An efficient parallel algorithm for the longest path problem in meshes

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Abstract The longest path problem is the problem of finding a simple path with the maximum number of vertices in a given graph, and so far it has been solved polynomially only for a few classes of graphs. This problem generalizes the well-known Hamiltonian path problem, hence it is NP-hard in general graphs. In this paper, first we give a sequential linear-time algorithm for the longest path problem in meshes. Then based on this algorithm, we present a constant-time parallel algorithm for the problem, which can be run on every parallel machine.

Keywords Grid graph · Longest path · Meshes · Sequential and parallel algorithms

1 Introduction

The longest path problem, i.e., the problem of finding a simple path with the maximum number of vertices, is one of the most important problems in graph theory. The well-known NP-complete Hamiltonian path problem, i.e., deciding whether there is a simple path that visits each vertex of the graph exactly once, is a special case of the longest path problem and has many applications [5, 7].

Only few polynomial-time algorithms are known for the longest path problem for special classes of graphs. This problem for trees began with the work of Dijkstra around 1960, and was followed by others [2, 9, 16, 18, 22]. In the area of approximation algorithms, it has been shown that the problem is not in APX, i.e., there is no polynomial-time constant factor approximation algorithm for the problem unless

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P = NP [9]. Also, it has been shown that finding a path of length \( n - n^\epsilon \) is not possible in polynomial-time unless P = NP [12]. For the background and some known results about approximation algorithms, we refer the reader to [1, 6, 24].

A grid graph is a graph in which vertices lie on integer coordinates and edges connect vertices that are separated by a distance of one. A solid grid graph is a grid graph without holes. A rectangular grid graph \( R(m, n) \) is the subgraph of \( G^\infty \) (the infinite grid graph) induced by \( V(R) = \{ v | 1 \leq v_x \leq m, 1 \leq v_y \leq n \} \), where \( v_x \) and \( v_y \) are respectively \( x \) and \( y \) coordinates of \( v \) (see Fig. 1). A mesh \( M(m, n) \) is a rectangular grid graph \( R(m, n) \). Grid graphs can be useful representations in many applications. Myers [19] suggests modeling city blocks in which street intersection are vertices and streets are edges. He studied enumeration of tours in Hamiltonian rectangular lattice graphs. Luccio and Mugnia [17] suggest using a grid graph to represent a two-dimensional array type memory accessed by a read/write head moving up, down or across. The vertices correspond to the center of each cell and edges connect adjacent cells. Finding a path in the grid corresponds to accessing all the data. They studied Hamiltonian paths on rectangular chessboards.

Itai et al. [11] have shown that the Hamiltonian path problem for general grid graphs, with or without specified endpoints, is NP-complete. The problem for rectangular grid graphs, however, is in P requiring only linear-time. Later, Chen et al. [3] improved the algorithm of [11] and presented a parallel algorithm for the problem in mesh architecture. Lenhart and Umans [15] have presented a polynomial-time algorithm for finding Hamiltonian cycles in solid grid graphs. Zamfirescu and Zamfirescu [23] have given sufficient conditions for a grid graph to be Hamiltonian and it is proved that all finite grid graphs of positive width have Hamiltonian line graphs.

Recently, Hamiltonian cycle (path) and longest path problems in grid graphs have received much attention. Salman [21] introduced a family of grid graphs, i.e., alphabet grid graphs, and determined classes of alphabet grid graphs that contain Hamiltonian cycles. Islam et al. [10] showed that the Hamiltonian cycle problem in hexagonal grid graphs is NP-complete. Gordon et al. [8] proved that all connected, and locally connected triangular grid graphs are Hamiltonian, and gave a sufficient condition for a connected graph to be fully cycle extendable and also showed that the Hamiltonian cycle problem for triangular grid graphs is NP-complete.

Zhang and Liu [25] gave an approximation algorithm for the longest path problem in grid graphs; their algorithm runs in quadratic time. Keshavarz-Kohjerdi et al. [13] studied the longest path problem for rectangular grid graphs; their algorithm is based on the divide and conquer technique and runs in linear time. Some other results about grid graphs are investigated in [14, 20].

In this paper, first we present a sequential and then a parallel algorithm for finding a longest path between two given vertices in a rectangular grid graph (mesh).

1.1 Our contribution

We present the first parallel algorithm for the longest path problem on meshes. This algorithm can be considered as an improvement and a parallel version of the algorithm in [13]. Our algorithm has improved the previous algorithm [13] by reducing the number of partition steps from \( O(m + n) \) to a constant.
Fig. 1 The rectangular grid graph $R(8, 7)$

1.2 Organization of the paper

In Sect. 2, some necessary definitions and previous results are given. A sequential algorithm for the longest path problem is given in Sect. 3. In Sect. 4, a parallel algorithm for the problem is introduced, which is based on the sequential algorithm. The conclusion is given in Sect. 5.

2 Preliminary definitions and previous results

In this section, we give a few definitions and introduce the corresponding notations. We then gather some previously established results on the Hamiltonian and the longest path problems in grid graphs which have been presented in [3, 11, 13].

The two-dimensional integer grid $G^\infty$ is an infinite graph with vertex set of all the points of the Euclidean plane with integer coordinates. In this graph, there is an edge between any two vertices of unit distance. For a vertex $v$ of this graph, let $v_x$ and $v_y$ denote $x$ and $y$ coordinates of its corresponding point, respectively (sometimes we use $(v_x, v_y)$ instead of $v$). We color the vertices of the two-dimensional integer grid as black and white. A vertex $v$ is colored white if $v_x + v_y$ is even, and it is colored black otherwise.

A grid graph $G_g$ is a finite vertex-induced subgraph of the two-dimensional integer grid. In a grid graph $G_g$, each vertex has degree at most four. Clearly, there is no edge between any two vertices of the same color. Therefore, $G_g$ is a bipartite graph. Note that any cycle or path in a bipartite graph alternates between black and white vertices. A rectangular grid graph $R(m, n)$ (or $R$ for short) is a grid graph whose vertex set is $V(R) = \{v \mid 1 \leq v_x \leq m, 1 \leq v_y \leq n\}$. A mesh $M(m, n)$ is a rectangular grid graph $R(m, n)$; see Fig. 1.

In the figures, we assume that $(1, 1)$ is the coordinates of the vertex in the upper left corner. The size of $R(m, n)$ is defined to be $mn$. $R(m, n)$ is called odd-sized if $mn$ is odd, and it is called even-sized otherwise.

Since even $\times$ odd rectangular grid graphs and odd $\times$ even rectangular grid graphs are isomorphic, so all rectangular grid graphs considered here are odd $\times$ odd, even $\times$ odd, and even $\times$ even. $R(m, n)$ is called a 1-rectangle if either $m = 1$ or $n = 1$, or a 2-rectangle if either $n = 2$ and $m > 1$, or $m = 2$ and $n > 1$.

The following lemma states a result about the Hamiltonicity of even-sized rectangular graphs.

Lemma 2.1 [3] $R(m, n)$ has a Hamiltonian cycle if and only if it is even-sized and $m, n > 1$.
Figure 2 shows a Hamiltonian cycle for an even-sized rectangular grid graph, found by Lemma 2.1. Every Hamiltonian cycle found by this lemma contains all the boundary edges on the three sides of the rectangular grid graph. This shows that for an even-sized rectangular grid graph $R$, we can always find a Hamiltonian cycle, such that it contains all the boundary edges, except of exactly one side of $R$ which contains an even number of vertices.

Two distinct vertices $v$ and $v'$ in $R(m, n)$ are called color-compatible if either both $v$ and $v'$ are white and $R(m, n)$ is odd-sized, or $v$ and $v'$ have different colors and $R(m, n)$ is even-sized. Let $(R(m, n), s, t)$ denote the rectangular grid graph $R(m, n)$ with two specified distinct vertices $s$ and $t$. Without loss of generality, we assume $s_x \leq t_x$.

$(R(m, n), s, t)$ is called Hamiltonian if there exists a Hamiltonian path between $s$ and $t$ in $R(m, n)$. An even-sized rectangular grid graph contains the same number of black and white vertices. Hence, the two end-vertices of any Hamiltonian path in the graph must have different colors. Similarly, in an odd-sized rectangular grid graph the number of white vertices is one more than the number of black vertices. Therefore, the two end-vertices of any Hamiltonian path in such a graph must be white. Hence, the color-compatibility of $s$ and $t$ is a necessary condition for $(R(m, n), s, t)$ to be Hamiltonian. Furthermore, Itai et al. [11] showed that if one of the following conditions hold, then $(R(m, n), s, t)$ is not Hamiltonian:

(F1) $R(m, n)$ is a 1-rectangle and either $s$ or $t$ is not a corner vertex (Fig. 3(a)).

(F2) $R(m, n)$ is a 2-rectangle and $(s, t)$ is a nonboundary edge, i.e., $(s, t)$ is an edge and it is not on the outer face (Fig. 3(b)).

(F3) $R(m, n)$ is isomorphic to a 3-rectangle grid graph $R'(m, n)$ such that $s$ and $t$ is mapped to $s'$ and $t'$ and all of the following three conditions hold:

1. $m$ is even,
2. $s'$ is black, $t'$ is white,
3. $s'_x = 2$ and $s'_y < t'_x$ (Fig. 3(c)) or $s'_y \neq 2$ and $s'_x < t'_x - 1$ (Fig. 3(d)).

They showed that $(R(m, n), s, t)$ is Hamiltonian if and only if $s$ and $t$ are color-compatible and $R(m, n), s$ and $t$ do not satisfy any of conditions (F1), (F2), and (F3). In the following, we use $P(R(m, n), s, t)$ to indicate the problem of finding a longest path between vertices $s$ and $t$ in a rectangular grid graph $R(m, n)$, $L(R(m, n), s, t)$ to show the length of longest paths between $s$ and $t$ and $U(R(m, n), s, t)$ to indicate the upper bound on the length of longest paths between $s$ and $t$. Keshavarz-Kohjerdi et al. [13] defined the following conditions:

(C0) $s$ and $t$ are color-compatible and none of (F1)–(F3) hold.

(C1) Neither (F1) nor (F2*) holds and either

1. $R(m, n)$ is even-sized and $s$ and $t$ are same-colored or
2. $R(m, n)$ is odd-sized and $s$ and $t$ are different-colored.

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Fig. 3 The rectangular grid graphs in which there is no Hamiltonian path between $s$ and $t$

(C2) 1. $R(m,n)$ is odd-sized and $s$ and $t$ are black-colored and neither (F1) nor (F2*) holds, or
2. $s$ and $t$ are color-compatible and (F3) holds,

where (F2*) is defined as follows:

(F2*) $R(m,n)$ is a 2-rectangle and $s_x = t_x$ or $(s_x = t_x - 1$ and $s_y \neq t_y$).

It is easy to show that any $(R(m,n), s, t)$ must satisfy one of conditions (C0), (C1), (C2), (F1) and (F2*). They proved the following upper bounds on the length of longest paths:

$$U(R(m,n), s, t) = \begin{cases} 
  t_x - s_x + 1, & \text{if (F1)}, \\
  \max(t_x + s_x, 2m - t_x - s_x + 2), & \text{if (F2*)}, \\
  mn, & \text{if (C0)}, \\
  mn - 1, & \text{if (C1)}, \\
  mn - 2, & \text{if (C2)}.
\end{cases}$$

**Theorem 2.1** [13] In a rectangular grid graph $R(m,n)$, a longest path between any two vertices $s$ and $t$ can be found in linear time and its length (i.e., $L(R(m,n), s, t)$) is equal to $U(R(m,n), s, t)$, which is defined before.

3 The sequential algorithm

In this section, we present our sequential algorithm for finding a longest path between two given vertices in a rectangular grid graph. This algorithm is the base of our parallel algorithm which is introduced in Sect. 4. First, we solve the problem for 1-rectangles and 2-rectangles.

**Lemma 3.1** [13] Let $P(R(m,n), s, t)$ be a longest path problem with $n = 1$ or $n = 2$, then $L(R(m,n), s, t) = U(R(m,n), s, t)$.

The proofs of lemmas are moved to the Appendix to improve readability of the paper.

**Definition 3.1** [13] A separation of a rectangular grid graph $R$ is a partition of $R$ into two disjoint rectangular grid graphs $R_1$ and $R_2$, i.e., $V(R) = V(R_1) \cup V(R_2)$, and $V(R_1) \cap V(R_2) = \emptyset$. 

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**Definition 3.2** [11] Let \( \upsilon \) and \( \upsilon' \) be two distinct vertices in \( R \). If \( \upsilon_x \leq 2 \) and \( \upsilon'_x \geq m - 1 \), then \( \upsilon \) and \( \upsilon' \) are called *antipodes*.

From now on, we assume that \( m, n > 2 \), so one of conditions (C0), (C1), and (C2) should hold. Following the technique used in [3], we develop an algorithm for finding longest paths.

Our sequential algorithm is given in Algorithm 3.1. In the following, we describe the steps of the algorithm in detail.

**Algorithm 3.1** The sequential longest path algorithm

```
procedure SeqLongestPath(R(m, n), s, t)
Input: A rectangular grid graph \( R(m, n) \) with two disjoint vertices \( s \) and \( t \)
Output: A longest path of \( G \)

Step 1. By a peeling operation, partitions \( R(n, m) \) into five disjoint rectangular grid subgraphs \( R_1 \) to \( R_5 \), such that \( s, t \in R_5 \)
Step 2. Find a longest path between \( s \) and \( t \) in \( R_5 \)
Step 3. Construct Hamiltonian cycles in rectangular grid subgraphs \( R_1 \) to \( R_4 \)
Step 4. Construct a longest path between \( s \) and \( t \) in \( R \) by combining the Hamiltonian cycles of \( R_1 \) to \( R_4 \) and the longest path of \( R_5 \)
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3.1 Peeling operation

In this section, we describe Step 1 of Algorithm 3.1. In this step, \( R(m, n) \) is partitioned into disjoint rectangular grid subgraphs.

**Definition 3.3** [3] Partitioning a rectangular grid graph \( R \) into five disjoint rectangular grid subgraphs \( R_1 - R_5 \) that is done by two horizontal and two vertical separations are called *peeling operation*, if the following two conditions hold:

1. \( s, t \in R_5 \), and \( s \) and \( t \) are antipodes.
2. Four rectangular grid subgraphs \( R_1 - R_4 \) are even-sized rectangular grid graphs whose boundary sizes are both greater than one, or are null rectangles.

Generally the two vertical separation of a peeling are done before the two horizontal separation. However, for an odd \( \times \) odd rectangular grid graph with \( s_x = t_x \), this order is reversed to guarantee that the boundary sizes of \( R_3 \) and \( R_4 \) are greater than one. Figure 4 shows a peeling operation on \( R(15, 11) \) where \( s \) is \( (6, 5) \) and \( t \) is \( (8, 9) \), and on \( R(9, 9) \) where \( s \) is \( (5, 3) \) and \( t \) is \( (5, 6) \).

The following lemma can be obtained directly from Definition 3.3.

**Lemma 3.2** [3] Let \( R_5(n_5, m_5) \) be the resulting rectangular grid subgraph of a peeling operation on \( R(n, m) \), where \( s, t \in V(R_5) \). Then

1. \( s \) and \( t \) keep the same colors in \( R_5 \) as in \( R \); and
2. \( R_5 \) has the same parity as \( R \), that is, \( m_5 \mod 2 = m \mod 2 \), and \( n_5 \mod 2 = n \mod 2 \).
Definition 3.4 A peeling operation on \( R \) is called proper if \(|R_1| + |R_2| + |R_3| + |R_4| + U(R_5, s, t) = U(R(n,m), s, t)\), where \(|R_i|\) denotes the number of vertices of \( R_i \).

Lemma 3.3 For the longest path problem \( P(R(n,m), s, t) \), any peeling operation on \( R(m,n) \) is proper if either:

1. Condition (C0) holds and \( m \mod 2 = n \mod 2 \) (i.e. \( R(m,n) \) is even \( \times \) even or odd \( \times \) odd), or
2. One of conditions (C1) and (C2) hold and \( R(m,n) \) is even \( \times \) odd or odd \( \times \) odd.

Nevertheless, a peeling operation on an even \( \times \) even rectangular grid graph \( R(m,n) \) may not be proper, see Fig. 5 where the dotted lines represent a peeling operation. In the two following cases, a peeling operation is not proper:

(F1') \( s \) is black, \( t_y = s_y + 1 \) and \( s_x \neq t_x \).
(F2') \( s \) is white, \( t_y = s_y - 1 \) and \( s_x \neq t_x \).

Lemma 3.4 For the longest path problem \( P(R(n,m), s, t) \), where \( R(m,n) \) is an even \( \times \) even rectangular grid graph, a peeling operation on \( R(m,n) \) is proper if and only if \( P(R(n,m), s, t) \) is not in cases (F1') and (F2').
When a peeling operation is not proper it can be made proper by adjusting the peeling boundaries. In that case, if $R_1$, $R_2$, $R_3$, and $R_4$ are empty, then $R_5$ is 2-rectangle that is in case (F2$^*$). Therefore, without loss of generality, we assume $R_1$, $R_2$, $R_3$, or $R_4$ is not empty. If $R_1$ or $R_2$ is not empty, then we move one column (or two columns when $R_1$ or $R_2$ is a 2-rectangle) from $R_1$ or $R_2$ to $R_5$ such that $R_1$ and $R_2$ are still even-sized rectangular grid graphs; see Fig. 6(a). If $R_3$ or $R_4$ is not empty, then we move one row (or two rows when $R_3$ or $R_4$ is 2-rectangle) from $R_3$ or $R_4$ to $R_5$ (Fig. 6(b)), or move the bottom row from $R_5$ to $R_4$ (Fig. 6(c)) or move the upper row from $R_5$ to $R_3$ (Fig. 6(d)), such that $R_3$ and $R_4$ are still even-sized rectangular grid graphs.

3.2 Finding a longest path in $R_5(m_5, n_5)$

In this section, we present Step 2 of Algorithm 3.1. Let $R(m, n)$ be an rectangular grid graph and let $R_5(m_5, n_5)$ be $R_5$ subrectangle of $R(m, n)$ constructed in Step 1. Now, we construct a longest path in $R_5(m_5, n_5)$. The following cases can be considered for $R_5(m_5, n_5)$:

(a) $m_5, n_5 \leq 3$.
(b) $m_5, n_5$ are even, and either $m_5 \geq 4$ or $n_5 \geq 4$;
(c) $m_5, n_5$ are odd, and either $m_5 \geq 5$ or $n_5 \geq 5$;
(d) $m_5$ is even and $n_5$ is odd, and either $m_5 \geq 4$ or $n_5 \geq 5$.

For case (a), we showed that when $n = 1, 2$ the problem can be solved easily. For $m, n = 3$ the longest paths of all the possible problems are depicted in Fig. 7 (the isomorphic cases are omitted). For cases (b), (c), and (d), we use the definition of trisecting.

**Definition 3.5** [3] Two separations of $R_5$ that partition it into three rectangular grid subgraphs $R_s^x$, $R_t^x$ and $R_m^x$ is called trisecting (see Fig. 8), if

(i) $R_s^x$ and $R_t^x$ are 2-rectangles, and
(ii) $s \in V(R_s^x)$ and $t \in V(R_t^x)$.

A trisecting can be done by two ways horizontally and vertically. If $n_5 \geq 4$, then trisecting is done horizontally, otherwise trisecting is done vertically.
Fig. 7 For $n = m = 3$, (a) $s$ and $t$ are white, then there is Hamiltonian path, (b) $s$ and $t$ have different colors, then there is a path with $U(R, s, t) = mn - 1$ and (c) $s$ and $t$ are black, then there is a path with $U(R, s, t) = mn - 2$

Fig. 8 A trisecting on $R(6, 6)$

Definition 3.6 The corner vertices $p$ and $q$ respectively on the boundary of $R^s_5$ and $R^t_5$ facing $R^m_5$ is called junction vertices if either

(i) Condition (C0) holds and they have different colors from $s$ and $t$, or

(ii) One of conditions (C1) and (C2) holds and $U(R^s_5, s, p) + U(R^m_5, m, m') + U(R^t_5, q, t) = U(R_5(m, n), s, t)$, where $m$ and $m'$ are two of the corner vertices of $R^s_5$ facing $R^s_5$ and $R^t_5$.

In Fig. 8, $p_1$ and $p_2$, $q_1$ and $q_2$, $m_1$, $m_2$, $m_3$ and $m_4$ are junction vertices in $R^s_5$, $R^t_5$, and $R^m_5$, respectively. Existence of junction vertices has been proved for condition (C0) in [3]; in this paper, we only consider conditions (C1) and (C2).

Lemma 3.5 Performing a trisecting on $R_5$, where $m, n > 3$, and assuming condition (C1) or (C2) holds, if $n_5 = 4$, and $s$ and $t$ are on the common border of $R^s_5$ and $R^t_5$ ($R^m_5$ is null), then there is no junction vertex for $R^s_5$ and $R^t_5$, otherwise $R^s_5$ and $R^t_5$ have at least one junction vertex.

After trisecting, we construct a longest path or Hamiltonian path in $R^s_5$, $R^m_5$ and $R^t_5$ between $s$ and $p$, $m$ and $m'$, and $q$ and $t$, respectively. In the case that none of $R^s_5$ and $R^t_5$ have junction vertices (when $n_5 = 4$ and both $s$ and $t$ are on the common border of $R^s_5$ and $R^t_5$), we construct a longest path in $R^s_5$ (resp. $R^t_5$) between $s$ (resp. $t$) and a noncorner vertex of the boundary facing $R^t_5$ (resp. $R^s_5$); see Fig. 9(a). At the end, the paths in $R_5$ are combined through the junction vertices to make a simple longest path in $R_5$; see Figs. 9(b), 9(c).
3.3 Construct Hamiltonian cycles in rectangular grid subgraphs and then a longest path in $R(m,n)$

In this section, we present Steps 3 and 4 of Algorithm 3.1. We construct Hamiltonian cycles in rectangular grid subgraphs $R_1$ to $R_4$ by Lemma 2.1; see Fig. 10(a), and merge all Hamiltonian cycles to a single Hamiltonian cycle.

Two nonincident edges $e_1$ and $e_2$ are parallel, if each end vertex of $e_1$ is adjacent to an end vertex of $e_2$. Using two parallel edges $e_1$ and $e_2$ of two Hamiltonian cycles, such as two darkened edges of Fig. 10(b), we can combine the cycles as illustrated in Fig. 10(c), and obtain a larger cycle.

Now, we describe combining a longest path in $R_5$ with the constructed cycle. Without loss of generality, suppose that at least one of $R_1, R_2, R_3,$ and $R_4$, say $R_i$, exists otherwise there is nothing to be merged with $R_5$. Note that the Hamiltonian cycle in $R_i$ is constructed such that it contains all the boundary edges facing $R_5$.

Furthermore, any path $P$ of length $U(R_5, s, t)$ in $R_5$ contains all the vertices of $R_5$ except one or two vertices as mentioned before. Therefore, $P$ should contain a boundary edge of $R_5$ that has a parallel edge in $R_i$ (see Fig. 11), except when $R_5$ is a 2-rectangle, in this case $R_5$ may have no boundary edge parallel to any edge of $R_i$ (see Figs. 12(a), 12(d)). But in this case, $s$ should be adjacent to $R_i$. Let an edge $(s, v)$ of $R_5$ be adjacent to $R_i$. Then $P$ must contain an edge $(s, u)$ such that $u$ is not adjacent to $R_i$ as depicted in Figs. 12(a), 12(d) (we may need to swap the roles of $s$ and $t$ to find such edges). Hence, replacing the role of $u$ with $v$ we can modify $P$.
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3.4 The time complexity of algorithm

Consider the pseudo-code of our algorithm in Algorithm 3.1. Step 1 does only a constant number of partitioning, during the peeling operation, which is done in constant
time. Step 2 trisects $R_5$, which requires also a constant number of partitioning operations and finds a longest path in $R_5$ by merging paths of the partitions which can be done in linear time. Step 3 finds Hamiltonian cycles of $R_1$ to $R_4$ which is done in linear time. Step 4 which combines the Hamiltonian cycles and the longest path requires only constant time. Therefore, in total, our sequential algorithm has linear time complexity.

4 The parallel algorithm

In this section, we present a parallel algorithm for the longest path problem. This algorithm is based on the sequential algorithm presented in the previous section. Our parallel algorithm runs on every parallel machine, we do not need any interprocessor communication in our algorithm. We assume there are $nm$ processors and they work in SPMD mode. For simplicity, we use a two-dimensional indexing scheme. Each vertex $v$ of the given rectangular grid graph $R(m,n)$ is mapped to processor $(v_x, v_y)$. Each processor knows its index, coordinates $s$ and $t$, and $m$ and $n$.

The peeling phase is parallelized easily, every processor calculates the following four variables, in parallel [3]:

$$
\begin{align*}
r_1 &= \begin{cases} 
    s_x - 2; & s_x \mod 2 = 0 \\
    s_x - 1; & \text{otherwise}
\end{cases} \\

r_2 &= \begin{cases} 
    t_x + 1; & t_x \mod 2 = m \mod 2 \\
    t_x + 2; & \text{otherwise}
\end{cases} \\

r_3 &= \begin{cases} 
    \min(s_y, t_y) - 2; & \min(s_y, t_y) \mod 2 = 0 \\
    \min(s_y, t_y) - 1; & \text{otherwise}
\end{cases} \\

r_4 &= \begin{cases} 
    \max(s_y, t_y) + 1; & \max(s_y, t_y) \mod 2 = n \mod 2 \\
    \max(s_y, t_y) + 2; & \text{otherwise}
\end{cases}
\end{align*}
$$

where variables $r_1$, $r_2$, $r_3$, and $r_4$ correspond to the right-most column number of $R_1$, the left-most column number of $R_2$, the bottom row number of $R_3$, and the top row number of $R_4$, respectively. Then a processor can identify its subrectangle by comparing its coordinates with these four variables. In cases (F1′) and (F2′), the boundary adjustment can be done by simply decrementing $R_1$, $R_2$, $R_3$, or $R_4$ or incrementing $R_3$ or $R_4$.

The trisecting phase is also parallelized in a similar manner. In the following, we describe how we parallelized the horizontal trisecting, in two cases $R(m,n)$ is even $\times$ odd (or odd $\times$ odd) and it is even $\times$ even. In the case $R(m,n)$ is even $\times$ odd or odd $\times$ odd, every processor simultaneously calculates the following two variables:

$$
\begin{align*}
b &= \begin{cases} 
    \min(s_y, t_y); & \min(s_y, t_y) \mod 2 = 0 \\
    \min(s_y, t_y) + 1; & \min(s_y, t_y) \mod 2 \neq 0
\end{cases}
\end{align*}
$$
where variables $b$ and $u$ correspond to the bottom row number of $R_5^s$ (resp. $R_5^t$), and the top row number of $R_5^t$ (resp. $R_5^s$), respectively.

In the case $R(m, n)$ is even $\times$ even, every processor simultaneously calculates the following two variables:

\[
\begin{align*}
b &= \begin{cases} 
\min(s_y, t_y) & \min(s_y, t_y) \mod 2 = 0 \\
\min(s_y, t_y) + 1 & \min(s_y, t_y) \mod 2 \neq 0
\end{cases} \\
u &= \begin{cases} 
\max(s_y, t_y) & \max(s_y, t_y) \mod 2 \neq 0 \\
\max(s_y, t_y) - 1 & \max(s_y, t_y) \mod 2 = 0
\end{cases}
\end{align*}
\]

A similar method can be used to parallelize the vertical trisecting.

After peeling and trisecting, all processors in the same subrectangles simultaneously construct either a longest path, Hamiltonian path or cycle according to the pattern associated with the subrectangle. For constructing a Hamiltonian path in a rectangular grid graph, we use the constant-time algorithm of [3]. For constructing a Hamiltonian cycle in an even-sized rectangle, we use the constant-time algorithm of [4] in which every processor computes its successor in the cycle. Such an algorithm is given in Algorithm 4.1; see Fig. 14(a).

For constructing a longest path between $s$ and $t$ in $R_5$, since $R_5^s$ and $R_5^t$ are 2-rectangles, then a Hamiltonian or longest path can be easily constructed using Lemma 3.1. For $R_5^m$, there are two cases:

Case 1. $R_5^m$ is odd sized. Since four vertices $m_1$, $m_2$, $m_3$, and $m_4$ are white, then we construct Hamiltonian path from $m_1$ or $m_2$ to $m_3$ or $m_4$ by algorithm of [3].

Case 2. $R_5^m$ is even sized. In this case, Figs. 14(b), 14(c) show that different patterns for constructing a longest subpath between vertices $(m, n)$ and $(m, 1)$ by Algorithm 4.2, and between $(1, n)$ and $(m, 1)$ by Algorithm 4.3, respectively. The algorithms for other patterns can be derived in a similar way.

Then the combining phase is parallelized as follows. The two processors at the two endpoints of an edge $e_1$ in a Hamiltonian cycle $c_1$ make sure whether an adjacent Hamiltonian cycle $c_2$ exists or not. Then their successors are modified to the adjacent processors in $c_2$ if $c_2$ exists. In the same way, the two processors at the endpoints of an edge in the longest path $P$ in $R_5$ as well make sure the existence of an adjacent edge in Hamiltonian cycle $C$, and modify their successors. Therefore, the combining phase can be parallelized in constant steps with no inter-processor communication.

5 Conclusion and future work

We presented a linear-time sequential algorithm for finding a longest path in a rectangular grid graph between any two given vertices. Based on the sequential algorithm, a constant-time parallel algorithm is introduced for the problem, which can be run on every parallel machine.
Algorithm 4.1 The parallel Hamiltonian cycle algorithm for an even-sized rectangular grid graphs

procedure HAM-CYCLE(R(m, n))
1: for each processor (x, y) in R(m, n) do in parallel
2: if y = 1 and x < m, then successor (x, y) ← (x + 1, y)
3: elseif (y = 2, x is odd and x ≠ 1) or (y = n and x even), then successor (x, y) ← (x - 1, y)
4: elseif x is even and y > 2, then successor (x, y) ← (x, y - 1)
5: elseif x is odd and y < n, then successor (x, y) ← (x, y + 1)
6: elseif (y is odd and x = 1) or (y = 2 and x is odd) or (y is even and x = m), then successor (x, y) ← (x, y - 1)

Algorithm 4.2 The parallel longest subpath algorithm for R^m_5 (Fig. 14(b))

procedure LongestSubPath(R^m_5, p, q)
1: for each processor (x, y) in R^m_5 do in parallel
2: if (x = m and y = 2), then successor (x, y) ← null
3: elseif y is odd and x > 1, then successor (x, y) ← (x - 1, y)
4: elseif (y is odd and x = 1) or (y = 2 and x is odd) or (y is even and x = m), then successor (x, y) ← (x, y - 1)
5: elseif y = 1 and x is even, then successor (x, y) ← (x, y + 1)
6: elseif (y is even and x < m) or (y = 2 and x is even) or (y = 1 and x is odd), then successor (x, y) ← (x + 1, y)

Algorithm 4.3 The parallel longest subpath algorithm for R^m_5 (Fig. 14(c))

procedure LongestSubPath(R^m_5, p, q)
1: for each processor (x, y) in R^m_5 do in parallel
2: if x = 1 and y = n - 1, then successor (x, y) ← null
3: elseif (y is odd and x = m) or (y is even and x = 1) or (y = n and x is even) then successor (x, y) ← (x, y - 1)
4: elseif (y is odd and x < m) or (y = n - 1 and x is even) or (y = n and x is odd) then successor (x, y) ← (x + 1, y)
5: elseif y is even and x > 1 then successor (x, y) ← (x - 1, y)
6: elseif y = n - 1 and x is odd then successor (x, y) ← (x, y + 1)

The algorithm initially divides R(m, n) into five subrectangles, then builds Hamiltonian cycles in the four subrectangles and a Hamiltonian path or longest path in the other subrectangle. Next, all Hamiltonian cycles are combined to a single Hamilto-
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Finally, it combines all the paths and the cycle to a longest path. Our algorithm is a parallel version of the previous sequential algorithm [13], and is an improvement of it by reducing the number of partitioning steps from $O(m + n)$ to a constant.

Since the longest path problem is NP-hard in general grid graphs [11], it remains open if the problem is polynomially solvable in solid grid graphs. Further study can be done on the Hamiltonian (or longest) path problem in other special classes of graphs.

Appendix

**Lemma 6.1** [13] Let $P(R(m, n), s, t)$ be a longest path problem with $n = 1$ or $n = 2$, then $L(R(m, n), s, t) = U(R(m, n), s, t)$.

**Proof** For a 1-rectangle obviously the lemma holds for the single possible path between $s$ and $t$ (see Fig. 15(a)). For a 2-rectangle, if removing $s$ and $t$ splits the graph into two components, then the path going through all vertices of the larger component has the length equal to $U(R(m, n), s, t)$ (see Fig. 4(b)). Otherwise, let $s'$ be the vertex adjacent to $s$ and $t'$ be the vertex adjacent to $t$ such that $s'_y \neq s_y$ and $t'_y \neq t_y$. Then we make a path from $s$ to $s'$ and a path from $t$ to $t'$ as shown in Figs. 15(c), 15(d), and connect $s'$ to $t'$ by a path such that at most one vertex remains out of the path as depicted in this figure. □

**Lemma 6.2** For the longest path problem $P(R(n, m), s, t)$, any peeling on $R(m, n)$ is proper if either:

1. Condition (C0) holds and $m \mod 2 = n \mod 2$ (i.e., $R(m, n)$ is even $\times$ even or odd $\times$ odd), or
2. One of conditions (C1) and (C2) hold and $R(m, n)$ is even $\times$ odd or odd $\times$ odd.

**Proof** The lemma has been proved for the case that (C0) holds (see [3]). So, we consider conditions (C1) and (C2). From Lemma 3.2, we know that $s$ and $t$ are still

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**Fig. 15** (a) The longest path between $s$ and $t$ in a 1-rectangle, (b) The longest path between $s$ and $t$ in a 2-rectangle, (c) and (d) Longest paths with length $2m$ and $2m - 1$ for a 2-rectangle, respectively.
color-compatible, and we are going to prove that $P(R_5(m_5, n_5), s, t)$ is not in cases $F_1$ and $F_2^*$. By Lemma 3.1, when $R(m, n)$ is an odd × odd rectangular grid graph, $R_5(m_5, n_5)$ is also an odd × odd rectangular grid graph, and hence $R_5(m_5, n_5)$ is not a 2-rectangle. If $R_5$ is a 1-rectangle, then $s_y = t_y$ or $s_x = t_x$ and then we have the following cases:

Case 1. (C1) holds, and $s$ and $t$ have different colors. In this case, one of $s_x$ and $t_x$ ($s_y$ and $t_y$) is even and the other is odd. Considering that $s$ and $t$ are antipodes and $R_5$ is odd × odd, one of $s$ and $t$ must be at the corner, and exactly one vertex goes out of the path.

Case 2. (C2) holds and both $s$ and $t$ are black. In this case, $s_x$, $s_y$, $t_x$ and $t_y$ are even. Hence, vertices $s$ and $t$ are before corner vertices and exactly two vertices go out of the path.

In a similar way, when $R(m, n)$ is an even × odd rectangular grid graph ((C1) holds), $R_5(m_5, n_5)$ is also an even × odd rectangular grid graph, and hence $R_5(m_5, n_5)$ is not a 2-rectangle. If $R_5$ is a 1-rectangle, then $s_y = t_y$. In this case, $s_x$ and $t_x$ are both odd or even. Hence, $s$ or $t$ are at the corner and exactly one vertex goes out of the path.

Therefore by Theorem 2.1, $U(R_5(m_5, n_5), s, t) = U(R(m, n), s, t)$ and the peeling of $R(m, n)$ is proper.

Lemma 6.3 For the longest path problem $P(R(n, m), s, t)$, where $R(m, n)$ is an even × even rectangular grid graph, a peeling operation on $R(m, n)$ is proper if and only if $P(R(n, m), s, t)$ is not in cases $(F_1')$ and $(F_2')$.

Proof The proof is a simple by case analysis.

Lemma 6.4 Performing a trisecting on $R_5$, where $m, n > 3$, and assuming condition (C1) or (C2) holds, if $n_5 = 4$, and $s$ and $t$ are on the common border of $R_5^s$ and $R_5^t$ ($R_5^{n_5}$ is null), then there is no junction vertex for $R_5^s$ and $R_5^t$, otherwise $R_5^s$ and $R_5^t$ have at least one junction vertex.

Proof Consider Figs. 16(a) and 16(b), where $n_5 = 4$ and two vertices $s$ and $t$ are on the common border of $R_5^s$ and $R_5^t$. In this case, the only two vertices $p_2$ and $q_1$ may be junction vertices. By Theorem 2.1, there exists a Hamiltonian path from $s$ to $p_2$ and from $q_1$ to $t$ in $R_5^s$ and $R_5^t$, respectively, and $U(R_5^s, s, p_2) + U(R_5^t, q_1, t) \neq U(R_5(m, n), s, t)$. Hence, neither $R_5^s$ nor $R_5^t$ has a junction vertex. Now for the other cases, we show that $R_5^s$ and $R_5^t$ have at least one junction vertex.

In case (b), $s$ and $t$ have the same colors (white or black), and two corner vertices on the boundary of $R_5^s$ (resp., $R_5^t$) facing $R_5^{n_5}$ have different colors and also $R_5^{n_5}$ is a $k$-rectangle, where $k$ is zero or $k \geq 2$ and even. We consider the following three cases for $s$ and $t$:

Case 1. Both $s$ and $t$ are the corner vertices on the boundary of $R_5^s$ and $R_5^t$ facing $R_5^{n_5}$; see Fig. 16(c). By Theorem 2.1, there exists a Hamiltonian path from $s$ to $p_2$ and from $q_1$ to $t$ in $R_5^s$ and $R_5^t$, respectively, and a path from $m_3$ to $m_2$ which does not contain a vertex in $R_5^{n_5}$. Therefore, $U(R_5^s, s, p_2) + U(R_5^{n_5}, m_3, m_2) + U(R_5^t, q_1, t) = U(R_5(m, n), s, t)$, and hence both $R_5^s$ and $R_5^t$ have a unique junction vertex.
Case 2. Only $s$ is the corner vertex on the boundary of $R_s^5$ facing $R_m^5$; see Fig. 16(d). By Theorem 2.1, there exists a Hamiltonian path from $s$ to $p_2$, from $q_1$ to $t$ and a path from $m_3$ to $m_2$ which does not contain one vertex, or a Hamiltonian path from $s$ to $p_2$, from $m_2$ to $m_4$ and a path from $q_2$ to $t$, which does not contain one vertex. Therefore, $U(R_s^5, s, p_2) + U(R_m^5, m_3, m_2) + U(R_s^5, q_1, t) = U(R_5(m, n), s, t)$ or $U(R_s^5, s, p_2) + U(R_m^5, m_2, m_4) + U(R_s^5, q_2, t) = U(R_5(m, n), s, t)$ and hence $R_s^5$ has a unique junction vertex and $R_m^5$ have two junction vertices (the same argument is also applied to $t$). In this case, where $n_5 = 4$, both $R_s^5$ and $R_m^5$ have a unique junction vertex; see Fig. 16(e).

Case 3. Neither $s$ nor $t$ are corner vertices on the boundary of $R_s^5$ and $R_m^5$ facing $R_m^5$; see Fig. 16(f). By Theorem 2.1, there exists a Hamiltonian path from $s$ to $p_1$, from $m_1$ to $m_3$ and a path from $q_1$ to $t$ which does not contain a vertex, or a Hamiltonian path from $s$ to $p_1$, from $q_2$ to $t$ and a path from $m_1$ to $m_4$, which does not contain a vertex, or a Hamiltonian path from $m_2$ to $m_4$, from $q_2$ to $t$ and a path from $s$ to $p_2$ which does not contain a vertex. Therefore, $U(R_s^5, s, p_1) + U(R_m^5, m_1, m_3) + U(R_s^5, q_1, t) = U(R_5(m, n), s, t)$, $U(R_s^5, s, p_1) + U(R_m^5, m_1, m_4) + U(R_1, q_2, t) = U(R_5(m, n), s, t)$ or $U(R_s^5, s, p_2) + U(R_m^5, m_2, m_4) + U(R_5, q_2, t) = U(R_5(m, n), s, t)$ and hence both $R_s^5$ and $R_s^5$ have two junction vertices.

In case (c), vertices $s$ and $t$ are black or have different colors, and the two corner vertices on the boundary of $R_s^5$ (resp., $R_s^5$) facing $R_m^5$ are black and also $R_m^5$ is a $k$-rectangle, where $k \geq 1$ and odd. There are three cases for $s$ and $t$:

Case 1. Both $s$ and $t$ are the corner vertices on the boundary of $R_s^5$ and $R_m^5$ facing $R_m^5$; see Fig. 17(a). Then $s$ and $t$ are black. By Theorem 2.1, there exists a path from $s$ to $p_2$ in $R_s^5$, and a path from $q_1$ to $t$ in $R_m^5$, which do not contain a vertex, and a Hamiltonian path from $m_3$ to $m_2$ in $R_m^5$. Therefore, $U(R_s^5, s, p_2) + U(R_m^5, m_3, m_2) + U(R_s^5, q_1, t) = U(R_5(m, n), s, t)$ and hence both $R_s^5$ and $R_s^5$ have a unique junction vertex.

Case 2. Only $s$ is the corner vertex on the boundary of $R_s^5$ facing $R_m^5$, then $s$ is black and $t$ is black or white; see Fig. 17(b). By Theorem 2.1, there exists a path from $s$ to $p_2$; and a path from $q_1$ (or $q_2$) to $t$, where $t$ is black, which does not contain a vertex, and a Hamiltonian path from and $m_2$ to $m_4$ (or from $m_3$
to \( m_2 \), or a Hamiltonian path from \( q_1 \) (or \( q_2 \)) to \( t \), where \( t \) is white and \( m_2 \) to \( m_4 \) (or from \( m_3 \) to \( m_2 \)) and a path from \( s \) to \( p_2 \), which does not contain a vertex. Therefore, \( U(R_s^5, s, p_2) + U(R_s^m, m_3, m_2) + U(R_t^l, q_1, t) = U(R_5(m, n), s, t) \) or \( U(R_s^5, s, p_2) + U(R_s^m, m_2, m_4) + U(R_t^l, q_2, t) = U(R_5(m, n), s, t) \) and hence \( R_s^5 \) has a unique junction vertex and \( R_t^l \) have two junction vertices (the same argument is also applied to \( t \)). In this case, where \( n_5 = 5 \), both \( R_s^5 \) and \( R_t^l \) have a unique junction vertex; see Fig. 17(c).

Case 3. Neither \( s \) nor \( t \) are corner vertices on the boundary of \( R_s^5 \) and \( R_t^l \) facing \( R_s^m \); see Fig. 17(d). By Theorem 2.1, there exists a Hamiltonian path from \( s \) to \( p \), \( m \) to \( m' \) and \( q \) to \( t \), where \( s \) (or \( t \)) is white, and a path from \( s \) to \( p \) and a path from \( q \) to \( t \) do not contain a vertex where \( s \) (or \( t \)) is black, \( p \) is \( p_1 \) or \( p_2 \), \( q \) is \( q_1 \) or \( q_2 \), \( m \) is \( m_1 \) or \( m_2 \) and \( m' \) is \( m_3 \) or \( m_4 \). Therefore, \( U(R_s^5, s, p) + U(R_s^m, m, m') + U(R_t^l, q, t) = U(R_5(m, n), s, t) \), and hence both \( R_s^5 \) and \( R_t^l \) have two junction vertices.

In case (d), if \( n_5 > 3 \), the trisecting is performed horizontally, and the claim is proved by applying the same argument for case (b); see Fig. 17(e). If \( n_5 = 3 \), the trisecting is performed vertically and also two corner vertices on the boundary of \( R_s^5 \) facing \( R_s^m \) are black. Therefore, the claim is proved by applying the same argument for case (c).

\[ \square \]

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