OPERATOR FORMALISM ON THE $\mathbb{Z}_n$ SYMMETRIC ALGEBRAIC CURVES

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ABSTRACT

In this paper it is shown that the $b - c$ systems on a Riemann surface $\Sigma_g$ of genus $g$ and with abelian group of internal automorphisms $\mathbb{Z}_n$ are equivalent to multivalued field theories on the complex plane $\mathbb{C}$. To this purpose, the fields $b$ and $c$ are expanded using suitable bases of tensors that are multivalued on $\mathbb{C}$ and singlevalued on $\Sigma_g$. The amplitudes of the $b - c$ systems on $\Sigma_g$ are then recovered exploiting simple normal ordering rules on the complex plane. Finally, we construct a conformal field theory in $\mathbb{C}$ having as primary operators twist fields and free ghosts. It turns out that the zero and two point functions of the $b - c$ systems on $\Sigma_g$ can be evaluated in terms of the operator product expansions between these primary fields.

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1. INTRODUCTION

In recent years there have been many physical applications of the theory of affine algebraic curves, not only in the traditional field of the string theories [1], [2], [3], [4], but also in general conformal field theories [5] and very different topics like solutions of the Einstein equations [6] integrable models [7] and the theory of defects [8]. The best known and simplest algebraic curves are the hyperelliptic ones, so that it is not a surprise that a big part of the publications quoted above deals with them. However, there are interesting applications in which more general curves are involved, e.g. the conformal field theories with \(Z_n\) symmetry ([9], [10]) which we investigate in the present paper.

In what follows we consider a conformal field theory on a closed and orientable Riemann surface \(\Sigma_g\) of genus \(g\). It is well known that such \(\Sigma_g\) can be represented as an algebraic curve in \(\mathbb{CP}_2\). Points of \(\Sigma_g\) are labeled by values of a pair of meromorphic functions \(z\) and \(y\) from \(\Sigma_g\) onto \(\mathbb{CP}_1\). They are defined up to a birational transformation. Hence, one can involve a multivalued function

\[
\tilde{y} \equiv y \circ z : \mathbb{CP}_1 \to \mathbb{CP}_1.
\]  

Thus we arrived at a description (a model) of \(\Sigma_g\) as a branch covering of the Riemann sphere. The same letter \(z\) will denote a function on the Riemann surface in question and a coordinate on the complex plane. It is very convenient and hopefully will not lead to any confusion. The mapping \(\tilde{y}(z)\) takes \(n\) values at a point except from a set of branch points \(a_1, \ldots, a_M\) where the number of values is less then \(n\). The local monodromy group \(G\) describes the way in which the branches of \(\tilde{y}\) are exchanged while moving around the branch points. In the rest of this paper we assume (to avoid extra purely technical complications) that the point at infinity is not a branch point. Therefore, it is sufficient to study the restriction of \(\tilde{y}\) to the complex plane \(\mathbb{C}\). By means of the inverse mapping \(\tilde{y}^{-1}\) one can project now the correlation functions of our field theory on the complex plane. In order to define a conformal field theory, one should introduce a set of primary fields with rational or irrational conformal weights. One example is provided by the screening charges of the minimal models [11]. Unfortunately, the presence of these primary fields modifies the monodromy behavior of the correlation functions [12] in a way which is difficult to handle on an algebraic curve. As a consequence, the best strategy is to start from the fermionic
$b - c$ systems with integer conformal weight $\lambda$ defined on $\Sigma_g$. Due to their dependence on $y$, the related correlation functions become in fact multivalued tensors on the complex plane and, using the techniques of the Riemann monodromy problem, $[8], [13], [14]$, can be expressed as a linear combination of a finite number of multivalued tensors. Motivated by the analogy of these tensors with the conformal blocks of conformal field theory, we will call them multivalued blocks. The coefficients appearing in the linear combination mentioned above are singlevalued functions on $C$. Moreover, the multivalued blocks should form a basis in which all the possible multivalued behaviors compatible with the local monodromy group $G$ of $\Sigma_g$ are represented. Normally, the elements of this basis are chosen in such a way that they build a set of independent solutions of the Riemann monodromy problem $[13], [14]$ associated with the group $G$. However, this is not a necessary requirement and one is allowed to consider other bases. Once a basis of multivalued blocks is known, it was conjectured in ref. $[15]$ that it is possible to treat the $b - c$ systems on an algebraic curve as a multivalued field theory on the complex plane. The proof of this statement in the case of the $Z_n$ symmetric curves is contained in Section 2. This is the main result of our paper. The method used for the proof is to construct an operator formalism $^1$ for the multivalued fields in such a way that their amplitudes coincide with the amplitudes of the $b - c$ systems on the Riemann surface. This approach does not exploit the peculiar properties of the $Z_n$ symmetric curves and can hopefully be extended to more general curves $[17]$.

Moreover, using the operator formalism presented here and the fact that the solutions of the Riemann-Hilbert problem can be expressed $^2$ as correlators of a conformal field theory containing free fermions and twist fields $[8], [13], [14]$, it is possible to show a deep connection between the $b - c$ systems on an algebraic curve and the conformal field theories on the complex plane. Of course, these cannot be standard conformal field theories since the amplitudes of the $b - c$ systems on $\Sigma_g$ are multivalued on $C$. This fact becomes evident

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$^1$ We notice at this point that operator formalisms on Riemann surfaces have been already intensively studied in the past (see for example $[16]$) from a point of view which is very different from the one developed here.

$^2$ The fact that the solutions of the Riemann monodromy problem can be described in terms of twist fields and free fermions has been first discussed in refs. $[13]$. The explicit form of the twist fields on a $Z_n$ symmetric curve has been given in refs. $[8]$ and $[10]$. The generalization to the algebraic curves with nonabelian monodromy group of symmetry $D_n$ has been discussed in $[15], [18]$ and $[19]$. Finally, the existence of primary fields on more general algebraic curves has been confirmed in $[2]$ and $[20]$. 

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in ref. [15], where algebraic curves with nonabelian group of internal authomorphisms $D_n$ are considered. In this case, in fact, the twist fields become nonlocal objects with a complicated statistics [19]. Despite of many efforts, curves with nonabelian group of symmetry remain very difficult to handle, but in the simpler situation of a $Z_n$ group it is indeed possible to construct a conformal field theory on $C$ whose amplitudes coincide with the amplitudes of the $b-c$ systems on $\Sigma_g$. This is done in Section 3, where the determinant of the zero modes and the two point function of the $b-c$ systems on a Riemann surface are exactly reproduced from the operator product expansions of free $b-c$ systems and twist fields on the complex plane.

The rest of the material presented in this paper is divided as follows. In the conclusion some possible extensions of our operator formalism to more general algebraic curves are discussed. Finally, in the appendix we prove proposition two of Section 2, which states that it is possible to compute the correlator $\langle b(z_1)\ldots b(z_{N_b}) \rangle$ of the $b-c$ systems on $\Sigma_g$ ($N_b = (2\lambda - 1)(g - 1)$) in terms of an analogous correlator containing only multivalued fields on the complex plane.

2. THE OPERATOR FORMALISM

Let us consider the $Z_n$ symmetric algebraic curves of the kind:

$$y^n = \prod_{i=1}^{nm} (z - a_i)$$

(2.1)

where $n$ and $m$ are integers, while $z \in \mathbb{CP}_1$. The points $a_i \in \mathbb{C}$ are the branch points of the curve. $y(z)$ can be viewed both as a function on the curve $\Sigma_g$ given by (2.1) and as a multivalued function on $\mathbb{CP}_1$ with $n$ branches (strictly speaking it should be then denoted as $\tilde{y}$ as in (1.1) but in what follows we shall omit writing $\tilde{y}$). The genus of the surface $\Sigma_g$ is given by:

$$g = 1 - n + \frac{nm(n - 1)}{2}.$$  

(2.2)

On $\Sigma_g$ we consider the theory of fermionic free $b-c$ fields of spin $\lambda \in \mathbb{Z}$:

$$S_{bc} = \int_{\Sigma_g} d^2 \xi \left( b\bar{\partial}c + c.c. \right)$$

(2.3)
where ξ and \( \bar{\xi} \) are coordinates on \( \Sigma_g \). Such theories on algebraic curves have already been discussed in refs. [4], [18], [13], [20], [1]. (see also refs. [12] and [21] where multivalued field theories have been treated in different contexts).

The theory defined by (2.3) can also be viewed as a multivalued field theory on the complex plane. A problem arises if it is possible to define both theories in such a way that correlation functions produced by them are related just by the projection from \( \Sigma_g \) to \( \mathbb{CP}_1 \). In the second theory \( y(z) \) is a multivalued function with \( n \) branches denoted by \( y^{(l)}(z) \), \( l = 0, \ldots, n-1 \). Any tensor \( T \) on \( \Sigma_g \) can be expressed by means of \( z, y(z) \) and \( dz \) and when projected becomes multivalued as well: \( T^{(l)}(z) \equiv T(z, y^{(l)}(z)) \). In the following we will usually omit the branch index \( l \) for simplicity. To establish a working operator formalism on \( \Sigma_g \), we have to find a suitable basis in which the \( b-c \) fields should be expanded. This is provided by techniques used in the Riemann monodromy problem (see ref. [22] for more details). First of all, we need a set of functions \( F_k(z, y(z)) \) on \( \Sigma_g \), \( k = 0, \ldots, n-1 \) being independent in the following way: They fulfill the condition \( F_k(z, y(z))/F_{k'}(z, y(z)) = f(z) \) only if \( k = k' \), where \( f(z) \) is a single-valued function on \( \mathbb{CP}_1 \). When \( k = k' \), of course, \( f(z) = 1 \). It is possible to show that the basis of functions \( F_k(z, y(z)) \) can have only \( n \) elements. Any meromorphic function \( g(z, y(z)) \) on \( \Sigma_g \) can be now represented as a linear combination of the functions \( F_k(z, y(z)) \) with coefficients being single-valued functions on \( \mathbb{CP}_1 \). There is still a large freedom in choosing particular bases \( F_k \). An extra requirement we impose is that for a given value of \( \lambda \) we would like to find two bases: \( f_k \) and \( \phi_k \) such that

\[
K(z, w)dz^\lambda dw^{1-\lambda} = \frac{1}{z-w} \sum_{k=0}^{n-1} f_k(z)\phi_k(w)dz^\lambda dw^{1-\lambda}
\]  

(2.4)

where the object on the LHS of (2.4) has just (as a differential in \( z \)) one single pole at \( w \) with residue \( n \) and presents itself as a basic building block of correlation function on Riemann surfaces (for details see [2]).

The solution of the classical equations of motion for (2.3)

\[
\bar{\partial}b = \bar{\partial}c = 0
\]  

(2.5)

are meromorphic tensors, which can be expanded in the following basis:

\[
b(z)dz^\lambda = \sum_{k=0}^{n-1} \sum_{i=-\infty}^{\infty} b_{k,i}z^{-i-\lambda}f_k(z)dz^\lambda
\]  

(2.6)
\begin{equation}
  c(z)dz^{1-\lambda} = \sum_{k=0}^{n-1} \sum_{i=-\infty}^{\infty} c_{k,i}z^{-i+\lambda-1}\phi_k(z)dz^{1-\lambda}
  \tag{2.7}
\end{equation}

where
\begin{equation}
  f_k(z)dz^\lambda = \frac{dz^\lambda}{[y(z)]^{-k+\lambda(n-1)}} \quad k = 0, \ldots, n - 1
  \tag{2.8}
\end{equation}

\begin{equation}
  \phi_k(z)dz^{1-\lambda} = \frac{dz^{1-\lambda}}{[y(z)]^{k-\lambda(n-1)}} \quad k = 0, \ldots, n - 1.
  \tag{2.9}
\end{equation}

The expansions (2.6) and (2.7) resemble the local expansions of the $b - c$ fields on the sphere. The only difference is provided by the presence of the functions $f_k(z)$ and $\phi_k(z)$. This is the contribution coming from the topology of the Riemann surface. It is easy to check that $f_k(z)$ and $\phi_k(z)$ form two different basis of rationally independent functions on $\Sigma_g$ satisfying the requirement (2.4). Eqs. (2.6) and (2.7) represent the most general possible expansions of the fields. Now we quantize the $b - c$ systems using these bases. In order to do it we have to postulate some anticommutation relations for classical degrees of freedom $b_{k,i}$ and $c_{k,i}$. It is convenient to split the fields in their components $b_k(z)$ and $c_k(z)$:

\begin{equation}
  b(z) = \sum_{k=0}^{n-1} b_k(z) \quad c(z) = \sum_{k=0}^{n-1} c_k(z)
  \tag{2.10}
\end{equation}

where

\begin{equation}
  b_k(z)dz^\lambda = f_k(z) \sum_{i=-\infty}^{\infty} b_{k,i}z^{-i-\lambda}dz^\lambda
  \tag{2.11}
\end{equation}

\begin{equation}
  c_k(z)dz^{1-\lambda} = \phi_k(z) \sum_{i=-\infty}^{\infty} c_{k,i}z^{-i+\lambda-1}dz^{1-\lambda}
  \tag{2.12}
\end{equation}

\begin{equation}
  \{b_{k,i}, c_{k',i'}\} = \delta_{kk'}\delta_{i+i',0}.
  \tag{2.13}
\end{equation}

Moreover, we introduce vacua $|0\rangle_k$, on which the operators $b_{k,n}$ and $c_{k',m}$ act. The total vacuum of the $b - c$ systems on $\Sigma_g$ is given by

\begin{equation}
  |0\rangle = \otimes_{k=0}^{n-1} |0\rangle_k.
  \tag{2.14}
\end{equation}

From now on, we will suppose that $\lambda > 1$. The case $\lambda = 1$ can be treated in an analogous way with an obvious complication coming from the $c$ field zero mode. We also consider only Riemann surfaces of genus $g \geq 2$ in order to avoid the exceptional algebraic curves. The zero modes occurring in a given $k-$component $b_k(z)$ of the fields $b(z)$ are of the form:

\begin{equation}
  \Omega_{k,j}dz^\lambda = \frac{z^{j-1}dz^\lambda}{[y(z)]^{-k+\lambda(n-1)}} \quad j = 1, \ldots, N_{b_k}
  \tag{2.15}
\end{equation}
where \( N_{bk} = -2\lambda + 1 + \lambda(n-1)m - km \). They correspond to the operators \( b_{k,i} \) of eq. (2.11) with \( \lambda - \lambda(n-1)m + km \leq i \leq -\lambda \). It is easy to check that \( \sum_{k=0}^{n-1} N_{bk} = (2\lambda - 1)(g - 1) \), giving exactly the Riemann-Roch theorem for a Riemann surface of genus \( g \), where \( g \) is given by eq. (2.2). A treatment of the \( b-c \) systems on \( \Sigma_g \) can be now performed using the techniques exploited in the case of genus zero. The annihilation operators are components of the fields \( b(z) \) and \( c(z) \) with negative powers of \( z \). Since there is the splitting in \( n \)-component of the fields showed by eqs. (2.11)-(2.13), this implies that:

\[
b_{k,i}^{-}\left|\right._{0}^{0} \equiv b_{k,i}\left|\right._{0}^{0} = 0 \quad \left\{ \begin{array}{l}
k = 0, \ldots, n - 1 \\
i \geq 1 - \lambda \end{array} \right. \quad (2.16)
\]

\[
c_{k,i}^{-}\left|\right._{0}^{0} \equiv c_{k,i}\left|\right._{0}^{0} = 0 \quad \left\{ \begin{array}{l}
k = 0, \ldots, n - 1 \\
i \geq \lambda \end{array} \right. \quad (2.17)
\]

We also introduce the "out" vacua \( \left|0\right>_k \) requiring:

\[
k\left|\right.\left|0\right>_k b_{k,i}^+ \equiv k\left|\right.\left|0\right>_k b_{k,i} = 0 \quad \left\{ \begin{array}{l}
k = 0, \ldots, n - 1 \\
i \leq - \lambda - N_{bk} \end{array} \right. \quad (2.18)
\]

\[
k\left|\right.\left|0\right>_k c_{k,i}^+ \equiv k\left|\right.\left|0\right>_k c_{k,i} = 0 \quad \left\{ \begin{array}{l}
k = 0, \ldots, n - 1 \\
i \leq \lambda - 1 \end{array} \right. \quad (2.19)
\]

By definition, the normal ordering of the product of two fields \( b(z)c(w) \) consists in putting the annihilation operators \( b_{k,n}^- \) and \( c_{k,n}^- \) to the right of the creation operators \( b_{k,n}^+ \) and \( c_{k,n}^+ \). Applying the definition (2.11) and (2.12) of the fields \( b \) and \( c \) and exploiting the commutation relations (2.13), it is possible to see that (we assume that \( |w/z| < 1 \)):

\[
b_k(z)c_k(w) =: b_k(z)c_k(w) : + \frac{1}{z - w}f_k(z)\phi_k(w) \quad (2.20)
\]

\[
c_k(z)b_k(w) =: c_k(z)b_k(w) : + \frac{1}{z - w}f_k(w)\phi_k(z). \quad (2.21)
\]

Moreover, in order to take into account also the zero modes, we have to impose the following conditions on the vacua \( \left|0\right>_k \):

\[
k\left|\right.\left|0\right>_k = 0 \quad \text{if} \quad N_{bk} \neq 0; \quad k\left|\right.\left|0\right>_k \prod_{i=1}^{N_{bk}} b_{k,i}\left|\right._{0}^{0} = 1. \quad (2.22)
\]
With the above definitions, the proof of the proposition below becomes straightforward and we do not report it.

**Proposition 1:**

\[ \langle 0 | b_k(z_1) \cdots b_k(z_{N_{b_k}}) | 0 \rangle_k = \text{Det} | \Omega_{k,j}(z_i) | \quad i, j = 1, \ldots, N_{b_k} \]  \hspace{1cm} (2.23)

where \( | \Omega_{k,j}(z_i) | \) is a matrix of the zero modes given by eq. (2.15).

The second step in order to set up an operator formalism is the following:

**Proposition 2:**

\[ \langle 0 | b(z_1) \cdots b(z_{N_{b}}) | 0 \rangle = \text{det} | \Omega_I(z_J) | \]  \hspace{1cm} (2.24)

where the vacuum \( | 0 \rangle \) and the field \( b(z) \) have been already defined in eqs. (2.14) and (2.6), (2.7) respectively. Moreover \( I, J = 1, \ldots, \sum_{k=0}^{n-1} N_{b_k} = N_b \), \( N_b \) describing the total number of the zero modes. Finally the \( \Omega_I(z)dz^\lambda \) represent all the possible zero modes with spin \( \lambda \):

\[ \Omega_I(z)dz^\lambda \in \{ \Omega_{k,i}(z)dz^\lambda | 1 \leq i \leq N_{b_k}, 0 \leq k \leq n - 1 \} . \]

Proposition 2 is proved in the Appendix A.

Now we are ready to compute the propagator of the \( b - c \) systems using the bases (2.6) and (2.7). In our operator formalism, the usual propagator of the \( b - c \) systems is the following ratio of correlators:

\[ G_\lambda(z, w) = \frac{\langle 0 | b(z)c(w) \prod_{I=1}^{N_b} b(z_I) | 0 \rangle}{\langle 0 | \prod_{I=1}^{N_b} b(z_I) | 0 \rangle} \]  \hspace{1cm} (2.25)

where the vacuum \( | 0 \rangle \) and the fields \( b \) and \( c \) are provided by eqs. (2.14) and (2.10) respectively. The denominator of eq. (2.25) can be easily evaluated using eq. (2.24). To compute the numerator, instead, we have to use the normal ordering given in eqs. (2.20) and (2.21). The normal ordering of two fields \( b \) and \( c \) becomes:

\[ b(z)c(w) =: b(z)c(w) : + K(z, w)dz^\lambda dw^{1-\lambda}. \]  \hspace{1cm} (2.26)

Since \( K(z, w) \) is a multivalued tensor, one should specify in which branches in \( z \) and \( w \) the singularity arises. It turns out that the pole at \( z = w \) occur only if the branches of \( K(z, w) \) in \( z \) and \( w \) are the same (see for example [4]). The fact that \( K(z, w)dz^\lambda dw^{1-\lambda} \) is
a tensor with only one singularity in \( z = w \) will play an important role in the following. The reason is that a tensor of this kind is one of the building blocks in the construction of the \( n \)-point functions of the \( b - c \) systems on an algebraic curve. The other building blocks are the zero modes.

Now we are ready to evaluate the propagator given in eq. (2.25). We suppose that the fields are radial ordered, i.e.

\[
|z| > |w| > |z_1| \ldots > |z_{N_b}|
\]

After simple calculations we find:

\[
\begin{align*}
\langle 0 | b^{(l)}(z) c^{(l')} (w) \prod_{I=1}^{N_b} b^{(l_I)}(z_I) | 0 \rangle
\nonumber
\end{align*}
\]

\[
\left( \prod_{I=1}^{N_b} b^{(l_I)}(z_I) \right) = K(z, w) (l') dz^\lambda dw^{1-\lambda} +
\]

\[
\sum_{J=1}^{N_b} (-1)^J K^{(l,l')}_\lambda(z_J, w) \langle b^{(l_1)}(z_1) \ldots b^{(l_{J-1})}(z_{J-1}) b^{(l)}(z) b^{(l_{J+1})}(z_{J+1}) \ldots b^{(l_{N_b})}(z_{N_b}) \rangle
\]

\[
\left( b^{(l_1)}(z_1) \ldots b^{(l_{N_b})}(z_{N_b}) \right)
\]

(2.27)

where \( l, l' \) and \( l_I, l_J \) denote the branches of fields and tensors in the variables \( z, w, z_I, z_J \) respectively. The residual correlation function in eq. (2.27) contain products of \( N_b \) fields \( b \) and therefore can be easily computed by means of eq. (2.24). The final result is the following propagator:

\[
\begin{align*}
\langle 0 | b^{(l)}(z) c^{(l')} (w) \prod_{I=1}^{N_b} b^{(l_I)}(z_I) | 0 \rangle
\nonumber
\end{align*}
\]

\[
\left( \prod_{I=1}^{N_b} b^{(l_I)}(z_I) \right) = \det \begin{vmatrix}
\Omega_1^{(l)}(z) & \ldots & \Omega_I^{(l)}(z) & K^{(l,l')}_{\lambda}(z,w) \\
\Omega_1^{(l_I)}(z_1) & \ldots & \Omega_I^{(l_I)}(z_1) & K^{(l_I,l')}_{\lambda}(z_1, w) \\
\vdots & \ddots & \vdots & \vdots \\
\Omega_1^{(l_{N_b})}(z_{N_b}) & \ldots & \Omega_I^{(l_{N_b})}(z_{N_b}) & K^{(l_{N_b},l')}_{\lambda}(z_{N_b}, w)
\end{vmatrix}
\]

(2.28)

In the above formula we have introduced the indices denoting the branches in order to facilitate the comparison with the propagator of the \( b - c \) systems on the \( Z_n \) symmetric curves that has been found in [23] using the method of the fermionic construction. It is easy to see that the two results coincide showing that the operator formalism here established...
is able to reproduce the two point function on a $Z_n$ symmetric algebraic curve. Moreover, starting from eq. (2.28), we can compute all the other $n$–point functions applying the Wick theorem. The Wick theorem for the $b−c$ systems has been rigorously studied in [24] and it is valid also in our case. One can check it inductively starting from eq. (2.28) and supposing that the Wick theorem has been checked for the correlator

$$G_{N-1,M-1}(z_1, \ldots, z_{N-1}; w_1, \ldots, w_{M-1}) = \langle 0 | b(z_1) \ldots b(z_{N-1}) c(w_1) \ldots c(w_{M-1}) | 0 \rangle$$

with $N - M = N_b$. Then, using eq. (2.26) we obtain:

$$\sum_{i=1}^{M} (-1)^i K_\lambda(z_N, w_i) G_{N-1,M-1}(z_1, \ldots, z_{N-1}; w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_{M-1}). \tag{2.29}$$

All the other possible contractions vanish due to the fact that the Wick theorem holds by hypothesis in the case of any product containing $N-1$ fields $b$ and $M-1$ fields $c$. As an upshot we obtain:

$$< \prod_{s=1}^{M} b^{(l_s)}(z_s) \prod_{t=1}^{N} c^{(l_t')} (w_t) > =$$

$$\det \begin{vmatrix} \Omega^{(l_1)}_1(z_1) & \ldots & \Omega^{(l_1)}_{N_b}(z_1) & K^{(l_1 l_2')}_\lambda(z_1, w_1) & \ldots & K^{(l_1 l_N')}_\lambda(z_1, w_N) \\ \vdots & \ddots & \vdots & \ddots & \ddots & \ddots \\ \Omega^{(l_M)}_1(z_M) & \ldots & \Omega^{(l_M)}_{N_b}(z_M) & K^{(l_M l_2')}_\lambda(z_M, w_1) & \ldots & K^{(l_M l_N')}_\lambda(z_M, w_N) \end{vmatrix} \tag{2.30}$$

where $M - N = (2\lambda - 1)(g - 1) = N_b$. The tensor $K^{(l')}_\lambda(z, w)$ has spurious poles in the limit $w \to \infty$. However one can show as in [2] and [25] that these poles do not contribute to the determinant (2.30). The important fact to be noted here is that both eqs. (2.28) and (2.30) were evaluated using the operator formalism explained above, which can be considered for this reason as an operator formalism valid on the Riemann surfaces of any genus with $Z_n$ symmetry.

3. THE $b−c$ SYSTEMS ON A RIEMANN SURFACE AS A CONFORMAL FIELD THEORY

Using the results found in the previous section, we express now the correlation functions of free fermionic $b−c$ systems on a Riemann surface by means of a suitable conformal
field theory on the sphere. The content of primary fields is the following. First of all, we need a set of \( n \) free \( b-c \) fields:

\[
\tilde{b}_k(z) dz^\lambda = \sum_{i=-\infty}^{\infty} \tilde{b}_{k,i} z^{-i-\lambda} dz^\lambda
\]  
(3.1)

\[
\tilde{c}_k(z) dz^{1-\lambda} = \sum_{i=-\infty}^{\infty} \tilde{c}_{k,i} z^{-i+\lambda-1} dz^{1-\lambda}
\]  
(3.2)

with

\[
\tilde{b}_k(z) \tilde{c}_{k'}(w) \sim \frac{\delta_{kk'}}{z-w} + \tilde{b}_k(z) \tilde{c}_{k'}(w) : .
\]  
(3.3)

Moreover, we introduce the spin fields \( V_k(a_l), l = 1, \ldots, mn \) such that:

\[
\tilde{b}_k(z) V_k(a_l) \sim (z-a_l)^{-q_k,a_l} \tilde{b}_k(z) V_k(a_l) : + \ldots
\]  
(3.4)

\[
\tilde{c}_k(z) V_k(a_l) \sim (z-a_l)^{q_k,a_l} \tilde{c}_k(z) V_k(a_l) : + \ldots
\]  
(3.5)

where:

\[
q_k,a_l = \frac{\lambda(n-1) - k}{n} \quad k = 0, \ldots, n-1.
\]  
(3.6)

Theories with spin fields of that kind have been already discussed in refs. [9] and [10]. In the first part of this section, therefore, we will follow [10] in order to compute the correlator:

\[
s_k = \langle \tilde{b}_k(z) \prod_{i=1}^{N_{bk}} \tilde{b}_k(z_i) \prod_{l=1}^{mn} V_k(a_l) \rangle.
\]  
(3.7)

The result will be an equation which is the equivalent of eq. (2.23). We notice that \( s_k \) has no singularities in the variables \( z_i \), therefore it should be proportional to a product of the zero modes (2.15). The correlator \( s_k \) can be computed using the energy momentum tensor method, i.e. exploiting the fact that if \( \phi_1(z_1) \ldots \phi_N(z_N) \) are primary fields then the following equation holds:

\[
\langle T(z) \phi_1(z_1) \ldots \phi(z_N) \rangle = \sum_{i=1}^{N} \left( \frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i} \partial_{z_i} \right) \langle \phi_1(z_1) \ldots \phi(z_N) \rangle
\]  
(3.8)

where \( T(z) \) is the energy momentum tensor. In our case it is splitted in \( n \) components \( T_k(z), k = 0, \ldots, n-1 \), where:

\[
T_{k,zz}(z) = -\lambda \partial_z \tilde{b}_k \tilde{c}_k + (1-\lambda) \tilde{b}_k \partial_z \tilde{c}_k.
\]  
(3.9)
Therefore, instead of eq. (3.8), we have to evaluate:

$$
\langle\langle T_k(z)\rangle\rangle = \frac{k \langle \bar{0} | T_k(z) \prod_{i=1}^{N_{b_k}} \bar{b}_k(z_i) \prod_{l=1}^{nm} V_k(a_l) | \bar{0} \rangle_k}{k \langle \bar{0} | \prod_{i=1}^{N_{b_k}} \bar{b}_k(z_i) \prod_{l=1}^{nm} V_k(a_l) | \bar{0} \rangle_k}.
$$

(3.10)

As a first step, we derive the following Green function:

$$
F_k(z, z_0) = \frac{k \langle \bar{0} | \bar{b}_k(z) \bar{c}_k(z_0) \prod_{i=1}^{N_{b_k}} \bar{b}_k(z_i) \prod_{l=1}^{nm} V_k(a_l) | \bar{0} \rangle_k}{k \langle \bar{0} | \prod_{i=1}^{N_{b_k}} \bar{b}_k(z_i) \prod_{l=1}^{nm} V_k(a_l) | \bar{0} \rangle_k}.
$$

(3.11)

As shown in refs. [18] and [15] in the more general case of algebraic curves with nonabelian group of symmetry $D_n$, the total ghost charge in the numerator and in the denominator of eq. (3.11) is zero, so that $F_k(z, z_0)$ is well defined. In particular, let us notice that the numerator of eq. (3.11) is the only possible correlator with the insertion of only one $c$ field that does not vanish. This fact will be useful below when we will compute the propagator of the $b-c$ systems on $\Sigma_g$ in terms of the Green function $F_k(z, z_0)$. As noticed in [10] the Green functions (3.11), which are tensors in $z$ and $z_0$, are completely determined by their poles and zeros in $z$, $z_0$ and $z_i$. The upshot is (see [10] and [3] for more details):

$$
F_k(z, z_0) = \frac{f_k(z) \phi_k(z_0)}{z-z_0} \prod_{i=1}^{N_{b_k}} \left( \frac{z-z_i}{z_0-z_i} \right).
$$

(3.12)

We note that $F_k(z, z_0)$ is normalized in such a way that $\lim_{z \to z_0} F_k(z, z_0) = 1$. The Green functions $F_k(z, z_0)$ will play the role of the multivalued blocks in our conformal field theory induced by the $b-c$ systems on $\Sigma_g$. In fact, $F_k(z, z_0)$ is proportional to the function $f_k(z)$ given in eq. (3.12) which describes all the possible monodromy properties compatible with the local monodromy group $Z_n$ of $\Sigma_g$. Now it is possible to evaluate $\langle\langle T_k(z)\rangle\rangle$ using the explicit form of the $F_k(z, z_0)$. As a matter of fact, from eq. (3.4) we get:

$$
\langle\langle T_k(z)\rangle\rangle = \lim_{z \to z_0} \left( -\lambda \partial_z F_k(z, z_0) + (1 - \lambda) \partial_{z_0} F_k(z, z_0) - \frac{1}{(z-z_0)} \right).
$$

(3.13)

Moreover, eq. (3.8) says that the residues of $\langle\langle T_k(z)\rangle\rangle$ in $z = a_l$ are:

$$
\text{Res} \langle\langle T_k(z)\rangle\rangle_{z=a_l} = \partial_{a_l} \log \left( k \langle \bar{0} | \prod_{i=1}^{N_{b_k}} \bar{b}(z_i) \prod_{l=1}^{nm} V_k(a_l) | \bar{0} \rangle_k \right).
$$

(3.14)
The left hand side becomes explicitly known after having combined with eqs \((3.12)\) and \((3.13)\). Finally, eq. \((3.14)\) can be solved as in \([10]\) giving:

\[
s_k = \det|\Omega_{k,jk}(z_{ik})| \prod_{l=1}^{nm} \prod_{l' \neq l=1}^{nm} (a_l - a_{l'})^{q_{k,a_l}q_{k,a_{l'}}}.
\]

This is the promised equivalent of eq. \((2.23)\).

We would like to show that the above conformal field theory with additional \(Z_n\) symmetry is equivalent to the \(b-c\) systems on the Riemann surface \(\Sigma_g\) defined in eq. \((2.1)\). To this purpose, we introduce the total vacuum

\[
|\tilde{0}\rangle = \prod_{k=0}^{n-1} |\tilde{0}\rangle_k
\]

and the total fields:

\[
\tilde{b}(z) = \sum_{k=0}^{n-1} \tilde{b}_k(z) \quad \tilde{c}(z) = \sum_{k=0}^{n-1} \tilde{c}_k(z).
\]

Finally, we define the spin fields

\[
V(a_l) = \prod_{k=0}^{n-1} V_k(a_l).
\]

As a first step, we prove the following equation:

\[
\langle \tilde{0}|\tilde{b}(z_1)\ldots \tilde{b}(z_{N_b}) \prod_{l=1}^{nm} V(a_l)|\tilde{0}\rangle = \det|\Omega_I(z_J)| \prod_{l=1}^{nm} \prod_{l' \neq l=1}^{nm} (a_l - a_{l'})^{\gamma_{ll'}}
\]

where \(\gamma_{ll'} = \sum_{k=0}^{n-1} q_{k,a_l}q_{k,a_{l'}}\) and \(I,J = 1,\ldots,N_b\). Eq. \((3.19)\) can be verified in the same way as eq. \((2.24)\). The only difference is that instead of eq. \((2.23)\) we have to exploit now eq. \((3.15)\) in order to compute the correlators \(s_k\). This is the reason for which eq. \((2.24)\) and eq. \((3.19)\) differ by the factor \(\prod_{l=1}^{nm} \prod_{l' \neq l=1}^{nm} (a_l - a_{l'})^{\gamma_{ll'}}\).

Now we are ready to compute the propagator \((2.28)\) of the \(b-c\) systems on a Riemann surface in terms of the multivalued blocks \((3.12)\). We have to evaluate a correlator of the kind

\[
\langle \tilde{0}|\tilde{c}(w) \prod_{l=1}^{N_b+1} \tilde{b}(z_I) \prod_{l=1}^{nm} V(a_l)|\tilde{0}\rangle = \sum_{k=0}^{n-1} \langle \tilde{0}|\tilde{c}_k(w) \prod_{l=1}^{N_b+1} \tilde{b}(z_I) \prod_{l=1}^{nm} V(a_l)|\tilde{0}\rangle.
\]
Following a strategy similar to that applied in the Appendix A, we use the expansions (3.17) and then apply the Lemma 1. We observe that for each value of $k$ there is only one partition of the number $N_b + 1$ giving a non-zero contribution. Taking into account (3.16) the structure of the expression we obtain is the following. It is a product of $(n-1)$ correlators of the type (3.7) multiplied by a correlator present in the numerator of (3.11). These $(n-1)$ correlators can be calculated using the formula (3.15). The last correlator to be computed in eq. (3.20) can be represented in the following very convenient form:

$$k\langle \tilde{0} | \tilde{c}_k(w) \prod_{i_k=\alpha(k)}^{\beta(k)} \tilde{b}_k(z_{\sigma(i_k)}) \prod_l V_k(a_l) | \tilde{0} \rangle_k =$$

$$= - \prod_{l=1}^{nm} \prod_{l' \neq l=1}^{nm} (a_l - a_{l'})^{q_{k,a_l} q_{k,a_{l'}}} \sum_{i_k=\alpha(k)}^{\beta(k)} \frac{\phi_k(w) f_k(z_{\sigma(i_k)})}{z_{\sigma(i_k)} - w} (-1)^{ik} \det|\Omega_{k,j_k}(z_{\sigma(l_k)})| (3.21)$$

where

$$l_k \in \{\alpha(k), \ldots, \beta(k)\} \quad \text{with} \quad l_k \neq i_k.$$

Here $\alpha(k) = 1 + \sum_{m=1}^{k-1} N_{b_m}$ and $\beta(k) = \alpha(k) + N_{b_k}$. The notation is taken from the Appendix A with a minor modification due to the presence of the field $\tilde{c}_k$ in (3.21). When $k = 0$, the sum in $\alpha(k)$ is by definition zero. We notice that a priori we should put a sign $(-1)^{\alpha(k)}$ in the LHS of eq. (3.21). This comes from the fact that, before to compute the correlator (3.21), we have to commute $\alpha(k)$ times the field $\tilde{c}_k(w)$ with the $b-$fields. However, this sign cancels with an equal factor appearing in the RHS, coming from the fact that the sum over $i_k$ starts from $\alpha(k)$. Using eq. (3.21) we obtain:

$$\langle \tilde{0} | \tilde{c}(w) \prod_{l=1}^{N_b+1} \tilde{b}(z_I) \prod_l V(a_l) | \tilde{0} \rangle = - \prod_{l=1}^{nm} \prod_{l' \neq l=1}^{nm} (a_l - a_{l'})^{\gamma_{l,l'}} \times$$

$$\sum_{k=0}^{n-1} \sum_{\sigma} \sum_{p \neq k} \det|\Omega_{p,j_p}(z_{\sigma(i_p)})| \prod_{i_k=\alpha(k)}^{\beta(k)} \frac{\phi_k(w) f_k(z_{\sigma(i_k)})}{z_{\sigma(i_k)} - w} (-1)^{ik} \det|\Omega_{k,j_k}(z_{\sigma(l_k)})|. (3.22)$$

It is now useful to observe that:

$$\sum_{i_k=\alpha(k)}^{\beta(k)} \frac{\phi_k(w) f_k(z_{\sigma(i_k)})}{z_{\sigma(i_k)} - w} (-1)^{ik} \det|\Omega_{k,j_k}(z_{\sigma(l_k)})| =$$
\[ \det \begin{pmatrix} \phi_k(w)f_k(z_\sigma(\alpha(k)))/z_\sigma(\alpha(k))-w \\ \vdots \\ \phi_k(w)f_k(z_\sigma(\beta(k)))/z_\sigma(\beta(k))-w \\ \Omega_1(z) \\ \vdots \\ \Omega_1(z) \\ \Omega_1(z) \\ \Omega_k,j_1(z_\sigma(\alpha(k))) \\ \vdots \\ \Omega_k,j_1(z_\sigma(\alpha(k))) \\ \cdots \\ \Omega_k,j_{N_bkh} \end{pmatrix} \]

It is easy to see that the factor \((-1)^{\alpha(k)}\) comes from the fact that the sum in the LHS of eq. (3.23) begins from \(\alpha(k)\). Using now the Lemma 2 from the Appendix A we obtain

\[ \langle \tilde{0} | \tilde{c}(w) \prod_{I=1}^{N_b+1} \tilde{b}(z_I) \prod_l V(a_l)|\tilde{0} \rangle = -\prod_{\ell=1}^{nm} \prod_{\ell' \neq \ell=1}^{nm} (a_\ell - a_{\ell'})^\gamma_{\mu\nu} \]

\[ \sum_{k=0}^{n-1} (-1)^{\alpha(k)} \det \begin{pmatrix} \Omega_1(z_1) \\ \vdots \\ \Omega_1(z_{N_b+1}) \\ \Omega_1(z_1) \\ \vdots \\ \Omega_1(z_{N_b+1}) \\ \cdots \\ \Omega_1(z_1) \\ \cdots \\ \cdots \\ \Omega_{N_b}(z_{N_b+1}) \end{pmatrix} \]

It is not difficult to convince oneself that with the help of eq. (2.4) one can reproduce the formula (2.28) up to the term coming from the ramification points (closely related to the chiral determinant of the appropriate Dirac operator). As a matter of fact, the wrong position of the column containing the tensors \(\phi_k(w)f_k(z_I)/(z_I-w)\), \(I = 1, \ldots , N_b\), is compensated by the factor \((-1)^{\alpha(k)}\). Moreover, the overall minus sign in the RHS of eq. (3.24) takes into account of the fact that, with respect to eq. (2.28), the field \(c(w)\) has been once commuted with a \(b\)-field. Thus we have proved that the two point function of the \(b - c\) systems can be computed starting from the conformal field theory with \(Z_n\) symmetry described at the beginning of this section.

4. CONCLUSIONS

In this paper we have shown that the \(b - c\) systems on a Riemann surface are equivalent to a multivalued theory of ghosts defined on the complex plane, in which the fields are expanded in the basis given by eqs. (2.11) and (2.12). It is clear that this basis has zeros and poles at the projections on the Riemann surface of the points \(z = 0\) and \(z = \infty\) respectively. In this sense, our basis represents an example of the generalized Krichever-Novikov bases [26] of the kind discussed in [27]. In the basis (2.11) and (2.12) the derivation
of the amplitudes of the $b-c$ systems becomes simpler, but we believe that it is possible to set up an operatorial formalism like that of section 2 also starting from any other equivalent basis provided that (2.4) is satisfied. Obviously, the operator formalism of section 2 is a general one and can be extended to other curves. If this is true, it implies that the $b-c$ systems on an arbitrary Riemann surface are equivalent to a multivalued field theory on the complex plane. The next step in order to verify this conjecture is provided by the $D_n$ symmetric curves discussed in ref. [15] and by the example of a quartic of genus three [4, 4].

On the other side, in section 3 we have also shown how the amplitudes of the $b-c$ systems on the $Z_n$ symmetric curves are determined by the operator product expansions of the primary fields of eqs. (3.3), (3.4) and (3.5). Explicit examples have been given in the case of the zero and two point functions (eqs. (3.19) and (3.24)). The extension of this result also to the $D_n$ symmetric curves of ref. [15] seems to be feasible. This is an interesting class of algebraic curves because the exchange algebra between the twist fields provides an example of nonabelian statistics involving multiparameter Yang–Baxter matrices [19].

Appendix A.

The proof of the proposition 2 becomes quite straightforward once two basic lemmas are established.

Lemma 1

\[ \prod_{i=1}^{N} \left( \sum_{j=1}^{M} a_{i,j} \right) = \sum_{r_1+\ldots+r_M=N} \sum_{\sigma} \text{sgn}(\sigma) \prod_{k=1}^{N} \prod_{l=\alpha(k)}^{\beta(k)} a_{\sigma(l),k} \]  

(A.1)

where $a_{k,j}$ are anticommuting Grassmann variables,

\[ r_j \geq 0; \quad r_0 = 0; \quad \alpha(k) = 1 + \sum_{m=1}^{k-1} r_m; \quad \beta(k) = \alpha(k) + r_k - 1. \]  

(A.2)
The symbol $\sum_\sigma$ in the (A.1) denotes the sum over all the permutations of numbers 1,...,N such that
\[ \sigma(\alpha(k)) < \ldots < \sigma(\beta(k)) \quad k = 1, \ldots, M. \] (A.3)

The simplest way to proof (A.1) is to perform an induction in M.

**Lemma 2**

Consider a $N \times N$ matrix $A$ with elements $a_{ik}$. Suppose we have a partition of $N$ into $M$ integers: $N = r_1 + r_2 + \ldots + r_M$ where $r_j \geq 0$. For each permutation $\sigma$ satisfying the condition (A.3) we define matrices:
\[ A^{(k)}_\sigma = \begin{pmatrix}
a_{\sigma(\alpha(k)), \alpha(k)} & \cdots & a_{\sigma(\alpha(k)), \beta(k)} \\
\vdots & \ddots & \vdots \\
a_{\sigma(\beta(k)), \alpha(k)} & \cdots & a_{\sigma(\beta(k)), \beta(k)}
\end{pmatrix} \] (A.4)

where $\alpha(k)$ and $\beta(k)$ are defined in (A.2). Then
\[ \det A = \sum_\sigma \text{sign}(\sigma) \prod_{j=1}^M \det |A^{(j)}| \] (A.5)

where $\sigma$ is an arbitrary permutation satisfying (A.3). This is a generalization of the usual way of computing determinants (see for example [28]).

The best known subcase of eq. (A.5) is that for which $r_1 = 1$, $r_2 = N - 1$ and remaining $r_j$ vanish.

With the help of Lemmas 1 and 2 we can check eq. (2.24). Our aim is to calculate the correlator $\langle 0 | \prod_{I=1}^{N_b} b(z_I) | 0 \rangle$. The strategy is to decompose the fields $b(z_I)$ according to (2.10). We apply now the Lemma 1 and obtain a sum over all possible partitions of the number $N_b$. According to properties of the vacuum expressed in (2.22) only one partition gives a nonzero contribution: the number of $b_k$ fields has to be equal $N_{b_k}$. Once we establish this fact we arrive immediately at the product of $n$ correlators of the type given in eq. (2.23). With the help of Lemma 2 we immediately complete the proof of the Proposition 2.
References

[1] D. J. Gross, P. F. Mende, Phys. Lett. 179B (1987), 129; Nucl. Phys. B303 (1988), 407; R. Iengo, Nucl. Phys. B15 (Proc. Suppl.) (1990), 67; E. Gava, R. Iengo, G. Sotkov, Phys. Lett. 207B (1988), 283; D. Lebedev, A. Morozov, Nucl. Phys. B302 (1988), 63; A. A. Belavin, V. G. Knizhnik, A. Yu. Morozov, A. M. Perelomov, JETP Lett. 43 (1986), 411.

[2] F. Ferrari, Int. Jour. Mod. Phys. A5 (1990), 2799.

[3] F. Ferrari, Int. Jour. Mod. Phys. A7 (1992), 5131.

[4] J. Sobczyk, W. Urbanik, Lett. Math. Phys. 21 (1991), 1.

[5] Al. B. Zamolodchikov, Nucl. Phys. B285 [FS19] (1987), 481.

[6] D. A. Korotkin, V. V. Matveev Leningrad Math. Jour. 1 (1990), 379.

[7] I. M. Krichever, Russian Math. Surveys 32 (6) (1977), 185; Funct. Anal. Appl. 11 (1) (1977), 12; H. P. McKean, E. Trubowitz, Comm. Pure Appl. Math. XXIX (1976), 143.

[8] E. I. Zverovich, Russ. Math. Surv. 21 (1971), 99.

[9] V. G. Knizhnik, Sov. Phys. Usp. 32(11) (1989) 945.

[10] M. A. Bershadsky and A. O. Radul, Int. Jour. Math. Phys. A2 (1987) 165.

[11] B.L. Feigin, D.B. Fuchs, Funct. Anal. Applic. 17 (1983), 241; A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B241 (1984), 333; V. S. Dotsenko, V. A. Fateev, Nucl. Phys. B240 [FS12] (1984), 312; G. Felder, Nucl. Phys. B317 (1989), 215.

[12] G. Felder, J. Fröhlich, G. Keller, Comm. Math. Phys. 124 (1989), 647; L. Bonora, M. Matone, F. Toppan and K. Wu, Phys. Lett. 224B (1989) 115; Nucl. Phys. B334 (1990) 717.

[13] M. Sato, T. Miwa and M. Jimbo. Holonomic quantum fields (Kyoto U.P. Kyoto), part I; 14 (1978) p. 223; II: 15 (1979) p. 201; III: 15 (1979) p. 577; IV: 15 (1979) p. 871; V; 16 (1980) p. 531.

[14] D. V. Chudnowski in: C. Bardos, D. Bessis (eds.), Bifurcation Phenomena in Mathematical Physics and Related Topics, D. Reidel Publishing Company 1980.

[15] F. Ferrari, Comm. Math. Phys. 156 (1993) 179

[16] C. Vafa, Phys. Lett. 190B (1987), 47; L. Alvarez Gaumé, C. Gomez, C. Reina, Phys. Lett. 190B (1987), 55.

[17] F. Ferrari, J. Sobczyk, W. Urbanik, in preparation.

[18] F. Ferrari, Phys. Lett. 277B (1992), 423.

[19] F. Ferrari, Multivalued Fields on the Complex Plane and Braid Group Statistics, Preprint LMU-TPW 93-24, to appear in Int. Jour. Mod. Phys. A.

[20] J. Sobczyk, Mod. Phys. Lett. A6 (1991), 1103.
[21] L. Dixon, D. Friedan, E. Martinec, S. Shenker, *Nucl. Phys.* **B282** (1987), 13; S. Hamidi, C. Vafa, *Nucl. Phys.* **B279** (1987), 465; J. J. Atick, A. Sen, *Nucl. Phys.* **B286** (1987), 189; E. Guadagnini, M. Martellini, M. Mintchev, *Jour. Math. Phys.* **31** (1990), 1226; J. Sobczyk, *Current-Current Correlation Functions on Algebraic Curves*, Preprint ITP UWr-791/91; *Insertion at Infinity in Conformal Field Theories on Algebraic Curves*, Preprint ITP UWr 792/91.

[22] B. Blok and S. Yankielowicz, *Nucl. Phys.* **B321** (1989) 327. B. Blok and S. Yankielowicz, *Phys. Lett.* **226B** (1989) 279.

[23] F. Ferrari, *Jour. Math. Phys.* **32** (1991), 2186.

[24] A. K. Raina, *Helv. Phys. Acta* **63** (1990), 694.

[25] M. Bonini and R. Iengo, *Int. Jour. Mod. Phys.* **A3** (1988) 841.

[26] I. M. Krichever, S. Novikov, *Funkt. Anal. i Pril.* **211** (1987), 46.

[27] R. Dick, *Lett. Math. Phys.* **18** (1989), 255; M. Schlichenmaier, *Lett. Math. Phys.* **19** (1990), 151.

[28] F. Neiß, H. Liermann, Determinanten und Matrizen, 8th edition, Springer Verlag, (in german).