POSITIVITY OF ZERO–REGULAR BUNDLES, CONTINUOUS CM–REGULARITY,
AND GENERIC VANISHING

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ABSTRACT. The purpose of this article is twofold. Firstly, we study the positivity properties of 0–regular bundles (in the sense of Castelnuovo–Mumford) on a polarized smooth projective variety \((X,\mathcal{O}_X(1))\). Secondly, we study the continuous CM–regularity of torsion–free coherent sheaves on \((X,\mathcal{O}_X(1))\), and its relation with the theory of generic vanishing. This continuous variant of CM–regularity was introduced by Mustopa, and he raised the question whether a continuously 1–regular such sheaf \(\mathcal{F}\) is GV. Here we answer the question in the affirmative for many pairs \((X,\mathcal{O}_X(1))\) which includes the case of any polarized abelian variety. Moreover, for these pairs, we show that if \(\mathcal{F}\) is continuously \(k\)–regular for some integer \(1 \leq k \leq \text{dim } X\), then \(\mathcal{F}\) is a GV\(_{(k-1)}\) sheaf. Further, we extend the notion of continuous CM–regularity to a real valued function on the \(\mathbb{Q}\)–twisted bundles on polarized abelian varieties \((X,\mathcal{O}_X(1))\), and we show that this function can be extended to a continuous function on \(N^2(X)\). We also provide syzygetic consequences of our results for \(\mathcal{O}_{\mathbb{P}^1}(1)\) on \(\mathbb{P}^1\) associated to a 0–regular bundle \(\mathcal{E}\) on polarized smooth projective varieties. In particular, when \(X\) is abelian, we show that \(\mathcal{O}_{\mathbb{P}^1}(1)\) satisfies \(N_p\) property if the base–point freeness threshold of the class of \(\mathcal{O}_X(1)\) in \(N^1(X)\) is less than \(\frac{1}{\text{dim } X}\). This result is obtained using a theorem in the Appendix A written by Atsushi Ito.

1. Introduction

Given a smooth projective variety \(X \subseteq \mathbb{P}^n\), it is well–known that the geometry of the embedding is reflected by the coherent sheaves on \(X\) with suitable positivity properties. One of the most fundamental notion of such properties that governs the complexity of a sheaf \(\mathcal{F}\) is given by its Castelnuovo–Mumford regularity with respect to the pair \((X,\mathcal{O}_X(1))\). In particular, a result of Mumford shows that a 0–regular coherent sheaf is globally generated. Among the interesting classes of 0–regular bundles on \(X\), one particular class, namely the class of Ulrich bundles, has been the focus of considerable activities in recent years. The interests centered around these bundles are partly due to their rich interactions with several topics in commutative algebra and in algebraic geometry (see [Bea18],[CMP21],[ESW03] and the references therein for a detailed overview).

1.1. Positivity of zero–regular bundles. The main motivation for the first topic of this article comes from the study of the positivity of Ulrich bundles that was initiated in [Lop21a]. Since an Ulrich bundle \(\mathcal{E}\) on \(X \subseteq \mathbb{P}^n\) is 0–regular by definitions, it is globally generated. In loc. cit., Lopez studied the positivity of \(c_1(\mathcal{E})\) for an Ulrich bundle \(\mathcal{E}\), and the following theorem was established by him which shows that the positivity of \(c_1(\mathcal{E})\) for a 0–regular bundle \(\mathcal{E}\) depends on the projective geometry of \(X\).

**Theorem 1.1.** ([Lop21a] Theorem 7.2) Let \((X, H)\) be a smooth polarized projective variety of dimension \(n\) with \(H\) very ample. Also, let \(\mathcal{E}\) be a 0–regular bundle of rank \(r\) for \((X, tH)\) for some \(t \geq 1\). Then for every \(x \in X\) and for every subvariety \(Z \subseteq X\) of dimension \(k\) passing through \(x\), the following inequality holds

\[
c_1(\mathcal{E})^k \cdot Z \geq t^k \text{mult}_x(Z).
\]

if either \(t \geq 2\) or if \(t = 1\) and \(x\) does not lie on a line contained in \(\varphi_H : X \to \mathbb{P}^{h^0(H)-1}\).

In [LS21], Lopez extended the above result with Sierra for Ulrich bundles. Given a globally generated vector bundle \(\mathcal{E}\) of rank \(r\) on \(X\), there are two natural maps governing the geometry of \(X\), namely \(\Phi_{\mathcal{E}} : X \to \mathbb{P}(r-1, \mathbb{P} H^0(\mathcal{E}))\) and \(\varphi_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \to \mathbb{P} H^0(\mathcal{E})\). The following is the precise result of Lopez–Sierra.

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Theorem 1.2. ([LS21] Theorem 1) Let $X \subseteq \mathbb{P}^N$ be a smooth projective variety and let $\mathcal{E}$ be an Ulrich bundle on $X$. Then we have the following equivalences.

1. $\mathcal{E}$ is very ample (i.e., $\varphi_\mathcal{E}$ is an embedding) $\iff$ $\mathcal{E}$ is ample $\iff$ either $X$ does not contain lines or $\mathcal{E}|_L$ is ample for any line $L$ contained in $X$.

2. $\varphi_\mathcal{E}$ is an embedding $\iff$ $\text{det}(\mathcal{E})$ is very ample $\iff$ $\text{det}(\mathcal{E})$ is ample $\iff$ either $X$ does not contain lines or $\mathcal{E}|_L$ is non-trivial for any line $L$ contained in $X$.

Inspired by these results, here we investigate how the positivity of a 0–regular bundle changes as the polarization becomes more positive. The first result along this direction that we observe is the following.

Proposition A. (= Proposition 2.6, Corollary 2.8) Let $(X, H)$ be a smooth polarized projective variety of dimension $n$. Also, let $\mathcal{E}$ be a 0–regular vector bundle for $(X, H)$. Then the following statements hold.

1. If $H$ is $t$–jet (resp. $t$–very) ample for some $t \geq 1$ then $\mathcal{E}$ is $(t – 1)$–jet (resp. $(t – 1)$–very) ample.

2. If $H$ is $t$–jet ample for some $t \geq 1$, then for every $x \in X$ and for every subvariety $Z \subseteq X$ of dimension $k$ passing through $x$, the following inequality holds where $r := \text{rank}(\mathcal{E})$

$$c_1(\mathcal{E})^k \cdot Z \geq r^k(t – 1)^k \text{mult}_{\text{cayley–chow}}(Z).$$

A word about our convention: when we say $(X, H)$ is polarized, we only mean that the line bundle $H$ is ample. We refer to §2 for the definitions of $k$–very ampleness and $k$–jet ampleness.

For a globally generated bundle $\mathcal{E}$, if we set $\text{amp}_v(\mathcal{E})$ to be the maximum $k$ such that $\mathcal{E}$ is $k$–very ample, then the result above gives the inequality $\text{amp}_v(\mathcal{E}) – \text{amp}_v(\mathcal{O}_X(1)) \geq – 1$ for 0–regular bundles $\mathcal{E}$ on $X \subseteq \mathbb{P}^N$. We also observe in Proposition 2.9 that an Ulrich bundle $\mathcal{E}$ on $X \subseteq \mathbb{P}^N$ satisfies the positivity upper bound $\text{amp}_v(\mathcal{E}) \leq \text{deg}(X)$. However, there are examples which shows that even if we assume $\mathcal{E}$ is Ulrich, the difference $\text{amp}_v(\mathcal{E}) – \text{amp}_v(\mathcal{O}_X(1))$ can be arbitrarily large. We produce these examples in Corollary 2.12 by studying Ulrich line bundles on bielliptic surfaces in Proposition 2.10. This result is of independent interest as the existence of Ulrich line bundles implies the associated Cayley–Chow form of $X$ is linear determinantal.

Next we turn to the study of the positivity of 0–regular bundles for divisible polarizations. In this case, perhaps the first question to ask is: given a 0–regular coherent sheaf $\mathcal{F}$ on $(X, H_1 + H_2)$, whether $\mathcal{F}(–H_1)$ is 0–regular for $(X, H_2)$. In Corollary 2.15, we prove a more general statement which answers the above question in the affirmative, and our proof uses a result of Totaro ([Tot13] Lemma 3.2) where he shows that partial CM–regularity behaves the same way as the usual CM–regularity. In particular, he shows that if $\mathcal{F}$ is $q$–partially $k$–regular, then it is $q$–partially $(k + m)$–regular for any $m \geq 0$ (see §2 for details). Corollary 2.15 and Proposition A, combined with Theorem 1.1 yield the following consequence.

Theorem B. Let $X$ be a smooth projective variety and let $H$ be an ample and globally generated line bundle on $X$. Also, let $\mathcal{E}$ be a 0–regular vector bundle for $(X, tH)$ of rank $r$ for some $t \in \mathbb{N}$, $t \geq 1$. Then $\mathcal{E}(–(t – 1)H)$ is 0–regular for $(X, H)$; in particular it is globally generated. Moreover, if $H$ is $s$–jet ample for some $s \in \mathbb{N}$, $s \geq 1$, then the following statements hold.

1. The vector bundle $\mathcal{E}$ is $(st – 1)$–jet ample.

2. For every $x \in X$ and for every subvariety $Z \subseteq X$ of dimension $k$ passing through $x$,

   (A) if $s \geq 2$ then $c_1(\mathcal{E})^k \cdot Z \geq r^k(s – 1)^k \text{mult}_{\text{cayley–chow}}(Z) + r^k(t – 1)^k H^k Z$,

   (B) if $s = 1$, then the following holds if $x$ does not lie on a line contained in $\varphi_H : X \xrightarrow{|H|} \mathbb{P}^h(\mathcal{H}) – 1$

   $$c_1(\mathcal{E})^k \cdot Z \geq r^k \text{mult}_{\text{cayley–chow}}(Z) + r^k(t – 1)^k H^k Z.$$ 

1.2. Continuous CM–regularity and generic vanishing. Now we turn to our second topic of study, which is continuous CM–regularity and generic vanishing. This variant of CM–regularity was introduced by Mustopa in [Mus16] for polarized irregular varieties $(X, \mathcal{O}_X(1))$. It is defined as follows: $\mathcal{F}$ is continuously $k$–regular if the cohomological support loci $V^i(\mathcal{F}(k – i)) \neq \phi$ for $i \geq 1$ (strictly speaking, this definition is slightly more general from that in [Mus16] as we are not assuming global generation of $\mathcal{O}_X(1)$). The
structures of these cohomological support loci are of great importance in the study of the geometry of irregular varieties. An important notion in this topic is the notion of generic vanishing (GV for short) introduced by Green and Lazarsfeld in the pioneering works [GL87], [GL91]. Fundamental contributions from Hacon ([Hac04]), Pareschi–Popa ([PP03] – [PP11b]), and Debarre ([Deb06]) through derived category approach and Fourier–Mukai transform functors led to subsequent developments of the theory of generic vanishing.

Turning to details, a coherent sheaf $\mathcal{F}$ on $X$ is said to be GV if $\text{codim} V^i(\mathcal{F}) \geq i$ for all $i$. This property is intimately related to the positivity of $\mathcal{F}$, in particular, a GV sheaf on an abelian variety is nef. In [Mus16], Mustopa asked the following question.

**Question 1.3.** ([Mus16], Question (∗)) Let $X$ be a smooth projective variety of dimension $d \geq 1$ and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on $X$. Let $\mathcal{F}$ be a torsion–free coherent sheaf on $X$. If $\mathcal{F}$ is continuously $1$–regular for $(X, \mathcal{O}_X(1))$, is $\mathcal{F}$ a GV sheaf?

The question above was motivated by Beauville’s construction in [Bea16] of rank 2 Ulrich bundles on abelian surfaces $(X, \mathcal{O}_X(1))$. It turns out that for these bundles $\mathcal{E}$, $\mathcal{E}(-1)$ is indeed GV. The answer of Question 1.3 is affirmative for polarized curves. It was shown in [Mus16] that the answer of the question is also affirmative for

− a large class of polarized surfaces that includes the case of any polarized abelian surface (loc. cit. Theorem B, Corollary C),
− certain polarizations on Cartesian and symmetric products of curves (loc. cit. Propositions 3.1, 3.2),
− some scrollar embeddings of ruled threefolds over a curve (loc. cit. Proposition 4.1).

Continuous CM–regularity for semihomogeneous bundles on abelian varieties has been studied by Küronya and Mustopa in [KM20], and later by Grieve in [Gri21]. In particular, Küronya and Mustopa showed in [KM20], that if $\mathcal{E}$ is a semihomogeneous bundle on abelian variety $(X, \mathcal{O}_X(1))$ of dimension $g$ as in the set–up of Question 1.3, and moreover if $c_1(\mathcal{E})$ is a rational multiple of $c_1(\mathcal{O}_X(1))$, then even more is true, namely $\mathcal{E}(1 – g)$ is GV. In [Gri21], Grieve established a description of continuous CM–regularity of semihomogeneous bundles on abelian varieties. This description was in terms of a normalized polynomial function studied in [Gri17], and obtained via the Wedderburn decomposition of the endomorphism algebra of the abelian variety. A further study of index and generic vanishing theory of simple semihomogeneous bundles was also carried out in [Gri21] building on and refining [Gri14a], [Gri14b].

Beside the notion of GV sheaves, a related notion in the generic vanishing theory is that of $M$–regularity of coherent sheaves. A coherent sheaf $\mathcal{F}$ on $X$ is said to be $M$–regular if $\text{codim} V^i(\mathcal{F}) > i$ for all $i > 0$. In this direction, Mustopa raised the following question.

**Question 1.4.** ([Mus16], Remark 1.6) Let $X$ be a smooth projective variety, and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on $X$. Let $\mathcal{F}$ be a torsion–free coherent sheaf on $X$. If $\mathcal{F}$ is continuously $0$–regular, is $\mathcal{F}$ an $M$–regular sheaf?

Following this train of thoughts, it is natural to propose the more general question: given a continuously $k$–torsion–free coherent sheaf on a smooth polarized variety $(X, \mathcal{O}_X(1))$ with $\mathcal{O}_X(1)$ globally generated and $1 \leq k \leq \dim X$, is it true that $\text{codim} V^i(\mathcal{F}) \geq i – k + 1$ for all $i$? Here we remark that the notion of generic vanishing was generalized in [PP11b] where Pareschi–Popa called a sheaf $\mathcal{F}$ $GV_{-k}$ for an integer $k \geq 0$, if $\text{codim}(V^i(\mathcal{F})) \geq i – k$ for all $i$. In view of this, here we will devote ourselves to find the answer of the following question.

**Question 1.5.** Let $X$ be a smooth projective variety of dimension $d \geq 1$ and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on $X$. Let $\mathcal{F}$ be a torsion–free coherent sheaf on $X$. Assume $\mathcal{F}$ is continuously $k$–regular for $(X, \mathcal{O}_X(1))$ for some integer $0 \leq k \leq d$.

1. If $1 \leq k \leq d$, is $\mathcal{F}$ a $GV_{-k}$ sheaf?
2. If $k = 0$, is $\mathcal{F}$ an $M$–regular sheaf?
The following is the main result of this article that answers the above question in the affirmative for many pairs \((X, \mathcal{O}_X(1))\). We also note that the result below does not require the hypothesis that the polarization \(\mathcal{O}_X(1)\) is globally generated.

**Theorem C.** Let \((X, H)\) be a polarized smooth projective variety. Assume that there exists a globally generated line bundle \(H_1\) on \(X\) and an ample line bundle \(H_2\) on \(\text{Alb}(X)\) such that \(H = H_1 + \text{alb}_X^* H_2\), where \(\text{alb}_X : X \rightarrow \text{Alb}(X)\) is the Albanese map. Let \(\mathcal{F}\) be a torsion-free coherent sheaf on \(X\) that is continuously \(k\)-regular for \((X, H)\) for some integer \(0 \leq k \leq \dim X\). Then the following statements hold.

1. \(V^i(\mathcal{F}) = \phi\) for \(i \geq k + 1\).
2. If \(k \neq 0\), then \(\text{codim}(V^k(\mathcal{F})) \geq 1\).

In particular, the answer of Question 1.5 is affirmative for the pair \((X, H)\).

Note that the above result answers Question 1.5 in the affirmative for any polarized abelian variety. Moreover, it also shows that the question has an affirmative answer in many cases that include

- \(X = Y \times A\) where \(Y\) is a regular smooth projective variety, \(A\) is an abelian variety, and \(\mathcal{O}_X(1)\) is ample and globally generated;
- \(X\) is a projective bundle \(\mathcal{P}(\mathcal{E})\) on an abelian variety \(A\) associated to an ample and globally generated vector bundle \(\mathcal{E}\), and \(\mathcal{O}_X(1) = T + F\) where \(T\) is the tautological bundle, and \(F\) is the pull-back of any ample line bundle on \(A\).

We will give several other variants of Theorem C in §4.3.

The proof of Theorem C is homological in nature and relies on an inductive argument, but the crucial ingredient of the proof is a relative set-up of continuous CM-regularity that we develop in this article.

In §4.4, we also give an alternate approach via Fourier–Mukai transforms to prove the generic vanishing of \(\mathcal{F}\) when we additionally assume that \(\text{alb}_X\) is finite onto its image in the statement of Theorem C. Let us briefly describe the argument when \(\mathcal{F}\) is locally free, \(k = 1\), and \(H_1 = \mathcal{O}_X\) as this is the argument that connects the two parts of this article. Given a morphism \(a : X \rightarrow A\) to an abelian variety \(A\), we call a sheaf \(\mathcal{F}\) \(a\)-continuously \(k\)-regular if the relative cohomological support loci \(V^i_a(\mathcal{F}((k - i)H) \neq \phi\) for all \(i \geq 1\).

We use the covering trick introduced by Pardini in [Par93]. More precisely, we set \(A := \text{Alb}(X)\), \(a := \text{alb}_X\), and let \(\hat{A}\) be the dual abelian variety. Let \(L\) be an ample line bundle on \(\hat{A}\) inducing the isogeny \(\varphi_L : \hat{A} \rightarrow A\) sending \(y \in \hat{A}\) to \(t_y^* L \otimes t_y^{* -1}\) where \(t_y : \hat{A} \rightarrow \hat{A}\) is the translation by \(y\). Consider the commutative diagram with Cartesian squares

\[
\begin{array}{ccc}
\hat{X}_n & \xrightarrow{\varphi_n} & X \\
\downarrow a_n & & \downarrow a \\
\hat{A} & \xrightarrow{\varphi_L} & A
\end{array}
\]

where \(n\hat{\lambda}\) is the multiplication map by \(n\). Since \(\mathcal{F}(H)\) is \(a\)-continuously \(0\)-regular by assumption, we observe in Lemma 3.8 that \(\varphi_n^*(\mathcal{F}(H)) = \mu_n^*(\hat{\mathcal{F}}(H))\) is \(a_n\)-continuously \(0\)-regular for \((\hat{X}_n, \mu_n^* H)\) where \(\hat{\mathcal{F}} := \varphi^* \mathcal{F}\) and \(\hat{H} := \varphi^* H\). Recall that for any line bundle \(L'\) on \(\hat{A}\), \(n^* A' = n^2 L' + \zeta L'\) for some \(\zeta L' \in \text{Pic}^0(\hat{A})\).

It is easy to see that this implies that there exists \(\zeta \in \text{Pic}^0(\hat{A})\) such that \(\mu_n^* \hat{\mathcal{F}} \otimes a_n^*(n^2 \hat{H}_2 + \zeta)\) is \(0\)-regular (in the usual sense) for \((\hat{X}_n, n^2 a_n^* \hat{H}_2)\) where \(\hat{H}_2 := \varphi_L^* H_2\). By the first assertion of Theorem B, we deduce that when \(n\) is \(2\)-divisible,

\[
\mu_n^* \hat{\mathcal{F}} \otimes a_n^*(n^2 \hat{H}_2 + \zeta) - \left(\frac{n^2}{2} - 1\right) a_n^*(2 \hat{H}_2) = \mu_n^* \hat{\mathcal{F}} \otimes a_n^*(2 \hat{H}_2 + \zeta)
\]

is \(0\)-regular for \((\hat{X}_n, a_n^*(2 \hat{H}_2))\). The key property of \(a_n^*(2 \hat{H}_2)\) is that for any \(\zeta' \in \text{Pic}^0(\hat{A})\), \(a_n^*(2 \hat{H}_2 + \zeta')\) is globally generated. We show in Proposition 4.7 that this ensures \(V^i_{a_n}(\mu_n^* \hat{\mathcal{F}} \otimes a_n^*(2 \hat{H}_2)) = \phi\) for all \(i \geq 1\). In other words, \((a_n)_* (\mu_n^* \hat{\mathcal{F}}) \otimes (2 \hat{H}_2)\) is IT_0 (see §3 for relevant definitions) as \(a_n\) is also finite onto its image. Thus for \(n \gg 0\), \(n^2 L - 2 \hat{H}_2\) will be ample, and consequently by a preservation of vanishing statement of

 spécialité.
Pareschi–Popa, we deduce that $a_{n+}(\mu_n^*(\mathcal{F} \otimes \mathcal{A}^* L))$ is IT$_0$, whence $H^i(\mu_n^*(\mathcal{F} \otimes \mathcal{A}^* L)) = 0$ for all $i \geq 1$ which in turn implies by the projection formula that $H^i(\mathcal{F} \otimes \mathcal{A}^* L) = 0$ for all $i \geq 1$. The fact that $\mathcal{F}$ is GV follows from here by an application of the works of Hacon, and Pareschi–Popa.

We mention the immediate consequences of Theorem C for continuously $k$–regular torsion–free coherent sheaves $\mathcal{F}$ on polarized abelian varieties $(X, \mathcal{O}_X(1))$. It follows immediately that

- (Corollary 4.16) if $k = 1$, then $\chi(\mathcal{F}) \geq 0$ with equality if and only if $V^0(\mathcal{F})$ is a divisor,
- (Corollary 4.19) if $k = 1$ then $\mathcal{F}$ is nef, and if $k = 0$ then it is ample.

We also show in Corollary 4.21 that for any torsion–free coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on a polarized abelian variety $(X, \mathcal{O}_X(1))$, the following inequality holds

$$
\text{reg}_{\mathcal{O}_X(1)}(\mathcal{E} \otimes \mathcal{F}) \leq \text{reg}_{\mathcal{O}_X(1)}(\mathcal{E}) + \text{reg}_{\mathcal{O}_X(1)}(\mathcal{F})
$$

as soon as one of the sheaves is locally free. Here $\text{reg}_{\mathcal{O}_X(1)}(\mathcal{H})$ stands for the (usual) CM–regularity. We remark that it was shown earlier by Totaro in \cite{Totaro} that \eqref{1.1} holds for arbitrary polarized smooth projective variety when $\mathcal{O}_X(1)$ is very ample and satisfies a certain Koszul hypothesis (see Theorem 6.8).

Finally, inspired by the recent development of the cohomological rank function by Jiang–Pareschi (see \cite{JP20}), we extend the notion of continuous CM–regularity to define a real–valued regularity $Q$-\text{reg}_{\mathcal{O}_X(1)}(\mathcal{H}(\mathcal{J}))$ for $Q$–twisted bundles $\mathcal{H}(\mathcal{J})$ where $\mathcal{J}$ is an ample class in $N^1(X)$ on an abelian variety $X$, and $\mathcal{J} \in N^1(X)$.

For them, we prove the following

**Theorem D.** Let $X$ be an abelian variety of dimension $g$, and let $\mathcal{E}$ be a vector bundle on $X$. If $f \in N^1(X)$ is a polarization, then the function $Q$-\text{reg}_{\mathcal{O}_X(1)}(\mathcal{E}(\mathcal{J})) : N^1(X)_{\mathbb{Q}} \to \mathbb{R}$ sending $N^1(X)_{\mathbb{Q}} \ni \mathcal{J} \mapsto Q$-\text{reg}_{\mathcal{O}_X(1)}(\mathcal{E}(\mathcal{J})) can be extended to a continuous function $\mathbb{R}$-\text{reg}_{\mathcal{O}_X(1)}(\mathcal{E}(\mathcal{J})) : N^1(X)_{\mathbb{R}} \to \mathbb{R}.$

**1.3. Application: syzygies of projective bundles.** We apply the results proven thus far to study the syzygies of $\mathcal{O}_{P(\mathcal{E})}(1)$ for a $0$–regular bundle $\mathcal{E}$ on $(X, tH)$. Given a polarized variety $(X, H)$, a resolution of the diagonal $\Delta \subset X \times X$ was constructed by Orlov, Kawamata, and Totaro (see \cite{Orlov97, Kawamata04, Totaro}) under an assumption on the Koszul property of the section ring $R(H) := \oplus_{q \geq 0} H^0(X, qH)$. Using this, we prove a criterion for surjectivity of the multiplication maps in Lemma 6.9. Combining that with the regularity inequality of Totaro that we mentioned earlier (c.f. Theorem 6.8), we prove the following result (see §6 for the relevant definitions).

**Theorem E.** Let $X$ be a smooth projective variety of dimension $n$ with a very ample line bundle $H$. Assume for some $p \geq 0$, $t \geq p + 1$ and $R(\mathcal{H})$ is $3n$–Koszul for all $1 \leq k \leq p + 1$. Let $\mathcal{E}$ be an ample and $0$–regular vector bundle for $(X, tH)$.

Then $\mathcal{O}_{P(\mathcal{E})}(1)$ satisfies $N_p$ property.

We also deduce syzygetic consequences from Theorem C for projective bundles on abelian varieties that are analogous to Theorem E but do not require any Koszul assumption. Before stating our result, we spend a few words on linear series on abelian varieties to set the context.

It was a conjecture of Lazarsfeld that on a polarized abelian variety $(X, H)$, $tH$ satisfies $N_p$ property if $t \geq p + 3$. Lazarsfeld conjecture was proven by Pareschi in \cite{Pareschi00}. The result was further extended by Pareschi–Popa in \cite{PP04} where it was shown that $tH$ satisfies $N_p$ property if $t \geq p + 3$ and $|H|$ contains no base–divisor. Further, let us denote the ideal sheaf at the origin of $X$ as $\mathcal{I}_0$, and define

$$r(H) := \inf\{ c \in \mathbb{Q} : \text{there exists an effective Q–divisor } D = cH \text{ such that } \mathcal{I}(X, D) = \mathcal{I}_0 \}$$

where $\mathcal{I}(X, D)$ is the multiplier ideal. Lazarsfeld–Pareschi–Popa proved in \cite{LPP11} that if $r(H) < \frac{1}{p+2}$ then $H$ satisfies $N_p$ property. Very recently, Jiang–Pareschi defined in \cite{JP20} the base–point freeness threshold $0 < \beta(h) \leq 1$ for a polarization $h \in N^1(X)$ on an abelian variety $X$ that can be characterized as follows: $\beta(h) < \frac{1}{p+2}$ for all $h \in N^1(X)$ on an abelian variety $X$ that can be characterized as follows: $\beta(h) < x \iff \mathcal{I}_0(xH)$ is IT$_0$ for $x \in \mathbb{Q}$. It was shown by Caucci in \cite{Caucci20} that $\beta(h) \leq r(H)$, and further the above results are unified to shown that $H$ satisfies $N_p$ property if $\beta(h) < \frac{1}{p+2}$. The study of a more general property $N'_p$ via base–point freeness threshold was carried out by Ito in \cite{Ito22a}. Sharp results on
projective normality and higher syzygies of general polarized abelian varieties were also established by Ito in [Ito21] and in [Ito22b].

The following is our result on syzygies of projective bundles associated to continuously 0–regular vector bundles on polarized abelian varieties \((X, H)\), which is an immediate consequence of Theorem C and Theorem A.1 in the Appendix A which is due to Atsushi Ito.

**Corollary F.** Let \((X, H)\) be a polarized abelian variety. Also let \(E\) be a continuously 0–regular vector bundle for \((X, H)\). Then \(\Theta_{\mathcal{O}(E)}(1)\) satisfies \(N_p\) property if \(\beta(h) < \frac{1}{p+2}\) where \(h\) is the class of \(H\) in \(N^1(X)\).

1.4. **Organization.** The structure of this article can be summarized as follows: in §2, we study the positivity of zero–regular bundles. In this section, we first prove Proposition A, then we discuss the case of Ulrich bundles. Later we study partial CM–regularity and prove Theorem B. In §3, we recall the preliminaries of continuous regularity. In §4 we first discuss the theory of generic vanishing in §4.1, and then we proceed to prove Theorem C. In §4.3 we prove a few variants of Theorem C, in §4.4 we give the alternate approach to prove generic vanishing when the Albanese map is finite onto its image, and in §4.5 we prove Corollaries 4.16, 4.19, 4.21. We define the Q CM–regularity §5 and prove Theorem D. Finally we deduce the syzygetic consequences of our results, i.e., Theorem E, and Corollary F in §6.

1.5. **Conventions.** We work over the field of complex numbers \(\mathbb{C}\). We use the additive and multiplicative notation interchangeably for tensor products of line bundles. The sign “\(\equiv\)” will be used for numerical equivalence and the sign “\(\sim\)” will be used for linear equivalence.

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2. **Positivity of zero–regular bundles: general results**

2.1. **Preliminaries.** We recall a few definitions that we will be using throughout the rest of this article.

2.1.1. The first one is the extension of the usual Castelnuovo–Mumford regularity (cf. [Laz04a] Definition 1.8.4 or [MS18] Section 5.4, and [Tot13] Lemma 3.2).

**Definition 2.1.** (CM–regularity) Let \(X\) be a smooth projective variety and let \(H\) be a line bundle on \(X\). Also, let \(q, k\) be integers with \(q \geq 0\). A coherent sheaf \(\mathcal{F}\) on \(X\) is called \(C_{q,k}\) for \((X, H)\) if \(H^{q+i}(\mathcal{F}((k-i)H)) = 0\) for all integers \(i \geq 1\). When \(H\) is ample, we will say that \(\mathcal{F}\) is \(k\)–regular for \((X, H)\) for \(k \in \mathbb{Z}\) if it is \(C_{0,k}\) for \((X, H)\).

The following important result in the study of partial regularity was proven by Totaro in [Tot13], and is well–known in the set–up of the usual (i.e., non–partial) Castelnuovo–Mumford regularity.

**Lemma 2.2.** ([Tot13] Lemma 3.2) Let \((X, H)\) be a smooth projective variety with \(H\) globally generated. Let \(\mathcal{F}\) be a coherent sheaf on \(X\), and \(q, k\) be integers with \(q \geq 0\). If \(\mathcal{F}\) is \(C_{q,k}\) for \((X, H)\) then it is \(C_{q,k+m}\) for \((X, H)\) for any integer \(m \geq 0\).

When \(H\) is ample and globally generated, define \(\text{reg}_H(\mathcal{F}) := \min \{m \in \mathbb{Z} \mid \mathcal{F} \text{ is } m\text{–regular for } (X, H)\}\). We record a consequence of Lemma 2.2 below that will later be useful for us.
Corollary 2.3. Let $f: X \to Y$ be a morphism between smooth projective varieties. Let $H'$ be an ample and globally generated line bundle on $Y$, and let $\mathcal{F}$ be a coherent sheaf that is $C_{0,k}$ for $(X, f^* H')$ for some integer $k$. Then $R^j f_* \mathcal{F} = 0$ for all integers $j \geq 1$.

Proof. We follow the proof of Grauert–Riemenschneider vanishing theorem (c.f. [Laz04a] Theorem 4.3.9). Choose $m \gg 0$ such that $m \geq k - 1$, and for all $j \geq 0$, the following two conditions are satisfied: (1) the coherent sheaves $R^j f_* \mathcal{F} \otimes (m H')$ is globally generated if non–zero, and (2) $H^j(R^j f_* \mathcal{F} \otimes (m H')) = 0$ for all $i \geq 1$. Then the Leray spectral sequence

$$E_2^{ij} := H^i(R^j f_* \mathcal{F} \otimes (m H')) \Rightarrow H^{i+j}(\mathcal{F} \otimes f^* (m H'))$$

degenerates to give $H^0(R^j f_* \mathcal{F} \otimes (m H')) = H^j(\mathcal{F} \otimes f^* (m H'))$. But we know that $H^j(\mathcal{F} \otimes f^* (m H')) = 0$ for all $m \geq k - 1$ if $j \geq 1$ by Lemma 2.2. Consequently, $H^0(R^j f_* \mathcal{F} \otimes (m H')) = 0$ for $j \geq 1$ which contradicts the global generation of $R^j f_* \mathcal{F} \otimes (m H')$ if it is non–zero. \hfill \Box

2.1.2. Next we recall the definitions of $k$–very ample line bundles and $k$–jet ampleness. The first one is the extension of the notion of $k$–very ample line bundles defined in [BS88]. For $k$–jet ampleness, see for example [BDS99], Section 1.2.

Definition 2.4. ($k$–very ampleness) Let $\mathcal{E}$ be a vector bundle on a smooth irreducible projective variety $X$. The bundle $\mathcal{E}$ is called $k$–very ample if for any $0$–dimensional subscheme $\xi$ of length $k + 1$, the evaluation map of global sections $H^0(\mathcal{E}) \to H^0(\mathcal{E} \otimes \mathcal{O}_X(\xi))$ surjects. We define the very ampleness of a globally generated bundle $\mathcal{E}$ to be $\text{amp}_v(\mathcal{E}) := \max \{k \text{ such that } \mathcal{E} \text{ is } k\text{–very ample} \}$.

Definition 2.5. ($k$–jet ampleness) Let $\mathcal{E}$ be a vector bundle on a smooth irreducible projective variety $X$. The bundle $\mathcal{E}$ is called $k$–jet ample if for every choice of $t$ distinct points $x_1, \cdots, x_t \in X$ and for every tuple $(k_1, \cdots, k_t)$ of positive integers with $\sum k_i = k + 1$, the evaluation map

$$H^0(\mathcal{E}) \to H^0(|\mathcal{E}_X/m_{x_1}^{k_1} \cdots \otimes m_{x_t}^{k_t}|) = \bigoplus_{i=1}^{t} H^0(|\mathcal{E}_X/m_{x_i}^{k_i}|)$$

surjects where $m_{x_i}$ is the maximal ideal sheaf of $x_i \in X$. We set the jet ampleness of of a globally generated bundle $\mathcal{E}$ to be $\text{amp}_j(\mathcal{E}) := \max \{k \text{ such that } \mathcal{E} \text{ is } k\text{–jet ample} \}$.

A vector bundle is 0–jet ample if and only if it is 0–very ample if and only if it is globally generated. Further, $\text{amp}_v(\mathcal{E}) \geq \text{amp}_j(\mathcal{E})$. Also, a line bundle is 1–jet ample if and only if it is very ample.

2.2. A lower bound on positivity. The next two results prove Proposition 1.1 stated in the introduction.

Proposition 2.6. Let $(X, H)$ be a smooth polarized projective variety of dimension $n$ with $H$ $t$–jet (resp. $t$–very) ample for some $t \geq 1$. Let $\mathcal{E}$ be a $0$–regular vector bundle for $(X, H)$. Then $\mathcal{E}$ is $(t–1)$–jet (resp. $(t–1)$–very) ample.

Proof. We only prove this when $H$ is $t$–jet ample, the other case is similar. Let $x_1, \cdots, x_t \in X$ be distinct points and let $Z = \text{Spec}(\mathcal{O}_X/m_{x_1}^{k_1} \cdots \otimes m_{x_t}^{k_t})$ where each $k_i$ is a positive integer and $\sum k_i = t$. For $1 \leq i \leq n$, let $s_i \in H^0(H)$ be such that $B_i := \langle s_{i1}, \cdots, s_{in} \rangle$ contains $Z$ for all $1 \leq i \leq n$. Since $H$ is $t$–jet ample, for general choices of $s_i$‘s, $Z' := B_1 \cap \cdots \cap B_n$ is 0–dimensional. Setting $V = \text{span}(s_1, \cdots, s_n)$, we have the Koszul resolution of $\mathcal{I}_{Z'/X}$

$$K_s: 0 \to \bigwedge^n V^* \otimes H^{0-n} \to \bigwedge^{n-1} V^* \otimes H^{0-(n-1)} \to \cdots \to V^* \otimes H^{0-1} \to \mathcal{I}_{Z'/X} \to 0.$$

Considering $\mathcal{E} \otimes K_s$, using 0–regularity and [Laz04a], Proposition B.1.1, we obtain $H^1(\mathcal{E} \otimes \mathcal{I}_{Z'/X}) = 0$. Consequently, $H^0(\mathcal{E}) \to H^0(\mathcal{E} \otimes \mathcal{I}_{Z'})$ surjects, whence $H^0(\mathcal{E}) \to H^0(\mathcal{E} \otimes \mathcal{I}_{Z'})$ also surjects. \hfill \Box

Remark 2.7. The above proposition shows that for a 0–regular bundle $\mathcal{E}$ on a polarized smooth projective variety $(X, H)$ with $H$ very ample, we have $\text{amp}_j(\mathcal{E}) \geq \text{amp}_j(H) - 1$, and $\text{amp}_v(\mathcal{E}) \geq \text{amp}_v(H) - 1$. 
Corollary 2.8. Let \((X, H)\) be a smooth polarized projective variety of dimension \(n\) with \(H\) \(t\)-jet ample for some \(t \geq 1\). Also, let \(\mathcal{E}\) be a \(0\)--regular vector bundle for \((X, H)\) of rank \(r\). Then, for every \(x \in X\) and for every subvariety \(Z \subseteq X\) of dimension \(k\) passing through \(x\), the following inequality holds

\[
c_1(\mathcal{E})^k \cdot Z \geq r^k (t - 1)^k \text{mult}_x(Z).
\]

Proof. Let \(x \in X\) and \(Z \subseteq X\) be as in the statement. Let \(\mu_x : \tilde{X} \to X\) be the blow–up of \(X\) at \(x\) and let \(E\) be the exceptional divisor on \(\tilde{X}\). Since \(\mathcal{E}\) is \((t - 1)\)--jet ample by Proposition 2.6, it follows from [BDS99] Proposition 2.7 that \(\mu_x^*\mathcal{E} \big((t - 1)E\big)\) is globally generated. The rest of the proof is exactly the same as [Lop21a] proof of Theorem 7.2, we reproduce it here. Let \(\tilde{Z}\) be the strict transform of \(Z\). We thus have

\[
(\mu_x^*c_1(\mathcal{E}) - r(t - 1)E)^k \cdot \tilde{Z} \geq 0.
\]

On the other hand, we have by [Laz04a] Lemma 5.1.10 that \((E)|_Z\) is \((-1)^{k+1}\text{mult}_x(Z)\), whence

\[
(\mu_x^*c_1(\mathcal{E}) - r(t - 1)E)^k \cdot \tilde{Z} = c_1(\mathcal{E})^k \cdot Z + (-1)^k r(t - 1)^{k+1} \text{mult}_x(Z) = c_1(\mathcal{E})^k \cdot Z - r(t - 1)^k \text{mult}_x(Z).
\]

Combining (2.1) and (2.2) we obtain the desired inequality. \(\square\)

2.3. **The case of Ulrich bundles.** We recall that a vector bundle \(\mathcal{E}\) on a smooth polarized projective variety \((X, H)\) with \(H\) very ample is called **Ulrich** if \(H^*(\mathcal{E} \big(-iH\big)) = 0\) for all \(1 \leq i \leq \dim X\).

2.3.1. Now we show that the positivity of an Ulrich bundle is bounded above by the degree of the variety.

**Proposition 2.9.** Let \((X, H)\) be a polarized smooth projective variety with very ample \(H\) carrying an Ulrich bundle \(\mathcal{E}\). Then \(\text{amp}_{\mu}(\mathcal{E}) \leq \deg(X)\).

Proof. For the sake of contradiction, assume \(\mathcal{E}\) is \(d + 1\) very ample where \(d := \deg(X)\). Let \(Z\) be a general length \(d\) subscheme obtained by intersecting \(n\) general sections of \(H\) where \(n = \dim X\). Choose \(x, x' \in X\) distinct from each other and not contained in the support of \(Z\). Let \(Z_x, Z_{x'}\) and \(Z' = Z \cup \{x\} Z \cup \{x'\}\) respectively. Given \(s \in H^0(\mathcal{E} \otimes \mathcal{O}_Z)\) we obtain lifts \(s_x, s_{x'}\) in \(H^0(\mathcal{E} \otimes \mathcal{O}_{Z_x})\) and in \(H^0(\mathcal{E} \otimes \mathcal{O}_{Z_{x'}})\). Since \(Z'\) is a length \(d + 2\) subscheme, we obtain two distinct elements \(s_1, s_2\) in \(H^0(\mathcal{E})\) mapping to the same element in \(H^0(\mathcal{E} \otimes \mathcal{O}_Z)\). But by [LS21] Lemma 3.1, \(H^0(\mathcal{E}) \to H^0(\mathcal{E} \otimes \mathcal{O}_Z)\) is an isomorphism which gives a contradiction. \(\square\)

2.3.2. We now show that given any \(N \in \mathbb{Z}\), there exists a polarized surface \((X, H)\) with very ample \(H\) carrying an Ulrich line bundle \(L\) such that \(\text{amp}_{\mu}(L) - \text{amp}_{\mu}(H) \geq N\). Recall that a bielliptic surface \(X\) is a minimal algebraic surface with Kodaira dimension zero that satisfies \(q(X) := h^1(\mathcal{O}_X) = 1\) and \(p_g(X) := h^2(\mathcal{O}_X) = 0\). Given a bielliptic surface \(X\), there exists two elliptic curves \(A\) and \(B\), and an abelian group \(G\) acting on \(A\) and \(B\) such that

1. \(A/G\) is elliptic, and \(B/G \cong \mathbb{P}^1\), and
2. \(X \cong (A \times B)/G\) where \(G\) acts on \(A \times B\) componentwise.

We will use the notations \(\varphi : X \cong (A \times B)/G \to A/G\) and \(\psi : X \cong (A \times B)/G \to B/G \cong \mathbb{P}^1\) to describe the two projections. The morphism \(\varphi\) is smooth with every fiber isomorphic to \(B\). The fibers of \(\psi\) are multiples of smooth elliptic curves, with all but finitely many fibers isomorphic to \(A\). We will denote by \(A\) and \(B\), the numerical equivalence classes of the fibers of \(\psi\) and \(\varphi\) respectively. The classes intersect as

\[
A^2 = 0, B^2 = 0, A \cdot B = |G|.
\]

There is a classical result due to Bagnara and de Franchis that classifies the bielliptic surfaces into seven types based the action of \(G\) on \(B\). We recall a theorem of Serrano which gives a \(Z\)--basis of the divisors on bielliptic surfaces \(S\) up to numerical equivalence for each of these seven types, along with the multiplicities of the singular fibers of \(\psi\). We denote the multiplicities of the singular fibers of \(\psi : X \cong (A \times B)/G \to \mathbb{P}^1\) by...
In this case, fix an effective divisor that is ample. There can be two cases. Further, if \( H^0(L) = 0 \), then \( a \geq 0 \) and \( b \geq 0 \) ([Ser90], Lemma 1.3). The second one says that a line bundle \( L \) is such a basis. The value of \( \mu \) is early equivalent. Setting \( b_1 = 3a \) and \( b_2 = 3b \), we get \( b_1 = 2b \). Consequently, \( L_r - H \equiv b \cdot (\mu/\gamma)B \) and \( L_r - H \equiv a \cdot A/\mu \). Since \( h^0(L_r - H) = 0 \), we see that \( b \mu/\gamma \) can not be an integer which proves the "only if" part. To prove the "if" part, notice that it is enough to show in the cases (1) and (2) that there is a line bundle numerically equivalent to \( a \cdot A/\mu \) that has no global sections. There can be two cases.

**Case 1:** \( a | b \). In this case, let \( D \) be a movable (hence base point free) line bundle with \( D \equiv a \cdot A/\mu \). Notice that if \( D \equiv D' \) and \( D \equiv D' \) then \( D' \) has a fixed component. Further let \( D_1, D_2 \equiv D \) be effective divisors with \( D_1 = M_1 + F_1 \) with \( M_1 \) movable and \( F_1 \) fixed. Assume \( D_1 \sim D \) and \( D_1 \sim D_2 \). Then we claim that \( F_1 \neq F_2 \). Indeed, if \( F_1 = F_2 \) then \( M_1 \sim M_2 \) and since both are movable, \( M_1 \sim M_2 \) which in turn will imply \( D_1 \sim D_2 \) which is a contradiction. Thus, any two non–linearly equivalent divisors that are numerically equivalent to \( D \) will have distinct fixed components. Also, the fixed components of any divisor \( D' \equiv D \) and \( D' + D \) can be uniquely expressed as \( \sum_{j=1}^n b_j A_j \) where \( A_j \)’s are the reduced parts of the multiple fibers of \( \psi \), and \( 1 \leq b_j \leq m_j - 1 \) (this follows since \( D' \cdot A = 0 \), support of \( D' \) is contained inside a finite union of fibers of \( \psi \)). Hence, for general \( \delta \in \text{Pic}^0(X) \), \( h^0(D + \delta) = 0 \).

**Case 2:** \( a \nmid b \). In this case, fix an effective divisor \( D \equiv a \cdot A/\mu \). Notice that in this case, any \( D' \equiv D \) has non–trivial fixed component \( F' \). Arguing just like before, for any two divisors \( D_1, D_2 \equiv D \) with \( D_1 \sim D_2 \), we have \( F_1 \neq F_2 \) where \( F_i \)’s are fixed components of \( D_i \). Now arguing as before, we obtain that for general \( \delta \in \text{Pic}^0(X) \), \( h^0(D + \delta) = 0 \). The proof is now complete.

**Remark 2.11.** The above result shows that Ulrich line bundles exist on a bielliptic surface only if it is of even type. It follows from [MP93], Theorem 3.3 and Theorem 3.4 that for a bielliptic surface \( X \) of even type and a line bundle \( H \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B \) is very ample if and only if

1. \( X \) is of type 2 or 4, \( a \geq 3/2 \) and \( b \geq 3 \); or
2. \( X \) is of type 6, and either \( a = 1, b \geq 5 \) or \( a \geq 2, b \geq 3 \).

**Corollary 2.12.** Given any \( N \in \mathbb{Z} \), there exists a polarized bielliptic surface \((X, H)\) with \( H \) very ample carrying an Ulrich line bundle \( L \) such that \( \text{amp}_v(L) - \text{amp}_v(H) \geq N \).

**Proof.** Given \( N \), choose \( a \) and \( b \) satisfying: (1) \( b \geq 4a \) and \( b \) is odd, (2) \( a \geq 2 \) and \( 2a \geq N \). Consider a bielliptic surface \( X \) of type 2 or 4 with \( H \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B \). We obtain by Proposition 2.10 that \((X, H)\) carries an Ulrich line bundle \( L \equiv 2a \cdot A/\mu + b \cdot (\mu/\gamma)B \). By [MP93] Theorem 3.3, \( \text{amp}_v(L) - \text{amp}_v(H) = 2a \geq N \). That completes the proof. \( \square \)
Finally we state an immediate consequence of Proposition 2.10. The existence of rank two Ulrich bundles on arbitrary embeddings of bielliptic surfaces is known (see for example [Bea18] Proposition 6, [Lop21b] Theorem 1 and Subsection 5.4). On the other hand, it is also known that an Ulrich bundle of rank two is stable if it can not be obtained as an extension of Ulrich line bundles (see for example [CKM12] Lemma 2.15). Thus we obtain the following

**Corollary 2.13.** Let \( X \) be a bielliptic surface and let \( H \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B \) be a very ample line bundle on \( X \) with \( a, b \in \mathbb{Z}_{>0} \). Then any Ulrich bundle of rank two on \( X \) is slope–(equivalently Gieseker–)stable with respect to \( H \) unless \((X, H)\) falls into one of the two cases described in Proposition 2.10.

### 2.4. Positivity for divisible polarization

We aim to prove Theorem B using the following result.

**Proposition 2.14.** Let \( X \) be a smooth projective variety and let \( H_1 \) and \( H_2 \) be globally generated line bundles on \( X \). Let \( t_1, t_2 \geq 1 \) be integers and let \( \mathcal{F} \) be a coherent sheaf on \( X \) that is \( C_{q,0} \) for \((X, H) := t_1 H_1 + t_2 H_2\) for some integer \( 0 \leq q \leq \dim X - 1 \). Then \( H^i(\mathcal{F}(-aH_1-bH_2)) = 0 \) for all integers \( i \geq q + 1, a \leq (i-q)t_1, b \leq (i-q)t_2 \).

**Proof.** The proof is via decreasing induction on \( i \). Clearly the statement holds if \( i = \dim X + 1 \). Assume that the statement holds for \( i \geq i' \geq q + 2 \), i.e., \( H^i(\mathcal{F}(-aH_1-bH_2)) = 0 \) for all integers \( a \leq (i-q)t_1, b \leq (i-q)t_2 \) where \( i \) is in the range mentioned above. We know that \( H^{i-1}(\mathcal{F}(-(i' - 1 - q)t_1 H_1 - (i' - 1 - q)t_2 H_2)) = 0 \). Thus, \( \mathcal{F}(-(i' - 1 - q)t_1 H_1 - ((i' - 1 - t_1)t_2 - 1) H_2) \) is \( C_{i',-2,0} \) for \((X, H_2)\), whence by Lemma 2.2, we obtain the vanishing \( H^{i'-1}(\mathcal{F}(-(i' - 1 - q)t_1 H_1 - b H_2)) = 0 \) for all integers \( b \leq (i' - 1 - q)t_2 \). Consequently, for \( b \) in the mentioned range, \( \mathcal{F}(-(i' - 1 - q)t_1 H_1 - b H_2) \) is \( C_{i'-2,0} \) for \((X, H_1)\). One more application of Lemma 2.2 now shows that \( H^{i'-1}(\mathcal{F}(-aH_1-bH_2)) = 0 \) for all integers \( a \leq (i' - 1 - q)t_1, b \leq (i' - 1 - q)t_2 \).

**Corollary 2.15.** Let \( X, H_1, H_2, t_1, t_2 \) be as in Proposition 2.14. Let \( \mathcal{F} \) be a coherent sheaf on \( X \) that is \( C_{0,0} \) for \((X, H) := t_1 H_1 + t_2 H_2\). Then, for any \( q \in \mathbb{N} \) with \( 1 \leq q \leq \dim X \), \( \mathcal{F}(-q t_1 H_1 - (q t_2 - 1) H_2) \) is \( C_{q,-1,0} \) for \((X, H_2)\).

**Proof of Theorem B.** It is immediate from Corollary 2.15 that \( \mathcal{E}(-(t-1)H) \) is 0–regular for \((X, H)\). The fact that it is globally generated now follows from [Laz04a], Theorem 1.8.5.

1. The vector bundle \( \mathcal{E} \) is \((st-1)\)–jet ample by [BDS99] Proposition 2.3 since \( \mathcal{E}(-(t-1)H) \) is \((s-1)\)–jet ample by Proposition 2.6.

2. \((A)\) We have \( c_1(\mathcal{E}(-(t-1)H))^k \cdot Z \geq r^k(s-1)^k \text{mult}_x(Z) \) by Corollary 2.8. Also, since we have the inequality \( c_1(\mathcal{E}(-(t-1)H)^{k-l}) H^lZ \geq 0 \) for any \( 0 \leq i \leq k \), we obtain
\[
c_1(\mathcal{E})^k Z = c_1(\mathcal{E}(-(t-1)H)) H^k Z = c_1(\mathcal{E}(-(t-1)H)) H^k Z \geq c_1(\mathcal{E}(-(t-1)H))^k \cdot Z + r^k(t-1)^k H^k Z \geq r^k(s-1)^k \text{mult}_x(Z) + r^k(t-1)^k H^k Z.
\]

\((B)\) In this case, by [Lop21a] Theorem 7.2, \( c_1(\mathcal{E}(-(t-1)H))^k \cdot Z \geq r^k \text{mult}_x(Z) \). Similar computation as above yields the result.

**Remark 2.16.** Theorem 1.1 further gives a lower bound on the Seshadri constant of \( c_1(\mathcal{E}) \). Recall that for a nef line bundle \( L \) on \( X \) and a point \( x \in X \), the Seshadri constant \( \epsilon(L, x) \) is defined as \( \epsilon(L, x) := \inf_{C \in \mathcal{C}} \frac{L \cdot C}{\text{mult}_x C} \), where the infimum is taken over all curves \( C \) through \( x \), and it measures the local positivity of \( L \). Thus, if \( H \) is very ample, then by Theorem 1.1 \((2)\), we obtain
\[
\epsilon(\text{det}(\mathcal{E}), x) \geq \begin{cases} r(s-1 + (t-1) \epsilon(H,x)) & \text{if } s \geq 2; \\ r(1 + (t-2) \epsilon(H,x)) & \text{if } s = 1 \text{ and } x \text{ does not lie on a line contained in } X. \end{cases}
\]

Using Theorem 1.1 \((2)\) we also deduce that when \( H \) is very ample, the following inequality holds
\[
c_1(\mathcal{E})^n \geq \begin{cases} r^n(s-1)^n + d(t-1)^n & \text{if } s \geq 2; \\ r^n(1 + d(t-1)^n) & \text{if } s = 1 \text{ and } X \text{ is not covered by lines}. \end{cases}
\]
3. Continuous CM–regularity of coherent sheaves

3.1. Definition and first properties. We start with the definition of the cohomological support loci (cf. [Par12] Definition 1.2) that are of fundamental importance in the study of irregular varieties.

**Definition 3.1. (Cohomological support loci)** Let $X$ be a smooth projective variety and let $a : X \to A$ be a morphism to an abelian variety $A$. Let $\mathcal{F}$ be a coherent sheaf on $X$. The $i$–th cohomological support locus $V^i_a(\mathcal{F})$ with respect to $a$ for $i \in \mathbb{N}$ is defined as

\[ V^i_a(\mathcal{F}) := \left\{ \xi \in \text{Pic}^0(A) \mid h^i(\mathcal{F} \otimes a^*\xi) \neq 0 \right\}. \]

We will simply write $V^i(\mathcal{F})$ for $V^i_{\text{alb}_X}(\mathcal{F})$ where $\text{alb}_X : X \to \text{Alb}(X)$ is the Albanese morphism of $X$.

As we discussed earlier, continuous CM–regularity is a slightly coarser measure of positivity than CM–regularity, and was introduced by Mustopa in [Mus16]. Further study of this invariant was carried out by Küronya and Mustopa in [KM20], and by Grieve in [Gri21].

For our purpose, we need to generalize the definition of continuous CM–regularity (cf. [Mus16] Definition 1.1) to a relative–partial set–up that we describe below.

**Definition 3.2. (Continuous CM–regularity)** Let $X$ be a smooth projective variety and let $H$ be a line bundle on $X$. Further, let $a : X \to A$ be a morphism to an abelian variety $A$. Let $\mathcal{F}$ be a coherent sheaf on $X$ and let $q, k$ be integers with $q \geq 0$. The sheaf $\mathcal{F}$ is called $\mathcal{C}^{i,a}_{q,k}$ for $(X, H)$ if $V^{q+i}_{a}(\mathcal{F}((k-i)H)) \neq \text{Pic}^0(A)$ for all integers $i \geq 1$. When $H$ is ample, we will say $\mathcal{F}$ is continuously $k$–regular for $(X, H)$ if it is $\mathcal{C}^{i,a}_{0,k}$ for $(X, H)$.

When $H$ is ample and globally generated, we define the continuous regularity of $\mathcal{F}$ as $\text{reg}^\text{cont}(\mathcal{F}) := \min \{ m \in \mathbb{Z} \mid \forall i > 0, V^{i}(\mathcal{F}((m-i)H)) \neq \text{Pic}^0(\text{Alb}(X)) \equiv \text{Pic}^0(X) \}$. We remark that in general, we have the inequality $\text{reg}^\text{cont}(\mathcal{F}) \leq \text{reg}_H(\mathcal{F})$.

We will see that in practice, it is often useful to work with relative continuous CM–regularity rather than CM–regularity. The following observation (where we use semicontinuity to see that the first three equivalent condition implies the fourth) highlights this and shows that the former property is stable under perturbations by elements of $a^*\text{Pic}^0(A)$.

**Observation 3.3. (see also [Mus16] Lemma 1.2)** Let $X$ be a smooth projective variety and let $H$ be a line bundle on $X$. Let $a : X \to A$ be a morphism to an abelian variety $A$, and let $\mathcal{F}$ be a coherent sheaf on $X$. Also, let $q, k$ be integers with $q \geq 0$. The following conditions are equivalent:

1. $\mathcal{F}$ is $\mathcal{C}^{i,a}_{q,k}$ for $(X, H)$,
2. $\mathcal{F} \otimes a^*\xi$ is $\mathcal{C}^{i}_{q,k}$ for $(X, H)$ for some (equivalently for all) $\xi \in \text{Pic}^0(A)$,
3. $\mathcal{F}$ is $\mathcal{C}^{i,a}_{q,k}$ for $(X, H + a^*\xi)$ for some (equivalently for all) $\xi \in \text{Pic}^0(A)$,
4. $\mathcal{F} \otimes a^*\xi$ is $\mathcal{C}^{i}_{q,k}$ for $(X, H)$ for some (equivalently for general) $\xi \in \text{Pic}^0(A)$.

Thus, the (partial) continuous CM–regularity is determined by the class of the line bundle $H$ in the Néron–Severi group $\text{Pic}(X)/\text{Pic}^0(X)$.

3.2. Behavior of continuous CM–regularity. In this subsection, we study the behavior of continuous CM–regularity of torsion–free coherent sheaves with respect to restriction and pull–back.

3.2.1. We introduce the property $(P_a)$ that is crucial for us since it allows us to produce smooth sections in the appropriate linear series by Bertini’s theorem.

**Definition 3.4. (Property $(P_a)$)** We will say that a polarized smooth projective variety $(X, H)$ satisfies $(P_a)$ where $a : X \to A$ is a morphism to an abelian variety if for all $\xi \in \text{Pic}^0(A)$, $H + a^*\xi$ is globally generated.
Observation 3.5. Let \((X, H)\) be a polarized smooth projective variety and let \(a : X \rightarrow A\) be a morphism to an abelian variety. Assume that \((X, H)\) satisfies \(P_\alpha\). Then for any \(\zeta \in \text{Pic}^0(A)\) and any smooth and irreducible member \(Y \in |H + a^*\zeta|\), the pair \((Y, H|_Y)\) satisfies \(P_{\alpha|_Y}\) where \(a|_Y : Y \rightarrow A\) is the restriction of \(a\).

It is important for us to note that continuous CM–regularity satisfy better restriction properties.

Lemma 3.6. Let \((X, H)\) be a polarized smooth projective variety and let \(a : X \rightarrow A\) be a morphism to an abelian variety \(A\). Assume that \((X, H)\) satisfies \(P_\alpha\). Let \(\mathcal{F}\) be a torsion–free coherent sheaf that is \(C^{q}_{q,k}\) for \((X, H)\) where \(q, k\) are integers with \(q \geq 0\). Then for any \(\zeta \in \text{Pic}^0(A)\) and any smooth and irreducible member \(Y \in |H + a^*\zeta|\), there exists \(\zeta' \in \text{Pic}^0(A)\) such that \(\mathcal{F} \otimes a^*\zeta'|_Y\) is \(C_{q,k}\) for \((Y, H|_Y)\).

Proof. It is safe to assume \(k = 0\). Consider the following part of the long exact sequence

\[ H^{q+i}(\mathcal{F}(-iH) \otimes a^*\zeta') \rightarrow H^{q+i}(\mathcal{F}(-iH)|_Y \otimes a^*\zeta') \rightarrow H^{q+i+1}(\mathcal{F}(-(i+1)H - a^*\zeta + a^*\zeta')) \]

obtained from twisting the restriction sequence by general \(\zeta' \in \text{Pic}^0(A)\) and passing to cohomology. The statement now follows from semicontinuity. \(\square\)

3.2.2. We now recall the definition of a strongly generating morphism as presented in the introduction of [BPS20].

Definition 3.7. (Strongly generating morphisms) Let \(a : X \rightarrow A\) be a morphism from a smooth projective variety \(X\) to an abelian variety \(A\). We call the morphism \(a\) strongly generating if the induced map \(a^* : \hat{\mathcal{A}} = \text{Pic}^0(A) \rightarrow \text{Pic}^0(X)\) is injective.

Inspired by [Par05], given a morphism \(a : X \rightarrow A\) as above, we will work with the covering trick i.e. we will consider the following base–change diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{a}} & X \\
\downarrow & & \downarrow a \\
\hat{A} & \xrightarrow{\mu} & A
\end{array}
\]

where \(\mu : \hat{A} \rightarrow A\) is an isogeny of abelian varieties. We recall that if \(a\) is strongly generating, then so is \(\tilde{a}\), and moreover \(\tilde{X}\) is smooth and irreducible (see the proof of Lemma 2.3 in [BPS16]). This last property is the reason we consider strongly generating morphisms. We show that continuous CM–regularity behaves well with respect to the above covering trick.

Lemma 3.8. Let \(X\) be a smooth projective variety and let \(H\) be a line bundle on \(X\). Also, let \(a : X \rightarrow A\) be a morphism to an abelian variety \(A\) that is strongly generating. Let \(\mathcal{F}\) be a coherent sheaf on \(X\) that is \(C^{q}_{0,k}\) for \((X, H)\). Let \(\mu : \hat{A} \rightarrow A\) be an isogeny and consider the base–change diagram (3.1). Then \(\tilde{\mu}^* \mathcal{F} = C^{q}_{0,k}\) for \((\tilde{X}, \tilde{H} := \tilde{\mu}^* H)\).

Proof. Without loss of generality, we may assume \(k = 0\). For \(\zeta \in \hat{A}\), we have by the projection formula

\[
H^1(\tilde{\mu}^* (\mathcal{F}(-iH + a^*\zeta))) = H^1(\mathcal{F}(-iH + a^*\zeta)) \oplus \bigoplus_{k=1}^{d-1} H^1(\mathcal{F}(-iH + a^*\zeta + a^*\zeta_k))
\]

where \(d := \text{deg}(\mu)\), \(\mu_* \mathcal{O}_{\hat{A}} = \mathcal{O}_A \oplus \bigoplus_{k=1}^{d-1} \mathcal{O}_{\hat{A}}\), and \(\zeta_k \in \hat{A}\) for \(1 \leq k \leq d - 1\). The conclusion now follows from semicontinuity and the commutativity of the diagram (3.1). \(\square\)

4. Continuous CM–regularity and generic vanishing

4.1. The theory of generic vanishing. Throughout this subsection, for an abelian variety \(A\), we denote by \(\hat{A}\) the dual abelian variety identified with \(\text{Pic}^0(A)\).
4.1.1. Let $X$ be a smooth projective variety and let $\text{alb}_X : X \to A := \text{Alb}(X)$ be the Albanese map. Consider $\hat{A} = \text{Pic}^0(A) \cong \text{Pic}^0(X)$ and let $\mathcal{P}$ be a Poincaré line bundle on $A \times \hat{A}$. Let $P := (\text{alb}_X \times \text{id}_A)^* \mathcal{P}$. Let $\mathcal{D}(X)$ and $\mathcal{D}(\hat{A})$ be the bounded derived categories of $\text{Coh}(X)$ and $\text{Coh}(\hat{A})$. In this situation, we have the following two Fourier–Mukai transform functors

$$R\Phi_p : \mathcal{D}(X) \to \mathcal{D}(\hat{A}), \quad R\Phi_p(-) := R\Phi_{p \hat{A}}(p_X^*(-) \otimes P),$$

$$R\Psi_p : \mathcal{D}(\hat{A}) \to \mathcal{D}(X), \quad R\Psi_p(-) := R\Psi_{p \hat{A}}(p_{\hat{A}}^*(-) \otimes P).$$

A reference of the following definitions can be found for example in [PP11a] Proposition/Definition 2.1, and Proposition/Definition 2.7.

**Definition 4.1. (Generic vanishing and M–regularity)** Let $X$ be a smooth projective variety and let $\mathcal{F}$ be a coherent sheaf on $X$.

- $\mathcal{F}$ is called *Generic Vanishing* (abbreviated as GV) if $\text{codim}(V^i(\mathcal{F})) \geq i$ for all integers $i > 0$. More generally, for an integer $k \geq 0$, $\mathcal{F}$ is called $\text{GV}_{-k}$ if $\text{codim}(V^i(\mathcal{F})) \geq i - k$ for all integers $i$.

- $\mathcal{F}$ is called *Mukai regular* (abbreviated as $\text{M–regular}$) if $\text{codim}(V^i(\mathcal{F})) > i$ for all $i > 0$.

Evidently, we have $\text{GV} = \text{GV}_0$. The following fundamental theorem is due to Hacon (see [Hac04]), Pareschi and Popa (see [PP11a]).

**Theorem 4.2. ([PP11b] Theorem 3.7, Corollary 3.11)** Let $X$ be a smooth projective variety with $\dim \text{Alb}(X) = g$. Let $\mathcal{F}$ be a coherent sheaf on $X$, and let $k \geq 0$ be an integer. Then the following are equivalent:

1. $\mathcal{F}$ is $\text{GV}_{-k}$,
2. $\text{codim} \text{Supp}(R^i \Phi_p \mathcal{F}) \geq i - k$ for all integers $i$,
3. for any sufficiently positive ample line bundle $L$ on $\hat{A}$, $H^i(\mathcal{F} \otimes R\Phi_p \mathcal{L} \otimes L^{g-1}) = 0$ for all integers $i > k$.

4.1.2. Now, let $X$ be an abelian variety of dimension $g$. We recall the notions of $\text{M–regularity}$ and Index Theorems with prescribed indices (cf. [PP03], the end of Section 1 on p. 5).

**Definition 4.3. (IT sheaves)** Let $X$ be an abelian variety and let $\mathcal{F}$ be a coherent sheaf on $X$. The sheaf $\mathcal{F}$ is said to satisfy *Index Theorem with index $k$* for some $k \in \mathbb{Z}$ (abbreviated as $\text{IT}_k$) if $V^i(\mathcal{F}) = \phi$ for all $i \neq k$.

It is clear that on an abelian variety $X$, a coherent sheaf $\mathcal{F}$ satisfies $\text{IT}_0 \implies \mathcal{F}$ is $\text{M–regular} \implies \mathcal{F}$ is GV. Also, notice that an ample line bundle on an abelian variety $X$ satisfies $\text{IT}_0$. In the abelian case, we will denote the Fourier–Mukai transform functors as

$$R\hat{\Phi} : \mathcal{D}(X) \to \mathcal{D}(\hat{X}), \quad R\hat{\mathcal{F}} : \mathcal{D}(\hat{X}) \to \mathcal{D}(X).$$

A fundamental result of Mukai (see [Muk81] Theorem 2.2) shows that $R\hat{\Phi} : \mathcal{D}(X) \to \mathcal{D}(\hat{X})$ is an equivalence of derived categories, and we have the following inversion formulae

$$R\hat{\Phi} \circ R\hat{\mathcal{F}} = (-1)^g_X[-g], \quad \text{and} \quad R\mathcal{F} \circ R\hat{\Phi} = (-1)^g_X[-g]$$

where $(-1)^g_X$ and $(-1)^g_X$ are multiplications by $(-1)$ on $\hat{X}$ and $X$ respectively.

It turns out that if $\mathcal{F}$ on $X$ is $\text{IT}_k$ for some $k \in \mathbb{Z}$, then $R\hat{\Phi} \mathcal{F} = R^k \hat{\Phi} \mathcal{F} [-k]$ and $R^k \hat{\Phi} \mathcal{F}$ is a locally free sheaf. In particular, if $L$ is an ample line bundle on $\hat{X}$, then $R\mathcal{F} \mathcal{L}^{g-1} = R^k \mathcal{F} \mathcal{L}^{g-1}[-g]$.

4.1.3. A result of Pareschi–Popa on *preservation of vanishing* says that on an abelian variety, a tensor product of a GV sheaf $\mathcal{F}$ and an $\text{IT}_0$ sheaf $\mathcal{G}$ is $\text{IT}_0$ if one of $\mathcal{F}$ and $\mathcal{G}$ is locally free ([PP11a] Proposition 3.1). A variation of their proof yields the following that suits the most for our purpose.

**Proposition 4.4.** Let $X$ be an abelian variety of dimension $g$. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on $X$ with one of them locally free. Assume $\mathcal{G}$ is $\text{IT}_0$ and $m \geq 0$ is an integer. If one of the following holds

1. either $V^i(\mathcal{F}) = \phi$ for all integers $i \geq m + 1$, or
2. $\mathcal{F}$ is $\text{GV}_{-m}$,

then $V^i(\mathcal{F} \otimes \mathcal{G}) = \phi$ for all integers $i \geq m + 1$. 
4.1.4. One of the most essential and useful tools in the study of irregular varieties is the notion of continuous global generation that we define next ([PP11a] Definition 5.2).

**Definition 4.6. (Continuous global generation)** Let $X$ be an irregular variety. A sheaf $\mathcal{F}$ on $X$ is continuously globally generated if for any non–empty open subset $U \subset \Pic^0(X)$, the following sum of evaluation maps is surjective:

$$\bigoplus_{\zeta \in U} H^0(\mathcal{F} \otimes \zeta) \otimes \zeta^* \to \mathcal{F}.$$ 

It was shown by Pareschi–Popa in [PP11a] Corollary 5.3 that an $M$–regular sheaf is continuously globally generated. Also, another result of them asserts that if $\mathcal{F}$ is a continuously globally generated sheaf and $L$ is a continuously globally generated line bundle on $X$, then $\mathcal{F} \otimes L$ is globally generated ([PP03] Proposition 2.12). Thus, if $H_1$ and $H_2$ are two ample line bundles on an abelian variety $X$, then $H_1 + H_2$ is globally generated.

4.2. **Proof of Theorem C.** We now aim to prove Theorem C stated in the introduction.

**Proposition 4.7.** Let $(X, H)$ be a polarized smooth projective variety and let $a : X \to A$ be a morphism to an abelian variety $A$. Assume that $(X, H)$ satisfies $(P_a)$ and let $\mathcal{F}$ be a torsion–free coherent sheaf on $X$ that is $C_{q-1,0}$ for $(X, H)$ for some integer $1 \leq q \leq \dim X$. Then $V^i_a(\mathcal{F}) = \phi$ for all integers $i \geq q$.

**Proof.** The proof is based on induction on $\dim X =: n$ and we divide the proof into the following steps.

**Step 1.** We first prove the statement for $i = n$. By hypothesis, we know that $H^n(\mathcal{F}(-(n-q+1)H) \otimes a^* \zeta) = 0$ for some $\zeta \in \Pic^0(A)$. Since $(X, H)$ satisfies $(P_a)$, for any given $\zeta' \in \Pic^0(A)$, we can choose a smooth and irreducible member $Y \in (n-q+1)H + a^*(\zeta' - \zeta)$. Consider the restriction exact sequence

$$0 \to \mathcal{F}(-(n-q+1)H) \otimes a^* \zeta' \to \mathcal{F} \otimes a^* \zeta' \to \mathcal{F} \otimes a^* \zeta'|_Y \to 0.$$ 

Passing to cohomology, we obtain the desired vanishing $H^i(\mathcal{F} \otimes a^* \zeta') = 0$.

**Step 2.** Now we prove the statement by induction and thanks to the previous step, we assume $n \geq 2$. Also, because of Step 1, we assume that $1 \leq q \leq i \leq n-1$. Let $\zeta \in \Pic^0(A)$ and we want to show that $H^i(\mathcal{F} \otimes a^* \zeta) =$
0. Our proof is inspired by [LS21] proof of Lemma 3.3. We know by Observation 3.3 that there exists \( \zeta' \in \text{Pic}^0(A) \) such that \( \mathcal{F} \otimes a^* \zeta' \) is \( C_{q-1,0} \) for \((X, H)\). Observe that by Lemma 2.2, \( H^i(\mathcal{F} \otimes a^* \zeta' \otimes (-jH)) = 0 \) for all integers \( j \leq \dim X - q + 1 \). To this end, consider the restriction exact sequence

\[
0 \to \mathcal{F} \otimes a^* \zeta \to \mathcal{F} \otimes (H + a^* \zeta') \to \mathcal{F} \otimes (H + a^* \zeta')|_Y \to 0
\]

where \( Y \in |H + a^* \zeta' - a^* \zeta| \) is a smooth and irreducible member (exists by Bertini thanks to \((P_a)\)). Passing to the cohomology of (4.3), we deduce that it is enough to prove that the restriction map

\[ H^{i-1}(\mathcal{F} \otimes (H + a^* \zeta')) \to H^{i-1}(\mathcal{F} \otimes (H + a^* \zeta')|_Y) \]

sursjects since \( H^i(\mathcal{F} \otimes (H + a^* \zeta')) = 0 \). On the other hand, choose a general smooth and irreducible member \( Z \in |H + a^* \zeta - a^* \zeta'| \) such that \( \mathcal{F}|_Z \) is torsion-free (such a section exists thanks again to the fact that \((X, H)\) satisfies property \((P_a)\)). Then we know that \( H^{i-1}(\mathcal{F} \otimes (H + a^* \zeta')) \to H^{i-1}(\mathcal{F} \otimes (H + a^* \zeta')|_{Y+Z}) \) surjects. Consequently, it is enough to show that the map \( H^{i-1}(\mathcal{F} \otimes (H + a^* \zeta')|_{Y+Z}) \to H^{i-1}(\mathcal{F} \otimes (H + a^* \zeta')|_Y) \) surjects. Consider the following commutative diagram with exact rows and exact left column

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_X(-Y - Z) & \to & \mathcal{O}_X & \to & \mathcal{O}_{Y+Z} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_X(-Y) & \to & \mathcal{O}_X & \to & \mathcal{O}_Y & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}_Z(-Y) & & & & & & & & 0 \\
\end{array}
\]

which by snake lemma yields the following short exact sequence

\[ 0 \to \mathcal{O}_Z(-Y) \to \mathcal{O}_{Y+Z} \to \mathcal{O}_Y \to 0. \]

Twisting the above by \( \mathcal{F} \otimes (H + a^* \zeta') \) and passing to cohomology, we deduce that it is enough to prove the following vanishing \( H^i(\mathcal{F} \otimes (H + a^* \zeta') \otimes (H + a^* \zeta' - a^* \zeta)|_Z) = H^i(\mathcal{F} \otimes a^* \zeta|_Z) = 0 \). But \( \mathcal{F}|_Z \) is torsion–free and \( C_{q-1,0}^a \) for \((Z, H|_Z)\) by Lemma 3.6, and \((Z, H|_Z)\) satisfies \((P_a|_Z)\) by Observation 3.5. Thus we are done by the induction hypothesis. \( \square \)

**Theorem 4.8.** Let \((X, H)\) be a polarized smooth projective variety and let \( a : X \to A \) be a morphism to an abelian variety \( A \). Assume that \((X, H)\) satisfies \((P_a)\). Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( X \) that is \( C_{q-1,0}^a \) for \((X, H)\) for some integer \( 1 \leq q \leq \dim X \). Then for any integer \( 0 \leq i \leq \dim X - q \), \( V_a^{q+i}(\mathcal{F}(-aH)) = \phi \) for all integers \( 0 \leq a \leq i \).

**Proof.** Clearly the statement holds by Proposition 4.7 if \( a = 0 \), in particular it holds if \( \dim X = n = 1 \). Consider the restriction sequence

\[ 0 \to \mathcal{F}(-aH + a^* \zeta) \to \mathcal{F}(-(a-1)H + a^* \zeta) \to \mathcal{F}(-(a-1)H + a^* \zeta)|_Y \to 0 \]

where \( Y \in |H| \) is a general smooth and irreducible member such that \( \mathcal{F}|_Y \) is torsion–free. The cohomology sequence of the above, and an easy induction on \( n \) and \( a \) finishes the proof (thanks to Lemma 3.6 and Observation 3.5). \( \square \)

**Corollary 4.9.** Let \((X, H)\) be a polarized smooth projective variety and let \( a : X \to A \) be a morphism to an abelian variety \( A \). Assume one of the following two conditions holds:

1. \((X, H)\) satisfies property \((P_a)\), or
(2) the morphism \( a \) is strongly generating, and there exists an isogeny \( \mu : \tilde{A} \to A \) such that \((\tilde{X}, \tilde{\mu}^* H)\) satisfies \((P_\mu)\), where \( \tilde{X}, \tilde{a}, \) and \( \tilde{\mu} \) are as in (3.1).

Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( X \) that is \( C_{0,k}^\alpha \) for some integer \( 0 \leq k \leq \dim X \). Then

(A) \( V^i_a(\mathcal{F}) = \phi \) for all integers \( i \geq k + 1 \),

(B) if \( k \neq 0 \), then \( \text{codim}(V^i_a(\mathcal{F})) \geq 1 \).

\textbf{Proof.} The assertion follows immediately from Theorem 4.8 if (1) holds. Now assume (2) holds. The assertion (B) is obvious, so it is enough to show (A), that is \( V^i_a(\mathcal{F}) = \phi \) for \( i \geq k + 1 \). By Lemma 3.8, we know that \( \tilde{\mu}^* \mathcal{F} \) is \( C_{0,k}^\alpha \) for \( (\tilde{X}, \tilde{\mu}^* H) \). Also, \( \tilde{\mu}^* \mathcal{F} \) is torsion–free and coherent. Since \( (\tilde{X}, \tilde{\mu}^* H) \) satisfies \((P_\mu)\), using Theorem 4.8 we conclude that \( V^i_a(\tilde{\mu}^* \mathcal{F}) = \phi \) for \( i \geq k + 1 \). It follows from the projection formula that \( \mu^* V^i_a(\mathcal{F}) \subseteq V^i_a(\tilde{\mu}^* \mathcal{F}) \) for all \( i \) where \( \mu^* : \text{Pic}^0(A) \to \text{Pic}^0(\tilde{A}) \) is the induced map, whence the assertion follows. \( \square \)

\textbf{Example 4.10.} Let \( X \) be a smooth projective variety and let \( H = K_X + sL \), where \( s \geq \dim X + 1 \) and \( L \) is an ample and globally generated line bundle on \( X \). By Corollary 4.9 (1), the answer of Question 1.5 is affirmative for the pair \((X, H)\) whenever \( H \) is ample.

\textbf{Proof of Theorem C.} This is an immediate consequence of Corollary 4.9 (2). Indeed, set \( a := \text{alb}_X, A := \text{Alb}(X) \), and let \( \mu = n_A : A \to A \) be the multiplication by \( n \) isogeny. Then \( a \) is strongly generating. Observe that \( n_A^* H = \tilde{n}_A^* H + \tilde{a}^*(n_A^* H) \), and consequently \((\tilde{X}, \tilde{n}_A^* H)\) satisfies \((P_\mu)\) for \( n \geq 2 \). \( \square \)

4.3. \textbf{A few variants.} In this subsection, we prove several other variants of Theorem C.

4.3.1. We start by showing that the moduli space of Gieseker–stable sheaves on abelian surfaces answers Question 1.5 in the affirmative.

\textbf{Example 4.11.} Let \( A \) be an abelian surface and let \( v \in H^{0,0}(A, \mathbb{Z}) \) be a primitive Mukai vector satisfying \( v > 0 \) and \( \langle v, v \rangle \geq 6 \) (see [Yos01] for details). Let \( L \) be a very general ample divisor on \( A \), and let \( M_L(v) \) be the moduli space of Gieseker–stable sheaves on \( A \) with respect to \( L \) with Mukai vector \( v \). By the results of \textit{loc. cit.}, we know that \( M_L(v) \) is a smooth projective variety, and \( \text{Alb}(M_L(v)) = A \times \tilde{A} \). Moreover, if we set \( n(v) = \frac{1}{2}\langle v, v \rangle \), and \( a := \text{alb}_{M_L(v)} \), then we have the base–change diagram (see \textit{loc. cit.} (4.10), (4.11))

\[
\begin{array}{ccc}
K_L(v) \times A \times \tilde{A} & \xrightarrow{n(v)} & M_L(v) \\
\downarrow \tilde{a} & & \downarrow a \\
A \times \tilde{A} & \xrightarrow{n(v)_A} & A \times \tilde{A}
\end{array}
\]

where \( K_L(v) \) is a regular smooth projective variety (it is a hyperkähler manifold deformation equivalent to generalized Kummer variety), and \( \tilde{a} = \text{pr}_{A \times \tilde{A}} \) is the projection which is also the Albanese map of the variety \( K_L(v) \times A \times \tilde{A} \) as \( q(K_L(v)) = 0 \). Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( M_L(v) \) that is continuously \( k \)–regular for \((X, H)\) where \( H \) is an ample and globally generated line bundle on \( M_L(v) \) and \( 0 \leq k \leq \dim M_L(v) = 2n(v) + 2 \). We claim that \( V^i(\mathcal{F}) = \phi \) for \( i \geq k + 1 \). Indeed, it is easy to verify that it is enough to show that \( V^i(\tilde{n}(v)^* (\mathcal{F})) = \phi \) for \( i \geq k + 1 \). Since \( K_L(v) \) is regular, we deduce that \( \tilde{n}(v)^* (H) = H_1 \oplus H_2 \) where \( H_1 \) and \( H_2 \) are ample and globally generated line bundles on \( K_L(v) \) and \( A \times \tilde{A} \) respectively. The assertion now follows from Theorem C.

We include a corollary whose proof is precisely what we argued in the example above.

\textbf{Corollary 4.12.} Let \( X \) be a smooth projective variety, and let \( \text{alb}_X : X \to A := \text{Alb}(X) \) be the Albanese map of \( X \). Assume there is an isogeny \( \mu : \tilde{A} \to A \) such that the following two conditions hold:

(1) \( X \times_{\text{alb}_X} \tilde{A} \cong Y \times \tilde{A} \) for a regular smooth projective variety \( Y \), and

(2) the induced map \( X \times_{\text{alb}_X} \tilde{A} \to \tilde{A} \) is \( \text{pr}_A \) under the identification in (1).

Let \( \mathcal{O}_X(1) \) is an ample and globally generated line bundle on \( X \). Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( X \) that is continuously \( k \)–regular for \((X, \mathcal{O}_X(1))\) where \( 0 \leq k \leq \dim X \) Then
(A) \( V^i(\mathcal{F}) = \phi \) for \( i \geq k + 1 \).
(B) If \( k \neq 0 \), then \( \text{codim}(V^k(\mathcal{F})) \geq 1 \).

In particular, Question 1.5 has an affirmative answer for \( X \).

4.3.2. The following result is a variant of Theorem C where we assume \( H_1 = K_X + Q \) for a nef line bundle \( Q \), but require that \( \text{alb}_X : X \rightarrow \text{Alb}(X) \) is finite onto its image.

**Theorem 4.13.** Let \( (X, H) \) be a polarized smooth projective variety. Assume that the Albanese map \( \text{alb}_X : X \rightarrow \text{Alb}(X) \) is finite onto its image. Further assume that there exists a nef line bundle \( H_2 \) on \( \text{Alb}(X) \) such that \( H = K_X + Q + \text{alb}_X^* H_2 \). Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( X \) that is continuously \( k \)-regular for \( (X, H) \), for some integer \( 0 \leq k \leq \dim X \). Then the following statements hold.

1. \( V^i(\mathcal{F}) = \phi \) for \( i \geq k + 1 \).
2. If \( k \neq 0 \), then \( \text{codim}(V^k(\mathcal{F})) \geq 1 \).

In particular, the answer of Question 1.5 is affirmative for the pair \( (X, H) \).

**Proof.** This is also an immediate consequence of Corollary 4.9 (2). Indeed, set \( a := \text{alb}_X \) and \( A := \text{Alb}(X) \), and notice that \( \tilde{n}_A^* H \equiv K_X + \tilde{n}_A^* (Q) + \tilde{a}^* (n^2 H_2) \) satisfies \( (P_a) \) for \( n \gg 0 \). □

4.3.3. The following result is not a consequence of Theorem C when \( a = \text{alb}_X \) and \( A = \text{Alb}(X) \) as we are not assuming that \( H \) is ample.

**Theorem 4.14.** Let \( X \) be a smooth projective variety and assume that \( a : X \rightarrow A \) is a strongly generating surjective morphism to an abelian variety \( A \). Further, assume that there exists a globally generated line bundle \( H_1 \) on \( X \) and an ample line bundle \( H_2 \) on \( A \), and set \( H := H_1 + a^* H_2 \). Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( X \) that is \( C_{0,0}^a \) for \( (X, H) \).

1. If \( k = 1 \), then \( V^1_a(\mathcal{F}) = \phi \) and \( \text{codim}(V^1_a(\mathcal{F})) \geq 1 \). In particular, when \( a \) is the Albanese morphism, \( \mathcal{F} \) is a GV sheaf.
2. If \( k = 0 \), then \( V^1_a(\mathcal{F}) = \phi \) for \( i \geq 1 \).

**Proof.** (1) It is clear that \( \text{codim}(V^1_a(\mathcal{F})) \geq 1 \), so we only need to show that \( V^1_a(\mathcal{F}) = \phi \) for \( i \geq 2 \). To start with, note that \( \mathcal{F}(H) \) is \( C_{0,0}^a \) for \( (X, H) \). Consider the base–change diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{\mu_n} & X \\
\downarrow{a_n} & & \downarrow{a} \\
A & \xrightarrow{n_A} & A
\end{array}
\]

where as before, \( n_A : A \rightarrow A \) is multiplication map by \( n \geq 2 \). Recall that \( X_n \) is smooth and irreducible as \( a \) is strongly generating. By Lemma 3.8, we obtain that \( \mu_n^*(\mathcal{F}(H)) = C_{0,0}^{a_n} \) for \( (X_n, \mu_n^* H) \). Rewriting this, we obtain that \( \mu_n^* \mathcal{F} \otimes a_n^*(n_A^*(H_2)) \otimes \mu_n^*(H_1) = C_{0,0}^{a_n} \) for \( (X_n, a_n^*(n_A^*(H_2)) + \mu_n^*(H_1)) \). Consequently, by Observation 3.3, there exists \( \zeta \in \hat{A} \) such that \( \mu_n^* \mathcal{F} \otimes a_n^*(n_A^*(H_2) + \zeta) \otimes \mu_n^*(H_1) = C_{0,0} \) for the pair \( (X_n, a_n^*(n_A^*(H_2)) + \mu_n^*(H_1)) \).

As \( a_n^*(n_A^*(H_2)) \equiv a_n^*(n^2 H_2) \) and \( \mu_n^*(H_1) \) are both globally generated for \( n \geq 2 \), Corollary 2.15 now implies that

\[
\mu_n^* \mathcal{F} \otimes \left( (a_n^*(n_A^*(H_2)) + \zeta) - (\mu_n^*(H_1)) \right) = \mu_n^* \mathcal{F} \otimes a_n^*(n_A^*(H_2) + \zeta)
\]

is \( C_{0,0} \) for \( (X_n, a_n^*(n_A^*(H_2)) \). Since \( a_n^*(n_A^*(H_2)) \) is globally generated, consequently by Corollary 2.3, we deduce that \( R^i(a_n)_* \left( (\mu_n^* \mathcal{F} \otimes a_n^*(n_A^*(H_2) + \zeta)) \right) = 0 \) for all integers \( i \geq 1 \). Thus, we obtain that, for all \( i \geq 1 \)

\[
H^i \left( (a_n)_* \left( (\mu_n^* \mathcal{F} \otimes a_n^*(n_A^*(H_2) + \zeta)) \right) \right) = H^i \left( (\mu_n^* \mathcal{F} \otimes a_n^*(n_A^*(H_2) + \zeta)) \otimes (\mu_n^*(H_1)) \right) = 0,
\]

i.e., \( (a_n)_* \left( (\mu_n^* \mathcal{F} \otimes a_n^*(n_A^*(H_2) + \zeta)) \right) = C_{0,0} \) for \( (A, n_A^*(H_2)) \). This is equivalent to the fact that \( (a_n)_* \left( (\mu_n^* \mathcal{F}) \right) \) is \( C_{0,0}^{\text{alb}_A} \) for \( (A, n_A^*(H_2)) \). Notice that \( (A, n_A^*(H_2)) \) satisfies \( (P_{\text{alb}_A}) \), and the sheaf \( (a_n)_* \left( (\mu_n^* \mathcal{F}) \right) \) is torsion–free since \( a_n \) is surjective. Thus, using Theorem 4.8, we deduce that \( V^i \left( (a_n)_* \left( (\mu_n^* \mathcal{F}) \right) \right) = \phi \) for all \( i \geq 2 \) i.e.,
\( V^i_{a_n}((\mu_n^*F)) = \phi \) for all \( i \geq 2 \). Thus, \( V^i_{a}(F) = \phi \) for all \( i \geq 2 \) since \( n^*(V^i_{a}(F)) \leq V^i_{a_n}(\mu_n^*F) \) by projection formula.

(2) In this case, \( F = C_{0,0}^a \) for \( (X,H_1 + a^*H_2) \). As before, \( \mu_n^*F \) is \( C_{0,0}^a \) for \( (X_n,\mu_n^*H_1 + a_n^*(n^*_AH_2)) \). Choose \( n \geq 2 \), and consequently \( \mu_n^*H_1 \) and \( n^*_AH_2 \) are both globally generated. Thus, by Observation 3.3 and Proposition 2.14, there exists \( \zeta \in \hat{A} \) such that \( \mu_n^*F \otimes a_n^*\zeta \) is \( C_{0,0} \) for \( (X_n,a_n^*(n^*_AH_2)) \). Arguing as before, we obtain \( V^i_{a_n}(\mu_n^*F) = \phi \) for \( i \geq 1 \), whence the assertion follows by projection formula.

4.4. An alternate approach via Fourier–Mukai transforms. Here we give an alternate way to prove the following statement only by using Proposition 4.7 and the Fourier–Mukai criterion of generic vanishing, i.e. Theorem 4.2, without using Theorem 4.8.

(\#) Let \( X \) be a smooth projective variety of dimension \( d \) and assume that its Albanese map \( \text{alb}_X : X \to \text{Alb}(X) \) is finite onto its image. Let \( H_2 \) be an ample line bundle on \( X \) and let \( H := H_1 + \text{alb}_X^*H_2 \) where \( H_1 \) is a globally generated line bundle on \( X \). Further, let \( F \) be a continuously \( k \)-regular torsion–free coherent sheaf for \( (X,H) \) for some integer \( 1 \leq k \leq d \). Then the sheaf \( F \) is \( GV_{-(k–1)} \).

Proof. Let us prove this in two steps, the proof of the first step uses the technique employed in the proof of Theorem 4.14.

Step 1. Let \( a : X \to A \) be a morphism from a smooth projective variety \( X \) to an abelian variety \( A \) that is finite onto its image and strongly generating. Let \( H_1 \) be a globally generated line bundle on \( X \) and let \( H'_2 \) be an ample line bundle on \( A \). Let \( H := H_1 + a^*H'_2 \). Let \( F \) be a torsion–free coherent sheaf on \( X \) that is \( C_{0,k}^a \) for \( (X,H) \) for some integer \( 1 \leq k \leq \dim X \). Then \( H^i(F \otimes a^*L) = 0 \) for any ample line bundle \( L \) and for any integer \( i \geq k \).

To prove the above statement, we note that \( F(kH) \) is \( C_{0,0}^a \) for \( (X,H) \), and we consider the base–change diagram (4.4). Once again, recall that \( X_n \) is smooth and irreducible as \( a \) is strongly generating, and fix an ample line bundle \( L \) on \( A \). We know that \( n^*_AH'_2 = n^2H'_2 + \zeta_{H'_2} \), \( n^*_AL = n^2L + \zeta_L \) for some \( \zeta_{H'_2},\zeta_L \in \hat{A} \). By Lemma 3.8, we obtain that \( \mu_n^*(F(kH)) = C_{0,0}^a \) for \( (X_n,\mu_n^*H) \). Rewriting this, we obtain that \( \mu_n^*F \otimes a_n^*(n^2H'_2 + k\zeta_{H'_2}) \otimes \mu_n^*(H_1) = C_{0,0}^a \) for \( (X_n,a_n^*(n^2H'_2 + \zeta_{H'_2} + \mu_n^*(H_1))) \). Consequently, by Observation 3.3, there exists \( \zeta'_{H'_2} \in \hat{A} \) such that \( \mu_n^*F \otimes a_n^*(n^2H'_2 + \zeta'_{H'_2}) \otimes \mu_n^*(H_1) = C_{0,0}^a \) for \( a_n^*(2H'_2) \) and \( \mu_n^*(H_1) \) are both globally generated, Corollary 2.15 now implies that

\[
\mu_n^*F \otimes \left( a_n^*(n^2H'_2 + \zeta'_{H'_2}) + \mu_n^*(H_1) \right) \cdot \left( k\mu_n^*(H_1) + \left( k^2/2 - 1 \right) a_n^*(2H'_2) \right) = \mu_n^*F \otimes a_n^*\left( 2H'_2 + \zeta'_{H'_2} \right)
\]

is \( C_{k–1,0} \) for \( (X_n,a_n^*(2H'_2)) \). Observe that \( (X_n,a_n^*(2H'_2)) \) satisfies \( (P_{a_n}) \). Consequently, by Proposition 4.7, we deduce that \( V^i_{a_n}(\mu_n^*F \otimes a_n^*(2H'_2)) = \phi \) for all integers \( i \geq k \). Since \( a_n \) is also finite onto its image, we obtain that \( V^i((a_n^*(\mu_n^*F \otimes (2H'_2))) = \phi \) for all \( i \geq k \). Observe that for \( n \gg 0 \), \( n^2L - 2H'_2 \) is ample, whence by Proposition 4.4, \( V^i((a_n^*(\mu_n^*F \otimes n^*_AL)) = \phi \) for all \( i \geq k \) if \( n \) is sufficiently large. This in particular implies by the commutativity of (4.4) that \( V^i_{a_n}(\mu_n^*(F \otimes a^*L)) = \phi \) for all \( i \geq k \) if \( n \gg 0 \). By projection formula, there is an injection \( H^i(F \otimes a^*L) \to H^i(\mu_n^*(F \otimes a^*L)) \) and consequently the desired vanishing follows.

Step 2. We finish the proof of (\#). By Theorem 4.2, it is enough to show that \( H^i(F \otimes R\Psi_{P|\hat{X}}L_{\hat{X}}) = 0 \) for all integers \( i \geq 1 \) and for any ample line bundle \( L \) on \( \hat{A} \) where \( A := \text{Alb}(X) \), \( g := \dim(\text{alb}_X) = q(X) \). Fix an ample line bundle \( L \) on \( \hat{A} \). We recall from [PP11b] proof of Theorem B that \( R\Psi_{P|\hat{X}}L_{\hat{X}} = \text{alb}_X^*(R^0F \hat{L})^* \).

Denoting translations by an element \( y \in \hat{A} \) by \( t_y : \hat{A} \to \hat{A} \), we obtain an isogeny \( q^*_L : \hat{A} \to A \) by sending
\[ y \in \hat{A} \text{ to } t_y^* L \otimes L^{\otimes -1}. \] To this end, consider the following base–change diagram.

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\phi}} & X \\
\downarrow & & \downarrow \text{alb}_X \\
\hat{A} & \xrightarrow{\phi} & A
\end{array}
\]

On the other hand, [Muk81] Proposition 3.11 shows that \( \phi_1^*(R^0\mathcal{I} L)^* = H^0(L) \otimes L. \) Consequently, we deduce using the projection formula that there is an injection

\[
H^1(\mathcal{F} \otimes R\Psi_{p|g|} L^{\otimes -1}) = H^i(\mathcal{F} \otimes \text{alb}^*_X (R^0\mathcal{I} L)^*) \to H^i(\hat{\phi}^*(\mathcal{F} \otimes \text{alb}^*_X (R^0\mathcal{I} L)^*)) = H^0(L) \otimes H^i((\phi^* \mathcal{F}) \otimes \hat{a}^* L).
\]

But \( \hat{\phi}^* \mathcal{F} \) is a torsion-free coherent sheaf that is \( C_{0,k}^{\text{alb}} \) for \((\hat{X}, \hat{\phi}^* H)\) by Lemma 3.8. Observe that \( \hat{\phi}^* H = \hat{\phi}^* H_1 + \hat{a}^*(\phi_1^* H_2) \), consequently, \( H^i((\phi^* \mathcal{F}) \otimes \hat{a}^* L) = 0 \) for all integers \( i \geq k \) by Step 1.

Similar approach (or by using the results proven before), we obtain the following

**Corollary 4.15.** Let \( a : X \to A \) be a strongly generating morphism from a smooth projective variety \( X \) to an abelian variety \( A \). Let \( H_1 \) be a line bundle on \( X \) and let \( H_2 \) be an ample line bundle on \( A \). Set \( H := H_1 + a^* H_2 \). Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( X \) that is \( C_{0,k}^{\text{alb}} \) for \((X, H)\). Assume one of the following conditions hold:

1. \( k = 1 \), \( a \) is surjective, and \( H_1 \) is globally generated; or
2. \( 1 \leq k \leq \dim X \), \( a \) is finite onto its image, and one of the following holds:
   - (A) \( H_1 \) is globally generated; or
   - (B) \( H_1 = K_X + Q \) for a nef line bundle \( Q \).

Then \( H^i(\mathcal{F} \otimes a^* L) = 0 \) for any ample line bundle \( L \) on \( A \) and for any integer \( i \geq k \). In particular, under the assumptions as above, when \( a \) is the Albanese map and \( g = q(X) \), \( H^i(\mathcal{F} \otimes R\Psi_{p|g|} L^{\otimes -1}) = 0 \) for all \( i \geq k \) and for any ample line bundle \( L \) on \( \hat{A} \).

**4.5. Immediate consequences.** We list three consequences of the generic vanishing theorems.

**Corollary 4.16.** Let \((X, H)\) be a polarized smooth projective variety. Assume that there exists a globally generated line bundle \( H_1 \) on \( X \) and an ample line bundle \( H_2 \) on \( \text{Alb}(X) \) such that \( H = H_1 + \text{alb}^*_X H_2 \), where \( \text{alb}_X : X \to \text{Alb}(X) \) is the Albanese map. Let \( \mathcal{F} \) be a torsion–free coherent sheaf on \( X \) that is continuously \( 1 \)–regular for \((X, H)\). Then \( \chi(\mathcal{F}) \geq 0 \) with equality if and only if any component \( V^0(\mathcal{F}) \) is of codimension \( 1 \).

**Proof.** We know that \( \mathcal{F} \) is GV from Theorem C, and moreover \( V^i(\mathcal{F}) = \phi \) for \( i \geq 2 \). It follows from the generic vanishing theory that \( \chi(\mathcal{F}) \geq 0 \) with equality if and only if \( \text{codim}(\mathcal{F}) \geq 1 \). Now, by [PP11b] Proposition 3.15, and [Par12] Lemma 1.8, we know that a codimension \( q \) component of \( V^0(\mathcal{F}) \) is also a component of \( V_q^\dagger(\mathcal{F}) \) whence the conclusion follows.

**Remark 4.17.** For a coherent sheaf \( \mathcal{F} \) on a smooth projective variety \( X \) of dimension \( d \), define \( R^\Delta \mathcal{F} := R\mathcal{H}om(\mathcal{F}, K_X) \). If \( \mathcal{F} \) is GV, then \( R\Phi_p(R^\Delta \mathcal{F}) \) is a sheaf concentrated in degree \( d \), i.e., \( R\Phi_p(R^\Delta \mathcal{F}) = R^d\Phi_p(R^\Delta \mathcal{F})/[-d] \). The generic vanishing theory tells us that for a GV sheaf \( \mathcal{F} \), \( \chi(\mathcal{F}) \) is the rank of \( R^\Delta \mathcal{F} \) and the support of \( R^\Delta \mathcal{F} \) is \( -V^0(\mathcal{F}) \) where \( R^\Delta \mathcal{F} = R^d\Phi_p(R^\Delta \mathcal{F}) \). Thus, if \( X, H, \mathcal{F} \) are as in Corollary 4.16, then \( \chi(\mathcal{F}) \geq 0 \) with equality if and only if any component of the support of \( R^\Delta \mathcal{F} \) is of codimension one.

**Remark 4.18.** The conclusion of Corollary 4.16 also applies in the set–up of Corollary 4.12, Theorem 4.13 and Theorem 4.14, all with \( k = 1 \).

**Corollary 4.19.** Let \((X, H)\) be a polarized abelian variety and let \( \mathcal{F} \) be torsion–free coherent sheaf on \( X \). Assume \( \mathcal{F} \) is continuously \( k \)–regular for \((X, H)\) for some \( k \in \mathbb{N} \). Then the following statements hold.

1. If \( 1 \leq k \leq \dim X \), then \( \mathcal{F} \) is a \( GV_{-(k-1)} \) sheaf. In particular, if \( k = 1 \), then \( \mathcal{F} \) is nef.
2. If \( k = 0 \), then \( \mathcal{F} \) is an IT_0 sheaf, in particular \( \mathcal{F} \) is ample.
Proof. Follows immediately from Theorem C combined with the facts that GV sheaves on abelian varieties are nef ([PP11a] Theorem 4.1), and IT₀ sheaves are ample ([Deb06] Corollary 3.2). □

Remark 4.20. Let \((X, H)\) be a polarized abelian variety and let \(\mathcal{E}\) be a vector bundle of rank \(r\) that is continuously 0–regular for \((X, H)\). Then for any subvariety \(Z \subseteq X\) of dimension \(k\), we have
\[
c_1(\mathcal{E})^k \cdot Z = (c_1(\mathcal{E}(-H)) + rH)^k \cdot Z \geq r^k H^k Z
\]
as \(\mathcal{E}(-H)\) is nef by the above corollary. In particular, \(c_1(\mathcal{E})^n \geq r^n H^n\) where \(n := \dim X\), and for any \(x \in X\), we have \(c(\det(\mathcal{E}), x) \geq r(c(H, x))\).

Corollary 4.21. Let \((X, H)\) be a polarized abelian variety. Let \(\mathcal{F}_1\) and \(\mathcal{F}_2\) be torsion–free coherent sheaves on \(X\) with one of them locally free. Assume \(\mathcal{F}_1\) is continuously \(k_1\)-regular and \(\mathcal{F}_2\) is continuously \(k_2\)-regular for \((X, H)\). Then \(\mathcal{F}_1 \otimes \mathcal{F}_2\) is \(C_{k_1+k_2}\) for \((X, H)\).

Proof. Without loss of generality, we may assume that \(k_1 = k_2 = 0\). For any integer \(1 \leq i \leq \dim X\), we have \(H^i(\mathcal{F}_1 \otimes \mathcal{F}_2(-iH)) = H^i((\mathcal{F}_1(-iH)) \otimes \mathcal{F}_2)\). By Theorem C, we know that \(\mathcal{F}_1(-iH)\) is GV–\((-i-1)\) and \(\mathcal{F}_2(-H)\) is GV. By the preservation of vanishing, \(\mathcal{F}_2\) is IT₀, whence by Proposition 4.4, we obtain the required vanishing \(H^i(\mathcal{F}_1 \otimes \mathcal{F}_2(-iH)) = 0\). □

Remark 4.22. We remark that the proof of the above corollary also shows the following: let \((X, H)\) be a polarized abelian variety and let \(\mathcal{F}_1, \mathcal{F}_2\) be torsion-free coherent sheaves on \(X\) with one of them is locally free. If \(\mathcal{F}_1\) is continuously 0–regular for \((X, H)\) and \(\mathcal{F}_2\) is IT₀, then \(\mathcal{F}_1 \otimes \mathcal{F}_2\) is 0–regular for \((X, H)\). This can also be thought of as a generalization of a result of Murty and Sastry ([MS18], Proposition 5.4.1). We will see a variant of this in Proposition 5.5.

5. Continuous \(\mathbb{Q}\) CM–regularity on abelian varieties

5.1. Continuous \(\mathbb{Q}\) CM–regularity for vector bundles. In view of the recent development of the cohomological rank function by Jiang–Pareschi (see [JP20]) that was motivated by the continuous rank function introduced and studied in [Bar15], [BPS20], it is natural to extend the notion of continuous CM–regularity to an \(\mathbb{R}\)-valued regularity function on \(\mathbb{Q}\)-twisted vector bundles.

Definition 5.1. (Cohomological rank function) Define \(h^i_\text{gen}(\mathcal{E})\) for \(i \in \mathbb{N}\) as the dimension of \(H^i(\mathcal{E} \otimes \zeta)\) for general \(\zeta \in \text{Pic}^0(X)\). Given a polarization \(L \in N^1(X)\), and \(x = a/b \in \mathbb{Q}\) with \(a, b \in \mathbb{Z}\) and \(b > 0\), following [JP20] Definition 2.1, define the cohomological rank function \(h^i_\text{gen}(xL) = b^{-2k}h^i_\text{gen}(bX^*\mathcal{E} \otimes L^{a/b})\) where \(L\) is a line bundle representing \(L\).

We note that if \(L\) is an ample line bundle, then \(\mathcal{E}\) being continuously \(k\)-regular for \((X, L)\) for \(k \in \mathbb{Z}\) is equivalent to the the condition \(h^i_\text{gen}(k-i) = 0\) for all integers \(i \geq 1\).

Let \(L \in N^1(X)\) be a polarization. Assume for some \(y = a/b \in \mathbb{Q}\) with \(a, b \in \mathbb{Z}\) and \(b > 0\), \(h^i_\text{gen}((y-i)L) = 0\) for all integers \(i \geq 1\). This means that \(h^i_\text{gen}(bX^*\mathcal{E} \otimes L^{a/b}) = 0\) for all integers \(i \geq 1\). This is equivalent to the condition that \(bX^*\mathcal{E} \otimes L^{a/b}\) is continuously 0–regular for \((X, L^{a/b})\). We claim that for any integers \(c, d > 0\), \(h^i_\text{gen}((a+c-i)L) = 0\) for all integers \(i \geq 1\). To see this, we need to show that \(h^i_\text{gen}((bd)cX^*\mathcal{E} \otimes L^{a+bc-i}) = 0\) for \(i \geq 1\), which is equivalent to \((bd)cX^*\mathcal{E} \otimes L^{a+bc-i}L^{bd}\) being continuously 0–regular for \((X, L^{bd})\). But the given condition implies that \((bd)cX^*\mathcal{E} \otimes L^{a+bd}\) is continuously 0–regular for \((X, L^{bd})\), and consequently the required continuous regularity follows by Corollary 4.15. Thus we define the following

\[
\text{Q-reg}_L(\mathcal{E}) := \inf \left\{ y \in \mathbb{Q} \mid h^i_\text{gen}(y-iL) = 0 \text{ for all integers } i \geq 1 \right\}.
\]
Example 5.2. \textit{(Continuous $\mathbb{Q}$ CM–regularity for Verlinde bundles)} We compute an example of $\text{Q-reg}_{\mathfrak{p}}(\mathfrak{E})$ when $X := J(C)$ is the Jacobian of a smooth projective curve $C$ of genus $g \geq 1$. $\mathfrak{p} = s\Theta$ is the class of $s\Theta$ where $\Theta$ is a symmetric theta–divisor on $X$, $s \geq 2$ is an integer, and $\mathfrak{E} := E_{r,k}$ is the Verlinde bundle. We recall the definition of these bundles. For a pair of positive integers $(r,k)$, let $U_{r,k}(r,0)$ be the moduli space of semistable bundles of rank $r$ and degree $0$ on $C$, and let $\det : U_{r,k}(r,0) \to X$ be the determinant map. The \textit{Verlinde bundle} associated to $(r,k)$ is by definition $E_{r,k} := \det_* \mathcal{O}_{U_{r,k}}(k\Theta)$ where $\Theta$ is the generalized theta–characteristic of $C$. For details about these bundles, we refer to [PO02] and [Opr11].

Küronya and Mustopa showed in [KM20] Proposition 3.2 that $\text{reg}^\text{cont}_{\mathcal{E}_X(x)}(E_{r,k}) = [g - \frac{1}{r^2}]$ if $2 \nmid \text{gcd}(r,k)$. We claim that $\text{Q-reg}_{\mathfrak{p}}(E_{r,k}) = g - \frac{k}{r^2}$ if $2 \nmid \text{gcd}(r,k)$.

We show this by following their proof. First of all, by [KM20] proof of Proposition 3.2, one can write $E_{r,k} = \bigoplus W_{a,b} \otimes \zeta_i$ for $\zeta_i \in X$ where $a = r/\text{gcd}(r,k)$, $b = k/\text{gcd}(r,k)$, and $W_{a,b}$ is semihomogeneous i.e., for any $x \in X$, there exists $\zeta_i \in X$ such that $t_{s,x}^\mathfrak{E} W_{a,b} = W_{a,b} \otimes \zeta_i$. We also know that $\text{rank}(W_{a,b}) = a^2$, $\text{det}(W_{a,b}) = \mathcal{O}_X(a^2 - b \Theta)$. Observe that we have the following equality:

$$\text{Q-reg}_{\mathfrak{p}}(E_{r,k}) = \inf \left\{ y \in \mathcal{Q} \mid h^i_{E_{r,k}}(\mathcal{O}(y - i) s\Theta) = 0 \forall i \geq 1 \right\} = \inf \left\{ y \in \mathcal{Q} \mid h^i_{W_{a,b}}(\mathcal{O}(y - i) s\Theta) = 0 \forall i \geq 1 \right\}$$

Since $(s')^* W_{a,b} \otimes (s\Theta) \otimes r's'$ is semihomogeneous and $c_1((s')^* W_{a,b} \otimes (s\Theta) \otimes r's') \in N^1(X)$ is $Q$–proportional to $s^2 s\Theta$. Thus, by [KM20] Proposition 2.8, $\text{Q-reg}_{\mathfrak{p}}(E_{r,k}) \leq \frac{t}{r^2}$ if and only if $(s')^* W_{a,b} \otimes (s\Theta) \otimes r's'(-gs^2 s\Theta)$ is nef, which in turn holds if and only if $a^{g-1}(s^2 b + a(r's' - gs^2 s)) \geq 0$ by [KM20] Proposition 2.7. A simple computation shows $a^{g-1}(s^2 b + a(r's' - gs^2 s)) \geq 0$ if and only if $\frac{t}{r^2} \geq g - \frac{k}{r^2}$.

5.2. \textit{Continuous $\mathbb{Q}$ CM–regularity for $\mathfrak{E}$–twisted bundles.} Given a class $n \in \text{Pic}(X)/\text{Pic}^0(X) =: N^1(X)$, following [La04b] Chapter 6, define the $\mathfrak{E}$–twisted bundle $\mathfrak{E}(x n)$ for $x \in \mathbb{Q}$ as the equivalence class of pairs $(\mathfrak{E}, xn)$ with respect to the equivalence relation generated by declaring $(\mathfrak{E}, M^e, y \mathfrak{E}) \sim (\mathfrak{E}, e \mathfrak{E} + x \mathfrak{E})$ where $M \in \text{Pic}(X)$, $m$ is its class in $N^1(X)$, $e \in \mathbb{Z}$ and $y \in \mathbb{Q}$. Now, given a polarization $L \in N^1(X)$, we define the \textit{continuous $\mathbb{Q}$ CM–regularity of $\mathfrak{E}(x n)$} as follows

$$\text{Q-reg}_{\mathfrak{E}}(\mathfrak{E}(x n)) := \text{Q-reg}_{\mathfrak{p}^2}(b^* \mathfrak{E} \otimes N^{\otimes ab})$$

where $x = a/b$, $a, b \in \mathbb{Z}$ and $b > 0$. It is a formal verification that this quantity is well–defined. We only show that it does not depend on the representation of the $\mathfrak{E}$–twist. The fact that it does not depend on its representation is trivial. It is easy to check that the equivalence described above can also be easily checked.

5.3. \textit{Generic vanishing for $\mathfrak{E}$–twisted bundles.} We mention an immediate corollary of Theorem C. In [JP20], Jiang and Pareschi extended the definitions of GV, M–regular and $\Pi_0$ sheaves to the $\mathfrak{E}$–twisted cases. In particular, according to their definitions, for an ample class $L \in N^1(X)$, a $\mathfrak{E}$–twisted vector bundle $\mathfrak{E}(x L)$ for $x = \frac{a}{b}$ with $b > 0$ is GV, M–regular, or $\Pi_0$ if so is $b^* \mathfrak{E} \otimes L^{\otimes ab}$.

\textbf{Corollary 5.3.} Let $X$ be an abelian variety. Let $L, L' \in N^1(X)$ be ample classes and let $\mathfrak{E}$ be a vector bundle on $X$. If $\text{Q-reg}_{\mathfrak{E}}(\mathfrak{E}(x L)) < 1$, then $\mathfrak{E}(x L')$ is $\Pi_0$, and if $\text{Q-reg}_{\mathfrak{E}}(\mathfrak{E}(x L')) = 1$, then $\mathfrak{E}(x L')$ is GV.
Proof. Let $x = \frac{L}{b}$ with $s > 0$. First, if $\text{Q-}\text{reg}_2(\mathcal{E}(xL')) < 1$, then we can find $a, b \in \mathbb{Z}$ with $b > 0$ such that we have $\text{Q-}\text{reg}_2(\mathcal{E}(xL')) < \frac{b}{s} < 1$. This means that $h^i_{b,s}\mathcal{E} \otimes L^{b^2s} = 0$ for all $i \geq 1$. But this means that the bundle $(b\mathcal{E})_X \otimes L^{b^2s} \otimes L^{ab^2}$ is continuously $0$–regular for $(X, L^{b^2x})$. Consequently, by Theorem C, we deduce that $(b\mathcal{E})_X \otimes L^{b^2s} \otimes ((abx^2 - b^2x^2)L)$ is GV, whence by preservation of vanishing, we conclude that $(b\mathcal{E})_X \otimes L^{b^2s}$ is GV. Thus, $s\mathcal{E}_X \otimes L^{b^2s}$ is GIT, and the conclusion follows. Finally, if $\text{Q-}\text{reg}_2(\mathcal{E}(xL)) = 1$, then similarly we see that $s\mathcal{E}_X \otimes L^{b^2s}$ is GV, and the conclusion follows.

5.4. Proof of Theorem D. The proof of Theorem D is based on the following two results.

Lemma 5.4. Let $X$ be an abelian variety of dimension $g$ and let $\mathcal{E}$ be a vector bundle on $X$. Let $a, b \in \mathbb{Z}$ with $b > 0$, and set $x = a/b \in \mathbb{Q}$. If $n, l \in N^1(X)$ with $l$ ample, and $\delta = c/d \in \mathbb{Q}$ with $c, d \in \mathbb{Z}$, $d > 0$, then

$$\text{Q-}\text{reg}_2(\mathcal{E}(xn + \delta l)) = \text{Q-}\text{reg}_2(\mathcal{E}(xn)) - \delta.$$

Proof. Let $r, s \in \mathbb{Z}$ with $s > 0$. It is enough to show the following equivalence

$$(5.1) \quad h^i_{(bd)s}(\mathcal{E} \otimes N^{ab^2} \otimes L^{b^2s})((r/s - i)b^2d^2) = 0 \forall i \geq 1 \iff h^i_{b,s}(\mathcal{E} \otimes N^{ab^2} \otimes ((r/s + c/d - i)b^2d^2) = 0 \forall i \geq 1.$$

But the left hand side of (5.1) is equivalent to the following condition

$$(5.2) \quad h^i_{\text{gen}}((bd)s)(\mathcal{E} \otimes N^{ab^2} \otimes L^{b^2c^2} \otimes L^{b(r-s)b^2}) = 0 \forall i \geq 1.$$

And the right hand side of (5.1) is equivalent to $h^i_{\text{gen}}((bd)s)(\mathcal{E} \otimes N^{ab^2} \otimes L^{(rd^2+c^2)bd^2}) = 0 \forall i \geq 1$, which under simplification boils down to (5.2). The proof is now complete.

Proposition 5.5. Let $X$ be an abelian variety of dimension $g$ and let $\mathcal{E}$ be a vector bundle on $X$. Let $l, l', n \in N^1(X)$ with $l, l'$ ample. Further, let $x = a/b, y = c/d \in \mathbb{Q}$ with $a, b, c, d \in \mathbb{Z}$, $b, c, d > 0$. Then

$$\text{Q-}\text{reg}_2(\mathcal{E}(xn + yl')) \leq \text{Q-}\text{reg}_2(\mathcal{E}(xn)).$$

Proof. Let $\beta = \beta_1/\beta_2 \in \mathbb{Q}$ with $\beta_1, \beta_2 \in \mathbb{Z}$, $\beta_2 > 0$. It is enough to show that if $\text{Q-}\text{reg}_2(\mathcal{E}(xn)) < \beta$ then $\text{Q-}\text{reg}_2(\mathcal{E}(xn + yl')) \leq \beta$. As before, $\text{Q-}\text{reg}_2(\mathcal{E}(xn)) < \beta \iff h^i_{b,s}(\mathcal{E} \otimes N^{ab^2} \otimes ((b - i)b^2d^2) = 0 \forall i \geq 1$, which is equivalent to $(bd)^2(\mathcal{E} \otimes N^{ab^2} \otimes L^{b^2c^2} \otimes L^{(b^2c^2)d})$ being continuously $0$–regular for $(X, L^{(b^2c^2)d})$. This in turn is equivalent to $\mathcal{E} := (bd)^2(\mathcal{E} \otimes N^{ab^2} \otimes L^{b^2c^2} \otimes L^{(b^2c^2)d})$ being continuously $0$–regular for $(X, L^{(b^2c^2)d})$.

We aim to show that $\text{Q-}\text{reg}_2(\mathcal{E}(xn + yl')) \leq \beta$ which is equivalent to showing the vanishing condition $h^i_{(bd)s}(\mathcal{E} \otimes N^{ab^2} \otimes L^{b^2c^2d})((\beta - i)b^2d^2) = 0 \forall i \geq 1$, which in turn is equivalent to showing that $\mathcal{E} \otimes L^{b^2c^2d}$ is continuously $0$–regular for $(X, L^{b^2c^2d})$. But this follows from Corollary 4.15.

Proof of Theorem D. We adapt an argument of Ito ([Ito20], proof of Proposition 2.9). First we show that $\text{Q-}\text{reg}_2(\mathcal{E}(\cdot)) : N^1(X)_Q \rightarrow \mathbb{R}$ is continuous. Let $\{\xi_i\}_{i \in \mathbb{N}}$ be a sequence in $N^1(X)_Q$ converging to $\xi \in N^1(X)_Q$. For any rational number $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $i \geq N_0$, $\xi_i - \xi + \delta_1$ and $\xi_i + \delta_1 - \xi_i$ are both ample. Thus, by Proposition 5.5, we deduce that for all $i \geq N_0$, we have the following inequality

$$\text{Q-}\text{reg}_2(\mathcal{E}(\xi_i + \delta_1)) \leq \text{Q-}\text{reg}_2(\mathcal{E}(\xi_i)) \leq \text{Q-}\text{reg}_2(\mathcal{E}(\xi - \delta_1)).$$

Now by Lemma 5.4, we deduce that for all $i \geq N_0$, $|Q-\text{reg}_2(\mathcal{E}(\xi_i)) - Q-\text{reg}_2(\mathcal{E}(\xi))| \leq \delta$ and that proves the claim. To finish the proof, we need to show that given a sequence $\{\xi_i\}_{i \in \mathbb{N}}$ in $N^1(X)_Q$ converging to $\xi \in N^1(X)_\mathbb{R}$, $\{\text{Q-}\text{reg}_2(\mathcal{E}(\xi_i))\}_{i \in \mathbb{N}}$ converges to a real number. As before, for any rational $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that for $j, k \geq N_0$, $\xi_j - \xi + \delta_1$ and $\xi + \delta_1 - \xi_k$ are both ample. Using Proposition 5.5, we deduce that $\text{Q-}\text{reg}_2(\mathcal{E}(\xi_j + \delta_1)) \leq \text{Q-}\text{reg}_2(\mathcal{E}(\xi_k - \delta_1))$ which by Lemma 5.4 implies that for all $j, k \geq N_0$, we have the inequality $\text{Q-}\text{reg}_2(\mathcal{E}(\xi_j)) - \delta \leq \text{Q-}\text{reg}_2(\mathcal{E}(\xi_k)) + \delta$. Thus we obtain

$$\lim_{j \to \infty} \text{Q-}\text{reg}_2(\mathcal{E}(\xi_j)) - \delta \leq \lim_{k \to \infty} \text{Q-}\text{reg}_2(\mathcal{E}(\xi_k)) + \delta.$$
Now, by letting $\delta \to 0$, we see that $\{ \text{Q-reg}_{t}(E \langle \xi_{i} \rangle) \}_{i \in \mathbb{N}}$ converges to a real number. $\square$

6. Syzygies of tautological bundles of zero–regular bundles

6.1. Background on syzygies. Let $X$ be a smooth projective variety and let $L$ be a very ample line bundle on $X$. Consider the embedding $X \subseteq \mathbb{P}^{r}$ given by the complete linear series $|L|$ where $r = h^{0}(L) - 1$. One has a minimal graded free resolution of $R(X, L) := \bigoplus_{q \geq 0} H^{q}(qL)$ as an $S := \text{Sym}^{*}(H^{0}(L))$ module as follows:

$$0 \to E_{r+1} = \bigoplus_{j} S(-a_{r+1,j}) \to E_{r} = \bigoplus_{j} S(-a_{r,j}) \to \cdots \to E_{1} = \bigoplus_{j} S(-a_{1,j}) \to E_{0} = \bigoplus_{j} S(-a_{0,j}) \to R(X, L) \to 0.$$ 

For a reference of the following definition, see for example [Laz04a] Definition 1.8.50.

**Definition 6.1. (Projective normality and $N_{p}$ property)** Suppose we are in the situation as above.

- The embedding given by the complete linear series $|L|$ is called *projectively normal* if $E_{0} = S$.
- We say that $L$ satisfies $N_{p}$ property if $a_{i,j} = i + 1$ for all $0 \leq i \leq p$.

In practice, to calculate the syzygies of a projective variety one needs to calculate cohomology groups involving the syzygy bundles that we will define next (see for example [Par07], Section 3).

**Definition 6.2. (Syzygy bundle)** Let $X$ be a smooth projective variety and let $E$ be a globally generated bundle on $X$. The *syzygy bundle* $M_{E}$ is the kernel of the map $H^{0}(E) \otimes \mathcal{O}_{X} \to E$ i.e. we have the exact sequence

$$0 \to M_{E} \to H^{0}(E) \otimes \mathcal{O}_{X} \to E \to 0. \quad (6.1)$$

In this article, we will only make use of the following proposition of Park that was derived from a criterion of $N_{p}$ property due to Ein and Lazarsfeld.

**Proposition 6.3. ([Par07], Proposition 3.2)** Let $X$ be a smooth projective variety and let $E$ be an ample and globally generated bundle on $X$. Then $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ satisfies $N_{p}$ property if $H^{k}\left(\bigwedge^{i} M_{E} \otimes E^{*} \otimes j\right) = 0$ for all $0 \leq i \leq p + 1, j \geq 1$.

6.2. $N_{p}$ property via Koszul embeddings. We aim to prove Theorem E stated in the introduction.

6.2.1. We start with the definition of the Koszul property (cf. [Tot13] Section 1).

**Definition 6.4. (Koszul property)** Let $X$ be a smooth projective variety and let $L$ be a very ample line bundle on $X$. Then $R := R(L)$ is called $N$–Koszul for $N \in \mathbb{N}$ if the resolution of $C$ as $R$ module is linear until $N$–th stage i.e., it looks as follows

$$\cdots \to E_{N} = \bigoplus R(-N) \to E_{N-1} = \bigoplus R(-(N-1)) \to \cdots \to E_{1} = \bigoplus R(-1) \to E_{0} = R \to C \to 0.$$ 

$R$ is called Koszul if it is $N$–Koszul for all $N$.

We take this opportunity to state a few immediate syzygetic consequences of Theorem B for the determinant bundles of 0–regular vector bundles of an embedded variety.

**Corollary 6.5.** Let $X$ be a smooth projective variety of dimension $n$ and let $H$ be an ample and globally generated line bundle on $X$. Also, let $E$ be a 0–regular vector bundle for $(X, tH)$ of rank $r$ for some $t \in \mathbb{N}$, $t \geq 1$. Then the following statements hold.

1. If $H$ is very ample and $r(t-1) \geq n+1$ then $K_{X} + \det(E)$ satisfies $N_{r(t-1)-n-1}$ property.
2. If $H$ is very ample, $(X, H, E) \neq (\mathbb{P}^{s}, \mathcal{O}_{\mathbb{P}^{s}}(1), \bigoplus \mathcal{O}_{\mathbb{P}^{s}}(t-1))$, then
   
   (a) if $r(t-1) \leq n$ then $K_{X} + \det(E)$ satisfies $N_{r(t-1)-n}$ property,
   
   (b) if $r(t-1) \geq n+1$ then the section ring $R(K_{X} + \det(E))$ is Koszul.
3. If $H$ is very ample and $r(t-1) \geq d+1$ then $\det(E)$ satisfies $N_{r(t-1)-d+1}$ property.
Proof. All of these follow from Theorem B combined with the works of other authors. (1) follows from [EL93] Corollary 2.2. In order to see (2), notice that if $X = \mathbb{P}^N$ and $\det(\mathcal{E}(-(t - 1)H))$ is trivial, then [OSS80], Corollary of Theorem 3.2.1 implies $\mathcal{E} = \bigoplus \mathcal{O}_{\mathbb{P}^N}((t - 1)H)$. The conclusion follows from [EL93] Proposition 3.1, and [Par93] Theorem C. Finally (3) follows from [EL93] Proposition 3.3. \qed

6.2.2. We provide a few necessary preliminaries. We first recall a resolution of the diagonal $\Delta \subset X \times X$ that was first constructed by Orlov in [Orl97] when $L$ is sufficiently ample, and by Kawamata in [Kaw04] when $R(L)$ is Koszul. Finally, Totaro generalized these earlier results by assuming $N$--Koszulity. To state the result, we need to introduce a few notations.

Given a very ample line bundle $L$ on $X$, the vector spaces $B_m$ are defined inductively for $m \geq 0$ by

$$B_0 = \mathbb{C}, \quad B_1 = H^0(L), \quad B_m = \ker \left(B_{m-1} \otimes H^0(L) \to B_{m-2} \otimes H^0(2L)\right).$$

Finally, we set $R_0 = \mathcal{O}_X$ and for $m \geq 0$, define $R_m = \ker(B_m \otimes \mathcal{O}_X \to B_{m-1} \otimes L)$.

**Theorem 6.6.** ([Tot13] Theorem 2.1) Let $X$ be a smooth projective variety with a very ample line bundle $L$. If $R(L)$ is $N$--Koszul for some $N \geq 1$, then we have an exact sequence of sheaves on $X \times X$ as follows

$$\Delta(L)_*: \cdots \to R_{N-1} \otimes L^\otimes(N-1) \to \cdots \to R_1 \otimes L^\otimes 1 \to R_0 \otimes \mathcal{O}_X \to \mathcal{O}_X \to 0.$$

We recall two more results. The following lemma is due to Arapura ([Ara06] Corollary 1.9) and Totaro.

**Lemma 6.7.** ([Tot13] Lemma 3.3) Let $X$ be a smooth projective variety with a very ample line bundle $L$. Let $\mathcal{F}$ be a $0$--regular coherent sheaf for $(X, L)$. If $R(L)$ is $(n - j + 1 + i)$--Koszul, then $H^j(\mathcal{F} \otimes \mathcal{O}_X) = 0$ for $j \geq 1$.

The regularity inequality of Totaro is one of the main ingredients of the proof of Theorem E.

**Theorem 6.8.** ([Tot13] Theorem 3.4) Let $X$ be a smooth projective variety of dimension $n$ with a very ample line bundle $L$. Let $\mathcal{E}$ be a vector bundle and let $\mathcal{F}$ be a coherent sheaf on $X$. If $R(L)$ is $2n$--Koszul, then $\text{reg}_L(\mathcal{E} \otimes \mathcal{F}) \leq \text{reg}_L(\mathcal{E}) + \text{reg}_L(\mathcal{F})$.

6.2.3. We deduce a criterion for surjectivity of multiplication maps using Theorem 6.6 and Lemma 6.7.

**Lemma 6.9.** Let $X$ be a smooth projective variety of dimension $n$ with a very ample line bundle $H$. Let $\mathcal{E}$ and $\mathcal{F}$ be a $0$--regular vector bundles for $(X, H)$. If $R(H)$ is $3n$--Koszul, then the multiplication map $H^0(\mathcal{E}) \otimes H^0(\mathcal{F}) \to H^0(\mathcal{E} \otimes \mathcal{F})$ surjects.

**Proof.** Tensoring $\Delta(H)_*$ with $\mathcal{E} \otimes \mathcal{F}$, we obtain the following resolution of $\mathcal{E} \otimes \mathcal{F}$:

$$\cdots \to (R_{2n} \otimes \mathcal{E}) \otimes (\mathcal{F} \otimes L^\otimes 2n) \to \cdots \to (R_1 \otimes \mathcal{E}) \otimes (\mathcal{F} \otimes L^\otimes 1) \to \mathcal{E} \otimes \mathcal{F} \to \mathcal{E} \otimes \mathcal{F} \to 0.$$ 

Thus, by [Laz04a] Proposition B.1.1, we need $H^i((\mathcal{F} \otimes \mathcal{O}_X)(\mathcal{F} \otimes L^\otimes i)) = 0$ for $1 \leq i \leq 2n$ which by Künneth formula, boils down to showing that for all $0 \leq b \leq i$, either

$$(A) \quad H^{i-b}(\mathcal{E} \otimes \mathcal{O}_X) = 0 \quad \text{or} \quad (B) \quad H^b(\mathcal{F} \otimes L^\otimes i) = 0.$$ 

By $0$--regularity of $\mathcal{F}$, we only need to show (A) when $0 \leq b \leq i - 1$ which follows from Lemma 6.7 since $R(H)$ is $(n + b + 1) \leq 3n$--Koszul. \qed

Proof of Theorem E is based on the following proposition.

**Proposition 6.10.** Let $X$ be a smooth projective variety of dimension $n$ with a very ample line bundle $H$. Assume for some $p \geq 0$, $t \geq p + 1$ and $R(kH)$ is $3n$--Koszul for all $1 \leq k \leq p + 1$. Let $\mathcal{E}$ be a $0$--regular vector bundle for $(X, tH)$. Then, for all $0 \leq p' \leq p$ and $m \geq 1$, $M_{\mathcal{E}}^{p'} \otimes \mathcal{E} \otimes m$ is $0$--regular for $(X, (p + p' + 1)H)$.

**Proof.** Induct on $p$ and $p'$. Clearly, by Theorem 6.8, the statement is true for all $p \geq 0$ if $p' = 0$ since $\mathcal{E}$ is $0$--regular for $(X, (p + 1)H)$ thanks to Corollary 2.15. Assume the statement holds for all $0 \leq p \leq p_0 - 1$ and for all $0 \leq p' \leq p_0' - 1$ where $1 \leq p_0' \leq p_0$. We aim to show that $M_{\mathcal{E}}^{p'} \otimes \mathcal{E} \otimes m$ is $0$--regular for $(X, (p_0 + p'' + 1)H)$ when $0 \leq p'' \leq p_0'$. By Corollary 2.15, we know this if $p'' \leq p_0' - 1$, so it is enough to show that $M_{\mathcal{E}}^{p_0''} \otimes \mathcal{E} \otimes m$ is
0–regular for \((X, (p_0 - p'_0 + 1)H)\). Also, thanks to Theorem 6.8, it is enough to show this for \(m = 1\). To this end, consider
\[
0 \to M^{p_0}_E \otimes \mathcal{E}(-i(p_0 - p'_0 + 1)H) \to H^0(\mathcal{E}) \otimes M^{p_0-1}_E \otimes \mathcal{E}(-i(p_0 - p'_0 + 1)H) \to M^{p_0-1}_E \otimes \mathcal{E} \otimes \mathcal{E}^2(-i(p_0 - p'_0 + 1)H) \to 0.
\]

**Case 1: \(i = 1\)**. By the induction hypothesis, \(M^{p_0-1}_E \otimes \mathcal{E}\) is 0–regular for \((X, (p_0 - p'_0 + 2)H)\). Consequently, Corollary 2.15 implies \(M^{p_0-1}_E \otimes \mathcal{E}(-(p_0 - p'_0 + 1)H)\) is 0–regular for \((X, H)\). Since \(R(H)\) is 3n–Koszul, by Lemma 6.9, we obtain \(H^1(M^{p_0}_E \otimes \mathcal{E}(-(p_0 - p'_0 + 1)H)) = 0\).

**Case 2: \(i \geq 2\)**. We have noted in the previous case that \(\text{reg}_H(M^{p_0}_E \otimes \mathcal{E}) \leq -(p_0 - p'_0 + 1)\). An application of Corollary 2.15 also shows that \(\text{reg}_H(\mathcal{E}) \leq -p_0\). Consequently, Theorem 6.8 yields
\[
\text{reg}_H(M^{p_0-1}_E \otimes \mathcal{E} \otimes \mathcal{E}^2) \leq -(2p_0 - p'_0 + 1).
\]

Notice that \((i - 1)(2p_0 - p'_0 + 1) \geq i(p_0 - p'_0 + 1)\) since \(p'_0 \geq 1\) and \(i \geq 2\), whence the cohomology group \(H^{i-1}(M^{p_0-1}_E \otimes \mathcal{E} \otimes \mathcal{E}^2(-(i(p_0 - p'_0 + 1)H)) = 0\). Since \(H^i(M^{p_0-1}_E \otimes \mathcal{E}(-(i(p_0 - p'_0 + 1)H)) = 0\) as well, we obtain the required vanishing.

**Proof of Theorem E**. By Proposition 6.3, we need to check \(H^i(M^{p'}_E \otimes \mathcal{E}^\otimes m) = 0\) for all \(i \geq 1, 0 \leq p' \leq p + 1\) and \(m \geq 1\). For \(p' \leq p\), the vanishing is an immediate consequence of Proposition 6.10. Consider
\[
0 \to M^{p+1}_E \otimes \mathcal{E}^\otimes m \to H^0(\mathcal{E}) \otimes M^{p}_E \otimes \mathcal{E}^\otimes m \to M^{p+1}_E \otimes \mathcal{E}^\otimes m+1 \to 0.
\]

Lemma 6.9 shows the above remains exact at the level of global sections as \(M^p_E \otimes \mathcal{E}^\otimes m\) is 0–regular for \((X, H)\). This shows the vanishings of higher cohomology groups of \(M^{p+1}_E \otimes \mathcal{E}^\otimes m\).

6.3. **The case of abelian varieties**. Finally we prove Corollary F which is an immediate consequence of Theorem A.1 due to Ito.

**Proof of Corollary F**. By Corollary 4.19, \(\mathcal{E}(-x\mathcal{L})\) is \(H_0\) for any \(x \in \mathbb{Q}\) with \(x < 1\). Since \(\beta(h) < \frac{1}{p+2}\), we have \((p+2)\beta(h_\mathbb{Q}) < 1\). Thus, for \(0 < c \ll 1\), we have \(x = 1 - c \geq (p+2)\beta(h_\mathbb{Q})\) whence the assertion follows immediately by Theorem A.1.

**APPENDIX A. ON SYZYGIES OF PROJECTIVE BUNDLES ON ABELIAN VARIETIES**

**BY ATSUSHI ITO**

In this appendix, we follow the notation in the previous sections. In particular, we work over the field of complex numbers \(\mathbb{C}\).

Let \((X, L)\) be a polarized abelian variety and \(\mathcal{L} \in \mathcal{N}^1(X)\) be the class of \(L\). In [JP20], Z. Jiang and G. Pareschi define the basepoint-freeness threshold \(\beta(\mathcal{L}) \in (0, 1]\). This invariant is quite useful to study syzygies of polarized abelian varieties since [JP20] and F. Caucci [Cau20] show that \(L\) satisfies \(N_p\) property if \(\beta(\mathcal{L}) < 1/(p+2)\).

The purpose of this short note is to prove the following theorem:

**Theorem A.1.** Let \((X, L)\) be a polarized abelian variety and \(\mathcal{E}\) be a vector bundle on \(X\). Let \(p \geq 0\) be an integer. Assume that there exists a rational number \(x \geq (p + 2)\beta(\mathcal{L})\) such that \(\mathcal{E}(-x\mathcal{L})\) is \(M\)-regular. Then \(\mathcal{O}_{\mathcal{P}(\mathcal{E})}(1)\) satisfies \(N_p\) property.

**Remark A.2.** In the case \(\mathcal{E} = L\), \(L(-x\mathcal{L})\) is \(M\)-regular if and only if \(x < 1\) (cf. [Ito22a, Example 2.1]). Hence the existence of a rational number \(x\) in Theorem A.1 is equivalent to \(\beta(\mathcal{L}) < 1/(p+2)\) in this case.

To prove this theorem, we use the following lemma.

**Lemma A.3.** Let \((X, L)\) be a polarized abelian variety and \(\mathcal{E}\) be a vector bundle on \(X\). Assume that there exists a rational number \(x \geq \beta(\mathcal{L})\) such that \(\mathcal{E}(-x\mathcal{L})\) is \(M\)-regular. Then
(1) \( E \) is IT\(_0\) and globally generated.

(2) Let \( M_E \) be the syzygy bundle of \( E \) defined by (6.1). For a rational number \( y > 0 \), \( M_E \langle y \rangle \) is IT\(_0\) if 
\[
\frac{1}{x} + \frac{1}{y} \leq \frac{1}{\beta \langle \ell \rangle}.
\]

Proof. (1) Since \( E \langle -\langle x \rangle \rangle \) is M-regular and \( x \geq \beta \langle \ell \rangle > 0 \), the bundle \( E \) is IT\(_0\). The global generation of \( E \) follows from [Ito22a, Theorem 1.2 (1)].

(2) Write \( y = \frac{a}{b} \) with integers \( a, b > 0 \). Then 
\[
M_E \langle \frac{a}{b} \rangle = M_E \langle \frac{N}{b} \rangle \text{ is IT}_0 \iff b_X^* M_E \otimes L^{a/b} \text{ is IT}_0 \iff b_X^* (b_X^* M_E \otimes L^{a/b}) = M_E \otimes b_X^* (L^{a/b}) \text{ is IT}_0,
\]
where the first equivalence follows from the definition and the second one holds since the property IT\(_0\) is preserved by the pushforward by isogenies (cf. [Ito22a, (2.4)]).

Consider the exact sequence
\[
0 \to M_E \otimes b_X^* (L^{a/b}) \to H^0(E) \otimes b_X^* (L^{a/b}) \to E \otimes b_X^* (L^{a/b}) \to 0
\]
obtained by tensoring \( b_X^* (L^{a/b}) \) with (6.1). Since \( L^{a/b} \) is IT\(_0\), so is the pushforward \( b_X^* (L^{a/b}) \). Since \( E \) is IT\(_0\) as well by (1), both \( H^0(E) \otimes b_X^* (L^{a/b}) \) and \( E \otimes b_X^* (L^{a/b}) \) are IT\(_0\). Thus \( H^i(M_E \otimes b_X^* (L^{a/b}) \otimes \zeta) = 0 \) for any \( i \geq 2 \) and \( \zeta \in \text{Pic}^0(X) \), and hence \( M_E \otimes b_X^* (L^{a/b}) \) is IT\(_0\) if and only if
\[
H^1(M_E \otimes b_X^* (L^{a/b}) \otimes \zeta) = 0
\]
for any \( \zeta \in \text{Pic}^0(X) \), which is equivalent to the surjectivity of
\[
(A.1) \quad H^0(E) \otimes H^0(b_X^* (L^{a/b}) \otimes \zeta) \to H^0(E \otimes b_X^* (L^{a/b}) \otimes \zeta).
\]

In conclusion, (2) follows from the surjectivity of (A.1) for any \( \zeta \in \text{Pic}^0(X) \). Set \( \mathcal{F} = b_X^* (L^{a/b}) \otimes \zeta \).

Recall that \( y = \frac{a}{b} \). Since 
\[
\frac{1}{x} + \frac{b}{a} = \frac{1}{x} + \frac{1}{y} \leq \frac{1}{\beta \langle \ell \rangle}
\]
by assumption, it suffices to show that
\[
\varphi^*_t \Phi(E) \otimes \varphi^*_t \Phi((-1)^*_X \mathcal{F}) \langle (\frac{1}{x} + \frac{b}{a}) \ell \rangle
\]
is M-regular by [Ito22a, Proposition 4.4], where \( \varphi_t : X \to \tilde{X} = \text{Pic}^0(X) \) is the isogeny induced by the the polarization \( L \Phi = R \mathcal{F} : \text{D}(X) \to \text{D} \tilde{X} \) is the Fourier-Mukai functor associated to the Poincaré line bundle on \( X \times \tilde{X} \), and \( (-1)^*_X \) is the multiplication map by \((-1)^*_X \mathcal{F}) \) locally free sheaves since \( E \), \( (-1)^*_X \mathcal{F} \) are IT\(_0\).

The rest is to check the M-regularity of \( \varphi_t^* \Phi(E) \otimes \varphi_t^* \Phi((-1)^*_X \mathcal{F}) \langle (\frac{1}{x} + \frac{b}{a}) \ell \rangle \), which can be shown by modifying the argument in the proof of [Ito22a, Proposition 4.4 (1)] as follows: The pullback of a \( \mathcal{Q} \)-twisted sheaf \( \mathcal{F} \langle t \ell \rangle \) by the multiplication map \( n_X \) on \( X \) for \( n \geq 1 \) is defined as 
\[
n_X^* (\mathcal{F} \langle t \ell \rangle) := n_X^* \mathcal{F} \langle t n_X^* \ell \rangle = n_X^* \mathcal{F} \langle n^2 t \ell \rangle.
\]
Since M-regularity is preserved by such pullbacks (cf. [Ito22a, (2.2)]), it suffices to show the M-regularity of
\[
(A.2) \quad a_X^* \left( \varphi_t^* \Phi(E) \otimes \varphi_t^* \Phi((-1)^*_X \mathcal{F}) \langle (\frac{1}{x} + \frac{b}{a}) \ell \rangle \right) = a_X^* \left( \varphi_t^* \Phi(E) \otimes \varphi_t^* \Phi((-1)^*_X \mathcal{F}) \langle (\frac{1}{x} + ab) \ell \rangle \right).
\]
Recall that \( \mathcal{F} = b_X^* (L^{a/b}) \otimes \zeta = b_X^* (L^{a/b}) \otimes b_X^* \zeta \). We take \( \zeta' \in \text{Pic}^0(X) \) such that \( \zeta' \otimes ab = b_X^* \zeta \) and set \( L' = (-1)^*_X (L \otimes \zeta') \). Then we have
\[
(-1)^*_X \mathcal{F} = (-1)^*_X b_X^* ((L \otimes \zeta') \otimes ab) = b_X^* ((-1)^*_X (L \otimes \zeta') \otimes ab) = b_X^* (L^{ab})
\]
and hence
\[
a_X^* \varphi_t^* \Phi((-1)^*_X \mathcal{F}) = a_X^* \varphi_t^* \Phi(b_X^* (L^{ab})) = a_X^* \varphi_t^* b_X^* \Phi(L^{ab}) = \varphi_{ab_X^*} \Phi(L^{ab}) = H^0(L^{ab}) \otimes L^{ab}.
\]
In fact, the second equality follows from [Muk81, (3.4)]. The third one follows from $b_X \circ \varphi_L \circ a_X = \varphi_{abl}$. The last one follows from [Muk81, Proposition 3.11 (1)] since the numerical class of $L'\otimes ab$ is $abl$ by $L' = (-1)^X (L \otimes \xi') \equiv L$. Hence it holds that

$$a_X^* \left( \varphi_L^* \Phi(\xi) \otimes \varphi_L^* \Phi((-1)^X \xi, \mathbb{R}) \otimes (\xi^2 + ab) \right) = a_X^* \varphi_L^* \Phi(\xi) \otimes H^0(L' \otimes ab) \otimes L' \otimes ab \left((\xi^2 + ab) \right) = H^0(L' \otimes ab) \otimes a_X^* \left( \varphi_L^* \Phi(\xi) \otimes (\xi^2 + ab) \right),$$

where the second equality follows from $L' \equiv L$. Since $\xi(-x\xi) \equiv M$-regular, $\varphi_L^* \Phi(\xi) \otimes (\xi^2 + ab) \equiv M$-regular as well by [Ito22a, Proposition 4.1]. Thus so is the pullback $a_X^* \left( \varphi_L^* \Phi(\xi) \otimes (\xi^2 + ab) \right)$ and hence we obtain the $M$-regularity of (A.2), which implies (2).

**Proof of Theorem A.1.** Since $x \geq (p + 2) \beta(l) \geq \beta(l)$, the bundle $\xi$ is globally generated by Lemma A.3 (1).

To prove $N_p$ property for $\mathcal{O}(\xi(l))$, it suffices to show that $H^k(M_{\xi}^\otimes \otimes \xi^\otimes j) = 0$ for all $0 \leq i \leq p + 1, j \geq 1$ by Proposition 6.3 since we work over $\mathbb{C}$, whose characteristic is zero. Hence this theorem holds if $M_{\xi}^\otimes \otimes \xi^\otimes j = \Gamma_0$ for all $0 \leq i \leq p + 1, j \geq 1$.

If $i = 0, M_{\xi}^\otimes \otimes \xi^\otimes j = \Gamma_0$ for $j \geq 1$ since so is $\xi$ by Lemma A.3 (1). If $i \geq 1, M_{\xi}^\otimes \otimes \xi^\otimes j$ is written as

(A.3)

$$M_{\xi}^\otimes \otimes \xi^\otimes j = (M_{\xi}^\otimes (\xi^2 \xi)) \otimes \xi(-x\xi) \otimes (\xi^\otimes j - 1)$$

as a $\mathbb{Q}$-twisted sheaf. For $1 \leq i \leq p + 1, M_{\xi}^\otimes (\xi^2 \xi) \equiv \Gamma_0$ by Lemma A.3 (2) since it holds that

$$\frac{1}{x} + \frac{i}{x} \leq \frac{p + 2}{x} \leq \frac{1}{\beta(l)}$$

by assumption. Furthermore, $\xi(-x\xi)$ is $M$-regular by assumption and $\xi$ is $M_\xi$ by [Cau20, Proposition 3.4].

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