An easy upper bound for Ramsey numbers

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Abstract: We prove easy upper bounds for Ramsey numbers. 1.

1 Introduction

A theorem of Ramsey, see [4], implies the existence of a smallest natural integer $R(n)$, now called the $n$–th Ramsey number, such that every (simple unoriented) graph $G$ with at least $R(n)$ vertices contains either a complete graph with $n$ vertices or $n$ pairwise non-adjacent vertices (defining a complete graph in the complementary graph of $G$).

The aim of this paper is to give a new simple proof of the following upper bound for Ramsey numbers:

Theorem 1.1. We have

$$R(n) \leq 2^{2n-3}$$

for $n \geq 2$.

The currently best asymptotic upper bound,

$$R(n+1) \leq \binom{2n}{n} n^{-C \log n / \log \log n},$$

(for a suitable constant $C$) is due to Conlon, see [2].

The standard proof of Ramsey’s theorem, due to Erdős and Szekeres (see [3] or Chapter 35 of [1]), uses a two parameter Ramsey number $R(a, b)$ defined as the smallest integer such that every graph with $R(a, b)$ vertices contains either a complete graph with $a$ vertices or a subset of $b$ non-adjacent vertices. It is slightly more involved than our proof and gives the upper bound $R(n + 1) \leq \binom{2n}{n}$ (based on the trivial values $R(a, 1) = R(1, a) = 1$ and on the inequality $R(a, b) \leq R(a - 1, b) + R(a, b - 1)$ for $a, b > 1$).

Simple graphs are equivalent to complete graphs with edges of two colours (encoding edges, respectively nonedges of simple graphs). There is a generalization of Ramsey’s theorem to an arbitrary finite number $m$.

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of colours as follows: There exists a smallest natural number $R_m(n)$ such that every complete graph on $R_m(n)$ vertices with edges of $m$ colours contains $n$ vertices belonging to a complete edge-monochromatic subgraph. The following result gives an upper bound for $R_m(n)$:

**Theorem 1.2.** We have

$$R_m(n) \leq 1 + \sum_{j=0}^{m(n-2)} m^j = 1 + \frac{m^{mn-2m+1} - 1}{m - 1}$$

for $m, n \geq 2$.

For $m = 2$, the upper bound $1 + \frac{m^{mn-2m+1} - 1}{m - 1}$ of Theorem 1.2 coincides with the upper bound $2^{2n-3} = 1 + \frac{2^{2n-4+1} - 1}{2-1} = 2^{2n-3}$ for $R_2(n) = R(n)$ given by Theorem 1.1.

This paper contains a simple proof of Theorem 1.1 (Section 2) and Theorem 1.2 (Section 3) which is a variation on the proofs usually found and is perhaps slightly simpler. In Section 4 we discuss a few generalizations of the numbers $R'(n)$ and $R'_m(n)$ playing a crucial role in the proofs.

## 2 Proof of Theorem 1.1

Given a finite graph $G$, we define $\rho'(G)$ to be the largest natural number such that $G$ contains two (not necessarily disjoint) subsets $A$ and $B$ of vertices satisfying the following two conditions:

1. All vertices of $A$ are adjacent to each other and no vertices of $B$ are adjacent.

2. $\sharp(A) + \sharp(B) = \rho'(G)$.

In this section, the letter $A$ always denotes a set of pairwise adjacent vertices and $B$ denotes a set of pairwise non-adjacent vertices. Two such subsets $A, B$ of vertices in a graph $G$ realize $\rho'(G)$ if $\rho'(G) = \sharp(A) + \sharp(B)$.

We define $R'(n)$ as the smallest natural integer such that $\rho'(G) \geq n$ for every graph $G$ with $R'(n)$ vertices.

**Lemma 2.1.** We have $R(n) \leq R'(2n - 1)$.

**Proof** A graph $G$ with $R'(2n - 1)$ vertices contains subsets $A$ and $B$ of vertices realizing $\rho'(G) \geq 2n - 1$. One of the subsets $A, B$ thus contains at least $n$ vertices. If $\sharp(A) \geq n$, the graph $G$ contains a complete subgraph of $n$ vertices, if $\sharp(B) \geq n$, the graph $G$ contains $n$ pairwise non-adjacent vertices. $\square$

**Lemma 2.2.** We have $R'(n + 1) \leq 2R'(n)$.
Proof We choose a vertex $v$ in a graph $G$ with $2R'(n)$ vertices. We denote by $G_v$ the subgraph of $G \setminus \{v\}$ defined by all neighbours of $v$. Up to replacing $G$ by its complementary graph (and exchanging the roles of the sets $A$ and $B$), we can suppose that $G_v$ has at least $\lceil(2R'(n) - 1)/2\rceil = R'(n)$ vertices. Hence we have $\rho'(G) \geq n$ and we can find subsets $A, B$ of vertices in $G_v$ which realize $\rho'(G)$. The subset $A \cup \{v\}$ contains thus $\#(A) + 1$ pairwise adjacent vertices of $G$ and we have $\rho'(G) \geq \#(A \cup \{v\}) + \#(B) = \rho'(G_v) + 1 \geq n + 1$.

Proposition 2.3. We have $R'(n) \leq 2^{n-2}$ for $n \geq 2$.

Proof If $n = 2$ we take $A = B = \{v\}$ where $v$ is the unique vertex of the trivial graph $G = \{v\}$ on one vertex $v$.

Induction on $n$ using Lemma 2.2 ends the proof.

Proposition 2.4. The value $R'(4) = 3$ can of course be used for improving the upper bound $R'(n) \leq 2^{n-2}$ in Theorem 1.1 to $3 \cdot 2^{n-5}$ for $n \geq 3$. More generally, any interesting upper bound on $R'(n)$ for $n > 4$ easily yields an improvement of Theorem 1.1.
3 Proof of Theorem 1.2

We define \( R'_m(n) \) to be the smallest integer such that every complete graph with \( R'_m(n) \) vertices and edges of \( m \) colours contains \( m \) (not necessarily disjoint) subsets of vertices \( A_1, \ldots, A_m \) with \( \sharp(A_1) + \sharp(A_2) + \cdots + \sharp(A_m) = n \) and with \( A_1, \ldots, A_m \) defining \( m \) complete edge-monochromatic graphs of different edge-colours.

Given a complete graph \( G \) with \( m \)–coloured edges, we denote by \( \rho'_m(G) \) the largest integer such that \( G \) contains \( m \) (not necessarily disjoint) subsets \( A_1, \ldots, A_m \) of vertices defining complete edge-monochromatic subgraphs of different colours and \( \rho'_m(G) = \sharp(A_1) + \cdots + \sharp(A_m) \). We say that \( m \) such subsets \( A_1, \ldots, A_m \) realize \( \rho'_m(G) \).

We have of course \( \rho'_m(G) \geq n \) if \( G \) contains at least \( R'_m(n) \) vertices.

Examples:

1. We have \( R'_m(m) = 1 \) by setting \( A_1 = A_2 = \cdots = A_m = \{v\} \) where \( v \) is the unique vertex of the trivial graph with one vertex (the empty sets of edges in \( A_1, \ldots, A_m \) have different colours by convention).

   The value \( R'_m(m) = 1 \) also follows from \( R_m(1) = 1 \) applied to the the trivial inequality \( R'_m(n + m - 1) \leq R_m(n) \) obtained by completing a complete edge-monochromatic subgraph on \( n \) vertices with \( m - 1 \) singletons representing complete edge-monochromatic subgraphs of the \( m - 1 \) remaining colours.

2. \( R'_m(m + 1) = 2 \) since an edge-coloured complete graph on \( 2 \) vertices is always monochromatic.

3. \( R'_m(m + 2) = 3 \) since every edge-coloured triangle is either edge-monochromatic or contains two edges of different colours.

Lemma 3.1. We have \( R_m(n) \leq R'(m(n - 1) + 1) \).

Proof A set of \( m \) integers summing up to \( m(n - 1) + 1 \) contains an element at least equal to \( n \). For every realization \( A_1, \ldots, A_m \) of \( \rho'(G) \geq m(n - 1) + 1 \) of a graph \( G \) with \( R'(m(n - 1) + 1) \) vertices there thus exists an index \( i \) such that \( A_i \) defines an edge-monochromatic complete graph on at least \( n \) vertices.

\[ \square \]

Lemma 3.2. We have \( R'_m(n + 1) \leq 2 + m(R'_m(n) - 1) \).

Proof Fixing a vertex \( v \) in a complete graph \( G \) with \( 2 + m(R'_m(n) - 1) \) vertices and edges of \( m \) colours, we get a partition \( V \setminus \{v\} = V_1 \cup \cdots \cup V_m \) of all vertices different from \( v \) by considering the set \( V_i \) of vertices joined by an edge of colour \( i \) to \( v \). Since \( V \setminus \{v\} \) has \( 1 + m(R'_m(n) - 1) \) elements, there exists a set \( V_i \) containing at least \( R'_m(n) \) vertices. The subgraph \( G_i \) with vertices \( V_i \) thus contains a realization \( A_1, \ldots, A_m \) of \( \rho'_m(G_i) \geq n \). Since \( v \) is
joined by edges of colour $i$ to all elements of $A_i$, the set of vertices $A_i \cup \{v\}$ defines a complete edge-monochromatic subgraph of colour $i$ in $G$. This proves $\rho'(G) \geq \sharp(A_1) + \cdots + \sharp(A_i \cup \{v\}) + \cdots + \sharp(A_m) = \rho(G_i) + 1 \geq n + 1$. □

**Proposition 3.3.** We have

$$R'_m(m + k) \leq 1 + \sum_{j=0}^{k-1} m^j = 1 + \frac{m^k - 1}{m - 1}$$

for every natural integer $k$ (using the convention $\sum_{j=0}^{-1} m^j = 0$ if $k = 0$).

**Proof** The formula holds for $k = 0$ with $A_1 = A_2 = \cdots = A_m = \{v\}$ the unique vertex of the trivial graph $\{v\}$ reduced to one vertex.

Using Lemma 3.2 and induction on $k$ we have

$$R'(m + k + 1) \leq 2 + m(R'_m(m + k) - 1)$$
$$\leq 2 + m \left( 1 + \sum_{j=0}^{k-1} m^j \right) - 1$$
$$= 1 + \sum_{j=0}^{k} m^j$$

which ends the proof. □

**Proof of Theorem 1.2** We have

$$R_m(n) \leq R'(m(n - 1) + 1) \leq 1 + \frac{m^{m(n-1)+1-m} - 1}{m - 1} = 1 + \frac{m^{mn-2m+1} - 1}{m - 1}$$

where the first inequality is Lemma 3.1 and the second inequality is Proposition 3.3. □

### 4 Generalizations of the number $R'_m(n)$

The number $R'_m(n)$ has two obvious generalizations.

The first one is given by considering $R'_{m,j}(n)$ with $j \in \{1, \ldots, m\}$ defined as the smallest integer such that every complete graph with $R'_{m,j}(n)$ vertices and edges of $m$ colours contains $j$ edge-monochromatic complete subgraphs of different edge-colours and of size $\alpha_1, \ldots, \alpha_j$ such that $\alpha_1 + \cdots + \alpha_j = n$. Therefore we consider only the $j$ colours corresponding to the $j$ largest edge-monochromatic complete subgraphs. For $j = 1$, we recover the usual Ramsey numbers $R_m(n)$, for $j = m$ we get the numbers $R'_m(n)$ introduced previously.

The second generalization depends on an unbounded function $s : G \rightarrow \mathbb{N}$ (one can also work with $m$ different unbounded functions $s_c : G \rightarrow \mathbb{N}$).
indexed by colours or replace the target-set of natural integers by the set of non-negative real numbers) on the set $G$ of all finite simple graphs.

For $n \geq 1$ we define $R'_m(n)$ as the smallest integer such that every complete graph on $R'_m(n)$ vertices contains $m$ (not necessarily complete) edge-monochromatic subgraphs $G_1, \ldots, G_m$ of colour $1, \ldots, m$ satisfying $s(G_1) + s(G_2) + \cdots + s(G_m) \geq n$ (respectively $s_1(G_1) + \cdots + s_m(G_m) \geq n$).

The numbers $R'_m(n)$ correspond to the choice $s(G) = n$ if $G$ is the complete graph on $n$ vertices and $s(G) = 0$ otherwise.

Other perhaps interesting choices are $s(G) = n$ if $G$ is an $n$-cycle and $s(G) = 0$ otherwise, or $s(G) = n$ if $G$ is a simple path (two endpoints of degree 1 and all other vertices of degree 2) with $n$ vertices.

It is of course possible to combine both generalizations by defining $R'_{m,j,s}(n)$ in the obvious way considering only the $j$ colours giving the largest contributions to the sum $s(G_1) + \cdots + s(G_m)$.

4.1 Analogues of $R'$ for van der Waerden numbers

Van der Waerden’s Theorem gives the existence of a function $W : \{2, 3, 4, \ldots \} \times \{2, 3, 4, \ldots \} \to \mathbb{N}$ associating to two integers $m, n \geq 2$ the smallest natural integer $W(m, n)$ such that every colouring of the $W(m, n)$ consecutive natural integers $1, 2, \ldots, W(m, n)$ with $m$ colours contains a monochromatic arithmetic progression with $n$ elements.

We define $W'(m, n)$ in the obvious way as the smallest natural integer such that every colouring of $1, 2, \ldots, W'(m, n)$ with $m$ colours contains $m$ (perhaps empty) monochromatic progressions of different colours and of lengths $\alpha_1, \ldots, \alpha_m$ summing up to $n = \alpha_1 + \cdots + \alpha_m$.

We have $W(m, n) \leq W'(m, m(n-1)+1)$ since a set of $m$ integers strictly smaller than $n$ sums up at most to $m(n-1)$. It is easy to check that $W'(2, 1) = 1, W'(2, 2) = 2$ and $W'(2, 3) = 3$.

For $W'(2, 4)$ we get $W'(2, 4) = 6$ as can be seen as follows: $W'(2, 4) > 5$ by inspection of the black-white colouring $bbwbb$ of $1, 2, 3, 4, 5$. Consider a black-white colouring of $1, \ldots, 6$ not containing a black progression of size $\alpha$ and a white progression of size $\beta$ such that $\alpha + \beta \geq 4$. Such a colouring cannot use only one colour (otherwise we can take $\alpha = 6$ or $\beta = 6$). It cannot use both colours twice (otherwise we can take $\alpha = 2$ and $\beta = 2$). It uses thus one colour, say white, only once and we have necessarily $\alpha \geq 3$ and $\beta = 1$ since either $1, 3, 5$ (for an even white element) or $2, 4, 6$ (for an odd white element) are all black.

It is of course also possible to consider the numbers $W'_j(m, n)$ defined by considering only the $j$ largest arithmetical progressions. For $j = 1$ we get the classical van der Waerden number $W(m, n)$.

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