Proof of the Strong Ivić Conjecture for the Cubic Moment of Maass-form $L$-functions

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In memory of Professor Aleksandar Ivić.

Abstract. In this paper, we prove the following asymptotic formula for the spectral cubic moment of central $L$-values:

$$
\sum_{t_f \leq T} \frac{2L\left(\frac{1}{2}, f\right)^3}{L(1, \text{Sym}^2 f)} + \frac{2}{\pi} \int_0^T \frac{|\zeta\left(\frac{1}{2} + it\right)|^6}{|\zeta(1 + 2it)|^2} \, dt = T^2 P_3(\log T) + O(T^{1+\varepsilon}),
$$

where $\zeta$ ranges in an orthonormal basis of (even) Hecke–Maass cusp forms, and $P_3$ is a certain polynomial of degree 3. It improves on the error term $O(T^{8/7 + \varepsilon})$ in a paper of Ivić and hence confirms his strong conjecture for the cubic moment. This is the first time that the (strong) moment conjecture is fully proven in a cubic case. Moreover, we establish the short-interval variant of the above asymptotic formula on intervals of length as short as $T^{\varepsilon}$.

1. Introduction

1.1. Ivić’s Moment Conjectures. Let $\mathcal{B}$ be an orthonormal basis of even Hecke–Maass cusp forms $f(z)$ for $SL_2(\mathbb{Z})$, with Laplacian eigenvalue $\frac{1}{4} + t_f^2$ ($t_f \geq 0$) and Fourier expansion

$$
f(z) = \sum_{n \neq 0} \rho_f(n) \sqrt{k} K_{it_f} (2\pi |n| y) e(nx), \quad z = x + iy, \quad y > 0.
$$

Let $L(s, f)$ and $L(s, \text{Sym}^2 f)$ be the standard and the symmetric square $L$-functions attached to $f(z)$. Define $\omega_f$ to be the harmonic weight

$$
\omega_f = \frac{|ho_f(1)|^2}{\cosh(\pi t_f)} = \frac{2}{L(1, \text{Sym}^2 f)}.
$$

Define the $k$-th moment of central $L$-values of height $T$ as follows:

$$
\mathcal{M}_k(T) = \sum_{t_f \leq T} \omega_f L\left(\frac{1}{2}, f\right)^k + \frac{2}{\pi} \int_0^T \frac{|\zeta\left(\frac{1}{2} + it\right)|^{2k}}{|\zeta(1 + 2it)|^2} \, dt.
$$

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In 2002, Ivič [Iv2] conjectured that
\begin{equation}
\mathcal{M}_k(T) = T^2 P_{(k^2-k)/2}(\log T) + O_\varepsilon(T^{1+c_k+\varepsilon}),
\end{equation}
where \( P_{(k^2-k)/2} \) is a suitable polynomial of degree \((k^2-k)/2\) whose coefficients depend on \( k \), and \( 0 \leq c_k < 1 \). He verified \((1.3)\) in the cases \( k = 3 \) and \( k = 4 \) with \( c_3 = 1/7 \) and \( c_4 = 1/3 \). Furthermore, a stronger conjecture of Ivič is that \( c_k = 0 \); namely
\begin{equation}
\mathcal{M}_k(T) = T^2 P_{(k^2-k)/2}(\log T) + O_\varepsilon(T^{1+\varepsilon}).
\end{equation}
For \( k = 0 \), it may be considered as the (weighted) Weyl law if the odd Maass forms are included (note that the central \( L \)-value vanishes if the form is odd), and Xiaoming Li [Li2] proved the currently best error bound \( O(T/\log T) \). For \( k = 1 \) and \( k = 2 \), \((1.4)\) is proven respectively by Ivič–Jutila [IJ] and Motohashi [Mot1] with sharper error terms \( O(T\log^2 T) \) and \( O(T\log^6 T) \). For \( k \geq 5 \), Ivič’s (weak) conjecture \((1.3)\) is still wide open.

For other typical families of \( L \)-functions, we refer the reader to the work of Conrey, Farmer, Keating, Rubinstein, and Snaith [CFK+ §1.3] for their moment conjectures and connection to the Random Matrix Theory. A common feature of these conjectures is the “square-root rule”—the magnitude of the error terms is the square root of that of the main terms. As such, one may consider the strong Ivič conjecture \((1.4)\) as the Maass-form analogue of these moment conjectures.

In this paper, we prove Ivič’s strong conjecture for \( k = 3 \). This is the first instance of the moment conjecture that is fully proven in the cubic case \( k = 3 \).

1.2. Main Results. More generally, we consider the cubic moment on short intervals. For \( T^\varepsilon \leq M \leq T/3 \) define
\begin{equation}
\mathcal{M}_3(T, M) = \sum_{\mid t_j - T \mid \leq M} \omega_j L\left(\frac{1}{3}, f\right)^3 + \frac{2}{\pi} \int_{T-M}^{T+M} \frac{\mid \zeta\left(\frac{1}{3} + it\right)\mid^6}{\mid \zeta(1 + 2it)\mid^2} dt,
\end{equation}
and its smoothed variant
\begin{equation}
\mathcal{M}_3^*(T, M) = \sum_{f \in \mathcal{H}} \omega_j L\left(\frac{1}{3}, f\right)^3 k_{T,M}(t_j) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mid \zeta\left(\frac{1}{3} + it\right)\mid^6}{\mid \zeta(1 + 2it)\mid^2} k_{T,M}(t) dt,
\end{equation}
where
\begin{equation}
k_{T,M}(t) = e^{-\left(t-T\right)^2/M^2} + e^{-\left(t+T\right)^2/M^2}.
\end{equation}
A well-known result for the cubic moment of \( L\left(\frac{1}{3}, f\right) \) on short intervals is the following average Lindelöf bound of Ivič [Iv1]:
\begin{equation}
\sum_{\mid t_j - T \mid \leq M} L\left(\frac{1}{3}, f\right)^3 \ll_\varepsilon M T^{1+\varepsilon},
\end{equation}
\footnotetext[1]{For \( k = 4 \), however, this result was first claimed in a preprint of Kuznetsov [Kuz2] in 1999, for which Ivič provided two proofs—the first is a correction and simplification of the original proof of Kuznetsov, and the second is an elaboration of a method of Jutila [Jut].}
\footnotetext[2]{There are abundant works related to the moments of \( L \)-functions for holomorphic modular forms, in either the level or the weight aspect. See for example [Duk] [IS] [KM] [Van] [BF] for \( k = 1, 2 \), [Cl] [Pen] [Pet] [You2] [PY1] [PY2] [Fro] for \( k = 3 \), [DFI1] [DFI2] [K MV] [BK] [KY] [Kha] for \( k = 4, 5 \), but some of which, especially for \( k = 3, 4, 5 \), are concerned with bounds instead of asymptotics.}
\footnotetext[3]{It is interesting to note the controversy for the cubic moment of quadratic Dirichlet \( L \)-functions: [CFK+] conjectured that the error term should have exponent \( 1/2 \), but [DGH] [Zha] suggested the existence of a lower-order term of exponent \( 3/4 \). See also [You1] and [AR].}
By the non-negativity of $L(\frac{1}{2}, f)$, its Weyl-type subconvex bound follows immediately from (1.8).

As shown in the next lemma, $\mathcal{M}_k(T, M)$ and $\mathcal{M}_k(T, M)$ are connected via an averaging process for the $T$-parameter.

**Lemma 1.1.** For $T^\epsilon \leq M^{1+\epsilon} \leq H \leq T/3$ we have

$$\mathcal{M}_3(T, H) = \frac{1}{\sqrt{\pi M}} \int_{T-H}^{T+H} \mathcal{M}_3(K, M) dK + O(\epsilon M^{1+\epsilon}).$$

This is a simple application of Lemma 5.3 and 5.4 in [LQ2] (the short-interval variant of the unsmoothing process in Ivić–Jutila [IJ §3]), along with Ivić’s bound (1.8), the non-negativity of $L(\frac{1}{2}, f)$, and also the Weyl subconvex bound for $\zeta(\frac{1}{2} + it)$.

Our main results are the following asymptotic formulae for $\mathcal{M}_6(T, M), \mathcal{M}_3(T, H)$, and $\mathcal{M}_3(T)$.

**Theorem 1.2.** For any $T^\epsilon \leq M \leq T^{1-\epsilon}$ we have

$$\mathcal{M}_6(T, M) = \sqrt{\pi} MT P_3'(\log T) + O(\epsilon T^{1+\epsilon}),$$

where $P_3'$ is an explicit cubic polynomial.

**Theorem 1.3.** For any $T^\epsilon \leq H \leq T/3$ we have

$$\mathcal{M}_3(T, H) = \int_{T-H}^{T+H} KP_3'(\log K) dK + O(\epsilon T^{1+\epsilon}).$$

**Corollary 1.4.** We have

$$\mathcal{M}_3(T) = T^2 P_3'(\log T) + O(\epsilon T^{1+\epsilon}),$$

for an explicit cubic polynomial $P_3$.

Although the error terms for $\mathcal{M}_6(T, M)$ and $\mathcal{M}_3(T, H)$ in Theorem 1.2 and 1.3 are of the same strength, the proof of the latter requires more refined analysis.

**1.3. Backgrounds and Our Method.** For the twisted second moment of central $L$-values for Maass forms, an explicit formula of Kuznetsov–Motohashi ([Kuz1, Mot1, Mot3]) is particularly useful. It was used by the “Troika”, Motohashi, Ivić, and Jutila [Mot1, Mot3, Ivi1, Ivi2, Jut] to study the second, third, and fourth moments, and recently by [Liu, BHS] to obtain lower bounds for the non-vanishing proportion.

In our recent work [LQ2], the results in [BHS] were recovered and extended to imaginary quadratic fields by the Kuznetsov–Voronoï approach rather than the Kuznetsov–Motohashi formula (such a formula is currently not available over imaginary quadratic fields). It naturally drives us to revisit the problem of Ivić for the cubic moment from this perspective.

The study of cubic moment for $\text{GL}_2$ via the (Petersson–Kuznetsov) trace formula and the (triple Poisson) summation formula was initiated in the groundbreaking work of Conrey and Iwaniec [CI]. Their focus is on the $q$-aspect, and, in the most simplified setting, their result in the Maass-form case reads as follows:

$$\sum_{t \leq T} L(\frac{1}{2}, f \times \chi_q)^3 + \int_0^T |L(\frac{1}{2} + it, \chi_q)|^6 dt \ll T^{1+\epsilon},$$

where $\chi_q$ is the quadratic character of square-free conductor $q$. 
For the spectral aspect, novel ideas were introduced by Xiaoqing Li \[LH1\] in her study of the first moment for $GL_3 \times GL_2$, and by Young \[You2\] for his hybrid version of Conrey and Iwaniec’s results: the former used the (Voronoi) summation formula twice, while the latter used the hybrid large sieve after the (triple Poisson) summation formula. Later, Nunes \[Num\] used Young’s idea to improve Xiaoqing Li’s subconvexity bounds for $GL_3$, replacing the second Voronoi by the large sieve, and the author \[Qi\] extended his results to arbitrary number fields.

Our approach, simply speaking, combines those of Xiaoqing Li and Young: After Kuznetsov, use Voronoi+Poisson, the second Voronoi, and finally the large sieve. The use of Voronoi+Poisson instead of the triple Poisson is beneficial for us to see the main term (see \[LQ2\] and the opening discussions in \[§\]. Similar ideas with the second Voronoi replaced by the $GL_3$ functional equation were used in the recent joint work \[LNQ\] to get strong subconvexity bounds for $GL_3$.

1.4. Remarks. The study of the $2k$-th moment of $\zeta(\frac{1}{2} + it)$ is quite a fascinating story on its own. More explicitly, the moment conjecture for $\zeta(\frac{1}{2} + it)$ reads

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt = TP_{k^2}((\log T)^2) + E_k(T),$$

with $P_{k^2}$ a certain polynomial of degree $k^2$ and $E_k(T) = O_<(T^{1/2+\epsilon})$.

See \[CFK+ \ §1.3\]. Along with the standard bound $1/|\zeta(1+it)| \leq \log t$ ($t > 3$) (see \[Tit\] Theorem 5.17), the moment conjecture for $\zeta(\frac{1}{2} + it)$ implies that in the asymptotic formula given by \[12\] and \[14\] the integral could be absorbed into the error term $O(T^{1+\epsilon})$. However, this can be done unconditionally only for $k = 1, 2$.

Our analysis suggests that the error term for the smooth cubic moment $M_3^k(T, M)$ is connected to the fourth moment of $\zeta(\frac{1}{2} + it)$ up to the height $U = T/M$ (see \[48\]), for which the bound $O(U^{1+\epsilon})$ is known as indicated above, so Theorem 1.2 is probably optimal. It is interesting to see whether a lower order term can be extracted from the error term.

The heuristic connection above should be regarded as the inverse of the Motohashi formula (see \[Mot2, Mot3\]), and it can be realized in explicit terms as the spectral moment formula of Chung-Hang Kwan \[Kwa\] or the spectral reciprocity formula in the on-going work of Humphries and Khan. However, in the $q$-aspect, this Motohashi-type connection was already visible in the work of Conrey and Iwaniec \[CI\], and it has been implemented by many successors \[MV, Pet, PY2, PY3, Nel, Wu, BFW\].

Finally, we remark that there seems to be substantial analytic obstacles (see Remark 4.3 and 4.6) that prevent us from obtaining asymptotic over any number field other than $\mathbb{Q}$.

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\footnote{The currently best estimates for $E_1(T)$ and $E_2(T)$ are $O(T^{1515/4816+\epsilon})$ and $O(T^{2/3} \log C T)$ ($C$ is an effective constant) due to Bourgain–Watt \[BW\] and Ivic–Motohashi \[IM\]. For $k = 6$, Heath-Brown \[HB\] proved that the 12th moment integral is $O(T^2 \log T)$. For $k = 3$ and 4, the best bounds are $O(T^{5/4} \log^{3/4} T)$ and $O(T^{3/2} \log^{21/2} T)$ (see \[LY1\]), which are trivial consequences of Heath-Brown’s bound and the Hölder (or Cauchy–Schwarz) inequality.}
2. Preliminaries

2.1. Bessel Kernel. Bessel functions (as in [Wat]) arise in both the Kuznetsov trace formula and the Voronoĭ summation formula. To make our exposition succinct, we introduce the Bessel kernel $B_s(x)$ defined as follows

\begin{equation}
B_s(x) = \frac{\pi}{\sin(\pi s)} (J_{-2s}(4\pi \sqrt{x}) - J_{2s}(4\pi \sqrt{x})), \quad B_s(-x) = 4 \cos(\pi s) K_{-2s}(4\pi \sqrt{x}),
\end{equation}

for real $x > 0$ and complex $s$.

For $t$ real, the Bessel kernel $B_{it}(x)$ appears in a certain Bessel integral $\mathcal{H}(x)$ in Kuznetsov which is well understood by the works of [LI, LI1, You2, LQ1]. Moreover, the Bessel kernel $B_0(x)$ arises in the Hankel transform in Voronoĭ, and the following formulae will be crucial in our analysis:

\begin{equation}
B_0(x) = \sum_{\pm} e(\pm(2\sqrt{x} + 1/8)) W_0(\pm \sqrt{x}), \quad B_0(-x) = O\left(\exp\left(-4\pi \sqrt{x}\right)\right),
\end{equation}

for $x > 1$, in which $x^j W_0^{(j)}(x) \ll_j 1$ (see [LQ2] §6).

2.2. Kuznetsov Trace Formula. Let $\mathbb{R}$ be an orthonormal basis of even Hecke–Maass forms for $\text{SL}_2(\mathbb{Z})$. For $f(z) \in \mathbb{R}$ let $\frac{1}{4} + t_f^2$ ($t_f \geq 0$) be its Laplacian eigenvalue, $\lambda_f(n)$ ($n \geq 1$) be its Hecke eigenvalues, and $\rho_f(n)$ ($n \neq 0$) be its Fourier coefficients. By definition, $f(z)$ is even in the sense that $f(-\bar{z}) = f(z)$, so that $\rho_f(-n) = \rho_f(n)$. It is known that $\rho_f(\pm n) = \rho_f(1)\lambda_f(n)$.

Now we recall the Kuznetsov trace formula for even Maass forms. See [LI1] §2 and [LQ2] §3.3.

**Lemma 2.1.** Let $h(t)$ be an even test function such that

1. $h(t)$ is holomorphic in $\Im(t) \leq \frac{1}{2} + \varepsilon$, and
2. $h(t) \ll (|t| + 1)^{-2-\varepsilon}$ in the above strip.

Then for $n_1, n_2 \geq 1$ we have

\begin{equation}
\sum_{f \in \mathbb{R}} h(t_f) \omega_f \lambda_f(n_1) \lambda_f(n_2) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \omega(t) \tau_{it}(n_1) \tau_{it}(n_2) dt
\end{equation}

\begin{equation}
= \delta_{n_1,n_2} \mathcal{H} + \sum_{\pm} \sum_{c=1}^{\infty} S(n_1, \pm n_2; c) \mathcal{H}\left(\pm \frac{n_1 n_2}{c^2}\right),
\end{equation}

where $\delta_{n_1,n_2}$ is Kronecker’s $\delta$-symbol,

\begin{equation}
\tau_s(n) = \tau_{-s}(n) = \sum_{ab=n} (a/b)^s,
\end{equation}

\begin{equation}
\omega_f = \frac{|\rho_f(1)|^2}{\cosh(\pi t_f)} = \frac{2}{L(1, \text{Sym}^2 f)} \quad \omega(t) = \frac{4}{|\zeta(1+2it)|^2},
\end{equation}

and

\begin{equation}
\mathcal{H} = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} h(t) \tanh(\pi t) dt, \quad \mathcal{H}(x) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} h(t) B_{it}(x) \tanh(\pi t) dt.
\end{equation}
2.3. Poisson and Voronoi Summation Formulae. The following Poisson formula is essentially a special case of [IK (4.25)].

**Lemma 2.2.** Let \( w \in C^c_{\infty}(-\infty, \infty) \). Let \( a, c \) be integers with \( c \geq 1 \). Then

\[
\sum_n e \left( \frac{-an}{c} \right) w(n) = \sum_{m \equiv a \pmod{c}} \hat{w} \left( \frac{m}{c} \right),
\]

where \( \hat{w} \) is the Fourier transform of \( w \) defined by

\[
\hat{w}(y) = \int_{-\infty}^{\infty} w(x)e(-xy)dx.
\]

Note that there is no zero frequency in the case that \( c > 1 \) and \( (a, c) = 1 \).

The following Voronoi formula for the divisor function \( \tau(n) = \sigma_0(n) \) is from [IK (4.49)]. Note that \( B_0(x) = -2\pi Y_0(4\pi \sqrt{x}) \) for \( x > 0 \) by [Wat 3.54 (1)].

**Lemma 2.3.** Let \( w \in C^c_{\infty}(0, \infty) \). Let \( a, \tilde{a}, c \) be integers with \( c \geq 1 \) and \( a \tilde{a} \equiv 1 \pmod{c} \). Then

\[
\sum_n \tau(n)e \left( \frac{-an}{c} \right) w(n) = 2(\gamma - \log c)\tilde{w}_0(0) + \tilde{w}_0'(0) + \sum_{m \neq 0} \tau(m)e \left( \frac{am}{c} \right) \tilde{w}_0 \left( \frac{m}{c} \right),
\]

where \( \gamma \) is Euler’s constant,

\[
\tilde{w}_0(0) = \int_{0}^{\infty} w(x)dx, \quad \tilde{w}_0'(0) = \int_{0}^{\infty} w(x) \log xdx,
\]

and \( \tilde{w}_0 \) is the Hankel transform of \( w \) (with kernel \( B_0 \)) defined by

\[
\tilde{w}_0(y) = \int_{0}^{\infty} w(x)B_0(xy)dx,
\]

for real \( y \neq 0 \).

2.4. Approximate Functional Equations. According to [LQ 4], with slightly altered notation, we have the following approximate functional equations:

\[
L \left( \frac{1}{2}, f \right) = 2 \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}} V_1(n; t_f), \quad L \left( \frac{1}{2}, f \right)^2 = 2 \sum_{n=1}^{\infty} \frac{\lambda_f(n)\tau(n)}{\sqrt{n}} V_2(n; t_f),
\]

and similarly

\[
|\zeta \left( \frac{1}{2} + it \right)|^2 = 2 \sum_{n=1}^{\infty} \frac{\tau(n)}{\sqrt{n}} V_1(n; t) + O(e^{-\epsilon^2/2}),
\]

\[
|\zeta \left( \frac{1}{2} + it \right)|^4 = 2 \sum_{n=1}^{\infty} \frac{\tau(n)\tau(n)}{\sqrt{n}} V_2(n; t) + O(e^{-\epsilon^2}),
\]

with

\[
V_1(y; t) = \frac{1}{2\pi i} \int_{(3)} \delta(v, t)e^y e^{-\epsilon v}dv, \quad V_2(y; t) = \frac{1}{2\pi i} \int_{(3)} \zeta(1 + 2v)\delta(v, t)^2 e^{2v} e^{-\epsilon v}dv,
\]

for \( y > 0 \), where

\[
\delta(v, t) = \frac{\gamma \left( \frac{1}{2} + v, t \right)}{\gamma \left( \frac{3}{2}, t \right)},
\]
Finally, for 
(2.16) \( \gamma(s, t) = \pi^{-s} \Gamma \left( \frac{s - it}{2} \right) \Gamma \left( \frac{s + it}{2} \right) \).

**Lemma 2.4.** For real \( t \) define

\[
C(t) = \sqrt{\frac{1}{4} + t^2}.
\]

Let \( U > 1 \). We have

(2.17) \[ V_1(y; t) \ll_A \left( 1 + \frac{y}{C(t)} \right)^{-A}, \quad V_2(y; t) \ll_A \left( 1 + \frac{y}{C(t)^2} \right)^{-A}, \]

(2.18) \[ V_1(y; t) = \frac{1}{2\pi i} \int_{\varepsilon-iU}^{\varepsilon+iU} \delta(v, t) e^{yv} \frac{\E}{v} + O \left( \frac{C(t)^e}{y^e e^{U^2/2}} \right), \]

and, furthermore, if \( |t| > U^2 \) and \( v \) is on the contour, then we may write \( \delta(v, t) = (t/2\pi)^e (1 + \delta(v, t)) \) so that

(2.19) \[ \frac{\partial^2 \delta(v, t)}{\partial v^2} \ll_{i} \frac{U^2}{|t|^{1+e}}. \]

Finally, for \( 1 \leq y < C(t)^2 \) we have

(2.20) \[ V_2(y; t) = \gamma + \psi_1(t) - \log \sqrt{y} + O_A \left( \left( \frac{y}{C(t)^2} \right)^A \right), \]

with \( \psi_1(t) = (\partial \log \gamma(s, t)/\partial s) |_{s=1/2} \).

**Proof.** (2.17) and (2.18) are essentially from [IK Proposition 5.4] and [Blo Lemma 1]; see also [LQ2 Lemma 4.1 (1)]. (2.19) follows readily from Stirling’s formulae for \( \log \Gamma \) and its derivatives (see for example [MOS §§1.1, 1.2]). (2.20) is from [LQ2 Lemma 4.1 (2)].

Q.E.D.

2.5. The Large Sieve. The following is Gallagher’s large sieve inequality [Gal Theorem 2] in the case \( q = 1 \).

**Lemma 2.5.** Let \( a_n \) be a sequence of complex numbers. For \( T > 1 \) we have

(2.21) \[ \int_{-T}^{T} \left| \sum_n a_n e^{it} \right|^2 dt \ll \sum_n (T+n)|a_n|^2, \]

provided that the sum of \( |a_n| \) is bounded.

3. Refined Analysis for the Bessel Integral

Subsequently, we shall always let \( U = \log T \). For \( v_1, v_2 \in \{ \varepsilon - iU, \varepsilon + iU \} \) define

(3.1) \[ h(t; v_1, v_2) = k(t) \delta(v_1, t) \delta(v_2, t)^2, \]

with

(3.2) \[ k(t) = e^{-(t-T)^2/M^2} + e^{-(t+T)^2/M^2}. \]

Since \( v_1 \) and \( v_2 \) are inessential to our analysis, we shall simply write \( h(t) = h(t; v_1, v_2) \) and let \( \mathcal{H}(x) \) be its associated Bessel integral as in (2.6).
For the analysis of $\mathcal{H}(\pm x^2)$, the most important are certain integral representations. The reader is referred to [Li1] §§4, 5, [You2] §7, [LQ1] Appendix], and [Qi] §8.1 for more details, and also [LQ2] §7] for a summary (although $h(t)$ may vary in different settings).

For the proof of Theorem 1.3, we shall change $T$ into $K$ as in Lemma 1.1 but keep $U = \log T$ for the range of $v_1, v_2$, so that $K$ and $v_1, v_2$ vary independently. Moreover, in order to perform the $K$-integration effectively, we wish to refine (the weight function in) the integral representations.

**Lemma 3.1.** We may write $\mathcal{H}(x) = \mathcal{H}_+(x) + \mathcal{H}_-(x) + O_A(T^{-A})$ for $|x| > 1$, with

\begin{align}
\mathcal{H}_±(x^2) &= M^{1+v} \int_{-M^v/M}^{M^v/M} g(Mr)e(Tr/\pi \mp 2x \cosh r)dr, \\
\mathcal{H}_±(-x^2) &= M^{1+v} \int_{-M^v/M}^{M^v/M} g(Mr)e(Tr/\pi \pm 2x \sinh r)dr,
\end{align}

for $x > 1$, where $v = v_1 + 2v_2$, and $g(r)$ is a Schwartz function of the form

\begin{equation}
g(r) = g^A(r) + \frac{MU}{T}g^0(r),
\end{equation}

with

\begin{equation}g^0(r) = \frac{2^{1-v}}{\pi^{1/2+v}} e^{-r^2},\end{equation}

\begin{equation}(d/dr)^i g^0(r) \ll_i 1, r^{-A}.
\end{equation}

Lemma 3.1 especially (3.5), can be proven by analyzing the arguments in [LQ1 A.2, A.4] more carefully with the aid of (2.19) (see also the asymptotic analysis in [You2 §5]).

It is important that neither the definition in (3.5) nor the implied constants in (3.7) depend on $T$ or $M$. Evidently, the derivatives of $g(r)$ also satisfy (3.7).

According to (3.5), for $|x| > 1$ we may write

\begin{equation}\mathcal{H}(x) = \mathcal{H}^0(x) + \frac{MU}{T}\mathcal{H}^0(x) + O_A(T^{-A}),\end{equation}

where $\mathcal{H}^0 = \mathcal{H}^0_+ + \mathcal{H}^0_-$ and $\mathcal{H}^0 = \mathcal{H}^0_+ + \mathcal{H}^0_-$, with $\mathcal{H}^0_+$ and $\mathcal{H}^0_-$ defined in the same manner by (3.3) and (3.4).

Moreover, the following lemmas tell us the ranges that we need to focus on.

**Lemma 3.2.** For $|x| \leq 1$ we have $\mathcal{H}(x) \ll_A M^{1-2A}\sqrt{|x|}$.

**Lemma 3.3.** For $|x| > 1$ we have $\mathcal{H}(x) \ll A(T^{-A})$ unless $|x| \gg T^2$. To be precise, $\mathcal{H}(x^2)$ and $\mathcal{H}(-x^2)$ are negligibly small unless $x > M^{1-\epsilon}T$ and $x = T$ respectively.

Of course, Lemma 3.3 also holds if $\mathcal{H}$ were replaced by $\mathcal{H}^0$ or $\mathcal{H}^0$.

**4. Refined Analysis for the Fourier–Hankel Transform**

In this section, we use the analysis in the work of Young [You2] (and the author [Qi]) to study the Fourier–Hankel transform of the Bessel integral.

Let $w_1(x), w_2(x) \in C^\infty_c(0, \infty)$ be such that $w_1^{(i)}(x), w_2^{(i)}(x) \ll_i U^i$. For $A \gg T^2$ define

\begin{equation}w^\pm(x_1, x_2; A) = w_1(x_1)w_2(x_2)\mathcal{H}(\pm Ax_1x_2).
\end{equation}
The Fourier–Hankel integral transform arising after Voronoï and Poisson will be of the form

$$\hat{w}^±(y_1, y_2; A) = \int_0^\infty \int_0^\infty w^±(x_1, x_2; A)e(-x_1y_1)B_0(x_2y_2)dx_1dx_2.$$  

Define $\hat{w}^o±(y_1, y_2; A)$ and $\hat{w}^\varphi±(y_1, y_2; A)$ in the same way.

To apply the analysis of Young, we reformulate (4.2) in the following way (with $x = x_1x_2$)

$$\hat{w}^±(y_1, y_2; A) = \int_0^\infty J(\pm Ax)dx,$$

with Fourier–Bessel integral kernel

$$J(x; y_1, y_2) = \int_0^\infty w_1(x_1)w_2(x/x_1)e(-x_1y_1)B_0(xy_2/x_1)dx_1.$$  

The next lemma provides a description of the asymptotic of $J(x; y_1, y_2)$ which is similar to those in [You2 Lemma 6.5] and [Q1 Lemma 4.12] as expected.

**Lemma 4.1.** Assume that $|y_2| > T^\varepsilon$. Then $J(x; y_1, y_2) = O(T^{-A})$ unless $y_2 > T^\varepsilon$ is positive and

$$|y_1| = \sqrt{y_2},$$

and under these conditions there is a smooth function $W(\lambda; y_1, y_2)$ with support in $\lambda = \sqrt[3]{y_1y_2}$, satisfying

$$\lambda^\varepsilon \frac{\partial^i W(\lambda; y_1, y_2)}{\partial \lambda^j} \ll_U U^i,$$

with the implied constants uniform in $y_1$ and $y_2$, such that

$$J(x; y_1, y_2) = \frac{e(-3\sqrt[3]{y_1y_2})}{\sqrt[3]{y_1y_2}}W(\sqrt[3]{y_1y_2}; y_1, y_2) + O(T^{-A}).$$

**Proof.** In view of (2.2), the integral $J(x; y_1, y_2)$ is exponentially small if $y_2 < -T^\varepsilon$ is negative, and it suffices to consider for $y_2 > T^\varepsilon$ integrals of the form

$$\frac{1}{\sqrt[3]{y_2}} \int_0^\infty w^±(x_1, x_2; y_2)e\left(\pm 2\sqrt[3]{y_2}/x_1 - x_1y_1\right)dx_1,$$

for suitable smooth $w^±(x_1, x_2; y_2)$ supported in $x_1, x = 1$, such that

$$\frac{\partial^{i+j} w^±(x_1, x_2; y_2)}{\partial x_1^i \partial x^j} \ll_{i,j} U^{i+j}.$$  

The first assertion follows immediately from repeated partial integration (for example, one can use [Q1 Lemma 7.1]). Moreover, the integral in (4.8) is also negligibly small if the ± does not match the sign of $y_1$. As for the second assertion, we assume for simplicity that $y_1 = \sqrt[3]{y_2}$ is positive, let $\lambda = \sqrt[3]{y_1y_2}$, and make the change of variable $x_1 \to \sqrt[3]{y_2}/y_1^2 \cdot x_1$ so that (4.8) is transformed into

$$\frac{e(-3\lambda)}{\sqrt[3]{|\lambda|}} \int_0^\infty v(x_1; \lambda; y_1, y_2)e(\lambda(3 - 2\sqrt[3]{x_1} - x_1))dx_1,$$

where $v(x_1; \lambda; y_1, y_2)$ is smooth, supported in $x_1 = 1$, such that

$$\lambda^\varepsilon \frac{\partial^{i+j} v(x_1; \lambda; y_1, y_2)}{\partial x_1^i \partial \lambda^j} \ll_{i,j} U^{i+j}.$$
Then (4.10) and (4.11) follow from Sogge’s version of stationary phase estimates (see [Sog] Theorem 1.1.1 and [Qi] Lemma 7.3)).

It should be stressed that the implied constants in (4.9) and (4.11) (also in (4.6)) do not depend on $y_1$ or $y_2$. Q.E.D.

The following lemma is essentially (8.16) in [You2] but stated in the fashion of Proposition 11.3 and Corollary 11.10 in [Qi] with necessary adaptions. It is obtained by the method of stationary phase.

**Lemma 4.2.** Let $y_2 > T^\varepsilon$ and $A >> T^2$. Assume $|y_1| = \sqrt{y_2}$. Then for $y_1 y_2 = \Lambda$ we have

\begin{equation}
(4.12) \quad e(\pm y_1 y_2 / A) \hat{\varphi}(y_1, y_2; A) = \frac{MT^{1+v}}{\sqrt{|y_1 y_2|}} \Phi^\pm(y_1 y_2 / A) + O(T^{-\alpha}),
\end{equation}

such that (for $x = Y / (\Lambda)$) $\Phi^+(x) = 0$ and $\Phi^-(x) = 0$ unless

\begin{equation}
(4.13) \quad \sqrt{|Y|} > \Lambda, \quad |Y| / \Lambda > TM^{1-\varepsilon},
\end{equation}

and

\begin{equation}
(4.14) \quad \sqrt{\Lambda} \sim T, \quad \sqrt{\Lambda} / \sqrt{|Y|} > M^{1-\varepsilon},
\end{equation}

respectively, in which cases

\begin{equation}
(4.15) \quad \Phi^+(x) = \int e(Tr / \pi - x \tanh^2 r) V^+(r) dr,
\end{equation}

provided that $|x| < T^{2-\varepsilon}$, and

\begin{equation}
(4.16) \quad \Phi^-(x) = \int e(Tr / \pi - x \coth^2 r) V^-(r) dr,
\end{equation}

where $V^+(r)$, $V^-(r)$ are supported in

\begin{equation}
(4.17) \quad r \approx TA / Y, \quad |r| = \sqrt{|Y|} / \sqrt{\Lambda},
\end{equation}

respectively, satisfying $r^2 (d/dr)V^\pm(r) \ll U^\pm$.

Now we write $\Phi^\pm(x) = \Phi^\pm_{T,M}(x)$ to indicate its dependence on $T$ and $M$. Moreover we add superscript like $\Phi_{T,M}^\alpha(x)$ or $\Phi_{T,M}^\beta(x)$ to indicate its origin from $\mathcal{H}^\alpha$ or $\mathcal{H}^\beta$. Lemma 4.2 remains valid if $0$ or $b$ were attached in the notation.

Let $M^{1+\varepsilon} \leq H \leq T / 3$ and $|K - T| \leq H$. We would like to change $T$ into $K$ and average $K^{1+v} \Phi_{K,M}^\pm(x)$ over $K$. However, it is only necessary to change $T$ into $K$ in (4.12), (4.13), and (4.16)—the $T$ in (4.13), (4.14), and (4.17) are kept because $K \approx T$. We stress that $V^\alpha(r)$ is independent on $K$ as it is just a smooth truncation of $g^\alpha(r)$ according to (4.17), while the definition of $g^\alpha(r)$ as in (3.6) does not involve $T$ (or $K$).

**Lemma 4.3.** Let notation be as above. Define

\begin{equation}
(4.18) \quad U^+ = T^2 \Lambda / |Y|, \quad U^- = T \sqrt{|Y| / \sqrt{\Lambda}}.
\end{equation}

Then

\begin{equation}
(4.19) \quad \int_{T-H}^{T+H} \Phi_{K,M}^\pm(x)dK = \frac{T}{U^\pm} ((T + H)^{1+v} \Phi_{T,H,M}^\pm(x) - (T - H)^{1+v} \Phi_{T-H,M}^\pm(x))
\end{equation}

\begin{align*}
&- \frac{(1 + v)T}{U^\pm} \int_{T-H}^{T+H} K^{1+v} \Phi_{K,M}^\pm(x)dK + MU \int_{T-H}^{T+H} K^{1+v} \Phi_{K,M}^\pm(x)dK,
\end{align*}
for \( \Phi_{K,M}^\pm(x) \) and \( \Phi_{K,M}^{0,\pm}(x) \) of the same shape as \( \Phi_{K,M}^\pm(x) \).

Proof. Express \( \Phi_{K,M}^\pm(x) \) on the left of (4.19) as the sum \( \Phi_{K,M}^{0,\pm}(x) + MU/K \cdot \Phi_{K,M}^{0,\pm}(x) \) according to (3.2). Then (4.19) readily follows from the simple identity:

\[
\int_{T-H}^{T+H} K^{1+\varepsilon}(Kr/\pi)dK = \frac{K^{1+\varepsilon}(Kr/\pi)}{2\varepsilon} \int_{T-H}^{T+H} K^{\varepsilon}(Kr/\pi)dK.
\]

Note that the factor \( 1/r \) arises on the right, while, in view of (4.17), \( |1/r| = U \pm/T \) on the support of \( V^0 \pm(r) \), so we can define the weight function in \( \Phi_{K,M}^{\pm}(x) \) to be \( V^{\pm}(r) = U \pm/2(Tr \cdot V^0 \pm(r)) \).

Q.E.D.

Remark 4.4. It is substantial that the weight function \( V^0 \pm(r) \) is localized as in (4.17), since \( r \to 0 \) is not allowed in view of the factor \( 1/r \) cause by the averaging process. By examining the analysis in [Qi], unfortunately, we find that it is no longer the case if the field is not \( \mathbb{Q} \).

Finally, we return to the setting of Lemma 4.2 and record here the following expression of \( \Phi_{K,M}^\pm(x) \) due to Young [You2, Lemma 8.3] (see also [Qi, Lemma 13.2]). It is obtained by the technique of Mellin transform.

Lemma 4.5. Let \( x = X \). Suppose that \(|X| > T^{\varepsilon}/A \) and \( A \gg T^2 \). For

(4.20) \(|X| \approx \sqrt{A}, \quad T < \sqrt{A}/M^{1-\varepsilon}\),

or

(4.21) \(T \approx \sqrt{A}, \quad |X| < \sqrt{A}/M^{3-\varepsilon}\),

in the \( \pm \)-case, respectively, we have

(4.22) \( \Phi^\pm(x) = \frac{1}{T} \int_{|t|=U^{\pm}} \lambda_{X,T}^\pm(t)|x|^itdt\),

with \( \lambda_{X,T}^\pm(t) \ll 1 \) (\( \lambda_{X,T}^\pm(t) \) depends on \( X,T \) but the implied constant does not) and

(4.23) \( U^+ = T^2/|X|, \quad U^- = \sqrt[3]{|X|/T^2}\),

provided that \(|X| < T^{2-\varepsilon}\).

Note that (4.20), (4.21), and (4.23) respectively are tantamount to (4.13), (4.14), (4.18) on letting \( X = Y/A \). Moreover, under the conditions in (4.20) or (4.21), we have

(4.24) \( T^{\varepsilon} < U^\pm < \frac{T}{M^{1-\varepsilon}}, \)

provided that \( T^{\varepsilon}/A < |X| < T^{2-\varepsilon}\).

Remark 4.6. Note that the product of \( 1/T \) in (4.22) and \( T/M^{1-\varepsilon} \) in (4.24) equals \( 1/M^{1-\varepsilon} \)—it will appear to be our saving in the error term for \( M_2^\pm(T,M) \). Again, by examining the analysis in [Qi], we find that there is no longer such a saving if the field is not \( \mathbb{Q} \).
5. A Simple Lemma for the Hankel Transform

Our last analytic lemma is on the Hankel transform of a special kind of functions that involve $x^{it}$.

**Lemma 5.1.** Let $t, y > T^\varepsilon$. For fixed $v(x) \in C_c^\infty(0, \infty)$ define

$$
\tilde{v}_0(0; t) = \int_0^\infty v(x) \frac{dx}{x^{1/2} + it}, \quad \tilde{v}_0'(0; t) = \int_0^\infty v(x) \log x \frac{dx}{x^{1/2} + it},
$$

Then $\tilde{v}_0(0; t), \tilde{v}_0'(0; t),$ and $\tilde{v}_0(-y; t)$ are all negligibly small, while $\tilde{v}_0(y; t)$ is bounded but negligibly small unless $y = t^2$.

**Proof.** The proof is standard. It is well-known that the Mellin integrals $\tilde{v}_0(0; t), \tilde{v}_0'(0; t)$ are negligibly small. In view of (2.2), it is clear that $\tilde{v}_0(-y; t)$ is exponentially small, while for $\tilde{v}_0(y; t)$ it is reduced to consider integrals of the form

$$
y^{1/4 + it} \int_0^\infty w(x)e((t/\pi) \log x + 2\sqrt{x}) \, dx,$$

for suitable $w(x) \in C_c^\infty(0, \infty)$. By repeated partial integration (again, one can use [Q1] Lemma 7.1), the integral is negligibly small unless the sign is + and $t = \sqrt{x}$, in which case, the second derivative test ([Hux] Lemma 5.1.3) may be applied to prove that the integral is bounded.

A direct consequence of Lemma 2.3 (in the special case $c = 1$) and Lemma 5.1 is the following truncated Voronoï summation formula.

**Corollary 5.2.** Let $t, N > T^\varepsilon$. We have

$$
\sum_{n > N} \frac{\tau(n)}{n^{1/2 - \varepsilon}} v(n/N) = \sum_{m > T^\varepsilon/N} \frac{\tau(m)}{m^{1/2 + \varepsilon}} \tilde{v}_0(Nm; t) + O(T^{-A}),
$$

for $v(x)$ fixed and $\tilde{v}_0(y; t)$ bounded.

6. Setup

As our motivation, we start with reviewing an asymptotic formula in [LQ2] for the twisted second moment of central $L$-values as follows:

$$
\sum_{f \in \mathcal{A}} \omega_f \lambda_f(n_1) L\left(\frac{1}{2}, f\right)^2 k(t_f) + \frac{1}{4\pi} \int_{-\infty}^\infty \omega(t) \tau_{it}(n_1) \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 k(t) \, dt
$$

$$
= \frac{4\tau(n_1)MT}{\pi \sqrt{\pi n_1}} \left( \log \frac{T}{\sqrt{n_1}} + \gamma - \log 2\pi \right) + O\left( \left( \frac{M^3}{\sqrt{n_1}T} + \frac{\sqrt{n_1}T}{\sqrt{M}} \right) T^{\varepsilon} \right),
$$

for any $n_1 \leq T^{2-\varepsilon}$, while the second error term can be removed in the case $n_1 \leq M^{2-\varepsilon}$. This formula is proven by the “Kuznetsov–Voronoï” approach, along with analysis for the Hankel and Mellin transforms of Bessel integrals. It should be stressed that the main term has two sources: half is from the diagonal term in Kuznetsov while the other half is from the zero frequency after Voronoï.

Heuristically, by summing up to $n_1 \leq T^{1+\varepsilon}$ according to the approximate functional equations (see (2.12), (2.13), and (2.17) [5] albeit with a weaker error term, we

\[\text{5}A\,\text{subtle\,issue\,is\,that\,the\,spectral\,}t_f\text{\,and\,}t\text{\,are\,involved\,in\,the\,weights\,}V_f(n_1; t_f)\text{\,and\,}V_f(n_1; t),\text{\,so\,the\,arguments\,and\,results\,in\,}LQ2\text{\,can\,not\,be\,applied\,directly\,and\,must\,be\,adapted\,slightly.}\]
may already deduce from (6.1) an asymptotic formula for the cubic moment of the form (1.10) in Theorem 1.2. To strengthen the error term, in addition to the Voronoï summation used in [LQ2], we need to also apply Poisson summation to the \( n_1 \)-variable as Conrey and Iwaniec did in [CI].

Now we turn to the smooth spectral cubic moment:

\[
\mathcal{M}_3 = \sum_{f \in \mathbb{B}} \omega_f L \left( \frac{1}{2}, f \right)^3 k(t_f) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \omega(t) \left| \zeta \left( \frac{1}{2} + it \right) \right|^6 k(t) \, dt.
\]

For brevity, we suppress \( T, M \) from our notation here, but keep in mind that we need to average the \( T \)-parameter later for the proof of Theorem 1.3.

By the approximate functional equations (2.12) and (2.13), we infer that

\[
\mathcal{M}_3 = 4 \sum_{n_1, n_2} \frac{\tau(n_2)}{\sqrt{n_1 n_2}} \left\{ \sum_{f \in \mathbb{B}} \omega_f \lambda_f(n_1) \lambda_f(n_2) V_1(n_1; t_f) V_2(n_2; t_f) k(t_f) \right. \\
+ \left. \frac{1}{4\pi} \int_{-\infty}^{\infty} \omega(t) \tau_{it}(n_1) \tau_{it}(n_2) V_1(n_1; t) V_2(n_2; t) k(t) \, dt \right\}.
\]

In view of (2.17) in Lemma 2.4, at the cost of a negligible error term, we may truncate the \( n_1 \)- and \( n_2 \)-sums to the ranges \( n_1 \leq T^{1+\varepsilon} \) and \( n_2 \leq T^{2+\varepsilon} \) respectively.

7. Applying the Kuznetsov Trace Formula

Next, we apply the Kuznetsov trace formula in Proposition 2.1 to the expression between the large brackets in (6.3), and then use (2.18) in Lemma 2.4 with \( U = \log T \) (so that the errors therein are negligible) to reformulate the resulting off-diagonal terms. More explicitly, we have

\[
\mathcal{M}_3 = \mathcal{D}_3 + \mathcal{O}_3 + O(T^{-1}),
\]

where \( \mathcal{D}_3 \) is the diagonal sum

\[
\mathcal{D}_3 = 4 \sum_{n \leq T^{1+\varepsilon}} \frac{\tau(n)}{n} \mathcal{H}_3(n),
\]

with

\[
\mathcal{H}_3(n) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} V_1(n; t) V_2(n; t) k(t) \tanh(\pi t) \, dt,
\]

while \( \mathcal{O}_3 \) is the off-diagonal contribution in the form

\[
\mathcal{O}_3 = -\frac{1}{\pi^2} \int_{\varepsilon-iU}^{\varepsilon+iU} \int_{\varepsilon-iU}^{\varepsilon+iU} \mathcal{O}_3(v_1, v_2) \zeta(1 + 2v_2) e^{v_1^2 + 2v_2^2} \frac{dv_1}{v_1} \frac{dv_2}{v_2},
\]

with

\[
\mathcal{O}_3(v_1, v_2) = \sum_{\pm} \sum_{n_1 \in T^{1+\varepsilon}} \sum_{n_2 \in T^{2+\varepsilon}} \frac{\tau(n_2)}{n_1^{1/2 + v_1} n_2^{1/2 + v_2}} \frac{S(n_1, \pm n_2; c)}{c} \mathcal{H} \left( \pm \frac{n_1 n_2}{c^2}; v_1, v_2 \right),
\]

(7.6) \( \mathcal{H}(x; v_1, v_2) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} h(t; v_1, v_2) B_{it}(x) \tanh(\pi t) \, dt \),

and

\[
h(t; v_1, v_2) = k(t) \delta(v_1, t) \delta(v_2, t)^2.
\]
It follows from \eqref{2.20} that
\begin{equation}
\mathcal{H}_3(n) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} V_1(n; t) \left( \gamma + \psi_1(t) - \log \sqrt{n} \right) k(t) \tanh(\pi t) dt + O(T^{-A}).
\end{equation}

By Lemma \ref{3.2} and \ref{3.3}, one may impose the condition \( n_1n_2/c^2 \gg T^2 \) to the summations in \eqref{7.3}, with the cost of a negligible error.

8. Applying the Voronoï and Poisson Summation Formulae

At this point, we introduce smooth dyadic partitions for the \( n_2 \)- and \( n_1 \)-sums prior to the application of Voronoï and Poisson. More explicitly, we split \( \mathcal{O}_3(v_1, v_2) \) into the sum of
\begin{equation}
\frac{1}{N_1^{1/2 + v_1} N_2^{1/2 + v_2}} \sum_{e \leq c} \sum_{n_1, n_2} \tau(n_2) \frac{S(n_1, \pm n_2; c)}{c} w^\pm \left( \frac{n_1}{N_1}, \frac{n_2}{N_2}, \frac{N_1N_2}{c^2}; v_1, v_2 \right),
\end{equation}
for dyadic \( 1/2 < N_1 \leq T^{1+\epsilon} \) and \( 1/2 < N_2 \leq T^{2+\epsilon} \), where
\begin{equation}
w^\pm(x_1, x_2; A; v_1, v_2) = w(x_1; v_1)w(x_2; v_1)\mathcal{H}_2(\pm Ax_1x_2; v_1, v_2), \quad w(x; v) = \frac{v(x)}{x^{1/2+\epsilon}},
\end{equation}
for suitable \( v(x) \in C_0^\infty[1, 2] \). For the moment, one may restrict the \( c \)-sum to the range \( c \ll \sqrt{N_1N_2}/T \).

For the \( n_2 \)-sum, we open the Kloosterman sum \( S(n_1, \pm n_2; c) \) and apply the Voronoï summation formula as in Lemma \ref{2.3}.

For the entire zero-frequency contribution \( \mathcal{F}_3 \), after reversing the procedures of truncation and partition (of sums and integrals), the arguments in \cite{LQ2} §12.2 can be easily adapted to prove
\begin{equation}
\mathcal{F}_3 = 2 \sum_{n_1 \leq T^{1+\epsilon}} \frac{1}{\sqrt{n_1}} \mathcal{F}_3(n_1) + O(T^{-A}),
\end{equation}
with
\begin{equation}
\mathcal{F}_3(n_1) = \frac{1}{2\pi^2} \frac{\tau(n_1)}{\sqrt{n_1}} \int_{-\infty}^{\infty} V_1(n_1; t) (2\gamma + 2\psi_1(t) - \log n_1) k(t) \tanh(\pi t) dt.
\end{equation}

We only remark that it is crucial to have the formula:
\begin{equation}
\int_{-\infty}^{\infty} B_{\tilde{t}}(x|x|^{s-1} dx = \frac{\gamma(s, t)}{\gamma(1-s, t)}.
\end{equation}

By comparing \eqref{7.2} and \eqref{7.8} with \eqref{8.1} and \eqref{8.2}, it is clear that \( \mathcal{O}_3 \) and \( \mathcal{F}_3 \) are equal to each other up to a negligible error.

For the dual \((m_2)\)-sum after Voronoï, the exponential sum turns into \( S(n_1 \mp m_2, 0; c) \). For the \( n_1 \)-sum, similarly, we open the Ramanujan sum \( S(n_1 \mp m_2, 0; c) \) and apply the Poisson summation formula as in Lemma \ref{2.2}. Now the dual exponential sum reduces to \( e(\pm m_1m_2/c) \) along with the condition \((m_1, c) = 1 \). Note that the zero frequency in the \( m_1 \)-sum exists only when \( c = 1 \) and is negligibly small by Lemma \ref{4.1}.

It is left to consider the sum \( \mathcal{F}_3^{\pm}(N_1, N_2) \) defined by
\begin{equation}
N_1^{1/2-v_1} N_2^{1/2-v_2} \sum_{e > 0} \frac{1}{e^2} \sum_{m_1, m_2 \neq 0 \atop (m_1, c) = 1} \tau(m_2)e \left( \pm \frac{m_1m_2}{c} \right) \tilde{w}^\pm \left( \frac{m_1N_1}{c}, \frac{m_2N_2}{c^2}, \frac{N_1N_2}{c^2} \right),
\end{equation}
where \( \hat{w}^\pm \) is the Fourier–Hankel transform defined by
\[
\hat{w}^\pm(y_1, y_2; A) = \int_0^\infty \int_0^\infty w^\pm(x_1, x_2; A)e(-x_1y_1)B_0(x_2y_2)dx_1dx_2,
\]
for
\[
w^\pm(x_1, x_2; A) = w_1(x_1)w_2(x_2)\mathcal{H}_2(\pm Ax_1x_2)
\]
with weight functions \( w_1, w_2 \in C_c^\infty[1, 2] \) such that \( w_1^{(i)}(x), w_2^{(i)}(x) \ll_i U^i_w(x) = w(x; v_1) \) and \( w_2(x) = w(x; v_2) \). For brevity, we have suppressed \( v_1 \) and \( v_2 \) from our notation.

9. Treating the Main Term

The purpose of this section is to prove that there is a certain cubic polynomial
\( P_3^\circ(X) \) such that
\[
\mathcal{D}_3 + \mathcal{L}_3 = \sqrt{\pi}MTP_3^\circ(\log T) + O(M^3 \log^2 T/T).
\]
In view of (7.2), (7.8), (8.1), (8.2), along with (2.17), (2.18), and (3.2), by completing the sum, truncating the integrals, and moving the sum inside the integrals, it is reduced to consider
\[
\frac{8}{\pi^2} \int_{T-M_1}^{T+M_1} \left((\gamma + \psi_1(t))\theta(t) + \theta_1(t)\right) e^{-(t-T)^2/M^2} \text{d}t,
\]
where \( \theta(t) \) and \( \theta_1(t) \) are the integrals
\[
\theta(t) = \frac{1}{2\pi i} \int_{\epsilon-iU}^{\epsilon+iU} \delta(v, t)\zeta(1+v)^2 e^{v^2} dv v,
\]
\[
\theta_1(t) = \frac{1}{2\pi i} \int_{\epsilon-iU}^{\epsilon+iU} \delta(v, t)\zeta(1+v)\zeta'(1+v)e^{v^2} dv v.
\]
Define \( \gamma, \gamma_1, \) and \( \gamma_2 \) by
\[
\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \frac{\gamma_2}{2}(s-1)^2 + \cdots, \quad s \to 1,
\]
so that
\[
\zeta(1+v)^2 = \frac{1}{v^2} + \frac{2\gamma}{v} + \frac{\gamma^2 - 2\gamma_1 + \cdots}{v^2}, \quad v \to 0,
\]
\[
\zeta(1+v)\zeta'(1+v) = -\frac{1}{v^2} - \frac{\gamma}{v^2} - \gamma_1 + \frac{\gamma_2}{2} + \cdots, \quad v \to 0.
\]
Define \( \psi_1(t), \psi_2(t), \) and \( \psi_3(t) \) by
\[
\delta(v, t) = 1 + \psi_1(t)v + \frac{\psi_2(t)}{2}v^2 + \frac{\psi_3(t)}{6}v^3 + \cdots, \quad v \to 0,
\]
By the Stirling formula for the derivatives of \( \log \Gamma(s) \), we have
\[
\psi_k(t) = (\log C(t) - \log(2\pi))^k + O(\log^{k-1} C(t)/C(t)^2).
\]
Now we shift the integral contour in (9.3) further down to \( \text{Re}(v) = -A \) and calculate the residues at \( v = 0 \) with the aid of (9.5)–(9.8). It follows that the integral in (9.2) turns into
\[
\frac{8}{\pi^2} \int_{T-M_1}^{T+M_1} S_3(\log C(t) - \log 2\pi)\delta(v, t)\theta(t)\theta_1(t) e^{-(t-T)^2/M^2} \text{d}t + O(M \log^2 T/T),
\]
where $S_3(X)$ is defined by

$$(9.10) \quad S_3(X) = \frac{1}{3} X^3 + 2\gamma X^2 + (3\gamma^2 - 2\gamma_1)X + \gamma_3 - 3\gamma_1 + \frac{\gamma_2}{2}. $$

Finally, by the change of variable $t \to T + Mt$, one can easily deduce (9.11) with

$$(9.11) \quad P_3^3(X) = \frac{8}{\pi^2} S_3(X - \log 2\pi). $$

10. Applying the Second Voronoï and the Large Sieve

In this section, we use the analysis of Young as in [4] the Voronoï as in [2,3] and the large sieve of Gallagher as in [3,5] to estimate the sum $S^\pm(N_1, N_2)$ defined by (8.3).

By Lemma 4.1 and 4.2, we infer that, up to a negligible error,

$$(10.1) \quad S^\pm(N_1, N_2) = \frac{MT^{1+v}}{N_1^{\frac{v}{2}}N_2} \sum_{c>0} \frac{1}{c} \sum_{|m_1|, |m_2| > 0} \sum_{(m_1, c) = 1} \frac{\tau(m_2)}{\sqrt{|m_1m_2|}} \Phi^\pm \left( \frac{m_1m_2}{c} \right),$$

where the summations are subject to the following conditions:

$$(10.2) \quad |m_1|N_1 = \sqrt{m_2}N_2,$$

and

$$(10.3) \quad c < \frac{\sqrt{N_1N_2}}{M^{1-\epsilon}T}, \quad |m_1m_2| = \sqrt{N_1N_2},$$

in the + case, or

$$(10.4) \quad c = \frac{\sqrt{N_1N_2}}{T}, \quad |m_1m_2| < \frac{\sqrt{N_1N_2}}{M^{3-\epsilon}},$$

in the – case. In particular, it follows that

$$y_2 = \frac{m_2N_2}{c^2} \geq \frac{N_2}{c^2} \geq \frac{T^2}{N_1} \geq T^{1-\epsilon},$$

and

$$|x| = \frac{|m_1m_2|}{c} \leq |m_1m_2| \leq \sqrt{N_1N_2} \leq T^{3/2+\epsilon},$$

by $N_1 \leq T^{1+\epsilon}$ and $N_2 \leq T^{2+\epsilon}$, so the assumptions in Lemma 1.2 are satisfied. Moreover, recall that $v = v_1 + 2v_2$ and $\text{Re}(v_1) = \text{Re}(v_2) = \epsilon$.

Next, we use the M"obius function to relax the condition $(c, m_1) = 1$, and then introduce dyadic partitions to the variables $c$, $m_1$, and $m_2$, it follows that if we define

$$(10.5) \quad S_{\delta t}(C) = \sum_{c \in C} \frac{1}{c^{1/2+it}}, \quad S_{\delta t}(L) = \sum_{m \sim L} \frac{1}{m^{1/2-it}}, \quad S_{\delta t}^\pm(L) = \sum_{m \sim L} \frac{v(m)}{m^{1/2-it}},$$

for suitable $v \in C_\infty[1, 2]$, then Lemma 1.5 implies that $S^\pm(N_1, N_2)$ is bounded by the supremum of

$$(10.6) \quad \mathcal{F}^\pm(C^\pm, L_1^\pm, L_2^\pm) = MT^\epsilon \int_{|t| = U^\pm} |S_{\delta t}(C^\pm) S_{\delta t}(L_1^\pm) S_{\delta t}^\pm(L_2^\pm)| \, dt,$$

for dyadic parameters $C^\pm$, $L_1^\pm$, and $L_2^\pm$ in the ranges

$$(10.7) \quad C^+ = \frac{\sqrt{N_1N_2}}{M^{1-\epsilon}T}, \quad L_1^+ = \frac{\sqrt{L_2}N_2}{N_1}, \quad L_2^+ = N_1,$$

and

$$(10.8) \quad C^- = \frac{\sqrt{N_1N_2}}{T}, \quad L_1^- = \frac{\sqrt{L_2}N_2}{N_1}, \quad L_2^- < \frac{N_1}{M^{2-\epsilon}},$$

in the ranges
and for
\begin{equation}
U^+ = \frac{C^+T^2}{L_1^2 L_2}, \quad U^- = \frac{\sqrt{L_1 L_2 T^2}}{\sqrt{C^-}}.
\end{equation}
Recall from (4.24) that
\begin{equation}
T^\varepsilon < U^\pm < \frac{T}{M^{1-\varepsilon}}.
\end{equation}

In view of Corollary 5.2, for the $m_2$-sum of length $L_2^\pm$, its dual sum is of length $U^\pm / L_2^\pm$, so we can always ensure that the length of summation does not exceed $U^\pm$. For the $c$-sum and the $m_1$-sum, we take the complex conjugate of the former and group them together so that the new sum is over $cm_1$ and of length $C^\pm L_1^\pm$. Furthermore, it follows from (10.7), (10.8), and (10.9), along with $N_2 \leq T^{2+\varepsilon}$, that
\begin{equation}
C^+ L_1^\pm \ll \frac{C^+ N_2}{L_1 N_1^2} \ll U^+ T^\varepsilon,
\end{equation}
and
\begin{equation}
C^- L_1^\pm \ll \frac{C^- \sqrt{L_1 L_2 N_2}}{\sqrt{N_1^2}} \ll U^- T^\varepsilon.
\end{equation}
We conclude by Cauchy–Schwarz and Lemma 2.5 that
\begin{equation}
\mathcal{F}^\pm (C^\pm, L_1^\pm, L_2^\pm) \ll MU^\pm T^\varepsilon,
\end{equation}
and hence by (10.10) that
\begin{equation}
\mathcal{F}^\pm (C^\pm, L_1^\pm, L_2^\pm) \ll T^{1+\varepsilon}.
\end{equation}

11. Conclusion

Firstly, Theorem 1.2 follows immediately from (9.1) and (10.12).

Next, we prove Theorem 1.3. To this end, we invoke (1.9) in Lemma 1.1, change $T$ into $K$, and average (9.1) and (10.1) over $K$ from $T-H$ to $T+H$. Clearly, (9.1) yields the main term in (1.11) along with an error $O(M^2H \log^2 T / T) = O(MHT^\varepsilon)$. As for (10.1), after the dyadic partitions, we use the expression (4.19) in Lemma 4.3 for $\mathcal{F}^\pm(C^\pm, L_1^\pm, L_2^\pm)$.

It follows from (10.11) and (10.12) that the contributions from the terms with $\Phi_{K,M}^\pm$ on the right of (4.19) are bounded by $O(T^{1+\varepsilon} + HT^\varepsilon) = O(T^{1+\varepsilon})$ and $O(MHT^\varepsilon)$ respectively; the cancellation of $U^\pm$ is crucial here. We conclude that
\begin{equation}
\int_{T-H}^{T+H} K^{1+\varepsilon} \Phi_{K,M}^\pm (m_1 m_2 / c) dK.
\end{equation}
Also note that there is an error term $O(MT^{1+\varepsilon})$ in (1.9) in Lemma 1.1. Consequently, we obtain (1.11) on choosing $M = T^\varepsilon$.

Finally, Corollary 1.4 (Ivić’s strong moment conjecture) follows from Theorem 1.3 if we choose $H = T/3$ and apply a dyadic summation. It is clear that $P_3$ and $P_5$ are related by
\begin{equation}
\frac{d}{dK} \left( K^2 P_3(\log K) \right) = K P_5(\log K).
\end{equation}
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