A new lower bound for eternal vertex cover number

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Abstract. We obtain a new lower bound for the eternal vertex cover number of an arbitrary graph \( G \), in terms of the cardinality of a vertex cover of minimum size in \( G \) containing all its cut vertices. The consequences of the lower bound includes a quadratic time algorithm for computing the eternal vertex cover number of chordal graphs.

1 Introduction

Eternal vertex cover number of a graph is the minimum number of guards required to successfully keep defending attacks on a graph, in a certain multi-round attack-defense game [1]. The rules to play the game with \( k \) guards on a graph \( G \) are the following. Initially, the defender places the \( k \) guards on a subset of vertices of \( G \). The positions of the guards defines an initial configuration. In each round of the game, the attacker chooses an edge \( e \) of \( G \) to attack. In response to the attack, the defender is free to move each of guard from its current position to an adjacent vertex or retain it in its current position. All guards are assumed to move in parallel, at the same time. The constraint to be satisfied is that at least one guard should move from an endpoint of \( e \) to the other. If the defender is able to successfully move the guards satisfying this constraint, we say that the attack in the current round is successfully defended. The resultant positions of the guards define the configuration from where the next round of the attack-defense game continues. If the defender can keep on successfully defending any sequence of attacks, we say that the defender has a defense strategy on this graph, with \( k \) guards. Eternal vertex cover number of a graph \( G \), denoted by evc(\( G \)) is the minimum integer \( k \) such that the defender has a defense strategy on \( G \), with \( k \) guards. When this game is played with \( k \)-guards, each configuration encountered in the game is equivalent to some function \( f \) from \( V \) to \( \{0, 1, 2, \ldots, k\} \) such that \( \sum_{v \in V} f(v) = k \) (where, for each \( v \in V \), \( f(v) \) will be the number of guards on \( v \)). A set of such configurations \( C \), such that the defender can start with any configuration in \( C \) as the initial configuration and keep moving between configurations in \( C \) for defending the attacks, is called an eternal vertex cover class of \( G \) and each configuration in \( C \) is an eternal vertex cover configuration. If \( C \) is an eternal vertex cover class of \( G \) such that the number of guards in the configurations in \( C \) is equal to evc(\( G \)), then \( C \) is a minimum eternal vertex cover class of \( G \). There are two popular versions of the game: the former in which in
any configuration, at most one guard is allowed on a vertex and the latter in which this restriction is not there. Since the main structural result in this paper is a lower bound for eternal vertex cover number, we will be assuming the first version of the game. It can be easily verified that our proofs work the same way in the other model of the game as well.

From the description of the game, it is clear that, in any configuration, if at least one of the endpoints of an edge is not occupied, the defender will not be able to successfully defend an attack on that edge. Therefore, \( \text{mvc}(G) \leq \text{evc}(G) \). This is the only general lower bound known for the parameter, so far in literature.

In this work, we prove that the size of a minimum sized vertex cover of \( G \) that contains all cut vertices of \( G \) is also a lower bound for \( \text{evc}(G) \). This improved lower bound has many algorithmic consequences, including a quadratic time algorithm for computing the eternal vertex cover number of chordal graphs and a PTAS for computing the eternal vertex cover number of internally triangulated planar graphs. These results generalize the results presented in [2].

2 A new lower bound

**Definition 1.** Let \( G \) be a graph and \( X \subseteq V(G) \). The smallest integer \( k \), such that \( G \) has a vertex cover \( S \) of cardinality \( k \) with \( X \subseteq S \), is denoted by \( \text{mvc}_X(G) \).

**Definition 2 (x-components and x-extensions).** Let \( x \) be a cut vertex in a graph \( G \) and \( H \) be a component of \( G \setminus x \). Let \( G' \) be the induced subgraph of \( G \) on the vertex set \( V(H) \cup \{x\} \). Then \( G' \) is called an \( x \)-component of \( G \) and \( G \) is called an \( x \)-extension of \( G' \).

To simplify the expressions that appear later, we introduce the following notation. For any graph \( G \) and any set \( X \), the notation \( X(G) \) will be used to denote the set \( X \cap V(G) \).

**Definition 3.** Let \( X \) be the set of cut vertices of a graph \( G \) and let \( x \in X \). The set of \( x \)-components of \( G \) will be denoted as \( C_x(G) \). If \( B \) is any block of \( G \), then the set of \( B \)-components of \( G \) is defined as \( C_B(G) = \{G_i : G_i \in C_x(G) \text{ for some } x \in X(B) \text{ and } G_i \text{ edge disjoint with } B\} \).

**Definition 4 (EVC-Cut-Property).** Let \( G' \) be a graph and let \( X' \) be the set of cut vertices of \( G' \). The graph \( G' \) is said to have the EVC-cut-property if for every graph \( G \) that is an \( x \)-extension of \( G' \) for some \( x \in V(G') \), it is true that in each eternal vertex cover configuration of \( G \), at least \( \text{mvc}_{X' \cup \{x\}}(G') \) guards are present on the vertices of \( G' \), out of which at least \( \text{mvc}_{X' \cup \{x\}}(G') - 1 \) guards are present on \( V(G') \setminus \{x\} \).

**Note 1.** For a graph \( G' \) to satisfy the EVC-Cut-Property, it is not necessary that the vertex \( x \) is occupied by a guard in every eternal vertex cover configuration of an \( x \)-extension \( G \) of \( G' \). All the \( \text{mvc}_{X' \cup \{x\}}(G') \) (or more) guards could be on vertices other than \( x \).
Note 2. Definition \[ \text{Definition 4} \] gives some lower bounds on the number of guards and not on the number of vertices with guards. Note that, if more than one guard is allowed on a vertex, then these two numbers could be different.

The following two lemmas are easy to obtain, using a straightforward counting argument.

**Lemma 1.** Let \( G \) be a graph and \( X \) be the set of cut vertices of \( G \). For any \( x \in X \),

\[
\operatorname{mvc}_{X \cup \{x\}}(G) = \operatorname{mvc}_X(G) = 1 + \sum_{G_i \in \mathcal{C}_X(G)} \left[ \operatorname{mvc}_{X(G_i)}(G_i) - 1 \right].
\]

**Lemma 2.** Let \( G \) be a graph and \( X \) be the set of cut vertices of \( G \). If \( B \) is a block of \( G \) and \( v \) is any vertex of \( B \) such that \( v \notin X(B) \), then

\[
\operatorname{mvc}_{X \cup \{v\}}(G) = \operatorname{mvc}_{(X \setminus \{v\}) \cup \{v\}}(B) + \sum_{G_i \in \mathcal{C}_B(G)} \left[ \operatorname{mvc}_{X(G_i)}(G_i) - 1 \right].
\]

**Lemma 3.** Every graph satisfies EVC-cut-property.

**Proof.** The proof is by induction on the number of blocks of the graph. First consider a graph \( G' \) with a single block. Let \( x \) be any vertex of \( G' \) and \( G \) be an \( x \)-extension of \( G' \). Let \( C \) be an eternal vertex cover configuration of \( G \) and let \( S \) be the set of vertices of \( G \) on which guards are present in \( C \). Since \( C \) is an eternal vertex cover configuration of \( G \), \( S \) must be a vertex cover of \( G \) and \( S \cap V(G') \) must be a vertex cover of \( G' \). Therefore, \( |S \cap V(G')| \geq \operatorname{mvc}(G') \). If \( |S \cap V(G')| = \operatorname{mvc}(G') \), then there are at least \( \operatorname{mvc}_{\{x\}}(G') \) guards on \( V(G') \) and at least \( \operatorname{mvc}_{\{x\}}(G') - 1 \) guards on \( V(G') \setminus \{x\} \), as we need to prove. Also, it is easy to see that \( \operatorname{mvc}_{\{x\}}(G') \leq \operatorname{mvc}(G') + 1 \). Therefore, we are left with the case when \( \operatorname{mvc}(G') = |S \cap V(G')| < \operatorname{mvc}_{\{x\}}(G') = \operatorname{mvc}(G') + 1 \). This implies that \( x \notin S \). Thus, in the remaining case to be handled, the number of vertices on which guards are present is exactly \( \operatorname{mvc}(G') \) and there is no guard on \( x \).

From this point, let us focus on the number of guards on \( V(G') \) and not just the number of vertices that are occupied. If there are more than \( \operatorname{mvc}(G') \) guards in \( V(G') \), then the conditions we need to prove are satisfied for the configuration \( C \). In the remaining case, we have exactly \( |S \cap V(G')| = \operatorname{mvc}(G') \) guards in \( V(G') \), with \( x \notin S \). In this case, we will derive a contradiction.

Consider an attack on an edge \( xy \) incident at \( x \), where \( v \in V(G') \). Let \( \tilde{C} \) be the new configuration, after defending this attack and \( \tilde{S} \) be the set of vertices on which guards are present in \( \tilde{C} \). In the transition from \( C \) to \( \tilde{C} \), a guard must have moved from \( v \) to \( x \). Also, \( x \) being a cut vertex, no guard can move from \( V(G') \setminus V(G') \) to \( V(G') \setminus \{x\} \). Therefore, \( |S \cap V(G')| = |S \cap V(G')| = \operatorname{mvc}(G') \). But, this is a contradiction because \( \tilde{S} \cap V(G') \) is a minimum vertex cover of \( G' \) containing \( x \), but we have \( \operatorname{mvc}(G') < \operatorname{mvc}_{\{x\}}(G') \).

Thus, the lemma holds for all graphs with only one block. Now, as induction hypothesis, assume that the lemma holds for any graph \( G' \) with at most \( k \) blocks. We need to show that the lemma holds for any graph with \( k + 1 \) blocks.
Let $G'$ be an arbitrary graph with $k+1$ blocks and let $x$ be an arbitrary vertex of $G'$. Let $X'$ be the set of cut vertices of $G'$ and let $G$ be an arbitrary $x$-extension of $G'$. Let $C$ be an arbitrary eternal vertex cover configuration of $G$ and let $S$ be the set of vertices on which guards are present in $C$. Let $l = \text{mvc}_{X \cup \{x\}}(G')$. We need to show that there are at least $l$ guards on $V(G')$ in $C$ and at least $l - 1$ guards on $V(G') \setminus \{x\}$. Let $t$ be the number of guards on $V(G')$ in $C$. We split our proof into two cases based on whether $x$ is a cut vertex in $G$ or not.

**Case 1.** $x$ is a cut vertex of $G'$:

In this case, by our induction hypothesis, for each $x$-component $G_i$ of $G'$, at least $\text{mvc}_{X(G_i)}(G_i)$ guards are on $V(G_i)$ in the configuration $C$. There are two possible sub-cases.

(a) If $x$ is not occupied by a guard in $C$, then by induction hypothesis,

\[ t \geq \sum_{G_i \in \mathcal{C}_x(G')} \text{mvc}_{X(G_i)}(G_i). \]

Since $\mathcal{C}_x(G')$ is non-empty, by Lemma 1, it follows that $t \geq \text{mvc}_{X \cup \{x\}}(G') = l$. The number of guards on $V(G') \setminus \{x\}$ is $t \geq l$.

(b) If $x$ is occupied by a guard in $C$, still, in order to satisfy the induction hypothesis for all $x$-components of $G$, the number of guards on $V(G') \setminus \{x\}$ must be at least $\sum_{G_i \in \mathcal{C}_x(G')} (\text{mvc}_{X(G_i)}(G_i) - 1)$. Therefore, by Lemma 1, it follows that the number of guards on $V(G') \setminus \{x\}$ is at least $l - 1$ and $t \geq l$.

**Case 2.** $x$ is not a cut vertex of $G'$:

Let $B$ be the block of $G'$ that contains $x$. By Lemma 2, we have:

\[ l = \text{mvc}_{X(B \cup \{x\})}(B) + \sum_{G_i \in \mathcal{C}_B(G)} (\text{mvc}_{X(G_i)}(G_i) - 1). \]  \hfill (1)

Before proceeding with the proof, we establish the following claim.

**Claim.** Suppose $C'$ is an eternal vertex cover configuration of $G$. Then the number of guards on $V(G') \setminus \{x\}$ in configuration $C'$ is at least $l - 1$.

**Proof.** To count the number of guards on $V(G') \setminus \{x\}$, we count the total number of guards on the $B$-components of $G'$ and the number of guards on the remaining vertices separately and add them up.

- First, we will count the total number of guards on the $B$-components of $G'$. For each $B$-component $G_i$ of $G'$, let $k_i = \text{mvc}_{X(G_i)}(G_i)$. For each cut vertex $v \in X(B)$, let $C_v$ denote the family of $B$-components of $G'$ that intersect at the cut vertex $v$ and let $n_v$ denote $|C_v|$. Consider a $B$-component $G_i$ of $G'$. By our induction hypothesis, the number of guards on $V(G_i)$ is at least $k_i$ in $C'$. Moreover, since $G_i$ is connected to $B$ by a single cut vertex, from the induction hypothesis it follows that the number of guards on $V(G_i) \setminus B$ is at least $k_i - 1$. Note that, for each cut vertex $v \in X(B)$, the total number of guards on $\bigcup_{G_i \in C_v} V(G_i)$ must be at least $1 + \sum_{G_i \in C_v} (k_i - 1)$, to satisfy the above requirement. By summing this over all cut vertices
in $X(B)$, the total number of guards on $\bigcup_{G_i \in C(B)} V(G_i)$ must be at least $|X(B)| + \sum_{G_i \in C(B)} (\text{mvc}_{X(G_i)}(G_i) - 1)$.

Now, we will count the number of guards on the remaining vertices. To cover the edges inside the block $B$ that are not incident at any vertex in $X(B)$, at least $\text{mvc}(B \setminus X(B))$ vertices of $B \setminus X(B)$ are to be occupied in $C'$. If $x$ is occupied in $C'$, then at least $\text{mvc}_{x}(B \setminus X(B))$ vertices of $B \setminus X(B)$ are occupied in $C'$. In either of these cases, the number of guards on $(V(B) \setminus X(B)) \setminus \{x\}$ is at least $\text{mvc}_{x}(B \setminus X(B)) - 1$.

Therefore, the total number of guards on $V(G') \setminus \{x\}$ is at least $\text{mvc}_{x}(B \setminus X(B)) - 1 + |X(B)| + \sum_{G_i \in C(B)} (\text{mvc}_{X(G_i)}(G_i) - 1)$. Since $\text{mvc}_{x}(B \setminus X(B)) + |X(B)| = \text{mvc}_{X(B \setminus \{x\})}(B)$, we can conclude that the number of guards on $V(G') \setminus \{x\}$ is equal to $\text{mvc}_{X(B \setminus \{x\})}(B) - 1 + \sum_{G_i \in C(B)} (\text{mvc}_{X(G_i)}(G_i) - 1)$. Comparing this expression with Equation 1, we can see that the number of guards on $V(G') \setminus \{x\}$ is at least $l - 1$.

Now, we continue with the proof of Lemma 3. There are two possible sub-cases.

(a) If $x$ is occupied by a guard in $C$, then by Claim 2 it follows that the number of guards on $V(G') \setminus \{x\}$ is at least $l - 1$ and the number of guards on $V(G')$ is at least $l$, as we require.

(b) If $x$ is not occupied in $C$, then by Claim 2, $t \geq l - 1$. If $t \geq l$, we are done. If $t = l - 1$, then we will derive a contradiction. Consider an attack on an edge $xu$ such that $u \in V(B)$. While defending this attack, a guard must move from $u$ to $x$. Let $\tilde{C}$ be the new configuration in $G$ and let $\tilde{S}$ be the set of vertices on which guards are present in $\tilde{C}$. Note that no guards from $V(G) \setminus V(G')$ can move to any vertex of $V(G') \setminus \{x\}$ in this transition from $C$ to $\tilde{C}$, because $x$ is a cut vertex in $G$. Therefore, in $\tilde{C}$, the total number of guards on $V(G') \setminus \{x\}$ is less than $l - 1$, contradicting Claim 2. Therefore, $t = l$ and the lemma holds for $G'$.

Thus, by induction, the lemma holds for every graph. \hfill \Box

Remark 1. Note that, the above lemma holds for both the models of the eternal vertex cover; the first model in which the number of guards permitted on a vertex in any configuration is limited to one and the second model, where this restriction is not there. However, it is possible that, in the second model, the number of vertices on which guards are present could be smaller than $\text{mvc}_{X \cup \{x\}}(G)$. An example for this is shown in Figure 1.

Theorem 1. For any connected graph $G$, $\text{evc}(G) \geq \text{mvc}_{X}(G)$, where $X$ is the set of cut vertices of $G$.

Proof. Let $C$ be an eternal vertex cover configuration of $G$ and $S$ be the set of all vertices of $G$ containing guards in $C$. Suppose $\text{evc}(G) < \text{mvc}_{X}(G)$. Then,
Fig. 1. Any vertex cover of the graph in (a) that contains vertex $v_7$ and both the cut vertices must be of size at least 5. The graph in (b) is a $v_7$-extension of the graph in (a). Positions of guards in an eternal vertex cover configuration of the graph in (b) are indicated using gray squares. This is a valid configuration. Note that, only four vertices of the graph in (a) are occupied in the configuration shown in (b).

there exists a vertex $x \in X$ such that $x \notin S$. Since every graph satisfies EVC-cut-property by Lemma 3 for each $x$-component $G_i$ of $G$, exactly $\text{mvc}_{X(G_i)}(G_i)$ guards are present on $V(G_i) \setminus \{x\}$. Therefore, the total number of guards is at least $\sum_{G_i \in C(G)} \text{mvc}_{X(G_i)}(G_i)$. Since there are at least two $x$-components, by comparing this expression with the RHS of the equation in Lemma 1 we can see that the total number of guards is more than $\text{mvc}_X(G)$. This contradicts our initial assumption. \(\square\)

**Observation 1** Let $G$ be a connected graph and let $X$ be the set of cut vertices of $G$. If $\text{evc}(G) = \text{mvc}_X(G)$, then in every minimum eternal vertex cover configuration of $G$, there are guards on each vertex of $X$.

**Proof.** For contradiction, assume that there exists a minimum eternal vertex cover configuration $C$ of $G$ with a cut vertex $x$ unoccupied. Rest of the proof is exactly the same as in the proof of Theorem 1. \(\square\)

For any graph $G$ and $S \subseteq V(G)$, let $\text{evc}_S(G)$ denote the minimum number $k$ such that $G$ has an eternal vertex cover class $C$ with $k$ guards in which all vertices of $S$ are occupied in every configuration of $C$. By Observation 1 we have the following generalization of Corollary 2 of [3].

**Theorem 2.** Let $G$ be a connected graph with at least two vertices and let $X$ be the set of cut vertices of $G$. Suppose that every vertex cover $S$ of $G$ of size $\text{mvc}_X(G)$, such that $X \subseteq S$, induces a connected subgraph in $G$. If for every vertex $v \in V(G) \setminus X$, $\text{mvc}_{X \cup \{v\}}(G) = \text{mvc}_X(G)$ then, $\text{evc}(G) = \text{evc}_X(G) = \text{mvc}_X(G)$. Otherwise, $\text{evc}(G) = \text{evc}_X(G) = \text{mvc}_X(G) + 1$.

**Proof.**
A graph $G$ is locally connected if for every vertex $v$ of $G$, its open neighborhood $N_G(v)$ induces a connected subgraph in $G$. If every block of a graph $G$ is locally connected, then every vertex cover of $G$ that contains all its cut vertices is connected. Hence, we have:

**Corollary 1.** Let $G$ be a connected graph with at least two vertices, such that each block of $G$ is locally connected and let $X$ be the set of cut vertices of $G$. Then, $\text{mvc}_v(G) \leq \text{evc}(G) \leq \text{mvc}_v(G) + 1$. Further, $\text{evc}(G) = \text{mvc}_v(G)$ if and only if for every vertex $v \in V(G) \setminus X$, $\text{mvc}_{X \cup \{v\}}(G) = \text{mvc}_v(G)$.

The following remark is a generalization of Remark 3 of [2].

**Remark 2.** Let $G$ be a connected graph with at least two vertices and let $X$ be the set of cut vertices of $G$. Suppose that for every vertex cover $S$ of $G$ of size $\text{mvc}_v(G)$ such that $X \subseteq S$, the induced subgraph $G[S]$ is connected. Then, $\text{evc}(G) = \min\{k : \forall v \in V(G), \ G \text{ has a vertex cover } S_v \text{ of size } k \text{ such that } X \cup \{v\} \subseteq S_v\}$.

## 3 Algorithmic implications

In this section, we provide generalizations of some algorithmic results in [2], using results obtained in the previous section.

We define $F$ to be the graph class that consists of connected graphs such that for each $G_i \in F$, it is true that for every vertex cover $S$ of $G_i$ such that $X_i \subseteq S$ and $|S| = \text{mvc}_{X_i}(G_i)$, where $X_i$ is the set of all cut vertices of $G_i$, the subgraph $G_i[S]$ is connected.

The following result is a generalization of Corollary 3 of [2].

**Observation 2** Given a graph $G \in F$ and an integer $k$, deciding whether $\text{evc}(G) \leq k$ is in $\text{NP}$.

**Proof.** Consider any $G \in F$ with at least two vertices and let $X$ be the set of cut vertices of $G$.

By Remark 2, $\text{evc}(G) = \min\{k : \forall v \in V(G), \ G \text{ has a vertex cover } S_v \text{ of size } k \text{ such that } X \cup \{v\} \subseteq S_v\}$. To check if $\text{evc}(G) \leq k$, the polynomial time verifiable certificate consists of at most $|V|$ vertex covers of size at most $k$ such that for each vertex $v \in V$, there exists a vertex cover in the certificate containing all vertices of $X \cup \{v\}$. \qed
### 3.1 Hereditary graph classes

The following theorem is obtained by generalizing Corollary 6 of [2], by applying Theorem 2.

**Theorem 3.** Let $C$ be a hereditary graph class such that each biconnected graph in $C$ is locally connected. If the vertex cover number of any graph in $C$ can be computed in $O(f(n))$ time, then the eternal vertex cover number of any graph $G \in C$ can be computed in $O(n.f(n))$ time.

*Proof.* Let $G$ be a graph in $C$. Since each block of $G$ is locally connected, by Corollary 1, \[ mvc_X(G) \leq evc(G) \leq mvc_X(G) + 1. \] Further, by Corollary 1, to check whether \[ evc(G) = mvc_X(G), \] it is enough to decide if for every vertex \[ v \in V \setminus X, mvc_{X \cup \{v\}}(G) = mvc_X(G). \] Minimum vertex cover computation can be done for graphs of $C$ in $O(f(n))$ time, for a vertex \[ v, \] checking whether \[ mvc_{X \cup \{v\}}(G) = mvc_X(G), \] takes only $O(f(n))$ time. Therefore, checking whether \[ evc(G) = mvc_X(G) \] can be done in $O(n.f(n))$ time. \qed

### 3.2 Chordal graphs

The following theorem is a special case of Theorem 3, using the fact that minimum vertex cover computation can be done for chordal graphs in $O(m + n)$ time [4], where $m$ is the number of edges and $n$ is the number of vertices of the input graph. This result is a generalization of a result for biconnected chordal graphs in [2].

**Theorem 4.** Let $G$ be a chordal graph and $X$ be the set of cut vertices of $G$. Then, \[ mvc_X(G) \leq evc(G) \leq mvc_X(G) + 1 \] and the value of $evc(G)$ can be determined in $O(n^2 + mn)$ time, where $m$ is the number of edges and $n$ is the number of vertices of the input graph.

### 3.3 Internally triangulated planar graphs

The following lemma is a generalization of a result in [2] for biconnected internally triangulated planar graphs.

**Lemma 4.** Given an internally triangulated planar graph $G$ and an integer $k$, deciding whether $evc(G) \leq k$ is in NP.

*Proof.* Since each block of an internally triangulated planar graph $G$ is locally connected, every vertex cover $S$ of $G$ that contains all its cut vertices induces a connected subgraph. Therefore, by Observation 2 deciding whether $evc(G) \leq k$ is in NP. \qed

The existence of a polynomial time approximation scheme for computing the eternal vertex cover number of biconnected internally triangulated planar graphs, given in [2], is generalized by the following result.
Lemma 5. There exists a polynomial time approximation scheme for computing the eternal vertex cover number of internally triangulated planar graphs.

Proof. Let $G$ be an internally triangulated planar graph. Let $X$ be the set of cut vertices of $G$. It is possible to compute $X$ in linear time, using a well-known depth first search based method. By Remark 2, $evc(G) = \max\{mvc_{X\cup\{v\}}(G) : v \in V(G)\}$. It is easy to see that for a vertex $v \in V(G)$, $mvc_{X\cup\{v\}}(G) = |X| + 1 + mvc(G \setminus (X \cup \{v\}))$. Using the PTAS designed by Baker et al. [5] for computing the vertex cover number of planar graphs, given any $\epsilon > 0$, it is possible to approximate $mvc(G \setminus (X \cup \{v\}))$ within a $1 + \epsilon$ factor, in polynomial time. Therefore, there exists a polynomial time approximation scheme for computing $evc(G)$. \qed

References

1. Klostermeyer, W., Mynhardt, C.: Edge protection in graphs. Australasian Journal of Combinatorics 45 (2009) 235 – 250
2. Babu, J., Chandran, L.S., Francis, M., Prabhakaran, V., Rajendraprasad, D., Warrier, J.N.: On graphs whose eternal vertex cover number and vertex cover number coincide (2019)
3. Babu, J., Chandran, L.S., Francis, M., Prabhakaran, V., Rajendraprasad, D., Warrier, J.N.: On graphs with minimal eternal vertex cover number. In: Conference on Algorithms and Discrete Applied Mathematics (CALDAM), Springer (2019) 263–273
4. Rose, D.J., Tarjan, R.E., Lueker, G.S.: Algorithmic aspects of vertex elimination on graphs. SIAM Journal on computing 5(2) (1976) 266–283
5. Baker, B.S.: Approximation algorithms for NP-complete problems on planar graphs. J. ACM 41(1) (January 1994) 153–180