Infrared and Ultraviolet Finiteness of Topological BF Theory in Two Dimensions

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ABSTRACT

The two–dimensional topological BF model is considered in the Landau gauge in the framework of perturbation theory. Due to the singular behaviour of the ghost propagator at long distances, a mass term to the ghost fields is introduced as infrared regulator. Relying on the supersymmetric algebraic structure of the resulting massive theory, we study the infrared and ultraviolet renormalizability of the model, with the outcome that it is perturbatively finite.
1 Introduction

The BF–systems \[1\] are the only known examples of topological quantum field theories of the Schwarz type, together with the three–dimensional Chern–Simons model, of which they represent the natural extension in an arbitrary number of spacetime dimensions. From this point of view they allow to define topological invariants which are the \(n\)–dimensional generalization of the \(3d\)–linking number \[2\].

The topological nature of these models has led to the conjecture that, at least perturbatively, they should be finite. The proof of this assertion is based on the fact that, besides the gauge invariance, there are additional symmetries whose algebraic structure becomes transparent in the Landau gauge. Exploiting this algebra of the supersymmetric type, the conjecture has been proved for the BF models in three \[3\] and four \[4, 5\] flat euclidean spacetime. Recently, this result has been extended to dimensions \(n > 4\) \[6\].

What is still lacking is an analogous discussion concerning the two–dimensional case, which is particularly interesting because it collects properties of quantum field theories at first sight very different one from the other.

For instance, it has been noticed in \[1, 7, 8, 9\] that the zero coupling constant limit of the Yang–Mills action in two dimensions can be reduced to a topological theory in which the field strength is coupled to a scalar field.

The same model is also obtained when considering the Chern–Simons three–form on a manifold \(M_3 = S^1 \times M_2\), where \(S^1\) is a circle and \(M_2\) is an oriented boundaryless surface \[7, 8\]. Indeed, the dimensional reduction on \(M_2\) leads to a metric independent action which can be recognized as a two–dimensional BF model.

Moreover, this model has been proposed in \[1, 7, 10\] as a gauge theory of two–dimensional topological quantum gravity. Indeed, in the particular case in which the gauge group is \(SO(2, 1)\), the field equations of motion describe a torsion–free spin connection living on a manifold with constant negative curvature. Another interesting feature is the deep relation existing
between the $SO(2, 1)$ BF field equations and the dimensionally reduced solutions of the Hitchin’s self–duality equations on a Riemann surface [11], which provide a mechanism to construct metrics with constant negative curvature.

Finally, the equations of motion of the connection for a generic simple nonabelian gauge group necessarily implies that the scalar field obeys the constraint typical of a nonlinear sigma model.

Thus, the theory described by a BF model in two dimensions seems to play the role of a trait d’union between several field theories, and the question arises spontaneously whether such a model maintains the finiteness to all order of perturbation theory.

Although a formal proof of this assertion may be also based on the algebraic constraints of the model in the Landau gauge, the discussion becomes more complex due to the singular infrared behaviour of the massless ghost propagator. On the other hand, the introduction of a mass term for the ghost fields breaks the symmetries which are at the root of the procedure to analyze the perturbative finiteness of the theory.

Therefore the first step consists in giving a mass to the Faddeev–Popov ghosts by means of a spontaneous symmetry breaking mechanism induced by a suitable set of external fields.

In this version of the BF system in two dimensions we have massless and massive propagators and in the discussion of the finiteness of the model we have to take into account also the infrared canonical dimensions of the fields [12].

The paper is organized as follows. In section two we describe the classical model and its supersymmetric algebraic structure. In section three we cure the infrared singularities of the theory by introducing a mass term for the ghost fields and we extend the algebraic structure by means of suitable external fields. In section four, we show that the symmetries are anomaly–free and that no counterterms are needed to compensate the divergences of the theory, i.e. we prove that the model is perturbatively finite. Finally, section five is devoted to a discussion of the obtained results.
We consider the metric independent action
\[ S_{\text{inv}} = \frac{1}{2} \int_M d^2x \, \varepsilon^{\mu\nu} F^{a}_{\mu\nu} \phi^a, \tag{2.1} \]
where \( M \) is the Euclidean spacetime, \( \varepsilon^{\mu\nu} \) is the completely antisymmetric Levi–Civita tensor (\( \varepsilon^{12} = +1 \)), \( \phi^a \) is a zero–form, \( F^{a}_{\mu\nu} \) is the usual field strengh
\[ F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + f^{abc} A^{b}_{\mu} A^{c}_{\nu}, \tag{2.2} \]
and \( f^{abc} \) are the completely antisymmetric real structure constants of some nonabelian gauge group \( G \), which we assume to be compact and simple.

The action \( S_{\text{inv}} \) is invariant under the infinitesimal gauge transformations
\[ \delta A^a_{\mu} = - \left( \partial_{\mu} \theta^a + f^{abc} A^b_{\mu} \theta^c \right) \equiv -(D_{\mu} \theta)^a, \]
\[ \delta \phi^a = - f^{abc} \phi^b \theta^c, \tag{2.3} \]
where \( \theta^a \) is a local parameter.

The field equations, derived from the action \( S_{\text{inv}} \) are
\[ \frac{\delta S_{\text{inv}}}{\delta A^a_{\mu}} = \varepsilon^{\mu\nu} (D_{\nu} \phi)^a = 0 \tag{2.4} \]
\[ \frac{\delta S_{\text{inv}}}{\delta \phi^a} = \frac{1}{2} \varepsilon^{\mu\nu} F^{a}_{\mu\nu} = 0. \tag{2.5} \]

The first of the above equations implies that the scalar field \( \phi^a \), in close similarity with a nonlinear \( \sigma \)-model, lives on a hypersphere
\[ \phi^a \phi^a = \text{constant}, \tag{2.6} \]
while \( (2.3) \) yields the vanishing curvature condition typical of topological models.

Following the BRS procedure to quantize the gauge theories, we introduce a ghost, an antighost and a Lagrange multiplier field \((c^a, \overline{c}^a, b^a)\) and
we write the nilpotent BRS transformations

\[ sA^a_{\mu} = -(D^c_{\mu})^a \]
\[ s\phi^a = f^{abc}c^b\phi^c \]
\[ sc^a = \frac{1}{2}f^{abc}b^c \]
\[ sc\bar{c}^a = b^a \]
\[ sb^a = 0. \]  

(2.7)

We then choose the Landau gauge by adding to the action \( S_{inv} \) the gauge–fixing term

\[ S_{gf} = \int d^2x \, s\bar{c}^a \partial A^a \]
\[ = \int d^2x \left( b^a \partial A^a - (\partial^\mu \bar{c}^a)(D_\mu c)^a \right). \]  

(2.8)

In Table 1 we list the canonical dimensions and Faddeev–Popov charges of the quantum fields

|       | A | \( \phi \) | c | \( \bar{c} \) | b |
|-------|---|-----|---|------|---|
| dim   | 1 | 0   | 0 | 0    | 0 |
| \( \Phi \Pi \) | 0 | 0   | 1 | -1   | 0 |

Table 1. Dimensions and Faddeev–Popov charges of the quantum fields.

The action

\[ S = S_{inv} + S_{gf} \]  

is, by construction, BRS invariant

\[ sS = 0. \]  

(2.10)

As a common feature of the topological models in the Landau gauge [3, 4, 3, 4, 13], the gauge–fixed action \( S \) is left invariant under a further transformation of the supersymmetric kind and carrying a vectorial index

\[ \delta_\mu A^a_{\nu} = 0 \]
\[ \delta_\mu \phi^a = -\varepsilon_{\mu\nu} \partial^\nu \bar{c}^a \]
\[ \delta_\mu c^a = -A^a_{\mu} \]
\[ \delta_\mu \bar{c}^a = 0 \]
\[ \delta_\mu b^a = \partial_\mu \bar{c}^a. \]  

(2.11)
Indeed one can easily verify that
\[ \delta_{\mu} S = 0 \quad . \] (2.12)

Moreover, the following algebraic structure holds
\[
\{ s, s \} = 0
\]
\[
\{ s, \delta_{\mu} \} = \partial_{\mu} + \text{equations of motion}
\]
\[
\{ \delta_{\mu}, \delta_{\nu} \} = 0
\]
which closes on–shell on the translations.

In order to write a Slavnov identity and to extend the formalism off–shell, we couple external sources to the nonlinear BRS variations in (2.7)
\[
S_{ext} = \int d^2x \left( \Omega^{a\mu}(sA^a_{\mu}) + L^a(s\alpha^a) + \rho^a(s\phi^a) \right),
\]
where \((\Omega^{a\mu}, L^a, \rho^a)\) are external sources whose canonical dimensions and Faddeev–Popov charges are reported in Table 2

|   | \(\Omega\) | \(L\) | \(\rho\) |
|---|---|---|---|
| dim | 1 | 2 | 2 |
| \(\Phi\Pi\) | −1 | −2 | −1 |

*Table 2. Dimensions and Faddeev–Popov charges of the external fields.*

The classical action
\[
\Sigma = S_{inv} + S_{gf} + S_{ext}
\]
(2.15)
satisfies the Slavnov identity
\[
S(\Sigma) = 0 \quad ,
\]
(2.16)
where
\[
S(\Sigma) = \int d^2x \left( \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta \Sigma}{\delta A^a_{\mu}} + \frac{\delta \Sigma}{\delta \rho^a} \frac{\delta \Sigma}{\delta \phi^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta \bar{c}^a} \right)
\]
(2.17)
and the corresponding linearized operator
\[
B_{\Sigma} = \int d^2x \left( \frac{\delta \Sigma}{\delta \Omega^{a\mu} \delta A^a_{\mu}} + \frac{\delta \Sigma}{\delta A^a_{\mu} \delta \Omega^{a\mu}} + \frac{\delta \Sigma}{\delta \rho^a} \frac{\delta \Sigma}{\delta \phi^a} + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta \Sigma}{\delta \rho^a} + \frac{\delta \Sigma}{\delta \Omega^{a\mu}} \frac{\delta \Sigma}{\delta A^a_{\mu}} \right)
\]
(2.18)
is nilpotent
\[ B_\Sigma B_\Sigma = 0 . \] (2.19)

Due to the introduction of the source term (2.14), the invariance (2.12) writes
\[ \mathcal{W}_\mu \Sigma = \Delta_\mu , \] (2.20)
where
\[ \mathcal{W}_\mu = \int d^2x \left( \varepsilon_{\mu\nu} \rho^a \frac{\delta}{\delta A^a_\nu} - \varepsilon_{\mu\nu} \left( \Omega^{a\nu} + \partial^\nu c^a \right) \frac{\delta}{\delta \phi^a} - A^a_\mu \delta_{c^a} + \left( \partial_\mu c^a \right) \delta_{b^a} - L^a \frac{\delta}{\delta \Omega^{a\mu}} \right) \] (2.21)
and the breaking \( \Delta_\mu \)
\[ \Delta_\mu = \int d^2x \left( L^a \left( \partial_\mu c^a \right) - \rho^a \partial_\mu \phi^a - \Omega^{a\mu} \left( \partial_\mu A^a_\nu \right) - \varepsilon_{\mu\nu} \rho^a \partial^\nu b^a \right) , \] (2.22)
being linear in the quantum fields, is only present at classical level.

The model, as it happens for all theories in the Landau gauge, has the further symmetry associated with the integrated ghost equation of motion [14]
\[ G^a \Sigma = \Delta^a , \] (2.23)
where
\[ G^a = \int d^2x \left( \frac{\delta}{\delta c^a} + f^{abc} \frac{\delta}{\delta b^c} \right) \] (2.24)
and \( \Delta^a \) is also a classical breaking
\[ \Delta^a = \int d^2x \ f^{abc} \left( \Omega^{b\mu} A^c_\mu - L^b c^c + \rho^b \phi^c \right) . \] (2.25)
Anticommuting the ghost equation (2.23) with the Slavnov identity (2.16), one gets the Ward identity for the rigid gauge invariance of the theory
\[ \mathcal{H}^a \Sigma = 0 , \] (2.26)
where
\[ \mathcal{H}^a = \sum_{\text{all fields } \psi} \int d^2x \ f^{abc} \psi^b \frac{\delta}{\delta \psi^c} , \] (2.27)
while the anticommutator between the Slavnov operator (2.17) and the susy operator \( \mathcal{W}_\mu \) (2.21) expresses the invariance under translations
\[ \mathcal{P}_\mu \Sigma = \sum_{\text{all fields } \psi} \int d^2x \ (\partial_\mu \psi) \frac{\delta \Sigma}{\delta \psi} = 0 . \] (2.28)
The set of constraints on the classical action $\Sigma$ is completed by the gauge condition
\[
\frac{\delta \Sigma}{\delta b^a} = \partial A^a
\] (2.29)
which, commuted with the Slavnov identity (2.16) yields the antighost equation of motion
\[
\frac{\delta \Sigma}{\delta \bar{c}^a} + \partial^\mu \frac{\delta \Sigma}{\delta \Omega^{a\mu}} = 0.
\] (2.30)

By the introduction of the external fields we have reabsorbed the terms proportional to the equations of motion in (2.13); the resulting nonlinear off–shell algebra for a generic even charged functional $\gamma$ is
\[
\begin{align*}
B_\gamma S(\gamma) &= 0 \\
\{\mathcal{W}_\mu, \mathcal{W}_\nu\} &= 0 \\
\mathcal{W}_\mu S(\gamma) + B_\gamma (\mathcal{W}_\mu \gamma - \Delta_\mu) &= P_\mu \gamma \\
\mathcal{G}^a S(\gamma) + B_\gamma (\mathcal{G}^a \gamma - \Delta^a) &= \mathcal{H}^a \gamma \\
\mathcal{G}^a (\mathcal{W}_\mu \gamma - \Delta_\mu) + \mathcal{W}_\mu (\mathcal{G}^a \gamma - \Delta^a) &= 0 \\
\{\mathcal{G}^a, \mathcal{G}^b\} &= 0 \\
[\mathcal{H}^a, \mathcal{G}^b] &= -f^{abc} \mathcal{G}^c \\
[\mathcal{H}^a, \mathcal{H}^b] &= -f^{abc} \mathcal{H}^c.
\end{align*}
\] (2.31)

To summarize, the model we are considering is characterized by

\begin{enumerate}
\item the Slavnov identity (2.16);
\item the susy identity (2.20);
\item the ghost equation (2.23);
\item the gauge–fixing condition (2.29),
\end{enumerate}

and, as by–products, the rigid gauge invariance (2.26), the Ward identity for the translations (2.28) and the antighost equation (2.30).

Now a quantum extension of the model cannot be directly defined from the action (2.15) due to the infrared problems in the ghost sector; we shall see how these can be solved by the introduction of a mass for the Faddeev–Popov ghost fields and how the functional differential operators describing the symmetries modify still preserving the off–shell algebraic structure in (2.31).

7
3 Infrared regularization

An important point to be noticed concerns the infrared behaviour of the theory. Indeed, the propagators are

\[
< A_\mu^a \phi^b >= -i \delta^{ab} \varepsilon_{\mu \nu} k^\nu \quad < A_\mu^a b^b >= i \delta^{ab} \frac{k_\mu}{k^2} \quad (3.1)
\]

\[
< \bar{c}^a c^b >= \delta^{ab} \frac{1}{k^2} . \quad (3.2)
\]

Dealing with a field theory defined on the two-dimensional flat spacetime, we see that the ghost propagator (3.2) is not integrable at small momenta. For a correct treatment of the model, we have to regularize it at long distances and, to do so, we introduce into the action an infrared regulator in the form of a ghost mass term, so that the ghost propagator becomes

\[
< \bar{c}^a c^b >_m = \delta^{ab} \frac{1}{k^2 + m^2} . \quad (3.3)
\]

In presence of both massive and massless propagators, we must distinguish between ultraviolet (d) and infrared (r) dimensions of the fields and composite operators of the model, which coincide in the completely massless case [12]. From the massive propagator (3.3) we obtain

\[
r(\bar{c} + c) = r(\bar{c}) + r(c) = 2 , \quad (3.4)
\]

and we have the freedom to choose

\[
r(\bar{c}) = 0 ; \quad r(c) = 2 . \quad (3.5)
\]

We summarize the ultraviolet and infrared dimensions of the quantum fields and sources of the model in Table 3.

|     | A | \phi | c | \bar{c} | b | \Omega | L | \rho |
|-----|---|------|---|--------|---|-------|---|-----|
| d   | 1 | 0    | 0 | 0      | 0 | 1     | 2 | 2   |
| r   | 1 | 0    | 0 | 0      | 0 | 1     | 2 | 2   |

**Table 3.** Ultraviolet (d) and infrared (r) dimensions.
Of course the presence of a ghost mass term may in principle destroy the algebraic structure (2.31), in particular it breaks both the Slavnov identity (2.16) and the susy (2.20), but we shall see that we can recover the same algebra in terms of modified operators obtained at the price of adding a few external sources.

Indeed, let us consider the following term

\[ S_m = \int d^2 x \left( (\tau_1 + m^2) \bar{c}^a c^a + \tau_2 (b^a c^a - \frac{1}{2} f^{abc} \bar{c}^b c^c) \right. \]

\[ \left. + \tau_3 \bar{c}^a A_\mu^a + \tau_4^{\mu} \left( b^a A_\mu^a + \bar{c}^a (D_\mu c)^a \right) \right), \]  

(3.6)

where \((\tau_1, \tau_2, \tau_3^{\mu}, \tau_4^{\mu})\) are external fields whose ultraviolet and infrared dimensions, as well as their Faddeev–Popov charges are displayed in Table 4.

|   | \(\tau_1\) | \(\tau_2\) | \(\tau_3\) | \(\tau_4\) |
|---|---|---|---|---|
| \(d\) | 2 | 2 | 1 | 1 |
| \(r\) | 2 | 2 | 2 | 2 |
| \(\Phi\Pi\) | 0 | -1 | 1 | 0 |

Table 4. Quantum numbers of the \(\tau\)-sources.

The action

\[ \Sigma_m = \Sigma + S_m \]  

(3.7)

satisfies the modified Slavnov identity

\[ \tilde{S}(\Sigma_m) = 0 , \]  

(3.8)

where

\[ \tilde{S}(\Sigma_m) = \int d^2 x \left( \frac{\delta \Sigma_m}{\delta \Omega^{\alpha\mu}} \frac{\delta \Sigma_m}{\delta A_\mu^a} + \frac{\delta \Sigma_m}{\delta \rho^a} \frac{\delta \Sigma_m}{\delta \phi^a} + \frac{\delta \Sigma_m}{\delta L^a} \frac{\delta \Sigma_m}{\delta \bar{c}^a} + b^a \frac{\delta \Sigma_m}{\delta c^a} \right. \]

\[ \left. - (\tau_1 + m^2) \frac{\delta \Sigma_m}{\delta \tau_2} + \tau_3^{\mu} \frac{\delta \Sigma_m}{\delta \tau_4^{\mu}} \right), \]  

(3.9)

whose corresponding linearized operator

\[ \tilde{B}_{\Sigma_m} = \int d^2 x \left( \frac{\delta \Sigma_m}{\delta \Omega^{\alpha\mu}} \frac{\delta}{\delta A_\mu^a} + \frac{\delta \Sigma_m}{\delta A_\mu^a} \frac{\delta}{\delta \Omega^{\alpha\mu}} + \frac{\delta \Sigma_m}{\delta \rho^a} \frac{\delta}{\delta \phi^a} + \frac{\delta \Sigma_m}{\delta \phi^a} \frac{\delta}{\delta \rho^a} + \frac{\delta \Sigma_m}{\delta L^a} \frac{\delta}{\delta \bar{c}^a} + \frac{\delta \Sigma_m}{\delta \bar{c}^a} \frac{\delta}{\delta L^a} \right. \]

\[ \left. + \frac{\delta \Sigma_m}{\delta \bar{c}^a} \frac{\delta}{\delta L^a} + b^a \frac{\delta}{\delta \bar{c}^a} - (\tau_1 + m^2) \frac{\delta}{\delta \tau_2} + \tau_3^{\mu} \frac{\delta}{\delta \tau_4^{\mu}} \right) \]  

(3.10)
is still nilpotent
\[ \mathcal{B}_{\Sigma_m} \mathcal{B}_{\Sigma_m} = 0 . \] (3.11)

Similarly, the susy identity (2.20) now becomes
\[ \mathcal{W}_\mu \Sigma_m = \tilde{\Delta}_\mu , \] (3.12)

where
\[ \mathcal{W}_\mu \Sigma_m = \int d^2x \left( \varepsilon_{\mu\nu} \rho^a \frac{\delta}{\delta A^{a}_\nu} - \varepsilon_{\mu\nu} (\Omega^{a\nu} + \partial^\mu c^a - \tau_4^\mu \bar{c}^a) \frac{\delta}{\delta \phi^a} - A^a_{\mu} \frac{\delta}{\delta c^a} \\
+ (\partial^\mu \bar{c}^a) \frac{\delta}{\delta b^a} - L^a \frac{\delta}{\delta \Omega^{a\mu}} - (\partial_\mu \tau_2) \frac{\delta}{\delta \tau_1} \right. \\
\left. + [\partial_\mu \tau_4^\nu - \delta_\mu^\nu (\tau_1 + m^2)] \frac{\delta}{\delta \tau_3^\nu} - \tau_2 \frac{\delta}{\delta \tau_4^\mu} \right) \] (3.13)

and
\[ \tilde{\Delta}_\mu = \int d^2x \left( L^a (\partial_\mu c^a) - \rho^a \partial_\mu \phi^a - \Omega^{a\nu} (\partial_\mu A^a_{\nu}) - \varepsilon_{\mu\nu} \rho^a \partial^\nu b^a \\
+ \varepsilon_{\mu\nu} \rho^a \tau_3^\nu \bar{c}^a + \varepsilon_{\mu\nu} \rho^a \tau_4^\nu b^a \right) \] (3.14)

In addition, the dependence of the classical action \( \Sigma_m \) on the massive parameter \( m^2 \) is controlled by the following shift equation
\[ \left( \frac{\partial}{\partial m^2} - \int d^2x \frac{\delta}{\delta \tau_1} \right) \Sigma_m = 0 . \] (3.15)

The gauge condition (2.29) reads now
\[ \frac{\delta \Sigma_m}{\delta b^a} = \partial A^a + \tau_2 c^a + \tau_4^\mu A^a_{\mu} , \] (3.16)

which, commuted with the Slavnov identity (3.8) gives the new “antighost” equation of motion
\[ \tilde{\mathcal{G}}^a(x) \Sigma_m = \Delta^a_{(g)}(x) , \] (3.17)

where
\[ \tilde{\mathcal{G}}^a(x) = \frac{\delta}{\delta \bar{c}^a} + \partial^\mu \frac{\delta}{\delta \Omega^{a\mu}} + \tau_4^\mu \frac{\delta}{\delta \Omega^{a\mu}} - \tau_2 \frac{\delta}{\delta L^a} , \] (3.18)

and \( \Delta^a_{(g)}(x) \) is the classical breaking given by
\[ \Delta^a_{(g)}(x) = (\tau_1 + m^2) c^a - \tau_3^\mu A^a_{\mu} . \] (3.19)
The ghost equation (2.23) acquires also an additional linear breaking

$$\mathcal{G}^{a}\Sigma_{m} = \tilde{\Delta}^{a}, \quad (3.20)$$

where

$$\tilde{\Delta}^{a} = \Delta^{a} - \int d^{2}x \left( (\tau_{1} + m^{2})\tilde{c}^{a} + \tau_{2}b^{a} \right). \quad (3.21)$$

One can verify that the algebraic structure (2.31), written for the modified operators, survives unaltered.

At this stage we can analyze the perturbative renormalizability of the massive model whose classical action is $\Sigma_{m}$ in (3.7). The infrared regulator $m^{2}$ and the satellite external fields needed to implement the breaking to the original symmetry, affect the ghost sector and we expect that there are no alterations induced by $m^{2}$ in the topological properties of the original massless model.

## 4 Perturbative finiteness

We prove the perturbative finiteness of the model by first showing the absence of counterterms, leaving as a next step the proof that the symmetries are not anomalous. Since we have a theory with both massive and massless propagators, we must take into account either the ultraviolet or the infrared dimensions of the possible counterterms arising from the analysis of the stability and of the absence of anomalies. Indeed, renormalizability of the theory requires the counterterms to be integrated local functionals with vanishing ghost number, ultraviolet dimensions $d \leq 2$ and infrared dimensions $r \geq 2$. In particular, we must check the absence of counterterms with infrared dimensions $r < 2$, which otherwise introduce incurable long-distance divergences in the model [12].

### 4.1 Absence of counterterms

For what concerns the stability of the model, we shall find that no counterterm is compatible with the algebraic constraints; indeed a local pertur-
bation $\Sigma^c$ of the classical action $\Sigma_m$ obeys
\[
\frac{\delta \Sigma^c}{\delta b^a(x)} = 0 , \quad (4.1)
\]
\[
G^a \Sigma^c = 0 , \quad (4.2)
\]
\[
G^a(x) \Sigma^c = 0 , \quad (4.3)
\]
and
\[
\bar{\mathcal{W}}_\mu \Sigma^c = 0 , \quad \bar{B}_\Sigma \Sigma^c = 0 ,
\]
moreover we impose
\[
\left. \frac{\delta \Sigma^c}{\delta \tau_1(x)} \right|_{\psi = 0} = 0 , \quad \psi = \text{all fields} , \quad (4.6)
\]
since the mass to the ghost fields in (3.6) is provided by the spontaneous symmetry breaking in the direction of the $\tau_1$–external field.

We first analyze (4.1)–(4.3); condition (4.1) is satisfied by a functional which does not depend on $b^a$, while the ghost condition (4.2) implies that $\Sigma^c$ does not depend on the undifferentiated ghost field $c^a$. Finally, the most general functional having ultraviolet dimensions $d \leq 2$ and vanishing Faddeev–Popov charge which obeys the stability conditions (4.1)–(4.3) and (4.6) is
\[
\Sigma^c = \Sigma^c_{(0)} + \Sigma^c_{(2)} , \quad (4.7)
\]
where, according to the ultraviolet dimensions,
\[
\Sigma^c_{(0)} = \sum_{n=2}^{+\infty} \int d^2x \, a_0^n \Phi P_n \quad (4.8)
\]
and
\[
\Sigma^c_{(2)} = \int d^2x \sum_{n=0}^{+\infty} \left( a_1^{(ab)n} (\partial^a \phi^a) (\partial^b \phi^b) + a_2^{[ab]n} \varepsilon^{\mu\nu} (\partial_\mu \phi^a) (\partial_\nu \phi^b)
\right.
\]
\[
\left. + a_3^{ab} A^a_\mu (\partial^a \phi^b) + a_4^{ab} A^a_\mu \varepsilon^{\mu\nu} A^b_\nu (\partial_\nu \phi^a) + a_5^{ab} \tau_4^\mu A^a_\mu (\partial_\nu \phi^a) + a_6^{(ab)n} A^a_\mu A^b_\mu + a_7^{[ab]} A^a_\mu \varepsilon^{\mu\nu} A^b_\nu A^c_\nu + a_8^{ab} \varepsilon^{\mu\nu} \tau_4^\mu \tau_4^\nu A^c_\mu A^a_\nu A^b_\nu
\right.
\]
\[
\left. + a_9^{ab} \varepsilon^{\mu\nu} \tau_4^\mu \tau_4^\nu A^c_\mu + a_{10}^{ab} \varepsilon^{\mu\nu} \tau_4^\mu \tau_4^\nu A^a_\mu + a_{11}^{(ab)n} \Omega^a_\mu (\partial_\nu \phi^b) + a_{12}^{[ab]} \varepsilon^{\mu\nu} \tau_3^a \tau_3^b \Omega^a_\mu + a_{13}^{ab} \varepsilon^{\mu\nu} \tau_3^\mu \Omega^a_\nu (\partial_\nu \phi^b) + a_{14}^{[ab]} \varepsilon^{\mu\nu} \tau_3^\mu \Omega^a_\nu (\partial_\nu \phi^b)
\right.
\]
\[
\left. + a_{15}^{ab} \varepsilon^{\mu\nu} \tau_3^\mu \tau_4^\nu \Omega^a_\mu (\partial_\nu \phi^b) + a_{16}^{(ab)n} \tau_1 \phi^a + a_{17}^{(ab)n} m^2 \phi^a \right) \Phi P_n , \quad (4.9)
\]
with the short–hand notation
\[ P_n \equiv (p_1 \ldots p_n) ; \quad \Phi^P_n \equiv \phi^{p_1} \ldots \phi^{p_n} , \tag{4.10} \]
and \((a_0, \ldots, a_{17})\) are invariant tensors. Furthermore
\[ a_2^{[ab]}(p_1\ldots p_n) + \frac{1}{2} \sum_{k=1}^{n} \left( a_2^{[ap_k]}(bp_1\ldots \hat{p}_k\ldots p_n) + a_2^{[p_kb]}(ap_1\ldots \hat{p}_k\ldots p_n) \right) = 0 , \tag{4.11} \]
where the hat means omission of the relative index. Moreover, condition (4.3) is solved by a the counterterm \( \Sigma^c \) depending on the combinations
\[ \widehat{\Omega}^{a\mu} = \Omega^{a\mu} + \partial^{\mu} \overline{c}^a - \tau_4^{\mu} \overline{c}^a , \]
\[ \widehat{L}^a = L^a - \tau_2 \overline{c}^a . \tag{4.12} \]

Now, it is sufficient to analyze the susy condition (4.5) to conclude that \( \Sigma^c \) vanishes, i.e.
\[ \overline{W}_\mu \Sigma^c = 0 \implies \Sigma^c = 0 . \tag{4.13} \]

As it happens for the Chern–Simons theory and for the BF models in higher dimensions [3, 4, 5], there is no need of the Slavnov condition (4.5) to show that these so–called Schwarz–type topological quantum field theories admit no counterterm. This is mainly due to the presence of the susy (3.12) and of the ghost equation (3.20), without which other techniques must be used [15] to reach the same goal (4.13). We do not report here the algebraic discussion of (4.13), which is as straightforward as it is lenghtly and tedious.

### 4.2 Absence of anomalies

To complete the proof of the perturbative finiteness of the model, we must show that the symmetries are not anomalous. Their quantum implementation must be discussed having in mind that we must control also the infrared dimensionality of the term compensating the breaking; this is due to the fact that renormalizability by power counting forbids the presence of counterterms having infrared dimensions strictly less than two.

There is no problem for the quantum extensions of the gauge condition (3.16) and of the antighost equation (3.17), which, written for the
quantum vertex functional $\Gamma$, read
\[
\frac{\delta \Gamma}{\delta b^a(x)} = \Delta^a(x) \tag{4.14}
\]
\[
\bar{G}^a(x) = \bar{\Delta}^a(x) \tag{4.15}
\]

The Quantum Action Principle (Q.A.P.) \cite{16} insures that the breakings $\Delta^a(x)$ and $\bar{\Delta}^a(x)$ are local functionals with the quantum numbers in Table 5.

|   | $\Delta$ | $\bar{\Delta}$ |
|---|---|---|
| $d$ | $\leq 2$ | $\leq 2$ |
| $r$ | $\geq 2$ | $\geq 2$ |
| $\Phi \Pi$ | $0$ | $+1$ |

Table 5. Quantum numbers of $\Delta^a$, $\bar{\Delta}^a$.

In particular, both $\Delta^a$ and $\bar{\Delta}^a$ have infrared dimensions greater than or equal to two, and they can easily be reabsorbed by counterterms with the correct infrared and ultraviolet dimensions
\[
\Delta^a(x) = \frac{\delta}{\delta b^a(x)} \Delta \ ; \ d(\Delta) \leq 2 \leq r(\Delta) \tag{4.16}
\]
\[
\bar{\Delta}^a(x) = \bar{G}^a(x) \bar{\Delta} \ ; \ d(\bar{\Delta}) \leq 2 \leq r(\bar{\Delta}) \tag{4.17}
\]

The nature of the problems that might arise when discussing a mixed theory can be tasted in the quantum extension of the ghost equation (3.20), which reads
\[
\bar{G}^a \Gamma = \Delta^a \tag{4.18}
\]

According to the Q.A.P., the breaking $\Delta^a$ is an integrated local functional with ghost number $-1$, ultraviolet dimension $d(\Delta^a) \leq 2$ and infrared dimension $r(\Delta^a) \geq 0$. In particular, the Q.A.P. does not forbid a breaking with vanishing infrared dimension
\[
\Delta^a_{(0)} = \int d^2x \bar{c}^a P(\phi) \tag{4.19}
\]

where $P(\phi)$ is a polynomial in the scalar field $\phi^a$. But such a breaking is not allowed by the antighost equation, under which the breaking $\Delta^a$ must be invariant. One can easily verify that
\[
\Delta^a = \bar{G}^a \Delta \ ; \ d(\Delta) \leq 2 \leq r(\Delta) \tag{4.20}
\]
To summarize, we assume that the quantum vertex functional $\Gamma$ satisfies

$$\frac{\delta \Gamma}{\delta b^a} = \partial A^a + \tau_2 c^a + \tau_4^\mu A^a_{\mu},$$

$$\mathcal{G}^a(x) \Gamma = \Delta^a_{(g)}(x)$$

$$\mathcal{G}^a \Gamma = \tilde{\Delta}^a,$$

(4.21)

which respectively are the quantum extension of (3.16), (3.17) and (3.20).

It is easy to show that the rigid gauge invariance also holds to all orders of perturbation theory

$$\mathcal{H}^a \Gamma = 0.$$  \hspace{1cm} (4.22)

To prove that the Slavnov identity (3.8) and the susy (3.12) are not anomalous, we adopt the method illustrated in [5], which consists in collecting the symmetries $s$ (2.7) and $\delta_\mu$ (2.11) and the translation operator $P_\mu$ (2.28) into a unique operator $Q$

$$Q \equiv s + \xi^\mu \delta_\mu + \eta^\mu P_\mu - \xi^\mu \frac{\partial}{\partial \eta^\mu}$$  \hspace{1cm} (4.23)

by means of two global anticommuting parameters $\xi^\mu$ and $\eta^\mu$, whose ghost numbers are respectively 2 and 1.

The operator $Q$ is nilpotent

$$Q^2 = 0,$$  \hspace{1cm} (4.24)

and describes a symmetry of the gauge–fixed action

$$Q(S_{inv} + S_{gf}) = 0.$$  \hspace{1cm} (4.25)

Once we have modified the source term as follows

$$S_{ext}^{(Q)} = \int d^2 x \left( - (D_\mu c)^a + \eta^\mu \partial_\nu A^a_\mu \right) + L^a \left[ \frac{1}{2} f^{abc} c^b c^c - \xi^\mu A^a_\mu + \eta^\mu \partial_\mu c^a \right]$$

$$+ \rho^a \left[ f^{abc} c^b \phi^c - \xi^\mu \varepsilon_{\mu\nu} (\partial^\nu \phi^a + \Omega^a_{\nu} - \tau^a_4 c^a) + \eta^\mu \partial_\mu \phi^a \right],$$  \hspace{1cm} (4.26)

the new action

$$\mathcal{I} = S_{inv} + S_{gf} + S_{ext}^{(Q)} + S_m$$  \hspace{1cm} (4.27)
satisfies the generalized Slavnov identity

\[ \mathcal{D}(\mathcal{I}) = 0 , \]  

(4.28)

where

\[
\mathcal{D}(\mathcal{I}) = \int d^2 x \left( \frac{\delta \mathcal{I}}{\delta \Omega^a \mu} \frac{\delta \mathcal{I}}{\delta A^a_\mu} + \frac{\delta \mathcal{I}}{\delta \rho^a \delta \phi^a} + \frac{\delta \mathcal{I}}{\delta L^a} \frac{\delta \mathcal{I}}{\delta c^a} + (Q b^a) \frac{\delta \mathcal{I}}{\delta b^a} 
+ (Q \bar{c}^a) \frac{\delta \mathcal{I}}{\delta \bar{c}^a} + (Q T_1) \frac{\delta \mathcal{I}}{\delta \tau_1} + (Q T_2) \frac{\delta \mathcal{I}}{\delta \tau_2} 
+ (Q T_3^\mu) \frac{\delta \mathcal{I}}{\delta \tau_3^\mu} + (Q T_4^\mu) \frac{\delta \mathcal{I}}{\delta \tau_4^\mu} \right) - \xi^\mu \frac{\partial \mathcal{I}}{\partial \eta^\mu} ,
\]

(4.29)

whose corresponding linearized Slavnov operator \( \mathcal{D}_\mathcal{I} \) is nilpotent

\[ \mathcal{D}_\mathcal{I} \mathcal{D}_\mathcal{I} = 0 . \]  

(4.30)

Besides the generalized Slavnov identity (4.28), the action \( \mathcal{I} \) satisfies the symmetries (4.21), (4.22) and also

\[ \frac{\partial \mathcal{I}}{\partial \xi^\mu} = \Delta_{\mu}^{(\xi)} ; \quad \frac{\partial \mathcal{I}}{\partial \eta^\mu} = \Delta_{\mu}^{(\eta)} , \]

(4.31)

where \( \Delta_{\mu}^{(\xi)} \) and \( \Delta_{\mu}^{(\eta)} \) are linear breakings

\[ \Delta_{\mu}^{(\xi)} = - \int d^2 x \left( L^a A^a_\mu + \varepsilon_{\mu \nu} \rho^a (\Omega^{a \nu} + \partial^{\nu} \bar{c}^a - \tau_4^{\nu} \bar{c}^a) \right) \]

(4.32)

\[ \Delta_{\mu}^{(\eta)} = - \int d^2 x \left( \Omega^{a \nu} \partial_\mu A^a_\nu - L^a \partial_\mu c^a + \rho^a \partial_\mu \phi^a \right) . \]

(4.33)

The following nonlinear algebra

\[ \frac{\partial}{\partial \xi^\mu} \mathcal{D}(\gamma) - D_\gamma \left( \frac{\partial \gamma}{\partial \xi^\mu} - \Delta_{\mu}^{(\xi)} \right) = (\bar{W}_\mu \gamma - \bar{\Delta}_\mu) - \left( \frac{\partial \gamma}{\partial \eta^\mu} - \Delta_{\mu}^{(\eta)} \right) 
+ \int d^2 x \ \varepsilon_{\mu \nu} \eta^\lambda \left( (\partial^{\nu} \rho^a)(\partial_\lambda \bar{c}^a) + \rho^a \bar{c}^a \partial_\lambda \tau_4^{\nu} + \rho^a \tau_4^{\nu} \partial_\lambda \bar{c}^a \right) 
+ \int d^2 x \ \varepsilon_{\mu \nu} \xi^\nu \rho^a \bar{c}^a \tau_2 \]

\[ \frac{\partial}{\partial \eta^\mu} \mathcal{D}(\gamma) + D_\gamma \left( \frac{\partial \gamma}{\partial \eta^\mu} - \Delta_{\mu}^{(\eta)} \right) = \mathcal{P}_{\mu \gamma} , \]

(4.34)

(4.35)
holds for any even ghost charged functional $\gamma$. If $\gamma$ is the quantum vertex functional $\Gamma^{(Q)}$ satisfying

$$\frac{\partial \Gamma^{(Q)}}{\partial \eta^\mu} = \Delta^{(\eta)}_\mu \quad ; \quad \frac{\partial \Gamma^{(Q)}}{\partial \xi^\mu} = \Delta^{(\xi)}_\mu$$

\begin{equation}
\mathcal{D}(\Gamma^{(Q)}) = 0 , \tag{4.37}
\end{equation}

from (4.34) and (4.35) we have

$$\bar{W}_\mu \Gamma^{(Q)} = \bar{\Delta}_\mu$$

\begin{equation}
\bar{\mathcal{S}}(\Gamma) = 0 \tag{4.41}
\end{equation}

\begin{equation}
\bar{\mathcal{P}}_\mu \Gamma^{(Q)} = 0 \tag{4.39}
\end{equation}

In particular, at vanishing global ghosts, the validity of equations (4.36) and (4.37) implies for the quantum vertex functional

$$\Gamma \equiv \Gamma^{(Q)}|_{\xi=\eta=0}$$

the identities

$$\bar{S}(\Gamma) = 0$$

\begin{equation}
\bar{W}_\mu \Gamma = \bar{\Delta}_\mu . \tag{4.42}
\end{equation}

Since the classical constraints (4.31) are easily implemented at the quantum level (4.36), the extension of the generalized Slavnov identity (4.28) to all orders of perturbation theory implies the identities (4.41), (4.42) and the translation quantum invariance, so that we recover the wanted result.

Let us now prove that the generalized Slavnov identity (4.28) is not anomalous.

According to the Q.A.P., the quantum extension of the classical identity (4.28) is

$$\mathcal{D}(\Gamma^{(Q)}) = \mathcal{A} \cdot \Gamma^{(Q)} = \mathcal{A} + O(\hbar \mathcal{A}) , \tag{4.43}
$$

where the breaking $\mathcal{A} \cdot \Gamma^{(Q)}$ is a quantum insertion whose lowest non-vanishing order in $\hbar$, $\mathcal{A}$, is an integrated local functional with Faddeev–Popov charge +1, ultraviolet dimension $d(\mathcal{A}) \leq 2$ and infrared dimension $r(\mathcal{A}) \geq 0$. 

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The breaking $\mathcal{A}$ must satisfy the following consistency conditions

\[
\frac{\delta \mathcal{A}}{\delta b^a} = 0 \quad (4.44)
\]

\[
\bar{\mathcal{G}}^a(x) \mathcal{A} = 0 \quad (4.45)
\]

\[
\mathcal{G}^a \mathcal{A} = 0 \quad (4.46)
\]

and

\[
\mathcal{D}_I \mathcal{A} = 0 \, . \quad (4.47)
\]

The constraints (4.44)–(4.46) imply that $\mathcal{A}$ does not depend on $b^a$ and on the undifferentiated ghost field $c^a$ and it depends only on the combinations (4.12). Let us write $\mathcal{A}$ as

\[
\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \mathcal{A}^{(2)} \, , \quad (4.49)
\]

according to the ultraviolet dimensions. Since the linearized Slavnov operator $\mathcal{D}_I$ does not alter the ultraviolet dimensions, the consistency condition (4.47) reads

\[
\mathcal{D}_I \mathcal{A}^{(0)} = 0 \quad (4.50)
\]

\[
\mathcal{D}_I \mathcal{A}^{(1)} = 0 \quad (4.51)
\]

\[
\mathcal{D}_I \mathcal{A}^{(2)} = 0 \, . \quad (4.52)
\]

For what concerns the first two conditions (4.50) and (4.51), we cannot find local functionals of the type (4.48) having ultraviolet dimension less than two and Faddeev–Popov charge +1, hence

\[
\mathcal{A}^{(0)} = \mathcal{A}^{(1)} = 0 \, . \quad (4.53)
\]

This implies not only the absence of anomalies with ultraviolet dimensions less than two, but also the absence of counterterms having infrared dimensions less than two, thus avoiding from now on the problems deriving from the infrared renormalization. Indeed, the general solution of equation (4.52) is

\[
\mathcal{A}^{(2)} = \mathcal{D}_I \tilde{\mathcal{A}}^{(2)} + \tilde{\mathcal{A}}^{(2)} \, . \quad (4.54)
\]
A rapid investigation reveals that
\[ r(\mathcal{A}^{(2)}) \geq 4 \]  
(4.55)
and, since the linearized Slavnov operator $\mathcal{D}_I$ can raise the infrared dimensions by at most two units, equation (4.55) implies for the counterterm
\[ r(\tilde{\mathcal{A}}^{(2)}) \geq 2, \]  
(4.56)
as required by the infrared power counting. We are then left with the task of proving the absence of anomalies
\[ \tilde{\mathcal{A}}^{(2)} = 0. \]  
(4.57)

To study the cohomology problem in equation (4.52), we filter the operator $\mathcal{D}_I$ with the filtering operator \[ \mathcal{N} = \xi^\mu \frac{\partial}{\partial \xi^\mu} + \eta^\mu \frac{\partial}{\partial \eta^\mu}, \]  
(4.58)
according to which $\mathcal{D}_I$ decomposes as
\[ \mathcal{D}_I = D_I^{(0)} + D_I^{(R)}, \]  
(4.59)
where
\[ D_I^{(0)} = \tilde{B}_m - \xi^\mu \frac{\partial}{\partial \eta^\mu}. \]  
(4.60)
The nilpotency of the linearized massive Slavnov operator (3.10) implies that
\[ D_I^{(0)} D_I^{(0)} = 0. \]  
(4.61)

We know \[ [17] \] that the space of solutions $\mathcal{A}_I^{(2)}$ of the equation (4.52) is isomorphic to a subspace of the solutions of the cohomology equation for the operator $D_I^{(0)}$
\[ D_I^{(0)} X = 0. \]  
(4.62)
Now, the cohomology sector of the operator $D_I^{(0)}$ coincides with that of the linearized ordinary Slavnov operator
\[ \tilde{B}_m \Delta = 0, \]  
(4.63)
where $\Delta$ does not depend on the global ghosts $(\xi^\mu, \eta^\mu)$, which appear in $D_I^{(0)}$ as BRS doublets [17].

Let us characterize $\Delta$ in terms of differential forms

$$
\Delta = \int \Delta^1_2(x) .
$$

Equation (4.63) can be casted into a local one

$$
\tilde{B}_\Sigma \Delta^1_2(x) + d\Delta^2_1(x) = 0 ,
$$

where $d$ is the exterior derivative and we adopted the usual notation

$$
\Delta^p_q(x) ; \begin{cases} p = \text{ghost number} \\ q = \text{form degree} \end{cases}
$$

The descent equations [18, 19] deriving from (4.63) are

$$
\tilde{B}_\Sigma \Delta^1_2 + d\Delta^2_1 = 0
$$

and

$$
\tilde{B}_\Sigma \Delta^2_1 + d\Delta^3_0 = 0
$$

The general solution of equation (4.69) is

$$
\Delta^3_0 = \sum_{n=0}^{+\infty} a_{(n)}^{[abc]} P_n c^a c^b c^c \Phi P_n + \tilde{B}_\Sigma \Delta^2_0 ,
$$

where $\{ a_{(n)}^{[abc]} P_n \}$ is a sequence of invariant tensors.

The cocycle $\Delta^3_0$ yields as candidate for the anomaly of the operator $D_I$

$$
A = \sum_{n=0}^{+\infty} \int d^2 x a_{(n)}^{[abc]}(p_1...p_n) \left( -6 \rho^a c^b c^c \phi^{p_1}... \phi^{p_n} - 6 \varepsilon^{\mu \nu} A^a_\mu A^b_\nu c^c \phi^{p_1}... \phi^{p_n} + 6 n A^a_\mu c^b c^c \tilde{\Omega}^p \phi^{p_1}... \phi^{p_n} - 2 n c^a c^b c^c L^p \phi^{p_1}... \phi^{p_n} + n(n-1) \varepsilon_{\mu \nu} c^a c^b c^c \tilde{\Omega}^p \tilde{\Omega}^{p_1} \phi^{p_2}... \phi^{p_n} \right),
$$

which does not satisfy the ghost equation (4.46), so that it must be

$$
a_{(n)}^{[abc]} P_n = 0 \implies \Delta^3_0 = \tilde{B}_\Sigma \Delta^2_0 .
$$
Equation (4.68) becomes now a problem of local cohomology
\[ \tilde{B}_{\Sigma_m}(\Delta_1^2 - d\Delta_0^2) = 0 . \] (4.73)

To solve it, we filter the operator \( \tilde{B}_{\Sigma_m} \) with
\[ \tilde{N} = \int d^2x \left( A^a_{\mu} \frac{\delta}{\delta A^a_{\mu}} + \phi^a \frac{\delta}{\delta \phi^a} + c^a \frac{\delta}{\delta c^a} + \tilde{\Omega}^{a\mu} \frac{\delta}{\delta \tilde{\Omega}^{a\mu}} + \tilde{L}^a \frac{\delta}{\delta L^a} + \rho^a \frac{\delta}{\delta \rho^a} + \tau^a_3 \frac{\delta}{\delta \tau^a_3} + \tau^a_4 \frac{\delta}{\delta \tau^a_4} \right) , \] (4.74)
according to which \( \tilde{B}_{\Sigma_m} \) decomposes as
\[ \tilde{B}_{\Sigma_m} = \tilde{B}_{\Sigma_m}^{(0)} + \tilde{B}_{\Sigma_m}^{(R)} , \] (4.75)
where
\[ \tilde{B}_{\Sigma_m}^{(0)} = \tilde{B}^{(0)} - (\tau_1 + m^2) \frac{\partial}{\partial \tau_2} , \] (4.76)
and
\[ \tilde{B}^{(0)} = \int d^2x \left( -\partial_{\mu} c^a \frac{\delta}{\delta A^a_{\mu}} + \varepsilon_{\mu\nu} \partial^\nu \phi^a \frac{\delta}{\delta \phi^a} - \partial \tilde{\Omega}^a \frac{\delta}{\delta \tilde{\Omega}^a} + \varepsilon^{\mu\nu} \partial_{\mu} A^a_{\nu} \frac{\delta}{\delta A^a_{\nu}} + \tau^a_3 \frac{\delta}{\delta \tau^a_3} \right) . \] (4.77)

The operator \( \tilde{B}_{\Sigma_m}^{(0)} \) is nilpotent
\[ \tilde{B}_{\Sigma_m}^{(0)} \tilde{B}_{\Sigma_m}^{(0)} = 0 . \] (4.78)

Now, it is easy to show \([17, 19]\) that the local cohomology of \( \tilde{B}_{\Sigma_m}^{(0)} \) can depend only on the undifferentiated fields \((c^a, \phi^a, \tau_1)\) thus solving the equations (4.67), (4.68) one finds
\[ \Delta_2^1(x) = \tilde{B}_{\Sigma_m}^{(0)} \Delta_2^0(x) + d\Delta_1^1(x) + \tilde{\Delta}_2^1(c^a, \phi^a, \tau_1) . \] (4.79)

Coming back to the integrated level and remembering that, due to the ghost condition (4.46), the anomaly cannot depend on the undifferentiated ghost \( c^a \), equation (4.79) implies that the cohomology of \( \tilde{B}_{\Sigma_m} \) is empty.

This concludes the proof of the absence of anomalies of the operator \( D_L \). Indeed, since the cohomology of \( D_L \) is isomorphic \([17]\) to a subspace of that of \( \tilde{B}_{\Sigma_m} \), we can affirm that
\[ D_L A^{(2)} = 0 \implies A^{(2)} = D_L \tilde{A}^{(2)} , \] (4.80)
which was our purpose.
5 Conclusions

The analysis we have presented shows that the infrared regularized (massive) two-dimensional BF model shares with its higher dimensionality partners the same “topological” peculiarity of being finite; hence the introduction of a mass parameter is, in this respect, harmless. Furthermore the classical relation (3.15) is easily shown to hold unaltered to all orders of perturbation theory by a straightforward application of the Quantum Action Principle [16], thanks to the fact that, lacking any counterterm, the classical and the effective action coincide. This proves that the Callan-Symanzik equation is completely trivial in this model. The mass parameter is implicitly identified by the integrated equation of motion of the ghost (3.20) which implies

$$\delta^2 \Gamma \frac{\delta \bar{c}_a(0) \delta \tilde{c}^a(0)}{\delta \psi|_{\psi=0}} = -m^2 , \ \psi = \text{all fields}$$

as a normalization condition at zero momentum and the tilde denotes the Fourier transform. The question remains of the zero mass limit of this model. There is an analogous situation in the two-dimensional nonlinear coset G/H sigma model, which also needs a mass term to be free of infrared singularities and where Elitzur conjectured [20], as was later proven [21], that the correlation functions of G invariant local operators have a well defined zero mass limit; these are identified as the local observables of the theory. In the present case, where there are no local observables, the conjecture may be transferred to the non-local ones (Wilson loops). Finally the classical equations of motion, i.e. the zero curvature constraint for the gauge field and the vanishing of the covariant derivative for the scalar, imply that $\phi^a$ lives on a sphere; this suggests that the infrared problem may appear only in perturbation theory, whereas it may not be present in the “true” model due to the compactness of the field manifold.

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References

[1] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Reports 209(1991)129 and references therein;

[2] D. Rolfsen, Knots and Links, Publish or Perish, Wilmington 1976;

[3] N. Maggiore and S. P. Sorella, Finiteness of the topological models in the Landau gauge, Lapp-Th-928-91, Nucl. Phys. B, to appear;

[4] E. Guadagnini, N. Maggiore and S. P. Sorella, Phys. Lett. B255 (1991) 65;

[5] N. Maggiore and S.P. Sorella, Perturbation theory for antisymmetric tensor fields in four dimensions, preprint GEF-TH-6/1992, UGVA-DPT 1992/04-761, Int. J. Mod. Phys. A, to appear;

[6] C. Lucchesi, O. Piguet and S.P. Sorella, Renormalizability and finiteness of topological BF theories, preprint UGVA-DPT 1992/7-773;

[7] M. Blau and G. Thompson, Ann. Phys. (N.Y.) 205 (1991) 130;

[8] E. Witten, Comm. Math. Phys. 141 (1991) 153;

[9] J. Soda, Phys. Lett. B267 (1991) 214;

[10] T. Fukuyama and K. Kamimura, Phys. Lett. B160 (1985) 259; K. Isler and C.A. Trugenberger. Phys. Rev. Lett. 63 (1989) 834; A.H. Chamseddine and D. Wyler, Phys. Lett. B228 (1989) 75; Nucl. Phys. B340 (1990) 595;
[11] N.J. Hitchin, *Proc. London Math. Soc.* 55(1987)59;

[12] O. Piguet and A. Rouet, *Phys. Reports* 76 (1981) 1;

[13] F. Delduc, F. Gieres and S. P. Sorella, *Phys. Lett.* B225 (1989) 367;  
C. Lucchesi and O. Piguet, *Phys. Lett.* B271 (1991) 350;  
C. Lucchesi and O. Piguet, *Local supersymmetry of the Chern-Simons theory and finiteness*, MPI-PH-91-101, UGVA-DPT-1991-11-754;

[14] A. Blasi, O. Piguet and S.P. Sorella, *Nucl. Phys.* B356 (1991) 154;

[15] F. Delduc, O. Piguet, C. Lucchesi and S. P. Sorella, *Nucl. Phys.* B346 (1990) 313;  
A. Blasi and R. Collina, *Nucl. Phys.* B345 (1990) 477;

[16] Y.M.P Lam, *Phys. Rev.* D6 (1972) 2145, 2161;  
T.E. Clark and J.H. Lowenstein, *Nucl. Phys.* B113 (1976) 109;

[17] J.A. Dixon, *Cohomology and Renormalization of gauge theories*, Imperial College preprint-1977;  
G. Bandelloni, *J. Math. Phys.* 28(1987)2775;

[18] J. Manes, R. Stora and B. Zumino, *Comm. Math. Phys.* 102 (1985) 157;  
B. Zumino, *Nucl. Phys.* B253 (1985) 477;

[19] O. Piguet and S. P. Sorella, *On the finiteness of the BRS modulo−d cocycle*, preprint UGVA–DPT 1992/3-759, *Nucl. Phys.* B, to appear;

[20] S. Elitzur, *Nucl. Phys.* B212 (1983) 536;

[21] C. Becchi, A. Blasi, G. Bonneau, R. Collina and F. Delduc, *Comm. Math. Phys* 120(1988)121.