COEFFICIENTS OF THE INVERSE OF FUNCTIONS FOR THE SUBCLASS OF THE CLASS $U(\lambda)$

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Abstract. Let $A$ be the class of functions $f$ that are analytic in the unit disk $D$ and normalized such that $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Let $0 < \lambda \leq 1$ and

$$U(\lambda) = \left\{ f \in A : \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, z \in D \right\}.$$

In this paper sharp upper bounds of the first three coefficients of the inverse function $f^{-1}$ are given in the case when $f(z) = 1 - z(1 - \lambda z)$.

Let $A$ denote the family of all analytic functions in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let $S$ denote the subclass of $A$ which consists of univalent functions in $D$ and let $U(\lambda)$, $0 < \lambda \leq 1$, denote the set of all $f \in A$ satisfying the condition

$$\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda \quad (z \in D).$$

For $\lambda = 1$ we put $U(1) = U$. More about these classes can be found in [5, 6, 7, 8, 10].

In [7] it was claimed that all functions $f$ from $U(\lambda)$ satisfy

$$\frac{f(z)}{z} < \frac{1}{(1 + z)(1 + \lambda z)}.$$  

Here "\(<" denotes the usual subordination, i.e., $F(z) < G(z)$, for $f$ and $G$ being analytic functions in $D$, means that there exists function $\omega(z)$, also analytic in $D$, such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in D$. Recently, in [3], the author gave a counterexample that subordination (2) is not necessarily satisfied by all functions from $U(\lambda)$.

For the functions $f$ from $U(\lambda)$ satisfying subordination (2) we have

$$\frac{f(z)}{z} = \frac{1}{(1 - \omega(z))(1 - \lambda \omega(z))},$$

where $\omega$ is a Schwarz function, i.e., it is analytic in $D$, $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in D$. Let’s put

$$\omega(z) = c_1 z + c_2 z^2 + \cdots.$$  

Later on we will use the fact due to Schur [9] that $|c_2| \leq 1 - |c_1|^2$ (can be found also in Carlson’s work [1]).

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Further, the inequality (1) for the function \( f \) from \( \mathcal{U}(\lambda) \) can be rewritten in the following, equivalent, form
\[
\left| \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' - 1 \right| < \lambda \quad (z \in \mathbb{D})
\]
and further
\[
\left| \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' - 1 \right| \leq \lambda |z|^2 \quad (z \in \mathbb{D}).
\]
From here, after some calculations we obtain
\[
|(1 + \lambda)c_2 - \lambda c_1^2 + (2(1 + \lambda)c_3 - 4\lambda c_1 c_2)z + \cdots| \leq \lambda
\]
for all \( z \in \mathbb{D} \), and next,
\[
(4) \quad |(1 + \lambda)c_2 - \lambda c_1^2| \leq \lambda, \quad |2(1 + \lambda)c_3 - 4\lambda c_1 c_2| \leq \lambda - \frac{1}{\lambda} |(1 + \lambda)c_2 - \lambda c_1^2|^2,
\]
for all \( z \in \mathbb{D} \). The last inequality follows from the result of Carlson for the second coefficient of Schwarz functions cited above.

If \( f \in \mathcal{S} \) and
\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,
\]
then the inverse of \( f \) has an expansion
\[
f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots
\]
near the origin (or precisely at least in \( |w| < \frac{1}{4} \)). By using the identity \( f(f^{-1}) = w \) and the representations for the functions \( f \) and \( f^{-1} \), we can obtain the next relations
\[
(7) \quad A_2 = -a_2, \quad A_3 = -a_3 + 2a_2^2, \quad A_4 = -a_4 + 5a_2 a_3 - 5a_3^2.
\]

The main result of this paper are the sharp upper bounds for the modulus of these three initial coefficients of \( f^{-1} \).

**Theorem 1.** Let \( f \in \mathcal{U}(\lambda) \), \( 0 < \lambda \leq 1 \), satisfies subordination (2), and let \( f \) and \( f^{-1} \) be given by (5) and (6), respectively. Then
\[
|A_2| \leq 1 + \lambda, \quad |A_3| \leq 1 + 3\lambda + \lambda^2, \quad |A_4| \leq (1 + \lambda)(1 + 5\lambda + \lambda^2).
\]
All these results are best possible.

**Proof.** For \( f \in \mathcal{U}(\lambda) \), from the relation (8) we have (see [2], [7])
\[
\sum_{n=1}^{\infty} a_{n+1} z^n = \sum_{n=1}^{\infty} \frac{1 - \lambda^{n+1}}{1 - \lambda} \omega^n(z).
\]
If we put \( \omega(z) = c_1 z + c_2 z^2 + \cdots \), then from (8) by comparing the coefficients we obtain
\[
\begin{align*}
a_2 &= (1 + \lambda)c_1, \\
a_3 &= (1 + \lambda)c_2 + (1 + \lambda + \lambda^2)c_1^2, \\
a_4 &= (1 + \lambda)c_3 + 2(1 + \lambda + \lambda^2)c_1 c_2 + (1 + \lambda + \lambda^2 + \lambda^3)c_1^3.
\end{align*}
\]
Using (7) and (9) we also have
\[
\begin{align*}
A_2 &= -(1 + \lambda)c_1, \\
A_3 &= -(1 + \lambda)c_2 + (1 + 3\lambda + \lambda^2)c_2^2, \\
A_4 &= -(1 + \lambda)c_3 + (3 + 8\lambda + 3\lambda^2)c_1c_2 - (1 + \lambda)(1 + 5\lambda + \lambda^2)c_1^3.
\end{align*}
\]
(10)

Since \(|c_1| \leq 1\) and \(|c_2| \leq 1 - |c_1|^2\), from (10) we receive
\[|A_2| \leq 1 + \lambda\]
and
\[
|A_3| \leq (1 + \lambda)|c_2| + (1 + 3\lambda + \lambda^2)|c_1|^2 \\
\leq (1 + \lambda)(1 - |c_1|^2) + (1 + 3\lambda + \lambda^2)|c_1|^2 \\
\leq (1 + \lambda) + (2\lambda + \lambda^2)|c_1|^2 \\
\leq 1 + 3\lambda + \lambda^2.
\]

Also, from (10) we obtain
\[A_4 = -\frac{1}{2} \left[ 2(1 + \lambda)c_3 - 4\lambda c_1c_2 - 6(1 + \lambda)c_1((1 + \lambda)c_2 - \lambda c_1^2) + 2(1 + \lambda)^3 c_1^3 \right],\]
and from here, by applying (4),
\[
|A_4| \leq \frac{1}{2} \left[ 2(1 + \lambda)c_3 - 4\lambda c_1c_2 + 6(1 + \lambda)c_1((1 + \lambda)c_2 - \lambda c_1^2) + 2(1 + \lambda)^3 |c_1|^3 \right] \\
\leq \frac{1}{2} \left[ \lambda - \frac{1}{\lambda} |(1 + \lambda)c_2 - \lambda c_1^2| + 6(1 + \lambda)c_1||(1 + \lambda)c_2 - \lambda c_1^2| + 2(1 + \lambda)^3 |c_1|^3 \right] \\
= \frac{1}{2} \left[ \lambda - \frac{1}{\lambda} t^2 + 6(1 + \lambda)|c_1|t + 2(1 + \lambda)^3 |c_1|^3 \right] \\
= \frac{1}{2} t h(t),
\]
where \(t = |(1 + \lambda)c_2 - \lambda c_1^2|\) and \(0 \leq t \leq \lambda\), since
\[(1 + \lambda)c_2 - \lambda c_1^2 \leq (1 + \lambda)|c_2| + \lambda|c_1|^2 \leq (1 + \lambda)(1 - |c_1|^2) + \lambda |c_1|^2 = \lambda.\]

As for the maximal value of function \(h\), we consider two cases:

Case 1: When \(0 \leq |c_1| \leq \frac{1}{3(1 + \lambda)}\) the function \(h\) attains its maximum for \(t_0 = 3(1 + \lambda)|c_1|\) and we have
\[h(t_0) \leq \lambda + 27\lambda(1 + \lambda)^2 |c_1|^2 + 2(1 + \lambda)^3 |c_1|^3 \leq 4\lambda + \frac{2}{27},\]
i.e.,
\[|A_4| \leq 2\lambda + \frac{1}{27}.\]

Case 2: For \(\frac{1}{3(1 + \lambda)} \leq |c_1| \leq 1\), the function \(h\) attains its maximum for \(t = \lambda\) and we have
\[h(t) \leq 6(1 + \lambda)\lambda |c_1| + 2(1 + \lambda)^3 |c_1|^3 \leq 2(1 + \lambda)(1 + 5\lambda + \lambda^2),\]
when \(0 \leq t \leq \lambda\). So,
\[|A_4| \leq (1 + \lambda)(1 + 5\lambda + \lambda^2).\]

From cases 1 and 2, since \((1 + \lambda)(1 + 5\lambda + \lambda^2) > 2\lambda + \frac{1}{27}\) when \(0 < \lambda \leq 1\), we receive the estimate for \(|A_4|\).
For the proof of sharpness of the theorem, let consider the function

\[ w = f_\lambda(z) = \frac{z}{(1-z)(1-\lambda z)}. \]

Then

\[ (11) \quad z = f_\lambda^{-1}(w) = w - (1 + \lambda)w^2 + (1 + 3\lambda + \lambda^2)w^3 - (1 + \lambda)(1 + 5\lambda + \lambda^2)w^4 - \cdots, \]

which shows that our results are the best possible. \(\square\)

Note that for \(\lambda = 1\) in Theorem 1 we have the estimates for class \(U\) and in that case the inverse of the Koebe function is extremal, as for the class \(S\) (see, for example Goodman’s book, Vol II, p.205, [2]).

In the next theorem we study the Fekete-Szegő functional for the inverse functions of the class \(U(\lambda)\). Namely, we have

**Theorem 2.** For the inverse functions of functions from \(U(\lambda), 0 < \lambda \leq 1,\) satisfying subordination (2), we have

\[ |A_3 - \mu A_2^2| \leq \lambda + |1 - \mu|(1 + \lambda)^2, \]

where \(\mu\) is complex number. The result is sharp for \(0 \leq \mu \leq 1\).

**Proof.** From the relations (10) and (4) we obtain

\[ |A_3 - \mu A_2^2| = |-(1 + \lambda)c_2 + (1 + 3\lambda + \lambda^2)c_1^2 - \mu(1 + \lambda)^2c_1^2| \]

\[ = |-(1 + \lambda)c_2 - \lambda c_1^2| + |1 - \mu|(1 + \lambda)^2c_1^2| \]

\[ \leq |(1 + \lambda)c_2 - \lambda c_1^2| + |1 - \mu|(1 + \lambda)^2|c_1^2| \]

\[ \leq \lambda + |1 - \mu|(1 + \lambda)^2. \]

The sharpness of the estimate in the case when \(0 \leq \mu \leq 1\) follows from the function \(f_\lambda^{-1}\) defined by (11). \(\square\)

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**Conflict of Interest**

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