Variable thermal conductivity approach for bioheat transfer during thermal ablation

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ABSTRACT
The aim of the article is to present a mathematical model that predicts tissue temperature during thermal therapy (thermal ablation). The mathematical formulation of variable thermal conductivity of thermal wave bioheat transfer model for a generalized coordinate system due to external heat source during thermal ablation has been given. The governing bioheat transfer equation is simplified by Kirchhoff’s transformation. The numerical solution of the problem has been achieved using a finite difference scheme to discretize in space coordinate; the reduced system of second order ordinary differential equation is solved by the finite element Legendre wavelet Galerkin method (FELWGM). The obtained result from FELWGM is compared with exact analytical solution and shows good agreement. Parametric study is performed to evaluate the effect of variable thermal conductivity on tissue temperature distribution for different physical parameters and illustrated graphically. Particular cases are also deduced from present investigation. The effect of variability of thermal conductivity parameter, variability of time, generalized coordinate system, lagging time and external heat source coefficient on tissue temperature is discussed in detail. Thus, the consideration of the variable thermal conductivity is extremely beneficial for the clinical therapeutic application in the treatment of cancerous cells.

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1. Introduction
For image-guided thermal treatments delivering of heat via focused ultrasound (FUS), radiofrequency (RF), microwave (MW), or laser, temperature induced changes in tissue properties are of relevance in relation to predicting tissue temperature profile, monitoring during treatment, and evaluation of treatment results. These thermal treatments cause reversible or irreversible changes in the properties of tissues. Therefore, for thermal therapies the adequate consideration of tissue properties and their temperature dependence is necessary to achieve accurate results.

Mathematical modelling of thermal therapies has been used extensively to predict and optimize clinical treatments and medical devices. Pennes (1948) model is widely used to model such problems. Kumar, Kumar, and Rai (2015a) studied the heat transfer process in biological tissue during thermal ablation in a generalized coordinate system. Non-linear dual-phase-lag model is utilized by Kumar, Kumar, and Rai (2016) in living tissue during thermal ablation. Kumar and Rai (2017) studied the time fractional DPL model in presence of temperature dependent metabolic and electromagnetic radiation heat source. Kashcooli, Salimpour, and Shirani (2017) investigated the effects of blood vessels on temperature distribution in a skin tissue subjected to various thermal therapy conditions.

Recently, Kumar, Damor, and Shukla, (2018) studied the heat transfer thermal damage in triple layer skin tissue using a fractional bioheat model. For therapeutic treatment of cancerous cells, Kumar, Singh, Sharma, and Rai (2018) used non-linear dual-phase-lag model under Dirichlet boundary conditions. Tan, Zou, Dong, Ding, and Zhao (2018) investigated the coupled effects of blood vessels and relaxation time on tissue temperature and thermal lesion region during HIFU hyperthermia.

Very few authors considered the temperature-dependent thermal parameters over wide temperature ranges in tissues. Valvano, Cochran, and Diller (1985) measured thermal conductivity and diffusivity of biomaterials over a wide temperature range. Bhattacharya and Mahajan (2003) presented the temperature-dependent thermal conductivity of sheep collagen and cow liver in vitro. Watanabe, Kobayashi, Hashizume, and Fujie (2009) investigated the effects of the temperature dependence of thermophysical properties of the liver on the temperature distribution during RFA. Trujillo and Berjano (2013) reviewed...
the mathematical functions most commonly used for modelling the temperature dependence of electrical and thermal conductivities of biological tissue in RFA.

Guntur, Lee, Paeng, Coleman, and Choi (2013) found that the thermal properties of ex vivo porcine liver tissue during RFA vary significantly with temperature. Rossmann and Haemmerich (2014) reviewed temperature-dependent thermal and electrical properties of biological tissues in the supraphysiological temperature regime. Guntur and Choi (2015) examined the effects of temperature-dependent thermal parameters on the temperature distribution in liver tissue exposed to a clinical high-intensity focused ultrasound (HIFU) field, and found that alteration of tissue thermal parameters significantly affects the prediction of temperature distributions and suggest that the dependence of thermal parameters should be considered in estimation of the thermal dose in tissues exposed to a clinical HIFU field.

Very useful work has been carried out by Ellahi et al. on thermal conductivity; notable among them are Bhatti, Zeeshan, and Ellahi (2017), Bhatti, Zeeshan, Ellahi, and Shit (2018), Bhatti, Zeeshan, Tripathi, and Ellahi (2018), Ellahi, Alamri, Sultan, Basit, and Majeed, (2018), Ellahi, Zeeshan, Shehzad, and Alamri (2018), Hassan, Marin, Ellahi, and Alamri (2018), Ijaz, Zeeshan, Bhatti, and Ellahi (2018). Akbar, Raza, and Ellahi (2016a) studied the effects of entropy and magnetic field in an endoscope for a thermal conductivity model. Akbar, Raza, and Ellahi (2016b) studied the anti-bacterial applications for new thermal conductivity model in arteries with CNT suspended nanofluid.

The present study aims to investigate the effects of variable thermal conductivity during thermal ablation for a thermal wave bioheat transfer model with modified Gaussian external heat source in a generalized coordinate system. The governing bioheat transfer equation is simplified by Kirchhoff's transformation and then finite difference scheme, and Legendre wavelet Galerkin approach is used to solve the problem. The problem is converted into a system of second order differential equation with initial conditions by using the finite difference scheme in space coordinate. Then, this reduced system is solved by the Wavelet Galerkin approach with Legendre wavelet as basis function, which reduces the problem into the Sylvester matrix equation. The solution of the Sylvester matrix equation gives the temperature distribution during thermal ablation with variable thermal conductivity and a significant effect of the variable conductivity has been observed on tissue temperature distribution.

2. Formulation of the problem

Consider a tissue of finite length ($L$), which is heated by modified Gaussian external heat source with variable thermal conductivity during thermal ablation in generalized coordinate system. Pennes (1948) performed a series of experiments that measured temperatures on human forearms and derived a thermal energy conservation equation known as bioheat transfer equation as follows:

From the principle of conservation of energy to tissue control volume ($V$), we have

$$U_{\text{gain}} = U_{\text{storage}} + U_{\text{loss}}.$$  \hspace{1cm} (1)

where $U_{\text{gain}}$, $U_{\text{storage}}$, and $U_{\text{loss}}$ are given below.
The general internal heat generation per unit volume due to metabolic heat \( q_m \) and the interior heat caused by outer heat source \( q_s \) is

\[
U_{\text{gain}} = \int_V (q_m + q_s) \, dV = \int_V q_v(x, t) \, dV. \tag{2}
\]

The total rate of stored thermal energy over the control volume is

\[
U_{\text{storage}} = \int_V \rho c \frac{\partial T(x, t)}{\partial t} \, dV. \tag{3}
\]

Heat loss to adjacent tissues by convection and conduction is given as

\[
U_{\text{loss}} = U_c - U_b, \tag{4}
\]

where

\[
U_c = \int_A q_n \, dA,
\]

is the rate production through the control volume and

\[
U_b = \int_V q_b \, dV,
\]

the total amount of thermal energy over the tissue control volume.

Using Equations (2)–(6) in Equation (1), yields

\[
\int_V q_v(x, t) \, dV = \int_V \rho c \frac{\partial T(x, t)}{\partial t} \, dV + \int_A q_n \, dA - \int_V q_b \, dV. \tag{7}
\]

Applying the divergence theorem to the surface integral and observing that Equation (7) must hold for any arbitrary volume element, we obtain

\[
\rho c \frac{\partial T}{\partial t} - q_b - q_m - q_t = - \frac{\partial q}{\partial r}. \tag{8}
\]

Cattaneo (1958), Vernotte (1958) proposed the following modified heat flux model that is a linear extension of the Fourier’s law in order to take in the wave-like behaviour in heat conduction process,

\[
\left(1 + \tau_q \frac{\partial}{\partial t}\right) q = -k(T) \frac{\partial T}{\partial r}. \tag{9}
\]

where \( k(T) \) is the thermal conductivity with variable temperature.

Using Equation (9) in Equation (8) with temperature dependent specific heat yields

\[
\left(1 + \tau_q \frac{\partial}{\partial t}\right) \left( \rho c(T) \frac{\partial T}{\partial t} - q_b - q_m - q_t \right) = \frac{\partial}{\partial r} \left( k(T) \frac{\partial T(r, t)}{\partial r} \right). \tag{10}
\]

Equation (10) in generalized coordinate system can be written as

\[
\left(1 + \tau_q \frac{\partial}{\partial t}\right) \left( \frac{k(T)}{\kappa} \frac{\partial T}{\partial t} - q_b - q_m - q_t \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 k(T) \frac{\partial T}{\partial r} \right), 0 < r < L, t > 0, \tag{11}
\]

where \( \Gamma = 0, 1, 2 \) corresponds to Cartesian, axis-symmetric, and spherical symmetric coordinate system and

\[
\rho c(T) = \frac{k(T)}{\kappa}. \tag{12}
\]

The term \( q_b \) can be expressed as

\[
q_b = \omega_b \rho_b c_b (T_a - T). \tag{13}
\]

The term \( q_m \) can be approximated as a linear function of local tissue temperature taken in Gupta, Singh, and Rai (2010),

\[
q_m = q_{m0} \left(1 + \frac{T - T_0}{T_0}\right). \tag{14}
\]

Following (Kumar, Kumar, & Rai, 2015b) modified Gaussian heat source \( q_s \) is taken as

\[
q_s(r, t) = \rho \mathcal{S} e^{-r^2/4(r-n)^2}. \tag{15}
\]

To solve the problem, the following initial and boundary conditions are taken:

**Initial conditions**

\[
T(r, 0) = T_0 \text{ and } \frac{\partial T(r, t)}{\partial t} = 0. \tag{16}
\]

**Boundary conditions**

\[
T(0, t) = T_w. \tag{17}
\]

**Symmetric condition**

\[
\frac{\partial T(L, t)}{\partial r} = 0. \tag{18}
\]

Considering the mapping (Kirchhoff’s transformation), following Hetnarski (1986) as

\[
\vartheta = \frac{1}{k_0} \int_0^T T(z) \, dz, \tag{19}
\]

yields

\[
k_0 \frac{\partial \vartheta}{\partial r} = k(T) \frac{\partial T}{\partial r}, \quad k_0 \frac{\partial^2 \vartheta}{\partial r^2} = \frac{\partial}{\partial r} \left( k(T) \frac{\partial T}{\partial r} \right), \quad k_0 \frac{\partial \vartheta}{\partial r} = k(T) \frac{\partial T}{\partial r}. \tag{20}
\]

An important factor to achieve realistic models is the use of mathematical functions to describe the temperature dependence of thermal properties of tissue. A commonly used approach for modelling the temperature dependence of thermal properties for temperature below 100°C is based on linear equations and employs constant temperature coefficient such as:

\[
k(T) = k_0(1 + k_1 T). \tag{21}
\]

From the above, we have the following relations:

\[
\vartheta = T + k_1 T^2 \frac{\partial \vartheta}{\partial r} \left(1 + k_1 T \right), \quad \frac{\partial \vartheta}{\partial r} = \frac{\partial T}{\partial r} \left(1 + k_1 T \right), \tag{22}
\]

and
\[ T = -\frac{1 + \sqrt{1 + 2k_1^3}}{k_1}. \]  

Equations (11) and (16–18) with the aid of Equations (19–21) in linear form become

\[ \begin{align*}
(1 + \tau_x \frac{\partial}{\partial r}) \left[ k_0 \frac{\partial}{\partial r} - q_b - q_m - q_s \right] &= k_0 \frac{\partial^2}{\partial r^2} + \frac{\Gamma}{r} k_0 \frac{\partial^2}{\partial r^2}, \\
\dot{\psi}(r, 0) &= T_0 + \frac{k_1}{2} T_0^2 \frac{\partial}{\partial r}(r, 0) = 0, \\
\dot{\psi}(0, t) &= T_w + \frac{k_1}{2} T_w^2 \frac{\partial}{\partial r}(0, t) = 0. 
\end{align*} \]

3. Solution of the problem

Introducing the non-dimensional variable and similarity criteria

\[ \begin{align*}
x &= \frac{x}{L}, & x^* &= \frac{x}{L}, & F_0 &= \omega_0 t, & \dot{\psi} &= \frac{\psi - T_0}{T_0}, & \dot{\theta} &= \frac{\theta - T_0}{T_0}, \\
\dot{\psi}_a &= \frac{T_a - T_0}{T_0}, & \dot{\psi}_w &= \frac{T_w - T_0}{T_0}, \\
K_1 &= k_1 T_0, & F_{0q} &= \omega_0 \tau_q, & a &= a_0 L, & D_0 &= \sqrt{\frac{W_0 C_0 L^2}{k_0}}, \\
P_{m0} &= \frac{q_0 a L^2}{T_0 k_0}, & P_{mP} &= \frac{q_0 a L^2}{T_0 k_0}, \\
W_0 &= \omega_b \rho_b, & \omega_0 &= \frac{k_0}{\rho C L^2}, & d &= .1 T_0. 
\end{align*} \]

With the help of Equation (27), Equations (24–26) reduce to the following form

\[ \begin{align*}
F_{0q} \frac{\partial^2 \Theta}{\partial F_0^2} + (1 + F_{0q} P_t^2 - F_{0q} P_{m0} d) \frac{\partial \Theta}{\partial F_0} - (P_{m0} d - P_t^2) \Theta &= \frac{\partial^2 \Theta}{\partial x^2} + \frac{\Gamma}{x} \frac{\partial \Theta}{\partial x} + P_t^2 \dot{\psi}_a + P_{m0} + P_{mP} e^{-2(x-x')^2}. 
\end{align*} \]
The Equation (34) is converted into a system of second order ordinary differential equation in vector form as follows:

\[
F_0 \frac{d^2 \Theta}{dF_0^2} + (1 + F_0 P_0^2 - F_0 P_0 d) \frac{d \Theta}{dF_0} - (P_0 d - P_1) \Theta - A \Theta = B,
\]

(37)

subjected to initial condition

\[
\Theta(0) = K_1, \quad \text{and} \quad \frac{d \Theta(0)}{dF_0} = 0,
\]

(38)

where \( n = 2^{k-1}M, I \) is the identity matrix of order \( n \), \( A \) is a square matrix of order \( 2^{k-1}M \times 2^{k-1}M \); \( B \), \( \Theta(0) \), and \( \frac{d \Theta(0)}{dF_0} \) are the column matrix of \( 2^{k-1}M \times 1 \) which are given as follows

\[
A = \frac{1}{h^2} \begin{bmatrix}
-2 & 1 + \frac{\Gamma h}{2x_1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 + \frac{\Gamma h}{2x_2} & -2 & 1 + \frac{\Gamma h}{2x_2} & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 + \frac{\Gamma h}{2x_3} & -2 & 1 + \frac{\Gamma h}{2x_3} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 - \frac{\Gamma h}{2^{2n-1}} & 17 & 1 + \frac{\Gamma h}{2x_n} \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & -9 & 1 + \frac{\Gamma h}{2^{2n-1}} & 13 & 1 + \frac{\Gamma h}{2x_n}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
V_0 \left( \phi \frac{1}{x_n} + \frac{1}{x_n} \right) + B_1 & B_2 & B_3 & \cdots & B_i & \cdots & B_n
\end{bmatrix}^T,
\]

where

\[
B_i = P_0^2 \Theta_i + P_{m0} + P_0 e^{-\rho^{2}(x_i-x)^2}, i = 1, 2, \ldots, n.
\]

\[
\Theta(0) = \begin{bmatrix} \Theta_1(0) & \Theta_2(0) & \cdots & \Theta_i(0) & \cdots & \Theta_n(0) \end{bmatrix}^T,
\]

and

\[
\frac{d \Theta(0)}{dF_0} = \begin{bmatrix} \frac{d \Theta_1(0)}{dF_0} & \frac{d \Theta_2(0)}{dF_0} & \cdots & \frac{d \Theta_i(0)}{dF_0} & \cdots & \frac{d \Theta_n(0)}{dF_0} \end{bmatrix}^T.
\]

3.2. Legendre wavelet Galerkin approach

To solve the differential Equation (37), let us assume that

\[
\frac{d^2 \Theta}{dF_0^2} = C^T \psi(F_0),
\]

(39)

where

\[
C^T \psi(F_0) = \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{M-1} C_{n,m} \psi_{n,m}(F_0)
\]

and

\[
C_{n,m} = \int_{F_0} f(F_0) \psi_{n,m}(F_0) dF_0,
\]

(40)

and \( C^T \) is unknown \( 2^{k-1}M \times 2^{k-1}M \) matrix, and \( \psi(F_0) \) is a \( 2^{k-1}M \times 1 \) matrix defined as

\[
C = \begin{bmatrix} C_{1,0}, \ldots, C_{1,M-1}, C_{2,0}, \ldots, C_{2,M-1}, \ldots, C_{2^{k-1},0}, \ldots, C_{2^{k-1},M-1} \end{bmatrix}^T.
\]

and

\[
\psi(F_0) = \begin{bmatrix} \psi_{1,0}, \ldots, \psi_{1,M-1}, \psi_{2,0}, \ldots, \psi_{2,M-1}, \ldots, \psi_{2^{k-1},0}, \ldots, \psi_{2^{k-1},M-1} \end{bmatrix}^T.
\]

The Legendre wavelet \( \psi_{m,n}(F_0) = \psi(k, \hat{n}, m, F_0) \), \( \hat{n} = 2n-1, m = 0, 1, \ldots, M-1, k = 1, 2, \ldots, n = 1, 2, \ldots, 2^k-1 \), \( k \) is any positive integer, defined on the interval \( [0,1] \) by Razzaghi, and Yousefi (2001)

\[
\psi_{m,n}(F_0) = \begin{cases} \sqrt{(m+1/2)2^{k-1}}P_m(2\hat{n}F_0-n), & \hat{n} \leq \frac{n+1}{2} \\ 0, & \text{otherwise} \end{cases}
\]

(41)

where \( P_m(F_0) \) is the well known Legendre polynomials of degree \( m \) which are given as follows:

\[
P_0(F_0) = 1,
\]

\[
P_1(F_0) = F_0,
\]

and

\[
P_{m+1}(F_0) = \frac{1}{m+1} \left[ (2m+1)P_0P_m(F_0) - mP_{m-1}(F_0) \right],
\]

\[
m = 1, 2, 3, \ldots M-1.
\]

Integrating Equation (39) with respect to \( F_0 \) from 0 to \( F_0 \), we have

\[
\frac{d \Theta(0)}{dF_0} = C^T \psi(F_0),
\]

(42)

where \( P \) is \( 2^{k-1}M \times 2^{k-1}M \) operational matrix of integration (Razzaghi, & Yousefi, 2001) given as follows

\[
P = \frac{1}{2^k} \begin{bmatrix} L & F & F & \cdots & F \\
0 & L & F & \cdots & F \\
0 & 0 & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & L & F \\
0 & 0 & \cdots & 0 & L
\end{bmatrix},
\]

(43)

where \( L \) and \( F \) are \( M \times M \) matrices given by:
\[
L = \begin{pmatrix}
1 & \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\
\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{15}} & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{\sqrt{15}} & 0 & \frac{1}{\sqrt{35}} & \cdots & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{35}} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \sqrt{2M-1}/(2M-1) & 0
\end{pmatrix}, \quad (44)
\]

and
\[
F = \begin{pmatrix}
2 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}. \quad (45)
\]

Again integrating Equation (42) with respect to \( F_0 \) from 0 to \( F_0 \),
\[
\Theta(F_0) = C^T P^2 \psi(F_0) + \Theta_0, \quad (46)
\]
where \( \Theta_0 = \Theta(0) \).

Using Equations (39), (42), and (46) in Equation (37), we have
\[
AC^T C_1 - C^T + D = 0, \quad (47)
\]
where
\[
D = (B + (P_{\text{mod}} - P_1^2)) \Theta_0 + A \Theta_0) d_1^T H^{-1}, \quad C_1 = P^2 H^{-1},
\]
\[
H = F_0 I_n + (1 + F_0 P^2 - F_0 P_{\text{mod}}) P_{\text{mod}} - (P_{\text{mod}} - P_1^2) P^2, \quad I_n \text{ is the identity matrix of order } n \text{ and } d_1 \text{ is } n \times 1
\]

matrix such that \( d_1^T \psi(F_0) = 1 \).

Equation (47) is Sylvester matrix equation, after solving this equation, we obtain the element of matrix \( C_1 \). Substituting the value of \( C_1 \) in Equation (46) gives tissue temperature distribution \( \Theta(F_0) \). The tissue temperature in terms of \( T \) can be obtained from Equation (23).

**Particular cases:**

i. If we take \( K_1 = 0 \) in Equation (47), the problem is reduced to the non-Fourier bioheat transfer model during thermal ablation and the corresponding results are the same as obtained by Kumar, Kumar, and Rai (2015a).

ii. If we take \( \tau_q = 0 \) in Equation (47), then we obtain results for the Pennes’ bioheat transfer model with variable thermal conductivity during thermal ablation.

iii. If we take \( \tau_q = 0 \) and \( K_1 = 0 \) in Equation (47), then we obtain results for the Pennes’ bioheat transfer model in the absence of variable thermal conductivity during thermal ablation.

**4. Numerical results and discussion**

Yadav, Kumar, and Rai (2014) showed that the Legendre wavelet Galerkin method produces a higher degree of accuracy and minimizes error. That is why we used this approach for numerical procedure. This approach combines both the multi-resolution and multi-scale computational property of Legendre wavelet with element-wise analysis. Multi-resolution analysis of Legendre wavelet localizes small scale variation of solution and fast switching of functional bases. A brief algorithm of the numerical method is given in Table 1.

The problem is solved analytically by Laplace transform in the absence of modified Gaussian external heat source, which is given as follows:

By taking the Laplace transform of Equation (28) and applying the initial conditions (29), the following ordinary differential equation is obtained

**Table 1.** Algorithm procedure.

| Algorithm MATLAB programming procedure of FELWGM |
|--------------------------------------------------|
| **Step 1:** Introduce time variable \( F_0 \) and other expressions required in computation; |
| **Step 2:** For \( F_0 \in [0, 1] \); |
| **Step 3:** Fixed the value of \( M \) and \( k \); |
| **Step 4:** Define Legendre Wavelet \( \varphi_{nm} \) given in Equation (41) and the matrices \( P, L, F \) given in Equations (43) (44) and (45); |
| **Step 5:** \( C_{\text{mod}} = \begin{bmatrix} C_{1,0} & \cdots & C_{1,M-1} & C_{2,0} & \cdots & C_{2,M-1} & \cdots & C_{P-1,0} & \cdots & C_{P-1,M-1} \end{bmatrix}^T \), where \( C \) is unknown matrix of order \( 2^{P-1}M \times 2^{P-1}M \); |
| **Step 6:** Solve Sylvester matrix Equation (47) to obtain \( C_1 \), where \( C_1 = \text{transpose}(C) \) by using the command `sylvester` |
| **Step 7:** \( \Theta(F_0) = \text{transpose}(C) \times P^2 \times \psi(F_0) + \Theta_0 \), where \( \Theta_0 \) is initial temperature given in Equation (38); |
| **Step 8:** end |
Figure 2. Dimensionless tissue temperature response along dimensionless depth with different values of time for $K_1 = 0.1$.

Figure 3. Dimensionless tissue temperature response along dimensionless depth for generalized coordinate systems.

Figure 4. Dimensionless tissue temperature profile for different value of relaxation time.
The exact solution of Equation (48), by using the Laplace transform of boundary conditions (30–31), is

\[ \Theta(x, s) = \frac{1}{\cosh(\sqrt{\beta})} \left( V_w - \frac{h}{\beta} \right) \cosh(\sqrt{\beta}(x - 1)) + \frac{h}{\beta s}, \]

(50)

The inverse Laplace transform of \( \Theta(x, s) \) can be obtained from the following Bromwich contour integration (Arpaci, 1996)

\[ \Theta(x, F_0) = \frac{1}{2\pi i} \lim_{C \to \infty} \int_{C-i\infty}^{C+i\infty} e^{st} \Theta(x, s) ds. \]

(51)

Using the inversion theorem, the inverse Laplace transform of Equation (50) is

\[ \Theta(x, F_0) = \frac{1}{2\pi i} \lim_{C \to \infty} \int_{C-i\infty}^{C+i\infty} e^{st} \Theta(x, s) ds. \]

(52)

By using Bromwich contour integration (Arpaci, 1996), temperature distribution in tissue is

\[ \Theta(x, F_0) = \frac{V_w \cosh(\sqrt{g}(x-1))}{\cosh(\sqrt{g})} + \sum_{m=1}^{2} \sum_{n=1}^{\infty} \frac{V_m \cos(\lambda_m(x-1)) 2ij\lambda_m}{s_{nm} \sin(\lambda_m)(2es_{nm} + f)} e^{es_{nm}F_0} + \sum_{m=1}^{2} \sum_{n=1}^{\infty} \frac{2h \cos(\lambda_m(x-1))}{s_{nm} \sqrt{\beta_{is_{nm}}}} e^{es_{nm}F_0}, \]

(53)

where

\[ e = F_0q, f = (1 + F_0q^2 - F_0qP_0m), g = -(P_0m - P_0^2), h = P_0^2 \beta_a + P_0m, s_{nm} = \frac{-\sqrt{\beta_{is_{nm}}}}{2\pi}, \]

\[ \lambda_m = 2n + 1, n = 1, 2, 3, \ldots \]

The approximate solution (FELWGM) has compared with analytical solution given by Equation (53). Figure 6 shows good agreement between the results obtained by the numerical and exact solution.
The value of thermo-physical parameters is similar to the values of parameters as in the literature (Kumar, Kumar, & Rai, 2015a).

\[ \rho = 1000 \text{ kg m}^{-3}, \quad C = 4.18 \times 10^{3} \text{ J kg}^{-1} \text{ K}^{-1}, \]

\[ T_{0} = T_{a} = T_{w} = 37^\circ \text{C}, \quad L = 0.05 \text{ m}, \]

\[ q_{r0} = 7.58 \times 10^{3} \text{ W m}^{-3}, \quad q_{m0} = 1.091 \times 10^{3} \text{ W m}^{-3}, \]

\[ W_{0} = 0.5 \text{ kg m}^{-1} \text{ s}^{-1}, \quad k_{0} = 0.5 \text{ W m}^{-1} \text{ K}, \quad a_{0} = 1 \times 10^{2} \text{ m}^{-1}, \]

\[ r_{0} = 0.025 \text{ m}, \quad t_{q} = 20 \text{ s}, \quad t = 6 \text{ min}. \]

MATLAB(R2016a) software has been used for all computational work. Thermal ablation refers to the destruction of tissue by extreme hyperthermia (elevated tissue temperatures) or hypothermia (depressed tissue temperatures). The temperature change is concentrated to a focal zone in and around the tumour. The high temperature occurs at the target region for four to six minutes during the process.

Figure 1 shows the effect of variable thermal conductivity parameter \( K_{1} \) on tissue temperature distribution. It is noted that as the value of \( K_{1} \) increases, the temperature increases and the temperature attains maxima at the heating position \( x^{*} = 0.5 \). Thus, to achieve more accuracy in clinical treatments variable thermal conductivity plays an important role.

Figure 2 shows the effect of time on tissue temperature along the space coordinate when \( K_{1} = 0.1 \). It is evident that temperature increases as the time increases from four minutes to six minutes during thermal ablation. Thus, it is noted that time duration affects the treatment process.

Figure 3 shows the effect of a generalized coordinate system on the treatment of tumour during thermal ablation for \( K_{1} = 0 \) and \( K_{1} = 0.1 \). It is noted that for all three coordinate systems, tissue temperature remains unaffected at the heating position \( x^{*} = 0.5 \) for both the cases, but for \( x>x^{*} \) and \( x<x^{*} \), it affects the tissue temperature.

Figure 4 shows the comparison of temperature distribution for Pennes (\( s_{q} = 0 \)) and thermal wave model (\( s_{q} = 20 \text{s} \)). It is observed that temperature profile for the thermal wave model is less than the temperature profile predicted by the Pennes’ model for both the cases of \( K_{1} = 0 \) and \( K_{1} = 0.1 \). But, independently, both models predict higher temperature with the variable thermal conductivity parameter. Thus, lagging time is affecting the temperature distribution during the treatment.

Figure 5 shows the effects of external heat source on temperature distribution for \( K_{1} = 0 \) and \( K_{1} = 0.1 \). It is clear that as the value of \( K_{1} \) and \( q_{r0} \) increases, temperature also increases. Thus, the consideration of external heat source is important during the treatment. Figure 7 depicts the absolute error of the solutions obtained by finite element Legendre wavelet transform.

Table 2. The absolute error between the solutions obtained by FELWGM and analytical method.

| Time  | FELWGM     | Exact solution | Absolute error |
|-------|------------|----------------|----------------|
| 0     | 37.219842251500538 | 37.219842341320123 | 0.0090 \times 1.0e−05 |
| 100   | 37.31919280829554  | 37.319183170819544 | 0.9638 \times 1.0e−05 |
| 200   | 37.35916731208197  | 37.35916631208289  | 0.0801 \times 1.0e−05 |
| 300   | 37.369163646412476 | 37.369163916414714 | 0.0275 \times 1.0e−05 |
| 400   | 37.398599167830516 | 37.39859955830201  | 0.0390 \times 1.0e−05 |
| 500   | 37.409153152968211 | 37.409153429124091 | 0.0276 \times 1.0e−05 |

Figure 7. Absolute error.
The Galerkin method (FELWGM) and analytical method. Temperature (T) for time (t) 0 to 500 s is evaluated by both the methods and results are given in Table 2.

Therefore, we can say that the consideration of variable thermal conductivity in bioheat transfer model is very beneficial for the clinical therapeutic application in the treatment of cancerous cells.

5. Conclusions

The problem of modelling the variable thermal conductivity of thermal wave bioheat transfer model in a generalized coordinate system due to external heat source during thermal ablation has been investigated. Kirchhoff’s transformation is used to formulate the governing bioheat transfer equation. The numerical solution of the problem is achieved by using a finite difference scheme and Legendre wavelet Galerkin approach. FELWGM converts the problem into a system of algebraic equation, so it reduces the computational complexity and avoids large storage requirements, as in the case of conventional discretize schemes. The results obtained from the FELWGM are compared with the analytical results and give good accuracy in the absence of external heat source. The significant effect of variable thermal conductivity parameters, variability of time, lagging times, and external heat source is observed on tissue temperature distribution. Therefore, the temperature dependent changes need to be considered to enhance the accuracy of predicting and monitoring pre-clinical and clinical treatments and consideration of this model is useful in image-guided thermal treatments (e.g. hyperthermia and thermal ablation) and other biomedical treatments. It will give new direction in clinical therapeutic application.

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Disclosure statement

The authors have no conflict of interest.

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