Saturation phenomena for some classes of nonlinear nonlocal eigenvalue problems

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Abstract

Let us consider the following minimum problem

\[ \lambda_\alpha(p, r) = \min_{u \in W^{1, p}_0(-1, 1) \setminus \{0\}} \frac{\int_{-1}^1 |u'|^p dx + \alpha \left| \int_{-1}^1 |u|^{r-1} u dx \right|^\frac{p}{r}}{\int_{-1}^1 |u|^p dx}, \]

where \( \alpha \in \mathbb{R} \), \( p \geq 2 \) and \( \frac{p}{2} \leq r \leq p \). We show that there exists a critical value \( \alpha_c = \alpha_c(p, r) \) such that the minimizers have constant sign up to \( \alpha = \alpha_c \) and then they are odd when \( \alpha > \alpha_c \).

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1 Introduction

In this paper we consider the problem:

\[ \lambda_\alpha(p, r) = \inf \left\{ Q_\alpha[u], \ u \in W^{1, p}_0(-1, 1), u \neq 0 \right\}, \tag{1} \]

where

\[ Q_\alpha[u] := \frac{\int_{-1}^1 |u'|^p dx + \alpha \left| \int_{-1}^1 |u|^{r-1} u dx \right|^\frac{p}{r}}{\int_{-1}^1 |u|^p dx}, \tag{2} \]

with \( \alpha \in \mathbb{R} \) and \( 1 \leq \frac{p}{2} \leq r \leq p \).

The problem we deal with has been treated by many authors both in the one-dimensional and in the n-dimensional case. For example, reaction-diffusion equations...
describing chemical processes (see [F], [S]) or Brownian motion with random jumps (see [P]).

The minimization problem (1) leads, in general, to a nonlinear eigenvalue problem with a nonlocal term. Supposing without loss of generality that \( y \) is a minimizer with
\[
\int_{-1}^{1} |y|^{r-1} y \, dx > 0,
\]
we have
\[
\begin{cases}
-\left((|y'|^{p-2}y')' + \alpha \left( \int_{-1}^{1} |y|^{r-1} y \, dx \right)^{\frac{p}{r}-1} \right) |y|^{r-1} = \lambda_\alpha(p, r) |y|^{p-2} y \quad \text{in } ]-1, 1[ \\
y(-1) = y(1) = 0
\end{cases}
\]
(see Section 2 for its precise statement).

The value \( \lambda_\alpha(p, r) \) is the optimal constant in the Sobolev-Poincaré-Wirtinger inequality
\[
\lambda_\alpha(p, r) \int_{-1}^{1} |u|^p \, dx \leq \int_{-1}^{1} |u'|^p \, dx + \alpha \left( \int_{-1}^{1} |u|^{r-1} u \, dx \right)^{\frac{p}{r}},
\]
which holds for any \( u \in W^{1,p}_0(-1, 1) \). Our aim is to study symmetry properties of the minimizers of (1) and, as a consequence, to give some informations on \( \lambda_\alpha(p, r) \). In the local case (\( \alpha = 0 \)), this inequality reduces to the classical one-dimensional Poincaré inequality; in particular,
\[
\lambda_0(p, r) = \left( \frac{\pi_p}{2} \right)^p
\]
for any \( p \) and \( r \), where
\[
\pi_p = 2 \int_{0}^{+\infty} \frac{1}{1 + \frac{1}{p-r} s^p} \, ds = 2 \pi \left( \frac{p - 1}{p} \right)^{\frac{1}{p}} \frac{1}{p \sin \frac{\pi}{p}}.
\]

Our problem is related to the study of the minimization of (2) under the assumption
\[
\int_{-1}^{1} |u|^{r-1} u = 0 \quad \text{(that is the limit case “} \alpha = \infty \text{”)}.
\]
This was studied by several authors (see for example [DGS, E, BKN, BK, N1, CD, GN]), considering various cases of the exponents \( p, q, r \). A very general case was studied recently in [GGR], where the authors studied the symmetry of the minimizers of
\[
\tilde{\Lambda}(p, q, r) := \min \left\{ \frac{\int_{-1}^{1} |u'|^p \, dx}{\left( \int_{-1}^{1} |u|^q \, dx \right)^{\frac{p}{q}}} \right\}, \quad u \in W^{1,p}_0(-1, 1), \quad \int_{-1}^{1} |u|^{r-1} u \, dx = 0, \quad u \neq 0,
\]
and showed that, when \( p, q > 1, \ r > 0 \) with \( q \leq (2r + 1)p \), these minimizers are odd functions. In particular, if \( 1 < p = q < \infty \), they showed that
\[
\Lambda(p, r) := \tilde{\Lambda}(p, p, r) = \pi_p^p
\]
for any \( r \). In [DP] we studied the problem (1) in the case \( p = 2 \). In this paper we consider the more general case \( p \geq 2 \). Recently, this problem was studied also in the multidimensional case, when \( \alpha \in \mathbb{R} \) and \( p = q = 2 \) in [BFNT] (\( r = 1 \)) and in [D] (\( r = 2 \)). For related problems we refer the reader to [FH, N2, BDNT, BCGM, CHP1, KN, Pi, BCGM].
In the present paper, we show that the nonlocal term affects the minimizer of problem (1) in the sense that it has constant sign up to a critical value of \( \alpha \) and, for \( \alpha \) larger than the critical value, it has to change sign, and a saturation effect occurs. More precisely, the first main result we obtain is the following.

**Theorem 1.1.** Let \( p \geq 2, \frac{p}{2} \leq r \leq p \). Then there exists a positive number \( \alpha_C = \alpha_C(p, r) \) such that:

1. If \( \alpha < \alpha_C \), then
   \[
   \lambda_\alpha(p, r) < \pi_p^p,
   \]
   and any minimizer \( y \) of \( \lambda_\alpha(p, r) \) has constant sign in \( (-1, 1) \).

2. If \( \alpha \geq \alpha_C \), then
   \[
   \lambda_\alpha(p, r) = \pi_p^p.
   \]

Moreover, if \( \alpha > \alpha_C \), the function \( y(x) = \sin_p \pi_p x, x \in [-1, 1] \), is the unique minimizer, up to a multiplicative constant, of \( \lambda_\alpha(p, r) \). Hence it is odd, \( \int_{-1}^1 |y(x)|^{r-1} y(x) \, dx = 0 \), and \( \pi = 0 \) is the only point in \( (-1, 1) \) such that \( y(\pi) = 0 \).

Moreover we analyze the behaviour of the minimizers associated to the critical values.

**Theorem 1.2.** Let \( p \geq 2, \frac{p}{2} \leq r \leq p \), if \( \alpha = \alpha_C(p, r) \), then \( \lambda_{\alpha_C}(p, r) \) admits both a positive minimizer and the minimizer \( y(x) = \sin_p \pi_p x \), up to a multiplicative constant. Moreover, if \( r > \frac{p}{2} \) any minimizer has constant sign or it is odd. Furthermore, if \( r = p \), then \( \alpha_C(p, p) = \frac{2^p-1}{2^{p-1}} \pi_p^p \).

**Remark 1.3.** When the interval is \( [a, b] \) instead of \( [-1, 1] \), we have

\[
\lambda_\alpha(p, r; [a, b]) = \left( \frac{2}{b - a} \right)^p \cdot \lambda_\alpha(p, r),
\]

with \( \bar{\alpha} = \left( \frac{b - a}{2} \right)^{\frac{p}{2} + p - 1} \alpha \). The outline of the paper follows. In Section 2 we show some properties of \( \lambda_\alpha(p, r) \), while in Section 3 we study the behavior of the changing-sign minimizers. Finally, in Section 4 we give the proof of the main results.

## 2 Preliminaries

### 2.1 The \( p \)-circular functions

Let \( p > 1 \) and let us consider the function \( F_p : [0, (p - 1)^{\frac{1}{p}}] \) defined as

\[
F_p(x) = \int_0^x \frac{dt}{[1 - t^p / (p - 1)]^{\frac{1}{p}}},
\]

Denote by \( z(s) \) the inverse function of \( F \) which is defined on the interval \( [0, \frac{\pi_p}{2}] \), where

\[
\pi_p = 2 \int_0^{(p - 1)^{\frac{1}{p}}} \frac{dt}{[1-t^p/(p-1)]^{\frac{1}{p}}} = 2(p - 1)^{\frac{1}{p}} \int_0^1 \frac{dx}{(1 - x^p)^{\frac{1}{p}}}.\]
We define $\sin_p$, the $p$-sine function, as the following periodic extension of $z(t)$:

$$
\sin_p(t) = \begin{cases} 
z(t) & \text{if } t \in \left[0, \frac{\pi p}{2}\right], \\
z(\pi_p - t) & \text{if } t \in \left[\frac{\pi p}{2}, \pi_p\right], \\
-\sin_p(-t) & \text{if } t \in [-\pi_p, 0]. 
\end{cases}
$$

It is extended periodically to all $\mathbb{R}$, with period $2\pi_p$. The $p$-cosine function is defined as

$$
\cos_p(t) = \sin_p \left(t + \frac{\pi p}{2}\right)
$$

and it is again an even function with period $2\pi_p$. Let us explicitly observe that these generalized sine and cosine functions coincide with the usual ones when $p = 2$ and that they have continuous second derivative if $1 < p < 2$ and continuous first derivative if $2 < p < \infty$ (see [Ô]). For further details we refer for example to [L]. The study of the $p$-circular functions is connected with the $1$-dimensional Dirichlet $p$-Laplacian eigenvalue problem. Indeed, the minimum $\lambda_p$ of the Rayleigh quotient

$$
Q_p(u) = \frac{\int_{-1}^{1} |u'(x)|^p \, dx}{\int_{-1}^{1} |u(x)|^p \, dx} \quad (1 < p < \infty),
$$

among all real valued functions $u \in W_0^{1,p}$, is the first eigenvalue $\lambda_p$ of the problem

$$
\begin{cases}
-(|y'|^{p-2}y')' + \alpha \gamma |y|^{r-1} = \lambda_p |y|^{p-2}y & \text{in } ]-1,1[ \\
y(-1) = y(1) = 0.
\end{cases}
$$

This first eigenvalue is just $(\frac{\pi}{2})^p$ and the first eigenfunction is represented by $\sin_p(\pi_p x)$, up to a multiplicative constant.

### 2.2 Some properties of the eigenvalue problem

Now we list some properties of the minimizers of problem (i). We argue similarly as in [DP], where some of these properties have been proved in the case when $p = 2$.

**Proposition 2.1.** Let $\alpha \in \mathbb{R}$, $p \geq 2$ and $\frac{p}{2} \leq r \leq p$, then the following properties hold.

(a) Problem (i) has a solution.

(b) Any minimizer $y$ of (i) satisfies the following boundary value problem

$$
\begin{cases}
-(|y'|^{p-2}y')' + \alpha \gamma |y|^{r-1} = \lambda_{\alpha}(p, r) |y|^{p-2}y & \text{in } ]-1,1[ \\
y(-1) = y(1) = 0,
\end{cases}
$$

where

$$
\gamma = \begin{cases} 
0 & \text{if both } r = p \text{ and } \int_{-1}^{1} |y|^{p-1}y \, dx = 0, \\
\left(\int_{-1}^{1} |y|^{r-1}y \, dx\right)^\frac{p-2}{p} \left(\int_{-1}^{1} |y|^{r-1}y \, dx\right) & \text{otherwise}.
\end{cases}
$$

Moreover, $y, y' |y|^{p-2} \in C^1[-1,1]$. 

(c) The function \( \lambda_\alpha(p, r) \) is Lipschitz continuous and non-decreasing with respect to \( \alpha \in \mathbb{R} \).

(d) If \( \alpha \leq 0 \), the minimizers of (1) do not change sign in \( ]-1,1[ \), and

\[
\lim_{\alpha \to -\infty} \lambda_\alpha(p, r) = -\infty.
\]

(e) We have that

\[
\lim_{\alpha \to +\infty} \lambda_\alpha(p, r) = \Lambda(p, r) = \pi_p^0.
\]

Proof. By the method of Calculus of Variations it is easily proved the existence of a minimizer. Furthermore, any minimizer satisfies (4). This follows in a standard way if \( r < p \), since the functional \( Q_\alpha \) in (2) is differentiable in \( u \). When \( r = p \), this functional is not differentiable if \( \int_{-1}^{1} |y|^{r-1}y \, dx = 0 \). Actually, in this case, the problem (1) coincides with the minimum of the functional \( Q_\alpha \) among the functions satisfying \( \int_{-1}^{1} |y|^{r-1}y \, dx = 0 \) and, by [DGS, Lem. 2.4], it follows that \( \gamma = 0 \). From (4) immediately follows that \( y, y'|y|^{p-2} \in C^1[-1,1] \) and hence (a)-(b) have been proved.

In order to get property (c), we stress that for all \( \varepsilon > 0 \), by Hölder inequality, it holds

\[
Q_{\alpha+\varepsilon}[u] \leq Q_\alpha[u] + \varepsilon \left( \frac{\left( \int_{-1}^{1} |u|^r \, dx \right)^{p/r}}{\int_{-1}^{1} |u|^p \, dx} \right) \leq Q_\alpha[u] + 2^{p/r} \varepsilon, \quad \forall \varepsilon > 0.
\]

Therefore the following chain of inequalities

\[
Q_\alpha[u] \leq Q_{\alpha+\varepsilon}[u] \leq Q_\alpha[u] + 2^{p/r} \varepsilon, \quad \forall \varepsilon > 0,
\]

implies, taking the minimum as \( u \in W^{1,p}_0(-1,1) \), that

\[
\lambda_\alpha(p, r) \leq \lambda_{\alpha+\varepsilon}(p, r) \leq \lambda_\alpha(p, r) + 2^{p/r} \varepsilon, \quad \forall \varepsilon > 0,
\]

that proves (c). If \( \alpha < 0 \), then

\[
Q_\alpha[u] \geq Q_\alpha[|u|],
\]

with equality if and only if \( u \geq 0 \) or \( u \leq 0 \). Hence any minimizer has constant sign in \( ]-1,1[ \). Finally, it is clear from the definition that \( \lim_{\alpha \to -\infty} \lambda_\alpha(p, r) = -\infty \). Indeed, by fixing a positive test function \( \varphi \) we get

\[
\lambda_\alpha(p, r) \leq Q_\alpha[\varphi].
\]

Being \( \varphi > 0 \) in \( ]-1,1[ \), then \( Q_\alpha[\varphi] \to -\infty \) as \( \alpha \to -\infty \), and the proof of (d) is completed. The problem (3) was studied, for example, in [CD, GN] and the minimum \( \Lambda(p, r) \) is equal to \( \pi_p^0 \). In particular, if there exists a minimizer \( y \) of \( \lambda_\alpha(p, r) \) such that \( \int_{-1}^{1} |y|^{r-1}y \, dx = 0 \), then it holds that \( \gamma = 0 \) in (4). Indeed, in such a case \( y \) is a minimizer also of the problem (3), whose Euler-Lagrange equation is

\[
\begin{cases}
-(y'|y|^{p-2}y') = \lambda_\alpha(p, r) |y|^{p-2}y & \text{in } ]-1,1[, \\
y(-1) = y(1) = 0.
\end{cases}
\]
Since \( \lambda(\alpha, p, r) \) is decreasing with respect to \( \alpha \), we have that \( \lambda(\alpha, p, r) \leq \Lambda(p, r) = \pi_p^n. \)

Now, let \( \alpha_k \geq 0, k_n \in \mathbb{N} \), be a positively divergent sequence. For any \( k \), we consider a minimizer \( y_k \in W_0^{1,p} \) of (1) such that \( \|y_k\|_{L^p} = 1 \). We have that

\[
\lambda_{\alpha_k}(p, r) = \int_{-1}^{1} |y_k'|^p \, dx + \alpha_k \left( \int_{-1}^{1} |y_k|^{r-1} y_k \, dx \right) \leq \Lambda(p, r).
\]

Then \( y_k \) converges (up to a subsequence) to a function \( y \in W_0^{1,p}(-1, 1) \), strongly in \( L^p \) and weakly in \( W_0^{1,p} \). Moreover \( \|y\|_{L^p} = 1 \) and

\[
\left( \int_{-1}^{1} |y_k|^{r-1} y_k \, dx \right)^{\frac{p}{p-1}} \leq \frac{\Lambda(p, r)}{\alpha_k} \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty
\]

which gives that \( \int_{-1}^{1} |y|^p \, dx = 0 \). On the other hand the weak convergence in \( W_0^{1,p} \) implies that

\[
\int_{-1}^{1} |y|^p \, dx \leq \liminf_{k \to +\infty} \int_{-1}^{1} |y_k|^p \, dx.
\]

Therefore, by the definitions of \( \Lambda(p, r) \) and \( \lambda(\alpha, p, r) \), and by (5) we have

\[
\Lambda(p, r) \leq \int_{-1}^{1} |y'|^p \, dx \leq \liminf_{k \to +\infty} \left[ \int_{-1}^{1} |y_k'|^p \, dx + \alpha_k \left( \int_{-1}^{1} |y_k|^{r-1} y_k \, dx \right)^{\frac{p}{p-1}} \right]
\]

\[
\leq \lim_{k \to +\infty} \lambda(\alpha_k, p, r) \leq \Lambda(p, r).
\]

and the property (e) follows.

\[\square\]

**Remark 2.2.** Let us observe that when \( \lambda(\alpha, p, r) = 0 \), we have (as in [DP]):

\[-\alpha = \min_{w \in W_0^{1,p}(-1,1)} \frac{\int_{-1}^{1} |w'|^p \, dx}{\left( \int_{-1}^{1} |w|^r \, dx \right)^{p/r}}.
\]

## 3 The Symmetry of the Solutions

The main result of this Section, contained in Proposition 3.5, consists in the fact that each minimizer of problem (1) is represented by a generalized sine function, that is symmetric and whose \( (r-1) \)-power has zero average. This result will allow us to prove, in the following Section, the existence of a critical value of the parameter for the problem (1) such that the minimizers are symmetric above this value.

A key role in the proof of the main results is played by the minimizers that change sign in \([-1, 1] \). In the following Proposition we find an expression of the first non-local eigenvalue \( \lambda(\alpha, p, r) \) with an auxiliary function \( H \), whose study leads us to show important properties of problem (1).

**Proposition 3.1.** Let \( p \geq 2, \frac{p}{2} \leq r \leq p \) and suppose that there exists \( \alpha > 0 \) such that \( \lambda(\alpha, p, r) \) admits a minimizer \( y \) that changes sign in \([-1, 1] \). Then the following properties hold.
(a) The minimizer $y$ has in $]-1,1[$ exactly one maximum point, $\eta_M$, and exactly one minimum point, $\eta_m$, and, up to a multiplicative constant, is such that $y(\eta_M) = 1$ and $y(\eta_m) = -\bar{m} \in ]0,1[$.

(b) If $y_+ \geq 0$ and $y_- \leq 0$ are, respectively, the positive and negative part of $y$, then $y_+$ and $y_-$ are, respectively, symmetric about $x = \eta_M$ and $x = \eta_m$.

(c) There exists a unique zero of $y$ in $]-1,1[$.

(d) In the minimum value $\bar{m}$ of $y$, it holds that

$$\lambda_\alpha(p, r) \equiv (p - 1) H(\bar{m}, p, r)^p,$$

where $H(m, p, r), (m, p, r) \in [0, 1] \times [2, +\infty[ \times \left[ \frac{p}{2}, p \right]$, is the function defined as

$$H(m, p, r) := \int_{-m}^{1} \frac{dy}{[1 - R(m, p, r)(1 - |y|^{r-1}y) - |y|^p]^{\frac{1}{r}}} = \int_{0}^{1} \frac{dy}{[1 - R(m, p, r)(1 - y^r) - y^p]^{\frac{1}{r}}} + \int_{0}^{1} \frac{mdy}{[1 - R(m, p, r)(1 + m^r y^r) - m^p y^p]^{\frac{1}{r}}}$$

and $R(m, p, r) = \frac{1 - m^p}{1 + m^r}$.

Proof. Let us suppose that $\lambda_\alpha(p, r)$ admits a minimizer $y$ that changes sign and that

$$\max_{[-1,1]} y(x) = 1, \quad \min_{[-1,1]} y(x) = -\bar{m}, \quad \text{with} \quad \bar{m} \in ]0,1[.$$

It is always possible to reduce to this condition by multiplying the solution for a suitable positive constant. Let us consider $\eta_M, \eta_m$ in $]-1,1[$ such that $y(\eta_M) = 1 = \max_{[-1,1]} y$, and $y(\eta_m) = -\bar{m} = \min_{[-1,1]} y$. For the sake of simplicity, we will write $\lambda = \lambda_\alpha(p, r)$. If we multiply the equation in (4) by $y'$ and integrate, we get

$$\frac{|y'|^p}{p'} + \lambda \frac{|y|^p}{p} \equiv \frac{\alpha y}{r} |y|^{r-1} y' + c \quad \text{in} \quad ]-1,1[,$$

for a suitable constant $c$ and $\frac{1}{p'} + \frac{1}{p} = 1$. Being $y'(\eta_M) = 0$ and $y'(\eta_m) = 1$, we have

$$c = \frac{\lambda}{p} \frac{\alpha}{r}.$$

Moreover, $y'(\eta_m) = 0$ and $y(\eta_m) = -\bar{m}$ give also that

$$c = \frac{\lambda}{p} \frac{\bar{m}^p}{p} + \frac{\alpha}{r} \bar{m}^r.$$

Joining (7) and (8), we obtain

$$\begin{cases} 
\gamma = \frac{\lambda}{p} \alpha R(m, p, r) \\
\bar{c} = \frac{\lambda}{p} \Gamma(m, p, r) 
\end{cases}$$

(9)

where

$$R(m, p, r) = \frac{1 - m^p}{1 + m^r} \quad \text{and} \quad \Gamma(m, p, r) = \frac{m^p + m^r}{1 + m^r} = 1 - R(m, p, r).$$
Then (6) can be written as

$$\frac{|y'|^p}{p'} + \lambda \frac{|y|^p}{p} = \frac{\lambda}{p} R(m, p, r)|y|^{r-1} y + \frac{\lambda}{p} (1 - R(m, p, r)) \quad \text{in } ]-1, 1[. \quad (10)$$

From (10), we have

$$|y'|^p = \frac{\lambda}{p - 1} (1 - R(m, p, r)(1 - |y|^{r-1} y) - |y|^p) \quad \text{in } ]-1, 1[.$$

It is easy to see that the number of zeros of \(y\) has to be finite, hence let

$$-1 = \zeta_1 < \ldots < \zeta_j < \zeta_{j+1} < \ldots < \zeta_n = 1$$

be the zeroes of \(y\). As observed in [CD], it is easy to show that

$$y'(x) = 0 \iff y(x) = -m \text{ or } y(x) = 1.$$

This implies that \(y\) has no other local minima or maxima in \(]-1, 1[\), and in any interval \(]\zeta_j, \zeta_{j+1}[\) where \(y > 0\) there is a unique maximum point, and in any interval \(]\zeta_j, \zeta_{j+1}[\) where \(y < 0\) there is a unique minimum point.

Now, we set

$$g(Y) := 1 - R(m, p, r)(1 - |Y|^{r-1} Y) - Y^p, \quad Y \in [-m, 1],$$

and we have

$$|y'|^p = \frac{\lambda}{p - 1} g(y). \quad (11)$$

Let us observe that \(g(-m) = g(1) = 0\). Being \(p > r\), it holds that \(g'(\tilde{Y}) = 0\) implies \(g(\tilde{Y}) > 0\). Hence, \(g\) does not vanish in \(]-m, 1[\). By (11), it holds that \(y'(x) \neq 0\) if \(y(x) \neq 1\) and \(y(x) \neq -m\).

Now, we will adapt the argument of [DGS, Lemma 2.6]. The following three claims below allow to complete the proof of (a), (b) and (c).

**Claim 1**: in any interval \(]\zeta_j, \zeta_{j+1}[\) given by two subsequent zeros of \(y\) and in which \(y = y^+ > 0\), has the same length; in any of such intervals, \(y^+\) is symmetric about \(x = \frac{\zeta_j + \zeta_{j+1}}{2}\);

**Claim 2**: in any interval \(]\zeta_j, \zeta_{j+1}[\) given by two subsequent zeros of \(y\) and in which \(y = y^- < 0\) has the same length; in any of such intervals, \(y^-\) is symmetric about \(x = \frac{\zeta_j + \zeta_{j+1}}{2}\);

**Claim 3**: there is a unique zero of \(y\) in \(]-1, 1[\).

This result was proved in the case \(p = 2\) in [DP] and following this proof, we can show the result in the hypothesis of the Proposition. Properties (a), (b) and (c) can be also proved by using a symmetrization argument, by rearranging the functions \(y^+\) and \(y^-\) and using the Pólya-Szegő inequality and the properties of rearrangements (see also, for example, [BFNT] and [D]).

Now denote by \(\eta_M\) and \(\eta_m\), respectively, the unique maximum and minimum point of \(y\). It is not restrictive to suppose \(\eta_M < \eta_m\). They are such that \(\eta_M - \eta_m = 1\), with \(y' < 0\) in \(]\eta_M, \eta_m[\). Then

$$\lambda^\frac{1}{p} = (p - 1)^\frac{1}{p - 1} \frac{-y'}{|1 - R(m, p, r)(1 - |y|^{r-1} y) - y^p|^\frac{1}{p}} \quad \text{in } ]\eta_M, \eta_m[.$$
Integrating between $\eta_M$ and $\eta_m$, we have
\[ \lambda = (p-1) \left[ \int_{\eta_m}^{\eta_M} \frac{dy}{m \left[ 1 - R(m, p, r)(1 - y^{p-1})y - y^p \right]^\frac{1}{p}} \right]^p = (p-1)H(m, p, r)^p, \]
and the proof of the Proposition is completed.

To prove the main result of this Section, we will show the monotonicity of the function $H(m, p, r)$, defined in Proposition 3.1, with respect to $r$ (Lemma 3.2) and with respect to $m$ (Lemma 3.3).

The proof of the monotonicity with respect to $r$ is based on the study of the integrand function that defines $H(m, p, r)$, that is
\[ h(m, p, r, y) := \frac{1}{[1 - R(m, p, r)(1 - y^p) - y^p]^\frac{1}{p}} + \frac{m}{[1 - R(m, p, r)(1 + m^r y^r) - m^p y^p]^\frac{1}{p}}, \]
for $y \in [0, 1]$. Let us explicitly observe that if $m = 1$, then $z(1, p, r) = 0$ and
\[ h(1, p, r, y) = \frac{2}{[1 - y^p]^\frac{1}{p}}, \]
that is constant in $r$. Moreover, if $y = 0$, then
\[ h(m, p, r, 0) = \frac{1 + m}{[1 - R(m, p, r)]^\frac{1}{p}} \]
that is strictly increasing in $r \in \left[ \frac{p}{2}, p \right]$.

**Lemma 3.2.** For any fixed $y \in [0, 1]$ and $m \in [0, 1]$, the function $h(m, p, r, y)$ is strictly increasing with respect to $r$ as $\frac{p}{2} \leq r \leq p$.

**Proof.** From the preceding observations, we may assume $m \in [0, 1]$ and $y \in [0, 1]$. Differentiating in $r$, we have, for $R = R(m, p, r)$, that
\[ \partial_r h = -\frac{1}{p F_1^{p+1}} \left[ - (1 + m^r y^r) \right] \partial_r R + R y^r \log y + \frac{m}{p F_1^{p+1}} \left[ - (1 + m^r y^r) \right] \partial_r R - R m^r y^r (\log m + \log y), \]
where
\[ F_1(m, p, r, y) := [1 - R(1 - y^r) - y^p]^\frac{1}{p} \leq [1 - y^p]^\frac{1}{p}, \tag{12} \]
and
\[ F_1(m, p, r, y) := [1 - R(1 + m^r y^r) - m^p y^p]^\frac{1}{p} \geq m [1 - y^p]^\frac{1}{p}. \tag{13} \]

Being
\[ R = \frac{1 - m^p}{1 + m^r}, \quad \partial_r R = -\frac{1 - m^p}{(1 + m^r)^2} m^r \log m, \]
we have that
\[ \partial_r h = \frac{1}{p} \frac{1 - m^p}{(1 + m^r)^2} \left\{ \frac{h_1(m, r, y)}{} \right\} \]
\[ + \left[ \frac{m^r - 1}{(1 - y^p)^{p - 1}} \right] \]
\[ + \left[ \frac{(y^r - 1) \log m + (1 + m^r) y^r \log y}{m^r - 1} \right] \]
\[ + \left[ \frac{(m^r + m^{r - p}) \log m + r(1 + m^r)(m^{r - p} - 1) \log y + (1 + m^r)(m^{r - p} - 1)}{} \right]. \]

Let us observe that \( h_1(m, p, r, y) \geq 0 \). Hence, in the set \( A \) of \((m, p, r, y)\) such that \( h_2(m, p, r, y) \) is nonnegative, we have that \( \partial_r h(m, p, r, y) \geq 0 \). Moreover, \( h_1(m, p, r, y) \) cannot vanish \((y < 1)\), then \( \partial_r h > 0 \) in \( A \).

Hence, let us consider the set \( B \) where
\[ h_2 = (y^r - 1) \log m + (1 + m^r) y^r \log y \leq 0 \]
(observe that in general \( A \) and \( B \) are nonempty). By \((12)\) and \((13)\) we have that
\[ \partial_r h \geq \frac{1}{p} \frac{1 - m^p}{(1 + m^r)^2} \left\{ \frac{h_1(m, r, y)}{} \right\} \]
\[ + \left[ \frac{m^r - 1}{(1 - y^p)^{p - 1}} \right] \]
\[ + \left[ \frac{(y^r - 1) \log m + (1 + m^r) y^r \log y}{m^r - 1} \right] \]
Hence, to show that \( \partial_r h > 0 \) in the set \( B \) it is sufficient to prove that
\[ g(p; m, r, y) := \left[ \frac{(m^r + m^{r - p}) \log m + r(1 + m^r)(m^{r - p} - 1) \log y + (1 + m^r)(m^{r - p} - 1)}{} \right] \]
\[ \text{when } m \in [0, 1], \ r \in [\frac{p}{2}, p] \text{ and } y \in [0, 1]. \]

**Claim 1.** For any \( r \in [\frac{p}{2}, p] \) and \( m \in [0, 1] \), the function \( g(m, r, \cdot) \) is strictly decreasing for \( y \in [0, 1] \).

To prove the Claim 1, we differentiate \( g \) with respect to \( y \), obtaining
\[ \partial_y g = \left[ \frac{r y^r - 1}{r y^r - 1} \log m - r y^r - 1 (1 + m^r) \log y - y^r (1 + m^r) \log y \right] + \]
\[ + \left[ \frac{r y^r - 1}{r y^r - 1} \log m + (1 + m^r) (r y^r - 1) \log y + y^r (1 + m^r) \log y \right] m^r - p = \]
\[ = y^r - 1 \right] \left[ 1 + m^r + m^{r - p} \log m + r(1 + m^r)(m^{r - p} - 1) \log y + (1 + m^r)(m^{r - p} - 1) \right]. \]

Then \( \partial_y g < 0 \) if and only if
\[ (1 + m^r)(m^{r - p} - 1)(r \log y + 1) < -r(m^r + m^{r - p}) \log m. \]
The above inequality is true, as we will show that \((\text{recall that } 0 < m < 1 \text{ and } \frac{p}{2} \leq r \leq p)\)
\[ \log y < -\frac{1}{r} + \left( \frac{m^r + m^{r - p}}{1 + m^r(1 - m^{r - p})} \right) =: -\frac{1}{r} + \ell(m, r). \]
If the the right-hand side of (15) is nonnegative, then for any \( y \in ]0, 1[ \) the inequality (15) holds.

**Claim 2.** For any \( r \in [\frac{p}{2}, p] \) and \( m \in ]0, 1[, \ell(m, r) > \frac{1}{r} \).

We will show that
\[
\ell(m, r) > \frac{1}{r}.
\]
We have
\[
\ell(m, r) = \frac{(m^r + m^{r-p}) \log m}{(1 + m^r)(1 - m^{r-p})} > \frac{1}{r}
\]
if and only if
\[
\mu(m, r) = (m^r + m^{r-p}) \log \frac{1}{m} - \frac{1}{r}(1 + m^r)(m^{r-p} - 1) = \left( m^r + m^{r-p} \right) \log \frac{1}{m} + \frac{1}{r}(1 + m^r - m^{r-p} - m^{2r-p}) = m^r \left( \log \frac{1}{m} + \frac{1}{r} \right) + m^{r-p} \left( \log \frac{1}{m} - \frac{1}{r} \right) + \frac{1}{r}(1 - m^{2r-p}) > 0.
\]
Then for \( m \in ]0, 1[ \) we have
\[
\mu(m, r) = m^r \left( \log \frac{1}{m} + \frac{1}{r} \right) + m^{r-p} \left( \log \frac{1}{m} - \frac{1}{r} \right) \geq m^r \left( \log \frac{1}{m} + \frac{1}{r} \right) + m^{r-p} \left( \log \frac{1}{m} - \frac{1}{r} \right) = m^{r-p} \left( m^p \left( \log \frac{1}{m} + \frac{1}{r} \right) + \log \frac{1}{m} - \frac{1}{r} \right) := m^{r-p} \eta(m, r) > 0.
\]

We prove that \( \mu(m, r) \) is positive by showing that \( \eta(m, r) \) is decreasing in \( m \):
\[
\partial_m \eta(m, q) = m^{p-1} \left( \log \frac{1}{m^p} - \frac{1}{m^p} + \frac{p}{r} - 1 \right).
\]
Since \( \log \frac{1}{m^p} < \frac{1}{m^p} - 1 \), we have that \( \partial_m \eta(m, q) < 0 \) when \( r > \frac{p}{2} \) and the Claim 2, and then the Claim 1, are proved. To conclude the proof of (14), it is sufficient to observe that
\[
g(m, r, y) > g(m, r, 1) = 0
\]
when \( m \in ]0, 1[ \), \( r \in [\frac{p}{2}, p] \) and \( y \in ]0, 1[ \).

The Claim 1 gives that \( \partial_r h(m, r, y) > 0 \) when \( m \in ]0, 1[, r \in [\frac{p}{2}, p] \) and \( y \in ]0, 1[ \). and this conclude the proof.

Now, to prove the monotonicity of \( H \) in \( m \), we argue similarly as in [GGR]. We show that, for any fixed \( p \geq 2 \) the function \( K(m) := H(m, p, \frac{p}{2}) \) is constant.

**Lemma 3.3.** Let \( p \geq 2 \), then \( K'(m) = 0 \), \( \forall \ m \in ]0, 1[ \).
Proof. For any fixed $p \geq 2$, we denote the following non negative function by:

$$A(m, y) := m^\frac{p}{2} + (1 - m^\frac{p}{2}) y^\frac{p}{2} - y^p, \quad \forall (m, y) \in [0, 1]^2;$$

$$B(m, y) := m^\frac{p}{2} - (1 - m^\frac{p}{2}) m^\frac{p}{2} y^\frac{p}{2} - m^p y^p, \quad \forall (m, y) \in [0, 1]^2.$$

Moreover, in this case

$$R\left( m, p, \frac{p}{2} \right) = 1 - m^\frac{p}{2}, \quad \forall m \in [0, 1].$$

Hence $K(m) = \int_0^1 \left( A(m, y)^{-\frac{p}{2}} + mB(m, y)^{-\frac{1}{2}} \right) \, dy$ and

$$K'(m) = -\frac{1}{p} \int_0^1 \left( A(m, y)^{-\frac{1}{2}} - 1 \right) \frac{\partial A(m, y)}{\partial m} \, dy + B(m, y)^{-\frac{1}{2} - 1} \left( -pB(m, y) + m\frac{\partial B(m, y)}{\partial m} \right) \, dy.$$

Differentiating with respect to $m$, we obtain

$$\frac{\partial A(m, y)}{\partial m} = \frac{p}{2} m^\frac{p}{2} - 1 \left( 1 - y^\frac{p}{2} \right),$$

$$-pB(m, y) + m\frac{\partial B(m, y)}{\partial m} = \frac{p}{2} m^\frac{p}{2} \left( 1 - y^\frac{p}{2} \right).$$

Hence

$$K'(m) = \frac{m^\frac{p}{2} - 1}{2} \int_0^1 \left( \frac{1 - y^\frac{p}{2}}{A(m, y)^{\frac{p}{2} + 1}} + \frac{m(1 - y^\frac{p}{2})}{B(m, y)^{\frac{p}{2} + 1}} \right) \, dy.$$

Now we study the sign of the right integral. We want to prove that

$$\int_0^1 \frac{1 - y^\frac{p}{2}}{A(m, y)^{\frac{p}{2} + 1}} \, dy = \int_0^1 \frac{m(1 - y^\frac{p}{2})}{B(m, y)^{\frac{p}{2} + 1}} \, dy \tag{16}$$

Following the ideas of [GGR], for all $m \in (0, 1)$, we set

$$\delta(y) := \left[ 1 - (1 - m^\frac{p}{2}) y^\frac{p}{2} \right]^\frac{1}{p} \quad \forall \in [0, 1]$$

and

$$h(y) := \frac{my}{\delta(y)} \quad \forall y \in [0, 1]. \tag{17}$$

It holds that $h(0) = 0$, $h(1) = 1$ and

$$h'(y) := \frac{m}{\delta(y)^{\frac{p}{2} + 1}}, \quad \forall y \in (0, 1).$$

Hence the function $h$ is strictly increasing and, keeping (17) into account, the result follows if we prove that

$$\int_0^1 \frac{1 - y^\frac{p}{2}}{\left( m^\frac{p}{2} + (1 - m^\frac{p}{2}) m^\frac{p}{2} y^\frac{p}{2} \delta(y)^{-\frac{p}{2}} - m^p y^p \delta(y)^{-p} \right)^{\frac{p}{2} + 1}} \, dy$$

$$= \int_0^1 \frac{1 - y^\frac{p}{2}}{\left( m^\frac{p}{2} - (1 - m^\frac{p}{2}) m^\frac{p}{2} y^\frac{p}{2} - m^p y^p \right)^{\frac{p}{2} + 1}} \, dy$$
Therefore (16) is proved if we show that
\[ m^p - (1 - m^p) m^p y^p - m^p y^p = \delta(y)^p \left( m^p + (1 - m^p) \frac{m^p y^p}{\delta(y)^p} - \frac{m^p y^p}{\delta(y)^p} \right), \]
and this is an equality that can be easily checked. □

Now, we are in position to state the main property of the function \( H(m, p, r) \).

**Lemma 3.4.** Let \( p \geq 2 \) and \( \frac{p}{2} \leq r \leq p \), then for all \( m \in [0, 1] \) it holds that
\[ H(m, p, r) \geq \frac{\pi p}{(p-1)^p}. \]

Moreover:
* when \( \frac{p}{2} < r \leq p \), then \( H(m, p, r) = \frac{\pi p}{(p-1)^p} \) if and only if \( m = 1 \);
* \( H\left(m, p, \frac{p}{2}\right) = \frac{\pi p}{(p-1)^p} \) for all \( m \in [0, 1] \).

**Proof.** If \( m = 1 \), we have that
\[ H(1, p, r) = 2 \int_0^1 \frac{dy}{1 - y^p} = \frac{\pi p}{(p-1)^p} \]
for \( 1 \leq \frac{p}{2} \leq r \leq p \). Moreover, by Lemma 3.3
\[ H\left(m, p, \frac{p}{2}\right) = H\left(1, p, \frac{p}{2}\right) = \frac{\pi p}{(p-1)^p}, \]
for any \( m \in [0, 1] \).

To study all the other cases, we first consider \( 0 < m < 1 \). Then for any \( p \geq 2 \), \( \frac{p}{2} < r \leq p \), by Lemma 3.2 we get we have
\[ H(m, p, r) > H\left(m, p, \frac{p}{2}\right) = \frac{\pi p}{(p-1)^p}. \]

When \( m = 0 \), simple calculations give
\[ H(0, p, r) = \int_0^1 \frac{dy}{|y^r - y^p|} > \int_0^1 \frac{dy}{|y^{\frac{p}{2}} - y^p|} = H\left(0, p, \frac{p}{2}\right) = \frac{\pi p}{(p-1)^p}, \]
and hence the result. □

**Proposition 3.5.** Let \( p \geq 2 \), \( \frac{p}{2} \leq r \leq p \) and suppose that there exists \( \alpha > 0 \) such that \( \lambda_\alpha(p, r) \)
admits a minimizer \( y \) that changes sign in \([-1, 1]\).

(i) If \( \frac{p}{2} \leq r \leq p \), then
\[ \lambda_\alpha(p, r) = \Lambda(p, r) = \pi p. \]

(ii) If \( \frac{p}{2} < r \leq p \), then
\[ \int_{-1}^1 |y|^{r-1} y \, dx = 0. \quad (18) \]
(iii) If \( \frac{p}{2} \leq r \leq p \) and (18) holds, then \( y(x) = C \sin_p(\pi_p x) \), with \( C \in \mathbb{R} \setminus \{0\} \). Hence the only point in \([-1, 1]\) where \( y \) vanishes is \( x = 0 \).

Proof. Let us consider a minimizer \( y \) of \( \lambda_\alpha(p, r) \) in \([-1, 1]\) that changes sign, with \( \max y = 1 \) and \( \bar{y} = -\min y \).

By (d) of Proposition 3.1 and Lemma 3.4, the eigenvalue \( \lambda_\alpha(p, r) \) has to satisfy the inequality

\[
\lambda_\alpha(p, r) \geq \pi_p^p.
\]

Hence, by (c) and (e) of Proposition 2.1, it follows that

\[
\lambda_\alpha(p, r) = \pi_p^p,
\]

that gives (i).

Now assume that \( \frac{p}{2} < r \leq p \). Again by Lemma 3.4 and (d) of Proposition 3.1, \( \lambda_\alpha(p, r) = \pi_p^p \) if and only if \( \bar{y} = 1 \). Hence, the first identity of (9) gives that

\[
\int_{-1}^{1} |y|^{r-1} y \, dx = 0,
\]

and (ii) follows. To prove (iii), let us explicitly observe that, when (18) holds, \( y \) solves

\[
\begin{cases}
|y'|^p - 2y' + \pi_p^p |y|^{p-2} y = 0 & \text{in } ]-1, 1[ \\
y(-1) = y(1) = 0.
\end{cases}
\]

Hence \( y(x) = C \sin_p(\pi_p x) \), with \( C \in \mathbb{R} \setminus \{0\} \). \( \square \)

4 PROOF OF THE MAIN RESULTS

In this Section we prove the main results by using the properties of Section 3.

Proof of Theorem 1.1. We prove that there exists a critical value of the parameter such that the minimizer is symmetric. Firstly we prove the following claim.

Claim. There exists a positive value of \( \alpha \) such that the minimum problem

\[
\lambda_\alpha(p, r) = \min_{u \in W^{1,p}_0([-1,1])} \frac{\int_{-1}^{1} |u'|^p \, dx + \alpha \left( \int_{-1}^{1} |u|^{r-1} \, dx \right)^{p/\gamma}}{\int_{-1}^{1} |u|^p \, dx}
\]

admits an eigenfunction \( y \) that satisfies \( \int_{-1}^{1} y|y|^{r-1} \, dx = 0 \). In such a case, \( \lambda_\alpha(p, r) = \pi_p^p \) and, up to a multiplicative constant, \( y = \sin_p(\pi_p x) \).

By Proposition 2.1 (e), if a minimizer \( y \) changes sign, then we may suppose that \( \alpha > 0 \). By contradiction, we suppose that for any \( k \in \mathbb{N} \), there exists a divergent sequence \( \alpha_k \), and a corresponding sequence of eigenfunctions \( \{y_k\}_{k \in \mathbb{N}} \) relative to \( \lambda_{\alpha_k}(p, r) \) such that \( \int_{-1}^{1} y_k |y_k|^{r-1} \, dx > 0 \) and \( \|y_k\|_{L^p([-1,1])} = 1 \). By Proposition 3.5, these eigenfunctions do not change sign and, as we have already observed, \( \lambda_{\alpha_k}(p, r) \leq \pi_p^p \). Hence, it holds that

\[
\int_{-1}^{1} |y_k'|^p \, dx + \alpha_k \left( \int_{-1}^{1} |y_k|^{r} \, dx \right)^{p/\gamma} \leq \pi_p^p.
\]
Therefore, $y_k$ converges (up to a subsequence) to a function $y \in W^{1,p}_0(-1,1)$, strongly in $L^p(-1,1)$ and weakly in $W^{1,p}_0(-1,1)$. Moreover $\|y\|_{L^p(-1,1)} = 1$ and $y$ is not identically zero. Therefore $\|y\|_{L^1(-1,1)} > 0$ and, letting $\alpha_k \to +\infty$ in (19) we have a contradiction and the claim is proved.

Now, let us recall that, by Proposition 2.1, $\lambda_\alpha(p,r)$ is a nondecreasing Lipschitz function in $\alpha$. Therefore we can define

$$\alpha_C = \min(\alpha \in \mathbb{R}: \lambda_\alpha(p,r) = \pi_\alpha^p) = \sup(\alpha \in \mathbb{R}: \lambda_\alpha(p,r) < \pi_\alpha^p).$$

We easily verify that this value of the parameter is positive and if $\alpha < \alpha_C$, then the minimizers corresponding to $\lambda_\alpha(p,r)$ have constant sign, otherwise $\lambda_\alpha(p,r) = \pi_\alpha^p$. When $\alpha > \alpha_C$, then any minimizer $y$ corresponding to $\alpha$ is such that $\int_{-1}^1 |y|^{r-1}y \, dx = 0$. Indeed, if we assume, by contradiction, that there exist $\tilde{\alpha} > \alpha_C$ and $\tilde{y}$ such that $\int_{-1}^1 |\tilde{y}|^{r-1}\tilde{y} \, dx > 0$, $|y|_{L^p} = 1$ and $Q_\alpha[\tilde{y}] = \lambda_\alpha(p,r)$, then

$$Q_{\tilde{\alpha}-\varepsilon}[\tilde{y}] = Q_\tilde{\alpha}[\tilde{y}] - \varepsilon \left( \int_{-1}^1 |\tilde{y}|^{r-1}\tilde{y} \, dx \right)^{\frac{p}{r}} = \lambda_\alpha(p,r) - \varepsilon \left( \int_{-1}^1 |\tilde{y}|^{r-1}\tilde{y} \, dx \right)^{\frac{p}{r}} < \lambda_\alpha(p,r).$$

Hence, for $\varepsilon$ sufficiently small, $\pi_\alpha^p = \lambda_\alpha(p,r) \leq \lambda_{\tilde{\alpha}-\varepsilon}(p,r) < \lambda_\alpha(p,r)$ and this is absurd. Finally, by (iii) of Proposition 3.5, the proof of Theorem 1.1 is completed.

**Proof of Theorem 1.2.** It is not difficult to see, by means of approximating sequences, that $\lambda_\alpha(p,r)$ admits both a nonnegative minimizer and a minimizer with vanishing $r$-average. To conclude the proof of Theorem 1.2, we have to study the behavior of the solutions when $r = p$. When $\alpha = \alpha_C(p,p)$, the corresponding positive minimizer $y$ is a solution of

$$\begin{cases}
(|y'|^{p-2}y')' + \pi_\alpha^p y^{p-1} = \alpha_C(p,p)y^{p-1} & \text{in } ]-1,1[ \\
y(-1) = y(1) = 0.
\end{cases}$$

The positivity of the eigenfunction guarantees that

$$\pi_\alpha^p - \alpha_C(p,p) = \lambda_0(p,p) = \left( \frac{\pi_2}{2} \right)^p,$$

hence $\alpha_C(p,p) = 2^{p-1}\frac{\pi_2}{2^p}$. \hfill $\square$

**Remark 4.1.** When $\frac{p}{2} \leq r < p$, we obtain the following lower bound on $\alpha_C(p,r)$:

$$\alpha_C(p,r) \geq \frac{2^p - 1}{2^p + p - 1} \pi_\alpha^p.$$  \hfill (20)

To get the estimate (20), we use the monotonicity of $\lambda_\alpha(p,r)$ with respect to $\alpha$, and consider the test function $u(x) = \cos_p(\frac{\pi_2}{2}x)$. Hence

$$\pi_\alpha^p = \lambda_{\alpha_C}(p,r) \leq Q[u, \alpha_C] = \left( \frac{\pi_2}{2} \right)^p + \alpha_C \left( \int_{-1}^1 u^r \, dx \right)^{\frac{p}{r}} \leq \frac{\pi_2^p}{2^p} + \alpha_C 2^{\frac{p}{r}-1}.$$
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REFERENCES

[BK] M. Belloni, B. Kawohl, A symmetry problem related to Wirtinger’s and Poincaré’s inequality. Journal of Diff. Eq. 156.1 (1999): 211-218.

[BDNT] B. Brandolini, F. Della Pietra, C. Nitsch, C. Trombetti, Symmetry breaking in a constrained Cheeger type isoperimetric inequality. ESAIM Control Optim. Calc. Var. 21.2 (2015): 359-371.

[BFN] B. Brandolini, P. Freitas, C. Nitsch, C. Trombetti, Sharp estimates and saturation phenomena for a nonlinear eigenvalue problem. Adv. Math. 228.4 (2011): 2352-2365.

[BCGM] F. Brock, G. Croce, O. Guibé, A. Mercaldo, Symmetry and asymmetry of minimizers of a class of noncoercive functionals. Adv. Calc. Var., in press.

[BKN] A. P. Buslaev, V. A. Kondrat’ev, A. I. Nazarov, On a family of extremal problems and related properties of an integral, Mat. Zametki 64.6 (1998): 830-838 (Russian); English transl.: Math. Notes, 64.5-6 (1998): 719-725.

[CD] G. Croce, B. Dacorogna, On a generalized Wirtinger inequality, Discr. Contin. Dyn. Syst. 9.5 (2003): 1329-1341.

[CHP1] G. Croce, A. Henrot, G. Pisante, An isoperimetric inequality for a nonlinear eigenvalue problem, Annales de l’IHP (C), Non Lin. Anal. 29 (2012): 21-34; Corrigendum to “An isoperimetric inequality for a nonlinear eigenvalue problem”, Annales de l’IHP (C), Non Lin. Anal. 32 (2015): 485-487.

[DGS] B. Dacorogna, W. Gangbo, N. Subía, Sur une généralisation de l’inégalité de Wirtinger. Annales de l’IHP Anal. non Lin. 9.1 (1992): 29-50.

[D] F. Della Pietra, Some remarks on a shape optimization problem. Kodai Mathematical Journal 37 (2014): 608-619.

[DP] F. Della Pietra, G. Piscitelli, A saturation phenomenon for a nonlinear nonlocal eigenvalue problem. NoDEA Nonlin. Diff. Eq. Appl. 23.6 (2016): 1-18.

[E] Y. V. Egorov, On a Kondratiev problem. C.R.A.S. Paris Ser. I 324 (1997): 503-507.

[F] P. Freitas, Nonlocal reaction-diffusion equations. Diff. Eq. Appl. Biol., Halifax, NS (1997). Fields Inst. Commun. 21, Amer. Math. Soc., Providence, RI (1999): 187-204.

[FH] P. Freitas, A. Henrot, On the first twisted Dirichlet eigenvalue. Comm. Anal. Geom.12 (2004): 1083-1103.

[GN] I. V. Gerasimov, A. I. Nazarov, Best constant in a three-parameter Poincaré inequality. Probl. Mat. Anal. 61 (2011): 69-86 (Russian). English transl.: J. Math. Sci. 179.1 (2011): 80-99.
[GGR] M. Ghisi, M. Gobbino, G. Rovellini. *Symmetry-breaking in a generalized Wirtinger inequality.* ESAIM: Control, Optimisation and Calculus of Variations (2018). 2, 11, 12

[KN] N. Kuznetsov, A. Nazarov. *Sharp constants in the Poincaré, Steklov and related inequalities (a survey).* Mathematika, 61(2), 328-344 (2015). 2

[L] P. Lindqvist, *Some remarkable sine and cosine functions,* Ric. Mat. 44 (1995): 269-290. 4

[N1] A. I. Nazarov, *On exact constant in the generalized Poincaré inequality.* Probl. Mat. Anal. 24 (2002): 155-180 (Russian). English transl.: J. Math. Sci. 112.1 (2002): 4029-4047. 2

[N2] A. I. Nazarov, *On symmetry and asymmetry in a problem of shape optimization,* preprint arXiv:1208:3640 (2012). 2

[P] R. Pinsky, *Spectral analysis of a class of non-local elliptic operators related to Brownian motion with random jumps.* Trans. Amer. Math. Soc. 361 (2009): 5041-5060. 2

[Pi] G. Piscitelli, *A nonlocal anisotropic eigenvalue problem.* Diff. Int. Eq. 29.11/12 (2016): 1001-1020. 2

[Ô] M. Ôtani, *A remark on certain nonlinear elliptic equations.* Proceedings of the Faculty of Science, Tokai University, 19 (1984): 23-28. 4

[S] R. P. Sperb, *On an eigenvalue problem arising in chemistry.* Z. Angew. Math. Phys. 32 4 (1981): 450-463. 2