TWO-DIMENSIONAL METRIC SPHERES FROM GLUING HEMISPHERES

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Abstract. We study metric spheres $(Z, d_Z)$ obtained by gluing two hemispheres of $S^2$ along an orientation-preserving homeomorphism $g: S^1 \to S^1$, where $d_Z$ is the canonical distance that is locally isometric to $S^2$ off the seam.

We show that if $(Z, d_Z)$ is quasiconformally equivalent to $S^2$, in the geometric sense, then $g$ is a welding homeomorphism with conformally removable welding curves. We also show that $g$ is bi-Lipschitz if and only if $(Z, d_Z)$ has a 1-quasiconformal parametrization whose Jacobian is comparable to the Jacobian of a quasiconformal mapping $h: S^2 \to S^2$. Furthermore, we show that if $g^{-1}$ is absolutely continuous and $g$ admits a homeomorphic extension with exponentially integrable distortion, then $(Z, d_Z)$ is quasiconformally equivalent to $S^2$.

1. Introduction

In this paper, we work in the unit sphere $S^2 \subset \mathbb{R}^3$. We denote the equator $S^2 \cap (\mathbb{R}^2 \times \{0\})$ by $S^1$ and endow $S^2$ with the length distance $\sigma$ induced by the Euclidean distance of $\mathbb{R}^3$. The open southern and northern hemispheres are denoted by $Z_1$ and $Z_2$, respectively. Here $(0, 0, 1) \in Z_2$.

Consider an orientation-preserving homeomorphism $g: S^1 \to S^1$, mapping the boundary of $Z_1$ to the boundary of $Z_2$. We identify each $z \in S^1$ with its image $g(z) \in S^1$. With this identification, we obtain a set $Z$ and inclusion maps $i_1: Z_1 \to Z$ and $i_2: Z_2 \to Z$. We call $S_Z = i_1(|S^1|) = i_2(|S^1|)$ the seam of $Z$.

We construct a pseudodistance $d_Z$ on $Z$, see Section 3, making the inclusion maps local isometries off the seam and 1-Lipschitz everywhere. We consider the quotient map $Q: Z \to \tilde{Z}$ identifying points $x, y \in Z$ whenever $d_Z(x, y) = 0$, and endow $\tilde{Z}$ with the associated quotient distance.

We are interested in this construction for the following reason: whenever the metric space $\tilde{Z}$ is quasiconformally equivalent to $S^2$, there exist Riemann maps $\phi_1: Z_1 \to \Omega_1$, $\phi_2: Z_2 \to \Omega_2$ onto the complementary components of a Jordan curve $\mathcal{C}$ with $g = \phi_2^{-1} \circ \phi_1|_{S^1}$; with the Carathéodory theorem we can make sense of the composition $\phi_2^{-1} \circ \phi_1|_{S^1}$ [GM05]. Any such $g$ is called a welding homeomorphism and $\mathcal{C}$ a welding curve. A long-standing problem is to understand which homeomorphisms $g$ satisfy $\phi_2^{-1} \circ \phi_1|_{S^1}$ for some Riemann maps. We refer to the survey articles [Ham02], [You15] for further background information.

We also investigate the properties of $\tilde{Z}$, given an arbitrary welding homeomorphism $g$. We show in Section 4 that the 1-dimensional Hausdorff measures on the seam $Q(S_Z)$ and on (the tangents of) $\mathcal{C}$ are closely connected, using results from classical complex analysis [GM05]. For example, our results show that a given subarc of the welding curve has tangents only in a set negligible to the 1-dimensional Hausdorff measure if and only if the quotient map $Q$ collapses the corresponding part of the seam to a point.

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We present in Sections 7.1 and 7.2 examples illustrating that for some homeomorphisms \( g \), after removing a portion \( E' \) of the seam \( Q(S_2) \), one can find a \( 1 \)-quasiconformal embedding \( \psi: \tilde{Z} \setminus E' \to S^2 \), but not necessarily a quasiconformal homeomorphism \( \Psi: \tilde{Z} \to S^2 \). A similar phenomenon was investigated in [Ham91] and [Bis07b] in more detail.

We now state our first result.

**Theorem 1.1.** Let \( g: S^1 \to S^1 \) be an orientation-preserving homeomorphism. The following are quantitatively equivalent.

1. \( g \) is \( L \)-bi-Lipschitz;
2. there exists an \( L' \)-bi-Lipschitz homeomorphism \( \Psi: \tilde{Z} \to S^2 \);
3. there exists \( C' \geq 0 \) such that for every \( y \in Q(S_2) \),
   \[
   \liminf_{r \to 0} \frac{\mathcal{H}_2^2(B(y, r))}{\pi r^2} \leq C'.
   \]

In the implications "(1) \( \Rightarrow \) (2)" we may take \( L' = L \), in "(2) \( \Rightarrow \) (3)" \( C' = (L')^4 \), and in "(3) \( \Rightarrow \) (1)" \( L = \pi C' \).

We prove "(1) \( \Rightarrow \) (2)" by observing that if \( g: S^1 \to S^1 \) admits an \( L \)-bi-Lipschitz extension \( \phi: \mathbb{Z}_2 \to \mathbb{Z}_2 \), the space \( \tilde{Z} \) has an \( L' \)-bi-Lipschitz parametrization. That we may take \( L' = L \) in "(1) \( \Rightarrow \) (2)", follows by applying a known planar extension result [Kal14] and stereographic projection.

The claim "(2) \( \Rightarrow \) (3)" is a straightforward consequence of the properties of Hausdorff measures. The implication "(3) \( \Rightarrow \) (1)" is proved by carefully analyzing the behaviour of the inclusion mappings \( i_t: \mathbb{Z}_2 \to \tilde{Z} \) at the equator \( S^1 \). Notice that the \( i_t \) are \( 1 \)-Lipschitz everywhere and local isometries outside the equator. This implies \( C' \geq 1 \) in (3). Remark 5.9 shows two ways to improve the bi-Lipschitz constant \( \pi C' \). The improvements imply that as \( C' \to 1^+ \) in (3), the bi-Lipschitz constant of \( g \) converges to one. In particular, (3) holds with \( C' = 1 \) if and only if \( g \) is an isometry.

Theorem 1.1 is closely related to the following result.

**Theorem 1.2.** If an orientation-preserving homeomorphism \( g: S^1 \to S^1 \) is \( L \)-bi-Lipschitz, there exists a \( 1 \)-quasiconformal homeomorphism \( \varphi: S^2 \to \tilde{Z} \) and a \( K \)-quasiconformal homeomorphism \( h: S^2 \to S^2 \) such that the Jacobsen satisfy

\[
C^{-1} J_h(x) \leq J_\varphi(x) \leq C J_h(x) \quad \text{for } H^2_\text{S} \text{-a.e. } x \in S^2
\]

for \( K = L^4 \) and \( C = L^2 \). Conversely, if there exists \( K, C, \) and \( h \) for which (1) holds, then \( g \) is \( \pi(KC)^2 \)-bi-Lipschitz.

The Jacobsen are defined in Section 2.3. We note that if \( h: S^2 \to S^2 \) is an orientation-preserving quasiconformal homeomorphism, the \( J_h \) coincides with the usual distributional Jacobian; see for example [AIM09, Section 3.8].

If \( g \) is \( L \)-bi-Lipschitz, the existence of \( \varphi \) and \( h \) is a straightforward consequence of the implication "(1) \( \Rightarrow \) (2)" in Theorem 1.1. If \( h \) and \( \varphi \) exist, we first check that \( \Psi = h \circ \varphi^{-1} \) is \( \text{bi-Lipschitz} \) and use the implications "(2) \( \Rightarrow \) (3) \( \Rightarrow \) (1)" from Theorem 1.1 to verify that \( g \) is \( \text{bi-Lipschitz} \).

Theorem 1.2 is a special case of the *quasiconformal Jacobian problem*: which weights \( \omega: S^2 \to [0, \infty] \) are comparable to the Jacobsen of quasiconformal homeomorphisms \( h: S^2 \to S^2 \); see [BHS04], [Bis07a], and references therein for further reading.

Given that (1) and (3) are equivalent in Theorem 1.1, it is not entirely clear for which classes of homeomorphisms one can expect \( \tilde{Z} \) to be quasiconformally
equivalent to $S^2$, or what kind of geometric properties one can expect from such a $\tilde{Z}$.

**Question 1.3.** Let $\tilde{Z}$ be the metric space obtained from a homeomorphism $g: S^1 \to S^1$. When can we find a quasiconformal homeomorphism $\psi: \tilde{Z} \to S^2$? What kind of restrictions does this impose on $g$?

As an example, if $g$ is a welding homeomorphism corresponding to the von Koch snowflake, then $d_2(x, y) = 0$ for every pair of points in the seam, see Remark 4.2. Hence $\tilde{Z}$ can fail to be quasiconformally equivalent, or homeomorphic, to $S^2$ when $g$ is a quasisymmetry. We show that a simple measure-theoretic assumption removes this obstruction.

**Proposition 1.4.** Let $g: S^1 \to S^1$ be a quasisymmetry whose inverse is absolutely continuous. Then $\tilde{Z}$ is quasiconformally equivalent to $S^2$.

The absolute continuity of $g^{-1}$ is used in two ways. First, it guarantees that $\tilde{Z} = (Z, d_2)$. Second, if $\psi: Z_2 \to Z_3$ is a quasisymmetric extension of $g$, we show that the homeomorphism $H: S^2 \to \tilde{Z}$ satisfying $H|_{Z_2} = \iota_1$ and $H|_{Z_3} = \iota_2 \circ \psi|_{Z_2}$ is quasiconformal. A key step in the proof is showing the Sobolev regularity $H^{-1} \in N^{1,2}(\tilde{Z}, S^2)$; the absolute continuity of $g^{-1}$ is applied here.

**Theorem 1.5.** Let $g: S^1 \to S^1$ be an orientation-preserving homeomorphism whose inverse is absolutely continuous. If $g$ extends to a homeomorphism $\psi: \overline{Z_2} \to Z_2$ for which $\psi|_{Z_2}$ has exponentially integrable distortion, then $\tilde{Z}$ is quasiconformally equivalent to $S^2$.

We now explain the main steps of the proof of Theorem 1.5. We first show that there exists a homeomorphism $H: S^2 \to \tilde{Z}$ with exponentially integrable distortion. We also have $H^{-1} \in N^{1,2}(\tilde{Z}, S^2)$; see Remark 6.8. The exponential integrability of distortion of $H$ is used to verify the reciprocity condition of $\tilde{Z}$, see Definition 2.5. Then [Raj17, Theorem 1.4] shows that $\tilde{Z}$ is quasiconformally equivalent to $S^2$. The key ingredients in the proof are the condenser estimates for mappings of exponentially distortion [KO06], applicable because $H^{-1} \in N^{1,2}(\tilde{Z}, S^2)$, and the Stoilow factorization theorem [AIM09, Chapter 20]. There are some known criteria which guarantee that $g$ admits an extension as in Theorem 1.5; see [Zak08] [KN21].

In Section 7.1, we present an example of $g: S^1 \to S^1$ that is locally bi-Lipschitz outside a single point, but for which $\tilde{Z}$ is not quasiconformally equivalent to $S^2$. This illustrates that the absolute continuity of $g^{-1}$ is not enough to guarantee that $\tilde{Z}$ is quasiconformally equivalent to $S^2$. This fact is a consequence of the following result, partially answering Question 1.3.

**Theorem 1.6.** Suppose that $g: S^1 \to S^1$ is an orientation-preserving homeomorphism for which there exists a quasiconformal homeomorphism $h: S^2 \to \tilde{Z}$. Then $\tilde{Z} = (Z, d_2)$ and there exists a 1-quasiconformal homeomorphism $\pi: S^2 \to \tilde{Z}$. Furthermore, $g$ is a welding homeomorphism whose welding curves are conformally removable.

The first step in the proof of Theorem 1.6 is showing that $h$ can be assumed to be 1-quasiconformal. Then, up to an orientation-reversing Möbius transformation, $\phi_i = h^{-1} \circ \iota_i: Z_i \to S^2$ are Riemann maps with welding curve $C = h^{-1}(Q(S_2))$ and welding homeomorphism $\phi_{z_1}^{-1} \circ \phi_1|_{S^1}$. The equality $\tilde{Z} = (Z, d_2)$
and the conformal removability of $C$ follow from a connection we show between the tangents of the welding curve $C$ and the Hausdorff 1-measure on the seam $Q(S_2)$; see Section 4. The equality $Z = (Z, d_Z)$ implies $g = \varphi_{S_1}^{-1} \circ \varphi_{S_1}$.

We recall that a compact proper subset $K \subset S^2$ is \textit{conformally removable} if every homeomorphism $M: S^2 \to S^2$ conformal in $S^2 \setminus K$ is Möbius. The von Koch snowflake example illustrates that conformal removability of a welding curve $C$ is not enough to guarantee even that $Z$ is homeomorphic to $S^2$. We refer the reader to [You15] and [You18] for further reading on conformal weldings and the connections to conformal removability. See [HK03] for some results in the context of Theorem 1.5.

The paper is structured as follows. In Section 2, we introduce our notations and some preliminary results. In Section 3, we analyze the distance $d_Z$ induced by any given homeomorphism $g: S^1 \to S^1$. When $g$ is a welding homeomorphism, we establish in Section 4 a connection between the geometry of the seam $S_Z$ and the tangents of the corresponding welding curves $C$. We also prove Theorem 1.6 in this section. In Section 5, we prove Theorems 1.1 and 1.2. Proposition 1.4 and Theorem 1.5 are proved in Section 6. In Section 7, we give some concluding remarks.

2. Preliminaries

2.1. Notation. Let $(Y, d_Y)$ be a metric space. We sometimes drop the subscript from $d_Y$ when there is no chance for confusion. For all $Q \geq 0$, the $Q$-dimensional Hausdorff measure, or a Hausdorff $Q$-measure, is defined by

$$H^Q_Y(B) = \frac{\alpha(Q)}{2Q} \sup_{\delta > 0} \left\{ \sum_{i=1}^\infty (\text{diam } B_i)^Q : B \subset \bigcup_{i=1}^\infty B_i, \text{diam } B_i < \delta \right\}$$

for all sets $B \subset Y$, where $\alpha(Q)$ is chosen so that $H^Q_{\text{leb}}$ coincides with the Lebesgue measure $\mathcal{L}^n$ for all positive integers.

The \textit{length} of a path $\gamma: [a, b] \to Y$ is defined as

$$\ell_d(\gamma) = \sup \sum_{i=1}^n d(\gamma(t_i), \gamma(t_{i+1})),$$

the supremum taken over all finite partitions $a = t_1 \leq t_2 \leq \cdots \leq t_{n+1} = b$. A path is \textit{rectifiable} if it has finite length.

The \textit{metric speed} of a path $\gamma: [a, b] \to Y$ at the point $t \in [a, b]$ is defined as

$$v_\gamma(t) = \lim_{t \neq k \to t} \frac{d(\gamma(s), \gamma(t))}{|s - t|}$$

whenever this limit exists. The limit exists $\mathcal{L}^1$-almost everywhere for every rectifiable path [Dud07, Theorem 2.1].

A rectifiable path $\gamma: [a, b] \to Y$ is \textit{absolutely continuous} if for all $a \leq s \leq t \leq b$,

$$d(\gamma(t), \gamma(s)) \leq \int_s^t v_\gamma(u) \, d\mathcal{L}^1(u)$$

with $v_\gamma \in L^1([a, b])$ and $\mathcal{L}^1$ the Lebesgue measure on the real line. Equivalently, the rectifiable path $\gamma$ is absolutely continuous if it maps sets of $\mathcal{L}^1$-measure zero to sets of $H^1_{\text{mez}}$-measure zero [Dud07, Section 3].

Let $\gamma: [a, b] \to X$ be an absolutely continuous path. Then the \textit{(path) integral} of a Borel function $\rho: X \to [0, \infty]$ over $\gamma$ is

$$(2) \quad \int_\gamma \rho \, ds = \int_a^b (\rho \circ \gamma) v_\gamma \, d\mathcal{L}^1.$$
If $\gamma$ is rectifiable, then the path integral of $\rho$ over $\gamma$ is defined to be the path integral of $\rho$ over the arc length parametrization $\gamma_s$ of $\gamma$; see [HKST15, Chapter 5] for further details.

Given a Borel set $A \subset Y$, the length of a path $\gamma: [a, b] \to Y$ in $A$ is defined as $\int_Y \chi_A(y)\#(\gamma^{-1}(y))\,d\mathcal{H}^1_\rho(y)$, where $\#(\gamma^{-1}(y))$ is the counting measure of $\gamma^{-1}(x)$. For $A = Y$, [Fed69, Theorem 2.10.13] states

$$
\ell(\gamma) = \int_Y \#(\gamma^{-1}(y))\,d\mathcal{H}^1_\rho(y).
$$

When $\gamma$ is rectifiable, for every Borel function $\rho: Y \to [0, \infty]$, 

$$
\int_{\gamma} \rho \, ds = \int_Y \rho(y)\#(\gamma^{-1}(y))\,d\mathcal{H}^1_\rho(y).
$$

The equality (4) follows from [Fed69, Theorem 2.10.13] via a standard approximation argument using simple functions.

### 2.2. Metric Sobolev spaces

In this section we give an overview of Sobolev theory in the metric surface setting, and refer to [HKST15] for a comprehensive introduction.

Let $\Gamma$ be a family of paths in $Y$. A Borel function $\rho: Y \to [0, \infty]$ is admissible for $\Gamma$ if the path integral $\int_{\gamma} \rho \, ds \geq 1$ for all rectifiable paths $\gamma \in \Gamma$. Given $1 \leq p < \infty$, the $p$-modulus of $\Gamma$ is

$$
\operatorname{mod}_p \Gamma = \inf \int_Y \rho^p \, d\mathcal{H}^1_\rho,
$$

where the infimum is taken over all admissible functions $\rho$. Observe that if $\Gamma_1$ and $\Gamma_2$ are path families and every path $\gamma_1 \in \Gamma_1$ contains a subpath $\gamma_2 \in \Gamma_2$, then $\operatorname{mod}_p \Gamma_1 \leq \operatorname{mod}_p \Gamma_2$. In particular, this holds if $\Gamma_1 \subset \Gamma_2$. When $p = 2$, and there is no chance for confusion, we omit the subscript from $\operatorname{mod}_2$.

If $\rho$ is admissible for a path family $\Gamma \setminus \Gamma_0$, where $\operatorname{mod}_p \Gamma_0 = 0$, we say that $\rho$ is $p$-weakly admissible for $\Gamma$. If a property holds for every path $\gamma \in \Gamma$ except in a subfamily of $p$-modulus zero, the property is said to hold on $p$-almost every path in $\Gamma$. We also refer to $2$-almost every path as almost every path.

We recall the following lemma [HKST15, Lemma 5.2.8].

**Lemma 2.1.** Let $1 \leq p < \infty$. A family of nonconstant paths $\Gamma$ satisfies $\operatorname{mod}_p \Gamma = 0$ if and only if there exists $\rho: Y \to [0, \infty], \rho \in L^p(Y)$ with

$$
\rho = \int_{\gamma} \rho \, ds \quad \text{for every } \gamma \in \Gamma.
$$

Let $\psi: (Y, d_Y) \to (Z, d_Z)$ be a mapping between metric spaces $Y$ and $Z$. A Borel function $\rho: Y \to [0, \infty]$ is an upper gradient of $\psi$ if

$$
d_Y(\psi(x), \psi(y)) \leq \int_{\gamma} \rho \, ds
$$

for every rectifiable path $\gamma: [a, b] \to Y$ connecting $x$ to $y$. The function $\rho$ is a $p$-weak upper gradient of $\psi$ if the same holds for $p$-almost every rectifiable path.

A $p$-weak upper gradient $\rho \in L^p_{\text{loc}}(Y)$ of $\psi$ is minimal if it satisfies $\rho \leq \tilde{\rho}$ almost everywhere for all $p$-weak upper gradients $\tilde{\rho} \in L^p_{\text{loc}}(Y)$ of $\psi$. If $\psi$ has a $p$-weak upper gradient $\rho \in L^p_{\text{loc}}(Y)$, then $\psi$ has a minimal $p$-weak upper gradient, which we denote by $\rho_\psi$. We refer to Section 6 of [HKST15] and Section 3 of [Will12] for further details. Minimal $2$-weak upper gradients are also referred to as minimal weak upper gradients.

Fix a point $z \in Z$, and let $d_z = d_Z(\cdot, z)$. The space $L^p(Y, Z)$ is defined as the collection of measurable maps $\psi: Y \to Z$ such that $d_z \circ \psi$ is in $L^p(Y)$. Moreover,
$L^p_{\text{loc}}(Y,Z)$ is defined as those measurable maps $\psi: Y \to Z$ for which, for all $y \in Y$, there is an open set $U \subset Y$ containing $y$ such that $\psi|_U$ is in $L^p(U,Z)$.

The metric Sobolev space $N^{1,p}_{\text{loc}}(Y,Z)$ consists of those maps $\psi: Y \to Z$ in $L^p_{\text{loc}}(Y,Z)$ that have a minimal $p$-weak upper gradient $\rho_\psi \in L^p_{\text{loc}}(Y)$.

For subsets $\emptyset \neq U \subset Y$, we say that $\psi \in N^{1,p}(U,Z)$ if $\psi|_U \in N^{1,p}_{\text{loc}}(U,Z)$, $\rho_\psi|_U \in L^p(U)$ and $\psi|_U \in L^p(U,Z)$. If $Z = \mathbb{R}$, we denote $N^{1,p}(U,Z) = N^{1,p}(U)$, and in the case $p = 2$,

$$E(\psi) := 2^{-1} \|\rho_\psi\|_{L^2(U)}^2.$$ 

We refer to $E(\psi)$ as the Dirichlet energy of $\psi$.

We repeatedly use the following technical lemma in later sections.

**Lemma 2.2.** Let $\psi: Y \to Z$ be continuous, $\rho: Y \to [0,\infty]$ a Borel function and $\gamma: [0,1] \to Y$ absolutely continuous with $\int_0^1 \rho \, ds < \infty$.

Suppose that $E \subset Y$ is compact, $\mathcal{H}^1_\gamma(\psi(E)) = 0$, and $\ell(\psi \circ \gamma|_I) \leq \int_{\gamma|_I} \rho \, ds$ for each closed interval $I \subset [0,1] \setminus \gamma^{-1}(E)$. Then $\ell(\psi \circ \gamma) \leq \int_\gamma \rho \, ds$.

**Proof.** First, for every closed interval $J \subset [0,1] \setminus \gamma^{-1}(E)$, $\psi \circ \gamma|_J$ is absolutely continuous with $\nu_{\psi \circ \gamma}(s) \leq (\rho \circ \gamma)(s)\nu_\gamma(s)$ for $\mathcal{L}^1$-almost every $s \in J$. This follows from [HKST15, Proposition 6.3.2].

Second, consider the connected components $\{I_i\}_{i=1}^\infty$ of $[0,1] \setminus (\psi \circ \gamma)^{-1}(\psi(E))$. Notice that $I_i \subset [0,1] \setminus \gamma^{-1}(E)$ for every $i$.

Let $J_i = \overline{I_i}$. Then $\nu_{\psi \circ \gamma}(s) \leq (\rho \circ \gamma|_{J_i})(s) \mathcal{L}^1$-almost everywhere on $J_i$ (on $I_i$).

This fact, the continuity of $\psi \circ \gamma$ and $\int_{J_i} \rho \, ds < \infty$ imply

$$\ell(\psi \circ \gamma|_{J_i}) \leq \int_{J_i} (\rho \circ \gamma)\nu_\gamma \, ds < \infty.$$ 

By summing over $i$, we conclude

$$\sum_{i=1}^\infty \ell(\psi \circ \gamma|_{J_i}) \leq \int_{\bigcup_{i=1}^\infty J_i} (\rho \circ \gamma)\nu_\gamma \, ds \leq \int_\gamma \rho \, ds.$$ 

Given $\mathcal{H}^1_\gamma(\psi(E)) = 0$, (3) and (4) imply

$$\ell(\psi \circ \gamma) = \int_{Z \setminus \psi(E)} \#((\psi \circ \gamma)^{-1}(x)) \, d\mathcal{H}^1_{\gamma}(x)$$

$$\leq \sum_{i=1}^\infty \int_{Z \setminus \psi(E)} \#((\psi \circ \gamma|_{J_i})^{-1}(x)) \, d\mathcal{H}^1_{\gamma}(x)$$

$$= \sum_{i=1}^\infty \ell(\psi \circ \gamma|_{J_i}) \leq \int_\gamma \rho \, ds.$$ 

Hence $\ell(\psi \circ \gamma) \leq \int_\gamma \rho \, ds$. \hfill $\Box$

2.3. **Measure theory.** Let $Y$ be a Borel subset of a complete and separable metric space. A Borel measure $\mu$ on $Y$ is $\sigma$-finite if there exists a Borel decomposition $\{B_i\}_{i=1}^\infty$ of $Y$ for which $\mu(B_i) < \infty$ for every $i$.

A pair of $\sigma$-finite Borel measures $\mu$ and $\nu$ on $Y$ are said to be mutually singular if there exists a Borel set $B \subset Y$ such that $\mu(B) = 0$ and $\nu(Y \setminus B) = 0$. The measure $\mu$ admits a Lebesgue decomposition (with respect to $\nu$), where $\mu = f \cdot \nu + \mu^\perp$, with $\mu^\perp$ and $\nu$ mutually singular and $f$ Borel measurable [Bog07, Sections 3.1-3.2 in Volume I]. We say that $\mu$ and $\nu$ are mutually absolutely continuous if $\mu = f \cdot \nu$ with density $f > 0$ $\nu$-almost everywhere.
Given a homeomorphism $\psi: Y \to Z$ and measures $\nu$ on $Y$ and $\mu$ on $Z$, the measure $\psi^*\mu(B) = \mu(\psi(B))$ is called the pullback measure. Such a measure admits a decomposition $\psi^*\mu = f \cdot \nu + \mu^+$ with $\nu$ and $\mu^+$ mutually singular. If $\nu = \mathcal{H}_K^2$ and $\mu = \mathcal{H}_2^2$, the density $f$ is called the Jacobian of $\psi$ and denoted by $J_\psi$.

2.4. Quasiconformal mappings. Here we define quasiconformal maps and recall some basic facts.

**Definition 2.3.** Let $(Y, d_Y)$ and $(Z, d_Z)$ be metric spaces with locally finite Hausdorff 2-measures. A homeomorphism $\psi: (Y, d_Y) \to (Z, d_Z)$ is quasiconformal if there exists $K \geq 1$ such that for all path families $\Gamma$ in $Y$

$$K^{-1} \text{mod} \Gamma \leq \text{mod} \psi \Gamma \leq K \text{mod} \Gamma,$$

where $\psi \Gamma = \{ \psi \circ \gamma : \gamma \in \Gamma \}$. If (5) holds with a constant $K \geq 1$, we say that $\psi$ is $K$-quasiconformal.

A special case of [Wil12, Theorem 1.1] yields the following.

**Theorem 2.4.** Let $Y$ and $Z$ be locally compact separable metric spaces with locally finite Hausdorff 2-measure and $\psi: Y \to Z$ a homeomorphism. The following are equivalent for the same constant $K > 0$:

1. $\text{mod} \Gamma \leq K \text{mod} \psi \Gamma$ for all path families $\Gamma$ in $Y$.
2. $\psi \in N^{1,2}_{\text{loc}}(Y, Z)$ and satisfies

$$p_\psi^2(y) \leq K J_\psi(y)$$

for $\mathcal{H}_Y^2$-almost every $y \in Y$.

The outer dilatation of $\psi$ is the smallest constant $K_O \geq 0$ for which the modulus inequality $\text{mod} \Gamma \leq K_O \text{mod} \psi \Gamma$ holds for all $\Gamma$ in $Y$. The inner dilatation of $\psi$ is the smallest constant $K_I \geq 0$ for which $\text{mod} \psi \Gamma \leq K_I \text{mod} \Gamma$ holds for all $\Gamma$ in $Y$. The number $K(\psi) = \max\{K_I(\psi), K_O(\psi)\}$ is the maximal dilatation of $\psi$.

For a set $G \subset Y$ and disjoint sets $F_1, F_2 \subset G$, let $\Gamma(F_1, F_2; G)$ denote the family of paths with each path starting at $F_1$, ending at $F_2$ and whose images are contained in $G$. A quadrilateral is a set $Q$ homeomorphic to $[0, 1]^2$ with boundary $\partial Q$ consisting of four boundary arcs, overlapping only at the end points, labelled $\xi_1, \xi_2, \xi_3, \xi_4$ in cyclic order.

A metric surface is a separable metric space $Y$ with locally finite Hausdorff 2-measure that is homeomorphic to a (connected) 2-manifold without boundary.

**Definition 2.5.** A metric surface $Y$ is reciprocal if there exists a constant $\kappa \geq 1$ such that

$$\kappa^{-1} \leq \text{mod} \Gamma(\xi_1, \xi_3; Q) \text{mod} \Gamma(\xi_2, \xi_4; Q) \leq \kappa$$

for every quadrilateral $Q \subset Y$, and

$$\lim_{r \to 0^+} \text{mod} \Gamma(\overline{B}_Y(y, r), Y \setminus B_Y(y, R); \overline{B}_Y(y, R)) = 0$$

for all $y \in Y$ and $R > 0$ such that $Y \setminus B_Y(y, R) \neq \emptyset$.

We note that for every metric surface,

$$\kappa_0^{-1} \leq \text{mod} \Gamma(\xi_1, \xi_3; Q) \text{mod} \Gamma(\xi_2, \xi_4; Q),$$

with $\kappa_0 = (4/\pi)^2$ [EP21] [RR19].

**Theorem 2.6** (Theorem 1.4 of [Raj17]). Let $(Y, d_Y)$ be a metric surface homeomorphic to $\mathbb{R}^2$ or to $\mathbb{S}^2$. Then there exists a quasiconformal embedding $\psi: (Y, d_Y) \to \mathbb{S}^2$ if and only if $Y$ is reciprocal.
Theorem 1.3 of [Ikoar] shows that if a metric surface \((Y, d_{Y})\) can be covered by quasiconformal images of domains \(V \subset \mathbb{R}^2\), then \((Y, d_{Y})\) is quasiconformally equivalent to a Riemannian surface. In particular, we have the following.

Theorem 2.7. Let \((Y, d_{Y})\) be a metric surface homeomorphic to \(S^2\). Then there exists a quasiconformal homeomorphism \(\psi: (Y, d_{Y}) \to S^2\) if and only if each point \(y \in Y\) is contained in an open set \(U\) from which there exists a quasiconformal homeomorphism \(\psi: U \to V \subset \mathbb{R}^2\).

Since (8) holds, we have the following result.

Proposition 2.8 (Corollary 12.3 of [Raj17]). Let \(Y\) be a metric surface, \(U \subset Y\) a domain, and \(\psi: U \to \Omega \subset \mathbb{R}^2\) a homeomorphism. If \(K_{O}(\psi) < \infty\), then \(\psi\) is \(K\)-quasiconformal for \(K = (2 \cdot k_{0}) \cdot K_{O}(\psi)\).

3. Hemispheres

We construct a (pseudo)distance \(d_{\tilde{Z}}\) on \(Z\) using a predistance \(D: Z \times Z \to [0, \infty]\) defined in the following way, with the identification \(S_{\tilde{Z}} \subset \tilde{Z}_{1}\) for the seam,

\[
D(x, y) = \begin{cases} 
\infty, & \text{if } (x, y) \in Z_{1} \times Z_{2} \cup Z_{2} \times Z_{1}, \\
\min \{\sigma(x, y), \sigma(g(x), g(y))\}, & \text{if } x, y \in S_{Z}, \\
\sigma(x, y), & \text{otherwise.}
\end{cases}
\]

Then we denote \(d_{\tilde{Z}}(x, y) = \inf \sum_{i=1}^{n} D(x_{i}, x_{i+1})\), the infimum taken over finite chains \((x_{i})\}_{i=1}^{n+1}\) for which \(x_{1} = x\) and \(x_{n+1} = y\). We obtain a metric space \(\tilde{Z}\) and a quotient map \(Q: Z \to \tilde{Z}\) by identifying \((x, y) \in Z \times Z\) whenever \(d_{\tilde{Z}}(x, y) = 0\), and setting \(d_{\tilde{Z}}(x, y) = d_{\tilde{Z}}(Q^{-1}(x), Q^{-1}(y))\) for each \(x, y \in \tilde{Z}\).

In this section, we focus on analyzing the distance \(d_{\tilde{Z}}\) on the seam \(S_{\tilde{Z}}\). The main results of this section are Lemmas 3.2 and 3.3 and Proposition 3.6.

In the following two lemmas we abuse notation and identify \(i_{1}(Z_{i})\) with \(Z_{i}\) when convenient.

Lemma 3.1. The following hold:

1. Let \(x, y \in S_{1} \subset \tilde{Z}_{1}\) and \((x_{i})_{i=1}^{n+1}\) a chain with \(x_{1} = x\), \(x_{n+1} = y\), and \(x_{i} \in \tilde{Z}_{1}\) otherwise. Then \(\sum_{i=1}^{n} D(x_{i}, x_{i+1}) \geq D(x, y)\).

2. Let \(x, y \in S_{1} \subset \tilde{Z}_{1}\) and \((x_{i})_{i=1}^{n+1}\) a chain with \(g(x_{1}) = g(x), g(x_{n+1}) = g(y)\), and \(x_{i} \in \tilde{Z}_{2}\) otherwise. Then \(\sum_{i=1}^{n} D(x_{i}, x_{i+1}) \geq D(x, y)\).

Proof. Given the chain from the claim (1), for every \(i\), \(D(x_{i}, x_{i+1}) = \sigma(x_{i}, x_{i+1})\). Thus, \(\sum_{i=1}^{n} D(x_{i}, x_{i+1}) \geq \sigma(x_{1}, x_{n+1}) \geq D(x_{1}, x_{n+1})\). The corresponding inequalities hold for the chain from (2).

Lemma 3.1 implies that when computing \(d_{\tilde{Z}}(i_{1}(x), i_{1}(y))\) for \(x, y \in S_{1}\), it is sufficient to consider chains with intermediate points staying within the seam.

Lemma 3.2. If \(x, y \in Z_{1}\), then

\[
d_{\tilde{Z}}(i_{1}(x), i_{1}(y)) = \begin{cases} 
\sigma(x, y), & \text{or there exist } w, w' \in S_{1} \text{ with} \\
\sigma(x, w) + d_{\tilde{Z}}(i_{1}(w), i_{1}(w')) + \sigma(w', y) \leq \sigma(x, y).
\end{cases}
\]

The corresponding identity holds for points \(x, y \in Z_{2}\).

Furthermore, if \(x \in Z_{1}\) and \(y \in Z_{2}\), there exist \(w, w' \in S_{1}\) such that

\[
d_{\tilde{Z}}(i_{1}(x), i_{2}(y)) = \sigma(x, w) + d_{\tilde{Z}}(i_{1}(w), i_{1}(w')) + \sigma(w', y).
\]
Proof. We show (9). Suppose that there exists a sequence $e_j \to 0^+$ and a sequence of chains $(x_{i,j})_{i=1}^{n_{i-1}}$ joining $x$ to $y$ with $d_Z(t_1(x), t_1(y)) \geq -e_j + \sum_{i=1}^{n_{i-1}} D(x_{i,j}, x_{i+1,j})$ so that every chain has an element in $S^1$. If $i_1$ is the first index for which $x_{i_1,j} \in S^1$ and $i_2$ the last one, then

$$
\sum_{i=1}^{n_{i-1}} D(x_{i,j}, x_{i+1,j}) \geq \sigma(x, x_{i_1,j}) + d_Z(t_1(x_{i_1,j}), t_1(x_{i_2,j})) + \sigma(x_{i_2,j}, y) \\
\geq \inf \{ \sigma(x, w) + d_Z(t_1(w), t_1(w')) + \sigma(w', y) \}.
$$

the infimum taken over every $w, w' \in S^1$. Observe that the infimum is realized by some $w, w' \in S^1$. Given such $w, w' \in S^1$, we pass to the limit $j \to \infty$ and conclude

$$d_Z(t_1(x), t_1(y)) \geq \sigma(x, w) + d_Z(t_1(w), t_1(w')) + \sigma(w', y).$$

Since "$\leq$" holds for every pair $w, w' \in S^1$, the lower equality in (9) follows.

If no such sequence of $e_j \to 0^+$ exists, then there exists $e_0 > 0$ such that for every $e_0 > e > 0$, any chain joining $x$ to $y$ with $d_Z(t_1(x), t_1(y)) \geq -e + \sum_{i=1}^{n_{i-1}} D(x_{i,j}, x_{i+1,j})$ does not intersect $S^1$. Hence $\sum_{i=1}^{n_{i-1}} D(x_{i,j}, x_{i+1,j}) \geq \sigma(x, y)$. So, either way, we obtain (9). The claims for each $x, y \in Z_2$ and $(x, y) \in Z_2 \times Z_2$ are proved in a similar manner. □

For $i = 1, 2$, we denote $\tilde{t}_i := Q \circ t_i: Z_i \to \tilde{Z}$. Lemma 3.2 implies that $\tilde{t}_i$ is 1-Lipschitz everywhere and a local isometry in $Z_i$. We also establish that $\tilde{t}_i$ is monotone, i.e., the preimage of a point is a compact and connected set.

Lemma 3.3. For $i = 1, 2$, the inclusion map $\tilde{t}_i: Z_i \to \tilde{Z}$ is 1-Lipschitz everywhere and a local isometry on $Z_i$. Moreover, for every $z \in \tilde{Z}$, the preimage $\tilde{t}_i^{-1}(z)$ is compact and connected. It contains two or more points only if $\tilde{t}_i^{-1}(z) \subset S^1$.

Before proving Lemma 3.3, we show two auxiliary results.

Lemma 3.4. Let $x, y \in S^1$ be distinct. Then there exists an arc $\gamma: [0, 1] \to S^1$ joining $x$ to $y$ with $D(t_1(x), t_1(y)) = \min \{ \ell(\gamma), \ell(g \circ \gamma) \}$. The arc satisfies

$$D(t_1(x), t_1(y)) \geq \sup_{(t_i)_{i=1}^{n}} \sum_{i=1}^{n} D(t_1(\gamma(t_i)), t_1(\gamma(t_{i-1}))),$$

the supremum taken over finite partitions of $[0, 1]$. In particular, $D(t_1(x), t_1(y)) \geq \ell(t_1(\gamma))$.

Proof. The existence of $\gamma$ with $D(t_1(x), t_1(y)) = \min \{ \ell(\gamma), \ell(g \circ \gamma) \}$ follows from the fact that $\sigma$ is geodesic on $S^1$. We identify $t_1(x)$ with $x$ for every $x \in S^1$ in the following computations.

The claim about the partitions is a consequence of the following observation and induction: If $0 \leq a < s < b \leq 1$, then

$$D(\gamma(a), \gamma(b)) \geq D(\gamma(a), \gamma(s)) + D(\gamma(s), \gamma(b)).$$

We first assume that $D(\gamma(a), \gamma(b)) = \sigma(\gamma(a), \gamma(b))$. Then $\gamma$ is a length-minimizing geodesic joining $\gamma(a)$ and $\gamma(b)$. Consequently,

$$\sigma(\gamma(a), \gamma(b)) = \sigma(\gamma(a), \gamma(s)) + \sigma(\gamma(s), \gamma(b)).$$

Since $c(c, d) \geq D(c, d)$ holds for every $c, d \in S^1$, the inequality (11) holds in this case. In the remaining case, $g \circ \gamma$ is a length-minimizing geodesic joining $g(\gamma(a))$ and $g(\gamma(b))$ and

$$\sigma(g(\gamma(a)), g(\gamma(b))) = \sigma(g(\gamma(a)), g(\gamma(s))) + \sigma(g(\gamma(s)), g(\gamma(b))).$$
Since \( c(g(c), g(d)) \geq D(c, d) \) for every \( c, d \in S^1 \), the inequality (11) holds also in this case.

The partition claim implies \( D(x, y) \geq \sum_{i=1}^n d_Z(i_1(x), i_1(y)) \) for every partition \( \{t_i\}_{i=1}^{n+1} \) of \([0, 1]\). The inequality \( D(x, y) \geq \ell(I_1(\gamma)) \) follows by taking the supremum over such partitions. \( \square \)

**Lemma 3.5.** Let \( x, y \in S^1 \) be distinct. Then there exists an arc \( \gamma: [0, 1] \to S^1 \) joining \( x \) to \( y \) such that \( d_Z(I_1(x), I_1(y)) = \ell(I_1(\gamma)) \).

**Proof.** Let \( \varepsilon > 0 \). The defining property of \( d_Z \) and Lemma 3.1 imply the existence of a chain \( \{x_i\}_{i=1}^{n+1} \subseteq S^1 \) joining \( x \) to \( y \) for which
\[
d_Z(I_1(x), I_1(y)) \geq -\varepsilon + \sum_{i=1}^{n} D(I_1(x_i), I_1(x_{i+1})).
\]
For each \( i \), Lemma 3.4 yields the existence of an arc \( \theta_i: [0, 1] \to S^1 \) joining \( x_i \) to \( x_{i+1} \) with \( D(I_1(x_i), I_1(x_{i+1})) \geq \ell(I_1(\theta_i)) \). Let \( \theta \) denote the concatenation of these paths. Then
\[
d_Z(I_1(x), I_1(y)) \geq -\varepsilon + \ell(I_1(\theta)).
\]
Let \( \theta': [0, 1] \to S^1 \) be an arc joining \( x \) to \( y \) within the image of \( \theta \). Applying (4) on \( \tilde{Z} \) with \( \rho = \chi_{\tilde{Z}} \) implies that \( \ell(I_1(\theta)) \geq \ell(I_1(\theta')) \). Such a \( \theta' \) is one of the arcs joining \( x \) to \( y \) within \( S^1 \).

Let \( \epsilon_j \to 0^+ \) and consider \( \theta'_j \) as above for every such \( \epsilon_j \). Up to passing to a subsequence and relabeling, we may assume that every such \( \theta'_j \) is the same arc \( \theta' \). Passing to the limit \( j \to \infty \) establishes \( d_Z(I_1(x), I_1(y)) \geq \ell(I_1(\theta')) \geq d_Z(I_1(x), I_1(y)) \). We set \( \gamma = \theta' \) to conclude the proof. \( \square \)

**Proof of Lemma 3.3.** The claimed 1-Lipschitz and local isometry properties follow from Lemma 3.2. The local isometry property implies that given \( z \in \tilde{Z} \), the preimage \( \tilde{\gamma}^{-1}(z) \) has more than two points only if the preimage is a subset of \( S^1 \).

Suppose the existence of a distinct pair \( x, y \in \tilde{\gamma}^{-1}(z) \). Then \( x, y \in S^1 \). Lemma 3.5 shows that there exists an arc \( \gamma \) joining \( x \) to \( y \) within \( S^1 \) satisfying
\[
0 = d_Z(I_1(x), I_1(y)) = \ell(I_1(\gamma)).
\]
This implies \( |\gamma| \subset \tilde{\gamma}^{-1}(z) \). Since \( x \) and \( y \) were arbitrary, we conclude that \( \tilde{\gamma}^{-1}(z) \) is path connected. Consequently, \( \tilde{\gamma}^{-1}(z) \) is a connected and compact subset of \( S^1 \).

The properties of \( \tilde{\gamma} \) follow from a symmetry in the argument. Hence the claim follows. \( \square \)

**Proposition 3.6.** Let \( g: (S^1, \mathcal{H}^1_{S^1}) \to (S^1, \mathcal{H}^1_{S^1}) \) be a homeomorphism with \( g^* \mathcal{H}^1_{S^1} = v_g \mathcal{H}^1_{S^1} + \mu^* \) mutually singular. Then, for every Borel set \( B \subseteq S^1 \),
\[
\mathcal{H}^1_{d_Z}(I_1(B)) = \int_B \min\{1, v_g\} \, d\mathcal{H}^1_{S^1} = \int_{I_1(B)} \#(\tilde{\gamma}^{-1}(z)) \, d\mathcal{H}^1_{S^1}(z).
\]

Moreover, for every \( x, y \in S^1 \), there exists an arc \( |\gamma| \subset S^1 \) joining \( x \) to \( y \) for which
\[
d_Z(I_1(x), I_1(y)) = \ell(I_1(\gamma)).
\]

Before proving Proposition 3.6, we first consider a Carathéodory construction on \( S^1 \). First, fix a Borel set \( B_0 \subseteq S^1 \) for which \( \mathcal{H}^1_{S^1}(B_0) = 0 \) and \( \mu^*(S^1 \setminus B_0) = 0 \). Set \( \nu^{ABS}(B) := \int_B \min\{1, v_g\} \chi_{S^1 \setminus B_0} \, d\mathcal{H}^1_{S^1} \) for all Borel sets \( B \subseteq S^1 \).

For every arc \( \gamma: [0, 1] \to S^1 \), we denote \( \xi^{ABS}(\gamma) := \nu^{ABS}(|\gamma|) \) and \( \xi(|\gamma|) := D(\gamma(0), \gamma(1)) \). The set function \( \xi^{ABS} \) and the family of arcs \( |\gamma| \subset S^1 \) yields Carathéodory premeasures \( \nu^{ABS} \) for each \( \delta > 0 \).
Lemma 3.7. For every Borel set $B \subset S^1$, we have $\nu^{ABS}(B) = \sup_{\delta > 0} \nu^{ABS}_\delta(B) \geq \mathcal{H}_2^1(\tilde{\gamma}_1(B))$.

Proof. The equality $\nu^{ABS}(B) = \sup_{\delta > 0} \nu^{ABS}_\delta(B)$ follows from the fact that $\nu^{ABS}$ is a finite Borel measure.

We denote $B_1 = \{x_0, x_1 \geq 1\} \cup B_0$ and $B_2 = S^1 \setminus B_1$. If $B \subset S^1$ is Borel, we have

$$\mathcal{H}_2^1(\tilde{\gamma}_1(B)) = \sum_{i=1}^2 \mathcal{H}_2^1(\tilde{\gamma}_1(B \cap B_i)) \leq \mathcal{H}_2^1(B \cap B_1) + \mathcal{H}_2^1(g(B) \cap g(B_2))$$

since $\tilde{\gamma}_1$ is 1-Lipschitz for $i = 1, 2$. The right-hand side equals $\nu^{ABS}(B)$. Therefore $\mathcal{H}_2^1(\tilde{\gamma}_1(B)) \leq \nu^{ABS}(B)$ holds for all Borel sets. □

Lemma 3.8. Let $x, y \in S^1$ be distinct and $\gamma : [0, 1] \to S^1$ an arc joining $x$ to $y$ such that $d_2(\tilde{\gamma}_1(x), \tilde{\gamma}_1(y)) = \ell(\tilde{\gamma}_1(\gamma))$. Then $\mathcal{H}_2^1(\tilde{\gamma}_1(\gamma)) = d_2(\tilde{\gamma}_1(x), \tilde{\gamma}_1(y)) = \nu^{ABS}(\gamma)$.

Proof. Let $\pi/2 > \delta_0 > 0$ be such that $D(t_1(a), t_1(b)) < \delta_0$ implies $\{\sigma(a, b), \sigma(g(a), g(b))\} < \pi/2$.

Given such a pair $a, b \in S^1$, the length-minimizing geodesic $\theta : [0, 1] \to S^1$ joining $a$ to $b$ satisfies $\xi(\theta) = \min \{\ell(\theta), \ell(g \circ \theta)\}$. Then $\xi(\theta) = \xi^{ABS}(\theta)$.

Let $\gamma$ be as in the claim. Let $0 < \delta < \delta_0$ and $0 < \epsilon < \delta/2$. We consider a partition $\{\gamma_i\}_{i=1}^{n+1}$ of $[0, 1]$ such that $\sigma(\gamma(t), \gamma(t+1)) < \delta/2$ for every $i$. Then there exists a chain $\{x_{ij}\}_{i=1}^{n+1} \subset S^1$ joining the ends of $\gamma_i$ so that

$$d_2(t_1 \circ \gamma(t), t_1 \circ \gamma(t+1)) \geq -\frac{\epsilon}{n} + \sum_{j=1}^{n_i} D(t_1(x_{ij}), t_1(x_{ij+1})).$$

In particular, $D(t_1(x_{ij}), t_1(x_{ij+1})) < \delta < \delta_0$ for every $j$. Hence the length-minimizing geodesic $\gamma_{ij}$ joining $x_{ij}$ to $x_{ij+1}$ satisfies the assumptions of Lemma 3.4. For every $i$, Lemma 3.4 implies that, up to further partitioning the paths $\gamma_{ij}$ and relabeling, we may assume $\sigma(x_{ij}, x_{ij+1}) < \delta$ for every $j$. Given this property, we conclude $D(t_1(x_{ij}), t_1(x_{ij+1})) = \xi(\gamma_{ij}) \geq \xi^{ABS}(\gamma_{ij})$ and

$$\ell(\vartheta_\gamma) = \sum_{i=1}^n d_2(t_1 \circ \gamma(t), t_1 \circ \gamma(t+1)) \geq -\epsilon + \nu^{ABS}_\delta \left( \bigcup_{i=1}^n \bigcup_{j=1}^{n_i} \gamma_{ij} \right).$$

Since the concatenation $\vartheta_\gamma$ of $\{\gamma_{ij}\}_{i=1}^{n}$ is a path joining $\gamma(t_i)$ to $\gamma(t_{i+1})$, the concatenation $\theta$ of $\{\theta_i\}_{i=1}^n$ is a path joining $x$ to $y$. Hence $\bigcup_{i=1}^{n-1} \bigcup_{j=1}^{n_i} \gamma_{ij} = |\theta|$ contains $\gamma$ or $S^1 \setminus |\gamma|$, and

$$\ell(\vartheta_\gamma) \geq -\epsilon + \min \left\{ \nu^{ABS}(\gamma), \nu^{ABS}(S^1 \setminus |\gamma|) \right\}.$$

After passing to $\epsilon \to 0^+$ and then to $\delta \to 0^+$, we conclude

$$\mathcal{H}_2^1(\vartheta_\gamma) = \ell(\vartheta_\gamma) \geq \min \left\{ \nu^{ABS}(\gamma), \nu^{ABS}(S^1 \setminus |\gamma|) \right\}.$$

If we had $\nu^{ABS}(\gamma) > \nu^{ABS}(S^1 \setminus |\gamma|)$, this would contradict Lemma 3.7 and the length-minimizing property of $\tilde{\gamma}_1(\gamma)$. Hence $\nu^{ABS}(\gamma) \leq \nu^{ABS}(S^1 \setminus |\gamma|)$, and $\mathcal{H}_2^1(\vartheta_\gamma) = d_2(\tilde{\gamma}_1(x), \tilde{\gamma}_1(y)) = \nu^{ABS}(\gamma)$ follows from Lemma 3.7. □

Proof of Proposition 3.6. The existence of $\gamma$ and equality in (13) already follows from Lemma 3.4.
We claim that (12) holds. To this end, we consider three arcs \( \gamma_i : [0,1] \to S^1 \) overlapping only at their end points, whose images cover \( S^1 \), with the arcs satisfying \( v^{ABS}(|\gamma_i|) \leq v^{ABS}(S^1 \setminus |\gamma_i|) \).

Lemmas 3.7 and 3.8 imply that \( i_1 \circ \gamma_i \) is a length-minimizing geodesic joining its end points and \( v^{ABS}(|\gamma_i|) = \mathcal{H}^1_2(i_1(|\gamma_i|)). \) Lemma 3.7 implies that the metric speed of \( \bar{n}_i|_{|\gamma_i|} \) is bounded from above by \( \min \{1, \nu_8\} \). Hence the equality \( v^{ABS}(|\gamma_i|) = \mathcal{H}^1_2(i_1(|\gamma_i|)) \) forces the metric speed of \( \bar{i}_1 \) to equal \( \min \{1, \nu_8\} \). The \( \mathcal{H}^1_{\tilde{\gamma}_i} \)-almost everywhere on \( |\gamma_i| \) for \( i = 1,2,3 \). The equality (12) follows from the area formula (4) and the fact that \( \#(\bar{i}_1^{-1}(x)) = 1 \mathcal{H}^1_{\tilde{\gamma}_i} \)-almost everywhere. The fact \( \#(\bar{i}_1^{-1}(x)) = 1 \mathcal{H}^1_{\tilde{\gamma}_i} \)-almost everywhere follows from the monotonicity of \( \bar{i}_1 \) and the integrability of the multiplicity. The integrability of the multiplicity follows from (3).

\[ \square \]

Remark 3.9. We consider a \( 2\pi \)-periodic doubling measure \( \mu \) on \( \mathbb{R} \) with \( 2\pi = \mu([0,2\pi]) \) such that for some Borel set \( B \subset [0,2\pi], \mathcal{L}^1(B) = 0 = \mu([0,2\pi] \setminus B) \), the existence of which is established by Ahlfors–Beurling [BA56, Section 7]. Then \( \psi(x) = \int_0^x d\mu \) is a homeomorphism and there exists a quasisymmetry \( g : S^1 \to S^1 \) with \( \theta \circ \psi = g \circ \theta \), where \( \theta(t) = (\cos(t), \sin(t), 0) \). Then \( \nu_8 \) in (12) is identically zero. Consequently, \( d_Z \equiv 0 \) on the seam \( S_Z \).

4. Harmonic measure and welding homeomorphisms

We consider a welding homeomorphism \( g : S^1 \to S^1 \) and a welding circle \( \mathcal{C} \) with complementary components \( \Omega_1 \) and \( \Omega_2 \). Kiemann maps \( \phi_i : Z_i \to \Omega_i \) for \( i = 1,2 \), and \( g = \phi_2^{-1} \circ \phi_1 \). In this section, we consider the harmonic measures \( \omega_i(E) = \phi_i^* \mathcal{H}^1_\Omega(E) / (2\pi) \) for all Borel sets \( E \subset \mathbb{S}^2 \).

We define a homeomorphism \( \pi : \mathbb{S}^2 \to (Z, d_Z) \) and a quotient map \( \bar{\pi} : \mathbb{S}^2 \to \bar{Z} \) via the formulas

\[
\pi(x) = \begin{cases} 
\phi_1^{-1}(x), & \text{when } x \in \overline{\Omega_1} \\
\phi_2^{-1}(x), & \text{when } x \in \overline{\Omega_2} 
\end{cases}
\]

and \( \bar{\pi} = Q \circ \pi \).

Recall that \( Q : Z \to \bar{Z} \) is the quotient map identifying \( x, y \in Z \) whenever \( d_Z(x,y) = 0 \). Lemma 3.3 implies that \( \bar{\pi} \) is monotone and \( \bar{\pi}^{-1}(x) \) contains two or more points only if \( x \) is a point of the seam \( Q(S_Z) \), and in such a case \( \bar{\pi}^{-1}(x) \subset \mathcal{C} \).

For \( a = 1,2 \), we denote, for every Borel set \( B \subset \mathbb{S}^2 \),

\[
\bar{\pi}^* \mathcal{H}^a_\Omega(B) := \int_B \#(\bar{\pi}^{-1}(x)) \, d\mathcal{H}^a_\mathbb{S}(x) = \mathcal{H}^a_\mathbb{S}(\bar{\pi}(B)),
\]

where the multiplicity can be ignored in the case \( a = 2 \) since it equals one outside the negligible set \( Q(S_Z) \). For \( a = 1 \), the multiplicity is two or more only when it is \( \infty \) and this happens in a set of negligible \( \mathcal{H}^1_{\mathbb{S}} \)-measure. Either way, the multiplicity is negligible in (15), so the second equality is justified.

Proposition 4.1. Let \( g \) be a welding homeomorphism with a welding circle \( \mathcal{C} \) and \( I \subset \mathcal{C} \) a subarc. Then \( d_Z(\bar{\pi}(x), \bar{\pi}(y)) = 0 \) for all \( x, y \in I \) if and only if \( \omega_1 | I \) and \( \omega_2 | I \) are mutually singular. If such an interval exists, then \( \bar{Z} \) is not quasiconformally equivalent to \( \mathbb{S}^2 \).

Remark 4.2. If \( g \) is a welding homeomorphism obtained from Remark 3.9 or any welding \( g \) corresponding to the von Koch snowflake [GM05, Example 4.3], Proposition 4.1 implies that \( Q(S_Z) \) is a singleton. In particular, \( \bar{Z} \) is not even
homeomorphic to the sphere. For a given $g$, this happens if and only if $g^*\mathcal{H}^1_{S^1}$ and $\mathcal{H}^1_{S^1}$ are mutually singular.

A key step in the proof of the conformal removability in Theorem 1.6 is the following.

**Proposition 4.3.** Let $g$ be a welding homeomorphism and $\tilde{\pi}$ as in (14). Then $\tilde{\pi}$ is continuous, monotone, and surjective. Moreover, for all path families $\Gamma$ on $S^2$, $\text{mod } \Gamma \leq \text{mod } \tilde{\pi}\Gamma$. The metric space $\tilde{Z}$ is quasiconformally equivalent to $S^2$ if and only if $\tilde{\pi}$ is a homeomorphism for which $\text{mod } \Gamma = \text{mod } \tilde{\pi}\Gamma$ for all path families.

The proof of Proposition 4.3 requires some preparatory work. Given the curve $C$, we say that $x_0 \in C$ is a tangent point if there exists a homeomorphism $\gamma: (-\epsilon, \epsilon) \to C' \subset C$ with $\gamma(0) = x_0$, and a tangent vector $v_0 \in T_{x_0}S^2$ with unit length such that for every smooth $f: S^2 \to \mathbb{R}$, its differential $df$ satisfies

$$df(v_0) = \lim_{t \to 0^+} \frac{f(\gamma(t)) - f(x_0)}{\sigma(\gamma(t), x_0)} \quad \text{and} \quad df(-v_0) = \lim_{t \to 0^-} \frac{f(\gamma(t)) - f(x_0)}{\sigma(\gamma(t), x_0)}.$$

If $v_0$ exists, the tangent vector $v_0$ is independent of the parametrization $\gamma$ and $C'$ up to multiplication by $-1$; see [GM05, Chapter II, Section 4]. The collection of tangents points of $C$ is denoted by $\text{Tr}(C)$. The key properties of $\text{Tr}(C)$ are self-contained in the following statement.

**Lemma 4.4.** The Borel set $T_n(C)$ has $\sigma$-finite Hausdorff 1-measure. Moreover, on the set $\text{Tr}(C)$, the measures $\omega_1$, $\omega_2$, and $\mathcal{H}^1_C$ are mutually absolutely continuous.

Given any Borel set $E \subset C$ with $\omega_1(E) \cdot \omega_2(E) > 0$, the restrictions $\omega_1|_E$ and $\omega_2|_E$ are mutually singular on $E$ if and only if $\mathcal{H}^1_C(\text{Tr}(C) \cap E) = 0$.

**Proof.** The Borel measurability of $\text{Tr}(C)$ follows from [GM05, Chapter II, Theorem 4.2] which connects the tangents of $C$ and the angular derivatives of any given Riemann map $\phi_1: Z_1 \to \Omega_1$, where $\partial \Omega_1 = C$. The fact that $\text{Tr}(C)$ has $\sigma$-finite Hausdorff 1-measure follows from [GM05, Chapter VI, Theorem 4.2].

Theorem 6.3 of [GM05, Chapter VI] states that if a Borel set $E \subset C$ is such that $\omega_1(E) \cdot \omega_2(E) > 0$, then $\omega_1|_E$ and $\omega_2|_E$ are mutually singular on $E$ if and only if $\mathcal{H}^1_C(\text{Tr}(C) \cap E) = 0$.

The fact that on the set $\text{Tr}(C)$ the measures $\omega_1$, $\omega_2$, and $\mathcal{H}^1_C$ are mutually absolutely continuous follows from [GM05, Chapter VI, Theorem 4.2 and the following discussion on p. 211].

**Lemma 4.5.** The measures $\chi_C\tilde{\pi}^*\mathcal{H}^1_Z$, $\chi_{\text{Tr}(C)}\omega_1$, $\chi_{\text{Tr}(C)}\omega_2$, and $\chi_{\text{Tr}(C)}\mathcal{H}^1_C$ are mutually absolutely continuous.

More precisely, a given Borel set $B \subset \text{Tr}(C)$ has positive 1-dimensional Hausdorff measure if and only if $\mathcal{H}^1_Z(\tilde{\pi}(B)) > 0$. Furthermore, if $B \subset C \setminus \text{Tr}(C)$, then $\mathcal{H}^1_Z(\tilde{\pi}(B)) = 0$.

**Proof.** We write $g^*\mathcal{H}^1_{S^1} = v_S\mathcal{H}^1_{S^1} + 2\pi \cdot \mu^\perp$ with $\mathcal{H}^1_{S^1}$ and $\mu^\perp$ are mutually singular. We recall from Proposition 3.6 that for every Borel set $B \subset C$,

$$\mathcal{H}^1_Z(\tilde{\pi}(B)) = \int_{\phi_1^{-1}(B)} \min \{1, v_8\} \, d\mathcal{H}^1_{S^1}. \quad (16)$$

We denote $h = v_S \circ \phi_1^{-1}$ and observe the equality $\omega_2 = h \omega_1 + (\phi_1)_* \mu^\perp$. Then (16) is equivalent to

$$2\pi^{-1}\mathcal{H}^1_Z(\tilde{\pi}(B)) = \int_B \min \{1, h\} \, d\omega_1. \quad (17)$$
Lemma 4.4 implies that the measures \( \chi_{C \setminus Tn(C)} \omega_1 \) and \( \chi_{C \setminus Tn(C)} \omega_2 \) are mutually singular. Consequently, \( h = 0 \) \( \omega_1 \)-almost everywhere in \( C \setminus Tn(C) \). In particular, if \( B = C \setminus Tn(C) \), the left-hand side equals zero in (17).

Lemma 4.4 yields that the measures \( \chi_{Tn(C)} \omega_1 \) and \( \chi_{Tn(C)} \omega_2 \) are mutually absolutely continuous. Hence \( \infty > h > 0 \) \( \omega_1 \)-almost everywhere in \( Tn(C) \). This implies that the measure in (17) is mutually absolutely continuous with the measures \( \chi_{Tn(C)} \omega_1 \) and \( \chi_{Tn(C)} \omega_2 \) and \( \chi_{Tn(C)} H_1^{1} \). The claim follows from the equalities (15) for \( \alpha = 1 \).

\[ \Box \]

**Proof of Proposition 4.1.** Fix a subarc \( I \subset C \). Proposition 3.6 implies that \( \hat{\pi}(I) \) has zero \( H_{\mathbb{Z}}^{1} \)-measure if and only if for every \( x, y \in I \), \( d_{\hat{\pi}}(\hat{\pi}(x), \hat{\pi}(y)) = 0 \) if and only if \( v_{\pi} = 0 \) \( H_{\mathbb{Z}}^{1} \)-almost everywhere on \( \phi_{1}^{-1}(I) \). Equivalently, \( \omega_{1, I} \) and \( \omega_{2, I} \) are mutually singular.

Lemma 3.3 shows that \( \hat{\pi} \neq (Z, d_{Z}) \) if and only if there exists a closed arc \( I \subset S^{1} \) such that \( y = \tilde{\pi}_{1}(I) \). Assume that such an \( I \) exists. Having fixed \( x_{0} \in Z_{1} \) and \( 0 < s < \sigma(x_{0}, S^{1}) \), there exists \( c = c(x_{0}, I, s) \) for which

\[ \text{mod} \Gamma(I, \mathcal{B}_{S^{2}}(x_{0}, s); I \cup Z_{1}) \geq c > 0; \]

a positive lower bound can be shown, for example, by estimating the modulus of all geodesics joining \( I \) to \( \mathcal{B}_{S^{2}}(x_{0}, s) \) in \( I \cup Z_{1} \).

When \( R > 0 \) is small enough, for every \( R > r > 0 \) and every path in \( \Gamma(I, \mathcal{B}_{S^{2}}(x_{0}, s)); I \cup Z_{1} \), we find a subpath \( \gamma' : [0, 1] \rightarrow Z_{1} \) so that \( \tilde{\gamma}_{1} \circ \gamma \) joins \( \mathcal{B}_{S^{2}}(y, r) \) to \( \mathcal{B}_{S^{2}}(y, R) \) within \( \mathcal{B}_{S^{2}}(y, R) \). Since \( \tilde{\gamma}_{1} \) is a local isometry off the seam, this implies

\[ \liminf_{r \to 0^{+}} \text{mod} \Gamma \left( \mathcal{B}_{S^{2}}(y, r), \mathcal{B}_{S^{2}}(y, R) \right) \geq c. \]

Recalling Theorem 2.6, we see that \( \hat{\pi} \) is not quasiconformally equivalent to \( S^{2} \). \[ \Box \]

**Lemma 4.6.** For \( i = 1, 2 \), let \( \rho_{i} : \Omega_{i} \rightarrow [0, \infty] \) denote the operator norm of the differential of \( D(\phi_{i}^{-1}) \). Then

\[ G = \chi_{\Omega_{1}} \rho_{1} + \chi_{\Omega_{2}} \rho_{2} + \infty \cdot \chi_{Tn(C)} \in L^{2}(S^{2}) \]

is a weak upper gradient of \( \hat{\pi} \).

**Proof.** The \( L^{2} \)-integrability of \( G \) follows from the change of variables formulas of the Riemann maps \( \phi_{1} \) and \( \phi_{2} \) and the fact that \( Tn(C) \) has negligible area. Hence, as a consequence of Lemma 2.1, \( G \) is integrable along almost every absolutely continuous path \( \gamma : [0, 1] \rightarrow S^{2} \). Given such a \( \gamma \), we claim that

\[ d_{\hat{\pi}}(\hat{\pi}(\gamma(0)), \hat{\pi}(\gamma(1))) \leq \int_{\gamma} G \, ds, \]

implying that \( G \) is a weak upper gradient of \( \hat{\pi} \).

Since \( G \) is integrable along \( \gamma \), \( \gamma \) has negligible length in \( Tn(C) \). Then (4) implies \( \mathcal{H}_{\mathbb{Z}}^{1}(Tn(C) \cap |\gamma|) = 0 \). We conclude \( \mathcal{H}_{\mathbb{Z}}^{1}(\hat{\pi}^{*}(C) \cap |\hat{\pi} \circ \gamma|) = 0 \) from Lemma 4.5. The assumptions of Lemma 2.2 are satisfied and the conclusion \( \ell(\hat{\pi} \circ \gamma) \leq \int_{\gamma} G \, ds \) follows. The inequality (19) is a consequence.

We define the Jacobian of \( \hat{\pi} \) to be the density of \( \hat{\pi}^{*} H_{\mathbb{Z}}^{2} \), defined in (15), with respect to \( H_{\mathbb{Z}}^{2} \).

**Lemma 4.7.** The mapping \( \hat{\pi} \) satisfies Lusin’s Condition (N) and the Jacobian \( J_{\hat{\pi}} \) coincides with \( G^{2} H_{\mathbb{Z}}^{2} \)-almost everywhere, with \( G \) being from (18).
Proof. The Lusin’s Condition (N) of \( \tilde{\pi} \) follows from the fact that \( \tilde{\pi}(C) \) has negligible \( H^2_{\mathcal{F}} \)-measure, the fact that \( i_t : Z_t \to \tilde{Z}_t \) is a local isometry, and as \( \phi_1^{-1} : \Omega_t \to Z_t \) satisfies Condition (N). Here \( J_{\tilde{\pi}} = 0 \) \( H^2_{\mathcal{F}} \)-almost everywhere on \( C \), so the equality \( J_{\tilde{\pi}} = G^2 \) follows from the fact that \( \phi_1 \) and \( \phi_2 \) are Riemann maps. \( \square \)

Proof of Proposition 4.3. The claimed topological properties of \( \tilde{\pi} \) were already verified at the beginning of this section. Lemmas 4.6 and 4.7 prove that \( J_{\tilde{\pi}} = G^2 \in L^1(\mathbb{S}^2) \) with \( G \) being a weak upper gradient of \( \tilde{\pi} \). This fact and the fact that the multiplicity of \( \tilde{\pi} \) is negligible for \( \tilde{\pi}^* H^2_{\mathcal{F}} \) imply \( \text{mod} \Gamma \leq \text{mod} \tilde{\pi} \Gamma \) for all path families \( \Gamma \).

Lastly, we argue that a \( K \)-quasiconformal map \( \psi : \tilde{Z} \to \mathbb{S}^2 \) exists (for some \( K \geq 1 \)) if and only if \( \tilde{\pi} \) is a \( 1 \)-quasiconformal homeomorphism. The "if"-direction is obvious.

In the "only if"-direction, the fact that \( \tilde{\pi} \) is a homeomorphism follows from Proposition 4.1. So \( h = \psi \circ \tilde{\pi} : \mathbb{S}^2 \to S^2 \) is a homeomorphism satisfying \( \text{mod} \Gamma \leq K \text{mod} \tilde{\pi} \Gamma \) for all path families \( \Gamma \). Theorem 2.4 and [AIM09, Definition 3.1.1 and Theorem 3.7.7] prove that \( h \) is \( K \)-quasiconformal. Consequently, \( \tilde{\pi} \) is \( K' \)-quasiconformal for some \( K' \leq K^2 \). This self-improves to \( K' = 1 \) due to Lemma 4.8 below. This yields \( \text{mod} \tilde{\pi} \Gamma = \text{mod} \Gamma \) for all path families. \( \square \)

Lemma 4.8. Suppose that \( \tilde{\pi} : \mathbb{S}^2 \to \tilde{Z} \) from (14) is a homeomorphism. Then \( \tilde{\pi} : \mathbb{S}^2 \to \tilde{Z} \) is \( 1 \)-quasiconformal if and only if for every 1-Lipschitz \( h : \mathbb{S}^2 \to \mathbb{R} \), \( h \circ \tilde{\pi}^{-1} \in N^{1,2}(\tilde{Z}) \).

Proof. The "only if"-claim is clear, given Theorem 2.4 (ii). In the "if"-direction, fix a 1-Lipschitz \( h : \mathbb{S}^2 \to \mathbb{R} \) for now.

Consider the Borel function \( G : \mathbb{S}^2 \to [0, \infty] \) defined on Lemma 4.6. Then \( \rho = 1/G \circ \tilde{\pi}^{-1} \) is such that \( \rho^2 \) is the Jacobian of \( \tilde{\pi}^{-1} \), as a consequence of Lemma 4.7. Hence \( \rho \in L^2(\tilde{Z}) \).

Given that \( h \circ \tilde{\pi}^{-1} \in N^{1,2}(\tilde{Z}) \) and \( H^2_{\mathcal{F}}(Q(S)) = 0 \), for almost every \( \gamma : [0,1] \to \tilde{Z} \), the composition \( (h \circ \tilde{\pi}^{-1}) \circ \gamma \) is absolutely continuous, \( \gamma \) has negligible length on the seam \( Q(S) \), and \( \int_\gamma \rho \, ds < \infty \). Indeed, the absolute continuity of \( (h \circ \tilde{\pi}^{-1}) \circ \gamma \) for almost every path follows from [HKST15, Proposition 6.3.2]. The fact that almost every path has negligible length on \( Q(S) \) follows from Lemma 2.1 and the \( L^2 \)-integrability of \( \rho \circ \tilde{\pi}^{-1} \). Similarly, the conclusion \( \int_\gamma \rho \, ds < \infty \) follows from Lemma 2.1 and the \( L^2 \)-integrability of \( \rho \).

If we denote \( E = (h \circ \tilde{\pi}^{-1})(|\gamma| \cap Q(S)) \), the absolute continuity of \( (h \circ \tilde{\pi}^{-1}) \circ \gamma \) implies \( H^1_{\mathcal{F}}(E) = 0 \). Then Lemma 2.2 yields \( \ell((h \circ \tilde{\pi}^{-1}) \circ \gamma) \leq \int_\gamma \rho \, ds \). We conclude that \( \rho \) is a weak upper gradient of \( h \circ \tilde{\pi}^{-1} \).

Since \( \rho \) is independent of \( h \) and \( h \) is an arbitrary 1-Lipschitz function, Theorem 7.1.20 [HKST15] shows that \( \rho \) is a weak upper gradient of \( \tilde{\pi}^{-1} \). Since \( \rho^2 \) is the Jacobian of \( \tilde{\pi}^{-1} \), we conclude \( K_O(\tilde{\pi}^{-1}) = 1 \). Recall \( K_O(\tilde{\pi}) = 1 \) from Proposition 4.3. \( \square \)

Remark 4.9. If the welding curve \( C \) happens to be rectifiable, the Hausdorff 1-measure on \( C \) and \( CH_{\text{Tir}}(C) \) are mutually absolutely continuous [GM05, Chapter VI, Theorem 1.2 (F. and M. Riesz)]. With this fact at hand, Lemma 4.5 implies that \( \tilde{\pi} \) is a homeomorphism. Moreover, one can show that \( h \circ \tilde{\pi}^{-1} \in N^{1,2}(\tilde{Z}) \) for every 1-Lipschitz \( h : \mathbb{S}^2 \to \mathbb{R} \). Hence \( \tilde{\pi} \) is 1-quasiconformal.
Proof of Theorem 1.6. Suppose the existence of a quasiconformal homeomorphism \( \psi : \tilde{Z} \to S^2 \). Up to postcomposing \( \psi \) by an orientation-reversing Möbius transformation of \( S^2 \), we may assume that \( \bar{\phi}_i := \psi \circ \bar{t}_i|_{Z_i} : Z_i \to S^2 \) is orientation-preserving for \( i = 1, 2 \). Let \( C = \psi(Q(S_Z)) \).

The set \( S^2 \setminus C \) is the disjoint union of Jordan domains \( \Omega_1 \) and \( \Omega_2 \), where \( \Omega_i \) is the image of \( \bar{\phi}_i \) for \( i = 1, 2 \). Proposition 4.1 implies \((Z, d_Z) = \tilde{Z} \).

Next, since \( \psi : \tilde{Z} \to S^2 \) is a quasiconformal homeomorphism, \( \psi \) satisfies Lusin’s Condition (N) [Raj17, Section 17]. Consequently, \( C \) has zero 2-dimensional Hausdorff measure.

We consider the Beltrami differential \( \mu = \chi\Omega_1\mu_1 + \chi\Omega_2\mu_2 \), where \( \mu_i \) is the Beltrami differential of \( \bar{\phi}_i^{-1} \). If \( h : S^2 \to S^2 \) is a normalized solution to the Beltrami equation induced by \( \mu \) [AIM09, Measurable Riemann mapping theorem], the mapping \( \tilde{\psi} = h \circ \psi \) is 1-quasiconformal. Since \( C \) has zero measure, this is readily verified by hand or by applying [Ikoar, Theorem 4.12].

We have verified that \((Z, d_Z) = \tilde{Z} \) and we may assume that \( \psi : (Z, d_Z) \to S^2 \) is 1-quasiconformal with \( \bar{\phi}_i = \psi \circ \bar{t}_i|_{Z_i} \) being Riemann maps [AIM09, Weyl’s lemma]. The definition of \( Z \) implies that \( g = \phi_2^{-1} \circ \phi_1|_{\mathcal{S}} \). Consequently, \( g \) is a welding homeomorphism.

In order to show the removability of \( C := \psi(S_Z) \), we are given an orientation-preserving homeomorphism \( M : S^2 \to S^2 \) conformal in the complement of \( C \). Then \( \pi' := \psi^{-1} \circ M^{-1} \) defines a mapping as in (14) for the curve \( \mathcal{C}' = M(\mathcal{C}) \). Proposition 4.3 implies that \( \pi' \) is 1-quasiconformal. Consequently, \( M^{-1} = \psi \circ \pi' \) is 1-quasiconformal, i.e., a Möbius transformation.

5. Mass upper bound

In this section, we prove Theorems 1.1 and 1.2. We first consider the implication "(3) \( \Rightarrow \) (1)". Recall that we are given an orientation-preserving homeomorphism \( g : S^1 \to S^1 \) and the canonical quotient map \( Q : Z \to \tilde{Z} \). We are assuming the existence of a constant \( C > 0 \) for which

\[
\liminf_{r \to 0^+} \frac{\mathcal{H}^2_2(\overline{B}_2(y, r))}{\pi r^2} \leq C \quad \text{for every } y \in Q(S_Z).
\]

In order to make transparent how \( C \) and the Lipschitz constants of \( g \) (resp. \( g^{-1} \)) are related to \( C \) in (20), we define \( C_1, C_2 \geq 0 \) to be the smallest constants for which

\[
\liminf_{r \to 0^+} \frac{\mathcal{H}^2_2(\overline{t}_1(Z_1) \cap B_2(y, r))}{\pi r^2} \leq C_1 \quad \text{for every } y \in Q(S_Z)
\]

and

\[
\liminf_{r \to 0^+} \frac{\mathcal{H}^2_2(\overline{t}_2(Z_2) \cap B_2(y, r))}{\pi r^2} \leq C_2 \quad \text{for every } y \in Q(S_Z).
\]

Recalling from Lemma 3.3 the fact that the inclusion maps are 1-Lipschitz and local isometries outside the seam, the limit infimums in (21) and (22) are bounded from below by 1/2. Hence, \( C_1, C_2 \geq 2^{-1} \). Since the seam is negligible, we have \( 1 \leq C_1 + C_2 \leq C. \)

We show that the constant \( C_1 \) in (21) and the Lipschitz constant \( L_1 \) of \( g^{-1} \) are connected via the following function

\[ f(\epsilon) := \frac{(\sin |0, \pi/2|)^{-1}(\epsilon)}{\pi} + \frac{\sqrt{1 - \epsilon^2}}{\pi \epsilon} \quad \text{for } 0 < \epsilon \leq 1. \]
Remark 5.2. We note that for every $0 < \epsilon \leq 1$, we have $f(\epsilon) \geq (\pi \epsilon)^{-1}$. We use this fact during the proof of Theorem 1.1.

Proposition 5.3. If (21) holds with constant $C_1$ and $L_1 = L(C_1)$ is as in Definition 5.1, then $g^{-1}$ is $L_1$-Lipschitz and $\tilde{\iota}_1: Z_1 \to \bar{Z}$ satisfies for every $x, y \in \mathbb{Z}_1$,

$$\sigma(x, y) \geq d_Z(\tilde{\iota}_1(x), \tilde{\iota}_1(y)) \geq \sigma(x, y) / L_1.$$  

The symmetry in the argument yields the following result.

Proposition 5.4. If (22) holds with constant $C_2$ and $L_2 = L(C_2)$ is as in Definition 5.1, then $g$ is $L_2$-Lipschitz and $\tilde{\iota}_2: Z_2 \to \bar{Z}$ satisfies for every $x, y \in \mathbb{Z}_2$,

$$\sigma(x, y) \geq d_Z(\tilde{\iota}_2(x), \tilde{\iota}_2(y)) \geq \sigma(x, y) / L_2.$$  

We start the proof of Proposition 5.3. We consider the decomposition $g^*H_{g^1}^1 = v_\Phi H_{g^1}^1 + \mu^+$ with $\mu^+$ and $H_{g^1}^1$ being singular. We fix a Borel representative of $v_\Phi$.

Let $f$ be as in (23). The following statement holds for every $\tilde{Z}$.

Proposition 5.5. Given $1 > \epsilon > 0$ and a $H_{g^1}^1$-density point $x_0 \in S^1$ of $E := \{v_\Phi \leq \epsilon\}$, we have

$$f(\epsilon) \leq \liminf_{r \to 0^+} \frac{\mathcal{H}_2^2(\tilde{\iota}_1(Z_1) \cap B_\rho(x_0, r))}{\pi r^2}.$$  

Proof. For the duration of the proof, we fix normal coordinates $F: B(0, \pi/2) \to S^2$ centered at $x_0$ in such a way that the preimage of $S^1 \cap B(x_0, \pi/2)$ is $(-\pi/2, \pi/2) \times \{0\}$. Recall that this means that $F$ is an isometry along radial geodesics and the metric has the expansion $g_{ij}(x) = \delta_{ij} + O(\|x\|^2)$ in these coordinates. In particular, as $r \to 0^+$, the bi-Lipschitz constant of $F |_{B(0, r)}$ is of the form $1 + O(r^2)$. We denote $\Gamma(s) := F(s, 0)$ for $|s| \leq \pi/2$.

We fix $0 < \eta < 1/\epsilon - 1$. Since $x_0$ is a density point of $E$, there exists $s_0 < \pi/2$ such that for every $0 < s \leq s_0$,

$$\mathcal{H}_2^1(\Gamma([-s, s]) \setminus E) \leq \epsilon \eta s.$$  

We fix $0 < r \leq \epsilon s_0$. Then, for every $0 < s < r/\epsilon$, Proposition 3.6 yields for both $I = [0, s]$ and $I = [-s, 0]$,

$$\ell(E \cap (\tilde{\iota}_1 \circ \Gamma | I)) \leq \epsilon s.$$  

Since $\tilde{\iota}_1$ is 1-Lipschitz according to Lemma 3.3, (25) and (26) imply

$$\ell(\tilde{\iota}_1 \circ \Gamma | I) \leq se + \epsilon \eta s = e(1 + \eta) s < s.$$  

We denote for every $|s| < r/(1 + \eta)e$, $\rho_s = r - e(1 + \eta)|s|$. For each $z \in Z_1 \cap B_{\rho_s}(F(s, 0), r)$, the inequality (27) implies $\tilde{\iota}_1(z) \in \overline{B}_\rho(\tilde{\iota}_1(x_0), r)$.

We estimate $A_r := \mathcal{H}_2^2(\tilde{\iota}_1(Z_1) \cap B_\rho(\tilde{\iota}_1(x_0), r))$ as $r \to 0^+$. In estimating $A_r$, we use the fact that the seam $Q(S^2)$ has negligible $H_2^2$-measure and that $\tilde{\iota}_1$ is a local isometry outside the seam. We claim that for each $0 < \theta < \pi/2$ the following holds:

$$A_r \geq (1 + O((r/\epsilon)^2))^{-2} \left(\theta^2 + \cos(\theta) \frac{r^2}{\pi(1 + O((r/\epsilon)^2))\epsilon(1 + \eta)}\right).$$  

The term $(1 + O((r/\epsilon)^2))^{-2}$ comes from estimating the Jacobian of $\tilde{\iota}_1 \circ F$. The first term in the brackets comes from the fact that $F$ preserves the speed of radial
geodesics, so
\[
(i_1 \circ F) \left( \left\{ (s, t) : \sqrt{s^2 + t^2} < r, 0 < t \right\} \right) \subset \overline{t_i(Z_1)} \cap \overline{B_Z(i_1(x_0), r)}.
\]
We use this inclusion in a circular sector \( C_0(r) \) which has a total angle \( 2\theta \) and an angle bisector \( \{ 0 \} \times \mathbb{R} \).

The second term in the brackets is twice the area of a suitable triangle. The factor of two comes from the symmetry of the estimate \( (27) \) with respect to the parameter \( s = 0 \). We consider a triangle \( T_0(r) \subset \mathbb{R}^2 \) foliated by line segments \( \ell(s) \), where \( 0 \leq s < r/(1 + \eta) \), with \( \ell(s) \) having the start point \( (s, 0) \), tangent in the direction \( (\sin(\theta), \cos(\theta)) \), and has length \( \rho_s/(1 + O((r/\varepsilon)^2)) \). The \( i_1 \circ F \) image of such a triangle \( T_0(r) \) contributes to \( A_r \). The inequality \( (28) \) follows.

We choose the angle \( \theta \) to satisfy \( \sin(\theta) = \epsilon(1 + \eta) \). We divide \( (28) \) by \( \pi r^2 \), pass to the limit \( r \to 0^+ \), and then \( \eta \to 0^+ \) and conclude
\[
\liminf_{r \to 0^+} \frac{A_r}{\pi r^2} \geq \frac{\left(\sin \left[ (0, \pi/2) \right]\right)^{-1}(\epsilon)}{\pi} + \frac{\sqrt{1 - \epsilon^2}}{\pi \epsilon} = f(\epsilon).
\]
The inequality \( (24) \) is the same as \( (29) \). \( \square \)

**Remark 5.6.** Given \( 0 < \epsilon < 1 \), the lower bound in \( (29) \) is sharp. This can be shown by considering a bi-Lipschitz \( g : S^1 \to S^1 \) with metric speed \( v_s \equiv \varepsilon \) everywhere in an open neighbourhood of \( x_0 \in S^1 \).

If the circular sector \( C_0(r) \) and triangle \( T_0(r) \) are defined as in the proof of the lower bound \( (29) \), with \( \eta = 0 \), and \( \theta = \left(\sin \left[ (0, \pi/2) \right]\right)^{-1}(\epsilon) \), we have
\[
\liminf_{r \to 0^+} \frac{A_r}{\pi r^2} = \frac{\mathcal{H}^2_{\mathbb{R}^2}(C_0(1)) + \mathcal{H}^2_{\mathbb{R}^2}(T_0(1))}{\pi} = f(\epsilon).
\]
This can be showed using Lemma 3.2 and Proposition 3.6. The key property of the angle \( \theta \) is that the line on \( \mathbb{R}^2 \) containing \( (r/\epsilon, 0) \) with tangent vector \((-\cos(\theta), \sin(\theta))\) intersects every ball \( \overline{B}_{\mathbb{R}^2}((s, 0), \rho_s) \) tangentially when \( 0 \leq s < 1/\epsilon \) and \( \rho_s = 1 - \epsilon s \).

**Proof of Proposition 5.3.** Given \( (21) \) and Proposition 5.5, we have \( v_s(x) \geq L_1^{-1} \) for \( \mathcal{H}^1_{\mathbb{S}^1} \)-almost every \( x \in \mathbb{S}^1 \). This implies that \( g^{-1} \) is absolutely continuous and \( v_{g^{-1}}(x) \leq L_1 \) for \( \mathcal{H}^1_{\mathbb{S}^1} \)-almost every \( x \in \mathbb{S}^1 \). Therefore \( g^{-1} \) is \( L_1 \)-Lipschitz.

The fact that \( i_1 \) \( 1 \)-Lipschitz follows from Lemma 3.3. Proposition 3.6 implies that
\[
d_G(i_1(x), i_1(y)) \geq \sigma(x, y)/L_1 \quad \text{for every } x, y \in \mathbb{Z}_1. \]
The equality Lemma 3.2 \( (9) \) implies the corresponding inequality for every pair \( x, y \in \mathbb{Z}_1 \). Hence \( i_1^{-1} \) is \( L_1 \)-Lipschitz. \( \square \)

Next, we verify a lemma about radial extensions of bi-Lipschitz maps, which we need during the proof of Theorem 1.1.

For the south pole \( P_1 \in \mathbb{Z}_1 \), we consider the (orientation-reversing) stereographic projection \( P : \mathbb{S}^2 \setminus \{ P_1 \} \to \mathbb{R}^2 \times \{ 0 \} \) fixing the equator and mapping the north pole \( P_2 = (0, 0, 1) \) to the origin. We identify \( \mathbb{R}^2 \times \{ 0 \} \) with \( \mathbb{R}^2 \). We note that \( P^{-1} \) has the explicit definition
\[
P^{-1}(x, y) = \left( \frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{1 - x^2 + y^2}{1 + x^2 + y^2} \right).
\]
The Riemannian tensor of \( S^2 \) in these coordinates is \( I = (4/(1 + r^2)^2)g_E \), where \( r \) is the distance to the origin and \( g_E \) the Euclidean inner product. In polar coordinates, \( g_E = dr^2 + r^2 d\theta^2 \). We see from the form of \( I \) that the bi-Lipschitz constant of \( \tilde{g} = P \circ g \circ (P|_{S^1})^{-1} \) and \( g : S^1 \to S^2 \) coincide.

We represent the polar coordinates using the complex notation \( re^{i\theta} \). We note that there exists a homeomorphism \( \tilde{G} : \mathbb{R} \to \mathbb{R} \) with \( \tilde{g}(re^{i\theta}) = e^{i\tilde{G}(\theta)} \) for every \( \theta \in \mathbb{R} \). For every \( 0 \leq r \leq 1 \) and \( \theta \in \mathbb{R} \), we set \( \tilde{\psi} \) as the radial extension of \( \tilde{g} \). We recall from [Kal14, Theorem 2.2] that the bi-Lipschitz constants of \( \tilde{g} \) and \( \tilde{\psi} \) coincide. Let \( \psi = P^{-1} \circ \tilde{\psi} \circ P|_{Z_2} : Z_2 \to Z_2 \).

We use the following fact during the proof of Lemma 5.8: see for example [DCJS16], [CS20].

**Lemma 5.7.** For every \( x, y \in S^2 \), \( 0 < \epsilon < 1 \), and \( 0 < 4r < \sigma(x, y) \), the modulus of the family of paths joining \( B_{S^2}(x, r) \) to \( B_{S^2}(y, r) \) with length \( (1 + \epsilon)\sigma(x, y) \) is positive.

**Lemma 5.8.** The map \( \psi : Z_2 \to Z_2 \) is \( L \)-bi-Lipschitz if \( g \) is \( L \)-bi-Lipschitz.

**Proof.** We refer the interested reader to [Kal14, Section 2] for the proof of the fact that \( \psi \) is bi-Lipschitz if \( \tilde{g} \) (equivalently \( g \)) is bi-Lipschitz. We take this as a given.

Since \( \psi \) is bi-Lipschitz, it has a differential at \( L^2 \)-almost every point in \( D \). Given this fact, the following computations are understood to hold at \( L^2 \)-almost every \( (x, y) = re^{i\theta} \) in the unit disk.

The pullback \( \psi^*I \) is a diagonal matrix with respect to the basis \( (dr, d\theta) \), with diagonal \( 4/(1 + r^2)^2 \) and \( 4|G'(\theta)|^2 r^2/(1 + r^2)^2 \). Hence the maximum of the operator norms of \( D\tilde{\psi} : (TD, I) \to (TD, I) \) and its inverse is equal to \( L(\tilde{r}^2 \theta) = \max \left\{ |G'(\theta)|, |G'(\theta)|^{-1} \right\} \). Then, if \( L' \) denotes the essential supremum of \( L(\tilde{r}^2 \theta) \), Lemma 5.7 implies that \( \psi \) is \( L' \)-bi-Lipschitz. On the other hand, \( L' \) is the bi-Lipschitz constant of \( g \).

**Proof of Theorem 1.1.** We first claim that "(1) \Rightarrow (2)". Lemma 5.8 provides us with an \( L \)-bi-Lipschitz \( \psi : Z_2 \to Z_2 \) extension of the given \( L \)-bi-Lipschitz \( g \). We define \( H(x) = i_1(x) \) for each \( x \in \mathbb{Z}_1^1 \) and \( H(x) = i_2 \circ \psi(x) \) otherwise. Proposition 3.6 implies that \( H \) is \( L \)-bi-Lipschitz at the seam, and Lemma 3.2 implies that \( H \) is \( L \)-bi-Lipschitz everywhere.

Notice that if \( H : S^2 \to \tilde{Z} \) is \( L' \)-bi-Lipschitz, we may choose \( C = (L')^4 \) as an upper bound for the 2-dimensional Hausdorff lower density. Hence "(2) \Rightarrow (3)" follows, quantitatively. Lastly, "(3) \Rightarrow (1)" follows from Propositions 5.3 and 5.4. In fact, given \( C \geq 1 \) for which the lower density bound (20) holds, \( g \) is \( L' \)-bi-Lipschitz for \( L' \) solving \( C = f(1/L') \). Since \( f(\epsilon) \geq 1/\pi \epsilon \) for every \( 0 < \epsilon \leq 1 \), we have \( C \pi \geq L' \). Hence \( g \) is \( C \pi \)-bi-Lipschitz.

**Remark 5.9.** The estimates between the constants in "(3) \Rightarrow (1)" in Theorem 1.1 can be improved in two ways. First, the constants \( C_1 \) and \( C_2 \) in (21) and (22) satisfy max \( \{C_1, C_2\} \leq C - 1/2 \), so \( g \) is \( (C - 1/2)\pi \)-bi-Lipschitz.

The second improvement is obtained by using the constant \( L' = L(C - 1/2) \) from Definition 5.1. Then \( g \) is \( L' \)-bi-Lipschitz, where \( L' \leq (C - 1/2)\pi \).

These improvements imply that the bi-Lipschitz constant of \( g \) converges to 1 as \( C \to 1^+ \). These facts also improve Theorem 1.2 and the following result, Proposition 5.10.

Before proving Theorem 1.2, we investigate a related problem. To this end, suppose that we are given Riemann maps \( \phi_i : Z_i \to \Omega_i \) with \( \Omega_1 \) and \( \Omega_2 \) denoting the complementary components of a welding curve \( \mathcal{C} \), and set \( g = \phi_2^{-1} \circ \phi_1 |_{S^1} \).
Proposition 5.10. Let $K, C \geq 1$. The welding homeomorphism $g$ is $\pi(KC)^2$-bi-Lipschitz if there exists a $K$-quasiconformal homeomorphism $h : S^2 \to S^2$ such that for both $i = 1, 2$,

\[(30) \quad C^{-1} J_h(x) \leq J_{\phi_i}(x) \leq C J_h(x) \quad \text{for } \mathcal{H}^2_{S^2}\text{-a.e. } x \in \Omega_i.
\]

Conversely, if $g$ is $L$-bi-Lipschitz, then there exists $L^4$-quasiconformal homeomorphism $h : S^2 \to S^2$ such that (30) holds for $C = L^2$.

Proof. We first assume that $g : S^1 \to S^1$ is $L$-bi-Lipschitz. Then Theorem 1.1 provides us with an $L$-bi-Lipschitz homeomorphism $\Psi : \overline{\mathcal{Z}} \to S^2$. Proposition 4.3 and (14) imply that $\tilde{\pi} : S^2 \to \overline{\mathcal{Z}}$ defined via the formula

\[(31) \quad \tilde{\pi}(x) = \begin{cases} i_1 \circ \phi_1^{-1}(x), & x \in \overline{\Omega_1}, \\ i_2 \circ \phi_2^{-1}(x), & x \in \Omega_2 \end{cases}
\]

is a $1$-quasiconformal homeomorphism. Therefore, $h := \Psi \circ \tilde{\pi} : S^2 \to S^2$ is $K$-quasiconformal for $K = L^4$, and as $\Psi$ is $L$-bi-Lipschitz, the Jacobians of $h$ and $\tilde{\pi}$ are comparable with comparison constant $C = L^2$.

Next, we are given a Jordan curve $C \subset S^2$ corresponding to a welding homeomorphism $g = \phi_2^{-1} \circ \phi_1|_{\mathcal{Z}}$, a $K$-quasiconformal homeomorphism $h : S^2 \to S^2$, and a constant $C \geq 1$ such that

\[(32) \quad C^{-1} J_h(x) \leq J_{\tilde{\pi}}(x) \leq C J_h(x) \quad \mathcal{H}^2_{S^2}\text{-a.e. } x \in S^2 \setminus C.
\]

For $i = 1, 2$, the composition $h \circ \phi_i$ is $K$-quasiconformal with Jacobian bounded from above $C$ and below by $C^{-1}$, respectively; here we apply (32). Theorem 2.4 (ii) and Hadamard’s inequality imply that $C^{-1} \leq \rho_{h \circ \phi_i}^2 \leq KC \mathcal{H}^2_{S^2}$-almost everywhere in $Z_i$. Lemma 5.7 implies that the homeomorphism $h \circ \phi_i$ is locally $L’$-bi-Lipschitz for $L’ = \sqrt{KC}$.

Since, for both $i = 1, 2$, $Z_i$ is geodesic, it is immediate that $h \circ \phi_i : Z_i \to S^2$ is $L’$-Lipschitz. Since this holds for both $i = 1, 2$, the construction of $d_{L’}$ implies that whenever $x, y \in S^1$, $\sigma(h \circ \phi_1(x), h \circ \phi_1(y)) \leq L’d_{L’}(i_1(x), i_1(y))$. Lemma 3.2 (9) establishes the same inequality for each $x, y \in Z_i$. Hence the mapping $\tilde{\pi}$ defined by the expression (31) is a homeomorphism and $\Psi := h \circ \tilde{\pi}^{-1}$ is $L’$-Lipschitz on the southern hemisphere. A similar argument shows that $\Psi$ is $L’$-Lipschitz on both of the hemispheres. Then Lemma 3.2 (10) implies that $\Psi$ is $L’$-Lipschitz everywhere.

Since mod $\Gamma \leq K \text{mod } \Psi^{-1} \Gamma$ for all path families (recall Proposition 4.3), we have $\Psi^{-1} \in N^{1, 2}(S^2, \overline{\mathcal{Z}})$. On the other hand, $\Psi(Q(S_Z))$ has negligible $\mathcal{H}^2_{S^2}$-measure and $\Psi^{-1}$ is locally $L’$-Lipschitz in the complement of that set. In particular, almost every absolutely continuous $\gamma : [0, 1] \to S^2$ has zero length in $\Psi(Q(S_Z))$ and $\Psi^{-1} \circ \gamma$ is absolutely continuous. As a consequence, $\mathcal{H}^2_{S^2}(Q(S_Z) \cap |\Psi^{-1} \circ \gamma|) = 0$.

Denoting $E = Q(S_Z) \cap |\Psi^{-1} \circ \gamma|$ and $\rho = L’ \chi_{E-Z}$, we conclude from Lemma 2.2 that $\ell(|\Psi^{-1} \circ \gamma|) \leq \int_{E} \rho \, ds \leq L’ \ell(\gamma)$. Lemma 5.7 implies that $\Psi^{-1}$ is $L’$-Lipschitz.

We have verified that $\Psi$ is $L’$-bi-Lipschitz. By applying the implications "(2) $\Rightarrow$ (3) $\Rightarrow$ (1)" in Theorem 1.1, we conclude that $g$ is $L$-bi-Lipschitz for $L = \pi(L’)^4 = \pi(KC)^2$.

Next, we prove Theorem 1.2. This essentially follows from Proposition 5.10.
Proof of Theorem 1.2. We claim that \( g: S^1 \to S^1 \) is bi-Lipschitz if and only if there exists a quasiconformal homeomorphism \( h: S^2 \to S^2 \) and a 1-quasiconformal homeomorphism \( \varphi: S^2 \to \tilde{Z} \) such that \( J_{\varphi} \) and \( J_h \) are comparable.

If such \( \varphi \) and \( h \) exist, we may assume that \( \varphi_i = \varphi^{-1} \circ i|Z_i \) is a Riemann map for both \( i = 1, 2 \). Then Proposition 5.10 shows that \( g \) is bi-Lipschitz.

Conversely, if \( g \) is bi-Lipschitz, Theorem 1.1 provides a bi-Lipschitz homeomorphism \( \Psi: \tilde{Z} \to S^2 \). Then Theorem 1.6 implies the existence of a 1-quasiconformal homeomorphism \( \pi: S^2 \to \tilde{Z} \) such that \( \varphi_i = \pi^{-1} \circ \iota_i|Z_i \) is a Riemann map for \( i = 1, 2 \). We may also assume that \( \Psi \circ \iota_i|Z_i \) is orientation-preserving for \( i = 1, 2 \), by post-composing \( \Psi \) with a suitable reflection, if need be. Defining \( h = \Psi \circ \pi \) implies that the assumptions of Proposition 5.10 hold for \( g \).

Since Theorem 1.1 and Proposition 5.10 are quantitative, so is Theorem 1.2. □

6. Mappings of finite distortion

In this section, we establish Proposition 1.4 and Theorem 1.5.

Definition 6.1. Let \( \Omega, \Omega' \subset S^2 \) be open. A homeomorphism \( \psi: \Omega \to \Omega' \) is a mapping of finite distortion if \( \psi \in N^{1,1}(\Omega, S^2) \); second, the determinant \( J(D\psi) \) of the differential \( D\psi \) is nonnegative and integrable; lastly, there exists a function \( 1 \leq K_\psi < \infty \) for which

\[
|D\psi|^2 \leq K_\psi J(D\psi) \quad \mathcal{H}^2_{S^2}\text{-a.e. in } \Omega.
\]

Here \( |D\psi|_g \) refers to the operator norm of the differential \( D\psi \). We let \( K_\psi \) denote a smallest Borel function which is bounded from below by \( \chi_\Omega \) and for which (33) holds.

Definition 6.2. A smooth strictly increasing function \( A: [1, \infty) \to [0, \infty) \) is admissible if

1. \( A(1) = 0 \),
2. \( \int_1^\infty t^{-2} A(t) \, d\mathcal{L}^1(t) = \infty \), and
3. \( t \mapsto t A(t) \) is increasing for large values \( t \), and converges to \( \infty \) as \( t \to \infty \).

We obtain the same class of admissible \( A \) if we replace (2) with the condition

\[
\int_1^\infty t^{-1} A'(t) \, d\mathcal{L}^1(t) = \infty.
\]

This follows from the fact that \( A(s)/s \leq 4 \int_s^2 t^{-2} A(t) \, d\mathcal{L}^1(t) \) whenever \( s \geq 1 \) and the integration by parts formula.

Definition 6.3. Let \( \Omega, \Omega' \subset S^2 \) be open, and \( \psi: \Omega \to \Omega' \) a homeomorphism. We say that \( \psi \) is admissible if \( \psi \) is a mapping of finite distortion and there exists an admissible \( A \) with

\[
\int_{\Omega} e^{A(K_\psi)} \, d\mathcal{H}^2_{S^2} < \infty.
\]

If \( A(t) = pt - p \) for some \( p > 0 \), we say that \( \psi \) has exponentially integrable distortion.

We recall some properties of such \( \psi \). First, \( \psi \) satisfies Lusin’s Condition (N) [KKM’03, Theorem 1.1]. Second, \( \psi^{-1} \in N^{1,1}(\Omega', \Omega) \) [KO06, Corollary 1.2]; this implies that \( \psi^{-1} \) satisfies Lusin’s Condition (N) [AIM09, Theorem 3.3.7]. Third, the Jacobian \( J(D\psi) \) appearing on the right-hand side of (33) coincides with the Jacobian \( J_\psi \) we defined in Section 2.2 [KKM’03].

In this section, we show the following theorem.
Theorem 6.4. Suppose that \( g : S^1 \to S^1 \) is a homeomorphism, \( g^{-1} \) absolutely continuous, and there exists a homeomorphism \( \psi : \mathbb{Z}_2 \to \mathbb{Z}_2 \) extending \( g \) with \( \psi|_{Z_2} \) admissible. Then \( \tilde{Z} \) is quasiconformally equivalent to \( S^2 \).

Note that Theorem 1.5 is a consequence of Theorem 6.4 so it suffices to verify Theorem 6.4.

Definition 6.5. Given \( x_0 \in S^1 \) and \( \pi > R_0 > 0 \), set \( \tilde{Q} := \overline{B}_{S^2}(x_0, R_0) \subset S^2 \). We define \( H(x) = \tilde{t}_1(x) \) if \( x \in \tilde{Q} \cap Z_1 \) and \( \tilde{t}_2 \circ \psi(x) \) if \( x \in \tilde{Q} \cap Z_2 \), and denote \( \tilde{R} = H(\tilde{Q}) \subset \tilde{Z} \).

Proposition 6.6. If \( \tilde{R} \) and \( H \) are as in Definition 6.5, then \( H \) is a homeomorphism and there exists a \( 1 \)-quasiconformal homeomorphism \( f = (u, v) : \tilde{R} \to [0, 1] \times [0, M] \) for some \( M > 0 \).

Proof of Theorem 6.4 assuming Proposition 6.6. We cover the seam in \( \tilde{Z} \) by the interiors of \( R \) as in Definition 6.5. This implies that \( \tilde{Z} \) can be covered by quasiconformal images of planar domains, and the quasiconformal equivalence of \( \tilde{Z} \) and \( S^2 \) follows from Theorem 2.7.

The following lemma is a key step in proving Proposition 6.6.

Lemma 6.7. The \( H \) from Definition 6.5 is a homeomorphism, \( H \in N^{1,1}(\tilde{Q}, \tilde{R}) \) and \( H^{-1} \in N^{1,2}(\tilde{R}, \tilde{Q}) \). Furthermore, \( H \) satisfies Lusin’s Conditions (N) and (N').

Proof. The absolute continuity of \( g^{-1} \) implies for the Lebesgue decomposition \( g^*H^i = v^iH^1 + \mu^i \) that \( \{v^i = 0\} \) has negligible \( H^1 \)-measure in an open neighbourhood of \( S^1 \cap \tilde{Q} \). Then Proposition 3.6 and Lemma 3.2 imply that \( H \) is a homeomorphism.

We recall from Lemma 3.3 the fact that the inclusion maps \( \tilde{t}_1|_{Z_1} : Z_1 \to \tilde{Z} \) and \( \tilde{t}_2|_{Z_2} : Z_2 \to \tilde{Z} \) are \( 1 \)-Lipschitz local isometries. This implies that \( H \) and its inverse are absolutely continuous in measure; the seam has negligible Hausdorff 2-measure.

In the following proof, we write \( \tilde{\rho}_i \) for functions defined on \( \tilde{Q} \cap Z_i \subset S^2 \) and \( \rho_i = (\tilde{\rho}_1 \circ \tilde{t}_1^{-1}) \) on \( \tilde{R} \cap \tilde{t}_1(Z_i) \subset \tilde{Z} \) for \( i = 1, 2 \).

Since \( \tilde{t}_1^{-1} \in N^{1,2}(\tilde{Q} \cap Z_2, S^2) \), for \( i = 1, 2 \), there exists an upper gradient \( \tilde{\rho}_i \in L^2(\tilde{Q} \cap Z_i) \) of \( H^{-1} \circ \tilde{t}_1|_{Z_i} \) for \( i = 1, 2 \). We fix such functions and denote \( \rho := \chi_{\tilde{R} \cap \tilde{t}_1(Z_1)}\rho_1 + \chi_{\tilde{R} \cap \tilde{t}_2(Z_2)}\rho_2 \in L^1(\tilde{R}) \).

Let \( \Gamma_0 \) denote the collection of non-constant paths on \( \tilde{R} \subset \tilde{Z} \) which have positive length in the seam \( Q(S_Z) \) or along which \( \rho \) fails to be integrable. Since \( \rho + \infty \cdot \chi_{Q(S_Z)} \) is \( L^2 \)-integrable, Lemma 2.1 yields mod \( \Gamma_0 = 0 \).

Consider next an absolutely continuous path \( \gamma : [0, 1] \to \tilde{R} \) in the complement of \( \Gamma_0 \). Then \( \theta = H^{-1} \circ \gamma \) is such that \( H^1_S(\{\theta| \cap S^1) = 0 \). Indeed, since \( \gamma \) has zero length in the seam, the area formula (4) implies \( H^1_S(\{\gamma| \cap Q(S_Z)) = 0 \). This implies \( H^1_S(\{\theta| \cap S^1) = 0 \) due to Proposition 3.6 and the absolute continuity of \( g^{-1} \). Since \( H^1_S(\{\theta| \cap S^1) = 0 \), the assumptions of Lemma 2.2 are satisfied. Hence

\[ \ell(\theta) \leq \int_\gamma \rho \, ds < \infty. \]

This implies that \( H^{-1} \) has an \( L^2 \)-integrable weak gradient, so \( H^{-1} \in N^{1,2}(\tilde{R}, \tilde{Q}) \).

Lastly, we claim that \( H \in N^{1,1}(\tilde{Q}, \tilde{R}) \). To this end, we observe that \( H|_{\tilde{Q} \cap Z_2} \) has an upper gradient \( \tilde{\rho}_i \in L^1(\tilde{Q} \cap Z_i) \), and denote \( \tilde{\rho} = \sum_{i=1}^2 \chi_{\tilde{Q} \cap Z_i} \tilde{\rho}_i \in L^1(\tilde{Q}) \). Now
\( \tilde{\rho} \) is integrable along 1-almost every absolutely continuous path \( \gamma: [0,1] \to \tilde{Q} \) and 1-almost every such path has zero length in \( S^1 \). Having fixed a path \( \gamma \) with these properties, Proposition 3.6 implies that \( \theta = H \circ \gamma \) has zero length in the seam. The inequality \( \ell(\theta) \leq \int_\gamma \rho \, ds \) follows from Lemma 2.2. This yields that \( H \in N^{1,1}(\tilde{Q}, \tilde{R}) \).

\[ \Box \]

**Remark 6.8.** The Sobolev regularity \( H^{-1} \in N^{1,2}(\tilde{Q}, \tilde{R}) \) is crucial in the following. Typically, the Sobolev regularity of the inverse of a Sobolev homeomorphism is a subtle issue in the metric surface setting.

To highlight the issue, we recall [IRar, Example 6.1]. There an example of a metric surface \( X \) was constructed for which there exists a 1-Lipschitz homeomorphism \( H: \mathbb{R}^2 \to X \) with \( \text{mod} \Gamma \leq \text{mod} HT \) for all path families, but \( H^{-1} \notin N^{1,2}(X, \mathbb{R}^2) \). In fact, \( H \) is a local isometry outside a Cantor set \( E \subset \mathbb{R} \times \{0\} \) of positive \( L^1 \)-measure and \( H(E) \) has negligible \( H_1^1 \)-measure. The key point is that \( X \) is not reciprocal; recall Definition 2.5.

We define the following auxiliary function for later use:

\[
P(t) := \begin{cases} t^2, & 0 \leq t < 1, \\ \frac{t^2}{A^{-1}(\log t)^2}, & t \geq 1. \end{cases}
\]

We note that for every \( a \in [0,\infty) \),

\[
P(a) \leq e^{A(K_{\Omega})} + \frac{a^2}{K_H} \quad \text{for } \mathcal{H}^2_\mathbb{S}^2 \text{-a.e. in } \tilde{Q}.
\]

This follows by first observing that \( a^2 < e^{A(K_{\Omega})} \) implies \( P(a) \leq e^{A(K_{\Omega})} \) and otherwise \( P(a) \leq \frac{a^2}{K_H} \).

Also, for any measurable function \( \tilde{\rho}: \tilde{Q} \to [0,\infty] \),

\[
\int_\tilde{Q} P(\tilde{\rho}) \, d\mathcal{H}^2_\mathbb{S}^2 < \infty \quad \text{implies} \quad \int_\tilde{Q} \tilde{\rho} \, d\mathcal{H}^2_\mathbb{S}^2 < \infty.
\]

The implication (36) follows since \( A'(t) \) is increasing for large \( t \) and converges to infinity as \( t \to \infty \). Consequently, there exists \( t_1 \geq 1 \) for which the derivative of \( h(t) = e^{A(t)}/t^2 \) is bounded from below by \( h(t)/t \) for every \( t \geq t_1 \). This implies the existence of \( t_0 \geq 1 \) such that \( h(t) \geq 1 \) for every \( t \geq t_0 \). This is equivalent to saying that \( P(t) \geq t \) for every \( t \geq t_0 \). This yields (36).

We set \( K_\psi(x) = |D\psi|_g^2 / |J(D(\psi))(x)| \) and \( K_{\psi^{-1}}(x) = |D(\psi^{-1})|_g^2 / |J(D(\psi^{-1}))| \).

Observe that \( K_\psi = K_{\psi^{-1}} \circ \psi \mathcal{H}^2_\mathbb{S}^2 \)-almost everywhere.

We set \( K_H(x) = 1 \) if \( x \in \tilde{Q} \cap \mathbb{Z}_1 \) and \( K_H(x) = K_\psi(x) \) in \( x \in \tilde{Q} \cap \mathbb{Z}_2 \). Then

\[
\int_\tilde{Q} e^{A(K_H)} \, d\mathcal{H}^2_\mathbb{S}^2 < \infty.
\]

Also, \( K_{H^{-1}} := \rho_{H^{-1}}^2 / |J_{H^{-1}}| \) satisfies \( K_H = K_{H^{-1}} \circ H \mathcal{H}^2_\mathbb{S}^2 \)-almost everywhere, since, outside a \( \mathcal{H}^2_\mathbb{S}^2 \)-negligible set, either the number is one, or \( \rho_{H^{-1}}^2 \circ \tilde{\iota}_2 = |D(\psi^{-1})|_g^2 \)

\( |J_{H^{-1}} \circ \tilde{\iota}_2| = |J(\psi^{-1})| \), and \( K_\psi = K_{\psi^{-1}} \circ \psi \).

For every \( z \in \tilde{Q} \) and every pair \( 0 < r < r_0 \), we denote \( \Gamma(z,r,r_0) := \Gamma(B_{\mathbb{S}^2}(z,r), \tilde{Q} \setminus B_{\mathbb{S}^2}(z,r_0); \tilde{Q}) \).

**Lemma 6.9.** For every \( z \in \tilde{Q} \) and \( 0 < r < r_0 \) with \( \tilde{Q} \setminus B_{\mathbb{S}^2}(z,r_0) \neq \emptyset \),

\[
\text{mod } HT(\gamma) \leq \inf \left\{ \int_{\tilde{Q}} \rho_{K_H} \, d\mathcal{H}^2_\mathbb{S}^2 : \tilde{\rho} \text{ is admissible for } \Gamma(z,r,r_0) \right\}.
\]
Proof. Fix an admissible function \( \tilde{\rho} \) for \( \Gamma(z,r,r_0) \). Then for almost every \( \gamma \in H\Gamma(z,r,r_0) \), \( H^{-1} \circ \gamma \) is absolutely continuous, and
\[
1 \leq \int_{H^{-1} \circ \gamma} \tilde{\rho} \, ds \leq \int_{\gamma} (\tilde{\rho} \circ H^{-1}) \rho_{H^{-1}} \, ds.
\]
In particular, \( \rho = (\tilde{\rho} \circ H^{-1}) \rho_{H^{-1}} \) is weakly admissible for \( H\Gamma(z,r,r_0) \). Consequently,
\[
\text{mod } H\Gamma(z,r,r_0) \leq \int_{R} \rho^2 \, d\mathcal{H}_2.
\]
The change of variables formula for \( H \) and the fact that the seam \( Q(S_2) \) is \( \mathcal{H}_2 \)-negligible establish the claim, after taking the infimum over such \( \tilde{\rho} \). \( \square \)

Having observed Lemma 6.9 and (37), the capacitary estimate [KO06, Theorem 5.3] implies that keeping \( r_0 \) fixed in (38), we obtain \( \text{mod } H\Gamma(z,r,r_0) \to 0 \) as \( r \to 0^+ \). A key point is that \( A \) in (37) is admissible. Since \( H \) is a homeomorphism, this implies that (7) holds for every \( y \in \text{int}(\bar{R}) \subset \bar{Z} \). By repeating the argument with a slightly larger \( Q \), we conclude the following.

**Lemma 6.10.** The identity (7) holds for every \( y \in \bar{R} \subset \bar{Z} \).

Fix a decomposition \( \tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4 \) of \( \partial Q \) of four arcs overlapping only at their end points, labelled in cyclic order consistently with the orientation of \( S^2 \). For each \( i \), we denote \( \xi_i = H(\tilde{\xi}_i) \).

Given the validity of (7) for each \( y \in \bar{R} \) and the universal lower bound (8), [Raj17, Proposition 9.1] yields the existence of a homeomorphism \( f = (u, v) : \bar{R} \to [0,1] \times [0,M] \) with the following properties:

- \( u \in N^{1,2}(\bar{R}) \) with \( 2E(u) = M \) [Raj17, Section 4];
- \( u^{-1}(0) = \xi_1, u^{-1}(1) = \xi_3, v^{-1}(0) = \xi_2, \) and \( v^{-1}(M) = \xi_4 \) [Raj17, Theorem 5.1 and Proposition 7.3];
- The minimal weak upper gradient \( \rho_u \) is weakly admissible for the path family \( \Gamma(\xi_1, \xi_3; \bar{R}) \) and is a minimizer, i.e., \( M = \text{mod } \Gamma(\xi_1, \xi_3; \bar{R}) \) [Raj17, Section 4-5];
- For every Borel set \( E \subset \bar{R} \), \( L^2(f(E)) = \int_E \rho_u^2 \, d\mathcal{H}_2^2 \). In particular, the Jacobian of \( f \) coincides with \( \rho_u^2 \) [Raj17, Proposition 8.2].

The third point implies that if \( u' \in N^{1,2}(\bar{R}) \) has the same boundary values as \( u \) in \( \xi_1 \cup \xi_3 \), the Dirichlet energies satisfy \( E(u) \leq E(u') \). Given this, we say that \( u \) is an energy minimizer for \( \Gamma(\xi_1, \xi_3; \bar{R}) \).

During the proof of Proposition 6.11, the Beltrami differential of \( H \) is defined to be zero in \( \text{int}(\bar{Q}) \cap \overline{Z_1} \), and coincide with the one of \( \psi \in \text{int}(\overline{Q}) \cap Z_2 \).

**Proposition 6.11.** The map \( f = (u, v) : \bar{R} \to [0,1] \times [0,M] \) is a 1-quasiconformal homeomorphism.

The proof of Proposition 6.11 is split into several lemmas.

**Lemma 6.12.** Let \( 0 < a < b < 1 \) and \( 0 < c < d < M \) for which
\[
Q^{0} = \left\{ x \in \bar{R} : f(x) \in [a,b] \times [c,d] \right\} \subset \text{int}(\bar{R}) \setminus Q(S_2).
\]
Then \( f|_{\text{int}(Q^{0})} \) is a 1-quasiconformal homeomorphism.

**Proof.** For the duration of the proof, we denote
\[
\begin{align*}
\tilde{\xi}_1^0 &= f^{-1}(\{a\} \times [c,d]), \\
\tilde{\xi}_2^0 &= f^{-1}(\{b\} \times [c,d]), \\
\tilde{\xi}_3^0 &= f^{-1}(\{a\} \times [c,d]), \\
\tilde{\xi}_4^0 &= f^{-1}(\{0,1\} \times \{d\}).
\end{align*}
\]
There exists a Jordan domain $V \subset \text{int}(Q) \cap Z_i$, for some $i = 1, 2$, such that $\tilde{\iota}(\overline{V}) = Q^0$. Equation (57) [Raj17, Lemma 10.2] states that
\[
\text{mod} \Gamma(\xi^0_1, \xi^0_3; Q^0) = \frac{d-c}{b-a}.
\]
Since $\tilde{\iota}$ is 1-Lipschitz and a local isometry in $\overline{V}$, we have for every quadrilateral $Q' \subset Q^0$,
\[
\text{mod} \Gamma(\xi^0_2, \xi^0_4; Q') \text{ mod } \Gamma(\xi^0_2, \xi^0_4; Q^0) = 1.
\]
In particular, we have
\[
\text{mod} \Gamma(\xi^0_1, \xi^0_3; Q^0) \text{ mod } \Gamma(\xi^0_2, \xi^0_4; Q^0) = 1.
\]
We wish to apply [Raj17, Proposition 11.1]. There Rajala assumes that (6) holds for some $\kappa > 1$ and concludes that 2000 $\cdot \sqrt{\kappa} \rho_\mu$ is a weak upper gradient of $f$. We do not assume this. However, a quick inspection of the proof shows that given any open set $\Omega \subset \text{int}(Q^0)$, the property (39) implies that 2000 $\cdot \chi_{\text{int}(Q^0)} \cdot \rho_\mu$ is a weak upper gradient of $f|_{\text{int}(Q^0)}$ in $\Omega$. By exhausting $\text{int}(Q^0)$ by such open sets, we conclude that $f|_{\text{int}(Q^0)} \in N^{1,2}(\text{int}(Q^0); \mathbb{R}^2)$.

Since $u \in N^{1,2}(\tilde{\mathcal{R}})$ is a continuous energy minimizer, the composition $u \circ \iota|_V$ is harmonic [AIM09, Weyl’s lemma]. The Riemann mapping theorem, the Sobolev regularity of $f|_{\text{int}(Q^0)}$, the boundary values of the components of $f|_{\partial Q^0}$, and (40) imply that $f \circ \iota|_V$ is a Riemann map. In particular, $f|_{\text{int}(Q^0)}$ is a 1-quasiconformal homeomorphism. □

**Lemma 6.13.** The composition $\tilde{f} = f \circ H : \tilde{Q} \to [0,1] \times [0,M]$ is an element of $N^{1,1}(\text{int}(\tilde{Q}), \mathbb{R}^2)$. Moreover, the Beltrami differential of $\tilde{f}$ coincides with the one of $H$ and (37) holds for $K_{\tilde{f}}$ in place of $K_H$.

**Proof.** Given Lemma 6.12, the Beltrami differential of $\tilde{f}$ and $H$ coincide $\mathcal{H}^2$-almost everywhere in $\text{int}(\tilde{Q}) \setminus S^1$, i.e., $\mathcal{H}^2$-almost everywhere in $\text{int}(\tilde{Q})$. The result also implies that the pointwise distortions of $\tilde{f}$ and $H$ coincide $\mathcal{H}^2$-almost everywhere in $\text{int}(\tilde{Q})$.

Next, we show that $\tilde{u} = u \circ H \in N^{1,1}(\tilde{Q})$. We recall that $H \in N^{1,1}(\tilde{Q}, \mathcal{R})$. Moreover, if $\rho_0 \in L^2(\mathcal{R})$ is an upper gradient of $u$, the function $\rho = (\rho_0 \circ H)\rho_H$ is a 1-weak upper gradient of $\tilde{u}$ with
\[
\int_{\tilde{Q}} P(\rho) \, d\mathcal{H}^2 \leq \int_{\tilde{Q}} e^{A(K_H)} \, d\mathcal{H}^2 + \|\rho_0\|_{L^2(\tilde{Q})}^2 < \infty,
\]
where we apply (35) and the distortion inequality $\rho_H^2 \leq K_H |f|$. The $L^1(\tilde{Q})$-integrability of $\rho$ follows from (36), so $\tilde{u} \in N^{1,1}(\tilde{Q})$.

Let $\tilde{v} = v \circ H$. Lemma 6.12 implies that $\rho = (\rho_0 \circ H)\rho_H \in L^1(\tilde{Q})$ is a 1-weak upper gradient of $\tilde{v}$ in every open $U \subset \text{int}(\tilde{Q}) \setminus S^1$. Therefore, $\tilde{v} \in N^{1,1}(\text{int}(\tilde{Q}) \setminus S^1)$. Given the continuity of $\tilde{v}$, we actually have $\tilde{v} \in N^{1,1}(\text{int}(\tilde{Q}))$. This is seen by verifying the ACL (absolute continuity on lines) property for $\tilde{v}|_{\text{int}(\tilde{Q})}$ on charts covering $S^1 \cap \text{int}(\tilde{Q})$. The ACL property on charts follows from a minor modification of the proof in [Väät71, Theorem 35.1] showing that closed sets with $\sigma$-finite Hausdorff 1-measure are quasiconformally removable. This implies that $\rho$ is a 1-weak upper gradient of $\tilde{v}$ on $\text{int}(\tilde{Q})$. The claim follows from this. □

**Lemma 6.14.** Let $u'$ denote the energy minimizer for $\Gamma(\xi^0_2, \xi^0_4; Q)$. Then $v = Mu'$. 


Proof. Similarly to \( f \) and \( \tilde{f} \), let \( f' = (u', v') \) and \( \tilde{f}' \) denote the homeomorphisms obtained from the energy minimizer \( u' \) for \( \Gamma(\xi_2, \xi_4; \tilde{R}) \). Let \( R' \) denote the image of \( f' \) and \( R \) the image of \( f \).

Lemma 6.13 shows that the Beltrami differentials of \( \tilde{f} \) and \( \tilde{f}' \) coincide with one another \( \mathcal{H}^2_{\xi} \)-almost everywhere and their distortion satisfies (37) for an admissible \( A \). Then the Stoilow factorization theorem [AIM09, Theorems 20.5.1, 20.5.2] implies that \( \varphi = \tilde{f}' \circ f^{-1} \) is conformal; note also that \( \varphi = f' \circ f^{-1} \).

Since \( \varphi \) is conformal, the energy minimizer \( \pi_1 \) for \( \Gamma(f'(\xi_2), f'(\xi_4); R') \) is such that \( \pi_1 \circ \varphi \) is the energy minimizer for \( \Gamma(f(\xi_2), f(\xi_4); R) \). On the other hand, here \( \pi_1 \) is the projection to the \( x \)-axis and \( \pi_1 \circ \varphi \) is \( M^{-1} \) times the projection to the \( y \)-axis. Since \( \varphi = f' \circ f^{-1} \), the equality \( u' = \pi_1 \circ \varphi \circ f = M^{-1} v \) follows. \( \square \)

Proof of Proposition 6.11. Lemma 6.14 implies that \( f = (u, v) \in L^{1,2}(\tilde{R}, \mathbb{R}^2) \). Furthermore, Lemma 6.12 implies \( \rho_f^2 = 1/f \in L^1(\tilde{R}) \). Hence \( \text{mod} \, \Gamma \leq \text{mod} \, f \Gamma \) for every path family in \( \tilde{R} \). This improves to \( K \)-quasiconformality for some \( K \geq 1 \) due to Proposition 2.8. As \( f(Q(S_2) \cap \tilde{R}) \) is negligible due to the change of variables formula for \( f \), and as \( f^{-1} \) is 1-quasiconformal outside \( f(S_2 \cap \tilde{R}) \), we immediately obtain \( \text{mod} \, \Gamma \leq \text{mod} \, f^{-1} \Gamma \) for every path family in \( f(\tilde{R}) \). Thus \( f \) is 1-quasiconformal. \( \square \)

Proof of Proposition 6.6. This is proved by Proposition 6.11. \( \square \)

Remark 6.15. Notice that if Lemma 6.10 holds for a given homeomorphism \( g : S^1 \to S^1 \) having an admissible extension, even without assuming the absolutely continuity of \( g^{-1} \), the rest of the proof of Proposition 6.11 (and Proposition 6.6) go through the same way.

Proof of Proposition 1.4. Given a quasisymmetry \( g : S^1 \to S^1 \), its Beurling–Ahlfors extension \( \psi : Z_2 \to \mathbb{Z}_2 \) is a quasisymmetry and \( \psi \vert_{Z_2} \) is \( K \)-quasiconformal for some \( K \geq 1 \) [BA56]. Thus, if \( g^{-1} \) is absolutely continuous, \( g \) satisfies the assumptions of Theorem 1.5. Alternatively, if \( H \) is as in Definition 6.5, Lemma 6.9 implies that \( H^{-1} \) has outer dilatation \( K_0(H^{-1}) \leq K \). Proposition 2.8 implies that \( H \) is quasiconformal; this self-improves to \( K \)-quasiconformality. Clearly \( H \) extends to a \( K \)-quasiconformal homeomorphism \( H : S^2 \to \tilde{Z} \). \( \square \)

7. Concluding remarks

7.1. A point of positive capacity. For a general orientation-preserving homeomorphism \( g : S^1 \to S^1 \), the \( \tilde{Z} \) can have points of positive capacity (in the sense that (7) can fail) even if \( g \) is locally bi-Lipschitz in the complement of a single point. For example, having fixed arbitrary \( 1 < \alpha < \beta \), we consider the homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) defined by

\[
\begin{cases}
   x^\alpha, & x \geq 0, \\
   -(-x)^\beta, & x < 0.
\end{cases}
\]

We construct a homeomorphism \( g : S^1 \to S^1 \) by restricting \( h \) to the interval \([-1, 1]\), extending the restriction to \( \mathbb{R} \) periodically, and by considering the covering map \( \theta(t) = (\cos(\pi t), \sin(\pi t), 0) \), and a homeomorphism \( g : S^1 \to S^1 \) satisfying \( g \circ \theta = \theta \circ h^{-1} \). Then \( g^{-1} \) is an \( L \)-Lipschitz homeomorphism for some \( L \geq 1 \), and one can check directly from the definition of \( d_2 \) that the inclusion map \( \tilde{t}_1 : \tilde{Z}_1 \to \tilde{Z} \) is \( L \)-bi-Lipschitz onto its image.
Let \( x_0 \in \tilde{Z} \) denote the point corresponding to \((1,0,0)\). By using the techniques from Section 5, we can show that \( \tilde{Z} \setminus \{x_0\} \) can be covered by bi-Lipschitz images of planar domains. Then [Ikoar, Theorem 1.3] implies that \( \tilde{Z} \setminus \{x_0\} \) is 1-quasiconformally equivalent to a Riemannian surface (that is homeomorphic to a planar domain). Such a Riemannian surface can be conformally embedded into \( S^2 \) [AS60, Section III.4]. Hence there exists a 1-quasiconformal embedding \( \psi: \tilde{Z} \setminus \{x_0\} \to S^2 \).

We claim that the complement of the image of \( \psi \) is a non-trivial continuum (which is equivalent to the failure of (7) at \( x_0 \)). Indeed, otherwise \( \psi \) would extend to a 1-quasiconformal homeomorphism and \( g \) would be a welding homeomorphism, as a consequence of Theorem 1.6. This would contradict both [Oik61, Example 1] and [Vai89, Theorem 3], where both of these result show that \( g \) is not a welding homeomorphism.

In contrast, if we set \( a = \beta \geq 1 \) in (41), the homeomorphism \( g \) is a quasisymmetry, so \( \tilde{Z} \) is quasiconformally equivalent to \( S^2 \), as a consequence of Proposition 1.4.

### 7.2. Points of positive capacity

We construct another example for which points of positive capacity occur. To this end, consider a Cantor set \( E \subset [0,1] \) and

\[
(42) \quad h(x) = \begin{cases} \left( (\mathcal{L}^1([0,1] \setminus E))^{-1} \int_{0}^{x} \chi_{\mathbb{R} \setminus E}(y) \, d\mathcal{L}^1(y) \right) \theta(x), & 0 \leq x \leq 1, \\ x, & \text{otherwise.} \end{cases}
\]

Then \( h: \mathbb{R} \to \mathbb{R} \) is a Lipschitz homeomorphism coinciding with the identity map outside \((0,1)\).

Next, consider the Möbius transformation \( \theta_1(z) = (z - i) / (z + i) \) from the upper half-space \( \mathbb{H} \) onto the Euclidean unit disk \( \mathbb{D} \). Let \( \theta_2(x,y) = \left( 2x / (1 + x^2 + y^2), 2y / (1 + x^2 + y^2), (1 - x^2 - y^2) / (1 + x^2 + y^2) \right) \). Then \( \theta := \theta_2 \circ \theta_1: \mathbb{H} \to \mathbb{D} \) defines a 1-quasiconformal homeomorphism, given that \( \theta_2^{-1} \) is an orientation-reversing stereographic projection.

There exists a unique homeomorphism \( \varphi: S^1 \to S^1 \) satisfying \( g \circ \theta = \theta \circ h^{-1} \). We see from (42) that \( g^{-1} \) is \( L \)-Lipschitz and \( \tilde{\gamma}_i \) is \( L \)-bi-Lipschitz with a constant \( L \) depending only on \( \mathcal{L}^1(E) \). In particular, \( \tilde{Z} = (\mathbb{Z}, d_Z) \).

We denote \( E' = \tilde{\gamma}_i(\theta(E)) \subset \tilde{Z} \), and apply [Ikoar, Theorem 1.3] as in Section 7.1, and find a 1-quasiconformal embedding \( \psi: \tilde{Z} \setminus E' \to S^2 \).

Consider on \( \mathbb{R}^2 \) the distance \( d_E \) obtained as follows: For each absolutely continuous \( \gamma: [0,1] \to \mathbb{R}^2 \), denote \( \ell_E(\gamma) := \int_{0}^{1} \chi_{\mathbb{R} \setminus E} \, ds \). We set \( d_E(x,y) = \inf \ell_E(\gamma) \), the infimum taken over all absolutely continuous paths joining \( x \) to \( y \).

We denote \( X = (\mathbb{R}^2, d_E) \). The change of distance map \( H: \mathbb{R}^2 \to X \) is a 1-Lipschitz homeomorphism that is a local isometry on \( \mathbb{R}^2 \setminus E \). Moreover, if \( \theta: [0,1] \to \mathbb{R}^2 \) is absolutely continuous, the metric speeds satisfy

\[
(43) \quad v_{H \circ \theta} = \left( \chi_{\mathbb{R}^2 \setminus E} \circ \theta \right) \cdot v_{\theta} \quad \mathcal{L}^1\text{-almost everywhere.}
\]

The composition \( G = \tilde{\gamma}_2 \circ \theta \circ \left( H |_{[-1,2] \times [0,1]} \right)^{-1} \) is a 1-quasiconformal homeomorphism. This follows from Lemma 2.2, the equalities \( \mathcal{H}_E^2(E') = 0 = \mathcal{H}_X^1(H(E)) \), together with Proposition 3.6 and (43).

We consider a Cantor set \( E \) obtained from [IRar, Example 6.1]. The key property of \( E \) is the following: there exists a path family \( \Gamma \) on \([0,1]^2\), each path joining \((0,0)\) to \((1,0)\), such that \( \text{mod} \, H \Gamma \geq (4\pi)^{-1} \) and \( \text{mod} \, \Gamma = 0 \). Given that \( G \) is
1-quasiconformal, the points \( \tilde{\tau}_2(\theta(x)) \), where \( x = (0,0), (1,0) \), fail (7). Consequently, \( \tilde{Z} \) is not quasiconformally equivalent to \( S^2 \), and the embedding \( \psi \) does not have a quasiconformal extension \( \Psi: \tilde{Z} \to S^2 \).

**Question 7.1.** Are there Cantor sets \( E \) with \( \mathcal{L}_1(E) > 0 \) such that a quasiconformal embedding \( \psi: \tilde{Z} \setminus E' \to S^2 \) extends to a quasiconformal homeomorphism \( \Psi: \tilde{Z} \to S^2 \)?

Given a compact set \( F \subset Y \) with \( Y = \mathbb{R}^2 \) or \( Y = S^2 \), we say that \( F \) has zero absolute area if every 1-quasiconformal embedding \( f: Y \setminus F \to S^2 \) satisfies \( \mathcal{H}^2_{S^2}(S^2 \setminus f(Y \setminus F)) = 0 \).

We expect that the quasiconformal extension \( \Psi \) exists if and only if the set \( F = \mathbb{R}^2 \setminus \psi(\tilde{Z} \setminus E') \) has zero absolute area; the "only if"-direction follows by applying the techniques used in Section 4, by noting that the composition \( (f \circ \psi)^{-1} \) has a continuous, monotone, and surjective extension \( \tilde{\pi} \) with \( \mod \Gamma \leq \mod \tilde{\pi} \Gamma \) for all path families. We expect that the "if"-direction follows from [IRar, Theorems 1.3 and 1.4, together with Lemma 5.1].

If \( E \) in Question 7.2 has zero absolute area, [IRar, Theorem 1.3] implies that the change of distance map \( H \) is a 1-quasiconformal homeomorphism. Given that the \( G \) above is 1-quasiconformal, one readily verifies that \( \tilde{\tau}_2 \) is a 1-quasiconformal homeomorphism onto its image. We ask the following.

**Question 7.2.** Let \( E, g, \) and \( \psi \) be as in Question 7.1. If \( \tilde{\tau}_2: \mathbb{Z}_2 \to \tilde{Z} \) is a 1-quasiconformal parametrization of its image, does \( \psi: \tilde{Z} \setminus E' \to S^2 \) extend to a quasiconformal homeomorphism \( \Psi: \tilde{Z} \to S^2 \)?

In particular, if \( E \) has zero absolute area, does \( F = \mathbb{R}^2 \setminus \psi(\tilde{Z} \setminus E') \) have zero absolute area?

It follows from [Iko20, Theorem 1.1 and Proposition 1.2] that the inclusion map \( \tilde{\tau}_2 \) is a 1-quasiconformal homeomorphism if and only if there exists a quasiconformal homeomorphism \( h: \tilde{\tau}_2(\mathbb{Z}_2) \to \mathbb{D} \), where \( \mathbb{D} \) is the closed Euclidean unit disk.

**7.3. Welding homeomorphisms.** We consider a welding homeomorphism \( g: S^1 \to S^1 \) with welding curve \( C \subset S^2 \). Consider the monotone mapping \( \tilde{\pi}: S^2 \to \tilde{Z} \) obtained from (14).

**Question 7.3.** If \( \tilde{\pi} \) is a homeomorphism, is it a 1-quasiconformal homeomorphism?

We showed in Proposition 4.1 that if \( \tilde{\pi} \) is not a homeomorphism, then \( \tilde{Z} \) is not quasiconformally equivalent to \( S^2 \); the collapsing creates points of positive capacity — by which we mean that (7) fails — in \( \tilde{Z} \). Question 7.3 asks if the collapsing is the only obstruction for quasiconformal uniformization. Lemma 4.8 reduces the question to understanding when \( \tilde{\pi}^{-1} \in N^{1,2}(\tilde{Z}, S^2) \).

**7.4. Quasisymmetries.** Observe that the assumptions of Proposition 1.4 are satisfied by every quasisymmetry \( g: S^1 \to S^1 \) that is strongly quasisymmetric [Sem86] [Bis88] [AZ91] [B94]: for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for every subarc \( I \subset S^1 \) and Borel set \( E \subset I \),

\[
\mathcal{H}^1_{S^1}(E) \leq \delta \mathcal{H}^1_{S^1}(I) \quad \text{implies} \quad \mathcal{H}^1_{S^1}(g(E)) \leq \varepsilon \mathcal{H}^1_{S^1}(g(I)).
\]

The welding curves corresponding to strongly quasisymmetric homeomorphisms are special cases of the asymptotically conformal quasicircles; see [Pom78]. One might ask whether or not \( \tilde{Z} \) is quasiconformally equivalent to \( S^2 \) whenever \( g \) is a
welding homeomorphism corresponding to such a curve. Corollary 4 of [Pom78] provides us with an example of asymptotically conformal quasicircle $C$ which has an uncountable number of tangent points, with the tangent points dense in $C$, but they also have zero 1-dimensional Hausdorff measure.

**Lemma 7.4.** There exists a quasisymmetric $g : S^1 \to S^1$ with asymptotically conformal welding curve $C$ such that $\tilde{Z}$ is not homeomorphic to $S^2$.

Lemma 7.4 follows from Proposition 4.1, Lemma 4.5, and the cited example.

**Question 7.5.** Is the answer to Question 7.3 yes if we also assume that $g : S^1 \to S^1$ is a quasisymmetry?

To answer Question 7.5 negatively, one needs to construct a quasisymmetry $\psi : Z_2 \to \tilde{Z}_2$, with $g = \psi|_{S^1}$, for which the measures $g^*H^1_{S^1}$ and $H^1_{S^1}$ are not mutually singular in any subarc $I \subset S^1$, yet the corresponding $\tilde{Z}$ is not quasiconformally equivalent to $\tilde{Z}$. Equivalently, one only needs to show that the homeomorphism $H : S^2 \to \tilde{Z}$, coinciding with $\tilde{t}_1$ in $Z_1$ and with $\tilde{t}_2 \circ \psi$ in $Z_2$, is not quasiconformal. By arguing as in the proof of Lemma 4.8, one sees that $H$ is quasiconformal if and only if $H^{-1} \in N^{1,2}(\tilde{Z}, S^2)$.

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