Comment on the higher derivative Lagrangians in relativistic theory

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We discuss the consequences of higher derivative Lagrangians of the form $\lambda_2 G_{\mu} \dot{x}^\mu$, $\lambda_3 B_{\mu} x^\mu$, $\lambda_4 G_{\mu} x^\mu$ in relativistic theory.

I. HIGHER DERIVATIVE LAGRANGIANS AND DYNAMIC EQUATIONS

Ostrogradsky was the first who initiated the idea of higher derivative Lagrangians in classical mechanics [1], and more recently were published some articles about this topic [2], [3], [4]. But in my knowledge, there is no article dealing with the relativistic consequences if we consider higher order Lagrangians of this type:

$$\tilde{L}(\dot{x}, \ddot{x}, \ldots, x^{(n)}) = \lambda_1 A_{\mu}(x) \dot{x}^\mu + \lambda_2 G_{\mu} \dot{x}^\mu + \lambda_3 B_{\mu} x^\mu + \lambda_4 G_{\mu} x^\mu$$

where $x^{(n)}(s) \equiv (d^n x(s)/ds^n)$ is the n-derivative of the position ($ds = c d\tau$, where $\tau$ is the proper time), and $U(n)\mu x^{(n)}\mu$ is the generalised field coupling linearly with the derivatives. One can see that we denote the field $G_{(1)}\mu(x) = A_{\mu}(x)$ to refer to be the electromagnetic potential. For $n \geq 2$, we will see their meaning in the sequel. Now, we set the action:

$$S = \int ds L_0(\dot{x}) + \int ds L(\dot{x}, \ddot{x}, \ldots, x^{(n)})$$

where $L_0(\dot{x}) \equiv \frac{m c^2}{2} \dot{x}^\mu \dot{x}_\mu$. We would not give an explicit general dynamic theory for a given $n$, we consider only $n = 2$ for the moment and we will discuss the general case later. By some obvious integral by part for $n = 2$ we will get the equivalent action:

$$S = \lambda_1 \int ds A_{\mu}(x) \dot{x}^\mu + \lambda_2 \int ds \partial_\nu G_{\mu} \dot{x}^\mu \dot{x}^\nu$$

and one can see that the first part of the action is similar to the electromagnetic whereas the second part is similar to the gravitational equation. Indeed, the generalised Euler-Lagrange equation is given by (we give the relativistic form of the equations in [1], [2], [3], [4]):

$$\frac{d^2}{ds^2} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{d}{ds} \frac{\partial L}{\partial \ddot{x}^\mu} + \frac{\partial L}{\partial x^\mu} = 0$$

for $L = L_0 + \tilde{L}$, and so we get:

$$mc^2 \eta_{\mu\nu} \ddot{x}^\mu - \lambda_2 \varepsilon_{\mu\nu \dot{x}^\mu \dot{x}^\nu} - \lambda_3 \Delta_{\mu\nu \dot{x}^\mu \dot{x}^\nu} = -\lambda_1 F_{\mu\nu} \dot{x}^\nu$$

where $\varepsilon_{\mu\nu}$ and $\Delta_{\mu\nu}$ are defined as:

$$\varepsilon_{\mu\nu} = \partial_\mu G_{\nu} + \partial_\nu G_{\mu}$$
$$\Delta_{\mu\nu} = \partial_\mu \partial_\nu G_{\mu} = \frac{1}{2} (\partial_\nu \varepsilon_{\mu\sigma} + \partial_\sigma \varepsilon_{\mu\nu} - \partial_\mu \varepsilon_{\sigma\nu})$$

and where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Of course one see the analogy with the geodesic equations for a charged particle in gravity field and in an electromagnetic field, but we can see also that the Minkowski metric is such a background and so that $G_{\mu}(x)$ is nothing but an additional field which can be seen as a deformation field and that $\varepsilon_{\mu\nu}$ can ve viewed as an infinitesimal strain field by analogy with the deformation theory of a continuous medium [5].

We would like to go further than the rank 2. Let’s take $n = 3$, we denote $G_{(3)}\mu(x) = B_{\mu}$. Then one has

$$- \frac{d^3}{ds^3} \frac{\partial L}{\partial \dot{x}^\mu} + \frac{d^2}{ds^2} \frac{\partial L}{\partial \ddot{x}^\mu} - \frac{d}{ds} \frac{\partial L}{\partial \dot{x}^\nu} + \frac{\partial L}{\partial x^\mu} = 0$$

and so one can easily get:

$$mc^2 \eta_{\mu\nu} \ddot{x}^\mu + \lambda_3 H_{\mu\nu} \ddot{x}^\nu - \lambda_3 \gamma_{\mu\nu\rho \dot{x}^\mu \dot{x}^\nu \dot{x}^\rho} - 3 \Sigma_{\mu\nu \dot{x}^\mu \dot{x}^\nu} - \lambda_3 \Delta_{\mu\nu \dot{x}^\mu \dot{x}^\nu \dot{x}^\rho} = -\lambda_1 F_{\mu\nu} \ddot{x}^\nu$$

where

$$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$\Sigma_{\mu\nu\rho} = \partial_\mu \partial_\nu B_\rho$$
$$\gamma_{\mu\nu\rho\sigma} = \partial_\mu \partial_\nu \partial_\rho B_\sigma$$

We can see that this field generalise the idea of electromagnetic field because of the asymmetry of $H_{\mu\nu}$. But there is some other gradient fields similar to the field $\Delta_{\mu\nu}$ coupling to the combination of the odd derivatives of $x^\mu u^\nu$, i.e. $x^\mu \dot{x}^\nu$ and $\dddot{x}^\mu \dot{x}^\nu$. On can easily consider that for $n = 4$ and even more. We denote the field $K_{\mu}(x) = G_{(4)}\mu(x)$. It would give a similar structure in the dynamic equation than $G_{\mu}(x)$, Is is known indeed that in the non-relativistic theory, the Lagrangian $\frac{mc^2}{2} x^\mu \dot{x}^\mu$ is equivalent to this Lagrangian $\frac{mc^2}{2} x^\mu \dot{x}^\mu$ which could be interpréted as a dynamic energy. [2], [3], [4]. Here the problem is similar: the Lagrangian $\lambda_4 K_{\mu} \dot{x}^\mu$ is equivalent to $\lambda_4 \partial_\mu K_{\nu} \dddot{x}^\mu + \lambda_4 \partial_\mu \partial_\nu K_{\mu} \dot{x}^\nu \dot{x}^\mu$ and hence for a more general field than the trivial one $K_{\mu} = x_\mu$, the equivalent Lagrangian seems to be more complicated and the dynamic equations have some similar structure that for $n = 2$.

More generally, for $G_{(n)}\mu(x)$, $n \geq 1$ we will get some term $\partial^{p} \ldots \partial^{p} G_{(n)}$, $p = 1, \ldots, n$ multiplied by the combination of the derivatives $x^{(1)} x^{(2)} \ldots x^{(r)}$, where $\sum_{j=1}^{r} l_j = n$. One can see that “even” $n$-field (for $n$ even) could be associated with the ”gravitational fields” and the “odd” $n$-field with the ”electromagnetic field” since in the dynamic equations the $n$-derivative term

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\[ x^\mu \text{ is multiplied symmetric (if } n \text{ is even) / antisymmetric (if } n \text{ is odd) first derivative of the field} \]

\[ (\partial_\mu U_\nu + (-1)^n \partial_\nu U_\mu) x^{(n)\nu} \]

where we denoted \( U_\mu \) the \( n \)-field to remove the index \( n \).

**II. GENERAL FIELDS HYPOTHESIS**

The point is to relate the possible existence of these field with some induction phenomena. Or in an other word, we will suppose some physical coupling, unknown yet but still relevant at least for future research.

As we noticed just above, the analogies with the electromagnetic field and gravity field are crucial to understand what is the physical meaning of these field. We give some example. Imagine an electrokinetic experiment where one set a constant intensity of electron current on a circuit (one can imagine a particle accelerator for a more modern discussion), whatever the shape of the circuit. Then one can expect that a serious student at the University could measure a magnetic field and check the well known Biot and Savart law. Now, imagine that we increase constantly the intensity such that

\[ \frac{\partial \epsilon_{\nu\sigma}(x)}{\partial \nu} + \frac{\partial \rho(x) c^2}{\partial \sigma} = \partial_\nu \epsilon_{\nu\sigma}(x) + \partial_\sigma \rho(x) c^2 \epsilon_{\nu\sigma} \]  

The equations (10) are analogous to the compatibility equations for the strain tensor in the three-dimensional non-relativistic theory of deformation of continuous media.[5]

Hence we get the following wave equations:

\[ \nabla^2 \epsilon_{\nu\sigma}(x) + \partial_\nu \partial_\sigma \epsilon_{\nu\sigma}(x) = -\kappa \xi^{(2)}_{\nu\sigma}(x) \]  

where we get a stress tensor \( \xi^{(2)}_{\nu\sigma}(x) \equiv \partial_\nu j^{(2)}_{\nu}(x) + \partial_\nu j^{(2)}_{\sigma}(x) \).

**B. Generalisation to the \( n \)-field equations**

It comes naturally that for the 2\( n \)-field \( G_{(2n)} \), so-called gravitational type field, the coupling is of the form:

\[ \frac{8\pi G_{(2n)}}{c^2} G_{(2n)\mu}(x) j^{(2n)\nu}(x) \]

while the 2\( n \) + 1-field \( G_{(2n+1)} \), so-called electromagnetic type field, the coupling is of the form:

\[ \frac{\mu_0 \lambda^{2n}}{c^2} G_{(2n+1)\mu}(x) j^{(2n+1)\mu}(x) \]

where \( \mu_0 \) is the vacuum permeability since we generalized for the 2\( n \) + 1-field the electromagnetic type field (the \( \lambda^{2n} \) come from a dimensional analysis) as well as we did for the 2\( n \)-gravitational fields. The generalized currents are defined for \( n = 1, 2, 3, \ldots \) by:

\[ j^{(n)\nu}(x) c^2 \frac{d^n x^{\nu}}{ds^n} \]

where \( \rho^{(n)}(x) \) give the density of mass for \( n \) even and the density of charge for \( n \) odd. From those rule we could
obtain similar wave equation than (11) but with higher order differential operator \( \Box \cdots \Box \). For example, for the 4-field we give the wave equation for the trace of the tensor \( \eta_{\mu\nu} \equiv \partial_\mu K_\nu + \partial_\nu K_\mu \):

\[
(\Box + 1)\Box \eta_{\mu\nu}(x) = -\kappa \lambda^2 \partial_\mu j^{(4)\mu}(x)
\]

and this means that the scalar field \( \eta_{\mu\nu} \) is a massive field.

### III. COMMENTS

The effect of the gravitation at the microscopic scale is not well known yet. We can expect that the current \( J^{(2)} \) of acceleration of mass has a contribution in particle physics as well as the energy-impulsion tensor. We can suggest that the constant \( \lambda \) is constraint by the current observations. The recent cosmological observations show that there is a lack of energy due to the cosmological constant \( \Lambda \) that might be related to our constant \( \lambda \) (we could expect \( \lambda = 1/\sqrt{\Lambda} \)). The interpretation of this relation is not so obvious and again speculative, but remember that we gave above an analogy between the field \( \varepsilon_{\mu\nu} \) with the strain-deformation tensor of a continuous media. This remark is not obsolet. Indeed, if we consider the covariant derivative for a Riemannian metric space \( (\varepsilon_{\mu\nu} = D_\mu G_\nu + D_\nu G_\mu) \), one can construct a stress field

\[
\sigma_{\mu\nu} = \rho G c^2 \varepsilon_{\mu\nu}
\]

where \( \rho G c^2 = \frac{e^4}{8\pi G \lambda^2} \) is the density of energy constant analogous to the Young modulus. The stress tensor could be adding to the Einstein field equations of gravity and we get the relation:

\[
D_\nu \sigma^{\mu\nu} + D_\mu T^{\mu\nu} = 0
\]

where \( T^{\mu\nu} \) is the energy-momentum tensor in the Einstein field equation. This last equation means that the total energy in the Universe is conserved but that the “visible” energy can be accelerated and this variation of inertie is compensate by the stress energy variation due to the strain of the internal structure of the continuous medium. There is no contradiction with the Einstein theory of gravity field and moreover this gives an new perspective on the Mach principle revisiting the “absolute” acceleration concept as a natural motion in a space-time deformed by the matter-energy contained therein, we refer the reader on the paper of Einstein on this related topic [7]. The relativistic theory of an Aether was discussed several time, see for e.g. [8], [9]. In this paper, our hypothesis is different and gives a relativistic theory of the deformation of continuous media (for which the geometry is described by the metric field).

Now we come back to the microscopic scales. Nothing prove today that at those scales the gravitation can be viewed as a metric field. It might be challenge to see if the higher derivative fields play a rule for particle physics in the domain of astroparticle or for the futur linear accelerator(s). The analogies between the \( 2n \)-fields (ex \( G_\mu \) and \( K_\mu \)) and between the \( 2n + 1 \)-fields (ex \( A_\mu \) and \( B_\mu \)) can be viewed as a unity between those fields (where the constant \( \lambda \) plays a rule again) and generalise the notion of gravitational and electromagnetic field for more general currents. Additionally, the unity between the odd and even field is an other tempting idea, which is less obvious but still interesting. Indeed, we could see the field \( G^{(2)}_{\mu} \) and \( G^{(1)}_{\mu} \) as two components of the same field \( G_\mu \) coupling to both currents \( j^{(1)} \) and \( j^{(2)} \). This is rather speculative but also shows the wide perspective.

About the 4-field \( K_\mu \), in a more general theory of gravitational relativity, we can expect that his notion of “dynamic energy” would have a sense in a way that we could make an analogy between the tensor \( \partial_\mu K_\nu \), with a more general tensor \( k_{\mu\nu} \) coupling with \( \partial_\mu \partial_\nu \) and its derivatives with the covariant form of \( \partial_\mu \partial_\nu \partial_\sigma \). This could be a way to include the dynamic energy in general relativity theory.

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