Low-Energy Spectrum of Toeplitz Operators with a Miniwell

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Abstract: We study the concentration properties of low-energy states for quantum systems in the semiclassical limit, in the setting of Toeplitz operators, which include quantum spin systems as a large class of examples. We establish tools proper to Toeplitz quantization to give a general subprincipal criterion for localisation. In addition, we build up symplectic normal forms in two particular settings, including a generalisation of Helffer–Sjöstrand miniwells, in order to prove asymptotics for the ground state and estimates on the number of low-lying eigenvalues.

1. Introduction

1.1. Quantum selection. The computation of ground states for quantum systems is an ubiquitous problem of great difficulty in the non-integrable case, such as antiferromagnetic spin models on lattices in several dimensions. On those systems, approaches in the large spin limit are commonly used [11,26,35,39], in an effort to reduce the problem to the study of the minimal set of the classical energy. A general procedure of semiclassical order by disorder was proposed by Douçot and Simon [15], in situations where this classical minimal set is not discrete.

In the mathematical setting of Schrödinger operators in the semiclassical limit, a general study of ground state properties was done by Helffer and Sjöstrand [22,23], including situations where the minimal set of the potential is a smooth submanifold. The classical phase space of spin systems, a product of spheres, is compact. In particular, spin systems are neither Schrödinger operators nor given by Weyl quantization. However, spin operators are example of Toeplitz operators, which allows to understand the large spin limit as a semiclassical limit. In a previous article [13], we studied semiclassical concentration of ground states in the context of semiclassical Toeplitz operators, when the minimal set of the classical energy (or symbol) is a finite set of non-degenerate
points, with results analogous to the Schrödinger case [22]. To this end we introduced the Melin value (see Definition 2.17 in the present article) associated with a quadratic form on $\mathbb{R}^{2n}$.

In frustrated antiferromagnetic spin systems, such as on the Kagome lattice, the minimal set of the classical energy does not form a smooth submanifold. The goal of this article is to not only to extend the degenerate case [23] to Toeplitz quantization, but also to generalise the geometrical conditions on the zero set of the classical energy.

We prove several results of quantum selection: not all points of classical phase space where the energy is minimal are equivalent for quantum systems; and the semiclassical quantum ground state localises only on a subset of the classical minimal set. To do so, on one hand we develop techniques which are proper to Toeplitz quantization; on the other hand we prove new symplectic normal forms which are also useful in the context of pseudodifferential calculus.

Theorem A gives a general criterion (in terms of the Melin value) for the localisation of low-energy states of an arbitrary (smooth) Toeplitz operator. This theorem is in the spirit of Melin’s inequality (see [30], or [24], Thm 22.2.3). Theorem B is more precise and applies in a particular setting which generalises the miniwells of [23]. Theorem C treats a degenerate case where the symbol is minimal on a set with a singular point. Theorem D analyses the relative role of regular and singular points in the low-energy Weyl law.

1.2. Main results.

Subprincipal localisation in general In order to state the main theorems we need to briefly introduce Toeplitz quantization, the criterion under which localisation takes place, and what localisation means in this context.

Toeplitz quantization takes place on quantizable Kähler manifolds [7], which are complex manifolds with a symplectic and a Riemannian structure. Let $M$ be such a manifold and let $h \in C^\infty(M, \mathbb{R})$, which might depend on $N \in \mathbb{N}$. Through Toeplitz quantization we associate to this function a sequence of self-adjoint operators $(T_N(h_N))_{N \geq 1}$ acting on a sequence of Hilbert spaces $(H_N)_{N \geq 1}$ (see Definitions 2.1 and 2.10). The semiclassical limit is $N \to +\infty$. In this article we are interested in the spectrum and eigenvectors of $T_N(h_N)$ for $N$ large. See Sect. 2 for a more detailed presentation of Toeplitz operators.

Let $h = \sum_{k=0}^{+\infty} N^{-i} h_i + O(N^{-\infty})$ be a classical symbol on a Kähler quantizable manifold $M$ and suppose $\min(h_0) = 0$. The selection criterion is a function $\mu : \{h_0 = 0\} \to \mathbb{R}^+$, which depends on the Hessian of $h_0$ and on $h_1$. It captures the effects of order $N^{-1}$ on the low-energy spectrum of $T_N(h)$. For each point $x$ such that $h_0(x) = 0$, we call $\mu(x)$ the Melin value at $x$ (see Definition 2.15).

The Hilbert spaces $H_N$ consist of $L^2$ functions on a circle bundle over $M$ (with projection $\pi$). In particular, for any Borel set $B \subset M$ and function $u \in H_N$, the microlocal mass of $u$ on $B$ is directly defined (as $\int_{\pi^{-1}(B)} |u|^2 dVol$), in contrast with Weyl quantization where this needs some work; see Definition 2.5.

**Theorem A.** Let $M$ be a compact Kähler quantizable manifold and let $h$ be a classical symbol on $M$. Suppose that $\min(h_0) = 0$. Let $\mu$ be the function associating to each point where $h_0$ vanishes the Melin value at this point. Let

$$\mu_{\min} = \min(\mu(x), x \in M, h_0(x) = 0).$$
Then, as $N \to +\infty$, one has

\[ |\min \text{Sp}(T_N(h)) - N^{-1}\mu_{\text{min}}| = o(N^{-1}). \]

Let $((\lambda_N, u_N))_{N \geq 1}$ be a sequence of eigenpairs of $(T_N(h))_{N \geq 1}$. If $\|u_N\|_{L^2} = 1$ and $\lambda_N \leq N^{-1}(\mu_{\text{min}} + C)$ for some $C > 0$, for any open set $U$ at positive distance from

\[ \{x \in M, h_0(x) = 0, \mu(x) \leq \mu_{\text{min}} + C\}, \]

as $N \to +\infty$ there holds

\[ \int_{\pi^{-1}(U)} |u_N|^2 dVol = O(N^{-\infty}). \]

Theorem A already appears in previous work [13], under the much stronger hypothesis that $\{h = 0\}$ is a finite set of regular critical points.

Two particular cases  A typical illustration of quantum selection is given by the following symbol on $\mathbb{C}^2$:

\[ h = \Im(z_1)^2 + (1 + \Re(z_1)^2)|z_2|^2. \]

This symbol vanishes on $Z = \{z_2 = 0, \Im(z_1) = 0\}$ but its Hessian varies along $Z$. It can be solved explicitly in Toeplitz quantization. The ground state of $T_N(h)$ is

\[ C(N) \exp[-N(|z_2|^2/2 + \Im(z_1)^2)] \exp(-\sqrt{N}z_1^2) \exp(\beta(N)z_2^2), \]

where $C(N)$ is a normalisation constant and

\[ \beta(N) = \frac{N}{2} \left(\sqrt{N} - 1\right)^2 - \frac{N}{2} + \sqrt{N} = O(1). \]

This ground state concentrates only at $\{z = 0\}$, but the speed of concentration is anisotropic: it is $N^{-\frac{1}{2}}$ in the directions transverse to $Z$, but only $N^{-\frac{1}{4}}$ along $Z$.

This behaviour is characteristic of the “miniwell” situation, which is the subject of the following theorem.

**Theorem B.** Under the hypotheses of Theorem A, suppose that the function $\mu$ reaches its minimum on a unique point $P_0$. Suppose further that, in a neighbourhood of $P_0$, the set $\{h_0 = 0\}$ is an isotropic submanifold of $M$, on which $h_0$ has non-degenerate transverse Hessian matrix. Then $\mu$ is a smooth function on this piece of isotropic submanifold. Finally, suppose that, near $P_0$, along $\{h_0 = 0\}$, the function $\mu$ reaches its minimum in a non-degenerate way.

Then for any sequence $(u_N)_{N \geq 1}$ of unit eigenfunctions corresponding to the first eigenvalue of $T_N(h)$, for any $\epsilon > 0$, one has

\[ \int_{\{\text{dist}(\pi(y), P_0) > N^{-\frac{1}{4} + \epsilon}\}} |u_N(y)|^2 dVol = O(N^{-\infty}). \]

Moreover, the first eigenvalue is simple and the gap between the two first eigenvalues is of order $N^{-\frac{3}{2}}$. There is a full expansion of the first eigenvalue and eigenvector in powers of $N^{-\frac{1}{4}}$. 


In all this article, we will informally call “piece of linear subspace” or “piece of submanifold”, near a point, the intersection of a linear subspace or a manifold with an open neighbourhood of the point.

If \( h_0 = 0 \) is an isotropic submanifold on which \( h_0 \) vanishes exactly at order 2, then \( \mu \) is a smooth function on \( h_0 = 0 \) (see Proposition 2.24), so that it makes sense to ask for \( \mu \) to reach its minimum in a non-degenerate way. Following [23], we will call \( P_0 \) a miniwell for \( h \).

In the situation of Theorem B, the first eigenvector concentrates rapidly on \( \{ h_0 = 0 \} \) (by Proposition 2.19, it is \( O(N^{-\infty}) \) outside a neighbourhood of size \( N^{-\frac{1}{4}+\epsilon} \)), and the speed of concentration towards the point which minimises \( \mu \) is much slower (only \( N^{-\frac{1}{4}+\epsilon} \)). In particular this state is more and more squeezed as \( N \) increases.

The expansion of the first eigenvector \( v_N \) in powers of \( N^{-\frac{1}{4}} \) is indirect. In Sect. 5 we prove that there exists a semiclassical Fourier Integral Operator \( U_N \) with classical symbol (in powers of \( N^{-1} \)), which unitarily maps (up to \( O(N^{-\infty}) \)) elements of \( H_N(M) \) localised near \( P_0 \) to elements of \( L^2(\mathbb{R}_x^r \times \mathbb{R}_x^{-r}) \), such that \( U_N v_N \) takes the form:

\[
U_N v_N(x, q) = N^{\frac{5}{2} - \frac{q}{4}} e^{-N^{\frac{1+r}{2}} e^{-\sqrt{N} \phi(q)}} \sum_{k=0}^{+\infty} N^{-\frac{k}{4}} b_k \left( N^{\frac{1}{2}} x, N^{\frac{1}{2}} q \right) + O(N^{-\infty}).
\]

Here \( \phi \) is a positive definite quadratic form and the \( b_k \)'s are polynomials; \( r \) is the dimension of \( \{ h_0 = 0 \} \) near \( P_0 \).

The Fourier Integral Operator \( U_N \) is the composition of the Bargmann transform (so that we deal with elements of \( L^2(\mathbb{R}_x^r \times \mathbb{R}_x^{-r}) \)) and a quantum map that quantizes a symplectic normal form (which we construct in Proposition 5.2). From the stationary phase lemma, a general expansion for \( v_N \), in a neighbourhood of \( P_0 \), is of the form

\[
v_N(z) = N^{\frac{5}{2} - \frac{q}{4}} \Phi_0^{\otimes N}(z) e^{-\sqrt{N} \varphi_1(z)} \sum_{k=0}^{+\infty} N^{-\frac{k}{4}} a_k(N^{\frac{1}{4}} q, N^{\frac{1}{4}} p, N^{\frac{1}{2}} x, N^{\frac{1}{2}} \xi) + O(N^{-\infty}),
\]

where

- \( \Phi_0 \) is a holomorphic section such that \( |\Phi_0(z)| \leq \exp(-c \text{ dist}(z, Z)) \)
- \( \varphi_1 \) is a holomorphic function such that, for all \( z \in Z \), \( \varphi_1(z) \geq c \text{ dist}(z, P_0)^2 \)
- the \( a_k \)'s are holomorphic functions
- \((q + ip, x + i\xi)\) are complex coordinates on a neighbourhood of \( P_0 \) such that \( Z = \{ p = 0, x = \xi = 0 \} \). (Since \( Z \) is isotropic, it is totally real.)

**Remark 1.1.** Theorem B is weaker than its counterparts in [23], in which the lowest-energy eigenvector admits a WKB ansatz. Here, in Equation (2), the rate of decay far from \( P_0 \) is only \( O(N^{-\infty}) \). One cannot hope for an expansion of \( v_N \) of the form

\[
N^{\frac{5}{2} - \frac{q}{4}} \Phi_0^{\otimes N}(z) e^{-\sqrt{N} \varphi_1(z)} \left[ \sum_{k=0}^{+\infty} N^{-\frac{k}{4}} a_k(N^{\frac{1}{4}} q, N^{\frac{1}{4}} p, N^{\frac{1}{2}} x, N^{\frac{1}{2}} \xi) + O(N^{-\infty}) \right]
\]

since the Szegö kernel itself does not admit an WKB expansion in general [10]. Under conditions of real-analyticity, such a result might hold in a neighbourhood of \( P_0 \).
Theorem C. Under the hypotheses of Theorem A, suppose that the function $\mu$ reaches its minimum on a unique point $P_0$ at which there is a simple crossing (see Definition 6.1).

Then for any sequence $(u_N)_{N \geq 1}$ of unit eigenfunctions corresponding to the first eigenvalue of $T_N(h)$, for any $\epsilon > 0$, one has

$$\int_{\text{dist}(\pi(y), P_0) > N^{-\frac{1}{2} + \epsilon}} |u_N(y)|^2 \, dVol = O(N^{-\infty}).$$

Moreover, the first eigenvalue is simple and the gap between the two first eigenvalues is of order $N^{-\frac{4}{3}}$. There is a full expansion of the first eigenvalue and eigenvector in powers of $N^{-\frac{1}{6}}$.

An example of symbol with a simple crossing, with dimensions $(1, 1)$, is the following function on $\mathbb{R}^4$:

$$h : (q_1, q_2, p_1, p_2) \mapsto p_1^2 + p_2^2 + q_1^2 q_2^2,$$

which reaches its minimum on the transverse union of two manifolds, $\mathbb{R} \times \{0, 0\}$ and $\{0\} \times \mathbb{R} \times \{0, 0\}$, intersecting at one point. More generally, simple crossing implies that, near $P_0$, the principal symbol $h_0$ reaches its minimum on a transverse union of isotropic manifolds, such that the sum at $P_0$ of the two transverse tangent spaces is still isotropic.

As in the case of Theorem B, the first eigenvector is more and more squeezed as $N \to +\infty$. Note that the speed of convergence, and the powers of $N$ involved in the expansions, differ between the two cases.

Again, the expansion of the first eigenvector $v_N$ is indirect: there exists a semiclassical Fourier Integral operator $U_N$ from $H_N(M)$ to $L^2(\mathbb{R}^{r_1 + r_2} \times \mathbb{R}^{n-r_1-r_2})$ so that

$$U_N v_N = N^{\frac{2}{n} - \frac{r_1 + r_2}{3}} e^{-N\frac{|x|^2}{2}} \sum_{k=0}^{+\infty} N^{-\frac{k}{6}} u_k \left(N^{\frac{1}{2}} x, N^{\frac{1}{2}} q\right) + O(N^{-\infty}).$$

Here, the functions $u_k$ have polynomial dependence in $x$ and are square-integrable (for fixed $x$) with respect to $q$; the dimensions of the pieces of isotropic submanifold crossing at $P_0$ are $r_1$ and $r_2$.

Weyl law The question now arises of the inverse spectral problem in our setting: given the high $N$ spectrum of a Toeplitz operator, is one able to distinguish the geometry of the set on which the Melin value $\mu$ is minimal?

In the situations of Theorems B and C, $\mu$ reaches a strict minimum at the miniwell or the crossing point, respectively. As in [13], one can build a symbol containing several miniwells or crossing points. From Theorem A, only those for which $\mu$ reaches a global minimum will contribute to low-energy states (of energy less than $N^{-1}(\mu_{\text{min}} + \epsilon)$). Since these miniwells or crossing points are at positive distance from each other, the low-energy spectrum of the full operator is (up to $O(N^{-\infty})$) the collection of the low-energy spectra of operators restricted to a neighbourhood of each of the minimal points for $\mu$. Indeed, one can build $O(N^{-\infty})$ almost eigenfunctions for the full operator, that are supported on a small neighbourhood of any of the minimal points for $\mu$. The next theorem studies the number of such modes in a given spectral window.

Theorem D (Local Weyl law). Under the hypotheses of Theorem A, there exist $0 < c \leq C$, $\epsilon > 0$ and $N_0 \geq 0$ such that the following is true. Let $\mu_{\text{min}}$ be the infimum of the Melin value, and $N \geq N_0$. 


1. For each regular miniwell with Melin value $\mu_{\text{min}}$ and dimension $r$, for each sequence $(\Lambda_N)$ with

$$N^{-\frac{1}{2} + \epsilon} \leq \Lambda_N \leq \epsilon,$$

in the spectral window $[0, N^{-1}(\mu_{\text{min}} + \Lambda_N)]$, the number of orthogonal almost eigenfunctions of $T_N(h)$ supported on a small neighbourhood of the miniwell belongs to the interval

$$\left[c(N^{\frac{1}{2}} \Lambda_N)^r, C(N^{\frac{1}{2}} \Lambda_N)^r\right].$$

2. For each simple crossing with Melin value $\mu_{\text{min}}$ and dimensions $(r, r)$, for each sequence $(\Lambda_N)$ with

$$N^{-\frac{1}{3} + \epsilon} \leq \Lambda_N \leq \epsilon,$$

in the spectral window $[0, N^{-1}(\mu_{\text{min}} + \Lambda_N)]$, the number of orthogonal almost eigenfunctions of $T_N(h)$ supported on a small neighbourhood of the crossing point belongs to the interval

$$\left[c(N^{\frac{1}{3}} \Lambda_N)^{3r} \log(N^{\frac{1}{3}} \Lambda_N), C(N^{\frac{1}{3}} \Lambda_N)^{3r} \log(N^{\frac{1}{3}} \Lambda_N)\right].$$

The notion of dimension of a miniwell and a simple crossing can be found in Definition 7.1. In Theorem D, cases 1 and 2 apply respectively in the settings of Theorems B and C.

Remark 1.2. • If $\Lambda_N < N^{-\epsilon}$, then there are more eigenvalues near a miniwell than near a crossing point (the ratio is of order $N^{\frac{2}{5}}$). If we look at eigenvalues in such windows, then a miniwell of dimension $r$ not only “hides” miniwells of smaller dimensions, but also crossing points of dimensions up to and including $(r, r)$.

If $\Lambda_N > \frac{r}{5}$, then there are more eigenvalues near a crossing point than near a miniwell (the ratio is of order $\log(N)$). In these windows, crossing points hide miniwells of dimension smaller or equal.

In particular, this proves that the spectral inverse problem allows, not only to recover the value of $\mu_{\text{min}}$, but also to determine the largest dimensions of the miniwells or crossing points achieving $\mu_{\text{min}}$, and to tell whether there are only miniwells, only crossing points, or both.

• In both cases, the number of eigenvalues in the window $[0, N^{-1}(\mu_{\text{min}} + \Lambda_N)]$ does not correspond at all to $N^n$ times the volume of $h_0^{-1}([0, N^{-1}(\mu_{\text{min}} + \Lambda_N)])$, which is always of order $N^{\frac{2}{5}}$, independently on $\Lambda_N$. There are far less eigenvalues than volume considerations would suggest.

Theorem D also allows to study low-temperature quantum states for a model on which there is a competition between a regular point and a crossing point with the same $\mu$. It shows a transition from temperature ranges similar to $N^{-1}$, for which the Gibbs measure concentrates on the crossing point, and temperature ranges of order $N^{-1-\epsilon}$, for which this measure concentrates on the regular point.

In this work only rapid decay estimates are obtained: quantities are controlled modulo an $O(N^{-\infty})$ error as $N \to +\infty$. The natural question of exponential decay [23] requires refined estimates on the Szegő kernel which are currently unknown for general compact Kähler manifolds.
The study of the function $\mu$ on examples of spin systems requires the full diagonalisation of matrices which size grows with the number of sites. Few theoretical results are known in this setting (see Sect. 8). The general conjecture is that $\mu$ should reach a minimum on planar configurations; up to now this is only supported by numerical evidence and the fact that planar configurations are local minima for $\mu$.

1.3. Pseudodifferential operators with degenerate minimal sets. While Theorems A to D are stated in the setting of concentration of eigenfunctions in a semiclassical limit, the first mathematical manifestation of quantum selection lies in the fact that some differential operators have compact resolvent because of subprincipal effects. In our setting, the phase space is compact, so that the spectrum always consist of eigenvalues with finite multiplicity, but the fact that the Weyl quantization of the symbol given by (3) has compact resolvent [36,38] for fixed $\hbar$ is already a form of quantum selection. A simple proof for this fact is recalled in Proposition 6.9.

A broad class of differential operators admitting a compact resolvent because of lower-order effects was identified in [20]. For such operators, and in particular for Schrödinger or magnetic Schrödinger operators with polynomial coefficients, one can then study Weyl laws [32,36,41] (in particular, the number of eigenvalues in a low-energy window is not given by the volume of its preimage by the symbol), speed of decay of eigenfunctions [6,21], and the construction of quasimodes [4,16–19,23,28,29,33,34,42].

Because of its higher degree of geometrical generality, the case of general Schrödinger or magnetic Schrödinger operators with a submanifold as classical minimal set is of greater interest in our discussion; let us present here it in detail. The article [23] treats the case of an operator of the form $-\hbar^2 \Delta + V$, on $L^2(\mathbb{R}^n)$, under the following hypotheses:

- $V \in C^\infty(\mathbb{R}^n, \mathbb{R})$; $V \geq 1$ at infinity.
- $\{V = 0\} = Z$ is a compact submanifold of $\mathbb{R}^n$.
- The transverse Hessian of $V$ on $Z$ is everywhere non-degenerate.
- The trace of the square root of the Hessian of $V$, as a function on $Z$, reaches a unique, non-degenerate minimum (“miniwell” condition)

Under these conditions, the authors show that the ground state of the Schrödinger operator is localised at the minimal point for this trace, and give asymptotic expansions for the ground state and its energy, as well as an exponential decay rate along $Z$.

The trace of the square root of a semidefinite form $Q$ coincides with the ground state energy of $-\Delta + Q$, giving a physical interpretation for the result of concentration: as in [13], the ground state only concentrates at the points near which the energy contributions of order $\hbar$ are the lowest. Note that, if $d$ is the Agmon distance associated with $-\Delta + V$ where $V = Q + O(|x|^3)$, then $\text{tr}(\sqrt{Q}) = \Delta d(0)$.

The geometry of the zero set of the symbol $|\xi|^2 + V$, under the hypotheses above, is as follows: it is a smooth submanifold, isotropic for the symplectic form, on which the symbol vanishes at order exactly 2.

Motivated by supraconductivity, a series of articles [4,16–19,34] consider the problem of “magnetic bottles”, that is, the analysis of the ground state of a purely magnetic Schrödinger operator

$$(i\hbar d + \alpha)^* (i\hbar d + \alpha)$$

acting on $L^2(\mathbb{R}^n)$, associated with a 1-form $\alpha$. The 2-form $d\alpha$ can be seen as an (antisymmetric) linear operator $B : TM \rightarrow TM$ using the standard metric.
The low-lying eigenvalues of the operator above are then linked to the behaviour of $\text{tr} \left[ (B^* B)^{1/2} \right]$. The parallel with the “miniwell” case is obvious from a geometrical perspective. Here the zero set of the symbol is the smooth manifold $\{ \xi = \alpha \}$, on which the symbol vanishes at order exactly 2. Moreover, the quantity $\text{tr} \left[ (B^* B)^{1/2} \right]$ again coincides with the ground state energy of the quadratic operator at the zero point. A particularity of this model is that the symplectic rank of $\{ \xi = \alpha \}$ is arbitrary and may vary with the base point; the most precise results (giving eigenfunction expansions) assume that the symplectic rank is constant (or at least good-behaved) at the points of interest. In Theorem B, we focus on symbols that are minimal on an isotropic submanifolds, but the classical normal form and the quasimode construction of Sect. 5 can be adapted to the case of a submanifold with constant symplectic rank.

An essential feature of the work above is a family of Melin-type estimates, which give a lower bound to the spectrum of an operator depending on the quantum ground state energy of the Hessian (Melin value) along the zero set. The original result by Melin [30] is concerned with general pseudodifferential operators (without a semiclassical parameter). In the magnetic case, a semiclassical version of the Melin estimate was given in [18].

Most of the results that we just described use not only the geometric (that is, microlocal) structure of the symbol near its minimal set, but also the specific form of the operator, which allows one, for instance, to conjugate the operator with multiplication operators of the form $\exp(\phi/\hbar^\alpha)$. The generalisation of these results to arbitrary pseudodifferential operators verifying the same geometric hypotheses is lacking. In this article, while restricting ourselves to compact geometries, we give a version of these results for general symbols, which we present in the formalism of Berezin-Toeplitz quantization but which applies in particular to pseudodifferential operators (if the principal symbol is confining, for instance).

Moreover, a common characteristic of the “miniwell” and “magnetic well” framework is the fact that the classical minimal set is a smooth submanifold, on which the symbol vanishes in a non-degenerate way. Lifting this hypothesis is necessary in order to understand quantum selection on the Kagome lattice (for which the classical minimal set is an algebraic manifold). In Theorem A we prove that the Melin value is, in general, a criterion for localisation of the ground state.

1.4. Application to spin systems. One of the main physical motivations for this study is the mathematical foundation of quantum selection in the context of spin systems. The search for materials with a non-conventional magnetic behaviour led experimental and theoretical physicists to consider frustrated antiferromagnetic spin systems, such as pyrochlore or the Kagome lattice. Order by disorder approaches in the large spin limit are commonly used in the physics literature, and the subprincipal effects presumably select a very small subset of configurations [11,15,26,35,39].

The term “order by disorder” embraces two different settings. In both settings, one starts with a function $f$ on a symplectic manifold $M$ such that $Z = \text{argmin}(f)$ does not consist of a single point, but this degeneracy does not come from a symmetry of the problem. Although all points of $Z$ form closed orbits for the Hamiltonian flow of $f$, the behaviour of a trajectory starting in the vicinity of a given point in $Z$ is unclear. A physically relevant quantity, in this setting, is the Gibbs measure at low temperature. This measure will concentrate on $Z$ as the temperature goes to zero, but not evenly: it might select only one point in $Z$. In this setting, adding disorder in the system leads to an ordering of the possible configurations at low energy [31,43].
In this article, rather than tuning a thermal noise, we add a quantum noise. The conclusion of Theorem A is that, similarly, the quantum noise selects one particular subset of $\mathbb{Z}$. At this point we note that there is no reason why thermal selection and quantum selection should give the same subset of $\mathbb{Z}$. Indeed, in Theorem D, we show that in the case of a competition between a miniwell and a crossing point with equal Melin value, the point selected by the quantum Gibbs measure depends on the relative scaling of the semiclassical parameter and the temperature.

Spin systems are particular cases of Toeplitz operators. In such systems the base manifold is a product of 2-spheres. Let $G = (V, E)$ be a finite graph and $M = (\mathbb{S}^2)^{\left| V \right|}$. $M$ is formed as follows: at each vertex $i \in V$ of the graph we associate a unit vector $e_i = (x_i, y_i, z_i) \in \mathbb{S}^2$, called spin at site $i$. In particular, there are $3|V|$ coordinate functions $(x_i, y_i, z_i)_{i \in V}$ on $M$. The standard symplectic structure on $\mathbb{S}^2$ gives raise to a natural symplectic structure on $M$. For this symplectic structure, one has $\{ x_i, y_j \} = \delta_{ij} z_i$, and two similar equations given by cyclic permutation. We introduce the antiferromagnetic Heisenberg symbol:

$$h : M \mapsto \mathbb{R} \quad (e_i)_{i \in V} \mapsto \sum_{(i,j) \in E} x_i x_j + y_i y_j + z_i z_j.$$  

The minima of this function correspond to situations where the sum of the scalar products between neighbouring spins is the smallest. If $G$ is bipartite, this minimum is reached in situations where neighbouring vectors are opposite. In frustrated systems, this is not possible. If for instance three vertices in the graph are linked with each other, then not all of them can be opposite to the other ones. This is the case of the Kagome lattice, and the Husimi tree, considered in [15] and depicted in Fig. 1.

We will consider a class of graphs made of triangles. A finite connected graph $G = (V, E)$ is made of triangles when there is a partition $E = \bigsqcup_{i \in J} E_i$ where, for every $i$, $E_i$ contains three edges that link together three vertices; in addition, we ask that the degree at any vertex does not exceed 4 (and is hence equal to either 2 or 4). We will call the $E_i$’s the triangles of the graph.

Pieces of the Kagome lattice and the Husimi tree, in Fig. 1, are made of triangles. In general, from a 3-regular finite graph $G = (V, E)$, one can build an associated graph made of triangles $\tilde{G} = (\tilde{V}, \tilde{E})$ which is the edge graph of $G$: the set of vertices is $\tilde{V} = E$ and two elements of $\tilde{V}$ are adjacent in $\tilde{G}$ when they are adjacent as edges of $G$ (i.e. when they share a common vertex). In this case the triangles of $\tilde{G}$ correspond to the vertices of $G$. The Kagome lattice is thus associated with the hexagonal lattice, and the infinite Husimi tree with the 3-regular tree.

The presence of the “frustration” by triangles leads to a large degeneracy of the classical minimal set. Indeed, $h$ is minimal if, on each triangle $V_i$, the sum of the three spins at the vertices of the triangle is zero (so that these elements of $\mathbb{S}^2$ must form a great equilateral triangle). This is not always possible as the example to the right of Fig. 1 shows. Those configurations exist on subsets of the Husimi tree and the Kagome lattice, and are highly degenerate: on the Husimi tree, once the spins on a triangle are chosen, there is an $\mathbb{S}^1$ degeneracy for each of its children; the set of minimal configurations is an isotropic torus whose dimension grows linearly with the number of triangles. On the Kagome lattice, the set of these configurations does not form a smooth submanifold, hence the need for Theorems A and C. It is currently unknown which minimal points of $h$ achieve $\mu_{\min}$.  

Proposition 1.3. For a loop of 6 triangles (the basic element of the Kagome lattice), the minimal set is not a smooth manifold.

For a loop of 4 triangles, the minimal set is the direct product of $SO(3)$ and the union of three circles, two of each having transverse intersection at exactly one point. Planar configurations are local minima for $\mu$.

The proof is presented in Sect. 8.

1.5. Outline. In Sect. 2 we recall the necessary material on Toeplitz operators, including a universality lemma proved in [13], and quantum maps as developed in [8]. In this section we also define and study the Melin value.

Section 3 contains the main tool in the proof of Theorem A, which is a Toeplitz version of the Melin estimates [24,30]. We give a global and a local version of these estimates, and use them to prove pseudolocality of the resolvent at a distance $\geq \varepsilon N^{-1}$ of the spectrum, for every $\varepsilon > 0$.

Section 4 concludes the proof of Theorem A, based on the Melin estimate.

Sections 5 and 6 respectively contain the proofs of Theorems B and C, following the same strategy. We first find a convenient symplectic normal form, then use quantum maps to reduce the problem to a normal form, then we use a particular perturbative argument to exhibit, for every $k \in \mathbb{N}$, an approximate eigenvector up to $O(N^{-k})$.

In Sect. 7 we prove Theorem D, using estimates developed in the previous sections, and especially Proposition 5.8 and Proposition 6.17.

Section 8 contains a discussion on frustrated spin systems.

2. Toeplitz Operators and Quadratic Symbols

Toeplitz operators are a generalisation of the Bargmann-Fock point of view on the quantum harmonic oscillator [2]. They realise a quantization on some symplectic manifolds, and are a particular case of geometric quantization [25,40]. Another particular case of geometric quantization is Weyl quantization which leads to pseudodifferential operators. Toeplitz operators were first studied from a microlocal point of view [5,12], and the study of the Szegő kernel was further motivated by geometrical applications [14,45]. Here we directly use the semiclassical point of view developed in [7,27,37].
In this section we recall the properties of Toeplitz operators, and we refer to earlier work on the topic [2, 3, 7, 13, 27, 37, 44] for the proofs of the exposed facts. We also prove several facts about the symplectic reduction of quadratic forms, and their quantizations.

2.1. Hardy spaces and the Szegő projector. If a symplectic manifold (phase space) $M$ has a complex structure, the idea behind Berezin-Toeplitz operators is to consider quantum states as holomorphic functions. If $M$ is compact, holomorphic functions on $M$ are all constant, so that one needs to consider sections of a convenient line bundle over $M$ or, by duality, holomorphic functions on a dual line bundle.

Let $M$ be a Kähler manifold of dimension $n$, with symplectic form $\omega$. If the Chern class of $\omega/2\pi$ is integral, there exists a Hermitian holomorphic line bundle $(L, h)$ over $M$, with curvature $-i\omega$ ([44], pp. 158–162).

Let $(L^*, h^*)$ be the dual line bundle of $L$, with dual metric. Let $D$ be the unit ball of $L^*$, that is:

$$\{D = (m, v) \in L^*, \|v\|_{h^*} < 1\}.$$  

The boundary of $D$ is denoted by $X$. It is a circle bundle over $M$, with projection $\pi$ and an $S^1$ action

$$r_\theta : X \mapsto X$$

$$(m, v) \mapsto (m, e^{i\theta}v).$$

$X$ inherits a Riemannian structure from $L^*$ so that $L^2(X)$ is well-defined. We are interested in the equivariant Hardy spaces on $X$, defined as follows:

**Definition 2.1.**

- The Hardy space $H(X)$ is the closure in $L^2(X)$ of $
\{f|_X, \ f \in C^\infty(D \cup X), \ f \text{ holomorphic in } D\}.$

- The Szegő projector $S$ is the orthogonal projection from $L^2(X)$ onto $H(X)$.

- Let $N \in \mathbb{N}$. The equivariant Hardy space $H_N(X)$ is:

$$H_N(X) = \{f \in H(X), \forall (x, \theta) \in X \times S^1, \ f(r_\theta x) = e^{iN\theta} f(x)\}.$$

- The equivariant Szegő projector $S_N$ is the orthogonal projection from $L^2(X)$ onto $H_N(X)$.

Throughout this paper, we will work with the sequence of spaces $(H_N(X))_{N \in \mathbb{N}}$. If $M$ is compact, then the spaces $H_N(X)$ are finite-dimensional spaces of smooth functions. (Note, however, that this dimension grows polynomially with $N$.) Hence, the Szegő projector has a Schwartz kernel, that we will also denote by $S_N$.

**Example 2.2** (The sphere). The sphere $S^2$ has a canonical Kähler structure as $(\mathbb{C}P^1, \omega_{FS})$, which is quantizable. Here $D$ is the unit ball in $\mathbb{C}^2$, blown up at zero, and $X = S^3$. One recovers the usual $S^1$ free action on $S^3$ with quotient $S^2$.

Here, $H_N(X)$ is the space of homogeneous polynomials of two complex variables, of degree $N$, with Hilbert structure the scalar product of the restriction to $X$ of these polynomials. A natural Hilbert basis corresponds to the normalized monomials

$$(z_1, z_2) \mapsto \sqrt{\frac{N + 1}{\pi}} \binom{N}{k} \frac{z_1^k z_2^{N-k}}{k!}.$$
In particular, the Szegő projector has kernel

\[ S_N^{CP_1} : (z, w) \mapsto \frac{N + 1}{\pi} (z \cdot \overline{w})^N. \]

**Example 2.3.** (\( \mathbb{C}^n \)) Another important example (though non compact) is the case \( M = \mathbb{C}^n \), with standard Kähler form. As \( \mathbb{C}^n \) is contractile, the bundle \( L \) is trivial, but the metric \( h \) is not. The curvature condition yields:

\[ (L, h) = \left( \mathbb{C}^n_z \times \mathbb{C}_v, e^{\frac{1}{2} |z|^2} |u|^2 \right). \]

This leads to the following identification [2]:

\[ H_N(X) \simeq B_N := L^2(\mathbb{C}^n) \cap \left\{ z \mapsto e^{-\frac{1}{2} |z|^2} f(z), \ f \text{ is an entire function} \right\}. \]

The space \( B_N \) is a closed subspace of \( L^2(\mathbb{C}^n) \). The orthogonal projector \( \Pi_N \) from \( L^2(\mathbb{C}^n) \) to \( B_N \) admits as Schwartz kernel the function

\[ \Pi_N : (z, w) \mapsto \left( \frac{N}{\pi} \right)^n \exp \left( -\frac{1}{2} N |z - w|^2 + iN \Im(z \cdot \overline{w}) \right). \]

As the case \( M = \mathbb{C}^n \) is of particular interest, we will keep separate notations for the Szegő kernel in this case, which will always be denoted by \( \Pi_N \).

The sequence of kernels \( (\Pi_N)_{N \geq 1} \) is rapidly decreasing outside the diagonal set. This key property also holds in the case of a compact Kähler manifold:

**Proposition 2.4** ([27], prop 4.1, or [3,7,37]). Let \( M \) be a compact Kähler manifold, and \( (S_N)_{N \geq 1} \) be the sequence of Szegő projectors of Definition 2.1. Let \( \delta \in [0, 1/2) \). For every \( k \geq 0 \) there exists \( C \) such that, for every \( N \geq 1 \), for every \( x, y \in X \) such that \( \text{dist}(\pi(x), \pi(y)) \geq N^{-\delta} \), one has

\[ |S_N(x, y)| \leq CN^{-k}. \]

This roughly means that, though the operators \( S_N \) are non-local, the “interaction range” shrinks with \( N \).

In the spirit of Proposition 2.4, we define what it means for a sequence of functions in \( H_N(X) \) to be localised on a set.

**Definition 2.5.** Let \( u = (u_N)_{N \geq 1} \) be a sequence of unit elements of \( L^2(X) \). Let \( dVol \) denote the Liouville volume form on \( M \). For every \( N \), the probability measure \( |u_N|^2 dVol \otimes d\theta \) is well-defined on \( X \), and we call \( \mu_N \) the push-forward of this measure on \( M \).

Let moreover \( Z \subset M \) be compact. We will say that the sequence \( u \) localises on \( Z \) when, for every open set \( U \subset M \) at positive distance from \( Z \), one has, as \( N \to +\infty \):

\[ \mu_N(U) = O\left(N^{-\infty}\right). \]

A corollary of this definition is that, if a sequence \( (u_N)_{N \geq 1} \) localises on a set \( Z \), then so does the sequence \( (S_N u_N)_{N \geq 1} \).

**Remark 2.6.** Elements in the Hardy space are functions on the whole phase space. Hence, Definition 2.5 corresponds to microlocalisation for elements of \( L^2(\mathbb{R}^n) \).
To complete Proposition 2.4, we have to describe how \( S_N \) acts on sequences of functions concentrated on a point. For this we need a convenient choice of coordinates.

Let \( P_0 \in M \). The real tangent space \( T_{P_0}M \) carries a Euclidian structure and an almost complex structure coming from the Kähler structure on \( M \). Then, we can (non-uniqely) identify \( T_{P_0}M \) with \( \mathbb{C}^n \) endowed with the standard metric.

The map \( \exp_M : T_{P_0}M \mapsto M \) on the Riemannian manifold \( M \), together with the identification \( \mathbb{C}^n \simeq T_{P_0}M \), leads to the notion of normal coordinates:

**Definition 2.7.** Let \( U \) be a neighbourhood of 0 in \( \mathbb{C}^n \) and \( V \) be a neighbourhood of a point \( P_0 \) in \( M \).

A smooth diffeomorphism \( \rho : U \times S^1 \to \pi^{-1}(V) \) is said to be a normal map or a map of normal coordinates if it satisfies the following conditions:

- \( \forall (z, v) \in U \times S^1, \exists \theta \in \mathbb{R}, \rho(z, ve^{i\theta}) = r_\theta \rho(z, v) \)
- Identifying \( \mathbb{C}^n \) with \( T_{P_0}M \) as previously, one has:
  \[ \pi(\rho(z, w)) = \exp_M(z). \]

**Remark 2.8.** The choice of a normal map around a point \( P_0 \) reflects the choice of an identification of \( \mathbb{C}^n \) with \( T_{P_0}(M) \) and a point over \( P_0 \) in \( X \). Hence, if \( \rho_1 \) and \( \rho_2 \) are two normal maps around the same point \( P_0 \), then \( \rho_1^{-1} \circ \rho_2 \in U(n) \times U(1) \).

Using Definition 2.7, one can compare, for \( N \) large, the Szegő kernel \( S_N \) with the flat case \( \Pi_N \). For this, we push by \( \rho \) the Bargmann kernel and multiply by the correct factor in the fibre to obtain an equivariant kernel on \( X \):

\[ \rho^* \Pi_N(\rho(z, \theta), \rho(w, \phi)) := e^{iN(\theta - \phi)} \Pi_N(z, w). \]

By convention, \( \rho^* \Pi_N \) is zero outside \( \pi^{-1}(V)^2 \).

**Proposition 2.9** ([13]). Let \( P_0 \in M \), and \( \rho \) a normal map around \( P_0 \). For every \( \epsilon > 0 \) there exists \( \delta \in (0, 1/2) \) and \( C > 0 \) such that for every \( N \in \mathbb{N} \), for every \( u \in L^2(X) \), if the support of \( u \) lies inside \( \rho(B(0, N^{1-\delta}) \times S^1) \), then

\[ \|(S_N - \rho^* \Pi_N)u\|_{L^2} < CN^{-\frac{1}{2} + \epsilon}\|u\|_{L^2}. \]

In a sense, Proposition 2.9 states that the operator \( S_N \) asymptotically looks like \( \Pi_N \) on small scales. The proof can be found in a previous paper [13], as a consequence of previously known results on the asymptotical behaviour of the Schwartz kernel of \( S_N \) near the diagonal set \([7,27,37]\).

### 2.2. Toeplitz operators.

**Definition 2.10.** Let \( M \) be a Kähler manifold, with equivariant Szegő projectors \( (S_N)_{N \geq 1} \). Let \( h \in C^\infty(M) \) be a smooth function on \( M \). For all \( N \geq 1 \), the Toeplitz operator \( T_N(h) : H_N(X) \to H_N(X) \) associated with the symbol \( h \) is defined as

\[ T_N(h) = S_N h S_N. \]

In this work we investigate the spectral properties of the operators \( T_N(f) \), for fixed \( f \) and \( N \to +\infty \).
Example 2.11. (Spin operators) Let us continue from Example 2.2. The sphere $S^2$ is naturally a submanifold of $\mathbb{R}^3$; as such, there are three coordinate functions $(x, y, z) : S^2 \mapsto \mathbb{R}^3$. They are closed under Poisson brackets: one has $\{x, y\} = z$ and two similar identities by cyclic permutation.

In the Hilbert basis given by the normalized monomials, the associated Toeplitz operators $T_N(x), T_N(y), T_N(z)$ are, up to a factor $\frac{N}{N+2}$, the usual spin matrices with spin $\frac{N}{2}$.

2.2.1. Toeplitz operators on $\mathbb{C}^n$ and the Melin value The manifold $\mathbb{C}^n$ is not compact. Let us release the condition that the symbol is bounded. This defines Toeplitz operators as unbounded operators on $B_N$.

Toeplitz operators on $\mathbb{C}^n$ whose symbols are semipositive definite quadratic forms play an important role in this work. If $Q$ is a quadratic form on $\mathbb{R}^{2n}$ identified with $\mathbb{C}^n$, then $T_N(Q)$ is essentially self-adjoint. This operator is related to the Weyl quantization $Op_\hbar^W(Q)$ with semi-classical parameter $\hbar = N^{-1}$. In fact, $T_N(Q)$ is conjugated, via the Bargmann transform $B_N$ [2], with the operator

$$Op_\hbar^{N^{-1}}(Q) + \frac{N^{-1}}{4} \text{tr}(Q).$$

If $Q$ is semi-definite positive, then it takes non-negative values as a function on $\mathbb{R}^{2n}$, hence $T_N(Q) \geq 0$ for all $N \geq 0$ since, for $u \in B_N$, one has

$$\langle u, \Pi_N Q \Pi_N u \rangle = \langle u, Qu \rangle \geq 0.$$

The infimum of the spectrum of $T_N(Q)$ is of utmost interest, since it leads to the notion of Melin value. As $Q$ is 2-homogeneous, and the Bargmann spaces are identified with each other through a scaling, one has $T_N(Q) \sim N^{-1} T_1(Q)$, and in particular

$$\inf(\text{Sp}(T_N(Q))) = N^{-1} \inf(\text{Sp}(T_1(Q))).$$

Definition 2.12. Let $Q$ be a semi-definite positive quadratic form on $\mathbb{R}^{2n}$, identified with $\mathbb{C}^n$.

We denote by $\mu(Q)$ the Melin value of $Q$, defined by

$$\mu(Q) := \inf(\text{Sp}(T_1(Q))).$$

Given $Q \geq 0$, how can one compute $\mu(Q)$? As stated above, it depends first on the trace of $Q$ (which is easy to compute), and second on the infimum of the spectrum of $Op_\hbar^1(Q)$. This second part is invariant through a symplectic change of variables, and the problem reduces to a symplectic diagonalisation of $Q$ (see Proposition 2.22). As an example:

Example 2.13. Let $\alpha, \beta \geq 0$. Then

$$\mu \left( (x, y) \mapsto \alpha x^2 + \beta y^2 \right) = \frac{1}{4} (2\sqrt{\alpha \beta} + \alpha + \beta).$$

The function $\mu$ itself is not invariant under symplectomorphisms (for example, in the previous example it does not only depend on $\alpha \beta$). However, it is invariant under unitary changes of variables.

The regularity of the map $\mu$ will be useful in the proof of Theorem A:
Proposition 2.14 ([30]). The function $Q \mapsto \mu(Q)$ is Hölder continuous with exponent $\frac{1}{2n}$ on the set of semi-definite positive quadratic forms of dimension $2n$.

If $Q$ is definite positive, then $T_N(Q)$ has compact resolvent, and the first eigenvalue is simple.

Definition 2.15. Let $M$ be a Kähler manifold and let $h$ be a classical symbol on $M$ with $h_0 \geq 0$. Let $P_0 \in M$ be such that $h_0(P_0) = 0$. Let $\rho$ be a normal map around $P_0$; the function $h_0 \circ \rho$ is well-defined and non-negative on a neighbourhood of 0 in $\mathbb{C}^n$, and the image of 0 is 0. Hence, there exists a semi-definite positive quadratic form $Q$ such that

$$h_0 \circ \rho(x) = Q(x) + O(|x|^3).$$

We define the Melin value $\mu(P_0)$ as $\mu(Q) + h_1(P_0)$.

2.2.2. Toeplitz operators on compact manifolds When the base manifold $M$ is compact and $h$ is real-valued, for fixed $N$ the operator $T_N(h)$ is a symmetric operator on a finite-dimensional space. In this setting, we will speak freely about eigenvalues and eigenfunctions of Toeplitz operators.

The composition of two Toeplitz operators can be written, in the general case, as a formal series of Toeplitz operators [7], that is:

$$T_N(f)T_N(g) = T_N(fg) + N^{-1}T_N(C_1(f, g)) + N^{-2}T_N(C_2(f, g)) + \ldots.$$

The composition properties of formal series of Toeplitz operators lead to the following property, which appears in previous work [13], and which is an important first step towards the study of the low-lying eigenvalues.

Proposition 2.16. Let $M$ be a compact Kähler manifold and $h$ a real non-negative smooth function on $M$.

Let $u = (u_N)_{N \geq 1}$ be a sequence of unit elements in the Hardy spaces such that, for every $N$, one has

$$T_N(h)u_N = \lambda_N u_N,$$

with $\lambda_N = O(N^{-1})$.

Then the sequence $u$ localises on $\{h = 0\}$. More precisely, for every $\epsilon > 0$, if

$$Z_N = \{m \in M, h(m) \geq N^{-1+\epsilon}\},$$

one has, as $N \to +\infty$,

$$\int_{\pi^{-1}(Z_N)} |u(x)|^2 dVol = O(N^{-\infty}).$$

On a point where $h$ is minimal, one can pull-back Definition 2.12 by normal coordinates of Definition 2.7:

Definition 2.17. Let $h \in C^\infty(M, \mathbb{R}^+)$. Let $P_0 \in M$ be such that $h(P_0) = 0$. Let $\rho$ be a normal map around $P_0$; the function $h \circ \rho$ is well-defined and non-negative on a neighbourhood of 0 in $\mathbb{C}^n$, and the image of 0 is 0. Hence, there exists a semi-definite positive quadratic form $Q$ such that

$$h \circ \rho(x) = Q(x) + O(|x|^3).$$

We define the Melin value $\mu(P_0)$ as $\mu(Q)$. 
Remark 2.18. A different choice of normal coordinates corresponds to a $U(n)$ change of variables for $Q$, under which $\mu$ is invariant. Hence, $\mu(P_0)$ does not depend on the choice of normal coordinates.

The function $P_0 \mapsto \mu(P_0)$ is $\frac{1}{2n}$-Hölder continuous as a composition of the smooth function $P_0 \mapsto Q$ and the Hölder continuous function $Q \mapsto \mu$.

We will use, throughout this article, a slight modification of Proposition 3.1 in [13], based on the calculus of Toeplitz operators.

Proposition 2.19. Let $h$ be a classical symbol on $M$ with $h_0 \geq 0$ and let $(u_N)_{N \geq 0}$ be a normalised sequence of eigenvectors of $T_N(h)$ with associated sequence of eigenvalues $O(N^{-1})$. Let $\delta > 0$.

Then

$$\int_{\{h \geq N^{-1+\delta}\}} |u_N|^2 = O(N^{-\infty}).$$

Proof. As in the proof of Proposition 3.1 in [13], if $h$ satisfies the claim, then so does $h - CN^{-1}$ for all $C \in \mathbb{R}$. In particular, without loss of generality $h_1 \geq 2$, and we can recover from the proof of Proposition 3.1 in [13] that, for every $k \geq 0$, there exists $C_k$ such that

$$|\langle u_N, h^k u_N \rangle| \leq C_k N^{-k}.$$

As $\delta > 0$, one has

$$\{h \geq N^{-1+\delta}\} \subset \left\{ h_0 \geq \frac{1}{2} N^{-1+\delta} \right\},$$

so that we obtain

$$\int_{\{h \geq N^{-1+\delta}\}} |u_N|^2 \leq \int_{\{h_0 \geq \frac{1}{2} N^{-1+\delta}\}} |u_N|^2 \leq 2^k N^{k-\delta} \int_M h^k |u_k|^2 \leq 2^k C_k N^{-k\delta}.$$

In particular,

$$\int_{\{h \geq N^{-1+\delta}\}} |u_N|^2 = O(N^{-\infty}).$$

\[\Box\]

Remark 2.20. Another scheme of proof for Proposition 2.19 consists in composing $T_N(h)$ with a test function $\chi_N$ which is 0 on $\{h \leq N^{-1+\delta/2}\}$ and 1 on $\{h \geq N^{-1+\delta}\}$. Such a function can be chosen temperate, meaning that $\|\nabla^k \chi_N\| \leq C N^{\frac{k(1-\delta)}{2}}$. See [9] for details on the composition of temperate symbols.
2.3. Symplectic reduction of quadratic forms. Let \( h \) be a classical symbol on \( M \) and suppose that \( Z = \text{argmin}(M) \) is a smooth submanifold on which \( h_0 \) vanishes at order 2 (the Morse-Bott condition). In order to study in detail the low-energy spectrum of \( T_N(h) \), a common strategy consists in a Born-Oppenheimer approximation, where one reduces to an operator on \( Z \) by the assumption that quantum states minimise their energy “transversally to \( Z \)” at every point. These considerations require a symplectic reduction (as far as one is able to do so) of the Hessian matrix at every point of \( Z \), with smooth dependence on the base point. In this subsection we perform this reduction, to prepare for Sect. 5 and the proof of Theorem B.

**Definition 2.21.** Let \((E, \omega)\) be a linear symplectic space and \( Q \) be a semi-positive quadratic form on \( E \). We let \( \tilde{Q} : E \to E^* \) be such that \([\tilde{Q}(e)](f) = Q(e, f)\) for any \((e, f) \in E^2\). We also let \( \tilde{\omega} : E \to E^* \) be such that \([\tilde{\omega}(e)](f) = \omega(e, f)\). Then, since \( \omega \) is non-degenerate, \( \tilde{\omega} \) admits an inverse. The **symplectic eigenvalues** of \( Q \) are defined as the elements of

\[
\sigma(i\tilde{\omega}^{-1}\tilde{Q}) \cap \mathbb{R}^+.
\]

The concept of symplectic eigenvalues, that is, eigenvalues of a positive definite quadratic form relatively to a symplectic form, is akin to the notion of eigenvalues of a quadratic form relatively to a euclidian metric (if \( g \) is a scalar product on \( E \), one can as above define the isomorphism \( \tilde{g} : E \mapsto E^* \) and consider \( \sigma(\tilde{g}^{-1}(\tilde{Q})) \)). In particular, the **symplectic trace** (sum of the symplectic eigenvalues with multiplicities) plays an essential role in this article along with the usual trace (taken with respect to the standard Euclidian structure on \( \mathbb{R}^{2n} \)). Indeed they allow one to compute the Melin value. The analogy is, however, uncomplete: while the trace of a quadratic form is easily computed, the symplectic trace admits no explicit formula.

We will need in Sect. 5 to reduce, as much as possible, quadratic forms which depend on a parameter. If a symmetric matrices depends smoothly on several parameters, it is in general not possible to diagonalise this matrices with smooth dependence in the parameters (this is only possible away from eigenvalue crossings). In our context, however, one can find a smooth way to reduce quadratic forms with respect to the symplectic form, so that the associated quantum ground state is fixed.

We begin with a statement of the result for families of positive quadratic forms.

**Proposition 2.22.** Let \( Q : \mathbb{R}^d \mapsto S^+_{2n}(\mathbb{R}) \) be a smooth \( n \)-parameter family of positive quadratic forms.

Then there is a smooth family \( S : \mathbb{R}^d \mapsto \text{Sp}(2n) \) of symplectic matrices, a parameter family \( U : \mathbb{R}^d \mapsto \text{Sp}(2n) \cap \text{O}(2n) \cong U(n) \) of unitary matrices, and a family \((\lambda_1, \ldots, \lambda_n) : \mathbb{R}^d \mapsto (\mathbb{R}^+)^n\) of values such that, letting

\[
(e_1, f_1, \ldots, e_n, f_n) = S(U(\text{canonical basis})),
\]

one has

\[
Q(t) \left( \sum_{i=1}^n q_i e_i(t) + p_i f_i(t) \right) = \sum_{i=1}^n \lambda_i(t)(p_i^2 + q_i^2).
\]

The symplectic eigenvalues of \( Q(t) \) are the family \((\lambda_i(t))_{1 \leq i \leq n}\).

In particular, for every \( t \in \mathbb{R}^d \), the ground state of \( T_1(Q(t) \circ S(t)) \) is the standard Gaussian \( z \mapsto \frac{N^n}{\sqrt{\pi}^n} e^{-\frac{|z|^2}{2}} \).
Proof. Let $M$ be the matrix of $Q$ in the (symplectic) canonical basis. Then $M^{\frac{1}{2}}$ is a smooth family of symmetric matrices, so that $M^{\frac{1}{2}}JM^{\frac{1}{2}}$ is a smooth family of antisymmetric matrices, where $J$ is the matrix of the standard symplectic form in the canonical basis. Hence, there is a family $V$ of orthogonal matrices, and a family $V$ of positive diagonal matrices, such that

$$V^T M^{\frac{1}{2}} J M^{\frac{1}{2}} V = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$ 

Note that, in general, $V$ and $D$ do not depend continuously on $M$.

In particular, with $A = \begin{pmatrix} D^{-\frac{1}{2}} & 0 \\ 0 & D^{-\frac{1}{2}} \end{pmatrix}$, one has

$$(AV^T M^{\frac{1}{2}} J)M(-JM^{\frac{1}{2}}V A) = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} + O(t^\infty),$$

and

$$(AV^T M^{\frac{1}{2}} J)(-JM^{\frac{1}{2}}V A) = J.$$ 

Hence, the matrix $-JM^{\frac{1}{2}}V A$ corresponds to a linear symplectic change of variables under which $Q$ is diagonal. It remains to write this matrix as $SV$, where $S$ depends smoothly on the parameters and $V \in U(n)$. Under such a decomposition, one has

$$SS^T = -JM^{\frac{1}{2}}VA \left(-JM^{\frac{1}{2}}VA\right)^T,$$

and

$$JM^{\frac{1}{2}}VAAV^T M^{\frac{1}{2}} J^T = -JM^{\frac{1}{2}}V \begin{pmatrix} D^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} V^T M^{\frac{1}{2}} J.$$

From the definition of $D$, there holds

$$\begin{pmatrix} D^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} = \left(D^2 \begin{pmatrix} 0 & 0 \\ 0 & D^2 \end{pmatrix}\right)^{-\frac{1}{2}} = (-V^T M^{\frac{1}{2}} J M J M^{\frac{1}{2}} V)^{-\frac{1}{2}}$$

$$= V^T \left(-M^{\frac{1}{2}} J M J M^{\frac{1}{2}}\right)^{-\frac{1}{2}} V.$$ 

Hence,

$$JM^{\frac{1}{2}}VAAV^T M^{\frac{1}{2}} J^T = -JM^{\frac{1}{2}} \left(-M^{\frac{1}{2}} J M J M^{\frac{1}{2}}\right)^{-\frac{1}{2}} M^{\frac{1}{2}} J.$$ 

We set

$$S = \left[-JM^{\frac{1}{2}} \left(-M^{\frac{1}{2}} J M J M^{\frac{1}{2}}\right)^{-\frac{1}{2}} M^{\frac{1}{2}} J\right]^{\frac{1}{2}}.$$ 

Then $S$ depends smoothly on $M$ since the square root is a smooth function on the set of positive quadratic matrices. It is symplectic (as the square root of a symmetric symplectic matrix), and by construction $(-JM^{\frac{1}{2}}V A)^{-1}S \in Sp(2n) \cap O(2n)$ is unitary.
To conclude the proof, we first observe that
\[ \sigma(M(0)^{1/2}JM(0)^{1/2}) = \sigma(JM(0)) = \{ \pm i\lambda_j(0), \ 1 \leq j \leq d \} \]

coincides with the construction of Definition 2.21.

Second, the standard Gaussian \( z \mapsto \frac{n^n}{\pi^n} e^{-\frac{|z|^2}{2}} \) is the ground state of \( T_N(Q \circ S \circ U) \), and unitary change of variables act in a simple way on Toeplitz quantization:
\[ T_N(Q \circ S \circ U) = U^{-1}T_N(Q \circ S)U. \]

Here \( U \) acts on elements in \( B_N \) by a change of variables.

Since the standard Gaussian is invariant under unitary change of variables, it follows that it is the ground state of \( T_N(Q \circ S) \), independently on the parameter. \( \square \)

We recall that the rank of a quadratic form \( Q \) is the maximum dimension of a subspace on which \( Q \) is non-degenerate; it coincides with the rank of the associated linear map \( \tilde{Q} : E \to E^* \).

**Definition 2.23.** Let \( E \) be a real vector space and \( \omega \) a real antisymmetric form on \( E \). As in Definition 2.21, let \( \tilde{\omega} : E \leftrightarrow E^* \) be such that \( \tilde{\omega}(e)(f) = \omega(e, f) \) for \( (e, f) \in E^2 \).

The symplectic rank of \((E, \omega)\) is the rank of \( \tilde{\omega} \). In particular, if \((E, \omega)\) is a symplectic linear space, the symplectic rank of a linear subspace \( F \) is the rank of the restricted map \( \tilde{\omega} : F \leftrightarrow F^* \).

**Proposition 2.24.** Let \( Q : \mathbb{R}^d \mapsto S^+_{2n}(\mathbb{R}) \) be a smooth \( d \)-parameter family of semi-positive quadratic forms. Suppose \( \text{rank } Q \) is constant and suppose that the space \( \text{Ker } Q \) has constant symplectic rank.

Let \( 2r_1 \) be the symplectic rank of \( \text{Ker } Q \) and \( 2r_1 + r \) be the dimension of \( \text{Ker } Q \).

Then there is a smooth \( d \)-parameter family \( S : \mathbb{R}^d \mapsto S^+_{2n}(\mathbb{R}) \) of symplectic matrices, a \( d \)-parameter family \( U : \mathbb{R}^d \mapsto U(n - r - r_1) \) of unitary matrices, and a \( d \)-parameter family \((\lambda_{r_1+r_1}, \ldots, \lambda_n) : \mathbb{R}^d \mapsto \mathbb{R}^{n-r-r_1} \) such that, letting
\[
U(n) \ni U' = \begin{pmatrix} 1d & 0 \\ 0 & U \end{pmatrix}
\]
\((e_1, f_1, \ldots, e_n, f_n) = S(U'(\text{canonical basis}))\),

one has
\[
Q(t) \left( \sum_{i=1}^{n} q_i e_i(t) + p_i f_i(t) \right) = \sum_{i=r_1+1}^{r_1+r} p_i^2 + \sum_{i=r_1+r+1}^{n} \lambda_i(t)(p_i^2 + q_i^2). \]

In the study of the Hamiltonian dynamics related to \( Q \), the vectors \( f_i \) for \( i \) ranging from \( r_1 + 1 \) to \( r_1 + r \) are called slow modes. They correspond to the motion of a free particle. The vectors \((e_i, f_i)\), for \( i \) ranging from \( r_1 + r + 1 \) to \( n \), are called fast modes and correspond to harmonic oscillations. Elements in the kernel of \( Q \) are called zero modes.

As before, the symplectic eigenvalues of \( Q \) are the \( \lambda_j \)'s, as well as \( 0 \) if \( Q \) is degenerate.

**Proof.** Let us construct a symplectic basis \((e_1, \ldots, e_n, f_1, \ldots, f_n)\) of \( \mathbb{R}^{2n} \), depending smoothly on the parameters, on which the quadratic form \( Q \) is diagonal up to an action of \( U(n - r_1 - r) \) on the last variables. We first can reduce to the case where \( \text{Ker } Q \) is isotropic. Indeed, let \( \Sigma \) denote a smooth family of symplectic subspaces such that \( \Sigma \subset \text{Ker } Q \) and...
Ker $Q/\Sigma$ is isotropic. The existence of such a smooth family is guaranteed by the fact that Ker $Q$ has constant dimension and constant symplectic rank.

In any symplectic basis adapted to $\Sigma$, the matrix of the quadratic form $Q$ takes the form

$$
\begin{pmatrix}
0 & 0 \\
0 & M
\end{pmatrix},
$$

and it remains to study the quadratic form on $\mathbb{R}^{2n}/\Sigma$.

From now on we suppose that Ker $Q$ isotropic, and we set $r = \dim\text{Ker } Q$. We proceed by induction: if $Q$ is degenerate, we construct the first pair $(e_1, f_1)$ with $e_1 \in \text{Ker } Q$, hence the reduction to $Q'$ on $\mathbb{R}^{2(n-1)}$ with $\dim Q' = r - 1$. If $Q$ is non-degenerate, we conclude using Proposition 2.22.

Suppose $r > 0$. Pick $e_1 \in \ker Q$ smoothly depending on the parameters. The quadratic form $Q$ is degenerate, but it is a well-known fact that it has no co-isotropic subspaces: if a subspace $F$ is such that

$$
\{e \in \mathbb{R}^{2n}, \forall f \in F, Q(e + f) = Q(e) + Q(f)\} \subset F,
$$

then $F = \mathbb{R}^{2n}$.

Hence, with $F = \{z \in \mathbb{R}^{2n}, \langle z, Je_1 \rangle = 0\}$ denoting the symplectic orthogonal of $e_1$, there exists $f_1$ such that:

$$
\langle e_1, Jf_1 \rangle = 1 \\
\forall z \in F, Q(z + f_1) = Q(z) + Q(f_1).
$$

The vector $f_1$ again depends smoothly on the parameters. As $\lambda = Q(f_1)$ is far from zero on compact sets (recall that ker $Q$ is a continuous family of isotropic subspaces), changing $e_1$ into $\sqrt{\lambda}e_1$ and $f_1$ into $f_1/\sqrt{\lambda}$ yields two smooth vectors with the supplementary condition that $Q(f_1) = 1$.

If one can find a smooth symplectic basis $(e_2, \ldots, e_n, f_2, \ldots, f_n)$ of the symplectic orthogonal of $\text{Span}(e_1, f_1)$, which diagonalises the restriction of $Q$ with diagonal values as above, then completing this basis with $e_1$ and $f_1$ concludes the proof.

If $r = 0$, we are reduced to Proposition 2.22. $\square$

2.4. Quantum maps. To a (local) symplectomorphism between Kähler manifolds, one can associate an almost unitary (local) transformation on the Hardy spaces, such that, at first order, the Toeplitz quantizations on both sides are related by the symplectic change of variables in the symbols [8]:

**Proposition 2.25.** Let $\sigma : (M, x) \mapsto (N, y)$ be a local symplectomorphism between two quantizable Kähler manifolds.

Let $U$ be a small open set around $x$. Then there exists, for every $N$, a linear map $\mathcal{S}_N : H_N(M, L) \mapsto H_N(N, K)$ and a sequence of differential operators $(L_j)_{j \geq 1}$, such that, for any sequence $(u_N)_{N \geq 1}$ of sections which are $O(N^{-\infty})$ outside of $U$, and for any symbol $a \in C^\infty(N)$, one has:

$$
\|\mathcal{S}_N u_N\|_{L^2} = \|u_N\|_{L^2} + O(N^{-\infty})
$$

$$
\mathcal{S}_N^{-1} T_N(a) \mathcal{S}_N u_N = T_N \left( a \circ \sigma + \sum_{k=1}^{\infty} N^{-i} L_j(a \circ \sigma) \right) u_N + O(N^{-\infty}).
$$

Moreover, for every $j \geq 1$, the differential operator $L_j$ is of degree $2j$. 
As a preliminary lemma for Sects. 5 and 6, let us show that quantum maps preserve concentration speed:

**Lemma 2.26.** Let \( \sigma : (M, x) \mapsto (N, y) \) a local symplectomorphism between two quantizable Kähler manifolds.

Let \( 0 < \delta < \frac{1}{2} \) and let \((u_N)_{N \in \mathbb{N}}\) a sequence of unit elements in the Hardy spaces \( H_N(M) \) such that

\[
\int \{ \text{dist}(\pi(y), x) \leq N^{-\frac{1}{2} + \delta} \} |u_N(y)|^2 = O(N^{-\infty}).
\]

Then

\[
\int \{ \text{dist}(\pi'(y), \sigma(x)) \leq N^{-\frac{1}{2} + \delta} \} |\mathcal{S}u_N(y)|^2 = O(N^{-\infty}).
\]

**Proof.** Let us observe that the condition on \((u_N)\) is equivalent to the following: for every \( k \in \mathbb{N} \), there exists \( C_k > 0 \) such that

\[
\langle u_N, T_N(\text{dist}(\cdot, x)^{2k})u_N \rangle \leq C_N N^{-k(1+2\delta)}.
\]

Let us prove, by induction on \( k \), the estimate

\[
\langle \mathcal{S}_N u_N, T_N(\text{dist}(\cdot, \sigma(x))^{2k})\mathcal{S}_N u_N \rangle \leq C_k N^{-k(1+2\delta)}.
\]

The case \( k = 0 \) is clearly true since \( \mathcal{S}_N \) is an almost unitary operator when acting on functions localised near \( x \).

Let us now apply Proposition 2.25 with \( a = \text{dist}(\cdot, x)^{2k} \), stopping the expansion at order \( k \).

For \( j \leq k \), the error terms are controlled:

\[
\left| N^{-j} L_j(a \circ \sigma) \right| \leq N^{-j} C_{j,k} \text{dist}(\cdot, \sigma(x))^{2(k-j)}.
\]

Hence, by induction,

\[
\langle \mathcal{S}_N u_N, T_N(\text{dist}(\cdot, \sigma(x))^{2k})\mathcal{S}_N u_N \rangle \\
\leq \sum_{j=0}^{2k} C_{j,k} \langle u_N, T_N(\text{dist}(\cdot, x)^{2(k-j)})u_N \rangle + O(N^{-k}) = O(N^{k(-1+2\delta)}).
\]

This ends the proof. \( \square \)

### 3. A Melin Estimate for Toeplitz Quantization

Before stating (and proving) the Melin estimate, let us give two lemmas.

**Lemma 3.1.** Let \( Y \) be a compact Riemannian manifold. There exist two positive constants \( C \) and \( a_0 \) such that, for every positive integrable function \( f \) on \( Y \), for every
0 < a < a_0 and t ∈ (0, 1), there exists a finite family \((U_j)_{j \in J}\) of open subsets covering \(Y\) with the following properties:

\[
\forall j \in J, \text{diam}(U_j) < a
\]

\[
\forall j \in J, \text{dist}\left(\left.Y \setminus U_j, Y \setminus \bigcup_{i \neq j} U_i\right)\right) \geq ta
\]

\[
\sum_{i \neq j} \int_{U_i \cap U_j} f \leq Ct \int_Y f.
\]

**Proof.** Let \(m \in \mathbb{N}\) be such that there exists a smooth embedding of differential manifolds from \(Y\) to \(\mathbb{R}^m\), and let \(\Phi\) be such an embedding. \(\Phi\) may not preserve the Riemannian structure, so let \(c_1\) be such that, for any \(\xi \in TY\), one has

\[
c_1 \|\Phi^*\xi\| \leq \|\xi\|.
\]

We now let \(L > 0\) be such that any hypercube \(H\) in \(\mathbb{R}^m\) of side \(2/L\) is such that \(\text{diam}(\Phi^{-1}(H)) < a\). Since \(\Phi^{-1}\) is uniformly Lipschitz continuous, then if \(a\) is small enough one has \(aL \leq C_1\) for some \(C_1\) depending only on \(Y\).

We then prove the claim with \(C = \frac{2mC_1}{c_1}\).

Let \(1 \leq k \leq m\), and let \(\Phi_k\) denote the \(k\)-th component of \(\Phi\). The function \(\Phi_k\) is continuous from \(Y\) onto a segment of \(\mathbb{R}\). Without loss of generality this segment is \([0, 1]\).

Let \(g_k\) denote the integral of \(f\) along the level sets of \(\Phi_k\). The function \(g_k\) is a positive integrable function on \([0, 1]\).

Let \(t' > 0\) be the inverse of an integer, and let \(0 \leq \ell \leq L - 1\). In the interval \([\ell/L, (\ell + 1)/L]\), there exists a subinterval \(I\), of length \(t'/L\), such that

\[
\int_I g_k \leq t' \int_{\ell/L}^{(\ell+1)/L} g_k.
\]

Indeed, one can cut the interval \([\ell/L, (\ell + 1)/L]\) into \(1/t'\) intervals of size \(t'/L\). If none of these intervals was verifying (4), then the total integral would be strictly greater than itself.

Let \(x_{k, \ell}\) denote the centre of such an interval. Then, let

\[
V_{k,0} = \left[0, x_{k,0} + \frac{t'}{2L}\right)
\]

\[
V_{k,\ell} = \left(x_{k,\ell-1} - \frac{t'}{2L}, x_{k,\ell} + \frac{t'}{2L}\right) \text{ for } 1 \leq \ell \leq L
\]

\[
V_{k,L+1} = \left(x_{k,L} - \frac{t'}{2L}, 1\right].
\]

Each open set \(V_{k,l}\) has a length smaller than \(2/L\). The overlap of two consecutive sets has a length \(t'/L\), and the sum over \(k\) of the integrals on the overlaps is less than

\[
t' \int_0^1 g_k = t' \int_Y f.
\]

Now let \(\nu\) denote a polyindex \((\nu_k)_{1 \leq k \leq m}\), with \(\nu_k \leq L + 1\) for every \(k\). Define

\[
U_\nu = \Phi^{-1}\left(V_{1,\nu_1} \times V_{2,\nu_2} \times \cdots \times V_{m,\nu_m}\right).
\]
Then the family \((U_\nu)_\nu\) covers \(Y\). For every polyindex \(\nu\) one has \(\text{diam } U_\nu \leq a\) since \(U_\nu\) is the pull-back of an open set contained in a hypercube of side \(2/L\). Moreover, one has

\[
\text{dist}\left( Y \backslash U_\nu, Y \backslash \bigcup_{\nu' \neq \nu} U_{\nu'} \right) \geq \frac{c_1 t'}{L}.
\]

To conclude, observe that

\[
\sum_{\nu \neq \nu'} \int_{U_\nu \cap U_{\nu'}} f = \sum_{k=1}^{m} \sum_{\ell=0}^{L} \int_{V_{k,\ell} \cap V_{k,\ell+1}} g_k \leq m t' \int_Y f.
\]

It only remains to choose \(t'\) conveniently. The fraction \(\frac{a L}{c_1}\) may not be the inverse of an integer; however the inverse of some integer lies in \([\frac{a L}{2c_1}, \frac{a L}{c_1}]\). This allows us to conclude.

**Remark 3.2.** In the previous Lemma, the number of elements of \(J\) is bounded by a polynomial in \(a\) that depends only on the geometry of \(Y\).

**Lemma 3.3.** Let \(f \in C^3(\mathbb{R}, \mathbb{R}^+)\) and suppose that \(|f^{(3)}| \leq K\). Then

\[
f'' \geq -\left( \frac{3K^2 f}{2} \right)^{\frac{1}{3}}.
\]

**Proof.** Let \(x_0 \in \mathbb{R}\). Without loss of generality \(f'(x_0) \leq 0\) (otherwise we compose \(f\) with \(x \mapsto 2x_0 - x\)). Since \(f''\) is uniformly Lipschitz-continuous, for all \(x \geq x_0\) there holds

\[
f''(x) \leq f''(x_0) + K(x - x_0).
\]

Integrating twice yields

\[
f(x) \leq f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + K \frac{6}{6}(x - x_0)^3.
\]

Since \(f'(x_0) \leq 0\), one has

\[
f(x) \leq f(x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + K \frac{6}{6}(x - x_0)^3.
\]

If \(f''(x_0) \geq 0\) there is nothing to prove. Otherwise, the function

\[
x \mapsto f(x_0) - \frac{f''(x_0)}{2}(x - x_0)^2 + K \frac{6}{6}(x - x_0)^3
\]

reaches a local minimum at

\[
x_1 = x_0 - \frac{2f''(x_0)}{K},
\]

and

\[
0 \leq f(x_1) = f(x_0) + \frac{3(f''(x_0))^3}{K^2} - 4(f''(x_0))^3 = f(x_0) + \frac{2f''(x_0)^3}{3K^2}.
\]
In particular,

\[ f''(x_0) \geq -\left( \frac{3K^2 f(x_0)}{2} \right)^{\frac{1}{3}}, \]

hence the claim. \( \square \)

We are now in position to state, and prove, a Toeplitz version of the well-known Melin estimate for pseudodifferential estimates. It requires a weak condition on the speed of growth of the symbol near its zero set.

**Proposition 3.4 (Melin estimate).** Let \( h \in C^\infty (M, \mathbb{R}^+) \) with \( \min(h) = 0 \). Let

\[ \mu_{\min} = \min_{h(x) = 0} (\mu(x)). \]

Then

\[ \min \text{Sp}(T_N(h)) \geq \mu_{\min} N^{-1} - o(N^{-1}). \]

The remainder depends on \( M \) and \( h \). In particular, if there exist \( C > 0 \) and \( \alpha > 0 \) such that, for every \( t \geq 0 \), one has

\[ \text{dist}_{\text{Hausdorff}} ([h \leq t], \{h = 0\}) \leq Ct^\alpha, \]

then there exist \( \varepsilon > 0 \), \( N_0 \) and \( C' > 0 \) such that, for every \( N \geq N_0 \), one has

\[ \min \text{Sp}(T_N(h)) \geq \mu_{\min} N^{-1} - C'N^{-1-\varepsilon}. \]

The more precise result is a generalisation of [23] (where \( \varepsilon = \frac{1}{2}, \frac{1}{4} \)), [18] (in which case \( \varepsilon = \frac{1}{4} \)), [17] (\( \varepsilon = 1 \)). The sharpest possible \( \varepsilon \) depends on the geometry of the problem. We will actually see that, in the settings of Theorems B and C, there holds \( T_N(h) \geq \mu_{\min} N^{-1} \). We don’t know whether or not \( T_N(h) \geq \mu_{\min} N^{-1} \) is true for any \( h \in C^\infty (M, \mathbb{R}^+) \).

**Proof.** We begin with a local result: for all \( \delta_0, \delta_1 \) small enough and a real sequence \( g(N) \xrightarrow{N \to +\infty} 0 \) such that, for every \( x \in M \) with \( h(x) < N^{-1+\delta_1} \), for every \( u \in L^2(X) \) supported on \( B(x, N^{-\frac{1}{2}+\delta_0}) \times S^1 \), one has

\[ \langle S_N u, hS_N u \rangle \geq (\mu_{\min} N^{-1} - N^{-1} g(N)) \|S_N u\|^2. \]

To this end, we modify \( h_0 \) near \( x \) into a convex function \( \tilde{h}_0 \) (so that, when comparing \( \tilde{h}_0 \) to its Hessian at a critical point, the Hessian will be semipositive and we will be able to consider its Melin value).

Indeed, by Lemma 3.3, one has

\[ \text{Hess}(h_0)(x) \geq -CN^{(-1+\delta_1)/3}. \]

The following perturbation of \( h_0 \) is convex on \( B(x, N^{-\frac{1}{2}+\delta_0}) \):

\[ \tilde{h}_0 : y \mapsto h_0(y) + CN^{\max((-1+\delta_1)/3, -1/2+\delta_0)} \text{dist}(y, x)^2. \]
If now \( u \in L^2(X) \) is normalized and supported on \( B(x, N^{-\frac{1}{2}+\delta_0}) \times \mathbb{S}^1 \), and if \( S_N u = u + O(N^{-\infty}) \), one has
\[
\left| \langle S_N u, (h_0 - \tilde{h}_0)S_N u \rangle \right| \leq C N^{-1+2\delta_0+\max((-1+\delta_1)/3,-1/2+\delta_0)}.
\]

As \( \sqrt{h_0} \) is Lipschitz-continuous (see Lemma 4.31 in \([46]\)), one has
\[
\sup \left( \sqrt{h_0}(y), \text{dist}(y, x) < 2N^{-\frac{1}{2}+\delta_0} \right) < C N^{-\frac{1}{2}} + C N^{-\frac{1}{2}+\delta_0}.
\]
Hence,
\[
\sup \left( h_0(y), \text{dist}(y, x) < 2N^{-\frac{1}{2}+\delta_0} \right) < C N^{-1+\max(\delta_1,2\delta_0)}.
\]

Recall from Proposition 2.9 that, for \( \delta \) small enough, for every \( x \in M \) with associated normal map \( \rho \), for every \( u \) with support inside \( \rho(B(0, N^{-\frac{1}{2}+\delta}) \times \mathbb{S}^1) \), one has
\[
\| (S_N - \rho^*\Pi_N) u \|_{L^2} < CN^{-\frac{1}{4}}.
\]

Hence, if \( \delta_0 < \delta \), then, for \( N \) large enough, by Proposition 2.9,
\[
\left| \langle (S_N - \Pi_N^*)u, \tilde{h}_0 S_N u \rangle \right| \leq C N^{-\frac{1}{4}} N^{-1+\max(\delta_1,2\delta_0)},
\]
\[
\left| \langle \Pi_N^* u, \tilde{h}_0 (S_N - \Pi_N^*) u \rangle \right| \leq C N^{-\frac{1}{4}} N^{-1+\max(\delta_1,2\delta_0)}.
\]

If the function \( \tilde{h}_0 \) reaches its minimum on \( B(x, N^{-\frac{1}{2}+\delta_0}) \) at an interior point \( x' \) and if \( Q \) denotes half of the Hessian matrix of \( \tilde{h}_0 \) at \( x' \), then
\[
\left| \langle \Pi_N u^*, \tilde{h}_0^* - Q, \Pi_N u^* \rangle \right| \leq C N^{-\frac{1}{4}+3\delta_0}.
\]

Here, the subscript \(^*\) denotes the pull-back by the normal map. Similarly \(^*\) will denote the push-forward by the normal map.

If \( \tilde{h}_0 \) reaches its minimum at a boundary point \( x' \), then if \( L \) denotes the differential of \( \tilde{h}_0 \) at \( x' \) one has, by convexity of the ball, for all \( y \in B(x, N^{-\frac{1}{2}+\delta_0}) \),
\[
L(y - x') \geq 0.
\]

In particular,
\[
\langle \Pi_N^* u, \tilde{h}_0 \Pi_N^* u \rangle \geq \langle \Pi_N^* u, \tilde{h}_0 - L \rangle \Pi_N^* u \rangle.
\]

Then \( y \mapsto \tilde{h}_0(y) - L(y - x') \) has a critical point at \( x' \). If \( Q \) denotes again half of the Hessian matrix of \( \tilde{h}_0 \) at \( x' \), then
\[
\left| \langle \Pi_N u^*, \tilde{h}_0^* - L - Q, \Pi_N u^* \rangle \right| \leq C N^{-\frac{3}{4}+3\delta_0}.
\]

In any case, \( x' \) is at distance at most \( N^{-\frac{1}{2}+\delta_0} \) of \( x \), and \( u \) is supported on a ball around \( x \) of same radius, so that
\[
N^{-1} \langle S_N u, h_1 S_N u \rangle - h_1(x') \| S_N u \|^2 \leq C N^{-\frac{3}{4}+\delta_0},
\]
\[
\left| \langle S_N u, \sum_{k=2}^{+\infty} N^{-k} h_k S_N u \rangle \right| \leq C N^{-2}.
\]
Since
\[ \text{dist}(x', \{h = 0\}) \leq \text{dist}_H(\{h_0 = 0\}, \{h \leq N^{-1+\delta_1}) + N^{-\frac{1}{2}+\delta_0} = g_0(N) \left[ N \to +\infty \right] \to 0, \]
the matrix $Q$ is $g_0(N)$-close to half of the Hessian matrix of $h_0$ at a zero point (recall we only added $CN^{-\epsilon} I$ to the Hessian matrix of $h$ at $x$.)

The Melin value $\mu$ is Hölder-continuous with exponent $(2n)^{-1}$ on the set of semi-positive quadratic forms $[30]$, hence
\[ \mu(Q) + h_1(x') \geq \mu_{\text{min}} + O((g_0(N))^{1/2n}). \]

To conclude, with $g(N) = C(g_0(N))^{1/2n}$, one has
\[ \langle S_N u, h S_N u \rangle \geq N^{-1} \mu_{\text{min}} - N^{-1} g(N). \]

Note that, if $h_0$ satisfies, for some $\alpha$, for all $t \geq 0$,
\[ \text{dist}_H(\{h_0 \leq t\}, \{h_0 = 0\}) \leq Ct^\alpha, \]
then $g(N) = N^{-\epsilon}$ for some $\epsilon$ depending on $\alpha, \delta_0, \delta_1$.

From this local estimate, we deduce a global estimate using Lemma 3.1 proved previously, and Proposition 2.19.

Indeed, let $(u_N)_{N \geq 1}$ be a sequence of normalised eigenfunctions for $T_N(h)$ with minimal eigenvalue. Either the associated sequence of eigenvalues is not $O(N^{-1})$, in which case the proposition holds, or it is, in which case, by Proposition 2.19, $u_N$ is $O(N^{-\infty})$ outside $\{h_0 \leq N^{-1+\delta_1}\}$ for every $\delta_1 > 0$.

We now invoke Lemma 3.1 with the following data:

- $Y = M$.
- $f = |u_N|^2$.
- $a = N^{-\frac{1}{2}+\delta_0}$.
- $t = N^{-\frac{2\delta_0}{2}}$.

The Lemma yields a sequence of coverings $(U_{j,N})_{j \in J_N, N \in \mathbb{N}}$. The proof also yields a sequence of coverings by slightly smaller open sets $(U'_{j,N})$, with

- $U'_{j,N} \subset U_{j,N}$.
- $d(M \setminus U_{j,N}, U'_{j,N}) > \frac{1}{3} N^{-\frac{1}{2}+\delta_0}$.

Let $(\chi_{j,N})_{j \in J_N, N \in \mathbb{N}}$ be a partition of unity associated with $(U'_{j,N})_{j \in J_N, N \in \mathbb{N}}$.

Before we proceed to the proof, let us show that, for all $g \in C^\infty(M, \mathbb{R}^+)$, as $N \geq 0$, one has
\[ \sum_{j \neq k \in J_N} |\langle \chi_{j,N} u_N, T_N(g) \chi_{k,N} u_N \rangle| \leq C N^{-\frac{\delta_0}{2}} \|u_N\|_{L^2}^2 \sup_{\{h \leq N^{-1+\delta_1}\}} (g) + O(N^{-\infty}). \]

First, let $U'_{j,N} \subset V_{j,N} \subset U_{j,N}$ be such that
\[ d(M \setminus U_{j,N}, V_{j,N}) > \frac{1}{6} N^{-\frac{1}{2}+\delta_0}, \quad d(M \setminus V_{j,N}, U'_{j,N}) > \frac{1}{6} N^{-\frac{1}{2}+\delta_0}. \]

Then $S_N \chi_{j,N} u_N$ is $O(N^{-\infty})$ outside $V_{j,N}$. It is also $O(N^{-\infty})$ outside $\{h \geq N^{-1+\delta_1}\}$ by Proposition 2.19.
In particular,
\[
\langle \chi_j, N u_N, T_N(g) \chi_k, N u_N \rangle = \langle S_N \chi_j, N u_N, g S_N \chi_k, N u_N \rangle
\]
\[
\leq \|S_N \chi_j, N u_N\|_{L^2(V_{j,N} \cap V_{k,N})} \|S_N \chi_k, N u_N\|_{L^2(V_{j,N} \cap V_{k,N})} \sup_{\{h \geq N^{-1+\epsilon_1}\}} (g).
\]

Now
\[
\|S_N \chi_j, N u_N\|^2_{L^2(V_{j,N} \cap V_{k,N})} = \int_{V_{j,N} \cap V_{k,N}} \left| \int_{U_{j,N}'} S_N(x, y) \chi_j, N u_N(y) \, dy \right|^2 \, dx
\]
\[
= \int_{V_{j,N} \cap V_{k,N}} \left| \int_{U_{j,N} \cap U_{k,N}'} S_N(x, y) u_N(y) \, dy \right|^2 \, dx + O(N^{-\infty})
\]
\[
\leq \|S_N \chi_j, N u_N\|^2 + O(N^{-\infty})
\]
\[
\leq \|u\|^2_{L^1(U_{j,N} \cap U_{k,N})} + O(N^{-\infty}).
\]

Then
\[
\sum_{j \neq k \in J_N} |\langle \chi_j, N u_N, T_N(g) \chi_k, N u_N \rangle| \leq \sup_{\{h \leq N^{-1+\epsilon_1}\}} (g) \sum_{j \neq k \in J_N} \|u\|^2_{L^1(U_{j,N} \cap U_{k,N})} + O(N^{-\infty})
\]
and one can conclude by Lemma 3.1. (The $O(N^{-\infty})$ can be summed over $J_N^2$ since the latter has a number of elements bounded by a polynomial in $N$ by Remark 3.2.)

In particular, with $g = h$, there holds
\[
\sum_{j \neq k \in J_N} |\langle \chi_j, N u_N, T_N(h) \chi_k, N u_N \rangle| \leq C N^{-1+\delta_1} N^{-\frac{\delta_0}{2}} + O(N^{-\infty}).
\]

In particular, if $\delta_1 < \delta_0/2$, then
\[
\sum_{j \neq k \in J_N} |\langle \chi_j, N u_N, T_N(h) \chi_k, N u_N \rangle| = O(N^{-1-\epsilon}).
\]

On the other hand,
\[
\sum_{j \in J_N} \langle \chi_j, N u_N, T_N(h) \chi_j, N u_N \rangle \geq (\mu_{\min} N^{-1} - N^{-1} g(N)) \sum_{j \in J_N} \|S_N \chi_j, N u_N\|^2_{L^2}.
\]

With $g = 1$, one has in turn
\[
\sum_{j \neq k \in J_N} |\langle \chi_j, N u_N, \chi_k, N u_N \rangle| \leq N^{-\frac{\delta_0}{2}}
\]
so that, since $\sum_j \chi_j, N = 1$,
\[
\sum_{j \in J_N} \|S_N \chi_j, N u_N\|^2_{L^2} \geq (1 - C N^{-\frac{\delta_0}{2}}).
\]

Then, choosing $\delta_1 < \frac{\delta_0}{2}$ allows us to conclude:
\[
\langle u_N, T_N(h) u_N \rangle \geq N^{-1} (\mu_{\min} - g(N))
\]
\[\square\]
Note that, in the last proof, it is essential that we know beforehand that $u_N$ is $O(N^{-\infty})$ on $\{h_0 \geq N^{-1+\delta}\}$ for every $\delta > 0$. This was achieved by picking $u_N$ as the unique minimizer of $\langle u, T_N(h)u \rangle$ under $\|u\| = 1$, in which case $u_N$ is an eigenfunction of $T_N(h)$.

Remark 3.5. Proposition 3.4 only relies on elementary properties of the Szegő kernel and Toeplitz operators (that is, Propositions 2.4 and 2.9). As such, it extends readily to more general contexts of quantizations, such as Spin$^c$-Dirac [27] (up to a modification in the definition of $\mu_{\min}$).

4. Quantum Selection in the General Setting

4.1. Pseudo-locality of the resolvent.

Proposition 4.1. Let $h$ and $\mu_{\min}$ be as in Proposition 3.4. Then, for every $c > 0$, the operator $T_N(h - N^{-1}(\mu_{\min} - c))$ is invertible (as a positive definite operator on a finite-dimensional space). Its inverse $R$ is pseudo-local: if $a$ and $b$ are smooth functions with supp$(a) \cap$ supp$(b) = \emptyset$, then

$$T_N(a)RT_N(b) = O_{L^2 \to L^2(N^{-\infty})}.$$

Proof. The proposition may be reformulated this way: if $U \subset\subset V$ are two open sets in $M$ and a sequence $(u_N)_{N \geq 1}$ of normalised states in $H_N(M)$ is such that

$$T_N(h - N^{-1}\mu_{\min} + cN^{-1})u_N = O_{L^2(N^{-\infty})}$$

on $V$, then we wish to prove that $u_N = O_{L^2(N^{-\infty})}$ on $U$. Here

$$\text{supp}(a) \subset\subset U \subset\subset V \subset\subset (M \setminus \text{supp}(b)).$$

We first remark that, for every $\delta$, and for every $U \subset\subset V_1 \subset\subset V$, the following holds:

$$\int_V \overline{u} T_N(h)u \geq C N^{-1+\delta} \int_{V_1 \cap \{h_0 \geq N^{-1-\delta}\}} |u|^2.$$

Hence, $u$ is $O(N^{-\infty})$ on $V_1 \cap \{h_0 \geq N^{-1-\delta}\}$ for every $\delta$.

We are now able to repeat the global part of the proof of Proposition 3.4 by cutting a neighbourhood of $U$ into small pieces, hence the claim. □

4.2. Upper estimate of the first eigenvalue.

Proposition 4.2. Let $h$ be a classical symbol on $M$ with $\min(h_0) = 0$ and let $\mu_{\min}$ be as in Proposition 3.4. Then there exists $\epsilon > 0$ such that

$$\inf Sp(T_N(h)) \leq N^{-1}\mu_{\min} + N^{-1-\epsilon}.$$

Proof. The spirit of the proof is to test $T_N(h)$ against an eigenstate of a quadratic operator $T_N(\text{Hess}(h_0)(P_0))$, where $\mu$ is minimal at $P_0$. However, since $\text{Hess}(h_0)(x_0)$ is only semi-positive, its ground state may have no sense as an $L^2$ function or fail to localise at $x_0$. We slightly modify $h_0$ in the neighbourhood of $P_0$ so that the Hessian is non-degenerate.
Let \( P_0 \in M \) achieve the minimal value \( \mu_{\min} \), let \( \rho \) be a normal map around \( P_0 \) and, following Proposition \( 2.9 \), let \( \delta > 0 \) and \( C > 0 \) be such that, for every \( N \), for every \( u \) supported on \( B(P_0, N^{-\frac{1}{2}+\delta}) \times S^1 \), one has \( \| (S_N - \rho^* \Pi_N) u \| \leq CN^{-\frac{1}{4}} \). Without loss of generality \( \delta < \frac{C}{8} \).

Pick \( \alpha < 2\delta \), and let \( Q \) denote half of the Hessian of \( h_0 \) at \( P_0 \). Then, since the function \( Q \mapsto \mu(Q) \) is H"older continuous with exponent \( \frac{1}{2n} \) [30], one has

\[
\mu(Q + N^{-\alpha} \cdot |^2) \leq \mu(Q) + CN^{-\frac{\alpha}{2n}}.
\]

Let \( v_N \) denote a normalised ground state of \( T_N(Q + N^{-\alpha} \cdot |^2) \), then \( v_N \) is \( O(N^{-\infty}) \) outside \( B(0, N^{-\frac{1}{2}+\delta}) \) by Proposition \( 2.19 \).

Then

\[
\langle \rho_* v_N, T_N(h_0) \rho_* v_N \rangle = \langle v_N, \Pi_N(h_0 \circ \rho) \Pi_N v_N \rangle + O(N^{-\frac{5}{4}+2\delta})
= \langle v_N, \Pi_N Q \Pi_N v_N \rangle + O(N^{-\frac{3}{4}+2\delta}) + O(N^{-\frac{3}{4}+3\delta})
\leq \langle v_N, \Pi_N (Q + N^{-\alpha} \cdot |^2) \Pi_N v_N \rangle + O(N^{-\frac{5}{4}+2\delta}) + O(N^{-\frac{3}{4}+3\delta})
= N^{-1} \mu(Q + N^{-\alpha} \cdot |^2) + O(N^{-\frac{5}{4}+2\delta}) + O(N^{-\frac{3}{4}+3\delta})
\leq N^{-1} \mu(Q) + O(N^{-1-\alpha/2n}) + O(N^{-\frac{5}{4}+2\delta}) + O(N^{-\frac{3}{4}+3\delta})
= N^{-1} \mu(Q) + O(N^{-1-\epsilon})
\]

for some \( \epsilon > 0 \).

In particular, since for all \( y \in B(P_0, N^{-\frac{1}{2}+\delta}) \) one has \( h_1(y) \leq h_1(x) - CN^{-\frac{1}{2}+\delta} \), one has

\[
\langle \rho_* v_N, T_N(h) \rho_* v_N \rangle \leq N^{-1} \mu_{\min} + O(N^{-1-\epsilon}).
\]

\( \square \)

### 4.3. End of the proof

We can now conclude the proof of Theorem \( A \). The estimate on the first eigenvalue consists in Propositions \( 3.4 \) and \( 4.2 \).

Let \( C > 0 \), and let \( \{u_N\}_{N \in \mathbb{N}} \) be a sequence of eigenfunctions of \( T_N(h) \), with eigenvalues \( \lambda_N \) smaller than \( N^{-1} (\mu_{\min} + C) \).

Let \( a \in C^\infty(M, \mathbb{R}) \) be supported away from \( \{ x \in M, h_0(x) = 0, \mu(x) \leq \mu_{\min} + C \} \). Let \( \hat{h} \) be a classical symbol on \( M \) such that \( \hat{h} = h \) on a neighbourhood of the support of \( a \), and such that \( \mu_{\min}(\hat{h}) > \mu_{\min}(h) = C \). Then \( T_N(\hat{h} - N^{-1} \lambda_N) \) is invertible because of the Melin estimate of Proposition \( 3.4 \). Its inverse \( R \) is pseudolocal, with norm \( O(N) \), by Proposition \( 4.1 \). In particular,

\[
T_N(a) u_N = T_N(a) RT_N(\hat{h} - N^{-1} \lambda_N) u_N
= T_N(a) RT_N(\hat{h} - h) u_N
= O(N^{-\infty}).
\]

This concludes the proof of Theorem \( A \).
5. Normal Form for Miniwells

In this section we prove Theorem B, and establish the necessary material for the Weyl asymptotics of Sect. 7.

We first study a problem of symplectic geometry, which consists in finding a normal form for a non-negative function $h_0$ vanishing at order 2 on an isotropic submanifold. Then, we apply a Quantum Map to find an expansion of the first eigenvalue and eigenfunction.

We let $M$ be a compact quantizable Kähler manifold and $h$ be a classical symbol on $M$ which satisfies the hypotheses of Theorem B.

5.1. A convenient chart. Recall the following well-known application of Moser’s principle:

**Proposition 5.1.** Let $S$ be a symplectic manifold and $Z \subset S$ be a smooth $d$-dimensional submanifold of constant symplectic rank. Then, in a neighbourhood (in $S$) of any point in $Z$, there is a symplectomorphism $\rho$ onto a neighbourhood of $0$ in $\mathbb{R}^{2n}$, such that $\rho(Z)$ is the intersection of a linear subspace with an open neighbourhood of zero in $\mathbb{R}^{2n}$.

Using Propositions 5.1 and 2.24, we will prove the normal form for miniwells on isotropic submanifolds:

**Proposition 5.2.** Let $h_0$ be a smooth non-negative function on $M$, which vanishes on an isotropic manifold $Z$ of dimension $r$ with everywhere non-degenerate transverse Hessian.

Near any point of $Z$, there is a symplectomorphism $\rho$ into $\mathbb{R}^{2r} \times \mathbb{R}^{2(n-r)}$, a smooth function $Q_S$ from $\mathbb{R}^r$ into the set of positive quadratic forms of dimension $r$, and $n-r$ smooth positive functions $(\lambda_i)_{1 \leq i \leq n-r}$ such that:

$$h_0 \circ \rho = Q_F^{red}(q)(x, \xi) + Q_S(q)(p) + O(|x|^3 + |\xi|^3 + |p|^3),$$

where, for every $q$ close to $0$, the ground state of $T_N((x, \xi) \mapsto Q_F^{red}(q)(x, \xi))$ is the standard Gaussian.

In particular, $Z$ is mapped into $\{(p, x, \xi) = 0\}$.

**Proof.** Let $P_0 \in Z$, and let $U$ be a small neighbourhood of $P_0$ in $M$. Let us use Proposition 2.24 with set of parameters $Z \cap U$ and quadratic form $\text{Hess}(h_0)$, which is semi-positive definite along $Z \cap U$, with kernel of constant symplectic rank.

This yields, at each point of $Z$ in a neighbourhood of $P_0$, a family of $2n$ vector fields which form a symplectic basis:

$$B = (Q_1, \ldots, Q_r, P_1, \ldots, P_r, X_1, \ldots, X_{n-r}, \Xi_1, \ldots, \Xi_{n-r}),$$

such that $\text{Span}(Q_1, \ldots, Q_r) = TZ$. In the general setting, this does not give a symplectic change of variables under which the quadratic form is diagonal along the whole zero set (indeed, $Q_1, \ldots, Q_r$ are prescribed by the $2n-r$ other vector fields, and do not commute in general). However, one can separate the slow variables and the fast variables (first step), then diagonalise the fast variables (second step).

**First step:** Let us define the distribution $\mathcal{F}$ on $Z \cap U$ as follows: for $x \in Z \cap U$,

$$\mathcal{F}_x = \text{Span}(Q_1, \ldots, Q_r, P_1, \ldots, P_r)(x).$$
Then $T(Z \cap U) \subset \mathcal{F}$. In particular, there is a piece $S$ of symplectic submanifold of $M$, containing $Z \cap U$, and tangent to $\mathcal{F}$ on $Z \cap U$.

Using Proposition 5.1, we let $\phi_0$ be a symplectomorphism from a neighbourhood of $P_0$ in $M$ into a neighbourhood of $0$ in $\mathbb{R}^{2r} \times \mathbb{R}^{2(n-r)}$, such that $S$ is mapped into $\mathbb{R}^{2r} \times \{0\}$. Using Proposition 5.1 again, let $\phi_1$ be a symplectomorphism on a neighbourhood of $0$ in $\mathbb{R}^{2r}$, that maps $\phi_0(Z)$ into $\mathbb{R}^r \times \{0\}$. Then the map $\tilde{\phi}_1$ acting on $\mathbb{R}^{2r} \times \mathbb{R}^{2(n-r)}$ by

$$\tilde{\phi}_1 (p, q, x, \xi) = (\phi_1(p, q), x, \xi)$$

is a symplectomorphism. We claim that $\rho = \tilde{\phi}_1 \circ \phi_0$ separates the fast variables from the slow variables up to $O((x, \xi, p)^3)$.

Indeed, consider $D\rho$ at a point of $Z$. Since $\rho$ sends $Z$ into $\mathbb{R}^r \times \{0\}$, and $S$ into $\mathbb{R}^{2r} \times \{0\}$, the matrix of $D\rho$, from the basis $B$ to the canonical basis, is of the form:

$$D\rho = \begin{pmatrix} A_{qq} & 0 & 0 & 0 \\ A_{pq} & A_{pp} & 0 & 0 \\ A_{xq} & A_{xp} & A_{xx} & A_{x\xi} \\ A_{\xi q} & A_{\xi p} & A_{\xi x} & A_{\xi \xi} \end{pmatrix}.$$ 

Moreover, $D\rho$ is symplectic, so that the bottom left part vanishes. Hence,

$$h_0 \circ \rho^{-1} = Q_F(q)(x, \xi) + Q_S(q)(p) + O(|p|^3 + |x|^3 + |\xi|^3),$$

for some quadratic forms $Q_F$ and $Q_S$.

Since $h_0$ vanishes at order exactly 2 on $Z$, the quadratic forms $Q_F$ and $Q_S$ are positive definite.

**Second step:** It only remains to diagonalise $Q_F$ with a symplectomorphism. In fact, this is possible without modifying $Q_S$. Indeed, let $\phi : (\mathbb{R}^r, 0) \mapsto S^p(2(n-r))$ be such that, for every $q$ near zero, the matrix $\phi(q)$ realises a symplectic reduction of $Q_F(q)$, as in Proposition 2.24. With $J$ the standard complex structure matrix on $\mathbb{R}^{2(n-r)}$ and $\langle \cdot, \cdot \rangle$ its standard Euclidian norm, we define, for every $1 \leq i \leq r$, the real function

$$f_i : (q, x, \xi) \mapsto \frac{1}{2} \langle (x, \xi), (\partial_{q_i} \phi(q) J \phi^t(q))(x, \xi) \rangle.$$ 

We then define $f : (\mathbb{R}^{2n-r}, 0) \rightarrow \mathbb{R}^r$ as the map with components $f_i$ in the canonical basis. Then a straightforward computation shows that the map

$$\Phi : (q, p, x, \xi) \mapsto (q, p + f, \phi(q)(x, \xi))$$

is a symplectomorphism. As $f = O((x, \xi) \mapsto 0((x, \xi)^2)$, the 2-jet of $h_0 \circ \Phi$ at $(q, 0, 0, 0)$ is the same as the 2-jet of $h_0 \circ ((q, p, x, \xi) \mapsto (q, p, \phi(q)(x, \xi)))$, i.e.

$$Q_S(q)(p) + Q_F^{red}(q)(x, \xi),$$

where the ground state of $T_N(Q_F^{red})$ is the standard Gaussian in $x$ and $\xi$. This concludes the proof. □

**Remark 5.3.** We corrected the map

$$(q, p, x, \xi) \mapsto (q, p, \phi_q(x, \xi))$$

into a symplectomorphism by only changing the second coordinate. This does not depend on the fact that $\phi_q$ acts linearly but relies only on $\phi_q(0, 0) = (0, 0)$. 
5.2. Approximate first eigenfunction. Let us quantize, using Proposition 2.25, the symplectic map of Proposition 5.2, and conjugate with pseudodifferential operators:

**Definition 5.4.** For any choice $\mathcal{S}_N$ of quantization of the map $\rho$ of Proposition 5.2, the classical symbol $g_{\mathcal{S}} \sim \sum N^{-i} g_i$ on a neighbourhood $U$ of 0 in $\mathbb{R}^{2n}$ is defined as follows: for any sequence $(v_N)_{N \geq 1}$ with microsupport in a compact set of $U$, the following holds:

$$B^{-1}_N \mathcal{S}^{-1}_N T_N(h) \mathcal{S}_N B_N v_N = Op_W^{-1}(g_{\mathcal{S}}) v_N + O(N^{-\infty}).$$

In what follows, we choose an arbitrary quantum map $\mathcal{S}_N$, and we write $g$ instead of $g_{\mathcal{S}}$. The reason we use Weyl quantization in this subsection is because we will rely heavily on squeezing operators. While one could, in principle, perform all computations in the Bargmann setting, the squeezing operators have a much simpler form in the Weyl representation.

The principal and subprincipal symbols of $g$ are explicit at the points of interest: $g_0 = h_0 \circ \rho$ by construction, and $g_1$ is prescribed on $\{g_0 = 0\}$ by the Melin estimates for Weyl and Toeplitz quantizations:

**Proposition 5.5.** For any $q$ close to 0, one has

$$g_1(q, 0, 0, 0) = \frac{1}{4} \text{tr(Hess}(h_0)(\rho(q, 0, 0, 0))) + h_1(\rho(q, 0, 0, 0)).$$

**Proof.** From the expression of $h_0 \circ \rho$ in Proposition 5.2, one has

$$\mu(\rho(q, 0, 0, 0)) = \frac{1}{4} \text{tr}(Q^{red}_F(q)) + \frac{1}{4} \text{tr(Hess}(h_0)(\rho(q, 0, 0, 0))) + h_1(\rho(q, 0, 0, 0)).$$

If $h_0(x) = 0$ and $\delta > 0$ is small enough, the value $\mu(x)$ has the following variational characterisation:

$$\mu(x) = \lim_{N \to +\infty} \left( N \inf \left( \int_M h|u|^2, u \in H_N(X), \int_{B(x, N^{-\frac{1}{2}+\delta})} |u|^2 = 1 \right) \right)$$

This variational problem can be read via the quantum map. If

$$\int_{B(x, 2N^{-\frac{1}{2}+\delta})} |u|^2 = O(N^{-\infty}),$$

then $B^{-1}_N \mathcal{S}^{-1}_N u$ microlocalises at speed $N^{-\frac{1}{2}+\delta}$ on $\rho^{-1}(x)$, and moreover,

$$\int_M h|u|^2 = \left( B^{-1}_N \mathcal{S}^{-1}_N u, Op_W^{-1}(g_0 + N^{-1} g_1) B^{-1}_N \mathcal{S}^{-1}_N u \right) + O(N^{-2}) \|u\|_2.$$

Now, if $x = \rho(q, 0, 0, 0)$, the usual Melin estimate yields

$$\lim_{N \to +\infty} \left( N \inf \left( \left( v Op_W^{-1}(g_0) v, \mathcal{S}_N B_N v \text{ as above} \right) \right) \right) = \frac{1}{4} \text{tr}(Q^{red}_F(q)),$$

hence, $g_1(\rho(q, 0, 0, 0))$ contains all the defect between $\mu(\rho(q, 0, 0, 0))$ and this estimate. \qed
Remark 5.6. In general, the subprincipal symbol is not unique after application of a quantum map. Indeed, if \( a \) is any smooth real-valued function on \( M \) then \( \exp(i T_N(a)) \) is a unitary operator, and composing \( \mathcal{G}_N \) with this operator changes the subprincipal term. Indeed, by the Baker-Campbell-Hausdorff formula, one has

\[
e^{i T_N(a)} T_N(h) = e^{i T_N(a)} T_N(h) e^{-i T_N(a)} e^{i T_N(a)} = T_N \left( h + N^{-1} [a, h] + O(N^{-2}) \right) e^{i T_N(a)}.
\]

Proposition 5.5 shows that on the points where the principal symbol vanishes, the subprincipal symbol is in fact rigid through any such transformations.

Let us find a candidate for an approximate first eigenfunction:

**Proposition 5.7.** Suppose that the function \( q \mapsto \mu \circ \rho(q, 0, 0, 0) \) reaches a non-degenerate minimum at 0. Let \( \phi \) be the positive quadratic form such that \( q \mapsto e^{-\phi(q)} \) is the ground state of the operator

\[
Q_S(0)(-i \nabla) + \frac{1}{2} \sum_{i,j=1}^{r} q_i q_j \frac{\partial^2}{\partial q_i \partial q_j} (\mu \circ \rho)(0, 0, 0, 0),
\]

with eigenvalue \( \mu_2 \).

Then there exists a sequence of polynomials \((b_i)_{i \geq 1}\), and a sequence of real numbers \((\mu_i)_{i \geq 1}\), with

\[
\mu_0 = \mu \circ \rho(0, 0, 0, 0) \\
\mu_1 = 0
\]

and \( \mu_2 \) as previously, such that, for every \( k \),

\[
f_N^k : (q, x) \mapsto N^{\frac{3}{2}} - \xi x N^{-\frac{3}{2}} e^{-\sqrt{N} \phi(q)} \left( 1 + \sum_{i=1}^{k} N^{-\frac{i}{2}} b_i (N^{\frac{1}{2}} q, N^{\frac{1}{2}} x) \right)
\]

is an approximate eigenvector to \( Op_{W}^{N^{-1}}(g) \), with eigenvalue

\[
\lambda_N^k = N^{-1} \sum_{i=0}^{k} N^{-\frac{i}{2}} \mu_i,
\]

in the sense that, for every \( K \) there exists \( k \) such that

\[
\| Op_{W}^{N^{-1}}(g) f_N^k - \lambda_N^k f_N^k \|_{L^2} = O(N^{-K}).
\]

This proposition provides an almost eigenfunction which we will show to be associated to the lowest eigenvalue (see Proposition 5.9). It is the main argument in the proof of Theorem B; the concentration speed of this eigenfunction on zero, which is \( N^{-\frac{1}{2}} \), is the concentration speed of the lowest eigenvector of \( T_N(h) \) on the miniwell \( P_0 \), because of Proposition 5.10.
Proof. The proof proceeds by a squeezing of $Op_{W}^{N-1}(g)$ by a factor $N^{\frac{1}{4}}$ along the $q$ variable.

Let

$$\tilde{g}_N = g(N^{-\frac{1}{4}}q, N^{-\frac{3}{4}}p, N^{-\frac{1}{2}}x, N^{-\frac{1}{2}}\xi).$$

Then $Op_{W}^{N-1}(g_N)$ is conjugated with $Op_{W}^{1}(\tilde{g}_N)$ through the unitary change of variables $u \mapsto N^{\frac{n}{2}-\frac{r}{4}}u(N^{-\frac{1}{4}}q, N^{-\frac{1}{2}}x)$.

Grouping terms in a Taylor expansion of $\tilde{g}_N$ yields

$$\tilde{g}_N = N^{-1} \sum_{i=0}^{K} N^{-\frac{i}{4}}a_i(q, p, x, \xi) + O(N^{-\frac{K+5}{4}}),$$

with first terms

$$a_0 = g_1(0, 0, 0, 0) + Q_{F}^{red}(0)(x, \xi)$$

$$a_1 = q \cdot \nabla_q \left( g_1(\cdot, 0, 0, 0) + Q_{F}^{red}(\cdot)(x, \xi) \right)(0)$$

$$a_2 = Q_{S}(p) + \frac{1}{2} \text{Hess}_q \left( g_1(\cdot, 0, 0, 0) + Q_{F}^{red}(\cdot)(x, \xi) \right)(0)(q)$$

$$+ R_3(x, \xi) + L(x, \xi).$$

Here $R_3$ is a homogeneous polynomial of degree 3 and $L$ is a linear form.

We further write $A_i = Op_{W}^{1}(a_i)$.

Recall from Proposition 5.5 that $g_1(q, 0, 0, 0) + \frac{1}{4} \text{tr}(Q_{F}^{red}(q)) = \mu \circ \rho(q, 0, 0, 0)$, and let $\phi$ be the positive quadratic form such that $e^{-\phi}$ is the ground state (up to a positive factor) of

$$Op_{W}^{1} \left( Q_{S}(p) + \frac{1}{2} \text{Hess}(\mu \circ \rho)(0)(q) \right).$$

Finally, let

$$u_0 : (q, x) \mapsto e^{-\frac{|x|^2}{2} - \phi(q)}.$$

We will provide a sequence of almost eigenfunctions of $Op_{W}^{1}(\tilde{g}_N)$, of the form

$$u_0(q, x) \left( 1 + \sum_{i=1}^{+\infty} N^{-\frac{i}{4}}b_i(q, x) \right),$$

with approximate eigenvalue

$$N^{-1} \sum_{i=0}^{+\infty} N^{-\frac{i}{4}}\mu_i.$$

We proceed by perturbation of the dominant order $A_0$, which does not depend on $q$. Our starting point is

$$u_0 = e^{-\frac{|q|^2}{2} - \phi(q)}, \mu_0 = \min \text{Sp}(A_0)$$

$$u_1 = 0, \mu_1 = 0.$$
Indeed, one has $A_0 u_0 = \mu_0 u_0$, and $A_1 u_0 = 0$ since
\[
\nabla \left( g_1(\cdot, 0, 0, 0) + \frac{1}{4} \text{tr} (Q^\text{red} (\cdot)) \right) (0) = 0,
\]
so that $u_0$ is an approximate eigenvector for $O_{\mathcal{W}}^1(\tilde{g})$.

Let us proceed by induction. Let $k \geq 1$ and suppose that we have already built $u_0, \ldots, u_k$ and $\mu_1, \ldots, \mu_k$ which solve the eigenvalue equation at order $k$; suppose further that there exists $C_{k+1} \in \mathbb{R}$ such that, for every $q \in \mathbb{R}^r$,
\[
\int_{\mathbb{R}^{n-r}} \overline{u}_0(x, q) \left( \sum_{i=1}^{k+1} [A_i u_{k+1-i}](q, x) - \sum_{i=1}^{k} \mu_i u_{k+1-i} (q, x) \right) dx = C_{k+1} |u_0(x, q)|^2.
\]

Then one can solve the equation
\[
(A_0 - \mu_0) u_{k+1} + \cdots + (A_{k+1} - \mu_{k+1}) u_0 = 0,
\]
up to a multiple of $e^{-\frac{|x|^2}{2}}$ in $u_{k+1}$. Indeed, if we write
\[
u_0(x, q) = v(q) e^{-\frac{|x|^2}{2}} + w(x, q),
\]
where for every $q \in \mathbb{R}^r$ one has $w(q, \cdot) \perp e^{-\frac{|x|^2}{2}}$, the equation reduces to
\[
(A_0 - \mu_0) w + \cdots + (A_{k+1} - \mu_{k+1}) u_0 = 0.
\]
Freezing $q$ and taking the scalar product with $x \mapsto e^{-\frac{|x|^2}{2}}$ yields
\[
\lambda_{k+1} = C_{k+1}.
\]

Then, with $q$ still frozen one has $(A_0 - \mu_0) w = \text{r.h.s}$ where the r.h.s is orthogonal to the ground state of $A_0$, which allows us to solve for $w$.

If the r.h.s is $u_0$ times a polynomial in $(q, x)$, then the same holds for $w$ (in particular, for all $i$ one has $A_i w \in L^2$ so that it makes sense to proceed with the induction).

It remains to choose $v$ so that $u_{k+1}$ satisfies the orthogonality constraint above, in order to be able to build the next terms.

Since $\mu_1 = 0$ and $A_1 u_0 = 0$, the terms $i = 1$ vanish so that the first integral in which $u_{k+1}$ appears is not the next one but the one after it:
\[
\int_{\mathbb{R}^{n-r}} \overline{u}_0(x, q) \left( \sum_{i=2}^{k+3} [A_i u_{k+3-i}] (q, x) - \sum_{i=2}^{k+2} \mu_i u_{k+3-i} (q, x) \right) dx.
\]

Hence, one wants to solve
\[
\int_{\mathbb{R}^{n-r}} e^{-\frac{|x|^2}{2}} [(A_2 - \mu_2)v e^{-\frac{|x|^2}{2}}] (q, x) = F(q) + C_{k+3} e^{-\phi(q)},
\]
with
\[
F(q) = - \int_{\mathbb{R}^{n-r}} e^{-\frac{|x|^2}{2}} \left( [(A_2 - \mu_2) w](x, q) + \sum_{i=3}^{k+3} [A_i u_{k+3-i}] (x, q) - \sum_{i=3}^{k+2} \mu_i u_{k+3-i} (x, q) \right) dx.
\]
The symbol \( a_2 \) decomposes into a quadratic symbol in \((q, p)\), and an odd polynomial in \((x, \xi)\). The latter does not contribute to the integral in the left-hand-side, and the former commutes with multiplication by \( e^{-\frac{|x|^2}{2}} \), so that

\[
\int_{\mathbb{R}^{n-r}} e^{-\frac{|x|^2}{2}} [(A_2 - \mu_2) v e^{-\frac{|x|^2}{2}}](q, x) = C_{n-r} \left( Q_S(i D) + \frac{1}{2} \text{Hess}(g_1(\cdot, 0, 0, 0)) + \frac{1}{4} \text{Tr}(Q^\text{red}_F(\cdot))(q) - \mu_2 \right) v.
\]

The equation on \( v \) is then

\[
\left( Q_S(i D) + \frac{1}{2} \text{Hess}(\mu \circ \rho)(0)(q) - \mu_2 \right) v = C_{n-r}^{-1} \left( F(q) + C_{k+3} e^{-\phi(q)} \right).
\]

With

\[ C_{k+1} = -\langle e^{-\phi(q)}, F(q) \rangle, \]

one has

\[ F - C_{k+1} e^{-\phi} \perp e^{-\phi}, \]

so that one can solve for \( v \).

Again, if \( u_0, \ldots, u_k \) and \( w \) are \( u_0 \) times a polynomial function in \((x, q)\), then \( F \) is \( e^{-\phi} \) times a polynomial function, so that the same is true for \( v \). This concludes the construction by induction.

The estimation of the error terms stems directly from the fact that the terms \( u_k \) are polynomials time a function with Gaussian decay. Hence, this formal construction yields approximate eigenfunctions. \( \square \)

Before we show that the almost eigenfunction computed in Proposition 5.7 corresponds indeed to the lowest eigenvalue, let us use the quantum maps \( \mathcal{S}_N \) to obtain upper and lower bounds for \( T_N(h) \), which will be useful in Sect. 7.

**Proposition 5.8.** Let \( A_N^{\text{reg}} \) be the following operator on \( L^2(\mathbb{R}^r) \):

\[
A_N^{\text{reg}} = \text{Op}_W^{-1} \left( |p|^2 + N^{-1} |q|^2 \right)
\]

Under the conditions of Proposition 5.7, there exists \( a_0 > 0 \), and two constants \( 0 < c < C \) such that, for any \( N \), for any \( a < a_0 \), for any normalised \( u \in L^2(X) \) supported in \( B(P_0, a) \times S^1 \) such that \( S_N u = u + O(N^{-\infty}) \), letting \( v = B_N^{-1} S_N^{-1} u \), one has:

\[
c \langle v, A_N^{\text{reg}} v \rangle + c \left( \langle v, \text{Op}_W^{-1} (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n - r}{2} \right) - C \langle v, \text{Op}_W^{-1} (|N^{-\frac{1}{2}}|, p, x, \xi |^3) v \rangle - O(N^{-\infty}) \leq \langle u, hu \rangle - N^{-1} \mu(P_0).
\]
In addition, the following bound holds:

\[
c \langle v, A^{reg}_N v \rangle + \langle v, Op^{-1}_W (Q^{red}_F (0)(x, \xi)) v \rangle - C \langle v, Op^{-1}_W (|N^{-\frac{1}{2}}, p, x, \xi|^3) v \rangle \\
- a C \left( \langle v, Op^{-1}_W (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n - r}{2} \right) - O(N^{-\infty})
\]

\[
\leq \langle u, hu \rangle - N^{-1} \mu(P_0) + \frac{N^{-1}}{4} \text{tr}(Q^{red}_F (0))
\]

\[
\leq C \langle v, A^{reg}_N v \rangle + \langle v, Op^{-1}_W (Q^{red}_F (0)(x, \xi)) v \rangle + C \langle v, Op^{-1}_W (|N^{-\frac{1}{2}}, p, x, \xi|^3) v \rangle
\]

\[
+ a C \left( \langle v, Op^{-1}_W (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n - r}{2} \right) + O(N^{-\infty}).
\]

Here, the notation \( O(|N^{-\frac{1}{2}}, p, x, \xi|^3) \) stands for \( O(|p, x, \xi|^3 + N^{-\frac{3}{2}}) \).

**Proof.** Let us prove the first lower bound. As

\[
go(q, p, x, \xi) = Q^{red}_F (q)(x, \xi) + Q_s(q)(p) + O(|p, x, \xi|^3),
\]

one has first, by a lower bound on \( Op^{-1}_W (Q^{red}_F (q)(x, \xi)) \),

\[
\langle v, Op^{-1}_W (g_0) v \rangle
\]

\[
\geq c \langle v, Op^{-1}_W (|p|^2) v \rangle + \frac{N^{-1}}{4} \langle v, \text{tr}(Q^{red}_F (q)), v \rangle + \langle v, Op^{-1}_W (|x, p, N^{-\frac{1}{2}}|) v \rangle.
\]

Let us make this bound more precise. Since all eigenvalues of \( Q^{red}_F (q) \) are positive, there exists \( c > 0 \) such that, for \( q \) small enough, one has

\[
Op^{-1}_W (Q^{red}_F (q)(x, \xi)) - \frac{N^{-1}}{4} \langle v, \text{tr}(Q^{red}_F (q)), v \rangle
\]

\[
\geq c Op^{-1}_W (|x|^2 + |\xi|^2) - \frac{N^{-1}}{2} c(n - r).
\]

Hence

\[
\langle v, Op^{-1}_W (g_0) v \rangle \geq c \langle v, Op^{-1}_W (|p|^2) v \rangle + \frac{N^{-1}}{4} \langle v, \text{tr}(Q^{red}_F (q)), v \rangle
\]

\[
+ c \langle v, Op^{-1}_W (|x|^2 + |\xi|^2) \rangle - \frac{N^{-1}}{2} (n - r) - C \langle v, Op^{-1}_W (|x, \xi, N^{-\frac{1}{2}}|^3) v \rangle.
\]

Recall from Proposition 5.5 that \( g_1 = \mu(q, 0, 0, 0) - \frac{1}{2} \text{tr}(Q^{red}_F (q)) + O(|x, p, \xi|) \).

Hence,

\[
\langle v, Op^{-1}_W (g) v \rangle \geq c \langle v, Op^{-1}_W (|p|^2) v \rangle + \frac{N^{-1}}{2} \langle v, \mu(q, 0, 0, 0), v \rangle
\]

\[
+ c \langle v, Op^{-1}_W (|x|^2 + |\xi|^2) \rangle - \frac{N^{-1}}{2} (n - r) - C \langle v, Op^{-1}_W (|N^{-\frac{1}{2}}, p, x, \xi|^3) v \rangle.
\]

As \( \mu(q, 0, 0, 0) \geq \mu(P_0) + c|q|^2 \), this yields the lower bound.
We now turn to the second estimate. This first requires a bound on 
\[ \langle v, Op_{W}^{N^{-1}} (Q_{F}(q)(x, \xi) - Q_{F}^{red}(0)(x, \xi))v \rangle. \]

Since \( Q_{F}(q) - Q_{F}(0) = O(|q|) \) and since the expression above vanishes only when \( v \) is a standard Gaussian in \( x \), one has
\[
\left| \langle v, Op_{W}^{N^{-1}} (Q_{F}(q)(x, \xi) - Q_{F}^{red}(0)(x, \xi))v \rangle \right| \\
\leq Ca \left( \langle v, Op_{W}^{N^{-1}} (|x|^{2} + |\xi|^{2})v - N^{-1} \frac{n-r}{2} \rangle \right).
\]

Moreover, since \( QS(q) > 0 \) and using the miniwell condition, one has
\[
c \langle v, A_{W}^{reg} v \rangle - C \langle v, Op_{W}^{N^{-1}} (|N^{-\frac{1}{2}}, p, x, \xi|^{3})v \rangle \\
\leq \langle v, Op_{W}^{N^{-1}} (QS(q)(p) + N^{-1} g_{1}(q, p, x, \xi)) \rangle \\
\leq C \langle v, A_{W}^{reg} v \rangle + C \langle v, Op_{W}^{N^{-1}} (|N^{-\frac{1}{2}}, p, x, \xi|^{3})v \rangle.
\]

This concludes the proof. \( \Box \)

5.3. Spectral gap. It only remains to show that the sequence of almost eigenfunctions given by Proposition 5.7 corresponds to the first eigenvalue of \( T_{N}(h) \).

**Proposition 5.9.** Let \( h \) be a classical symbol with \( h_{0} \geq 0 \), such that the minimum of the Melin value \( \mu \) is only reached at one point, which is a miniwell for \( h \).

Let \( (\mu_{i}) \) be the real sequence constructed in the previous proposition, and let \( \lambda_{\min} \) be the first eigenvalue of \( T_{N}(h) \).

Then
\[
\lambda_{\min} \sim N^{-1} \sum_{i=0}^{\infty} N^{-\frac{i}{2}} \mu_{i}.
\]

Moreover, there exists \( c > 0 \) such that, for every \( N \), one has
\[
\text{dist}(\lambda_{\min}, Sp(T_{N}(h))\setminus \{\lambda_{\min}\}) \geq c N^{-\frac{3}{2}}.
\]

**Proof.** Let us show that any function orthogonal to the one proposed in Proposition 5.7 has an energy which is larger by at least \( c N^{-\frac{3}{2}} \).

Let \( (v_{N}) \) be a sequence of unit vectors in \( L^{2}(\mathbb{R}^{n}) \). If
\[
\langle v_{N}, Op_{W}^{1}(g_{N})v_{N} \rangle \leq N^{-1} \mu_{0} + CN^{-\frac{3}{2}}
\]
for some \( C \), then \( v_{N} = e^{-\frac{|x|^{2}}{2}} w_{N}(q) + O(N^{-\frac{1}{2}}) \), with \( \|w_{N}\|_{L^{2}} = 1 + O(N^{-\frac{1}{2}}) \).

If \( C - \mu_{2} \) is strictly smaller than the spectral gap of the quadratic operator
\[
Op_{W}^{1} \left( QS(0)(p) + \frac{1}{2} \text{Hess} \mu \circ \rho(\cdot, 0, 0, 0)(q) \right),
\]
then \( \langle w_{N}, e^{-\phi(q)} \rangle \geq a \) for some \( a > 0 \) independent of \( N \), which concludes the proof. \( \Box \)
To conclude the proof of Theorem B, let us show that quantum maps preserve concentration speed:

**Lemma 5.10.** Let $\sigma : (M, x) \mapsto (M', y)$ a local symplectomorphism between two quantizable Kähler manifolds.

Let $0 < \delta < 1/2$ and let $(u_N)_{N \in \mathbb{N}}$ a sequence of unit elements in the Hardy spaces $H^0(M, L^{\otimes N})$ such that

$$\int \left\{ \text{dist}(\pi(y), x) \leq N^{-\frac{1}{2} + \delta} \right\} |u_N(y)|^2 = O(N^{-\infty}).$$

Then

$$\int \left\{ \text{dist}(\pi'(y), \sigma(x)) \leq N^{-\frac{1}{2} + \delta} \right\} |\mathcal{G}u_N(y)|^2 = O(N^{-\infty}).$$

**Proof.** Let us observe that the condition on $(u_N)$ is equivalent to the following: for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$\langle u_N, T_N(\text{dist}(\cdot, x)^{2k})u_N \rangle \leq C_k N^{-k(1+2\delta)}.$$

Let us prove, by induction on $k$, the estimate

$$\langle \mathcal{G}_N u_N, T_N(\text{dist}(\cdot, \sigma(x))^{2k})\mathcal{G}_N u_N \rangle \leq \tilde{C}_k N^{-k(1+2\delta)}.$$

The case $k = 0$ is clearly true since $\mathcal{G}_N$ is an almost unitary operator when acting on functions localised near $x$.

Let us now apply Proposition 2.25 with $a = \text{dist}(\cdot, x)^{2k}$, stopping the expansion at order $k$.

For $j \leq k$, the error terms are controlled:

$$|N^{-j} L_j(a \circ \sigma)| \leq N^{-j} C_{j,k} \text{dist}(\cdot, \sigma(x))^{2(k-j)}.$$

Hence, by induction,

$$\langle \mathcal{G}_N u_N, T_N(\text{dist}(\cdot, \sigma(x))^{2k})\mathcal{G}_N u_N \rangle \leq \sum_{j=0}^{2k} C_{j,k} \langle u_N, T_N(\text{dist}(\cdot, x)^{2(k-j)})u_N \rangle + O(N^{-k}) = O(N^k(-1+2\delta)).$$

This ends the proof. $\square$

**Remark 5.11 (Excited states).** The expansion in Proposition 5.7 corresponds to the ground state. As in the case of non-degenerate wells (see Theorem B in [13]), higher-energy states do not admit, in general, a complete asymptotic expansion, because the perturbed operator (in our case, the quadratic operator defined in Proposition 5.7) might have multiple eigenvalues. Thus, one cannot hope for more than the first few terms in the expansion for excited states.
In this section we treat a case in which the zero set of the symbol is not a submanifold. The local hypotheses on the symbol are as follows:

**Definition 6.1.** Let \( h \) be a classical symbol on \( M \) with \( h_0 \geq 0 \) and let \( P_0 \in M \). The zero set of \( h_0 \) is said to have a *simple crossing* at \( P_0 \) if there is an open set \( U \) containing \( P_0 \) such that:

- \( \{ h_0 = 0 \} \cap U = Z_1 \cup Z_2 \), where \( Z_1 \) and \( Z_2 \) are two pieces of smooth isotropic submanifolds of \( M \).
- \( Z_1 \cap Z_2 = \{ P_0 \} \) and \( T_{P_0}Z_1 \cap T_{P_0}Z_2 = \{0\} \).
- \( T_{P_0}Z_1 \oplus T_{P_0}Z_2 \) is isotropic.
- For \( i = 1, 2 \), on all of \( Z_i \setminus \{ P_0 \} \), \( h_0 \) vanishes at order exactly 2 on \( Z_i \).
- There is \( c > 0 \) such that, for all \( x \in Z_1 \cup Z_2 \), one has:
  \[ \mu(x) - \mu(P_0) \geq c \operatorname{dist}(P_0, x). \]

The last condition may seem very strong. However, \( \mu \) is typically only Lipschitz-continuous at the intersection. A typical example is

\[
h(q_1, q_2, p_1, p_2) = p_1^2 + p_2^2 + q_1^2 q_2^2,
\]

where along \( \{ q_1, 0, 0, 0 \} \) one has \( \mu(q_1) = |q_1| + 1 \). We exclude on purpose situations like \( \mu(q_1) = 1 + |q_1| - q_1 + q_1^2 \), which grows like \( |q_1| \) for \( q_1 < 0 \) but grows like \( q_1^2 \) for \( q_1 > 0 \).

Under the hypotheses of Definition 6.1, we first give a symplectic normal form of \( h_0 \) near \( P_0 \), then a description of the first eigenvector and eigenvalue of \( T_N(h) \) in the following Subsections.

The symplectic normal form for \( h_0 \) does not depend on the hypothesis on \( \mu \). However, the pseudodifferential operator \( P \) associated to the first Taylor coefficients in this normal form, which we study in Sect. 6.2, has compact resolvent under this assumption, and its inverse is well-behaved (in particular, it preserves fast decay, see Proposition 6.11).

### 6.1. Symplectic normal form.

Let \( Q \geq 0 \) be a semidefinite positive quadratic form on \((\mathbb{R}^{2n}, \omega)\), and \((e_i, f_i)\) a symplectic basis of \( \mathbb{R}^{2n} \) which diagonalises \( Q \):

\[
Q \left( \sum_{i=1}^{n} q_i e_i + p_i f_i \right) = \sum_{i=r+1}^{r'} p_i^2 + \sum_{i=r'+1}^{n} \lambda_i (q_i^2 + p_i^2),
\]

\( \forall i, \lambda_i \neq 0. \)

Let \( M \) denote the matrix of \( Q \) in the canonical basis. Then

\[ \{ \pm i \lambda_{r'+1}, \ldots, \pm i \lambda_n \} = \sigma(JM) \setminus \{0\}. \]

More precisely, if \( E_\lambda \) denotes the (complex) eigenspace of \( JM \) with eigenvalue \( \lambda \), then

\[ E_{i \lambda, j} \oplus E_{-i \lambda, j} = \operatorname{Span}_\mathbb{C}(\{e_k, f_k\}, k > r', \lambda_k = \lambda_j). \]

Moreover, Jordan blocks never occur for nonzero eigenvalues. Hence,
Proposition 6.2. If $Q : \mathbb{R}^m \mapsto S^+_{2n}(\mathbb{R})$ is a smooth parameter-dependent semipositive quadratic form on $(\mathbb{R}^{2n}, \omega)$, of constant symplectic rank $2d$, then the span of the non-zero symplectic eigenspaces of $Q$ (whose dimension is $2d$), depends smoothly on $Q$.

Using the result above, one can build a symplectic normal form for functions with crossing points. Let us first “flatten” the geometry near a crossing point.

Proposition 6.3. Let $h_0 \in C^{\infty}(M, \mathbb{R}^+)$ be such that the zero set of $h_0$ has a simple crossing at $P_0 \in M$ and let

$$r_1 = \dim(Z_1)$$
$$r_2 = \dim(Z_2).$$

Then there exists a symplectomorphism $\sigma$ from a neighbourhood of $P_0$ to a neighbourhood $V$ of 0 in $\mathbb{R}^{2n} = \mathbb{R}^{2r_1} \times \mathbb{R}^{2r_2} \times \mathbb{R}^{2(n-r_1-r_2)}$ such that

1. $\sigma((h_0 = 0)) = V \cap [\mathbb{R}^{r_1} \times \{0, 0, 0, 0\} \cup \{0, 0\} \times \mathbb{R}^{r_1} \times \{0, 0, 0\}]$.
2. $\ker \text{Hess}(h_0 \circ \sigma^{-1})(0) = \mathbb{R}^{r_1} \times \{0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\}$.
3. $\forall q_1 \in \mathbb{R}^{r_1}, \ker \text{Hess}(h_0 \circ \sigma^{-1})(q_1, 0, 0, 0, 0) = \mathbb{R}^{r_1} \times \{0, 0, 0, 0\}$.
4. $\forall q_2 \in \mathbb{R}^{r_2}, \ker \text{Hess}(h_0 \circ \sigma^{-1})(0, 0, q_2, 0, 0) = \{0, 0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\}$.
5. $\forall z \in \sigma((h_0 = 0)), z = 0$ in $\mathbb{R}^{2(n-r_1-r_2)}$ is symplectically invariant under $\text{Hess}(h_0 \circ \sigma^{-1})(z)$.

Proof. From Proposition 6.2 applied to the Hessian matrix of $h_0$, the span $\mathcal{F}$ of the “fast modes”, corresponding to the $n - r_1 - r_2$ largest symplectic eigenvalues, vary smoothly along $Z_1$ and along $Z_2$. Its symplectic orthonal $\mathcal{S}$ then varies smoothly along $Z_1$ and along $Z_2$.

First part: let us prove that there exists a piece of symplectic manifold $\Sigma$, tangent to $\mathcal{S}$ along $Z_1$ and along $Z_2$ near 0. The existence of such a symplectic manifold, as we will see, depends on an integrability condition at $P_0$ which is satisfied in our setting.

We first push, using any smooth diffeomorphism, a neighbourhood of $P_0$ in $M$ to $\mathbb{R}^{2n}$, in a way which sends $Z_1$ to $\mathbb{R}^{r_1} \times \{0, 0, 0, 0\}$ and $Z_2$ to $\{0, 0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\}$ (we will also call these subspaces $Z_1$ and $Z_2$ to avoid cumbersome notation). Since $T_0Z_1 \cup T_0Z_2 \subset \mathcal{S}(0)$, after another (linear) change of variables, one has

$$\mathcal{S}(0) = \mathbb{R}^{2(r_1+r_2)} \times \{0, 0\}.$$

In this chart, we will construct $\Sigma$ as the graph of a smooth function

$$f : \mathbb{R}^{2(r_1+r_2)} \rightarrow \mathbb{R}^{2(n-r_1-r_2)},$$

such that $df$ is prescribed along $Z_1$ and along $Z_2$, where $f = 0$. Since symplectic manifolds are stable by deformation, and $f$ is a smooth deformation of the zero function, this graph will be symplectic in a neighbourhood of 0 (for the pulled-back symplectic structure).

Since $TZ_1 \subset \mathcal{S}$, the prescription of $df$ on $Z_1$ is of the form

$$[df(q_1, 0, 0, 0, 0)](dq_1, dp_1, dq_2, dp_2) = L_1(q_1)(dp_1, dq_2, dp_2).$$

Here $L_1$ is a linear form with smooth dependence in $q_1$.

Let

$$f_1 : (q_1, p_1, q_2, p_2) \mapsto L_1(q_1)(p_1, q_2, p_2).$$
Then the graph of \( f_1 \) is tangent to \( S \) at 0. Since \( d f(0) = 0 \), one has \( L_1(0) = 0 \) so that \( f_1 \) vanishes on \( Z_2 \).

Let us prove that \( d f_1 = 0 \) on this set, that is, for all \((q_1, q_2) \in \mathbb{R}^{r_1+r_2}, \)
\[
\partial_{q_1} L_1(0)(0, q_2, 0) = 0.
\]

The form \( L_1 \) is determined by the Hessian of \( h_0 \) along \( Z_1 \). In a Taylor expansion of \( h_0 \) near 0, there are no terms in \((q_1, q_2)\) of degree less than 4. Indeed, when restricted on \( \mathbb{R}^{r_1} \times \{0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\} \), the function \( h_0 \) and its differential vanishes along \( \mathbb{R}^{r_1} \times \{0\} \), hence one can write
\[
h_0 = \sum_{j,k=1}^{r_2} q_2, j q_2, k g_{j,k}(q_1, q_2)
\]
where \( g_{j,k} \) is a smooth function: there are no terms of order less than 2 in \( q_2 \) in the Taylor expansion of \( h_0 \) near 0. Symmetrically, there are no terms of order less than 2 in \( q_1 \) in this expansion, so that there are no terms of total degree less than 4.

In particular, when \( h_0 \) is restricted on \( \mathbb{R}^{r_1} \times \{0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\} \), its Hessian at \((q_1, 0)\) is \( O(q_1^2) \). This means that the subspace \( \{0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\} \) is at distance \( O(q_1^2) \) from its projection on \( S(q_1, 0, 0, 0, 0) \), so that \( \partial_{q_1} L_1(0)(0, q_2, 0) = 0 \).

Symmetrically, we can construct a smooth function \( f_2 \), tangent to \( S \) on \( Z_2 \), such that \( f_2 \) and \( d f_2 \) vanish on \( Z_1 \).

The function \( f = f_1 + f_2 \) then satisfies all requirements.

**Second part.** From Proposition 5.1, one can flatten the symplectic submanifold \( \Sigma \): after a symplectic change of variables, one has \( \Sigma = \mathbb{R}^{2r_1+2r_2} \times \{0, 0\} \). Inside \( \Sigma \), one can use Proposition 5.1 again to flatten \( Z_1 \) into \( \mathbb{R}^{r_1} \times \{0, 0, 0\} \). At this stage of the proof, we obtain a symplectomorphism which satisfies conditions 3, and 5 in Proposition 6.3.

It only remains to flatten \( Z_2 \) inside \( \Sigma \). After a linear change of variables, condition 2 is satisfied, that is,
\[
T_0 Z_2 = \{0, 0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\}.
\]

Inside \( \Sigma \), the manifold \( Z_2 \) is then the graph of a smooth function
\[
\{g_1(q_2), g_2(q_2), q_2, g_3(q_2)\}
\]
with \( g_i(0) = 0, d g_i(0) = 0 \) for all \( i = 1, 2, 3 \). Since \( Z_2 \) is isotropic, one has, for all \( 1 \leq i, k \leq r_2 \),
\[
\frac{\partial g_{3,i}}{\partial q_{2,k}} = \sum_{j=1}^{r_1} \left( \frac{\partial g_{1,j}}{\partial q_{2,k}} \frac{\partial g_{2,j}}{\partial q_{2,i}} - \frac{\partial g_{1,j}}{\partial q_{2,i}} \frac{\partial g_{2,j}}{\partial q_{2,k}} \right).
\]
In particular, the left-hand side in the equation above is an antisymmetric matrix.

Let us first remove the two first components, that is, apply a symplectomorphism of the form
\[
(q_1, p_1, q_2, p_2) \mapsto (q_1 - g_1(q_2), p_1 - g_2(q_2), q_2, p_2 + G(q_1, p_1, q_2))
\]
where it remains to find \( G \) such that this is indeed a symplectomorphism and such that \( G(q_1, 0, 0) = 0 \) (so that \( Z_1 \) is left invariant).
The requirement of a symplectomorphism is equivalent to the following system of differential equations on $G$, for all $1 \leq i, k \leq r_2$, $1 \leq j \leq r_1$:

$$\frac{\partial G_i}{\partial q_1,k} = \frac{\partial g_{2,k}}{\partial q_2,i},$$

$$\frac{\partial G_i}{\partial p_1,k} = -\frac{\partial g_{1,k}}{\partial q_2,i},$$

$$\frac{\partial G_i}{\partial q_2,k} - \frac{\partial G_k}{\partial q_2,i} = \sum_{j=1}^{r_1} \left( \frac{\partial g_{1,j}}{\partial q_2,k} \frac{\partial g_{2,j}}{\partial q_2,i} - \frac{\partial g_{1,j}}{\partial q_2,i} \frac{\partial g_{2,j}}{\partial q_2,k} \right) = \frac{\partial g_{3,i}}{\partial q_2,k} = \frac{1}{2} \left( \frac{\partial g_{3,i}}{\partial q_2,k} - \frac{\partial g_{3,k}}{\partial q_2,i} \right),$$

All equations are satisfied by letting:

$$G_i : (q_1, p_1, q_2) \mapsto \sum_{j=1}^{r_1} \left( q_1.j \frac{\partial g_{2,j}}{\partial q_2,i}(q_2) - p_1.j \frac{\partial g_{1,j}}{\partial q_2,i}(q_2) \right) + \frac{1}{2} g_{3,i}(q_2).$$

Notice that one has indeed $G(q_1, 0, 0) = 0$.

After this change of variables, $Z_2$ is a Lagrangean subspace of $\{0, 0\} \times \mathbb{R}^{2r_2}$. By Proposition 5.1, it can be flattened into $\{0, 0\} \times \mathbb{R}^{2r_2} \times \{0\}$, which concludes the proof. □

Once that the zero set near a crossing point is conveniently flattened, one can perform a reduction of $h_0$ near this crossing point.

**Proposition 6.4.** Let $h$ satisfy the simple crossing conditions of Definition 6.1, and let

$$r_1 = \dim(Z_1)$$

$$r_2 = \dim(Z_2).$$

Then there is an open set $V \subset U$, containing $P_0$, and a symplectic map

$$\sigma : V \mapsto \mathbb{R}^{2r_1} \times \mathbb{R}^{2r_2} \times \mathbb{R}^{2(n-r_1-r_2)}$$

such that $h_0$, read in the map $\sigma$, takes the form

$$h_0 \circ \sigma^{-1} = Q_{F, ed}^{\text{red}}(q_1, q_2)(x, \xi) + Q_S(q_1, q_2)(p_1, p_2) + \sum_{i=1}^{r_1} \sum_{j=1}^{r_2} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l}$$

$$+ O(\|x, \xi, p_1, p_2\|^3) + O(\|p_1, p_2, x, \xi \|q_1, q_2\|^3) + O(\|q_1\|^2 \|q_2\|^2(\|q_1\| + \|q_2\|)).$$

Here, for every $(q_1, q_2)$ close to 0, the ground state of $T_N((x, \xi)) \mapsto Q_{F, ed}^{\text{red}}(q_1, q_2)(x, \xi)$ is the standard Gaussian. $Q_S$ is a quadratic form in $(p_1, p_2)$ with smooth dependence in $(q_1, q_2)$. Moreover, for every $(q_1, q_2) \in (\mathbb{R}^{r_1} \setminus \{0\}) \times (\mathbb{R}^{r_2} \setminus \{0\})$ small enough, the matrices given by $\left[ \sum_{i,j} \alpha_{ijkl} q_{1,i} q_{1,j} \right]_{k,l}$ and $\left[ \sum_{k,l} \alpha_{ijkl} q_{2,k} q_{2,l} \right]_{i,j}$ are positive.
Proof. The first step is Proposition 6.3. A symplectic change of variables \( \sigma_1 \) sends \( Z_1 \) to \( \mathbb{R}^{r_1} \times \{0, 0, 0, 0\} \), \( Z_2 \) to \( \{0, 0\} \times \mathbb{R}^{r_2} \times \{0, 0, 0\} \), and such that, at each point of \( Z_1 \cup Z_2 \), the fast modes of \( \text{Hess}(h_0) \) span \( \{0, 0, 0\} \times \mathbb{R}^{2(n-r_1-r_2)} \). At this stage, one has

\[
h \circ \sigma_1 = Q_F(q_1, q_2)(x, \xi) + Q_S(q_1, q_2)(p_1, p_2) + O(\|p_1, p_2, x, \xi\|^3) + O(\|q_1, q_2\|^4).
\]

As in Proposition 5.2, the next step is to reduce \( Q_F \) into a quadratic form which is unitarily equivalent to a symplectically diagonal form, with smooth dependence on \( (q_1, q_2) \). This change of variables can again be corrected into a symplectic change of variables up to a negligible modification of \( (p_1, p_2) \). Thus there exists a symplectic change of variables \( \sigma_2 \) such that

\[
h \circ \sigma_1 = Q^{\text{red}}_F(q_1, q_2)(x, \xi) + Q_S(q_1, q_2)(p_1, p_2) + O(\|p_1, p_2, x, \xi\|^3) + O(\|q_1\|^2\|q_2\|^2).
\]

with \( Q^{\text{red}}_F \) as requested.

Continuing the expansion yields

\[
h \circ \sigma_1 = Q^{\text{red}}_F(q_1, q_2)(x, \xi) + Q_S(q_1, q_2)(p_1, p_2) + \sum_{i,j,k,l} \alpha_{ijkl}q_1,i q_1,j q_2,k q_2,l
\]

\[
+ O(\|p_1, p_2, x, \xi\|q_1, q_2\|^3)
\]

\[
+ O(\|p_1, p_2, x, \xi\|^3) + O(\|q_1\|^2\|q_2\|^2\|q_1, q_2\|).
\]

The positivity conditions on the tensor \( \alpha \) are then directly given by the fact that \( h_0 \) vanishes at order 2 on \( Z_1 \setminus \{P_0\} \) and \( Z_2 \setminus \{P_0\} \). \( \square \)

One can easily adapt Definition 6.1 to the case of a crossing along a submanifold.

**Definition 6.5** (Crossing along a submanifold). Let \( h \) be a classical symbol on \( M \) with \( h_0 \geq 0 \). The zero set of \( h_0 \) is said to cross along a submanifold near \( P_0 \) if there is an open set \( U \) containing \( P_0 \) such that:

- \( \{h_0 = 0\} \cap U = Z_1 \cup Z_2 \), where \( Z_1 \) and \( Z_2 \) are two pieces of smooth isotropic submanifolds of \( M \).
- \( Z_1 \cap Z_2 = Z_3 \) is a piece of smooth submanifold containing \( P_0 \). For each \( x \in Z_3 \), one has \( T_xZ_3 = T_xZ_1 \cap T_xZ_2 \).
- For each \( x \in Z_3 \), the space \( T_xZ_1 + T_xZ_2 \) is isotropic.
- For \( i = 1, 2 \), on all of \( Z_i \setminus Z_3 \), \( h_0 \) vanishes at order exactly 2 on \( Z_i \).
- There is \( c > 0 \) such that, for all \( x \in Z_1 \cup Z_2 \), one has:

\[
\mu(x) - \mu(P_0) \geq c \cdot \text{dist}(Z_3, x).
\]

With this definition one can find a normal form as previously:

**Proposition 6.6.** Let \( h \) satisfy the conditions of Definition 6.5, and let

\[
r_1 = \dim(Z_1) - \dim(Z_3)
\]

\[
r_2 = \dim(Z_2) - \dim(Z_3)
\]

\[
r_3 = \dim(Z_3).
\]
Then there is an open set $V \subset U$, containing $P_0$, and a symplectic map

$$\sigma : V \mapsto \mathbb{R}^{2r_1} \times \mathbb{R}^{2r_2} \times \mathbb{R}^{2r_3} \times \mathbb{R}^{2(\nu - r_1 - r_2 - r_3)}$$

such that

$$h_0 \circ \sigma^{-1}(q_1, p_1, q_2, p_2, q_3, p_3, x, \xi) = Q^{red}_F(q_1, q_2, q_3)(x_i^2 + \xi_i^2) + Q_S(q_1, q_2, q_3)(p_1, p_2, p_3) + \sum_{i,j=1}^{r_1} \sum_{k,l=1}^{r_2} a_{ijkl}(q_3)q_{1,i}q_{1,j}q_{2,k}q_{2,l} + O(||x||^3) + O(||p_1, p_2, x, \xi||^3) + O(||q_1||^2||q_2||^2 \cdot ||(q_1, q_2)||).$$

Here $Q^{red}_F$ and $Q_S$ are positive quadratic forms with smooth dependence on $(q_1, q_2, q_3)$, and $Q^{red}_F$ has the standard Gaussian as a quantum ground state Moreover, for every $q_3 \in \mathbb{R}^{r_3}$ small enough, for every $(q_1, q_2) \in (\mathbb{R}^{r_1} \setminus \{0\}) \times (\mathbb{R}^{r_2} \setminus \{0\})$ small enough, the matrices given by $[\sum_{i,j} a_{ijkl}q_{1,i}q_{1,j}]_{k,l}$ and $[\sum_{k,l} a_{ijkl}q_{2,k}q_{2,l}]_{i,j}$ are positive.

**Proof.** As in Proposition 6.3 the first step is to transform, with a symplectomorphism, $Z_1$ into $(q_1, 0, 0, 0, q_3, 0, 0, 0, (q_1, q_3) \in \mathbb{R}^{r_1+r_3})$ and, similarly, $Z_2$ into $(0, 0, q_2, 0, q_3, 0, 0, 0, (q_2, q_3) \in \mathbb{R}^{r_2+r_3})$ while respecting the decomposition between fast and slow modes along $Z_1$ and $Z_2$. The piece of symplectic submanifold $\Sigma$ containing $Z_1 \cup Z_2$ and tangent to the slow modes can be built as previously (in particular, the integrability condition is satisfied along $Z_3$). Within this manifold, one can flatten the isotropic submanifold $Z_1$ into $(q_1, 0, 0, 0, q_3, 0, 0, 0, (q_1, q_3) \in \mathbb{R}^{r_1+r_3})$. Without loss of generality, $Z_2$ is then of the form

$$\{q_1(q_2, q_3), p_1(q_2, q_3), q_2, p_2(q_2, q_3), q_3, p_3(q_2, q_3)\}.$$

One can flatten this manifold in three steps. For the first step, consider the following map, with smooth dependence on $q_2$:

$$\sigma_{q_2} : (q_3, p_3) \mapsto (q_3, p_3 - p_3(q_2, q_3)).$$

Since $Z_2$ is isotropic, for all $i, j$ one has $\frac{\partial p_3}{\partial q_j} = -\frac{\partial p_3}{\partial q_j}$, so that $\sigma_{q_2}$ is a symplectic change of variables, which maps 0 to 0. Then, as in the proof of Proposition 5.2, there exists $f$ such that

$$(q_2, p_2, q_3, p_3) \mapsto (q_2, p_2 + f(q_2, q_3, p_3), \sigma_{q_2}(q_3, p_3))$$

is a symplectic change of variables.

After this first step, $Z_2$ is of the form

$$\{(q_1(q_2, q_3), p_1(q_2, q_3), q_2, p_2(q_2, q_3), q_3, 0, (q_2, q_3) \in (\mathbb{R}^{2+r_3}, 0))\}.$$

Now, for fixed $q_3$, following Proposition 6.3 there is a change of variables which flattens $Z_2$. The second step is to apply this change of variables (and add a correction to $p_3$ in order to have a symplectic change of variables in all coordinates). After this step, $Z_2$ is of the form

$$\{(0, 0, q_2, 0, q_3, p_3'(q_2, q_3), (q_2, q_3) \in (\mathbb{R}^{2+r_3}, 0))\}.$$
Since $Z_2$ is isotropic, $p_3'$ does not, in fact, depend on $q_2$, and one can simply flatten this isotropic manifold into
\[ \{(0, 0, q_2, 0, q_3, 0), (q_2, q_3) \in (\mathbb{R}^{r_2+r_3}, 0)\} . \]
One can then repeat the proof of Proposition 6.4. This yields the desired result. \(\square\)

Remark 6.7. (More general degenerate crossings) Simple crossings (and crossings along submanifolds) are not stable by Cartesian products, which leads to a slightly more general situation (see Remark 6.8).

On the other hand, one could try to deal with symbols whose zero set form a stratified manifold, which are defined recursively: a stratified manifold is a union of smooth manifolds with clean intersections, such that the union of all intersections is itself a stratified manifold. The boundary of a hypercube is an instance of a stratified manifold.

In this respect, a model case for a stratified situation of degree three is
\[ p_1^2 + p_2^2 + p_3^2 + q_1^2 q_2^2 q_3^2 , \]
with zero set \{ $p_1 = 0$, $p_2 = 0$, $p_3 = 0$, $q_i = 0$ \} for every $i = 1, 2, 3$.

For this operator, the ground state is rapidly decreasing at infinity [21] but this is not due to subprincipal effects. Indeed, in this setting, $\mu$ is constant along the three axes. If we add a generic transverse quadratic operator $Q_{q_i}(x, \xi)$, the subprincipal effect will dominate and has no reason to select the point \{ $q = 0$ \}, as opposed to the simple crossing case where an open set of symbols sharing the same minimal set have minimal Melin value at the crossing point.

6.2. Study of the model operator. As Proposition 6.4 suggests, the following operators play an important role in the study of the crossing case:
\[ P = Q(iD) + \sum_{i,j=1}^{r_1} \sum_{k,l=1}^{r_2} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} + \sum_{i=1}^{r_1} L_{1,i} q_{1,i} + \sum_{i=1}^{r_2} L_{2,i} q_{2,i} , \]
acting on $L^2(\mathbb{R}^{r_1+r_2})$, where $D$ is the differentiation operator and $Q > 0$ is a quadratic form. The linear form $L$ will appear as an effect of the subprincipal symbol, as we will see later.

Let $Q_1$ and $Q_2$ denote the restrictions of the quadratic form $Q$ on $\mathbb{R}^{r_1} \times \{0\}$ and $\{0\} \times \mathbb{R}^{r_2}$, respectively. Throughout this subsection we impose the following conditions on $P$:

- For every $(q_1, q_2)$, one has
  \[ \sum_{i,j=1}^{r_1} \sum_{i,j=1}^{r_2} \alpha_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} \geq 0 . \]

- For every $q_1 \neq 0$, one has
  \[ Q_2(iD) + \sum_{ijkl} q_{1,i} q_{1,j} q_{2,k} q_{2,l} > \sum_{i=1}^{r_1} L_{1,i} q_{1,i} , \]
• For every \( q_2 \neq 0 \), one has
\[
Q_1(iD) + \sum_{ijkl} q_{i,j} q_{k,l} > \sum_{i=1}^{r_1} L_{i,j} q_{i,j},
\]

Remark 6.8. These conditions are weaker than what Definition 6.1 calls for. There does not need to be a simple crossing in this case as the following example illustrates:
\[
P = -\Delta + q_{1,1} q_{2,2} + q_{1,2} q_{2,2}.
\]
There, the zero set of the symbol is a union of four isotropic surfaces in \( \mathbb{R}^8 \), i.e. \( \{ p = 0, q_{1,i} = 0, q_{2,j} = 0 \} \) for all \( (i, j) \in \{1, 2\}^2 \).

Proposition 6.9. Under the previous conditions, there exists \( c > 0 \) such that
\[
P \geq c(Q(iD) + |q|).
\]
Proof. Let \( Q_2 \) be the restriction of the quadratic form \( Q \) to \( [0] \times \mathbb{R}^2 \). One has \( Q \geq Q_2 \), hence \( Q(iD) \geq Q_2(iD) \). By hypothesis,
\[
Q_2(iD) + \sum_{ijkl} q_{i,j} q_{k,l} > \sum_{i=1}^{r_1} L_{i,j} q_{i,j},
\]
and the infimum of the spectrum of the left hand side is 1-homogeneous in \( q_1 \), so that
\[
Q_2(iD) + \sum_{ijkl} q_{i,j} q_{k,l} \geq (1 - c) \sum_{i=1}^{r_1} L_{i,j} q_{i,j} + 2c|q_1|
\]
for some \( c > 0 \). In particular,
\[
P \geq cQ(iD) + 2c|q_1|.
\]
The same reasoning applies to \( Q_2 \), hence
\[
2P \geq 2cQ(iD) + 2c|q_1| + 2c|q_2|,
\]
which allows us to conclude. \( \square \)

One deduces immediately:

Proposition 6.10. The operator \( P \) has compact resolvent. Its first eigenvalue is positive.

We are now able to use Agmon estimates. In the particular case where \( Q \) is diagonal, the following result is contained in the Helffer–Nourrigat theory [21], see also the related results in [33].

Proposition 6.11. Let \( \lambda_0 \) be the first eigenvalue of \( P \). There exists \( c > 0 \) such that, if \( u \in L^2(\mathbb{R}^{r_1+r_2}) \), and \( (C_\beta)_{\beta \in \mathbb{N}^{r_1+r_2}} \) are such that \( |\beta^\beta u(q)| \leq C_\beta e^{-c|q|^{3/2}} \) for all \( q \in \mathbb{R}^{r_1+r_2}, \beta \in \mathbb{N}^{r_1+r_2} \), then for any \( f \in L^2(\mathbb{R}^{r_1+r_2}) \) such that \( (P - \lambda_0)f = u \), there exists \( (C_\beta)_{\beta \in \mathbb{N}^{r_1+r_2}} \) such that \( |\beta^\beta f(q)| \leq C_\beta e^{-c|q|^{3/2}} \) for every \( q \in \mathbb{R}^{r_1+r_2}, \beta \in \mathbb{N}^{r_1+r_2} \).

Proof. With \( \phi(q) = c|q|^{3/2} \), one has \( Q(\nabla \phi) \leq c'|q| \). Hence \( P - \lambda_0 - Q(\nabla \phi) \) is positive far from zero, and one can use Agmon estimates as developed in [1]. \( \square \)
We will also need the following two facts. Proposition 6.12 is an essential ingredient of Sect. 6.3 and Proposition 6.13 is necessary to compare the Weyl asymptotics with the regular case.

**Proposition 6.12.** The first eigenvalue $\lambda_0$ of $P$ is simple.

**Proof.** This follows from an argument which is standard in the case $Q = Id$. Let $u_0 \in L^2(\mathbb{R}^{r_1+r_2})$ be such that $Pu_0 = \lambda_0 u_0$. Then $u_0$ is a minimizer of the Courant-Hilbert problem

$$\min_{\|u\|_{L^2} = 1, u \in H^1} \int Q(\nabla u) + V|u|^2.$$

The set $\{u_0 = 0\}$ has zero Lebesgue measure from a standard Unique Continuation argument. The function $|u_0|$ is then also a minimizer of this quantity, since $\nabla |u_0| = \pm \nabla u_0$ whenever $u_0 \neq 0$.

Then $|u_0|$ itself belongs to the eigenspace of $P$ with value $\lambda_0$, which is (a priori) a finite-dimensional space of real analytic (complex-valued) functions. Hence, $|u_0|$ is real analytic so that $u_0 = |u_0|e^{i\theta_0}$, with $\theta_0$ real analytic.

Now

$$\int u_0^* P u_0 = \int |u_0|(P - Q(\nabla \theta_0))|u_0| = \lambda_0 - \int Q(\nabla \theta_0)|u_0|^2.$$

As $\{u_0 = 0\}$ has zero Lebesgue measure and $Q > 0$, the function $\theta_0$ is constant, so that $u_0$ and $|u_0|$ are colinear.

To conclude, if $u_0$ and $u_1$ are two orthogonal eigenfunctions of $P$ with eigenvalue $\lambda_0$, then $|u_0|$ and $|u_1|$ are orthogonal with each other, and both have $\mathbb{R}^{r_1+r_2}$ as support, so that either $u_0 = 0$ or $u_1 = 0$. $\square$

**Proposition 6.13.** Suppose $P$ satisfies the following two supplementary conditions:

- $r_1 = r_2$.
- For every $(q_1, q_2) \in (\mathbb{R}^{r_1} \setminus \{0\}) \times (\mathbb{R}^{r_2} \setminus \{0\})$, the matrices given by

$$\left[\sum_{i,j} \alpha_{ijk} q_1^i q_1^j \right]_{k,l} \text{ and } \left[\sum_{k,l} \alpha_{ijk} q_2^k q_2^l \right]_{i,j}$$

are positive.

Let $\Lambda > 0$ and let $N_\Lambda$ denote the number of eigenvalues of $P$ less than $\Lambda$ (with multiplicity).

Then there are $C > c > 0$ such that, as $\Lambda \to +\infty$, one has

$$c\Lambda^{\frac{3}{2}r_1} \log(\Lambda) \leq N_\Lambda \leq C\Lambda^{\frac{3}{2}r_1} \log(\Lambda).$$

**Proof.** Under the second supplementary condition, the quartic part of the potential is greater than $c|q_1|^2|q_2|^2$ for some $c > 0$. Hence, for some $C > 0$ one has $N_\Lambda \geq \tilde{N}_\Lambda$, where $\tilde{N}_\Lambda$ counts the eigenvalues less than $\Lambda$ of

$$-\Delta + |q_1|^2|q_2|^2 + |q_1| + |q_2|.$$

On the other hand one clearly has $P \leq C(-\Delta + |q_1|^2|q_2|^2 + |q_1| + |q_2|)$ for some $C > 0$.

Thus, the problem boils down to Weyl asymptotics for the elliptic operator

$$-\Delta + |q_1|^2|q_2|^2 + |q_1| + |q_2|.$$
It suffices to control the volume of the sub-levels of its symbol:

\[((q_1, q_2, p_1, p_2) \in \mathbb{R}^{4r_1}, |p_1|^2 + |p_2|^2 + |q_1|^2|q_2|^2 + |q_1| + |q_2| \leq \Lambda)\].

We first study

\[A_{\Lambda} = \{(q_1, q_2) \in \mathbb{R}^{r_1}, |q_1|^2|q_2|^2 + |q_1| + |q_2| \leq \Lambda\} \]

Then, decomposing \(A_{\Lambda}\) into \(A_{\Lambda} \cap B(0, \frac{\Lambda}{2})\) and its complement set yields

\[\text{Vol}(A_{\Lambda}) \leq C\Lambda^{r_1} + 2\int_{|q_1| \geq \Lambda^{1/4}} \text{Vol}\{q_2, |q_1|^2|q_2|^2 + |q_1| + |q_2| \leq \Lambda\}.\]

On the other hand,

\[\text{Vol}(A_{\Lambda}) \geq 2\int_{|q_1| \geq \Lambda^{1/4}} \text{Vol}\{q_2, |q_1|^2|q_2|^2 + |q_1| + |q_2| \leq \Lambda\}.\]

Integrating yields

\[\text{Vol}((q_1, q_2, p_1, p_2) \in \mathbb{R}^{4r_1}, |p_1|^2 + |p_2|^2 + |q_1|^2|q_2|^2 + |q_1| + |q_2| \leq \Lambda) \in \left[c\Lambda^{3/2} \log(\Lambda), C\Lambda^{3/2} \log(\Lambda)\right],\]

hence the claim. \(\square\)

6.3. Approximate first eigenfunction. In this subsection we give an expansion for the first eigenfunction and eigenvalue in a crossing case, following the same strategy as Sect. 5.2. We quantize the symplectic map of Proposition 6.4 and we use the Bargmann transform to reformulate the problem in the pseudodifferential algebra, in which we squeeze the operator. This time, the squeezing is of order \(N^{-\frac{1}{6}}\) along \((q_1, q_2)\), with a concentration speed of \(N^{-\frac{1}{3}+\epsilon}\) along the zero set, instead of \(N^{-\frac{1}{4}+\epsilon}\) as was seen in the regular case. We then apply a perturbative argument to obtain the full expansion of the first eigenvalue and eigenvector.
**Definition 6.14.** For any choice $S_N$ of quantization of the map $\sigma$ of Proposition 6.4, the classical symbol $\varphi \sim \sum N^{-1} g_i$ on a neighbourhood $U$ of 0 in $\mathbb{R}^{2n}$ is defined as follows: for any sequence $(u_N)_{N \geq 1}$ with microsupport in a compact set of $U$, the following holds:

$$B_N^{-1} \cdot S_N^{-1} T_N(h) S_N B_N u_N = \text{Op}^{N^{-1}}_W (g_S) u_N + O(N^{-\infty}).$$

In what follows, we choose an arbitrary quantum map $S_N$, and we write $g$ instead of $g_S$.

The subprincipal part $g_1$ is prescribed on $Z_1 \cup Z_2$ by the local Melin estimates.

**Proposition 6.15.** Along $\sigma(Z_1)$, for $q_1$ close to zero, one has

$$g_1(q_1, 0, 0, 0, 0, 0) = \frac{1}{4} \text{tr} Q \text{red}_F (q_1, 0) + \left( \frac{1}{4} \text{tr} (\text{Hess}(h_0)) + h_1 \right) (q_1, 0, 0, 0, 0, 0).$$

Along $\sigma(Z_2)$, for $q_2$ close to zero, one has

$$g_1(0, 0, q_2, 0, 0, 0) = \frac{1}{4} \text{tr} Q \text{red}_F (q_2, 0) + \left( \frac{1}{4} \text{tr} (\text{Hess}(h_0)) + h_1 \right) (0, 0, q_2, 0, 0, 0).$$

The proof is exactly the same as for Proposition 5.5.

Let us define

$$P = Q_S(0) (-iD_{q_1}, -iD_{q_2}) + \sum_{ijkl} \alpha_{ijklq_1,iq_1,jq_2,kq_2,l}$$

$$\quad + \nabla \left( \frac{1}{4} \text{tr} Q \text{red}_F + \frac{1}{4} \text{tr} Q_S \right)_{q_1=q_2=0} \cdot (q_1, q_2).$$

Then $P$ satisfies the hypotheses of Sect. 6.2 and Proposition 6.13.

**Proposition 6.16.** Under the conditions of Definition 6.1, there exists $c > 0$, a sequence $(u_i) \in (L^2(\mathbb{R}^{1+r+2})(X_1, \ldots, X_n-r_1-r_2))^N$ of square-integrable functions of $q$ with polynomial dependence on $x$, and a family of real values $(C_{i,\alpha,\beta})$ with

$$\forall (i, \alpha, \beta, q) \in \mathbb{N} \times \mathbb{N}^{n-r_1-r_2} \times \mathbb{N}^{r_1+r_2} \times \mathbb{R}^{r_1+r_2-2}, \ |\alpha^\beta \mu_i(q)| \leq C_{i,\alpha,\beta} e^{-c|q|^{3/2}},$$

and a sequence $(\mu_i) \in \mathbb{R}^N$ with $\mu_0 = \mu(P_0)$, $\mu_1 = 0$ and $\mu_2 = \min \text{Sp}(P)$, so that

$$N^{\frac{2}{3}} e^{-\frac{1}{6} q} e^{-N|q|^{1/2}} \sum_{i=0}^{+\infty} N^{-\frac{i}{6}} u_i(N^{\frac{1}{2}} x, N^{\frac{1}{2}} q)$$

is an $O(N^{-\infty})$-eigenfunction of $\text{Op}^{N^{-1}}_W(g)$, with eigenvalue

$$N^{-\frac{1}{3}} \sum_{i=0}^{+\infty} N^{-\frac{i}{6}} \mu_i.$$ 

This proposition provides an almost eigenfunction which we will show to be associated to the lowest eigenvalue (see Proposition 6.18). It is the main argument in the proof of Theorem C; the concentration speed of this eigenfunction on zero, which is $N^{-\frac{1}{3}}$, is the concentration speed of the lowest eigenvector of $T_N(h)$ on the miniwell, because of Proposition 5.10.
Proof. As announced, let us squeeze \( g \) by computing
\[
\tilde{g} = g(N^{-\frac{1}{3}}q_1, N^{-\frac{2}{3}}p_1, N^{-\frac{1}{3}}q_2, N^{-\frac{2}{3}}p_2, N^{-\frac{1}{2}}x, N^{-\frac{1}{2}}\xi).
\]
Grouping terms in the Taylor expansion yields, for any fixed \( K \in \mathbb{N} \),
\[
Op^1_W(\tilde{g}) = N^{-1} \sum_{i=0}^{K} N^{-\frac{i}{6}} Op^1_W(a_i) + O(N^{-\frac{K+2}{6}}).
\]
The first terms are:
\[
a_0 = Q^F_{\text{red}}(0)(x, \xi) + \frac{1}{4} \text{tr}(\text{Hess}(h_0)(0)) + h_1(0)
\]
\[
a_1 = 0
\]
\[
a_2 = \sigma(P).
\]
Here \( P \) is as above.

With \( A_i = Op^1_W(a_i) \) let us solve by induction on \( k \) the following equation, where \((u_k)_{k \in \mathbb{N}}\) is as in the claim:
\[
\left( \sum N^{-\frac{i}{6}} (A_i - \mu_i) \right) \left( \sum N^{-\frac{i}{6}} u_i \right) = 0.
\]
If \( v_0 \) is the (unique) ground state of \( P \) then our starting point is
\[
u_0 = e^{-\frac{|x|^2}{2}} v_0, \mu_0 = \text{min Sp } A_0,
\]
\[
u_1 = 0, \mu_1 = 0.
\]
Indeed \( u_0 \) is an almost eigenvector for \( Op^1_W(\tilde{g}) \), with eigenvalue \( N^{-1} \mu_0 + O(N^{-\frac{4}{3}}) \).

Let us start an induction at \( k = 1 \). Suppose we have constructed the first \( k \) terms of the expansion \( u_0, \ldots, u_k \) and \( \mu_0, \ldots, \mu_k \), with \( u_i \perp u_0 \) for every \( i \), and suppose that, for some \( C_k \in \mathbb{R} \), one has, for every \( q \in \mathbb{R}^{1+r_2} \),
\[
\int_{\mathbb{R}^{n-r_1-r_2}} \overline{u_0}(q, x) \left( \sum_{i=2}^{k+1} [A_i u_{k+1-i}](q, x) - \sum_{i=2}^{k} [\mu_i u_{k+1-i}](q, x) \right) dx = C_{k+1}|v_0(q)|^2.
\]

Then the eigenvalue problem yields \( u_{k+1} \) up to a function of the form \( v(q)e^{-\frac{|x|^2}{2}} \). Indeed, writing
\[
u_{k+1}(q, x) = v(q) e^{-\frac{|x|^2}{2}} + w(q, x),
\]
where for every \( q \) one has \( w(q, \cdot) \perp e^{-\frac{|x|^2}{2}} \), the eigenvalue equation is
\[
(A_0 - \mu_0) u_{k+1} + (A_2 - \mu_2) u_{k-1} + \cdots + (A_{k+1} - \mu_{k+1}) u_0 = 0
\]
for \( u_{k+1} \) and \( \mu_{k+1} \). First \( (A_0 - \mu_0)v(q)e^{-\frac{|x|^2}{2}} = 0 \) so that
\[
(A_0 - \mu_0) w + (A_2 - \mu_2) u_{k-1} + \cdots + (A_{k+1} - \mu_{k+1}) u_0 = 0
\]
By hypothesis, freezing the $q$ variable and taking the scalar product of this equation with $x \mapsto e^{-\frac{|x|}{2}}$ yields $(C_{k+1} - \mu_{k+1})|v_0(q)|^2 = 0$. Let $\mu_{k+1} = C_{k+1}$. Then, for every $q \in \mathbb{R}^{r_1+r_2}$, the function

$$f_{k+1} : x \mapsto \sum_{i=2}^{k+1} [(A_i - \mu_i)u_{k+1-i}](q, x)$$

is orthogonal to $x \mapsto e^{-\frac{|x|}{2}}$. Hence $w = (A_0 - \mu_0)^{-1} f_{k+1}$ is well-defined and satisfies the eigenvalue equation.

Moreover, from Proposition 6.11, if by induction $f_{k+1}$ is $e^{-\frac{|x|}{2}}$ times a polynomial in $x$, and if any derivative of any coefficient decays as fast as $e^{-c|q|^{3/2}}$, then the same is true for $w$.

At this point we need to check that, after the first step $k = 1$, the value $\mu_2$ is indeed $\min \text{Sp}(P)$.

If $k = 1$ then we are interested in the integral

$$\int_{\mathbb{R}^{n-r_1-r_2}} e^{-\frac{|x|}{2}} v_0(q)[A_2u_0](q, x) dx = \min \text{Sp}(P)|v_0(q)|^2,$$

since $v_0$ is a ground state of $P$. This is indeed a constant function times $|v_0(q)|^2$, so that the induction hypothesis is satisfied at the first step, and $\mu_2 = \min \text{Sp}(P)$ as required.

Now recall $u_k(q, x) = v(q)e^{-\frac{|x|}{2}} + w(q, x)$. The eigenvalue equation in itself does not state any condition on $v$; however, to compute the second next order, one needs to satisfy an orthogonality condition, i.e.

$$\int_{\mathbb{R}^{n-r_1-r_2}} \mu_0(q, x) \left( \sum_{i=2}^{k+3} [A_iu_{k+3-i}](q, x) - \sum_{i=2}^{k+2}[\mu_iu_{k+3-i}](q, x) \right) dx = C_{k+3}|v_0(q)|^2.$$

This is equivalent to

$$\int_{\mathbb{R}^{n-r_1-r_2}} e^{-\frac{|x|}{2}} \left[ (A_2 - \mu_2)ve^{-\frac{|x|}{2}} \right](x, q) dx = F(q) + C_{k+3}v_0(q).$$

Now $a_2$ has no terms in $x$ or $\xi$ so the equation reduces to

$$(A_2 - \mu_2)v = F(q) + C_{k+3}v_0(q).$$

Here,

$$F(q) = \int_{\mathbb{R}^{n-r_1-r_2}} e^{-\frac{|x|}{2}} \left( \sum_{i=3}^{k+3} [A_iu_{k+3-i}](q, x) - \sum_{i=3}^{k+2}[\mu_iu_{k+3-i}](q, x) \right) dx,$$

so that $|\partial_\beta F(q)| \leq C_\beta e^{-c|q|^{3/2}}$.

To solve this equation, one takes $C_{k+3} = -(v_0, F)$, then the r.h.s is orthogonal to $v_0$, so that one can solve for $v$ (indeed, $\mu_2$ is a simple eigenvalue of $A_2$ by Proposition 6.12).

Then, by Proposition 6.11, one has, for all $\beta \in \mathbb{N}^{r_1+r_2}$, for some $C_\beta$, that $|\partial_\beta v(q)| \leq C_\beta e^{-c|q|^{3/2}}$ for all $q \in \mathbb{R}^{r_1+r_2}$. This ends the induction.

The previous considerations were formal, but the decay properties of the functions $u_k$ imply that $A_ju_k \in L^2$ for every $j$ and $k$, which concludes the proof. □
Proposition 6.17. Let $A^\text{cross}_N$ be the following operator on $L^2(\mathbb{R}^{r_1+r_2})$:

$$A^\text{cross}_N = Op_W^{N^{-1}} \left( |p|^2 + |q_1|^2 |q_2|^2 \right)$$

Under the conditions of Definition 6.1 and Proposition 6.13, there exists $a_0 > 0$, and two constants $0 < c < C$ such that, for any $N$, for any $a < a_0$, for any normalized $u \in L^2(X)$ supported in $B(P_0, a) \times \mathbb{S}^1$ such that $S_N u = u + O(N^{-\infty})$, with $v = B^{-1}_N \mathcal{S}^{-1}_N u$, one has:

$$c \langle v, A^\text{cross}_N v \rangle + c \left( \langle v, Op_W^{N^{-1}} (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n-r}{2} \right)$$

$$- C \langle v, Op_W^{N^{-1}} (|N^{-\frac{1}{2}} x, \xi|^3) v \rangle - CN^4$$

$$\leq \langle u, hu \rangle - N^{-1} \mu(P_0).$$

In addition, the following bound holds:

$$c \langle v, A^\text{cross}_N v \rangle + \langle v, Op_W^{N^{-1}} (Q_F(0)(x, \xi)) v \rangle - C \langle v, Op_W^{N^{-1}} (|N^{-\frac{1}{2}} x, \xi|^3) v \rangle$$

$$- aC \left( \langle v, Op_W^{N^{-1}} (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n-r}{2} \right) - CN^{-\frac{3}{4}}$$

$$\leq \langle u, hu \rangle - N^{-1} \mu(P_0) + \frac{N^{-1}}{4} \text{tr } Q_F^{\text{red}}(0)$$

$$\leq C \langle v, A^\text{cross}_N v \rangle + \langle v, Op_W^{N^{-1}} (Q_F(0)(x, \xi)) v \rangle + C \langle v, Op_W^{N^{-1}} (|N^{-\frac{1}{2}} x, \xi|^3) v \rangle$$

$$+ aC \left( \langle v, Op_W^{N^{-1}} (|x|^2 + |\xi|^2) v \rangle - N^{-1} \frac{n-r}{2} \right) + CN^{-\frac{4}{3}}.$$

Proof. The proof follows the exact same lines as for Proposition 5.8: the difficulty lies in handling the $(x, \xi)$ terms which take a similar form as above.

The supplementary $N^{-\frac{4}{3}}$ terms are due to positivity estimates for the Weyl quantization: from $c \sigma(A^\text{cross}_N) \leq g_0$ we can only deduce $c A^\text{cross}_N \leq Op_W^{N^{-1}} (g_0) + O(N^{-\frac{4}{3}})$. □

6.4. Spectral gap. As before, we show that the almost eigenfunction found previously corresponds to the first eigenvalue.

Proposition 6.18. Let $h$ be a classical symbol with $h_0 \geq 0$, and such that the minimum of the Melin value $\mu$ is only reached at one point, which is a simple crossing point of $h$.

Let $(\mu_i)$ be the real sequence constructed in Proposition 6.16, and let $\lambda_{\text{min}}$ be the first eigenvalue of $T_N(h)$.

Then

$$\lambda_{\text{min}} \sim N^{-\frac{1}{2}} \sum_{i=0}^{\infty} N^{-\frac{1}{2}} \mu_i.$$

Moreover, there exists $c > 0$ such that, for every $N$, one has

$$\text{dist}(\lambda_{\text{min}}), Sp(T_N(h)) \setminus \{\lambda_{\text{min}}\} \geq c N^{-\frac{4}{3}}.$$
Proof. Let us show that any function orthogonal to the one proposed in Proposition 6.16 has an energy which is larger by at least \(cN^{-\frac{4}{3}}\).

Let \((v_N)_{N \geq 1}\) be a sequence of unit vectors in \(L^2(\mathbb{R}^n)\). If
\[
\langle v_N, Op_1^W(g_N)v_N \rangle \leq N^{-1} \mu_0 + CN^{-\frac{4}{3}}
\]
for some \(C\), then
\[
v_N = e^{-\frac{|x|^2}{2}} w_N(q) + O(N^{-\frac{1}{3}})
\]
with \(\|w_N\|_{L^2} = 1 + O(N^{-\frac{1}{3}})\).

If \(C - \mu_2\) is strictly smaller than the spectral gap of the operator \(P\) then, for some \(a > 0\), one has \(\langle w_N, v_0 \rangle \geq a\), which concludes the proof. \(\square\)

7. Weyl Asymptotics

Definition 7.1. We will say a miniwell has dimension \(r\) when the dimension of the zero set of \(h_0\) around the miniwell is \(r\). Similarly, we will say a crossing point has dimensions \((r_1, r_2)\) when the dimensions of the two manifolds \(Z_1\) and \(Z_2\) around the point are \(r_1\) and \(r_2\), respectively.

Proof of Theorem D. 1. Let \(u\) be a sequence of \(O(N^{-\infty})\)-quasimodes for \(T_N(h)\) in the spectral window above, that localise at a miniwell \(P_0\). Let \(\mathcal{G}_N\) be the quantum map quantizing the symplectic change of variables constructed in Proposition 5.2. Let \(v_N = B_N^{-1} \mathcal{G}_N^{-1} u_N\). The first lower bound on Proposition 5.8 yields
\[
\langle v_N, Op_1^W(|x|^2 + |\xi|^2)v_N \rangle - N^{-1} \frac{h - r}{2} \leq C \Lambda N N^{-1}.
\]

From Proposition 4.1, for every \(\delta > 0\), if \(\epsilon > 0\) is small enough then \(u\) localises on \(B(P_0, \delta)\). If \(\delta\) is small enough then
\[
\delta C \left( \langle v_N, Op_1^W(|x|^2 + |\xi|^2)v_N \rangle - N^{-1} \frac{h - r}{2} \right) \leq \frac{N^{-1} \Lambda N}{2}.
\]

Let us prove an upper bound in the number of eigenvalues of \(T_N(h)\). The second lower bound in Proposition 5.8 leads to
\[
c \langle v_N, A_{reg}^N v_N \rangle + \langle v_N, Op_1^W(Q_F(0)(x, \xi))v_N \rangle - \frac{N^{-1}}{2} \sum \lambda_i(0) \leq \langle u_N, hu_N \rangle - N^{-1} \mu(P_0) + \frac{N^{-1} \Lambda N}{2}.
\]

For \(\epsilon\) smaller than the spectral gap of \(Q_F(0)(x, D)\), the left-hand side has less than \(C \Lambda N\) eigenvalues smaller than \(\frac{3N^{-1} \Lambda N}{2}\), hence the claim.

The lower bound proceeds along the same lines. The upper bound in Proposition 5.8 yields
\[
C \langle v_N, A_{reg}^N v_N \rangle + \langle v_N, Op_1^W(Q_F(0)(x, \xi))v_N \rangle - \frac{N^{-1}}{2} \sum \lambda_i(0) \geq \langle u_N, hu_N \rangle - N^{-1} \mu(P_0) - \frac{N^{-1} \Lambda N}{2}.
\]
The left-hand side has always more than $c \Lambda_N$ eigenvalues smaller than $\frac{N^{-1} \Lambda_N}{2}$, hence the claim.

2. The proof for crossing points is the same except for the actual count of eigenvalues of the reference operator, which stems from Proposition 6.13. □

8. Examples: Frustrated Spin Systems

In this Section, we discuss the class of examples introduced in Sect. 1.4. We first describe the minimal set in the general setting, and we prove that, for a loop of six triangles, the classical minimal set is not a smooth manifold; then we prove that the choice of the vectors on triangle “leaves” does not affect $\mu$; to conclude we treat numerically a simple case supporting the general conjecture that $\mu$ is minimal on planar configurations.

8.1. Description of the zero set. If a graph is made of triangles $(V_i)_{i \in J}$, and if we denote by $(u_i, v_i, w_i)$ the three elements of $S^2$ at the vertices of $V_i$, we write

$$h((e_i)_{i \in V}) = \sum_{i \in J} u_i \cdot v_i + u_i \cdot w_i + v_i \cdot w_i.$$ 

Moreover, for all $u, v, w \in S^2$ one has

$$u \cdot v + u \cdot w + v \cdot w = \frac{1}{2} \| u + v + w \|^2 - \frac{3}{2}.$$ 

A way to minimize the symbol is thus to try to choose the vectors such that, for each triangle in the graph, the vectors at the vertices form a great equilateral triangle on $S^2$ (this is equivalent to the requirement that their sum is the zero vector). As the example of the Husimi tree shows, this minimal set can be degenerate: once the vector at a vertex is chosen, there is an $S^1$ degeneracy in the choice of the vectors at its children.

In the general case this solution is not always possible as can be seen on the right of Fig. 1. Moreover, even if this solution is possible, the minimal set is not a submanifold, as we will see in an example.

A subset of interest of these minimal configurations consists in the case where all vectors are coplanar. This corresponds to colouring the graph with three colours (indeed, if all spins are coplanar then there are only three possibilities for each spins; then the global constraint is that no neighbouring spins share the same value). For some graphs made of triangles, there is no 3-colouring. Conversely, if the size of the graph grows the number of 3-colourings may grow exponentially fast, as for pieces of the Kagome lattice.

A common conjecture in the physics literature, which appears in [11, 15, 26, 35, 39], is that the Melin value $\mu$ (as in Definition 2.15) is always minimal on planar configurations, except for a leaf degeneracy (see Proposition 8.3): in other terms, in the semiclassical limit, the quantum state presumably selects only planar configurations. Of course, this conjecture may only holds in situations where $G$ can be 3-coloured. It is unclear whether a study of the sub-subprincipal effects would discriminate further between planar configurations, but numerical evidence suggests that the quantum ground state is not distributed evenly on them at large spin.

Other selection effects tend to select the planar configurations: consider for instance the classical Gibbs measure, at a very small temperature. This measure concentrates on
the points of the minimal set where the Hessian has a maximal number of zero eigenvalues (thermal selection); in this case it always corresponds to planar configurations, if any.

We conclude this subsection with a general statement about the isotropy of the classical minimal set.

**Proposition 8.1.** Let $G = (V, E)$ be a graph made of triangles, and let $e \in (S^2)^{|V|}$ be such that $h(e) = -|V|$, where $h$ is the classical antiferromagnetic energy. Let $F \subset T_e M$ be the kernel of $\text{Hess}(h)(e)$. Then $F$ is isotropic.

**Proof.** Let $(u, v, w) \in V^3$ be a triangle in the graph, and let $\pi : (S^2)^{|V|} \to (S^2)^3$ be the projection map which keeps only the spin coordinates corresponding to $(u, v, w)$. We will prove that $\pi(F)$ is isotropic. Since $\text{Hess}(h)(e) \geq 0$, one has $\pi(F) \subset \ker(\text{Hess}(h_{u,v,w})(\pi(e)))$ where

$$h_{u,v,w}(e_u, e_v, e_w) = e_u \cdot e_v + e_u \cdot e_w + e_v \cdot e_w.$$  

The problem then reduces to the case of one triangle. With the choice of coordinates on Fig. 3, the Hessian of $h_{u,v,w}$ at a minimal point reads

$$(q_1 - q_2)^2 + (q_1 - q_3)^2 + (q_2 - q_3)^2 + 2(p_1 + p_2 + p_3)^2.$$  

The kernel of this quadratic form is

$$\text{Span}((1, 1, 0, 0, 0), (0, 0, 1, 1, 0), (0, 0, 1, 0, -1)),$$  

which is isotropic. □

8.2. Irregularity of the zero set. One of the key examples of frustrated spin systems is the Kagome lattice. We restrict our study to the case of one loop of six triangles.

**Proposition 8.2.** For a loop of six triangles (as in Fig. 2), the minimal set is not smooth.

**Proof.** Let us try to find one vector at each node of a loop of six triangles, such that the sum over each triangle of the vectors is zero.

The problem is invariant under rotation of all vectors by a common element of $SO(3)$. Up to such a rotation, the two vectors drawn on the left in Fig. 2 are fixed. Moreover, the position of the six inner vectors determines the position of the six outer vectors in a unique and smooth way, so we will forget about the latter.

The space of configurations of the pair $(a, a')$ is a subset of a two-dimensional torus; indeed the choice for $a'$ is made along a circle having its center on the lower-left vector, and the choice for $a$ is similarly made along a circle with center $a'$. The above applies to the pair $(b, b')$. Hence, the set of global configurations is a subset of a four-dimensional torus: the subset on which the angle between $a$ and $b$ is exactly $\frac{2\pi}{3}$. This cannot be an open set of the four-dimensional torus, as every coordinate and function involved is real analytic. Hence, if this set is a smooth manifold, its dimension does not exceed three.

On the other hand, consider the particular case on the right of Fig. 2 which represents a particular configuration. Starting from this configuration, there are four linearly independent moves along which one stays in the minimal set: by moving $a'$ along a circle with center $a$; moving $a$ along a circle with center $a'$ only, or moving $b$ only, or moving $b'$ only. The set of possible smooth moves from this configuration spans a set of dimension at least four, hence the contradiction. □

In the previous example, the algebraic manifold of minimal configurations has dimension three, and has a critical point (presumably a crossing point) at the configuration on the right of Fig. 2.
Low-Energy Spectrum of Toeplitz Operators with a Miniwell

Fig. 2. On the left, a graph with 6 triangles and two prescribed vectors. On the right, a particular (planar) configuration

Fig. 3. General minimal configuration for one triangle (left) and an triangle leaf (right) of spins, with choice of tangent coordinates. On the left, $e_1, e_2, e_3$ form a great equilateral triangle on the sphere; the associated great circle is drawn in dotted lines. From this configuration, the coordinate $q_i$ corresponds to an infinitesimal displacement of $e_i$ tangent to the circle, and $p_i$ corresponds to an infinitesimal displacement orthogonal to the circle. On the right, $e_1, e_2, e_3$ and $e_3, e_4, e_5$ form two great equilateral triangles, and the angle between the associated great circles is $\theta$. The coordinates are chosen in the same way as on the left

8.3. Degeneracy for triangle leaves. The simplest example of a frustrated system is a triangle with three vertices, connected with each other. In this setting the degeneracy of the minimal set (which is exactly the set of configurations such that the sum of the three vectors is zero) corresponds to a global $SO(3)$ symmetry of the problem; in this case the function $\mu$ is constant.

Consider the left part of Fig. 3. The three elements $e_1, e_2, e_3$ lie on the same large circle. We choose the coordinate $q_i$ along this circle and the coordinate $p_i$ orthogonal to it. In these coordinates, the half-Hessian of the classical symbol can be written as:

$$2(p_1 + p_2 + p_3)^2 + (q_1 - q_2)^2 + (q_1 - q_3)^2 + (q_2 - q_3)^2.$$ 

Since this quadratic form does not depend on the positions of $e_1, e_2, e_3$, the function $\mu$ is constant.

In the following Proposition we consider a slightly more general situation.

Proposition 8.3. Consider a graph with a “triangle leaf” as in the inset on the right of Fig. 2. In order to find a classical minimum for such a graph, once all vectors except for $e_4$ and $e_5$ are chosen, then $e_4$ and $e_5$ are fixed except for a rotation of centre $e_3$. The Melin value $\mu$ does not depend on this choice.
Fig. 4. The two general configurations for a loop of 4 triangles. On the left, $e_3 = e_1$ so $e_{12} = e_{23}$ and $e_{14} = e_{34}$. The great circle passing through $e_1$, $e_2$, $e_{12}$ and the great circle passing through $e_1$, $e_4$, $e_{14}$ make an angle $\theta$. On the right, one has $e_1 \neq e_3$, and the great circle through $e_1$ and $e_3$ is the smallest bisector of the two others. $e_3$ is at (spherical) distance $\phi$ from the circle $\{e \cdot e_1 = -\frac{1}{2}\}$, where $\tan(\pi/3 - \phi/2) = 2\cos(\theta/2)$. We omitted to draw $e_{12}, e_{23}, e_{34}, e_{41}$ for simplicity.

Proof. Denoting $c = \cos(\theta)$ and $s = \sin(\theta)$, and using local coordinates as in the right part of Fig. 3, the 2-jet of the Hamiltonian reads, in local coordinates:

$$Q(p_1, p_2, q_1, q_2, ...) + 2(p_4 + p_5)^2 + (q_4 - q_5)^2 + q_4^2 + q_5^2 + 4q_3^2 + 4p_3^2$$
$$+ 4p_3(p_1 + p_2) - 2q_3(q_1 + q_2)$$
$$+ 4cp_3(p_4 + p_5) - 4sq_3(p_4 + p_5)$$
$$- 2cq_3(q_4 + q_5) - 2sp_3(q_4 + q_5).$$

The trace of this quadratic form does not depend on $\theta$. Hence, in order to prove that $\mu$ does not depend on $\theta$ it is sufficient to find symplectic coordinates in which this quadratic form does not depend on $\theta$. A first symplectic change of variables leads to:

$$Q(p_1, p_2, q_1, q_2, ...) + 4p_4^2 + q_4^2 + 3q_5^2 + 4q_3^2 + 4p_3^2$$
$$+ 4p_3(p_1 + p_2) - 2q_3(q_1 + q_2)$$
$$+ 4\sqrt{2}cp_3p_4 - 4\sqrt{2}sq_3p_4 - 2\sqrt{2}cq_3q_4 - 2\sqrt{2}sp_3q_4.$$

Let us make the following change of variables:

$$p_4 \mapsto cp_4 - s \frac{q_4}{2}$$
$$q_4 \mapsto cq_4 + 2sp_4.$$

This change of variables is symplectic, and preserves $4p_4^2 + q_4^2$. The quadratic form becomes:

$$Q(p_1, p_2, q_1, q_2, ...) + 4p_3(p_1 + p_2) - 2q_3(q_1 + q_2)$$
$$+ 4p_4^2 + q_4^2 + 3q_5^2 + 4p_3^2 + 4q_3^2 + 8p_3p_4 - 4q_4q_3.$$

Since this quadratic form does not depend on $\theta$, the function $\mu$ does not depend on $\theta$. $\Box$
Fig. 5. Numerical plot of $\mu$, for a loop of 4 triangles (inset), along $C_1$ (left) and $C_2$ (right).
8.4. A numerical example. The last example we treat is the case of a loop of 4 triangles. In this setting, the minimal set is not a submanifold but a union of three submanifolds, with transverse intersection. The general configuration is shown in Fig. 4. We believe that the intersections correspond in fact to the case of crossing along a submanifold (see Definition 6.5). From Proposition 8.1, the three first conditions in Definition 6.5 are satisfied, and an explicit computation of the Hessian matrix yields condition 4. We only have numerical proof for the behaviour of $\mu$ near the crossing submanifold (see Fig. 5), since we cannot give an explicit expression for $\mu$ in this setting.

In this example, the crossing submanifolds correspond to the coplanar configurations, so that Fig. 5 is a strong indication that $\mu$ is, in this example, minimal along coplanar configurations.

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