ON SOME LOCAL COHOMOLOGY INVARIANTS OF
LOCAL RINGS

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Abstract. Let \( A \) be a commutative Noetherian local ring containing
a field of characteristic \( p > 0 \). The integer invariants \( \lambda_{i,j}(A) \) have been
introduced in an old paper of ours. In this paper we completely describe
\( \lambda_{d,d}(A) \) where \( d = \dim A \) in terms of the topology of \( \text{Spec}A \).

1. Introduction

All rings in this paper are commutative and Noetherian. Let \( A \) be a
local ring that admits a surjection from a regular local \( n \)-dimensional ring
\( R \) containing a field. Let \( I \subset R \) be the kernel of the surjection, let \( m \subset R \) be the maximal ideal and let \( k = R/m \) be the residue field of \( R \). The Bass
numbers \( \lambda_{i,j}(A) = \dim_k \text{Ext}_R^i(k, H^{n-j}_I(R)) \) of the local cohomology module
\( H^{n-j}_I(R) \) are all finite and depend only on \( A, i \) and \( j \), but neither on \( R \), nor
on the surjection \( R \to A \). This has been proven in our old paper [10, Sec.
4] and these invariants of local rings have since been studied by a number
of authors [1, 2, 8, 9, 15].

Let \( d = \dim A \). We have proven in our paper [10 4.4i,ii,iii] that \( \lambda_{i,j} = 0 \)
if \( j > d \) or \( i > j \) while \( \lambda_{d,d}(A) \neq 0 \). Kawasaki [8] and Walther [15] have
completely described \( \lambda_{d,d}(A) \) for \( d \leq 2 \), while Kawasaki [9 Sec. 3, Cor. 1]
proved that \( \lambda_{d,d}(A) = 1 \) if \( A \) is \( S_2 \). In [12 Sec. 7] we stated a question
which we reproduce here in a more precise form:

Question 1.1. Is \( \lambda_{d,d}(A) \) equal to the number of connected components of
the graph \( \Gamma_B \) where \( B = \widehat{A}^{sh} \) is the completion of the strict Henselization
of the completion of \( A \) ?

The graph \( \Gamma_B \) for any local ring \( B \) has been introduced by Hochster and
Huneke [3.4]. We reproduce their definition:

Definition 1.2. Let \( B \) be a local ring. The graph \( \Gamma_B \) is defined as follows.
Its vertices are the top-dimensional minimal primes of \( B \) (i.e. primes \( P \)
such that \( \dim(B/P) = \dim B \)) and two distinct vertices \( P \) and \( Q \) are joined
by an edge if and only if the ideal \( P + Q \) has height one.

We expect that Question 1.1 has a positive answer and the goal of this
paper is to prove this in characteristic \( p \). Our main result is the following.

NSF support is gratefully acknowledged.
Theorem 1.3. Let $A$ be a local $d$-dimensional ring containing a field of characteristic $p > 0$. Let $B = \hat{A}^{sh}$ be the completion of the strict Henselization of the completion of $A$. Then $\lambda_{d,d}(A)$ equals the number of connected components of the graph $\Gamma_B$.

In particular, if $A$ is complete and has a separably closed residue field, then $B = A$ and so in this case we get the simpler statement that $\lambda_{d,d}(A)$ equals the number of connected components of the graph $\Gamma_A$.

Let $V$ be a projective variety over a separably closed field and let $A$ be the local ring at the vertex of the affine cone of some projective embedding of $V$. In [12, p. 133] we asked whether $\lambda_{i,j}(A)$ depends only on $V$, $i$ and $j$ but not on the embedding. Our Theorem 1.3 provides some supporting evidence for a positive answer to this question by showing that in characteristic $p > 0$ the integer $\lambda_{d,d}(A)$ where $d = \dim A = 1 + \dim V$ indeed is independent of the embedding and only depends on the dimensions of the pairwise intersections of the irreducible components of $V$.

Some of our arguments are characteristic-free and they might be used in an eventual proof of the characteristic zero case. We have collected them in Section 2. Theorem 1.3 is proven in Section 4.

2. Characteristic-free results

The main result of this section is Corollary 2.4 which reduces Question 1.1 to the case where $A$ is complete with separably closed residue field (i.e. $A = B$) and $\Gamma_B$ is connected, in which case we expect that $\lambda_{d,d}(A) = 1$.

Proposition 2.1. Let $R$ be a regular local ring containing a field, let $I \subset R$ be an ideal and let $A = R/I$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ be the connected components of $\Gamma_A$. Assume $\text{height} I = h$. Let $I_i$ be the intersection of the minimal primes of $I$ that are the vertices of $\Gamma_i$. Then $H^h_I(R) \cong \oplus_i H^h_{I_i}(R)$.

Proof. Let $I_0$ be the intersection of the minimal primes of $I$ of height bigger than $h$, so the radical of $I$ is $\cap_{i \geq 0} I_i$. Let $J = \cap_{i = 0}^{t-1} I_i$. Then $I = J \cap I_t$ up to radical and Mayer-Vietoris yields an exact sequence

$$H^h_{J+I_t}(R) \to H^h_J(R) \oplus H^h_{I_t}(R) \to H^h_I(R) \to H^{h+1}_{J+I_t}(R).$$

Since every minimal prime of $I_0$ has height bigger than $h$ and does not contain any minimal prime of $I_t$, the height of $I_0 + I_t$ is bigger than $h + 1$. The height of $I_i + I_t$ for $1 \leq i \leq t - 1$ also is bigger than $h + 1$ because these ideals are intersections of primes corresponding to the vertices of two different connected components of $\Gamma_A$. Hence the ideal $J + I_t$ has height bigger than $h + 1$ and therefore $H^h_{J+I_t}(R) \cong H^{h+1}_{J+I_t}(R) \cong 0$. Thus the above exact sequence implies an isomorphism $H^h_I(R) \cong H^h_J(R) \oplus H^h_{I_t}(R)$. Now we are done by induction on $t$ considering that $H^h_{J_0}(R) = 0$ because the height of every minimal prime of $I_0$ is bigger than $h$. \qed
Lemma 2.2. Let $R$ be a regular local ring containing a field, let $I \subset R$ be an ideal, let $A = R/I$, let $m \subset R$ be the maximal ideal of $R$ and let $s = \lambda_{i,j}(A)$. Then $H^i_m(H^{n-j}_I(R)) \cong E^s$, a direct sum of $s$ copies of $E$ where $E$ is the residue field of the residue field $k = R/m$ in the category of $R$-modules.

Proof. By [10, 3.6a], $H^i_m(H^{n-j}_I(R))$ is an injective $R$-module. Since it is supported only on the maximal ideal $m$, we have that $H^i_m(H^{n-j}_I(R)) \cong E^s$, a direct sum of $s$ copies of $E$, where $s$ is some finite or infinite cardinal. We quote [10] 1.4:

Let $P$ be a prime of $R$ and let $M$ be an $R$-module such that $(H^i_P(M))_P$ are injective for all $i$. Let $K(R/P)$ be the fraction field of $R/P$. Then $\text{Ext}^i_{R_k}(K(R/P), MP) \cong \text{Hom}_{R_P}(K(R/P), (H^i_P(M))_P)$.

We set $M = H^{n-j}_I(R)$ and $P = m$. Since $H^i_m(H^{n-j}_I(R)) \cong E^s$ while $K(R/P) = k$ and $\text{Hom}_R(k,E) \cong k$, we get $\lambda_{i,j}(A) = \dim_k \text{Hom}_R(k,E^s) = s$.

If a local ring $A$ containing a field does not admit a surjection from a regular local ring, one sets $\lambda_{i,j}(A) \overset{\text{def}}{=} \lambda_{i,j}({\hat A})$ where $\hat A$ is the completion of $A$ with respect to the maximal ideal. A complete local ring containing a field always admits a surjection from a complete regular local ring containing a field. In this way $\lambda_{i,j}(A)$ are defined for every local ring $A$ containing a field and one always has $\lambda_{i,j}(A) = \lambda_{i,j}({\hat A})$ [10] pp. 53-54. For this reason we do not include the assumption that $A$ is a surjective image of a regular local ring in Theorem 1.3 and in the following proposition.

Proposition 2.3. Let $A$ be a local ring containing a field and let $B = \hat A^{\text{sh}}$ be the completion of the strict Henselization of the completion of $A$. Then $\lambda_{i,j}(A) = \lambda_{i,j}({\hat B})$.

Proof. Since $\lambda_{i,j}(A) = \lambda_{i,j}({\hat A})$, we can assume that $A$ is complete, i.e. there is a surjection $R \to A$ from a regular local ring $R$ containing a field. Let $I$ be the kernel of the surjection and let $n = \dim R$. By Lemma 2.2 $H^i_m(H^{n-j}_I(R)) \cong E^s$ where $\lambda_{i,j}(A) = s$.

Let $T = \hat R^{\text{sh}}$ be the completion of the strict Henselization of the completion of $R$. Then $T/IT = B$. Since $T$ is flat over $R$, we have that $H^i_m(H^{n-j}_I(T)) \cong T \otimes_R H^i_m(H^{n-j}_I(R)) \cong (T \otimes_R E)^s$. It is enough to prove that $T \otimes_R E \cong E_T$ where $E_T$ is the injective hull of the residue field of $T$ in the category of $T$-modules for then $H^i_m(T) \cong E^s_T$ and $\lambda_{i,j}(B) = s$ by Lemma 2.2.

But $E \cong H^0_m(R)$ and $E_T \cong H^0_m(T)$ since $R$ and $T$ are regular, $mT$ is the maximal ideal of $T$ and $n = \dim R = \dim T$. Considering that $H^0_m(T) = H^0_m(T \otimes_R R) = T \otimes_R H^0_m(R)$, we are done.

Corollary 2.4. Let $A$ be a local ring of dimension $d$ containing a field and let $B = \hat A^{\text{sh}}$ be the completion of the strict Henselization of the completion...
of $A$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ be the connected components of $\Gamma_B$. Let $I_j$ be the intersection of the minimal primes of $B$ that are the vertices of $\Gamma_i$. Let $B_j = B/I_j$. Then

(a) $\lambda_{i,d}(A) = \sum_{j=1}^{r} \lambda_{i,d}(B_j)$ for every $i$.

(b) Question 1.1 has a positive answer for $A$ (i.e. $\lambda_{d,d}(A) = r$) if and only if it has a positive answer for every $B_j$ (i.e. if and only if $\lambda_{d,d}(B_j) = 1$ for every $j$).

Proof. By Proposition 2.3, $\lambda_{i,j}(A) = \lambda_{i,j}(B)$. Since $B$ is complete, there is a surjection $T \to B$ from a regular local ring $T$ containing a field. Let $\tilde{I}$ be the kernel of the surjection and let $\tilde{I}_j \subset T$ be the preimage of $I_j$, so $T/\tilde{I}_j = B_j$. By Proposition 2.1, $H_{T}^{n-d}(T) \cong \oplus_j H_{I_j}^{n-d}(T)$. Hence $\text{Ext}_{T}^{i}(k, H_{I_j}^{n-d}(T)) \cong \oplus_j \text{Ext}_{T}^{i}(k, H_{I_j}^{n-d}(T))$ where $k$ is the residue field of $T$. Hence the dimensions over $k$ of the two sides of this equation are the same. This implies (a). And (b) is immediate from (a). \[\square\]

We note that each $B_j$ is complete, reduced, equidimensional, has a separably closed residue field and $\Gamma_{B_j}$ is connected (since $\Gamma_{B_j} = \Gamma_j$). Thus Corollary 2.4 reduces Question 1.1 to rings of this type.

Finally, it is worth pointing out that the graph $\Gamma_B$ where $B = \hat{A}_{sh}$ is realized by a substantially smaller ring than $\hat{A}_{sh}$. Namely, let $k \subset \hat{A}$ be a coefficient field. It follows from \[7, 4.2\] that there exists a finite separable field extension $K$ of $k$ such that the graphs $\Gamma_{K \otimes_k \hat{A}}$ and $\Gamma_B$ are isomorphic.

3. $F_T$-modules, $B\{f\}$-modules and the functor $\mathcal{H}_{T,B}$.

In this section we review some facts from our old paper \[11\] that are used in our proof of Theorem \[13\]. Throughout this section $T$ is a complete local regular ring containing a field of characteristic $p > 0$.

3.1. The Frobenius functor $F_T$. Let $T'$ be the additive group of $T$ regarded as a $T$-bimodule with the usual left $T$-action and with the right $T$-action defined by $t' t = t'^p t$ for all $t \in T, t' \in T'$. The Frobenius functor $F_T = F : T\text{-mod} \to T\text{-mod}$ of Peskine-Szpiro \[14, I.1.2\] is defined by

$F(M) = T' \otimes_T M$

$F(M \xrightarrow{h} N) = (T' \otimes_R M \xrightarrow{id \otimes_{T'} h} T' \otimes_T N)$

for all $T$-modules $M$ and all $T$-module homomorphisms $h$, where $F(M)$ acquires its $T$-module structure via the left $T$-module structure on $T'$. For a summary of basic properties of the Frobenius functor \[11, \text{Remarks 1.0}\] may be consulted.
3.2. $F_T$-modules. An $F$-module \[\text{[11, 1.1]}\] (more precisely, an $F_T$-module) is a $T$-module $\mathcal{M}$ equipped with a $T$-module isomorphism $\theta: \mathcal{M} \to F(\mathcal{M})$ which we call the structure morphism of $\mathcal{M}$. A homomorphism of $F$-modules is a $T$-module homomorphism $f: \mathcal{M} \to \mathcal{M}'$ such that the following diagram commutes (where $\theta$ and $\theta'$ are the structure morphisms of $\mathcal{M}$ and $\mathcal{M}'$).

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{f} & \mathcal{M}' \\
\downarrow{\theta} & & \downarrow{\theta'} \\
F(\mathcal{M}) & \xrightarrow{F(f)} & F(\mathcal{M}')
\end{array}
\]

3.3. Generating morphisms. A generating morphism of an $F$-module $\mathcal{M} \text{[11, 1.9]}$ is a $T$-module homomorphism $\beta: \mathcal{M} \to F(\mathcal{M})$, where $\mathcal{M}$ is some $T$-module, such that $\mathcal{M}$ is the limit of the inductive system in the top row of the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\beta} & F(M) \\
\downarrow{\beta} & & \downarrow{F(\beta)} \\
F(M) & \xrightarrow{F(\beta)} & F^2(M) \\
\downarrow{F(h)} & & \downarrow{F^2(h)} \\
F^2(M) & \xrightarrow{F^2(\beta)} & F^3(M) \\
\downarrow{F^2(h)} & & \downarrow{F^3(h)} \\
\vdots & & \vdots
\end{array}
\]

and $\theta: \mathcal{M} \to F(\mathcal{M})$, the structure isomorphism of $\mathcal{M}$, is induced by the vertical arrows in this diagram. Since the tensor product commutes with direct limits, the limit of the inductive system of the bottom row is indeed $F(\mathcal{M})$, so this definition of $\theta$ makes sense.

3.4. Morphisms of $F$-modules in terms of generating morphisms. If $\beta: M \to F(M)$ and $\beta': M' \to F(M')$ are generating morphisms of $F$-modules $\mathcal{M}$ and $\mathcal{M}'$ respectively, then any $R$-module homomorphism $h: M \to M'$ that makes the leftmost square in the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\beta} & F(M) \\
\downarrow{h} & & \downarrow{F(h)} \\
M' & \xrightarrow{\beta'} & F(M')
\end{array}
\]

commutative, makes the whole diagram commutative and the vertical arrows of this diagram induce an $F$-module homomorphism $\mathcal{H}: \mathcal{M} \to \mathcal{M}' \text{[11, 1.10b]}$.

3.5. Localization in terms of generating morphisms. If $g \in T$ is an element and $\mathcal{M}$ is an $F$-module, the localization $\mathcal{M}_g$ carries a natural structure of $F$-module such that the natural localization map $\ell: \mathcal{M} \to \mathcal{M}_g$ is a homomorphism of $F$-modules $\text{[11, 1.3b]}$. If $\beta: M \to F(M)$ is a generating morphism of $\mathcal{M}$, then $\beta \circ g^{p-1}: M \to F(M)$ is a generating morphism of

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\beta} & F(\mathcal{M}) \\
\downarrow{\theta} & & \downarrow{F(\beta)} \\
F(\mathcal{M}) & \xrightarrow{F(\beta)} & F^2(\mathcal{M}) \\
\downarrow{F(h)} & & \downarrow{F^2(h)} \\
F^2(\mathcal{M}) & \xrightarrow{F^2(\beta)} & F^3(\mathcal{M}) \\
\downarrow{F^2(h)} & & \downarrow{F^3(h)} \\
\vdots & & \vdots
\end{array}
\]

and $\theta': \mathcal{M} \to F(\mathcal{M})$, the structure isomorphism of $\mathcal{M}$, is induced by the vertical arrows in this diagram. Since the tensor product commutes with direct limits, the limit of the inductive system of the bottom row is indeed $F(\mathcal{M})$, so this definition of $\theta'$ makes sense.
\( \mathcal{M}_g \) and the natural localization map \( \ell : \mathcal{M} \to \mathcal{M}_g \) is induced by the vertical arrows in the commutative diagram

\[
\begin{array}{cccccc}
M & \xrightarrow{\beta} & F(M) & \xrightarrow{F(\beta)} & F^2(M) & \xrightarrow{F^2(\beta)} & \cdots \\
\downarrow{\text{mult by } g} & & \downarrow{\text{mult by } F(g)=g^p} & & \downarrow{\text{mult by } F^2(g)=g^{p^2}} & & \\
M & \xrightarrow{\beta g^{p-1}} & F(M) & \xrightarrow{F(\beta g^{p-1})} & F^2(M) & \xrightarrow{F^2(\beta g^{p-1})} & \cdots \\
\end{array}
\]

Let \( x \in F^s(M) \) (we mean the copy of \( F^s(M) \) in the top row) and let \( \tilde{x} \in \mathcal{M} \) be its image under the maps in the top row. Then the image of \( x \in F^s(M) \) (here we mean the copy of \( F^s(M) \) in the bottom row) in \( \mathcal{M}_g \) under the maps in the bottom row is \( \tilde{x}' = \frac{\ell(\tilde{x})}{g^{ps}} \). The \( s \)-th vertical map which is the multiplication by \( g^{ps} \) on \( F^s(M) \) induces a map from the image of the top \( F^s(M) \) in \( \mathcal{M} \) to the image of the bottom \( F^s(M) \) in \( \mathcal{M}_g \) that sends \( \tilde{x} \in \mathcal{M} \) to \( g^{ps}\tilde{x}' = \frac{\ell(\tilde{x})}{g^{ps}} \in \mathcal{M}_g \).

3.6. \( B\{f\} \)-modules. Let \( B \) be a surjective image of \( T \) and let \( B\{f\} \) be the ring extension of \( B \) obtained by adjoining a variable \( f \) subject to relations \( fb = b^p f \) for every \( b \in B \). A (left) \( B\{f\} \)-module is a \( B \)-module \( N \) with a map \( f : N \to N \) called the action of the Frobenius on \( N \) such that \( f(bx) = b^p f(x) \) for every \( x \in N \). We call \( N \) a cofinite \( B\{f\} \)-module if it is cofinite (i.e. Artinian) as a \( B \)-module.

3.7. The functor \( \mathcal{H}_{T,B} \). The surjective homomorphism \( T \to B \) gives every \( B \)-module a structure of \( T \)-module and it gives every \( T \)-module annihilated by the kernel of the surjection a structure of \( B \)-module. If \( N \) is a \( B\{f\} \)-module, we get a \( T \)-module homomorphism

\[
\gamma_N : F(N) = T' \otimes_T N' \xrightarrow{t' \otimes x \mapsto f(x)} N.
\]

Let \( D = \text{Hom}_T(-, E) \) be the Matlis duality functor where \( E \) is the injective hull of the residue field of \( T \) in the category of \( T \)-modules. Assume \( N \) is a cofinite \( B\{f\} \)-module. Applying \( D \) to the map \( \gamma_N \) and considering that there is a functorial \( T \)-module isomorphism \( \tau : D(F(N)) \to F(D(N)) \) \([11, 4.1]\) we get a \( T \)-module homomorphism

\[
\beta_N = \tau \circ D(\gamma_N) : D(N) \to F(D(N)).
\]

\( \mathcal{H}_{T,B}(N) \) is the \( F \)-module with generating morphism \( \beta_N \). A homomorphism of \( B\{f\} \)-modules \( \theta : N \to N' \) induces a homomorphism of \( T \)-modules \( D(\theta) : D(N') \to D(N) \). It is straightforward to check that the left square in the diagram

\[
\begin{array}{cccccc}
D(N') & \xrightarrow{\beta_{N'}} & F(D(N')) & \xrightarrow{F(\beta_{N'})} & F^2(D(N')) & \xrightarrow{F^2(\beta_{N'})} & \cdots \\
\downarrow{D(\theta)} & & \downarrow{F(D(\theta))} & & \downarrow{F^2(D(\theta))} & & \\
D(N) & \xrightarrow{\beta_N} & F(D(N)) & \xrightarrow{F(\beta_N)} & F^2(D(N)) & \xrightarrow{F^2(\beta_N)} & \cdots \\
\end{array}
\]
is commutative, hence the whole diagram is commutative, hence it induces an $F$-module homomorphism $\mathcal{H}_{T,B}(\theta): \mathcal{H}_{T,B}(N') \to \mathcal{H}_{T,B}(N)$ \[11\text{.}10b\]. The functor $\mathcal{H}_{T,B}$ thus defined is a functor from the category of cofinite $B\{f\}$-modules to the category of $F$-modules supported on $V(I)$ where $I \subset T$ is the kernel of the surjection $T \to B$. This functor is additive, contravariant and exact \[11\text{.}4.2\].

3.8. $H^i_m(B)$ as a $B\{f\}$-module. The local cohomology modules $H^i_m(B)$ have a natural structure of $B\{f\}$-modules \[11\text{.}1.2a,b\]. If the dimension of $T$ is $n$, then $\mathcal{H}_{T,B}(H^i_m(B)) \cong H^n_{I}^i(T)$ \[11\text{.}4.8\].

3.9. The stable part of a cofinite $B\{f\}$-module. Since $B$ is complete, it contains a coefficient field $k \subset B$. Let $N$ be a cofinite $B\{f\}$-module, let $\text{im} f^j$ be the set of elements of $N$ of the form $f^j(x)$ where $x \in N$, and let $k\text{-im} f^j$ be the $k$-vector subspace of $N$ spanned by $\text{im} f^j$. We set the stable part of $N$ to be the $k$-vector space $N_s = \cap_j k\text{-im} f^j$. This definition depends on the choice of the coefficient field $k$; nevertheless we denote the stable part simply by $N_s$. According to \[11\text{.}4.9\], $N_s$ is finite-dimensional over $k$, $f: N_s \to N_s$ is injective and the $k$-vector subspace of $N$ spanned by $f(N_s)$ coincides with $N_s$; this implies that if a set $\{y_1, \ldots, y_q\} \subset N_s$ is a $k$-basis of $N_s$, then the set $\{f^v(y_1), \ldots, f^v(y_q)\} \subset N_s$, for every $v$, is also a $k$-basis of $N_s$. The dimension of $N_s$ as a $k$-vector space is independent of the choice of the coefficient field $k$ \[11\text{.}4.11\].

If the dimension of the support of $\mathcal{H}_{T,B}(N)$ is zero, it follows from \[11\text{.}4.10\] that $\mathcal{H}_{T,B}(N) \cong E^r$ where $r = \dim_k N_s$ (because in this case the maximal zero-dimensional quotient of $\mathcal{H}_{T,B}(N)$ in the category of $F$-modules, denoted $\mathcal{L}$ in \[11\text{.}\text{p. 110}\], is $\mathcal{H}_{T,B}(N)$ itself).

4. Proof of Theorem \[11\text{.}3\]

We deduce Theorem \[11\text{.}3\] from Corollary 2.4 by proving that $\lambda_{d,d}(B_j) = 1$ for every $B_j$. First we express $\lambda_{i,j}(B)$ in terms of the Frobenius action on $H^i_m(B)$ (Proposition 4.1 and Corollary 4.2). We then use this to prove that if $B$ is one the rings $B_j$ appearing in Corollary 2.4, then $\lambda_{j,d}(B) = \lambda_{d,d}(S)$ where $S$ is the $S_2$-ification of $B$ (Proposition 4.3) and conclude by appealing to a result of Kawasaki to the effect that $\lambda_{d,d}(S) = 1$.

Let $B$ be a complete local ring containing a field of characteristic $p > 0$. If $N$ is a cofinite $B\{f\}$-module and $g \in B$, we let $N(g)$ to be the cofinite $B\{f\}$-module defined as follows. The underlying $B$-module of $N(g)$ is $N$ and the action of the Frobenius on $N(g)$ is defined by $f(x) = g^{p-1} f(\text{id}(x))$ for every $x \in N(g)$ where $\text{id}: N(g) \to N$ is the identity map of the underlying $B$-modules. Clearly $N(gg') = N(g)(g')$ for all $g, g' \in B$.

Since $B$ is complete, there is a surjection $T \to B$ where $T$ is a complete regular local ring containing a field.

**Proposition 4.1.** Let $B$ and $T \to B$ be as above. Let $g = \{g_1, \ldots, g_d\} \subset B$ be a system of parameters of $B$ and let $N$ be a cofinite $B\{f\}$-module. Let
$K_\bullet(g;N)$ be the Koszul complex of $N$ on $g$, namely
\[ 0 \to K_d(g;N) \to K_{d-1}(g;N) \to \cdots \to K_0(g;N) \to 0 \]
where $K_r(g;N) = \oplus_{1 \leq i_1 < i_2 < \cdots < i_r \leq d} N(g_{i_1} \cdots g_{i_r})$ and the differentials are defined as follows. The image of $x \in N(g_{i_1} \cdots g_{i_r}) \subset K_r(g;N)$ under the corresponding differential is $\sum_j (-1)^j g_j \cdot \bar{g}_{i_1} \cdots \bar{g}_{i_j} \cdots \bar{g}_{i_r}(x) \in K_{r-1}(g;N)$.

(i) $K_\bullet(g;N)$ is a complex in the category of cofinite $B\{f\}$-modules. Hence the cohomology modules $H_i(K_\bullet(g;N))$ of this complex are in a natural way cofinite $B\{f\}$-modules.

(ii) $\mathcal{H}_{T,B}(H_i(K_\bullet(g;N))) \cong H^i_m(\mathcal{H}_{T,B}(N))$ where $m$ is the maximal ideal of $T$.

(iii) $H^i_m(\mathcal{H}_{T,B}(N)) \cong E^r$ where $r = \dim_k H_i(K_\bullet(g;N))$, $E$ is the injective hull of $T/m$ in the category of $T$-modules and $H_i(K_\bullet(g;N))$ is the stable part of $H_i(K_\bullet(g;N))$ with respect to each coefficient field $k$ of $B$ (see (3.9)).

Proof. (i) The map $\delta : N(g) \to N$ which is the multiplication by $g$ and the underlying $B$-module is a $B\{f\}$-module homomorphism because $\delta(f(x)) = g \cdot g^{p-1}f(id(x)) = g^p f(id(x)) = f(\delta(x))$ for every $x \in N(g)$. Since $N(g_{i_1} \cdots g_{i_r}) \cong N(g_{i_1} \cdots \bar{g}_{i_j} \cdots g_{i_r})$, the map $N(g_{i_1} \cdots g_{i_r}) \to N(g_{i_1} \cdots \bar{g}_{i_j} \cdots g_{i_r})$ is the multiplication by $g_{i_j}$ which is a morphism of $B\{f\}$-modules. Hence the differentials of $K_\bullet(g;N)$ are morphisms of $B\{f\}$-modules. This proves (i).

(ii) Let $\delta : N(g) \to N$ be the multiplication by $g$ as before and let $\bar{g} \in T$ be a lifting of $g$. The map $\delta$ may be viewed as the multiplication by $\bar{g}$. Associated to this map is a commutative diagram like in (3.7):

\[
\begin{array}{ccccccccc}
D(N) & \xrightarrow{\beta_N} & F(D(N)) & \xrightarrow{F(\beta_N)} & F^2(D(N)) & \xrightarrow{F^2(\beta_N)} & \cdots \\
\downarrow D(\delta) & & \downarrow F(D(\delta)) & & \downarrow F^2(D(\delta)) & & \\
D(N(g)) & \xrightarrow{\beta_N(g)} & F(D(N(g))) & \xrightarrow{F(\beta_N(g))} & F^2(D(N(g))) & \xrightarrow{F^2(\beta_N(g))} & \cdots 
\end{array}
\]

Since $\delta$ is the multiplication by $\bar{g}$, we conclude that $F^s(D(\delta))$ is the multiplication by $\bar{g}^p$. Since the underlying $B$-modules of $N(g)$ and $N$ are the same, $D(N(g)) \cong D(N)$. Identifying the underlying $B$-modules of $N(g)$ and $N$ and viewing the maps $\gamma_N$ and $\gamma_{N(g)}$ of (3.7) as two maps with the same source and target, we get $\gamma_{N(g)} = \bar{g}^{p-1} \gamma_N$, hence $\beta_N(g) = \beta_N \circ \bar{g}^{p-1}$. Putting all of this together we get that the above commutative diagram takes the following form

\[
\begin{array}{ccccccccc}
D(N) & \xrightarrow{\beta_N} & F(D(N)) & \xrightarrow{F(\beta_N)} & F^2(D(N)) & \xrightarrow{F^2(\beta_N)} & \cdots \\
\downarrow \text{mult by } \bar{g} & & \downarrow \text{mult by } \bar{g}^p & & \downarrow \text{mult by } \bar{g}^{p^2} & & \\
D(N) & \xrightarrow{\beta_N \circ \bar{g}^{p-1}} & F(D(N)) & \xrightarrow{F(\beta_N \circ \bar{g}^{p-1})} & F^2(D(N)) & \xrightarrow{F^2(\beta_N \circ \bar{g}^{p-1})} & \cdots 
\end{array}
\]
The limit of the inductive system in the top row is by definition $\mathcal{H}_{T,B}(N)$. Comparing this commutative diagram with the one in (3.5) we see that the limit of the inductive system of the bottom row is $\mathcal{H}_{T,B}(N)_{\hat{g}}$ and the vertical arrows induce the natural localization map $\ell : \mathcal{H}_{T,B}(N) \to \mathcal{H}_{T,B}(N)_{\hat{g}}$. In other words, $\mathcal{H}_{T,B}(N(g)) \cong \mathcal{H}_{m,B}(N)_{\hat{g}}$ and $\mathcal{H}_{T,B}(\delta) \cong \ell$.

Let $\tilde{g} = \{\tilde{g}_1, \ldots, \tilde{g}_d\} \subset T$ be a lifting of $g$, i.e. $\tilde{g}_i$ is a lifting of $g_i$ for every $i$. Considering that $N(g_i \cdots g_r) \cong N(\tilde{g}_i \cdots \tilde{g}_r)(\tilde{g}_j)$ the above implies that $\mathcal{H}_{T,B}$ transforms the map $N(g_i \cdots g_r) \to N(\tilde{g}_i \cdots \tilde{g}_r \cdots g_r)$ which is the multiplication by $g_j$ on the underlying $B$-module into the natural localization map $\mathcal{H}_{T,B}(N)_{\tilde{g}_i \cdots \tilde{g}_r} \to \mathcal{H}_{T,B}(N)_{\tilde{g}_i \cdots \tilde{g}_r}$.

Hence the complex $\mathcal{H}_{T,B}(K_\bullet(g;N))$ is nothing but the Čech complex $C^\bullet(\tilde{g};\mathcal{H}_{T,B}(N))$ of $\mathcal{H}_{T,B}(N)$ with respect to $\tilde{g}$, namely

$$0 \to C^0(\tilde{g};\mathcal{H}_{T,B}(N)) \to C^1(\tilde{g};\mathcal{H}_{T,B}(N)) \to \cdots \to C^d(\tilde{g};\mathcal{H}_{T,B}(N)) \to 0$$

where $C^i(\tilde{g};\mathcal{H}_{T,B}(N)) = \mathcal{H}_{T,B}(K_i(g;N)) = \oplus_{1 \leq i_1 < \cdots < i_r \leq d}\mathcal{H}_{T,B}(N)_{\tilde{g}_i_1 \cdots \tilde{g}_i_r}$. The $i$-th cohomology module of this Čech complex is $H_j^i(\mathcal{H}_{T,B}(N))$ where $J$ is the ideal of $T$ generated by $\tilde{g}$ [10, 1.3]. Since the functor $\mathcal{H}_{T,B}$ is exact, it commutes with the operation of taking the cohomology of complexes. Hence $\mathcal{H}_{T,B}(H_i(K_\bullet(g;N))) \cong H^i(\mathcal{H}_{T,B}(N)) \cong H_j^i(\mathcal{H}_{T,B}(N))$. It remains to show that $H^i_j(\mathcal{H}_{T,B}(N)) \cong H^i_m(\mathcal{H}_{T,B}(N))$.

Since $\tilde{g}$ is a lifting of a system of parameters of $B$, the ideal $J + I$ ($I$ is the kernel of the surjection $T \to B$) is $m$-primary, i.e. $H^i_m(M) \cong H^i_{I + J}(M)$ for every $T$-module $M$. The composition of functors $H^0_m(-) = H^0_J(H^0_I(-))$ yields a spectral sequence $E^{p,q}_2 = H^p_J(H^q_I(M)) \Rightarrow H^{p+q}_m(M)$. If $M$ is supported on $V(I)$ then $H^p_J(M) = 0$ for $q > 0$ and $H^q_I(M) = M$, i.e. $E^{p,0}_2 = 0$ for $q > 0$ and $E^{0,0}_2 = H^p_J(M)$, so the spectral sequence degenerates at $E_2$ and implies that $H^i_m(M) = H^i_J(M)$ for all $i$. Since $\mathcal{H}_{T,B}(N)$ is supported on $V(I)$, this completes the proof of (ii).

(iii) Since the dimension of the support of $H^i_m(\mathcal{H}_{T,B}(N))$ is zero, we are done by (3.9) considering (ii).

The following corollary is crucial. It expresses $\lambda_{i,j}(B)$ in terms of the Frobenius action on $H^i_m(B)$ without any reference to the surjection $T \to B$.

**Corollary 4.2.** Let $B$ be a local ring containing a field of characteristic $p > 0$. Let $g = \{g_1, \ldots, g_d\} \subset B$ be a system of parameters of $B$ and let $m$ be the maximal ideal of $B$. Let $K_\bullet(g;H^i_m(B))$ be the Koszul complex of $H^i_m(B)$ on $g$. Let $H_i(K_\bullet(g;H^i_m(B)))$ be its $i$-th cohomology module. Then $\lambda_{i,j}(B) = \dim_k H_i(K_\bullet(g;H^j_m(B)))$.

**Proof.** The local cohomology module $H^i_m(B)$ and the action of the Frobenius on it remain the same after replacing $B$ by its completion with respect to $m$. Hence we may assume that $B$ is complete with respect to its maximal ideal and therefore admits a surjection $T \to B$ from a complete regular local ring $T$ containing a field of characteristic $p > 0$. Let $I \subset T$ be the kernel
of this surjection and let \( n = \dim T \). Then \( \mathcal{H}_{T,B}(H^0_m(B)) \cong H^n_{m,j}(T) \) by (3.8). Hence \( H^i_m(H^{a-j}_I(T)) \cong H^i_m(\mathcal{H}_{T,B}(H^j_m(B))) \cong E^s \) where \( s = \lambda_{i,j}(B) \) (by Lemma 2.2 with \( A = B \) and \( R = T \)). Now we are done by Proposition 4.1iii with \( N = H^1_m(B) \). 

If \( B \) is complete, it has a canonical module. If in addition it is reduced and equidimensional, the existence of a canonical module implies that it has an \( S_2 \)-ification which we denote \( S \). It is a ring extension of \( B \) that is module-finite over \( B \) [6, 2.4]. A central result of [6] says that \( S \) is a local ring if and only if \( \Gamma_B \) is connected [6, 3.6c,e].

**Proposition 4.3.** Let \( B \) be a complete local ring containing a field of characteristic \( p > 0 \). Assume \( B \) is reduced, equidimensional of dimension \( d \), has a separably closed residue field, and the graph \( \Gamma_B \) is connected. Let \( S \) be the \( S_2 \)-ification of \( B \). Then \( \lambda_{i,d}(B) = \lambda_{i,d}(S) \) for every \( i \).

**Proof.** Since \( \Gamma_B \) is connected, \( S \) is local, as is pointed out above. Hence \( \lambda_{i,d}(S) \) makes sense.

Let \( m_B \) and \( m_S \) be the maximal ideals of \( B \) and \( S \) respectively. The ideal \( m_B S \) of \( S \) is \( m_S \)-primary since \( S \) is module-finite over \( B \). This implies that \( H^i_{m_B}(B^S) \cong B^i_{(m_B)S}(S) \cong_B H^i_{m_S}(S) \) where the subscript \( B(–) \) means that the corresponding \( S \)-module is viewed as a \( B \)-module via ”restriction of scalars”.

Let \( Q = S/B \). The short exact sequence \( 0 \to B \to S \to Q \to 0 \) in the category of \( B \)-modules yields an exact sequence \( H^{d-1}_B(Q) \to H^n_{m_B}(B) \to H^n_{m_B}(S) \to H^n_{m_B}(Q) \). But for every \( a \in S \) the ideal \( \{ b \in B \mid ba \in B \} \) of \( B \) has height at least two [6, 2.4], so the natural inclusion \( B \to S \) becomes an isomorphism after localization at every prime of \( B \) of height at most one. Therefore the dimension of the \( B \)-module \( Q \) is at most \( d - 2 \). Hence \( H^j_{m_B}(Q) = 0 \) for \( j = d, d - 1 \) and the above exact sequence implies an isomorphism of \( B \)-modules \( H^d_{m_B}(B) \cong H^d_{m_B}(S) \cong_B H^d_{m_S}(S) \) induced by the natural inclusion \( B \to S \).

We claim that under this \( B \)-module isomorphism the natural action of the Frobenius on \( H^d_{m_B}(B) \) coincides with the natural action of the Frobenius on \( H^d_{m_S}(S) \), i.e. we have an isomorphism of \( B\{ f \}-\)modules, not just \( B \)-modules. Indeed, let \( g = \{ g_1, \ldots, g_d \} \subset B \) be a system of parameters of \( B \). Then \( H^i_{m_B}(B) \) is the \( i \)-th cohomology of the \( \check{\text{C}} \)ech complex \( C^\bullet(g; B) \) and the natural action of the Frobenius on \( H^i_{m_B}(B) \) is induced by the action of the Frobenius on the complex \( C^\bullet(g; B) \), namely, if \( x \in B_{g_1 \cdots g_r} \subset C^r(g; B) \), then \( f(x) = x^p \in B_{g_1 \cdots g_r} \subset C^r(g; B) \). This commutes with the differentials and therefore induces an action of the Frobenius on cohomology. Since \( S \) is module-finite over \( B \), the set \( g \) is a system of parameters for \( S \) as well. Hence \( H^i_{m_S}(S) \) is the \( i \)-th cohomology of the \( \check{\text{C}} \)ech complex \( C^\bullet(g; S) \) and the natural action of the Frobenius on \( H^i_{m_S}(S) \) is induced by the action of the Frobenius on the complex \( C^\bullet(g; S) \), namely, if \( x \in S_{g_1 \cdots g_r} \), then
$f(x) = x^p \in S_{g_1 \ldots g_n}$. The natural inclusion map $B \to S$ induces a map of complexes $C^\bullet(g; B) \to_B C^\bullet(g; S)$ which commutes with the action of the Frobenius on both sides, as is easy to see. Hence it induces a map on the $i$-th cohomology groups that commutes with the action of the Frobenius. Since we have already seen that this map is an isomorphism of $B$-modules for $i = d$, the claim is proven.

Now let $N = H_i(K\bullet(g; H_m^d(B)))$. By Corollary 4.2, $\lambda_i(N) = \dim_k N_{s,k}$ and $\lambda_i(N) = \dim_k N_{s,k}$ where $k \subset B$ and $K \subset S$ are coefficient fields of $B$ and $S$ respectively and $N_{s,k}$ and $N_{s,K}$ are the stable parts of $N$ with respect to $k$ and $K$ respectively, i.e. $N_{s,k} = \cap_j k \cdot \text{im} f^j$ and $N_{s,K} = \cap_j K \cdot \text{im} f^j$. Thus it remains to prove that $\dim_k N_{s,k} = \dim_k N_{s,K}$. This equality trivially holds if the residue fields of $B$ and $S$ coincide (which happens for example if the residue field of $B$ is algebraically closed) for in this case a coefficient field of $B$ is automatically a coefficient field of $S$, i.e. we may put $K = k$. In the general case we are done by the following lemma. □

**Lemma 4.4.** Let $B$ and $S$ be as in Proposition 4.3. Let $N$ be a cofinite $S\{f\}$-module. Let $k \subset B$ and $K \subset S$ be coefficient fields of $B$ and $S$ respectively. Let $N_{s,k}$ and $N_{s,K}$ be the stable parts of $N$ with respect to $k$ and $K$ respectively, i.e. $N_{s,k} = \cap_j k \cdot \text{im} f^j$ and $N_{s,K} = \cap_j K \cdot \text{im} f^j$. Then $\dim_k N_{s,k} = \dim_k N_{s,K}$.

**Proof.** Viewing $B$ as a subring of $S$ we let $\tilde{B} = B + m_S$ where $m_S$ is the maximal ideal of $S$. Clearly $\tilde{B}$ is a subring of $S$ containing $B$. Since $S$ is module-finite over $B$, so is $\tilde{B}$. Hence $\tilde{B}$ is a complete local ring with maximal ideal $m_{\tilde{B}} = m_S$ and $N$ is a cofinite $\tilde{B}\{f\}$-module. Clearly, $k \subset \tilde{B}$ and $k$ is a coefficient field of $\tilde{B}$.

Let $\tilde{k} \subset K$ be the preimage of the image of $k$ in $S/m_S$ under the natural map $k \cong B/m_B \to S/m_S \cong K$. We claim $\tilde{k} \subset \tilde{B}$. Indeed, for every $c \in \tilde{k}$ there is $c' \in k \subset B$ such that their images in $S/m_S$ are the same. Hence $c - c' \in m_S$. Therefore $c = c' + (c - c') \in \tilde{B}$ which proves the claim.

Let $N_{s,\tilde{k}}$ be the stable part of $N$ with respect to $\tilde{k}$. As is pointed out in (3.9), the dimension of the stable part is independent of the choice of the coefficient field. Since $k$ and $\tilde{k}$ are two coefficient fields of the same ring $\tilde{B}$ and $N$ is a cofinite $\tilde{B}\{f\}$-module, $\dim_{\tilde{k}} N_{s,\tilde{k}} = \dim_{\tilde{k}} N_{s,\tilde{k}}$. It remains to show that $\dim_k N_{s,k} = \dim_{\tilde{k}} N_{s,\tilde{k}}$.

Since $\tilde{k} \subset K$, we have that $\tilde{k} \cdot \text{im} f^j \subset K \cdot \text{im} f^j$ for every $j$, so $N_{s,\tilde{k}} \subset N_{s,K}$. Since $S$ is module-finite over $B$, the residue field of $S$ is a finite field extension of the residue field of $B$. Since the residue field of $B$ is separably closed, the extension is purely inseparable. Thus $K \cong S/m_S$ is a finite purely inseparable extension field of $\tilde{k} \cong k \cong B/m$. Let $u$ be an integer such that $\sigma^u \in \tilde{k}$ for every $c \in K$.

Let $\dim_{\tilde{k}} N_{s,\tilde{k}} = r$ and let $x_1, \ldots, x_r \in N_{s,\tilde{k}}$ be a $\tilde{k}$-basis of $N_{s,\tilde{k}}$. We claim $x_1, \ldots, x_r$ are linearly independent over $K$. Indeed, let $\sum_j c_j x_j = 0$
be a linear dependency relation where \( c_j \in K \) are not all zero. Applying \( f^u \) to this relation we get \( \sum_j c_j \phi^u f^u(x_j) = 0 \) where \( \phi^u_j \in \hat{k} \) for every \( j \). Thus \( f^u(x_1),\ldots,f^u(x_r) \) are linearly dependent over \( \hat{k} \). But this is impossible because the \( \hat{k} \)-linear span of these elements coincides with the \( \hat{k} \)-linear span of \( f^u(N,\tilde{z}) \) which according to (3.9) is just \( N_{s,\hat{k}} \) and \( N_{s,\hat{k}} \) has dimension \( r \) over \( \hat{k} \). This proves the claim and implies that \( r = \dim_{\hat{k}} N_{s,\hat{k}} \leq \dim_K N_{s,K} \) since \( N_{s,\hat{k}} \subset N_{s,K} \). It remains to show that \( \dim_K N_{s,K} \leq r \).

Let \( q = \dim_K N_{s,K} \) and let \( y_1,\ldots,y_q \in N_{s,K} \) be a \( K \)-basis of \( N_{s,K} \). According to (3.9), \( f^u(y_1),\ldots,f^u(y_q) \) is a \( K \)-basis of \( N_{s,K} \) for every \( v \). Consequently \( f^u(y_1),\ldots,f^u(y_q) \) are linearly independent over \( \hat{k} \) which is a subfield of \( K \).

Let \( N' \) be the \( \hat{k} \)-linear span of \( f^u(y_1),\ldots,f^u(y_q) \). We claim that \( f \) sends \( N' \) to itself. Indeed, \( f(y_j) \in N_{s,K} \) for every \( j \) according to (3.9), \( f \) sends \( N_{s,K} \) to itself. Hence \( f(y_j) = \sum_i c_i y_i \) where \( c_i \in K \). Applying \( f^u \) we get \( f^{u+1}(y_j) = \sum_i c_i \phi^u_i f^u(y_i) \in N' \) since \( \phi^u_i \in \hat{k} \) for every \( i \). Since \( f^{u+1}(y_1),\ldots,f^{u+1}(y_q) \) span the \( \hat{k} \)-linear span of \( f(N') \), the claim is proven.

Since \( f^{u+1}(y_1),\ldots,f^{u+1}(y_q) \) are linearly independent over \( \hat{k} \), the \( \hat{k} \)-linear span of \( f(N') \) has dimension at least \( q = \dim_{\hat{k}} N' \), i.e. the \( \hat{k} \)-linear span of \( f(N') \) coincides with \( N' \). Hence \( N' \subset \hat{k} \)-im\( f^j \) for every \( j \), i.e. \( N' \subset N_{s,\hat{k}} \).

This implies \( q = \dim_{\hat{k}} N' \leq \dim_{\hat{k}} N_{s,\hat{k}} = r \) and completes the proof of the lemma. \( \square \)

Now let \( B \) be one of the rings \( B_j \) that appear in Corollary 2.4. Then \( B \) is complete, local, contains a field of characteristic \( p > 0 \), has a separably closed residue field, is reduced, equidimensional, \( d \)-dimensional and the graph \( \Gamma_B \) is connected. Hence it has an \( S_2 \)-ification which is a local ring \( S \) and by Proposition 4.3, \( \lambda_{d,d}(B) = \lambda_{d,d}(S) \). Now we appeal to a result of Kawasaki \[ 9 \] Sec 3, Prop.1] to the effect that if \( S \) is a \( d \)-dimensional \( S_2 \) local ring that admits a surjection from a regular local ring containing a field, then \( \lambda_{d,d}(S) = 1 \) (see Proposition 4.5 below). This implies that \( \lambda_{d,d}(B_j) = 1 \) for every \( j \), so Theorem 1.3 now follows from Corollary 2.4. \( \square \)

Finally, following a referee’s suggestion, we include, for the reader’s convenience, a proof of Kawasaki’s result mentioned in the preceding paragraph.

**Proposition 4.5.** \[ 9 \] Sec 3, Prop.1] Let \( S \) be a \( d \)-dimensional \( S_2 \) local ring that admits a surjection from a regular local ring \( B \) containing a field. Then \( \lambda_{d,d}(S) = 1 \).

**Proof.** Let \( B \to S \) be the surjection in question and let \( I \) be its kernel. Let \( \mathfrak{m} \) be the maximal ideal of \( B \). The composition of functors \( \Gamma_\mathfrak{m}(-) = \Gamma_\mathfrak{m}(\Gamma_I(-)) \) leads to the spectral sequence

\[
E_2^{p,q} = H_{\mathfrak{m}}^p(H_I^q(B)) \Rightarrow H_{\mathfrak{m}}^{p+q}(B).
\]

Let \( n \) be the dimension of \( B \). Since \( B \) is regular, \( H_{\mathfrak{m}}^n(B) \cong E \) where \( E \) is the injective hull of the residue field of \( B \) in the category of \( B \)-modules.
According to Lemma 2.2, all we need to show is that $H^d_m(H^l_{I^d}(B)) \cong E$. But $H^d_m(H^{n-d}_I(B)) \cong E$ and the abutment in total degree $n$ is $H^n_m(B) \cong E$. Hence it is enough to prove that all differentials going out of and coming into $E^r_{n-d}$ are zero for $r \geq 2$, and all terms $E^p_q$ with $p+q = n$ vanish unless $p = d$ and $q = n-d$. For this would imply that $H^d_m(H^{n-d}_I(B))$ is isomorphic to the abutment which is $E$.

The outgoing differentials $d_r : E^d_{r,n-d} \to E^d_{r+n-d-r+1}$ are zero for all $r \geq 2$ since the target module $E^d_{r+n-d-r+1}$ is a subquotient of $E^d_{2r,n-d-r+1}$, and the abutment in total degree $n$ is $E$. Indeed, this target module is a subquotient of $E^d_{2r,n-d-r+1}$, so it is enough to prove that the dimension of the support of $E^d_{2r,n-d-r+1}$ is $\geq d + r - 1$, while the dimension of $E^d_{2r,n-d-r+1}$ is $\geq d + r - 2$ (the latter inequality holds because $r \geq 2$). Finally, if $\delta = d - r$, then the dimension of $E^d_{2r,n-d-r+1}$ is $n - d$. If $\delta = d - r + 1$, then the dimension of $E^d_{2r,n-d-r+1}$ is $n - d - 1$, hence $H^n_{I_P}(B_P) = 0$. Since $A = B/I$ is $S_2$ and catenary, it is equidimensional, hence the height of $IB_P$ is $n - d$. If $\delta = d - r$, then the dimension of $E^d_{2r,n-d-r+1}$ is $n - d - 1$, and $H^n_{I_P}(B_P) = 0$ by the Hartshorne-Lichtenbaum local vanishing theorem [5, 3.1] considering that the height of $\hat{I}_{\hat{B}_P}$, where $\hat{B}_P$ is the completion of the regular local ring $B_P$ with respect to its maximal ideal, is $n - d$, while the dimension of $\hat{B}_P$ is $n - d + r - 1 > n - d$ (the latter inequality holds because $r \geq 2$). Finally, if $\delta = d - r$, then the dimension of $\hat{B}_P$ is $n - d + r \geq n - d + 2$, the height of $I_P$ is $n - d$, hence the dimension of $\hat{B}_P/I_P$ is $\geq 2$. The fact that $A = B/I$ is $S_2$ implies that the depth of $\hat{B}_P/I_P$ is $\geq 2$. Since $\hat{B}_P/I_P$ is the completion of $\hat{B}_P/I_P$, it is faithfully flat over $B_P/I_P$, the depth of $\hat{B}_P/I_P$ is $\geq 2$, which by [11, 2.1] implies that the punctured spectrum of $\hat{B}_P/I_P$ is connected. These two facts, namely, $\dim B_P/I_P \geq 2$ and the connectedness of the punctured spectrum of $\hat{B}_P/I_P$, imply that $H^{n-d-r+1}_{I_P}(B_P) = 0$ by [13, Cor. 2.11], [14, III, 5.5], [7, 2.9] (the first two of these three references prove this result in characteristic 0 and $p > 0$ respectively, while the third one gives a characteristic-free proof). All of this shows that if $\delta \geq d - r$, then $H^{n-d-r+1}_{I_P}(B_P) = 0$, i.e. the dimension
of the support of $H_{I}^{n-d+r-1}(B)$ is less than $d - r$. This completes the proof that all the incoming differentials are indeed zero.

It remains to show that if $p + q = n$, then $E_{2}^{p,q} = 0$ unless $p = d$ and $q = n - d$. Indeed, if $q < n - d$, then $H_{I}^{q}(B) = 0$ since $B$ is regular and $q$ is smaller than the height of every minimal prime over $I$, which is $n - d$. Hence $E_{2}^{p,q} = H_{m}^{p}(H_{I}^{q}(B)) = 0$ in this case. If $q > n - d$, i.e. $q = n - d + r - 1$ for some $r \geq 2$, it has been shown in the preceding paragraph that the dimension of the support of $H_{I}^{q}(B)$ is less than $d - r = n - q - 1 = p - 1$, hence $E_{2}^{p,q} = H_{m}^{p}(H_{I}^{q}(B)) = 0$. $\square$

**References**

[1] M. Blickle and R. Bondud, *Local cohomology multiplicities in positive characteristic*, preprint, (2004).

[2] R. Garcia Lopez and C. Sabbah, *Topological computation of local cohomology multiplicities*, Dedicated to the memory of Fernando Serrano. Collect. Math. 49 (1998), no. 2-3, 317–324.

[3] A. Grothendieck, Local cohomology, Lecture Notes in Mathematics, v. 41, Springer-Verlag, Heidelberg, 1967.

[4] R. Hartshorne, Complete intersections and connectedness, Amer. J. Math. 84 497–508 (1962).

[5] R. Hartshorne, Cohomological Dimension of Algebraic Varieties, Ann. Math., 88 403-450 (1968).

[6] M. Hochster and C. Huneke, Indecomposable canonical modules and connectedness, Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992), 197–208, Contemp. Math., 159, Amer. Math. Soc., Providence, RI, 1994.

[7] C. Huneke and G. Lyubeznik, On the vanishing of local cohomology modules, Invent. Math. 102 (1990), no. 1, 73–93.

[8] K.-I. Kawasaki, On the Lyubeznik number of local cohomology modules, Bull. Nara Univ. Ed. Natur. Sci. 49 (2000) no. 2, 5-7.

[9] K.-I. Kawasaki, On the highest Lyubeznik number, Math. Proc. Camb. Phil. Soc., 132 (2002), no. 3, 409–417.

[10] G. Lyubeznik, Finiteness properties of local cohomology modules (an application of $D$-modules to commutative algebra) Invent. Math. 113 (1993), no. 1, 41–55.

[11] G. Lyubeznik, $F$-modules: applications to local cohomology and $D$-modules in characteristic $p > 0$, J. reine angew. Math. 491 (1997), 65 - 130.

[12] G. Lyubeznik, A partial survey of local cohomology, Local cohomology and its applications (Guanajuato, 1999), 121–154, Lecture Notes in Pure and Appl. Math., 226, Marcel Dekker, Inc., New York, 2002.

[13] A. Ogus, Local Cohomological Dimension of Algebraic Varieties, Ann. Math., 98 327-365 (1973).

[14] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale. Applications la demonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck, Inst. Hautes tudes Sci. Publ. Math. No. 42 (1973), 47–119.

[15] U. Walther, On the Lyubeznik numbers of a local ring, Proceedings of the AMS, 129 (6) (2001) 1631-1634.

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