**HOPF-CYCLIC HOMOLOGY WITH CONTRAMODULE COEFFICIENTS**

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**ABSTRACT.** A new class of coefficients for the Hopf-cyclic homology of module algebras and coalgebras is introduced. These coefficients, termed *stable anti-Yetter-Drinfeld contramodules*, are both modules and *contramodules* of a Hopf algebra that satisfy certain compatibility conditions.

1. **Introduction.** It has been demonstrated in [8], [9] that the Hopf-cyclic homology developed by Connes and Moscovici [5] admits a class of non-trivial coefficients. These coefficients, termed *anti-Yetter-Drinfeld modules* are modules and comodules of a Hopf algebra satisfying a compatibility condition reminiscent of that for cross modules. The aim of this note is to show that the Hopf-cyclic (co)homology of module coalgebras and module algebras also admits coefficients that are modules and *contramodules* of a Hopf algebra with a compatibility condition.

All (associative and unital) algebras, (coassociative and counital) coalgebras in this note are over a field $k$. The coproduct in a coalgebra $C$ is denoted by $\Delta_C$, and counit by $\varepsilon_C$. A Hopf algebra $H$ is assumed to have a bijective antipode $S$.

We use the standard Sweedler notation for coproduct $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$, $\Delta_C^2(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc., and for the left coaction $N\varrho$ a $C$-comodule $N$, $N\varrho(x) = x_{(-1)} \otimes x_{(0)}$ (in all cases summation is implicit). $\text{Hom}(V, W)$ denotes the space of $k$-linear maps between vector spaces $V$ and $W$.

2. **Contramodules.** The notion of a *contramodule* for a coalgebra was introduced in [6], and discussed in parallel with that of a comodule. A *right contramodule* of a coalgebra $C$ is a vector space $M$ together with a $k$-linear map $\alpha : \text{Hom}(C, M) \rightarrow M$ rendering the following diagrams commutative

$$
\begin{array}{c}
\text{Hom}(C, \text{Hom}(C, M)) \xrightarrow{\text{Hom}(C, \alpha)} \text{Hom}(C, M) \\
\downarrow \Theta \\
\text{Hom}(C \otimes C, M) \xrightarrow{\text{Hom}(\Delta_C, M)} \text{Hom}(C, M) \xrightarrow{\alpha} M, \\
\end{array}
$$

$$
\begin{array}{c}
\text{Hom}(k, M) \xrightarrow{\text{Hom}(\varepsilon_C, M)} \text{Hom}(C, M) \\
\downarrow \simeq \\
M, \\
\end{array}
$$

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where $\Theta$ is the standard isomorphism given by $\Theta(f)(c \otimes c') = \Theta(f)(c)(c')$. Left contramodules are defined by similar diagrams, in which $\Theta$ is replaced by the isomorphism $\Theta'(f)(c \otimes c') = f(c')(c)$ (or equivalently, as right contramodules for the co-opposite coalgebra $C^{op}$). Writing blanks for the arguments, and denoting by matching dots the respective functions $\alpha$ and their arguments, the definition of a right $C$-contramodule can be explicitly written as, for all $f \in \text{Hom}(C \otimes C, M)$, $m \in M$,

$$\hat{\alpha}(\hat{\alpha}(f(-) \otimes -)) = \alpha(f((-)(1) \otimes (-)(2))), \quad \alpha(\varepsilon_C(-)m) = m.$$  

With the same conventions the conditions for left contramodules are

$$\hat{\alpha}(\hat{\alpha}(f(-) \otimes -)) = \alpha(f((-)(1) \otimes (-)(2))), \quad \alpha(\varepsilon_C(-)m) = m.$$  

If $N$ is a left $C$-comodule with coaction $\delta^N : N \to C \otimes N$, then its dual vector space $M = N^* := \text{Hom}(N, k)$ is a right $C$-contramodule with the structure map

$$\alpha : \text{Hom}(C, M) \cong \text{Hom}(C \otimes N, k) \to \text{Hom}(N, k) = M, \quad \alpha = \text{Hom}(\delta^N, k).$$  

Explicitly, $\alpha$ sends a functional $f$ on $C \otimes N$ to the functional $\alpha(f)$ on $N$,

$$\alpha(f)(x) = f(x_{(-1)} \otimes x_{(0)}), \quad x \in N.$$  

The dual vector space of a right $C$-comodule $N$ with a coaction $\rho^N : N \to N \otimes C$ is a left $C$-contramodule with the structure map $\alpha = \text{Hom}(\rho^N, k)$. The reader interested in more detailed accounts of the contramodule theory is referred to [1], [14].

3. **Anti-Yetter-Drinfeld contramodules.** Given a Hopf algebra $H$ with a bijective antipode $S$, anti-Yetter-Drinfeld contramodules are defined as $H$-modules and $H$-contramodules with a compatibility condition. Very much the same as in the case of anti-Yetter-Drinfeld modules [7] they come in four different flavours.

1. A **left-left anti-Yetter-Drinfeld contramodule** is a left $H$-module (with the action denoted by a dot) and a left $H$-contramodule with the structure map $\alpha$, such that, for all $h \in H$ and $f \in \text{Hom}(H, M)$,

$$h \cdot \alpha(f) = \alpha(h_{(2)} \cdot f(S^{-1}(h_{(1)})(-))h_{(3)}).$$

$M$ is said to be stable, provided that, for all $m \in M$, $\alpha(r_m) = m$, where $r_m : H \to M$, $h \mapsto h \cdot m$.

2. A **left-right anti-Yetter-Drinfeld contramodule** is a left $H$-module and a right $H$-contramodule, such that, for all $h \in H$ and $f \in \text{Hom}(H, M)$,

$$h \cdot \alpha(f) = \alpha(h_{(2)} \cdot f(S(h_{(3)})(-))h_{(1)}).$$

$M$ is said to be stable, provided that, for all $m \in M$, $\alpha(r_m) = m$.

3. A **right-left anti-Yetter-Drinfeld contramodule** is a right $H$-module and a left $H$-contramodule, such that, for all $h \in H$ and $f \in \text{Hom}(H, M)$,

$$\alpha(f) \cdot h = \alpha(f(h_{(3)}(-))S(h_{(1)})) \cdot h_{(2)}.$$  

$M$ is said to be stable, provided that, for all $m \in M$, $\alpha(\ell_m) = m$, where $\ell_m : H \to M$, $h \mapsto m \cdot h$. 


(4) A right-right anti-Yetter-Drinfeld contramodule is a right $H$-module and a right $H$-contramodule, such that, for all $h \in H$ and $f \in \text{Hom}(H, M)$,
\[
\alpha(f) \cdot h = \alpha \left( f \left( h(1)(-S^{-1}(h(3))) \right) \cdot h(2) \right).
\]
$M$ is said to be stable, provided that, for all $m \in M$, $\alpha(\ell_m) = m$.

In a less direct, but more formal way, the compatibility condition for left-left anti-Yetter-Drinfeld contramodules can be stated as follows. For all $h \in H$ and $f \in \text{Hom}(H, M)$, define $k$-linear maps $\ell \ell_{f,h} : H \to M$, by
\[
\ell \ell_{f,h} : h' \mapsto h(2) \cdot f \left( S^{-1}(h(1)) h' h(3) \right).
\]
Then the main condition in (1) is
\[
h \cdot \alpha(f) = \alpha \left( \ell \ell_{f,h} \right), \quad \forall h \in H, \ f \in \text{Hom}(H, M).
\]
Compatibility conditions between action and the structure maps $\alpha$ in (2)–(4) can be written in analogous ways.

If $N$ is an anti-Yetter-Drinfeld module, then its dual $M = N^*$ is an anti-Yetter-Drinfeld contramodule (with the sides interchanged). Stable anti-Yetter-Drinfeld modules correspond to stable contramodules. For example, consider a right-left Yetter-Drinfeld module $N$. The compatibility between the right action and left coaction $N^\ell$ thus is, for all $x \in N$ and $h \in H$,
\[
N^\ell (x \cdot h) = S(h(3)) x_{(-1)} h(1) \otimes x_{(0)} h(2).
\]
The dual vector space $M = N^*$ is a left $H$-module by $h \otimes m \mapsto h \cdot m$,
\[
(h \cdot m)(x) = m(x \cdot h),
\]
for all $h \in H, m \in M = \text{Hom}(N, k)$ and $x \in N$, and a right $H$-contramodule with the structure map $\alpha(f) = f \circ N^\ell, f \in \text{Hom}(H \otimes N, k) \simeq \text{Hom}(H, M)$. The space $\text{Hom}(H \otimes N, k)$ is a left $H$-module by $(h \cdot f)(h' \otimes x) = f(h' \otimes x \cdot h)$. Hence
\[
(h \cdot \alpha(f))(x) = \alpha(f)(x \cdot h) = f \left( N^\ell (x \cdot h) \right),
\]
and
\[
\alpha \left( h(2) \cdot f \left( S(h(3))(-h(1)) \right) \right)(x) = h(2) \cdot f \left( S(h(3)) x_{(-1)} h(1) \otimes x_{(0)} \right) = f \left( S(h(3)) x_{(-1)} h(1) \otimes x_{(0)} \cdot h(2) \right).
\]
Therefore, the compatibility condition in item (2) is satisfied. The $k$-linear map $r_m : H \to M$ is identified with $r_m : H \otimes N \to k, r_m(h \otimes x) = m(x \cdot h)$. In view of this identification, the stability condition comes out as, for all $m \in M$ and $x \in N$,
\[
m(x) = \alpha(r_m)(x) = r_m(x_{(-1)} \otimes x_{(0)}) = m(x_{(0)} \cdot x_{(-1)}),
\]
and is satisfied provided $N$ is a stable right-left anti-Yetter-Drinfeld module. Similar calculations establish connections between other versions of anti-Yetter-Drinfeld modules and contramodules.

4. Hopf-cyclic homology of module coalgebras. Let $C$ be a left $H$-module coalgebra. This means that $C$ is a coalgebra and a left $H$-comodule such that, for all $c \in C$ and $h \in H$,
\[
\Delta_C(h \cdot c) = h(1) \cdot c_{(1)} \otimes h(2) \cdot c_{(2)}, \quad \varepsilon_C(h \cdot c) = \varepsilon_H(h) \varepsilon_C(c).
\]
The multiple tensor product of $C$, $C^\otimes n+1$, is a left $H$-module by the diagonal action, that is

$$h \cdot (c^0 \otimes c^1 \otimes \ldots \otimes c^n) := h(1) \cdot c^0 \otimes h(2) \cdot c^1 \otimes \ldots \otimes h(n+1) \cdot c^n.$$  

Let $M$ be a stable left-right anti-Yetter-Drinfeld contramodule. For all positive integers $n$, set $C^H_n(C, M) := \text{Hom}_H(C^\otimes n+1, M)$ (left $H$-module maps), and, for all $0 \leq i, j \leq n$, define $d_i : C^H_n(C, M) \rightarrow C^H_{n-1}(C, M)$, $s_j : C^H_n(C, M) \rightarrow C^H_{n+1}(C, M)$, $t_n : C^H_n(C, M) \rightarrow C^H_n(C, M)$, by

$$d_i(f)(c^0, \ldots, c^{n-1}) = f(c^0, \ldots, \Delta_C(c^i), \ldots, c^{n-1}), \quad 0 \leq i < n,$$

$$d_n(f)(c^0, \ldots, c^{n-1}) = \alpha \left( f \left( c^0, c^1, \ldots, c^{n-1}, (-) \cdot c^0(1) \right) \right),$$

$$s_j(f)(c^0, \ldots, c^{n+1}) = \varepsilon_C(c^{j+1}) f(c^0, \ldots, c^j, c^{j+2}, \ldots, c^{n+1}),$$

$$t_n(f)(c^0, \ldots, c^n) = \alpha \left( f \left( c^1, \ldots, c^n, (-) \cdot c^0 \right) \right).$$

It is clear that all the maps $s_j, d_i, i < n$, are well-defined, i.e. they send left $H$-linear maps to left $H$-linear maps. That $d_n$ and $t_n$ are well-defined follows by the anti-Yetter-Drinfeld condition. To illustrate how the anti-Yetter-Drinfeld condition enters here we check that the $t_n$ are well defined. For all $h \in H$,

$$t_n(f)(h \cdot (c^0, \ldots, c^n)) = t_n(f)(h(1) \cdot c^0, \ldots, h(n+1) \cdot c^n)$$

$$= \alpha \left( f \left( h(2) \cdot c^1, \ldots, h(n+1) \cdot c^n, (-) \cdot h(1) \cdot c^0 \right) \right)$$

$$= \alpha \left( f \left( c^2, \ldots, c^{n-1}, h(n+1) \cdot c^n, h(n+2) S(h(n+3)) (-) \cdot h(1) \cdot c^0 \right) \right)$$

$$= \alpha \left( h \cdot \alpha \left( f \left( c^1, \ldots, c^n, (-) \cdot c^0 \right) \right) \right) = h \cdot t_n(f)(c^0, \ldots, c^n),$$

where the third equation follows by the properties of the antipode and counit, the fourth one is a consequence of the $H$-linearity of $f$, while the anti-Yetter-Drinfeld condition is used to derive the penultimate equality.

**Theorem 1.** Given a left $H$-module coalgebra $C$ and a left-right stable anti-Yetter-Drinfeld contramodule $M$, $C^H_n(C, M)$ with the $d_i$, $s_j$, $t_n$ defined above is a cyclic module.

**Proof.** One needs to check whether the maps $d_i, s_j, t_n$ satisfy the relations of a cyclic module; see e.g. [12] p. 203. Most of the calculations are standard, we only display examples of those which make use of the contramodule axioms. For example,

$$(t_{n-1} \circ d_{n-1}) (f)(c^0, \ldots, c^{n-1}) = \alpha \left( d_{n-1}(f) \left( c^1, \ldots, c^{n-1}, (-) \cdot c^0 \right) \right)$$

$$= \alpha \left( f \left( c^1, \ldots, c^{n-1}, \Delta_C \left( (-) \cdot c^0 \right) \right) \right)$$

$$= \alpha \left( f \left( c^1, \ldots, c^{n-1}, (-) \cdot c^0(1), (-) \cdot c^0(2) \right) \right)$$

$$= \alpha \left( t_n(f) \left( c^0(2), c^1, \ldots, c^{n-1}, (-) \cdot c^0(1) \right) \right)$$

$$= (d_n \circ t_n)(f)(c^0, \ldots, c^{n-1}),$$

where the third equality follows by the module coalgebra property of $C$, and the fourth one is a consequence of the associative law for contramodules. In a similar way, using compatibility of $H$-action on $C$ with counits of $H$ and $C$, and that $\alpha(\varepsilon_C(-)m) = m$, for all $m \in M$, one easily shows that $d_{n+1} \circ s_n$ is the identity.
map on \( C_n^H(C, M) \). The stability of \( M \) is used to prove that \( t_{n+1}^n \) is the identity. Explicitly,
\[
t_{n+1}^n(f)(c^0, \ldots, c^n) = \alpha^{n+1}(f((-)_1 \cdot c^0, \ldots, (-)_n \cdot c^n)) = \alpha(f(c^0, \ldots, c^n),
\]
where the second equality follows by the \( n \)-fold application of the associative law for contramodules, and the penultimate equality is a consequence of the \( H \)-linearity of \( f \). The final equality follows by the stability of \( M \). \( \square \)

Let \( N \) be a right-left stable anti-Yetter-Drinfeld module, and \( M = N^* \) the corresponding left-right stable anti-Yetter-Drinfeld contramodule, then
\[
C_n^H(C, M) = \text{Hom}_H(C^\otimes n+1, \text{Hom}(N, k)) \simeq \text{Hom}(N \otimes_H C^\otimes n+1, k).
\]
With this identification, the cycle module \( C_n^H(C, N^*) \) is obtained by applying functor \( \text{Hom}(-, k) \) to the cyclic module for \( N \) described in \([8, \text{Theorem 2.1}]\).

5. Hopf-cyclic cohomology of module algebras. Let \( A \) be a left \( H \)-module algebra. This means that \( A \) is an algebra and a left \( H \)-module such that, for all \( h \in H \) and \( a, a' \in A \),
\[
h \cdot (aa') = (h_{(1)} \cdot a)(h_{(2)} \cdot a), \quad h \cdot 1_A = \varepsilon_H(h)1_A.
\]

**Lemma 1.** Given a left \( H \)-module algebra \( A \) and a left \( H \)-contramodule, \( \text{Hom}(A, M) \) is an \( A \)-bimodule with the left and right \( A \)-actions defined by
\[
(a \cdot f)(b) = f(ba), \quad (f \cdot a)(b) = \alpha(f((-)_1 \cdot a) b),
\]
for all \( a, b \in A \) and \( f \in \text{Hom}(A, M) \).

**Proof.** The definition of left \( A \)-action is standard, compatibility between left and right actions is immediate. To prove the associativity of the right \( A \)-action, take any \( a, a', b \in A \) and \( f \in \text{Hom}(A, M) \), and compute
\[
((f \cdot a) \cdot a')(b) = \alpha \left( \tilde{\alpha} \left( f \left( \left((-)_1 \cdot a \right) \left((-)_2 \cdot a' \right) b \right) \right) \right) = \alpha(f((-)_1 \cdot a) ((-)_2 \cdot a') b) = \alpha(f(((-)_1 \cdot (aa')) b) = ((aa') \cdot f)(b),
\]
where the second equality follows by the definition of a left \( H \)-contramodule, and the third one in a consequence of the module algebra property. The unitality of the right \( A \)-action follows by the triangle diagram for contramodules and the fact that \( h \cdot 1_A = \varepsilon_H(h)1_A \). \( \square \)

For an \( H \)-module algebra \( A \), \( A^{\otimes n+1} \) is a left \( H \)-module by the diagonal action
\[
h \cdot (a^0 \otimes a^1 \otimes \ldots \otimes a^n) := h_{(1)} \cdot a^0 \otimes h_{(2)} \cdot a^1 \otimes \ldots \otimes h_{(n+1)} \cdot a^n.
\]
Take a stable left-left anti-Yetter-Drinfeld contramodule \( M \), set \( C^n_H(A, M) \) to be the space of left \( H \)-linear maps \( \text{Hom}_H(A^{\otimes n+1}, M) \), and, for all \( 0 \leq i, j \leq n \), define \( \delta_i : C^{n-1}_H(A, M) \to C^n_H(A, M) \), \( \sigma_j : C^{n+1}_H(A, M) \to C^n_H(C, M) \), \( \tau_n : C^n_H(A, M) \to C^n_H(A, M) \).
Similarly to the module coalgebra case, the above maps are well-defined by the anti-Yetter-Drinfeld condition. Explicitly, using the aforementioned condition as well as the fact that the inverse of the antipode is the antipode for the co-opposite Hopf algebra, one computes

\[
\tau_n(f)(h \cdot (a^0, \ldots, a^n)) = \alpha \left( (h_{(2)})^{-1} (h_{(1)})^{-1} \cdot a^n, h_{(1)}^{-1} \cdot a^0, h_{(2)}^{-1} \cdot a^1, \ldots, h_{(n+2)}^{-1} \cdot a^{n-1}) \right)
\]

Analogous calculations ensure that also \(\delta_n\) is well-defined.

**Theorem 2.** Given a left \(H\)-module algebra \(A\) and a stable left-left anti-Yetter-Drinfeld contramodule \(M\), \(C_H^n(A, M)\) with the \(\delta_i\), \(\sigma_j\), \(\tau_n\) defined above is a (co)cyclic module.

**Proof.** In view of Lemma \(\mathbb{[1]}\) and taking into account the canonical isomorphism \(\text{Hom}(A^\otimes n, M) \simeq \text{Hom}(A^\otimes n, \text{Hom}(A, M))\),

\[
\text{Hom}(A^\otimes n, M) \ni f \mapsto \left[ a^1 \otimes a^2 \otimes \ldots \otimes a^n \mapsto f (-, a^1, a^2, \ldots, a^n) \right],
\]

the simplicial part comes from the standard \(A\)-bimodule cohomology. Thus only the relations involving \(\tau_n\) need to be checked. In fact only the equalities \(\tau_n \circ \delta_n = \delta_{n-1} \circ \tau_{n-1}\) and \(\tau_{n+1} = \text{id}\) require one to make use of definitions of a module algebra and a left contramodule. In the first case, for all \(f \in C_H^n(A, M)\),

\[
(\tau_n \circ \delta_n)(f)(a^0, \ldots, a^n) = \alpha \left( (\cdots) \cdot a^n, a^0, \ldots, a^{n-1} \right)
\]

where the second equality follows by the associative law for left contramodules and the third one by the definition of a left \(H\)-module algebra. The equality \(\tau_{n+1} = \text{id}\) follows by the associative law of contramodules, the definition of left \(H\)-action on \(A^\otimes n\), and by the stability of anti-Yetter-Drinfeld contramodules. \(\square\)

In the case of a contramodule \(M\) constructed on the dual vector space of a stable right-right anti-Yetter-Drinfeld module \(N\), the complex described in Theorem \(\mathbb{[2]}\) is the right-right version of Hopf-cyclic complex of a left module algebra with coefficients in \(N\) discussed in \(\mathbb{[3]}\) Theorem 2.2.

6. **Anti-Yetter-Drinfeld contramodules and hom-connections.** Anti-Yetter-Drinfeld modules over a Hopf algebra \(H\) can be understood as comodules of an \(H\)-coring; see \(\mathbb{[2]}\) for explicit formulae and \(\mathbb{[4]}\) for more information about corings. These are corings with a group-like element, and thus their comodules can be interpreted.
as modules with a flat connection; see [2] for a review. Consequently, anti-Yetter-Drinfeld modules are modules with a flat connection (with respect to a suitable differential structure); see [10].

Following similar line of argument anti-Yetter-Drinfeld contramodules over a Hopf algebra $H$ can be understood as contramodules of an $H$-coring. This is a coring of an entwining type, as a vector space built on $H \otimes H$, and its form is determined by the anti-Yetter-Drinfeld compatibility conditions between action and contra-action. The coring $H \otimes H$ has a group-like element $1_H \otimes 1_H$, which induces a differential graded algebra structure on tensor powers of the kernel of the counit of $H \otimes H$. As explained in [3] Section 3.9] contramodules of a coring with a group-like element correspond to flat hom-connections. Thus, in particular, anti-Yetter-Drinfeld contramodules are flat hom-connections. We illustrate this discussion by the example of right-right anti-Yetter-Drinfeld contramodules.

First recall the definition of hom-connections from [3]. Fix a differential graded algebra $\Omega A$ over an algebra $A$. A hom-connection is a pair $(M, \nabla_0)$, where $M$ is a right $A$-module and $\nabla_0$ is a $k$-linear map from the space of right $A$-module homomorphisms $\text{Hom}_A(\Omega^1 A, M)$ to $M$, $\nabla_0 : \text{Hom}_A(\Omega^1 A, M) \to M$, such that, for all $a \in A$, $f \in \text{Hom}_A(\Omega^1 A, M)$,

$$\nabla_0(f \cdot a) = \nabla_0(f) \cdot a + f(da),$$

where $f \cdot a \in \text{Hom}_A(\Omega^1 A, M)$ is given by $f \cdot a : \omega \mapsto f(\omega a)$, and $d : \Omega^* A \to \Omega^{*+1} A$ is the differential. Define $\nabla_1 : \text{Hom}_A(\Omega^2 A, M) \to \text{Hom}_A(\Omega^1 A, M)$, by $\nabla_1(f)(\omega) = \nabla_0(f(\omega)) + f(d\omega)$, where, for all $f \in \text{Hom}_A(\Omega^2 A, M)$, the map $f \omega \in \text{Hom}_A(\Omega^1 A, M)$ is given by $\omega \mapsto f(\omega \omega')$. The composite $F = \nabla_0 \circ \nabla_1$ is called the curvature of $(M, \nabla_0)$. The hom-connection $(M, \nabla_0)$ is said to be flat provided its curvature is equal to zero.

Consider a Hopf algebra $H$ with a bijective antipode, and define an $H$-coring $C = H \otimes H$ as follows. The $H$ bimodule structure of $C$ is given by

$$h \cdot (h' \otimes h'') = h(1)h' S^{-1}(h(3)) \otimes h(2)h'', \quad (h' \otimes h'') \cdot h = h' \otimes h'h,$$

the coproduct is $D \otimes \text{id}_H$ and counit $\varepsilon_H \otimes \text{id}_H$. Take a right $H$-module $M$. The identification of right $H$-linear maps $H \otimes H \to M$ with $\text{Hom}(H, M)$ allows one to identify right contramodules of the $H$-coring $C$ with right-right anti-Yetter-Drinfeld contramodules over $H$.

The kernel of the counit in $C$ coincides with $H^+ \otimes H$, where $H^+ = \ker \varepsilon_H$. Thus the associated differential graded algebra over $H$ is given by $\Omega^n H = (H^+ \otimes H)^{\otimes n} \simeq (H^+)^{\otimes n} \otimes H$, with the differential given on elements $h$ of $H$ and one-forms $h' \otimes h' \in H^+ \otimes H$ by

$$dh = 1_H \otimes h - h(1) S^{-1}(h(3)) \otimes h(2),$$

$$d(h' \otimes h) = 1_H \otimes h' \otimes h - h'(1) \otimes h'(2) \otimes h + h' \otimes h_1 S^{-1}(h(3)) \otimes h(2).$$

Take a right-right anti-Yetter-Drinfeld contramodule $M$ over a Hopf algebra $H$ and identify $\text{Hom}_H(\Omega^1 H, M)$ with $\text{Hom}(H^+, M)$. For any $f \in \text{Hom}(H^+, M)$, set $\bar{f} : H \to M$ by $\bar{f}(h) = f(h - \varepsilon_H(h)1_H)$, and then define

$$\nabla_0 : \text{Hom}(H^+, M) \to M, \quad \nabla_0(f) = \alpha(\bar{f}).$$

$(M, \nabla_0)$ is a flat hom-connection with respect to the differential graded algebra $\Omega H$. 


7. Final remarks. In this note a new class of coefficients for the Hopf-cyclic homology was introduced. It is an open question to what extent Hopf-cyclic homology with coefficients in anti-Yetter-Drinfeld contramodules is useful in studying problems arising in (non-commutative) geometry. The answer is likely to depend on the supply of (calculable) examples, such as those coming from the transverse index theory of foliations (which motivated the introduction of Hopf-cyclic homology in [5]). It is also likely to depend on the structure of Hopf-cyclic homology with contramodule coefficients. One can easily envisage that, in parallel to the theory with anti-Yetter-Drinfeld module coefficients, the cyclic theory described in this note admits cup products (in the case of module coefficients these were foreseen in [8] and constructed in [11]) or homotopy formulae of the type discovered for anti-Yetter-Drinfeld modules in [13]. Alas, these topics go beyond the scope of this short note. The author is convinced, however, of the worth-whileness of investigating them further.

REFERENCES

[1] Böhm, G., Brzeziński, T., Wisbauer, R., Monads and comonads in module categories, arXiv:0804.1460 (2008).
[2] Brzeziński, T., Flat connections and (co)modules, in New Techniques in Hopf Algebras and Graded Ring Theory, S. Caenepeel and F. Van Oystaeyen (eds.), Universa Press, Wetteren, pp. 35–52. (2007).
[3] Brzeziński, T., Non-commutative connections of the second kind, arXiv:0802.0445. J. Algebra Appl., in press (2008).
[4] Brzeziński, T., Wisbauer, R., Corings and Comodules. Cambridge University Press, Cambridge (2003). Erratum: [http://www-maths.swan.ac.uk/staff/tb/Corings.htm](http://www-maths.swan.ac.uk/staff/tb/Corings.htm)
[5] Connes, A., Moscovici, H., Cyclic cohomology and Hopf algebra symmetry. Lett. Math. Phys. 52, 1–28 (2000).
[6] Eilenberg, S., Moore, J.C., Foundations of relative homological algebra, Mem. Amer. Math. Soc. 55 (1965).
[7] Hajac, P.M., Khalkhali, M., Rangipour, B., Sommerhäuser, Y., Stable anti-Yetter-Drinfeld modules, C. R. Math. Acad. Sci. Paris, 338, 587–590 (2004).
[8] Hajac, P.M., Khalkhali, M., Rangipour, B., Sommerhäuser, Y., Hopf-cyclic homology and cohomology with coefficients, C. R. Math. Acad. Sci. Paris, 338, 667–672 (2004).
[9] Jara, P., Ştefan, D., Cyclic homology of Hopf-Galois extensions and Hopf algebras, Proc. London Math. Soc. 93, 138–174 (2006).
[10] Kaygun, A., Khalkhali, M., Hopf modules and noncommutative differential geometry, Lett. Math. Phys. 76, 77–91 (2006).
[11] Khalkhali, M., Rangipour, B., Cup products in Hopf-cyclic cohomology, C. R. Math. Acad. Sci. Paris 340, 9–14 (2005).
[12] Loday, J.-L., Cyclic Homology 2nd ed., Springer, Berlin (1998).
[13] Moscovici, H., Rangipour, B., Cyclic cohomology of Hopf algebras of transverse symmetries in codimension 1, Adv. Math. 210, 323–374 (2007).
[14] Positselski, L., Homological algebra of semimodules and semicontramodules, arXiv:0708.3398 (2007).

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