An infinite natural sum

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As far as algebraic properties are concerned, the usual addition on the class of ordinal numbers is not really well behaved; for example, it is not commutative, nor left cancellative etc. In a few cases, the natural Hessenberg sum is a better alternative, since it shares most of the usual properties of the addition on the naturals.

A countably infinite iteration of the natural sum has been used in a recent paper by Väänänen and Wang, with applications to infinitary logics. We present a detailed study of this infinitary operation, showing that there are many similarities with the ordinary infinitary sum, providing an order theoretical characterization, and finding connections with certain kinds of infinite mixed sums.

1 Introduction

There are different ways to extend the addition operation from the set $\omega$ of natural numbers to the class of ordinals. The standard way is to take $\alpha + \beta$ as the ordinal which represents the order type of $\alpha$ with a copy of $\beta$ added at the top. This operation can be introduced by the customary inductive definition and satisfies only few of the familiar properties shared by the addition on the naturals.

On the other hand, again on the class of the ordinals, one can define the (Hessenberg) natural sum $\alpha \# \beta$ of $\alpha$ and $\beta$ by expressing $\alpha$ and $\beta$ in Cantor normal form and summing linearly. Cf. below for further details. The resulting operation $\#$ is commutative, associative and cancellative. It can be given an inductive definition as follows.

\[
0 \# 0 := 0 \\
\alpha \# \beta := \sup \{S(\alpha \# \beta'), S(\alpha' \# \beta)\} \quad (1)
\]

where $S$ denotes successor.

It is relevant that the natural sum, too, admits an order theoretical definition. If $\alpha$, $\beta$, and $\gamma$ are ordinals, $\gamma$ is said to be a mixed sum of $\alpha$ and $\beta$ if there are disjoint subsets $A$ and $B$ of $\gamma$ such that $\gamma = A \cup B$ and $A$, $B$ have order type, respectively, $\alpha$ and $\beta$, under the order induced by $\gamma$. Carruth showed that $\alpha \# \beta$ is the largest mixed sum of $\alpha$ and $\beta$; he also found many applications [3].

Väänänen and Wang define a countably infinite extension of $\#$ by taking supremum at the limit stage [11]. They provide applications to infinitary logics. Subsequently, we have found applications to compactness of topological spaces in the spirit of [7], in particular, with respect to Frolík sums.

Carruth’s Theorem, as it stands, cannot be generalized to such an infinite natural sum. Indeed, every countably infinite ordinal is an infinite mixed sum of countably many 1’s, hence in the infinite case the maximum is not necessarily attained. Cf. Definition 4.1 and the comment after Theorem 4.2.

However, we show that Carruth’s Theorem can indeed be generalized, provided we restrict ourselves to certain well behaved infinite mixed sums. In order to provide this generalization, we need a finer description of the countably infinite natural sum. We show that any infinite natural sum can be computed in two steps: in the first step one takes the natural sum of some sufficiently large finite set of summands. In the second step one adds the infinite ordinary sum of the remaining summands. In other words, the infinite natural sum and the more usual

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infinite sum differ only for a finite head and they agree on the remaining tail. This is used in order to show that the infinite natural sum of a sequence is the maximum of all possible infinite mixed sums made of elements from the sequence, provided one restricts himself only to mixed sums satisfying an appropriate finiteness condition.

In the end, we show that the infinite natural sum can be actually computed as some finite natural sum, and can be expressed in terms of the Cantor normal forms of the summands. We show that a countable sequence has only a finite number of mixed sums satisfying an additional convexity property. This extends a classical theorem by Sierpiński [9], asserting that one gets only a finite number of values for the sum of some fixed countable sequence of ordinals, by changing their order.

2 Natural sums

We now give more details about the definitions hinted above and list some simple facts about the natural sums. Here and below sums, products and exponentiations will be always intended in the ordinal sense. Cf., e.g., [2, 10] for a detailed introduction to ordinal operations. Recall that every ordinal \( \alpha > 0 \) can be expressed in a unique way in Cantor normal form as follows
\[
\alpha = \omega^\xi r_k + \omega^\xi r_{k-1} + \cdots + \omega^\xi r_1 + \omega^\xi r_0
\]
for some integers \( k \geq 0, r_k, \ldots, r_0 > 0 \) and ordinals \( \xi_k > \xi_{k-1} > \cdots > \xi_1 > \xi_0 \).

**Definition 2.1** The natural sum \( \alpha \# \beta \) of two ordinals \( \alpha \) and \( \beta \) is the only operation satisfying
\[
\alpha \# \beta = \omega^\xi (r_k + s_k) + \cdots + \omega^\xi (r_1 + s_1) + \omega^\xi (r_0 + s_0)
\]
whenever
\[
\alpha = \omega^\xi r_k + \cdots + \omega^\xi r_1 + \omega^\xi r_0
\]
\[
\beta = \omega^\xi s_k + \cdots + \omega^\xi s_1 + \omega^\xi s_0
\]
and \( k, r_k, \ldots, r_0, s_k, \ldots, s_0 < \omega, \xi_k > \cdots > \xi_1 > \xi_0 \).

The definition is justified by the fact that we can represent every nonzero \( \alpha \) and \( \beta \) in Cantor normal form and then insert some more null coefficients for convenience just in order to make the indices match. The null coefficients do not affect the ordinals, hence the definition is well-posed. Cf., e.g., [2, 10] for further details.

An elegant way to introduce the natural sum is obtained by expressing equation (2) in a conventional way as \( \alpha = \sum_{\xi \in F} \omega^\xi \), where \( F \) is the finite *multiset* which contains each \( \xi \) exactly \( r_k \) times. This is justified by the fact that, say, \( \omega^\xi + \omega^\xi = \omega^\xi \cdot 2 \). In this way, \( \alpha = 0 \) is expressed by summing over the empty multiset. Of course, when expanding the above summation, one should be careful to consider the terms with larger exponents first, that is, write them on the left. If \( \alpha = \sum_{\xi \in F} \omega^\xi \) and \( \beta = \sum_{\xi \in T} \omega^\xi \), then \( \alpha \# \beta \) is defined as \( \sum_{\xi \in F \cup T} \omega^\xi \), where in the union \( F \cup T \) we take into account multiplicities. In this note, however, we shall follow the more conventional notations.

It can be shown by induction on \( (\max(\alpha, \beta), \min(\alpha, \beta)) \), ordered lexicographically, that Definition 2.1 is equivalent to the definition given by means of equations (1). This shall not be needed in what follows.

Notice that the assumption \( \xi_k > \ldots > \xi_1 > \xi_0 \) in Definition 2.1 is necessary, since, for example, \((1 + \omega) \# (1 + \omega^0) = \omega \# 1 = \omega + 1\), while summing linearly we would obtain \( 2 + \omega = \omega \).

**Proposition 2.2** Let \( \alpha, \beta \) and \( \eta \) be ordinals.

1. The operation \# is commutative, associative, both left and right cancellative and strictly monotone in both arguments.
2. \( \sup(\alpha, \beta) \leq \alpha + \beta \leq \alpha \# \beta \).
3. If \( \alpha, \beta < \omega^\eta \), then \( \alpha \# \beta < \omega^\eta \).
4. If \( \beta < \omega^\eta \), then \( \alpha \# \beta < \alpha + \omega^\eta \).
5. If \( \beta < \omega^\eta \), then \( (\alpha \# \beta) + \omega^\eta = \alpha + \omega^\eta \).

**Proof.** Everything is almost immediate from Definition 2.1.

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For example, to prove (4), let \( \alpha = \omega^i r_k + \cdots + \omega^j r + \cdots \) with, as usual, the exponents of \( \omega \) in decreasing order, and where we can allow \( r \) to be 0. Then \( \alpha + \omega^k = \omega^i r_k + \cdots + \omega^j r + \omega^k (r + 1) \), while, if \( \beta < \omega^k \), then \( \alpha \# \beta = \omega^i r_k + \cdots + \omega^j r + \cdots \), since \( \beta \) does not contribute to summands where the exponent of \( \omega \) is \( \geq \eta \). Thus surely \( \alpha + \omega^k > \alpha \# \beta \), with no need to compute explicitly those summands which are \( < \omega^k \).

Parentheses are usually necessary in expressions involving both + and #; for example, \( (1 \# 0) + \omega = \omega \neq \omega + 1 = 1 \# (0 + \omega) \), or \( (1 + 0) \# \omega = \omega + 1 \neq \omega = 1 + (0 \# \omega) \).

**Definition 2.3** Suppose that \((\alpha_i)_{i < \omega}\) is a countable sequence of ordinals, and set \( S_n = \alpha_0 \# \cdots \# \alpha_{n-1} \), for every \( n < \omega \). The natural sum of \((\alpha_i)_{i < \omega}\) is

\[
\#_{i < \omega} \alpha_i := \sup_{n < \omega} S_n
\]

The above natural sum is denoted by \( \sum_{i < \omega} \alpha_i \) in [11].

In the above notation \( \alpha_0 \# \cdots \# \alpha_{n-1} \) we conventionally allow \( n = 0 \), and assume that 0 is the outcome of such an empty sum. Note that the notation is not ambiguous, in view of Proposition 2.2(1).

**Proposition 2.4** Let \( \alpha_i, \beta_i \) be ordinals and \( n, m < \omega \).

1. \( \sum_{i < \omega} \alpha_i = \sum_{i < \omega} \alpha_i \)
2. If \( \beta_i < \alpha_i \), for every \( i < \omega \), then \( \sum_{i < \omega} \beta_i < \sum_{i < \omega} \alpha_i \)
3. If \( n < m \), then \( S_n \leq S_m \); equality holds if and only if \( \alpha_0 = \cdots = \alpha_{m-1} = 0 \).
4. \( S_n \geq \sum_{i < \omega} \alpha_i \); equality holds if and only if \( \alpha_i = 0 \), for every \( i \geq n \).
5. If \( \pi \) is a permutation of \( \omega \), then \( \sum_{i < \omega} \alpha_i = \sum_{i < \omega} \alpha_{\pi(i)} \).
6. More generally, suppose that \((F_i)_{i < \omega}\) is a partition of \( \omega \) into finite subsets, say, \( F_i = \{j_1, \ldots, j_{r(i)}\} \), for every \( h \in \omega \). Then

\[
\#_{i < \omega} \alpha_i = \sum_{h < \omega} \sum_{j \in F_h} \alpha_j = \sum_{h < \omega} (\alpha_{j_1}, \# \alpha_{j_1}, \cdots, \# \alpha_{j_{r(h)}})
\]

**Proof.** Clauses (1) to (4) are immediate from the definitions and Proposition 2.2 (1) & (2). Clause (5) is a remark in the proof of [11, Proposition 4.4]. Moreover, clause (5) is the particular case of clause (6) when all the \( F_i \)’s are singletons.

To prove clause (6), define, for \( h < \omega \)

\[
T_h = \left( \#_{j \in F_h} \alpha_j \right) \# \cdots \# (\#_{j \in F_{h-1}} \alpha_j)
\]

Thus the right-hand of the equation in (6) is \( \sup_{h < \omega} T_h \). For \( h < \omega \), let \( m = \max_{0 \leq i < h} F_i \). The maximum exists since each \( F_i \) is finite, and we are considering only a finite number of \( F_i \)’s at a time. Then each summand in the expansion of \( T_h \) appears in \( S_{m+1} \) (taking into account multiplicity), hence, by (4) and monotonicity of the natural sum, \( \#_{i < \omega} \alpha_i \geq S_{m+1} \geq T_h \). Hence \( \#_{i < \omega} \alpha_i \geq \sup_{h < \omega} T_h \). The reverse inequality is similar and easier.

The assumption that each \( F_h \) is finite in condition (6) above is necessary. For example, take \( \alpha_i = 1 \), for every \( i < \omega \), thus \( \#_{i < \omega} \alpha_i = \omega \). Suppose that there is some infinite \( F_h \). Then \( \#_{j \in F_h} \alpha_j = \omega \). If \( \omega \backslash F_h \neq \emptyset \), then \( \#_{h < \omega} (\#_{j \in F_h} \alpha_j) \geq \omega \# 1 > \omega \).

Not everything from Proposition 2.2 generalizes to infinite sums. For example, the operation \( \#_{i < \omega} \alpha_i \), though monotone, as stated in (2) above, is not strictly monotone. E.g., \( \#_{i < \omega} 2 = \#_{i < \omega} 1 = \omega \). Actually, \( \#_{i < \omega} \alpha_i = \omega \), for every choice of the \( \alpha_i \)’s such that \( i < \omega \), for every \( i < \omega \), and such that there are infinitely many nonzero \( \alpha_i \)’s.

Condition (5) above can be interpreted as a version of commutativity, and (6) as a version of the generalized commutative-associative law. However, not all forms of associativity hold. We have seen that we cannot associate infinitely many summands inside some natural sum. Similarly, we are not allowed to associate inside out. Indeed, \( \omega + 1 = 1 \# \#_{i < \omega} 1 \neq \#_{i < \omega} 1 = \omega \). This is of course a general and well-known fact. For infinitary operations, some very weak form of generalized associativity implies some form of absorption.
Example 2.5 Suppose that $\oplus$ is a binary operation on some set $X,$ and $a \in X$ is such that $a \oplus x \neq x,$ for every $x \in X.$ There is no infinitary operation $\bigoplus$ on $X$ such that

$$x_0 \oplus \bigoplus_{i \in \omega} x_{i+1} = \bigoplus_{i \in \omega} x_i$$

for every sequence $(x_i)_{i \in \omega}$ of elements of $X.$ Indeed, taking $x_i = a,$ for every $i \in \omega,$ and letting $x = \bigoplus_{i \in \omega} x_i,$ we get $a \oplus x = x,$ a contradiction.

3 Computing the infinite natural sum

Theorem 3.1 If $(a_i)_{i < \omega}$ is a sequence of ordinals, then there is $m < \omega$ such that the following hold, for every $n \geq m.$

$$\#_{n \leq i < \omega} a_i = \sum_{n \leq i < \omega} a_i$$  \hspace{1cm} (3)

$$\#_{i < \omega} a_i = (a_0 \# \ldots \# a_{n-1}) + \#_{n \leq i < \omega} a_i$$

$$= (a_0 \# \ldots \# a_{n-1}) + \sum_{n \leq i < \omega} a_i$$  \hspace{1cm} (4)

Proof. Let $\xi$ be the smallest ordinal such that the set $\{i \in \omega \mid a_i \geq \omega^\xi\}$ is finite. Let $m$ be the smallest index such that $a_i < \omega^\xi,$ for every $i \geq m.$ The definition of $\xi$ assures the existence of such an $m.$ If $\xi = 0,$ then all but finitely many $a_i$'s are 0 and the proposition is trivial.

Suppose that $\xi$ is a successor ordinal, say $\xi = \varepsilon + 1.$ By the minimality of $\xi,$ the set $\{i \in \omega \mid a_i \geq \omega^\varepsilon\}$ is infinite, hence unbounded in $\omega.$ It follows that $\#_{n \leq i < \omega} a_i \geq \sum_{n \leq i < \omega} a_i \geq \omega^\varepsilon \omega = \omega^{\varepsilon+1} = \omega^\xi.$ Suppose that $\xi$ is limit. By the definition of $\xi,$ we have that, for every $\varepsilon < \xi,$ there are infinitely many $i < \omega$ such that $a_i \geq \omega^\varepsilon.$ In particular, we can choose such an $i$ with $i \geq n.$ Then $\sum_{n \leq i < \omega} a_i \geq a_i \geq \omega^\varepsilon.$ Since this holds for every $\varepsilon < \xi,$ we get $\#_{n \leq i < \omega} a_i \geq \sum_{n \leq i < \omega} a_i \geq \sup_{\varepsilon < \xi} \omega^\varepsilon = \omega^\xi.$ The inequality $\#_{i \leq \omega} a_i \geq \sum_{n \leq i < \omega} a_i \geq \omega^\xi$ is proved, no matter whether $\xi$ is successor or limit.

On the other hand, because of the definition of $m,$ if $i \geq n \geq m,$ then $a_i < \omega^\xi.$ By Proposition 2.2(3), $a_n \# \ldots \# a_{\xi-1} < \omega^\xi,$ for every $\xi \geq n.$ Hence $\sum_{n \leq i < \omega} a_i \leq \#_{i < \omega} a_i = \sup_{\varepsilon < \omega}(a_0 \# \ldots \# a_{\xi-1}) \leq \omega^\xi.$ In conclusion,

$$\#_{n \leq i < \omega} a_i = \sum_{n \leq i < \omega} a_i = \omega^\xi$$ \hspace{1cm} (5)

thus we have proved (3).

Let us now prove (4). The inequality $\#_{i \leq \omega} a_i \geq (a_0 \# \ldots \# a_{\xi-1}) + \#_{n \leq i < \omega} a_i$ is trivial, since every partial sum on the right is bounded by the partial sum on the left having the same length, by Proposition 2.2(2). For the other direction, and recalling that $S_i$ denotes $a_0 \# \ldots \# a_{\xi-1},$ observe that, by associativity, for every $\ell \geq n,$ we have $S_{\ell} = S_n \# a_n \# \ldots \# a_{\xi-1} < S_n + \omega^\xi = S_n + \#_{n \leq i < \omega} a_i,$ where the strict inequality follows from repeated applications of Proposition 2.2(4), since $a_n, \ldots, a_{\xi-1} < \omega^\xi.$ The last identity is from equation (5). Since $\#_{i \leq \omega} a_i = \sup_{i < \omega} S_i$ and since $S_i$ is increasing, we get $\#_{i \leq \omega} a_i \leq S_n + \#_{n \leq i < \omega} a_i.$

The identity $\#_{i < \omega} a_i = (a_0 \# \ldots \# a_{n-1}) + \sum_{n \leq i < \omega} a_i$ is now immediate from (3). It can be also proved in a way similar to above. \hfill \Box

Notice that the sum + in equation (4) cannot be replaced by a natural sum #, that is, we do not have, in general, $\#_{i \leq \omega} a_i = S_n \# \#_{n \leq i < \omega} a_i,$ nor we have $\#_{i \leq \omega} a_i = S_n \# \sum_{n \leq i < \omega} a_i.$ This is similar to the argument in Example 2.5: just take $a_i = 1,$ for every $i \in I,$ then $\#_{i \leq \omega} a_i = \omega$ but $S_n \# \#_{n \leq i < \omega} a_i = S_n \# \sum_{n \leq i < \omega} a_i = n \# \omega = \omega + n.$ However, in Corollary 5.1 we shall show that the computation of a countable natural sum can be actually reduced to the computation of some finite natural sum.
Remark 3.2 Notice that equation (3) in Theorem 3.1, together with Proposition 2.4(5), imply that if \((\alpha_i)_{i<\omega}\) is a sequence of ordinals, \(m\) is given by Theorem 3.1, and \(n \geq m\), then \(\sum_{n \leq i < \omega} \alpha_i = \#_{n \leq i < \omega} \alpha_i = \sum_{n \leq i < \omega} \alpha_{\pi(i)}\), for every permutation \(\pi\) of \([n, \omega)\). Actually, equation (5) in the proof of Theorem 3.1 shows that it is enough to assume that \(\pi\) is a bijection from \([n, \omega)\) to \([n', \omega)\), for some \(n' \geq m\) (equation (5) does not hold if \(\xi = 0\), but this case is trivial).

The result in the present remark can be obtained also as a consequence of a theorem by Sierpiński, asserting that a countable sum of nondecreasing ordinals is invariant under permutations [9]. Just notice that every sequence of ordinals is nondecreasing from some point on. On the other hand, Sierpiński’s result is immediate from equation (5) and Proposition 2.4(5), since in the case at hand \(m = 0\). Thus parts of the present note can be seen as an extension of results from [9] to natural sums.

4 Some kinds of mixed sums

The definition of a mixed sum of two ordinals can be obviously extended to deal with infinitely many ordinals.

Definition 4.1 Let \((\alpha_i)_{i \in I}\) be any sequence of ordinals (with no restriction on the cardinality of \(I\)). We say that an ordinal \(\gamma\) is a mixed sum of \((\alpha_i)_{i \in I}\) if there are pairwise disjoint subsets \((A_i)_{i \in I}\) of \(\gamma\) such that \(\bigcup_{i \in I} A_i = \gamma\) and, for every \(i \in I\), \(A_i\) has order type \(\alpha_i\), with respect to the order induced on \(A_i\) by \(\gamma\).

In the above situation, we say that \(\gamma\) is a mixed sum of \((\alpha_i)_{i \in I}\) realized by \((A_i)_{i \in I}\), or simply that \((A_i)_{i \in I}\) is a realization of \(\gamma\). Notice that \(\alpha_i\) can be recovered by \(A_i\), as embedded in \(\gamma\).

Notice that we could have given the above definition just under the assumption that \(\gamma\) and the \(\alpha_i\)'s are linearly ordered sets, not necessarily well ordered. In this respect, notice that any finite mixed sum of well ordered sets is itself necessarily well ordered; however, in case \(I\) is infinite, the \(\alpha_i\)'s could mix themselves to a non well ordered set. For example, starting with countably many 1's, we could obtain every countably infinite linear order as a mixed sum. Throughout this note, however, and no matter how interesting the general case of linear orders is, we shall always assume that \(\gamma\) is an ordinal, that is, well ordered.

Theorem 4.2 (Carruth, Neumer; [3, 8]) For every \(n < \omega\) and ordinal numbers \(\alpha_0, \ldots, \alpha_n\), the largest mixed sum of \((\alpha_i)_{i \leq n}\) exists and is \(\alpha_0 \# \alpha_1 \# \ldots \# \alpha_n\).

As we hinted in the introduction, and contrary to the finite case, the set of all the mixed sums of an infinite sequence of ordinals need not have a maximum. As above, if we take \(\alpha_i = 1\) for every \(i < \omega\), then every finite countable ordinal is a mixed sum of \((\alpha_i)_{i \leq \omega}\), thus the supremum of all the mixed sums of \((\alpha_i)_{i \leq \omega}\) is \(\alpha_\omega\), which is not a mixed sum of \((\alpha_i)_{i \leq \omega}\). Hence there is some interest in restricting ourselves to well-behaved mixed sums.

Definition 4.3 We say that \(\gamma\) is a left-finite mixed sum of \((\alpha_i)_{i \in I}\) if \(\gamma\) can be realized as a mixed sum by \((A_i)_{i \in I}\) in such a way that, for every \(\delta < \gamma\), the set \(\{i \in I \mid A_i \cap [0, \delta) \neq \emptyset\}\) is finite; in words, for every \(\delta < \gamma\), the predecessors of \(\delta\) are all taken from finitely many \(A_i\)'s.

Given a realization \((A_i)_{i \in I}\) of \(\gamma\) and \(i \in I\), we say that \(A_i\) is convex (in \(\gamma\)) if \([a, a']_\gamma = \{\delta \in \gamma \mid a \leq \delta \leq a'\} \subseteq A_i\), whenever \(a < a' \in A_i\).

We say that \(\gamma\) is a piecewise convex (an almost piecewise convex) mixed sum of \((\alpha_i)_{i \in I}\) if \(\gamma\) can be realized in such a way that all the \(A_i\)'s (all but a finite number of the \(A_i\)'s) are convex in \(\gamma\).

If \(\gamma\) is a piecewise convex mixed sum of \((\alpha_i)_{i \in I}\), as realized by \((A_i)_{i \in I}\), then, for every \(i \neq j \in I\) and \(\delta, \varepsilon \in A_i, \delta', \varepsilon' \in A_j\), we have that \(\delta < \delta'\) if and only if \(\varepsilon < \varepsilon'\). In this way, if each \(A_i\) is nonempty, the order on \(\gamma\) induces an order (in fact, a well order) on \(I\). Hence we can reindex \((A_i)_{i \in I}\) as \((A_{\pi(i)}(i)_{i \in \omega}\) for some ordinal \(\vartheta\) and some bijection \(\pi : \vartheta \to I\) in such a way that \(\delta < \delta'\), whenever \(\delta \in A_{\pi(i)}, \delta' \in A_{\pi(i)}\) and \(i' < i\). Then an easy induction shows that \(\gamma = \sum_{i < \omega} A_{\pi(i)}\). If in addition the realization is left finite, then necessarily \(\vartheta \leq \omega\).

Conversely, if \(\gamma = \sum_{i < \omega} A_{\pi(i)}\), for some reindexing of the \(\alpha_i\)'s, then trivially \(\gamma\) is a piecewise convex mixed sum of \((\alpha_i)_{i \in I}\) and if \(\vartheta \leq \omega\), then \(\gamma\) is also left finite. We have proved the next proposition.

Proposition 4.4 Suppose that \((\alpha_i)_{i \in I}\) is a sequence of ordinals, and \(\alpha_i > 0\), for every \(i \in I\). Then \(\gamma\) is a piecewise convex (is realized as a piecewise convex and left-finite) mixed sum of \((\alpha_i)_{i \in I}\) if and only if there are some ordinal \(\vartheta\) (with \(\vartheta \leq \omega\)) and a bijection \(\pi : \vartheta \to I\) such that \(\gamma = \sum_{i < \omega} A_{\pi(i)}\).
Remark 4.5 There might be infinitely many left-finite mixed sums of the same sequence. Indeed, take \( \alpha_i = \omega \), for every \( i < \omega \). Since \( \omega \) is the union of countably many disjoint countably infinite sets, we see that \( \omega \) is a (necessarily left-finite) mixed sum of \( (\alpha_i)_{i < \omega} \). By moving just one copy of \( \omega \) to the bottom we get that also \( \omega + \omega \) is a left-finite mixed sum of \( (\alpha_i)_{i < \omega} \). Iterating, for every \( n < \omega \) we get \( \omega^n \) as a left-finite mixed sum of \( (\alpha_i)_{i < \omega} \).

Also \( \omega^2 \) is a left-finite mixed sum of \( (\alpha_i)_{i < \omega} \), by Proposition 4.4 it is the only one which is left-finite and piecewise convex; actually, it is the only one which is left-finite and almost piecewise convex. We shall see in Theorem 5.3 below that a sequence has at most a finite number of mixed sums which are simultaneously realized as left-finite and almost piecewise convex.

Remark 4.6 Let \( \alpha_0 = \omega^\alpha \) and \( \alpha_1 = \alpha_2 = \cdots = \omega \). Then \( \omega^3 + \omega \) can be realized as a left-finite mixed sum of \( (\alpha_i)_{i < \omega} \); just put \( \omega^3 \) at the bottom and join all the copies of \( \omega \) into one, as in the above remark. Moreover, \( \omega^\alpha + \omega \) can be realized also as a piecewise convex mixed sum of \( (\alpha_i)_{i < \omega} \). However, there is no realization of \( \omega^3 + \omega \) which is simultaneously left-finite and piecewise convex. This is an immediate application of Proposition 4.4.

A similar comment applies to \( \omega^n + \omega^m \), for every \( n > 0 \).

Theorem 4.7 If \( (\alpha_i)_{i < \omega} \) is a sequence of ordinals, then \( \#_{i < \omega} \alpha_i \) is a mixed sum of \( (\alpha_i)_{i < \omega} \). In fact, \( \#_{i < \omega} \alpha_i \) is the largest left-finite mixed sum of \( (\alpha_i)_{i < \omega} \), and also the largest left-finite and almost piecewise convex mixed sum of \( (\alpha_i)_{i < \omega} \).

Proof. By equation (4) in Theorem 3.1, we have \( \#_{i < \omega} \alpha_i = (\alpha_0 \# \cdots \# \alpha_{n-1}) + \sum_{i < j < \omega} \alpha_i \), for some \( n < \omega \). By Theorem 4.2, \( \gamma_1 = \alpha_0 \# \cdots \# \alpha_{n-1} \) is a mixed sum of \( \alpha_0, \ldots, \alpha_{n-1} \). By the easy part of Proposition 4.4, \( \gamma_2 = \sum_{i < j < \omega} \alpha_i \) is a left-finite piecewise convex mixed sum of \( (\alpha_i)_{i < \omega} \). Putting the members of the realization of \( \gamma_1 \) at the bottom, and (isomorphic copies of) the members of the realization of \( \gamma_2 \) at the top, we realize \( \#_{i < \omega} \alpha_i = \gamma_1 + \gamma_2 \) as a left-finite and almost piecewise convex mixed sum of \( (\alpha_i)_{i < \omega} \).

To finish the proof of the theorem it is enough to show that if \( \gamma \) is any left-finite mixed sum of \( (\alpha_i)_{i < \omega} \), then \( \gamma \leq \#_{i < \omega} \alpha_i \). Let \( \gamma \) be a left-finite mixed sum of \( (\alpha_i)_{i < \omega} \) as realized by \( (A_i)_{i < \omega} \). If all but a finite number of the \( A_i \)'s are 0, then the result is immediate from Theorem 4.2. Otherwise, left finiteness implies that \( \gamma \) is a limit ordinal. If \( \gamma' < \gamma \), then \( (0, \gamma') \cap A_i \), \( i < \omega \), witnesses that \( \gamma' \) is a mixed sum of \( \beta_i \), \( i < \ell \), where, for every \( i < \omega \), \( \beta_i \) is the order type of \( (0, \gamma') \cap A_i \); thus, \( \beta_i \leq \alpha_i \). Left finiteness implies that only a finite number of the \( \beta_i \)'s are nonzero, thus, again by Theorem 4.2, \( \gamma' \leq \beta_i \# \cdots \# \beta_1 \), for certain distinct indices \( i_1, \ldots, i_\ell \). Taking \( n = \sup\{i_1, \ldots, i_\ell \} \), we get \( \gamma' \leq \beta_i \# \cdots \# \beta_1 \leq \alpha_i \# \cdots \# \alpha_i \leq \alpha_0 \# \cdots \# \alpha_n \leq \#_{i < \omega} \alpha_i \). Since \( \gamma \) is limit and \( \gamma' \leq \#_{i < \omega} \alpha_i \), for every \( \gamma' < \gamma \), we get \( \gamma \leq \#_{i < \omega} \alpha_i \).

\[ \] 5 Expressing sums in terms of the normal form

The proof of Theorem 3.1 gives slightly more. Let \( \alpha \) and \( \xi \) be ordinals, and express \( \alpha \) in Cantor normal form as \( \omega^{\eta_0} r_\ell + \cdots + \omega^{\eta_0} r_0 \). The ordinal \( \alpha^{\lfloor \xi \rfloor} \), in words, \( \alpha \) truncated at the \( \xi \)th exponent of \( \omega \), is \( \omega^{\eta_0} r_\ell + \cdots + \omega^{\eta_0} r_0 \), where \( \ell \) is the smallest index such that \( \eta_\ell \geq \xi \). The above definition should be intended in the sense that \( \alpha^{\lfloor \xi \rfloor} = 0 \) in case that \( \alpha < \omega^{\lfloor \xi \rfloor} \).

Corollary 5.1 Suppose that \( (\alpha_i)_{i < \omega} \) is a sequence of ordinals, and let \( \xi \) be the smallest ordinal such that \( \{ i < \omega \mid \alpha_i \geq \omega^{\lfloor \xi \rfloor} \} \) is finite. Enumerate those \( \alpha_i \)'s such that \( \alpha_i \geq \omega^{\lfloor \xi \rfloor} \) as \( \alpha_{i_0}, \ldots, \alpha_{i_\ell} \), with \( i_0 < \cdots < i_\ell \) (the sequence might be empty). If \( \xi > 0 \), then

\[
\#_{i < \omega} \alpha_i = (\alpha_{i_0} \# \cdots \# \alpha_{i_\ell}) + \omega^{\xi} = \alpha_{i_0}^{\lfloor \xi \rfloor} \# \cdots \# \alpha_{i_\ell}^{\lfloor \xi \rfloor} \# \omega^{\xi} \quad \text{and}
\]

\[
\sum_{i < \omega} \alpha_i = \alpha_{i_0} + \cdots + \alpha_{i_\ell} + \omega^{\xi} = \alpha_{i_0}^{\lfloor \xi \rfloor} + \cdots + \alpha_{i_\ell}^{\lfloor \xi \rfloor} + \omega^{\xi};
\]

moreover, for every \( \varepsilon < \xi \), we have

\[
\#_{i < \omega} \alpha_i = \#_{i < \omega} \alpha_i^{\lfloor \varepsilon \rfloor} \quad \text{and} \quad \sum_{i < \omega} \alpha_i = \sum_{i < \omega} \alpha_i^{\lfloor \varepsilon \rfloor}.
\]
Proof. The $\xi$ defined in the statement of the present corollary is the same as the $\xi$ defined in the proof of Theorem 3.1; and the $\alpha_i$ defined here is the same as $a_{m-1}$ in that proof (if the sequence of the $\alpha_i$’s is not empty). Equation (5) in the proof of Theorem 3.1 gives $\#_{m\leq i<\omega} \alpha_i = \omega^\xi$. By commutativity and associativity of $\#$, and using Proposition 2.2(5), equation (4) in Theorem 3.1 becomes exactly the first identity in equation (6). The second identity is easy ordinal arithmetic, noticing that $\alpha + \omega^\xi = \alpha^\xi + \omega^\xi$ and $(\alpha \# \beta)^\xi = \alpha^\xi \# \beta^\xi$, for every $\alpha$ and $\beta$.

The proof of (7) is similar, using the fact that $\sum_{i<\omega} \alpha_i = \alpha_0 + \cdots + \alpha_{m-1} + \sum_{m \leq i<\omega} \alpha_i$. Then one should use the identity $\beta + \gamma + \omega^\xi = \gamma + \omega^\xi$, holding whenever $\beta < \omega^\xi$. Indeed, if $\gamma < \omega^\xi$, then all sides are equal to $\omega^\xi$; otherwise, if $\gamma \geq \omega^\xi$, then $\beta$ is absorbed by $\gamma$, since it is already absorbed by the leading term in the Cantor normal expression of $\gamma$. The whole argument for (7) is similar to [9].

To prove the last two identities, notice that if $\varepsilon < \xi$, then $\xi$ is also the least ordinal such that $\{i < \omega | \alpha_i^{\varepsilon} \geq \omega^\xi\}$ is finite. Hence we can apply (6) twice to get $\#_{1<\omega} \alpha_i^{\varepsilon} = (\alpha_{i_0}^{\varepsilon})^\xi \# \cdots \# (\alpha_{i_m}^{\varepsilon})^\xi \# \omega^\xi = \alpha_{i_0}^{\varepsilon} \# \cdots \# \alpha_{i_m}^{\varepsilon} \# \omega^\xi = \#_{i<\omega} \alpha_i$. The last identity is proved in the same way, using equation (7).

Notice that Corollary 5.1 furnishes a method to compute $\#_{1<\omega} \alpha_i$ and $\sum_{i<\omega} \alpha_i$ in terms of the Cantor normal forms of the $\alpha_i$’s, in fact, of just finitely many $\alpha_i$’s, once $\xi$ has been determined.

One cannot expect that, for every sequence $(\alpha_i)_{i<\omega}$ of ordinals, there is some permutation of $\omega$ such that $\#_{1<\omega} \alpha_i$ is a mixed sum of $\alpha_i$’s, once $\xi$ is determined. Indeed, if $\alpha_i = \varepsilon$ for all $i$, and $\varepsilon < \xi$, then $\sum_{1<\omega} \alpha_i^{\varepsilon} = \#_{1<\omega} \alpha_i^{\varepsilon} = \omega^\xi$.

Lemma 5.2 Suppose that $\gamma$ is a mixed sum of $\alpha_1$ and $\alpha_2$ with $\alpha_1 < \alpha_2 = \omega^\xi$, for some $\xi > 0$, and the mixed sum is realized by $A_1, A_2$ in such a way that $A_2$ is cofinal in $\gamma$. Then $\gamma = \omega^\xi$.

Proof. Trivially, $\gamma \geq \alpha_2 = \omega^\xi$. For the other direction, let $\delta < \gamma$. Thus $\delta$ is a mixed sum of $\beta_1, \beta_2$, where $\beta_1, \beta_2$ are the order types of, respectively, $A_1 \cap [0, \delta)$, $A_2 \cap [0, \delta)$. Since $\delta < \gamma$, $A_2$ is cofinal in $\gamma$ and $A_2$ has order type $\omega^\xi$, then $A_2 \cap [0, \delta)$ has order type $< \omega^\xi$. Moreover $A_1 \cap [0, \delta)$ has order type $\leq \alpha_1 < \omega^\xi$. Since $\delta$ is a mixed sum of $\beta_1$ and $\beta_2$, then, by Theorem 4.2, $\delta \leq \beta_1 \# \beta_2$. We have showed that $\beta_1, \beta_2 < \omega^\xi$, hence $\delta \leq \beta_1 \# \beta_2 < \omega^\xi$, by Proposition 2.2(3). Since $\delta < \omega^\xi$, for every $\delta < \gamma$, then $\gamma \leq \omega^\xi$.

Theorem 5.3 If $(\alpha_i)_{i<\ell}$ is a sequence of ordinals, then there are at most a finite number of ordinals which can be realized as a mixed sum of $(\alpha_i)_{i<\ell}$ in such a way that the realization is both left-finite and almost piecewise convex.

Proof. If there is some left-finite mixed sum of $(\alpha_i)_{i<\ell}$, then necessarily all but countably many $\alpha_i$’s are 0. Hence it is no loss of generality to assume that $|I| \leq \omega$. If all but a finite number of the $\alpha_i$’s are 0, then the corollary follows from a theorem by Läuchli, asserting that a finite set of ordinals has only a finite number of mixed sums [6]. In conclusion, we can assume that $I = \omega$ and $\alpha_i \neq 0$, for every $i < \omega$.

Suppose that $\gamma$ is a left-finite and almost piecewise convex mixed sum of $(\alpha_i)_{i<\omega}$, as realized by $(A_i)_{i<\omega}$. Assume the notation in Corollary 5.1. We shall show that $\gamma$ is a mixed sum of $\alpha_{i_0}, \ldots, \alpha_{i_b}, \omega^\xi$. This will give the result by the mentioned theorem of Läuchli’s [6].

Since $\gamma$ is realized as an almost piecewise convex mixed sum, there is $n$ such that $A_i$ is convex, for every $i \geq n$. Without loss of generality, choose $n > i_n$, or, which is the same, $n \geq m$, where $m$ is given by Theorem 3.1. Let $A = \bigcup_{i \geq n} A_i$, thus $\gamma$ is realized as a finite mixed sum by $A_0, \ldots, A_{n-1}, A$. Moreover, by the left finiteness of the realization $(A_i)_{i<\ell}$, and since we have assumed that the $A_i$’s are nonempty, we get that $A$ is cofinal in $\gamma$. The order type of $A$ is $\sum_{i<\omega} \alpha_{\pi(i)}$, for some permutation of $\pi$ of $[n, \omega)$, by Proposition 4.4, since $A = \bigcup_{i> I_0} A_i$, the $A_i$’s are convex (in $\gamma$, hence, a fortiori, in $A$), for $i \geq n$, and the realization $(A_i)_{i<\omega}$ is left-finite. By Remark 3.2 and equation (5) in the proof of Theorem 3.1, we get that $A$ has order type $\omega^\xi$. Now let $F = \{0, \ldots, n-1\}\setminus\{i_0, \ldots, i_b\}$, and set $A' = A \setminus \bigcup_{i \in F} A_i$. By repeated applications of Lemma 5.2, we get that $A'$ has order type $\omega^\xi$. Since $\gamma = A' \cup A_{i_0} \cup \cdots \cup A_{i_b}$, then $\gamma$ is a mixed sum of $\omega^\xi, \alpha_{i_0}, \ldots, \alpha_{i_b}$, what we wanted to prove.

In view of Proposition 4.4, Theorem 5.3 extends another classical result from Sierpiński, which asserts that $\sum_{i<\omega} \alpha_{\pi(i)}$ assumes only finitely many values, $\pi$ varying among all permutations of $\omega$ [9].
By Remark 4.5, the assumption of almost piecewise convexity is necessary in Theorem 5.3.

Remark 5.4 A shifted sum is defined in the same way as a mixed sum, except that we do not require the $A_i$’s to be pairwise disjoint. Cf. [7] for applications of finite shifted sums. Since every mixed sum is a shifted sum and, on the other hand, given a shifted sum of $(a_i)_{i \in I}$, there is always some larger mixed sum of $(a_i)_{i \in I}$, we get that Theorem 4.2 holds for shifted sums in place of mixed sums. Then the proof of Theorem 4.7, too, carries over to get the result for shifted sums in place of mixed sums. On the other hand, the analogue of Theorem 5.3 does not hold for shifted sums. Indeed, let $a_i = \omega$, for $i < \omega$. Then $\omega$ can be realized as a left-finite piecewise convex shifted sum of $(a_i)_{i < \omega}$ by $A_i = \{i, \omega\}$. Then we can get infinitely many left-finite piecewise convex shifted sums of $(a_i)_{i < \omega}$ arguing as in Remark 4.5.

It is clear that, when restricted to the class of ordinals, the surreal number addition is the natural sum. Cf. [1, 4, 5] for information about the surreal numbers. Corollary 5.1 suggests the possibility of defining an infinitary natural sum for an $\omega$-indexed sequence of surreal numbers which have positive coefficients in their Conway normal representation. Recall that every surreal number $x$ can be uniquely expressed in Conway normal form as $x = \sum_{r \in S} \omega^i r_i$, where $S$ is a reverse-well-ordered set of surreal numbers and the $r_i$ are nonzero real numbers. In case $x$ is an ordinal the Conway and the Cantor normal forms coincide.

Let $(x_i)_{i < \omega}$ be a countable sequence of surreal numbers with normal forms $x_i = \sum_{s \in S} \omega^i r_{s,i}$ and such that all the $r_{s,i}$’s are positive (we shall show that this request can be somewhat weakened). Let $S = \bigcup_{i < \omega} S_i$ and define a subset $S^*$ of $S$ by declaring $s \in S^*$ if and only if $\{t \in S \mid t \geq s\}$ is reverse-well-ordered. It might well happen that $S^* = \emptyset$; however, in any case, $S^*$ is reverse-well-ordered. For $s \in S$, let $c_s = \sum_{i} r_{s,i}$, where the sum is taken among all $i < \omega$ such that $s \in S_i$. This might be either a finite sum or a countably infinite sum of positive real numbers; in the latter case we consider it as an infinite sum in the sense of classical analysis. We allow the possibility $c_s = \infty$, that is, $c_s = \omega$ in the surreal sense.

Suppose first that $S = S^*$. In this case we define

$$\#_{i < \omega} x_i := \sum_{s \in S} \omega^i c_s \quad (8)$$

This is a well-defined surreal number, since $S = S^*$, hence $S$ is reverse-well-ordered. Strictly speaking, equation (8) does not necessarily give a normal form representation, due to the possibility that some $c_s$ is $\omega$. Formally, $\#_{i < \omega} x_i = \sum_{i < \omega} \omega^i c_i + \sum_{s \in U} \omega^{s+1}$, where $U = \{s \in S \mid c_s = \omega\}$ and $T = S \setminus U$.

Notice also that, in order to give the definition in equation (8), we only need that $S = S^*$ and that the sums $\sum_i r_{s,i}$ are well-defined in the sense of classical analysis, no matter whether all the $r_{s,i}$ are positive.

Now suppose that $S \neq S^*$ and let $\bar{s}$ be the surreal number $[S\setminus S^* \mid \emptyset]$. In this case we define

$$\#_{i < \omega} x_i := \omega^{\bar{s}} + \sum_{s \in S^*} \omega^i c_s \quad (9)$$

The definition in equation (9) makes sense just under the assumption that $c_s = \sum_i r_{s,i}$ is well-defined, for every $s \in S^*$. However, it is natural to ask that there is $s \in S \setminus S^*$ such that $\sum_i r_{s,i}$ is well-defined and strictly positive, for every $t \in S \setminus S^*$ with $t \geq s$. In case that there is $s \in S \setminus S^*$ such that $\sum_i r_{s,i}$ is well-defined and strictly negative, for every $t \in S \setminus S^*$ with $t \geq s$, a more natural definition would be $\#_{i < \omega} x_i = -\omega^{\bar{s}} + \sum_{s \in S^*} \omega^i c_s$.

Notice that the definitions given by (8) and (9) do not always coincide with Definition 2.3, when each $x_i$ is an ordinal.

The definition given in the present remark is quite tentative, and is not the only possible one. Notice that the definition of $\#_{i < \omega} x_i$, given here for surreal numbers does not satisfy the analogue of Proposition 2.4(2). Indeed, let $x_i = \omega^i$ and $y_i = \omega^{i-2}$, for $i < \omega$. Then $x_i < y_i$, for every $i < \omega$; however, $\#_{i < \omega} x_i = \omega^{\omega} > \omega^{\omega-1} = \omega^{\omega-2}\omega = \#_{i < \omega} y_i$.

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