A Sparse Random Graph Model for Sparse Directed Networks

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Abstract

An increasingly urgent task in analysis of networks is to develop statistical models that include contextual information in the form of covariates while respecting degree heterogeneity and sparsity. In this paper, we propose a new parameter-sparse random graph model for density-sparse directed networks, with parameters to explicitly account for all these features. The resulting objective function of our model is akin to that of the high-dimensional logistic regression, with the key difference that the probabilities are allowed to go to zero at a certain rate to accommodate sparse networks. We show that under appropriate conditions, an estimator obtained by the familiar penalized likelihood with an \(\ell_1\) penalty to achieve parameter sparsity can alleviate the curse of dimensionality, and crucially is selection and rate consistent. Interestingly, inference on the covariate parameter can be conducted straightforwardly after the model fitting, without the need of the kind of debiasing commonly employed in \(\ell_1\) penalized likelihood estimation. Simulation and data analysis corroborate our theoretical findings. In developing our model, we provide the first result highlighting the fallacy of what we call data-selective inference, a common practice of artificially truncating the sample by throwing away nodes based on their connections, by examining the estimation bias in the Erdős-Rényi model theoretically and in the stochastic block model empirically.

Keywords: \(\beta\)-model; Data-selective inference; Degree heterogeneity; Model-selective inference; Sparse networks.

1 Introduction

The study of inter-relationship of multiple entities in data is taking a central role in modern science and society. In the past few decades, this trend has been largely driven by the rapid deployment of information and measurement technology and the growing availability of virtual connectivity, giving rise to an increasing array of data in the form of networks comprising multiple components that interact. Often these interactions are conveniently represented as graphs that exhibit entities as nodes and interactions as edges. Biological networks, social networks, financial networks, computer networks, and transportation networks are all examples of this network data deluge.

This paper concerns a new random graph model for describing networks with directed edges. As a motivating example, Figure 1 depicts the lawyer friendship data in Lazega (2001) in which each of 71 lawyers was asked to name their friends: An edge from node \(i\) to node \(j\) exists if and only if lawyer \(i\) indicated in a survey that they socialized with lawyer \(j\) outside work. While it may seem intuitive to treat friendships as undirected, Figure 1 shows that some lawyers have vastly different numbers of incoming and outgoing connections, prompting us to model this friendship network as directed. Mathematically, we denote the adjacency matrix of a network as a binary matrix \(A \in \mathbb{R}^{n \times n}\), where \(n\) is the number of nodes and \(A_{ij} = 1\) if \(i\) points to \(j\) and 0 otherwise. Many network data often come with covariate information. For the lawyer dataset, these covariates include a lawyer's status (partner or associate), their gender (man or woman), which of three offices they worked in, the number of years

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they had spent with the firm, their age, their practice (litigation or corporate) and the law school they had visited (Harvard and Yale, UConn or other). The main interest here is to understand how links were formed.

![Figure 1: Lazega’s lawyers friendship network. The size of the nodes corresponds to their in-degrees. For better visibility all nodes with an in-degree of five or less are plotted with the same size. The 71 lawyers are colour-coded by their age group: The lawyers aged 20-29 are represented in orange, those aged 30-39 in light-blue, those aged 40-49 in green, followed by the lawyers aged 50-59 in yellow and finally those lawyers aged 60 or older in dark-blue. The eight nodes in black correspond to lawyers with either zero in- or out-degree or both.](image)

To develop a realistic model for the lawyer network, we start by summarizing its main features. These features are also commonly observed in many real-life networks.

1. **Degree heterogeneity**, which refers to the different tendency that the nodes in a network have in participating in network activities. Degree heterogeneity is a hallmark of networks in observational studies, often manifested as nodes having different, sometimes vastly, numbers of links known as degrees. To appreciate degree heterogeneity in the lawyer data, we note that the maximum and minimum out-degree are 25 and 0 respectively, with 22 and 0 for the in-degrees respectively.

2. **Sparsity.** Real-life networks are typically sparse in the sense that the observed number of ties does not scale proportionally to the total number of possible links. In Figure 1 for example, the average in- (and out-) degree is 8.1, where the maximum possible value is 70. The sparsity of a network is often reflected in it having nodes not well connected. In the lawyer data for example, there are 8 nodes having either no incoming edges or no outgoing edges or both, as colored in black in Figure 1.

3. **Covariates.** Covariates are often useful to explain the linking patterns. For our motivating data, whether two lawyers are connected by a friendship depends naturally on their covariates. For example, lawyers working in the same office or practice tend to befriend each other. Developing regression models that incorporate covariates is at the core of statistical modelling. In network science, we have only started to see statistical models very recently that involve covariates. See Section 1.3 for references.

This paper proposes a new parameter-Sparse Random Graph Model (SRGM) that can effectively deal with all the above features in directed networks. Specifically, our model postulates that links are independently made with the linking probability between node $i$ and $j$ specified as

$$p_{ij} = P(A_{ij} = 1 | Z_{ij}) = \frac{\exp(\alpha_i + \beta_j + \mu + \gamma^T Z_{ij})}{1 + \exp(\alpha_i + \beta_j + \mu + \gamma^T Z_{ij})},$$  \hspace{1cm} (1)
where $Z_{ij} \in \mathbb{R}^p$ is a $p$-dimensional covariate vector specific to the link from node $i$ to $j$, and $\mu \in \mathbb{R}$, $\alpha_i \in \mathbb{R}$, $\beta_j \in \mathbb{R}$ and $\gamma \in \mathbb{R}^p$ are parameters. For identifiability, we assume $\min_i \{\alpha_i\} = \min_j \{\beta_j\} = 0$. Importantly, we assume that the two parameters $\alpha = (\alpha_1, ..., \alpha_n)^T$ and $\beta = (\beta_1, ..., \beta_n)^T$ are both sparse and thus SRGM is parameter-sparse. Without the sparsity constraints on $\alpha$ and $\beta$, the SRGM becomes the model in Yan et al. (2019) by absorbing $\mu$ into $\alpha_i$ and $\beta_j$ as $\mu/2 + \alpha_i$ and $\mu/2 + \beta_j$ respectively. The SRGM in (1) possesses the following attractive properties.

- It handles degree heterogeneity explicitly with two node-specific parameters $\alpha_i$ and $\beta_i$ for each node. In particular, we interpret a positive $\alpha_i$ as the (excess) sociability of node $i$ to make outgoing links and a positive $\beta_i$ as its (excess) popularity to attract connections, relative to $\mu$. In this sense, $\alpha$ and $\beta$ are local parameters.

- It models sparse networks in two means. First, SRGM includes $\mu$, which can be interpreted as the global density parameter, and handles sparse networks by allowing $\mu$ to go to $-\infty$ when $n \to \infty$. Second, by imposing a sparsity assumption on $\alpha$ and $\beta$, SRGM can have a much smaller number of heterogeneity parameters than $2n$ and thus can overcome the curse of dimensionality. Intuitively, the smaller the number of parameters to estimate, the sparser the network that SRGM can fit. In the extreme case when all the heterogeneity parameters are zero, we only have $1 + p$ parameters to estimate as compared to $2n + p$ for the model in Yan et al. (2019). Stein & Leng (2020) showed that statistical inference can be conducted for this sub-model of SRGM as long as the total number of links of a network is in the order $O(n^\xi)$ for any $\xi \in (0, 2]$. That is, this submodel can model networks that are arbitrarily sparse. The sparsity assumption on $\alpha$ and $\beta$ immediately motivates the use of the penalized likelihood approach for their estimation. Our particular estimator is presented in (5) using an $\ell_1$ penalty.

- The model handles covariates by including the term $\gamma^T Z_{ij}$. When a covariate encodes the similarity of a node attribute, a positive $\gamma$ implies homophily, the tendency of nodes similar in attributes to connect. Homophily is a widely observed phenomenon in real-life networks.

- When both heterogeneity and covariate parameters are absent, the model in (1) can be interpreted as a null model in which a directed link between any pair of nodes is formed with the same probability $\exp(\mu)/(1 + \exp(\mu))$. SRGM builds on this null model by including a regression component in the covariates and node-specific nonnegative sender and receiver effects for the vertices. Note that to avoid overparametrizing the model, we assume that $\alpha$ and $\beta$ are sparse, and estimate them by differentially assigning node-specific incoming and outgoing parameters.

Before we proceed further with our network model, we highlight the fallacy of what we call data-selective inference — a common practice employed for analyzing networks by focusing on selected nodes, often times in a giant component, while leaving out a non-negligible portion; see Section 1.3 for a representative list of datasets — and emphasize the need for coherent modelling frameworks that can handle all the nodes in a network. The data-selective inference may be due to convenience or, more significantly, due to the nature of the model and its associated algorithms not working in case of disconnected networks or very sparse networks. Frequently it is argued that those nodes and edges in the giant component, or those nodes that are better connected with their edges, are the only nodes and edges that matter. However, this data-selective inference selects nodes via a non-random sampling scheme and thus can be questionable. In the next subsection, we highlight the problem of data-selective inference by quantifying the bias of the parameter estimates for the simplest network model theoretically and that for a widely used model via simulation. As an alternative, we advocate the use of model-selective inference adopted in this paper by fitting a model to all the data before conducting inference based on a selected model.

### 1.1 Data-selective inference produces bias

Statistical models are motivated by data in real life. When it comes to applying many statistical network models to analyze data, a common pattern has emerged in the kind of networks they analyze:
There often exists a non-negligible fraction of nodes either disconnected from the largest connected component or not having enough links, which are discarded from analysis. The motivating lawyer network is an example, having eight nodes with zero in- or out-degrees, as coloured in black in Figure 1, excluded from the analysis in Yan et al. (2019). More examples can be found in Section 1.3. The resulting data-selective inference, the exercise of applying a model to a sub-network excluding nodes based on their links, is a special case of biased sampling, because nodes in a giant component or well-connected nodes are systematically favoured over other nodes. This immediately raises the following fundamental question:

Does data-selective inference provide valid inference?

Before we provide an answer, we note that an argument to nullify the question above is simply to assume that the intended statistical model only works for the nodes in a giant component or those nodes that are well connected. While this argument is acceptable for mathematical convenience, it is not logically coherent or correct from a statistical or practical point of view. The selection of nodes is based entirely on the links or the response variable in a network model, and thus is non-random. Intuitively, biased sampling in such data-selective inference may produce artificial signal that does not exist at all or mitigate existing signals or both, leading to problematic or even completely wrong findings. The fact that a non-negligible fraction of non-random nodes are removed from analysis suggests that systematic bias will occur as a result.

We remark that the practice of ignoring selected nodes for modelling a network appears to originate from physics and computer science communities where the intention was to find meaningful clusters of nodes and hence is not statistical model-based (Girvan & Newman 2002, Newman 2006). Later statisticians injected rigour into this line of research notably by introducing likelihood-based estimators for statistical network models (Bickel & Chen 2009). On one hand, statistical modelling is extremely attractive because it provides a proper probabilistic framework for statistical inference and allows easy generalization of a model to more complex situations. On the other hand, however, issues such as sampling and asymptotics, including consistency and limiting distributions as the size of a network grows, inevitably arise. In particular, it is no longer appropriate for a statistical framework to ignore the non-random sampling issue in data-selective inference that removes nodes based on their links. We note that our discussion should be distinguished from the asymptotic framework in Shalizi & Rinaldo (2013) where their interest is to characterize consistency of parameter estimation when the network size goes to infinity under random sampling.

We now quantify the bias caused by data-selective inference. In what follows, we assume that an observed network is the realization of some statistical network model \( f \in \mathcal{F} \), with \( \mathcal{F} \) the family of candidate models. Crucially, we will assume that \( f \) would have produced the whole network, including any isolated vertices or small components. Given a realized network from the model, we want to quantify the bias of the estimator of the unknown parameter(s) in \( f \), if we only use those nodes in the giant component. As a reminder, this is what we refer to as data-selective inference. Motivated by the lawyer data and several other widely used datasets in the literature as discussed in Section 1.3, we will specify the parameter(s) in \( f \) such that a fixed proportion of nodes are not in the giant component. We will derive theoretically the bias of data-selective inference in the Erdős-Rényi model (Erdős & Rényi 1959, 1960) and use simulation to study the bias in estimating the parameters in a simple stochastic block model (Holland et al. 1983). The Erdős-Rényi model is interesting because the methods and approaches developed for understanding this model and the insight it brings provide the foundation for the study of more general graphs. On the other hand, the stochastic block model is one of the most popular network models that are widely applied, studied and extended in the literature.

The Erdős-Rényi model. As a reminder, the Erdős-Rényi model deals with undirected graphs in which edges are independently formed with the same probability \( p \). A sufficient condition for a realized network from this model to have a giant component and smaller components with probability tending to one is to take \( p = p(n) = \lambda/n \) for some fixed \( \lambda > 1 \) (van der Hofstad 2016, Chapter 4, e.g.). We will denote this family of models by \( ER(\lambda/n) \) and we are interested in estimating \( p \) given a network generated from this model. Notice that \( \lambda \) is very close to the expected degree of each node, which
Asymptotic bias is given by $\lambda \cdot (n-1)/n$. Consequently $ER(\lambda/n)$ produces sparse networks with the expected total degree scaling linearly in $n$.

Denote $\eta_\lambda$ as the unique solution that is smaller than one to the equation

$$
\eta_\lambda = e^{\lambda(\eta_\lambda - 1)},
$$

which exists if and only if $\lambda > 1$. It is known that in this regime $ER(\lambda/n)$ will produce giant components with size tightly concentrating around $(1 - \eta_\lambda) \cdot n$ (van der Hofstad 2016, Theorem 4.8, e.g.). We can interpret $\eta_\lambda$ as the survival probability of $P(\lambda)$, the Poisson branching process with mean offspring $\lambda$, whose behaviour is closely linked to the connectivity behaviour of $ER(\lambda/n)$ (van der Hofstad 2016, Chapters 3 and 4). Consider the estimation of $p$ by only focusing on the induced subgraph $G_{\text{max}}$ of $ER(\lambda/n)$ consisting of the giant component, that is $G_{\text{max}} = (C_{\text{max}}, E(C_{\text{max}}))$, where $C_{\text{max}}$ denotes the giant component and $E(C_{\text{max}})$ contains only those edges between the nodes contained in $C_{\text{max}}$. Denote by $\hat{p}_{\text{max}}$ the maximum likelihood estimator of $p$ based on $G_{\text{max}}$, that is

$$
\hat{p}_{\text{max}} = \frac{|E(C_{\text{max}})|}{(|C_{\text{max}}|)^2}.
$$

We have the following results.

**Proposition 1.** Fix any $\lambda > 1$ and consider the models $ER(\lambda/n)$. Let $G_{\text{max}} = (C_{\text{max}}, E(C_{\text{max}}))$ be the induced subgraph of $ER(\lambda/n)$ consisting only of the giant component. Let $\hat{p}_{\text{max}}$ as in (3), $p = p(n) = \lambda/n$ and $\eta_\lambda$ as in (2). Then,

$$
\frac{\hat{p}_{\text{max}}}{p} \xrightarrow{p} \frac{1 + \eta_\lambda}{1 - \eta_\lambda}.
$$

The estimator $\hat{p}_{\text{max}}$ is unbiased asymptotically only when $(1 + \eta_\lambda)/(1 - \eta_\lambda) = 1$, which never holds for fixed $\lambda$ unfortunately. Figure 2 shows how the asymptotic bias $(1 + \eta_\lambda)/(1 - \eta_\lambda)$ deviates from one as a function of $\lambda$. We can see that the bias increases when $\lambda$ decreases.

A few remarks are in order. Firstly, notice that the asymptotic factor by which we overestimate $p$, $(1 + \eta_\lambda)/(1 - \eta_\lambda)$, is strictly larger than one for any $\lambda > 1$. Thus, the incurred bias will not disappear as $n$ grows large. Secondly, the larger $\lambda$, the more nodes in the giant component and the smaller the probability $\eta_\lambda$ and thus the smaller the bias. On the other hand, when $\lambda$ approaches one, it is easy to see that $\eta_\lambda$ will approach one, making the bias in Proposition 1 approach $+\infty$. Indeed, for the limit case $\lambda = 1$, $C_{\text{max}}$ will have size of order $n^{2/3}$ (van der Hofstad 2016, Chapter 5), which in light

![Figure 2: Asymptotic bias $(1 + \eta_\lambda)/(1 - \eta_\lambda)$ of $\hat{p}_{\text{max}}$ as a function of $\lambda$. For better visibility we only display values of $\lambda$ ranging from 1.3 to 7 since the bias diverges to $+\infty$ when $\lambda$ approaches 1.](image-url)
of the proposition means that we must abandon all hope of recovering \( p \) if we only focus on the giant component.

While Proposition 1 follows readily from results in the random network literature, to the best of our knowledge, it has not been stated in this form in the literature before. In particular, the results we draw upon for its proof are mostly rooted in probability theory and as far as we know have not been used before to explicitly quantify the biases incurred by statistical procedures.

**The stochastic block model.** This model postulates that nodes in a network can be grouped into communities where the probability of any pair of nodes making connections depends only on their community membership. We here focus on what is called the symmetric stochastic block model with two communities, for which the probability matrix of making connections is

\[
P = \frac{1}{n} \begin{pmatrix} a & b \\ b & a \end{pmatrix},
\]

where \( a \) and \( b \) are constants. For this model, a pair of nodes link with probability \( a/n \) within the same community and with probability \( b/n \) between communities. The scaling \( 1/n \) ensures that a resulting network from this model will have a giant component with a fixed, smaller-than-one proportion of the nodes with high probability. See Figure 3a for the proportion of the nodes in the giant components produced under this parametrization. The symmetric stochastic block model is widely studied and relatively well understood as reviewed by Abbe (2018). Because of the sparsity of any resulting network, many clustering methods for community detection including spectral methods based on the adjacency matrix or the graph Laplacian, as well as their semidefinite relaxations, do not work well under this parametrization. Indeed, Zhang & Zhou (2016) showed that under our scaling no consistent algorithm exists to achieve vanishingly small misclassification. In view of this, we take an oracle approach by assuming that the community membership of each node is known a priori, and focus on what happens if we estimate \( a \) and \( b \) when nodes not in the giant component are removed as in data-selective inference.

In our simulation, we fix the number of nodes to be \( n = 10,000 \) and set the size of each community as \( n/2 \). We consider a fine grid of values \((a, b)\) by taking their values from 0.05 to 8.05 in steps of 0.05, resulting in 25,921 distinct combinations of \((a, b)\). For each such pair \((a, b)\) we sample a network from the symmetric stochastic block model and calculate the maximum likelihood estimate for \( a \) and \( b \) using only the nodes in the giant component. We repeat this process \( M = 1,000 \) times for every pair \((a, b)\). Denote the estimator as \( \hat{a}, \hat{b} \) and \( \hat{P} \). We measure the accuracy of the estimator by calculating the ratio of the spectral norm of the estimated probability matrix and that of the true probability matrix defined as \( \rho = \|\hat{P}\|_2/\|P\|_2 \), where \( \|P\|_2 = (a + b)/n \) for the \( 2 \times 2 \) matrix. Note that we have purposefully chosen a large \( n \) and \( M \) so that the resulting averages of the estimates will be close to their true limit. Figure 3a shows the average proportion of nodes in the giant component for each pair \((a, b)\) and Figure 3b the average value \( \rho \). When the proportion is one, the giant component contains all the nodes. The closer the average value of \( \rho \) is to one, the less biased the estimates are. In addition to the figures, Table 4 in appendix D provides the average values of the size of the giant component and \( \rho \) and their standard deviations, for selected values of \( \|P\|_2 \).

The simulations show that the incurred bias and the size of the giant component behave similarly when \( a + b \) is a constant. We highlight the bias of the parameter estimates more closely. When \( a + b = 2.5 \) the giant component on average contains 37\% of all nodes and we overestimate \( \|P\|_2 \) by a factor of 4.4. For \( a + b = 4 \) (bottom black line in Figure 3b), the giant component on average contains around 80\% of all nodes, with an average bias \( \rho = 1.5 \). For \( a + b = 6 \) (second black line in Figure 3b), the giant component contains on average 94\% of all nodes, while the average estimated \( \rho \) is 1.13, still significantly larger than one. Even for \( a + b = 8 \) (middle black line in Figure 3b), the giant component contains on average 98\% of all nodes, we overestimate \( \|P\|_2 \) by a factor of 1.04. Only once \( a + b \geq 10.70 \), where the giant component contains 99.5\% of all nodes is the incurred bias smaller than 1\%.

The above results illustrate that even if the statistician has perfect knowledge of the underlying communities, and even if only a seemingly insignificant fraction of say, 0.5\% of the nodes is removed, parameter estimation based solely on the giant component will be biased regardless of the network
size. This casts severe doubt on the suitability of the stochastic block model and its variants for fitting many popular datasets if only the nodes in giant components are retained. Having biased estimators will have consequences in all aspects of any downstream statistical inference including consistency, model selection, hypothesis testing, and so on; see Section 1.3 for some results that may be affected.

1.2 Model-selective inference for the lawyer data

We come back to the motivating example by comparing our estimates of the regression coefficients with those in Yan et al. (2019). For the seven covariates in this dataset, we followed Yan et al. (2019) in using the absolute differences of the continuous variables and the indicators whether the categorical variables are equal as our covariates.

Practically to fit their model, Yan et al. (2019) had to remove the eight nodes in black in Figure 1 that have zero in-degree or out-degree, because otherwise their maximum likelihood estimates (MLEs) would be $-\infty$ for $\alpha_i$ if node $i$ has no outgoing connections or for $\beta_i$ if the node has no incoming links. Another interesting aspect of the model in Yan et al. (2019) lies in the inference for the fixed-dimensional parameter $\gamma$. Because the rate of convergence of its MLE is slowed down by the MLE of the growing-dimensional heterogeneity parameters $\alpha$ and $\beta$, the estimator of $\gamma$ requires a bias correction to be asymptotically normal. In contrast, by making a sparsity assumption on $\alpha$ and $\beta$ in SRGM, we estimate the parameters via penalized likelihood and prove that inference for the estimated $\gamma$ can be read off after fitting the model via a model selection procedure, as seen in Theorem 3 and thus is straightforward. This result is surprising, because in high-dimensional statistics, it is often found that a different debiasing procedure must be conducted for the valid inference for the estimated parameters due to the bias incurred by regularization (Zhang & Zhang 2014).

To summarize, for our approach, rather than throwing away data at the beginning of the modelling process as in data-selective inference, we fit a model to all the data by judiciously assigning heterogeneity parameters to the nodes using a model selection criterion. For this reason, we refer to our modelling framework as model-selective inference.

When the Bayesian information criterion (BIC) is used to choose the tuning parameter in the penalized likelihood estimation, SRGM gives a model with 7 nonzero $\alpha_i$'s and 7 nonzero $\beta_i$'s. Four pairs of these nonzeros come from the same nodes. In Table 1, we present the estimated $\gamma$ and their standard errors when our model and the model in Yan et al. (2019) are fitted to the lawyers’ dataset. We remark that since Yan et al. (2019) employed data-selective inference by removing eight nodes,
their estimates can be biased as we have discussed. In terms of the parameter estimates themselves, although generally similar, we can see a few differences. First we can see that the standard errors of our estimates are smaller than those in Yan et al. (2019), reflecting that our estimates are based on a large sample size (a network with 71 nodes compared to one with 63 nodes in the latter paper) with fewer parameters (22 versus 132). Second, the effect of age difference is not significant in our model while it is in the model in Yan et al. (2019). To explore the age effect graphically, we colour-coded the lawyers by their age group in Figure 1. We can see that plenty of connections are made between age groups and across the circle in the circle layout of the network, i.e. between lawyers with a large difference in age, suggesting that age may not have played an important role. Indeed, a third (33.9%) of all friendships are formed between lawyers with an age difference of ten or more years. Third, we estimate the effect of attending the same law school as positive, implying that the lawyers tend to befriend those who graduated from the same school, while Yan et al.’s model states the opposite. The former conforms better to our intuition about social networks.

![Table 1](image)

| Covariate               | This Paper Estimate (SE) | Yan et al. (2019) Estimate (SE) |
|-------------------------|--------------------------|---------------------------------|
| Same status             | 1.52(0.10)               | 1.76(0.16)                      |
| Same gender             | 0.44(0.09)               | 0.96(0.14)                      |
| Same office             | 2.02(0.10)               | 3.23(0.18)                      |
| Same practice           | 0.58(0.09)               | 1.11(0.12)                      |
| Same Law School         | 0.29(0.10)               | -0.48(0.12)                     |
| Years with firm difference | -0.01(0.006)           | -0.064(0.014)                   |
| Age difference          | 0.003(0.006)             | -0.027(0.011)                   |

Table 1: Estimated regression coefficients and their standard errors (SE) for Lazega’s Lawyer friendship network.

1.3 Previous work and contributions

Statistical analysis of random networks has attracted enormous research attention in recent years thanks to a deluge of network data (Kolaczyk 2009, Fienberg 2012, Kolaczyk 2017). Although social scientists had long realized the importance of network modelling, it was until the late 50s that mathematicians started to develop random graph theory for the Erdös-Rényi model to rigorously study the random mechanism of network formation. This model provides much insight into more complex models but is overly simple for capturing most features in an observed network.

There are several classes of statistical models that have been successfully applied to model degree heterogeneity, one of the most important features of networks. Intuitively, degree heterogeneity can be modelled explicitly by associating parameters to nodes. As one of the most celebrated class of models, the stochastic block model aims to identify groups of nodes as communities for which linking probabilities depend on the community memberships of the nodes (Holland et al. 1983). This class of models has undergone rapid development in the last decades. For an incomplete list, see Bickel & Chen (2009), Karrer & Newman (2011), Rohe et al. (2011), Zhao et al. (2012), Lei & Rinaldo (2015), Gao et al. (2017), Amini & Levina (2018), Abbe (2018), Zhang et al. (2021) and references therein for recent developments. Another class of explicit models to model heterogeneity is to associate each node with its own parameters, giving rise to the $\beta$-model and its extensions (Chatterjee et al. 2011, Yan & Xu 2013, Yan et al. 2016, Chen et al. 2020). Degree heterogeneity can also be modelled implicitly. In the class of exponential random graph models (Holland & Leinhardt 1981, Frank & Strauss 1986, Robins et al. 2007), this heterogeneity is often encoded in some sufficient statistics of networks including, for example, degree sequences and other subnetworks or motifs of a network.

The increasing prevalence of covariates in network data calls for models that can effectively account for their effects. Towards this, we have started to see regression analysis incorporated in network models that allows inference including consistency and asymptotic normality. For stochastic block models, we point to Binkiewicz et al. (2017), Zhang et al. (2016), Huang & Feng (2018), Yan & Sarkar (2020). For the $\beta$-model, see Graham (2017), Yan et al. (2019), and Stein & Leng (2020). Additional references
include Ma, Ma & Yuan (2020) where a latent space model approach is investigated and Zhang et al. (2021) that proposes a model-free approach to study the dependence of links on covariates.

When it comes to applying models to data, many papers in the literature choose to ignore the modelling of those nodes in smaller components, or isolated nodes, or nodes with small degrees. Many if not all real-life networks have such nodes. Sometimes this is due to restrictions on a model from the outset, for example, the degree-corrected stochastic block model (Karrer & Newman 2011) and the $\beta$-model (Chatterjee et al. 2011) cannot handle nodes with zero degree for obvious reasons. At other times, this is due to the fact that the model can not fit isolated components. As we have argued Section 1.1, this practice is highly problematic. Here we list several popular datasets that are routinely related to data-selective inference. The first is the lawyer dataset discussed so far, for which Yan et al. (2019) chose to work with 63 out of 71 nodes (i.e., 11% of the nodes are removed). Below are two statisticians’ favorite datasets:

- **Political blog data.** This is a dataset recorded during the 2004 U.S. Presidential Election in the form of a directed network of hyperlinks between 1494 political blogs (Adamic & Glance 2005). Depending on their political views, these blogs can be liberal or conservative. Often converted to an undirected graph for analysis, this dataset has become a testbed for many network models especially the stochastic block model and its generalizations. In practice, most papers chose to focus on 1222 blogs that appear in a giant component or 1224 nodes which have at least one connection. Either way, this amounts to removing about 18% of the nodes in this network. See Amini et al. (2013), Olhede & Wolfe (2014), Jin (2015), Cai & Li (2015), Caron & Fox (2017), Chen & Lei (2018), Huang & Feng (2018), Ma, Ma & Yuan (2020), among many others.

- **Statistician citation network.** This dataset, collected by Ji & Jin (2016), contains rich citation information about all papers published between 2003 and 2012 in four statistics journals. The original dataset has 3607 authors or nodes based on which various networks have been constructed, but almost all attempts to use this data have chosen to examine subnetworks with fewer than 3607 nodes. Ji & Jin (2016) applied various community detection methods to three networks constructed from this dataset. The first one is a co-authorship network with 236 nodes (7% of all nodes), in which a link is formed between two authors if they wrote at least 2 papers together. See also Jin et al. (2021). The second one is another co-authorship network with 2236 nodes (63% of all nodes), in which a link is formed between two authors if they wrote at least 1 paper together. The third one is a directed citation network with 2654 authors (74% of all nodes). See also Zhang et al. (2021). Other attempts to use this dataset include Li et al. (2020) in which a network with 706 authors (20% of all nodes) was formed by repeatedly deleting nodes with less than 15 mutual citations and their corresponding edges, Jin et al. (2021) examined a citee network with 1790 (50% of all nodes) constructed by tying an edge between two authors if they have been cited at least once by the same author other than themselves.

In addition to the datasets above, there is a large growing body of works opting for data-selective inference that remove nodes before analysis. Among many others, see Chen et al. (2018) and Ma, Ma & Yuan (2020) for the Simmons College and Caltech data, two datasets on friendship networks in universities, Sengupta & Chen (2018) for the British MPs network (where 329 out of 360 MPs belonging to the giant component were analyzed), and Ma, Su & Zhang (2020) for Pokec social network for which only those nodes with no fewer than 10 links were retained for analysis. We emphasize that a notable feature of the analyses in these papers is that non-negligible portions of the nodes are excluded.

We now illustrate the fallacy of data-selective inference with the stochastic block model when it is applied to detecting communities in the political blog data network, to highlight a wider problem in statistical modelling of networks. If one assumes that this model generates all the 1494 nodes, the mere existence of more than 200 nodes not in the giant component suggests imposing a scaling of $1/n$ on the connectivity probability matrix of the data generating process. This is similarly done in Section 1.1 to ensure that the resulting giant component contains a positive fraction of the vertices. Under this regime, however, there is no way to separate all the vertices and thus no algorithm can
provide consistent community detection or parameter estimation. If instead one focuses on the giant component and assumes consistent community estimation for the nodes in the giant component, the connectivity probability matrix will be estimated with bias, as we have illustrated. The estimation bias of course is just the tip of a larger problem in biased sampling. By focusing on a non-random sample, we have no idea whether an intended model truly reflects the data generating process, or rather is merely an artefact of biased sampling. Equally importantly, the bias problem incurred via this data-selective inference will have knock-on effects of all aspects of any downstream analysis, including goodness-of-fit measures of a model, hypothesis testing and model selection; see, for example, Bickel & Sarkar (2016), Lei (2016), Wang & Bickel (2017), and Hu et al. (2020), for additional use of data-selective inference for data analysis.

Thus, there is a fundamental choice that we statisticians need to make. If we assume the stochastic block model generates the whole network including all the nodes in the political blog, consistent community detection and parameter estimation will be impossible. On the other hand, if we make the unreal assumption that the model only generates a subnetwork consisting only of those nodes in the giant component, we face the problem of data-selective inference. Although this fallacy seems ubiquitous in statistical applications of many network models, we are not aware of any systematic study of the effect of omitting nodes due to their degrees on model fitting.

**Main contributions.** Our major methodological contribution is to propose SRGM as a general model for directed networks that handles all the three main features of a real-life network (degree heterogeneity, sparsity and covariates) within the model-selective inference framework without artificially throwing away nodes. Our paper provides affirmative answers to three important questions for the modelling of density-sparse networks:

- Can we identify the right amount of heterogeneity such that the model is properly parametrized for inference?
- Can we consistently estimate these heterogeneity parameters?
- Can we develop statistical inference for the estimation of the covariates?

We remark that a model similar to SRGM has been developed for undirected networks by Stein & Leng (2020). The SRGM model in this paper is more complex due to the presence of two sets of heterogeneity parameters compared to one in Stein & Leng (2020). This added complexity makes theoretical development much more involved, which is also evidenced in Yan et al. (2016) and Yan et al. (2019), and thus SRGM for directed networks warrants a separate study. More importantly, this paper focuses more on variable selection while Stein & Leng (2020) focused exclusively on estimation consistency. Compared to the existing theory to prove selection consistency of similar estimators in the literature, one major novelty of our proof lies in the fact that we allow $p_{ij}$ to go to zero, and thus need to take care of the delicate situation that most quantities are not bounded away from zero. In this vein, our results substantially generalize those in the literature (Ravikumar et al. 2010, van de Geer & Buehlmann 2011) in a different context (network modelling versus regression modelling). As another major contribution of this paper, we expose the fallacy of data-selective inference, a widely employed inferential technique for many popular network data, in theory and via simulations. Our formal exposition of this fallacy is the first in the literature to our best knowledge and can be of independent interest. As an alternative, we recommend the use of model-selective inference that does not remove any nodes prior to analysis.

### 1.4 Notation and the plan

Denote $N = n(n-1)$ and $[n] := \{1, \ldots, n\}$. For any $a, b \in \mathbb{R}$ we use the notation $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Denote $\mathbb{R}_+ = [0, +\infty)$ as the nonnegative real line. For a vector $v \in \mathbb{R}^n$, we use $S(v) = \{i : v_i \neq 0\}$ to denote its support and $\text{diag}(v) \in \mathbb{R}^{n \times n}$ the $n$-by-$n$ diagonal matrix with $v$ on the diagonal. Let $\| \cdot \|_1, \| \cdot \|_2, \| \cdot \|_\infty$ denote the vector $\ell_1$, $\ell_2$ and $\ell_\infty$-norm respectively.
and \( \|v\|_0 \) denotes the \( \ell_0 \)-"norm". That is, \( \|v\|_0 = |S(v)| \). For any subset \( S \subset [n] \), we denote by \( v_S \) the vector \( v \) with components not belonging to \( S \) set to zero. When dealing with a vector \( v \in \mathbb{R}^N \), we will number its elements as \( v = (v_{ij})_{i\neq j} \). For a matrix \( A \in \mathbb{R}^{d \times d} \) and a subset \( S \subseteq [d] \), denote by \( A_{S,S} \in \mathbb{R}^{|S| \times |S|} \) the submatrix of \( A \) obtained by only taking the rows and columns belonging to \( S \). Denote by \( A_{-,S} \in \mathbb{R}^{d \times |S|} \) the submatrix obtained by keeping all the rows and taking only those columns belonging to \( S \) and by \( A_{S,-} \in \mathbb{R}^{|S| \times d} \) the submatrix obtained by taking only those rows belonging to \( S \) and keeping all the columns. For any square matrix \( A \), we denote by \( \text{maxeval}(A) \) its maximum eigenvalue and by \( \text{mineval}(A) \) its minimum eigenvalue.

For brevity, we denote \( \vartheta = (\alpha^T, \beta^T)^T \) as the vector of the degree heterogeneity parameters and \( \theta = (\vartheta^T, \mu, \gamma^T)^T \) with its true value denoted as \( \theta_0 = (\vartheta_0^T, \mu_0, \gamma_0^T)^T \). We also use \( \xi = (\mu, \gamma^T)^T \in \mathbb{R}^{p+1} \) to refer to the global parameters. We write \( S_0 = \mathcal{S}(\theta_0) \) and denote its cardinality as \( s_0 = |S_0| \). We write \( S_{0,+} := S_0 \cup \{2n+1, 2n+2, \ldots, 2n+1+p\} \) with cardinality \( s_{0,+} = |S_{0,+}| = s_0 + p + 1 \) to refer to all active indices including \( \mu \) and \( \gamma \). Thus, \( s_0 \) and \( s_{0,+} \) can be understood as the parameter sparsity of our model. When we want to make the dependence of the link probabilities given \( Z_{ij} \) on different values of \( \theta = (\vartheta^T, \mu, \gamma^T)^T \) explicit, we write \( p_{ij}(\theta) = \frac{\exp(\alpha_i + \beta_j + \mu + \gamma^T Z_{ij})}{1 + \exp(\beta_j + \mu + \gamma^T Z_{ij})} \). Finally, we use \( C \) for some generic, strictly positive constant that may change between displays.

The rest of the paper is organized as follows. In Section 2, we discuss how to estimate the parameters in the SRGM. In Section 3, we present the main theory including both model and rate consistency of our estimates, as well as the asymptotic normality of the estimated regression coefficients. An extensive simulation study is presented in Section 4 and we provide discussion in Section 5. All the technical details are found in the appendix.

## 2 Estimation

A directed network on \( n \) nodes is represented as a directed graph \( G_n = (V,E) \), consisting of a node set \( V \) with cardinality \( n \) and an edge set \( E \subseteq V \times V \). Without loss of generality we assume \( V = [n] \) and that \( G_n \) is simple, having no self-loops nor multiple edges between any pair of nodes. Such a graph \( G_n \) is represented as a binary adjacency matrix \( A \in \mathbb{R}^{n \times n} \), where \( A_{i,j} = 1 \), if \( (i,j) \in E \) and \( A_{i,j} = 0 \) otherwise. Note by assumption \( A_{ii} = 0 \) for all \( i \).

Given the adjacency matrix \( A \) and the covariates \( \{Z_{ij}\}_{i \neq j} \), it is easily seen that the negative log-likelihood of the SRGM in (1) is

\[
\mathcal{L}(\alpha, \beta, \mu, \gamma) = -\sum_{i=1}^{n} \alpha_i b_i - \sum_{i=1}^{n} \beta_i d_i - d - \mu - \sum_{i,j=1}^{n} \sum_{i \neq j} (\gamma^T Z_{ij}) A_{ij} + \sum_{i,j=1}^{n} \sum_{i \neq j} \log(1 + \exp(\alpha_i + \beta_j + \mu + \gamma^T Z_{ij})),
\]

(4)

where \( b_i = \sum_{j=1, j \neq i}^{n} A_{ij} \) is the out-degree of vertex \( i \) and \( d_i = \sum_{j=1, j \neq i}^{n} A_{ij} \) its in-degree. Write \( d = (d_1, \ldots, d_n)^T \) and \( b = (b_1, \ldots, b_n)^T \) as the corresponding degree sequences. Denote \( d_+ := \sum_i d_i \) and \( b_+ := \sum_i b_i \) for which we have \( b_+ = d_+ \). Note that a network is sparse if \( E[d_+] \sim n^\kappa \) for some \( \kappa \in (0, 2) \), where \( E \) is the expectation with regard to the data generating process. A network is dense if \( E[d_+] \sim n^2 \). It is easy to see that \( \theta_0 = \text{arg min} \mathbb{E}[\mathcal{L}(\theta)] \).

To estimate the parameter \( \theta = (\alpha^T, \beta^T, \mu, \gamma^T)^T \) and identify the support of \( \vartheta = (\alpha^T, \beta^T)^T \), a natural idea is to resort to the method of penalized likelihood by solving

\[
\text{arg min}_{\theta \in \Theta} \frac{1}{N} \mathcal{L}(\alpha, \beta, \mu, \gamma) + \lambda(\|\alpha\|_1 + \|\beta\|_1),
\]

(5)

where \( \lambda \) is a tuning parameter and \( \Theta = \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R} \times \mathbb{R}^P \) is the parameter space. Note that we have used the same amount of penalty on \( \alpha \) and \( \beta \) because \( b_+ = d_+ \). The objective function in (5) is similar to the penalized logistic regression with an \( \ell_1 \) penalty and thus can be easily solved thanks to the many algorithms developed to compute the latter. In this paper, we use the solver in the \texttt{R} package...
and propose to perform estimation via a sufficiently large constant with \( \rho \) parameter. The above constraint is equivalent to \( D \) matrixalent to \( \rho \) since a smaller \( \rho \) will also be reflected in the different rates of convergence we obtain in Theorem 2 below. and \( \beta \) and \( \beta \), and \( \beta \) is only

\[
X \sim \mathcal{N}(2^{n^2} + 1), \quad \text{with rows } X_{ij} \in \mathbb{R}^{1 \times n}, \quad i \neq j, \quad \text{such that for each component } k = 1, \ldots, n, \quad X_{i,j,k} = 1 \text{ if } k = i \text{ and zero otherwise.}
\]

Likewise, define the in-matrix \( X^{in} \in \mathbb{R}^{N \times n} \) with rows \( X^{in}_{ij} \in \mathbb{R}^{1 \times n}, \quad i \neq j, \quad \text{such that for each component } k = 1, \ldots, n, \quad X^{in}_{i,j,k} = 1 \text{ if } k = j \text{ and zero otherwise.} \)

Let \( Z = (Z^{i,j}_{ij})_{i,j} \in \mathbb{R}^{N \times p} \) be the matrix of the covariate vectors written below each other. Then, the design matrix \( D \) consists of four blocks, written next to each other:

\[
D = [ X^{out} | X^{in} | \mathbf{1} | Z ] \in \mathbb{R}^{N \times (2n + p + 1)},
\]

where \( \mathbf{1} \in \mathbb{R}^{n} \) is a vector of all ones. We use the shorthand \( X = [ X^{out} | X^{in} ] \in \mathbb{R}^{N \times 2n} \).

The design matrix \( D \) reveals an important property of our model (1). While the columns of the the global parameters \( \mu \) and \( \gamma \) have non-zero entries in all \( N \sim n^2 \) rows of \( D \), the local parameters \( \alpha \) and \( \beta \) only have \( n \) non-zero entries in their respective columns. Thus, the effective sample size for \( \alpha \) and \( \beta \) is only \( n \), whereas it is \( N \), that is, of order \( n \) larger, for the global parameters \( \mu \) and \( \gamma \), which will also be reflected in the different rates of convergence we obtain in Theorem 2 below.

A key quantity in the theory of high-dimensional statistics is the population Gram matrix \( \Sigma \), which is closely linked to the hessian of \( \mathcal{L} \) and the precision matrix. Informally speaking, it is given by

\[
\Sigma = \frac{1}{\text{sample size}} \mathbb{E}[D^T D].
\]

Were we to naively ignore the differing sample sizes between local and global parameters and choose \( \Sigma = 1/N \cdot \mathbb{E}[D^T D] \), our proofs would fail, due to the top-left corner of \( \Sigma \) rapidly converging to the zero-matrix, making \( \Sigma \) singular in the limit. In particular, the compatibility condition (cf. Section 3.2), crucial for proofs for LASSO-type problems, would not hold. We need to account for this fact and therefore have to use a sample-size adjusted Gram matrix. To that end, we introduce the matrix

\[
T = \begin{bmatrix} \sqrt{n - 1}/2 & 0 \\ 0 & \sqrt{N}/p \end{bmatrix},
\]

\textbf{glmnet} (Friedman et al. 2010). The similarity of our estimator to the LASSO estimator makes our estimation approach extremely scalable.

Since our focus is on sparse networks, some link probabilities \( p_{ij} \) will tend to zero as \( n \) tends to infinity. It is then natural to impose restrictions on how fast this decay can be. Therefore, we assume the existence of a non-random sequence \( \rho_{n,0} \in (0, 1/2] \) such that \( \rho_{n,0} \to 0 \) as \( n \to \infty \) such that for all \( i, j \), almost surely,

\[
1 - \rho_{n,0} \geq p_{ij} \geq \rho_{n,0}.
\]

Since a smaller \( \rho_{n,0} \) allows sparser networks, we sometimes refer to \( \rho_{n,0} \) as the network sparsity parameter. The above constraint is equivalent to

\[
|\alpha_{0,i} + \beta_{0,j} + \mu_0 + \gamma_{0}^T Z_{ij}| \leq -\text{logit}(\rho_{n,0}) =: r_{n,0}, \quad \forall i, j,
\]

with \( r_{n,0} \geq 0 \) since \( \rho_n \leq 1/2 \). The previous inequality can also be expressed in terms of the design matrix \( D \) associated with the corresponding logistic regression problem, as in (8) below, and is equivalent to \( \|D\theta_0\|_{\infty} \leq r_{n,0} \). This motivates the following tweak to the estimation procedure in (5): Given a sufficiently large constant \( r_n \) we define the local parameter space

\[
\Theta_{loc} = \Theta_{loc}(r_n) := \{ \theta \in \Theta : \|D\theta\|_{\infty} \leq r_n \}
\]

and propose to perform estimation via

\[
\hat{\theta} = (\hat{\alpha}^T, \hat{\beta}^T, \hat{\mu}^T, \hat{\gamma}^T)^T = \arg \min_{\theta=(\alpha^T, \beta^T, \mu^T, \gamma^T)^T \in \Theta_{loc}} \frac{1}{N} \mathcal{L}(\alpha, \beta, \mu, \gamma) + \lambda(\|\alpha\|_1 + \|\beta\|_1),
\]

which is more amenable for theoretical analysis. Notice that \( \Theta_{loc} \) is convex.

We now give an explicit form of the associated design matrix \( D \). We have to consider the presence/absence of \( N = n(n - 1) \) directed edges and want to perform inference on \( 2n + 1 + p \) parameters. That means, our design matrix \( D \) will have dimension \( n(n - 1) \times (2n + 1 + p) \). Define the out-matrix \( X^{out} \in \mathbb{R}^{N \times n} \) with rows \( X_{ij}^{out} \in \mathbb{R}^{1 \times n}, i \neq j, \) such that for each component \( k = 1, \ldots, n, \quad X_{i,j,k}^{out} = 1 \) if \( k = i \) and zero otherwise. Likewise, define the in-matrix \( X^{in} \in \mathbb{R}^{N \times n} \) with rows \( X_{ij}^{in} \in \mathbb{R}^{1 \times n}, i \neq j, \) such that for each component \( k = 1, \ldots, n, \quad X_{i,j,k}^{in} = 1 \) if \( k = j \) and zero otherwise. Let \( Z = (Z_{ij}^{out})_{i,j} \in \mathbb{R}^{N \times p} \) be the matrix of the covariate vectors written below each other. Then, the design matrix \( D \) consists of four blocks, written next to each other:

\[
D = [ X^{out} | X^{in} | \mathbf{1} | Z ] \in \mathbb{R}^{N \times (2n + p + 1)},
\]
where $I_m$ is the $m \times m$ identity matrix and define the sample size adjusted Gram matrix $\Sigma$ as

$$
\Sigma = T^{-1}E[D^T D]T^{-1}.
$$

(9)

It will be convenient to cast problem (7) in terms of re-scaled parameters $\bar{\theta}$ which adjust for the discrepancy in effective sample sizes. This new formulation is equivalent to the one in (7), but gives us a unified framework for treating convergence properties of our estimators. We will rely heavily on that rescaled version in our proofs. Precisely, define the sample size adjusted design matrix $\bar{D}$ as

$$
\bar{D} = \left[ \begin{array}{cc} \bar{X} & 1 \end{array} \right] \in \mathbb{R}^{N \times (2n+p+1)},
$$

where

$$
\bar{X} = \left[ \begin{array}{cc} X^{\text{out}} & X^{\text{in}} \end{array} \right] = \left[ \begin{array}{cc} \sqrt{n}X^{\text{out}} & \sqrt{n}X^{\text{in}} \end{array} \right],
$$

is blowing up the entries in $D$ belonging to $\bar{\theta}$. For any parameter $\theta = (\theta^T, \mu, \gamma)^T \in \Theta$, we introduce the notation

$$
\bar{\theta} = (\bar{\theta}, \mu, \gamma) = \left( \frac{1}{\sqrt{n}} \bar{\theta}, \mu, \gamma \right).
$$

(10)

In particular we use the notation $\bar{\theta}_0 = (\bar{\theta}_0^T, \mu_0, \gamma_0^T)^T$, to denote the re-parametrized true parameter value. The blow-up factor $\sqrt{n}$ was chosen precisely such that we can now reformulate our problem as a problem in which each parameter effectively has sample size $N$ in the sense that

$$
\Sigma = \frac{1}{N}E[D^T \bar{D}].
$$

Now, our original penalized likelihood problem can be rewritten as

$$
\hat{\theta} = (\hat{\theta}, \hat{\mu}, \hat{\gamma}) = \arg\min_{\theta \in \Gamma} \frac{1}{N} \left( -\frac{n}{2} \sum_{i=1}^{n} \sqrt{n} \bar{\alpha}_i b_i - \sum_{i=1}^{n} \sqrt{n} \bar{\beta}_i d_i - d_+ \mu - \sum_{i \neq j} (Z_{ij}^T \gamma) A_{ij} \right.
$$

$$
+ \sum_{i \neq j} \log \left( 1 + \exp \left( \sqrt{n} \bar{\alpha}_i + \sqrt{n} \bar{\beta}_j + \mu + Z_{ij}^T \gamma \right) \right) \left. \right) + \bar{\lambda} \| \bar{\theta} \|_1,
$$

(11)

where $\bar{\lambda} = \sqrt{n} \lambda$ and the argmin is taken over $\Theta_{\text{loc}} = \{ \theta : \theta \in \Theta, \| D \bar{\theta} \|_\infty \leq r_n \}$. Note that by the same arguments as before, $\Theta_{\text{loc}}$ is convex. Then, given a solution $\hat{\theta}$ for a given penalty parameter $\bar{\lambda}$ to this modified problem (11), we can obtain a solution to our original problem (7) with penalty parameter $\lambda = \bar{\lambda}/\sqrt{n}$, by setting

$$
(\hat{\theta}, \hat{\mu}, \hat{\gamma}) = \left( \frac{\sqrt{n} \bar{\theta}}{\sqrt{n} \lambda}, \hat{\mu}, \hat{\gamma} \right).
$$

Note that for any $\theta \in \Theta$, $D \theta = D \bar{\theta}$, and hence the bound $r_n$ is the same in the definitions of $\Theta_{\text{loc}}$ and $\Theta_{\text{loc}}$. Note also that $\theta \in \Theta_{\text{loc}}$ if and only if $\hat{\theta} \in \Theta_{\text{loc}}$. For any $\theta = (\bar{\theta}^T, \mu, \gamma)^T$, denote the negative log-likelihood function corresponding to the rescaled problem (11) as $\bar{L}(\bar{\theta})$. Then, clearly $\bar{L}(\bar{\theta}) = L(\theta)$ and $E[\bar{L}(\bar{\theta})] = E[L(\theta)]$. Thus, $\bar{\theta}_0$ satisfies that $\bar{\theta}_0 = \arg\min_{\theta \in \Theta} E[L(\theta)]$.

### 3 Theory

We outline the main assumptions first. For the covariates, we focus on the case when $Z_{ij}$ is finite dimensional following a random design which is somewhat more interesting than a fixed design. We assume that its associated parameter $\gamma_0$ is a fixed vector.

**Assumption 1.** Assume that $Z_{ij}$ are independent with $E[Z_{ij}] = 0$ and that $|Z_{ij}|$ is uniformly bounded. We also assume that the covariate parameter $\gamma_0$ lies in some compact, convex set $\Gamma \subset \mathbb{R}^p$ and that $p$ remains fixed. As a result, there exist constants $\kappa, c > 0$ such that $|Z_{ij}^T \gamma| \leq \kappa$ for all $1 \leq i \neq j \leq n$ and...
Assumption 2. We assume that \( \theta_0 \in \Theta_{\text{loc}} \) or equivalently \( r_n \geq r_{n,0} \). Therefore, without loss of generality we assume \( r_n = r_{n,0} \) and consequently \( \rho_n = \rho_{n,0} \). Indeed, this can always be achieved by simply increasing \( r_n \) as needed.

Assumption 1 is quite standard. Note that \( Z_{ij} \)'s are not necessarily i.i.d., possibly having correlated entries and that \( Z_{ij} \) can be asymmetric in that \( Z_{ij} \neq Z_{ji} \). We note that we could have made a fixed-design assumption but the random-design assumption is somewhat more interesting. Assumption 2 is rather harmless as it simply states that the parameter we are estimating is actually contained in the space over which we are optimizing. Results that take a potential model-misspecification when \( r_n < r_{n,0} \) into account and quantify the resulting bias can be derived similarly to the results in Stein & Leng (2020), but are omitted for reasons of space. Indeed, in practice, it will not be necessary to choose \( r_n \) explicitly, as discussed in Section 4 and the existence of \( r_{n,0} \) and \( \rho_{n,0} \) are technical artefacts that encode the permissible sparsity of the networks we study and enter our rates of convergence.

Assumption B1. \( \sqrt{n}s_n^2 \rho_n^{-2} \to 0, n \to \infty \).

For all of our theorems striking the right balance between parameter sparsity \( s_+ \), network sparsity \( \rho_n \) and penalty parameter \( \lambda \) is crucial. The restrictiveness of these balancing assumptions will depend on the complexity of the results being proven and we number them separately from the general assumptions as "Assumption Bi", \( i = 1, 2, 3 \), to make their special standing explicit in our notation. Our main result on model selection consistency, Theorem 1, is the most refined of our theorems and hence Assumption B1 is the strongest such balancing assumption. In particular, the weaker balancing assumptions required to establish parameter estimation consistency, Theorem 2 (assumption B2), and asymptotic normality of the homophily parameter estimator \( \gamma \), Theorem 3 (assumption B3), follow from assumptions B1 and 3. Thus, our estimator \( \hat{\theta} \) in (7) is capable of simultaneously recovering the correct support, consistently estimating the parameter values and producing an asymptotically normal estimate of \( \gamma_0 \).

3.1 Model selection consistency

In this section we study under which conditions our estimator (7) identifies the correct subset of active variables \( S_0 \). Our main result, Theorem 1, is that under the appropriate conditions, our estimator \( \hat{\theta} \) will correctly exclude all the truly inactive parameters and correctly include all those truly active parameters whose value exceeds a certain threshold. The latter "\( \hat{\theta}\)-min"-condition is typical for model selection in high-dimensional logistic regression type problems (Ravikumar et al. 2010, Chen et al. 2020, e.g.).

Recall that we use \( S_0 \) to refer to the active set of indices associated with \( \theta = (\alpha^T, \beta^T)^T \), whereas \( S_{0,+} \) refers to the active indices including those of \( \mu \) and \( \gamma \), that is \( S_{0,+} = S_0 \cup \{2n+1, \ldots, 2n+1+p\} \). In the following derivations it will be crucial to distinguish the two correctly. To make the representation cleaner, in this section only we will simply write \( S \) for \( S_0 \) and \( S_+ \) for \( S_{0,+} \). We use \( S_0^c \) to denote the complement of \( S_+ \) in \( [2n+1+p] \), that is \( S_0^c = [2n+1+p] \setminus S_+ \). Importantly, we will use \( S^c \) to refer to the complement of \( S \) in \( [2n] \) only, that is, ignoring the indices \( 2n+1, \ldots, 2n+1+p \): \( S^c = [2n] \setminus S \).

While this may seem like a potential notational pitfall, this allows for much cleaner notation in our proofs in Appendix C. Let \( S_\alpha = \{ i : \alpha_{0,i} > 0 \} \), \( S_\beta = \{ j : \beta_{0,j} > 0 \} \) and \( s_\alpha = |S_\alpha|, s_\beta = |S_\beta| \).

We first state the main theorem of this section before giving more details on its derivation. Recall that \( \lambda \) is the penalty parameter in the rescaled version (11) of our problem (7). Also notice that \( \tilde{S} := \{ i : \tilde{\theta}_i > 0 \} = \{ i : \tilde{\theta}_i > 0 \} \), that is the estimator (7) and (11) will always select the same active set of parameters.
where $W_1$. The minimum threshold for an index to be included by Remark.
in (11) must be at least of order $\sqrt{\log(n)/N}$, or, informally speaking, that the penalty parameter in Assumption 3.

The name dependency condition the population Hessian of $\bar{\theta}$ imposed in our model. Rather, a careful argument is needed to prove that analogous properties hold in Ravikumar et al. (2010) imposed on the Hessian of the negative log-likelihood and random columns in Ravikumar et al. (2010). It is important to point out, however, that due to the mixture of deterministic the columns of it are not perfectly correlated. For our specific case, we let prescribing form as is the case here), imposing this condition would be equivalent to demanding that $\rho_n = o(n^{-1/4})$.

Our tool of choice for proving Theorem 1 is a primal-dual witness construction, similar to the one in Ravikumar et al. (2010). It is important to point out, however, that due to the mixture of deterministic and random columns in $D$ and the differing sample sizes between $\vartheta$ and $\xi$, the standard assumptions in Ravikumar et al. (2010) imposed on the Hessian of the negative log-likelihood $\mathcal{L}$ cannot simply be imposed in our model. Rather, a careful argument is needed to prove that analogous properties hold for sufficiently large $n$. The first condition is the so-called dependency condition which demands that the population Hessian of $\bar{\mathcal{L}}$ with respect to the variables contained in the active set $\bar{S}$ is invertible. The name dependency condition comes from the fact that if we had a random design matrix as is usually assumed in LASSO theory (or a deterministic design matrix that does, however, not have a prescribed form as is the case here), imposing this condition would be equivalent to demanding that the columns of it are not perfectly correlated. For our specific case, we let

$$Q := \frac{1}{n-1}X^TW_0^2X = H_{\bar{\vartheta}}X \bar{\mathcal{L}}(\bar{\vartheta}) \in \mathbb{R}^{2n \times 2n},$$

where $W_0 = \text{diag}(\sqrt{p_{ij}(\theta_0)(1 - p_{ij}(\theta_0)), i \neq j})$, be the Hessian of $\bar{\mathcal{L}}$ with respect to $\bar{\vartheta}$ only.
Lemma 2 (Dependency condition). For any $n$,
\[
\min_{\alpha} (Q_{S,\tilde{S}}) \geq \frac{1}{2} \rho_n \cdot \left( 1 - \frac{\max\{s_\alpha, s_\beta\}}{n - 1} \right) > 0.
\]

Lemma 2 is proved in Appendix C. The next condition we need is the so-called incoherence condition, which is also proved in Appendix C.

Lemma 3 (Incoherence condition). For any $n$,
\[
\|Q_{S,\tilde{S}} Q_{S,\tilde{S}}^{-1}\|_\infty \leq \frac{1}{2} \rho_n \cdot \frac{\max\{s_\alpha, s_\beta\}}{n - \max\{s_\alpha, s_\beta\}}.
\]

Notice that by Lemma 2 the left-hand side of Lemma 3 is well-defined. Furthermore, under assumption B1, $\sqrt{n}s_\alpha^2 \rho_n^{-2} = n^{-1/2} \rho_n^{-1}s_\alpha - n\rho_n s_\lambda \to 0$ as, $n \to \infty$. On the other hand, by Assumption 3, $n\rho_n s_\lambda \to C\rho_n^{-1}s_\lambda \log(n) \to \infty$. Therefore it must hold that $n^{-1/2} \rho_n^{-1}s_\lambda \to 0$, which in particular implies that the right-hand side in Lemma 3 tends to zero as $n$ tends to infinity.

Our starting point for proving Theorem 1 is the KKT conditions (Bertsekas 1995, Chapter 5, e.g.): Equation (11) is a convex optimization problem. Hence, by subdifferential calculus, a vector $\tilde{\theta}$ is a minimiser of (11) if and only if zero is contained in the subdifferential of $\frac{1}{N} \mathcal{L}(\tilde{\theta}) + \bar{\lambda}\|\tilde{\theta}\|_1$ at $\tilde{\theta}$. That is, if and only if there is a vector $\bar{z} \in \mathbb{R}^{2n+1+p}$ such that
\[
0 = \frac{1}{N} \nabla \mathcal{L}(\tilde{\theta}) + \bar{\lambda}\bar{z}, \tag{13}
\]
and
\[
\bar{z}_i = 1, \text{ if } \bar{\theta}_i > 0, i = 1, \ldots, 2n, \tag{14a}
\]
\[
\bar{z}_i \in [-1, 1], \text{ if } \bar{\theta}_i = 0, i = 1, \ldots, 2n, \tag{14b}
\]
\[
\bar{z}_i = 0, i = 2n + 1, \ldots, 2n + 1 + p. \tag{14c}
\]

Notice that in the first $2n$ components of $\nabla \mathcal{L}$ we are taking the derivative with respect to $\bar{\theta}$ instead of $\vartheta$, which means we will need to pay attention to additional $\sqrt{n}$ factors. We call such a pair $(\bar{\theta}, \bar{z}) \in \mathbb{R}^{2n+1+p} \times \mathbb{R}^{2n+1+p}$ primal-dual optimal for the rescaled problem (11). Notice that for such a pair to identify the correct support $S$, it is sufficient for
\[
\bar{\theta}_i > 0, \text{ for all } i \in S, \text{ and } \tag{15a}
\]
\[
\|\bar{z}_{S^c}\|_\infty < 1 \tag{15b}
\]

to hold. Where (15a) ensures that all truly active indices are included and (15b) ensures that all truly inactive indices are excluded (due to (14a)). We call (15b) the strict feasibility condition as in Ravikumar et al. (2010). We will proceed to construct a pair $(\bar{\theta}, \bar{z})$ that satisfies condition (13), (14a) - (14c) and (15a) - (15b) with high probability and for sufficiently large $n$. We say the construction succeeds, if $(\bar{\theta}, \bar{z})$ fulfils (13) - (15b), which in particular implies that $\bar{\theta}$ identifies the correct support $S$ and also is a solution to (11).

In the construction of $(\bar{\theta}, \bar{z})$, we make use of knowledge of the true active set $S$, which makes it infeasible to use in practice. However, by the following lemma, if the construction succeeds, any solution to (11) must have the same support as $\bar{\theta}$. In summary, if the construction succeeds, our estimator $\hat{\theta}$ must identify the correct support $S$, too. In fact, the following Lemma is even slightly more nuanced.

Lemma 4. Suppose the construction $(\bar{\theta}, \bar{z})$ fulfils equations (13) and (14a) - (14c) and (15b). Let $S^\dagger = \{i : \bar{\theta}_i > 0\}$. Then,
\[
\hat{S} = S^\dagger.
\]
In particular, if $(\bar{\theta}, \bar{z})$ additionally fulfils (15a), then $S^\dagger = S$, and thus,
\[
\hat{S} = S.
\]
Lemma 4 is proved in Appendix C. We now give a detailed description of the primal-dual witness construction.

Primal-dual witness construction.

1. Solve the restricted penalised likelihood problem

\[ \tilde{\theta}^* = (\tilde{\varphi}^*, \mu^*, \gamma^*) = \arg \min \frac{1}{N} \bar{L}(\tilde{\theta}) + \lambda \|\tilde{\varphi}\|_1, \] (16)

where the argmin is taken over all \( \tilde{\theta} = (\tilde{\varphi}^T, \mu, \gamma^T) \) with support \( S_+ \), i.e., \( \tilde{\theta}_S^* = \tilde{\theta}^* \) or equivalently \( \tilde{\theta}_{S^*} = 0 \). Thus, by construction, \( \tilde{\theta}^* \) correctly excludes all inactive indices.

2. Set \( \bar{z}_i^* = 1 \), if \( \tilde{\varphi}_i^* > 0 \), such that (14a) holds and \( \bar{z}_i^* = 0 \), \( i = 2n + 1, \ldots, 2n + p \), such that (14c) holds.

3. Plug \( \tilde{\theta}^* \) and \( \bar{z}^* \) into (13) and solve for the remaining components of \( \bar{z}^* \), such that (13) holds for \((\tilde{\theta}^*, \bar{z}^*)\).

The challenge will be proving that (15a) and (15b) also hold, which together ensure that (14b) holds, too. This will be proved in Appendix C.

### 3.2 Consistency

After having seen in Theorem 1 that our estimator \( \hat{\theta} \) will recover the true set of active indices \( S_0 \) with high probability for sufficiently large \( n \), in this section we will show that under similar assumptions it will also be consistent in terms of excess risk and \( \ell_1 \)-error.

Following the empirical risk literature (cf. Greenshtein & Ritov (2004), Koltchinskii (2011)) we will analyze the performance of our estimator in terms of excess risk. To that end, let \( P_n \) denote the empirical measure with respect to our observations \((A_{ij}, Z_{ij})\), that is, for any suitable function \( g \),

\[ P_n g := \frac{1}{N} \sum_{i \neq j} g(A_{ij}, Z_{ij}). \]

In particular, if we let for each \( \theta \in \Theta \), \( l_{\theta}(A_{ij}, Z_{ij}) = -A_{ij}(\alpha_i + \beta_j + \mu + \gamma^T Z_{ij}) + \log(1 + \exp(\alpha_i + \beta_j + \mu + \gamma^T Z_{ij})) \), then \( P_n l_{\theta} = L(\theta)/N \). Similarly, we define the theoretical risk as \( P = \mathbb{E} P_n \). In particular,

\[ P l_{\theta} = \mathbb{E} P_n l_{\theta} = \frac{1}{N} \mathbb{E}[L(\theta)], \]

where we suppress the dependence of the theoretical risk on \( n \) in our notation. As mentioned above, the truth \( \theta_0 \) fulfils

\[ \theta_0 = \arg \min_{\theta} P l_{\theta} = \arg \min_{\theta \in \Theta_{\text{loc}}(r_n, 0)} P l_{\theta}, \]

where the second equality follows from Assumption 2. We define the excess risk as

\[ \mathcal{E}(\theta) := P(\theta - l_{\theta_0}). \]

Analogously, we define the excess risk for the sample-size adjusted parameters \( \tilde{\theta} \) as

\[ \tilde{\mathcal{E}}(\tilde{\theta}) = \frac{1}{N} \mathbb{E}[L(\tilde{\theta}) - L(\tilde{\theta}_0)]. \]

By construction, \( \tilde{\mathcal{E}}(\tilde{\theta}) = \mathcal{E}(\theta) \).

A compatibility condition. A crucial identifiability assumption in LASSO theory is the so called compatibility condition (van de Geer & Bühlmann 2011, van de Geer et al. 2014). It relates the quantities \( \| (\hat{\theta} - \theta_0) S_{0, i} \|_1 \) and

\[ (\hat{\theta} - \theta_0) \Sigma(\hat{\theta} - \theta_0), \]
in a suitable sense made precise below and is crucial for deriving consistency results. In SRGM, similar to the sparse $\beta$-model in Stein & Leng (2020), the classical compatibility condition as for example defined for generalized linear models in van de Geer et al. (2014) does not hold. The reason for this is that $\vartheta$ and $(\mu, \gamma^T)^T$ have different effective sample sizes. Therefore, it is crucial that we use the sample size adjusted Gram matrix (9). Using similar techniques as in Stein & Leng (2020), we can now show that the sample size adjusted Gram matrix fulfills the compatibility condition.

**Proposition 5 (Compatibility condition).** Under Assumption 1, for $s_0 = o(\sqrt{n})$ and $n$ large enough, it holds for every $\theta \in \mathbb{R}^{2n + 1 + p}$ with $\|\theta_{s_0^+}\|_1 \leq 3\|\theta_{s_0^+}\|_1$ that

$$\|\theta_{s_0^+}\|_1^2 \leq \frac{2s_0 + \vartheta^T \Sigma \theta}{c_{\min}}.$$

Parameter estimation consistency is the most lenient of our theorems in terms of restrictions that we have to impose on the parameter sparsity $s_0$ and the network sparsity $\rho_n$. We may replace the stricter assumption B1 by the following.

**Assumption B2.** $\sqrt{n}s_0\rho_n^{-1}\bar{\lambda} \to 0$, $n \to \infty$.

Theorem 2 below suggests a choice of $\bar{\lambda} \asymp \sqrt{\log(n)/N}$. Under these conditions, Assumption B2 becomes $s_0\rho_n^{-1}\sqrt{\log(n)/n} \to 0$. That is, up to an additional factor $\rho_n^{-1}$, which is the price we have to pay for allowing our link probabilities to go to zero, the permissible sparsity for $\vartheta_0$ is the permissible sparsity in classical LASSO theory for an effective sample size of order $n$. This makes sense, considering the discussion of the differing effective sample sizes in Section 3. Also, this choice of $\bar{\lambda}$ together with Assumption B2 implies $s_0 = o(\sqrt{n})$, as is required by Proposition 5 and which thus is not a restriction.

**Theorem 2.** Let assumptions 1, 2 and B2 hold. Fix a confidence level $t$ and let

$$a_n := \sqrt{\frac{2\log(2(2n + p + 1))}{N}}(1 \vee c).$$

Choose $\lambda_0 = \lambda_0(t, n)$ as

$$\lambda_0 = 8a_n + 2\sqrt{\frac{t}{N}(11(1 \vee (c^2p)) + 16(1 \vee c)\sqrt{s_0a_n})} + \frac{4t(1 \vee c)\sqrt{n}}{3N}.$$

Let $\bar{\lambda} = \sqrt{n}\lambda \geq 8\lambda_0$. Then, with probability at least $1 - \exp(-t)$ we have

$$\mathcal{E}(\hat{\vartheta}) + \bar{\lambda} \left(\frac{1}{\sqrt{n}}\|\hat{\vartheta} - \vartheta^*\|_1 + |\hat{\mu} - \mu^*| + \|\hat{\gamma} - \gamma^*\|_1\right) \leq \frac{64}{c_{\min}} \frac{s_{0^+}\bar{\lambda}^2}{\rho_n}.$$

Notice that the construction of $\lambda_0$ in Theorem 2 imply $\lambda_0 \asymp \sqrt{\log(n)/N}$, suggesting that we may choose $\bar{\lambda}$ of the same order. Thus, up to the additional factor $\rho_n^{-1}$, we obtain the classical LASSO rates of convergence for a parameter of effective sample size $N$ for $\mu$ and $\gamma$ and those for a parameter of effective sample size $n$ for $\alpha$ and $\beta$. Notice that this regime for the penalty parameter $\lambda$ is compatible with Assumption 3. Thus, if we additionally impose assumption B1, Theorem 2 ensures that with probability tending to one we are capable of simultaneously recovering the correct support and consistently estimating the parameter $\theta_0$. Finally, notice that if $s_0$ is a lower order term, such as growing logarithmically or constant, then up to log-factors assumption B2 requires that $\rho_n$ tend to zero at rate at most as fast as $1/\sqrt{n}$, which is faster and thus allows for sparser networks than what we had for model selection consistency in Theorem 1.

### 3.3 Inference

Finally, we will derive the limiting distribution of our estimator of the covariates weights, $\hat{\gamma}$. We will see that the same arguments used for deriving the limiting distribution for $\hat{\gamma}$ also work for $\hat{\mu}$ and as a by-product of our proofs we also obtain an analogous limiting result for $\hat{\mu}$.
Our strategy will be inverting the KKT conditions, similar to van de Geer et al. (2014). Recall our discussion of the KKT conditions in Section 3.1. By the same arguments, we find that 0 has to be contained in the subdifferential of $\frac{1}{N}L(\theta) + \lambda \| \beta \|_1$ at $\hat{\theta}$, where this time we consider the KKT conditions with respect to the original parameters $\theta$. That is, there exists a $\hat{z} \in \mathbb{R}^{2n+1+p}$ such that

$$0 = \frac{1}{N} \nabla L(\theta)_{\theta=\hat{\theta}} + \lambda \hat{z}; \quad (17)$$

where $\nabla L(\theta)_{\theta=\hat{\theta}}$ is the gradient of $L(\theta)$ evaluated at $\hat{\theta}$ and for $i = 1, \ldots, 2n$, $\hat{z}_i = 1$ if $\hat{\theta}_i > 0$ and $\hat{z}_i \in [-1,1]$ if $\hat{\theta}_i = 0$, and for $i = 2n+1, \ldots, 2n+1+p$, $\hat{z}_i = 0$.

Denoting $\nabla_\xi L(\theta)_{\theta=\hat{\theta}} \in \mathbb{R}^{p+1}$ the gradient of $L$ with respect to the unpenalized parameters $\xi = (\mu, \gamma^T)^T$ only, evaluated at $\hat{\theta}$, we have

$$0 = \nabla_\xi L(\theta)_{\theta=\hat{\theta}}. \quad (18)$$

Denote by $H(\hat{\theta}) := H_{\xi \times \xi}(\theta)_{\theta=\hat{\theta}}$ the Hessian of $\frac{1}{N}L(\theta)$ with respect to $\xi$ only, evaluated at $\hat{\theta}$. Consider the entries of $H(\hat{\theta})$: For all $k, l = 1, \ldots, (p+1)$,

$$H(\hat{\theta})_{k,l} = \frac{1}{N} \partial_{\xi_k,\xi_l} L(\hat{\theta}) = \frac{1}{N} \sum_{i \neq j} D_{ij,2n+k} D_{ij,2n+l} \cdot p_{ij}(\hat{\theta})(1 - p_{ij}(\hat{\theta})), \quad \text{where } D_{ij}^k \text{ is the } (i,j)-\text{th row of the design matrix } D, \text{ i.e. in particular } D_{ij,2n+k} = 1 \text{ if } k = 1 \text{ and } D_{ij,2n+k} = Z_{ij,k-1} \text{ for } k = 2, \ldots, (p+1).$$

In particular, we have the following matrix representation of $H(\hat{\theta})$: Let $D_\xi = [1 | Z]$ be the part of the design matrix $D$ corresponding to $\xi$ with rows $D_{\xi,ij}^k = (1, Z_{ij}^k), i \neq j$. Also, let $W = \text{diag}(\sqrt{p_{ij}(\hat{\theta})(1 - p_{ij}(\hat{\theta}))), i \neq j)$. Then we have

$$H(\hat{\theta}) = \frac{1}{N} D_\xi^T W^2 D_\xi.$$ Let $W_0 = \text{diag}(\sqrt{p_{ij}(\theta_0)(1 - p_{ij}(\theta_0))), i \neq j)$ and consider the corresponding population version:

$$\mathbb{E}[H(\theta_0)] = \frac{1}{N} \mathbb{E}[D_\xi^T W_0^2 D_\xi].$$

To be consistent with commonly used notation, call $\hat{\Sigma}_\xi = H(\hat{\theta}) = \frac{1}{N} D_\xi^T W^2 D_\xi$, and $\Sigma_\xi = \mathbb{E}[H(\theta_0)] = \frac{1}{N} \mathbb{E}[D_\xi^T W_0^2 D_\xi]$ and

$$\hat{\Theta}_\xi := \hat{\Sigma}_\xi^{-1}, \quad \Theta_\xi := \Sigma_\xi^{-1}.$$

For the proof of asymptotic normality we need to invert $\hat{\Sigma}_\xi$ and $\Sigma_\xi$ and show that these inverses are close to each other in an appropriate sense. It is commonly assumed in LASSO theory (cf. van de Geer et al. (2014)) that the minimum eigenvalues of these matrices stay bounded away from zero. In our case, however, such an assumption is invalid. Since we allow for the lower bound $\rho_{n,0}$ on the link probabilities to go to zero, any lower bound on the entries in $W_0$ will go to zero with $n$ and as a consequence our lower bound on the minimum eigenvalue of $\Sigma_\xi$ will tend to zero as $n$ goes to infinity as well. The best we can achieve is a strict positive definiteness of $\Sigma_\xi$ for finite $n$, but not uniformly in $n$. Since these lower bounds tend to zero with increasing $n$, a careful argument is needed and we have to impose stricter assumptions than for our consistency result alone. Precisely, we need a slightly stricter balancing assumption than Assumption B2.

**Assumption B3.** $\sqrt{n}s_0 \rho_n^{-2} \lambda \to 0, n \to \infty.$

**Theorem 3.** Under assumptions 1, 2 and B3, with $\lambda \asymp \sqrt{\log(n)/N}$ fulfilling the conditions of Theorem 2, we have for any $k = 1, \ldots, p$, as $n \to \infty$,

$$\sqrt{N} \frac{\hat{\gamma}_k - \gamma_{0,k}}{\sqrt{\hat{\Theta}_{\theta,k+1,k+1}}} \overset{d}{\to} \mathcal{N}(0,1).$$
We also have for our estimator of the global sparsity parameter, \( \hat{\mu} \), as \( n \to \infty \),

\[
\sqrt{N} \frac{\hat{\mu} - \mu_0}{\sqrt{\hat{\Theta}_{\hat{\theta},1,1}}} \xrightarrow{d} \mathcal{N}(0, 1).
\]

Notice that contrary to what is commonly seen in the penalized likelihood literature (Zhang & Zhang 2014, van de Geer et al. 2014), no debiasing of \( \hat{\gamma} \) and \( \hat{\mu} \) is needed. The reason for this is that columns of \( D \) pertaining to those parameters which are indeed biased, that is to \( \hat{\theta} \), and those pertaining to \( \xi = (\mu, \gamma^T) \) become asymptotically orthogonal, meaning that the bias in \( \xi \) vanishes fast enough for the derivation of Theorem 3 to be possible.

Notice in particular, that if we choose \( \tilde{\lambda} \propto \sqrt{\log(n)/N} \), then the stricter Assumption B1 implies assumptions B2 and B3 and Theorems 1, 2 and 3 hold simultaneously under the assumptions of Theorem 1. In particular, with probability tending to one, the estimator \( \hat{\theta} \) from (7) will identify the correct support \( S_{0,+} \), estimate the true parameter \( \theta_0 \) at the classical LASSO rate of convergence up to an additional factor \( n^{-1} \) and produce asymptotically normal estimators for the global parameters \( \xi_0 = (\mu_0, \gamma_0^T) \). Also notice that for a lower order \( s_0 \), assumption B3 essentially allows for the same level of network sparsity as assumption B1, up to lower order factors.

4 Simulation

In this section we demonstrate the effectiveness of our estimator (7) in performing simultaneous parameter estimation and model selection consistently. To this end, we test its performance on networks of varying sizes. Specifically, we let \( n \) vary between 150 and 800 in steps of 50 and choose the sparsity level \( s_0 \) to be close to \( \sqrt{n}/2 \). We let \( s_0 = 6, 6, 6, 8, 8, 10, 10, 10, 12, 12, 12, 12, 14 \) and chose \( s_0 = s_0 = s_0/2 \) in each case. We selected a heterogeneous configuration for the assignment of non-zero \( \alpha \) and \( \beta \) values. That is, we included dedicated ‘spreader’ nodes, with large \( \alpha \) and zero \( \beta \) value as well as ‘attractor’ nodes with large \( \beta \) and zero \( \alpha \) as well as some nodes with both active \( \alpha \) and \( \beta \). In detail, we let

\[
\alpha = (2, 1.5, 1, 0.8, \ldots, 0.8, 0, \ldots, 0),
\beta = (0, \ldots, 0, 2, 1.5, 1, 0.8, \ldots, 0.8, 0, \ldots, 0),
\]

where the number of entries with value 0.8 was chosen to match the aforementioned sparsity level (zero for the first three values of \( n \)) and the number of leading zeros in \( \beta \) was chosen such that there were exactly two nodes with both active \( \alpha \) and \( \beta \). We let the networks get progressively sparser and set \( \mu = -1.2 \cdot \log(\log(n)) \). In all cases we used \( p = 2 \) and sampled the covariate values \( Z_{ij,k}, k = 1, 2, i \neq j \) from a centred Beta(2, 2) distribution, that is

\[
Z_{ij,k} \sim \text{Beta}(2, 2) - 1/2,
\]

and weighted the covariates with \( \gamma = (1, 0.8)^T \). For each value of \( n \) we drew \( M = 500 \) realizations of this model. The observed median edge density, as well as median minimum and maximum link probabilities \( p_{ij} \) are recorded in Table 2.

Our estimator requires us to choose a tuning parameter \( \lambda \) and we explored the use of the Bayesian Information Criterion (BIC) as well as a heuristic based on our developed theory for model selection. While the former criterion is purely data-driven, the use of the latter is to ensure that our theoretical results are right in terms of the rates. We will see that, while the two model selection procedures perform similarly in terms of parameter estimation and inference for \( \gamma \), with BIC achieving slightly better results, the heuristic based on our developed theory is superior to BIC in terms of model selection consistency.

To make the dependence of our estimator (7) on the penalty parameter explicit, we denote the solution of (7) when using penalty \( \lambda \) by \( \hat{\theta}(\lambda) = (\hat{\alpha}(\lambda)^T, \hat{\beta}(\lambda)^T, \hat{\mu}(\lambda), \hat{\gamma}(\lambda)^T)^T \) and write \( s(\lambda) = |\{i :
| $n$ | Median edge density | $\min p_{ij}$ | $\max p_{ij}$ |
|-----|---------------------|--------------|--------------|
| 150 | 0.140               | 0.059        | 0.888        |
| 200 | 0.130               | 0.055        | 0.886        |
| 250 | 0.123               | 0.052        | 0.879        |
| 300 | 0.119               | 0.050        | 0.873        |
| 350 | 0.115               | 0.048        | 0.868        |
| 400 | 0.112               | 0.047        | 0.867        |
| 450 | 0.109               | 0.046        | 0.864        |
| 500 | 0.107               | 0.045        | 0.861        |
| 550 | 0.105               | 0.044        | 0.859        |
| 600 | 0.104               | 0.043        | 0.858        |
| 650 | 0.102               | 0.043        | 0.854        |
| 700 | 0.100               | 0.042        | 0.856        |
| 750 | 0.099               | 0.041        | 0.851        |
| 800 | 0.098               | 0.041        | 0.853        |

Table 2: Network summary statistics for directed network model for various values of $n$. 

The value of the BIC at $\lambda$ is given by

$$\text{BIC} = -2\mathcal{L}(\hat{\theta}(\lambda)) + s(\lambda) \log(N)$$

and the penalty $\lambda$ was chosen to minimize BIC.

On the other hand, our heuristic is motivated by the theory developed in the previous sections. We have two restrictions on the choice of the penalty parameter $\lambda$, namely the ones specified in Theorems 1 and 2. While both theorems demand $\lambda$ to be of the same order, upon inspection we see that the conditions imposed by Theorem 2 demand $\lambda$ to be larger in terms of the leading constant, which is why we take that theorem as starting point for the construction of our heuristic. In detail, recall that Theorem 2 suggests that based on a confidence level $t$ picked by us, we should first define

$$a_n := \sqrt{\frac{2 \log(2(2n + p + 1))}{N}} (1 \lor c),$$

and then based on that, choose $\lambda_0 = \lambda_0(t, n)$ as

$$\lambda_0 = 8a_n + 2\left(\frac{t}{N} (1 \lor (c^2 p)) + 16(1 \lor c) \sqrt{na_n}\right) + \frac{4t(1 \lor c)\sqrt{n}}{3N}.$$ 

Finally, the consistency results derived hold for any $\bar{\lambda} \geq 8\lambda_0$, where $\bar{\lambda}$ is the penalty parameter in the rescaled penalized likelihood problem, which relates to the penalty parameter $\lambda$ in the original penalized problem (7) as $\bar{\lambda} = \sqrt{n} \cdot \lambda$. Looking back at the proof of Theorem 2, we see that the factor eight in the relation between $\lambda_0$ and $\bar{\lambda}$ is a technical artifact we had to introduce to prove that the sample-size adjusted estimator $\hat{\theta}$ was close enough to the truth $\bar{\theta}_0$. If we assume that our estimator is close enough to the truth, we may ignore that factor and set $\lambda = \frac{1}{\sqrt{n}} \lambda_0$. We pick $t = 3$ and set $c$ to the maximum observed covariate value. It is known that in high-dimensional settings the penalty values prescribed by mathematical theory in practice tend to over-penalize the parameter values, see, for example, Yu et al. (2019). Decreasing the penalty by removing the factor eight is thus in line with these empirical findings.

We drew $M = 500$ realizations for each value of $n$ and recorded the mean absolute error for estimation of $(\alpha^T, \beta^T)^T$, the absolute error for estimation of $\mu$ and the $\ell_1$-error for estimation of $\gamma$. We also constructed confidence intervals as prescribed by Theorem 3 and recorded the empirical coverage at the nominal 95% level. Finally, we studied how well BIC and our heuristic did in terms of identifying the correct model.

Consistency We display the various error statistics for estimation of $\theta_0 = (\alpha_0^T, \beta_0^T)^T, \mu_0$ and $\gamma_0$ in figures 4a, 4b and 4c respectively. We see that the error decreases with increasing network size for both
model selection procedures. We see that especially for small \( n \), BIC outperforms the heuristic for \( \vartheta_0 \) and \( \mu_0 \), while they both give essentially the same results for estimation of \( \gamma_0 \). The better performance of BIC is less prominent as \( n \) increases. BIC selects the penalty in a purely data driven manner, which allows it to adapt to differing degrees of sparsity in the network, while for the heuristic the penalty value only depends on \( n \) and \( p \). This additional flexibility is what allows BIC to achieve lower error values.

**Asymptotic normality** We construct confidence intervals at the nominal 95% level for our estimators of \( \gamma_{0,1} \) and \( \gamma_{0,2} \) as prescribed by Theorem 3. Table 3 shows the results for \( \gamma_{0,1} \) across the values of \( n \). The results for \( \gamma_{0,2} \) are similar and are omitted to save space. The coverage is very close to the 95%-level across all network sizes, independent of which model selection criterion we use. This is to be expected, considering that there was hardly any difference for the estimation of \( \gamma \) between our two model selection criteria. This empirically illustrates the validity of the asymptotic results derived in Theorem 3. As expected, the median length of the confidence interval decreases with increasing network size.

| \( n \) | Coverage | CI | Coverage | CI |
|---|---|---|---|---|
| Pre-determined \( \lambda \) | BIC |
| 150 | 0.960 | 0.342 | 0.962 | 0.344 |
| 200 | 0.952 | 0.265 | 0.962 | 0.266 |
| 250 | 0.952 | 0.217 | 0.954 | 0.218 |
| 300 | 0.944 | 0.183 | 0.948 | 0.184 |
| 350 | 0.946 | 0.160 | 0.952 | 0.160 |
| 400 | 0.950 | 0.141 | 0.964 | 0.141 |
| 450 | 0.962 | 0.127 | 0.960 | 0.127 |
| 500 | 0.940 | 0.115 | 0.944 | 0.115 |
| 550 | 0.952 | 0.106 | 0.950 | 0.106 |
| 600 | 0.954 | 0.097 | 0.956 | 0.097 |
| 650 | 0.960 | 0.091 | 0.964 | 0.091 |
| 700 | 0.946 | 0.085 | 0.950 | 0.085 |
| 750 | 0.944 | 0.079 | 0.946 | 0.079 |
| 800 | 0.946 | 0.075 | 0.952 | 0.075 |

Table 3: Empirical coverage under nominal 95% coverage and median lengths of confidence intervals.

**Model selection** Finally we compare model selection performance between BIC and our heuristic. Figure 5a shows the empirical probability of selecting the correct model versus the various network sizes. We can see very clearly that asymptotically, as \( n \) grows, our heuristic outperforms BIC, achieving correct model selection almost all the time. Nonetheless, it is worth pointing out that even though BIC may not select the exact correct model, the number of misclassifications it does on average is not very large, as shown in figure 5b. Figure 5b also shows that the heuristic, by virtue of selecting a
larger penalty than BIC, will on average incur more false negatives for small $n$. On the other hand, as $n$ grows, BIC will incur false positives, resulting in the decreasing probability of selecting the correct subset.

5 Conclusion

We have proposed a parameter-sparse random graph model to analyze density-sparse directed networks. Estimated via penalized likelihood with an $\ell_1$ penalty, our model consistently identifies the support of the parameter, consistently estimates their values, and provides valid inference for the global density parameter and the covariates parameter. The computation of our estimator capitalizes on the tremendous progress made in the algorithmic development of LASSO-type estimators and thus is straightforward to implement and extremely scalable. To push the scalability of our model even further, we plan to exploit the sparse structure of $D$ in (8) in future work. Our model operates in the model-selective inference framework where no artificial removal of data is made before analysis, as opposed to the common problematic practice of employing data-selective inference that is popular in the literature.

In this paper, we have assumed that, given the covariates, directed links are formed independently between and within node-pairs. The latter is a limitation because empirically reciprocity, a measure of the likelihood of vertices in a directed network to be mutually linked, may be present. In the motivating lawyers data for example, lawyer $j$ will more likely call lawyer $i$ a friend if the converse is true. To address this layer of sophistication, the next natural step is to add a reciprocity parameter to the model. One approach to do this is to add an extra term $\delta \sum_{i<j} A_{ij} A_{ji}$ in the model, where $\delta \in \mathbb{R}$ is an unknown parameter, as adopted in the $p_1$ model (Holland & Leinhardt 1981). For the models in Yan et al. (2016) and Yan et al. (2019), including this extra reciprocity parameter turns out very challenging for their theoretical framework, because their analytical tool needed to approximate the inverse of a Fisher information matrix accurately is no longer applicable. On the other hand, by assuming parameter sparsity in $\alpha$ and $\beta$, our model will deal with a much smaller number of parameters and thus some of the theoretical argument in this paper may go through. We are currently investigating this and will report the progress elsewhere.

Many network models have been applied to real networks such as the political blog data and the statistician citation data. Often times these networks contain nodes that are not well connected. A popular exercise in the literature is to focus on a sub-network induced by nodes with better connectivity, leaving out a substantial portion of the nodes. We have highlighted for the first time the fallacy of the resulting data-selective inference for analysis. Its associated non-random sampling often brings biased estimates as we have demonstrated in two fundamental network models, under the idealistic scenario when the assumed model does produce the realized data. As a result of data-selective inference, it is not clear whether any findings are genuine or artefact of biased sampling – Statisticians are well
aware of the pitfalls of what systematic sampling bias can bring to data analysis. The apparent gap between the growing need for statistical methodologies applicable to many real-life data and the simple conceptual principle of having a coherent modelling framework that deals with sampling issues calls for attention to re-examine many popular models developed for network analysis. Our discussion on data-selective inference is an attempt to make it easy for researchers to identify "model-selective inference" opportunities in their work. It will be interesting to see how the statistical network community respond to this call.

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Supplementary Materials

A Proof of Theorem 2

A.1 A compatibility condition

In this section we first prove a sample compatibility condition before providing a proof for the population compatibility condition in Proposition 5. That is, we first want to find a suitable relation between the quantities \( \| \hat{\theta} - \theta_0 \|_1 \) and \( (\hat{\theta} - \theta_0) \Sigma (\hat{\theta} - \theta_0) \) where \( \Sigma = T^{-1} D^T DT^{-1} \) is the sample version of the sample size adjusted Gram matrix \( \Sigma \).

We make this mathematically precise now: For a general matrix \( A \in \mathbb{R}^{(2n+1+p) \times (2n+1+p)} \) we say the compatibility condition holds, if \( A \) has the following property: There is a constant \( b \) independent of \( n \) such that for every \( \theta \in \mathbb{R}^{2n+1+p} \) with \( \| \theta_{S_0^c} \|_1 \leq \| \theta_{S_0^+} \|_1 \) it holds that

\[
\| \theta_{S_0^+} \|_1^2 \leq \frac{s_0^+ + \theta^T A \theta}{b}.
\]

Notice that the compatibility condition is clearly equivalent to the condition that

\[
\kappa^2(A, s_0) := \min_{\theta \in \mathbb{R}^{2n+1+p} \setminus \{0\}} \frac{\theta^T A \theta}{\| \theta_{S_0^+} \|_1^2} \leq \frac{1}{s_0^+} \| \theta_{S_0^+} \|_1^2
\]

stays bounded away from zero.

We first show that the compatibility condition holds for the matrix

\[
\Sigma_A := \begin{bmatrix} I_{2n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E[Z^T Z/N] \end{bmatrix} \in \mathbb{R}^{(2n+1+p) \times (2n+1+p)},
\]

where \( I_{2n} \) is the \((2n) \times (2n)\) identity matrix.

Recall that by assumption 1, the minimum eigenvalue \( \lambda_{\min} = \lambda_{\min}(n) \) of \( \frac{1}{N} E[Z^T Z] \) stays uniformly bounded away from zero. That is, there is a finite constant \( c_{\min} > 0 \) independent of \( n \), such that \( \lambda_{\min} > c_{\min} > 0 \) for all \( n \). Then, clearly, for any \( \theta = (\theta^T, \mu, \gamma^T)^T \),

\[
\theta^T \Sigma_A \theta = \| \theta \|_2^2 + \mu^2 + \gamma^T \frac{1}{N} E[Z^T Z] \gamma \geq \| \theta \|_2^2 + \mu^2 + c_{\min} \| \gamma \|_2^2 \geq (1 \wedge c_{\min}) \| \theta \|_2^2.
\]

Thus, \( \Sigma_A \) is strictly positive definite. Furthermore, by Cauchy-Schwarz’ inequality, for any \( \theta \in \mathbb{R}^{2n+1+p} \) with \( \| \theta_{S_0^+} \|_1 \leq 3 \| \theta_{S_0^+} \|_1 \),

\[
\frac{1}{s_0^+} \| \theta_{S_0^+} \|_1^2 \leq \| \theta_{S_0^+} \|_2^2 \leq \| \theta \|_2^2.
\]

Thus,

\[
\kappa^2(\Sigma_A, s_0) = \min_{\theta \in \mathbb{R}^{2n+1+p} \setminus \{0\} \atop \| \theta_{S_0^+} \|_1 \leq 3 \| \theta_{S_0^+} \|_1} \frac{\theta^T \Sigma_A \theta}{\| \theta_{S_0^+} \|_1^2} \geq \frac{(1 \wedge c_{\min}) \| \theta \|_2^2}{\| \theta \|_2^2} > 0.
\]

We conclude that the compatibility condition holds for \( \Sigma_A \). Now, we need to show that with high probability \( \kappa(\Sigma, s_0) \geq \kappa(\Sigma_A, s_0) \), which would imply that the compatibility condition holds with high probability for \( \Sigma \). To that end, we have the following auxiliary lemma found in Kock & Tang (2019).

For completeness, we give the short proof of it. The notation is adapted to our setting.

Lemma 6 (Lemma 6 in Kock & Tang (2019)). Let \( A \) and \( B \) be two positive semi-definite \((2n+1+p) \times (2n+1+p)\) matrices and \( \delta = \max_{i,j} |A_{ij} - B_{ij}| \). For any set \( S \subseteq \{ 1, \ldots, 2n \} \) with cardinality \( s_0 \), one has

\[
\kappa^2(B, s_0) \geq \kappa^2(A, s_0) - 16\delta(s_0 + p + 1).
\]
Proof. Denote by \( S_{0,+} = S_0 \cup \{2n+1, \ldots, 2n+1+p\} \) and \( s_{0,+} = s_0 + (1 + p) \). Let \( \theta \in \mathbb{R}^{2n+1+p} \setminus \{0\} \), with \( \|\theta s_{0,+}\|_1 \leq 3\|\theta s_{0,+}\|_1 \). Then,

\[
|\theta^T A\theta - \theta^T B\theta| = |\theta^T (A - B)\theta| \leq \|\theta\|_1 \|(A - B)\theta\|_\infty \leq \delta \|\theta\|_1^2
\]

\[
= \delta (\|\theta s_{0,+}\|_1 + \|\theta s_{0,+}\|_1)^2 \leq \delta (\|\theta s_{0,+}\|_1 + 3\|\theta s_{0,+}\|_1)^2
\]

\[
\leq 16\delta \|\theta s_{0,+}\|_1^2.
\]

Hence, \( \theta^T B\theta \geq \theta^T A\theta - 16\delta\|\theta s_{0,+}\|_1^2 \) and thus

\[
\frac{\theta^T B\theta}{\|\theta s_{0,+}\|_1^2} \geq \frac{\theta^T A\theta}{\|\theta s_{0,+}\|_1^2} - 16\delta s_{0,+} \geq \kappa^2(A, s_0) - 16\delta s_{0,+}.
\]

Minimizing the left-hand side over all \( \theta \neq 0 \) with \( \|\theta s_{0,+}\|_1 \leq 3\|\theta s_{0,+}\|_1 \) proves the claim. \( \square \)

This shows that to control \( \kappa^2(\hat{\Sigma}, s_0) \), we need to control the maximum element-wise distance between \( \hat{\Sigma} \) and \( \Sigma_A \): \( \max_{ij} |\hat{\Sigma}_{ij} - \Sigma_{A,ij}|. \) Introduce the set

\[
\mathcal{J} = \left\{ \max_{ij} |\hat{\Sigma}_{ij} - \Sigma_{A,ij}| \leq \frac{c_{\min}}{32s_{0,+}} \right\}.
\]

On the set \( \mathcal{J} \), by Lemma 6, we have \( \kappa^2(\hat{\Sigma}, s_0) \geq \kappa(\Sigma_A, s_0) - \frac{c_{\min}}{2} \geq \frac{c_{\min}}{2} > 0 \) and thus the compatibility condition holds for \( \hat{\Sigma} \) on \( \mathcal{J} \).

**Lemma 7.** If \( s_0 = o(\sqrt{n}) \), for \( n \) large enough, with \( \delta = \frac{c_{\min}}{32s_{0,+}} \) and \( \bar{c} = c^2 \vee (2c^4) \), where \( c > 0 \) is the universal constant such that \( |Z_{k,ij}| \leq c \) for all \( k, i, j \), we have

\[
P(\mathcal{J}) = P \left( \max_{ij} |\hat{\Sigma}_{ij} - \Sigma_{A,ij}| \leq \frac{c_{\min}}{32s_{0,+}} \right) \geq 1 - p(p + 3) \exp \left( -N \frac{c_{\min}^2}{2048s_{0,+}^2} \right).
\]

**Proof.** To make referencing of sections of \( \hat{\Sigma} \) easier, we number its blocks as follows

\[
\hat{\Sigma} = T^{-1} \begin{bmatrix}
X^T X & X^T 1 & X^T Z \\
1 & X & 1 \\
Z^T X & Z^T 1 & Z^T Z
\end{bmatrix}
T^{-1}.
\]

For block \((1)\), i.e. \( i, j = 1, \ldots, 2n \), notice that \( (X^{out})^T X^{out} = (X^{in})^T X^{in} = (n - 1)I_n \) and \( (X^{out})^T X^{in} \) is a matrix with zero on the diagonal and ones everywhere else. Therefore, we have either \( \hat{\Sigma}_{ij} = \Sigma_{A,ij} \) or

\[
|\hat{\Sigma}_{ij} - \Sigma_{A,ij}| = \frac{1}{n - 1} < \frac{c_{\min}}{32s_{0,+}},
\]

for \( n \) large enough, since \( s_{0,+} = o(\sqrt{n}) \). Blocks \((2)\) and \((4)\) are a \( 2n \) dimensional column and row vector respectively in which each entry is equal to \( n - 1 \). Thus, for \( i, j \) corresponding to these blocks,

\[
|\hat{\Sigma}_{ij} - \Sigma_{A,ij}| = \frac{n - 1}{\sqrt{(n - 1)N}} = \frac{1}{\sqrt{n}} \leq \frac{c_{\min}}{32s_{0,+}},
\]

for \( n \) large enough, since \( s_{0,+} = o(\sqrt{n}) \). For \( i, j \) corresponding to blocks \((3)\) and \((7)\), we have

\[
|\hat{\Sigma}_{ij} - \Sigma_{A,ij}| = \frac{c}{\sqrt{n}} < \frac{c_{\min}}{32s_{0,+}}.
\]

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for $n$ large enough. Block (5) is a single real number and equal for $\hat{\Sigma}$ and $\Sigma_A$.

The only cases left to consider are those entries corresponding to blocks (6), (8) and (9). For the blocks (6) and (8), that is for $i = 2n+1$, $j = 2n+2, \ldots, 2n+1+p$ and $i = 2n+2, \ldots, 2n+1+p$, $j = 2n+1$, $\hat{\Sigma}_{ij} - \Sigma_{A,ij} = \Sigma_{ij}$ is the scaled sum of all the entries of some column $Z_k$ of the matrix $Z$ for an appropriate $k$. That is, there is a $1 \leq k \leq p$ such that

$$
\hat{\Sigma}_{ij} - \Sigma_{A,ij} = \frac{1}{N} Z_k^T 1 = \frac{1}{N} \sum_{s \neq l} Z_{k,st}.
$$

Note, that thus by model assumption $\mathbb{E}[\hat{\Sigma}_{ij} - \Sigma_{A,ij}] = 0$. We know that for each $k, s, t : Z_{k,st} \in [-c, c]$. Hence, by Hoeffding’s inequality, for all $\delta > 0$,

$$
P \left( |\hat{\Sigma}_{ij} - \Sigma_{A,ij}| \geq \delta \right) = P \left( \left| \sum_{s \neq l} Z_{k,st} \right| \geq N\delta \right) \leq 2 \exp \left( -\frac{2N^2\delta^2}{\sum_{i \neq j}(2c)^2} \right) = 2 \exp \left( -N \frac{\delta^2}{2c^2} \right).
$$

For block (9), that is for $i, j = 2n+2, \ldots, 2n+1+p$, a typical element has the form

$$
\hat{\Sigma}_{ij} - \Sigma_{A,ij} = \frac{1}{N} \sum_{s \neq t} \{Z_{k,st} Z_{l,st} - \mathbb{E}[Z_{k,st} Z_{l,st}]\},
$$

for appropriate $k, l$. In other words, $\hat{\Sigma}_{ij} - \Sigma_{A,ij}$ is the inner product of two columns of $Z$, minus their expectation, scaled by $1/N$. Since $Z_{k,st} Z_{l,st} \in [-c^2, c^2]$ for all $k, l, s, t$, we have that for all $k, l, s, t$: $Z_{k,st} Z_{l,st} - \mathbb{E}[Z_{k,st} Z_{l,st}] \in [-2c^2, 2c^2]$. Thus, by Hoeffding’s inequality, for all $\delta > 0$,

$$
P \left( |\hat{\Sigma}_{ij} - \Sigma_{A,ij}| \geq \delta \right) = P \left( \left| \sum_{s \neq t} \{Z_{k,st} Z_{l,st} - \mathbb{E}[Z_{k,st} Z_{l,st}]\} \right| \geq N\delta \right) \leq 2 \exp \left( -N \frac{\delta^2}{8c^4} \right).
$$

Thus, with $\delta = c^2 \vee (2c^4)$, we have for any entry in blocks (6), (8), (9), that for any $\delta > 0$,

$$
P \left( |\hat{\Sigma}_{ij} - \Sigma_{A,ij}| \geq \delta \right) \leq 2 \exp \left( -N \frac{\delta^2}{2c^2} \right).
$$

Choosing $\delta = \frac{c_{\min}}{32s_{0,+}}$, by the exposition above we know that all entries in blocks (1) - (5) and (7) are bounded by $\delta$ for $n \gg 0$. Also, because block (6) is the transpose of block (8), it is sufficient to control one of them. By symmetry of block (9) it suffices to control the upper triangular half, including the diagonal, of block (9). Thus, we only need to control the entries $\hat{\Sigma}_{ij} - \Sigma_{A,ij}$ for $i, j$ in the following index set

$$
\mathcal{A} = \{i, j : i, j \text{ belong to block (8) or the upper triangular half or the diagonal of block (9)} \} = \{(i, j) \in (n+2, \ldots, n+1+p) \times \{n+1\} \cup \{i \leq j : i, j = n+2, \ldots, n+1+p\}.
$$

Keep in mind that block (8) has $p$ elements, while the upper triangular part of block (9) plus its diagonal has $\binom{p}{2} + p = \binom{p+1}{2}$ elements. Thus, for $n \gg 0$,

$$
P(\mathcal{J}^c) = P \left( \max_{i,j} |\hat{\Sigma}_{ij} - \Sigma_{A,ij}| \geq \frac{c_{\min}}{32s_{0,+}} \right) \leq \sum_{i,j \in \mathcal{A}} P \left( |\hat{\Sigma}_{ij} - \Sigma_{A,ij}| \geq \frac{c_{\min}}{32s_{0,+}} \right) \leq 2p \exp \left( -N \frac{\delta^2}{2c^2} \right) + 2 \left( \frac{p+1}{2} \right) \exp \left( -N \frac{\delta^2}{8c^4} \right) \leq 2 \left( \frac{p+1}{2} \right) \exp \left( -N \frac{\delta^2}{2c^2} \right) = p(p+3) \exp \left( -N \frac{\delta^2}{2c^2} \right).
$$

This proves the claim. \hfill \Box
We summarize these results in the following proposition

**Proposition 8.** Under assumption 1, for $s_0 = o(\sqrt{n})$ and $n$ large enough, with $\tilde{c} = c^2 \lor (2c^4)$, where $c > 0$ is the universal constant such that $|Z_{k,i,j}| \leq c$ for all $k, i, j$: With probability at least

$$1 - p(p + 3) \exp \left( -N \frac{c_{\min}^2}{2048 s_0^2 \tilde{c}} \right)$$

it holds that for every $\theta \in \mathbb{R}^{2n+1+p}$ with $\|\theta_{S_0^c,S_0^c}\|_1 \leq 3\|\theta_{S_0^c}\|_1$,

$$\|\theta_{S_0^c}\|^2_1 \leq \frac{2s_0 + \theta^T \hat{\Sigma} \theta}{c_{\min}}.$$

**Proof.** This follows from Lemma 7. \(\square\)

**Proof of Proposition 5.** To prove that the compatibility condition holds for the population sample-size adjusted Gram matrix $\Sigma$ we may follow the same steps as in the proof of Proposition 8: Number the blocks of $\Sigma$ as $1^\circ - 9^\circ$ as we did for $\hat{\Sigma}$. $\Sigma$ and $\Sigma_A$ are equal on blocks $3^\circ, 5^\circ, 6^\circ, 7^\circ, 8^\circ$ and $9^\circ$. For blocks $1^\circ, 2^\circ$ and $4^\circ$ we use the exact same arguments as in the proof of Proposition 8 to find that for $n$ sufficiently large, almost surely,

$$\max_{ij} |\Sigma_{ij} - \Sigma_{A,ij}| \leq \frac{c_{\min}}{32s_0^2}.$$

The claim follows from Lemma 6. \(\square\)

### A.2 A basic Inequality

A key result in the consistency proofs in classical LASSO settings is the so called basic inequality (cf. van de Geer & Buehlmann (2011), Chapter 6). We define the empirical process as

$$\{v_n(\theta) = (P_n - P)_{l_\theta} : \theta \in \Theta\}.$$

**Lemma 9** (Basic Inequality). For any $\theta = (\beta^T, \mu, \gamma^T)^T \in \Theta_{loc}$ it holds

$$\mathcal{E}(\hat{\theta}) + \lambda\|\hat{\beta}\|_1 \leq -[v_n(\hat{\theta}) - v_n(\theta)] + \mathcal{E}(\theta) + \lambda\|\beta\|_1.$$

**Proof.** By plugging in the definitions and rearranging, we see that the above equation is equivalent to

$$\frac{1}{N} \mathcal{L}(\hat{\theta}) + \lambda\|\hat{\beta}\|_1 \leq \frac{1}{N} \mathcal{L}(\theta) + \lambda\|\beta\|_1,$$

which is true by definition of $\hat{\theta}$. \(\square\)

Notice that since the basic inequality in Lemma 9 only relies on the argmin property of the estimator $\hat{\theta}$, an analogous result follows line by line for the rescaled parameter $\hat{\theta}$. Writing

$$\hat{v}_n(\hat{\theta}) := \frac{1}{N} (\hat{\mathcal{L}}(\hat{\theta}) - \mathbb{E}[\hat{\mathcal{L}}(\hat{\theta})]) = v_n(\theta).$$

for the rescaled empirical process, we have the following.

**Lemma 10.** For any $\hat{\theta} \in \Theta_{loc}$ it holds

$$\mathcal{E}(\hat{\theta}) + \lambda\|\hat{\theta}\|_1 \leq -[\hat{v}_n(\hat{\theta}) - v_n(\theta)] + \mathcal{E}(\theta) + \lambda\|\theta\|_1.$$
Remark. For any $0 < t < 1$ and $\theta \in \Theta_{\text{loc}}$, let $\tilde{\theta} = t\theta + (1 - t)\bar{\theta}$. Since $\Gamma$ is convex, $\tilde{\theta} \in \Theta_{\text{loc}}$ and since $\theta \rightarrow I_{\theta}$ and $\| . \|_1$ are convex functions, we can replace $\hat{\theta}$ by $\bar{\theta}$ in the basic inequality and still obtain the same result. Plugging in the definitions, we see that the basic inequality is equivalent to the following:

$$E(\tilde{\theta}) + \lambda \| \tilde{\beta} \|_1 \leq -[v_n(\tilde{\theta}) - v_n(\theta)] + \lambda \| \beta \|_1 + E(\theta)$$

$$\iff \frac{1}{N} L(\tilde{\theta}) + \lambda \| \tilde{\beta} \|_1 \leq \frac{1}{N} L(\theta) + \lambda \| \beta \|_1$$

and by convexity

$$\frac{1}{N} L(\tilde{\theta}) + \lambda \| \tilde{\beta} \|_1 \leq \frac{1}{N} t L(\tilde{\theta}) + \frac{1}{N} (1 - t) L(\theta) + t\lambda \| \tilde{\beta} \|_1 + (1 - t)\lambda \| \beta \|_1 \leq \frac{1}{N} L(\theta) + \lambda \| \beta \|_1,$$

where the last inequality follows by definition of $\hat{\theta}$. In particular, for any $M > 0$, choosing

$$t = \frac{M}{M + \| \hat{\theta} - \theta \|_1},$$

gives $\| \hat{\theta} - \theta \|_1 \leq M$. The completely analogous result holds for $\bar{\theta}$.

A.3 Two norms and one function space

To give us a more compact way of writing, for any $\bar{\theta} \in \Theta$ we introduce functions $f_{\bar{\theta}} : \mathbb{R}^{2n+1+p} \rightarrow \mathbb{R}$, $f_{\bar{\theta}}(v) = v^T \bar{\theta}$ and denote the function space of all such $f_{\bar{\theta}}$ by $\mathcal{F} := \{ f_{\bar{\theta}} : \bar{\theta} \in \Theta \}$. We endow $\mathcal{F}$ with two norms as follows:

Denote the law of the rows of $\hat{D}$ on $\mathbb{R}^{2n+1+p}$, i.e. the probability measure induced by $(\hat{X}_{ij}, 1, Z_{ij})^T$, $i \neq j$, by $\hat{Q}$. That is, for a measurable set $A = A_1 \times A_2 \subset \mathbb{R}^{2n+1} \times \mathbb{R}^p$,

$$\hat{Q}(A) = \frac{1}{N} \sum_{i \neq j} P(D_{ij} \in A) = \frac{1}{N} \sum_{i \neq j} \delta_{ij}(A_1) \cdot P(Z_{ij} \in A_2),$$

where $\delta_{ij}(A_1) = 1$ if $(\hat{X}_{ij}, 1)^T \in A_1$ and zero otherwise, is the Dirac-measure. We are interested in the $L_2$ and $L_{\infty}$ norm on $\mathcal{F}$ with respect to the measure $\hat{Q}$ on $\mathbb{R}^{2n+1} \times \mathbb{R}^p$. Denote the $L_2(\hat{Q})$-norm of $f \in \mathcal{F}$ simply by $\| . \|_{\hat{Q}}$ and let $\mathbb{E}_Z$ be the expectation with respect to $Z$:

$$\| f \|_{\hat{Q}}^2 := \| f \|_{L_2(\hat{Q})}^2 = \int_{\mathbb{R}^{2n+1} \times \mathbb{R}^p} f(v)^2 \hat{Q}(dv) = \frac{1}{N} \sum_{i \neq j} \mathbb{E}_Z[f((\hat{X}_{ij}, 1, Z_{ij}^T)^T]^2$$

and define the $L_{\infty}(\hat{Q})$-norm as usual as the $\hat{Q}$-a.s. smallest upper bound of $f$:

$$\| f \|_{\hat{Q}, \infty} = \inf \{ C \geq 0 : | f(v) | \leq C \text{ for } \hat{Q}\text{-almost every } v \in \mathbb{R}^{2n+1+p} \}.$$

Notice in particular, that for any $f_{\bar{\theta}} \in \mathcal{F}$, $\bar{\theta} \in \Theta_{\text{loc}}$: $\| f_{\bar{\theta}} \|_{\infty} \leq \sup_Z \| D\bar{\theta} \|_{\infty} \leq r_n$.

We make the analogous definitions for the unscaled design matrix. Let $Q$ denote the probability measure induced by the rows of $D$. Since $D\hat{\theta} = D\bar{\theta}$, for any $\theta$ with rescaled version $\hat{\theta}$, we have

$$\| f_{\bar{\theta}} \|_{L_2(Q)} = \| f_{\theta} \|_{L_2(Q)}, \quad \| f_{\bar{\theta}} \|_{Q, \infty} = \| f_{\theta} \|_{Q, \infty}.$$  

We want to apply the compatibility condition to vectors of the form $\theta = \theta_1 - \theta_2, \theta_1, \theta_2 \in \Theta_{\text{loc}}$.

Notice, that we have the following relation between the $L_2(Q)$-norm and the sample size adjusted Gram matrix $\Sigma$: For any $\theta$ we have

$$\| f_{\theta} \|_{\hat{Q}}^2 = \mathbb{E}_Z \left[ \frac{1}{N} \sum_{i \neq j} (D_{ij}^T \theta)^2 \right] = \bar{\theta}^T \Sigma \bar{\theta}. \quad (19)$$

We have the following corollary which follows immediately from Proposition 5 (see e.g. van de Geer & Buehlmann (2011), section 6.12 for a general treatment).
Corollary 11. Under assumption 1, for \( s_0 = o(\sqrt{n}) \) and \( n \) large enough and with \( \tilde{c} = c^2 \vee (2c^4) \), where \( c > 0 \) is the universal constant such that \( |Z_{k,i,j}| \leq c \) for all \( k, i, j \), it holds that for every \( \theta = \theta_1 - \tilde{\theta}_2, \tilde{\theta}_1, \tilde{\theta}_2 \in \tilde{\Theta}_{\text{loc}} \) with \( \|\tilde{\theta}_{S_{0,+}}\|_1 \leq 3\|\theta_{S_{0,+}}\|_1 \),

\[
\|\tilde{\theta}_{S_{0,+}}\|^2 \leq \frac{s_0+}{C} \|f_{\theta_1} - f_{\theta_2}\|^2_Q,
\]

where \( C = c_{\min}/2 \).

Proof. By Proposition 5,

\[
\|\tilde{\theta}_{S_{0,+}}\|^2 \leq \frac{s_0+}{c_{\min}} \|\theta\Sigma\tilde{\theta}\|.
\]

The claim follows from (19) and the fact that \( \theta \mapsto f_\theta \) is linear. \( \square \)

A.4 Lower quadratic margin for \( \mathcal{E} \)

In this section we will derive a lower quadratic bound on the excess risk \( \mathcal{E}(\theta) \) if the parameter \( \theta \) is close to the truth \( \theta_0 \). This is a necessary property for the proof to come and is referred to as the margin condition in classical LASSO theory (cf. van de Geer & Buehlmann (2011)).

The proof mainly relies on a second order Taylor expansion of the function \( l_\theta \) of introduced in section 3. Given a fixed \( \theta \), we treat \( l_\theta \) as a function in \( \theta^T x \) and define new functions \( l_{ij} : \mathbb{R} \to \mathbb{R}, i \neq j \),

\[
l_{ij}(a) = \mathbb{E}[l_\theta(A_{ij}, a)|Z_{ij}] = -p_{ij}a + \log(1 + \exp(a)),
\]

where \( p_{ij} = P(A_{ij} = 1|Z_{ij}) \) and by slight abuse of notation we use \( l_\theta(A_{ij}, a) := -A_{ij}a+\log(1+\exp(a)) \). Taking derivations, it is easy to see that

\[
f_{\theta_0}((X_{i,j}^T, 1, Z_{ij}^T)^T) \in \arg\min_a l_{ij}(a).
\]

All \( l_{ij} \) are clearly twice continuously differentiable with derivative

\[
\frac{\partial^2}{\partial a^2} l_{ij}(a) = \frac{\exp(a)}{(1 + \exp(a))^2} > 0, \forall a \in \mathbb{R}.
\]

Using a second order Taylor expansion around \( a_0 = f_0((X_{i,j}^T, 1, Z_{ij}^T)^T) \) we get

\[
l_{ij}(a) = l_{ij}(a_0) + l'(a_0)(a - a_0) + \frac{l''(a)}{2}(a - a_0)^2 = l_{ij}(a_0) + \frac{l''(\bar{a})}{2}(a - a_0)^2,
\]

with an \( \bar{a} \) between \( a \) and \( a_0 \). Note that \( |a_0| \leq r_n \). Then, for any \( a \) with \( |a| \leq r_n \), we must have that for any intermediate point \( \bar{a} \) between \( a_0 \) and \( a \) it also holds that \( |\bar{a}| \leq r_n \). Also note that \( \frac{\exp(a)}{(1 + \exp(a))^2} \) is symmetric and monotone decreasing for \( a \geq 0 \). Thus, for any \( a \) with \( |a| \leq r_n \),

\[
l_{ij}(a) - l_{ij}(a_0) = \frac{\exp(\bar{a})}{(1 + \exp(\bar{a}))^2} \frac{(a - a_0)^2}{2}
\]

\[= \frac{\exp(|\bar{a}|)}{(1 + \exp(|\bar{a}|))^2} \frac{(a - a_0)^2}{2}, \text{ by symmetry}
\]

\[\geq \frac{\exp(r_n)}{(1 + \exp(r_n))^2} \frac{(a - a_0)^2}{2}.
\]

In particular, if we pick any \( \theta \) and let \( a = f_\theta((X_{i,j}^T, 1, Z_{ij}^T)^T) \), we have

\[
l_{ij}(f_\theta((X_{i,j}^T, 1, Z_{ij}^T)^T)) - l_{ij}(f_\theta((X_{i,j}^T, 1, Z_{ij}^T)^T)) \geq \frac{\exp(r_n)}{(1 + \exp(r_n))^2} \frac{(f_\theta((X_{i,j}^T, 1, Z_{ij}^T)^T) - f_\theta((X_{i,j}^T, 1, Z_{ij}^T)^T))^2}{2}.
\]
Let
\[ K_n = \frac{2(1 + \exp(r_n))^2}{\exp(r_n)}. \] (21)

Define a subset \( F_{\text{local}} \subset F \) as \( F_{\text{local}} = \{ f_\theta : \theta \in \Theta_{\text{loc}} \} \). Now, for all \( f_\theta \in F_{\text{local}} \):
\[
E(\theta) = \frac{1}{N} \sum_{i \neq j} [E[l_\theta(A_{ij}, D_{ij}) - l_{\theta_0}(A_{ij}, D_{ij})]]
\]
\[
= \frac{1}{N} \sum_{i \neq j} [E[(l_\theta(f_\theta(D_{ij}) - l_{ij}(f_\theta(D_{ij})))]]]
\]
\[
\geq \frac{1}{K_n} \cdot \frac{1}{N} (\theta - \theta_0)^T E[Z(D^T D)(\theta - \theta_0)]
\]
\[
= \frac{1}{K_n} \cdot \|f_\theta - f_0\|^2_Q.
\]

Thus, we have obtained a lower bound for the excess risk given by the quadratic function \( G_n(\|f_\theta - f_0\|) \) where \( G_n(u) = \frac{1}{K_n} \cdot u^2 \). Recall that the convex conjugate of a strictly convex function \( G \) on \([0, \infty)\) with \( G(0) = 0 \) is defined as the function
\[
H(v) = \sup_u \{uv - G(u)\}, \quad v > 0,
\]
and in particular, if \( G(u) = cu^2 \) for a positive constant \( c \), we have \( H(v) = v^2/(4c) \). Hence, the convex conjugate of \( G_n \) is
\[
H_n(v) = \frac{v^2 K_n}{4}.
\]

Keep in mind that by definition for any \( u, v \)
\[
uv \leq G(u) + H(v).
\]

### A.5 Consistency on a special set

In this section we will show that the penalized likelihood estimator is consistent. We will first define a set \( I \) and show that consistency holds on \( I \). It will then suffice to show that the probability of \( I \) tends to one as well. The proof follows in spirit van de Geer & Buehlmann (2011), Theorem 6.4.

We define some objects that we will need for the proof of consistency. We want to use the quadratic margin condition derived in section A.4. Recall that the quadratic margin condition holds for any \( \theta \in \Theta_{\text{loc}} \). Define
\[
\epsilon^* = H_n \left( \frac{4\sqrt{2} \sqrt{\gamma_0 + \bar{\lambda}}}{\sqrt{c_{\min}}} \right).
\]

Recall the definition of \( \bar{\theta} \) in equation (10) and let for any \( M > 0 \)
\[
Z_M := \sup_{\theta \in \Theta_{\text{loc}}, \|\theta - \theta_0\|_1 \leq M} |v_n(\theta) - v_n(\theta_0)|,
\]
where \( v_n \) denotes the empirical process. The set over which we are maximizing in the definition of \( Z_M \) can be expressed in terms of parameters \( \theta \) on the original scale as
\[
\left\{ \theta = (\theta^T, \mu, \gamma^T)^T \in \Theta_{\text{loc}} : \frac{1}{\sqrt{n}} \|\theta - \theta_0\|_1 + |\mu - \mu_0| + \|\gamma - \gamma_0\|_1 \leq M \right\}.
\]

Set
\[
M^* := \epsilon^*/\lambda_0,
\]
where \( \lambda_0 \) is a lower bound on \( \bar{\lambda} \) that will be made precise in the proof showing that \( I \) has large probability. Define
\[
I := \{Z_M \leq \lambda_0 M^* \} = \{Z_M \leq \epsilon^* \}. \tag{22}
\]
Theorem 4. Assume that assumptions 1 and B2 hold and that $\lambda \geq 8\lambda_0$. Then, on the set $\mathcal{I}$, we have

$$E(\hat{\theta}) + \hat{\lambda} \left( \frac{1}{\sqrt{n}} \|\hat{\theta} - \hat{\theta}_0\|_1 + |\hat{\mu} - \mu_0| + \|\hat{\gamma} - \gamma_0\|_1 \right) \leq 4\epsilon^* = 4H_n \left( \frac{4\sqrt{2} \sqrt{S_0,+,\lambda}}{\sqrt{c_{\min}}} \right).$$

Proof of Theorem 4. We assume that we are on the set $\mathcal{I}$ throughout. Set

$$t = \frac{M^*}{M^* + \|\hat{\theta} - \hat{\theta}_0\|_1}$$

and $\tilde{\theta} = (\tilde{\theta}^T, \tilde{\mu}, \tilde{\gamma}^T)^T = t\hat{\theta} + (1 - t)\hat{\theta}_0$. Then,

$$\|\tilde{\theta} - \hat{\theta}_0\|_1 = t\|\hat{\theta} - \hat{\theta}_0\|_1 \leq M^*. $$

Since $\tilde{\theta}, \hat{\theta}_0 \in \Theta_{\text{loc}}$ and by the convexity of $\Theta_{\text{loc}}$, $\tilde{\theta} \in \Theta_{\text{loc}}$, and by the remark after Lemma 10, the basic inequality holds for $\tilde{\theta}$. Also, recall that $E(\hat{\theta}_0) = 0$:

$$E(\tilde{\theta}) + \hat{\lambda}\|\tilde{\theta}\|_1 \leq - (\bar{v}_n(\tilde{\theta}) - \bar{v}_n(\hat{\theta}_0)) + E(\hat{\theta}_0) + \hat{\lambda}\|\tilde{\theta}_0\|_1$$

$$\leq ZM^* + \hat{\lambda}\|\tilde{\theta}_0\|_1$$

$$\leq \epsilon^* + \hat{\lambda}\|\tilde{\theta}_0\|_1.$$ 

From now on write $\tilde{E} = E(\tilde{\theta})$. Note, that $\|\tilde{\theta}\|_1 = \|\tilde{\theta}_S\|_1 + \|\tilde{\theta}_0\|_1$ and thus, by the triangle inequality,

$$\tilde{E} = \tilde{E} + \hat{\lambda}\|\tilde{\theta}_S\|_1 \leq \epsilon^* + \hat{\lambda}(\|\tilde{\theta}_0\|_1 - \|\tilde{\theta}_S\|_1)$$

$$\leq \epsilon^* + \hat{\lambda}(\|\tilde{\theta}_0 - \tilde{\theta}_S\|_1)$$

$$\leq \epsilon^* + \hat{\lambda}(\|\tilde{\theta} - \tilde{\theta}_S\|_1 + \|\mu_0, \gamma_0^T - \mu_0, \gamma_0^T\|_1)$$

$$= \epsilon^* + \hat{\lambda}(\|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1).$$

Case i) If $\hat{\lambda}(\|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1 > \epsilon^*$, then

$$\hat{\lambda}(\|\tilde{\theta}_S\|_1 \leq \tilde{E} + \hat{\lambda}(\|\tilde{\theta}_S\|_1 \leq 2\hat{\lambda}(\|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1. $$

Since $\|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1 = \|\tilde{\theta}_S\|_1$, we may thus apply the compatibility condition corollary 11 (note that $\tilde{\theta}_0 = \tilde{\theta}_{0,S}$) to obtain

$$\|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1 \leq 2C \sqrt{2} \sqrt{S_0,+,} \|f_\tilde{\theta} - f_{\tilde{\theta}_0}\|_Q,$$

where we have used that $\theta \mapsto f_\theta$ is linear and hence $f_{\tilde{\theta} - \tilde{\theta}_0} = f_\tilde{\theta} - f_{\tilde{\theta}_0}$. Observe that

$$\|\tilde{\theta} - \tilde{\theta}_0\|_1 = \|\tilde{\theta}_S\|_1 + \|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1.$$ 

Hence,

$$\tilde{E} + \hat{\lambda}(\|\tilde{\theta} - \tilde{\theta}_0\|_1 = \tilde{E} + \hat{\lambda}(\|\tilde{\theta}_S\|_1 + \|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1)$$

$$\leq \epsilon^* + 2\hat{\lambda}(\|\tilde{\theta} - \tilde{\theta}_0\|_{S,+,} \|_1)$$

$$\leq \epsilon^* + 2\sqrt{2} \sqrt{S_0,+,} \sqrt{c_{\min}} \|f_\tilde{\theta} - f_{\tilde{\theta}_0}\|_Q.$$
Recall that for a convex function $G$ and its convex conjugate $H$ we have $uv \leq G(u) + H(v)$. Thus, we obtain

$$2\sqrt{2\lambda} \frac{\sqrt{S_{0,+}}}{\sqrt{c_{\min}}} \| \hat{f}_\theta - \hat{f}_{\bar{\theta}_0} \|_Q = 4\sqrt{2\lambda} \frac{\sqrt{S_{0,+}}}{\sqrt{c_{\min}}} \| f_\theta - f_{\bar{\theta}_0} \|_Q$$

$$\leq H_n \left( 4\sqrt{2\lambda} \frac{\sqrt{S_{0,+}}}{\sqrt{c_{\min}}} \right) + G_n \left( \frac{\| f_\theta - f_{\bar{\theta}_0} \|_Q}{2} \right)$$

$$\| f_\theta - f_{\bar{\theta}_0} \|_Q \leq H_n \left( 4\sqrt{2\lambda} \frac{\sqrt{S_{0,+}}}{\sqrt{c_{\min}}} \right) + \tilde{E} \leq \frac{\tilde{E}}{2}.$$ 

It follows

$$\tilde{E} + \hat{\lambda} \| \hat{\theta} - \bar{\theta}_0 \|_1 \leq \epsilon^* + H_n \left( 4\sqrt{2\lambda} \frac{\sqrt{S_{0,+}}}{\sqrt{c_{\min}}} \right) + \frac{\tilde{E}}{2} = 2\epsilon^* + \frac{\tilde{E}}{2}.$$

and therefore

$$\frac{\tilde{E}}{2} + \hat{\lambda} \| \hat{\theta} - \bar{\theta}_0 \|_1 \leq 2\epsilon^*.$$

Finally, this gives

$$\| \hat{\theta} - \bar{\theta}_0 \|_1 \leq \frac{2\epsilon^*}{\lambda} = \frac{2\lambda_0 M^*}{\lambda} \leq \frac{M^*}{\frac{\lambda}{2}}.$$

From this, by using the definition of $\hat{\theta}$, we obtain

$$\| \hat{\theta} - \bar{\theta}_0 \|_1 = \epsilon \| \hat{\theta} - \bar{\theta}_0 \|_1 = \frac{M^*}{M^* + \| \hat{\theta} - \bar{\theta}_0 \|_1} \| \hat{\theta} - \bar{\theta}_0 \|_1 \leq \frac{M^*}{\frac{\lambda}{2}}.$$

Rearranging gives

$$\| \hat{\theta} - \bar{\theta}_0 \|_1 \leq M^*.$$

**Case ii)** If $\hat{\lambda} \| (\hat{\theta}_0 - \hat{\bar{\theta}})_{S_{0,+}} \|_1 \leq \epsilon^*$, then from (23)

$$\tilde{E} + \hat{\lambda} \| \hat{\theta}_{S_{\bar{\theta}}} \|_1 \leq 2\epsilon^*.$$

Using once more (25), we get

$$\hat{E} + \hat{\lambda} \| \hat{\theta} - \hat{\bar{\theta}}_0 \|_1 = \hat{E} + \hat{\lambda} \| \hat{\theta}_{S_{\bar{\theta}}} \|_1 + \hat{\lambda} \| (\hat{\theta} - \hat{\bar{\theta}}_0)_{S_{0,+}} \|_1 \leq 3\epsilon^*.$$

Thus,

$$\| \hat{\theta} - \hat{\bar{\theta}}_0 \|_1 \leq 3 \frac{\epsilon^*}{\lambda} = \frac{3\lambda_0 M^*}{\lambda} \leq \frac{M^*}{\frac{\lambda}{2}}.$$

by choice of $\lambda \geq 6\lambda_0$. Again, plugging in the definition of $\hat{\theta}$, we obtain

$$\| \hat{\theta} - \hat{\bar{\theta}}_0 \|_1 \leq M^*.$$

Hence, in either case we have $\| \hat{\theta} - \hat{\bar{\theta}}_0 \|_1 \leq M^*$. That means, we can repeat the above steps with $\hat{\theta}$ instead of $\hat{\bar{\theta}}$. Writing $\hat{E} := \hat{E}(\hat{\theta})$, following the same reasoning as above we arrive once more at (23):

$$\hat{E} + \hat{\lambda} \| \hat{\theta}_{S_{\bar{\theta}}} \|_1 \leq \epsilon^* + \hat{\lambda} \| \hat{\theta} - \hat{\bar{\theta}}_0 \|_1 \leq 2\epsilon^* + \hat{\lambda} \| (\hat{\theta} - \hat{\bar{\theta}}_0)_{S_{0,+}} \|_1.$$

From this, in **case i)** we obtain (24) which allows us to use the compatibility assumption to arrive at (26):

$$\frac{\hat{E}}{2} + \hat{\lambda} \| \hat{\theta} - \hat{\bar{\theta}}_0 \|_1 \leq 2\epsilon^*.$$
resulting in
\[ \hat{\mathcal{E}} + \lambda \| \hat{\theta} - \theta_0 \|_1 \leq 4 \epsilon^* . \]

In case ii) on the other hand, we arrive directly at (27), and hence
\[ \hat{\mathcal{E}} + \lambda \| \hat{\theta} - \theta_0 \|_1 \leq 3 \epsilon^* . \]

Plugging in the definitions of \( \hat{\theta} \) and \( \theta_0 \) and using the fact that \( \hat{\mathcal{E}} = \mathcal{E}(\hat{\theta}) = \mathcal{E}(\hat{\theta}) \) proves the claim. \( \square \)

### A.6 Controlling the special set \( \mathcal{I} \)

We now show that \( \mathcal{I} \) has probability tending to one. Recall some results on concentration inequalities.

#### A.6.1 Concentration inequalities

We first recall some probability inequalities that we will need. This is based on chapter 14 in van de Geer & Bühlmann (2011). Throughout let \( Z_1, \ldots, Z_n \) be a sequence of independent random variables in some space \( Z \) and \( G \) be a class of real valued functions on \( Z \).

**Definition 12.** A Rademacher sequence is a sequence \( \epsilon_1, \ldots, \epsilon_n \) of i.i.d. random variables with \( P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2 \) for all \( i \).

**Theorem 5** (Symmetrization Theorem as in van der Vaart & Wellner (1996), abridged). Let \( \epsilon_1, \ldots, \epsilon_n \) be a Rademacher sequence independent of \( Z_1, \ldots, Z_n \). Then
\[
\mathbb{E} \left( \sup_{g \in G} \left| \frac{1}{n} \sum_{i=1}^{n} \{ g(Z_i) - \mathbb{E}[g(Z_i)] \} \right| \right) \leq 2 \mathbb{E} \left( \sup_{g \in G} \left| \sum_{i=1}^{n} \epsilon_i g(Z_i) \right| \right) .
\]

**Theorem 6** (Contraction Theorem as in Ledoux & Talagrand (1991)). Let \( z_1, \ldots, z_n \) be non-random elements of \( Z \) and let \( F \) be a class of real-valued functions on \( Z \). Consider Lipschitz functions \( g_i : \mathbb{R} \rightarrow \mathbb{R} \) with Lipschitz constant \( L = 1 \), i.e. for all \( i \)
\[ |g_i(s) - g_i(s')| \leq |s - s'|, \forall s, s' \in \mathbb{R}. \]

Let \( \epsilon_1, \ldots, \epsilon_n \) be a Rademacher sequence. Then for any function \( f^* : Z \rightarrow \mathbb{R} \) we have
\[
\mathbb{E} \left( \sup_{f \in F} \left| \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \{ g_i(f(z_i)) - g_i(f^*(z_i)) \} \right| \right) \leq 2 \mathbb{E} \left( \sup_{f \in F} \left| \sum_{i=1}^{n} \epsilon_i \{ f(z_i) - f^*(z_i) \} \right| \right) .
\]

The last theorem we need is a concentration inequality due to Bousquet (2002). We give a version as presented in van de Geer (2008).

**Theorem 7** (Bousquet’s concentration theorem). Suppose \( Z_1, \ldots, Z_n \) and all \( g \in G \) satisfy the following conditions for some real valued constants \( \eta_n \) and \( \tau_n \)
\[ \| g \|_\infty \leq \eta_n, \forall g \in G \]
and
\[ \frac{1}{n} \sum_{i=1}^{n} \text{Var}(g(Z_i)) \leq \tau_n^2, \forall g \in G. \]

Define
\[ Z := \sup_{g \in G} \left| \frac{1}{n} \sum_{i=1}^{n} g(Z_i) - \mathbb{E}[g(Z_i)] \right| . \]

Then for any \( z > 0 \)
\[
P \left( Z \geq \mathbb{E}[Z] + z \sqrt{2(\tau_n^2 + 2\eta_n \mathbb{E}[Z]) + \frac{2z^2\eta_n}{3}} \right) \leq \exp(-nz^2). \]
Lemma 14. \( f(Z_i) = (g(Z_i) - \mathbb{E}[g(Z_i)])/(2\eta_n) \), \( \tilde{Z}_k = \sup_f |\sum_{i \neq k} f(Z_i)| \),

\[ f_k = \arg \sup_f |\sum_{i \neq k} f(Z_i)|, \quad \tilde{Z}_k = |\sum_{i=1}^{n} f_k(Z_i)| - \tilde{Z}_k \]

\[ \bar{Z} = \frac{2\eta_n}{n} \bar{Z}. \]

Now apply Theorem 2.1 in Bousquet (2002), choosing for their \((Z, Z_1, \ldots, Z_n)\) the above defined \((\bar{Z}, \bar{Z}_1, \ldots, \bar{Z}_n)\), for their \((\bar{Z}_1', \ldots, \bar{Z}_n')\) the above defined \((\bar{Z}_1', \ldots, \bar{Z}_n')\) and setting \(u = 1\) and \(\sigma^2 = \frac{\bar{n}^2}{4n^2}\) in their theorem: The result is exactly Theorem 7 above.

Finally we have a Lemma derived from Hoeffding’s inequality. The proof can be found in van de Geer & Buehlmann (2011), Lemma 14.14 (here we use the special case of their Lemma for \(m = 1\)).

Lemma 13. Let \( \mathcal{G} = \{g_1, \ldots, g_p\} \) be a set of real valued functions on \(Z\) satisfying for all \(i = 1, \ldots, n\) and all \(j = 1, \ldots, p\)

\[ \mathbb{E}[g_j(Z_i)] = 0, \quad |g_j(Z_i)| \leq c_{ij} \]

for some positive constants \(c_{ij}\). Then

\[ \mathbb{E} \left[ \max_{1 \leq j \leq p} \left| \sum_{i=1}^{n} g_j(Z_i) \right| \right] \leq \left[ 2 \log(2p) \right]^{1/2} \max_{1 \leq j \leq p} \left[ \sum_{i=1}^{n} c_{ij}^2 \right]^{1/2}. \]

A.6.2 The expectation of \(Z_M\)

Recall the definition of \(Z_M\)

\[ Z_M := \sup_{\theta \in \Theta_{loc}, \|\theta - \bar{\theta}_0\|_1 \leq M} |\bar{v}_n(\theta) - \bar{v}_n(\bar{\theta}_0)|, \]

where \(\bar{v}_n\) denotes the re-parametrized empirical process. Recall, that there is a constant \(c \in \mathbb{R}\) such that uniformly \(|Z_{ij,k}| \leq c, 1 \leq i \neq j \leq n, k = 1, \ldots, p\).

Lemma 14. For any \(M > 0\) we have

\[ \mathbb{E}[Z_M] \leq 8M(1 + c)\sqrt{\frac{2 \log(2n + p + 1)}{N}}. \]

Proof. Let \(\epsilon_{ij}, i \neq j\), be a Rademacher sequence independent of \(A_{ij}, Z_{ij}, i \neq j\). We first want to use the symmetrization theorem 5: For the random variables \(Z_1, \ldots, Z_n\) we choose \(T_{ij} = (A_{ij}, \bar{X}_{ij}, 1, Z_{ij})^T \in \{0,1\} \times \mathbb{R}^{2n+1+p}\). For any \(\bar{\theta} \in \Theta_{loc}\) we consider the functions

\[ g_{\bar{\theta}}(T_{ij}) = \frac{1}{N} \left\{ -A_{ij}D_{ij}^T(\bar{\theta} - \bar{\theta}_0) + \log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \bar{\theta}_0)) \right\} \]

and the function set \(\mathcal{G} = \mathcal{G}(M) := \{g_{\bar{\theta}} : \bar{\theta} \in \Theta_{loc}, \|\bar{\theta} - \bar{\theta}_0\|_1 \leq M\}\). Note, that

\[ \bar{v}_n(\bar{\theta}) - \bar{v}_n(\bar{\theta}_0) = \sum_{i \neq j} \{g_{\bar{\theta}}(T_{ij}) - \mathbb{E}[g_{\bar{\theta}}(T_{ij})]\}. \]

Then, the symmetrization theorem gives us

\[ \mathbb{E}[Z_M] = \mathbb{E} \left[ \sup_{g_{\bar{\theta}} \in \mathcal{G}} \left| \sum_{i \neq j} g_{\bar{\theta}}(T_{ij}) - \mathbb{E}[g_{\bar{\theta}}(T_{ij})]\right| \right] \leq 2 \mathbb{E} \left[ \sup_{g_{\bar{\theta}} \in \mathcal{G}} \left| \sum_{i \neq j} \epsilon_{ij}g_{\bar{\theta}}(T_{ij})\right| \right]. \]
Next, we want to apply the contraction Theorem 6. Denote $T = (T_{ij})_{i,j}$ and let $\mathbb{E}_T$ be the conditional expectation given $T$. We need the conditional expectation at this point, because Theorem 6 requires non-random arguments in the functions. This does not hinder us, as later we will simply take iterated expectations, cancelling out the conditional expectation, see below. For the functions $g_i$ in Theorem 6 we choose

$$g_{ij}(x) = \frac{1}{2} \{ -A_{ij}x + \log(1 + \exp(x)) \}$$

Note, that $\log(1 + \exp(x))$ has derivative bounded by one and thus is Lipschitz continuous with constant one by the Mean Value Theorem. Thus, all $g_{ij}$ are also Lipschitz continuous with constant $1$:

$$|g_{ij}(x) - g_{ij}(x')| \leq \frac{1}{2} \{ |A_{ij}(x - x')| + |\log(1 + \exp(x)) - \log(1 + \exp(x'))| \} \leq |x - x'|.$$ 

For the function class $\mathcal{F}$ in Theorem 6 we choose $\mathcal{F} = \mathcal{F}_M := \{ f_\theta : \theta \in \mathcal{O}_{loc}, \|\theta - \theta_0\|_1 \leq M \}$ and pick $f^* = f_{\tilde{\theta}_0}$. Then, by Theorem 6

$$\mathbb{E}_T \left[ \sup_{\tilde{\theta} \in \mathcal{O}_{loc}, \|\tilde{\theta} - \theta_0\|_1 \leq M} \left| \frac{1}{N} \sum_{i \neq j} e_{ij}(g_{ij}(f_\theta((\bar{X}_{ij}^T, 1, Z_{ij}^T)^T)) - g_{ij}(f_{\tilde{\theta}_0}((\bar{X}_{ij}^T, 1, Z_{ij}^T)^T))) \right| \right]$$

$$\leq 2 \mathbb{E}_T \left[ \sup_{\tilde{\theta} \in \mathcal{O}_{loc}, \|\tilde{\theta} - \theta_0\|_1 \leq M} \left| \frac{1}{N} \sum_{i \neq j} e_{ij}(f_\theta((\bar{X}_{ij}^T, 1, Z_{ij}^T)^T) - f_{\tilde{\theta}_0}((\bar{X}_{ij}^T, 1, Z_{ij}^T)^T)) \right| \right].$$

Recall that we can express the functions $f_\theta = f_{\alpha,\beta,\mu,\gamma}$ as

$$f_{\alpha,\beta,\mu,\gamma}(.) = \sum_{i=1}^{n} a_i e_i(\cdot) + \sum_{i=n+1}^{2n} \beta_i \tilde{e}_i(\cdot) + \mu e_{2n+1}(\cdot) + \sum_{i=1}^{p} \gamma_i e_{2n+1+i}(\cdot),$$

where $e_i(\cdot)$ is the projection on the $i$-th coordinate. Consider any $\bar{\theta} \in \mathcal{O}_{loc}$ with $\|\bar{\theta} - \theta_0\|_1 \leq M$. For the sake of a compact representation we use our shorthand notation $\bar{\theta} = (\bar{\theta}_1)_{i=1}^{2n+1+p}$ where the components $\theta_i$ are defined in the canonical way and we also simply write $e_k(\bar{X}_{ij}, 1, Z_{ij})$ for the projection of the vector $(\bar{X}_{ij}^T, 1, Z_{ij}^T)^T \in \mathbb{R}^{2n+1+p}$ to its $k$-th component, i.e. instead of $e_k((\bar{X}_{ij}^T, 1, Z_{ij}^T)^T)$. Then,

$$\left| \frac{1}{N} \sum_{i \neq j} e_{ij}(f_\theta((\bar{X}_{ij}^T, 1, Z_{ij}^T)^T) - f_{\tilde{\theta}_0}((\bar{X}_{ij}^T, 1, Z_{ij}^T)^T)) \right|$$

$$= \left| \frac{1}{N} \sum_{i \neq j} e_{ij} \left( \sum_{k=1}^{2n+p+1} (\bar{\theta}_{k} - \tilde{\theta}_{0,k})e_k(\bar{X}_{ij}, 1, Z_{ij}) \right) \right|$$

$$\leq \frac{1}{N} \sum_{k=1}^{2n+p+1} \max_{1 \leq l \leq 2n+p+1} \left| \sum_{i \neq j} e_{ij} e_l(\bar{X}_{ij}, 1, Z_{ij}) \right|$$

$$\leq M \max_{1 \leq l \leq 2n+p+1} \left| \frac{1}{N} \sum_{i \neq j} e_{ij} e_l(\bar{X}_{ij}, 1, Z_{ij}) \right|. $$

Note, that the last expression no longer depends on $\bar{\theta}$. To bound the right hand side in the last expression we use Lemma 13: In the language of the Lemma, choose $Z_1, \ldots, Z_n$ as $T_{ij} = (e_{ij}, \bar{X}_{ij}^T, 1, Z_{ij}^T)^T$. We choose for the $p$ in the formulation of the Lemma $2n + p + 1$ and pick for our functions

$$g_k(T_{ij}) = \frac{1}{N} e_{ij} e_k(\bar{X}_{ij}, 1, Z_{ij}), k = 1, \ldots, 2n + p + 1.$$ 

Note, that then $\mathbb{E}[g_k(T_{ij})] = 0$. We want to employ Lemma 13 which requires us to bound $|g_k(T_{ij})| \leq c_{ij,k}$ for all $i \neq j$ and $k = 1, \ldots, n + 1 + p$. 

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For any fixed $1 \leq k \leq n$ we have
\[
|g_k(T_{ij})| \leq \begin{cases} \frac{\sqrt{n}}{N} = \frac{1}{(n-1)\sqrt{n}}, & \text{if } i \neq j \\ 0, & \text{otherwise.} \end{cases}
\]

Note that the first case occurs exactly $(n-1)$ times for each $k$. Thus, for any $k \leq 2n$,
\[
\sum_{i \neq j} c_{ij,k}^2 = \left( \frac{1}{(n-1)\sqrt{n}} \right)^2 (n-1) = \frac{1}{N}.
\]

If $k = 2n + 1$, $|g_k(T_{ij})| = 1/N$ and hence
\[
\sum_{i \neq j} c_{ij,2n+1}^2 = \frac{1}{N}.
\]

Finally, if $k > 2n + 1$, $|g_k(T_{ij})| \leq c/N$ and therefore,
\[
\sum_{i \neq j} c_{ij,k}^2 \leq \frac{c^2}{N}.
\]

In total, this means
\[
\max_{1 \leq k \leq 2n+1+p} \sum_{i \neq j} c_{ij,k}^2 \leq \frac{1 \lor c^2}{N}.
\]

Therefore, an application of Lemma 13 results in
\[
\mathbb{E} \left[ \max_{1 \leq l \leq 2n+p+1} \left| \frac{1}{N} \sum_{i \neq j} \epsilon_{ij} e_l(\tilde{X}_{ij}, Z_{ij}) \right| \right] \leq \sqrt{2 \log(2(2n+1+p))} \max_{1 \leq k \leq 2n+1+p} \left[ \sum_{i \neq j} c_{ij,k}^2 \right]^{1/2}
\]
\[
\leq \sqrt{2 \log(2(2n+1+p))} \frac{1 \lor c^2}{N}
\]
\[
= \sqrt{2 \log(2(2n+1+p))} \frac{1 \lor c}{N}.
\]

Putting everything together, we obtain
\[
\mathbb{E}[Z_M] \leq 2\mathbb{E} \left[ \sup_{\theta \in \Theta_{loc}, \|\theta - \theta_0\|_1 \leq M} \left| \frac{1}{N} \sum_{i \neq j} \epsilon_{ij} (-A_{ij}(f_\theta(\tilde{X}_{ij}, 1, Z_{ij}) - f_{\tilde{\theta}_0}(\tilde{X}_{ij}, 1, Z_{ij})) \right| \right]
\]
\[
= 2\mathbb{E} \left[ \mathbb{E}_T \left[ \sup_{\theta \in \Theta_{loc}, \|\theta - \theta_0\|_1 \leq M} \left| \frac{1}{N} \sum_{i \neq j} \epsilon_{ij} (-A_{ij}(f_{\theta}(\tilde{X}_{ij}, 1, Z_{ij}) - f_{\tilde{\theta}_0}(\tilde{X}_{ij}, 1, Z_{ij})) \right| \right] \right]
\]
\[
\leq 8\mathbb{E} \left[ \mathbb{E}_T \left[ \sup_{\theta \in \Theta_{loc}, \|\theta - \theta_0\|_1 \leq M} \left| \frac{1}{N} \sum_{i \neq j} \epsilon_{ij} (f_{\theta}(\tilde{X}_{ij}, 1, Z_{ij}) - f_{\tilde{\theta}_0}(\tilde{X}_{ij}, 1, Z_{ij})) \right| \right] \right]
\]
\[
\leq 8M \mathbb{E} \left[ \mathbb{E}_T \left[ \max_{1 \leq l \leq 2n+p+1} \left| \frac{1}{N} \sum_{i \neq j} \epsilon_{ij} e_l(\tilde{X}_{ij}, 1, Z_{ij}) \right| \right] \right]
\]
\[
\leq 8M \sqrt{\frac{2 \log(2(2n+1+p))}{N}} (1 \lor c).
\]

This concludes the proof.
We now want to show that $Z_M$ does not deviate too far from its expectation. The proof relies on the concentration theorem due to Bousquet, Theorem 7.

**Corollary 15.** Pick any confidence level $t > 0$. Let
\[
a_n := \sqrt{\frac{2 \log(2(n + p + 1))}{N}} (1 + c)
\]
and choose $\lambda_0 = \lambda_0(t, n)$ as
\[
\lambda_0 = 8a_n + 2\sqrt{\frac{t}{N} (11(1 \vee (c^2 p)) + 16(1 \vee c) \sqrt{n}a_n)} + \frac{4t(1 \vee c) \sqrt{n}}{3N}
\]
Then, we have the inequality
\[
P(Z_M \geq M\lambda_0) \leq \exp(-t).
\]

**Proof.** We want to apply Bousquet’s concentration Theorem 7. For the random variables $Z_i$ in the formulation of the theorem we choose once more $T_{ij} = (A_{ij}, \bar{X}_{ij}, 1, Z_{ij}), i \neq j$, and as functions we consider
\[
g_\bar{\theta}(T_{ij}) = -A_{ij} \bar{D}^T_{ij}(\bar{\theta} - \bar{\theta}_0) + \log(1 + \exp(\bar{D}^T_{ij}\bar{\theta})) - \log(1 + \exp(\bar{D}^T_{ij}\bar{\theta}_0)),
\]
\[
G = G_M := \{g_\bar{\theta}: \bar{\theta} \in \Theta_{loc}, \|\bar{\theta} - \bar{\theta}_0\|_1 \leq M\}.
\]
Then, we have
\[
Z_M = \sup_{g_\bar{\theta} \in G} \frac{1}{N} \left| \sum_{i \neq j} \{g_\bar{\theta}(T_{ij}) - \mathbb{E}[g_\bar{\theta}(T_{ij})]\} \right|.
\]
To apply Theorem 7, we need to bound the infinity norm of $g_\bar{\theta}$. Recall that we denote the distribution of $[\bar{X} \mid 1 \mid Z]$ by $\bar{Q}$ and the infinity norm is defined as the $\bar{Q}$-almost sure smallest upper bound on the value of $g_\bar{\theta}$. We have for any $g_\bar{\theta} \in G$, using the Lipschitz continuity of $\log(1 + \exp(x))$:
\[
|g_\bar{\theta}(T_{ij})| \leq |\bar{D}^T_{ij}(\bar{\theta} - \bar{\theta}_0)| + |\log(1 + \exp(\bar{D}^T_{ij}\bar{\theta})) - \log(1 + \exp(\bar{D}^T_{ij}\bar{\theta}_0))| \\
\leq 2|\bar{D}^T_{ij}(\bar{\theta} - \bar{\theta}_0)| \\
\leq 2\|\bar{\theta} - \bar{\theta}_0\|_1 + |\mu - \mu_0| + c\|\gamma - \gamma_0\|_1.
\]
Thus,
\[
\|g_\bar{\theta}\|_\infty \leq 2\|\bar{\theta} - \bar{\theta}_0\|_1 + |\mu - \mu_0| + c\|\gamma - \gamma_0\|_1 \\
\leq 2(1 \vee c)\|\bar{\theta} - \bar{\theta}_0\|_1 \\
\leq 2(1 \vee c)\sqrt{n}M := \eta_n.
\]
For the last inequality we used that for any $\theta$ with $\|\bar{\theta} - \bar{\theta}_0\|_1 \leq M$ it follows that $\|\bar{\theta} - \theta_0\|_1 \leq \sqrt{n}M$, which is possibly a very generous upper bound. This does not matter, however, as the term associated with the above bound will be negligible, as we shall see.

The second requirement of Theorem 7 is that the average variance of $g_\bar{\theta}(T_{ij})$ has to be uniformly bounded. To that end we calculate
\[
\frac{1}{N} \sum_{i \neq j} \text{Var}(g_\bar{\theta}(T_{ij})) = \frac{1}{N} \sum_{i \neq j} \text{Var}(-A_{ij} \bar{D}^T_{ij}(\theta - \theta_0)) \\
+ \frac{1}{N} \sum_{i \neq j} \text{Var}(\log(1 + \exp(\bar{D}^T_{ij}\bar{\theta})) - \log(1 + \exp(\bar{D}^T_{ij}\bar{\theta}_0))) \\
+ \frac{2}{N} \sum_{i \neq j} \text{Cov}(-A_{ij} \bar{D}^T_{ij}(\theta - \theta_0), \log(1 + \exp(\bar{D}^T_{ij}\bar{\theta})) - \log(1 + \exp(\bar{D}^T_{ij}\bar{\theta}_0))).
\]
Let us look at these terms in term. For the first term, we obtain

\[ \frac{1}{N} \sum_{i \neq j} \text{Var}(-A_{ij}D^T_{ij}(\theta - \theta_0)) \leq \frac{1}{N} \sum_{i \neq j} \mathbb{E}[-A_{ij}D^T_{ij}(\theta - \theta_0)^2] \leq \mathbb{E} \left[ \frac{1}{N} \sum_{i \neq j} (D^T_{ij}(\theta - \theta_0))^2 \right]. \]

For the second term we get

\[ \frac{1}{N} \sum_{i \neq j} \text{Var}( \log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0))) \leq \frac{1}{N} \sum_{i \neq j} \mathbb{E}[\log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0))] \leq \mathbb{E} \left[ \frac{1}{N} \sum_{i \neq j} (D^T_{ij}(\theta - \theta_0))^2 \right]. \]

The last term decomposes as

\[ \frac{2}{N} \sum_{i \neq j} \text{Cov}(-A_{ij}D^T_{ij}(\theta - \theta_0), \log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0))) \]

\[ = \frac{2}{N} \sum_{i \neq j} \mathbb{E}[-A_{ij}D^T_{ij}(\theta - \theta_0) \cdot (\log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0)))] \]

\[ - \frac{2}{N} \sum_{i \neq j} \mathbb{E}[-A_{ij}D^T_{ij}(\theta - \theta_0)] \cdot \mathbb{E}[\log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0))] \]

For the first term in that decomposition we have

\[ \frac{2}{N} \sum_{i \neq j} \mathbb{E}[A_{ij}D^T_{ij}(\theta - \theta_0) \cdot (\log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0)))] \]

\[ \leq \frac{2}{N} \sum_{i \neq j} \mathbb{E}[|D^T_{ij}(\theta - \theta_0)| \cdot |\log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0))|] \]

\[ \leq \frac{2}{N} \sum_{i \neq j} \mathbb{E}[|D^T_{ij}(\theta - \theta_0)|^2] \]

and for the second term using the same arguments, we get

\[ \frac{2}{N} \sum_{i \neq j} \mathbb{E}[-A_{ij}D^T_{ij}(\theta - \theta_0)] \cdot \mathbb{E}[\log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0))] \leq \frac{2}{N} \sum_{i \neq j} \mathbb{E}[|D^T_{ij}(\theta - \theta_0)|^2]. \]

Meaning that in total

\[ \frac{2}{N} \sum_{i \neq j} |\text{Cov}(-A_{ij}D^T_{ij}(\theta - \theta_0), \log(1 + \exp(D^T_{ij}\bar{\theta})) - \log(1 + \exp(D^T_{ij}\bar{\theta}_0)))| \]

\[ \leq \frac{2}{N} \sum_{i \neq j} \mathbb{E}[|D^T_{ij}(\theta - \theta_0)|^2] + \frac{2}{N} \sum_{i \neq j} \mathbb{E}[|D^T_{ij}(\theta - \theta_0)|^2]. \]

In total, we thus get

\[ \frac{1}{N} \sum_{i \neq j} \text{Var}(g_{\bar{\theta}}(T_{ij})) \leq 4 \cdot \mathbb{E} \left[ \frac{1}{N} \sum_{i \neq j} (D^T_{ij}(\theta - \theta_0))^2 \right] + \frac{2}{N} \sum_{i \neq j} \mathbb{E}[|D^T_{ij}(\theta - \theta_0)|^2]. \]
Furthermore,
\[
\frac{1}{N} \sum_{i \neq j} (D_{ij}^T(\theta - \theta_0))^2 = \frac{1}{N} \sum_{i \neq j} (\alpha_i + \beta_j + \mu - \alpha_{0,i} - \beta_{0,j} - \mu_0 + (\gamma - \gamma_0)^T Z_{ij})^2
\]
\[
\leq \frac{4}{N} \sum_{i \neq j} \left\{ (\alpha_i - \alpha_{0,i})^2 + (\beta_j - \beta_{0,j})^2 + (\mu - \mu_0)^2 + ((\gamma - \gamma_0)^T Z_{ij})^2 \right\}.
\]
Recall that for any \( x \in \mathbb{R}^p, \|x\|_2 \leq \|x\|_1 \leq \sqrt{p}\|x\|_2 \) and note that
\[
\|(\gamma - \gamma_0)^T Z_{ij}\| \leq c\|\gamma - \gamma_0\|_1 \leq c\sqrt{p}\|\gamma - \gamma_0\|_2.
\]
Then, from the above
\[
\frac{1}{N} \sum_{i \neq j} (D_{ij}^T(\theta - \theta_0))^2 \leq \frac{4}{N} \sum_{i \neq j} \left\{ (\alpha_i - \alpha_{0,i})^2 + (\beta_j - \beta_{0,j})^2 + (\mu - \mu_0)^2 + c^2\|\gamma - \gamma_0\|_2^2 \right\}
\]
\[
= 4 \left( (\mu - \mu_0)^2 + c^2\|\gamma - \gamma_0\|_2^2 + \frac{1}{N} \sum_{i \neq j} \left\{ (\alpha_i - \alpha_{0,i})^2 + (\beta_j - \beta_{0,j})^2 \right\} \right)
\]
\[
= 4 \left( (\mu - \mu_0)^2 + c^2\|\gamma - \gamma_0\|_2^2 + \frac{1}{N}(n-1)\|\hat{\theta} - \theta_0\|_2^2 \right)
\]
\[
= 4 \left( (\mu - \mu_0)^2 + c^2\|\gamma - \gamma_0\|_2^2 + \left\| \frac{1}{\sqrt{n}}(\hat{\theta} - \theta_0) \right\|_2^2 \right)
\]
\[
\leq 4(1 \vee (c^2\mu))\|\hat{\theta} - \theta_0\|_2^2
\]
\[
\leq 4(1 \vee (c^2\mu))\|\hat{\theta} - \theta_0\|_1^2
\]
\[
\leq 4(1 \vee (c^2\mu))M^2.
\]
Notice that for the second summand on the right-hand side in (28), we have
\[
\frac{2}{N} \sum_{i \neq j} \mathbb{E}||D_{ij}^T(\theta - \theta_0)|||^2 = \frac{2}{N} \sum_{i \neq j} \mathbb{E}(\alpha_i + \beta_j + \mu - \alpha_{0,i} - \beta_{0,j} - \mu_0 + (\gamma - \gamma_0)^T \mathbb{E}[Z_{ij}])^2
\]
\[
= \frac{2}{N} \sum_{i \neq j} (\alpha_i + \beta_j + \mu - \alpha_{0,i} - \beta_{0,j} - \mu_0)^2.
\]
So that we may use the same steps as in (29) to conclude that
\[
\frac{2}{N} \sum_{i \neq j} \mathbb{E}||D_{ij}^T(\theta - \theta_0)|||^2 \leq 6(1 \vee (c^2\mu))M^2.
\]
Such that in total,
\[
\frac{1}{N} \sum_{i \neq j} \text{Var}(g_0(T_{ij})) \leq 22(1 \vee (c^2\mu))M^2 := \tau_n^2.
\]
Applying Bousquet’s concentration Theorem 7 with \( \eta_n, \tau_n \) defined above, we obtain for all \( z > 0 \)
\[
\exp \left( -Nz^2 \right) \geq P \left( Z_M \geq \mathbb{E}[Z_M] + z\sqrt{2(\tau_n^2 + 2\eta_n\mathbb{E}[Z_M]) + \frac{2z^2\eta_n}{3}} \right)
\]
\[
= P \left( Z_M \geq \mathbb{E}[Z_M] + z\sqrt{2(22(1 \vee (c^2\mu))M^2 + 4(1 \vee c)\sqrt{nM}\mathbb{E}[Z_M]) + \frac{4z^2(1 \vee c)\sqrt{nM}}{3}} \right).
\]
From Lemma 14, we know
\[
\mathbb{E}[Z_M] \leq 8M \sqrt{\frac{2\log(2(2n + p + 1))}{N}}(1 \vee c) = 8Ma_n.
\]
We now want to show that the same holds with high probability for the sample matrix invertible with high probability, from which the desired properties of the minimum eigenvalue of the tools deployed in the proofs of Lemma 6 and 7 we can now show that with high probability.

Hence, for finite $\tilde{\Sigma}$ in (21), which simplifies to

\[ \mathcal{E}(\tilde{\theta}) + \tilde{\lambda} \left( \frac{1}{\sqrt{N}} \| \tilde{\theta} - \theta^* \|_1 + |\tilde{\mu} - \mu^*| + \| \tilde{\gamma} - \gamma^* \|_1 \right) \leq C \frac{s_p^* \tilde{\lambda}^2}{\rho_{n,0}}. \]

with constant $C = 64/c_{\min}$. \hfill \Box

## A.7 Putting it all together

**Proof of Theorem 2.** Theorem 2 now follows from Theorem 4 and Corollary 15. Recall the definition of $K_n$ in (21), which simplifies to

\[ K_n = 2 \left( \frac{1 + \exp(r_{n,0})^2}{\exp(r_{n,0})} \right) = 2 \left( \frac{1 + \exp(-\logit(\rho_{n,0}))^2}{\exp(-\logit(\rho_{n,0}))} \right) = \frac{2}{\rho_{n,0}}. \]

Thus, under the conditions of Theorem 2, we have with high probability by Theorem 4 and Corollary 15,

\[ \mathcal{E}(\tilde{\theta}) + \tilde{\lambda} \left( \frac{1}{\sqrt{N}} \| \tilde{\theta} - \theta^* \|_1 + |\tilde{\mu} - \mu^*| + \| \tilde{\gamma} - \gamma^* \|_1 \right) \leq C \frac{s_p^* \tilde{\lambda}^2}{\rho_{n,0}}. \]

with constant $C = 64/c_{\min}$. \hfill \Box

## B Proof of Theorem 3

### B.1 Inverting population and sample Gram matrices

Note that the function $f(x) = x(1-x)$ is monotonically increasing in $x$ for $x \leq 1/2$ and monotonically decreasing in $x$ for $x \geq 1/2$. Thus, by considering the cases $p_{ij} \leq 1/2$ and $p_{ij} \geq 1/2$ separately and using that $p_n \leq 1/2$, we may employ the following lower bound for all $i \neq j$: $p_{ij}(\theta_0) / \rho_{ij}(\theta_0) \geq 1/2\rho_n$. Also, recall that by assumption 1, the minimum eigenvalue $\lambda_{\min}$ of $\mathbb{E}[Z^T Z]/N$ stays uniformly bounded away from zero for all $n$. Then, for any $n$ and $v \in \mathbb{R}^{p+1} \setminus \{0\}$ with components $v = (v_1, v_R)^T$, $v_R \in \mathbb{R}^p$, we have

\[ v^T \tilde{\Sigma} v \geq \frac{1}{2} \rho_n v^T \frac{1}{N} \mathbb{E}[D_\xi^T D_\xi] v = \frac{1}{2} \rho_n v^T \left( \frac{1}{N} \mathbb{E}[Z^T Z] \right) v \]
\[ = \frac{1}{2} \rho_n \left( v_1^2 + v_R^T \frac{1}{N} \mathbb{E}[Z^T Z] v_R \right) \]
\[ \geq \frac{1}{2} \rho_n (v_1^2 + \lambda_{\min} \| v_R \|_2^2) \geq \frac{1}{2} \rho_n (1 \land c_{\min}) \| v \|_2^2 > 0. \]

Hence, for finite $n$ all eigenvalues of $\tilde{\Sigma}$ are strictly positive and consequently this matrix is invertible. We now want to show that the same hold with high probability for the sample matrix $\hat{\Sigma}_\xi$. Using the tools deployed in the proofs of Lemma 6 and 7 we can now show that with high probability the minimum eigenvalue of $D_\xi^T D_\xi / N$ is also strictly larger than zero, which means that $D_\xi^T D_\xi / N$ is invertible with high probability, from which the desired properties of $\hat{\Sigma}_\xi$ follow. More precisely, recall
the definition of $\kappa(A, m)$ for square matrices $A$ and dimensions $m$. We want to consider the expression

$$
\kappa^2 \left( \frac{1}{N} \mathbb{E}[D_x^T D_x], p + 1 \right)
$$

which simplifies to

$$
\kappa^2 \left( \frac{1}{N} \mathbb{E}[D_x^T D_x], p + 1 \right) := \min_{v \in \mathbb{R}^{p+1}\{0\}} \frac{v^T \frac{1}{N} \mathbb{E}[D_x^T D_x] v}{\|v\|_2}
$$

and compare it to $\kappa^2 \left( \frac{1}{N} D_x^T D_x, p + 1 \right)$. By assumption 1 and the argument above, we have

$$
\kappa^2 \left( \frac{1}{N} \mathbb{E}[D_x^T D_x], p + 1 \right) \geq C > 0
$$

for a universal constant $C$ independent of $n$. With $\delta = \max_{kl} \left| \left( \frac{1}{N} D_x^T D_x \right)_{kl} - \left( \frac{1}{N} \mathbb{E}[D_x^T D_x] \right)_{kl} \right|$, by Lemma 6, we have

$$
\kappa^2 \left( \frac{1}{N} D_x^T D_x, p + 1 \right) \geq \kappa^2 \left( \frac{1}{N} \mathbb{E}[D_x^T D_x], p + 1 \right) - 16\delta(p + 1).
$$

By looking at the proof of Lemma 6, we see that in this particular case we do not even need the factor $16(p + 1)$ on the right hand side above, but this does not matter anyways, so we keep it. By the exact same arguments we have used in the proof of Lemma 7 for the blocks $\{5, 6, 8\}$ and $\{9\}$, we now get

$$
\delta = O_p \left( N^{-1/2} \right).
$$

Thus, for $n$ large enough, we have with high probability $\delta \leq \frac{\lambda_{\min}}{32}$. Then, by Lemma 6, with high probability and uniformly in $n$,

$$
\kappa^2 \left( \frac{1}{N} D_x^T D_x, p + 1 \right) \geq \kappa^2 \left( \frac{1}{N} \mathbb{E}[D_x^T D_x], p + 1 \right) - 16\delta(p + 1) \geq \frac{\lambda_{\min}(p + 1)}{2} \geq C > 0.
$$

Yet, if $\kappa^2 \left( \frac{1}{N} D_x^T D_x, p + 1 \right) \geq C > 0$ uniformly in $n$, then for any $v \neq 0$, $v^T \frac{1}{N} D_x^T D_x v \geq C\|v\|_2^2$. But we also know that the minimum eigenvalue of $\frac{1}{N} D_x^T D_x$ is the largest possible $C$ such that this bound holds (it is actually tight with equality for the eigenvectors corresponding to the minimum eigenvalue). Therefore, with high probability, the minimum eigenvalue of $\frac{1}{N} D_x^T D_x$ stays uniformly bounded away from zero. Thus, for any $v \in \mathbb{R}^{p+1}\{0\}$ and any finite $n$:

$$
\frac{1}{N} v^T D_x^T \tilde{W}^2 D_x v \geq \min_{i \neq j} \{ p_{ij}(\hat{\theta})(1 - p_{ij}(\hat{\theta})) \} \left( v^T \frac{1}{N} D_x^T D_x v \right) \geq C\rho_n \|v\|_2^2 > 0.
$$

Thus, $\text{minval} \left( \frac{1}{N} D_x^T \tilde{W}^2 D_x \right) \geq C\rho_n \text{minval} \left( \frac{1}{N} D_x^T D_x \right) > 0$. That is, for every finite $n$, $\frac{1}{N} D_x^T \tilde{W}^2 D_x$ is invertible with high probability.

### B.2 Goal and approach

**Goal:** We want to show that for $k = 1, \ldots, p + 1$,

$$
\sqrt{N} \frac{\xi_k - \xi_{0,k}}{\hat{\theta}_{k,k}} \to \mathcal{N}(0, 1).
$$

**Approach:** Recall the definition of the "one-sample-version" of $\mathcal{L}$, i.e. $l_\theta : \{0, 1\} \times \mathbb{R}^{2n+1+p} \to \mathbb{R}$, for $\theta = (\alpha^T, \beta^T, \mu, \gamma^T)^T \in \Theta$,

$$
l_\theta(y, x) := -y\theta^T x + \log(1 + \exp(\theta^T x)).
$$
Then, the negative log-likelihood is given by
\[ \mathcal{L}(\theta) = \sum_{i \neq j} l_\theta(A_{ij}, D_{ij}^T) \]
and
\[ \nabla \mathcal{L}(\theta) = \sum_{i \neq j} \nabla l_\theta(A_{ij}, D_{ij}^T), \quad H \mathcal{L}(\theta) = \sum_{i \neq j} Hl_\theta(A_{ij}, D_{ij}^T), \]
where \( H \) denotes the Hessian with respect to \( \theta \). Consider \( l_\theta \) as a function in \( \theta^T x \) and introduce:
\[ l(y, a) := -ya + \log(1 + \exp(a)), \quad \text{(31)} \]
with second derivative: \( \tilde{l}(y, a) = \partial_a^2 l(y, a) = \frac{\exp(a)}{(1 + \exp(a))^2} \). Note, that \( \partial_a^2 l(y, a) \) is Lipschitz continuous (it has bounded derivative \(|\partial_a^2 l(y, a)| \leq 1/(6\sqrt{3})\); Lipschitz continuity then follows by the Mean Value Theorem). Doing a first order Taylor expansion in \( a \) of \( \tilde{l}(y, a) = \partial_a l(y, a) \) in the point \((A_{ij}, D_{ij}^T \theta_0)\) evaluated at \((A_{ij}, D_{ij}^T \tilde{\theta})\), we get
\[ \partial_a l(A_{ij}, D_{ij} \tilde{\theta}) = \partial_a l(A_{ij}, D_{ij}^T \theta_0) + \partial_a^2 l(A_{ij}, \alpha) D_{ij}^T(\tilde{\theta} - \theta_0), \quad \text{(32)} \]
for an \( \alpha \) between \( D_{ij}^T \tilde{\theta} \) and \( D_{ij}^T \theta_0 \). By Lipschitz continuity of \( \partial_a l \), we also find
\[ |\partial_a^2 l(A_{ij}, \alpha) D_{ij}^T(\tilde{\theta} - \theta_0) - \partial_a^2 l(A_{ij}, D_{ij}^T \tilde{\theta}) D_{ij}^T(\tilde{\theta} - \theta_0)| \leq |\alpha - D_{ij}^T \tilde{\theta}| |D_{ij}^T(\tilde{\theta} - \theta_0)| \leq |D_{ij}^T(\tilde{\theta} - \theta_0)|^2, \quad \text{(33)} \]
where the last inequality follows, because \( \alpha \) is between \( D_{ij}^T \tilde{\theta} \) and \( D_{ij}^T \theta_0 \).

Consider the vector \( P_n \nabla l_\theta \): By equation (32), with \( \alpha_{ij} \) between \( D_{ij}^T \tilde{\theta} \) and \( D_{ij}^T \theta_0 \),
\[ P_n \nabla l_\theta = \frac{1}{N} \sum_{i \neq j} \left( \partial_{\alpha_{ij}} l(A_{ij}, D_{ij}^T \tilde{\theta}) \right)_{k=1,...,2n+1+p}, \quad \text{as a } (2n+1+p) \times 1 \text{-vector} \]
\[ = \frac{1}{N} \sum_{i \neq j} \tilde{l}(A_{ij}, D_{ij}^T \tilde{\theta}) D_{ij} \]
\[ = \frac{1}{N} \sum_{i \neq j} \left( \tilde{l}(A_{ij}, D_{ij}^T \theta_0) + \tilde{l}(A_{ij}, \alpha_{ij}) D_{ij}^T(\tilde{\theta} - \theta_0) \right) D_{ij} \]
which by (33) gives
\[ P_n \nabla l_\theta + \frac{1}{N} \sum_{i \neq j} D_{ij} \left\{ \tilde{l}(A_{ij}, D_{ij}^T \tilde{\theta}) D_{ij}^T(\tilde{\theta} - \theta_0) + O(|D_{ij}^T(\tilde{\theta} - \theta_0)|^2) \right\}. \]
Noticing that \( \tilde{l}(A_{ij}, D_{ij}^T \tilde{\theta}) = p_{ij}(\tilde{\theta})(1-p_{ij}(\tilde{\theta})) \) and thus \( \sum_{i \neq j} \tilde{l}(A_{ij}, D_{ij}^T \tilde{\theta}) D_{ij} = D^T \tilde{W}^2 D(\tilde{\theta} - \theta_0) \):
\[ = P_n \nabla l_\theta + P_n Hl_\theta(\tilde{\theta} - \theta_0) + O \left( \frac{1}{N} \sum_{i \neq j} D_{ij} |D_{ij}^T(\tilde{\theta} - \theta_0)|^2 \right) \]
\[ = P_n \nabla l_\theta + \frac{1}{N} D^T \tilde{W}^2 D(\tilde{\theta} - \theta_0) + O \left( \frac{1}{N} \sum_{i \neq j} D_{ij} |D_{ij}^T(\tilde{\theta} - \theta_0)|^2 \right), \]
where the \( O \) notation is to be understood componentwise. Above, we have equality of two \((2n+1+p) \times 1\)-vectors. We are only interested in the portion relating to \( \xi = (\mu, \gamma^T)^T \), that is, in the last \( p+1 \) entries. Introduce the \(((2n+1+p) \times (2n+1+p))\)-matrix
\[ A = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\theta} \xi \end{pmatrix}, \]
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where 0 are zero-matrices of appropriate dimensions. Multiplying the above with A on both sides gives:

\[ AP_n \nabla \hat{\theta} = AP_n \nabla l_{\theta_0} + A \frac{1}{N} D^T \hat{W}^2 D(\hat{\theta} - \theta_0) + \mathcal{O} \left( \frac{1}{N} \sum_{i \neq j} D_{ij} |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \right). \quad (34) \]

Let us consider these terms in turn: Multiplication by A means that the first \( n \) entries of any of the vectors above are zero. Hence we only need to consider the last \( p + 1 \) entries. The left-hand side of (34) is equal to zero by (18). The last \( p + 1 \) entries of the first term on the right-hand side are \( \hat{\Theta}_\xi P_n \nabla \xi l_{\theta_0} \). For the second term on the right-hand side, notice that

\[ \frac{1}{N} D^T \hat{W}^2 D = \frac{1}{N} \begin{bmatrix} X^T \hat{W}^2 X & X^T \hat{W}^2 \hat{Z} \\ \hat{Z}^T \hat{W}^2 X & \hat{Z}^T \hat{W}^2 \hat{Z} \end{bmatrix}. \]

\( \hat{\Theta}_\xi \) is the exact inverse of \( \hat{\Sigma}_\xi \) which is the lower-right \((p+1) \times (p+1)\) block of above matrix. Thus,

\[ A \frac{1}{N} D^T \hat{W}^2 D = \begin{bmatrix} 0 & 0 \\ \hat{\Theta}_\xi \frac{1}{N} D^T \hat{W}^2 X & I_{(p+1) \times (p+1)} \end{bmatrix}. \]

Then, for the last \( p + 1 \) entries of \( A \frac{1}{N} D^T \hat{W}^2 D(\hat{\theta} - \theta_0) \)

\[ \left( A \frac{1}{N} D^T \hat{W}^2 D(\hat{\theta} - \theta_0) \right)_{\text{last } p+1 \text{ entries}} = \hat{\Theta}_\xi \frac{1}{N} D^T \hat{W}^2 X(\hat{\theta} - \theta_0) + \left( \frac{\hat{\mu}}{\hat{\gamma}} - \frac{\mu_0}{\gamma_0} \right). \]

Thus, (34) implies

\[ 0 = \hat{\Theta}_\xi P_n \nabla \gamma l_{\theta_0} + \hat{\Theta}_\xi \frac{1}{N} D^T \hat{W}^2 X(\hat{\theta} - \theta_0) + \left( \frac{\hat{\mu}}{\hat{\gamma}} - \frac{\mu_0}{\gamma_0} \right) + \mathcal{O} \left( \hat{\Theta}_\xi \frac{1}{N} \sum_{i \neq j} \left( \frac{1}{Z_{ij}} \right) |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \right), \]

which is equivalent to

\[ \left( \frac{\hat{\mu}}{\hat{\gamma}} - \frac{\mu_0}{\gamma_0} \right) = -\hat{\Theta}_\xi P_n \nabla \xi l_{\theta_0} - \hat{\Theta}_\xi \frac{1}{N} D^T \hat{W}^2 X(\hat{\theta} - \theta_0) + \mathcal{O} \left( \hat{\Theta}_\xi \frac{1}{N} \sum_{i \neq j} \left( \frac{1}{Z_{ij}} \right) |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \right). \quad (35) \]

Our goal is now to show that for each component \( k = 1, \ldots, p + 1 \),

\[ \sqrt{\frac{N}{N}} \frac{\hat{\xi}_k - \xi_{0,k}}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \overset{d}{\to} \mathcal{N}(0, 1). \]

as described in the **Goal** section. To that end, by equation (35), we now need to solve the following three problems: Writing \( \hat{\Theta}_{\xi,k} \) for the \( k \)-th row of \( \hat{\Theta}_\xi \),

1. \( \sqrt{\frac{N}{N}} \frac{\hat{\Theta}_{\xi,k} P_n \nabla \xi l_{\theta_0}}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \overset{d}{\to} \mathcal{N}(0, 1) \),

2. \( \frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \hat{\Theta}_{\xi,k} \frac{1}{N} D^T \hat{W}^2 X(\hat{\theta} - \theta_0) = o_P \left( N^{-1/2} \right) \),

3. \( \mathcal{O} \left( \frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \hat{\Theta}_{\xi,k} \frac{1}{N} \sum_{i \neq j} \left( \frac{1}{Z_{ij}} \right) |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \right) = o_P \left( N^{-1/2} \right). \)
B.3 Bounding inverses

The problems (1) - (3) above suggest that it will be essential to bound the norm and the distance of \( \Theta_\xi \) and \( \Theta_\xi \) in an appropriate manner. Notice that for any invertible matrices \( A, B \in \mathbb{R}^{m \times m} \) we have

\[
A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.
\]

Thus, for any sub-multiplicative matrix norm \( \| \cdot \| \), we get

\[
\|A^{-1} - B^{-1}\| \leq \|A^{-1}\| \|B^{-1}\| \|B - A\|.
\] (36)

We are particularly interested in the matrix \( \infty \)-norm, defined as

\[
\|A\|_{\infty} := \sup \left\{ \frac{\|Ax\|_{\infty}}{\|x\|_{\infty}}, \ x \neq 0 \right\} = \sup \{\|Ax\|_{\infty}, \|x\|_{\infty} = 1\} = \max_{1 \leq i \leq m} \sum_{j=1}^{m} |A_{ij}|,
\]

i.e. \( \|A\|_{\infty} \) is the maximal row \( \ell_1 \)-norm of \( A \). It is well-known, that any such matrix norm induced by a vector norm is sub-multiplicative (\( \|AB\|_{\infty} \leq \|A\|_{\infty} \|B\|_{\infty} \)) and consistent with the inducing vector norm (\( \|Ax\|_{\infty} \leq \|A\|_{\infty} \|x\|_{\infty} \) for any vector \( x \) of appropriate dimension). We first want to bound the matrix \( \infty \)-norm in terms of the largest eigenvalue.

**Lemma 16.** For any symmetric, positive semi-definite \((m \times m)\)-matrix \( A \) with maximal eigenvalue \( \lambda > 0 \), we have \( \|A\|_{\infty} \leq \sqrt{m} \lambda \).

**Proof.**

\[
\|A\|_{\infty} = \sup \{\|Ax\|_{\infty}, \|x\|_{\infty} = 1\} \\
\leq \sup \{\|Ax\|_{2}, \|x\|_{\infty} = 1\}, \quad \|Ax\|_{\infty} \leq \|Ax\|_{2} \\
= \sup \left\{ \frac{\|Ax\|_{2}}{\|x\|_{2}}, \|x\|_{\infty} = 1 \right\} \\
\leq \sqrt{m} \sup \left\{ \frac{\|Ax\|_{2}}{\|x\|_{2}}, \|x\|_{\infty} = 1 \right\}, \quad \text{if} \ \|x\|_{\infty} = 1, \ \text{then} \ \|x\|_{2} \leq \sqrt{m}, \\
\leq \sqrt{m} \sup \left\{ \frac{\|Ax\|_{2}}{\|x\|_{2}}, x \neq 0 \right\} \\
= \sqrt{m} \|A\|_{2} = \sqrt{m} \lambda,
\]

where \( \|A\|_{2} \) is the spectral norm of the matrix \( A \) and we have used that for symmetric matrices, the spectral norm is equal to the modulus of the largest eigenvalue of \( A \).

Also, recall that the inverse of a symmetric matrix \( A \) is itself symmetric:

\[
I = AA^{-1} = A^T A^{-1} \xrightarrow{\text{transpose}} I = (A^{-1})^T A^{\text{symmetry}} (A^{-1})^T A \xrightarrow{\text{uniqueness of inverse}} (A^{-1})^T = A^{-1}.
\]

Hence, \( \hat{\Theta}_\xi \) and \( \Theta_\xi \) are symmetric and we may apply Lemma 16. Using that \( \lambda_{\max}(\Sigma_\xi^{-1}) = \frac{1}{\lambda_{\min}(\Sigma_\xi)} \), we get

\[
\|\Theta_\xi\|_{\infty} \leq \sqrt{p} \lambda_{\max}(\Sigma_\xi^{-1}) \leq C \frac{1}{\rho_n},
\]

and with high probability

\[
\|\hat{\Theta}_\xi\|_{\infty} \leq \sqrt{p} \lambda_{\max}(\hat{\Sigma}_\xi^{-1}) \leq C \frac{1}{\rho_n},
\]

with some absolute constant \( C \). Finally, by (36),

\[
\|\hat{\Theta}_\xi - \Theta_\xi\|_{\infty} \leq \|\hat{\Theta}_\xi\|_{\infty} \|\Theta_\xi\|_{\infty} \|\hat{\Sigma}_\xi - \Sigma_\xi\|_{\infty} \leq \frac{C}{\rho_n} \|\hat{\Sigma}_\xi - \Sigma_\xi\|_{\infty}.
\]
It remains to control $\|\hat{\Sigma} - \Sigma\|_{\infty}$. We have

$$\hat{\Sigma} - \Sigma_{\xi} = \frac{1}{N} \left( D_{\xi}^T \hat{W}^2 D_{\xi} - \mathbb{E}[D_{\xi}^T W_0^2 D_{\xi}] \right) = \frac{1}{N} \left( D_{\xi}^T (\hat{W}^2 - W_0^2) D_{\xi} \right) + \frac{1}{N} \left( D_{\xi}^T W_0^2 D_{\xi} - \mathbb{E}[D_{\xi}^T W_0^2 D_{\xi}] \right).$$

Recall that $\hat{w}_{ij}^2 = p_{ij}(\hat{\theta})(1 - p_{ij}(\hat{\theta})) = \frac{\exp(D_{\xi}^T \hat{\theta})}{(1 + \exp(D_{\xi}^T \hat{\theta}))^2} = \partial_{\alpha l}(A_{ij}, D_{ij}^T \hat{\theta})$, with the function $l$ defined in (31). Also recall that $\partial_{\alpha l}$ is Lipschitz with constant one, by the Mean Value Theorem and the fact that it has derivative $\partial_{\alpha l} l$ bounded by one. Thus, considering the $(k, l)$-th element of $(I)$ above, we get:

$$\left| \frac{1}{N} \left( D_{\xi}^T (\hat{W}^2 - W_0^2) D_{\xi} \right)_{kl} \right| = \left| \frac{1}{N} \sum_{i \neq j} D_{ij,n+k} D_{ij,n+l} (\hat{w}_{ij}^2 - w_{0,ij}^2) \right| \leq C \left| \frac{1}{N} \sum_{i \neq j} |\hat{w}_{ij}^2 - w_{0,ij}^2|, \right. \text{ by uniform boundedness of } Z_{ij}$$

$$\leq C \left| \frac{1}{N} \sum_{i \neq j} |D_{ij}^T (\hat{\theta} - \theta_0)|, \text{ by Lipschitz continuity} \right| \leq C \left| \sum_{i \neq j} \left\{ |\hat{\alpha}_i - \alpha_{0,i}| + |\hat{\beta}_j - \beta_{0,j}| + |\hat{\mu} - \mu_0| + |Z_{ij}^T (\hat{\gamma} - \gamma_0)| \right\} \right| \leq C \left\{ \sum_{i \neq j} |\hat{\alpha}_i - \alpha_{0,i}| + |\hat{\beta}_j - \beta_{0,j}| \right\} + C|\hat{\mu} - \mu_0| + C|\hat{\gamma} - \gamma_0|_1 \leq C \left\{ \frac{1}{n} \|\hat{\theta} - \theta_0\|_1 + |\hat{\mu} - \mu_0| + \|\hat{\gamma} - \gamma_0\|_1 \right\} = O_P \left( s_+^* \sqrt{\frac{\log(n)}{N}} \rho_n^{-1} \right), \text{ under the conditions of theorem 2.}$$

Since the dimension of $(I)$ is $(p + 1) \times (p + 1)$ and thus remains fixed, any row of $(I)$ has $\ell_1$ norm of order $O_P \left( s_+^* \sqrt{\frac{\log(n)}{N}} \rho_n^{-1} \right)$ and thus

$$\|(I)\|_{\infty} = O_P \left( s_+^* \sqrt{\frac{\log(n)}{N}} \rho_n^{-1} \right).$$

Taking a look at the $(k, l)$-th element in $(II)$:

$$\left| \frac{1}{N} \left( D_{\xi}^T W_0^2 D_{\xi} - \mathbb{E}[D_{\xi}^T W_0^2 D_{\xi}] \right)_{kl} \right| = \left| \frac{1}{N} \sum_{i \neq j} \left\{ D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2 - \mathbb{E}[D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2] \right\} \right|.$$

Note that the random variables $D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2$ are bounded uniformly in $i, j, k, l$. Thus, by Hoeffding’s inequality, for any $t \geq 0$,

$$P \left( \left| \frac{1}{N} \sum_{i \neq j} \left\{ D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2 - \mathbb{E}[D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2] \right\} \right| \geq t \right) \leq 2 \exp \left( -CNt^2 \right).$$
This means, \( \frac{1}{N} \left( D_\xi W_0^2 D_\xi - E[D_\xi W_0^2 D_\xi] \right)_{kl} \) = \( O_P \left( N^{-1/2} \right) \). Again, since the dimension \( p + 1 \) is fixed, we get by a simple union bound
\[
\| (II) \|_\infty = O_P \left( N^{-1/2} \right).
\]
In total, we thus get
\[
\| \tilde{\Sigma}_\xi - \Sigma_\xi \|_\infty = O_P \left( s_+ \sqrt{\frac{\log(n)}{N}} \rho_n^{-1} + \frac{1}{\sqrt{N}} \right) = O_P \left( s_+ \sqrt{\frac{\log(n)}{N}} \rho_n^{-1} \right).
\]
We can now obtain a rate for \( \| \tilde{\Theta}_\xi - \Theta_\xi \|_\infty \).
\[
\| \tilde{\Theta}_\xi - \Theta_\xi \|_\infty \leq C \| \tilde{\Sigma}_\xi - \Sigma_\xi \|_\infty = O_P \left( s_+ \sqrt{\frac{\log(n)}{N}} \rho_n^{-3} \right).
\]
By assumption B3, we have \( s_+ \sqrt{\frac{\log(n)}{\rho_n^2}} \to 0, n \to \infty \), which in particular also implies that the above is \( o_P(1) \). Notice in particular, that we have now managed to get for \( k = 1, \ldots, p + 1 \),
\[
\bullet \| \tilde{\Theta}_{\xi,k} - \Theta_{\xi,k} \|_1 = o_P(1),
\]
\[
\bullet \tilde{\Theta}_{\xi,k} = \Theta_{\xi,k} + o_P(1).
\]

### B.4 Problem 1

We can now take a look at the problems (1) - (3) outlined above. For problem (1), we want to show:
\[
\sqrt{N} \frac{\tilde{\Theta}_{\xi,k} P_n \nabla_{\xi} l_{\theta_0}}{\sqrt{\tilde{\Theta}_{\xi,k,k}}} \to \mathcal{N}(0, 1).
\]

**Step 1:** Show that
\[
\tilde{\Theta}_{\xi,k} P_n \nabla_{\xi} l_{\theta_0} = \Theta_{\xi,k} P_n \nabla_{\xi} l_{\theta_0} + o_P \left( N^{-1/2} \right).
\]

We have
\[
\left\| (\tilde{\Theta}_{\xi,k} - \Theta_{\xi,k}) P_n \nabla_{\xi} l_{\theta_0} \right\| \leq \left\| \tilde{\Theta}_{\xi,k} - \Theta_{\xi,k} \right\|_1 \left\| \frac{1}{N} \sum_{i \neq j} \left( Z_{ij} \right) (p_{ij}(\theta_0) - A_{ij}) \right\|_\infty
\]
\[
\leq \left\| \tilde{\Theta}_\xi - \Theta_\xi \right\|_\infty \left\| \frac{1}{N} \sum_{i \neq j} D_{\xi,ij}(p_{ij}(\theta_0) - A_{ij}) \right\|_\infty.
\]

Consider the vector \( \sum_{i \neq j} D_{\xi,ij}(p_{ij}(\theta_0) - A_{ij}) \in \mathbb{R}^{p+1} \). The \( k \)-th component of it has the form \( \sum_{i \neq j} Z_{ij,k-1}(p_{ij}(\theta_0) - A_{ij}) \), \( k = 2, \ldots, p + 1 \). Notice that for these components are all centred:
\[
E[D_{\xi,ij,k}(p_{ij}(\theta_0) - A_{ij})] = E[D_{\xi,ij,k}E[(p_{ij}(\theta_0) - A_{ij})|Z_{ij}]] = E[D_{\xi,ij,k} \cdot 0] = 0,
\]
as well as \( |D_{\xi,ij,k}(p_{ij}(\theta_0) - A_{ij})| \leq c \), where \( c > 1 \) is a universal constant bounding \( |Z_{ij,k}| \) for all \( i, j, k \). Thus, by Hoeffding’s inequality, for any \( t > 0 \),
\[
P \left( \left\| \frac{1}{N} \sum_{i \neq j} D_{\xi,ij,k}(p_{ij}(\theta_0) - A_{ij}) \right\| \geq t \right) \leq 2 \exp \left( -2 \frac{Nt^2}{c^2} \right).
\]
and thus,

\[
\frac{1}{N} \sum_{i \neq j} D_{\xi,ij}(p_{ij}(\theta_0) - A_{ij}) = O_P \left( N^{-1/2} \right).
\]

Since we have \( \| \hat{\Theta}_Z - \Theta_Z \|_\infty = o_P(1) \), by section B.3, step 1 is now concluded.

**Step 2:** Show that

\[
\hat{\Theta}_{\xi,k,k} = \Theta_{\xi,k,k} + o_P(1).
\]

Since \( \| \hat{\Theta}_\xi - \Theta_\xi \|_\infty = o_P(1) \), by section B.3, for all \( k \)

\[
|\hat{\Theta}_{\xi,k,k} - \Theta_{\xi,k,k}| \leq \| \hat{\Theta}_\xi - \Theta_\xi \|_\infty = o_P(1)
\]

and step 2 is concluded.

**Step 3:** Show that

\[
\left| \frac{1}{\Theta_{\xi,k,k}} \right| \leq C < \infty,
\]

for some universal constant \( C > 0 \). Then, we may conclude from step 1 and step 2 that

\[
\frac{\sqrt{N} \hat{\Theta}_{\xi,k} P_n \nabla_{\ell_0} \Theta_{\xi,k}}{\sqrt{\Theta_{\xi,k,k}}} = \frac{\sqrt{N} \Theta_{\xi,k} P_n \nabla_{\ell_0} \Theta_{\xi,k}}{\sqrt{\Theta_{\xi,k,k}}} + o_P(1).
\]

To prove step 3, notice that \( \Theta_\xi \) is symmetric and hence has only real eigenvalues. Therefore it is unitarily diagonalizable and for any \( x \in \mathbb{R}^{p+1} \), we have \( x^T \Theta_\xi x \geq \lambda_{\min}(\Theta_\xi) \| x \|_2^2 \). We also know that

\[
\lambda_{\min}(\Theta_\xi) = \frac{1}{\lambda_{\max}(\Sigma_\xi)}.
\]

Under assumption 1 we can now deduce an upper bound on the maximum eigenvalue of \( \Sigma_\xi \): For any \( x \in \mathbb{R}^p \),

\[
x^T \Sigma_\xi x = x^T \frac{1}{N} \mathbb{E}[D_\xi^T W_0^2 D_\xi] x \leq x^T \frac{1}{N} \mathbb{E}[D_\xi^T D_\xi] x \leq (1 \lor \lambda_{\max}) \| x \|_2^2,
\]

where we have used that any entry in \( W_0^2 \) is bounded above by one. Since \( x^T \Sigma_\xi x \leq \lambda_{\max}(\Sigma_\xi) \| x \|_2^2 \) and since this bound is tight (we have equality if \( x \) is an eigenvector corresponding to \( \lambda_{\max} \), we can conclude by assumption 1 that \( \lambda_{\max}(\Sigma_\xi) \leq (1 \lor \lambda_{\max}) \leq C < \infty \) for some universal constant \( C > 0 \).

In particular, since \( \Theta_{\xi,k,k} = e_k^T \hat{\Theta}_\xi e_k \), we get

\[
\Theta_{\xi,k,k} \geq \lambda_{\min}(\Theta_\xi) \| e_k \|_2^2 = \frac{1}{\lambda_{\max}(\Sigma_\xi)} \geq C > 0,
\]

uniformly for all \( n \). Consequently,

\[
0 < \frac{1}{\Theta_{\xi,k,k}} \leq C < \infty.
\]

Step 3 is thus concluded.

**Step 4:** Finally, show that

\[
\frac{\sqrt{N} \hat{\Theta}_{\xi,k} P_n \nabla_{\ell_0} \Theta_{\xi,k}}{\sqrt{\Theta_{\xi,k,k}}} \overset{d}{\rightarrow} \mathcal{N}(0, 1),
\]

Such that by all the above

\[
\frac{\sqrt{N} \hat{\Theta}_{\xi,k} P_n \nabla_{\ell_0} \Theta_{\xi,k}}{\sqrt{\Theta_{\xi,k,k}}} \overset{d}{\rightarrow} \mathcal{N}(0, 1).
\]
For brevity, we write $p_{ij}$ for the true link probabilities $p_{ij}(\theta_0)$. Also keep in mind that $\Theta_{\xi,k}$ denotes the $k$-th row of $\Theta_\xi$, while $D_{\xi,ij}$ denote $((p+1) \times 1)$-column vectors. We want to apply the Lindeberg-Feller Central Limit Theorem. The random variables we study are the summands in

$$\sqrt{N}\Theta_{\xi,k} P_n \nabla \xi \theta_0 = \sum_{i \neq j} \left\{ \frac{1}{\sqrt{N}} \Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij}) \right\}.$$

First, notice that these random variables are centred:

$$\mathbb{E} \left[ \frac{1}{\sqrt{N}} \Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij}) \right] = \mathbb{E} \left[ \frac{1}{\sqrt{N}} \Theta_{\xi,k} D_{\xi,ij} \mathbb{E}[p_{ij} - A_{ij} | Z_{ij}] \right] = \mathbb{E} \left[ \frac{1}{\sqrt{N}} \Theta_{\xi,k} D_{\xi,ij} \cdot 0 \right] = 0.$$

For the Lindeberg-Feller CLT we need to sum up the variances of these random variables. We claim that

$$\sum_{i \neq j} \operatorname{Var} \left( \frac{1}{\sqrt{N}} \Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij}) \right) = \Theta_{\xi,k,k}.$$

Indeed, consider the vector-valued random variable $\sum_{i \neq j} \left\{ \frac{1}{\sqrt{N}} D_{\xi,ij} (p_{ij} - A_{ij}) \right\} \in \mathbb{R}^{p+1}$. It has covariance matrix

$$\mathbb{E} \left[ \sum_{i \neq j} \left\{ \frac{1}{\sqrt{N}} D_{\xi,ij} (p_{ij} - A_{ij}) \right\} \sum_{i \neq j} \left\{ \frac{1}{\sqrt{N}} D_{\xi,ij} (p_{ij} - A_{ij}) \right\}^T \right]$$

$$= \mathbb{E} \left[ \sum_{i \neq j} \frac{1}{\sqrt{N}} D_{\xi,ij} (p_{ij} - A_{ij}) \frac{1}{\sqrt{N}} D_{\xi,ij}^T (p_{ij} - A_{ij}) \right], \quad \text{by independence across } i, j$$

$$= \frac{1}{N} \sum_{i \neq j} \left[ \mathbb{E}[D_{\xi,ij} D_{\xi,ij}^T (p_{ij} - A_{ij})^2] \right]_{k,l=1,\ldots,p+1}, \quad \text{as a } ((p+1) \times (p+1))-\text{matrix}$$

$$= \frac{1}{N} \mathbb{E}[D_{\xi}^T W_0^2 D_{\xi}]$$

$$= \Sigma_{\xi}.$$

Thus, by independence across $i, j$,

$$\sum_{i \neq j} \operatorname{Var} \left( \frac{1}{\sqrt{N}} \Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij}) \right) = \operatorname{Var} \left( \Theta_{\xi,k} \sum_{i \neq j} \frac{1}{\sqrt{N}} D_{\xi,ij} (p_{ij} - A_{ij}) \right) = \Theta_{\xi,k} \Sigma_{\xi} \Theta_{\xi,k}^T = \Theta_{\xi,k,k},$$

where for the last equality we have used that $\Theta_{\xi}$ is the inverse of $\Sigma_{\xi}$ and thus, $\Sigma_{\xi} \Theta_{\xi,k}^T = \epsilon_k$. Now, we need to show that the Lindeberg condition holds. That is, we want that for any $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\Theta_{\xi,k,k}} \sum_{i \neq j} \mathbb{E} \left[ \left\{ \frac{1}{\sqrt{N}} \Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij}) \right\}^2 \mathbb{1} \left( |\Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij})| > \epsilon \sqrt{N} \Theta_{\xi,k,k} \right) \right] = 0. \quad (38)$$

We have

$$|\Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij})| \leq p \cdot c \cdot \|\Theta_{\xi,k}\|_1 \leq C \|\Theta_{\xi}\|_\infty \leq C \rho_n^{-1}.$$

At the same time, we know from step 3 that $\Theta_{z,k,k} \geq C > 0$ for some universal $C$. Then, as long as $\rho_n^{-1}$ goes to infinity at a rate slower than $n$, which is enforced by assumption B3, we must have for $n$ large enough

$$|\Theta_{\xi,k} D_{\xi,ij} (p_{ij} - A_{ij})| < \epsilon \sqrt{N} \Theta_{\xi,k,k}$$

uniformly in $i, j$. Thus, the indicator function and therefore each summand in (38) is equal to zero for $n$ large enough. Hence, (38) holds. Then, by the Lindeberg-Feller CLT,

$$\sqrt{N} \frac{\Theta_{\xi,k} P_n \nabla \xi \theta_0}{\sqrt{\Theta_{\xi,k,k}}} \overset{d}{\to} \mathcal{N}(0,1).$$
Now, by steps 1-4,

$$\sqrt{N} \frac{\hat{\Theta}_{\xi,k} P_{\xi} \nabla \ell_0}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \xrightarrow{d} \mathcal{N}(0, 1).$$

This concludes solving problem 1.

### B.5 Problem 2

For problem 2 we must show

$$\frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \hat{\Theta}_{\xi,k} \frac{1}{N} D_{\xi}^T \hat{W}^2 X (\hat{\vartheta} - \vartheta_0) = o_P \left( N^{-1/2} \right).$$

Since we have $$\|\hat{\Theta}_\xi - \Theta_\xi\|_\infty = o_P(1),$$ we do not need to worry about $$\frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}}$$, because $$\hat{\Theta}_{\xi,k,k} = \Theta_{\xi,k,k} + o_P(1)$$ and $$\frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \leq C < \infty$$, i.e. $$\frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}} = O_P(1)$$. By Theorem 2 we also have a high-probability error bound on $$\|\hat{\vartheta} - \vartheta_0\|_1$$. The problem will be bounding the corresponding matrix norms.

$$\left| \hat{\Theta}_{\xi,k} \frac{1}{N} D_{\xi}^T \hat{W}^2 X (\hat{\vartheta} - \vartheta_0) \right| \leq \left\| \frac{1}{N} X^T \hat{W}^2 D_{\xi} \hat{\Theta}_{\xi,k}^T \right\|_\infty \left\| \hat{\vartheta} - \vartheta_0 \right\|_1.$$

Notice that in the display above we have the vector $$\ell_\infty$$-norm. Also,

$$\left\| \frac{1}{N} X^T \hat{W}^2 D_{\xi} \hat{\Theta}_{\xi,k}^T \right\|_\infty \leq \|\hat{\Theta}_{\xi,k}\|_\infty \left\| \frac{1}{N} X^T \hat{W}^2 D_{\xi} \right\|_\infty.$$

Here we used the compatibility of the matrix $$\ell_\infty$$-norm with the vector $$\ell_\infty$$-norm. The first term is the vector norm, the second the matrix norm. We know,

$$\|\hat{\Theta}_{\xi,k}\|_\infty \leq \|\hat{\Theta}_\xi\|_\infty \leq C \rho_n^{-1},$$

where on the left hand side we have the vector norm and in the middle display the matrix norm. Finally, $$1/N \cdot X^T \hat{W}^2 D_{\xi}$$ is a $$(n \times (p+1))$$-matrix. The $$(k,l)$$-th element looks like $$1/N \cdot S_{k,l},$$ where $$S_{k,l}$$ is the sum of $$n-1$$ terms of the form $$D_{\xi,i,l} \hat{w}_{ik}^2$$, summed over the appropriate indices $$i,j$$, all of which are uniformly bounded. Thus,

$$\left| \left( \frac{1}{N} X^T \hat{W}^2 D_{\xi} \right)_{k,l} \right| \leq \frac{1}{N} \cdot (n-1) \cdot c = \frac{C}{n}.$$

Thus, the $$\ell_1$$-norm of any row of $$\frac{1}{N} X^T \hat{W}^2 D_{\xi}$$ is bounded by $$pC/n$$ and thus

$$\left\| \frac{1}{N} X^T \hat{W}^2 D_{\xi} \right\|_\infty \leq \frac{C}{n}.$$

Recall that $$\|\hat{\vartheta} - \vartheta_0\|_1 = O_P \left( s^*_+ \frac{\sqrt{\log(n)}}{\sqrt{n}} \rho_n^{-1} \right)$$ by Theorem 2. Then,

$$\left| \hat{\Theta}_{\xi,k} \frac{1}{N} X^T \hat{W}^2 D_{\xi} (\hat{\vartheta} - \vartheta_0) \right| \leq \|\hat{\Theta}_{\xi,k}\|_\infty \left\| \frac{1}{N} D_{\xi}^T \hat{W}^2 X \right\|_\infty \|\hat{\vartheta} - \vartheta_0\|_1$$

$$= O_P \left( \frac{s^*_+}{\rho_n^2} \cdot \frac{\sqrt{\log(n)}}{\sqrt{n}} \right).$$

Multiplying by $$\sqrt{N} = O(n)$$, gives

$$\sqrt{N} \left| \hat{\Theta}_{\xi,k} \frac{1}{N} D_{\xi}^T \hat{W}^2 X (\hat{\vartheta} - \vartheta_0) \right| = O_P \left( \frac{s^*_+}{\rho_n^2} \cdot \frac{\sqrt{\log(n)}}{\sqrt{n}} \right),$$

which is $$o_P(1)$$ under Assumption B3.
Finally, we must show
\[ O\left( \frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \frac{1}{N} \sum_{i \neq j} \left( \frac{1}{Z_{ij}} \right) |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \right) = o_P\left( N^{-1/2} \right). \]

Again, since \( \hat{\Theta}_{\xi,k,k} = \Theta_{\xi,k,k} + o_P(1) \) and \( \Theta_{\xi,k,k} \geq C > 0 \) uniformly in \( n \), we do not need to worry about the factor \( \frac{1}{\sqrt{\hat{\Theta}_{\xi,k,k}}} \) and it remains to show
\[ O\left( \hat{\Theta}_{\xi,k} \frac{1}{N} \sum_{i \neq j} D_{ij}^T(\hat{\theta} - \theta_0) \right) = o_P\left( N^{-1/2} \right). \]

We have
\[
\left| \hat{\Theta}_{\xi,k} \frac{1}{N} \sum_{i \neq j} D_{ij}^T(\hat{\theta} - \theta_0) \right| \leq \frac{1}{N} \sum_{i \neq j} |\hat{\Theta}_{\xi,k} D_{ij}| |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \\
\leq c\|\hat{\Theta}_{\xi,k}\| \frac{1}{N} \sum_{i \neq j} |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \\
\leq C\frac{1}{\rho_n} \frac{1}{N} \sum_{i \neq j} |D_{ij}^T(\hat{\theta} - \theta_0)|^2,
\]
where for the last inequality we have used that \( \|\hat{\Theta}_{\xi,k}\| \leq \|\hat{\Theta}_{\xi}\| \leq C\frac{1}{\rho_n} \). Now remember from (29) that
\[
\frac{1}{N} \sum_{i \neq j} |D_{ij}^T(\hat{\theta} - \theta_0)|^2 \leq C\|\hat{\theta} - \bar{\theta}_0\|_1^2,
\]
where we make use of the fact that \( \bar{D}\hat{\theta} = D\theta \). From Theorem 2 we know that under the assumptions of Theorem 3, \( \|\hat{\theta} - \bar{\theta}_0\|_1 = O_P\left( s_0 + \sqrt{\frac{\log(n)}{N}} \rho_n^{-1} \right) \). Thus,
\[
\sqrt{N} \left| \hat{\Theta}_{\xi,k} \frac{1}{N} \sum_{i \neq j} D_{ij}^T(\hat{\theta} - \theta_0) \right| = O_P\left( (s_0 +) \frac{\sqrt{\log(n)}}{\sqrt{N}} \rho_n^{-3} \right).
\]

We see that this is \( o_P(1) \) by applying assumption B3 twice. Problem 3 is solved.

Proof of Theorem 3. Theorem 3 now follows from the solved problems (1) - (3). \( \Box \)

C Proof of Theorem 1

C.1 Proof of Lemmas

We begin by providing proofs of the lemmas in section 3.1.

Proof of Lemma 2. Notice that
\[
\frac{1}{n-1} X^T X = \begin{bmatrix} I_n & B \\ B & I_n \end{bmatrix} \in \mathbb{R}^{2n \times 2n},
\]
where $I_n$ is the $(n \times n)$ identity matrix and $B$ is a matrix with zeros on the diagonal and $1/(n-1)$ everywhere else. Now consider the submatrix with only those rows and columns belonging to $S$

$$P := \frac{1}{n-1} (X^T X)_{S \times S} = \begin{bmatrix} I_{s_\alpha} & B_{S_\alpha, S_\alpha} \\ B_{S_\beta, S_\alpha} & I_{s_\beta} \end{bmatrix} \in \mathbb{R}^{s \times s}.$$  

This matrix $P$ is strictly diagonally dominant. Indeed,

$$\sum_{j \in S, j \neq i} P_{ij} = \frac{s_\beta}{n-1} < 1 = P_{ii}, \quad i \in S_\alpha$$

$$\sum_{j \in S, j \neq i} P_{ij} = \frac{s_\alpha}{n-1} < 1 = P_{ii}, \quad i \in S_\beta,$$

where the strict inequalities hold because $\min_i \{\alpha_{0,i}\} = \min_j \{\beta_{0,j}\} = 0$. Thus, $P$ is strictly positive definite. More, by the Gershgorin Circle Theorem, all the eigenvalues of $P$ must lie in one of the discs $D(P_{ii}, R_i)$, where $R_i = \sum_{j \in S, j \neq i} P_{ij}$ and $D(P_{ii}, R_i)$ is the disc with radius $R_i$ centred at $P_{ii}$. In particular,

$$\text{minval}(P) \geq 1 - \frac{\max\{s_\alpha, s_\beta\}}{n-1}.$$  

But now, for any $v \in \mathbb{R}^s$,

$$v^T Q_{S,S} v \geq \frac{1}{2} \rho_n \cdot v^T P v \geq \frac{1}{2} \rho_n \left(1 - \frac{\max\{s_\alpha, s_\beta\}}{n-1}\right) \|v\|_2^2$$

and the claim follows. \qed

Proof of Lemma 3. We make use of the following bound of a the infinity norm of the inverse of a diagonally dominant matrix (see for example Varah (1975))

$$\|Q_{S,S}^{-1}\|_\infty \leq \max_{i \in S} \left\{\frac{1}{|q_{ii}| - R_i}\right\},$$

where $q_{ii}$ is the $i$th diagonal entry of $Q_{S,S}$ and $R_i$ is the sum of the off-diagonal elements of the $i$th row of $Q_{S,S}$. That is, for $i \in S_\alpha$,

$$q_{ii} - R_i = \frac{1}{n-1} \sum_{j=1, j \neq i}^n p_{ij} (1 - p_{ij}) - \frac{1}{n-1} \sum_{j \in S_\beta} p_{ij} (1 - p_{ij}) \geq \frac{1}{2(n-1)} \rho_n (n - s_\beta),$$

and analogously for $i \in S_\beta$,

$$q_{ii} - R_i \geq \frac{1}{2(n-1)} \rho_n (n - s_\alpha).$$

Thus,

$$q_{ii} - R_i \geq \frac{1}{2(n-1)} \rho_n (n - \max\{s_\alpha, s_\beta\})$$

and therefore,

$$\|Q_{S,S}^{-1}\|_\infty \leq 2 \rho_n^{-1} \cdot \frac{n-1}{n - \max\{s_\alpha, s_\beta\}}.$$  \hspace{1cm} (39)

Furthermore, notice that any row of $Q_{S^c,S}$ has either $s_\alpha$ or $s_\beta$ non-zero entries, each of the form $1/(n-1) \cdot p_{ij} (1 - p_{ij}) \leq 1/(4(n-1))$. Hence,

$$\|Q_{S^c,S}\|_\infty \leq \frac{\max\{s_\alpha, s_\beta\}}{4(n-1)}.$$  

The claim follows by the submultiplicativity of the matrix infinity norm. \qed
Proof of Lemma 4. Since $\tilde{\theta}^\dagger$ and $\tilde{\theta}$ both solve (11), we must have
\[
\frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}^\dagger) + \lambda \|\tilde{\theta}^\dagger\|_1 = \frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}) + \lambda \|\tilde{\theta}\|_1.
\]
Denote by $\tilde{z}_\delta^\dagger$ the first 2n components of $\tilde{z}^\dagger$. Then, by (14a) and (14b), $\langle \tilde{z}_\delta^\dagger, \tilde{\theta}^\dagger \rangle = \|\tilde{\theta}^\dagger\|_1$. Thus,
\[
\frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}^\dagger) + \lambda \langle \tilde{z}_\delta^\dagger, \tilde{\theta}^\dagger \rangle = \frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}) + \lambda \|\tilde{\theta}\|_1.
\]
Hence, using that the last $p + 1$ components of $\tilde{z}^\dagger$ are zero,
\[
\frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}^\dagger) + \lambda \langle \tilde{z}_\delta^\dagger, \tilde{\theta}^\dagger - \tilde{\theta} \rangle = \frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}) + \lambda \left( \|\tilde{\theta}\|_1 - \langle \tilde{z}_\delta^\dagger, \tilde{\theta} \rangle \right).
\]
But by (13), $\lambda \tilde{z}^\dagger = -1/N \cdot \nabla \tilde{\mathcal{L}}(\tilde{\theta})$ and therefore
\[
\frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}) - (1/N \cdot \nabla \tilde{\mathcal{L}}(\tilde{\theta}), \tilde{\theta}^\dagger - \tilde{\theta}) - \frac{1}{N} \tilde{\mathcal{L}}(\tilde{\theta}) = \lambda \left( \|\tilde{\theta}\|_1 - \langle \tilde{z}^\dagger, \tilde{\theta} \rangle \right).
\]
By the convexity of $\tilde{\mathcal{L}}$, the left-hand side in the above display is negative. Therefore,
\[
\|\tilde{\theta}\|_1 \leq \langle \tilde{z}_\delta^\dagger, \tilde{\theta} \rangle = \langle \tilde{z}_\delta^\dagger, \tilde{\theta} \rangle \leq \|\tilde{z}_\delta^\dagger\|_1 \|\tilde{\theta}\|_1 \leq \|\tilde{\theta}\|_1.
\]
Hence, $\langle \tilde{z}_\delta^\dagger, \tilde{\theta} \rangle = \|\tilde{\theta}\|_1$. But since $\|\tilde{z}^\dagger_{S^c}\|_\infty < 1$ by (15b), this can only hold if $\tilde{\theta}_{S^c} = 0$. The claim follows.

C.2 General strategy

The proof of Theorem 1 hinges on the construction of $(\tilde{\theta}^\dagger, \tilde{z}^\dagger)$ succeeding with high probability and the challenge in proving this is proving that $(\tilde{\theta}^\dagger, \tilde{z}^\dagger)$ fulfills conditions (15a) and (15b). Our proof relies on the following derivations. From (13) we obtain
\[
0 = \frac{1}{N} \nabla \tilde{\mathcal{L}}(\tilde{\theta}^\dagger) + \lambda \tilde{z}^\dagger - \frac{1}{N} \nabla \tilde{\mathcal{L}}(\tilde{\theta}_0) + \frac{1}{N} \nabla \tilde{\mathcal{L}}(\tilde{\theta}_0).
\]
Doing a Taylor expansion along the same lines as (32) and (33), we obtain
\[
\frac{1}{N} \nabla \tilde{\mathcal{L}}(\tilde{\theta}^\dagger) - \frac{1}{N} \nabla \tilde{\mathcal{L}}(\tilde{\theta}_0) = \frac{1}{N} D^T W_0^2 D(\tilde{\theta}^\dagger - \tilde{\theta}_0) + O \left( \frac{1}{N} \sum_{i \neq j} \tilde{D}_{ij} |D^T_{ij}(\tilde{\theta}^\dagger - \tilde{\theta}_0)|^2 \right),
\]
where we have used the fact that we are taking derivatives with respect to $\tilde{\theta}$ and used $\tilde{D}_{ij}\tilde{\theta}_0$ in (33), to obtain $W_0^2$ instead of $W^2$ above. Combining the last two equations, we obtain
\[
\frac{1}{N} D^T W_0^2 D(\tilde{\theta}^\dagger - \tilde{\theta}_0) = -\lambda \tilde{z}^\dagger - \frac{1}{N} \nabla \tilde{\mathcal{L}}(\tilde{\theta}_0) + O \left( \frac{1}{N} \sum_{i \neq j} \tilde{D}_{ij} |D^T_{ij}(\tilde{\theta}^\dagger - \tilde{\theta}_0)|^2 \right).
\]
Taking only the first 2n entries of that equation we obtain
\[
\frac{1}{N} X^T W_0^2 X(\tilde{\theta}^\dagger - \tilde{\theta}_0) = -\frac{1}{N} \nabla \tilde{\mathcal{L}}(\tilde{\theta}_0) + \frac{1}{N} X^T W_0^2 \left[ \begin{array}{c} 1 \\ Z \end{array} \right] (\xi^\dagger - \xi_0) - \lambda \tilde{z}^\dagger_{1:2n} + \tilde{R} \tag{40}
\]
where we use $\tilde{z}^\dagger_{1:2n}$ to refer to the first 2n components of $\tilde{z}^\dagger_{1:2n}$, use our shorthand notation $\xi = (\mu, \gamma^T)^T$ and let
\[
\tilde{R} = O \left( \frac{1}{N} \sum_{i \neq j} X_{ij} |D^T_{ij}(\tilde{\theta}^\dagger - \tilde{\theta}_0)|^2 \right).
\]
Notice that we the left-hand side in (40) is equal to
\[
Q(\tilde{\theta}^\dagger - \tilde{\theta}_0) = Q_{-S}(\tilde{\theta}^\dagger - \tilde{\theta}_0)_S + Q_{-S^c}(\tilde{\theta}^\dagger - \tilde{\theta}_0)_{S^c} = 0.
\]
Plugging this into (40) and splitting up by rows, we get

\begin{align}
Q_{S,S}(\bar{\vartheta}^t - \bar{\vartheta}_0)_S &= - \frac{1}{N} (\nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0))_S + \frac{1}{N} \bar{X}_S^T W_0^2 [1 \mid Z ] (\xi^t - \xi_0) - \bar{\lambda} \bar{z}^t_{1:2n,S} + \bar{R}_S \\
Q_{S^c,S}(\bar{\vartheta}^t - \bar{\vartheta}_0)_S &= - \frac{1}{N} (\nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0))_{S^c} + \frac{1}{N} \bar{X}_{S^c}^T W_0^2 [1 \mid Z ] (\xi^t - \xi_0) - \bar{\lambda} \bar{z}^t_{1:2n,S^c} + \bar{R}_{S^c},
\end{align}

where it is important to remember that \( S^c = [2n]\setminus S \) refers to the complement of \( S \) in \([2n]\). We solve (41a) for \((\bar{\vartheta}^t - \bar{\vartheta}_0)_S \) and plug the result into (41b). Finally we rearrange for \(-\bar{\lambda} \bar{z}^t_{1:2n,S^c} \),

\[-\bar{\lambda} \bar{z}^t_{1:2n,S^c} = Q_{S^c,S} Q_S^{-1} \bar{z}^t_{S,S} \left\{ - \frac{1}{N} (\nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0))_{S^c} + \frac{1}{N} \bar{X}_{S^c}^T W_0^2 [1 \mid Z ] (\xi^t - \xi_0) - \bar{\lambda} \bar{z}^t_{1:2n,S} + \bar{R}_{S^c} \right\}
\]

Now, divide by \( \bar{\lambda} \) and take the \( \infty \)-norm on both sides. Rearrange corresponding terms.

\[
\| \bar{z}^t_{1:2n,S^c} \|_\infty \leq \frac{1}{\bar{\lambda}} \left\{ \| Q_{S^c,S} Q_S^{-1} \|_\infty + 1 \right\} \left\| \frac{1}{N} \nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0) \right\|_\infty (I)
\]
\[
+ \frac{1}{\bar{\lambda}} \left\{ \| Q_{S^c,S} Q_S^{-1} \|_\infty + 1 \right\} \| \bar{R} \|_\infty (II)
\]
\[
+ \frac{1}{\bar{\lambda}} \left\{ \| Q_{S^c,S} Q_S^{-1} \|_\infty + 1 \right\} \left\| \frac{1}{N} \bar{X}_{S^c}^T W_0^2 [1 \mid Z ] (\xi^t - \xi_0) \right\|_\infty (III)
\]
\[
+ \| Q_{S^c,S} Q_S^{-1} \|_\infty (IV).
\]

By appropriately bounding the terms \((I) - (IV)\) on the right-hand side, we will proceed to show that for sufficiently large \( n \), with high probability, \( \| \bar{z}^t_{1:2n,S^c} \|_\infty < 1 \), which is clearly equivalent to (15b). Notice that we already may control term \((IV)\) as well as the terms \( \| Q_{S^c,S} Q_S^{-1} \|_\infty + 1 \) by the incoherence condition, Lemma 3.

### C.3 Controlling term \((I)\)

Notice that the \( i \)th component of \( \frac{1}{N} \nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0) \) is of the form

\[
\frac{1}{N} \sqrt{n} \sum_{j=1,j\neq i} \left( A_{ij} - p_{ij} \right) = \frac{1}{\sqrt{n}} \cdot \frac{1}{n-1} \sum_{j=1,j\neq i} \left( A_{ij} - p_{ij} \right).
\]

In particular, each summand is a centred, bounded random variable. By Hoeffding’s inequality, we have for every \( t > 0 \),

\[
P \left( \left| \frac{1}{n-1} \sum_{j=1,j\neq i} \left( A_{ij} - p_{ij} \right) \right| \geq t \right) \leq 2 \exp \left( -\frac{n-1}{2} t^2 \right).
\]

Thus, for any \( \epsilon > 0 \), picking \( t = \epsilon \sqrt{n} \bar{\lambda} \), gives

\[
P \left( \frac{1}{\bar{\lambda}} \left| \frac{1}{N} \nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0)_i \right| \geq \epsilon \right) \leq 2 \exp \left( -\frac{N \bar{\lambda}^2}{2} \epsilon^2 \right).
\]

Taking a union bound over all \( 2n \) components of \( \nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0) \), leads to

\[
P \left( \frac{1}{\bar{\lambda}} \left\| \frac{1}{N} \nabla_{\vartheta} \bar{\mathcal{L}}(\bar{\vartheta}_0) \right\|_\infty \geq \epsilon \right) \leq 4n \cdot \exp \left( -\frac{N \bar{\lambda}^2}{2} \epsilon^2 \right) = 4 \cdot \exp \left( -\frac{N \bar{\lambda}^2}{2} \epsilon^2 + \log(n) \right).
\]
In the next section, when controlling term $(II)$, we will also need a similar bound on the components of $\frac{1}{N}\nabla \tilde{L}(\hat{\theta}_0)$ corresponding to $\xi = (\mu, \gamma^T)^T$, which is why we derive the respective bounds now. Using analogous arguments to the above, we obtain

$$
P \left( \frac{1}{N} \left\| \frac{1}{N} \nabla \tilde{L}(\hat{\theta}_0) \right\|_\infty \geq \epsilon \right) \leq 2(p + 1) \cdot \exp \left( -\frac{N\lambda^2}{2(1 + c^2)} \epsilon^2 \right).$$

(43)

Combining (42) and (43), we obtain a bound on the infinity norm of the full gradient,

$$
P \left( \frac{1}{N} \left\| \nabla \tilde{L}(\hat{\theta}_0) \right\|_\infty \geq \epsilon \right) \leq 4 \cdot \exp \left( -\frac{N\lambda^2}{2} \epsilon^2 + \log(n) \right) + 2(p + 1) \cdot \exp \left( -\frac{N\lambda^2}{2(1 + c^2)} \epsilon^2 \right),$$

(44)

which tends to zero, as long as $-\frac{N\lambda^2}{2} \epsilon^2 + \log(n) \to \infty$, as $n$ tends to infinity.

C.4 Controlling term $(II)$

Controlling term $(II)$ is by far the most involved step in controlling $\|\bar{z}^t_{1,2n,S}\|_\infty$. We start by controlling the $\ell_2$-error between our construction $\bar{\theta}^t$ and the truth $\hat{\theta}_0$.

**Lemma 17.** Under assumptions 1, 2, B1, 3, for $n$ large enough, for any $\epsilon > 0$, with probability at least

$$
1 - 4 \cdot \exp \left( -\frac{N\lambda^2}{2} \epsilon^2 + \log(n) \right) - 2(p + 1) \cdot \exp \left( -\frac{N\lambda^2}{2(1 + c^2)} \epsilon^2 \right) - p(p + 3) \exp \left( -N \cdot \frac{c_n^2}{2048 \cdot s_\lambda \lambda} \right),
$$

which tends to one as long as $-\frac{N\lambda^2}{2} \epsilon^2 + \log(n) \to -\infty$, as $n$ tends to infinity, we have

$$
\| \bar{\theta}^t - \hat{\theta}_0 \|_1 \leq (1 + \epsilon) \cdot \frac{9}{c_n} \rho_n^{-1} s_\lambda \lambda.
$$

**Proof.** Keep in mind that $\bar{\theta}^t - \hat{\theta}_0 = \bar{\theta}^t_{S_\lambda} - \hat{\theta}_0_{S_\lambda}$. Define a function $G: \mathbb{R}^{s_+ + p} \to \mathbb{R}$,

$$
G(u) = \frac{1}{N} \left( \tilde{L}(\hat{\theta}_{0,S_\lambda} + u) - \tilde{L}(\hat{\theta}_{0,S_\lambda}) \right) + \tilde{\lambda} \left( \| \hat{\theta}_{0,S} + uS \|_1 - \| \hat{\theta}_{0,S} \|_1 \right),
$$

where for the addition $\hat{\theta}_{0,S_\lambda} + u$ to be well-defined, we use the canonical embedding of $\mathbb{R}^{s_+ + p} \to \mathbb{R}^{2n + s_+ + p}$, by setting the components not contained in $S$ to zero. In the following we will make use of that embedding without explicitly mentioning it if there is no chance of confusion. Also pay close attention to the distinction between $S_\lambda$ and $S$ in above display. Clearly, $G(0) = 0$ and $G$ is minimized at $\bar{u} = \bar{\theta}^t_{S_\lambda} - \hat{\theta}_{0,S_\lambda}$, which implies that $G(\bar{u}) \leq 0$. Also, $G$ is convex.

Now suppose we manage to find some $B \in \mathbb{R}, B > 0$, such that for all $u \in \mathbb{R}^{s_+ + p}$ with $\|u\|_1 = B$ it holds $G(u) > 0$. We claim that in that case it must hold $\|\bar{u}\|_1 \leq B$. Indeed, if $\|\bar{u}\|_1 > B$, then there exists a $t \in (0,1)$ such that for $\bar{u} = t \bar{u}$ we have $\|\bar{u}\|_1 = B$. But then, by convexity of $G$, $G(\bar{u}) \leq tG(\bar{u}) + (1 - t)G(0) = tG(\bar{u}) \leq 0$. A contradiction.

Thus, we need to find an appropriate $B$. Let $B > 0$, the correct form to be determined later. Now, pick any $u \in \mathbb{R}^{s_+ + p}$ with $\|u\|_1 = B$. We do a first order Taylor expansion of $\tilde{L}$ in the point $\hat{\theta}_{0,S_\lambda}$, evaluated at $\hat{\theta}_{0,S_\lambda} + u$. This yields

$$
G(u) = \frac{1}{N} \left\{ \nabla S_\lambda \tilde{L}(\hat{\theta}_{0,S_\lambda})^T (\hat{\theta}_{0,S_\lambda} + u - \hat{\theta}_{0,S_\lambda}) + \frac{1}{2} \cdot u^T H_{S_\lambda,S_\lambda} \tilde{L}(\hat{\theta}_{0,S_\lambda} + u\alpha)u \right\}
+ \tilde{\lambda} \left( \| \hat{\theta}_{0,S} + uS \|_1 - \| \hat{\theta}_{0,S} \|_1 \right),
$$

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for some $\alpha \in [0, 1]$. Now, using (44), we know that with high-probability,

$$\left\| \frac{1}{N} \nabla S_{\lambda} \bar{L}(\bar{\theta}_{0,S_{\lambda}})^T u \right\| \leq \left\| \frac{1}{N} \nabla S_{\lambda} \bar{L}(\bar{\theta}_{0,S_{\lambda}}) \right\| \| u \|_1 \leq \epsilon \bar{\lambda} B \quad (45)$$

with the $\epsilon$ from (44). Furthermore, by using the triangle inequality, we obtain

$$\bar{\lambda} \left( \| \bar{\theta}_{0,S} + u_S \|_1 - \| \bar{\theta}_{0,S} \|_1 \right) \geq -\bar{\lambda} \| u \|_1 = -\bar{\lambda} B \quad (46)$$

Clearly, the canonical embedding of $u$ into $\mathbb{R}^{2n+1+p}$ fulfills the condition of the empirical compatibility condition, Proposition 8. Also, keep in mind that assumptions B1 and 3 together imply $n^{-1/2} \rho_n^{-1}s_+ \to 0$, which in particular implies $s_+ = o(\sqrt{n})$. Thus, Proposition 8 is applicable and with high probability as prescribed in Proposition 8, we have

$$\frac{1}{2} \cdot u^T H_{S_{\lambda},S_{\lambda}} \bar{L}(\bar{\theta}_{0,S_{\lambda}} + u\alpha) u \geq \frac{1}{4} \rho_n u^T \left\{ \frac{1}{N} \bar{D}^T \bar{D} \right\}_{S_{\lambda},S_{\lambda}} u$$

$$= \frac{1}{4} \rho_n u^T \Sigma u$$

$$\geq \frac{1}{8} \rho_n \frac{c_{\min}}{s_+} \| u \|_1^2$$

$$= \frac{1}{8} \rho_n \frac{c_{\min}}{s_+} B^2 \quad (47)$$

Combining (45), (46), (47), we find

$$G(u) \geq -\epsilon \bar{\lambda} B - \bar{\lambda} B + \frac{1}{8} \rho_n \frac{c_{\min}}{s_+} B^2.$$  

The right-hand side of this equation is strictly larger zero, whenever

$$B > (1 + \epsilon) \frac{8}{c_{\min}} \rho_n^{-1}s_+ \bar{\lambda}.$$  

Thus, the claim follows from picking

$$B = (1 + \epsilon) \frac{9}{c_{\min}} \rho_n^{-1}s_+ \bar{\lambda}.$$

\[\square\]

**Lemma 18.** Under assumptions 1, 2, B1, 3, for $n$ large enough, for any $\epsilon > 0$, with probability at least

$$1 - 4 \cdot \exp \left( -\frac{N \bar{\lambda}^2}{2} \epsilon^2 + \log(n) \right) - 2(p + 1) \cdot \exp \left( -\frac{N \bar{\lambda}^2}{2(1 + \epsilon)^2} \epsilon^2 \right)$$

$$- p(p + 3) \exp \left( -\frac{N \bar{\lambda}^2}{2048c_{\min}^2} \epsilon^2 \right),$$

which tends to one as long as $-\frac{N \bar{\lambda}^2}{2} \epsilon^2 + \log(n) \to -\infty$, as $n$ tends to infinity, we have

$$\frac{1}{\bar{\lambda}} \| \bar{R} \|_{\infty} \leq \frac{324(1 + (\epsilon^2 p))(1 + \epsilon)^2}{c_{\min}^2} \cdot \sqrt{n} \rho_n^{-2}s_+ \bar{\lambda}.$$
Proof. Consider the \( i \)th component of \( \bar{R}_i \), for \( i \in S_\alpha \). Similar to (29) we obtain,

\[
\bar{R}_i = \frac{1}{N} \sum_{j=1, j \neq i}^{n} X_{ij} |D_{ij}^T(\bar{\theta}^\top - \bar{\theta}_0)|^2
\]

\[
= \frac{1}{\sqrt{n}} \cdot \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} |D_{ij}^T(\bar{\theta}^\top - \bar{\theta}_0)|^2
\]

\[
= \frac{1}{\sqrt{n}} \cdot \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} \left( \alpha_i^\top - \alpha_{0,i} + \beta_j^\top - \beta_{0,j} + \mu^\top - \mu_0 + Z_{ij}^T(\gamma^\top - \gamma_0) \right)^2
\]

\[
\leq \frac{4}{\sqrt{n}} \cdot \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} \left( (\alpha_i^\top - \alpha_{0,i})^2 + (\beta_j^\top - \beta_{0,j})^2 + (\mu^\top - \mu_0)^2 + c^2 p \|\gamma^\top - \gamma_0\|_2^2 \right)
\]

\[
= \frac{4}{\sqrt{n}} \left\{ (\alpha_i^\top - \alpha_{0,i})^2 + (\mu^\top - \mu_0)^2 + c^2 p \|\gamma^\top - \gamma_0\|_2^2 \right\} + \frac{4}{\sqrt{n}} \cdot \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} (\beta_j^\top - \beta_{0,j})^2
\]

\[
\leq \frac{4}{\sqrt{n}} \left( 1 \vee (c^2 p) \right) \left\{ (\alpha_i^\top - \alpha_{0,i})^2 + (\mu^\top - \mu_0)^2 + \|\gamma^\top - \gamma_0\|_2^2 \right\} + \frac{n}{n-1} \frac{4}{\sqrt{n}} \mu^\top - \bar{\theta}_0\|_2^2.
\]

We have

\[
(\alpha_i^\top - \alpha_{0,i})^2 = n(\alpha_i^\top - \bar{\alpha}_{0,i})^2 \leq n\|\alpha^\top - \bar{\alpha}_0\|_2^2.
\]

Thus, by Lemma 17, with at least the prescribed probability and for all \( i \in S_\alpha \),

\[
\frac{R_i}{\lambda} \leq 4(1 \vee (c^2 p)) \sqrt{n} \|\bar{\theta}^\top - \bar{\theta}_0\|_2^2 \leq 4(1 \vee (c^2 p)) \sqrt{n} \|\bar{\theta}^\top - \bar{\theta}_0\|_2^2
\]

\[
\leq \frac{324(1 \vee (c^2 p))(1 + \epsilon)^2}{c_{\text{min}}^2} \cdot \sqrt{n} \rho_{-1}^{-1} s_+^2 \bar{\lambda}.
\]

The same bound is found for all \( i \in S_\beta \) using the exact same steps. Since the right-hand side above does not depend on \( i \) the claim follows. \( \square \)

C.5 Controlling term \( (\text{III}) \)

Lemma 19. Under assumptions 1, 2, B1, 3 for \( n \) large enough, for any \( \epsilon > 0 \), with probability at least

\[
1 - 4 \cdot \exp \left( -\frac{N \lambda^2}{2} \epsilon^2 + \log(n) \right) - 2(p+1) \cdot \exp \left( -\frac{N \lambda^2}{2(1 \vee c^2)} \epsilon^2 \right)
\]

\[
- p(p+3) \cdot \exp \left( -N \cdot \frac{c_{\text{min}}^2}{2048 s_+^2 \epsilon^2} \right),
\]

which tends to one as \( -\frac{N \lambda^2}{2} \epsilon^2 + \log(n) \rightarrow -\infty \), as \( n \) tends to infinity, we have

\[
\frac{1}{\lambda} \left\| \frac{1}{N} \bar{X}^T W_0^2 \left[ 1 \mid Z \right] (\xi^\top - \xi_0) \right\|_{\infty} \leq \frac{9(1 \vee c)(1 + \epsilon)(p+1)}{4c_{\text{min}}} \cdot \frac{1}{\sqrt{n}} \rho_{-1}^{-1} s_+.
\]

Proof. We have

\[
\left\| \frac{1}{N} \bar{X}^T W_0^2 \left[ 1 \mid Z \right] (\xi^\top - \xi_0) \right\|_{\infty} \leq \left\| \frac{1}{N} \bar{X}^T W_0^2 \left[ 1 \mid Z \right] \right\|_{\infty} \| (\xi^\top - \xi_0) \|_{\infty}
\]

\[
\leq \left\| \frac{1}{N} \bar{X}^T W_0^2 \left[ 1 \mid Z \right] \right\|_{\infty} \| (\bar{\theta}^\top - \bar{\theta}_0) \|_{1}.
\]

Consider the \( i \)th row of the matrix \( \frac{1}{N} \bar{X}^T W_0^2 \left[ 1 \mid Z \right] \),

\[
\left\| \left( \frac{1}{N} \bar{X}^T W_0^2 \left[ 1 \mid Z \right] \right)^T \right\|_{1} \leq \frac{1}{N} \sqrt{n}(n-1) \cdot \frac{1}{4} (1 \vee c)(p+1) = \frac{2+1}{4} (1 \vee c) \frac{1}{\sqrt{n}}.
\]
where we have used that the $i$th column of $X$ has exactly $(n-1)$ non-zero entries, each with value $\sqrt{n}$, each entry of $W_0^2$ is upper bounded by $1/4$ and any row of $[1 \mid Z]$ has $p+1$ entries, each of which is upper bounded by $1 \lor c$. Thus, by Lemma 17, with the prescribed probability,

$$\frac{1}{\lambda} \left\| \frac{1}{N} \tilde{X}^T W_0^2 [1 \mid Z] (\xi^\dagger - \xi_0) \right\|_\infty \leq \frac{9(1 \lor c)(1 + \epsilon)(p + 1)}{4c_{\text{min}}} \cdot \frac{1}{\sqrt{n}} \rho_n^{-1} s_+.$$

\hfill \Box

### C.6 Condition 15b

**Lemma 20.** Under assumptions 1, 2, B1, 3, for $n$ large enough, with probability at least

$$1 - 4 \cdot \exp \left( -\frac{N \bar{\lambda}^2}{18} + \log(n) \right) - 2(p + 1) \cdot \exp \left( -\frac{N \bar{\lambda}^2}{18(1 \lor c^2)} \right) - p(p + 3) \exp \left( -\frac{N}{2048 s_+^2 c} \right),$$

which tends to one as long as $-\frac{N \bar{\lambda}^2}{18} + \log(n) \rightarrow -\infty$, as $n$ tends to infinity, we have

$$\|z_1^\dagger_{1:2nS^c}\|_\infty < 1.$$

**Proof.** By equation (42), Lemmas 3, 18, 19, with the probability given in those Lemmas, for any $\epsilon > 0$,

$$\|z_1^\dagger_{1:2nS^c}\|_\infty \leq \left\{ \|Q_{S^c, S}Q_{S,S}^{-1}\|_\infty + 1 \right\} \epsilon + \left\{ \|Q_{S^c, S}Q_{S,S}^{-1}\|_\infty + 1 \right\} \frac{324(1 \lor (c^2 p))(1 + \epsilon)^2}{c_{\text{min}}^2} \cdot \sqrt{n} \rho_n^{-2} s_+^2 \bar{\lambda} + \left\{ \|Q_{S^c, S}Q_{S,S}^{-1}\|_\infty + 1 \right\} \frac{9(1 \lor c)(1 + \epsilon)(p + 1)}{4c_{\text{min}}} \cdot \frac{1}{\sqrt{n}} \rho_n^{-1} s_+ + \frac{1}{2} \|Q_{S^c, S}Q_{S,S}^{-1}\|_\infty.$$

By Lemma 3, for $n$ sufficiently large, we have $\|Q_{S^c, S}Q_{S,S}^{-1}\|_\infty < 1/2$. Thus, by equation (42), Lemmas 18 and 19, for $n$ sufficiently large, with the prescribed probability,

$$\|z_1^\dagger_{1:2nS^c}\|_\infty \leq \frac{3}{2} \epsilon + \frac{1}{4} \cdot \frac{486(1 \lor (c^2 p))(1 + \epsilon)^2}{c_{\text{min}}^2} \cdot \sqrt{n} \rho_n^{-2} s_+^2 \bar{\lambda} + \frac{27(1 \lor c)(1 + \epsilon)(p + 1)}{8c_{\text{min}}} \cdot \frac{1}{\sqrt{n}} \rho_n^{-1} s_+.$$

Pick $\epsilon = 1/3$, to obtain

$$\|z_1^\dagger_{1:2nS^c}\|_\infty \leq \frac{3}{4} + \frac{486(1 \lor (c^2 p))(4/3)^2}{c_{\text{min}}^2} \cdot \sqrt{n} \rho_n^{-2} s_+^2 \bar{\lambda} + \frac{27(1 \lor c)(4/3)(p + 1)}{8c_{\text{min}}} \cdot \frac{1}{\sqrt{n}} \rho_n^{-1} s_+.$$

The second and third term go to zero as $n$ tends to infinity by assumption B1. Indeed, the second term is assumption B1 exactly. For the third term note that by assumption B1, $\sqrt{n} s_+^2 \bar{\lambda} \rho_n^{-2} = n^{-1/2} \rho_n^{-1} s_+ \cdot n \rho_n s_+ \cdot \bar{\lambda} \rightarrow 0$ as, $n \rightarrow \infty$. On the other hand, by assumption 3, $n \rho_n s_+ \bar{\lambda} \geq C \rho_n^{-1} s_+ \log(n) \rightarrow \infty$. Therefore it must hold that $n^{-1/2} \rho_n^{-1} s_+ \rightarrow 0$. The claim follows. \hfill \Box

### C.7 Proof of Theorem 1

**Proof of Theorem 1.** By Lemma 20, we know that with probability at least as large as

$$1 - 4 \cdot \exp \left( -\frac{N \bar{\lambda}^2}{18} + \log(n) \right) - 2(p + 1) \cdot \exp \left( -\frac{N \bar{\lambda}^2}{18(1 \lor c^2)} \right) - p(p + 3) \exp \left( -\frac{N c_{\text{min}}^2}{2048 s_+^2 c} \right),$$

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property (15b) holds for the construction \((\tilde{\vartheta}^\dagger, \tilde{z}^\dagger)\). Thus, by Lemma 4, \(\hat{S} = S^\dagger\) and in particular \(\hat{S} \cap S^c = \emptyset\).

For the second part of Theorem 1, recall that by equation (41a),

\[
\bar{\vartheta}^\dagger = \bar{\vartheta}_{0,S} + Q_{S,S}^{-1} \left\{ -\frac{1}{N} (\nabla_{\vartheta} \tilde{L}(\bar{\vartheta}_0))_S + \frac{1}{N} \bar{X}^T S W_0^2 [1 \mid Z] (\xi^\dagger - \xi_0) - \tilde{\lambda}^\dagger_{1:2n,S} + \tilde{R}_S \right\}. \tag{48}
\]

Thus, \(S^\dagger\) contains all those indices \(i\) with

\[
\left\| Q_{S,S}^{-1} \left\{ -\frac{1}{N} (\nabla_{\vartheta} \tilde{L}(\bar{\vartheta}_0))_S + \frac{1}{N} \bar{X}^T S W_0^2 [1 \mid Z] (\xi^\dagger - \xi_0) - \tilde{\lambda}^\dagger_{1:2n,S} + \tilde{R}_S \right\} \right\|_\infty < \bar{\vartheta}_{0,i}.
\]

Hence, consider

\[
\left\| Q_{S,S}^{-1} \left\{ -\frac{1}{N} (\nabla_{\vartheta} \tilde{L}(\bar{\vartheta}_0))_S + \frac{1}{N} \bar{X}^T S W_0^2 [1 \mid Z] (\xi^\dagger - \xi_0) - \tilde{\lambda}^\dagger_{1:2n,S} + \tilde{R}_S \right\} \right\|_\infty 
\leq \|Q_{S,S}\|_\infty \left\| -\frac{1}{N} (\nabla_{\vartheta} \tilde{L}(\bar{\vartheta}_0))_S + \frac{1}{N} \bar{X}^T S W_0^2 [1 \mid Z] (\xi^\dagger - \xi_0) - \tilde{\lambda}^\dagger_{1:2n,S} + \tilde{R}_S \right\|_\infty 
\leq 2\rho_n^{-1} \cdot \frac{n - 1}{n - \max\{s_\alpha, s_\beta\}} 
\left\{ \epsilon \bar{\lambda} + \bar{\lambda} 
+ \frac{9(1 + c)(1 + \epsilon)(\rho + 1)}{4c_{\min}} \cdot \frac{1}{\sqrt{n}} \rho_n^{-1} s + \bar{\lambda} 
+ \frac{324(1 + 2\epsilon)(\rho + 1)}{c_{\min}^2} \cdot \sqrt{n} \rho_n^{-2} s^2 \bar{\lambda}^2 \right\}
\]

where we used (39), (44) and Lemmas 18 and 19.

By assumption \(\bar{\lambda} \leq C \cdot \sqrt{\log(n)/N}\) for some \(C > 0\), thus the first two terms in the bracket may be upper bound by \(C \cdot \sqrt{\log(n)/N}\), for a possibly different \(C\). The third term is \(o(1) \cdot 1/n\) by assumption B1 and the last term is \(o(1) \cdot \sqrt{\log(n)/n}\) by assumption B1. Since \((n - 1)/(n - \max\{s_\alpha, s_\beta\}) = O(1)\), the entire right-hand side is less or equal

\[
C \rho_n^{-1} \frac{\sqrt{\log(n)}}{n}.
\]

Multiply (48) by \(\sqrt{n}\) to transition to the unscaled parameters \(\vartheta^\dagger\) and the claim follows. In particular,

\[
C \rho_n^{-1} \frac{\sqrt{\log(n)}}{n}
\]

goes to zero as \(n\) tends to infinity, which implies that for \(n\) large enough, with at least the prescribed probability the construction fulfils (15a) and thus

\[
\hat{S} = S^\dagger = S.
\]

\[\square\]

**D Proofs of Section 1.1 and the simulation results**

Denote the Poisson branching process with mean offspring \(\lambda\) by \(\mathcal{P}(\lambda)\). That is, each vertex in \(\mathcal{P}(\lambda)\) has offspring distribution \(\text{Poi}(\lambda)\). It is well-known that the behaviour of \(ER(\lambda/n)\) is intimately linked to the properties of \(\mathcal{P}(\lambda)\). The following results on Poisson branching processes can be found, for
example, in van der Hofstad (2016), chapter 3, and the links between $P(\lambda)$ and $ER(\lambda/n)$ can be found in chapter 4 of van der Hofstad (2016).

We know that $P(\lambda)$ with $\lambda > 1$ has a positive probability of surviving forever. More precisely, denote by $\eta_\lambda$ the probability that $P(\lambda)$ dies out. Recall from equation (2) that we may calculate $\eta_\lambda$ explicitly by finding the unique solution that is smaller than one to the equation (van der Hofstad 2016, equation 3.6.2)

$$\eta_\lambda = e^{\lambda(\eta_\lambda - 1)}.$$ 

Notice that a solution smaller than one to (2) exists if and only if $\lambda > 1$. Conversely, we define the probability that $P(\lambda)$ survives forever as

$$\zeta_\lambda = 1 - \eta_\lambda.$$ (49)

Denote by $C_{\text{max}}$ the largest connected component of $ER(\lambda/n)$, where we suppress the dependence of $C_{\text{max}}$ on $\lambda$ and $n$ in our notation as it will be clear from the context. Then, for $\lambda > 1$, the size $|C_{\text{max}}|$ of the giant component will concentrate closely around $\zeta_\lambda n$ (van der Hofstad 2016, Theorem 4.8).

We now prove Proposition 1. Proposition 1 follows from the following deep result on the phase transition of $ER(\lambda/n)$, which can be found in van der Hofstad (2021).

**Theorem 8** (Phase transition in Erdős-Rényi random graphs, Theorem 2.33 in van der Hofstad (2021), abbreviated). Fix $\lambda > 0$ and let $C_{\text{max}}$ be the largest connected component of the Erdős-Rényi graph $ER(\lambda/n)$. Then,

$$\frac{|C_{\text{max}}|}{n} \xrightarrow{P} \zeta_\lambda,$$ (50)

where $\zeta_\lambda$ is the survival probability of a Poisson branching process with mean offspring $\lambda$. In particular $\zeta_\lambda > 0$ precisely when $\lambda > 1$. Further, for $\lambda > 0$, with $\eta_\lambda = 1 - \zeta_\lambda$,

$$\frac{|E(C_{\text{max}})|}{n} \xrightarrow{P} \frac{1}{2} \lambda(1 - \eta_\lambda^2).$$ (51)

**Proof of Proposition 1.** Proposition 1 follows from Theorem 8 by repeated application of Slutzký’s Theorem. First, by (50) and Slutzký,

$$\frac{|C_{\text{max}}|-1}{n} \xrightarrow{P} \zeta_\lambda.$$ 

Thus, by Slutzký,

$$\left( \frac{|C_{\text{max}}|}{2} \right)^2 \xrightarrow{P} \frac{1}{2} \cdot \zeta_\lambda^2.$$ 

Now, a final application of Slutzký’s Theorem together with (50) and (51) yields,

$$\frac{\hat{p}_{\text{max}}}{\hat{p}} = \frac{|E(C_{\text{max}})|}{\binom{|C_{\text{max}}|}{2}} \cdot \frac{n}{\lambda} = \frac{|E(C_{\text{max}})|}{\binom{|C_{\text{max}}|}{2}} \cdot \frac{n^2}{\binom{|C_{\text{max}}|}{2}} \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda} \cdot \frac{1 - \eta_\lambda^2}{\zeta_\lambda^2} \xrightarrow{P} 1 + \eta_\lambda \cdot \frac{1}{1 - \eta_\lambda},$$

where we used the definition of $\zeta_\lambda$ in the last step. \(\square\)

Below in Table 4, the detailed results of the simulation on the stochastic block model are provided.
| $\|P\|_2$ | Size giant | $\rho(\hat{\theta}_{max})$ |
|---------|------------|-----------------|
| 1       | 0.0018 (0e+00) | 2208 (163.9)    |
| 2       | 0.0433 (1e-03) | 60.57 (2.1281)  |
| 3       | 0.5824 (4e-04) | 2.4345 (0.0015) |
| 4       | 0.7968 (2e-04) | 1.5102 (0.0004) |
| 5       | 0.8927 (1e-04) | 1.2406 (0.0003) |
| 6       | 0.9405 (1e-04) | 1.1265 (0.0002) |
| 7       | 0.9660 (1e-04) | 1.0704 (0.0002) |
| 8       | 0.9802 (0e+00) | 1.0404 (0.0002) |
| 9       | 0.9883 (0e+00) | 1.0236 (0.0002) |
| 10      | 0.9930 (0e+00) | 1.0140 (0.0002) |
| 11      | 0.9958 (0e+00) | 1.0084 (0.0002) |
| 12      | 0.9975 (0e+00) | 1.0050 (0.0002) |

Table 4: Average size of the giant component and biases $\rho$ for different values of $\|P\|_2$. The respective standard deviations, rounded to four significant digits, are given in parenthesis.