Scaling and Renormalization Group in Replica Symmetry Breaking space: Evidence for a simple analytical solution of the SK model at zero temperature

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Using numerical self-consistent solutions of a sequence of finite replica symmetry breakings (RSB) and Wilson’s renormalization group but with the number of RSB-steps playing a role of decimation scales, we report evidence for a non-trivial \( T \to 0 \)-limit of the Parisi order function \( q(x) \) for the SK spin glass. Supported by scaling in RSB-space, the fixed point order function is conjectured to be \( q^*(a) = \frac{a}{\pi \xi} \text{erf}\left(\frac{a}{\xi}\right) \) on \( 0 \leq a \leq \infty \) where \( \frac{a}{\xi} \to a \) at \( T = 0 \) and \( \xi \approx 1.13 \pm 0.01 \). \( \xi \) plays the role of a correlation length in \( a \)-space. \( q^*(a) \) may be viewed as the solution of an effective 1D field theory.

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It is now 30 years since, stimulated by the seminal work of Edwards and Anderson\textsuperscript{1,21}, one of the authors (DS) devised the canonical soluble mean-field spin glass model\textsuperscript{22} now known as the Sherrington-Kirkpatrick (or SK) model. The first published report\textsuperscript{2} already exhibited the ‘smoking gun’ which a few years later led Parisi to his remarkable predictions of a hierarchy of replica symmetry breaking (RSB)\textsuperscript{3} and its associated implications for macrostate complexity\textsuperscript{4} in disordered frustrated systems. These studies further stimulated new fields of investigation across physics, computer science, biology, econophysics and probability theory\textsuperscript{23}. There has been great progress since then, in studies of the original model and its many extensions; the concept of RSB is now well-accepted for mean field models\textsuperscript{24} and the behaviour of the Parisi order function is well studied perturbatively near the transition temperature; there exist self-consistency equations from which the Parisi function can be obtained in principle; and some features of RSB are considered known at least semi-quantitatively down towards zero temperature. However, here we demonstrate that there appear to be unappreciated subtleties in the limit as zero temperature is approached. In particular, we exhibit evidence for a new zero-temperature limiting Parisi order function with a new ‘correlation length’ in RSB-space and also unconventional behaviour of the \( T = 0 \) local field distribution in the zero field limit.

The models we consider are the usual SK one\textsuperscript{2}

\[
\mathcal{H} = \sum_{i<j} J_{ij} \sigma_i \sigma_j; \quad \sigma = \pm 1
\]

and the Fermionic Ising Spin Glass (FISG)\textsuperscript{10,11}

\[
\mathcal{H} = \sum_{i,j} J_{ij} \sigma_i^+ \sigma_j^+ - \mu \sum_i (n_{i+} + n_{i-}) - \mu \sum_i n_{i,s} = 0, 1
\]

in the half-filling limit, where the spin variables are given by \( \sigma^+ = a_i^+ a_i^\dagger - a_i^\dagger a_i^+ \) and \( \sigma^+ \) and \( a_i^\dagger \) and \( a_i^+ \) are fermion creation and annihilation operators.\textsuperscript{25} The \( J_{ij} \) are quenched disordered and frustrated interactions independently distributed over all pairs of sites as

\[
P(J) = \exp(-NJ^2_j/(2J^2))/\sqrt{2\pi J^2/N}
\]

where we measure temperature in units of \( J \) and for notational convenience set the energy scale \( J = 1 \). We shall consider its properties in replica theory and assume the Parisi ansatz as an infinite sequence of replica symmetry breaking (RSB) steps for an order function \( q(x) \) defined on \( 0 \leq x \leq 1 \).\textsuperscript{3,12} Denoting the steps as \( x_k \) it is believed\textsuperscript{8,14} that in the limit \( T \to 0 \) and small \( x \leq O(T) \) the steps scale with \( T \) and the natural variables are \( T \)-normalized steps \( a_k \) given by

\[
a_k \equiv \lim_{T \to 0} (J/T)x_k(T),
\]

which become self-consistently distributed on the interval \( 0 \leq a \leq \infty \). We focus on the \( T = 0 \) limit and express the free energy functional of the Parisi scheme at \( \alpha \)-RSB in terms of the variational parameter set \( \{ q_k+1, a_k; k = 1...\alpha \} \); we also have constrained \( \{ q_1 = 1, q_{\alpha+2} = 0 \} \). The extremization has been performed for \( \alpha = 1...5 \) to an accuracy of 4 decimal places. The resulting values of \( q_k,a_k \) are shown in Fig.\textsuperscript{11} for \( \alpha = 2 \) to 5-RSB, together with fitting functions of the form

\[
q^\alpha_{\text{model}}(a) = \frac{\sqrt{\pi}}{2} \frac{a}{\xi(\alpha)} \text{erf}\left[\frac{\xi(\alpha)}{\sqrt{\pi}f(\alpha)}\right].
\]

where \( f(\alpha) \) is chosen as \( (a^2 + w(\alpha))^{1/2} \) in a 2-parameter model and as \( (a^2 + c(\alpha)a + w(\alpha)) \) in a refined 3-parameter model. Both choices derive from the ansatz of a Gaussian distribution of error-functions which (in contrast to a single error function) fit well the numerical data. The \( \xi(\alpha) \), \( w(\alpha) \) and \( c(\alpha) \) are viewed as parameters in an RG decimation-type flow space with the \( \alpha \) playing the role of the degree of decimation. Fig.\textsuperscript{2} shows these flows together with analytic fit functions which extrapolate as \( \alpha \to \infty \) to \( \lim_{\alpha \to \infty} \xi(\alpha) = 1.14 \) and \( \lim_{\alpha \to \infty} w(\alpha) = 0 \) (and for the 3-parameter case also \( \lim_{\alpha \to \infty} c(\alpha) = 0 \)). Fig.\textsuperscript{4} shows the quality of the fits. Moreover, if the indices \( k \) are considered as potentially real continuous variables, single parameter rescalings \( R_{\alpha,k} \) and \( T_{\alpha,k} \) are found to map \( a \) and \( q \) respectively almost onto single curves, as Fig.\textsuperscript{5} shows, with small decreasing deviations as \( \alpha \) increases. This scaling behaviour in RSB-space of
the k-parametrized order parameter function supports a single parameter limiting form. Hence we conjecture that the infinite RSB limiting form of the $T \to 0$ Parisi order function is

$$q^*(a) = \frac{\sqrt{\pi}}{2} \frac{a}{\xi} \text{erf} \left( \frac{\xi}{a} \right).$$  

Thus we predict that there is a finite intrinsic correlation length $\xi$ in RSB-space. As noted above, our numerical results fit $\xi(\alpha = \infty) = 1.138$. Assuming low T PatT-scaling $q(x, T) = q(x, T)$ up to $x = x_{\text{max}}(T)$ and $q(T) \sim 1 - O(T^2)$ for $x > x_{\text{max}}$, fitting the Gibbs susceptibility $\chi(0) = \lim_{T \to 0} \beta \int_0^1 dx (1 - q(x, T)) = \int_0^\infty da (1 - q(a)) = \frac{1}{2}\sqrt{\pi}a$ to its expected value $\chi(T = 0) = 1$ in the $\alpha = \infty$-limit requires $\xi = 2/\sqrt{\pi} \approx 1.128$. Applying the same scaling assumption to the internal energy $U(T) = -\frac{1}{2} \beta \int_0^1 dx (1 - q(x, T))$ we obtain $\xi \approx 1.1255$ by equating $U(0)$ and the $T = 0$-limit of the free energy $\chi$. The smallness of this length suggests why already low orders ($a > 2$) of RSB produce good results. Fig. 1 compares our prediction with the finite-LSB results.

The overlap probability distribution (OPDF) for finite temperatures is given by $\mathcal{P}(q) = \int_0^1 dx \delta(q - q(x, T)) = dx/dq$, where $x(q, T)$ is the inverse of the original Parisi function $q(x, T)$. The formulation for $q^*(a)$ which we have determined can be considered valid only for low $x$ scaling as $T$ but suffices to give $\mathcal{P}(q)$ for $q < O(1)$. Rescaling $a$ to a unit interval via $a \to \zeta = a/(1+a)$, the contribution of $q^*(\zeta)$ to $\mathcal{P}(q)$ is $T(1 - \zeta^*(q))^2 \mathcal{P}^*(q)$ where $\mathcal{P}^*(q) = \int_0^1 d\zeta \delta(q - q^*(\zeta)) = d\zeta^*(q)/dq$ is the $T = 0$ analog OPDF. This function obeys a non-algebraic equation yet allows one to extract an analytical result for the small-$q$ gap and a $q = 1$ divergence as

$$\mathcal{P}^*(q = 0) = 4/\pi, \quad \mathcal{P}^*(q \approx 1) = \frac{\sqrt{3}}{4\sqrt{\pi}} \frac{1}{\sqrt{1-q}}$$

(7)

FIG. 1: Order parameter functions of the SK-model at zero temperature. Shown as points are the numerical solutions of the finite RSB approximations from 2nd (lowest curve) up to 5th order, together with fitting functions $q^*_\text{model}(a)$ with best-fit $\xi$ and $w(\alpha)$ and also the predicted limiting function $q^*(a)$ for $\xi = 2/\sqrt{\pi}$ ($\xi = 1.138$ is indistinguishable on the given scale). Small dots, approximating the smallest $q_0(a), a_0(a)$ by extrapolating the $a$-dependence beyond the calculated 5th order, up to 10th order, confirm the linear rise of $q^*(a)$. The insert shows the overlap probability distribution function $\mathcal{P}^*(q) \equiv dq(\zeta)/dq$ derived from $q^*(a = \zeta/(1 - \zeta))$.

FIG. 2: Numerical renormalization group RSB-flow of the replica correlation length $\xi(\alpha)$ (insert) and of the spreading parameter $w(\alpha)$ for the 2-parameter model (3-parameter model shows an equivalent flow) with analytic fitting curves extrapolating to $\lim_{\alpha \to \infty} \xi(\alpha) \approx 1.138$ and $\lim_{\alpha \to \infty} w(\alpha) = 0$ are included. Note that the evaluation of the points shown requires a numerical accuracy far beyond that visible at the scale of Fig. 1.
where \( \tilde{a} \equiv a/\xi \). It can also be conveniently expressed as a simple generalized equation of motion in \( a \)-space through a substitution, to yield (dropping the tilde on \( a \))

\[
\frac{d}{da} \psi(a) = -\frac{\delta \mathcal{H}(\psi)}{\delta \psi(a)}, \quad \psi(a) = e^{1/2} d \log(Q(a))
\]

with a cubic ‘Hamiltonian’ \[38\]

\[
\mathcal{H}(\psi) = \int da \left( 2a^{-4} e^{1/2} \psi(a) + \frac{1}{3} e^{-1/2} \psi(a)^3 \right).
\]

Now we turn to the local field distribution

\[
P(h) = N^{-1} \sum_i (\delta(h - \sum_j J_{ij} \sigma_j)),
\]

again in the limit \( T \to 0 \). The results for \( P(h) \) of 0-5 RSB show that for any finite RSB there is a gap around \( h = 0 \) which reduces with RSB order but can be made invariant by rescaling the field and (inversely) the probability \( P \) with an \( \alpha \)-dependent factor \( \lambda_\alpha \)

\[
\tilde{P}_\alpha(h) = \lambda_\alpha P_{\alpha+1}(h/\lambda_\alpha)
\]

with \( \lambda_\alpha \) chosen to yield the same gap for each \( \alpha \). This results in a set of \( \tilde{P}_\alpha(h) \) with good overlap with one another as \( h \) tends towards the gap edge; this is illustrated in Fig. 5. Extrapolating to \( \infty \)-RSB yields the \( h \to 0 \) fixed point equation

\[
P^*(h) = \lambda_\infty P^*(h/\lambda_\infty).
\]

The numerical results for the rescaling parameter \( \lambda \) show an almost linear increase with \( \alpha \), confirming that in the \( \infty \)-RSB limit \( P^*(0) = 0 \).

A crude look at the limiting \( P(h) \) for \( h \) less than order 0.5 shows a linear slope of \( P(h) \) of order 0.3, which has been suggested as the correct limit for many years \[15\, 16\, 17\, 18\, 19\]. However a close look at the approach to the gap edge suggests that the situation is more subtle. This is illustrated in Fig. 6 where it is seen that the limit of \( h \to 0 \) has a much smaller slope of approximately 0.17 with a slope of order 0.3 taking over only at slightly larger \( h \). The figure insert for \( dP(h)/dh \) amplifies

**FIG. 3**: Almost perfect mapping of \( a_k \) parameters (top) onto a single curve (below) by rescaling with \( a^{-k+1} \rightarrow R_\alpha(a^{-k+1}) \) in RSB space; \( R_\alpha \) is shown in insert as a function of RSB order. A similar transformation exists for the \( q \) parameters.

**FIG. 4**: Quality of fits illustrated by the distance between numerical data and fit curves for the 2- and 3-parameter models (2pm and 3pm, dashed). Labels indicate RSB order. Misfits decrease from \( O(10^{-3}) \) for the 2pm to \( O(10^{-4}) \) (3pm) and improve with RSB order for both, while the numerical inaccuracy of \( q \) and \( a \)-parameters is \( O(10^{-5}) \) and \( O(10^{-4}) \) resp.

**FIG. 5**: Rescaled field distributions \( \tilde{P}_\alpha(h) \) (top; full scale, RSB orders labelled, bottom: gap region zoomed). The Insert shows the rescaling parameter \( \lambda_\alpha \) as a function of RSB order.
this observation as (i) the limit of the gap-edge values of 
\( dP(h)/dh \) at 0.17 and (ii) a peak in \( dP(h)/dh \) around 0.3 at slightly higher \( h \). 

In summary, via scaling and a new renormalization group in RSB space, for the first time a surprisingly simple analytical model solution \( q^*(a) \) and a corresponding solvable one-dimensional effective field theory are proposed for spin glasses in the low temperature limit, inviting the design of approximations which may cope with more complicated problems of related but non-solvable natures. Similar considerations have raised novel issues concerning the pseudogap in the distribution of local fields and densities of states.

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\[ \begin{align*}
    &\text{FIG. 6: Field distribution } P(h) \text{ close to } h = 0; \text{ the lines end at the gaps in the distributions. The inset shows } dP(h)/dh \text{ with solid lines corresponding to regions of non-zero } P(h) \text{ and dotted lines showing their linear extrapolation to } h = 0.
\end{align*} \]

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[21] DS thanks Prof Edwards for discussing his discoveries with him in 1974, before their publication.
[22] This model was exposed at an informal ‘brown-bag’ seminar at Imperial College in the spring of 1975.
[23] For further descriptions of both the history and the developments see the articles by Anderson, Sherrington, Parisi and Mézard in [3], and also [4, 5].
[24] Rigorous proofs are however just being developed [1, 2] and the corresponding situation for finite-range models remains controversial.
[25] The thermodynamics [10] and the spectral density functions [24] of the \( \mu = 0 \) FISG map onto properties of the SK model; they have an identical Parisi \( q(x) \) and the one particle density of states \( \rho(x) \) of the FIRSG is given by the SK \( \rho(h) \) with \( h = \epsilon \)
[26] Details of the expression for the free energy and results, as well as a discussion of the logical steps guiding our analysis, are deferred to a longer paper [13].
[27] The slight differences in these figures for \( \xi \) probably reflect on an imprecision of the PaT-hypothesis [14].
[28] Our results are also qualitatively and semi-quantitatively comparable with results of a study of \( q(\beta T) \) using a numerical analysis of high order perturbation theory in the reduced temperature \( (T_c - T)/T_c \), where \( T_c \) is the spin glass transition temperature [14]. Both these methods can be considered limited but in complementary, though possibly related, ways; our analysis is explicit for zero temperature but extrapolates from a finite number of steps of replica symmetry breaking, whereas the perturbation method is based on full replica symmetry breaking but is limited by the perturbation being around \( T_c \).
[29] Note that the negativity of the slope of \( P(h) \) at near \( q = 0 \) is removed by the prefactor \( 1 - \zeta(q) \), which yields a flat \( P(q)/T \) at \( q = 0 \) in agreement with Ref [11].
[30] Contributions from higher orders of \( \exp(-1/a^2) \psi(a) \) cannot be distinguished neither for small nor for large \( a \) due to numerical accuracy limitations.
[31] It is presumably this peak which is responsible for the larger slope observed in earlier studies. It is however impossible at the RSB order we have studied to be sure whether the peak in \( P(h) \) around 0.3 saturates at a finite \( h \) or tends towards zero field, while equally, from Fig it is difficult to believe that the slope at the gap edge does not tend to the lower value.