QUANTUM SYMMETRIC SPACES AND RELATED $q$-ORTHOGONAL POLYNOMIALS

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ABSTRACT

A class of quantum analogues of compact symmetric spaces of classical type is introduced by means of constant solutions to the reflection equations. Their zonal spherical functions are discussed in connection with $q$-orthogonal polynomials.

The following two naive questions are the main motives of this paper:

(1) How can one define the analogue of homogeneous spaces $G/K$ in the framework of quantum groups?

(2) What do their spherical functions look like?

To be more specific, we will introduce a class of quantum analogues of compact symmetric spaces $G/K$ of classical type and study their zonal spherical functions associated with finite dimensional representations. It is natural to expect that they could provide a good class of $q$-orthogonal polynomials in many variables. Unfortunately, we have not yet reached an abstract definition of quantum symmetric spaces. In this paper we will propose instead a practical method to construct examples of quantum symmetric spaces of classical type, by means of constant solutions to the reflection equations. This method works well in fact and it turns out in many examples that the zonal spherical functions are expressed by the Macdonald polynomials associated with root systems or by Koornwinder's Askey-Wilson polynomials for $BC_\ell$.

1. Recalls on compact quantum groups.

Let $G$ be one of the compact classical groups $SU(N)$, $SO(N)$, $Sp(N)$ and let $\mathfrak{g}$ be the complexification of the Lie algebra of $G$. For such a compact group $G$, we already have a (more or less) standard definition of the quantum group $\mathcal{G}_q$ (Woronowicz\textsuperscript{1}, Reshetikhin-Takhtajan-Faddeev\textsuperscript{2}, see also Hayashi\textsuperscript{3}, Dijkhuizen-Koornwinder\textsuperscript{4}). One can define in fact two Hopf $\ast$-algebras (over $\mathbb{C}$) $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{A}_q(G)$ which are $q$-deformations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ and of the algebra of regular functions $\mathcal{A}(G)$ on $G$, respectively. For $\mathcal{U}_q(\mathfrak{g})$, we take the quantized universal enveloping algebra of Drinfeld and Jimbo endowed with the $\ast$-operation corresponding to the compact real form. Hereafter we always assume that $q$ is a real number with $0 < q < 1$. The algebra $\mathcal{A}_q(G)$ is the subalgebra of the algebraic dual $\mathcal{U}_q(\mathfrak{g})'$ of $\mathcal{U}_q(\mathfrak{g})$ generated by
the matrix elements of the vector representation. Hence we have the natural pairing of Hopf $*$-algebras

\[(\ ,\ ) : U_q(g) \times A_q(G) \to \mathbb{C}. \tag{1.1} \]

We say that a Hopf algebra $A$ is a Hopf $*$-algebra if it has a $*$-operation (involutive, conjugate linear antiautomorphism of $R$-algebra) such that the coproduct $\Delta : A \otimes_R A \to A$ and the counit $\varepsilon : A \to R$ are $*$-homomorphisms. When $U$ and $A$ are two Hopf $*$-algebras, a $C$-bilinear mapping $(\ ,\ ) : U \times A \to C$ is called a pairing of Hopf $*$-algebras if the following conditions are satisfied:

\begin{enumerate}
\item $(ab, \varphi) = (a \otimes b, \Delta(\varphi))$, \quad $(1, \varphi) = \varepsilon(\varphi)$,
\item $(a, \varphi \psi) = (\Delta(a), \varphi \otimes \psi)$, \quad $(a, 1) = \varepsilon(a)$,
\item $(S(a), \varphi) = (a, S(\varphi))$,
\item $(a^\ast, \varphi) = (a, S(\varphi)^\ast)$,
\end{enumerate}

for any $a, b \in U$ and $\varphi, \psi \in A$.

From the pairing between $U_q(g)$ and $A_q(G)$ we obtain a natural structure of $U_q(g)$-bimodule on $A_q(G)$, corresponding to the right and the left regular representations of $G$. The left and the right actions of an element $a \in U_q(g)$ on $A_q(G)$ are defined by

\[
a \cdot \varphi = (a, \Delta(\varphi))_1, \quad \varphi \cdot a = (a, \Delta(\varphi))_2 \quad (\varphi \in A_q(G)), \tag{1.2}
\]

where $(a, \cdot)_i$ stands for the contraction with respect to the $i$-th tensor component. It should be noted here that these actions of $U_q(g)$ on $A_q(G)$ are compatible with the multiplication of $A_q(G)$: If $a \in U_q(g)$ and $\Delta(a) = \sum (a_{(1)} \otimes a_{(2)})$, then we have

\[
a \cdot (\varphi \psi) = \sum (a_{(1)}, \varphi) (a_{(2)}, \psi), \quad (\varphi \psi).a = \sum (\varphi, a_{(1)})(\psi, a_{(2)}), \tag{1.3}
\]

for any $\varphi, \psi \in A_q(G)$ and $a.1 = 1.a = \varepsilon(a)1$.

The most fundamental fact about the structure of the $U_q(g)$-bimodule $A_q(G)$ is that $A_q(G)$ has the following irreducible decomposition of Peter-Weyl type:

\[
A_q(G) = \bigoplus_{\lambda \in P^+_G} W(\lambda), \quad W(\lambda) \simeq V(\lambda)^\vee \otimes \mathbb{C} V(\lambda). \tag{1.4}
\]

Here $P^+_G$ denotes the cone of dominant integral weights corresponding to the $G$-rational representations. For each $\lambda \in P^+_G$, $V(\lambda)$ is the finite dimensional irreducible left $U_q(g)$-module (or right $A_q(G)$-comodule) with highest weight $\lambda$, and $W(\lambda)$ is the vector subspace of $A_q(G)$ spanned by the matrix elements of $V(\lambda)$.

Another characterization of $W(\lambda)$ is given by the action of the center $\mathbb{Z}U_q(g)$ of $U_q(g)$. For each $\lambda \in P^+_G$, we denote the central character of $V(\lambda)$ by $\chi_\lambda : \mathbb{Z}U_q(g) \to \mathbb{C}$: $C_{V(\lambda)} = \chi_\lambda(C)\text{id}_{V(\lambda)}$ ($C \in \mathbb{Z}U_q(g)$). Then $W(\lambda)$ is the following simultaneous eigenspace of $\mathbb{Z}U_q(g)$:

\[
W(\lambda) = \{ \varphi \in A_q(G) | C.\varphi = \chi_\lambda(C)\varphi \text{ for all } C \in \mathbb{Z}U_q(g) \}. \tag{1.5}
\]

We denote by $h : A_q(G) \to W(0) = \mathbb{C}$ the projection to the trivial representation in the decomposition (1.4). Then the functional $h$ gives the unique invariant functional with $h(1) = 1$, which we call the normalized Haar functional of the quantum group $G_q$. By this functional, we can define a scalar product on $A_q(G)$:

\[
\langle \varphi | \psi \rangle = h(\varphi^* \psi) \quad (\varphi, \psi \in A_q(G)). \tag{1.6}
\]

It is known that $\langle \cdot | \cdot \rangle$ is in fact a positive definite Hermitian form and that the Peter-Weyl decomposition (1.4) is orthogonal under this scalar product.
2. Quantum analogue of $G/K$ and $K\backslash G/K$.

In this section, we discuss how one can define the quantum analogue of homogeneous spaces $G/K$ and double coset spaces $K\backslash G/K$ in the sense of $q$-deformation of algebras of regular functions. Suppose now a closed subgroup $K$ of $G$ is given and let $\mathfrak{k}$ be the complexification of the Lie algebra of $K$. The question we have to discuss first is:

- **What is the appropriate definition of a quantum subgroup?**

We will take here the infinitesimal approach of quantized universal enveloping algebras rather than the global approach of quantized algebras of functions. Given a pair $(\mathfrak{g}, \mathfrak{k})$ of a Lie algebra $\mathfrak{g}$ and its Lie subalgebra $\mathfrak{k}$, it is natural to ask:

- **What should be the quantum analogue of the pair $(U(\mathfrak{g}), U(\mathfrak{k}))$?**

We have at least two possible versions of quantum analogue of the pair $(U(\mathfrak{g}), U(\mathfrak{k}))$.

(A) **Regular version** (or a quantum subgroup in the *strict* sense).

Suppose we have a Hopf subalgebra $\mathcal{V}$ of $\mathcal{U} = U_q(\mathfrak{g})$ such that $\mathcal{V}$ “tends” to $U(\mathfrak{t})$ as $q \to 1$. In such a case we might write $\mathcal{V} = U_q(\mathfrak{t})$ as well and might expect also that there would be a quotient Hopf algebra, say $A_q(K)$, of $A_q(G)$ representing the quantum subgroup $K_q$. As for quantized universal enveloping algebras of Drinfeld-Jimbo, a class of pairs $(U_q(\mathfrak{g}), U_q(\mathfrak{k}))$ of Hopf algebras arises naturally from the embedding of root systems. This class is, however, not so large as to cover all symmetric pairs. In fact, no natural embedding $U_q(\mathfrak{so}(N)) \to U_q(\mathfrak{sl}(N))$, nor $U_q(\mathfrak{sp}(N)) \to U_q(\mathfrak{sl}(N))$ seems to be known. In relation to this point, non-existence of Hopf algebra homomorphisms $A_q(G) \to A_q(K)$ for pairs $(G, K)$ of classical groups is discussed by Hayashi. In our context, we should probably say that the condition

$$\Delta(V) \subset V \otimes V$$

for a Hopf subalgebra $\mathcal{V} \subset \mathcal{U} = U_q(\mathfrak{g})$ is too restrictive.

(B) **Twisted version** (or a quantum subgroup in the *broader* sense).

We propose to consider subalgebras $\mathcal{V} \subset \mathcal{U} = U_q(\mathfrak{g})$ which are not Hopf subalgebras, as well. Instead of (2.1), we assume that the subalgebra $\mathcal{V}$ satisfies the *coideal property*

$$\Delta(V) \subset V \otimes \mathcal{U} + \mathcal{U} \otimes V.$$

This condition (2.2) is equivalent to saying that $\mathcal{V}$ is generated by a coideal of $\mathcal{U}$. One might call a subalgebra $\mathcal{V}$ of $\mathcal{U}$ satisfying (2.2) a *coideal subalgebra*. Note that, in this context of subalgebras, it is not natural to impose the condition on the counit. Subalgebras of this type are already considered by Olshanski in the case of Yangians under the name of *twisted Yangians*. As for $U_q(\mathfrak{g})$, the $q$-deformation of $\mathfrak{so}(N)$ proposed by Gavrilik and Klimyk can be regarded as a subalgebra of $U_q(\mathfrak{sl}(N))$ satisfying (2.2) (see Section 2.4 of Noumi). We remark that, in these examples, the subalgebras in question are actually *one-sided* in the sense

$$\Delta(V) \subset V \otimes \mathcal{U} \quad \text{or} \quad \Delta(V) \subset \mathcal{U} \otimes V.$$

When we can consider a subalgebra $\mathcal{V} \subset \mathcal{U} = U_q(\mathfrak{g})$ satisfying (2.2) as a $q$-deformation of $U(\mathfrak{t})$, we will use the notation $\mathcal{V} = U_q^{tw}(\mathfrak{t})$ to remember that $\mathcal{V}$ is may not be a Hopf subalgebra.

We now suppose that a subalgebra $\mathcal{V} = U_q^{tw}(\mathfrak{t})$ of $\mathcal{U} = U_q(\mathfrak{g})$ satisfying (2.2) is given and define the left ideal $\mathcal{J}$ of $\mathcal{U}$ by

$$\mathcal{J} = \sum_{a \in \mathcal{V}} a \mathcal{U}(a - \varepsilon(a)).$$  

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We now suppose that a subalgebra $\mathcal{V} = U_q^{tw}(\mathfrak{t})$ of $\mathcal{U} = U_q(\mathfrak{g})$ satisfying (2.2) is given and define the left ideal $\mathcal{J}$ of $\mathcal{U}$ by

$$\mathcal{J} = \sum_{a \in \mathcal{V}} a \mathcal{U}(a - \varepsilon(a)).$$
Then we define the quantum analogue of the algebra of right $K$-invariant regular functions on $G$ to be the subspace of all elements in $A_q(G)$ annihilated by the left action of $\mathcal{J}$. Namely, we set

$$A_q(G/K) := \{ \varphi \in A_q(G) \mid \mathcal{J} \varphi = 0 \}. \tag{2.5}$$

The coideal property (2.2) of $V = U_q^w(t)$ guarantees that this subspace $A_q(G/K)$ is actually a subalgebra of $A_q(G)$. Note also that $A_q(G/K)$ is a right $U_q(\mathfrak{g})$-submodule of $A_q(G)$. Assume furthermore that the left ideal $J$ associated with $V$ satisfies the condition

$$S(\mathcal{J})^* \subset J. \tag{2.6}$$

Then one can show that the subalgebra $A_q(G/K)$ of (2.5) becomes a $*$-subalgebra of $A_q(G)$.

We will mainly consider the case when the pair $(U, V)$ is a Gelfand pair in the sense that

$$\dim_{\mathbb{C}} V(\lambda)^J \leq 1 \quad \text{for all} \quad \lambda \in P^+, \tag{2.7}$$

where $V(\lambda)^J = \{ v \in V(\lambda) \mid \mathcal{J} v = 0 \}$. Then the Peter-Weyl decomposition (1.4) implies the following multiplicity free decomposition of $A_q(G/K)$ as a right $U_q(\mathfrak{g})$-module:

$$A_q(G/K) \simeq \bigoplus_{\lambda \in P^+_G \cap J} V(\lambda)^V, \tag{2.8}$$

where $P^+_G \cap J$ is the subset of $P^+_G$ consisting of all $\lambda$ for which $V(\lambda)$ has nonzero $\mathcal{J}$-fixed vectors.

The next object we have to consider is the quantum analogue of the double coset space $K \backslash G/K$. By using the group-like element $q^\rho$ of $U_q(\mathfrak{g})$, corresponding the half sum of positive roots, we modify the $*$-operation of $U_q(\mathfrak{g})$ as $a^* = q^\rho a^* q^{-\rho}$. With this notation, we take the left ideal $\mathcal{J}$ of (2.4) and the right ideal $\overline{\mathcal{J}}$ of $U = U_q(\mathfrak{g})$ and set

$$A_q(K \backslash G/K) := \{ \varphi \in A_q(G) \mid \mathcal{J} \varphi = \varphi, \overline{\mathcal{J}} = 0 \}. \tag{2.9}$$

Under the condition (2.6), $A_q(K \backslash G/K)$ becomes a $*$-subalgebra of $A_q(G)$, which we regard as the quantum analogue of the algebra of $K$-biinvariant regular functions on $G$. In what follows, we also use the notation $\mathcal{H} = A_q(K \backslash G/K)$ for simplicity. Under the condition (2.7) of Gelfand pair, this algebra $\mathcal{H} = A_q(K \backslash G/K)$ is decomposed into one-dimensional subspaces as

$$\mathcal{H} = \bigoplus_{\lambda \in P^+_G \cap J} \mathcal{H}(\lambda), \quad \mathcal{H}(\lambda) = \mathcal{H} \cap W(\lambda). \tag{2.10}$$

From (1.5), we see that (2.10) also gives the decomposition of $\mathcal{H}$ into simultaneous eigenspaces of the action of the center $ZU_q(\mathfrak{g})$. We call a nonzero element $\varphi_\lambda$ of $\mathcal{H}(\lambda)$ ($\lambda \in P^+_G \cap J$) a zonal spherical function on the quantum homogeneous space $(G/K)_q$, associated with the representation $V(\lambda)$. In other words, the zonal spherical function is characterized by the conditions $\mathcal{J} \varphi_\lambda = \varphi_\lambda, \overline{\mathcal{J}} = 0$ and

$$C \varphi_\lambda = \chi_\lambda(C) \varphi_\lambda \quad \text{for all} \quad C \in ZU_q(\mathfrak{g}), \tag{2.11}$$

for $C$.
up to a constant multiple. Note also that we have Schur’s orthogonality relations

$$\langle \varphi_\lambda | \varphi_\mu \rangle = 0 \quad (\lambda \neq \mu)$$

(2.12)

for $\lambda, \mu \in P_{G,J}^+$, under the scalar product (1.6) defined by the Haar functional.

At this stage, our problems can be stated as follows:

1. Find a systematic way to construct a coideal subalgebra $U_q^{tw}(\mathfrak{t})$ of $U_q(\mathfrak{g})$ for a given Gelfand pair $(\mathfrak{g}, \mathfrak{t})$.

2. Describe the structure of subalgebras $A_q(G/K)$ and $A_q(K \setminus G/K)$ of invariants in $A_q(G)$.

In many examples, the subalgebra $\mathcal{H} = A_q(K \setminus G/K)$ turns out to be commutative. If we have a subalgebra $U_q^{tw}(\mathfrak{t})$ of $U_q(\mathfrak{g})$ with the desired properties and if we can describe the structure of $\mathcal{H}$, then the zonal spherical functions $\varphi_\lambda$ ($\lambda \in P_{G,J}^+$) would give rise to a family of $q$-orthogonal polynomials in many variables. Schur’s orthogonality relations (2.12) would then guarantee their orthogonality relations. Furthermore, the action of the center $Z U_q(\mathfrak{g})$ on $\mathcal{H}$ would provide a commuting family of $q$-difference operators for which the $q$-orthogonal polynomials should be simultaneous eigenfunctions. We will see below that this is the case in many examples and that zonal spherical functions on quantum homogeneous spaces generate in fact various $q$-orthogonal polynomials.

There have been a lot of works related to the quantum analogue of symmetric spaces of rank one and their spherical functions. See Vaksman-Soibelman, Masuda et al., Koornwinder, and also Vilenkin-Klimyk, for the interpretation of little $q$-Jacobi polynomials as the matrix elements of unitary representations of $SU_q(2)$. Several extensions of this result are discussed by Noumi-Mimachi, Koornwinder, Koelink in relation to Podles’s quantum spheres and Askey-Wilson polynomials. For spherical functions related to $SU_q(2)$, we refer to the survey papers Koornwinder and Noumi. For spherical functions on higher dimensional quantum spheres, see Noumi-Yamada-Mimachi, Vaksman-Soibelman and Sugitani.

In the rank one case, a typical example of the regular version is the quantum sphere $(SU(2)/S(U(1) \times U(1)))_q$ obtained as the quotient space of $SU_q(2)$ by its diagonal subgroup, whose zonal spherical functions are expressed by little $q$-Legendre polynomials. The first example of the twisted version is $(SU(2)/SO(2))_q$ and its zonal spherical functions are expressed then by continuous $q$-Legendre polynomials. It should be noted that both $(SU(2)/S(U(1) \times U(1)))_q$ and $(SU(2)/SO(2))_q$ are two extreme cases of quantum 2-spheres of Podles. The zonal spherical functions on $(SU(2)/SO(2))_q$ were discussed by Koornwinder for the first time, in which he used the infinitesimal approach of twisted primitive elements in $U_q(\mathfrak{sl}(2))$. This work of Koornwinder can be regarded as the starting point of quantum subgroups in the broader sense that we described above.

From the examples studied so far, it seems natural to suspect that quantum homogeneous spaces of the regular version would provide $q$-orthogonal polynomials with respect to discrete measures of $q$-integral. On the other hand, quantum homogeneous spaces of the twisted version would be related in general to $q$-orthogonal polynomials with respect to measures involving continuous parts. The quantum analogues of symmetric spaces that we discuss below belong to this second category.

3. Quantum symmetric spaces and reflection equations.

Recall first we have the following seven series of compact Riemannian symmetric
spaces $G/K$ of classical type (see Loos\textsuperscript{22}):

AI: $SU(n)/SO(n)$ ($N = n$),

AII: $SU(2n)/Sp(2n)$ ($N = 2n$),

AIII: $U(n)/U(\ell) \times U(n - \ell)$ ($N = n, \ell \leq \lfloor \frac{N}{2} \rfloor$),

BDI: $SO(N)/SO(\ell) \times SO(N - \ell)$ ($N = 2n + 1$ or $2n, \ell \leq \lfloor \frac{N}{2} \rfloor$) \quad (3.1)

CI: $Sp(2n)/U(n)$ ($N = 2n$),

CII: $Sp(2n)/Sp(2\ell) \times Sp(2(n - \ell))$ ($N = 2n, \ell \leq \lfloor \frac{N}{2} \rfloor$),

DIII: $SO(2n)/U(n)$ ($N = 2n$).

In this section, we will propose a method to construct twisted quantized universal enveloping algebra $U_q^{\text{tw}}(\mathfrak{g})$ for $G/K$ of these series of symmetric spaces other than AIII. The procedure of our “quantization” can be described in a unified manner by using constant solutions of the reflection equations. Although we will not treat the case of type AIII in this paper, a similar argument can be carried out as well, by using a different type of reflection equations. We will not consider here the cases of group manifolds $G \times G/\Delta(G)$ either.

We remark that the quantization of symmetric spaces of type AI was discussed by Ueno-Takebayashi\textsuperscript{23} and, on the quantum $SU(3)/SO(3)$, they gave the interpretation of Macdonald polynomials of type $A_2$ as zonal spherical functions. In our previous paper\textsuperscript{8} we discussed the quantization of AI and AII as well as the relation with the Macdonald polynomials of type $A$. The method we use here is essentially the same as that of the paper\textsuperscript{8}

Let $(G, K)$ be the pair of compact Lie groups corresponding to one of the symmetric spaces $G/K$ listed in (3.1). We assume that $G/K$ is not of type AIII. We denote by $V = \mathbb{C}^{\mathbb{N}}$ the underlying vector space of the vector representation of $G$. We will use the following $R$-matrix $R \in \text{End}_\mathbb{C}(V \otimes \mathbb{C} V)$ for the vector representation of $U_q(\mathfrak{g})$:

$$R = q^{-1/N} \left( \sum_{1 \leq i,j \leq N} e_{ii} \otimes e_{jj} q^{\delta_{ij}} + (q - q^{-1}) \sum_{1 \leq i < j \leq N} e_{ij} \otimes e_{ji} \right) \quad (3.2)$$

if $G = SU(N)$, and

$$R = \sum_{1 \leq i,j \leq N} e_{ii} \otimes e_{jj} q^{\delta_{ij} - \delta_{ij}} + (q - q^{-1}) \sum_{1 \leq j < i \leq N} (e_{ij} \otimes e_{ji} - \kappa_i \kappa_j q^{\rho_i - \rho_j} e_{ij} \otimes e_{ji}) \quad (3.3)$$

if $G = SO(N)$ or $G = Sp(N)$, where the $e_{ij}$’s are the matrix units corresponding to the canonical basis of $V$. For $G = SO(N), Sp(N)$, we use the notation $j' = N + 1 - j$ and $q^{\rho_j}$ is the $(j,j)$-component of the diagonal matrix representing $q^{\rho}$ in $U_q(\mathfrak{g})$ on the vector representation $V$. If $G = SO(N)$, then $\kappa_j = 1$ for all $j$, and if $G = Sp(N)$ ($N = 2n$), then $\kappa_j = 1$ or $-1$ according as $j \leq n$ or $j > n$. We also use the $L$-operators $L^+, L^- \in \text{End}_\mathbb{C}(V) \otimes U_q(\mathfrak{g})$ such that

$$(\text{id}_V \otimes \rho_V)(L^\pm) = R^\pm \quad \text{with} \quad R^+ = P R P, \ R^- = R^{-1}, \quad (3.4)$$

where $P \in \text{End}_\mathbb{C}(V \otimes \mathbb{C} V)$ is the flip $u \otimes v \mapsto v \otimes u$. 


The first step of our method is to find an appropriate constant solution to the following reflection equation for \( J \in \text{End}_C(V) \):

\[
R_{12} J_1 R_{12}^t J_2 = J_2 R_{12} J_1 R_{12},
\]

(3.5)

where \( R_{12}^t \) stands for the matrix obtained from \( R = R_{12} \) by transposition in the first tensor component. Reflection equations have been discussed in various contexts related to quantum groups. We mention here only a few references: Cherednik\(^{23}\), Sklyanin\(^{24}\), Olshanski\(^{6}\), Kulish-Sasaki-Schwiebert\(^{25}\), . . .

Suppose that we have an invertible matrix \( J \in \text{End}_C(V) \) satisfying the reflection equation (3.5). Then we define the matrix \( K \in \text{End}_C(V) \otimes_C U_q(\mathfrak{g}) \) by

\[
K = S(\mathcal{L}^+) JS(\mathcal{L}^-)^t.
\]

(3.6)

By using this matrix \( K = (K_{ij})_{ij} \), we define the twisted subalgebra \( U_q^{tw}(\mathfrak{t}) \) of \( U_q(\mathfrak{g}) \) to be the subalgebra generated by the matrix elements \( K_{ij} \) \((1 \leq i, j \leq N)\):

\[
U_q^{tw}(\mathfrak{t}) = \mathbb{C}[K_{ij} \ (1 \leq i, j \leq N)].
\]

(3.7)

It is easily checked that \( U_q^{tw}(\mathfrak{t}) \) has the coideal property (2.2). We remark that the matrix \( K \) also satisfies the reflection equation similar to (3.5). As in Section 2, we denote by \( J \) the left ideal associated with this subalgebra \( U_q^{tw}(\mathfrak{t}) \):

\[
J = \sum_{1 \leq i, j \leq N} U_q(\mathfrak{g})(K_{ij} - \varepsilon(K_{ij})).
\]

(3.8)

We define a tensor \( w_J \in V \otimes_C V \) of degree 2 associated with \( J \) by

\[
w_J = \sum_{1 \leq i, j \leq N} v_i \otimes J_{ij}v_j,
\]

(3.9)

where \( \{v_j\}_j \) is the canonical basis of \( V \). Then the reflection equation (3.5) implies that \( J \cdot w_J = 0 \). In this sense, the twisted subalgebra \( U_q^{tw}(\mathfrak{t}) \) can be understood as the stabilizer of this quadratic tensor \( w_J \).

In the following, we will give a list of constant solutions \( J \) to (3.5) that we take for the construction of \( U_q^{tw}(\mathfrak{t}) \) corresponding to each symmetric space \( G/K \).

**AI:**

\[
J = \sum_{k=1}^{n} e_{kk}a_k \quad J = \sum_{k=1}^{n} (-e_{2k,2k-1}a_{2k-1} + e_{2k-1,2k}a_{2k}) \quad (a_{2k-1} = qa_{2k} \ (1 \leq k \leq n))
\]

**AII:**

\[
J = \sum_{1 \leq j,j' \leq \ell} e_{jj'}a_j + \sum_{\ell < j < \ell'} e_{jj'}q^{-\rho_j} + \sum_{j=1}^{\ell} e_{jj'}(1 - q^{2\rho_j})q^{-\rho_j} \quad (a_1a_1 = \cdots = a_\ell a_\ell = q^{2\rho})
\]

**BDI:**

\[
J = \sum_{1 \leq j,j' \leq \ell} e_{jj'}a_j + \sum_{\ell < j < \ell'} e_{jj'}q^{-\rho_j} + \sum_{j=1}^{\ell} e_{jj'}(1 - q^{2\rho_j})q^{-\rho_j} \quad (a_1a_1 = \cdots = a_\ell a_\ell = q^{2\rho})
\]
\[ J = \sum_{k=1}^{2n} c_{kk} a_k \]  

(1) 

\[ J = \sum_{k=1}^{\ell} (-e_{2k,2k-1} a_{2k-1} + e_{2k-1,2k} a_{2k} - e_{(2k-1)'}(2k)' a_{(2k)'} + e_{(2k)'}(2k-1)' a_{(2k-1)'} + e_{j,j'}(2k)' a_{(2k)'} + e_{j,j'}(2k-1)' a_{(2k-1)'} + \sum_{2l<j\leq n} e_{j,j'} q^{-\rho_j}) \]

\[ (a_{2k-1} = qa_{2k}, a_{(2k)'} = qa_{(2k-1)'} (1 \leq k \leq \ell), \quad a_1 a_1' = \cdots = a_{2\ell} a_{(2\ell)'} = -q^{2(\rho_2-2)} \]

(2) 

\[ J = \sum_{k=1}^{\ell} (-e_{2k,2k-1} a_{2k-1} + e_{2k-1,2k} a_{2k} - e_{(2k-1)'}(2k)' a_{(2k)'} + e_{(2k)'}(2k-1)' a_{(2k-1)'} + e_{j,j'} q^{-\rho_j}) \]

\[ (a_{2k-1} = qa_{2k}, a_{(2k)'} = qa_{(2k-1)'} (1 \leq k \leq \ell), \quad a_1 a_1' = \cdots = a_{n} a_{n'} \]

(3) 

\[ J = \sum_{k=1}^{\ell} (-e_{2k,2k-1} a_{2k-1} + e_{2k-1,2k} a_{2k} - e_{(2k-1)'}(2k)' a_{(2k)'} + e_{(2k)'}(2k-1)' a_{(2k-1)'} + e_{j,j'} q^{-\rho_j}) \]

\[ (a_{2k-1} = qa_{2k}, a_{(2k)'} = qa_{(2k-1)'} (1 \leq k \leq \ell), \quad a_1 a_1' = \cdots = a_{n} a_{n'}, a_n = a_{n'} \]

In the list above, the \( a_k \)'s are nonzero real parameters.

**Theorem 1.**

1. Each matrix \( J \in \text{End}_C(V) \) listed above satisfies the reflection equation (3.5) for the corresponding \( R \)-matrix.
2. For any \( \lambda \in P_G^+ \), one has \( \dim_C V(\lambda)_{\mathcal{J}} \leq 1 \). Furthermore, the set \( P_G^{+,\mathcal{J}} \) consisting of all \( \lambda \in P_G^+ \) such that \( V(\lambda)_{\mathcal{J}} \neq 0 \) coincides with that of the case of \( G/K \).

Since \( U_q^{\text{tw}}(\mathfrak{g}) \) has the coideal property, we obtain the following quantum analogue of the algebra of right \( K \)-invariant regular functions on \( G \):

\[ A_q(G/K) := \{ \varphi \in A_q(G) \mid \mathcal{J}.\varphi = 0 \}. \quad (3.10) \]

Furthermore, as a right \( U_q(\mathfrak{g}) \)-module, this algebra has the multiplicity free irreducible decomposition

\[ A_q(G/K) \simeq \bigoplus_{\lambda \in P_G^{+,\mathcal{J}}} V(\lambda)^\vee, \quad (3.11) \]

exactly in the same way as in the setting of \( G/K \). In each case, one can specify an appropriate set of parameters \( a_k \) so that \( \mathcal{J} \) associated with \( U_q^{\text{tw}}(\mathfrak{g}) \) satisfies the condition
S(\mathcal{J})^* \subset \mathcal{J}. For such special values of a_k, the subalgebra A_q(G/K) actually becomes a *-subalgebra of A_q(G).

**Remark 1.** As far as invariant elements are concerned, the left ideal \mathcal{J} is essential rather than the algebra U_q(\mathfrak{k}) itself. Actually, one can take some different matrices, say \mathcal{K}', to get the same left ideal \mathcal{J}. One can take \mathcal{K}' = \mathcal{L}^+J(\mathcal{L}^-)^t, for instance; for other choices, see Section 2.4 of Noumi\(^8\). In our previous paper \(^8\), we used an additive way to quantize \(\mathfrak{k} \subset U(\mathfrak{g})\). Define a matrix \(\mathcal{M} = (\mathcal{M}_{ij})_{ij} \in \text{End}_\mathbb{C}(V) \otimes_\mathbb{C} U_q(\mathfrak{g})\) by

\[
\mathcal{M} = \mathcal{L}^+ - JS(\mathcal{L}^-)^tJ^{-1},
\]

and take the coideal

\[
\mathfrak{k}_q = \sum_{1 \leq i,j \leq N} \mathbb{C} \mathcal{M}_{ij} \subset U_q(\mathfrak{g}).
\]

Then it is easily seen that \(U_q(\mathfrak{g})\mathfrak{k}_q = \mathcal{J}\). In this sense, the additive and the multiplicative approaches make no essential differences in defining the invariant subalgebra \(A_q(G/K)\).

**Remark 2.** The definition \(U_q(\mathfrak{g})\) of Drinfeld-Jimbo depends on a fixed Cartan subalgebra \(\mathfrak{k}\) of \(\mathfrak{g}\). For this reason, we have various possible quantizations of the symmetric pair \((\mathfrak{g}, \mathfrak{k})\), depending on the “position” of \(\mathfrak{k}\) inside \(\mathfrak{g}\), in relation to the fixed \(\mathfrak{k}\). We remark that, in the quantum analogues of this section, the Lie subalgebra \(\mathfrak{k} \subset \mathfrak{g}\) is taken so that, in the Cartan decomposition \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\), \(\mathfrak{p} \cap \mathfrak{k} = \mathfrak{a}\) gives a maximal abelian subalgebra in \(\mathfrak{p}\). This choice of \(\mathfrak{k}\) is quite opposite to the case of \(\mathfrak{k} \subset \mathfrak{g}\) corresponding to the embedding of root systems.

4. Zonal spherical functions.

In the rest of this paper, we consider the quantum analogue \(\mathcal{H} = A_q(K\backslash G/K)\) of the algebra of \(K\)-biinvariant regular functions on \(G\), and discuss zonal spherical functions associated with finite dimensional representations of \(U_q(\mathfrak{g})\). Until now we have checked that the zonal spherical functions for the quantum analogue of the following symmetric spaces are expressed by Macdonald polynomials associated with root systems \(^{26}\) or by Koornwinder’s Askey-Wilson polynomials for \(BC_\ell\) \(^{27}\):

\[
\begin{array}{cccc}
(1) & SU(n)/SO(n) & \cdots & AI & A_{n-1} & 1 \\
(2) & SU(2n)/Sp(2n) & \cdots & AII & A_{n-1} & 4 \\
(3) & SO(2n)/SO(n) \times SO(n) & (n = \ell) & \cdots & DI & D_\ell & 1 \\
(4) & Sp(2n)/U(n) & (n = \ell) & \cdots & CI & C_\ell & 1, 1 \\
(5) & Sp(2n)/Sp(2\ell) \times Sp(2\ell) & (n = 2\ell) & \cdots & CII & C_\ell & 4, 3 \\
(6) & SO(2n)/U(n) & (n = 2\ell) & \cdots & DIII & C_\ell & 4, 1 \\
(7) & SO(2n)/U(n) & (n = 2\ell + 1) & \cdots & DIII & BC_\ell & 4, 1, 4 \\
\end{array}
\]

The last two columns of this table indicate the type of the restricted root system of each symmetric space and the multiplicity of restricted roots \(\alpha\) with \((\alpha, \alpha) = 2, 4, 1\).

For the symmetric spaces listed in (4.1), we take the following constant solution of the reflection equation:

\[
J = \text{diag}(q^{\alpha_1}, \cdots, q^{\alpha_N}) \quad \text{for} \quad (1), (3), (4),
\]

\[
J = J_0 \text{diag}(q^{\alpha_1}, \cdots, q^{\alpha_N}) \quad \text{for} \quad (2), (5), (6), (7),
\]

9
where \( J_0 = \sum_{k=1}^{n}(e_{2k,2k-1} - e_{2k,2k-1}) \). The condition \( S(\mathcal{J})^* \subset \mathcal{J} \) for the left ideal \( \mathcal{J} \subset U_q(\mathfrak{g}) \) is also fulfilled for this \( J \). Hence we get the \(*\)-subalgebra

\[
\mathcal{H} = A_q(K\backslash G/K) = \{ \varphi \in A_q(G) | J.\varphi = \varphi.J = 0 \}. \tag{4.3}
\]

Furthermore, it has the simultaneous eigenspace decomposition

\[
\mathcal{H} = \bigoplus_{\lambda \in P^+_{G,J}} \mathcal{H}(\lambda), \quad \mathcal{H}(\lambda) = \{ \varphi \in \mathcal{H} | C.\varphi = \chi_\lambda(C)\varphi \ (C \in ZU_q(\mathfrak{g})) \} \tag{4.4}
\]

under the action of the center \( ZU_q(\mathfrak{g}) \), where \( \dim_C \mathcal{H}(\lambda) = 1 \) for all \( \lambda \in P^+_{G,J} \).

As for the cases (1) and (2), a detailed description of the structure of \( \mathcal{H} \) and the zonal spherical functions is already given in Ueno-Takebayashi\(^{28}\) and Noumi \(^8\). Hereafter, we will consider the remaining cases (3) – (7), so that \( G = SO(2n) \) or \( G = Sp(2n) \).

In order to describe the algebra \( \mathcal{H} \), recall that the quantum group \( G_q \) has the diagonal subgroup \( \mathbb{T} \) which is isomorphic to the \( n \)-dimensional torus. Namely, we have a canonical surjective \(*\)-homomorphism \( \varphi : A_q(G) \rightarrow \mathcal{H}(\mathbb{T}) \), where \( A(\mathbb{T}) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \) with \(*\)-operation \( z_j^* = z_j^{-1} \). By means of this “restriction mapping”, we consider the composition

\[
\mathcal{H} \ni \mathcal{H} = A_q(K\backslash G/K) \hookrightarrow A_q(G) \rightarrow A(\mathbb{T}). \tag{4.5}
\]

Let \( W(\Sigma) \) be the Weyl group of the restricted root system \( \Sigma \) of the symmetric space \( G/K \), and \( P(\Sigma) \) the lattice of integral weights of \( \Sigma \). We define the elements \( x_1, \ldots, x_\ell \) in \( A(\mathbb{T}) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \) by

\[
x_j = z_j^2 \quad (1 \leq j \leq \ell) \quad \text{for} \ (3), (4),
\]

\[
x_j = z_{2j-1} z_{2j} \quad (1 \leq j \leq \ell) \quad \text{for} \ (5), (6), (7), \tag{4.6}
\]

and take the following subalgebra \( \mathcal{A} \) of \( A(\mathbb{T}) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \):

\[
\mathcal{A} = \bigoplus_{\mu \in P(\Sigma)} \mathbb{C} \ x^\mu. \tag{4.7}
\]

(We understand \( x_j^\pm = z_j \) in the case (3).)

**Theorem 2.**

1. The restriction mapping \( \mathcal{H} \ni \mathcal{H} \rightarrow A(\mathbb{T}) \) is injective. Hence \( \mathcal{H} \subset A_q(G) \) is a commutative subalgebra.

2. The image \( \mathcal{H}|_\mathbb{T} \) of the restriction mapping coincides with the subalgebra \( \mathcal{A}^{W(\Sigma)} \) consisting of \( W(\Sigma) \)-invariant elements in \( \mathcal{A} \) above. Namely we have \( \mathcal{H} \simeq \mathcal{A}^{W(\Sigma)} \).

We recall now the Macdonald polynomials, restricting ourselves to those associated with root systems of type \( B_\ell, C_\ell, D_\ell \). Taking a root system \( \Sigma \) of type \( B_\ell, C_\ell, D_\ell \), let \( (m_\alpha)_{\alpha \in \Sigma} \) be a set of nonnegative real numbers, invariant under the action of the Weyl group \( W = W(\Sigma) \) of \( \Sigma \). Setting \( t_\alpha = q_\alpha^{m_\alpha/2} \) with \( q_\alpha = q^{(\alpha,\alpha)/2} \), we take the function \( \Delta^+(x) \) on \((\mathbb{C}^*)^\ell \) defined by

\[
\Delta^+(x) = \prod_{\alpha \in \Sigma^+} \frac{(x^\alpha;q_\alpha)^\infty}{(t_\alpha x^\alpha;q_\alpha)^\infty}. \tag{4.8}
\]
where \( (a;q)_\infty = \prod_{i=0}^{\infty} (1-aq^i) \), and set \( \Delta(x) = \Delta^+(x) \Delta^+(x)^* \). With the *-operation \( x_j^* = x_j^{-1} \), define the scalar product \( \langle \cdot | \cdot \rangle \) on the subalgebra \( \mathcal{A}^{W(\Sigma)} \) of \( W(\Sigma) \)-invariants in \( \mathcal{A} = \bigoplus_{\mu \in P(\Sigma)} \mathbb{C} x^\mu \) by
\[
\langle f | g \rangle = \frac{1}{|W(\Sigma)|} \int_P f(x)^* g(x) \Delta(x), \quad (f, g \in \mathcal{A}^{W(\Sigma)}),
\]
where \( \int_P \) is the normalized Haar measure of the torus \( T = \mathbb{R}^\ell/Q^\vee(\Sigma) \), and \( x^\mu \) are regarded as functions on \( T \) by \( x^\mu(\nu) = e^{2\pi i (\mu, \nu)} \). Then it is known by Macdonald that the subalgebra \( \mathcal{A}^{W(\Sigma)} \) has a unique basis \( \{ P_\mu(x) \}_{\mu \in P^+(\Sigma)} \) orthogonal under the scalar product (4.9), such that
\[
P_\mu(x) = m_\mu + \sum_{\nu < \mu} u_{\mu \nu} m_\nu(x)
\]
for each \( \mu \in P^+(\Sigma) \), where \( m_\mu(x) = \sum_{\nu \in W(\Sigma) \mu} x^\nu \) is the orbit sum and \( < \) denotes the dominance order of weights. These \( P_\mu(x) \) are called the Macdonald polynomials associated with the root system \( \Sigma \) (or, to be more precise, associated with the pair of root systems \( (\Sigma, \Sigma^\vee) \)).

An extension of Macdonald polynomials of type \( BC_\ell \) has been introduced by Koornwinder. Koornwinder’s Askey-Wilson polynomials for \( BC_\ell \) are defined similarly by taking the root system \( \Sigma \) of type \( BC_\ell \) and the function
\[
\Delta^+(x) = \prod_{k=1}^{\ell} \frac{(x_k^2;q)_\infty}{(ax_k,bx_k,cx_k,dx_k;q)_\infty} \prod_{1 \leq i < j \leq \ell} \frac{(x_i/x_j,x_ix_j;q)_\infty}{(tx_i/x_j,tx_ix_j;q)_\infty},
\]
instead of \( \Delta^+(x) \) of (4.8), where \( a, b, c, d, t \) are real parameters and \( (a,b,\cdots;q)_\infty = (a;q)_\infty (b;q)_\infty \cdots \). Among many works related to Macdonald polynomials, we only refer to Cherednik for the relation to affine Hecke algebras and to Etingof-Kirillov for a realization of Macdonald polynomials of type \( A \) by vector-valued characters of \( U_q(\mathfrak{g}(N)) \).

We now return to the setting of quantum symmetric spaces of type (4.1), (3)–(7) and consider the zonal spherical functions \( \varphi_\lambda \ (\lambda \in P^+_{G,\mathcal{J}}) \) on \( (G/K)_q \) defined by the constant solution \( J \) of (4.2). From our definition of \( U^w_q(\mathfrak{t}) \), it turns out that the zonal spherical function \( \varphi_\lambda \) can be normalized so that its restriction to \( T \) takes the form
\[
\varphi_\lambda|_T = z^\lambda + \sum_{\mu < \lambda} a_{\lambda \mu} z^\mu,
\]
where \( < \) stands for the dominance order of weights. One can show that the restriction \( \varphi_\lambda|_T \) are expressed by Macdonald polynomials (or Koornwinder’s Askey-Wilson polynomials for \( BC_\ell \)) in the variables \( x = (x_1, \cdots, x_\ell) \) of (4.6). For \( \lambda \in P^+_{G,\mathcal{J}} \), we take the dominant integral weight \( \mu \in P^+(\Sigma) \) of the corresponding restricted root system such that \( z^\lambda = x^\mu \) under (4.6). Then we have

**Theorem 3.** If the symmetric space \( G/K \) is of type (3)–(6), then the restriction \( \varphi_\lambda|_T \) (\( \lambda \in P^+_{G,\mathcal{J}} \)) of the zonal spherical function on our quantum symmetric space \( (G/K)_q \) coincides with the Macdonald polynomial \( P_\mu(x) \) associated with the restricted root system \( \Sigma \) and and the set \( \{ m_\alpha \}_{\alpha \in \Sigma} \) of multiplicities of roots of \( G/K \). They have the base \( q^4 \) in the cases (3),(4) and \( q^2 \) in the cases (5),(6), respectively. If \( G/K \) is of type (7), \( \varphi_\lambda|_T \)
coincides with Koornwinder’s Askey-Wilson polynomial $P_\mu(x)$ for $BC_\ell$ with parameters $(a, b, c, d; q, t)$ replaced by $(q^3, q^3, -q, -q; q^2, q^4)$.

By means of the function $\Delta^+(x)$ of (4.8) (or (4.11)), we define the $q$-difference operator $D_\sigma$ for the weight $\sigma = \epsilon_1$ as follows:

$$D_\sigma = \frac{1}{|W_\sigma|} \sum_{w \in W} w\Phi_\sigma(x)(T_{w\sigma} - 1) \text{ with } \Phi_\sigma(x) = \frac{T_\sigma \Delta^+(x)}{\Delta^+(x)},$$

(4.13)

where $T_\mu$ is the $q$-shift operator such that $T_\mu x^\lambda = q^{(\lambda, \mu)} x^\mu$. This $q$-difference operator $D_\sigma$ is selfadjoint with respect to the scalar product of Macdonald (or Koornwinder). Furthermore, the Macdonald polynomials (or Koornwinder’s Askey-Wilson polynomials) $P_\mu(x)$ are eigenfunctions of the operator $D_\sigma$.

In our context, the operator $D_\sigma$, with the parameters described in Theorem 3, arises as the radial component of the central element

$$C_\sigma = \sum_{1 \leq i, j \leq N} q^{2\rho_{ij}} L^+_i L^-_j$$

(4.14)

of $U_q(\mathfrak{g})$ defined by Reshetikhin-Takhtajan-Faddeev. For a central element $C \in Z U_q(\mathfrak{g})$, we mean by the radial component of $C$ an operator $D : A^W(\Sigma) \to A^W(\Sigma)$ that makes the following diagram commutative:

$$\begin{array}{cccc}
\mathcal{H} & C & \rightarrow & \mathcal{H} \\
| \downarrow \tau | & & \downarrow & | \downarrow \tau |
\end{array}$$

$$\begin{array}{cccc}
A^W(\Sigma) & D & \rightarrow & A^W(\Sigma)
\end{array}$$

(4.15)

In fact it turns out that the radial component of $C_\sigma - \chi_\lambda(C_\sigma)$ is a constant multiple of an operator of the form $D_\sigma - a_\sigma(\mu)$ for some $a_\sigma(\mu) \in \mathbb{C}$. Hence the restriction $\varphi_\lambda|_\tau$ is an eigenfunction of $D_\sigma$ for each $\lambda \in P^+_G \cap J$. By using this fact, one can show that, on the $*$-subalgebra $\mathcal{H} = A_q(K \setminus G / K)$, the scalar product defined by the Haar functional of $G_q$ in (1.6) corresponds to that of Macdonald (or Koornwinder) in (4.9), up to a scalar multiple. From this it follows that the basis $\{ \varphi_\lambda|_\tau \}_{\lambda \in P^+_G \cap J}$ of $A^W(\Sigma)$ coincides with the orthogonal basis of Macdonald (or Koornwinder) described above.

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**Notes:** This article is based on the talk given at the conference by one of the authors (M.N.). On that occasion, Prof. Allan Solomon kindly made the following two comments:

1. It is a fairly strong abuse of language to describe as a “quantum group” an object which is not a group; but you have defined as “quantum subgroups” objects which are not even quantum groups!
2. The commutation $ef - fe = (t - t^{-1})/(q - q^{-1})$ would indicate a Haar measure based on the $q$-integral of Nelson et al. (1988) and not that of Jackson (1909).