ON THE DUALITY BETWEEN $\ell^1$-HOMOLOGY AND BOUNDED COHOMOLOGY

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Abstract. We modify the definition of $\ell^1$-homology and argue why our definition is more adequate than the classical one. While we cannot reconstruct the classical $\ell^1$-homology from the new definition for various reasons, we can reconstruct its Hausdorffification so that no information concerning semi-norms is lost. We obtain an axiomatic characterization of our $\ell^1$-homology as a universal $\delta$-functor and prove that it is pre-dual to our definition of bounded cohomology. We thus answer a question raised by Löh in her thesis. Moreover, we prove Gromov’s theorem and the Matsumoto-Morita conjecture in our context.

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1. INTRODUCTION

Gromov introduced $\ell^1$-homology and bounded cohomology for topological spaces in the late seventies [Gro82]. The initial purpose of these exotic (co-)homology theories was to provide topological invariants which control the minimal volume of a smooth manifold which, by definition, is an invariant of the differentiable structure. One of Gromov’s deeper theorems asserts that the bounded cohomology of a countable and connected CW-complex is an invariant of its fundamental group. In order to make this statement precise, he needed to introduce $\ell^1$-homology and bounded cohomology for discrete groups, which apparently was developed in unpublished work of Trauber.

Matsumoto-Morita raised the question whether the analog of Gromov’s theorem holds true for $\ell^1$-homology [MM85, Remark 2.6]. After some flawed attempts to prove this true, see [Par04] and [Bou04], the question was finally answered affirmatively by Löh [Löh07] and the present author [Büh08] independently.

The variants of $\ell^1$-homology and bounded cohomology for groups were studied by [MM85] and bounded cohomology was given a “functorial approach” by Brooks [Bro81], Ivanov [Iva85, Iva88] and Noskov [Nos90, Nos92], see [Gri95, Gri96] and [Löh07] for further references. The theory was substantially improved and generalized to topological groups by Burger and Monod, see [BM99, Mon01, BM02]. While the Burger-Monod theory proved to be extremely fruitful in the context of...
rigidity theory, the algebraic underpinning remained rather undeveloped. In particular, it was unknown whether bounded cohomology could be interpreted as a derived functor. The main purpose of [Buh08] is to close this gap and to give an interpretation of \( \ell^1 \)-homology and bounded cohomology in the context of modern homological algebra in order to benefit from the power of its proper language, i.e., category theory.

Let us turn to mathematics proper. Let \( \text{Ban} \) be the additive category of Banach spaces and continuous linear maps. It is well-known that \( \text{Ban} \) is quasi-abelian and that there are enough projectives and enough injectives. If \( G \) is a group, we denote the category of isometric representations of \( G \) on Banach spaces and \( G \)-equivariant continuous linear maps by \( G-\text{Ban} \). It is easy to prove that \( G-\text{Ban} \) is quasi-abelian and has enough projectives and enough injectives, hence the formalism of derived categories allows us to derive functors defined on \( G-\text{Ban} \). In order to speak of homology, the theory of \( t \)-structures and their hearts is virtually forced upon us.

For every quasi-abelian category there are two canonical \( t \)-structures, which we call the left and right \( t \)-structures, see Definition 2.5. The left \( t \)-structure on \( \text{D}(\mathcal{A}^{\text{op}}) \equiv (\text{D}(\mathcal{A}))^{\text{op}} \) is dual to the right \( t \)-structure on \( \text{D}(\mathcal{A}) \) in the sense of [BBDS92] 1.3.2 (iii). In particular the heart \( \mathcal{C}_t(\mathcal{A}^{\text{op}}) \) of the left \( t \)-structure on \( \text{D}(\mathcal{A}^{\text{op}}) \) is equivalent to the opposite category of the heart \( \mathcal{C}_r(\mathcal{A}) \) of the right \( t \)-structure on \( \text{D}(\mathcal{A}) \). We write \( H^*_t : \text{D}(\mathcal{A}) \to \mathcal{C}_t(\mathcal{A}) \) and \( H^*_r : \text{D}(\mathcal{A}) \to \mathcal{C}_r(\mathcal{A}) \) for the associated homological functors.

There is the following explicit description of the left heart \( \mathcal{C}_t(\mathcal{A}) \) on \( \text{D}(\mathcal{A}) \): objects are represented by a monic \( (A^{-1} \to A^0) \) in \( \mathcal{A} \) while the morphisms are obtained from the morphisms of pairs by dividing out the homotopy equivalence relation and inverting quasi-isomorphisms (bicartesian squares) formally. By the aforementioned duality, the right heart \( \mathcal{C}_r(\mathcal{A}) \) has a dual description.

There are exact inclusion functors
\[
\iota_t : \mathcal{A} \to \mathcal{C}_t(\mathcal{A}) \quad \text{and} \quad \iota_r : \mathcal{A} \to \mathcal{C}_r(\mathcal{A})
\]
given on objects by \( \iota_t(A) = (0 \to A) \) and \( \iota_r(A) = (A \to 0) \). The functor \( \iota_t \) has a left adjoint \( \iota_t^{-1} \) given on objects by \( \iota_t^{-1}(d : A^{-1} \to A^0) = \text{Coker}_{\mathcal{A}}(A) \). Similarly, \( \iota_r \) has a right adjoint \( \iota_r^{-1} \) induced by the kernel functor in \( \mathcal{A} \).

Let us specialize to the category \( G-\text{Ban} \). The trivial module functor (augmentation) \( (-)^G : \text{Ban} \to G-\text{Ban} \) has a left adjoint given by the co-invariants \( (-)^G \) and a right adjoint given by the invariants \( (-)^G \). Underlying our definition of \( \ell^1 \)-homology and bounded cohomology are the derived functors
\[
\text{L}^-(-)^G : \text{D}^- (G-\text{Ban}) \to \text{D}^- (\text{Ban})
\]
and
\[
\text{R}^+(-)^G : \text{D}^+ (G-\text{Ban}) \to \text{D}^+ (\text{Ban}).
\]

By considering \( G-\text{Ban} \) as the full subcategory of complexes concentrated in degree zero we define for each \( M \in G-\text{Ban} \) the \( \ell^1 \)-homology of \( G \) with coefficients in \( M \) as
\[
\mathcal{H}_{ht}^n(G,M) := H_{ht}^{-n}(\text{L}^-(-)^G(M)) \in \mathcal{C}_r(\text{Ban})
\]
and the bounded cohomology of \( G \) with coefficients in \( M \) as
\[
\mathcal{H}_{hb}^n(G,M) := H_{ht}^n(\text{R}^+(-)^G(M)) \in \mathcal{C}_t(\text{Ban}).
\]

**Theorem.**

(i) The \( \ell^1 \)-homology functors assemble to a universal homological \( \delta \)-functor
\[
\mathcal{H}_{ht}^1(G,-) : G-\text{Ban} \to \mathcal{C}_r(\text{Ban}),
\]

moreover, \( \mathcal{H}_{ht}^1(G,M) = (M_G \to 0) \).
Theorem. The bounded cohomology functors assemble to a universal cohomological \( \delta \)-functor

\[
H^\delta_b(G, -) : G\text{-Ban} \to \mathcal{C}_\ell(G\text{-Ban}),
\]

moreover, \( H^0_b(G, M) \cong (0 \to M^G) \).

Proof. A more precise statement for \( \ell^1 \)-homology is given in Theorem 3.6 and the (entirely dual) statement for bounded cohomology is given in [B"uh08, p.xiv]. \( \square \)

Remark. While it may be perfectly plausible that for duality reasons one should choose to use both the left heart and the right heart for defining \( \ell^1 \)-homology and bounded cohomology, it is natural to wonder whether one could interchange “left” and “right” in the definition. In brief, the answer is “yes, one could, but only at the cost of a reasonable duality theory”. We will discuss this matter in Section 5.

The duality functor on \( \text{Ban} \) which is exact by Hahn-Banach, yields an exact duality functor on \( G\text{-Ban} \) and a duality functor

\[
(-)^* : \mathcal{C}_r(G\text{-Ban}) \to \mathcal{C}_\ell(G\text{-Ban})
\]

which is explicitly given on objects by \((f : A \to B)^* = (f^* : B^* \to A^*)\). We will prove the following result as Proposition 3.7.

Proposition. The duality functor \((-)^* : \mathcal{C}_r(G\text{-Ban}) \to \mathcal{C}_\ell(G\text{-Ban})\) is well-defined, exact and there is a natural isomorphism of functors on \( D(G\text{-Ban}) \)

\[
H^p_r((-)^*) \cong (H^{-n}_r(-))^*.
\]

One principal motivation for our definition is that one cannot interchange “left” and “right” in the previous proposition, see Remark 3.8. Theorem 3.10 is:

Theorem. The duality functor \((-)^* : \mathcal{C}_r(\text{Ban}) \to \mathcal{C}_\ell(\text{Ban})\) yields a natural isomorphism

\[
\left( \mathcal{H}^\ell_n(G, M) \right)^* \cong \mathcal{H}^n_b(G, M^*).
\]

To end this introductory section, we pass from groups to spaces. Following Gromov we associate to a topological space \( X \) its \( \ell^1 \)-singular chain complex \( C_*^\ell(X) \) and its bounded singular cochain complex \( C^\ell_b(X) \), see [B"uh08, p.xxi] for the precise definition. We define \( \ell^1 \)-homology of \( X \) as

\[
\mathcal{H}^n(X) := H^{-n}_r(C^\ell_b(X)) \in \mathcal{C}_r(\text{Ban})
\]

and bounded cohomology as

\[
\mathcal{H}^n_b(X) := H^p_r(C^\ell_b(X)) \in \mathcal{C}_\ell(\text{Ban}).
\]

If \( X \) is a countable and connected CW-complex, let \( G = \pi_1(X) \) be its fundamental group. We proved that \( C_*^\ell(X) \) considered as complex in \( \mathcal{C}_\ell(G\text{-Ban}) \) is a projective resolution of the ground field \( k \), see [B"uh08] p.xxii. Dually, considered as a complex in \( \mathcal{C}_b^\ell(X) \) the bounded cochain complex \( C_*^\ell_b(X) \) is an injective resolution of the ground field. Our proof of these facts relies on one of the main results of Ivanov’s proof of Gromov’s theorem, whence the hypothesis that \( X \) be countable. Since

\[
C^\ell_b(X) \cong (C_*^\ell(X))_G \cong L^- (\cdot)_G(k) \quad \text{and} \quad C^\ell_b(X) \cong (C_*^\ell(X))^G \cong R^+ (\cdot)^G(k)
\]

we obtain the following variant of Gromov’s theorem and the Matsumoto-Morita conjecture:

Theorem. Let \( X \) be a connected and countable CW-complex and let \( G = \pi_1(X) \) be its fundamental group. There are canonical isomorphisms:

\[
\mathcal{H}^\ell_b(X) \cong \mathcal{H}^\ell_b(G, k) \quad \text{and} \quad \mathcal{H}^\ell_b(X) \cong \mathcal{H}^\ell_b(G, k).
\]
Remark. Notice that we deduced the theorem from the fact that the complexes computing $\ell^1$-homology and bounded cohomology are invariants of the fundamental group in the derived category $\mathbf{D}(\mathbf{Ban})$.

Remark. For connected (countable) CW-complexes, Löh introduced $\ell^1$-homology and bounded cohomology with twisted coefficients, see [Löh07, p.27]. Let $M$ be a Banach $G$-module, equip the projective tensor product complex $C^\ell_*(\widetilde{X}) \otimes M$ with the diagonal $G$-action and apply the co-invariants. In other words, she considers

$$C^\ell_*(\widetilde{X}) \otimes_G M \cong k \otimes_G^L M,$$

where the right hand side shows that this complex is an invariant of the fundamental group in $\mathbf{D}(\mathbf{Ban})$. Similarly, for bounded cohomology, she considers the complex

$$\text{Hom}_G(C^\ell_*(\widetilde{X}), M) \cong R^+ \text{Hom}_G(k, M).$$

Using the facts that $k \otimes_G - \cong (-)^G_G$ and $\text{Hom}_G(k, -) \cong (-)^G_G$ as well as the balance of the derived tensor product and derived Hom, we immediately conclude that these complexes compute $H^\ell_*(G, M)$ and $H^*_b(G, M)$.

Remark. The previous remark and our duality theorem constitute a rather trivial universal coefficient theorem for $\ell^1$-homology and bounded cohomology with twisted coefficients of countable and connected CW-complexes—provided that one is willing to accept our definition of $\ell^1$-homology as the correct one.

2. Cohomology in Quasi-Abelian Categories

Let $\mathcal{A}$ be an abelian category and consider a complex

$$A^\bullet = (A' \xrightarrow{f} A \xrightarrow{g} A'')$$

in $\mathcal{A}$, that is, $gf = 0$. Since the compositions $\text{Im } f \hookrightarrow A \twoheadrightarrow A''$ and $A' \twoheadrightarrow A \hookrightarrow \text{Im } g$ are both zero we obtain a commutative diagram

$$
\begin{array}{cccccc}
\text{Im } f & & \phi & & \text{Ker } g \\
\downarrow & & \downarrow \varphi & & \downarrow \\
A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' \\
\downarrow & & \downarrow \psi & & \downarrow \psi \\
\text{Coker } f & \xrightarrow{\psi} & \text{Ker } g & \xrightarrow{\phi} & \text{Im } g \\
\end{array}
$$

and the (co)homology of $A^\bullet$ is defined to be any one of the isomorphic objects

$$H(A^\bullet) \cong \text{Coker } \phi \cong \text{Ker } \psi \cong \text{Im } u,$$

where $u$ is the morphism $\text{Ker } g \to \text{Coker } f$, see e.g. [KS06, p.178].

Recall the notion of a quasi-abelian category in the sense of Yoneda [Yon60] (see also Prosmans [Pro00] and Schneiders [Sch99]): an additive category $\mathcal{A}$ is called quasi-abelian if

(i) every morphism has a kernel and a cokernel,

(ii) the class of all kernel-cokernel pairs in $\mathcal{A}$ is an exact structure in the sense of Quillen [Qui73]: every kernel is the kernel of its cokernel, the class of kernels is closed under composition and push-outs along arbitrary morphisms and, dually, every cokernel is the cokernel of its kernel, the class of cokernels is closed under composition and pull-backs along arbitrary morphisms.
If $\mathcal{A}$ is quasi-abelian but not abelian, the situation is no longer as straightforward as before. Assume for simplicity that $\mathcal{A}$ has enough projectives and enough injectives. We obtain the diagram

```
\begin{array}{c}
\text{Coker } f \\
\downarrow \psi \\
\text{Im } g
\end{array}
\quad
\begin{array}{c}
\text{Coim } f \\
\downarrow \phi \\
\text{Ker } g
\end{array}
\quad
\begin{array}{c}
A' \\
\downarrow f \\
A \\
\downarrow g \\
A''
\end{array}
```

in which the dotted arrows are categorical monics or epics (here we use that there are enough projectives and enough injectives) that may or may not be kernels or cokernels.

**Remark 2.1** (Huber). The morphism $u : \text{Ker } g \to \text{Coker } f$ is strict in the sense that it factors as $\text{Ker } g \to X \to \text{Coker } f$, so that $X \cong \text{Coim } u \cong \text{Im } u$.

Since $f$ factors over $\text{Im } f$ and $gf = 0$, the morphism $\text{Im } f \to A$ factors over $\text{Ker } g \to A$. The morphism $v : \text{Im } f \to \text{Ker } g$ is an admissible monic by Quillen’s “obscure axiom”, see [Kel90, A.1, c)](op). Let $X = \text{Coker } v$ and form the following push-out diagram

```
\begin{array}{c}
\text{Ker } u \\
\downarrow \\
A \\
\downarrow \\
Y
\end{array}
```

which by [Kel90, A.1, 1st step] is bicartesian. It is easy to see that $A \to Y$ is the cokernel of $\text{Im } f \to A$ so that $Y \cong \text{Coker } f$ (it is a general fact that in an exact category the push-out of an admissible epic along an admissible monic yields an admissible epic). From this diagram one readily reads off that

$$\text{Ker } u \cong \text{Im } f \quad \text{and} \quad \text{Coker } u \cong \text{Coim } g,$$

so $X \cong \text{Coim } u \cong \text{Im } u$ as claimed.

**Remark 2.2.** The object $X$ constructed in the previous remark is at the same time the cokernel of $\text{Im } f \to \text{Ker } g$ and the kernel of $\text{Coker } f \to \text{Coim } g$. If the quasi-abelian category $\mathcal{A}$ is such that for each morphism $h$ the morphism $\text{Coim } h \to \text{Im } h$ is categorically monic and epic then it follows that $\text{Coker } \varphi \cong X \cong \text{Ker } \psi$. This is the case if $\mathcal{A}$ has enough projective and enough injective objects, however, the author does not know whether this is true in general.

**Example 2.3.** Let $\mathcal{A} = \text{Ban}$ be the category of Banach spaces and consider the complex

$$\ell^1 \left[ \begin{array}{c} i \\ 0 \end{array} \right] \to \ell^1 \left[ \begin{array}{c} 0 \\ i \end{array} \right] \to c_0$$

where $i : \ell^1 \to c_0$ is the obvious inclusion. We have

$$\text{Coom } [i] = \ell^1, \quad \text{Ker } [0 \ i] = c_0, \quad \text{Coker } [i] = \ell^1, \quad \text{Im } [0 \ i] = c_0,$$

which shows that the dotted morphisms are indeed not kernels or cokernels in general.

By the theory of $t$-structures, both $\varphi$ and $\psi$ yield legitimate notions of cohomology: $\varphi$ represents $H^0_\ell(A^\bullet)$ in the left heart $\mathcal{C}_\ell(\mathcal{A})$ and $\psi$ represents $H^0_r(A^\bullet)$ in the right heart $\mathcal{C}_r(\mathcal{A})$ of the derived category $D(\mathcal{A})$ if $A^\bullet$ is considered as a complex concentrated in degrees $-1, 0, 1$. To be more specific, we need two definitions.
Definition 2.4. Let $A^\bullet = (\cdots \to A^{-2} \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \to \cdots)$ be a complex in the quasi-abelian category $\mathcal{A}$. The left truncation functors are defined by
\[ \tau^\leq_\ell A^\bullet = (\cdots \to A^{-2} \xrightarrow{d^{-2}} A^{-1} \to \text{Ker}d^0 \to 0 \to 0 \to \cdots) \]
and
\[ \tau^\geq_\ell A^\bullet = (\cdots \to 0 \to \text{Coim}d^{-1} \to A^0 \xrightarrow{d^0} A^1 \to \cdots) \]
while the right truncation functors are given by
\[ \tau^\leq_r A^\bullet = (\cdots \to A^{-1} \xrightarrow{d^{-1}} A^0 \to \text{Im}d^0 \to 0 \to \cdots) \]
and
\[ \tau^\geq_r A^\bullet = (\cdots \to 0 \to \text{Coker}d^{-1} \to A^1 \xrightarrow{d^1} A^2 \to \cdots). \]
The truncation functors yield endofunctors of the derived category $D(\mathcal{A})$. As usual, we put for $n \in \mathbb{Z}$
\[ \tau^\leq_n = \Sigma^{-n} \circ \tau^\leq_\ell \circ \Sigma^n, \]
etc.

Definition 2.5. Denote by $D^{\leq}_\ell (\mathcal{A})$ the essential image of $\tau^\leq_\ell$, etc. It is not difficult to prove that $(D^{\leq}_\ell (\mathcal{A}), D^{\geq}_\ell (\mathcal{A}))$ is a $t$-structure, see [BBDS82, 1.3.1, 1.3.22], which we call the left $t$-structure. By duality $(D^{\leq}_r (\mathcal{A}), D^{\geq}_r (\mathcal{A}))$ is a $t$-structure as well and we call it the right $t$-structure. The corresponding (left and right) hearts are
\[ \mathcal{C}_\ell (\mathcal{A}) = D^{\leq}_\ell (\mathcal{A}) \cap D^{\geq}_\ell (\mathcal{A}) \quad \text{and} \quad \mathcal{C}_r (\mathcal{A}) = D^{\leq}_r (\mathcal{A}) \cap D^{\geq}_r (\mathcal{A}), \]
they are admissible abelian subcategories of $D(\mathcal{A})$. The associated homological functors are $H^0_\ell = \tau^\leq_\ell \tau^\geq_\ell : D(\mathcal{A}) \to \mathcal{C}_\ell (\mathcal{A})$ and $H^0_r = \tau^\leq_r \tau^\geq_r$.

There is the following explicit description of $\mathcal{C}_\ell (\mathcal{A})$: objects are represented by a monic $(A^{-1} \hookrightarrow A^0)$ in $\mathcal{A}$ while the morphisms are obtained from the morphisms of pairs by dividing out the homotopy equivalence relation and inverting quasi-isomorphisms (bicartesian squares) formally, see [BBDS82, 1.3.22], [Lau83, 1.5.7] or [Buh08, Construction 2.2.1, p.35].

Proposition 2.6. The inclusion functor $\iota_\ell : \mathcal{A} \to \mathcal{C}_\ell (\mathcal{A})$ given on objects by $A \mapsto (0 \leftarrow A)$ preserves monics, is fully faithful, exact and reflects exactness. Its image is closed under extensions in $\mathcal{C}_\ell (\mathcal{A})$. It has a left adjoint $q_\ell$ given on objects by $\text{Coker}(A^{-1} \hookrightarrow A^0)$. Every exact and monic-preserving functor $\mathcal{A} \to \mathcal{B}$ to an abelian category factors uniquely over an exact functor $\mathcal{C}_\ell (\mathcal{A}) \to \mathcal{B}$.

Proof. This is all well-known, see e.g. [Buh08, Chapter III.2]. \hfill \Box

3. $\ell^1$-HOMOLOGY AND BOUNDED COHOMOLOGY

Let $G$ be a group and let $G-\text{Ban}$ be the category of isometric representations of $G$ on Banach spaces and $G$-equivariant bounded linear maps. It is a simple consequence of the open mapping theorem that $G-\text{Ban}$ is quasi-abelian.

Notation 3.1. Let $\ell^1(G)$ be the Banach group algebra and let $E$ be a Banach space. The induced Banach $G$-module is
\[ \uparrow E = \ell^1(G) \hat{\otimes} E \cong \ell^1(G, E) \]
with the left $G$-action on the factor $\ell^1(G)$. The coinduced Banach $G$-module is
\[ \uparrow E := \text{Hom}_{\text{Ban}}(\ell^1(G), E) \cong \ell^\infty(G, E) \]
with the action coming from the right action of $G$ on $\ell^1(G)$. 

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**Notation 3.2.** Let $M \in G\text{-Ban}$ be a Banach $G$-module. The *module of coinvariants* of $M$ is the Banach space

$$M_G = M / \text{span}\{m - gm : m \in M, g \in G\}$$

and the *module of invariants* is the Banach space

$$M^G = \{m \in M : gm = m \text{ for all } g \in G\}.$$

At the heart of the homological algebra of $\ell^1$-homology and bounded cohomology is the following simple result which is proved by direct inspection:

**Theorem 3.3 (Fundamental Adjunctions [Büh08, p.xviii]).** Let $\downarrow: G\text{-Ban} \to \text{Ban}$ be the forgetful functor and let $\varepsilon(-): \text{Ban} \to G\text{-Ban}$ be the trivial module functor. There are two adjoint triples of functors

$$\begin{array}{ccc}
G\text{-Ban} & \downarrow & G\text{-Ban} \\
\epsilon(-) & \| & \epsilon(-)^G
\end{array}$$

and

$$\begin{array}{ccc}
\text{Ban} & \downarrow & \text{Ban} \\
(-) & \| & (\varepsilon(-))^G
\end{array}$$

that is to say $\downarrow$ is left adjoint to $\downarrow$ and $\downarrow$ is left adjoint to $\uparrow$, etc.

The forgetful functor, induction, coinduction are all exact as well as the trivial module functor. □

The most important consequence for the present work is:

**Corollary 3.4 ([Büh08, p.xviii]).** There are enough projectives and enough injectives in $G\text{-Ban}$. □

This allows us to consider the derived functors

$$L^-(-)^G: D^- (G\text{-Ban}) \to D^- (\text{Ban})$$

and

$$R^+(\varepsilon-)^G: D^+ (G\text{-Ban}) \to D^+ (\text{Ban})$$

which underlie $\ell^1$-homology and bounded cohomology.

**Definition 3.5.** Let $M \in G\text{-Ban}$. We define $\ell^1$-homology as

$$\mathcal{H}^\ell_n (G,M) := H_{-n} (L^-(-)^G(M))$$

and bounded cohomology as

$$\mathcal{H}^b_n (G,M) := H^p (R^+(\varepsilon-)^G(M)).$$

**Theorem 3.6.** Up to unique isomorphism of $\delta$-functors there is a unique family of functors

$$\mathcal{H}^\ell_n (G,-): G\text{-Ban} \to \mathcal{C}_r (\text{Ban}), \quad n \in \mathbb{Z},$$

having the following properties:

(i) (Normalization) $\mathcal{H}^\ell_n (G,0) = (M_G \to 0)$ for all $M \in G\text{-Ban}$.

(ii) (Vanishing) $\mathcal{H}^\ell_n (G,P) = 0$ for all projective objects $P \in G\text{-Ban}$ and all $n > 0$.

(iii) (Long exact sequence) Associated to each short exact sequence $0 \to M' \to M \to M'' \to 0$ in $G\text{-Ban}$ there are morphisms $\delta_{n+1}: \mathcal{H}^\ell_n (G,M'') \to \mathcal{H}^\ell_n (G,M')$ depending naturally on the sequence and fitting into a long exact sequence

$$\cdots \to \mathcal{H}^\ell_n (G,M') \to \mathcal{H}^\ell_n (G,M) \to \mathcal{H}^\ell_n (G,M'') \xrightarrow{\delta} \mathcal{H}^\ell_{n-1} (G,M') \to \cdots$$

in $\mathcal{C}_r (\text{Ban})$. 

ON THE PROOF. This follows from dualizing the proof of the theorem on page xiv of [Bühl08]. Notice that $(-)_G$ and $\iota_G : \mathcal{B}an \to \mathcal{C}_r(\mathcal{B}an)$, $E \mapsto (E \to 0)$ both have a right adjoint. An existence proof is also given in Section 4. □

Now consider the duality functor $(-)^* : \mathcal{C}l(-) \to \mathcal{C}l(-)$ and recall that it is exact, hence it extends to the derived category $D(\mathcal{G}^{-\text{Ban}})$. It induces a (contravariant) duality functor

$$(-)^* : \mathcal{C}_r(\mathcal{G}^{-\text{Ban}}) \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$$

which is explicitly given on objects by $(e : A \to B)^* = (e^* : B^* \to A^*)$.

**Proposition 3.7.** The duality functor $(-)^* : \mathcal{C}_r(\mathcal{G}^{-\text{Ban}}) \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$ is well-defined, exact and there is a natural isomorphism of functors on $D(\mathcal{G}^{-\text{Ban}})$

$$H^n_r((-)^*) \cong (H^{-n}_r(-))^*.$$

**Proof.** First, the duality functor $\mathcal{C}_r(\mathcal{G}^{-\text{Ban}}) \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$ is well-defined since the duality functor on $\mathcal{G}^{-\text{Ban}}$

(i) maps epics (morphisms with dense range) to monics (injective morphisms) by [Rud91, 4.12, Corollaries (b), p.99],

(ii) preserves the homotopy equivalence relation since it is additive,

(iii) preserves bicartesian squares because it is exact.

Let us prove that the duality functor $\mathcal{C}_r(\mathcal{G}^{-\text{Ban}}) \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$ is exact. Points (i) and (iii) yield that the duality functor $(-)^* : \mathcal{G}^{-\text{Ban}} \to \mathcal{G}^{-\text{Ban}}$ is exact and preserves monics. The same holds true for $\iota_G : \mathcal{G}^{-\text{Ban}} \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$, hence also for the composition $F = \iota_G \circ (-)^*$. By [Bühl08, 2.2.3, p.37] the universal property of the inclusion functor $\iota_G : \mathcal{G}^{-\text{Ban}} \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$ yields a unique exact prolongation $\tilde{F} : \mathcal{C}_r(\mathcal{G}^{-\text{Ban}}) \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$. The construction of $\tilde{F}$ given in [Bühl08, p.40] together with [Bühl08, 2.2.8, p.39] yield that

$$\tilde{F}(f : A \to B) = (f^* : B^* \to A^*).$$

so that $\tilde{F}$ coincides with the above description of the duality functor under the equivalence $\mathcal{C}_r(\mathcal{G}^{-\text{Ban}}) \cong \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$.

In order to see that there is a natural isomorphism $H^n_r((-)^*) \cong (H^{-n}_r(-))^*$, it suffices to notice that for a morphism $f$ of $\mathcal{G}^{-\text{Ban}}$ there are natural isomorphisms

$$(\text{Coker } f)^* \cong \text{Ker } (f^*) \quad \text{and} \quad (\text{Im } f)^* \cong \text{Coim } (f^*),$$

which is a straightforward consequence of [Rud91, 4.12, Theorem, p.99]. □

**Remark 3.8.** The dual of a monic in $\mathcal{G}^{-\text{Ban}}$ is not in general an epic, the range is weak*-dense by [Rud91, 4.12, Corollaries, (c), p.99] but not necessarily norm-dense: consider for instance the inclusion $\ell^1 \hookrightarrow c_0$ whose dual is the inclusion $\ell^1 \hookrightarrow \ell^\infty$ the range of which is clearly not norm-dense. It follows in particular that there is no duality functor $\mathcal{C}_r(\mathcal{G}^{-\text{Ban}}) \to \mathcal{C}_r(\mathcal{G}^{-\text{Ban}})$ as constructed above.

In a similar vein, $\text{(Coim } f)^*$ does not in general coincide with $\text{Im } (f^*)$ but rather with its weak*-closure and $\text{(Ker } f)^*$ is isomorphic to the codomain modulo the weak*-closure of $\text{Im } (f^*)$, hence it may be distinct from $\text{Coker } (f^*)$.

Recall the main properties of the duality functor on $\mathcal{G}^{-\text{Ban}}$:

**Proposition 3.9 (Bühl08, p.65).** The duality functor $(-)^* : \mathcal{G}^{-\text{Ban}} \to \mathcal{G}^{-\text{Ban}}$ is exact, reflects exactness and sends projective objects to injective objects. Moreover, there is a natural isomorphism $(-)^* \circ (-)_G \cong (-)_G \circ (-)^*$. □
Theorem 3.10. The duality functor $(-)^* : \mathcal{C}_\ell(Ban) \to \mathcal{C}_\ell(Ban)$ yields a natural isomorphism

$$\left( \mathcal{H}_n^\ell(G,M) \right)^* \cong \mathcal{H}_b^n(G,M^*).$$

Proof. To compute $\mathcal{H}_n^\ell(G,M)$ choose a projective resolution $P_\bullet \to M$, apply the coinvants $(-)^*_G$ to $P_\bullet$ and then the right cohomology functor $H^\bullet_n$ to the resulting complex. Now the two previous propositions give natural isomorphisms

$$H^\bullet_n((P_\bullet)_G)^* \cong H^\bullet_n(((P_\bullet)_G)^*) \cong H^\bullet_n((P_\bullet)^*_G)$$

and it remains to notice that $M^* \to (P_\bullet)^*$ is an injective resolution of $M^*$, so that the right hand side computes bounded cohomology in degree $n$. □

4. Canonical Resolutions

Using the canonical resolution associated to the induction comonad we give a relatively elementary proof of the existence of the $\ell^1$-homology functors as described in Theorem 3.6. In the next section we will make use of this construction in order to relate our theory to the classical one.

Recall the fundamental adjunction of induction $! = \ell^1(G) \otimes - : Ban \to BanG$ to the forgetful functor $! : BanG \to Ban$, see Theorem 3.3. The latter functor is obviously exact while the former is exact since $\ell^1(G)$ is projective and hence flat as a Banach space. Every adjoint pair of functors gives rise to a comonad and a monad, see [Wei94] 8.6, 8.7, as follows:

Let $L : \mathcal{A} \to \mathcal{B} : R$ be an adjoint pair and let $\varepsilon : LR \Rightarrow id_\mathcal{A}$ and $\eta : id_\mathcal{A} \Rightarrow RL$ be the adjunction morphisms. Write $\perp = LR$ and $\top = RL$, as well as $\delta_B = L(\eta_{RB})$ and $\mu_A = R(\varepsilon_{LA})$, it is then a simple fact that $(\perp, \varepsilon, \mu)$ is a comonad and $(\top, \eta, \delta)$ is a monad, see [Wei94] 8.6.2. The simplicial object associated to the comonad $\perp$ is described in [Wei94] 8.6.4, it gives rise to a simplicial resolution $\perp, B \to B$, where $\perp_n B := (\perp)^{n+1} B$.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are additive. By taking the alternating sum of the face maps one obtains a complex which we still denote by $\perp, B$, and it yields the canonical resolution $\perp, B \to B$. This parlance is justified since it is well-known and easy to check [Wei94] 8.6.8, 8.6.10 that $R(\perp, B) \to R(B)$ as well as $\perp, L(A) \to L(A)$ are chain homotopy equivalences for all $B \in \mathcal{B}$ and all $A \in \mathcal{A}$.

We apply this to the induction comonad $\perp = ! !$ and obtain in particular for each $M \in BanG$ the canonical resolution

$$\perp, M \to M,$$

which has the property that for all $M \in G-Ban$ and all $E \in Ban$ the complexes

$$\perp (\cdots \to \perp_1 M \to \perp_0 M \to M)$$

and

$$\cdots \to \perp_1 E \to \perp_0 E \to E$$

are split exact in $Ban$ and $G-Ban$, respectively.

Since $\perp$ is exact, we obtain for each short exact sequence $M' \to M \to M''$ a short exact sequence of complexes

$$\perp, M' \to \perp, M \to \perp, M''$$

in $\text{Ch}^{\leq 0}(G-Ban)$. Writing temporarily $\perp_{-1} = id_{G-Ban}$ we have for all $n \geq 0$

$$(\perp_n M)_G \cong \ell^1(G) \otimes_G \perp_{n-1} M \cong \perp_{n-1} M,$$

so we get a short exact sequence of complexes in $\text{Ch}^{\leq 0}(Ban)$

$$(\perp_n M')_G \to (\perp_n M)_G \to (\perp_n M'')_G.$$
Since the inclusion functor \( \iota_r : \text{Ban} \to \mathcal{C}_r(\text{Ban}) \) is exact, the snake lemma provides us with a long exact sequence
\[
\cdots \to H^n_r(\iota_r(\bot_{\ast}M')) \to H^n_r(\iota_r(\bot_{\ast}M)) \to H^n_r(\iota_r(\bot_{\ast}M'')) \to H^{n+1}_r(\iota_r(\bot_{\ast}M')) \to \cdots
\]
which is obviously natural in the short exact sequence \( M' \to M \to M'' \) so that we have constructed a \( \delta \)-functor.

Because the complexes involved are concentrated in non-positive degrees and because \( \iota_r \) and \( (-)_G \) are left adjoints and hence commute with taking cokernels, we have that
\[
H^0_r(\iota_r(\bot_{\ast}M)) = \text{Coker} (\iota_r(\bot_{\ast}M \to \bot_{0}M)_G))
\]
\[
\cong \iota_r \circ (-)_G \circ \text{Coker} (\bot_{\ast}M \to \bot_{0}M)
\]
\[
\cong (M_G \to 0).
\]

For each Banach space \( E \) the sequence
\[
\cdots \to \bot_{\ast}E \to \bot_{0}E \to \uparrow E
\]
is split exact, so the map
\[
(\bot_{\ast}E)_G \to (\uparrow E)_G \cong E
\]
is a quasi-isomorphism and hence the cohomology of the complex \( \iota_r(\bot_{\ast}E)_G \) vanishes outside degree zero. Finally, the morphism \( \lfloor \varepsilon_M : \bot_{\ast}M \to \uparrow M \) is a split epic for each \( M \in G-\text{Ban} \), hence \( \bot_{\ast}M \to \uparrow M \) is an admissible epic and it follows that every projective \( P \in G-\text{Ban} \) is a direct summand of \( \bot P = \uparrow \uparrow P \). Consequently, our \( \delta \)-functor vanishes on projectives outside degree zero and we conclude from Theorem 4.6 that:

**Theorem 4.1.** There is a canonical isomorphism \( \mathcal{H}^{\ell^1} (G, -) \cong H^{-\ast}_r(\iota_r(\bot_{\ast}(-))_G) \).

**Remark 4.2.** The complex \( \bot_{\ast}M \) is of course nothing but the bar resolution as given e.g. in [Loh07, (2.13), p.20], Call a Banach \( G \)-module induced if it is of the form \( \uparrow E \) for some \( E \in \text{Ban} \). By [Wei94, 8.6.7, Exercise 8.6.3] the direct summands of induced modules are precisely the \( \bot \)-projective objects, or, equivalently, the projective objects with respect to the exact structure \( \mathcal{C}_r(\text{Ban}) \) consisting of short sequences \( \sigma \) such that \( \lfloor \sigma \rfloor \) is split exact. This notion is closely related to relative projectivity as defined in [Loh07, (A.1), p.104] but it is somewhat less restrictive.

In particular we have shown:

**Corollary 4.3.** Every \( \bot \)-projective object is \( \mathcal{H}^{\ell^1} (G, -) \)-acyclic. \( \square \)

**Remark 4.4.** The acyclicity of \( \bot \)-projective objects implies by dimension-shifting that one may compute \( \ell^1 \)-cohomology with coefficients in \( M \) using any resolution \( P_{\bullet} \to M \) with \( \bot \)-projective components. Requiring that \( \to \) is more than just a quasi-isomorphism (e.g., a strong resolution) is only necessary if one is concerned with ensuring that the resolution can be used to compute the canonical semi-norms.

**Remark 4.5.** The construction given here shows in particular that \( \ell^1 \)-homology is the derived functor of the induction comonad with coefficient functor \( \iota_r \) in the sense of Barr and Beck, see e.g. [Wei94, 8.7.1].

**Remark 4.6.** Putting \( \top = \uparrow \uparrow \) we obtain the coinduction monad which we will not discuss further because the arguments given in this section are straightforward to dualize.
5. Remarks on Our Definition of $\ell^1$-Homology

Our first and main motivation for our definition of $\ell^1$-homology is purely utilitarian in nature: we want to have a smooth duality between $\ell^1$-homology and bounded cohomology in order to save a lot of work.

Second, we want to show that no information concerning semi-norms is lost: For this we need to describe the classical $\ell^1$-homology as defined e.g. in [Löh07]. An object of $\mathcal{C}_\ell(\text{Ban})$ can be considered as a morphism of the category $\text{Csn}$ of complete seminormed spaces and continuous linear maps. Taking the cokernel in $\text{Csn}$ gives a realization functor $\text{real}: \mathcal{C}_\ell(\text{Ban}) \to \text{Csn}$ which is exact in the sense that it transforms exact sequences to sequences in $\text{Csn}$ whose underlying sequence of vector spaces is exact, see e.g. [Büh08, p.xv, Lemma]. It is thus easy to see that $\ell^1$-homology as defined e.g. in [Löh07] coincides with

$$H^1_\ell(G, M) = \text{real} H^n_\ell(\mathcal{L}, M)$$

Notice that we use the left homology functor $H^1_\ell$ instead of the right one. We have

$$H^0_\ell(G, M) \cong \text{real} H^n_\ell(\mathcal{L}^*, M) \cong \mathcal{H}^0_\ell(G, M).$$

The complications involved in the development of a reasonable duality between the two classical theories is discussed at length in [Löh07, Chapter 3].

Recall that the inclusion functor $\iota_\ell: \text{Ban} \to \mathcal{C}_\ell(\text{Ban})$ has a left adjoint $q_\ell$ defined on objects by taking the cokernel in $\text{Ban}$, see Proposition 2.6. Dually, the inclusion functor $\iota_r$ has a right adjoint given by taking the kernel in $\text{Ban}$. Remark 2.2 implies that there is a natural isomorphism $q_\ell H^n_\ell \cong q_r H^n_\ell$ on the derived category $D(\text{Ban})$. From this all we deduce easily:

**Theorem 5.1.** The functor $q_\ell \circ \mathcal{H}^1_\ell(G, -)$ coincides with Hausdorffification of classical $\ell^1$-homology $H^1_\ell(G, -)$. Similarly, $q_\ell \circ \mathcal{H}^0_\ell(G, -)$ coincides with the Hausdorffification of classical bounded cohomology $H^0_\ell(G, -).$ \hfill $\Box$

**Remark 5.2.** The main interest of the theorem is of course that it shows that as far as semi-norms are concerned one may as well work with our version of the $\ell^1$-homology functors since Hausdorffification only consists of quotienting out the space of vectors of semi-norm zero in $H^1_\ell(G, M)$.

**Remark 5.3.** It is important to notice that Hausdorffification as well as $q_r$ both fail to be “exact”, so that the Hausdorffified long exact sequence of $\ell^1$-homology and bounded cohomology is no longer exact in general.

**Remark 5.4.** The dual of the kernel of $f: M \to N$ in $\text{Ban}$ is *not* the cokernel of the dual map $f^*: N^* \to M^*$ in general but the quotient of $M^*$ by the weak*-closure of the range of $f^*$. So there is only a natural quotient map

$$q_\ell \circ \mathcal{H}^0_\ell(G, M^*) \to (q_r \circ \mathcal{H}^1_\ell(G, M))^*$$

as is well-known in the classical context, see e.g. [MMS5, p.540]. This map is of course not an isomorphism in general.

**References**

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR751966 (86g:32015)

[BM99] M. Burger and N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. (JEMS) 1 (1999), no. 2, 199–235. MR1694584 (2000d:57058a)

[BM02] _____, *Continuous bounded cohomology and applications to rigidity theory*, Geom. Funct. Anal. 12 (2002), no. 2, 219–290. MR1911660 (2003d:53065a)
[Bou04] Abdesselam Bouarich, *Théorèmes de Zilber-Eilenberg [sic!] et de Brown en homologie*, Proyecciones 23 (2004), no. 2, 151–186. MR2142264 (2006h:55008)

[Bro81] Robert Brooks, *Some remarks on bounded cohomology*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978) (Princeton, N.J.), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, pp. 53–63. MR624804 (83a:57038)

[Büh08] Theo Bühler, *On the algebraic foundation of bounded cohomology*, Ph.D. thesis, ETH Zürich, 2008.

[Gri95] R. I. Grigorchuk, *Some results on bounded cohomology*, Combinatorial and geometric group theory (Edinburgh, 1993), London Math. Soc. Lecture Note Ser., vol. 204, Cambridge Univ. Press, Cambridge, 1995, pp. 111–163. MR1320279 (96j:20073)

[Gri96] , *Bounded cohomology of group constructions*, Mat. Zametki 59 (1996), no. 4, 546–550. MR1445197 (98f:20037)

[Gro82] Michael Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982), no. 56, 5–99 (1983). MR686042 (84h:53053)

[Iva85] N. V. Ivanov, *Foundations of the theory of bounded cohomology*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 143 (1985), 69–109, 177–178. MR806562 (87b:53070)

[Iva88] , *The second bounded cohomology group*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988), no. Issled. Topol. 6, 117–120, 191. MR964260 (90a:55015)

[Kel90] Bernhard Keller, *Chain complexes and stable categories*, Manuscripta Math. 67 (1990), no. 4, 379–417. MR1052551 (91h:18006)

[KS06] Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006. MR2182076 (2006k:18023)

[Lau83] G. Laumon, *Sur la catégorie dérivée des D-modules filtrés*, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 151–237. MR726427 (85d:32022)

[Löh07] Clara Löh, $\ell^1$-Homology and Simplicial Volume, Ph.D. thesis, Westfälische Wilhelms-Universität Münster, 2007.

[MM85] Shigenori Matsumoto and Shigeyuki Morita, *Bounded cohomology of certain groups of homeomorphisms*, Proc. Amer. Math. Soc. 94 (1985), no. 3, 539–544. MR787909 (87e:57006)

[Mon01] Nicolas Monod, *Continuous bounded cohomology of locally compact groups*, Lecture Notes in Mathematics, vol. 1758, Springer-Verlag, Berlin, 2001. MR1840942 (2002i:46121)

[Nos90] G. A. Noskov, *Bounded cohomology of discrete groups with coefficients*, Algebra i Analiz 2 (1990), no. 5, 146–164. MR1086449 (92b:57014)

[Nos92] , *The Hochschild-Serre spectral sequence for bounded cohomology*, Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989) (Providence, RI), Contemp. Math., vol. 131, Amer. Math. Soc., 1992, pp. 613–629. MR1175809 (93g:20096)

[Par04] HeeSook Park, *Foundations of the theory of $l_1$ homology*, J. Korean Math. Soc. 41 (2004), no. 4, 591–615. MR2068142 (2005c:55011)

[Pro00] Fabienne Prosmans, *Derived categories for functional analysis*, Publ. Res. Inst. Math. Sci. 36 (2000), no. 1, 19–83. MR1749013 (2001g:46156)

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR0338129 (49 #2895)

[Rud91] Walter Rudin, *Functional analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991. MR1157815 (92k:46001)

[Sch99] Jean-Pierre Schneiders, *Quasi-abelian categories and sheaves*, Mém. Soc. Math. Fr. (N.S.) (1999), no. 76, vi+134. MR1779315 (2001i:18023)

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001)

[Yon60] Nobuo Yoneda, *On Ext and exact sequences*, J. Fac. Sci. Univ. Tokyo Sect. I 8 (1960), 507–576 (1960). MR0225854 (37 #1445)