MULTIDIMENSIONAL VAN DER CORPUT SETS AND SMALL FRACTIONAL PARTS OF POLYNOMIALS

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Abstract. We establish Diophantine inequalities for the fractional parts of generalized polynomials \( f \), in particular for sequences \( \nu(n) = \lfloor n^c \rfloor + n^k \) with \( c > 1 \) a non-integral real number and \( k \in \mathbb{N} \), as well as for \( \nu(p) \) where \( p \) runs through all prime numbers. This is related to classical work of Heilbronn and to recent results of Bergelson et al.

1. Introduction

By Dirichlet’s approximation theorem for any real number \( \xi \) and any positive integer \( N \),

\[
\min_{1 \leq n \leq N} \| n \xi \| \leq \frac{1}{N+1},
\]

where \( \| \cdot \| \) denotes the distance to the nearest integer. Hardy and Littlewood [22] conjectured a similar result for the distances \( \| n^k \xi \| \) (with given exponent \( k \in \mathbb{N} \)). Vinogradov [54] proved the following

**Theorem 1.1.** Let \( \xi \in \mathbb{R} \) be a given real number and \( N \in \mathbb{N} \), a given positive integer. Then for every \( k \in \mathbb{N} \), there exists an exponent \( \eta_k > 0 \) such that

\[
\min_{1 \leq n \leq N} \| n \xi^k \| \ll_k N^{-\eta_k}.
\]

Throughout this paper we use the standard notation \( \ll \), where the index denotes the dependence of the implicit constant; furthermore we use instead of \( \ll \) sometimes the \( O \)-notation.

In the case of squares Heilbronn [24] improved Vinogradov’s exponent to \( -\frac{1}{2} + \varepsilon \) (with arbitrary \( \varepsilon > 0 \)). For the related literature up to 1986 we refer to the classical monograph of Baker [1]. The best known exponent in the quadratic case is \( -\frac{1}{4} + \varepsilon \) due to Zaharescu [59]. However, his method is not applicable to higher powers. It is an open conjecture that \( \eta_k \) can be taken as \( 1 - \varepsilon \) with arbitrary \( \varepsilon > 0 \).

Generalizations to arbitrary polynomials \( f \in \mathbb{Z}[X] \) with \( f(0) = 0 \) are due to Davenport [17] and Cook [13]. Let \( 2 \leq k_1 \leq \cdots \leq k_s \) be integers, \( \xi_i \in \mathbb{R} \) for \( 1 \leq i \leq s \) and \( \ell = 2^{1-k_1} + \cdots + 2^{1-k_s} \). Then Cook [15] also showed that \( \| \xi_1 n_1^{k_1} + \cdots + \xi_s n_s^{k_s} \| \ll_{k_1, \ldots, k_s, \varepsilon} N^{-s/(s+(1-\varepsilon)2^{1-k_s})+\varepsilon} \). Wooley [57] considered the Diophantine inequality over smooth numbers to obtain an improvement. The proofs of these results are based on a sophisticated treatment of the occurring exponential sums. In a recent paper Lê and Spencer [29] proved the following

**Theorem 1.2 (29, Theorem 3).** Let \( N \in \mathbb{N} \) and \( h \in \mathbb{Z}[X] \) be a polynomial with integer coefficients such that for every non-zero integer \( q \) there exists a solution \( n \) to the congruence \( h(n) \equiv 0 \pmod{q} \). Then there is an exponent \( \eta > 0 \) depending only on the degree of \( h \) such that

\[
\min_{1 \leq n \leq N} \| h(n) \| \ll_h N^{-\eta}
\]

for arbitrary \( \xi \in \mathbb{R} \).
A recent proof of a related well-known conjecture concerning the Vinogradov integral is also due to Wooley \[58\] for \(k = 3\) and to Bourgain et al. \[12\] in the general case. Baker \[3\] used this approach to improve on Diophantine inequalities as considered in Theorem 1.2.

Danicic \[16\] considered two dimensional extensions of the above problem and showed that
\[
\min_{1 \leq n \leq N} \max_{1 \leq i \leq k} \left( \left\| \alpha n^2 \right\| + \left\| \beta n^2 \right\| \right) \ll N^{-\eta}
\]
uniformly in \(N, \alpha\) and \(\beta\). Higher powers were investigated by Liu \[31\]. Cook \[13, 14\] generalized these results to a system of polynomials without constant term.

**Definition 1.1.** Let \(h_1, \ldots, h_k\) be a system of polynomials in \(\mathbb{Z}[X]\). This system is called jointly intersective if for every \(q \neq 0\), there exists an \(n \in \mathbb{Z}\) such that \(h_i(n) \equiv 0 \mod q\) for \(i = 1, \ldots, k\).

**Remark 1.3.** The concept of jointly intersective polynomials was introduced independently (under different names) in different areas by Bergelson et al. \[6\], Lê \[28\], Rice \[45\] and Wierdl \[56\].

Note that the common root condition in Definition 1.1 is necessary which is shown by a simple counter example in the case \(\xi = a/q\).

Now we link the concept of jointly intersective polynomials with intersective sets. In the case \(k = 1\) an arbitrary subset \(I \subset \mathbb{Z} \setminus \{0\}\) is called intersective if
\[
I \cap (S - S) \neq \emptyset
\]
whenever the upper density of \(S \subset \mathbb{Z}\) is positive. A famous result, independently established by Furstenberg \[19\] and Sárközy \[17\] states that the set of \(k\)-th powers and the set of shifted primes \(p + 1\) and \(p - 1\) are intersective sets. Moreover, a sufficient condition for a set being intersective is due to Kamae and Mendès-France \[25\] and was later extended by Nair \[41–43\]. Lê and Spencer \[29\] established the following

**Theorem 1.4 (\[29, Theorem 4\]).** Let \(\ell\) be a positive integer, \(h_1, \ldots, h_k\) be jointly intersective polynomials, and let \(A = (a_{ij})\) be an arbitrary \(\ell \times k\) matrix with real entries and let \(N \in \mathbb{N}\). Then there is an exponent \(\eta > 0\) depending only on \(\ell\) and on the polynomials \(h_i\) such that
\[
\min_{1 \leq n \leq N} \max_{1 \leq i \leq \ell} \left\| \sum_{j=1}^{k} a_{ij} h_j(n) \right\| \ll \ell, h_1, \ldots, h_k, N^{-\eta},
\]
where the bound is uniform in \(A\).

Harman \[23\] considered the sequence \(\alpha p^k\) for \(\alpha > 0\) and \(k\) a positive integer, where \(p\) runs through the prime numbers. Baker and Kolesnik \[2\] considered the distribution modulo one of the more general sequence \(\alpha p^k\). Improvements of the latter for the case \(\theta = 1\) have been established by Matomäki \[34\]. Recent refinements of the statement are given by Baker \[4\]. Motivated by the above observations concerning intersective sets, Lê and Spencer \[30\] also proved an extension of Theorem 1.4 to polynomials evaluated at prime numbers.

The aim of our paper is to establish such Diophantine inequalities for the Piatetski-Shapiro sequence and for pseudo-polynomial sequences, for instance for the sequence \(n^c + n^k\) with \(c > 0\) a non-integral real number and \(k \in \mathbb{N}\). In Section 2 we introduce the concept of van der Corput sets and we formulate our results in detail. In Section 3 we prove that also in the multi-variate setting every van der Corput set is a Heilbronn set. This extends a classical one-dimensional result of Montgomery \[35\]. Section 4 is devoted to exponential sum estimates, the final Sections 6 and 7 deal with single and multiple pseudo-polynomials, respectively.

2. Van der Corput sets and statement of Results

In the following we introduce van der Corput sets for multi-parameter systems that is for \(\mathbb{Z}^k\)-actions (in the terminology of ergodic theory). For more details see Bergelson and Lesigne \[8\].
Definition 2.1. A subset $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$ is a van der Corput set (vdC-set) if for any family $(u_n)_{n \in \mathbb{Z}^k}$ of complex numbers of modulus 1 such that
\[
\forall h \in \mathcal{H}, \quad \lim_{N_1, \ldots, N_k \to \infty} \frac{1}{N_1 \cdots N_k} \sum_{0 \leq n < (N_1, \ldots, N_k)} u_{n+h} u_n = 0
\]
we have
\[
\lim_{N_1, \ldots, N_k \to \infty} \frac{1}{N_1 \cdots N_k} \sum_{0 \leq n < (N_1, \ldots, N_k)} u_n = 0.
\]
Here in the limit $N_1, \ldots, N_k$ tend to infinity independently and $<$ stands for the product order.

Equivalently, $\mathcal{H}$ is a vdC-set if any family $(x_n)_{n \in \mathbb{N}^k}$ of real numbers having the property that for all $h \in \mathcal{H}$ the family $(x_{n+h} - x_n)_{n \in \mathbb{N}^k}$ is uniformly distributed mod 1, is itself uniformly distributed mod 1. The concept of uniform distribution for multi-parameter systems (so called multi-sequences) was investigated in various papers, see for instance Losert and Tichy [32], Kirschenhofer and Tichy [26], Tichy and Zeiner [52] and the book of Drmota and Tichy [18].

Van der Corput’s difference theorem states that in the case $k = 1$ the full set $\mathbb{N}$ of positive integers is a van der Corput set (cf. [27, Theorem 3.1]). However, this is only a sufficient condition. Therefore the question of the necessary “size” of vdC sets arises. For various aspects of Van der Corput’s difference theorem we refer to the recent paper of Bergelson and Moreira [9]. Delange observed that also the sets $q\mathbb{N}$, where $q \geq 2$ is an integer, are van der Corput sets. More general examples like the $4k$th powers or shifted primes $p+1$ and $p-1$ are due to Kamae and Mendès-France [25]. In particular, they proved in the case $k = 1$ that each van der Corput set is also intersective. The converse does not hold true as it was shown by Bourgain [11].

In his seminal paper Ruzsa [46] gave four equivalent definitions of vdC-sets. We refer the interested reader to chapter 2 of Montgomery [38] or the important work of Bergelson and Lesigne [8] for a detailed account on vdC-sets.

Definition 2.2. Let $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$. We call $\mathcal{H}$ a Heilbronn set if for every $\xi \in \mathbb{R}^k$ and every $\varepsilon > 0$ there is an $h \in \mathcal{H}$ such that
\[
\|h \cdot \xi\| < \varepsilon
\]
where $\cdot$ denotes the standard inner product.

Following Montgomery we want to analyze quantitative aspects of Heilbronn sets. This is related to the spectral definition of van der Corput sets given by Kamae and Mendès-France [25] and its multidimensional variant by Bergelson and Lesigne [8]. For each subset $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$ we denote by $T = T(\mathcal{H})$ the set of real trigonometric polynomials
\[
T(x) = a_0 + \sum_{h \in \mathcal{H}} a_h \cos(2\pi h \cdot x)
\]
with $T(x) \geq 0$ for all $x \in \mathbb{R}^k$ and $T(0) = 1$. Furthermore we set
\[
\delta(\mathcal{H}) := \inf_{T \in T(\mathcal{H})} a_0.
\]

In order to provide a quantitative result on Heilbronn sets we introduce the following quantity.
\[
(2.1) \quad \gamma = \gamma(\mathcal{H}) = \sup_{\xi \in \mathbb{R}^k} \inf_{h \in \mathcal{H}} \|h \cdot \xi\|.
\]

Then our first result is the following

Theorem 2.1. Let $\mathcal{H} \subset \mathbb{Z}^k \setminus \{0\}$.
\begin{enumerate}
\item $\mathcal{H}$ is a van der Corput set if and only if $\delta(\mathcal{H}) = 0$.
\item $\mathcal{H}$ is a Heilbronn set if and only if $\gamma(\mathcal{H}) = 0$.
\item $\gamma(\mathcal{H}) \leq \delta(\mathcal{H})$.
\item Any van der Corput set is a Heilbronn set.
\end{enumerate}
Remark 2.2. In the one dimensional case the if-statement of Assertion (1) was already proved by Kamae and Mendès-France [23] and the only-if-part is due to Ruzsa [19]. The general case was shown Bergelson and Lesigne [8]. Assertion (2) is obvious. The one dimensional version of Assertion (3) was already shown by Montgomery [38]. In Section 3 we prove the multidimensional case of Assertion (3). Finally Assertion (4) is a direct consequence of Assertion (3).

In Montgomery [38] one can also find counter examples which show that in general the converse is false. Former results by Bergelson, Boshernitzan, Kolesnik, Lesigne, Madritsch, Quas, Son, Tichy and Wierdl provide us with many examples of vdC-sets:

- Let $f, g \in \mathbb{Z}[X]$ such that for any $q \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $q$ divides $f$ and $g$. Then the 2-dimensional set $\{(f(n), g(n)) : n \in \mathbb{N}\}$ is a vdC-set (see [8]).
- Let $f, g \in \mathbb{Z}[X]$ be polynomials with zero constant term. Then the 2-dimensional set $\{(f(p-1), g(p-1)) : p \text{ prime}\}$ is a vdC-set (see [8]).
- Let $b, d \neq 0$. Then the set $\{[bn^c] : n \in \mathbb{N}\}$ is a vdC-set (see [10]).
- Let $b, d \neq 0$ such that $b/d$ is irrational, $c \geq 1$, $a > 0$ and $a \neq c$. Then the set $\{[bn^c + dx^a] : n \in \mathbb{N}\}$ is a vdC-set (see [10]).
- Let $b \neq 0$, $c > 1$ be irrational and $d$ any real number. Then the set $\{[bn^c(\log n)^d] : n \in \mathbb{N}\}$ is a vdC-set (see [10]).
- Let $b \neq 0$, $c > 1$ be rational and $d \neq 0$. Then the set $\{[bn^c(\log n)^d] : n \in \mathbb{N}\}$ is a vdC-set (see [10]).
- Let $b, d \neq 0$, $c \geq 1$ and $a > 1$. Then the set $\{[bn^c + d(\log n)^a] : n \in \mathbb{N}\}$ is a vdC-set (see [10]).
- Let $\alpha_i$ be positive integers and $\beta_i$ be positive non-integral reals. Then the $(k + \ell)$-dimensional sets
  
  \[
  \{((p-1)^{\alpha_1}, \ldots, (p-1)^{\alpha_k}, [(p-1)^{\beta_1}], \ldots, [(p-1)^{\beta_\ell})] : p \text{ prime}\}
  \]

  and

  \[
  \{((p+1)^{\alpha_1}, \ldots, (p+1)^{\alpha_k}, [(p+1)^{\beta_1}], \ldots, [(p+1)^{\beta_\ell})] : p \text{ prime}\}
  \]

are vdC-sets (see [7] and [33]).

Furthermore Bergelson et al. [5] showed under some mild conditions that for a function $f$ from a Hardy-field the sequence $(f(p))_{p \text{ prime}}$ is uniformly distributed mod 1. From this they deduced general classes of vdC-sets and by our Theorem 2.1 they are Heilbronn sets, too.

In Sections 3 and 4 we want to show that for sets of the form $\{[n^c] + n^k : n \in \mathbb{N}\}$, where $c > 1$ is not an integer and $k \in \mathbb{N}$, and multidimensional variants thereof, we may replace $\varepsilon$ by some negative power $N^{-\eta}$ depending only on the exponents $c$ and $k$. Therefore we want to introduce the concept of pseudo-polynomials.

Definition 2.3. Let $\alpha_1, \alpha_2, \ldots, \alpha_d, \theta_1, \theta_2, \ldots, \theta_d$ be positive reals such that $1 \leq \theta_1 < \cdots < \theta_d$ and at least one $\theta_j \notin \mathbb{Z}$ for $1 \leq j \leq d$. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form

\[
f(x) = \alpha_1 x^{\theta_1} + \cdots + \alpha_d x^{\theta_d}
\]

is called a pseudo-polynomial. By abuse of notation we write $\deg f = \theta_d$.

We investigate Diophantine inequalities for sequences of the form $([f(n)])_{n \geq 1}$. A simple example is the Piatetski-Shapiro type sequence $([n^c] + n^k)_{n \geq 1}$ with $c > 1$ a non-integral real and $k \in \mathbb{N}$.

Theorem 2.3. Let $\xi$ be a real number, $N \in \mathbb{N}$ sufficiently large and $f$ be a pseudo-polynomial. Then there exists an exponent $\eta > 0$ depending only on $f$ such that

\[
\min_{1 \leq n \leq N} \|\xi [f(n)]\| < f N^{-\eta}.
\]

When applying our result to the sequence $([n^c] + n^k)_{n \geq 1}$ we obtain the following
Corollary 2.4. Let \( \xi \) be real, \( c > 1 \) be a non-integral real number, \( N \in \mathbb{N} \) sufficiently large and \( k \in \mathbb{N} \). Then for arbitrary \( \varepsilon > 0 \)

\[
\min_{1 \leq p \leq N} \| \xi (n^c + n^k) \| \ll \begin{cases} N^{-\frac{1}{c} + \frac{1}{k} + \frac{1}{c} + \varepsilon} & \text{if } c > k, \\ N^{-\frac{1}{2c} + \frac{1}{2k} + \varepsilon} & \text{if } c < k. \end{cases}
\]

Remark 2.5. Using the methods of Mauduit and Rivat [35,36] and of Morgenbesser [39] combined with Spiegelhofer [51] and Müller and Spiegelhofer [40], who considered Beatty sequences, one could improve this exponent in the case \( c > k \).

Similarly to the results above we may consider sequences over the primes.

Theorem 2.6. Let \( \xi \) be a real number, \( N \in \mathbb{N} \) sufficiently large and \( f \) be a pseudo-polynomial. Then there exists an exponent \( \eta > 0 \) depending only on \( f \) such that

\[
\min_{1 \leq p \leq N} \| \xi [f(p)] \| \ll_f N^{-\eta}.
\]

Corollary 2.7. Let \( \xi \) be real, \( c > 1 \) be a non-integral real number, \( N \in \mathbb{N} \) sufficiently large and \( k \in \mathbb{N} \). Then for arbitrary \( \varepsilon > 0 \)

\[
\min_{1 \leq p \leq N} \| \xi (p^c + p^k) \| \ll \begin{cases} N^{-\frac{1}{c} + \frac{1}{k} + \varepsilon} & \text{if } c > k, \\ N^{-\frac{1}{2c} + \frac{1}{2k} + \varepsilon} & \text{if } c < k. \end{cases}
\]

Finally we state multidimensional variants of these estimates.

Theorem 2.8. Let \( \ell \) be a positive integer, \( f_1, \ldots, f_k \) be \( \mathbb{Q} \)-linearly independent pseudo-polynomials, let \( A = (a_{ij}) \) be an arbitrary \( \ell \times k \) matrix with real entries and let \( N \in \mathbb{N} \). Then there is an exponent \( \eta > 0 \) depending only on \( \ell \) and on the polynomials \( f_i \) such that

\[
\min_{1 \leq n \leq N} \max_{1 \leq i \leq \ell} \left| \sum_{j=1}^k a_{ij} [f_j(n)] \right| \ll_{\ell,f_1,\ldots,f_k} N^{-\eta},
\]

where the bound is uniform in \( A \).

Again we may consider sequences of integer parts of pseudo-polynomials over the primes.

Theorem 2.9. Let \( \ell \) be a positive integer, \( f_1, \ldots, f_k \) be \( \mathbb{Q} \)-linearly independent pseudo-polynomials, let \( A = (a_{ij}) \) be an arbitrary \( \ell \times k \) matrix with real entries and let \( N \in \mathbb{N} \). Then there is an exponent \( \eta > 0 \) depending only on \( \ell \) and on the polynomials \( f_i \) such that

\[
\min_{1 \leq n \leq N} \max_{1 \leq i \leq \ell} \left| \sum_{j=1}^k a_{ij} [f_j(p)] \right| \ll_{\ell,f_1,\ldots,f_k} N^{-\eta},
\]

where the bound is uniform in \( A \).

3. Every van der Corput set is also a Heilbronn set

We first present the following easy proof\(^1\) that any van der Corput set is a Heilbronn set. Let \( \mathcal{H} \subset \mathbb{Z}^k \setminus \{0\} \) be a van der Corput set. It is an easy deduction that \( \mathcal{H} \) must be intersective in the multidimensional sense. Then for a given \( \xi \in \mathbb{R}^k \) we consider the set

\[ A = \{ h \in \mathbb{Z}^k : \| h \cdot \xi \| < \varepsilon/2 \}. \]

Since \( \mathcal{H} \) is intersective and \( A \) has positive density we deduce that

\[ \mathcal{H} \cap (A - A) \neq \emptyset, \]

which proves Theorem 2.1.

A quantitative analysis of van der Corput and Heilbronn sets can be provided using the parameter \( \delta \) introduced in Section 2. Let \( N \) be a positive real. Then for a fixed van der Corput set \( \mathcal{H} \) we

\(^1\)Personal communication by Imre Rusza
set $\mathcal{H}_N := \mathcal{H} \cap [-N, N]^d$. Then $\delta(\mathcal{H}_N)$ tends to 0 as $N$ tends to infinity. Only few upper bounds for this quantity are known. In particular, for the set of shifted primes Slijepčević [31] could show that

$$\delta((p - 1 \leq N; \ p \ \text{prime})) \ll (\log N)^{-1+o(1)}.$$  

Similar results are known for squares [19]. In case of Heilbronn sets one has to consider the parameter $\gamma$ introduced in Section 2 instead of $\delta$.

The following proposition immediately yields a complete proof of Theorem 2.1.

**Proposition 3.1.** Let $\mathcal{H} \in \mathbb{Z}^k \setminus \{0\}$. Then $\gamma(\mathcal{H}) \leq \delta(\mathcal{H})$.

**Proof.** Our proof follows Montgomery’s proof for $k = 1$ in [38]. Let $T \in \mathcal{T}(\mathcal{H})$ as above, let $0 < \varepsilon \leq \frac{1}{2}$ and set $f(x) = \max (0, 1 - \|x\|/\varepsilon)$. Then we consider

$$g(\xi) := a_0 + \sum_{h \in \mathcal{H}} a_h f(h \cdot \xi),$$  

where $\xi \in \mathbb{R}^k$. Since $f$ is continuous and of bounded variation, its Fourier transform converges absolutely to $f$. Thus

$$g(\xi) = a_0 + \sum_{h \in \mathcal{H}} a_h \sum_{m \in \mathbb{Z}} \hat{f}(m)e(mh \cdot \xi).$$  

The function $f$ is even. Hence its Fourier coefficients $\hat{f}(m)$ are real. Moreover $g(\xi)$ is real, hence

$$g(\xi) = a_0 + \sum_{h \in \mathcal{H}} a_h \sum_{m \in \mathbb{Z}} \hat{f}(m) \cos(2\pi m h \cdot \xi).$$  

Inverting the order of summation yields

$$g(\xi) = \sum_{m \in \mathbb{Z}} \hat{f}(m) T(m\xi).$$  

A simple calculation shows that $\hat{f}(m) = \frac{1}{\varepsilon} \left( \frac{\sin(\pi m \varepsilon)}{\pi m} \right) \geq 0$ and $T(m \theta) \geq 0$ for all $m$. Thus $g(\xi)$ is greater than the contribution of the term for $m = 0$ in the above sum. Since $\hat{f}(0) = \varepsilon$ and $T(0) = 1$ we get

$$g(\xi) \geq \varepsilon.$$  

Now, if $\varepsilon > a_0$, then there must be at least one $h \in \mathcal{H}$ such that $a_h > 0$ and $f(h \cdot \xi) > 0$. Hence $\|h \cdot \xi\| < \varepsilon$. Since this holds for every $\varepsilon > a_0$ we obtain $\inf_{h \in \mathcal{H}} \|h \cdot \xi\| \leq a_0$. Furthermore, since this holds for every polynomial $T \in \mathcal{T}(\mathcal{H})$, we get $\inf_{h \in \mathcal{H}} \|h \cdot \xi\| \leq \delta$. Finally, since this holds for every $\xi \in \mathbb{R}^k$, it follows that $\gamma \leq \delta$ which proves the proposition. 

**Remark 3.2.** Applying the results of Slijepčević [19, 50] Proposition 3.1 implies upper bounds for $\gamma$ in case of shifted primes and squares.

4. **Exponential sum estimates for the case $\theta_r > k$**

Before stating the proofs of the main theorems we collect some well-known facts on exponential sums which will occur in the sequel. Let $f$ be a pseudo-polynomial. Then there exists a real function $g$ and a polynomial $P$ such that $f(x) = g(x) + P(x)$, $g(x) = \sum_{i=1}^m a_j x \theta_i$ with $1 < \theta_1 < \ldots < \theta_r$ and $\theta_j \notin \mathbb{Z}$ for $j = 1, \ldots, r$. Let $k$ be the degree of $P$, and we set $k = 0$ if $P \equiv 0$. By abuse of notation we write

$$\deg f = \begin{cases} \theta_r & \text{if } \theta_r > k \\ k & \text{otherwise.} \end{cases}$$  

We only consider the one dimensional case, since the multidimensional case is similar. The proof is by supposing that $\|\xi [f(n)]\| > M^{-1}$ for every $1 \leq n \leq N$. Then using Vinogradov’s method we approximate the indicator function and by our assumption this yields a lower bound for an exponential sum of the form

$$\sum_{1 \leq n \leq N} e(\beta f(n)),$$  

where $\beta > 0$ is some real number.
The aim of this section is to provide upper bounds for this exponential sum that subsequentially violates the lower bounds yielding a contradiction. In the proof we use different arguments but essentially the same exponential sum estimate if $\beta$ is large or $\beta$ is small respectively, where these sizes have to be understood modulo 1. In both cases we use Weyl differencing which is equivalent to derivation. If $g$ is the dominant part of $f$, this means that $\deg f \notin \mathbb{Z}$, we may differentiate as often as we wish till the resulting function has the desired behavior. On the other hand if the polynomial part $P$ is dominant we cannot differentiation that freely since after $k$ steps we lose the polynomial and therefore the dominant part. Therefore we have to further consider two cases for $k > \theta_r$ or not.

Let $\rho$ be a real satifying

$$0 < \rho < \frac{1}{\deg f + 3}.$$  

Then we distinguish the following cases:

* If $\deg f = \theta_r$ is not an integer ($\theta_r > k$), then we only have

$$N^{\theta_r - k} < |\beta| \leq N^{1/10}.$$  

* If $\deg f = k$ is an integer ($k > \theta_r$), then in the following section we distinguish the cases

$$N^{\theta_r - k} < |\beta| \leq N^{\theta_r}, \quad N^{\theta_r - k} < |\beta| \leq N^{1/10}.$$  

Note that in the case of

$$0 < |\beta| < N^{\deg f}$$

we apply a different argument that allows us to reuse the estimates for bigger $\beta$. Furthermore we note that the exponent $\frac{1}{10}$ is an artifact of Lemma 2.3 of [7] which we use in the proof.

If $\theta_r > k$ we may apply Weyl-differencing sufficiently often till the sum does not rotate to much.

**Lemma 4.1.** [7 Lemma 2.5] Let $X, k, q \in \mathbb{N}$ with $k, q \geq 0$ and set $K = 2^k$ and $Q = 2^q$. Let $P(x)$ be a polynomial of degree $k$ with real coefficients. Let $g(x)$ be a real $(q + k + 2)$ times continuously differentiable function on $[X, 2X]$ such that $|g^{(r)}(x)| \asymp GX^{-r}$ ($r = 1, \ldots, q + k + 2$). Then, if $G = o(X^{q+2})$ for $G$ and $X$ large enough, we have

$$\sum_{X < n \leq 2X} e(g(n) + P(n)) \ll X^{1 - \frac{1}{K}} + X \left(\frac{\log X}{G}\right)^{\frac{2}{K}} + X \left(\frac{G}{X^{q+2}}\right)^{\frac{2}{K(q+2)}},$$

where $A \asymp B$ means that $A$ is of the same order as $B$, i.e. $A \ll B$ and $B \ll A$.

**Proposition 4.2.** Let $P(x)$ be a polynomial of degree $k$ and $g(x) = \sum_{j=0}^{r} d_j x^{\theta_j}$ with $r \geq 1$, $d_r \neq 0$, $d_j$ real, $0 < \theta_1 < \theta_2 < \cdots < \theta_r$ and $\theta_j \notin \mathbb{N}$. Assume that $\ell < \theta_r < \ell + 1$ for some integer $\ell$. Let $N^{\epsilon - \theta_r} \leq |\xi| \leq N^{-\theta_r}$. Then for arbitrary $\varepsilon > 0$

$$\sum_{n \leq N} e(g(n) + \xi P(n)) \ll N^{1 - \frac{1}{K}} \left(\frac{1}{\log^{L+1}(X)}\right),$$

where $K = 2^k$ and $L = 2^q$.

**Proof.** We split the sum into $\leq \log N$ sub-sums of the form

$$\sum_{X \leq n \leq 2X} e(g(n) + \xi P(n)),$$

and denote by $S$ a typical one of them. Because of the polynomial structure of $g$ and since $\theta_r \notin \mathbb{Z}$ we get for $j \geq 1$ that $g^{(j)}(x) \asymp X^{\theta_r - j}$. Thus an application of Lemma 4.1 with $q = \ell$ yields

$$S \ll X^{1 - \frac{1}{K}} + X \left(\frac{\log X}{|\xi| X^{\theta_r}}\right)^{\frac{2}{K}} + X \left(\frac{|\xi| X^{\theta_r}}{X^{q+2}}\right)^{\frac{2}{K(q+2)}}.$$
Summing over all sub-sums and using the above bounds on $|\xi|$ we get
\[
\sum_{n \leq N} e(\xi g(n) + \xi P(n)) \ll N (|\xi| N^{\theta_e})^{-\frac{1}{2}} + N^{1 - \frac{1}{\max(2,\delta)}} \ll N^{1 - \frac{1}{\max(2,\delta)}},
\]
which yields the desired bound. \qed

Before we will prove the corresponding estimate for sums over primes we need some standard tools. The first one is the von Mangoldt’s function defined by
\[
\Lambda(n) = \begin{cases} 
\log p, & \text{if } n = p^k \text{ for some prime } p \text{ and an integer } k \geq 1; \\
0, & \text{otherwise}.
\end{cases}
\]

**Lemma 4.3 ([7, Lemma 11]).** Let $g$ be a function such that $|g(n)| \leq 1$ for all integers $n$. Then
\[
\left| \sum_{p \leq P} g(p) \right| \ll \frac{1}{\log P} \max_{t \leq P} \left| \sum_{n \leq t} \Lambda(n) g(n) \right| + \sqrt{P}.
\]

Next we introduce the $s$-fold divisor sum $d_s(n)$, i.e.
\[
ds(n) = \sum_{x_1 \cdots x_s = n} 1.
\]

The central tool in the treatment of the exponential sum over primes is Vaughan’s identity which we use to subdivide the weighted exponential sum into several sums of Type I and II.

**Lemma 4.4 ([7, Lemma 2.3]).** Assume $F(x)$ to be any function defined on the real line, supported on $[X, 2X]$ and bounded by $F_0$. Let further $U, V, Z$ be any parameters satisfying $3 \leq U < V < Z < 2X$, $Z \geq 4U^2$, $X \geq 32Z^2U$, $V^3 \geq 32X$ and $Z - \frac{1}{2} \in \mathbb{N}$. Then
\[
\left| \sum_{X < n \leq 2X} \Lambda(n) F(n) \right| \ll K \log P + F_0 + L(\log X)^8,
\]
where $K$ and $L$ are defined by
\[
K = \max_M \sum_{m=1}^{\infty} d_3(m) \left| \sum_{Z < n \leq M} F(mn) \right| \quad \text{(Type I)},
\]
\[
L = \sup_m \sum_{m=1}^{\infty} d_4(m) \left| \sum_{U < n < V} b(n) F(mn) \right| \quad \text{(Type II)},
\]
where the supremum is taken over all arithmetic functions $b(n)$ satisfying $|b(n)| \leq d_3(n)$.

Using these tools we obtain a similar estimate for the sum over primes.

**Proposition 4.5.** Let $P(x)$ be a polynomial of degree $k$ and $g(x) = \sum_{j=1}^{r} d_j x^{\theta_j}$ with $r \geq 1$, $d_r \neq 0$, $d_j$ real, $0 < \theta_1 < \theta_2 < \cdots < \theta_r$ and $\theta_j \notin \mathbb{N}$. Assume that $\ell < \theta_r < \ell + 1$ for some integer $\ell$. Let $N^{\rho - \theta} \leq |\xi| \leq N^{\frac{1}{\ell+1}}$. Then for arbitrary $\varepsilon > 0$
\[
\left| \sum_{p \leq N} e(\xi g(p) + \xi P(p)) \right| \ll N^{1 - \rho + \varepsilon} + N^{1 - \frac{1}{\ell+1} + \varepsilon} + N^{1 - \frac{1}{\max(2,\delta)} + \varepsilon},
\]
where $K = 2^k$ and $L = 2^k$.

**Proof.** We start with an application of Lemma [5,3] which transforms the sum over the primes into the weighted sum
\[
\left| \sum_{p \leq N} e(\xi g(p) + \xi P(p)) \right| \ll \frac{1}{\log N} \max_{n \leq N} \left| \sum_{n \leq N} \Lambda(n) e(\xi (g(n) + P(n))) \right| + N^{\frac{1}{\ell+1}}.
\]
Then we split the inner sum into \( \leq \log N \) subsums of the form

\[
\left| \sum_{X < n \leq 2X} \Lambda(n) e (\xi(g(n) + P(n))) \right|
\]

with \( 2X \leq N \) and we denote by \( S \) a typical one of them. We may assume that \( X \geq N^{1 - \rho} \).

Applying Vaughan’s identity (Lemma 4.3) with the parameters \( U = \frac{1}{4} X^{1/5} \), \( V = 4X^{1/3} \) and \( Z \) the unique number in \( \frac{1}{2} + N \), which is closest to \( \frac{1}{2}X^{2/5} \), yields

(4.2)

\[
S \ll 1 + (\log X) S_1 + (\log X)^8 S_2,
\]

where

\[
S_1 = \sum_{x < \frac{X}{2}} d_3(x) \sum_{y > \frac{X}{2}, \frac{X}{2} < y < \frac{X}{2}} e (\xi(g(xy) + P(xy)))
\]

\[
S_2 = \sum_{\frac{X}{2} < x \leq \frac{X}{2}} d_4(x) \sum_{U < y < V, \frac{X}{2} < y \leq \frac{X}{2}} b(y) e (\xi(g(xy) + P(xy))).
\]

We start estimating \( S_1 \). Since \( d_3(x) \ll x^2 \) we have

\[
|S_1| \ll X^\varepsilon \sum_{x \leq \frac{X}{2}} \left| \sum_{\frac{X}{2} < y \leq \frac{X}{2}} e (\xi(g(xy) + P(xy))) \right|.
\]

We fix \( x \) and write \( Y = \frac{X}{x} \) for short. Since \( \theta_r \notin \mathbb{Z} \), we obtain for \( j \geq 1 \) that

\[
|\frac{\partial^j g(y)}{\partial y^j}| \asymp X^\theta_r Y^{-j}.
\]

For \( j \geq 5(\ell + 1) \) we get

\[
\xi X^\theta_r Y^{-j} \ll N^{1/2} X^{\theta_r - \frac{1}{4j}} \ll X^{-\frac{1}{2j}}.
\]

Thus an application of Proposition 4.1 yields the following estimate:

(4.3)

\[
|S_1| \ll X^\varepsilon \sum_{x \leq \frac{X}{2}} Y \left[ Y^{-\frac{1}{2j}} + (\log Y)^{\delta (\xi)} X^\theta_r \right] \ll X^{1 + \varepsilon} (\log X) \left[ X^{-\frac{1}{2j}} + X^{-\frac{1}{4(\delta (\xi) + 1)}} \right],
\]

where we have used that \( \frac{1}{2j} < 1 \) and \( \rho(\theta_r + 1) < \rho[(\deg f) + 2] < 1 \) by (4.1).

Now we turn our attention to the second sum \( S_2 \). We split the range \((\frac{X}{2}, \frac{X}{2})\) into \( \leq \log X \) subintervals of the form \((X_1, 2X_1)\). Thus

\[
|S_2| \leq (\log X) X^\varepsilon \sum_{X_1 < x \leq 2X_1} \left| \sum_{U < y < V, \frac{X}{2} < y \leq \frac{X}{2}} b(y) e (\xi(g(xy) + P(xy))) \right|.
\]

An application of Cauchy’s inequality together with the estimate \( |b(y)| \ll X^\varepsilon \) yields

\[
|S_2|^2 \ll (\log X)^2 X^{2\varepsilon} X_1 \sum_{X_1 < x \leq 2X_1} \left| \sum_{U < y < V, \frac{X}{2} < y \leq \frac{X}{2}} b(y) e (\xi(g(xy) + P(xy))) \right|^2
\]

\[
\ll (\log X)^2 X^{4\varepsilon} X_1 \times \left( X_1 \frac{X}{X_1} + \sum_{X_1 < x \leq 2X_1} \sum_{A < y_1 < y_2 \leq B} e (\xi(g(xy_1) - g(xy_2) + P(xy_1) - P(xy_2))) \right),
\]
where $A = \max\{U, \frac{x}{x}\}$ and $B = \min\{U, \frac{2x}{x}\}$. A change of the order of summation yields

$$|S_2|^2 \ll (\log X)^2 X^{4\varepsilon} X_1 \times \left( X + \sum_{A < y_1 \leq y_2 \leq B} \left| \sum_{X_1 \leq x \leq 2X_1} e \left( \xi \left( g(xy_1) - g(xy_2) + P(xy_1) - P(xy_2) \right) \right) \right| \right).$$

We fix $y_1$ and $y_2 \neq y_1$. Similarly as above we get

$$\left| \frac{\partial^j}{\partial x^j} \left( g(xy_1) - g(xy_2) + P(xy_1) - P(xy_2) \right) \right| \ll \frac{|y_1 - y_2|}{y_1} X^\theta X_1^{-j}.$$

In this case we suppose that $j \geq 2|\theta_r| + 3$ in order to obtain

$$\xi \frac{|y_1 - y_2|}{y_1} X^{\theta_r} X_1^{-j} \ll X^{\frac{1}{3} + \theta_r - \frac{3}{2}j} \ll X^{-\frac{1}{2}}.$$

Thus again an application of Lemma 4.1 yields

$$|S_2|^2 \ll (\log X)^2 X^{4\varepsilon} X_1 \left( X + \sum_{A < y_1 \leq y_2 \leq B} X_1 \left( X_1^{-\frac{1}{3}} + (|\xi| X^{\theta_r})^{-\frac{1}{3}} + X^{-\frac{1}{3} - \frac{16}{5}} \right) \right)$$

$$\ll (\log X)^2 X^{4\varepsilon} \left( X^{2 - \frac{1}{3} \varepsilon} + X^{2 - \frac{1}{3} \varepsilon} + X^{2 - \frac{16}{5} - \frac{1}{3} \varepsilon} \right).$$

Plugging the two estimates (4.3) and (4.4) into (4.2) together with a summation over all intervals proves the proposition.

5. Exponential sum estimates for the case $\theta_r < k$

As above we write $f(x) = g(x) + P(x)$, where $P$ is a polynomial of degree $k$ with real coefficients and $g(x) = \sum_{j=1}^r \alpha_j x^{\theta_j}$ with $1 < \theta_1 < \ldots < \theta_r$ and $\theta_j \not\in \mathbb{Z}$ for $j = 1, \ldots, r$. Then we consider two cases according to the size of $|\beta|$

$$N^{1-\rho-k} < |\beta| \leq N^{\rho-\theta_r} \quad \text{and} \quad N^{\rho-\theta_r} < |\beta| \leq N^{1/10},$$

where $\rho$ is as in (4.1).

The "large" coefficient case may be treated as in Section 4. For the case of smaller coefficients we need a completely different approach. On the one hand the coefficients are too big ($|\xi| > N^{1-\rho-k}$) to use the method of van der Corput and on the other hand the polynomial part is the dominant one ($k > \theta_r$) and an application of Weyl differencing as in the case of the large coefficients would make the polynomial disappear.

The idea is to separate the polynomial $P$ and the real function $g$ by means of a partial summation. Then for the sum over the polynomial we use standard estimates due to Weyl [35] and Harman [24] for the integer and prime case, respectively. Since these are standard methods we present only the non-standard steps and refer the interested reader to Chapter 3 of Montgomery [35], Chapter 4 of Nathanson [44] or the monograph of Graham and Kolesnik [21] for a more complete account on Weyl-van der Corput’s method.

Proposition 5.1. Let $f$ be a pseudo polynomial and $\varepsilon > 0$. Suppose that $\rho(k + 3) < 1$ and that

$$N^{3\rho-k} \leq |\xi| \leq N^{\rho-\theta_r}.$$

Then for sufficiently large $N$

$$\sum_{n \leq N} e(\xi f(n)) \ll N^{1-\rho \theta_r^3 + \varepsilon}.$$
For \( \ell \geq 2 \), we define the iterated difference operator \( \Delta_{d_{\ell},d_{\ell-1},\ldots,d_1} \) by

\[
\Delta_{d_{\ell},d_{\ell-1},\ldots,d_1} = \Delta_{d_{\ell}} \circ \Delta_{d_{\ell-1},\ldots,d_1} = \Delta_{d_{\ell}} \circ \Delta_{d_{\ell-1}} \circ \cdots \circ \Delta_{d_1}.
\]

The following lemma describes the idea of “Weyl differencing”.

**Lemma 5.2** (**[54] Lemma 4.13**). Let \( N_1,N_2,N, \) and \( \ell \) be integers such that \( \ell \geq 1, N_1 < N_2, \) and \( N_2 - N_1 \leq N \). Let \( f(n) \) be a real-valued arithmetic function. Then

\[
\left| \sum_{n=N_1+1}^{N_2} e(f(n)) \right| \leq (2N)^{2\ell - \ell - 1} \sum_{|d_1|<N} \cdots \sum_{|d_\ell|<N} e(\Delta_{d_\ell,\ldots,d_1}(f(n))).
\]

where \( I(d_1,\ldots,d_1) \) is an interval of consecutive integers contained in \([N_1 + 1, N_2]\).

In order to estimate the innermost sum of the previous lemma we use the following observation.

**Lemma 5.3** (**[54] Lemma 4.7**). For every real number \( \alpha \) and all integers \( N_1 < N_2 \),

\[
\sum_{n=N_1+1}^{N_2} e(\alpha n) \ll \min \left( N_2 - N_1, \|\alpha\|^{-1} \right).
\]

The final tool is the following estimate for sums of minima.

**Lemma 5.4** (**[54] Lemma 4.9**). Let \( \alpha \) be a real number. If

\[
|\alpha - \frac{a}{b}| \leq \frac{1}{b^2},
\]

where \( b \geq 1 \) and \((a,b) = 1\), then for any negative real numbers \( H \) and \( N \) we have

\[
\sum_{h=1}^{H} \min \left( N, \frac{1}{\|\alpha h\|} \right) \ll \left( b + H + N + \frac{HN}{b} \right) \max \{1, \log b\}.
\]

The following lemma links the estimate of the exponential sum with the leading coefficient of the polynomial part and non-polynomial part.

**Lemma 5.5.** Let \( X, k \in \mathbb{N} \) with \( k \geq 0 \) and set \( K = 2^{k-1} \). Let \( P \) be a polynomial of degree \( k \) with real coefficients and let \( \alpha \) be the leading coefficient. Let \( g(x) \) be a real \( k \) times continuously differentiable function on \([X, 2X]\) such that \( |g^{(r)}(x)| \geq GX^{-r} \) \((r = 1,\ldots,k)\). Then, for \( G \) and \( X \) large enough, we have

\[
\left| \sum_{X < n \leq 2X} e(\alpha n + g(n)) \right| \ll X^{2^{k-1} - 1} + (1 + G)X^{2^{k-1} - k + \varepsilon} \sum_{t=1}^{k} \min \left( X, \frac{1}{\|\alpha t\|} \right)
\]

with arbitrary \( \varepsilon > 0 \).

**Proof.** Our first tool is **Lemma 5.2** with \( \ell = k - 1 \) to get

\[
\left| \sum_{X < n \leq 2X} e(\alpha n + g(n)) \right| \ll X^{2^{k-1} - 1} + X^{2^{k-1} - k} \sum_{1 \leq |d_1|<X} \cdots \sum_{1 \leq |d_{k-1}|<X} e(\Delta_{d_{k-1},\ldots,d_1}(P + g)(n)).
\]

Now we want to separate the polynomial and non-polynomial part. To this end we set

\[
a_n = e(\Delta_{d_{k-1},\ldots,d_1}P(n)) \quad \text{and} \quad b_n = e(\Delta_{d_{k-1},\ldots,d_1}g(n)).
\]

Then an application of partial summation yields

\[
\sum_{X < n \leq 2X} a_n b_n \leq \left| \sum_{X < n \leq 2X} a_n \right| + \sum_{X < h \leq 2X} \left| b_h - b_{h+1} \right| \sum_{X < n \leq h} a_n.
\]
For the non-polynomial part $b_h - b_{h+1}$ we note the following representation for the forward difference operator (cf. Lemma 2.7 of Graham and Kolesnik [21])

$$\Delta_{d_k-1} \cdots d_1 (g)(n) = \int_0^1 \cdots \int_0^1 \frac{\partial^{k-1}}{\partial t_1 \cdots \partial t_{k-1}} g(n + t_1 d_1 + \cdots + t_{k-1} d_{k-1}) dt_1 \cdots dt_{k-1}.$$ 

Together with the mean value theorem we get

$$|b_h - b_{h+1}| \leq X^{k-1} G X^{-k} = GX^{-1}.$$

Now we turn our attention to the polynomial part, which means to the sum of $\xi P(n)$. Since $\Delta_{d_k-1} \cdots d_1 (P)(n) = k! d_1 \cdots d_{k-1} + m(d_1, \ldots, d_{k-1})$, where $m$ is a function not depending on $n$, an application of Lemma 5.6 yields

$$\left| \sum_{X<n \leq h} e(\Delta_{d_k-1} \cdots d_1 P(n)) \right| \ll \min \left(h, \frac{1}{\|k! d_1 \cdots d_{k-1} \xi\|}\right).$$

Putting the two estimates together we get

$$\left| \sum_{X<n \leq 2X} e(P(n) + g(n)) \right|^{2^{k-1}} \ll X^{2^{k-1}-1} + X^{2^{k-1}-k} \sum_{1 \leq |d_1|<X} \cdots \sum_{1 \leq |d_{k-1}|<X} (1 + G) \min \left(1, \frac{1}{\|k! d_1 \cdots d_{k-1} \xi\|}\right).$$

Noting that for every $t \leq k! X^{k-1}$ there are $X^\varepsilon$ solutions $d_1, \ldots, d_{k-1}$ to

$$t = k! d_1 \cdots d_{k-1},$$

we get that

$$\left| \sum_{X<n \leq 2X} e(P(n) + g(n)) \right|^{2^{k-1}} \ll X^{2^{k-1}-1} + (1 + G) X^{2^{k-1}-k+\varepsilon} \sum_{t=1}^{X^{k-1}} \min \left(1, \frac{1}{\|t\|}\right),$$

which proves the lemma. \qed

We have seen that the estimates reduce to an approximation of the leading coefficient of the polynomial part of $f$. The following lemma shows that the leading coefficient of $\xi P(x)$ can always be approximated well provided that $\xi$ is in the “medium” range.

**Lemma 5.6.** Let $N, \alpha, \rho$ and $\xi$ be positive reals and let $k \geq 2$ be a positive integer. Suppose that $\rho < (k + 3)^{-1}$ and that

$$N^{3\rho - k} < |\xi| \leq N^{\rho - \theta},$$

Then there exist coprime $a, b \in \mathbb{Z}$ such that

$$|\xi \alpha - a| \leq b^{-2} \quad \text{and} \quad N^{2\rho} \leq b \leq N^{k-2\rho}$$

provided that $N$ is sufficiently large.

**Proof.** By Dirichlet’s approximation theorem there exist coprime $a, b \in \mathbb{Z}$ such that

$$|\xi \alpha - a| \leq N^{-k+2\rho} \quad \text{and} \quad 1 \leq b \leq N^{k-2\rho}.$$

If $b \geq N^{2\rho}$, then there is nothing to show. Suppose the contrary. We distinguish different cases for the size of $b$.

**Case 1.** $2 \leq b < N^{2\rho}$. In this case we get

$$N^{\rho - \theta} \gg |\xi \alpha| \geq \left| \frac{a}{b} \right| - \frac{1}{b^2} \geq \frac{1}{2b} \geq \frac{1}{2} N^{-2\rho}.$$

Since $3\rho < 1 < \theta$, this contradicts our lower bound for sufficiently large $N$.

**Case 2.** $b = 1$. This case requires a further distinction according to whether $a = 0$ or not.
Case 2.1. \(|\xi \alpha| \geq \frac{1}{2}\). It follows that

\[ N^{\rho - \theta r} \gg |\xi \alpha| \geq \frac{1}{2} \]

which is absurd for \(N\) sufficiently large.

Case 2.2. \(|\xi \alpha| < \frac{1}{2}\). This implies that \(a = 0\) which yields

\[ N^{3\rho - k} \ll |\xi \alpha| \leq N^{-k + 2\rho}, \]

which again is absurd for sufficiently large \(N\). \(\square\)

Now we have all tools at hand to prove the estimate in the integer case.

**Proof of Proposition 5.1.** We divide the sum into \(\leq (\log N)\) subsums of the form

\[ \sum_{X < n \leq 2X} e(\xi f(n)) \]

and denote by \(S\) a typical one of them. Without loss of generality we may suppose that \(X \geq N^{1-\rho}\). For \(r = 1, \ldots, k\) we have

\[ |\xi g^{(r)}(n)| = |\xi| X^{\theta r - \ell}. \]

Thus an application of Lemma 5.5 yields

\[ S^K \ll X^{K-1} + (1 + |\xi| X^{\theta r}) X^{K-k+\epsilon} \sum_{t=1}^{4X^{k-1}} \min \left( X, \frac{1}{\|\xi \alpha\|} \right). \]

Let \(a, b\) be positive integers such that

\[ |\xi \alpha - \frac{a}{b}| \leq \frac{1}{b^2}. \]

Using Lemma 5.4 we get for the sum

\[ S^K \ll X^{K-1} + (1 + |\xi| X^{\theta r}) X^{K-k+\epsilon} \left( b + \frac{X^k}{b} \right). \]

Taking the \(K\)th root and summing over all dyadic intervals \([X, 2X]\) we get

\[ \sum_{n \leq N} e(\xi f(n)) \ll N^{1 - \frac{K}{2} + \epsilon} + N^{\frac{K}{2}} N^{1 - \frac{k}{2} + \epsilon} \left( b + \frac{X^k}{b} \right)^\frac{K}{2}, \]

where we used that \(|\xi| \leq N^{\rho - \theta}\). Finally we derive the bounds for \(b\) of Lemma 5.6

\[ \sum_{n \leq N} e(\xi f(n)) \ll N^{1 - \frac{K}{2} + \epsilon} + N^{\frac{K}{2}} N^{1 + \epsilon} N^{-\frac{2K}{2}} \ll N^{1 - \frac{K}{2} + \epsilon}, \]

which is the desired result. \(\square\)

Now we turn our attention to the prime case. Therefore we will reuse the central idea of the good approximation but we have to adopt the Weyl differencing part. In particular, we will obtain bilinear forms (sums over products) instead of the usual exponential sums.

**Proposition 5.7.** Let \(N\) and \(\rho\) be positive reals and \(f\) be a pseudo-polynomial. If \(\rho(k + 3) < 1\) and \(\xi\) is such that

\[ N^{3\rho - k} < |\xi| \leq N^{\rho - \theta r} \]

holds, then

\[ \sum_{p \leq N} e(\xi f(p)) \ll N^{1 - \rho 4^{k+\epsilon}} \]

with arbitrary \(\epsilon > 0\).
For the prime case we need two further tools dealing with the bilinear forms appearing after an application of Vaughan’s identity (Lemma 4.4). These are adoptions of the corresponding Lemmas 3 and 4 of Harman [23] to our setting of pseudo-polynomial functions. Let \( \varphi \) and \( \psi \) be two real functions. Then for \( U, V \) and \( X \) reals, we consider sums of the form

\[
\sum_{u=1}^{U} \sum_{v=1}^{V} \varphi(u)\psi(v)e(P(uv) + g(uv)),
\]

where \( P \) is a polynomial of degree \( k \) and \( g \) is a \( k \) times continuously differentiable real function. Furthermore we define \( \Psi \) by

\[
\Psi(n, y_1, \ldots, y_s) = \psi(n) \prod_{i=1}^{s} \psi(n + y_i) \prod_{1 \leq i < j \leq s} \psi(n + y_i + y_j) \cdot \prod_{i=1}^{s} \varphi \left( n + \sum_{j \neq i} y_j \right) \psi \left( n + \sum_{i=1}^{s} y_i \right).
\]

The first Lemma of Harman [23] is the following.

**Lemma 5.8.** Let \( P \) be a polynomial of degree \( k \) with real coefficients and let \( \alpha \) be its leading coefficient. Let \( g(x) \) be a real \((2k+1)\) times continuously differentiable function on \([X, 2X]\) such that \( |g^{(r)}(x)| \leq GX^{-r} \) \((r = 1, \ldots, k)\). Set

\[
T = \max |\psi(v)|, \quad \text{and} \quad F = \left( \frac{1}{U} \left( \sum_{u \leq U} \varphi(u)^2 \right) \right)^{\frac{1}{2}}.
\]

For positive integers \( U, V, X \) write

\[
(5.1) \quad S = \sum_{u=1}^{U} \sum_{v=1}^{V} \varphi(u)\psi(v)e(P(uv) + g(uv)).
\]

Suppose that there exist \( a, b \in \mathbb{Z} \) such that

\[
|\alpha - \frac{a}{b}| \leq \frac{1}{b^2}.
\]

Then we have

\[
\left( \frac{S}{TF} \right)^K \ll (UV)^{K^2+\varepsilon} \left( V^{-K} + (1 + G) (U^{-1} + b^{-1} + (UV)^{-k}) \right),
\]

where \( K = 2^{k-1} \).

**Proof.** We may assume that \( T = F = 1 \) and \( \psi(v) \geq 0 \) for all \( v \) as well as omit the restriction \( X \leq uv \leq 2X \) for the moment. Then an application of the Cauchy-Schwarz inequality yields

\[
S^2 \ll U \sum_{v_1=1}^{V} \sum_{v_2=1}^{V} \psi(v_1)\psi(v_2) \sum_{u=1}^{U} \left( P(uv_1) - P(uv_2) + g(uv_1) - g(uv_2) \right)
\]

\[
\ll US_1 + E_1.
\]

For positive integers \( s \) we write for short

\[
S_s = \sum_{v=1}^{V-1} \cdots \sum_{v=1}^{V-1} \sum_{v} \sum_{u=1}^{U} \psi(v, d_1, \ldots, d_s) e(\Delta_{d_1, \ldots, d_s}(P + g)(uv)),
\]

where the forward difference operator \( \Delta_{d_1, \ldots, d_s} \) acts on \( v \) not on \( u \) and the range of summation over \( v \) being \( X < v < v + d_1 + \cdots + d_s \leq 2X \), and

\[
E_s = U^{2s}V^{2s-1}.
\]

An easy induction shows for \( s = 2, \ldots, k-1 \) that

\[
S^2 \ll E_s + U^{2s-1}V^{2s-1} |S_s|.
\]
Now we look at the innermost sum of $S_g$. Since (cf. Lemma 10B of Schmidt [48])
\[
\Delta_{d_1-1, \ldots, d_k}(P)(uv) = d_1 \cdots d_k \left( \frac{1}{k!} \log^k(2v + d_1 + \cdots + d_k) + (k - 1)! u^{k-1} \right) \\
= u^k h(d_1, \ldots, d_k, v) + u^{k-1}(k - 1)! d_1 \cdots d_k,
\]
we can see $\Delta_{d_1-1, \ldots, d_k}(P)(uv)$ as a polynomial of degree $k$ with leading coefficient $h(d_1, \ldots, d_k, v)$. Furthermore $\Delta_{d_1-1, \ldots, d_k}(g)(uv)$ is a $k$ times differentiable function and we may apply Lemma 5.5. Thus after $k - 1$ iterations of the Cauchy-Schwarz inequality we obtain
\[
S K^2 \ll (UV)^K V^{-K} + U^{K^2 - K} V^{K^2 - k} \sum_{d_1 = 1}^{V} \cdots \sum_{d_k = 1}^{V} \sum_{v = 1}^{U} \left| \sum_{u = 1}^{U} e \left( \Delta_{d_1, \ldots, d_k}(P + g)(uv) \right) \right|^K.
\]

As second we adapt Lemma 4 of Harman [23].

**Lemma 5.9.** Suppose we have the hypotheses of Lemma 5.8 but either
\[
\varphi(x) = 1, \text{ for all } x,
\]
or \[
\varphi(x) = \log x, \text{ for all } x.
\]
Then
\[
S \ll (UV)^{1 + \gamma} V^{\frac{2k}{k + 1}} (1 + G)^{1 + \frac{1}{k - 1}} ((UV)^{1 + \gamma} V^{\frac{2k}{k + 1}} + U^{-1} + b^{-1})^{1 + \frac{1}{k - 1}}.
\]

**Proof.** By an application of partial summation we may easily remove the log factor. Therefore without loss of generality we assume that $\varphi(x) = 1$. Using Hölder’s inequality we obtain
\[
S K \ll V^{K - 1} \sum_{v = 1}^{V} \sum_{u = 1}^{U} e \left( P(uv) + g(uv) \right),
\]
where this time the forward difference operator $\Delta_{d_1, \ldots, d_k}$ is with respect to $u$.

Now an application of Lemma 5.5 for the innermost sum yields
\[
S K \ll V^{K - 1} \sum_{v = 1}^{V} U^{K - 1} V^{k + 1} (1 + G) \sum_{t = 1}^{V} \min \left( U, \frac{1}{\|tv^k\alpha\|} \right)
\ll U^{K - 1} V^{k + 1} (1 + G) \sum_{t = 1}^{V} \min \left( U, \frac{1}{\|tv\alpha\|} \right).
\]

An application of Lemma 5.4 proves the lemma. \qed

Finally we have to combine the two lemmas as in the proof of Harman.

**Proof of Proposition 5.7.** This proof starts along the same lines as the proof of Proposition 4.9.

An application of Lemma 4.3 transforms the sum over the primes into the weighted sum
\[
\left| \sum_{p \leq N} e(\xi g(p) + \xi P(p)) \right| \ll \frac{1}{\log N} \max_{n \leq N} \sum_{\Lambda(n) e(\xi g(n) + P(n))} + N^{\frac{1}{2}}.
\]
Then we split the inner sum into \( \leq \log N \) subsums of the form

\[
\left| \sum_{X < n \leq 2X} A(n) e \left( \xi(g(n) + P(n)) \right) \right|
\]

with \( 2X \leq N \) and we denote by \( S \) a typical one of them. We may assume that \( X \geq N^{1-\rho} \).

Applying Vaughan’s identity (Lemma 4.4) with the parameters \( U = \frac{1}{4} X^{1/5} \), \( V = 4X^{1/3} \) and \( Z \) the unique number in \( \frac{1}{2} + N \), which is closest to \( \frac{1}{2}X^{2/5} \), yields

\[
S \ll 1 + (\log X)S_1 + (\log X)^8 S_2,
\]

where

\[
S_1 = \sum_{x < \frac{4X}{3}} d_3(x) \sum_{y > Z, \frac{4X}{3} < y < \frac{5X}{4}} e \left( \xi(g(xy) + P(xy)) \right)
\]

\[
S_2 = \sum_{x < \frac{4X}{3}} d_4(x) \sum_{U < y < V, \frac{4X}{3} < y < \frac{5X}{4}} b(y) e \left( \xi(g(xy) + P(xy)) \right).
\]

We consider these two sums as variants of the following general sum

\[
S_3 = \sum_{u \leq \frac{4X}{3}} \sum_{v \leq V} \varphi(u) \psi(v) e \left( \xi f(uv) \right),
\]

where \( V \ll X^{1/3} \) or \( V \ll X^{2/3} \).

Similar as in Proposition 5.1 we get that

\[
\left| \xi f^{(\ell)}(n) \right| \asymp |\xi| X^{\theta_\ell - \ell}.
\]

Furthermore let \( a, b \in \mathbb{Z} \) be as in Lemma 5.6 i.e.

\[
|\xi a - \frac{a}{b}| \leq b^{-2} \quad \text{and} \quad N^{2\rho} \leq b \leq N^{k-2\rho}.
\]

Now we apply Lemma 5.8 and Lemma 5.9 whether

\[
V^K \geq N^{-\rho} \min \left( b, N^{1/2}, N^{k} b^{-1} \right)
\]

holds or not respectively. Suppose that (5.2) holds. In this case an application of Lemma 5.8 yields

\[
S_3^{K^2} \ll X^{K^2 + \varepsilon} \left( V^K + |\xi| X^{\theta_{\ell}} \left( X^{-\frac{1}{2}} + b^{-1} + X^{-\frac{1}{2}} b^{-1} \right) \right)
\]

On the contrary, if (5.2) does not hold, then an application of Lemma 5.9 yields

\[
S_3 \ll X^{1+\varepsilon} V^{\frac{1}{2+\varepsilon}} (1 + |\xi| X^{\theta_{\ell}}) \left( X^{-\frac{1}{2}} b + X^{-\frac{1}{2}} + b^{-1} \right)^{\frac{1}{2+\varepsilon}}
\]

where we have used that \( 1/K - (k-1)/K^2 \geq 1/K^2 \).

Summing over the intervals \([X, 2X]\) using both estimates together with \( |\xi| \leq N^{\rho-\theta_{\ell}} \) yields the desired bound.

\[ \square \]

6. The case of a single pseudo-polynomial

In the present section we want to prove Theorems 2.3 and 2.6. The main tool originates from large sieve estimates due to Montgomery which provides us with a lower bound if all elements are sufficiently far away from an integer.

**Lemma 6.1.** Let \( M \) and \( N \) be positive integers. Consider a sequence of real numbers \( x_1, \ldots, x_N \) and weights \( c_1, \ldots, c_N \geq 0 \). Suppose \( \|x_j\| \geq M^{-1} \) for all \( j = 1, \ldots, N \). Then there exists \( 1 \leq m \leq M \) such that

\[
\left| \sum_{n=1}^N c_n e(mx_n) \right| \geq \frac{1}{6M} \sum_{n=1}^N c_n.
\]
Proof. This is a weighted version of [1] Theorem 2.2.

The second tool are Vaaler polynomials which we need to deal with the floor function.

Lemma 6.2 ([53 Theorem 19]). Let \( I \subset [0, 1] \) be an interval and \( \chi_I \) its indicator function. Then for each positive integer \( H \) there exist coefficients \( a_H(h) \) and \( C_h \) for \( -H \leq h \leq H \) with \( |a_H(h)| \leq 1 \) and \( |C_h| \leq 1 \) such that the trigonometric polynomial

\[
\chi_{I,H}^*(t) = |I| + \frac{1}{\pi} \sum_{|h| \leq H} \frac{a_H(h)}{|h|} e(ht)
\]

satisfies

\[
|\chi_I(t) - \chi_{I,H}^*(t)| \leq \frac{1}{H + 1} \sum_{|h| \leq H} C_h \left( 1 - \frac{|h|}{H + 1} \right) e(ht).
\]

Remark 6.3. The coefficients \( a_H(h) \) and \( C_h \) are explicitly given in Vaaler's proof. However, the stated bounds are sufficient for our purposes.

Now we are able to state the proof of Theorem 2.3.

Proof of Theorem 2.3. Let \( M = \lfloor N^n \rfloor \) where \( \eta \) is a sufficiently small exponent. We conduct the proof by supposing that

\[
(6.1) \quad \min_{1 \leq n \leq N} \| \xi \cdot |f(n)| \| \geq M^{-1}
\]

and deducing a contradiction. Since

\[
M^{-1} \leq \| \xi \cdot |f(n)| \| \leq \| \xi \| \cdot \| f(n) \| \ll \| \xi \|
\]

we get \( |\xi| \gg M^{-1} \). Furthermore, by Lemma 6.1 there exists \( 1 \leq m \leq M \) such that

\[
(6.2) \quad \sum_{n=1}^N e(m \xi \cdot |f(n)|) \gg \frac{N}{M}.
\]

The aim is to establish an upper bound of this exponential sum contradicting the lower bound for sufficiently small \( \eta \).

We suppose for the moment that

\[
\|m \xi\| \geq \lambda^{1-\rho-\deg f}.
\]

The idea is to use digital expansion and Vaaler polynomials to get rid of the floor function. Then we are left with an exponential sum and use the tools from above.

Let \( q \geq 2 \) be an integer, which is chosen later. Then we denote by \( I_d \) with \( 0 \leq d < q - 1 \) the interval of all reals in \([0, 1)\) whose initial \( q \)-adic digit is \( d \), i.e.

\[
I_d := \left[ \frac{d}{q}, \frac{d + 1}{q} \right).
\]

If \( \{f(n)\} \in I_d \), then there exists \( 0 \leq \vartheta < 1 \) such that \( \{f(n)\} = \frac{d}{q} + \frac{\vartheta}{q} \). Thus

\[
e(m \xi \cdot |f(n)|) = e\left( m \xi f(n) - m \xi \frac{d}{q} \right) \left( 1 + \mathcal{O}\left( \frac{1}{q} \right) \right),
\]

yielding

\[
\left| \sum_{n=1}^N e(m \xi \cdot |f(n)|) \right| = \sum_{d=0}^{q-1} \left| \sum_{n \leq N} e(m \xi f(n)) \chi_{I_d}(f(n)) \right| + \mathcal{O}\left( \frac{N}{q} \right).
\]

Hence for a fixed \( d \in \{0, \ldots, q - 1\} \) we may write

\[
\left| \sum_{n \leq N} e(m \xi f(n)) \chi_{I_d}(f(n)) \right| \leq \left| \sum_{n \leq N} e(m \xi f(n)) \chi_{I_d}^*(f(n)) \right| + \left| \sum_{n \leq N} \chi_{I_d}(f(n)) - \chi_{I_d,H}(f(n)) \right|,
\]
where we used the notation from Lemma 6.2. Using the estimates there we get for the first part that
\[
\left| \sum_{n \leq N} e(m \xi f(n)) \chi_{I_d,H}(f(n)) \right| = \left| \sum_{n \leq N} e(m \xi f(n)) \left( \frac{1}{q} + \frac{1}{\pi} \sum_{1 \leq |h| \leq H} \frac{a_f(h)}{|h|} e(h f(n)) \right) \right|
\]
\[
\leq \frac{1}{q} \left| \sum_{n \leq N} e(m \xi f(n)) \right| + \frac{1}{\pi} \sum_{0 < |h| \leq H} \frac{1}{|h|} \left| \sum_{n \leq N} e((m \xi + h) f(n)) \right|.
\]

For the second part we again use the estimates in Lemma 6.2 and arrive at
\[
\left| \sum_{n \leq N} \chi_{I_d}(f(n)) - \chi_{I_d,H}(f(n)) \right| \leq \frac{1}{H + 1} \sum_{|h| \leq H} \left( 1 - \frac{|h|}{H + 1} \right) \left| \sum_{n \leq N} e(h f(n)) \right|.
\]

The different exponential sums in (6.3) and (6.4) are of the form
\[
\sum_{n \leq N} e(m \xi f(n)), \quad \sum_{n \leq N} e((m \xi + h) f(n)) \quad \text{and} \quad \sum_{n \leq N} e(h f(n)),
\]
respectively. We write them in the form \(\sum_{n \leq N} e(\beta f(n))\) for short.

Since \(\|m \xi\| \geq N^{1 - \rho - \deg f}\) we get \(\beta \geq N^{1 - \rho - \deg f}\) and we may distinguish the following two cases.

- Suppose we have \(N^{\rho - \theta_r} \leq \beta \leq N^{\frac{K}{L}}\). Then we write \(f(x) = P(x) + g(x)\) where \(P\) is a polynomial of degree \(k\) and \(g(x) = \sum_{j=1}^r d_j x^{\theta_j}\) with \(1 < \theta_1 < \cdots < \theta_r\) and \(\theta_j \notin \mathbb{N}\) for \(1 \leq j \leq r\). Now an application of Proposition 4.2 yields
  \[
  \sum_{n \leq N} e(\beta f(n)) \ll N^{1 - \frac{1}{k} + \varepsilon},
  \]
where \(K = 2^k\) and \(L = q^{[\theta_r]}\).

- Now we suppose that \(N^{1 - \rho - \deg f} \leq \beta \leq N^{\nu - \theta_r}\). Then an application of Proposition 5.1 yields
  \[
  \sum_{n \leq N} e(\beta f(n)) \ll N^{1 - \rho - 1 - \frac{1}{k} + \varepsilon}
  \]
with \(\varepsilon > 0\).

We set \(\gamma = \min \left( \rho 2^{1-k}, (8KL - 4K)^{-1} \right)\). Thus
\[
\left| \sum_{n \leq N} e(m \xi f(n)) \chi_{I_d}(f(n)) \right| \ll N^{1 - \gamma} \left( \frac{1}{q} + \sum_{0 < |h| \leq H} \frac{1}{|h|} + \frac{1}{H + 1} \sum_{|h| \leq H} \left( 1 - \frac{|h|}{H + 1} \right) \right)
\]
\[
\ll N^{1 - 2\gamma + \varepsilon},
\]
where we have chosen \(q = H = N^{\gamma}\). Finally we obtain
\[
\left| \sum_{n \leq N} e(m \xi f(n)) \right| \ll N^{1 - \gamma}.
\]
Plugging this upper bound into the lower bound in (6.2) we get a contradiction as soon as \(\eta < \gamma\).

Now we turn our attention to the case of \(\|m \xi\| \leq N^{1 - \rho - \deg f}\). Then there is some \(h \in \mathbb{Z}\) such that
\[
|\beta| = |m \xi + h| \leq N^{1 - \rho - \deg f}
\]
and the coefficient in the second sum of (6.3) might be arbitrarily small destroying our argument. Using the existence of a multiple of \(m\) in the sequence \(\lfloor f(n) \rfloor\) yields a contradiction to (6.1).

In particular, for sufficiently large \(X\) (a small power of \(N\) we will fix later) we will show that there exists \(1 \leq n \leq X\) such that \(\lfloor f(n) \rfloor\) is a multiple of \(m\).

Let
\[
f(n) = a_m m^k + \cdots + a_1 m + a_0 + a_{-1} m^{-1} + \cdots\]
be the $m$-adic expansion of $f(n)$. Then $\lfloor f(n) \rfloor$ is a multiple of $m$ if and only if $a_0 = 0$ and this is the case if and only if $m^{-1} f(n) \in [0, \frac{1}{m})$.

Let $N(f, X)$ be the number of $1 \leq n \leq X$ such that $\lfloor f(n) \rfloor$ is a multiple of $m$. Furthermore let $\chi$ be the indicator function of the interval $[0, \frac{1}{m})$. Then an application of Vaaler polynomials (Lemma 6.2) yields

\[ N(f, X) - \frac{X}{m} \leq \sum_{1 \leq n \leq X} \left| \chi \left( \frac{f(n)}{m} \right) - \chi^* \left( \frac{f(n)}{m} \right) \right| + \sum_{1 \leq n \leq X} \left| \chi^* \left( \frac{f(n)}{m} \right) - \frac{1}{m} \right| \]

\[ \leq \frac{1}{H+1} \sum_{|h| \leq H} C_h \left( 1 - \frac{|h|}{H+1} \right) \sum_{1 \leq n \leq X} e \left( \frac{h f(n)}{m} \right) + \frac{1}{\pi} \sum_{0 < |h| \leq H} a_H(h) \sum_{1 \leq n \leq X} e \left( \frac{h f(n)}{m} \right). \]

Setting $X = M^{1-\varepsilon}$ and $H = X^{\varepsilon}$ we note that

\[ X^{\rho-\varepsilon} \leq X^{\rho-1} = \frac{1}{M} \leq \frac{h}{m} \leq X^{\frac{1}{\varepsilon}}. \]

Thus an application of Proposition 4.2 yields

\[ N(f, X) - \frac{X}{m} \ll X^{1-\frac{1}{m}} + \varepsilon. \]

Then an application of Proposition 6.2 yields

\[ N(f, X) - \frac{X}{m} \leq \| f \|_1 \leq | m \xi | \cdot | r | \leq N^{1-\rho-\deg f} M^{\frac{\deg f}{1-\rho}} \]

yielding a contradiction as long as

\[ \eta < \frac{\deg f + \rho - 1}{1 + \frac{\deg f}{1-\rho}}. \]

Putting both cases together we get a contradiction if

\[ \eta < \min \left( \rho^{2^{1-k}}, \frac{1}{8KL-4K} \cdot \frac{\deg f + \rho - 1}{1 + \frac{\deg f}{1-\rho}} \right), \]

proving the theorem. \hfill \Box

**Proof of Theorem 2.6** This runs very much along the same lines as the proof of Theorem 2.3 above. Let $M = \lfloor N^\eta \rfloor$ for a sufficiently small $\eta$ which we choose later. Suppose that $\| f(p) \| \geq M^{-1}$ for all primes $2 \leq p \leq N$. An application of Lemma 6.1 yields

\[ (6.6) \quad \sum_{p \leq N} e \left( \frac{m \xi}{f(p)} \right) \gg \frac{\pi(N)}{M}, \]

where $\pi$ is the prime-counting function. As in the integer case we are looking for an upper bound for the exponential sum yielding conditions on $\eta$.

We start with the case of $\| m \xi \| \geq N^{1-\rho-\deg f}$. Following the lines of the integer case we have to find upper bounds for exponential sums of the form

\[ \sum_{p \leq N} e \left( \beta f(p) \right) \]

with $\beta = m \xi$, $\beta = m \xi + h$ and $\beta = h$, respectively. We again distinguish two cases:

- Either we have $N^{1-\rho-\deg f} \leq |\beta| \leq N^{\rho-\varepsilon}$. Then an application of Proposition 5.7 yields
  \[ \sum_{p \leq N} e \left( \beta f(p) \right) \ll N^{1-\rho + \varepsilon}. \]
• Or we have $N^{ho - \theta_r} \leq |\beta| \leq N^{\frac{\rho}{2}}$. Again we write $f(x) = P(x) + g(x)$ where $P$ is a polynomial of degree $k$ and $g(x) = \sum_{j=1}^{r} d_j x^{\theta_j}$ with $1 < \theta_1 < \cdots < \theta_r$ and $\theta_j \notin \mathbb{N}$ for $1 \leq j \leq r$. Then an application of Proposition 4.5 yields
  $$\sum_{p \leq N} e(\beta f(p)) \ll N^{1 - \frac{\deg f}{4KL^b} - 4r\varepsilon},$$
  where $K = 2^k$ and $L = 2^{[\theta_k, 1]}$.

Now we turn our attention to the case of $\|m \xi\| \leq N^{1 - \rho - \deg f}$. Again following the integer case above together with Proposition 4.5 we get the existence of a prime $2 \leq p \leq M^{\frac{\rho}{2}}$ such that $[f(p)]$ is a multiple of $m$. Repeating the steps from above we get a contradiction provided
  $$\eta \leq \frac{\deg f + \rho - 1}{1 + \frac{\deg f}{1 - \rho}}.$$  
  Together with the first case we get a contradiction provided that
  $$\eta < \min \left( \rho^{4^{1-k}}, \frac{1}{64KL^b}, \frac{\deg f + \rho - 1}{1 + \frac{\deg f}{1 - \rho}} \right)$$
  proving the theorem. \hfill \Box

7. The multi-dimensional case

In this section we turn our attention to the case of simultaneous approximation. The equivalent to Theorem 2.8 is the following more general result.

**Theorem 7.1.** Let $f_1, \ldots, f_k$ be $\mathbb{Q}$-linear independent pseudo-polynomials, $\ell \in \mathbb{N}$ and $N \in \mathbb{N}$ sufficiently large. Then there exists $\theta > 0$ such that for any lattice $\Lambda$ with determinant $|\det(\Lambda)| \leq N^{\theta}$, and any $\ell \times k$ matrix $A$ there exists $n \in \mathbb{N}$ with $1 \leq n \leq N$ such that
  $$A \begin{pmatrix} f_1(n) \\ \vdots \\ f_k(n) \end{pmatrix} \in \Lambda + B_\ell,$$
  where $B_\ell$ is the Euclidean unit ball in $\mathbb{R}^\ell$.

The role of the lower bound similar to Lemma 6.1 is played by the following multidimensional counterpart.

**Lemma 7.2 (\cite[Theorem 14A]{R}).** Suppose $\Lambda$ is a lattice of full rank in $\mathbb{R}^\ell$ such that $\Lambda \cap B_\ell = \{0\}$, where $B_\ell$ denotes the Euclidean unit ball in $\mathbb{R}^\ell$. Suppose that $x_1, \ldots, x_N \in \mathbb{R}^\ell$ are not in $\Lambda + B_\ell$. Let $\varepsilon > 0$ and
  $$S_\varepsilon = \sum_{n=1}^{N} e(x_n \cdot p).$$
  Then, provided $N$ is sufficiently large in terms of $\varepsilon$, there is a point $p$ in a basis of the dual lattice $\Pi$ of $\Lambda$ such that $|p| \leq N^\varepsilon$ and an integer $1 \leq t \leq \frac{N^{\varepsilon}}{|p|}$ such that
  $$|S_\varepsilon p| \geq N^{1 - \varepsilon} |\det(\Lambda)|^{-1}.$$

**Proof of Theorem 7.1.** As in the one-dimensional case we use Lemma 7.2 to transform the problem into an estimation of an exponential sum. In particular, suppose that $x_1, \ldots, x_N$ are not in $\Lambda + B_\ell$. Then by Lemma 7.2 there exists $p$ with $|p| \leq N^\varepsilon$ and an integer $1 \leq m \leq \frac{N}{|p|}$ such that
  $$(7.1) \sum_{n=1}^{N} e(mp_1 f_1(n) + \cdots + mp_k f_k(n)) \geq N^{1 - \varepsilon} |\det(\Lambda)|^{-1}.$$ 
  Similar to above we derive an estimate for the exponential sum contradicting this lower bound.
We start by getting rid of the floor function. Let \( q \geq 2 \) be an integer chosen later. Again we denote by \( I_d \) the interval of all reals in \([0, 1]\) whose \( q\)-adic expansion starts with \( d \), i.e.

\[
I_d := \left[ \frac{d}{q}, \frac{d+1}{q} \right).
\]

Then if \( \{f_i(n)\} \in I_d \) there exists \( 0 \leq \vartheta < 1 \) such that \( \{f_i(n)\} = \frac{d}{q} + \frac{\vartheta}{q} \). Thus

\[
e(mp_i \xi_i [f_i(n)]) = e \left( mp_i \xi_i f_i(n) - mp_i \xi_i \frac{d}{q} \right) \left( 1 + O \left( \frac{1}{q} \right) \right).
\]

Hence

\[
\left| \sum_{n \leq t} \left( m \sum_{i=1}^{k} p_i \xi_i [f_i(n)] \right) \right| = \sum_{d_1=0}^{q-1} \cdots \sum_{d_k=0}^{q-1} \sum_{n \leq t} e \left( m \sum_{i=1}^{k} p_i \xi_i f_i(n) \right) \prod_{j=1}^{k} \chi_{I_{d_j}}(f_j(n)) \right|.
\]

As above we want to approximate the occurring indicator function by suitable functions. We follow Grabner [20] (cf. Section 1.2.2 of Drmota and Tichy [18]) who considered multidimensional variants of Vaaler polynomials. We fix a vector of digits \( \mathbf{d} = (d_1, \ldots, d_k) \). Thus

\[
(7.2) \quad \sum_{n \leq N} e (m_1 \xi_1 f_1(n) + \cdots + m_k \xi_k f_k(n)) \prod_{j=1}^{k} \chi_{I_{d_j}}(f_j(n)) \leq \sum_{n \leq N} e (m_1 \xi_1 f_1(n) + \cdots + m_k \xi_k f_k(n)) \prod_{j=1}^{k} \chi_{I_{d_j}}(f_j(n)) \prod_{j=1}^{k} \chi_{I_{d_j}}(f_j(n)) + R(H),
\]

where

\[
(7.3) \quad R(H) = \sum_{n \leq N} \left| \prod_{j=1}^{k} \chi_{I_{d_j}}(f_j(n)) - \prod_{j=1}^{k} \chi_{I_{d_j}}(f_j(n)) \right|.
\]

We start estimating \( R(H) \). Using the inequality

\[
\left| \prod_{j=1}^{k} b_j - \prod_{j=1}^{k} a_j \right| \leq \sum_{\emptyset \neq J \subseteq \{1, \ldots, k\}} \prod_{j \in J} |a_j| \prod_{j \in J} |b_j - a_j|
\]

we get

\[
\left| \prod_{j=1}^{k} \chi_{I_j}(f_j(n)) - \prod_{j=1}^{k} \chi_{I_{d_j}}(f_j(n)) \right| \leq \sum_{\emptyset \neq J \subseteq \{1, \ldots, k\}} \prod_{j \in J} \left| \chi_{I_j} - \chi_{I_{d_j}} \right| = \prod_{j=1}^{k} \left( 1 + \left| \chi_{I_j} - \chi_{I_{d_j}} \right| \right) - 1.
\]
Plugging this into (7.3) together with Lemma 6.2 yields
\[ R(H) = \sum_{n \leq N} \left| \prod_{j=1}^{k} \chi_{\ell_{ij}} \left( f_j(n) \right) - \prod_{j=1}^{k} \chi_{\ell_{ij}, H} \left( f_j(n) \right) \right| \]
\[ \leq \sum_{n \leq N} \left( \prod_{j=1}^{k} \left( 1 + \left| \chi_{\ell_{ij}} - \chi_{\ell_{ij}, H} \right| \right) - 1 \right) \]
\[ \leq \sum_{n \leq N} \left( \prod_{j=1}^{k} \left( 1 + \frac{1}{H + 1} + \frac{1}{H + 1} \sum_{1 \leq |h_j| \leq H} C_{h_j} \left( 1 - \frac{|h_j|}{H + 1} \right) e \left( h_j f_j(n) \right) \right) - 1 \right) \]
\[ = N \left( \left( 1 + \frac{1}{H + 1} \right)^k - 1 \right) + \sum_{0 \leq |h|_{\infty} \leq H} \left( \frac{1}{H + 1} \right)^{k - \delta(h)} \left( 1 + \frac{1}{H + 1} \right)^{\delta(h)} \sum_{n \leq N} e(h \cdot f(n)) \],
where \( \delta(h) = \sum_{j=1}^{k} \delta_{h_j} \) counts the number of coordinates of \( h = (h_1, \ldots, h_k) \) which are zero; \( \|h\|_{\infty} = \max\{|h_1|, \ldots, |h_k|\} \).

Now we consider the first part of (7.2). Again using Lemma 6.2 we have
\[ \left| \sum_{n \leq N} e(m\xi_1 f_1(n) + \cdots + m\xi_k f_k(n)) \prod_{j=1}^{s} \chi_{\ell_{ij}, H} \left( f_j(n) \right) \right| \]
\[ = \left| \sum_{n \leq N} e(m\xi_1 f_1(n) + \cdots + m\xi_k f_k(n)) \prod_{j=1}^{s} \left( 1 + \frac{1}{q} + \frac{1}{\pi} \sum_{1 \leq |h_j| \leq H} a_{h_j} e(h_j f_j(n)) \right) \right| \]
\[ \leq \frac{1}{q^s} \sum_{n \leq N} e(m\xi \cdot f(n)) + \sum_{0 \leq |h|_{\infty} \leq H} \frac{1}{\pi h} \sum_{n \leq N} e((m\xi + h) \cdot f(n)) , \]
where \( \xi = (\xi_1, \ldots, \xi_k) \) and \( f(n) = (f_1(n), \ldots, f_k(n)) \).

The occurring exponential sums are of the form
\[ \sum_{n \leq N} e(h \cdot f(n)), \sum_{n \leq N} e((m\xi + h) \cdot f(n)) \text{ or } \sum_{n \leq N} e((m\xi + h) \cdot f(n)) . \]

We consider these three sums simultaneously and denote them by
\[ \sum_{n \leq N} e \left( \sum_{i=1}^{k} \beta_i f_i(n) \right) \).

By reordering if necessary we may suppose that \( \deg f_1 < \deg f_2 < \cdots < \deg f_k \). We split each \( f_i(n) = g_i(n) + P_i(n) \) where \( P_i \) is a polynomial of degree \( k_i \) and \( g_i(n) = \sum_{j=1}^{k_i} a_{j \ell_{ij}} \) with \( 1 < \theta_{i,1} < \cdots < \theta_{i,r} \) and \( \ell_{i,j} \not\in \mathbb{Z} \) for \( j = 1, \ldots, r \).

Let \( \gamma_0 = 6p \) and suppose that \( \|\beta_i\| > N^{\gamma_0/2 - \deg f_i} \). Then an application of Proposition 4.2 and Proposition 5.5, respectively, yields
\[ \sum_{n \leq N} e(\beta_1 f_1(n)) \ll N^{1-\gamma_1} , \]
where
\[ \gamma_1 = \min \left( \rho 2^{-k_i} \cdot (8KL - 4K)^{-1} \right) . \]

Now we recursively define \( \gamma_j \) for \( 2 \leq j \leq k \) as follows. Suppose that \( \|\beta_i\| > N^{\gamma_j - 1/2 - \deg f_i} \), then by Proposition 4.2 and Proposition 5.5, respectively, there exists \( \gamma_j > 0 \) such that
\[ \sum_{n \leq N} e \left( \sum_{i=1}^{j} \beta_i f_i(n) \right) \ll N^{1-\gamma_j} . \]
For the moment we suppose that there exists $1 \leq s \leq k$ such that $|m\xi_j| \leq N^{\gamma_j-1/2-\deg f_i}$ for $s < j \leq k$ and $|m\xi_i| > N^{\gamma_s-\deg f_i}$. Using partial summation we split those $j > s$ apart. To this end we set

$$\phi(n) = e\left(m \sum_{i=1}^{s} p_i \xi_i [f_i(n)]\right) \quad \text{and} \quad \psi(n) = e\left(m \sum_{i=s+1}^{k} p_i \xi_i f_i(n)\right).$$

Since $\gamma_{j-1} > \gamma_j$ for $s < j \leq k$ we get

$$|\psi(n+1) - \psi(n)| \leq \sum_{i=s+1}^{k} N^{\gamma_{i-1/2-\deg f_i}} N^{\deg f_i-1} \ll N^{\gamma_s/2-1}.$$  

Thus

$$\sum_{n \leq N} \phi(n)\psi(n) \leq \psi(N) \sum_{n \leq N} \phi(n) + N^{\gamma_s/2-1} \sum_{t \leq 1} \sum_{n \leq t} \phi(n).$$

By the definition of $\gamma_s$ we have

$$\sum_{n \leq N} \phi(n) = \sum_{n \leq N} e\left(\sum_{i=1}^{s} \beta_i f_i(n)\right) \ll N^{1-\gamma_s}.$$

Putting everything together we get that

$$\sum_{n \leq N} e\left(\sum_{i=1}^{k} mp_i \xi_i [f_i(n)]\right) \ll N^{1-\gamma_s/2}.$$

Thus contradicting the lower bound for $\eta > \gamma_s/2$.

Now we return to the case that there is no $1 \leq s \leq k$ such that $|m\xi_s| \geq N^{\gamma_s-\deg f_s}$. Similar to the one-dimensional case we may not apply the indicator function. Therefore we show that there is a joint multiple of $m$ directly contradicting that there is no element near a point of $\Lambda$. To this end let

$$f_1(n) = a_1 \xi_1 m^f + \ldots + a_{1,1} m + a_{1,0} + a_{1,-1} m^{-1} + \ldots$$

$$\vdots$$

$$f_s(n) = a_s \xi_s m^f + \ldots + a_{s,1} m + a_{s,0} + a_{s,-1} m^{-1} + \ldots$$

Then $m | [f_i(n)]$ for $i = 1, \ldots, k$ if and only if $a_{i,0} = 0$ for $i = 1, \ldots, k$ if and only if $m^{-1} f_i(n) \in [0, \frac{1}{m}]$ for $i = 1, \ldots, k$.

Let $N(f, X)$ denote the number of $1 \leq n \leq X$ such that $[f_i(n)]$ is a multiple of $m$ for $1 \leq i \leq k$. As above $X$ will be a power of $M$ and thus of $N$. If $\chi$ denotes the indicator function of the interval $[0, \frac{1}{m}]$, then an application of Vaaler polynomials yields

$$\left|N(f, X) - \frac{X}{m^k}\right| \leq \sum_{1 \leq n \leq X} \left|\prod_{i=1}^{k} \chi\left(\frac{f_i(n)}{m}\right) - \prod_{i=1}^{k} \chi^*\left(\frac{f_i(n)}{m}\right)\right| + \sum_{1 \leq n \leq X} \left|\prod_{i=1}^{k} \chi^*\left(\frac{f_i(n)}{m}\right) - \frac{1}{m^k}\right|$$

$$\leq N \left(1 + \frac{1}{H+1}\right)^k - 1 \quad + \quad \sum_{0 < ||h|| \leq H} \frac{1}{r(\pi h)} \sum_{n \leq X} e\left(\frac{1}{m} \sum_{i} h_i f_i(n)\right)$$

$$\quad + \quad \sum_{0 < ||h|| \leq H} \left(1 + \frac{1}{H+1}\right)^{k-\beta(h)} \left(1 + \frac{1}{H+1}\right)^{\beta(h)} \sum_{n \leq X} e\left(\frac{1}{m} \sum_{i} h_i f_i(n)\right).$$

Setting $X = M^{\frac{1}{1+\delta}}$ and $H = X^{\frac{1}{1+\delta}}$ we get by the same reasoning as in the one-dimensional case that for sufficiently large $X$ we have $N(f, X) \geq 1$. Thus similar to the one-dimensional case there
is a $1 \leq n \leq N$ such that $\|\xi_i \lfloor f_i(n) \rfloor\|$ is very close to an integer. Multiplying by the primitive vector $p$ we get that this is very close to a point in the lattice $\Lambda$ violating the assumption. $\Box$

Proof. The proof of Theorem 2.9 follows along the same lines as the proof of Theorem 7.1. The only change is in the exponentials sums which run over the primes and the corresponding estimates. $\Box$

Final remarks

In the present paper we have considered one of the many examples of van der Corput sets provided in Section 2. Each of these examples lead to different exponential sums whose treatment is interesting on their own. In the vain of Bergelson et al. [7], where mixtures of polynomials and pseudo-polynomials were considered, similar results should hold for Heilbronn sets. For example, let $f$ be a polynomial with real coefficients. Then we suppose that one can prove the existence of an $\eta > 0$ such that

$$\min_{1 \leq n \leq N} \|\xi \lfloor f(n) \rfloor\| \ll N^{-\eta}$$

for any given $\xi \in \mathbb{R}$ and any given $N \in \mathbb{N}$. Similar statements should be true for the sequence $\lfloor f(p) \rfloor$ and multi-dimensional variates thereof.

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References

[1] R. C. Baker, Diophantine inequalities, London Mathematical Society Monographs. New Series, vol. 1, The Clarendon Press, Oxford University Press, New York, 1986. Oxford Science Publications. MR865981
[2] R. C. Baker and G. Kolesnik, On the distribution of $p^\alpha$ modulo one, J. Reine Angew. Math. 356 (1985), 174–193. MR779381 (86m:11053)
[3] R. Baker, Small fractional parts of polynomials, Funct. Approx. Comment. Math. 55 (2016), no. 1, 131–137. MR3549017
[4] FRACTIONAL PARTS OF POLYNOMIALS OVER THE PRIMES, Mathematika 63 (2017), no. 3, 715–733. MR3731301
[5] V. Bergelson, G. Kolesnik, and Y. Son, Uniform distribution of subpolynomial functions along primes and applications, ArXiv e-prints (March 2015), available at [1503.04963]
[6] V. Bergelson, A. Leibman, and E. Lesigne, Intersective polynomials and the polynomial Szemerédi theorem, Adv. Math. 219 (2008), no. 1, 369–388. MR2435427
[7] V. Bergelson, G. Kolesnik, M. Madritsch, Y. Son, and R. Tichy, Uniform distribution of prime powers and sets of recurrence and van der Corput sets in $\mathbb{Z}^d$, Israel J. Math. 201 (2014), no. 2, 729–760. MR3265301
[8] V. Bergelson and E. Lesigne, Van der Corput sets in $\mathbb{Z}^d$, Colloq. Math. 110 (2008), no. 1, 1–49. MR2353898 (2008j:11089)
[9] V. Bergelson and J. Moreira, Van der Corput’s difference theorem: some modern developments, Indag. Math. (N.S.) 27 (2016), no. 2, 437–479. MR3479166
[10] M. Boshernitzan, G. Kolesnik, A. Quas, and M. Wierdl, Ergodic averaging sequences, J. Anal. Math. 95 (2005), 63–103. MR2145587 (2006b:37011)
[11] J. Bourgain, Ruzsa’s problem on sets of recurrence, Israel J. Math. 59 (1987), no. 2, 150–166. MR920079 (89d:11012)
[46] I. Z. Ruzsa, *Connections between the uniform distribution of a sequence and its differences*, Topics in classical number theory, Vol. I, II (Budapest, 1981), 1984, pp. 1419–1443. MR781190 (86e:11062)

[47] A. Sárközy, *On difference sets of sequences of integers, I*, Acta Math. Acad. Sci. Hungar. 31 (1978), no. 1–2, 125–149. MR0466059 (57 #5942)

[48] W. M. Schmidt, *Small fractional parts of polynomials*, American Mathematical Society, Providence, R.I., 1977. Regional Conference Series in Mathematics, No. 32. MR0457360

[49] S. Slijepčević, *On van der Corput property of squares*, Glas. Mat. Ser. III 45(65) (2010), no. 2, 357–372. MR2753306 (2012c:11017)

[50] __________, *On van der Corput property of shifted primes*, Funct. Approx. Comment. Math. 48 (2013), no. part 1, 37–50. MR3086959

[51] L. Spiegelhofer, *Piatetski-Shapiro sequences via Beatty sequences*, Acta Arith. 166 (2014), no. 3, 201–229. MR3283620

[52] R. Tichy and M. Zeiner, *Baire results of multisequences*, Unif. Distrib. Theory 5 (2010), no. 1, 13–44. MR2804660

[53] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 2, 183–216. MR776471 (86g:42005)

[54] I. M. Vinogradov, *Analytischer Beweis des Satzes über die Verteilung der Bruchteile eines ganzen Polynoms*, Bull. Acad. Sci. USSR 21 (1927), no. 6, 567–578.

[55] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins.*, Math. Ann. 77 (1916), 313–352 (German).

[56] M. Wierdl, *Almost everywhere convergence and recurrence along subsequences in ergodic theory*, ProQuest LLC, Ann Arbor, MI, 1989. Thesis (Ph.D.)–The Ohio State University. MR2638457

[57] T. D. Wooley, *New estimates for smooth Weyl sums*, J. London Math. Soc. (2) 51 (1995), no. 1, 1–13. MR1310717

[58] __________, *The cubic case of the main conjecture in Vinogradov's mean value theorem*, Adv. Math. 294 (2016), 532–561. MR3479572

[59] A. Zaharescu, *Small values of $n^a$ (mod 1)*, Invent. Math. 121 (1995), no. 2, 379–388. MR1346212 (96d:11079)

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