Environment-induced quantum coherence spreading of a qubit

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We make a thorough study of the spreading of quantum coherence (QC), as quantified by the $l_1$-norm QC, when a qubit (a two-level quantum system) is subjected to noise quantum channels commonly appearing in quantum information science. We notice that QC is generally not conserved and that even incoherent initial states can lead to transitory system-environment QC. We show that for the amplitude damping channel the evolved total QC can be written as the sum of local and non-local parts, with the last one being equal to entanglement. On the other hand, for the phase damping channel (PDC) entanglement does not account for all non-local QC, with the gap between them depending on time and also on the qubit’s initial state. Besides these issues, the possibility and conditions for time invariance of QC are regarded in the case of bit, phase, and bit-phase flip channels. Here we reveal the qualitative dynamical inequivalence between these channels and the PDC and show that while the PDC requires initial coherence of the qubit, for some other channels non-zero population of the excited state (i.e., energy) is sufficient. Related to that, considering the depolarizing channel we notice the qubit’s ability to act as a catalyst for the creation of joint QC and entanglement, without need for nonzero initial QC or excited state population.

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I. INTRODUCTION

Although the main conceptual ideas related to quantum coherence (QC) have been present in the literature for decades \cite{1–4}, its formal quantification from a resource theory (RT) point of view have been addressed only very recently \cite{5–17}. Free states (FS) and free operations (FO) are two basic ingredients in a RT \cite{18}, with the resource states allowing us to overcome the limitations imposed by the FO. Any function (leading quantum states into non-negative real numbers) satisfying some basic conditions and not increasing under the FO may be regarded as a monotone measure for the resource under consideration. For instance, in the RT of entanglement \cite{19}, the FS are the separable ones while LOCC (local quantum operations and classical communication) are the FO. Concurrence \cite{20} and negativity \cite{21} are two famous entanglement monotones; and we shall present their definitions and use them in the next section.

In this article, we apply some developments achieved in the RT of quantum coherence \cite{5}. In this RT, the FS, the incoherent states, are those density matrices which are diagonal in some reference basis $|j\rangle$, i.e.,

$$\tau = \sum_{j=1}^{d} \iota_j |j\rangle\langle j|,$$

(1)

with $\iota_j$ being a probability distribution and $d = \text{dim } \mathcal{H}$ is the dimension of the system’s Hilbert space. We shall use $\mathcal{I}$ to denote the set of all density matrices of the type (1). The motivation to choose a particular reference basis usually originates from the characteristic physical properties of the system under analysis. For example, in transport phenomena in quantum biology, the eigenvectors of the system’s hamiltonian form a natural choice for the reference basis \cite{22}. In this article, $|j\rangle$, with $j = 0, 1, \cdots, d - 1$, is always assumed to be the standard-computational basis in the regarded Hilbert state.

Arguably, the most involved part of a RT is the identification of its FO. While it is clear that the FO must not create resource states from FS and should not increase the value of a resource quantifier in the general case, which additional constraints one should add to the FO is a subtle matter. The RT of coherence is a typical example of this situation, where there exists several different levels of restrictions one can impose on the FO \cite{15, 16}. In this work,

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we will use the $l_1$-norm quantum coherence, which is defined and given by \[ C(\rho) := \min_{\iota \in \mathbb{I}} ||\rho - \iota||_{l_1} = \sum_{j \neq k} |\langle j | \rho | k \rangle|, \] (2)

where the $l_1$-norm of an operator $M$ is $||M||_{l_1} := \sum_{j \neq k} |\langle j | M | k \rangle|$. This coherence monotone satisfies most of the required properties. For example, it is is zero if and only if $\rho$ is an incoherent state and it does not increase under mixing or by (maximally) incoherent operations. For more details, see Ref. \[5\].

Recently, several articles have investigated the relation between coherence (or other quantumnesses) and entanglement (or other quantum correlations) and their inter-conversions via (incoherent) unitary operations \[23-32\]. Here, we address a somewhat related question, but we regard the natural dynamics generated due to the interaction of a two-level quantum system (a qubit) with the environment. Besides its practical relevance regarding the preservation of QC and its conversion to quantum correlations, this kind of analysis is important e.g. to better understanding the transition to classicality. This process is normally accompanied by the loss of the system’s QC, which is induced by the spontaneous creation of correlations between it and its surroundings. It is worthwhile mentioning that similar issues have been investigated previously, but mainly with regard to decorrelating dynamics for composite systems interacting with local or global environments, or in the thermodynamical, non-Markovianity, and sensing contexts \[33-47\]. The goal of the present contribution is to provide a thorough description of the dynamical flow of a quantum coherence monotone, the $l_1$-norm QC, during the time evolution of a qubit interacting with environments modeled by quantum channels important for quantum information science \[48-50\].

The reminder of this article is organized as follows. In Sec. II, we start describing the initial conditions that shall be considered in all subsequent calculations. Then we mention briefly the Kraus’ operator-sum representation and its corresponding unitary mapping. Afterwards, we apply this formalism to study the $l_1$-norm quantum coherence flow for a quantum bit evolving under the action of environments modeled by the amplitude damping (Sec. II A), phase damping (Sec. II B), bit flip (Sec. II C), phase flip (Sec. II D), bit-phase flip (Sec. II E), and depolarizing (Sec. II F) channels. Some final remarks about our findings are included in Sec. III.

II. DYNAMICAL FLOW OF QUANTUM COHERENCE FOR QUANTUM CHANNELS

Throughout this article, we consider a qubit prepared initially in a generic quantum state \[48, 49\]:

$$\rho^S = 2^{-1} (\sigma_0 + \vec{r} \cdot \vec{\sigma}) . \tag{3}$$

In the last equation, $\sigma_0$ is the identity matrix and the components of the Bloch vector $\vec{r} = (r_1, r_2, r_3)$ are the system’s polarizations: $r_j = \text{Tr}(\rho^S \sigma_j)$, with $\sigma_j$ being the Pauli matrices. It is straightforward seeing that the $l_1$-norm quantum coherence of a qubit-like system represented by a Bloch vector $\vec{r}$ (as in Eq. (3)) is given by:

$$C(\rho^S) = \sqrt{r_1^2 + r_2^2} . \tag{4}$$

The positivity of the density operator requires $\sum_{j=1}^{3} r_j^2 \leq 1$, which leads to $C(\rho^S) \in [0, 1]$. For a $d$-level system, a qudit, we have $C(\rho^S) \in [0, d-1]$ with the maximally coherent state being $|\psi_{mc}\rangle = \sum_{j=0}^{d-1} |j\rangle/\sqrt{d}$.

After being prepared in the state (3), the qubit is let to interact with the surroundings. We assume that initially the two are uncorrelated, that the environment is in the vacuum state $|E_0\rangle$, and that it is described by one of the quantum channels regarded in the next sub-sections. Qubit plus environment form an isolated system which evolves unitarily: $\rho_t = U_t(\rho^S \otimes |E_0\rangle \langle E_0|)U^\dagger_t$. One can show that, if the matrix elements of the Kraus operators $K_j(t)$ are $\langle S_l | K_j(t) | S_m \rangle = \langle S_l | \otimes \langle E_j |)U_t(|S_m \rangle \otimes |E_0\rangle)$, the system’s evolved state can be cast in the form \[51\]:

$$\rho^S_t = \text{Tr}_E(\rho_t) = \sum_{j} K_j(t) \rho^S K_j^\dagger(t) , \tag{5}$$

with $\text{Tr}_E(\cdot)$ being the partial trace function \[52\].

If, as is most frequently the case, the Kraus’ operators describing the action of a noise channel are known from phenomenological or quantum process tomography means \[53, 54\], we can model the system-environment dynamics via the following isometric map \[49\]:

$$U_t |S_l\rangle \otimes |E_0\rangle = \sum_{j} (K_j(t) |S_l\rangle) \otimes |E_j\rangle , \tag{6}$$

which also leads to the dynamics in Eq. (5). It is worthwhile mentioning that the set of Kraus’ operators inducing a certain evolution on the system state is not unique. In the context we are interested in this article, $\{K_i(t)\}$ and
\{K_j'(t) = \sum_i V_{ji} K_i(t)\} lead to the same \(\rho_j^S\) if \(V\) is a unitary transformation applied to the environment after its interaction with the system ceased [51]. As our main goal is to study the quantum coherence spreading during and because of the system-environment interaction, in the sequence we shall not regard this issue anymore.

We observe that \(|E_k\rangle\) are used to denote distinguishable states of the environment on its \(k\)-excitations subspaces. Thus, when we talk about the coherence of the environment and about the system-environment coherence, we might be referring to subspaces, instead of simply to some of the environment’s bases states. In addition to that, it is important stressing that although this formalism (Eq. (6)) does not utilize details about the environment’s “structure”, it is mathematically and physically correct and has been applied in numerous previous works [33–35, 38, 39, 55–59]. In the sequence we will apply it considering some relevant quantum channels.

A. Amplitude damping channel

In this sub-section, we study a qubit embodied in a two-level atom-like system[77] with ground state \(|0\rangle\) and excited state \(|1\rangle\). In the last equations and in the rest of this article, we use the notation:

\[ |S_j\rangle \otimes |E_k\rangle = |jk\rangle. \]  

(7)

The environment is the electromagnetic field, which is initially in the vacuum state (with no excitations). The amplitude damping channel (ADC), represented by the Kraus’ operators \(K_0^{ad}(p) = |0\rangle\langle 0| + \sqrt{q}|1\rangle\langle 1|\) and \(K_1^{ad}(p) = \sqrt{p}|0\rangle\langle 1|\), is a phenomenological-approximate representation of the spontaneous emission process, that will take place because of the interaction of the atomic system with the vacuum fluctuations [60]. Above we defined \(q := 1 - p\), with \(p \in [0, 1]\) being the probability for the atom to emit a photon, also dubbed parametrized time. For instance, in liquid state nuclear magnetic resonance (NMR), the explicit probability-time relation is: \(p = 1 - \exp(-t/T_1)\), with \(T_1\) being the so called longitudinal relaxation time [61]. The unitary representation of the ADC is [50]: \(U_{ad}(p)|0\rangle = |0\rangle\) and \(U_{ad}(p)|1\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle\). From this unitary map, we obtain the system-environment evolved state

\[ \rho_p = 2^{-1}[(1 + r_3)|00\rangle\langle 00| + (1 - r_3)(p|01\rangle\langle 01| + q|10\rangle\langle 10|) + (1 - r_3)\sqrt{pq}|01\rangle\langle 01| + (r_1 - ir_2)(\sqrt{p}|00\rangle\langle 01| + \sqrt{q}|01\rangle\langle 00|) + c.t.]. \]  

(8)

Above, and hereafter, we use c.t. to denote the conjugate transpose (adjoint) of the preceding term(s) in the parenthesis, bracket or line (which one should be clear from the context). By using the partial trace [52], we can write the reduced states of the system and environment as in Eq. (3), but with the respective Bloch vector being:

\[ \bar{r}(p) = (r_1 \sqrt{q}, r_2 \sqrt{q}, r_3 q + p), \]  

(9)

\[ \tilde{R}(p) = (r_1 \sqrt{p}, r_2 \sqrt{p}, r_3 p + q). \]  

(10)

So the system-environment, system, and environment quantum coherences at time \(p\) are given, respectively, by:

\[ C(\rho_p) = (\sqrt{q} + \sqrt{p})C(\rho^S) + \sqrt{pq}(1 - r_3), \]  

(11)

\[ C(\rho_p^S) = \sqrt{q}C(\rho^S), \]  

(12)

\[ C(\rho_p^E) = \sqrt{p}C(\rho^S). \]  

(13)

Throughout this article, when computing the total coherence, we use as reference basis the tensor product of local computational bases: \(|S_j\rangle \otimes |E_k\rangle \equiv |jk\rangle\), with \(j = 0, 1\) and \(k = 0, \cdots, d_E - 1\). We notice that the effective dimension of the environment, \(d_E\), is equal to the number of Kraus’ operators in a given representation of a quantum channel.

The expressions in Eqs. (11), (12), and (13) show nicely the splitting of the total coherence into its local,

\[ C_l(\rho_p) := C(\rho_p^S) + C(\rho_p^E), \]  

(14)

and nonlocal,

\[ C_{nl}(\rho_p) := C(\rho_p) - C_l(\rho_p), \]  

(15)

parts[78]. We also see that QC is not conserved in general; note for instance that \(C_l(\rho_p) = (\sqrt{q} + \sqrt{p})C(\rho^S) \geq C(\rho^S)\). Besides, even if \(C(\rho^S) = 0\), any initial state of the qubit with non-null population of the excited state \(|1\rangle\) (i.e., \(r_3 \neq 1\)) shall lead to transitory system-environment non-local QC, which will disappear only at the asymptotic time \(p = 1\).
obtain of the qubit. We also notice that, as effectively as a three-level quantum system (a qutrit). In the case of liquid state NMR, the environment can not exchange energy with the environment. However, each one of these states leaves a unique “fingerprint” in the environment, causing it to conditionally jump to a different configuration. Besides, here the environment is modeled effectively as a three-level quantum system (a qutrit). In the case of liquid state NMR, the environment can be characterized by independent amplitude and phase damping channels. The parametrized time for this last case is $p = 1 - \exp(-t/T_2)$, with the transverse relaxation time $T_2$ being usually much smaller than the longitudinal relaxation time $T_1$.

For the same initial condition as stated above, the system-environment, system, and environment evolved states are given by:

\[ \rho_p = 2^{-1}\{ (1 + r_3)q|00\rangle\langle 00| + p|01\rangle\langle 01| + (1 - r_3)q|10\rangle\langle 10| + p|12\rangle\langle 12| \}
+ \sqrt{pq}(1 + r_3)|00\rangle\langle 01| + (1 - r_3)|10\rangle\langle 12| + (r_1 - ir_2)q|00\rangle\langle 10| + p|01\rangle\langle 12| + c.t.
+ (r_1 - ir_2)\sqrt{pq}|00\rangle\langle 12| + |01\rangle\langle 10| + c.t., \]

\[ \rho_p^E = 2^{-1}\{ (1 + r_3)|00\rangle\langle 00| + (1 - r_3)|10\rangle\langle 10| + [(r_1 - ir_2)q|0\rangle\langle 1| + c.t.]. \]

\[ \rho_p^S = 2^{-1}\{ 2q|0\rangle\langle 0| + p[(1 + r_3)|1\rangle\langle 1| + (1 - r_3)|2\rangle\langle 2|] + \sqrt{pq}[(1 + r_3)|0\rangle\langle 1| + (1 - r_3)|0\rangle\langle 2| + c.t.]. \]

Summing up the absolute value of the off-diagonal elements of these three density matrices, we obtain their QC:s:

\[ C(\rho_p) = (1 + 2\sqrt{pq})C(\rho^S) + 2\sqrt{pq}, \]

\[ C(\rho_p^S) = qC(\rho^S), \]

\[ C(\rho_p^E) = 2\sqrt{pq}. \]

We see that the PDC destroys the coherence of the system faster than the ADC does, a fact well known in decoherence theory [62]. Besides, for the PDC the QC of the environment is transient and does not depend on the initial state of the qubit. We also notice that, as $C(\rho_p^S) = C(\rho_p^E) = 0$, any non-zero initial QC present in the system shall be asymptotically ($p = 1$) transformed into non-local quantum coherence (NLQC).

The next question we want to address is if, for the PDC, entanglement is also, at least up to a constant factor, equivalent to this NLQC:

\[ C_{nl}(\rho_p) = (p + 2\sqrt{pq})C(\rho^S). \]

As the Peres’ criterion gives a necessary and sufficient condition for separability of qubit-qutrit systems [63, 64], for this channel we shall analyze the entanglement negativity, whose definition is [21]:

\[ E_n(\rho_p) = 2^{-1}(|\langle T_E(\rho_p) |1\rangle - 1) = 2^{-1}\sum_j(|\lambda_j| - \lambda_j). \]

In the last equation $T_E(\rho_p)$ stands for the partial transpose of $\rho_p$, $\lambda_j$ are the eigenvalues of $T_E(\rho_p)$, and $||X||_1 = \text{Tr}\sqrt{XX^T}$ is the trace norm [48]. In the general case, the characteristic polynomial for $T_E(\rho_p)$ is $\lambda^2(\lambda^4 - \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0) = 0$, and

\[ \rho_p = \frac{1}{4} (a X + b Y + c Z + d), \]

where $a$, $b$, $c$, and $d$ are real numbers, and $X$, $Y$, $Z$ are the Pauli matrices. The eigenvalues of $\rho_p$ are given by:

\[ \lambda_j = \frac{1}{2} (a_j + \alpha_j), \]

where $a_j$ and $\alpha_j$ are the eigenvalues of $\rho_p$. The eigenvalues of $\rho_p$ are:

\[ \{0, 1\}, \]

and the eigenvalues of $\rho_p^E$ are:

\[ \{0, 1\}, \]

and the eigenvalues of $\rho_p^S$ are:

\[ \{0, 1\}. \]

Therefore, we arrive at the following dynamical coherence-entanglement relation:

\[ C(\rho_p) = C_{nl}(\rho_p) + E_c(\rho_p). \]

We also notice that $C(\rho_p^S) = E(\rho_p^E) = 0$ and $C(\rho_p^E) = C(\rho_p^S)$.

\[ C(\rho_p^S) = E(\rho_p^E) = 0 \quad \text{and} \quad C(\rho_p^E) = C(\rho_p^S). \]

That is to say, for this channel all QC initially present in the qubit is transferred to the environment in the asymptotic time ($p = 1$).

B. Phase damping channel

The phase damping channel (PDC), which can be represented by the Kraus’ operators $K_{nl}^{pd}(p) = \sqrt{pq}\sigma_0$, $K_{nl}^{pd}(p) = \sqrt{pq}0\rangle\langle 0|$, and $K_{nl}^{pd}(p) = \sqrt{pq}|1\rangle\langle 1|$, is a phenomenological description of the principal cause of coherence loss in several quantum systems. The unitary map for this kind of qubit-environment interaction is[79]: $U_{pd}(p)|00\rangle = \sqrt{pq}|00\rangle + \sqrt{pq}|12\rangle$. We see from this map that the system base states do not change, i.e., it does not exchange energy with the environment. However, each one of these states leaves a unique “fingerprint” in the environment.

The parametrized time for this last case is $p = 1 - \exp(-t/T_2)$, with the transverse relaxation time $T_2$ being usually much smaller than the longitudinal relaxation time $T_1$. For simplicity, in order to relate the non-local quantum coherence with quantum correlations, let us begin by computing the concurrence. For two-qubit systems, this entanglement quantifier can be written as [20]

\[ E_c(\rho_p) = \max \left(0, \sqrt{\lambda_1 - \sqrt{\lambda_2 - \sqrt{\lambda_3 - \sqrt{\lambda_4}}}} \right), \]

where $\lambda_j$ are the eigenvalues, in decreasing order, of $\rho_p\rho_p$ with $\rho_p := \sigma_2 \otimes \sigma_2 \rho_p^* \sigma_2 \otimes \sigma_2$ and $\rho_p^*$ is the complex conjugate of $\rho_p$. One can verify that $\rho_p\rho_p$ possess only one non-null eigenvalue: $\lambda_1 = pq(1 - r_3)^2$. Therefore, as $\lambda_1 \geq 0$, we obtain $E_c(\rho_p) = \sqrt{pq}(1 - r_3) = C_{nl}(\rho_p)$. Hence, we arrive at the following dynamical coherence-entanglement relation:

\[ C(\rho_p) = C_{nl}(\rho_p) + E_c(\rho_p). \]
states: U

parametrized time p

c1λ + c0 = 0, with c2 = 2\textsuperscript{-2}(1 - r_1^2 - r_2^2 - r_3^2), c1 = 2\textsuperscript{-2}p(2 - p)(r_1^2 + r_3^2), and c0 = 2\textsuperscript{-4}p\textsuperscript{2}(2 - p)(r_1^2 + r_3^2)(r_2^2 - 1).

Although analytical expressions for the eigenvalues λ\textsubscript{j} can be obtained [65], they have a cumbersome form which does not help in addressing the issue under analysis here. In contrast, it is not difficult verifying that the equality $C_{nl}(\rho_{p=1}) = 2E_n(\rho_{p=1})$ holds asymptotically (and for $p = 0$). Nevertheless, by generating some random initial states $\rho^S$ [66, 67], we checked that for $p \in (0, 1)$ there exists a gap, $\Delta = C_{nl} - 2E_n$, between NLQC and entanglement which depends both on p and on $\rho^S$. Some examples are shown in Fig. 1. Therefore, for the PDC the coherence-entanglement relation is of the kind:

$$C(\rho_p) = C_l(\rho_p) + [2E_n(\rho_p) + \Delta(p, \rho^S)],$$

with $\Delta(0, \rho^S) = \Delta(1, \rho^S) = 0$. Moreover, we verified that other quantum correlations [68], such as for example the amended Hilbert-Schmidt quantum discord [69], measurement-induced disturbance [70], and measurement-induced nonlocality [71], do worse than entanglement in accounting for the PDC-induced NLQC. That is to say, in addition to show a different qualitative behavior from the NLQC, these correlations cannot be made to coincide with the NLQC even in the asymptotic time $p = 1$. So, we leave open the question of the possible description of this quantity via a quantum correlation function.

### C. Bit flip channel

The bit flip channel (BFC) is the most common error in classical information, where a bit can be flipped, 0 $\rightarrow$ 1, due to random noise. This type of state modification can also take place for a quantum bit, where the computational base states can be left alone, $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow |1\rangle$, with probability $1 - p$ or can flipped, $|0\rangle \rightarrow |1\rangle$, with a probability $p$. The Kraus operators for these transformations are [50]: $K_0^B(p) = \sqrt{p}\sigma_0$ and $K_1^B(p) = \sqrt{p}\sigma_1$. And the induced unitary map is: $U_{bf}(p)|0\rangle = \sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$ and $U_{bf}(p)|1\rangle = \sqrt{p}|1\rangle + \sqrt{1-p}|0\rangle$. Using this map, we find the evolved states:

$$\rho_p = 2^{-1}\{(1 + r_3)(q|00\rangle\langle00| + p|11\rangle\langle11|) + (1 - r_3)(q|10\rangle\langle10| + p|01\rangle\langle01|) + (r_1 - ir_2)(q|00\rangle\langle00| + |11\rangle\langle11|) + c.t. \}

\,+ (r_1 + ir_2)\sqrt{p}|00\rangle\langle00| + |11\rangle\langle11| + c.t.\},$$

$$\rho_p^S = 2^{-1}\{(1 + r_3(1 - 2p))[0\rangle\langle0| + (1 - r_3(1 - 2p))[1\rangle\langle1| + [(r_1 - ir_2(1 - 2p))0\rangle\langle1| + c.t.\}],$$

$$\rho_p^E = q|0\rangle\langle0| + p|1\rangle\langle1| + r_1\sqrt{p}|0\rangle\langle1| + c.t.\}. \hspace{1cm} (29)$$

The formula for the total QC is equal to that obtained for the PDC (Eq. (21)). On the other hand, the local QCs
FIG. 2: (color online) Coherences and entanglement for the time evolution generated by the bit flip channel applied to a qubit prepared in the state \( \vec{r} = (-0.11, -0.61, 0.77) \). The inset shows the same quantities for \( \vec{r} = (0.37, -0.08, -0.49) \). In this last case, even though the quantum coherence of the system is almost constant, a considerable amount of transitory non-local quantum coherence and entanglement is generated. As seen in these figures and easily verified with Eqs. (30) and (31), for any initial state, at \( p = 1/2 \) the coherences of the system and of the environment are equal to \( |r_1| \).

are given by:

\[
C(\rho_p^S) = \sqrt{r_1^2 + r_2^2(1-2p)^2},
\]

\[
C(\rho_p^E) = 2\sqrt{pq}|r_1|,
\]

which are not only quantitatively, but also qualitatively different from those obtained for the ADC and PDC. Actually, for the BFC, the distribution of QC is seen to be even more involved than for the previous two channels. For instance, in contrast to the ADC and PDC, for the BFC the QC can be made constant (freezed) with time if the qubit initial state has \( r_2 = 0 \). For a in-depth investigation of frozen quantum coherence, see Ref. [72]. On the other side, if \( r_1 = 0 \) the QC of the environment is always null and the initial QC of the qubit is altogether transformed into NLQC (for \( p = 1 \)). So, depending on \( \rho^S \), the BFC can lead to qualitative behaviors typical of the ADC or of the PDC, or to a “mixture” of their characteristics traits.

Now, let’s analyze the NLQC and entanglement generated by the BFC. Here, to compute the concurrence, we notice that the non-null eigenvalues of \( \rho_p \) are: \( \lambda_{\pm} = pq(\sqrt{x} \pm \sqrt{y})^2 \), where we used \( x = r_2^2 + r_3^2 \) and \( y = 1 - r_1^2 \). So, the concurrence of \( \rho_p \) can be written as follows:

\[
E_c(\rho_p) = \frac{\sqrt{pq} \max \left(0, \sqrt{x} + \sqrt{y} - |\sqrt{x} - \sqrt{y}|\right)}{2\sqrt{pq} \min(\sqrt{x}, \sqrt{y})}.
\]

By its turn, the created NLQC reads:

\[
C_{nl}(\rho_p) = (1 + 2\sqrt{pq})C(\rho^S) - C(\rho_p^S) + 2\sqrt{pq}(1 - |r_1|).
\]

Examples of the dynamics of coherences and entanglement are shown in Fig. 2. As is indicated by the formulae, entanglement is seem not to capture, in general, the quantitative behavior of the NLQC generated by the BFC. Actually, the creation of system-environment entanglement is shown here not to necessarily imply in the destruction of the system’s QC.

D. Phase flip channel

The phase flip channel (PFC) is a kind of error which only happens in the quantum realm. The computational base states acquire random phases differing by \( \pi \). Or, disregarding global phases, one may say that the state \( |0\rangle \) is
is shown an example of the time dependence of these coherences and entanglement.

From these density matrices, we compute the quantum coherences. For the system we have

\[ C(\rho_p) = |1 - 2p|C(\rho^S). \]  

The total coherence is once more equal to that obtained for the PDC (Eq. (21)). The QC of the environment is obtained from the corresponding expression for the BFP, Eq. (31), by substituting \( r_1 \) with \( r_3 \).

We see thus that, in contrast to the BFC, for the PFC we cannot obtain a constant value for the QC of the system. The qubit’s QC monotonically decreases up to \( p = 1/2 \), which is the time (probability) for which our uncertainty about what happened with the system is maximal. For \( p \in (1/2, 1) \), the qubit’s QC increases, going back to its initial value when \( p = 1 \). Besides, for the PFC the environment’s QC does not depend on the initial QC of the system. In fact, the environment gains transitory coherence whenever the qubit’s ground and excited states populations are not equal. We observe also that the concurrence of \( \rho_p \) for the PFC can be obtained from that in Eq. (32) by the exchange of \( r_1 \) and \( r_3 \). The NLQC for the PFC is

\[ C_{nl}(\rho_p) = (1 + 2\sqrt{|p|} - |1 - 2p|)C(\rho^S) + 2\sqrt{|p|(1 - |r_3|)}. \]  

So, NLQC is created for all but the incoherent and excited state unpopulated initial states. In Fig. 3 is shown an example of the time dependence of these coherences and entanglement.

E. Bit-phase flip channel

The bit-phase flip channel (BPFC) is used to describe the situation in which both the bit flip and the phase flip errors happen with probability \( p \), but simultaneously, i.e., \( |0\rangle \rightarrow e^{i\pi/2}|1\rangle \) and \( |1\rangle \rightarrow e^{i(-\pi/2)}|0\rangle \). To describe this kind
of noise interaction, we can use the Kraus operators: \( K_{bp}^f(p) = \sqrt{p}\sigma_0 \) and \( K_{bp}^f(p) = \sqrt{p}\sigma_2 \), which lead to the unitary mapping: \( U_{bp}(p)|00\rangle = \sqrt{p}|00\rangle + i\sqrt{1-p}|11\rangle \) and \( U_{bp}(p)|10\rangle = \sqrt{p}|10\rangle - i\sqrt{1-p}|01\rangle \). The density operators generated by this map are:

\[
\rho_p = 2^{-1}[(1 + r_3)(q|00\rangle\langle 00| + p|11\rangle\langle 11|) + (1 - r_3)(q|10\rangle\langle 10| + p|01\rangle\langle 01|)] + i\sqrt{pq}[(1 - r_3)|10\rangle\langle 01| - (1 + r_3)|00\rangle\langle 11|] + (r_1 - ir_2)(q|00\rangle\langle 10| - p|11\rangle\langle 01|) + c.t. + (r_1 - ir_2)\sqrt{pq}(|00\rangle\langle 01| + |11\rangle\langle 10|) + c.t.],
\]

\[
\rho_p^S = 2^{-1}[(1 + r_3(1 - 2p))|00\rangle\langle 00| + (1 - r_3(1 - 2p))|11\rangle\langle 11| + [(r_1(1 - 2p) - ir_2)|00\rangle\langle 10| + c.t.|],
\]

\[
\rho_p^E = (q|00\rangle\langle 00| + p|11\rangle\langle 11| + r_2\sqrt{pq}(|00\rangle\langle 11| + c.t.).
\]

The expressions for the quantum coherences and entanglement associated with these three density matrices are obtained from those in Eqs. (21), (30), (31), and (32) by (when needed) exchanging \( r_1 \) and \( r_2 \). Therefore, the analysis of the QC flow for the BPFC is similar to that for the BFC. It is noticeable though that, in contrast to the PFC, if the phase flip error happens accompanied by a bit flip error, we regain the possibility for freezeed QC, which is obtained for the BPFC when \( r_1 = 0 \).

\section{Depolarizing channel}

The depolarizing channel (DC) describes the situation in which the interaction of the system with the surroundings mixes its state with the maximally entropic one with a probability \( p \), i.e., \( \rho^S \to (1 - p)\rho^S + p\sigma_0/2 \). This kind of environment appears, for instance, in teleportation with arbitrary mixed entangled resources [73]. The DC can be described using the following set of Kraus’ operators: \( K_2^q(p) = \sqrt{1 - 3p/4}\sigma_0 \), \( K_1^q(p) = \sqrt{p/4}\sigma_1 \), and \( K_2^q(p) = \sqrt{p/4}\sigma_2 \), which lead to the unitary map: \( U_d(p)|00\rangle = \sqrt{1 - 3p/4}|00\rangle + \sqrt{p/4}(|11\rangle + i|12\rangle + |03\rangle) \) and \( U_d(p)|10\rangle = \sqrt{1 - 3p/4}|10\rangle + \sqrt{p/4}(|01\rangle - i|02\rangle - |13\rangle) \). So, this environment is modeled effectively as a four-level system (a quartat). Now, if we define \( u = p/4 \) and \( v = 1 - 3u \), the evolved states take the form:

\[
\rho_p = 2^{-1}[(1 + r_3)|v|00\rangle\langle 00| + u(|03\rangle\langle 03| + |11\rangle\langle 11| + |12\rangle\langle 12|)] + (1 - r_3)|v|10\rangle\langle 10| + u(|01\rangle\langle 01| + |02\rangle\langle 02| + |13\rangle\langle 13|)] + (1 + r_3)\sqrt{uv}(|00\rangle\langle 00| + |03\rangle\langle 03| - i|00\rangle\langle 12|)] + u(|03\rangle\langle 11| - i|03\rangle\langle 12| - i|11\rangle\langle 12|)] + c.t. + (1 - r_3)\sqrt{uv}(|01\rangle\langle 01| - i|02\rangle\langle 02| - |10\rangle\langle 13|)] + u(|01\rangle\langle 02| + |02\rangle\langle 01| + |01\rangle\langle 13| + i|02\rangle\langle 13|)] + c.t. + (r_1 - ir_2)|v|00\rangle\langle 01| + \sqrt{uv}(|00\rangle\langle 01| + i|00\rangle\langle 02| - |00\rangle\langle 02| + |13\rangle\langle 10| + |i|12\rangle\langle 10| + |03\rangle\langle 10|)] + c.t. + (r_1 - ir_2)|v|11\rangle\langle 01| + i|02\rangle\langle 13| + |12\rangle(|01\rangle - |02\rangle - |13\rangle) + |03\rangle(|01\rangle + i|02\rangle - |13\rangle)] + c.t.,
\]

\[
\rho_p^S = 2^{-1}[(1 + r_3q)|00\rangle\langle 00| + (1 - r_3q)|11\rangle\langle 11| + [(r_1 - ir_2)q|00\rangle\langle 10| + c.t.|],
\]

\[
\rho_p^E = (u|00\rangle\langle 00| + u\sum_{j=1}^3|j\rangle\langle j| + \sqrt{uv}(\sum_{j=1}^3r_j|0\rangle\langle j| + c.t.|) + u[-i(r_3|1\rangle\langle 2| - r_2|1\rangle\langle 3| + r_1|2\rangle\langle 3|)] + c.t.].
\]

The total coherence can be written as follows:

\[
C(\rho_p) = (3\sqrt{u} + \sqrt{v})^2C(\rho^S) + 6(\sqrt{uv} + u).
\]

The expression for the QC of the system is equal to that for the PDC (Eq. (22)). So the DC is seen to be as effective as the PDC at erasing the QC of the qubit. The QC of the environment at \( p \) is given by:

\[
C(\rho_p^E) = 2(\sqrt{uv} + u)\sum_{j=1}^3|r_j|.
\]

As \( \sqrt{uv} + u \) for \( p = 1/2 \), we notice that, asymptotically, QC is generated in the environment for all coherent and/or population unbalanced initial states. For the DC the NLQC reads:

\[
C_{nl}(\rho_p) = [(3\sqrt{u} + \sqrt{v})^2 - q]C(\rho^S) + 2(\sqrt{uv} + u)(3 - \sum_{j=1}^3|r_j|).
\]

From these equations, we readily identify a somewhat puzzling phenomenon. The DC, even for a maximally mixed or ground initial states of the qubit, leads to the creation of NLQC (no energy or QC required). So, for this channel, the system can be a catalyst whose state is not modified but whose presence helps in creating joint QC and quantum correlations. Examples of these dynamic behaviors are presented in Fig. 4.
III. CONCLUDING REMARKS

In this article, we performed a detailed analysis of the dynamical flow of the $l_1$-norm quantum coherence for a qubit interacting with environments modeled by quantum channels relevant for quantum information science. Our investigation provided several insights about how these different kinds of system-environment interaction can affect the quantum properties of the system, of the environment, and of their correlations. We noticed that quantum coherence is not conserved in general and that even incoherent initial states may lead to the creation of transitory QC. For the amplitude damping channel, the non-local QC created during the time evolution was shown to be completely captured by the entanglement concurrence. This is not the case for the phase damping channel. Besides, asymptotically the PDC transforms all initial coherence of the qubit into non-local system-environment QC, while the ADC only transfer it to the environment. The dynamic spreading of QC due to the bit, phase, or bit-phase flip channels was shown to be more diverse than for the ADC and PDC. Actually, by tuning the initial state of the qubit we can observe dynamical behaviors typical of these two channels or a mixture of them. We showed with this that, contrary to what have been suggested previously (see e.g. Ref. [55]), the PDC is qualitatively distinct from the flip channels. Moreover, for the flip channels, we have identified the possibility and conditions for frozen QC and for a sudden modification in its rate of change with time. Besides, the creation of system-environment entanglement was shown here not necessarily to imply in the decaying of the system’s QC. We also investigated the qubit’s initial resources needed for creating non-local QC, showing that while the PDC requires QC the ADC and phase flip channel need only nonzero population of the excited state (i.e., energy). What is more, the potential of the qubit for acting as a catalyst for the creation of joint QC and entanglement by the depolarizing channel was reported.

Most of the literature on decoherence theory acknowledges that this process takes place due to the creation of correlations between the system and its surroundings. How much and what kind of correlation is dynamically generated depends on the specificities of the system-environment interaction Hamiltonian. Modeling these interactions in a general setting is extremely difficult and surely out of reach. Quantum channels are tools capable of providing such a description for a wide range of physical situations. In this article, we used this tool to present the most complete study of the environment-induced quantum coherence flow up to now. Besides the several interesting findings described in the main text and summarized in the last paragraph, our work may contribute for two distinct directions of research. On one side, it may be interesting to emulate the distinct dynamical behaviors induced by distinct quantum channels in order to transform coherence into quantum correlations in a prescribed and desired way. On the other hand, understanding how the system loses its coherence and how this coherence is transferred to the environment and/or transformed into nonlocal properties (entanglement, discord, etc) is the essence of what one might call understanding the decoherence process. Of course, in quantum information science we would like to ultimately use this knowledge to control, prevent or mitigate such kind of unwanted transformations. We hope that our results and the solutions for the questions raised here can shed some light for understanding and tackling this intricate problem.
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Another relevant context in which this channel appears is in quantum communication with spin chains [74].

While completing this article we became aware of two related works: Ref. [75] investigated the distribution of multipartite QC while Ref. [76] defined an entanglement monotone via the NLQC of extensions of a bipartite state.

We observe that the minimal isometric extension (Stinespring dilation) [49] of the PDC uses only two Kraus' operators. However, because of its nice phenomenological interpretation [50], here we apply the isometric extension of the PDC with three Kraus' operators.