THE GROMOV-WITTEN AND DONALDSON-THOMAS
CORRESPONDENCE FOR TRIVIAL ELLIPTIC FIBRATIONS

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Abstract. We study the Gromov-Witten and Donaldson-Thomas correspondence conjectured in [MNOP1, MNOP2] for trivial elliptic fibrations. In particular, we verify the Gromov-Witten and Donaldson-Thomas correspondence for primary fields when the threefold is $E \times S$ where $E$ is a smooth elliptic curve and $S$ is a smooth surface with numerically trivial canonical class.

1. Introduction

The correspondence between the Gromov-Witten theory and Donaldson-Thomas theory for threefolds was conjectured and studied in [MNOP1, MNOP2]. Since then, it has been investigated extensively (see [MP, JL, Kat, KLQ, Beh, BF2] and the references there). A relationship between the quantum cohomology of the Hilbert scheme of points in the complex plane and the Gromov-Witten and Donaldson-Thomas correspondence for local curves was proved in [OP2, OP3]. The equivariant version was proposed and partially verified in [BP, GS]. In this paper, we study the Gromov-Witten and Donaldson-Thomas correspondence when the threefold admits a trivial elliptic fibration.

To state our results, we introduce some notation and refer to Subsect. 2.1 and Subsect. 3.1 for details. Let $X$ be a complex threefold, $\gamma_1, \ldots, \gamma_r \in H^*(X; \mathbb{Q})$, $\beta \in H_2(X; \mathbb{Z}) \setminus \{0\}$, $k_1, \ldots, k_r$ be nonnegative integers, and $u, q$ be formal variables. Let

$$Z'_{GW}(X; u | \prod_{i=1}^{r} \tau_{k_i}(\gamma_i))_\beta, \quad Z'_{DT}(X; q | \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i))_\beta$$

be the reduced degree-$\beta$ partition functions for the descendent Gromov-Witten invariants and Donaldson-Thomas invariants of $X$ respectively.

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Conjecture 1.1. Let $\beta \in H_2(X;\mathbb{Z}) \setminus \{0\}$ and $\vartheta = -\int K_X$. Then after the change of variables $e^{iu} = -q$, we have
\[
(-iu)^{\vartheta} Z_{GW}^{\beta} \left( X; u \prod_{i=1}^{r} \tau_0(\gamma_i) \right)_\beta = (-q)^{-\vartheta/2} Z_{DT}^{\beta} \left( X; q \prod_{i=1}^{r} \tilde{\tau}_0(\gamma_i) \right).
\]

(1.1)

Theorem 1.2. Let $f : X = E \times S \to S$ be the projection where $E$ is an elliptic curve and $S$ is a smooth surface. Then the Gromov-Witten/Donaldson-Thomas correspondence holds if either $\int_{\beta} K_X = \int_{\beta} f^* K_S = 0$, or
\[
\gamma_1, \ldots, \gamma_r \in f^* H^*(S; \mathbb{Q}) \subset H^*(X; \mathbb{Q}).
\]

Proof. The conclusion follows from Proposition 2.6 and Proposition 3.6 when
\[
\int_{\beta} K_X = \int_{\beta} f^* K_S = 0.
\]

It follows from Proposition 2.7 and Proposition 3.7 when $\gamma_1, \ldots, \gamma_r \in f^* H^*(S; \mathbb{Q}) \subset H^*(X; \mathbb{Q})$.

Corollary 1.3. Let $E$ be an elliptic curve and $S$ be a smooth surface with numerically trivial canonical class $K_S$. Then the Gromov-Witten/Donaldson-Thomas correspondence holds for the threefold $X = E \times S$.

In fact, when $\gamma_1, \ldots, \gamma_r \in f^* H^*(S; \mathbb{Q}) \subset H^*(X; \mathbb{Q})$, Proposition 2.7 and Proposition 3.7 state that after the change of variables $e^{iu} = -q$,
\[
(-iu)^{\vartheta - \sum_i k_i} Z_{GW}^{\beta} \left( X; u \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \right)_\beta = (-q)^{-\vartheta/2} Z_{DT}^{\beta} \left( X; q \prod_{i=1}^{r} \tilde{\tau}_{k_i}(\gamma_i) \right).
\]

This is consistent with (and partially sharpens) the Conjecture 4 in [MNOP2] which is about the Gromov-Witten and Donaldson-Thomas correspondence for descendent fields. It would be interesting to see whether this sharpened version holds for general cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X; \mathbb{Q})$.

Proposition 2.7 and Proposition 3.7 are proved in Sect. 2 and Sect. 3, respectively. The idea is to view the elliptic curve $E$ as an algebraic group and to use the action of $E$ on the moduli space $\mathfrak{M}_{g,r}(X, \beta)$ of stable maps and the moduli space $\mathcal{J}_n(X, \beta)$ of ideal sheaves. The $E$-action on $\mathfrak{M}_{g,r}(X, \beta)$ has no fixed points when $r \geq 1$, or $g \neq 1$, or $\beta \neq d\vartheta_0$. It follows from Lemma 2.5 that the corresponding Gromov-Witten invariants are zero. The only exception is $\langle \gamma, d\vartheta_0 \rangle$ which can be computed directly by using the work of Okounkov-Pandharipande [OP1] on the Gromov-Witten invariants of an elliptic curve and Göttsche’s formula for the Euler characteristics of the Hilbert scheme $S^{[d]}$ of points on a smooth surface $S$. Similarly, the $E$-action on $\mathcal{J}_n(X, \beta)$ has no fixed points when $n \geq 1$ or $\beta \neq d\vartheta_0$. It follows from Lemma 3.5 that the corresponding Donaldson-Thomas invariants are also zero. The only exception is $\langle \gamma, d\vartheta_0 \rangle$ which can be computed directly by determining the obstruction bundle over the moduli space $\mathcal{J}_0(X, d\vartheta_0) \cong S^{[d]}$. 
It is expected that our approach can be used to handle the relative Gromov-Witten and Donaldson-Thomas correspondence (see [MNOP2]) for trivial elliptic fibrations. In another direction, one might attempt to study the (absolute and relative) Gromov-Witten and Donaldson-Thomas correspondence for nontrivial elliptic fibrations. We leave these to the interested readers.

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2. Gromov-Witten theory

2.1. Gromov-Witten invariants.

Let $X$ be a smooth projective complex variety. Fix $\beta \in H_2(X; \mathbb{Z})$. Let $\overline{M}_{g,r}(X, \beta)$ be the moduli space of stable maps from connected genus-$g$ curves with $r$ marked points to $X$ representing the class $\beta$. The virtual fundamental class $[\overline{M}_{g,r}(X, \beta)]^{\text{vir}}$ has been constructed in [BF1, LT]. By ignoring the extra notation of stacks, the virtual fundamental class $[\overline{M}_{g,r}(X, \beta)]^{\text{vir}}$ is defined by the element

$$R(\pi_{g,r})_* (\text{ev}_{r+1})^* T_X$$

in the derived category $\mathcal{D}_{\text{coh}}(\overline{M}_{g,r}(X, \beta))$ of coherent sheaves on $\overline{M}_{g,r}(X, \beta)$, where $\text{ev}_i : \overline{M}_{g,r+1}(X, \beta) \to X$ is the $i$-th evaluation map, and $\pi_{g,r}$ stands for the morphism:

$$\pi_{g,r} : \overline{M}_{g,r+1}(X, \beta) \to \overline{M}_{g,r}(X, \beta)$$

forgetting the $(r + 1)$-th marked point. Let $\mathcal{L}_i$ be the cotangent line bundle on $\overline{M}_{g,r}(X, \beta)$ associated to the $i$-th marked point. Put

$$\psi_i = c_1(\mathcal{L}_i).$$

For $\gamma_1, \ldots, \gamma_r \in H^*(X; \mathbb{Q})$ and nonnegative integers $k_1, \ldots, k_r$, define

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = \int_{[\overline{M}_{g,r}(X, \beta)]^{\text{vir}}} \prod_{i=1}^r \psi_i^{k_i} (\text{ev}_i^* (\gamma_i)).$$

Define the reduced Gromov-Witten potential of $X$ by

$$F'_{\text{GW}} \left( X; u, v | \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right) = \sum_{\beta \neq 0} \sum_{g \geq 0} \left\langle \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta} u^{2g-2} v^\beta$$

omitting the constant maps. For $\beta \neq 0$, the reduced partition function

$$Z'_{\text{GW}} \left( X; u | \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right)_\beta$$

of degree-$\beta$ Gromov-Witten invariants is defined by setting:

$$1 + \sum_{\beta \neq 0} Z'_{\text{GW}} \left( X; u | \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right)_\beta v^\beta = \exp F'_{\text{GW}} \left( X; u, v | \prod_{i=1}^r \tau_{k_i}(\gamma_i) \right).$$
Alternatively, let $\mathcal{M}_{g,r}(X,\beta)$ be the moduli space of stable maps from possibly disconnected curves $C$ of genus-$g$ with $r$ marked points and with no collapsed connected components. Here the genus of a possibly disconnected curve $C$ is

$$1 - \chi(O_C) = 1 - \ell + \sum_{i=1}^{\ell} g_{C_i}$$

where $C_1,\ldots,C_\ell$ denote all the connected components of $C$. For $\gamma_1,\ldots,\gamma_r \in H^*(X;\mathbb{Q})$ and $k_1,\ldots,k_r \geq 0$, define the reduced Gromov-Witten invariant by

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle'_{g,\beta} = \int_{[\mathcal{M}_{g,r}(X,\beta)]^{\text{vir}}} \prod_{i=1}^{r} \psi_i^{k_i} \text{ev}_i^*(\gamma_i).$$

Then the reduced partition function of degree-$\beta$ invariants is also given by

$$Z'_{GW}(X; u) = \sum_{g \in \mathbb{Z}} \langle \prod_{i=1}^{r} \tau_{k_i}(\gamma_i) \rangle'_{g,\beta} u^{2g-2}.$$ (2.7)

When $\dim(X) = 3$, the expected dimensions of $\mathcal{M}_{g,r}(X,\beta)$ and $\mathcal{M}'_{g,r}(X,\beta)$ are

$$- \int_{\beta} K_X + r.$$ (2.8)

Remark 2.1. By the Fundamental Class Axiom, Divisor Axiom and Dilation Axiom of the descendent Gromov-Witten invariants, if $\beta \neq 0$ and $\int_{\beta} K_X = 0$, then

$$Z'_{GW}(X; u)$$

can be reduced to the case $r = 0$, i.e., to the reduced partition function

$$Z'_{GW}(X; u).$$ (2.9)

2.2. The computations.

We begin with the Gromov-Witten invariants of a smooth elliptic curve $E$. Let $d \geq 1$ and $[E] \in H_2(E;\mathbb{Z})$ be the fundamental class. We use

$$\mathcal{M}_{g,r}(E,d), \mathcal{M}'_{g,r}(E,d)$$

to denote the moduli spaces $\mathcal{M}_{g,r}(E,d)$, $\mathcal{M}'_{g,r}(E,d)$ respectively. The expected dimension of the moduli spaces $\mathcal{M}_{1,0}(E,d)$ and $\mathcal{M}'_{1,0}(E,d)$ is zero. So

$$\langle 1, d[E] \rangle = \deg[\mathcal{M}_{1,0}(E,d)]^{\text{vir}},$$

$$\langle 1', d[E] \rangle = \deg[\mathcal{M}'_{1,0}(E,d)]^{\text{vir}}.$$ (2.10) (2.11)

Note that if $C$ is the (possibly disconnected) domain curve of a stable map in $\mathcal{M}_{1,0}(E,d)$, then every connected component of $C$ must be of genus-1. Therefore,
as in (2.4), (2.5) and (2.7), we obtain the following relation:

\[ 1 + \sum_{d=1}^{+\infty} \langle 1, d[H] \rangle v^d = \exp \sum_{d=1}^{+\infty} \langle 1, d[H] \rangle v^d. \] (2.12)

By the Theorem 5 in [OP1] (replacing \( n \) and \( q \) there by 0 and \( v \) respectively),

\[ 1 + \sum_{d=1}^{+\infty} \langle 1, d[H] \rangle v^d = \frac{1}{\prod_{m=0}^{+\infty} (1 - v^m)}. \] (2.13)

In the rest of this section, we adopt the following notation.

**Notation 2.2.**

(i) Let \( X = E \times S \) where \( E \) is an elliptic curve and \( S \) is a smooth surface. Let \( \beta_0 \in H_2(X; \mathbb{Z}) \) be the fiber class of the fibration \( f : X = E \times S \rightarrow S \).

We use \( K_X \) to denote both the canonical class and the canonical line bundle of \( X \).

(ii) For \( d \geq 0 \), let \( S^{[d]} \) be the Hilbert scheme which parametrizes the length-\( d \)-0-dimensional closed subschemes of the surface \( S \).

(iii) Fix \( O \in E \) as the zero element for the group law on \( E \). For \( p \in E \), let \( \phi_p : E \rightarrow E \) be the automorphism of \( E \) defined via translation \( \phi_p(e) = p + e \). We have an action of \( E \) on \( X = E \times S \) via the automorphisms \( \phi_p \times \text{Id}_S \), \( p \in E \).

**Lemma 2.3.** Let \( X \) be from Notation 2.2 and \( d \geq 1 \). Then, we have

\[ \langle 1, d[\beta_0] \rangle = \chi(S^{[d]}). \]

**Proof.** First of all, let \( H^E_1 \) be the rank-1 Hodge bundle over \( \overline{\mathcal{M}}_{1,0}(E, d) \), i.e.,

\[ H^E_1 = (\pi_{1,0})_* \omega_{1,0} \]

where \( \omega_{1,0} \) is the relative dualizing sheaf of the forgetful map \( \pi_{1,0} \) in (2.2).

Next, by the universal property of moduli spaces, we have

\[ \overline{\mathcal{M}}_{1,0}(X, d[\beta_0]) \cong \overline{\mathcal{M}}_{1,0}(E, d) \times S. \] (2.15)

By the definitions of virtual fundamental classes and the Hodge bundle,

\[ [\overline{\mathcal{M}}_{1,0}(X, d[\beta_0])]^{\text{vir}} = e(\pi_1^*(H^E_1)^v \otimes \pi_2^* T_S) \cap \pi_1^*[\overline{\mathcal{M}}_{1,0}(E, d)]^{\text{vir}} \] (2.16)

where \( \pi_1 \) and \( \pi_2 \) are the two projections of \( \overline{\mathcal{M}}_{1,0}(X, d[\beta_0]) \) via the isomorphism (2.15), and \( e(\cdot) \) denotes the Euler class (or the top class). Note that

\[ e(\pi_1^*(H^E_1)^v \otimes \pi_2^* T_S) = \pi_2^* e(S) + \pi_2^* K_S \cdot \pi_1^* c_1(H^E_1) + \pi_1^* c_1(H^E_1)^2. \]
By (2.16), \( \langle 1_{1,d} \rangle = \chi(S) \cdot \langle 1_{d[E]} \rangle \). Therefore, we obtain
\[
1 + \sum_{d=1}^{+\infty} \langle 1_{1,d} \rangle v^d = \exp \sum_{d=1}^{+\infty} \langle 1_{d[E]} \rangle v^d
\]
\[
= \exp \left( \chi(S) \cdot \sum_{d=1}^{+\infty} \langle 1_{d[E]} \rangle v^d \right)
\]
\[
= \left( 1 + \sum_{d=1}^{+\infty} \langle 1_{1,d} \rangle v^d \right)^{\chi(S)}
\]
\[
= \frac{1}{\prod_{m=0}^{+\infty} (1 - v^m)^{\chi(S)}}
\]
(2.17)
by (2.12) and (2.13). By Göttsche’s formula in [Got] for \( \chi(S^{[d]}) \), we have
\[
\sum_{d=0}^{+\infty} \chi(S^{[d]}) v^d = \frac{1}{\prod_{m=0}^{+\infty} (1 - v^m)^{\chi(S)}}.
\]
Combining this with (2.17), we conclude that \( \langle 1_{1,d} \rangle = \chi(S^{[d]}) \).

Let \( X \) be from Notation 2.2 and \( \beta \in H_2(X; \mathbb{Z}) \backslash \{0\} \). For any \( p \in E \),
\[
(\phi_p \times \text{Id}_S), \beta = \beta
\]
(2.18)
since \( \{\phi_p \times \text{Id}_S\}_{p \in E} \) form a connected algebraic family of automorphisms of \( X \). Thus the algebraic group \( E \) acts on the stack of \( r \)-pointed degree-\( \beta \) stable maps to \( X \) (see [Kon]). The universal properties of moduli spaces imply that there is a corresponding action of \( E \) on the moduli space \( \overline{M}_{g,r}(X, \beta) \). For \( p \in E \), let
\[
\Psi_p : \overline{M}_{g,r}(X, \beta) \to \overline{M}_{g,r}(X, \beta)
\]
be the corresponding automorphism. Then we see that the automorphism \( \Psi_p \) maps a point \([\mu : (C, w_1, \ldots, w_r) \to X] \in \overline{M}_{g,r}(X, \beta)\) to the point
\[
[(\phi_p \times \text{Id}_S) \circ \mu : (C, w_1, \ldots, w_r) \to X] \in \overline{M}_{g,r}(X, \beta).
\]
(2.19)

Lemma 2.4. With the notation as above, the algebraic group \( E \) acts without fixed points on \( \overline{M}_{g,r}(X, \beta) \) if \( \beta \neq d\beta_0 \), or \( r \geq 1 \), or \( g \neq 1 \).

Proof. Assume that \([\mu : (C, w_1, \ldots, w_r) \to X] \in \overline{M}_{g,r}(X, \beta)\) is fixed by the action of \( E \). By definition, for every \( p \in E \), there is an automorphism \( \tau_p \) of \( C \) such that
\[
\mu \circ \tau_p = (\phi_p \times \text{Id}_S) \circ \mu
\]
(2.20)
and \( \tau_p(w_i) = w_i \) for all \( 1 \leq i \leq r \). In particular, for every \( p \in E \), we have
\[
\mu(C) = (\phi_p \times \text{Id}_S)(\mu(C)).
\]
So \( \mu(C) \) is a fiber of the elliptic fibration \( f \), and \( \beta = d\beta_0 \) for some \( d \geq 1 \). By our assumption, either \( r \geq 1 \) or \( g \geq 2 \). By (2.20), we get
\[
\mu \circ \tau_p(C) = \phi_p(\mu(C)).
\]
(2.21)
Since $\phi_p$ acts freely on the fiber $\mu(C)$, (2.21) implies that the automorphisms $\tau_p$ of the marked curve $(C; w_1, \ldots, w_r)$ are different for different points $p \in E$. Hence the automorphism group of the marked curve $(C; w_1, \ldots, w_r)$ is infinite. This is impossible since either $g \geq 2$ or $g = 1$ and $r \geq 1$. □

Lemma 2.5. Let $\beta \in H_2(X; \mathbb{Z}) \setminus \{0\}$. Assume that $\gamma_1, \ldots, \gamma_r \in f^*H^*(S; \mathbb{Q}) \subset H^*(X; \mathbb{Q})$. If $\beta \neq d\beta_0$, or $r \geq 1$, or $g \neq 1$, then we have

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = 0.$$ 

Proof. First of all, note that it suffices to show that

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = 0 \quad (2.22)$$

if $\beta \neq d\beta_0$, or $r \geq 1$, or $g \neq 1$. In the following, we prove (2.22).

By Lemma 2.4, $E$ acts without fixed points on $\overline{\mathcal{M}}_{g,r}(X, \beta)$. Since $E$ is an elliptic curve, any proper algebraic subgroup is finite. Thus the stabilizer of any point for the $E$ action on $\overline{\mathcal{M}}_{g,r}(X, \beta)$ is finite. Since $\overline{\mathcal{M}}_{g,r}(X, \beta)$ is finite type, the order of the stabilizer subgroup at any point is bounded by some number $N$. Thus, if $G$ is a cyclic subgroup of $E$ of prime order $p > N$, then $G$ acts freely on $\overline{\mathcal{M}}_{g,r}(X, \beta)$. We fix such a cyclic subgroup $G$ of $E$ in the rest of the proof.

The complex $R(\pi_{g,r})_*\text{ev}_{E+r+1}^*T_X$ from (2.1) is equivariant for the action of any algebraic automorphism group of $X$. Thus for some positive integer $m$ (independent of $G$), the cycle $m[\overline{\mathcal{M}}_{g,r}(X, \beta)]^\text{vir}$ defines an element of the integral equivariant Borel-Moore homology group $H^*_G(\overline{\mathcal{M}}_{g,r}(X, \beta))$. Likewise if $\gamma_i \in f^*H^*(S; \mathbb{Q})$, then the cycle $\gamma_i$ is invariant under the action of $E$ on $X$. Hence some positive multiple $m_i\gamma_i$ defines an element of $H^*_G(X)$, where $m_i$ is independent of $G$. Note from (2.14) that the evaluation map $\text{ev}_i : \overline{\mathcal{M}}_{g,r}(X, \beta) \to X$ is $G$-equivariant, so the pullback $\text{ev}_i^*(m_i\gamma_i)$ determines an element of $H^*_G(\overline{\mathcal{M}}_{g,r}(X, \beta))$. In addition, the cotangent line bundles $L_i$ ($1 \leq i \leq r$) over $\overline{\mathcal{M}}_{g,r}(X, \beta)$ are equivariant for the action of $G$. It follows from the definition (2.8) that the cycle

$$mm_1 \cdots m_r \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta}$$

defines an element in the degree-0 Borel-Moore homology $H^0_G(\overline{\mathcal{M}}_{g,r}(X, \beta))$.

Since $G$ is a cyclic subgroup of order $p$ which acts freely on $\overline{\mathcal{M}}_{g,r}(X, \beta)$, any element of $H^0_G(\overline{\mathcal{M}}_{g,r}(X, \beta))$ is represented by a $G$-invariant 0-cycle whose degree is a multiple of $p$ (possibly 0). Since $p$ can be taken to be arbitrarily large,

$$mm_1 \cdots m_r \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = 0.$$ 

Therefore, $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_r}(\gamma_r) \rangle_{g,\beta} = 0$. This completes the proof of (2.22). □

We define the cohomology degree $|\gamma| = \ell$ when $\gamma \in H^\ell(X; \mathbb{Q})$. 

Proposition 2.6. Let $\beta \in H_2(X; \mathbb{Z}) \setminus \{0\}$. Assume $\int_{\beta} K_X = \int_{\beta} f^* K_S = 0$. Then,

$$Z'_{GW} \left( X; u | \prod_{i=1}^r \tau_0(\gamma_i) \right)_{\beta} = \left\{ \prod_{i=1}^r \int_{\beta} \gamma_i \cdot \chi(S^{[d]}) \right\}
\begin{cases} 0 & \text{if } |\gamma_i| = 2 \text{ for every } i \text{ and } \beta = d\beta_0 \text{ for some } d \geq 1; \\
\sum_{i=1}^r |\gamma_i| = 2r & \text{otherwise.}
\end{cases}$$

Proof. By (2.8) and the degree condition on Gromov-Witten invariants,

$$\sum_{i=1}^r |\gamma_i| = 2r.$$ 

By the Fundamental Class Axiom and Divisor Axiom of Gromov-Witten invariants,

$$\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \rangle_{g,\beta} = \left\{ \prod_{i=1}^r \int_{\beta} \gamma_i \cdot \chi(S^{[d]}) \right\}
\begin{cases} 0 & \text{if } |\gamma_i| = 2 \text{ for every } i; \\
\sum_{i=1}^r |\gamma_i| = 2r & \text{otherwise.}
\end{cases}$$

So by Lemma 2.3 and by taking $r = 0$ in (2.22), we conclude that

$$\langle \tau_0(\gamma_1) \cdots \tau_0(\gamma_r) \rangle'_{g,\beta} = \left\{ \prod_{i=1}^r \int_{\beta} \gamma_i \cdot \chi(S^{[d]}) \right\}
\begin{cases} 0 & \text{if } |\gamma_i| = 2 \text{ for every } i, g = 1, \beta = d\beta_0; \\
\sum_{i=1}^r |\gamma_i| = 2r & \text{otherwise.}
\end{cases}$$

Now our proposition follows directly from the identity (2.7). \hfill \square

Proposition 2.7. Let $X$ be from Notation 2.2 and $\beta \in H_2(X; \mathbb{Z}) \setminus \{0\}$. Assume that $\gamma_1, \ldots, \gamma_r \in f^* H^*(S; \mathbb{Q}) \subset H^*(X; \mathbb{Q})$. Then,

$$Z'_{GW} \left( X; u | \prod_{i=1}^r \tau_k(\gamma_i) \right)_{\beta} = \left\{ \chi(S^{[d]}) \right\}
\begin{cases} 0 & \text{if } r = 0 \text{ and } \beta = d\beta_0 \text{ with } d \geq 1; \\
\sum_{i=1}^r |\gamma_i| = 2r & \text{otherwise.}
\end{cases}$$

Proof. Follows from the identity (2.7), Lemma 2.3 and Lemma 2.5 \hfill \square

3. Donaldson-Thomas theory

3.1. Donaldson-Thomas invariants.

Let $X$ be a smooth projective complex threefold. For a fixed class $\beta \in H_2(X; \mathbb{Z})$ and a fixed integer $n$, following the definition and notation in [MNOP1, MNOP2], we define $\mathcal{I}_n(X, \beta)$ to be the moduli space parametrizing the ideal sheaves $I_Z$ of 1-dimensional closed subschemes $Z$ of $X$ satisfying the conditions:

$$\chi(O_Z) = n, \quad [Z] = \beta$$

where $[Z]$ is the class associated to the dimension-1 component (weighted by their intrinsic multiplicities) of $Z$. Note that $\mathcal{I}_n(X, \beta)$ is a special case of the moduli spaces of Gieseker semistable torsion-free sheaves over $X$. When the anti-canonical divisor $-K_X$ is effective, perfect obstruction theories on the moduli spaces $\mathcal{I}_n(X, \beta)$
have been constructed in [Tho]. This result has been generalized in [MP]. By the Lemma 1 in [MNOP2], the virtual dimension of $\mathcal{J}_n(X,\beta)$ is

$$-\int_\beta K_X. \quad (3.2)$$

The Donaldson-Thomas invariant is defined via integration against the virtual fundamental class $[\mathcal{J}_n(X,\beta)]^{vir}$ of the moduli space $\mathcal{J}_n(X,\beta)$. More precisely, let $\gamma \in H^\ell(X;\mathbb{Q})$ and $\mathcal{I}$ be the universal ideal sheaf over $\mathcal{J}_n(X,\beta) \times X$. Let

$$\chi_{k+2}(\gamma) : H_*(\mathcal{J}_n(X,\beta);\mathbb{Q}) \to H_{*-2k+2-\ell}(\mathcal{J}_n(X,\beta);\mathbb{Q}) \quad (3.3)$$

be the operation on the homology of $\mathcal{J}_n(X,\beta)$ defined by

$$\chi_{k+2}(\gamma)(\xi) = \pi_1^*(\chi_{k+2}(\mathcal{I}) \cdot \pi_2^*\gamma \cap \pi_1^*\xi) \quad (3.4)$$

where $\pi_1$ and $\pi_2$ be the two projections on $\mathcal{J}_n(X,\beta) \times X$. Define

$$\langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) \rangle_{n,\beta}$$

$$= \int_{[\mathcal{J}_n(X,\beta)]^{vir}} \prod_{i=1}^r (-1)^{k_i+1} \chi_{k_i+2}(\gamma_i)$$

$$= (-1)^{k_1+1} \chi_{k_1+2}(\gamma_1) \circ \cdots \circ (-1)^{k_r+1} \chi_{k_r+2}(\gamma_r) ([\mathcal{J}_n(X,\beta)]^{vir}). \quad (3.5)$$

The partition function for these descendent Donaldson-Thomas invariants is

$$Z_{DT}'(X; q | \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i))_{\beta} = \sum_{n \in \mathbb{Z}} \langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) \rangle_{n,\beta} q^n. \quad (3.6)$$

The partition function for the degree-0 Donaldson-Thomas invariants of $X$ is

$$Z_{DT}(X; q)_0 = M(-q)^{\chi(X)} \quad (3.7)$$

by [JLi BF2] (this formula was conjectured in [MNOP1 MNOP2]), where

$$M(q) = \prod_{n=1}^{+\infty} \frac{1}{(1-q^n)^n}$$

is the McMahon function. The reduced partition function is defined to be

$$Z_{DT}'(X; q | \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i))_{\beta} = \frac{Z_{DT}(X; q | \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i))_{\beta}}{Z_{DT}(X, q)_0} \quad (3.8)$$

In the next two lemmas, we study the operators $\chi_2(\gamma)$ and $\chi_3(1_X)$ respectively, where $1_X \in H^*(X;\mathbb{Q})$ is the fundamental cohomology class. The results will be used in Subsect. 3.2. Note that the first lemma is the analogue to the Fundamental Class Axiom and Divisor Axiom of Gromov-Witten invariants, while the second one is the analogue to the Dilaton Axiom of Gromov-Witten invariants. By [333],

$$\chi_2(\gamma) : H_b(\mathcal{J}_n(X,\beta);\mathbb{Q}) \to H_{b-2+\ell}(\mathcal{J}_n(X,\beta);\mathbb{Q}),$$
Lemma 3.1. (i) Let $\beta \in H_2(X;\mathbb{Z})$ and $\gamma \in H^\ell(X;\mathbb{Q})$. Then,
\[
\operatorname{ch}_2(\gamma)|_{H^\ell_{\text{alg}}(\mathcal{J}(X,\beta))} = \begin{cases} 
0 & \text{if } \ell = 0 \text{ or } 1; \\
-\int_{\overline{\mathcal{J}}} \gamma \cdot \text{Id} & \text{if } \ell = 2.
\end{cases}
\]

(ii) If the moduli space $\mathcal{J}(X,\beta)$ is smooth, then
\[
\operatorname{ch}_2(\gamma) = \begin{cases} 
0 & \text{if } \ell = 0 \text{ or } 1; \\
-\int_{\overline{\mathcal{J}}} \gamma \cdot \text{Id} & \text{if } \ell = 2.
\end{cases}
\]

Proof. (i) Let $\mathcal{J} = \mathcal{J}(X,\beta)$. By [FG], there is a proper morphism
\[
p : \mathcal{J} \to \mathcal{J}
\]
with $\mathcal{J}$ smooth and $p_\ast : H^\ell_{\text{alg}}(\mathcal{J}) \to H^\ell_{\text{alg}}(\mathcal{J})$ surjective. Such a morphism $p$ is called a nonsingular envelope (see p.299 of [FG]). Let $\bar{\pi}_1$ and $\bar{\pi}_2$ be the projections from $\mathcal{J} \times X$ to the first and second factors respectively.

Let $\xi \in H^\ell_{\text{alg}}(\mathcal{J})$. Then $\xi = p_\ast \bar{\xi}$ for some $\bar{\xi} \in H^\ell_{\text{alg}}(\mathcal{J})$. Define
\[
\bar{\operatorname{ch}}_2(\gamma)(\bar{\xi}) = \bar{\pi}_1\ast \left( \operatorname{ch}_2((p \times \text{Id}_X)^\ast \mathcal{I}) \bar{\pi}_2^\ast \gamma \cap \bar{\pi}_1^\ast \bar{\xi} \right)
\]
(3.9)

where $\mathcal{I}$ denotes the universal ideal sheaf over $\mathcal{J} \times X$. Using the projection formula and the fact that $(p \times \text{Id}_X)_\ast \bar{\pi}_\ast \bar{\xi} = \pi^\ast p_\ast \bar{\xi} = \pi^\ast \xi$, we have
\[
p_\ast (\bar{\operatorname{ch}}_2(\gamma)(\bar{\xi})) = p_\ast \bar{\pi}_1\ast \left( \operatorname{ch}_2((p \times \text{Id}_X)^\ast \mathcal{I}) \bar{\pi}_2^\ast \gamma \cap \bar{\pi}_1^\ast \bar{\xi} \right)
\]
\[
= \pi^\ast (p \times \text{Id}_X)_\ast \left( (p \times \text{Id}_X)^\ast (\operatorname{ch}_2(\mathcal{I}) \bar{\pi}_2^\ast \gamma) \cap \bar{\pi}_1^\ast \bar{\xi} \right)
\]
\[
= \pi^\ast \left( \operatorname{ch}_2(\mathcal{I}) \bar{\pi}_2^\ast \gamma \cap (p \times \text{Id}_X)_\ast \bar{\pi}_1^\ast \bar{\xi} \right)
\]
\[
= \operatorname{ch}_2(\gamma)(\xi).
\]
(3.10)

Since $\mathcal{J}$ is smooth, the Poincaré duality holds and we see from (3.9) that
\[
\bar{\operatorname{ch}}_2(\gamma)(\bar{\xi}) = \bar{\pi}_1\ast (\operatorname{ch}_2((p \times \text{Id}_X)^\ast \mathcal{I}) \bar{\pi}_2^\ast \gamma) \cap \bar{\xi}
\]
where $\bar{\pi}_1\ast (\operatorname{ch}_2((p \times \text{Id}_X)^\ast \mathcal{I}) \bar{\pi}_2^\ast \gamma)$ is the cohomology class Poincaré dual to
\[
\bar{\pi}_1\ast \left( \operatorname{ch}_2((p \times \text{Id}_X)^\ast \mathcal{I}) \bar{\pi}_2^\ast \gamma \cap [\mathcal{J} \times X] \right).
\]
Thus by (3.10), to prove the lemma, it suffices to show that
\[
\bar{\pi}_1\ast \left( \operatorname{ch}_2((p \times \text{Id}_X)^\ast \mathcal{I}) \bar{\pi}_2^\ast \gamma \cap [\mathcal{J} \times X] \right) = \begin{cases} 
0 & \text{if } \ell = 0 \text{ or } 1; \\
-\int_{\overline{\mathcal{J}}} \gamma \cdot [\mathcal{J}] & \text{if } \ell = 2.
\end{cases}
\]
(3.11)

Let $\mathcal{Z} \subset \mathcal{J} \times X$ be the universal closed subscheme. Set-theoretically,
\[
\mathcal{Z} = \{(I_{\mathcal{Z}}, x) \in \mathcal{J} \times X | x \in \text{Supp}(\mathcal{Z})\}.
\]
Let $\widetilde{Z} = (p \times \text{Id}_X)^{-1} \mathcal{Z}$. Then, $\mathcal{I} = \mathcal{I}_Z$, $(p \times \text{Id}_X)^* \mathcal{I} = (p \times \text{Id}_X)^* \mathcal{I}_Z = I_{\widetilde{Z}}$, and
\[
\text{ch}_2((p \times \text{Id}_X)^* \mathcal{I}) = \text{ch}_2(I_{\widetilde{Z}}) = -c_2(I_{\widetilde{Z}}) = c_2(O_{\widetilde{Z}}).
\] 
(3.12)

If $\beta = 0$, then $\mathcal{Z}$ is of codimension-3 in $\mathcal{I} \times X$, and $\widetilde{Z}$ is of codimension-3 in $\tilde{\mathcal{I}} \times X$ as well. By (3.12), $\text{ch}_2((p \times \text{Id}_X)^* \mathcal{I}) = 0$. Therefore, (3.11) holds.

Next, we assume $\beta \neq 0$. Then, $\mathcal{Z}$ is of codimension-2 in $\mathcal{I} \times X$, and $\widetilde{Z}$ is of codimension-2 in $\tilde{\mathcal{I}} \times X$. By (3.12), $\text{ch}_2((p \times \text{Id}_X)^* \mathcal{I}) = -[\widetilde{Z}]$. So
\[
\pi_{1*}\left(\text{ch}_2((p \times \text{Id}_X)^* \mathcal{I}) \cdot [\pi_2^* \gamma] \cap (\tilde{\mathcal{I}} \times X]\right) = -\pi_{1*}([\widetilde{Z}] \cdot [\pi_2^* \gamma]).
\] 
(3.13)

When $\ell = 0$ or 1, we get $\pi_{1*}([\widetilde{Z}] \cdot [\pi_2^* \gamma]) = 0$ by degree reason. Hence (3.11) holds.

We are left with the case $\ell = 2$. In this case, $\pi_{1*}([\widetilde{Z}] \cdot [\pi_2^* \gamma])$ is a multiple of $[\tilde{\mathcal{I}}]$. Let $m$ be the multiplicity, and $\tilde{w} \in \tilde{\mathcal{I}}$ be a point. Then, we have
\[
m = \deg ( [\widetilde{Z}] \cdot [\pi_2^* \gamma] ) |_{(\tilde{w}) \times X} = \int_\beta \gamma.
\]
Therefore, we conclude from (3.13) that (3.11) holds when $\ell = 2$.

(ii) Follows from the proof of (i) by taking $\tilde{\mathcal{I}} = \mathcal{I}$ and $p = \text{Id}_3$. \qed

**Lemma 3.2.** (i) Let $\beta \in H_2(X; \mathbb{Z})$. Then, we have
\[
\text{ch}_3(1_X)|_{H^{\text{alg}}_2(\mathcal{I}_n(X, \beta))} = -\left(n + \int_\beta K_X\right) \cdot \text{Id}.
\]

(ii) If the moduli space $\mathcal{I}_n(X, \beta)$ is smooth, then
\[
\text{ch}_3(1_X) = -\left(n + \int_\beta K_X\right) \cdot \text{Id}.
\] 
(3.14)

**Proof.** Note that (i) follows from the proof of (ii) and the similar trick of using a nonsingular envelope as in the proof of Lemma 3.1 (i). To prove (ii), we adopt the notation in (3.1). Using the projection formula, we get
\[
\text{ch}_3(1_X)(\xi) = \pi_{1*}\left(\text{ch}_3(\mathcal{I}) \cap [\pi^*_1 \xi]\right) = \pi_{1*}\text{ch}_3(\mathcal{I}) \cdot \xi
\] 
(3.15)
since our moduli space $\mathcal{I}_n(X, \beta)$ is smooth. Note that $\pi_{1*}\text{ch}_3(\mathcal{I})$ is a multiple of the fundamental cycle of $\mathcal{I}_n(X, \beta)$. Let $m$ be the multiplicity. Then,
\[
m = \deg \text{ch}_3(\mathcal{I})|_{[I_x] \times X} = \deg \text{ch}_3(I_x) = -\deg \text{ch}_3(O_Z) = -\frac{1}{2} \deg c_3(O_Z)
\]
where $[I_x]$ denotes a point in $\mathcal{I}_n(X, \beta)$. Since $c_1(O_Z) = 0$ and $c_2(O_Z) = -[Z] = -\beta$, we see from (3.11) and the Hirzebruch-Riemann-Roch Theorem that
\[
m = -\frac{1}{2} \deg c_3(O_Z) = -\left(n + \int_\beta K_X\right).
\]
Now combining this with (3.15), we immediately obtain formula (3.14). \qed
Remark 3.3. Let \( \beta \in H_2(X; \mathbb{Z}) \) and \( \gamma \in H^2(X; \mathbb{Q}) \). We expect that both Lemma 3.1 and Lemma 3.2 can be sharpened, i.e., we expect in general that

\[
\chi_2(\gamma) = \begin{cases} 
0 & \text{if } \ell = 0 \text{ or } 1; \\
-\int_\beta \gamma \cdot \text{Id} & \text{if } \ell = 2;
\end{cases}
\]

\[
\chi_3(1_X) = -\left(n + \int_\beta K_X\right) \cdot \text{Id}.
\]

3.2. The computations.

In the rest of this section, we adopt the notation in Notation 2.22. We begin with the case when \( n = 0 \) and \( \beta = d\beta_0 \) with \( d \geq 0 \). Note that

\[
\mathcal{I}_0(X, d\beta_0) \cong S^{[d]}.
\]

However, the expected dimension of \( \mathcal{I}_0(X, d\beta_0) \) is zero by 3.2.

Lemma 3.4. (i) The obstruction bundle over the moduli space \( \mathcal{I}_0(X, d\beta_0) \cong S^{[d]} \)

is isomorphic to the tangent bundle \( T_{S^{[d]}} \) of the Hilbert scheme \( S^{[d]} \).

(ii) The Donaldson-Thomas invariant \( \langle \rangle_{0, d\beta_0} \) is equal to \( \chi(S^{[d]}) \).

Proof. It is clear that (ii) follows from (i). To prove (i), let

\[
\psi = \text{Id}_{S^{[d]}} \times f : S^{[d]} \times X \to S^{[d]} \times S
\]

and \( \phi : S^{[d]} \times S \to S^{[d]} \) be the projections. Let \( \pi = \phi \circ \psi : S^{[d]} \times X \to S^{[d]} \). Let \( \mathcal{J} \) be the universal ideal sheaf over \( S^{[d]} \times S \). Then the universal ideal sheaf over

\[
\mathcal{I}_0(X, d\beta_0) \times X \cong S^{[d]} \times X
\]

is \( \mathcal{I} = \psi^* \mathcal{J} \). The Zariski tangent bundle and obstruction bundle over the moduli space \( \mathcal{I}_0(X, d\beta_0) \cong S^{[d]} \) are given by the rank-2d bundles

\[
\mathcal{E}xt^1_\pi(\psi^* \mathcal{J}, \psi^* \mathcal{J})_0, \mathcal{E}xt^2_\pi(\psi^* \mathcal{J}, \psi^* \mathcal{J})_0
\]

respectively (see, for instance, the Theorem 3.28 in [T]).

In the following, we will prove the local version of (3.17), i.e., for every point \( I_{f^*} \in \mathcal{I}_0(X, d\beta_0) \) with \( \xi \in S^{[d]} \), we show that there exists a canonical isomorphism:

\[
\mathcal{E}xt_\pi^1(I_{f^*}, I_{f^*})_0 \cong \mathcal{E}xt_\pi^2(I_{f^*}, I_{f^*})_0.
\]

The argument for the global version (3.17) follows from that for the local version (3.18) and the isomorphisms via relative duality (see the Proposition 8.14 in [LeP]):

\[
\mathcal{E}xt_\pi^2(\psi^* \mathcal{J}, \psi^* \mathcal{J})_0 \cong \mathcal{E}xt_\pi^1(\psi^* \mathcal{J}, \psi^* \mathcal{J} \otimes \tilde{\rho}^* K_S)_0
\]

\[
\mathcal{E}xt_\phi^1(\mathcal{J}, \mathcal{J})_0 \cong \mathcal{E}xt_\phi^1(\mathcal{J}, \mathcal{J} \otimes \rho^* K_S)_0
\]

where \( \tilde{\rho} : S^{[d]} \times X = S^{[d]} \times S \times E \to S \) and \( \rho : S^{[d]} \times S \to S \) are the projections.

Here is an outline for (3.18). We apply the Serre duality twice: once on \( X \) with

\[
\mathcal{E}xt^2(I_{f^*}, I_{f^*})_0 \cong \mathcal{E}xt^1(I_{f^*}, I_{f^*} \otimes K_X)_0
\]

\[
\cong \mathcal{E}xt^1(I_{f^*}, I_{f^*} \otimes f^* K_S)_0.
\]
and the other on $S$ with $\text{Ext}^1(I_\xi, I_\xi)_0 \cong \text{Ext}^1(I_\xi, I_\xi \otimes K_S)_0$. Note from (3.16) that
\[ \text{Ext}^1(I_{f*\xi}, I_{f*\xi})_0 \cong \text{Ext}^1(I_\xi, I_\xi)_0. \] (3.19)
The main part of our argument is to prove that there is a natural isomorphism:
\[ \text{Ext}^1(I_{f*\xi}, I_{f*\xi} \otimes f^*K_S)_0 \cong \text{Ext}^1(I_\xi, I_\xi \otimes K_S)_0. \]

For simplicity, we assume that $\text{Supp}(\xi) = \{s\} \subset S$. Note that the vector spaces $\text{Ext}^1(I_\xi, I_\xi)_0, \text{Ext}^1(I_{f*\xi}, I_{f*\xi})_0, \text{Ext}^2(I_{f*\xi}, I_{f*\xi})_0$ all have dimension $2d$.

Applying the local-to-global spectral sequence to $\text{Ext}^1(I_\xi, I_\xi)$, we obtain
\[ 0 \to H^1(S, \mathcal{O}_S) \to \text{Ext}^1(I_\xi, I_\xi) \to H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi)) \to H^2(S, \mathcal{O}_S). \]
It follows that we have an exact sequence
\[ 0 \to \text{Ext}^1(I_\xi, I_\xi)_0 \to H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi)) \to H^2(S, \mathcal{O}_S). \] (3.20)
Since the second term can be computed locally, by taking $S = \mathbb{P}^2$, we see that
\[ h^0(S, \mathcal{E}xt^1(I_\xi, I_\xi)) = 2d \]
for an arbitrary surface $S$. So we conclude from (3.20) that
\[ \text{Ext}^1(I_\xi, I_\xi)_0 \cong H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi)) \] (3.21)
since $\dim \text{Ext}^1(I_\xi, I_\xi)_0 = 2d$. Similarly, we have canonical isomorphisms:
\[ \text{Ext}^1(I_{f*\xi}, I_{f*\xi})_0 \cong H^0(X, \mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi})) \cong H^0(S, f_*\mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi})). \] (3.22)

As in (3.20), we have an injection
\[ 0 \to \text{Ext}^1(I_\xi, I_\xi \otimes K_S)_0 \to H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi \otimes K_S)). \]
Note that $H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi \otimes K_S)) \cong H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi)) \otimes \mathcal{O}_S$ since $\mathcal{E}xt^1(I_\xi, I_\xi)$ is supported at $\text{Supp}(\xi) = \{s\}$, where $K_S|_s$ is the fiber of $K_S$ at $s \in S$. So we get
\[ 0 \to \text{Ext}^1(I_\xi, I_\xi \otimes K_S)_0 \to H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi)) \otimes \mathcal{O}_S. \]
By (3.21) and the Serre duality, $\text{Ext}^1(I_\xi, I_\xi \otimes K_S)_0$ and $H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi))$ have the same dimension. Hence, we get an isomorphism
\[ \text{Ext}^1(I_\xi, I_\xi \otimes K_S)_0 \cong H^0(S, \mathcal{E}xt^1(I_\xi, I_\xi)) \otimes \mathcal{O}_S. \] (3.23)
Again as in (3.20), we have another injection:
\[ 0 \to \text{Ext}^1(I_{f*\xi}, I_{f*\xi} \otimes f^*K_S)_0 \to H^0(X, \mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi} \otimes f^*K_S)). \]
By the Serre duality, $\text{Ext}^1(I_{f*\xi}, I_{f*\xi} \otimes f^*K_S)_0 \cong \text{Ext}^2(I_{f*\xi}, I_{f*\xi})_0$. Also,
\[ H^0(X, \mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi} \otimes f^*K_S)) \cong H^0(S, f_*\mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi} \otimes f^*K_S)) \cong H^0(S, f_*\mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi})) \otimes \mathcal{O}_S \]
\[ \cong H^0(S, f_*\mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi})) \otimes \mathcal{O}_S \]
since $f_*\mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi})$ is supported on $\text{Supp}(\xi) = \{s\}$. Therefore, we obtain
\[ 0 \to \text{Ext}^2(I_{f*\xi}, I_{f*\xi})_0 \to H^0(S, f_*\mathcal{E}xt^1(I_{f*\xi}, I_{f*\xi})) \otimes \mathcal{O}_S. \] (3.24)
Since $\text{Ext}^2(I_{f^\ast \xi}, I_{f^\ast \xi})_0$ and $\text{Ext}^1(I_{f^\ast \xi}, I_{f^\ast \xi})_0$ have the same dimension, we obtain

$$\text{Ext}^2(I_{f^\ast \xi}, I_{f^\ast \xi})_0 \cong H^0(S, f_*\text{Ext}^1(I_{f^\ast \xi}, I_{f^\ast \xi})) \otimes_{\mathbb{C}} K_S|_s$$

$$\cong \text{Ext}^1(I_{f^\ast \xi}, I_{f^\ast \xi})_0 \otimes_{\mathbb{C}} K_S|_s$$

from (3.22) and (3.24). Combining this with (3.19) and (3.21), we get

$$\text{Ext}^2(I_{f^\ast \xi}, I_{f^\ast \xi})_0 \cong H^0(S, \text{Ext}^1(I_{f^\ast \xi}, I_{f^\ast \xi})) \otimes_{\mathbb{C}} K_S|_s.$$  \hfill (3.25)

In view of (3.23), the Serre duality and (3.19), we conclude that

$$\text{Ext}^2(I_{f^\ast \xi}, I_{f^\ast \xi})_0 \cong \text{Ext}^1(I_{f^\ast \xi}, I_{f^\ast \xi})_0 \cong \text{Ext}^1(f^\ast I_{\xi}, f^\ast I_{\xi})_0.$$ 

This completes the proof of the isomorphism (3.18). \hfill \Box

Next, we consider the case when either $n \neq 0$ or $\beta \neq d\beta_0$ with $d \geq 0$. We further assume that the moduli space $\mathcal{I}_n(X, \beta)$ is nonempty. For simplicity, put

$$\mathcal{J} = \mathcal{I}_n(X, \beta).$$

Let $\mathcal{I}$ be the universal ideal sheaf over $\mathcal{J} \times X$. Denote the trace-free part of the element $R\text{Hom}(\mathcal{I}, \mathcal{I})$ in the derived category $\mathcal{D}_{\text{coh}}(\mathcal{J} \times X)$ by

$$R\text{Hom}(\mathcal{I}, \mathcal{I})_0.$$ 

Let $\pi : \mathcal{J} \times X \to \mathcal{J}$ be the projection. By (1.9), the virtual fundamental class $[\mathcal{I}]^\text{vir}$ is defined via the following element in the derived category $\mathcal{D}_{\text{coh}}(\mathcal{J})$:

$$\mathcal{E} = R\pi_* (R\text{Hom}(\mathcal{I}, \mathcal{I})_0).$$ \hfill (3.26)

Let $p \in E$, and consider the sheaf $(\text{Id}_\mathcal{J} \times \phi_p \times \text{Id}_S)^* \mathcal{I}$ over

$$\mathcal{J} \times X = \mathcal{J} \times E \times S.$$ 

We see from (2.18) that $(\text{Id}_\mathcal{J} \times \phi_p \times \text{Id}_S)^* \mathcal{I}$ is a flat family of ideal sheaves whose corresponding 1-dimensional closed subschemes satisfy (3.1). By the universal property of the moduli space $\mathcal{I}$, there is an automorphism

$$\Phi_p : \mathcal{J} \to \mathcal{J}$$ \hfill (3.27)

such that $(\Phi_p \times \text{Id}_X)^* \mathcal{I} = (\text{Id}_\mathcal{J} \times \phi_p \times \text{Id}_S)^* \mathcal{I} \cong \mathcal{I}$. In particular, $E$ acts on $\mathcal{J}$.

**Lemma 3.5.** Let $n \neq 0$ or $\beta \neq d\beta_0$ with $d \geq 0$. Then,

$$\langle \bar{\tau}_{k_1}(\gamma_1) \cdots \bar{\tau}_{k_r}(\gamma_r) \rangle_{n, \beta} = 0$$ \hfill (3.28)

whenever $\gamma_1, \ldots, \gamma_r \in f^* H^*(S; \mathbb{Q}) \subseteq H^*(X; \mathbb{Q})$.

**Proof.** The proof is similar to that of Lemma 2.5. Assume that the moduli space

$$\mathcal{J} = \mathcal{I}_n(X, \beta)$$

is nonempty. If $\beta \neq d\beta_0$ with $d \geq 0$, then the algebraic group $E$ acts on $\mathcal{J}$ with finite stabilizers. If $\beta = d\beta_0$ with $d \geq 0$ and if $I_{Z} \in \mathcal{J}$, then $Z$ consists of a curve $f^*(\xi)$ for some $\xi \in S^{\text{vir}}$ and of some (possibly embedded) points of length $n \neq 0$. So again $E$ acts on the moduli space $\mathcal{J}$ with finite stabilizers.

As in the proof of Lemma 2.5, there exists some number $N$ such that if $G$ is a cyclic subgroup of $E$ of prime order $p > N$, then $G$ acts freely on $\mathcal{J}$. Fix such
cyclic subgroups $G$ of $E$. Since the complex $R\pi_*(R\text{Hom}(\mathcal{I}, \mathcal{I}))$ from (3.20) is equivariant for the action of any algebraic automorphism group of $X$, the cycle $[\mathcal{I}]^\text{vir}$ defines an element of the equivariant Borel-Moore homology group $H^*_G(\mathcal{I})$. For $1 \leq i \leq r$, choose a positive integer $m_i$ such that the multiple $m_i \gamma_i$ defines an element of $H^*_G(X)$. It follows from (3.4) and (3.5) that the cycle

$$m_1 \cdots m_r \langle \tilde{\tau}_{k_1} (\gamma_1) \cdots \tilde{\tau}_{k_r} (\gamma_r) \rangle_{n, \beta}$$

defines an element in the degree-0 Borel-Moore homology $H^*_0(\mathcal{I})$. Again as in the proof of Lemma 2.5, we conclude that $\langle \tilde{\tau}_{k_1} (\gamma_1) \cdots \tilde{\tau}_{k_r} (\gamma_r) \rangle_{n, \beta} = 0$. \qed

**Proposition 3.6.** Let $\beta \in H_2(X; \mathbb{Z}) \setminus \{0\}$. Assume $\int_\beta K_X = \int_\beta f^* K_S = 0$. Then,

$$Z'_{DT} \left( X; q \prod_{i=1}^{r} \tilde{\tau}_0 (\gamma_i) \right)_\beta = \left\{ \begin{array}{ll} \prod_{i=1}^{r} \int_\beta \gamma_i \cdot \chi (S^{[d]}) & \text{if } |\gamma_i| = 2 \text{ for every } i \text{ and } \beta = d\beta_0 \text{ for some } d \geq 1; \\ 0 & \text{otherwise}. \end{array} \right.$$ \hspace{1cm} (3.29)

Proof. First of all, since $\chi (X) = 0$, we see from (3.8) and (3.6) that

$$Z'_{DT} \left( X; q \prod_{i=1}^{r} \tilde{\tau}_{k_i} (\gamma_i) \right)_\beta = \sum_{n \in \mathbb{Z}} \langle \tilde{\tau}_{k_1} (\gamma_1) \cdots \tilde{\tau}_{k_r} (\gamma_r) \rangle_{n, \beta} q^n. \hspace{1cm} (3.29)$$

Next, in view of (3.2) and the condition on degrees, we have

$$\sum_{i=1}^{r} |\gamma_i| = 2r, \quad |\gamma_r| \leq 2.$$ 

Therefore, we conclude from (3.2) and Lemma 3.4 (i) that

$$\langle \tilde{\tau}_0 (\gamma_1) \cdots \tilde{\tau}_0 (\gamma_r) \rangle_{n, \beta} = \left\{ \begin{array}{ll} \prod_{i=1}^{r} \int_\beta \gamma_i \cdot \langle \rangle_{n, \beta} & \text{if } |\gamma_i| = 2 \text{ for every } i; \\ 0 & \text{otherwise}. \end{array} \right.$$ 

By Lemma 3.4 (ii) and Lemma 3.5 we obtain

$$\langle \tilde{\tau}_0 (\gamma_1) \cdots \tilde{\tau}_0 (\gamma_r) \rangle_{n, \beta} = \left\{ \begin{array}{ll} \prod_{i=1}^{r} \int_\beta \gamma_i \cdot \chi (S^{[d]}) & \text{if } |\gamma_i| = 2 \text{ for every } i, n = 0, \beta = d\beta_0; \\ 0 & \text{otherwise}. \end{array} \right.$$ 

Now the proposition follows immediately from (3.29). \qed

**Proposition 3.7.** Let $X$ be from Notation 2.2 and $\beta \in H_2(X; \mathbb{Z}) \setminus \{0\}$. Assume that $\gamma_1, \ldots, \gamma_r \in f^* H^* (S; \mathbb{Q}) \subset H^* (X; \mathbb{Q})$. Then,

$$Z'_{DT} \left( X; q \prod_{i=1}^{r} \tilde{\tau}_{k_i} (\gamma_i) \right)_\beta = \left\{ \begin{array}{ll} \chi (S^{[d]}) & \text{if } r = 0 \text{ and } \beta = d\beta_0 \text{ with } d \geq 1, \\ 0 & \text{otherwise}. \end{array} \right.$$


Proof. If $\beta \neq d\beta_0$, then the proposition follows from (3.29) and Lemma 3.3. In the rest of the proof, we let $\beta = d\beta_0$ with $d \geq 1$. By (3.29) and Lemma 3.5 again,

$$Z_{DT}'\left( X; q \prod_{i=1}^r \tilde{\tau}_{k_i}(\gamma_i) \right)_{d\beta_0} = \langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) \rangle_{0,d\beta_0}.$$ 

Thus we see from Lemma 3.4 (ii) that the proposition holds if $r = 0$.

To prove our proposition, it remains to verify that if $r \geq 1$, then

$$\langle \tilde{\tau}_{k_1}(\gamma_1) \cdots \tilde{\tau}_{k_r}(\gamma_r) \rangle_{0,d\beta_0} = 0.$$  (3.30)

Since the expected dimension of $I_0(X, d\beta_0)$ is zero, (3.30) holds unless

$$\sum_{i=1}^r (2k_i - 2 + |\gamma_i|) = 0, \quad (2k_r - 2 + |\gamma_r|) \leq 0.$$  (3.31)

W.l.o.g., we may assume that $k_{\tilde{r}+1} = k_{\tilde{r}+2} = \ldots = k_r = 0$ and

$$k_1, \ldots, k_{\tilde{r}-1}, k_{\tilde{r}} \geq 1$$  (3.32)

for some $\tilde{r}$ with $0 \leq \tilde{r} \leq r$. Then we see from (3.30), Lemma 3.4 (i) and (3.31) that (3.30) holds unless $\tilde{r} = r$, $k_1 = \ldots = k_r = 1$, and $|\gamma_1| = \ldots = |\gamma_r| = 0$. When

$$k_1 = \ldots = k_r = 1, \quad |\gamma_1| = \ldots = |\gamma_r| = 0,$$

(3.30) follows from Lemma 3.2 (ii) since the moduli space $I_0(X, d\beta_0)$ is smooth. □

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