Analytic Berezin–Toeplitz operators

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Abstract
We introduce new tools for analytic microlocal analysis on Kähler manifolds. As an application, we prove that the space of Berezin–Toeplitz operators with analytic contravariant symbol is an algebra. We also give a short proof of the Bergman kernel asymptotics up to an exponentially small error.

1 Introduction

Asymptotic behavior of Bergman kernels on Kähler manifolds has been studied in many papers after the pioneer works of Bouche [3], Tian [17], Zelditch [18] and Catlin [7]. New asymptotics with exponentially small error terms have been obtained recently by Rouby–Sjöstrand–Vu Ngoc [14]. In the same vein, Deleporte [9] started to develop analytic microlocal techniques for Berezin–Toeplitz operators on Kähler manifolds.

Our goal in this paper is to further develop this program in two ways. First, we introduce new methods to simplify the complicated estimates of [9] and obtain a short proof of the Bergman kernel estimates of [14]. Second, we define the algebra of Berezin–Toeplitz operators whose contravariant symbols are analytic symbols in the sense of [16].

Before we present our main results and methods, let us start with two typical expansions in this theory: the Bergman kernel on the diagonal and the composition of two Toeplitz operators. In both cases, we will compare the (smooth) usual version of the result with its (analytic) improvement.

Let $M$ be a compact complex manifold having dimension $n$ and equipped with two holomorphic Hermitian line bundles $L$, $L'$. Assume $L$ is positive and for any $k \in \mathbb{N}$, let $\mathcal{H}_k$ be the space of holomorphic section of $L^k \otimes L'$. The Bergman kernel of $L^k \otimes L'$ is defined by $\Pi_k(x, \overline{y}) = \sum_{i=1}^{d_k} \Psi_i(x) \otimes \overline{\Psi_i(y)}$, for any $x, y \in M$, where $(\Psi_i)_{i=1, \ldots, d_k}$ is any orthonormal basis of $\mathcal{H}_k$. It was proved in [7,18] from [6] that the Bergman kernel has the following asymptotic expansion on the diagonal

$$
\Pi_k(x, \overline{x}) = \left(\frac{k}{2\pi}\right)^n \sum_{\ell=0}^{N} k^{-\ell} \rho_\ell(x) + \mathcal{O}(k^{n-N-1}), \quad \forall N \in \mathbb{N}
$$

(1)
with smooth coefficients $\rho_\ell \in C^\infty(M)$. A lot can be said on these coefficients but our aim here is to improve the remainder. Let us assume from now on that the metrics of $L$ and $L'$ are analytic. Then if we replace the finite sum by a partial sum over all integers $\ell$ smaller than $\epsilon k$ with $\epsilon$ sufficiently small, the remainder becomes exponentially small. The precise result, proved in [14], is that there exist $\epsilon > 0$ and $C > 0$ such that

$$\Pi_k(x, \bar{x}) = \left( \frac{k}{2\pi} \right)^n \sum_{\ell=0}^{\lfloor \epsilon k \rfloor} k^{-\ell} \rho_\ell(x) + \mathcal{O}(e^{-k/C})$$

(2)

with a $\mathcal{O}$ uniform on $M$. The case of surfaces with constant curvature was done before by Berman [2]. Let us consider now Toeplitz operators. For any $f \in C^\infty(M)$, let $T_k(f) : \mathcal{H}_k \rightarrow \mathcal{H}_k$ be the operator sending $\Psi$ into $\Pi_k(f \Psi)$ with $\Pi_k$ the orthogonal projection of $C^\infty(M, L^k \otimes L')$ onto $\mathcal{H}_k$. By Boutet de Monvel–Guillemin [4,10], for any $f, g \in C^\infty(M)$,

$$T_k(f)T_k(g) = \sum_{\ell=0}^{N} k^{-\ell} T_k(h_\ell) + \mathcal{O}(k^{-N-1}), \quad \forall N \in \mathbb{N}$$

(3)

for some coefficients $h_\ell \in C^\infty(M)$. In the case $f$ and $g$ are analytic, we will prove that there exist $\epsilon > 0$ and $C > 0$ such that

$$T_k(f)T_k(g) = \sum_{\ell=0}^{\lfloor \epsilon k \rfloor} k^{-\ell} T_k(h_\ell) + \mathcal{O}(e^{-k/C})$$

(4)

where the $\mathcal{O}$ is in uniform norm.

These two results, Bergman kernel expansion and Toeplitz composition, belong actually to the same theory. Let us explain this first in the usual smooth setting and then in the analytic case. Define a Berezin–Toeplitz operator as any family $(S_k) \in \prod_{k \geq 1} \text{End}(\mathcal{H}_k)$ such that

$$S_k = \sum_{\ell=0}^{N} k^{-\ell} T_k(f_\ell) + \mathcal{O}(k^{-N-1}), \quad \forall N \in \mathbb{N}$$

(5)

for a sequence $(f_\ell) \in C^\infty(M)$. By [4,10], the space $\mathcal{T}$ of Berezin–Toeplitz operator is a subalgebra of $\prod_k \text{End} \mathcal{H}_k$. In [8], we proved the following characterization of the Schwartz kernel of Berezin–Toeplitz operators. Here, the Schwartz kernel of $S_k \in \text{End}(\mathcal{H}_k)$ is the holomorphic section of $(L^k \otimes L') \otimes (\overline{L}^k \otimes \overline{L}')$ given by

$$S_k(x, \bar{y}) = \sum_{i=1}^{d_k} (S_k \Psi_i)(x) \otimes \overline{\Psi}_i(y), \quad x, y \in M$$

where $(\Psi_i)_{i=1,\ldots,d_k}$ is any orthonormal basis of $\mathcal{H}_k$. Then $(S_k) \in \prod \text{End}(\mathcal{H}_k)$ is a Berezin–Toeplitz operator if and only if for any compact set $K$ not intersecting the diagonal $[S_k(x, y_c)] = \mathcal{O}(k^{-N})$ on $K$ for any $N$ and on a neighborhood $W$ of the diagonal

$$S_k(x, y_c) = \left( \frac{k}{2\pi} \right)^n E^k(x, y_c)E'(x, y_c) \sum_{\ell=0}^{N} k^{-\ell} \bar{\tilde{\tau}}_\ell(x, y_c) + \mathcal{O}(k^{n-N-1})$$

(6)

for any $N \in \mathbb{N}$, where $E, E'$ are holomorphic sections of $L \otimes \overline{L}$ and $L' \otimes \overline{L}'$ on $W$ such that their restrictions to the diagonal are the canonical sections determined by the metrics, and the $\tilde{\tau}_\ell$’s are smooth functions on $W$ such that $\partial \bar{\partial} \tilde{\tau}_\ell$ vanishes to infinite order along the diagonal.
In particular, since the Bergman kernel is the Schwartz kernel of the Berezin–Toeplitz operator \((\text{id} \circ \tau_k)\), \(\Pi_k(x, y_c)\) satisfies the expansion (6), which extends the expansion (1) outside the diagonal.

For the analytic version, we need the notion of analytic symbol [16]. We say that a formal series \(\sum h^\ell a_\ell \in \mathcal{C}^\infty(M)[[h]]\) is an analytic symbol if there exist a neighborhood \(W \subset M \times \overline{M}\) of the diagonal and \(C > 0\) such that each \(a_\ell\) has a holomorphic extension \(\tilde{a}_\ell\) to \(W\) satisfying \(|\tilde{a}_\ell| \leq C^{\ell+1} \ell!\). When \(\epsilon < 1/C\), we set
\[
a(\epsilon, k) = \sum_{\ell=0}^{[\epsilon k]} a_\ell k^{-\ell}, \quad \tilde{a}(\epsilon, k) = \sum_{\ell=0}^{[\epsilon k]} \tilde{a}_\ell k^{-\ell}.
\]
Note that these partial sums already appeared in the previous expansions of the Bergman kernel (2) and the Toeplitz operator composition (4). To complete these results, the series \(\sum h^\ell \rho_\ell\) and \(\sum h^\ell h_\ell\) appearing there are analytic symbols.

An analytic Berezin–Toeplitz operator is a family \((S_k) \in \prod_k \text{End} \mathcal{H}_k\) such that
\[
S_k = T_k(f(\epsilon, k)) + \mathcal{O}(e^{-k/C})
\]
for an analytic symbol \(\sum h^\ell f_\ell\) and \(C > 0\).

**Theorem 1.1**

1. The space of analytic Berezin–Toeplitz operator is a subalgebra of \(\mathcal{T}\).
2. A family \((S_k) \in \prod_k \text{End} \mathcal{H}_k\) is an analytic Berezin–Toeplitz operator if and only if for any compact set \(K\) of \(M \times \overline{M}\) not intersecting the diagonal \(|S_k(x, y_c)| = \mathcal{O}(e^{-k/C})\) on \(K\) for some \(C_K > 0\) and on a neighborhood of the diagonal
\[
S_k(x, y_c) = \left(\frac{k}{2\pi}\right)^n E^k(x, y_c) E'(x, y_c) \tilde{\tau}(\epsilon, k)(x, y_c) + \mathcal{O}(e^{-k/C})
\]
where \(E, E'\) are the same sections as in (6), \(\tilde{\tau}(\epsilon, k)\) is the partial sum associated to an analytic symbol \(\sum h^\ell \tau_\ell\) as in (7) and \(C > 0\).

**Remark 1.2**

1. By the first assertion, \((\text{id} \circ \tau_k)\) is an analytic Berezin–Toeplitz operator and thus the second assertion describes the Bergman kernel up to \(\mathcal{O}(e^{-k/C})\). This description was the main result of [14, Theorem 6.1].
2. In [9], the operators \((S_k) \in \prod_k \text{End} \mathcal{H}_k\) whose Schwartz kernel satisfies the condition given in the second assertion are called covariant Toeplitz operators. The main results of [9] are first that these operators are closed under product and second that the operators with an elliptic symbol have an inverse in the same class of operators.
3. Given the previous remarks, the original result in Theorem 1.1 is its second assertion: the covariant Toeplitz operators in the sense of [9] are the analytic Berezin–Toeplitz operators defined by (8). But of course, our proof of the whole theorem is completely independent of [9,14].

As already mentioned, the main contribution of this work is in our proofs. Recall first that the basic tools of analytic microlocal analysis, including analytic symbols and stationary phase lemma, have been introduced a long time ago in [5,16] for the theory of analytic pseudo-differential operators. When we try to apply these techniques to Berezin–Toeplitz operators, we face the difficulty that the symbol product corresponding to operator composition is only partially known, unlike the Moyal–Weyl product for the pseudodifferential operators. A large literature exists on these products describing them in terms of the Kähler metric, cf. [13] for instance, but the application to our problem is not straightforward, as is attested by the attempt to prove that \(\sum h^\ell \rho_\ell\) is analytic in [11], or the complicated estimates of [9].
So we have to identify the characteristics of these products which allow the analytic calculus. The convenient property we found is a particular growth of some coefficients. To be more specific, let us define the symbol of a Berezin–Toeplitz operator \( \hat{S}_k \) as the series \( \sum h^\ell \tau_\ell \) whose coefficients \( \tau_\ell \in C^\infty(M) \) are the restrictions to the diagonal of the \( \tilde{\tau}_\ell \) in (6). In [8], these symbols were called non normalized covariant symbols, because the Bezerin covariant symbol is obtained by normalising them. In this same paper we proved that the product of these symbols, corresponding to the operator composition, has the form \( \sum h^\ell f_\ast \sum h^m g_m = \sum h^{\ell+m+p} A_p(f_\ell, g_m) \) where the \( A_p \) are bidifferential operators with a local expression

\[
\frac{1}{p!} A_p(f, g) = \sum_{|\alpha|, |\beta| \leq p} a_{p, \alpha, \beta} \frac{\partial^\beta \bar{z} f}{\alpha! \beta!} \frac{\partial^\alpha z g}{.}
\]

The unit of \((C^\infty(M)[[h]], \ast)\) is nothing else than the series \( \sum h^\ell \rho_\ell \) associated to the Bergman kernel (1).

The main estimate we will establish is that a holomorphic extension of the \( a_{p, \alpha, \beta} \)’s satisfies

\[
|\tilde{a}_{p, \alpha, \beta}| \leq C^{p+1}
\]

where \( C \) does not depend on \( p, \alpha, \beta \); cf. Theorem 2.3. From this, we will deduce that \( \sum h^\ell \rho_\ell \) is analytic, cf. Theorem 3.1, and that the space of analytic symbols is closed under \( \ast \), cf. Proposition 4.1.

To do that, the two essential tools we will use are first the explicit computation of the \( A_p \) given in [8] and second a family of seminorms already used in [15]. These seminorms are useful to estimate coefficients of holomorphic differential operators and of their compositions without using Leibniz formula. They were introduced in [15] to prove that the inverse of an analytic symbol of a pseudodifferential operator is analytic, a problem similar to ours. However there is the slight difference that these seminorms are defined in [15] for the symbols themselves, whereas we use them for the operators acting on symbols.

To prove the second part of Theorem 1.1, that is the characterization of analytic Berezin–Toeplitz operators in terms of their Schwartz kernel, a similar difficulty arises: define the contravariant symbol of a Berezin–Toeplitz \( \hat{S}_k \) operator as the formal series \( \sum h^\ell f_\ell \) whose coefficients are given in (5); then we have to show that the isomorphism \( B \) of \( C^\infty(M)[[h]] \) sending the contravariant symbol into the non normalised covariant symbol restricts to a bijection of the space of analytic symbols. To do this, we will prove with the same methods as before, that \( B \) and its inverse have the form \( B = \sum h^\ell B_\ell \) where the \( B_\ell \) are differential operators satisfying estimates similar to (9).

The paper is organised as follows. In Sect. 2, we recall the computation of the bidifferential operators \( A_p \) from [8] and prove our main estimates (9). In Sect. 3, we deduce that the symbol of the Bergman projector is analytic. In Sect. 4, we introduce analytic Berezin–Toeplitz operators by their Schwartz kernels, and prove that they form an algebra with unit the Bergman kernel. In Sect. 5, we prove that analytic Berezin–Toeplitz operators can equivalently be defined by multipliers (8). Last section is an appendix in two parts: in the first part, we prove basic facts on analytic symbols which are essentially known [5,16]; in the second part, we systematize some of the techniques used in the previous parts.
2 Symbolic calculus

2.1 Complexification

Let $M$ be a complex manifold. We denote by $\overline{M}$ the complex manifold, which has the same underlying real manifold as $M$ but the opposite almost complex structure. If $x \in M$, we denote by $\overline{x}$ the corresponding point of $\overline{M}$.

The product $M \times \overline{M}$ is a complexification of the diagonal $\Delta_M = \{(x, \overline{x})/ x \in M\}$, in the sense that for any $x \in M$, there exists a holomorphic chart $\psi : W \to \mathbb{C}^{2n}$ of $M \times \overline{M}$ at $(x, \overline{x})$ such that $\psi(W \cap \Delta_M) = \psi(W) \cap \mathbb{R}^{2n}$. This has the consequence that any analytic function $f : \Delta_M \to \mathbb{C}$ has a holomorphic extension $\tilde{f} : W \to \mathbb{C}$ on a neighborhood $W$ of $\Delta_M$ in $M \times \overline{M}$. We will often add a tilde to denote a holomorphic extension. We will also often identify $\Delta_M$ with $M$.

Similarly, if $L \to M$ is a holomorphic line bundle, $\overline{L} \to \overline{M}$ is the conjugate holomorphic line bundle, and any analytic section of the restriction of $L \otimes \overline{L}$ to the diagonal has a holomorphic extension to a neighborhood of the diagonal. Consider a Hermitian metric of $L$, which is analytic in the sense that for any local holomorphic frame $s : U \to L$, $x \to |s(x)|^2$ is an analytic function of $U$. Then the section of $L \otimes \overline{L} \to \Delta_M$, sending $(x, \overline{x})$ to $|u|^{-2}u \otimes \overline{u}$, $u \in L_x$ is analytic, so it has a holomorphic extension $E : W \to L \otimes \overline{L}$. If $s$ is a holomorphic frame as above and $|s(x)|^2 = \exp(-\phi(x))$, then we have the local expression for $E$

$$E(x, \overline{y}) = e^{\tilde{\phi}(x, \overline{y})} s(x) \otimes \overline{s(y)}$$

(10)

where $\tilde{\phi}$ is a holomorphic extension of $\phi$. Sometimes, it will be convenient to identify the tensor product $L_x \otimes \overline{L}_x$ with $\mathbb{C}$ through the metric, so that the restriction of $L \otimes \overline{L}$ to the diagonal becomes the trivial line bundle of $\Delta_M$. With this convention, the restriction of $E$ to the diagonal is simply the constant function equal to 1.

2.2 Product formula

Consider a compact complex manifold $M$, with two Hermitian holomorphic line bundles $L$ and $L'$. We assume that $L$ is positive. Let $f$ and $g$ be two analytic functions of $M$.

Introduce as in Sect. 2.1, holomorphic extensions $E : W \to L \otimes \overline{L}$, $E' : W \to L' \otimes \overline{L}'$, $\tilde{f} : W \to \mathbb{C}$, $\tilde{g} : W \to \mathbb{C}$ defined on the same neighborhood $W$ of $\Delta_M$. The fact that $L$ is positive has the consequence that $|E(x, y_c)| < 1$ when $y_c \neq \overline{x}$ and $(x, y_c)$ is sufficiently close to the diagonal [8, Proposition 1]. So restricting $W$ if necessary, we can assume that $|E| < 1$ on $W \setminus \Delta_M$. For any integer $k \geq 1$ and $(x, y_c) \in W$, set

$$E_k(x, y_c) := \left(\frac{k}{2\pi}\right)^n E^k(x, y_c) \otimes E'(x, y_c)$$

(11)

Choose a smooth compactly supported function $\rho : W \to \mathbb{R}$ which is equal to 1 on a neighborhood of $\Delta_M$. Define

$$T_k(x, y_c) = \rho(x, y_c) E_k(x, y_c) \tilde{f}(x, y_c),$$

$$S_k(x, y_c) = \rho(x, y_c) E_k(x, y_c) \tilde{g}(x, y_c).$$

So $T_k$ and $S_k$ are smooth sections of $(L^k \otimes L') \otimes (\overline{L}^k \otimes \overline{L}')$. They are Schwartz kernels of operators $T_k$, $S_k$ acting of $C^\infty(M, L^k \otimes L')$. Our convention for operator kernels is

$$(T_k \psi)(x) = \int_M T_k(x, y) \cdot \psi(y) \, d\mu(y), \quad \psi \in C^\infty(M, L^k \otimes L')$$

(12)
where $\mu$ is the Liouville measure of $M$ and the the dot stands for the scalar product of $(L^k \otimes L')_y$. Let $r_k$ be the restriction of the Schwartz kernel of $S_k \circ T_k$ to the diagonal

$$r_k(x) = \int_M S_k(x, y) \cdot T_k(y, x) \, d\mu(y)$$

By [8], the sequence $(r_k)$ has an asymptotic expansion

$$r_k(x) = \left( \frac{k}{2\pi} \right)^n \sum_{\ell=0}^N k^{-\ell} A_\ell(f, g)(x) + O(k^{N-1}), \quad \forall N$$

and we can compute explicitly the coefficients $A_\ell(f, g)(x)$ as follows.

For any $x_0 \in M$, choose holomorphic frames $s, s'$ of $L$ and $L'$ over a connected open set $U \ni x_0$. Set $|s|^2 = \exp(-\varphi)$ and $|s'|^2 = \exp(-\varphi')$. Restricting $U$ if necessary, we have holomorphic extensions $\tilde{\varphi}, \tilde{\varphi}'$ on $U \times \overline{U} \subset W$. By (10), $|E_k|^2 = (k/2\pi)^{2n} \exp(-k\psi - \psi')$ where

$$\psi(x, y) = -\tilde{\varphi}(x, y) - \tilde{\varphi}(y, \overline{x}) + \tilde{\varphi}(x, \overline{x}) + \tilde{\varphi}(y, \overline{y})$$

$$\psi'(x, y) = -\tilde{\varphi}'(x, y) - \tilde{\varphi}'(y, \overline{x}) + \tilde{\varphi}'(x, \overline{x}) + \tilde{\varphi}'(y, \overline{y})$$

With the identification $L_x \otimes \overline{L}_x \simeq \mathbb{C}$ given by the metric, $E_k(x, y) \cdot E_k(y, x) = |E_k(x, y)|^2$. So

$$r_k(x) = \left( \frac{k}{2\pi} \right)^{2n} \int_M e^{-k\psi(x, y) - \psi'(x, y)} \tilde{f}(x, \overline{y}) \tilde{g}(y, \overline{x}) \rho'(x, y) \delta(y) \, d\mu_L(y).$$

where $\rho'(x, y) = \rho(x, y) \rho(x, \overline{y}) \mu_L$ is the Lebesgue measure of $\mathbb{C}^n$ and $\mu(x) = \delta(x) \mu_L(x)$. Choosing a holomorphic chart with domain $U$, we consider $U$ as an open set of $\mathbb{C}^n$. Let $G_{ij} = \partial^2 \varphi / \partial z_i \partial \overline{z}_j$ and recall that for any $x \in U$, $(G_{ij}(x))$ is positive definite because $L$ is a positive line bundle.

**Theorem 2.1** For any $x \in U$, we have

$$A_\ell(f, g)(x) = b(x) \sum_{m=0}^{2\ell} \frac{\Delta^\ell+m(c^{m}d^{c'})}{m!(\ell+m)!}(x, 0)$$

where for $u \in \mathbb{C}^n$ sufficiently small

$$b(x) = (\det(G_{ij}(x)))^{-1}, \quad c(x, u) = \sum_{i,j=1}^n G_{ij}(x) u_i \overline{u}_j - \psi(x, x + u)$$

$$d(x, u) = \tilde{f}(x, \overline{x} + \overline{u}) \tilde{g}(x + u, \overline{x}), \quad c'(x, u) = e^{-\psi'(x, x + u)} \delta(x + u)$$

and $\Delta = \sum_{i,j=1}^n G^{ij}(x) \partial_{u_i} \partial_{\overline{u}_j}$, ($G^{ij}(x)$) being the inverse of $(G_{ij}(x))$.

**Proof** We can rewrite the integral in terms of the functions $c, c', d$

$$r_k(x) = \left( \frac{k}{2\pi} \right)^{2n} \int e^{-k \sum G_{ij}(x) u_i \overline{u}_j e^{kc(x, u)}} d(x, u) c'(x, u) \rho'(x, x + u) d\mu(u)$$

Notice that $c(x, u) = O(|u|^3)$. The result follows from Laplace’s method, cf. for instance [12, Theorem 7.7.5].

The proof in [8] was much longer because we computed there the Schwartz kernel of $S_k \circ T_k$ on $M \times M$ up to a $O(k^{-\infty})$, which requires the stationary phase lemma with complex valued...
phase depending on parameters. In Sect. 4, we will see the similar result in the analytic setting.

Since \( c'(x, 0) = \delta(x) = \det(G_{ij}(x)) \), we have

\[
A_0(f, g) = fg.
\]

In the sequel, we will use the following holomorphic extension of (13). For any holomorphic function \((u, v) \to \tilde{d}(u, v)\) defined on a neighborhood of the origin in \(\mathbb{C}^n \times \mathbb{C}^n\), we set

\[
P_\ell(\tilde{d})(x, y_c) = \tilde{b}(x, y_c) \sum_{m=0}^{2\ell} \frac{\hat{\Delta}^{\ell+m}(\tilde{c}^m \tilde{d}^{\ell'})}{m!(\ell + m)!} (x, y_c, 0, 0) \tag{14}
\]

Here \(\tilde{b}, \tilde{c}, \tilde{c}'\) are holomorphic extensions of \(b, c, c'\) respectively. So \(\tilde{b}\) depends holomorphically on \(x, y_c\) and \(b(x) = \tilde{b}(x, \overline{x})\), \(\tilde{c}\) and \(c\) are holomorphic functions of the variables \((x, y_c, u, v)\) such that \(c(x, u) = \tilde{c}(x, \overline{x}, u, \overline{u})\), \(c'(x, u) = \tilde{c}'(x, \overline{x}, u, \overline{u})\). Similarly, \(\hat{\Delta}\) is the operator \(\sum_{i, j=1}^{n} \tilde{G}^{ij}(x, y_c) \partial_{u_i} \partial_{v_j}\). Then the function \(A_\ell(f, g)\) of Theorem 2.1 has the holomorphic extension

\[
\tilde{A}_\ell(f, g)(x, y_c) = P_\ell(\tilde{d}(x, y_c))(x, y_c)
\]

with \(\tilde{d}(x, y_c)(u, v) = \tilde{f}(x, y_c + v) \tilde{g}(x + u, y_c)\).

Even if we don’t need it, let us observe that everything can be explicitly computed in terms of \(\tilde{\varphi}\) and \(\tilde{\varphi}'\). Indeed, \((\tilde{G}^{ij}(x, y_c))\) is the inverse of \(\tilde{G}_{ij}(x, y_c) = (\partial^2 \tilde{\varphi}/\partial x_i \partial y_{c,j})(x, y_c)\), \(\tilde{b} = \det \tilde{G}_{ij}, \tilde{c}(x, y_c, u, v) = \tilde{\varphi}(x, y_c + v) + \tilde{\varphi}(x + u, y_c) - \tilde{\varphi}(x, y_c)\). Similarly, \(\hat{\Delta}\) is the operator \(\sum_{i, j=1}^{n} \breve{G}^{ij}(x, y_c) \partial_{u_i} \partial_{v_j}\) and there is a similar formula for \(\tilde{c}'(x, y_c, u, v)\).

2.3 Main estimates

Consider the local expression of \(A_\ell(f, g)\) given in Theorem 2.1.

Lemma 2.2 We have

\[
\frac{1}{\ell!} A_\ell(f, g) = \sum_{|\alpha|, |\beta| \leq \ell} a_{\ell, \alpha, \beta} \frac{\partial^{\beta} f(\tilde{\varphi}) \partial^{\beta} g(\tilde{\varphi})}{\alpha! \beta!}
\]

where \(\alpha, \beta\) are multi-indices of \(\mathbb{N}^n\), \(a_{\ell, \alpha, \beta}\) are analytic function of \(U\), the derivatives are \(\partial_\alpha z = \partial_{z_1}^{\alpha_1} \ldots \partial_{z_n}^{\alpha_n}\), \(\partial_\beta z = \partial_{z_1}^{\beta_1} \ldots \partial_{z_n}^{\beta_n}\).

Proof Observe first that \((\partial_\alpha u \partial_\beta u)(d)(x, 0) = (\partial_\beta \tilde{f})(\tilde{\varphi})(\partial_\alpha \tilde{g})(\tilde{\varphi})(x, 0)\). Second, we have \((\partial_\alpha c)(x, 0) = (\partial_\alpha \tilde{c})(x, 0)\). Thus the Taylor expansion of \(u \to c(x, u)\) at 0, only appear monomials of the form \(u^\alpha \tilde{u}^\beta\) with \(|\alpha|, |\beta| \geq 1\). Thus the Taylor expansion of \(u \to c^m(x, u)\) has only monomials of the form \(u^\alpha \tilde{u}^\beta\) with \(|\alpha|, |\beta| \geq m\). Now \(\Delta^{m+\ell}\) is a linear combination of \(\partial_\alpha \partial_\beta u\) with \(|\alpha|, |\beta| \geq m\). By the previous remark, expanding \(\Delta^{m+\ell}(c^m d c')(x, 0)\) with the Leibniz rule, only the terms with at least \(m\) holomorphic derivatives and \(m\) antiholomorphic derivatives on \(c^m\) will not be zero, so it remains at most \(\ell\) holomorphic and \(\ell\) antiholomorphic derivatives on \(d\).

The functions \(a_{\ell, \alpha, \beta}\) can be computed by expanding (13). Since the functions \(b, c, c'\) in (13) have holomorphic extensions to \((x, y_c) \in U \times \overline{U}\), each \(a_{\ell, \alpha, \beta}\) has a holomorphic extension \(\tilde{a}_{\ell, \alpha, \beta} : U \times \overline{U} \to \mathbb{C}\). We can now state our main estimate.
For any compact subset $K$ of $U \times \overline{U}$, there exists $C > 0$ such that for any $\ell$, $\alpha$, $\beta$,

$$|\tilde{a}_{\ell,\alpha,\beta}(x, y_c)| \leq C^{\ell+1}, \quad \forall (x, y_c) \in K. \quad (16)$$

It is possible to deduce Theorem 2.3 directly from Theorem 2.1 by repeated applications of Leibniz formula. Instead, we will present a less computational argument based on a family of seminorms considered in [15]. Let $\Omega_r$ be the open ball of $\mathbb{C}^n \times \mathbb{C}^n$ centered at the origin with radius $r$. Let $\mathcal{B}(\Omega_r)$ be the space of holomorphic bounded functions of $\Omega_r$, with the norm $\|f\|_r = \sup_{\Omega_r} |f|$. For any bounded operator $P : \mathcal{B}(\Omega_r) \rightarrow \mathcal{B}(\Omega_s)$, we denote by $\|P\|_{t,s} = \sup\{\|Pf\|_s / \|f\|_t \leq 1\}$ the corresponding norm.

**Lemma 2.4**

1. There exists $C > 0$ such that for any $\gamma \in \mathbb{N}^{2n}$ and $0 < s < t$ we have

$$\|\partial^\gamma\|_{t,s} \leq \frac{C^{|\gamma|+1}\gamma!}{(t-s)^{|\gamma|}}. \quad (17)$$

2. Let $t_0 > 0$, $p$, $q \in \mathbb{N}$, $C > 0$ and $P$, $Q$ be two operators $\mathcal{B}(\Omega_{t_0}) \rightarrow \mathcal{B}(\Omega_{t_0})$ such that for any $0 < s < t \leq t_0$, $\|P\|_{t,s} \leq (Cp)^p/(t-s)^p$ and $\|Q\|_{t,s} \leq (Cq)^q/(t-s)^q$. Then for any $0 < s < t \leq t_0$,

$$\|P \circ Q\|_{t,s} \leq \frac{(C(p+q))^{p+q}}{(t-s)^{p+q}}. \quad (18)$$

**Proof** First assertion follows from Cauchy inequality. For the second assertion, we follow [15, page 69]: let $r \in [s, t]$ be such that $r - s = \frac{p}{p+q} (t-s)$ and $t - r = \frac{q}{p+q} (t-s)$. Then

$$\|P \circ Q\|_{t,s} \leq \|P\|_{r,s} \|Q\|_{t,r} \leq \frac{(Cp)^p}{(r-s)^p} \frac{(Cq)^q}{(t-s)^q} = \frac{(C(p+q))^{p+q}}{(t-s)^{p+q}}$$

as was to be proved. \qed

**Proof of Theorem 2.3**

The $\tilde{a}_{\ell,\alpha,\beta}$ are given in terms of the operator $P_\ell$ defined in (14) by

$$\tilde{a}_{\ell,\alpha,\beta}(x, y_c) = P_\ell(u^\beta v^\alpha)(x, y_c) \quad (19)$$

For any $0 < r \leq 1$, $\|u^\beta v^\alpha\|_r \leq 1$. We will estimate the norm of $\mathcal{B}(\Omega_r) \rightarrow \mathbb{C}$, $\tilde{a} \rightarrow P_\ell(\tilde{a})(x, y_c)$. In the sequel, the constants $C$, $C'$ depend on $(x, y_c)$, but remain bounded as $(x, y_c)$ stays in a compact set.

Since the function $\tilde{c}(x, y_c, u, v)$ vanishes when $u = v = 0$, the multiplication operator

$$\mathcal{B}(\Omega_r) \rightarrow \mathcal{B}(\Omega_r), \quad \tilde{a}(u, v) \rightarrow \tilde{c}(x, y_c, u, v)\tilde{a}(u, v) \quad (18)$$

has a norm smaller than 1 when $r$ is sufficiently small. So the same holds for the multiplication by $\tilde{c}^m$.

If $r$ is sufficiently small, there exists a constant $C$ such that for any $m$,

$$\mathcal{B}(\Omega_r) \rightarrow \mathbb{C}, \quad \tilde{a}(u, v) \rightarrow (\tilde{a}^m\tilde{a})(0, 0) \quad (19)$$

has a norm smaller than $C^m m^{2m}$. Indeed, by assertion 1 of Lemma 2.4, $\|\tilde{a}\|_{t,s} \leq C/(t-s)^2$ when $0 < s < t$ are sufficiently small. So by assertion 2 of Lemma 2.4, for any $m \in \mathbb{N}$, $\|\tilde{a}^m\|_{t,s} \leq C^m m^{2m}/(t-s)^{2m}$ with the same constant $C$. To conclude, choose $(t, s) = (r, 0)$ and replace $C/r^2$ by $C$. \hfill $\square$
The norm estimates of (18) and (19) imply through (17) that
\[ |\hat{\alpha}_{\ell,\alpha,\beta}(x, x_c)| \leq C' \sum_{m=0}^{2\ell} C^{\ell+m} (\ell + m)^{(\ell+m)!} \]
where \( C' \) is the product of upper bounds of \( |\hat{b}| \) and \( |\hat{c}'| \). Using that \( p^p \leq e^p p! \) and \( (p+q)! \leq 2^{p+q} p! q! \), we get that
\[ \frac{(\ell + m)^{(\ell+m)!}}{(\ell + m)!} \leq \left( \frac{e}{2} \right)^{(\ell+m)} \leq e^{2(\ell+m)} \frac{(\ell + m)!}{m!} \leq (2e^2)^{\ell+m} \]
so \( |\hat{\alpha}_{\ell,\alpha,\beta}(x, x_c)| \leq C'(2\ell+1)(2Ce^2)^{3\ell} \) and we conclude easily. \( \square \)

3 Symbol of the Bergman kernel

The Bergman projector \( \Pi_k \) of \( L^k \otimes L' \) is the projector of \( C^\infty(M, L^k \otimes L') \) onto the subspace \( \mathcal{H}_k = H^0(M, L^k \otimes L') \) consisting of holomorphic sections. The Bergman kernel \( (x, y) \to \Pi_k(x, y) \) is the Schwartz kernel of \( \Pi_k \), so
\[ \Pi_k(x, y_c) = \sum_{i=1}^{d_k} \Psi_i(x) \otimes \overline{\Psi}_i(y_c). \]
where \((\Psi_i)_{i=1,...,d_k}\) is any orthonormal basis of \( H^0(M, L^k \otimes L') \).

It was proved in [18] that the restriction to the diagonal of the Bergman kernel has an asymptotic expansion
\[ \Pi_k(x, x) = \left( \frac{k}{2\pi} \right)^n \sum_{\ell=0}^{N} k^{-\ell} \rho_\ell(x) + O(k^{n-N-1}), \quad \forall N \tag{20} \]
with smooth coefficients \( \rho_\ell \in C^\infty(M) \).

**Theorem 3.1** There exist an open neighborhood \( W \) of \( \Delta_M \) and a constant \( C > 0 \) such that for any \( \ell \), the function \( \rho_\ell \) has a holomorphic extension \( \tilde{\rho}_\ell \) to \( W \) satisfying
\[ |\tilde{\rho}_\ell(x, y_c)| \leq C^{\ell+1} \ell^\ell, \quad \forall (x, y_c) \in W. \]

As already mentioned in the introduction, this result was proved in [14]. For the proof we will compute the functions \( \rho_\ell \) from the bidifferential operators \( A_\ell \) considered previously, by using that \( (\Pi_k) \) is the unit of an algebra of Berezin–Toeplitz operators.

These operators will be defined as families \( (T_k \in \text{End}(\mathcal{H}_k), \ k \in \mathbb{N}) \) whose Schwartz kernel has a particular form. Here the Schwartz kernel is given by
\[ T_k(x, y_c) = \sum_{i=1}^{d_k} (T_k \Psi_i)(x) \otimes \overline{\Psi}_i(y_c), \quad (x, y_c) \in M \times \overline{M}. \tag{21} \]
where \((\Psi_i)\) is an orthonormal basis of \( \mathcal{H}_k \) as above. Conversely, we recover \( T_k \Psi \) from its Schwartz kernel with the integral (12).

\[ ^1 \text{When } \Psi \text{ is smooth, the right-hand side of (12) is equal to } T_k \Pi_k \Psi. \text{ So we have implicitly identified the endomorphisms of } \mathcal{H}_k \text{ with the operators } T_k \text{ acting on } C^\infty(M, L^k \otimes L') \text{ and satisfying } \Pi_k T_k \Pi_k = T_k. \]
Following [8], we define a Berezin–Toeplitz operator as any family $(T_k) \in \prod_k \text{End } \mathcal{H}_k$ with Schwartz kernels of the form

$$T_k(x, y) = E_k(x, y) \sum_{\ell=0}^N k^{-\ell} \tau_{\ell}(x, y) + O(k^{n-N-1}), \quad \forall N$$

(22)

where the sections $E_k$ are defined in (11) on $W \subset M \times \bar{M}$ and for any $\ell$, $\tau_{\ell}$ is a smooth function of $M \times \bar{M}$ supported in $W$ such that $\bar{\partial} \tau_{\ell}$ vanishes to infinite order along the diagonal. Denote by $\tau_{\ell}$ the restriction of $\tau_{\ell}$ to the diagonal. We call the formal series $\sum \hbar^\ell \tau_{\ell}$ the symbol of $(T_k)$. Let $T$ be the space of Berezin–Toeplitz operators and define the application

$$\sigma : T \to C^\infty(M)[[\hbar]], \quad (T_k) \to \sum_{\ell=0}^\infty \hbar^\ell \tau_{\ell}.$$ 

We deduced in [8] from [6] that the Bergman kernel itself has the form (22). In other words, $(\text{id}_{\mathcal{H}_k})$ belongs to $T$. By (20), its symbol if $\sum \hbar^\ell \rho_{\ell}$. We also proved in [8] that $T$ is closed under product and the map $\sigma$ is onto with kernel the space of families $(T_k \in \text{End } \mathcal{H}_k)$ such that $\| T_k \| = O(k^{-\infty})$. So $T$ is a subalgebra of $\prod_k \mathcal{H}_k$ and $O(k^{-\infty})$ being an ideal of this subalgebra, $C^\infty(M)[[\hbar]]$ inherits a product $\star$. This product has been actually computed in Theorem 2.1

$$\sum_m h^m f_m \star \sum_p h^p g_p = \sum_{\ell, m, p} h^{\ell+m+p} A_{\ell}(f_m, g_p).$$

(23)

So $\sum \hbar^\ell \rho_{\ell}$ is the unit of the associative algebra $(C^\infty(M)[[\hbar]], \star)$. By Theorem 2.1, the $A_{\ell}$ have analytic coefficients, so $C^\infty(M)[[\hbar]]$ is a subalgebra of $(C^\infty(M)[[\hbar]], \star)$, in particular the $\rho_{\ell}$’s are analytic. For an operator whose symbol has analytic coefficients, we can simplify slightly the formula (22) by choosing for $\tau_{\ell}$ a function holomorphic on a neighborhood of the diagonal. This has the advantage that the right hand side of (22) is uniquely determined on a neighborhood of the diagonal by its restriction to the diagonal. Furthermore, these operators form a subalgebra of $T$, so we could have considered only them. But in the theory developed in [8], this was not possible because we worked with a smooth metric not necessarily analytic for the bundle $L$.

**Lemma 3.2** We have for each $m \geq 1$,

$$\rho_m = \sum_{\ell \geq 1, (i_1, \ldots, i_\ell) \in \mathbb{Z}_{\geq 0}, \sum_{i_1+\ldots+i_\ell = m} Q_{i_1} \ldots Q_{i_\ell}(1)$$

(24)

where for any $i \in \mathbb{Z}_{\geq 0}$, $Q_i$ is the differential operator $Q_i(f) = -A_i(f, 1)$. 

**Proof** Since $\sum \hbar^\ell \rho_{\ell}$ is the unit of $\star$, we have $\sum \hbar^{\ell+m} A_{\ell}(\rho_m, 1) = 1$. Using that $A_0(f, g) = fg$, we obtain

$$\rho_0 = 1, \quad \rho_1 = Q_1(\rho_0), \quad \rho_2 = Q_2(\rho_0) + Q_1(\rho_1)$$

and more generally $\rho_m = Q_m(\rho_0) + Q_{m-1}(\rho_1) + \cdots + Q_1(\rho_{m-1})$. This can be solved inductively by

$$\square$$

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\[\rho_0 = 1, \quad \rho_1 = Q_1(1), \]

\[\rho_2 = (Q_2(1) + Q_1(Q_1(1))) = (Q_2 + Q^2_1)(1)\]

\[\rho_3 = Q_3(1) + Q_2(Q_1(1)) + Q_1((Q_2 + Q^2_1)(1)) = (Q_3 + Q_2Q_1 + Q_1Q_2 + Q^3_1)(1)\]

and more generally we obtain (24).

**Proof of Theorem 3.1** Locally, we can define the holomorphic extension \(\widetilde{\mathcal{Q}}_\ell(\tilde{f}) = \tilde{A}_\ell(\tilde{f}, 1)\) of \(Q_\ell(f)\) with \(\tilde{A}\) given in (15). If in the right-hand side of Equation (24), we replace each \(Q_{i_j}\) by \(\tilde{Q}_{i_j}\), we obtain a holomorphic extension of \(\rho_m\). We have

\[\tilde{Q}_\ell(\tilde{f})(x, y_c) = -\ell! \sum_{|\alpha| \leq \ell} \frac{1}{\alpha!} \tilde{a}_{\ell, \alpha, 0}(x, y_c) \partial_{y_c}^\alpha \tilde{f}(x, y_c).\]

Assume the coordinates are centered at \(x_0\) and use them to identify a neighborhood of \((x_0, \tilde{x}_0)\) with a neighborhood of the origin in \(\mathbb{C}^n \times \mathbb{C}^n\). Introduce the same seminorms \(\|\cdot\|_{t, s}\) as before Lemma 2.4. We have for any \(\ell \geq 1\) and \(0 < s < t\) sufficiently small

\[\|\widetilde{\mathcal{Q}}_\ell\|_{t, s} \leq \frac{(C'\ell)^\ell}{(t-s)^\ell} \quad (25)\]

for some \(C' > 0\). Indeed choose \(C\) so that first part of Lemma 2.4 holds and Theorem 2.3 as well. So we have

\[\frac{1}{\alpha!} \|\tilde{a}_{\ell, \alpha, 0}\|_{t, s} \leq \frac{C^{\ell+|\alpha|+1}}{(t-s)^{|\alpha|}}.\]

Assuming that \(C \geq 1\) and \(|\alpha| \leq \ell\), we have \((C/(t-s))^{\ell} \leq (C/(t-s))^\ell\). Furthermore the number of multiindices \(\alpha \in \mathbb{N}^n\) with \(|\alpha| \leq \ell\) is \((\ell+n-1)\) which is \(\leq 2^{\ell+n-1}\). So the number of \(\alpha\) with \(|\alpha| \leq \ell\) is smaller that \(2^{\ell+n}\), so

\[\frac{1}{\ell!} \|\widetilde{\mathcal{Q}}_\ell\|_{t, s} \leq \sum_{|\alpha| \leq \ell} \frac{C^{\ell+|\alpha|+1}}{(t-s)^{|\alpha|}} \leq 2^{\ell+n} \frac{C^{\ell+1}}{(t-s)^\ell}\]

We obtain (25) by using that \(\ell! \leq \ell^\ell\) and setting \(C' = 2^{n+1}C^3\).

Now (25) implies by assertion 2 of Lemma 2.4 that for any \((i_1, \ldots, i_\ell) \in \mathbb{Z}_{>0}^\ell\)

\[\|\tilde{Q}_{i_1} \circ \cdots \circ \tilde{Q}_{i_\ell}\|_{t, s} \leq \frac{(C'm)^m}{(t-s)^m} \quad (26)\]

where \(m = i_1 + \cdots + i_\ell\). Recall that

\[\tilde{\rho}_m = \sum_{\ell \geq 1, (i_1, \ldots, i_\ell) \in \mathbb{Z}_{>0}^\ell, i_1 + \cdots + i_\ell = m} \tilde{Q}_{i_1} \cdots \tilde{Q}_{i_\ell}(1).\]

Since the number of terms in the sum is \(2^{m-1}\), we deduce from (26) with \(t = 2s\) and \(s\) sufficiently small that

\[\|\tilde{\rho}_m\|_s \leq 2^{m-1} \frac{(C'm)^m}{s^m},\]

which concludes the proof. \qed
4 Analytic Berezin–Toeplitz operators

4.1 Analytic symbols

An analytic symbol of $M$ is a formal series $\sum_{\ell=0}^{\infty} \hat{h}^\ell f_\ell$ with coefficients in $C^0(M)$ such that there exist a neighborhood $W$ of $\Delta_M$ in $M \times \overline{M}$ and a constant $C > 0$ so that each $f_\ell$ has a holomorphic extension $\tilde{f}_\ell$ to $W$ satisfying

$$|\tilde{f}_\ell(x)| \leq C^{\ell+1} \| |f_\ell| \|, \quad \forall x \in W. \quad (27)$$

Let $S^0$ be the subspace of $C^\infty(M[[\hbar]]$ consisting of analytic symbols. Recall the product $\star$ of $C^\infty(M[[\hbar]])$ introduced in (23).

**Proposition 4.1** $S^0$ is a subalgebra of $(C^\infty(M)[[\hbar]], \star)$.

**Proof** By Theorem 3.1, the unit $\sum \hbar^\ell \rho_\ell$ of $\star$ is an analytic symbol. Let us prove that the product of two analytic symbols $\sum \hbar^\ell g_\ell$ and $\sum \hbar^\ell h_\ell$ is analytic. Recall the holomorphic extension $\tilde{A}_\ell$ on $\tilde{U} = U \times \overline{U}$ defined in (15). Restricting $U$ if necessary, each $f_\ell$, $g_\ell$ has a holomorphic extension to $\tilde{U}$ and we have to prove that

$$\tilde{f}_\ell = \sum_{m+p+q=\ell} \tilde{A}_m(g_p, h_q)$$

satisfies (27) on an open neighborhood $\tilde{V}$ of $(x_0, \overline{x}_0)$. This follows from the second part of Lemma 5.4 and the fact that if $\tilde{V}$ has a compact closure in $\tilde{U}$, then $\tilde{A}_\ell$ is continuous $\mathcal{B}(\tilde{U}) \times \mathcal{B}(\tilde{U}) \to \mathcal{B}(\tilde{V})$ with a norm smaller than $(C')^{\ell+1}$. Here $\mathcal{B}(\tilde{V})$ is the space of bounded holomorphic function of $\tilde{V}$ with the sup norm $| \cdot |_{\tilde{V}}$.

To prove the continuity and the norm estimate of $\tilde{A}_\ell$, choose $C$ as in Theorem 2.3. Replacing $C$ by a larger constant, we have by Cauchy’s inequality $\frac{1}{\gamma!} |\partial^\gamma \tilde{f}|_{\tilde{V}} \leq C^{\gamma} |f|_{\tilde{U}}$. So

$$|\tilde{a}_{\ell, \alpha, \beta}(\partial^\gamma \tilde{f})(\partial^\gamma \tilde{g})|_{\tilde{V}} \leq C^{3\ell+1} |\tilde{f}|_{\tilde{U}} |\tilde{g}|_{\tilde{U}}$$

for $|\alpha|, |\beta| \leq \ell$. The number of $\alpha, \beta$ satisfying this condition being $\leq 4^{\ell+n-1}$, we obtain

$$|\tilde{A}_\ell(\tilde{f}, \tilde{g})|_{\tilde{V}} \leq \ell!4^{\ell+n-1}C^{3\ell+1} |\tilde{f}|_{\tilde{U}} |\tilde{g}|_{\tilde{U}}$$

which proves the claim with $C' = \max(4C^3, 4^{n-1}C)$. \square

4.2 Berezin–Toeplitz operators

**Definition 4.2** An analytic Berezin–Toeplitz operator is a family $(T_k) \in \prod_k \text{End}(\mathcal{H}_k)$ whose Schwartz kernels satisfy

i. for any compact subset $K$ of $M \times \overline{M}$ not intersecting $\Delta_M$, there exists $C_K > 0$ such that $|T_k(x, y_c)| \leq e^{-k/C_K}$ on $K$.

ii. on a neighborhood $W$ of the diagonal

$$T_k(x, y_c) = E_k(x, y_c) \sum_{\ell=0}^{\lfloor k \rfloor} k^{-\ell} \tilde{f}_\ell(x, y_c) + \mathcal{O}(e^{-k/C'})$$

(28)

where the sections $E_k$ are defined in (11), $\sum \hbar^\ell f_\ell$ is an analytic symbol, each $\tilde{f}_\ell$ is a holomorphic extension of $f_\ell$ to $W$ satisfying (27) for a constant $C, 0 < \epsilon < 1/C, C' > 0$ and the $\mathcal{O}$ is uniform on $W$. \square
Comparing to the introduction, the definition here is the characterization in theorem 1.1. The fact that this definition is equivalent to the expansion (8) will be proved in Sect. 5.

Let \( T^\omega \) be the space of analytic Berezin–Toeplitz operators and \( \sigma : T^\omega \to S^\omega \) be the map sending \((T_k)\) to the symbol \( \sum \hat{h}^\ell f_\ell \). Recall the algebra \( \mathcal{T} \) of Berezin–Toeplitz operators and the symbol map \( \sigma : \mathcal{T} \to \mathcal{C}^\infty(M)[[\hbar]] \) defined in Sect. 3.

**Theorem 4.3**
1. \( T^\omega \) is a subspace of \( \mathcal{T} \) and \( \sigma : T^\omega \to S^\omega \) is the restriction of \( \sigma : \mathcal{T} \to \mathcal{C}^\infty(M)[[\hbar]] \).
2. \( \sigma : T^\omega \to S^\omega \) is surjective and its kernel consists of the families \((T_k) \in \prod_k \text{End}(\mathcal{H}_k)\)
   such that \( \|T_k\| = \mathcal{O}(e^{-k/C}) \) for some \( C > 0 \).
3. \( T^\omega \) is closed under product, the corresponding product of symbols of \( S^\omega \) is \(*\).
4. \((\text{id}_{\mathcal{H}_k})\) belongs to \( T^\omega \), with symbol \( \sum \hat{h}^\ell \rho_\ell \).

Since the Schwartz kernel of \( \text{id}_{\mathcal{H}_k} \) is the Bergman kernel \( \Pi_k \), the last assertion is equivalent to the fact \([\Pi_k(x, y_c)]\) is a \( \mathcal{O}(e^{-k/C_k}) \) on any compact set \( K \) of \( M \times \overline{M} \) not intersecting the diagonal and that
\[
\Pi_k(x, y_c) = E_k(x, y_c) \sum_{\ell=0}^{E(\ell)} k^{-\ell} \tilde{\rho}_\ell(x, y_c) + \mathcal{O}(e^{-k/C}).
\]

As already mentioned in the introduction, the third assertion has been proved in [9], the last one in [14]. The proof of the third assertion relies on an analytic version of the stationary phase lemma [16]. The fact that \( \mathcal{T} \) is closed under product was already proved in [8] by using a stationary phase lemma for smooth complex valued phase. The last assertion can certainly be deduced from Theorem 3.1 by using the method of [1]. Actually, it is a simple corollary of the third assertion as was noticed in [9]. We will reproduce this short proof here.

### 4.3 Proof of Theorem 4.3

By Proposition 5.3, \( \tilde{f}(\epsilon, k) := \sum_{\ell=0}^{\lfloor \epsilon k \rfloor} k^{-\ell} \tilde{f}_\ell \) has an asymptotic expansion
\[
\tilde{f}(\epsilon, k) = \sum_{\ell=0}^{N} k^{-\ell} \tilde{f}_\ell + \mathcal{O}(k^{-N-1}), \quad \forall N
\]

This implies immediately that \( T^\omega \subset \mathcal{T} \) and that the definitions of the symbols are the same for Berezin–Toeplitz operators and analytic Berezin–Toeplitz operators. Another basic property of the partial sums \( f(\epsilon, k) \) is that for \( \epsilon' < \epsilon \), \( f(\epsilon', k) = f(\epsilon, k) + \mathcal{O}(e^{-k/C'}) \) for some \( C' > 0 \), cf. Proposition 5.3. This simple remark is actually needed to check that \( T^\omega \) is a linear subspace of \( \prod_k \text{End} \mathcal{H}_k \).

To prove that \( \sigma : T^\omega \to S^\omega \) is surjective, we define for an analytic symbol \( \sum \hat{h}^\ell f_\ell \) as above the kernel
\[
T_k(x, y_c) := \rho(x, y_c) E_k(x, y_c) \tilde{f}(\epsilon, k)(x, y_c)
\]
where \( \rho \in \mathcal{C}^\infty(W) \) is equal to 1 on a neighborhood of \( \Delta_M \). The family \((T_k)\) certainly satisfies (28) and also \( |T_k| = \mathcal{O}(e^{-k/C_k}) \) on any compact subset \( K \) of \( M \times \overline{M} \) not intersecting \( \Delta_M \). But \( T_k \) is not the Schwartz kernel of an endomorphism of \( \mathcal{H}_k \) because \( T_k \) is not holomorphic. Actually, since \( \bar{\partial} T_k = (\partial \bar{\partial}) E_k \tilde{f}(\epsilon, k) \), we have that \( |\bar{\partial} T_k| = \mathcal{O}(e^{-k/C'}) \) uniformly on \( M \times \overline{M} \) for some \( C' > 0 \). By the Kodaira-Nakano-Hörmander inequality, this has the consequence that the orthogonal projection \( S_k \) of \( T_k \) onto the space of holomorphic sections of \( (L^k \otimes L^\prime) \otimes \mathcal{O}(H) \)
\((L^k \otimes L')\) is equal to \(T_k\) up to a uniform \(\mathcal{O}(e^{-k/C'})\) with a larger \(C'\). So the corresponding family of operators is an analytic Berezin–Toeplitz operator with symbol \(\sum h^f_\ell f_\ell\).

The kernel of \(\sigma : T^{\omega} \to S^{\omega}\) consists of operators whose Schwartz kernel is uniformly in \(\mathcal{O}(e^{-k/C'})\) for some \(C > 0\). This condition is equivalent to the fact that the operator norm \(\|T_k\|\) is in \(\mathcal{O}(e^{-k/C'})\) for some \(C' > 0\). To prove this, we can in one direction use the Schur criterion and on the other direction that

\[
T_k(x, y) = (T_k \xi_{x,k}, \xi_{y,k}), \quad \|\xi_{x,k}\|, \|\xi_{y,k}\| = \mathcal{O}(k^{n/2})
\]

where \(\xi_{x,k}, \xi_{y,k}\) are the coherent states of \(L^k \otimes L'\) at \(x\) and \(y\) respectively.

We now prove that \(T^{\omega}\) is closed under product. First observe that if \((T_k), (S_k) \in \prod_k \mathcal{H}_k\) satisfy both the condition \(i)\) of Definition 4.2 and their Schwartz kernels are uniformly bounded independently of \(k\), then the same holds for \((T_k S_k)\). Furthermore, for any open subsets \(V\) and \(W\) of \(M\) such that \(V\) has a compact closure contained in \(W\), we have

\[
(T_k S_k)(x, x_c) = \int_W T_k(x, y) S_k(y, x_c) \, d\mu(y) + \mathcal{O}(e^{-k/C})
\]

for any \((x, x_c) \in V \times \mathring{V}\) with a uniform \(\mathcal{O}\).

Let us assume from now on that \((T_k)\) and \((S_k)\) are both analytic Berezin–Toeplitz operators with symbols \(\sum h^f_\ell f_\ell\) and \(\sum h^g_\ell g_\ell\). By Theorems 2.1 and 4.3, we already know that the restriction to the diagonal of the Schwartz kernel of \(T_k S_k\) has an asymptotic expansion

\[
(T_k S_k)(x, x) = \sum_{\ell=0}^N k^{-\ell} h_\ell(x) + \mathcal{O}(k^{-N-1}), \quad \forall N
\]

where \(\sum h^f_\ell h_\ell\) is the analytic symbol \(\sum h^f_\ell f_\ell \cdot \sum h^g_\ell g_\ell\). So we have to prove that \(T_k S_k = E_k h_0 + \mathcal{O}(e^{-k/C})\) on a neighborhood of the diagonal.

Introduce the same local data \(U \ni x_0, \varphi, \varphi'\) as in Sect. 2.2. Let

\[
\tilde{\psi}(x, y, y_c) = -\tilde{\psi}(x, y_c) - \varphi(y, x_c) + \varphi(x, x_c) + \varphi(y, yc) \\
\tilde{\psi}'(x, y, y_c) = -\tilde{\psi}'(x, y_c) - \varphi'(y, x_c) + \varphi'(x, x_c) + \varphi'(y, y_c)
\]

If \(\tilde{f}, \tilde{g}\) are two holomorphic functions of \(\tilde{U} = U \times \mathring{U}\), we set

\[
I_k(\tilde{f}, \tilde{g})(x, x_c) = \left(\frac{2\pi}{k}\right)^n \int_B e^{-k \tilde{\psi}(x, y, y_c)} \tilde{f}(x, y) \tilde{g}(y, x_c) \, d\mu(y)
\]

where \(B\) is a compact neighborhood of \(x_0\) in \(U\).

On a neighborhood of \((x_0, \mathring{x}_0)\), the Schwartz kernel of \(T_k S_k\) is equal to \(I_k(\tilde{f}(\epsilon, k), \tilde{g}(\epsilon, k))\) \(E_k + \mathcal{O}(e^{-k/C})\). It will from Lemma 4.4 and Lemma 4.5 that \(I_k(\tilde{f}(\epsilon, k), \tilde{g}(\epsilon, k)) = h_0(\epsilon, k) + \mathcal{O}(e^{-k/C})\), which ends the proof that \((T_k S_k)\) is an analytic Berezin–Toeplitz operator.

**Lemma 4.4** There exist a neighborhood \(V\) of \(x_0\) in \(U\) and \(\epsilon > 0\) such that

\[
I_k(\tilde{f}, \tilde{g})(x, x_c) = \sum_{\ell=0}^{|\epsilon k|} k^{-\ell} \tilde{A}_\ell(\tilde{f}, \tilde{g})(x, x_c) + \mathcal{O}(e^{-\epsilon k} |\tilde{f}|_\infty |\tilde{g}|_\infty)
\]

(29)

for any \((x, x_c) \in V \times \mathring{V}\), with a uniform \(\mathcal{O}\) and \(|\tilde{f}|_\infty, |\tilde{g}|_\infty\) the sup norms of \(\tilde{f}\) and \(\tilde{g}\) on \(\tilde{U}\).

**Lemma 4.5** \(\sum_{\ell=0}^{|\epsilon k|} k^{-\ell} \tilde{A}_\ell(\tilde{f}(\epsilon, k), \tilde{g}(\epsilon, k)) = h_0(\epsilon, k) + \mathcal{O}(e^{-k/C})\).
Proof of Lemma 4.4 This is a consequence of the analytic version of stationary phase lemma proved in [16]. When \( x_c = \bar{x} \), the phase \( y \to \tilde{\psi}(x, \bar{y}, y, \bar{x}) \) has a critical point at \( y = x \) as was used in Sect. 2.2 to establish Theorem 2.1. In the case where \( x_t \neq \bar{x} \), we consider the holomorphic extension \((y, y_c) \to \tilde{\psi}(x, y_c, y, x_c) \). As it was already noticed in [8], the critical point is now at \( y = x, y_c = x_c \). Indeed, we compute easily from the definition of \( \tilde{\psi} \) that
\[
\tilde{\psi}(x, x_c + \bar{u}, x + u, x_c) = \sum_{i,j} \frac{\partial^2 \tilde{\psi}}{\partial x_i \partial x_{c,j}}(x, x_c)u_i \bar{u}_j + O(|u|^3). \tag{30}
\]
Furthermore, \((\partial^2 \tilde{\psi}/\partial x_i \partial x_{c,j})(x, \bar{x}) = G_{i,j}(x) \) is definite positive. In this situation, we can apply [16, Theorem 2.8]. The first step is a deformation which can be made explicit in our case. We identify \( U \) with an open set of \( \mathbb{C}^n \), \( x_0 \) being sent to the origin. We denote by \( B_r \), the closed ball with radius \( r \) of \( \mathbb{C}^n \). We choose an open neighborhood \( \hat{V} \) of the origin in \( U \times \bar{U} = \hat{U} \) and \( r > 0 \) such that
1. \((x, x_c) \in \hat{V}, u \in B_r, \) and \( t \in [0, 1] \) implies that \((tx + u, tx_c + \bar{u}) \in \hat{U} \).
2. for any \((x, x_c) \in \hat{V}, \) \( \Re \tilde{\psi}(x, x_c, 1, u) > 0 \) if \( u \in B_r \setminus \{0\} \),
3. for any \((x, x_c) \in \hat{V}, \) \( \Re \tilde{\psi}(x, x_c, t, u) > 0 \) if \( u \in \partial B_r \) and \( t \in [0, 1] \).

Then, by this last condition and Stokes Lemma, for any holomorphic \( b : \hat{U} \to \mathbb{C} \), the integral
\[
J_k(t, b)(x, x_c) := \int_{B_r} e^{-k\tilde{\psi}(x, tx_c + \bar{u}, tx + u, x_c)} \tilde{b}(tx + u, tx_c + \bar{u}) \, d\mu_L(u),
\]
is independent of \( t \in [0, 1] \) up to a term in \( O(e^{-k/C} \sup_{\hat{U}} |b|) \) uniformly with respect to \((x, x_c) \in \hat{V} \). The second step is to prove that
\[
J_k(1, b)(x, x_c) = \sum_{\ell=0}^{E(\epsilon k)} k^{-\ell} c_{\ell}(x, x_c) + O(e^{-k/C} \sup_{\hat{U}} |b|)
\]
with a \( O \) uniform for \((x, x_c) \in \hat{V} \). This follows from a holomorphic version of Morse Lemma and a careful application of Laplace method. The assumption for this second step is the condition 2 above and the fact that the Hessian \( \partial^2 \tilde{\psi}/\partial x_i \partial x_{c,j} \) in (30) has a positive definite real part. This concludes the proof except for the computation of the coefficients in (29). Actually we already know these coefficients for \( x_c = \bar{x} \), and by [16, Remarque 2.10], they depend holomorphically on \((x, x_c) \).

Proof of Lemma 4.5 This follows from the second part of Lemma 5.4 and the continuity of the \( \tilde{A}_k \) established in the proof of Proposition 4.1. \( \square \)

We now prove the fourth assertion of Theorem 4.3 by following [9]. Choose \((T_k) \) in \( T^{w} \) with symbol \( \sum h^t b_{t^*} \). Replacing \( T_k \) by \( \frac{1}{2}(T_k + T_k^*) \), we can assume that \( T_k \) is self-adjoint. Since \((T_k) \) belongs to \( T \) and has the same symbol of \((\text{id} \mathcal{H}_k) \), we already know that \( T_k = \text{id} \mathcal{H}_k + O(k^{-\infty}) \). Our goal is to prove that \( T_k^2 = \text{id} \mathcal{H}_k + O(e^{-k/C}) \). The symbol of \((T_k) \) is idempotent, so \( T_k^2 = T_k + O(e^{-k/C}) \), so the spectrum of \( T_k \) splits in two parts, concentrating at 0 and 1 respectively, more precisely
\[
\text{sp}(T_k) \subset [-Ce^{-k/C}, Ce^{-k/C}] \cup [1 - Ce^{-k/C}, 1 + Ce^{-k/C}]
\]
with a larger \( C \). If \( k \) is sufficiently large, these two intervals are disjoint. Let \( n_k \) be the number of eigenvalues counted with multiplicity in the second interval. Then \( \text{tr} T_k = n_k + O(e^{-k/C}) \).
The fact that $T_k = \text{id}_{\mathcal{H}_k} + \mathcal{O}(k^{-\infty})$ implies that
\[
\text{tr } T_k = \dim \mathcal{H}_k + \mathcal{O}(k^{-1}),
\]
so $n_k = \dim \mathcal{H}_k$ when $k$ is sufficiently large, so $T_k = \text{id}_{\mathcal{H}_k} + \mathcal{O}(e^{-k/C})$ as was to be proved.

5 Multipliers

For $f \in C^\infty(M)$, we define the operator $T_k(f) : \mathcal{H}_k \to \mathcal{H}_k$ sending $\psi$ into $\Pi_k(f\psi)$. By [8], any Berezin–Toeplitz operator $(S_k)$ has an expansion of the form
\[
S_k = \sum_{\ell=0}^{N} k^{-\ell} T_k(f_\ell) + \mathcal{O}(k^{-N-1}), \quad \forall N. \tag{31}
\]
where $(f_\ell)$ is a sequence of $C^\infty(M)$, and the $\mathcal{O}$ is in uniform norm. Conversely, for any sequence $(f_\ell)$, there exists a Toeplitz operator $S_k$ satisfying (31). Furthermore,
\[
\sigma(S_k) = \sum_{\ell,m} h^\ell m B_\ell(f_m)
\]
where the $B_\ell$ are differential operators of $C^\infty(M)$, $B_0$ being the identity. The map sending $\sum h^\ell f_\ell$ into the symbol of $(S_k)$ is an isomorphism of $C^\infty(M)[[\hbar]]$.

Theorem 5.1 1. For any analytic symbol $\sum \hbar^\ell f_\ell$ and $\epsilon > 0$ sufficiently small, the family $(T_k(\sum_{\ell=0}^{\lfloor k\epsilon \rfloor} k^{-\ell} f_\ell))$ is an analytic Berezin–Toeplitz operator.
2. For any $f \in C^\alpha(M)$, $B_\ell(f)$ has a holomorphic extension $\tilde{B}_\ell(\tilde{f})$ given locally by
\[
\tilde{B}_\ell(\tilde{f})(x, y_x) = \sum_{p+q+r=\ell \atop |\alpha| \leq p, |\beta| \leq q} \frac{\partial_p \partial_{\alpha} \partial_{\beta}(\tilde{\rho}_q(x, y_x) f(x, y_x) \tilde{\rho}_r(y, y_c))}{y_x \partial_{y_x}} y_x = x_c \tag{32}
\]
where the $a_{p,\alpha,\beta}$'s are the functions introduced in Lemma 2.2.
3. The map sending $\sum \hbar^m f_m$ into $\sum \hbar^\ell + m B_\ell(f_m)$ is an isomorphism of $S^\alpha$. Consequently, any analytic Berezin–Toeplitz operator $(S_k)$ has the form
\[
S_k = T_k(\sum_{\ell=0}^{\lfloor k\epsilon \rfloor} k^{-\ell} f_\ell) + \mathcal{O}(e^{-k/C})
\]
for an analytic symbol $\sum h^\ell f_\ell$.

The proof is long, but actually a variation of what we did before.

Proof 1 and 2. We compute the Schwartz kernel of the product
\[
(\Pi_k f \Pi_k)(x, y_c) = \int_M \Pi_k(x, y) f(y) \Pi_k(y, x) \, d\mu(y)
\]
exactly as we did for the product of two Toeplitz operators in Sect. 4.3. The only change is the factor $f$. The computation of the symbol is the same as in Theorem 2.1, where we replace $d$ by the series $\sum_q r \hbar^{q+r} \tilde{\rho}_q(x, \bar{x} + \tilde{u}) f(x + u, \bar{x} + \tilde{u}) \tilde{\rho}_r(x + u, \bar{x})$. This leads to the formula
of $\tilde{B}_\ell(f)$. The estimates of $a_{\ell,\alpha,\beta}$ given in Theorem 2.3 and the fact that $\sum h^\ell \rho_\ell$ is analytic, Theorem 3.1, imply that the new symbol $\sum h^{\ell+m} B_\ell(f_m)$ is analytic when $\sum h^m f_m$ is.

3. We already know that $B = \sum h^\ell B_\ell$ sends $S^\omega$ into itself. We have to prove that the same holds for $B^{-1}$. It is as difficult as proving that $\sum h^\ell \rho_\ell$ is analytic. Fortunately, we can follow the same method. First, we deduce from Theorems 2.3 and 3.1 that

$$\|\tilde{B}_\ell\|_{t,s} \leq \frac{(C \ell)^\ell}{(t-s)^\ell}.$$  

The proof is the same as the one of (25) except that we now use Formula (32). We deduce that the inverse $B^{-1} = \sum h^\ell C_\ell$ satisfies

$$\|\tilde{C}_\ell\|_{t,s} \leq 2^{\ell-1} \frac{(C \ell)^\ell}{(t-s)^\ell}$$

with exactly the same proof as in Theorem 3.1. We deduce that $\sum h^\ell C_\ell$ sends $S^\omega$ into itself.

□

Remark 5.2 We can also prove that the differential operators $B_\ell$ have locally the form

$$B_\ell = \sum_{|\alpha|,|\beta| \leq \ell} b_{\ell,\alpha,\beta} \partial^\alpha \overline{\partial}^\beta$$

with analytic coefficients admitting holomorphic extensions on $U \times \overline{U}$ such that on any compact set $|b_{\ell,\alpha,\beta}| \leq C^{\ell+1} \ell!$. The coefficients $C_\ell$ of $B^{-1} = \sum h^\ell C_\ell$ have exactly the same property, cf. Sect. 5.2.

Acknowledgements I would like to thank Alix Deleporte for helpful discussions about his work.

Appendix

In the first part, we discuss the asymptotic expansion of analytic symbols. We work in an abstract setting where the symbols are not functions but belong to a normed space, because it makes the discussion simpler. One goal is to compare some remainder estimates (34) coming from [5] with the partial sums (35) introduced in [16]. Even if we haven’t found such a discussion in the literature, we do not claim that these results are original.

The second part is a digression on the method we use to prove Theorems 3.1 and 5.1. We propose alternative arguments and some generalisation.

5.1 Analytic asymptotic expansion

Let $(E, | \cdot |)$ be a normed space. Consider a sequence $(u(k))$ of $E$ having an asymptotic expansion of the form

$$u(k) = \sum_{\ell=0}^{N-1} a_\ell k^{-\ell} + O(k^{-N}), \quad \forall N$$  

with coefficients $a_\ell \in E$. Two important facts are that $(u(k))$ is determined modulo $O(k^{-\infty})$ by the $a_\ell$’s, and for any sequence $(a_\ell)$, there exists a sequence $(u(k))$ satisfying (33).

We are interested in a particular class of asymptotic expansions where the remainder in (33) has the following explicit upper bound

\[
\text{Remark 5.2} \quad \text{We can also prove that the differential operators } B_\ell \text{ have locally the form}
\[
B_\ell = \sum_{|\alpha|,|\beta| \leq \ell} b_{\ell,\alpha,\beta} \partial^\alpha \overline{\partial}^\beta
\]

\[
\text{with analytic coefficients admitting holomorphic extensions on } U \times \overline{U} \text{ such that on any compact set } |b_{\ell,\alpha,\beta}| \leq C^{\ell+1} \ell!. \text{ The coefficients } C_\ell \text{ of } B^{-1} = \sum h^\ell C_\ell \text{ have exactly the same property, cf. Sect. 5.2.}
\]

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\]

\[
\text{We are interested in a particular class of asymptotic expansions where the remainder in (33) has the following explicit upper bound}
\]
\[ |u(k) - \sum_{\ell=0}^{N-1} a_\ell k^{-\ell}| \leq C^{N+1} k^{-N} N!, \quad \forall \ k \in \mathbb{N}^*, \ N \in \mathbb{N} \] (34)

for a constant \( C \) independent of \( k \) and \( N \). Unlike the expansion (33), the sequence \( (u(k)) \) is uniquely determined up to a \( O(e^{-\epsilon k}) \) by (34). Furthermore, the coefficients \( a_\ell \) have a particular growth. The precise result is as follows.

**Proposition 5.3**  
1. If a sequence \((u(k))\) satisfies (34), then the coefficients \((a_\ell)\) satisfy \(|a_\ell| \leq C^{\ell+1}!\) with the same constant \( C \).
2. Assume that \((u(k))\) satisfies (34). Then \((u'(k))\) satisfies (34) with the same coefficients \( a_\ell \) and possibly a larger constant \( C \) if and only if there exists \( \epsilon > 0 \) such that \( u(k) = u'(k) + O(e^{-\epsilon k}) \).
3. If \(|a_\ell| \leq (C')^{\ell+1}! \) for \( C' > 0 \) and \( \epsilon > 0 \) is such that \( \epsilon C' < 1 \), then

\[ u(k) := \sum_{\ell=0}^{[\epsilon k]} a_\ell k^{-\ell} \] (35)

satisfies (34) for some \( C > 0 \).

**Proof**  
1. (34) implies that \(|a_N k^{-N}| \leq C^{N+1} k^{-N} N! + C^{N+2} k^{-N-1} (N + 1)! \). Multiplying by \( k^N \) and taking the limit \( k \to \infty \), we get \(|a_N| \leq C^{N+1} N! \).

2. Assume that \((u(k))\) and \((u'(k))\) satisfy both (34). Then

\[ |u(k) - u'(k)| \leq 2C^{N+1} k^{-N} N! \leq 2C(CN/k)^N. \]

Choose \( \epsilon \) such that \( C \epsilon < 1 \) and set \( N = [\epsilon k] \). Then \( N \leq \epsilon k \) so \( CN/k \leq C \epsilon \) so \( (CN/k)^N \leq (C \epsilon)^N \leq (C \epsilon)^{\epsilon k} - 1 \) because \( N \geq \epsilon k - 1 \). So \( |u(k) - u'(k)| \leq 2e^{-1} \exp(\epsilon k \ln(C \epsilon)) \). Hence \( u(k) = u'(k) + O(e^{-\epsilon k}) \) with \( \epsilon' = -\epsilon \ln(C \epsilon) \). Conversely, we have to prove that for any \( \epsilon \), there exists \( C > 0 \) such that \( e^{-\epsilon k} \leq C(CN/k)^N \) for any \( k \in \mathbb{N}^* \) and \( N \in \mathbb{N} \). The function \( x \to \ln x - \epsilon x \) is bounded above on \( \mathbb{R}_{>0} \), so

\[ \ln(k/N) - (\epsilon k/N) \leq \ln C, \quad \forall \ k, N \in \mathbb{N}^* \]

if \( C \) is sufficiently large. Multiplying with \( N \) and taking the exponential, we get \( e^{-\epsilon k} \leq (CN/k)^N \). We conclude easily.

3. By assumption \(|a_\ell k^{-\ell}| \leq C'(C'/k)^\ell \ell! =: b_\ell \). We will use that

\[ \frac{b_\ell}{b_{\ell-1}} = C'\ell/k. \]

Let \( N' = [\epsilon k] \). Assume that \( N \leq \ell \leq N' \). Then \( \ell \leq \epsilon k \), so \( b_\ell / b_{\ell-1} \leq C' \epsilon < 1 \), so

\[ \sum_{\ell=N}^{N'} b_\ell \leq b_N (1 + \cdots + (C'\epsilon)^{N'-N}) \leq \frac{b_N}{1-C'\epsilon} = \frac{C'}{1-C'\epsilon} (C'/k)^N N! \]

Assume now that \( N' < \ell \leq N \). Then \( \epsilon k \leq \ell \), so \( b_\ell / b_{\ell+1} \leq (C'\epsilon)^{-1} =: r \). Since \( r > 1 \),

\[ \sum_{\ell=N+1}^{N-1} b_\ell \leq \sum_{\ell=N'+1}^N b_\ell \leq b_N (1 + r + \cdots + r^{N-N'-1}) \leq b_N \frac{r^{N-N'}-1}{r-1} \leq b_N \frac{r^N}{r-1} = \frac{C'}{r-1} (\epsilon k)^N N! \]

which concludes the proof. \( \square \)
Let us call a formal series \( a = \sum h^\ell a_\ell \) of \( E[[\hbar]] \) an analytic symbol if \( |a_\ell| \leq C^{\ell+1}\ell! \) for some \( C > 0 \). For any \( \varepsilon > 0 \), we set \( a(\varepsilon, k) := \sum_{\ell=0}^{[\varepsilon k]} k^{-\ell} a_\ell \). Choose a second normed space \((E', |\cdot'|)\) and let \( L(E, E') \) (resp. \( B(E, E') \)) be the space of bounded linear maps \( E \to E' \) (resp. bounded bilinear maps \( E \times E \to E' \)) with its natural norm.

**Lemma 5.4** 1. For any analytic symbols \( a \in E[[\hbar]] \) and \( P \in L(E, E')[[\hbar]] \), the series \( b = \sum_{\ell, m} h^{\ell+m} P_\ell(a_m) \) is an analytic symbol of \( E'[[\hbar]] \). Furthermore, if \( \varepsilon > 0 \) is sufficiently small, then there exists \( C > 0 \) such that

\[
P(\varepsilon, k)(a(\varepsilon, k)) = b(\varepsilon, k) + O(e^{-k/C}).
\]

2. For any analytic symbols \( a, a' \in E[[\hbar]] \) and \( B \in B(E, E')[[\hbar]] \), the series \( b = \sum_{\ell, m, p} h^{\ell+m+p} B_\ell(a_m, a'_p) \) is an analytic symbol of \( E'[[\hbar]] \). Furthermore if \( \varepsilon > 0 \) is sufficiently small, then there exists \( C > 0 \) such that

\[
B(\varepsilon, k)(a(\varepsilon, k), a'(\varepsilon, k)) = b(\varepsilon, k) + O(e^{-k/C}).
\]

**Proof** Assume that \( \|P_\ell\| \leq C^{\ell+1}\ell! \) and \( |a_m| \leq C^{m+1}m! \). Then \( |P_\ell a_m| \leq C^2 C^{\ell+m}(\ell + m)! \), so \( |b_p| \leq (p + 1)C^2 + p! \) and we conclude that \( b \) is analytic. Furthermore, for \( N = [\varepsilon k] \), we have

\[
|P(\varepsilon, k)(a(\varepsilon, k)) - b(\varepsilon, k)| \leq \sum_{N < \ell < \ell + m} k^{-\ell-m}|P_\ell(a_m)| \leq N \sum_{p=1}^{2N} C^2(Cp/k)^p
\]

by the previous estimate. \( N < p \leq 2N \) implies that \( \varepsilon k \leq p \leq 2\varepsilon k \) so that \( (Cp/k)^p \leq (2Ce)^p \leq (2Ce)^{\varepsilon k} \) where we have assumed that \( 2Ce < 1 \). It follows that

\[
|P(\varepsilon, k)(a(\varepsilon, k)) - b(\varepsilon, k)| \leq N^2 C^2(2Ce)^{\varepsilon k} \leq (\varepsilon C)^2 k^2(2Ce)^{\varepsilon k} = O(e^{-k/C'})
\]

for a sufficiently large \( C' \). The proof of the second part is similar. \( \square \)

### 5.2 Estimates for holomorphic star-products

Let \( \text{End}(E) \) be the algebra of endomorphisms of a vector space \( E \). Let \( (\| \cdot \|_\ell, \ell \in \mathbb{N}) \) be a family of seminorms of \( \text{End} E \) satisfying

\[
\|P \circ Q\|_{p+q} \leq \|P\|_p \|Q\|_q, \quad \forall P, Q \in \text{End} E
\]

(36) for any \( p, q \in \mathbb{N} \). Consider a formal series \( \text{id} - \sum_{\ell \geq 1} \hbar^\ell F_\ell \) of \( (\text{End} E)[[\hbar]] \) with inverse \( \text{id} + \sum_{\ell \geq 1} \hbar^\ell G_\ell \).

**Lemma 5.5** If there exists \( C > 0 \) such that \( \|F_\ell\|_\ell \leq C^\ell \) for any \( \ell \), then there exists \( C' > 0 \) such that \( \|G_\ell\|_\ell \leq (C')^\ell \) for any \( \ell \).

We give two proofs, the first one is a direct generalization of the proof of Theorem 3.1, the second is inspired from [5,16].

**Proof** A first proof is to establish the formula

\[
G_m = \sum_{\ell \geq 1, (i_1, \ldots, i_\ell) \in \mathbb{Z}_+^\ell \atop i_1 + \cdots + i_\ell = m} F_{i_1} \ldots F_{i_\ell}
\]

(37) and we conclude easily by using that the number of terms in the sum is \( 2^{m-1} \). Another proof less precise but interesting as well is to introduce for any formal series \( R = \sum \hbar^\ell R_\ell \) and \( \rho > 0 \) the series \( f(R, \rho) = \sum \rho^\ell \|R_\ell\|_\ell. \) Then
1. \( f(R, \rho) \) converges for some \( \rho > 0 \) if and only there exists \( C > 0 \) such that \( \|R_\ell\|_\ell \leq C^{\ell+1} \) for any \( \ell \).
2. \( f(RS, \rho) \leq f(R, \rho) f(S, \rho) \) by (36).
3. if \( (R^{(n)})_n \) is a sequence of \( (\text{End } E)[[h]] \) such that \( R^{(n)} = O(h^n) \) for any \( n \), then \( f(\sum_n R^{(n)}, \rho) \leq \sum_n f(R^{(n)}, \rho) \) by triangle inequality.

Then we can argue as follows. Set \( R = \sum_{\ell \geq 1} h^\ell F_\ell \). The assumption on the \( F_\ell \)'s implies that \( f(R, \rho) = O(\rho) \), so \( f(R, \rho) \leq \delta < 1 \) when \( \rho \) is sufficiently small. Applying the previous properties we have

\[
f(\sum R^n, \rho) \leq \sum f(R^n, \rho) \leq \sum \delta^n < \infty
\]

which concludes the proof because \( \text{id} + \sum h^\ell G_\ell = \sum R^n \).

We apply this to holomorphic differential operators of an open set \( \Omega \) of \( \mathbb{C}^n \). Consider a formal series \( \text{id} - \sum_{\ell \geq 1} h^\ell F_\ell \) where for any \( \ell \),

\[
F_\ell = \sum_{|\alpha| \leq N\ell} a_{\ell,\alpha} \frac{1}{\alpha!} \partial^{\alpha}
\]

and the \( a_{\ell,\alpha} \)'s are holomorphic functions of \( \Omega \). Here \( N \) is any positive integer, \( N = 1 \) or 2 in our applications. Then the inverse \( \text{id} + \sum_{\ell \geq 1} h^\ell G_\ell \) has the same form \( G_\ell = \sum_{|\alpha| \leq N\ell} b_{\ell,\alpha} \frac{1}{\alpha!} g^{\alpha} \) as follows for instance from (37).

**Lemma 5.6** Assume that on any compact set \( K \) of \( \Omega \), there exists \( C_K \) such that \( |a_{\ell,\alpha}| \leq C^\ell K \ell^\ell \) for any \( \alpha, \ell \). Then the family \( (b_{\ell,\alpha}) \) satisfies the same estimates, with different constants \( C_K \).

**Proof** Let \( x_0 \in \Omega \) and \( 0 < t_0 \leq 1 \) be such that the closure of the ball \( B(x_0, t_0) \) is contained in \( \Omega \). For any \( 0 < s < t < t_0 \), define as in Sect. 2.3 \( \|P\|_{r,s} \) as the norm of the restriction \( P : B(B(x_0, t)) \rightarrow B(B(x_0, s)) \). Set

\[
\|P\|_\ell := \sup(\ell^{-\ell}(t-s)^\ell \|P\|_{r,s} / 0 < s < t < t_0).
\]

The submultiplicativity (36) follows from Lemma 2.4. By the first part of Lemma 2.4 and the assumption on the \( a_{\ell,\alpha} \)'s, we have \( \|F_\ell\|_\ell \leq C^\ell \). This implies by Lemma 5.5 that \( \|G_\ell\|_\ell \leq C^\ell \) with a larger \( C \). For any \( x \in B(x_0, t_0/2) \), the sup norm of \( (z-x)^\alpha \) on \( B(x_0, t_0) \) is smaller than \( (3t_0/2)^{|\alpha|} \leq (3/2)^\ell \). So

\[
|b_{\ell,\alpha}(x)| = |G_\ell((z-x)^\alpha)(x)| \leq (3/2)^\ell \|G_\ell\|_{t_0, t_0/2} \leq (3/2)^\ell \ell^\ell (t_0/2)^{-\ell} \|G_\ell\|_\ell \leq (3C/t_0)^\ell \ell^\ell.
\]

as was to be proved.

A first application is the proof of the claim in Remark 5.2. It is also easy to recover Theorem 3.1. A third application is to prove that the inverse for the product \( \star \) of an analytic elliptic symbol \( f = \sum h^\ell f_\ell \) is analytic. Here, elliptic means that \( f_0 \) does not vanish anywhere, and thus has an inverse \( f_0^{-1} \) for the usual product. The inverse \( g \) of \( f \) satisfies \( f \star g = \rho \) where \( \rho \) is the Bergman kernel symbol. So \( (f_0^{-1} \star f) \star g = f_0^{-1} \star \rho \). Now \( (f_0^{-1} \star f) \star g = (\text{id} - \sum h^\ell F_\ell) (g) \) and Theorem 2.3 implies that the coefficients of the \( F_\ell \)'s satisfy the assumption of Lemma 5.6. On the other hand \( f_0^{-1} \star \rho \) is analytic by Proposition 4.1. Thus \( g = (\text{id} + \sum h^\ell G_\ell) (f_0^{-1} \star \rho) \) is analytic by Lemma 5.6.
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