\textit{n–ary star product: construction and integral representation}

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Abstract

This paper addresses a construction of an \textit{n–ary star product}. Relevant identities are given. Besides, the formalism is illustrated by a computation of eigenvalues and eigenfunctions for a physical system of coupled oscillators in an \textit{n} dimensional phase space.

Keywords \textit{n–ary star product; integral representation; Schwartz space.}

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1 Introduction

Several works were devoted to generalizations of Lie algebras to various types of \textit{n}-ary algebras. To cite a few, see the works by Filippov, Hanlon, Vinogradov, Takhtajan and collaborators in \cite{1}-\cite{2}-\cite{3}. In the same time, and intended to physical applications, the new algebraic structures were considered in the case of the algebra \( C^\infty(M) \) of functions on a \( C^\infty \)–manifold \( M \), under the assumption that the operation is a derivation of each entry separately. In this way one got the Nambu-Poisson brackets, see e.g. \cite{4}. The same versality was observed for generalized Poisson brackets in \cite{5} (and references therein) providing unexpected algebraic structures on vector fields, which played an essential role in the construction of universal enveloping algebras of Filippov algebras (\textit{n}-Lie algebras). See, for instance, \cite{6} and references therein. For many other applications, especially to theoretical physics, see a nice and interesting survey of \textit{n–ary analogues of Lie algebras} written by Azcárraga and Izquierdo \cite{7}.

This work intends to provide a construction of an \textit{n–ary star product}, to investigate some identities related to it, and to give a concrete illustration on a physical system of coupled oscillators in an \textit{n}–dimensional phase space.

The paper is organized as follows. In section 2 we focus on the study of a 3–ary star product in a 3-dimensional Euclidean space. We prove that the 3–ary star product is distributive, associative and satisfies the Jacobi identity. We also construct its integral representation which should likely allow to establish new classes of solvable actions in the context of field theories. Section 3 is devoted to the generalization of this star product in higher dimension, i. e. for
3. In section 4 we provide a simple application of such a star product on a physical system of coupled oscillators for which the eigenvalues and eigenfunctions are explicitly computed. In the section 5, we give some concluding remarks.

2 3–ary star product

2.1 Definition and properties

We start by the following definition:

**Definition 2.1** Consider \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Let \( \sigma_c \) be the cyclic permutation over the set \( \{1, 2, 3\} \) such that \( \sigma_c(1) = 2, \sigma_c(2) = 3, \sigma_c(3) = 1 \), and \( \mathcal{A} = \left( \mathcal{S}(\mathbb{R}^3), \star \right) \) be the Schwartz space of (smooth, rapidly decreasing, together with all their derivatives, faster than the reciprocal of any polynomial at \( \infty \)) real valued functions on \( \mathbb{R}^3 \), endowed with a 3-ary star product defined at a point \( x \) as follows:

\[
(f \star g \star h)(x) := m\left[ e^{\mathcal{P}(\theta_1, \theta_2, \theta_3)}(f \otimes g \otimes h)(x) \right]
\] (2.1)

where

\[
\mathcal{P}(\theta_1, \theta_2, \theta_3) = \sum_{k=1}^{3} \frac{i\theta_k}{2} \left( \partial_k \otimes \partial_{\sigma_c(k)} \otimes \partial_{\sigma_c \circ \sigma_c(k)} - \partial_k \otimes \partial_{\sigma_c \circ \sigma_c(k)} \otimes \partial_{\sigma_c(k)} \right),
\] (2.2)

\( m(f \otimes g \otimes h) = fgh \); the parameters \( \theta_i \), \( i = 1, 2, 3 \), are real numbers.

The conjugate of \( (f \star g \star h)(x) \), denoted by \( (f \overline{\star} g \overline{\star} h)(x) \) is

\[
(f \overline{\star} g \overline{\star} h)(x) := m\left[ e^{-\mathcal{P}(\theta_1, \theta_2, \theta_3)}(f \otimes g \otimes h)(x) \right].
\] (2.3)

Note that, if the space \( \mathbb{R}^3 \) is the space of position coordinates, then the parameters \( \theta_i \) must be of order of magnitude of the Planck volume, i.e. \( [\theta_i] = [L^3] \). There exist several constructions of \( n \)–ary star products in the literature. For instance in [16], see remark (1.1) where a more general star product with tensor \( \Theta^{\mu_1 \mu_2 \cdots \mu_n} \) is examined. In (2.2), we adopt a specific choice of \( \Theta \) motivated by the fact that the resulting \( n \)–ary product has nice properties like the associativity, the distributivity, a consistent Jacobi identity, and so on. Moreover, it well behaves in concrete applications like for the example of coupled oscillators exhibited in this work.

Let \( f_i, g_i, h_i \in \mathcal{A} \), \( i = 1, 2, 3 \). The following properties are satisfied for the defined 3–ary star product:

**Property 2.1 (Distributivity)** The distributivity of the 3–ary star product (2.1) is given by the three following relations:

\[
(f_1 + f_2) g_1 \star h_1 = f_1 g_1 \star h_1 + f_2 g_1 \star h_1,
\] (2.4)

\[
f_1 (g_1 + g_2) \star h_1 = f_1 g_1 \star h_1 + f_1 g_2 \star h_1,
\] (2.5)

\[
f_1 g_1 \star (h_1 + h_2) = f_1 g_1 \star h_1 + f_1 g_1 \star h_2.
\] (2.6)
Property 2.2 (Associativity) The associativity of the $3$–ary star product \( (2.1) \) can be defined as:

\[
(f_1 \ast h_1) \ast h_2 = f_1 \ast (h_1 \ast h_2).
\]

(2.7)

Besides, it appears natural to define a $3$–ary star bracket as below:

**Definition 2.2 (3–ary star bracket)**

\[
\{ f, h \}_* := f \ast h - h \ast f
\]

(2.8)

with the following properties:

**Property 2.3 (Skew-symmetry)**

\[
\{ f, h \}_* = -\{ h, f \}_*
\]

(2.9)

**Property 2.4 (Jacobi identity)**

\[
\{ l, \{ f, h \}_* \}_g + \{ l, \{ f, h \}_*, g \}_g + \{ h, \{ l, f \}_* \}_g + \{ h, \{ l, f \}_*, g \}_g = 0.
\]

(2.10)

The proof of the Jacobi identity stems from the relation

\[
\{ l, \{ f, h \}_* \}_g = (l \ast f)^{g_1}_1 h - (l \ast h)^{g_1}_1 f - (f \ast h)^{g_2}_1 l + (h \ast f)^{g_2}_1 l.
\]

(2.11)

A thorough analysis of the defined star product and bracket properties is not concerned here and will be in the core of a forthcoming work. Let us now sketch some interesting computations with this star product on the Euclidean coordinates. Indeed, the star composition of functions in \( \mathcal{A} \) with coordinate functions gives rise to the following results:

**Proposition 2.1 (3–ary star product of two functions in \( \mathcal{A} \) and one coordinate function)**

\[
x_k f \ast g = x_k f g + \frac{i\theta_k}{2} (\partial_{\sigma_c(k)} f \partial_{\sigma_c(c)}(k) g - \partial_{\sigma_c(c)}(k) f \partial_{\sigma_c(k)} g)
\]

(2.12)

\[
g \ast f = x_k f g - \frac{i\theta_{\sigma_c(k)}}{2} \partial_{\sigma_c(k)} g \partial_{\sigma_c(c)}(k) f + \frac{i\theta_{\sigma_c(c)}(k)}{2} \partial_{\sigma_c(c)}(k) g \partial_{\sigma_c(k)} f
\]

(2.13)

\[
f \ast x_k = x_k f g + \frac{i\theta_{\sigma_c(k)}}{2} \partial_{\sigma_c(k)} f \partial_{\sigma_c(c)}(k) g - \frac{i\theta_{\sigma_c(c)}(k)}{2} \partial_{\sigma_c(c)}(k) f \partial_{\sigma_c(k)} g.
\]

(2.14)

**Proposition 2.2 (3–ary star product of one function in \( \mathcal{A} \) and two coordinate functions)**

\[
x_k \ast x_{\sigma_c(k)} f = x_k f \partial_{\sigma_c(c)}(k) f + \frac{i\theta_k}{2} \partial_{\sigma_c(c)}(k) f
\]

(2.15)

\[
x_k \ast x_{\sigma_c(k)} = x_k f \partial_{\sigma_c(c)}(k) f - \frac{i\theta_k}{2} \partial_{\sigma_c(c)}(k) f
\]

(2.16)

\[
x_k \ast x_{\sigma_c(c)}(k) f = x_k f \partial_{\sigma_c(c)}(k) f - \frac{i\theta_k}{2} \partial_{\sigma_c(c)}(k) f
\]

(2.17)

\[
x_k \ast x_{\sigma_c(c)}(k) = x_k f \partial_{\sigma_c(c)}(k) f + \frac{i\theta_k}{2} \partial_{\sigma_c(c)}(k) f.
\]

(2.18)
Therefore the following interesting results hold for a 3-ary star product of any function \( f \in \mathcal{A} \) with two coordinate functions:

**Proposition 2.3 (3-ary star product complex conjugation)** Provided (2.1), we have, \( \forall f \in \mathcal{A} \),

\[
x_k^{\star a}(k) f = x_k f, \quad x_k^{\star a}(k) f = x_k f \quad (2.19)
\]

In opposite, when any two functions \( f, g \in \mathcal{A} \) enter in the 3-ary star product with a unique coordinate function, the star noncommutativity is clearly expressed, i.e.

\[
x_k^{g} f \neq f^{g} x_k, \quad x_k^{g} f \neq f^{g} x_k, \quad g^{f} \neq f^{g}.
\]

Furthermore, introducing complex variables \( a_{kl} \) and their conjugate \( \bar{a}_{kl} \) by

\[
a_{kl} = \frac{x_k + ix_l}{\sqrt{2}}, \quad \bar{a}_{kl} = \frac{x_k - ix_l}{\sqrt{2}}, \quad k, l = 1, 2, 3, \ l \neq k \quad (2.20)
\]

and using the equations (2.12) (2.13) and (2.14), we establish the relations given in the three next propositions:

**Proposition 2.4 (3-ary star product of two functions in \( \mathcal{A} \) and one complex coordinate function)**

\[
a_{ij} f^{g} = a_{ij} f g + i\theta_i \left( \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \partial_{\sigma^2(ii)} f \partial_{\sigma(i)} g \right) - \frac{\theta_i}{4} \left( \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \partial_{\sigma^2(ii)} f \partial_{\sigma(i)} g \right) \quad (2.21)
\]

\[
\bar{a}_{ij} f^{g} = \bar{a}_{ij} f g + i\theta_i \left( \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \partial_{\sigma^2(ii)} f \partial_{\sigma(i)} g \right) + \frac{\theta_i}{4} \left( \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \partial_{\sigma^2(ii)} f \partial_{\sigma(i)} g \right) \quad (2.22)
\]

**Proposition 2.5 (3-ary star product of two functions in \( \mathcal{A} \) and one complex coordinate function)**

\[
g^{f} a_{ij} = a_{ij} f g + i\theta_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \frac{i\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \frac{\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} g \partial_{\sigma^2(ii)} f + \frac{\theta_{\sigma^2(ii)}}{4} \partial_{\sigma^2(ii)} g \partial_{\sigma(i)} f \quad (2.23)
\]

\[
g^{f} \bar{a}_{ij} = \bar{a}_{ij} f g + i\theta_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \frac{i\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \frac{\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} g \partial_{\sigma^2(ii)} f + \frac{\theta_{\sigma^2(ii)}}{4} \partial_{\sigma^2(ii)} g \partial_{\sigma(i)} f \quad (2.24)
\]

**Proposition 2.6 (3-ary star product of two functions in \( \mathcal{A} \) and one complex coordinate function)**

\[
f^{a_{ij}} g = a_{ij} f g - \frac{i\theta_{\sigma(i)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g + \frac{i\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g + \frac{\theta_{\sigma(i)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \frac{\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g \quad (2.25)
\]

\[
f^{\bar{a}_{ij}} g = \bar{a}_{ij} f g - \frac{i\theta_{\sigma(i)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g + \frac{i\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g + \frac{\theta_{\sigma(i)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g - \frac{\theta_{\sigma^2(ii)}}{4} \partial_{\sigma(i)} f \partial_{\sigma^2(ii)} g \quad (2.26)
\]
Evidently, any two of these last results cannot be straightforwardly obtained by complex conjugation of each other. Indeed, we well get:

\[ a_{ij} f g \neq \bar{a}_{ij} \bar{f} \bar{g}, \quad g \ast a_{ij} \neq g \ast \bar{a}_{ij}, \quad a_{ij} f g \neq f a_{ij}, \quad f a_{ij} \neq \bar{f} \bar{a}_{ij}. \]

2.2 Integral representation

To construct the integral representation of the 3–ary star product (2.1), consider \( s, x \in \mathbb{R}^3 \) and the plane wave function of the form:

\[ \exp(isx) = \exp\left[i(s_1x_1 + s_2x_2 + s_3x_3)\right]. \] (2.27)

Then, their 3–ary star product gives

\[ e^{ikx} e^{iqx} \ast e^{irx} = e^{\sum_{j=1}^{3} \frac{\theta_j}{2} (k_j q_{\sigma_e(j)} r_{\sigma_e(j)} - k_j q_{\sigma_e(j)} r_{\sigma_e(j)}) + i(k+q+r)x}. \] (2.28)

Defining the quantity \( \Omega_j^{qr} \) by

\[ \Omega_j^{qr} = q_{\sigma_e(j)} r_{\sigma_e(j)} - q_{\sigma_e(j)} r_{\sigma_e(j)}, \] (2.29)

which satisfies the conditions

\[ \Omega_j^{qr} = -\Omega_j^{rq}, \quad p\Omega_j^{qr} = r\Omega_j^{pq} = q\Omega_j^{rp}, \] (2.30)

the integral representation of the 3–ary star product of functions can be expressed as follows:

\[ (f \ast g \ast h)(x) = \int d^3k d^3q d^3r \tilde{f}(k) \tilde{g}(q) \tilde{h}(r) \left( e^{ikx} e^{iqx} e^{irx} \right) \]

\[ = \frac{1}{(2\pi)^9} \int d^3k d^3q d^3r d^3y d^3z d^3t f(y) g(z) h(t) \times e^{\frac{i}{2} \sum j \theta_j k_j \Omega_j^{qr}} e^{ik(x-y)} e^{iq(x-z)} e^{ir(x-t)}. \] (2.31)

It can be simplified, using the identity

\[ \int d^3k e^{ik(x-y)} e^{i\frac{i}{2} \Omega_j^{qr}} = (2\pi)^3 \delta^{(3)}(x - y - \frac{i\theta}{2} \Omega_j^{qr}), \] (2.32)

into the form

\[ (f \ast g \ast h)(x) = \frac{1}{(2\pi)^6} \int d^3q d^3r d^3y d^3z d^3t f(y) g(z) h(t) \times \delta^{(3)}(x - y - \frac{i}{2} \Omega_j^{qr}) e^{iq(x-z)} e^{ir(x-t)}. \] (2.33)

3 Generalization to \( n \)--ary star product

In this section, we deal with the generalization of the 3–ary star product (2.1) into an \( n \)--ary star product.
**Definition 3.1** Consider \( x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \). Let \( \sigma_c \) be the cyclic permutation over the set \( \{1, 2, \cdots, d\} \), i.e. \( \sigma_c(k) = k + 1 \), \( k + 1 \leq n \), and let \( A = \( \mathcal{S}(\mathbb{R}^n), \star \) \) be the Schwartz space of (smooth, rapidly decreasing, together with all their derivatives, faster than the reciprocal of any polynomial at \( \infty \)) real valued functions on \( \mathbb{R}^n \), endowed with an \( n \)-ary star product defined at a point \( x \) as follows:

\[
\star \{ \cdot, \cdot, \cdots, \cdot \} : \quad \overbrace{A \times A \times \cdots \times A}^{n \text{ times}} \longrightarrow A \\
(f_1, f_2, \cdots, f_n) \longmapsto \star\{f_1, f_2, \cdots, f_n\} := \star\{f_i\}_{i=1}^n \tag{3.1}
\]

where

\[
\star\{f_i\}_{i=1}^n(x) = m\left[ e^{P(\theta_1, \theta_2, \cdots, \theta_n)} (f_1 \otimes f_2 \otimes \cdots \otimes f_n)(x) \right], \tag{3.2}
\]

\[
m(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \prod_{i=1}^n f_i, \tag{3.3}
\]

and

\[
P(\theta_1, \theta_2, \cdots, \theta_n) = \sum_{k=1}^n \frac{i\theta_k}{2} \left( \partial_k \otimes \partial_{\sigma_c(k)} \otimes \cdots \otimes \partial_{\sigma_n(k)} \right) - \partial_k \otimes \partial_{\sigma_c^{-1}(k)} \otimes \cdots \otimes \partial_{\sigma_n^{-1}(k)}. \tag{3.4}
\]

For \( f_i, g_i \in A, i \in \mathbb{N} \), we obtain:

**Proposition 3.1** \((n\text{-ary star product of functions in } A \text{ and coordinate functions})\)

\[
\star\{f_i, x_p, g_j\}_{i=1, \cdots, m-1, j=m+1, \cdots, n-1} = x_p \prod_{i=1}^{m-1} f_i \prod_{j=m+1}^n g_j \\
+ \frac{i\theta_{p-m+1}}{2} \prod_{i=1}^{m-1} \partial_{\sigma_c^{-1}(i)} f_i \prod_{i=m+1}^n \partial_{\sigma_n^{-1}(i)} g_i \\
- \frac{i\theta_{p+m-n-1}}{2} \prod_{i=1}^{m-1} \partial_{\sigma_c^{-1}(i)} f_i \prod_{i=m+1}^n \partial_{\sigma_n^{-1}(i)} g_i \tag{3.5}
\]

**Proposition 3.2** \((n\text{-ary star product of functions in } A \text{ and coordinate functions})\)

\[
\star\{f_i, x_p, g_j\}_{i=1, \cdots, m-1, j=m+1, \cdots, n-1} = x_p \prod_{i=1}^{m-1} f_i \prod_{j=m+1}^n g_j + \frac{i\theta_{p-m+1}}{2} \prod_{i=1}^{m-1} \partial_{\sigma_c^{-1}(i)} f_i \prod_{i=m+1}^n \partial_{\sigma_n^{-1}(i)} g_i \\
- \frac{i\theta_{p+m-n-1}}{2} \prod_{i=1}^{m-1} \partial_{\sigma_c^{-1}(i)} f_i \prod_{i=m+1}^n \partial_{\sigma_n^{-1}(i)} g_i. \tag{3.6}
\]

Therefore, for \( p, q \in \{1, 2, \cdots, n\} \), these relations show that any two of them cannot be given by complex conjugation. In fact, we have:

\[
\star\{a_{pq}, f_i\}_{i=1, \cdots, n-1} \neq \star\{a_{pq}, f_i\}_{i=1, \cdots, n-1}, \\
\star\{f_i, a_{pq}\}_{i=1, \cdots, n-1} \neq \star\{f_i, a_{pq}\}_{i=1, \cdots, n-1}
\]
\[ \{ f_i, a_{pq}, g_j \}_{i=1, \ldots, m-1, j=m+1, \ldots, n-1} \neq \{ f_i, a_{pq}, g_j \}_{i=1, \ldots, m-1, j=m+1, \ldots, n-1}. \]

The \( n \)-ary star product integral representation, for \( n \) arbitrary points, can also be computed by the same method as in the previous section \( 3 \). By considering the plane wave functions
\[ e^{i s x} = e^{i (s_1 x_1 + \cdots + s_n x_n)}, \quad s, x \in \mathbb{R}^n, \] (3.7)

the following result can be proved:

**Proposition 3.3** (\( n \)-ary star product integral representation)
\[ \{ f_i \}_{i=1}^n \{ \} = \frac{1}{(2\pi)^2 n} \int \prod_{j=1}^{n-1} d^n q_j \prod_{j=1}^{n} d^n y_j f(y_j) \delta^{(n)}(x - y - i \frac{q}{2} \Omega^r \theta) \prod_{j=1}^{n-1} e^{iq(x - y_j)}. \] (3.8)

4 Application

Consider a physical system described by the Hamiltonian model:
\[ H = \sum_{j=1}^{n} x_j^2 + \sum_{i<j} (\epsilon_{ij} \lambda_{ij} x_i x_j) + \sum_{i<j<k<l} (\epsilon_{ijkl} \lambda_{ijkl} x_i x_j x_k x_l) + \cdots \]
(4.1)

where \( \lambda_{ij}, k \leq n \), are the coupling constants and \( \epsilon_{ij} \) the Levi-Civita tensor of rank \( k \); \( k = n \) if \( n \) is even and \( n - 1 \) if \( n \) is odd. Using the orthogonal transformation \( \mathcal{R} \) such that \( \mathcal{R}_{kl} x_l = X_k \), the Hamiltonian (4.1) can be re-expressed in the new coordinates \( X \) as follows:
\[ H = \sum_{i=1}^{n} \lambda_i^{(0)} X_i^2 + \sum_{i=1}^{n} \lambda_i^{(2)} X_i^4 + \cdots \] (4.2)

allowing a re-formulation of terms in previous quantities \( a_{ij} \) and \( \tilde{a}_{ij} \), defined in (2.20), as follows:
\[ H = \sum_{i,j=1}^{n} \sum_{p=0}^{n-1} \left( \lambda_i^{(2p)} a_{ij} \tilde{a}_{ij} \right)^{p+1}. \] (4.3)

For \( \psi^m \in \mathcal{A} \), the eigenvalue problem is given by the system of equations:
\[ \{ H, \psi^m_1, \ldots, \psi^m_n \} = E_{1,n} \left\{ 1, \psi^m_1, \cdots, \psi^m_n \right\} \] (4.4)
\[ \{ \psi^m_1, H, \psi^m_2, \ldots, \psi^m_n \} = E_{2,n} \left\{ \psi^m_1, 1, \psi^m_2, \cdots, \psi^m_n \right\} \] (4.5)
\[ \cdots \]
\[ \{ \psi^m_1, \psi^m_2, \ldots, \psi^m_n, H \} = E_{n,n} \left\{ \psi^m_1, \psi^m_2, \cdots, \psi^m_n, 1 \right\}. \] (4.6)
\( n \in \mathbb{N}^n \) is an \( n \)-vector characterizing the quantum number associated to the Hamiltonian (4.1) while \( \psi^m_j, j = 1, 2, \ldots n - 1; m = 1, 2, \ldots, n \), are the eigenstates diagonalizing it. The ground state satisfies the equation
\[ \{ a_{ij}, \psi^0_1, \cdots, \psi^0_n \} = 0 \] (4.7)
which can be explicitly solved to give:
\[
\psi^0_k = C e^{-|x|^2/2}H_k(|x|^2/2)f(\lambda_k, |x|), \quad C \in \mathbb{R}
\]

where \(H_k\) is the Hermite polynomial; the functions \(f(\lambda_k, |x|) := f_k\) are orthogonal with the orthogonality condition
\[
\star \{f_1, f_2, \cdots, f_n\} = C^n \in \mathbb{R}.
\]

The excited states can be computed by using the well-known harmonic oscillator algebraic method performed with the raising operator, acting here as follows:
\[
\star \{\bar{a}_{ij}, \psi^0_1, \cdots, \psi^0_{n-1}\} = f_1(n)(\star \{1, \psi^1_1, \cdots, \psi^1_{n-1}\}),
\]

where \(f_1(n)\) is a function depending on the parameter \(n\).

There result the following expressions for the eigenvalues and eigenfunctions characterizing the considered physical model:
\[
E_k^n = \theta_k \left( \lambda^{(0)}_k |\bar{n}| + \sum_{i=1}^n \lambda^{(k)}_i |\bar{n}|^2 + \sum_{i,j=1}^n \lambda^{(k)}_i \lambda^{(k)}_j |\bar{n}|^3 + \cdots + \frac{n}{2} \right),
\]

and
\[
\psi^n_k = C e^{-|x|^2/2}H_k^n(|x|^2/2)f(\lambda_k, |x|).
\]

\(H_k^n\) stands for the \((n, k)\)-order Hermite polynomial.

5 Concluding remarks

In this work, we have given a method of constructing an \(n\)-ary star product. Relevant identities have been provided and discussed. A physical problem of coupled oscillators has been treated. The associated eigenvalues and eigenfunctions have been explicitly computed.

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