EXTREMIZABILITY OF FOURIER RESTRICTION TO THE PARABOLOID

BETSY STOVALL

ABSTRACT. In this article, we prove that nearly all valid, scale-invariant Fourier restriction inequalities for the paraboloid in $\mathbb{R}^{1+d}$ have extremizers and that $L^p$-normalized extremizing sequences are precompact modulo symmetries. This result had previously been established for the case $q = 2$. In the range where the boundedness of the restriction operator is still an open question, our result is conditional on improvements toward the restriction conjecture.

1. Introduction

There has been substantial recent attention paid to the problem of determining whether equality is possible for various inequalities in harmonic analysis. In the case of Fourier restriction inequalities, essentially all of the known results are for $L^2$-based Fourier restriction, wherein it is possible to use Hilbert space techniques and Plancherel; an excellent survey of recent results in this vein may be found in [10]. An exception is a result of Christ–Quilodrán [7] stating that Gaussian functions are not maximizers for Fourier restriction to the paraboloid except in the $L^1$ case and possibly in the Stein–Tomas–Strichartz case. The result of [7], however, leaves open the question of whether maximizers actually exist for the intermediate Lebesgue space bounds. The purpose of this article is to establish the existence of extremizers and precompactness of extremizing sequences for all valid, nonendpoint, $L^p$ to $L^q$ restriction inequalities for the paraboloid, including, conditionally, the conjectural ones. We note that the existence of a second endpoint restriction inequality, i.e. other than trivial one at $L^1$, would be rather a surprise to the harmonic analysis community, and thus it is expected that our result is sharp.

We start with a quick recap of the current state of the restriction problem, which will give us an opportunity to define the notation and terminology needed to state our results.

In the late 1960s, Stein conjectured that the restriction operator

$$\mathcal{R} f(\xi) := \hat{f}(|\xi|^2, \xi)$$

extends as a bounded operator from $L^p(\mathbb{R}^{1+d})$ into $L^q(\mathbb{R}^d)$ for all pairs $(p, q)$ satisfying

$$p = \frac{dq}{d+2} \quad \text{and} \quad q > p.$$  \hspace{1cm} (1.1)

An equivalent formulation is that the extension operator

$$\mathcal{E} f(t, x) = \int_{\mathbb{R}^d} e^{i(t,x)(|\xi|^2, \xi)} f(\xi) \, d\xi,$$

Key words and phrases. Fourier restriction, Fourier extension, extremizer, maximizer, profile decomposition.
extends as a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^{1+d})$ for all pairs $(p, q)$ satisfying
\[ q = \frac{d+2}{d} p' \quad \text{and} \quad q > p. \tag{1.2} \]
Conditions (1.1) and (1.2) are known to be necessary for boundedness of $\mathcal{R}$ and $\mathcal{E}$, respectively. As of this writing, the above described restriction/extension conjecture is settled when $d = 1$, by Stein and Fefferman, and is open in all higher dimensions. More precisely, in higher dimensions, it is solved for $q > q(d) = \frac{2(d+3)}{d+1}$ when $d = 2$, and for $q > q(d)$, for some (explicit, yet complicated) $q(d) = \frac{2(d+3)}{d+1}$ when $d \geq 3 \ [25, \ 26, \ 28, \ 29].$

Classical symmetries of both the Fourier transform and the paraboloid lead to a wealth of symmetries for the extension and restriction operators. These symmetries, in turn, are of paramount importance in the study of uniqueness and compactness questions for maximizers and near maximizers of the Lebesgue bounds for $\mathcal{E}$ and $\mathcal{R}$.

To be more precise, by a symmetry of the extension operator $\mathcal{E} : L^p_\xi \to L^q_{t,x}$, we mean a pair $(S, T)$ with $S$ an isometry of $L^p_\xi$, $T$ an isometry of $L^q_{t,x}$, and $\mathcal{E} \circ S = T \circ \mathcal{E}$. We let $\mathcal{S}_p$ denote the group of all symmetries of $\mathcal{E} : L^p_\xi \to L^q_{t,x}$. For simplicity, we will often abuse notation by associating the symmetry $(S, T)$ with its first coordinate, $S$. Under this convention, $\mathcal{S}_p$ contains the dilations, $f \mapsto \lambda f$ for $\lambda \neq 0$, the frequency translations, $f \mapsto f(\cdot - \xi_t)$, the space-time modulations, $f \mapsto e^{i(t_0 x_0)(|\cdot|^2 - \cdot)} f$, and compositions of these three. There are other symmetries, such as rotations and multiplication by unimodular constants, but these generate compact subgroups of $\mathcal{S}_p$, and therefore play no role in our analysis. We let $\hat{\mathcal{S}}_p$ denote the subgroup of $\mathcal{S}_p$ generated by the aforementioned noncompact symmetry groups. Finally, we note that if $(S, T)$ is a symmetry of $\mathcal{E} : L^p_\xi \to L^q_{t,x}$, then $(T^*, S^*)$ is a symmetry of the corresponding restriction operator, $\mathcal{R} : L^q_{t,x} \to L^{p'}_{\xi}$. Fix a pair $(p, q)$ for which the extension operator is bounded, and let $A_p := \|\mathcal{E}\|_{L^p_\xi \to L^q_{t,x}}$. In this article we take up two natural questions: Do there exist nonzero functions that achieve equality in the estimates
\[ \|\mathcal{E}f\|_q \leq A_p \|f\|_p, \quad \text{and} \quad \|\mathcal{R}g\|_{p'} \leq A_p \|g\|_{q'}, \tag{1.3} \]
and, Must a function that nearly achieves equality be close to one that achieves equality? Under the additional condition that there exists an exponent pair $(\bar{p}, \bar{q})$ satisfying $\bar{p} > p$ at which $\mathcal{E}$ is bounded, we answer both of these questions in the affirmative, and show in addition that the intersection of the $L^p_\xi$ (resp., $L^q_{t,x}$) unit sphere with the set of all $f$ (resp. $g$) achieving equality in (1.3) is compact modulo symmetries.

To state our result more precisely, we call a nonzero $L^p_\xi$ function $f$ an (extension) extremizer if it achieves equality in (1.3) and a nonzero $L^p_\xi$ sequence $\{f_n\}$ extremizing if $\lim_{n \to \infty} \|\mathcal{E}f_n\|_q = A_p$; restriction extremizers are defined analogously. For questions of compactness, it is most natural to work with $L^p_\xi$-normalized extremizing sequences, that is, extremizing sequences $\{f_n\}$ with $\|f_n\|_p \equiv 1$.

**Theorem 1.1.** Assume that the extension operator extends as a bounded operator from $L^p_\xi$ to $L^q_{t,x}$ for some $1 < p_0 < q_0 = \frac{d+2}{d} p_0'$. Let $1 < p < p_0$ and $q = \frac{d+2}{d} p'$. Define $A_p$ to be the $L^p_\xi \to L^q_{t,x}$ operator norm of $\mathcal{E}$. If $\{f_n\} \subseteq L^p_\xi$ is an $L^p_\xi$-normalized extremizing sequence, then, after passing to a subsequence, there exist
symmetries \( \{S_n\} \subseteq \tilde{S}_p \) such that \( S_n f_n \to f \), in \( L^p_{t,x} \), for some extremizer \( f \) of the extension inequality in (1.3).

By duality and strict convexity of \( L^p \) when \( 1 < p < \infty \), we have the analogous result for the restriction operator. Details of this deduction are given in Section 6.

**Corollary 1.2.** Assume that the restriction operator extends as a bounded operator from \( L^p_{t,x} \) to \( L^q_{t,x} \), for some \( \infty > q > p_0 = \frac{d+2}{d+1} p' \). Let \( 1 < p < p_0 \) and \( q = \frac{d+2}{d+1} p' \). Then any \( L^p_{t,x} \)-normalized extremizing sequence for \( R \) has a subsequence that, modulo symmetries, converges in \( L^p_{t,x} \) to an extremizer of the restriction inequality in (1.3).

That \( \mathcal{E} : L^1_\xi \to L^\infty_{t,x} \) and \( \mathcal{R} : L^1_{t,x} \to L^\infty_\xi \) possess extremizers and that extremizing sequences for these operators need not be compact are both elementary; indeed, for both, one need only consider sequences of the form \( \{\phi + \phi(\cdot + ne_1)\} \), for some \( 0 \leq \phi \in L^1 \) with \( \phi \neq 0 \).

The Stein–Tomas–Strichartz case \( p = 2 \) has been well-studied, and extremizers are known to exist in all dimensions [3, 9, 15, 25]. It is conjectured that radial Gaussians are, up to symmetries, the unique extremizers of the \( L^2_\xi \to L^2_{t,x} \) inequality, but this is only known in dimensions \( d = 1, 2 \) [9], wherein the exponent \( q = \frac{d+2}{d+1} 2' \) is an even integer. Curiously, it is known in all dimensions that radial Gaussians are not extremal for any \( L^p_\xi \to L^q_{t,x} \) inequality for \( \mathcal{E} \) unless \( p \in \{1, 2\} \) [7], but outside of the cases \( p = 1, 2 \), extremizers had not been previously shown to exist.

Unfortunately, the aforementioned proofs of existence of extremizers to the Stein–Tomas–Strichartz inequality rely on Plancherel and the Hilbert space structure of \( L^2_\xi \). The proof that seems most amenable to generalization is that of [25] (see also [20]), which applies the linear profile decomposition of [2, 5, 21].

We turn to a heuristic overview of the statement and proof of the \( L^2_\xi \)-based linear profile decomposition in general dimensions. (We give a formal statement later.)

This overview will allow us to explain some of the difficulties in adapting the \( L^2_\xi \)-based arguments to the general case. Set \( q_2 := \frac{2(d+2)}{d} \). By Tao’s bilinear extension estimate for the paraboloid [28] and an adaptation of the bilinear to linear argument of Tao–Vargas–Vega [29], one can prove an “improved Strichartz inequality,” which implies that if \( \{f_n\} \) is an \( L^2_\xi \)-normalized sequence with \( \|\mathcal{E} f_n\|_{q_2} \not\to 0 \), then there is a nontrivial contribution coming from a portion of \( f_n \) well-adapted to a ball [2]. After passing to a subsequence and applying a suitable sequence \( \{S_n\} \) of symmetries, there is a nontrivial weak limit: \( S_n^{-1} f_n \rightharpoonup f \neq 0 \). For large \( n \), \( f_n - S_n f_n \) has a smaller \( L^2_\xi \) norm, via arguments that are elementary in Hilbert spaces. By repeating this process, we can, after passing to a subsequence, write the extension \( \mathcal{E} f_n \) as a sum of a finite number of asymptotically (pairwise) orthogonal profiles, together with an error that is small in \( L^2_{t,x} \). One can show, either directly [3, 21] or by using local smoothing estimates [17], that the \( L^p_{t,x} \) norms of asymptotically orthogonal bubbles decouple. Thus, after passing to a subsequence, for each \( J \geq 1 \) we may
decompose \( f_n = \sum_{j=1}^{J} S_n^j \phi^j + w_n^f \), where
\[
\lim_{n \to \infty} \|f_n\|_2^2 - (\sum_j \|\phi_j\|_2^2 + \|w_n^f\|_2^2) = 0
\]
\[
\lim_{n \to \infty} \|E f_n\|_{q_2}^2 - (\sum_j \|E \phi_j\|_{q_2}^2 + \|w_n^f\|_{q_2}^2) = 0,
\]
and, moreover, \( \lim_{J \to \infty} \lim_{n \to \infty} \|E w_n^f\|_{q_2} \to 0 \).

If the sequence \( f_n \) is extremizing, strict convexity (coming from \( 2 < q_2 \)) dictates that in fact there is only one profile and the error \( w_n \) tends to zero in \( L_{\xi}^2 \) \[28\].

Unfortunately, for any \( p \neq 2 \), we hit a snag early on, because it is possible to increase the limit of the norms of a sequence in \( L_p^\xi \) by subtracting the weak limit of the sequence; a simple example is given in \[29\]. It is natural to try to subtract a “positive” quantity from the sequence to reduce the \( L_p^\xi \) norm (such as the portion of each \( f_n \) that is well-adapted to a ball), but this presents some challenges since \( E \) is not a positive operator; in particular, we must keep the spacetime modulations under control, and in order to use convexity, it is crucial that we have equality in both estimates in \((1.4)\). The need for this precision is also why the general framework for profile decompositions in Banach spaces found in \[27\] does not seem to directly yield our result.

Our approach is to first control the positive symmetries: the dilations and frequency translations. To gain this control, we generalize the improved Strichartz inequality of \[2, 5, 21\] to \( L_p^\xi \) (Lemma 2.2). This inequality controls the extension with a nontrivial positive operator (2.5), whose norm can be reduced by deleting the portions of each \( f_n \) well-adapted to balls. The portion of each \( f_n \) which contributes to \( E f_n \) is, by convexity, carried on a controllable number of balls (Proposition 2.1), and we then use convexity again to show that the major portion of each \( f_n \) is, in fact, carried on a single ball (Proposition 3.1). Applying a symmetry, we then have an extremizing sequence in \( L_p^\xi \) that is nearly bounded with compact support, which means that it is almost in \( L^2_{\xi} \). (This part of the argument is closely related to Lieb’s method of missing mass \[18\]; see also \[11\].)

Truncating introduces some error, but lets us apply the \( L_p^\xi \)-based profile decomposition. The only profiles that can arise are spacetime translates, because the compact support and boundedness of our truncations rule out dilations and frequency translations. In order to reduce both the \( L_p^\xi \) and the \( L_{q,t,x}^q \) norm, we need to take care with how we extract \( L_p^\xi \) bubbles from this decomposition; we do this by carefully truncating on the spacetime side and bounding a vector-valued operator (Lemma 4.3). Finally, this yields an \( L_p^\xi \)-based profile decomposition (Proposition 4.1) in which both the \( L_p^\xi \) norms of the profiles and the \( L_{q,t,x}^q \) norms of the extensions are sufficiently decoupled that we can, in Section 5, prove Theorem 1.1.

**Terminology.** For nonnegative numbers \( A \) and \( B \), we will write \( A \lesssim B \) to mean that \( A \leq CB \) for a constant \( C \) that depends only on \( d, p_0, p, A_{p_0} \) but that is otherwise allowed to change from line to line. A dyadic interval is an interval of the form \([m2^{-n}, (m+1)2^{-n}]\), \( m, n \in \mathbb{Z} \), and a dyadic cube is a product of dyadic intervals all having the same length. We denote the set of all dyadic cubes of side length \( 2^{-k} \) by \( D_k \), and an individual dyadic cube will typically be denoted \( \tau \). To simplify later statements, we will consider the empty set to be a dyadic cube, \( \emptyset \in D_\infty \). We
Acknowledgements. The author would like to thank Mike Christ and Taryn Flock for valuable conversations in the early stages of this project. These discussions helped inspire the eventual idea for the proof. She would also like to thank the anonymous referee for a number of thoughtful and extremely helpful comments.

This work was supported in part by NSF DMS-1600458; in addition, part of this research was supported by NSF DMS-1440140, while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring Semester of 2017.

2. Positive profiles

In this section, we prove the following proposition, which allows us to nibble away at the absolute value of a function, while reducing its extension in a quantitative way.

Proposition 2.1. There exist a sequence \( \rho_j \searrow 0 \) such that for every \( f \in L^p \), there exists a sequence \( \tau_j \) of dyadic cubes such that if the sequences \( g_j, r_j \) are defined inductively by setting

\[
r^0 := f, \quad g^j := r^{j-1} \chi_{\tau_j \cap \{|f| < \rho_j^{-1} \|f\|_p |\tau_j|^{-1/p}\}}, \quad r^j := r^{j-1} - g^j,
\]

then \( \| \mathcal{E}_h \|_q \leq \rho_j \|f\|_p \) for all \( j \) and all measurable functions \( |h| \leq |r^j| \).

The main step in proving the proposition is to prove the following lemma.

Lemma 2.2. Let \( f \in L^p \). For some \( 0 < \theta < 1 \) and \( c_0 > 0 \),

\[
\|\mathcal{E} f\|_q \lesssim \sup_{k \in \mathbb{Z}} \sup_{\tau \in D_k} \sup_{l \geq 0} 2^{-c_0 l} \|f_{\tau,l}\|_p \|f\|_p^{1-\theta}, \tag{2.1}
\]

where \( f_{\tau,l} \) equals \( f \) multiplied by the characteristic function of

\( \tau \cap \{|f| < 2^l \|f\|_p |\tau|^{-1/p}\} \).

Assuming the lemma for a moment, we give the short proof of Proposition 2.1.

Proof of Proposition 2.1. Given \( 0 \neq f \in L^p \) and a sequence \( \{\rho_j\} \subseteq (0, \infty) \), we may, by a routine application of the Dominated Convergence Theorem, select dyadic cubes \( \tau^j \) so that

\[
\|g^j\|_p = \max_{\text{dyadic}} \|r^{j-1} \chi_{\{|f| < \rho_j^{-1} \|f\|_p |\tau|^{-1/p}\}}\|_p,
\]
in the notation of the proposition. We will inductively construct a sequence \( \rho_j \searrow 0 \) such that the conclusion of the proposition holds for every \( f \) with this choice of dyadic cubes.

We assume that we are given \( k \geq 0 \) and integers \( 0 = J_0 < \cdots < J_k \) such that the conclusions of the proposition hold for the sequence

\[
\rho_{J_i+1} := \cdots := \rho_{J_i+1} := 2^{-i} A_p, \quad i < k, \quad \rho_{J_j} := 2^{-k} A_p, \quad j > J_k \tag{2.2}
\]

and corresponding dyadic cubes. (In the base case, \( k = 0 \), the assumption follows from the definition of \( A_p \).)

Let \( J_{k+1} > J_k \) be a sufficiently large integer, to be determined in a moment, and consider a function \( f \) with \( \|f\|_p = 1 \). By monotonicity of the remainder terms, the
inductive step will be complete once we prove that for $J_{k+1}$ sufficiently large, the extension of a measurable $|h| \leq |r_{J_{k+1}}|$ obeys a better bound, $\|\mathcal{E}h\|_q \leq 2^{-(k+1)}A_p$.

By the maximality property of our cubes,

$$\max_{\tau} \| p^{J_{k+1}} \chi_{\tau}(\{ |f| < \rho_{J_{k+1}}^{-1} |r|^{-\frac{1}{p}} \}) \|_p \leq \min_{J_k < J \leq J_{k+1}} \| g_j \|_p.$$  

By construction, the $g_j$ have pairwise disjoint supports and $\sum_j |g_j| \leq |f|$. Therefore

$$(J_{k+1} - J_k)^{\frac{1}{p}} \min_{J_k < J \leq J_{k+1}} \| g_j \|_p \leq \left( \sum_{j=J_{k+1}}^{J_{k+1}} \| g_j \|_p \right)^{\frac{1}{p}} \leq \| f \|_p,$$

so

$$\max_{\tau} \| p^{J_{k+1}} \chi_{\tau}(\{ |f| < \rho_{J_{k+1}}^{-1} |r|^{-\frac{1}{p}} \}) \|_p \leq (J_{k+1} - J_k)^{-\frac{1}{p}}. \quad (2.3)$$

Now let $L = L_k$ be an integer, sufficiently large that $B2^{-c_0L} \leq 2^{-(k+1)}A_p$, with $B$ the implicit constant in (2.1). We may assume that $2^{-L} \leq 2^{-k}A_p = \rho_{J_{k+1}}$. Let $|h| \leq |p^{J_{k+1}}|$ be a measurable function. By (2.1),

$$\|\mathcal{E}h\|_q \leq B \max_{\text{dyadic}} \| h \chi_{\tau}(\{ |f| < 2^{-L} |r|^{-\frac{1}{p}} \}) \|_p.$$

Each dyadic $\tau$ is covered by at most $C(2^L \rho_{J_{k+1}})^{pd}$ dyadic $\tau'$ whose diameter equals $(2^{-L} \rho_{J_{k+1}})^p$ times the diameter of $\tau$, so

$$\max_{\text{dyadic}} \| h \chi_{\tau}(\{ |f| < 2^{-L} |r|^{-\frac{1}{p}} \}) \|_p \lesssim (2^L \rho_{J_{k+1}})^d \max_{\text{dyadic}} \| h \chi_{\tau'}(\{ |h| < \rho_{J_{k+1}} |r|^{-\frac{1}{p}} \}) \|_p.$$

Thus

$$\|\mathcal{E}h\|_q \leq \max\{2^{-(k+1)}A_p, C_k (J_{k+1} - J_k)^{-\frac{1}{p}} \},$$

and taking $J_{k+1}$ sufficiently large completes the proof.  

The proof of Lemma 2.2 is an adaptation of the argument of Bézout–Vargas [2], which was carried out there in the case $p = 2$. See also [4] for earlier results in a similar vein.

**Proof of Lemma 2.2** Throughout the proof, we assume that $\|f\|_p = 1$.

Tao’s bilinear adjoint restriction theorem [28] for the paraboloid states that

$$\|\mathcal{E}f_1 \mathcal{E}f_2\|_r \lesssim \|f_1\|_2 \|f_2\|_2,$$

for $r > \frac{d}{d+2}$ and $f_1, f_2$ supported on cubes in $D_0$ that are separated by a distance of at least 1, while any valid $L^p_x$ to $L^2_{t,x}$ linear estimate trivially yields a bilinear estimate

$$\|\mathcal{E}f_1 \mathcal{E}f_2\|_r \lesssim \|f_1\|_p \|f_2\|_p.$$

Interpolating these two bilinear estimates and rescaling implies that for any nonendpoint pair $(p, q)$ for which [13] holds, there exists some $s < p$ such that

$$\|\mathcal{E}f_1 \mathcal{E}f_{\tau'}\|_{q/2} \lesssim 2^{2^k(d+4-s)} \|f_1\|_s \|f_{\tau'}\|_s,$$

whenever $f_1, f_{\tau'}$ are supported on cubes $\tau, \tau' \in D_k$ separated by a distance of at least $2^{-k}$.  


Recalling \[29\], for \(\tau, \tau' \in D_k\), we say that \(\tau \sim \tau'\) if \(C_d2^{-k} \leq \text{dist}(\tau, \tau') \leq 2C_d2^{-k}\), with \(C_d\) sufficiently large. For every \(\xi \neq \xi'\), there exist \(k\) and \(\tau \sim \tau' \in D_k\) such that \(\xi \in \tau\) and \(\xi' \in \tau'\). Thus
\[
\|E_f\|_q^2 = \|EfEf\|_q^2 = \| \sum_{\tau \sim \tau' \in D_k} Ef\tau Ef\tau' \|_2^4,
\]
where \(f_\tau = \chi_\tau f\), with \(\chi_\tau\) a cutoff supported on \(\tau\). The product \(Ef\tau Ef\tau'\) has frequency support in
\[
\{((|\xi'|^2 + |\xi'|^2), \xi + \xi') : \xi \in \tau, \xi' \in \tau'\},
\]
which, for \(C_d\) sufficiently large, is contained in a parallellelepiped
\[
R_{\tau, \tau'} \subseteq \{(t, \zeta) : \zeta \in \tau + \tau', ||t\zeta| - \frac{1}{2}|\xi|^2| \sim 2^{-2k}\}.
\]
These parallellepipeds are finitely overlapping as \(k\) and \(\tau \sim \tau'\) vary, so by the almost orthogonality lemma (Lemma 6.1) of \[29\],
\[
\|Ef\|_q^2 \lesssim \left( \sum_{k} \sum_{\tau \sim \tau' \in D_k} \|Ef\tau Ef\tau'\|_2^1 \right)^t,
\]
where \(t = \min\{\frac{q}{q'}, \frac{q}{2q'}\}\). From our bilinear restriction inequality (2.4),
\[
\|Ef\tau Ef\tau'\|_4 \lesssim 2^{kd(\frac{1}{2} - \frac{1}{q})} \|f_\tau\|_s \|f_\tau\|_s \lesssim 2^{kd(\frac{1}{2} - \frac{1}{q})} \|f_{\tau'}\|_s^2,
\]
where \(\tau''\) is a slightly larger cube containing both \(\tau\) and \(\tau'\). Thus, after reindexing,
\[
\|Ef\|_q^2 \lesssim \left( \sum_{k} \sum_{\tau \in D_k} 2^{kd(\frac{1}{2} - \frac{1}{q})} \|f_\tau\|_s^{2t} \right)^\frac{1}{t}.
\]
Arithmetic shows that \(2t > p\), and we recall that \(p > s\).

This completes the proof of Lemma 2.2 modulo the inequality
\[
\left( \sum_{k} \sum_{\tau \in D_k} |\tau|^{-2t(\frac{1}{2} - \frac{1}{p})} \|f_\tau\|_s^{2t} \right)^\frac{1}{t} \lesssim \sup_{k \in \mathbb{Z}} \sup_{\tau \in D_k} \sup_{l \geq 0} 2^{-c_0l} \|f_{\tau, l}\|_p \|f\|_p^{1 - \theta},
\]
which we take up in the next lemma.

\[\square\]

**Lemma 2.3.** Assume that \(\|f\|_p = 1\). If \(2t > p > s\) and \(\max\{\frac{q'}{q}, \frac{q'}{2q'}\} < \theta < 1\), then for some \(c_0 > 0,\)
\[
\left( \sum_{k} \sum_{\tau \in D_k} |\tau|^{-2t(\frac{1}{2} - \frac{1}{p})} \|f_\tau\|_s^{2t} \right)^\frac{1}{t} \lesssim \sup_{k \in \mathbb{Z}} \sup_{\tau \in D_k} \sup_{l \geq 0} 2^{-c_0l} \|f_{\tau, l}\|_p^\theta. \tag{2.6}
\]

**Proof of Lemma 2.3** We will prove the slightly stronger (since \(|f_\tau| \leq \|f_{\tau, l}\|\)) inequality
\[
\left( \sum_{k} \sum_{\tau \in D_k} |\tau|^{-2t(\frac{1}{2} - \frac{1}{p})} \|f_\tau\|_s^{2t} \right)^\frac{1}{t} \lesssim \sup_{k \in \mathbb{Z}} \sup_{\tau \in D_k} \sup_{l \geq 0} 2^{-c_0l} \|f_{\tau, l}\|_p^\theta, \tag{2.7}
\]
where the \(f_{\tau, l}^i\) are defined inductively by \(f_{\tau, 0}^i := f_{\tau, 0}\) and \(f_{\tau, l}^i := f_{\tau, l} - f_{\tau, l+1}^{-1}, l \geq 1\).

For any \(c_1 > 0\), by using disjointness of the supports of the \(f_{\tau, l}^i\), then H"older, then two more applications of H"older (together with summability of \(2^{-t(\frac{1}{2} - \frac{1}{p})}(2t-s)\)),
\[
\begin{align*}
\sum_{k} \sum_{\tau \in D_k} |\tau|^{-2t(\frac{1}{2} - \frac{1}{p})} \|f_\tau\|_s^{2t} & \lesssim \sum_{k} \sum_{\tau \in D_k} |\tau|^{-2t(\frac{1}{2} - \frac{1}{p})} \left( \sum_{l \geq 0} \|f_{\tau, l}^i\|_s^{2t} \right)^\frac{2t}{2t - (2t - s)} \\
& \lesssim \sup_{k} \sup_{\tau \in D_k} \sup_{l \geq 0} \left( 2^{-c_1l} \|f_{\tau, l}\|_p^\theta \right)^{1 - \theta}.
\end{align*}
\]
Proposition 3.1. For each $\varepsilon > 0$, there exist $\delta > 0$ and $R < \infty$ such that for all $f \in L^p$ satisfying $\|E f\|_q > (A_p - \delta)\|f\|_p$, there exists a symmetry $S \in \mathcal{S}_p$ such that

$$\|Sf\|_{L^p(\{|f| > R\} \cup \{|Sf| > R\})} < \varepsilon \|f\|_p.$$ 

The symmetry $S$ may be chosen to depend only on $f$, and not on $\varepsilon$.

Proof of Proposition 3.1. We begin with the post hoc deduction of the independence of the symmetry from $\varepsilon$. We fix a function $f \in L^p_\varepsilon$, which we may assume has $\|f\|_p = 1$ and $\|E f\|_p \geq \frac{1}{2} A_p$. By Lemma 2.2, there exists a dyadic cube $\tau$ with

$$\|f_\tau\|_p := \|f \chi_{\{f| \leq |\tau|^{-\frac{1}{2}}\}}\|_p \gtrsim 1.$$  

By applying a symmetry to $f$, we may assume that $\tau$ is the unit cube. Now suppose that another symmetry, $Sf$ were to satisfy

$$\|Sf\|_{L^p(\{|f| > R\} \cup \{|Sf| > R\})} < \varepsilon,$$  

3. **Frequency localization**

By Proposition 2.1 for each $0 < A < A_p$ and $\varepsilon > 0$, there exists an integer $J$ such that for all $0 \neq f \in L^p$ with $\|E f\|_q \geq A\|f\|_p$, $f = \sum_{j=1}^J g^j + r^j$, where the remainder $r^j$ contributes an $\varepsilon$ portion of the extension, $\|E r^j\|_q \leq \varepsilon \|f\|_p$, and the $g^j$ are $\varepsilon$-well-adapted to dyadic cubes $\tau^j$, in the sense that $\supp g^j \subseteq \varepsilon^{-1} \tau^j$ and $\|g^j\| \leq \varepsilon^{-1} \|r^j\| f\|_p$. Our next task is to control these cubes. The following proposition states that in the special case $A = A_p$, we can take $j$ to be 1, and the cube to be independent of $\varepsilon$. (This is easily seen to be false for other values of $A$.)

We recall from the introduction that $\mathcal{S}_p$ denotes the subgroup of the group $\mathcal{G}_p$ of symmetries of $E : L^p_\varepsilon \to L^p_\varepsilon$ generated by the dilations, the frequency translations, and the space-time translations.

Proposition 3.1. For each $\varepsilon > 0$, there exist $\delta > 0$ and $R < \infty$ such that for all $f \in L^p$ satisfying $\|E f\|_q > (A_p - \delta)\|f\|_p$, there exists a symmetry $S \in \mathcal{S}_p$ such that

$$\|Sf\|_{L^p(\{|f| > R\} \cup \{|Sf| > R\})} < \varepsilon \|f\|_p.$$ 

The symmetry $S$ may be chosen to depend only on $f$, and not on $\varepsilon$.

Proof of Proposition 3.1. We begin with the post hoc deduction of the independence of the symmetry from $\varepsilon$. We fix a function $f \in L^p_\varepsilon$, which we may assume has $\|f\|_p = 1$ and $\|E f\|_p \geq \frac{1}{2} A_p$. By Lemma 2.2, there exists a dyadic cube $\tau$ with

$$\|f_\tau\|_p := \|f \chi_{\{f| \leq |\tau|^{-\frac{1}{2}}\}}\|_p \gtrsim 1.$$  

By applying a symmetry to $f$, we may assume that $\tau$ is the unit cube. Now suppose that another symmetry, $Sf$ were to satisfy

$$\|Sf\|_{L^p(\{|f| > R\} \cup \{|Sf| > R\})} < \varepsilon,$$  


for some sufficiently small $\varepsilon$. We will show that that (3.2) holds with $S$ equal to the identity and $R$ replaced by some slightly larger $R'$ (given below). As modulations leave the absolute value invariant, we may assume that $Sf = \lambda^{\frac{d}{2}} f(\lambda(\cdot - \xi_0))$. Since $|Sf_0| \leq |f_0|$, 

$$
\|f_0\|_{L^p(|\{\xi + \lambda \xi_0 > \lambda R\} \cup \{|f_0| > \lambda^{-\frac{d}{2}} R\})} = \|Sf_0\|_{L^p(|\{\xi > R\} \cup \{|Sf_0| > R\})} < \varepsilon. \tag{3.3}
$$

Hence by (3.1) and the triangle inequality, $\|f_0\|_{L^p(\bigcap_{n=1}^\infty E_n)} \sim 1$, where

$$
E_1 := \tau, \quad E_2 := \{|f| \leq 1\}, \quad E_3 := \{|\xi + \lambda \xi_0| < \lambda R\}, \quad E_4 := \{|f| < \lambda^{-\frac{d}{2}} R\}.
$$

By Hölder’s inequality, $\|f_0\|_{L^p(E_2 \cap E_4)} \lesssim \lambda^{-\frac{d}{2}} R$ and $\|f_0\|_{L^p(E_2 \cap E_3)} \lesssim (\lambda R)^{\frac{1}{p}}$, so $R^{-1} \lesssim \lambda \lesssim R^{\frac{d}{p}}$. Since $E_1 \cap E_3 \neq \emptyset$, $|\xi_0| \lesssim R + \lambda^{-1}$. Therefore

$$
\{|\xi + \lambda \xi_0| > \lambda R\} \supseteq \{|\xi| > CR^{\frac{d}{p} + 1}\} \quad\text{and}\quad \{|f| > \lambda^{-\frac{d}{2}} R\} \supseteq \{|f| > CR^{\frac{d}{p} + 1}\}.
$$

Inequality (3.2) (see also (3.3)) thus implies that

$$
\|f\|_{L^p(|\{\xi > R'\} \cup \{|f| > R'\})} \lesssim \lambda R^{\frac{d}{p} + 1}.
$$

with $R' = CR^{\frac{d}{p} + 1}$.

Now we turn to the proof of the main conclusion of the proposition.

Were the proposition to fail, there would exist $\varepsilon > 0$ and a sequence $\{f_n\} \subseteq L^p$, satisfying $\|f_n\|_{L^p} \equiv 1$ and $\|\mathcal{E} f_n\|_{L^p} \to A_p$, but such that

$$
\liminf\|S_n f_n\|_{L^p(|\{\xi > n\} \cup \{|S_n f_n| > n\})} > \varepsilon, \tag{3.4}
$$

for every sequence $\{S_n\} \subseteq \tilde{S}_p$ of symmetries of $\mathcal{E}$.

By Proposition 2.1 there exist $J \in \mathbb{N}$ and dyadic cubes $\tau_j$, $n \in \mathbb{N}$, $1 \leq j \leq J$, such that if we inductively define

$$
\tau_n^0 := f_n, \quad g_n^0 := \tau_n^{-1} \chi_{\{f_n < C \rho^{-1} \rho\}} \tau_n^{-1} \chi_{\{f_n < C \rho^{-1} \rho\}}, \quad r_n^j := r_n^{j-1} - g_n^j, \quad 1 \leq j \leq J,
$$

then for all $n$ and all functions $|h_n^j| \leq |r_n^j|$, 

$$
\|\mathcal{E} h_n^j\|_q < \rho,
$$

with $\rho$ to be determined in a moment.

Let $F_n := f_n - r_n^J$. By our hypotheses and the disjointness of the supports of $F_n$ and $r_n^J$, 

$$
A_p - \rho \leq \liminf \|\mathcal{E} F_n\|_q \leq \liminf A_p \|F_n\|_p \leq \liminf A_p (1 - \|r_n^J\|_p)^{\frac{1}{p}} \leq A_p - c_p \limsup \|r_n^J\|_p,
$$

so, after passing to a subsequence, we may assume that for all $n$, $\|r_n^J\|_p \leq \rho^{\frac{1}{p}}$, which for $\rho$ sufficiently small implies 

$$
\|f_n - F_n\|_p < \frac{\rho^{\frac{1}{p}}}{2}. \tag{3.5}
$$

Since $(f_n)$ is extremizing, for each sufficiently large $n$, $\|g_n^0\|_p \gtrsim 1$. Indeed, that $\|g_n^0\|_p \ll 1$ implies $\|\mathcal{E} f_n\|_q \ll 1$ follows from the proof of Proposition 2.1 (See (?), in particular.) Applying symmetries if needed, we may assume that $\tau_n^1 = [0, 1]^d$ for all $n$. The remaining cubes may be written

$$
r_n^j = \xi_n^j + 2^{-k_n^j}[0, 1]^d, \quad n \in \mathbb{N}, \quad 1 \leq j \leq J,
$$

with $k_n^j \in \mathbb{Z}$ and $\xi_n^j \in 2^{-k_n^j} \mathbb{Z}^d$. 

Passing to a subsequence, we may assume that for each \( j \), either \( k_n \) remains bounded or \( |k_n| \to \infty \) and that either \( \xi_n \) remains bounded or \( |\xi_n| \to \infty \). Since our \( \tau_n \) are dyadic (and \( j \leq J < \infty \)), if \( k_n \) and \( \xi_n \) both remain bounded, after passing to a further subsequence, they are constant in \( n \). We say that an index \( 1 \leq j \leq J \) is good if the parameters \( k_n \) and \( \xi_n \) are constant in \( n \), and that it is bad otherwise. We decompose

\[
F_n = G_n + B_n, \quad G_n := \sum_{j \text{ good}} g_n^j, \quad B_n := F_n - G_n.
\]

It follows from our hypothesis (3.4) and the estimate (3.5) that \( \lim \inf \|B_n\|_p > \frac{c}{2} \), so, after passing to a subsequence,

\[
\|G_n\|_p \lesssim (1 - (\varepsilon/2)^p)^{1/2} \leq 1 - c\varepsilon^p.
\]

(3.6)

Since \( 1 \lesssim \|g_n^1\|_p \leq \|G_n\|_p \),

\[
\|B_n\|_p \leq (\|F_n\|_p - \|G_n\|_p)^{1/2} \leq 1 - c_0,
\]

(3.7)

for some \( c_0 > 1 \).

We claim that after passing to a subsequence, \( (EB_n) \) converges to zero a.e. Indeed, \( B_n = \sum_{bad \ j \leq J} g_n^j \), so it suffices to prove that a subsequence of each bad \( \mathcal{E}g_n^j \) tends to zero a.e., as \( n \to \infty \). If \( k_n \to \infty \), then \( g_n^j \to 0 \) in \( L^1_{\xi} \), so \( \mathcal{E}g_n^j \to 0 \) uniformly. If \( k_n \to -\infty \), then \( g_n^j \to 0 \) in \( L^p_{\xi} \), so \( \mathcal{E}g_n^j \to 0 \) in \( L^p_{\xi} \). In the remaining case, \( k_n \) is bounded, but \( |\xi_n| \to \infty \). Thus \( g_n^j \) is bounded in \( L^2_{\xi} \) and \( \mathcal{E}g_n^j \to 0 \) weakly.

By the Rellich–Kondrashov compactness theorem and the local smoothing estimate \[8, 26, 30\]

\[
\int |(|\nabla_x|^{1/2} + |\partial_t|^{1/2}) \mathcal{E}g_n^j(t, x)|^2|\phi(t, x)| dt \, dx \lesssim_\phi \|g_n^j\|_2 \lesssim 1, \quad \phi \in \mathcal{S}(\mathbb{R}^{1+d}),
\]

a subsequence of \( \mathcal{E}g_n^j \) converges to some function \( H \) in \( L^2_{\loc} \). As \( EB_n \) converges weakly to zero, \( H \equiv 0 \).

By a result of Brézis–Lieb \[11\], the a.e. convergence to zero of \( (EB_n) \) implies that

\[
\lim_{n \to \infty} \|\mathcal{E}F_n\|_q - \|\mathcal{E}G_n\|_q - \|\mathcal{E}B_n\|_q = 0.
\]

Thus by \[8, 10, 11\], our hypothesis that \( (f_n) \) is an \( L^p_{\xi} \)-normalized extremizing sequence, (3.5), (3.6), (3.7), and the fact that \( q > p \),

\[
A_p - \rho \leq \lim \inf \|\mathcal{E}F_n\|_q = \lim \inf \left(\|\mathcal{E}G_n\|_q + \|\mathcal{E}B_n\|_q\right)^{1/2} \leq A_p \left(\|G_n\|_q + \|B_n\|_p\right)^{1/2} \leq A_p \max\{1 - c\varepsilon^p, 1 - c_0\}^{1 - \frac{q}{p}} \|F_n\|_p^{1 - \frac{q}{p}}.
\]

Choosing \( \rho \) sufficiently small gives a contradiction. (This approach via Brézis–Lieb and local smoothing is due to Killip–Vişan, \[17\].)

\[\square\]

4. Space-time localization

In the previous sections, we used the bilinear theory to prove that near-extremizers have good frequency localization modulo symmetries. In this section, we take a first step toward localization in spacetime by applying the \( L^2_{\xi} \) theory to prove an \( L^p_{\xi} \)-based profile decomposition for frequency localized sequences.
Proposition 4.1. Let $R > 0$ and let $(f_n)$ be a sequence of measurable functions, supported on $\{ |\xi| < R \}$, and satisfying $|f_n| < R$. After passing to a subsequence, there exist $J_0 \in \mathbb{N} \cup \{\infty\}$, $(t_n^j, x_n^j) \in \mathbb{R}^{1+d}$, bounded, measurable functions $\phi^j$ supported on $\{ |\xi| < R \}$, and remainders $r_n^j$ such that for each $J < J_0$, $f_n = \sum_{j=1}^J e^{i(t_n^j \cdot x_n^j)|\xi|^2} \phi^j + r_n^j$, and

(i) For all $j \neq j'$, $\lim_{n \to \infty} (|t_n^j - t_n^{j'}| + |x_n^j - x_n^{j'}|) = \infty $,

(ii) If $\tilde{p} := \max \{p, p'\}$, then $\liminf_{n \to \infty} (\|f_n\|_p - (\sum_{j=1}^J \|\phi^j\|_p^{\tilde{p}})) \geq 0$,

(iii) For $J < J_0$, $\lim_{n \to \infty} (\|\mathcal{E} f_n\|_q - \sum_{j=1}^J \|\mathcal{E} \phi^j\|_q - \|\mathcal{E} r_n^j\|_q) = 0$,

(iv) The extensions of the errors tend to zero: $\lim_{J \to J_0} \limsup_{n \to \infty} \|\mathcal{E} r_n^j\| = 0$,

(v) For all $\phi^j$, $\phi' \in \text{wk-lim}_{n \to \infty} e^{i(t_n^j \cdot x_n^j)|\xi|^2} \phi^j f_n$.

Before beginning the proof, we recall the $L^2$-based profile decomposition for $\mathcal{E}$.

Theorem 4.2 (25, 5, 27). Let $\{f_n\}$ be a bounded sequence in $L^2$. After passing to a subsequence, there exist $J_0 \in \mathbb{N} \cup \{\infty\}$, symmetries $S^j_n \in \mathcal{S}_2$, nonzero profiles $\phi^j \in L^2$, and errors $r_n^j \in L^2$, such that for each $J < J_0$, $f_n = \sum_{j=1}^J S^j_n \phi^j + r_n^j$, and

(i) For all $j \neq j'$, $(S^j_n)^{-1} S^{j'}_n \to 0$ in the weak operator topology,

(ii) For $J < J_0$, $\lim_{n \to \infty} (\|f_n\|_2^2 - \sum_{j=1}^J \|\phi^j\|_2^2 - \|r_n^j\|_2^2) = 0$,

(iii) For $J < J_0$, $\lim_{n \to \infty} (\|\mathcal{E} f_n\|_{q_2}^2 - \sum_{j=1}^J \|\mathcal{E} \phi^j\|_{q_2}^2 - \|\mathcal{E} r_n^j\|_{q_2}^2) = 0$,

(iv) For all $j, \phi^j \to \text{wk-lim}(S^j_n)^{-1} f_n$,

(v) The extensions of the errors tend to zero: $\lim_{J \to J_0} \limsup_{n \to \infty} \|\mathcal{E} r_n^j\|_{q_2} = 0$.

In proving Proposition 4.1, we may assume that $p \neq 2$. Let $p_2 := 2$, and choose some $(p_1, q_1)$ at which (4.1) holds, such that $p$ lies strictly between $p_1$ and $p_2$. Set $q_i := \frac{4+2}{d} p_i$, $i = 1, 2$.

The main difficulty is in proving (ii), for which we will use the following technical lemma.

Lemma 4.3. Let $\phi, \psi$ be smooth, compactly supported functions on $\mathbb{R}^d$, with $0 \leq \phi, \psi \leq 1$ and $\phi(0) = 1$. Let $\{t_n^j, x_n^j\} : n \in \mathbb{N}, j \in \mathbb{N}\} \subset \mathbb{R}^{1+d}$, with $\lim_{n \to \infty} (|t_n^j - t_n^{j'}|, |x_n^j - x_n^{j'}|) = \infty$, for all $j \neq j'$. For $j \in \mathbb{N}$, define an operator

$$\pi_n f(\xi) := e^{i(t_n^j \cdot x_n^j)|\xi|^2} \psi(\eta) * (\phi(\eta) e^{-i(t_n^j \cdot x_n^j)|\eta|^2} \eta f(\eta)),$$

and for $J \in \mathbb{N}$, define a vector-valued operator $\Pi_n^j := (\pi_n^j)^j_{i=1}$. Then for each $J$,

$$\limsup_{n \to \infty} \|\Pi_n^j\|_{L^p_t L^q_x} \leq 1, \quad \tilde{p} := \max\{p, p'\}.$$ (4.1)

Proof of Lemma 4.3. For each $j, n$, and $1 \leq p \leq \infty$, $\pi_n^j$ is a bounded operator on $L^p_t$ with norm at most 1, so (4.1) is elementary for $p = 1, \infty$. By (complex) interpolation, this leaves us to prove the inequality in the case $p = 2$. By duality, it suffices to prove that $\limsup_{n \to \infty} \|\Pi_n^j\|^*_{L^2_t L^2_x} \leq 1$. We write

$$\|\Pi_n^j\|^*_{L^2_t L^2_x} = \sum_j \int |(\pi_n^j)^* f_j|^2 + \sum_{j \neq j'} \int (\pi_n^j)^* f_j (\pi_n^j)^* f_j d\xi.$$

It is elementary to bound the first term by $\sum_j \|f_j\|_2^2$, so it remains to prove that $\|\pi_n^j (\pi_n^j)^*\|_{L^2_t L^2_x} \to 0$. Abusing notation slightly, it thus suffices to prove that the
sequence \((T_n)\), defined by
\[
T_n g := \psi * \zeta (e^{i(t_n \cdot x_n) + |\xi|^2 \cdot \zeta}) \phi(\zeta) \psi * g(\zeta), \quad g \in L^2_{\zeta},
\]
tends to zero in \(L(L^2, L^2)\), whenever \(|t_n| + |x_n| \to \infty\). By the support condition on \(\phi, \psi\) and stationary phase,
\[
\|T_n g\|_2 \lesssim \|T_n g\|_{L^\infty} \lesssim (1 + |(t_n, x_n)|)^{-\frac{\theta}{4}} \|\psi * g\|_{C^2} \lesssim (1 + |(t_n, x_n)|)^{-\frac{\theta}{4}} \|g\|_2,
\]
whence \(\|T_n\|_{2 \to 2} \lesssim (1 + |(t_n, x_n)|)^{-\frac{\theta}{4}} \to 0.\)

**Proof of Proposition 4.1**. As the sequence \((f_n)\) is bounded in \(L^2\) (albeit with an \(R\)-dependent bound), we may apply the profile decomposition in Theorem 4.2. Each symmetry \(S^j_n\) arising therein may be written as a composition of a dilation with parameter \(\lambda^j_n\), a frequency translation with parameter \(\xi^j_n\), and a spacetime translation with parameter \((t^j_n, \chi^j_n)\). By the size and support conditions on the \(f_n\), as well as the definition of the \(\phi^j\) and their non-triviality, the dilation parameters are bounded away from 0 and \(\infty\), and the frequency parameters are bounded. Thus, after passing to a subsequence, for each \(j\) the dilations and frequency translations converge in the strong operator topology. Putting the limit on the \(\phi^j\) if needed, we may assume that
\[
S^j_n \phi^j = e^{i(t^j_n \cdot \xi^j_n + |\xi|^2 \cdot \xi)} \phi^j.
\]
Conclusions (i) and (v) follow.

Conclusion (iii) follows from local smoothing and the Brézis–Lieb inequality as in the proof of Proposition 3.1. Of course, \(f_n\) is bounded in \(L^p_{\xi}\), and Brézis–Lieb also yields (iii) with \(q\) replaced by \(q_1\). Thus, after passing to a subsequence, \(\|E r^j_n\|_{q_1}\) is uniformly bounded for all \(n\) and \(J\). We already know that
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|E r^j_n\|_{q_2} = 0,
\]
so (iv) follows from Hölder’s inequality.

This leaves us to prove (ii). Fix \(J \in \mathbb{N}\) with \(J \leq J_0\). Choose smooth, compactly supported \(\tilde{\psi}, \tilde{\phi}\) with \(0 \leq \tilde{\psi}, \tilde{\phi} \leq 1\) and \(\tilde{\phi}(0) = \int \tilde{\psi} = 1\) and \(\|\psi * (\phi \phi^j) - \phi^j\|_p < \varepsilon\). We claim that
\[
\lim_{n} \|\pi^j_n f_n - e^{i(t^j_n \cdot \xi^j_n + |\xi|^2 \cdot \xi)} \psi * (\phi \phi^j)\|_p = 0,
\]
where \(\pi^j_n\) is defined as in Lemma 4.3. To this end, it suffices to prove that for all \(1 \leq j \neq j' \leq J\),
\[
\lim_{n \to \infty} \|\pi^j_n (e^{i(t^j_n \cdot \xi^j_n + |\xi|^2 \cdot \xi)} \phi^j)\|_p = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\pi^{j'}_n r^j_n\|_p = 0. \quad (4.2)
\]
By (v), the claimed limits amount to proving that \(\lim \|\psi * (\phi g_n)\|_p = 0\), whenever \((g_n)\) is a sequence in \(L^p\) converging weakly to zero. This is an immediate consequence of the Dominated Convergence Theorem and the compact support of \(\phi, \psi\).

We send \(\varepsilon \searrow 0\), and (ii) follows from Lemma 4.3. \(\square\)

5. **Proof of Theorem 1.1**

Let \((f_n)\) be an \(L^p\)-normalized extremizing sequence. By Proposition 5.1, after applying a symmetry,
\[
\|E f^R_n\|_q \geq A_p - \varepsilon(n, R),
\]
where \( f^R_n := f \chi_{\{(|t| < R) \cup \{|f| < R\})} \), and \( \lim_{R \to \infty} \limsup_{n \to \infty} \varepsilon(n, R) = 0 \). We consider the integer truncations, \( f^m_n \) with \( R = m \in \N \). By Proposition \[4.1\] after passing to a subsequence in \( n \) (which is independent of \( m \)), we may decompose

\[
f^m_n := \sum_{j=1}^J e^{i(t^m_n \cdot x^m_n)(|t|^2 \xi)} \phi^m_j + r^m_n, \quad 1 \leq J < J_0 \in \N \cup \{\infty\},
\]

where the decomposition on the right satisfies the conclusions of that proposition.

By conclusions (iii) and (iv), then (ii) of Proposition \[4.1\] and \( q > \tilde{p} \),

\[
A_p^R - o_m(1) \leq \limsup_{n \to \infty} \|E f^m_n\|_q^q = \sum_{j=1}^J \|E \phi^m_j\|_q^q \leq A_p^R \max_{j \geq J_0} \|E \phi^m_j\|_q^q \sum_{j=1}^{J_0} \|\phi^m_j\|_{\tilde{p}}^{\tilde{p}}
\]

\[
\leq A_p^R \max_{j \geq J_0} \|\phi^m_j\|_{q^q - \tilde{p}}^{\tilde{p}} \leq A_p^R \max_{j \leq J_0} \|\phi^m_j\|_{q^q - \tilde{p}}^{\tilde{p}}.
\]

Choose \( j = j_m \) to maximize \( \|E \phi^m_j\|_q \), and set

\[
\phi^m := \phi^m_{j_m}, \quad (t^m_n, x^m_n) := (t^m_{j_m}, x^m_{j_m}).
\]

Then

\[
1 - o_m(1) \leq \|\phi^m\|_{\tilde{p}} \leq 1, \quad \text{and} \quad A_p - o_m(1) \leq \|E \phi^m\|_p.
\]

Since

\[
e^{-i(t^m_n \cdot x^m_n)(|t|^2 \xi)} f^m_n \to \phi^m, \quad \text{weakly in } L^p_{\xi},
\]

and

\[
\|\phi^m\|_p \geq (1 - o_m(1)) \limsup_{n \to \infty} \|f^m_n\|_p, \quad \text{as } m \to \infty,
\]

strict convexity of \( L^p \) implies

\[
\limsup_{n \to \infty} \|f^m_n - e^{-i(t^m_n \cdot x^m_n)(|t|^2 \xi)} \phi^m\|_p = o_m(1), \quad \text{as } m \to \infty.
\]

(See Theorem 2.5 and the proof of Theorem 2.11 in \[19\].)

By Proposition \[3.1\] and the triangle inequality,

\[
\limsup_{n \to \infty} \|f^m_n - e^{-i(t^m_n \cdot x^m_n)(|t|^2 \xi)} \phi^m\|_p = o_m(1), \quad \text{as } m \to \infty,
\]

whence

\[
\limsup_{n \to \infty} \|e^{i(t^m_n \cdot x^m_n)(|t|^2 \xi)} \phi^m - e^{i(t^m_n \cdot x^m_n)(|t|^2 \xi)} \phi^m\|_p = o_{\min\{m, m'\}}(1).
\]

Applying the projection \( \pi^m_n \) and using \[4.2\] and \[5.2\], for sufficiently large \( m, m' \), \([|t^m_n - t^m_m|, x^m_n - x^m_m]\) remains bounded as \( n \to \infty \). Applying a spacetime modulation to \( f_n \), we may assume that \((t^m_n, x^M_n) \equiv 0\), for some fixed, sufficiently large \( M \). Passing to a subsequence, we may thus assume that \((t^m_n, x^m_n) \to (t^m, x^m)\) for all \( m \geq M \), whence, replacing \( \phi^m \) with \( e^{i(t^m_n \cdot x^m_n)(|t|^2 \xi)} \phi^m \), we may assume that \((t^m_n, x^m_n) \to 0\) for all \( m \geq M \). Thus \( f \to e^{i(t^m_n \cdot x^m_n)(|t|^2 \xi)} f \) converges to the identity in the strong operator topology on \( \text{Aut}(L^p_{\xi}) \), for all \( m \). In summary, we knew that

\[
\limsup_{n \to \infty} \|f_n - e^{i(t^m_n \cdot x^m_n)(|t|^2 \xi)} \phi^m\|_p = o_m(1), \quad \text{as } m \to \infty;
\]

we now know that

\[
\limsup_{n \to \infty} \|f_n - \phi^m\|_p = o_m(1), \quad \text{as } m \to \infty.
\]

By \[5.3\] and the triangle inequality, \( \phi^m \) is Cauchy, hence convergent, in \( L^p_{\xi} \) as \( m \to \infty \), and \( \{f_n\} \) converges to the limit, which is an extremizer, as \( n \to \infty \).
6. Proof of the Corollary: Extremizers for the Restriction Operator

If \( \{g_n\} \) is an \( L_{1,x}^2 \)-normalized extremizing sequence for the restriction operator \( R \), by duality, \( f_n := |Rg_n|^{p'-2}Rg_n \in L_{p}^p \) is extremizing for \( E \), with \( \|f_n\|_p \to \|A_p\|_{p'}^{-1} \).

By Theorem 1.1 after passing to a subsequence, there exist extension symmetries \( S_n \in \hat{S}_p \) such that \( S_nf_n \) converges in \( L^p_\xi \) to an extension extremizer \( f \). As \( S_nf_n = |RT_ng_n|^{p'-2}RT_ng_n \), for a corresponding sequence \( \{T_n\} \) of restriction symmetries, we may assume, replacing \( g_n \) with \( T_ng_n \), that \( f_n \to f \) in \( L_{p}^p \). Passing to a subsequence, \( \{g_n\} \), being bounded, has a weak limit: \( g_n \to g \) in \( L_{1,x}^2 \). We claim that \( g \) is a restriction extremizer and that this weak convergence is in fact strong. Indeed, 

\[
A_p^p \|g\|_{q'} = A_p \|g\|_q \|f\|_p \geq |\langle g, Ef \rangle| = \lim |\langle g_n, Ef_n \rangle| = \lim \|Rg\|_{p'}^{p'} = A_p^p.
\]

By Theorem 2.11 of [19], weak convergence combined with convergence of norms implies strong convergence, \( g_n \to g \) in \( L_{1,x}^2 \). By continuity of \( R \), it follows that \( g \) is a restriction extremizer.

References

[1] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals. Proc. Amer. Math. Soc. 88 (1983), no. 3, 486–490.
[2] P. Béqout, A. Vargas, Mass concentration phenomena for the \( L^2 \)-critical nonlinear Schrödinger equation. Trans. Amer. Math. Soc. v. 359, n. 11 (2007), pp. 5257–5282.
[3] J. Bennett, N. Bez, A. Carbery, D. Hundertmark, Heat-flow monotonicity of Strichartz norms. Anal. PDE 2 (2009), no. 2, 147–158.
[4] J. Bourgain, Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity, Internat. Math. Res. Notices 5 (1998) 253–283.
[5] R. Carles, S. Keraani, On the role of quadratic oscillations in nonlinear Schrödinger equation. J. Amer. Math. Soc. 29 (2016), no. 4, 739–774.
[6] M. Christ, Extremizers of a Radon transform inequality. Advances in analysis: the legacy of Elias M. Stein, 84–107, Princeton Math. Ser., 50, Princeton Univ. Press, Princeton, NJ, 2014.
[7] M. Christ, R. Quilodrán, Gaussians rarely extremize adjacent Fourier restriction inequalities for paraboloids. Proc. Amer. Math. Soc. 142 (2014), no. 3, 887–896.
[8] P. Constantin, J.-C. Saut, Local smoothing properties of dispersive equations. J. Amer. Math. Soc. 1 (1988), no. 2, 413–439.
[9] D. Foschi, Maximizers for the Strichartz inequality. J. Eur. Math. Soc. (JEMS) 9 (2007), no. 4, 739–774.
[10] D. Foschi, D. O. Silva, Some recent progress on sharp Fourier restriction theory. Preprint [arXiv:1701.00695].
[11] R. Frank, E. H. Lieb, J. Sabin, Maximizers for the Stein-Tomas inequality. Geom. Funct. Anal. 26 (2016), no. 4, 1095–1134.
[12] L. Guth, A restriction estimate using polynomial partitioning. J. Amer. Math. Soc. 29 (2016), no. 2, 371–413.
[13] L. Guth, Restriction estimates using polynomial partitioning II. Preprint [arXiv:1603.04250].
[14] J. Hickman, K. M. Rogers, Improved Fourier restriction estimates in higher dimensions. Preprint [arXiv:1807.10940].
[15] D. Hundertmark, V. Zharnitsky, On sharp Strichartz inequalities in low dimensions. Int. Math. Res. Not. 2006, Art. ID 34080, 18 pp.
[16] S. Keraani, On the blow up phenomenon of the critical nonlinear Schrödinger equation. J. Funct. Anal. 235 (2006), 171–192.
[17] R. Killip, M. Vişan, Nonlinear Schrödinger equations at critical regularity. Evolution equations, 325–437, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.
[18] E. H. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. Annals of Math. 118 (1983), 349–374.
[19] E. H. Lieb, M. Loss, *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[20] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201.

[21] F. Merle, L. Vega *Compactness at blow-up time for $L^2$ solutions of the critical non-linear Schrödinger equation in 2D*, International Math. Research Notices 8 (1998) 399–425.

[22] A. Moyua, A. Vargas, L. Vega, *Schrödinger maximal function and restriction properties of the Fourier transform*, International Math. Research Notices 16 (1996) 793–815.

[23] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*. Bull. Amer. Math. Soc. 73 1967 591–597.

[24] B. Shayya, *Weighted restriction estimates using polynomial partitioning*. Proc. Lond. Math. Soc. (3) 115 (2017), no. 3, 545–598.

[25] S. Shao, *Maximizers for the Strichartz and the Sobolev-Strichartz inequalities for the Schrödinger equation*. Electron. J. Differential Equations 2009, No. 3, 13 pp.

[26] P. Sjölin, *Regularity of solutions to the Schrödinger equation*. Duke Math. J. 55 (1987), no. 3, 699–715.

[27] S. Solimini, C. Tintarev, *Concentration analysis in Banach spaces*. Commun. Contemp. Math. 18 (2016), no. 3, 1550038, 33 pp.

[28] T. Tao, *A sharp bilinear restriction estimate for paraboloids*, Geom. Funct. Anal., v. 13, (2003), pp. 1359–1384.

[29] T. Tao, A. Vargas, L. Vega, *A bilinear approach to the Restriction and Kakeya Conjectures*, J. Amer. Math. Soc., v. 11, n. 4, (1998), pp. 967–1000.

[30] L. Vega, *Schrödinger equations: pointwise convergence to the initial data*. Proc. Amer. Math. Soc. 102 (1988), no. 4, 874–878.

[31] H. Wang, *A restriction estimate in $\mathbb{R}^3$ using brooms*. Preprint, arXiv:1802.04312

E-mail address: stovall@math.wisc.edu

480 Lincoln Drive, Madison, WI 53706