LARGE DEVIATION FOR TWO-TIME-SCALE STOCHASTIC BURGERS EQUATION

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ABSTRACT. A Freidlin-Wentzell type large deviation principle is established for stochastic partial differential equations with slow and fast time-scales, where the slow component is a one-dimensional stochastic Burgers equation with small noise and the fast component is a stochastic reaction-diffusion equation. Our approach is via the weak convergence criterion developed in [3].

1. Introduction

In this paper, we study the large deviation principle (LDP) for the following stochastic slow-fast system on the interval [0, 1]:

\[
\begin{aligned}
\frac{\partial}{\partial t}X_t^{\varepsilon, \delta} (\xi) &= \frac{\partial^2}{\partial \xi^2} X_t^{\varepsilon, \delta} (\xi) + \frac{1}{2} \frac{\partial}{\partial \xi} \left( X_t^{\varepsilon, \delta} (\xi) \right)^2 + f \left( X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta} \right) (\xi), \\
+ \sqrt{\varepsilon} \sigma_1 \left( X_t^{\varepsilon, \delta} \right) (\xi) Q_1^{1/2} \frac{\partial W}{\partial t} (t, \xi), \\
\frac{\partial}{\partial t} Y_t^{\varepsilon, \delta} (\xi) &= \frac{1}{\delta} \frac{\partial^2}{\partial \xi^2} Y_t^{\varepsilon, \delta} (\xi) + g \left( X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta} \right) (\xi) + \frac{1}{\sqrt{\delta}} \sigma_2 \left( X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta} \right) (\xi) Q_2^{1/2} \frac{\partial W}{\partial t} (t, \xi), \\
X_t^{\varepsilon, \delta} (0) &= X_t^{\varepsilon, \delta} (1) = Y_t^{\varepsilon, \delta} (0) = Y_t^{\varepsilon, \delta} (1) = 0, \quad t > 0, \\
X_0^{\varepsilon, \delta} &= x, \quad Y_0^{\varepsilon, \delta} = y,
\end{aligned}
\]

where \( \varepsilon > 0, \delta = \delta(\varepsilon) > 0 \) are small parameters describing the ratio of time scales between the slow component \( X_t^{\varepsilon, \delta} \) and fast component \( Y_t^{\varepsilon, \delta} \). The functions \( f, g, \sigma_1 \) and \( \sigma_2 \) satisfy some appropriate conditions, \( \{W_t\}_{t \geq 0} \) is a standard cylindrical Wiener process on the Hilbert space \( L^2(0, 1) \), and \( Q_1, Q_2 \) are both trace class operators.

The motivation for the study of multi-scale processes can be founded, for example, in stochastic mechanics (see Freidlin and Wentzell [16, 17]), where a polar change (or an appropriate change linked to the considered Hamiltonian) may give an amplitude evolving slowly whereas the phase is on an accelerated time scale; or in climate models (see Kiefer [27]), where climate-weather interactions may be studied within an averaging framework, climate being the slow motion and weather the fast one; or in genetic switching models (see Ge et al. [19]), which involves fast switching of DNA states between active and inactive states and the transcriptional and translational processes with different rates depending on the DNA states.

The study of the averaging principle has been extensively developed in both the deterministic (i.e., \( \sigma_1 = 0 \)) and the stochastic context: see, for example, Bogoliubov and
Mitropolsky [1] for the deterministic case; Khasminskii [28] for a finite dimensional stochastic system; Cerrai and Freidlin [5] for an infinite dimensional stochastic reaction-diffusion systems. For more interesting results on this topic, we refer the reader to the recent works [5, 4, 23, 2, 37, 11, 36, 38, 18, 6, 9, 24].

There have been a large body of researchers working on large deviation problems of multi-scale diffusions, just to list a few but far from being complete: Freidlin and Wentzell [16, Chapter 7], Liptser [26], Veretennikov [35] and Puhalskii [31]. Several well known large deviation approaches have been applied in this direction, see [30, 13, 20] for Dupuis-Ellis’ weak convergence method [12], and [29] for the viscosity method developed by Feng et al. [14]. As a special multi-scale stochastic system, slow-fast dynamics and its large deviation have also been studied in [39, 32, 22, 20] and the references therein.

The novelty of our research is to apply a well known weak convergence criterion developed in [3] to prove the LDP of the slow-fast system (1.1). Thanks to this criterion, under some appropriate condition, we can simplify the delicate analysis in [39, 32, 22, 20] to be an asymptotic behaviour study of the following controlled process \((X^{\varepsilon, \delta, u^\varepsilon}, Y^{\varepsilon, \delta, u^\varepsilon}):\)

\[
\begin{align*}
\frac{\partial}{\partial t} X_t^{\varepsilon, \delta, u^\varepsilon}(\xi) &= \frac{\partial^2}{\partial \xi^2} X_t^{\varepsilon, \delta, u^\varepsilon}(\xi) + \frac{\partial}{\partial \xi} \left( X_t^{\varepsilon, \delta, u^\varepsilon}(\xi) \right)^2(\xi) + f \left( X_t^{\varepsilon, \delta, u^\varepsilon}, Y_t^{\varepsilon, \delta, u^\varepsilon} \right)(\xi), \\
&+ \sigma_1 \left( X_t^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} u_t^\varepsilon(\xi) + \sqrt{\varepsilon} \sigma_1 \left( X_t^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} \frac{\partial W}{\partial t}(t, \xi), \\
\frac{\partial}{\partial t} Y_t^{\varepsilon, \delta, u^\varepsilon}(\xi) &= \frac{1}{\delta} \left[ \frac{\partial^2}{\partial \xi^2} Y_t^{\varepsilon, \delta, u^\varepsilon}(\xi) + g \left( X_t^{\varepsilon, \delta, u^\varepsilon}, Y_t^{\varepsilon, \delta, u^\varepsilon} \right)(\xi) \right] + \frac{1}{\sqrt{\varepsilon}} \sigma_2 \left( X_t^{\varepsilon, \delta, u^\varepsilon}, Y_t^{\varepsilon, \delta, u^\varepsilon} \right) Q_2^{1/2} u_t^\varepsilon(\xi) \\
&+ \frac{1}{\sqrt{\delta}} \sigma_2 \left( X_t^{\varepsilon, \delta, u^\varepsilon}, Y_t^{\varepsilon, \delta, u^\varepsilon} \right)(\xi) Q_2^{1/2} \frac{\partial W}{\partial t}(t, \xi), \\
X_t^{\varepsilon, \delta, u^\varepsilon}(0) &= X_t^{\varepsilon, \delta, u^\varepsilon}(1) = Y_t^{\varepsilon, \delta, u^\varepsilon}(0) = Y_t^{\varepsilon, \delta, u^\varepsilon}(1) = 0, \
X_0^{\varepsilon, \delta, u^\varepsilon} &= x, \quad Y_0^{\varepsilon, \delta, u^\varepsilon} = y, \
x \in [0, T], \
y \in [0, T],
\end{align*}
\]

where \(u^\varepsilon\) is a square integrable process often called control in sequel.

Another finding of this paper is that, as \(\delta/\varepsilon \rightarrow 0\) and \(\varepsilon \rightarrow 0\), it is surprising to see that the controlled slow processes \(X_t^{\varepsilon, \delta, u^\varepsilon}\) would converge to a specific process \(\bar{X}^u\) (see skeleton equation (2.4) below). This leads us to use the criterion in [3, Theorem 4.4] directly to get our large deviation result. We also need to stress that when \(\delta/\varepsilon \rightarrow 0\) as \(\varepsilon \rightarrow 0\), the problem turns to be much more complicated and this straightforward method does not work anymore. For more applications of weak convergence criterion, we refer the reader to [42, 40, 41, 43].

In order to study the convergence of the controlled slow processes \(X_t^{\varepsilon, \delta, u^\varepsilon}\), we will use the classical Khasminskii’s time discretization approach, which is widely used in the proof of averaging principle. One difficulty here is from the non-linear term \(\frac{1}{2} \frac{\partial}{\partial \xi} \left( X_t^{\varepsilon, \delta} \right)^2(\xi)\) in the Burgers equation, when proving the convergence of \(X_t^{\varepsilon, \delta, u^\varepsilon}\), we have to use a stopping time technique developed in [9, 7] to handle this non-linearity. Moreover, to estimate the controlled system, we need to introduce an auxiliary process \((\hat{X}^{\varepsilon, \delta}, \hat{Y}^{\varepsilon, \delta})\) (see Eq. (4.17) and Eq. (4.18) below), which plays a crucial role for obtaining the weak convergence of controlled slow process. We believe that the method presented in this paper will be useful for studying the LDP for other types of slow-fast stochastic partial differential equations.
The paper is organized as follows. In the next section, we introduce some notations and assumptions used throughout the paper, then give the main result and the outline of the proof. Section 3 is devoted to the study of averaged equation and skeleton equation. In section 4, we finish the proof of LDP by the weak convergence approach. In Appendix, we recall some well-known results about the LDP and the Burgers equation.

In this paper, $C$ and $C_{p_1, \ldots, p_k}$ denote positive constants which may change from line to line along this paper, where $C_{p_1, \ldots, p_k}$ is used to emphasize that constant depends on $p_1, \ldots, p_k$, $k \geq 1$.

2. Notations and main results

Let $\mathbb{H} := L^2(0, 1)$ be the space of square integrable real-valued functions on $[0, 1]$. The norm and the inner product on $\mathbb{H}$ are denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. For positive integer $k$, let $\mathbb{H}_k(0, 1)$ be the Sobolev space of all functions in $\mathbb{H}$ whose all derivatives up to the order $k$ also belong to $\mathbb{H}$. $\mathbb{H}_0^1(0, 1)$ is the subspace of $\mathbb{H}^1(0, 1)$ of all functions whose values at 0 and 1 vanish.

Let $A$ be the Laplace operator on $\mathbb{H}$:

$$Ax := \frac{\partial^2}{\partial \xi^2} x(\xi), \quad x \in D(A) = \mathbb{H}^2(0, 1) \cap \mathbb{H}_0^1(0, 1).$$

It is well known that $A$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}\}_{t \geq 0}$. Let $\{e_k(\xi) := \sqrt{2} \sin(k\pi \xi)\}_{k \geq 1}$ be an orthonormal basis of $\mathbb{H}$ consisting of the eigenvectors of $A$, i.e.,

$$Ae_k = -\lambda_k e_k \quad \text{with} \quad \lambda_k = k^2\pi^2.$$

For any $\sigma \in \mathbb{R}$, let $\mathbb{H}_\sigma$ be the domain of the fractional operator $(-A)^{\sigma/2}$, i.e.,

$$\mathbb{H}_\sigma := D\left((−A)^{\sigma/2}\right) = \left\{ x = \sum_{k \geq 1} x_k e_k : (x_k)_{k \geq 1} \in \mathbb{R}, \sum_{k \geq 1} \lambda_k^\sigma |x_k|^2 < \infty \right\},$$

with norm

$$\|x\|_\sigma := \left(\sum_{k \geq 1} \lambda_k^\sigma |x_k|^2\right)^{1/2}.$$

Then, for any $\sigma > 0$, $\mathbb{H}_\sigma$ is densely and compactly embedded in $\mathbb{H}$. Particularly, $\mathbb{V} := \mathbb{H}_1 = \mathbb{H}_0^1(0, 1)$, whose dual space is $\mathbb{V}^{-1}$. The norm and the inner product on $\mathbb{V}$ are denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle_{\mathbb{V}}$, respectively.

Define the bilinear operator $B(x, y) : \mathbb{H} \times \mathbb{V} \rightarrow \mathbb{V}^{-1}$ by

$$B(x, y) := x \cdot \partial_\xi y,$$

and the trilinear operator $b(x, y, z) : \mathbb{H} \times \mathbb{V} \times \mathbb{H} \rightarrow \mathbb{R}$ by

$$b(x, y, z) := \int_0^1 x(\xi) \partial_\xi y(\xi) z(\xi) d\xi.$$

For convenience, set $B(x) := B(x, x)$, for $x \in \mathbb{V}$. The related properties about operators $e^{tA}$, $b$ and $B$ are listed in Section 5.
With the above notations, the system (1.1) can be rewritten as:

\[
\begin{align*}
\frac{dX^\varepsilon,\delta_t}{dt} &= AX^\varepsilon,\delta_t + B\left(X^\varepsilon,\delta_t\right) + f\left(X^\varepsilon,\delta_t, Y^\varepsilon,\delta_t\right) + \sqrt{\varepsilon}\sigma_1\left(X^\varepsilon,\delta_t\right)Q_1^{1/2}\,dW_t, \\
\frac{dY^\varepsilon,\delta_t}{dt} &= \frac{1}{\delta}\left[A Y^\varepsilon,\delta_t + g\left(X^\varepsilon,\delta_t, Y^\varepsilon,\delta_t\right)\right] + \frac{1}{\sqrt{\delta}}\sigma_2\left(X^\varepsilon,\delta_t, Y^\varepsilon,\delta_t\right)Q_2^{1/2}\,dW_t, \\
X^\varepsilon,\delta_0 &= x, \quad Y^\varepsilon,\delta_0 = y.
\end{align*}
\]

(2.1)

Here, \( W \) denotes a standard cylindrical Wiener process on \( \mathbb{H} \). Since \( Q_1 \) and \( Q_2 \) are trace class operators, the embedding of \( Q_i^{1/2} \mathbb{H} \) in \( \mathbb{H} \) is Hilbert-Schmidt for \( i = 1, 2 \). Let \( \mathcal{L}_2(\mathbb{H}; \mathbb{H}) \) denote the space of Hilbert-Schmidt operators from \( \mathbb{H} \) to \( \mathbb{H} \), endowed with the Hilbert-Schmidt norm \( \|G\|_{HS} = \sqrt{\text{Tr}(GG^*)} = \sqrt{\sum_k |G_{kk}|^2} \).

Suppose that \( f, g : \mathbb{H} \times \mathbb{H} \to \mathbb{H}, \sigma_1 Q_1^{1/2} : \mathbb{H} \to \mathcal{L}_2(\mathbb{H}; \mathbb{H}), \sigma_2 Q_2^{1/2} : \mathbb{H} \times \mathbb{H} \to \mathcal{L}_2(\mathbb{H}; \mathbb{H}) \) satisfy the following conditions:

\textbf{A1.} \( f, g, \sigma_1 \) and \( \sigma_2 \) are Lipschitz continuous, i.e., there exist some positive constants \( L_f, L_g, \sigma_1, \sigma_2 \) and \( C > 0 \) such that for any \( x_1, x_2, y_1, y_2 \in \mathbb{H}, \)

\[
|f(x_1, y_1) - f(x_2, y_2)| \leq L_f (|x_1 - x_2| + |y_1 - y_2|); \quad \|\sigma_1(x_1) - \sigma_1(x_2)\|_{HS} \leq L_{\sigma_1} |x_1 - x_2|;
\]

\[
\|\sigma_2(x_1, y_1) - \sigma_2(x_2, y_2)\|_{HS} \leq C |x_1 - x_2| + L_{\sigma_2} |y_1 - y_2|.
\]

\textbf{A2.} There exists \( C > 0 \) such that for any \( x \in \mathbb{H}, \)

\[
\sup_{y \in \mathbb{H}} \|\sigma_2(x, y) Q_2^{1/2}\|_{HS} \leq C(|x| + 1).
\]

\textbf{A3.} The smallest eigenvalue \( \lambda_1 \) of \( -\Delta \) and the Lipschitz constants \( L_g, L_{\sigma_2} \) satisfy

\[
\lambda_1 - L_g > 0 \quad \text{and} \quad \frac{L_{\sigma_2}^2}{\lambda_1} + \frac{L_g}{\lambda_1 - L_g} < 1.
\]

\textbf{A4.} \( \lim_{\varepsilon \to 0} \delta(\varepsilon) = 0 \) and \( \lim_{\varepsilon \to 0} \frac{\delta^2}{\varepsilon} = 0. \)

\textbf{Remark 2.1.} Condition (\textbf{A3}) is not a sharp condition and can be weakened by more accurate calculus. Condition (\textbf{A4}) is a very important condition to make sure that the additional controlled term in \( Y^{\varepsilon,\delta, w} \) converges to 0 (see Remark 4.5 for details).

Following the standard approach developed in [8], one can prove that under Condition (\textbf{A1}), there exists a unique mild solution to the system (2.1). More specifically, for any given initial value \( x, y \in \mathbb{H} \) and \( T > 0 \), there exists a unique solution \( (X^{\varepsilon,\delta}, Y^{\varepsilon,\delta}) \) such that \( X^{\varepsilon,\delta}, Y^{\varepsilon,\delta} \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \) satisfying that

\[
\begin{align*}
X_t^{\varepsilon,\delta} &= e^{tA}x + \int_0^t e^{(t-s)A}B\left(X_s^{\varepsilon,\delta}\right)ds + \int_0^t e^{(t-s)A}f\left(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}\right)ds \\
&\quad + \sqrt{\varepsilon} \int_0^t e^{(t-s)A}\sigma_1\left(X_s^{\varepsilon,\delta}\right)Q_1^{1/2}dW_s, \\
Y_t^{\varepsilon,\delta} &= e^{tA/\delta}y + \frac{1}{\delta} \int_0^t e^{(t-s)A/\delta}g\left(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}\right)ds + \frac{1}{\sqrt{\delta}} \int_0^t e^{(t-s)A/\delta}\sigma_2\left(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}\right)Q_2^{1/2}dW_s, \\
X_0^{\varepsilon,\delta} &= x, \quad Y_0^{\varepsilon,\delta} = y.
\end{align*}
\]

(2.2)
Let \( \Gamma^\varepsilon \) be the functional from \( C([0, T]; \mathbb{H}) \) into \( C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \) satisfying that
\[
\Gamma^\varepsilon(W) := X^{\varepsilon, \delta}.
\] (2.3)

Consider the following skeleton equation:
\[
\begin{aligned}
\{ \ d\bar{X}^u_t &= \left[ AX^u_t + B \left( \bar{X}^u_t \right) + \bar{f} \left( \bar{X}^u_t \right) \right] dt + \sigma_1 \left( \bar{X}^u_t \right) Q_1^{1/2} u(t) dt, \\
\bar{X}^u_0 &= x,
\end{aligned}
\] (2.4)

where \( u \in L^2([0, T]; \mathbb{H}) \) and
\[
\bar{f}(x) = \int_{\mathbb{H}} f(x, y) \mu^x(dy), \quad x \in \mathbb{H},
\]
with \( \mu^x(\cdot) \) being the unique invariant measure of the transition semigroup for the corresponding frozen equation (see Eq. (3.1) below). By Lemma 3.4 below, Eq. (2.4) admits a unique solution, and we denote the solution as follows
\[
\Gamma^0 \left( \int_0^T u(s) ds \right) := \bar{X}^u.
\] (2.5)

The main result of this paper is the following theorem.

**Theorem 2.2.** Under \((A1)-(A4)\), \( \{X^{\varepsilon, \delta}\}_{\varepsilon > 0} \) satisfies the LDP in \( C([0, T]; \mathbb{H}) \) with the rate function \( I \) given by
\[
I(g) := \inf_{\{u \in L^2([0, T]; \mathbb{H}) : g = \Gamma^0(\int_0^T u(s) ds)\}} \left\{ \frac{1}{2} \int_0^T |u(s)|^2 ds \right\}, \quad g \in C([0, T]; \mathbb{H}).
\]

**Proof of Theorem 2.2:** According to the weak convergence criteria in Theorem 5.3, we just need to prove that two conditions (a) and (b) in Theorem 5.3 are fulfilled. Condition (b) will be established in Proposition 3.5 in the following section, and the verification of Condition (a) will be given by Propositions 4.6 and 4.7 in Section 4.

### 3. The frozen equation and skeleton equation

In this section, we will prove Condition (b) in Theorem 5.3 to prove the LDP. Before proving the compactness of solutions \( \{\bar{X}^u\} \) to the skeleton equation (2.4), a frozen equation is also introduced. The unique invariant measure of the frozen equation is applied to define the coefficient \( \bar{f} \) in the skeleton equation, and the Lipschitz continuity of \( \bar{f} \) is used a lot in the following discussion. Note that we assume conditions \((A1)-(A3)\) hold in this section.

#### 3.1. The frozen and skeleton equations

For any fixed \( x \in \mathbb{H} \), we first consider the following frozen equation associated with the fast component:
\[
\begin{aligned}
dY_t &= AY_t dt + g(x, Y_t) dt + \sigma_2(x, Y_t)Q_2^{1/2} d\bar{W}_t, \\
Y_0 &= y,
\end{aligned}
\] (3.1)

where \( \bar{W}_t \) is a standard cylindrical Wiener process independent of \( W_t \). Since \( g(x, \cdot) \) and \( \sigma_2(x, \cdot)Q_2^{1/2} \) are Lipschitz continuous, it is easy to prove that for any fixed \( y \in \mathbb{H} \), Eq. (3.1) has a unique mild solution denoted by \( Y_t^{x,y} \). Moreover, \( Y_t^{x,y} \) is a homogeneous Markov process, and let \( P_t^x \) be the transition semigroup of \( Y_t^{x,y} \), that is, for any bounded measurable function \( \varphi \) on \( \mathbb{H} \),
\[
P_t^x \varphi(y) := \mathbb{E} \left[ \varphi \left( Y_t^{x,y} \right) \right], \quad y \in \mathbb{H}, \quad t > 0.
\]
Under Condition (A3), it is easy to prove that $\sup_{t \geq 0} E \left[ |Y_t^{x,y}|^2 \right] \leq C(1 + |x|^2 + |y|^2)$ and $P_t^x$ has unique invariant measure $\mu^x$. We here give the following asymptotic behavior of $P_t^x$ proved in [4].

**Proposition 3.1.** [4, (2.13)] There exist $C, \eta > 0$ satisfying that for any Lipschitz continuous function $\varphi : \mathbb{H} \to \mathbb{R}$, 

$$
\left| P_t^x \varphi(y) - \int_{\mathbb{H}} \varphi(z) \mu^x(dz) \right| \leq C(1 + |x| + |y|) e^{-\eta t} \|\varphi\|_{\text{Lip}}, \quad \forall x, y \in \mathbb{H}, t > 0,
$$

where $\|\varphi\|_{\text{Lip}} := \sup_{x,y \in \mathbb{H}, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|}$.

**Lemma 3.2.** There exists a constant $C > 0$ satisfying that for any $x_1, x_2, y \in \mathbb{H},$

$$
\sup_{t \geq 0} E \left[ |Y_t^{x_1,y} - Y_t^{x_2,y}|^2 \right] \leq C|x_1 - x_2|^2.
$$

**Proof.** Note that $Y_0^{x_1,y} - Y_0^{x_2,y} = 0$ and

$$
d(Y_t^{x_1,y} - Y_t^{x_2,y}) = g\left(\int_{0}^{t} a(s, Y_s^{x_1,y} - Y_s^{x_2,y}) ds + \int_{0}^{t} \eta(s, Y_s^{x_1,y} - Y_s^{x_2,y}) dW_s\right),
$$

By Itô’s formula and Condition (A1), we get

$$
\frac{d}{dt} E\left[ |Y_t^{x_1,y} - Y_t^{x_2,y}|^2 \right] = -2E\left[ |Y_t^{x_1,y} - Y_t^{x_2,y}|^2 \right] + 2E\left[ \int_{0}^{t} a(s, Y_s^{x_1,y} - Y_s^{x_2,y}) ds + \int_{0}^{t} \eta(s, Y_s^{x_1,y} - Y_s^{x_2,y}) dW_s \right],
$$

where $a = \frac{1}{2} \frac{\partial g}{\partial y}, \quad \eta = \frac{1}{2} \frac{\partial g}{\partial y} - \frac{1}{2} \frac{\partial^2 g}{\partial y^2} \frac{\partial y}{\partial y}.$

By Condition (A3), it is easy to see that $2 \lambda_1 - 2 L \sigma_\mu > 0$. Then by Young’s inequality, there exists a constant $\gamma > 0$ such that

$$
\frac{d}{dt} E\left[ |Y_t^{x_1,y} - Y_t^{x_2,y}|^2 \right] \leq -\gamma E\left[ |Y_t^{x_1,y} - Y_t^{x_2,y}|^2 \right] + C|x_1 - x_2|^2.
$$

Hence, the comparison theorem implies for any $t > 0$,

$$
E\left[ |Y_t^{x_1,y} - Y_t^{x_2,y}|^2 \right] \leq C|x_1 - x_2|^2 \int_{0}^{t} e^{-\gamma(t-s)} ds \leq C|x_1 - x_2|^2 / \gamma.
$$

The proof is complete. \qed

Let $\mathcal{K}$ be a Hilbert space endowed with norm $\|\cdot\|_{\mathcal{K}}$. For $p > 1$, $\alpha \in (0, 1)$, let $W^{\alpha,p}([0, T]; \mathcal{K})$ be the Sobolev space of all $u \in L^p([0, T]; \mathcal{K})$ such that

$$
\int_{0}^{T} \int_{0}^{T} \frac{\|u(t) - u(s)\|^p_{\mathcal{K}}}{|t-s|^{1+\alpha p}} dt ds < \infty,
$$

endowed with the norm

$$
\|u\|^p_{W^{\alpha,p}([0,T];\mathcal{K})} := \int_{0}^{T} \|u(t)\|^p_{\mathcal{K}} dt + \int_{0}^{T} \int_{0}^{T} \frac{\|u(t) - u(s)\|^p_{\mathcal{K}}}{|t-s|^{1+\alpha p}} dt ds.
$$

The following result represents a variant of the criteria for compactness proved in [25, Sect. 5, Ch. I], and [34, Sect. 13.3].
Lemma 3.3. Let $S_0 \subset S \subset S_1$ be Banach spaces, $S_0$ and $S_1$ reflexive, with compact embedding of $S_0$ in $S$. For $p \in (1, \infty)$ and $\alpha \in (0, 1)$, let $\Lambda$ be the space

$$\Lambda = L^p([0, T]; S_0) \cap W^{\alpha, p}([0, T]; S_1)$$

endowed with the natural norm. Then the embedding of $\Lambda$ in $L^p([0, T]; S)$ is compact.

Let $S = L^2([0, T]; H)$, and $A$ denotes the class of $\{F_t\}$-predictable processes taking values in $H$ a.s.. Let $S_N = \{u \in S; \int_0^T |u(s)|^2 ds \leq N\}$. The set $S_N$ endowed with the weak topology is a Polish space. Define $A_N = \{u \in A; u(\omega) \in S_N, \mathbb{P}\text{-a.s.}\}$.

Recall $\bar{X}^u$ given in the skeleton equation (2.4). The existence and uniqueness of the solution of Eq. (2.4) is given in the following lemma.

Lemma 3.4. For any $x \in H$, $u \in S$, Eq. (2.4) admits a unique mild solution $\bar{X}^u \in C([0, T]; H) \cap L^2([0, T]; V)$. Moreover, for any $N > 0$ and $\alpha \in (0, 1/2)$, there exist constants $C_{N,T}$ and $C_{\alpha,N,T}$ such that

$$\sup_{u \in S_N} \left\{ \sup_{t \in [0, T]} \left| \bar{X}^u_t \right|^2 + \int_0^T \left\| \bar{X}^u_s \right\|^2 ds \right\} \leq C_{N,T} (1 + |x|^2), \quad (3.2)$$

and

$$\sup_{u \in S_N} \left\| \bar{X}^u \right\|_{W^{\alpha, 2}([0, T]; V^{-1})} \leq C_{\alpha,N,T} (1 + |x|^2). \quad (3.3)$$

Proof. Step 1. (Existence and uniqueness of the solution): If $\tilde{f}$ is Lipschitz continuous, the existence and uniqueness of the solution can be proved similarly as in the case of the Burgers equation.

In fact, for any $x_1, x_2, y \in H$ and $t > 0$, by Proposition 3.1 and Lemma 3.2, we have

$$\left| \tilde{f}(x_1) - \tilde{f}(x_2) \right| \leq \left| \int_H f(x_1, z) \mu^{x_1}(dz) - \int_H f(x_2, z) \mu^{x_2}(dz) \right|$$

$$\leq \left| \int_H f(x_1, z) \mu^{x_1}(dz) - \mathbb{E} \left[ f(x_1, Y_{t}^{z_1,y}) \right] \right| + \left| \mathbb{E} \left[ f(x_2, Y_{t}^{z_2,y}) \right] - \int_H f(x_2, z) \mu^{x_2}(dz) \right|$$

$$\leq C (1 + |x_1| + |x_2| + |y|) e^{-yt} + C (|x_1 - x_2| + \mathbb{E} |Y_{t}^{z_1,y} - Y_{t}^{z_2,y}|)$$

$$\leq C (1 + |x_1| + |x_2| + |y|) e^{-yt} + C |x_1 - x_2|.$$
Step 2. (Proof of (3.2)): For any \( u \in S_N \), by (A1), (3.4) and Lemma 5.5, we have
\[
|X^u_t|^2 + 2 \int_0^t \|X^u_s\|^2 \, ds
\]
\[
\leq |x|^2 + C \int_0^t \left(1 + |X^u_s|^2\right) \, ds + 2 \int_0^t \langle \bar{f}(\bar{X}^u_s), \bar{X}^u_s \rangle \, ds + 2 \int_0^t \left\langle \sigma_1(\bar{X}^u_s)Q_1^{1/2}u(s), \bar{X}^u_s \right\rangle \, ds
\]
\[
\leq |x|^2 + C \int_0^t \left(1 + |X^u_s|^2\right) \, ds + 2 \int_0^t \|u(s)\| \cdot \|\sigma_1(\bar{X}^u_s)Q_1^{1/2}\|_{HS} \cdot |\bar{X}^u_s| \, ds
\]
\[
\leq |x|^2 + C \int_0^t \|X^u_s\|^2 \, ds + Ct.
\]
Since \( u \in S_N \), by Gronwall’s inequality, we get
\[
\sup_{t \in [0,T]} |X^u_t|^2 + \int_0^T \|X^u_s\|^2 \, ds \leq C_T \left(1 + |x|^2\right) \exp \left\{ \int_0^T C \left(1 + |u(s)|^2\right) \, ds \right\}
\]
\[
\leq \left(1 + |x|^2\right) C_{N,T} < \infty,
\]
which yields (3.2).

Step 3. (Proof of (3.3)): Notice that
\[
\bar{X}^u_t = x + \int_0^t A\bar{X}^u_s \, ds + \int_0^t B(\bar{X}^u_s) \, ds + \int_0^t \bar{f}(\bar{X}^u_s) \, ds + \int_0^t \sigma_1(\bar{X}^u_s)Q_1^{1/2}u(s) \, ds
\]
\[
= x + I_1(t) + I_2(t) + I_3(t) + I_4(t).
\]
Using the same arguments as that in the proof of [15, Theorem 3.1], we have
\[
\|I_4\|^2_{W^{\alpha,2}([0,T];V^{-1})} \leq C. \tag{3.6}
\]
By Corollary 5.7 and the Cauchy-Schwartz inequality, for any \( 0 \leq s \leq t \leq T \),
\[
\left\| \int_s^t B(\bar{X}^u_r) \, dr \right\|^2 \leq \left( \int_s^t \left\| B(\bar{X}^u_r) \right\|_1 \, dr \right)^2 \leq C \left( \int_s^t |\bar{X}^u_r| \cdot \|\bar{X}^u_r\| \, dr \right)^2
\]
\[
\leq C \left( \int_0^T |\bar{X}^u_r|^2 \, dr \right) \cdot \left( \int_s^t |\bar{X}^u_r|^2 \, dr \right).
\]
Thus,
\[
\int_0^T \|I_2(s)\|^2 \, ds \leq C_T \left( \int_0^T |\bar{X}^u_r|^2 \, dr \right) \left( \int_0^T \|\bar{X}^u_r\|^2 \, dr \right) < +\infty, \tag{3.7}
\]
and
\[
\int_0^T \int_0^T \|I_2(t) - I_2(s)\|^2 \, dsdt \leq C_T \left( \int_0^T |\bar{X}^u_r|^2 \, dr \right) \times \int_0^T \int_0^T \int_s^t \frac{|\bar{X}^u_r|^2}{|t-s|^{1+2\alpha}} \, drdsdt. \tag{3.8}
\]
By the Cauchy-Schwartz inequality and Fubini’s theorem, there exists \( C_{\alpha,T} > 0 \) such that
\[
\int_0^T \int_0^t \frac{|\bar{X}^u_r|^2}{|t-s|^{1+2\alpha}} \, drdsdt \leq C_{\alpha,T} \int_0^T \|\bar{X}^u_r\|^2 \, dr. \tag{3.9}
\]
Combining (3.2), (3.7), (3.8) and (3.9), we have
\[ \|I_2\|_{W^{\alpha,2}([0,T];V^{-1})} \leq C_{\alpha,T}(1 + |x|^2). \] 
(3.10)

Similarly, we also have
\[ \|I_3\|_{W^{\alpha,2}([0,T];V^{-1})} \leq C_{\alpha,T}(1 + |x|^2). \] 
(3.11)

It remains to deal with the last term \( I_4 \). Since \( u \in S_N \), by (A2), we have
\[
\int_0^T \left\| \int_0^t \sigma_1(\tilde{X}_s^u) Q_1^{1/2} u(s) ds \right\|_{L^2}^2 dt \leq C \int_0^T \left( \int_0^t \left\| \sigma_1(\tilde{X}_s^u) Q_1^{1/2} \right\|_{HS} \cdot |u(s)| ds \right)^2 dt \\
\leq C_T \int_0^T c \left( 1 + |\tilde{X}_s^u|^2 \right) ds \cdot \int_0^T |u(s)|^2 ds \\
\leq C_{N,T}
\]
and
\[
\left\| \int_s^t \sigma_1(\tilde{X}_r^u) Q_1^{1/2} u(r) dr \right\|_{L^2}^2 \leq C \int_s^t \left\| \sigma_1(\tilde{X}_r^u) Q_1^{1/2} \right\|_{HS}^2 dr \cdot \int_s^t |u(r)|^2 dr \\
\leq C_N \int_s^t \left( 1 + |\tilde{X}_r^u|^2 \right) dr.
\]

Similar to (3.10), the above two inequalities imply that
\[ \|I_4\|_{W^{\alpha,2}([0,T];V^{-1})} \leq C_{\alpha,N,T}(1 + |x|^2). \] 
(3.12)

By (3.6), (3.10), (3.11) and (3.12), we obtain (3.3). The proof is complete. \( \square \)

3.2. Compactness of solutions to skeleton equations. Recall that for \( u \in S \), \( \tilde{X}^u \) is the solution of the skeleton equation (2.4) and
\[ \Gamma^0 \left( \int_0^T u(s) ds \right) = \tilde{X}^u. \] 
(3.13)

**Proposition 3.5.** For any \( N < \infty \), the family
\[ K_N := \left\{ \Gamma^0 \left( \int_0^T u(s) ds \right) ; u \in S_N \right\} \]
is compact in \( C([0,T];H) \cap L^2([0,T];V) \).

**Proof.** Choose a sequence \( \{u_n \in S_N; n \geq 1\} \), and let \( \{\tilde{X}^{u_n} = \Gamma^0(\int_0^T u_n(s) ds); n \geq 1\} \) be a sequence of elements in \( C([0,T];H) \cap L^2([0,T];V) \). Lemma 3.3, together with (3.2) and (3.3), enables us to assert that there exist a subsequence \( \{n'\} \) and \( u \in S_N \) such that

(a) \( u_{n'} \to u \) in \( S_N \) weakly, as \( n' \to \infty \);
(b) \( \tilde{X}^{u_{n'}} \to \tilde{X}^u \) in \( L^2([0,T];H) \) strongly;
(c) \( \sup_{n' \geq 1} \sup_{0 \leq t \leq T} |\tilde{X}^{u_{n'}}(t)| < \infty \).

Using the same argument as in the proof of [33, Theorem 3.1], we can conclude that \( \tilde{X}^u = \Gamma^0(\int_0^T u(s) ds) \). Next, we will prove that \( \tilde{X}^{u_{n'}} \to \tilde{X}^u \) in \( C([0,T];H) \cap L^2([0,T];V) \).
Using Lemma 5.8, we obtain
\[
\begin{align*}
&\left|\tilde{X}_t^{u_{n'}} - \tilde{X}_t^u\right|^2 + 2 \int_0^t \left\|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right\|^2 ds \\
= &2 \int_0^t \left\langle B\left(\tilde{X}_s^{u_{n'}}\right) - B\left(\tilde{X}_s^u\right), \tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right\rangle ds \\
&+ 2 \int_0^t \left\langle \tilde{f}\left(\tilde{X}_s^{u_{n'}}\right) - \tilde{f}\left(\tilde{X}_s^u\right), \tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right\rangle ds \\
&+ 2 \int_0^t \left\langle \sigma_1\left(\tilde{X}_s^{u_{n'}}\right) Q_1^{1/2} \left[u_{n'}(s) - u(s)\right], \tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right\rangle ds \\
&+ 2 \int_0^t \left\langle \sigma_1\left(\tilde{X}_s^{u_{n'}}\right) - \sigma_1\left(\tilde{X}_s^u\right), Q_1^{1/2} u(s), \tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right\rangle ds \\
=: &I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t). 
\end{align*}
\]
(3.14)

For the first term, by the elementary inequality $2ab \leq a^2 + b^2$ for $a, b > 0$, we have
\[
|I_1^n(t)| \leq 2c \int_0^t \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right| \cdot \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right| \cdot \left(\left\|\tilde{X}_s^u\right\| + \left\|\tilde{X}_s^{u_{n'}}\right\|\right) ds \\
\leq \int_0^T \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 ds + C \int_0^t \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 \cdot \left(\left\|\tilde{X}_s^u\right\|^2 + \left\|\tilde{X}_s^{u_{n'}}\right|^2\right) ds. 
\]
(3.15)

For the second term, by the Lipschitz continuity of $\tilde{f}$ and (b), we have
\[
\sup_{t \in [0, T]} |I_2^n(t)| \leq C \int_0^T \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 ds \to 0, 
\]
(3.16)

For the third term, by the linear growth condition of $\sigma_1 Q_1^{1/2}$, (b) and (c),
\[
\begin{align*}
\sup_{t \in [0, T]} |I_3^n(t)| &\leq 2 \int_0^T \left\|\sigma_1\left(\tilde{X}_s^{u_{n'}}\right) Q_1^{1/2}\right\|_{\text{HS}} \cdot \left|u_{n'}(s) - u(s)\right| \cdot \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right| ds \\
&\leq C \left(1 + \sup_{0 \leq s \leq T} \left|\tilde{X}_s^{u_{n'}}\right|\right) \left(\int_0^T \left|u_{n'}(s) - u(s)\right|^2 ds\right)^{1/2} \cdot \left(\int_0^T \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 ds\right)^{1/2} \\
&\leq 2CN \left(1 + \sup_{0 \leq s \leq T} \left|\tilde{X}_s^{u_{n'}}\right|\right) \cdot \left(\int_0^T \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 ds\right)^{1/2} \to 0. 
\end{align*}
\]
(3.17)

For the last term, by Condition (A1), we have
\[
|I_4^n(t)| \leq C \int_0^t \left|u(s)\right| \cdot \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 ds. 
\]
(3.18)

By (3.14)-(3.18), we have
\[
\begin{align*}
\sup_{s \in [0, t]} \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 + \int_0^t \left\|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right\|^2 ds \\
\leq C \int_0^t \left(\left\|\tilde{X}_s^u\right|^2 + \left\|\tilde{X}_s^{u_{n'}}\right|^2 + \left|u(s)\right|\right) \left|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right|^2 ds + \sup_{0 \leq s \leq T} \left(I_2^n(s) + I_3^n(s)\right). 
\end{align*}
\]

By Gronwall’s inequality and (a)-(c), we have
\[
\sup_{t \in [0, T]} \left|\tilde{X}_t^{u_{n'}} - \tilde{X}_t^u\right|^2 + \int_0^T \left\|\tilde{X}_s^{u_{n'}} - \tilde{X}_s^u\right\|^2 ds \to 0, \text{ as } n' \to \infty.
\]
This implies that $\mathbb{K}_N$ is compact in $C([0, T]; \mathbb{H}) \cap L^2([0, T]; \mathbb{V})$. The proof is complete. □

4. Convergence of the controlled slow processes

In this section we will finish the proof of main result by verifying Condition (a) in Theorem 5.3. Before that, a series of auxiliary results are needed to prove the convergence of the process $X^{\varepsilon, \delta, u^\varepsilon}$. Note that we assume Conditions (A1)-(A4) hold in this section.

4.1. The auxiliary controlled equation. For every fixed $N \in \mathbb{N}, \varepsilon > 0, \delta > 0$, let $u^\varepsilon \in A_N$ and $\Gamma^\varepsilon$ be given by (2.3). By Girsanov’s theorem, we know that

$$X^{\varepsilon, \delta, u^\varepsilon} := \Gamma^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot u^\varepsilon(s) ds \right)$$

is a part of the solution $\left( X^{\varepsilon, \delta, u^\varepsilon}, Y^{\varepsilon, \delta, u^\varepsilon} \right)$ of the following controlled equation:

$$\begin{align*}
\left\{ \begin{array}{l}
dX^{\varepsilon, \delta, u^\varepsilon}_t &= \left[ AX^{\varepsilon, \delta, u^\varepsilon}_t^\varepsilon + B \left( X^{\varepsilon, \delta, u^\varepsilon}_t^\varepsilon \right) + f \left( X^{\varepsilon, \delta, u^\varepsilon}_t^\varepsilon, Y^{\varepsilon, \delta, u^\varepsilon}_t \right) \right] dt + \sigma_1 \left( X^{\varepsilon, \delta, u^\varepsilon}_t \right) Q_1^{1/2} u^\varepsilon(t) dt \\
&\quad + \sqrt{\varepsilon} \sigma_1 \left( X^{\varepsilon, \delta, u^\varepsilon}_t \right) Q_1^{1/2} dW_t,

dY^{\varepsilon, \delta, u^\varepsilon}_t &= \frac{1}{\delta} \left[ AY^{\varepsilon, \delta, u^\varepsilon}_t + g \left( X^{\varepsilon, \delta, u^\varepsilon}_t, Y^{\varepsilon, \delta, u^\varepsilon}_t \right) \right] dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2 \left( X^{\varepsilon, \delta, u^\varepsilon}_t, Y^{\varepsilon, \delta, u^\varepsilon}_t \right) Q_2^{1/2} u^\varepsilon(t) dt \\
&\quad + \frac{1}{\sqrt{\delta}} \sigma_2 \left( X^{\varepsilon, \delta, u^\varepsilon}_t, Y^{\varepsilon, \delta, u^\varepsilon}_t \right) Q_2^{1/2} dW_t,
X^{\varepsilon, \delta, u^\varepsilon}_0 = x, \quad Y^{\varepsilon, \delta, u^\varepsilon}_0 = y.
\end{array} \right.
\end{align*}$$

We first prove the uniform boundedness of the solutions $\left( X^{\varepsilon, \delta, u^\varepsilon}, Y^{\varepsilon, \delta, u^\varepsilon} \right)$ to the system (4.1) for all $\varepsilon, \delta \in (0, 1)$.

Lemma 4.1. For any $x, y \in \mathbb{H}, T > 0$ and $\{u^\varepsilon; \varepsilon > 0\} \subset A_N$, there exists a constant $C_T > 0$ such that for all $\varepsilon, \delta \in (0, 1)$,

$$\mathbb{E} \left( \sup_{t \leq T} \left\| X^{\varepsilon, \delta, u^\varepsilon}_t \right\|^2 \right) + \mathbb{E} \int_0^T \left\| X^{\varepsilon, \delta, u^\varepsilon}_t \right\|^2 dt \leq C_T \left( 1 + |x|^2 + |y|^2 \right)$$

and

$$\mathbb{E} \int_0^T \left\| Y^{\varepsilon, \delta, u^\varepsilon}_t \right\|^2 dt \leq C_T \left( 1 + |x|^2 + |y|^2 \right).$$

Proof. According to Itô’s formula, we have

$$\begin{align*}
\mathbb{E} \left[ \left| Y^{\varepsilon, \delta, u^\varepsilon}_t \right|^2 \right] &= |y|^2 - \frac{2}{\delta} \mathbb{E} \int_0^t \left\| Y^{\varepsilon, \delta, u^\varepsilon}_s \right\|^2 ds + \frac{2}{\delta} \mathbb{E} \int_0^t \left\langle g \left( X^{\varepsilon, \delta, u^\varepsilon}_s, Y^{\varepsilon, \delta, u^\varepsilon}_s \right), Y^{\varepsilon, \delta, u^\varepsilon}_s \right\rangle ds \\
&\quad + \frac{2}{\sqrt{\varepsilon \delta}} \mathbb{E} \int_0^t \left\langle \sigma_2 \left( X^{\varepsilon, \delta, u^\varepsilon}_s, Y^{\varepsilon, \delta, u^\varepsilon}_s \right) Q_2^{1/2} u^\varepsilon(s), Y^{\varepsilon, \delta, u^\varepsilon}_s \right\rangle ds \\
&\quad + \frac{1}{\sqrt{\delta}} \mathbb{E} \int_0^t \left\| \sigma_2 \left( X^{\varepsilon, \delta, u^\varepsilon}_s, Y^{\varepsilon, \delta, u^\varepsilon}_s \right) Q_2^{1/2} \right\|_{HS}^2 ds.
\end{align*}$$

By Poincaré’s inequality and (A1) and (A2), it follows from (4.4) that

$$\frac{d}{dt} \mathbb{E} \left[ \left| Y^{\varepsilon, \delta, u^\varepsilon}_t \right|^2 \right] \leq -\frac{2\lambda_1}{\delta} \mathbb{E} \left[ \left| Y^{\varepsilon, \delta, u^\varepsilon}_t \right|^2 \right] + \frac{2}{\delta} \mathbb{E} \left( C \left| Y^{\varepsilon, \delta, u^\varepsilon}_t \right| + C \left| X^{\varepsilon, \delta, u^\varepsilon}_t \right| \cdot \left| Y^{\varepsilon, \delta, u^\varepsilon}_t \right| + L_g \left| Y^{\varepsilon, \delta, u^\varepsilon}_t \right|^2 \right)$$

$$\quad + \frac{CL_\delta^2}{\sqrt{\varepsilon \delta}} \mathbb{E} \left[ \left( 1 + \left| X^{\varepsilon, \delta, u^\varepsilon}_t \right| \right) \cdot \left| Y^{\varepsilon, \delta, u^\varepsilon}_t \right| \right] + \frac{CL_\delta^2}{\delta} \mathbb{E} \left( 1 + \left| X^{\varepsilon, \delta, u^\varepsilon}_t \right|^2 \right).$$
Using \((A3)\) and Young’s inequality, we deduce that
\[
\frac{d}{dt} \mathbb{E} \left[ |Y_t^{\varepsilon, \delta, u^\varepsilon}|^2 \right] \leq -\frac{\lambda_1 - L_{\delta}}{\delta} \mathbb{E} \left[ |Y_t^{\varepsilon, \delta, u^\varepsilon}|^2 \right] + \frac{C}{\delta} \mathbb{E} \left( |X_t^{\varepsilon, \delta, u^\varepsilon}|^2 + 1 \right) + \frac{C}{\sqrt{\varepsilon}} \mathbb{E} \left[ \left( 1 + |X_t^{\varepsilon, \delta, u^\varepsilon}|^2 \right) |u^\varepsilon(t)|^2 \right].
\]

By the comparison theorem, we have
\[
\mathbb{E} \left[ |Y_t^{\varepsilon, \delta, u^\varepsilon}|^2 \right] \leq |y|^2 e^{-\frac{\lambda_1 - L_{\delta}}{\delta} t} + \frac{C}{\delta} \int_0^t e^{-\frac{\lambda_1 - L_{\delta}}{\delta} (t-s)} \left( \mathbb{E} \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 + 1 \right) ds
+ \frac{C}{\sqrt{\varepsilon}} \mathbb{E} \int_0^t e^{-\frac{\lambda_1 - L_{\delta}}{\delta} (t-s)} \left( 1 + |X_s^{\varepsilon, \delta, u^\varepsilon}|^2 \right) |u^\varepsilon(s)|^2 ds.
\]

Then we have
\[
\mathbb{E} \int_0^T |Y_t^{\varepsilon, \delta, u^\varepsilon}|^2 dt \leq |y|^2 \int_0^T e^{-\frac{\lambda_1 - L_{\delta}}{\delta} t} dt + \frac{C}{\delta} \int_0^T \int_0^t e^{-\frac{\lambda_1 - L_{\delta}}{\delta} (t-s)} \left( \mathbb{E} \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 + 1 \right) ds dt
+ \frac{C}{\sqrt{\varepsilon}} \mathbb{E} \left\{ \left( 1 + \sup_{s \in [0,T]} \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 \right) \int_0^T \int_0^t e^{-\frac{\lambda_1 - L_{\delta}}{\delta} (t-s)} |u^\varepsilon(s)|^2 ds dt \right\}
\leq C \left( 1 + |y|^2 \right) + C \int_0^T \mathbb{E} \left| X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 dt + \frac{C N \sqrt{\delta}}{\sqrt{\varepsilon}} \mathbb{E} \left[ \sup_{s \in [0,T]} \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 \right]. \tag{4.5}
\]

Applying Itô’s formula, we have
\[
\left| X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 = \left| x \right|^2 - \int_0^t 2 \left\langle B \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right), X_s^{\varepsilon, \delta, u^\varepsilon} \right\rangle ds + 2 \int_0^t \left\langle f \left( X_s^{\varepsilon, \delta, u^\varepsilon}, Y_s^{\varepsilon, \delta, u^\varepsilon} \right), X_s^{\varepsilon, \delta, u^\varepsilon} \right\rangle ds + 2 \sqrt{\varepsilon} \int_0^t \left\langle X_s^{\varepsilon, \delta, u^\varepsilon}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} dW_s \right\rangle
+ 2 \int_0^t \left\langle X_s^{\varepsilon, \delta, u^\varepsilon}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} u^\varepsilon(s) \right\rangle ds + \varepsilon \int_0^t \left\| \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} \right\|_{HS}^2 ds.
\]

By Lemma 5.5, \((A1)\) and \((A2)\), we obtain that
\[
\left| X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 + \int_0^t \left\| X_s^{\varepsilon, \delta, u^\varepsilon} \right\|^2 ds
\leq C + |x|^2 + C \int_0^t \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds + C \int_0^t \left| Y_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds + 2 \sqrt{\varepsilon} \int_0^t \left\langle X_s^{\varepsilon, \delta, u^\varepsilon}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} dW_s \right\rangle
+ C \int_0^t \left( 1 + \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 \right) |u^\varepsilon(s)| \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right| ds + \varepsilon C \int_0^t \left( 1 + \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 \right) ds
\leq C + |x|^2 + C \int_0^t \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds + C \int_0^t \left| Y_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds + 2 \sqrt{\varepsilon} \int_0^t \left\langle X_s^{\varepsilon, \delta, u^\varepsilon}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} dW_s \right\rangle
+ \frac{1}{4} \sup_{s \in [0,t]} \left| X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2.
\]
Then by Burkholder-Davis-Gundy’s inequality and (4.5), we have
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X_t^{\varepsilon,\delta,u^\varepsilon}|^2 \right) + \mathbb{E} \int_0^T \left\| X_s^{\varepsilon,\delta,u^\varepsilon} \right\|^2 ds \\
\leq C \left( 1 + |x|^2 \right) + C \mathbb{E} \int_0^T |X_s^{\varepsilon,\delta,u^\varepsilon}|^2 ds + C \mathbb{E} \int_0^T |Y_s^{\varepsilon,\delta,u^\varepsilon}|^2 ds \\
+ C \varepsilon^2 \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \left( X_s^{\varepsilon,\delta,u^\varepsilon}, \sigma_1 \left( X_s^{\varepsilon,\delta,u^\varepsilon} \right) Q_1^{1/2} dW_s \right) \right| \right] \\
\leq C \left( 1 + |x|^2 + |y|^2 \right) + C \int_0^T \left| X_s^{\varepsilon,\delta,u^\varepsilon} \right|^2 ds + \frac{C \sqrt{\varepsilon}}{\sqrt{\varepsilon}} \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^{\varepsilon,\delta,u^\varepsilon}|^2 \right) \\
+ C \mathbb{E} \left[ \int_0^T \left( 1 + |X_s^{\varepsilon,\delta,u^\varepsilon}|^2 \right) \cdot \left| X_s^{\varepsilon,\delta,u^\varepsilon} \right|^2 ds \right]^{1/2} \\
\leq C \left( 1 + |x|^2 + |y|^2 \right) + C \int_0^T \mathbb{E} \left[ \left| X_s^{\varepsilon,\delta,u^\varepsilon} \right|^2 \right] ds + \left( \frac{1}{4} + \frac{C \sqrt{\varepsilon}}{\sqrt{\varepsilon}} \right) \mathbb{E} \left( \sup_{t \in [0,T]} |X_t^{\varepsilon,\delta,u^\varepsilon}|^2 \right).}

By (A4), taking \( \varepsilon \) small enough such that \( \delta/\varepsilon \leq \frac{1}{4} \) we have,
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t^{\varepsilon,\delta,u^\varepsilon}|^2 \right] + \mathbb{E} \int_0^T \left\| X_s^{\varepsilon,\delta,u^\varepsilon} \right\|^2 ds \leq C \left( 1 + |x|^2 + |y|^2 \right) + C \mathbb{E} \int_0^T \left| X_s^{\varepsilon,\delta,u^\varepsilon} \right|^2 ds.
\]
By Gronwall’s inequality, we have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |X_t^{\varepsilon,\delta,u^\varepsilon}|^2 \right] + \mathbb{E} \int_0^T \left\| X_s^{\varepsilon,\delta,u^\varepsilon} \right\|^2 ds \leq C_T \left( 1 + |x|^2 + |y|^2 \right). \tag{4.6}
\]
The inequality (4.3) follows by combining (4.5) and (4.6). The proof is complete. \( \square \)

Because the approach based on the time discretization will be used later, we need the following lemma, which is inspired from [21, Lemma 3.2] and plays an important role in the proof. Meanwhile, it will be very helpful to weaken the regularity requirement of initial value \( x \), i.e., we drop the regularity of initial value \( x \in \mathbb{H}^{3/2}_\theta \) with \( \theta \in (1,3/2) \) in [9] and only assume \( x \in \mathbb{H} \) here. To this purpose, we first construct the following stopping time, for any \( R, \varepsilon > 0 \),
\[
\tau_{R}^{\varepsilon} := \inf \left\{ t > 0 \mid \left| X_t^{\varepsilon,\delta,u^\varepsilon} \right| > R \right\}.
\]

**Lemma 4.2.** For any \( x, y \in \mathbb{H} \), \( R, T > 0 \) and \( \varepsilon, \Delta > 0 \) small enough, there exists a constant \( C_{R,T} > 0 \) such that
\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_{R}^{\varepsilon}} \left| X_t^{\varepsilon,\delta,u^\varepsilon} - X_{t(\Delta)}^{\varepsilon,\delta,u^\varepsilon} \right|^2 dt \right] \leq C_{R,T} \Delta^{1/2} \left( 1 + |x|^2 + |y|^2 \right), \tag{4.7}
\]
where \( t(\Delta) := \left\lceil \frac{\Delta}{\Delta} \right\rceil \Delta \) and \([s]\) denotes the largest integer which is not greater than \( s \).
Proof. By a straightforward compute,

\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_R^\varepsilon} \left| X_t^{\varepsilon, \delta, u^\varepsilon} - X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 dt \right] \\
\leq \mathbb{E} \left( \int_0^{\Delta} \left| X_t^{\varepsilon, \delta, u^\varepsilon} - x \right|^2 1_{\{t \leq \tau_R^\varepsilon \}} dt \right) + \mathbb{E} \left[ \int_T^{T \wedge \tau_R^\varepsilon} \left| X_t^{\varepsilon, \delta, u^\varepsilon} - X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 1_{\{t \leq \tau_R^\varepsilon \}} dt \right] \\
\leq C_R \left( 1 + |x|^2 \right) \Delta + 2 \mathbb{E} \left( \int_0^{\Delta} \left| X_t^{\varepsilon, \delta, u^\varepsilon} - X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 1_{\{t \leq \tau_R^\varepsilon \}} dt \right) \\
+ 2 \mathbb{E} \left( \int_0^{T} \left| X_t^{\varepsilon, \delta, u^\varepsilon} - X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 1_{\{t \leq \tau_R^\varepsilon \}} dt \right). \quad (4.8)
\]

Firstly, we estimate the second term on the right-hand side of (4.8). Applying Itô's formula to \( Z_u := X_t^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon} \), we have the increment of \( |Z_u|^2 \) over interval \([t-\Delta, t]\) as follows,

\[
\left| X_t^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon} \right|^2 \\
= 2 \int_{t-\Delta}^{t} \left\langle AX_s^{\varepsilon, \delta, u^\varepsilon} + B \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right), X_s^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon} \right\rangle ds \\
+ 2 \int_{t-\Delta}^{t} \left\langle f \left( X_s^{\varepsilon, \delta, u^\varepsilon}, Y_s^{\varepsilon, \delta, u^\varepsilon} \right), X_s^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon} \right\rangle ds \\
+ 2 \int_{t-\Delta}^{t} \left\langle \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} u_s^{\varepsilon, \delta, u^\varepsilon}, X_s^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon} \right\rangle ds + \varepsilon \int_{t-\Delta}^{t} \left\| \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} \right\|_{HS}^2 ds \\
+ 2 \sqrt{\varepsilon} \int_{t-\Delta}^{t} \left\langle X_s^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{1/2} dW_s \right\rangle \\
:= L_1(t) + L_2(t) + L_3(t) + L_4(t) + L_5(t). \quad (4.9)
\]

For the term \( L_1(t) \), by Hölder's inequality, Corollary 5.7 and the definition of stopping time \( \tau_R^\varepsilon \), we have

\[
\mathbb{E} \left( \int_\Delta^{T} |L_1(t)| 1_{\{t \leq \tau_R^\varepsilon \}} dt \right) \\
\leq C \mathbb{E} \left( \int_\Delta^{T} \int_{t-\Delta}^{t} \left\| AX_s^{\varepsilon, \delta, u^\varepsilon} + B \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) \right\|_{-1} \left\| X_s^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon} \right\| ds 1_{\{t \leq \tau_R^\varepsilon \}} dt \right) \\
\leq C \left[ \mathbb{E} \left( \int_\Delta^{T} \int_{t-\Delta}^{t} \left\| AX_s^{\varepsilon, \delta, u^\varepsilon} - B \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) \right\|_{-1} ds 1_{\{t \leq \tau_R^\varepsilon \}} dt \right]^{1/2} \left[ \mathbb{E} \left( \int_\Delta^{T} \int_{t-\Delta}^{t} \left\| X_s^{\varepsilon, \delta, u^\varepsilon} - X_{t-\Delta}^{\varepsilon, \delta, u^\varepsilon} \right\|^2 ds 1_{\{t \leq \tau_R^\varepsilon \}} dt \right] \right]^{1/2} \right) \\
\leq C \Delta \mathbb{E} \int_0^{T} \left\| X_s^{\varepsilon, \delta, u^\varepsilon} \right\|^2 \left( 1 + \left\| X_s^{\varepsilon, \delta, u^\varepsilon} \right\|^2 \right) 1_{\{s \leq \tau_R^\varepsilon \}} ds \right]^{1/2} \cdot \left[ \Delta \mathbb{E} \int_0^{T} \left\| X_s^{\varepsilon, \delta, u^\varepsilon} \right\|^2 ds \right]^{1/2} \\
\leq C_R T \Delta (1 + |x|^2 + |y|^2), \quad (4.10)
\]

where we use the Fubini theorem and (4.2) in the third and fourth inequalities respectively.
For the term $L_2(t)$, by Condition (A1) and (4.3), we get 
\[
\mathbb{E} \left( \int_{\Delta} |L_2(t)| 1_{\{t \leq \tau_{R,\Delta}^\varepsilon \}} dt \right) 
\leq C\mathbb{E} \left( \int_\Delta \int_{t-\Delta}^t \left( 1 + |x_s^{\varepsilon,\delta,u^\varepsilon}| \right) \left( |X_{s-\Delta}^{\varepsilon,\delta,u^\varepsilon}| + |X_{t-\Delta}^{\varepsilon,\delta,u^\varepsilon}| \right) ds 1_{\{t \leq \tau_{R,\Delta}^\varepsilon \}} dt \right) 
\leq C_{R,T}\Delta + C_R \mathbb{E} \int_\Delta \int_{t-\Delta}^t |u_s^{\varepsilon}| ds dt 
\leq C_{R,T}\Delta + C_{R,T} \mathbb{E} \left[ \int_{\Delta} \left| Y_s^{\varepsilon,\delta,u^\varepsilon} \right|^2 ds \right]^{1/2} 
\leq C_{R,T}\Delta (1 + |x|^2 + |y|^2). \quad (4.11)
\]

For the terms $L_3(t)$ and $L_4(t)$, it is easy to see 
\[
\mathbb{E} \left( \int_\Delta |L_3(t)| 1_{\{t \leq \tau_{R,\Delta}^\varepsilon \}} dt \right) 
\leq C\mathbb{E} \left( \int_\Delta \int_{t-\Delta}^t \left( 1 + |x_s^{\varepsilon,\delta,u^\varepsilon}| \right) ds 1_{\{t \leq \tau_{R,\Delta}^\varepsilon \}} dt \right) 
\leq C_{R,T}\Delta + C_R \mathbb{E} \int_\Delta \int_{t-\Delta}^t |u_s^{\varepsilon}| ds dt 
\leq C_{R,T}\Delta + C_{R,T} \mathbb{E} \left[ \int_0^T \left| u_s^{\varepsilon} \right|^2 ds \right]^{1/2} 
\leq C_{R,T}\Delta, \quad (4.12)
\]

and 
\[
\mathbb{E} \left( \int_\Delta |L_4(t)| 1_{\{t \leq \tau_{R,\Delta}^\varepsilon \}} dt \right) 
\leq C\mathbb{E} \left( \int_\Delta \int_{t-\Delta}^t \left( 1 + |x_s^{\varepsilon,\delta,u^\varepsilon}| \right) 1_{\{s \leq \tau_{R,\Delta}^\varepsilon \}} ds dt \right) 
\leq C_{R,T}\Delta. \quad (4.13)
\]

For the term $L_5(t)$, Burkholder-Davies-Gundy’s inequality implies 
\[
\mathbb{E} \left( \int_\Delta \left| L_5(t) \right| 1_{\{t \leq \tau_{R,\Delta}^\varepsilon \}} dt \right) 
\leq C\mathbb{E} \left[ \int_\Delta \left[ \int_{t-\Delta}^t \left| \sigma_1(X_s^{\varepsilon,\delta,u^\varepsilon}) Q_1^{\varepsilon,\delta,u^\varepsilon} \right|^{1/2} \left| X_s^{\varepsilon,\delta,u^\varepsilon} - X_{t-\Delta}^{\varepsilon,\delta,u^\varepsilon} \right|^2 ds \right]^{1/2} dt \right] 
\leq C_T \mathbb{E} \left[ \int_\Delta \int_{t-\Delta}^t \left( 1 + |x_s^{\varepsilon,\delta,u^\varepsilon}|^2 \right) \left( |X_s^{\varepsilon,\delta,u^\varepsilon}|^2 + |X_{t-\Delta}^{\varepsilon,\delta,u^\varepsilon}|^2 \right) 1_{\{s \leq \tau_{R,\Delta}^\varepsilon \}} ds dt \right]^{1/2} 
\leq C_{R,T}\Delta^{1/2}. \quad (4.14)
\]

Combining estimates (4.9)-(4.14) together, we can deduce that 
\[
\mathbb{E} \left( \int_\Delta \left| X_{t}^{\varepsilon,\delta,u^\varepsilon} - X_{t-\Delta}^{\varepsilon,\delta,u^\varepsilon} \right|^2 1_{\{t \leq \tau_{R,\Delta}^\varepsilon \}} dt \right) \leq C_{R,T}\Delta^{1/2}(1 + |x|^2 + |y|^2). \quad (4.15)
\]
By the similar argument above, we can also get
\[
\mathbb{E} \left( \int_{\Delta} |X_{t}^{\varepsilon,\delta,u^{\varepsilon}} - X_{t-\Delta}^{\varepsilon,\delta,u^{\varepsilon}}|^2 1_{\{t \leq \tau_{R}^{\varepsilon}\}} dt \right) \leq C_{R,T} \Delta^{1/2}(1 + |x|^2 + |y|^2).
\] (4.16)
Hence, (4.8), (4.15) and (4.16) implies (4.7) holds. The proof is complete. □

4.2. Some priori estimates on auxiliary processes. Following the idea inspired by Khasminski [28], we introduce an auxiliary process. Specifically, we split the interval \([0,T]\) into some subintervals of size \(\Delta > 0\), and we will let \(\Delta = \delta^{1/2}\) later. With the initial value \(\hat{Y}^\varepsilon_{0} = Y^\varepsilon_{0} = y\), we construct the process \(\hat{Y}^\varepsilon_{t}\) as follows:
\[
d\hat{Y}^\varepsilon_{t} = \frac{1}{\delta} \left[ A\hat{X}^\varepsilon_{t} + g\left(X_{t(\Delta)}^{\varepsilon,\delta,u^{\varepsilon}}\right) \right] dt + \frac{1}{\sqrt{\delta}} \sigma_{2}\left(X_{t(\Delta)}^{\varepsilon,\delta,u^{\varepsilon}}\right) Q_{2}^{1/2} dW_{t},
\]
which satisfies
\[
\hat{Y}^\varepsilon_{t} = e^{t A/\delta} y + \frac{1}{\delta} \int_{0}^{t} e^{(t-s)A/\delta} g\left(X_{s(\Delta)}^{\varepsilon,\delta,u^{\varepsilon}}\right) ds
\]
\[
+ \frac{1}{\sqrt{\delta}} \int_{0}^{t} e^{(t-s)A/\delta} \sigma_{2}\left(X_{s(\Delta)}^{\varepsilon,\delta,u^{\varepsilon}}\right) Q_{2}^{1/2} dW_{s},
\] (4.17)
where \(t(\Delta) = \lfloor \frac{t}{\Delta} \rfloor \Delta\) is the nearest breakpoint proceeding \(t\). We construct the process \(\hat{X}^\varepsilon_{t}\) as follows:
\[
\frac{d}{dt} \hat{X}^\varepsilon_{t} = A\hat{X}^\varepsilon_{t} + B\left(\hat{X}^\varepsilon_{t}\right) + f\left(X_{t(\Delta)}^{\varepsilon,\delta,u^{\varepsilon}}\right) + \sigma_{1}\left(\hat{X}^\varepsilon_{t}\right) Q_{1}^{1/2} u^{\varepsilon}(t), \quad \hat{X}^\varepsilon_{0} = x.
\]
Then
\[
\hat{X}^\varepsilon_{t} = e^{t A} x + \int_{0}^{t} e^{(t-s)A} B\left(\hat{X}^\varepsilon_{s}\right) ds + \int_{0}^{t} e^{(t-s)A} f\left(X_{s(\Delta)}^{\varepsilon,\delta,u^{\varepsilon}}\right) ds
\]
\[
+ \int_{0}^{t} e^{(t-s)A} \sigma_{1}\left(\hat{X}^\varepsilon_{s}\right) Q_{1}^{1/2} u^{\varepsilon}(s) ds.
\] (4.18)

The following Lemma gives a control of the auxiliary processes \((\hat{X}^\varepsilon_{t}, \hat{Y}^\varepsilon_{t})\). Since the proof can be carried out almost the same way as in the proof of Lemma 4.1, we omit the proof here.

Lemma 4.3. For any \(x, y \in \mathbb{H}\) and \(T > 0\), there exists a constant \(C_{T} > 0\) such that for all \(\varepsilon, \delta \in (0,1]\)
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |\hat{X}^\varepsilon_{t}|^2 \right) + \mathbb{E} \int_{0}^{T} \|\hat{X}^\varepsilon_{t}\|^2 dt \leq C_{T} \left( 1 + |x|^2 + |y|^2 \right)
\] (4.19)
and
\[
\sup_{t \in [0,T]} \mathbb{E} |\hat{Y}^\varepsilon_{t}|^2 \leq C_{T} \left( 1 + |x|^2 + |y|^2 \right). \] (4.20)

Lemma 4.4. For any \(x, y \in \mathbb{H}, R, T > 0\), there exists a constant \(C_{R,T} > 0\) such that for all \(\varepsilon, \delta \in (0,1]\)
\[
\mathbb{E} \int_{0}^{T^{\wedge} R} |Y_{s}^{\varepsilon,\delta,u^{\varepsilon}} - \hat{Y}^\varepsilon_{s}|^2 ds \leq C_{R,T} (1 + |x|^2 + |y|^2) \Delta^{1/2} + \frac{C_{R,T} \sqrt{\delta}}{\sqrt{\varepsilon}}.
\] (4.21)
Proof. Let \( \rho_t := Y_t^{\varepsilon, \delta, u^\varepsilon} - \dot{Y}_t^{\varepsilon, \delta} \) and \( \Lambda_t := \rho_t - V_t - M_t \), with

\[
V_t := \frac{1}{\sqrt{\delta}} \int_0^t e^{(t-s)\frac{\lambda_1}{\delta}} \sigma_2 \left( X_s^{\varepsilon, \delta, u^\varepsilon}, Y_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_t^{1/2} u^\varepsilon(s) ds
\]

and

\[
M_t := \frac{1}{\sqrt{\delta}} \int_0^t e^{(t-s)\frac{\lambda_1}{\delta}} \left[ \sigma_2 \left( X_s^{\varepsilon, \delta, u^\varepsilon}, Y_s^{\varepsilon, \delta, u^\varepsilon} \right) - \sigma_2 \left( X_s^{\varepsilon, \delta, u^\varepsilon}, \dot{Y}_s^{\varepsilon, \delta} \right) \right] Q_t^{1/2} dW_s.
\]

Then it is easy to see that \( \Lambda_t \) satisfies the following equation:

\[
d\Lambda_t = \frac{1}{\delta} \left[ A \Lambda_t + g \left( X_t^{\varepsilon, \delta, u^\varepsilon}, Y_t^{\varepsilon, \delta, u^\varepsilon} \right) - g \left( X_t^{\varepsilon, \delta, u^\varepsilon}, \dot{Y}_t^{\varepsilon, \delta} \right) \right] dt, \quad \Lambda_0 = 0.
\]

Thus,

\[
\frac{d}{dt} |\Lambda_t|^2 = -\frac{2}{\delta} |\Lambda_t|^2 + \frac{2}{\delta} \left\langle g \left( X_t^{\varepsilon, \delta, u^\varepsilon}, Y_t^{\varepsilon, \delta, u^\varepsilon} \right) - g \left( X_t^{\varepsilon, \delta, u^\varepsilon}, \dot{Y}_t^{\varepsilon, \delta} \right), \Lambda_t \right\rangle
\]

\[
\leq - \frac{2\lambda_1}{\delta} |\Lambda_t|^2 + \frac{C}{\delta} |X_t^{\varepsilon, \delta, u^\varepsilon} - X_t^{\varepsilon, \delta, u^\varepsilon}| \cdot |\Lambda_t| + \frac{2L_g}{\delta} |\rho_t| \cdot |\Lambda_t|
\]

\[
\leq - \frac{2\lambda_1}{\delta} |\Lambda_t|^2 + \frac{C}{\delta} \left| X_t^{\varepsilon, \delta, u^\varepsilon} - X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 + \frac{(\lambda_1 - L_g)}{\delta} |\Lambda_t|^2 + \frac{2L_g}{\delta} |\Lambda_t|^2 + \frac{L_g}{2\delta} |\rho_t|^2
\]

\[
\leq - \frac{(\lambda_1 - L_g)}{\delta} |\Lambda_t|^2 + \frac{C}{\delta} \left| X_t^{\varepsilon, \delta, u^\varepsilon} - X_t^{\varepsilon, \delta, u^\varepsilon} \right|^2 + \frac{L_g}{2\delta} |\rho_t|^2.
\]

By the comparison theorem, we have

\[
|\Lambda_t|^2 \leq \frac{C}{\delta} \int_0^t e^{-(\lambda_1 - L_g)(t-s)\frac{\lambda_1}{\delta}} \left| X_s^{\varepsilon, \delta, u^\varepsilon} - X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds + \frac{L_g}{2\delta} \int_0^t e^{-(\lambda_1 - L_g)(t-s)\frac{\lambda_1}{\delta}} |\rho_s|^2 ds.
\]

Then by Fubini’s Theorem, for any \( T > 0 \),

\[
\int_0^T |\Lambda_t|^2 dt \leq \frac{C}{\delta} \int_0^T \int_0^t e^{-(\lambda_1 - L_g)(t-s)\frac{\lambda_1}{\delta}} \left| X_s^{\varepsilon, \delta, u^\varepsilon} - X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 dtds + \frac{L_g}{2\delta} \int_0^T \int_0^t e^{-(\lambda_1 - L_g)(t-s)\frac{\lambda_1}{\delta}} |\rho_s|^2 dtds
\]

\[
= \frac{C}{\delta} \int_0^T \left( \int_0^t e^{-(\lambda_1 - L_g)(t-s)\frac{\lambda_1}{\delta}} dt \right) \left| X_s^{\varepsilon, \delta, u^\varepsilon} - X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds + \frac{L_g}{2\delta} \int_0^T \left( \int_0^t e^{-(\lambda_1 - L_g)(t-s)\frac{\lambda_1}{\delta}} dt \right) |\rho_s|^2 ds
\]

\[
\leq C \int_0^T \left| X_s^{\varepsilon, \delta, u^\varepsilon} - X_s^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds + \frac{L_g}{2(\lambda_1 - L_g)} \int_0^T |\rho_s|^2 ds.
\]

By Lemma 4.2, we obtain

\[
\mathbb{E} \int_0^{T \wedge \tau^*_R} |\Lambda_t|^2 dt \leq \frac{L_g}{2(\lambda_1 - L_g)} \mathbb{E} \int_0^{T \wedge \tau^*_R} |\rho_s|^2 ds + C_{R, T} \left( 1 + |x|^2 + |y|^2 \right) \Delta^{1/2}.
\]
Now let’s estimate term $V_t$:

$$|V_t| \leq \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{-\lambda s} s \sigma_2 \left( X_{s}^{\varepsilon, \delta, u_s}, Y_{s}^{\varepsilon, \delta, u_s} \right) Q_2^{1/2} u_s(s) ds$$

$$= \frac{1}{\sqrt{\varepsilon}} \left( \int_0^t e^{-2\lambda_1(t-s)} ds \right)^{1/2} \left( \int_0^t \left| \sigma_2 \left( X_{s}^{\varepsilon, \delta, u_s}, Y_{s}^{\varepsilon, \delta, u_s} \right) Q_2^{1/2} u_s(s) \right|^2 ds \right)^{1/2}$$

$$\leq \frac{C \sqrt{\delta}}{\sqrt{\varepsilon}} \left( \int_0^t \left( 1 + \left| X_{s}^{\varepsilon, \delta, u_s} \right|^2 \right) \left| u_s(s) \right|^2 ds \right)^{1/2}$$

By the definition of $\tau_{R_t}$, we have

$$\mathbb{E} \int_0^{T^{\wedge} \tau_{R_t}} |V_t|^2 dt \leq \frac{C_{R,T} \sqrt{\delta}}{\sqrt{\varepsilon}}. \quad (4.22)$$

For term $M_t$, noting that $\frac{L^2}{\lambda_1} + \frac{L_0}{\lambda_1 - L_g} < 1$, there exist $\gamma_1, \gamma_2 > 1$ such that $\gamma_2 \left( \frac{\gamma_1 L^2}{\lambda_1} + \frac{L_0}{\lambda_1 - L_g} \right) < 1$. Then, by Lemma 4.2,

$$\mathbb{E} \int_0^{T^{\wedge} \tau_{R_t}} |M_t|^2 dt$$

$$= \mathbb{E} \int_0^{T^{\wedge} \tau_{R_t}} \left[ \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{\left( \frac{\lambda_1 \tau_0}{\varepsilon} \right) \Delta} \left[ \sigma_2 \left( X_{s}^{\varepsilon, \delta, u_s}, Y_{s}^{\varepsilon, \delta, u_s} \right) - \sigma_2 \left( X_{s(\Delta)}^{\varepsilon, \delta, u_s}, Y_{s(\Delta)}^{\varepsilon, \delta, u_s} \right) \right] Q_2^{1/2} dW_s \right]^2 dt$$

$$\leq \mathbb{E} \int_0^T \left[ \frac{1}{\sqrt{\varepsilon}} \int_0^{T^{\wedge} \tau_{R_t}} e^{\left( \frac{\lambda_1 \tau_0}{\varepsilon} \right) \Delta} \left[ \sigma_2 \left( X_{s}^{\varepsilon, \delta, u_s}, Y_{s}^{\varepsilon, \delta, u_s} \right) - \sigma_2 \left( X_{s(\Delta)}^{\varepsilon, \delta, u_s}, Y_{s(\Delta)}^{\varepsilon, \delta, u_s} \right) \right] Q_2^{1/2} dW_s \right]^2 dt$$

$$\leq \frac{1}{\delta} \mathbb{E} \int_0^T \left( \int_0^{T^{\wedge} \tau_{R_t}} e^{\left( \frac{\lambda_1 \tau_0}{\varepsilon} \right) \Delta} \left( C \left| X_{s}^{\varepsilon, \delta, u_s} - X_{s(\Delta)}^{\varepsilon, \delta, u_s} \right|^2 + \gamma_1 \frac{L^2}{\lambda_1} \left| \rho_s \right|^2 \right) ds \right) dt$$

$$\leq \frac{1}{\delta} \mathbb{E} \int_0^T \int_0^{T^{\wedge} \tau_{R_t}} e^{\left( \frac{\lambda_1 \tau_0}{\varepsilon} \right) \Delta} \left( C \left| X_{s}^{\varepsilon, \delta, u_s} - X_{s(\Delta)}^{\varepsilon, \delta, u_s} \right|^2 + \gamma_1 \frac{L^2}{\lambda_1} \left| \rho_s \right|^2 \right) ds \right) dt$$

$$\leq C_{\lambda_1} \mathbb{E} \left[ \int_0^{T^{\wedge} \tau_{R_t}} \left| X_{s}^{\varepsilon, \delta, u_s} - X_{s(\Delta)}^{\varepsilon, \delta, u_s} \right|^2 ds \right] + \gamma_1 \frac{L^2}{2\lambda_1} \mathbb{E} \int_0^{T^{\wedge} \tau_{R_t}} \left| \rho_s \right|^2 ds$$

Using the following inequality,

$$\rho_t^2 \leq \gamma_2 \left( \Lambda_t + M_t \right)^2 + C_{\gamma_2} V_t^2 \leq 2\gamma_2 \left( \Lambda_t^2 + M_t^2 \right) + C_{\gamma_2} V_t^2,$$

and by (4.22) and (4.23), we finally obtain

$$\mathbb{E} \int_0^{T^{\wedge} \tau_{R_t}} \left| \rho_t \right|^2 dt \leq \gamma_2 \left( \frac{\gamma_1 L_{\sigma^2}}{\lambda_1} + \frac{L_0}{\lambda_1 - L_g} \right) \mathbb{E} \int_0^{T^{\wedge} \tau_{R_t}} \left| \rho_t \right|^2 dt$$

$$+ C_{\lambda_1, R,T} \left( 1 + \left| x \right|^2 + \left| y \right|^2 \right) \Delta^{1/2} + \frac{C_{R,T} \sqrt{\delta}}{\sqrt{\varepsilon}},$$

which implies (4.21). The proof is complete. \qed
Remark 4.5. By comparing the equations of $Y_t^{\varepsilon, \delta, u^\varepsilon}$ and $\hat{Y}_t^{\varepsilon, \delta}$, it is easy to see the additional controlled term including $u^\varepsilon$ in $Y_t^{\varepsilon, \delta, u^\varepsilon}$ disappears in $\hat{Y}_t^{\varepsilon, \delta}$. Lemma 4.4 implies additional controlled term takes no effect as $\varepsilon \to 0$, which is the main reason why we assume (A4) holds.

Combining the following two propositions, we prove the convergence of controlled sequence $X_t^{\varepsilon, \delta, u^\varepsilon}$ to averaged process $\hat{X}^u_t$. This finally proves Condition (a) in Theorem 5.3, so that the large derivation principle in the main result Theorem 2.2 is obtained.

Proposition 4.6. For every fixed $N \in \mathbb{N}$, $\{u^\varepsilon\}_{\varepsilon > 0} \in \mathcal{A}_N$,

$$X_t^{\varepsilon, \delta, u^\varepsilon} - \hat{x}_t^{\varepsilon, \delta} \text{ converges to } 0 \text{ in distribution }$$

in $C([0, T]; \mathbb{H}) \cap L^2([0, T]; V)$ as $\varepsilon \to 0$.

Proof. Define $Z_t^{\varepsilon, \delta} := X_t^{\varepsilon, \delta, u^\varepsilon} - \hat{X}_t^{\varepsilon, \delta}$. According to Itô’s formula and Lemma 5.8, we have

$$|Z_t^{\varepsilon, \delta}|^2 = -2 \int_0^t \|Z_s^{\varepsilon, \delta}\|^2 ds + 2 \int_0^t \langle B \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) - B \left( \hat{X}_s^{\varepsilon, \delta} \right), Z_s^{\varepsilon, \delta} \rangle ds$$

$$+ 2 \int_0^t \left( f \left( X_s^{\varepsilon, \delta, u^\varepsilon}, Y_s^{\varepsilon, \delta, u^\varepsilon} \right) - f \left( X_s^{\varepsilon, \delta, u^\varepsilon}, \hat{Y}_s^{\varepsilon, \delta} \right), Z_s^{\varepsilon, \delta} \right) ds$$

$$+ 2 \int_0^t \left[ \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) - \sigma_1 \left( \hat{X}_s^{\varepsilon, \delta} \right) \right] Q_1^{\frac{1}{2}} \sigma_1(s), Z_s^{\varepsilon, \delta} \rangle ds$$

$$+ 2 \sqrt{\varepsilon} \int_0^t \left( Z_s^{\varepsilon, \delta}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{\frac{1}{2}} ds + \varepsilon \int_0^t \left\| \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{\frac{1}{2}} \right\|_{HS}^2 ds$$

$$\leq -2 \int_0^t \|Z_s^{\varepsilon, \delta}\|^2 ds + C \int_0^t \|Z_s^{\varepsilon, \delta}\| \left( \|X_s^{\varepsilon, \delta, u^\varepsilon}\| + \|\hat{X}_s^{\varepsilon, \delta}\| \right) \|Z_s^{\varepsilon, \delta}\| ds$$

$$+ C \int_0^t \|X_s^{\varepsilon, \delta, u^\varepsilon} - X_s^{\varepsilon, \delta, u^\varepsilon}\| \cdot |Z_s^{\varepsilon, \delta}| ds + C \int_0^t \|Y_s^{\varepsilon, \delta, u^\varepsilon} - \hat{Y}_s^{\varepsilon, \delta}\| \cdot |Z_s^{\varepsilon, \delta}| ds + C \int_0^t |u^\varepsilon(s)| \cdot |Z_s^{\varepsilon, \delta}|^2 ds$$

$$+ 2 \sqrt{\varepsilon} \int_0^t \left( Z_s^{\varepsilon, \delta}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{\frac{1}{2}} ds + \varepsilon \int_0^t \left\| \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{\frac{1}{2}} \right\|_{HS}^2 ds.$$

By Young’s inequality,

$$|Z_t^{\varepsilon, \delta}|^2 + \int_0^t \|Z_s^{\varepsilon, \delta}\|^2 ds \leq C \int_0^t \|Z_s^{\varepsilon, \delta}\|^2 \left( 1 + |u^\varepsilon(s)|^2 + \|X_s^{\varepsilon, \delta, u^\varepsilon}\|^2 + \|\hat{X}_s^{\varepsilon, \delta}\|^2 \right) ds$$

$$+ C \int_0^t \|X_s^{\varepsilon, \delta, u^\varepsilon} - X_s^{\varepsilon, \delta, u^\varepsilon}\|^2 ds + C \int_0^t \|Y_s^{\varepsilon, \delta, u^\varepsilon} - \hat{Y}_s^{\varepsilon, \delta}\|^2 ds$$

$$+ 2 \sqrt{\varepsilon} \int_0^t \left( Z_s^{\varepsilon, \delta}, \sigma_1 \left( X_s^{\varepsilon, \delta, u^\varepsilon} \right) Q_1^{\frac{1}{2}} ds + \varepsilon C \int_0^t \left( 1 + \|X_s^{\varepsilon, \delta, u^\varepsilon}\|^2 \right) ds.$$

For any $\varepsilon, R > 0$, we define a stopping time

$$\tau^\varepsilon_R := \inf \left\{ t > 0 \mid |X_t^{\varepsilon, \delta, u^\varepsilon}| + \int_0^t \|X_s^{\varepsilon, \delta, u^\varepsilon}\|^2 ds + \int_0^t \|\hat{X}_s^{\varepsilon, \delta}\|^2 ds > R \right\}.$$ 

(4.24)
By Gronwall’s inequality,

\[
\sup_{t \in [0,T]} \left| Z_{t}^{\varepsilon, \delta} \right|^2 + \int_{0}^{T} \left\| Z_{s}^{\varepsilon, \delta} \right\|^2 ds \\
\leq C \int_{0}^{T} \left\| Y_{s}^{\varepsilon, \delta, u_{s}} - Y_{s}^{\delta, \delta} \right\|^2 ds + C \int_{0}^{T} \left( X_{s}^{\varepsilon, \delta, u_{s}} - X_{s}^{\varepsilon, \delta, s(\Delta)} \right)^2 ds + \varepsilon C \int_{0}^{T} \left( 1 + \left| X_{s}^{\varepsilon, \delta, u_{s}} \right|^2 \right) ds \\
+ 2\sqrt{\varepsilon} \sup_{t \in [0,T]} \left\langle Z_{t}^{\varepsilon, \delta}, \sigma_1 \left( X_{s}^{\varepsilon, \delta, u_{s}} \right) Q^{1/2}_{1} dW_{s} \right\rangle e^{C \varepsilon T}.
\]

Note that \( \tau_{\varepsilon}^{R} \leq \tau_{\delta}^{R} \), then by Lemmas 4.2 and 4.4, and Burkholder-Davis-Gundy’s inequality, it follows that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| Z_{t}^{\varepsilon, \delta} \right|^2 \right] + \mathbb{E} \int_{0}^{T} \left\| Z_{s}^{\varepsilon, \delta} \right\|^2 ds \leq C_{R,T} \left( 1 + |x|^2 + |y|^2 \right) \left( \Delta^{1/2} + \frac{\sqrt{\varepsilon}}{\sqrt{\delta}} + \sqrt{\varepsilon} \right). \tag{4.25}
\]

For any \( r > 0 \), by the definition of stopping time \( \tau_{\varepsilon}^{R} \) in (4.24), we have

\[
\mathbb{P} \left( \sup_{t \in [0,T]} \left| Z_{t}^{\varepsilon, \delta} \right|^2 + \int_{0}^{T} \left\| Z_{s}^{\varepsilon, \delta} \right\|^2 ds \geq r \right) \\
\leq \mathbb{P} \left( T > \tau_{\varepsilon}^{R} \right) + \mathbb{P} \left( \sup_{t \in [0,T]} \left| Z_{t}^{\varepsilon, \delta} \right|^2 + \int_{0}^{T} \left\| Z_{s}^{\varepsilon, \delta} \right\|^2 ds \geq r, T \leq \tau_{\varepsilon}^{R} \right) \\
\leq \mathbb{P} \left( \sup_{t \in [0,T]} \left| X_{t}^{\varepsilon, \delta, u_{s}} \right| + \int_{0}^{T} \left\| X_{s}^{\varepsilon, \delta, u_{s}} \right\|^2 ds + \int_{0}^{T} \left\| \hat{X}_{s}^{\varepsilon, \delta} \right\|^2 ds \geq R \right) \\
+ \mathbb{P} \left( \sup_{t \in [0,T]} \left| Z_{t}^{\varepsilon, \delta} \right|^2 + \int_{0}^{T} \left\| Z_{s}^{\varepsilon, \delta} \right\|^2 ds \geq r \right).
\]

By Lemma 4.1, we can choose and fix \( R \) large enough to make the first term on the right hand side of the above inequality small enough, and for fixed \( R \) and (4.25), the second term can also be small enough by choosing \( \Delta = \delta^{1/2} \) and small \( \varepsilon \). Thus, we proved \( \sup_{t \in T} \left| Z_{t}^{\varepsilon, \delta} \right|^2 + \int_{0}^{T} \left\| Z_{s}^{\varepsilon, \delta} \right\|^2 ds \) converges to 0 in probability. The proof is complete. \( \square \)

**Proposition 4.7.** For any fixed \( N \in \mathbb{N} \), let \( \{u^\varepsilon\}_{\varepsilon > 0} \in \mathcal{A}_{N} \) such that that \( u^\varepsilon \) converges to \( u \) in distribution, as \( \varepsilon \to 0 \). It holds that

\[
\hat{X}^{\varepsilon, \delta} - \bar{X}^{u} \text{ converges to } 0 \text{ in distribution, in } C([0,T]; \mathbb{H}) \text{ as } \varepsilon \to 0, \text{ where } \bar{X}^{u} \text{ is the solution to skeleton equation (2.4).}
\]

**Proof.** By the Skorokhod representation theorem, we may assume that \( u^\varepsilon \to u \) in \( L^2([0,T]; \mathbb{H}) \) almost surely in the weak topology. The proof is divided into three steps.

**Step 1. (Splitting into three terms):** Let \( \tilde{Z}^{\varepsilon} := \hat{X}^{\varepsilon, \delta} - \bar{X}^{u} \) and set \( \tilde{X}^{\varepsilon} := \tilde{Z}^{\varepsilon} - L^{\varepsilon} - N^{\varepsilon} \), where

\[
L_{t}^{\varepsilon} := \int_{0}^{t} e^{(t-s)A} \left[ f \left( X_{s}^{\varepsilon, \delta, u_{s}}, Y_{s}^{\varepsilon, \delta} \right) - \hat{f} \left( \bar{X}^{u}_{s} \right) \right] ds,
\]
and

\[
N_{t}^{\varepsilon} := \int_{0}^{t} e^{(t-s)A} \sigma_{1} \left( \bar{X}^{u}_{s} \right) Q^{1/2}_{1} \left[ u_{s}^{\varepsilon} - u_{s} \right] ds.
\]
Then it is easy to see \( \bar{\Lambda}_t^\varepsilon \) satisfies the following equation

\[
\frac{d\bar{\Lambda}_t^\varepsilon}{dt} = A\bar{\Lambda}_t^\varepsilon + [B(\hat{X}_t^\varepsilon,\delta) - B(\hat{X}_t^u)] + [\sigma_1(\hat{X}_t^\varepsilon,\delta) - \sigma_1(\hat{X}_t^u)] \sigma_1^1/2 u_1^\varepsilon(t), \quad \bar{\Lambda}_0^\varepsilon = 0.
\]

By chain’s rule, we have

\[
|\bar{\Lambda}_t^\varepsilon|^2 = -2 \int_0^t \|\bar{\Lambda}_s^\varepsilon\|^2 \, ds + 2 \int_0^t \langle B(\hat{X}_s^\varepsilon,\delta) - B(\hat{X}_s^u), \bar{\Lambda}_s^\varepsilon \rangle \, ds
+ 2 \int_0^t \left[ \sigma_1(\hat{X}_s^\varepsilon,\delta) - \sigma_1(\hat{X}_s^u) \right] \sigma_1^1/2 u_1^\varepsilon(s), \bar{\Lambda}_s^\varepsilon \rangle \, ds
\leq -2 \int_0^t \|\bar{\Lambda}_s^\varepsilon\|^2 \, ds + 2 \int_0^t \|B(\hat{X}_s^\varepsilon,\delta) - B(\hat{X}_s^u)\|_{-1} \cdot \|\bar{\Lambda}_s^\varepsilon\| \, ds + C \int_0^t |\bar{Z}_s^\varepsilon| \cdot |u_1^\varepsilon(s)| \cdot |\bar{\Lambda}_s^\varepsilon| \, ds.
\]

Then by Young’s and Poincaré’s inequalities, we have

\[
|\bar{\Lambda}_t^\varepsilon|^2 \leq -2 \int_0^t \|\bar{\Lambda}_s^\varepsilon\|^2 \, ds + C \int_0^t \|B(\hat{X}_s^\varepsilon,\delta) - B(\hat{X}_s^u)\|^2 \, ds + \int_0^t \|\bar{\Lambda}_s^\varepsilon\|^2 \, ds + C \int_0^t |\bar{Z}_s^\varepsilon|^2 \cdot |u_1^\varepsilon(s)|^2 \, ds
\leq - \int_0^t \|\bar{\Lambda}_s^\varepsilon\|^2 \, ds + C \int_0^t \|\bar{Z}_s^\varepsilon\|^2 \left( \|\hat{X}_s^\varepsilon,\delta\|^2 + \|\hat{X}_s^u\|^2 + |u_1^\varepsilon(s)|^2 \right) \, ds.
\]

Then we obtain

\[
\sup_{t \in [0,T \wedge \bar{\tau}_R^\varepsilon]} \|\bar{\Lambda}_t^\varepsilon\|^2 + \int_0^{T \wedge \bar{\tau}_R^\varepsilon} \|\bar{\Lambda}_s^\varepsilon\|^2 \, ds \leq C \int_0^{T \wedge \bar{\tau}_R^\varepsilon} \|\bar{Z}_s^\varepsilon\|^2 \left( \|\hat{X}_s^\varepsilon,\delta\|^2 + \|\hat{X}_s^u\|^2 + |u_1^\varepsilon(s)|^2 \right) \, ds. \tag{4.26}
\]

**Step 2. (The estimate on \( N_\varepsilon^\varepsilon \)):** For term \( N_\varepsilon^\varepsilon \), we shall prove that it converges to 0 in \( C([0,T], \mathbb{H}) \) almost surely, for which we firstly prove its tightness, and then its convergence. For any \( \theta \in (0,1) \), by Lemma 5.4, we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| N_\varepsilon(t) \|_\theta^2 \right] \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| e^{(t-s)A} \sigma_1(\hat{X}_s^u) \right\|_\theta^2 \right]^2 \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t (t-s)^{-\theta/2} |\sigma_1(\hat{X}_s^u)| \sigma_1^1/2 (u_1^\varepsilon - u_t) \, ds \right]^2
\leq C_T \mathbb{E} \left[ \left( 1 + \sup_{t \in [0,T]} \|\hat{X}_s^u\|^2 \right) \cdot \int_0^T |u_1^\varepsilon - u_t|^2 \, dt \right]
\leq C_{N,T} (1 + |x|^2 + |y|^2),
\]

where \( C_{N,T} \) is independent of \( \varepsilon \).
For any $0 \leq s \leq t \leq T$, by Lemma 5.4, we have
\[
\mathbb{E} \left[ |N_t^\varepsilon - N_s^\varepsilon|^2 \right] = \mathbb{E} \left[ \int_0^t e^{(t-r)A} \sigma_1 \left( \tilde{X}^\varepsilon_r \right) Q_1^{1/2} (u^\varepsilon_r - u_r) \, dr - \int_s^t e^{(s-r)A} \sigma_1 \left( \tilde{X}^\varepsilon_r \right) Q_1^{1/2} (u^\varepsilon_r - u_r) \, dr \right]^2 \leq 2\mathbb{E} \left[ \int_s^t e^{(t-r)A} \sigma_1 \left( \tilde{X}^\varepsilon_r \right) Q_1^{1/2} (u^\varepsilon_r - u_r) \, dr \right]^2 + 2\mathbb{E} \left[ \int_0^s \left( e^{(t-r)A} - e^{(s-r)A} \right) \sigma_1 \left( \tilde{X}^\varepsilon_r \right) Q_1^{1/2} (u^\varepsilon_r - u_r) \, dr \right]^2 \leq C\mathbb{E} \left[ \left( 1 + \sup_{r \in [0,T]} |\tilde{X}^\varepsilon_r|^2 \right) \cdot \int_0^T |u^\varepsilon_r - u_r|^2 \, dr \right] t - s \leq C_T (1 + |x|^2 + |y|^2) |t - s|^2.
\]

Applying an Arzela-Ascoli’s argument, we can show that $\{N_t^\varepsilon \}_{\varepsilon \in (0,1]}$ is tight in $C([0,T]; \mathbb{H})$. Thus, there exist a subsequence $\{N_{t_n}^\varepsilon \}_{n \geq 1}$ being the Cauchy sequence, whose limit is denoted by $N_0$. By chain rule, we know that
\[
|N_{t_n}^\varepsilon|^2 + 2 \int_0^{t_n} \left\lVert N_{s_n}^\varepsilon \right\rVert^2 \, ds = 2 \int_0^{t_n} \left\langle N_{s_n}^\varepsilon, \sigma_1 \left( \tilde{X}^\varepsilon_{s_n} \right) Q_1^{1/2} (u^\varepsilon_{s_n} - u_{s_n}) \right\rangle \, ds \leq 2 \int_0^{t_n} \left\langle N_{s_n}^\varepsilon - N_0^0, \sigma_1 \left( \tilde{X}^\varepsilon_{s_n} \right) Q_1^{1/2} (u^\varepsilon_{s_n} - u_{s_n}) \right\rangle \, ds + \int_0^{t_n} \left\langle N_0^0, \sigma_1 \left( \tilde{X}^\varepsilon_{s_n} \right) Q_1^{1/2} (u^\varepsilon_{s_n} - u_{s_n}) \right\rangle \, ds \leq 2 \sup_{s \in [0,t_n]} |N_{s_n}^\varepsilon - N_0^0| \cdot \int_0^{t_n} \left( 1 + |\tilde{X}^\varepsilon_{s_n}| \right) \cdot |u^\varepsilon_{s_n} - u_{s_n}| \, ds + 2 \int_0^{t_n} \left\langle Q_1^{1/2} \sigma_1^* \left( \tilde{X}^\varepsilon_{s_n} \right) |N_0^0|, u^\varepsilon_{s_n} - u_{s_n} \right\rangle \, ds \rightarrow 0, \text{ a.s.,}
\]

where $\sigma_1^*$ is the transpose of $\sigma_1$ and we have used the facts of $N_{t_n}^\varepsilon \rightarrow N_0^0$ in $C([0,T], \mathbb{H})$, $u_{t_n}^\varepsilon \rightarrow u$ in $\mathbb{S}_N$ and $Q_1^{1/2} \sigma_1^*(\tilde{X}^\varepsilon_{s_n})N_0^0$ belongs to $L^2([0,T]; \mathbb{H})$. By the uniqueness of the limit, we know that $N^\varepsilon \rightarrow 0$ in $C([0,T]; \mathbb{H})$ almost surely.

**Step 3. (The estimate on $L_t^\varepsilon$):** For term $L_t^\varepsilon$,
\[
L_t^\varepsilon = \int_0^t e^{(t-s)A} \left\lbrack f \left( X^\varepsilon_{s(D)}^{\varepsilon,\delta, u^\varepsilon}, \tilde{X}^\varepsilon_{s(D)} \right) - \tilde{f} \left( X^\varepsilon_{s(D)}^{\varepsilon,\delta, u^\varepsilon} \right) \right\rbrack \, ds = I_1^\varepsilon(t) + I_2^\varepsilon(t) + I_3^\varepsilon(t).
\]

By Lipschitz property of $\tilde{f}$, we have
\[
\sup_{t \in [0,T]} |I_2^\varepsilon(t)|^2 \leq C \int_0^T \left\lVert f \left( X^\varepsilon_{s(D)}^{\varepsilon,\delta, u^\varepsilon} \right) - \tilde{f} \left( X^\varepsilon_{s(D)}^{\varepsilon,\delta, u^\varepsilon} \right) \right\rVert^2 \, ds \leq C \int_0^T \left\lVert X^\varepsilon_{s(D)}^{\varepsilon,\delta, u^\varepsilon} - \tilde{X}^\varepsilon_{s(D)}^{\varepsilon,\delta} \right\rVert^2 \, ds,
\]

where $X^\varepsilon_{s(D)}^{\varepsilon,\delta, u^\varepsilon}$ is the solution of the SDE with drift $\varepsilon f(u^\varepsilon, \cdot)$ and diffusion $\varepsilon^2 A$.
and
\[
\sup_{t \in [0, T^{\tau_R^\varepsilon}]} |I_3^\varepsilon(t)|^2 - C \int_0^{T^{\tau_R^\varepsilon}} \left| f \left( \tilde{X}_s^{\varepsilon, \delta} \right) - f \left( \tilde{X}_s \right) \right|^2 ds \leq C \int_0^{T^{\tau_R^\varepsilon}} \left| \tilde{Z}_s^\varepsilon \right|^2 ds.
\]

Then it follows that
\[
\sup_{t \in [0, T^{\tau_R^\varepsilon}]} |L_t^\varepsilon|^2 \leq C \sup_{t \in [0, T^{\tau_R^\varepsilon}]} |I_1^\varepsilon(t)|^2 + C \int_0^{T^{\tau_R^\varepsilon}} \left| X_{s,\delta}^{\varepsilon, u^\varepsilon} - \tilde{X}_{s,\delta}^{\varepsilon} \right|^2 ds + C \int_0^{T^{\tau_R^\varepsilon}} \left| \tilde{Z}_s^{\varepsilon} \right|^2 ds.
\]

By (4.26) and (4.27), we obtain
\[
\sup_{t \in [0, T^{\tau_R^\varepsilon}]} \left| \tilde{Z}_t^\varepsilon \right|^2 \leq C \int_0^{T^{\tau_R^\varepsilon}} \left| \tilde{Z}_s^\varepsilon \right|^2 \left( 1 + \left\| \tilde{X}_s^{\varepsilon, \delta} \right\|^2 + \left\| \tilde{X}_s^{u^\varepsilon} \right\|^2 + |u^\varepsilon(s)|^2 \right) ds + C \sup_{t \in [0, T]} |I_1^\varepsilon(t)|^2 + C \sup_{t \in [0, T]} |N_1^\varepsilon(t)|^2 + C \int_0^{T^{\tau_R^\varepsilon}} \left| X_{s,\delta}^{\varepsilon, u^\varepsilon} - \tilde{X}_{s,\delta}^{\varepsilon} \right|^2 ds.
\]

By (3.2) and the definition \( \tau_R^\varepsilon \), Gronwall’s inequality implies that
\[
\sup_{t \in [0, T^{\tau_R^\varepsilon}]} \left| \tilde{Z}_t^\varepsilon \right|^2 \leq \left[ \sup_{t \in [0, T]} |I_1^\varepsilon(t)|^2 + \sup_{t \in [0, T]} |N_1^\varepsilon(t)|^2 + \int_0^{T^{\tau_R^\varepsilon}} \left| X_{s,\delta}^{\varepsilon, u^\varepsilon} - \tilde{X}_{s,\delta}^{\varepsilon} \right|^2 ds \right] e^{C R, N, T}. (4.28)
\]

Next, we estimate \( I_1^\varepsilon(t) \). Let \( n_t := \left[ \frac{t}{\Delta} \right] \). Denote
\[
I_1^\varepsilon(t) = J_1^\varepsilon(t) + J_2^\varepsilon(t) + J_3^\varepsilon(t),
\]
where
\[
J_1^\varepsilon(t) := \sum_{k=0}^{n_t-1} \int_{k\Delta}^{(k+1)\Delta} e^{(t-s)A} \left[ f \left( X_{k\Delta}^{\varepsilon, \delta, u^\varepsilon} , \tilde{Y}_s^{\varepsilon, \delta} \right) - f \left( X_{k\Delta}^{\varepsilon, \delta, u^\varepsilon} \right) \right] ds,
\]
\[
J_2^\varepsilon(t) := \sum_{k=0}^{n_t-1} \int_{k\Delta}^{(k+1)\Delta} e^{(t-s)A} \left[ f \left( X_{k\Delta}^{\varepsilon, \delta, u^\varepsilon} \right) \right] ds,
\]
\[
J_3^\varepsilon(t) := \int_{n_t\Delta}^{t} e^{(t-s)A} \left[ f \left( X_{n_t\Delta}^{\varepsilon, \delta, u^\varepsilon} , \tilde{Y}_s^{\varepsilon, \delta} \right) - f \left( X_{n_t\Delta}^{\varepsilon, \delta, u^\varepsilon} \right) \right] ds.
\]

For \( J_2^\varepsilon(t) \), noticing that \( \tau_R^\varepsilon \leq \tau_R^\varepsilon \), by Lemma 4.2, we have
\[
\mathbb{E} \left( \sup_{t \in [0, T^{\tau_R^\varepsilon}]} |J_2^\varepsilon(t)|^2 \right) \leq C \mathbb{E} \int_0^{T^{\tau_R^\varepsilon}} \left| X_{s(\Delta)}^{\varepsilon, \delta, u^\varepsilon} - X_{s(\Delta)}^{\varepsilon, \delta, u^\varepsilon} \right|^2 ds \leq C R, T \left( 1 + |x|^2 + |y|^2 \right) \Delta^{1/2}. (4.29)
\]

For the term \( J_3^\varepsilon(t) \), by (4.2) and (4.20), we have
\[
\mathbb{E} \left( \sup_{t \in [0, T^{\tau_R^\varepsilon}]} |J_3^\varepsilon(t)|^2 \right) \leq C \Delta \int_{n_t\Delta}^{t} \mathbb{E} \left( 1 + \left| X_{n_t\Delta}^{\varepsilon, \delta, u^\varepsilon} \right|^2 + \left| \tilde{Y}_s^{\varepsilon, \delta} \right|^2 + \left| X_{s(\Delta)}^{\varepsilon, \delta, u^\varepsilon} \right|^2 \right) ds \leq C T \left( 1 + |x|^2 + |y|^2 \right) \Delta^2. (4.30)
\]
For the term $J^x_t(t)$, by the construction of $\tilde{Y}_t^{\varepsilon, \delta}$, we obtain that, for any $k \in \mathbb{N}$, and $s \in [0, \Delta)$,

$$
\begin{align*}
\tilde{Y}_{s+k\Delta}^{\varepsilon, \delta} &= \tilde{Y}_{k\Delta}^{\varepsilon, \delta} + \frac{1}{\delta} \int_{k\Delta}^{k\Delta+s} A \tilde{Y}_{r}^{\varepsilon, \delta} dr + \frac{1}{\delta} \int_{k\Delta}^{k\Delta+s} g \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r}^{\varepsilon, \delta} \right) dr \\
&\quad + \frac{1}{\sqrt{\delta}} \int_{k\Delta}^{k\Delta+s} \sigma_2 \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r}^{\varepsilon, \delta} \right) Q_2^{1/2} dW_r \\
&= \tilde{Y}_{k\Delta}^{\varepsilon, \delta} + \frac{1}{\delta} \int_0^{s} A \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta} dr + \frac{1}{\delta} \int_0^{s} g \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta} \right) dr \\
&\quad + \frac{1}{\sqrt{\delta}} \int_0^{s} \sigma_2 \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta} \right) Q_2^{1/2} d\tilde{W}_r,
\end{align*}
$$

(4.31)

where $\tilde{W}_t := W_{t+k\Delta} - W_{k\Delta}$ is the shift version of $W_t$. Recall that $\tilde{W}_t$ is a standard cylindrical Wiener process independent of $(X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{k\Delta}^{\varepsilon, \delta})$. Denote by $\tilde{W}_t = \delta^{1/2} \tilde{W}_t$. We construct a process $Y_{t}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{k\Delta}^{\varepsilon, \delta}}$ by means of $Y_{t}^{x,y} \big|_{(x,y) = (X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{k\Delta}^{\varepsilon, \delta})}$, where $Y_{t}^{x,y}$ is the solution to Eq. (3.1). Specifically, that is

$$
\begin{align*}
Y_{t}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{k\Delta}^{\varepsilon, \delta}} &= \tilde{Y}_{k\Delta}^{\varepsilon, \delta} + \int_0^{t} A Y_{r+k\Delta}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta}} dr + \int_0^{t} g \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, Y_{r+k\Delta}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta}} \right) dr \\
&\quad + \int_0^{t} \sigma_2 \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, Y_{r+k\Delta}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta}} \right) Q_2^{1/2} d\tilde{W}_r \\
&= \tilde{Y}_{k\Delta}^{\varepsilon, \delta} + \frac{1}{\delta} \int_0^{s} A Y_{r+k\Delta}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta}} dr + \frac{1}{\delta} \int_0^{s} g \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, Y_{r+k\Delta}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta}} \right) dr \\
&\quad + \frac{1}{\sqrt{\delta}} \int_0^{s} \sigma_2 \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, Y_{r+k\Delta}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{r+k\Delta}^{\varepsilon, \delta}} \right) Q_2^{1/2} d\tilde{W}_r.
\end{align*}
$$

(4.32)

The uniqueness of the solutions to Eq. (4.31) and Eq. (4.32) implies that the distribution of $(X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{s+k\Delta}^{\varepsilon, \delta})$ coincides with the distribution of $(X_{k\Delta}^{\varepsilon, \delta, u^s}, Y_{t_0}^{X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{k\Delta}^{\varepsilon, \delta}})$.

Next, we try to control $|J^x_t(t)|$:

$$
\begin{align*}
&\mathbb{E} \left[ \sup_{t \in [0, T]} |J^x_t(t)|^2 \right] \\
&= \mathbb{E} \sup_{t \in [0, T]} \left\{ \sum_{k=0}^{n_t-1} e^{(t-(k+1)\Delta)A} \int_{k\Delta}^{(k+1)\Delta} e^{((k+1)\Delta-s)A} \left[ f \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{s}^{\varepsilon, \delta} \right) - \tilde{f} \left( X_{k\Delta}^{\varepsilon, \delta, u^s} \right) \right] ds \right\}^2 \\
&\leq \mathbb{E} \sup_{t \in [0, T]} \left\{ \sum_{k=0}^{n_t-1} \left| \int_{k\Delta}^{(k+1)\Delta} e^{((k+1)\Delta-s)A} \left[ f \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{s}^{\varepsilon, \delta} \right) - \tilde{f} \left( X_{k\Delta}^{\varepsilon, \delta, u^s} \right) \right] ds \right|^2 \right\} \\
&\leq \left[ \frac{T}{\Delta} \right] \sum_{k=0}^{n_t-1} \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} e^{((k+1)\Delta-s)A} \left[ f \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{s}^{\varepsilon, \delta} \right) - \tilde{f} \left( X_{k\Delta}^{\varepsilon, \delta, u^s} \right) \right] ds \right]^2 \\
&\leq \frac{C_T}{\Delta^2} \max_{0 \leq k \leq \left\lfloor \frac{T}{\Delta} \right\rfloor - 1} \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} e^{((k+1)\Delta-s)A} \left[ f \left( X_{k\Delta}^{\varepsilon, \delta, u^s}, \tilde{Y}_{s}^{\varepsilon, \delta} \right) - \tilde{f} \left( X_{k\Delta}^{\varepsilon, \delta, u^s} \right) \right] ds \right]^2.
\end{align*}
$$
Then by changing variable, we get
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |J_1^ε(t)|^2 \right] \leq C_T \frac{δ^2}{Δ^2} \max_{0 ≤ k ≤ \left[ \frac{T}{Δ} \right] - 1} \mathbb{E} \left| \int_0^δ e^{(Δ-s)A} \left[ f \left( X_k^{ε,δ, u^ε} , \hat{Y}_{sδ+kΔ}^{ε,δ} \right) - \hat{f} \left( X_k^{ε,δ, u^ε} \right) \right] ds \right|^2
\]
\[
= 2C_T \frac{δ^2}{Δ^2} \max_{0 ≤ k ≤ \left[ \frac{T}{Δ} \right] - 1} \int_0^δ \int_r^δ Ψ_k(s, r) ds dr,
\]
where
\[
Ψ_k(s, r) = \mathbb{E} \left\langle e^{(Δ-s)A} \left[ f \left( X_k^{ε,δ, u^ε} , \hat{Y}_{sδ+kΔ}^{ε,δ} \right) - \hat{f} \left( X_k^{ε,δ, u^ε} \right) \right], e^{(Δ-r)A} \left[ f \left( X_k^{ε,δ, u^ε} , \hat{Y}_{rδ+kΔ}^{ε,δ} \right) - \hat{f} \left( X_k^{ε,δ, u^ε} \right) \right] \right\rangle.
\]
Now, let’s estimate Ψ_k(s, r). Define \( \tilde{F}_s := \sigma\{Y_{u^s}^{x,y}, u ≤ s\} \). Then for \( s > r \), by the Markov property and Proposition 3.1,
\[
Ψ_k(s, r) = \mathbb{E} \left\langle e^{(Δ-s)A} \left[ f \left( x, Y_s^{x,y} \right) - \tilde{f}(x) \right], e^{(Δ-r)A} \left[ f \left( x, Y_r^{x,y} \right) - \tilde{f}(x) \right] \right\rangle |_{(x,y) = (X_k^{ε,δ, u^ε}, \hat{Y}_{sδ+kΔ}^{ε,δ})}
\]
\[
= \mathbb{E} \left\langle e^{(Δ-s)A} \mathbb{E} \left[ f \left( x, Y_s^{x,y} \right) - \tilde{f}(x) \right] | \tilde{F}_s \right\rangle, e^{(Δ-r)A} \left[ f \left( x, Y_r^{x,y} \right) - \tilde{f}(x) \right] \right\rangle |_{(x,y) = (X_k^{ε,δ, u^ε}, \hat{Y}_{sδ+kΔ}^{ε,δ})}
\]
\[
\leq C_T \mathbb{E} \left\langle 1 + |x|^2 + |Y_r^{x,y}|^2 \right\rangle e^{-(s-r)η} |_{(x,y) = (X_k^{ε,δ, u^ε}, \hat{Y}_{sδ+kΔ}^{ε,δ})}
\]
\[
\leq C_T \left( 1 + |X_k^{ε,δ, u^ε}|^2 + |\hat{Y}_{sδ+kΔ}^{ε,δ}|^2 \right) e^{-(s-r)η}
\]
where the last two inequalities are deduced by (4.2) and (4.20). Then we get
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |J_1^ε(t)|^2 \right] \leq C_T \frac{δ^2}{Δ^2} \left( 1 + |x|^2 + |y|^2 \right) \int_0^δ \int_r^δ e^{-\frac{1}{2}(s-r)η} ds dr
\]
\[
\leq C_T \frac{δ}{Δ} \left( 1 + |x|^2 + |y|^2 \right). \quad (4.33)
\]
Thus, combining (4.29), (4.30) and (4.33), we get
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |J_1^ε(t)|^2 \right] \leq C_{R,T} \left( 1 + |x|^2 + |y|^2 \right) \left( Δ^{1/2} + \frac{δ}{Δ} \right). \quad (4.34)
\]
According to the estimates (4.28) and (4.34), we obtain
\[
\mathbb{E}\left[ \sup_{t \in [0,T^\wedge \hat{\tau}_R]} |\hat{X}^{\varepsilon,\delta}_t - \hat{X}_t^u|^2 \right] \\
\leq C_{R,N,T} \left( 1 + |x|^2 + |y|^2 \right) \left( \Delta^{1/2} + \frac{\delta}{\Delta} \right) + C \mathbb{E} \int_0^{T^\wedge \hat{\tau}_R} |X^{\varepsilon,\delta,u\varepsilon}_s - \hat{X}^{\varepsilon,\delta}_s|^2 \, ds.
\]
By (4.25) and choosing \( \Delta = \delta^{1/2} \), we have
\[
\lim_{\varepsilon \to 0} \mathbb{E}\left[ \sup_{t \in [0,T^\wedge \hat{\tau}_R]} |\hat{X}^{\varepsilon,\delta}_t - \hat{X}_t^u|^2 \right] = 0. \tag{4.35}
\]
For any \( r > 0 \), by the definition of stopping time \( \hat{\tau}_R \) and (4.35)
\[
\mathbb{P}\left( \sup_{t \in [0,T]} |\hat{X}^{\varepsilon,\delta}_t - \hat{X}_t^u| \geq r \right) \\
\leq \mathbb{P}\left( T > \hat{\tau}^{\varepsilon}_R \right) + \mathbb{P}\left( \sup_{t \in [0,T]} |\hat{X}^{\varepsilon,\delta}_t - \hat{X}_t^u| \geq r, T \leq \hat{\tau}^{\varepsilon}_R \right) \\
\leq \mathbb{P}\left( \sup_{t \in [0,T]} |X^{\varepsilon,\delta,u\varepsilon}_t| + \int_0^T \|X^{\varepsilon,\delta,u\varepsilon}_s\|^2 \, ds + \int_0^T \|\hat{X}^{\varepsilon,\delta}_s\|^2 \, ds > R \right) \\
+ \mathbb{P}\left( \sup_{t \in [0,T^\wedge \hat{\tau}_R]} |\hat{X}^{\varepsilon,\delta}_t - \hat{X}_t^u| \geq r \right).
\]
By Lemmas 4.1 and 4.3, we can choose an fixed \( R \) large enough to make the first term on the right hand side of the above inequality small enough, and for fixed \( R \) and (4.35), the second term can also be small enough by choosing small \( \varepsilon \). Thus, we proved \( \sup_{t \leq T} |\hat{X}^{\varepsilon,\delta}_t - \hat{X}_t^u| \to 0 \) in probability. The proof is complete. \( \square \)

5. Appendix

5.1. A weak convergence criteria for LDP. In this part, we will recall the general criteria for a LDP given in [3]. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with an increasing family \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) of the sub-\(\sigma\)-fields of \(\mathcal{F}\) satisfying the usual conditions. Let \(\mathcal{E}\) be a Polish space with the Borel \(\sigma\)-field \(\mathcal{B}(\mathcal{E})\).

**Definition 5.1. (Rate function)** A function \( I : \mathcal{E} \rightarrow [0, \infty] \) is called a rate function on \(\mathcal{E}\), if for each \( M < \infty \), the level set \( \{x \in \mathcal{E} : I(x) \leq M\} \) is a compact subset of \(\mathcal{E}\).

**Definition 5.2. (LDP)** Let \( I \) be a rate function on \(\mathcal{E}\). A family \(\{X^\varepsilon\}\) of \(\mathcal{E}\)-valued random elements is said to satisfy the LDP on \(\mathcal{E}\) with rate function \( I \) if the following two conditions hold.

(a) (Upper bound) For each closed subset \( F \) of \(\mathcal{E}\),
\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x \in F} I(x).
\]

(b) (Lower bound) For each open subset \( G \) of \(\mathcal{E}\),
\[
\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq -\inf_{x \in G} I(x).
\]

Let \(\mathcal{A}\) denote the class of \(\{\mathcal{F}_t\}\)-predictable processes \( u \) belonging to \(\mathbb{S}\) a.s.. Let \(\mathbb{S}_N = \{ u \in L^2([0,T], \mathbb{H}); \int_0^T |u(s)|^2 \, ds \leq N \} \). The set \(\mathbb{S}_N\) endowed with the weak topology is a Polish space. Define \(\mathcal{A}_N = \{ \phi \in \mathcal{A}; u(\omega) \in \mathbb{S}_N, \mathbb{P}\text{-a.s.} \}\).

Recall the following result from Budhiraja and Dupuis [3].
Corollary 5.7. Let \( \{\Gamma^\varepsilon\}_{\varepsilon > 0} \) be a family of measurable mappings from \( C([0,T],\mathbb{H}) \) into \( \mathcal{E} \). Suppose that there exists a measurable map \( \Gamma^0 : C([0,T],\mathbb{H}) \rightarrow \mathcal{E} \) such that

(a) for every \( N < +\infty \) and any family \( \{u^\varepsilon ; \varepsilon > 0\} \subset \mathcal{A}_N \) satisfying that \( u^\varepsilon \) converges in distribution as \( S_N \)-valued random elements to \( u \) as \( \varepsilon \rightarrow 0 \), \( \Gamma^\varepsilon \left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^t u^\varepsilon(s)ds\right) \) converges in distribution to \( \Gamma^0(f_0u(s)ds) \) as \( \varepsilon \rightarrow 0 \);

(b) for every \( N < +\infty \), the set \( \{\Gamma^0(f_0u(s)ds); u \in S_N\} \) is a compact subset of \( \mathcal{E} \).

Then the family \( \{\Gamma^\varepsilon(W)\}_{\varepsilon > 0} \) satisfies a LDP in \( \mathcal{E} \) with the rate function \( I \) given by

\[
I(g) := \inf_{u \in \mathbb{S}, g = \Gamma^0(f_0u(s)ds)} \left\{ \frac{1}{2} \int_0^T |u(s)|^2 ds \right\}, \quad g \in \mathcal{E},
\]

with the convention \( \inf \emptyset = \infty \).

5.2. Some estimates about the Burgers equation. We recall some properties of the semigroup \( \{e^{tA}\}_{t \geq 0} \) and the nonlinear operators \( b \) and \( B \), for example see [2], [10].

Lemma 5.4. For the semigroup \( \{e^{tA}\}_{t \geq 0} \), we have:

1. for any \( \theta \leq \gamma, x \in \mathbb{H} \),

\[ \|e^{tA}x\|_\gamma \leq Ct^{-\frac{\theta}{\gamma}}\|x\|_\theta; \]

2. for any \( \sigma \in [0,1] \) there exists \( C_\sigma > 0 \) such that for any \( 0 < s < t \) and \( x \in \mathbb{H} \),

\[ |e^{tA}x - e^{sA}x| \leq C_\sigma \frac{(t-s)^\sigma}{s^{\sigma}}|x|; \]

3. for any \( \sigma \in [0,2] \) there exists \( C_\sigma > 0 \) such that for any \( 0 \leq s < t \) and \( x \in \mathbb{H}_\sigma \),

\[ |e^{tA}x - e^{sA}x| \leq C_\sigma (t-s)^{\sigma/2}\|x\|_\sigma. \]

Lemma 5.5. For any \( x, y \in \mathbb{V} \),

\[ b(x, x, y) = -b(x, y, x), \quad b(x, x, x) = 0. \]

Lemma 5.6. Suppose \( \alpha_i \geq 0 \) \( (i = 1, 2, 3) \) satisfies one of the following conditions:

1. \( \alpha_i \neq \frac{1}{2} (i = 1, 2, 3), \alpha_1 + \alpha_2 + \alpha_3 \geq \frac{1}{2}; \)
2. \( \alpha_i = \frac{1}{2} \) for some \( i \), \( \alpha_1 + \alpha_2 + \alpha_3 > \frac{1}{2} \),

then \( b \) is continuous from \( \mathbb{H}_{\alpha_1} \times \mathbb{H}_{\alpha_2+1} \times \mathbb{H}_{\alpha_3} \) to \( \mathbb{R} \), i.e.

\[ |b(x, y, z)| \leq C\|x\|_{\alpha_1} \cdot \|y\|_{\alpha_2+1} \cdot \|z\|_{\alpha_3}. \]

The following inequalities can be derived by the above lemma.

Corollary 5.7. For any \( x \in \mathbb{V} \), we have:

1. \( |B(x)| \leq C\|x\|^2; \)
2. \( \|B(x)\|_{-1} \leq C|x| \cdot \|x\|. \)

Lemma 5.8. For any \( x, y \in \mathbb{V} \), we have:

1. \( |B(x) - B(y)| \leq C\|x - y\|(\|x\| + \|y\|); \)
2. \( \|B(x) - B(y)\|_{-1} \leq C|x - y| (\|x\| + \|y\|). \)
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