PARTITIONS INTO PRIME POWERS AND OTHER RESTRICTED PARTITION FUNCTIONS

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Abstract. For a subset $A \subset \mathbb{N}$, let $p_A(n)$ denote the restricted partition function, which counts partitions of $n$ with all parts lying in $A$. In this paper, we use a variation of the Hardy-Littlewood circle method to provide an asymptotic formula for $p_A(n)$, where $A$ is the set of $k$-th powers of primes (for fixed $k$). This combines Vaughan’s work on partitions into primes with the author’s previous result about partitions into $k$-th powers. This new asymptotic formula is an extension of a pattern indicated by several results about restricted partition functions over the past few years. Comparing these results side-by-side, we discuss a general strategy by which one could analyze $p_A(n)$ for a given set $A$.

1. Introduction and background

A partition of a number $n$ is a non-increasing sequence of positive integers whose sum is equal to $n$. The number of partitions of $n$ is denoted by the partition function $p(n)$. The asymptotic study of partitions began in 1918 with the seminal result of Hardy and Ramanujan, showing that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2}{3}} n^{\frac{1}{2}} \right),$$

as $n \to \infty$. For a subset $A \subset \mathbb{N}$, we let $p_A(n)$ denote the restricted partition function, which counts partitions of $n$ with all parts lying in $A$. In this paper, we study partitions into $k$-th powers of primes. The main result (Theorem 1.3) is an asymptotic formula for the number of such partitions. Because of the sparsity and irregularity of powers of primes, the formula is given in terms of quite complicated auxiliary functions. The result can be simplified to the following asymptotic equivalence:

**Theorem 1.1.** Fix $k \in \mathbb{N}$ and let $P_k = \{p^k : p \text{ prime} \}$. There exist positive constants $C_1, C_2$, depending only on $k$, such that the number of partitions of $n$ with all parts lying in $P_k$ satisfies

$$p_{P_k}(n) \sim C_1 n^{-\frac{2k+1}{2k+2}} (\log n)^{-\frac{k}{2k+2}} \exp \left( C_2 \frac{n^{1/k+1}}{(\log n)^{1/k+1}} (1 + o(1)) \right),$$

as $n \to \infty$.

**Remark.** This may seem to be an unusual statement, since it is possible that the negative powers of $n$ and $\log n$ may be dwarfed by the error term in the exponential. We state the result this way in order to illustrate the similarity in form between this asymptotic and the

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*Date:* October 8, 2020.

*2010 Mathematics Subject Classification.* 11P55, 11P82.

*Key words and phrases.* partitions, restricted partition functions, prime powers, exponential sums, Hardy-Littlewood circle method.
results below. The error term in the exponential is a result of the fact that the prime number theorem cannot be expressed in terms of elementary functions.

1.1. A pattern of results. Theorem [1] is an extension of a pattern indicated by several results over the past few years. In a previous paper, the author proved the following asymptotic for partitions into \( k \)-th powers:

**Theorem.** [1] Let \( A_k = \{x^k : x \in \mathbb{N}\}. \) Then

\[
p_{A_k}(n) \sim C_1 \exp \left( C_2 n^{\frac{1}{k+1}} \right) n^{-\frac{3k+1}{2(k+1)}},
\]

where \( C_1, C_2 \) are positive constants depending only on \( k \).

Berndt, Malik, and Zaharescu generalized that result to partitions into \( k \)-th powers in a residue class:

**Theorem** (Berndt-Malik-Zaharescu, [2]). Let \( A_{k,(a,b)} = \{x^k : x \equiv a \pmod{b}, x > 0\}. \) Then

\[
p_{A_{k,(a,b)}}(n) \sim C_1 \exp \left( C_2 n^{\frac{1}{k+1}} \right) n^{-\frac{6bk+2a}{20(k+1)}}
\]

where \( C_1, C_2 \) are positive constants depending only on \( k, a, \) and \( b \).

The sets \( A_k \) and \( A_{k,(a,b)} \) can be thought of as integer values of the polynomials \( x^k \) and \( (bx+a)^k \), respectively. Dunn and Robles further extended this idea to study partitions into integer values of a polynomial:

**Theorem** (Dunn-Robles, [3]). Let \( f \) be a polynomial and let \( A_f = \{f(x) : x \in \mathbb{N}\}. \) If \( A_f \subset \mathbb{N} \) and \( \gcd(A_f) = 1 \), then

\[
p_{A_f}(n) \sim C_1 \exp \left( C_2 n^{\frac{1}{d+1}} \right) n^{-\frac{2d(1-\zeta(0,\alpha)+1)}{2(d+1)}}
\]

where \( d = \deg(f) \), \( \zeta(0,\alpha) \) is a value of an appropriate Matsumoto-Weng \( \zeta \) function, and \( C_1, C_2 \) are positive constants depending only on the polynomial \( f \).

We can also consider \( p_A(n) \) for sets that are not induced by polynomials. In 2008, Vaughan [8] proved the following result about partitions into primes:

**Theorem** (Vaughan, [8]). Let \( A = \mathbb{P} \) be the set of primes. Then

\[
p_{\mathbb{P}}(n) \sim C_1 \exp \left( C_2 \frac{n^{\frac{1}{2}}}{(\log n)^{\frac{7}{2}}} (1 + o(1)) \right) n^{-\frac{3}{2}(\log n)^{-\frac{1}{2}}}
\]

where \( C_1, C_2 \) are positive constants.

**Remark.** Different constants \( C_1, C_2 \) are used in each of the results above. The original papers state more explicit versions of the asymptotic formulae, and it is possible to compute the constants from those theorems. We omit the explicit expressions for \( C_1, C_2 \) here because they are quite complicated and do not provide significant insight to this discussion.

\(^1\)The case \( k = 2 \) was proved by Vaughan [9].
Comparing the above results we see that the dominant term of $\log p_A(n)$ is given only in terms of the growth of $A$. Indeed, for partitions into polynomial values we have

$$\log p_A(n) \sim C_2 n^{\frac{1}{d+1}} = C_2 (n^{\frac{1}{d}})^{\frac{d}{d+1}},$$

and for partitions into powers of primes we have

$$\log p_P(n) \sim C_2 \frac{n^{\frac{1}{d+1}}}{(\log n)^{\frac{d}{d+1}}} = C \left( \frac{n^{\frac{1}{d}}}{\frac{d}{d} \log n} \right)^{\frac{d}{d+1}}.$$

All of these results are proved using the same variation of the Hardy-Littlewood circle method. In Section 2 we outline the common technique and discuss what information is needed to prove an analogous result for a given set $A$.

It should be noted that the results above are not the first formulas for restricted partition functions. Throughout the 20th Century, restricted partitions were studied extensively by a number of mathematicians, including Wright [10], Roth and Szekeres [6], and Bateman and Erdős [1]. The generality and strength of these results vary, as do the methods employed. The results stated above are highlighted because they are proved using a parallel framework and they exhibit the possibility of a pattern for the analysis of other restricted partition functions.

1.2. The full asymptotic formula. The generating function for partitions into $k$-th powers of primes is

$$\Psi(z) = \sum_{n \geq 0} p_P(n) z^n = \prod_{p \text{ prime}} (1 - z^{p^k})^{-1}.$$

It will be more convenient to write this as

$$\Psi(z) = \exp(\Phi(z)),$$

where

$$\Phi(z) = \sum_{j=1}^{\infty} \sum_{p \text{ prime}} \frac{1}{j} z^{jp^k}.$$

By Cauchy’s integral formula, we have

$$p_P(n) = \rho^{-n} \int_{-1/2}^{1/2} \Psi(\rho e(\alpha)) e(-n\alpha) d\alpha = \rho^{-n} \int_{-1/2}^{1/2} \exp(\Phi(\rho e(\alpha))) e(-n\alpha) d\alpha,$$

for any positive real number $\rho < 1$. Let $x \in \mathbb{R}$ be large. (Eventually we will set $x = n$.) We choose $\rho = \rho(x)$ so that

$$x = \rho \phi(x).$$

It will follow from Lemma 3.1 that the relationship between $x$ and $\rho$ is well-defined and injective, and that $\rho \to 1^{-}$ as $x \to \infty$.

In order to interpret the result of Theorem 1.3 and see that it implies Theorem 1.1 we will need estimates for the auxiliary functions involved.

**Proposition 1.2.** As $x \to \infty$, we have

$$x \log \frac{1}{\rho(x)} = \left( \frac{k+1}{k} \zeta(k+1) \Gamma(k+1) x^k \right)^{\frac{1}{k+1}} \left( 1 - \frac{k}{k+1} \frac{\log \log x}{\log x} + O \left( \frac{1}{\log x} \right) \right).$$
\begin{equation}
(1.4) \quad \Phi(\rho(x)) = k \left( \frac{k+1}{k} \zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{k+1}{k} \right) x^{\frac{k}{k+1}} \right) \left( 1 - \frac{k}{k+1} \log \log x + O \left( \frac{1}{\log x} \right) \right),
\end{equation}

and, for \( m \geq 1 \),
\begin{equation}
(1.5) \quad \Phi_m(\rho(x)) = x^{\frac{mk+1}{k+1}} \left( \frac{\log x}{\zeta \left( \frac{k+1}{k} \right) \Gamma \left( \frac{k+1}{k} \right)} \right)^{\frac{k(m-1)}{k+1}} \frac{\Gamma(m + \frac{1}{k})}{\Gamma(1 + \frac{1}{k})} \left( 1 + O \left( \frac{\log \log x}{\log x} \right) \right),
\end{equation}

where
\[ \Phi_m(\rho) = \left( \frac{d}{d\rho} \right)^m \Phi(\rho). \]

We are now ready to state the asymptotic formula.

**Theorem 1.3.** Using the notation defined above with \( \rho = \rho(n) \), we have
\[ p_{\varphi_k}(n) = \frac{\rho^{-n} \Psi(\rho)}{\sqrt{2\pi \Phi(\rho)}} \left( 1 + O(n^{-\frac{1}{2k+3}}) \right). \]

**1.3. Proof of Theorem 1.1 given Theorem 1.3.** By Proposition 1.2 we have
\[ \rho^{-n} \Psi(\rho) = \exp \left( n \log \frac{1}{\rho(n)} + \Phi(\rho(n)) \right) \]
\[ = \exp \left( (k+1) \left( \zeta(1 + \frac{1}{k}) \Gamma(2 + \frac{1}{k}) \right) \frac{n^{\frac{1}{k+1}}}{(\log n)^{\frac{1}{k+1}}} (1 + o(1)) \right), \]

and

\[ \sqrt{\Phi(\rho(n))} = n^{\frac{2k+1}{2k+2}} \left( \frac{\log n}{\zeta(1 + \frac{1}{k}) \Gamma(2 + \frac{1}{k})} \right)^{\frac{k}{2k+2}} \left( 1 + \frac{1}{k} \right)^{\frac{1}{2}} (1 + o(1)). \]

Therefore, Theorem 1.3 implies that
\[ p_{\varphi_k}(n) = \frac{\rho^{-n} \Psi(\rho)}{\sqrt{2\pi \Phi(\rho)}} \left( 1 + O(n^{-\frac{1}{2k+3}}) \right) \]
\[ \sim C_1 n^{-\frac{2k+1}{2k+2}} (\log n)^{-\frac{k}{2k+2}} \exp \left( C_2 \frac{n^{\frac{1}{k+1}}}{(\log n)^{\frac{1}{k+1}}} (1 + o(1)) \right) \]

where
\[ C_1 = \sqrt{2\pi} \left( \zeta(1 + \frac{1}{k}) \Gamma(2 + \frac{1}{k}) \right)^{\frac{k}{2k+2}} \left( 1 + \frac{1}{k} \right)^{\frac{1}{2}}, \quad C_2 = (k+1) \left( \zeta(1 + \frac{1}{k}) \Gamma(2 + \frac{1}{k}) \right)^{\frac{k}{k+1}}. \]

**1.4. The difference function.** The methods used to prove the asymptotic formula in Theorem 1.3 can also be used to estimate the growth of \( p_{\varphi_k}(n) \). This yields the following:

**Theorem 1.4.** Using the notation defined above with \( \rho = \rho(n) \), we have
\[ p^k(n+1) - p^k(n) \sim \frac{\rho^{-n} \log(\frac{1}{\rho}) \Psi(\rho)}{\sqrt{2\pi \Phi(\rho)}} \left( 1 + O(n^{-\frac{1}{2k+3}}) \right). \]

From Theorem 1.4 and Proposition 1.2 we immediately deduce an asymptotic equivalence:
Corollary 1.1. Let $\rho(n)$ be as defined above. Then
\[ p^k(n+1) - p^k(n) \sim \left( \left( 1 + \frac{1}{k} \right) \zeta \left( 1 + \frac{1}{k} \right) \Gamma \left( 1 + \frac{1}{k} \right) \right)^{1/k} \frac{p_{\mathcal{A}}(n)}{(n \log n)^{1/k+1}}. \]
as $n \to \infty$.

1.5. Organization and notation. In Section 2 we give an outline of a method to study $p_{\mathcal{A}}(n)$ for a given set $\mathcal{A}$. Section 3 provides an analysis of the function $\rho(x)$ and a proof of Proposition 1.2. The main result (Theorem 1.3) is proved in Sections 4, 5, and 6. We prove Theorem 1.4 in Section 7. Section 8 contains some additional analysis of the exponential sum $S_k^*(q,a)$, included for the sake of completeness.

We use the the Vinogradov notation $f \ll g$ to mean that there exists a positive constant $C$ such that $|f| \leq C|g|$. We write $f = g + O(h)$ to denote that $|f - g| \ll h$. The asymptotic equivalence $f \sim g$ means that $\lim f/g = 1$. We also use the standard notation $e(\alpha) = e^{2\pi i \alpha}$.

Acknowledgements. The author is grateful to Amita Malik for many fruitful conversations related to this paper.

2. A general method for studying restricted partitions

The results listed in Section 1 exhibit a pattern for the growth of restricted partition functions in general. In fact, the proofs of these results are parallel in many respects and give way to a method by which one could analyze $p_{\mathcal{A}}(n)$ for a range of different sets $\mathcal{A}$. In this section we summarize that method and identify the information about $\mathcal{A}$ required to implement the method.

The technique is based on the Hardy-Littlewood circle method (See [7]). The generating function for $p_{\mathcal{A}}(n)$ is
\[ \Psi_{\mathcal{A}}(z) := \sum_{n=0}^{\infty} p_{\mathcal{A}}(n) z^n = \prod_{a \in \mathcal{A}} (1 - z^a)^{-1}. \]

It is more convenient to work with an infinite sum than an infinite product, so we write
\[ \Psi_{\mathcal{A}}(\rho e^{2\pi i \theta}) = \exp(\Phi_{\mathcal{A}}(e^{-1/X} e(\theta))), \]
where $\rho = e^{-1/X}$ and
\[ \Phi_{\mathcal{A}}(z) = \sum_{j=1}^{\infty} \sum_{a \in \mathcal{A}} z^{aj} \bar{z}^{aj}. \]

We then see that
\[ (2.1) \quad p_{\mathcal{A}}(n) = \int_0^1 \rho^{-n} \exp(\Phi_{\mathcal{A}}(\rho e(\theta))) e(\theta) d\theta. \]

In a typical implementation of the circle method, one would split the integral (2.1) into the major arcs and the minor arcs, and all of the major arcs would contribute main term of the asymptotic formula. However, in the case of restricted partitions functions, the contribution from the major arc at the origin is significantly greater than the contribution from the rest of the major arcs. So, we split the integral into three main parts, namely
\[ p_{\mathcal{A}}(n) = \left\{ \int_{\mathfrak{M}(1,0)} + \int_{\overline{\mathfrak{N}}(1,0)} + \int_{\mathfrak{M}} \right\} \rho^{-n} \exp(\Phi_{\mathcal{A}}(\rho e(\theta))) e(\theta) d\theta. \]
We treat the remaining major arcs in the traditional way, but the major arcs \( M(q, a) \) with \( q > 1 \) do not contribute to the main term of the asymptotic formula. Rather they are “thrown away” into the error term. The main term of the asymptotic formula comes exclusively from the first part of the integral, when \( \theta \) is close to the origin.

To analyze (2.1), we need to understand the behavior of \( \Phi_A(\rho e(\theta)) \) on the principal major arc \( M(1, 0) \), the non-principal major arcs, and the minor arcs. Each of these settings requires different analytic information about the set \( A \).

To evaluate the integral (2.1) on the principal major arc, we need a precise estimate for

\[
\Phi_A(\rho) = \sum_{j=1}^{\infty} \sum_{a \in A} \frac{1}{j} e^{-ja/X}.
\]

Using a Mellin transform and interchanging the sums with the integral, we have

\[
\Phi_A(\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \zeta(s+1) X^s \left( \sum_{a \in A} a^{-s} \right) ds.
\]

The main term of the formula for \( p_A(n) \) comes from residue analysis of this integral over an appropriate contour. To obtain this, we need analytic information about about the Dirichlet series

\[
f_A(s) := \sum_{a \in A} a^{-s},
\]

including convergence properties, analytic continuation, zeros, singularities, and residues at poles.

To handle the non-principal major arcs, we need an estimate for

\[
\Phi_A(\rho e\left(\frac{r}{q} + \beta\right)) = \sum_{j=1}^{\infty} \sum_{a \in A} \frac{1}{j} e^{\left(\frac{ajr}{q}\right)} \exp (aj(2\pi i\beta - 1/X))
\]

\[
= \sum_{j=1}^{\infty} \sum_{q \in \mathbb{Z}} \sum_{\ell=1}^{q} e\left(\frac{rj\ell}{q}\right) \sum_{a \in A \ (\text{mod } q)} \exp (aj(2\pi i\beta - 1/X)).
\]

To analyze (2.3), we need to understand the distribution of \( A \) in residue classes.

Finally, on the minor arcs we encounter

\[
\Phi_A(\rho e(\theta)) = \sum_{j=1}^{\infty} \sum_{a \in A} \frac{1}{j} e^{-a j / X} e(ja\theta)
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{j} \int_0^{\infty} a j X^{-1} e^{-ajy/X} \sum_{a \leq y a \in A} e(ja\theta) dy.
\]

In order to analyze this further, we need an estimate for the Weyl sum

\[
(2.4) \quad \sum_{a \leq y a \in A} e(ja\theta).
\]

All together, we need the following ingredients in order to analyze \( p_A(n) \):

1. Analytic information about the Dirichlet series over \( A \) in (2.2),
(2) distribution of \( A \) in residue classes,
(3) bounds for the Weyl sum over \( A \) in \( (2.4) \).

If we understand these three ingredients for a given set \( A \), then we can analyze \( p_A(n) \). However, this information is quite elusive and it is rare to find a set \( A \) for which we have a good understanding of all three pieces. As we will see in the proof of Theorem 1.3, the current state of knowledge about the distribution of prime powers is only just sufficient to make this process work for \( A = \mathbb{P}_k \).

3. PROOF OF PROPOSITION 1.3

In this section we obtain estimates on \( \Phi(\rho) \) that will be necessary for the main result. For convenience, we define a parameter \( X = \frac{1}{\log \rho} \), so that \( \rho = e^{-1/X} \).

**Lemma 3.1.** Let \( \rho = e^{-1/X} \). Then, for \( m \in \mathbb{Z}_{\geq 0} \), we have

\[
(3.1) \quad (\rho \frac{d}{d\rho})^m \Phi(\rho) = \frac{\zeta(1 + \frac{1}{k})\Gamma(m + \frac{1}{k})X^{m+\frac{1}{k}}}{\log X} \left(1 + O\left(\frac{1}{\log X}\right)\right)
\]

and

\[
(3.2) \quad \Phi^{(m)}(\rho) = \frac{\zeta(1 + \frac{1}{k})\Gamma(m + \frac{1}{k})X^{m+\frac{1}{k}}}{\log X} \left(1 + O\left(\frac{1}{\log X}\right)\right),
\]

as \( \rho \to 1^- \).

**Proof.** We have

\[
(\rho \frac{d}{d\rho})^m \Phi(\rho) = \sum_{j=1}^{\infty} \sum_{p} j^{m-1} p^{km} \rho^{jp} = \sum_{j=1}^{\infty} \sum_{p} j^{m-1} p^{km} e^{-jp/X}.
\]

Using a Mellin transform and interchanging the sums with the integral, we obtain

\[
(3.3) \quad (\rho \frac{d}{d\rho})^m \Phi(\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X^s \mathcal{P}(k(s - m)) \zeta(s + 1 - m) \Gamma(s) \, ds,
\]

where \( \mathcal{P}(s) = \sum_p p^{-s} \) and \( c > m + 1 \). Note that \( \mathcal{P}(s) = \log \zeta(s) - D(s) \), where

\[
D(s) = \sum_{j=2}^{\infty} \frac{1}{j} \sum_p \frac{1}{p^{js}}.
\]

For any \( \delta > 0 \), \( D(s) \) converges absolutely and uniformly for \( \Re(s) \geq \frac{1}{2} + \delta \). If we replace \( \mathcal{P}(k(s - m)) \) by \( D(k(s - m)) \) in \( (3.3) \), then we can move the line of integration to the line \( \Re(s) = c_0 \) for any \( c_0 > m + \frac{1}{2k} \). We can then crudely bound the contribution from this piece of the integral by

\[
(3.4) \quad \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} X^s D(k(s - m)) \zeta(s + 1 - m) \Gamma(s) \, ds \ll X^{c_0}.
\]

We now consider the rest of the integral in \( (3.3) \), namely

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} X^s \log \left(\zeta(k(s - m))\right) \zeta(s + 1 - m) \Gamma(s) \, ds.
\]
The integrand here is analytic in the zero-free region for \( \zeta(k(s - m)) \) except for a logarithmic singularity at \( s = m + \frac{1}{k} \). Choosing \( T = \exp(\sqrt{\log X}) \), the integral can be truncated at height \( T \) with an acceptable error. The remaining part of the integral can be moved left to the line \( \Re(s) = m + \frac{1}{k} - \frac{c}{\log T} \) (where \( c \) is a suitably small positive constant), except for a keyhole loop around the singularity at \( s = m + \frac{1}{k} \). The loop runs along the bottom and top edges of the branch cut over \( \{ s = \sigma : \sigma \leq m + \frac{1}{k} \} \). Aside from the parts along the cut, all pieces of this contour are well-controlled and will contribute only to the error term. Along the bottom and top edges of the cut, the value of \( \log \zeta(s) \) differs by \( 2\pi i \) while the value of \( \zeta(s + 1 - m)X^s\Gamma(s) \) remains the same. Thus the contribution from the pieces along the cut is

\[
\frac{1}{2\pi i} \int_0^{\log T} 2\pi i \zeta \left( 1 + \frac{1}{k} - u \right) X^{m+\frac{1}{k}-u} \Gamma \left( m + \frac{1}{k} - u \right) du.
\]

For any \( m \), we have

\[
\zeta \left( 1 + \frac{1}{k} - u \right) \Gamma \left( m + \frac{1}{k} - u \right) = \zeta \left( 1 + \frac{1}{k} \right) \Gamma \left( m + \frac{1}{k} \right) + O(u)
\]

uniformly for \( u \in [0, 1/2] \), so the contribution from the cuts is

\[
\frac{X^{m+\frac{1}{k}}}{\log X} \zeta \left( 1 + \frac{1}{k} \right) \Gamma \left( m + \frac{1}{k} \right) \left( 1 + O \left( \frac{1}{\log X} \right) \right).
\]

Choosing \( c_0 = m + \frac{3}{4k} \), say, in (3.4), gives \( (3.1) \). To obtain \( (3.2) \), we first notice that the case \( m = 0 \) is immediate. By induction on \( m \), we have

\[
\rho^m \Phi^{(m)}(\rho) = \sum_{i=1}^m c_{i,m} \left( \rho \frac{d}{d\rho} \right)^i \Phi(\rho),
\]

where the \( c_{i,m} \) are real numbers with \( c_{m,m} = 1 \). Since \( \rho = 1 + O(1/X) \), the result follows. □

3.1. Proof of Proposition \[1.2\] Suppose that \( x \) is sufficiently large. Then \( \rho \) is close to 1 and so \( X = \frac{1}{\log(\rho)} \) is also large. By Lemma 3.1 we see that

\[
(3.5) \quad x = \rho \frac{d}{d\rho} \Phi(\rho) = \frac{\zeta(1 + \frac{1}{k})\Gamma(1 + \frac{1}{k})X^{\frac{k+1}{k}}}{\log X} \left( 1 + O\left( \frac{1}{\log X} \right) \right)
\]

Thus

\[
\log x = \log(X^{\frac{k+1}{k}}) - \log \log X + O(1)
\]

Since \( \log x \ll \log X \ll \log x \), we have that \( \log \log X = \log \log x + O(1) \). Thus

\[
\log X = \frac{k}{k + 1} \log x + \frac{k}{k + 1} \log \log x + O(1)
\]

Putting this into (3.5) and solving for \( X \), we find

\[
(3.6) \quad X = \left( \frac{k}{k + 1} \log x \right)^{\frac{k}{k+1}} \left( 1 + \frac{k}{k + 1} \frac{\log \log x}{\log x} + O\left( \frac{1}{\log x} \right) \right).
\]

The fact that \( x \log \frac{1}{\rho} = xX^{-1} \) yields (1.3). Combining (3.5) with Lemma 3.1, we obtain

\[
(3.7) \quad \Phi^{(m)}(\rho) = \left( \rho \frac{d}{d\rho} \right)^m \Phi(\rho) = x X^{m-1} \frac{\Gamma(m + \frac{1}{k})}{\Gamma(1 + \frac{1}{k})} \left( 1 + O\left( \frac{1}{\log x} \right) \right).
\]
Plugging (3.6) into (3.7) gives (1.4) when $m = 0$ and (1.5) when $m \geq 1$.

4. Proof of Theorem 1.3: The Setup

We define the major and minor arcs as follows. Set
\[
\delta_q = q^{-1}X^{-1}(\log X)^{12k}, \quad Q = (\log X)^{12k},
\]
and for $1 \leq a \leq q \leq Q$ with $(a, q) = 1$, define
\[
\mathcal{M}(q, a) = \left(\frac{a}{q} - \delta_q, \frac{a}{q} + \delta_q\right).
\]
The major arcs $\mathcal{M}$ and the minor arcs $\mathcal{m}$ are given by
\[
\mathcal{M} = \bigcup_{1 \leq a \leq q \leq Q, (a, q) = 1} \mathcal{M}(q, a), \quad \mathcal{m} = [-1/2, 1/2) \setminus \mathcal{M}.
\]

As discussed in Section 2, we divide the integral (1.2) into three pieces: the principal major arc $\mathcal{M}(1, 0)$, the non-principal major arcs $\mathcal{M}(q, a)$ with $q > 1$, and the minor arcs $\mathcal{m}$. The main term will come exclusively from principal major arc, which is analyzed in Section 6. Our bound for the denominators in the major arcs is limited by the scope of the Siegel-Walfisz theorem. We choose the exponent of $\log X$ to be $12k$ to obtain a satisfactory bound on the minor arcs.

We begin with some preliminary analysis of $\Phi(\rho e(\alpha))$ that will be used throughout the proof of Theorem 1.3. From the definition, we have
\[
\Phi(\rho e(\alpha)) = \sum_{j=1}^{\infty} \frac{1}{j} \sum_p e^{-p^k/jX} e(jp^k\alpha).
\]
We write
\[
e^{-p^k/jX} = \int_{p}^{\infty} k y^{k-1} j X^{-1} e^{-y^k/jX} dy.
\]
Thus
\[
\sum_p e^{-p^k/jX} e(jp^k\alpha) = \int_{2}^{\infty} k y^{k-1} j X^{-1} e^{-y^k/jX} \sum_{p \leq y} e(jp^k\alpha) dy.
\]
It is useful to observe the crude bound
\[
\int_{2}^{\infty} k y^{k-1} j X^{-1} e^{-y^k/jX} \sum_{p \leq y} e(jp^k\alpha) dy \ll \int_{0}^{\infty} y k y^{k-1} j X^{-1} e^{-y^k/jX} dy.
\]
Using integration by parts, we have that for any $\lambda > 0$,
\[
\int_{2}^{\infty} y^\lambda \left(k y^{k-1} j X^{-1} e^{-y^k/jX}\right) dy \ll \left(\frac{X}{j}\right)^{\lambda/k}.
\]
Let $J$ be a parameter at our disposal. Then
\[
\sum_{j=J+1}^{\infty} \frac{1}{j} \int_{2}^{\infty} k y^{k-1} j X^{-1} e^{-y^k/jX} \sum_{p \leq y} e(jp^k\alpha) dy \ll \sum_{j=J+1}^{\infty} \frac{1}{j} \left(\frac{X}{j}\right)^{1/k} \ll \left(\frac{X}{J}\right)^{1/k}.
\]
In summary, for any $J \geq 1$ we have

\begin{equation}
\Phi(\rho e(\alpha)) = \sum_{j=1}^{J} \left( \frac{1}{j} \sum_{p} e^{-y^j/X} e(j p^k \alpha) + O \left( \left( \frac{X}{J} \right)^{1/k} \right) \right)
\end{equation}

\begin{equation}
\sum_{j=1}^{J} \frac{1}{j} \int_{2}^{\infty} k y^{k-1} j X^{-1} e^{-y^j/X} \sum_{p \leq y} e(j p^k \alpha) \, dy + O \left( \left( \frac{X}{J} \right)^{1/k} \right).
\end{equation}

We conclude this section with an estimate of $\Phi(\rho e(\alpha))$ on the minor arcs.

**Lemma 4.1.** For $\alpha \in \mathfrak{m}$,

\[ \Phi(\rho e(\alpha)) \ll X^{\frac{1}{k}} (\log X)^{-2+\varepsilon}. \]

**Proof.** Fix $j \leq \sqrt{X}$ and consider

\begin{equation}
\int_{2}^{\infty} k y^{k-1} j X^{-1} e^{-y^j/X} \sum_{p \leq y} e(j p^k \alpha) \, dy.
\end{equation}

We use Dirichlet’s theorem to choose $a \in \mathbb{Z}$, $q \in \mathbb{N}$, so that

\[ \left| j \alpha - \frac{a}{q} \right| \leq q^{-1} X^{-1} (\log X)^{12k}. \]

Note that since $\alpha \in \mathfrak{m}$, we must have $q > j^{-1}(\log X)^{12k}$. By Kawada and Wooley [3], we have

\[ \sum_{p \leq x} e(j p^k \alpha) \ll x^{1-\eta(k)+\varepsilon} + \frac{q^{-\frac{1}{2}+\varepsilon} x (\log x)^{4}}{(1 + x^k \left| j \alpha - \frac{a}{q} \right|)^{1/2}}, \]

where $0 < \eta(k) < \frac{1}{2}$ is a constant that can be made explicit. Recalling (4.1) and observing that

\[ \int_{2}^{\infty} y (\log y)^{4} \left( k y^{k-1} j X^{-1} e^{-y^j/X} \right) \, dy \ll \left( \frac{X}{j} \right)^{1/k} \left( \log \left( \frac{X}{j} \right) \right)^{4}, \]

we see that the expression in (4.3) is

\[ \int_{2}^{\infty} k y^{k-1} j X^{-1} e^{-y^j/X} \sum_{p \leq y} e(j p^k \alpha) \, dy \ll \left( \frac{X}{j} \right)^{1-\eta(k)+\varepsilon} + q^{-\frac{1}{2}+\varepsilon} \left( \frac{X}{j} \right)^{1/k} \left( \log \left( \frac{X}{j} \right) \right)^{4} \]

\begin{align*}
& \ll \left( \frac{X}{j} \right)^{1-\eta(k)+\varepsilon} + \left( \frac{\log X}{j} \right)^{-\frac{1}{2}+\varepsilon} \left( \frac{X}{j} \right)^{1/k} \left( \log \left( \frac{X}{j} \right) \right)^{4}.
\end{align*}

Putting this into (4.2) with $J = \lfloor \sqrt{X} \rfloor$, we obtain

\[ \Phi(\rho e(\alpha)) = \sum_{j=1}^{J} \frac{1}{j} \int_{2}^{\infty} k y^{k-1} j X^{-1} e^{-y^j/X} \sum_{p \leq y} e(j p^k \alpha) \, dy + O \left( \left( \frac{X}{J} \right)^{1/k} \right) \]

\begin{align*}
& \ll X^{\frac{1-\eta(k)+\varepsilon}{k}} \left( \sum_{j=1}^{J} j^{-\left( 1 + \frac{1+\varepsilon-\eta(k)}{k} \right)} \right) + X^{\frac{1}{k}} \log X^{4+\varepsilon} \left( \sum_{j=1}^{J} j^{-\left( 1 + \frac{1+\varepsilon-\frac{1}{2}}{k} \right)} \right) + X^{\frac{1}{k}} \log X^{4+\varepsilon} \]

\begin{align*}
& \ll X^{\frac{1}{k}} (\log X)^{-2+\varepsilon},
\end{align*}

as desired. \[\square\]
5. Proof of Theorem 1.3: Major arc estimates

In this section we investigate the behavior of $\Phi(\rho e(\alpha))$ for $\alpha \in \mathcal{M}$. The goal is to obtain an estimate for $\Re(\Phi(\rho e(\alpha)))$ so that we can bound the integrand of (1.2) on the major arcs with $q > 1$.

**Lemma 5.1.** Suppose that $A$ is a positive real number, and that $X$ satisfies $X > X_0(A)$. Suppose also that $\alpha, \beta \in \mathbb{R}$, $q \in \mathbb{N}$, and $a \in \mathbb{Z}$ are such that $(a, q) = 1$, $q \leq (\log X)^A$, $\beta = \alpha - \frac{\pi}{q}$, $|\beta| \leq q^{-1}X^{-1}(\log X)^A$. Then

$$\Phi(\rho e(\alpha)) = \frac{X^{1/k} \Gamma(1/k)}{(1 - 2\pi i \beta X)^{1/k} \log X} \sum_{j \leq \sqrt{X}} \frac{S_k^*(q_j, a_j)}{j^{1+1/k} \varphi(q_j)} + O\left(\frac{X^{1/k} \log \log X}{(\log X)^2}\right).$$

where $q_j = q/(q, j)$, $a_j = aj/(q, j)$, and

$$S_k^*(q_j, a_j) = \sum_{\ell=1}^q e\left(\frac{a\ell^k}{q}\right).$$

**Proof.** Recalling (4.2), we note that for any integer $J$

$$\Phi\left(\rho e\left(\frac{a}{q} + \beta\right)\right) = \sum_{j=1}^J \frac{1}{\pi} \sum_{p \text{ prime}} e\left(\frac{ajp^k}{q}\right) \exp\left(jp^k(2\pi i \beta - 1/X)\right) + O\left(\left(\frac{X}{J}\right)^{\frac{1}{2}}\right)$$

(5.1)

$$= \sum_{j=1}^J \frac{1}{\pi} \sum_{\ell=1}^{q_j} e\left(\frac{aj\ell^k}{q_j}\right) \sum_{\ell \equiv \ell (mod q_j)} \exp\left(jp^k(2\pi i \beta - 1/X)\right) + O\left(\left(\frac{X}{J}\right)^{\frac{1}{2}}\right).$$

Let $J = \lceil \sqrt{X} \rceil$. By Abel summation, (5.2)

$$\sum_{p \equiv \ell (mod q_j)} \exp\left(jp^k(2\pi i \beta - 1/X)\right) = \int_{2}^{\infty} \pi(t; q_j, \ell) \left(\frac{j}{X} - 2\pi i j \beta\right) k t^{k-1} \exp(j t^k(2\pi i \beta - 1/X)) dt.$$

For $t > (X/j)^{1/k}$, we apply the Siegel-Walfisz theorem to obtain

$$\pi(t; q_j, \ell) = \frac{Li(t)}{\varphi(q_j)} + O\left(t \exp\left(-c\sqrt{k \log t}\right)\right) = \frac{Li(t)}{\varphi(q_j)} + O\left(t \exp\left(-c\sqrt{k \log (X/j)}\right)\right)$$

for some constant $c$ depending only on $A$. When $t \leq (X/j)^{1/k}$ we clearly have

$$\pi(t; q_j, \ell) = \frac{Li(t)}{\varphi(q_j)} + O\left(\left(\frac{X}{j}\right)^{1/k} \exp\left(-c\sqrt{k \log (X/j)}\right)\right).$$

Thus the integral in (5.2) is (via integration by parts)

$$\int_{2}^{\infty} \frac{\exp(j t^k(2\pi i \beta - 1/X))}{\varphi(q_j) \log t} dt + O\left(1 + |\beta|X \left(\frac{X}{j}\right)^{1/k} \exp\left(-c\sqrt{k \log (X/j)}\right)\right).$$
Putting this into (5.1), we have

\[
\begin{aligned}
\Phi(\rho e^{a/q} + \beta) - \sum_{j \leq \sqrt{X}} S_k(q, a_j) \frac{1}{j} \int_2^\infty \frac{\exp(jt^k(2\pi i\beta - 1/X))}{\log t} \, dt \\
\ll \sum_{j \leq \sqrt{X}} \varphi(q_j) (1 + |\beta|X) \left( \frac{X}{j} \right)^{1/k} \exp \left( -c \sqrt{k \log(X/j)} \right) + X^{1/2k}
\end{aligned}
\]

\[
\ll \varphi(q) (1 + |\beta|X)^{1/k} \exp \left( -c \sqrt{k \log(X)} \right) \ll X^{1/k} \log \log X \frac{1}{(\log X)^2},
\]

since \(\varphi(q) |\beta|X \leq (\log X)^A\), and \(\exp \left( -c \sqrt{k \log(X)} \right) \ll (\log X)^{-(A+2)}\).

It remains to evaluate the integral

\[
\int_2^\infty \frac{\exp(jt^k(2\pi i\beta - 1/X))}{\log t} \, dt.
\]

Note that the integrand has absolute value less than 1 for all \(t\). Thus

\[
\int_2^{\left(\frac{X}{j}\right)^{1/k} (\log X)^{-2}} \frac{\exp(jt^k(2\pi i\beta - 1/X))}{\log t} \, dt \ll \left(\frac{X}{j}\right)^{1/k} \frac{1}{(\log X)^2}.
\]

Meanwhile, the contribution from \(t \geq \left(\frac{X}{j}\right)^{1/k} \log \log X\) is

\[
\int_2^\infty \frac{\exp(-jt^k/X)}{\log t} \, dt \ll \left(\frac{X}{j}\right)^{1/k} \frac{1}{(\log X)} \int_0^\infty \log_e u \, du = \left(\frac{X}{j}\right)^{1/k} \frac{1}{(\log X)^2}.
\]

If \(\left(\frac{X}{j}\right)^{1/k} (\log X)^{-2} \leq t \leq \left(\frac{X}{j}\right)^{1/k} \log \log X\), then we have

\[
\frac{1}{\log t} = \frac{k}{\log X} + O \left( \frac{\log j + \log \log X}{(\log X)^2} \right).
\]

Furthermore

\[
\int_2^{\left(\frac{X}{j}\right)^{1/k} (\log X)^{-2}} \exp(-jt^k/X) \, dt \leq \int_0^\infty \exp(-jt^k/X) \, dt \ll \left(\frac{X}{j}\right)^{1/k}.
\]

So we have

\[
\int_2^\infty \frac{\exp(jt^k(2\pi i\beta - 1/X))}{\log t} \, dt
\]

\[
= \frac{k}{\log X} \int_0^\infty \exp(-t^k jX^{-1}(1 - 2\pi i\beta X)) \, dt + O \left( \left(\frac{X}{j}\right)^{1/k} \log j + \log \log X \right) \frac{1}{(\log X)^2}.
\]

We make the substitution \(z = (jX^{-1}(1 - 2\pi i\beta X))^{1/k} t\). Choose \(\phi\) so that \(|\phi| < 1/2\) and

\[
\frac{1 - 2\pi i\beta X}{|1 - 2\pi i\beta X|^2} = e^{2\pi i\phi}.
\]
We thus obtain
\[ z = (jX^{-1}|1 - 2\pi i\beta X|)^{1/k} e^{2\pi i\phi / k t}. \]

This gives
\[
\int_{0}^{\infty} \exp(-t^k jX^{-1}(1 - 2\pi i\beta X)) \, dx = \left( \frac{X}{j(1 - 2\pi i\beta X)} \right)^{1/k} \int_{L} e^{-z^k} \, dz,
\]
where \( L \) is the ray \( \{ z = te^{2\pi i\phi / k} : 0 \leq t < \infty \} \). By Cauchy’s theorem, the integral here is equal to \( \Gamma(k+1) \). Inserting this into (5.3) we obtain
\[ \Phi(\rho e^{(\alpha)}) = k \Gamma(k+1) \frac{X^{1/k}}{(1 - 2\pi i\beta X)^{1/k}} \sum_{j \leq \sqrt{X}} S^*_k(q, a_j) + O \left( \frac{X^{1/k} \log \log X}{(\log X)^2} \right). \]

The result follows upon noticing that \( k \Gamma(k+1) = \Gamma(\frac{1}{k}) \).

Ultimately, we need to bound the real part of \( \Phi(\rho e^{(\alpha)}) \) on the major arcs. Lemma 5.1 shows that an understanding of \( S^*_k(q, a) \) is central to that goal. In Section 8, we analyze \( S^*_k(q, a) \) asymptotically and show there exists a constant \( C_k \) such that \( |S^*_k(q, a)| \leq C_k q^{-\frac{1}{2}} \varphi(q) \), for all \((q, a) = 1\). From this theory and the following lemma, we can achieve the bound that we need.

**Lemma 5.2.** For all \( q > 2 \), we have
\[ \left| \Re \left( \frac{S^*_k(q, a)}{(1 - 2\pi i\beta X)^{1/k}} \right) \right| \leq (1 - \delta_k) \varphi(q), \]
where
\[ \delta_k = \frac{\pi^2}{2^{2k+3}C_k^2}. \]

and
\[ C_k = \begin{cases} 128 & \text{if } k = 2, \\ \prod_{p \leq k^2} p & \text{if } k \geq 3. \end{cases} \]

**Proof.** If \( q > (2C_k)^k \), then by Proposition 8.1 we have \( |S^*_k(q, a)| \leq \frac{1}{2} \varphi(q) \). Thus we restrict our attention to \( q \leq (2C_k)^k \). For notational convenience, let \( C = (2C_k)^k \).

As in the proof of Lemma 5.1, we let \( \phi \) be the real number satisfying \(-\frac{1}{2} < \phi < \frac{1}{2}\) and
\[ 1 - 2\pi i\beta X \bigg|_{1 - 2\pi i\beta X} = e^{2\pi i \phi}. \]

We also let \( \Delta := |1 - 2\pi i\beta X| = \sqrt{1 + 4\pi^2 |\beta|^2 X^2}. \)

We have
\[ \frac{S^*_k(q, a)}{(1 - 2\pi i\beta X)^{1/k}} = \Delta^{-1/k} e(-\phi / k) \sum_{\ell=1}^{q} e \left( \frac{a\ell^k}{q} \right) = \Delta^{-1/k} \sum_{\ell=1}^{q} e \left( \frac{a\ell^k}{q} - \frac{\phi}{k} \right). \]

Since \((a, q) = 1\), we have \((a\ell^k, q) = 1\). Let \( \|x\| \) denote the distance to the nearest integer. We have
\[ \left\| \frac{a\ell^k}{q} \right\| \geq \frac{1}{q} \geq \frac{1}{C} \quad \text{and} \quad \left\| \frac{a\ell^k}{q} - \frac{\ell}{q} \right\| \geq \frac{1}{2q} \geq \frac{1}{2C}. \]
It follows that each term \( e(\frac{ak}{q}) \) is on the unit circle with argument bounded away from 0 and \( \pi \). The main idea is to show that \( S_k^*(q, a) \) stays bounded away from \( \pm 1 \) when multiplied by \( e(-\phi/k)\Delta^{-1/k} \).

If \( |\phi| < \frac{k}{4C} \), then \( e(-i\phi) \) doesn’t twist the terms of \( S(q, a) \) much and so arguments are still bounded away from 0 and \( \pi \). More precisely,

\[
\left| \Re \left( e \left( \frac{ak}{q} - \frac{\phi}{k} \right) \right) \right| = \left| \cos \left( 2\pi \left( \frac{ak}{q} - \frac{\phi}{k} \right) \right) \right| < \cos \left( \frac{\pi}{2C} \right) < 1 - \frac{2}{5} \left( \frac{\pi}{2C} \right)^2 ,
\]

since \( \frac{2}{3} \leq (1 - \cos \omega)\omega^{-2} < \frac{1}{2} \) for \( 0 < \omega < \pi/2 \). Hence

\[
\left| \Re \left( \frac{S_k^*(q, a)}{(1 - 2\pi i\beta X)^{1/k}} \right) \right| < \Delta^{-1/k} \left( 1 - \frac{\pi^2}{10C^2} \right) \varphi(q) < \left( 1 - \frac{\pi^2}{10C^2} \right) \varphi(q)
\]

Now consider \( |\phi| \geq \frac{k}{4C} \). Since \( 2\pi\phi = \arg(1 - i2\pi\beta X) \), we have that

\[
2\pi|\beta|X = \tan|2\pi\phi| \geq \tan \left( \frac{\pi k}{2C} \right).
\]

So

\[
\Delta = |1 - i(2\pi\beta X)| \geq 1 - i \tan \left( \frac{\pi k}{2C} \right) = \sqrt{1 + \tan^2 \left( \frac{\pi k}{2C} \right)}.
\]

By the binomial theorem and the fact that \( \tan^2 y > y^2 \) for \( |y| \leq \pi/2 \), we thus have

\[
\Delta^{-1/k} \leq \left( 1 + \tan^2 \left( \frac{\pi k}{2C} \right) \right)^{-\frac{1}{2\pi}} \leq 1 - \frac{1}{2k} \left( \frac{\pi k}{2C} \right)^2
\]

So for \( |\phi| \geq \frac{k}{4C} \),

\[
\left| \Re \left( \frac{S_k^*(q, a)}{(1 - 2\pi i\beta X)^{1/k}} \right) \right| \leq \Delta^{-1/k} |S_k^*(q, a)| \leq \left( 1 - \frac{\pi^2 k}{8C^2} \right) \varphi(q).
\]

In either case,

\[
\left| \Re \left( \frac{S_k^*(q, a)}{(1 - 2\pi i\beta X)^{1/k}} \right) \right| \leq \left( 1 - \frac{\pi^2}{8C^2} \right) \varphi(q) = (1 - \delta_k) \varphi(q).
\]

We are now ready to complete the analysis of the non-principal major arcs.

**Lemma 5.3.** Suppose that \( A \) is a positive real number, and that \( X \) satisfies \( X > X_0(A) \). Suppose also that \( \alpha, \beta \in \mathbb{R}, q \in \mathbb{N}, \) and \( a \in \mathbb{Z} \) are such that \( (a, q) = 1, 1 < q \leq (\log X)^A, \beta = \alpha - \frac{a}{q}, |\beta| \leq q^{-1}X^{-1}(\log X)^A. \) Then

\[
|\Re (\Phi(\rho e(\alpha)))| \leq \left( 1 - \frac{\delta_k}{3} \right) \Phi(\rho) \left( 1 + O \left( \frac{\log \log X}{\log X} \right) \right),
\]

where \( \delta_k \) is the constant defined in Lemma 5.2.
Proof. By Lemma 5.1 we have (5.4)
\[ \Re (\Phi(\rho e(\alpha))) = \frac{X^{1/k} \Gamma(1/k)}{\log X} \Re \left( \frac{1}{(1 - 2\pi i \beta X)^{1/k}} \sum_{j \leq \sqrt{X}} \frac{S_k^*(q_j, a_j)}{j + \frac{1}{k}} \varphi(q_j) \right) + O \left( \frac{X^{1/k} \log \log X}{(\log X)^2} \right) \]

We first handle the case \( q = 2 \). Then \( q_j = 1, 2 \) depending on whether \( j \) is odd or even, respectively. So \( S_k^*(q_j, a_j) = (-1)^j \varphi(q_j) \). Hence
\[
\left| \sum_{j \leq \sqrt{X}} \frac{S_k^*(q_j, a_j)}{j + \frac{1}{k}} \varphi(q_j) \right| \leq \left| \sum_{j \leq \sqrt{X}} \frac{(-1)^j}{j + \frac{1}{k}} \right| \leq \sum_{j \leq \sqrt{X}} \frac{1}{j + \frac{1}{k}} \leq \left( 1 - 2^{-\frac{k+1}{k}} \right) \zeta \left( \frac{k+1}{k} \right) < \left( 1 - \frac{\delta_k}{3} \right) \zeta \left( \frac{k+1}{k} \right).
\]

Now consider \( q > 2 \). If \( q \nmid 2j \), then \( q_j > 2 \) and Lemma 5.2 tells us that
\[
\left| \Re \left( \frac{S_k^*(q_j, a_j)}{(1 - 2\pi i \beta X)^{1/k}} \right) \right| \leq (1 - \delta_k) \varphi(q_j).
\]
Thus
\[
\left| \Re \left( \frac{1}{(1 - 2\pi i \beta X)^{1/k}} \sum_{j \leq \sqrt{X}} \frac{S_k^*(q_j, a_j)}{j + \frac{1}{k}} \varphi(q_j) \right) \right| \leq \sum_{j \leq \sqrt{X}} \frac{1}{j + \frac{1}{k}} + (1 - \delta_k) \sum_{j \leq \sqrt{X}} \frac{1}{j + \frac{1}{k}} \leq \left( 1 - \frac{\delta_k}{3} \right) \zeta \left( \frac{k+1}{k} \right). \]

Putting this into (5.4), we have (for any \( q \geq 2 \)),
\[
|\Re (\Phi(\rho e(\alpha)))| \leq \left( 1 - \frac{\delta_k}{3} \right) \zeta \left( \frac{k+1}{k} \right) \frac{\Gamma(1/k)X^{1/k}}{\log X} + O \left( \frac{X^{1/k} \log \log X}{(\log X)^2} \right) \leq \left( 1 - \frac{\delta_k}{3} \right) \Phi(\rho) \left( 1 + O \left( \frac{\log \log X}{\log X} \right) \right).
\]

6. PROOF OF THEOREM 1.3

In this section, we conclude the proof of Theorem 1.3 by analyzing the contribution from \( \mathfrak{M}(1, 0) \). Recall that \( p_{\mathfrak{M}}(n) \) is given by the integral in (1.2). For \( \alpha \not\in \mathfrak{M}(1, 0) \) and \( n \) sufficiently large, we have
\[
|\Re (\Phi(\rho e(\alpha)))| \leq (1 - \frac{\delta_k}{3}) \Phi(\rho),
\]
by Lemma 4.1 and Lemma 5.3. Thus
\[
p_{\mathfrak{M}}(n) = \rho^{-n} \int_{\mathfrak{M}(1, 0)} \exp(\Phi(\rho e(\alpha)))e(-n\alpha)\,d\alpha + O \left( \rho^{-n} \Psi(\rho)n^{-B} \right),
\]
for any constant $B$.

If $\alpha \in \mathbb{R}(1, 0)$ and $|\alpha| > X^{-1}(\log X)^{-1/4}$, then by Lemma 5.1 we have

$$|\Phi(\rho e(\alpha))| = \frac{X^{1/k} \Gamma(1/k) \zeta(1 + 1/k)}{(1 + 4\pi^2 \alpha^2 X^2)^{1/2k} \log X} + O\left(\frac{X^{1/k} \log \log X}{(\log X)^2}\right),$$

and by the binomial theorem

$$(1 + 4\pi^2 \alpha^2 X^2)^{-1/2k} < (1 - \frac{2\pi^2}{k} \alpha^2 X^2) < 1 - \frac{2\pi^2}{k}(\log X)^{-1/2}.$$ Thus for $X$ sufficiently large, we have

$$|\Re\Phi(\rho e(\alpha))| < (1 - (\log X)^{-1})\Phi(\rho).$$

The contribution to the integral in (6.1) from such $\alpha$ is then

$$\leq \rho^{-n} \Psi(\rho) \exp\left(\frac{\Phi(\rho)}{\log X}\right) \ll \rho^{-n} \Psi(\rho) n^{-B}.$$ We are left to study the integral

$$\rho^{-n} \int_{-\tau}^{\tau} \exp(\Phi(\rho e(\alpha))) e(-n\alpha) d\alpha,$$

where

$$\tau = X^{-1}(\log X)^{-1/4}.$$ For $\alpha \in \mathbb{R}$, let $R(\alpha)$ and $I(\alpha)$ denote the real and imaginary parts of $\Phi(\rho e(\alpha))$, respectively. Applying Taylor’s theorem to each of $R(\alpha)$ and $I(\alpha)$, we have

$$\Phi(\rho e(\alpha)) = R(\alpha) + iI(\alpha)$$

$$= (R(0) + iI(0)) + \alpha(R'(0) + iI'(0)) + \frac{\alpha^2}{2}(R''(0) + iI''(0))$$

$$+ \frac{\alpha^3}{6}(R'''(c_R \alpha) + iI'''(c_I \alpha)),$$

where $0 < c_R, c_I < 1$ may depend on $\alpha$.

Now, for any real $\beta$ we have

$$R'(\beta) + iI'(\beta) = \frac{d}{d\beta} \Phi(\rho e(\beta)) = 2\pi i e(\beta) \rho \Phi'(\rho e(\beta)),$$

$$R''(\beta) + iI''(\beta) = -4\pi^2 e(\beta) \rho \Phi'(\rho e(\beta)) - 4\pi^2 e(2\beta) \rho^2 \Phi''(\rho e(\beta)),$$

and

$$R'''(\beta) + iI'''(\beta) = -8\pi^3 i e(\beta) \rho \Phi'(\rho e(\beta)) - 24\pi^3 i e(2\beta) \rho^2 \Phi''(\rho e(\beta))$$

$$- 8\pi^3 i e(3\beta) \rho^3 \Phi'''(\rho e(\beta)).$$

Thus

$$\sup(|R'''(\beta)|, |I'''(\beta)|) \leq 8\pi^3 \left(\rho \Phi'(\rho) + 3\rho^2 \Phi''(\rho) + \rho^3 \Phi'''(\rho)\right).$$ Hence, there exists $w \in \mathbb{C}$ (depending on $\alpha$) such that $|w| \leq 1$ and

$$\alpha^3 (R'''(c_R \alpha) + iI'''(c_I \alpha)) = 16\pi^3 w |\alpha|^3 \left(\rho \Phi'(\rho) + 3\rho^2 \Phi''(\rho) + \rho^3 \Phi'''(\rho)\right).$$
Putting this into (6.4), we have
\[
\Phi(\rho e(\alpha)) = \Phi(\rho) + 2\pi i\alpha \rho \Phi'(\rho) - 2\pi^2 \alpha^2 (\rho \Phi'(\rho) + \rho^2 \Phi''(\rho)) + \frac{8}{3} \pi^3 w|\alpha|^3 (\rho \Phi'(\rho) + 3\rho^2 \Phi''(\rho) + \rho^3 \Phi'''(\rho)).
\]

Hence the contribution to the integral in (6.1) from these
\[
\rho^{-n} \int_{-\tau}^{\tau} \exp(g(\rho, \alpha)) d\alpha,
\]
where
\[
\rho(\rho, \alpha) = \Phi(\rho) - 2\pi^2 \alpha^2 (\rho \Phi'(\rho) + \rho^2 \Phi''(\rho)) + \frac{8}{3} \pi^3 w|\alpha|^3 (\rho \Phi'(\rho) + 3\rho^2 \Phi''(\rho) + \rho^3 \Phi'''(\rho)).
\]

Recalling that \( n = \rho \Phi'(\rho) \), the integral in (6.3) becomes
\[
\left| \frac{8}{3} \pi^3 w|\alpha|^3 (\rho \Phi'(\rho) + 3\rho^2 \Phi''(\rho) + \rho^3 \Phi'''(\rho)) \right| \\
\leq C_1 \alpha^2 X^{2+\frac{1}{8}} (\log X)^{-5/4} \leq C_2 \alpha^2 X^{2+\frac{1}{8}} (\log X)^{-1} \\
\leq \pi^2 \alpha^2 (\rho \Phi'(\rho) + \rho^2 \Phi''(\rho)).
\]

Thus there exist positive constants \( C_1, C_2 \), such that for \( X \) sufficiently large and \( |\alpha| \leq \tau = X^{-1} (\log X)^{-1/4} \) we have
\[
|\Re(g(\rho, \alpha))| \leq \Phi(\rho) - \pi^2 \alpha^2 (\rho \Phi'(\rho) + \rho^2 \Phi''(\rho)).
\]

For \( |\alpha| \geq X^{-(1+\frac{3}{16})} (\log X)^{2} \), there is a positive constant \( C_3 \) such that
\[
|\Re(g(\rho, \alpha))| \leq \Phi(\rho) - C_3 (\log X)^3.
\]

Hence the contribution to the integral in (6.1) from these \( \alpha \) is
\[
\leq \rho^{-n} \Psi(\rho) X^{-C_3 (\log X)^3} \ll \rho^{-n} \Psi(\rho) n^{-B}.
\]

Finally, we need to estimate the integral over the interval \([-\eta, \eta] \), where \( \eta = X^{-(1+\frac{3}{16})} (\log X)^{2} \).

For \( \alpha \) in this interval, we have
\[
|\alpha|^3 (\pi^2 \rho \Phi'(\rho) + 3\pi^3 \rho^2 \Phi''(\rho) + \pi^3 \rho^3 \Phi'''(\rho)) \\
\ll (\log X)^0 X^{-(3+\frac{3}{16})} (\log X)^{-1} X^{2+\frac{1}{8}} = (\log X)^5 X^{-\frac{1}{8}}.
\]
Recall that \( n = x \asymp X^{\frac{k+1}{k+1}}(\log X)^{-1} \). So,

\[
X^{\frac{1}{2\pi k+1}}(\log X)^{-5} = \left( X^{\frac{1}{2\pi k+1}} \right)^{n^{\frac{1}{k+1}}} (\log X)^{-5} \\
\gg n^{\frac{1}{2\pi k+1}} (\log X)^{-5 + \frac{1}{2\pi k+1}} \gg n^{\frac{1}{2\pi k+1} + \varepsilon} \\
\gg n^{\frac{1}{2\pi k+1}}.
\]

Hence

\[
|\alpha|^3 (\pi^3 \rho \Phi'(\rho) + 3\pi^3 \rho^2 \Phi''(\rho) + \pi^3 \rho^3 \Phi'''(\rho)) \ll n^{\frac{1}{2\pi k+1}},
\]

and therefore

\[
\exp \left( \frac{8}{3} \pi^3 w |\alpha|^3 (\rho \Phi'(\rho) + 3\rho^2 \Phi''(\rho) + \rho^3 \Phi'''(\rho)) \right) = 1 + O(n^{-\frac{1}{2\pi k+1}})
\]

Putting this into (6.5), and noticing that \( \Phi(2)(\rho) = \rho \Phi'(\rho) + \rho^2 \Phi''(\rho) \), we thus have

\[
\int_{-\eta}^{\eta} \exp(\Phi(\rho e(\alpha)))e(-n\alpha) \, d\alpha = \int_{-\eta}^{\eta} \exp(g(\rho, \alpha)) \, d\alpha \\
= (1 + O(n^{-\frac{1}{2\pi k+1}})) \Psi(\rho) \int_{-\eta}^{\eta} \exp(-\alpha^2 \pi^2 \Phi(2)(\rho)) \, d\alpha.
\]

Recall that \( \eta \Phi(2)(\rho) \gg (\log X)^3 \). Through a routine polar coordinates integration, we have

\[
\left( \int_{-\eta}^{\eta} \exp(-\alpha^2 \pi^2 \Phi(2)(\rho)) \, d\alpha \right)^2 = \frac{1}{2\pi \Phi(2)(\rho)} \left( 1 - \exp(-\eta^2 \pi^2 \Phi(2)(\rho)) \right) \\
= \frac{1}{2\pi \Phi(2)(\rho)} \left( 1 + O(\epsilon^{-\log X^2}) \right).
\]

Therefore

\[
\int_{-\eta}^{\eta} \exp(\Phi(\rho e(\alpha)))e(-n\alpha) \, d\alpha = \frac{\Psi(\rho)}{\sqrt{2\pi \Phi(2)(\rho)}} \left( 1 + O(n^{-\frac{1}{2\pi k+1}}) \right) \left( 1 + O(\epsilon^{-\log X^2}) \right).
\]

Combining this with (6.1), (6.2), and (6.6) gives the desired result.

7. Proof of Theorem 1.4

Let \( \rho = \rho(n) \) and let \( X \) satisfy \( \rho = e^{-1/X} \). By (1.2), we have

\[
p_{F_k}(n+1) - p_{F_k}(n) = \int_{-1/2}^{1/2} \rho^{-n} \exp(\Phi(\rho e(\alpha)) - 2\pi i n \alpha)(\rho^{-1} e^{-2\pi i \alpha} - 1) \, d\alpha.
\]

Note that \( |\rho^{-1} e^{-2\pi i \alpha} - 1| \leq e^{1/X} + 1 \leq 4 \). From the proof of Theorem 1.3 we see that the contribution from \( |\alpha| > \eta = X^{-(1+\frac{1}{\pi})} (\log X)^2 \) is

\[
\ll \frac{\rho^{-n} \Psi(\rho)}{\sqrt{2\pi \Phi(2)(\rho)}} \eta^{-B}
\]

for any positive constant \( B \). On the other hand, when \( |\alpha| \leq \eta \), we have

\[
\rho^{-1} e^{-2\pi i \alpha} - 1 = \exp \left( \frac{1}{X} - 2\pi i \alpha \right) - 1 = \frac{1}{X} + O(\eta) = \frac{1}{X} + O \left( X^{-(1+\frac{1}{\pi})} (\log X)^2 \right).
\]
From the proof of Theorem 1.3, we have
\[
\int_{-\eta}^{\eta} \rho^{-n} \exp(\Phi(\rho e(\alpha)) - 2\pi i n \alpha) \, d\alpha = \frac{\rho^{-n} \Psi(\rho)}{\sqrt{2\pi \Phi(2)(\rho)}} (1 + O(n^{-\frac{1}{2k+3}})).
\]
Recalling (3.6), we thus have
\[
p_{F_k}(n + 1) - p_{F_k}(n) = \rho^{-n} \Psi(\rho) \left( \frac{1}{X} + O \left( X^{-1+\frac{1}{2k}} (\log X)^2 \right) \right)
= \frac{\rho^{-n} \log(\frac{1}{\rho}) \Psi(\rho)}{\sqrt{2\pi \Phi(2)(\rho)}} (1 + O(n^{-\frac{1}{2k+3}})),
\]
as desired. \(\square\)

8. Analysis of \(S_k^*(q, a)\)

In Section 5, we introduced the exponential sum \(S_k(q, a)\). Here we provide some analysis of these exponential sums. For any \(a, q \in \mathbb{N}\), we define
\[
S_k(q, a) = \sum_{\ell=1}^{q} e \left( \frac{a\ell}{q} \right), \quad S_k^*(q, a) = \sum_{\ell=1}^{q} e \left( \frac{a\ell^k}{q} \right).
\]
Note that
\[
S_k^*(q, a) = \sum_{\nu|q} \mu(\nu) S_k \left( \frac{q}{\nu}, a\nu^{k-1} \right).
\]
We begin by showing that \(S_k^*(q, a)\) exhibits a certain multiplicative behavior. We then show that \(S_k^*(p^\ell, a) = 0\) for large powers \(\ell\). Finally, we obtain an effective upper bound for \(S_k^*(q, a)\), which is provided by Proposition 8.1. The arguments here are similar to the treatment of \(S_k(q, a)\) in [7, Ch. 4].

Lemma 8.1. If \((q, r) = (a, r) = (a, q) = 1\), then
\[
S_k^*(qr, a) = S_k^*(q, ar^{k-1}) S_k^*(r, aq^{k-1}).
\]
Proof. We have
\[
S_k^*(q, ar^{k-1}) S_k^*(r, aq^{k-1}) = \sum_{\nu|q} \sum_{\eta|r} \mu(\nu\eta) S_k \left( \frac{q}{\nu}, ar^{k-1}\nu^{k-1} \right) S_k \left( \frac{r}{\eta}, aq^{k-1}\eta^{k-1} \right)
\]
(8.1)
\[
= \sum_{\nu|q} \sum_{\eta|r} \mu(\nu\eta) \sum_{m=\frac{\nu}{\eta}} \sum_{\ell=\frac{\nu}{\eta}} e \left( a \frac{(rvm)^k + (q\eta\ell)^k}{qr} \right).
\]
By Euclid’s algorithm, for each residue class \(h \pmod{\frac{qr}{\nu\eta}}\), there exists a unique pair \((m, \ell)\) with \(m \leq \frac{\nu}{\eta}\) and \(\ell \leq \frac{\nu}{\eta}\) such that
\[
h = \frac{\nu}{\eta} m + \frac{\nu}{\eta} \ell.
\]
Let \(\lambda = \nu\eta\). Then \(h\lambda = rvm + q\eta\ell\), and (8.1) is equal to
\[
\sum_{\lambda|qr} \mu(\lambda) \sum_{h=\frac{\nu\lambda}{\eta}} e \left( \frac{a \lambda h \lambda^k}{qr} \right) = S_k^*(qr, a).
\]
Lemma 8.2. For each prime $p$, define $\tau = \tau(p)$ so that $p^\tau || k$, and define

$$
\gamma = \gamma(p) = \begin{cases} 
\tau + 2, & \text{if } p = 2 \text{ and } \tau > 0 \\
\tau + 1, & \text{otherwise.}
\end{cases}
$$

Then $S_k^*(p^\ell, a) = 0$ whenever $\ell > \gamma$ and $p \nmid a$.

It is useful to note that $\gamma \leq k$ unless $p = k = 2$, in which case $\gamma = 3$.

Proof. We have

$$
S_k^*(p^\ell, a) = S_k(p^\ell, a) - S_k(p^{\ell-1}, ap^{k-1}).
$$

If $\ell \leq k$, then $S_k(p^\ell, a) = p^{\ell-1}$ by Lemma 4.4 of [7] and

$$
S_k(p^{\ell-1}, ap^{k-1}) = \sum_{m \leq p^{\ell-1}} e\left(\frac{ap^km^k}{p^\ell}\right) = p^{\ell-1}.
$$

If $\ell > k$ then $S_k(p^\ell, a) = p^{k-1}S_k(p^{\ell-k}, a)$ by Lemma 4.4 of [7] and

$$
S_k(p^{\ell-1}, ap^{k-1}) = \sum_{m \leq p^{\ell-1}} e\left(\frac{am^k}{p^{\ell-k}}\right) = p^{k-1}\sum_{r \leq p^{\ell-k}} e\left(\frac{ar^k}{p^{\ell-k}}\right) = p^{k-1}S_k(p^{\ell-k}, a).
$$

Proposition 8.1. For all $q, a \in \mathbb{N}$ with $(q, a) = 1$, we have

$$
|S_2^*(q, a)| \leq 8q^{-1/4}\varphi(q),
$$

and for $k > 2$

$$
|S_k^*(q, a)| \leq C_k q^{-\frac{1}{k}}\varphi(q),
$$

where

$$
C_k = \prod_{p \leq k^6} k
$$

Proof. Write $q = \prod p^\ell$. Then by Lemma 8.1 we have

$$
S_k^*(q, a) = \prod_{p \mid q} S_k^*(p^\ell, a(qp^{-\ell})^{k-1}).
$$

We first consider $k = 2$. If $p^2 \mid q$ for $p \geq 3$, or if $16 \mid q$, then $S_2^*(q, a) = 0$ by Lemma 8.2. So we may write $q = 2^\ell b$, where $0 \leq \ell \leq 3$ and $b$ is odd, squarefree.
It is easy to see that $|S_2^*(2^\ell, a)| = 2^{\ell-1}$ for $1 \leq \ell \leq 3$. For odd primes $p$

$$|S_2^*(p, a)|^2 = \sum_{x=1}^{p-1} \sum_{y=1}^{p} e \left( a \left( \frac{y^2 - x^2}{p} \right) \right) - \sum_{x=1}^{p-1} e \left( \frac{-ax^2}{p} \right)$$

$$= \sum_{x=1}^{p-1} \sum_{h=1}^{p} e \left( \frac{a(2x + h)h}{p} \right) - S_2^*(p, -a)$$

$$= \sum_{h=1}^{p} e \left( \frac{ah^2}{p} \right) \sum_{x=1}^{p-1} e \left( \frac{2ahx}{p} \right) - S_2^*(p, -a)$$

$$= \sum_{h=1}^{p-1} (-1)e \left( \frac{ah^2}{p} \right) + (p - 1) - S_2^*(p, -a)$$

$$= (p - 1) - S_2^*(p, a) - S_2^*(p, -a).$$

Thus

$$|S_2^*(p, a)| \leq \sqrt{p} + 1 = p^{-\frac{1}{4}}(p - 1) \left( \frac{p^\frac{1}{4}}{\sqrt{p} - 1} \right) < \begin{cases} p^{-\frac{1}{4}}(p - 1), & p \geq 7 \\ 2p^{-\frac{1}{4}}(p - 1), & p = 3, 5. \end{cases}$$

All together,

$$|S_k^*(q, a)| \leq 2^{\ell-1}4b^{-\frac{1}{4}}\varphi(b) \leq 8q^{-\frac{1}{4}}\varphi(q).$$

Now let $k > 2$. We consider $S_k^*(p^\ell, a)$. If $\ell > \gamma$ then $S_k^*(p^\ell, a) = 0$. So we may assume $\ell \leq \gamma(p) \leq k$. If $p \leq k$, then

$$|S_k^*(p^\ell, a)| \leq \varphi(p^\ell) = kp^{-1}\varphi(p^\ell) \leq kp^{-\ell/k}\varphi(p^\ell).$$

If $p > k$, then $\gamma = 1$ so $\ell = 1$. By Lemma 4.3 of [7], we have

$$|S_k^*(p^\ell, a)| = |S_k^*(p, a) - 1| \leq (k - 1)p^{\frac{1}{2}} + 1 \leq kp^{\frac{1}{2}} \leq kp^{-1/k}p^{\frac{5}{6}}$$

$$= kp^{-1/k} \frac{p - 1}{p^{\frac{1}{2}} - p^{-\frac{5}{6}}} \leq \begin{cases} p^{-1/k}(p - 1), & p > k^6 \\ kp^{-1/k}(p - 1), & p \leq k^6. \end{cases}$$

\[\square\]

References

1. P T Bateman and Paul Erdos, Monotonicity of partition functions, Mathematika 3 (1956), no. 1, 1–14.
2. Bruce C. Berndt, Amita Malik, and Alexandru Zaharescu, Partitions into $k$th-powers of terms in an arithmetic progression, Math. Zeitschrift 290 (2018), 1277–1307.
3. Alexander Dunn and Nicolas Robles, Polynomial partition asymptotics, J. Math. Anal. Appl. 459 (2018), 359–384.
4. Ayla Gafni, Power partitions, J. Number Theory 163 (2016), 19–42.
5. Koichi Kawada and Trevor D Wooley, On the Waring–Goldbach Problem for Fourth and Fifth Powers, Proc. London Math. Soc. 83 (2001), no. 1, 1–50.
6. K F Roth and G Szekeres, Some asymptotic formulae in the theory of partitions, Q. J. Math. 5 (1954), no. 1, 241–259.
7. Robert C. Vaughan, The Hardy-Littlewood Method, 2nd ed., Cambridge Tracts in Mathematics, Cambridge University Press, 1997.
8. , On the number of partitions into primes, Ramanujan J. 15 (2008), no. 1, 109–121.
9. , Squares: Additive questions and partitions, Int. J. Number Theory (2015), 1–43.
10. E. Maitland Wright, *Asymptotic partition formulae. III. Partitions into k-th powers*, Acta Math. 63 (1934), no. 1, 143–191.

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