Non-Abelian Stokes Theorem and Quark-Monopole Interaction

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Abstract

We derive a new non-abelian Stokes theorem by rewriting the Wilson loop as a gauge-invariant area integral, at the price of integrating over an auxiliary field from the coset $SU(N)/[(U(1)]^{N-1}$ space. We then introduce the relativistic quark-monopole interaction as a Wess–Zumino-type action, and extend it to the non-abelian case. We show that condensation of monopoles and confinement can be investigated in terms of the behaviour of the monopole world lines. One can thus avoid hard problems of how to introduce monopole fields and dual Yang–Mills potentials.

1 Introduction

On the way to describe confinement as due to the monopole condensation several serious technical problems arise. Until now monopole condensation has been fully theoretically understood only in theories undergoing spontaneous breaking of colour symmetry down to the $U(1)$ subgroup(s). We mean a) the Georgi–Glashow model in 2+1 dimensions by Polyakov and b) the supersymmetric model in 3+1 dimensions by Seiberg and Witten. In both cases there is an elementary Higgs field in the adjoint representation, whose non-zero vacuum expectation value mercilessly breaks the gauge group down to the $U(1)$. The compactness of the remaining $U(1)$ group allows Polyakov–’t Hooft monopoles. In 2+1 dimensions they are ‘pseudoparticles’ and cannot condense; in 3+1 dimensions they can. In both cases a linear confining potential is obtained for particles which are electrically charged in respect to the unbroken $U(1)$ subgroup.

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In principle, in QCD a similar scenario could take place: the role of the elementary Higgs field breaking the gauge symmetry could be played by some composite gluon operator belonging to the adjoint representation; its v.e.v. might break the colour $SU(3)$ down to $U(1) \times U(1)$. However, such a possibility seems to be ruled out by what we know experimentally: we would have far more hadrons than in reality. Therefore, one should probably assume that the colour group remains unbroken – in contrast to the supersymmetric example of Seiberg and Witten. An introduction of monopoles without elementary or dynamical Higgs mechanism is a difficult task by itself. To override the difficulty, ’t Hooft has suggested to visualize the $U(1)$ monopoles by partially fixing the gauge, up to the $U(1)$ transformations. This procedure is called abelian projection. Monopoles are then objects similar to those which one would find in the symmetry-broken case. Of course, such objects are to a great extent dependent on the concrete choice of the gauge, or the concrete choice of the abelian projection used. Their desirable condensation is even more obscure: even if it happens in one gauge, it need not necessarily happen in another. Therefore, it would be helpful to introduce a gauge-invariant description of monopoles and a gauge-invariant formulation of the monopole condensation, if there is any. This paper is a step in that direction.

It should be noted that, if the colour group remains unbroken, the mere notion of “monopole condensation” becomes somewhat ambiguous. It seems, at least, that it can not be the usual Higgs mechanism (applied to dual Yang–Mills potentials), for the same reasons: there would be a proliferation of hadron states. Indeed, two (dual) gluons out of eight would be ‘electrically neutral’ in respect to the two $U(1)$ subgroups where the monopoles live, as well as two new kinds of mesons with the colour structure $\bar{\psi}\lambda^{3,8}\psi$ and five new kinds of baryons which happen to have the colour $T_3$ and $Y$ equal to zero. All these states (multiplied by the variety of quantum numbers) are not affected by the dual Meissner effect, therefore they are not confined and should be thus observable as new types of hadrons. It is difficult to imagine that this new realm of hadrons has somehow avoided registration, the more so that, as recently observed in ref. an additional discrete symmetry makes some of these unusual hadrons stable.

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2There is an endless philological discussion whether the Higgs mechanism actually breaks gauge symmetry. Fortunately, there exists a gauge-invariant formulation, for example: Consider a two-point correlation function $C(x-y) = \langle \phi^a(x) P_{ab}(x,y) \phi^b(y) \rangle$, where $\phi^a$ is a scalar (possibly composite) field and $P_{ab}$ is a $P$-exponent in the adjoint representation. $C(x-y)$ is perfectly gauge invariant. If $C(x-y)$ decays exponentially at large separations, there is no symmetry breaking; if it tends to a constant, the situation is referred to as Higgs mechanism and spontaneous gauge symmetry breaking; if it decays as a power of the separation, it can witness the Berezinsky–Kosterlitz–Thouless phase.
under strong interactions.

If confinement is not due to the dual Higgs effect, then what is due to? There is increasing evidence from lattice investigations that monopoles extracted by the maximal abelian projection are important in getting the area law for the Wilson loop. Therefore, a direct theoretical study of the relation between monopoles and confinement is highly desirable, without referring to such evasive quantities as the monopole field and the dual Yang–Mills potentials. For that reason we prefer the first-quantization formalism for monopoles (i.e. path integrals over monopole loops) rather than the field-theoretical one. Trying to avoid the notion of the dual fields, we derive the direct (but of course non-local) interaction between non-abelian charges and non-abelian monopoles.

We start by rewriting the Wilson loop as the ordinary exponent of a certain flux through the surface spanned on the closed contour of a heavy quark loop. The price one has to pay for that is an additional integration over an auxiliary scalar field \( n \) from the \( SU(N)/[U(1)]^{(N-1)} \) coset space. (In the \( N = 2 \) case \( n \) is a unit 3-vector, \( n^2 = 1 \)).

We call it the non-abelian Stokes theorem. It is a smart formula, in fact. In order to fix the representation to which the probe quark of the Wilson loop belongs, we have to add a Wess–Zumino-type term for the \( n \) field. With that term, the flux becomes that of a gauge-invariant field strength introduced earlier by Polyakov \(^7\) and 't Hooft \(^8\) in relation to monopoles, and the Stokes theorem applies to that particular field strength. The auxiliary \( n \) field plays the role of the direction of the elementary Higgs field in colour space.

We next present a relativistic formula for the interaction of a point charge and a point monopole. It is also a Wess–Zumino-type formula, but in three dimensions. Using the results of our previous work \(^9\) we are able to formulate the quark-monopole interaction for the non-abelian case.

This paper deals with "kinematical" problems, leaving aside the hard dynamical one: what is the actual driving force for confinement. However, we hope that this paper gives a framework for a gauge-invariant description of the confinement mechanism based on monopole condensation, whatever it means.
2 Non-abelian Stokes theorem

The path ordering in the Wilson loop,

\[ L = \text{Tr } P \exp \left( i \oint A^a_\mu T^a dx_\mu \right), \]

(1)
can be eliminated at the price of introducing integration over all gauge transformations along the loop [10]. Let \( \tau \) parametrize the loop and \( A(\tau) \) be the tangent component of the Yang–Mills field along the loop in the fundamental representation of the gauge group, \( A(\tau) = A^a_\mu t^a dx_\mu / d\tau, \quad \text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab} \). The gauge transformation of \( A(\tau) \) is

\[ A(\tau) \rightarrow S(\tau)A(\tau)S^{-1}(\tau) + iS(\tau) \frac{d}{d\tau} S^{-1}(\tau). \]

(2)

Let \( H_i \) be the generators from the Cartan subalgebra \((i = 1, ..., r; \ r \text{ is the rank of the gauge group})\) and the \( r \)-dimensional vector \( \mathbf{m} \) be the eldest weight of the representation in which the Wilson loop is considered. The formula for the Wilson loop derived in ref. [10] is a path integral over all gauge transformations \( S(\tau) \) which should be periodic along the contour:

\[ L = \int DS(\tau) \exp \left[ i \int d\tau \text{Tr} m_i H_i (SAS^{-1} + iS\dot{S}^{-1}) \right]. \]

(3)

For example, in the simple case of the \( SU(2) \) group eq. (3) reads:

\[ L = \int DS(\tau) \exp \left[ iJ \int d\tau \text{Tr} \tau^3 (SAS^\dagger + iS\dot{S}^\dagger) \right] \]

(4)

where \( \tau^3 \) is the third Pauli matrix and \( J = \frac{1}{2}, \ 1, \frac{3}{2}, ... \) is the ‘spin’ of the representation of the Wilson loop considered. In what follows we shall concentrate for simplicity on the \( SU(2) \) gauge group.

The second term in the exponent of eqs. (3,4) is in fact a Wess–Zumino-type term, and it can be artificially rewritten not as a line but as a surface integral inside the closed contour of the Wilson loop. Indeed, let us parametrize the \( SU(2) \) matrix \( S \) from eq. (4) by the Euler angles,
\[ S = \exp(i\psi \tau^3/2) \exp(i\theta \tau^2/2) \exp(i\phi \tau^3/2). \] (5)

The second term in the exponent of eq. (4) is then

\[ iJ \int d\tau \text{Tr}(\tau^3 iS S'^\dagger) = iJ \int d\tau (\cos \theta \dot{\phi} + \dot{\psi}) \] (6)

where the last term is in fact zero due to the periodicity of \( S(\tau) \).

Introducing a unit three vector

\[ n = \frac{1}{2} \text{Tr}(S\tau S'^\dagger \tau^3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \] (7)

we can rewrite (6) as

\[ (6) = i\frac{J}{2} \int d\tau d\sigma \epsilon^{abc} \epsilon_{ij} n^a \partial_i n^b \partial_j n^c, \] (8)

where one integrates over any surface spanned on the contour. Indeed, the integrand of eq. (8) is known to be a full derivative; using the Stokes theorem (the standard one!) one reproduces eq. (6). Let us note that the r.h.s. of eq. (8) is the ‘topological charge’ of the \( n \) field at the surface:

\[ Q = \frac{1}{8\pi} \int d\tau d\sigma \epsilon^{abc} \epsilon_{ij} n^a \partial_i n^b \partial_j n^c. \] (9)

Eq. (8) can be also rewritten in a form which is invariant under the reparametrizations of the surface. Introducing the invariant element of a surface,

\[ d^2 \sigma_{\mu\nu} = d\sigma d\tau \left( \frac{\partial x_\mu}{\partial \tau} \frac{\partial x_\nu}{\partial \sigma} - \frac{\partial x_\mu}{\partial \sigma} \frac{\partial x_\nu}{\partial \tau} \right) = \epsilon_{\mu\nu} d^2(\text{Area}), \] (10)

one can rewrite eq. (8) as

\[ (6) = i\frac{J}{2} \int d^2 \sigma_{\alpha\beta} \epsilon^{abc} n^a \partial_\alpha n^b \partial_\beta n^c. \] (11)

We get for the Wilson loop

\[ L = \int Dn(\sigma, \tau) \exp \left[ iJ \int d\tau (A^a n^a) + \frac{iJ}{2} \int d^2 \sigma_{\alpha\beta} \epsilon^{abc} n^a \partial_\alpha n^b \partial_\beta n^c \right]. \] (12)

The interpretation of this formula is obvious: the unit vector \( n \) plays the role of the instant direction of the colour “spin” in colour space; however, multiplying its length by \( J \) does not yet guarantee that we deal with a true quantum state from a representation labelled by \( J \) – that is achieved only by introducing the Wess–Zumino term in eq. (12): it fixes the representation to which the probe quark of the Wilson loop belongs to be exactly \( J \).
Finally, we can rewrite the exponent in this formula so that both terms appear to be surface integrals:

\[ L = \int \mathbf{n} \exp \left( - \int d^2 \sigma \partial_\alpha F_\alpha^a n^a + \int d^2 \sigma \varepsilon^{abc} n^a (D_\alpha n)^b (D_\beta n)^c \right), \quad (13) \]

where \( F_\alpha^a = \partial_\alpha A_\mu^a - \partial_\mu A_\alpha^a + \varepsilon^{abc} A_\alpha^b A_\beta^c \) and \( D_\alpha = \partial_\alpha - \varepsilon^{abc} A_\alpha^b n^c \) is the covariant derivative. Indeed, expanding the exponent of eq. (13) in powers of \( A_\alpha^a \) one observes that the quadratic term cancels out while the linear one is a full derivative reproducing the \( A^a n^a \) term in eq. (12); the zero-order term is the Wess–Zumino term (9). Note that both terms in eq. (13) are explicitly gauge invariant. We call eq. (13) the new non-abelian Stokes theorem. Another version of a non-abelian Stokes theorem has been suggested some time ago by Simonov [13].

One can introduce a gauge-invariant field strength,

\[ G_{\alpha\beta} = F_{\alpha\beta}^a n^a - \varepsilon^{abc} n^a (D_\alpha n)^b (D_\beta n)^c , \quad (14) \]

which, as a matter of fact, coincides in form with the gauge-invariant field strength introduced by Polyakov [7] and ’t Hooft [8] in connection with monopoles. In that case the unit-vector field \( n \) had the meaning of the direction of the elementary Higgs field, \( \vec{\phi} / |\phi| \).

### 3 Quark–monopole interaction

We start with considering a non-relativistic abelian electric charge \( e \) moving in the field of a magnetic monopole sitting at the origin and having the magnetic field \( B = (g/4\pi) r/|r|^3 \). The equation of motion for the charge is given by the Lorentz force:

\[ m \ddot{x}_i = \frac{eg}{4\pi} \varepsilon_{ijk} \frac{x_j \dot{x}_k}{|x|^3}. \quad (15) \]

It is known that the Dirac quantization condition requires \( eg = 4\pi n \); we shall choose \( n = 1 \). The relativistic generalization of eq. (15), when the charge is moving along some world line \( x_\mu(\tau_1) \) and the monopole is moving along some world line \( y_\mu(\tau_2) \), is

\[ m \frac{d}{d\tau_1} \frac{\dot{x}_\mu}{\sqrt{\dot{x}_\mu^2}} = \frac{2}{\pi} \int d\tau_2 \varepsilon_{\mu\alpha\beta\gamma} \frac{(x(\tau_1) - y(\tau_2))_\alpha}{|x(\tau_1) - y(\tau_2)|^4} \frac{dx_\beta(\tau_1)}{d\tau_1} \frac{dy_\gamma(\tau_2)}{d\tau_2}. \quad (16) \]

Indeed, taking the monopole sitting at the origin so that its world line is \( y_\mu(\tau_2) = (\tau_2, 0, 0, 0) \), and a non-relativistic charge with \( x_\mu(\tau_1) = (\tau_1, x_i(\tau_1)) \), \( |\dot{x}_i| \ll 1 \), we return, after integrating over \( \tau_2 \) from minus to plus infinity, to eq. (15). We note that
eq. (16) is Lorentz invariant and also invariant under the re-parametrization of both world lines, \( x_\mu(\tau_1) \) and \( y_\mu(\tau_2) \), as it should be.

The l.h.s. of eq. (16) is obviously the variation of the standard relativistic action of a free particle,

\[
S_{\text{free}} = m \int d\tau \sqrt{\dot{x}_\mu^2};
\]
what about the right-hand side? The appropriate interaction term appears to be a tricky thing. It can be written only as a non-uniquely defined Wess–Zumino action. Let us introduce a unit 4-vector

\[
u_\mu(\tau_1, \tau_2) = \frac{x_\mu(\tau_1) - y_\mu(\tau_2)}{|x(\tau_1) - y(\tau_2)|}, \quad u_\mu^2 = 1,
\]
and define its analytical continuation to a third dimension labelled by \( \sigma \), \( 0 \leq \sigma \leq 1 \), so that

\[
u_\mu(\tau_1, \tau_2, \sigma = 1) = u_\mu(\tau_1, \tau_2), \quad \nu_\mu(\tau_1, \tau_2, \sigma = 0) = w_\mu,
\]
where \( w_\mu \) is some constant 4-vector of unit length. One can also introduce an \( SU(2) \) unitary matrix

\[
V(\tau_1, \tau_2, \sigma) = \nu_\mu(\tau_1, \tau_2, \sigma) \cdot \sigma^- \quad \sigma^- = (1_2, -i \vec{\tau})
\]
where \( \vec{\tau} \) are the three Pauli matrices and \( 1_2 \) is the unity \( 2 \times 2 \) matrix, and define three anti-hermitean matrices,

\[
L_A = V^\dagger \partial_A V, \quad A = \tau_1, \ \tau_2, \ \sigma.
\]

The needed relativistic charge–monopole interaction term can be written as the winding number of \( V \) (times \( 2\pi \), to make \( \exp(iS_{\text{int}}) \) uniquely defined):

\[
S_{\text{int}} = \frac{2\pi}{24\pi^2} \int d\tau_1 d\tau_2 d\sigma \epsilon_{ABC} \text{Tr} \left( L_A L_B L_C \right)
\]
\[
= -\frac{1}{6\pi} \int d\tau_1 d\tau_2 d\sigma \epsilon_{ABC} \epsilon^{\alpha\beta\gamma\delta} \partial_A v_\alpha \partial_B v_\beta \partial_C v_\gamma v_\delta.
\]
Indeed, let us vary this \( S_{\text{int}} \) in respect to the trajectory of the charge \( x_\mu(\tau_1) \). To that end we first find the change of \( S_{\text{int}} \) due to the arbitrary variation \( \delta v_\mu(\tau_1, \tau_2, \sigma) \). We have

\[
\delta \left[ \epsilon_{ABC} \epsilon^{\alpha\beta\gamma\delta} \partial_A v_\alpha \partial_B v_\beta \partial_C v_\gamma v_\delta \right] = 3 \epsilon_{ABC} \epsilon^{\alpha\beta\gamma\delta} \partial_A \delta v_\alpha \partial_B v_\beta \partial_C v_\gamma v_\delta
\]
\[
+ \epsilon_{ABC} \epsilon^{\alpha\beta\gamma\delta} \partial_A v_\alpha \partial_B v_\beta \partial_C \delta v_\gamma v_\delta.
\]
The last term is zero here since partial derivatives as well as the variation of $v_\mu$ must be all orthogonal to $v_\mu$ and to each other (because of the antisymmetric $\epsilon^{\alpha\beta\gamma\delta}$), which is impossible in four dimensions. Therefore, we are left with the first term in eq. (23), which, for the same reason, can be written as a full derivative,

$$ (23) = 3\partial_A \left[ \epsilon_{ABC} \epsilon^{\alpha\beta\gamma\delta} \delta v_\alpha \partial_B v_\beta \partial_C v_\gamma v_\delta \right]. $$

Integrals over full derivatives in $\tau_1, 2$ are zero, if we assume that the trajectories $x_\mu(\tau_1)$ and $y_\mu(\tau_2)$ are closed, so that we are left only with the full derivative in the auxiliary dimension labelled by $\sigma$. Therefore we get

$$ \delta S_{\text{int}} = -\frac{1}{\pi} \int d\tau_1 d\tau_2 \int d\sigma \frac{\partial}{\partial \sigma} \left[ \epsilon^{\alpha\beta\gamma\delta} \delta v_\alpha \partial_\tau_1 v_\beta \partial_\tau_2 v_\gamma v_\delta \right]. $$

The value of the square brackets at $\sigma = 0$ is zero since we have chosen the continuation (19) in such a way that $v_\mu$ at $\sigma = 0$ is equal to a constant vector $w_\mu$ and is thus $\tau_1, 2$-independent. Therefore, the integral over $\sigma$ reduces to the value of the square bracket at $\sigma = 1$ where $v_\mu$ assumes its physical value $u_\mu(\tau_1, \tau_2)$, as defined by eq. (18). We have thus demonstrated a well-known general fact that, though the Wess–Zumino term is not uniquely defined, its variation is.

We now take the variation $\delta v_\mu$ (equal to $\delta u_\mu$ at the physical surface $\sigma = 1$) to be due to the variation of the charge trajectory $x_\mu(\tau_1)$. In the environment of eq. (25) it means that

$$ \delta v_\mu = \frac{\delta x_\mu(\tau_1)}{|x(\tau_1) - y(\tau_2)|}. $$

We get finally

$$ \frac{\delta S_{\text{int}}}{\delta x_\mu(\tau_1)} = \frac{2}{\pi} \int d\tau_2 \epsilon_{\mu\alpha\beta\gamma}(x(\tau_1) - y(\tau_2))_\alpha \frac{dx_\beta(\tau_1)}{|x(\tau_1) - y(\tau_2)|^2} \frac{dy_\gamma(\tau_2)}{d\tau_2}, $$

that is the r.h.s of the equation of motion \([10], q.e.d.\) Actually, eq. (22) is a generalization of the non-relativistic charge-monopole interaction, see, e.g., refs. [11, 12].

The main point is that one cannot write down the action whose variation is given by eq. (27), in a unique way. This is because it is impossible to write the electric charge – magnetic charge interaction without introducing a string (or other more complicated objects) which take away the magnetic flux so that the QED equation, $\text{div} \mathbf{B} = 0$, is satisfied identically. The concrete form of the ‘string’ depends on the concrete way one parametrizes the continuation of the physical vector $u_\mu(\tau_1, \tau_2)$ to the unphysical dimension labelled by $\sigma$. Since the action is given by the ‘winding number’ eq. (22), different parametrizations may differ only by multiples of $2\pi$. Let us stress that the
correct coefficient in \((27)\) representing the Dirac quantization condition, follows from the normalization factor \(1/24\pi^2\) of the winding number, see eq. \((22)\).

4 Non-abelian monopoles

Writing down the charge–monopole interaction in the Wess–Zumino form we have actually integrated off the gauge fields, leaving only the charge and monopole trajectories as the dynamical variables of the theory. Therefore, \(\exp(iS_{\text{int}})\) is, in fact, what is called the generating functional of the theory, depending on the external charge current \(j^c_\mu(x)\) determined by the charge trajectory \(x_\mu(\tau)\). In case of the Yang–Mills theory the external colour current \(j^c_\mu(x)\) is an adjoint matrix. Some time ago we have shown \([9]\) that the Yang–Mills generating functional has specific properties which follow from gauge invariance. It can be written as a sum of two pieces: one is gauge invariant (call it \(W_1\)) and can depend only on the diagonal part of the current \(d^c_\mu\),

\[
d^c_\mu(\tau) = S(\tau) j^c_\mu(\tau) S^\dagger(\tau)
\]

(one can always locally rotate the adjoint matrix to make it diagonal), while the second piece is gauge non-invariant, and depends on the ‘angle’ variables \(S(\tau)\). According to ref.\([9]\) the general form of the generating functional in the Yang–Mills theory for a closed loop of a point-like current can be written as

\[
W[j^c_\mu] = W_1[d^c_\mu(\tau)] + J \int d\tau S \dot{S}^\dagger,
\]

In calculating the vacuum average of the Wilson loop one actually has to substitute the appropriate colour current \(j^c_\mu\), as induced by the specific loop considered, into the generating functional of the theory. The appropriate current of the Wilson loop can be read off from \((4)\) (or eq. \((3)\) for a general gauge group). It follows from eq. \((4)\) that the diagonal part of the probe current is nothing but

\[
d^c_\mu(x) = 2J \int d\tau \dot{x}_\mu(\tau) \delta^{(4)}(x_\alpha - x_\alpha(\tau)),
\]

where \(x_\alpha(\tau)\) is the trajectory of the colour charge, while its ‘angle’ part \(S(\tau)\) is nothing but the gauge transformation matrix over which one has to integrate in eq. \((4)\). Comparing eqs.\((4,29)\) one observes that the second (gauge non-invariant) pieces are exactly cancelled. This has been the conclusion of ref. \([9]\). Therefore, the integration over all possible gauge transformations \(S(\tau)\) in eq. \((4)\) becomes trivial, so that one is left only with the gauge-invariant piece of the generating functional, \(W_1\). If one assumes...
that the dynamics of large Wilson loops is governed by monopoles, this \( W_1 \) is just the Wess–Zumino-type charge–monopole interaction term, eq. (22), where one has to use the diagonal part of the colour charge current, given by eq. (30).

It is less evident how to introduce colour monopoles, however one can think of proceeding in a similar way as with the colour charges. Namely, the interaction of colour magnetic charges with the dual Yang–Mills potential \( B_\mu \) is given by a dual Wilson loop (cf. eq. (4)),

\[
L^m = \int DS(\tau) \exp \left[ iK \int d\tau \, \text{Tr} \, \tau^3 (SBS^\dagger + iS\dot{S}^\dagger) \right],
\]

where \( B(\tau) \) is the tangent component of the dual Yang–Mills potential along the monopole loop, \( S(\tau) \) is the unitary matrix of the instant colour orientation of the monopole, \( K = \frac{1}{2}, 1 \frac{1}{2}, \ldots \) is the colour ‘spin’ of the monopole. As in the case of the similar eq. (4) one has to integrate actually over unit orientation vectors \( m(\tau) \) from the coset space, rather than over the group elements \( S(\tau) \).

One can introduce the field theory of monopoles in the path-integral formalism and write the partition function as a sum over arbitrary numbers of monopole loop trajectories, with their colour orientation vectors \( m(\tau) \) living on those trajectories:

\[
Z^m_K = \sum_n \frac{1}{n!} \prod_i \int Dm_i(\tau) \int Dy_{i\mu}(\tau) \, A_K[y_{i\alpha}(\tau)],
\]

where \( A_K[y_{i\alpha}(\tau)] \) are some weight functions. For the free particles of mass \( m_K \) the weight function is \( A_K(l) = \exp(-m_K l) \) where \( l = \int d\tau \sqrt{\dot{y}_{i\alpha}^2} \) is the monopole loop length. With such a weight functional eq. (32) is equivalent to the partition function of the field theory of free scalar Yang–Mills particles with mass \( m_K \) and colour ‘spin’ belonging to a certain representation labelled by \( K \). One can take into account that monopoles can, in principle, belong to different representations of the gauge group, however for simplicity we consider only some particular \( K \). In a theory of interacting monopoles the weight functional is, of course, more complicated.

The main idea of this paper is to avoid the notion of monopole fields and of dual Yang–Mills potentials (as well as the usual ones), and to work directly with charge and monopole trajectories. The Wess–Zumino-type interaction term (22) solves the problem of eliminating the dual as well as the usual Yang–Mills potential, therefore it can be understood as the generating functional – not only for the electric charges but also for the magnetic ones, since they enter the interaction term on equal footing. Therefore, everything said above about the non-abelian electric currents applies to the non-abelian magnetic currents. The Wess–Zumino term containing the instant colour orientation of monopoles (the second term in eq. (31)) has to be cancelled exactly by
the gauge non-invariant term of the generating functional similar to that of eq. (29). Then the integration over \( m_i \) in the monopole partition function \( (32) \) becomes trivial, and one is left with the abelian charge–monopole interaction term where the abelian currents are given by eq. (30) for the electric charges and by

\[
d_m^\mu(x) = 2K \int d\tau \dot y_\mu(\tau) \delta^{(4)}(x_\alpha - y_\alpha(\tau)),
\]

for the magnetic currents, where \( y_\alpha(\tau) \) are the trajectories of monopoles.

We arrive thus to a simple recipe to compute the non-abelian electric Wilson loop in the background of fluctuating quantum monopole fields – without introducing such fields at all. It is given by the exponent of the charge–monopole interaction (22), averaged with the monopole partition function (32). We obtain for the averaged Wilson loop whose trajectory is denoted by \( x_\mu(\tau) \):

\[
\langle L[x_\mu(\tau)] \rangle = \frac{1}{Z_m} \sum_n \frac{1}{n!} \prod_i \int D y_i(\tau) A_K[y_i(\tau)] \times \exp \{(i(2J)(2K)S_{int}[x_\mu, y_i]) \}.
\]

The generalization to arbitrary gauge groups and representations of both charges and monopoles is straightforward, according to eq. (3): one has to replace \( 2J^3 \) (and \( 2K^3 \)) by \( \sum m_i H_i \) where \( H_i \) are \( r \) generators from the Cartan subalgebra and \( m \) is the \( r \)-dimensional eldest weight of the representation to which the electric charge belongs (and similarly for the magnetic charge).

Eq. (34) relates directly the Wilson loop to the integral over monopole paths. If confinement, i.e. the area behaviour of the Wilson loop, is due to the monopole condensation, it should be seen from this formula. Both monopole condensation and confinement can be thus formulated in completely gauge-invariant terms: it is a statement about the weight functional \( A_K[y_i(\tau)] \). For example, one can easily check that the weight \( A_K \sim \exp(-ml) \) corresponding to monopoles being free massive particles, does not lead to the area law: monopoles should be at least effectively massless. Long loops are absolutely necessary to derive the area law from eq. (34) [14].

5 Conclusions

We have derived a new non-abelian Stokes theorem for the Wilson loop. The path ordering of the Wilson loop is exchanged for the integration over an auxiliary scalar field from the coset \( SU(N)/[U(1)]^{(N-1)} \) space; its meaning is the instant direction of the colour charge. With the help of that field one can construct a gauge-invariant field
strength to which the Stokes theorem applies. The construction needs an introduction of a non-uniquely defined Wess–Zumino term for the auxiliary field, which fixes the group representation of the Wilson loop considered.

We have presented the interaction between quark and monopole in a Lorentz-invariant, gauge-invariant and reparametrization-invariant form. It is also given by a Wess–Zumino-type action; its non-uniqueness is due to the arbitrariness of how one draws the Dirac string. However, under the change of the string direction it can only produce phase factors $\exp(2\pi in) = 1$.

The average of the quark Wilson loop over the vacuum filled with fluctuating monopoles has been related directly to path integrals over monopole trajectories. This enables one to formulate in a gauge invariant way – in the language of monopole world lines – what precisely does the alleged monopole condensation mean, and how does one get the area behaviour of the Wilson loop out of that [14].

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