Intrinsic properties of the bars formed by the bar instability in flat stellar discs

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ABSTRACT

The properties of the bars formed by the bar instability are examined for flat stellar discs. The initial mass models chosen are Kuzmin–Toomre discs, for which two types of exact equilibrium distribution function (DF) are employed in order to realize different distributions of Toomre’s $Q$ values along the radius. First, the most linearly unstable, global two-armed modes (MLUGTAMs) of these disc models are determined by numerically solving the linearized collisionless Boltzmann equation. Next, we carry out $N$-body simulations whose models are constructed from the DFs adopted above. The latter simulations unravel that the MLUGTAMs corresponding to those obtained from the former modal calculations are excited in the early phases of evolution, finally being deformed into bars in the nonlinear regime by the bar instability. We show that for simulated bars, the length increases and the axis ratio, in essence, decreases as the amplitude increases. These correlations are almost similar to those of the observed bars. In addition, we find that these bar properties are tightly correlated with the initial typical $Q$ value, irrespective of the DF. In conclusion, a disc with a smaller typical $Q$ value produces a bar which is smaller in amplitude, shorter in length and rounder in shape. This finding might suggest that the Hubble sequence for barred galaxies is the sequence of decreasing $Q$ from SBa to SBc or SBd. The implied correlations between the initial typical $Q$ value and each of the bar properties are discussed on the basis of the characteristics of the MLUGTAMs.

Key words: methods: numerical – galaxies: bar – galaxies: disc – galaxies: kinematics and dynamics – galaxies: structure.

1 INTRODUCTION

Spiral and barred structures are the outstanding features in disc galaxies. Above all, bars can play an important role in the central activities and the secular evolution of disc galaxies as well as the dynamical one. For example, a bar is regarded as a key agent effective in extracting angular momentum from the gas in a disc, and as a result, also effective in fuelling the gas into the nucleus (Athanassoula 1992; Wada & Habe 1995; Patsis & Athanassoula 2000; Jogee, Scoville & Kenney 2005; Sheth et al. 2005). Eventually, such gas inflow could drive the central activities of disc galaxies (Friedli, Benz & Kennicutt 1994; Ellison et al. 2011). In addition, a bar as a wave pattern can cause wave-particle interactions with halo particles, so that the disc angular momentum is transferred to the halo (Athanassoula 2002, 2003; Debattista et al. 2006; Sellwood 2014). In this manner, the existence of a bar propels the secular evolution of a disc through such an angular momentum redistribution. The details of the phenomena mentioned above will hinge on the bar properties. In fact, as demonstrated from numerical simulations, the efficiency of transferring the gas to the centre of a galaxy along its bar depends sensitively on the bar strength that is one of the significant bar properties (Athanassoula 1992; Friedli & Benz 1993; Regan & Teuben 2004). Accordingly, elucidating the origin of the bar properties will finally lead to a better understanding of the complete picture of how barred galaxies evolve inherently.

From an observational point of view, the fraction of barred galaxies amounts to more than two-thirds of the entire disc galaxies, if not only strongly but also weakly barred galaxies are included (Knapen, Shlosman & Peletie 2000; Eskridge et al. 2000; Laurikainen, Salo & Buta 2004; Menéndez-Delmestre et al. 2007; Marinova & Jogee 2007; Barazza, Jogee & Marinova 2008; Aguerri, Méndez-Abreu & Corsini 2009; Buta et al. 2015; Díaz-García et al. 2016a; Díaz-García, Salo & Laurikainen 2016b). As a note, Block & Wainscoat (1991) found that the bar fraction is more enhanced in near infrared than in optical wavelengths, and subsequent related studies reinforce their finding (Grosbøl & Patsis 1998; Márquez et al. 1999; Eskridge et al. 2000; Menéndez-Delmestre et al. 2007; Laurikainen et al. 2011). In addition, Sheth et al. (2008) and Melvin et al. (2014) have revealed that the bar fraction increases with time over the last 7 or 8 Gyr. On the other hand, Elmegreen, Elmegreen & Hirst (2004) and Jogee et al. (2004) advocate that the observed bar fraction is nearly constant over time, and Simmons et al. (2014) have also reported that the bar fraction in the selected redshift range of $0.5 \leq z \leq 2$ shows no significant evolution. In either case, barred galaxies outnumber non-barred galaxies. This fact may indicate that galactic discs are prone to form bars, the reason of which might be found out by investigating how the bar properties are acquired.

On the background of an increasing number of observed barred galaxies, some properties of galactic bars have been uncovered on a statistically meaningful level of confidence. For instance, Erwin (2005) has shown that bars in S0-Sb galaxies are longer than those in...
Sc-Sd galaxies, which was pointed out before with relatively small samples in the literature (e.g. Martin 1995; Elmegreen & Elmegreen 1985; Chapelon, Contini & Davoust 1999; Laine et al. 2002; Laurikainen & Salo 2002; Laurikainen, Salo & Rautiainen 2002). Subsequently, Elmegreen et al. (2007) noticed a positive correlation between the length and the amplitude of bars. Thereafter, similar results have been presented by Díaz-García et al. (2016a), Guo et al. (2019) and Cuomo et al. (2019, 2020). Moreover, Menéndez-Delmestre et al. (2007) and Hoyle et al. (2011) unravelled that larger amplitude bars are more elongated in shape. These correlations might be formed by the difference in evolving speed. That is, because galaxies with higher central densities transfer angular momentum faster from a bar to the surrounding components such as a disc and a halo, a bar changes its properties over time (Elmegreen et al. 2007; Menéndez-Delmestre et al. 2007; Hoyle et al. 2011). However, this line of explanation does not necessarily exclude the view that the bar properties are different at the epoch of bar formation owing to the difference in the physical properties of the galactic discs that lead to barred structures in the end.

As mentioned above, recent observations have exposed the detailed properties of bars for each class of disc galaxies along the Hubble sequence, while the mechanism of bar formation itself is not yet fully understood. Empirically, we know from numerical simulations that galactic discs are dynamically unstable to result in bars via the bar instability (e.g. Hohl 1971; Sellwood 1981; Athanassoula & Sellwood & Wilkinson 1993). On the other hand, as noted above, we also know that non-barred galaxies do exist at no small fraction of disc galaxies in the real Universe. Therefore, the central issue of the disc dynamics in the past was to disclose how the discs were stabilized to survive as unbarred. As a result, some stability criteria of flat discs against bar formation were proposed (Ostriker & Peebles 1973; Efstathiou, Lake & Negroponte 1982), although they stood on an empirical basis. In a reflection of this circumstance, we did not pay, in essence, any attention to the bar properties generated by the bar instability at that time. Recent N-body simulations using reasonably realistic models with a huge number of particles have made it possible to compare simulated bars with observed ones, and they have revealed that the buckling instability deforms a bar which is caused by the bar instability into a boxy/peanut-shaped or X-shaped bulge. In fact, so-called pseudobulges which exhibit boxy/peanut/X-shaped features have been observed in real barred galaxies (Bureau & Freeman 1999; Lütticke, Dettmar & Pohlen 2000a,b; Yoshino & Yamauchi 2015; Erwin & Debattista 2017; Patsis et al. 2021). In particular, our Galaxy and M31 are considered to have such pseudobulges (Dwek et al. 1995; Athanassoula & Beaton 2006; Gerhard & Martinez-Valpuesta 2012; Portail et al. 2017; Ciambur, Graham & Bland-Hawthorn 2017). In this way, morphological and kinematical similarities between simulated and observed bars have been indicated in many studies (e.g. O’Neill & Dubinski 2003; Patsis & Xilouris 2006; Athanassoula et al. 2015; Xiang et al. 2021). However, we have not sufficiently understood the origin of the bar properties such as the correlation between the length and the amplitude and that between the axis ratio and the amplitude.

In relation to the destruction of a bar that is induced by a massive central black hole, Hozumi (2012) has revealed that the bar properties arising from the bar instability are well-correlated with a typical Toomre’s (1964) \( Q \) value for a given functional form of velocity distribution. That is, as a typical \( Q \) value is smaller, the resulting bar becomes lower in amplitude, shorter in length, and rounder in shape. However, the origin of these bar properties still remains unclear.

In this paper, we show that the properties of the bars are closely connected to those of the most linearly unstable, global two-armed modes (MLUGTAMs), on the basis of which their origin is considered. In Section 2, the disc models constructed from exact equilibrium distribution functions (DFs) are described. In Section 3, we explain how to determine the MLUGTAM and how to follow the evolution of an \( N \)-body model which is realized from the DF. Results including correlations between the MLUGTAMs and formed bars are presented in Section 4. We discuss the origin of the bar properties in Section 5. Conclusions are given in Section 6.

2 MODELS

In order to highlight the bar properties arising from the bar instability, we need to reduce likely factors which influence the formation and evolution of bars. If approximate equilibrium models were used, initial transients, for example, like those revealed by Fujii et al. (2011), could emerge in the early evolving phases of numerical simulations, so that the subsequent disc evolution and bar formation could be altered to some degree. Therefore, the appropriate disc models should be constructed on the basis of mathematically consistent and exact equilibrium DFs. This reasoning forces us to handle two-dimensional zero-thickness discs, although present-day computers have enabled us to easily simulate three-dimensional disc galaxies with a huge number of particles (e.g. Dubinski, Berentzen & Shlosman 2009; D’Onghia, Vogelsberger & Hernquist 2013; Fujii et al. 2018, 2019). This is because exact equilibrium DFs are known only for infinitesimally thin discs.

We adopt razor-thin Kuzmin–Toomre (K–T) discs (Kuzmin 1956; Toomre 1963), whose surface density, \( \mu \), and potential, \( \Phi \), are, respectively, given by

\[
\mu(r) = \frac{M_d}{2\pi a^2} \left(1 + \frac{r^2}{a^2}\right)^{-3/2}
\]

and

\[
\Phi(r) = -\frac{G M_d}{a} \left(1 + \frac{r^2}{a^2}\right)^{-1/2},
\]

where \( M_d \) and \( a \) are, respectively, the mass and scale length of the disc, \( G \) is the gravitational constant, and \( r \) is the distance from the disc centre.

The equilibrium DFs of directly rotating (prograde) stars for the K–T discs have been derived by Miyamoto (1971) and by Kalnajs (1976). In general, for a flat axisymmetric galaxy, the equilibrium DF of prograde stars, \( F^+ \), is represented by \( F^+(\epsilon, j) \), where \( \epsilon \) and \( j \) are, respectively, the energy and the angular momentum of a star per unit mass. On the other hand, there is no definite way of prescribing the distribution of retrograde stars. Then, we introduce them with the method devised by Nishida et al. (1984), on the ground that it includes no additional parameter. Consequently, the equilibrium DF for both prograde and retrograde stars, \( f_0(\epsilon, j) \), is written by

\[
f_0(\epsilon, j) = \begin{cases} (1/2) F_0^+(\epsilon) + F_1^+(\epsilon, j) & j \geq 0, \\ (1/2) F_0^+(\epsilon) & j < 0, \end{cases}
\]

where the functions \( F_0^+(\epsilon) \) and \( F_1^+(\epsilon, j) \) are derived from the expansion of \( F^+(\epsilon, j) \) with respect to the angular momentum such that

\[
F^+(\epsilon, j) = F_0^+(\epsilon) + F_1^+(\epsilon, j).
\]

We construct \( F^+(\epsilon, j) \) by employing the approach of Miyamoto (1971) and that of Kalnajs (1976). In both types of DFs, there is a model parameter which prescribes the kinematic structure of a disc. When denoting this parameter as \( m_K \) for Kalnajs’s DFs and
as $m_M$ for Miyamoto’s DFs, we take $m_K = 6, 7, 8, 9,$ and 10, which are termed models TK6, TK7, TK8, TK9, and TK10, respectively, and $m_M = 2, 3, 4,$ and 5, which are named models TM2, TM3, TM4, and TM5, respectively. These DFs are used for determining the MLUGTAMs with no explicit truncation of the discs, as described in Subsection 3.1, while the initial phase-space coordinates are realized with $N = 10,000,000$ particles of equal mass by truncating the discs at $r = 10a$ in order to examine the time evolution of these models with the method explained in Subsection 3.2.

In Fig. 1, we present Toomre’s (1964) $Q$ profiles for all the above-mentioned models realized with $N$ particles, along with those calculated from the analytic forms of the corresponding DFs. The models generated from Kalnaj’s DFs have slightly declining $Q$ distributions with the radius except for model TK6 that shows a slight increase in $Q$ from $r \sim 1.3a$ with the radius, while those from Miyamoto’s DFs show steeply rising $Q$ distributions with the radius such that $Q \propto (r/a)^2 + 4$. At any rate, as the model parameters, $m_M$ and $m_K$, increase, the $Q$ values at all radii decrease for both DFs. In addition, as found from Fig. 1, for all models but models TK9 and TK10, the $Q$ values are larger than unity throughout the radius, so that these discs are stable against local axisymmetric Jeans instabilities at all radii, while the $Q$ values for models TK9 and TK10 are lower than unity at large radii, so that these two models are unstable at the corresponding radii. In reality, regardless of the locally axisymmetric stability, all models are bar-unstable as shown in Subsection 4.2.

3 METHOD

The origin of the bar properties formed by the bar instability is considered to have a close connection to the MLUGTAMs, since such modes are known to deform into bars in nonlinear stages (e.g. Nishida et al. 1984). Then, we first determine the MLUGTAMs for the DFs of the models constructed in Section 2 as a basis for the

![Figure 1. Toomre's $Q$ parameters as a function of radius which is normalized by the scale length of the disc, $a$, for the models with Kalnaj's DFs (thick lines) and those with Miyamoto's DFs (thin lines). Solid lines show the $Q$ values calculated from $N$-body realization of each model, while dashed lines represent those calculated from corresponding analytical DFs.](image)

Properties of the bars by the bar instability

3.1 Determination of the fastest growing two-armed modes

The MLUGTAMs are determined by numerically integrating the linearized collisionless Boltzmann equation, which is represented by

$$\frac{df_m}{dt} = \frac{\partial \psi_m}{\partial r} \frac{\partial F_0}{\partial u} + im \psi_m \frac{\partial F_0}{\partial f_i}.$$  \hspace{1cm} (5)

where the functions $f_m$ and $\psi_m$ are the $m$th Fourier components of the perturbed part of the DF and potential, respectively, $u$ is the radial velocity, and $t$ is the time.

Equation (5) is solved for $m = 2$ modes as an initial value problem. That is, we continue integrating equation (5) with respect to the time until an imposed perturbation has reached exponentially growing phases in $f_m$. The details of the method for finding the MLUGTAMs are described by Hozumi, Fujiwara & Nishida (1987) and by Hozumi & Fujiwara (1989).

3.2 Evolution of stellar discs

The time evolution of the K–T discs is followed with a self-consistent field (SCF) method (Hernquist & Ostriker 1992; Hozumi & Hernquist 1995; Hozumi 1997). As shown analytically by Miller (1971, 1974) and numerically by Earn & Sellwood (1995), a softening length introduced in conventional $N$-body techniques has a stabilizing effect on stellar discs, so that, for example, the growth rate of the MLUGTAM is reduced considerably as compared with that obtained using unsoftened gravity, even though a relatively small softening length is assigned. Since no such gravitational softening is explicitly included in the SCF approach, we can directly compare the growth rate and pattern speed of the MLUGTAM estimated in an SCF simulation with those determined by a softening-free phase-space method of integrating equation (5).

For two-dimensional SCF simulations, we use the same method as that adopted by Hozumi & Hernquist (2005), in which it is necessary to prepare a pair of the density and potential basis functions, $(\mu_{nm}, \Phi_{nm})$, such that each pair satisfies Poisson’s equation given by

$$\nabla^2 \Phi_{nm}(r) = 4\pi G \mu_{nm}(r) \delta(z).$$  \hspace{1cm} (6)

where $r$ is the position vector in the disc plane, $\delta$ is the delta function with $z$ being the vertical coordinate to the disc plane, and $n$ and $m$ are those positive integers or zero which indicate the order in the radially and azimuthally expanded terms, respectively. In a set of these basis functions, the density and potential of the system can be expanded, respectively, in the forms

$$\mu(r) = \sum_{n=0}^{n_{max}} \sum_{m=0}^{m_{max}} A_{nm}(t) \mu_{nm}(r),$$  \hspace{1cm} (7)

and

$$\Phi(r) = \sum_{n=0}^{n_{max}} \sum_{m=0}^{m_{max}} A_{nm}(t) \phi_{nm}(r),$$  \hspace{1cm} (8)

where $A_{nm}(t)$ are the expansion coefficients at time $t$, and $n_{max}$ and $m_{max}$ are the maximum numbers of the radial and azimuthal expansion terms, respectively. We operate $\Phi_{nm}$ to the particle distribution at every time-step to obtain $A_{nm}(t)$ with the help of the
bi-orthogonality between $\mu_{nm}$ and $\Phi_{nm}$. Consequently, the accelerations, $a(r)$, are provided by

$$a(r) = -\nabla \Phi(r) = - \sum_{n=0}^{n_{\text{max}}} \sum_{m=0}^{m_{\text{max}}} A_{nm}(t) \nabla \Phi_{nm}(r),$$

(9)

where $\nabla \Phi_{nm}(r)$ can be calculated analytically in advance, when a basis set is specified. Once the accelerations for all particles are evaluated, the equations of motion are integrated in Cartesian coordinates with a time-centred leap-frog algorithm (e.g. Press et al. 1986).

The amplitude and pattern speed of the MLUGTAM at time $t$ are estimated in the same way as that carried out by Bland-Hawthorn & Hernquist (2005), although the bar amplitude itself will be defined in Subsection 4.3. Then, the amplitude is obtained from the absolute value of the expansion coefficient, $|A_{22}(t)|$, and the pattern speed is calculated from half the time change in the phase of $A_{22}(t)$.

We adopt Aoki & Iye’s (1978) basis set, which is suitable for flat discs and is constructed on the basis of the K–T discs, represented by

$$\mu_{nm}(r) = \frac{M_4}{2n a_b^2} (2n + 1) \left( 1 - \xi \right)^{3/2} P_n(\xi) \exp(i m \varphi)$$

(10)

and

$$\Phi_{nm}(r) = \frac{G M_4}{a_b} \left( 1 - \xi \right)^{1/2} P_n(\xi) \exp(i m \varphi),$$

(11)

where $P_n(\xi)$ are the Legendre functions, $\varphi$ is the azimuthal angle, and $\xi$ stands for the radial transformation defined as

$$\xi = \frac{r^2 - a_b^2}{r^2 + a_b^2}.$$  

(12)

In equations (10), (11), and (12), the scale length of the basis set, $a_b$, is not necessarily required to be equal to that of the disc, $a$. However, since we adopt the K–T discs whose functional forms are identical to those of the lowest-order members of the basis functions, we choose $a_b = a$. For SCF simulations, we set $n_{\text{max}} = 24$ and $m_{\text{max}} = 12$. In the angular expansion, only even $m$-values are retained to extract the properties of a bar easily. Furthermore, we have parallelized the SCF code in accordance with the prescription given by Hernquist, Sigurdsson & Bryan (1995) (see also Holley-Bockelmann et al. 2001) to reduce computation time.

We present results in the system of units such that $G = 1$, $M_4 = 1$, and $a = 1$. If unit time and velocity are converted into those scaled to physical values appropriate for the Milky Way, they become $9.77 \times 10^9$ yr and $260$ km s$^{-1}$, respectively, using $M_4 = 4.1 \times 10^{10}$ $M_\odot$ and $a = 2.6$ kpc (Bland-Hawthorn & Gerhard 2016). In this case, the unit of angular speed is $100$ km s$^{-1}$ kpc$^{-1}$. These values should be viewed as a reference, because the surface density profile of the K–T discs is different from that of real disc galaxies. The models are evolved forward in time until $t = 800$ at which the bar phases are sufficiently long to be relaxed, with a time-step of 0.05 for all simulations. As a result, the relative energy error was, in all cases, smaller than 0.062 per cent.

4 RESULTS

4.1 Growth of linearly unstable two-armed modes

We have measured the growth rate, $\gamma$, and pattern speed, $\Omega_p$, for each disc model from the linearly growing stages of the MLUGTAM in the SCF simulations using a least-squares fitting technique. The results are compared with those determined from the linear modal calculations based on the linearized collisionless Boltzmann equation (see equation 5). In Fig. 2, we show the results obtained with both methods. This figure indicates that the growth rates and pattern speeds measured from the SCF simulations are in considerably good agreement with those from the linear modal calculations. It thus follows that the growing features in the early stages of the SCF simulations are the MLUGTAMs inherent in the individual disc models. In addition, all the $N$-body models were deformed into bars at late times.
Figure 4. Time evolution of bar amplitude, $|A_{22}|$, for (a) the models with Kalnajs’s DFs and (b) those with Miyamoto’s DFs. The labels denote the model names. Note that the scaling of the abscissa is changed at $t = 60$ from the left to the right panel for each model sequence, and that in accordance with the change, the ordinate is also re-scaled.

Figure 5. Time evolution of the pattern speed of the two-armed mode, $\Omega_p$, which is equivalent to the bar pattern speed after the bar instability has occurred, for (a) the models with Kalnajs’s DFs and (b) those with Miyamoto’s DFs. The labels denote the model names.

Figure 6. Growth rate, $s$, of the most linearly unstable, global two-armed mode, obtained from the linear modal calculations as a function of the value of the Toomre’s $Q$ parameter at the scale length, $Q(1)$. The red dashed line represents a power-law fit for Kalnajs’s DFs while the blue one denotes that for Miyamoto’s DFs.

Figure 7. Pattern speed, $\Omega_p$, of the most linearly unstable, global two-armed mode, obtained from the linear modal calculations as a function of the value of the Toomre’s $Q$ parameter at the scale length, $Q(1)$. The red dashed line represents a power-law fit for Kalnajs’s DFs while the blue one denotes that for Miyamoto’s DFs.

(see Fig. 8), so that these bars formed in the SCF simulations can be regarded practically as the end products of the MLUGTAMS.

We see from Fig. 2 that $\Omega_p$ is almost exactly proportional to $s$ for each type of DF. In a strict sense, $\Omega_p \propto s^{0.59}$ for Kalnajs’s DFs while $\Omega_p \propto s^{0.62}$ for Miyamoto’s DFs, if we fit the data obtained using the linear modal calculations by a power-law. Furthermore, we find that for a given $s$, Kalnajs’s DFs lead to a higher $\Omega_p$ than Miyamoto’s DFs. In any case, for the mass profile of the K–T disc, regardless of the DF employed for a model, a larger $s$ leads to a higher $\Omega_p$, and vice versa.

In order to characterize the spiral patterns of the eigenmodes (see Fig. 8), we calculate pitch angles along the radius. In Figs 3(a) and (b), the pitch angle profiles are presented for the models with Kalnajs’s DFs and those with Miyamoto’s DFs, respectively. These figures show that for each model sequence, the pitch angle at a given radius within the dominant region of the spiral pattern decreases as the model parameter, $m_K$ or $m_M$, increases. In addition, the spiral patterns are confined in a smaller range of radii with increasing model parameter (again see Fig. 8). Since the growth rate becomes higher as the model parameter is larger, a more unstable disc against the
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4.2 Formation and evolution of bars

In Figs 4(a) and (b), we show the time evolution of the amplitude of the two-armed (or the bar) mode, \( |A_{22}| \), calculated from the SCF simulations for the models with Kalnajs’s DFs and those with Miyamoto’s DFs, respectively. In the early stages of evolution, a two-armed global mode develops and its amplitude grows exponentially with time (see the left panels of Figs 4a and b). At around the peak amplitude of each two-armed mode, the bar instability occurs as a symbolic event of the onset of nonlinear evolution, so that the excited two-armed spiral mode is deformed into a bar. In the nonlinear evolution stages, the bar amplitudes remain nearly constant after they exhibit some fluctuating changes. On the other hand, after the bars have fully grown up, their pattern speeds decrease over time as presented in Figs 5(a) and (b) for the models with Kalnajs’s DFs and those with Miyamoto’s DFs, respectively, because of the angular momentum exchange between the bar pattern and disc stars (Little & Carlberg 1991).

4.3 Correlations

In Figs 6 and 7, we show, respectively, the growth rate, \( s \), and pattern speed, \( \Omega_p \), of the MLUGTAM as a function of the Toomre’s \( Q \) value at the scale length, \( Q(1) \). Power-law fits indicate that \( s \propto Q^{-2.3} \) and \( \Omega_p \propto Q^{-1.4} \) for Kalnajs’s DFs while \( s \propto Q^{-2.5} \) and \( \Omega_p \propto Q^{-1.6} \) for Miyamoto’s DFs. We find from these figures that in the linear regime, both \( s \) and \( \Omega_p \) of the MLUGTAM decrease as \( Q(1) \) increases. We also note that for a given \( Q(1) \), Kalnajs’s DFs show smaller \( s \) and \( \Omega_p \) than Miyamoto’s DFs. Thus, for each model sequence, the eigenvalue representing the growth rate and pattern speed of the MLUGTAM is specified by \( Q(1) \). In order to examine the correspondence between the linear and nonlinear regimes, the final density contours of the bars are presented in Fig. 8, together

Figure 8. Density contours of the most linearly unstable, global two-armed modes (upper rows) and those of final bar patterns (lower rows) for (a) the models with Kalnajs’s DFs, and for (b) those with Miyamoto’s DFs. The contours of the most linearly unstable, global two-armed modes are drawn in white at the 90 per cent, 70 per cent, 50 per cent, \( \cdots \), 10 per cent levels of the peak amplitude on logarithmic scales, while those of the bar patterns are delineated in white at the 90 per cent, 80 per cent, 70 per cent, \( \cdots \), 40 per cent levels of the peak amplitude on logarithmic scales. White dashed circles in the upper rows show corotation radii. All the spiral and bar patterns rotate counterclockwise.
indicates that while the blue one denotes that for Miyamoto’s DFs. The red dashed line represents a power-law fit for Kalnajs’s DFs while the blue one denotes that for Miyamoto’s DFs.

Figure 9. Bar amplitude normalized by the amplitude of the axisymmetric ring mode averaged over the last 20 time units, $A_{\text{bar}}$, obtained from the SCF simulations against the growth rate of the most linearly unstable, global two-armed mode, $s$, obtained by solving the linearized collisionless Boltzmann equation. The red dashed line represents a power-law fit for Kalnajs’s DFs while the blue one denotes that for Miyamoto’s DFs.

Figure 10. Bar pattern speed, $\Omega_{\text{bar}}$, obtained from the SCF simulations against the pattern speed of the most linearly unstable, global two-armed mode, $\Omega_p$, obtained by solving the linearized collisionless Boltzmann equation. The red dashed line represents a power-law fit for Kalnajs’s DFs while the blue one denotes that for Miyamoto’s DFs.

Figure 11. Final smallest axis ratio, $(b/a)_{\text{min}}$, of the bar along the major axis as a function of the normalized bar amplitude, $A_{\text{bar}}$. The bar amplitude is obtained as described in Fig. 11. The red dashed line represents a power-law fit for Kalnajs’s DFs while the blue one denotes that for Miyamoto’s DFs without model TM2.

Figure 12. Final smallest axis ratio, $(b/a)_{\text{min}}$, of the bar along the major axis as a function of the normalized bar amplitude, $A_{\text{bar}}$. The bar amplitude is obtained as described in Fig. 11. The red dashed line represents a power-law fit for Kalnajs’s DFs while the blue one denotes that for Miyamoto’s DFs without model TM2.

with those of the corresponding MLUHTAMs. By comparing these paired density contours along each model sequence, it appears that the properties of the bars reflect those of the spiral patterns of the MLUHTAMs. Since the MLUHTAM is identified with the growth rate and pattern speed, both of which are correlated with $Q(1)$ as exhibited in Figs 6 and 7, the correspondence between the properties of the bars and those of the spiral density patterns illustrated in Fig. 8 suggests that the bars can be characterized by the typical Toomre’s $Q$ value. In what follows, we will reveal correlations in the quantities that prescribe the bar, finally demonstrating how these quantities are correlated with $Q(1)$.

In Fig. 9, we show a correlation between the growth rate obtained from the linear modal calculations and the bar amplitude, $A_{\text{bar}}$, from the SCF simulations. In plotting this figure, we first define the bar pattern speed, $\Omega_{\text{bar}}$, as an average over that period of the last 20 time units (from $t = 780$ to $t = 800$) which corresponds to roughly half the bar rotation period of each model. Next, we calculate the bar amplitude, $|A_{[22]}|$, as an average over the bar rotation period derived from $\Omega_{\text{bar}}$ and then normalize it by the amplitude of the axisymmetric ring mode, $|A_{[00]}|$, again averaged for that same period. Hereafter, we refer to the bar amplitude as that calculated this way, unless otherwise mentioned, that is, $A_{\text{bar}} = |A_{[22]}|/|A_{[00]}|$. Fig. 9 indicates that for each type of DF, the bar amplitude decreases almost linearly with increasing growth rate. More precisely, a power-law fit leads to $A_{\text{bar}} \propto s^{-0.89}$ for Kalnajs’s DFs while it results in $A_{\text{bar}} \propto s^{-0.66}$ for Miyamoto’s DFs. We see from Fig. 9 that for a given growth rate, Kalnajs’s DFs produce a lower amplitude bar than Miyamoto’s DFs.

In Fig. 10, we present a correlation between $\Omega_p$ of the MLUHTAM obtained from the linear modal calculations, and $\Omega_{\text{bar}}$ from the SCF simulations. We find from this figure that $\Omega_{\text{bar}}$ is almost proportional to $\Omega_p$. In reality, power-law fits show that $\Omega_{\text{bar}} \propto \Omega_p^{1.2}$ for Kalnajs’s DFs while $\Omega_{\text{bar}} \propto \Omega_p^{0.99}$ for Miyamoto’s DFs. In addition, for a given $\Omega_p$, Kalnajs’s DFs generate a higher $\Omega_{\text{bar}}$ than Miyamoto’s DFs.

Figs 9 and 10 imply that $A_{\text{bar}}$ and $\Omega_{\text{bar}}$ are closely connected to the eigenvalue of the disc model, from which $s$ and $\Omega_p$ are calculated. Consequently, from a quantitative point of view, the properties of a bar reflect those of its corresponding MLUHTAM.

As a bar property, we first measure the bar length, $L_{\text{bar}}$. Following Ohta, Hamabe & Wakamatsu (1990), we calculate, as a bar half-size, the radius of the bar region in which the value of $(l_0 + l_2 + l_4 + l_6)/(l_0 - l_2 + l_4 - l_6)$ exceeds 2.0, where $l_m$ ($m = 0, 2, 4, 6$) is the $m$th

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Fourier component of the disc surface density. We regard this radius as \( L_{\text{bar}} \). In Fig. 11, \( L_{\text{bar}} \) at \( t = 800 \) is plotted against \( A_{\text{bar}} \). From this figure, we find that \( L_{\text{bar}} \) is roughly proportional to \( A_{\text{bar}} \), irrespective of the type of DF. As a precise description made by power-law fits, \( L_{\text{bar}} \propto A_{\text{bar}}^{-0.52} \) for Kalnaj’s DFs while \( L_{\text{bar}} \propto A_{\text{bar}}^{-0.48} \) for Miyamoto’s DFs. Thus it turns out that a lower amplitude bar is shorter in length.

As another bar property, we measure the roundness of the bar, which is defined as the shortest axis ratio along the bar major axis, \((b/a)_{\text{min}}\). In doing so, the axis ratios of a bar along the radius are determined by calculating the principal moment of inertia tensor for particles included within a specified radius, and then, the value of this moment is assigned to the axis ratio at that radius. Fig. 12 shows the roundness of the bar as a function of \( A_{\text{bar}} \). This figure indicates that \((b/a)_{\text{min}}\) is roughly linearly decreasing as \( A_{\text{bar}} \) increases except for that model TM2 in the sequence of Miyamoto’s DFs which is the hottest model characterized by the largest typical \( Q \) value. A power-law fit tells us that \((b/a)_{\text{min}}\propto A_{\text{bar}}^{-0.13} \) for Kalnaj’s DFs while \((b/a)_{\text{min}}\propto A_{\text{bar}}^{-0.065} \) for Miyamoto’s DFs without model TM2. Unlike \( L_{\text{bar}} \), \((b/a)_{\text{min}}\) depends on the functional form of DF in the sense that Miyamoto’s DFs lead to a more elongated bar than Kalnaj’s DFs for a given \( A_{\text{bar}} \). At any rate, for each type of DF, a lower amplitude bar is rounder in shape except for model TM2 whose bar shows the highest amplitude in the sequence of Miyamoto’s DFs.

As exhibited in Fig. 9, \( A_{\text{bar}} \) is correlated with \( s \), while \( L_{\text{bar}} \) and \((b/a)_{\text{min}}\) are correlated with \( A_{\text{bar}} \), which is shown in Figs 11 and 12, respectively. Consequently, it follows that \( L_{\text{bar}} \) and \((b/a)_{\text{min}}\) are, respectively, also correlated with \( s \). On the other hand, as revealed in Fig. 6, \( s \) is well-correlated with \( Q(1) \), and so, we can infer that \( A_{\text{bar}} \), \( L_{\text{bar}} \) and \((b/a)_{\text{min}}\) are, respectively, specified by \( Q(1) \). In fact, as demonstrated in Fig. 13, in which \( A_{\text{bar}} \), \( L_{\text{bar}} \) and \((b/a)_{\text{min}}\) are plotted as a function of \( Q(1) \), we see that \( A_{\text{bar}} \) and \( L_{\text{bar}} \) are well-correlated with \( Q(1) \), almost regardless of the functional form of DF, while \((b/a)_{\text{min}}\) is roughly well-correlated with \( Q(1) \) and depends on the DF used. With the help of a power-law fit, we find that \( A_{\text{bar}} \propto Q^{-2.1} \), \( L_{\text{bar}} \propto Q^{-3.2} \), and \((b/a)_{\text{min}}\propto Q^{-0.28} \) for Kalnaj’s DFs while \( A_{\text{bar}} \propto Q^{-1.7} \), \( L_{\text{bar}} \propto Q^{-2.5} \), and \((b/a)_{\text{min}}\propto Q^{-0.15} \) for Miyamoto’s DFs, where model TM2 is excluded for fitting \((b/a)_{\text{min}}\). We note again that for \((b/a)_{\text{min}}\), model TM2 deviates from the correlation with \( Q(1) \).

As an additional remark, no tight correlation has been found between the final \( Q \) and the bar properties.

5 DISCUSSION

We have found that as a disc evolves, the MLUGTAM is growing in the linear phases, and that it eventually turns into a bar via the bar instability. We can thus infer that the properties of a bar such as the amplitude, length, and axis ratio are related to those of the MLUGTAM. In fact, as presented in Figs 9 and 10, the amplitude and pattern speed of a bar are, respectively, well-correlated with the growth rate and pattern speed of the MLUGTAM. Therefore, we discuss below the relation between the MLUGTAM and the bar properties, and thereby we will try to unravel the origin of the correlation, revealed in Fig. 13, between the initial \( Q(1) \) and the bar properties.

Fig. 8 demonstrates that the MLUGTAM is confined to a smaller radius as the initial \( Q(1) \) decreases. A more confined two-armed pattern could be considered to be formidable to provide gravitational influences farther away, unless its amplitude is sufficiently large. Consequently, a bar which is produced by a more confined MLUGTAM would result in a lower amplitude state owing to the difficulty in attracting many more masses in the bar, leading to the correlation that the bar amplitude decreases with decreasing \( Q(1) \), as exhibited in Fig. 13a.

Regarding the bar length, \( L_{\text{bar}} \), bar-supporting orbits do not exist beyond the bar corotation radius (Contopoulos 1980), \( r_{\text{CR}} \), at which the bar pattern speed, \( \Omega_{\text{bar}} \), is equal to the angular speed of a star on a circular orbit, \( \Omega \). Consequently, assuming that the orbital content is similar in all cases, we can use \( r_{\text{CR}} \) as a measure of \( L_{\text{bar}} \). In fact, from observations, Cuomo et al. (2020) have found the correlation that longer bars have larger corotation radii. Since \( \Omega \) is a decreasing function of the radius, \( r_{\text{CR}} \) becomes smaller with increasing \( \Omega_{\text{bar}} \). Cuomo et al. (2020) have found this relation in real barred galaxies as well. In addition, as presented in Fig. 10, \( \Omega_{\text{bar}} \) is approximately proportional to the pattern speed of the MLUGTAM, \( \Omega_{p} \). Accordingly, as \( \Omega_{p} \) is smaller, \( \Omega_{\text{bar}} \) is smaller, so that \( r_{\text{CR}} \), as a result \( L_{\text{bar}} \) also, is larger as \( \Omega_{p} \) is smaller. On the other hand, Fig. 7 indicates that \( \Omega_{p} \) decreases as the initial \( Q(1) \) increases. In this way, \( L_{\text{bar}} \) increases with increasing \( Q(1) \), as shown in Fig. 13b.
Fig. 8 demonstrates also that a more rounder bar is produced by a more tightly wrapped MLUGTAM. From the density wave theory, the m-armed wave satisfies the dispersion relation (e.g. Binney & Tremaine 2008) represented by

\[(\omega - m\Omega_p)^2 = c^2 k^2 - 2\pi G\mu |k| + \kappa^2\]

\[= c^2 \left( |k| - \frac{\pi G\mu}{c^2} \right)^2 + \kappa^2 \left( 1 - \frac{1}{Q^2} \right),\]

where \(\omega\) is the angular frequency, \(k\) is the radial wavenumber, \(c\) is the radial velocity dispersion, and the Toomre's \(Q\) is defined with a fluid approximation as

\[Q = \frac{\kappa c}{\pi G\mu}.\]

If the disc is unstable, equation (13) suggests that the most unstable wavelength, \(\lambda_u\), corresponding to the most unstable wavenumber, \(|k_u| = \pi G\mu/c^2\), is provided by

\[\lambda_u = \frac{2\pi}{|k_u|} = \frac{2c^2}{G\mu} = \frac{2\pi^2 G\mu}{\kappa^2} Q^2.\]

Since the mass profiles used are those of the K–T discs, \(\mu\) and \(\kappa\) do not change from model to model. It thus follows that \(\lambda_u\) is shorter with decreasing \(Q\). The shorter radial wavelength means a more tightly wrapped spiral. Therefore, the pitch angle becomes smaller as \(Q\) decreases. Indeed, from Fig. 3, we find that the pitch angle decreases as \(Q(1)\) decreases for each model sequence. The deviation of the force field from the axisymmetry is considered to be smaller for a tightly wrapped spiral than for a loosely wrapped one. Thus, it is conceivable that a more tightly wrapped spiral could result in a less violent change in the force field along the azimuthal direction when the bar instability occurs, and so, the produced bar could be rounder as the MLUGTAM is more tightly wrapped. Consequently, the correlation between the axis ratio and the initial \(Q(1)\) might emerge as revealed in Fig. 13c.

We now know from observations that the bar length decreases from early- to late-type barred galaxies (Erwin 2005) while longer bars have larger amplitudes (Elmegreen et al. 2007; Díaz-García et al. 2016a; Guo et al. 2019; Cuomo et al. 2019, 2020). It thus follows that the bar amplitude decreases from SBa to SbC or SbD. Furthermore, observations show that larger amplitude bars are more morphologically elongated (Menéndez-Delmestre et al. 2007; Hoyle et al. 2011), which means that the minor-to-major axis ratio of a bar becomes larger from SBa to SbC or SbD. Combining these conclusions with the correlations exhibited in Figs 13(b) and (c), we can infer that the Hubble sequence for barred galaxies could be the sequence of decreasing \(Q\) from SBa to SbC or SbD.

On the other hand, if we rely on the observations that the fraction of barred galaxies increases with time (Sheth et al. 2008; Melvin et al. 2014), as described in Section 1, the spiral structure in non-barred galaxies that we observe at present might be the MLUGTAM still on the stage of the linear growth, which would be deformed into barred galaxies sometime in the future. Indeed, Bertin et al. (1989) demonstrated that unstable global modes can at least generate the spiral morphology of all Hubble types, irrespective of non-barred or barred spirals. Actually, large-scale smooth symmetric arms which are reminiscent of a globally unstable mode are detected in the disc for the old stellar population at near-infrared wavelengths (Block et al. 1994). In addition, as shown by observations, if the difference in the properties between non-barred and barred galaxies is basically whether a bar exists or not, the Hubble sequence for non-barred galaxies might also represent the sequence of decreasing \(Q\) from Sa to Sc or Sd. If this is the case, the \(Q\) value at some radius like the disc scale length might give a clue to the initial typical \(Q\) for barred galaxies under the assumption that the spiral structure in non-barred galaxies is indicative of the MLUGTAM before transforming itself into a bar. In this sense, it will be important to obtain \(Q\) values observationally for non-barred galaxies along the Hubble sequence. However, our results lead to the consequence that the pitch angle of the MLUGTAM decreases with decreasing \(Q\) (see Figs 3 and 8), which indicates that if the Hubble sequence is the sequence of decreasing \(Q\), the spiral arm is more tightly wrapped from Sa to Sc or Sd, contrary to the real Hubble sequence. This discrepancy suggests that the MLUGTAM itself might not correspond directly to the observed spiral pattern. In fact, recent numerical simulations have revealed that spiral arms are not global steady patterns like the MLUGTAMs studied here but transient features (Fujii et al. 2011; Wada, Baba & Saitoh 2011; Baba, Saitoh & Wada 2013) repeatedly excited by swing amplification (Toomre 1981). Therefore, considering that modal calculations expose the existence of numerous unstable modes in self-gravitating discs (e.g. Ambastha & Varma 1983), we will need to make clear the relation between the spiral structure and the MLUGTAM in order to confirm whether the Hubble sequence is the sequence of decreasing \(Q\). Even though the initial typical \(Q\) is the key ingredient only to the Hubble sequence for barred galaxies, it will be significant to verify our findings using those realistic disc models with finite thickness which are embedded in live dark matter haloes.

Our results are obtained from extremely ideal disc models which are different from the galactic discs in the real Universe. For example, we have adopted K–T discs as a mass profile because the exact equilibrium DFs are known, in spite of the fact that real galaxy discs are represented by exponential surface density profiles (Freeman 1970). In particular, each of our disc models lacks a dark matter halo which is assumed to surround a disc, so that the effects of wave-particle interactions between a bar mode and halo particles (Athanassoula 2002) are neglected in the present study. Such resonant interactions can amplify the bar strength, especially for a massive, centrally concentrated halo (Athanassoula 2002). In addition, the motions of stars in our simulations are restricted to a single plane. As a result, the effects of buckling instabilities intrinsic in three-dimensional discs (Raha et al. 1991; Debattista et al. 2004; Martinez-Valpuesta & Shlosman 2004) are not taken into account, although such instabilities reduce the bar strength and can dissolve a bar in non-violent buckling cases (Collier 2020). Thus, the initial typical \(Q\) would not be the only factor that determines the bar amplitude. Accordingly, it is likely that the correlation between the bar amplitude and the initial typical \(Q\) will be altered from that shown in Fig. 13a for realistic disc galaxy models. Similarly, in addressing real barred galaxies, a certain modification might be required for our findings that the bar properties such as the length and axis ratio are closely connected to the initial typical \(Q\), which are represented in Figs 13(b) and (c). However, the correlations, which we have revealed here, between \(L_{\text{bar}}\) and the bar amplitude, \(A_{\text{bar}}\), depicted in Fig. 11 and between the axis ratio, \((b/a)_{\text{min}}\), and \(A_{\text{bar}}\), illustrated in Fig. 12 are consistent with those observed in real barred galaxies (Erwin 2005; Elmegreen et al. 2007; Menéndez-Delmestre et al. 2007; Hoyle et al. 2011). Therefore, the correlations shown in Figs 13(b) and (c) might hold for real barred galaxies.

6 CONCLUSIONS

We have obtained the MLUGTAMs of razor-thin K–T disc models constructed with the exact equilibrium DFs by solving the linearized collisionless Boltzmann equation as an initial value problem. In ad-

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dition, we have carried out $N$-body simulations with a softening-free SCF code using the same disc models. Putting both results together, we have identified that the growing feature in the early phases of disc evolution, which is finally disfigured to form a bar through the bar instability, is the MLUGTAM.

From the SCF simulations, we have confirmed that the resulting bars show the correlations observed in real barred galaxies such that the length increases and the axis ratio decreases as the amplitude increases. By demonstrating that the amplitude and pattern speed of a bar are, respectively, well-correlated with the growth rate and pattern speed of the MLUGTAM, we have shown that the correlations found in the simulated bars root in the eigenvalue of that MLUGTAM. The properties of the bar formed by the bar instability thus reflect those of its corresponding MLUGTAM. Furthermore, we have also shown that the growth rate and pattern speed of the MLUGTAM are well-correlated with the initial $Q$ value at the scale length of the disc. Consequently, we have revealed that the amplitude and the length increases while the ratio decreases in itself decreases, as that $Q$ value increases. Therefore, we conclude that this typical $Q$ value is the determinant of the bar properties. On the basis of this finding, we suggest that the Hubble sequence for barred galaxies, SB, might be the sequence of decreasing $Q$.

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DATA AVAILABILITY

The source code for solving the linearized collisionless Boltzmann equation and the SCF code used for the simulations as well as the models and the simulation data underlying this article will be shared on reasonable request to the corresponding author.

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