Upper Bounds on the Spanning Ratio of Constrained Theta-Graphs

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Abstract. We present tight upper and lower bounds on the spanning ratio of a large family of constrained θ-graphs. We show that constrained θ-graphs with 4k + 2 (k ≥ 1 and integer) cones have a tight spanning ratio of 1 + 2 sin(θ/2), where θ is 2π/(4k + 2). We also present improved upper bounds on the spanning ratio of the other families of constrained θ-graphs.

1 Introduction

A geometric graph G is a graph whose vertices are points in the plane and whose edges are line segments between pairs of points. Every edge is weighted by the Euclidean distance between its endpoints. The distance between two vertices u and v in G, denoted by \(d_G(u,v)\), is defined as the sum of the weights of the edges along the shortest path between u and v in G. A subgraph H of G is a t-spanner of G (for \(t ≥ 1\)) if for each pair of vertices u and v, \(d_H(u,v) ≤ t \cdot d_G(u,v)\). The smallest value \(t\) for which \(H\) is a t-spanner is the spanning ratio or stretch factor.

The graph G is referred to as the underlying graph of H. The spanning properties of various geometric graphs have been studied extensively in the literature (see [4,9] for a comprehensive overview of the topic). We look at a specific type of geometric spanner: θ-graphs. Introduced independently by Clarkson [6] and Keil [8], θ-graphs partition the plane around each vertex into \(m\) disjoint cones, each having aperture \(θ = 2π/m\).

The \(θ_m\)-graph is constructed by, for each cone of each vertex u, connecting u to the vertex v whose projection along the bisector of the cone is closest. Ruppert and Seidel [10] showed that the spanning ratio of these graphs is at most \(1/(1-2 \sin(θ/2))\), when \(θ < π/3\), i.e. there are at least seven cones. Recent results include a tight spanning ratio of \(1 + 2 \sin(θ/2)\) for θ-graphs with \(4k + 2\) cones [1], where \(k ≥ 1\) and integer, and improved upper bounds for the other three families of θ-graphs [5].

Most of the research, however, has focused on constructing spanners where the underlying graph is the complete Euclidean geometric graph. We study this problem in a more general setting with the introduction of line segment constraints. Specifically, let \(P\) be a set of points in the plane and let S be a set

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of line segments between two vertices in $P$, called *constraints*. The set of constraints is planar, i.e. no two constraints intersect properly. Two vertices $u$ and $v$ can see each other if and only if either the line segment $uv$ does not properly intersect any constraint or $uv$ is itself a constraint. If two vertices $u$ and $v$ can see each other, the line segment $uv$ is a *visibility edge*. The *visibility graph* of $P$ with respect to a set of constraints $S$, denoted $Vis(P, S)$, has $P$ as vertex set and all visibility edges as edge set. In other words, it is the complete graph on $P$ minus all edges that properly intersect one or more constraints in $S$.

This setting has been studied extensively within the context of motion planning amid obstacles. Clarkson [6] was one of the first to study this problem and showed how to construct a linear-sized $(1+\epsilon)$-spanner of $Vis(P, S)$. Subsequently, Das [7] showed how to construct a spanner of $Vis(P, S)$ with constant spanning ratio and constant degree. The Constrained Delaunay Triangulation was shown to be a $2.42$-spanner of $Vis(P, S)$ [3]. Recently, it was also shown that the constrained $\theta_{4k}$-graph is a $2$-spanner of $Vis(P, S)$ [2]. In this paper, we generalize the recent results on unconstrained $\theta$-graphs to the constrained setting. There are two main obstacles that differentiate this work from previous results. First, the main difficulty with the constrained setting is that induction cannot be applied directly, as the destination need not be visible from the vertex closest to the source (see Figure 5, where $w$ is not visible from $v_0$, the vertex closest to $u$). Second, when the graph does not have $4k + 2$ cones, the cones do not line up as nicely as in [2], making it more difficult to apply induction.

In this paper, we overcome these two difficulties and show that constrained $\theta$-graphs with $4k + 2$ cones have a spanning ratio of at most $1 + 2 \sin(\theta/2)$, where $\theta$ is $2\pi/(4k + 2)$. Since the lower bounds of the unconstrained $\theta$-graphs carry over to the constrained setting, this shows that this spanning ratio is tight. We also show that constrained $\theta$-graphs with $4k + 4$ cones have a spanning ratio of at most $1 + 2 \sin(\theta/2)/(\cos(\theta/2) - \sin(\theta/2))$, where $\theta$ is $2\pi/(4k + 4)$. Finally, we show that constrained $\theta$-graphs with $4k + 3$ or $4k + 5$ cones have a spanning ratio of at most $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$, where $\theta$ is $2\pi/(4k + 3)$ or $2\pi/(4k + 5)$.

2 Preliminaries

We define a *cone* $C$ to be the region in the plane between two rays originating from a vertex referred to as the apex of the cone. When constructing a (constrained) $\theta_{(4k+1)}$-graph, for each vertex $u$ consider the rays originating from $u$ with the angle between consecutive rays being $\theta = 2\pi/(4k + x)$, where $k \geq 1$ and integer and $x \in \{2, 3, 4, 5\}$. Each pair of consecutive rays defines a cone. The cones are oriented such that the bisector of some cone coincides with the vertical halfline through $u$ that lies above $u$. Let this cone be $C_0$ of $u$ and number the cones in clockwise order around $u$. The cones around the other vertices have the same orientation as the ones around $u$. We write $C^u_i$ to indicate the $i$-th cone of a vertex $u$. For ease of exposition, we only consider point sets in general position: no two points lie on a line parallel to one of the rays that define the cones, no two points lie on a line perpendicular to the bisector of a cone, and no three points are collinear.
Let vertex $u$ be an endpoint of a constraint $c$ and let the other endpoint $v$ lie in cone $C^n_i$. The lines through all such constraints $c$ split $C^n_i$ into several subcones. We use $C^n_{i,j}$ to denote the $j$-th subcone of $C^n_i$. When a constraint $c = (u, v)$ splits a cone of $u$ into two subcones, we define $v$ to lie in both of these subcones. We consider a cone that is not split to be a single subcone.

We now introduce the constrained $\theta_{(4k+\alpha)}$-graph: for each subcone $C^n_{i,j}$ of each vertex $u$, add an edge from $u$ to the closest vertex in that subcone that can see $u$, where distance is measured along the bisector of the original cone (not the subcone). More formally, we add an edge between two vertices $u$ and $v$ if $v$ can see $u$, $v \in C^n_{i,j}$, and for all points $w \in C^n_{i,j}$ that can see $u$, $|uv'| \leq |uw'|$, where $v'$ and $w'$ denote the projection of $v$ and $w$ on the bisector of $C^n_i$ and $|xy|$ denotes the length of the line segment between two points $x$ and $y$.

Note that our assumption of general position implies that each vertex adds at most one edge for each of its subcones.

Given a vertex $w$ in the cone $C_i$ of vertex $u$, we define the canonical triangle $T_{uw}$ to be the triangle defined by the borders of $C^n_i$ and the line through $w$ perpendicular to the bisector of $C^n_i$. Note that subcones do not define canonical triangles. We use $m$ to denote the midpoint of the side of $T_{uw}$ opposing $u$ and $\alpha$ to denote the unsigned angle between $uw$ and $um$ (see Figure 1). Note that for any pair of vertices $u$ and $w$, there exist two canonical triangles: $T_{uw}$ and $T_{wu}$. We say that a region is empty if it does not contain any vertex of $P$.

3 Some Useful Lemmas

In this section, we list a number of lemmas that are used when bounding the spanning ratio of the various graphs. Note that these lemmas are not new, as they are already used in [2,5], though some are expanded to work for all four families of constrained $\theta$-graphs. We start with a nice property of visibility graphs from [2].

Lemma 1. Let $u$, $v$, and $w$ be three arbitrary points in the plane such that $uw$ and $vw$ are visibility edges and $w$ is not the endpoint of a constraint intersecting the interior of triangle $uvw$. Then there exists a convex chain of visibility edges from $u$ to $v$ in triangle $uvw$, such that the polygon defined by $uw$, $vw$ and the convex chain is empty and does not contain any constraints.
Next, we use two lemmas from \cite{5} to bound the length of certain line segments. Note that Lemma 2 is extended such that it also holds for the constrained $\theta_{(4k+2)}$-graph. We use $\angle xyz$ to denote the smaller angle between line segments $xy$ and $yz$.

**Lemma 2.** Let $u$, $v$ and $w$ be three vertices in the $\theta_{(4k+2)}$-graph, $x \in \{2, 3, 4, 5\}$, such that $w \in C^u_x$ and $v \in T_{uw}$, to the left of $uw$. Let $a$ be the intersection of the side of $T_{uw}$ opposite $u$ and the left boundary of $C^v_x$. Let $C^v_i$ denote the cone of $v$ that contains $w$ and let $c$ and $d$ be the upper and lower corner of $T_{vw}$. If $1 \leq i \leq k-1$, or $i = k$ and $|cw| \leq |dw|$, then $\max\{|vc| + |cw|, |vd| + |dw|\} \leq |va| + |aw|$ and $\max\{|cw|, |dw|\} \leq |aw|$.

![Fig. 3. The situation where we apply Lemma 2](image1)

![Fig. 4. The situation where we apply Lemma 3](image2)

**Lemma 3.** Let $u$, $v$ and $w$ be three vertices in the $\theta_{(4k+2)}$-graph, $x \in \{2, 3, 4, 5\}$, such that $w \in C^u_x$, $v \in T_{uw}$ to the left of $uw$, and $w \notin C^v_x$. Let $a$ be the intersection of the side of $T_{uw}$ opposite $u$ and the line through $v$ parallel to the left boundary of $T_{uw}$. Let $y$ and $z$ be the corners of $T_{vw}$ opposite to $v$. Let $\beta = \angle awv$ and let $\gamma$ be the unsigned angle between $vw$ and the bisector of $T_{vw}$. Let $c$ be a positive constant. If $c \geq \frac{\cos \gamma - \sin \beta}{\cos \left(\frac{\theta}{2}\right) - \sin \left(\frac{\theta}{2} + \gamma\right)}$, then $|vp| + c \cdot |pw| \leq |va| + c \cdot |aw|$, where $p$ is $y$ if $|yw| \geq |zw|$ and $z$ if $|yw| < |zw|$.

4 Constrained $\theta_{(4k+2)}$-Graph

In this section we prove that the constrained $\theta_{(4k+2)}$-graph has spanning ratio at most $1 + 2 \cdot \sin(\theta/2)$. Since this is also a lower bound \cite{5}, this proves that this spanning ratio is tight.
Theorem 1. Let \( u \) and \( w \) be two vertices in the plane such that \( u \) can see \( w \). Let \( m \) be the midpoint of the side of \( T_{uw} \) opposing \( u \) and let \( \alpha \) be the unsigned angle between \( uw \) and \( um \). There exists a path connecting \( u \) and \( w \) in the constrained \( \theta_{(4k+2)} \)-graph of length at most

\[
\left( \frac{1 + \sin \left( \frac{\theta}{2} \right)}{\cos \left( \frac{\theta}{2} \right)} \right) \cdot \cos \alpha + \sin \alpha \cdot |uw|.
\]

Proof. We assume without loss of generality that \( w \in C^n_0 \). We prove the theorem by induction on the area of \( T_{uw} \). Formally, we perform induction on the rank, when ordered by area, of the triangles \( T_{xy} \) for all pairs of vertices \( x \) and \( y \) that can see each other. Let \( a \) and \( b \) be the upper left and right corner of \( T_{uw} \), and let \( A \) and \( B \) be the triangles \( uaw \) and \( ubw \) (see Figure 5).

Our inductive hypothesis is the following, where \( \delta(u, w) \) denotes the length of the shortest path from \( u \) to \( w \) in the constrained \( \theta_{(4k+2)} \)-graph:

\[- \text{ If } A \text{ is empty, then } \delta(u, w) \leq |ab| + |bw|. \]
\[- \text{ If } B \text{ is empty, then } \delta(u, w) \leq |ua| + |uw|. \]
\[- \text{ If neither } A \text{ nor } B \text{ is empty, then } \delta(u, w) \leq \max\{|ua| + |uw|, |ub| + |bw|\}. \]

We first show that this induction hypothesis implies the theorem: \( |wm| = |uw| \cdot \cos \alpha, |mw| = |uw| \cdot \sin \alpha, |am| = |bm| = |uw| \cdot \cos \alpha \cdot \tan(\theta/2), \) and \( |ua| = |ub| = |uw| \cdot \cos \alpha / \cos(\theta/2) \). Thus the induction hypothesis gives that \( \delta(u, w) \) is at most \( |uw| \cdot ((1 + \sin(\theta/2))/\cos(\theta/2)) \cdot \cos \alpha + \sin \alpha \).

Base case: \( T_{uw} \) has rank 1. Since the triangle is a smallest triangle, \( w \) is the closest vertex to \( u \) in that cone. Hence the edge \( (u, w) \) is part of the constrained \( \theta_{(4k+2)} \)-graph, and \( \delta(u, w) = |uw| \). From the triangle inequality, we have \( |uw| \leq \min\{|ua| + |uw|, |ub| + |bw|\} \), so the induction hypothesis holds.

Induction step: We assume that the induction hypothesis holds for all pairs of vertices that can see each other and have a canonical triangle whose area is smaller than the area of \( T_{uw} \).

If \( (u, w) \) is an edge in the constrained \( \theta_{(4k+2)} \)-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between \( u \) and \( w \), let \( v_0 \) be the vertex closest to \( u \) in the subcone of \( u \) that contains \( w \), and let \( a_0 \) and \( b_0 \) be the upper left and right corner of \( T_{uv_0} \) (see Figure 5). By definition, \( \delta(u, w) \leq |uw_0| + \delta(v_0, w) \), and by the triangle inequality, \( |uw_0| \leq \min\{|ua_0| + |a_0v_0|, |wb_0| + |b_0v_0|\} \).

We assume without loss of generality that \( v_0 \) lies to the left of \( uw \), which means that \( A \) is not empty.

Since \( uw \) and \( uv_0 \) are visibility edges, by applying Lemma 4 to triangle \( v_0uw \), a convex chain \( v_0, \ldots, v_l = w \) of visibility edges

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Fig. 5. A convex chain from \( v_0 \) to \( w \)
connecting \( v_0 \) and \( w \) exists (see Figure 5). Note that, since \( v_0 \) is the closest visible vertex to \( u \), every vertex along the convex chain lies above the horizontal line through \( v_0 \).

We now look at two consecutive vertices \( v_{j-1} \) and \( v_j \) along the convex chain. There are four types of configurations (see Figure 6): (i) \( v_j \in C_{k-1}^{v_{j-1}} \), (ii) \( v_j \in C_{i}^{v_{j-1}} \) where \( 1 \leq i < k \), (iii) \( v_j \in C_{i}^{v_{j-1}} \) and \( v_j \) lies to the right of or has the same \( x \)-coordinate as \( v_{j-1} \), (iv) \( v_j \in C_{0}^{v_{j-1}} \) and \( v_j \) lies to the left of \( v_{j-1} \).

By convexity, the direction of \( v_jv_{j+1} \) is rotating counterclockwise for increasing \( j \). Thus, these configurations occur in the order Type (i), Type (ii), Type (iii), Type (iv) along the convex chain from \( v_0 \) to \( w \). We bound \( \delta(v_{j-1}, v_j) \) as follows:

**Type (i):** If \( v_j \in C_{k}^{v_{j-1}} \), let \( a_j \) and \( b_j \) be the upper and lower left corner of \( T_{v_jv_{j-1}} \) and let \( B_j = v_{j-1}b_jv_j \). Note that since \( v_j \in C_{k}^{v_{j-1}} \), \( a_j \) is also the intersection of the left boundary of \( C_{k}^{v_{j-1}} \) and the horizontal line through \( v_j \). Triangle \( B_j \) lies between the convex chain and \( uw \), so it must be empty. Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_jv_{j-1}} \) has smaller area than \( T_{uw} \), the induction hypothesis gives that \( \delta(v_{j-1}, v_j) \) is at most \( |v_{j-1}a_j| + |a_jv_j| \).

**Fig. 6.** The four types of configurations

**Type (ii):** If \( v_j \in C_{i}^{v_{j-1}} \) where \( 1 \leq i < k \), let \( c \) and \( d \) be the upper and lower right corner of \( T_{v_jv_{j-1}} \). Let \( a_j \) be the intersection of the left boundary of \( C_{i}^{v_{j-1}} \) and the horizontal line through \( v_j \). Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_jv_{j-1}} \) has smaller area than \( T_{uw} \), the induction hypothesis gives that \( \delta(v_{j-1}, v_j) \) is at most \( |v_{j-1}c| + |v_j| + |v_jd| + |duv_j| \). Since \( v_j \in C_{i}^{v_{j-1}} \) where \( 1 \leq i < k \), we can apply Lemma 1 (which states that \( v_j \) and \( a_j \) from Lemma 2 are \( v_{j-1}, v_j \), and \( a_j \)), which gives us that \( \delta(v_{j-1}, v_j) \) is at most \( |v_{j-1}c| + |cv_j| + |v_jd| + |duv_j| \).

**Type (iii):** If \( v_j \in C_{0}^{v_{j-1}} \) and \( v_j \) lies to the right of or has the same \( x \)-coordinate as \( v_{j-1} \), let \( a_j \) and \( b_j \) be the left and right corner of \( T_{v_jv_{j-1}} \) and let \( A_j = v_{j-1}a_jv_j \) and \( B_j = v_{j-1}b_jv_j \). Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_jv_{j-1}} \) has smaller area than \( T_{uw} \), we can apply the induction hypothesis. Regardless of whether \( A_j \) and \( B_j \) are empty or not, \( \delta(v_{j-1}, v_j) \) is at most \( |v_{j-1}a_j| + |a_jv_j| + |v_jb_j| + |b_jv_j| \). Since \( v_j \) lies to the right of or has the same \( x \)-coordinate as \( v_{j-1} \), we know that \( |v_{j-1}a_j| + |a_jv_j| \geq |v_{j-1}b_j| + |b_jv_j| \), so \( \delta(v_{j-1}, v_j) \) is at most \( |v_{j-1}a_j| + |a_jv_j| \).

**Type (iv):** If \( v_j \in C_{0}^{v_{j-1}} \) and \( v_j \) lies to the left of \( v_{j-1} \), let \( a_j \) and \( b_j \) be the left and right corner of \( T_{v_jv_{j-1}} \) and let \( A_j = v_{j-1}a_jv_j \) and \( B_j = v_{j-1}b_jv_j \). Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_jv_{j-1}} \) has smaller area than \( T_{uw} \), we can apply the
induction hypothesis. Thus, if $B_j$ is empty, $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1}a_j| + |a_jv_j|$ and if $B_j$ is not empty, $\delta(v_{j-1}, v_j)$ is at most $|v_{j-1}b_j| + |b_jv_j|$.

To complete the proof, we consider three cases: (a) $\angle awu \leq \pi/2$, (b) $\angle awu > \pi/2$ and $B$ is empty, (c) $\angle awu > \pi/2$ and $B$ is not empty.

Case (a): If $\angle awu \leq \pi/2$, the convex chain cannot contain any Type (iv) configurations; for Type (iv) configurations to occur, $v_j$ needs to lie to the left of $v_{j-1}$. However, by construction, $v_j$ lies on or to the right of the line through $v_{j-1}$ and $w$. Hence, since $\angle awv_{j-1} < \angle awu \leq \pi/2$, $v_j$ lies to the right of or has the same x-coordinate as $v_{j-1}$. We can now bound $\delta(u, w)$ by using these bounds: $\delta(u, w) \leq |uwb| + \sum_{j=1}^l \delta(v_{j-1}, v_j) \leq |uav_0| + |a_0v_0| + \sum_{j=1}^l (|v_{j-1}a_j| + |a_jv_j|) = |ua| + |aw|$.

Case (b): If $\angle awu > \pi/2$ and $B$ is empty, the convex chain can contain Type (iv) configurations. However, since $B$ is empty and the area between the convex chain and $uw$ is empty (by Lemma 1), all $B_j$ are also empty. Using the computed bounds on the lengths of the paths between the points along the convex chain, we can bound $\delta(u, w)$ as in the previous case.

Case (c): If $\angle awu > \pi/2$ and $B$ is not empty, the convex chain can contain Type (iv) configurations and since $B$ is not empty, the triangles $B_j$ need not be empty. Recall that $v_0$ lies in $A$, hence neither $A$ nor $B$ are empty. Therefore, it suffices to prove that $\delta(u, w) \leq \max \{|ua| + |aw|, |ub| + |bw|\} = |ub| + |bw|$. Let $T_{v_jw}$ be the first Type (iv) configuration along the convex chain (if it has any), let $a'$ and $b'$ be the upper left and right corner of $T_{uw}$, and let $b''$ be the upper right corner of $T_{v_jw}$. We now have that $\delta(u, w) \leq |uw_0| + \sum_{j=1}^l \delta(v_{j-1}, v_j) \leq |ua'| + |a'v_j| + |v_jb''| + |b''w| \leq |ub| + |bw|$ (see Figure 7).

Since $((1 + \sin(\theta/2))/\cos(\theta/2)) \cdot \cos \alpha + \sin \alpha$ is increasing for $\alpha \in [0, \theta/2]$, for $\theta \leq \pi/3$, it is maximized when $\alpha = \theta/2$, and we obtain the following corollary:

**Corollary 1.** The constrained $\theta_{(4k+2)}$-graph is a $(1 + 2 \cdot \sin(\theta/2))$-spanner of $Vis(P, S)$. 

![Fig. 7. Visualization of the paths (thick lines) in the inequalities of case (c)](image-url)
5 Generic Framework for the Spanning Proof

Next, we modify the spanning proof from the previous section and provide a generic framework for the spanning proof for the other three families of $\theta$-graphs. After providing this framework, we fill in the blanks for the individual families.

**Theorem 2.** Let $u$ and $w$ be two vertices in the plane such that $u$ can see $w$. Let $m$ be the midpoint of the side of $T_{uw}$ opposing $u$ and let $\alpha$ be the unsigned angle between $uw$ and $um$. There exists a path connecting $u$ and $w$ in the constrained $\theta_{(4k+x)}$-graph of length at most

$$
\left(\frac{\cos \alpha}{\cos \left(\frac{\theta}{2}\right)} + \left(\cos \alpha \cdot \tan \left(\frac{\theta}{2}\right) + \sin \alpha\right) \cdot c\right) \cdot |uw|,
$$

where $c \geq 1$ is a constant that depends on $x \in \{3, 4, 5\}$. For the constrained $\theta_{(4k+4)}$-graph, $c$ equals $1/(\cos(\theta/2) - \sin(\theta/2))$ and for the constrained $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph, $c$ equals $\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4))$.

**Proof.** We prove the theorem by induction on the area of $T_{uw}$. Formally, we perform induction on the rank, when ordered by area, of the triangles $T_{xy}$ for all pairs of vertices $x$ and $y$ that can see each other. We assume without loss of generality that $w \in C_0^u$. Let $a$ and $b$ be the upper left and right corner of $T_{uw}$ (see Figure 5).

Our inductive hypothesis is the following, where $\delta(u, w)$ denotes the length of the shortest path from $u$ to $w$ in the constrained $\theta_{(4k+x)}$-graph: $\delta(u, w) \leq \max\{|ua| + |aw| \cdot c, |ub| + |bw| \cdot c\}$.

We first show that this induction hypothesis implies the theorem. Basic trigonometry gives us the following equalities: $|um| = |uw| \cdot \cos \alpha$, $|mw| = |uw| \cdot \sin \alpha$, $|am| = |bm| = |uw| \cdot \cos \alpha \cdot \tan(\theta/2)$, and $|ua| = |ub| = |uw| \cdot \cos \alpha / \cos(\theta/2)$. Thus the induction hypothesis gives that $\delta(u, w)$ is at most $|uw| \cdot (\cos \alpha / \cos(\theta/2)) + (\cos \alpha \cdot \tan(\theta/2) + \sin \alpha \cdot c)$.

**Base case:** $T_{uw}$ has rank 1. Since the triangle is a smallest triangle, $w$ is the closest vertex to $u$ in that cone. Hence the edge $(u, w)$ is part of the constrained $\theta_{(4k+x)}$-graph, and $\delta(u, w) = |uw|$. From the triangle inequality and the fact that $c \geq 1$, we have $|uw| \leq \min\{|ua| + |aw| \cdot c, |ub| + |bw| \cdot c\}$, so the induction hypothesis holds.

**Induction step:** We assume that the induction hypothesis holds for all pairs of vertices that can see each other and have a canonical triangle whose area is smaller than the area of $T_{uw}$

If $(u, w)$ is an edge in the constrained $\theta_{(4k+x)}$-graph, the induction hypothesis follows by the same argument as in the base case. If there is no edge between $u$ and $w$, let $v_0$ be the vertex closest to $u$ in the subcone of $u$ that contains $w$, and let $a_0$ and $b_0$ be the upper left and right corner of $T_{uv_0}$ (see Figure 5). By definition, $\delta(u, w) \leq |uv_0| + \delta(v_0, w)$, and by the triangle inequality, $|uv_0| \leq \min\{|um| + |a_0v_0|, |ub| + |b_0v_0|\}$. We assume without loss of generality that $v_0$ lies to the left of $uw$. 
Since \( uv \) and \( uv_0 \) are visibility edges, by applying Lemma 1 to triangle \( v_0uw \), a convex chain \( v_0, ..., v_i = w \) of visibility edges connecting \( v_0 \) and \( w \) exists (see Figure 3). Note that, since \( v_0 \) is the closest visible vertex to \( u \), every vertex along the convex chain lies above the horizontal line through \( v_0 \).

We now look at two consecutive vertices \( v_{j-1} \) and \( v_j \) along the convex chain. When \( v_j \not\in C_0^{i-1} \), let \( c \) and \( d \) be the upper and lower right corner of \( T_{v_{j-1},v_j} \). We distinguish four types of configurations: (i) \( v_j \in C_0^{i-1} \) where \( i > k \), or \( i = k \) and \( |cw| > |dw| \), (ii) \( v_j \in C_0^{i-1} \) where \( 1 \leq i \leq k - 1 \), or \( i = k \) and \( |cw| \leq |dw| \), (iii) \( v_j \in C_0^{i-1} \) and \( v_j \) lies to the right of or has the same \( x \)-coordinate as \( v_{j-1} \), (iv) \( v_j \in C_0^{i-1} \) and \( v_j \) lies to the left of \( v_{j-1} \). By convexity, the direction of \( v_jv_{j+1} \) is rotating counterclockwise for increasing \( j \). Thus, these configurations occur in the order Type (i), Type (ii), Type (iii), Type (iv) along the convex chain from \( v_0 \) to \( w \). We bound \( \delta(v_{j-1}, v_j) \) as follows:

**Type (i):** \( v_j \in C_0^{i-1} \) where \( i > k \), or \( i = k \) and \( |cw| > |dw| \). Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_{j-1},v_j} \) has smaller area than \( T_{uw} \), the induction hypothesis gives that \( \delta(v_{j-1}, v_j) \) is at most \( \max \{ |v_{j-1}c| + |cv_j| \cdot c, |v_{j-1}d| + |dv_j| \cdot c \} \).

Let \( a_j \) be the intersection of the left boundary of \( C_0^{i-1} \) and the horizontal line through \( v_j \). We aim to show that \( \max \{ |v_{j-1}c| + |cv_j| \cdot c, |v_{j-1}d| + |dv_j| \cdot c \} \leq |v_{j-1}a_j| + |a_jv_j| \cdot c \). We use Lemma 3 to do this. However, since the precise application of this lemma depends on the family of \( \theta \)-graphs and determines the value of \( c \), this case is discussed in the spanning proofs of the three families.

**Type (ii):** \( v_j \in C_0^{i-1} \) where \( 1 \leq i \leq k - 1 \), or \( i = k \) and \( |cw| \leq |dw| \). Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_{j-1},v_j} \) has smaller area than \( T_{uw} \), the induction hypothesis gives that \( \delta(v_{j-1}, v_j) \) is at most \( \max \{ |v_{j-1}c| + |cv_j| \cdot c, |v_{j-1}d| + |dv_j| \cdot c \} \).

Let \( a_j \) be the intersection of the left boundary of \( C_0^{i-1} \) and the horizontal line through \( v_j \). Since \( v_j \in C_0^{i-1} \) where \( 1 \leq i \leq k - 1 \), or \( i = k \) and \( |cw| \leq |dw| \), we can apply Lemma 2 in this case (where \( v, w, \) and \( a \) from Lemma 2 are \( v_{j-1}, v_j, \) and \( a_j \)) and we get that \( \max \{ |v_{j-1}c| + |cv_j|, |v_{j-1}d| + |dv_j| \} \leq |v_{j-1}a_j| + |a_jv_j| \) and \( \max \{ |cv_j|, |dv_j| \} \leq |a_jv_j| \). Since \( c \geq 1 \), this implies that \( \max \{ |v_{j-1}c| + |cv_j|, c, |v_{j-1}d| + |dv_j| \cdot c \} \leq |v_{j-1}a_j| + |a_jv_j| \cdot c \).

**Type (iii):** If \( v_j \in C_0^{i-1} \) and \( v_j \) lies to the right of \( a \) but has the same \( x \)-coordinate as \( v_{j-1} \), let \( a_j \) and \( b_j \) be the left and right corner of \( T_{v_{j-1},v_j} \). Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_{j-1},v_j} \) has smaller area than \( T_{uw} \), we can apply the induction hypothesis. Thus, since \( v_j \) lies to the right of \( a \) but has the same \( x \)-coordinate as \( v_{j-1} \), \( \delta(v_{j-1}, v_j) \) is at most \( |v_{j-1}a_j| + |a_jv_j| \cdot c \).

**Type (iv):** If \( v_j \in C_0^{i-1} \) and \( v_j \) lies to the left of \( v_{j-1} \), let \( a_j \) and \( b_j \) be the left and right corner of \( T_{v_{j-1},v_j} \). Since \( v_j \) can see \( v_{j-1} \) and \( T_{v_{j-1},v_j} \) has smaller area than \( T_{uw} \), we can apply the induction hypothesis. Thus, since \( v_j \) lies to the left of \( v_{j-1} \), \( \delta(v_{j-1}, v_j) \) is at most \( |v_{j-1}b_j| + |b_jv_j| \cdot c \).

To complete the proof, we consider two cases: (a) \( \angle awu \leq \frac{\pi}{2} \), (b) \( \angle awu > \frac{\pi}{2} \).

**Case (a):** We need to prove that \( \delta(u, w) \leq \max \{ |ua| + |aw|, |ub| + |bw| \} = |ua| + |aw| \). We first show that the convex chain cannot contain any Type (iv) configurations: for Type (iv) configurations to occur, \( v_j \) needs to lie to the left of \( v_{j-1} \). However, by construction, \( v_j \) lies on or to the right of the line through \( v_{j-1} \) and \( w \). Hence, since \( \angle awv_{j-1} < \angle awu \leq \frac{\pi}{2} \), \( v_j \) lies to the right of \( v_{j-1} \). We can
now bound \( \delta(u, w) \) by using these bounds: \( \delta(u, w) \leq |uw_0| + \sum_{j=1}^{l} \delta(v_{j-1}, v_{j}) \leq |ua_0| + |a_0v_0| + \sum_{j=1}^{l} (|v_{j-1}a_j| + |a_jv_j| \cdot c) \leq |ua| + |aw| \cdot c.

**Case (b):** If \( \angle uw > \pi/2 \), the convex chain can contain Type (iv) configurations. We need to prove that \( \delta(u, w) \leq \max\{|ua| + |aw|, |ub| + |bw|\} \). Let \( T_{v_j, v_{j+1}} \) be the first Type (iv) configuration along the convex chain (if it has any), let \( a' \) and \( b' \) be the upper left and right corner of \( T_{uv_{j-1}} \), and let \( b'' \) be the upper right corner of \( T_{uv_{j}} \). We now have that \( \delta(u, w) \leq |uw_0| + \sum_{j=1}^{l} \delta(v_{j-1}, v_{j}) \leq |ua'| + |a'v_j| \cdot c + |v_jb''| + |b''w| \cdot c \leq |ub| + |bw| \cdot c \) (see Figure 7). \( \square \)

6 The Constrained \( \theta_{(4k+4)} \)-Graph

In this section we complete the proof of Theorem 2 for the constrained \( \theta_{(4k+4)} \)-graph.

**Theorem 3.** Let \( u \) and \( w \) be two vertices in the plane such that \( u \) can see \( w \). Let \( m \) be the midpoint of the side of \( T_{uw} \) opposite \( u \) and let \( \alpha \) be the unsigned angle between \( uw \) and \( um \). There exists a path connecting \( u \) and \( w \) in the constrained \( \theta_{(4k+4)} \)-graph of length at most

\[
\left( \frac{\cos \alpha}{\cos \left( \frac{\theta}{2} \right)} + \frac{\cos \alpha \cdot tan \left( \frac{\theta}{2} \right) + \sin \alpha}{\cos \left( \frac{\theta}{2} \right) - \sin \left( \frac{\theta}{2} \right)} \right) \cdot |uw|.
\]

**Proof.** We apply Theorem 2 using \( c = 1/(\cos(\theta/2) - \sin(\theta/2)) \). The assumptions made in Theorem 2 still apply. It remains to show that for the Type (i) configurations, we have that \( \max\{|v_{j-1}c| + |cv_j| \cdot c, |v_{j-1}d| + |dv_j| \cdot c, |v_{j-1}a_j| + |a_jv_j| \cdot c \} \leq |v_{j-1}a_j| + |a_jv_j| \cdot c \), where \( c \) and \( d \) are the upper and lower right corner of \( T_{v_{j-1}, v_j} \) and \( a_j \) is the intersection of the left boundary of \( C_{v_{j-1}}^{v_j} \) and the horizontal line through \( v_j \).

We distinguish two cases: (a) \( v_j \in C_{k}^{v_{j-1}} \) and \( |cw| > |dw| \), (b) \( v_j \in C_{k+1}^{v_{j-1}} \). Let \( \beta \) be \( \angle a_jv_{j-1}v_j \) and let \( \gamma \) be the angle between \( v_jv_{j-1} \) and the bisection of \( T_{v_{j-1}, v_j} \).

**Case (a):** When \( v_j \in C_{k}^{v_{j-1}} \) and \( |cw| > |dw| \), the induction hypothesis for \( T_{v_{j-1}, v_j} \) gives \( \delta(v_{j-1}, v_j) \leq |v_{j-1}c| + |cv_j| \cdot c \). We note that \( \gamma = \theta - \beta \). Hence Lemma 8 gives that the inequality holds when \( c \geq (\cos(\theta - \beta) - \sin(\beta)/\cos(\theta/2 - \beta) - \sin(\theta/2 - \beta)) \). As this function is decreasing in \( \beta \) for \( \theta/2 \leq \beta \leq \theta \), it is maximized when \( \beta \) equals \( \theta/2 \). Hence \( \gamma \) needs to be at least \( (\cos(\theta/2) - \sin(\theta/2))/(1 - \sin(\theta/2)) \), which can be rewritten to \( 1/(\cos(\theta/2) - \sin(\theta/2)) \).

**Case (b):** When \( v_j \in C_{k+1}^{v_{j-1}} \), \( v_j \) lies above the bisection of \( T_{v_{j-1}, v_j} \) and the induction hypothesis for \( T_{v_{j-1}, v_j} \) gives \( \delta(v_{j-1}, v_j) \leq |v_{j-1}d| + |dv_{j-1}| \cdot c \). We note that \( \gamma = \beta \). Hence Lemma 8 gives that the inequality holds when \( c \geq (\cos(\theta - \beta) - \sin(\beta)/\cos(\theta/2 - \beta) - \sin(\theta/2 + \beta)) \). As this function is decreasing in \( \beta \) for \( 0 \leq \beta \leq \theta/2 \), it is maximized when \( \beta = 0 \). Hence \( c \) needs to be at least \( 1/(\cos(\theta/2) - \sin(\theta/2)) \). \( \square \)

Since \( \cos \alpha / \cos(\theta/2) + (\cos \alpha \cdot \tan(\theta/2) + \sin \alpha)/(\cos(\theta/2) - \sin(\theta/2)) \) is increasing for \( \alpha \in [0, \theta/2] \), for \( \theta \leq \pi/4 \), it is maximized when \( \alpha = \theta/2 \), and we obtain the following corollary:
Corollary 2. The constrained $\theta_{(4k+4)}$-graph is a \((1 + \frac{2\sin(\frac{\theta}{2})}{\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})})\)-spanner of Vis\((P, S)\).

7 The Constrained $\theta_{(4k+3)}$-Graph and $\theta_{(4k+5)}$-Graph

In this section we complete the proof of Theorem 2 for the constrained $\theta_{(4k+3)}$-graph and $\theta_{(4k+5)}$-graph.

Theorem 4. Let $u$ and $w$ be two vertices in the plane such that $u$ can see $w$. Let $m$ be the midpoint of the side of $T_{uw}$ opposite $u$ and let $\alpha$ be the unsigned angle between $uw$ and $um$. There exists a path connecting $u$ and $w$ in the constrained $\theta_{(4k+3)}$-graph of length at most

\[
\left(\cos(\frac{\theta}{2}) + \frac{\cos(\alpha \cdot \tan(\frac{\theta}{2})) + \sin(\alpha)}{\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})}\right) \cdot |uw|.
\]

Proof. We apply Theorem 2 using $\beta = \cos(\frac{\theta}{4})/(\cos(\frac{\theta}{2}) - \sin(\frac{3\theta}{4}))$. The assumptions made in Theorem 2 still apply. It remains to show that for the Type (i) configurations, we have that \(\max\{|v_j-1| + |c|, |v_j-1|d| + |dv_j| \cdot c \leq |v_j-1a_j| + |a_jv_j - c|\), where $c$ and $d$ are the upper and lower right corner of $T_{v_j-1v_j}$ and $a_j$ is the intersection of the left boundary of $C_k^{v_j-1}$ and the horizontal line through $v_j$.

We distinguish two cases: (a) $v_j \in C_k^{v_j-1}$ and $|uw| > |dv_j|$, (b) $v_j \in C_k^{v_j-1}$. Let $\beta$ be $\angle a_jv_jv_j-1$ and let $\gamma$ be the angle between $v_jv_j-1$ and the bisector of $T_{v_j-1v_j}$.

Case (a): When $v_j \in C_k^{v_j-1}$ and $|uw| > |dv_j|$, the induction hypothesis for $T_{v_j-1v_j}$ gives $\delta(v_j-1, v_j) \leq |v_j-1|c| + |cv_j| \cdot c$. We note that $\gamma = 3\theta/4 - \beta$. Hence Lemma 3 gives that the inequality holds when $c \geq (\cos(3\theta/4 - \beta) - \sin(\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 - \beta)))$. As this function is decreasing in $\beta$ for $\theta/4 \leq \beta \leq 3\theta/4$, it is maximized when $\beta$ equals $\theta/4$. Hence $c$ needs to be at least $(\cos(\theta/2) - \sin(\theta/4))/(\cos(\theta/2) - \sin(3\theta/4))$, which is equal to $(\cos(\theta/2) - \sin(\theta/4))/(\cos(\theta/2) - \sin(3\theta/4))$.

Case (b): When $v_j \in C_k^{v_j-1}$, $v_j$ lies above the bisector of $T_{v_j-1v_j}$ and the induction hypothesis for $T_{v_j-1v_j}$ gives $\delta(v_j-1, v_j) \leq |v_j|d| + |dv_j| \cdot c$. We note that $\gamma = \theta/4 + \beta$. Hence Lemma 3 gives that the inequality holds when $c \geq (\cos(\theta/4 + \beta) - \sin(\beta))/(\cos(\theta/2 - \beta) - \sin(3\theta/4 + \beta))$, which is equal to $(\cos(\theta/2) - \sin(\theta/4))/(\cos(\theta/2) - \sin(3\theta/4))$. \qed

Theorem 5. Let $u$ and $w$ be two vertices in the plane such that $u$ can see $w$. Let $m$ be the midpoint of the side of $T_{uw}$ opposite $u$ and let $\alpha$ be the unsigned angle between $uw$ and $um$. There exists a path connecting $u$ and $w$ in the constrained $\theta_{(4k+5)}$-graph of length at most

\[
\left(\cos(\frac{\theta}{2}) + \frac{\cos(\alpha \cdot \tan(\frac{\theta}{2})) + \sin(\alpha)}{\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})}\right) \cdot |uw|.
\]
We apply Theorem 2 using Proof. 12 Prosenjit Bose and André van Renssen

The spanning ratio of the constrained \( \alpha \)-graph is equal, i.e. when \( \alpha \) for \( \beta \sin c \) gives the inequality holds when \( \beta \). Hence Lemma 3 gives that the inequality holds when

\[
\frac{\sin(3\theta/4)}{(\cos(\theta/4) - \sin\theta)} \leq \theta/4.
\]

\( \beta \) is the intersection of the left boundary of \( C_{k+1}^{v_j} \) and the horizontal line through \( v_j \).

We distinguish two cases: (a) \( v_j \in C_{k}^{v_j} \) and \( |cw| > |dv| \), (b) \( v_j \in C_{k+1}^{v_j} \). Let \( \beta \) be \( \angle a_j v_j v_{j-1} \) and let \( \gamma \) be the angle between \( v_j v_{j-1} \) and the bisector of \( T_{v_{j-1} v_j} \).

**Case (a):** When \( v_j \in C_{k}^{v_j} \) and \( |cw| > |dv| \), the induction hypothesis for \( T_{v_{j-1} v_j} \) gives \( \delta(v_{j-1}, v_j) \leq |v_{j-1} c| + |c v_j| \cdot c \). We note that \( \gamma = 5\theta/4 - \beta \). Hence Lemma 3 gives that the inequality holds when \( c \geq (\cos(\theta/4 - \beta) - \sin\beta)/(\cos(\theta/4 - \beta) - \sin(5\theta/4 - \beta)) \). As this function is decreasing in \( \beta \) for \( 30\theta/4 \leq \beta \leq 5\theta/4 \), it is maximized when \( \beta \) equals \( 30\theta/4 \). Hence \( c \) needs to be at least \( (\cos(\theta/2) - \sin(3\theta/4))/(\cos(\theta/4) - \sin(3\theta/4)) \).

**Case (b):** When \( v_j \in C_{k+1}^{v_j} \), the induction hypothesis for \( T_{uw} \) gives

\[
\delta(v_{j-1}, v_j) \leq \max\{ |v_{j-1} c| + |c v_j| \cdot c, |v_{j-1} d| + |d v_j| \cdot c \}. \]

If \( \delta(v_{j-1}, v_j) \leq |v_{j-1} c| + |c v_j| \cdot c \), we note that \( \gamma = \theta/4 + \beta \). Hence Lemma 3 gives that the inequality holds when \( c \geq (\cos(\beta - \theta/4) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 - \beta)) \). As this function is decreasing in \( \beta \) for \( 0 \leq \beta \leq \theta/4 \), it is maximized when \( \beta \) equals \( 0 \). Hence \( c \) needs to be at least \( \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4)) \).

If \( \delta(v_{j-1}, v_j) \leq |v_{j-1} d| + |d v_j| \cdot c \), we note that \( \gamma = \theta/4 + \beta \). Hence Lemma 3 gives that the inequality holds when \( c \geq (\cos(\beta - \theta/4) - \sin\beta)/(\cos(\theta/2 - \beta) - \sin(3\theta/4 - \beta)) \), which is equal to \( \cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4)) \). \( \square \)

When looking at two vertices \( u \) and \( w \) in the constrained \( \theta_{(4k+3)} \)-graph and \( \theta_{(4k+5)} \)-graph, we notice that when the angle between \( uw \) and the bisector of \( T_{uw} \) is \( \alpha \), the angle between \( uw \) and the bisector of \( T_{uw} \) is \( \theta/2 - \alpha \). Hence the worst case spanning ratio becomes the minimum of the spanning ratio when looking at \( T_{uw} \) and the spanning ratio when looking at \( T_{uw} \).

**Theorem 6.** The constrained \( \theta_{(4k+3)} \)-graph and \( \theta_{(4k+5)} \)-graph are

\[
\cos(\theta/4)/(\cos(\theta/2) - \sin(3\theta/4)) - \text{spanners of } Vis(P, S).
\]

**Proof.** The spanning ratio of the constrained \( \theta_{(4k+3)} \)-graph and \( \theta_{(4k+5)} \)-graph is at most:

\[
\min \left\{ \frac{\cos(\alpha/2)}{\cos(\theta/4 - \sin(3\theta/4))}, \frac{\cos(\alpha/2) + \cos(\alpha/2) \tan(\theta/2) + \sin(\theta/2)}{\cos(\theta/4) - \sin(3\theta/4)} \right\}
\]

Since \( \cos(\alpha/2)/(\cos(\theta/2) + \cos(\alpha/2) \tan(\theta/2) + \sin(\theta/2)) \cdot e \) is increasing for \( \alpha \in [0, \theta/2) \), for \( \theta \leq 2\pi/7 \), the minimum of these two functions is maximized when the two functions are equal, i.e. when \( \alpha = \theta/4 \). Thus the constrained \( \theta_{(4k+3)} \)-graph and
\[ \theta_{(4k+5)} \text{-graph has spanning ratio at most:} \]

\[
\frac{\cos \left( \theta \frac{2}{4} \right)}{\cos \left( \frac{\theta}{2} \right)} + \frac{\cos \left( \theta \frac{2}{4} \right) \cdot \tan \left( \frac{\theta}{2} \right) + \sin \left( \theta \frac{2}{4} \right) \cdot \cos \left( \theta \frac{2}{4} \right)}{\cos \left( \frac{\theta}{2} \right) - \sin \left( \theta \frac{2}{4} \right)} = \frac{\cos \left( \theta \frac{2}{4} \right) \cdot \cos \left( \theta \frac{2}{4} \right)}{\cos \left( \frac{\theta}{2} \right) \cdot \left( \cos \left( \frac{\theta}{2} \right) - \sin \left( \theta \frac{2}{4} \right) \right)}
\]

\[
\square
\]

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