Vacuum Fluctuations of a Scalar Field in a Rectangular Waveguide

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Abstract

An analysis of one-loop vacuum fluctuations associated with a scalar field confined in the interior of an infinite waveguide of rectangular cross section is presented. We first consider the massless scalar field defined in a four-dimensional Euclidean space. To identify the infinities of the vacuum fluctuations we use a combination of dimensional and zeta function analytic regularization procedures. The infinities which occur in the one-loop vacuum fluctuations fall into two distinct classes: local divergences that are renormalized by the introduction of the usual bulk counterterms,
and surface and edge divergences that demand counterterms concentrated on the boundaries. We present the detailed form of the surface and edge divergences. Finally we discuss how to generalize our calculations for a confined massive scalar field defined in a higher dimensional Euclidean space.

PACS numbers:03.70+k,04.62.+v.

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1 Introduction

In 1948 Casimir [1] predicted that uncharged, parallel, perfectly conducting plates should attract with a force per unit area, \( F(L) \propto \frac{1}{L^4} \), where \( L \) is the distance between the plates. This force can be interpreted as the manifestation of the zero-point energy of the electromagnetic field in the presence of the plates. Complete reviews of this effect can be found in refs. [2] [3] [4].

The aim of this paper is to generalize the results obtained by Fulling [5] and others, for the renormalized vacuum fluctuations of a massless scalar field calculated between two parallel flat mirrors. Our purpose is to analyse the vacuum fluctuations near surfaces where the field satisfies some classical boundary condition, the boundary being defined by two of the coordinates which we assume are closed intervals. A special case of this situation is the infinite waveguide of rectangular cross section. We are thus interested in obtaining the renormalized one-loop vacuum fluctuations associated with a scalar field defined in the interior of a infinite waveguide of rectangular cross section. We first consider a massless scalar field in a four-dimensional Euclidean space, and then extend our calculations to a massive scalar field defined in a higher dimensional Euclidean space.

As stated by Deutsch and Candelas [6] and also Candelas [7] the calculation of the renormalized one-loop vacuum fluctuations contains all the important steps that we need to calculate the renormalized stress-energy tensor associated with the scalar field.

It is well known that there are two quantities which might be expected to correspond to the
The total renormalized energy of quantum fields \[3\]. The first is the mode sum energy, defined as

\[
\langle E \rangle_{\text{mode}}^{\text{ren}} = \int_0^\infty d\omega \frac{1}{2} \omega [N(\omega) - N_0(\omega)],
\]

where \(\frac{1}{2} \omega\) is the zero-point energy for each mode, \(N(\omega)d\omega\) is the number of modes with frequencies between \(\omega\) and \(\omega + d\omega\) in the presence of boundaries, and \(N_0(\omega)d\omega\) is the corresponding quantity evaluated in empty space. In this case the divergences that appear in the regularized energy are given by Weyl’s theorem and its generalization, that relate the asymptoptic distribution of eigenvalues of some elliptic differential operator with geometric parameters of the surface where the fields satisfy some boundary condition \[8\] \[9\]. The second is the volume integral of the renormalized energy density \(\langle E \rangle_{\text{vol}}^{\text{vol}}\) obtained by the Green’s functions method \[10\] \[11\] \[12\] \[13\].

For special configurations where the modes and the eigenfrequencies of the electromagnetic field in the presence of the boundaries can be found, it is possible to find the Casimir energy and the Casimir force. For the case of cylindrical geometry, Balian and Duplantier and also others calculated the Casimir energy of the electromagnetic field \[14\] \[15\] \[16\] \[17\]. For the important case of scalar or fermionic fields in the presence of a spherical shell, Bender and Hays \[13\], studying the global problem, found the renormalized zero-point energy of these fields assuming that the fields are confined in a spherical region of the space. Many years before Bender and Hays, Boyer \[18\] and Davies \[19\] also studying the global problem, obtained the Casimir energy of an electromagnetic field in the presence of a perfectly conducting spherical shell. A systematic study of the spherical shell configuration was made by Milton \[20\]. He calculated the zero-point energy for gluons and fermions, assuming that the fields are confined in the interior of the shell. He later calculated
the Casimir energy of massless fermions in the presence of a spherical shell, in this case taking into account the external modes of the fermions [21]. More recently, working in a generic flat d-dimensional spacetime Bender and Milton [22] obtained the Casimir energy associated with a massless scalar field taking into account the interior and exterior contributions of the modes in the presence of the hypersphere. Still studying the spherical configuration, Romeo [23] investigated the Casimir energy of a massless scalar field and for QED assuming that the field is confined in the interior of the spherical shell. Finally Bordag et.al. [24] have calculated the Casimir energy associated with a massive scalar field in the presence of a spherical shell assuming that the interior and the exterior modes give contributions to the energy.

Although the main interest in the literature is the global approach, where the Casimir energy can be found and from which one can derive the force on the boundaries, the necessity of studying the local problem has often been suggested. First, the local properties of the vacuum fluctuations can in principle be observed by measuring the energy level shift of atoms interacting with an electromagnetic field [25] [26]. Second, the distortion of the vacuum fluctuations due to the presence of classical boundaries can also be measured by studying the spontaneous and induced emission of excited atoms in the presence of classical boundaries [27] [28]. For a update discussion of QED vacuum effects, see for example [29]. Finally, it is clear that local results contain more information than the global ones. A few years ago, Actor [30] and also Actor and Bender considered this kind of problem [31]. These authors studied the use of the generalized zeta function method [32] [33] [34] [35] to find the one-loop effective action associated with a scalar field defined in
the interior of a infinite waveguide of rectangular cross section. The local problem has also been considered by other authors. For the case of parallel plates geometric configuration Brown and Maclay [12], using the Green’s function method obtained the renormalized vacuum expectation value of the stress-energy tensor associated with an electromagnetic field. Deutsch and Candelas [6] evaluated the renormalized stress-energy tensors associated with a conformally coupled scalar field and also with an electromagnetic field in the wedge-shaped region formed by two plane boundaries. Recently, Brevik et al [36] repeated these calculations using Schwinger’s source theory. Their results agree with those of Deutsch and Candelas. In the spherical geometry, the local problem was also investigated by Olaussen and Ravndal [37], who studied the vacuum fluctuations of an electromagnetic field within a perfectly conducting spherical cavity. They found that the vacuum expectation value of the squared electric and magnetic fields diverge as one approaches the boundary. This result has also been obtained by Deutsch and Candelas [6], DeWitt [38], and Kennedy et al [39]. Olaussen and Ravndal [40] and also Milton [41] generalized this result to the non-abelian gluon fields in the MIT bag model. It has often been suggested that these surface divergences are related to the uncertainty relation between the field and the canonical conjugate momentum associated with the field [37] [12] [43]. In other local calculations, Ford [44] and Ford and Svaiter [45] discuss the possibility of amplification of the vacuum fluctuations. These authors studied the renormalized vacuum fluctuations associated with a scalar and electromagnetic field near the focus of a parabolic mirror. Using the geometric optics approximation they found that the parabolic mirror geometry can produce large vacuum fluctuations near the focus, much as
what happens in the classical focusing effect by the parabolic mirror geometry.

In our study of the one-loop vacuum fluctuations associated with a scalar field we develop a method adequate to deal with rectangular geometries. In this paper, using analytic regularization procedures, we first calculate the regularized vacuum fluctuations associated with a massless scalar field, confined in the interior of a infinite waveguide with rectangular cross-section, in a four-dimensional Euclidean space. We first rederive the well-known result that there are surface and edge divergences that require the introduction of surface and edge counterterms in the renormalization procedure. Then, we show how it is possible to generalize our results to a massive scalar field in a higher dimensional Euclidean space. Preliminary calculations of the renormalized vacuum expectation value of the stress-energy tensor in the rectangular waveguide were performed by Dowker and Banach. Also in rectangular geometries has been calculated in refs. A seminal paper studying these geometries was made by Ambjorn and Wolfram. More recently Milton and Ng studied the Casimir effect in (2 + 1) Maxwell-Chern-Simons electrodynamics in a rectangular domain. Also Hayan et al and Maclay respectively studied the vacuum fluctuations of the electromagnetic field and the Casimir force in the interior of a rectangular waveguide.

When studying interacting field theories for translationally invariant systems, one usually goes from coordinate to momentum space representation, a more convenient framework to analyse the divergences of the theory. In this representation translational invariance is expressed by momentum conservation conditions. Because the system of interest for this work possesses translational
invariance along two directions, it is more convenient to use a mixed coordinate-momentum representation for the Green’s functions [55]. For a recent treatment of systems without translational invariance, see for example [56] and also [57]. As we discussed, the fundamental problem in the infinite waveguide of rectangular cross section is the lack of translational invariance, which manifests itself by the fact that the Green’s functions associated with the scalar field are expressed in terms of infinite double summations. Although the one-loop vacuum fluctuation is written in terms of expressions involving double summations, a simple trigonometric identity allows us to obtain expressions with only one summation. The advantage is that all the calculations can then be done analytically. We show that the form of surface and edge counterterms that we have to introduce to renormalize the one-loop vacuum fluctuations, can be explicitly calculated and we discuss some different physical arguments that support the introduction of surface counterterms that remove these divergences.

Throughout the paper we use the term bulk counterterms to those counterterms that are not related to the existence of the boundaries. For instance, in the case of periodic boundary conditions we have these counterterms. As well as a second group that we call surface and edge counterterms that are related to the existence of the boundaries.

The organization of the paper is the following: In the section II we study the one-loop vacuum fluctuations of a massive scalar field confined within a rectangular waveguide. In section III we use an analytic regularization method to identify the infinities that appear in the expression of the vacuum fluctuations of the massless field near the boundaries. In section IV, we show how it
is possible to generalize our calculations to a massive scalar field defined in a higher dimensional Euclidean space. Conclusions are given in section V. Finally in the appendix we present the formal relation between the one-loop vacuum fluctuations and the local generalized zeta function. In this paper we use $\hbar = c = 1$.

2 Vacuum fluctuations of a scalar field confined in a rectangular waveguide

In this section and in the next one we will investigate the one-loop vacuum fluctuations associated with a scalar field defined in the interior of an infinite waveguide with rectangular cross section.

First, we assume that the scalar field is defined in a four-dimensional Euclidean space, where the last two coordinates are unbounded, while the first two, which we call $x_1$ and $x_2$, lie in the interval $[0, a]$ and $[0, b]$, respectively. We assume Dirichlet boundary conditions on the boundaries. The free field is defined in the region

$$\Omega = \mathbf{x} \equiv (x_1, x_2, x_3, x_4) : 0 < x_1 < a, \quad 0 < x_2 < b \subset \mathbb{R}^4,$$

with Dirichlet boundary conditions at $x_1 = 0$ and $x_1 = a$, and also $x_2 = 0$ and $x_2 = b$. As stressed previously, the lack of translational invariance introduces surfaces and edges divergences. One way to reduce the degree of these divergences is to smooth out the surface of the plates, for example by using soft, hard or semi-hard boundary conditions [58] [59] [60]. The one-loop graphs will depend on the ad-hoc model assumptions, and consequently we prefer to maintain the
hard walls. Instead of smoothing the plates surfaces, another way to avoid surface divergences, discussed by Kennedy et al. [39], is to treat the boundary as a quantum mechanical object. This approach was used recently, by Ford and Svaiter in the case of parallel flat plates, to solve a long standing paradox concerning the renormalized energy of minimally and conformally coupled scalar fields [43]. As we stressed before, we prefer, at least at the moment to keep only a hard classical boundary conditions.

We would like to stress that we are studying first the massless four-dimensional case since this simpler case will give us an indication of the behavior that may be expected in the most general case of a massive scalar field in a higher dimensional Euclidean space. A logical going is to use analytic regularization procedures to identify the divergent contributions that appear in the one-loop vacuum fluctuations. Let us start using dimensional regularization, working at the begining in a d-dimensional Euclidean space. Assuming Dirichlet boundary conditions on the walls we calculate first the two-point Schwinger function at coincident points in the interior of the waveguide. Then we use analytic regularization procedures to identify the form of the surface and edge divergences. In the one-loop vacuum fluctuations, \( \langle \varphi^2(x_1, x_2, a, b) \rangle \), we will change the notation of the previous section to \( x_1 = x, \ x_2 = y \) (note that in our notation \( \langle \varphi^2(x, y) \rangle \) means the one-loop vacuum fluctuations in one point with cartesian coordinates \( (x, y) \) of the rectangular cross section of the waveguide). Let the waveguide be oriented along the z axis with walls at \( x = 0 \) and \( a \) and \( y = 0 \) and \( b \). In this mixed representation, since we are assuming Dirichlet boundary conditions, the expression for the one-loop vacuum fluctuations, \( T_{DD}(x, y, a, b, d) = \langle \varphi^2(x, y) \rangle \) can
be written as:

\[
T_{DD}(x, y, a, b, d) = \frac{4}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \sin^2\left(\frac{n\pi x}{a}\right) \sin^2\left(\frac{n'\pi y}{b}\right)
\]

\[
\int d^{d-2}p \frac{1}{(p^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + m^2)}. \tag{3}
\]

There are two points that we would like to stress. First is the fact that to perform analytic
regularizations, we have to introduce a parameter \(\mu\) with dimension of mass in order to have
dimensionless quantities raised to a complex power. For sake of simplicity, we omit the \(\mu\) factor
in the following. Second is the fact that the generalization for the case of Neumann boundary
conditions is straightforward, although in this case infrared divergences associated with the \(n = 0\)
mode will appear in the case of massless scalar field. To circumvent this situation we must have
a finite Euclidean volume to regularize the model in the infrared.

Using trigonometric identities, it is convenient to write the one-loop vacuum fluctuation in the
following way:

\[
T_{DD}(x, y, a, b, d) = T(a, b, d) + T(x, a, b, d) + T(y, a, b, d) + T(x, y, a, b, d), \tag{4}
\]

where each expression of the above equation are given by the following. For \(T(a, b, d)\) we have

\[
T(a, b, d) = \frac{4}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{1}{(p^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + m^2)}. \tag{5}
\]

The second term in Eq.(4), \(T(x, a, b, d)\) is given by

\[
T(x, a, b, d) = -\frac{1}{2}T(a, b, d) + \frac{2}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{\cos\left(\frac{2n\pi x}{a}\right)}{(p^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + m^2)}. \tag{6}
\]
The expression for $T(y, a, b, d)$ has the same functional form of the above equation only changing $x$ by $y$ and $a$ by $b$. Consequently we have:

$$T(y, a, b, d) = -\frac{1}{2} T(a, b, d) + \frac{2}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{\cos(\frac{2n'\pi y}{b})}{\left(\vec{p}^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2 + m^2\right)},$$

(7)

and finally:

$$T(x, y, a, b, d) = \frac{1}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \frac{\cos\left(\frac{2n\pi x}{a}\right) \cos\left(\frac{2n'\pi y}{b}\right)}{\left(\frac{\vec{p}^2}{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + m^2\right)} - d^2 - 2ab \Gamma(2 - \frac{d}{2}) \sum_{n,n'=1}^{\infty} \frac{1}{\left(m^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2\right)^{\frac{d}{2}}}.$$

(8)

Thus the contribution given by Eq.(8) contains the same contributions given by Eqs.(5),(6),(7), as well as a contribution that contains edge divergences that we will define as $N(x, y, a, b, d)$. It is given by

$$N(x, y, a, b, d) = \frac{1}{(2\pi)^{d-2}ab} \sum_{n,n'=1}^{\infty} \int d^{d-2}p \left[1 - \cos\left(\frac{2n\pi x}{a}\right) - \cos\left(\frac{2n'\pi y}{b}\right) + \cos\left(\frac{2n\pi x}{a}\right) \cos\left(\frac{2n'\pi y}{b}\right)\right].$$

(9)

Let us study each contribution separately. Using dimensional regularization on Eq.(4) it is possible to write $T(a, b, d)$ in terms of the Epstein zeta function. Thus we have

$$T(a, b, d) = \frac{4}{(2\sqrt{\pi})^{d-2}ab} \Gamma(2 - \frac{d}{2}) \sum_{n,n'=1}^{\infty} \frac{1}{\left(m^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2\right)^{\frac{d}{2}}}.$$

(10)

The contribution given by $T(a, b, d)$ is one part of the vacuum field fluctuations which does not depend from the distance to the boundaries and in the renormalization procedure will require only a usual bulk counterterm. The form of the counterterm is given by the principal part of the
Laurent expansion of Eq. (10) around some \(d\), which must be given by the analytic extension of the Epstein zeta function in the complex \(d\) plane. The structure of the divergences of the Epstein zeta function is well known in the literature \cite{51} \cite{61} \cite{62} \cite{63}. Since the polar structure of \(T(a, b, d)\) can be found in the literature, to calculate the analytic structure of \(T(x, a, b, d)\) we will concentrate only on the position dependent divergent part given by \(T(x, a, b, d) + \frac{1}{2} T(a, b, d)\). This expression is given by

\[
T(x, a, b, d) + \frac{1}{2} T(a, b, d) = \frac{2}{(2\pi)^{d-2}ab} \sum_{n, n' = 1}^{\infty} \int d^{d-2}p \frac{\cos\left(\frac{2n\pi x}{a}\right)}{(\vec{p}^2 + (\frac{n\pi}{a})^2 + (\frac{n'\pi}{b})^2 + m^2)}.
\]  

(11)

Although Eq. (11) is written in terms of two sums, one of the sums can be easily done using a trigonometric expression given by \cite{64} \cite{65}:

\[
\sum_{n=1}^{\infty} \frac{\cos nt}{n^2 + A^2} = -\frac{1}{2A^2} + \frac{\pi}{2A} \frac{\cosh A(\pi - t)}{\sinh \pi A}.
\]  

(12)

Using Eq. (12), it is possible to write Eq. (11) as

\[
T(x, a, b, d) + \frac{1}{2} T(a, b, d) = R_1(a, b, d) + R_2(x, a, b, d)
\]  

(13)

where:

\[
R_1(a, b, d) = -\frac{1}{(2\pi)^{d-2}ab} \sum_{n' = 1}^{\infty} \int d^{d-2}p \frac{1}{(\vec{p}^2 + m^2 + (\frac{n'\pi}{b})^2)}
\]  

(14)

and

\[
R_2(x, a, b, d) = \frac{1}{(2\pi)^{d-2}b} \sum_{n' = 1}^{\infty} \int d^{d-2}p \frac{1}{\sqrt{\vec{p}^2 + m^2 + (\frac{n'\pi}{b})^2}} \frac{\cosh((a - 2x)\sqrt{\vec{p}^2 + m^2 + (\frac{n'\pi}{b})^2})}{\sinh(a\sqrt{\vec{p}^2 + m^2 + (\frac{n'\pi}{b})^2})}.
\]  

(15)
It is clear that to calculate the analytic structure for the case of the position dependent divergent part $T(y, a, b, d)$ we can use the same method that we use for $T(x, a, b, d)$. Consequently the expression for $T(y, a, b, d) + \frac{1}{2} T(a, b, d)$ is:

$$T(y, a, b, d) + \frac{1}{2} T(a, b, d) = I_1(a, b, d) + I_2(y, a, b, d)$$

(16)

where:

$$I_1(a, b, d) = -\frac{1}{(2\pi)^{d-2}} \sum_{n=1}^{\infty} \int d^{d-2}p \frac{1}{p^2 + m^2 + \left(\frac{n\pi}{a}\right)^2}$$

(17)

and

$$I_2(y, a, b, d) = \frac{1}{(2\pi)^{d-2}} \sum_{a=1}^{\infty} \int d^{d-2}p \frac{1}{\sqrt{p^2 + m^2 + \left(\frac{n\pi}{a}\right)^2}} \frac{\cosh((b - 2y)\sqrt{p^2 + m^2 + \left(\frac{n\pi}{a}\right)^2})}{\sinh(b\sqrt{p^2 + m^2 + \left(\frac{n\pi}{a}\right)^2})}.$$  

(18)

Using dimensional regularization in both expressions $I_1(a, b, d)$ and $R_1(a, b, d)$, we obtain a special Epstein-Hurwitz zeta function. The analytic extension of this function for general $d$ in the massive and massless case can be found in the literature. For the massive case see, for example [66]. For the massless case, the duplication formula for the Gamma function allows us to write

$$I_1(a, b, d)|_{m=0} = \frac{1}{a^{d-3}b} f_1(d) \zeta(4 - d) \Gamma(4 - d),$$

(19)

where

$$f_1(d) = -\frac{1}{2} \frac{\pi^{\frac{d-5}{2}}}{\Gamma(\frac{5-d}{2})},$$

(20)

is an entire function of $d$. Since $\zeta(z)$ can be analytically continued from an open connected set of points in the complex $z$ plane into the entire domain of $z$, it is easy to find the analytic structure
of $I_1(a, b, d)|_{m=0}$ in the complex $d$ plane. It is clear that $I_1(a, b, d)$ and $R_1(a, b, d)$ demand bulk counterterms. To find the analytic structure of $I_2(y, a, b, d)$ and $R_2(x, a, b, d)$, let us concentrate on $I_2(y, a, b, d)$. Integrating over the solid angle in Eq.(18), i.e using the fact that $d^d p = p^{d-1} dp d\Omega_d$ and $\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ we have:

$$I_2(y, a, b, d) = \frac{1}{a} h(d) \sum_{n=1}^{\infty} \int dp \frac{1}{\sqrt{\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2}} \cosh((b - 2y)\sqrt{\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2}) \sinh(b\sqrt{\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2})$$

(21)

where the function $h(d)$ is given by:

$$h(d) = \frac{2}{(2\sqrt{\pi})^{d-2}\Gamma(\frac{d}{2})}$$

(22)

Performing a change of variables $v = \left(\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2\right)^{\frac{1}{2}}$, and now going back to the four-dimensional case it is possible to write $I_2(y, a, b, d)|_{d=4} \equiv I_2(y, a, b)$ as :

$$I_2(y; a, b) = \frac{1}{2\pi a} \sum_{n=1}^{\infty} \int_{\alpha}^{\infty} dv \frac{\cosh((b - 2y)v)}{\sinh bv}$$

(23)

where the lower limit of the above integral is given by

$$\alpha = \sqrt{m^2 + \left(\frac{n\pi}{a}\right)^2}.$$ 

(24)

It is important to keep in mind that the situation is completely different for $d \neq 4$, since after change of variables the term $(v^2 - m^2 - \left(\frac{n\pi}{a}\right)^2)^{\frac{d-4}{2}}$ it will appear in the integrand of Eq.(21). As a consequence of this fact, it is more difficult to perform algebraic manipulations that allow us to analytically regularize $R_2(x, a, b, d)$ and $I_2(y, a, b, d)$ and also other expressions. Nevertheless, utilizing the same techniques used in refs. [67] [68] [69], it is possible to perform all the calculations
for a higher dimensional Euclidean space. This generalization will be presented in section IV. Going back to Eq. (23) and using trigonometric identities, we have:

\[ I_2(y, a, b) = I_{21}(y, a) + I_{22}^+(y, a, b) + I_{22}^-(y, a, b) \]  

(25)

where:

\[ I_{21}(y, a) = \frac{1}{4\pi ya} \sum_{n=1}^{\infty} \exp \left( -2y\sqrt{m^2 + \left( \frac{n\pi}{a} \right)^2} \right) \],

(26)

\[ I_{22}^+(y, a, b) = \frac{1}{4\pi ya} \sum_{n=1}^{\infty} \int_{b\alpha}^{\infty} dq \left( \coth q - 1 \right) e^{\frac{2ny}{b}}, \]

(27)

and finally

\[ I_{22}^-(y, a, b) = \frac{1}{4\pi ya} \sum_{n=1}^{\infty} \int_{b\alpha}^{\infty} dq \left( \coth q - 1 \right) e^{-\frac{2ny}{b}}. \]

(28)

An exact expression in the massless case can be obtained from \( I_{21}(y, a) \). Summing the geometric series \( I_{21}(y, a) \)|\( m=0 \) can be written as

\[ I_{21}(y, a) \mid_{m=0} = \frac{1}{4\pi ay} \frac{1}{e^{\frac{2\pi}{b}} - 1}. \]

(29)

We will use the Bernoulli polynomials and numbers in the next sections. Consequently, let us introduce this Laurent expansion to study the behavior of \( I_{21}(y, a) \)|\( m=0 \) in the neighborhood of \( y = 0 \). To find the principal part of the Laurent series of \( I_{21}(y, a) \)|\( m=0 \) around \( y = 0 \) let us use the Bernoulli polynomials, which are defined by the generating function

\[ \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \]

(30)

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and the Bernoulli numbers $B_n = B_n(x = 0), (B_0 = 1, B_1 = -\frac{1}{2}, \ldots)$. Using this Laurent expansion one finds
\[
I_{21}(y, a)|_{m=0} = \left( \frac{B_0}{8\pi^2 y^2} + \frac{B_1}{4\pi ya} + \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{B_n}{n!} \left( \frac{2\pi y}{a} \right)^{n-1} \right). \tag{31}
\]

We note that we still have to calculate $N(x, y, a, b)$. Using again the trigonometric identity given by Eq.\((12)\), it is possible to write
\[
N(x, y, a, b, d) = N_1(x, a, b, d) + N_2(x, y, a, b, d)
\]
where we have:
\[
N_1(x, a, b, d) = -\frac{1}{2(2\pi)^{d-2}ab} \sum_{n=1}^{\infty} \int d^{d-2}p \frac{\cos\left(\frac{2n\pi x}{a}\right)}{\left(\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2\right)} \tag{32}
\]
and
\[
N_2(x, y, a, b, d) = \frac{1}{(2\pi)^{d-1}a} \sum_{n=1}^{\infty} \int d^{d-2}p \frac{\cos\left(\frac{2n\pi x}{a}\right)}{\left(\sqrt{\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2}\right)}
\]
\[
\cosh\left(\frac{b - 2y}{\sqrt{\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2}}\right) \sinh\left(\frac{b}{\sqrt{\vec{p}^2 + m^2 + \left(\frac{n\pi}{a}\right)^2}}\right). \tag{33}
\]

Let us study the expression given by $N_1(x, a, b, d)$. It is possible to write $N_1(x, a, b, d) = N_{11}(a, b, d) + N_{12}(x, a, b, d)$, where $N_{11}(a, b, d)$ and $N_{12}(x, a, b, d)$ are given by
\[
N_{11}(a, b, d) = \frac{1}{4(2\pi)^{d-2}ab} \int d^{d-2}p \frac{1}{(\vec{p}^2 + m^2)} \tag{34}
\]
and
\[
N_{12}(x, a, b, d) = -\frac{1}{4(2\pi)^{d-2}b} \int d^{d-2}p \frac{1}{\sqrt{\vec{p}^2 + m^2}} \frac{\cosh\left(\frac{(a - 2x)\sqrt{(\vec{p}^2 + m^2)}}{\sinh\left(a\sqrt{\vec{p}^2 + m^2}\right)}\right)}{\Gamma\left(2 - \frac{d}{2}\right)(m^2)^{\frac{d}{2}-2}}. \tag{35}
\]
The expression given by Eq.\((34)\) can be easily calculated using dimensional regularization and can be written as
\[
N_{11}(a, b, d) = \frac{1}{4ab(2\sqrt{\pi})^{d-2}} \Gamma\left(2 - \frac{d}{2}\right)(m^2)^{\frac{d}{2}-2}. \tag{36}
\]
This contribution for the vacuum fluctuations also demands bulk counterterm. We have to deal with the expression of \( N_{12}(x, y, a, b, d) \). Integrating over the solid angle, changing the variables and using the fact that we are in \( d = 4 \), i.e. defining \( N_{12}(x, a, b, d) \) we have

\[
N_{12}(x, a, b) = -\frac{1}{8\pi b} \int_{\alpha}^{\infty} dv \frac{\cosh((a - 2x)v)}{\sinh av}. \tag{37}
\]

Again, using trigonometric identities, we have:

\[
N_{12}(x, a, b) = -\frac{1}{8\pi ab} \left[ \frac{a}{2x} e^{-2xm} + \frac{1}{2} \int_{\alpha m}^{\infty} dq \, (\coth q - 1) e^{2q \frac{x}{a}} + \frac{1}{2} \int_{\alpha m}^{\infty} dq \, (\coth q - 1) e^{-2q \frac{x}{a}} \right]. \tag{38}
\]

We note that Eq.(38) has surface divergences when \( x \to 0 \) and \( x \to a \). The structure of the divergences of \( N_{12}(x, a, b) \) will be studied further.

As a next step in the discussion, let us investigate the part of the one-loop vacuum fluctuations that contains edge divergences that is given by \( N_2(x, y, a, b) \). Again, integrating over the solid angle, changing the variables and using the fact that we are in \( d = 4 \), the expression for \( N_2(x, y, a, b) \) is given by

\[
N_2(x, y, a, b) = \frac{1}{4a} \sum_{n=1}^{\infty} \cos(\frac{2n\pi x}{a}) \int_{\alpha}^{\infty} dv \frac{\cosh((b - 2y)v)}{\sinh bv}, \tag{39}
\]

where the lower limit of the above integral is given by \( \alpha \), defined in Eq.(24). Using trigonometric identities it is possible to write \( N_2(x, y, a, b) \) as

\[
N_2(x, y, a, b) = N_{21}(x, y, a) + N_{22}(x, y, a, b), \tag{40}
\]

where:

\[
N_{21}(x, y, a) = -\frac{1}{8ay} \sum_{n=1}^{\infty} \cos(\frac{2n\pi x}{a}) e^{-2y\alpha} \tag{41}
\]
and
\[ N_{22}(x, y, a, b) = \frac{1}{4a} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{a}\right) \int_{0}^{\infty} dv\ (\coth(bv) - 1) \cosh 2vy. \] (42)

From now we will put \( m = 0 \). Using the Poisson’s kernel given by
\[ 1 + 2 \sum_{n=1}^{\infty} r^{n} \cos n\phi = \frac{1 - r^{2}}{1 - 2r \cos \phi + r^{2}}, \quad 0 \leq r < 1, \] (43)
it is possible to obtain a closed expression for Eq.(11), for points outside the boundaries. Note that \( N_{21}(x, y, a, b) \) diverges for \( y \to 0 \) for any \( x \). Let us study the behavior of \( N_{21}(x, y, a, b) \) near the edges \( x = 0 \), \( y = 0 \) and also \( x = a, y = 0 \). Using the Poisson’s kernel, the function \( N_{21}(x, y, a, b) \) in \( x = 0 \) or \( x = a \) is given by
\[ N_{21}(x, y, a) \bigg|_{x=0} = \frac{1}{16ay} - \frac{1}{64\pi y^{2}} + \frac{1}{16ay} e^{\frac{4\pi y}{a}} + \frac{1}{16ay} e^{-\frac{4\pi y}{a}} + O(y). \] (44)

Again using the generating function of the Bernoulli numbers it is possible to find the principal part of the Laurent expansion of \( N_{21}(x, y, a, b) \big|_{x=0} \) around \( y = 0 \), and also \( N_{21}(x, y, a, b) \big|_{x=a} \) around \( y = 0 \). The Laurent expansion around \( y = 0 \) allow us to write \( N_{21}(x, y, a, b) \big|_{x=0} \) as
\[ N_{21}(x, y, a) \bigg|_{x=0} = \frac{B_{0}}{64\pi y^{2}} + \frac{B_{1}}{16ay} + O(y). \] (45)

We note that Eq.(11) has two edge divergences, one at \( x = y = 0 \) and the other at \( x = a, y = 0 \). To investigate the divergences of Eq.(12) let us rewrite \( N_{22}(x, y, a, b) \) as
\[ N_{22}(x, y, a, b) = N_{22}^{+}(x, y, a, b) + N_{22}^{-}(x, y, a, b) \] (46)
where
\[ N_{22}^+(x, y, a, b) = \frac{1}{8ab} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{a}\right) \int_{-b}^{b} dq \ (\coth q - 1) e^{\frac{2qy}{b}} \] (47)
\[ N_{22}^-(x, y, a, b) = \frac{1}{8ab} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{a}\right) \int_{-b}^{b} dq \ (\coth q - 1) e^{-\frac{2qy}{b}}. \] (48)

In the next section, we will show that \( N_{22}^+(x, y, a, b) \) has surface divergences when \( y \to b \) for all \( x \), and also two edge divergences, one at \( x = 0, y = b \) and the other at \( x = a, y = b \). We note also that \( N_{22}^-(x, y, a, b) \) is finite everywhere. Using a combination of dimensional and zeta function regularization, in the next section we will present the general method to regularize the one-loop vacuum field fluctuations in the rectangular waveguide.

3 Analysis of the surface divergences in the one-loop vacuum fluctuations.

The purpose of this section is to present a general method to analytic regularize the one-loop vacuum fluctuations associated with the confined scalar field. We first present the structure of the surface and edge divergences of the vacuum fluctuations associated with a massless scalar field in a four dimensional Euclidean space. As we discussed in the previous section, it is possible to write the one-loop vacuum fluctuations as
\[ T_{DD}(x, y, a, b, d) = T(a, b, d) + T(x, a, b, d) + T(y, a, b, d) + T(x, y, a, b, d) \] (49)

The first expression that we have to deal with is \( T(a, b, d) \). As we discussed in the previous
section, the analytic structure of $T(a, b, d)$ was carefully analysed by Kirsten and it is not necessary to repeat the calculation again. The second term of the above equation and third one can be written respectively in terms of $R_1(a, b, d)$, $R_2(x, a, b, d)$, $I_1(a, b, d)$ and finally $I_2(y, a, b, d)$. The polar structure of $R_1(a, b, d)$ and $I_1(a, b, d)$ can be found in the literature, and we will not repeat the analysis that was done in these papers. The next quantity that we have to regularize is $R_2(x, a, b, d)$ and also $I_2(y, a, b, d)$. Since both cases are equivalent, let us study only the expression given by $R_2(x, a, b, d)$. It is instructive to study first the simpler case $b \gg a$ which will give us an indication of the behavior that may be encountered. Let us now proceed with the calculations in a general d-dimensional Euclidean space. Defining $R_2(x, a, b, d)|_{b \gg a} = r_2(x, a, d)$, we have:

$$r_2(x, a, d) = \frac{1}{2(2\pi)^{d-1}} \int d^{d-1}p \frac{1}{\sqrt{p^2 + m^2}} \frac{\cosh((a - 2x)\sqrt{p^2 + m^2})}{\sinh(a\sqrt{p^2 + m^2})}. \quad (50)$$

We will again use the fact that $d^{d-1}p = p^{d-2}dp d\Omega_{d-1}$ and $d\Omega_{d-1} = \frac{2\pi^{d-1}}{\Gamma(d/2)}$. Let us choose now $m = 0$. Defining $h_2(d)$ by:

$$h_2(d) = \frac{1}{2^{d-2}\pi^{d/2-1}\Gamma(d/2)}, \quad (51)$$

it is possible to write $r_2(x, a, d)|_{m=0}$ as

$$r_2(x, a, d)|_{m=0} = \frac{1}{2} h_2(d) \left[ \int_0^\infty dk \, k^{d-3}(\coth ka - 1) \cosh 2kx 
+ \int_0^\infty dk \, k^{d-3}e^{-2kx} \right]. \quad (52)$$

In the first integral for large $k$, $(\coth ka - 1)$ has the behavior: $(\coth ka - 1) \sim e^{-2ka}$. Moreover, the second integral in the above equation is ultraviolet finite for $x \neq 0$. Let us define $t = ka$ and
\[ q = kx \text{ in the first and second integrals above respectively. Then Eq.}(52) \text{ becomes:} \]

\[
r_2(x, a, d)|_{m=0} = \frac{1}{2a^{d-2}} h_2(d) \int_0^\infty dt \ t^{d-3} \left( \coth t - 1 \right) \cosh\left(\frac{2xt}{a}\right) \\
+ \frac{1}{2x^{d-2}} h_2(d) \int_0^\infty dq \ q^{d-3} e^{-2q}. \tag{53}
\]

The second term in the above equation gives us the well known result that for a massless scalar field \( \langle \varphi^2(x) \rangle \) diverges as \( \frac{1}{x^2} \) (in a four-dimensional space) as we approach the plate \[3\]. In order to analyze the behavior of \( r_2(x, a, d)|_{m=0} \) around \( x = a \), let us use the following integral representation of the gamma function,

\[
\int_0^\infty dt \ t^{\mu-1} e^{-\nu t} = \frac{1}{\nu^\mu} \Gamma(\mu), \quad Re(\mu) > 0, \ Re(\nu) > 0 \tag{54}
\]

and also the following integral representation of the product of the Gamma function times the Hurwitz zeta function

\[
\int_0^\infty dt \ t^{\mu-1} e^{-\alpha t} (\coth t - 1) = 2^{1-\mu} \Gamma(\mu, \frac{\alpha}{2} + 1) \quad Re(\alpha) > 0, \ Re(\mu) > 1, \tag{55}
\]

where \( \zeta(s, u) \) is the Hurwitz zeta function defined by \[64\]

\[
\zeta(s, u) = \sum_{n=0}^{\infty} \frac{1}{(n + u)^s}, \quad Re(s) > 1, \quad u \neq 0, -1, -2... \tag{56}
\]

Then, using Eqs.\( (54) \), \( (53) \) and \( (56) \) in Eq.\( (53) \) we have that:

\[
r_2(x, a, d)|_{m=0} = \frac{1}{2} h_2(d) \frac{1}{a^{d-2}} \left[ 2^{2-d} \Gamma(d-2) \left( \zeta(d-2, \frac{x}{a} + 1) + \zeta(d-2, -\frac{x}{a} + 1) \right) \right] \\
+ \frac{1}{(2x)^{d-2}} h_2(d) \Gamma(d-2). \tag{57}
\]
Using the definition of the Hurwitz zeta function, it is evident that:

\[
\frac{1}{a^{d-2}} \left( \zeta(d-2, \frac{x}{a} + 1) + \zeta(d-2, -\frac{x}{a} + 1) \right) = \\
\frac{1}{a^{d-2}} \sum_{n=0}^{\infty} \frac{1}{\left(n + \left(1 + \frac{x}{a}\right)\right)^{d-2}} + \frac{1}{(a-x)^{d-2}} + \frac{1}{a^{d-2}} \sum_{n=1}^{\infty} \frac{1}{\left(n + \left(1 - \frac{x}{a}\right)\right)^{d-2}}.
\]

We see that the regularized $r_2(x, a, d)|_{m=0}$ has two poles of order $(d-2)$ in $x = 0$ and in $x = a$. Note that the residues of the poles in $x = 0$ and in $x = a$ are $a$-independent. The same analysis can be done for $I_2(y, a, b, d)$ assuming $a \gg b$. In the next section the general case will be studied.

Let us finally analyze $N_{21}(x, y, a)$ and $N_{22}(x, y, a, b)$, given respectively by

\[
N_{21}(x, y, a) = -\frac{1}{8ay} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{a}\right)e^{-\frac{3n\pi y}{a}}
\]

and

\[
N_{22}(x, y, a, b) = \frac{1}{4ab} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{a}\right) \int_{-\infty}^{\infty} dq \left( \coth q - 1 \right) \cosh\left(\frac{2qy}{b}\right).
\]

To find the analytic structure of $N_{21}(x, y, a)$ we can expand the general term in the sum in power series, commute the two summations, and use analytic continuation in the zeta function that will appear. The process will in general produce an extra term, which is generated by commuting the convergent exponential summation $\sum_m$ with the new divergent summation $\sum_k$ (for details see for e.g. refs. [70] [71] [72]). In our case this term vanishes due to the power of $n$. Let us express the sum that appears in Eq.(59) in terms of the complex variable $z = ix - y$:

\[
N_{21}(x, y, a) = -\frac{1}{8ay} \text{Re} \left\{ \sum_{n=1}^{\infty} \exp\left(\frac{2n\pi z}{a}\right) \right\}.
\]

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Expanding around $z = 0$, (using the Bernoulli expansion) will produce :

$$N_{21}(x, y, a) = -\frac{1}{8ay} Re \left\{ \frac{-a}{2\pi z} + \frac{1}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!} \left( -\frac{2\pi z}{a} \right)^k \right\}. \quad (62)$$

We see that the edge divergence appears in the term $1/z$. Taking the real part :

$$N_{21}(x, y, a) = -\frac{1}{16\pi (y^2 + x^2)} - \frac{1}{8ay} Re \left\{ f_1(z) \right\}, \quad (63)$$

where $f_1(z)$ is an entire function of $z$ and is given by :

$$f_1(z) = Re \left\{ \frac{1}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!} \left( -\frac{2\pi z}{a} \right)^k \right\}. \quad (64)$$

Expanding around $z = ia$ will produce :

$$N_{21}(x, y, a) = -\frac{1}{8ay} Re \left\{ \sum_{k=0}^{\infty} \left( \frac{2\pi}{a} \right)^k \frac{(z - ia)^k}{k!} \zeta(-k) - \left( \frac{a}{2\pi} \right)^{a+1} (z - ia)^{-1} \right\}. \quad (65)$$

Taking the real part of $1/(z - ia)$, we have :

$$N_{21}(x, y, a) = -\frac{1}{16\pi (y^2 + (x - a)^2)} - \frac{1}{8ay} f_2(z), \quad (66)$$

where $f_2(z)$ is also an entire function of $z$, and is given by:

$$f_2(z) = Re \left\{ \sum_{k=0}^{\infty} \left( \frac{2\pi}{a} \right)^k \frac{(z - ia)^k}{k!} \zeta(-k) \right\}. \quad (67)$$

To find the analytic structure of $N_{22}(x, y, a, b)$, it is enough to analyse the quantity $N_{22}^+(x, y, a, b)$, which is given by the Eq.(47) (the other quantity $N_{22}^-(x, y, a, b)$ is finite everywhere). To calculate the integral in Eq.(47), we can express coth$(bv)$ in exponential functions and expand the integrand.
in power series. The integral can be easily evaluated, and the result is given by

\[ N_{22}^+(x, y, a, b) = -\frac{1}{8ab} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{a}\right) \sum_{k=0}^{\infty} \frac{e^{-2\left(1 - \frac{y}{b} + k\right)a b}}{(1 - \frac{y}{b} + k)}. \] (68)

This result can be written in terms of the Lerch’s transcendent function \( \Phi(z, c, v) \) which is defined as follows [73]:

\[ \Phi(z, c, v) = \sum_{k=0}^{\infty} \frac{z^k}{(v + k)^c}. \] (69)

which is valid for \(|z| < 1\). By analytic continuation it can be extended to the whole complex plane. This function has singularities at \( z = 1 \) and \( c = 0 \) or \( c = 1 \) and when \( v \) is a non-positive integer and \( \text{Re}(c) \) is also non-positive. We note that only the term \( k = 0 \) has a surface divergence at \( y = b \). The remaining part of the series in \( k \) is finite everywhere and we call it \( F(x, y, a, b) \). Using the same procedure used to analyse \( N_{21}(x, y, a) \), we can define \( z = (b - y) + ix \) and expand the term \( k = 0 \) around \( z = 0 \) or around \( z = ia \). Therefore we have

\[ N_{22}^+(x, y, a, b) = B(x, y, a, b) + F(x, y, a, b), \] (70)

where the first expansion \( B(x, y, a, b) \) is given by

\[ B(x, y, a, b) = -\frac{1}{16\pi (x^2 + (y - b)^2)} - \frac{1}{8a(y - b)} h_1(z), \] (71)

where \( h_1(z) \) is an entire function given by

\[ h_1(z) = \text{Re} \left\{ \sum_{k=0}^{\infty} \left(\frac{2\pi}{a}\right)^k \frac{(z)^k}{k!} \zeta(-k) \right\}. \] (72)
We note that in this case $B(x, y, a, b)$ has a edge divergence for $x = 0$, $y = b$. For the second expansion, $B(x, y, a, b)$ is given by

$$B(x, y, a, b) = -\frac{1}{16\pi((x-a)^2 + (y-b)^2)} - \frac{1}{8a(y-b)}h_2(z), \quad (73)$$

where $h_2(z)$ is an entire function given by

$$h_2(z) = \text{Re}\left\{ \sum_{k=0}^{\infty} \left(\frac{2\pi}{a}\right)^k \frac{(z-ia)^k}{k!} \zeta(-k) \right\}. \quad (74)$$

We note that in this case $B(x, y, a, b)$ has a edge divergence for $x = a$, $y = b$. A general picture that emerges from the above discussion is the following: we have found that in order to eliminate the ultraviolet divergences of the one-loop vacuum fluctuations we have to introduce not only the usual bulk counterterms, but also counterterms concentrated on the boundaries. In the next section we will show how it is possible to generalize our local analysis for a massive field defined in a higher dimensional Euclidean space.

4 The one-loop vacuum fluctuation in the waveguide in a higher dimensional Euclidean space.

In this section we will discuss how to generalize our calculations to a higher dimensional Euclidean space. We are interested in investigating the one-loop vacuum fluctuations associated with a massive scalar field defined in a generic $d$-dimensional Euclidean space, where $d - 2$ coordinates are unbounded while the first two lie in some finite interval. As discussed in the previous sections,
we found that the vacuum fluctuations can be expressed as:

\[
\langle \varphi^2(x,y) \rangle = \frac{1}{4} T(a,b,d) + \frac{1}{2} \left( R_1(a,b,d) + I_1(a,b,d) \right) + \frac{1}{2} \left( R_2(x,a,b,d) + I_2(y,a,b,d) \right) + N(x,y,a,b,d). \tag{75}
\]

In the previous sections, we studied the expression given by \( T(a,b,d), R_1(a,b,d) \) and also \( I_1(a,b,d) \) in the massless case, which demands bulk counterterms. In the following, we will present a detailed analysis of the contributions of the vacuum fluctuations that demand counterterms concentrated on the boundaries. We will show that some of the terms of the one-loop vacuum fluctuations, can be expressed in terms of Bessel functions and also Hurwitz zeta functions.

From the previous analysis for the massless field in a four-dimensional Euclidean space, we found that to perform our calculations in a generic \( d \)-dimensional space, for the massive field, it is natural to write \( I_2(y,a,b,d) \) in the following way:

\[
I_2(y,a,b,d) = I_{21}(y,a,d) + I_{22}(y,a,b,d), \tag{76}
\]

where the functions \( I_{21}(y,a,d) \) and \( I_{22}(y,a,b,d) \) are given by:

\[
I_{21}(y,a,d) = \frac{1}{a} h(d) \sum_{n=1}^{\infty} \int_{\alpha}^{\infty} dv \left( v^2 - m^2 - \left( \frac{n\pi}{a} \right)^2 \right) \frac{d-4}{2} e^{-2vy}, \tag{77}
\]

and

\[
I_{22}(y,a,b,d) = \frac{1}{a} h(d) \sum_{n=1}^{\infty} \int_{\alpha}^{\infty} dv \left( v^2 - m^2 - \left( \frac{n\pi}{a} \right)^2 \right) \frac{d-4}{2} (\coth bv - 1) \cosh 2vy. \tag{78}
\]

The lower limit of both integrals is given by \( \alpha \) defined in Eq.(24) and the coefficient \( h(d) \) is an entire function defined in Eq.(22). Using the following integral representation of the modified
Bessel functions of third kind, or the Macdonald’s functions $K_{\nu}(x)$:

$$
\int_u^{\infty} (x^2 - u^2)^{\nu - 1} e^{-\mu x} \, dx = \frac{1}{\sqrt{\pi}} \left( \frac{2u}{\mu} \right)^{\nu - \frac{1}{2}} \Gamma(\nu) K_{\nu - \frac{1}{2}}(u\mu),
$$

(79)

which is valid for for $u > 0$, $Re \mu > 0$ and $Re \nu > 0$, $I_{21}(y, a, b, d)$ can be written in terms of these functions. A simple substitution gives

$$
I_{21}(y, a, b, d) = \frac{1}{a} \left( \frac{1}{2\sqrt{\pi}} \right)^{d-1} \sum_{n=1}^{\infty} \left( \frac{\alpha}{y} \right)^{-\frac{d-3}{2}} K_{\frac{d-3}{2}}(2\alpha y).
$$

(80)

Equation (80) can not be evaluated exactly, even for the massless case, but for small arguments of the Bessel function it is possible to show how does $I_{21}(y, a, d)$ contributes to the surface divergences of the one-loop vacuum fluctuations.

The expression $I_{22}(y, a, b, d)$ can also be computed in the both cases, the massive and the massless one. To illustrate the method that we discussed in previous sections, let us present the massless case calculation. The massive case follows the same procedure. To calculate $I_{22}(y, a, b, d)|_{m=0}$ note that after change of variables $I_{22}(y, a, b, d)|_{m=0}$, contains a power of a binomial. When $d$ is even (for $d > 4$) the power is an integer and the use of the Newton’s binomial theorem will give a very direct way to generalize our further results. When $d$ is odd ($d > 4$), the expansion on the binomial yields an infinite power series. The same technique was used in refs. [67] [68] [69]. It is worthwhile to remark that the study of even case follows the same procedure, only with a finite numbers of terms in the binomial expansion. Consequently, let us study the most interesting case, i.e the odd dimensional case. Let us use the following series representation for $(1 - \frac{v\pi}{va})^{\frac{d-4}{2}}$:

$$
(1 - \frac{v\pi}{va})^{\frac{d-4}{2}} = \sum_{k=0}^{\infty} (-1)^k C_{d-4}^k \left( \frac{v\pi}{va} \right)^{2k},
$$

(81)
where the $C_p^k$ are the generalizations of the binomial coefficients, given by: $C_p^0 = 1, C_p^1 = \frac{p}{1}, C_p^2 = \frac{1}{2!}p(p - 1)\ldots, C_p^k = \frac{1}{k!}(p(p - 1)..(p - k + 1))$ and $p = \frac{d - 4}{2}$. Note that the generalization of the binomial series is valid even for any complex exponent $p$. In other words, for $v > \frac{n\pi}{a}$ we have an everywhere convergent power series in $p$, hence a continuous function on $p$ in the complex $d$ plane. In the following, it is convenient to define the quantities:

$C^{(1)}(d, k) = (-1)^k C_p^k h(d)$, $C^{(2)}(d, k) = \pi^{2k} C^{(1)}(d, k)$ and finally $C^{(3)}(d, k) = \frac{\Gamma(d - 3 - 2k)}{2^{d - 3 - 2k}} C^{(2)}(d, k)$. After a change of variables, using Eq. (81) in Eq. (78), $I_{22}(y; a, b, d)|_{m=0}$ becomes

$$I_{22}(y, a, b, d)|_{m=0} = \frac{1}{ab^{d-3}} \sum_{k=0}^{\infty} C^{(2)}(d, k) (\frac{b}{a})^{2k} \sum_{n=1}^{\infty} n^{2k} \int_{\frac{n\pi b}{a}}^{\infty} du \ u^{d-4-2k}(\coth u - 1) \cosh(\frac{2uy}{b}) \int_{\frac{n\pi b}{a}}^{\infty} du \ u^{d-4}(\coth u - 1) \cosh(\frac{2uy}{b}), \quad (82)$$

where $C^{(2)}(d, k)$ is an entire function in the complex $d$ plane. The integral that appear in Eq. (82) cannot be evaluated explicitly in terms of well known functions. Nevertheless it is possible to write Eq. (82) in a convenient way where the structure of the divergences near the plate when $y \to b$ appear. Let us split the $I_{22}(y, a, b, d)|_{m=0}$ in the following way:

$$I_{22}(y, a, b, d)|_{m=0} = I_{22}^<(y, a, b, d) + I_{22}^>(y, a, b, d), \quad (83)$$

where

$$I_{22}^<(y, a, b, d) = \frac{1}{ab^{d-3}} \sum_{k=0}^{k<\frac{d-4}{2}} C^{(2)}(d, k) (\frac{b}{a})^{2k} \sum_{n=1}^{\infty} n^{2k} \int_{\frac{n\pi b}{a}}^{\infty} du \ u^{d-4-2k}(\coth u - 1) \cosh(\frac{2uy}{b}), \quad (84)$$

and

$$I_{22}^>(y, a, b, d) = \frac{1}{ab^{d-3}} \sum_{k=\frac{d-4}{2}}^{\infty} C^{(2)}(d, k) (\frac{b}{a})^{2k} \sum_{n=1}^{\infty} n^{2k} \int_{\frac{n\pi b}{a}}^{\infty} du \ u^{d-4-2k}(\coth u - 1) \cosh(\frac{2uy}{b}). \quad (85)$$
Although the integral that appears in Eq. (85) cannot be expressed in terms of known functions, the leading divergences of $I_{22}(y, a, b, d)_{m=0}$ are contained in the Eq. (84). Consequently, let us analyze the divergences that appear in Eq. (84). Using the fact that

$$f_{\alpha} \int_{\alpha}^{\infty} f(u) du = \int_{0}^{\infty} f(u) du - \int_{0}^{\alpha} f(u) du,$$

we reexpress Eq. (84) as

$$I_{22}^< (y, a, b, d) = I_{22}^a (y, a, b, d) + I_{22}^b (y, a, b, d), \quad (86)$$

where

$$I_{22}^a (y, a, b, d) = \frac{1}{ab^{d-3}} \sum_{k=0}^{k<d-\frac{d}{2}} C^{(2)}(d, k) \left(\frac{b}{a}\right)^{2k} \sum_{n=1}^{\infty} n^{2k} \int_{0}^{\infty} \int_{0}^{\infty} u^{d-4-2k} (\coth u - 1) \cosh\left(\frac{2uy}{b}\right), \quad (87)$$

and

$$I_{22}^b (y, a, b, d) = -\frac{1}{ab^{d-3}} \sum_{k=0}^{k<d-\frac{d}{2}} C^{(2)}(d, k) \left(\frac{b}{a}\right)^{2k} \sum_{n=1}^{\infty} n^{2k} \int_{0}^{\infty} \int_{0}^{\infty} u^{d-4-2k} (\coth u - 1) \cosh\left(\frac{2uy}{b}\right). \quad (88)$$

Expanding $\cosh\left(\frac{2uy}{b}\right)$ in powers series, the integral that appears in $I_{22}^b (y, a, b, d)$ is a Debye integral. Although this can be evaluated, the study of $I_{22}^b (y, a, b, d)$ is sufficient to obtain the divergences of $I_{22}^< (y, a, b, d)$. Again, we note that the integral that appear in Eq. (87) can be written in terms of products of zeta, Gamma function and Hurwitz zeta function. Thus we have:

$$I_{22}^< (y, a, b, d)_{m=0} = -\frac{1}{ab^{d-3}} \sum_{k=0}^{k<d-\frac{d}{2}} C^{(2)}(d, k) \left(\frac{b}{a}\right)^{2k} \sum_{n=1}^{\infty} n^{2k} \int_{0}^{\infty} \int_{0}^{\infty} u^{d-4-2k} (\coth u - 1) \cosh\left(\frac{2uy}{b}\right)$$

$$+ \frac{1}{2ab^{d-3}} C^{(3)}(d, 0) \left(\zeta(d - 3, -\frac{y}{b} + 1) + \zeta(d - 3, \frac{y}{b} + 1)\right). \quad (89)$$

Since $C^{(3)}(d, 0)$ is an entire function in the complex $d$ plane, it is clear that we have the same surface divergences that we studied before. To study the massive case $I_{22}(y, a, b, d)$ we have to
use that same procedures. Let us show how our procedure can be systematically applied in the other expressions. From the previous calculations we express the contribution to the vacuum fluctuations that contains edge divergences giving by \( N(x, y, a, b, d) \) in the following way:

\[
N(x, y, a, b, d) = N_{11}(a, b, d) + N_{12}(x, a, b, d) + N_2(x, y, a, b, d),
\]

(90)

where the \( N_{11}(a, b, d) \) term contains contributions to the vacuum fluctuations that demands bulk counterterms in order to renormalize the one-loop vacuum fluctuations. The next contribution that we have to study is \( N_{12}(x, a, b, d) \). Starting from Eq.(35) integrating over the solid angle and changing variables, it is also convenient to split \( N_{12}(x, a, b, d) \) in the following way:

\[
N_{12}(x, a, b, d) = N_{12}^a(x, a, b, d) + N_{12}^b(x, a, b, d),
\]

(91)

where each of the terms of Eq.(91) are given respectively by

\[
N_{12}^a(x, b, d) = -\frac{1}{4b} \frac{h(d)}{1} \int_m^\infty dv \left(v^2 - m^2\right) \frac{d}{dv} e^{-2vx},
\]

(92)

and

\[
N_{12}^b(x, a, b, d) = -\frac{1}{4b} \frac{h(d)}{1} \int_m^\infty dv \left(v^2 - m^2\right) \frac{d}{dv} (\coth av - 1) \cosh 2vx.
\]

(93)

We can again use the integral representation of the modified Bessel functions of the third kind given by Eq.(79), to write Eq.(92) as

\[
N_{12}^a(x, b, d) = -\frac{1}{b} \frac{1}{(2\sqrt{\pi})^{d-1}} \left(\frac{m}{x}\right)^{\frac{d-3}{2}} K_{\frac{d-3}{2}}(2mx).
\]

(94)
To calculate $N_{12}^b(x, a, b, d)$, let us study first the massless case. In this case, first changing variables and using Eq.(55) we get

$$N_{12}^b(x, a, b, d)|_{m=0} = -\frac{1}{a^{d-3} b} \frac{h(d)}{2^d} \Gamma(d - 3) \left( \zeta(d - 3, \frac{x}{a} + 1) + \zeta(d - 3, -\frac{x}{a} + 1) \right).$$

(95)

Next we consider the massive case. Note that we have exactly the same situation discussed before. After a change of variables, $N_{12}^b(x, a, b, d)$ also contains a power of a binomial. Let us discuss again the most interesting situation, the odd dimensional case. First, let us the same power series expansion that we used before for the quantity $(1 - \frac{m^2 v^2}{v^2})^{4d-4}$ given by

$$(1 - \frac{m^2 v^2}{v^2})^{4d-4} = \sum_{k=0}^{\infty} (-1)^k C_p^k \left( \frac{m}{v} \right)^{2k}.$$  

(96)

Substituting Eq.(96) in Eq.(93) and again changing variable, we have

$$N_{12}^b(x, a, b, d) = -\frac{1}{4a^{d-3} b} h(d) \sum_{k=0}^{\infty} (-1)^k C_p^k (ma)^{2k} \int_{ma}^{\infty} du u^{d-4-2k} (\coth u - 1) \cosh \left( \frac{2ux}{a} \right).$$

(97)

A natural way to obtain $N_{12}^b(x, a, b, d)$ expressed in terms of the Hurwitz zeta function is the following. Let us split $N_{12}^b(x, a, b, d)$ in the following way:

$$N_{12}^b(x, a, b, d) = N_{12}^{<b}(x, a, b, d) + N_{12}^{>b}(x, a, b, d),$$

(98)

where

$$N_{12}^{<b}(x, a, b, d) = -\frac{1}{4a^{d-3} b} \sum_{k=0}^{\frac{d-4}{2}} C^{(1)}(d, k)(am)^{2k} \int_{am}^{\infty} du u^{d-4-2k} (\coth u - 1) \cosh \left( \frac{2ux}{a} \right),$$

(99)

and

$$N_{12}^{>b}(x, a, b, d) = -\frac{1}{4a^{d-3} b} \sum_{k=\frac{d-4}{2}}^{\infty} C^{(1)}(d, k)(am)^{2k} \int_{am}^{\infty} du u^{d-4-2k} (\coth u - 1) \cosh \left( \frac{2ux}{a} \right).$$

(100)
We have the same situation that we studied before. Let us investigate first the $N_{12}^{b<}(x, a, b, d)$. Using that $f_α^∞ f(u)du = f_0^∞ f(u)du - f_0^α f(u)du$, it is possible to write Eq.(99) in the following way:

\[
N_{12}^{b<}(x, a, b, d) = -\frac{1}{4a^{d-3}b} \sum_{k=0}^{k<\frac{d-4}{2}} C^{(3)}(d, k) \left( \frac{am}{\pi} \right)^{2k} \left( \zeta(d - 3 - 2k, -\frac{x}{a} + 1) + \zeta(d - 3 - 2k, \frac{x}{a} + 1) \right) \\
+ \frac{1}{4a^{d-3}b} \sum_{k=0}^{k<\frac{d-4}{2}} C^{(1)}(d, k)(am)^{2k} \int_0^am du \frac{u^{d-4-2k}}{\coth u - 1} \cosh(\frac{2ux}{a}),
\]

(101)

where the singularities of $N_{12}^{b<}(x, a, b, d)$ appear at $x \rightarrow a$. We have to analyze the $N_{12}^{b>}(x, a, b, d)$. It is clear that in an even dimensional Euclidean space for $x < a$ the integral that appear in Eq.(100) is convergent. The odd dimensional case also can be studied. In this case we have to expand $\cosh(\frac{2uy}{a})$ in power series and it is clear that in the new integral that appears in Eq.(100) demands a generalization of the Debye integrals. The calculations that we presented can also be used to write $N_2(x, y, a, b, d)$ in the same form that we obtained for the other expressions. Unfortunately one cannot perform the summation in $n$. For completeness, we will only write the remaining expressions. Let us start from:

\[
N_2(x, y, a, b) = N_{21}(x, y, a, d) + N_{22}(x, y, a, b, d),
\]

(102)

where each term of the above equation is given by

\[
N_{21}(x, y, a, d) = \frac{1}{2\pi a} h(d) \sum_{n=1}^{\infty} \cos(\frac{2n\pi x}{a}) \int_α^∞ dv(v^2 - m^2 - (\frac{n\pi}{a})^2)^{\frac{d-4}{2}} e^{-2vy},
\]

(103)

and

\[
N_{22}(x, y, a, b, d) = \frac{1}{2\pi a} h(d) \sum_{n=1}^{\infty} \cos(\frac{2n\pi x}{a})
\]

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\[
\int_{\alpha}^{\infty} dv (v^2 - m^2 - \left(\frac{n\pi}{a}\right)^2) \frac{d-4}{2} (\coth bv - 1) \cosh 2vy.
\] (104)

Note that the lower limit of both integrals are given by \(\alpha\) defined in Eq.(24) and \(h(d)\) was defined in Eq.(22). Let us present the \(N_{21}(x, y, a, b, d)\) in the massless case. Using the same integral representation of the Bessel functions that we used before gives

\[
N_{21}(x, y, a, b, d)|_{m=0} = \frac{2}{a(2\sqrt{\pi})^{d+1}} \sum_{n=1}^{\infty} \cos\left(\frac{2n\pi x}{a}\right) \left(\frac{n\pi}{ay}\right)^{\frac{d-1}{2}} K_{d-\frac{3}{2}}\left(\frac{2n\pi y}{a}\right).
\] (105)

The massive case can also be presented but in both cases it is not possible to perform the summation in \(n\). Finally, let us present the expression for \(N_{22}(x, y, a, b, d)\) in the massless case and for the odd dimensional case :

\[
N_{22}(x, y, a, b, d)|_{m=0} = \frac{1}{2\pi a b^{d-3}} \sum_{k=0}^{\infty} C^{(2)}(d, k) \left(\frac{b}{a}\right)^{2k} \sum_{n=1}^{\infty} n^{2k} \cos\left(\frac{2n\pi x}{a}\right)
\int_{n\pi b/a}^{\infty} du \frac{u^{d-4-2k} (\coth u - 1) \cosh \left(\frac{2uy}{b}\right)}{u^2}.
\] (106)

The same procedure that we used in \(I_{22}(y, a, b, d)\), can be repeated again. Unfortunately one can not perform the summation in \(n\) that appears in Eq.(106). Nevertheless the same analysis that we did in the end of section III can be performed to study the edge divergences that appear in the one-loop vacuum fluctuations associated with the scalar field confined in the interior of a waveguide in a higher dimensonal Euclidean space.
5 Discussions and conclusions

In this paper, we first obtained the regularized one-loop vacuum fluctuations associated with a massless scalar field defined in the interior of an infinity waveguide of rectangular cross section in a four-dimensional Euclidean space. Then, we discussed how it is possible to generalize our results for a massive field defined in a higher dimensional Euclidean space.

Let us summarize our motivations and the results. First, in the rectangular waveguide configuration the calculations can be done analytically, and since in this geometric configuration the electromagnetic field can be described by two massless scalar fields with Dirichlet and Neumann boundary conditions, our calculations can be used to describe the vacuum fluctuations associated with the electromagnetic field in the interior of the waveguide. Also, the rectangular geometries are very convenient to carry out experiments on the effect of boundaries on the atomic levels of atoms and on the rate of spontaneous emission. It is well known that in waveguides, it is possible to obtain situations where the spontaneous emission of atoms can be suppressed and also enhanced. Finally, the calculation of the regularized one-loop vacuum fluctuations in the infinite waveguide of rectangular cross section using the combination of dimensional and zeta function analytic regularization has not been discussed in the literature, at least as far as we know.

We first rederive a well-know result that surface and edge divergences appear in the one-loop vacuum fluctuations as consequence of the uncertainty principle. There are at least two different possible solutions that can eliminate these divergences. The first is to take into account that real materials have imperfect conductivity at high frequencies. As was stressed by many authors,
the infinities that appear in renormalized values of local observables for the ideal conductor (or perfect mirror) represent a breakdown of the perfect-conductor approximation. A wavelength cutoff corresponding to the finite plasma frequency must be included. The second would be given by a quantum mechanical treatment of the boundary conditions [43]. It was shown [43] that position fluctuations of a reflecting boundary also remove divergences in the renormalized values of local observables at least in the flat plate configuration. A logical going is to use an analytic regularization procedure to identify these divergent terms. As we discussed counterterms concentrated on the boundaries produce a finite one-loop vacuum fluctuations in the interior of the waveguide. A natural extension of this paper is to go beyond the one-loop approximation, investigating interacting field models in the presence of a waveguide in a d-dimensional Euclidean space. This topic is under investigation by the authors.

6 Acknowledgement

We would like to thank B.Schroer and L.H.Ford for several helpful discussions. N.F.Svaiter would like to acknowledge the hospitality of the Center of Theoretical Physics, Laboratory for Nuclear Science and Department of Physics of the Massachusetts Institute of Technology, where part of this work was carried out. This paper was supported in part by Conselho Nacional de Desenvolvimento Cientifico e Tecnologico do Brazil (CNPq) and also by funds provided by the U.S.Department of Energy(D.O.E) under cooperative research agreement DF-FC02-94ER40810.

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A The one-loop vacuum fluctuations and the generalized zeta function method

In aim of this appendix is to discuss the link between the one-loop vacuum fluctuations and the generalized zeta function method [33][34][35], as a standard formalism to regularize products and determinants. Since our prime goal in section IV was to calculate the one-loop vacuum fluctuations in the massive model, we will present the formal relation between the one-loop vacuum fluctuations and the effective action in the one-loop approximation. We are following the treatment used by Dittrich and Reuter [74].

Let us consider the generating functional of the complete Schwinger functions for a scalar field in a d-dimensional Euclidean space:

\[ Z(J) = \int D\varphi \ e^{-S[\varphi] + \int d^d x J(x) \varphi(x)}, \]  

(A.1)

where \( D\varphi \) is the appropriate measure, and \( S[\varphi] \) is the classical action associated with the scalar field. The quantity \( Z(J) \) can be regarded as the functional integral representation for the imaginary time evolution operator \( \langle \varphi_2 | U(t_2, t_1) | \varphi_1 \rangle \) with the boundary conditions \( \varphi(t_1, \vec{x}) = \varphi_1(\vec{x}) \) and \( \varphi(t_2, \vec{x}) = \varphi_2(\vec{x}) \). The quantity \( Z(J) \) gives the transition amplitude from the initial state \( |\varphi_1 \rangle \) to a final state \( |\varphi_2 \rangle \) in the presence of some scalar source of compact support. As usual \( W(J) \), the generating functional of the connected correlation functions shall be given by \( W(J) = \ln Z(J) \). In a free theory the partition function \( Z(J) \) and also \( W(J) \) can be calculated exactly. In the presence of the waveguide we must assume that the path integral must be taken over the space of functions.
that vanish in the boundaries of the waveguide. Since our aim is to discuss the non-interacting theory, let us assume that \( J = 0 \). Thus, defining the \( d \)-dimensional laplacian by \( \Delta \), we have:

\[
Z(J)|_{J=0} = \int_{\text{Dirichlet}} D\varphi e^{-\frac{1}{2} \int d^d x \varphi(x)(-\Delta + m^2)\varphi(x)}, \quad (A.2)
\]

"Dirichlet" in the path integral means that we are performing the path integral over functions that vanish on the boundaries. Thus the generating functional can be written as

\[
Z(J)|_{J=0} = N(\det(-\Delta + m^2))^{-\frac{1}{2}}, \quad (A.3)
\]

where \( N \) is a normalization factor that does not contributes to the free energy. For simplicity, let us discuss the four-dimensional case. The global generalized zeta function is defined by

\[
\zeta_{-\Delta + m^2}(s) = N' \int dk_0 \int dk_1 \sum_{n,n'=1}^{\infty} \sin^2\left(\frac{n\pi x}{a}\right) \sin^2\left(\frac{n'\pi y}{b}\right) (k_0^2 + k_1^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + m^2)^{-s}, \quad (A.4)
\]

where \( N' \) is a normalization constant. It follows that

\[
Z(J = 0) = \exp\left(\frac{1}{2} \zeta'_0(0)\right), \quad (A.5)
\]

where \( \zeta'(0) = \frac{d}{ds} \zeta(s)|_{s=0} \) and the operator \((-\Delta + m^2)\) has the spectrum

\[
(k_0^2 + k_1^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + m^2), k_0, k_1 \in \mathbb{R}, n, n' \in \mathbb{N}. \quad (A.6)
\]

We can use the same procedure to define the local generalized zeta function which is related with the effective action. The local generalized zeta function is given by

\[
\zeta_{-\Delta + m^2}(s, x, y) = N' \int dk_0 \int dk_1 \sum_{n,n'=1}^{\infty} \frac{\sin^2\left(\frac{nx}{a}\right) \sin^2\left(\frac{n'y}{b}\right)}{(k_0^2 + k_1^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{n'\pi}{b}\right)^2 + m^2)^s}, \quad (A.7)
\]

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where $x$ and $y$ are two cartesian coordinates of one point of the infinite waveguide with rectangular crosss section. Thus we have the formal expression:

$$(-1)^s \left( \frac{\partial}{\partial m^2} \right)^s \langle \varphi^2(x, y) \rangle = \zeta(s, x, y).$$

(A.8)

To give a precise meaning of the derivative one can use for example the Liouville’s concept of fractional derivative \cite{75}. This result show that the one-loop vacuum fluctuations determines the effective action in the one-loop level.

References

[1] H.G.B.Casimir, Proc.K.Ned.Akad.Wet. 51, 793 (1948).

[2] G.Plunien, B.Muller and W.Greiner, Phys. Rep. 134, 87 (1986).

[3] A.A.Grib, S.G.Mamayev and V.M.Mostepanenko, ”Quantum Vacuum Effects in Strong Fields”, Friedmann Laboratory Publishing, St. Petersburg (1994).

[4] V.M.Mostepanenko and N.N.Trunov, ”The Casimir Effect and its Applications”, Clarendon Press, Oxford, (1997).

[5] S.A.Fulling, ”Aspects of Quantum Field Theory in Curved Spacetime”, Cambridge University Press, Cambridge (1989), pp. 95-104.

[6] D.Deutsch and P.Candelas, Phys. Rev. D20, 3063 (1979).
[7] P.Candelas, Phys. Rev. D21, 2185 (1980).

[8] R.Courant and D.Hilbert, “Methods of Mathematical Physics, Intercience Publishers Inc. N.Y. (1953), pp. 429-445.

[9] R.Balian and C.Bloch, Ann. Phys. 60, 401 (1970).

[10] W.Lukosz, Z. Phys. 258, 99 (1973).

[11] W.Lukosz, Z. Phys. 262, 327 (1973).

[12] L.H.Brown and G.J.Maclay, Phys. Rev. 184, 1272 (1969).

[13] C.M.Bender and P.Hays, Phys. Rev. D14, 2622 (1976).

[14] R. Balian and R.Duplantier, Ann. Phys. 117, 165 (1978).

[15] L.L.Raad Jr. and K.Milton, Ann. Phys. 136, 229 (1981).

[16] P.Gosdzinsky and A.Romeo, Phys. Lett. B441, 265 (1998).

[17] K.Milton, A.V.Nesterenko and V.V.Nesterenko, Phys. Rev. D59, 105009 (1999).

[18] T.H.Boyer, Phys. Rev. 174, 1764 (1968); J. Math. Phys. 10, 1729 (1969).

[19] B.Davies, J. Math. Phys. 13, 1325 (1972).

[20] K.A.Milton, Phys. Rev. D22, 1441 (1980); Phys. Rev. D22, 1444 (1980).
[21] K.A.Milton, Ann. Phys. 150, 432 (1983).

[22] C.M.Bender and K.A.Milton, Phys. Rev. D50, 6547 (1994).

[23] A.Romeo, Phys. Rev. D52, 7308 (1995); ibid., Phys. Rev. D53, 3392 (1996).

[24] M.Bordag, K.Elizalde, K.Kirsten and S.Leseduarte, Phys. Rev. D56, 4896 (1997).

[25] T.Weldon, Phys. Rev. 74, 1157 (1948).

[26] C.A.Lutken and F.Ravndal, Phys. Rev. A31, 2082 (1985).

[27] D.Meschede, Phys. Rep. 5, 201 (1992).

[28] L.H.Ford, N.F.Svaiter and M.L.Lyra, Phys. Rev. A49, 1378 (1994).

[29] P.W.Millonni, “The Quantum Vacuum”, Academic Press, Inc. NY (1994).

[30] A.A.Actor, Ann. Phys. 230, 303 (1994); Fortschr. Phys. 43, 141 (1995).

[31] A.A.Actor and I.Bender, Fortschr. Phys. 44, 281 (1996).

[32] J.S.Dowker and R.Critchley, Phys. Rev. D 13, 3224 (1976).

[33] S.W.Hawking, Comm. Math. Phys. 55, 133 (1977).

[34] J.S.Dowker and G.Kennedy, J. Phys. A11, 895 (1978).

[35] A.Voros, Comm. Math. Phys. 110, 439 (1987).
[36] I.Brevik and M.Lygren, Ann. Phys. 251, 157 (1996), I. Brevik, M.Lygren and V. N. Marachevsky, Ann. Phys. 267, 134 (1998).

[37] K.Olaussen and F.Ravndal, Nucl. Phys. B192, 237 (1981).

[38] B.S.DeWitt, Phys. Rep. 19, 259 (1975).

[39] G.Kennedy, R.Critchley and J.S.Dowker, Ann. Phys. 125, 346 (1980).

[40] K.Olaussen and F.Ravndal, Phys. Lett, 100B, 497 (1981).

[41] K.Milton, Phys. Lett, 104B, 49 (1981).

[42] K.Symanzik, Nucl. Phys. B190, 1 (1980).

[43] L.H.Ford and N.F.Svaiter, Phys. Rev. D58, 065007-1, (1998).

[44] L.H.Ford, Phys. Rev. A58, 4279 (1998).

[45] L.H.Ford and N.F.Svaiter, Phys. Rev. A62, 062105-1 (2000).

[46] H.D.Diehl and S. Dietrich, Phys. Rev. B 24, 2878 (1981); Phys. Lett. 70A, 408 (1981), Y.Y.Goldschmidt, Phys. Rev. B28, 4052 (1983); K.Krech and S. Dietrich, Phys. Rev. A 46, 1886 (1992).

[47] A.A.Nemirovsky and K.F.Freed, J. Phys. A18L, 319 (1985); Nucl. Phys. B270, 423 (1986).

[48] J.S.Dowker and R.Banach, J.Phys. A 11, 2255 (1978).
[49] F.Caruso, N.P.Neto, B.F.Svaiter and N.F.Svaiter, Phys. Rev. D43, 1300 (1991).

[50] N.F.Svaiter and B.F.Svaiter, J.Phys. A25, 979 (1992).

[51] J.Ambjorn and S. Wolfram, Ann. Phys. 147, 1 (1983).

[52] K.A.Milton and Y.Jack Ng, Phys. Rev. D42, 2875 (1990).

[53] S.Hacyan, R.Jauregui and C.Villarreal, Phys. Rev. A47, 4204 (1993).

[54] G.J.Maclay, Phys.Rev. A61, 052110 (2000).

[55] T.H.Hansson and R.L.Jaffe, Phys. Rev. D28, 882 (1983); Ann. Phys. 151, 204 (1983).

[56] C.D.Fosco and N.F.Svaiter, J. Math. Phys. 42, 5185, (2001).

[57] R.B.Rodrigues, N.F.Svaiter and R.De Paola. "Vacuum Stress Tensor of a Scalar Field in a Rectangular Waveguide." CBPF pre-print NF-015/01, hep-th 0110290.

[58] A.A.Actor and I.Bender, Phys. Rev. D52, 358 (1995).

[59] L.C.Albuquerque, Phys. Rev. D55, 7752 (1997)

[60] F.Caruso, R.De Paola and N.F.Svaiter, Int. Journ. Mod. Phys. A14, 2077 (1999).

[61] L.H.Ford, Phys. Rev. D21, 933 (1980).

[62] L.H.Ford and N.F.Svaiter, Phys. Rev. D51, 6981 (1995).
[63] K.Kirsten, J. Math. Phys. 32, 3008 (1991)

[64] I.S. Gradshteyn and I.M.Ryshik, in Table of Integrals and Products, edited by A. Jeffrey, Academic Press, Inc., New York (1980).

[65] G.Tolstov, Fourier Series, Dover Publications, Inc. New York, (1962).

[66] E.Elizalde and A.Romeu, J. Math. Phys. 30, 1133 (1989).

[67] N.F.Svaiter and B.F.Svaiter, J. Math. Phys. 32, 175 (1991).

[68] C.De Calan, A.P.C.Malbouisson and N.F.Svaiter, Mod. Phys. Lett. A13, 1357 (1998).

[69] G.N.J.Ananos, A.P.C.Malbouisson and N.F.Svaiter, Nucl. Phys. B547, 221 (1999).

[70] A.Actor, Fortschr. Phys. 41, 461 (1993).

[71] M.Weldon, Nucl. Phys. B270, 79 (1986).

[72] E.Elizalde and A.Romeo, Phys. Rev. D40, 436 (1989).

[73] Higher Transcedental Functions (Bateman Manuscript Project), edited by A. Erdelyi, W.Magnus, F.Oberhettinger and F.Tricomi, McGraw-Hill, New York, (1953), Vol.I, p.27.

[74] W.Dittrich and M.Reuter, “Effective Lagrangians in Quantum Electrodynamics”, Springer Verlag (1985), pp. 195-206.

[75] I.M.Guelfand and G.E.Shilov, “Les Distributions” Dunod, Paris (1972), p.112.