The $\mathcal{N} = 2$ cascade revisited
and the enhançon bearings

Francesco Benini$^{1,2}$, Matteo Bertolini$^2$, Cyril Closset$^3$
and Stefano Cremonesi$^{2,4}$

$^1$ Department of Physics, Princeton University
Princeton, NJ 08544, USA
$^2$ SISSA and INFN - Sezione di Trieste
Via Beirut 2; I 34014 Trieste, Italy
$^3$ Physique Théorique et Mathématique and International Solvay Institutes
Université Libre de Bruxelles, C.P. 231, 1050 Bruxelles, Belgium
$^4$ Raymond and Beverly Sackler Faculty of Exact Sciences, School of Physics and Astronomy
Tel-Aviv University, Ramat-Aviv 69978, Israel

Abstract

Supergravity backgrounds with varying fluxes generated by fractional branes at non-isolated Calabi-Yau singularities had escaped a precise dual field theory interpretation so far. In the present work, considering the prototypical example of such models, the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold, we propose a solution for this problem, and show that the known cascading solution corresponds to a vacuum on the Coulomb branch of the corresponding quiver gauge theory involving a sequence of strong coupling transitions reminiscent of the baryonic root of $\mathcal{N} = 2$ SQCD. We also find a slight modification of this cascading vacuum which upon mass deformation is expected to flow to the Klebanov-Strassler cascade. Finally, we discuss an infinite class of vacua on the Coulomb branch whose RG flows include infinitely coupled conformal regimes, and explain their gravitational manifestation in terms of new geometric structures that we dub enhançon bearings. Repulson-free backgrounds dual to all the vacua we analyse are explicitly provided.
Contents

1 Introduction and summary 1

2 D3 branes on the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold and a cascading solution 5

3 The enhançon and the Seiberg-Witten curve 10

4 The cascading vacuum in field theory 16
   4.1 One cascade step: $\mathcal{N} = 2$ SQCD 18
   4.2 The cascading vacuum in the quiver gauge theory 19
   4.3 The infinite cascade limit 22
   4.4 Mass deformation 24

5 More supergravity duals: enhançon bearings 28
   5.1 Reconstructing the cascading vacuum at the baryonic roots 35
   5.2 More bearings: the enhançon plasma 36

6 Excisions, warp factors and the cure of repulson singularities 39

7 Conclusions and outlook 44

A Effective field theory approach to the cascading SW curve 46

1 Introduction and summary

Supergravity solutions with running fluxes are ubiquitous in non-conformal versions of the gauge/gravity correspondence. In fact, they occur whenever fractional branes are present. The decrease of such fluxes as a function of the holographic coordinate is believed to correspond to a reduction in the number of degrees of freedom of the dual gauge theory as the latter flows towards the IR.

The most widely known example is the famous Klebanov-Tseytlin-Strassler model, arising from fractional branes at a conifold singularity $[1, 2, 3, 4]$. In such a context the dual gauge theory interpretation of running fluxes is in terms of a renormalization group (RG) flow described by a cascade of Seiberg dualities occurring at subsequent strong coupling scales and lowering the rank of the strongly coupled gauge group (see
for a review). This flow takes place along the baryonic branch of the gauge theory. The low energy dynamics involves confinement and chiral symmetry breaking, which geometrically translate into a complex structure deformation of the singularity. This behavior is prototypical of any isolated singularity admitting complex structure deformation.

By placing fractional branes at isolated singularities with obstructed complex structure deformation \cite{6,7,8} one obtains theories whose RG flow is expected to be similarly described by a cascade of Seiberg dualities, but where the geometric obstruction translates into a runaway along a baryonic direction \cite{9}.

The case of fractional branes at non-isolated singularities, which involves twisted sector fields propagating along the complex line singularity, was less understood so far. The simplest such example, which is a $\mathcal{N} = 2$ model obtained considering fractional branes at a $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold (also known as $A_1$ singularity) \cite{10,11}, has been interpreted in various ways in the literature \cite{11,12,13}. Consideration of probe fractional branes in the supergravity solutions \cite{11} and recent methods based on the computation of Page charges \cite{14,15} suggest that the RG flow of the dual theories involves strong coupling transitions where the rank of the non-abelian factor in a gauge group with an adjoint chiral superfield drops according to the same numerology as in Seiberg duality, leading to a cascade. Since Seiberg-like dualities do not hold in this case, such strong coupling transitions cry for an explanation. It is worth stressing that such a phenomenon is not specific to $\mathcal{N} = 2$ models, but instead appears quite generically in any $\mathcal{N} = 1$ setup admitting non-isolated singularities together with isolated ones: the RG flow, as read from the gravity solution, is described by suitable combinations of Seiberg duality cascades and $\mathcal{N} = 2$-like transitions \cite{15}. Therefore, clarifying which field theory dynamics governs these transitions is instrumental to understanding how string theory UV-completes field theories arising on systems of fractional branes at rather generic CY singularities.

To that aim, in this paper we reconsider the cascading solution describing regular and fractional D3 branes at the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold, as a prototype of the more general class of branes at non-isolated singularities, and provide a solution for this problem. Our proposal elaborates on previous ones \cite{11,12}, and solves a number of problems raised there. The dual gauge theory is a $SU(N + M) \times SU(N)$ $\mathcal{N} = 2$ quiver with bifundamental matter, where $N$ is the number of regular branes and $M$ the number of fractional ones, and its dual supergravity solution is known \cite{10}. The structure of such a gauge theory has many similarities with the conifold one, and the two are indeed
related by a $\mathcal{N} = 1$-preserving mass deformation [16]. In order to provide a precise interpretation of the cascading RG flow, we start approximating the dynamics around scales where one of the two gauge coupling diverges with an effective $\mathcal{N} = 2$ SQCD, treating the other group as global. This allows us to claim that the transition occurs at the baryonic root (i.e., the point of the quantum moduli space of $\mathcal{N} = 2$ SQCD where the baryonic branch meets the Coulomb branch), where the strongly coupled $SU(N + M)$ group is effectively broken to $SU(N - M)$ (plus abelian factors). As in the $\mathcal{N} = 1$ conifold model, this is an iterative process which has the effect of lowering the effective ranks of the two gauge groups as the energy decreases, in a way which is exactly matched by the dual supergravity solution. On the other hand, the power of the Seiberg-Witten (SW) curve technology allows us to check our claim exactly, in the full quiver theory.

Models arising from branes at non-isolated singularities have the distinctive property of having, besides a Higgs branch, also a Coulomb branch. This allows for a rather mundane UV completion of the cascading quiver theory, starting with the conformal $SU(N + M) \times SU(N + M)$ theory engineered by $N + M$ D3 branes at the orbifold singularity, and Higgsing it at some scale $z_0$ [11]. This stops the cascade in the UV as the theory is in a superconformal phase at energies higher than $z_0$ (notice that such a simple SCFT completion is not possible for the $\mathcal{N} = 1$ conifold model; see [17] for alternative ways to UV-complete the $\mathcal{N} = 1$ cascade with a SCFT). We first discuss the case where the cutoff is at finite energy: by means of the relevant Seiberg-Witten curves [18, 19], we provide a detailed analysis of several vacua on the Coulomb branch, together with the corresponding supergravity duals. For vacua at the origin of the Coulomb branch, there is in fact no cascade at all [13], while we show that the smaller is the number of adjoints fields having vanishing VEV, the larger is the number of steps in the cascade.

We then consider the case where the cutoff is sent to infinity, corresponding to the infinite cascade limit. This setup is the one which makes contact with the conifold cascade, as the two are expected to be related by a mass deformation. Actually, only specific vacua of the $\mathcal{N} = 2$ theory survive such a mass deformation [20], and we provide the corresponding SW curve, with a parametrically high level of accuracy. To find the supergravity solution interpolating from the $\mathcal{N} = 2$ to the $\mathcal{N} = 1$ cascade is left to future research.

Our analysis also allows us to provide a description of an infinite class of new vacua along the Coulomb branch, where the RG flow alternates energy ranges where the
theory runs, and others where the theory is in a superconformal phase. The borders between these subsequent regions are described by enhançon-like rings and we naturally dub the corresponding geometric structures enhançon bearings. We provide the corresponding supergravity duals and show, both from the gauge theory and supergravity points of view, how such vacua interpolate between the non-cascading and the cascading vacua.

The original supergravity solution of [10], which is the building block for all supergravity duals along the Coulomb branch that we analyse, presents an unphysical repulsive region around the origin. Another distinctive property of $\mathcal{N} = 2$ models is the peculiar way in which such a singularity is cured. Models with $\mathcal{N} = 2$ supersymmetry are not confining, and the resolution of the IR singularity is associated to the enhançon mechanism [21] which excises the unphysical region giving back a singularity-free solution. The scale at which the excision occurs depends on the dual gauge theory vacuum one is studying [12, 13], and therefore the excised solutions will differ for different vacua. We work out the enhançon mechanism for all gauge theory vacua mentioned above, computing explicitly the warp factors of the excised solutions. It is worth noticing that the way the enhançon mechanism works here is qualitatively different from the original one discussed in [21], since in the present case the enhançon shell is not of real codimension one, i.e. it is not a domain wall: the modification of the solution corresponds to an actual excision for the twisted fields but not for the untwisted ones, most notably the metric and the RR 5-form field strength. In turn, the corrected warp factor and 5-form depend on the excised configuration of twisted fields and fractional branes dual to the field theory vacuum under consideration. We find that around the origin the metric is free of singularities and the new solutions we find perfectly match, within the supergravity approximation, the dual gauge theory expectations.

The paper is organized as follows. In section 2 we briefly recall the $\mathcal{N} = 2$ quiver gauge theory at the $A_1$ singularity, the structure of its moduli space and that of the known supergravity duals, both for the conformal and non-conformal models. In section 3 we recall how the non-perturbative dynamics of the model can be studied through Seiberg-Witten curves, and review the enhançon mechanism. Section 4, which includes the main result of this work, is devoted to the analysis of the cascading vacua, while in section 5 we discuss the new class of vacua characterized by the presence of subsequent enhançon bearings. Finally, in section 6 we work out the excision procedure and the corresponding warp factors for all the gauge theory vacua previously discussed. Conclusions, outlook and an appendix follow.
Figure 1: Quiver diagram of the $U(N)_L \times U(N)_R$ $\mathcal{N} = 2$ theory, in $\mathcal{N} = 1$ notation. Nodes correspond to gauge factors, arrows connecting different nodes represent bifundamental chiral superfields while arrows going from one node to itself represent adjoint chiral superfields.

2 D3 branes on the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold and a cascading solution

The low energy theory on $N$ D3 branes placed at the origin of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold is a four-dimensional $U(N) \times U(N)$ $\mathcal{N} = 2$ gauge theory with two bifundamental hypermultiplets. The field content is summarized in the quiver diagram of figure 1. The beta functions of both $SU(N)$ factors vanish, the diagonal $U(1)$ is decoupled, while the anti-diagonal $U(1)$ becomes free in the IR and gives rise to a global symmetry, the baryonic symmetry $U(1)_B$.

The classical moduli space agrees precisely with the possible configurations of regular and fractional D3 branes on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. In terms of $\mathcal{N} = 1$ superfields, the tree level superpotential (dictated by $\mathcal{N} = 2$ supersymmetry) reads

$$W = (B_1 \Phi A_1 - B_2 \Phi A_2) - (A_1 \tilde{\Phi} B_1 - A_2 \tilde{\Phi} B_2),$$

where contractions over gauge indices are implied. The corresponding F-term equations are

$$\Phi A_i - A_i \tilde{\Phi} = 0 \quad B_i \Phi - \tilde{\Phi} B_i = 0 \quad A_1 B_1 - A_2 B_2 = B_1 A_1 - B_2 A_2 = 0 .$$

The holomorphic gauge invariant operators, which descend to local coordinates on the moduli space, are given by traces of products of the operators $A_i B_j \equiv \varphi_{ij}$ and $\Phi$ for the first gauge group, and $B_i A_j \equiv \bar{\varphi}_{ij}$ and $\tilde{\Phi}$ for the second one.

The moduli space consists of several branches. First we have the so-called Higgs branches, where the hypermultiplets obtain vacuum expectation values (VEV’s). These VEV’s result in the Higgsing of the quiver to a subgroup of the diagonal $U(N)$ gauge group, and the theory has an accidental $\mathcal{N} = 4$ supersymmetry in the IR. The Higgs
branch has \((\mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_2)^N / S_N\) geometry, corresponding to the displacement of regular D3 branes in the full transverse space, up to permutations. Because of \(\mathcal{N} = 2\) supersymmetry, the Kähler metric on the Higgs branch is protected against any quantum corrections. Next we have the Coulomb branch, on which the hypermultiplet VEV’s vanish while the VEV’s for the two adjoint scalars can take arbitrary values: at a generic point on this branch, the surviving gauge group is \(U(1)^{2N}\). The Coulomb branch has the form \(\mathbb{C}^N / S_N \times \mathbb{C}^N / S_N\), which corresponds to the displacement of the two types of fractional D3 branes, each of them associated to one gauge factor, along the orbifold singularity line. The quantum corrected metric on the Coulomb branch is exactly calculable thanks to Seiberg-Witten theory \[22\]. Finally, there are mixed branches, where some hypermultiplet VEV’s and some adjoint VEV’s are turned on.

In the large \(N\) and large \( \text{‘t Hooft} \) coupling limit, the low energy superconformal \(SU(N) \times SU(N)\) sector is better described by its type IIB supergravity dual \[23\]. The full Higgs branch is dual to a family of supergravity solutions corresponding to D3 branes at arbitrary positions on the 6-dimensional transverse space,

\[
d s^2 = Z^{-1/2} \eta_{\mu\nu} d x^\mu d x^\nu + Z^{1/2} \delta_{mn} d x^m d x^n 
\]

\[
g_s F_5 = (1 + \ast) dv_{3,1} \wedge d Z^{-1} ,
\]

where \(\mu, \nu = 0, \ldots, 3, m, n = 4, \ldots, 9\) and the orbifold identification \(x = (x^m) \simeq (\tilde{x}) \equiv (x^{4,5}, -x^{6,7,8,9})\) is understood. \(Z\) is a harmonic function of \(x\),

\[
Z = 4\pi g_s \alpha'^2 \sum_{j=1}^{N} \left( \frac{1}{|x - x_j|^4} + \frac{1}{|x - \tilde{x}_j|^4} \right).
\]

The function contains the D3 branes and their images. Notice that the total 5-form flux on \(S^5 / \mathbb{Z}_2\) at infinity is \(N\). The relation between the parameters \(x_j\) and the field theory moduli is \(x_j = 2\pi \alpha' \phi_j\), where \(\phi_j\) is an eigenvalue of the VEV of some field. \(\Phi\) and \(\tilde{\Phi}\) are mapped to \(x^4 + i x^5\), while \(\varphi_{ij}\) are mapped to algebraic coordinates \(z_{ij}\) on \(\mathbb{C}^2 / \mathbb{Z}_2\), such that \(z_{12} z_{21} - z_{11}^2 = 0\) and \(z_{22} = z_{11}\). The supergravity axio-dilaton \(\tau = C_0 + i e^{-\Phi} = C_0 + \frac{i}{g_s}\) is constant \(^1\) as D3 branes do not couple to it. It is related to the field theory gauge couplings and theta angles by

\[
\tau = \tau_1 + \tau_2 \quad \text{where} \quad \tau_j = \frac{\theta_j}{2\pi} + \frac{4\pi i}{g_j^2}, \quad j = 1, 2 .
\]

In the following we will take \(\tau = i / g_s\) unless otherwise stated.

\(^1\)We work in the string frame. Here \(\Phi\) is the full dilaton, which is constant in all the solutions under consideration, not to be confused with one of the adjoint chiral superfields. From now on we will rather use \(g_s = e^\Phi\).
As noticed in [12], for a generic point on the Higgs branch (and more generally on any branch), the supergravity solution has large curvature. However, configurations where all the branes are in big clumps have a good supergravity description, and configurations where only a small number of branes are isolated are well described by probe branes in the background generated by the other branes.

The Coulomb branch of our $\mathcal{N} = 2$ quiver is described by fractional D3 branes along the orbifold singularity. In this case supergravity solutions include a non-trivial profile for the twisted field fluxes. Indeed, fractional D3 branes source magnetically the twisted scalar $c$ and by supersymmetry they also source its NSNS partner, the twisted scalar $b$. This can be easily understood recalling [24] that fractional D3 branes are D5 branes wrapped on the exceptional 2-cycle $C$ which lives at the orbifold singularity. The twisted scalars are simply the reduction of the RR and NSNS 2-form potentials, $C_2$ and $B_2$, on $C$. They can be organized in a complex field as

$$\gamma \equiv c + \tau b = c + \frac{i}{g_s} b = \frac{1}{4\pi^2\alpha'} \int_C \left( C_2 + \frac{i}{g_s} B_2 \right), \quad (2.6)$$

while

$$G_3 = F_3 + \frac{i}{g_s} H_3 = 4\pi^2\alpha' d\gamma \wedge \omega_2 \quad (2.7)$$

is the complexified 3-form field strength, where $\omega_2$ is a closed anti-selfdual $(1,1)$-form with delta-function support at the orbifold plane, normalized as $\int_C \omega_2 = 1$. Regular D3 branes do not couple to the twisted sector, hence the profile of $\gamma$ is affected solely by fractional branes. The complex twisted scalar $\gamma$ is then subject to a two-dimensional Laplace equation in $\mathbb{C}$ with sources at the positions of the fractional branes. Supersymmetric solutions [25] have primitive, imaginary self-dual and $(2,1)$ $G_3$ flux, which implies that $\gamma = \gamma(z)$ is a meromorphic function of $z = x^4 + ix^5$, such that $d\gamma(z)$ has simple poles at the locations of sources. For a bunch of $N$ fractional and $N$ anti-fractional branes at positions $z_j$ and $\tilde{z}_j$, respectively, we have

$$\gamma = \frac{i}{\pi} \left[ \sum_{j=1}^N \log(z - z_j) - \sum_{j=1}^N \log(z - \tilde{z}_j) \right] + \gamma^{(0)} . \quad (2.8)$$

Here $\gamma^{(0)}$ is an integration constant: its imaginary part sets the value of $b$ at large $|z|$ or in the theory at the origin of the moduli space, while the real part does not really have a physical meaning in the dual theory because of the presence of the axial anomaly,

---

With some abuse of language, following [13] we call ‘anti-fractional branes’ D5 branes wrapped on $C$ with the opposite orientation, with some worldvolume flux through $C$ in order to preserve the same supercharges as the fractional branes.
and we will set it to zero. The positions of the fractional branes \( z_j \) and \( \tilde{z}_j \) are classically identified with the eigenvalues \( \Phi_j, \tilde{\Phi}_j \) of the field theory adjoint scalars. Corrections to this identification arise at quantum level and will be discussed in the next section.

The holographic relations between the Yang-Mills couplings and theta angles and the supergravity fields are

\[
\tau_1 + \tau_2 = \tau \quad \quad \tau_1 - \tau_2 = 2\gamma - \tau = 2\left[c + \tau\left(b - \frac{1}{2}\right)\right],
\]

but we will often set \( \tau = i/g_s \). In particular, when \( b = 0 \) the imaginary part of \( \tau_1 \) vanishes and \( g_1 \) diverges, whereas for \( b = 1 \) it is \( g_2 \) which diverges. What we face in such cases is obviously a peculiar field theory, a SCFT with one divergent gauge coupling, in which instanton corrections dominate even in the large \( N \) limit \([26]\), and about which not much is known. Although from the Seiberg-Witten curve analysis one does not expect extra massless fields in general, the supergravity description is a very incomplete description for this phase. When \( c \in \mathbb{Z} \) as well, extra massless states do appear, and the theory enters a tensionless string phase, as originally suggested in \([27]\) from consistency of \( T \)-duality with type IIA string theory.

So far, we have only discussed the superconformal \( SU(N) \times SU(N) \) theory\(^3\) which has a well behaved UV limit and whose stringy realization through AdS/CFT is unambiguous. However, what we are really interested in is the non-conformal \( SU(N+M) \times SU(N) \) gauge theory. This can be easily obtained through Higgsing from the superconformal \( SU(N+M) \times SU(N+M) \) theory, which can be engineered placing \( N+M \) regular D3 branes at the origin of the orbifold: taking \( M \) VEV’s of the second adjoint scalar to be at a scale \( |z_0|/2\pi\alpha' \) produces an effective \( SU(N+M) \times SU(N) \times U(1)^M \) theory below \( |z_0| \) where the \( U(1) \) factors are IR free and decouple. In the dual picture, this corresponds to placing \( M \) anti-fractional branes at, say, the roots of \( \tilde{z}_M = -z_0^M \), while the other \( N \) anti-fractional branes and \( N+M \) fractional branes sit classically at the origin. The twisted scalar in this configuration is then

\[
\gamma = \frac{i}{\pi} \log \frac{z^M}{z^M + z_0^M} + \gamma^{(0)}.
\]

\(^3\)Actually \( b \in [0, 1] \) is the only range of validity of the formulas, because otherwise one would have negative square couplings. As noticed in \([4]\) and extensively discussed in \([14, 15]\), when \( b \) is outside this range one has to perform a large gauge transformation to shift it to the interval where (2.9) can be applied.

\(^4\)From now on, we will often consciously forget the additional \( U(1) \times U(1) \) factor which decouples at low energies.

\(^5\)In the following, when speaking about scales we will often omit the \( 2\pi\alpha' \) factor.
For the sake of simplicity, unless differently specified, in the following we will set the orbifold point value \( \gamma^{(0)} = \frac{i}{2g_s} \), so that in the UV \( \tau_1 = \tau_2 = \frac{i}{2g_s} \). In the large \( M \) limit in which we work, (2.10) can be traded for its limiting behavior

\[
\gamma = \begin{cases} 
\frac{i}{M} \log \frac{z}{z_0} + \frac{i}{2g_s} & \text{if } |z| < |z_0| \\
\gamma^{(0)} & \text{if } |z| > |z_0|
\end{cases}
\]  

(2.11)

where we set \( z_1 = e^{\frac{i\pi}{M}\gamma^{(0)}} z_0 = e^{-\frac{i\pi}{2g_s M} z_0} \). Note that the twisted fluxes break the \( U(1) \) isometry corresponding to rotation in the \( z \)-plane to a discrete subgroup \( \mathbb{Z}_{2M} \). This is dual to the breaking of the \( U(1) \) R-symmetry because of anomalies in the gauge theory \[30\].

The gauge invariant D3 brane charge (Maxwell charge) carried by the fluxes of the solution is proportional to the 5-form flux; it is found by integrating the Bianchi identity in the absence of sources \( dF_5 = -H_3 \wedge F_3 \) on the angular \( S^5/\mathbb{Z}_2 \) of radius \( r \) and reads, for \( r < \rho_0 = |z_0| \),

\[
-\frac{1}{(4\pi^2\alpha')^2} \int F_5 = N + \frac{g_s M^2}{\pi} \log \frac{r}{\rho_1}
\]  

(2.12)

with \( \rho_1 = |z_1| \).

We see from eqs. (2.11)-(2.12) that, similarly to the Klebanov-Tseytlin (KT) solution \[3\], the solution enjoys logarithmically varying B field and 5-form flux below the cutoff: this naturally suggests that the dual field theory might enjoy a cascading RG flow with subsequent infinite coupling transitions reducing the rank of the infinitely coupled non-abelian gauge group by 2 \( M \) at scales \( \rho_k = e^{-\frac{(2k-1)\pi}{2g_s M}} \rho_0 \), \( k = 1, \ldots, l \), where \( l \equiv \lfloor N/M \rfloor \). This will be dealt with in section 4, where the \( N = 2 \) cascading nature of the solution will be discussed in great detail.

Before attacking this problem, though, we have to deal with another phenomenon, which always arises in supergravity solutions dual to non-conformal supersymmetric gauge theories with eight supercharges. By analyzing the explicit form of the warp factor, it was shown in \[10\] that the ten-dimensional metric obtained using (2.10), besides the obvious singularity on the orbifold fixed plane, displays an unphysical repulsive region near the origin, at a scale of order \( e^{-\frac{\pi N}{g_s M^2} / \rho_1} \). One expects that,
as suggested in [10], an enhançon-like mechanism [21] might be at work here, which excises the unphysical region rendering back a repulsion-free solution. We will show that this is indeed the case, discussing in the next section the specific way in which the enhançon mechanism manifests in this context, and providing in section 6 an excised and singularity-free solution.

3 The enhançon and the Seiberg-Witten curve

The quantum corrections to the Coulomb branch constrain the (anti)fractional D3 brane positions, \( z_j \) and \( \tilde{z}_j \), in the gravity dual. The full quantum corrected moduli space is exactly encoded in the full family of Seiberg-Witten (SW) curves [18, 19]. The SW curves for the \( \mathcal{N} = 2 \) superconformal field theory at hand were found in [22]. At the classical level, the fractional brane positions \( z_j \) and \( \tilde{z}_j \) correspond to the eigenvalues of the VEV’s of the adjoint scalars \( \Phi \) and \( \tilde{\Phi} \). In the quantum theory this identification cannot survive because the VEV’s parametrize the moduli space and are unconstrained, whereas fractional brane positions are constrained. That is, in the large \( N \) limit one expects [12] quantum corrections and the consequent constraints on \( z_j \) and \( \tilde{z}_j \) to be bound, because of supersymmetry, to a non-negative 5-form flux (that means non-negative enclosed D3-charge) for all allowed configurations on the quantum moduli space, at least whenever the supergravity approximation is valid. This property is in fact at the core of the enhançon mechanism.

Let us detail this point by first considering a simplified example. Consider the theory discussed previously with \( N = 0 \): this is an \( SU(M) \times SU(M) \) superconformal theory which can be engineered by \( M \) regular D3 branes. Below the UV scale \( |z| = |z_0| \), the theory is effectively Higgsed to \( SU(M) \) \( \mathcal{N} = 2 \) pure SYM (plus IR free \( U(1) \) factors). The dual supergravity solution is the one in (2.11)-(2.12) with \( N = 0 \), and it corresponds to the \( M \) fractional branes classically at the origin. The quantum moduli space can be studied with a good approximation by means of the SW curves for \( SU(M) \) [32, 33]

\[
y^2 = \prod_{a=1}^M (v - \phi_a)^2 + 4\Lambda^{2M},
\]

where \( \Lambda \) is the strong coupling scale of \( \mathcal{N} = 2 SU(M) \) SYM and \( \phi_a \) are the eigenvalues of the adjoint scalar \( \Phi \) parametrizing a family of hyperelliptic curves in \( \mathbb{C}^2 = \{(v, y)\} \).

The curves could also be written in terms of gauge invariant symmetric polynomials. Classically (\( \Lambda = 0 \)) the eigenvalues \( \phi_a \) coincide with the double branch points of (3.1).
and correspond to the fractional brane positions on the $z$ plane in the gravity description. An elegant way to see this is the following: type IIB string theory on the orbifold is T-dual to type IIA on a circle (with coordinate $x^6$) with two parallel NS5 branes along $x^0,\cdots,x^5$, separated in the compact direction $x^6$ (see [34] for a review). Fractional D3 branes are T-dual to D4 branes stretched along $x^6$ between the two NS5’s. The classical Coulomb branch is then given by all the possible configurations of D4 branes on the plane $v = x^4 + ix^5$. The system can be further uplifted to M-theory, where the NS5’s and the D4’s are just part of a single M5 brane. The M5 brane seen as a Riemann surface is identified with the SW curve [22]. At the quantum level, the eigenvalues $\phi_a$ still parametrize the whole moduli space (up to Weyl gauge identifications), but they no longer correspond to double branch points nor fractional brane positions, strictly speaking. In the perturbative regime of the theory, $|\phi_a| \gg |\Lambda|$, the branch points still appear in pairs close to $\phi_a$: in the M-theory picture the D4 branes are inflated into small tubes. As soon as the VEV’s get into the non-perturbative region (at scales comparable with $\Lambda$), the branch points get well separated and it does not make much sense to talk about fractional brane positions anymore.

At the origin of the moduli space ($\Phi = 0$), the hyperelliptic curve (3.1) becomes $y^2 = v^{2M} + 4\Lambda^{2M}$, which has $2M$ separate branch points at $v^{2M} = -4\Lambda^{2M}$. In the large $M$ limit, the branch points densely fill a ring of radius $2^{1/M}|\Lambda|$. It is also possible to see that, adding a probe fractional brane (in field theory terms, consider the $SU(M+1)$ theory with one additional VEV $\phi$), in which case the SW curves are

$$y^2 = v^{2M}(v - \phi)^2 + 4\Lambda^{2M+2},$$

the probe can freely move in the semi-classical region outside the ring, but it cannot penetrate it. For $|\phi| \gg |\Lambda|$, the two extra branch points are placed near $\phi$, with a small separation of order $\Lambda(\Lambda/\phi)^M$, while the other $2M$ branch points are still on the ring. As $|\phi|$ approaches $|\Lambda|$ and then goes to zero, the branch points split and melt into the ring.

As anticipated, the dual string theory picture of this is the famous enhançon mechanism [21]. The tension of BPS fractional D3 branes is equal to their gauge invariant Maxwell D3-charge, which is $\gamma$

$$T_{nf} = \frac{\mu_3}{g_s} |g_s \Im \gamma + n_f| = \frac{\mu_3}{g_s} |b + n_f|,$$

where $n_f$ is the number of units of worldvolume flux on the exceptional 2-cycle $C$ (notice that neither $b$ or $n_f$ are gauge invariant, while their sum is). This turns out
to be proportional to the perturbative moduli space metric on the Coulomb branch of the $SU(M) \mathcal{N} = 2$ pure SYM theory\footnote{There is a matching with the perturbative result because in the large $M$ limit instanton corrections are strongly suppressed, and abruptly show up at the scale $\Lambda$ \cite{26}.}. At the scale $|\Lambda| = \rho_1$, $b$ vanishes and fractional D3 branes, which are wrapped D5 branes with no worldvolume flux, become tensionless; below that scale they would be non-supersymmetric and they would feel a repulsive potential. Notice also that the enclosed D3 brane charge would become negative for smaller scales, which could hardly be the case if fractional D3 branes were at the origin. Moreover, a massive particle probe would experience an unphysical gravitational repulsion close to the origin. The resolution of this puzzle is that fractional branes cannot be brought all at the same place, but rather melt into a thin ring of radius $\rho_1$: the *enhançon ring*. This changes the twisted fields distribution in the geometry: inside the ring, $b = 0$ (more generally it is integer), $c$ is constant, and there is no D3 brane charge. The warp factor needs to be re-computed using the correct configuration of fractional branes and twisted field, and the result is that the suspicious repulsive region disappears, as will be shown in section 6.

In some sense, the whole region defined by $b = 0$ (more generally $b \in \mathbb{Z}$) behaves like a conductor: D5 charges (recall that the D3 charge vanishes along with the tension inside the enhançon) are pushed to the boundary and there is no field inside. We will call such a region the enhançon plasma. We already noticed in section 2 that the IR field theory dual to the interior region is quite peculiar: it is a conformal $SU(N) \times SU(N)$ theory with one divergent gauge coupling. However, in this particular case $N = 0$ and the dynamics is trivial inside the enhançon plasma: $SU(M)$ is simply broken by instantons to $U(1)^{M-1}$.

As discussed in \cite{13}, exactly the same kind of behavior can be found in the most generic situation, i.e. when $N \neq 0$ and the theory has product gauge group $SU(N + M) \times SU(N)$. Since the second gauge group is not asymptotically free, one should embed the theory into the $SU(N + M) \times SU(N + M)$ conformal one, properly Higgsed, as sketched at the end of Section 2. One can then exploit the power of the Seiberg-Witten technology. In order to write down the SW curve, let us define the complex coordinate

$$ u = i \frac{x^6 + ix^{10}}{2\pi R_{10}}, \quad (3.4) $$

which parametrizes the M-theory torus defined by the identifications $u \simeq u + 1 \simeq u + \tau$. The complex structure $\tau$ is identified with the type IIB axio-dilaton. Let us also define the parameter $q = e^{2\pi i \tau}$ and the coordinate $t = e^{2\pi i u}$; note that $t \simeq qt$ on the torus.
For concreteness, let us stick again to the case of equal gauge couplings in the UV CFT: $\tau_1 = \tau_2 = \tau/2$. In terms of the quasi-modular Jacobi $\theta$-functions

\[
\theta_2(2u|2\tau) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2} t^{2n-1},
\]
\[
\theta_3(2u|2\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} t^{2n}, \quad \theta_4(2u|2\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} t^{2n}, \tag{3.5}
\]

the SW curve for the conformal theory can be written as \[35\]

\[
\frac{S(v) + R(v)}{S(v) - R(v)} = f(u|\tau), \quad \text{with} \quad f(u|\tau) \equiv \frac{\theta_3(u|\tau/2) + \theta_2(2u|2\tau)}{\theta_4(u|\tau/2) - \theta_2(2u|2\tau)}, \tag{3.6}
\]

or alternatively

\[
\frac{R(v)}{S(v)} = g(u|\tau), \quad \text{with} \quad g(u|\tau) \equiv \frac{f - 1}{f + 1} = \frac{\theta_2(2u|2\tau)}{\theta_3(2u|2\tau)}. \tag{3.7}
\]

Here $R(v) = \prod_{a=1}^{N+M} (v - \phi_a)$ and $S(v) = \prod_{a=1}^{N+M} (v - \tilde{\phi}_a)$ are degree $N + M$ polynomials whose zeros $\phi_a$ and $\tilde{\phi}_a$ are the eigenvalues for the adjoint scalars of the first and second gauge group, respectively.

Following [13], let us choose a $\mathbb{Z}_M$-invariant configuration for the anti-fractional branes Higgsing the CFT at large $|z|$ (i.e. large $|v|$ for the corresponding D4 branes), and consider the origin of the moduli space of the low energy $SU(N+M) \times SU(N)$ theory,

\[
R(v) = v^{N+M} \quad \text{and} \quad S(v) = v^N (v^M - z_0^M). \tag{3.8}
\]

The $N$ common zeros of $R(v)$ and $S(v)$ factor out of the curve, without affecting the RG flow. They correspond to $N$ D3 branes, whose moduli space is flat (apart from orbifold singularities when several branes coincide) and not quantum corrected. We are then left to consider an $SU(M) \times SU(M)$ theory, spontaneously broken to $SU(M) \times U(1)^{M-1}$ at the scale $z_0$. Hence, if the IR dynamics is not much affected by the UV Higgsing, as it is natural to expect, the low energy physics should be similar to the enhançon mechanism previously discussed, but with $N$ leftover regular D3 branes.

Let us give further evidence for the above claim. As explained in [22], we can extract the running of the gauge coupling from the bending of the two NS5 branes due to the unbalanced D4 branes tension. In the M-theory picture, the gauge couplings at a scale $v$ can be extracted from the SW curve looking at the corresponding two values of $u$; we have that

\[
\Delta u = \tau_1, \quad \tau - \Delta u = \tau_2, \tag{3.9}
\]
Figure 2: RG flow of the theory at the enhançon vacuum (origin of the moduli space). The low energy theory below \( \Lambda \) is a peculiar one, with one formally diverging coupling.

while the map between the type IIB twisted scalars \((c, b)\) and the field theory couplings \((\tau_1, \tau_2)\) was given in (2.9). In particular, the curve (3.6) at the point (3.8) on the Coulomb branch reads

\[
1 - 2 \left( \frac{v}{z_0} \right)^M = f(u|\tau) .
\]

One can check [13] that in the UV regime \(|v| > |z_0|\), the theory is conformal with equal gauge couplings. Comparing (3.10) with (3.1), one can see [10] that the dynamically generated scale is at \( \Lambda = q^{1/4} z_0 \). In the range \(|\Lambda| < |v| < |z_0|\), the two gauge couplings are running with opposite \( \beta \)-functions

\[
\beta = \frac{\partial}{\partial \log |v|} \left( \frac{8\pi^2}{g^2_{1,2}(|v|)} \right) = \pm 2M .
\]

For \(|v| < |\Lambda|\) the gauge couplings are constant with \(8\pi^2/g^2_{1,2} = 0, 2\pi/g_s\) respectively. The RG flow is sketched in figure 2. At the scale \( \Lambda \), the gauge group is effectively broken by instantons from \( SU(N+M) \times SU(N) \times U(1)^M \) to \( SU(N) \times SU(N) \times U(1)^2 M \), the latter being conformal up to an IR free abelian sector.

Further information is gained from the computation of branch points of the SW curve, which correspond to double points of the function \( f(u|\tau) \): they are at \( u_* = 0, 1/2, \tau/2, (1 + \tau)/2 \) where \( f(u_*|\tau) = f_0, 1/f_0, -f_0, -1/f_0 \) respectively, and \( f_0 = 1 + 4q^{1/4} + \mathcal{O}(q^{1/2}) \). The first set is located at

\[
u = \frac{\tau}{2}, \frac{\tau + 1}{2} : \quad v \simeq v_h^\pm = z_0e^{2\pi in/M} \left[ 1 \pm \frac{2}{M} \left( \frac{\Lambda}{z_0} \right)^M \right] \quad h = 1, \ldots, M .
\]

[10] Notice that in the supergravity approximation, \( g_s \to 0 \) with \( g_s N \) large, the parameter \( q = e^{2\pi i \tau} \) has exponentially small modulus \(|q| = e^{-2\pi/g_s}\), allowing for a series expansion of \( f(t|q) \) in positive powers of \( q \).
These are almost double branch points, which correspond to the $M$ anti-fractional branes located near $|z_0|$, corresponding to the VEV’s of $\tilde{\Phi}$ we used to Higgs the conformal theory. The second set is located at

$$u = 0, \frac{1}{2}: \quad v \simeq v_k = 2^{1/M} e^{2\pi ik/2M} \Lambda \quad k = 1, \ldots, 2M . \quad (3.13)$$

These branch points correspond to $M$ fractional branes melted into an enhançon ring at scale $\Lambda$.

As in the pure SYM case, probe fractional branes can be studied on this background by means of the SW curves for the $SU(N + M + 1) \times SU(N + M + 1)$ theory

$$\frac{R(v)}{S(v)} = \frac{v^M (v - \phi)}{(v^M - z_0^M) (v - \tilde{\phi})} = g(u|\tau) , \quad (3.14)$$

where $\phi$ and $\tilde{\phi}$ parametrize the extra VEV for $\Phi$ and $\tilde{\Phi}$. The branch points corresponding to the eigenvalue $\phi$ (the fractional D3 probe) can freely move outside the enhançon ring, but as they approach it and $\phi$ goes to 0, the two branch points split and melt into the enhançon ring. The two branch points corresponding to the eigenvalue $\tilde{\phi}$ (the anti-fractional D3 probe) can instead penetrate the enhançon ring; when this happens, they unchain two branch points from the ring which follow them inside: an anti-fractional brane eats a melted fractional brane from the ring, forming a regular D3 brane free to move everywhere.

From this analysis, one concludes that, no matter the value of $N$, the fluxes in eqs. (2.11) and (2.12) do describe the physics of the $SU(N + M) \times SU(N)$ theory at the origin of its moduli space, provided that they are excited at radius $\rho_1 \simeq |\Lambda|$ by an enhançon mechanism. The solution should also be cut off at a radius $|z_0|$, or completed with $M$ anti-fractional branes, providing a conformal $AdS_5$ UV completion.

As already stressed, the warp factor needs to be recomputed in the presence of the correct configuration of fractional branes and excised twisted fields. This will be done in section 6.

Notice, however, that the supergravity solution of eqs. (2.11) and (2.12) does not seem to have any pathology below $\rho_1$, at least down to a scale of order $e^{-\pi l^\frac{N}{M} / \rho_1}$, where the 5-form flux (2.12) vanishes and the problematic repulsive region starts. The question arises whether there is any field theory interpretation for such a solution, suitably excited only at a radius

$$\rho_{\text{min}} = \rho_{l+1} \equiv e^{-\pi l^\frac{N}{M}} \rho_1 \quad \text{with} \quad l \equiv \lfloor N/M \rfloor , \quad (3.15)$$

the smallest infinite coupling scale outside the region of negative D3 brane charge. As already noticed, the presence of a constant 3-form flux and the logarithmic running of

15
the 5-form flux strongly suggests a cascading behavior, as for the Klebanov-Tseytlin-
Strassler $\mathcal{N} = 1$ model \cite{4,3}, properly adapted to a $\mathcal{N} = 2$ setting. An interpretation
of the would-be $\mathcal{N} = 2$ RG flow that can be extracted from the supergravity solution
in terms of some sort of Seiberg duality cascade was in fact argued for in \cite{11}, but the
existence of an appropriate $\mathcal{N} = 2$ duality had not been clarified, so far. On the other
hand, in \cite{12} the reduction of 5-form flux was interpreted as due to a distribution of D3
branes and/or wrapped D5 branes. It was further suggested that a suitable distribution
of D3 branes only (Higgs branch) could perhaps account for it. However, the latter
proposal encounters some problems in reproducing the running of gauge couplings
and decrease of nonabelian gauge group ranks that is suggested by the supergravity
solution.

Drawing on well established results about $\mathcal{N} = 2$ SQCD, we propose that there exist
field theory vacua, not at the origin of the Coulomb branch, which display a *cascading*
behavior. They are dual to the solution in (2.11) and (2.12), valid well below the first
infinite coupling radius $\rho_1$ down to some much lower scale, at most until the so-called
true enhançon scale $\Lambda_{\text{min}} = \rho_{\text{min}}$, where the twisted fields are excised. $\rho_{\text{min}}$ is named
the true enhançon radius since it is the scale at which the excision is performed. All the
higher infinite coupling scales, $\rho_j$ with $j = 1, \ldots, l$, will be called generalized enhançon
radii \cite{12}.

We provide a precise identification of these vacua in the next section. The excision
of the twisted fields by means of the enhançon mechanism and the disappearance of the
naive singularity will be discussed in section 6. Depending on the field theory vacua
one is studying, the excision can take place at different scales, for instance at $\rho = \rho_1$,
as in the vacua discussed in this section, or at the bottom of the cascade, at the scale
$\rho = \rho_{\text{min}}$, as for the cascading vacua to be discussed in section 4.

4 The cascading vacuum in field theory

The perturbative RG flow of the $SU(N + M) \times SU(N)$ theory, given in (3.11), is such
that the largest group goes to strong coupling at a scale $\Lambda$. The supergravity solution
we are considering suggests that, in the dual vacuum, a mechanism effectively reduces
the gauge group to $SU(N - M) \times SU(N)$ below $\Lambda$, plus possible $U(1)$ factors. This
statement can be supported by a computation of Page charges in supergravity, in the
gauge that gives sensible field theory couplings (as extensively discussed in [14,15]).
The value of \( b \), in the gauge in which \( b \in [0,1] \), is found from (2.11) to be
\[
b = g_s \text{Im} \gamma = \frac{g_s M}{\pi} \log \frac{\rho}{\rho_1} - \left[ \frac{g_s M}{\pi} \log \frac{\rho}{\rho_1} \right] ,
\]
where \( \rho = |z| \). The D5 and D3 brane Page charges at radius \( r \) are evaluated to be
\[
Q_{Page}^5 = -\frac{1}{4\pi^2 \alpha'} \int F_3 = 2M
\]
\[
Q_{Page}^3 = -\frac{1}{(4\pi^2 \alpha')^2} \int (F_5 + B_2 \wedge F_3) = N + M \left[ \frac{g_s M}{\pi} \log \frac{r}{\rho_1} \right] .
\]
This shows that the non-abelian factors in the gauge group drop as \( SU(N+M) \times SU(N) \rightarrow SU(N-M) \times SU(N) \) not only at the first strong coupling scale \( \rho_1 = \Lambda_1 \equiv \Lambda \), but actually at each generalized enhançon, which occurs at a scale
\[
\rho_k = \Lambda_k = e^{-\frac{\pi (k-1)}{g_s M}} \Lambda_1 = e^{-\frac{\pi (2k-1)}{2g_s M}} \rho_0 \quad k = 1, \ldots, l
\]
where recall that \( l = \lfloor N/M \rfloor \) and we also set \( N = lM + p \). Finally, at \( \Lambda_{l+1} \equiv \Lambda_{\text{min}} \equiv e^{-\frac{\pi l}{g_s M}} \Lambda_1 \) there is a true enhançon ring with \( M \) tensionless fractional branes, and the non-abelian factors in the gauge group reduce according to \( SU(M+p) \times SU(p) \rightarrow SU(p) \times SU(p) \), with one infinite gauge coupling. Twisted fields have to be excised there so as to avoid negative D3-charge in the interior region.

In passing let us stress, as in [12], that even though their dynamics takes place at arbitrarily low energies, the possible additional \( U(1) \) factors are described in the holographic setup by modes at a finite radius where the corresponding fractional D3 branes lie.

In order to have an intuition on the strong coupling dynamics at hand, let us first focus on the first such generalized enhançon, which occurs at the scale \( \Lambda_1 = \Lambda \). This will clearly be a prototype for any generalized enhançons. As already stressed, at the scale \( \Lambda \), the coupling of the largest gauge group diverges (and instantonic corrections dominate), while the other gauge coupling reaches the value \( g_{\text{min}}^2 = 4\pi g_s \). As a first step toward the understanding of the precise mechanism taking place, we can consider a corner of the parameter space of the gauge theory where \( Ng_{\text{min}}^2 \rightarrow 0 \). In this limit, the gauge dynamics of the second factor decouples and it effectively becomes a global symmetry: the theory around \( \Lambda \) is simply \( SU(N+M) \) SQCD with \( 2N \) flavors. Moreover, possible VEV’s for the smaller group adjoint scalar effectively behave as masses

---

\(^{11}\)The 3-cycle where the D5 charge integration is performed is the product of the exceptional 2-cycle \( C \) and an \( S^1 \) on the orbifold line. Since the intersection number is \( (D,C) = -2 \) where \( D \) is the cone over the 3-cycle, the D5 charge is twice the number of wrapped D5 branes.
for the larger group hypermultiplets. In this case we are out of the supergravity approximation but this analysis will give us some good insight. Hence, let us quickly review some results about the moduli space of \( \mathcal{N} = 2 \) SQCD.

### 4.1 One cascade step: \( \mathcal{N} = 2 \) SQCD

The moduli space of \( \mathcal{N} = 2 \) SQCD \[20\] with \( N_c \) colors and \( N_f \) flavors consists of a Coulomb branch and of various Higgs branches. The Coulomb branch \[37, 20\] is parametrized by the vacuum expectation value of the adjoint scalar field \( \Phi \) in the \( \mathcal{N} = 2 \) vector multiplet,

\[
\Phi = \text{Diag}(\phi_1, \ldots, \phi_{N_c}) \quad \sum_a \phi_a = 0 ,
\]

and is thus given by the \( N_c - 1 \) dimensional complex space of \( \phi_a \)'s modulo permutations (Weyl gauge transformations). The VEV’s generically break the \( SU(N_c) \) gauge group to its Cartan subgroup \( U(1)^{N_c-1} \). However, at special submanifolds where the Higgs branches meet the Coulomb branch a non-abelian gauge symmetry survives. Higgs branches can be divided into a baryonic branch and various non-baryonic branches (according to whether baryonic operators acquire VEV’s or not); the corresponding intersections with the Coulomb branch were dubbed roots. \[12\] Higgs branches are not quantum corrected, however their intersections among themselves and with the Coulomb branch are modified at quantum level.

The SW curve describing the Coulomb branch for vanishing masses is \[37, 38\]

\[
y^2 = \prod_{a=1}^{N_c} (x - \phi_a)^2 + 4\Lambda^{2N_c-N_f}x^{N_f} .
\]

Nonbaryonic branches are labeled by an integer \( 1 \leq r \leq \min([N_f/2]_-, N_c-2) \). The low energy effective theory at the roots are the IR free or finite \( SU(r) \times U(1)^{N_c-r} \) SQCD with \( N_f \) hypermultiplets in the fundamental representation and charged under one of the \( U(1) \) factors. At special points along these submanifolds, the SW curve shows that \( N_c - r - 1 \) additional massless singlet hypermultiplets arise, each one charged under one of the remaining \( U(1) \) factors. It is important that there are \( 2N_c - N_f \) such vacua, related by the broken \( \mathbb{Z}_{2N_c-N_f} \) non-anomalous R-symmetry acting on the Coulomb branch.

\[12\] Issues related to the baryonic root of \( \mathcal{N} = 2 \) SQCD and the mass deformation to \( \mathcal{N} = 1 \) were recently discussed in \[36\].
The baryonic branch exists for $N_c \leq N_f$, and the baryonic root is a single point, invariant under the $\mathbb{Z}_{2N_c-N_f}$ R-symmetry. Thus its coordinates on the Coulomb branch are
\[ \Phi_{bb} = (0, \ldots, 0, \phi \omega^2, \ldots, \phi \omega^{2N_c-N_f}) , \] (4.6)
where $\omega = \exp\{2\pi i/(2N_c-N_f)\}$, for some value of $\phi$ (and $\phi = 0$ classically). The gauge group is thus broken to $SU(N_f-N_c) \times U(1)^{2N_c-N_f}$, which is IR free. The requirement that a Higgs branch originates from this root implies the presence of $2N_c-N_f$ massless hypermultiplets charged only under the $U(1)$ factors; this singles out a point in the submanifold described by (4.6). The result is $\phi = \Lambda$, so that the SW curve takes the singular form
\[ y^2 = x^{2(N_f-N_c)}(x^{2N_c-N_f} + \Lambda^{2N_c-N_f})^2 . \] (4.7)

The $x^{2(N_f-N_c)}$ factor corresponds to an unbroken $SU(N_f-N_c)$ gauge group. The remaining $2(2N_c-N_f)$ branch points show up in coincident pairs, located at $x_k = \Lambda \omega^{k-\frac{1}{2}}$ with $k = 1, \ldots, 2N_c-N_f$, corresponding to the $2N_c-N_f$ mutually local massless hypermultiplets.

The reason for this detour should be clear by now: the non-perturbative dynamics at the baryonic root preserves the same $\mathbb{Z}_{2N_c-N_f} = \mathbb{Z}_M$ R-symmetry as the supergravity solution we are discussing, and its low energy effective theory possesses an $SU(N_f-N_c) = SU(N-M)$ non-abelian gauge symmetry precisely matching the numerology of the cascading interpretation. Hence, iterating the above procedure at the subsequent generalized enhançons $\Lambda_k$ (where the higher rank gauge group coupling diverges), it is natural to propose the supergravity solution in (2.11) and (2.12) (excised only down at the true enhançon $\rho_{\text{min}}$) to be dual to a cascading $SU(N+M) \times SU(N)$ quiver gauge theory at subsequent baryonic roots of the strongly coupled gauge groups. In what follows, we will provide several checks for the validity of our proposal.

4.2 The cascading vacuum in the quiver gauge theory

Let us now turn to the full quiver gauge theory $SU(N+M) \times SU(N)$. The vacuum we propose as the dual of the full cascading solution is a vacuum in which, at each step

---

13For $N_f > 3N_c/2$ there are other $\mathbb{Z}_{2N_c-N_f}$-invariant submanifolds. However the baryonic root is just one point, and one can show that it in fact belongs to the submanifold [20].

14We assume $N_f < 2N_c$ so that the microscopic theory is UV free. This bound is satisfied in the cascading quiver theory.

15We should mention that a proposal for an $\mathcal{N} = 2$ cascade at the baryonic root has been alluded to in [39], in the context of the M-theory realization of this elliptic model.
along the resulting cascade, the largest of the two gauge groups goes to strong coupling with a behavior analogous to the the baryonic root of SQCD. This vacuum is invariant under the same non-anomalous $\mathbb{Z}_{2M}$ subgroup of the R-symmetry as the supergravity solution we started with. Moreover, not only has it the correct spontaneous symmetry breaking pattern but also the correct RG flow, including the beta functions and the separation of scales where the transitions occur, as can be extracted from supergravity.

It is worth stressing that our vacuum does not sit exactly at the baryonic roots, as there are no baryonic roots in the quiver theory (see section 4.4 for an exception). However, it does approximate them in the supergravity limit in which $q \to 0$, which is the limit of interest to us.

Let us start for concreteness with an $SU((2K + 1)M) \times SU((2K + 1)M)$ conformal theory in the UV and then break the gauge group to $SU((2K + 1)M) \times SU(2KM)$ by giving VEV’s of order $z_0$ in a $\mathbb{Z}_M$-invariant way to $M$ eigenvalues of the adjoint scalar $\tilde{\Phi}$. We choose a vacuum in which, at each step of the RG flow, the most strongly coupled group is at its baryonic root (in the $q \to 0$ limit). Let us write the SW curve as $R(v)/S(v) = g(u|\tau)$ as in (3.7), where $u$ is the coordinate on a torus of complex structure $\tau$. We choose the polynomials $R(v)$ and $S(v)$ of degree $(2K + 1)M$, as

$$R(v) = v^M \prod_{j=0}^{K-1} (v^{2M} + q^{2j+2j} z_0^{2M})$$

$$S(v) = (v^M - z_0^{2M}) \prod_{j=0}^{K-1} (v^{2M} + q^{2j+2j} z_0^{2M}).$$

The polynomial $R(v)$ is related to the $SU((2K + 1)M)$ group that starts flowing toward strong coupling at the cutoff scale $z_0$, whereas the polynomial $S(v)$ is related to the $SU((2K + 1)M)$ group which is spontaneously broken to $SU(2KM)$ there\textsuperscript{16}. The eigenvalues of the two adjoint scalar fields are put, in an alternating manner, at energies corresponding to their subsequent strong coupling scales along the cascade: in the limit in which the dynamics of the weakly coupled group decouples at those scales, the vacua mimic the SQCD baryonic root. In agreement with the cascading RG flow of the supergravity solution, the hierarchy of strong coupling scales is controlled by $q = e^{2\pi i \tau}$. Because of the large $M$ limit, the running is led by the perturbative beta functions except at the successive strong coupling scales, where instantonic corrections sharply

\textsuperscript{16}Very similarly, we can also describe a cascade with an $SU(2KM) \times SU(2KM)$ UV completion: it amounts to putting the cutoff and the vanishing eigenvalues in the same adjoint field/polynomial in (4.8), otherwise preserving the structure of the polynomials. Finally, the generalization to the cascade with $N = lM + p$ can be achieved by multiplying $R$ and $S$ by the same degree $p$ polynomial.
Figure 3: RG flow of the theory at the cascading vacuum (taking $p = 0$, for definiteness). Here, as well as in figures 4 and 5, the horizontal axis is logarithmic and we have omitted the $SU$ factors for the gauge groups, to avoid clutter.

appear. This field theory running can be explicitly checked either numerically using the exact SW curve we wrote, or analytically by expanding the polynomials energy range by energy range, in an effective field theory approach (see Appendix A). A plot of the resulting RG flow is shown in figure 3.

We now move on to the study of the branch points of the curve. Recall that branch points are double solutions in $v$ at fixed $u$. In the dual type IIA construction, a pair of coincident branch points at $v$ corresponds to a D4 brane stretched between the two NS5’s, while in type IIB it corresponds to a fractional brane at position $z \simeq v$ on the orbifold singularity line. When the branch points are not in pairs, the full M-theory description is needed, fractional branes are no longer perturbative states in type IIB and their wavefunction is spread over the whole $b \in \mathbb{Z}$ region [21] (at least in the large $M$ limit).

It turns out that the branch points for $u = 0, 1/2$, up to corrections of higher order in $q$, lie at

$$v^M \simeq \mp q^{n+1/4} z_0^M, \quad n = 0, \ldots, K - 1 \quad \text{and} \quad v^M \simeq \mp 2q^{K+1/4} z_0^M. \quad (4.9)$$

The former class of points consists of $K$ sets of $2M$ double points (which are double up to an accuracy discussed at the end of the next subsection), corresponding to the $K$ baryonic-root-like VEV’s of the first gauge group, whereas the latter are $2M$ well separated branch points, corresponding to the true enhancement of the low energy $SU(M)$ theory. The branch points for $u = \tau/2, (\tau + 1)/2$ lie at

$$v^M = \mp q^{n+3/4} z_0^M, \quad n = 0, \ldots, K - 1 \quad \text{and} \quad v^M = (1 \pm 2q^{1/4}) z_0^M. \quad (4.10)$$

The first class of points consists again of $K$ sets of $2M$ (almost) double points, corresponding to the $K$ baryonic-root-like VEV’s of the second gauge group, while the
second set of points are the almost paired branch points associated to semiclassical fractional branes at the cutoff scale \( z_0 \).

### 4.3 The infinite cascade limit

In this subsection we analyse the case of an infinite cascade, created as the cutoff anti-fractional branes are sent to infinity. We are interested in this limit for two main reasons: first of all, this limit allows us to describe the field theory vacuum and the SW curve dual to the infinite cascade solution of [10], where there are no cutoff anti-fractional branes; secondly, this infinite cascade bears strong connections and similarities, that we will specify in the following, with the Klebanov-Tseytlin-Strassler \( \mathcal{N} = 1 \) cascade [3, 4], which is necessarily unbounded in the UV since fractional branes are stuck at an isolated conifold singularity.

In order to properly define this limit, we should keep fixed the IR enhançon scale \( \Lambda_{\text{min}} \), as well as the generalised enhançon scales defined in (4.3). It is thus convenient to rewrite the two polynomials as

\[
R_K(v) = v^M \prod_{j=1}^{K} (v^{2M} + q^{-2j} \Lambda_{\text{min}}^{2M})
\]

\[
S_K(v) = (v^M - q^{-\frac{1}{2}} \Lambda_{\text{min}}^M) \prod_{j=1}^{K} (v^{2M} + q^{1-2j} \Lambda_{\text{min}}^{2M}).
\]

The limit of infinite cascade is formally \( K \to \infty \). Let us define \( x = (v/\Lambda_{\text{min}})^M \), obtaining the SW curve

\[
T_K(x) \equiv \frac{R_K(v)}{S_K(v)} = \frac{x}{x - q^{-1/4-K} \prod_{j=1}^{K} (x^2 + q^{-2j})} = g(u|\tau) .
\]

Note that

\[
T_K(x) = \frac{x}{x - q^{-1/4-K} \prod_{j=1}^{K} (1 + q^{2j} x^2)}
\]

converges pointwise as \( K \to \infty \) for any fixed value of \( x \) (possibly with poles) since \(|q| < 1\), even though it does not converge uniformly.

We can then show that the approximate double points become exact at any order in \( q \) at large enough \(|v|\) (i.e. the monopoles become exactly massless in the upper reach
of the cascade). We will make use of the following property of \( g \) at its double points:
\[
g(0|\tau) = -g(1/2|\tau) = 1/\theta(\tau/2|\tau) = -1/g((1+\tau)/2|\tau).
\]
Moreover, the value of the periodic function at these points is given by
\[
g_0(q) \equiv g(0|\tau) = \frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)} = 2q^4 \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^2}{(1 + q^{2j-1})^2}.
\]
(4.14)

Let us start with the branch points at \( u \) and \( T \) periodic function at these points is given by
\[
\text{Let us start with the branch points at } u \text{ and } T \text{ periodic function at these points is given by}
\]
\[
g_0(q) \equiv g(0|\tau) = \frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)} = 2q^4 \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^2}{(1 + q^{2j-1})^2}.
\]

(4.14)

Let us start with the branch points at \( u = 0, 1/2 \) and \( x = -\epsilon q^{-n} \), where \( n = 1, \ldots, K \) and \( \epsilon = \pm 1 \). After some manipulations one gets
\[
T_K(-\epsilon q^{-n}) = \frac{2\epsilon q^{1/4}}{(1 + \epsilon q^{1/4}+K-n)} \prod_{j=1}^{\min(n-1,K-n)}(1 + q^{2j})^2 \prod_{j=1}^{\max(n-1,K-n)+1}(1 + q^{2j}) \prod_{j=\min(n-1,K-n)+1}^{\max(n-1,K-n)+1}(1 + q^{2j-1}) \prod_{j=1}^{\min(n-1,K-n)}(1 + q^{2j})^2 \prod_{j=\min(n-1,K-n)+1}^{\max(n-1,K-n)+1}(1 + q^{2j-1})
\]

(4.15)

Consequently, the equation \( T_K(x) = \epsilon g_0(q) \) is solved up to corrections \( O(q^{2\min(n,K-n)+1}) \), \( O(q^{2\min(n-1,K-n)+2}) \) and \( O(q^{1/4+K-n}) \). In particular, in the case \( K \geq 3n \) which is the lower part of the cascade we get
\[
\frac{T_K(-\epsilon q^{-n})}{\epsilon g_0(q)} = 1 + O(q^{2n})
\]

(4.16)

and the branch points we found are correct up to \( O(q^{2n}) \). Similarly, for the branch points at \( u = \tau/2, (\tau + 1)/2 \) and \( x = -\epsilon q^{-n+1/2} \) with \( n = 1, \ldots, K \), we get
\[
T_K(-\epsilon q^{-n+1/2}) = \frac{1}{2\epsilon q^{1/4} (1 + sq^{3/4+K-n})} \times
\]

\[
\prod_{j=1}^{\min(n-1,K-n)}(1 + q^{2j})^2 \prod_{j=1}^{\max(n-1,K-n)+1}(1 + q^{2j-1}) \prod_{j=\min(n-1,K-n)+1}^{\max(n-1,K-n)+1}(1 + q^{2j})
\]

(4.17)

and in particular, for \( K \geq 3n \)
\[
\frac{T_K(-\epsilon q^{-n+1/2})}{(\epsilon g_0(q))^{-1}} = 1 + O(q^{2n-1})
\]

(4.18)

In order to show that these two sets of branch points are double, we compute
\[
\frac{dT_K}{dx}(x) = T_K(x) \left\{ \frac{1}{x} + \sum_{j=1}^{K} \frac{2x}{x^2 + q^{-2j}} - \frac{1}{x - q^{-1/4-K}} - \sum_{j=1}^{K} \frac{2x}{x^2 + q^{-1-2j}} \right\}
\]

(4.19)

One can show that \( T_K(-\epsilon q^{-n}) = O(q^{n+1/2}) \) and \( T_K'(-\epsilon q^{-n+1/2}) = O(q^n) \) so that the points are double, up to sub-leading corrections (from numerical studies it seems that the corrections actually appear at some much higher order).

In a similar way, one shows that the non-double branch points at \( u = 0, 1/2 \) and \( x = -2\epsilon \) (enhançon) are correct up to \( O(q) \), whereas the almost double ones at \( u = \tau/2, (\tau + 1)/2 \) and \( x = (1 + 2\epsilon q^{1/4})q^{-1/4-K} \) (cutoff) are correct up to \( O(q^{1/4}) \).
Summarizing, our analysis shows that the SW curve \((4.12)\) for the finite cascade has a well defined infinite cascade limit as we send \(K \to \infty\). We also evaluated to which degree the approximate double points in the \(q \to 0\) limit, appearing at all the strong coupling scales except the smallest one, depart from being exactly double; we find that in the infinite cascade limit the mass of the corresponding monopoles goes to 0 for any value of \(q\) as we consider higher and higher scales up in the cascade, that is large \(n\). Finally, only at the bottom of the infinite cascade do we find equally separated double points (in the \(q \to 0\) limit), filling a true enhançon ring in the large \(M\) limit.

### 4.4 Mass deformation

A not completely satisfactory feature of the cascading vacua we proposed is that, although they preserve the \(Z_{2M}\) R-symmetry as the baryonic root of SQCD, the extra light monopoles are strictly massless only in the \(q \to 0\) limit or for very large \(n\). At finite \(q\) and \(n\), our vacua are not really singled out as very special points in the moduli space. Surely this is enough for our purpose of finding the field theory vacua dual to the supergravity solutions in \((2.11)\) and \((2.12)\). However, it will be useful to argue for the existence of a cascading vacuum with exactly massless monopoles.

The task can be related to mass deformation of the \(\mathcal{N} = 2\) theory to \(\mathcal{N} = 1\), after the addition of a mass term for the adjoint scalars

\[
W_{\text{mass}} = \frac{m}{2}(\Phi^2 - \bar{\Phi}^2) .
\]

(4.20)

In the case of \(\mathcal{N} = 2\) SQCD, a mass deformation lifts the moduli space and only the points on the Coulomb branch with \(2N_c - N_f\) extra massless monopoles survive, that is the baryonic root and the \(2N_f - N_c\) special points along the non-baryonic roots. The reason is that in the dual M-theory picture a mass deformation corresponds to a relative rotation of the two extended M5 branches (NS5-branes in IIA), and this is possible only if the curve has genus zero (because in the \(\mathcal{N} = 1\) theory confinement breaks completely the gauge group, and the genus of the M-theory/SW curve equals the rank of the left over group). On the other hand, moduli space points with massless monopoles are singular points where the genus of the curve reduces, and a maximal number of them is needed to reach zero genus.

This suggests that a special point on the moduli space of the quiver theory should be found after a mass deformation. There are two main problems however. The first is that the cascading theory is obtained from the conformal theory by spontaneous breaking at the cutoff \(z_0\); this is no longer a solution after mass deformation. A possible
solution is to consider an infinite cascade, as in the case of the conifold theory. From a more conservative point of view, one could consider an unstable time-dependent field configuration with a finite cascade (with a large number of steps) in which the VEV’s for the spontaneous breaking are very large but collapsing to zero. In this case the dimensionless parameter controlling the time evolution of the field is $\dot{\Phi}/\dot{\Phi}^3 = -(m/\Phi)^2$, which is in fact very small for $\Phi \gg m$. This mechanism would “freeze” the cutoff in this limit. The other problem is that, unlike the SQCD case, after mass deformation the far IR is $SU(M) \mathcal{N} = 1$ pure SYM, whose $M$ vacua break $\mathbb{Z}_{2M}$ to $\mathbb{Z}_2$.

These observations suggest that we should look for a genus zero SW curve which breaks $\mathbb{Z}_{2M}$ to $\mathbb{Z}_2$, mimicking the curve for $SU(M)$, and which describes an infinite cascade. Let us start from one of the $M$ genus zero curves of $\mathcal{N} = 2$ $SU(M)$ SYM: being of genus zero they are parametrized by a complex coordinate $\lambda$, from which one constructs two rational functions $v$ and $t$:

\[ v = \lambda + \frac{\Lambda^2}{\lambda}, \quad t = \lambda^M \quad \Rightarrow \quad t^2 - P_M(v)t + \Lambda^{2M} = 0, \quad (4.21) \]

where $P_M(v)$ is a particular polynomial of degree $M$ in $v$. In the following we will set $\Lambda = 1$; then $P_M(v)$ is a Chebyshev polynomial:

\[ P_M(v) = \left[ \frac{v + \sqrt{v^2 - 4}}{2} \right]^M + \left[ \frac{v - \sqrt{v^2 - 4}}{2} \right]^M. \quad (4.22) \]

The genus zero curve for the infinite cascade vacuum in the quiver theory is simply obtained by wrapping the SYM curve on the torus,

\[ Q = \lim_{K \to \infty} Q_K = \lim_{K \to \infty} \prod_{j=-K}^{K} F(q^j t, v) = 0 \quad \text{with} \quad F(t, v) = t - P_M(v) + \frac{1}{t}, \quad (4.23) \]

where $t = e^{2\pi i u}$. This definition is mainly formal, as the infinite product above does not converge. However its zero locus in $T^2 \times \mathbb{C}$ (the curve itself) is well defined, and it consists of the SYM curve wrapped infinitely many times on the torus. It is clear that it has genus zero (being non-compact, we mean that it is parametrized by $\lambda$) and that it reproduces the correct IR behavior of $SU(M)$ SYM.

In order to make sense of it, and to check that it is the limit of a sequence of SW curves for longer and longer cascades, with the correct hierarchy of scales as expected from the RG flow at the baryonic roots, we consider finite $K$ (eventually sent to $\infty$) and rewrite the curve as

\[ \tilde{Q}_K = q^{K(K+1)} f(q) \cdot Q_K = f(q) \left( t - P + \frac{1}{t} \right) \prod_{j=1}^{K} \left( 1 - Ptq^j + t^2 q^{2j} \right) \left( 1 - \frac{P}{t} q^j + \frac{q^{2j}}{t^2} \right) = 0, \quad (4.24) \]
where \( f(q) = \prod_{j=1}^{\infty} (1 - q^{2j})(1 - q^{2j-1})^2 \). The zero locus is the same as before, but now the product converges as \( K \to \infty \). Then, we define a sequence of SW curves for \( SU((2K + 1)M) \times SU((2K + 1)M) \) given by

\[
Q_K \equiv -\tilde{R}_K \theta_3(2u|2\tau) + \tilde{S}_K \theta_2(2u|2\tau) = 0 ,
\]

with the polynomials \( \tilde{R}_K \) and \( \tilde{S}_K \) chosen as

\[
\tilde{R}_K(v) = P(v) \prod_{j=1}^{K} (q^{2j}P(v)^2 + 1 - 2q^{2j} + q^{4j}) \]

\[
\tilde{S}_K(v) = q^{-1/4}(1 - q^{K+1/4}P(v)) \prod_{j=1}^{K} (q^{2j-1}P(v)^2 + 1 - 2q^{2j-1} + q^{4j-2}) .
\]

Using the identities

\[
\theta_3(2u|2\tau) = \prod_{j=1}^{\infty} (1 - q^{2j}) \left( 1 + t^2q^{2j-1} \right) \left( 1 + t^{-2}q^{2j-1} \right) \]

\[
\theta_2(2u|2\tau) = q^{1/4}(t + t^{-1}) \prod_{j=1}^{\infty} (1 - q^{2j}) \left( 1 + t^2q^{2j} \right) \left( 1 + t^{-2}q^{2j} \right) ,
\]

one can explicitly verify that

\[
\tilde{Q}_K = Q_K \quad \text{up to orders } O(q^{K+1/4}) .
\]

Moreover, since the polynomials \( P_M(v) \) behave as \( v^M \) for \( v \gg 1 \), one can check that the hierarchy of scales of the cascading vacuum of subsection 4.3 is reproduced, up to IR corrections related to the different unbroken R-symmetries.

Let us comment on this result. Eq. (4.23)-(4.24) defines a genus zero curve with exactly double branch points for any value of \( q \), which describes a theory with infinitely long cascade and exactly massless monopoles, dual to a specific type IIB supergravity solution with no \( AdS \) asymptotics. One could think of realizing the theory by wrapping an M5 brane along the curve, and then computing observables from it. However one could object that, unlike the \( \mathcal{N} = 1 \) infinite KS cascade which makes sense as a field theory through holographic renormalization [42], an infinite \( \mathcal{N} = 2 \) cascade probably does not. The reason is that as we cascade down the IR-free \( U(1) \) factors accumulate, and an infinite cascade would require an infinite number of photons at finite energies, which does not make much sense. Thus in (4.25)-(4.26) we constructed a sequence of legitimate SW curves for any value of \( K \), describing larger and larger field theories with cascade which, although not having genus zero because of the UV cutoff, approximate
the genus zero curve (4.24) with arbitrary precision, for any value of $q$ and $M$. We could compute observables in the sequence, getting in the limit the same answer as from (4.24). Therefore this procedure makes sense of the infinite cascade theory, in the sense that observables in finite sectors are insensible to the (possibly infinite number of) decoupled photons.

Eventually, notice that the sequence in (4.25)-(4.26) contains the finite $q$ corrections to the $\mathcal{N} = 2$ cascade that are required to have exactly massless monopoles and that were missing in (4.11) because those were not visible in supergravity.

The mass deformation of this $\mathcal{N} = 2$ vacuum is particularly interesting because it induces a flow from the cascading $\mathcal{N} = 2$ theory to the $\mathcal{N} = 1$ Klebanov-Strassler (KS) cascade. This is expected on the field theory side because the adjoint fields have to be integrated out at the scale of the deformation mass parameter, leaving the Klebanov-Strassler field theory at smaller energies.

This is clear also in M-theory. The genus zero SW curve we proposed is the one of $\mathcal{N} = 2 \ SU(M) \ SYM$, rewritten on the torus so as to create an elliptic model. Similarly to the $M$ genus zero points on the Coulomb branch of $\mathcal{N} = 2$ SYM which survive mass deformation and flow to the $M$ confining vacua of $\mathcal{N} = 1$ SYM, the $M$ genus zero $\mathcal{N} = 2$ curves we proposed flow to the $M$ cascading vacua of the $\mathcal{N} = 1$ KS theory, whose IR is in fact $\mathcal{N} = 1$ SYM.

The rotated $\mathcal{N} = 1$ curve in the limit $m \to \infty$ is easily written. As before, we start rotating the SW curve for $SU(M) \ SYM$, exploiting the rational parametrization in terms of $\lambda$ [41]

\[
\begin{align*}
v = & \lambda \\
t = & \lambda^M \\
w = & \zeta \lambda^{-1}
\end{align*}
\Rightarrow
\begin{align*}
t = & v^M \\
vw = & \zeta
\end{align*}
\]

(4.29)

where the low energy strong coupling scale $\zeta = \Lambda_{\mathcal{N}=1}^2 = m \Lambda_{\mathcal{N}=2}^2$ is kept fixed in the limit, and a suitable rescaling of variables is performed [40]. The curve for the quiver theory is obtained by wrapping the curve on the M-theory torus: $0 = \prod_j (q^j t - v^M)$. After a rescaling to make the product converge, we get

\[
0 = (t - v^M) \prod_{j=1}^{K-\infty} \left( tv^M - q^j(t^2 + v^{2M}) + q^{2j} tv^M \right), \quad vw = \zeta .
\]

(4.30)

Note however that while in the $\mathcal{N} = 2$ case the M5 brane embedding can be interpreted as the exact SW curve for the field theory, which encodes the prepotential and the full dynamics, after breaking to $\mathcal{N} = 1$ this is no longer the case. The theory on the M5
brane reduces to the field theory of interest only when, for particular choices of the parameters, the unwanted modes are decoupled, and we refer to \[41, 43\] for details.

It should be possible to reproduce this interpolating flow in supergravity, so as to gain insight also on the Kähler data of these $N = 1$ vacua. In particular, if the mass deformation is much larger than the enhançon scale $\Lambda$, the solution should interpolate to the Klebanov-Tseytlin (KT) solution (before chiral symmetry breaking takes place in the IR). We leave the analysis of such an interpolating solution, which should be performed along the lines of \[44\], to the future.

5 More supergravity duals: enhançon bearings

In this section we study other vacua of the $SU(N + M) \times SU(N)$ theory, focusing on a class preserving the same $Z_{2M}$ R-symmetry as the supergravity solution of section 2. We will start from the non-cascading enhançon vacuum of section 3 and gradually construct the cascading vacuum discussed previously by pulling VEV’s out of the origin. In this process, we will observe new nontrivial vacua, for which we will propose novel type IIB dual backgrounds.

Let us consider the following family of polynomials for the SW curves of the $SU(N + M) \times SU(N + M)$ theory, parametrized by $\phi$

$$R(v) = v^{N-M} (v^{2M} - \phi^{2M}) \quad S(v) = v^N (v^M - z_0^M).$$

An overall $v^{N-M}$ factor (interpreted as $N - M$ D3 branes at the origin) decouples from the SW curve \([3.7]\), so that we will effectively reduce to the $SU(2M) \times SU(2M)$ case, with

$$R(v) = v^{2M} - \phi^{2M} \quad S(v) = v^M (v^M - z_0^M).$$

(5.2)

For $\phi = 0$ we are at the origin of the moduli space of the $SU(2M) \times SU(M)$ effective theory, where the enhançon mechanism takes place. We want to study the branch points of the SW curve as we vary $\phi$ continuously, in the supergravity approximation of small $q$, so that $g_0(q) = 2q^{1/4} + O(q^{5/4})$. We will use the shorthand notation $\xi = v^M$ and define the enhançon scale $\Lambda = 2^{1/4} m^{1/4} z_0$.

Let us first consider the branch points at $u = 0, 1/2$, related to the polynomial $R$. Depending on the value of $|\phi|$, we find\[17\]

- $|\phi^M| < |q^{1/4} z_0^M|

$$\xi \approx \pm \Lambda^M, \quad \xi \approx \pm \left(\frac{\phi^2}{\Lambda}\right)^M,$$

(5.3)

\[17\]We write the first corrections only when they are necessary to split double branch points.
namely $2M$ equally separated branch points at the enhançon ring and $2M$ equally spaced branch points at a ring of radius $|\phi^2/\Lambda|$;

- $|\phi^M| > |q^{1/4}z_0^M|$

\[
\xi \simeq \pm (1 + \epsilon q^{1/4}) \phi^M, \quad \epsilon = \pm 1,
\]

namely $2M$ pairs of branch points on a circle of radius $|\phi|$.

The branch points at $u = \tau/2, (1 + \tau)/2$ related to the polynomial $S$, as long as $|\phi^M| < |q^{-1/4}z_0^M|$ which will always be the case if $|\phi| < |z_0|$, are

\[
\xi \simeq (1 \pm 2q^{1/4}) z_0^M, \quad \xi \simeq \pm 4 q^{1/2} \left( \frac{\phi^2}{\Lambda} \right)^M,
\]

namely $M$ pairs of branch points along a circle of radius $|z_0|$ and $2M$ equally spaced branch points on a ring of radius $4^{1/M} q^{1/(2M)} |\phi^2/\Lambda|$.

In order to understand what the supergravity solutions dual to these vacua are, it will be useful to recall what are the BPS fractional branes at our disposal. They are obtained by wrapping D5 branes or anti-D5’s ($\eta = \pm 1$ below, respectively) on the exceptional 2-cycle with $n_f$ units of worldvolume flux. Their Wess-Zumino action reads

\[
S_{WZ} = \eta \mu_3 \int_{M^{3,1}} \tilde{c}_4 + (b + n_f) C_4,
\]

where $\tilde{c}_4$ is a twisted potential dual to $c$. We will use the notation $D5_{n_f}$ and $\overline{D5}_{n_f}$ for the fractional branes with flux (recalling that $n_f$ is gauge dependent while the D3-charge is gauge invariant). The BPS objects are those whose worldvolume flux ensures positive D3-charge $\eta (b + n_f)$, which then equals the tension (3.3); notice that when the D3 charge exceeds one, we simply have a marginally stable bound state of a fractional D3 brane with a number of regular D3 branes.

The picture which stems from the branch points of the curve and from the study of the RG flow is the following.

First, in the case $|\Lambda| < |\phi| < |z_0|$, whose corresponding RG flow is depicted in figure 4, the theory is conformal in the UV, down to $z_0$ where $M$ eigenvalues of one adjoint scalar break the gauge group to $SU(N + M) \times SU(N) \times U(1)^M$, triggering the RG flow. They correspond to $M$ semiclassical $\overline{D5}_{-1}$’s in the type IIB picture. At the scale $\phi$ there are $2M$ pairs of branch points at the positions of the $2M$ VEV’s of the other adjoint scalar, which break further to $SU(N - M) \times SU(N) \times U(1)^M$ and invert the RG flow. They correspond to $2M$ semiclassical D5’s in the geometry, which invert the twisted fluxes; in particular $b$ starts to grow as the radius decreases. At a lower energy scale $q^{1/(2M)} \phi^2/\Lambda$ the $SU(N)$ coupling diverges, instantons break
the gauge group further to the conformal $SU(N - M) \times SU(N - M)$ theory with one divergent coupling (times the $U(1)^{4M}$ factor), and we find $2M$ branch points equally spaced along a ring. In type IIB, $b$ reaches the value 1 at the ring and there leaves $M$ tensionless $\overline{D5}_{-1}$’s smeared over the enhançon ring. It is possible to see by adding a $\overline{D5}_{-1}$ probe that it cannot penetrate into the interior, whereas a $D5_0$ can penetrate the enhançon ring, unchaining a $\overline{D5}_{-1}$ from it and making a D3 brane, which is free to move inside.

There is a more interesting behavior in the case of $|\phi| < |\Lambda|$. If $\phi = 0$ we are at the enhançon vacuum of Section 3. When $\phi$ does not vanish, the branch points follow
the pattern of figure 5 whereas the RG flow is the one depicted in figure 6. As before, \( M \, \overline{\text{D}5}_{-1} \)'s are placed at the cutoff scale \( z_0 \). From that scale downwards there is a flow with decreasing \( b \) towards smaller radii, and an enhançon ring with \( 2M \) equally spaced branch points at \( \Lambda \), where \( b \) reaches 0 and \( M \) tensionless \( \text{D}5_i \)'s are melted on the ring. At lower energies the theory includes the conformal \( SU(N) \times SU(N) \) factor with one divergent coupling: \( b = 0 \) in the dual supergravity solution, because of the \( M \) fractional branes at the enhançon ring. One could have expected that a new flow would start at a scale \( \phi \) because of the VEV’s, but it does not: it actually starts only at a lower scale \( \phi^2/\Lambda \), where there are \( 2M \) additional equally spaced branch points; below this energy scale, the gauge group with divergent coupling starts running towards weak coupling again, whereas the other one runs towards strong coupling. We enter a new perturbative regime, which ends with a final ring of equally spaced branch points at scale \( q^{1/(2M)} \phi^2/\Lambda \) where one gauge coupling diverges; in the interior we find a new conformal \( SU(N - M) \times SU(N - M) \) sector, with one divergent coupling, down to the IR.

We will call the ring at scale \( \phi^2/\Lambda \) an anti-enhançon. From the supergravity point of view it is indistinguishable from a usual enhançon. However from the field theory point of view it is quite peculiar: it represents instantonic effects that break the upper conformal theory to a running one. These effects at the scale \( \phi^2/\Lambda \) are triggered by VEV’s at the scale \( \phi \): they take some “affine RG time” to break the group; moreover this means that the effective conformal theory must have some remnant of the scale \( \Lambda \). These issues deserve further investigations.
We dub the regions between enhançon and anti-enhançon rings, where $b \in \mathbb{Z}$ and the theory enjoys a superconformal phase, *enhançon bearings*.

It turns out that one can construct two different type IIB solutions that describe this RG flow. The first one, say Higgsing-inspired (H), by continuity with the case $|\phi| > |\Lambda|$ where a perturbative Higgs mechanism takes place, interprets the ring of branch points at $\phi^2/\Lambda$ as an anti-enhançon made of $M$ tensionless $\overline{D5}_0$'s (like the ones at $\Lambda$), which therefore force $b$ to grow as the radius decreases, so that it remains bounded by 0 and 1. The innermost ring, placed where $b$ reaches 1, is an enhançon ring made of smeared tensionless $\overline{D5}_{-1}$. In this picture the $D5_0$'s ($\overline{D5}_{-1}$'s) are always associated to the first (second) gauge group.

The second, say cascade-inspired (C), works by analogy with the Klebanov-Tseytlin-Strassler $\mathcal{N} = 1$ cascade and interprets the ring of branch points at $\phi^2/\Lambda$ as an anti-enhançon made of $M$ tensionless $\overline{D5}_0$, and $b$ becomes negative at smaller radii. Then $b$ is monotonic, and the innermost ring at $b = -1$ is interpreted as an enhançon ring made of $M$ tensionless $D5_1$. This is the picture that matches with the solution in (2.11)-(2.12) and which is usually considered in the literature. The association between fractional branes and gauge groups is such that wrapped (anti)D5 branes always correspond to the larger (smaller) gauge group.

Type IIB solutions like the two we are discussing here can be explicitly constructed by excising and gluing twisted fields of the solution in (2.11)-(2.12) (possibly generated by one or the other kind of fractional branes) and of a fluxless solution, with suitable sources accounting for the discontinuities at the glued surfaces, along the lines of [45]. As already stressed in the case of the ordinary enhançon ring, this excision and gluing procedure works for twisted fields, which are constrained to the orbifold fixed plane. Instead, untwisted fields like the metric can propagate also in the four dimensions of the orbifold, and must be computed once the twisted fields and fractional brane configuration is specified; this will be done in section 6. It should be remarked that they turn out to be the same in the two pictures. One immediately realizes that all gauge invariant quantities one could compute from the two solutions will give the same answer, and in the field theory moduli space we have only one vacuum to match with the two solutions. This suggests that an ambiguity must be at work.

The ambiguity is particularly apparent in the T-dual type IIA/M-theory description. In type IIA, on each NS5-brane there is some worldvolume $G_1 = dA_0$ flux. Space-time filling I3 brane intersections of codimension two, where D4 branes end on an NS5 brane, are magnetic sources for $A_0$; the flux $\oint G_1$ through any closed path in
Figure 7: IIA description and ambiguity. (a) a point of the moduli space where $2M$ D4 branes are stretched between two NS5 branes. (b) another point where the D4 branes have collapsed to zero length. In the H-picture we interpret the D4 branes as still present, providing bending tension and flux jump; in the C-picture, the D4 branes are simply not there.

the 2 dimensions of the NS5 worldvolume parametrized by $v$, in which I3 branes are points, jumps by one unit whenever the path crosses one of these points. In what follows we will consider circular paths centered in the origin of the $v$ plane. One direction transverse to the NS5’s, say $x^6$, is compact of radius $R$ and the distance between the two NS5-branes is $2\pi b R$. In figure 7(a) we plotted the local geometry around a ring where the perturbative Higgsing takes place as in the RG flow of figure 4: the NS5 on the left has a flux $\oint G_1 = -M$ (in suitable units) below the stretched D4 branes, that jumps to $M$ above the D4’s, while the opposite happens to the NS5 on the right whose flux jumps from $M$ to $-M$. Along a generalized enhançon ring $b$ is integer valued, so that the stretched D4’s are degenerate and the NS5’s touch, as in figure 7(b). This interpretation leads to the H-picture in IIB: $b$ has a saw-shaped profile bounded by $[0, 1]$ and there are $2M$ fractional branes of one kind in the enhançon bearing, $M$ on each boundary. But the same IIA configuration can be equally well interpreted as two NS5 branes that just cross, without any D4 branes between them and without any
jump in the flux. This leads to the C-picture in IIB: $b$ is monotonic, and the bearing has fractional branes on one side and anti-fractional on the other side, which cancel their charge. In the type IIA picture there is clearly a single configuration (dual to a single vacuum in field theory) which gives rise to two pictures in IIB.

In type IIB, the ambiguity is related to S-duality: the duality group $PSL(2, \mathbb{Z})$ acts covariantly on the parameter space, whilst the left over $\mathbb{Z}_2$ that acts as $(B_2, C_2) \to (-B_2, -C_2)$ and $(b, c) \to (-b, -c)$ on the twisted fields, is gauged. The novel feature here is that the enhançon bearings are domain walls on the C orbifold line, and the $\mathbb{Z}_2$ can act on each domain separately. At the same time, as already stressed, the ambiguity does not affect the untwisted fields: $F_5$ and the warp factor are the same in the two pictures, since they depend on the twisted fields only quadratically in their field strengths; $B_2$ and $C_2$ are zero in the bulk.

We can keep playing the same game of adding suitable VEV’s, explained so far in this section, to the newly found solutions, so as to generate longer and longer RG flows with more and more transitions and reductions of degrees of freedom. Of course the number of steps is at most $[N/M]_-$. In this way we produce a class of vacua with a sort of cascading behavior, with cascades of different lengths.

We conclude discussing the behavior of probes through the enhançon bearing, as extracted from the branch points of the SW curve with a pair of VEV’s added in the perturbative regime outside the bearing, and interpreting it in the C-picture (the other one is equivalent). Consider first moving the VEV for the adjoint scalar of the gauge group related to the branch points of the bearing, keeping the VEV for the other adjoint fixed. As we decrease the VEV towards the outer enhançon scale, the two branch points reach the ring and there split and melt into it. Nothing happens until the VEV becomes smaller than the scale of the inner anti-enhánçon scale, when two branch points escape from this ring, pair up and then continue their motion as almost double branch points. In the C-picture, this corresponds to a $D5_0$ which melts at the outer enhançon, and later comes out of the inner anti-enhánçon as a $\overline{D5}_0$. Similarly, we can move the VEV for the adjoint scalar of the other gauge group. The corresponding two branch points cross the outer enhançon ring, unchaining two of its branch points. When they reach the inner ring, they leave two branch points there and move on. In the C-picture, this corresponds to a $\overline{D5}_{-1}$ that captures a $D5_0$ at the enhançon and becomes a D3-brane, free to move inside the bearing; then it leaves a $\overline{D5}_0$ at the anti-enhánçon and becomes a $D5_1$ which is a minimal BPS object in the region $b \in [-1, 0]$ below the anti-enhánçon ring. This behavior of probes through the enhançon bearings in the case
of monotonic $b$ precisely accounts for the non-trivial rearrangement of minimal objects in BPS bound states claimed in [11].

5.1 Reconstructing the cascading vacuum at the baryonic roots

We can now connect the enhançon bearing vacua discussed so far with the cascading vacuum at the baryonic roots of section 4. Such a cascading vacuum has the property that all the complexified strong coupling scales along the cascade are related by the same hierarchy $q^{1/2M}$, which ensures that, at least for $q \to 0$, the branch points pair up.

We start from a vacuum with an enhançon bearing and send the thickness of the bearing to zero sending $|\phi| \to |\Lambda|$ for the relevant strong coupling scale $\Lambda$. So doing, we end up with a single circle at scale $\Lambda$ where $4M$ branch points lie, $2M$ coming from inside and $2M$ coming from outside. For generic phases of $\phi$, these branch points do not pair up, and on the type IIB side we end up with a source term at the glued surface, accounting for a discontinuity of $c$. If instead the phase of $\phi$ is suitably tuned, branch points coming from the outer boundary and branch points coming from the inner boundary of the bearing collide, hence forming double branch points. Repeating the game with a vacuum with many enhançon bearings, we can obtain the cascading vacuum along the baryonic roots sending the thickness of each bearing to zero, see figure 8.

In type IIB, as we reduce the bearing to zero thickness we make the two smeared sources at the inner and outer boundaries of the bearing coincide. Following [12], we call the resulting shell a generalized enhançon ring. In the H-picture, this is made of $2M$ tensionless fractional branes, which account for the $U(1)^{2M}$ factor left over by the gauge breaking. The presence of the $2M$ massless hypermultiplets is more difficult
to be claimed: one could think of them as arising at the $2M$ points along the ring where $\gamma \in \mathbb{Z} + \tau \mathbb{Z}$; however they should only be massless for the correct tuning of the phase of $\phi$. Our belief is just that the IIB supergravity description is incomplete at the enhançon bearings. On the contrary, in the M-theory description the mass of BPS hypermultiplet states is given by the mass (proportional to the area) of M2 disks ending on the M5 brane [22, 46, 47, 48] which is the same as the SW curve; it is easy to see that the $2M$ double branch points corresponds to massless hypermultiplets.

In the C-picture the generalized enhançon is made of $M$ fractional and $M$ anti-fractional branes, both tensionless and D3-chargeless. When the phase of $\phi$ is suitably chosen and the inner and outer branch points coincide as we shrink the bearing, the D5-charges locally cancel leaving the continuous supergravity solution of Section 2 otherwise a source remains accounting for the discontinuity of $c$, and one might think of smeared dipoles of fractional/anti-fractional branes. In this picture the identification of the field theory modes is even subtler: even when a perfect annihilation seems to occur, this cannot be the case as the $U(1)^{2M}$ factor must still be there.

Let us conclude commenting on how the cascading vacuum at subsequent baryonic roots naturally arises as the dual of the supergravity solution of section 2. Such supergravity solution was constructed imposing rotational isometry on the $\mathbb{C}$ orbifold line and without introducing any source. Rotational isometry translates to $\mathbb{Z}_{2M}$ symmetry in field theory, whilst absence of sources requires all the VEV’s to be at a strongly coupled scale. Among these vacua, only the cascading vacuum in the C-picture avoid seeming discontinuities of $c$ (theta angles) and $\theta$.

### 5.2 More bearings: the enhançon plasma

So far we have described a class of $\mathbb{Z}_{2M}$-symmetric solutions of IIB supergravity, corresponding to vacua of the dual field theory with the same property, characterized by the presence of the enhançon plasma in the shape of fat rings (that we called enhançon bearings). From a simple numerical inspection of the field theory Coulomb branch, one discovers that the enhançon plasma can take quite different shapes (see for instance figure 9). We give here a general characterization of such vacua, in the large $N$ limit.

We will show that from the point of view of IIB supergravity any choice of the enhançon plasma domains, with the only constraint of charge quantization, leads to an actual solution and represents a field theory vacuum. For definiteness, we will study the $SU(N) \times SU(N)$ conformal theory with $b = \frac{1}{2}$, spontaneously broken to non-conformal theories. Thus first of all we distribute some number of anti-fractional branes in a
circular ring of radius $\rho_0$ in the $\mathbb{C}$-plane orbifold singularity. Then we will arbitrarily specify the enhançon plasma domains, without any restriction on the number of their holes and allowing nested domains.

The strategy to construct IIB supergravity solution is to solve for the twisted potentials $b$ and $c$ first, and then for $F_5$ and the warp factor.

The enhançon plasma domains behave as conductors for the objects carrying D5 charge, so that charges distribute themselves on the boundaries and inside there are no fields: $b$ and $c$ are constant with $b \in \mathbb{Z}$. Outside the plasma domains there are regions $\mathcal{D}_i$ where $b$ and $c$ are non-trivial. Consider one of these regions, with its boundary given by a collection of curves $\mathcal{C}_{i,\alpha}$: there is one external curve $\mathcal{C}_{i,E}$ while we call the internal ones $\mathcal{C}_{i,\alpha}$. The boundary conditions in $\mathcal{D}_i$ are that $b \in \mathbb{Z}$ on each curve $\mathcal{C}_{i,\alpha}$, and since we choose not to have generalized enhançon rings nor tensionful fractional branes around (they both can be obtained by sending to zero thickness an enhançon plasma with fat ring or circular shape), up to gauge transformations and picture ambiguity $b = 1$ on $\mathcal{C}_{i,E}$ and $b = 0, 1$ on $\mathcal{C}_{i,\alpha}$. The only exception is the outermost region $\mathcal{D}_E$ where $b = \frac{1}{2}$ on the external ring at $|z| = \rho_0$ and the branes are tensionful, while $b = 0$ on $\mathcal{C}_{E,\alpha}$. Supersymmetry constrains $\gamma(z) = c + \frac{1}{g_s} b$ to be a local meromorphic function, and after our choice of boundary conditions actually holomorphic. To be precise, $e^{-i\gamma}$ must be a holomorphic section of a $\mathbb{C}^*$ bundle. Rephrasing, we look for a harmonic real function $b$ with fixed boundary conditions, and a local harmonic real function $c$ which satisfies the Cauchy-Riemann relations.

The problem of finding a harmonic function $b$ with prescribed values on the bound-
aries $\mathcal{C}_{i,\alpha}$ has one and only one solution. It can be found by minimizing the functional

$$D[u] = \int_{\mathcal{D}_i} |\partial u|^2$$

amongst all $u \in C^{(1)}(\mathcal{D}_i \setminus \bigcup_\alpha \mathcal{C}_{i,\alpha}) \cap C^{(0)}(\mathcal{D}_i)$ with $u|_{\mathcal{C}_{i,\alpha}} = b(C_{i,\alpha})$. A local harmonic function that satisfies the Cauchy-Riemann relation can be constructed as

$$c(z) = \frac{1}{g_s} \int_{p_0}^z \left( \partial_y b \, dx - \partial_x b \, dy \right), \quad g_s \, dc = \partial_y b \, dx - \partial_x b \, dy = - \ast db,$$

where $z = x + iy$, $p_0$ is an arbitrary reference point and $\ast$ is constructed with the flat metric on $\mathbb{C}$.

As we will explain in section 6, the warp factor is obtained by solving a Poisson equation (6.1) on the orbifold $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, with two kinds of D3-charge source terms, both localized along the orbifold line. One is proportional to $|\partial_z \gamma|^2$ and comes from the twisted fluxes. The other one is localized on the tensionful fractional branes in the external ring and represents their D3-charge. In general the brane density per unit length $\omega$ along the ring is not constant but rather given by

$$\omega = -\frac{1}{2} \Re \partial_t \gamma,$$

where the derivative is taken tangent to the boundary. This comes from the Bianchi identity $dF_3 \sim \delta_{D5}^{(4)}$. On the boundaries of the enhançon plasma domains there are fractional branes too with density (5.9), but they are tensionless as $b \in \mathbb{Z}$ inside. Thus the only contribution of the latter kind comes from the circular cutoff ring at $|z| = \rho_0$.

So far we showed that for any choice of the enhançon plasma domains, we can in principle solve the supergravity equations. The last constraint is the D5-charge quantization, which amounts to the monodromy of $c(z)$ being quantized

$$\oint dc \in 2\mathbb{Z},$$

or in other terms $e^{-i\pi\gamma}$ being a single-valued function. A basis of monodromies is given by $\Delta c(\tilde{\mathcal{C}}_{i,\alpha})$ on the internal boundaries $\tilde{\mathcal{C}}_{i,\alpha}$, and the integral is the total number of fractional branes on them. As the solution only depends on the choice of the enhançon
plasma boundaries (and the value of \( b \) on them), (5.10) descends to a constraint (in fact the only one) for them.

The total D3-charge of the system is then easily determined. The contribution from the fluxes in all the regions \( \mathcal{D}_i \) is

\[
Q^{\text{flux}}_3 = \frac{1}{2} \sum_i \int_{\mathcal{D}_i} dc \wedge db = -\frac{1}{2} \sum_i \sum_\alpha b_c \int_{\mathcal{C}_{i,\alpha}} b dc = -\frac{1}{2} \sum_{i,\alpha} b(\mathcal{C}_{i,\alpha}) \Delta c(\mathcal{C}_{i,\alpha}) . \tag{5.11}
\]

The contribution from the anti-fractional branes on the external cutoff ring can be read from (5.9) to be: \( Q^{\text{cutoff}}_3 = -\frac{1}{4} \oint_{\rho_0} dc \), because \( b = \frac{1}{2} \) there. Since the external ring is the external boundary \( \mathcal{C}_{E,E} \) of the outermost region \( \mathcal{D}_E \), this contribution can be added to (5.11) by formally considering \( b(\mathcal{C}_{E,E}) = 1 \) instead of \( 1/2 \). Notice that (5.11) is gauge and picture invariant. However, for our choice of gauge and picture the total charge is

\[
Q^{\text{total}}_3 = \sum_{i,\alpha} \left( 1 - b(\tilde{\mathcal{C}}_{i,\alpha}) \right) \Delta c(\tilde{\mathcal{C}}_{i,\alpha}) \equiv N , \tag{5.12}
\]

where we used that \( \mathcal{C}_{i,E} = -\sum_\alpha \tilde{\mathcal{C}}_{i,\alpha} \) in homology, and \( b(\mathcal{C}_{i,E}) = 1 \). This expression counts the number of fractional (as opposed to anti-fractional) branes. And in fact the solution we constructed is dual to a vacuum of the \( SU(N) \times SU(N) \) theory. It is clear that if we want to embed this vacuum in a larger theory, we can simply add regular D3 branes.

Summarizing, we have shown that any choice of enhançon plasma domains, up to the charge quantization constraint, gives rise to a solution of IIB supergravity with sources. Taking the limit of zero thickness, we can also include generalized enhançons and isolated bunches of fractional branes; bunches of regular branes are easily included as well. Each of these solutions is dual to a vacuum on the Coulomb branch of the \( SU(N) \times SU(N) \) SCFT. Even though we cannot be more specific about the exact map (it should be worked out by computing operator VEV’s holographically), this huge class of solutions helps in covering the moduli space of the dual field theory.

6 Excisions, warp factors and the cure of repulson singularities

In this section we take into account the excision of twisted fields inside the enhançon ring and bearings and work out the correct warp factor for a quite general rotationally symmetric configuration of fractional branes, which will be useful to describe the enhançon vacuum of section 3, the cut off cascading vacuum of section 2, the infinite
cascade vacuum of section 4.3 and the vacua with rotationally symmetric bearings of section 5.

We stress once again that consistency of the configuration of fractional branes, in agreement with the dual field theory picture encoded in the SW curve, implies an excision of the naive twisted field solution at enhançon rings. Unlike the situation of [21], where there is an enhançon shell of codimension 1 in the non-compact part of the internal geometry, here we face enhançon rings having codimension 1 only for the twisted fields which are constrained to live on the orbifold plane, but not for the bulk fields which propagate also in the four additional dimensions of the orbifold. Consequently, the usual excision of [21, 45] works for twisted fields but not for untwisted fields; in particular, the warp factor has to be computed once and for all, once the correct configuration of fractional branes and twisted fields describing some gauge theory vacuum is specified.

The equation which determines the warp factor $Z$ follows from the modified Bianchi identity for $F_5$ in the presence of sources at the locations of tensionful (anti-)fractional branes; it is a Poisson’s equation which reads [10]

$$\Delta_6 Z + (4\pi^2\alpha')^2 g_s^2 |\partial z|^2 \delta^{(4)}(\vec{x}) + 2(4\pi^2\alpha')^2 g_s \sum_i Q(x_i) \delta^{(6)}(x - x_i) = 0 \ , \quad (6.1)$$

where $\Delta_6$ is the 6-dimensional Laplacian and $x = (\vec{y}, \vec{x})$ a 6-dimensional vector, $\vec{y} \equiv (\Re z, \Im z) = (x^4, x^5)$ being a vector on the orbifold fixed plane $\mathbb{R}^2$ and $\vec{x} = (x^6, \ldots, x^9)$ being a vector in the covering space $\mathbb{R}^4$ of the orbifold. In the previous formula, $Q(x_i)$ is the gauge invariant D3 brane charge of a regular or (anti-)fractional D3 brane placed at $x_i$, which depends on the object and on the value of fields at its position (in the case of fractional branes). The sum runs over all tensionful fractional D3 branes as well as regular D3 branes along with their images.

We will first consider $M$ tensionless fractional branes melted in an enhançon ring of radius $\rho_e$ in the fixed plane parametrized by $z$, together with $M$ ‘cutoff’ anti-fractional branes at the $M$ roots of $z^M = -z_0^M$, which are used to Higgs the conformal UV theory at the scale $\rho_0 = |z_0|$. Here and in the following, $\rho_0$ is the scale at which the excision should be performed and its actual value depends, case by case, on the vacuum one is actually considering. We will also impose that the total gauge invariant D3 brane charge of the configuration be $N + M$, adding regular D3 branes at the origin when needed, so that the dual gauge theory is $SU(N + M) \times SU(N + M)$ in the UV. Using the freedom of shifting the axion $b$ by an integer via a large gauge transformation, we will also set $b(\rho) = 0$ for $\rho < \rho_e$. Finally, we will be general and place the cutoff anti-
fractional branes at a scale \( \rho_0 \) such that \( b(\rho_0) \) can acquire any positive value; the gauge invariant D3 brane charge supported by each of the anti-fractional branes is therefore

\[
- n_f - b(\rho_0) = [b(\rho_0)]_+ - b(\rho_0) .
\] (6.2)

In other words, these anti-fractional branes are D5 branes wrapped on \(-C\), with \(-[b(\rho_0)]_+\) units of worldvolume flux on it. Being in the large \(M\) limit, we can safely approximate the cutoff anti-fractional branes with a ring.

The warp factor gets different contributions. First of all, if there are some regular D3 branes at the origin, they source the usual term according to (2.4). Secondly, the \(M\) cutoff anti-fractional branes, because of their tension, contribute the following term in the ring approximation

\[
Z_{\text{ring},M}(\rho, \sigma; \rho_0) = 8\pi g_s M^\alpha \left( [b(\rho_0)]_+ - b(\rho_0) \right) \frac{\sigma^2 + \rho^2 + \rho_0^2}{[\sigma^2 + \rho^2 + \rho_0^2]^2 - 4\rho_0^2 \rho^2}^{13/2} ,
\] (6.3)

where \(\rho = |\vec{y}|\) and \(\sigma = |\vec{x}|\). Fractional branes at the enhançon ring, being tensionless, do not contribute directly to the warp factor. Finally, there is a term sourced by the twisted field strengths

\[
d\gamma = \frac{i M}{\pi} dz \Theta(|z| - |z_0|) \Theta(|z_0| - z) .
\] (6.4)

In general it takes the form

\[
Z_{\text{fI}}(\vec{y}, \vec{x}) = 4\pi \alpha'^2 g_s^2 \int d^2 z \left| \partial_z \gamma \right|^2 \frac{1}{[|\vec{x}|^2 + |\vec{y} - \vec{z}|^2]^2} ,
\] (6.5)

which in the case under consideration reduces to

\[
Z_{\text{fI},M}(\rho, r; \rho_c, \rho_0) = \frac{2(g_s M \alpha')}{r^4} \left\{ 2 \log \frac{r^4 + \left( \rho_c^2 + \sqrt{(r^2 + \rho_c^2)^2 - 4\rho_c^2 \rho_0^2} \right) r^2 - 2\rho_c^2 \rho_0^2}{r^4 + \left( \rho_0^2 + \sqrt{(r^2 + \rho_0^2)^2 - 4\rho_0^2 \rho^2} \right) r^2 - 2\rho_0^2 \rho^2} + 2 \log \frac{\rho_0^2}{\rho_c^2} + \frac{r^2}{r^2 - \rho^2} \left[ \frac{3(r^2 - \rho^2) + \rho_0^2 - \rho^2}{\sqrt{(r^2 + \rho_0^2)^2 - 4\rho_0^2 \rho^2}} \right] \right\} ,
\] (6.6)

where \(r^2 = \rho^2 + \sigma^2\).

Notice that the total D3 brane charge, which is \(N + M\) if the UV theory has gauge group \(SU(N + M) \times SU(N + M)\), gets sectioned in different pieces. The flux term carries a charge \(M b(\rho_0)\), since \(b(\rho_0) = \frac{g_s M}{\pi} \log \frac{\rho_c}{\rho_0}\); the cutoff anti-fractional branes carry a charge \(M [b(\rho_0)]_+ - b(\rho_0)\); finally, there are \(N - [b(\rho_0)]_- M\) regular D3 branes at the origin. This can be checked via the large \(r\) asymptotics of the different terms in the warp factor.
The vacuum considered in [13] and described in section 3 has \(N\) regular D3 branes at the origin, the enhançon ring at \(\rho_1 = e^{-\frac{\pi}{g_s M} \rho_0}\), and \(M\) cutoff anti-fractional branes at \(\rho_0\), where \(b(\rho_0) = \frac{1}{2}\), carrying \(M/2\) units of D3 charge; the twisted fluxes between fractional and anti-fractional branes carry other \(M/2\) units of D3 charge.

The vacuum with a finite cascade starting at \(z_0\) and reaching \(SU(M)\) in the infrared has no regular D3 branes at the origin, \(M\) fractional branes with no D3 charge melted at an enhançon ring at \(\rho_{\text{min}} = \rho_{N/M} \equiv e^{-\frac{\pi N}{g_s M} \rho_1}\), and \(M\) cutoff anti-fractional branes at \(\rho_0\), where \(b(\rho_0) = \frac{N}{M} + \frac{1}{2}\), carrying again \(M/2\) units of D3 charge; this time the twisted fluxes between fractional and anti-fractional branes carry \(N + M/2\) units of D3 charge. As we explained in detail in section 5, what happens is that at each generalized enhançon ring scale along the cascade (where \(b \in \mathbb{Z}\)) melted tensionless fractional and anti-fractional branes are left, naively annihilating if \(c\) is continuous crossing radially the generalized enhançon ring. In case \(N = lM + p\) is not a multiple of \(M\), then \(\rho_{\text{min}} = e^{-\frac{\pi}{g_s M} \rho_1}\), \(b(\rho_0) = l + \frac{1}{2}\) and there are \(p\) D3 branes at the origin: the IR theory below the enhançon scale is the \(SU(p) \times SU(p)\) theory with one infinite coupling.

The infinite cascade limit can even be defined continuously: it is enough to send continuously the cutoff \(\rho_0 \to \infty\) keeping \(\rho_{\text{min}}\) fixed and \(b(\rho_{\text{min}}) = 0\). This can be achieved if \(b(\rho_0) = \frac{g_s M}{\pi} \ln \frac{\rho_0}{\rho_{\text{min}}}:\) as we change the cutoff \(\rho_0\), we also change the value of the gauge couplings at the cutoff (and on the string side the tension of the cutoff branes) so that low energy physics is not modified. Notice that every time a \(b(\rho_0) \in \mathbb{Z}\) threshold is crossed, the total D3 brane charge of the configuration (the ranks of the UV CFT) jumps by \(M\) units, and the cutoff anti-fractional branes change. The warp factor for the infinite cascade with no regular D3 branes is nothing but \(Z_{fl,M}(\rho, r; \rho_{\text{min}}, \infty)\), see eq. (6.6). If needed, the addition of \(p\) regular D3 branes is straightforward.

We can also find the warp factor for a configuration with any number of rotationally symmetric bearings. The total warp factor is sourced by twisted fluxes and possibly by cutoff anti-fractional branes, if there is no infinite cascade in the UV. Inside bearings fluxes vanish, whereas outside they take the usual form \(|d\gamma| = \frac{M}{\pi} \frac{dp}{p}\). Therefore fluxes contribute to the warp factor by a sum of terms taking the schematic form \(Z_{fl,M}(\rho, r; \rho_1^{(i+1)}, \rho_1^{(i)})\), where \(\rho_1^{(i+1)}\) is the outer radius of the \((i + 1)\)-th bearing and \(\rho_1^{(i)}\) is the inner radius of the \(i\)-th bearing, if the ordering points inwards. The requirement that \(\rho_1^{(i)}\) and \(\rho_1^{(i+1)}\) be boundaries of subsequent bearings translates into \(\rho_1^{(i)} = e^{\frac{\pi n_i}{g_s M}} \rho_1^{(i+1)}\), for some \(n_i \in \mathbb{N}\).

Finally, by now it should also be clear how to write the warp factor in the case of perturbative Higgsings by backreacting rings of tensionful fractional and anti-fractional
branes, adding terms like $6.3$ sourced at suitable radii and with the suitable normalizations.

We end this section with some important remarks about the backreacted geometries. For concreteness, we concentrate on solutions without bearings nor perturbative Higgsing except at the cutoff, since the generalization of the statements we are about to make should be clear.

The warp factor diverges (and the gravitational potential felt by a massive particle has an absolute minimum) only at the locations of sources for it (fractional branes and twisted field strengths), namely on the orbifold plane $\sigma = 0$ and for $\rho \in [\rho_e, \rho_0]$. There are no repulsive regions even when the D3 brane charge vanishes at some IR scale, as occurs at the enhançon scale in the vacuum of $13$ with $N = 0$ and in the finite or infinite cascade solution with $p = 0$. Massive objects (but BPS ones) are always attracted by the sources of stress-energy: they want to go where twisted fluxes and fractional branes (and possibly regular D3 branes) lie. For concreteness, we report in figure 10 the shape of the effective potential $V(\rho, \sigma)$ felt by a massive particle: it is proportional to $Z^{-1/2}$, once the kinetic terms are normalized to be $(\frac{d\rho}{d\tau})^2 + (\frac{d\sigma}{d\tau})^2$, $\tau$ being the worldline proper time.

In these solutions the curvature diverges approaching the domain where twisted fluxes have support. Therefore, strictly speaking, the gravity solution cannot be trusted

Figure 10: Potential $V = V(\rho, \sigma)$ felt by a massive particle in the background dual to a vacuum with finite cascade. The origin is a saddle point while the absolute minimum is on the $\sigma = 0$ axis all along the range where the dual field theory undergoes a RG flow, from the enhançon radius $\rho_e$ up to the UV cut-off $\rho_0$. 

\[ V \sim Z^{-1/2} \]
in that region and string theory is needed to resolve the curvature singularity. Still, the M-theory picture suggests that the form of the twisted fields will remain unchanged.

Finally, if there are no D3 branes at the origin the geometry smoothly approaches flat space at $r = 0$, where the warp factor approaches

$$Z(0) = 2(g_s M_{\alpha'})^2 \left( \frac{1}{\rho_c^4} - \frac{1}{\rho_0^4} \right) + 8\pi g_s M_{\alpha'}^2 \left( [b(\rho_0)]_+ - b(\rho_0) \right) \frac{1}{\rho_0^4},$$

(6.7)

signaling that excitations in the non-abelian sector have a minimal energy (consistently with the $SU(M)$ factor being broken to $U(1)^{M-1}$). If instead there are regular D3 branes at the origin, they dominate the IR asymptotics which is $AdS_5 \times S^5 / \mathbb{Z}_2$, signaling a non-abelian fixed point.

7 Conclusions and outlook

In this paper, we filled a gap in the understanding of the gauge theory dual interpretation of supergravity solutions with running fluxes, arising when considering fractional branes at generic Calabi-Yau singularities. It has been known for some time that fractional branes at isolated singularities describe RG flows which can be described in terms of cascades of Seiberg dualities. A similar interpretation was not possible for branes at non-isolated singularities, since their effective dynamics is intrinsically $\mathcal{N} = 2$.

The basic outcome of our analysis is that, for branes at non-isolated singularities, the reduction of the gauge group ranks along the RG flow can be understood in terms of a sequence of strong coupling transitions reminiscent of the low energy description of the baryonic root of $\mathcal{N} = 2$ SQCD. The energy range spanned by the cascade depends on the point in the Coulomb branch one is sitting at; specifically, on the number of non-vanishing VEV’s for the adjoint scalars.

We were also able to provide a gravity dual description of a new set of infinitely many vacua, characterized by new geometric structures, the enhançon bearings, where the dual gauge theory alternates energy ranges where it runs, with ranges in which it is in a strongly coupled superconformal phase.

For all these vacua, an enhançon mechanism takes place in the far IR. This changes the twisted fields configuration and ultimately the metric, whose correct repulson-free expression we provided for all vacua we have been studying.

Our analysis focused, for definiteness, on the $A_1$ singularity, but our results have a much wider validity. First, they trivially extend to any $\mathcal{N} = 2$ singularity, as for instance the full ADE series. Second, any Calabi-Yau cone with non-isolated singularities, which upon the inclusion of branes generically gives rise to a $\mathcal{N} = 1$ theory, should
present the same behavior. This is suggested from the supergravity solution and it is a rather non-trivial claim since SW techniques are not available in the $\mathcal{N} = 1$ context.

More complicated flows occur when fractional branes at isolated and non-isolated singularities are both present, which is in fact the most generic situation. Such a setup was recently considered in [15] for a $\mathcal{N} = 1$ non-chiral $Z_2$ orbifold of the conifold, where the complete RG flow of the dual cascading theory was extracted from supergravity. The analysis revealed that, while most of the rank reductions are understood in terms of Seiberg duality, some of them cannot as the theory, due to the presence of adjoint fields, exhibits at some energies an effective $\mathcal{N} = 2$ behavior. We conjecture that in those cases too the rank reduction is due to the adjoint fields being at baryonic-root-like points of its moduli space. One can easily follow on the field theory side all the cascades extracted from supergravity in [15], finding perfect agreement with our proposal.

It has been proposed in [49, 50] that gravity duals of metastable dynamical supersymmetry breaking models involve systems where fractional branes at isolated and non-isolated singularities are both present. The repulson-free warp factors we have found may prove useful to check those claims further, since the dynamics of $\mathcal{N} = 2$-like branes seems crucial to describe the metastable vacua.

Finally, in view of our results, it would also be interesting to reconsider $\mathcal{N} = 2$ D3-D7 fractional brane systems and the corresponding gauge theory RG flows suggested by the known gravity duals [30].

Acknowledgments

We would like to single out Riccardo Argurio for collaboration at the beginning of this project, continuous exchange of ideas and comments, and Alberto Zaffaroni for important remarks and insights. We are also grateful to Marco Billò, Jarah Evslin, Igor Klebanov, Stanislav Kuperstein, Andrei Mikhailov, Yaron Oz, Rodolfo Russo, Cobi Sonnenschein and Shimon Yankielowicz for useful discussions. Finally, we are grateful to Ofer Aharony for suggestions and comments on a preliminary version of this paper. F.B. acknowledges the support of the US Department of Energy under Grant No. DE-FG02-91ER40671. C.C. is a Boursier FRIA-FNRS. The research of C.C. is also supported by IISN - Belgium (convention 4.4505.86) and by the “Interuniversity Attraction Poles Programme Belgian Science Policy”. The work of S.C. was supported in part by a center of excellence supported by the Israel Science Foundation (grant No. 1468/06), by a grant (DIP H52) of the German Israel Project Cooperation, by a BSF United States-Israel binational science foundation grant 2006157 and German Israel
A Effective field theory approach to the cascading SW curve

Let us check the statements of section 4.2 concerning the RG flow and the double points (4.9)-(4.10), using an effective field theory approach for the Seiberg-Witten curve between two strong coupling transitions. Defining $\xi \equiv v^M$ and $\alpha \equiv z_0^M$, we have the following Seiberg-Witten curve,

$$
\frac{\xi \prod_{j=0}^{h-1}(\xi^2 + q^{2j+2}\alpha^2)}{(\xi - \alpha) \prod_{j=0}^{h-1}(\xi^2 + q^{2j+2}\alpha^2)} = g(t|q) = q^{\frac{1}{4}}(t + \frac{1}{t}) + O(q^{\frac{3}{4}})
$$

(A.1)

Now, defining $\Lambda_2^M \equiv q^{j+\frac{1}{2}M}z_0^M$, we can look at the curve in the range $\Lambda_2 n < v < \Lambda_2 n - 1$, where, at small $q$,

$$
\frac{R}{S} \approx -q^{\frac{1}{4}}(\xi^2 + \Lambda_2^M) \Lambda_2^M \xi = g(t, q) \approx q^{\frac{1}{4}}(t + \frac{1}{t}),
$$

(A.2)

which gives

$$
\xi \Lambda_2^M t^2 + (\xi^2 + \Lambda_2^M)t + \xi \Lambda_2^M = 0,
$$

(A.3)

This is a SW curve for a $SU(2M)$ gauge group with $2M$ massless flavors [22], at the baryonic root (hence it has exact double points). Extracting the roots for $t$ (and neglecting $\Lambda_n/v$ because of large $M$), one finds

$$
\frac{u_1 = -u_2 = -\frac{M}{2\pi i} \log \left( \frac{v}{\Lambda_{2n} e^{-2\pi i k}} \right) = -\frac{M}{2\pi i} \log \frac{v}{\Lambda_{2n}} + \frac{1}{2},}
$$

(A.4)

where $k = 0, 1$. We see that at $v = \Lambda_{2n}$, $u_1 = u_2 = 0, \frac{1}{2}$ (that is, the two NS5’s intersect at $x^6 = 0$, but in fact the corresponding M5 brane also self-intersects at two distinct points on the torus). Since $\tau_1 = u_2 - u_1 = \tau - \tau_2$, we have reproduced the correct perturbative running of the gauge couplings. Also notice that this effective field theory for the first node is valid only up to $v = \Lambda_{2n-1}$, where according to (A.4)

$$
u_1 = u_2 = \frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}
$$

(that is when the coupling of the second gauge group hits a Landau pole).

One can perform the same analysis for the second gauge group, i.e. for the double points at $u = \frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}$, obtaining (A.10).
References

[1] S. S. Gubser and I. R. Klebanov, “Baryons and domain walls in an $\mathcal{N} = 1$ superconformal gauge theory,” Phys. Rev. D 58, 125025 (1998) [arXiv:hep-th/9808075].

[2] I. R. Klebanov and N. A. Nekrasov, “Gravity duals of fractional branes and logarithmic RG flow,” Nucl. Phys. B 574, 263 (2000) [arXiv:hep-th/9911096].

[3] I. R. Klebanov and A. A. Tseytlin, “Gravity duals of supersymmetric $SU(N) \times SU(N + M)$ gauge theories,” Nucl. Phys. B 578, 123 (2000) [arXiv:hep-th/0002159].

[4] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and $\chi$SB-resolution of naked singularities,” JHEP 0008, 052 (2000) [arXiv:hep-th/0007191].

[5] M. J. Strassler, “The duality cascade,” [arXiv:hep-th/0505153].

[6] D. Berenstein, C. P. Herzog, P. Ouyang and S. Pinansky, “Supersymmetry breaking from a Calabi-Yau singularity,” JHEP 0509, 084 (2005) [arXiv:hep-th/0505029].

[7] S. Franco, A. Hanany, F. Saad and A. M. Uranga, “Fractional branes and dynamical supersymmetry breaking,” JHEP 0601, 011 (2006) [arXiv:hep-th/0505040].

[8] M. Bertolini, F. Bigazzi and A. L. Cotrone, “Supersymmetry breaking at the end of a cascade of Seiberg dualities,” Phys. Rev. D 72, 061902 (2005) [arXiv:hep-th/0505055].

[9] K. A. Intriligator and N. Seiberg, “The runaway quiver,” JHEP 0602, 031 (2006) [arXiv:hep-th/0512347].

[10] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, “Fractional D-branes and their gauge duals,” JHEP 0102, 014 (2001) [arXiv:hep-th/0011077].

[11] J. Polchinski, “$\mathcal{N} = 2$ gauge-gravity duals,” Int. J. Mod. Phys. A 16, 707 (2001) [arXiv:hep-th/0011193].

[12] O. Aharony, “A note on the holographic interpretation of string theory backgrounds with varying flux,” JHEP 0103, 012 (2001) [arXiv:hep-th/0101013].
[13] M. Petrini, R. Russo and A. Zaffaroni, “$\mathcal{N} = 2$ gauge theories and systems with fractional branes,” Nucl. Phys. B 608, 145 (2001) [arXiv:hep-th/0104026].

[14] F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, “Backreacting Flavors in the Klebanov-Strassler Background,” JHEP 0709, 109 (2007) [arXiv:0706.1238[hep-th]].

[15] R. Argurio, F. Benini, M. Bertolini, C. Closset and S. Cremonesi, “Gauge/gravity duality and the interplay of various fractional branes,” Phys. Rev. D 78, 046008 (2008) [arXiv:0804.4470 [hep-th]].

[16] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536, 199 (1998) [arXiv:hep-th/9807080].

[17] T. J. Hollowood and S. Prem Kumar, JHEP 0412, 034 (2004) [arXiv:hep-th/0407029].

[18] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in $\mathcal{N} = 2$ supersymmetric QCD,” Nucl. Phys. B 431, 484 (1994) [arXiv:hep-th/9408099].

[19] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory,” Nucl. Phys. B 426, 19 (1994) [Erratum-ibid. B 430, 485 (1994)] [arXiv:hep-th/9407087].

[20] P. C. Argyres, M. R. Plesser and N. Seiberg, “The Moduli Space of $\mathcal{N} = 2$ SUSY QCD and Duality in $\mathcal{N} = 1$ SUSY QCD,” Nucl. Phys. B 471, 159 (1996) [arXiv:hep-th/9603042].

[21] C. V. Johnson, A. W. Peet and J. Polchinski, “Gauge theory and the excision of repulson singularities,” Phys. Rev. D 61, 086001 (2000) [arXiv:hep-th/9911161].

[22] E. Witten, “Solutions of four-dimensional field theories via M-theory,” Nucl. Phys. B 500, 3 (1997) [arXiv:hep-th/9703166].

[23] S. Kachru and E. Silverstein, “4d conformal theories and strings on orbifolds,” Phys. Rev. Lett. 80, 4855 (1998) [arXiv:hep-th/9802183].

[24] D. E. Diaconescu, M. R. Douglas and J. Gomis, “Fractional branes and wrapped branes,” JHEP 9802, 013 (1998) [arXiv:hep-th/9712230].
[25] M. Grana and J. Polchinski, “Gauge / gravity duals with holomorphic dilaton,” Phys. Rev. D 65, 126005 (2002) [arXiv:hep-th/0106014].

[26] M. R. Douglas and S. H. Shenker, “Dynamics of SU(N) supersymmetric gauge theory,” Nucl. Phys. B 447, 271 (1995) [arXiv:hep-th/9503163].

[27] E. Witten, “Some comments on string dynamics,” arXiv:hep-th/9507121.

[28] P. S. Aspinwall, “K3 surfaces and string duality,” arXiv:hep-th/9611137.

[29] J. D. Blum and K. A. Intriligator, “Consistency conditions for branes at orbifold singularities,” Nucl. Phys. B 506, 223 (1997) [arXiv:hep-th/9705030].

[30] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, “N = 2 gauge theories on systems of fractional D3/D7 branes,” Nucl. Phys. B 621 (2002) 157 [arXiv:hep-th/0107057].

[31] M. Billo, L. Gallot and A. Liccardo, “Classical geometry and gauge duals for fractional branes on ALE orbifolds,” Nucl. Phys. B 614, 254 (2001) [arXiv:hep-th/0105258].

[32] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, “Simple singularities and \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory,” Phys. Lett. B 344, 169 (1995) [arXiv:hep-th/9411048].

[33] P. C. Argyres and A. E. Faraggi, Phys. Rev. Lett. 74, 3931 (1995) [arXiv:hep-th/9411057].

[34] A. Giveon and D. Kutasov, “Brane dynamics and gauge theory,” Rev. Mod. Phys. 71, 983 (1999) [arXiv:hep-th/9802067].

[35] I. P. Ennes, C. Lozano, S. G. Naculich and H. J. Schnitzer, “Elliptic models and M-theory,” Nucl. Phys. B 576 (2000) 313 [arXiv:hep-th/9912133].

[36] S. Bolognesi, “A Coincidence Problem: How to Flow from N=2 SQCD to N=1 SQCD,” arXiv:0807.2456 [hep-th].

[37] A. Hanany and Y. Oz, “On the quantum moduli space of vacua of \( \mathcal{N} = 2 \) supersymmetric SU\( (N_c) \) gauge theories,” Nucl. Phys. B 452, 283 (1995) [arXiv:hep-th/9505075].
[38] P. C. Argyres, M. R. Plesser and A. D. Shapere, “The Coulomb phase of $\mathcal{N} = 2$ supersymmetric QCD,” Phys. Rev. Lett. 75, 1699 (1995) [arXiv:hep-th/9505100].

[39] J. Evslin, “The cascade is a MMS instanton,” [arXiv:hep-th/0405210].

[40] K. Hori, H. Ooguri and Y. Oz, “Strong coupling dynamics of four-dimensional $\mathcal{N} = 1$ gauge theories from M-theory fivebrane,” Adv. Theor. Math. Phys. 1, 1 (1998) [arXiv:hep-th/9706082].

[41] E. Witten, “Branes and the dynamics of QCD,” Nucl. Phys. B 507, 658 (1997) [arXiv:hep-th/9706109].

[42] O. Aharony, A. Buchel and A. Yarom, “Holographic renormalization of cascading gauge theories,” Phys. Rev. D 72, 066003 (2005) [arXiv:hep-th/0506002].

[43] J. de Boer, K. Hori, H. Ooguri and Y. Oz, “Kaehler potential and higher derivative terms from M theory five-brane,” Nucl. Phys. B 518, 173 (1998) [arXiv:hep-th/9711143].

[44] N. Halmagyi, K. Pilch, C. Romelsberger and N. P. Warner, “The complex geometry of holographic flows of quiver gauge theories,” JHEP 0609, 063 (2006) [arXiv:hep-th/0406147].

[45] C. V. Johnson, R. C. Myers, A. W. Peet and S. F. Ross, “The enhançon and the consistency of excision,” Phys. Rev. D 64, 106001 (2001) [arXiv:hep-th/0105077].

[46] A. Fayyazuddin and M. Spalinski, “The Seiberg-Witten differential from M-theory,” Nucl. Phys. B 508, 219 (1997) [arXiv:hep-th/9706087].

[47] M. Henningson and P. Yi, “Four-dimensional BPS-spectra via M-theory,” Phys. Rev. D 57, 1291 (1998) [arXiv:hep-th/9707251].

[48] A. Mikhailov, “BPS states and minimal surfaces,” Nucl. Phys. B 533, 243 (1998) [arXiv:hep-th/9708068].

[49] R. Argurio, M. Bertolini, S. Franco and S. Kachru, “Gauge/gravity duality and meta-stable dynamical supersymmetry breaking,” JHEP 0701, 083 (2007) [arXiv:hep-th/0610212].

[50] R. Argurio, M. Bertolini, S. Franco and S. Kachru, “Metastable vacua and D-branes at the conifold,” JHEP 0706, 017 (2007) [arXiv:hep-th/0703236].