GLOBAL REGULARITY FOR 2D WATER WAVES WITH SURFACE TENSION

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ABSTRACT. We consider the full irrotational water waves system with surface tension and no gravity in dimension two (the capillary waves system), and prove global regularity and modified scattering for suitably small and localized perturbations of a flat interface. An important point of our analysis is to develop a sufficiently robust method (the “quasilinear I-method”) which allows us to deal with strong singularities arising from time resonances in the applications of the normal form method (the so-called “division problem”). As a result, we are able to consider a suitable class of perturbations with finite energy, but no other momentum conditions.

Part of our analysis relies on a new treatment of the Dirichlet-Neumann operator in dimension two which is of independent interest. As a consequence, the results in this paper are self-contained.

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1. Introduction

1.1. Free boundary Euler equations and water waves. The evolution of an inviscid perfect fluid that occupies a domain $\Omega_t \subset \mathbb{R}^n$, for $n \geq 2$, at time $t \in \mathbb{R}$, is described by the free boundary incompressible Euler equations. If $v$ and $p$ denote respectively the velocity and the pressure of the fluid (with constant density equal to 1) at time $t$ and position $x \in \Omega_t$, these equations are:

$$
\begin{cases}
  v_t + v \cdot \nabla v = -\nabla p - ge_n & x \in \Omega_t \\
  \nabla \cdot v = 0 & x \in \Omega_t \\
  v(x, 0) = v_0(x) & x \in \Omega_0,
\end{cases}
$$

(1.1)
where \( g \) is the gravitational constant. The free surface \( S_t := \partial \Omega_t \) moves with the normal component of the velocity according to the kinematic boundary condition:

\[
\partial_t + v \cdot \nabla \text{ is tangent to } \bigcup_t S_t \subset \mathbb{R}^{n+1}.
\]

In the presence of surface tension the pressure on the interface is given by

\[
p(x,t) = \sigma \kappa(x,t), \quad x \in S_t,
\]

where \( \kappa \) is the mean-curvature of \( S_t \) and \( \sigma > 0 \). In the case of irrotational flows, i.e.

\[
curl v = 0,
\]

one can reduce (1.1)-(1.2) to a system on the boundary. Such a reduction can be performed identically regardless of the number of spatial dimensions, but here we only focus on the two dimensional case – which is the one we are interested in – and moreover assume that \( \Omega_t \subset \mathbb{R}^2 \) is the region below the graph of a function \( h : \mathbb{R}^x \times \mathbb{R}^t \to \mathbb{R} \), that is \( \Omega_t = \{(x,y) : y \leq h(x,t)\} \) and \( S_t = \{(x,y) : y = h(x,t)\} \).

Let us denote by \( \Phi \) the velocity potential: \( \nabla \Phi(x,y,t) = v(x,y,t) \), for \( (x,y) \in \Omega_t \). If \( \phi(x,t) := \Phi(x,h(x,t),t) \) is the restriction of \( \Phi \) to the boundary \( S_t \), the equations of motion reduce to the following system for the unknowns \( h, \phi \):

\[
\begin{align*}
\partial_t h &= G(h) \phi \\
\partial_t \phi &= -gh + \sigma \frac{\partial_x^2 h}{(1 + h_x^2)^{3/2}} - \frac{1}{2} \frac{\partial_x^2 h}{1 + h_x^2} + \frac{(G(h)\phi + h_x \phi_x)^2}{2(1 + |h_x|^2)}
\end{align*}
\]

with

\[
G(h) := \sqrt{1 + |h_x|^2} \mathcal{N}(h)
\]

where \( \mathcal{N}(h) \) is the Dirichlet-Neumann map associated to the domain \( \Omega_t \). We refer to [46, chap. 11] or [19] for the derivation of the water waves equations (1.4). This system describes the evolution of an incompressible perfect fluid of infinite depth and infinite extent, with a free moving (one-dimensional) surface, and a pressure boundary condition given by the Young-Laplace equation. One generally refers to (1.4) as the gravity water waves system when \( g > 0 \) and \( \sigma = 0 \), and as the capillary water waves system when \( g = 0 \) and \( \sigma > 0 \).

The system (1.1)-(1.2) has been under very active investigation in recent years. Without trying to be exhaustive, we mention the early works on the wellposedness of the Cauchy problem in the irrotational case and with gravity by Nalimov [40], Yoshihara [49], and Craig [17]; the first works on the wellposedness for general data in Sobolev spaces (for irrotational gravity waves) by Wu [50, 51]; and subsequent works on the gravity problem by Christodoulou-Lindblad [12], Lannes [37], Lindblad [39], Coutand-Shkoller [15], Shatah-Zeng [44, 45], and Alazard-Burq-Zuily [2, 3]. Surface tension effects have been considered in the works of Beyer-Gunther [8], Ambrose-Masmoudi [7], Coutand-Shkoller [15], Shatah-Zeng [44, 45], Christianson-Hur-Staffilani [11], and Alazard-Burq-Zuily [1]. Recently, some blow-up scenarios have also been investigated [10, 16, 30].

The question of long time regularity of solutions with irrotational, small and localized initial data was also addressed in a few works, starting with [52], where Wu showed almost global existence for the gravity problem (\( g > 0 \), \( \sigma = 0 \)) in two dimensions (1d interfaces). Subsequently, Germain-Masmoudi-Shatah [23] and Wu [53] proved global existence of gravity waves in three dimensions (2d interfaces). Global regularity in 3d was also proven in the case of surface tension and no gravity (\( g = 0 \), \( \sigma > 0 \)) by Germain-Masmoudi-Shatah [24].
Global regularity for the gravity water waves system in dimension 2 (the harder case) has been proved by the authors in [32] \(^2\) Independently, a similar result was proved by Alazard-Delort [4]. More recently, a different proof of Wu’s 2d almost global existence result was given by Hunter-Ifrim-Tataru [27], and later complemented to a proof of global regularity in [28].

After the results of this paper were announced, Ifrim-Tataru [29] observed that if one assumes strong enough momentum conditions at low frequencies, then the capillary waves system (1.6) is similar to the gravity water waves system, and obtained independently a global existence result in this simpler case.

As explained below (see Remark 1.3 and subsection 1.5) the gravity and capillary systems are significantly different in the case of general data without momentum conditions, due to different types of singularities generated by the normal form method. Some of the main difficulties in this problem have been addressed in our previous work [34] for a model equation.

1.2. The main results. Our results in this paper concern the capillary water waves system

\[
\begin{aligned}
\partial_t h &= G(h)\phi, \\
\partial_t \phi &= \frac{\partial^2 h}{(1 + h_x^2)^{3/2}} - \frac{1}{2}|\phi_x|^2 + \frac{(G(h)\phi + h_x\phi_x)^2}{2(1 + |h_x|^2)}.
\end{aligned}
\]

This is the system (1.4) when gravity effects are neglected \((g = 0)\) and the surface tension coefficient \(\sigma\) is, without loss of generality, taken to be 1. The system admits the conserved Hamiltonian

\[
\mathcal{H}(h, \phi) := \frac{1}{2} \int_{\mathbb{R}} G(h)\phi \cdot \phi \, dx + \int_{\mathbb{R}} \frac{(\partial_x h)^2}{1 + \sqrt{1 + h_x^2}} \, dx \approx \left\| \partial_x \right\|^{1/2}_{L^2} + \left\| \partial_x h \right\|^2_{L^2}.
\]

To describe our results we first introduce some basic notation. Let

\[
C_0 := \{ f : \mathbb{R} \rightarrow \mathbb{C}, f \text{ continuous and } \lim_{|x| \rightarrow \infty} |f(x)| = 0 \}, \quad \| f \|_{C_0} := \| f \|_{L^\infty}.
\]

For any \(N \geq 0\) let \(H^N\) denote the standard Sobolev space of index \(N\). More generally, if \(N \geq 0, b \in [-1, N]\), and \(f \in C_0\) then we define

\[
\begin{aligned}
\| f \|_{\dot{H}^N,b} &:= \left\{ \sum_{k \in \mathbb{Z}} \| P_k f \|_{L^2}^2 \left( 2^{2Nk} + 2^{2kb} \right) \right\}^{1/2} \approx \left\| (|\partial_x|^N + |\partial_x|^b) f \right\|_{L^2}, \\
\| f \|_{\dot{W}^N,b} &:= \sum_{k \in \mathbb{Z}} \| P_k f \|_{L^\infty} \left( 2^{Nk} + 2^{bk} \right),
\end{aligned}
\]

where \(P_k\) denote standard Littlewood-Paley projection operators (see subsection 2.1 for precise definitions). Notice that the norms \(\dot{H}^N,b\) define natural spaces of distributions for \(b < 1/2\) in dimension 1, but not for \(b \geq 1/2\). This is the reason for the assumption \(f \in C_0\) in the definition. Our main result is the following:

**Theorem 1.1** (Global Regularity). Let

\[
N_0 = N_t := 9, \quad N_1 = N_S := 3, \quad N_2 = N_\infty := 4.5, \quad 0 < p_1 \leq 10^{-4} p_0 \leq 10^{-10}.
\]

Assume that \((h_0, \phi_0) \in (C_0 \cap \dot{H}^{N_0+1,p_1+1/2}) \times \dot{H}^{N_0+1/2,p_1}\) satisfies

\[
\| h_0 \|_{\dot{H}^{N_0+1,p_1+1/2}} + \| \phi_0 \|_{\dot{H}^{N_0+1/2,p_1}} + \| (\partial_x h_0) \|_{\dot{H}^{N_1+1,p_1+1/2}} + \| (\partial_x \phi_0) \|_{\dot{H}^{N_1+1/2,p_1}} = \varepsilon_0 \leq \varepsilon_0,
\]

\(\varepsilon_0\) We refer the reader to our earlier paper [31] for the analysis of a simplified model (a fractional cubic Schrödinger equation), and to [33] for an alternative description of the asymptotic behavior of the solutions constructed in [32].
where $\varepsilon_0$ is a sufficiently small constant. Then, there is a unique global solution 

$$(h, \phi) \in C([0, \infty) : (C_0 \cap \dot{H}^{N_0+1/2, p_1+1/2}) \times \dot{H}^{N_0+1/2, p_1})$$

of the system (1.6), with $(h(0), \phi(0)) = (h_0, \phi_0)$. In addition, with $S := (3/2)t\partial_t + x\partial_x$, we have 

$$
(1 + t)^{-p_0} \|f(t)\|_{\dot{H}^{N_0+1/2, p_1}} + (1 + t)^{-4p_0} \|Sf(t)\|_{\dot{H}^{N_1+1/2, p_1}} 
+ (1 + t)^{1/2} \|f(t)\|_{W^{N_2+1/2, 2, 5} \lesssim \varepsilon_0,}
$$

for any $t \in [0, \infty)$, where $f \in \{\partial_x^{-1/2} h, \phi\}$. Furthermore, the solution possesses modified scattering behavior as $t \to \infty$.

Remark 1.2 (Modified scattering). A precise statement of modified scattering can be found in Lemma 8.6, where we give the asymptotic behavior of a suitably modified profile associated to our solution in the variables $(\xi, t)$, where $\xi$ is the Fourier space variable dual to $x$. See (8.60) in Lemma 8.6 together with (8.57), (8.22), Lemma 8.2 and (3.17) with (3.1).

Thanks to the bounds in Lemmas 8.2 and 8.6 and the refined linear estimate (2.20), we can also obtain, via standard arguments, asymptotics in the physical space variables $(x, t)$. In particular, we can show that there exists a unique asymptotic profile $f_\infty$, with $\|((\xi)^{-1/10} + |\xi|^4)f_\infty\|_{L^\infty} \lesssim \varepsilon_0$, such that for all $t \geq 1$

$$
\left|(|\partial_x|h - i|\partial_x|^{1/2}\phi)(t, x) - \frac{e^{-it(4/27)|x/t|^3}}{\sqrt{1 + t}} f_\infty(x/t) e^{-id_0(x/t)^3 |f_\infty(x/t)|^2 \log(1+|t|)} \right| \lesssim \varepsilon_0 (1 + t)^{-1/2 - a}.
$$

for $d_0 = 1/54$, and some $0 < a < 1/20$.

Remark 1.3 (Low frequencies and momentum conditions). The proof of the main theorem becomes substantially easier if one makes the stronger low-frequency assumption

$$
\|h_0\|_{\dot{H}^{N_0+1, 1/2-}} + \|(x\partial_x)h_0\|_{\dot{H}^{N_1+1, 1/2-}} + \|\phi_0\|_{\dot{H}^{N_0+1/2, 0-}} + \|(x\partial_x)\phi_0\|_{\dot{H}^{N_1+1/2, 0-}} \ll 1,
$$

which is the same condition as (1.11), but taking $p_1 < 0$. Such low frequency norms are propagated by the flow, due to a suitable null structure at low frequencies of the nonlinearity. However, the finiteness of the norm (1.14) requires unwanted momentum conditions on the natural Hamiltonian variables $(|\partial_x|h, |\partial_x|^{1/2}\phi)$, compare with (1.7).

The choice of norm in (1.11) accomplishes our main goals. On one hand it is strong enough to allow us to control the singular terms arising from the resonances of the normal form transformation and deal with the “division problem”, see (1.5) below. On the other hand, it is weak enough to avoid making a momentum assumption on the natural energy variable $|\partial_x|h$ and $|\partial_x|^{1/2}\phi$.

As a byproduct of our energy estimates in section 4, we also obtain the following:

---

3One can, for example, follow the same arguments given in the case of gravity water waves in [33].

4In order to control energy norms one can take $P = 1$ in (2.20). In particular, the multipliers in the low frequency energy functionals $K_{\Theta, I}, \Theta \in \{I, S\}$, defined in (2.20) do not depend on the variable $t$, and this simplifies significantly the analysis in sections 8 and 9. Most importantly, the vanishing of the denominators generated by the application of the natural form method is matched by the strong condition on the zero frequency of the solution, and the resulting operators are nonsingular. In other words, the “division problem” largely disappears. The dispersive analysis in sections 8 and 9 also becomes simpler if one allows stronger low-frequency assumptions. See also [29] for a different proof in this case in a different system of coordinates.

5Similar assumptions at low frequencies, designed to avoid momentum conditions on the energy variables, were used by Germain-Masmoudi-Shatah [24] in their work on the capillary system in three dimensions.
**Theorem 1.4** (Long-time existence in Sobolev spaces). Assume that 

\[(h_0, \phi_0) \in (C_0 \cap \dot{H}^{N_0+1,p_1+1/2}) \times \dot{H}^{N_0+1/2,p_1}\]

satisfies

\[\|h_0\|_{\dot{H}^{N_0+1,p_1+1/2}} + \|\phi_0\|_{\dot{H}^{N_0+1/2,p_1}} = \varepsilon_0 \leq 1. \quad (1.15)\]

Then there is a unique solution 

\[(h, \phi) \in C([0,T_{\varepsilon_0}]: (C_0 \cap \dot{H}^{N_0+1,p_1+1/2}) \times \dot{H}^{N_0+1/2,p_1})\]

of the system \((1.6)\), with \((h(0), \phi(0)) = (h_0, \phi_0)\) and \(T_{\varepsilon_0} \gtrsim \varepsilon_0^{-2}\). Moreover,

\[\|h(t)\|_{\dot{H}^{N_0+1,p_1+1/2}} + \|\phi(t)\|_{\dot{H}^{N_0+1/2,p_1}} \lesssim \varepsilon_0 \quad (1.16)\]

for any \(t \in [0,T_{\varepsilon_0}]\).

A similar result also holds in the case of periodic solutions. The result in Theorem 1.4 is the first of this type for the capillary water waves system with general initial conditions in the Schwartz class (in the sense explained in remark 1.3 above). Under more restrictive assumptions this follows from the analysis in [29]. For gravity water waves such a result was obtained in [6, 47] in 2d, and in [48] in 3d, as a key step to proving the modulation approximation in infinite depth. By analogy, Theorem 1.4 should be considered an important step towards the rigorous justification of approximate models and scaling limits for the water waves system in the presence of surface tension. We refer the reader to [18, 42], the book [38], and references therein, for works dealing with the long-time existence of non-localized solutions of the water waves system and their modulation and scaling regimes.

### 1.3. Main ideas of the proof

The system \((1.6)\) is a time reversible quasilinear system. In order to prove global regularity for solutions of the Cauchy problem for this type of equations, one needs to accomplish two main tasks:

1) Propagate control of high frequencies (high order Sobolev norms);
2) Prove pointwise decay of the solution over time.

In this paper we use a combination of improved energy estimates and asymptotic analysis to achieve these two goals. This is a natural continuation of our work on the gravity water waves system [32]. However, here we adopt a more robust framework for most of the arguments. Unlike our previous work [32] where we used both Lagrangian and Eulerian coordinates, here we perform our proof entirely in Eulerian coordinates.\footnote{Besides being the natural coordinates associated to the Hamiltonian formulation of the water waves problem, Eulerian coordinates are in general more flexible because the analysis can be generalized to higher dimensions.}

We carry out both main parts of our analysis in Fourier space. More precisely, we use a “quasilinear I-method”, originally introduced in our recent work [34] on a simplified model (see also [21] for a different application of the method in a 2d problem), to construct high order energy functionals which can be controlled for long times. In order to set up the equations for the energy estimates, we perform a careful paralinearization in the spirit of [1, 5], but better adapted for our purposes. An important part of the argument here relies on new bounds on the Dirichlet-Neumann operator in two dimensions, consistent with the limited low-frequency structure we assume on the interface \(h\), see \((1.11)\). To complete our proof we then use the Fourier transform method to obtain sharp time decay rates for our solutions, and prove modified scattering. We elaborate on all these main aspects of our proof below.

\(\varepsilon_0\) is also a corollary of the results in [32, 37].
1.4. Paralinearization and the Dirichlet-Neumann operator. One of the main difficulties
in the analysis of the water waves system (1.4) comes from the quasilinear nature of the equations,
and their non-locality, due to the presence of the Dirichlet-Neumann operator $G(h)\phi$, see (1.5).
In a series of papers [1] [2] [3] Alazard-Burq-Zuily proposed a systematic approach to these
issues in the context of the local-in-time Cauchy problem, based on para-differential calculus. See also
the earlier works of Alazard-Métivier [6] and Lannes [37]. This approach was then extended and
adapted to study the problem of global regularity for gravity waves by Alazard-Delort in [5].

Our first step towards the proof of Theorem 1.1 is a paralinearization of the system (1.6) in-
spired by the works cited above. For our problem, however, we need bounds that depend only on
$\|\partial_x|^{1/2}+p_1 h\|_{L^2}$, not on $\|h\|_{L^2}$. More precisely, let

$$(h, \phi) \in C(C_0 \cap H^{N_0+1,p_1+1/2}(\mathbb{R}) \times H^{N_0+1/2,p_1}(\mathbb{R}))$$

be a real-valued solution of (1.6) on some time interval. Define

$$B := \frac{G(h)\phi + h_x \phi_x}{(1 + h_x^2)}, \quad V := \phi_x - Bh_x, \quad \omega := \phi - TB P_{\geq 1} h,$$

where, for any $a, b \in L^2(\mathbb{R})$, we have denoted by $T_0 b$ the operator

$$(V, B)$$

is the restriction of the velocity field $v$ to the boundary of $\Omega_t$, and the function $\omega$ is a variant
of the so-called “good-unknown” of Alinhac.

In Appendix B we prove the following formula for the Dirichlet-Neumann operator:

$$G(h)\phi = |\partial_x| \omega - \partial_x TV h + G_2 + G_{\geq 3},$$

where $G_2$ is an explicit semilinear quadratic term, and $G_{\geq 3}$ denotes cubic and higher order terms.
This formula was already derived and used by Alazard–Delort [5]: the new aspect here is that the
remainder $G_{\geq 3}$ satisfies better trilinear bounds of $L^2$, weighted $L^2$, and $L^\infty$-type, under the sole
assumptions (1.17). In particular we only need to assume that $|\partial_x|^{1/2}+p_1 h$ and $|\partial_x|^{p_1} \phi$ are in $L^2$,
for some $p_1 > 0$.

In section 3 using (1.20), we diagonalize and symmetrize the system (1.6) reducing it to a single
scalar equation for one complex unknown $u$ of the form

$$u \approx |\partial_x|h - i|\partial_x|^{1/2} \omega + \text{higher order corrections},$$

see the formulas (3.16)-(3.17). The capillary system (1.6) then takes the form

$$\partial_t u - i \Lambda(\partial_x) u - i \Sigma(u) = -TV \partial_x u + N_u + R_u,$$

where: $\Lambda(\xi) := |\xi|^{3/2}$ is the dispersion relation for linear waves of frequency $\xi$; $\Sigma$ can be thought of
as a self-adjoint differential operator of order 3/2 which is cubic in $u$; $N_u$ are semilinear quadratic
terms; and $R_u$ contains (semilinear) cubic and higher order terms in $u$. See Proposition 3.2 for details.
Equation (1.22) is our starting point in establishing energy estimates for $u$, hence for
$(|\partial_x|h, |\partial_x|^{1/2} \phi)$. 

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1.5. The “division problem” and energy estimates via a “Quasilinear I-Method”. In order to construct high order energy functionals controlling Sobolev norms of our solution, we first apply the natural differentiation operator associated to the equation (1.22). More precisely, we look at \( W_k := D^k u \), with \( D := |\partial_x|^{3/2} + \Sigma \), and show that
\[
\partial_t W_k - i\Lambda(\partial_x)W_k - i\Sigma(W_k) = -\Delta \partial_x W_k + \mathcal{N}_{W_k} + \mathcal{R}_{W_k}.
\]
(1.23)
Here \( W_k \approx |\partial_x|^{3k/2} u \), \( \mathcal{N}_{W_k} \) denotes semilinear quadratic terms, and \( \mathcal{R}_{W_k} \) are semilinear cubic and higher order terms in \( W_k \). If one looks at the basic functional \( E(t) = \|W_{N_0}(t)\|_2^2 \) associated to (1.23), it is easy to verify that, as long as solutions are of size \( \varepsilon \), such energy functional is controlled for \( O(\varepsilon^{-1}) \) times.

In order to go past this local existence time, one needs to rely on the dispersive properties of solutions. One of the main difficulties in dealing with a one dimensional problem such as (1.23) is the slow time decay, which is \( t^{-1/2} \) for linear solutions. A classical idea used to overcome the difficulties associated to weak dispersion is the use of normal forms [43], which, wherever possible, eliminate the slow decaying quadratic terms from the nonlinearity.

The implementation of the method of normal forms is delicate in quasilinear problems, due to the potential loss of derivatives. Nevertheless this has been done in some cases, for example by performing appropriate symmetrizations to avoid losses of derivatives, we obtain
\[
\partial_t E^{(2)}(t) = \text{semilinear cubic terms}.
\]
(1.27)
We then define a cubic energy functional \( E^{(3)} \) which is a sum of cubic terms of the form
\[
\int_{\mathbb{R} \times \mathbb{R}} \tilde{F}(\xi, t)\tilde{G}(\eta, t)\tilde{H}(\xi - \eta, t)m(\xi, \eta) d\xi d\eta
\]
(1.28)

For comparison, in the gravity water waves case the dispersion relation is \( \Lambda(\xi) = |\xi|^{1/2} \) and one has \( |\Lambda(\xi) \pm \Lambda(\xi - \eta) - \Lambda(\eta)| \approx |\xi - \eta|^{1/2} \) in the case \( |\xi| \approx |\eta| \approx 1 \gg |\xi - \eta| \).
where $F, G, H$ can be any of the functions $W_{N_0}, SW_{N_1}, u, Su$ or their complex conjugates, and the symbols $m$ are obtained by diving the symbols of the cubic expressions in (1.27) by the appropriate resonant phase function (1.24). The “division problem” mentioned above, see (1.24)–(1.25), manifests itself in the fact that some of these symbols have singularities of the form $(\text{low frequency})^{-1/2}$. This is ultimately due to the lack of symmetries in the equation for $Su$ (or $SW_{N_1}$). See the discussion in subsection 1.6 below for more on this. Despite these singularities, the cubic functional $E^{(3)}$ that we define is a perturbation of (1.26) on each fixed time slice. Moreover, by construction

$$
\partial_t \left( E^{(2)} + E^{(3)} \right) (t) = \text{singualr semilinear quartic terms.} \tag{1.29}
$$

To deal with the singularities of the form $(\text{low frequency})^{-1/2}$ in the above quartic bulk terms, and prove that they decay fast enough, we need to need to make our suitable low frequency assumptions (1.11) on the solutions. Using these low frequency assumptions, and anticipating a sharp decay rate of $t^{-1/2}$ for our solution, we then control the energy functional $E^{(2)} (t)$ for all times, allowing a slow growth of $t^{2p_0}$ (in fact, due to weaker symmetries, we need a faster growth rate on the weighted energies). This is done in sections 4 and 5 for the Sobolev and the weighted Sobolev norm.

The additional assumptions made in order to close the energy estimates above are then recovered by low frequency energy estimates in sections 5 and 6. More precisely, by following a similar argument to the one above, we control for long times a quadratic energy of the form

$$
E_{\text{low}}^{(2)} := \int_{\mathbb{R}} \left( \left| \hat{u}(\xi, t) \right|^2 + \left| \hat{Su}(\xi, t) \right|^2 \right) : |\xi|^{-1} P_t(\xi) \, d\xi, \tag{1.30}
$$

where $P_t : \xi \in \mathbb{R} \rightarrow [0, 1]$ is a smooth function that vanishes if $|\xi| \geq 20$, equals 1 if $(1+t)^{-\frac{1}{2}} \leq |\xi| \leq 1$, and is equal to $[(1+t)^2 |\xi|]^{2p_1}$, if $2|\xi| \leq (1+t)^{-2}$. The key to control these low frequency energy functionals is the null structure for low frequency outputs in the water waves system (1.6). In essence, here we exploit the fact that the nonlinear part of the system has a better low frequency behavior than the linear evolution, so that, without imposing moment conditions on the initial data, we can still recover strong enough low frequency information for the nonlinear evolution.

1.6. Compatible vector-field structures. As in other quasilinear problems, to prove global regularity we propagate control not only of high Sobolev norms but also of other $L^2$ norms defined by vector-fields (36). In our case, a natural vector-field to propagate is the scaling vector-field $S = (3/2)t \partial_t + x \partial_x$ which (essentially) commutes with the linear part of the equation.

In the context of the quasilinear I-method, propagating $L^2$ control of vector-fields carrying weights is challenging. The reason for this is simple: as in the semilinear case, the success of the I-method ultimately depends on exploiting certain symmetries of the equation, which are related to its Hamiltonian structure. Once weighted vector-fields, such as $S$, are applied to the equation, these symmetries are weakened. Moreover, every weighted vector-field requires its own energy functional, and its own set of cubic corrections. At the very least, this increases considerably the amount of work needed to prove weighted energy estimates.

In this paper we are able to use energy estimates to propagate control of a specific compatible vector-field structure, namely the vector-fields

$$
\partial_x^{m_0}, \quad S \partial_x^{m_1}, \quad m_0 \in \{0, \ldots, N_0\}, \quad m_1 \in \{0, \ldots, N_1\}. \tag{1.31}
$$

We also propagate control of a suitable low-frequency structure described by an energy functional as in (1.30).

The compatible vector-field structure (1.31), using at most one weighted vector-field $S$ and many vector-fields $\partial_x$, was introduced, in the setting of water waves, by the authors in [32]. It was then used, and played a critical role, in all the later papers on the subject such as [27, 28, 34, 29]. The point of this choice is that:
(i) The compatible vector-field structure \(1.31\) can be propagated in time, with small \(t^\varepsilon\) loss and reasonably manageable computations;
(ii) It is still strong enough to provide almost optimal \(t^{-1/2}\) dispersive decay, since we work in dimension one (see Lemmas \(2.2\) and \(2.3\)).

1.7. Decay and modified scattering. Having established the \(L^2\) bounds described above, we eventually prove a pointwise sharp decay rate of \(t^{-1/2}\) for our solutions. This is done in sections 8 and 9 following a similar strategy as in our previous works \([31, 32, 34]\). We write Duhamel’s formula in Fourier space and study the nonlinear oscillations in the spirit of \([23, 25, 31]\). After a normal form transformation, a stationary phase analysis reveals that a correction to the asymptotic behavior is needed similarly to the case of gravity waves \([32]\).

However, the analysis here is more complicated because of the low frequency singularities introduced by the strong quadratic time resonances. This is especially evident in this part of the argument where non \(L^2\)-based norms need to be estimated, and meaningful symmetrization cannot be performed. Using the carefully chosen norm \(1.11\), we are able to control uniformly, over time and frequencies, an appropriate norm of our solution; this gives us the desired decay, as well as modified scattering.

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2. Preliminaries

2.1. Notation and basic lemmas. In this subsection we summarize some of our main notation and recall several basic formulas and estimates. We fix an even smooth function \(\varphi : \mathbb{R} \to [0, 1]\) supported in \([-8/5, 8/5]\) and equal to 1 in \([-5/4, 5/4]\), and define, for any \(k \in \mathbb{Z}\),
\[
\varphi_k(x) := \varphi(x/2^k) - \varphi(x/2^{k-1}), \quad \varphi_{\leq k}(x) := \varphi(x/2^k), \quad \varphi_{\geq k}(x) := 1 - \varphi(x/2^{k-1}).
\]
For \(k \in \mathbb{Z}\) we denote by \(P_k, P_{\leq k},\) and \(P_{\geq k}\) the operators defined by the Fourier multipliers \(\varphi_k, \varphi_{\leq k},\) and \(\varphi_{\geq k}\) respectively. Moreover, let
\[
P_k' := P_{k-1} + P_k + P_{k+1} \quad \text{and} \quad \varphi_k' := \varphi_{k-1} + \varphi_k + \varphi_{k+1}. \tag{2.1}
\]

Given \(s \geq 0\) let \(H^s\) denote the usual space of Sobolev functions on \(\mathbb{R}\). Recall the space \(C_0\) defined in \((1.8)\). We often use 3 other main norms: assume \(N \geq 0, b \in [-1, N]\), and \(f \in C_0\) then
\[
\|f\|_{H^N,b} := \left\{ \sum_{k \in \mathbb{Z}} \|P_k f\|_{L^2}^2 (2^{2Nk} + 2^{2kb}) \right\}^{1/2},
\]
\[
\|f\|_{\tilde{H}^N,b} := \sum_{k \in \mathbb{Z}} \|P_k f\|_{L^\infty} (2^{Nk} + 2^{kb}), \tag{2.2}
\]
\[
\|f\|_{\tilde{W}^N} := \|f\|_{L^\infty} + \sum_{k \geq 0} \|P_k f\|_{L^\infty} 2^{nk}.
\]
Notice that \(\|f\|_{\tilde{H}^N,0} \approx \|f\|_{H^N}\) and \(\|f\|_{\tilde{W}^N} \lesssim \|f\|_{\tilde{H}^N,0}\). The spaces \(\tilde{W}^N\) are often used in connection with Lemma \(B.1\) to prove iterative bounds on products of functions.

\[\text{We refer the reader to} \quad [23, 20, 33, 32, 41] \text{ and references therein, for more works related to modified scattering in dispersive equations.}\]
2.1.1. Multipliers and associated operators. We will often work with multipliers $m : \mathbb{R}^2 → \mathbb{C}$ or $m : \mathbb{R}^3 → \mathbb{C}$, and operators defined by such multipliers. We define the class of symbols

$$S^\infty := \{ m : \mathbb{R}^2 → \mathbb{C} : m \text{ continuous and } \|m\|_{S^\infty} := \|\mathcal{F}^{-1}(m)\|_{L^1} < \infty \}. \quad (2.3)$$

Given a suitable symbol $m$ we define the associated bilinear operator $M$ by

$$\mathcal{F}[M(f, g)](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} m(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta,$$ \quad (2.4)

We often use the identity

$$SM(f, g) = M(Sf, g) + M(f, Sg) + \widetilde{M}(f, g) \quad (2.5)$$

where $S = (3/2)\partial_\xi + x\partial_x$, $f, g$ are suitable functions defined on $I \times \mathbb{R}$, and the symbol of the bilinear operator $\widetilde{M}$ is given by

$$\widetilde{m}(\xi, \eta) = -(\xi\partial_\xi + \eta\partial_\eta)m(\xi, \eta). \quad (2.6)$$

This follows by direct calculations and integration by parts.

Lemma 2.1 below summarizes some of the basic properties of symbols and associated operators.

**Lemma 2.1.** (i) We have $S^\infty ⊆ L^\infty(\mathbb{R} \times \mathbb{R})$. If $m, m' \in S^\infty$ then $m \cdot m' \in S^\infty$ and

$$\|m \cdot m'|_{S^\infty} \lesssim \|m\|_{S^\infty} \|m'|_{S^\infty}. \quad (2.7)$$

Moreover, if $m \in S^\infty$, $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, $v \in \mathbb{R}^2$, and $m_{A,v}(\xi, \eta) := m(A(\xi, \eta) + v)$ then

$$\|m_{A,v}\|_{S^\infty} = \|m\|_{S^\infty}. \quad (2.8)$$

(ii) Assume $p, q, r \in [1, \infty]$ satisfy $1/p + 1/q + 1/r = 1$, and $m \in S^\infty$. Then, for any $f, g \in L^2(\mathbb{R})$,

$$\|M(f, g)\|_{L^r} \lesssim \|m\|_{S^\infty} \|f\|_{L^p} \|g\|_{L^q}. \quad (2.9)$$

In particular, if $1/p + 1/q + 1/r = 1$,

$$\left| \int_{\mathbb{R}^2} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \hat{h}(\xi - \eta) \, d\xi d\eta \right| \lesssim \|m\|_{S^\infty} \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}. \quad (2.10)$$

(iii) If $p_1, p_2, p_3, p_4 \in [1, \infty]$ are exponents that satisfy

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 1$$

then

$$\left| \int_{\mathbb{R}^3} \hat{f}_1(\xi) \hat{f}_2(\eta) \hat{f}_3(\rho - \xi) \hat{f}_4(\rho - \eta)m(\xi, \eta, \rho) \, d\xi d\eta d\rho \right| \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}} \|f_4\|_{L^{p_4}} \|\mathcal{F}^{-1}m\|_{L^1}. \quad (2.11)$$

See [32, Lemma 5.2] for the proof.

Given any multiplier $m : \mathbb{R}^d → \mathbb{C}$, $d \in \{2, 3\}$, and any $k, k_1, k_2, k_3, k_4 \in \mathbb{Z}$, we define

$$m^{k,k_1,k_2}(\xi, \eta) := m(\xi, \eta) \cdot \varphi_k(\xi) \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta),$$

$$m^{k_1,k_2,k_3,k_4}(\xi, \eta, \rho) := m(\xi, \eta, \rho) \cdot \varphi_{k_1}(\xi) \varphi_{k_2}(\eta) \varphi_{k_3}(\rho - \xi) \varphi_{k_4}(\xi - \rho). \quad (2.12)$$

Let

$$X := \{(k, k_1, k_2) \in \mathbb{Z}^3 : \max(k, k_1, k_2) - \operatorname{med}(k, k_1, k_2) \leq 6\},$$

$$Y := \{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 : 2^{k_1} + 2^{k_2} + 2^{k_3} + 2^{k_4} \geq (1 + 2^{-10})2^\max(k_1,k_2,k_3,k_4)\}. \quad (2.13)$$
and notice that $m^{k,k_1,k_2} \equiv 0$ unless $(k, k_1, k_2) \in \mathcal{X}$, and $m^{k_1,k_2,k_3,k_4} \equiv 0$ unless $(k, k_1, k_2) \in \mathcal{Y}$. Moreover, we will often use the notation

$$m(\xi, \eta) = O(f(|\xi|, |\xi - \eta|, |\eta|)) \iff \|m^{k,k_1,k_2}(\xi, \eta)\|_{\mathcal{S}^\infty} \lesssim f(2^k, 2^{k_1}, 2^{k_2}) 1_{\mathcal{X}}(k, k_1, k_2) \tag{2.14}$$

So, for example,

$$m(\xi, \eta) = O(|\xi - \eta|^{3/2}) \quad \text{means} \quad \|m^{k,k_1,k_2}(\xi, \eta)\|_{\mathcal{S}^\infty} \lesssim 2^{3k_1/2} 1_{\mathcal{X}}(k, k_1, k_2).$$

We use a similar notation for symbols of three variables,

$$m(\xi, \eta, \rho) = O(f(|\xi|, |\eta|, |\rho - \xi|, |\rho + \eta|)) \iff \|\mathcal{F}^{-1}(m^{k_1,k_2,k_3,k_4})\|_{L^1} \lesssim f(2^k, 2^{k_2}, 2^{k_3}, 2^{k_4}) 1_{\mathcal{Y}}(k_1, k_2, k_3, k_4). \tag{2.15}$$

2.1.2. Paraproducts. For any $a, b \in L^2(\mathbb{R})$ we define the paraproduct $T_a b$ by the formula

$$\mathcal{F}(T_a b)(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{a}(\xi - \eta) \tilde{b}(\eta) \chi(\xi - \eta, \eta) \, d\eta,$$

$$\chi(x, y) := \sum_{k \in \mathbb{Z}} \varphi_k(y) \varphi_{k-10}(x). \tag{2.16}$$

We also use the general formulas

$$ab = T_a b + T_b a + R(a, b),$$

$$F(a) = T_{F(a)} a + R_F(a), \tag{2.17}$$

where $R(a, b)$ and $R_F(a)$ are (substantially) more smooth remainders. The precise bounds on the remainders $R(a, b)$ and $R_F(a)$ depend on the context.

2.1.3. A dispersive estimate and an interpolation lemma. The following is our main linear dispersive estimate, which we use to control pointwise decay of solutions.

**Lemma 2.2.** For any $t \in \mathbb{R} \setminus \{0\}$, $k \in \mathbb{Z}$, and $f \in L^2(\mathbb{R})$ we have

$$\|e^{it\Lambda} P_k f\|_{L^\infty} \lesssim |t|^{-1/2} 2^{k/4} \|\hat{f}\|_{L^\infty} + |t|^{-3/2} 2^{-2k/5} \|g\|_{L^2} + \|\hat{f}\|_{L^2} \tag{2.18}$$

and

$$\|e^{it\Lambda} P_k f\|_{L^\infty} \lesssim |t|^{-1/2} 2^{k/4} \|f\|_{L^1}. \tag{2.19}$$

Moreover, the following refinement of (2.18) holds true: with $\xi_0 := -4x^2/(9t^2) \text{sign}(x/t)$ we have

$$\left|e^{it\Lambda} f(x) - \frac{\sqrt{2}}{\sqrt{3\pi |t|}} e^{-it(4/27)|x/t|^3} |\xi_0|^{1/4} \hat{f}(\xi_0)\right| \lesssim |t|^{-1/2 - a} (\|\xi\|^{1/4} \hat{f}\|_{L^\infty})$$

$$+ (\|\xi\|^{3/5} \partial_\xi \hat{f}\|_{L^2} + \|\xi\|^{-2/5} \hat{f}\|_{L^2} + \|f\|_{H^3}), \tag{2.20}$$

for $0 < a \leq 1/20$.

We refer to [34] Lemma A.1 for the proof of (2.18) and (2.19). The estimate (2.20) is a simple variant of (2.18) that can be obtained by adapting the proof of Lemma A.1 in [34], or the analogous proof in the case $\Lambda = |\xi|^{1/2}$ in Lemma 3.2 of [33]. We also use the following simple interpolation lemma:

**Lemma 2.3.** For any $k \in \mathbb{Z}$, and $f \in L^2(\mathbb{R})$ we have

$$\|\widehat{P_k f}\|_{L^\infty}^2 \lesssim \|P_k f\|_{L^1}^2 \lesssim 2^{-k} \|\hat{f}\|_{L^2}^2 \left[2^{k} \|\partial_\xi \hat{f}\|_{L^2} + \|\hat{f}\|_{L^2}\right]. \tag{2.21}$$

See [34] Lemma A.2 for the proof.
2.2. The main proposition. Given \( p_1 \in [0, 10^{-3}] \) we fix \( \mathcal{P} = \mathcal{P}_{p_1} : [0, \infty) \to [0, 1] \) an increasing function, smooth on \((0, \infty)\), such that

\[
\mathcal{P}(x) = x^{2p_1} \quad \text{if} \quad x \leq 1/2, \quad \mathcal{P}(x) = 1 \quad \text{if} \quad x \geq 1, \quad x\mathcal{P}'(x) \leq 10p_1\mathcal{P}(x). \tag{2.22}
\]

Our main theorem follows using the local existence theory and a continuity argument from the following main proposition:

**Proposition 2.4** (Main bootstrap). Assume that

\[
N_0 = N_I := 9, \quad N_1 = N_S := 3, \quad N_2 = N_\infty := 4.5, \quad 0 < p_1 \leq p_0/100 \leq 2^{-8},
\]

\[
0 < \epsilon_0 \leq \epsilon_1 \leq \epsilon_0^{2/3} \ll 1.
\]

Assume \( T \geq 1 \) and \((h, \phi) \in C([0, \infty) : (C_0 \cap \dot{H}^{N_0+1/p_1+1/2}) \times \dot{H}^{N_0+1/2,p_1})\) is a real-valued solution of the system

\[
\begin{aligned}
\partial_t h &= G(h)\phi, \\
\partial_t \phi &= \frac{\partial^2 h}{(1 + h_x^2)^{3/2}} - \frac{1}{2} \phi_x^2 + \frac{(G(h)\phi + h_x \phi_x)^2}{2(1 + h_x^2)}.
\end{aligned}
\tag{2.24}
\]

Let \( h_1 := |\partial_x|h, \phi_1 := |\partial_x|^{1/2}\phi \) and assume that, for any \( t \in [0, T]\),

\[
(1 + t)^{-p_0} \left[ \mathcal{K}_{I,t}(t) + \mathcal{K}_{I,I}(t) \right] + (1 + t)^{-4p_0} \left[ \mathcal{K}_{S,t}(t) + \mathcal{K}_{S,I}(t) \right] + (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{N_k} + 2^{-k/10}) \left\| P_k(h_1, \phi_1)(t) \right\|_{L^\infty} \leq \epsilon_1,
\tag{2.25}
\]

where, for \( O \in \{I, S\}\),

\[
\mathcal{K}_{O,h}(t) := \left\| P_{\geq -4}(Oh_1, O\phi_1)(t) \right\|_{H^{N_0}},
\]

\[
\left[ \mathcal{K}_{O,I}(t) \right]^2 := \int_{\mathbb{R}} (|\widehat{Oh_1}(\xi, t)|^2 + |\widehat{O\phi_1}(\xi, t)|^2) \cdot |\xi|^{-1/2} P((1 + t)^2|\xi|)^{\varphi_{\leq 4}}(\xi) d\xi.
\tag{2.26}
\]

Assume also that the initial data \((h_1(0), \phi_1(0))\) satisfy the stronger bounds

\[
\sum_{O \in \{I, S\}} \left\{ \left\| P_{\geq -4}(Oh_1, O\phi_1)(0) \right\|_{H^{N_0}} + \left\| |\partial_x|^{-1/2+p_1}(Oh_1, O\phi_1)(0) \right\|_{L^2} \right\} \leq \epsilon_0.
\tag{2.27}
\]

Then we have the improved bound

\[
(1 + t)^{-p_0} \left[ \mathcal{K}_{I,h}(t) + \mathcal{K}_{I,I}(t) \right] + (1 + t)^{-4p_0} \left[ \mathcal{K}_{S,h}(t) + \mathcal{K}_{S,I}(t) \right] + (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{N_k} + 2^{-k/10}) \left\| P_k(h_1, \phi_1)(t) \right\|_{L^\infty} \leq \epsilon_1/2 + \overline{C}\epsilon_0,
\tag{2.28}
\]

for any \( t \in [0, T]\), for some absolute constant \( \overline{C} \).

The rest of the paper is concerned with the proof of Proposition 2.4. We will always work under the assumptions (2.25)-(2.26).
3. Derivation of the main scalar equation

As in Proposition 2.4, assume \( T \geq 1 \) and \((h, \phi) \in C([0, T] : H^{N_0+1} \times H^{p_1, N_0+1/2})\) is a real-valued solution of the system (2.24). Let

\[
B := \frac{G(h)\phi + h_x \phi_x}{1 + h^2_x}, \quad V := \phi_x - Bh_x,
\]

\[
\omega := \phi - T_B P_{\geq 1} h = \phi - T_B h + T_B P_{\leq 0} h,
\]

\[
\sigma := (1 + h^2_x)^{-3/2} - 1,
\]

where \( T \) is defined in (2.16). Using (3.1) we calculate

\[
- \frac{1}{2} \partial_x^2 h + \frac{(G(h)\phi + h_x \phi_x)^2}{2(1 + h^2_x)} = \frac{1}{2} \left[ B^2(1 + h^2_x) - (V + Bh_x)^2 \right] = \frac{1}{2} \left[ B^2 - 2VBh_x - V^2 \right].
\]

Moreover, using the formula in the second line of (2.17) and standard paradifferential calculus,

\[
\frac{\partial_x^2 h_{\phi}}{(1 + h_x^2)^{3/2}} = \partial_x^2 h + \partial_x \left( \frac{\partial_x h}{(1 + h_x^2)^{1/2}} - h_x \right) = \partial_x^2 h + \partial_x T_B h_x + \partial_x E_{\geq 3, h},
\]

where \( E_{\geq 3, h} \) is a more smooth cubic error, satisfying, for any \( t \in [0, T] \),

\[
(1 + t)^{1-p_0} \| E_{\geq 3, h}(t) \|_{H^{N_0+2}} + (1 + t)^{1-4p_0} \| SE_{\geq 3, h}(t) \|_{H^{N_1+2}} \lesssim \varepsilon^3_1.
\]

We will also use the formula (see Proposition 3.9)

\[
G(h)\phi = |\partial_x|\omega - |\partial_x|T_B P_{\leq 0} h - \partial_x T_V h + G_2 + G_{\geq 3},
\]

where

\[
G_2 := |\partial_x|T_3 \partial_x h - |\partial_x|(|h|\partial_x h) + \partial_x T_3 \partial_x h - \partial_x (h \partial_x \phi),
\]

and \( G_{\geq 3} \) is a cubic error, satisfying, for any \( t \in [0, T] \),

\[
(1 + t)^{1-p_0} \| G_{\geq 3}(t) \|_{H^{N_0+1}} + (1 + t)^{1-4p_0} \| SG_{\geq 3}(t) \|_{H^{N_1+1}} + (1 + t)^{11/10} \| G_{\geq 3}(t) \|_{W^{N_2+1}} \lesssim \varepsilon^3_1.
\]

The function \( G(h) \phi \) satisfies linear estimates with derivative loss (see (C.28) for a stronger bound)

\[
(1 + t)^{-p_0} \| G(h) \phi \|_{H^{N_0+1}} + (1 + t)^{1/2} \| G(h) \phi \|_{W^{N_2+1}} + (1 + t)^{-4p_0} \| SG(h) \phi \|_{H^{N_1+1}} \lesssim \varepsilon_1.
\]

For simplicity of notation, for \( \alpha \in [-2, 2] \) let \( O_{3, \alpha} \) denote generic functions \( F \) on \([0, T]\) that satisfy the “cubic” bounds (see also Definition C.1)

\[
(1 + t)^{1-p_0} \| F(t) \|_{H^{N_0+\alpha}} + (1 + t)^{1-4p_0} \| SF(t) \|_{H^{N_1+\alpha}} + (1 + t)^{11/10} \| F(t) \|_{W^{N_2+\alpha}} \lesssim \varepsilon^3_1.
\]

In this section we will proceed with the formal calculations, without proving that the various cubic errors terms that will appear satisfy indeed the desired bounds. All the claimed cubic bounds will follow from the assumptions (2.25) and the definitions, by elliptic estimates. Detailed proofs are provided in Appendix C.

The first equation in (2.24) becomes

\[
\partial_t h = |\partial_x|\omega - |\partial_x|T_3 \partial_x P_{\leq 0} h - \partial_x T_V h + G_2 + G_{\geq 3},
\]

with \( G_{\geq 3} = O_{3,1} \), while the second equation in (2.24) gives

\[
\partial_t \omega = \frac{\partial_x^2 h}{(1 + h_x^2)^{3/2}} + \left( \partial_t \phi - \frac{\partial_x^2 h}{(1 + h_x^2)^{3/2}} \right) - T_{\partial_t B P_{\geq 1} h} - T_B \partial_t P_{\geq 1} h
\]

\[
= \frac{\partial_x^2 h}{(1 + h_x^2)^{3/2}} + I + II + III,
\]
where 
\[ I = \frac{(B^2 - V^2)}{2} - VBh_x \]
\[ = TB_B + R(B, B)/2 - TVV - R(V, V)/2 - TVBh_x - T_Bh_x V + O_{3,1/2} \]
\[ = TB_B - T_Bh_x V - TV\phi_x + R(\partial_x |\omega|, |\partial_x \omega|)/2 - R(\partial_x \omega, \partial_\omega \omega)/2 + O_{3,1/2}, \]
\[ II = T_{|\partial_x|^h} P_{\geq 1} h + O_{3,1/2}, \]
\[ III = -TB(G(h)\phi) + T_B P_{\leq 0}(G(h)\phi) = -TB(G(h)\phi) + T_{|\partial_x|^h} P_{\leq 0} |\partial_x \omega| + O_{3,1/2}. \]

Notice that
\[ B - G(h)\phi - VH_x = B - G(h)\phi - \phi_x h_x + B h_x^2 = 0. \]

Therefore
\[ I + III = TB(Vh_x) - TBh_x V - TV\phi_x + T_{|\partial_x|^h} P_{\leq 0} |\partial_x \omega| + O_{3,1/2} \]
\[ = TB(Vh_x) - TBh_x V - TV(\partial_x TBh) - TV\omega_x + T_{|\partial_x|^h} P_{\leq 0} |\partial_x \omega| + O_{3,1/2}. \]

Finally, notice that
\[ TB(Vh_x) - TBh_x V - TV(\partial_x TBh) = TB TVh_x + TBh_x V - TVTBh_x + O_{3,1/2} \]
\[ = O_{3,1/2}. \]

Therefore, the system (2.24) becomes
\[ \begin{cases} \partial_t h = |\partial_x|\omega - |\partial_x| T_{|\partial_x|^h} P_{\leq 0} |\partial_x \omega| - \partial_x TVh + G_2 + G_{\geq 3}, \\ \partial_t \omega = |\partial_x|^2 h + \partial_x T_{\sigma} h_x - TV\omega_x + H_2 + T_{|\partial_x|^h} P_{\geq 1} h + \Omega_{\geq 3}, \end{cases} \quad (3.11) \]

where
\[ \begin{cases} G_2 := |\partial_x| T_{|\partial_x|^h} - |\partial_x| ph - |\partial_x| (h|\partial_x| \phi) + \partial_x T_{\partial_\omega} h - \partial_\omega (h_2 \partial_\omega \phi), \\ H_2 := T_{|\partial_x|^h} P_{\leq 0} |\partial_x \omega| + R(|\partial_x|\omega, |\partial_x|\omega|)/2 - R(\partial_x \omega, \partial_\omega \omega)/2, \end{cases} \quad (3.12) \]

and
\[ G_{\geq 3} = O_{3,1}, \quad \Omega_{\geq 3} = O_{3,1/2}. \]

Let \( \tilde{\chi}(x, y) := 1 - \chi(x, y) - \chi(y, x), \)
\[ m_2(\xi, \eta) := \tilde{\chi}(\xi - \eta, \eta)|\xi(\xi - \eta)| - |\xi|\xi - \eta|| - \chi(\xi - \eta, \eta)\xi||\xi - \eta|\varphi_{\leq 0}(\eta), \]
\[ q_2(\xi, \eta) := \tilde{\chi}(\xi - \eta, \eta)|\eta(\xi - \eta) + |\eta|\xi - \eta||/2 + \chi(\xi - \eta, \eta)|\eta|\xi - \eta|\varphi_{\leq 0}(\eta), \quad (3.13) \]
and recall the notation (2.4). To summarize, we proved the following:

**Proposition 3.1.** Let \( (h, \phi) \) be a solution of (2.24) satisfying the bootstrap assumption (2.25), and let \( \omega, \sigma, V \) be as in (3.1). Then
\[ \begin{cases} \partial_t h = |\partial_x|\omega - \partial_\omega TVh + M_2(\omega, h) + G_{\geq 3}, \\ \partial_t \omega = |\partial_x|^2 h + \partial_\omega T_{\sigma} h_x - TV\omega_x + T_{|\partial_x|^h} P_{\geq 1} h + Q_2(\omega, \omega) + \Omega_{\geq 3}. \end{cases} \quad (3.14) \]

where \( M_2 \) and \( Q_2 \) are the bilinear operators associated to the multipliers \( m_2 \) and \( q_2 \) in (3.13), and
\[ G_{\geq 3} = O_{3,1}, \quad \Omega_{\geq 3} = O_{3,1/2}. \quad (3.15) \]
3.1. Symmetrization of the equations. Recall the water waves system \((3.14)\) for the surface elevation \(h\) and Alinhac’s good unknown \(\omega\), and the definitions \((3.1)\). In this section we aim to diagonalize and symmetrize this system, and write it as a single scalar equation for a complex valued unknown \(u\). The main result can be summarized as follows:

**Proposition 3.2.** We define the real-valued functions \(\gamma, \ p_1, \ p_0\) by

\[
\gamma := \sqrt{1 + \sigma} - 1, \quad p_1 := \gamma, \quad p_0 := -\frac{3}{4} \partial_x \gamma, \tag{3.16}
\]

where \(\sigma\) is as in \((3.1)\), and the main complex-valued unknown

\[
u := |\partial_x| h - i|\partial_x|^{1/2} \omega + T_{p_1} P_{\geq 1} |\partial_x| h + T_{p_0} P_{\geq 1} |\partial_x|^{-1} \partial_x h. \tag{3.17}
\]

Then \(u\) satisfies the evolution equation

\[
\partial_t u - i|\partial_x|^{3/2} u - i\Sigma_\gamma(u) = -\partial_x T_V u + N_2(h, \omega) + \bar{U}_{\geq 3}, \tag{3.18}
\]

where

\[
\bar{U}_{\geq 3} = |\partial_x|^{1/2} O_{3,1/2}, \tag{3.19}
\]

the operator \(\Sigma_\gamma\) is given by

\[
\Sigma_\gamma(u) = T_{p_1} P_{\geq 1} |\partial_x|^{3/2} u - \frac{3}{4} T_{\partial_x} \gamma P_{\geq 1} |\partial_x|^{-1/2} u, \tag{3.20}
\]

and the quadratic terms (expressed in \(h\) and \(\omega\)) are

\[
N_2(h, \omega) = -[|\partial_x|, \partial_x T_{\partial_x} \omega] h - i T_{\partial_x} \gamma [\partial_x]^{1/2} \omega + i[|\partial_x|^{1/2}, T_{\partial_x} \omega] |\partial_x| \omega + |\partial_x| M_2(\omega, h) - i|\partial_x|^{1/2} Q_2(\omega, \omega) - i|\partial_x|^{1/2} T_{\partial_x} |\partial_x| h P_{\geq 1} h. \tag{3.21}
\]

We will express these quadratic terms as functions of \(u\) and \(\pi\) via \((3.17)\) later on.

**Proof.** We start by calculating:

\[
\partial_t u - \partial_x T_V u - [-|\partial_x|, \partial_x T_V] h - T_{p_1} P_{\geq 1} |\partial_x| T_V h + T_{p_0} P_{\geq 1} |\partial_x| T_V h + |\partial_x| M_2(\omega, h)
\]

\[
+ i|\partial_x| T_V \partial_x \omega - i|\partial_x|^{1/2} Q_2(\omega, \omega) - i|\partial_x|^{1/2} T_{\partial_x} |\partial_x| h
\]

\[
+ T_{p_1} P_{\geq 1} |\partial_x|^{2} \omega + T_{p_0} P_{\geq 1} \partial_x \omega + i|\partial_x|^{3/2} T_{\partial_x} |\partial_x| h
\]

\[
- i|\partial_x|^{3/2} T_{p_1} P_{\geq 1} |\partial_x| h - i|\partial_x|^{3/2} T_{p_0} P_{\geq 1} |\partial_x|^{-1} \partial_x h + |\partial_x|^{1/2} O_{3,1/2}. \tag{3.22}
\]

Gathering appropriately the above terms we can write

\[
\partial_t u - i|\partial_x|^{3/2} u = -[|\partial_x|, \partial_x T_V] h + T_{p_1} P_{\geq 1} |\partial_x| T_V h + T_{p_0} P_{\geq 1} |\partial_x| T_V h + |\partial_x| M_2(\omega, h)
\]

\[
+ i|\partial_x| T_V \partial_x \omega - i|\partial_x|^{1/2} Q_2(\omega, \omega) - i|\partial_x|^{1/2} T_{\partial_x} |\partial_x| h
\]

\[
+ T_{p_1} P_{\geq 1} |\partial_x|^{2} \omega + T_{p_0} P_{\geq 1} \partial_x \omega + i|\partial_x|^{3/2} T_{\partial_x} |\partial_x| h
\]

\[
- i|\partial_x|^{3/2} T_{p_1} P_{\geq 1} |\partial_x| h - i|\partial_x|^{3/2} T_{p_0} P_{\geq 1} |\partial_x|^{-1} \partial_x h + |\partial_x|^{1/2} O_{3,1/2}. \tag{3.23}
\]

We observe that the expression in the first two lines in the right-hand side of \((3.22)\) is equal to

\[
- \partial_x T_V u - [||\partial_x|, \partial_x T_V| h - i T_{\partial_x} V |\partial_x|^{1/2} \omega + i[|\partial_x|^{1/2}, T_V \partial_x \omega] |\partial_x| h - [T_{p_1} |\partial_x| P_{\geq 1}, \partial_x T_V] h
\]

\[
- [T_{p_0} |\partial_x|^{-1} P_{\geq 1}, \partial_x T_V] h + |\partial_x| M_2(\omega, h) - i|\partial_x|^{1/2} Q_2(\omega, \omega) - i|\partial_x|^{1/2} T_{\partial_x} |\partial_x| h P_{\geq 1} h
\]

\[
= -\partial_x T_V u - [||\partial_x|, \partial_x T_V| h - i T_{\partial_x} V |\partial_x|^{1/2} \omega + i[|\partial_x|^{1/2}, T_V \partial_x \omega] |\partial_x| h
\]

\[
+ |\partial_x| M_2(\omega, h) - i|\partial_x|^{1/2} Q_2(\omega, \omega) - i|\partial_x|^{1/2} T_{\partial_x} |\partial_x| h P_{\geq 1} h + |\partial_x|^{1/2} O_{3,1/2}. \tag{3.23}
\]
Using the definition of $V$ and $\omega$ in (3.1), we see that the above quadratic terms coincide up to $|\partial_x|^{1/2}O_{3,1/2}$ with the quadratic terms appearing in (3.18) with $\partial_x^{1/2}O_{3,1/2}$.

We then look at the cubic and higher order terms in the last two lines in the right-hand side of (3.22). Our aim is to show that they are of the form $i\Sigma_\gamma(u)$, see (3.20), up to acceptable errors. For this purpose we first use the definition of $u$ in (3.17) and write

$$
i\Sigma_\gamma(u) = iT_\gamma P_{\geq 1}|\partial_x|^{5/2}h + T_\gamma P_{\geq 1}|\partial_x|^2\omega + iT_\gamma P_{\geq 1}|\partial_x|^{3/2}T_p P_{\geq 1}|\partial_x|h
+ iT_\gamma P_{\geq 1}|\partial_x|^{3/2}T_p P_{\geq 1}|\partial_x|h
- \frac{3i}{4}T_{\partial_x\gamma}P_{\geq 1}|\partial_x|^{-1/2}T_p P_{\geq 1}|\partial_x|h
- \frac{3i}{4}T_{\partial_x\gamma}P_{\geq 1}|\partial_x|^{-1/2}T_p P_{\geq 1}|\partial_x|h. $$

The last term on the last line is $|\partial_x|^{1/2}O_{3,1/2}$ so we can disregard it. We then compare the expression in (3.24) above and the last two lines of (3.22). Our proposition will be proven if the identities

$$
i\partial_x|^{3/2}T_\sigma|\partial_x|h - i\partial_x|^{3/2}T_p P_{\geq 1}|\partial_x|h - i\partial_x|^{3/2}T_p P_{\geq 1}|\partial_x|^{-1}\partial_x|h

= iT_\gamma P_{\geq 1}|\partial_x|^{5/2}h + iT_\gamma P_{\geq 1}|\partial_x|^{3/2}T_p P_{\geq 1}|\partial_x|h + iT_\gamma P_{\geq 1}|\partial_x|^{3/2}T_p P_{\geq 1}|\partial_x|^{-1}\partial_x|h

- \frac{3i}{4}T_{\partial_x\gamma}P_{\geq 1}|\partial_x|^{-1/2}T_p P_{\geq 1}|\partial_x|h + |\partial_x|^{1/2}O_{3,1/2}

$$

and

$$T_p P_{\geq 1}|\partial_x|^2 \omega + T_p P_{\geq 1}|\partial_x|\omega = T_\gamma P_{\geq 1}|\partial_x|^2 \omega - \frac{3i}{4}T_{\partial_x\gamma}P_{\geq 1}|\partial_x|\omega. $$

hold true. We immediately notice that the second equation (3.26) is satisfied by imposing $p_1 = \gamma$ and $p_0 = -3\partial_x\gamma/4$, as in (3.16). We then need to verify that (3.25) can be satisfied for an appropriate choice of the function $\gamma$.

We notice that all the multipliers $P_{\geq 1}$ in (3.25) can be dropped, at the expense of acceptable errors. Therefore (3.25) holds provided one has the following two identities for the symbols:

$$\sigma - p_1 = \gamma + \gamma p_1, \quad (3.27)$$

$$\frac{3}{2}\partial_x\sigma - \frac{3}{2}\partial_x p_1 + p_0 = \frac{3}{2}\gamma \partial_x p_1 - \gamma p_0 + \frac{3}{4}\partial_x\gamma + \frac{3}{4}\partial_x p_1. \quad (3.28)$$

Since $p_1 = \gamma$, (3.27) becomes $2\gamma + \gamma^2 = \sigma$, which is satisfied by imposing the first identity in (3.16). One can then verify that the last equation (3.28) is automatically satisfied.

**Remark 3.3.** We notice that the symmetrization obtained in Proposition 3.2 and the formulas (3.16) for $p_1, p_0$ and $\gamma$, are simpler than the ones of [1]. This is not only because we are considering the 1 dimensional case, but also because of our choice of the main variables in which we express the system, that is the energy variables $|\partial_x|h$ and $|\partial_x|^{1/2}\omega$.

### 3.1.1. The quadratic terms.

We analyze now the quadratic terms in (3.21),

$$N_2(h, \omega) = \sum_{k=1}^6 N_2^k$$

\begin{align*}
N_2^1 &:= -|\partial_x| |\partial_x| T_{\partial_x} |\partial_x| h, & N_2^2 &:= -iT_{\partial_x|\partial_x|} |\partial_x|^{1/2} |\partial_x|, & N_2^3 &:= i|\partial_x|^{1/2} T_{\partial_x} |\partial_x| \omega, \\
N_2^4 &:= -i|\partial_x|^{1/2} T_{\partial_x} |\partial_x| h, & N_2^5 &:= |\partial_x| M_2(\omega, h), & N_2^6 &:= -i|\partial_x|^{1/2} Q_2(\omega, \omega).
\end{align*}

(3.29)
Since $\gamma = \sqrt{1 + \sigma} - 1$, using (3.17) we have
\[ h = \frac{1}{2} |\partial_x|^{-1}(u + \overline{u}) + P_{2\rightarrow 4}O_{3,1}, \quad \omega = -\frac{1}{2i} |\partial_x|^{-1/2}(u - \overline{u}). \quad (3.30) \]

Using these relations we can express the quadratic terms (3.29) in terms of $u$ and $\overline{u}$, more precisely,
\[
N_2^1 = \frac{1}{4i[|\partial_x|^{-1/2}(u - \overline{u})]} |\partial_x|^{-1}(u + \overline{u}) + |\partial_x|^{1/2}O_{4,1,2}, \\
N_2^2 = \frac{i}{4i} |\partial_x|^{1/2}(u - \overline{u}), \\
N_2^3 = \frac{i}{4i} |\partial_x|^{1/2}T_{\partial_x|^{-1/2}(u - \overline{u})} |\partial_x|^{-1/2}(u - \overline{u}), \\
N_2^4 = \frac{i}{4i} |\partial_x|^{1/2}P_{2\rightarrow 4} |\partial_x|^{-1}(u + \overline{u}) + |\partial_x|^{1/2}O_{4,1,2}, \\
N_2^5 = \frac{i}{4i} |\partial_x|^{1/2}M_2(\partial_x|^{-1/2}(u - \overline{u}), |\partial_x|^{-1}(u + \overline{u}) + |\partial_x|^{1/2}O_{4,1,2}, \\
N_2^6 = \frac{i}{4i} |\partial_x|^{1/2}Q_2(\partial_x|^{-1/2}(u - \overline{u}), |\partial_x|^{-1/2}(u - \overline{u}), \\
\]

where we are using the definition [C.1] for the quartic terms $O_{4,1,2}$.

We divide these terms into 8 groups, by distinguishing the different types of interactions with respect to the specific pairing of $u$ and $\overline{u}$, and the type of frequency interactions $(\text{Low} \times \text{High} \rightarrow \text{High}$ interactions associated to the symbol $\chi(\xi - \eta, \eta)$ and $\text{High} \times \text{High} \rightarrow \text{Low}$ interactions associated to the symbol $\overline{\chi}(\xi - \eta, \eta))$. Recall the formulas (3.13). We define
\[
a_{++}(\xi, \eta) := \frac{1}{4i} \chi(\xi - \eta, \eta) \left[ - \xi(\xi - \eta) \left( \frac{\xi}{|\eta|^{1/2}} - 1 \right) + |\xi - \eta|^{3/2} + \frac{\eta(\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right) \\
+ |\xi - \eta|^2 \frac{\xi^{1/2} \varphi_{\geq 1}(\eta)}{|\eta|} + \frac{\xi^2 - |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \varphi_{\leq 0}(\eta), \right. \quad (3.31) \\
a_{+-}(\xi, \eta) := \frac{1}{4i} \chi(\xi - \eta, \eta) \left[ - \xi(\xi - \eta) \left( \frac{\xi}{|\eta|^{1/2}} - 1 \right) - |\xi - \eta|^{3/2} - \frac{\eta(\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right) \\
+ |\xi - \eta|^2 \frac{\xi^{1/2} \varphi_{\geq 1}(\eta)}{|\eta|} + \frac{\xi^2 + |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \varphi_{\leq 0}(\eta), \right. \quad (3.32) \\
a_{-+}(\xi, \eta) := \frac{1}{4i} \chi(\xi - \eta, \eta) \left[ - \xi(\xi - \eta) \left( \frac{\xi}{|\eta|^{1/2}} - 1 \right) - |\xi - \eta|^{3/2} - \frac{\eta(\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right) \\
+ |\xi - \eta|^2 \frac{\xi^{1/2} \varphi_{\geq 1}(\eta)}{|\eta|} + \frac{\xi^2 + |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \varphi_{\leq 0}(\eta), \right. \quad (3.33) \\
a_{- -}(\xi, \eta) := \frac{1}{4i} \chi(\xi - \eta, \eta) \left[ - \xi(\xi - \eta) \left( \frac{\xi}{|\eta|^{1/2}} - 1 \right) + |\xi - \eta|^{3/2} + \frac{\eta(\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right) \\
+ |\xi - \eta|^2 \frac{\xi^{1/2} \varphi_{\geq 1}(\eta)}{|\eta|} + \frac{\xi^2 - |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \varphi_{\leq 0}(\eta), \right. \quad (3.34) \\
\]

and
\[
b_{++}(\xi, \eta) = \frac{i}{4i} \frac{\xi}{\xi - \eta} |\partial_x|^{1/2} \varphi_{\geq 1}(\eta) + \frac{i}{4i} \frac{\xi}{|\xi - \eta|^{1/2}} |\partial_x|^{1/2} \varphi_{\leq 0}(\eta), \quad (3.35) \\
\]
\[ b_{+}(\xi, \eta) = \frac{i |\xi|\tilde{m}_2(\xi, \eta)}{4|\xi - \eta|^{1/2}|\eta|} - \frac{i |\xi|^{1/2}\tilde{q}_2(\xi, \eta)}{4|\xi - \eta|^{1/2}|\eta|^{1/2}}, \] (3.36)
\[ b_{+}(\xi, \eta) = \frac{i |\xi|\tilde{m}_2(\xi, \eta)}{4|\xi - \eta|^{1/2}|\eta|} - \frac{i |\xi|^{1/2}\tilde{q}_2(\xi, \eta)}{4|\xi - \eta|^{1/2}|\eta|^{1/2}}, \] (3.37)
\[ b_{-}(\xi, \eta) = \frac{i |\xi|\tilde{m}_2(\xi, \eta)}{4|\xi - \eta|^{1/2}|\eta|} + \frac{i |\xi|^{1/2}\tilde{q}_2(\xi, \eta)}{4|\xi - \eta|^{1/2}|\eta|^{1/2}}, \] (3.38)

where
\[ \tilde{m}_2(\xi, \eta) := \tilde{\chi}(\xi - \eta, \eta)[|\xi(\xi - \eta) - |\xi||\xi - \eta|], \]
\[ \tilde{q}_2(\xi, \eta) := \tilde{\chi}(\xi - \eta, \eta)\frac{|\eta(\xi - \eta) + |\eta||\xi - \eta|}{2}. \] (3.39)

Using the operator-symbol notation (2.4) and (2.16), we notice that
\[ \sum_{k=1}^{6} A^k_{2} = \sum_{X \in \{A, B\}} X_{++}(u, \bar{u}) + X_{+-}(\bar{u}, u) + X_{-+}(\bar{u}, u) + X_{--}(\bar{u}, \bar{u}) + |\partial_x|^{1/2}O_{4,1/2}. \] (3.40)

Let us also denote
\[ \sum_{(\epsilon_1, \epsilon_2) \in \{(+,+), (+,-), (-,+), (-,-)\}} \] (3.41)

For any complex-valued function \( f \), we use the notation
\[ f_+ := f, \quad f_- := \overline{f}. \]

We summarize the above computations in the following proposition:

**Proposition 3.4.** Let \( (h, \phi) \) be a solution of (2.24) satisfying the bootstrap assumption (2.25), and let \( u \) be defined as in (3.17) with \( \omega, \sigma, V \) given by (3.1). Then we have
\[ \partial_t u - i|\partial_x|^{3/2}u - i\Sigma_\gamma(u) = -\partial_x T_V u + \mathcal{N}_u + U_{\geq 3} \] (3.42)
with \( U_{\geq 3} = |\partial_x|^{1/2}O_{3,1/2}, \)
\[ \Sigma_\gamma = T_\gamma P_{\geq 1}|\partial_x|^{3/2} - \frac{3}{4}T_\partial_\gamma P_{\geq 1}|\partial_x|^{-1/2}, \quad \gamma = \sqrt{1 + \sigma - 1}, \] (3.43)
and
\[ \mathcal{N}_u = \sum_{(\epsilon_1, \epsilon_2)} [A_{\epsilon_1\epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}) + B_{\epsilon_1\epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2})]. \] (3.44)

The symbols of the quadratic operators are given in (3.31)–(3.38).

We notice that the only quasilinear quadratic contributions to the nonlinearity in (3.42) come from the term \( -\partial_x T_V u \) on the right-hand side of (3.42). All of the other quadratic contributions do not lose derivatives. The term \( -i\Sigma_\gamma(u) \) arises from the presence of surface tension. This term loses 3/2 derivative but it is essentially a self-adjoint operator. We will exploit this structure below to perform energy estimates.

We have thus reduce the water waves system (2.24) to the equation (3.42) above for a single complex valued unknown \( u \). From now on we will work with \( u \) as our main variable. We will also keep \( V \) as a variable and keep in mind that, in view of (3.1) and (3.30),
\[ V = -\frac{1}{2i}\partial_x|\partial_x|^{-1/2}(u - \overline{u}) + V_2, \quad V_2 := \partial_x T_B P_{\geq 1}h - Bh_x. \] (3.45)
3.2. Higher order derivatives and weights. To implement the energy method we need to control the increment of higher order Sobolev norms of the main variable \( u \). Because of the presence of the operator \( \Sigma \), which is of order 3/2, one cannot construct higher order energies by applying regular derivatives to the equation. We apply instead suitably modified versions of derivatives to the equation. The differential operator we will use, dictated by the structure of the equation, is given by

\[
\mathcal{D} = |\partial_x|^{3/2} + \Sigma,
\]

see the definition (3.43). Let \( k \in [1, 2N_0/3] \) be an even integer and define

\[
W = W_k := \mathcal{D}^k u.
\]

Below we derive the equation satisfied by \( W \).

**Proposition 3.5.** Let \( u \) be the solution of (3.42)–(3.44), and let \( W = W_k \) be defined by (3.47). Let \( N := 3k/2 \). Then we have

\[
\partial_t W - i|\partial_x|^{3/2} W - i\Sigma(W) = Q_W + |\partial_x|^N N_W + O_W
\]

where \( \Sigma \) is as in (3.43), and the nonlinearities are

\[
\hat{Q}_W(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} -i\xi \frac{|\xi|^N}{|\eta|^N} \chi(\xi - \eta) \hat{V}(\xi - \eta) \hat{W}(\eta) \, d\eta
\]

and

\[
N_W = \sum_\ast [A_{\epsilon_1\epsilon_2}(u_{\epsilon_1}, |\partial_x|^{-N} W_{\epsilon_2}) + B_{\epsilon_1\epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2})].
\]

The cubic nonlinearity \( O_W \) satisfies the bounds

\[
(1 + t)^{1-p_0} \|O_W(t)\|_{H^{N_0-N}} + (1 + t)^{1-4p_0} \|SO_W(t)\|_{H^{N_1-N}} \lesssim \varepsilon_1^3,
\]

\[
(1 + t)^{1-p_0} \|O_W(t)\|_{H^{N_0-N}} \lesssim \varepsilon_1^3,
\]

if \( N \leq N_1 \), if \( N \in [N_1, N_0] \).

**Proof.** The starting point is the equation (3.42). Applying \( \mathcal{D}^k \) to this equation we see that

\[
\partial_t \mathcal{D}^k u - i\mathcal{D}^k u = [\partial_t, \mathcal{D}^k] u - \mathcal{D}^k \partial_x T \mathcal{D}^k u + \mathcal{D}^k N_u + \mathcal{D}^k |\partial_x|^{1/2} O_{3,1/2}.
\]

Since \( W = \mathcal{D}^k u \), we can rewrite this equation as

\[
\partial_t W - i\mathcal{D}W = -|\partial_x|^N \partial_x T \mathcal{D}^k u + |\partial_x|^N N_W
\]

\[
+ [\partial_t, \mathcal{D}^k] u + [\partial_x T, |\partial_x|^N] (u - |\partial_x|^{-N} W) + [\partial_x T, \mathcal{D}^k - |\partial_x|^N] u
\]

\[
+ (\mathcal{D}^k - |\partial_x|^N) N_u + |\partial_x|^N (N_u - N_W) + \mathcal{D}^k |\partial_x|^{1/2} O_{3,1/2}.
\]

To prove the proposition it suffices to show that all the terms in the last two lines of (3.52) satisfy the cubic bounds (3.51). This is proved in Proposition C.3. \( \square \)

Define the weighted variable

\[
Z := S \mathcal{D}^{2N_1/3} u, \quad \mathcal{D} = |\partial_x|^{3/2} + \Sigma,
\]

where \( S = (3/2)t\partial_t + x\partial_x \). The next lemma gives the evolution equation for \( Z \).

**Proposition 3.6.** We have

\[
\partial_t Z - i|\partial_x|^{3/2} Z - i\Sigma Z = Q_Z + N_{Z,1} + N_{Z,2} + N_{Z,3} + O_Z,
\]
where the quasilinear quadratic nonlinearity $Q_Z$ is given by
\[ Q_Z := Q_{N_1}(V, Z) = \frac{1}{2\pi} \int_{\mathbb{R}} q_{N_1}(\xi, \eta) \hat{V}(\xi - \eta) \hat{Z}(\eta) \, d\eta, \]

\[ q_{N_1}(\xi, \eta) := -i \xi \frac{\partial N_1}{\partial \eta} \chi(\xi - \eta). \tag{3.55} \]

The quadratic semilinear terms are given by
\[ N_{Z,1} := |\partial_x|^{N_1} \sum_{\epsilon} A_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, |\partial_x|^{-N_1} Z_{\epsilon_2}), \]
\[ N_{Z,2} := (i/2)Q_{N_1}(\partial_x|\partial_x|^{-1/2} (Su - \overline{Su}), |\partial_x|^{N_1} u) \]
\[ + |\partial_x|^{N_1} \sum_{\epsilon} [A_{\epsilon_1 \epsilon_2}(Su_{\epsilon_1}, u_{\epsilon_2}) + B_{\epsilon_1 \epsilon_2}(Su_{\epsilon_1}, u_{\epsilon_2}) + B_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, Su_{\epsilon_2})], \tag{3.56} \]
\[ N_{Z,3} := (i/2)Q_{N_1}(\partial_x|\partial_x|^{-1/2} (u - \overline{u}), |\partial_x|^{N_1} u) \]
\[ + (i/2)\tilde{Q}_{N_1}(\partial_x|\partial_x|^{-1/2} (u - \overline{u}), |\partial_x|^{N_1} u) + |\partial_x|^{N_1} (3/2 - N_1) N_u \]
\[ + |\partial_x|^{N_1} \sum_{\epsilon} [N_1 A_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}) + \tilde{A}_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}) + \tilde{B}_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2})]. \]

Here we are using the definition \[2.6\] - \[2.6\] for a bilinear operator $\tilde{M}$ with symbol $\tilde{m}$. The remainder term $O_Z$ is cubic and satisfies
\[ \|O_Z(t)\|_{L^2} \lesssim \varepsilon_1^3(1 + t)^{-1+4p_0}. \tag{3.57} \]

**Proof.** We take $N_1 = N$ in Proposition 3.5 and let $k_1$ be such that $k_1 = (2/3)N_1$. Equation (3.48) then reads
\[ \partial_t D^{k_1} u - i|\partial_x|^{3/2} D^{k_1} u - i\Sigma_\gamma D^{k_1} u = Q_{N_1}(V, D^{k_1} u) + |\partial_x|^{N_1} N_u + O_{W_{k_1}}, \tag{3.58} \]
with $Q_{N_1}$ and $q_{N_1}$ is in (3.55), and with a remainder $O_{W_{k_1}}$ satisfying, in view of (3.51),
\[ (1 + t)^{1-p_0} \|O_{W_{k_1}}(t)\|_{L^2} + (1 + t)^{1-4p_0} \|SO_{W_{k_1}}(t)\|_{L^2} \lesssim \varepsilon_1^3. \]

Next, observe that $[S, \partial_t - i\Lambda] = -(3/2)(\partial_t - i\Lambda)$, where $\Lambda = |\partial_x|^{3/2}$. Therefore, applying $S$ to (3.58), and commuting it with the left-hand side, we obtain
\[ \partial_t Z - i|\partial_x|^{3/2} Z - i\Sigma_\gamma Z = SQ_{N_1}(V, D^{k_1} u) + S|\partial_x|^{N_1} N_u \]
\[ + (3/2)(\partial_t - i\Lambda) D^{k_1} u + i[S, \Sigma_\gamma]D^{k_1} u + SO_{W_{k_1}}. \tag{3.59} \]

Recall also the formulas
\[ V = -\frac{1}{2i} |\partial_x|^{-1/2}(u - \overline{u}) + V_2, \quad V_2 := \partial_x T_B P_{\geq 1} h - Bh_x, \]
and
\[ SQ_{N_1}(V, D^{k_1} u) = Q_{N_1}(V, Z) + Q_{N_1}(SV, D^{k_1} u) + \tilde{Q}_{N_1}(V, D^{k_1} u). \]

Using also the commutation identities
\[ [S, |\partial_x|^{-1/2}] = -(1/2)|\partial_x|^{-1/2}, \quad [S, |\partial_x|^{N_1}] = -N_1|\partial_x|^{N_1}, \]
it follows that
\[ S|\partial_x|^{N_1} N_u = -N_1|\partial_x|^{N_1} N_u + |\partial_x|^{N_1} \sum_{\epsilon} [A_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, Su_{\epsilon_2}) + A_{\epsilon_1 \epsilon_2}(Su_{\epsilon_1}, u_{\epsilon_2}) \]
\[ + \tilde{A}_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}) + B_{\epsilon_1 \epsilon_2}(Su_{\epsilon_1}, u_{\epsilon_2}) + B_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, Su_{\epsilon_2}) + \tilde{B}_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2})]. \tag{3.60} \]
Notice that some of these terms can all found in \( N_{Z,2} \) and \( N_{Z,3} \).

The desired formula \( (3.51) \) follows from \( (3.59) \) provided that

\[
O_Z := i[S, \Sigma]\partial^{k_1} u + SO_{W_k} + \left[ Q_Ni(V, \partial^{k_1} u) - \frac{i}{2} \tilde{Q}_N (\partial_x |\partial_x|^{1/2}(u - \bar{u}), |\partial_x|^N u) \right] + \frac{3}{2} Q_Ni(V, \partial^{k_1} u) - \frac{3i}{4} Q_Ni(\partial_x |\partial_x|^{1/2}(u - \bar{u}), |\partial_x|^N u) + \left[ Q_Ni(SV, \partial^{k_1} u) - \frac{i}{2} Q_Ni(\partial_x |\partial_x|^{1/2}(u - \bar{u}), |\partial_x|^N u) \right] + \frac{i}{4} Q_Ni(\partial_x |\partial_x|^{1/2}(u - \bar{u}), |\partial_x|^N u) + \frac{3i}{2} \Sigma \partial^{k_1} u + \frac{3}{2} O_{W_k} + |\partial_x|^N \sum_{\lambda} [A_{e_1 e_2}(u_{e_1}, S u_{e_2}) - A_{e_1 e_2}(u_{e_1}, |\partial_x|^{-N_i} Z_{e_2}) - N_i A_{e_1 e_2}(u_{e_1}, u_{e_2})].
\]

The elliptic cubic bound \( (3.57) \) is verified in Proposition \( C.3 \). \( \square \)

4. Energy estimates I: High Sobolev estimates

Let us recall the a priori assumptions on the solution \( u \):

\[
\sup_{t \in [0,T]} (1 + t)^{-p_0} \|u(t)\|_{H^{N_0}} \leq \varepsilon_1,
\]

and

\[
\sup_{t \in [0,T]} (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{-k/10} + 2^{N_k}) \|P_k u(t)\|_{L^\infty} \leq \varepsilon_1,
\]

where \( 0 < p_0 \leq 10^{-6} \), \( 0 < \varepsilon_0 \ll \varepsilon_1 \leq \varepsilon_0^{2/3} \ll 1 \).

4.1. Construction of the quadratic higher order Energy. The equation \( (3.42) \) admits a basic quadratic energy

\[
E_0 = \int_\mathbb{R} |u(x,t)|^2 \, dx.
\]

Our aim is to construct a higher order quadratic energy. For this we fix \( k \) such that \( N_0 = 3k/2 \), where \( N_0 \) is the index for the Sobolev regularity of our solution in \( (2.23) \), and define

\[
W := \partial^{k} u, \quad \mathcal{D} = \partial_x^{3/2} + \Sigma_{\gamma}.
\]

Let us rewrite for future reference the equations \( (3.42) \) and \( (3.38) \) for \( u \) and \( W \), see Propositions \( 3.4 \) and \( 3.5 \):

\[
\partial_t u - i|\partial_x|^{3/2} u - i\Sigma_{\gamma} u = -\partial_x T_V u + N_u + |\partial_x|^{1/2} O_{3,1/2},
\]

\[
V = -\frac{1}{2i} \partial_x |\partial_x|^{-1/2}(u - \bar{u}) + O_{2,-1/2},
\]

\[
\partial_t W - i|\partial_x|^{3/2} W - i\Sigma_{\gamma} W = Q_W + |\partial_x|^N W + O_W
\]

\[
\Sigma_{\gamma} = T_{\gamma} P_{\geq 1} |\partial_x|^{3/2} - \frac{3}{4} T_{\partial_x \gamma} P_{\geq 1} |\partial_x|^{-1/2} \partial_x, \quad \gamma = \sqrt{1 + \sigma} - 1,
\]

where \( N_u \) is defined in \( (3.44) \), \( Q_W \) is in \( (3.49) \), \( N_W \) is defined in \( (3.50) \) together with \( (3.31) \)–\( (3.34) \), \( (3.35) \)–\( (3.38) \), and \( O_W \) satisfies the cubic bounds \( (3.51) \). Notice that \( (3.3) \) and \( (4.4) \) imply

\[
u = |\partial_x|^{-N_0} W + O_{3,0},
\]

\[
(4.4)
\]
We define the basic quadratic energy functional associated to the second equation in (4.4) by

\[
E^{(2)}_{N_0}(t) = \frac{1}{2} \int_{\mathbb{R}} |W(x, t)|^2 \, dx = \frac{1}{4\pi} \int_{\mathbb{R}} \bar{W}(\xi, t) W(\xi, t) \, d\xi.
\]

(4.7)

One can verify that, by construction, the energy \( E^{(2)}_{N_0} \) is a priori controlled on time scales of order \( \varepsilon_0^{-1} \). We then want to appropriately modify it to obtain a better almost conservation property.

4.2. The cubic higher order Energy. Based on the equation (4.4) we define the following cubic energy functional:

\[
E^{(3)}_{m_{N_0}}(t) := \frac{1}{4\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \bar{W}(\xi, t) W(\eta, t) m_{N_0}(\xi, \eta) \bar{u}(\xi - \eta, t) \, d\xi d\eta,
\]

\[
m_{N_0}(\xi, \eta) := \frac{\langle \xi - \eta \rangle [\xi |N_0|^{1/2} \chi(\xi - \eta, \eta) - \eta |N_0|^{1/2} \chi(\eta - \xi, \xi)]}{2|\xi - \eta|^{3/2}}.
\]

(4.8)

Given the symbols \( a_{\xi_1 \xi_2} \) and \( b_{\xi_1 \xi_2} \) in (3.31)-(3.38), we also define the cubic functionals

\[
a^{(3)}_{\alpha_{\xi_1 \xi_2}}(\xi, \eta) := \frac{-i \xi |N_0|^{1/2} \alpha_{\xi_1 \xi_2}(\xi, \eta)}{|\xi|^{3/2} - |\eta|^{3/2} - \varepsilon_1 |\xi - \eta|^{3/2}},
\]

\[
b^{(3)}_{\beta_{\xi_1 \xi_2}}(\xi, \eta) := \frac{-i \xi |N_0|^{1/2} \beta_{\xi_1 \xi_2}(\xi, \eta)}{|\xi|^{3/2} - |\eta|^{3/2} - \varepsilon_1 |\xi - \eta|^{3/2}}.
\]

(4.9)

(4.10)

where, for any function \( f \), we use the notation \( f_+ := f, f_- := \overline{f} \).

Then the cubic correction to the energy is given by

\[
E^{(3)}_{N_0}(t) := E^{(3)}_{m_{N_0}}(t) + \sum_{*} \left( E^{(3)}_{a_{\xi_1 \xi_2}}(t) + E^{(3)}_{b_{\xi_1 \xi_2}}(t) \right),
\]

(4.11)

and the total energy is

\[
E_{N_0}(t) := E^{(2)}_{N_0}(t) + E^{(3)}_{N_0}(t).
\]

(4.12)

In the remaining of this section we are going to use the energy \( E_{N_0} \) above to prove the following main proposition:

Proposition 4.1. Assume that \( u \) satisfies

\[
\sup_{t \in [0, T]} \left[ (1 + t)^{-p_0} \| u(t) \|_{H^{N_0}} + \sup_{k \in \mathbb{Z}} (1 + t)^{1/2} (2^{-k/10} + 2^{N_2 k}) \| P_k u(t) \|_{L^\infty} \right] \leq \varepsilon_1,
\]

(4.13)

then

\[
\sup_{t \in [0, T]} (1 + t)^{-p_0} \| u(t) \|_{H^{N_0}} \lesssim \varepsilon_0.
\]

(4.14)
Thanks to Proposition 4.1, we strengthen the control on the Sobolev norm assumed in Proposition 2.3, bounding as desired the first term in (2.28). Proposition 4.1 will follow from the two lemmas below:

**Lemma 4.2.** Assuming the bounds (4.13), for any \( t \in [0, T] \) we have
\[
|E_{N_0}^{(3)}(t)| \lesssim \varepsilon_1^4(1 + t)^{2p_0}.
\]

The above lemma essentially establishes the equivalence of \( E_{N_0} \) and \( E_{N_0}^{(2)} \) at every fixed time slice. The next lemma provides improved control on the increment of \( E_{N_0} \).

**Lemma 4.3.** Assuming the bounds (4.13), for any \( t \in [0, T] \) we have
\[
\frac{d}{dt}E_{N_0}(t) \lesssim \varepsilon_1^4(1 + t)^{-1+2p_0}.
\]

**Remark 4.4.** A byproduct of the proof of Lemma 4.3 is the estimate
\[
\frac{d}{dt}E_{N_0}(t) \lesssim E_{N_0}(t)^2.
\]

This immediately gives the result in Theorem 1.4.

**Proof of Proposition 4.1.** Using (4.12) and (4.16), we see that
\[
|E_{N_0}^{(2)}(t) + E_{N_0}^{(3)}(t)| \leq |E_{N_0}^{(2)}(0) + E_{N_0}^{(3)}(0)| + \int_0^t \varepsilon_1^3(1 + s)^{-1+2p_0} ds
\]
for any \( t \in [0, T] \). In view of (4.15) we then have
\[
E_{N_0}^{(2)}(t) \lesssim E_{N_0}^{(2)}(0) + \varepsilon_1^3(1 + t)^{2p_0},
\]
for any \( t \in [0, T] \). The assumptions on the initial data \( u_0 \), and (4.5), show that \( E_{N_0}^{(2)}(0) \lesssim \varepsilon_0^2 \). Moreover, in view of (3.1), (3.17), and the conservation of the physical energy (1.7), we have
\[
\|P_{\leq -5} u(t)\|_{L^2} \lesssim \|\partial_x|h(t) - i|\partial_x|^1/2 \phi\|_{L^2}^2 \lesssim \varepsilon_0^2.
\]
Therefore, using (4.6), for any \( t \in [0, T] \)
\[
\|u(t)\|_{H^{N_0}} \lesssim \|P_{\leq -5} u(t)\|_{L^2} + \|W\|_{L^2} + \varepsilon_1^3 \lesssim \varepsilon_0^2 + E_{N_0}^{(2)}(t) \lesssim \varepsilon_0^2(1 + t)^{2p_0},
\]
as desired. \( \square \)

### 4.3. Analysis of the symbols and proof of Lemma 4.2

In order to prove Lemmas 4.2 and 4.3, we need to establish the bounds on the symbols of the cubic energy functionals in (1.8), (4.9), and (1.10). With the definition (2.3), inspecting the symbol \( m_{N_0} \) in (1.8), using (A.17), and standard integration by parts, one can see that
\[
\|m_{N_0}^{k,k_1,k_2}\|_{S^\infty} \lesssim 2^{k_1/2}2^{-\max(k,k_2)/2}1_{\chi}(k,k_1,k_2)1_{[2,\infty)}(k_2 - k_1).
\]

Looking at the definition of the symbols in (4.9), and using the bounds (A.12) in Lemma A.1, we see that
\[
\left\| \frac{\partial_{N_0}}{\xi}^{k,k_1,k_2} \right\|_{S^\infty} \lesssim 2^{k_1/2}2^{-k/2}1_{\chi}(k,k_1,k_2)1_{[2,\infty)}(k_2 - k_1),
\]
\[
\left\| \frac{\partial_{N_0}}{\xi}^{k,k_1,k_2} \right\|_{S^\infty} \lesssim (2^{3k_1/2}2^{-3k/2}1_{[2,\infty)}(k) + 2^{k_1/2}2^{-k/2}1_{(-\infty,1)}(k))
\]
\times 1_{\chi}(k,k_1,k_2)1_{[2,\infty)}(k_2 - k_1),
\]
while (A.27) and (4.10) give
\[
\left\| \frac{(b_{N_0}^{k_1,k_2})_{k_1,k_2}}{\xi^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2}} \right\|_{S_{\infty}} \lesssim 2^{(N_0+1/2)k_2-k_2/2}1_{X}(k, k_1, k_2)1_{[-10,10]}(k_2 - k_1). \quad (4.20)
\]

We now apply Lemma 2.1(ii), together with the bounds established above, to prove (4.15). Using Lemma 2.1(ii), and the notation (2.1), we can estimate the term in (4.8) as follows:
\[
|E^{(3)}_{m_{N_0}}(t)| \lesssim \sum_{k,k_1,k_2 \in \mathbb{Z}} \|m_{N_0}^{k,k_1,k_2}\|_{S_{\infty}}\|P'_{k}W\|_{L^2}\|P'_{k_2}W\|_{L^2}\|P'_{k_1}u\|_{L^\infty}.
\]

Using the bound (4.18) on the symbol \(m_{N_0}\), the a priori assumptions in (4.13), and (4.5), we obtain
\[
|E^{(3)}_{m_{N_0}}(t)| \lesssim \sum_{(k,k_1,k_2) \in \mathcal{X}, |k-k_2| \leq 5} 2^{k_1/2-2-k_2/2}\|P'_{k_2}W\|_{L^2}\|P'_{k_2}W\|_{L^2}\epsilon_1 2^{k_1/10} (1 + t)^{-1/2}
\]
\[
\lesssim \epsilon_1^3 (1 + t)^{-1/3}.
\]

The cubic energies (4.9) can be dealt with in an identical fashion, since the bounds (4.19) on their symbols are analogous to the one for \(m_{N_0}\) in (4.18). The cubic corrections in (4.10) can also be treated similarly. We use Lemma 2.1(ii), (4.20), the a priori assumptions in (4.13) and (4.5), to obtain, for all \(\epsilon_1, \epsilon_2 \in \{+,-\},\)
\[
|E^{(3)}_{b_{\epsilon_1\epsilon_2}}(t)| \lesssim \sum_{k,k_1,k_2 \in \mathbb{Z}} \|b_{\epsilon_1\epsilon_2}^{k,k_1,k_2}\|_{S_{\infty}}\|P'_{k}W\|_{L^2}\|P'_{k}W\|_{L^2}\|P'_{k_1}u\|_{L^\infty}.
\]
\[
\lesssim \sum_{(k,k_1,k_2) \in \mathcal{X}, |k-k_2| \leq 10} 2^{(N_0+1/2)k_2-k_2/2}\|P'_{k_2}W\|_{L^2}\|P'_{k_2}W\|_{L^2}\epsilon_1 2^{k_1/10} (1 + t)^{-1/2}
\]
\[
\lesssim \epsilon_1^3 (1 + t)^{-1/3}.
\]

This concludes the proof of (4.15).

4.4. Proof of Lemma 4.3. Using the equation for \(W\) in (4.4) we can calculate
\[
\frac{d}{dt} E^{(2)}_{N_0}(t) = \frac{1}{2\pi} \mathfrak{Re} \int_{\mathbb{R}} \overline{W}(\xi, t) \partial_t \overline{W}(\xi, t) d\xi = A_1(t) + A_2(t) + A_3(t) + A_4(t),
\]
where
\[
A_1 := \frac{1}{2\pi} \mathfrak{Re} \int_{\mathbb{R}} \overline{W}(\xi) i\Sigma_{\gamma} \overline{W}(\xi) d\xi, \quad (4.21)
\]
\[
A_2 := \frac{1}{4\pi^2} \mathfrak{Re} \int_{\mathbb{R}^2} \overline{W}(\xi) \left( - i\xi_i |\xi|^{N_0} |\eta|^{N_0} \tilde{V}(\xi - \eta) \right) \overline{W}(\eta) d\xi d\eta, \quad (4.22)
\]
\[
A_3 := \frac{1}{2\pi} \mathfrak{Re} \int_{\mathbb{R}} \overline{W}(\xi) |\xi|^{N_0} \tilde{N}_W(\xi) d\xi, \quad (4.23)
\]
\[
A_4 := \frac{1}{2\pi} \mathfrak{Re} \int_{\mathbb{R}} \overline{W}(\xi) \partial_{\xi} \overline{W}(\xi) d\xi. \quad (4.24)
\]

All cubic contributions coming from the above integrals are matched, up to acceptable quartic remainder terms, with the contributions from the time evolution of \(E^{(3)}_{N_0}\), see (4.11) and (4.8)-(4.10). This fact is established through the following series of lemmas, which will also prove the desired estimate (4.16).
Lemma 4.5. Under the a priori assumptions \([4.13]\), we have
\[
|A_2(t) + \frac{d}{dt} E_{mN_0}^{(3)}(t)| \lesssim \varepsilon_1^4 (1 + t)^{-1+2p_0}. \tag{4.25}
\]

Lemma 4.6. Under the a priori assumptions \([4.13]\), we have
\[
|A_3(t) + \frac{d}{dt} \sum_x (E_{a,b,c_1}^{(3)}(t) + E_{b,c_1}^{(3)}(t))| \lesssim \varepsilon_1^4 (1 + t)^{-1+2p_0}. \tag{4.26}
\]

Lemma 4.7. Under the a priori assumptions \([4.13]\), we have
\[
|A_1(t) + A_4(t)| \lesssim \varepsilon_1^4 (1 + t)^{-1+2p_0}. \tag{4.27}
\]

The remaining of this section is dedicated to the proofs of the above lemmas.

4.4.1. Proof of Lemma 4.5. In what follows we will denote by \(R\) any quartic and higher order terms that satisfies
\[
|R(t)| \lesssim \varepsilon_1^4 (1 + t)^{-1+2p_0}. \tag{4.28}
\]

We start by symmetrizing the term \(A_2\) in \((4.22)\) using the fact that \(V\) is real-valued:
\[
A_2 = \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \overline{W(\xi)} \hat{W}(\eta) \hat{V}(\xi - \eta)
\times \left( -i\xi|\xi|^{N_0} |\eta|^{-N_0} \chi(\xi - \eta, \eta) + i\eta|\eta|^{N_0} |\xi|^{-N_0} \chi(\eta - \xi, \xi) \right) d\xi d\eta. \tag{4.29}
\]

Recall, see \((3.45)\), that
\[
V = -\frac{1}{2t} \partial_x |\partial_x|^{-1/2}(u - \overline{u}) + V_2, \quad V_2 = O_{2,-1/2}. \tag{4.30}
\]

Thus, we can write
\[
A_2 = A_{2,1} + A_{2,2}
\]

where
\[
A_{2,1} := \frac{1}{4\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \overline{W(\xi)} \hat{W}(\eta) \hat{u}(\xi - \eta) q_{N_0}(\xi, \eta) d\xi d\eta
\]
\[
q_{N_0}(\xi, \eta) := \frac{i(\xi - \eta)}{2|\xi - \eta|^{1/2}} \left[ |\xi|^{N_0} \chi(\xi - \eta, \eta) - \frac{\eta|\eta|^{N_0}}{|\xi|^{N_0}} \chi(\eta - \xi, \xi) \right], \tag{4.31}
\]

and
\[
A_{2,2} := \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \overline{W(\xi)} \hat{W}(\eta) \hat{V}_2(\xi - \eta) a_{2,2}(\xi, \eta) d\xi d\eta,
\]
\[
a_{2,2}(\xi, \eta) := -i\frac{\xi|\xi|^{N_0}}{|\eta|^{N_0}} \chi(\xi - \eta, \eta) + \frac{i\eta|\eta|^{N_0}}{|\xi|^{N_0}} \chi(\eta - \xi, \xi). \tag{4.32}
\]

According to \((4.31)\), the definition of \(E_{mN_0}^{(3)}\) in \((1.8)\) reads
\[
E_{mN_0}^{(3)} := \frac{1}{4\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \overline{W(\xi)} \hat{W}(\eta) m_{N_0}(\xi, \eta) \hat{u}(\xi - \eta) d\xi d\eta,
\]
\[
m_{N_0}(\xi, \eta) := \frac{q_{N_0}(\xi, \eta)}{i(|\xi|^{3/2} - |\xi - \eta|^{3/2} - |\eta|^{3/2})}. \tag{4.33}
\]
Using the equation (4.31) we can calculate
\[ \frac{d}{dt} E^{(3)}_{m_{N_0}} = I_1 + I_2 + I_3 + I_4 \]
where
\[ I_1 := \frac{1}{8\pi^2} \mathcal{R} \int_{\mathbb{R} \times \mathbb{R}} \hat{W}(\xi) \hat{\tilde{W}}(\eta) \hat{u}(\xi - \eta) \left[ -i|\xi|^{3/2} + i|\xi| - i|\eta|^{3/2} + i|\eta|^{3/2} \right] m_{N_0}(\xi, \eta) \, d\xi d\eta, \]
\[ I_2 := \frac{1}{8\pi^2} \mathcal{R} \int_{\mathbb{R} \times \mathbb{R}} \left[ i\Sigma \hat{W}(\xi) \hat{\tilde{W}}(\eta) + \hat{W}(\xi) i\Sigma \hat{\tilde{W}}(\eta) \right] \hat{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta, \]
\[ I_3 := \frac{1}{8\pi^2} \mathcal{R} \int_{\mathbb{R} \times \mathbb{R}} \left[ \hat{Q}_W(\xi) \hat{\tilde{W}}(\eta) + \hat{W}(\xi) \hat{\tilde{Q}_W}(\eta) \right] \hat{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta, \]
and
\[ I_4 := \frac{1}{8\pi^2} \mathcal{R} \int_{\mathbb{R} \times \mathbb{R}} \left[ |\xi|^{N_0} \hat{\Sigma} \hat{W}(\xi) \hat{\tilde{W}}(\eta) + \hat{W}(\xi) |\eta|^{N_0} \hat{\Sigma} \hat{\tilde{W}}(\eta) \right] \hat{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta \]
\[ + \frac{1}{8\pi^2} \mathcal{R} \int_{\mathbb{R} \times \mathbb{R}} \hat{W}(\xi) \hat{\tilde{W}}(\eta) \mathcal{F}(\partial_t u - i|\partial_z|^{3/2} u)(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta \]
\[ + \frac{1}{8\pi^2} \mathcal{R} \int_{\mathbb{R} \times \mathbb{R}} \left[ \hat{Q}_W(\xi) \hat{\tilde{W}}(\eta) + \hat{W}(\xi) \hat{\tilde{Q}_W}(\eta) \right] \hat{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta. \]

Using (4.31), (4.33) and (4.34) we see that \( A_{2,1} + I_1 = 0 \), and, therefore,
\[ A_2 + \frac{d}{dt} E^{(3)}_{m_{N_0}} = A_{2,2} + I_2 + I_3 + I_4. \]

It then suffices to show that
\[ |A_{2,2}(t)| + |I_2(t)| + |I_3(t)| + |I_4(t)| \lesssim \varepsilon_1^4 (1 + t)^{-1+2p_0}. \]

**Estimate of \( A_{2,2} \).** Using standard integration by parts one can see that the symbol in (4.32) satisfies
\[ a_{2,2}(\xi, \eta) = O\left( |\xi - \eta|^{1/2} |\eta|^{1/2} \right), \]
where we are using the notation (2.14) for bilinear symbols. Then, using Lemma 2.1(ii), (4.30), and (4.13), we have:
\[ |A_{2,2}| \lesssim \sum_{k,k_1,k_2 \in \mathbb{Z}} \| (a_{2,2})^{k,k_1,k_2} \|_{S^\infty} \| P_{k_1}^t W \|_{L^2} \| P_{k_2}^t V_2 \|_{L^2} \| P_{k_2}^t W \|_{L^2} \]
\[ \lesssim \sum_{(k,k_1,k_2) \in \mathcal{N}, |k-k_2| \leq 10} 2^{k_1} \| P_{k_2}^t W \|_{L^2} \epsilon_2^2 2^{-(N_2-1) \max(k_1,0)} (1 + t)^{-1} \| P_{k_2}^t W \|_{L^2} \]
\[ \lesssim \varepsilon_1^4 (1 + t)^{-1+2p_0}. \]

**Estimate of \( I_2 \).** The term \( I_2 \) in (4.35) presents a potential loss of 3/2 derivatives. However, exploiting the structure of \( \Sigma_\gamma \) and of the symbol \( m_{N_0} \), one can recover this loss. Recall the definition of \( \Sigma_\gamma \) from (4.4):
\[ \Sigma_\gamma W = T_\gamma |\partial_z|^{3/2} P_{\geq 1} W - \frac{3}{4} T_{\partial_x \gamma} P_{\geq 1} |\partial_z|^{-1/2} \partial_x W. \]

We can then estimate
\[ |I_2| \lesssim I_{2,1} + I_{2,2} + I_{2,3}, \]
where

\[ I_{2,1} = \left| -\int_{\mathbb{R}^3} \tilde{\gamma}(\rho - \xi) \chi(\xi - \rho, \rho)|\rho|^{3/2} \overline{P_{\geq 1}}(\rho) \tilde{W}(\eta) \tilde{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta d\rho \right| \]

\[ + \int_{\mathbb{R}^3} \overline{W}(\xi) \tilde{\gamma}(\rho - \xi) \chi(\eta - \rho, \rho)|\rho|^{3/2} \overline{P_{\geq 1}}(\rho) \tilde{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta d\rho, \tag{4.40} \]

\[ I_{2,2} = \left| \int_{\mathbb{R}^3} \tilde{\gamma_x}(\rho - \xi) \chi(\xi - \rho, \rho)|\rho|^{-1/2} \overline{P_{\geq 1}}(\rho) \tilde{W}(\eta) \tilde{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta d\rho \right|, \tag{4.41} \]

\[ I_{2,3} = \left| \int_{\mathbb{R}^3} \overline{W}(\xi) \tilde{\gamma_x}(\rho - \xi) \chi(\eta - \rho, \rho)|\eta|^{-1/2} \overline{P_{\geq 1}}(\rho) \tilde{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi d\eta d\rho \right|. \tag{4.42} \]

Since \( m_{N_0} \) recovers half a derivative, see \([4,18]\), the last two terms \( I_{2,2} \) and \( I_{2,3} \) do not lose derivatives. To see this more in detail, we first rewrite

\[ I_{2,2} = \left| \int_{\mathbb{R}^3} \overline{W}(\xi) \tilde{W}(\eta) \tilde{u}(\rho - \eta) \tilde{\gamma_x}(\xi - \rho) m_{2,2}(\xi, \eta, \rho) \, d\xi d\eta d\rho \right|, \tag{4.43} \]

Then, notice that

if \( f(\xi, \eta, \rho) = f_1(\xi, \rho) f_2(\eta, \rho) \) then \( \|F^{-1} f\|_{L^1(\mathbb{R}^3)} \lesssim \|F^{-1} f_1\|_{L^1(\mathbb{R}^2)} \|F^{-1} f_2\|_{L^1(\mathbb{R}^2)}. \tag{4.44} \)

Using this fact, and the bound for \( m_{N_0} \) in \([4,18]\), we can see that

\[ \|m_{2,2}^{k_1,k_2,k_3,k_4}\|_{S_0} \lesssim 2^{k_1/2} 2^{k_1/2} 2 - k_2/2 1_{[3,\infty)}(k_2 - k_3) 1_{[3,\infty)}(k_1 - k_3) 1_{[3,\infty)}(k_1, k_2, k_3, k_4). \tag{4.45} \]

Moreover, from the definition of \( \gamma \) in \([3,16]\), \([3,1]\) and \([3,30]\), Bernstein’s inequality, and the a priori bounds \([4,13]\), we have

\[ \|P_k \gamma\|_{L^\infty} \lesssim \varepsilon_2^{2} 2^{-(N_2 - 1) \max(k,0)} (1 + t)^{-1}, \]

\[ \|P_k \gamma\|_{L^\infty} \lesssim \varepsilon_2^{2} 2^{k/10} (1 + t)^{-2/3}, \quad k \leq 0. \tag{4.46} \]

We can then apply Lemma \(2.1(iii)\) together with the bound \([4,45]\), and use \([4,40]\) and the a priori bounds \([4,13]\), to estimate

\[ |I_{2,2}| \lesssim \sum_{k_1,k_2,k_3,k_4 \in \mathbb{Z}} \|m_{2,2}^{k_1,k_2,k_3,k_4}\|_{S_0} \|P_{k_1} W\|_{L^2} \|P_{k_2} W\|_{L^2} \|P_{k_3} \gamma_x\|_{L^\infty} \|P_{k_4} u\|_{L^\infty} \]

\[ \lesssim \sum_{|k_1 - k_2| \leq 10, k_1 \geq k_3, k_2 \geq 2, k_4} 2^{k_1/2} 2^{k_4/2} 2^{-k_2/2} \|P_{k_1} W\|_{L^2} \|P_{k_2} W\|_{L^2} \]

\[ \times \varepsilon_2^{2} 2^{k/10} 2^{-(N_2 - 2) \max(k_3,0)} (1 + t)^{-2/3} \varepsilon_1^{2} 2^{k_4/10} 2^{-N_2 \max(k_4,0)} (1 + t)^{-1/2} \]

\[ \lesssim \varepsilon_2^2 (1 + t)^{-7/6} \|W\|^2_{L^2}. \]

This is more than sufficient to show the desired bound for \( I_{2,2} \). The estimate for \( I_{2,3} \) is similar so we can skip it.

We then need to look at \( I_{2,1} \) in \([4,40]\). Changing variables in the integral, we write

\[ I_{2,1} = \left| \int_{\mathbb{R}^3} \overline{W}(\xi) \tilde{W}(\eta) \tilde{u}(\xi - \rho) \tilde{\gamma}(\rho - \eta) m_{2,1}(\xi, \rho, \eta) \, d\xi d\eta d\rho \right|, \]

\[ m_{2,1}(\xi, \rho, \eta) := \chi(\eta - \rho, \xi) |\xi|^{3/2} \varphi_{\geq 1}(\xi) m_{N_0}(\xi + \eta - \rho, \eta) \]

\[ + \chi(\rho - \eta, \eta) |\eta|^{3/2} \varphi_{\geq 1}(\eta) m_{N_0}(\xi, \rho). \tag{4.47} \]
We then want to establish a bound for the symbol in the above expression, showing that it does not cause any derivatives loss. Let us write

\[ m_{2,1} = n_1 + n_2 + n_3, \]
\[ n_1(\xi, \eta, \rho) := \left[ \chi(\rho - \eta, \eta) - \chi(\eta - \rho, \xi) \right] |\xi|^{3/2} \varphi_{\geq 1}(\xi)m_{N_0}(\xi + \eta - \rho, \eta), \]
\[ n_2(\xi, \eta, \rho) := \left[ \chi(\rho - \eta, \eta) \right] |\eta|^{3/2} \varphi_{\geq 1}(\eta) - |\xi|^{3/2} \varphi_{\geq 1}(\xi) \right] m_{N_0}(\xi, \rho), \]
\[ n_3(\xi, \eta, \rho) := \left[ \chi(\rho - \eta, \eta) \right] |\xi|^{3/2} \varphi_{\geq 1}(\xi) \right] \left[ m_{N_0}(\xi, \rho) - m_{N_0}(\xi + \eta - \rho, \eta) \right]. \]

Since the function \( \chi(\rho - \eta, \eta) - \chi(\eta - \rho, \xi) \) is supported in the region \( |\xi| \leq 2^{10} \max(|\xi - \rho|, |\eta - \rho|) \), the symbol \( n_1 \) does not lose derivatives, and, in particular, using (4.44) and the bound (4.18) for \( m_{N_0} \), we see that

\[ n_1(\xi, \eta, \rho) = O\left( |\xi|^{3/2} |\xi - \rho|^{1/2} |\eta|^{-1/2} 1_{[2^{-10}, \infty]}(|\xi - \rho|/|\xi|) 1_{[2^2, \infty]}(|\eta|/|\xi - \rho|) \right). \]

Here we are using the notation (2.15), with (2.12)-(2.13). Then, using again Lemma 2.1(iii) with the bound (4.49), the a priori decay assumption in (4.13), and (4.46), it is not hard to show

\[ \left| \int_{\mathbb{R}^3} \nabla(\xi) \nabla(\eta) \hat{u}(\xi - \rho) \hat{\gamma}(\rho - \eta) n_1(\xi, \eta, \rho) \, d\xi d\eta \right| \lesssim \|W\|_{L^2}^2 \varepsilon^3 (1 + t)^{-7/6}. \]

Using again (4.18) one sees that also \( n_2 \) does not cause derivatives losses:

\[ n_2(\xi, \eta, \rho) = O\left( (|\xi - \rho|^{3/2} + |\rho - \eta|^{3/2}) 1_{[2^2, \infty]}(|\eta|/|\rho - \eta|) 1_{[2^2, \infty]}(|\xi|/|\xi - \rho|) \right). \]

As was done before for the term (4.43), one can use Lemma 2.1(iii), followed by (4.46) and the a priori bounds (4.13), to obtain

\[ \left| \int_{\mathbb{R}^3} \nabla(\xi) \nabla(\eta) \hat{u}(\xi - \rho) \hat{\gamma}(\rho - \eta) n_2(\xi, \eta, \rho) \, d\xi d\eta \right| \lesssim \|W\|_{L^2}^2 \varepsilon^3 (1 + t)^{-7/6}. \]

Next we look at \( n_3 \). Recall the definition of \( m_{N_0} \) and \( q_{N_0} \) from (4.33) and (4.31), and write

\[ m_{N_0}(\xi, \rho) - m_{N_0}(\xi + \eta - \rho, \eta) = -ir_1(\xi, \eta, \rho) - ir_2(\xi, \eta, \rho), \]

\[ r_1(\xi, \eta, \rho) := q_{N_0}(\xi, \rho) - q_{N_0}(\xi + \eta - \rho, \eta) \]

\[ r_2(\xi, \eta, \rho) := q_{N_0}(\xi + \eta - \rho, \eta) \left[ \frac{1}{|\xi|^{3/2} - |\xi - \rho|^{3/2}} - \frac{1}{|\xi + \eta - \rho|^{3/2} - |\xi - \rho|^{3/2} - |\eta|^{3/2}} \right]. \]

Inspecting the formula (4.31) we see that

\[ q_{N_0}(x, y) - (N_0 - 1/2)|x - y|^{3/2} = O\left( \frac{|x - y|^{5/2}}{|x| + |y|} 1_{[2^2, \infty]}(|y|/|x - y|) \right). \]

It then follows that

\[ q_{N_0}(\xi, \rho) - q_{N_0}(\xi + \eta - \rho, \eta) = O\left( |\xi - \rho|^{5/2}(|\xi|^{-1} 1_{[2^2, \infty]}(|\xi|/|\xi - \rho|) + |\eta|^{-1} 1_{[2^2, \infty]}(|\eta|/|\xi - \rho|) \right), \]

and, therefore,

\[ r_1(\xi, \eta, \rho) = O\left( |\xi - \rho|^{3/2} (|\xi|^{-3/2} 1_{[2^2, \infty]}(|\xi|/|\xi - \rho|) + |\eta|^{-1} |\xi|^{-1/2} 1_{[2^2, \infty]}(|\eta|/|\xi - \rho|) \right). \]
Moreover, one can directly verify that for $|\eta| \geq 2^3 \max(|\xi - \rho|, |\eta - \rho|)$,
\[
|\xi + \eta - \rho|^{3/2} - |\xi - \rho|^{3/2} - |\eta|^{3/2} - (|\xi|^{3/2} - |\xi - \rho|^{3/2} - |\rho|^{3/2})
= O\left(\frac{|\xi - \rho|^2 + |\xi - \eta|^2}{|\eta|^{1/2} + |\xi|^{1/2}}\right).
\] (4.52)

It follows that
\[
r_2(\xi, \eta, \rho) = O\left(\frac{|\xi - \rho|^{3/2} + |\eta - \rho|^{3/2}}{|\eta|^{3/2} + |\xi|^{3/2}}\right)\mathbf{1}_{[2^2, \infty)}(|\eta|/|\xi - \rho|),
\]
whenever $|\eta| \geq 2^2 |\eta - \rho|$. This gives
\[
n_3(\xi, \eta, \rho) = O\left((|\xi - \rho|^{3/2} + |\eta - \rho|^{3/2})\mathbf{1}_{[2^2, \infty)}(|\eta|/|\xi - \rho|)\mathbf{1}_{[2^2, \infty)}(|\eta|/|\rho - \eta|)\right).
\] (4.53)

We then use Lemma (4.3) with (4.53), and the a priori bounds (4.3) and (4.4), to get
\[
\left|\int_{\mathbb{R}^3} \mathbf{W}(\xi) \mathbf{W}(\eta) \hat{u}(\xi - \eta) \gamma(\rho - \eta) n_3(\xi, \eta, \rho) \, d\xi d\eta d\rho\right|
\lesssim \sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}} \|n_3^{k_1, k_2, k_3, k_4}\|_{L^\infty} \|P_{k_1}^r W\|_{L^2} \|P_{k_2}^r W\|_{L^2} \|P_{k_3}^r \gamma\|_{L^\infty} \|P_{k_4}^r u\|_{L^\infty}
\lesssim (2^{3k_3/2} + 2^{3k_4/2}) \|P_{k_1}^r W\|_{L^2} \|P_{k_2}^r W\|_{L^2}
\times \varepsilon_2^2 (2^{k_3/10} 2^{-(N_2 - 1) \max(k_3,0)} (1 + t)^{-2/3} \varepsilon_1 2^{k_4/10} 2^{-(N_2 - 1) \max(k_4,0)} (1 + t)^{-1/2}
\lesssim \varepsilon_1^2 (1 + t)^{-7/6} \|W\|_{L^2}^2.
\]

This shows $|I_{2,1}| \lesssim \varepsilon_1^2 (1 + t)^{-1}$, and concludes the proof of the desired bound (4.38) for $I_2$.

From the above proof we can deduce Lemma (4.8) below, which we will use later on to estimate two other expressions with a potential loss of 3/2 derivatives, similar to the term $I_2$ in (4.3), namely, $I_6, i_1$ in (4.7A), and $K_{1,1}$ in (6.28).

**Lemma 4.8.** Consider the expression
\[
I(t) = \int_{\mathbb{R}^3} \mathbf{W}(\xi) \mathbf{W}(\eta) \hat{u}(\xi - \eta) \mathbf{m}(\xi, \eta) \, d\xi d\eta,
\] (4.54)
with a symbol of the form
\[
\mathbf{m}(\xi, \eta) = \frac{q(\xi, \eta)}{|\xi|^{3/2} + |\xi - \eta|^{3/2} - |\eta|^{3/2}}.
\] (4.55)
which is supported on a region where $2^2 |\xi - \eta| \leq |\eta|$. Here $u_+ = u$, $u_- = \mathbf{1}$, and $\gamma$ is in (4.2), (4.1). Assume that the following two properties hold:
\[
q(\xi, \eta) = O\left(|\eta - \rho|^{3/2} \mathbf{1}_{[2^2, \infty)}(|\eta|/|\xi - \eta|)\right),
\] (4.56)
and, whenever $|\eta| \geq 2^2 |\eta - \rho|$ and $|\xi| \geq 2^2 |\xi - \rho|$, 
\[
q(\xi, \rho) - q(\xi + \eta - \rho, \eta)
= O\left(\frac{|\xi - \rho|^{5/2} + |\eta - \rho|^{5/2}}{|\xi| + |\eta|} \mathbf{1}_{[2^2, \infty)}(|\xi|/|\xi - \rho|) \mathbf{1}_{[2^2, \infty)}(|\eta|/|\xi - \rho|)\right).
\] (4.57)

Then,
\[
|I(t)| \lesssim \|F\|_{L^2}^2 (1 + t)^{-7/6}.
\] (4.58)
Estimate of $I_3$. Directly from the definition of $I_3$ in (4.30) we have
\[
|I_3| \lesssim \left| \int_{\mathbb{R}^3} \left[ \overline{Q_W}(\xi) \widehat{W}(\eta) + \overline{W}(\xi) \overline{Q_W}(\eta) \right] \widehat{u}(\xi - \eta)m_{N_0}(\xi, \eta) \, d\xi d\eta \right| \tag{4.59}
\]
where $Q_W$ is defined in (3.49). This term can be estimated in the same way as in our previous paper [34], see Lemma 2.6 there. For completeness we give some details below. Using (3.49) we have
\[
|I_3| \lesssim \left| \int_{\mathbb{R}^3} \left[ \xi |\xi|^{N_0} |\rho|^{N_0} \lambda(\xi - \rho, \rho) \overline{W}(\xi - \rho) \overline{W}(\rho) \overline{W}(\eta) \right. \right.
\]
\[
- \left. \overline{W}(\xi) \eta \right| \left. \eta |\eta|^{N_0} |\rho|^{N_0} \lambda(\eta - \rho, \rho) \overline{W}(\eta - \rho) \overline{W}(\rho) \right] \widehat{u}(\xi - \eta)m_{N_0}(\xi, \eta) \, d\xi d\eta d\rho \bigg|.
\]
Applying some changes of variables we get
\[
|I_3| \lesssim \left| \int_{\mathbb{R}^3} \overline{W}(\xi) \widehat{W}(\eta) \widehat{u}(\xi - \rho) \overline{V}(\rho - \eta)m_3(\xi, \eta, \rho) \, d\xi d\eta d\rho \right|,
\]
\[
m_3(\xi, \eta, \rho) := (\xi + \eta - \rho)|\xi + \rho|^{N_0} |\xi|^{-N_0} \lambda(\xi - \rho, \eta)m_{N_0}(\xi + \eta - \rho, \eta)
\]
\[
- \rho|\rho|^{N_0} |\rho|^{-N_0} \lambda(\rho - \eta, \eta)m_{N_0}(\xi, \rho).
\]
Estimating directly with the aid of Lemma 2.1(iii), (4.18), and (4.13) some smoother terms that can easily be dealt with, we can write
\[
|I_3(t)| \lesssim I_3^4(t) + \varepsilon_1^2(1 + t)^{-1} \|W(t)\|^2_{L^2} \tag{4.60}
\]
where
\[
I_3 = \left| \int_{\mathbb{R}^3} \overline{W}(\xi) \widehat{W}(\eta) \widehat{u}(\xi - \rho) \overline{V}(\rho - \eta)m_3(\xi, \eta, \rho) \, d\xi d\eta d\rho \right|,
\]
\[
m_3^4(\xi, \eta, \rho) := \rho \left[ \chi(\eta - \rho, \xi)m_{N_0}(\xi + \eta - \rho, \eta) - \chi(\rho - \eta, \eta)m_{N_0}(\xi, \rho) \right].
\]
The main observation is that the symbol $m_3^4$ above has a similar structure to the symbol $n_3$ in (4.48). In particular, the same computations done after (4.50), that used the property (4.51) and gave (4.53), show that
\[
m_3^4(\xi, \eta, \rho) = O \left( |\xi - \rho|^{3/2} + |\rho - \eta|^{3/2} \right) \chi(\eta - |\eta|) |\eta|/ |\rho - \eta| \right) \|W(t)\|^2_{L^2} \tag{4.61}
\]
Using this bound in combination with Lemma 2.1(iii), recalling the identity (4.30) for $V$, and using the a priori bounds (4.13), we get
\[
|I_3| \lesssim \sum_{k_1 - k_2 \leq 5, k_1 \geq k_3, k_2 \geq k_4} \left( 2^{k_3} + 2^{k_4} \right) \|P_{k_1}W\|_{L^2} \|P_{k_2}W\|_{L^2} \|P_{k_3}V\|_{L^\infty} \|P_{k_4}u\|_{L^\infty}
\]
\[
\lesssim \|W\|^2_{L^2} \sum_{k_1 - k_2 \leq 5, k_1 \geq k_3, k_2 \geq k_4} \left( 2^{k_3} + 2^{k_4} \right) \|P_{k_1}W\|_{L^2} \|P_{k_2}W\|_{L^2} \|P_{k_3}V\|_{L^\infty} \|P_{k_4}u\|_{L^\infty}
\]
\[
\lesssim \varepsilon_1^4 \|W\|^2_{L^2} (1 + t)^{-1}.
\]
This gives $|I_3| \lesssim \varepsilon_1^4(1 + t)^{-1+2\rho_0}$.

We conclude also this paragraph by recording a lemma that follows from the computations done above.

Lemma 4.9. Let $Q_F$ be defined according to (3.49), and let
\[
J(t) = \int_{\mathbb{R}^3} \left[ \overline{Q_F}(\xi) \widehat{F}(\eta) + \overline{F}(\xi) \overline{Q_F}(\eta) \right] \widehat{u}(\xi - \eta,t)m(\xi, \eta) \, d\xi d\eta,
\]
\[
\tag{4.63}
\]
where the symbol has the form
\[ m(\xi, \eta) = \frac{q(\xi, \eta)}{|\xi|^{3/2} + |\xi - \eta|^{3/2} - |\eta|^{3/2}}, \]
and is supported on a region where \(2^2|\xi - \eta| \leq |\eta|\). Assume that
\[ q(\xi, \eta) = O(|\xi - \eta|^{3/2} 1_{[2^2, \infty)}(|\eta|/|\xi - \eta|)), \]
and
\[ q(\xi, \rho) - q(\xi + \eta - \rho, \eta) = O\left(\frac{|\xi - \rho|^{5/2} + |\eta - \rho|^{5/2}}{|\xi| + |\eta|} 1_{[2^2, \infty)}(|\xi|/|\xi - \rho|) 1_{[2^2, \infty)}(|\eta|/|\xi - \rho|)\right), \]
whenever \(|\eta| \geq 2^2|\eta - \rho|\) and \(|\xi| \geq 2^2|\xi - \rho|\). Then, under the a priori decay assumption \(4.13\) on \(u\), we have
\[ |J(t)| \lesssim \|F\|^2 L^2 (1 + t)^{-1}. \]

**Lemma 4.9** above will be used later on to estimate terms of the form \(4.63\), like \(I_3\) in \(4.36\), that have a potential loss of one derivative, namely, \(I_{7, \epsilon_{1+}}\) in \(4.71\) and \(K_{1,2}\) in \(6.29\).

### Estimate of \(I_4\)
All of the terms in \(4.37\) do not lose derivatives and are not hard to estimate, given the symbol bound \(4.14\) on \(m_{N_0}\), the estimates on the nonlinear terms in Lemma \(C.5\) and \(4.51\). We just show how to estimate the term
\[ I_{4,1} := \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{N_0} \mathcal{N}_W(\xi) \hat{W}(\eta) \hat{u}(\xi - \eta) m_{N_0}(\xi, \eta) \, d\xi \, d\eta, \]
the other terms being similar or easier. Applying Lemma \(2.1(ii)\) followed by the symbol bound \(4.14\), and using the a priori estimates \(4.13\), and the estimate on the nonlinearity \(C.57\), we see that
\[
|I_{4,1}| \lesssim \sum_{k, k_1, k_2 \in \mathbb{Z}} \|m_{N_0}^{k, k_1, k_2}\|_{L^\infty} 2^{N_0 k} \|P'_{k_2} W\|_{L^2} \|P'_{k_1} W\|_{L^2} \|P'_{k_1} u\|_{L^\infty}
\]
\[ \lesssim \min_{(k, k_1, k_2) \in \mathbb{Z}, |k - k_2| \leq 5} 2^{k_1/2} \xi_1 2^{2 \min(k, 0)} (1 + t)^{-1/2 + p_0} \xi_1 (1 + t)^{p_0} 2^{k_1/10} 2^{-N_2 \max(k_1, 0)} (1 + t)^{-1/2}
\]
\[ \lesssim \xi_1^4 (1 + t)^{-1 + 2p_0}. \]

#### 4.4.2. Proof of Lemma 4.6
Recall from \(4.23\) that
\[ A_3 = \frac{1}{2\pi} \Re \int_{\mathbb{R}} \hat{W}(\xi) |\xi|^{N_0} \mathcal{N}_W(\xi) \, d\xi, \]
where \(\mathcal{N}_W\) is defined in \(3.50\). Recall our definitions of the energies in \(E_{a, \epsilon_{1\epsilon_2}}(t)\) and \(E_{b, \epsilon_{1\epsilon_2}}(t)\) in \(4.9\) and \(4.10\), and the notation \(3.41\). Our aim is to show
\[
|A_3(t) + \frac{d}{dt} \sum_{\epsilon} \left( E_{a, \epsilon_{1\epsilon_2}}(t) + E_{b, \epsilon_{1\epsilon_2}}(t) \right) | \lesssim \xi_1^4 (1 + t)^{-1 + 2p_0}. \]

Calculating as in the previous section, using the evolution equations for \(W\) in \(4.4\), we see that for each \((\epsilon_1, \epsilon_2) \in \{(+, +), (+, -), (-, +), (-, -)\}\)
\[
\frac{d}{dt} E_{a, \epsilon_{1\epsilon_2}} = I_{5, \epsilon_{1\epsilon_2}} + I_{6, \epsilon_{1\epsilon_2}} + I_{7, \epsilon_{1\epsilon_2}} + I_{8, \epsilon_{1\epsilon_2}}
\]
where
\[
I_{5, \epsilon_1 \epsilon_2} = \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \mathcal{W}(\xi) \mathcal{W}_{\epsilon_2}(\eta) \hat{u}_{\epsilon_1}(\xi - \eta) d\xi d\eta
\]
(4.69)
\[
I_{6, \epsilon_1 \epsilon_2} = \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \left[ i\Sigma_\gamma \mathcal{W}(\xi) \mathcal{W}_{\epsilon_2}(\eta) + \mathcal{W}(\xi) (i\Sigma_\gamma) \mathcal{W}_{\epsilon_2}(\eta) \right] \hat{u}_{\epsilon_1}(\xi - \eta) a_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta,
\]
(4.70)
\[
I_{7, \epsilon_1 \epsilon_2} = \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \left[ \mathcal{Q} \mathcal{W}(\xi) \mathcal{W}_{\epsilon_2}(\eta) + \mathcal{W}(\xi) \mathcal{Q}_{\epsilon_2}^\gamma(\eta) \right] \hat{u}_{\epsilon_1}(\xi - \eta) a_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta,
\]
(4.71)
\[
I_{8, \epsilon_1 \epsilon_2} = \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \left[ \mathcal{W}(\xi) \mathcal{W}_{\epsilon_2}(\eta) \mathcal{F}(\partial_\xi u_{\epsilon_1} - \epsilon_1 i|\partial_\xi|^{3/2} u_{\epsilon_1}) (\xi - \eta) a_{\epsilon_1 \epsilon_2}^N(\xi, \eta) \right] d\xi d\eta
\]
(4.72)
\[
+ \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \mathcal{W}(\xi) \mathcal{W}_{\epsilon_2}(\eta) \mathcal{F}(\partial_\xi u_{\epsilon_1} - \epsilon_1 i|\partial_\xi|^{3/2} u_{\epsilon_1}) (\xi - \eta) a_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta
\]
\[
+ \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \mathcal{W}(\xi) \mathcal{W}_{\epsilon_2}(\eta) \mathcal{F}(\partial_\xi u_{\epsilon_1} - \epsilon_1 i|\partial_\xi|^{3/2} u_{\epsilon_1}) (\xi - \eta) a_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta.
\]
Here \( \mathcal{N}^+_W = \mathcal{N}_W \) and \( \mathcal{N}^-_W = \overline{\mathcal{N}_W} \). Similarly
\[
\frac{d}{dt} \mathcal{F}_{\gamma}(\xi, \eta) = I_{9, \epsilon_1 \epsilon_2} + I_{10, \epsilon_1 \epsilon_2}
\]
where
\[
I_{9, \epsilon_1 \epsilon_2} = \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \mathcal{W}(\xi) \hat{u}_{\epsilon_2}(\eta) \hat{u}_{\epsilon_1}(\xi - \eta)
\]
(4.73)
\[
I_{10, \epsilon_1 \epsilon_2} = \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \mathcal{F}(\partial_\xi W - i|\partial_\xi|^{3/2} W) (\xi) \hat{u}_{\epsilon_2}(\eta) \hat{u}_{\epsilon_1}(\xi - \eta) b_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta
\]
(4.74)
\[
+ \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \mathcal{W}(\xi) \mathcal{F}(\partial_\xi u_{\epsilon_1} - \epsilon_2 i|\partial_\xi|^{3/2} u_{\epsilon_1}) (\eta) \hat{u}_{\epsilon_1}(\xi - \eta) b_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta
\]
\[
+ \frac{1}{8\pi^2} \Re \int_{\mathbb{R} \times \mathbb{R}} \mathcal{W}(\xi) \hat{u}_{\epsilon_2}(\eta) \mathcal{F}(\partial_\xi u_{\epsilon_1} - \epsilon_1 i|\partial_\xi|^{3/2} u_{\epsilon_1}) (\xi - \eta) b_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta.
\]
We then see that
\[
A_3 + \frac{d}{dt} \sum_\star \left( \mathcal{F}_{
u_{\epsilon_1 \epsilon_2}}^3 + \mathcal{F}_{b_{\epsilon_1 \epsilon_2}}^3 \right) = \sum_\star \left( I_{6, \epsilon_1 \epsilon_2} + I_{7, \epsilon_1 \epsilon_2} + I_{8, \epsilon_1 \epsilon_2} + I_{10, \epsilon_1 \epsilon_2} \right).
\]
We show below how to estimate all the terms on the right-hand side above by \( \mathcal{E}_\delta^4(1 + t)^{-1+2p_0} \).

**Estimate of \( I_{6, \epsilon_1 \epsilon_2} \).** We start by looking at the case when \( \epsilon_2 = -1 \). We see from (1.19) that the symbols \( a_{\epsilon_1 \epsilon_2}^N \) have a strong ellipticity. We will now show how the potential loss of \( 3/2 \) derivatives coming from \( \Sigma_\gamma W \) is depleted by such smoothing property of the symbols. Looking at the term (4.70), and in view of the symmetries, we see that it is enough to estimate the expression
\[
I_6' := \left| \int_{\mathbb{R} \times \mathbb{R}} \Sigma_\gamma \mathcal{W}(\xi) \hat{W}(\eta) \hat{u}_{\epsilon_1}(\xi - \eta) a_{\epsilon_1 \epsilon_2}^N(\xi, \eta) d\xi d\eta \right|.
\]
(4.75)
Writing out \( \Sigma_\gamma \) explicitly, and making two simple changes of variables, we have
\[
I_6' = \left| \int_{\mathbb{R}^3} \mathcal{W}(\xi) \hat{W}(\eta) \hat{u}_{\epsilon_1}(\rho + \eta) \hat{\gamma}(\xi - \rho) \chi(\rho - \xi, \xi) |\xi|^{3/2} a_{\epsilon_1 \epsilon_2}^N(\rho, -\eta) d\xi d\eta d\rho \right|.
\]
This is a quartic expression with symbol
\[ m(\xi, \eta, \rho) = \chi(\rho - \xi, \xi)|\xi|^{3/2}a_{\epsilon_1}(\rho, -\eta) \]
Using the second bound in (4.19), and integration by parts, we can see that
\[
\|m^{k_1,k_2,k_3,k_4}\|_{S^\infty} \lesssim 2^{3k_1/2}2^{3k_2/2}2^{3k_3/2}2^{3k_4/2}1_{[2,\infty)}(k_2) + 2^{k_1/2}2^{k_2/2}2^{k_3/2}2^{k_4/2}1_{(-\infty,1)}(k_2)
\times 1_{[3,\infty)}(k_2 - k_4)1_{[3,\infty)}(k_1 - k_3)1_3(k_1,k_2,k_3,k_4),
\] (4.76)
where we are using the notation (2.12)-(2.13). We then apply Lemma 2.1(iii) together with the a priori bounds (4.13), to estimate
\[
|I_6'| \lesssim \sum_{k_1,k_2,k_3,k_4 \in \mathbb{Z}}\|m^{k_1,k_2,k_3,k_4}\|_{S^\infty}\|P'_{k_1}W\|_{L^2}\|P'_{k_2}W\|_{L^2}\|P'_{k_3}\gamma\|_{L^\infty}\|P'_{k_4}u\|_{L^\infty}
\lesssim \varepsilon_1^2(1 + t)^{-7/6}\sum_{|k_1-k_2| \leq 5, k_1 \geq k_3, k_2 \geq k_4, k_2 \geq 2}2^{3(k_1+k_4-k_2)/2}\|P'_{k_1}W\|_{L^2}\|P'_{k_2}W\|_{L^2}
+ \varepsilon_1^4(1 + t)^{-7/6+2\rho_0}\sum_{|k_1-k_2| \leq 5, k_1 \geq k_3, k_2 \geq k_4, k_2 \leq 1}2^{3k_1/2}(k_4-k_2)/2 2^{k_3/10}2^{k_4/10}
\lesssim \varepsilon_1^4(1 + t)^{-1}.\]
This is more than sufficient to show the desired bound
\[
|I_{6,\epsilon_1-}(t)| \lesssim \varepsilon_1^4(1 + t)^{-1+2\rho_0}.\] (4.77)
In the cases \((\epsilon_1, \epsilon_2) \in \{(+,+,(-,+))\}\) we have
\[
|I_{6,\epsilon_1+}| \lesssim \left|\int_{\mathbb{R} \times \mathbb{R}} \left[ -\Sigma \tilde{W}(\xi)\tilde{W}(\eta) + \tilde{W}(\xi)\Sigma \tilde{W}(\eta) \right] \tilde{a}_{\epsilon_1}(\xi - \eta) a_{\epsilon_1+}^N(\xi, \eta) d\xi d\eta \right|.\] (4.78)
Notice that this is of the form (4.54)-(4.55), with \(F = W, m = a_{\epsilon_1+}^N, q(\xi, \eta) = |\xi|^N|\eta|^{-N}a_{\epsilon_1+}(\xi, \eta)\).
We can then apply Lemma 4.8 provided we verify its assumptions for \(\epsilon_{\epsilon_1+}^N\). Observe first that the symbols \(a_{\epsilon_1+}^N(\xi, \eta)\) are supported on a region where \(2^0|\xi - \eta| \leq |\eta|\). The bound (A.10) for the symbol \(a_{\epsilon_1+}(\xi, \eta)\) gives the property in (4.56). Moreover, the properties (A.14) show that the second assumption (4.57) is satisfied. Applying Lemma 4.8 to \(I_{6,\epsilon_1+}\) gives us the desired bound
\[
|I_{6,\epsilon_1+}(t)| \lesssim \varepsilon_1^4(1 + t)^{-1+2\rho_0}.\]

*Estimate of \(I_{7,\epsilon_1+}\).* We first look at the case \(\epsilon_2 = -1\). Using the formula for \(I_{7,\epsilon_1+}\) in (4.71), the definition of the symbols \(a_{\epsilon_1+}^N\) in (4.72), and the second bound in (4.19), it is not hard to see that an argument similar to the one that gave (4.77) can be applied to obtain
\[
|I_{7,\epsilon_1-}(t)| \lesssim \varepsilon_1^4(1 + t)^{-1+2\rho_0}.\]
In the cases \((\epsilon_1, \epsilon_2) \in \{(+,+,(-,+))\}\) one has
\[
|I_{7,\epsilon_1+}| \lesssim \left|\int_{\mathbb{R} \times \mathbb{R}} \left[ \Sigma \tilde{W}(\xi)\tilde{W}(\eta) + \tilde{W}(\xi)\Sigma \tilde{W}(\eta) \right] \tilde{a}_{\epsilon_1}(\xi - \eta) a_{\epsilon_1+}^N(\xi, \eta) d\xi d\eta \right|.\]
This term is of the form (4.63), with \(F = W, m = a_{\epsilon_1+}^N\), and \(q(\xi, \eta) = |\xi|^N|\eta|^{-N}a_{\epsilon_1+}(\xi, \eta)\). Since the symbols \(a_{\epsilon_1+}^N\) are supported on the region \(2^0|\xi - \eta| \leq |\eta|\), and that the bounds (A.10) for \(a_{\epsilon_1+}^N\)...
hold, together with the properties (A.14), the hypotheses of Lemma 4.9 are verified. (4.67) then gives us

\[ |I_{\gamma}(t)| \lesssim \varepsilon_1^4(1 + t)^{-1+2p_0}. \]

**Estimate of I_{8,\epsilon_1,\epsilon_2}**. Observe that all the terms in (4.72) do not lose derivatives, and moreover they are similar to the terms in (4.37). By performing the same estimates above one sees that

\[ |I_{8,\epsilon_1,\epsilon_2}(t)| \lesssim \varepsilon_1^4(1 + t)^{-1+2p_0}. \]

**Estimate of I_{10,\epsilon_1,\epsilon_2}**. We notice that the symbols \( b_{i\epsilon_2}^N \) are smoothing and non-singular, see the definition (4.10) and the bound (4.20). Therefore, there are no losses of derivatives in (4.74) and all these terms are straightforward to estimate, so we can skip the details.

**4.4.3. Proof of Lemma 4.7** The term \( A_1 \) in (4.21) can bounded as desired because \( \Sigma_\gamma \) has been constructed as a symmetric operator up to order \(-1/2\). To see that this is indeed the case, recall the definition of \( \Sigma_\gamma \) in (4.4), and write

\[
2\pi A_1 = \Re \int_{\mathbb{R}} \overline{W}(\xi) i P_{\geq 1} W(\eta) \hat{\gamma}(\xi - \eta) |\eta|^{3/2} \chi(\xi - \eta, \eta) \, d\eta d\xi \\
+ \Re \int_{\mathbb{R}} \overline{W}(\xi) i P_{\geq 1} W(\eta) \hat{\gamma}(\xi - \eta) \frac{3}{4} \frac{(\xi - \eta)\eta}{|\eta|^{1/2}} \chi(\xi - \eta, \eta) \, d\eta d\xi \\
= \Re \int_{\mathbb{R}} \overline{W}(\xi) W(\eta) \hat{\gamma}(\xi - \eta) i \left[ \frac{1}{2} \left( |\eta|^{3/2} \varphi_{\geq 1}(\eta) \chi(\xi - \eta, \eta) - |\xi|^{3/2} \varphi_{\geq 1}(\xi) \chi(\xi - \xi, \xi) \right) \\
+ \frac{3}{4} \frac{(\xi - \eta)\eta}{|\eta|^{1/2}} \chi(\xi - \eta, \eta) \right] \, d\eta d\xi.
\]

Notice that the symbol in the above expression is \( O((\xi - \eta)^2(|\xi| + |\eta|)^{-1/2}) \cdot 1_{[2^3, \infty]}(|\eta|/|\xi - \eta|) \). We can then proceed, as done several times before, using Lemma 2.1(ii) and (4.46), and obtain

\[
|A_1| \lesssim \sum_{(k,k_1,k_2) \in X, |k-k_2| \leq 5} 2^{2k_1} 2^{-k/2} \| P_k^W \|_{L^2} \| P_{k_2}^W \|_{L^2} \| P_k \gamma \|_{L^\infty} \\
\lesssim \| W(t) \|_{L^2} \sum_{(k,k_1,k_2) \in X, |k-k_2| \leq 5} 2^{2k_1} 2^{-k/2} 2^{k_1/10} 2^{-N_2 \max(k,0)} \varepsilon_1^2(1 + t)^{-1} \tag{4.79}
\]

Using the cubic estimates on the term \( O_W \) in (3.51), it is easy to see that also the term \( A_4 \) in (4.23) satisfies the desired bound. We have then completed the proof of Lemma 4.7 and hence of Lemma 4.3 Proposition 4.1 is proved. \( \square \)

5. Energy estimates II: Low frequencies

5.1. The basic low frequency energy. In this section we exploit the null structure of the equation to control the low frequency component of the solution \( u \) which we denote by

\[
u_{\text{low}} := P_{\leq -10} u. \tag{5.1}
\]

For this purpose we define the Energy

\[
E^{(2)}_{\text{low}}(t) = \frac{1}{4\pi} \int_{\mathbb{R}} |\hat{u}_{\text{low}}(t, \xi)|^2 |\xi|^{-1} \mathcal{P}((1 + t)^2 |\xi|) \, d\xi
\]
where we recall from (2.22) that \( P : [0, \infty) \rightarrow [0, 1] \) is an increasing function, smooth on \((0, \infty), \) such that
\[
P(x) = x^{2p_1} \quad \text{if} \quad x \leq 1/2, \quad P(x) = 1 \quad \text{if} \quad x \geq 1, \quad xP'(x) \leq 10p_1P(x). \quad (5.3)
\]
With this definition we have
\[
E^{(2)}_{\text{low}}(0) \approx \|\varphi_{\leq -10}(\xi)|\xi|^{-1/2 + p_1}w_0\|^2_{L^2} \leq \varepsilon^2_0. \quad (5.4)
\]

5.2. **The cubic low frequency energy.** Recall from Proposition 3.4 the equation (3.42)-(3.43), from which it follows that
\[
\partial_t u_{\text{low}} - i|\partial_x|^{3/2} u_{\text{low}} = P_{\leq -10}(-\partial_x T_V u + \mathcal{N}_u) + |\partial_x|^{1/2}O_{3,1/2}. \quad (5.5)
\]
According to (5.5), we naturally define the cubic correction to the basic energy \( E^{(2)}_{\text{low}} \) as follows:
\[
E^{(3)}_{\text{low}}(t) := \sum_k \frac{1}{4\pi^3} \Re \int_{\mathbb{R} \times \mathbb{R}} \overline{u}(t, \xi)|\xi|^{-1/2} \varphi_{\xi_1}(t, \xi-\eta) \varphi_\eta(t, \eta)m^{\text{low}}_{\epsilon_1 \epsilon_2}(\xi, \eta) d\xi d\eta, \quad (5.6)
\]
where the symbol is
\[
m^{\text{low}}_{\epsilon_1 \epsilon_2}(\xi, \eta) := (1 + \epsilon_1)(1 + \epsilon_2) \frac{|\xi|^{1/2}(\xi - \eta)|\xi|^{-1/2} \varphi_\xi(\xi - \eta, \eta) - \eta|\eta|^{-1} \varphi_\eta(\eta - \xi, \xi)}{8|\xi - \eta|^{1/2}(|\xi|^{3/2} - |\xi - \eta|^{3/2} - |\eta|^{3/2})}
- i \frac{\varphi_\xi(\xi, \eta) + b_{\epsilon_1 \epsilon_2}(\xi, \eta)}{|\xi|^{1/2}(|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2 |\eta|^{3/2})}, \quad (5.7)
\]
and we have denoted
\[
\varphi_\xi(\xi) = \varphi_{\leq -2 \log_2(2+t)}(\xi) \varphi_{\leq -10}(\xi).
\]
The first part of \( m^{\text{low}}_{\epsilon_1 \epsilon_2} \) is non-zero only for \( \epsilon_1 = \epsilon_2 = 1, \) and takes into account the nonlinear term \(-\partial_x T_V u.\) The symbol in the second line of (5.7) is needed to correct the nonlinear terms in \( \mathcal{N}_u.\) Notice that no correction is needed if \(|\xi| \ll (1 + t)^{-2}.\)

The total energy for the low frequency part of the solution is given by
\[
E_{\text{low}} := E^{(2)}_{\text{low}} + E^{(3)}_{\text{low}}(t). \quad (5.8)
\]
The main proposition in this section is the following:

**Proposition 5.1.** Assume that \( u \) satisfies the a priori assumptions
\[
\sup_{t \in [0, T]} [(1 + t)^{-p_0}\|u(t)\|_{H^{N_0}} + (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{-k/10} + 2^{N_2k})\|P_ku(t)\|_{L^\infty}) \leq \varepsilon_1, \quad (5.9)
\]
and the initial data assumption
\[
\|\partial_x|^{-1/2 + p_1}u_0\|_{L^2} \leq \varepsilon_0. \quad (5.10)
\]
Assume also that
\[
\sup_{t \in [0, T]} (1 + t)^{-2p_0}E^{(2)}_{\text{low}}(t) \leq \varepsilon_1^2. \quad (5.11)
\]
Then there is some constant \( C \) such that
\[
\sup_{t \in [0, T]} (1 + t)^{-2p_0}E^{(2)}_{\text{low}}(t) \leq C \varepsilon^2_0 + \varepsilon^2_1/100. \quad (5.12)
\]
Lemma 5.3. Under the assumptions of Proposition 5.1, for any $k, k'$

$$||\varphi_{\geq -2\log_2(2+t)}(\cdot)\cdot|^{1/2}\hat{u}(t, \cdot)||_{L^2} \lesssim \left[E^{(2)}_{\text{low}}(t)\right]^{1/2} + ||u(t)||_{L^2} \lesssim \varepsilon_1(1 + t)^{p_0},$$

(5.13)

Thus, the conclusion (5.12) implies

$$\sup_{t \in [0, T]} (1 + t)^{-p_0} ||\varphi_{\geq -2\log_2(2+t)}(\cdot)\cdot|^{1/2}\hat{u}(t)||_{L^2} \lesssim \varepsilon_0,$$

(5.14)

$$\sup_{t \in [0, T]} (1 + t)^{-p_0 + 2p_1} ||\varphi_{\leq -2\log_2(2+t)}(\cdot)\cdot|^{1/2}\hat{u}(t)||_{L^2} \lesssim \varepsilon_0,$$

(5.15)

which gives us the desired improved control.

As in section 4, Proposition 5.1 follows from two main lemmas.

Lemma 5.3. Under the assumptions of Proposition 5.1, for any $t \in [0, T]$, we have

$$|E_{\text{low}}^{(3)}(t)| \lesssim \varepsilon_1^3(1 + t)^{2p_0}.$$  

(5.16)

Lemma 5.4. Under the assumptions of Proposition 5.1, for any $t \in [0, T]$, we have

$$\frac{d}{dt}E_{\text{low}}(t) \leq C\varepsilon_1^3(1 + t)^{-1+2p_0} + 40p_1\varepsilon_1^2(1 + t)^{-1+2p_0}.$$  

(5.17)

5.3. Analysis of the symbols and proof of Lemma 5.3. We first show that the symbol in

(5.7) satisfies the bounds

$$||m_{\epsilon_{1 \leq 2}}^{\text{low}}(k, k_1, k_2)||_{S^\infty} \lesssim 2^{-\max(k_1, k_2)/2} \chi(k, k_1, k_2)1_{[-2\log_2(2+t) - 10, 0]}(k)$$

(5.18)

and

$$||m_{\epsilon_{1 \leq 2}}^{\text{low}}(\xi - \eta)\varphi_{k_2}(\eta)||_{S^\infty} \lesssim 2^{-\max(k_1, k_2)/2},$$

(5.19)

for all $k, k_1, k_2 \in \mathbb{Z}$.

Let us start with the first component of $m_{\epsilon_{1 \leq 2}}^{\text{low}}$, which we denote by

$$n_1(\xi, \eta) := \frac{|\xi|^{1/2}(\xi - \eta)|\xi|^{-1}\varphi_t(\xi)\chi(\xi - \eta, \eta) - \eta|\xi|^{-1}\varphi_t(\eta)\chi(\eta - \xi, \xi)|}{2|\xi - \eta|^{1/2}|\xi|^{3/2} - |\xi|^3/2 - |\eta|^{3/2}}.$$  

Since $\chi(\xi - \eta, \eta)$ forces $2^3|\xi - \eta| \leq |\eta|$, through Taylor expansions and standard integration by parts, we see that

$$\frac{\xi|\xi|^{-1}\varphi_t(\xi)\chi(\xi - \eta, \eta) - \eta|\xi|^{-1}\varphi_t(\eta)\chi(\eta - \xi, \xi)|}{|\xi|^{3/2} - |\xi - \eta|^{3/2} - |\eta|^3/2} = O(|\eta|^{-3/2}1_{[2^3, \infty)}(|\eta|/|\xi - \eta|)1_{(1+t)^{-2}, 0}(\xi)).$$

and deduce

$$n_1(\xi, \eta) = O(|\xi - \eta|^{1/2})|\eta|^{1}1_{[2^3, \infty)}(|\eta|/|\xi - \eta|)1_{(1+t)^{-2}, 0}(\xi)).$$

Let us denote

$$n_2(\xi, \eta) := -i\frac{\varphi_t(\xi)\alpha_{\epsilon_{1 \leq 2}}(\xi, \eta)}{|\xi|^{1/2}(|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2})}.$$  

Using the bound (A.12) in Lemma A.1, one can see that

$$n_2(\xi, \eta) = O(|\xi - \eta|^{1/2})|\xi|^{-1}1_{[2^3, \infty)}(|\eta|/|\xi|)1_{((1+t)^{-2}, 0}(\xi)).$$
which is a better bound than what we need. We then look at
\[ n_3(\xi, \eta) := -i \frac{\varphi_\xi(\xi) b_{\epsilon_1 \epsilon_2}(\xi, \eta)}{|\xi|^{1/2}(|\xi|^{3/2} - \epsilon_1 |\xi - \eta|^{3/2} - \epsilon_2 |\eta|^{3/2})}, \]
where the symbols \( b_{\epsilon_1 \epsilon_2} \) are defined in (3.35)–(3.39). Using the bound (A.26) in Lemma A.2, we know that on the support of \( b_{\epsilon_1 \epsilon_2} \) the sizes of \( |\eta| \) and \( |\xi - \eta| \) are comparable, and so it follows that
\[ n_3(\xi, \eta) = O(\max(|\xi - \eta|, |\xi|)^{-1/2} 1_{|\eta|/|\xi - \eta| < 2^{10}}(1_{|\xi|/|\eta| < 2^{10}})). \] (5.20)
This suffices to obtain (5.18). Similar arguments, using the formulas (A.5)–(A.8), (A.21)–(A.24), and integration by parts, also give (5.19).

We now use (5.18) to prove (5.16). Let us denote
\[ k_- := \min(k_1, k_2), \quad k_+ := \max(k_1, k_2). \] (5.21)
Using the formula (5.6), the bound (5.18) together with Lemma 2.1(i), we estimate
\[ |E^{(3)}_{\text{low}}(t)| \lesssim \sum_{k, k_1, k_2} \sum_{(1+t)^{-2} \leq |k|^{3/2} \leq 2|k| \leq 2^{10}} \left( ||(m_{\epsilon_1 \epsilon_2})^{\text{low}} ||_{\infty} 2^{-k/2} \left\| P^j_k u \right\|_{L^2} \left\| P^j_{k_+} u \right\|_{L^2} \right) \]
Using the a priori assumptions (5.9) and (5.11), and (5.13), we obtain
\[ |E^{(3)}_{\text{low}}(t)| \lesssim \sum_{(k, k_1, k_2) \in \mathcal{K}, (1+t)^{-2} \leq 2k^{10}} 2^{-k/2} \left\| P^j_k u \right\|_{L^2} 2^{-k_+/2} \left\| P^j_{k_+} u \right\|_{L^2} 2^{k/10} 2^{-N_2 \max(k_+, 0)} (1 + t)^{-1/2} \lesssim \eta^3 (1 + t)^{-1/3}. \]

5.4. Proof of Lemma 5.4. Using the definitions (5.2), (5.3), (5.6), the equation (5.5), and a symmetrization argument similar to the one performed for the term (1.22) and leading to (1.31), we calculate
\[ \frac{d}{dt}(E^{(2)}_{\text{low}} + E^{(3)}_{\text{low}}) = \frac{1}{4\pi} J_1 + \Re \left[ \frac{1}{2\pi} J_2 + \frac{1}{4\pi^2} \sum_{*} (J_{3,\epsilon_1 \epsilon_2} + J_{4,\epsilon_1 \epsilon_2} + J_{5,\epsilon_1 \epsilon_2} + J_{6,\epsilon_1 \epsilon_2}) \right] + R, \]
where
\[ J_1 = \int_{\mathbb{R}} |\widehat{u_{\text{low}}}(\xi)|^2 \varphi'((1 + t)^2|\xi|) 2(1 + t) d\xi, \]
\[ J_2 = \int_{\mathbb{R}} \overline{u_{\text{low}}(\xi)} \mathcal{F}(\partial_t u_{\text{low}} - iA u_{\text{low}})(\xi) \varphi_{\leq -2 \log_2(2+t)^{-1}}(\xi) \xi^{-1/2} \eta^{\epsilon_1 \epsilon_2} (\xi - \eta) \eta \eta d\xi d\eta, \]
\[ J_{3,\epsilon_1 \epsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \eta^\epsilon_2 (\xi) \partial_t m_{\epsilon_1 \epsilon_2}^{\text{low}}(\xi, \eta) \widehat{u}_{\epsilon_1}(\xi - \eta) \widehat{u}_{\epsilon_2}(\eta) d\xi d\eta, \]
\[ J_{4,\epsilon_1 \epsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \mathcal{F}(\partial_t u - i\Lambda u)(\xi) m_{\epsilon_1 \epsilon_2}^{\text{low}}(\xi, \eta) \widehat{u}_{\epsilon_1}(\xi - \eta) \widehat{u}_{\epsilon_2}(\eta) d\xi d\eta, \]
\[ J_{5,\epsilon_1 \epsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \varphi(\xi) m_{\epsilon_1 \epsilon_2}^{\text{low}}(\xi, \eta) \mathcal{F}(\partial_t u_{\epsilon_1} - i\lambda_{\epsilon_1} \Lambda u_{\epsilon_1})(\xi - \eta) \widehat{u}_{\epsilon_2}(\eta) d\xi d\eta, \]
\[ J_{6,\epsilon_1 \epsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \varphi(\xi) m_{\epsilon_1 \epsilon_2}^{\text{low}}(\xi, \eta) \widehat{u}_{\epsilon_1}(\xi - \eta) \mathcal{F}(\partial_t u_{\epsilon_2} - i\lambda_{\epsilon_2} \Lambda u_{\epsilon_2})(\eta) d\xi d\eta. \]
Here \( R \) denotes a quartic term which includes the contribution from \( |\partial_x|^{1/2} O_{3,1/2} \) in (5.5), and from the quadratic part of \( V \), that is \( V_2 \) in (4.30), and can be easily seen to satisfy \( |R(t)| \lesssim \eta^4 (1 + t)^{-1/2} \). The term \( J_1 \) comes from differentiating the cutoff \( \varphi \) in the quadratic energy, whereas \( J_2 \) is obtained when we differentiate \( |\widehat{u_{\text{low}}}|^2 \) in the region \( |\xi| \leq (1 + t)^{-2} \). The term \( J_{3,\epsilon_1 \epsilon_2} \) comes from differentiating the symbol in the cubic energy functional. The remaining three terms in
are the net result of differentiating the quadratic energy when \(|\xi| \geq (1 + t)^{-2}\), and its cubic correction, after symmetrizations and cancellations. To obtain Lemma 5.4 it suffices to prove that each of the terms in (5.22) is bounded by \(\varepsilon_1^3(1 + t)^{-1+2p}\).

5.4.1. Estimate of \(J_1\). Using (5.3), the definition of \(E_{low}^{(2)}\) in (5.2), and the a priori assumption (5.11), we immediately see that

\[
\frac{1}{4\pi} |J_1| \leq \frac{1}{4\pi} \int_R \left| \overline{w_{low}}(t, \xi) \right|^2 \frac{20p_1}{(1 + t)} |\xi| \mathcal{P}((1 + t)^2|\xi|) d\xi \leq \frac{20p_1}{1 + t} E_{low}^{(2)} \leq 20p_1 \varepsilon_1^2 (1 + t)^{-1+2p} ,
\]

as desired.

5.4.2. Estimate of \(J_2\). Notice that the term \(J_2\) in (5.22) is supported on a region where \(|\xi| \lesssim (1 + t)^{-2}\). Then, we can bound the nonlinearity (C.56), and the second bound in (5.13), to see that

\[
\begin{align*}
|J_2| &\lesssim \sum_{2^k \leq (1 + t)^{-2}} 2^{(-1+2p_1)k} \|P_k u(t)\|_{L^2} \|P_k' (\partial_t u - i\Lambda u)(t)\|_{L^2} (1 + t)^{4p_1} \\
&\lesssim (1 + t)^{4p_1} \sum_{2^k \leq (1 + t)^{-2}} 2^{(-1+2p_1)k} \|P_k u(t)\|_{L^2} \varepsilon_1^2 2^{k/2} (1 + t)^{-1/2+p_0} \\
&\lesssim (1 + t)^{-1/2+p_0+4p_1} \varepsilon_1^3 \sum_{2^k \leq (1 + t)^{-2}} 2^{2p_1 k} \\
&\lesssim \varepsilon_1^3 (1 + t)^{-1+2p_0} ,
\end{align*}
\]

5.4.3. Estimate of \(J_{3, \epsilon_1 \epsilon_2}\). Recall the notation \(k_- := \min(k_1, k_2)\), \(k_+ := \max(k_1, k_2)\). Using the definition of \(\tilde{m}_{k_1 k_2}^{low}\) in (5.7), and the same arguments in the proof of Lemma 5.3 that gave us (5.18), one can see that

\[
\| (\partial_t m_{\epsilon_1 \epsilon_2})^{k_1, k_2} \|_{S^\infty} \lesssim 1_{X}(k_1, k_2)(1 + t)^{-2} 2^{-k_+/2} 1_{(-2 \log_{2}(2+t)-10,0)}(k).
\]

We can then use Lemma 2.4 (ii) with the above bound, to obtain, for every \(\epsilon_1, \epsilon_2 = \pm\),

\[
\begin{align*}
|J_{3, \epsilon_1 \epsilon_2}| &\lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \| (\partial_t m_{\epsilon_1 \epsilon_2})^\epsilon \|_{S^\infty} 2^{-k} \|P_k' u\|_{L^2} \|P_{k_1} u\|_{L^2} \|P_{k_-} u\|_{L^\infty} \\
&\lesssim \varepsilon_1^2 (1 + t)^{-2} \sum_{(k_1, k_2) \in X, (1 + t)^{-1} \leq 2^k \leq 1} 2^{-k/2} \|P_k' u\|_{L^2} 2^{-k_+/2} \|P_{k_+} u\|_{L^2} \\
&\quad \times \varepsilon_1 2^{k-10} 2^{-N_2 \max(k_-,0)} (1 + t)^{-1/2} \\
&\lesssim \varepsilon_1^3 (1 + t)^{-1} ,
\end{align*}
\]

having also used (5.9), and the first inequality in (5.13).

5.4.4. Estimate of \(J_{4, \epsilon_1 \epsilon_2}\). We use Lemma 2.1 (ii) to estimate, for every \(\epsilon_1, \epsilon_2 = \pm\),

\[
|J_{4, \epsilon_1 \epsilon_2}| \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \| m_{\epsilon_1 \epsilon_2}^{low} \|_{S^\infty} 2^{-k} \|P_k' \partial_k - i\Lambda u\|_{L^2} \|P_{k_1} u\|_{L^2} \|P_{k_-} u\|_{L^\infty} .
\]

Using the bound (5.18) for the symbol, (C.56) for the nonlinearity, and (5.13), we have

\[
|J_{4, \epsilon_1 \epsilon_2}| \lesssim \varepsilon_1^3 (1 + t)^{-1+p_0} \sum_{(k_1, k_2) \in X, (1 + t)^{-2} \leq 2^k \leq 1} 2^{-k_+/2} \|P_{k_+} u\|_{L^2} 2^{-k/10} 2^{-N_2 \max(k_-,0)} \\
\quad \lesssim \varepsilon_1^4 (1 + t)^{-1+2p_0} .
\]
5.4.5. Estimates of $J_{5,\epsilon_1 \epsilon_2}$ and $J_{6,\epsilon_1 \epsilon_2}$. Since the terms $J_{5,\epsilon_1 \epsilon_2}$ and $J_{6,\epsilon_1 \epsilon_2}$ are similar, we only estimate the first one. We begin again by using Lemma \ref{lemma2.2}(ii), and estimate, for every $\epsilon_1, \epsilon_2 = \pm$,

$$|J_{5,\epsilon_1 \epsilon_2}| \lesssim \|\varphi_{\geq -2 \log_2(1+t) - 10}|^{-1/2} \hat{u}\|_{L^2} \sum_{k_1, k_2} \|m_{\epsilon_1 \epsilon_2} \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta)\|_{S_{\infty}} \times \|P_{k_1}(\partial_t - i\Lambda) u\|_{L^\infty} \|P_{k_2} u\|_{L^2}.$$ 

We then use the symbol bound \ref{5.1}, the estimates \ref{5.56}, \ref{5.13} and the a priori bounds \ref{5.9} and \ref{5.11}, to obtain

$$|J_{5,\epsilon_1 \epsilon_2}| \lesssim \epsilon_1 (1 + t)^{-1 + 2p_0} \sum_{k_1, k_2} 2^{-k_1/10} 2^{k_1/2} 2^{-2 \log_2(1+t)} 2^{2N_2 - 2} \max(k_1, 0) \epsilon_1 2^{(1/2 - p_0)k_2} 2^{-N} \max(k_2, 0) \lesssim \epsilon_1^4 (1 + t)^{-1 + 2p_0}.$$ 

The proof of Lemma \ref{5.4} is completed, and Proposition \ref{5.1} follows. \hfill \Box

6. Energy estimates III: Weighted estimates for high frequencies

Let us recall the main a priori assumptions

$$\sup_{t \in [0, T]} \left[ (1 + t)^{-p_0} \|u(t)\|_{H^{N_0}} + (1 + t)^{-4p_0} \|Su(t)\|_{H^{N_1}} + (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{-k/10} + 2^{N_2 k}) \|P_k u(t)\|_{L^\infty} \right] \leq \epsilon_1, \tag{6.1}$$

and the low frequency a priori assumptions

$$\sup_{t \in [0, T]} (1 + t)^{-p_0} \left[ \|\varphi_{\geq -2 \log_2(2 + t)} \hat{u}(t)\|_{L^2} + (1 + t)^{2p_1} \|\varphi_{\leq -2 \log_2(2 + t)} \hat{u}(t)\|_{L^2} \right] \leq \epsilon_1. \tag{6.2}$$

In this section we want to control the high frequency component of the weighted Sobolev norm of our solution $u$. In other words, under the a priori assumptions \ref{6.1}--\ref{6.2} we strengthen the control on $\|Su\|_{H^{N_1}}$. In the first subsection below we derive the evolution equation for the weighted quantity corresponding to $|\partial_x|^N Su$. Then, in \ref{6.1} we define the quadratic energy \ref{6.3} associated to this equation, and state our main proposition of this section, Proposition \ref{6.1}, which gives control of this energy. Similarly to before, this key proposition will follow from the construction of a cubic correction for the basic quadratic energy, and two main lemmas, Lemma \ref{6.4} and Lemma \ref{6.5} which are proven in the remaining of the section.

6.1. The weighted energy.

6.1.1. The quadratic weighted energy. The quadratic energy we associate to the equation \ref{5.1} is

$$E^{(2)}_Z(t) = \frac{1}{2} \int_{\mathbb{R}} |P_{\geq -10} Z(t, x)|^2 dx = \frac{1}{4\pi} \int_{\mathbb{R}} \overline{Z(t, \xi)} \mathcal{Z}(t, \xi) \varphi^2_{\geq -10}(\xi) d\xi. \tag{6.3}$$

The main result of this section is the following weighted energy estimate:

**Proposition 6.1.** Assume that $u$ satisfies \ref{6.1}--\ref{6.2} and the initial data assumption

$$\|\partial_x\|^{-1/2 + p_1} u_0\|_{L^2} + \|\partial_x\|^{-1/2 + p_1} Su_0\|_{L^2} \leq \epsilon_0. \tag{6.4}$$
Moreover, assume that the energy $E^{(2)}_{Z}$ in (6.3) is a priori bounded:

$$
\sup_{t \in [0,T]} (1 + t)^{-8p_0} E^{(2)}_{Z}(t) \leq \varepsilon_1^2.
$$

(6.5)

Then

$$
\sup_{t \in [0,T]} (1 + t)^{-8p_0} E^{(2)}_{Z}(t) \lesssim \varepsilon_0^2.
$$

(6.6)

**Remark 6.2** (Control of the high frequency weighted norm). In (7.4) we are going to define a weighted energy functional for low frequencies, $E^{(2)}_{Z_{\text{low}}}$, such that, in particular,

$$
\|Su(t)\|_{L^2}^2 \lesssim E^{(2)}_{Z_{\text{low}}}(t).
$$

Using the definition of $Z$ in (3.53), one can verify that

$$
\|Su(t)\|_{H^N_1}^2 \lesssim E^{(2)}_{Z}(t) + E^{(2)}_{Z_{\text{low}}}(t) + E^{(2)}_{N}(t),
$$

(6.7)

where $E^{(2)}_{N}$ is the Sobolev energy (4.7).

**Remark 6.3** (Control of the low frequency weighted norm). The a priori hypotheses on the (non-weighted) $L^2$ norm in (6.2) are satisfied under the assumption that the low frequency energy $E^{(2)}_{\text{low}}$ verifies

$$
\sup_{t \in [0,T]} (1 + t)^{-2p_0} E_{\text{low}}(t) \leq \varepsilon_1^2.
$$

(6.8)

This bound was already improved in Proposition 5.1. The hypotheses on the weighted norms in (6.2) are satisfied provided that the weighted energy $E^{(2)}_{Z_{\text{low}}}$ that we are going to define later in (7.4), satisfies

$$
\sup_{t \in [0,T]} (1 + t)^{-8p_0} E_{Z_{\text{low}}}(t) \leq \varepsilon_1^2,
$$

(6.9)

see (7.21). The a priori bound (6.9) will be improved in Proposition 7.1 below, see (7.10). Using (6.7) together with (6.6), (4.14) and (7.10), we are able to improve the a priori bounds on the weighted quantities in (2.26), and conclude the proof of (2.28), hence of Proposition 2.4.

6.1.2. *The cubic weighted energy.* The cubic energy associated to (3.54)-(3.56) is constructed as the sum of three energy functionals

$$
E^{(3)}_{Z}(t) = E_{Z,1}(t) + E_{Z,2}(t) + E_{Z,3}(t).
$$

(6.10)

The first energy functional is is the natural correction associated to the nonlinearities $Q_Z$ and $N_{Z,1}$ in (3.55)-(3.56):

$$
E_{Z,1}(t) := \frac{1}{4\pi^2} \sum_{*} \Re \int_{\mathbb{R} \times \mathbb{R}} \hat{Z}(\xi, t) \hat{Z}(\eta, t) m_{1,\epsilon_1\epsilon_2}(\xi, \eta) \hat{u}(\xi - \eta, t) \, d\xi d\eta,
$$

$$
m_{1,\epsilon_1\epsilon_2}(\xi, \eta) :=
\frac{(\xi - \eta) [\xi |\eta|^{-N_1} \varphi_{\leq -10}^2(\xi) \chi(\xi - \eta, \eta) - \eta |\eta|^{N_1} |\xi|^{-N_1} \varphi_{\leq -10}^2(\eta) \chi(\eta - \xi, \xi)]}{8 |\xi - \eta|^{1/2} (|\xi|^{3/2} - |\xi - \eta|^{3/2} - |\eta|^{3/2})}
- \frac{\xi |\eta|^{-N_1} \varphi_{\leq -10}^2(\xi) a_{\epsilon_1\epsilon_2}(\xi, \eta)}{|\xi|^{3/2} - \epsilon_1 |\xi - \eta|^{3/2} - \epsilon_2 |\eta|^{3/2}},
$$

(6.11)
where the first part of the symbol, which is only present for \( \epsilon_1 = \epsilon_2 = 1 \), is associated to \( Q_Z \), while the second one corresponds to \( N_{Z,1} \). The second functional takes into account \( N_{Z,2} \) in (6.11)-(6.13) and is defined as

\[
E_{Z,2}(t) := \frac{1}{4\pi^2} \sum_{\ast} \Re \int_{\mathbb{R} \times \mathbb{R}} \overline{Z}(\xi,t) \overline{u_{12}}(\eta,t) m_{2,\epsilon_1 \epsilon_2}(\xi,\eta) \overline{S_{12}}(\xi - \eta, t) \, d\xi d\eta,
\]

\[
m_{2,\epsilon_1 \epsilon_2}(\xi,\eta) := \frac{\epsilon_1(1 + \epsilon_2)(\xi - \eta) \xi |N_1| \varphi_{-10}^2(\xi) \chi(\xi - \eta, \eta)}{4|\xi - \eta|^{1/2}(|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - |\eta|^{3/2})}
\]

\[
+ \frac{\eta}{|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2}}
\]

The last functional is

\[
E_{Z,3}(t) := \frac{1}{4\pi^2} \sum_{\ast} \Re \int_{\mathbb{R} \times \mathbb{R}} \overline{Z}(\xi,t) \overline{u_{12}}(\eta,t) m_{3,\epsilon_1 \epsilon_2}(\xi,\eta) \overline{u_{12}}(\xi - \eta, t) \, d\xi d\eta,
\]

\[
m_{3,\epsilon_1 \epsilon_2}(\xi,\eta) := \frac{i\epsilon_1(1 + \epsilon_2)(\xi - \eta) \varphi_{-10}^2(\xi) (\eta N_1(\xi,\eta)|\eta|^{N_1} + q_{N_1}(\xi,\eta)|\eta|^{N_1})}{4|\xi - \eta|^{1/2}(|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - |\eta|^{3/2})}
\]

\[
+ \frac{\eta}{|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2}}
\]

and is associated to the nonlinear term \( N_{Z,3} \) in (3.50). The total weighted energy is

\[
E_Z(t) := E_Z^{(2)}(t) + E_Z^{(3)}(t).
\]

Proposition 6.4 will follow from two main lemmas:

**Lemma 6.4.** Under the assumptions of Proposition 6.4, for any \( t \in [0,T] \) we have

\[
|E_Z^{(3)}(t)| \lesssim \epsilon_3^2(1 + t)^{8\rho_0}.
\]

**Lemma 6.5.** Under the assumptions of Proposition 6.4, for any \( t \in [0,T] \) we have

\[
\frac{d}{dt} E_Z(t) \lesssim \epsilon_4^{-4}(1 + t)^{-1+8\rho_0}.
\]

### 6.2. Analysis of the symbols and proof of Lemma [6.4]

To prove Lemma 6.4 and 6.5 we need bounds for the symbols of the cubic energy functionals in (6.11)-(6.13). The first part of the symbol \( m_{1,\epsilon_1 \epsilon_2} \) in (6.11) is similar to the symbol \( m_{N_1} \) in (4.8), and satisfies a similar bound

\[
\frac{(\xi - \eta)|N_1|\eta^{-N_1} \chi(\xi - \eta, \eta) - \eta|N_1|\eta^{N_1} \chi(\eta - \xi, \xi)}{|\xi - \eta|^{1/2}(|\xi|^{3/2} - |\xi - \eta|^{3/2} - |\eta|^{3/2})}
\]

\[
= O(|\xi - \eta|^{-1/2} |\xi|^{1/2} \mathbf{1}_{[2^3,\infty]}(|\eta|/|\xi - \eta|)).
\]

The second component of \( m_{1,\epsilon_1 \epsilon_2} \) in (6.11) can be estimated by using directly (A.12):

\[
\|m_{k_1 k_2}^{1,1,1} \|_{S^{\infty}} \lesssim 2^{k_1/2 - k_2/2} \mathbf{1}_X(k, k_1, k_2) \mathbf{1}_{[2,\infty]}(k_2 - k_1),
\]

\[
\|m_{k_1 k_2}^{1,1,2} \|_{S^{\infty}} \lesssim (2^{3k_1/2 - 3k_2/2} + 2^{k_1/2 - k_2/2} \mathbf{1}_{(-\infty,1]}(k)) \mathbf{1}_X(k, k_1, k_2) \mathbf{1}_{[2,\infty]}(k_2 - k_1).
\]
Using the notation (2.14), we see that the first part of the symbol $m_{2,\epsilon_1\epsilon_2}$ in (6.12) satisfies

$$
\frac{(\xi - \eta)\xi|\chi(\xi - \eta, \eta)}{|\xi - \eta|^{1/2}(|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - |\eta|^{3/2})} = O(|\xi|^{N_1+1/2}|\xi - \eta|^{-1/2}1_{[2^4, \infty)}(|\eta|/|\xi - \eta|)).
$$

Notice that the above estimate presents a singularity of the type $|\xi - \eta|^{-1/2}$. Using the bounds (A.12) and (A.27) we also see that

$$
\frac{|\xi|^{N_1}(a_{\epsilon_1\epsilon_2}(\xi, \eta) + b_{\epsilon_1\epsilon_2}(\xi, \eta) + b_{k_1\epsilon_2}(\xi, \xi - \eta))}{|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2}} = O(|\xi|^{N_1+1/2}|\xi - \eta|^{-1/2}1_{[2^4, \infty)}(|\eta|/|\xi - \eta|)) + O(|\xi|^{N_1+1/2}|\eta|^{-1/2}1_{[2^{-9}, 2^9]}(|\eta|/|\xi - \eta|)).
$$

It follows that $m_{2,\epsilon_1\epsilon_2}$ satisfies:

$$
\|m_{2,\epsilon_1\epsilon_2}\|_{S^\infty} \lesssim 1_\chi(k, k_1, k_2)(2^{(N_1+1/2)k_2-1/2}1_{[2^3, \infty)}(k_2 - k_1) + 2^{N_1}1_{[-10, 10]}(k_2 - k_1)). \tag{6.18}
$$

Moreover, one can verify that

$$
\left\|\frac{|\xi|^{1/2}(\xi - \eta)\chi(\xi - \eta, \eta)}{|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2}} \varphi_k(\xi) \varphi_{k_2}(\eta)\right\|_{S^\infty} \lesssim 1, \tag{6.19}
$$

for all $k, k_2 \in \mathbb{Z}$. Using (6.19) for the first component of the symbol in (6.12), standard integration by parts and the formulas (3.31)-(3.34), (3.35)-(3.38) for the second component, we see that we satisfy:

$$
\left\|\frac{|\xi|^{1/2}(\xi - \eta)\chi(\xi - \eta, \eta)}{|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2}} \varphi_k(\xi) \varphi_{k_2}(\eta)\right\|_{S^\infty} \lesssim 2^{(N_1+1/2)k_2-1/2}1_{[-20, \infty)}(k_2 - k). \tag{6.20}
$$

We eventually look at the last symbol $m_{3,\epsilon_1\epsilon_2}$ in (6.13). Recalling the definition of $q_{N_1}$ from (6.53), the definition (2.6), and the bounds (A.12) and (A.27), we see that $m_{3,\epsilon_1\epsilon_2}$ has similar properties to $m_{2,\epsilon_1\epsilon_2}$, and satisfies

$$
\|m_{3,\epsilon_1\epsilon_2}\|_{S^\infty} \lesssim 1_\chi(k, k_1, k_2)(2^{(N_1+1/2)k_2-1/2}1_{[2^3, \infty)}(k_2 - k_1) + 2^{N_1}1_{[-10, 10]}(k_2 - k_1)), \tag{6.21}
$$

and

$$
\left\|\frac{|\xi|^{1/2}m_{3,\epsilon_1\epsilon_2}\varphi_k(\xi) \varphi_{k_2}(\eta)}{|\xi|^{3/2} - \epsilon_1|\xi - \eta|^{3/2} - \epsilon_2|\eta|^{3/2}}\right\|_{S^\infty} \lesssim 2^{(N_1+1/2)k_2-1/2}1_{[-20, \infty)}(k_2 - k). \tag{6.22}
$$

The proof of (6.15) is similar to the proof of (1.15) in Lemma 1.2 and (5.16) in Lemma 5.3. In particular, (6.15) can be proved, as already done similarly in sections 4.3 and 5.3 by using the bounds (6.17)-(6.21) on the symbols, and Lemma 2.1(ii). We leave the details to the reader.

6.3. Proof of Lemma 6.5. We start by computing the time evolution of $E_Z$ in (6.14). Using the definitions of the energies $E_{Z}^{(2)}$ in (6.3), and $E_{Z}^{(3)}$ in (6.10)-(6.13), and the evolution equation for $Z$ derived in Lemma 3.6 see (3.54)-(3.57), we can calculate:

$$
\frac{d}{dt}(E_{Z}^{(2)} + E_{Z}^{(3)}) = K_0 + \frac{1}{4\pi} \Re \sum_{*} (K_{1,\epsilon_1\epsilon_2} + K_{2,\epsilon_1\epsilon_2} + K_{3,\epsilon_1\epsilon_2}) + R, \tag{6.23}
$$

where

$$
K_0 := \frac{1}{2\pi} \Re \int_{\mathbb{R}} \overline{Z}(\xi) \xi \overline{\varphi}_{\xi=0}^2 d\xi, \tag{6.24}
$$


\[ K_{1,\epsilon_1\epsilon_2} := \int_{\mathbb{R} \times \mathbb{R}} \overline{F(\partial_t Z - i\Lambda Z)}(\xi) m_{1,\epsilon_1\epsilon_2}(\xi, \eta) \widehat{u_{\epsilon_1}}(\xi - \eta) \widehat{Z_{\epsilon_2}}(\eta) \, d\xi d\eta \]
\[ + \int_{\mathbb{R} \times \mathbb{R}} \overline{Z(\xi)} m_{1,\epsilon_1\epsilon_2}(\xi, \eta) F(\partial_t u_{\epsilon_1} - i\epsilon_1 \Lambda u_{\epsilon_1})(\xi - \eta) \widehat{Z_{\epsilon_2}}(\eta) \, d\xi d\eta \]
\[ + \int_{\mathbb{R} \times \mathbb{R}} \overline{\widehat{Z(\xi)}} m_{1,\epsilon_1\epsilon_2}(\xi, \eta) \widehat{u_{\epsilon_1}}(\xi - \eta) F(\partial_t Z_{\epsilon_2} - i\epsilon_2 \Lambda Z_{\epsilon_2})(\eta) \, d\xi d\eta, \]

where \( m_{1,\epsilon_1\epsilon_2} \) is defined in (6.11).

\[ K_{2,\epsilon_1\epsilon_2} = K_{2,\epsilon_1\epsilon_2}^1 + K_{2,\epsilon_1\epsilon_2}^2 + K_{2,\epsilon_1\epsilon_2}^3, \]

\[ K_{2,\epsilon_1\epsilon_2}^1 := \int_{\mathbb{R} \times \mathbb{R}} \overline{F(\partial_t Z - i\Lambda Z)}(\xi) m_{2,\epsilon_1\epsilon_2}(\xi, \eta) \widehat{u_{\epsilon_1}}(\xi - \eta) \widehat{u_{\epsilon_2}}(\eta) \, d\xi d\eta, \]

\[ K_{2,\epsilon_1\epsilon_2}^2 := \int_{\mathbb{R} \times \mathbb{R}} \overline{\widehat{Z(\xi)}} m_{2,\epsilon_1\epsilon_2}(\xi, \eta) F(\partial_t S u_{\epsilon_1} - i\epsilon_1 \Lambda S u_{\epsilon_1})(\xi - \eta) \widehat{u_{\epsilon_2}}(\eta) \, d\xi d\eta, \]

\[ K_{2,\epsilon_1\epsilon_2}^3 := \int_{\mathbb{R} \times \mathbb{R}} \overline{\widehat{Z(\xi)}} m_{2,\epsilon_1\epsilon_2}(\xi, \eta) \widehat{S u_{\epsilon_1}}(\xi - \eta) F(\partial_t u_{\epsilon_2} - i\epsilon_2 \Lambda u_{\epsilon_2})(\eta) \, d\xi d\eta, \]

where \( m_{2,\epsilon_1\epsilon_2} \) is defined in (6.12), and

\[ K_{3,\epsilon_1\epsilon_2} = K_{3,\epsilon_1\epsilon_2}^1 + K_{3,\epsilon_1\epsilon_2}^2 + K_{3,\epsilon_1\epsilon_2}^3, \]

\[ K_{3,\epsilon_1\epsilon_2}^1 := \int_{\mathbb{R} \times \mathbb{R}} \overline{F(\partial_t Z - i\Lambda Z)}(\xi) m_{3,\epsilon_1\epsilon_2}(\xi, \eta) \widehat{u_{\epsilon_1}}(\xi - \eta) \widehat{u_{\epsilon_2}}(\eta) \, d\xi d\eta, \]

\[ K_{3,\epsilon_1\epsilon_2}^2 := \int_{\mathbb{R} \times \mathbb{R}} \overline{\widehat{Z(\xi)}} m_{3,\epsilon_1\epsilon_2}(\xi, \eta) F(\partial_t u_{\epsilon_1} - i\epsilon_1 \Lambda u_{\epsilon_1})(\xi - \eta) \widehat{u_{\epsilon_2}}(\eta) \, d\xi d\eta, \]

\[ K_{3,\epsilon_1\epsilon_2}^3 := \int_{\mathbb{R} \times \mathbb{R}} \overline{\widehat{Z(\xi)}} m_{3,\epsilon_1\epsilon_2}(\xi, \eta) \widehat{u_{\epsilon_1}}(\xi - \eta) F(\partial_t u_{\epsilon_2} - i\epsilon_2 \Lambda u_{\epsilon_2})(\eta) \, d\xi d\eta, \]

where \( m_{3,\epsilon_1\epsilon_2} \) is defined in (6.13). The remainder \( R \) comes from quartic terms involving the remainder \( O_Z \) in (3.54), and quartic terms involving the quadratic part of the function \( V \) as it appears in (3.53), that is \( V_2 \) in (4.30). Using the estimate (3.57) for the first, and arguments similar to the one used for \( A_{2,2} \), see (4.32) and (4.39), for the second, we see that

\[ |R(t)| \lesssim \varepsilon_1^4 (1 + t)^{-1+8p_0}. \]

We now show that all of the terms in (6.25)-(6.27) are bounded by \( \varepsilon_1^4 (1 + t)^{-1+8p_0} \) as well.

6.3.1. \textit{Estimate of } \( K_0 \). The term in (6.24) is like to \( A_1 \) in (4.21), with \( Z \) instead of \( W \). We can then estimate it in the same way we estimated \( A_1 \) in (4.43). In particular, (4.79) shows \( |K_0(t)| \lesssim \|Z(t)\|_{L^2}^2 (1 + t)^{-1} \), which gives us the desired bound.

6.3.2. \textit{Estimate of } \( K_{1,\epsilon_1\epsilon_2} \). We first want to rearrange the integrals in (6.25), using the equation (3.54) for \( Z \), and distinguish between the terms that could potentially lose derivatives and need additional arguments, and does that do not. Let \( K_{1,\epsilon_1\epsilon_2} := K_{1,1} + K_{1,2} + K_{1,3} \), see (5.58), and write

\[ \sum_{ \ast } K_{1,\epsilon_1\epsilon_2} = K_{1,1} + K_{1,2} + K_{1,3} \]
where

\[
K_{1,1} := \sum_{\epsilon_1 = \pm} \int_{R \times R} \left[ i \Sigma_1 Z(\xi) \hat{Z}(\eta) + \overline{Z(\xi) i \Sigma_1 Z(\eta)} \right] m_{1,\epsilon_1}(\xi, \eta) \hat{u}_{\epsilon_1}(\xi - \eta) \, d\xi d\eta,
\]

\[
K_{1,2} := \sum_{\epsilon_1 = \pm} \int_{R \times R} \left[ \overline{Q_1(\xi) \hat{Z}(\eta)} + \overline{Z(\xi) \overline{Q_1(\eta)}} \right] m_{1,\epsilon_1}(\xi, \eta) \hat{u}_{\epsilon_1}(\xi - \eta) \, d\xi d\eta,
\]

and

\[
K_{1,3} := \sum_{\epsilon_1 = \pm} \int_{R \times R} \left[ \overline{N_1(\xi) \hat{Z}(\eta)} + \overline{Z(\xi) \overline{N_1(\eta)}} \right] m_{1,\epsilon_1}(\xi, \eta) \hat{u}_{\epsilon_1}(\xi - \eta) \, d\xi d\eta,
\]

\[
+ \sum_{\epsilon_1 = \pm} \int_{R \times R} \overline{Z(\xi)} m_{1,\epsilon_1}(\xi, \eta) \mathcal{F}(\partial_t u_{\epsilon_1} - i \epsilon_1 \Lambda u_{\epsilon_1})(\xi - \eta) \hat{Z}(\eta) \, d\xi d\eta,
\]

\[
+ \sum_{\epsilon_1 = \pm} \left( \int_{R \times R} \mathcal{F}(\partial_t Z - i \Lambda Z)(\xi) m_{1,\epsilon_1}(\xi, \eta) \hat{u}_{\epsilon_1}(\xi - \eta) \hat{Z}(\eta) \, d\xi d\eta \right).
\]

**Estimate of \( K_{1,1} \).** To estimate \( K_{1,1} \), which is a term of the form \( I \) as in (4.54)-(4.55), we want to apply Lemma 4.8 with \( F = Z, m = m_{1,\epsilon_1}, q(\xi, \eta) = (|\xi|^{3/2} - \epsilon_1 |\xi - \eta|^{3/2} - |\eta|^{3/2})m_{1,\epsilon_1}(\xi, \eta), \) see (6.11). The condition on the support of the symbols, and the first assumption in (4.56) are easily seen to be verified after a direct inspection of the first component of the symbol \( m_{1,\epsilon_1} \) in (6.11), and using (A.10). The property (4.57) can also be easily seen to hold true by a direct inspection of the formula for \( m_{1,\epsilon_1} \), using (4.51) with \( N_1 \) instead of \( N_0 \) (compare with (4.31)) for the first part of the symbol, and (A.14) for the second part. The conclusion (4.58) in Lemma 4.8 then gives

\[
|K_{1,1}| \lesssim \varepsilon_1^2 \|Z(t)\|_{L^2}^2 (1 + t)^{-7/6},
\]

which is more than sufficient.

**Estimate of \( K_{1,2} \).** This term is of the form \( J \) as in (4.63) in Lemma 4.9. Having already verified above the assumptions in the statement of this Lemma, thanks to (4.67) we obtain

\[
|K_{1,2}| \lesssim \varepsilon_1^2 \|Z(t)\|_{L^2}^2 (1 + t)^{-1} \lesssim \varepsilon_1(1 + t)^{-1 + 8p_0}.
\]

**Estimate of \( K_{1,3} \).** In all the terms appearing in (6.30) there are no losses of derivatives since \( N_Z \) is a semilinear term, and the symbols \( m_{3,\epsilon_1} \) are strongly elliptic, as we can see from the second bound in (6.17).

One can use Lemma 2.1(ii), the first bound in (6.17), and (C.61), to estimate the first line of (6.30) by the following:

\[
\sum_{k, k_1, k_2 \in Z} \|m_{1,k,k_1}^{k_2}\|_{S^\infty} \left( \|P_{k_1}^t N_Z\|_{L^2} \|P_{k_2}^t Z\|_{L^2} + \|P_{k_1}^t Z\|_{L^2} \|P_{k_2}^t N_Z\|_{L^2} \|P_{k_1}^t u\|_{L^\infty} \right)
\]

\[
\lesssim \sum_{(k, k_1, k_2) \in \mathcal{X}, |k_2 - k| \leq 5} 2^{k_1} t^{-1/2} |2^{-k/2} \varepsilon_1^2 \min(k,0) (1 + t)^{-1/2 + 4p_0} \varepsilon_1(1 + t)^{4p_0} \varepsilon_1 2^{-N_2 \max(k_1,0)} (1 + t)^{-1/2}
\]

\[
\lesssim \varepsilon_1^4 (1 + t)^{-1 + 8p_0}.
\]
Similarly, again Lemma \[2.1\text{(ii)}\] together with the first bound in \(6.17\) and \(C.61\), can be used to bound the second term in \(6.30\) by

\[
\sum_{k,k_1,k_2 \in \mathbb{Z}} \|m_{k,k_1,k_2}^{k,k_1,k_2} \|_{L^\infty} \|P_{k_1} Z\|_{L^2} \|P_{k_2} Z\|_{L^2} \|P_{k_1}' (\partial_t - i\Lambda) u\|_{L^\infty} \lesssim \sum_{(k,k_1,k_2) \in \mathcal{K}, |k_2-k| \leq 5} 2^{k_1/2} 2^{-k_2/2} \epsilon_1^2 (1+t)^{8p_0} \epsilon_1^2 2^{k_1/2} 2^{-(N_2-2) \max(k_1,0)} (1+t)^{-1}
\]

Finally, we look at the last two terms in \(6.30\). Using the equation \(3.54\) to express \(\partial_t Z - i\Lambda Z\) gives several contributions. All the contributions coming from the nonlinear terms on the right-hand side of \(3.54\) can be estimated in a straightforward fashion using Lemma \[2.1\text{(ii)}\] together with the second bound in \(6.17\), \(C.61\), and the a priori assumptions on \(Z\) and \(u\). The contribution coming from \(\Sigma_2 Z\) gives an integral which is similar to the term \(I_6'\) in \(4.75\), that can be estimated in the same ways, so that we can skip the details.

### 6.3.3. Estimate of \(K_{2,\epsilon_1 \epsilon_2}\)

The main difficulty in estimating the terms in \(6.26\) comes from the singularity in the symbol \(m_{2,\epsilon_1 \epsilon_2}\), see \(6.18\). We can overcome this using the low frequencies information in \(6.2\). We begin with the first term in \(6.26\), and split it into two pieces depending on the size of the frequency \(\xi - \eta\):

\[
K_{2,\epsilon_1 \epsilon_2}^1 = I + II,
\]

\[
I := \int_{\mathbb{R} \times \mathbb{R}} \mathcal{F}(\partial_t Z - i\Lambda Z)(\xi) m_{2,\epsilon_1 \epsilon_2}(\xi, \eta) \varphi_{\leq -2 \log_2 (1+t)}(\xi - \eta) \widehat{Su}_1(\xi - \eta) \widehat{u}_{\epsilon_2}(\eta) \, d\xi d\eta,
\]

\[
II := \int_{\mathbb{R} \times \mathbb{R}} \mathcal{F}(\partial_t Z - i\Lambda Z)(\xi) m_{2,\epsilon_1 \epsilon_2}(\xi, \eta) \varphi_{\geq -2 \log_2 (1+t)}(\xi - \eta) \widehat{Su}_1(\xi - \eta) \widehat{u}_{\epsilon_2}(\eta) \, d\xi d\eta.
\]

We observe that, in view of \(6.2\), we have

\[
\|P_1 Su\|_{L^\infty} \lesssim 2^t (1-p_1) \epsilon_1 (1+t)^{4p_0-p_1}, \quad \text{for} \quad 2^t \leq (1+t)^{-2}.
\]

Then, to estimate the first term it suffices to use Lemma \[2.1\text{(ii)}\] together with the bound on \(m_{2,\epsilon_1 \epsilon_2}\) in \(6.18\), followed by \(C.60\), and the a priori assumptions \(6.1\):

\[
|I| \lesssim \sum_{k,k_1,k_2 \in \mathbb{Z}} \|m_{2,\epsilon_1 \epsilon_2}^{k,k_1,k_2} \|_{L^\infty} \|P_{k_2} Z\|_{L^2} \|P_{k_1} Su\|_{L^\infty} \|P_{k_1}' u\|_{L^2} \lesssim \sum_{(k,k_1,k_2) \in \mathcal{K}, 2^{k_1} \leq (1+t)^{-2}, k_2 \geq k-20} 2^{N_1 k (2k/2 - k_1/2 + 1)} \epsilon_1^2 (1+t)^{-1/2+4p_0} 2^{k/2} \max(k_0,0)
\]

\[
\times \epsilon_1 2^{k_1 (1-p_1)} (1+t)^{4p_0-2p_1} \epsilon_1 2^{-N_2 \max(k_2,0)} \lesssim \epsilon_1^4 (1+t)^{-1+8p_0}.
\]
To estimate $II$ we use Lemma 2.1(ii) together with the symbol bound in (6.20), (C.60) and the a priori assumption (6.2):

$$
|II| \lesssim \|P_{\geq -2 \log_2(1+t)-5}|\partial_x|^{-1/2}Su\|_{L^2} \sum_{k,k_2 \in \mathbb{Z}} \|\xi - \eta\|^{1/2} m_{2,\epsilon_1 \epsilon_2} \varphi_k(\xi) \varphi_{k_2}(\eta)\|_{S^{\infty}}
\times \|P'_k(\partial_t - i\Lambda)Z\|_{L^2} \|P'_{k_2}u\|_{L^\infty}
\lesssim \epsilon_1(1 + t)^{4p_0} \sum_{k,k_2 \in \mathbb{Z}, k_2 \geq k-20} 2^{(N_1 + 1/2)k_2^2/2} (1 + t)^{-1/2 + 4p_0} 2^{k/2} 2^{\max(k,0)}
\times \epsilon_1 2^{-N_2 \max(k_2,0)} (1 + t)^{-1/2}
\lesssim \epsilon_1^4(1 + t)^{-1+8p_0}.
$$

The term $K_{2,\epsilon_1 \epsilon_2}^2$ in (6.20) is easier to estimate. We can use again Lemma 2.1(ii) with (6.18), and the estimate (C.58), to obtain:

$$
|K_{2,\epsilon_1 \epsilon_2}^2| \lesssim \sum_{k,k_1,k_2 \in \mathbb{Z}} \|m_{2,\epsilon_1 \epsilon_2}^{k,k_1,k_2}\|_{S^{\infty}} \|P'_kZ\|_{L^2} \|P'_{k_1}(\partial_t - i\Lambda)Su\|_{L^2} \|P'_{k_2}u\|_{L^\infty}
\lesssim \epsilon_1^4(1 + t)^{-1+8p_0} \sum_{k,k_1,k_2 \in \mathbb{Z}, k_2 \geq k-20} 2^{N_1k(2^{k/2}2^{-k_1/2} + 1)} 2^{k_1/2} 2^{\max(k_1,0)} 2^{-N_2 \max(k_2,0)}
\lesssim \epsilon_1^4(1 + t)^{-1+8p_0}.
$$

We then look at $K_{2,\epsilon_1 \epsilon_2}^3$ in (6.20). This term is similar to $K_{2,\epsilon_1 \epsilon_2}^1$ above. We first write

$$
K_{2,\epsilon_1 \epsilon_2}^3 = III + IV,
$$

$$
III := \int_{\mathbb{R} \times \mathbb{R}} Z(\xi) m_{2,\epsilon_1 \epsilon_2}(\xi, \eta) \varphi_{\leq -2 \log_2(1+t)}(\xi - \eta) \hat{Su}_1(\xi - \eta) \mathcal{F}(\partial_t u_{\epsilon_2} - ie_2 \Lambda u_{\epsilon_2})(\eta) \, d\xi d\eta,
$$

$$
IV := \int_{\mathbb{R} \times \mathbb{R}} Z(\xi) m_{2,\epsilon_1 \epsilon_2}(\xi, \eta) \varphi_{\leq -2 \log_2(1+t)}(\xi - \eta) \hat{Su}_1(\xi - \eta) \mathcal{F}(\partial_t u_{\epsilon_2} - ie_2 \Lambda u_{\epsilon_2})(\eta) \, d\xi d\eta.
$$

Then we use Lemma 2.1(ii), the symbol bound (6.18), the estimates (C.58), and the a priori assumptions (6.1) to obtain:

$$
|III| \lesssim \sum_{k,k_1,k_2 \in \mathbb{Z}, 2^{k_1}(1+t)^{-2}} \|m_{2,\epsilon_1 \epsilon_2}^{k,k_1,k_2}\|_{S^{\infty}} \|P'_kZ\|_{L^2} \|P'_{k_1}Su\|_{L^\infty} \|P'_{k_2}(\partial_t - i\Lambda)u\|_{L^2}
\lesssim \epsilon_1^4(1 + t)^{-1/2 + 9p_0 - 2p_1} \sum_{(k,k_1,k_2) \in \mathcal{X}, 2^{k_1}(1+t)^{-2}, k_2 \geq k-20} 2^{N_1k(2^{k_2/2}2^{-k_1/2} + 1)}
\times 2^{k_1/2} 2^{-N_0+2 \max(k_2,0)}
\lesssim \epsilon_1^4(1 + t)^{-1}.
$$
To estimate IV we use again Lemma 2.4 (ii), this time together with the bound (6.20), the low frequency assumption in (6.2), and the estimates (C.56), and see that

\[
|IV| \lesssim \|P_{\geq -2 \log_2(1+t)-5} \partial_x^{-1/2} Su\|_{L^2} \sum_{k, k_2 \in \mathbb{Z}} \|\xi - \eta|^{1/2} m_{2, \epsilon_1 \epsilon_2} \varphi_k(\xi) \varphi_{k_2}(\eta)\|_{S^\infty} \|P_k Z\|_{L^2} \\
\lesssim \varepsilon_1 (1 + t)^{4p_0} \sum_{k, k_2 \in \mathbb{Z}, k_2 \geq k - 20} 2^{(N_1 + 1/2)k} \varepsilon_1 (1 + t)^{4p_0} \varepsilon_2 2^{k_2/2} 2^{-(N_2 - 3/2) \max(k_2, 0)} (1 + t)^{-1} \\
\lesssim \varepsilon_1^4 (1 + t)^{-1 + 8p_0}.
\]

This concludes the desired estimate for the integrals in (6.26).

6.3.4. Estimate of $K_{3, \epsilon_1 \epsilon_2}$. We first observe that the symbol $m_{3, \epsilon_1 \epsilon_2}$ appearing in the three expressions in (6.27) satisfies the same bound as the symbol $m_{2, \epsilon_1 \epsilon_2}$, see (6.18)-(6.20) and (6.21)-(6.22). Moreover, we notice that terms $K_{3, \epsilon_1 \epsilon_2}^j$, $j = 1, 2, 3$, in (6.27) are trilinear functions of $(Z, u, u)$, while the terms $K_{2, \epsilon_1 \epsilon_2}^j$, $j = 1, 2, 3$, in (6.26) are trilinear functions of $(Z, u, Z)$. Thus, it is clear that estimating the terms in (6.27) is easier than estimating those in (6.26), because the information we have on $u$ are stronger than those we have on $Z$. Since we have already estimated $K_{2, \epsilon_1 \epsilon_2}$ in the previous paragraph, we can skip the estimates for $K_{3, \epsilon_1 \epsilon_2}$, and eventually deduce (6.16). This concludes the proof of Proposition 6.1.

7. Energy estimates IV: Weighted estimates for low frequencies

In this section we improve our control on the low frequency component of $Su$.

7.1. Set up. Starting from the equation (3.42) for $u$, we apply $S$ to it, commute it with the linear part by using $[S, \partial_t - i\Lambda] = -(3/2)(\partial_t - i\Lambda)$, commute it with the nonlinearities by using (2.5)-(2.6), and obtain, after projecting onto low frequencies,

\[
P_{\leq -5}(\partial_t - i\Lambda) Su = P_{\leq -5} Q_0 (V, Su) + P_{\leq -5} N_{Z_{\text{low}},1} + P_{\leq -5} N_{Z_{\text{low}},2} + |\partial_x|^{1/2} O_{Z_{\text{low}}}.  
\]

(7.1)

Here the symbol of $Q_0$ is $q_0(\xi, \eta) = -i\xi(\xi - \eta, \eta)$, the nonlinear terms are

\[
N_{Z_{\text{low}},1} := (i/2) Q_0(\partial_x |\partial_x|^{-1/2} (Su - \overline{u}), u) \\
+ \sum_{*} A_{\epsilon_1 \epsilon_2}(Su_{\epsilon_1}, u_{\epsilon_2}) + A_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, Su_{\epsilon_2}) + B_{\epsilon_1 \epsilon_2}(Su_{\epsilon_1}, u_{\epsilon_2}) + B_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, Su_{\epsilon_2}),
\]

\[
N_{Z_{\text{low}},2} := (i/2) \widetilde{Q}_0(\partial_x |\partial_x|^{-1/2} (u - \overline{u}), u) + (i/2) Q_0(\partial_x |\partial_x|^{-1/2} (u - \overline{u}), u) \\
+ \sum_{*} \widetilde{A}_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}) + (3/2) A_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}) + B_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}) + (3/2) B_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, u_{\epsilon_2}),
\]

(7.2)

and the remainder satisfies

\[
\|O_{Z_{\text{low}}}(t)\|_{L^2} \lesssim \varepsilon_1^4 (1 + t)^{-1 + 4p_0}.
\]

(7.3)

7.2. Low frequency weighted energy. The basic quadratic energy that we associate to the equation (7.1)-(7.2) is

\[
E_{Z_{\text{low}}}^{(2)}(t) = \frac{1}{4\pi} \int_{\mathbb{R}} |\widehat{Su}(t, \xi)|^2 |\xi|^{-1} P((1 + t)^2 |\xi|) \varphi_{\leq -10}(\xi) d\xi.
\]

(7.4)
where $P$ is defined in (5.3). The natural cubic energy associated to (7.1) is given by
\[ E^{(3)}_{Z_{\text{low}}} = E_{Z_{\text{low}},1} + E_{Z_{\text{low}},2}; \]
where
\[ E_{Z_{\text{low}},1}(t) := \frac{1}{4\pi^2} \sum_{\star} \Re \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{S_u(\xi, t)} \overline{u_{\epsilon t}}(\eta, t) u_{\epsilon_1}(\xi - \eta, t) q_{\epsilon_1 \epsilon t}(\xi, \eta) d\xi d\eta, \]
\[ q_{\epsilon_1 \epsilon t}(\xi, \eta) := (1 + \epsilon_1)(1 + \epsilon_2) |\xi|^{1/2}(\xi - \eta) \left[ |\xi|^{-1} \varphi_t(\xi) \chi(\xi - \eta, -\eta) - \eta |\eta|^{-1} \varphi_t(\eta) \chi(\eta - \xi, \xi) \right] \]
\[ + \frac{i}{8} |\eta|^{1/2} |\xi - \eta|^{-1/2} \varphi_t(\xi) \eta \left( \xi - \eta \right) \eta \]
\[ - \frac{i}{2} \left( a_{\epsilon_1 \epsilon t}(\xi, \eta) + b_{\epsilon_1 \epsilon t}(\xi, \eta) \right) + b_{\epsilon_1 \epsilon t}(\xi, \eta), \]
\[ \varphi_t(\xi) := \varphi_{\leq 2 \log_d(2 + t)}(\xi) \varphi_{\leq -10}(\xi), \]
and
\[ E_{Z_{\text{low}},2}(t) := \frac{1}{4\pi^2} \sum_{\star} \Re \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{S_u(\xi, t)} \overline{u_{\epsilon_1 \epsilon t}}(\eta, t) r_{\epsilon_1 \epsilon t}(\xi, \eta) d\xi d\eta, \]
\[ r_{\epsilon_1 \epsilon t}(\xi, \eta) := \frac{i \epsilon_1 (1 + \epsilon_2) \varphi_t(\xi) \eta \left( \xi - \eta \right) \eta \left( \xi - \eta \right) \eta \]
\[ + \frac{i}{2} \left( a_{\epsilon_1 \epsilon t}(\xi, \eta) + b_{\epsilon_1 \epsilon t}(\xi, \eta) \right) + b_{\epsilon_1 \epsilon t}(\xi, \eta), \]
\[ \text{where we are using the notation (2.6). The first part of the symbol } q_{\epsilon_1 \epsilon t} \text{ in (7.6) is needed to correct the quadratic interaction } Q_0(V, S_u) \text{ in (7.1), after the proper symmetrization. This is similar to the symbols in (4.3) and (6.11). The rest of the symbol } q_{\epsilon_1 \epsilon t} \text{ takes into account the nonlinear term } N_{Z_{\text{low}},1} \text{ in (7.2). The symbol } r_{\epsilon_1 \epsilon t} \text{ is naturally associated to the nonlinear term } N_{Z_{\text{low}},2} \text{ in (7.2).} \]

We will prove the following:

**Proposition 7.1.** Assume that $u$ satisfies
\[ \sup_{t \in [0, T]} \left[ (1 + t)^{-\rho_0} \|u(t)\|_{H^{N_0}} + (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{-k/10} + 2^{N_2 k}) \|P_k u(t)\|_{L^\infty} \right] \leq \varepsilon_1, \]
\[ (7.8) \]
Moreover, assume that
\[ \sup_{t \in [0, T]} (1 + t)^{-8\rho_0} \left[ E^{(2)}_{Z_{\text{low}}}(t) + E^{(2)}_{Z_{\text{low}}}(t) + E^{(2)}_{Z_{\text{low}}}(t) \right] \leq \varepsilon_1^2, \]
\[ (7.9) \]
where $E^{(2)}_{Z_{\text{low}}}$ is the weighted energy (6.3), and $E^{(2)}_{Z_{\text{low}}}$ is the low frequency energy (7.4). Then
\[ \sup_{t \in [0, T]} (1 + t)^{-8\rho_0} E^{(2)}_{Z_{\text{low}}}(t) \leq C \varepsilon_0^2 + \varepsilon_1^2 / 100. \]
\[ (7.10) \]

The conclusion (7.10) above improves the a priori bound on the weighted low frequency energy in (7.9). The a priori bound on the weighted energy in (7.9) was already improved in Proposition 6.4 under the same assumptions of Proposition 7.1 above. Therefore, see also remark (6.3), by proving (7.10) we will conclude all our a priori bounds on $L^2$ based norms.

The estimate (7.10) will follow from the two lemmas below.
Lemma 7.2. Under the assumptions of Proposition 7.1 for any \( t \in [0, T] \) we have
\[
|E_{Z_{\text{low}}}^{(3)}(t)| \lesssim \varepsilon_1^3(1 + t)^{8p_0}.
\] (7.11)

Lemma 7.3. Under the assumptions of Proposition 7.1 for any \( t \in [0, T] \) we have
\[
\frac{d}{dt} E_{Z_{\text{low}}}(t) \leq C\varepsilon_1^3(1 + t)^{-1 + 8p_0} + 40p_1\varepsilon_1^2(1 + t)^{-1 + 8p_0}.
\] (7.12)

7.3. Analysis of the symbols and proof of Lemma 7.2. To prove Lemma 7.2 and 7.3 we need appropriate bounds for the symbols in (7.6) and (7.7). These can be obtained as already done before in section 6.3 and 6.2. In particular, using the bounds (A.12) and (A.27), it is not hard to verify that
\[
\|q_{\epsilon_1, \epsilon_2}|^2 \|_{S^\infty} \lesssim 2^{-k_2/2} 1_{X}(k, k_1, k_2) 1_{[-2 \log_2(1+t) - 10, 0]}(k).
\] (7.13)

This is a somewhat rougher bound than what actually holds true, but it will be sufficient for our estimates. Furthermore, using also
\[
\|\eta|^{1/2} \varphi_{k_1}(\xi) \varphi_{k_2}(\xi - \eta)\|_{S^\infty} \lesssim 1,
\]
one can verify that
\[
\|\eta|^{1/2} q_{\epsilon_1, \epsilon_2} \varphi_{k_1}(\xi) \varphi_{k_2}(\xi - \eta)\|_{S^\infty} \lesssim 1_{[-2 \log_2(1+t) - 10, 0]}(k).
\] (7.14)

We also have
\[
\|\eta|^{1/2} q_{\epsilon_1, \epsilon_2} \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta)\|_{S^\infty} \lesssim 1.
\] (7.15)

Similarly, using again (A.12) and (A.27) one can see that
\[
\|q_{\epsilon_1, \epsilon_2}|^2 \|_{S^\infty} \lesssim 2^{-k_2/2} 1_{X}(k, k_1, k_2) 1_{[-2 \log_2(1+t) - 10, 0]}(k).
\] (7.16)

and
\[
\|\eta - \eta|^{1/2} r_{\epsilon_1, \epsilon_2} \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta)\|_{S^\infty} \lesssim 1.
\] (7.17)

The proof of (7.11) can then be done in a similar fashion to what was done before in section 5 in the proof of Lemma 5.3 by using the bounds (7.13)- (7.17) above. We skip the details.

7.4. Proof of Lemma 7.3. As in section 5.4, we use the definition of the quadratic energy (7.3), the equation (7.1)- (7.2), the formulas for the cubic energies (7.5)- (7.7), and a symmetrization argument like the one performed for the terms (4.22) and leading to (4.31), to calculate
\[
\frac{d}{dt} (E_{Z_{\text{low}}}^{(2)} + E_{Z_{\text{low}}}^{(3)}) = \frac{1}{2\pi} L_1 + \frac{1}{2\pi} R L_2 + \frac{1}{4\pi^2} \Re \sum_* \left( L_{3, \epsilon_1 \epsilon_2} + L_{4, \epsilon_1 \epsilon_2} + L_{5, \epsilon_1 \epsilon_2} + L_{6, \epsilon_1 \epsilon_2} \right)
\]
\[
+ \frac{1}{4\pi^2} \Re \sum_* \left( L_{7, \epsilon_1 \epsilon_2} + L_{8, \epsilon_1 \epsilon_2} + L_{9, \epsilon_1 \epsilon_2} + L_{10, \epsilon_1 \epsilon_2} \right) + R,
\]

where
\[
L_1 = \int_{\mathbb{R}} |\hat{S}\zeta(\xi)|^2 (1 + t)^P ((1 + t)^2|\xi|) \varphi_{\leq -10}(\xi) d\xi,
\]
\[
L_2 = \int_{\mathbb{R}} \overline{\hat{S}\zeta}(\xi) \mathcal{F} \left( \partial_S S - iAS \right)(\xi) \varphi_{\leq -2 \log_2(1+t) - 1}(\xi) |\xi|^{-1+2p_1}(1 + t)^{4p_1} d\xi,
\]
\[ L_{3, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \widehat{Su}(\xi) \partial_t q_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \hat{u}_{\varepsilon_1}(\xi - \eta) \hat{Su}_{\varepsilon_2}(\eta) \, d\xi d\eta, \]

\[ L_{4, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{F(\partial_t Su - i\Lambda Su)}(\xi) q_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \hat{u}_{\varepsilon_1}(\xi - \eta) \hat{Su}_{\varepsilon_2}(\eta) \, d\xi d\eta, \]

\[ L_{5, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{Su}(\xi) q_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \mathcal{F}(\partial_t u_{\varepsilon_1} - ie_1 \Lambda u_{\varepsilon_1})(\xi - \eta) \hat{Su}_{\varepsilon_2}(\eta) \, d\xi d\eta, \]

\[ L_{6, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{Su}(\xi) q_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \hat{u}_{\varepsilon_1}(\xi - \eta) \mathcal{F}(\partial_t Su_{\varepsilon_2} - ie_2 \Lambda Su_{\varepsilon_2})(\eta) \, d\xi d\eta, \]  

(7.18)

\[ L_{7, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{Su}(\xi) \partial_t r_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \hat{u}_{\varepsilon_1}(\xi - \eta) \hat{Su}_{\varepsilon_2}(\eta) \, d\xi d\eta, \]

\[ L_{8, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{F(\partial_t Su - i\Lambda Su)}(\xi) r_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \hat{u}_{\varepsilon_1}(\xi - \eta) \hat{Su}_{\varepsilon_2}(\eta) \, d\xi d\eta, \]

\[ L_{9, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{Su}(\xi) r_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \mathcal{F}(\partial_t u_{\varepsilon_1} - ie_1 \Lambda u_{\varepsilon_1})(\xi - \eta) \hat{Su}_{\varepsilon_2}(\eta) \, d\xi d\eta, \]

\[ L_{10, \varepsilon_1 \varepsilon_2} = \int_{\mathbb{R} \times \mathbb{R}} |\xi|^{-1/2} \overline{Su}(\xi) r_{\varepsilon_1 \varepsilon_2}(\xi, \eta) \hat{u}_{\varepsilon_1}(\xi - \eta) \mathcal{F}(\partial_t u_{\varepsilon_2} - ie_2 \Lambda u_{\varepsilon_2})(\eta) \, d\xi d\eta, \]  

(7.19)

and the remainder \( R \) satisfies \(|R(t)| \lesssim \varepsilon_1^2 (1 + t)^{-1 + 8p_0} \), in view of (7.3).

Recall that under the assumption (7.9) we have

\[ \|\varphi_{\geq 2 \log_2(1+t)}(\cdot)\| \cdot |\frac{-1/2}{\hat{u}}\|_{L^2} \leq \varepsilon_1 (1 + t)^{p_0}, \]

\[ \|\varphi_{\leq 2 \log_2(1+t)}(\cdot)\| \cdot |\frac{-1/2}{\hat{u}}\|_{L^2} \leq \varepsilon_1 (1 + t)^{p_0 - 2p_1}, \]  

(7.20)

and

\[ \|\varphi_{\geq 2 \log_2(1+t)}(\cdot)\| \cdot |\frac{-1/2}{\hat{Su}}\|_{L^2} \leq \varepsilon_1 (1 + t)^{4p_0}, \]

\[ \|\varphi_{\leq 2 \log_2(1+t)}(\cdot)\| \cdot |\frac{-1/2}{\hat{Su}}\|_{L^2} \leq \varepsilon_1 (1 + t)^{4p_0 - 2p_1}. \]  

(7.21)

7.4.1. Estimate of \( L_1 \). Using (2.22) the definition of \( E_{Z_1}^{(2)} \) in (4.3), and the a priori assumption (7.9), we see that

\[ \frac{1}{2\pi} |L_1| \leq \frac{1}{4\pi} \int_{\mathbb{R}} \left| \widehat{Su}(t, \xi) \right|^2 \frac{20p_1}{(1 + t)^2} \mathcal{P}(\xi) |\varphi_{\leq 10}(\xi)| \, d\xi \leq 20p_1 \varepsilon_1^2 (1 + t)^{-1 + 8p_0}, \]

as desired.

7.4.2. Estimate of \( L_2 \). Notice that the integrand in \( L_2 \) is supported on a region where \(|\xi| \lesssim (1 + t)^{-2} \), so that we can use the bound in (4.58), and (7.21), to obtain

\[ |L_2| \lesssim \sum_{k \in \mathbb{Z}, 2^k \leq t \lesssim (1 + t)^{-2}} 2^{2k(-1 + 2p_1)} \left\| P_k' Su(t) \right\|_{L^2} \left\| P_k' (\partial_t - i\Lambda) Su(t) \right\|_{L^2} (1 + t)^{4p_1} \]

\[ \lesssim \sum_{k \in \mathbb{Z}, 2^k \leq t \lesssim (1 + t)^{-2}} 2^{2k(-1 + 2p_1)} \left\| P_k' Su(t) \right\|_{L^2} \varepsilon_1^2 (1 + t)^{-1 + 4p_0 + 4p_1} \]

\[ \lesssim \varepsilon_1 (1 + t)^{-1 + 8p_0 + 2p_1} \sum_{k \in \mathbb{Z}, 2^k \leq t \lesssim (1 + t)^{-2}} 2^{pk} \lesssim \varepsilon_1^3 (1 + t)^{-1 + 8p_0}. \]
7.4.3. Estimate of $L_{3,\epsilon_1 \epsilon_2}$. Looking at the definition of $q_{\epsilon_1 \epsilon_2}$ in (7.4), and using (7.13), one can see that
\[ \| (\partial_t q_{\epsilon_1 \epsilon_2})^{k_1,k_2} \|_{S^\infty} \lesssim (1+t)^{-2} - 2^{-k_2/2} \| P_k^t u \|_{L^2} \| P_k u \|_{L^\infty} \| P_k^t S u \|_{L^2}. \]
We can then use Lemma 2.1 (ii) to estimate, for every $\epsilon_1, \epsilon_2 \in \{+, -\}$,
\[ |L_{3,\epsilon_1 \epsilon_2}| \lesssim L_{3,1} + L_{3,2}, \]
\[ L_{3,1} := (1+t)^{-2} \sum_{(k,k_1,k_2) \in \mathcal{X}, k \leq k_2 + 10, 2^{k_1-10} \leq (1+t)^{-2}} 2^{-k_2/2} - 2^{-k_2/2} \| P_k^t S u \|_{L^2} \| P_k u \|_{L^\infty} \| P_k^t S u \|_{L^2}, \]
\[ L_{3,2} := (1+t)^{-2} \sum_{(k,k_1,k_2) \in \mathcal{X}, k \leq k_2 + 10, 2^{k_1-10} \leq (1+t)^{-2}} 2^{-k_2/2} - 2^{-k_2/2} \| P_k^t S u \|_{L^2} \| P_k u \|_{L^\infty} \| P_k^t S u \|_{L^2}. \]
Using (7.21) and (7.4) we see that
\[ L_{3,1} \lesssim \epsilon_1^3 (1+t)^{-2+4p_0} \sum_{k_1,k_2 \in \mathbb{Z}, 2^{k_2+20} \geq (1+t)^{-2}} 2^{k_1/10} - N_2 \max(k_1,0) (1+t)^{-1/2 - k_2/2} \| P_k^t S u \|_{L^2} \lesssim \epsilon_1^3 (1+t)^{-1}. \]
For the second term we can use (7.21), and the inequality
\[ \| P_t S u \|_{L^\infty} \lesssim 2^{(1-p_1) \| P_{0} S u \|_{\mathcal{H}^0}} \| P_{0} S u \|_{\mathcal{H}^0}, \]
if $2^l \leq (1+t)^{-2}$, to obtain:
\[ L_{3,2} \lesssim \epsilon_1^3 (1+t)^{-2+4p_0} \sum_{k_1,k_2 \in \mathbb{Z}, 2^{k_2} \geq (1+t)^{-2}} 2^{-k_2/2} (1+t)^{p_0} \min \left( 2^{(1/2-p_1)k_1}, 2^{-N_0 \max(k_1,0)} \right) \times \epsilon_1 2^{(1-p_1)k_2} (1+t)^{4p_0} \lesssim \epsilon_1^3 (1+t)^{-2}. \]

7.4.4. Estimate of $L_{4,\epsilon_1 \epsilon_2}$. To deal with the term $L_{4,\epsilon_1 \epsilon_2}$ in (7.18) we first use Lemma 2.1 (ii) to estimate
\[ |L_{4,\epsilon_1 \epsilon_2}| \lesssim L_{4,1} + L_{4,2} + L_{4,3}, \]
\[ L_{4,1} := \sum_{k,k_1,k_2 \in \mathbb{Z}, 2^{k_2+10} \leq (1+t)^{-2}} \| q_{\epsilon_1 \epsilon_2}^{k_1,k_2} \|_{S^\infty} 2^{-k_2/2} \| P_k^t (\partial_t - i\Lambda) S u \|_{L^2} \| P_k u \|_{L^\infty} \| P_k^t S u \|_{L^2}, \]
\[ L_{4,2} := \| \varphi_{\geq 2 \log_2 (1+t) - 10} \|_{L^2} \| \varphi_{\geq 2 \log_2 (1+t) - 10} \|_{L^2} \sum_{k,k_1 \in \mathbb{Z}, |k-k_1| \leq 10} \| \varphi_{\epsilon_1 \epsilon_2}^{k_1,k_2} \|_{S^\infty} 2^{-k_2/2} \| P_k^t (\partial_t - i\Lambda) S u \|_{L^2} \| P_k u \|_{L^\infty} \| P_k^t S u \|_{L^2}, \]
\[ L_{4,3} := \sum_{k,k_1,k_2 \in \mathbb{Z}, k_2+10 \leq k_1} \| q_{\epsilon_1 \epsilon_2}^{k_1,k_2} \|_{S^\infty} 2^{-k_2/2} \| P_k^t (\partial_t - i\Lambda) S u \|_{L^2} \| P_k u \|_{L^\infty} \| P_k^t S u \|_{L^2}. \]
Using the symbol bound (7.13), (5.58), (7.3), and (7.22), we see that:
\[ L_{4,1} \lesssim \sum_{(k,k_1,k_2) \in \mathcal{X}, 2^{k_2} \leq (1+t)^{-2}, |k-k_1| \leq 10} 2^{-k_2/2} \| (1+t)^{-1/2+4p_0} \|_{\mathcal{H}^0} 2^{(1/2-p_1)k_1} 2^{-N_0 \max(k_1,0)} (1+t)^{p_0} \times \epsilon_1 2^{(1-p_1)k_2} (1+t)^{4p_0} \lesssim \epsilon_1^4 (1+t)^{-1}. \]
Using (7.21), the symbol bound (7.14), and (C.58), we obtain
\[
L_{4.2} \lesssim \varepsilon_1 (1 + t)^{4p_0} \sum_{k, k_1 \in \mathbb{Z}, |k - k_1| \leq 10} \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0} \varepsilon_1 2^{k_1/10 + 2 - N_2 \max(k_1, 0)} (1 + t)^{-1/2} \lesssim \varepsilon_1^4 (1 + t)^{-1 + 8p_0}.
\]

To estimate the last term we use (7.21), (7.13), and (C.58), and see that
\[
L_{4.3} \lesssim \sum_{(k, k_1, k_2) \in \mathcal{X}, (1 + t)^{-2} \leq 2^k \leq 2^{10}, k_2 + 10 \geq k_1} \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0} (2^{k/2} + (1 + t)^{-1/2})
\times \varepsilon_1 2^{k_1/10 + 2 - N_2 \max(k_1, 0)} (1 + t)^{-1/2} \varepsilon_1 (1 + t)^{4p_0} \lesssim \varepsilon_1^4 (1 + t)^{-1 + 8p_0}.
\]

7.4.5. Estimate of $L_{5,\epsilon_1\epsilon_2}$. We first use Lemma 2.1(ii) to bound
\[
|L_{5,\epsilon_1\epsilon_2}| \lesssim L_{5.1} + L_{5.2} + L_{5.3},
\]
\[
L_{5.1} := \sum_{k, k_1, k_2 \in \mathbb{Z}, 2^{k_2 + 10} \leq (1 + t)^{-2}, |k - k_1| \leq 10} \| \xi_{t, \epsilon_1\epsilon_2}^{k, k_1, k_2} \|_{L^\infty_\epsilon} 2^{k/2} \| P_k' Su \|_{L^2} \| P_k' (\partial_t - i \Lambda) u \|_{L^2} \| P_k' S u \|_{L^\infty_\epsilon},
\]
\[
L_{5.2} := \| \varphi_{\geq -2 \log_2 (1 + t) - 10} \|_{L^2} \| \varphi_{\geq -2 \log_2 (1 + t) - 10} \|_{L^2} \sum_{k, k_1 \in \mathbb{Z}, |k - k_1| \leq 10} \| \eta \|_{L^\infty_\epsilon} \| q_{\epsilon_1 \epsilon_2} \varphi (\xi) \varphi (\xi - \eta) \|_{L^\infty_\epsilon}
\times \| P_k' (\partial_t - i \Lambda) u \|_{L^\infty_\epsilon} \| P_k' Su \|_{L^2} \| P_k' Su \|_{L^2},
\]
\[
L_{5.3} := \| \varphi_{\geq -2 \log_2 (1 + t) - 10} \|_{L^2} \sum_{k, k_1, k_2 \in \mathbb{Z}, k_2 + 10 \geq k_1} \| \eta \|_{L^\infty_\epsilon} \| q_{\epsilon_1 \epsilon_2} \varphi (\xi - \eta) \|_{L^\infty_\epsilon} \| P_k' (\partial_t - i \Lambda) u \|_{L^\infty_\epsilon} 2^{-k/2} \| P_k' S u \|_{L^2}.
\]

Using the symbol bound (7.13), (C.56), (7.8), and (7.22), we get:
\[
L_{5.1} \lesssim \sum_{k, k_1, k_2 \in \mathbb{Z}, 2^{k_2 + 10} \leq (1 + t)^{-2}, |k - k_1| \leq 10} 2^{-k/2} \varepsilon_1 (1 + t)^{4p_0} \varepsilon_1 2^{k_1/2} 2^{-(N_0 - 2) \max(k_1, 0)} (1 + t)^{-1/2 + p_0}
\times \varepsilon_1 2^{k_1/10 + 2 - (N_0 - 2) \max(k_1, 0)} (1 + t)^{-1} \lesssim \varepsilon_1^4 (1 + t)^{-1}.
\]

Using (7.21), the symbol bound (7.13), and (C.56), we obtain
\[
L_{5.2} \lesssim \varepsilon_1 (1 + t)^{4p_0} \sum_{k, k_1 \in \mathbb{Z}, |k - k_1| \leq 10} \varepsilon_1 (1 + t)^{4p_0} \varepsilon_1 2^{k_1/2} 2^{-(N_2 - 2) \max(k_1, 0)} (1 + t)^{-1} \lesssim \varepsilon_1^4 (1 + t)^{-1 + 8p_0}.
\]

Similarly, using (7.21), (7.15), and (C.56), we see that
\[
L_{5.3} \lesssim \varepsilon_1 (1 + t)^{4p_0} \sum_{k_1, k_2 \in \mathbb{Z}, k_2 + 10 \geq k_1} \varepsilon_1 2^{k_1/2} 2^{-(N_2 - 2) \max(k_1, 0)} (1 + t)^{-1} \varepsilon_1 2^{-p_1 k_2} (1 + t)^{4p_0} \lesssim \varepsilon_1^4 (1 + t)^{-1 + 8p_0}.
\]
7.4.6. Estimate of $L_{6, \epsilon_1 \epsilon_2}$.

Finally we look at $L_{6, \epsilon_1 \epsilon_2}$ in (7.18). Using Lemma 2.1(ii) we can bound

$$|L_{6, \epsilon_1 \epsilon_2}| \lesssim L_{6,1} + L_{6,2},$$

$$L_{6,1} := \sum_{k, k_1, k_2 \in \mathbb{Z}, \pm 2^{k_2} \leq (1+t)^{-2}} \|q_{k_1 k_2}^{|k_1, k_2|} \|_{L^2} 2^{-k/2} \|P_k^{|k_1, k_2|} Su \|_{L^2} \|P_k^{|k_1, k_2|} (\partial_t - i\Lambda) Su \|_{L^\infty},$$

$$L_{6,2} := \|\varphi_{\geq -2\log_2(1+t)-10} \cdot (-1/2)^{\sum_{k_1, k_2 \in \mathbb{Z}, \pm 2^{k_2} \geq (1+t)^{-2}} \| \varphi_{k_1} (\xi - \eta) \varphi_{k_2} (\eta) \|_{L^\infty} \times \|P_k^{|k_1, k_2|} Su \|_{L^\infty} 2^{-k_2/2} \|P_k^{|k_1, k_2|} (\partial_t - i\Lambda) Su \|_{L^2}.$$}

Using (7.13), (7.21), the decay in (7.8), and Bernstein’s inequality followed by (C.58), we have

$$L_{6,1} \lesssim \sum_{k, k_1, k_2 \in \mathbb{Z}, \pm 2^{k_2} \leq (1+t)^{-2}, k_2 - 10 \leq k \leq 10} 2^{-k_2/2} \varepsilon_1 (1 + t)^{4p_0} \varepsilon_1 2^{(1/2-p_1)k_1} 2^{-N_0 \max(k_1, 0)} \varepsilon_1 \varepsilon_1 2^{k_2/2} (1 + t)^{-1/2 + 4p_0} \lesssim \varepsilon_1^4 (1 + t)^{-1}.$$}

Using (7.21), (7.15), (7.8), and (C.58), we have

$$L_{6,2} \lesssim \varepsilon_1 (1 + t)^{4p_0} \sum_{k_1, k_2 \in \mathbb{Z}, \pm 2^{k_2} \geq (1+t)^{-2}} \varepsilon_1 2^{k_1/10} \varepsilon_1 2^{-N_2 \max(k_1, 0)} (1 + t)^{-1/2} \times \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0} (2^{k_2/2} + (1 + t)^{-1/2}) 2^{-\max(k_2, 0)} \lesssim \varepsilon_1^4 (1 + t)^{-1 + 8p_0}.$$}

7.4.7. Estimate of $L_{j, \epsilon_1 \epsilon_2}$, $j = 7, \ldots, 10$.

Observe that the terms $L_{j, \epsilon_1 \epsilon_2}$, for $j = 7, \ldots, 10$ in (7.19) are easier to estimate than the terms $L_{j, \epsilon_1 \epsilon_2}$, for $j = 3, \ldots, 6$ in (7.18). The is because the bounds for the symbols are essentially the same, see (7.13)–(7.17), but we have stronger information on $u$ than on $Su$. Therefore, the integrals $L_{j, \epsilon_1 \epsilon_2}$, for $j = 7, \ldots, 10$ can be treated similarly to the ones that we have just estimated. We can then conclude the desired bound of $\varepsilon_1^4 (1 + t)^{-1 + 8p_0}$ for the evolution of $E_{Z_{low}}$. This gives Lemma 7.3 and hence Proposition 7.1.

8. Decay estimates

The aim of this section is to prove the following:

**Proposition 8.1.** If $u$ satisfies the main bootstrap assumptions (2.25)–(2.26) in Proposition 2.4 then

$$\sup_{t \in [0, T]} (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{-k/10} + 2^{N_2 k}) \|P_k u(t)\|_{L^\infty} \lesssim \varepsilon_0. \quad (8.1)$$

The above Proposition improves the control on the decaying norm in (2.25) giving us (2.28) and closing our bootstrap argument, thereby completing the proof of Theorem 1.1. The proof of Proposition 8.1 will be given through a series of steps below. The strategy follows the general approach of [32, 33].
8.1. **Set up.** Recall that for any suitable multiplier \( m : \mathbb{R}^d \to \mathbb{C} \) we define the associated bilinear and trilinear operators \( M \) by the formulas

\[
\mathcal{F}[M(f,g)](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} m(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) \, d\eta,
\]

\[
\mathcal{F}[M(f,g,h)](\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R} \times \mathbb{R}} m(\xi, \eta, \sigma) \hat{f}(\xi - \eta) \hat{g}(\eta - \sigma) \hat{h}(\sigma) \, d\eta d\sigma.
\]

Recall from Proposition 3.4 our main equation (3.42):

\[
\partial_t u - i |\partial_x|^{3/2} u - i \Sigma \gamma(u) = - \partial_x T_V u + N u + U \geq 3. \tag{8.2}
\]

Also recall that we are only interested in establishing decay of a lower number of derivatives than what is allowed by the highest order Sobolev norm.

We invoke Lemma C.4 and recast (8.2) into the form

\[
\mathcal{L} u = Q_u + C_u + R_{\geq 4}, \quad \mathcal{L} := \partial_t - i \Lambda, \quad \Lambda := |\partial_x|^{3/2}, \tag{8.3}
\]

where:

- The quadratic nonlinear terms are
  
  \[
  Q_u := Q_0(u, u) - Q_0(\overline{u}, u) + \sum_{X \in \{A, B\}} X_{++}(u, u) + X_{+-}(u, \overline{u}) + X_{-+}(u, \overline{u}) + X_{--}(u, \overline{u}, \overline{u}), \tag{8.4}
  \]

  where the symbol of \( Q_0 \) is
  
  \[
  q_0(\xi, \eta) := \frac{i \xi (\xi - \eta)}{2|\xi - \eta|^{1/2}} \chi(\xi - \eta, \eta), \tag{8.5}
  \]

  and the other symbols are given in (3.31) – (3.38).

- The cubic terms have the form
  
  \[
  C_u := M_{++}(u, u, \overline{u}) + M_{+++}(u, u, u) + M_{+-+}(\overline{u}, \overline{u}, u) + M_{--+}(\overline{u}, \overline{u}, \overline{u}), \tag{8.6}
  \]

  with symbols such that
  
  \[
  \|m_{k_1k_2k_3k_4}^{i_1i_2i_3}(\xi, \eta, \sigma)\|_{S^\infty} \lesssim 2^{k/2}2^{\max(k_1, k_2, k_3, k_4, 0)} \tag{8.7}
  \]

  for all \((i_1i_2i_3) \in \{(++-), (-+-), (+++), (-+-)\}.

- We have, see Definition C.1
  
  \[
  R_{\geq 4} = O_{4,-2}, \quad C_u + R_{\geq 4} = |\partial_x|^{1/2}O_{3,-1}. \tag{8.8}
  \]

8.2. **The “semilinear” normal form transformation.** We follow the classical normal form approach of Shatah [43] to define a modified variable which is a quadratic perturbation of \( u \) and satisfies a cubic equation. Let

\[
v := u + M_{++}(u, u) + M_{+-}(u, \overline{u}) + M_{-+}(\overline{u}, u) + M_{--}(\overline{u}, \overline{u}), \tag{8.9}
\]
where, for any $\epsilon_1, \epsilon_2 \in \{1, -1\}$, the bilinear operators $M_{\epsilon_1 \epsilon_2}$ are defined by the multipliers

$$m_{++}(\xi, \eta) := -i \frac{q_0(\xi, \eta) + a_{++}(\xi, \eta) + b_{++}(\xi, \eta)}{|\xi|^{3/2} - |\xi - \eta|^{3/2} - |\eta|^{3/2}},$$

$$m_{+-}(\xi, \eta) := -i \frac{a_{+-}(\xi, \eta) + b_{+-}(\xi, \eta)}{|\xi|^{3/2} - |\xi - \eta|^{3/2} + |\eta|^{3/2}},$$

$$m_{-+}(\xi, \eta) := -i \frac{-q_0(\xi, \eta) + a_{-+}(\xi, \eta) + b_{-+}(\xi, \eta)}{|\xi|^{3/2} + |\xi - \eta|^{3/2} - |\eta|^{3/2}},$$

$$m_{--}(\xi, \eta) := -i \frac{a_{--}(\xi, \eta) + b_{--}(\xi, \eta)}{|\xi|^{3/2} + |\xi - \eta|^{3/2} + |\eta|^{3/2}}.$$  \hspace{1cm} (8.10)

A direct computation shows that $v$ solves

$$\mathcal{L}v = \sum_{*} M_{\epsilon_1 \epsilon_2}((\mathcal{L}u)_{\epsilon_1}, u_{\epsilon_2}) + M_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, (\mathcal{L}u)_{\epsilon_2}) + \mathcal{C}u + \mathcal{R}_{\geq 4},$$  \hspace{1cm} (8.11)

having used the notation (3.41), and $f_+ := f, f_- := \overline{f}$. In view of (C.53) we can also write

$$\mathcal{L}v = \sum_{*} M_{\epsilon_1 \epsilon_2}((\mathcal{L}u)_{\epsilon_1}, u_{\epsilon_2}) + M_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, (\mathcal{L}u)_{\epsilon_2}) + |\partial_x|^{1/2}O_{3,-1}.$$  \hspace{1cm} (8.12)

We now prove several bounds on the new variable $v$.

**Lemma 8.2.** Let $v$ be defined by (8.9)-(8.10), then for any $t \in [0, T]$ and $k \in \mathbb{Z}$ we have

$$\|P_k(u(t) - v(t))\|_{L^\infty} \lesssim \epsilon_1^2 \min \left(2^k/2, 2^{-(N_2-1/2)k}(1 + t)^{-3/4+2p_0}\right),$$  \hspace{1cm} (8.13)

$$\|P_k(u(t) - v(t))\|_{L^2} \lesssim \epsilon_1^2 2^{-(N_0-1/2)\max(k,0)}(1 + t)^{-1/4+3p_0},$$  \hspace{1cm} (8.14)

$$\|P_k S(u(t) - v(t))\|_{L^2} \lesssim \epsilon_1^2 (2^{k/2}, 2^{-(N_1-1/2)k})(1 + t)^{-1/4+6p_0}.$$  \hspace{1cm} (8.15)

Furthermore, we have:

$$\|P_k(u(t) - v(t))\|_{L^2} \lesssim \epsilon_1^2 (1 + t)^{-1/2+3p_0} \min \left(2^{k/2}, 2^{-(N_2-1/2)k}\right).$$  \hspace{1cm} (8.16)

The bounds (8.13)-(8.15) show that $v$ and $u$ have the same relevant norms, and that all the a priori assumptions on $u$ transfer without significant losses to $v$. The bound (8.16) is a variant of the $L^2$ bound (8.14) which provides more decay in time, but less decay at high frequencies. This will be used later on to bound quartic remainder terms.

**Remark 8.3.** Observe that (8.13), (8.14) and Sobolev's embedding, imply

$$\|P_k(u(t) - v(t))\|_{L^\infty} \lesssim \epsilon_1^2 (1 + t)^{-1/2} \min \left(2^{-(N_2+1)k}, 2^{k/8}\right).$$  \hspace{1cm} (8.17)

This shows that in order to obtain (8.1) is suffices to show

$$\sup_{t \in [0, T]} (1 + t)^{1/2} \sum_{k \in \mathbb{Z}} (2^{-k/10} + 2^{N_2k}) \|P_k v(t))\|_{L^\infty} \lesssim \epsilon_0.$$  \hspace{1cm} (8.18)

**Proof of Lemma 8.2** In view of (8.9), we see that we have to estimate the bilinear terms $M_{\epsilon_1 \epsilon_2}$. Notice that from the formulas (8.10), together with the definition (8.5) and the bounds (A.12), (A.27), we have

$$\|M_{\epsilon_1 \epsilon_2}^{k,k_2} \|_{S^\infty} \lesssim 2^{k/2-2k_1/2} X(k, k_1, k_2) 1_{[-10, \infty)}(k_2 - k_1).$$  \hspace{1cm} (8.19)
We recall that from the a priori bounds (2.25)-(2.26) we know, for any \( l \in \mathbb{Z} \) and \( t \in [0, T] \),
\[
\|P_t u(t)\|_{L^2} \lesssim \varepsilon_1(1 + t)^{p_0} \min \left( 2^{l(1/2-p_1)}, 2^{-N_0 l} \right),
\]
\[
\|P_t u(t)\|_{L^\infty} \lesssim \varepsilon_1(1 + t)^{-1/2} \min \left( 2^{l/10}, 2^{-N_1 l} \right),
\]
\[
\|P_t S u(t)\|_{L^2} \lesssim \varepsilon_1(1 + t)^{4p_0} \min \left( 2^{(1/2-p_1)l}, 2^{-N_1 l} \right).
\]

Therefore, using Lemma 2.1(ii), (8.19), and (8.20), we estimate, for any \( k \in \mathbb{Z} \),
\[
\|P_k M_{\varepsilon_1}(u_{\varepsilon_1}, u_{\varepsilon_2})(t)\|_{L^\infty} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \|m_{k_1, k_2}^1\|_{S^\infty} \|P_{k_1} u(t)\|_{L^\infty} \|P_{k_2} u(t)\|_{L^\infty}
\]
\[
\lesssim \varepsilon_1^2 (1 + t)^{-1} \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1 + t)^{-1/2}} 1\chi(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{-N_2 k_1}, 2^{k_1/10}) \min(2^{-N_0 k_2}, 2^{k_2/10})
\]
\[+ \varepsilon_1^2 (1 + t)^{-1/2+4p_0} \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1 + t)^{-1/2}} 1\chi(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{-N_0 k_2}, 2^{k_2/10}) \min(2^{-N_0 k_2}, 2^{k_2/10})
\]
\[\lesssim \varepsilon_1^2 \min \left( 2^{k_2/2}, 2^{-(N_0-1/2)k_1} \right) (1 + t)^{-3/4+2p_0},
\]

which is more than sufficient. We proceed similarly to obtain (8.14):
\[
\|P_k M_{\varepsilon_1}(u_{\varepsilon_1}, u_{\varepsilon_2})(t)\|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \|m_{k_1, k_2}^1\|_{S^\infty} \|P_{k_1} u(t)\|_{L^\infty} \|P_{k_2} u(t)\|_{L^2}
\]
\[
\lesssim \varepsilon_1^2 (1 + t)^{-1/2+4p_0} \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1 + t)^{-1/2}} 1\chi(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{-N_0 k_2}, 2^{(1/2-p_1)k_2})
\]
\[+ \varepsilon_1^2 (1 + t)^{2p_0} \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1 + t)^{-1/2}} 1\chi(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{-N_0 k_2}, 2^{(1/2-p_1)k_2})
\]
\[\lesssim \varepsilon_1^2 \min \left( 2^{k_2/2}, 2^{-(N_0-1/2)k_1} \right) (1 + t)^{-1/4+3p_0},
\]

Using (2.5)-(2.6) we have
\[
SM_{\varepsilon_1}(f, g) = M_{\varepsilon_1}(S f, g) + M_{\varepsilon_1}(f, S g) + \tilde{M}_{\varepsilon_1}(f, g)
\]
where \( \tilde{M}_{\varepsilon_1} \) is the operator associated to \( \tilde{m}_{\varepsilon_1}(x, \eta) = -\{\xi \partial_x + \eta \partial_\eta\} m_{\varepsilon_1}(x, \eta) \). It is not hard to verify that the symbols \( \tilde{m}_{\varepsilon_1} \) satisfy the same bounds (8.14) as the symbols \( m_{\varepsilon_1} \) and, therefore, estimating as above
\[
\|\tilde{M}_{\varepsilon_1}(u_{\varepsilon_1}, u_{\varepsilon_2})\|_{L^2} \lesssim \varepsilon_1^2 \min \left( 2^{k_2/2}, 2^{-(N_0-1/2)k_1} \right) (1 + t)^{-1/4+3p_0}.
\]

Thus, to prove (8.15) it suffices to estimate \( M_{\varepsilon_1}(Su_{\varepsilon_1}, u_{\varepsilon_2}) \) and \( M_{\varepsilon_1}(u_{\varepsilon_1}, Su_{\varepsilon_2}) \). As in the proof of (8.14), we use Lemma 2.1(ii), followed by (8.19), and (8.20), to obtain
\[
\|P_k M_{\varepsilon_1}(u_{\varepsilon_1}(t), Su_{\varepsilon_2}(t))\|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \|m_{k_1, k_2}^1\|_{S^\infty} \|P_{k_1} u(t)\|_{L^\infty} \|P_{k_2} S u(t)\|_{L^2}
\]
\[
\lesssim \varepsilon_1^2 (1 + t)^{-1/2+4p_0} \sum_{2^{k_1} \geq (1 + t)^{-1/2}} 1\chi(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{-N_1 k_1}, 2^{(1/2-p_1)k_2})
\]
\[+ \varepsilon_1^2 (1 + t)^{5p_0} \sum_{2^{k_1} \leq (1 + t)^{-1/2}} 1\chi(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{-N_1 k_1}, 2^{(1/2-p_1)k_2})
\]
\[\lesssim \varepsilon_1^2 \min \left( 2^{k_2/2}, 2^{-(N_1-1/2)k_1} \right) (1 + t)^{-1/4+6p_0},
\]
and
\[ \|P_k M_{e_1 e_2} (S u_1 (t), u_2 (t))\|_{L^2} \]
\[ \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \|r_{e_1 e_2}^{k_1 k_2}\|_{S^\infty} \min (\|P'_{k_1} S u(t)\|_{L^2}, \|P'_{k_2} u(t)\|_{L^2}, \|P'_{k_1} S u(t)\|_{L^\infty}, \|P'_{k_2} u(t)\|_{L^\infty}) \]
\[ \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0} \sum_{2^{k_1} \geq (1 + t)^{-1/2}} 1_{X} (k, k_1, k_2) 2^{(k-k_1)/2 - N_1 \max (0, k_1, 0)} \min (2^{-N_2 k_2}, 2^{k_2/10}) \]
\[ + \varepsilon_1^2 (1 + t)^{2p_0} \sum_{k_1, k_2 \in \mathbb{Z}} 1_{X} (k, k_1, k_2) 2^{(k-k_1)/2 - 1/2 - (1-p_1)k_1} \min (2^{-N_0 k_2}, 2^{-1/2 - p_1}k_2) \]
\[ \lesssim \varepsilon_1^2 \min (2^{k/2}, 2^{-(N_1 - 1/2)k}) (1 + t)^{-1/2 + 3p_0}. \]

To prove (8.16), which gives better time decay than (8.14), we use again Lemma 2.1 ii), the symbol bounds (8.19), and the a priori bounds on \( u \) in (8.20):
\[ \|P_k M_{e_1 e_2} (u_1 (t), u_2 (t))\|_{L^2} \]
\[ \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \|r_{e_1 e_2}^{k_1 k_2}\|_{S^\infty} \min (\|P'_{k_1} u(t)\|_{L^2}, \|P'_{k_2} u(t)\|_{L^2}, \|P'_{k_1} u(t)\|_{L^\infty}, \|P'_{k_2} u(t)\|_{L^\infty}) \]
\[ \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + p_0} \sum_{2^{k_1} \geq (1 + t)^{-1/2}} 1_{X} (k, k_1, k_2) 2^{(k-k_1)/2 - 1/2 - (1-p_1)k_1} \min (2^{-N_0 k_2}, 2^{k_2/10}) \]
\[ + \varepsilon_1^2 (1 + t)^{2p_0} \sum_{k_1, k_2 \in \mathbb{Z}} 1_{X} (k, k_1, k_2) 2^{(k-k_1)/2 - 1/2 - (1-p_1)k_1} \min (2^{-N_0 k_2}, 2^{-1/2 - p_1}k_2) \]
\[ \lesssim \varepsilon_1^2 \min (2^{k/2}, 2^{-(N_2 - 1/2)k}) (1 + t)^{-1/2 + 3p_0}. \]

This completes the proof of the Lemma. \( \square \)

8.3. The profile \( f \). For \( t \in [0, T] \) we define the profile of the solution of (8.11) as
\[ f (t) := e^{itA} v (t). \] (8.22)

In the next proposition we summarize the main properties of the function \( f \).

Proposition 8.4 (Bounds for the profile). If \( u \) satisfies (2.24) - (2.26) and \( v \) and \( f \) are defined as above, then
\[ e^{itA} \partial_t f = (\partial_t - iA) v = N' \]
\[ N' := \sum_{\nu} M_{e_1 e_2} ((L u)_e_1, u_{e_2}) + M_{e_1 e_2} (u_{e_1}, (L u)_{e_2}) + C_u + R_{\geq 4}, \] (8.23)

where the bilinear operators \( M_{e_1 e_2} \) are defined via (8.10), and \( C_u \) is in (8.6).

Moreover, for any \( t \in [0, T] \) and \( k \in \mathbb{Z} \), we have the estimates
\[ \|P_k (e^{itA} f (t))\|_{L^\infty} \leq \varepsilon_1 \min (2^{k/10}, 2^{-N_2 k}) (1 + t)^{-1/2}, \] (8.24)
\[ \|P_k f (t)\|_{L^2} \leq \varepsilon_0 (1 + t)^{6p_0} \min (2^{(1/2 - p_1)k}, 2^{-(N_0 - 1/2)k}), \] (8.25)
\[ \|P_k (x \partial_x f (t))\|_{L^2} \leq \varepsilon_0 (1 + t)^{6p_0} \min (2^{(1/2 - p_1)k}, 2^{-(N_1 - 1/2)k}). \] (8.26)

Proof. The equation (8.23) follows just from the definition (8.22) and the equation (8.11). The \( L^\infty \) bound (8.24) follows from (8.17) and the a priori decay assumption in (2.25). The \( L^2 \) bound (8.25) follows from the energy estimate (8.14) in Proposition 4.1, the low frequency energy estimate (5.14) - (5.15), and the bounds (8.13) and (8.16).
To prove (8.29) we start from the identity
\[ Su = e^{it\Lambda}(x\partial_x f) + (3/2)t e^{i\Lambda}(\partial_t f), \]
which is a consequence of the commutation identity \([S, e^{it\Lambda}] = 0\). Therefore, for any \( t \in [0, T] \) and \( k \in \mathbb{Z} \),
\[
\| P_k(x\partial_x f(t)) \|_{L^2} \lesssim \| P_k(Su(t)) \|_{L^2} + (1 + t) \| P_k(\partial_t - i\Lambda) v(t) \|_{L^2}.
\]
(8.27)
Using (6.6) in Proposition 6.1 and (6.6) in Proposition 7.1 we know, in particular, that
\[
\| P_k Su(t) \|_{L^2} \lesssim \varepsilon_0 \min(2^{(1/2-p_1)k}, 2^{-N_1k})(1 + t)^{4p_0},
\]
(8.28)
for all \( k \in \mathbb{Z} \). Together with (8.15), and (8.27), this gives
\[
\| P_k(x\partial_x f(t)) \|_{L^2} \lesssim (1 + t) \| P_k(\partial_t - i\Lambda) v(t) \|_{L^2} + \varepsilon_1^2(2^k/2, 2^{-(N_1-1/2)k})(1 + t)^{-1/4 + 6p_0}
+ \varepsilon_0 \min(2^{(1/2-p_1)k}, 2^{-N_1k})(1 + t)^{4p_0}.
\]
It then suffices to show
\[
\| P_k(\partial_t - i\Lambda) v(t) \|_{L^2} \lesssim (1 + t)^{-1/2 + 6p_0} \varepsilon_1^2(2^{(1/2-p_1)k}, 2^{-(N_1-1/2)k}).
\]
(8.29)
The cubic term \(|\partial_x|O_{3,-1}\) in (8.12) is already bounded by the right-hand side above, according to the definition of \(O_{3,a}\), see (3.34). To complete the proof of (8.29) it suffices to estimate \( M_{\epsilon_1 \epsilon_2}((Su)_{\epsilon_1}, u_{\epsilon_2}) \) and \( M_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, (Lu)_{\epsilon_2}) \).

Let us recall from Lemma C.5 that \( Lu = (\partial_t - i\Lambda) u \) satisfies the bounds:
\[
\| P_k Lu(t) \|_{L^2} \lesssim \varepsilon_1^2 \min(2^k/2, 2^{-(N_0-2)k})(1 + t)^{-1/2 + p_0},
\]
\[
\| P_k Lu(t) \|_{L^\infty} \lesssim \varepsilon_1^2 \min(2^k/2, 2^{-(N_2-2)k})(1 + t)^{-1}.
\]
(8.30)
Then, using Lemma 2.3 (ii) with the symbol bounds (8.19), and (8.30), we can estimate:
\[
\| P_k M_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}(t), (Lu)_{\epsilon_2}(t)) \|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \| m_{\epsilon_1 \epsilon_2}^{k_1, k_2} \|_{S^\infty} \min \left( \| P_{k_1} u(t) \|_{L^2}, \| P_{k_2} Lu(t) \|_{L^\infty}, \| P_{k_1} u(t) \|_{L^\infty}, \| P_{k_2} Lu(t) \|_{L^2} \right)
\]
\[
\lesssim \varepsilon_1^2(1 + t)^{-1/2 + p_0} \sum_{2^k_1 \geq (1 + t)^{-4}} 1_x(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{(1/2-p_1)k_1}, 2^{-N_0k_1}) \min(2^{k_2/2}, 2^{-(N_2-2)k_2})
+ \varepsilon_1^2(1 + t)^{-1/2 + 2p_0} \sum_{2^k_1 \leq (1 + t)^{-4}} 1_x(k, k_1, k_2) 2^{(k-k_1)/2} \min(2^{(1/2-p_1)k_1}, 2^{-N_0k_1}) \min(2^{k_2/2}, 2^{-(N_2-2)k_2})
\]
\[
\lesssim \varepsilon_1^2 \min(2^{k/2}, 2^{-(N_0-5/2)k})(1 + t)^{-1/2 + 3p_0}.
\]
and, similarly,
\[
\| P_k M_{\epsilon_1 \epsilon_2}((Lu)_{\epsilon_1}(t), u_{\epsilon_2}(t)) \|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} \| m_{\epsilon_1 \epsilon_2}^{k_1, k_2} \|_{S^\infty} \| P_{k_1} u(t) \|_{L^2} \| P_{k_2} Lu(t) \|_{L^\infty}
+ \sum_{k_1, k_2 \in \mathbb{Z}} \| m_{\epsilon_1 \epsilon_2}^{k_1, k_2} \|_{S^\infty} \| P_{k_1} Lu(t) \|_{L^\infty} \| P_{k_2} u(t) \|_{L^2}
\]
\[
\lesssim \varepsilon_1^2 \min(2^{k/2}, 2^{-(N_2-1/2)k})(1 + t)^{-1/2 + 2p_0}.
\]
These last two estimates complete the proof of (8.29) and give us (8.26). □
8.4. The $Z$-norm and proof of Proposition 8.1. For any function $h \in L^2(\mathbb{R})$ let
\[
\|h\|_Z := \left\| \left( |\xi|^{1/10} + |\xi|^{N_2+1} \right) \hat{h}(\xi) \right\|_{L^\infty}. \tag{8.31}
\]

Proposition 8.5. Let $f$ be defined as in (8.22) and assume that for $T' \in [0, T]$
\[
\sup_{t \in [0,T']} \|f(t)\|_Z \leq \varepsilon_1. \tag{8.32}
\]
Then
\[
\sup_{t \in [0,T']} \|f(t)\|_Z \lesssim \varepsilon_0. \tag{8.33}
\]

We now show how to prove Proposition 8.1 using Proposition 8.5 above.

Proof of Proposition 8.1. Define
\[
z(t) := \|f(t)\|_Z,
\]
and notice that $z : [0, T] \to \mathbb{R}_+$ is a continuous function.

We show first that
\[
z(0) \lesssim \varepsilon_0. \tag{8.34}
\]
Indeed, using the definitions and Lemma 2.23, we get
\[
z(0) \lesssim \sup_{k \in \mathbb{Z}} \left( 2^{k/10} + 2^{(N_2+1)k} \right) \|\hat{P}_k v(0)\|_{L^\infty}
\lesssim \sup_{k \in \mathbb{Z}} \left( 2^{k/10} + 2^{(N_2+1)k} 2^{-k/2} \|\hat{P}_k v(0)\|_{L^2} \right) \left( 2^k \|\partial \hat{P}_k v(0)\|_{L^2} + \|\hat{P}_k v(0)\|_{L^2} \right)^{1/2}.
\]

Thanks to (8.14)-(8.16) with $t = 0$, and the initial data assumptions (2.27), we have
\[
\|\hat{P}_k v(0)\|_{L^2} \lesssim \min \varepsilon_0 \left( 2^{(1/2-p_1)k}, 2^{-(N_0-1)k} \right),
\]
\[
2^k \|\partial \hat{P}_k v(0)\|_{L^2} \lesssim \varepsilon_0 \left( 2^{(1/2-p_1)k}, 2^{-(N_1-1)k} \right),
\]
so that (8.34) follows, using also $(N_0 + N_1)/2 \geq N_2 + 1$, see (2.23).

We apply now Proposition 8.5. By continuity, $z(t) \lesssim \varepsilon_0$ for any $t \in [0, T]$, provided that $\varepsilon_0$ is sufficiently small and $\varepsilon_0 \ll \varepsilon_1 \ll \varepsilon_0^{2/3} \ll 1$ as in (2.23). Therefore, for any $k \in \mathbb{Z}$ and $t \in [0, T]$,
\[
(2^{k/10} + 2^{(N_2+1)k}) \|\hat{P}_k f(t)\|_{L^\infty} \lesssim \varepsilon_0. \tag{8.35}
\]

Recall that we aim to prove, for all $t \in [1, T]$, the decay bound
\[
\sup_{t \in [0,T]} \left( 1 + t \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \left( 2^{-k/10} + 2^{N_2k} \right) \|P_k v(t)\|_{L^\infty} \right) \lesssim \varepsilon_0, \tag{8.36}
\]
which, as already observed, implies (8.1) via (8.17). Also, observe that Lemma 2.22 applied to $v = e^{it\Lambda} f$ gives
\[
\|P_k v(t)\|_{L^\infty} \lesssim t^{-1/2} 2^{k/4} \|\hat{P}_k f(t)\|_{L^\infty} + t^{-3/5} 2^{-2k/5} \left( 2^k \|\partial \hat{P}_k f\|_{L^2} + \|\hat{P}_k f(t)\|_{L^2} \right) \tag{8.37}
\]
and
\[
\|P_k v(t)\|_{L^\infty} \lesssim t^{-1/2} 2^{k/4} \|P_k f\|_{L^1}. \tag{8.38}
\]

Recall that from (8.25) and (8.26) we have
\[
2^k \|\partial \hat{P}_k f\|_{L^2} + \|\hat{P}_k f(t)\|_{L^2} \lesssim \varepsilon_0 \left( 1 + t \right)^{6p_0} 2^{(1/2-p_1)k} \tag{8.39}
\]
We can then use (8.35) and (8.39) in (8.37), to obtain
\[
\|P_k v(t)\|_{L^\infty} \lesssim t^{-1/2} \frac{2^{k/10} + \varepsilon_0}{2^{N_2+1}k} + t^{-3/5} 2^{-2k/5} \varepsilon_0 \left( 1 + t \right)^{6p_0} 2^{(1/2-p_1)k},
\]
which is enough to show
\[ \sum_{k \in \mathbb{Z}, 2^k \leq (1+t)^{-10p_0}} (2^{-k/10} + 2^{N_2k}) \|P_k v(t)\|_{L^\infty} \lesssim \varepsilon_0 (1 + t)^{-1/2} \quad (8.40) \]

Combining (8.38), with Lemma 2.3 and (8.25)-(8.26) we see that
\[ \|P_k v(t)\|_{L^\infty} \lesssim t^{-1/2} 2^{k/4} \varepsilon_0 (1 + t)^{p_0} \min(2^{-kp_1}, 2^{-(N_0+N_1)k/2}). \]

Using also (2.23) it follows that
\[ \sum_{k \in \mathbb{Z}, 2^k \leq (1+t)^{-10p_0}} 2^{-k/10} \|P_k v(t)\|_{L^\infty} + \sum_{k \in \mathbb{Z}, 2^k \geq (1+t)^{10p_0}} 2^{N_2k} \|P_k v(t)\|_{L^\infty} \lesssim \varepsilon_0 (1 + t)^{-1/2}. \]
(8.42)

The estimates (8.40) and (8.42) give us (8.36), and conclude the proof of Proposition 8.1.

8.5. The equation for \(v\) and proof of Proposition 8.5. We now derive an equation for \(v\) with cubic terms that only involve \(v\) itself. Recall that \(u\) solves (8.3)-(8.8), or the variant (C.53) in Remark C.4, \(v\) solves (8.23), and they are related via (8.9)-(8.10).

According to (8.4) and (8.5) we define
\[ Q_v := Q_0 (v - \overline{v}, v) + \sum_{X \in \{A,B\}} X_{++}(v, v) + X_{+-}(v, \overline{v}) + X_{-+}(\overline{v}, v) + X_{--}(\overline{v}, \overline{v}). \]
(8.43)

Let us also adopt the notation
\[ \sum_{**} := \sum_{(\epsilon_1 \epsilon_2 \epsilon_3) \in \{(+++), (++-), (+-+), (-++), (-+-), (---)\}}. \]

Using (8.11) with (8.6), and (C.53), we rewrite (8.23) as:
\[ \partial_t f = e^{-itA} (N'' + R'_{\geq 4}) \]
(8.44)

where
\[ N'' := \sum_{**} M_{\epsilon_1 \epsilon_2 \epsilon_3} ((Q_v)_{\epsilon_1}, v_{\epsilon_2}) + M_{\epsilon_1 \epsilon_2 \epsilon_3} (v_{\epsilon_1}, (Q_v)_{\epsilon_2}) + \sum_{**} M_{\epsilon_1 \epsilon_2 \epsilon_3} (v_{\epsilon_1}, v_{\epsilon_2}, v_{\epsilon_3}) \]
(8.45)

and
\[ R'_{\geq 4} := \sum_{**} [M_{\epsilon_1 \epsilon_2} ((Q_v)_{\epsilon_1}, u_{\epsilon_2}) + M_{\epsilon_1 \epsilon_2} (u_{\epsilon_1}, (Q_v)_{\epsilon_2})] \]
\[ + \sum_{**} [M_{\epsilon_1 \epsilon_2} (u_{\epsilon_1}, (Q_v)_{\epsilon_2}) - M_{\epsilon_1 \epsilon_2} (v_{\epsilon_1}, (Q_v)_{\epsilon_2})] \]
\[ + \sum_{**} [M_{\epsilon_1 \epsilon_2 \epsilon_3} (u_{\epsilon_1}, u_{\epsilon_2}, u_{\epsilon_3}) - M_{\epsilon_1 \epsilon_2 \epsilon_3} (v_{\epsilon_1}, v_{\epsilon_2}, v_{\epsilon_3})] + R_{\geq 4} \]
\[ + \sum_{**} M_{\epsilon_1 \epsilon_2} ((|\partial_x|^{1/2} O_{3,-1})_{\epsilon_1}, u_{\epsilon_2}) + M_{\epsilon_1 \epsilon_2} (u_{\epsilon_1}, (|\partial_x|^{1/2} O_{3,-1})_{\epsilon_2}) \].
(8.46)

The point of the above decomposition is to identify \(N''\) as the main “cubic” part of the nonlinearity, which can be expressed only in terms of \(v(t) = e^{itA} f(t)\). \(R'_{\geq 4}\) can be thought of as a quartic remainder, due to the quadratic nature of \(u - v\), see Lemma 8.2.

To analyze the equation (8.44), and identify the asymptotic logarithmic phase correction, we need to distinguish among different types of interactions in the nonlinearity \(N''\). In order to do
this, we first define the bilinear operators

\[ Q_{++}(v, v) := Q_0(v, v) + A_{++}(v, v) + B_{++}(v, v), \]

\[ Q_{+-}(v, \overline{v}) := -Q_0(v, \overline{v}) + A_{+-}(v, \overline{v}) + A_{--}(\overline{v}, v) + B_{+-}(v, \overline{v}) + B_{--}(\overline{v}, v), \]

\[ Q_{--}(\overline{v}, \overline{v}) := A_{--}(\overline{v}, \overline{v}) + B_{--}(\overline{v}, \overline{v}), \]

with respective symbols \( q_{++}, q_{+-} \) and \( q_{--} \). Notice, see (8.43), that

\[ Q_v = Q_{++}(v, v) + Q_{+-}(v, \overline{v}) + Q_{--}(\overline{v}, \overline{v}). \]

We then write

\[ \mathcal{N}'' = C^{++} + C^{++} + C^{-+} + C^{--}, \]

where

\[ C^{++} = C^{++}(v, v, v) := M_{++}(Q_{--}(v, v) + M_{--}(v, Q_{--}(v, v)) + M_{--}(Q_{--}(v, v), v) + M_{--}(v, Q_{--}(v, v))) + M_{--}(v, Q_{--}(v, v), v) \]

\[ C^{-+} = C^{-+}(v, v, v) := M_{++}(Q_{--}(v, v) + M_{--}(v, Q_{--}(v, v)) + M_{--}(Q_{--}(v, v), v) + M_{--}(v, Q_{--}(v, v))) + M_{--}(v, Q_{--}(v, v), v) \]

\[ C^{--} = C^{--}(v, v, v) := M_{++}(Q_{--}(v, v) + M_{--}(v, Q_{--}(v, v)) + M_{--}(Q_{--}(v, v), v) + M_{--}(v, Q_{--}(v, v))) + M_{--}(v, Q_{--}(v, v), v) \]

Notice that

\[ \overline{Q_{\xi\eta\overline{\xi}\overline{\eta}}(g_1, g_2)} = -Q_{\xi\eta\overline{\xi}\overline{\eta}}(g_1, g_2), \]

since \( Q_0(g_1, g_2) = -Q_0(\overline{g_1}, \overline{g_2}) \), and \( X_{\xi\eta\overline{\xi}\overline{\eta}}(g_1, g_2) = -X_{\xi\eta\overline{\xi}\overline{\eta}}(g_1, g_2) \), for \( X \in \{ A, B \} \). Letting \( v^+ = v, v^- = \overline{v} \), we expand

\[ \tilde{C}^{\xi_{123}}(\xi) = \frac{i}{4\pi^2} \int_{\mathbb{R} \times \mathbb{R}} c_1 c_2 c_3(\xi, \eta, \sigma) \overline{v_1^2}(\eta - \sigma) \overline{v_2}(\eta - \sigma) \overline{v_3}(\sigma) d\eta d\sigma \]
for \((\nu_1, \nu_2, \nu_3) \in \{(+ + -), (+ + +), (- - +), (- - -)\}\), where

\[
\begin{align*}
    ic^{++-}(\xi, \eta, \sigma) := m_{++}(\xi, \xi - \eta)q_{++}(\eta, \sigma) + m_{++}(\xi, \eta)q_{+-}(\eta, \sigma) \\
    + m_{+-}(\xi, \sigma)q_{+-}(\xi - \eta, \xi - \eta) - m_{+-}(\xi, \eta)q_{+-}(\eta, \eta - \sigma) \\
    - m_{--}(\xi, \xi - \eta)q_{--}(\eta, \eta - \sigma) + m_{--}(\xi, \xi - \eta)q_{--}(\eta, \xi - \eta) \\
    - m_{--}(\xi, \xi - \eta)q_{--}(\xi, \xi - \eta) - m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) \\
    + m_{++}(\xi, \eta)q_{++}(\eta, \eta - \sigma) - m_{++}(\xi, \eta)q_{++}(\eta, \xi - \eta) \\
    - m_{++}(\xi, \xi - \eta)q_{++}(\xi, \eta - \sigma) + m_{++}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) \\
    + m_{++}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) - m_{++}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta)
\end{align*}
\]

\((8.52)\)

\[
\begin{align*}
    ic^{+++}(\xi, \eta, \sigma) := m_{++}(\xi, \xi - \eta)q_{++}(\eta, \sigma) + m_{++}(\xi, \eta)q_{++}(\eta, \sigma) \\
    - m_{--}(\xi, \xi - \eta)q_{--}(\eta, \sigma) + m_{--}(\xi, \xi - \eta)q_{--}(\eta, \sigma) \\
    + m_{--}(\xi, \xi - \eta)q_{--}(\xi, \xi - \eta) - m_{--}(\xi, \xi - \eta)q_{--}(\xi, \xi - \eta) \\
    - m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) + m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) \\
    - m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) + m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta)
\end{align*}
\]

\((8.53)\)

\[
\begin{align*}
    ic^{+++}(\xi, \eta, \sigma) := m_{++}(\xi, \xi - \eta)q_{++}(\eta, \sigma) + m_{++}(\xi, \eta)q_{++}(\eta, \sigma) \\
    - m_{--}(\xi, \xi - \eta)q_{--}(\eta, \sigma) + m_{--}(\xi, \xi - \eta)q_{--}(\eta, \sigma) \\
    + m_{--}(\xi, \xi - \eta)q_{--}(\xi, \xi - \eta) - m_{--}(\xi, \xi - \eta)q_{--}(\xi, \xi - \eta) \\
    - m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) + m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) \\
    - m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta) + m_{--}(\xi, \xi - \eta)q_{++}(\xi, \xi - \eta)
\end{align*}
\]

\((8.54)\)

Using the definitions of the quadratic symbols \((8.10)\) and \((8.43)\), with \((8.5)\) and \((3.31)-(3.38)\), we see that the cubic symbols \(c^{11243}\) are real-valued. Recalling the formulas

\[
\begin{align*}
    \hat{w}^{\pm}(\xi, \tau) = \hat{f}(\xi, \tau)e^{\pm i\xi^3/2}, & \quad \hat{w}^{-}(\xi, \tau) = \hat{f}(\xi, \tau)e^{-i\xi^3/2},
\end{align*}
\]

we can rewrite

\[
\begin{align*}
    F(e^{-it\Delta}\Psi''(\tau))(\xi) = \frac{i}{4\pi^2}\left[ I^{++-}(\xi, \tau) + I^{+++}(\xi, \tau) + I^{-+\pm}(\xi, \tau) + I^{--\pm}(\xi, \tau) \right],
\end{align*}
\]

\((8.55)\)

where

\[
I^{11243}(\xi, \tau) := \int_{R \times \mathbb{R}} e^{it(-|\xi|^{3/2} + |\xi - \eta|^{3/2} + 3i\tau|\eta - \sigma|^{3/2} + i\tau_3|\sigma|^{3/2})}
\times c^{11243}(\xi, \eta, \sigma)\int_{\mathbb{R}} f''(\xi)\int_{\mathbb{R}} f''(\tau)\,d\eta\,d\sigma
\]

\((8.56)\)

for \((\nu_1, \nu_2, \nu_3) \in \{(+ + -), (+ + +), (- - +), (- - -)\}\). The formula \((8.44)-(8.46)\) then becomes

\[
(\partial_t\hat{f})(\xi, \tau) = \frac{i}{4\pi^2}[I^{++-}(\xi, \tau) + I^{+++}(\xi, \tau) + I^{-+\pm}(\xi, \tau) + I^{--\pm}(\xi, \tau)] + e^{-i|\xi|^{3/2}}\widetilde{R}'(\xi, \eta, \sigma).
\]

\((8.55)\)

In analyzing the formula \((8.55)\), the main contribution comes from the stationary points of the phase functions \((t, \eta, \sigma) \rightarrow t\Phi^{11243}(\xi, \eta, \sigma)\), where

\[
\Phi^{11243}(\xi, \eta, \sigma) := -|\xi|^{3/2} + \nu_1|\xi - \eta|^{3/2} + \nu_2|\eta - \sigma|^{3/2} + \nu_3|\sigma|^{3/2}.
\]

\((8.56)\)

More precisely, one needs to understand the contribution of the \textit{spacetime resonances}, i.e., the points where

\[
\Phi^{11243}(\xi, \eta, \sigma) = (\partial_\eta \Phi^{11243})(\xi, \eta, \sigma) = (\partial_\sigma \Phi^{11243})(\xi, \eta, \sigma) = 0.
\]

In our case, it can be easily verified that the only spacetime resonances correspond to \((\nu_1 \nu_2 \nu_3) = (+ + -)\) and the points \((\xi, \eta, \sigma) = (\xi, 0, -\xi)\). Moreover, the contribution from these points is not
absolutely integrable in time, and we have to identify and eliminate its leading order term using a suitable logarithmic phase correction. More precisely, see also (A.28), we define

$$
\tilde{c}(\xi) := \frac{8\pi |\xi|^{1/2}}{3} e^{++-}(\xi, 0, -\xi) = \frac{\pi |\xi|^2}{3},
$$

$$
L(\xi, t) := \frac{\tilde{c}(\xi)}{4\pi^2} \int_0^t |\hat{f}(\xi, s)|^2 \frac{1}{s+1} \, ds,
$$

$$
g(\xi, t) := e^{iL(\xi, t)} \hat{f}(\xi, t).
$$

The formula (8.55) then becomes

$$
(\partial_t g)(\xi, t) = \frac{i}{4\pi^2} e^{iL(\xi, t)} \left[ I^{++-}(\xi, t) + \tilde{c}(\xi) \frac{\hat{f}(\xi, t)^2}{t+1} \hat{f}(\xi, t) \right]
+ \frac{i}{4\pi^2} e^{iL(\xi, t)} \left[ I^{+++}(\xi, t) + I^{+++}(\xi, t) + I^{---}(\xi, t) \right]
+ e^{-it|\xi|^{3/2}} e^{iL(\xi, t)} \hat{r}_{\geq 4}(\xi, t).
$$

Notice that the phase $L$ is real-valued. Therefore, to complete the proof of Proposition 8.5, it suffices to prove the following main lemma:

**Lemma 8.6.** Recall that from (8.25)-(8.26) in Proposition 8.4, and the apriori assumption (8.32), we have for any $t \in [0, T']$ and $k \in \mathbb{Z}$,

$$
\|\hat{P}_k \hat{f}(t)\|_{L^\infty} \lesssim \varepsilon_k \min \left( 2^{-k/10}, 2^{-(N_2+1)k} \right)
\|\hat{P}_k \hat{f}(t)\|_{L^2} \lesssim \varepsilon_0 (1 + t)^{6p_0} \min \left( 2^{k(1/2 - p_0)}, 2^{-(N_0-1)k} \right),
\|\hat{P}_k \hat{f}(t)\|_{L^2} + 2^k \|\partial \hat{P}_k \hat{f}(t)\|_{L^2} \lesssim \varepsilon_0 (1 + t)^{6p_0} \min \left( 2^{k(1/2 - p_0)}, 2^{-(N_1-1)k} \right).
$$

Under these assumptions, for any $m \in \{1, 2, \ldots\}$ and any $t_1 \leq t_2 \in [2^m - 2, 2^{m+1}] \cap [0, T']$, we have

$$
\|\left( |\xi|^{1/10} + |\xi|^{N_2+1} \right) |g(\xi, t_2) - g(\xi, t_1)|\|_{L^\infty} \lesssim \varepsilon_0 2^{-p_0 m}.
$$

**9. Proof of Lemma 8.6**

In this section we provide the proof of Lemma 8.6, which is the analogue of Lemma 4.6 in [34]. We first notice that the desired conclusion can be easily proved for large and small enough frequencies. Indeed, for any $t \in [2^m - 2, 2^{m+1}] \cap [0, T']$, and any $|\xi| \approx 2^k$ with $k \in \mathbb{Z}$ and

$$
k \in (-\infty, -80p_0) \cup [20p_0, \infty),
$$

we can use the interpolation inequality (2.21) and the bounds (8.59) to obtain

$$
\left( |\xi|^{1/10} + |\xi|^{N_2+1} \right) |g(\xi, t)| \lesssim \left( 2^{k/10} + 2^{(N_2+1)k} \right) \|\hat{P}_k \hat{f}(t)\|_{L^\infty}
\lesssim \left( 2^{k/10} + 2^{(N_2+1)k} \right) \left[ 2^{-k} \|\hat{P}_k \hat{f}\|_{L^2} \left( 2^k \|\partial \hat{P}_k \hat{f}\|_{L^2} + \|\hat{P}_k \hat{f}\|_{L^2} \right) \right]^{1/2}
\lesssim \varepsilon_0 (1 + t)^{6p_0} \min \left( 2^{k(1/10 - p_0)}, 2^{-k/2} \right),
\lesssim \varepsilon_0,
$$

having also used $(N_0 + N_1)/2 \geq N_2 + 3/2$, see (2.23).
It remains to prove (8.60) in the intermediate range $|\xi| \in [(1 + t)^{-80p_0}, (1 + t)^{20p_0}]$. For $k \in \mathbb{Z}$ let $f_k^+ := P_k f$ and $f_k^- := P_k \overline{f}$ and, for any $k_1, k_2, k_3 \in \mathbb{Z}$, let

$$I_{k_1, k_2, k_3}^{114t3} (\xi, t) := \int_{\mathbb{R} \times \mathbb{R}} e^{it(-|\xi|^{3/2} + t_1 |\xi - \eta|^{3/2} + t_2 |\eta - \sigma|^{3/2} + t_3 |\sigma|^{3/2})}$$

\[ \times e^{i14t3} (\xi, \eta, \sigma) f_{k_1}^{\pm} (\xi - \eta) f_{k_2}^{\pm} (\eta - \sigma) f_{k_3}^{\pm} (\sigma) \, d\eta d\sigma. \] (9.1)

Using (8.59) and Lemma 2.2 we know that for any $m$ frequencies (relative to $m$),

$$\int_{[0, T]} |f_k^+ (t)|^2 \, dt \lesssim 2^{1/2-p} (1 + t)^{6p_0},$$

$$\int_{[0, T]} |e^{itA} f_k^+ (t)|^2 \, dt \lesssim 2^{1/10} (1 + t)^{-1/2},$$

wheras, for $l \geq 0$

$$\int_{[0, T]} |f_k^+ (t)|^2 \, dt \lesssim 2^{-(N_0 - 1)} (1 + t)^{6p_0},$$

$$\int_{[0, T]} |e^{itA} f_k^+ (t)|^2 \, dt \lesssim 2^{-(N_1 - 1)} (1 + t)^{-1/2},$$

$$\int_{[0, T]} |f_k^+ (t)|^2 \, dt \lesssim 2^{-(N_2 + 1)}.$$ (9.3)

Using (8.52) and the definitions (8.10) and (8.47), it is not hard to see that

$$\left| e^{i14t3} (\xi, \eta, \sigma) \cdot \varphi_{k_1} (\xi - \eta) \varphi_{k_2} (\eta - \sigma) \varphi_{k_3} (\sigma) \right| \lesssim 2^{2 \max(k_1, k_2, k_3)} 2^{\min(k_1, k_2, k_3)/2}. $$ (9.4)

Using this one can decompose the integrals $I_{k_1, k_2, k_3}^{114t3}$ into sums of the integrals $I_{k_1, k_2, k_3}^{114t3}$, and then estimate the terms corresponding to large frequencies, and the terms corresponding to small frequencies (relative to $m$), using only the bounds (9.3) - (9.4). As in [34, Section 5], we can then reduce matters to proving the following:

**Lemma 9.1.** Assume that $k \in [-80p_0m, 20p_0m]$, $|\xi| \in [2^k, 2^{k+1}] \cap \mathbb{Z}$, $m \geq 1$, $t_1 \leq t_2 \in [2^m - 2, 2^{m+1}] \cap [0, T']$, and $k_1, k_2, k_3$ are integers satisfying

$$k_1, k_2, k_3 \in [-3m, 3m/N_0 - 1000],$$

$$\min(k_1, k_2, k_3)/2 + 3\overline{\text{med}(k_1, k_2, k_3)}/2 \geq m(1 + 10p_0).$$ (9.5)

Then

$$\left| \int_{t_1}^{t_2} e^{iL(\xi, s)} \left[ I_{k_1, k_2, k_3}^{114t3} (\xi, s) + e^{c(\xi)} \frac{f_{k_1}^{\pm} (\xi, s) f_{k_2}^{\pm} (\xi, s) f_{k_3}^{\pm} (-\xi, s)}{s + 1} \right] \, ds \right| \lesssim \varepsilon_1^3 2^{-200p_0},$$ (9.6)

and, for $(t_1, t_2, t_3) \in \{(++, +), (-, -), (++, -), (-, +)\},$

$$\left| \int_{t_1}^{t_2} e^{iL(\xi, s)} I_{k_1, k_2, k_3}^{114t3} (\xi, s) \, ds \right| \lesssim \varepsilon_1^3 2^{-200p_0}. $$ (9.7)

Moreover

$$\left| \int_{t_1}^{t_2} e^{iL(\xi, s)} e^{-is|\xi|^{3/2}} \mathcal{R}^{\leq 4}_{\geq 3} (\xi, s) \, ds \right| \lesssim \varepsilon_1^3 2^{-200p_0}. $$ (9.8)
The rest of this section is concerned with the proof of this lemma. We will often use the alternative formulas
\[ I_{k_1,k_2,k_3}^{\xi_1\xi_2\xi_3}(\xi,t) = \int_{\mathbb{R}^2} e^{it\Phi^{\xi_1\xi_2\xi_3}(\xi,\eta,\sigma)} c^{\xi_1\xi_2\xi_3}(\xi,k_1,k_2,k_3) \tilde{f}_{k_1}(\xi + \eta) \tilde{f}_{k_2}(\xi + \sigma) \tilde{f}_{k_3}(-\xi - \eta - \sigma) \, d\eta d\sigma, \tag{9.9} \]
where
\[ \Phi^{\xi_1\xi_2\xi_3}(\xi,x,y) := -\Lambda(\xi) + t_1 \Lambda(\xi + x) + t_2 \Lambda(\xi + y) + t_3 \Lambda(\xi + x + y), \]
\[ c^{\xi_1\xi_2\xi_3}(\xi,k_1,k_2,k_3)(x,y) := c^{\xi_1\xi_2\xi_3}(\xi,-x,-\xi - y) \cdot \varphi_{k_1}(\xi + x) \varphi_{k_2}(\xi + y) \varphi_{k_3}(\xi + x + y). \tag{9.10} \]
These formulas follow from (9.1) via simple changes of variables.

We now recall that the symbols \( c^{\xi_1\xi_2\xi_3} \) are given by the explicit formulas \( \Phi \) for \( m_{\xi_1\xi_2} \), and \( \Phi_{\xi_1\xi_2} \xi_3 \) for \( q_{\xi_1\xi_2} \). Using these formulas, the bound (8.19) for \( m_{\xi_1\xi_2} \), and the bounds (8.9) and (8.26), one can verify that the symbols \( c^{\xi_1\xi_2\xi_3} \) satisfy the estimates
\[ \| F^{-1}(c^{\xi_1\xi_2\xi_3}) \|_{L^1(\mathbb{R}^2)} \lesssim 2^4 \max(l_{\text{max}},0) 2^{(l_{\text{med}}-l_{\text{min}})/2}, \tag{9.11} \]
and
\[ \| F^{-1}((\partial_\eta c^{\xi_1\xi_2\xi_3})_{\xi,k_1,k_2,k_3}(\eta)) \|_{L^1(\mathbb{R}^2)} \lesssim 2^4 \max(l_{\text{max}},0) 2^{(l_{\text{med}}-l_{\text{min}})/2} 2^{-l_{\text{med}}/2} \lesssim 2^4 \max(l_{\text{max}},0) 2^{(l_{\text{med}}-l_{\text{min}})/2} 2^{-l_{\text{med}}/2}, \tag{9.12} \]
for any \( \xi \in \mathbb{R} \) and any \( l_1,l_2,l_3 \in \mathbb{Z} \), where \( l_{\text{max}} := \max(l_1,l_2,l_3) \), \( l_{\text{med}} := \min(l_1,l_2,l_3) \), \( l_{\text{min}} := \min(l_1,l_2,l_3) \).

9.1. Proof of (9.6). We divide the proof of the bound (9.6) into several lemmas. Since in this subsection we will only deal with interactions of the type \( ++ \), for simplicity of notation we denote \( \Phi := \Phi^{++} \) and \( c^k := c^{k,++} \).

**Lemma 9.2.** The bound (9.6) holds provided that (9.5) holds, and
\[ \max(|k-k_1|,|k-k_2|,|k-k_3|) \leq \frac{1}{20}. \]

**Proof.** This is the case which gives the precise form of the correction. However, the proof is similar to the proof of Lemma 6.4 in [32]. The only difference is that in the present case \( \Lambda(\xi) = |\xi|^{3/2} \) instead of \( |\xi|^{1/2} \). Therefore one has the expansion
\[ \Lambda(\xi) - \Lambda(\xi + \eta) - \Lambda(\xi + \sigma) + \Lambda(\xi + \eta + \sigma) - \frac{3\eta \sigma}{4|\xi|^{3/2}} \lesssim 2^{-3k^2/2(|\eta|^{3} + |\sigma|^{3}}. \]
One can then follow the same argument in [32] Lemma 6.4 to obtain the desired bound. \( \square \)

**Lemma 9.3.** The bound (9.6) holds provided that (9.5) holds and, in addition,
\[ \max(|k-k_1|,|k-k_2|,|k-k_3|) \geq \frac{1}{21}, \]
\[ \max(|k_1-k_3|,|k_2-k_3|) \geq \frac{1}{5} \quad \text{and} \quad \min(k_1,k_2,k_3) \geq -\frac{3}{100} m. \tag{9.13} \]

**Proof.** In this case we will show the stronger bound
\[ |I_{k_1,k_2,k_3}^{\xi_1\xi_2\xi_3}(\xi,s)| \lesssim \varepsilon_1 2^{-m/2} 2^{-200|m|}. \tag{9.14} \]
Without loss of generality, by symmetry we can assume that \( |k_1-k_3| \geq 5 \) and \( k_2 \leq \max(k_1,k_3) + 5 \). Under the assumptions (9.13) we have
\[ |(\partial_\eta \Phi)(\xi,\eta,\sigma)| = |\Lambda'(\xi + \eta) - \Lambda'(\xi + \eta + \sigma)| \gtrsim 2^{k_{\text{max}}}. \tag{9.15} \]
Therefore we can integrate by parts in $\eta$ in the integral expression (9.9) for $I_{k_1,k_2,k_3}^{++}$. This gives
\[
|I_{k_1,k_2,k_3}^{++}(\xi,s)| \lesssim |K_1(\xi,s)| + |K_2(\xi,s)| + |K_3(\xi,s)| + |K_4(\xi,s)|,
\]
where
\[
K_1(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i \Phi(\xi,\eta,\sigma)} m_1(\eta,\sigma) c_k(\eta,\sigma) (\partial \tilde{f}_{k_1}^+)(\xi + \eta) \tilde{f}_{k_2}^+(\xi + \sigma) \tilde{f}_{k_3}^-(\xi - \eta - \sigma) \, d\eta d\sigma,
\]
\[
K_2(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i \Phi(\xi,\eta,\sigma)} m_1(\eta,\sigma) c_k(\eta,\sigma) \tilde{f}_{k_1}^+(\xi + \eta) \tilde{f}_{k_2}^+(\xi + \sigma) (\partial \tilde{f}_{k_3}^-)(\xi - \eta - \sigma) \, d\eta d\sigma,
\]
\[
K_3(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i \Phi(\xi,\eta,\sigma)} (\partial_{\eta} m_1)(\eta,\sigma) c_k(\eta,\sigma) \tilde{f}_{k_1}^+(\xi + \eta) \tilde{f}_{k_2}^+(\xi + \sigma) \tilde{f}_{k_3}^-(\xi - \eta - \sigma) \, d\eta d\sigma,
\]
\[
K_4(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i \Phi(\xi,\eta,\sigma)} m_1(\eta,\sigma) (\partial_{\eta} c_k)(\eta,\sigma) \tilde{f}_{k_1}^+(\xi + \eta) \tilde{f}_{k_2}^+(\xi + \sigma) \tilde{f}_{k_3}^-(\xi - \eta - \sigma) \, d\eta d\sigma,
\]
having denoted
\[
m_1(\eta,\sigma) := \frac{1}{s \partial_\eta \Phi(\xi,\eta,\sigma)} \varphi_{k_1}'(\xi + \eta) \varphi_{k_3}'(\xi + \eta + \sigma).
\]

Observe that, under the restrictions (9.13), we have
\[
\|m_1(\eta,\sigma)\|_{S^\infty} \lesssim 2^{-m} 2^{-k_{\max}/2}.
\]

We can then estimate $K_1$ using Lemma 2.11(ii), the estimate on $c_k^*$ in (9.11), the bounds (9.2)-(9.3), and the last constraint in (9.13). More precisely, if $k_1 \leq k_3$ (so that $2^{k_1} \approx 2^{k_{\max}}$) then we estimate
\[
|K_1(\xi,s)| \lesssim \|m_1(\eta,\sigma)\|_{S^\infty} \|c_k^*(\eta,\sigma)\|_{S^\infty} \|	ilde{f}_{k_1}^+(s)\|_{L^2} \|e^{i \lambda \tilde{f}_{k_2}^+(s)}\|_{L^\infty} \|	ilde{f}_{k_3}^-(s)\|_{L^2}
\]
\[
\lesssim 2^{-m} 2^{-k_{\max}/2} \cdot 2^5 \max(k_3,0) 2^{-k_{\min}/2} \cdot \varepsilon_1 2^{-k_1(1/2+p_0)} e^{6 m_0} 2^{-2-(N_0-1)k_3}
\]
\[
\lesssim \varepsilon_1^2 2^{-3m/2} 2^{12 p_0 m} 2^{-k_{\min}/2} 2^{-k_1(1/2+p_0)}
\]
\[
\lesssim \varepsilon_1^2 2^{-3m/2} 2^{15 p_0 m} 2^{-2k_{\min}}.
\]

This suffices to prove (9.14) because of the last inequality in (9.3). On the other hand, if $k_1 \geq k_3$ (so that $2^{k_1} \approx 2^{k_{\max}} \approx 2^{k_3}$) then
\[
|K_1(\xi,s)| \lesssim \|m_1(\eta,\sigma)\|_{S^\infty} \|c_k^*(\eta,\sigma)\|_{S^\infty} \|	ilde{f}_{k_1}^+(s)\|_{L^2}
\]
\[
\times \min \left( e^{i \lambda \tilde{f}_{k_2}^+(s)}\right)_{L^\infty} \|	ilde{f}_{k_3}^-(s)\|_{L^2} \|	ilde{f}_{k_2}^+(s)\|_{L^2} \|e^{-i \lambda \tilde{f}_{k_3}^-(s)}\|_{L^\infty}
\]
\[
\lesssim 2^{-m} 2^{-k_{\max}/2} \cdot 2^{4.5 \max(k_1,0)} 2^{-k_{\min}/2} \cdot \varepsilon_1 2^{-N_1 k_1} 6 m_0
\]
\[
\times \varepsilon_1^2 2^{-3m/2} 2^{12 p_0 m} 2^{-k_{\min}/2} 2^{k_{\max} 2^{-2-(N_0-1) \max(k_2,k_3,0)}}
\]
\[
\lesssim \varepsilon_1^2 2^{-3m/2} 2^{14 p_0 m} 2^{-k_{\min}/2} 2^{k_{\max} 2^{-2-(N_0-1) \max(k_2,k_3,0)}}.
\]

This suffices to prove (9.14), since $2^{k_{\max} 2^{-6}} \max(k_2,k_3,0) \leq 2^{30 p_0 m}$ as a consequence of the assumption $k \leq 20 p_0 m$. The estimate for $|K_2(s,\xi)|$ is identical. Using
\[
\|\partial_{\eta} m_1(\eta,\sigma) c_k^*(\eta,\sigma)\|_{S^\infty} + \|m_1(\eta,\sigma) \partial_{\eta} c_k^*(\eta,\sigma)\|_{S^\infty}
\]
\[
\lesssim 2^{-m} 2^{-k_{\max}/2} 2^{4 \max(k_{\max},0)} 2^{(k_{\max}-k_{\min})/2} 2^{-2 \min(k_1,k_3)},
\]
one can estimate similarly $|K_3(s,\xi)|$, and $|K_4(s,\xi)|$. \qed
Lemma 9.4. The bound (9.6) holds provided that (9.3) holds and, in addition,
\[
\max(|k - k_1|, |k - k_2|, |k - k_3|) \geq 21,
\]
\[
\max(|k_1 - k_3|, |k_2 - k_3|) \geq 5 \quad \text{and} \quad \min(k_1, k_2) \leq -\frac{48m}{100}.
\] (9.19)

Proof. By symmetry we may assume that \( k_2 = \min(k_1, k_2) \). The main observation is that we still have the strong lower bound
\[
|\langle \partial_\eta \Phi \rangle(\xi, \eta, \sigma)| = |\Lambda'(\xi + \eta) - \Lambda'(\xi + \eta + \sigma)| \gtrsim 2^{-k_{\max}/2} k^k.
\]
This is easy to verify since \( |\xi| \geq 2^{-80p_0m} \), \( |\xi + \sigma| \leq 2^{k_2+1} \leq 2^{-48m/100+1} \). We can then integrate by parts in \( \eta \) and estimate the resulting integrals as in Lemma 9.3 by placing \( f_{k_2} \) in \( L^2 \), and using the restriction \( k_{\min} + 3k_{\med} \geq -2m(1 + 10p_0) \) in (9.3).

Lemma 9.5. The desired bound (9.6) holds provided that (9.5) holds and, in addition,
\[
\max(|k - k_1|, |k - k_2|, |k - k_3|) \geq 21,
\]
\[
\max(|k_1 - k_3|, |k_2 - k_3|) \geq 5 \quad \text{and} \quad \min(k_1, k_2) \geq -\frac{48m}{100} \quad \text{and} \quad k_3 \leq -\frac{49m}{100}.
\] (9.20)

Proof. In this case we need to integrate by parts in time. Without loss of generality, we may again assume \( k_2 = \min(k_1, k_2) \). Therefore, also in view of (9.5), we have
\[
k_3 \leq -49m/100 \leq -48m/100 \leq k_2 \leq k_1, \quad k_1 \geq k - 10 \geq -80p_0m - 10. \] (9.21)

Recall that
\[
\Phi(\xi, \eta, \sigma) = -\Lambda(\xi) + \Lambda(\xi + \eta) + \Lambda(\xi + \sigma) - \Lambda(\xi + \eta + \sigma).
\]
For \( |\xi + \eta| \approx 2^{k_1} \), \( |\xi + \sigma| \approx 2^{k_2} \), \( |\xi + \eta + \sigma| \approx 2^{k_3} \), with \( k_1, k_2, k_3 \) satisfying (9.21), one can use (9.2) to show that
\[
|\Phi(\xi, \eta, \sigma)| \gtrsim | - \Lambda(\xi) + \Lambda(\xi + \eta) + \Lambda(\eta) - 2^{10}|\xi + \eta + \sigma|^{k_2/2} \geq 2^{k_2/2}. \] (9.22)

Thanks to this lower bound we can integrate by parts in \( s \) to obtain
\[
\left| \int_{t_1}^{t_2} e^{iL(\xi, s)} f_{k_1, k_2, k_3}(\xi, s) \, ds \right| \lesssim |N_1(\xi, t_1)| + |N_1(\xi, t_2)|
\]
\[
+ \int_{t_1}^{t_2} |N_2(\xi, s)| + |N_3(\xi, s)| + |N_4(\xi, s)| + |(\partial_s L)(\xi, s)||N_1(\xi, s)| \, ds,
\] (9.23)

where
\[
N_1(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} \frac{c_\xi^\eta(\eta, \sigma)}{i\Phi(\xi, \eta, \sigma)} f_{k_1}^+ f_{k_2}^+ f_{k_3}^+(-\xi - \eta - \sigma) \, d\eta d\sigma,
\]
\[
N_2(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} \frac{c_\xi^\eta(\eta, \sigma)}{i\Phi(\xi, \eta, \sigma)} f_{k_1}^+ f_{k_2}^+ f_{k_3}^+(-\xi - \eta - \sigma) \, d\eta d\sigma,
\]
\[
N_3(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} \frac{c_\xi^\eta(\eta, \sigma)}{i\Phi(\xi, \eta, \sigma)} f_{k_1}^+ f_{k_2}^+ f_{k_3}^+(-\xi - \eta - \sigma) \, d\eta d\sigma,
\]
\[
N_4(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} \frac{c_\xi^\eta(\eta, \sigma)}{i\Phi(\xi, \eta, \sigma)} f_{k_1}^+ f_{k_2}^+ f_{k_3}^+(-\xi - \eta - \sigma) \, d\eta d\sigma.
\] (9.24)

To estimate the first term in (9.23) we first notice that we have the pointwise bound
\[
\left| \frac{c_\xi^\eta(\eta, \sigma)}{\Phi(\xi, \eta, \sigma)} \right| \lesssim 2^{k_1} 2^{-k_2/2} 2^{-k_3/2} 2^{-k_1/2},
\] (9.25)
see \((9.22)\) and \((9.11)\). Using also \((9.2)\), \((9.3)\) we can obtain, for any \(s \in [t_1, t_2]\),
\[
|N_1(\xi, s)| \lesssim 2^{4\max(k_1,0)}2^{-k/2}2^{-k_1/2}2^{-k_3/2}||\hat{f}^+_{k_1}(s)||_{L^\infty}2^{k_3/2}||\hat{f}^+_{k_2}(s)||_{L^2}2^{k_3/2}||\hat{f}^+_{k_3}(s)||_{L^2} \\
\lesssim \varepsilon_1^22^{-k_3(1/2-p_1)}2^{100p_0m} \\
\lesssim \varepsilon_1^22^{-m/10}.
\]
Moreover, the definition of \(L\) in \((8.57)\), and the a priori assumptions \((9.2)-(9.3)\), show that
\[
|\langle \partial_s L(\xi, s) \rangle| \lesssim \varepsilon_1^22^{-m}.
\]
Therefore
\[
|N_1(\xi, t_1)| + |N_1(\xi, t_2)| + \int_{t_1}^{t_2} \langle \partial_s L(\xi, s) \rangle|N_1(\xi, s)| ds \lesssim \varepsilon_1^22^{-m/10}.
\]
Using the equation for \(\partial_t f\) in \((8.23)\), and the estimate \((8.29)\), we have
\[
\|\langle \partial_s \hat{f}^+ \rangle(s) \|_{L^2} \lesssim \varepsilon_1^2 \min(2^{(1/2-p_1)}, 2^{-2})2^{-m+6p_0m}.
\]
Using this \(L^2\) bound, and the pointwise bound \((9.25)\) on the symbol, the term \(N_2\) can be estimated as follows:
\[
|N_2(\xi, s)| \lesssim 2^{4\max(k_1,0)}2^{-k/2}2^{-k_2/2}2^{-k_1/2}||\partial_s \hat{f}^+_{k_1}(s)\|_{L^2}||\hat{f}^+_{k_2}(s)\|_{L^2}||\hat{f}^+_{k_3}(s)\|_{L^2}2^{k_3/2} \\
\lesssim \varepsilon_1^22^{-m+100p_0m}2^{-p_1k_2}2^{k_3(1/2-p_1)} \\
\lesssim \varepsilon_1^22^{-m}2^{-m/10}.
\]
The integral \(|N_3(\xi, s)|\) can be estimated in the same way. To deal with the last term in \((9.24)\) we use again \((9.25)\), \((9.27)\), and the a priori bounds \((9.2)-(9.3)\) to obtain:
\[
|N_4(\xi, s)| \lesssim 2^{4\max(k_1,0)}2^{-k/2}2^{-k_2/2}2^{-k_1/2}||\hat{f}^+_{k_1}(s)\|_{L^2}||\hat{f}^+_{k_2}(s)\|_{L^2}||\partial_s \hat{f}^+_{k_3}(s)\|_{L^2}2^{k_3/2} \\
\lesssim \varepsilon_1^22^{-m+100p_0m}2^{-p_1k_2}2^{(1/2-p_1)} \\
\lesssim \varepsilon_1^22^{-m}2^{-m/10}.
\]
We deduce that
\[
\int_{t_1}^{t_2} |N_2(\xi, s)| + |N_3(\xi, s)| + |N_4(\xi, s)| ds \lesssim \varepsilon_1^22^{-m/10},
\]
and the lemma follows from \((9.23)\) and \((9.26)\).

\textbf{Lemma 9.6.} The bound \((9.6)\) holds provided that \((9.5)\) holds and, in addition,
\[
\max(|k-k_1|, |k-k_2|, |k-k_3|) \geq 21 \quad \text{and} \quad \max(|k_1-k_3|, |k_2-k_3|) \leq 4.
\]
\textbf{Proof.} In this case we have \(\min(k_1, k_2, k_3) \geq k+10\), so that, in particular, \(|\sigma| \approx 2k^2\). Since all input frequencies are comparable in view of the second assumption in \((9.28)\), we can see that
\[
|\langle \partial_\eta \Phi \rangle(\xi, \eta, \sigma)| = |\Lambda'(\xi + \eta) - \Lambda'(\xi + \eta + \sigma)| \gtrsim 2^{\max/2},
\]
which is the same lower bound as in \((9.13)\). We can then integrate by parts in \(\eta\) similarly to what was done before in Lemma 9.3. This gives
\[
|I_{k_1k_2k_3}^+(\xi, s)| \lesssim |K_1(\xi, s)| + |K_2(\xi, s)| + |K_3(\xi, s)| + |K_4(\xi, s)|
\]
where the term \(K_j, j = 1, \ldots, 4\) are defined in \((9.16)-(9.17)\), and the bound \((9.17)\) is satisfied. Then, the same estimates that followed \((9.18)\) show that \(|I_{k_1k_2k_3}^+(\xi, s)| \lesssim \varepsilon_1^22^{-m}2^{-200p_0m}\), which suffices to prove the lemma.
9.2. Proof of (9.7). We divide the proof of the bound (9.7) into several lemmas. We only consider in detail the case \((\iota_1\iota_2\iota_3) = (-+-),\) since the cases \((\iota_1\iota_2\iota_3) = (+++)\) or \((-+-)\) are very similar. In the rest of this subsection we let \(\Phi := \Phi^{++}\) and \(c_k := c_{\xi,k_1,k_2,k_3}^{++}\).

**Lemma 9.7.** The bound (9.7) holds provided that (9.5) holds and, in addition,

\[
\max(|k_1 - k_3|, |k_2 - k_3|) \geq 5 \quad \text{and} \quad \min(k_1, k_2, k_3) \geq -\frac{49}{100}m. \quad (9.29)
\]

**Proof.** This case is similar to the proof of Lemma 9.3. Without loss of generality, by symmetry we can assume that \(|k_1 - k_3| \geq 5\) and \(k_2 \leq \max(k_1, k_3) + 5\). Under the assumptions (9.29) we still have the strong lower bound

\[
|\partial_\eta \Phi(\xi, \eta, \sigma)| = |\Lambda'(\xi + \eta) - \Lambda'(\xi + \eta + \sigma)| \gtrsim 2^{k_{\max}/2}.
\]

The proof can then proceed exactly as in Lemma 9.3, using integration by parts in \(\eta\). \(\square\)

**Lemma 9.8.** The bound (9.7) holds provided that (9.5) holds and, in addition,

\[
\max(|k_1 - k_3|, |k_2 - k_3|) \geq 5 \quad \text{and} \quad \text{med}(k_1, k_2, k_3) \leq -48m/100. \quad (9.30)
\]

**Proof.** This is similar to the situation in Lemma 9.4. By symmetry we may assume that \(k_2 = \min(k_1, k_3)\). The main observation is that in this case we must have \(|k_1 - k_3| \geq 5\), because of the second assumption in (9.30) and \(k \geq -80p_0m\). We then have the lower bound

\[
|\partial_\eta \Phi(\xi, \eta, \sigma)| = |\Lambda'(\xi + \eta) - \Lambda'(\xi + \eta + \sigma)| \gtrsim 2^{k_{\max}/2}.
\]

Thus, we integrate by parts in \(\eta\) and estimate the resulting integrals as in Lemma 9.3 by placing \(\hat{f}_{k_2}\) in \(L^2\), and using the restriction \(k_{\min} + 3k_{\text{med}} \geq -2m(1 + 10p_0)\), see (9.5). \(\square\)

**Lemma 9.9.** The bound (9.7) holds provided that (9.5) holds and, in addition,

\[
\max(|k_1 - k_3|, |k_2 - k_3|) \geq 5 \quad \text{and} \quad k_{\text{med}} \geq -\frac{48m}{100} \quad \text{and} \quad k_{\min} \leq -\frac{49m}{100}.
\]

**Proof.** This situation is similar to the one in Lemma 9.5. The main observation is that we have the lower bound

\[
|\Phi(\xi, \eta, \sigma)| \gtrsim 2^{k_{\text{med}}/2} 2^{k/2},
\]

so that we can integrate by parts in time and estimate the resulting integrals as in the proof of Lemma 9.5. \(\square\)

**Lemma 9.10.** The bound (9.7) holds provided that (9.5) holds and, in addition,

\[
\max(|k - k_1|, |k - k_2|, |k - k_3|) \geq 21 \quad \text{and} \quad \max(|k_1 - k_3|, |k_2 - k_3|) \leq 4.
\]

**Proof.** This case is similar to that of Lemma 9.6. Observing that \(k + 10 \leq \min(k_1, k_2, k_3)\), we see that

\[
|\partial_\eta \Phi(\xi, \eta, \sigma)| = |\Lambda'(\xi + \eta) - \Lambda'(\xi + \eta + \sigma)| \gtrsim 2^{k_{\max}/2}.
\]

We can then use again integration by parts in \(\eta\) to obtain the desired bound. \(\square\)

**Lemma 9.11.** The bound (9.7) holds provided that (9.5) holds and, in addition,

\[
\max(|k - k_1|, |k - k_2|, |k - k_3|) \leq 20. \quad (9.31)
\]
Proof. This is the main case where there is a substantial difference between the integrals $I_{k_1, k_2, k_3}^{++, +}$ and $I_{k_1, k_2, k_3}^{--+}$. The main point is that the phase function $\Phi^{--+}$ does not have any spacetime resonances, i.e. there are no $(\eta, \sigma)$ solutions of the equations

$$\Phi^{--+}(\xi, \eta, \sigma) = (\partial_\eta \Phi^{--+})(\xi, \eta, \sigma) = (\partial_\sigma \Phi^{--+})(\xi, \eta, \sigma) = 0.$$ 

For any $l, j \in \mathbb{Z}$ satisfying $l \leq j$ define

$$\varphi_j(l) := \begin{cases} \varphi_j & \text{if } j \geq l + 1, \\ \varphi_l & \text{if } j = l. \end{cases}$$

Let $\mathfrak{t} := k - 20$ and decompose

$$I_{k_1, k_2, k_3}^{--+} = \sum_{l_1, l_2 \in [\mathfrak{t} + 40]} J_{l_1, l_2},$$

$$J_{l_1, l_2}(\xi, t) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} c_k^*(\eta, \sigma) \varphi_{l_1}(\eta, \sigma) \varphi_{l_2}(\sigma) f_{k_1}\widehat{f}_{k_2}(\xi + \eta) \widehat{f}_{k_3}(\xi + \sigma) \widehat{f}_{k_3}^+(-\xi - \eta - \sigma) d\eta d\sigma.$$ 

The contributions of the integrals $J_{l_1, l_2}$ for $(l_1, l_2) = (\mathfrak{t}, \mathfrak{t})$ can be estimated by integration by parts either in $\eta$ or in $\sigma$ (depending on the relative sizes of $l_1$ and $l_2$), since the $(\eta, \sigma)$ gradient of the phase function $\Phi$ is bounded from below by $c^{2k/2}$ in the support of these integrals.

On the other hand, to estimate the contribution of the integral $J_{\mathfrak{t}+\mathfrak{t}}$ we notice that

$$|\Phi(\xi, \eta, \sigma)| \gtrsim 2^{3k/2}$$

in the support of the integral, so that we can integrate by parts in $s$. This gives

$$\left| \int_{t_1}^{t_2} e^{iL_4(\xi, s)} J_{\mathfrak{t}+\mathfrak{t}}(\xi, s) \, ds \right| \lesssim |L_4(\xi, t_1)| + |L_4(\xi, t_2)|$$

$$+ \int_{t_1}^{t_2} \left| L_1(\xi, s) \right| + |L_2(\xi, s)| + |L_3(\xi, s)| + |(\partial_s L)(\xi, s)||L_4(\xi, s)| \, ds,$$

where

$$L_1(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} c_k^*(\eta, \sigma) \frac{\varphi_{\leq l}(\eta) \varphi_{\leq l}(\sigma)}{i\Phi(\xi, \eta, \sigma)} (\partial_s f_{k_1})(\xi + \eta) \widehat{f}_{k_2}(\xi + \sigma) \widehat{f}_{k_3}^+(-\xi - \eta - \sigma) d\eta d\sigma,$$

$$L_2(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} c_k^*(\eta, \sigma) \frac{\varphi_{\leq l}(\eta) \varphi_{\leq l}(\sigma)}{i\Phi(\xi, \eta, \sigma)} \widehat{f}_{k_1}(\xi + \eta) (\partial_s \widehat{f}_{k_2})(\xi + \sigma) \widehat{f}_{k_3}^+(-\xi - \eta - \sigma) d\eta d\sigma,$$

$$L_3(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} c_k^*(\eta, \sigma) \frac{\varphi_{\leq l}(\eta) \varphi_{\leq l}(\sigma)}{i\Phi(\xi, \eta, \sigma)} \widehat{f}_{k_1}(\xi + \eta) \widehat{f}_{k_2}(\xi + \sigma) (\partial_s f_{k_3})(-\xi - \eta - \sigma) d\eta d\sigma,$$

$$L_4(\xi) := \int_{\mathbb{R} \times \mathbb{R}} e^{i\Phi(\xi, \eta, \sigma)} c_k^*(\eta, \sigma) \frac{\varphi_{\leq l}(\eta) \varphi_{\leq l}(\sigma)}{i\Phi(\xi, \eta, \sigma)} \widehat{f}_{k_1}(\xi + \eta) \widehat{f}_{k_2}(\xi + \sigma) \widehat{f}_{k_3}^+(-\xi - \eta - \sigma) d\eta d\sigma.$$ 

To estimate the integrals $L_1, L_2, L_3, L_4$ we notice that using (9.11) and (9.31) we have

$$\left\| \frac{c_k^*(\eta, \sigma) \varphi_{\leq l}(\eta) \varphi_{\leq l}(\sigma)}{i\Phi(\xi, \eta, \sigma)} \right\|_{S^{\infty}} \lesssim 2^{3\max(k, 0)}.$$ 

Therefore, using Lemma 2.1(ii) and the a priori bounds (9.2)–(9.3) we see that

$$|L_4(\xi, s)| \lesssim \left\| \frac{c_k^*(\eta, \sigma) \varphi_{\leq l}(\eta) \varphi_{\leq l}(\sigma)}{i\Phi(\xi, \eta, \sigma)} \right\|_{S^{\infty}} \left\| f_{k_1}(s) \right\|_{L^2} \left\| e^{isA} f_{k_2}(s) \right\|_{L^\infty} \left\| f_{k_3}(s) \right\|_{L^2} \lesssim \varepsilon^3 2^{-m/4}. $$
Using also (9.27) we obtain
\[ |L_1(\xi, s)| \lesssim \left\| \frac{\mathcal{A}^* (\eta, \sigma) \varphi_{\leq \ell}(\eta) \varphi_{\leq \ell}(\sigma)}{i \Phi (\xi, \eta, \sigma)} \right\|_{S^{\infty}} \left\| (\partial_s \tilde{f}_{k_1}) (s) \right\|_{L^2} \|e^{isA} f_{k_2} (s)\|_{L^\infty} \left\| f_{k_3}^* (s) \right\|_{L^2} \lesssim \varepsilon_1^{3/2 - 5m/4} . \]

The bounds on \(|L_2(\xi, s)|\) and \(|L_3(\xi, s)|\) are similar. Recalling also the bound \(|(\partial_s L)(\xi, s)| \lesssim \varepsilon_1^{2 - m} \), see the definition (8.57), it follows that the right-hand side of (9.32) is dominated by \(C \varepsilon_1^{2 - m/10} \), which completes the proof of the lemma.

\[ \square \]

9.3. Proof of (9.8). We now prove the estimate (9.8) involving the quartic remainder term \( R''_{\geq 4} \) which is defined in (8.36). Our main aim is to show the following \( L^2 \) estimates:

**Lemma 9.12.** For any \( t \in [0, T'] \) and \( l \in \mathbb{Z} \) we have
\[ \| P_t R''_{\geq 4} (t) \|_{L^2} \lesssim \varepsilon_1^4 (1 + t)^{-9/8 + 10p_0} , \]
\[ \| P_t S R''_{\geq 4} (t) \|_{L^2} \lesssim \varepsilon_1^4 (1 + t)^{-1 + 2p_0} . \]

The desired conclusion (9.8) can then obtained as a consequence of (9.33) and the interpolation inequality (2.21) in Lemma (2.3) similarly to the proof of Lemma 5.14 in [31].

To prove (9.33) we first recall that from the a priori assumptions on \( u \) and Lemma 8.12 we have the linear bounds
\[ \| P_t u(t) \|_{L^2} + \| P_t v(t) \|_{L^2} \lesssim \varepsilon_1 \min \left\{ 2^{(1/2 - p_1)l}, 2^{-(N_0 - 1/2)l} \right\} (1 + t)^{p_0} , \]
\[ \| P_t u(t) \|_{L^\infty} + \| P_t v(t) \|_{L^\infty} \lesssim \varepsilon_1 \min \left\{ 2^{l/10}, 2^{-(N_2 - 1/2)l} \right\} (1 + t)^{-1/2} , \]
\[ \| P_t S u(t) \|_{L^2} + \| P_t S v(t) \|_{L^2} \lesssim \varepsilon_1 \min \left\{ 2^{(1/2 - p_1)l}, 2^{-(N_1 - 1/2)l} \right\} (1 + t)^{4p_0} , \]
and the quadratic bounds
\[ \| P_t (u(t) - v(t)) \|_{L^2} \lesssim \varepsilon_1^2 \min \left\{ 2^{l/2}, 2^{-(N_2 - 1/2)l} \right\} (1 + t)^{-1/2 + 6p_0} , \]
\[ \| P_t (u(t) - v(t)) \|_{L^\infty} \lesssim \varepsilon_1^2 \min \left\{ 2^{l/2}, 2^{-(N_2 - 1/2)l} \right\} (1 + t)^{-3/4 + 2p_0} , \]
\[ \| P_t S (u(t) - v(t)) \|_{L^2} \lesssim \varepsilon_1^2 \min \left\{ 2^{l/2}, 2^{-(N_1 - 1/2)l} \right\} (1 + t)^{-1/4 + 6p_0} . \]

Using these we now prove a few more nonlinear estimates.

**Lemma 9.13.** For any \( t \in [0, T] \) and \( l \in \mathbb{Z} \) we have the quadratic-type bounds
\[ \| P_t Q_\alpha(t) \|_{L^2} + \| P_t Q_\nu(t) \|_{L^2} \lesssim \varepsilon_1^2 \min \left\{ 2^{l/2}, 2^{-(N_0 - 3/2)l} \right\} (1 + t)^{-1/2 + p_0} , \]
\[ \| P_t Q_\alpha(t) \|_{L^\infty} + \| P_t Q_\nu(t) \|_{L^\infty} \lesssim \varepsilon_1^2 \min \left\{ 2^{l/2}, 2^{-3l} \right\} (1 + t)^{-1} , \]
\[ \| P_t S Q_\alpha(t) \|_{L^2} + \| P_t S Q_\nu(t) \|_{L^2} \lesssim \varepsilon_1^2 \min \left\{ 2^{l/2}, 2^{-l} \right\} (1 + t)^{-1/2 + 4p_0} , \]
and the cubic-type bounds
\[ \| P_t (Q_\alpha(t) - Q_\nu(t)) \|_{L^2} \lesssim \varepsilon_1^3 (1 + t)^{-7/8 + 6p_0} \min \{ 2^{l/2}, 2^{-3l} \} , \]
\[ \| P_t (Q_\alpha(t) - Q_\nu(t)) \|_{L^\infty} \lesssim \varepsilon_1^3 (1 + t)^{-5/4 + 2p_0} \min \{ 2^{l/2}, 2^{-3l} \} , \]
\[ \| P_t S (Q_\alpha(t) - Q_\nu(t)) \|_{L^2} \lesssim \varepsilon_1^3 (1 + t)^{-3/4 + 8p_0} \min \{ 2^{l/2}, 2^{-l} \} . \]

**Proof.** To obtain (9.36) is suffices to recall the definition of \( Q_\alpha \), respectively \( Q_\nu \), in (8.4), respectively (8.43), use the symbol bounds (A.19), (A.26) and
\[ \| q_{0}^{l_{1}, k_{1}} \|_{S^{\infty}} \lesssim 1 \chi(l, k_1, k_2) 2^{l l_{1} k_2} 1_{[2, \infty]} (l - k_1) , \]
the commutation identity (2.5)-(2.6), and the linear bounds (9.34).
We examine the formula (8.46) for $\mathcal{R}_n''$ and begin by looking at the terms in the second line:

$$M_{\epsilon_1\epsilon_2} (u_{\epsilon_1}, (Q_u)_{\epsilon_2}) - M_{\epsilon_1\epsilon_2} (v_{\epsilon_1}, (Q_v)_{\epsilon_2})$$

$$= M_{\epsilon_1\epsilon_2} (u_{\epsilon_1} - v_{\epsilon_1}, (Q_u)_{\epsilon_2}) + M_{\epsilon_1\epsilon_2} (v_{\epsilon_1}, (Q_u)_{\epsilon_2} - (Q_v)_{\epsilon_2}).$$

(9.40)
We recall the bound (9.19):
\[
\|m^{l,k_1,k_2}_{l_1l_2}\|_{S^\infty} \lesssim 2^{l/2-2k_1/2}1_{\chi}(l, k_1, k_2)1_{[-10,\infty)}(k_2 - k_1),
\]
and remark that the difficulty in estimating the terms in (9.40) comes from the low frequency singularity in this estimate. Using Lemma 2(iii) we have
\[
I := \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} \|m^{l,k_1,k_2}_{l_1l_2}\|_{S^\infty} \|P'_{k_1}(u - v)(t)\|_{L_\infty} \|P'_{k_2}Q_u(t)\|_{L_2},
\]
\[
II := \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} \|m^{l,k_1,k_2}_{l_1l_2}\|_{S^\infty} \|P'_{k_1}(u - v)(t)\|_{L_2} \|P'_{k_2}Q_u(t)\|_{L_\infty}.
\]
Using (9.41), Bernstein’s inequality, and the $L^2$ estimates in (9.35) and (9.36), we have
\[
I \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} 2^{k_2/2}\|P'_{k_1}(u - v)(t)\|_{L_2} \|P'_{k_2}Q_u(t)\|_{L_2}^{7/8+6\rho_0} \epsilon_1 \min(2^{k_2/2}, 2^{-k_2/2}(1+t)^{-1/2}+\rho_0) \lesssim \epsilon_1^4(1+t)^{-3/2}.
\]
Using the $L^2$ bound in (9.35), and the $L^\infty$ bound in (9.36), we get
\[
II \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} 2^{k_2/2}2^{-k_1/2}\|P'_{k_1}(u - v)(t)\|_{L_2} \|P'_{k_2}Q_u(t)\|_{L_\infty}^{7/8+6\rho_0} \epsilon_1 \min(2^{k_2/2}, 2^{-k_2/2})(1+t)^{-1} \lesssim \epsilon_1^4(1+t)^{-5/4}.
\]
Similarly,
\[
\|P_M{\epsilon_1\epsilon_2}(v_{\epsilon_1}, (Q_u)_{\epsilon_2} - (Q_v)_{\epsilon_2})\|_{L_2} \lesssim III + IV,
\]
\[
III := \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} \|m^{l,k_1,k_2}_{l_1l_2}\|_{S^\infty} \|P'_{k_1}v(t)\|_{L_\infty} \|P'_{k_2}(Q_u - Q_v)(t)\|_{L_2},
\]
\[
IV := \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} \|m^{l,k_1,k_2}_{l_1l_2}\|_{S^\infty} \|P'_{k_1}v(t)\|_{L_2} \|P'_{k_2}(Q_u - Q_v)(t)\|_{L_\infty}.
\]
so that using (9.34) and (9.37), we have
\[
III \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} 2^{k_2/2}\|P'_{k_1}v(t)\|_{L_2} \|P'_{k_2}(Q_u - Q_v)(t)\|_{L_2}
\]
\[
\lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} \epsilon_1(2^{(1/2-\rho_0)k_1}(1+t)^{6\rho_0}) \epsilon_1^3 \min(2^{k_2/2}, 2^{-k_2/2})(1+t)^{-7/8+6\rho_0}
\]
\[
\lesssim \epsilon_1^4(1+t)^{-3/2}.
\]
and

\[ IV \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} 2^{k_2/2} 2^{-k_1/2} \| P_{k_1} v(t) \|_{L^2} \| P_{k_2} (Q_u - Q_v)(t) \|_{L^\infty} \]

\[ \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} \epsilon_1^2 2^{-p_1 k_1} (1+t)^{6p_0} \epsilon_1^3 \min(2^{k_2/2}, 2^{-k_2/2})(1+t)^{-5/4+2p_0} \]

\[ \lesssim \epsilon_1^4 (1+t)^{-5/4+10p_0}. \]

We have obtained

\[ \| P_t [M_{\epsilon_1 \epsilon_2}(u_{\epsilon_1}, (Q_u)_{\epsilon_2}) - M_{\epsilon_1 \epsilon_2}(v_{\epsilon_1}, (Q_v)_{\epsilon_2})] \|_{L^2} \lesssim \epsilon_1^4 (1+t)^{-9/8}. \] (9.42)

Using (9.40) and the commutation identity (2.5)-(2.6) we compute

\[ S(\epsilon_1 \epsilon_2)(u_{\epsilon_1}, (Q_u)_{\epsilon_2}) - M_{\epsilon_1 \epsilon_2}(v_{\epsilon_1}, (Q_v)_{\epsilon_2}) = M_{\epsilon_1 \epsilon_2}(S(u_{\epsilon_1} - v_{\epsilon_1}), (Q_u)_{\epsilon_2}) + M_{\epsilon_1 \epsilon_2}(u_{\epsilon_1} - v_{\epsilon_1}, (S(Q_u)_{\epsilon_2}) + \tilde{M}_{\epsilon_1 \epsilon_2}(u_{\epsilon_1} - v_{\epsilon_1}, (Q_u)_{\epsilon_2}) \]

\[ + M_{\epsilon_1 \epsilon_2}(S(v_{\epsilon_1}, (Q_u - Q_v)_{\epsilon_2}) + M_{\epsilon_1 \epsilon_2}(v_{\epsilon_1}, S(Q_u - Q_v)_{\epsilon_2}) + \tilde{M}_{\epsilon_1 \epsilon_2}(v_{\epsilon_1}, (Q_u - Q_v)_{\epsilon_2}). \] (9.43)

Using the bounds (9.35) and (9.36), we can estimate similarly to above

\[ \| M_{\epsilon_1 \epsilon_2}(S(u_{\epsilon_1} - v_{\epsilon_1}), (Q_u)_{\epsilon_2}) \|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} 2^{k_2/2} \| P_{k_1} S(u - v) \|_{L^2} \| P_{k_2} Q_u \|_{L^2} \]

\[ + \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} 2^{-k_1/2} 2^{k_2/2} \| P_{k_1} S(u - v) \|_{L^2} \| P_{k_2} Q_u \|_{L^\infty} \]

\[ \lesssim \epsilon_1^4 (1+t)^{-1}, \]

and

\[ \| M_{\epsilon_1 \epsilon_2}(u_{\epsilon_1} - v_{\epsilon_1}, (S Q_u)_{\epsilon_2}) \|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} 2^{k_2/2} \| P_{k_1} (u - v) \|_{L^2} \| P_{k_2} S Q_u \|_{L^2} \]

\[ + \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} 2^{-k_1/2} 2^{k_2/2} \| P_{k_1} (u - v) \|_{L^\infty} \| P_{k_2} S Q_u \|_{L^2} \]

\[ \lesssim \epsilon_1^4 (1+t)^{-1}. \]

Using instead (9.34) and (9.37) we see that

\[ \| M_{\epsilon_1 \epsilon_2}(S(v_{\epsilon_1}, (Q_u - Q_v)_{\epsilon_2}) \|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} 2^{k_2/2} \| P_{k_1} S v \|_{L^2} \| P_{k_2} (Q_u - Q_v) \|_{L^2} \]

\[ + \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} 2^{-k_1/2} 2^{k_2/2} \| P_{k_1} S v \|_{L^2} \| P_{k_2} (Q_u - Q_v) \|_{L^\infty} \]

\[ \lesssim \epsilon_1^4 (1+t)^{-1}, \]
Furthermore, using (9.42) and the fact the the symbols \( \tilde{m}_{\epsilon_{1}\epsilon_{2}} \) give us the desired bound for the second line of (8.46). Therefore, we can skip the estimates of these terms.

These are easier to bound than the terms in the second line of (8.46), which we have already estimated, because the singularity produced by the symbols \( m_{\epsilon_{1}\epsilon_{2}} \) is eliminated in all cases by the low frequency improved behavior of the arguments \( Q_{\epsilon} \), see (9.36), and \( Q_{\epsilon} - Q_{\epsilon} \), see (9.37). Therefore, we can skip the estimates of these terms.

Next, we look at the terms in the third line of (8.46):

\[
M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}(u_{e_{1}}, u_{e_{2}}, u_{e_{3}}) - M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}(v_{e_{1}}, v_{e_{2}}, v_{e_{3}}) = M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}((Q_{\epsilon} - Q_{\epsilon}), u_{e_{2}}) + M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}((Q_{\epsilon}), u_{e_{2}} - v_{e_{2}}). 
\] (9.44)

Recall that the symbols \( m_{\epsilon_{1}\epsilon_{2}\epsilon_{3}} \) satisfy the strong (non-singular) bounds (8.7). Using (9.34) and (9.35), one can estimate

\[
\|P_{1}[M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}(u_{e_{1}}, u_{e_{2}}, u_{e_{3}}) - M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}(v_{e_{1}}, v_{e_{2}}, v_{e_{3}})]\|_{L^{2}} \lesssim \epsilon_{\epsilon_{1}}^{4}(1 + t)^{-9/8}.
\]

The analogue of (2.5) for symbols of three variables is

\[
SM(f, g, h) = M(Sf, g, h) + M(f, Sg, h) + M(f, g, Sh) + \tilde{M}(f, g, h) \quad \tilde{m}(\xi, \eta, \sigma) = -\xi \partial_{\xi} + \eta \partial_{\eta} + \sigma \partial_{\sigma})m(\xi, \eta, \sigma). 
\] (9.46)

By homogeneity, the symbols \( \tilde{m}_{\epsilon_{1}\epsilon_{2}\epsilon_{3}} \) will satisfy the same bounds (8.7). Then, applying \( S \) to the identity (9.44) above, and using (9.34) and (9.35), we can obtain

\[
\|S[M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}(u_{e_{1}}, u_{e_{2}}, u_{e_{3}}) - M_{\epsilon_{1}\epsilon_{2}\epsilon_{3}}(v_{e_{1}}, v_{e_{2}}, v_{e_{3}})]\|_{L^{2}} \lesssim \epsilon_{\epsilon_{1}}^{4}(1 + t)^{-9/8}.
\]

Since the term \( R_{\epsilon} \geq 4 \) in (8.46) already satisfies the desired bounds, see (8.8), the only remaining terms that we need to look at are \( M_{\epsilon_{1}\epsilon_{2}}(F_{e_{1}}, u_{e_{2}}) \) and \( M_{\epsilon_{1}\epsilon_{2}}(u_{e_{1}}, F_{e_{2}}) \), \( F := \|\partial_{\xi}\|^{1/2}O_{3,-1} \).

The \( O_{3,0} \) notation in (8.9) implies that we have the following estimate

\[
\|P_{1}F(t)\|_{L^{2}} \lesssim \epsilon_{\epsilon_{1}}^{3} \min (2^{1/2}, 2^{-(N_{0} - 2)^{l}})(1 + t)^{-1+p_{0}},
\]

\[
\|P_{1}SF(t)\|_{L^{2}} \lesssim \epsilon_{\epsilon_{1}}^{3} \min (2^{1/2}, 2^{-(N_{1} - 2)^{l}})(1 + t)^{-1+4p_{0}}.
\]
Using these estimates, Lemma 2.1(ii), the symbol bound (9.41), and (9.34), we can bound
\[
\|P_{1}M_{\psi_1\epsilon_2}(F_{1}, u_{\epsilon_2})\|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \leq (1+t)^{-2}} 2^{k_2/2}\|P_{1}^{'} F(t)\|_{L^2}\|P_{k_2}^{'} u(t)\|_{L^2} + \sum_{k_1, k_2 \in \mathbb{Z}, 2^{k_1} \geq (1+t)^{-2}} 2^{k_2/2-1/2}k_1\|P_{1}^{'} F(t)\|_{L^2}\|P_{k_2}^{'} u(t)\|_{L^\infty} \lesssim \epsilon_1^4 (1+t)^{-5/4}.
\]
The same estimate also holds for $P_{\widetilde{1}}M_{\psi_1\epsilon_2}(F_{1}, u_{\epsilon_2})$. Similarly, we have
\[
\|P_{1}M_{\psi_1\epsilon_2}(SF_{\epsilon_1}, u_{\epsilon_2})\|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} 2^{k_2/2}\|P_{1}^{'} SF(t)\|_{L^2}\|P_{k_2}^{'} u(t)\|_{L^2} \lesssim \epsilon_1^4 (1+t)^{-1+5\delta_0},
\]
\[
\|P_{1}M_{\psi_1\epsilon_2}(F_{\psi_1}, Su_{\epsilon_2})\|_{L^2} \lesssim \sum_{k_1, k_2 \in \mathbb{Z}} 2^{k_2/2}\|P_{1}^{'} F(t)\|_{L^2}\|P_{k_2}^{'} Su(t)\|_{L^2} \lesssim \epsilon_1^4 (1+t)^{-1+5\delta_0}.
\]
The analogous estimates for the terms $M_{\psi_1\epsilon_2}(u_{\epsilon_1}, F_{\epsilon_2})$ are very similar to the ones above, so we can skip them. The proof of (9.33) is completed. 

**Appendix A. Analysis of symbols**

**A.1. Notation.** Recall the definition of the class of symbols
\[
S^\infty := \{ m : \mathbb{R}^2 \rightarrow \mathbb{C} : m \text{ continuous and } \|m\|_{S^\infty} := \|\mathcal{F}^{-1}(m)\|_{L^1} < \infty \}, \tag{A.1}
\]
and the notation
\[
m_{k,k_1,k_2}(\xi, \eta) := m(\xi, \eta)\varphi_k(\xi)\varphi_k(\xi - \eta)\varphi_{k_2}(\eta),
\]
\[
m_{k_1,k_2,k_3,k_4}(\xi, \eta, \rho) := m(\xi, \eta, \rho) \cdot \varphi_{k_1}(\xi)\varphi_{k_2}(\eta)\varphi_{k_3}(\rho - \xi)\varphi_{k_4}(\rho - \eta), \tag{A.2}
\]
\[
\mathcal{X} := \{(k, k_1, k_2) \in \mathbb{Z}^3 : \max(k, k_1, k_2) - \text{med}(k, k_1, k_2) \leq 6\},
\]
\[
\mathcal{Y} := \{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 : 2^{k_1} + 2^{k_2} + 2^{k_3} + 2^{k_4} \geq (1 + 2^{-10})2^{\max(k_1,k_2,k_3,k_4)} \}. \tag{A.3}
\]
Also recall that we denote
\[
m(\xi, \eta) = O(f(|\xi|, |\xi - \eta|, |\eta|)) \iff \|m_{k,k_1,k_2}(\xi, \eta)\|_{S^\infty} \lesssim f(2^{k_1}, 2^{k_2})1_{\mathcal{X}}(k_1, k_2), \tag{A.4}
\]
and use the analogous notation (2.13) for symbols of three variables.

**A.2. Quadratic symbols.** For ease of reference we recall here the definitions (3.31)–(3.34):
\[
a_{++}(\xi, \eta) := \frac{1}{4i} \chi(\xi - \eta, \eta)\left[-\frac{\xi(\xi - \eta)}{\xi - \eta} \frac{|\xi|^3}{|\eta|^2} - 1 + |\xi - \eta|^3/2 + \frac{\eta(\xi - \eta)}{\xi - \eta} \left(1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}}\right) + |\xi - \eta| \frac{|\xi|^{1/2} \varphi_{\geq 1}(\eta)}{|\eta|} + \frac{|\xi|^2 - |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \varphi_{\leq 0}(\eta), \right] \tag{A.5}
\]
\[
a_{+-}(\xi, \eta) := \frac{1}{4i} \chi(\xi - \eta, \eta)\left[-\frac{\xi(\xi - \eta)}{\xi - \eta} \frac{|\xi|^3}{|\eta|^2} - 1 - |\xi - \eta|^3/2 - \frac{\eta(\xi - \eta)}{\xi - \eta} \left(1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}}\right) + |\xi - \eta| \frac{|\xi|^{1/2} \varphi_{\geq 1}(\eta)}{|\eta|} + \frac{|\xi|^2 + |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \varphi_{\leq 0}(\eta), \right] \tag{A.6}
\]
As a consequence, we have

\[ a_{\pm}(\xi, \eta) := \frac{1}{4k} \chi(\xi-\eta, \eta) \left[ \frac{\xi(\xi - \eta)}{\xi - \eta} - 1 \right] - |\xi - \eta|^{3/2} - \frac{\eta(\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right) \]

(A.7)

Furthermore, for \( \kappa \), the following bounds holds

\[ a_{\pm}(\xi, \eta) := \frac{1}{4k} \chi(\xi-\eta, \eta) \left[ \frac{\xi(\xi - \eta)}{\xi - \eta} - 1 \right] + |\xi - \eta|^{3/2} + \frac{\eta(\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right) \]

(A.8)

In the following lemma we collect several estimates on the above symbols that are used throughout the paper.

**Lemma A.1.** Let \( \epsilon_1, \epsilon_2 \in \{+, -, \} \), and \( a_{\epsilon_1 \epsilon_2} \) be the symbols in (A.3)–(A.8), then

\[
\left\| a_{\epsilon_1 \epsilon_2}^{k_1, k_2} \right\|_{S^\infty} \lesssim 2^{3k_1/2} \chi(k, k_1, k_2)1_{[2, \infty)}(k_2 - k_1) \\
+ 2^{k_1/2} 2^k \chi_{(-\infty, 1)}(k)1_{[2, \infty)}(k_2 - k_1).
\]

(A.9)

Moreover, for \( \epsilon_1 \in \{+, -, \} \), the following bounds holds

\[
\left\| a_{\epsilon_1}^{k_1, k_2} \right\|_{S^\infty} \lesssim 2^{3k_1/2} \chi(k, k_1, k_2)1_{[2, \infty)}(k_2 - k_1),
\]

(A.10)

\[
\left\| a_{\epsilon_1}^{k_1, k_2} \right\|_{S^\infty} \lesssim 2^{3k_1/2} \chi_{[2, \infty)}(k) + 2^{k_1/2} 2^k \chi_{(-\infty, 1)}(k)1_{[2, \infty)}(k_2 - k_1).
\]

(A.11)

As a consequence, we have

\[
\left\| \frac{a_{\epsilon_1}^{k_1, k_2}(\xi, \eta) \chi^{|\xi|^{3/2} - \epsilon_1 |\xi - \eta|^{3/2} - |\eta|^{3/2}}}{a_{\epsilon_1}^{k_1, k_2}(\xi, \eta) \chi^{|\xi|^{3/2} - \epsilon_1 |\xi - \eta|^{3/2} + |\eta|^{3/2}}} \right\|_{S^\infty} \lesssim 2^{k_1/2} 2^{-k/2} \chi(k, k_1, k_2)1_{[2, \infty)}(k_2 - k_1),
\]

(A.12)

Furthermore, for \( \epsilon_1 \in \{+, -, \} \) let us define the symbol

\[
\alpha_{\epsilon_1}^+(\xi, \eta, \rho) := a_{\epsilon_1}^+(\xi, \rho) - a_{\epsilon_1}^+(\xi + \eta - \rho, \eta).
\]

(A.13)

Then, using the notation (A.2)–(A.3), we have the following: whenever \( k_3 \leq k_1 - 3, k_4 \leq k_2 - 3, \) and \( k_1 \geq 5, \)

\[
\left\| \alpha_{\epsilon_1}^{k_1, k_2, k_3, k_4} \right\|_{S^\infty} \lesssim (2^{5k_3/2} + 2^{5k_4/2})(2^{k_1} + 2^{k_2})^{-1} \chi(k_1, k_2, k_3, k_4).
\]

(A.14)
Proof. We begin by recalling that the cutoff \( \chi \), see (A.19), is supported on a region where \( 2^3|\xi - \eta| \leq |\eta| \). Using integration by parts, one can verify that

\[
\frac{\xi (\xi - \eta)}{|\xi - \eta|^{1/2}} \left( \frac{|\xi|}{|\eta|} - 1 \right) \chi (\xi - \eta, \eta) = O \left( |\xi - \eta|^{3/2} 1_{[2^3, \infty)} (|\eta|/|\xi - \eta|) \right),
\]

\[
\frac{\eta (\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right) \chi (\xi - \eta, \eta) = O \left( |\xi - \eta|^{3/2} 1_{[2^3, \infty)} (|\eta|/|\xi - \eta|) \right),
\]

\[
\frac{|\xi - \eta|^{1/2} |\xi - \eta|}{|\eta|} \chi (\xi - \eta, \eta) = O \left( |\xi - \eta|^2 |\eta|^{-1/2} 1_{[2^3, \infty)} (|\eta|/|\xi - \eta|) \right),
\]

\[
\frac{|\xi|^2 - |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \chi (\xi - \eta, \eta) = O \left( |\xi - \eta|^{3/2} 1_{[2^3, \infty)} (|\eta|/|\xi - \eta|) \right)
\]

\[
\frac{|\xi|^2 + |\xi|^{1/2} |\eta|^{3/2}}{|\eta|} |\xi - \eta|^{1/2} \varphi \lesssim (\xi - \eta) \chi (\xi - \eta, \eta) = O \left( |\xi||\xi - \eta|^{1/2} 1_{[2^3, \infty)} (|\eta|/|\xi - \eta|) \right) 1_{(0,1)} (|\xi|),
\]

(A.15)

where we are using the notation (A.11)-(A.4). Since the bound for \( |\xi - \eta|^{3/2} \) is obvious, using (A.15), and inspecting the formulas (A.5)-(A.8), one immediately obtains (A.9). The bound (A.10) follows from the first four identities in (A.15). (A.11) follows directly from (A.9).

To prove (A.12) we notice first that

\[(a + b)^{3/2} - b^{3/2} - a^{3/2} \in [ab^{1/2}/4, 4ab^{1/2}] \quad \text{if} \quad 0 \leq a \leq b.\]

Therefore, using standard integration by parts, we see that

\[
\left\| \frac{\varphi_k (\xi) \varphi_k (\xi - \eta) \varphi_k (\eta)}{|\xi|^{3/2} - |\xi - \eta|^{3/2} - |\eta|^{3/2}} \right\|_{S^\infty \geq 1} \lesssim \frac{1}{2 \min (k_1, k_2) 2 \max (k_1, k_2)^{1/2}};
\]

(A.16)

for all \( k, k_1, k_2 \in \mathbb{Z} \). In particular, whenever \( k \geq k_1 + 3 \), we have

\[
\left\| \frac{\varphi_k (\xi) \varphi_k (\xi - \eta) \varphi_k (\eta)}{|\xi|^{3/2} - \epsilon \xi - |\xi - \eta|^{3/2} - |\eta|^{3/2}} \right\|_{S^\infty \geq \xi} \lesssim \frac{1}{2 k_1 2 \max (k_1, k_2)^{1/2}},
\]

(A.17)

\[
\left\| \frac{\varphi_k (\xi) \varphi_k (\xi - \eta) \varphi_k (\eta)}{|\xi|^{3/2} - |\xi - \eta|^{3/2} + |\eta|^{3/2}} \right\|_{S^\infty \geq \xi} \lesssim \frac{1}{2 k_1 2 \max (k_1, k_2)^{1/2}}.
\]

(A.18)

Thus, (A.10) and (A.17) give the first inequality in (A.12). The second inequality in (A.12) is a consequence of (A.11) and (A.18).

To prove (A.14) we write down the four pieces that constitute the symbols (A.5) and (A.7):

\[
a_1 (\xi, \eta) := \frac{\xi (\xi - \eta)}{|\xi - \eta|^{1/2}} \left( \frac{|\xi|}{|\eta|} - 1 \right), \quad a_2 (\xi, \eta) := |\xi - \eta|^{3/2},
\]

\[
a_3 (\xi, \eta) := \frac{\eta (\xi - \eta)}{|\xi - \eta|^{1/2}} \left( 1 - \frac{|\xi|^{1/2}}{|\eta|^{1/2}} \right), \quad a_4 (\xi, \eta) := \frac{|\xi - \eta|^2 |\xi|^{1/2}}{|\eta|}.
\]

(A.19)

Notice that the last piece in the formulas (A.5) and (A.7) has been disregarded, since we are only interested in the case \( k_1 \geq 5 \) in (A.14). It suffices to prove that

\[
\|a_k (k_1, k_2, k_3, k_4)\|_{S^\infty \geq 1} \lesssim (2^{5k_3/2} + 2^{5k_4/2})(2^{k_1} + 2^{k_2})^{-1} 1_{y} (k_1, k_2, k_3, k_4),
\]

for \( a_j (\xi, \eta) := a_j (\xi, \rho) - a_j (\xi + \eta - \rho, \eta), \quad j = 1, \ldots, 4, \)

(A.20)

whenever \( k_3 \leq k_1 - 3 \) and \( k_4 \leq k_2 - 3 \).
By Taylor expansion, one easily sees that for $4|\xi - \eta| \leq |\eta|$

$$a_1(\xi, \eta) = |\xi - \eta|^{3/2} \frac{|\xi|}{|\eta|} + O\left(|\xi - \eta|^{5/2} |\eta|^{-1}\right).$$

It follows that if $4|\rho - \eta| \leq |\eta|$, $4|\rho - \xi| \leq |\xi|$, we have

$$\alpha_1(\xi, \eta, \rho) = |\xi - \rho|^{3/2} \left(\frac{|\xi|}{|\rho|} - \frac{|\xi - \eta|}{|\eta|} \right) + O\left(|\xi - \rho|^{5/2} (|\rho|^{-1} + |\eta|^{-1})\right) = O\left(\frac{|\xi - \rho|^{5/2}}{|\eta| + |\xi|}\right).$$

The bound (A.20) for $j = 2$ is trivial. For $j = 3$ we can use a Taylor expansion to see that

$$a_3(\xi, \eta) = -\frac{1}{2}|\xi - \eta|^{3/2} + O\left(|\xi - \eta|^{5/2} |\eta|^{-1}\right),$$

when $4|\xi - \eta| \leq |\eta|$, and so deduce

$$\alpha_3(\xi, \eta, \rho) = O\left(\frac{|\xi - \rho|^{5/2}}{|\eta| + |\xi|}\right),$$

whenever $4|\rho - \eta| \leq |\eta|$, $4|\rho - \xi| \leq |\xi|$. Similarly, since

$$a_4(\xi, \eta) = |\xi - \eta|^2 |\eta|^{-1/2} + O\left(|\xi - \eta|^3 |\eta|^{-3/2}\right),$$

for $4|\xi - \eta| \leq |\eta|$, we have

$$\alpha_4(\xi, \eta, \rho) = |\xi - \rho|^2 (|\rho|^{-1/2} - |\eta|^{-1/2}) + O\left(\frac{|\xi - \rho|^3}{|\rho|^{3/2}}\right) + O\left(\frac{|\xi - \rho|^3}{|\eta|^{3/2}}\right)$$

$$= O\left(\frac{|\xi - \rho|^3 + |\rho - \xi|^3}{(|\eta| + |\xi|)^{3/2}}\right),$$

whenever $4|\rho - \eta| \leq |\eta|$, $4|\rho - \xi| \leq |\xi|$. This is more than sufficient to obtain (A.20) when $j = 4$, and therefore complete the proof of (A.14). \hfill \Box

Now we analyze the symbols $b_{k_1 k_2}$ in (5.35)—(5.38). Recall the definitions:

$$b_{++}(\xi, \eta) = \frac{i}{4} \frac{|\xi|}{|\eta|^{1/2}} \bar{m}_2(\xi, \eta) + \frac{i}{4} \frac{|\eta|^{1/2}}{|\eta|^{1/2}} \bar{q}_2(\xi, \eta)$$

(A.21)

$$b_{+-}(\xi, \eta) = \frac{i}{4} \frac{|\xi|}{|\eta|^{1/2}} \bar{m}_2(\xi, \eta) - \frac{i}{4} \frac{|\eta|^{1/2}}{|\eta|^{1/2}} \bar{q}_2(\xi, \eta)$$

(A.22)

$$b_{-+}(\xi, \eta) = -\frac{i}{4} \frac{|\xi|}{|\eta|^{1/2}} \bar{m}_2(\xi, \eta) - \frac{i}{4} \frac{|\eta|^{1/2}}{|\eta|^{1/2}} \bar{q}_2(\xi, \eta)$$

(A.23)

$$b_{--}(\xi, \eta) = -\frac{i}{4} \frac{|\xi|}{|\eta|^{1/2}} \bar{m}_2(\xi, \eta) + \frac{i}{4} \frac{|\eta|^{1/2}}{|\eta|^{1/2}} \bar{q}_2(\xi, \eta)$$

(A.24)

where

$$\bar{m}_2(\xi, \eta) := \bar{\chi}(\xi - \eta, \eta) \left[\xi(\xi - \eta) - |\xi||\xi - \eta|\right],$$

$$\bar{q}_2(\xi, \eta) := \bar{\chi}(\xi - \eta, \eta) \left[\xi(\xi - \eta) + |\eta||\xi - \eta|\right] / 2,$$

with $\bar{\chi}(x, y) = 1 - \chi(x, y) - \chi(y, x)$.

**Lemma A.2.** The following symbol bound holds true:

$$\|b_{k_1 k_2}\|_{S_\infty} \lesssim 2^{2k/2} 1_{\chi(k, k_1, k_2)} 1_{[-10, 10]}(k_1 - k_2),$$

(A.26)
and
\[ \left\| \frac{b_{k_1,k_2}}{\xi^{3/2} - \epsilon_1} \right\| \lesssim 2^{k/2}2^{-k_1/2} \chi(k,k_1,k_2)1_{[-10,10]}(k_1 - k_2). \] (A.27)

**Proof.** Inspecting the formula (A.25), and recalling the definition of \( \tilde{\gamma}_2 \) above (3.13), we see that the symbols \( \tilde{m}_2 \) and \( \tilde{\gamma}_2 \) are supported on a region where \( 2^{-8} |\xi - \eta| \leq |\eta| \leq 2^8 |\xi - \eta| \). Then it is easy to see that
\[
\tilde{m}_2(\xi, \eta) = O\left( |\xi| |\xi - \eta| 1_{[2^{-8}, 2^8]}(|\xi - \eta|/|\eta|) \right),
\]
and therefore
\[
\frac{|\xi|^{3/2} \tilde{m}_2(\xi, \eta)}{|\xi - \eta|^{1/2}} = O\left( |\xi|^{3/2} 1_{[2^{-8}, 2^8]}(|\xi - \eta|/|\eta|) \right),
\]
which is consistent with the bound on the right-hand side of (A.26). Since \( \tilde{\gamma}_2 \equiv 0 \) for \( |\xi| < |\eta| \), we have
\[
\tilde{\gamma}_2(\xi, \eta) = O\left( |\eta| |\xi - \eta| 1_{[2^{-8}, 2^8]}(|\xi - \eta|/|\eta|) 1_{[2^{-8}, 210]}(|\xi|/|\eta|) \right),
\]
and can deduce
\[
\frac{|\xi|^{1/2} \tilde{\gamma}_2(\xi, \eta)}{|\xi - \eta|^{1/2}} = O\left( |\xi|^{3/2} 1_{[2^{-8}, 2^8]}(|\xi - \eta|/|\eta|) \right).
\]
The desired conclusion (A.26) follows from the above bounds, the formulas (A.21)-(A.24), and the notation (A.4), (A.27) follows from (A.26) and (A.16). \( \square \)

**Computing the resonant value.** We now compute the resonant value \( c^{++}(\xi, 0, -\xi) \), see (8.37), of the symbol in (8.52), from the formulas (8.10), (A.5)-(A.8), (8.21)-(8.24), and (8.5). Since \( q_+(0, -\xi) = q_+(0, \xi) = m_+(-\xi, -\xi) = m_-(\xi, -\xi) = 0 \), and \( m_+(-\xi, 0, -\xi) = id_1 |\xi|^{3/2} \) we have
\[
\begin{align*}
  ic^{++}(\xi, 0, -\xi) &= m_+(\xi, \xi) q_+(-0, -\xi) + m_+(-\xi, 0) q_+(-0, -\xi) \\
  &\quad + m_+(-\xi, -\xi) q_++(2 \xi, \xi) - m_+(\xi, 0) q_+(-0, 0) \\
  &\quad - m_+(-\xi, \xi) q_++(0, 0) + m_+(\xi, 2 \xi) q_++(2 \xi, \xi) \\
  &\quad - m_-(\xi, -\xi) q_-(2 \xi, \xi) - m_-(\xi, 0) q_-(2 \xi, 0) + m_-(\xi, 0, -\xi) \\
  &= m_+(\xi, 2 \xi) q_++(2 \xi, \xi) - m_-(\xi, 2 \xi) q_-(2 \xi, 0) + id_1 |\xi|^{3/2}.
\end{align*}
\]
Since \( a_{\epsilon_1 \epsilon_2}(\xi, 2 \xi) = a_{\epsilon_1 \epsilon_2}(2 \xi, \xi) = q_0(\xi, 2 \xi) = q_0(2 \xi, \xi) = 0 \), we obtain
\[
  e^{++}(\xi, 0, -\xi) = -b_{++}(\xi, 2 \xi) b_++(2 \xi, \xi) + b_-(\xi, 2 \xi) b_-(2 \xi, \xi) + d_1 |\xi|^{3/2} = d_2 |\xi|^{3/2} \tag{A.28}
\]
having used (A.21)-(A.25) and \( \tilde{m}_2(\xi, 2 \xi) = -2|\xi|^2, \tilde{m}_2(2 \xi, \xi) = \tilde{\gamma}_2(2 \xi, \xi) = 0, \tilde{\gamma}_2(2 \xi, \xi) = |\xi|^2 \).

**Appendix B. The Dirichlet-Neumann operator**

Recall the spaces \( C_0, H^N, W^N, \) and \( \tilde{W}^N \) defined in (1.3) and (2.2). To estimate products we often use the following simple general lemma:

**Lemma B.1.** Assume \( \lambda \geq 1, a_0, a_2 \in [1/100, 100], A_0, A_2 \in (0, \infty) \), and \( f, g \in L^2(\mathbb{R}) \) satisfy
\[
A_0^{-1} \left( \|f\|_{H^{a_0}} + \|g\|_{H^{a_0}} \right) + A_2^{-1} \left( \|f\|_{\tilde{W}^{a_2}} + \|g\|_{\tilde{W}^{a_2}} \right) \leq 1. \tag{B.1}
\]

Then
\[
A_0^{-1} \|fg\|_{H^{a_0}} + A_2^{-1} \|fg\|_{\tilde{W}^{a_2}} \lesssim A_2. \tag{B.2}
\]
Lemma B.2. The operators

\[ (R_n f)(\alpha) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(h(\alpha) - h(\beta))}{\alpha - \beta} \, d\beta, \]  

for integers \( n \geq 0 \). Assume that \( b \in (0, 2/5] \) and \( f \) satisfies the bound

\[ \|f\|_{W^{0,0,-b} \cap L^p} \leq A^{1/2}. \]  

(i) We have

\[ \|R_0 f\|_{W^{N_0-1/2}} \leq \varepsilon_1 A^{p_0-1/2} \quad \text{and} \quad \|R_0 f\|_{W^{N_2+1, b}} \leq b \varepsilon_1 A^{-1}. \]
(ii) Moreover, there is a constant $C \geq 1$ such that
\[
\|R_1 f\|_{H^{N_0+1}} \leq (C\varepsilon)^2 \Lambda^{n-1} \quad \text{and} \quad \|R_1 f\|_{W^{N_2+1,1/10}} \leq (C\varepsilon)^2 \Lambda^{-11/10}
\] \hspace{1cm} (B.13)
and, for any $n \geq 2$,
\[
\|R_n f\|_{H^{N_0+1}} \leq (C\varepsilon)^{n+1} \Lambda^{-5/4}.
\] \hspace{1cm} (B.14)

**Proof.** We rewrite first the operators $R_n$. We take the Fourier transform in $\alpha$ and make the change of variables $\alpha \to \beta + \rho$ to write
\[
\mathcal{F}[R_n f](\xi) = \frac{1}{\pi} \int_{\mathbb{R}^n} \frac{f(\beta)}{\rho} \left( h(\beta + \rho) - h(\beta) \right) \frac{n h(\beta + \rho) - h(\beta) - \rho h'(\beta)}{\rho} e^{-i\xi \beta} e^{-i\rho} d\beta d\rho.
\]
Notice that
\[
\frac{h(\beta + \rho) - h(\beta)}{\rho} = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{h}(\eta) e^{in\beta} e^{in\rho} - 1 d\eta.
\]
Therefore
\[
\mathcal{F}[R_n f](\xi) = \frac{1}{\pi(2\pi)^{n+1}} \int_{\mathbb{R}^2 \times \mathbb{R}^n} \frac{f(\beta)}{\rho} e^{-i\xi \beta} e^{-i\rho} e^{in(\eta_1 + \ldots + \eta_{n+1})\beta}
\times \tilde{h}(\eta_1) \ldots \tilde{h}(\eta_{n+1}) e^{in\eta_{n+1}\beta} - 1 - i\eta_{n+1} e^{i\rho} \prod_{l=1}^{n} e^{in\rho} - 1 \frac{d\beta d\rho d\eta_1 \ldots d\eta_{n}}{\rho}.
\]
This can be rewritten in the form
\[
\mathcal{F}[R_n f](\xi) = \frac{1}{\pi(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \tilde{M}_{n+1}(\xi; \eta_1, \ldots, \eta_{n+1}) \tilde{f}(\xi - \eta_1 - \ldots - \eta_{n+1})
\times \tilde{h}(\eta_1) \ldots \tilde{h}(\eta_{n+1}) d\eta_1 \ldots d\eta_{n+1},
\] \hspace{1cm} (B.15)
where
\[
\tilde{M}_{n+1}(\xi; \eta_1, \ldots, \eta_{n+1}) := \int_{\mathbb{R}} \frac{e^{-i\xi \rho}}{\rho} \frac{e^{in(\eta_{n+1}\beta) - 1 - i\eta_{n+1}\beta}}{\rho} \prod_{l=1}^{n} \frac{e^{in\rho} - 1}{\rho} d\rho.
\] \hspace{1cm} (B.16)

Using the formula
\[
\frac{d}{d\rho} \frac{e^{-in\rho} - 1}{\rho} = \frac{1 - e^{-in\rho} - in\rho e^{-in\rho}}{\rho^2},
\]
and integration by parts in $\rho$ in (B.16), we have
\[
\tilde{M}_{n+1}(\xi; \eta_1, \ldots, \eta_{n+1}) = - \int_{\mathbb{R}} \frac{e^{-in(\eta_{n+1}\beta)} - 1}{\rho} \frac{d}{d\rho} \left[ e^{-i(\xi - \eta_{n+1})\rho} \prod_{l=1}^{n} \frac{e^{in\rho} - 1}{\rho} \right] d\rho
\]
\[
= -i(\xi - \eta_{n+1}) \int_{\mathbb{R}} e^{-i\xi \rho} \prod_{l=1}^{n} \frac{e^{in\rho} - 1}{\rho} d\rho
\]
\[
+ \sum_{j=1}^{n} \int_{\mathbb{R}} e^{-i\xi \rho} \frac{i\eta_{n+1} \rho e^{in\rho} - e^{in\rho} + 1}{\rho^2} \prod_{l=1, l \neq j}^{n} \frac{e^{in\rho} - 1}{\rho} d\rho.
\] \hspace{1cm} (B.17)

For $j \in \{1, \ldots, n\}$ let $\pi_j : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the map that permutes the variables $\eta_j$ and $\eta_{n+1}$,
\[
\pi_j(\eta_1, \ldots, \eta_j, \ldots, \eta_{n}, \eta_{n+1}) := \pi_j(\eta_1, \ldots, \eta_{n+1}, \ldots, \eta_n, \eta_j).
\]
The formulas (B.16) and (B.17) show that
\[
\tilde{M}_{n+1}(\xi; \eta) + \sum_{j=1}^{n} \tilde{M}_{n+1}(\xi; \pi_j(\eta)) = -i(\xi - \eta_{n+1}) \int_{\mathbb{R}} e^{-i\xi \rho} \prod_{l=1}^{n+1} \frac{e^{i\eta_l \rho} - 1}{\rho} d\rho \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}} e^{-i\xi \rho} \frac{\eta_j e^{i\eta_j \rho} - e^{\eta_j \rho} + 1}{\rho^2} \prod_{l=1, l \neq j}^{n+1} \frac{e^{\eta_l \rho} - 1}{\rho} d\rho \\
+ \sum_{j=1}^{n} \int_{\mathbb{R}} e^{-i\xi \rho} \frac{1 - i\eta_j \rho}{\rho} \prod_{l=1, l \neq j}^{n+1} \frac{e^{\eta_l \rho} - 1}{\rho} d\rho \\
= -i(\xi - \eta_1 - \ldots - \eta_{n+1}) \int_{\mathbb{R}} e^{-i\xi \rho} \prod_{l=1}^{n+1} \frac{e^{i\eta_l \rho} - 1}{\rho} d\rho,
\]
where \( \eta := (\eta_1, \ldots, \eta_{n+1}) \). Letting
\[
M_{n+1}(\xi; \eta) := \frac{1}{n+1} \left[ \tilde{M}_{n+1}(\xi; \eta) + \sum_{j=1}^{n} \tilde{M}_{n+1}(\xi; \pi_j(\eta)) \right]
\]
(B.18)
it follows by symmetrization from (B.15) that
\[
\mathcal{F}[R_0 f](\xi) = \frac{1}{\pi (2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} M_{n+1}(\xi; \eta) \hat{f}(\xi - \eta_1 - \ldots - \eta_{n+1}) \times \hat{h}(\eta_1) \ldots \hat{h}(\eta_{n+1}) d\eta.
\]
(B.19)

(i) We prove now the bounds (B.12). Recall the formulas
\[
\int_{\mathbb{R}} e^{-i\xi x} \, dx = 2\pi \delta_0(\xi), \quad \int_{\mathbb{R}} e^{-i\xi x} \frac{1}{x} \, dx = -i\pi \text{sgn}(\xi), \quad \int_{\mathbb{R}} e^{-i\xi x} \frac{1}{x^2} \, dx = -\pi |\xi|.
\]
(B.20)

for any \( \xi \in \mathbb{R} \). Using these formulas, the symbol \( M_1(\xi; \eta_1) \) can be calculated easily,
\[
M_1(\xi; \eta_1) = \pi(\xi - \eta_1) \left[ \text{sgn}(\xi) - \text{sgn}(\xi - \eta_1) \right].
\]
(B.21)

Therefore
\[
\mathcal{F}[R_0 f](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} (\xi - \eta_1) \left[ \text{sgn}(\xi) - \text{sgn}(\xi - \eta_1) \right] \hat{f}(\xi - \eta_1) \hat{h}(\eta_1) \, d\eta_1.
\]
(B.22)

The bounds (B.12) can be proved easily: it follows from (B.22) and (B.11) that
\[
\|P_k R_0 f\|_{L^p} \lesssim \sum_{k_2 + 10 \geq \max(k, k_1)} 2^{k_1} \|P_{k_1}^* f\|_{L^\infty} \|P_k h\|_{L^p} \lesssim \Lambda^{-1/2} \sum_{k_2 + 10 \geq k} 2^{(1-b) \min(k_2, 0)} \|P_{k_2}^* h\|_{L^p},
\]
for any \( k \in \mathbb{Z} \) and \( p \in (2, \infty) \). Moreover,
\[
\|R_0 f\|_{L^2} \lesssim \sum_{k_2 + 10 \geq k_1} 2^{k_1} \|P_{k_1}^* f\|_{L^\infty} \|P_k h\|_{L^2} \lesssim \varepsilon \Lambda^{k_0 - 1/2}.
\]

The desired bounds in (B.12) follow using the assumptions (B.3).

(ii) We prove now the bounds (B.13). Using (B.20) again
\[
M_2(\xi; \eta_1, \eta_2) := \frac{i\pi}{2} (\xi - \eta_1 - \eta_2) \left[ |\xi - \eta_1 - \eta_2| + |\xi| - |\xi - \eta_1| - |\xi - \eta_2| \right].
\]
(B.23)
The main observation is that $M_2(\xi; \eta_1, \eta_2) = 0$ if $|\eta_1| + |\eta_2| \leq |\xi|$. It is easy to see that
\[ \left\| F^{-1} \left[ M_2(\xi; \eta_1, \eta_2) \varphi_k(\eta_1) \varphi_k(\eta_2) (\xi - \eta_1 - \eta_2) \right] \right\|_{L^1(\mathbb{R}^3)} \lesssim 2^k_2 2^{k_1} \]
if $k_2 \geq k_1$. Therefore, using Lemma 2.1 and the bounds (B.3) and (B.11),
\[ \| P_k R_1 f \|_{L^p} \lesssim \sum_{k_2 + 10 \geq \max(k, k_1, k_3)} 2^{k_1 + k_3} \| P'_{k_1} h \|_{L^\infty} \| P'_{k_2} h \|_{L^p} \| P'_{k_3} f \|_{L^\infty} \lesssim \varepsilon_1 \Lambda^{-1} \sum_{k_2 + 10 \geq k} 2^{3 \min(k_2, 0) / 5} \| P'_{k_2} h \|_{L^p}, \tag{B.24} \]
for $p \in \{2, \infty\}$ and any $k \in \mathbb{Z}$. In particular,
\[ \| P_{\geq 0} R_1 f \|_{H_{N_0+1}} \lesssim \varepsilon_1^2 \Lambda^{-1/2} \quad \text{and} \quad \sum_{k \geq 0} 2^{(N_2+1)k} \| P_k R_1 f \|_{L^\infty} \lesssim \varepsilon_1^2 \Lambda^{-3/2}. \tag{B.25} \]
Moreover
\[ \| R_1 f \|_{L^2} \lesssim \sum_{k_2 + 10 \geq \max(k, k_1, k_3)} 2^{k_1 + k_3} \| P'_{k_1} h \|_{L^\infty} \| P'_{k_2} h \|_{L^2} \| P'_{k_3} f \|_{L^\infty} \lesssim \varepsilon_1^2 \Lambda^{p_0-1}. \tag{B.26} \]
Finally, as a consequence of (B.3), (B.24), and Sobolev embedding,
\[ 2^{-k/10} \| P_k R_1 f \|_{L^\infty} \lesssim 2^{2k/5} \| P_k R_1 f \|_{L^2} \lesssim 2^{2k/5} \varepsilon_1^2 \Lambda^{p_0-1}, \]
\[ 2^{-k/10} \| P_k R_1 f \|_{L^\infty} \lesssim 2^{-k/10} \varepsilon_1 \Lambda^{-1} \cdot 2^{-2k/5} \varepsilon_1 \Lambda^{-1/2} \lesssim \varepsilon_1^2 \Lambda^{-3/2} 2^{-k/2}, \tag{B.27} \]
for any integer $k \leq 0$. The desired bounds (B.13) follow from (B.25) - (B.27).

To prove (B.14) we would like to use induction over $n$. For this we need to prove slightly stronger bounds. The desired conclusion follows from Lemma 2.3 below with $f_\mathbf{k} = -\partial_x f$. \qed

**Lemma B.3.** For any $\mathbf{k} = (k_1, \ldots, k_{n+1}) \in \mathbb{Z}^{n+1}$ and $n \geq 1$ let $F_{n; \mathbf{k}}(f_\mathbf{k})$ be defined by
\[ F(F_{n; \mathbf{k}}(f_\mathbf{k}))(\xi) := \frac{1}{\pi (2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} M'_{n+1}(\xi; \eta) \varphi_{k_1}(\eta_1) \cdots \varphi_{k_{n+1}}(\eta_{n+1}) \]
\[ \times \hat{f}_\mathbf{k}(\xi - \eta_1 - \cdots - \eta_{n+1}) \cdot \hat{h}(\eta_1) \cdots \hat{h}(\eta_{n+1}) d\eta, \tag{B.28} \]
where $\eta = (\eta_1, \ldots, \eta_{n+1})$ and
\[ M'_{n+1}(\xi; \eta) := \int_{\mathbb{R}} e^{-i\xi \rho} \prod_{l=1}^{n+1} \frac{e^{i\eta_l \rho} - 1}{\rho} d\rho. \tag{B.29} \]
Assume that the functions $f_\mathbf{k}$ satisfy the uniform bounds
\[ \| f_\mathbf{k} \|_{W^{1/5, 0}} \leq \Lambda^{-1/2}. \tag{B.30} \]
Then there is a constant $C_0 \geq 1$ such that if $n \geq 2$ then
\[ \left\| \sum_{\mathbf{k} \in \mathbb{Z}^{n+1}} F_{n; \mathbf{k}}(f_\mathbf{k}) \right\|_{H_{N_0+1}} \leq (C_0 \varepsilon_1)^{n+1} \Lambda^{-5/4}. \tag{B.31} \]

**Proof.** Notice that
\[ M'_{n+1}(\xi; \eta_1, \ldots, \eta_{n+1}) = 0 \quad \text{if} \quad \eta_1 \cdot \cdots \eta_{n+1} = 0. \tag{B.32} \]
Taking partial derivatives in $\eta$ it follows that
\[ M'_{n+1}(\xi; \eta_1, \ldots, \eta_{n+1}) = 0 \quad \text{if} \quad |\eta_1| + \cdots + |\eta_{n+1}| \leq |\xi|. \tag{B.33} \]
Let $\chi_0 := F^{-1}(\varphi_0)$ and let $\chi_0'$ denote the derivative of $\chi_0$. Let $\tilde{k}_1 \leq \ldots \leq \tilde{k}_{n+1}$ denote the increasing rearrangement of the integers $k_1, \ldots, k_{n+1}$. In view of (B.33),

$$F_{n,k}(f_k) = F_{n,k}(P_{\leq \tilde{k}_{n+1} + 5n} f_k).$$  \hfill (B.34)

Using the formula

$$\frac{1}{2\pi} \int_{\mathbb{R}} \varphi_k(\mu)e^{iy\mu}e^{i\mu\rho} - 1 \, d\mu = \frac{2^k}{\rho} \left[ \chi_0(2^k(y + \rho)) - \chi_0(2^k y) \right],$$

and the definition (B.28), we rewrite

$$F_{n,k}(f_k)(x) = \frac{1}{\pi} \int_{\mathbb{R}^n+2} f_k(x - \rho) \prod_{l=1}^{n+1} P_{k_l} h(x - \rho - y_l) \cdot \prod_{l=1}^{n+1} K_{k_l}(y_l, \rho) \, d\rho dy,$$  \hfill (B.35)

where $y = (y_1, \ldots, y_{n+1})$ and

$$K_{k_l}(y_l, \rho) := \frac{2^k}{\rho} \left[ \chi_0(2^k(y_l + \rho)) - \chi_0(2^k y_l) \right].$$  \hfill (B.36)

In view of (B.34) we may also assume that $f_k = P_{\leq \tilde{k}_{n+1} + 5n} f_k$.

We notice that

$$\int_{\mathbb{R}} |K_k(y, \rho)| \, dy \lesssim \min (|\rho|^{-1}, 2^k).$$  \hfill (B.37)

The inequalities (B.37) show that

$$\int_{\mathbb{R}^n+2} \left| \prod_{l=1}^{n+1} K_{k_l}(y_l, \rho) \right| \, d\rho dy \leq C \cdot 2^{\tilde{k}_1 + \ldots + \tilde{k}_n} (1 + |\tilde{k}_{n+1} - \tilde{k}_n|),$$  \hfill (B.38)

for some constant $C \geq 1$ (which is allowed to change from now on from line to line). Therefore, using (B.35),

$$\|F_{n,k}(f_k)\|_{L^2} \leq C \cdot (1 + |\tilde{k}_{n+1} - \tilde{k}_n|) \cdot \|f_k\|_{L^\infty} \|P_{\tilde{k}_{n+1}}' h\|_{L^\infty} \prod_{l=1}^{n} 2^{\tilde{k}_l} \|P_{k_l}' h\|_{L^\infty}. $$  \hfill (B.39)

Therefore

$$\sum_{(k_1, \ldots, k_{n+1}) \in J_1} \|F_{n,k}(f_k)\|_{H^{N_0+1}} \leq (C \varepsilon_1)^{n+1} \Lambda^{-(n+1)/2 + p_0},$$  \hfill (B.40)

and

$$\sum_{(k_1, \ldots, k_{n+1}) \in J_2} \|F_{n,k}(P_{\leq \tilde{k}_{n+1} - 10n+1} f_k)\|_{H^{N_0+1}} \leq (C \varepsilon_1)^{n+1} \Lambda^{-(n+1)/2 + p_0},$$  \hfill (B.41)

where

$$J_1 := \{ (k_1, \ldots, k_{n+1}) \in \mathbb{Z}^{n+1} : \tilde{k}_{n+1} \leq \max(0, \tilde{k}_n) + 10n \},$$

$$J_2 := \{ (k_1, \ldots, k_{n+1}) \in \mathbb{Z}^{n+1} : k_1 = \tilde{k}_{n+1} \geq \max(0, \tilde{k}_n) + 10n \}. $$  \hfill (B.42)

It remains to prove that if $n \geq 1$ then

$$\left\| \sum_{(k_1, \ldots, k_{n+1}) \in J_2} F_{n,k}(P_{\leq k_1 - 10n} f_k) \right\|_{H^{N_0+1}} \leq (C_1 \varepsilon_1)^{n+1} \Lambda^{-\varepsilon_n}$$  \hfill (B.43)

for some constant $C_1 \geq 1$, where $\varepsilon_1 = 1 - p_0$ and $\varepsilon_n = 5/4$ if $n \geq 2$. 


We prove the inequalities in (B.42) using induction over $n$ (the case $n = 1$ was already proved in Lemma [B.2]). We decompose

$$K_{k_{n+1}}(y_{n+1}, \rho) = \tilde{K}_{k_{n+1}}(y_{n+1}, \rho) + 2^{2k_{n+1}} \chi_0'(2^{k_{n+1}} y_{n+1})$$

where

$$\tilde{K}_k(y, \rho) := \frac{2^k [\chi_0(2^k (y + \rho)) - \chi_0(2^k y) - 2^k \rho \chi_0'(2^k y)]}{\rho}.$$ 

Then we decompose

$$F_{n,k}(P_{\leq k_1-10n} f_k) = F_{n,k}^1(P_{\leq k_1-10n} f_k) + F_{n,k}^2(P_{\leq k_1-10n} f_k),$$

where

$$F_{n,k}^1(g)(x) := \frac{1}{\pi} \int_{\mathbb{R}^{n+2}} g(x - \rho) \prod_{l=1}^{n+1} P_{k_l}' h(x - \rho - y_l) \cdot \tilde{K}_{k_{n+1}}(y_{n+1}, \rho) \prod_{l=1}^n K_{k_l}(y_l, \rho) \, d\rho \, dy,$$  

and

$$F_{n,k}^2(g)(x) := \frac{1}{\pi} \int_{\mathbb{R}^{n+2}} g(x - \rho) \prod_{l=1}^{n+1} P_{k_l}' h(x - \rho - y_l) \cdot 2^{2k_{n+1}} \chi_0'(2^{k_{n+1}} y_{n+1}) \prod_{l=1}^n K_{k_l}(y_l, \rho) \, d\rho \, dy.$$  

We notice that

$$\int_{\mathbb{R}} |\tilde{K}_k(y, \rho)| \, dy \lesssim 2^k \rho \min(|\rho|^{-1}, 2^k).$$

Using also (B.37) it follows that if $(k_1, \ldots, k_{n+1}) \in J_2$ then

$$\int_{\mathbb{R}^{n+2}} |\tilde{K}_{k_{n+1}}(y_{n+1}, \rho) \prod_{l=1}^n K_{k_l}(y_l, \rho)| \, d\rho \, dy \leq C^n(1 + |\bar{k}_n - \bar{k}_{n-1}|) 2^{\bar{k}_1 + \cdots + \bar{k}_n},$$

which is slightly stronger than the inequality (B.38). Therefore

$$\left\| F_{n,k}^1(P_{\leq k_1-10n} f_k) \right\|_{L^2} \leq C^n(1 + |\bar{k}_n - \bar{k}_{n-1}|) \cdot \|f_k\|_{L\infty} \|P_{k_1}' h\|_{L^2} \prod_{l=1}^n 2^{\bar{k}_l} \|P_{k_l}' h\|_{L\infty}.$$ 

Therefore, for any $l \geq 0$,

$$\left\| P_l \left( \sum_{k \in J_2} F_{n,k}^1(P_{\leq k_1-10n} f_k) \right) \right\|_{L^2} \leq C^n(\varepsilon_1 \Lambda^{-1/2})^n \Lambda^{-1/2} \sum_{|k_1 - l| \leq 10} \|P_{k_1}' h\|_{L^2}.$$ 

Therefore

$$\left\| \sum_{k \in J_2} F_{n,k}^1(P_{\leq k_1-10n} f_k) \right\|_{H^{n_0+1}} \leq (C\varepsilon_1)^{n+1} \Lambda^{-5/4}. \quad (B.45)$$

We estimate now the contributions of the terms $F_{n,k}^2(P_{\leq k_1-10n} f_k)$. We integrate the variable $y_{n+1}$ in the defining formula (B.44). Let

$$g_{(k_1, \ldots, k_n)}(z) := \sum_{k_{n+1} \leq k_1-10n} f_{(k_1, \ldots, k_n, k_{n+1})}(z) \int_{\mathbb{R}} P_{k_{n+1}}' h(z - y_{n+1}) 2^{2k_{n+1}} \chi_0'(2^{k_{n+1}} y_{n+1}) \, dy_{n+1}.$$ 

Using the assumptions (B.30) and (B.3), it is easy to see that

$$\sum_{l \in \mathbb{Z}} (2^l/5 + 1) \|P_l g_{(k_1, \ldots, k_n)}\|_{L\infty} \lesssim \Lambda^{-1/2} \cdot \varepsilon_1 \Lambda^{-2/5}.$$
The induction hypothesis shows that
\[ \left\| \sum_{k \leq l} F_{n,k}^2 (P_{\leq n} + \Delta_{n+1} f_k) \right\|_{H^{n_0+1}} \leq (C_1 \epsilon_1)^n \Lambda^{-1+p_0} \cdot C \epsilon_1 \Lambda^{-2/5}. \]

The desired conclusion follows, using also (B.45) provided that \( C_1 \) is sufficiently large.

\( \square \)

**B.2. Decomposition of the\textsuperscript{Dirichlet–Neumann operator}.** The main result in this subsection is the decomposition of the operator \( G(h) \phi \) proved in Proposition [B.7]. For \( \alpha \in [-1, 1] \) let
\[ \mathcal{E}^\alpha := \{ f \in C_0 : \|f\|_{\mathcal{E}^\alpha} := \Lambda^{-p_0} \|f\|_{H^{n_0+\alpha,p_1}} + \Lambda^{1/2} \|f\|_{W^{2+\alpha,-2/5}} < \infty \}. \]  

We consider first a suitable decomposition of the operators \( \mathcal{H}_\gamma \).

**Lemma B.4.** Let
\[ (H_0 f)(\alpha) := \frac{1}{\pi i} \text{p.v.} \int_\mathbb{R} \frac{f(\beta)}{\alpha - \beta} d\beta, \]  
de note the unperturbed Hilbert transform, and consider the operators \( T_1 \) and \( T_2 \) defined by
\[ (T_1 f)(\alpha) := \frac{1}{\pi i} \text{p.v.} \int_\mathbb{R} \frac{h(\alpha) - h(\beta) - \beta' h(\beta)(\alpha - \beta)}{|\gamma(\alpha) - \gamma(\beta)|^2} f(\beta) d\beta, \]
\[ (T_2 f)(\alpha) := \frac{1}{\pi i} \text{p.v.} \int_\mathbb{R} \frac{h(\alpha) - h(\beta) - \beta' h(\beta)(\alpha - \beta) h(\alpha) - h(\beta)}{\alpha - \beta} f(\beta) d\beta. \]

Then
\[ \mathcal{H}_\gamma = H_0 - T_1 + i T_2, \]  
and
\[ T_1 = \sum_{n \geq 0} (-1)^n R_{2n}, \quad T_2 = \sum_{n \geq 0} (-1)^n R_{2(n+1)}. \]  

Moreover, if \( f \in \mathcal{E}^{-1} \) then
\[ \| (T_1 - R_0) f \|_{H^{n_0+1}} + \| T_2 f \|_{H^{n_0+1}} \lesssim \epsilon_1^2 \Lambda^{p_0-1} \| f \|_{\mathcal{E}^{-1}}, \]
\[ \| (T_1 - R_0) f \|_{W^{2,1/10}} \| T_2 f \|_{W^{2,1/10}} \lesssim \epsilon_1^2 \Lambda^{-11/10} \| f \|_{\mathcal{E}^{-1}}, \]  
and
\[ \| \mathcal{H}_\gamma f \|_{\mathcal{E}^{-1}} \lesssim \| f \|_{\mathcal{E}^{-1}}, \quad \| R_0 f \|_{\mathcal{E}^1} \lesssim \epsilon_1 \Lambda^{-1/2} \| f \|_{\mathcal{E}^{-1}}. \]

As a consequence
\[ \| T_1 f \|_{\mathcal{E}^1} + \| T_2 f \|_{\mathcal{E}^1} \lesssim \epsilon_1 \Lambda^{-1/2} \| f \|_{\mathcal{E}^{-1}}. \]  

**Proof.** The identities (B.49) and (B.50) follow directly from definitions. Notice that the condition (B.11) is verified if \( \| f \|_{\mathcal{E}^{-1}} \leq 1 \), so the conclusions of Lemma [B.2] can be applied here. The bounds (B.51) follow from (B.13)–(B.14). Notice that
\[ \tilde{H}_g(\xi) = - \text{sgn}(\xi) \tilde{g}(\xi). \]

The bounds (B.52) follow using also the formula (B.49), and the bounds (B.51) and (B.12). \( \square \)

We are now ready to define the conjugate pair \((\phi, \psi)\).

**Lemma B.5.** (i) We have
\[ \mathcal{H}_\gamma^2 = I \]  
on \( \mathcal{E}^{-1} \).

Moreover, if \( \phi \in \mathcal{E}^{-1} \) is a real-valued function then there is a unique real-valued function \( \psi \in \mathcal{E}^{-1} \) with the property that
\[ (I - \mathcal{H}_\gamma)(\phi + i \psi) = 0. \]  

\( \square \)
(ii) The function $F : \Omega \to \mathbb{C}$,

$$F(z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(\phi + i\psi)(\beta)\gamma'(\beta)}{z - \gamma(\beta)} \, d\beta$$

(B.57)

in a bounded analytic function in $\Omega$, which extends to a $C^1$ function in $\overline{\Omega}$ with the property that $F(x + ih(x)) = (\phi + i\psi)(x)$ for any $x \in \mathbb{R}$.

Proof. The identity (B.55) is a standard consequence of the Cauchy integral formula applied to the analytic function $F_f$ defined in (B.6), and the second limit in (B.9).

The uniqueness of $\psi$ satisfying (B.56) follows from Lemma B.4: if $\psi_1, \psi_2$ are real-valued solutions of (B.56) then

$$(I - H\gamma)(\psi_1 - \psi_2) = 0.$$ 

Using the formula (B.56) and taking the real part, it follows that $(I + T_1)(\psi_1 - \psi_2) = 0$. Since $\|T_1\|_{\mathcal{E}^{-1} \to \mathcal{E}^{-1}} \lesssim \varepsilon_1$ (see (B.53), this shows that $\psi_1 - \psi_2 = 0$.

To prove existence, let $P := (I - H\gamma)(-i\phi + \psi)$ and notice that $(I + H\gamma)P = 0$ (as a consequence of (B.55)). Therefore, using that $T_1$ is a contraction on $\mathcal{E}^{-1}$ as before, $P = 0$ if and only if $\Re P = 0$. This is equivalent to

$$(I - T_1)\psi = (-iH_0 + T_2)\phi.$$ (B.58)

The existence of $\psi \in \mathcal{E}^{-1}$ satisfying (B.58) is a consequence of Lemma B.4. This completes the proof of part (i). The claims in (ii) follow from the second identity in (B.9). □

Assume $\phi, \psi$ and $F = G + iH$ are as in Lemma B.5. Let

$$v_1 := \partial_x G = \partial_y H, \quad v_2 := \partial_y G = -\partial_x H.$$ 

Let

$$V(x) := v_1(x + ih(x)), \quad B(x) := v_2(x + ih(x)), \quad x \in \mathbb{R}.$$ 

Notice that

$$G(x + ih(x)) = \phi(x), \quad H(x + ih(x)) = \psi(x).$$

Taking derivatives, we have

$$\phi_x = V + h_x B, \quad -\psi_x = B - h_x V.$$ (B.59)

Therefore

$$V = \frac{\phi_x + h_x \psi_x}{1 + h_x^2}, \quad B = \frac{h_x \phi_x - \psi_x}{1 + h_x^2}.$$ (B.60)

Notice that

$$\partial_x H\gamma f = \gamma_x H\gamma(f_x/\gamma_x).$$

Therefore the identity (B.56) gives

$$(I - H\gamma)\left(\frac{\phi_x + i\psi_x}{\gamma_x}\right) = 0.$$ 

Notice also that

$$\phi_x + i\psi_x = (1 + ih_x)(V - iB),$$

as a consequence of (B.59). Therefore

$$(I - H\gamma)(V - iB) = 0.$$ 

To summarize, we have the following lemma:
Lemma B.6. Assume $\phi \in \mathcal{E}^{1/2}$ and define $\psi$ as in Lemma B.5. Then $\psi \in \mathcal{E}^{1/2}$ and
\[
\|\psi\|_{\mathcal{E}^{1/2}} \lesssim \|\phi\|_{\mathcal{E}^{1/2}}. \tag{B.61}
\]
Let
\[
V := \frac{\phi_x + h_x \psi_x}{1 + h_x^2}, \quad B := \frac{h_x \phi_x - \psi_x}{1 + h_x^2}. \tag{B.62}
\]
Then
\[
\phi_x = V + h_x B, \quad -\psi_x = B - h_x V, \quad (I - \mathcal{H}_\gamma)(V - iB) = 0. \tag{B.63}
\]
In addition, for any $g \in \{\phi_x, \psi_x\}$,
\[
\Lambda^{-p_0}\|g\|_{H^{N_0-1/2}} + \Lambda^{1/2} \sum_{k \in \mathbb{Z}} (2^{(N_2-1/2)k} + 2^{-k/10}) \|P_k g\|_{L^\infty} \lesssim \|\phi\|_{\mathcal{E}^{1/2}}, \tag{B.64}
\]
and, for any $f \in \{\phi_x, \psi_x, V, B\}$,
\[
\Lambda^{-p_0}\|f\|_{H^{N_0-1/2}} + \Lambda^{1/2} \sum_{k \in \mathbb{Z}} (2^{N_2k} + 2^{-k/10}) \|P_k f\|_{L^\infty} \lesssim \|\phi\|_{\mathcal{E}^{1/2}}. \tag{B.65}
\]

Proof. The inequality (B.61) follows from (B.53) and (B.58). The formulas (B.63) were derived earlier. The bounds (B.64) follow as a consequence of (B.61) and the definition. The function $h_x$ satisfies similar bounds,
\[
\Lambda^{-p_0}\|h_x\|_{H^{N_0}} + \Lambda^{1/2} \sum_{k \in \mathbb{Z}} (2^{N_2k} + 2^{-k/10}) \|P_k h_x\|_{L^\infty} \lesssim \varepsilon_1,
\]
see (B.3). The desired bounds (B.65) for $V, B$ follow from (B.62) and Lemma B.1. \hfill \Box

We are now ready to state the main result in this section.

Proposition B.7. Assume $\phi \in \mathcal{E}^{1/2}$, and define $\psi, V, B$ as in Lemma B.6. Let
\[
G(h) \phi := -\psi_x, \tag{B.66}
\]
and
\[
N_2 := |\partial_x| T_B h + \partial_x T_V h - G_2, \quad G_{\geq 3} := |\partial_x| \phi + G(h) \phi + N_2, \tag{B.67}
\]
where
\[
\widehat{G_2}(\xi) = \frac{1}{2\pi} \int_R \widehat{h}(\eta) \widehat{\phi}(-\xi - \eta) [1 - \chi(\xi - \eta, \eta)] [\xi(\xi - \eta) - |\xi||\xi - \eta|] \, d\eta. \tag{B.68}
\]
Then
\[
\Lambda^{-p_0} \|G_{\geq 3}\|_{H^{N_0+1}} + \Lambda^{11/10} \|G_{\geq 3}\|_{W^{N_2+1}} \lesssim \varepsilon_1^2 \|\phi\|_{\mathcal{E}^{1/2}}. \tag{B.69}
\]

Proof. We may assume $\|\phi\|_{\mathcal{E}^{1/2}} = 1$. The last identity in (B.63) gives
\[
(I - H_0)(V - iB) + (T_1 - iT_2)(V - iB) = 0
\]
Taking real and imaginary parts we have
\[
V + iH_0 B = -T_1 V + T_2 B, \quad -B + iH_0 V = T_2 V + T_1 B. \tag{B.70}
\]
Using also the formulas $-\psi_x = B - h_x V$, see (B.63), and $|\partial_x| \phi = iH_0(\phi_x)$, we have
\[
|\partial_x| \phi - G(h) \phi = iH_0(\phi_x) - B + h_x V
\]
\[
= T_2 V + T_1 B + h_x V + iH_0(h_x B). \tag{B.71}
\]
Let
\[
D := T_2 V + T_1 B = iH_0 V - B. \tag{B.72}
\]
Therefore
\[
|\partial_x \phi - G(h)\phi - N_2| = I + II,
\]
\[
I := \dot{h}_x V - H_0(h_x H_0 V) - |\partial_x| T_{h_0}^* h - \partial_x T_V h + G_2 + R_0(iH_0 V),
\]
\[
II := D - R_0(iH_0 V) - iH_0(h_x D) + |\partial_x| T_D h.
\]
Using the definitions and (B.22), we write
\[
\hat{I}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(\eta) \hat{V}(\xi - \eta)p(\xi, \eta) \, d\eta + G_2
\]
where
\[
p(\xi, \eta) := i\eta - i\eta \text{sgn}(\xi) \text{sgn}(\xi - \eta) + i|\xi| \text{sgn}(\xi - \eta) \chi(\xi - \eta, \eta) - i\xi \chi(\xi - \eta, \eta)
\]
\[
+ i(\xi - \eta)[1 - \text{sgn}(\xi) \text{sgn}(\xi - \eta)]
\]
\[
= i\xi[1 - \chi(\xi - \eta, \eta)][1 - \text{sgn}(\xi) \text{sgn}(\xi - \eta)].
\]
Using also the formulas (B.68) and (B.59), we have
\[
\hat{I}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(\eta) F(V - \phi_\xi)(\xi - \eta)p(\xi, \eta) \, d\eta = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}(\eta) F(h_x B)(\xi - \eta)p(\xi, \eta) \, d\eta.
\]
In view of (B.65), (B.3), and Lemma B.1
\[
\|h_x B\|_{L^\infty} + \sum_{k\geq 0} 2^{(N_2 - 1/2)k} \|P_k(h_x B)\|_{L^\infty} \lesssim \varepsilon_1 \Lambda^{-1}.
\]
Therefore, for \(p \in \{2, \infty\}\) and any \(k \in \mathbb{Z}\),
\[
\|P_k I\|_{L^p} \lesssim \sum_{k' \geq k - 10} 2^k \|P_{k'} h\|_{L^p} \|P_{k'}(h_x B)\|_{L^\infty} \lesssim \varepsilon_1 \Lambda^{-1} \sum_{k' \geq k - 10} 2^{k - (N_2 - 1/2) \max(k, 0)} \|P_{k'} h\|_{L^p}.
\]
This shows that
\[
\|I\|_{H^{N_0 + 1}} \lesssim \varepsilon_1^2 \Lambda^{-1 + \rho_0} \quad \text{and} \quad \|I\|_{W^{-N_2 + 1}} \lesssim \varepsilon_1^2 \Lambda^{-11/10}.
\]
It follows from (B.13)–(B.14) and (B.65) that
\[
\|(T_1 - R_0) B\|_{H^{N_0 + 1}} + \|T_2 V\|_{H^{N_0 + 1}} \lesssim \varepsilon_1^2 \Lambda^{\rho_0 - 1},
\]
\[
\|(T_1 - R_0) B\|_{L^\infty} + \|T_2 V\|_{L^\infty} + \sum_{k \geq 0} 2^{(N_2 + 1)k} \|P_k(T_1 - R_0) B\|_{L^\infty} + \|P_k T_2 V\|_{L^\infty} \lesssim \varepsilon_1^2 \Lambda^{-11/10}.
\]
Therefore, using the formula (B.72),
\[
\|D - R_0 B\|_{H^{N_0 + 1}} \lesssim \varepsilon_1^2 \Lambda^{-1 + \rho_0} \quad \text{and} \quad \|D - R_0 B\|_{W^{-N_2 + 1}} \lesssim \varepsilon_1^2 \Lambda^{-11/10}.
\]
Moreover,
\[
F[-iH_0(h_x D) + |\partial_x| T_D h](\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{D}(\xi - \eta) \hat{h}(\eta) \text{sgn}(\xi)|\xi\chi(\xi - \eta, \eta) - \eta| \, d\eta.
\]
Using (B.3) and (B.76), it follows easily that
\[
\| - iH_0(h_x D) + |\partial_x |T D h \|_{\dot{H}^{N_0+1}} \lesssim \varepsilon_1^2 \Lambda^{-1/p_0},
\]
\[
\| - iH_0(h_x D) + |\partial_x |T D h \|_{\dot{W}^{N_2+1}} \lesssim \varepsilon_1^2 \Lambda^{-11/10}.
\]
(B.78)

The desired bound (B.69) follows from (B.73) and the bounds (B.74), (B.75), (B.77) and (B.78).

\subsection{B.3. Weighted estimates}

Assume now that \( h \in C([0, T] : \mathcal{C}_0 \cap \dot{H}^{N_0+1/2+p_1}) \) satisfies the stronger bounds
\[
\| h(t) \|_{\dot{H}^{N_0+1/2+p_1}} \leq \varepsilon_1(1 + t)^{p_0},
\]
\[
\| h(t) \|_{\dot{W}^{N_2+1/10}} \leq \varepsilon_1(1 + t)^{-1/2},
\]
\[
\| Sh(t) \|_{\dot{H}^{N_1+1/2+p_1}} \leq \varepsilon_1(1 + t)^{4p_0},
\]
(B.79)

for any \( t \in [0, T] \), compare with the hypothesis of Proposition 2.4. We define the curve \( \gamma = \gamma(x, t) \), its associated Hilbert transform \( \mathcal{H}_\gamma \), and the operators \( R_n \) as before, see (B.4), (B.5), and (B.10).

For \( \alpha \in [-1, 1] \) let
\[
\mathcal{E}_{\alpha} := \left\{ f \in C([0, T] : \dot{H}^{N_0+\alpha,p_1}) : \| f \|_{\mathcal{E}_{\alpha}} := \sup_{t \in [0, T]} \| f(t) \|_{\mathcal{E}_{\alpha}} \right\},
\]
where
\[
\| f(t) \|_{\mathcal{E}_{\alpha}} := (1 + t)^{-p_0} \| f(t) \|_{\dot{H}^{N_0+\alpha,p_1}} + (1 + t)^{1/2} \| f(t) \|_{\dot{W}^{N_2+\alpha,1/5}}
\]
\[
+ (1 + t)^{-4p_0} \| Sh(t) \|_{\dot{H}^{N_1+\alpha,p_1}},
\]
(B.81)

compare with the definition (B.46). We have the following weighted analogue of Lemma B.2

\begin{lemma}
Assume that \( f \in \mathcal{E}_{\alpha}^{-1} \). There is a constant \( C \geq 1 \) such that, for any \( t \in [0, T] \),
\[
\| R_0 f(t) \|_{\mathcal{E}_{\alpha}^{-1}} \leq C\varepsilon_1 (1 + t)^{-1/2} \| f \|_{\mathcal{E}_{\alpha}^{-1}}.
\]
(B.82)

Moreover, for any \( t \in [0, T] \),
\[
\| R_1 f(t) \|_{\dot{H}^{N_0+1}} \leq (C\varepsilon_1)^2 (1 + t)^{p_0-1} \| f \|_{\mathcal{E}_{\alpha}^{-1}},
\]
\[
\| R_1 f(t) \|_{\dot{W}^{N_2+1/10}} \leq (C\varepsilon_1)^2 (1 + t)^{-11/10} \| f \|_{\mathcal{E}_{\alpha}^{-1}},
\]
\[
\| SR_1 f(t) \|_{\dot{H}^{N_1+1}} \leq (C\varepsilon_1)^2 (1 + t)^{4p_0-1} \| f \|_{\mathcal{E}_{\alpha}^{-1}},
\]
(B.83)

and, for any \( n \geq 2 \),
\[
\| R_n f(t) \|_{\dot{H}^{N_0+1}} \leq (C\varepsilon_1)^{n+1} (1 + t)^{-5/4} \| f \|_{\mathcal{E}_{\alpha}^{-1}},
\]
\[
\| SR_n f(t) \|_{\dot{H}^{N_1+1}} \leq (C\varepsilon_1)^{n+1} (1 + t)^{-5/4} \| f \|_{\mathcal{E}_{\alpha}^{-1}}.
\]
(B.84)

\end{lemma}

\begin{proof}
We may assume that \( \| f \|_{\mathcal{E}_{\alpha}^{-1}} = 1 \). Given the results of Lemma B.2, we only need to prove the weighted \( L^2 \) bounds in (B.82) – (B.84),
\[
\| SR_n f(t) \|_{\dot{H}^{N_1+1}} \leq (C\varepsilon_1)^{n+1} (1 + t)^{4p_0-d_n},
\]
(B.85)

for any \( n \geq 0 \) and \( t \in [0, T] \), where \( d_0 = 1/2 \), \( d_1 = 1 \), and \( d_n = 5/4 + 4p_0 \) for \( n \geq 2 \).

\end{proof}
Starting from the formula (B.19), we have, as in (2.5),
\[ F[SR_\eta f](\xi) = \frac{c_n}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} M_{n+1}'(\xi; \eta) \hat{S}_x f(\xi - \eta_1 - \ldots - \eta_{n+1}) \hat{h}(\eta_1) \ldots \hat{h}(\eta_{n+1}) d\eta \]
\[ + \frac{(n+1)c_n}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} M_{n+1}'(\xi; \eta) \partial_x f(\xi - \eta_1 - \ldots - \eta_{n+1}) \hat{h}(\eta_1) \ldots \hat{S}_x h(\eta_{n+1}) d\eta \]
\[ + \frac{c_n}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \tilde{M}_{n+1}'(\xi; \eta) \partial_x f(\xi - \eta_1 - \ldots - \eta_{n+1}) \hat{h}(\eta_1) \ldots \hat{h}(\eta_{n+1}) d\eta, \tag{B.86} \]
where \( c_n := -1/(\pi(n+1)) \),
\[ M_{n+1}'(\xi; \eta) = \int_{\mathbb{R}} e^{-i\xi_0} \prod_{i=1}^{n+1} e^{in_\rho} - \frac{1}{\rho} d\rho \tag{B.87} \]
as in (B.18), and
\[ \tilde{M}_{n+1}'(\xi; \eta) := -\left( \xi \partial_x + \sum_{j=1}^{n+1} \eta_j \partial_{\eta_j} \right) (M_{n+1}')(\xi; \eta). \]
We notice that \( M_{n+1}'(\lambda \xi; \lambda \eta) = \lambda^n M_{n+1}'(\xi; \eta) \), if \( \lambda > 0 \). Taking the derivative in \( \lambda \) it follows that
\[ \tilde{M}_{n+1}'(\xi; \eta) = -n M_{n+1}'(\xi; \eta). \tag{B.88} \]

Given the formulas (B.15)–(B.17), the estimates (B.85) can be proved as in Lemma B.2, using dyadic decompositions and the bounds (B.79). As a general rule, we always estimate the factor that has the vector-field \( S \) in \( L^2 \) and the remaining factors in \( L^\infty \). \( \square \)

We recall the definition of the operators \( T_1 \) and \( T_2 \) in (B.48) and the decomposition \( \mathcal{H}_\gamma = H_0 - T_1 + iT_2 \). As in Lemma B.4, see (B.51)–(B.53), as a consequence of Lemma B.8 we have
\[ \left\| (T_1 - R_0) f(t) \right\|_{H^{N_0+1}} + \left\| T_2 f(t) \right\|_{H^{N_0+1}} \lesssim \varepsilon_1^2 (1 + t)^{p_0 - 1} \| f \|_{E_{w,T}}, \]
\[ \left\| (T_1 - R_0) f(t) \right\|_{W^{N_2+1,-1/10}} + \left\| T_2 f(t) \right\|_{W^{N_2+1,-1/10}} \lesssim \varepsilon_1^2 (1 + t)^{-11/10} \| f \|_{E_{w,T}}, \tag{B.89} \]
\[ \left\| S(T_1 - R_0) f(t) \right\|_{H^{N_0+1}} + \left\| ST_2 f(t) \right\|_{H^{N_0+1}} \lesssim \varepsilon_1^2 (1 + t)^{4p_0 - 1} \| f \|_{E_{w,T}}, \]
and
\[ \left\| \mathcal{H}_\gamma f(t) \right\|_{E^{-1}} \lesssim \| f \|_{E^{-1}}, \quad \left\| R_0 f(t) \right\|_{E^{-1}} \lesssim \varepsilon_1 (1 + t)^{-1/2} \| f \|_{E^{-1}} T, \tag{B.90} \]
for any \( t \in [0, T] \). In particular
\[ \sup_{t \in [0, T]} (1 + t)^{1/2} \left[ \left\| T_1 f(t) \right\|_{E_{w,T}}^2 + \left\| T_2 f(t) \right\|_{E_{w,T}}^2 \right] \lesssim \varepsilon_1 \| f \|_{E^{-1}_{w,T}}. \tag{B.91} \]

Given \( \phi \in E^{1/2}_{w,T} \) we can define \( \psi \in E^{1/2}_{w,T} \) such that
\[ (I - T_1) \psi = (-iH_0 + T_2) \phi, \quad \left\| \psi \right\|_{E^{1/2}_{w,T}} \lesssim \| \phi \|_{E^{1/2}_{w,T}}, \tag{B.92} \]
see (B.58). As before, see Lemma B.6 let
\[ V = \frac{\phi_x + h_x \psi_x}{1 + h_x^2}, \quad B = \frac{h_x \phi_x - \psi_x}{1 + h_x^2}, \tag{B.93} \]
such that
\[ \phi_x = V + h_x B, \quad -\psi_x = B - h_x V, \quad (I - \mathcal{H}_\gamma)(V - iB) = 0. \tag{B.94} \]
Then, as in (B.65), for any \( f \in \{ \phi_x, \psi_x, V, B \} \) and \( t \in [0, T] \)
\[
(1 + t)^{-p_0} \| f \|_{H^{N_0 - 1/2}} + (1 + t)^{1/2} \| f \|_{\overline{W}_2^{-1/2}} + (1 + t)^{-4p_0} \| Sf \|_{H^{N_1 - 1/2}} \lesssim \| \phi \|_{\mathcal{C}^{1/2}_{w, T}}. \tag{B.95}
\]

We have the following analogue of Proposition B.7

**Proposition B.9.** Assume \( \phi \in \mathcal{E}^{1/2}_{w, T} \), and define \( \psi, V, B \) as before. Let \( G(h) \phi := -\psi_x \), see (B.66), and decompose
\[
G(h) \phi = |\partial_x| \phi - |\partial_x| T_B h - \partial_x T_V h + G_2 + G_{\geq 3},
\]

see (B.67)–(B.68). Then, for any \( t \in [0, T] \),
\[
(1 + t)^{1-p_0} \| G_{\geq 3}(t) \|_{H^{N_0 + 1}} + (1 + t)^{11/10} \| G_{\geq 3} \|_{\overline{W}_2^{-1/2}} + (1 + t)^{-4p_0} \| SG_{\geq 3}(t) \|_{H^{N_1 + 1}} \lesssim \varepsilon_1^2 \| \phi \|_{\mathcal{C}^{1/2}_{w, T}}. \tag{B.96}
\]

**Proof.** We may assume \( \| \phi \|_{\mathcal{C}^{1/2}_{w, T}} = 1 \) and use the formulas derived in the proof of Proposition B.7. In particular, see (B.72)–(B.73),
\[
-G_{\geq 3} = I + II,
\]
where
\[
I = h_x V - H_0(h_x H_0 V) - |\partial_x| T_{1H_0 V} h - \partial_x T_V h + G_2 + R_0(iH_0 V),
\]
\[
II = D - R_0(iH_0 V) - iH_0(h_x D) + |\partial_x| T_{Dh},
\]

where
\[
D = T_2 V + T_1 B = iH_0 V - B.
\]

The proof of the bounds (B.74)–(B.78) can be extended to the weighted setting, using the bounds (B.89)–(B.95) above. More precisely, as in Proposition B.7, one can verify that
\[
(1 + t)^{-4p_0} \| SF(t) \|_{H^{N_1 + 1}} \lesssim \varepsilon_1^2
\]
where
\[
F \in \{ I, D - R_0 B, R_0 B - R_0(iH_0 V), -iH_0(h_x D) + |\partial_x| T_{Dh} \}.
\]

The desired conclusion (B.96) follows. \( \square \)

### Appendix C. Elliptic bounds

In this appendix we prove elliptic-type bounds on several multilinear expressions that appear in the course of the proofs, mainly in the derivation of the equations done in section 3 and in the energy estimates in sections 4-7. To organize these bounds efficiently, we start with a definition:

**Definition C.1.** Assume \( \alpha \in [-2, 2] \) and let \( b := -1/10 \). Let \( O_{1, \alpha} \) denote generic functions \( f_1 \) on \([0, T]\) that satisfy the “linear” bounds
\[
(1 + t)^{-p_0} \| f_1(t) \|_{H^{N_0 + 1 + \alpha + b}} + (1 + t)^{-4p_0} \| Sf_1(t) \|_{H^{N_1 + 1 + \alpha + b}} + (1 + t)^{1/2} \| f_1(t) \|_{\overline{W}^{2 + \alpha}_2} \lesssim \varepsilon_1
\]
for any \( t \in [0, T] \). Let \( O_{2, \alpha} \) denote generic functions \( f_2 \) on \([0, T]\) that satisfy the “quadratic” bounds
\[
(1 + t)^{1/2-p_0} \| f_2(t) \|_{H^{N_0 + \alpha}} + (1 + t)^{1/2-4p_0} \| Sf_2(t) \|_{H^{N_1 + \alpha}} + (1 + t) \| f_2(t) \|_{\overline{W}^{2 + \alpha}_2} \lesssim \varepsilon_1^2
\]
for any \( t \in [0, T] \). Let \( O_{3, \alpha} \) denote generic functions \( f_3 \) on \([0, T]\) that satisfy the “cubic” bounds
\[
(1 + t)^{-p_0} \| f_3(t) \|_{H^{N_0 + \alpha}} + (1 + t)^{-4p_0} \| Sf_3(t) \|_{H^{N_1 + \alpha}} + (1 + t)^{11/10} \| f_3(t) \|_{\overline{W}^{2 + \alpha}_2} \lesssim \varepsilon_1^3
\]
for any \( t \in [0, T] \). Let \( O_{4, \alpha} \) denote generic functions \( f_4 \) on \([0, T]\) that satisfy the “quartic” bounds
\[
(1 + t)^{5/4} \| f_4(t) \|_{H^{N_0 + \alpha}} + (1 + t)^{5/4} \| Sf_4(t) \|_{H^{N_1 + \alpha}} \lesssim \varepsilon_1^4
\]
for any $t \in [0, T]$.

In other words, the generic notation $O_{m, \alpha}$ measures (1) the degree of the function (linear, quadratic, cubic, and quartic) represented by the exponent $m$, and (2) the number of derivatives under control, relative to the Hamiltonian variables $|\partial_x h, |\partial_x|^3/2 \phi$ which correspond to $\alpha = 0$. Notice that $\{f : f = O_{4, \alpha}\} \subseteq \{f : f = O_{3, \alpha}\} \subseteq \{f : f = O_{2, \alpha}\};$ in the linear case $O_{1, \alpha}$ we make slightly stronger assumptions on the low frequency part of the $L^2$ norms.

We will often use the following lemma to estimate products and paraproducts of functions.

**Lemma C.2.** Assume that $m_1, m_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ are continuous functions with

$$
\text{supp} m_1 \subseteq \{(\xi, \eta) \in \mathbb{R}^2 : |\xi - \eta| \leq |\eta|/2^4\},
$$

$$
\text{supp} m_2 \subseteq \{(\xi, \eta) \in \mathbb{R}^2 : |\xi - \eta|/|\eta| \in [2^{-20}, 2^{20}]\}.
$$

Assume that

$$
\left\| \mathcal{F}^{-1} [m_1(\xi, \eta) \varphi_k(\xi)] \right\|_{L^1} \lesssim 1, \quad \left\| \mathcal{F}^{-1} [m_2(\xi, \eta) \varphi_k(\xi - \eta) \varphi_k(\eta)] \right\|_{L^1} \lesssim 1,
$$

for any $k, k_1, k_2 \in \mathbb{Z}$. Let $\tilde{m}_i := -(\xi \partial_x + \eta \partial_y)m_i$, see (2.6), and assume that the multipliers $\tilde{m}_1, \tilde{m}_2$ satisfy the bounds (C.6) as well. Let $M_1$ and $M_2$ denote the bilinear operators associated to $m_1$ and $m_2$ respectively, see (2.1). Then, for any $\alpha \in [-2, 2]$,

$$
\begin{align*}
\text{if } f_1 &= O_{1, -2}, g_1 = O_{1, \alpha} \quad \text{then} \quad M_1(f_1, g_1) = O_{2, \alpha}, \quad M_1(f_1, g_1) = O_{1, \alpha}; \\
\text{if } f_1 &= O_{1, -2}, f_2 = O_{2, \alpha} \quad \text{then} \quad M_1(f_1, f_2) = O_{3, \alpha}, \quad M_1(f_1, f_2) = O_{1, \alpha}; \\
\text{if } f_2 &= O_{2, -2}, f_1 = O_{1, \alpha} \quad \text{then} \quad M_1(f_1, f_2) = O_{3, \alpha}, \quad M_1(f_1, f_2) = O_{1, \alpha}; \\
\text{if } f_1 &= O_{1, -2}, f_3 = O_{3, \alpha} \quad \text{then} \quad M_1(f_1, f_3) = O_{4, \alpha}, \quad M_1(f_1, f_3) = O_{1, \alpha}; \\
\text{if } f_3 &= O_{3, -2}, f_1 = O_{1, \alpha} \quad \text{then} \quad M_1(f_3, f_1) = O_{4, \alpha}, \quad M_1(f_3, f_1) = O_{1, \alpha}; \\
\text{if } f_2 &= O_{2, -2}, g_2 = O_{2, \alpha} \quad \text{then} \quad M_1(f_2, g_2) = O_{4, \alpha}, \quad M_1(f_2, g_2) = O_{1, \alpha}.
\end{align*}
$$

Moreover, for any $\alpha \in [-2, 0]$,

$$
\begin{align*}
\text{if } f_1 &= O_{1, \alpha}, g_1 = O_{1, \alpha} \quad \text{then} \quad M_2(f_1, g_1) = O_{2, \alpha+2}, \quad M_2(f_1, g_1) = O_{1, \alpha+2}; \\
\text{if } f_1 &= O_{1, \alpha}, f_2 = O_{2, \alpha} \quad \text{then} \quad M_2(f_1, f_2) = O_{3, \alpha+2}, \quad M_2(f_1, f_2) = O_{1, \alpha+2}; \\
\text{if } f_1 &= O_{1, \alpha}, f_3 = O_{3, \alpha} \quad \text{then} \quad M_2(f_1, f_3) = O_{4, \alpha+2}, \quad M_2(f_1, f_3) = O_{1, \alpha+2}; \\
\text{if } f_2 &= O_{2, \alpha}, g_2 = O_{2, \alpha} \quad \text{then} \quad M_2(f_2, g_2) = O_{4, \alpha+2}, \quad M_2(f_2, g_2) = O_{1, \alpha+2}.
\end{align*}
$$

**Proof.** We only show in detail how to prove (C.7) and (C.13), since the other bounds can be proved in a similar manner. We assume that $t \in [0, T]$ is fixed and sometimes drop it from the notation. Using Lemma 2.1, the definition (C.1), and the assumption (C.6),

$$
\left\| P_k M_1(f_1, g_1) \right\|_{L^p} \lesssim \| P_k g_1 \|_{L^p} \| P_{\leq k-8} f_1 \|_{L^\infty} \lesssim \varepsilon_1 (1+t)^{-1/2} \| P_k' g_1 \|_{L^p}
$$

for $p \in \{2, \infty\}$ and for any $k \in \mathbb{Z}$. Therefore

$$
\left\| M_1(f_1, g_1) \right\|_{\dot{H}^{\varepsilon_1/2}} \lesssim \varepsilon_1^2 (1+t)^{-1/2+p_0},
$$

$$
\left\| M_1(f_1, g_1) \right\|_{\dot{H}^{\varepsilon_1}} \lesssim \varepsilon_1^2 (1+t)^{-1}.
$$

Similarly, using Lemma 2.1, the definition (C.1), and the assumption (C.6),

$$
\left\| P_k M_1(S f_1, S g_1) \right\|_{L^2} \lesssim \| P_k' S g_1 \|_{L^2} \| P_{\leq k} S f_1 \|_{L^\infty} \lesssim \varepsilon_1 (1+t)^{-1/2} \| P_k' S g_1 \|_{L^2},
$$

and

$$
\left\| P_k M_1(S f_1, g_1) \right\|_{L^2} \lesssim \| P_k' g_1 \|_{L^\infty} \| P_{\leq k} S f_1 \|_{L^2} \lesssim \varepsilon_1 (1+t)^{-1/2} \min(2^{-2k}, 2^{-k(N_2+\alpha)}).
$$
Therefore
\[ \| M_1(f_1, S g_1) \|_{H^{N_1+\alpha,b}} + \| M_1(S f_1, g_1) \|_{H^{N_1+\alpha,b}} \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0}. \] (C.18)
Moreover, since \( \tilde{m}_1 \) satisfies the same bounds as \( m_1 \), we have as in the proof of (C.17),
\[ \| \tilde{M}_1(f_1, g_1) \|_{H^{N_1+\alpha,b}} \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0}. \] (C.19)
The desired identities (C.7) follow from the bounds (C.17)–(C.19) and the identity (2.5).

We consider now the operator \( M_2 \) and prove (C.13). We estimate first, using as before Lemma 2.4 the definition (C.1), and the assumption (C.6),
\[ \| M_2(f_1, g_1) \|_{L^2} \lesssim \sum_{|k_1 - k_2| \leq 30} \| P_{k_1} f_1 \|_{L^2} \| P_{k_2} g_1 \|_{L^\infty} \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + p_0} \] (C.20)
and, similarly,
\[ \| M_2(f_1, g_1) \|_{L^\infty} \lesssim \varepsilon_1^2 (1 + t)^{-1}, \] (C.21)
\[ \| M_2(f_1, g_1) \|_{L^1} \lesssim \varepsilon_1^2 (1 + t)^{2p_0}, \] (C.22)
\[ \| M_2(S f_1, g_1) \|_{L^2} + \| M_2(f_1, S g_1) \|_{L^2} + \| \tilde{M}_2(f_1, g_1) \|_{L^2} \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0}, \] (C.23)
\[ \| M_2(S f_1, g_1) \|_{L^1} + \| M_2(f_1, S g_1) \|_{L^1} + \| \tilde{M}_2(f_1, g_1) \|_{L^1} \lesssim \varepsilon_1^2 (1 + t)^{8p_0}. \] (C.24)
These estimates and Sobolev embedding (in the form \( \| P_k h \|_{L^2} \lesssim 2^{k/2} \| P_k h \|_{L^1} \)) provide the desired estimates on low frequencies,
\[
(1 + t)^{1/2 - p_0} \| P_{\leq 4} M_2(f_1, g_1) \|_{H^{N_0 + \alpha}} + (1 + t)^{1/2 - 4p_0} \| P_{\leq 4} S M_2(f_1, g_1) \|_{H^{N_1 + \alpha}} + (1 + t) \| P_{\leq 4} M_2(f_1, g_1) \|_{\tilde{W}^{N_2 + \alpha}} \lesssim \varepsilon_1^2
\]

and
\[
(1 + t)^{-p_0} \| P_{\leq 4} M_2(f_1, g_1) \|_{H^{N_0 + \alpha,b}} + (1 + t)^{-4p_0} \| P_{\leq 4} S M_2(f_1, g_1) \|_{H^{N_1 + \alpha,b}} + (1 + t)^{1/2} \| P_{\leq 4} M_2(f_1, g_1) \|_{\tilde{W}^{N_2 + \alpha,b}} \lesssim \varepsilon_1.
\]
To estimate the high frequencies we notice that, for \( k \geq 0 \) and \( p \in \{2, \infty\} \)
\[ \| P_k M_2(f_1, g_1) \|_{L^p} \lesssim \sum_{k_1 \geq k - 30} \| P_{k_1} f_1 \|_{L^p} \cdot \varepsilon_1 (1 + t)^{-1/2 - k_1(N_2 + \alpha)}. \]
Therefore
\[ \| P_{\geq 0} M_2(f_1, g_1) \|_{H^{N_0 + \alpha + 2}} \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + p_0}, \]
\[ \sum_{k \geq 0} 2^{(N_0 + \alpha + 2)k} \| P_k M_2(f_1, g_1) \|_{L^\infty} \lesssim \varepsilon_1^2 (1 + t)^{-1}. \] (C.25)
Similarly
\[ \| P_{\geq 0} M_2(S f_1, g_1) \|_{H^{N_1 + \alpha + 2}} + \| P_{\geq 0} M_2(f_1, S g_1) \|_{H^{N_1 + \alpha + 2}} + \| \tilde{M}_2(f_1, g_1) \|_{H^{N_1 + \alpha + 2}} \lesssim \varepsilon_1^2 (1 + t)^{-1/2 + 4p_0}. \] (C.26)
The desired identities in (C.13) follow. \( \square \)
C.1. The cubic remainders \( G_{\geq 3}, \Omega_{\geq 3}, U_{\geq 3}, \mathcal{O}_W, \mathcal{O}_Z \). In this subsection we prove the elliptic bounds on the quadratic and the cubic terms used implicitly to justify the calculations in section 3. Recall the formulas derived in section 3:

\[
\sigma = (1 + h_x^2)^{-3/2} - 1, \quad \gamma = (1 + h_x^2)^{-3/4} - 1, \quad p_1 = \gamma, \quad p_0 = -(3/4)\gamma_x,
\]

\[
B = \frac{G(h)\phi + h_x\phi_x}{(1 + h_x^2)}, \quad V = \phi_x - Bh_x, \quad \omega = \phi - T_B P_{\geq 1} h.
\]

(C.27)

Recall the bounds (2.25). With the notation in Definition C.1 and using also (B.92), and Lemma C.2, we have

\[
h_x = O_{1,0}, \quad \phi_x = O_{1,-1/2}, \quad G(h)\phi = O_{1,-1/2}, \quad B = O_{1,-1/2}, \quad V = O_{1,-1/2}.
\]

(C.28)

Using the identities (C.28) and Lemma C.2 it follows that

\[
\sigma = O_{2,0}, \quad \gamma = O_{2,0}, \quad p_1 = O_{2,0}, \quad p_0 = O_{2,-1}, \quad \omega_x - \phi_x = O_{2,0}, \quad |\partial_x|\omega - |\partial_x|\phi = O_{2,0}, \quad V - \phi_x = O_{2,-1/2}, \quad B - G(h)\phi = O_{2,-1/2}.
\]

(C.29)

Using Proposition B.9 it follows that

\[
G(h)\phi - |\partial_x|\phi = O_{2,0}, \quad \mathcal{N}_2 = O_{2,0}, \quad G_{\geq 3} = O_{3,1}.
\]

(C.30)

The main evolution system (2.24) shows that

\[
\partial_t h = O_{1,-1/2}, \quad \partial_t \phi = O_{1,-1}.
\]

The formula (B.58) \((I - T_1)\psi = (-iH_0 + T_2)\phi\) now shows that

\[
\partial_t G(h)\phi - iH_0 \partial_t \phi_x = O_{2,-2}.
\]

Using again the main system (2.24) and previous identities,

\[
\partial_t h - |\partial_x|\phi = O_{2,-1}, \quad \partial_t \phi - h_{xx} = O_{2,-1}, \quad \partial_t (G(h)\phi) + |\partial_x|^3 h = O_{2,-2},
\]

\[
\partial_t B + |\partial_x|^3 h = O_{2,-2}, \quad \partial_t V - |\partial_x|^3 h = O_{2,-2}.
\]

(C.31)

Finally, the formula (3.17) shows that

\[
u = O_{1,0}.
\]

(C.32)

The following proposition is the main result in this subsection.

**Proposition C.3.** Let \((G_{\geq 3}, \Omega_{\geq 3}), U_{\geq 3}, \mathcal{O}_W,\) and \(\mathcal{O}_Z\) denote the cubic remainders in Proposition 3.1, Proposition 3.4, Proposition 3.5, and Lemma 3.6 respectively. Then, with \(N = 3k/2,\)

\[
G_{\geq 3} = O_{3,1}, \quad \Omega_{\geq 3} = O_{3,1/2}, \quad U_{\geq 3} = |\partial_x|^{1/2}O_{3,1/2},
\]

(C.33)

\[
(1 + t)^{1-p_0}\|\mathcal{O}_W(t)\|_{H^{N_0-N}} + (1 + t)^{1-4p_0}\|S\mathcal{O}_W(t)\|_{H^{N-1-N}} \lesssim \varepsilon_1^3, \quad \text{if } N \leq N_1,
\]

\[
(1 + t)^{1-p_0}\|\mathcal{O}_W(t)\|_{H^{N_0-N}} \lesssim \varepsilon_1^3, \quad \text{if } N \in [N_1, N_0],
\]

(C.34)

and

\[
(1 + t)^{1-4p_0}\|\mathcal{O}_Z(t)\|_{L^2} \lesssim \varepsilon_1^3.
\]

(C.35)
Proof. The desired conclusion $G \geq 3 = O_{3,1}$ was already proved in Proposition 3.9. We examine now the term $\Omega \geq 3$ in Proposition 3.1. Inspecting the calculations that lead from (3.10) to (3.11), we see that

$$\Omega \geq 3 = \sum_{j=1}^{5} \Omega_{j} \geq 3,$$

$$\Omega_{1} := \Sigma(h) - (h_{xx} + \partial_{x} T_{0} \partial_{x} h),$$

$$\Omega_{2} := \left[R(B, B) - R(\partial_{x} h, \partial_{x} h)\right] / 2 - \left[R(V, V) - R(\partial_{x} h, \partial_{x} h)\right] / 2 - R(V, B_{0} h_{x}),$$

$$\Omega_{3} := -T_{B x} P_{0}^{2} (h - T_{B x} P_{1}^{2} h),$$

$$\Omega_{4} := T_{B} P_{0}^{2} (h - T_{B x} P_{1}^{2} h),$$

$$\Omega_{5} := T_{B} (V h_{x}) - T_{B h_{x}} V - T_{V} \partial_{x} (T_{B} P_{1}^{2} h).$$

We examine the terms and use (C.28)–(C.31) and Lemma C.2 to see

$$\Omega_{1}^3 = O_{3,1}, \quad \Omega_{2}^3 = O_{3,1}, \quad \Omega_{3}^3 = O_{3,1}, \quad \Omega_{4}^3 = O_{3,1}.$$

Moreover, an argument similar to the proof of (C.7) in Lemma C.2 shows that

$$T_{f_{1}} f_{2} g - T_{f_{1}} T_{f_{2}} g = O_{3,1} \quad \text{if} \quad f_{1} = O_{1,-1}, \ f_{2} = O_{1,-1}, \ g = O_{1,-1}.$$

Therefore, we can rewrite

$$\Omega_{5} = [T_{B} T_{h_{x}} V - T_{B h_{x}} V] + T_{B} P_{0}^{2} (h - T_{B x} P_{1}^{2} h) - T_{V} T_{B} P_{1}^{2} h,$$

and conclude that $\Omega_{5} = O_{3,1}$. Therefore $\Omega_{5} = O_{3,1}$ as desired.

We now move on to examine the cubic terms appearing in (3.12) in Proposition 3.1. For this we inspect the computations in the proof of Proposition 3.2 to retrieve all cubic terms that have been incorporated in $U_{\geq 3}$. We find

$$U_{\geq 3} := \sum_{j=1}^{4} U_{\geq 3}^{j} + |\partial_{x}|^{1/2} O_{4,1/2},$$

where

$$U_{\geq 3}^{2} := |\partial_{x}| G_{\geq 3} - i |\partial_{x}|^{1/2} \Omega_{\geq 3} + T_{0} P_{1}^{2} |\partial_{x}| h + T_{0} P_{1}^{2} |\partial_{x}|^{-1} \partial_{x} h$$

$$+ T_{p_{1}} P_{1}^{2} |\partial_{x}| (h - |\partial_{x}| h + T_{p_{1}} P_{1}^{2} |\partial_{x}|^{-1} \partial_{x} h - |\partial_{x}| h - T_{p_{1}} P_{1}^{2} |\partial_{x}|^{-1} \partial_{x} h + T_{p_{1}} P_{1}^{2} |\partial_{x}|^{-1} \partial_{x} h),$$

$$U_{\geq 3}^{3} := - \left[T_{p_{1}} P_{1}^{2} |\partial_{x}|, \partial_{x} T_{V} h\right] - \left[T_{p_{1}} P_{1}^{2} |\partial_{x}|^{-1} \partial_{x} h, \partial_{x} T_{V} h\right] + \left[|\partial_{x}|, \partial_{x} T_{V} h\right] + i \left[|\partial_{x}|^{1/2}, \partial_{x} T_{V} h\right]$$

$$+ i \left[|\partial_{x}|^{1/2}, \partial_{x} T_{V} h\right] + \left[|\partial_{x}|^{1/2}, \partial_{x} T_{V} h\right],$$

$$U_{\geq 3}^{4} := i |\partial_{x}|^{3/2} T_{\sigma} P_{\leq 0} |\partial_{x}| h + i \left(|\partial_{x}|^{3/2} T_{\sigma} P_{1}^{2} |\partial_{x}| h - T_{\sigma} P_{1}^{2} |\partial_{x}|^{5/2} h + \frac{3}{2} T_{\sigma} P_{1}^{2} |\partial_{x}|^{1/2} \partial_{x} h\right)$$

$$- i \left(|\partial_{x}|^{3/2} T_{\sigma} P_{1}^{2} |\partial_{x}|^{5/2} h + \frac{3}{2} T_{\sigma} P_{1}^{2} |\partial_{x}|^{1/2} \partial_{x} h\right)$$

$$- i \left(|\partial_{x}|^{3/2} T_{p_{0}} P_{1}^{2} |\partial_{x}|^{-1} \partial_{x} h - T_{p_{0}} P_{1}^{2} |\partial_{x}|^{1/2} \partial_{x} h\right).$$
\[ U_{\geq 3}^4 := -\frac{3i}{4} T_{\partial_x} P_{\geq 1} |\partial_x|^{-1/2} \partial_x T_{p_0} |\partial_x|^{-1} \partial_x h \]
\[ + i \left( T_{\gamma} P_{\geq 1} |\partial_x|^{3/2} T_{p_1} P_{\geq 1} |\partial_x| h - T_{\gamma p_1} P_{\geq 1} |\partial_x|^{5/2} h + \frac{3}{2} T_{\gamma \partial_x p_1} P_{\geq 1} |\partial_x|^{1/2} \partial_x h \right) \]
\[ + i \left( T_{\gamma} P_{\geq 1} |\partial_x|^{3/2} T_{p_0} P_{\geq 1} |\partial_x|^{-1} \partial_x h - T_{\gamma p_0} P_{\geq 1} |\partial_x|^{1/2} \partial_x h \right) \]
\[ - \frac{3i}{4} \left( T_{\partial_x} P_{\geq 1} |\partial_x|^{-1/2} \partial_x T_{p_1} P_{\geq 1} |\partial_x| h - T_{\partial_x \gamma p_1} P_{\geq 1} |\partial_x|^{1/2} \partial_x h \right) \]  
\[ \text{(C.42)} \]

The first term comes from the remainders in the first chain of identities in the proof of Proposition 3.2. This is the same remainder that appears also at the end of (3.22). The second term above comes from (3.23). The term \( U_{\geq 3}^4 \) contains cubic (and higher order) terms that are discarded in (3.25). Quartic terms that have been discarded in (3.25), and the last term on the right-hand side of (3.52), are in \( U_{\geq 3}^4 \). The term \( |\partial_x|^{1/2} O_{4,1/2} \) comes from the formulas after (3.30).

We examine the formulas (C.38)–(C.42) and use Lemma C.2 and (C.37) to conclude that \( U_{\geq 3} = |\partial_x|^{1/2} O_{3,1/2} \), as desired. This completes the proof of (C.33).

We turn now to the proof of (C.34). We examine the formula (3.52) and observe that
\[ [\partial_t, \mathcal{D}^k] = \sum_{j=0}^{k-1} \mathcal{D}^j [\partial_t, \mathcal{D}] \mathcal{D}^{k-j-1} = \sum_{j=0}^{k-1} \mathcal{D}^j [\partial_t, \Sigma_\gamma] \mathcal{D}^{k-j-1}. \]  
\[ \text{(C.43)} \]

In view of (C.31) and Lemma C.2 \([\partial_t, \Sigma_\gamma]\) is an operator of order 3/2 that transforms linear terms into cubic terms. Moreover
\[ \mathcal{D}^k - |\partial_x|^N = \sum_{P_1, \ldots, P_k \in \{|\partial_x|^{3/2}, \Sigma_\gamma\}} P_1 \ldots P_k, \]
where the sum above is taken over all possible choices of operators \( P_1, \ldots, P_k \in \{|\partial_x|^{3/2}, \Sigma_\gamma\} \), not all of them equal to \( |\partial_x|^{3/2} \). Therefore \( \mathcal{D}^k - |\partial_x|^N \) is an operator of order \( 3k/2 \) that transforms linear terms into cubic terms. Finally, using Lemma A.1, Lemma A.2 and Lemma C.2 \( N_u \) is a quadratic term that does not lose derivatives,
\[ N_u = O_{2,0}. \]  
\[ \text{(C.44)} \]

The desired conclusion follows by applying elliptic estimates, as in the proof of Lemma C.2 to the terms in the last two lines of (3.52).

The proof of the elliptic bound (C.35) is similar, using the formula (3.61), Lemma C.2 and (C.43).

C.2. More on the cubic terms. Recall the notation for trilinear operators
\[ \mathcal{F}[M(f_1, f_2, f_3)](\xi) := \frac{1}{4\pi^2} \int_{\mathbb{R} \times \mathbb{R}} m(\xi, \eta, \sigma) \hat{f}_1(\xi - \eta) \hat{f}_2(\eta - \sigma) \hat{f}_3(\sigma) \, d\eta d\sigma. \]  
\[ \text{(C.45)} \]

The semilinear analysis in sections 8 and 9 leading to pointwise decay relies on the main equation for the function \( u \) in Proposition 3.4. To use this equation we need to expand the nonlinearity as a sum of quadratic, cubic, and higher order terms. At this point we are not interested in the potential loss of derivatives. The following is the main result in this subsection.

Lemma C.4. Let \( u \) be a solution of the main scalar equation (3.32), with the notation of Propositions 3.2 and 3.4. Then we can write
\[ \partial_t u - i|\partial_x|^{3/2} u = Q_u + C_u + R_{\geq 4}. \]  
\[ \text{(C.46)} \]

where the following holds true:
• The quadratic nonlinear terms are
  \[ Q_u := Q_0(u, u) - Q_0(\overline{u}, u) + N_u, \]  
  where the symbol of \( Q_0 \) is
  \[ q_0(\xi, \eta) = \frac{i\xi(\xi - \eta)}{2|\xi - \eta|^{1/2}}\chi(\xi - \eta, \eta), \]  
  and \( N_u \) is in \((3.44)\).

• The cubic terms have the form
  \[ C_u := M_{+++}(u, u, \overline{u}) + M_{++-}(u, u, u) + M_{+-+}(u, \overline{u}, u) + M_{-+-}(u, \overline{u}, \overline{u}), \]  
  with symbols \( m_{i_1i_2i_3} \) such that
  \[ \|F^{-1}[m_{i_1i_2i_3}(\xi, \eta, \sigma) \cdot \varphi_k(\xi)\varphi_k(\eta - \sigma)\varphi_k(\sigma)]\|_{L^1} \lesssim 2^{k/2}2^\max(k_1,k_2,k_3) \]  
  for all \((i_1i_2i_3) \in \{(++-), (+-+), (---), (++), (-+-)\} \). Moreover
  \[ m_{+++}(\xi, 0, -\xi) = id_1|\xi|^{3/2}, \]  
  for some constant \( d_1 \in \mathbb{R} \).

• \( R_{\geq 4} \) is a quartic remainder
  \[ R_{\geq 4} = O_{4,-2}. \]  

Moreover
  \[ C_u + R_{\geq 4} = |\partial_x|^{1/2}O_{3,-1}. \]

Proof. We start from the equation \((3.42)\)
  \[ \partial_t u - i|\partial_x|^{3/2}u = i\Sigma_{\gamma}(u) - \partial_x T_{\gamma}u + N_u + U_{\geq 3}. \]

Notice that
  \[ h = \frac{1}{2}|\partial_x|^{-1}(u + \overline{u}) + P_{\geq 4}O_{3,0}; \quad \omega = -\frac{1}{2i}|\partial_x|^{-1/2}(u - \overline{u}), \]  
  \[ V = \frac{i}{2}|\partial_x|^{-1/2}(u - \overline{u}) + V_2; \quad V_2 := \partial_x T_BP_{\geq 1}h - Bh_x. \]

Notice that
  \[ Q_0(u, u) - Q_0(\overline{u}, u) = -\partial_x T_{(i/2)|\partial_x|^{-1/2}(u - \overline{u})}u \]  
  which gives the additional quadratic term in \((3.47)\).

To extract the cubic terms in \((C.49)\) we use the definition \( \gamma = (1 + h_x^2)^{-3/4} - 1 \) and write
  \[ \gamma = -(3/4)h_x^2 + O_{3,0}. \]

Then
  \[ i\Sigma_{\gamma}(u) = -(3i/4)T_{h_x^2}P_{\geq 1}|\partial_x|^{3/2}u + (9i/16)T_{\partial_x h_x^2}P_{\geq 1}|\partial_x|^{-1/2}\partial_x u + O_{4,-3/2}. \]

The cubic terms in \( C_u \) come from several places: the terms \( U_{\geq 3}^1, U_{\geq 3}^2, U_{\geq 3}^3 \) in \((3.39)\)–\((3.41)\), the terms in the formula \( i\Sigma_{\gamma}(u) \) above, and the terms corresponding to the difference between \(-\partial_x T_{\gamma}u \) and \( Q_0(u - \overline{u}, u) \). By inspection it is easy to see that all these terms are homogeneous of degree \( 3/2 \) (if one makes the assumption that \( \chi \) is homogeneous of degree \( 0 \)), with the output frequency always appearing at least with a power of \( 1/2 \), and suitable symbols satisfying \((C.50)\). \( \square \)
C.3. Bounds on nonlinear terms. The next lemma gives bounds on the nonlinear terms in Propositions 3.4, 3.5 and 3.6.

Lemma C.5. With the notation in definition C.1 we have
\[
\begin{align*}
\partial_t u - i\Lambda u &= |\partial_x|^{1/2}O_{2,-1}, \\
N_u &= |\partial_x|^{3/2}O_{2,3/2}, \\
N_W &= |\partial_x|^{3/2}O_{2,3/2}.
\end{align*}
\] (C.56)
Moreover
\[
\|P_k(\partial_t - i\Lambda)Su\|_{L^2} \lesssim \varepsilon_1^{2}2^{k/2}(1+t)^{-1/2+4p_0}(2^{k/2} + (1+t)^{-1/2}).
\] (C.58)
Furthermore, with notation of Proposition 3.6 let
\[
N_Z := N_{Z,1} + N_{Z,2} + N_{Z,3}.
\] (C.59)
Then, we have
\[
\begin{align*}
\|P_k(\partial_t - i\Lambda)Z\|_{L^2} &\lesssim \varepsilon_1^{2}2^{k/2}2^{\max(k,0)}(1+t)^{-1/2+4p_0}, \\
\|P_kN_Z\|_{L^2} &\lesssim \varepsilon_1^{2}2^{\min(k,0)}(1+t)^{-1/2+4p_0}.
\end{align*}
\] (C.60)

Proof. The bounds (C.57) follow from the definitions of $N_u$ in (3.44) and $N_W$ in (3.50), the bounds on the symbols in Lemmas A.1 and A.2 and Lemma 2.1(ii). Bound (C.56) follows using also (C.53). The bound (C.58) is a more refined version of (C.56), that is obtained directly from Lemma (C.4).

The estimate (C.61) follows by examining the terms in (3.56) and the same symbols bounds used before. (C.60) follows from the equations (3.54) (for $k \geq 0$) and (7.1) (for $k \leq 0$). □

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