Darboux transformations and exact soliton solutions of integrable coupled spin systems related with the Manakov system

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Abstract

We construct a Darboux transformation of a general $su(3)$-valued spin system called the $\Gamma$-spin system. Using this Darboux transformation we derive a recursive formula for the soliton solutions of this spin system. Then using these results we present explicit expressions for the 1-soliton solution of the coupled $su(2)$-valued spin systems, namely, of the coupled M-LIII equation.

1 Introduction

Darboux transformations (DT) constitute one of the most fruitful approaches to the construction of soliton solutions of integrable nonlinear equations. DT transform solutions of differential equations into solutions of same class differential equations. There are a key role plays the so-called Darboux matrix: $L$. Here we briefly explain the notion of the Darboux matrix. For this aim, let us consider the spatial part of the Lax representation

$$\mathcal{R}_x = U \mathcal{R}. \quad (1.1)$$

Consider the transformation

$$\mathcal{R}' = L \mathcal{R}, \quad (1.2)$$

where Darboux matrix $L$ satisfies the equation

$$L_x = U'L - LU. \quad (1.3)$$

Then the $\mathcal{R}'$ obeys the equation

$$\mathcal{R}'_x = U' \mathcal{R}'. \quad (1.4)$$

DT originated in the pioneering work of Darboux and others on the differential geometry of surfaces. Also we recall that some particular examples of such transformations were known to Euler and Laplace. In the 1970s, the DT were rediscovered in the soliton theory. In fact, the role of DT is more important than as mere technical methods for obtaining solutions of some differential equations.
Integrable spin systems are an important part of integrable systems \[1\]-\[44\]. They describe nonlinear dynamics of magnetic systems. This article is a continuation of our previous papers devoted to study the integrable coupled spin systems related with the Manakov system and their connections with the integrable motion of curves and surfaces (see e.g. \[39\]-\[44\] and references therein). In this paper we consider the DT for integrable coupled spin systems namely for the coupled M-LIII equation and for the Γ-spin system \[39\]-\[44\]. We recall that these spin systems are equivalent to the one and same equation namely to the Manakov system. These three equations - the coupled M-LIII equation, the Γ-spin system and the Manakov system are integrable by the inverse scattering method. The coupled M-LIII equation is one of natural candidates to be integrable multilayer generalizations of the classical continuous Heisenberg ferromagnetic equation.

The structure of the paper is as follows. In the next two sections we gather a number of definitions and basic results on the Manakov system and the M-LIII equation. These are needed to construct the DT and the exact soliton solutions of these two integrable systems. Section 2 provides a basic review of the Manakov system. In Section 3, we recall key results on the coupled M-LIII equation. In particular we give the LR for the coupled M-LIII equation. In Section 4, we establish the gauge equivalence between the coupled M-LIII equation and the Manakov system. Sections 5 and 6 are devoted to the DT for the Γ-spin system and for the coupled M-LIII equation. Finally, in Sec. 7, we present our conclusions.

2 The Manakov system. Preliminaries

In this section we recall some general facts about the Manakov system. The Manakov system has many physical significant applications such as the modeling crossing sea waves and for propagation in elliptically birefringent optical fibers.

2.1 The equation

The Manakov system is the particular case of the vector nonlinear Schrödinger equation and has the form

\[
\begin{align*}
    iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 &= 0, \\
    iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 &= 0.
\end{align*}
\]

2.2 Lax representation

The Manakov equation is integrable by the inverse scattering method. Its Lax representation reads as

\[
\begin{align*}
    \Phi_x &= U \Phi, \\
    \Phi_t &= V \Phi,
\end{align*}
\]

where \(\Phi = (\phi_1, \phi_2, \phi_3)\) and

\[
U = -i\lambda \Sigma + U_0, \quad V = -2i\lambda^2 \Sigma + 2MU_0 + V_0.
\]

Here

\[
\Sigma = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & -1 & 0 \\
    0 & 0 & -1
\end{pmatrix}, \quad U_0 = \begin{pmatrix}
    0 & q_1 & q_2 \\
    -\bar{q}_1 & 0 & 0 \\
    -\bar{q}_2 & 0 & 0
\end{pmatrix}, \quad V_0 = i \begin{pmatrix}
    |q_1|^2 + |q_2|^2 & q_{1x} & q_{2x} \\
    q_{1x} & -|q_1|^2 & -q_1q_2 \\
    q_{2x} & -q_2q_1 & -|q_2|^2
\end{pmatrix}.
\]

3 The coupled M-LIII equation. Preliminaries

The coupled M-LIII equation describe nonlinear dynamics of the coupled (two-layer) magnetic systems. It is integrable by the inverse scattering method. The coupled M-LIII equation plays important role in differential geometry of curves and surfaces. It related with the fact that this coupled equation induced the integrable class of two interacting curves and surfaces. In this section we give some main facts on this equation.
3.1 The equation

Consider two spin vectors \( \mathbf{A} = (A_1, A_2, A_3) \) and \( \mathbf{B} = (B_1, B_2, B_3) \), where \( \mathbf{A}^2 = \mathbf{B}^2 = 1 \). Then the coupled M-LIII equation reads as

\[
\begin{align*}
iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + F &= 0, \\
iB_t + \frac{1}{2}[B, B_{xx}] + iu_2 B_x + E &= 0,
\end{align*}
\]

where \( u_k \) are real functions, \( F \) and \( E \) are matrix functions,

\[
A = \begin{pmatrix} A_3^+ & A^- \\ A^+ & -A_3^- \end{pmatrix}, \quad B = \begin{pmatrix} B_3^- & B^- \\ B^+ & -B_3^+ \end{pmatrix}, \quad F = \begin{pmatrix} F_3^- & F^- \\ F^+ & -F_3^+ \end{pmatrix}, \quad E = \begin{pmatrix} E_3^- & E^- \\ E^+ & -E_3^+ \end{pmatrix},
\]

with \( A^\pm = A_1 \pm iA_2 \), \( B^\pm = B_1 \pm iB_2 \), \( A^2 = B^2 = I \), \( F^\pm = F_1 \pm iF_2 \), \( E^\pm = E_1 \pm iE_2 \). Here \( A \in su(2) \), \( B \in su(2) \). In this paper, we assume that \( F \) and \( E \) have the form

\[
F = v_1[\sigma_3, A], \quad E = v_2[\sigma_3, B],
\]

where \( v_j \) are some real functions (potentials). Then the coupled M-LIII equation takes the form

\[
\begin{align*}
iA_t + \frac{1}{2}[A, A_{xx}] + iu_1 A_x + v_1[\sigma_3, A] &= 0, \\
iB_t + \frac{1}{2}[B, B_{xx}] + iu_2 B_x + v_2[\sigma_3, B] &= 0,
\end{align*}
\]

where \( u_j \) and \( v_j \) are coupling potentials and have the forms

\[
\begin{align*}
u_1 &= \frac{2i}{\Delta_1} (q_2 g_1 \bar{g}_3 - q_2 \bar{g}_1 g_3), \\
u_2 &= \frac{2i}{\Delta_2} (q_1 g_1 \bar{g}_2 - q_1 \bar{g}_1 g_2), \\
v_2 &= \frac{2i}{\Delta_2} (q_1 g_1 \bar{g}_2 - q_1 \bar{g}_1 g_2), \\
v_2 &= \frac{2i}{\Delta_2} (q_1 g_1 \bar{g}_2 - q_1 \bar{g}_1 g_2),
\end{align*}
\]

Here

\[
\Delta_1 = |g_1|^2 + |g_2|^2, \quad \Delta_2 = |g_1|^2 + |g_3|^2, \quad \Delta = |g_1|^2 + |g_2|^2 + |g_3|^2.
\]

In elements, the coupled M-LIII equation (3.5)-(3.6) reads as

\[
\begin{align*}
iA^+_t + (A^+ A^+_{3xx} - A^- A^-_{xx} A_3) + iu_1 A^+_x - 2v_1 A^+ &= 0, \\
iA^-_t - (A^- A^-_{3xx} - A^+ A^+_{xx} A_3) + iu_1 A^-_x + 2v_1 A^- &= 0, \\
iA_{3t} + \frac{1}{2} (A^- A^+_{xx} - A^+ A^-_{xx}) + iu_1 A_{3x} &= 0, \\
iB^+_t + (B^+ B^+_{3xx} - B^- B^-_{xx} B_3) + iu_2 B^+_x + 2v_2 B^+ &= 0, \\
iB^-_t - (B^- B^-_{3xx} - B^+ B^+_{xx} B_3) + iu_2 B^-_x + 2v_2 B^- &= 0, \\
iB_{3t} + \frac{1}{2} (B^- B^+_{xx} - B^+ B^-_{xx}) + iu_2 B_{3x} &= 0.
\end{align*}
\]

3.2 Lax representation

The coupled M-LIII equation is integrable by the inverse scattering method. It admits the Lax representation of the form

\[
\begin{align*}
Y_x &= (-i\lambda A + S_1)Y + H_1, \\
Y_t &= -2i\lambda^2 AY + 2\lambda N_1 Y + X_1 Y + J_1,
\end{align*}
\]
\begin{align*}
Z_x &= (-i\lambda B + S_4)Z + H_2 \\
Z_t &= -2i\lambda^2 BZ + 2\lambda N_2 Z + X_2 Z + J_1,
\end{align*}
where \(Y = (\psi_1, \psi_2)^T\) and \(Z = (\psi_4, \psi_3)^T\), \(S_j\) and \(H_j\) are matrix functions. The compatibility conditions \(Y_{xt} = Y_{tx}\) and \(Z_{xt} = Z_{tx}\) give the coupled M-LIII equation (3.5)-(3.6).

4 Gauge equivalence between the coupled M-LIII equation and the Manakov system

Let \(Y = (\psi_1, \psi_2)^T\) and \(Z = (\psi_4, \psi_3)^T\) are the solutions of the set (3.20)-(3.21). Now let us introduce the new functions \(\phi_j\) as
\[
\begin{align*}
\phi_1 &= g_1\psi_1 + \bar{g}_2\psi_2, \\
\phi_2 &= g_2\psi_1 - \bar{g}_1\psi_2, \\
\phi_3 &= g_1\psi_1 + \bar{g}_2\psi_3.
\end{align*}
\]
(4.1)
The straight calculations show that these new functions \(\phi_j\) satisfy the set of equations (2.3)-(2.4). That is they give the Lax representation of the Manakov system. This result proves the gauge equivalence between the coupled M-LIII equation (3.5)-(3.6) and the Manakov system (2.1)-(2.2) [39]-[44].

5 Gauge equivalence between the \(\Gamma\)-spin system and the Manakov system

We recall that there is another spin system which is the gauge equivalent to the Manakov system (2.1)-(2.2). This spin system called by us as, the \(\Gamma\)-spin system, reads as [38]
\[
i\Gamma_t + \frac{1}{2}[\Gamma, \Gamma_{xx}] = 0.
\]
(5.1)
It is integrable and admits the following Lax representation (see e.g. [38] and references therein)
\[
\begin{align*}
\Psi_x &= U'\Psi, \\
\Psi_t &= V'\Psi,
\end{align*}
\]
where
\[
U' = -i\lambda \Gamma, \quad V' = -2i\lambda^2 \Gamma + \frac{1}{2}\lambda [\Gamma, \Gamma_x].
\]
(5.4)
Here
\[
\Gamma = g^{-1}\Sigma g, \quad \Gamma^2 = I
\]
(5.5)
and
\[
\Gamma = \begin{pmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\
\Gamma_{31} & \Gamma_{32} & \Gamma_{33}
\end{pmatrix} \in su(3).
\]
(5.6)

6 Darboux transformation and exact solutions of the \(\Gamma\)-spin system

6.1 Darboux transformation of the \(\Gamma\)-spin system

In this section, we construct the DT for the equation (5.1). To do this, let us consider the following transformation of solutions of the equations (5.2)-(5.3)
\[
\Psi' = L\Psi,
\]
(6.1)
where
\[ L = \lambda N - I. \] (6.2)

We require that $\Phi'$ satisfies the same Lax representation as (5.2)-(5.3) so that
\[ \Phi'_{x} = U'\Phi', \] (6.3)
\[ \Phi'_{t} = V'\Phi', \] (6.4)

where $U' - V'$ depend on $\Gamma'$ as $U - V$ on $\Gamma$. The matrix $L$ obeys the following equations
\[ L_{x} + LU = U'L, \] (6.5)
\[ L_{t} + LV = V'L. \] (6.6)

These equations yield the following equations for $N$
\[ N_{x} = i\Gamma' - i\Gamma, \] (6.7)
\[ N_{t} = -\Gamma'\Gamma'_{x} + \Gamma\Gamma_{x} \] (6.8)

and
\[ \Gamma' = N\Gamma N^{-1}. \] (6.9)

Also we have the following useful second form of the DT for $S$
\[ \Gamma' = \Gamma - iN_{x}. \] (6.10)

Let us consider the following set of equations
\[ H_{x} = -i\Gamma HA, \] (6.11)
\[ H_{t} = -2i\Gamma HA^{2} + \Gamma\Gamma_{x}HA, \] (6.12)

where
\[ \Lambda = \begin{pmatrix} \lambda_{1} & 0 & 0 \\ 0 & \lambda_{2} & 0 \\ 0 & 0 & \lambda_{3} \end{pmatrix}, \] (6.13)

\[ \det H \neq 0 \] and $\lambda_{k}$ are complex constants. We now assume that the matrix $N$ can be written as:
\[ N = H\Lambda^{-1}H^{-1} = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}. \] (6.14)

The inverse matrix we write as
\[ N^{-1} = H\Lambda H^{-1} = \frac{1}{n} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \] (6.15)

or
\[ N^{-1} = \frac{1}{n} \begin{pmatrix} n_{22}n_{33} - n_{23}n_{32} & -(n_{12}n_{33} - n_{13}n_{32}) & n_{12}n_{23} - n_{13}n_{22} \\ -(n_{21}n_{33} - n_{23}n_{31}) & n_{11}n_{33} - n_{13}n_{31} & -(n_{11}n_{23} - n_{21}n_{13}) \\ n_{32}n_{21} - n_{31}n_{22} & -(n_{11}n_{32} - n_{31}n_{12}) & n_{11}n_{22} - n_{21}n_{12} \end{pmatrix}. \] (6.16)

where $n = \det N$ and has the form
\[ n = n_{11}n_{22}n_{33} + n_{12}n_{23}n_{31} + n_{13}n_{32}n_{21} - n_{31}n_{22}n_{13} - n_{12}n_{21}n_{33} - n_{11}n_{23}n_{32}. \] (6.17)
From these equations follow that $N$ obeys the equations

$$
N_x = i\Gamma N^{-1} - i\Gamma, \quad (6.18)
$$

and

$$
N_i = \Gamma \Gamma_x - N \Gamma \Gamma^{-1}, \quad (6.19)
$$

which are equivalent to Eqs. (6.7)-(6.8) as we expected. The $\Gamma$ and matrix solutions of the system (5.2)-(5.3) obey the condition

$$
\Phi^\dagger = \Phi^{-1}, \quad \Gamma^\dagger = \Gamma, \quad (6.20)
$$

which follow from the equations

$$
\Phi^\dagger = i\lambda \Phi^\dagger, \quad (\Phi^{-1})^\dagger = i\lambda \Phi^{-1}\Gamma^{-1}. \quad (6.21)
$$

Here $\dagger$ denote an Hermitian conjugate. After some calculations we came to the formulas

$$
\lambda_2 = \lambda_3 = \lambda_1^*, \quad H = \begin{pmatrix}
\psi_3(\lambda_1; t, x, y) & \psi_3^*(\lambda_1; t, x, y) & \psi_3^*(\lambda_1; t, x, y) \\
\psi_2(\lambda_1; t, x, y) & -\psi_1^*(\lambda_1; t, x, y) & 0 \\
\psi_1(\lambda_1; t, x, y) & 0 & -\psi_1^*(\lambda_1; t, x, y)
\end{pmatrix}, \quad (6.22)
$$

and

$$
H^{-1} = \frac{1}{\delta} \begin{pmatrix}
\psi_1^2 & \psi_1 \psi_2 & \psi_1 \psi_3 \\
\psi_1 \psi_2 & (|\psi_1|^2 + |\psi_2|^2) & -\psi_2 \psi_3 \\
\psi_1 \psi_3 & -\psi_2 \psi_3 & (|\psi_1|^2 + |\psi_2|^2)
\end{pmatrix}, \quad (6.23)
$$

where

$$
\delta = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2. \quad (6.24)
$$

So finally for the matrix $N$ we get the following expression

$$
N = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\
n_{21} & n_{22} & n_{23} \\
n_{31} & n_{32} & n_{33} \end{pmatrix} = \frac{1}{\delta} \begin{pmatrix}
n_{11} \delta & |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \\
n_{12} \delta & -|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 \\
n_{13} \delta & 0
\end{pmatrix}\begin{pmatrix}
\psi_1 \psi_2 \psi_3 \\
\psi_1 \psi_2 \psi_3 \psi_3 \\
\psi_1 \psi_2 \psi_3 \psi_3 \psi_3
\end{pmatrix}, \quad (6.25)
$$

where $\delta_{ij} = \lambda_i^{-1} - \lambda_j^{-1}$,

$$
n_{11} \delta = \frac{|\psi_1|^2}{\lambda_1} + \frac{|\psi_2|^2}{\lambda_2} + \frac{|\psi_3|^2}{\lambda_3}, \quad (6.26)
n_{12} \delta = \frac{\psi_1 \psi_2}{\lambda_1} + \frac{\psi_1 \psi_2}{\lambda_2} + \frac{\psi_2 \psi_3}{\lambda_1} (\lambda_1^{-1} - \lambda_2^{-1}), \quad (6.27)
n_{13} \delta = \frac{\psi_1 \psi_3}{\lambda_1} + \frac{\psi_3 \psi_3}{\lambda_3} + \frac{\psi_2 \psi_3}{\lambda_1} (\lambda_1^{-1} - \lambda_3^{-1}). \quad (6.28)
$$

Hence we can write the DT in terms of the eigenfunctions of the Lax representations (5.2)-(5.3) as

$$
\Gamma^{[1]} = \frac{1}{\delta} \begin{pmatrix} n_{11} m_{11} - n_{12} m_{21} - n_{13} m_{31} & n_{11} m_{12} - n_{12} m_{22} - n_{13} m_{32} & n_{11} m_{13} - n_{12} m_{23} - n_{13} m_{33} \\
n_{21} m_{11} - n_{22} m_{21} - n_{23} m_{31} & n_{21} m_{12} - n_{22} m_{22} - n_{23} m_{32} & n_{21} m_{13} - n_{22} m_{23} - n_{23} m_{33} \\
n_{31} m_{11} - n_{32} m_{21} - n_{33} m_{31} & n_{31} m_{12} - n_{32} m_{22} - n_{33} m_{32} & n_{31} m_{13} - n_{32} m_{23} - n_{33} m_{33} \end{pmatrix}, \quad (6.29)
$$

6.2 1-Soliton solutions for $\Gamma$ - spin system

To construct the 1-soliton solution of the $\Gamma$-spin system (5.1), now we consider a seed solution

$$
\Gamma^{[0]} = \Sigma. \quad (6.30)
$$

In our case the eigenfunctions are given by

$$
\psi_1 = e^{-i\lambda x - 2\lambda^3 t + i\delta_1} = e^{-\theta + i\delta_1}, \quad \psi_2 = e^{\theta + i\delta_2}, \quad \psi_3 = e^{\theta + i\delta_3}, \quad (6.31)
$$
where $\delta_i$ are complex constants and

$$\theta = \theta_1 + i\theta_2 = -i\lambda_1 x - 2i\lambda_2^2 t.$$  \hspace{1cm} (6.32)

Then we get

$$\Gamma^{[1]} = \begin{pmatrix}
\Gamma^{[1]}_{11} & \Gamma^{[1]}_{12} & \Gamma^{[1]}_{13} \\
\Gamma^{[1]}_{21} & \Gamma^{[1]}_{22} & \Gamma^{[1]}_{23} \\
\Gamma^{[1]}_{31} & \Gamma^{[1]}_{32} & \Gamma^{[1]}_{33}
\end{pmatrix},$$  \hspace{1cm} (6.33)

where $n_{ij}$ and $m_{ij}$ are given by the equations (6.15)-(6.16) and (6.25).

### 7 The 1-soliton solutions of the coupled M-LIII equation

Let us now we present the formulas for the 1-soliton solution of the coupled M-LIII equation (3.5)-(3.6). Its seed solution we write as

$$A^{[0]} = \sigma_3, \quad B^{[0]} = \sigma_3.$$  \hspace{1cm} (7.1)

To find the 1-soliton solution of the coupled M-LIII equation here we use the inverse M-transformation [39]-[44]. The inverse M-transformation allows us to fine solutions of the coupled M-LIII equation (3.5)-(3.6), if we know the solutions of the $\Gamma$-spin system (5.1). So the 1-soliton solution of the coupled M-LIII equation has the form

$$A^{[1]} = \frac{1}{1 - \Gamma^{[1]}_{33}} \begin{pmatrix}
\Gamma^{[1]}_{11} - \Gamma^{[1]}_{22} & 2\Gamma^{[1]}_{12} \\
2\Gamma^{[1]}_{21} & \Gamma^{[1]}_{22} - \Gamma^{[1]}_{11}
\end{pmatrix},$$  \hspace{1cm} (7.2)

$$B^{[1]} = \frac{1}{1 - \Gamma^{[1]}_{22}} \begin{pmatrix}
\Gamma^{[1]}_{11} - \Gamma^{[1]}_{33} & 2\Gamma^{[1]}_{13} \\
2\Gamma^{[1]}_{31} & \Gamma^{[1]}_{33} - \Gamma^{[1]}_{11}
\end{pmatrix},$$  \hspace{1cm} (7.3)

where $\Gamma^{[1]}_{ij}$ are given by the formulas (6.33) and (6.29).

### 8 Conclusion

In this paper we have presented the DT for the $\Gamma$-spin system which is the integrable $su(3)$-valued spin system. In particular, we have given the explicit formula for its 1-soliton solution. Then we have shown how construct soliton solutions of the coupled M-LIII equation for the two coupled $su(2)$-valued spin systems. For this purpose we have used the DT formulas of the $\Gamma$-spin system. Also the Lax representation of the coupled M-LIII equation is presented. Using this Lax representation, the gauge equivalence between the coupled M-LIII equation and the Manakov system is established. The results obtained in this paper will be useful in the study of nonlinear dynamics of multi-layer magnetic systems. Also they will be useful in differential geometry of curves and surfaces to find their integrable deformations of interacting curves and surfaces.

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