Abstract

At a linear collider of the next generation the large event rates expected from Bhabha and Møller scattering may be used to determine simultaneously $\sin^2 \theta_w$ and the polarization of both beams with very high accuracy. These measurements can be performed in parallel to the other tasks of the linear collider as a free by-product. A high degree of polarization and a good polar angle coverage of the detectors turn out to be major assets.
While the LHC offers an entry into the the high energy regime of the standard model with a significant opportunity for discovering new phenomena, the linear electron colliders of the next generation \[1\] will provide a complementary program of experiments with unique opportunities for both discoveries and precision measurements. A major asset to fulfill this purpose is the versatility of the linear colliders, as they can be operated in the four $e^+e^-$, $e^-e^-$, $e^−γ$ and $γγ$ modes, with highly polarized electron and photon beams. Moreover, starting from a center of mass energy of several hundred GeV, it will later be possible to upgrade these machines into the TeV range.

An important feature of the linear colliders is the high degree of polarization which can be obtained for the electron beams. Beam polarizations exceeding 80% are by now routinely obtained at SLAC and are steadily improving. A final 90% electron polarization seems a quite sensible assumption \[2\]. Concerning the positron beam, although at present no scheme for polarizing positrons has been proven to be implementable, there are reasonable hopes that some practicable technology may be available by the time a linear collider is operating. This ingredient is an important additional lever arm to increase the sensitivity of the searches for new phenomena, and the precision measurement which are the main goal of a linear collider. It is therefore of utter importance to be able to measure the degree of polarization with great accuracy.

We propose here a simple method to determine the polarization of both beams in $e^+e^-$ and $e^-e^-$ collisions \[3\]. This procedure takes advantage of the large cross sections of Bhabha and Møller scattering to obtain a good analyzing power, competitive with Compton polarimetry \[4\]. Moreover, as the polarizations are measured from the distributions of the final state electrons and positrons, we are guaranteed to take into account all depolarizing effects which can spoil the initial beam polarization at the interaction point. A similar procedure has been illustrated for the $Z^0$ peak in Ref. \[5\].

The interesting feature of this measurement is that it simultaneously provides a very accurate determination of $\sin^2 \theta_w$. At present, parity violating asymmetry measurements in $Z^0$ decays have allowed its most precise determination: combining the SLD measurement of the left-right asymmetries with the various asymmetries from LEP, the effective leptonic $\sin^2 \theta_w$ is now constrained to $0.2314 \pm 0.0003$ \[6\]. An early discussion of the determination of the weak mixing angle from Bhabha scattering at LEP1 can be found in \[7\]. After the end of operation of the $e^+e^-$ colliders on the $Z^0$ peak, the situation is unlikely to improve significantly, although interesting proposals have been put forward, for both low \[8\] and at high energy \[9\] experiments. It is therefore particularly interesting to study the potential of a high energy linear collider in this respect.

The typical linear collider designs aim at an integrated yearly $e^+e^-$ luminosity
\[ \mathcal{L} \] scaling with the squared center of mass energy \( s \) like

\[ \mathcal{L}_{e^+e^-} \ [\text{fb}^{-1}] \approx 80 \ s \ [\text{TeV}^2], \tag{1} \]

or \( \mathcal{L}_{e^+e^-} \approx 3 \times 10^7 \ s \) in \( c = \hbar = 1 \) units. For the luminosity of the \( e^-e^- \) mode we take

\[ \mathcal{L}_{e^-e^-} \approx \frac{1}{2} \mathcal{L}_{e^+e^-}, \tag{2} \]

because this mode will suffer to some extent from the anti-pin effect \[10]. \] If needed, it is straightforward to modify our results for a different scaling relation.

Using \( N^+ \) and \( N^- \) for the number of particles in a beam which are longitudinally polarized parallel or antiparallel to their momentum, we define the polarization of an electron or positron beam to be

\[ P_{e^-} = \frac{N^+ - N^-}{N^+ + N^-}, \quad P_{e^+} = \frac{N^- - N^+}{N^+ + N^-}. \tag{3} \]

With this definition \( P = +\langle h \rangle \) for electrons and \( P = -\langle h \rangle \) for positrons, where \( \langle h \rangle \) is the helicity mean value.

We assume the integrated luminosities to be equally distributed over the four possible combinations of beam polarizations \( LL, RR, LR \) and \( RL \) (with \( R \) and \( L \) referring to positive and negative polarizations of the beams, respectively). From the physics point of view there is no difference between the last two combinations in the \( e^+e^- \) mode. However, since the electron guns may have different efficiencies, it is important to consider them both in order to measure this hardware asymmetry. It is essential that the polarization of the beams be flipped randomly at short time intervals, a technique in use at SLC \[11]. \] In this case, if the absolute value of the polarization is on average constant, random and systematic fluctuations cancel out.

Neglecting the \( Z^0 \) width, the polarized differential Bhabha and Møller scattering cross sections are

\[ \frac{d\sigma_{e^+e^-}}{dt} = \frac{4\pi\alpha^2}{s^2} \times \]

\[ \left\{ \frac{1 + P_1 + P_2 + P_1P_2}{4} \right\} \left[ \left( \sum_i R_i^2 \left( \frac{u}{s - m_i^2} + \frac{u}{t - m_i^2} \right) \right)^2 + \left( \sum_i L_iR_i \frac{t}{s - m_i^2} \right)^2 \right] \tag{4} \]
\[
\frac{d\sigma^{e^-e^-}}{dt} = \frac{2\pi\alpha^2}{s^2} \times \\
\left\{ \begin{aligned}
\frac{1 + P_1 + P_2 + P_1 P_2}{4} & \left( \sum_i R_i^2 \left( \frac{s}{t - m_i^2} + \frac{u}{u - m_i^2} \right) \right)^2 \\
\frac{1 - P_1 - P_2 + P_1 P_2}{4} & \left( \sum_i L_i^2 \left( \frac{s}{t - m_i^2} + \frac{u}{u - m_i^2} \right) \right)^2 \\
\frac{1 - P_1 P_2}{2} & \left[ \left( \sum_i L_i R_i \frac{t}{s - m_i^2} \right)^2 + \left( \sum_i L_i R_i \frac{u}{t - m_i^2} \right)^2 \right] 
\end{aligned} \right\},
\]

where \( \alpha \) is the fine structure constant, \( P_1 \) stands for the positron polarization in the case of Bhabha scattering, \( s, t, u \) are the Mandelstam variables, the summations are over \( i = \gamma, Z^0 \) and the couplings are defined by

\[
R_\gamma = L_\gamma = 1 , \quad R_Z = -\frac{\sin \theta_w}{\cos \theta_w} , \quad L_Z = \frac{1 - 2\sin^2 \theta_w}{2\sin \theta_w \cos \theta_w} .
\]

It is our purpose to use these cross sections to determine as precisely as possible the values of the weak mixing angle and the polarization of each beam:

\[
\sin^2 \theta_w \quad P_1 \quad P_2 .
\]

The most natural choice of observables for determining these parameters are the differential events rates

\[
\delta n_{LL} \quad \delta n_{RR} \quad \delta n_{LR} \quad \delta n_{RL} ,
\]

where for each bin

\[
\delta n = \mathcal{L} \int d \cos \theta \frac{d\sigma}{d\cos \theta} .
\]

However, the experimental determination of the absolute cross sections is hindered by the systematic error on luminosities, acceptances and efficiencies, which dominate the statistical errors when the event rates are as large as in
Bhabha and Møller scattering. It is therefore of great advantage to use three independent differential polarization asymmetries, for example

\[ A_1 = \frac{\delta n_{LL} - \delta n_{RR}}{\delta n_{LL} + \delta n_{RR}} \]  
(10)

\[ A_2 = \frac{\delta n_{RR} - \delta n_{LR}}{\delta n_{RR} + \delta n_{LR}} \]  
(11)

\[ A_3 = \frac{\delta n_{LR} - \delta n_{RL}}{\delta n_{LR} + \delta n_{RL}} \]  
(12)

for which the systematic errors cancel out to a very large extent. As long as the correlations between the three asymmetries are correctly taken into account and the statistical errors dominate, it does not matter which triplet of independent asymmetries is chosen. Any choice other than (10–12) yields the same results.

We have chosen to normalize the $Z^0$ couplings by the fine structure constant $\alpha$. In this way the asymmetries depend solely on $\sin^2 \theta_w$ and the beam polarizations, which effectively parametrize all the available information. Moreover, when it will come to compute the radiative corrections, in the framework of an $\overline{MS}$ scheme [12] this choice has the additional advantage of avoiding large electroweak corrections, such as $m_t^2$ corrections.

We have checked that the accuracy with which the parameters (7) can be measured is such that we can safely assume a linear dependence of the cross sections (4,5) in the region of interest, i.e., within the error bands around the central values. The error bands corresponding to one standard deviation are therefore given by the quadratic form

\[
\begin{pmatrix}
\Delta \sin^2 \theta_w & \Delta P_1 & \Delta P_2
\end{pmatrix}
W^{-1}
\begin{pmatrix}
\Delta \sin^2 \theta_w \\
\Delta P_1 \\
\Delta P_2
\end{pmatrix}
= 1 ,
\]  
(13)

where the inverse covariance matrix $W^{-1}$ is given by

\[
W_{ij}^{-1} = \sum_{k,l=1}^{3} \sum_{\text{bins}} V_{kl}^{-1} \left( \frac{\partial A_k}{\partial \epsilon_i} \right) \left( \frac{\partial A_l}{\partial \epsilon_j} \right) 
\]  
(14)

\[
\epsilon_i = \sin^2 \theta_w , P_1 , P_2 .
\]  
(15)

In contrast to the polarized cross sections, the asymmetries are correlated.
Their covariance matrix $V$ contains therefore off-diagonal terms and is given by

$$V_{kl} = \langle (A_k - \bar{A}_k)(A_l - \bar{A}_l) \rangle,$$

$$= \sum_{i=1}^{4} (\Delta n_i)^2 \left( \frac{\partial A_k}{\partial n_i} \right) \left( \frac{\partial A_l}{\partial n_i} \right) + (\Delta \theta)^2 \left( \frac{\partial A_k}{\partial \theta} \right) \left( \frac{\partial A_l}{\partial \theta} \right),$$

$$n_i = \delta n_{LL}, \delta n_{RR}, \delta n_{LR}, \delta n_{RL},$$

where the statistical errors originating from the uncorrelated polarized event rates in each bin are given by

$$\Delta n_i = \sqrt{n_i},$$

whereas the systematic error (second term in Eq. (16)) stems from the inaccurate measurement of the scattering angle. A realistic value that we employ in our analysis is

$$\Delta \theta = 0.5 \, \text{mrad}.$$  

Since the small angle singularities of the differential cross sections cancel out in the asymmetries, the latter have a rather smooth angular dependence. As a result, the contribution of the second term in Eq. (16) is almost negligible.

The quadratic form (13) defines a 3-dimensional ellipsoid in the $(\sin^2 \theta_w, P_1, P_2)$ parameter space. The inverse square root of the diagonal elements of the inverse covariance matrix $W^{-1}$ are the values of the intersections of the error ellipsoid with the corresponding parameter axes. These correspond to the one-standard-deviation errors on this parameter, assuming the other two parameters are known exactly. In contrast, the square roots of the diagonal elements of the covariance matrix $W$ are the values of the projections of this ellipsoid onto the corresponding parameter axes. These correspond to the one-standard-deviation errors on this parameter, whatever values the other two parameters assume. When presenting our results we choose the latter for our predictions of the errors on $\sin^2 \theta_w$ and the beam polarizations.

Unless stated otherwise, we assume from now on the following values for the expectation values of the parameters and the angular acceptance of the detec-
\[
\begin{align*}
\sin^2 \theta_w &= .2315 \\
P_1 &= P_2 = 90\% \\
|\cos \theta| &< .995
\end{align*}
\] (20)

To take into account the angular dependence of the asymmetries, we have chosen to work with 200 equal size bins in \(\cos \theta\) over the angular range \((20)\). This is easy to implement experimentally, as the scattering angles can be measured with very high accuracy \((19)\). Since the asymmetries have a relatively smooth angular behaviour, increasing the number of bins beyond 50 does not significantly improve the accuracy of the measurement. We have checked that, as expected, the results approach very closely the Cramér-Rao minimum variance bound \([13]\).

For the sake of illustration, we have plotted in Fig. 1 the 3-dimensional ellipsoid defined by the quadratic form \((13)\) for the \(e^+e^-\) experiment at 500 GeV. This figure provides some interesting insight. For instance, it is clear that the two polarization measurements are highly correlated, in the sense that the average polarization can be determined much more precisely than the polarization difference of the two beams. In contrast, \(\sin^2 \theta_w\) is only weakly correlated to the beam polarizations.

Because we assume the luminosities to scale proportionally to the square of the collider energy \((1,2)\), the resolution of the measurement improves at higher energies. This is displayed in Fig. 3, where we plot the center of mass energy dependence of the one standard deviation errors on the measurements of \(\sin^2 \theta_w\) and the electron beam polarization. We observe a clear saturation beyond 1 TeV for both Bhabha and Møller scattering.

At \(\sqrt{s} = 500\) GeV, \(\sin^2 \theta_w\) can be measured with an error of about \(4 \times 10^{-4}\). Although this will not improve the combined LEP-SLC accuracy, it may provide an independent check. On the other hand, at 2 TeV the resolution on \(\sin^2 \theta_w\) can reach up to \(1 \times 10^{-4}\). Similarly, the polarization can be determined at 500 GeV down to 2\% in Bhabha and 1.5\% in Møller scattering. Compton polarimetry currently yields a similar accuracy of 1.7\% \([4]\) and is constantly improving. However, at 2 TeV both Bhabha and Møller scattering can measure the polarization down to 0.5\%, a very promising result.

As we mainly rely on the \(\gamma - Z^0\) interferences to measure \(\sin^2 \theta_w\), it is essential to probe small scattering angles. This is well depicted in Fig. 3, where we display the errors as a function of the polar angle coverage. The slight decrease in sensitivity observed for very small polar angle is due to the finite bin size. Improving the angular coverage beyond 5\(^\circ\) does not appear to be very useful.
The error on the polarization is not very sensitive to the detector acceptance, especially for Möller scattering.

The dependence of the errors on the polarization of both beams is displayed in Fig. [I]. Clearly, high degrees of polarization are an important asset, especially at lower energies. This should not present any problem for the electron beams and the Möller scattering experiment. To gauge, however, the importance of the positron polarization in Bhabha scattering, we plot in Fig. [I] the errors as a function of the positron beam polarization. It appears that at 500 GeV the resolution degrades significantly for positron polarizations less than 50%. For 2 TeV collisions positron polarizations as small as 30% still yield interesting results.

In the event the positrons cannot be polarized at all, a strong correlation develops between $\sin^2 \theta_w$ and the electron polarization so that these two parameters remain effectively unconstrained. Still, $\sin^2 \theta_w$ can be determined accurately if the electron polarization is also known precisely from the onset (from Compton polarimetry for instance) and its resolution is treated as a systematic error. In this case we observe in Fig. [I] that the resolution on $\sin^2 \theta_w$ is approximately degraded by as little as a factor $\sqrt{2}$. The systematic error stemming from the measurement of the electron polarization is not very important.

The bounds to be obtained for a few realistic energies and polarizations are summarized in Table [1]. They assume of course the validity of the luminosities stated in Eqs (1,2). For different values of the integrated luminosity the results can be easily corrected, since the statistical errors largely dominate the systematic errors included here and scale like $1/\sqrt{L}$. We also note that, with respect to the processes studied here, the $e^+e^- \rightarrow \mu^+\mu^-$ mode yields much less interesting bounds, about one order of magnitude worse. This is obviously expected because of the absence of the forward Coulomb peak in this case.

The present preliminary analysis has been carried out at the tree level only. Electroweak radiative corrections to Bhabha scattering off the $Z^0$ peak have been first calculated in [14], and updated to leading two-loop order in [15]. In general, electroweak corrections can be included in the Bhabha amplitudes by means of three complex-valued gauge invariant form factors explicitly depending on $\theta$ [15]. We expect a similar factorization of radiative corrections in Möller scattering, for which a full one-loop computation, to the best of our knowledge, is still missing at high energies [8]. The inclusion of these calculable radiative effects and a discussion of the problems connected to matching the required accuracy of less than one permille are beyond the scope of this paper, but they should not affect significantly our estimates of the statistical error, particularly because they are dominated by events in the forward peak, where electroweak corrections become less relevant. QED effects are generally quite sizable in large angle Bhabha and Möller scattering [13], and could in
principle introduce additional uncertainties. However, soft photons and other
QED effects factorize and cancel in the parity violating asymmetries, and we
do not expect dramatic effects on our error estimates.

To conclude, we have demonstrated how the large Bhabha and Møller scattering
cross sections can be advantageously used at a high energy linear collider
to measure the polarization of the incoming electron or positron beams down
to the percent level or better. The method we propose measures the polar-
ization of the interacting beams through the final states, so that it takes into
account all depolarizing effects due to beamstrahlung and disruption.

Simultaneously, the value of \( \sin^2 \theta_w \) can be determined in 500 GeV collisions
with an absolute error of about \( 4 \times 10^{-4} \). This error can be further reduced
down to almost \( 1 \times 10^{-4} \), by increasing the center of mass energy up to 2
TeV. Beyond this energy, however, there is little gain unless the luminosity is
increased with respect to Eq. (1,2).

These precision measurements can be easily carried out and do not interfere
with the main tasks of the linear collider. To reach the abovementioned accuracies, though, it is essential to have a good polar angle coverage of the
detector as well as highly polarized beams.

If electron and positron beams can be polarized with the same efficiency,
both Bhabha and Møller scattering yield very similar results. At high en-
ergies Bhabha scattering performs marginally better, because of the higher
luminosity of the \( e^+e^- \) mode with respect to the \( e^-e^- \) mode (2). However, if
positron beams cannot be polarized, the resolving power of Bhabha scattering
is approximately reduced by 30%.

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One standard deviation error bounds on the measurements of $\sin^2 \theta_w$ and the beam polarizations in Bhabha and Møller scattering for various values of energy and polarization. In the case of Bhabha scattering $P_1$ stands for the positron polarization. When the positrons are not polarized, the polarization of the electrons is assumed to have been determined with a precision of 1.5\% by other means (*).

| reaction               | $\sqrt{s}$ [TeV] | $P_1$ | $P_2$ | $\Delta \sin^2 \theta_w \times 10^4$ | $\Delta P_1 / P_1$ [%] | $\Delta P_2 / P_2$ [%] |
|------------------------|------------------|------|------|-----------------------------------|--------------------------|--------------------------|
| $e^+e^- \rightarrow e^+e^-$ | .5               | 0    | .9   | 5.2                               | –                        | 1.5\*                    |
|                         |                  | .3   | .9   | 8.5                               | 4.5                      | 4.7                      |
|                         |                  | .6   | .9   | 5.0                               | 2.5                      | 2.5                      |
|                         |                  | .9   | .9   | 4.0                               | 1.9                      | 1.9                      |
| $e^-e^- \rightarrow e^-e^-$ | .5               | .9   | .9   | 4.1                               | 1.4                      | 1.4                      |
| $e^+e^- \rightarrow e^+e^-$ | 2                | 0    | .9   | 1.9                               | –                        | 1.5\*                    |
|                         |                  | .3   | .9   | 2.4                               | 1.2                      | 1.3                      |
|                         |                  | .6   | .9   | 1.5                               | .65                      | .65                      |
|                         |                  | .9   | .9   | 1.2                               | .45                      | .45                      |
| $e^-e^- \rightarrow e^-e^-$ | 2                | .9   | .9   | 1.6                               | .55                      | .55                      |

Table 1

10
Fig. 1. One standard deviation error bounds on the measurement of $\sin^2 \theta_w$ and the beam polarizations for Bhabha scattering at 500 GeV center of mass energy.
Fig. 2. Energy dependence of the errors on $\sin^2 \theta_w$ and the beam polarizations in Møller and Bhabha scattering.
Fig. 3. Polar angle acceptance dependence of the errors on $\sin^2 \theta_w$ and the beam polarizations in Møller and Bhabha scattering. The upper and lower pairs of curves correspond to 500 GeV and 2 TeV center of mass energy collisions.
Fig. 4. Polarization dependence of the errors on $\sin^2 \theta_w$ and the beam polarizations in Møller and Bhabha scattering. The upper and lower pairs of curves correspond to 500 GeV and 2 TeV center of mass energy collisions.
Fig. 5. Positron polarization dependence of the errors on $\sin^2 \theta_w$ and the electron beam polarization in Bhabha scattering. The upper and lower curves correspond to 500 GeV and 2 TeV center of mass energy collisions.
Fig. 6. Electron polarization dependence of the errors on $\sin^2 \theta_w$ in Bhabha scattering. The positron beam is unpolarized and the polarization of the electrons is assumed to have been determined with a precision of 1.5% by other means. The upper and lower curves correspond to 500 GeV and 2 TeV center of mass energy collisions. The dotted curves indicate the expectations with 1%, 0.5% and no error on the electron polarization. They can almost not be resolved on this scale from the 500 GeV curve.