LIFTING OF CURVES WITH AUTOMORPHISMS

ANDREW OBUS

Abstract. The lifting problem for curves with automorphisms asks whether we can lift a smooth projective characteristic $p$ curve with a group $G$ of automorphisms to characteristic zero. This was solved by Grothendieck when $G$ acts with prime-to-$p$ stabilizers, and there has been much progress over the last few decades in the wild case. We survey the techniques and obstructions for this lifting problem, aiming at a reader whose background is limited to scheme theory at the level of Hartshorne’s book. Throughout, we include numerous examples and clarifying remarks. We also provide a list of open questions.

1. Introduction

A fundamental success of algebraic geometry is its ability to reason about algebraically defined objects in characteristic $p$ using geometric intuition gleaned from our experience in the real world. For instance, not only can we define concepts such as smoothness and tangent spaces in characteristic $p$, but such concepts also turn out to reflect our characteristic zero intuition surprisingly well.

Of course, characteristic $p$ geometry does not behave exactly like geometry in characteristic zero! The differences are too numerous to count, but let us quickly mention one example, which will be motivating for the discussion to come. Consider the affine line $X = \mathbb{A}^1_k$. There are certainly no nontrivial finite étale covers $Y \to X$, because any such cover would give a topological cover $Y(\mathbb{C}) \to X(\mathbb{C})$ with the complex topology (e.g., [Sza09, Proposition 4.5.6]), contradicting the fact that $X(\mathbb{C}) \cong \mathbb{C}$ is simply connected. However, in characteristic $p$ we have the following example.

Example 1.1. Let $k$ be a field of characteristic $p$ (say, algebraically closed, to preserve the analogy as well as possible). The map from the zero-locus $Y$ of $y^p - y = x$ in $\mathbb{A}^2_k$ to $\mathbb{A}^1_k$ given by projecting to the $x$-coordinate is a finite étale cover. Indeed, the group $\mathbb{Z}/p$ acts freely on $Y$ by sending $(x, y)$ to $(x, y + 1)$, and the projection is nothing but the quotient map.

In fact, there are a wealth of finite étale covers of $\mathbb{A}^1_k$. The famous Abhyankar’s conjecture (now a result of Raynaud ([Ray94]) and later generalized by Harbater ([Har94])) shows that for every $G$ generated by its $p$-Sylow subgroups (in particular,

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2010 Mathematics Subject Classification: Primary 14H37, 12F10; Secondary 11G20, 12F15, 13B05, 13K05, 14G22, 14H30.

Date: September 14, 2018.

Key words and phrases. branched cover, lifting, Galois group, Oort conjecture, wild ramification, curves with automorphisms.

The author was supported by NSF Grants DMS-1265290 and DMS-1602054.

1For our purposes, a branched cover $f : Y \to X$ is a finite, flat, generically étale morphism of geometrically connected, normal schemes. If $f$ is unramified, we say it is an étale cover.
for every simple group $G$ with order divisible by $p$), there exists a finite étale $G$-Galois cover $Y \to \mathbb{A}^1_k$. The lifting problem and local lifting problem that are the subject of this paper are inspired by understanding the relation between branched covers in characteristic $p$ and in characteristic zero.

1.1. Riemann’s existence theorem. Our starting point is the following theorem (see [Har03, Theorem 2.1.1]).

**Theorem 1.2. (Riemann’s existence theorem)** Let $X$ be a smooth, projective algebraic curve over $\mathbb{C}$, and let $f^{\text{top}} : Y^{\text{top}} \to X(\mathbb{C})$ be a finite topological branched cover \footnote{Recall that a *topological branched cover* is a continuous map that is a topological cover away from a nowhere dense set of points.} with finitely many branch points. Then there exists a branched cover $f : Y \to X$, unique up to isomorphism, such that the corresponding topological cover $f^{\text{an}} : Y(\mathbb{C}) \to X(\mathbb{C})$ (in the complex topology) is isomorphic to $f^{\text{top}}$ (that is, there is a homeomorphism $i_Y : Y(\mathbb{C}) \to Y^{\text{top}}$ such that $f^{\text{top}} \circ i_Y = f^{\text{an}}$).

Since topological covers correspond to quotients of fundamental groups, and fundamental groups of Riemann surfaces are well understood, this leads to combinatorial/group-theoretic parameterizations of branched covers of algebraic curves over $\mathbb{C}$. For instance, degree $n$ covers of $\mathbb{P}^1$ branched at $r + 1$ points $\{x_0, \ldots, x_r\}$ correspond to index $n$ subgroups of the topological fundamental group of $\mathbb{P}^1 \setminus \{x_0, \ldots, x_r\}$, which is the free group $F_r$.

**Example 1.3. (Belyi maps)** A famous result of Belyi ([Bel79]) states that every algebraic curve $X$ defined over $\overline{\mathbb{Q}}$ can be expressed as a branched cover of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ étale outside $\{0, 1, \infty\}$. Such covers are called *three-point covers*. By Riemann’s existence theorem, these three-point covers correspond to finite index subgroups $N$ of $F_2$. The cover is *Galois* with group $G$ if and only $N$ is normal in $F_2$ and $F_2/N \cong G$.

**Remark 1.4.** The situation in characteristic $p$ is quite different. In fact, every smooth geometrically connected curve $X$ defined over a perfect field $k$ of characteristic $p$ has a map to $\mathbb{P}^1_k$ étale outside $\{\infty\}$, see [Kat88, Lemma 16], or [Abh57, Remark 4] for the case when $k$ is algebraically closed. In fact, if there exists an exact differential form on $X$ whose divisor is supported at one $k$-point and $k$ is algebraically closed, then one can force the map to have only one ramification point over $\infty$ ([Zap08, Theorem 1]).

The earliest results on the lifting problem were motivated by the search for a characteristic $p$ version of Riemann’s existence theorem. That is, is there a way to parameterize branched covers of curves in characteristic $p$ in terms of well-understood group theory? In some sense, obtaining a full answer is hopeless. For instance, even if we restrict ourselves to branched covers of $\mathbb{P}^1$ branched only at $\infty$, there are infinitely many linearly disjoint $\mathbb{Z}/p$-covers (take the smooth projective completion of the affine cover $V(y^p - y - x^N) \subseteq \mathbb{A}^2 \to \mathbb{A}^1_k$ given by projecting to the $x$-axis, as $N$ ranges through $\mathbb{N}\setminus p\mathbb{N}$). This tells us that the fundamental group of $\mathbb{A}^1_k$, for $k$ an algebraically closed field of characteristic $p$, is not finitely generated (since it has an infinitely large elementary abelian $p$-quotient).
However, the situation is better when we restrict our attention to tame covers, that is, covers where $p$ does not divide the ramification indices. In this situation, Grothendieck showed that each branched cover in characteristic $p$ is the reduction of a branched cover in characteristic zero, which is more or less unique.

**Theorem 1.5.** (Grothendieck’s “tame Riemann existence converse” theorem in characteristic $p$) Let $R$ be a complete discrete valuation ring with algebraically closed residue field $k$ of characteristic $p$ and fraction field $K$ with algebraic closure $\overline{K}$. Let $X_R$ be a smooth, projective, relative $R$-curve with special fiber $X$ and generic fiber $X_K$, and let $x_1,R,\ldots,x_n,R$ be pairwise disjoint sections of $X_R \to \text{Spec } R$. Write $x_i$ for the intersection of $x_i,R$ with $X$.

If $f : Y \to X$ is a tamely ramified finite cover, étale above $X\setminus\{x_1,\ldots,x_n\}$, then there is a unique finite flat branched cover $f_R : Y_R \to X_R$, étale above $X_R\setminus\{x_1,R,\ldots,x_n,R\}$ with the same ramification indices as that of $f$, such that the special fiber of $f_R$ is $f$. If $f$ is $G$-Galois, then so is $f_R$.

**Proof.** If $f$ is étale, the theorem follows from Grothendieck’s theory of étale lifting ([SGA03 I, Corollaire 8.4], combined with [SGA03 III, Prop. 7.2]). In general, we can use Grothendieck’s theory of tame lifting ([SGA03 XIII, Corollaire 2.12], or [Wew99] for an exposition). See also [Ful69] for an alternate proof. If $f$ is a Galois cover, the theorem also follows from the local-global principle (Theorem 3.1), along with Example 1.3.

**Remark 1.6.** The cover $f_R$ is called a lift of the cover $f$ over $R$. Thus, the tame Riemann existence converse can be stated succinctly as “tame covers lift over $R$ (uniquely once the branch locus is fixed).” See 1.2.

**Remark 1.7.** Assume without loss of generality that $k$ is countable (any morphism of varieties in characteristic $p$, being given by finitely many equations, can be defined over some finitely generated extension of $\mathbb{F}_p$, whose algebraic closure is thus countable). If $R$ is a complete discrete valuation ring with residue field $k$ and fraction field $K$, then it is an easy exercise to show that $|R| = |K| = |\overline{K}| = |\mathbb{C}|$, and thus that char($R$) = 0 implies $\overline{K} \cong \mathbb{C}$, since they are both algebraically closed fields of characteristic zero with the same cardinality.

Suppose char($R$) = 0 above. Then Theorem 1.5 shows that once the branch locus is fixed, each tame branched cover in characteristic $p$ gives rise to a branched cover over $R$, thus a branched cover over $\overline{K} \cong \mathbb{C}$ via base change, and thus to a topological branched cover of $X(\mathbb{C})$. This is why we call the theorem a Riemann existence converse theorem: It shows that any tame algebraic cover in characteristic $p$ comes from a topological cover over $\mathbb{C}$. Example 1.3 shows that this is not true for wild covers!

**Remark 1.8.** Theorem 1.5 has the following interpretation in terms of fundamental groups. Write $U = X\setminus\{x_1,\ldots,x_n\}$ and write $U_{\overline{K}} = (X_R\setminus\{x_1,R,\ldots,x_n,R\}) \times_R \overline{K}$. Let $\pi_1(U)$ be the tame fundamental group of $U$ (that is, the automorphism group of the pro-universal tame cover of $X$, étale above $U$), and let $\pi_1(U_{\overline{K}})$ be the $p$-tame fundamental group of $U_K \times_K \overline{K}$ (same definition, but replace “tame” with “ramified of prime-to-$p$ index”). Then there is a natural surjection $\phi : \pi_1(U_{\overline{K}}) \to \pi_1(U)$, well defined up to conjugation. See [Obu12] Remark 2.4 for details on this surjection.

In particular, when char($R$) = 0, this shows that the tame fundamental group of a characteristic $p$ curve is a quotient of the $p$-tame fundamental group of a characteristic zero curve of the same genus with the same number of missing points.
Remark 1.9. The surjection $\phi$ from Remark 1.8 descends to an isomorphism
\[ \phi' : \pi_1^{(p')} (U_{\overline{K}}) \to \pi_1^{(p')} (U), \]
where $\pi_1^{(p')}$ represents the maximal prime-to-$p$ quotient of $\pi_1$ ([SGA03 XIII, Corollaire 2.12]). This could be called the “prime-to-$p$ Riemann existence theorem” in characteristic $p$, as $\phi'$ gives a bijection on finite-index normal subgroups, thus placing the prime-to-$p$ Galois covers of $X_{\overline{K}}$, étale above $U_{\overline{K}}$, in natural one-to-one correspondence with the prime-to-$p$ Galois covers of $X$, étale above $U$.

Of course, we would like to know whether wild (i.e., not tame) covers in characteristic $p$ come from characteristic zero as well. This will be our main focus.

1.2. Lifting of covers of curves. Let $k$ be an algebraically closed field of characteristic $p$. We formulate the lifting problem precisely:

Question 1.10. (The lifting problem for covers of curves) Let $G$ be a finite group acting on a smooth, projective curve $Y$ over $k$, so that $f : Y \to X = Y/G$ is a branched $G$-cover of smooth, projective curves over $k$. Is there a characteristic zero discrete valuation ring $R$ with residue field $k$ and a branched $G$-cover $f_R : Y_R \to X_R$ of smooth (in particular, flat) relative $R$-curves with special fiber $f$?

If the answer to the question above is “yes,” then we say that $f$ lifts to characteristic zero (or lifts over $R$), and that $f_R$ is a lift of $Y$ (with $G$-action).

In what follows, we will refer to the lifting problem for covers of curves simply as “the lifting problem.”

Remark 1.11. It is clearly equivalent to ask whether $Y$ lifts to a curve $Y_R$ with an action of $G$ by $R$-automorphisms whose restriction to the special fiber is the original $G$-action on $Y$. This is how the lifting problem was originally formulated in [Oor87]. One speaks of $(Y,G)$ lifting to characteristic zero.

Remark 1.12. Of course, one cannot solve the lifting problem by simply taking equations for $f$, $Y$, $X$, and the $G$-action and lifting the coefficients to $R$. There is no reason to expect that you end up with a $G$-action on $Y_R$ at all in this case. For instance, suppose $Y = \mathbb{P}^1_k$ and the nontrivial element of $G = \mathbb{Z}/2$ acts by sending $z \to -z$, where $z$ is a coordinate. Taking $Y_R = \mathbb{P}^1_R$ and having the nontrivial element of $G$ send $z$ to $az$ clearly does not give a $G$-action on $Y_R$ if $a \in R$ is congruent to $-1$ modulo the maximal ideal but is not equal to $\pm 1$.

Remark 1.13. In fact, naïvely lifting as in Remark 1.12 does not work, even when $G$ is trivial. For instance, take $Y \subseteq \mathbb{P}^3_k$ to be the twisted cubic given by the ideal $(xz - y^2, yw - z^2, xw - yz)$. If we lift these equations to characteristic zero, most choices of coefficients will give a variety $Y_R$ where the generic fiber is zero-dimensional, in which case $Y_R$ will not be flat over $R$!

The example above may seem silly, since it is easy to lift the equations correctly (just keep all the coefficients 0’s and 1’s). But in fact there do exist smooth varieties over $k$ that cannot be lifted to characteristic zero. The first example of this is a threefold and is due to Serre ([Ser61]).

Remark 1.14. A smooth projective curve always lifts over any complete discrete valuation ring $R$ with residue field $k$ ([SGA03 III, Corollaire 6.10 and Proposition 7.2]). Thus Question 1.10 for tame covers of curves over $R$ has a positive answer:
We first lift $X$ to a curve $X_R$, we lift the branch points of $f$ to $R$-points of $X_R$ using the valuative criterion for properness, and then we apply Theorem 1.5.

In spite of Remark 1.14, the lifting problem does not always have a solution (see Examples 2.4, 2.5, and 2.6). However, one of the major open conjectures in the field, giving a positive lifting result, has been recently solved by Obus-Wewers and Pop ([OW14], [Pop14]).

**Theorem 1.15. (The Oort conjecture)** The lifting problem for cyclic covers of curves has a solution (for some discrete valuation ring).

In §2 we discuss a number of basic examples of the lifting problem. It turns out that determining whether the lifting problem can be solved reduces to a local lifting problem on extensions of power series rings (Question 1.1). This reduction is thanks to a local-global principle, which we state and prove in §3.

The local lifting problem is the main approach to the lifting problem today. In §4 we define the local lifting problem, and give some obstructions to solving it. In §5 we summarize the positive results for the local lifting problem known at the present, which is followed by §6 where we give an overview of the various techniques that are used to construct lifts. Although, historically, positive results for the local lifting problem were discovered before the discovery of systematic obstructions, we feel that it may be beneficial for the reader to first have a limited set of cases of the local lifting problem to think about, before seeing what is known. The interested reader can certainly skip to §5 right after reading the introduction to §4.

In §7 we sketch the deformation-theoretic approach to the local lifting problem, which aims not only to solve it, but to understand what the space of solutions looks like (this section assumes familiarity with general deformation theory, and is written at a somewhat higher level of sophistication than the rest of the chapter). We close with §8 which is a list of open problems. Appendix A includes some algebraic results that are used in our proofs.

1.3. A remark on this exposition. The paper [Obu12] is an earlier exposition that I wrote on the local lifting problem. In what follows, I have attempted to minimize duplication of [Obu12] and to cite it for proofs whenever possible. At the same time, I have striven to keep notation as identical as possible to that of [Obu12]. This chapter contains a great deal of material not present in [Obu12], such as a proof of the local-global principle, the (differential) Hurwitz tree obstruction, the “Mumford method” of using equicharacteristic deformations to build lifts and its application in the resolution of the Oort conjecture, all the deformation-theoretic material, several results of the last few years, and many examples and open problems. There is also quite a bit of material in [Obu12] not included here, in particular a detailed account of lifting for cyclic extensions. That being said, this chapter has the same basic structure as [Obu12], and to make it readable on its own, some repetition is necessary. In some sense, this chapter and [Obu12] are companion papers that can be read together if desired. In particular, I will often give a reference to [Obu12], even when a result is originally from another source (which I will also cite).

In order to keep the level of exposition relatively basic, we do not include details on differential/deformation data, although they are mentioned in §4.2.
1.4. **Notation and conventions.** Throughout this paper, $p$ represents a (fixed) prime number, $k$ is (unless otherwise noted) an algebraically closed field of characteristic $p$, and $W(k)$ is the ring of Witt vectors of $k$, that is, the unique complete discrete valuation ring in characteristic zero with uniformizer $p$ whose residue field is $k$ (see, e.g., [Ser79, II, §6]).

If $\Gamma$ is a group of automorphisms of a ring $A$, we write $A^\Gamma$ for the fixed ring under $\Gamma$. For a finite group $G$, a $G$-Galois extension (or $G$-extension) of rings is a finite extension $A \hookrightarrow B$ (also written $B/A$) of integrally closed integral domains such that the associated extension of fraction fields is $G$-Galois. We do not require $B/A$ to be étale.

If $x$ is a scheme-theoretic point of a scheme $X$, then $\mathcal{O}_{X,x}$ is the local ring of $x$ in $X$. If $R$ is any local ring, then $\hat{R}$ is the completion of $R$ with respect to its maximal ideal. A $G$-Galois cover (or $G$-cover) is a branched cover $f : Y \to X$ with an isomorphism $G \cong \text{Aut}(Y/X)$ such that $G$ acts transitively on each geometric fiber of $f$. Note that $G$-covers of affine schemes give rise to $G$-extensions of rings, and vice versa.

Suppose $f : Y \to X$ is a branched cover, with $X$ and $Y$ locally noetherian. If $x \in X$ and $y \in Y$ are smooth codimension 1 points such that $f(y) = x$, then the ramification index of $y$ is the ramification index of the extension of complete local rings $\hat{O}_{X,x} \to \hat{O}_{Y,y}$. If $f$ is Galois, then the ramification index of a smooth codimension 1 point $x \in X$ is the ramification index of any point $y$ in the fiber of $f$ over $x$. If $x \in X$ (resp. $y \in Y$) has ramification index greater than 1, then it is called a branch point (resp. ramification point).

If $R$ is any ring with a non-archimedean absolute value $|\cdot|$, then $R\{T\}$ is the ring of power series $\sum_{i=0}^{\infty} c_i T^i$ such that $\lim_{i \to \infty} |c_i| = 0$. Throughout the paper, we normalize the valuation on $R$ and $K$ so that $p$ has valuation 1.

If $X$ is a smooth curve over a complete discrete valuation field $K$ with valuation ring $R$, then a semistable model for $X$ is a relative curve $X_R \to \text{Spec} R$ with $X_R \times_R K \cong X$ and semistable special fiber (i.e., the special fiber is reduced with only ordinary double points for singularities).

If $R$ is a discrete valuation ring with residue field $k$ and fraction field $K$, and $A$ is an $R$-algebra, we write $A_k$ and $A_K$ for $A \otimes_R k$ and $A \otimes_R K$, respectively.

Suppose $S$ is a ring of characteristic zero, with an ideal $I$ such that $S/I$ has characteristic $p$. If an indeterminate in $S$ is given by a capital letter, our convention (which we will no longer state explicitly) will be to write its reduction in $S/I$ using the respective lowercase letter. For example, if $I \subset W(k)[[U]]$ is the ideal generated by $p$, then $W(k)[[U]]/I \cong k[[u]]$, and $u$ is the reduction of $U$.

The group $D_n$ is the dihedral group of order $2n$. The symbol $\zeta_p$ represents a primitive $p$th root of unity. The genus of a curve $X$ is written $g(X)$.

2. **Global results**

In this section, $f : Y \to X$ is a branched $G$-Galois cover of smooth projective curves over $k$, where $G$ is a finite group. Recall that the lifting problem asks if there is a characteristic zero discrete valuation ring $R$ with residue field $k$ and a branched $G$-Galois cover $f_R : Y_R \to X_R$ of smooth relative $R$-curves whose special fiber is $f$. 


2.1. Tame covers. As we have seen in Remark 2.13 (based on Theorem 1.5), if \( R \) is complete and \( f \) is furthermore tamely ramified, then \( f \) lifts over \( R \). In particular, \( f \) lifts to characteristic zero whenever \([G]\) is not divisible by \( p \). It is not always easy to write down tame covers and their lifts explicitly, but one can do this for cyclic (or abelian) covers, as shown in the next example.

Example 2.1. Suppose \( f : Y \to \mathbb{P}^1_k \) is a cyclic \( \mathbb{Z}/n \)-cover over \( k \) corresponding to the function field embedding

\[
k(t) \to k(t)[z]/(z^n - \prod_{i=1}^{r}(t - a_i)^{c_i})
\]

for some pairwise distinct \( a_i \in k \), where \( p \nmid n \) and all \( c_i > 0 \). Let \( \mathbb{P}^1_k \) be the standard lift of \( \mathbb{P}^1_k \) with coordinate \( T \) reducing to \( t \). Let \( R \) be a complete characteristic zero discrete valuation ring with residue field \( k \). We claim that a lift \( f_R \) of \( f \) can be given by taking the normalization \( Y_R \) of \( \mathbb{P}_R^1 \) in the function field

\[
K := K(T)[Z]/(Z^n - \prod_{i=1}^{r}(T - A_i)^{c_i}),
\]

where \( A_i \) is any lift of \( a_i \) to \( R \). Here \( K = \text{Frac}(R) \) and a generator of the Galois group takes \( Z \) to \( \zeta_n Z \) (note that \( R \), being complete, contains \( \zeta_n \)).

The map \( f_R \) is flat by Proposition A.3. Since the normalization of \( A_1R \) in \( K \) contains \( \text{Spec} \, R[T, Z]/(Z^n - \prod_{i=1}^{r}(T - A_i)^{c_i}) \), it is clear that \( Y_R \times_R K \) is birationally equivalent to \( Y \). But we must check that it is smooth (the normal scheme \( Y_R \) can, in theory, have singularities in codimension 2!). The generic fiber \( f_k := f_R \times_R K \) is branched of index \( e_i := n / \gcd(n, c_i) \) above \( A_i \), so the Riemann-Hurwitz formula gives that the genus \( g_n \) of \( Y_R \times_R K \) satisfies

\[
2g - 2 = -2n + \sum_{i=1}^{r}(n - e_i).
\]

The same is true for the genus \( g_Y \) of \( Y \). Since \( f_R \) is flat, the arithmetic genus \( p_a(Y_R \times_R K) = g_n = g_Y \) (e.g., [Har77 II, Theorem 9.13]). Thus \( Y_R \times_R K \) is smooth (see, e.g., [Har77 IV, Exercise 1.8]).

2.2. Wild covers. In stark contrast to the case of tame covers, the lifting problem for wild covers contains a great deal of mystery. Even in the most basic example when a wild cover lifts, writing the lift down is less straightforward than in Example 2.1.

Example 2.2. Let \( f : Y = \mathbb{P}^1 \to X = \mathbb{P}^1 \) be the Artin-Schreier \( \mathbb{Z}/p \)-cover over \( k \) given by the equation \( z \mapsto z^p - z \), where \( z \) is a coordinate on \( \mathbb{P}^1 \). The Galois action is generated by \( z \mapsto z + 1 \). Let \( R \) be a complete discrete valuation ring with residue field \( k \) containing \( \zeta_p \) and let \( \lambda = \zeta_p - 1 \). A lift of \( f \) to characteristic zero is \( f_R : Y_R = \mathbb{P}_R^1 \to X_R = \mathbb{P}_R^1 \), where \( f_R \) is given by

\[
Z \mapsto \frac{(1 + \lambda Z)^p - 1}{\lambda^p}
\]

(note that this is defined over \( R \), and reduces to \( z \mapsto z^p - z \) over \( k \)). The \( \mathbb{Z}/p \)-Galois action on \( Y_R \) is generated by \( Z \mapsto \zeta_p Z + 1 \) (which has order \( p \) — the reader should check this!), and this action reduces to \( z \mapsto z + 1 \) over \( k \).
Remark 2.3. Notice that the generic fiber of \( Y_R \) above has two ramification points (at \( Z = -1/\lambda \) and \( Z = \infty \)), both of which specialize to the unique ramification point \( z = \infty \) of \( Y \). This phenomenon of a ramification point “splitting” into several ramification points on a lift to characteristic zero happens whenever the ramification point is wild (indeed, the Riemann-Hurwitz formula necessitates this, as a wild ramification point of index \( e \) contributes more than \( e - 1 \) to the degree of the ramification divisor).

There are also examples of wild covers that do not lift to characteristic zero, such as the following.

Example 2.4. Let \( Y \cong \mathbb{P}^1_k \). The group \( G = (\mathbb{Z}/p)^n \) (for any \( n \)) embeds into the additive group of \( k \), and acts on \( Y \) by the additive action, fixing \( \infty \). Let \( f : Y \to X \cong \mathbb{P}^1 \) be the induced \( G \)-cover. If \( f_R : Y_R \to X_R \) is a lift of \( f \) to characteristic zero and \( K = \text{Frac}(R) \), then flatness of \( Y_R \to \text{Spec} R \) implies that the generic fiber \( Y_K \) of \( Y_R \) is a genus zero curve in characteristic zero with \( G \)-action. However, the automorphism group of \( Y_K \) embeds into \( \text{PGL}_2(K) \), which does not contain \((\mathbb{Z}/p)^n\) if \( n > 1 \) and \( p^n \neq 4 \). So the \( G \)-action on \( Y \) cannot lift to characteristic zero in these cases.

Along the same lines, and more simply, a cover might not lift simply because the automorphism group is too large for characteristic zero. The following example is from [Roq70] §4.

Example 2.5. Consider the smooth projective model \( Y \) of the curve \( y^2 = x^p - x \) over \( k \), where \( p \geq 5 \). The genus \( g_Y \) of \( Y \) is \((p - 1)/2\). The group \( G := \text{Aut}(Y) \) is generated by

\[
\begin{align*}
\sigma & : x \mapsto x + 1, \quad y \mapsto y \\
\tau & : x \mapsto ax, \quad y \mapsto \sqrt{a}y \quad (a \text{ a generator of } \mathbb{F}_p^\times) \\
v & : x \mapsto -\frac{1}{x}, \quad y \mapsto \frac{y}{x^{(p+1)/2}}
\end{align*}
\]

This group contains \( \text{PGL}_2(p) \) as an index 2 subgroup (considering only the action on \( x \)), so \(|G| = 2p(p^2 - 1)\). If the \( G \)-cover \( f : Y \to Y/\text{Aut}(Y) \) lifted to characteristic zero, the generic fiber of the lift would be a genus \( g_Y \) curve with at least \(|G|\) automorphisms. Since \(|G| > 84(g - 1)\), this violates the Hurwitz bound on automorphisms of curves in characteristic zero (see, e.g., [Har77] IV, Ex. 2.5). Indeed, this is the case even when \( G \) is replaced by its index 2 subgroup \( \text{PGL}_2(p) \).

Lastly, here is an example from [Oor87] §2 of a wild cover that is non-liftable for reasons other than the size of the automorphism group.

Example 2.6. Suppose \( p = 5 \), and let \( G \) be the group of order 20 with presentation \( \langle \sigma, \tau | \sigma^5 = 1, \tau^4 = 1, \sigma \tau = \tau \sigma^{-1} \rangle \). Let \( f : Y \to X = \mathbb{P}^1 \) be the \( G \)-cover corresponding to the embedding of function fields \( k(t) \hookrightarrow k(t)[x, y]/(x^4 - t, y^5 - y - x^{-2}) \), where the \( G \)-action is given by

\[
\sigma^*y = y + 1, \quad \sigma^*x = x, \quad \tau^*y = 4y, \quad \tau^*x = 3x.
\]

If \( Z \) corresponds to the subfield \( k(t, x) \), then \( Z \to X \) is a \( \mathbb{Z}/4 \)-cover of genus zero curves ramified at \( x = 0 \) and \( x = \infty \), and \( Y \to Z \) is an Artin-Schreier \( \mathbb{Z}/5 \)-cover branched only at \( x = 0 \). If \( P \in Y \) is the point above \( x = 0 \), then \( xy^2 \) is a
uniformizing parameter at $P$, and the ramification divisor $D$ of $Y \to Z$ can be calculated by taking
\[
\frac{dx}{d(xy^2)} = -\frac{dx}{2xydy} = -\frac{x^2}{y},
\]
whose divisor has $P$-part $12[P]$. Thus $D = 12[P]$, and the Riemann-Hurwitz formula shows that the genus of $Y$ is 2.

Now, suppose that $f$ has a lift $f_R : Y_R \to X_R$ over a discrete valuation ring $R$ in characteristic zero. There is an intermediate $\mathbb{Z}/5$-cover $Y_R \to Z_R$ lifting $Y \to Z$.

By flatness, the generic fiber $Y_K \to Z_K$ over $K := \text{Frac}(R)$ is a $\mathbb{Z}/5$-cover of a genus 0 curve by a genus 2 curve, and $Z_K \to X_K$ is a $\mathbb{Z}/4$-cover of genus zero curves. Since all ramification points of $Y_K \to Z_K$ have ramification index 5, the Riemann-Hurwitz formula yields
\[
2(2) - 2 = 5(-2) + 4r,
\]
where $r$ is the number of branch points. Thus $r = 3$, and these three points are permuted by the $\mathbb{Z}/4$-action on $Z_K$. Since $Z_K \to X_K$ is a cover of genus zero curves, this action is free apart from two fixed points. So no set of three points is stable under this action, yielding a contradiction.

The examples above motivate the following obstruction to lifting, known as the Katz-Gabber-Bertin (or KGB) obstruction (cf. [CGH11, §1], where the definition is given in a slightly different context).

**Definition 2.7. (The KGB Obstruction)** If $f : Y \to X$ is a branched $G$-cover of smooth projective curves over $k$, then there is a KGB obstruction to lifting $f$ if there exists no curve $C$ in characteristic zero with faithful $G$-action such that, for all $H \leq G$, the genus of $C/H$ equals the genus of $Y/H$. If there does exist such a curve, we say that the KGB obstruction vanishes.

**Remark 2.8.** The KGB obstruction above motivates the local KGB obstruction (Definition 4.5), which will be the form we primarily use.

By flatness, it is clear that if $f$ has a KGB obstruction to lifting, then it does not lift to characteristic zero. Furthermore, the KGB obstruction explains the failure of lifting in Examples 2.4, 2.5, and 2.6.

**Remark 2.9.** In general, obstructions to the lifting problem come from looking at all conceivable lifts of characteristic $p$ branched covers with certain properties, seeing what properties the lifts would have, and attempting to encode these properties in an abstract structure. There will then be an obstruction to lifting if this abstract structure can’t exist. As we have seen, the KGB obstruction rests on the simple observation that a lift of a curve with a given genus has the same genus. The abstract structure in this case can be viewed as a function from subgroups $H \leq G$ to integers (representing the genus of $Y/H$). This function can exist only when it obeys the constraints given by the Riemann-Hurwitz formula in characteristic zero.

2.3. **Oort groups and the Oort conjecture.** Remark 1.14 has the consequence that every $G$-Galois cover $f : Y \to X$ of smooth projective curves over $k$, where $p \nmid |G|$, lifts to characteristic zero. This motivates the natural question of which groups $G$ have the property that all $G$-covers lift. After [CCH08], we call a finite group $G$ with this property an Oort group for $p$ (we sometimes suppress $p$ when it is implicit). Thus prime-to-$p$ groups are Oort groups. By the examples of 2.2...
(\mathbb{Z}/p^n) is not an Oort group when \( n > 1 \) and \( p^n > 4 \), the group \( PGL_2(p) \) is not an Oort group for \( p > 3 \), and the group of order 20 with presentation \( \langle \sigma, \tau \mid \sigma^5 = 1, \tau^4 = 1, \sigma \tau = \tau \sigma^{-1} \rangle \) is not an Oort group for 5. From this, one can easily obtain more examples of non-Oort groups (for instance, it is an exercise to show that any direct product of a non-Oort group with another group will be a non-Oort group).

On the other hand, it is generally not easy to prove that a group is an Oort group. The first major result for a group with order divisible by \( p \) is due to Sekiguchi-Oort-Suwa.

**Theorem 2.10** ([SOS89]). If \( G \) is a cyclic group of order \( mp \), with \( p \nmid m \), then \( G \) is an Oort group.

Around the same time, Oort ([Oor87, §7]) made the statement that “it seems reasonable to expect that \( \text{lifting} \) is possible for every automorphism of an algebraic curve.” This is equivalent to saying that all cyclic groups are Oort groups (for all \( p \)). This statement has come to be known as the Oort conjecture, and has driven much of the progress to date in the lifting problem. The Oort conjecture was recently proven by the combined work of Obus-Wewers ([OW14]) and Pop ([Pop14]). The proof makes heavy use of the local-global principle, which is the subject of the next section. In fact, the local-global principle underlies virtually all progress on the lifting problem since [SOS89]. Thus, we will discuss it before saying anything further about Oort groups.

3. The local-global principle

In this section, \( R \) is a complete characteristic zero discrete valuation ring with residue field \( k \).

Suppose a finite group \( G \) acts on a smooth, projective curve \( Y/k \), giving rise to a \( G \)-cover \( f : Y \to X := Y/G \). Let \( y \in Y \) be a closed point, and let \( I_y \leq G \) be the inertia group of \( y \). The \( G \)-action on \( Y \) induces an \( I_y \)-action on the complete local ring \( \hat{O}_{Y,y} \), which is isomorphic to a power series ring \( k[[z]] \), since \( Y \) is smooth (Lemma A.4). The fixed ring \( \hat{O}_{Y,y}^{I_y} \) is the complete local ring \( \hat{O}_{X,f(y)} \), so is also a power series ring in one variable, say \( k[[t]] \).

Now, suppose further that \( f : Y \to X := Y/G \) lifts to a characteristic zero \( G \)-cover \( f_R : Y_R \to X_R \). We identify the special fiber of \( f_R \) with \( f \). Then each \( y \in Y \) as above gives rise to an extension of complete local rings \( \hat{O}_{Y_R,y}/\hat{O}_{X_R,f(y)} \). The group \( I_y \) acts on \( \hat{O}_{Y_R,y} \), this action lifts the action of \( I_y \) on \( \hat{O}_{Y,y} \), and \( \hat{O}_{X_R,f(y)} = \hat{O}_{Y_R,y}^{I_y} \). In fact, there exist isomorphisms \( \hat{O}_{Y_R,y} \cong R[[Z]] \) and \( \hat{O}_{X_R,f(y)} \cong R[[T]] \), where \( Z \) and \( T \) can be chosen to be any lifts of \( z \) and \( t \) (Lemma A.5). Thus, it makes sense to say that the \( I_y \)-extension \( R[[Z]]/R[[T]] \) is a lift of the \( I_y \)-extension \( k[[z]]/k[[t]] \) to characteristic zero. In other words, any lift of the \( G \)-cover \( f \) over \( R \) gives rise to a lift of the \( I_y \)-extension \( \hat{O}_{Y,y}/\hat{O}_{X,f(y)} \) over \( R \).

Amazingly, the converse of the above statement holds as well! This is called the local-global principle for lifting.

**Theorem 3.1.** (Local-global principle) Let \( f : Y \to X \) be a \( G \)-cover of smooth, projective, \( k \)-curves. For each closed point \( y \in Y \), let \( I_y \leq G \) be the inertia group. If, for all \( y \), the \( I_y \)-extension \( \hat{O}_{Y,y}/\hat{O}_{X,f(y)} \) lifts over \( R \), then \( f \) lifts over \( R \).

**Remark 3.2.** If \( y \) is unramified in \( f \) above, then \( I_y \) is trivial, so the extension \( \hat{O}_{Y,y}/\hat{O}_{X,f(y)} \) lifts automatically. Thus, in light of Theorem 3.1, one need only...
check the (finitely many) ramified points \( y \in Y \) in order to show that \( f \) lifts to characteristic zero.

Our proof of the local-global principle uses formal patching in the spirit of [Sa12 §1.2] and [Hen02 §3] (see [Har03] for a detailed introduction to patching, and [Har77 §II.9] for an introduction to formal schemes). The idea is as follows: Suppose we are given a branched Galois cover \( f : Y \to X \) and lifts of the Galois covers \( \hat{O}_{Y,y}/\hat{O}_{X,f(y)} \) over \( R \) for all ramification points \( y \). Let \( U \subseteq X \) and \( V \subseteq Y \) be the complements of the branch and ramification loci, respectively. The cover \( f : V \to U \) is étale, and thus admits a lift to a formal scheme over \( R \) due to Grothendieck’s theory of étale morphisms. The individual Galois covers \( \hat{O}_{Y,y}/\hat{O}_{X,f(y)} \) admit lifts over \( R \) by assumption. These can be “patched” together to create a Galois branched cover of proper formal schemes \( \mathcal{F} : \mathcal{Y} \to \mathcal{X} \) over \( R \). Grothendieck’s Existence Theorem (also known as “formal GAGA”) shows that \( \mathcal{F} \) is actually the formal completion of a \( G \)-Galois branched cover of \( R \)-curves \( f_R : Y_R \to X_R \), which is the lift we seek.

The rest of §3 will be devoted to the details of this proof.

3.1. Preliminaries and étale lifting. We start with two basic lemmas on formal lifting.

**Lemma 3.3.** Let \( X \) be a smooth curve over \( k \) and let \( R \) be a complete discrete valuation ring with residue field \( k \). There exists a smooth formal \( R \)-curve \( \mathcal{X} \) whose special fiber is \( X \). Furthermore, if \( U \subseteq X \) is an open subscheme, there is a smooth formal \( R \)-curve \( \mathcal{U} \subseteq \mathcal{X} \) whose special fiber is \( U \).

**Proof.** Let \( \overline{X} \) be a smooth projective completion of \( X \). By Remark 1.14 there is a lift of \( \overline{X} \) to a smooth projective curve \( \overline{X}_R \) over \( R \). By Hensel’s lemma, we can lift each of the points of \( \overline{X} \setminus X \) to an \( R \)-point of \( \overline{X}_R \). Let \( X_R \) be the affine \( R \)-curve given by the complement of these \( R \)-points. Now take \( \mathcal{X} \) to be the formal completion of \( X_R \) at \( X \). This process suffices as well for the construction of \( \mathcal{U} \). \( \square \)

**Lemma 3.4** ([SGA03 I, Corollaire 8.4]). Let \( f : Y \to X \) be an étale cover of smooth \( k \)-curves and let \( R \) be a complete discrete valuation ring with residue field \( k \). Let \( \mathcal{X} \) be a smooth formal \( R \)-curve with special fiber \( X \). Then there is a unique smooth formal \( R \)-curve \( \mathcal{Y} \) with a finite map \( \mathcal{F} : \mathcal{Y} \to \mathcal{X} \) such that \( \mathcal{F} \) has special fiber \( f \). If \( f \) is \( G \)-Galois, then so is \( \mathcal{F} \), and the \( G \)-action on \( \mathcal{Y} \) lifts that on \( Y \).

3.2. A patching result.

**Proposition 3.5** (cf. [Hen02 Proposition 4.1]). Let \( R \) be a complete discrete valuation ring with residue field \( k \) and uniformizer \( \pi \). Let \( \mathcal{X} \) be a smooth formal affine \( R \)-curve with special fiber \( X \). Suppose \( f : Y \to X \) is a finite, dominant, separable morphism. Let \( x \in X \) be a closed point, let \( U = X \setminus \{x\} \), let \( V = f^{-1}(U) \), and assume that \( f|_V \) is étale. Let \( \mathcal{U} \subseteq \mathcal{X} \) be a formal lift of \( U \) over \( R \) as in Lemma 3.3 and let \( \Phi : \mathcal{V} \to \mathcal{U} \) be the lift of \( f|_V : V \to U \) guaranteed by Lemma 3.4. Lastly, let \( A \) be a finite normal \( \hat{O}_{\mathcal{X},x} \)-algebra such that there is an isomorphism \( A/\pi \to \prod_{y \in f^{-1}(x)} \hat{O}_{\mathcal{Y},y} \).

(i) There is a smooth formal \( R \)-curve \( \mathcal{Y} \) containing \( V \) and a finite cover \( \mathcal{F} : \mathcal{Y} \to \mathcal{X} \) such that \( \mathcal{F}|_V = \Phi \), that the special fiber of \( \mathcal{F} \) is \( f \), and that the restriction of \( \mathcal{F} \) above \( \text{Spec} \hat{O}_{\mathcal{X},x} \) gives the extension \( A/\hat{O}_{\mathcal{X},x} \).
(ii) If \( f \) is \( G \)-Galois, \( G \) acts on \( A \) with \( A^G \cong \hat{O}_{X,x} \), and the morphism \( A \to \prod_{y \in f^{-1}(x)} \hat{O}_{Y,y} \) is \( G \)-equivariant, then \( \mathcal{F} \) is \( G \)-Galois, and the \( G \)-Galois action on \( \mathcal{Y} \) lifts that on \( Y \).

**Proof.** We adapt the proof of Henrio. Write \( f^{-1}(x) = \{ y_1, \ldots, y_n \} \). Fix isomorphisms \( \hat{O}_{Y,y_i} \cong k[[z_i]] \) and \( \hat{O}_{X,x} \cong k[[t]] \). Then

\[
\hat{O}_{Y,y_i} \cong k[[z_i]] \cong \hat{O}_{X,x}[z_i]/p(z_i),
\]

where \( p_i \) is a separable monic Eisenstein polynomial with coefficients in \( \hat{O}_{X,x} \cong k[[t]] \).

Fix an isomorphism \( \hat{O}_{X,x} \cong R[[T]] \), with \( T \) lifting \( t \) (Lemma A.3). This fixes an isomorphism \( (\hat{O}_{X,x})^\wedge_{(x)} \cong R[[T]]/\{T^{-1}\} \) of complete discrete valuation rings. Now, \( B := A \otimes_{R[[T]]} R[[T]]/\{T^{-1}\} \) is finite over \( R[[T]]/\{T^{-1}\} \), and reducing modulo \( \pi \) yields \( B/\pi \cong \prod_{i=1}^n k((z_i)) \cong \prod_{i=1}^n k((t_i))[z_i]/p_i(z_i) \), where \( n = |f^{-1}(x)| \). Let \( P_i \in R[[T]]/\{T^{-1}\} \) be a monic polynomial lifting \( p_i \). By Hensel’s lemma, \( B \) is a product of finite \( R[[T]]/\{T^{-1}\} \)-algebras \( B_i \) \((1 \leq i \leq n)\) where \( B_i \) contains a root \( z_i \) of \( P_i \) lifting \( z_i \) and \( B_i/\pi \cong k((z_i)) \) as a \( k((t)) \)-algebra. One obtains homomorphisms

\[
u_i : R[[T]]/\{T^{-1}\}[Z_i]/P_i(Z_i) \to B_i.
\]

This homomorphism is an isomorphism modulo \( \pi \), and is thus an isomorphism by Remark A.2. Note that, under the assumptions in (ii), \( G \) acts on \( B \) by \( R[[T]]/\{T^{-1}\} \)-homomorphisms.

On the other hand, let \( C \) be the finite \( R[[T]]/\{T^{-1}\} \)-algebra given by \( \mathcal{O}(\mathcal{V}) \otimes_{\mathcal{O}(\mathcal{U})} R[[T]]/\{T^{-1}\} \), where \( \mathcal{O}(\mathcal{U}) \subseteq R[[T]]/\{T^{-1}\} \) is the natural inclusion. Then

\[
C/\pi \cong \mathcal{O}(\mathcal{V}) \otimes_{\mathcal{O}(\mathcal{U})} k((t)) \cong \prod_{i=1}^n k((z_i)).
\]

Proceeding as with \( B \), we can write \( C = \prod_{i=1}^n C_i \), with isomorphisms

\[
u_i : R[[T]]/\{T^{-1}\}[Z_i]/P_i(Z_i) \to C_i.
\]

Note that, under the assumptions in (ii), Lemma 5.4 shows that \( \mathcal{V} \to \mathcal{U} \) is \( G \)-Galois, and thus that \( G \) acts on \( C \) by \( R[[T]]/\{T^{-1}\} \)-homomorphisms.

The maps \( \mu_i := u_i \circ w_i^{-1} : C_i \to B_i \) are \( R[[T]]/\{T^{-1}\} \)-isomorphisms, giving rise to a product isomorphism \( \mu : C \to B \). Then, \( \mu \) modulo \( \pi \) is the identity on \( \prod_{i=1}^n k((z_i)) \). Thus \( \mu \) is \( G \)-equivariant, since roots of \( P_i \) are in one-to-one correspondence with roots of \( p_i \).

We can define an \( \mathcal{O}(\mathcal{X}) \)-module homomorphism \( \theta : \mathcal{O}(\mathcal{V}) \times A \to B \) by

\[
\theta(h, g) = \mu(h \otimes 1) - g \otimes 1.
\]

Since one can find a rational function on an affine curve (namely, \( V \)) with specified principal part at finitely many points (exercise!), the map \( \theta \) is surjective modulo \( \pi \). By Remark A.2, \( \theta \) is surjective. Under the assumptions in (ii), \( \theta \) is clearly \( G \)-equivariant.

Let \( \mathcal{A} = \ker(\theta) \), which is an \( \mathcal{O}(\mathcal{X}) \)-algebra. We claim that we can take \( \mathcal{V} = \text{Spf} \mathcal{A} \).

Since \( \mathcal{A}/\pi \cong \mathcal{O}(\mathcal{Y}) \) is a finite \( \mathcal{O}(\mathcal{X}) \)-algebra, \( \mathcal{A} \) is a finite \( \mathcal{O}(\mathcal{X}) \)-algebra (this follows from Lemma A.1 applied to a lift of a presentation for \( \mathcal{A}/\pi \)). Thus we obtain a finite morphism \( \mathcal{F} : \mathcal{Y} \to \mathcal{X} \). Furthermore, we have \( A \otimes_{\mathcal{O}(\mathcal{X})} \mathcal{O}(\mathcal{U}) \cong \mathcal{O}(\mathcal{V}) \), using Remark A.2 and the fact that the isomorphism holds modulo \( \pi \) (where it simply expresses the fact that \( \mathcal{Y} \otimes_{\mathcal{X}} \mathcal{U} \cong \mathcal{V} \)). Geometrically, this means that
Remark 4.2. As one sees in a group cyclic-by-$k$ to the given $G$, does there exist a characteristic zero discrete valuation ring $R$?

Question 4.1. Suppose $G$ is a finite group. Does there exist a faithful $G$-extension of the form $P \rtimes \mathbb{Z}/m$, where $P$ is a $p$-group. We will call such a group cyclic-by-$p$. In particular, cyclic-by-$p$ groups are solvable. This is one of...
the benefits of the local-global principle — it converts a problem that potentially
deals with all finite groups into a problem dealing only with solvable groups.

An $G$-extension $k[[z]]/k[[t]]$ will be known as a local $G$-extension. Since the lifting
problem does not always have a solution, neither does the local lifting problem. So
we ask: Can we find necessary and sufficient criteria for a local $G$-extension to lift
to characteristic zero? Over some particular $R$?

A cyclic-by-$p$ group for which every local $G$-extension lifts to characteristic zero
is called a local Oort group for $k$. If there exists a local $G$-extension lifting to
characteristic zero, $G$ is called a weak local Oort group. The question of whether a
group $G$ is a weak local Oort group has been called “the inverse Galois problem”
for the local lifting problem ([Mat99]).

**Example 4.3.** The group $\mathbb{Z}/m$ is a local Oort group for all $p \nmid m$. This is because,
up to a change of variable, every local $\mathbb{Z}/m$-extension is $k[[z]]/k[[t]]$ where $z^m = t$
and the Galois action is generated by $z \mapsto \zeta_m z$. The lift is simply $R[[Z]]/R[[T]]$
with Galois action generated by $Z \mapsto \zeta_m Z$.

**Example 4.4.** By the local-global principle, every $G$-cover that does not lift to
characteristic zero is a local $H$-extension that does not lift to characteristic
zero, where $H$ is the inertia group of some ramification point. For instance, let
$\iota : G = (\mathbb{Z}/p)^n \rightarrow k^+$ be an embedding. Example 2.4 gives rise to the $\mathbb{Z}/p^n$-
action on $k[[u]]$ such that $\sigma \in G$ sends $u$ to $u/(1 + \iota(\sigma)u)$ (here, $u = 1/z$, where $z$
is the coordinate on $\mathbb{P}^1$). Let $k[[t]] = k[[u]]^G$. Then, for $n > 1$ and $p^n > 4$, the local
$G$-extension $k[[u]]/k[[t]]$ does not lift to characteristic zero. In particular, $(\mathbb{Z}/p)^n$
is not a local Oort group (although it is a weak local Oort group, by [Mat99]).

### 4.1. The (local) KGB obstruction
The KGB obstruction (Definition 2.7) has a local counterpart called the local KGB obstruction (often, one drops the word
“local,” which should not cause much confusion). This is the main tool used to show that local $G$-extensions do not lift to characteristic zero.

Let $k[[z]]/k[[t]]$ be a local $G$-extension, where $G \cong P \rtimes \mathbb{Z}/m$, with $P$ a $p$-group
and $p \nmid m$. A theorem of Harbater ([Har80]) in the case $m = 1$ and Katz and Gabber
([Kat86, Theorem 1.4.1]) in the general case states that there exists a unique $G$-
cover $Y \rightarrow \mathbb{P}_k^1$ that is étale outside $t \in \{0, \infty\}$, tamely ramified of index $m$ above
t = $\infty$, and totally ramified above $t = 0$ such that the $G$-extension of complete
local rings at $t = 0$ is given by $k[[z]]/k[[t]]$. This is called the Harbater-Katz-Gabber
(HKG) cover associated to $k[[z]]/k[[t]]$. By Theorem 4.1 and Example 4.3 the $G$-
cover $f : Y \rightarrow \mathbb{P}_k^1$ lifts to characteristic zero if and only if the extension $k[[z]]/k[[t]]$
does.

**Definition 4.5.** In the context above, we say that a local $G$-extension has a (local)
KGB obstruction to lifting if the associated HKG-cover does.

**Remark 4.6.** One can also formulate the local KGB obstruction in a purely local
way (not using HKG-covers), using the different and Proposition 6.2 below as a replacement for the fact that the genus of a lift of a curve equals the genus of the
original curve. We leave this as an exercise.

**Remark 4.7.** The Bertin obstruction of [Ber98] is strictly weaker than the local
KGB obstruction, so we do not discuss it further. That the local KGB obstruction
is at least as strict as the Bertin obstruction is proven in [CGH11, Theorem 4.2].
4.1.1. Global consequences. Suppose $G$ is an arbitrary finite group and fix a prime $p$. Let us revisit the question of whether $G$ is an Oort group (for $p$). By the local-global principle, it is clear that if every cyclic-by-$p$ subgroup of $G$ is local Oort, then $G$ is an Oort group. In fact, the converse is true as well.

**Proposition 4.9** ([CGH08, Theorem 2.4]). If $G$ is a finite group, then it is an Oort group for $p$ if and only if every cyclic-by-$p$ subgroup of $G$ is a local Oort group.

**Proof.** The “if” direction follows from the local-global principle. To prove the “only if” direction, let $I \leq G$ be a cyclic-by-$p$ subgroup, and let $k[[z]]/k[[t]]$ be a local $I$-extension. By [CGH08, Lemma 2.5], there is a $G$-Galois cover $f : Y \rightarrow \mathbb{P}^1$ such that there is a point $x \in \mathbb{P}^1$ and a point $y \in Y$ above $X$ for which the $I$-Galois extension $\mathcal{O}_{Y,y}/\mathcal{O}_{\mathbb{P}^1,x}$ is isomorphic (as an $I$-extension) to $k[[z]]/k[[t]]$. Since $G$ is an Oort group, $f$ lifts to characteristic zero. By the easy direction of the local-global principle, $k[[z]]/k[[t]]$ lifts to characteristic zero as well. So $I$ is a local Oort group.

The following example is contained in [Obu16, Corollary 1.4].

An example of a local $\mathbb{Z}/3 \times \mathbb{Z}/3$-extension with vanishing Bertin obstruction but non-vanishing local KGB obstruction is given in [CGH11, Example B.2].

Clearly, if $k[[z]]/k[[t]]$ lifts to characteristic zero, its local KGB obstruction vanishes. If $G$ is cyclic-by-$p$ and the local KGB obstruction vanishes for all local $G$-extensions, then $G$ is called a local KGB group for $p$. If the local KGB obstruction vanishes for some local $G$-extension, then $G$ is called an weak local KGB group for $p$.

The following classification of the local KGB groups is due to Chinburg, Guralnick, and Harbater.

**Theorem 4.8** ([CGH11, Theorem 1.2]). The local KGB groups for $k$ consist of the cyclic groups, the dihedral group $D_p^n$ for any $n$, the group $A_4$ (for char($k$) = 2), and the generalized quaternion groups $Q_{2^m}$ of order $2^m$ for $m \geq 4$ (for char($k$) = 2).

**Sketch of proof.** We briefly outline the negative direction (i.e., that there are no local KGB groups aside from the ones on the list). The first observation is that if $G$ is a local KGB group for $p$, then any quotient of $G$ is as well. This is because any local $G/H$-extension can be extended to a local $G$-extension ([CGH08, Lemma 2.10]), and if the local $G/H$-extension has nontrivial local KGB obstruction, then the $G$-extension clearly has one too. Thus, to show that a group is not local KGB, it suffices to show it has a quotient that is not local KGB. There is an explicit list of types of groups that can be shown not to be local KGB (this list includes, for example, $\mathbb{Z}/p \times \mathbb{Z}/p$ for $p$ odd), and it can be further shown that any cyclic-by-$p$-group either has a quotient on this list, or is one of the groups in Theorem 4.8 (see [CGH11, §11-12]). This completes the proof.

For examples illustrating the KGB obstruction for $\mathbb{Z}/p^n \times \mathbb{Z}/m$ and additional examples for $\mathbb{Z}/p \times \mathbb{Z}/p$, see [Obu12, Propositions 5.8, 5.9].

Since any local Oort group is a local KGB group, the search for local Oort groups is restricted to the groups in Theorem 4.8. The generalized quaternion groups were shown not to be local Oort groups in [BW09]. The obstruction developed in [BW09] to show this is called the Hurwitz tree obstruction, and will be discussed in §4.2 (see specifically Example 4.17).

4.1.1. Global consequences. Suppose $G$ is an arbitrary finite group and fix a prime $p$. Let us revisit the question of whether $G$ is an Oort group (for $p$). By the local-global principle, it is clear that if every cyclic-by-$p$ subgroup of $G$ is local Oort, then $G$ is an Oort group. In fact, the converse is true as well.

**Proposition 4.9** ([CGH08, Theorem 2.4]). If $G$ is a finite group, then it is an Oort group for $p$ if and only if every cyclic-by-$p$ subgroup of $G$ is a local Oort group.

**Proof.** The “if” direction follows from the local-global principle. To prove the “only if” direction, let $I \leq G$ be a cyclic-by-$p$ subgroup, and let $k[[z]]/k[[t]]$ be a local $I$-extension. By [CGH08, Lemma 2.5], there is a $G$-Galois cover $f : Y \rightarrow \mathbb{P}^1$ such that there is a point $x \in \mathbb{P}^1$ and a point $y \in Y$ above $X$ for which the $I$-Galois extension $\mathcal{O}_{Y,y}/\mathcal{O}_{\mathbb{P}^1,x}$ is isomorphic (as an $I$-extension) to $k[[z]]/k[[t]]$. Since $G$ is an Oort group, $f$ lifts to characteristic zero. By the easy direction of the local-global principle, $k[[z]]/k[[t]]$ lifts to characteristic zero as well. So $I$ is a local Oort group.

The following example is contained in [Obu16, Corollary 1.4].
Example 4.10. The group $A_5$ is an Oort group for every prime. Indeed, its only cyclic-by-$p$ groups (for any $p$) are $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/5$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $S_3$, $A_4$, and $D_5$, all of which are local Oort groups for their respective primes (see [5]).

In light of Proposition 4.9 classifying the Oort groups for $p$ consists of two parts: classifying the local Oort groups for $p$, and classifying the groups whose cyclic-by-$p$ subgroups are on this list. Following Chinburg-Guralnick-Harbater ([CGH15]), we call a group $G$ an $O$-group for $p$ if every cyclic-by-$p$ subgroup of $G$ is either cyclic, $D_{p^n}$, or $A_4$ if $p = 2$. Since these are the only cyclic-by-$p$ groups that can be local Oort groups, Proposition 4.9 shows that a group $G$ can be an Oort group only if it is an $O$-group. In particular, if $D_{p^n}$ is shown to be local Oort for all $p$ and all $n$, then the list of $O$-groups is the same as the list of Oort groups. The list of all $O$-groups has been computed by Chinburg-Guralnick-Harbater.

Theorem 4.11 ([CGH15 Theorems 2.4, 2.6]). If $p$ is an odd prime, then $G$ is an $O$-group for $p$ if and only if a $p$-Sylow subgroup $P$ of $G$ is cyclic and either

- The normalizer $N_G(P)$ and centralizer $Z_G(P)$ of $P$ in $G$ are equal, or,
- $|N_G(P)/Z_G(P)| = 2$, the order $p$ subgroup $Q \leq P$ has abelian centralizer, and every element of $N_G(Q)/Z_G(Q)$ acts as an involution inverting $Z_G(Q)$.

If $p = 2$, then $G$ is an $O$-group if and only if $P$ is cyclic or $P$ is dihedral, with $Z_G(K) = K$ for all elementary abelian subgroups $K$ of order 4.

More explicit lists of groups satisfying these criteria are given in [CGH15 Theorems 2.7, 3.8].

4.1.2. Consequences of obstructions for weak local Oort groups. Suppose $G := \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ is non-cyclic (equivalently, non-abelian). It turns out that a local $G$-extension $k[[z]]/k[[t]]$ whose $\mathbb{Z}/p^n$-subextension has first positive lower jump $h$ has vanishing KGB obstruction precisely when $h \equiv -1 \pmod{m}$. If this happens, the conjugation action of $\mathbb{Z}/m$ on $\mathbb{Z}/p^n$ is faithful, or equivalently, $G$ is center-free ([Obu12 Proposition 5.9]). So groups $G$ of this form that are neither cyclic nor center-free cannot be weak local Oort.

Furthermore, Green-Matignon showed that if $G$ contains an abelian subgroup that is neither cyclic nor a $p$-group, then $G$ is not a weak local Oort group (see [Gre03 Proposition 3.3], which shows that no abelian group can be weak local Oort unless it is cyclic or a $p$-group — it is more or less trivial to see that if a group contains a subgroup that is not weak local Oort, then the group itself is not weak local Oort). For a somewhat stronger statement, see [CGH11 Theorem 1.8].

4.2. The (differential) Hurwitz tree obstruction. Recall that the (local) KGB obstruction comes from exploiting the fact that the genus of a curve does not change when it is lifted to characteristic zero. However, if we have a local $G$-extension $k[[z]]/k[[t]]$ and a lift of its corresponding HKG-cover, then only remembering the genus of this cover and its subcovers means that we are actually throwing out a great deal of other information. In particular, we are forgetting the $p$-adic geometry of the branch locus (that is, the distances of the branch points from each other). It turns out that these distances satisfy subtle constraints, and the information about these constraints can be packaged in a combinatorial structure called a Hurwitz tree. The local $G$-extension $k[[z]]/k[[t]]$ will have a Hurwitz tree obstruction if no Hurwitz tree exists that properly reflects the higher ramification filtration of $k[[z]]/k[[t]]$. 
Henrio gave the first major exposition of Hurwitz trees ([Hen00a]), dealing with the case of local \( \mathbb{Z}/p \)-extensions. The concept was extended by Bouw and Wewers to encompass local \( \mathbb{Z}/p \times \mathbb{Z}/m \)-extensions ([BW06]). Later, it was extended by Brewis and Wewers to arbitrary groups ([BW09]). A detailed overview of the construction of Hurwitz trees for \( \mathbb{Z}/p \times \mathbb{Z}/m \)-actions is also given in [Obu12] §7.3.1, §7.3.2. We will give a much briefer overview below, and then we will mention how one generalizes to the case of arbitrary local \( G \)-extensions.

4.2.1. Building a Hurwitz tree. Suppose \( k[[z]]/k[[t]] \) is a local \( G \)-extension that lifts to a \( G \)-extension \( R[[Z]]/R[[T]] \), where \( R \) is a characteristic zero complete discrete valuation ring. The key observation is that we should think of \( R[[Z]] \) as the ring of \( R \)-valued functions on the \( p \)-adic open unit disc (since, if \( x \) is algebraic over \( R \) and has positive valuation, all power series converge on \( x \)). Thus we will think of \( D := \text{Spec} R[[Z]] \) as the open unit disc. By assumption, \( G \) acts faithfully on \( D \) by \( R \)-automorphisms with no inertia above a uniformizer of \( R \) (that is, the action reduces to a faithful \( G \)-action on \( \text{Spec} k[[z]] \)). The idea of a Hurwitz tree is to understand the geometry of this action in combinatorial form. We will write \( D_K \) for the generic fiber of \( D \), where \( K = \text{Frac}(R) \). Clearly, \( G \) acts on \( D_K \). We assume \( R \) and \( K \) are large enough for all ramification points of \( D_K \to D_K/G \) to be defined over \( K \).

Let \( D' = D/G = \text{Spec} R[[T]] \), and write \( D'_K \) analogously to \( D_K \). Let \( y_1, \ldots, y_s \) (resp. \( z_1, \ldots, z_r \)) be the branch (resp. ramification) points of \( D_K \to D'_K \). The first step in building the Hurwitz tree is to let \( Y^{st} \) (resp. \( Z^{st} \)) be the stable model of \( \mathbb{P}^1_k \) corresponding to \( (D; y_1, \ldots, y_s) \) (resp. \( (D'; z_1, \ldots, z_r) \)—we assume \( r, s \geq 2 \), which is automatic as long as \( G \) has nontrivial \( p \)-Sylow subgroup). This is the minimal semistable \( R \)-curve with generic fiber \( \mathbb{P}^1_k \) that separates the specializations of the \( y_i \) (resp. \( z_i \)) and \( \infty \), where \( D_K \) (resp. \( D'_K \) ⊆ \( \mathbb{P}^1_k \)) is viewed as the open unit disc centered at 0 (see, e.g., [Obu12] Appendix A for more details). One way to think about this is that we extend the coordinate \( Z \) (resp. \( T \)) on \( D_K \) (resp. \( D'_K \)) to \( \mathbb{P}^1_k \). Let \( \mathbb{Y} \) (resp. \( \mathbb{Z} \)) be the special fiber of \( Y^{st} \) (resp. \( Z^{st} \)). Note that there is no natural map \( Y^{st} \to Z^{st} \), but the group \( G \) does act naturally on the irreducible components of \( \mathbb{Y} \) (via lifting a point on an irreducible component to \( D_K \) and looking at the specialization of its image under an element of \( G \)), and associates an irreducible component of \( \mathbb{Z} \) to each irreducible component of \( \mathbb{Y} \) (its "image" under the "quotient map"). Furthermore, if \( \mathbb{V} \) is an irreducible component of \( \mathbb{Y} \) with generic point \( \eta_{\mathbb{V}} \) lying above an irreducible component \( \mathbb{W} \) of \( \mathbb{Z} \) with generic point \( \eta_{\mathbb{W}} \), the map \( D \to D' \) induces Galois extensions of complete local rings \( \mathcal{O}_{Y^{st}, \eta_{\mathbb{V}}}/\mathcal{O}_{Z^{st}, \eta_{\mathbb{W}}} \).

The underlying tree of the Hurwitz tree is built from the dual graph \( \Gamma \) of \( \mathbb{Z} \). Specifically, the edges \( E(\Gamma) \) and vertices \( V(\Gamma) \) correspond to the irreducible components and nodes of \( \mathbb{Z} \), respectively. An edge connects two vertices if the corresponding node is the intersection of the two corresponding components. We append another vertex and edge \( v_0 \) and \( e_0 \) so that \( e_0 \) connects \( v_0 \) to the vertex corresponding to the component containing the specialization of \( \infty \). Lastly, we append a vertex
for each branch point \( y_i \), and connect it via an edge \( e_i \) to the vertex representing the irreducible component of \( \mathbb{Z} \) to which \( y_i \) specializes. This gives us a tree \( \Gamma' \).

To each edge \( e \in E(\Gamma') \setminus \{e_0, \ldots, e_t\} \), we attach a positive number \( \epsilon_e \) equal to the thickness of the corresponding node. That is, if the edge corresponds to a point \( z \in \mathbb{Z} \), then the complete local ring \( \hat{O}_{\mathbb{Z}^{\pi}} \) is isomorphic to \( R[[U,V]]/(UV - \rho) \) where \( v(\rho) \in \mathbb{Q} (\text{Hen00b}) \), and we set \( \epsilon_e = v(\rho) \). We set \( \epsilon_{e_0} \) equal to \( v(z_i) \), where \( z_i \) is a branch point of smallest possible valuation (thought of as an element of \( R \)). Such a branch point will specialize to the same component of \( \mathbb{Z} \) as \( \infty \). Lastly, we set \( \epsilon_{e_i} = 0 \) for all \( i > 0 \).

Now, suppose the \( p \)-Sylow subgroup of \( G \) is \( \mathbb{Z}/p \). In this case we attach the following data to each vertex \( v \in V(\Gamma') \).

- A nonnegative integer \( \delta_v \) called the depth at the vertex \( v \), which is a measure of inseparability of the map \( D \to D' \) above the irreducible component \( W \) of \( \mathbb{Z} \) corresponding to \( v \) (for \( v \in V(\Gamma') \)). Specifically, let \( \eta_v \) be the generic point of this irreducible component and let \( \eta_w \) be the generic point of the (always unique) component of \( Y \) above it. The extension \( \hat{O}_{\mathbb{Z}^{\pi},\eta_w}/\hat{O}_{\mathbb{Z}^{\pi},\eta_v} \) has inertia group \( \mathbb{Z}/p \) because of its inseparable residue field extension, and one can define its depth \( \delta_v \) as in [Obu12 Appendix B.1]. One also sets \( \delta_{v_0} = 0 \) and \( \delta_{v_i} = 1 \) for \( 1 \leq i \leq r \).
- A meromorphic differential form \( \omega_v \) on \( W \) associated to the extension \( \hat{O}_{\mathbb{Z}^{\pi},\eta_w}/\hat{O}_{\mathbb{Z}^{\pi},\eta_v} \) called a differential datum (or deformation datum). These are only associated to the vertices in \( V(\Gamma) \). By definition, they are either exact (equal to \( df \) for some \( f \in k(W) \)) or logarithmic (equal to \( df/f \) for some \( f \in k(W) \)). See [Hen00a, Corol. 1.8] or [Obu12 Appendix B.2].

The tree \( \Gamma' \), along with the \( \epsilon_e, \delta_v, \) and \( \omega_v \) together form a Hurwitz tree. It turns out that the data satisfy a host of conditions (see, e.g., [BW06 Definition 3.2 and Proposition 3.4] or [Obu12 Proposition 7.9]). A (general) Hurwitz tree for a group \( G \) with \( p \)-Sylow subgroup \( \mathbb{Z}/p \) is a tree \( \mathcal{T} \), along with data \( \epsilon_e, \delta_v, \) and \( \omega_v \) satisfying these conditions (see [BW06 Definition 3.2] or [Obu12 Definition 7.10]).

If \( G \) is a general finite group of the form \( P \times \mathbb{Z}/m \) with \( P \) a \( p \)-group and \( p \mid m \), the Hurwitz tree construction of [BW09] starts in the same way, and one produces the same tree \( \Gamma' \) and the same edge thicknesses \( \epsilon_e \). However, one makes the following changes.

- Each vertex \( v \in V(\Gamma') \) now has a decomposition conjugacy class \([G_v]\), which is the conjugacy class (inside \( G \)) of the Galois group of \( \hat{O}_{\mathbb{Z}^{\pi},\eta_w}/\hat{O}_{\mathbb{Z}^{\pi},\eta_v} \).
- Instead of being a number, the depth \( \delta_v \) for \( v \in V(\Gamma') \) is now a \( \mathbb{Q} \)-valued character on \( G \). It is induced from \( G_v \) (see [BW09 §3.2]). It is equal to 0 on \( v_0 \).
- Each edge \( e \in E(\Gamma') \) now has an Artin character \( a_e \), which is an (integral) character on \( G \) (see [BW09 §3.3]). In fact, the Artin character \( a_{e_0} \) is the Artin character of the higher ramification filtration of the original local \( G \)-extension (this result is from [BW09 Definition 3.5], and the definition of the Artin character can be found on [Ser79 p. 99]).
- There are no differential data \( \omega_v \).

Remark 4.12. The depth character is a useful way of encoding the depth information for all of the \( \mathbb{Z}/p \)-subextensions of \( R[[Z]]/R[[T]] \).
Remark 4.13. Similarly, the Artin character is a useful way of encoding the differential data for all of the \( \mathbb{Z}/p \)-subextensions of \( R[[t]]/R[[T]] \). In the case where the \( p \)-Sylow subgroup \( P \) of \( G \) is \( \mathbb{Z}/p \) (so that we can define differential data and Artin characters), the Artin character tells us about the zeroes and poles of the differential data. Indeed, for an edge \( e \in E(\Gamma) \) corresponding to a node \( \varpi \) of \( \mathbb{Z} \), the value of \( a_e \) on a nontrivial element of \( P \) is exactly the order of the divisor of \( \omega_v \) at \( \varpi \), where \( v \) is one of the vertices incident to \( e \) and thus corresponds to an irreducible component \( \varpi \) of \( \mathbb{Y} \) passing through \( \varpi \).

Remark 4.14. One would like to have a full theory of differential data for general Hurwitz trees. In particular, for a vertex \( v \in V(\Gamma) \), one would like to associate a meromorphic differential form \( \omega_v(\chi) \) to each character \( \chi \) of \( G \). The thesis of Brewis ([Bre09, Chapter 1]) shows how to use Kato’s Swan conductors to do this when \( \chi \) has rank 1. However, if \( \chi \) has higher rank, one can obtain a tensor product of differential forms rather than just one form. Furthermore, even when one does obtain just one form, it is not clear exactly how to categorize what forms can be obtained (unlike in the case where \( G \) has \( p \)-Sylow subgroup of order \( p \), in which case the form is exact or logarithmic).

Remark 4.15. In [Hen00a], [BW06], and [Obu12], the graph \( \Gamma \) and thicknesses \( \epsilon_e \) come from \( Y^{st} \) instead of \( Z^{st} \) (in the case where \( G = \mathbb{Z}/p \), this has the effect of dividing the \( \epsilon_e \) by \( p \)). We use \( Z^{st} \) for our definitions since that is what the more general Hurwitz trees of [BW09] use.

Example 4.16. Let \( R \) be a complete discrete valuation ring containing \( \zeta_p \), and let \( \lambda = \zeta_p - 1 \), which has valuation \( 1/(p-1) \). Pick \( N \in \mathbb{N} \setminus p\mathbb{N} \). In Example 6.5 it is shown that taking the integral closure \( A \) of \( R[[T]] \) in \( \text{Frac}(R[[T]])[W]/(W^p - 1 - \lambda^p T^{-N}) \) gives a lift of the local \( \mathbb{Z}/p \)-extension given by taking the integral closure of \( k[[t]] \) in \( k((t))[y]/(y^p - y - t^{-N}) \). We write down the (differential) Hurwitz tree for this lift, using the notation above freely.

The branch points \( z_1, \ldots, z_{N+1} \) of \( D_K \to D'_K \) are at \( T = 0 \) and at \( T \) any \( N \)-th root of \( -\lambda^p \). Since these points are all mutually equidistant, we have that \( \mathbb{Z} \) is isomorphic to \( \mathbb{P}^1 \), and the graph \( \Gamma \) is simply a point, which we call \( v \). To build the Hurwitz tree \( \Gamma' \), we add a root vertex \( v_0 \) and vertices \( v_1, \ldots, v_{N+1} \) corresponding to the branch points, with each \( v_i \) connected to \( v \) by an edge \( e_i \). The thickness \( \epsilon(v_0) \) is just the valuation \( p/N(p-1) \) of \( \sqrt[p]{-\lambda^p} \). If \( i > 0 \), then \( \epsilon(e_i) = 0 \).

By definition, we have \( \delta_v = 0 \) and \( \delta_{v_i} = 1 \) for \( 1 \leq i \leq N + 1 \). It turns out that \( \delta_v = 1 \) as well (e.g., [BW06, Definition 3.2(ii) and Lemma 3.3(iii)]). Lastly, let \( \alpha \) be an \( N \)-th root of \( -\lambda^p \). Then, taking \( X = T/\alpha \), we have that \( X \) reduces to a coordinate \( x \) on \( \mathbb{Z} \) (the specializations of the branch points to \( \mathbb{Z} \) correspond to \( x = 0 \) and all the \( N \)-th roots of unity). Since \( 1 + \lambda^p T^{-N} = 1 - X^{-N} \), it follows from [BW06 §3.2] (see also [Hen00a Corollaire 1.8(A)]) that

\[
\omega_v = \frac{d(1 - x^{-N})}{1 - x^{-N}} = -N \frac{dx}{x(x^N - 1)}.
\]

4.2.2. The Hurwitz tree obstruction. In the context of a general Hurwitz tree, the way one gets an obstruction is now clear. Let \( k[[z]]/k[[t]] \) be a local \( G \)-extension whose higher ramification filtration has Artin character \( \chi \). We say there is a Hurwitz tree obstruction to lifting if no Hurwitz tree for \( G \) can be constructed having Artin character \( \chi \) on the vertex \( v_0 \).
Example 4.17. Let $G = Q_{2m}$, the generalized quaternion group of order $2^m$, which can be presented as

$$G = \langle \sigma, \tau \mid \tau^{2^{m-1}} = 1, \tau^{2^{m-2}} = \sigma^2, \sigma \tau \sigma^{-1} = \tau^{-1} \rangle.$$  

If $m = 3$, this is the standard quaternion group, and we assume $m \geq 3$. Let $G = G/\langle \tau^2 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Then $\overline{G}$ is generated by the images $\overline{\sigma}$ and $\overline{\tau}$ of $\sigma$ and $\tau$. Consider the $\overline{G}$-action on $k[[z]]$ such that

$$\overline{\sigma}(z) = \frac{1}{1 + z} \quad \text{and} \quad \overline{\tau}(z) = \frac{1}{1 + \mu z}$$

with $\mu \in k \setminus \mathbb{F}_2$. This action can be lifted to a faithful $G$-action on $k[[t]]$, giving a local $G$-extension ([CGH08, Lemma 2.10]). By [BW09, Theorem 4.8], this local $G$-extension has a Hurwitz tree obstruction to lifting. In particular, $Q_m$ is not a local Oort group (cf. Theorem 4.8).

4.2.3. The differential Hurwitz tree obstruction. What should be the definition of the differential Hurwitz tree obstruction? Well, if $k[[z]]/k[[t]]$ is a local $G$-extension whose higher ramification filtration has Artin character $\chi$, there should be a differential Hurwitz tree obstruction to lifting if no differential Hurwitz tree for $G$ can be constructed having Artin character $\chi$ on the vertex $v_0$. Of course, we have not defined a differential Hurwitz tree (aside from when $G$ has $p$-Sylow subgroup $\mathbb{Z}/p$, in which case a differential Hurwitz tree is just our usual definition of a Hurwitz tree with differential data). But it should be a Hurwitz tree enriched with meromorphic differential forms attached to each vertex $v$ (aside from the leaves) such that the divisors of these forms have some sort of compatibility with the Artin characters, and such that the forms themselves are of a certain type (which will in general be more complicated than just “exact” or “logarithmic,” see Remark 4.14).

We elaborate somewhat in the case $G = (\mathbb{Z}/p)^n$. Suppose we have a Hurwitz tree for $G$ whose underlying graph has only one non-leaf vertex. Suppose further that the Artin character for the edge going from this vertex to the root of the tree (i.e., the vertex with depth zero) gives the value $m + 1$ for all nontrivial elements of $G$. In this case, the extra “differential data” should be an $n$-dimensional $\mathbb{F}_p$-vector space of logarithmic differential forms, each with a (unique) zero at $\infty$ of order $m - 1$ and $m + 1$ simple poles. Such a vector space is called an $E_{m+1,n}$ in the introduction to [Pag02a].

Example 4.18. Pagot has shown ([Pag02a, Théorème 2]) that for $p \geq 3$, no $E_{m+1,2}$ exists when $m + 1 = ap$ with $a \in \{1, 2, 3\}$, unless $a = 2$ and $p = 3$ (Turchetti has also shown nonexistence for $a = 5$ and $p = 3$, see [Tur15, Theorem 0.3]). As a consequence of this, he shows that a local $(\mathbb{Z}/p^2)$-extension $k[[z]]/k[[t]]$ where every degree $p$-subextension of $k[[t]]$ has ramification jump $p - 1$ (equivalently, where the only ramification jump for $k[[z]]/k[[t]]$ is $p - 1$) cannot lift to characteristic zero when $p \geq 3$ ([Pag02a, Théorème 3]). The proof proceeds by first observing that a Hurwitz tree for this lift has only one non-leaf vertex ([CM99, Theorem III, 3.1]—this corresponds to the branch points of the lift all being equidistant from each other), and then invoking the nonexistence of an $E_{p,2}$.

In fact, there is no KGB obstruction to lifting this type of extension ([Obu12, Proposition 5.8]), and one can show that the Hurwitz tree obstruction vanishes as well. Thus it is justified to say that there is a differential Hurwitz tree obstruction to lifting in this case.
Remark 4.19. In fact, differential Hurwitz trees can be used to get positive results for the local lifting problem. See §6.4.

5. Summary of present local lifting problem results

5.1. Local Oort groups. From Theorem 4.8 and the discussion following it, we know that the only possible local Oort groups are cyclic, $D_p^n$, and $A_4$ for $p = 2$. The local Oort conjecture says that cyclic groups are local Oort groups. The local-global principle shows that the local Oort conjecture implies the Oort conjecture, and the existence of HKG-covers shows that the Oort conjecture implies the local Oort conjecture. Here is the state of current knowledge.

- Cyclic groups $G$ are local Oort (i.e., the Oort conjecture is true). This is the main result of Obus-Wewers and Pop ([OW14], [Pop14]). Earlier cases had been proven by Sekiguchi-Oort-Suwa ([SOS89], $v_p(|G|) \leq 1$) and Green-Matignon ([GM98], $v_p(|G|) \leq 2$).
- Bouw and Wewers showed that $D_p$ is local Oort for $p$ odd ([BW06]), and Pagot showed the same for $p = 2$ ([Pag02a], [Pag02b]).
- Obus proved that $D_3$ is local Oort for $p = 3$ ([Obu15]), and Weaver proved the same for $D_4$ and $p = 2$ ([Wea]). These are the only dihedral groups not of the form $D_p$ for which this is known.
- The group $A_4$ was announced to be a local Oort group for $p = 2$ by Bouw in [BW06]. A proof has been written up by Obus in [Obu16].

The results above will be discussed further in §6.5 and §6.6.

5.2. Weak local Oort groups. We have seen in §4.1.2 that there are certain obstructions to being a weak local Oort group. On the positive side, aside from the local Oort groups from §5.1, we have the following results.

- Matignon showed that $(\mathbb{Z}/p)^n$ is weak local Oort for all $p$ and $n$ ([Mat99]). This is done using explicit methods (see §6.2).
- Obus showed that $G = \mathbb{Z}/p^n \times \mathbb{Z}/m$ is weak local Oort whenever it is center-free ([Obu15, Corollary 1.19]). This is also a necessary condition as long as $G$ is not cyclic, as we saw in §4.1.2.
- Brewis showed that the group $D_4$ is a weak local Oort group for 2 ([Bre08]). Weaver’s proof that $D_4$ is local Oort for 2 uses Brewis’s result as a base case (see §6.6.1).

Important open questions about the local lifting problem will be discussed in §8.

6. Lifting techniques and examples

In this section, we will survey a variety of techniques that have been used to construct lifts of local $G$-extensions.

6.1. Birational lifts and the different criterion. Usually, when dealing with Galois extensions of $k[[t]]$, it will be more convenient to deal with extensions of fraction fields than extensions of rings. For instance, by Artin-Schreier theory, one knows that any $\mathbb{Z}/p$-extension $L/k((t))$ is given by an equation of the form $y^p - y = f(t)$. But writing down equations for the integral closure of $k[[t]]$ in $L$ is much more difficult. So we will often want to think of a Galois ring extension in terms of the associated extension of fraction fields. In particular, we define a birational lift as follows:
Definition 6.1. Let \( k[[z]]/k[[t]] \) be a local \( G \)-extension. A birational lift over a complete characteristic zero discrete valuation ring \( R \) with residue field \( k \) is a \( G \)-extension \( M/\text{Frac}(R[[T]]) \) such that:

1. If \( A \) is the integral closure of \( R[[T]] \) in \( M \), then the integral closure of \( A_k \) is isomorphic to (and identified with) \( k[[z]] \) (equivalently, \( \text{Frac}(A_k) \cong k((z)) \)).
2. The \( G \)-action on \( k((z)) = \text{Frac}(A_k) \) induced from that on \( A \) restricts to the given \( G \)-action on \( k[[z]] \).

In fact, Garuti has shown (Gar96) that any local \( G \)-extension has a birational lift to characteristic zero.

The following criterion, which saves one from the effort of making explicit computations with integral closures, is extremely useful for seeing when a birational lift is actually a lift.

Proposition 6.2 (The different criterion, GM98 I, 3.4). Suppose \( A/R[[T]] \) is a birational lift of the local \( G \)-extension \( k[[z]]/k[[t]] \). Let \( K = \text{Frac}(R) \), let \( \delta_\eta \) be the degree of the different \( D_\eta \) of \( A_K/R[[T]]_K \) (i.e., the length of \( A_K/D_\eta \) as a \( K \)-module), and let \( \delta_\nu \) be the degree of the different \( D_\nu \) of \( k[[z]]/k[[t]] \) (i.e., the length of \( A_k/D_\nu \) as a \( k \)-module). Then \( \delta_\nu \leq \delta_\eta \), and equality holds if and only if \( A/R[[T]] \) is a lift of \( k[[z]]/k[[t]] \) (that is, \( A \cong R[[Z]] \)).

Remark 6.3. Replacing \( R \) and \( K \) by finite extensions does not affect the degree of \( D_\eta \) above, so we may assume that the ramified ideals in \( A_K/R[[T]]_K \) have residue field \( K \).

Remark 6.4. The different criterion is also valid when \( R \) is an equicharacteristic complete discrete valuation ring (i.e., \( R = k[[\varpi]] \)). This will be used in §6.3.

6.2. Explicit lifts. Sometimes, the simplest way of lifting a local \( G \)-extension is to write down explicit equations. We give two examples in this section.

Example 6.5. (\( Z/p \)-extensions) The following argument shows that all local \( \mathbb{Z}/p \)-extensions lift to characteristic zero. It is a simplified version of arguments originally from SOS89, and can also be found in Obu12, Theorem 6.8. Since it is the most basic example, one would be remiss not to include it here.

The key observation is that any \( \mathbb{Z}/p \)-extension of a characteristic zero field containing a \( p \)th root of unity is a Kummer extension, given by extracting a \( p \)th root. The trick is then to assume \( \zeta_p \in R \) and to find an element of \( \text{Frac}(R[[T]]) \) such that normalizing \( R[[T]] \) in the corresponding Kummer extension yields the original local Artin-Schreier extension.

Say that an element of a field \( L \) of characteristic \( p \) is a \( \phi \)th power if it is expressible as \( x^p - x \) for \( x \in L \). By Artin-Schreier theory, any \( \mathbb{Z}/p \)-extension of \( k((t)) \) is given by \( k((t))[y]/(y^p - y - g(t)) \), and is well defined up to adding a \( \phi \)th power to \( g(t) \). In particular, we may assume that \( g(t) \in t^{-1}k[t^{-1}] \), as any element \( u \in k[[t]] \) can be written as \( x^p - x \), where \( x = -u - u^p - u^{2p} - \cdots \). Similarly, we may assume that \( g(t) \) has no terms of degree divisible by \( p \). If \( g(t) = t^{-N}h(t) \), where \( h(t) \in k[t] \) has nonzero constant term, then \( h(t) \) is an \( N \)th power in \( k((t)) \), so replacing \( t \) with an \( N \)th root of \( 1/g(t) \) (which is a uniformizer), we may assume that \( g(t) = t^{-N} \), with \( p \nmid N \).

Given a local \( \mathbb{Z}/p \)-extension \( k[[z]]/k[[t]] \), we may thus assume without loss of generality that it is the integral closure of \( k[[t]] \) in the Artin-Schreier extension of \( k((t))[y]/(y^p - y - t^{-N}) \) given by \( t^{-N} \). Let \( R = W(k)[\zeta_p] \), let \( \lambda = \zeta_p - 1 \), and let...
$K = \text{Frac}(R)$. Then $\nu(\lambda^{p-1} + p) > 1$. Consider the integral closure $A$ of $R[[T]]$ in the Kummer extension of $\text{Frac}(R[[T]])$ given by
\begin{equation}
W^p = 1 + \lambda^p T^{-N}.
\end{equation}
Making the substitution $W = 1 + \lambda Y$, we obtain
\[ (\lambda Y)^p + p\lambda Y + o(p^{\nu(p-1)}) = \lambda^p T^{-N}, \]
where $o(p^{\nu(p-1)})$ represents terms with coefficients of valuation greater than $p/(p-1)$. This reduces to $y^p - y = t^{-N}$. So we have constructed a birational lift.

The jump in the ramification filtration for $k[[z]]/k[[t]]$ occurs at $N$ (Exercise! This can be done explicitly by writing a uniformizer in terms of $t$ and $y$). By (A.1), the degree of the different of $k[[z]]/k[[t]]$ is $(N + 1)(p - 1)$. On the other hand, the generic fiber of $\text{Spec } A \to \text{Spec } R[[T]]$ is branched at exactly $N + 1$ points in the unit disc (at $T = 0$ and $T = \nu$ as $\nu$ ranges through the $N$th roots of $-\lambda^p$). Since the ramification is tame, the degree of the different of $A_K/R[[T]]_K$ is $(N + 1)(p - 1)$ as well. By Proposition 6.2 our birational lift is an actual lift.

Remark 6.6. It is easy to use Example 6.5 to show that all local $\mathbb{Z}/pm$-extensions lift to characteristic zero, when $p \nmid m$. See [Oba12, Proposition 6.3].

Remark 6.7. The case $N = 1$ is the local version of Example 2.2 above. Note that in this case, $y^{-1}$ is a uniformizer of $k[[z]]$, and we can set $z = y^{-1}$. Taking $Z = Y^{-1} = \lambda/(W - 1)$, Remark A.6 shows that $A$ can be written as $R[[Z]]$, once we verify that $Z \in A$. This is true because expanding out the equation $(1 + \lambda Z^{-1})^p = 1 + \lambda^p T^{-1}$ coming from (6.1), and multiplying both sides by $T Z^p/\lambda^p$ gives an integral equation for $Z$ over $R[[T]]$.

Example 6.8. (Some $\mathbb{Z}/2 \times \mathbb{Z}/2$-extensions) For odd $p$, it is an open problem in general to determine exactly which local $\mathbb{Z}/p \times \mathbb{Z}/p$-extensions lift to characteristic zero (see Example 6.4 and also [Oba12, Proposition 5.8], which is an exposition of material in [GM98, I, Theorem 5.1]). However, some local $\mathbb{Z}/p \times \mathbb{Z}/p$-extensions can be lifted explicitly. For example, suppose $p = 2$, and consider the local $\mathbb{Z}/2 \times \mathbb{Z}/2$-extension $k[[z]]$ of $k[[t]]$ given by normalizing $k[[t]]$ in $k((t))[y, w]/(y^2 - y - t - 1, w^2 - w - t - N)$ for any odd $N > 1$. Letting $R = W(\mathbb{F}_2)$, I claim that normalizing $R[[T]]$ in
\[ L = \text{Frac}(R[[T]])/(U, V)/(U^2 - 1 - 4T^{-1}, V^2 - 1 - 4T^{-N}) \]
gives an extension $A/R[[T]]$ lifting $k[[z]]/k[[t]]$. Since the field extension $L/\text{Frac}(R[[T]])$ is the compositum of two $\mathbb{Z}/2$-extensions, each giving rise to a lift of the component $\mathbb{Z}/2$-extensions of $k[[t]]$ (see (6.1), and note that $\lambda^p = 4$), we have that $L/\text{Frac}(R[[T]])$ certainly gives rise to a birational lift. In order to show that it is actually a lift, we apply the different criterion (Proposition 6.2).

Since the upper numbering is preserved under taking quotients ([Ser79, IV, Proposition 14], the upper numbering of $k[[z]]/k[[t]]$ appear at $1$ and $N$. This means that $G_0 = G_1 = \mathbb{Z}/2 \times \mathbb{Z}/2$, and $G_2 = \cdots = G_{2N - 1} = \mathbb{Z}/2$, with the rest of the filtration trivial. By (A.1), the degree of the different is equal to $2N + 4$.

On the other hand, since ramification groups in characteristic zero are cyclic, every ramification index of the $\mathbb{Z}/2 \times \mathbb{Z}/2$-extension $A_K/R[[T]]_K$ is $2$, and thus each ramified ideal of $R[[T]]_K$ with residue field $K$ (which, after a finite extension of $K$, we assume is every ramified ideal) contributes $2$ to the degree of the different. These ramified ideals correspond to the zeroes and poles of $1 + 4T^{-1}$ and $1 + 4T^{-N}$.
in the open unit disc. There are a total of $N + 1$ zeroes and 1 pole, showing that the degree of the different is $2N + 4$. We are done.

**Remark 6.9.** In fact, every local $\mathbb{Z}/2 \times \mathbb{Z}/2$-extension lifts to characteristic zero, but writing down the lift is not generally as straightforward as above. For more on this, see [Pag02a] and [Pag02b].

**Remark 6.10.** Matignon ([Mat99]) has shown that $(\mathbb{Z}/p)^n$ is a weak local Oort group for any $p$ and $n$, by writing down an explicit example and an explicit lift.

For an example of a local $D_p$-extension for any odd $p$ with an explicit lift to characteristic zero, see [GM99] IV, Proposition 2.2.1 (or [Obu12] Proposition 7.3 for the same example). As in the case of $\mathbb{Z}/2 \times \mathbb{Z}/2$, all local $D_p$-extensions lift to characteristic zero ([BW06]), but it is not in general easy to write down the lift explicitly.

For an example of explicit lifts of some local $A_4$-extensions to characteristic zero, see [Obu16, Propositions 5.1, 5.2]. The paper [Bre08] gives examples of explicit lifts of some local $D_4$-extensions.

### 6.3. Sekiguchi-Suwa Theory

One potential way of obtaining explicit lifts for cyclic local extensions is the Kummer-Artin-Schreier-Witt theory, or Sekiguchi-Suwa theory (developed in [SS94], [SS99], with [MRT14] being a nice survey). Here, we will limit ourselves to mentioning that Kummer-Artin-Schreier theory, as developed by Sekiguchi, Oort, and Suwa in [SOS89], gives an explicit group scheme $G$ defined over $\mathbb{Z}_p[z]/\mathbb{Z}_p$ whose special fiber is $G_a$ and whose generic fiber is $G_m$. Furthermore, the theory exhibits the (more or less unique) degree $p$ isogeny on $G$ explicitly. Any lift of a local $\mathbb{Z}/p$-extension (which is Artin-Schreier) to a Kummer extension is a torsor under the kernel of this isogeny, and knowing the explicit equations cutting out this kernel leads one to discover the Kummer extension used in Example 6.5. The Kummer-Artin-Schreier-Witt theory generalizes this story to isogenies of degree $p^n$. We refer the reader to [Obu12, §4.8] for a brief exposition, and then to [MRT14] if deeper knowledge is desired.

We note that Green and Matignon were able to use the Kummer-Artin-Schreier-Witt theory to show that $\mathbb{Z}/p^2$ is a local Oort group ([GM98], or [Obu12, §6.5] for an overview). The equations involved in Kummer-Artin-Schreier-Witt theory for $\mathbb{Z}/p^n$ become very complicated when $n > 2$, and the theory has not been successfully applied to the local lifting problem for these groups.

### 6.4. Hurwitz trees

The Hurwitz trees discussed in §4.2 have been used to obtain positive results for the local lifting problem in the case $G = \mathbb{Z}/p$ ([Hen00a]—of course, this is already proven in Example 6.5) and $G = \mathbb{Z}/p \times \mathbb{Z}/m$ ([BW06], [BWZ09]). The process is outlined in some detail in [Obu12] §7.3.3 and §7.3.4, and we will not repeat it here. We content ourselves with stating a (lightly paraphrased) version of the theorem of Bouw, Wewers and Zapponi.

**Theorem 6.11** ([BWZ09], Theorem 2.1). Suppose $p$ is a prime not dividing $m$. A nonabelian $\mathbb{Z}/p \times \mathbb{Z}/m$-extension $k[[z]]/k[[t]]$ lifts to characteristic zero iff its KGB obstruction vanishes.

**Remark 6.12.** By [Obu12, Proposition 5.9], Theorem 6.11 is equivalent to stating that $k[[z]]/k[[t]]$ lifts to characteristic zero if and only if the ramification jump of the $\mathbb{Z}/p$-subextension is congruent to $-1 \pmod{m}$. This condition holds when $m = 2$ (e.g., [Pri02, Lemma 1.4.1(iv)]), so the dihedral group $D_p$ is local Oort for $p$ odd.
6.5. Successive approximation. The method of successive approximation gives a new approach to lifting local G-extensions, which has been successful in the case that \( G = \mathbb{Z}/p^n \) (OW14) and \( \mathbb{Z}/p^n \times \mathbb{Z}/m \), with \( p \nmid m \) (Obu15). The method has been described in detail in Obu12 §6.6 when \( G = \mathbb{Z}/p^n \), and is similar when \( G = \mathbb{Z}/p^n \times \mathbb{Z}/m \) with \( p \nmid m \), so as in [6.6.6] we give only an extremely brief overview.

In both cases, one constructs a lift inductively. Suppose \( G = \mathbb{Z}/p \times \mathbb{Z}/m \), and \( k[[z]]/k[[t]] \) is a local \( G \)-extension with vanishing KGB obstruction. If \( G \) is cyclic (resp. non-abelian), then the extension lifts by Example 6.5 and Remark 6.6 (resp. Theorem 6.11). Now, by induction, assume that the \( G/(\mathbb{Z}/p) \)-subextension \( k[[s]]/k[[t]] \) has a lift to characteristic zero. If \( G \) is cyclic, we start by guessing the form of a \( \mathbb{Z}/p^n \)-Kummer extension in characteristic zero such that the \( \mathbb{Z}/p^{n-1} \)-subextension is a lift of \( k[[s]]/k[[t]] \), and the full extension is a lift of some local \( \mathbb{Z}/p^n \)-extension \( k[[z']]/k[[t]] \) with smallest possible upper jump. Our guess will generally be incorrect, but one can show that it can be gradually deformed into something closer to a lift (closeness of \( A/R[[T]] \) to being a lift is measured by the different of \( A/R[[T]] \) localized at a uniformizer of \( R \)). The main result of OW16 shows that this deformation process eventually stops, and we get a lift of \( k[[z']]/k[[t]] \). A further deformation gives a lift of \( k[[z]]/k[[t]] \), as desired. If \( G \) is metacyclic, the idea is the same, but we try to lift \( k[[z]]/k[[t]] \) from the beginning, without going through the intermediate step involving the extension \( k[[z']]/k[[t]] \).

The method of successive approximation gives rise to the following two results.

**Theorem 6.13 (OW14 Theorem 1.4, Obu12 Theorem 6.28).** Suppose \( k[[z]]/k[[t]] \) is a local \( \mathbb{Z}/p^n \)-extension with upper jumps \( u_1, \ldots, u_n \), such that \( u_{i+1} < pu_i + p \) for \( 1 \leq i \leq n-1 \). Then \( k[[z]]/k[[t]] \) lifts to characteristic zero.

**Remark 6.14.** (i) The criterion on the \( u_i \) stated in OW14 and Obu12 is slightly different from what appears above, but it holds whenever the criterion above holds by Pop14 Key Lemma 4.15. Pop calls the condition above no essential ramification.

(ii) By Example A.8, one has \( u_{i+1} \geq pu_i \) for any local \( \mathbb{Z}/p^n \)-extension.

(iii) In fact, in OW14, one only needs that \( u_{i+1} < pu_i + p \) for \( 2 \leq i \leq n-2 \). Since the full Oort conjecture has been proven in any case (using the techniques of §6.4), this is not so important.

**Theorem 6.15 (Obu15 Theorem 1.14).** Suppose \( k[[z]]/k[[t]] \) is a local \( \mathbb{Z}/p^n \times \mathbb{Z}/m \)-extension whose (unique) \( \mathbb{Z}/p^n \)-subextension has upper jumps \( u_1, \ldots, u_n \), such that all \( u_i \equiv -1 \pmod{m} \), that \( u_{i+1} < pu_i + mp \) for \( 1 \leq i \leq n-1 \), and that \( u_1 < mp \). If a certain criterion called the “isolated differential data criterion” is satisfied by \((u_1, \ldots, u_n)\), then \( k[[z]]/k[[t]] \) lifts to characteristic zero.

**Remark 6.16.** (i) The isolated differential data criterion is defined in Obu15 §1.4 and Definition 7.23. The definition is somewhat technical, so we do not discuss it here beyond saying that it has to do with the existence of certain logarithmic differential forms on \( \mathbb{P}^1_k \), and can be phrased as asserting the existence of a solution to an equidetermined system of (non-linear!) equations over \( k \). In the context of Theorem 6.15 it is satisfied if \( G = D_{9g} \), if \( G = D_{p^2} \) with \( u_1 = 1 \), or if

\[
(u_1, \ldots, u_n) = (m-1, p(m-1), \ldots, p^{n-1}(m-1))
\]

(Obu15 Propositions 8.1, 8.2, 8.4).
The isolated differential data criterion is related to the differential Hurwitz trees from [4.23]. It is somewhat stronger than what should be meant by “there exists a differential Hurwitz tree for \( G \) whose Artin character matches that of \( k[[z]]/k[[t]] \).”

The condition that the jumps be congruent to \(-1 \pmod{m}\) is required because otherwise there is a local KGB obstruction to lifting ([Obu12 Proposition 5.9]).

The condition \( u_{i+1} < pu_i + mp \) is also called no essential ramification.

### 6.6. The “Mumford method”

A technique that has been tremendously successfully applied to the local lifting problem in recent years is the method of deforming a local \( G \)-extension within characteristic \( p \) to one that has nicer properties, showing that one can lift this nicer extension to characteristic zero, and then showing that this implies the original extension can be lifted. Oort has called this the “Mumford method” because a similar idea, originally due to Mumford, was used to show that all abelian varieties over algebraically closed fields of characteristic \( p \) lift to characteristic zero. Namely, by the Serre-Tate theory of canonical liftings, it was known that ordinary abelian varieties lift to characteristic zero (see, e.g., [Kat81]). Norman and Oort ([NOS81]) showed that any characteristic \( p \) abelian variety deforms to an ordinary abelian variety, that can then be lifted by the Serre-Tate theory. They then showed that this implies the original abelian variety can be lifted. In this case, “nice” means ordinary. In the case of the local lifting problem, “nice” will be related to having limited ramification in some sense. This sense might vary depending on the group; for \( \mathbb{Z}/p^n \times \mathbb{Z}/m \), “nice” will mean “no essential ramification.”

#### 6.6.1. Equicharacteristic deformations

Let \( k[[z]]/k[[t]] \) be a local \( G \)-extension. An equicharacteristic deformation of \( k[[z]]/k[[t]] \) is a \( G \)-extension \( k[[\varpi,z]]/k[[\varpi,t]] \), where \( \varpi \) is a transcendental parameter, such that the \( G \)-action on \( k[[\varpi,z]] \) reduces to the original \( G \)-action on \( k[[z]] \) modulo \( \varpi \). This is the equicharacteristic version of lifting to characteristic zero (of course, in the equicharacteristic case we have the trivial deformation, so the behavior is somewhat different)! We think of \( \varpi \) as the deformation parameter, and the generic fiber of the deformation is \( k[[\varpi,z]][\varpi^{-1}]/k[[\varpi,t]][\varpi^{-1}] \).

**Remark 6.17.** It is useful to think of \( k[[\varpi,t]][\varpi^{-1}] \) as the ring of functions on the open unit disc over \( k((\varpi)) \) with parameter \( t \) (as an exercise to understand this, think about which elements \( t-a \) are units and which are not, as \( a \) ranges over \( k((\varpi)) \)). Observe that \( k((\varpi))[t] \) is strictly bigger than \( k[[\varpi,t]][\varpi^{-1}] \)!

Let us give a nontrivial example of an equicharacteristic deformation, which is the inspiration for all examples of equicharacteristic deformations we will mention (see Proposition 6.23).

**Example 6.18.** Let \( k[[z]]/k[[t]] \) be the \( \mathbb{Z}/p \)-extension given by taking the integral closure of \( k[[t]] \) in

\[
k((t))y/(y^p - y - t^N),
\]

where \( N > p \) is not a multiple of \( p \). We claim that the integral closure \( \mathcal{A} \) of \( k[[\varpi,t]] \) in

\[
k((\varpi,t))y/(y^p - y - t^p(t - \varpi)^{N+p})
\]

is an equicharacteristic deformation. Setting \( \varpi = 0 \) clearly yields \( k((z))/k((t)) \) after taking fraction fields, but we must show that \( \mathcal{A} \cong k[[\varpi,z]] \).
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To do this we use the different criterion (Proposition 6.2) along with Remark 6.4. It suffices to show that the degree \( \delta_s \) of the different of the original \( G \)-extension is equal to the degree \( \delta_y \) of the different of the generic fiber of the deformation. We have \( \delta_s = (N + 1)(p - 1) \) (see Example 6.3). On the generic fiber, the two ramified ideals are \((t)\) and \((t - \varpi)\). The function \( g = t^p(t - \varpi)^{-N+p} \) has a pole of order \( N - p \) when expanded out in \( k((\varpi))(t - \varpi) \), and thus the ideal \((t - \varpi)\) gives a contribution of \((N - p + 1)(p - 1)\) to \( \delta_y \). On the other hand, \( g \) has a pole of order \( p \) when expanded out in \( k((\varpi))((t)) \). Thus, by replacing \( g \) with \( g + x^p - x \) for some \( x \in k((\varpi), t) \) (which doesn’t change the Artin-Schreier extension), we may assume that \( g \) has a pole of order less than \( p \), and thus that \((t)\) contributes at most \( p(p - 1) \) to \( \delta_y \). So \( \delta_y \leq (N + 1)(p - 1) \). By Proposition 6.2 we in fact have equality, and thus \( A \cong k[[\varpi, z]] \).

Remark 6.19. Notice that the ramification jumps on the generic fiber are smaller than the ramification jumps of the original extension. In fact, based on the example above, it is an easy exercise to show that for a local \( \mathbb{Z}/p \)-extension, one can always find an equicharacteristic deformation such that the ramification jumps on the generic fiber are less than \( p \).

6.6.2. Lifting via equicharacteristic deformations. In order to apply the Mumford method, we need to show that being able to lift the generic fiber of an equicharacteristic deformation to characteristic zero allows us to do the same for the original local \( G \)-extension. First, we must say what we mean by “being able to lift the generic fiber.” Take \( k[[\varpi, z]][[\varpi^{-1}]]/k[[\varpi]][[\varpi^{-1}]] \) and tensor over \( k((\varpi)) \) with the algebraic closure \( \bar{k}((\varpi)) \). We obtain a \( G \)-extension of Dedekind \( \bar{k}((\varpi)) \)-algebras, and localizing at any branched maximal ideal gives a \( G \)-extension of \( \bar{k}((\varpi))[s] \) for some parameter \( s \) (for instance, one could have \( s = t \) or \( s = t - \varpi \)). This is a local \( G \)-extension (with the field \( \bar{k}((\varpi)) \) replacing \( k \)). We say that the generic fiber lifts to characteristic zero if all of the local \( G \)-extensions obtained this way lift to characteristic zero.

The following theorem says more or less that being able to lift the generic fiber of an equicharacteristic deformation implies being able to lift the original local \( G \)-extension to characteristic zero. The argument comes from Pop'14 and a conversation with Pop, but was only written in Pop'14 for \( G \) cyclic. The papers Obu15 and Obu16 use similar arguments, but do not directly cite Pop'14 since they deal with non-cyclic groups. Our statement here is intended to be clickable for general \( G \).

Theorem 6.20. Suppose that \( k[[z]]/k[[t]] \) is a local \( G \)-extension that admits an equicharacteristic deformation whose generic fiber lifts to characteristic zero after base change to the algebraic closure. Then \( k[[z]]/k[[t]] \) lifts to characteristic zero.

Proof. Let \( k[[\varpi, z]]/k[[\varpi, t]] \) be an equicharacteristic deformation of \( k[[z]]/k[[t]] \). Let \( Y \to W = \mathbb{P}^1_k \) be the HKG-cover associated to \( k[[z]]/k[[t]] \). Let \( \mathcal{W} = \mathbb{P}^1_{k[[\varpi]]} \) with coordinate \( t \). There is a \( G \)-cover of flat relative \( k[[\varpi]] \)-curves \( Y \to W = \mathbb{P}^1_{k[[\varpi]]} \) such that \( Y \times \mathcal{W} \text{Spec } k[[\varpi, t]] \to \text{Spec } k[[\varpi, t]] \) corresponds to \( k[[\varpi, z]]/k[[\varpi, t]] \) via Spec, and that this cover is unramified outside \( \text{Spec } k[[\varpi, t]] \) (this follows from Pop’s argument deducing Pop'14, Theorem 3.6 from Pop'14, Theorem 3.2). Write \( \overline{Y} \to \overline{W} \) for the base change of \( Y \to W \) to the integral closure of \( k[[\varpi]] \) in \( k((\varpi)) \), and let \( Y_\eta \to \mathcal{W}_\eta \) be the generic fiber of \( \overline{Y} \to \overline{W} \). Since the generic fiber of
We now show that \( Y \rightarrow W \) lifts over a characteristic zero discrete valuation ring. Since the \( G \)-cover \( Y_\mathcal{O} \rightarrow W_\mathcal{O} \) can be described using finitely many equations, it descends to a cover \( Y_A \rightarrow W_A \) over some subring \( A \subseteq \mathcal{O} \) that is finitely generated over \( W(k) \). Let \( m = A \cap m_\mathcal{O} \), where \( m_\mathcal{O} \) is the maximal ideal of \( \mathcal{O} \). Then \( A \) is a domain, and \( A/m \cong k \). Furthermore, the base change of \( Y_A \rightarrow W_A \) to \( A/m \) is the original cover \( Y \rightarrow W \). By Lemma [A.9], there is an ideal \( I \subseteq m \subseteq A \) such that \( A/I \) is a finite extension \( R \) of \( W(k) \). Base changing \( Y_A \rightarrow W_A \) to \( A/I \) gives a lift of \( Y \rightarrow W \) over \( R \). Applying the easy direction of the local-global principle, we obtain a lift of \( k[[z]]/k[[t]] \) over \( R \), which concludes the proof.

**Remark 6.21.** In practice, the affine space constraint in Theorem [6.20] does not cause trouble. For example, suppose \( \mathcal{F} \) is the family of local \( \mathbb{Z}/p \)-extensions of \( k[[t]] \) with ramification jump at most \( N \), for some \( N \) not divisible by \( p \). These extensions can be parameterized by polynomials \( f \) in \( t^{-1} \), where \( f \) has degree at most \( N \) and no terms of degree divisible by \( p \). These coefficients vary over an affine space, and the ramification jump is \( N \) on the complement of a hyperplane.

In general, if \( G \) is a \( p \)-group, one can use [Har80] Theorem 1.2 and Proposition 2.1] to show that affine spaces parameterize local \( G \)-extensions of \( k[[t]] \) whose higher ramification filtrations are subject to a given bound.

**Remark 6.22.** In the notation of the proof of Theorem [6.20] if one only cares about lifting the extension \( k[[z]]/k[[t]] \) to characteristic zero without any uniformity over \( \mathcal{F} \) of the discrete valuation ring \( R \), then one can invoke the completeness of the theory of characteristic zero algebraically closed valued fields with residue characteristic \( p \) ([Rob77, III]) to see that the \( \text{Frac}(\mathcal{O}) \) is an elementary extension of \( \text{Frac}(W(k)) \), which means that any first-order sentence in the language of characteristic zero algebraically closed valued fields with residue characteristic \( p \) that is true in \( \text{Frac}(\mathcal{O}) \) is also true in \( \text{Frac}(W(k)) \). This can be shown to include the sentence “\( k[[z]]/k[[t]] \) has a lift over the valuation ring.” So \( k[[z]]/k[[t]] \) has a lift over the algebraic closure of \( W(k) \), which means it has a lift over some finite extension of \( W(k) \). This is a lift over a discrete valuation ring.

6.6.3. **Consequences for specific groups.** For applications to the local lifting problem we need to know: How nice can we hope to make local \( G \)-extensions via equicharacteristic deformations? Here is the current state of knowledge. In all cases, assume \( k[[z]]/k[[t]] \) is a local \( G \)-extension.

**Proposition 6.23.** (i) If \( G = \mathbb{Z}/p^n \), then there is an equicharacteristic deformation of \( k[[z]]/k[[t]] \) whose generic fiber has no essential ramification (i.e., the upper jumps \( u_1, \ldots, u_m \) at any ramified point of the generic fiber satisfy \( u_{i+1} < pu_i + p \), see Remark [6.14(iv)]).

(ii) If \( G = \mathbb{Z}/p^n \times \mathbb{Z}/m \) is center-free, then there is an equicharacteristic deformation of \( k[[z]]/k[[t]] \) whose generic fiber has one branch point with inertia group
G and no essential ramification (in the sense of Remark 6.10 (iii)), and the rest of the branch points have cyclic inertia groups. Furthermore, the upper jumps of the \( \mathbb{Z}/p^n \)-subextension of the generic fiber at the point with inertia group G are congruent to the original upper jumps of this subextension modulo \( mp \).

(iii) Suppose \( G = A_4 \) and all \( \mathbb{Z}/2 \)-subextensions \( k[[v]]/k[[u]] \) of \( k[[z]]/k[[t]] \), with \( k[[v]] \neq k[[z]] \), have ramification jump \( \nu \geq 6 \). Then there is an equicharacteristic deformation of \( k[[z]]/k[[t]] \) whose generic fiber has one branch point with inertia group G and corresponding ramification jump \( \nu - 6 \), and all of the other branch points have inertia group \( \mathbb{Z}/2 \times \mathbb{Z}/2 \).

(iv) Suppose \( G = D_4 \), and let \( \nu \) be the maximal number such that \( |G^\nu| > 2 \) for \( k[[z]]/k[[t]] \) (we can think of \( \nu \) as the “second upper jump” if we count jumps with multiplicity in case the ramification filtration jumps from \( D_4 \) straight to \( \mathbb{Z}/2 \)). If \( \nu > 1 \), then there is an equicharacteristic deformation of \( k[[z]]/k[[t]] \) such that the branch points of the generic fiber with inertia group G have corresponding upper jump less than \( \nu \), and all of the other branch points have inertia group \( \mathbb{Z}/4 \) or \( \mathbb{Z}/2 \times \mathbb{Z}/2 \).

Proof. Parts (i), (ii), (iii), and (iv) are due to Pop (Proposition 6.23), Obus (Lemma 6.10 Proposition 3.1), Obus (Lemma 6.10 Proposition 3.2), and Weaver (Theorem 6.13), respectively.

The equicharacteristic deformations above have significant consequences for the local lifting problem.

**Theorem 6.24.**

(i) The Oort conjecture is true.

(ii) Theorem 6.15 holds even when the isolated differential data criterion is only satisfied for \((u_1', \ldots, u_n')\) where each \( u_i' \equiv u_i \pmod{mp} \), where \( u_1 < mp \), and where \( pu_i \leq u_{i+1} < pu_i + mp \) for \( i > 1 \).

(iii) \( A_4 \) is a local Oort group.

(iv) \( D_4 \) is a local Oort group.

Proof. To prove (i), we first reduce to the case \( G = \mathbb{Z}/p^n \) by [Obu12, Proposition 6.3]. By Proposition 6.23(i) and Theorem 6.20, it is enough to show that cyclic extensions with no essential ramification lift to characteristic zero. But this is Theorem 6.13.

Part (ii) follows from Proposition 6.23(ii), Theorem 6.20 and part (i) of this theorem.

To prove (iii), we proceed by induction on \( \nu \), where \( \nu \) is as in Propsoition 6.23(iii). The base cases \( \nu < 6 \) can be handled explicitly, see [Obu12] Propositions 5.1, 5.2. Then (iii) follows from Proposition 6.23(iii), Theorem 6.20 and the fact that \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) is a local Oort group for 2 ([Pag02]).

The proof of (iv) proceeds by induction on \( \nu \) as in (iii), but here the base case is \( \nu = 1 \) which is [Bre08, Theorem 4], we use Proposition 6.23(iv), and we also use that \( \mathbb{Z}/4 \) and \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) are local Oort groups, by [GM08] and [Pag02] respectively.

7. Approach using deformation theory

The previous sections of this chapter have been concerned with whether or not a Galois branched cover or a local \( G \)-extension over \( k \) lifts to characteristic zero.
In this section, we will take a more general approach and try to understand the deformation spaces of Galois branched covers and local $G$-extensions. Our first goal is to sketch a proof of a more refined version of the local-global principle, due to Bertin and Mézard. This reduces the global deformation problem to a local one. Then, we will give some results on local deformation spaces.

We assume familiarity with deformation theory à la Schlessinger ([Sch68]) throughout, and we will not generally cite specific results. Another good reference for basic deformation theory is [FGI*05] Chapter 6.

7.1. Setup. For our global deformation problem, we start with a smooth, projective, connected $k$-curve $Y$ acted upon by a finite group $G$ of automorphisms (when dealing with deformation theory, it will often be more convenient to think of things this way than in terms of branched covers). Let $\hat{C}$ be the category of complete local noetherian $W(k)$-algebras with residue field $k$, and let $\mathcal{C}$ be the full subcategory consisting of finite length $W(k)$-algebras. A deformation of $(Y,G)$ over $A$ is a relative smooth $A$-curve $Y_A$ with special fiber $Y$, such that $G$ acts on $Y_A$ by $A$-automorphisms and this action restricts to the original $G$-action on $Y$. We define a global deformation functor $D_{gl} : \mathcal{C} \to \text{Sets}$ such that $\mathcal{D}_{gl}(A)$ is the set of $G$-equivariant isomorphism classes of deformations of $(Y,G)$ over $A$. If $f : A \to B$ is a $W(k)$-algebra homomorphism, then $D_{gl}(f) : \mathcal{D}_{gl}(A) \to \mathcal{D}_{gl}(B)$ is given by base change.

Similarly, suppose we have an injection of a finite group $G$ into $\text{Aut}_k k[[z]]$ (we will call this a local $G$-action). For $A \in \hat{C}$, we define a deformation of $G \to \text{Aut}_k k[[z]]$ over $A$ to be a map $G \to \text{Aut}_A A[[Z]]$ such that the $G$-action on $A[[Z]]$ reduces to the given $G$-action on $k[[z]]$. We define a local deformation functor $D_{loc} : \mathcal{C} \to \text{Sets}$ such that $\mathcal{D}_{loc}(A)$ is the set of classes of morphisms $G \to \text{Aut}_A A[[Z]]$ lifting $G \to \text{Aut}_k k[[z]]$, considered up to conjugation by elements of $\text{Aut}_A A[[Z]]$ reducing to the identity on $k[[z]]$.

Let $D : \mathcal{C} \to \text{Sets}$ be a functor such that $D(k)$ has one element. Recall that a miniversal deformation ring for $D$ is a ring $R \in \hat{C}$ such that there is a smooth (in particular, surjective) natural transformation $\xi : \text{Hom}_W(k)(R,\cdot) \to D$ of functors on $\hat{C}$ inducing an isomorphism $\text{Hom}_W(k)(R,k[[\epsilon]]/\epsilon^2) \cong D(k[[\epsilon]]/\epsilon^2)$. Note that most works we cite below call this a versal deformation ring, but we will use the term miniversal so as not to conflict with usages of “versal” in the literature that mean only that $\text{Hom}_W(k)(R,\cdot) \to D$ is smooth. If $D$ is $D_{gl}$ or $D_{loc}$ above, we will simply say that $R$ is a miniversal deformation ring for $(Y,G)$ or $G \to \text{Aut}_k k[[z]]$, respectively. The ring $R$ is a universal deformation ring if the natural transformation $\xi$ is an isomorphism, and in this case the functor $D$ extends to $\hat{C}$ and is isomorphic to the extension of $\text{Hom}_W(k)(R,\cdot)$ to $\hat{C}$. We say that $D$ is pro-representable, and that the element of $D(R)$ corresponding to the identity morphism in $\text{Hom}_W(k)(R,R)$ is the universal deformation. Miniversal and universal deformation rings are unique up to isomorphism when they exist. It is not hard to show, using Schlessinger’s criteria, that both $D_{gl}$ and $D_{loc}$ have miniversal deformation rings. Furthermore, if $g(Y) \geq 2$, then $D_{gl}$ has a universal deformation ring, as deformations of $(Y,G)$ have finite automorphism groups, and thus no infinitesimal automorphisms.

The tangent space to a functor $D : \mathcal{C} \to \text{Sets}$ is defined to be $D(k[[\epsilon]]/\epsilon^2)$. 
7.2. The local-global principle via deformation theory. It is clear that a global deformation gives rise to local deformations. More specifically, let $Y$ be a smooth, projective, connected $k$-curve with an action of a finite group $G$, and let $(Y_A, G)$ be a deformation of $Y$ over $A \in \mathcal{C}$. Let $\{x_1, \ldots, x_n\}$ be the set of branch points of $Y \to Y/G$, and for each $i$, choose an inertia group $G_i \subseteq G$ of a ramification point $y_i$ above $x_i$ (it does not matter which point is chosen). As we have seen at the beginning of §3 the deformation $(Y_A, G)$ gives rise to deformations of $G_i \to \text{Aut}_k \hat{O}_{Y,y_i}$ over $A$. In particular, if $D_{gl}$ is the deformation functor associated to $(Y, G)$ and $D_{loc,i}$ is the deformation functor associated to $G_i \to \text{Aut}_k \hat{O}_{Y,y_i}$, we have natural transformations

$$D_{gl} \to D_{loc,i}$$

for all $i$. If $R_{gl}$ and $R_i$ are the respective miniversal deformation rings, we obtain morphisms $R_i \to R_{gl}$.

If $D_{loc}$ is the direct product of the deformation functors $D_{loc,i}$, then by general categorical principles, the miniversal deformation ring of $D_{loc}$ is $R_{loc} := \bigotimes R_i$. The natural transformation of functors $D_{gl} \to D_{loc}$ thus gives rise to a ring homomorphism

$$\phi : R_{loc} \to R_{gl}.$$ 

The statement of the deformation theoretic local-global principle is as follows.

**Theorem 7.1 ([BM00 Corollaire 3.3.5]).** We have $R_{gl} \cong R_{loc}[[U_1, \ldots, U_M]]$ for some $M$, and the map $R_{loc} \to R_{gl}$ is the natural inclusion. In other words, the map of deformation functors $D_{gl} \to D_{loc}$ is smooth.

The following corollary generalizes the local-global principle of Theorem 3.1 to complete noetherian local rings when the genus of the curve is at least 2.

**Corollary 7.2.** Let $Y$ be a smooth projective $k$-curve with an action of a finite group $G$. Let $A$ be a complete local noetherian ring with residue field $k$. Let $y_1, \ldots, y_r$ be the ramification points of $Y \to Y/G$ with inertia groups $G_1, \ldots, G_r$. Then $(Y, G)$ has a deformation over $A$ if and only if each local $G_i$-action on $\hat{O}_{Y,y_i}$ has a deformation over $A$.

**Proof.** The “only if” direction is obvious. So suppose that each local $G_i$-action has a deformation over $A$. Without loss of generality, suppose that the set $y_1, \ldots, y_s$, for some $s \leq r$, consists of one ramification point above each branch point of $Y \to Y/G$. By miniversality, we have a homomorphism from $\bigotimes_{i=1}^r R_i = R_{loc} \to A$, where $R_i$ is the miniversal deformation ring corresponding to the local $G_i$-action on $\hat{O}_{Y,y_i}$. By Theorem 7.1 this gives rise to a morphism $R_{gl} \to A$, by choosing the images of $U_1, \ldots, U_M$ in the maximal ideal of $A$ arbitrarily (these choices parameterize the different global deformations with given local behavior). This gives rise to a global deformation over $A$. \qed

7.2.1. Brief sketch of the proof of Theorem 7.1. Maintain the notation of §3. The key to the proof of Theorem 7.1 is to understand the tangent spaces and obstruction spaces to the relevant functors $D_{gl}$ and $D_{loc,i}$.

By [BM00 Théorème 2.2], the tangent space to $D_{loc,i}$ is isomorphic to $H^1(G_i, \Theta_i)$, where $\Theta_i$ is the $G_i$-module $k[[z_i]] \frac{dz_i}{z_i}$ of formal derivations at the ramification point $y_i$ (here $z_i$ is a local parameter of $Y$ at $y_i$). The obstruction space associated to $D_{loc,i}$ is $H^2(G_i, \Theta_i)$ ([BM00 Remarque 2.3]).
Describing the global counterparts requires equivariant cohomology. If $X$ is a finite-type $k$-scheme with $G$-action and $\mathcal{F}$ is a $(G, \mathcal{O}_X)$-module, then we can define the equivariant cohomology groups $H^q(G, \mathcal{F})$ to be the right derived functors of the left-exact functor $\Gamma(X, \mathcal{F})^G$. If $T_Y$ is the tangent sheaf of $Y$ with its natural $G$-action, then the tangent and obstruction spaces associated to $D_{gl}$ are $H^1(G, T_Y)$ and $H^2(G, T_Y)$, respectively (see [BM00] Propositions 3.2.1, 3.2.3) — see [FGI+05] Definition 6.1.21 for the definition of an obstruction space).

In fact, one can show that the map $\phi : D_{gl} \to D_{loc}$ of deformation functors induces a surjection $d\phi : H^1(G, T_Y) \to \bigoplus_i H^1(G_{x_i}, \Theta_i)$ on tangent spaces, whose kernel is the tangent space for the functor of locally trivial deformations (Definition 6.1.21). Furthermore, $\phi$ induces an isomorphism of obstruction spaces $H^2(G, T_Y) \to \bigoplus_i H^2(G_i, \Theta_i)$ (see [BM00] Lemma 3.3.1). A standard deformation theory argument now shows that $\phi$ is smooth.

By definition, $\text{Hom}_{W(k)}(R_{gl}, \cdot)$ and $\text{Hom}_{W(k)}(R_{loc}, \cdot)$ have the same tangent spaces as $D_{gl}$ and $D_{loc}$, respectively. The obstruction theories are the same as well, by [FGI+05] Lemma 6.3.3. So the same deformation theory argument shows that $\text{Hom}_{W(k)}(R_{gl}, \cdot)$ is smooth over $\text{Hom}_{W(k)}(R_{loc}, \cdot)$. Theorem 7.1 now follows from [Sch08] Proposition 2.5(ii).

**Remark 7.3.** Bertin and Mézard have proven a similar local-global principle for equivariant deformations of reduced curves that are locally complete intersections, but not necessarily smooth ([BM00, Théorème 4.3]).

### 7.3. Examples of local miniversal deformation rings

In view of Theorem 7.1 understanding global deformations is tantamount to calculating miniversal deformation rings for local $G$-actions. Since we will be dealing with one local action at a time in this section, we will use $R_{loc}$ to mean the miniversal deformation ring of a local $G$-action (this should not cause confusion with the usage of $R_{loc}$ in §7.2). If $G = \mathbb{Z}/m$ with $p \nmid m$, then it is an easy exercise to see that $R_{loc} = W(k)$.

The following theorem of Bertin and Mézard gives information on the miniversal deformation ring $R_{loc}$ for local $\mathbb{Z}/p$-actions.

**Theorem 7.4 ([BM00 Théorèmes 4.2.8, 4.3.7, 5.3.3]).** Let $k[[z]]/k[[t]]$ be a local $\mathbb{Z}/p$-extension with upper jump $N$, and let $R_{loc}$ be the miniversal deformation ring of the corresponding local $\mathbb{Z}/p$-action.

(i) If $p = 2$, then $R_{loc} = W(k)[[x_1, \ldots, x_{(N+1)/2}]]$.

(ii) If $N = 1$ and $p = 3$, then $R_{loc} = W(k)$.

(iii) If $N = 1$ and $p > 3$, then $R_{loc}$ is isomorphic to $W(k)[[X]]/\psi(X)$, where

$$
\psi(X) = \sum_{i=0}^{(p-1)/2} \binom{p-1-i}{i} (-1)^i (X + 4)^{(p-1)/2-i},
$$

and is not formally smooth.

(iv) If $N > 1$ and $p$ is odd, then there is a surjection

$$R_{loc} \to W(k)[[\zeta_p]][[x_1, \ldots, x_{(N+1)/p}]].$$

Furthermore, the Krull dimension of $R_{loc}$ equals $1 + \lfloor (N+1)/p \rfloor$.

**Remark 7.5.** Theorem 7.4(iv) shows that the formal spectrum of the miniversal deformation ring contains a smooth component of maximal dimension. Bertin and Mézard call this the “Oort-Sekiguchi-Suwa” component.
Remark 7.6. We see that when $p = 2$, the deformation space is formally smooth. This is somewhat surprising, considering that the obstruction space $H^2(\mathbb{Z}/2, \Theta)$ is nontrivial ([BM00 Proposition 4.1.1]). In any case, combining this result with Theorem [7.1] shows that the deformation space for a hyperelliptic curve in characteristic 2 is formally smooth ([BM00 Remarque 3.3.8]). This is originally a result of Laudal and Lønsted ([LL78]).

Example 7.7. Suppose $\text{char}(k) = 3$. Consider the local $\mathbb{Z}/3$-extension $k[[z]]/k[[t]]$ given by the integral closure of $k[[t]]$ in

$$k((t))[y]/(y^3 - y - t^{-1}),$$

with Galois action generated by $\sigma(y) = y + 1$ (which leads to $\sigma(z) = z/(z + 1)$, if we set the uniformizer $z$ equal to $y^{-1}$). As we have seen from Example 6.5, this can be lifted over $R := W(k)[\zeta_3]$ by taking the integral closure $A$ of $R[[T]]$ in

$$\text{Frac}(R[[T]])[W]/(W^3 - (1 + \lambda^3 T^{-1})),
$$

where $\lambda = \zeta_3 - 1$. Here, $Z := \lambda/(W - 1) \in A$ is a lift of $z$ and $A = R[[Z]]$ (Remark 7.6). Furthermore $\sigma(W) = \zeta_3 W$ and therefore $\sigma(Z) = Z/(Z + \zeta_3)$ (where we abuse notation and use $\sigma$ for the lift of $\sigma$ to $R[[Z]]$).

By Theorem 7.4(ii), however, this lift is definable over $W(k)!$ To see how to do this, let

$$V = \frac{\bar{\lambda} - \lambda W}{W - 1} \in R[[Z]],
$$

where $\bar{\lambda} \neq \lambda$ is Galois conjugate to $\lambda$. One calculates $V = \zeta_3 Z - \lambda$. If $v$ is the reduction of $V$, then $v = z$ and we have $R[[Z]] = R[[V]]$ by Remark A.6. A straightforward calculation yields

$$\sigma(V) = \frac{V - 3}{V - 2},$$

which is defined over $W(k)$ and reduces to the original action $\sigma(z) = z/(z + 1)$. So $R[[V]]/R[[T]]$ with this action is a lift of the original local $\mathbb{Z}/3$-extension that is conjugate to the original lift $R[[Z]]/R[[T]]$.

There is not a great deal known about miniversal deformation rings of local $G$-extensions when $G$ is not cyclic. One case where there has been some progress is the weakly ramified case (i.e., when the second lower ramification group $G_2$ is trivial). One reason such extensions are of interest is the following result of Nakajima ([Nak87, Theorem 2(i)]): Any local extension that arises from a Galois cover $X \to \mathbb{P}^1_k$, where $X$ is an ordinary curve (i.e., the $p$-rank of $\text{Jac}(X)$ is the genus of $X$) is weakly ramified. This in turn implies that $G \cong (\mathbb{Z}/p)^t \rtimes \mathbb{Z}/m$ for some $t$ and $p \nmid m$, by basic ramification theory ([A.3].

Theorem 7.8 ([CK03 Theorem 4.5]). Let $k[[z]]/k[[t]]$ be a weakly ramified local $G$-extension with miniversal deformation ring $R_{\text{loc}}$ and equicharacteristic miniversal deformation ring $S_{\text{loc}} := R_{\text{loc}}/p$. Suppose $G = (\mathbb{Z}/p)^t \rtimes \mathbb{Z}/m$, with $p \nmid m$. If $p \neq 2, 3$, then $S_{\text{loc}}$ has dimension $t/s - 1$, where $s = [\mathbb{F}_p(\zeta_m) : \mathbb{F}_p]$.

Remark 7.9. In fact, Cornelissen and Kato compute $S_{\text{loc}}$ explicitly ([CK03 §4.4]). The case where $G \cong \mathbb{Z}/p$ in Theorem 7.8 is covered by Theorem [7.3(i)-(iii)].

Remark 7.10. The corresponding HKG-cover to a weakly ramified local extension $k[[z]]/k[[t]]$ is genus zero (exercise!). This means that, after an appropriate change of
variables, the relevant automorphisms of $k[[z]]$ can be expressed as fractional linear transformations. This is heavily exploited in calculating deformation rings, and is one of the reasons the assumption of weak ramification makes these calculations more tractable.

**Remark 7.11.** Suppose $f : X \to \mathbb{P}^1_k$ is a weakly ramified Galois cover, where $g(X) \geq 2$ and char($k$) $> 3$. Let $n$ be the number of branch points of $f$, and $w$ the number of those that are wildly branched. Using the local-global principle (Theorem 7.1), Cornelissen and Kato show that the dimension of the equicharacteristic global miniversal deformation ring is

$$3g(X) - 3 + n + \sum_{i=1}^{w} \frac{t_i}{s(n_i)},$$

where the inertia group above point $i$ is $(\mathbb{Z}/p)^t \rtimes \mathbb{Z}/n_i$, and $s(n_i) = [\mathbb{F}_p(\zeta_{n_i}) : \mathbb{F}_p]$ ([CK03 Main Algebraic Theorem]). The $t_i/s(n_i)$ terms can be thought of as “correction” terms for the wild ramification (note that these terms vanish if the cover is tamely ramified and one recovers the classical characteristic zero result).

In [CK03], this is applied to Drinfeld modular curves, and comparison is made to an analytic deformation functor. Exploring this would take us somewhat far afield, so we direct the interested reader to the (very readable) paper [CK03].

Other papers dealing with equicharacteristic deformation of wildly ramified covers include [PZ12], [Pri02], [FM02] (non-Galois case!), and [CK05].

In addition to calculating the equicharacteristic miniversal deformation rings for weakly ramified local extensions, we can also calculate the general miniversal deformation ring (and thus, whether the extension lifts to characteristic zero).

**Theorem 7.12 ([CM06 Théorème 1.2]).** If $R_{\text{loc}}$ is the local miniversal deformation ring of a weakly (but wildly) ramified local $G$-extension, then the characteristic of $R_{\text{loc}}$ is either 0 or $p$. In particular, the characteristic of $R_{\text{loc}}$ is 0 when $G \in \{\mathbb{Z}/p, D_p^{-1}, A_4\}$, and is $p$ otherwise.

**Remark 7.13.** The exact formulas for the miniversal deformation rings above are given in [CM06 Corollaire 4.1].

**Example 7.14.** Consider the smooth projective curve $Y$ with affine equation

$$(y^p - y)(x^p - x) = 1$$

([CM06 p. 240]). The group $G = (\mathbb{Z}/p)^2 \rtimes D_{p^{-1}}$ acts on this as follows: the generators $\sigma$ and $\tau$ of the normal $(\mathbb{Z}/p)^2$ subgroup send $x$ to $x + 1$ and $y$ to $y + 1$, respectively, an element $\alpha$ of order 2 in a copy $H$ of $D_{p^{-1}}$ inside $G$ exchanges $x$ and $y$, and a generator $\beta$ of the order $p - 1$ cyclic normal subgroup of $H$ sends $y$ to $cy$ and $x$ to $c^{-1}x$, for some $c$ generating $\mathbb{F}_p^\times$.

Viewing $Y$ as a $\mathbb{Z}/p$-cover of the projective $x$-line, we see that it is ramified above the $p$ poles of $1/(x^p - x)$, and the ramification is weak in each case (it is unramified at $\infty$, as $1/(x^p - x) = 0$ at $x = \infty$). In fact, the inertia group $I$ inside $G$ at the ramification point above $x = 0$ is generated by $\tau$ and $\beta$, and is thus isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/(p - 1)$. Since the lower numbering respects subgroups by definition, the local action of $I$ at this point is weakly ramified. This, combined with the local-global principle, gives a KGB obstruction to lifting the $G$-cover $Y \to Y/G$ to characteristic zero when $p > 3$ ([Obu12 Proposition 5.9]). By Theorem 7.12, $Y \to Y/G$ cannot even lift to characteristic $p^n$ for any $n > 1$. 


Remark 7.15. In the example above, if \( p \geq 41 \), then one can also use the Hurwitz bound to show that lifting is impossible.

Next, we give a result on whether miniversal deformation rings are in fact universal. Unsurprisingly, the strongest results are in the weakly ramified case.

Proposition 7.16 ([BC09, Theorems on p. 879], [BCK12]). Let \( k[[z]]/k[[t]] \) be a local \( G \)-extension with miniversal deformation ring \( R_{\text{loc}} \) and equicharacteristic miniversal deformation ring \( S_{\text{loc}} = R_{\text{loc}}/p \).

(i) If \( k[[z]]/k[[t]] \) is weakly ramified, then \( R_{\text{loc}} \) is not universal if and only if \( \text{char}(k) = 2 \) and \( G \) is \( \mathbb{Z}/2 \) or \( (\mathbb{Z}/2)^2 \).

(ii) If \( k[[z]]/k[[t]] \) is weakly ramified, then \( S_{\text{loc}} \) is not universal for equicharacteristic deformations if and only if \( \text{char}(k) = 2 \) and \( G \cong \mathbb{Z}/2 \).

(iii) If \( G \cong \mathbb{Z}/5 \) and \( k[[z]]/k[[t]] \) has ramification jump 2 (so necessarily \( \text{char}(k) = 5 \)), then \( R_{\text{loc}} \) is universal.

Lastly, we mention a potential application of understanding miniversal and universal deformation rings, due to Byszewski. Suppose \( k[[z]]/k[[t]] \) is a local \( G \)-extension, and \( H \) is a normal subgroup of \( G \) with corresponding subextension \( k[[z]]/k[[s]] \). One would like to be able to “stack” a lift of the \( H \)-extension \( k[[z]]/k[[s]] \) above a lift of the \( G/H \)-extension \( k[[s]]/k[[t]] \) to form a lift of \( k[[z]]/k[[t]] \). Of course, this is in general difficult (otherwise the Oort conjecture would follow from the \( \mathbb{Z}/p \) case). However, we have the following result.

Theorem 7.17 ([Bys11, Definition 2.1, Theorem 2.8]). In the context above, let \( R_G \) and \( R_H \) be the respective miniversal deformation rings for the corresponding local extensions, with \( D_G \) and \( D_H \) the respective deformation functors. There is a natural \( G/H \)-action on \( D_H \), and a natural transformation of functors from \( D_G \) to \( D_{G/H} \) coming from restriction. If \( p \nmid |G/H| \) and \( R_H \) is universal, then this natural transformation of functors is an isomorphism, and \( R_G \) is universal. In particular, \( R_G \cong (R_H/I_H) \), where \( I_H \) is the ideal generated by all \( gx - x \) for \( g \in G \) and \( x \in R_H \) (universality of \( R_H \) gives a natural \( G/H \)-action on \( R_H \)).

In the weakly ramified case, Theorem 7.17 gives an alternate way of recovering some of the miniversal deformation rings from Remark 7.13 ([Bys11 Proposition 2.10]).

8. Open Problems

Some open problems are collected below. Anything called a “conjecture” is something that I strongly believe to be true. If I am less confident, I will use the word “question.” Some of the questions are open-ended. Unless otherwise mentioned, \( k \) is an algebraically closed field of characteristic \( p \).

8.1. Existence of local lifts. Recall the the list of local Oort groups is (at most) the cyclic groups, the dihedral groups \( D_{pn} \), and \( A_4 \) for \( p = 2 \). The most basic question about the local lifting problem is

Question 8.1. Is each group above a local Oort group?

As we have seen in [X] this question is only open for \( D_{pn} \) where \( n > 1 \) (and is known for \( D_4 \) and \( D_5 \)). It has been referred to as the “strong Oort conjecture” ([CGH05 Conjecture 1.2]).
For groups that have obstructions to being local Oort (for instance, the KGB obstruction), we can still ask if the KGB obstruction is the only thing that goes wrong. The following conjecture of mine generalizes Theorem 6.15.

**Conjecture 8.2.** The KGB obstruction is the only obstruction to the local lifting problem for local $G$-extensions where $G$ has cyclic $p$-Sylow subgroup. In particular, $D_{p^n}$ is a local Oort group when $p$ is odd.

Recall that the local KGB obstruction vanishes exactly when all the upper jumps $u_1, \ldots, u_n$ of the $\mathbb{Z}/p^n$-subextension of the local $G$-extension $k[[z]]/k[[t]]$ are congruent to $-1$ (mod $m$). By Theorem 6.24(ii), one way of proving Conjecture 8.2 is by showing that the isolated differential data criterion holds for all such $(u_1, \ldots, u_n)$ where $u_1 \leq m_p$ and $pu_i \leq u_{i+1} \leq pu_i + mp$. Since the criterion is about solving equidetermined systems of equations over an algebraically closed field (Remark 6.16), its truth is plausible. Furthermore, Conjecture 8.2 is true whenever the $p$-Sylow subgroup of $G$ is $\mathbb{Z}/p$, by Theorem 6.11.

The following question is much more speculative.

**Question 8.3.** Is every $p$-group a weak local Oort group?

Any $p$-group $G$ is a so-called Green-Matignon group ([CGH11, Definition 1.7]), which means, by [CGH11, Theorem 1.8], that there is at least one local $G$-extension whose Bertin obstruction vanishes ([Ber98] — this is a weaker obstruction than the KGB obstruction). However, the set of $p$-groups is too vast and our evidence in §5 too sparse to venture an opinion on the truth of Question 8.3. Indeed, it is not even known for $Q_8$! On the other hand, with the proof of Brewis and Wewers that $D_4$ is weak local Oort, one can be more confident in the following conjecture.

**Conjecture 8.4.** The group $D_{2^n}$ is a weak local Oort group for 2.

Lastly we mention the following question (see §4.2.3).

**Question 8.5.** For what groups $G$ can one give an explicit definition of the differential Hurwitz tree obstruction?

This should not be extremely difficult when the $p$-Sylow subgroup of $G$ is abelian, and Example 4.18 provides some guidance. It may be trickier to obtain a clean expression for the differential Hurwitz tree obstruction for arbitrary groups $G$.

### 8.2. Rings of definition.

The proofs that $\mathbb{Z}/p$ and $\mathbb{Z}/p^2$ are local Oort groups give explicit formulas for the lifts, and in the case of $\mathbb{Z}/p$ (resp. $\mathbb{Z}/p^2$), it is shown that lifting is possible over $W(k)[\zeta_p]$ (resp. $W(k)[\zeta_{p^2}]$) by [SOS89] (resp. [GM98]), where $k$ is the field of the local extension. These explicit formulas come from the Sekiguchi-Suwa theory ([Sek93]).

**Question 8.6.** Can all local $\mathbb{Z}/p^n$-extensions over $k$ be lifted over $W(k)[\zeta_{p^n}]$?

Sekiguchi-Suwa theory fundamentally takes place over $W(k)[\zeta_{p^n}]$, so if one could somehow lift all $\mathbb{Z}/p^n$-extensions using Sekiguchi-Suwa theory, then one would get a positive answer to Question 8.6. Alternatively, one could try to be more careful about the coefficients that show up in the current proof ([Sek93]), but this seems quite hard. We mention that a “weak” version of Question 8.6 has a positive answer. Namely, for any $n$, there exists a local $\mathbb{Z}/p^n$-extension over $k$ that can be lifted over $W(k)[\zeta_{p^n}]$ ([Gre04, Theorem 4.2], and implicit in [GM99, p. 280]).
8.3. Moduli/deformations/geometry of local lifts. We begin with a question of Gunther Cornelissen [CO05].

**Question 8.7.** Is there a local $G$-extension over $k$ that lifts to a ring of characteristic $p^n$ for $n > 1$, but does not lift to characteristic zero? That is, can the miniversal deformation ring of a local $G$-extension have characteristic other than 0 or $p$?

As we have seen in Theorem 7.12, this is not possible for a weakly ramified local $G$-extension.

We can also ask about obstructions to lifting to non-prime characteristic.

**Question 8.8.** Is it possible to write down any general obstruction to lifting a local $G$-extension to characteristic $p^n$ (for some $n > 1$), in the spirit of the KGB or Hurwitz tree obstructions?

Of course, if the answer to Question 8.7 is negative, then obstructions to lifting to characteristic zero work equally well as obstructions to lifting to characteristic $p^n$ for $n > 1$.

In another direction, we have seen in Example 7.7 that the relationship between Artin-Schreier equations, Kummer equations, and deformation rings of local $\mathbb{Z}/p$-actions is not straightforward (although Bertin and Mézard’s proof of Theorem 7.4(iv) is based on a partial understanding). Having a Kummer equation for a lift of a $\mathbb{Z}/p$-extension allows one to write down its Hurwitz tree and the geometry of its branch locus.

**Question 8.9.** In the case where $G = \mathbb{Z}/p$, can one give an explicit link between deformation parameters and Hurwitz trees of lifts? To what extent can Hurwitz trees be used to distinguish isomorphism classes of lifts? Can Hurwitz trees be used to identify whether or not a deformation lies in the Oort-Sekiguchi-Suwa component (Remark 7.6)?

An answer to the following question would reduce the inexplicitness of the Mumford method.

**Question 8.10.** Suppose a local $G$-extension is shown to lift to characteristic zero using the Mumford method via a specific equicharacteristic deformation. Can one say anything about the geometry of the branch locus of the resulting lift? In particular, can one say anything about the resulting Hurwitz tree?

It would be natural to attack Question 8.10 by first assuming $G \cong \mathbb{Z}/p$, where the structure of Hurwitz trees is well-understood.

Saidi has called the following conjecture the “Oort conjecture revisited” [Saï12].

**Conjecture 8.11.** Local cyclic extensions are liftable in towers. That is, given a local $G$-extension with $k[[z]]/k[[t]]$ with $G$ cyclic and a lift $R[[S]]/R[[T]]$ of a subextension $k[[s]]/k[[t]]$ to characteristic zero, there is a lift of $k[[z]]/k[[t]]$ to characteristic zero containing $R[[S]]/R[[T]]$ as a subextension.

The proof of the Oort conjecture in [OW14] already proceeds by induction, and one could in theory prove Conjecture 8.11 by making the induction process more flexible. In particular, the induction argument of [OW14] only works if one can lift a subextension so that the branch points all have high enough valuation, see [OW14 Theorem 3.4(i)]. If one could remove the valuation restriction, Conjecture...
would follow (and one would additionally get a proof of the Oort conjecture without using the Mumford method). Removing this restriction directly seems more promising than trying to use deformation theory in towers as in [Byss11] (see discussion before Theorem 7.12). In any case, though, it would be interesting to understand the deformation theory of local \(\mathbb{Z}/p^n\)-actions for \(n > 1\).

### 8.4. Non-algebraically closed residue fields.

Throughout this entire paper, we have considered the local lifting problem over an algebraically closed field of characteristic \(p\). There is no reason that one cannot consider local \(G\)-extensions \(k[[z]]/k[[t]]\) where \(k\) is an arbitrary field of characteristic \(p\), and try to lift them to a characteristic zero mixed characteristic local domain with residue field \(k\). The following is a question of Oort.

**Question 8.12.** Does there exist a characteristic \(p\) field \(k\) and a cyclic branched cover of curves defined over \(k\) that does not lift over a characteristic zero local normal domain \(R\) with residue field \(k\)? What if \(k\) is perfect, or finite? What if we relax the assumption that \(R\) is normal?

One can also ask the above question in the local context, i.e., whether there is a local cyclic extension over a characteristic \(p\) field that does not lift over any characteristic zero local normal domain. It is not clear to me whether this question is necessarily equivalent to Question 8.12 (i.e., whether there is a local-global principle in this context).

### Appendix A. Some algebraic preliminaries

#### A.1. Homological Algebra.

**Lemma A.1** (cf. [Hen02, §3, Lemma 1.1]). Let \(R\) be a complete local noetherian ring with residue field \(k\) and maximal ideal \(m\). Suppose \(0 \to M_1 \to M_2 \to M_3 \to 0\) is a complex of \(m\)-adically separated \(R\)-modules, with \(M_1\) and \(M_2\) complete, and \(M_2\) and \(M_3\) flat over \(R\). For \(i = 1, 2, 3\), let \(\overline{M}_i = M_i/mM_i\). If \(0 \to \overline{M}_1 \to \overline{M}_2 \to \overline{M}_3 \to 0\) is exact, then so is \(0 \to M_1 \to M_2 \to M_3 \to 0\).

**Proof.** To prove exactness on the right, take \(\beta_0 \in M_1\). There exists \(\alpha_0 \in M_2\) and \(\beta_1 \in M_3\) such that \(v(\alpha_0) = \beta_0 - m_1\beta_1\), with \(m_1 \in m\). Similarly, define \(\alpha_n \in M_2\) and \(\beta_{n+1} \in M_3\) so that \(v(\alpha_n) = \beta_n - m_{n+1}\beta_{n+1}\) for some \(m_{n+1} \in m\). Letting \(\alpha = \sum_{n=0}^{\infty} m_1 \cdots m_n \alpha_n\), the separatedness of \(M_3\) yields that \(v(\alpha) = \beta\).

To prove exactness on the left, let \(N = \ker(u)\). It is a closed submodule of \(M_1\), thus separated. Since \(0 \to N \to M_1 \to u(M_1) \to 0\) is exact and \(u(M_1)\), being torsion-free, is flat, we have that \(0 \to \overline{N} \to \overline{M}_1 \to u(\overline{M}_1) \to 0\) is exact. So \(\overline{N} = 0\), and \(N = mN\). Since \(N\) is separated, we have \(N = \bigcap_{n=1}^{\infty} m^n N = 0\).

Exactness in the middle follows from surjectivity of \(\overline{M}_1 \to \ker(v)\) and the result on right-exactness above.

**Remark A.2.** Taking \(M_1 = 0\) in Lemma A.1 one sees immediately that if \(v : M_2 \to M_3\) is a morphism of \(m\)-adically separated flat \(R\)-modules, with \(M_2\) complete, then \(v\) is surjective (resp. an isomorphism) if and only if its reduction modulo \(\pi\) is.

**Proposition A.3.** Let \(X\) be a regular, connected scheme of dimension \(\leq 2\) with function field \(K(X)\), let \(L/K(X)\) be a finite extension, and let \(f : Y \to X\) be the normalization of \(X\) in \(L\). Then \(f\) is flat.
The proof above shows that if $R_0$ and uniformizer $p$ are satisfied and the map $\phi$ is an isomorphism modulo the maximal ideal of $R$. We conclude that $\phi$ is an isomorphism.

Remark A.6. The proof above shows that if $T'$ is any element of $R[[T]]$ such that $T'$ reduces to a uniformizer of $k[[t]]$, then $R[[T']] = R[[T]]$.

A.3. Ramification theory. The following facts are from [Ser79, IV]. Recall that if $k$ is a field of characteristic $p$, one can form the ring $W(k)$ of Witt vectors over $k$. If $k$ is perfect, this is the unique complete characteristic zero discrete valuation ring with residue field $k$. Let $F$ be a complete DVR with algebraically closed residue field of characteristic $p > 0$ and uniformizer $\pi$. If $L/F$ is a finite $G$-Galois extension, then $G$ is of the form $P \rtimes \mathbb{Z}/m$, where $P$ is a $p$-group and $m$ is prime to $p$ (if $p = 0$, then $P$ is trivial). In particular, $G$ is solvable. The group $G$ has a filtration $G = G_0 \supseteq G_i$ ($i \in \mathbb{R}_{\geq 0}$) defined by

$$g \in G_i \iff v\left(\frac{g(\pi)}{\pi} - 1\right) \geq i + 1$$

(here $v$ is defined so that $v(\pi) = 1$. There is also a filtration $G \supseteq G^i$ for the upper numbering $(i \in \mathbb{R}_{\geq 0})$ given by $G^i = G_{\psi(i)}$, where $\psi$ is the inverse of the Herbrand function $\varphi$, given by $\varphi(u) = \int_0^u dt/[G_0 : G_1]$. If $i \leq j$, then $G_i \supseteq G_j$ and $G^i \supseteq G^j$.

The subgroup $G_i$ (resp. $G^i$) is known as the $i$th higher ramification group for the lower numbering (resp. the upper numbering).

One knows that $G_0 = G^0 = G$, and that $G_1 = G^1 = P$ (in particular, if $p = 0$ then $G_1$ is trivial). For sufficiently large $i$, $G_i = G^i = \{id\}$. Any $i$ such that $G^i \supseteq G^{i+\epsilon}$ for all $\epsilon > 0$ is called an upper jump of the extension $L/F$. Likewise, if $G_i \supseteq G_{i+\epsilon}$ for $\epsilon > 0$, then $i$ is called a lower jump of $L/F$. If $i$ is a lower (resp. upper) jump, $i > 0$, and $\epsilon > 0$ is sufficiently small, then $G_{i}/G_{i+\epsilon}$ (resp. $G^i/G^{i+\epsilon}$) is an elementary abelian $p$-group. The lower jumps are clearly all integers. The Hasse-Arf theorem says that the upper jumps are integers whenever $G$ is abelian (in general, the upper jumps need only be rational). The extension $L/F$ is called tamely ramified if $G_1 = \{id\}$ (equivalently, $G \cong \mathbb{Z}/m$), and wildly ramified otherwise.
Example A.7. Suppose $L/F$ is a $G$-Galois extension as above, with residue field of characteristic $p > 0$. Let $M$ be the subextension corresponding to $P < G$. By the definition of the lower numbering, we have $P_i = G_i$ for $i > 0$. By the definition of the upper numbering, we have $P^i = G^{i/m}$. In particular, if $P$ is abelian then the upper jumps for $L/F$ lie in $\frac{1}{m}\mathbb{Z}$.

The degree $\delta$ of the different of a $G$-extension $L/F$ is given by the formula

$$(A.1) \quad \delta = \sum_{i=0}^{\infty} (|G_i| - 1).$$

Note that $\delta \geq |G| - 1$, with strict inequality holding if and only if $L/F$ is wildly ramified.

Example A.8. Suppose $k[[z]]/k[[t]]$ is a $\mathbb{Z}/p^n$-extension with upper jumps $u_1 \prec \cdots \prec u_n$. For $i > 1$, we have $u_i \geq pu_{i-1}$, with $p \nmid u_i$ whenever strict inequality holds (see, e.g., [Gar02, Theorem 1.1]).

Now, let $f : Y \to X$ be a degree $d$ branched cover of curves over an algebraically closed field $k$. The cover is called tamely (resp. wildly) ramified at a point $y$ if the corresponding extension $\hat{O}_{Y,y}/\hat{O}_{X,f(y)}$ is. The Riemann-Hurwitz formula ([Har77, IV, §2]) states that

$$2g_Y - 2 = d(2g_X - 2) + \sum_{y \in Y} \delta_y,$$

where $\delta_y$ is the degree of the different of $\hat{O}_{Y,y}/\hat{O}_{X,f(y)}$, and $g_Y$ (resp. $g_X$) is the genus of $Y$ (resp. $X$). The Riemann-Hurwitz formula, combined with (A.1), shows that if $Y \to X$ is a wildly ramified cover of curves over $k$, then the genus of $Y$ is higher than it would be if the cover had the same ramification points and indices, but was in characteristic zero.

A.4. Miscellaneous. The following lemma is used in the proof of Theorem 6.20.

Lemma A.9. Let $k$ be an algebraically closed field of characteristic $p$, and let $A$ be a finitely generated $W(k)$-algebra that is a domain. If $\mathfrak{m} \subseteq A$ is an ideal such that $A/\mathfrak{m} \cong k$ as a $W(k)$-algebra, then there is an ideal $I \subseteq \mathfrak{m} \subseteq A$ such that $A/I$ is a finite extension of $W(k)$.

Proof. Embed Spec $A$ into $\mathbb{A}^n_{W(k)} \subseteq \mathbb{P}^n_{W(k)}$ for some $n$, and let $Z$ be the projective closure of Spec $A$ in $\mathbb{P}^n_{W(k)}$. Let $x \in \text{Spec } A \subseteq Z$ be the point corresponding to $\mathfrak{m}$. Since $A$ is a domain, $x$ is in the closure of the generic fiber of Spec $A$, and thus of $Z$. By [Mum99] II, §8, Theorem 1], $x$ is the specialization of some geometric point on the generic fiber of $Z$. Let $y \in Z$ be the image of this point. Since $x$ does not lie in the hyperplane at infinity, neither does $y$. So $y$ is in the generic fiber of Spec $A$. Since the closure $\overline{\{y\}}$ of $y$ is finite over Spec $W(k)$ and contains $x$, taking $I = I(\overline{\{y\}}) \subseteq A$ gives the desired ideal.

Acknowledgements

I thank Frans Oort for guidance and help in preparing this chapter, and Florian Pop for useful conversations.
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University of Virginia, 141 Cabell Drive, Charlottesville, VA 22904
E-mail address: andrewobus@gmail.com