.4. A different discrete Morse function defined on a non-strongly collapsible triangulation of the 2-disc.
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\[ \phi, \psi: K \to L \] are contiguous, denoted \( \phi \sim_c \psi \), if for any simplex \( \sigma \in K \), we have that \( \phi(\sigma) \cup \psi(\sigma) \) is a simplex of \( L \). Because this relation is reflexive and symmetric but not transitive, we say that \( \phi \) and \( \psi \) are in the same contiguity class, denoted \( \phi \sim \psi \), if there is a sequence \( \phi = \phi_0 \sim_c \phi_1 \sim_c \ldots \sim_c \phi_n = \psi \) of contiguous simplicial maps \( \phi_i: K \to L, 0 \leq i \leq n \). A map \( \phi: K \to L \) is a strong equivalence if there exists \( \psi: L \to K \) such that \( \psi \phi \sim \text{id}_K \) and \( \phi \psi \sim \text{id}_L \). In this case, we say that \( K \) and \( L \) are strongly equivalent, denoted by \( K \sim L \).

There is a nice link between strong equivalences and strong collapses.

**Theorem 4.2.** [3, Cor. 2.12] Two complexes \( K \) and \( L \) have the same strong homotopy type if and only if \( K \sim L \).

**Definition 4.3.** [8] Let \( K \) be a simplicial complex. We say that the subcomplex \( U \subseteq K \) is categorical in \( K \) if there exists a vertex \( v \in K \) such that the inclusion \( i: U \to K \) and the constant map \( c_v: U \to K \) are in the same contiguity class. The simplicial L–S category, denoted \( \text{scat}(K) \), of the simplicial complex \( K \), is the least integer \( m \geq 0 \) such that \( K \) can be covered by \( m + 1 \) categorical subcomplexes.

One of the basic results of classic Lusternik-Schnirelmann theory states that the L–S category is homotopy invariant. Next result shows that simplicial L–S category satisfies the analogous property in the discrete setting.

**Theorem 4.4.** [8, Theorem 3.4] Let \( K \sim L \) be two strongly equivalent complexes. Then \( \text{scat}(K) = \text{scat}(L) \).

We refer the interested reader to the papers [8, 7] for a detailed study of this topic.

4.1. Simplicial Lusternik–Schnirelmann theorem. Our main result is the following simplicial version of the Lusternik-Schnirelmann Theorem:

**Theorem 4.5.** Let \( f: K \to \mathbb{R} \) be a discrete Morse function. Then

\[ \text{scat}(K) + 1 \leq \sharp(\text{scrit}(f)). \]

**Proof.** For any natural number \( n \), define \( c_n := \min\{a \in \mathbb{R} : \text{scat}(K(a)) \geq n - 1\} \). We claim that \( c_n \) is a strong critical value of \( f \). If \( c_n \) is a regular value, then it is either contained in a strong interval or it is not. If \( c_n \) is contained in a strong interval \( I(c_v) \), then by Theorem 3.11 we have \( \text{scat}(K(c_v)) = \text{scat}(K(c_n)) \), contradicting the minimality of \( c_n \). Otherwise, \( c_n \) is outside a strong interval. Then, by Theorem 3.11 there exists \( \epsilon > 0 \) such that \( K(c_n) \backslash \backslash K(c_n - \epsilon) \). By Theorem 4.4 \( \text{scat}(K(c_n)) = \text{scat}(K(c_n - \epsilon)) \). But \( c_n > c_n - \epsilon \) and \( c_n \) was the minimum value such that \( \text{scat}(K(c_n)) = n - 1 \), which is a contradiction. Thus each \( c_n \) is a strong critical value of \( f \).

We now prove by induction on \( n \) that \( K(c_n) \) must contain at least \( n \) critical objects. By the well-ordering principle, the set \( \text{Im}(f) \) has a minimum, say \( f(v) = 0 \) for some 0-simplex \( v \in K \). For \( n = 1 \), \( c_1 = 0 \) so that \( K(c_1) \)
contains 1 critical object. For the inductive hypothesis, suppose that $K(c_n)$ contains at least $n$ strong critical objects. Since all the strong critical values of $f$ are distinct, $c_n < c_{n+1}$ so that there is at least one new critical object in $f^{-1}(c_{n+1})$. Thus $K(c_{n+1})$ contains at least $n + 1$ critical objects. Hence if $c_1 < c_2 < \ldots < c_{\text{scat}(K)+1}$ are the critical objects, then $K(c_{\text{scat}(K)+1}) \subseteq K$ contains at least $\text{scat}(K) + 1$ critical objects. Thus $\text{scat}(K) + 1 \leq \sharp(\text{scrit}(f))$.

**Example 4.6.** We give an example where the upper bound of Theorem 4.5 is attained. Let $A$ be the several times considered non-strongly collapsible triangulation of the 2-disc with the discrete Morse function $f$ given in Example 3.10. This satisfies $\sharp(\text{scrit}(f)) = 2$. Since $A$ has no dominating vertex, $\text{scat}(A) > 0$, whence $\text{scat}(A) + 1 = \sharp(\text{scrit}(f)) = 2$.

Notice that just adding one simplex to $A$, the above equality turns into a strict inequality and thus, the number of critical objects may increase while the simplicial category keeps the same. To see this, let us consider $A'$ as the clique complex of $A$, that is, we add to $A$ the triangle generated by its three bounding vertices. This is a simplicial 2-sphere, so by means of Morse inequalities, it follows that every discrete Morse function $f$ defined on $A'$ has at least two critical simplices: one critical vertex (global minimum) and a critical triangle (global maximum). In addition, since $A'$ is a simplicial 2-sphere, then it does not contain any dominated vertex. Moreover, after removing the critical triangle (and hence obtaining $A$) no new dominated vertices appear, so at least one critical pair arises in order to collapse $A$ to a subcomplex containing dominated vertices. Hence, we conclude that that any discrete Morse function $f: A' \to \mathbb{R}$ must have at least 3 critical objects, which could be considered optimal in the sense that it minimizes the number of critical objects. Furthermore, it is easy to cover $A'$ with 2 categorical sets so that $\text{scat}(A') = 1$. Hence $\text{scat}(A') + 1 = 2 < 3 \leq \sharp(\text{scrit}(f))$.

**References**

[1] S. Aaronson, N. Scoville, Lusternik-Schnirelmann for simplicial complexes, *Illinois J. Math.* 57 (3) (2013), 743–753.

[2] J.A. Barmak, *Algebraic topology of finite topological spaces and applications*, Lecture Notes in Mathematics, 2032, Springer, Heidelberg, 2011.

[3] J.A. Barmak and E.G. Minian, Strong homotopy types, nerves and collapses, *Discrete Comput. Geom.* 47 (2) (2012), 301–328.

[4] M. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics, Vol. 10. Springer-Verlag, New York-Berlin, 1973.

[5] O. Cornea, G. Lupton, J. Oprea, D. Tanrè, *Lusternik-Schnirelmann Category*, Mathematical Surveys and Monographs, 103. American Mathematical Society, Providence, RI, 2003.

[6] J. Curry, R. Ghrist, V. Nanda, Discrete Morse Theory for Computing Cellular Sheaf Cohomology, *Found. Comput. Math.* 16 (2016), 875–897.

[7] D. Fernández-Ternero, E. Macías-Virgós, E. Minuz and J.A. Vilches, Simplicial Lusternik-Schnirelmann category, [arXiv:1605.01322 [math.AT]], 2016.
[8] D. Fernández-Ternero, E. Macías-Virgós, J.A. Vilches, Lusternik-Schnirelmann category of simplicial complexes and finite spaces, Topology Appl. 194 (2015), 37–50.
[9] R. Forman, Morse Theory for cell complexes, Adv. Math. 134 (1) (1998), 90–145.
[10] L. Lusternik and L. Schnirelmann, Méthodes Topologiques dans les Problèmes Variationnels, Hermann, Paris, 1934.
[11] J. Milnor, Morse theory, Based on lecture notes by M. Spivak and R. Wells, Annals of Mathematics Studies, No. 51 Princeton University Press, Princeton, N.J. 1963.
[12] J. R. Munkres, Elements of Algebraic Topology, Addison Wesley, Menlo Park, CA. 1984.
[13] R. S. Palis, Lusternik–Schnirelmann theory on Banach manifolds, Topology 5 (1966), 115–132.
[14] R. Reina-Molina, D. Díaz-Pernil, P. Real and A. Berciano, Membrane parallelism for discrete Morse theory applied to digital images, Appl. Algebra Engy. Comm. Comput. 26 (1-2) (2015), 49–71.

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