Measurement-based quantum phase estimation algorithm for finding eigenvalues of non-Hermitian matrices

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We propose a new quantum algorithm for finding eigenvalues of non-Hermitian matrices. We show how to construct, through interactions in a quantum system and partial measurements, a non-Hermitian matrix and obtain its eigenvalues and eigenvectors. This proposal combines ideas of frequent partial measurement, measured quantum Fourier transform, and quantum state tomography. Similar to the conventional phase estimation algorithm, our algorithm provides a speed-up over its classical counterpart. Moreover, in our approach the controlled unitary operation is implemented only once. In addition, the algorithm provides a high success probability.

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Introduction.— One of the most important tasks for a quantum computer is to efficiently obtain eigenvalues and eigenvectors of high-dimensional matrices. It has been suggested [1] that the quantum Phase Estimation Algorithm (PEA) [2] can be used to obtain eigenvalues of a Hermitian matrix or Hamiltonian. For a quantum system with a Hamiltonian $H$, a phase factor, which encodes the information of eigenvalues of $H$, is generated via the unitary evolution $U = e^{-iH\tau}$. By evaluating this phase, we can obtain the eigenvalues of $H$. The standard PEA consists of four steps: preparing an initial approximate eigenstate of the Hamiltonian $H$, implementing unitary evolution operation, performing the inverse quantum Fourier transformation (QFT), and measuring binary digits of the index qubits.

The PEA is at the heart of a variety of quantum algorithms, including Shor’s factoring algorithm [3]. A number of applications of PEA have been developed, including generating eigenstates associated with an operator [4], evaluating eigenvalues of differential operators [5], and it has been generalized using adaptive measurement theory to achieve a quantum-enhanced measurement precision at the Heisenberg limit [6]. The PEA has been applied in quantum chemistry to obtain eigenenergies of molecular systems [7, 8]. This application has been demonstrated in a recent experiment [9].

PEA is one of the few quantum algorithms that provide speed-up over their classical counterparts. It is only designed for finding eigenvalues of a Hermitian matrix. However, in modern engineering and science, people are often more interested in eigen-problems of non-Hermitian matrices. In this paper, we propose a measurement-based phase estimation algorithm (MPEA) to evaluate eigenvalues of non-Hermitian matrices. Our proposal concatenates ideas in conventional PEA, partial measurement, and current techniques in one-qubit state tomography. The method in this proposal can be used to design useful quantum algorithms apart from those based on the standard unitary circuit model. Like the conventional PEA, we expect our MPEA to play a key role in discovering new quantum algorithms.

Constructing non-Hermitian matrices.—Now we describe how to construct non-Hermitian matrices for a quantum system. A bipartite system, composed of subsystems $A$ and $B$, evolves under $H = H_A + H_B + H_{AB}$, where $H_A(B)$ is the Hamiltonian of subsystem $A(B)$ and $H_{AB}$ is their interaction. If subsystem $A$ is subject to a projective measurement $M = |\varphi^A\rangle \langle \varphi^A|$ applied during a time interval $\tau$, where $|\varphi^A\rangle$ is a pure state of subsystem $A$, this is equivalent to driving subsystem $B$ with an evolution matrix $V_B(\tau) = |\varphi^A\rangle e^{-iH\tau} |\varphi^A\rangle$. This evolution matrix is in general neither unitary nor Hermitian. We can construct such a matrix $V_B$ by choosing an appropriate basis for the partial measurements.

The Hamiltonian $H$ of the whole quantum system can be spanned as: $H = \sum_{j=1}^{D} E_j |E_j\rangle \langle E_j|$, with eigenenergies $E_j$. The eigenvector $|E_j\rangle$ can be spanned in terms of tensor products of basis vectors $\{|\psi_A^1\rangle\}$ and $\{|\psi_B^j\rangle\}$ of Hilbert spaces of subsystems $A$ and $B$, which are of dimensions $D_A$ and $D_B$, respectively, and $D = D_A \times D_B$. Using basis for $A$ and $B$, one can expand $|E_j\rangle = \sum_{k,l} \sum_{r,s} f_{kr}^j f_{ls}^j |\psi_A^k\rangle \otimes |\psi_B^s\rangle$, and the evolution matrix on subsystem $B$, after a measurement on $A$ during a time interval $\tau$, becomes

$$V_B(\tau) = \langle \varphi^A | e^{-iH\tau} | \varphi^A \rangle = \sum_{r,s=1}^{D_B} V_{rs} |\psi_B^r\rangle \langle \psi_B^s|,$$

where $V_{rs} = \sum_{j=1}^{D} e^{-iE_j \tau} \sum_{k,l=1}^{D_A} f_{kr}^j f_{ls}^j c_k^A c_l^A$, and $c_k^A = \langle \varphi^A | \psi_A^k \rangle$. We can thus construct any non-Hermitian matrix that can be written in this form on a quantum system for any given arbitrary $f_{kr}^j$ and $|\varphi^A\rangle$.

More generally, we can construct a different evolution matrix by performing measurements on subsystem $A$ with different time intervals and/or different measurement bases. For example, by sequentially performing partial measurements with time intervals $\tau_1, \tau_2, \tau_3$, an evolution matrix $V_B(\tau_1, \tau_2, \tau_3) = V_B(\tau_3) V_B(\tau_2) V_B(\tau_1)$ is constructed.
can also combine unitary evolution matrices with the non-Hermitian transformations on subsystem $B$ to construct some desired evolution matrices.

**Measurement-based quantum phase estimation algorithm.**—

We now present a MPEA for evaluating the eigenvalues and eigenvectors of a non-Hermitian matrix $V_B(\tau)$. There are three primary steps in the MPEA: (1) preparing an initial state; (2) constructing controlled successive binary powers of the evolution matrix $V_B(\tau)$; and (3) retrieving the eigenvalue of the evolution matrix $V_B(\tau)$ using either quantum state tomography (QST) \[13\] or a measured quantum Fourier transform (mQFT) combined with projective measurements.

We now prepare the system in a separable state $\rho_0 = |\varphi_A\rangle\langle\varphi_A| \otimes \rho_B$, with $A$ in a pure state $|\varphi_A\rangle$ and $B$ in an arbitrary (mixed) state $\rho_B$. Constructing the controlled evolution matrix of $V_B(\tau)$ on $B$ is achieved by implementing the Controlled Unitary transformation ($C-U$), for the whole quantum system and performing the periodic measurement $M = |\varphi_A\rangle\langle\varphi_A|$ on $A$ with period $\tau$. In order to resolve the phase factor up to $n$ binary digits using the inverse QFT, one has to construct a series of evolution matrices of successive binary powers, from $(n-1)$ to 0. In the MPEA, this is done by implementing the $C-U$ operation for the whole system once and performing a series of $2^k$ periodic measurements in time intervals $\tau$ for a number of successive binary powers of $k$. Then we can obtain a series of transformation matrices in binary powers of $V_B(\tau)$, $[V_B(\tau)]^2^k$, $k = (n-1), (n-2), \ldots, 0$.

It can be shown that after $m$ successful partial measurements on $A$, the evolution operation on the Hilbert space of subsystem $B$ is $[V_B(\tau)]^m$ and the state of $B$ evolves to $|1\rangle$,

$$\rho_B^{(m)}(m) = [V_B(\tau)]^m \rho_B [V_B(\tau)]^m / P_m,$$

where $P_m = \text{Tr}_B\{|V_B(\tau)]^m \rho_B [V_B(\tau)]^m\}$ is the probability of finding subsystem $A$ still in its initial state $|\varphi_A\rangle$ after $m$ measurements. The evolution matrix $V_B(\tau)$ can be spanned as $V_B(\tau) = \sum_{\lambda_k} \lambda_k |u_k\rangle\langle v_k|$, where $|u_k\rangle$ and $|v_k\rangle$ are the right- and left-eigenvectors and $\lambda_k$ is the corresponding eigenvalue satisfying $0 \leq |\lambda_k| \leq 1$ for arbitrary $k \[11\]$. In the large $m$ limit, the operator $[V_B(\tau)]^m$ is dominated by a single term $\lambda_{\text{max}}^m |u_{\text{max}}\rangle\langle v_{\text{max}}|$, provided the largest eigenvalue $\lambda_{\text{max}}$ is unique, discrete, and nondegenerate. In the limit of large $m$ and finite $\tau$, $\rho_B^{(m)}$ tends to a pure state, independent of the initial (mixed) state of system $B$. The final state of $\rho_B^{(m)}$ is dominated by $|u_{\text{max}}\rangle$, and this outcome is found with probability $P_m \to |\lambda_{\text{max}}|^2 |u_{\text{max}}\rangle\langle u_{\text{max}}| \rho_B |v_{\text{max}}\rangle$. The state of subsystem $B$ evolves to $|u_{\text{max}}\rangle$, after performing a number of operations of $V_B(\tau)$.

Then we can evaluate $\lambda_{\text{max}}$ by resolving the phase of the state.

The circuit for the MPEA is shown in Fig. 1. In Fig. 1(a), we show the circuit for $4\theta$ partial measurement with period $\pi$, $W(k)$, where a measurement $M = |\varphi_A\rangle\langle\varphi_A|$ is performed $2^k$ times in the period $\pi$, as the whole system evolves under the controlled unitary operation. Fig. 1(b) shows the MPEA circuit; from top to bottom: three quantum registers are prepared: an index register, a target register and an interacting register. The index register is a single qubit, which is used as control qubit and to obtain the final results; the target register is used to represent the state of subsystem $B$; the interacting register is used to represent the state of subsystem $A$, and to perform measurements.

The initial state of the MPEA circuit is prepared in state $|0\rangle + |1\rangle |\varphi_A\rangle + |\varphi_A\rangle |\varphi_A\rangle$, after performing the $(n-1)$th periodic measurements $W(n-1)$ on the interacting register, the state of the system is transformed to $(|0\rangle + |1\rangle [V_B(\tau)]^{2^n-1}) (|0\rangle + |1\rangle [V_B(\tau)]^{2^n-1}) \otimes \rho_B \otimes |\varphi_A\rangle\langle\varphi_A|.$

The dominating term is $(|0\rangle + |\lambda_{\text{max}}^{2^n-1}|0\rangle) |\lambda_{\text{max}}| |\varphi_A\rangle$. There are two approaches to resolve $\lambda_{\text{max}}$ at this stage. In QST, we prepare a large number of identical copies of the state on the index qubit, we can obtain $\lambda_{\text{max}}$ by determining the index qubit state. In the second approach, we combine the mQFT technique and projective measurement. The phases which encode the eigenvalues of $V_B(\tau)$ in general are complex numbers, the QFT can only help resolving the real part of the phase. In this approach, we perform measurements on subsystem $A$ for $n$ series in sequence of $W(n-1), W(n-2), \cdots, W(0)$, then we obtain single qubit states $(|0\rangle + |\lambda_{\text{max}}^{2^n-1}|1\rangle), (|0\rangle + |\lambda_{\text{max}}^{2^n-2}|1\rangle), \cdots, (|0\rangle + |\lambda_{\text{max}}|1\rangle)$ on the index qubit. The real part of the phase factor of $\lambda_{\text{max}}$ can be obtained by performing mQFT. This procedure is shown in Fig. 1, where the circuit in the dashed square in Fig. 1(b) is replaced by circuits in the dashed squares shown in Fig. 1(c) sequentially to obtain $n$ binary digits of the phase. The technique for resolving the imaginary part of the phase factor will be discussed in the following sections. Next we give details of obtaining all eigenvalues of $V_B(\tau)$.

**Obtaining complex eigenvalues through state tomography.**—

QST can fully characterize the quantum state of a particle or particles through a series of measurements in different bases $\[13\]$. In MPEA, after performing a large number, $m$, of partial measurements, the state of the index qubit is dominated by $|\psi_{\text{ind}}\rangle = |0\rangle + (|\lambda_{\text{max}}|)^m |1\rangle$. In general $\lambda_{\text{max}}$ can be written as $\lambda_{\text{max}} = e^{i\phi} = e^{i(\alpha + ib)}$. We can obtain $\lambda_{\text{max}}$ by retrieving the phase factor $\phi = (\alpha + ib)$.

Using QST, we prepare a large number of identical copies of the single qubit state $|\psi_{\text{ind}}\rangle = |0\rangle + e^{-imb} e^{ima} |1\rangle$. To determine this state, we can perform a projective measurement in the basis $|1\rangle$ to obtain the probability of finding the state as $|1\rangle$, thus obtaining the value of $b$. Performing a $\pi/2$ rotation around the $x$-axis and a measurement in the basis of the Pauli matrix $\sigma_z$ on the index qubit, we can obtain the observable $|\psi_{\text{ind}}\rangle e^{-i\phi} \sigma_z e^{i\phi} |\psi_{\text{ind}}\rangle$ and thus obtain the value of $a$.

The measurement errors of QST, from counting statistics, obey the central limit theorem. To obtain more accurate results, we have to prepare a larger ensemble of the single qubit states.

**Obtaining complex eigenvalues through mQFT and projective measurements.**—

In the MPEA circuit shown in Fig. 1, we resolve the real part of the phase factor using mQFT. The mQFT technique implements QFT using only a single qubit $\[10\]$. It uses the fact that gates within the Fourier trans-
form are applied sequentially on qubits. This modification of the QFT algorithm preserves the probabilities of all measurements \[12\]. We employ a projective measurement to resolve the imaginary part of the phase factor.

In the MPEA, each time when we perform the periodic partial measurements \(W(k)\), we obtain a state \(|0\rangle + (\lambda_{\text{max}})^k |1\rangle\) on the index qubit. We rewrite this state as \(|\psi_{\text{ind}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i(\alpha + ib)}|1\rangle)\). Let \((1/\sqrt{2})^2 + (e^{-b}/\sqrt{2})^2 = r^2\), then we have \(|\psi_{\text{ind}}\rangle = \frac{1}{r\sqrt{2}}(|0\rangle + e^{-b}e^{ia}|1\rangle)\). We now prepare a large number of identical copies of \(|\psi_{\text{ind}}\rangle\) by running the MPEA for many times. Then, performing a projective measurement \(|0\rangle \langle 0|\) on the ensemble of qubits, we can evaluate the probability for observing the qubit in state \(|0\rangle\), and therefore obtain \(r\) and \(b\).

After we obtain the value of \(b\), we run MPEA again and perform a single qubit operation on the index qubit such that the index qubit is rotated to the state \(|\psi_{\text{ind}}\rangle = \frac{1}{\sqrt{2}}((0) + e^{i\alpha}|1\rangle)\). Then we apply the mQFT technique to resolve the real part \(\alpha\) of the phase factor \(\alpha + ib\). We therefore obtain the eigenvalue \(e^{i(a+ib)}\) of the non-Hermitian matrix \(V_B(\tau)\).

In the above procedure, the state on the target register is driven to the eigenstate with the largest eigenvalue of \(V_B(\tau)\) by performing periodic partial measurements. The target state can evolve to the eigenstate with the second largest eigenvalue of \(V_B(\tau)\) by projecting out \(|u_{\text{max}}\rangle\) in preparing the initial state. If we prepare the initial state on the target state as \(\rho_B(1 - |u_{\text{max}}\rangle \langle u_{\text{max}}|\rho_B|u_{\text{max}}\rangle)\), and feed it into the MPEA circuit, we obtain the second largest eigenvalue of \(V_B(\tau)\). All eigenvalues of \(V_B(\tau)\) can be obtained by repeating this procedure.

The MPEA resolves the \(n\)th binary digit of the phase factor first, and the partial measurement on \(A\) is performed in sequence of \(2^{n-1} \rightarrow 2^{n-2} \rightarrow 2^0\) times. This procedure provides high fidelity for the state of subsystem \(B\) since each measurement drives the state of \(B\) closer to the eigenstate of \(V_B(\tau)\). The success probability of the MPEA is the product of the success probability of the MPEA measurements on subsystem \(A\). Since this fidelity is close to one, the success probability only depends on the probability of successful measurements on \(A\).

An example.-- Let us now use a simple model to show how MPEA works. We consider a quantum system consisting of two subsystems \(A\) and \(B\), where \(B\) contains two non-interacting spin qubits, \(B_1\) and \(B_2\); and \(A\) is a photon. The whole system can be described by the Jaynes-Cummings Hamiltonian \[13\]

\[
H = w_0b^\dagger b + w_1(\sigma_z^1 + \sigma_z^2) + \frac{J}{2}[b(\sigma_z^+1 + \sigma_z^+2) + b^\dagger(\sigma_z^-1 + \sigma_z^-2)],
\]

where \(b (b^\dagger)\) is a bosonic annihilation (creation) operator of the photons. Consider the case \(w_0 = w_1\), and perform partial measurements in the basis of a single photon state \(|\varphi_A\rangle = |1\rangle\). We thus have \(V_B(\tau) = \text{diag}\{1, [3 + 2 \cos(\sqrt{10}\tau J)/5 \times e^{-2\sin \tau \cos(\sqrt{6}/2)e^{-i\tau J}/2}, \cos(\sqrt{6}/2)e^{-i\tau J}/2, \cos(\sqrt{2}/2)e^{-i\tau J}/2]\}, \text{in the ordered basis}\{1, |s\rangle, |t_+\rangle, |t_0\rangle, |t_-\rangle\}\), where \(|s\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), |t_+\rangle = |11\rangle, |t_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\) and \(|t_-\rangle = |00\rangle\). Let \(\tau = 1/2\) and \(w_0 = w_1 = J = 1\), then, the evolution matrix \(V_B(\tau)\) becomes \(V_B(\tau) = \text{diag}\{1, [3 + 2 \cos(\sqrt{10}/2)/5 \times e^{-i\tau}, \cos(\sqrt{6}/2)e^{-i\tau/2}, \cos(\sqrt{2}/2)e^{-i\tau/2}]\}\).

Set now the initial state of subsystem \(B\) as a mixed state: \(\rho_B = \frac{1}{2}(|s\rangle \langle s| + |t_+\rangle \langle t_+| + |t_0\rangle \langle t_0| + |t_-\rangle \langle t_-|), and apply the MPEA to this system. After a series of partial measurements in the basis of \(|\varphi_A\rangle = |1\rangle\), the state of subsystem \(B\) evolves to a singlet state \(|s\rangle\), which corresponds to the largest eigenvalue of \(V_B(\tau)\). We resolve the corresponding phase as zero, thus the eigenvalue is one. The survival probability, \(P_m\), of the state \(|\varphi_A\rangle = |1\rangle\) on subsystem \(A\) after \(m\) successful measurements, and the fidelity, \(F(m)\), for subsystem \(B\) to be in state \(|u_{\text{max}}\rangle\) are shown in Fig. 2. Since \(F(m)\) is close to one, the success probability is determined by \(P_m\).

If we start from a pure initial state of subsystem \(B\) and ap-
MPEA, the state of $B$ can evolve to other eigenstates of $V_B$. For example, applying the MPEA to the above system using $|t_\pm\rangle$ as the initial state for subsystem $B$, by performing partial measurements in terms of $|\varphi_A\rangle = |1\rangle$ on $A$, the state of the system $B$ would remain in the $|t_\pm\rangle$ state. We can retrieve the real part of the phase factor of the corresponding eigenvalue up to an accuracy of 2, 8, and 16 binary digits, respectively, and obtain the eigenvalues as $\exp[-0.5177 - i2\pi(0.25 \pm 0.25)], \exp[-0.5177 - i2\pi(0.160 \pm 0.008)],$ and $\exp[-0.5177 - i2\pi(0.15918 \pm 0.00003)]$, assuming we obtain the imaginary part through QST and projective measurement exactly. The true eigenvalue is $\exp(-0.5177 - i)$.

Constructing the Hamiltonian for performing the controlled unitary operation.— On the MPEA circuit, we set the control qubit as a single spin and label it as subsystem $C$. Thus, the controlled Hamiltonian of the whole system becomes

$$\bar{H} = \frac{1}{2}(1-\sigma^z)H = \frac{1}{2}(1-\sigma^z)\{\omega b^\dagger b + w_1(\sigma^+_1 + \sigma^-_2) + \frac{J}{2}[b(\sigma^+_1 + \sigma^+_2) + b^\dagger(\sigma^-_1 + \sigma^-_2)]\}. \quad (4)$$

This Hamiltonian contains three-body interactions and cannot be implemented directly. One could decompose the three-body interaction into two-body interactions \[16\] \[17\] and then implement them. Generally, an arbitrary unitary evolution matrix $U = e^{-i\bar{H}t}$ can be decomposed into tensor products of unitary matrices of $4 \times 4$ and $2 \times 2$ \[18\] \[19\], which correspond to two- and single-qubit operations respectively, and can be implemented on a universal quantum computer.

Conclusion.—We have presented a measurement-based quantum phase estimation algorithm to obtain the eigenvalues and corresponding eigenvectors of non-Hermitian matrices. In MPEA, we implement the unitary transformation of the whole system only once; the non-Hermitian matrix is constructed as the evolution matrix on the target register. By performing periodic partial measurements on the interacting register, the state of the target register is driven automatically to an eigenstate of the transformation matrix. Using single-qubit tomography and mQFT, we can obtain the complex eigenvalues of the non-Hermitian matrix. This algorithm can be used to study open quantum system and in developing other new quantum algorithms.

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