Entanglement is the feature of quantum mechanics that renders it distinctly different from a classical theory [1–5]. It is at the heart of quantum information science and technology as a resource that is used to accomplish task and is increasingly also seen as an important concept in condensed-matter physics [6]. Given its significance in protocols of quantum information, it hardly surprises that already early in the development of the field, questions were asked how one form of entanglement could be transformed into another. It was one of the early main results of the field of quantum information theory to show that all pure bipartite states could be asymptotically reversibly transformed to maximally entangled states with local operations and classical communication (LOCC) at a rate that is determined by a single number [3]: the entanglement entropy, the von-Neumann entropy of each reduced state. This insight makes the resource character of bipartite entanglement most manifest: The entanglement content is given simply by its content of maximally entangled states, and each form can be transformed reversibly into another and back.

The situation in the multi-partite setting is significantly more intricate, however [7–10]. The rates that can be achieved when aiming at asymptotically transforming one multi-partite state into another with LOCC are far from clear. It is not even understood what the “ingredients” of multi-partite entanglement theory are [8, 11], so the basic units of multi-partite entanglement from which any other pure state can be asymptotically reversibly prepared. This state of affairs is unfortunate, and even more so since multi-partite states come again more into the focus of attention in the light of the observation that elements of the vision of a quantum network – or the “quantum internet” [12, 13] – may become an experimental reality in the not too far future. It is not that multi-partite entanglement ceases to have a resource character: For example, Greenberger-Horne-Zeilinger (GHZ) states are known to constitute a resource for quantum secret sharing [14, 15], the probably best known multi-partite cryptographic primitive. Progress on stochastic conversion for several copies of multi-partite states was made recently [16]. However, given a collection of arbitrary pure states, it is not known at what rate such states could be asymptotically distilled under LOCC.

In this work, we report surprisingly substantial progress on the old question of the rate at which GHZ and other multi-partite states can be asymptotically distilled from arbitrary pure states. Surprisingly, in that much of the technical substance can be delegated to the powerful machinery of entanglement combing [17], putting it here into a fresh context, which in turn can be seen to derive from quantum state merging [18], assisted entanglement distillation [19, 20], and time-sharing, meaning, using resource states in different roles in the asymptotic protocol. The basic insight underlying the analysis is that entanglement combing provides a reference, a helpful normal form rooted in the better understood theory of bipar-
tite entanglement, that can be used in order to assess rates of asymptotic multi-partite state conversion. Basically, putting entanglement combing to good work, therefore, we are in the position to make significant progress on the question of entanglement transformation rates in a general setting.

**Multi-partite state conversion.** We consider the problem of converting an $N$-partite state $\rho$ into $\sigma$ via $N$-partite LOCC. In particular, we are interested in the optimally achievable asymptotic rate for this procedure, which can be formally defined as

$$R(\rho \rightarrow \sigma) = \sup \left\{ r : \lim_{n \to \infty} \left( \inf_n \Lambda \left( \rho^{\otimes n} - \sigma^{\otimes n} \right) \right) = 0 \right\}.$$  
(1)

Here, $\Lambda$ reflects an $N$-partite LOCC operation and $||M||_1 = \text{Tr} |M|^2$ denotes the trace norm. This problem has a known solution in the bipartite case $N = 2$ for conversion between arbitrary pure states $\psi^{AB} \rightarrow \phi^{AB}$, rooted in Shannon theory. The corresponding rate in this case can be written as

$$R(\psi^{AB} \rightarrow \phi^{AB}) = \frac{S(\psi)}{S(\phi)}.$$  
(2)

where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. Moreover, $\psi^{AB}$ indicates a pure state shared between parties referred to as Alice and Bob, while $\phi^{AB}$ reflects the reduced state of Alice.

This simple picture ceases to hold in any setting beyond the bipartite one. Indeed, significantly less is known in the multi-partite setting for $N \geq 3$ [7]. While we will mainly focus on the tri-partite scenario here to be concise, most of our ideas can be generalized to any $N$, as we will hint at below. Needless to say, the bipartite solution (2) readily gives upper bounds on the rates in multi-partite settings. For example, for conversion between tri-partite pure states $\psi^{ABC} \rightarrow \phi^{ABC}$, it must be true that

$$R(\psi^{ABC} \rightarrow \phi^{ABC}) \leq \min \left\{ \frac{S(\psi^A)}{S(\phi^A)}, \frac{S(\psi^B)}{S(\phi^B)}, \frac{S(\psi^C)}{S(\phi^C)} \right\}.$$  
(3)

This follows from the fact that any tri-partite LOCC protocol is also bipartite with respect to any of the bipartitions. If the desired final state $\phi^{ABC}$ is the GHZ state with state vector $|\text{GHZ}\rangle = (|0, 0, 0\rangle + |1, 1, 1\rangle)/\sqrt{2}$, the bound in Eq. (3) is known to be achievable whenever one of the reduced states $\psi^{AB}, \psi^{BC}$ or $\psi^{AC}$ is separable [20].

We also note that for some states the bound in Eq. (3) is a strict inequality. This can be seen by considering the scenario where each of the parties, holds two qubits respectively. Consider now the transformation

$$|\text{GHZ}\rangle^{A_1 B_1 C_1} \otimes |\text{GHZ}\rangle^{A_2 B_2 C_2} \rightarrow |\Phi^+\rangle^{A_1 B_1} \otimes |\Phi^+\rangle^{A_2 C_1} \otimes |\Phi^+\rangle^{B_2 C_2},$$  
(4)

e.g., the parties aim to transform two GHZ states into three Bell state vectors $|\Phi^+\rangle = (|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$, which are equally distributed among all the parties. It is straightforward to check that in this case the bound in Eq. (3) becomes $R \leq 1$. However, the bound is not achievable, as the aforementioned transformation cannot be performed with unit rate [21].

**Lower bound on conversion rates for three parties.** The above discussion suggests that the bound in Eq. (3) is a very rough estimate for general transformations and is saturated only for very specific sets of states, having zero volume in the set of all pure states. Quite surprisingly, we will see below that this is not the case: there exist large families of tri-partite pure states which saturate the bound (3). This will follow from a very general and surprisingly simple lower bound on conversion rate, which will be presented below in Theorem 2.

The methods developed here build upon the machinery of entanglement combing, which was introduced and studied for general $N$-partite scenarios in Ref. [17]. In the specific tri-partite scenario, entanglement combing aims to transform the initial state $\psi^{ABC}$ into a state of the form $\mu^{A_1 B} \otimes \nu^{A_2 C}$ with pure bipartite states $\mu$ and $\nu$. The following Lemma restates the results from Ref. [17] in a form which will be suitable for the purpose of this work.

**Lemma 1 (Conditions from tri-partite entanglement combing).** The transformation

$$\psi^{ABC} \rightarrow \mu^{A_1 B} \otimes \nu^{A_2 C}$$  
(5)

is possible via asymptotic LOCC if and only if

$$E(\mu^{A_1 B}) + E(\nu^{A_2 C}) \leq S(\psi^A),$$  
(6a)
$$E(\mu^{A_1 B}) \leq S(\psi^B),$$  
(6b)
$$E(\nu^{A_2 C}) \leq S(\psi^C).$$  
(6c)

We refer to Appendix A for the proof of the Lemma. Using this result, we are now in position to present a tight lower bound on the transformation rate between tri-partite pure states [22].

**Theorem 2 (Lower bound for state transformation).** For tri-partite pure states $\psi^{ABC}$ and $\phi^{ABC}$, the LOCC conversion rate is bounded from below as

$$R(\psi^{ABC} \rightarrow \phi^{ABC}) \geq \min \left\{ \frac{S(\psi^A)}{S(\phi^A) + S(\phi^B) + S(\phi^C)}, \frac{S(\psi^B)}{S(\phi^B) + S(\phi^C)}, \frac{S(\psi^C)}{S(\phi^C)} \right\}.$$  
(7)

**Proof.** We prove this bound by presenting an explicit protocol achieving the bound. In the first step, the parties apply entanglement combing $\psi^{ABC} \rightarrow \mu^{A_1 B} \otimes \nu^{A_2 C}$ in such a way that the following equalities are fulfilled for some $r \geq 0$.

$$E(\mu^{A_1 B}) = rS(\phi^A), \quad E(\nu^{A_2 C}) = rS(\phi^C).$$  
(8)

The significance of this specific choice will become clear in a moment. In the next step, Alice and Charlie apply LOCC for transforming the state $\nu^{A_2 C}$ into the desired final state $\phi^{A_2 A_3 C}$. Since this is a bipartite LOCC protocol, the rate for this process is given by $E(\nu^{A_3 C})/S(\phi^C)$. Note that due to Eqs. (8), this rate is equal to $r$.

In a next step, Alice applies what is called Schumacher compression [23] to her register $A_3$. The overall compression rate per copy of the initial state $\psi^{ABC}$ is given as

$$\bar{r} = r S(\phi^A) = r S(\phi^B),$$  
(9)
where in the last equality we used the fact that $S(\phi^A) = S(\phi^B)$. Due to Eqs. (8), this rate interestingly coincides with the entanglement of the state $\mu^{A:B}$,

$$r = E(\mu^{A:B}).$$

(10)

In a final step, Alice and Bob distill the states $\mu^{A:B}$ into maximally entangled bipartite singlets, and use them to teleport [24, 25] the (compressed) particle $A_3$ to Bob. Due to Eq. (10), Alice and Bob share exactly the right amount of entanglement for this procedure, i.e., the process is possible with rate one and no entanglement is left over. In summary, the overall protocol transforms the state $\psi^{ABC}$ into $\phi^{ABC}$ at rate $r$.

For completing the proof, we will now show that $r$ can be chosen such that

$$r = \min \left\{ \frac{S(\psi^{A})}{S(\phi^{A}) + S(\phi^{C})}, \frac{S(\psi^{B})}{S(\phi^{B}) + S(\phi^{C})}, \frac{S(\psi^{C})}{S(\phi^{C})} \right\}. \tag{11}$$

This can be seen directly by inserting Eqs. (8) in Eqs. (6). In particular, the rate $r$ can attain any value which is simultaneously compatible with inequalities

$$r \leq \frac{S(\psi^{A})}{S(\phi^{A}) + S(\phi^{C})}, \quad r \leq \frac{S(\psi^{B})}{S(\phi^{B}) + S(\phi^{C})}, \quad r \leq \frac{S(\psi^{C})}{S(\phi^{C})}. \tag{12}$$

This completes the proof of the theorem.

We stress some important aspects and implications of this theorem. Whenever the minimum in Eq. (7) is attained on the second or third entry, the lower bound coincides with the upper bound in Eq. (3). This means that in all these instances the conversion problem is completely solved, giving rise to the rate

$$R(\psi^{ABC} \rightarrow \phi^{ABC}) = \min \left\{ \frac{S(\psi^{A})}{S(\phi^{A}) + S(\phi^{C})}, \frac{S(\psi^{B})}{S(\phi^{B}) + S(\phi^{C})}, \frac{S(\psi^{C})}{S(\phi^{C})} \right\}. \tag{13}$$

Moreover, the bound in Eq. (7) can be immediately generalized by interchanging the roles of the parties, i.e.,

$$R(\psi^{ABC} \rightarrow \phi^{ABC}) \geq \min \left\{ \frac{S(\psi^{A})}{S(\phi^{A}) + S(\phi^{C})}, \frac{S(\psi^{B})}{S(\phi^{B}) + S(\phi^{C})}, \frac{S(\psi^{C})}{S(\phi^{C})} \right\}. \tag{14}$$

$$R(\psi^{ABC} \rightarrow \phi^{ABC}) \geq \min \left\{ \frac{S(\psi^{C})}{S(\phi^{A}) + S(\phi^{B})}, \frac{S(\psi^{A})}{S(\phi^{A}) + S(\phi^{B})}, \frac{S(\psi^{B})}{S(\phi^{A}) + S(\phi^{B})} \right\}. \tag{15}$$

The best bound is obtained by taking the maximum of Eqs. (7), (14) and (15).

Our results also shed new light on reversibility questions for tri-partite state transformations. In general, a transformation $\psi \rightarrow \phi$ is said to be asymptotically reversible if the conversion rates fulfill the relation

$$R(\psi \rightarrow \phi) = R(\phi \rightarrow \psi)^{-1}. \tag{16}$$

Let now $\psi$ and $\phi$ be two states for which the bound in Theorem 2 is tight, e.g., $R(\psi \rightarrow \phi) = S(\phi^{A})/S(\psi^{A})$. Due to Eq. (3) it must be that $S(\psi^{A})/S(\psi^{B}) \leq S(\phi^{C})/S(\phi^{C})$ in this case. If this inequality is strict (which will be the generic case), we obtain for the inverse transformation $\phi \rightarrow \psi$

$$R(\phi \rightarrow \psi) \leq \frac{S(\phi^{C})}{S(\psi^{C})} \frac{S(\phi^{B})}{S(\psi^{B})} = R(\psi \rightarrow \phi)^{-1}, \tag{17}$$

where the first inequality follows from Eq. (3). These results show that those states which saturate the bound (3) do not allow for reversible transformations in the generic case. The question of reversibility for multipartite LOCC transformations has been previously studied for specific classes of pure states, in particular in the context of minimal reversible entanglement generating sets [11, 21, 26, 27].

We will now comment on the limits of the approach presented here. In particular, it is important to note that the lower bound in Theorem 2 is not optimal in general. This can be seen in the most simple way by considering the trivial transformation which leaves the state unchanged, i.e., $\psi^{ABC} \rightarrow \phi^{ABC}$. Clearly, this can be achieved with unit rate $R = 1$. However, if we apply the lower bound in Theorem 2 to this transformation, we get $R \geq S(\psi^{A})/[S(\phi^{B}) + S(\phi^{C})]$. Due to subadditivity of von Neumann entropy, it follows that our lower bound is in general below the achievable unit rate in this case.

**Generalization to multi-partite pure states.** In the discussion so far, we have focused on tri-partite pure states. However, the presented tools can readily be applied to more general scenarios involving an arbitrary number of parties. Pars pro toto we will make the case very explicit in which we convert four-partite pure state $\psi = \psi^{ABCD}$ into four-partite pure state $\phi = \phi^{ABCD}$, providing the tools needed for any $N$. The general idea for this procedure follows the same line of reasoning as in the tri-partite scenario discussed above. In the first step, entanglement combing is applied to the initial state $\psi$, i.e., the transformation

$$\psi \rightarrow \mu^{A:B} \otimes \nu^{C:D} \otimes \nu^{A:D} \tag{18}$$

with pure states $\mu, \nu$, and $r$. In the next step, Alice and Bob transform their state $\mu^{A:B}$ into the desired final state $\phi$ via bipartite LOCC. In the final step, Alice applies Schumacher compression to parts of her state $\phi$, and sends these parts to each of the remaining parties $C$ and $D$ by using entanglement obtained in the first step of this protocol.

As in the tri-partite case, this protocol can be further optimized by interchanging the roles of the parties and applying the time-sharing technique. Moreover, we will allow for catalytic use of singlets in the following, and the corresponding catalytic conversion rate will be denoted by $R_{\text{cat}}$. In general, it holds that $R_{\text{cat}} \geq R$, where $R$ is the conversion rate defined in Eq. (1). We refer to Appendix B for the rigorous definition of $R_{\text{cat}}$ and the proof of the following theorem.

**Theorem 3 (Lower bound for multi-partite state conversion).** For four-partite pure states $\psi^{ABCD}$ and $\phi^{ABCD}$, the catalytic LOCC conversion rate is bounded from below as

$$R_{\text{cat}}(\psi^{ABCD} \rightarrow \phi^{ABCD}) \geq \min \left\{ \frac{S(\psi^{T})}{\sum_{X \in T} S(\phi^{X})} \right\}, \tag{19}$$

where $T$ denotes a part of the total system $ABCD$. 
Some remark about this theorem are in place. The sum \( \sum_{WX} S(\phi^X) \) in Eq. (19) is performed over the individual subsystems of \( W \), e.g. for \( W = ACD \) the sum takes the form \( \sum_{WX} S(\phi^X) = S(\phi^A) + S(\phi^C) + S(\phi^D) \). Further, it is assumed that the states \( \psi \) and \( \phi \) are not product states with respect to the same cut: otherwise we can apply the results for bipartite and tri-partite state conversion discussed above in this letter.

By using similar arguments as below Eq. (3), an upper bound to the catalytic conversion rate is found to be

\[
R_{\text{cat}}(\psi^{ABCD} \to \phi^{ABCD}) \leq \min_{T} \frac{S(\psi^T)}{S(\phi^T)},
\]

(20)

If the bounds in Eqs. (19) and (20) coincide, we obtain a full solution of the conversion problem, and the corresponding rate is given by

\[
R_{\text{cat}}(\psi^{ABCD} \to \phi^{ABCD}) = \min_{T} \frac{S(\psi^T)}{S(\phi^T)}.
\]

(21)

Finally, we note that the lower bound in Theorem 3 also holds for the conversion between tri-partite pure states \( \psi^{ABC} \to \phi^{ABC} \), which is also a direct consequence of Theorem 2. We conjecture that Theorem 3 holds for an arbitrary number of parties \( N \geq 4 \). Such a generalization to conversion of \( N \)-partite pure states follows essentially the structure of a proof by induction and the proof strategy laid out here; however, the details of this argument will be presented elsewhere.

**Generalization to multi-partite mixed states.** In the most general case, we are dealing with \( N \)-partite mixed states \( \rho \) and \( \sigma \), and the corresponding conversion rate \( R(\rho \to \sigma) \) can be defined in the same way as in Eq. (1), where \( \Lambda \) is now an arbitrary asymptotic \( N \)-partite LOCC protocol. We first generalize the upper bound (3) for the transformation rate \( R \). As we show in Appendix C, a generalized upper bound can be given as

\[
R(\rho \to \sigma) \leq \min_{\mathcal{P}} \frac{E_{c}(\rho)}{E_{c}(\sigma)},
\]

(22)

Here, \( E_{c}(\rho) = \lim_{n \to \infty} E_{c}(\rho^{\otimes n})/n \) is the regularized relative entropy of entanglement [28, 29], and \( \mathcal{P} \) denotes a bipartition of all the \( N \) subsystems [30].

We will now show that the ideas which led to lower bounds on conversion rates in the previous sections can also be used in this mixed-state scenario. We will demonstrate this on a specific example, considering the transformation

\[
|\text{GHZ}\rangle|\text{GHZ}\rangle \to \sigma,
\]

(23)

where \( |\text{GHZ}\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2} \) denotes an \( N \)-partite GHZ state vector, and \( \sigma \) is an arbitrary \( N \)-partite mixed state. As we show in Appendix D, by using similar methods as in previous sections, we obtain a lower bound on the transformation rate,

\[
R(|\text{GHZ}\rangle|\text{GHZ}\rangle \to \sigma) \geq \frac{1}{E_{c}(\sigma) + \sum_{X \in \Lambda} S(\sigma^{X})},
\]

(24)

where \( E_{c}(\sigma) \) denotes the entanglement cost [31] between Alice and the rest of the system, and the sum \( \sum_{X \in \Lambda} S(\sigma^{X}) \) is performed over all subsystems \( X \) apart from Alice and Bob. Again, this bound can be further optimized by interchanging the roles of the parties.

**Applications in quantum networks.** It should be clear that the results established here readily allow to assess how resources for multi-partite protocols can be prepared from multi-partite states given in some form. In particular, GHZ states readily provide a resource for quantum secret sharing [14, 15] in which a message is split into parts so that no subset of parties is able to access the message, while at the same time the entire set of parties is. It also gives rise to an efficient scheme of quantum secret sharing requiring purely classical communication during the reconstruction phase [32].

The significance in the established results on multi-partite entanglement transformations hence lies in the way they help understanding how multi-partite resources for protocols beyond point-to-point schemes in quantum networks can be prepared and manipulated. We expect this to be particularly important when thinking of applications of transforming resources into the desired form in quantum networks: Here, multi-partite entanglement is conceived to be created by local processes and bi-partite transmissions involving pairs of nodes, followed by steps of entanglement manipulation, which presumably involve instances of classical routing techniques. Hence, we see this work as a significant contribution to how a quantum internet [12, 13] can possibly be conceived.

**Conclusions.** In this work, we have reported significant progress on the notorious problem of multi-partite entanglement transformations, a problem that was identified as an important problem already early on in the development of quantitative entanglement theory [8]. Similar techniques may also prove helpful in the study of other quantum resource theories different from entanglement, such as the resource theory of quantum coherence [33] and quantum thermodynamics [34, 35].

Putting notions of entanglement combing into a fresh light, we have been able to derive stringent bounds on multi-partite entanglement transformations. This progress seems particularly relevant in the light of the advent of quantum networks and the quantum internet in which multi-partite features are directly exploited beyond point-to-point architectures. It is the hope that the present work stimulates further progress in the understanding of multi-partite protocols.

**Acknowledgements.** We acknowledge discussions with Paweł Horodecki and Dong Yang, and financial support by the Alexander von Humboldt-Foundation, the National Science Center in Poland (POLONEZ UMO-2016/21/P/ST2/04054), the BMBF (Q.com), and the ERC (TAQ).
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[18] In Theorem 2 we assume that the initial state \( \psi_{ABC} \) and the final state \( \psi'_{ABC} \) are not product with respect to the same cut. Otherwise, we can apply results for bipartite pure state conversion, see Eq. (2).

Appendix A: Proof of Lemma 1

The proof presented below will be based on the protocol known as entanglement combing [17]. We will review this protocol for a tri-partite state \( \psi = \psi_{ABC} \). In this case, entanglement combing transforms the state \( \psi_{ABC} \) into \( \mu_{ABC}^B \otimes \gamma_{ABC}^C \) with pure states \( \mu \) and \( \nu \). Clearly, the transformation is not possible if any of the inequalities (6) is violated. We will now show the converse, i.e., any pair of pure states \( \mu_{ABC}^B \) and \( \gamma_{ABC}^C \) which fulfill the inequalities (6) can be obtained from \( \psi_{ABC} \) via LOCC in the asymptotic limit. For this, we will distinguish between the following cases.

**Case 1:** \( S(\psi^A) \geq S(\psi^B) \geq S(\psi^C) \). In this case, Bob can send his part of the state \( \psi \) to Alice by applying quantum state merging [18]. This procedure is possible by using LOCC operations between Alice and Bob. Additionally, Alice and Bob gain singlets at rate \( S(\psi^A) - S(\psi^B) = S(\psi^A) - S(\psi^C) \). The overall process thus achieves the transformation (5) with

\[
E(\mu_{A:B}^B) = S(\psi^A) - S(\psi^C), \\
E(\nu_{A:C}^C) = S(\psi^C). \tag{A1}
\]

Alternatively, Charlie can send his part of the state \( \psi \) to Alice, thus gaining singlets at rate \( S(\psi^A) - S(\psi^B) \). In this way they achieve the transformation (5) with

\[
E(\mu_{A:B}^B) = S(\psi^B), \\
E(\nu_{A:C}^C) = S(\psi^A) - S(\psi^B). \tag{A2}
\]

In the next step we apply-time sharing, i.e., the first procedure is performed with probability \( p \) and the second with probability \( 1-p \). In this way, we see that the transformation (5) is possible for any pair of states \( \mu_{ABC}^B \) and \( \gamma_{ABC}^C \) with the property

\[
E(\mu_{A:B}^B) = p \left( S(\psi^A) - S(\psi^C) \right) + (1-p)S(\psi^B), \\
E(\nu_{A:C}^C) = pS(\psi^C) + (1-p)\left( S(\psi^A) - S(\psi^B) \right). \tag{A3}
\]

By using subadditivity of von Neumann entropy, it is now straightforward to check that for a suitable choice of \( p \), the
quantities $E(\mu^{A:B})$ and $E(\nu^{A:C})$ can attain any value compatible with conditions

$$E(\mu^{A:B}) + E(\nu^{A:C}) = S(\psi^A), \quad (A4a)$$
$$E(\mu^{A:B}) \leq S(\psi^B), \quad (A4b)$$
$$E(\nu^{A:C}) \leq S(\psi^C). \quad (A4c)$$

This completes the proof of Lemma 1 for Case 1.

**Case 2:** $S(\psi^B) \geq S(\psi^A) \geq S(\psi^C)$. In this case, Alice, Bob, and Charlie apply assisted entanglement distillation [19, 20], with Charlie being the assisting party. This procedure achieves the transformation (5) with

$$E(\mu^{A:B}) = \min \left\{ S(\psi^A), S(\psi^B) \right\} = S(\psi^A),$$
$$E(\nu^{A:C}) = 0. \quad (A5)$$

Alternatively, they can apply assisted entanglement distillation with Bob being the assisting party, thus achieving

$$E(\mu^{A:B}) = 0,$$
$$E(\nu^{A:C}) = \min \left\{ S(\psi^A), S(\psi^C) \right\} = S(\psi^A). \quad (A6)$$

By applying time-sharing, we see that we can achieve the transformation (5) with any states $\mu^{A:B}$ and $\nu^{A:C}$ fulfilling

$$E(\mu^{A:B}) = pS(\psi^A),$$
$$E(\nu^{A:C}) = (1 - p)S(\psi^A). \quad (A7a)$$

This completes the proof of Lemma 1 for Case 2.

**Case 3:** $S(\psi^B) \geq S(\psi^A) \geq S(\psi^C)$. Here, we will apply a combination of protocols used in Case 1 and 2. In particular, Bob can send his part of the state $\psi$ to Alice by quantum state merging, see Eq. (A1). Alternatively, they can apply assisted entanglement distillation, see Eq. (A5). By time-sharing we obtain

$$E(\mu^{A:B}) = S(\psi^A) - pS(\psi^C),$$
$$E(\nu^{A:C}) = pS(\psi^C). \quad (A8)$$

By a suitable choice of the probability $p$ it is now possible to obtain any pair of states $\mu^{A:B}$ and $\nu^{A:C}$ such that

$$E(\mu^{A:B}) + E(\nu^{A:C}) = S(\psi^A),$$
$$E(\mu^{A:B}) \leq S(\psi^A),$$
$$E(\nu^{A:C}) \leq S(\psi^C). \quad (A9)$$

This completes the proof of Lemma 1 for Case 3. Note that any other case can be obtained from the above three cases by interchanging the role of Bob and Charlie. Thus, the proof of the Lemma is complete.

**Appendix B: Proof of Theorem 3**

Here, we present the proof of Theorem 3 for the catalytic conversion rate, defined as

$$R_{\text{cat}}(\rho \rightarrow \sigma) = \sup \left\{ \epsilon : \lim_{n \rightarrow \infty} \left( \inf_{A} \left\| \left( \rho_{A}^{n} \otimes \Psi_{A}^{n} \right)^{\otimes} - \sigma_{A}^{(\epsilon n)} \otimes \Psi_{A}^{n} \right\|_{1} \right) = 0 \right\} \quad (B1)$$

Here, the catalyst state $\Psi$ is a pure state, composed of maximally entangled states of sufficiently large finite dimension shared between any two parties. The catalyst state $\Psi$ is required in the following protocol to provide bipartite entanglement in the intermediate steps.

In what follows, we will denote the parties by Alice ($A$) and three Bobs ($B_i$). We will now prove that the catalytic rate for conversion is bounded below by

$$R_{\text{cat}}(\psi^{B_1, B_2, B_3} \rightarrow \phi^{B_1, B_2, B_3}) \geq \min \left\{ \frac{S(\psi^X)}{\sum_{B_i \in X} S(\phi^{B_i})} \right\}, \quad (B2)$$

where $X$ denotes a subset of all the Bobs. The proof of Theorem 3 is then completed by interchanging the role of Alice and Bobs.

The ideas presented in the following generalize the proof of Theorem 2 for tri-partite pure state conversion. In particular, starting with the four-partite state $\psi = \psi^{A, B_1, B_2, B_3}$, we will apply quantum state merging [18] between Alice and all other parties (here referred to as “all the Bobs”) in different orders, aiming to distill entanglement between Alice and each of the parties $B_i$. If $E_i$ denotes the entanglement rate between Alice and $i$-th Bob after this procedure, the rate for state conversion from $\psi$ to $\phi = \phi^{A, B_1, B_2, B_3}$ is bounded below as

$$R(\psi \rightarrow \phi) \geq \min_{i} \frac{E_i}{S(\phi^{B_i})}. \quad (B3)$$

To achieve conversion at rate $\min_{i} \{E_i/S(\phi^{B_i})\}$, Alice locally prepares the state $\phi^{A, B_1, B_2, B_3}$, applies Schumacher compression [23] to the registers $\tilde{A}_i$, and distributes them among the Bobs by using entanglement which has been distilled in the previous procedure. For proving Theorem 3, we will now show that by application of quantum state merging between Alice and Bobs in different orders it is possible to achieve entanglement rates $E_i$ which fulfill

$$\min_{i} \left\{ \frac{E_i}{S(\phi^{B_i})} \right\} \geq \min_{X} \left\{ \frac{S(\psi^X)}{\sum_{B_i \in X} S(\phi^{B_i})} \right\}, \quad (B4)$$

where $X$ denotes a subset of all the Bobs.

In the first step of the proof we will consider all possible ways to merge Bobs’ parts of the state $B_i$ with Alice. Since in the scenario considered here we have three Bobs, there are six different ways to achieve this, depending on the order of the Bobs in the merging procedure. In the following, we will consider entanglement triples $(E_1, E_2, E_3)$, where $E_i$ denotes the amount of entanglement shared between Alice and $i$-th Bob after the merging procedure. The aforementioned six merging procedures give rise to the six triples

$$\begin{align*}
&\left( S(\psi^A) - S(\psi^{AB_1}), S(\psi^{AB_2}), S(\psi^{AB_3}) \right), \\
&\left( S(\psi^A) - S(\psi^{AB_1}), S(\psi^{AB_2}), S(\psi^{AB_3}) \right), \\
&\left( S(\psi^{AB_1}) - S(\psi^{AB_2}), S(\psi^{AB_3}) \right), \\
&\left( S(\psi^{AB_1}) - S(\psi^{AB_2}), S(\psi^{AB_3}) \right), \\
&\left( S(\psi^{AB_1}) - S(\psi^{AB_2}), S(\psi^{AB_3}) \right), \\
&\left( S(\psi^{AB_1}) - S(\psi^{AB_2}), S(\psi^{AB_3}) \right).
\end{align*}$$

(B5)
We note that some of these rates can be negative, implying that entanglement is consumed in this case. This entanglement is then provided by the catalyst state $\Psi$ in Eq. (B1).

In the next step we will use the time-sharing technique, which implies that any convex combination of the above six entanglement triples also leads to an achievable entanglement triple. As we will show in the following, in this way it is indeed possible to obtain a triple $(E_1, E_2, E_3)$ which fulfills Eq. (B4). In the following, we will denote the $i$-th element in the $j$-th line of Eq. (B5) by $E_{ij}$. It is now crucial to note that the third elements in each line $E_{3j}$ fulfill
\begin{align}
E_{13} &\geq E_{23} \geq E_{33}, \\
E_{43} &\geq E_{53} \geq E_{63}.
\end{align}
\tag{B6a} \tag{B6b}
This can be proven by using strong subadditivity of the von Neumann entropy, see Appendix E for more details. Moreover, it holds that
\begin{align}
E_{13} = E_{43}, \quad E_{33} = E_{63}, 
\end{align}
\tag{B7}
as can be immediately checked from Eqs. (B5).

To simplify the notation, we will denote the right-hand side of the inequality (B4) by $G$, i.e.,
\begin{align}
G := \min_{X} \left\{ \frac{S(\psi^X)}{\sum_{B\in X} S(\phi^B)} \right\}.
\end{align}
\tag{B8}
From Eq. (B5) we see that $E_{13} = E_{43} = S(\psi^B)$, which immediately implies the inequality
\begin{align}
E_{13} = E_{43} \geq GS(\phi^B).
\end{align}
\tag{B9}
The further procedure will depend on whether the rates $E_{ij}$ are smaller or larger than $GS(\phi^B)$. Due to Eqs. (B6), (B7), and (B9), there are five possible cases, which we list in the following.

**Case 1:**
\begin{align}
E_{13} &\geq GS(\phi^B) \geq E_{33} \geq E_{13}, \\
E_{43} &\geq GS(\phi^B) \geq E_{53} \geq E_{63}.
\end{align}
\tag{B10a} \tag{B10b}

**Case 2:**
\begin{align}
E_{13} &\geq GS(\phi^B) \geq E_{33} \geq E_{33}, \\
E_{43} &\geq E_{53} \geq GS(\phi^B) \geq E_{63}.
\end{align}
\tag{B11a} \tag{B11b}

**Case 3:**
\begin{align}
E_{13} &\geq E_{33} \geq GS(\phi^B) \geq E_{13}, \\
E_{43} &\geq GS(\phi^B) \geq E_{53} \geq E_{63}.
\end{align}
\tag{B12a} \tag{B12b}

**Case 4:**
\begin{align}
E_{13} &\geq E_{33} \geq GS(\phi^B) \geq E_{13}, \\
E_{43} &\geq E_{53} \geq GS(\phi^B) \geq E_{63}.
\end{align}
\tag{B13a} \tag{B13b}

**Case 5:**
\begin{align}
E_{13} &\geq E_{23} \geq E_{33} > GS(\phi^B), \\
E_{43} &\geq E_{53} \geq E_{63} > GS(\phi^B).
\end{align}
\tag{B14a} \tag{B14b}

We will now discuss each of the above cases in more detail. Depending on the particular case, we will apply the time-sharing technique to the different lines of Eq. (B5), i.e., we will study convex combinations of rate triples in Eq. (B5). In particular, we will show that in each of the aforementioned cases there exists a convex combination of the rate triples such that Eq. (B4) is fulfilled. This will complete the proof of the theorem.

**Case 1.** In this case we consider convex combinations of the first and second line of Eq. (B5), aiming for an entanglement triple which has third element equal to $GS(\phi^B)$. Such a triple exists due to Eq. (B10a). Noting that the rates in every line in Eq. (B5) sum up to $S(\psi^A)$, the resulting rate triple must have the form
\begin{align}
\left( S(\psi^A) - S(\psi^{AB}), S(\psi^{AB}) - GS(\phi^B), GS(\phi^B) \right).
\end{align}
\tag{B15}
Moreover, we also consider convex combinations of the fourth and fifth line of Eq. (B5). Also in this case we aim for an entanglement triple which has third element equal to $GS(\phi^B)$. This rate triple exists due to Eq. (B10b), and it has the form
\begin{align}
\left( S(\psi^{AB}) - GS(\phi^B), S(\psi^A) - S(\psi^{AB}), GS(\phi^B) \right).
\end{align}
\tag{B16}
In the following, we will denote the rates in Eq. (B15) by $(\tilde{E}_{13}, \tilde{E}_{23}, \tilde{E}_{33})$, and the rates in Eq. (B16) will be denoted by $(\bar{E}_{13}, \bar{E}_{23}, \bar{E}_{33})$.

In the next step, we will make use of the inequalities
\begin{align}
\bar{E}_{13} \geq GS(\phi^B), \\
\bar{E}_{13} \geq GS(\phi^B), \\
\bar{E}_{13} \geq E_{23}.
\end{align}
\tag{B17a} \tag{B17b} \tag{B17c}
Eqs. (B17a) and (B17b) follow directly from the definition of $G$ in Eq. (B8). Eq. (B17c) follows from the definition of $G$ together with strong subadditivity of von Neumann entropy. In particular, Eq. (B8) implies that
\begin{align}
S(\psi^{AB};B_1) \geq GS(\phi^B).
\end{align}
\tag{B18}
By using strong subadditivity of von Neumann entropy we arrive at the following result:
\begin{align}
S(\psi^{AB}) + S(\psi^B) - S(\psi^A) \geq GS(\phi^B).
\end{align}
\tag{B19}
This inequality is equivalent to $\bar{E}_{13} \geq \bar{E}_{23}$, and thus completes the proof of Eq. (B17c).

Our protocol for Case 1 will depend on whether $\bar{E}_{23}$ is larger or smaller than $GS(\phi^B)$. If $\bar{E}_{23} < GS(\phi^B)$, due to Eq. (B17b) there exists a convex combination of the triples in Eqs. (B15) and (B16) such that the second rate is equal to $GS(\phi^B)$. Since the sum of all rates in a triple must be equal to $S(\psi^A)$, the resulting rate triple has the form
\begin{align}
\left( S(\psi^A) - GS(\phi^B) - GS(\phi^B), GS(\phi^B), GS(\phi^B) \right).
\end{align}
\tag{B20}
The first rate of this triple is bounded below by $GS(\phi^{B_1})$, as follows from the definition of $G$ in Eq. (B8). Thus, this triple indeed fulfills the desired inequality (B4), and the proof of the theorem is complete in this case. Finally, if $E_2^2 \geq GS(\phi^{B_2})$, due to Eq. (B17a) the triple in Eq. (B16) fulfills the desired inequality (B4), and thus the proof of the theorem is complete for Case 1.

**Case 2.** The procedure in this and the following cases will be similar to Case 1. Again, we consider convex combinations of the first and second line of Eqs. (B5), aiming for a rate triple which has third element equal to $GS(\phi^{B_3})$. Such a rate triple exists due to Eq. (B11a), and it has the same form as Eq. (B15). For clarity, we will repeat the expression here, as

$$
(S(\psi^A) - S(\psi^{AB_1})) > S(\psi^{AB_2}) - GS(\phi^{B_2}), GS(\phi^{B_3})) \tag{B21}
$$

We will also consider convex combinations of the fifth and sixth line of Eqs. (B5). Due to Eq. (B11b), there exists a convex combination of these lines with the third element equal to $GS(\phi^{B_3})$. The corresponding rate triple has the form

$$
(S(\psi^{AB_1}), S(\psi^A) - S(\psi^{AB_2}), GS(\phi^{B_2}), GS(\phi^{B_3})) \tag{B22}
$$

Similar as in Case 1, we will denote the rates in Eq. (B21) by $(E_1^1, E_2^1, E_3^1)$, and the rates in Eq. (B22) will be denoted by $(E_1^2, E_2^2, E_3^2)$.

It is now important to note that Eqs. (B17) also hold in this case. Eqs. (B17a) and (B17b) follow from the definition of $G$ in Eq. (B8). Eq. (B17c) is a direct consequence of the inequality

$$
S(\psi^{AB_2}) + S(\psi^{AB_2}) \geq S(\psi^A) \tag{B23}
$$

which follows from subadditivity of von Neumann entropy.

Similar as in Case 1, the protocol for Case 2 will depend on whether $E_2^2$ is larger or smaller than $GS(\phi^{B_3})$. If $E_2^2 < GS(\phi^{B_3})$, we can apply the same arguments as in Case 1, arriving at the rate triple (B20), which fulfills the desired inequality (B4). If $E_2^2 \geq GS(\phi^{B_3})$, the rate triple in Eq. (B22) fulfills the desired inequality (B4) by the same arguments as in Case 1. This completes the proof of the theorem for Case 2.

**Case 3.** In this case we consider convex combinations of the second and third line of Eqs. (B5), aiming for a rate triple with third rate equal to $GS(\phi^{B_3})$. The existence of such a convex combination is guaranteed by Eq. (B12a), and it has the form

$$
(S(\psi^A) - S(\psi^{AB_1}), GS(\phi^{B_2}), S(\psi^{AB_2}), GS(\phi^{B_3})) \tag{B24}
$$

We will also consider convex combinations of the fourth and fifth line of Eqs. (B5). Also in this case due to Eq. (B12b) it is possible to achieve a rate triple with third rate equal to $GS(\phi^{B_3})$. The corresponding rate triple has the same form as Eq. (B16), and we repeat it here for clarity,

$$
(S(\psi^{AB_2}) - GS(\phi^{B_2}), S(\psi^A) - S(\psi^{AB_1}), GS(\phi^{B_3})) \tag{B25}
$$

Also in this case we denote the rates in Eq. (B24) by $(E_1^1, E_2^1, E_3^1)$, and the rates in Eq. (B25) will be denoted by $(E_1^2, E_2^2, E_3^2)$.

These rates fulfill Eqs. (B17) in this case as well. In particular, Eqs. (B17a) and (B17b) follow from the definition of $G$ in Eq. (B8), and Eq. (B17c) is a consequence of the inequality

$$
S(\psi^{AB_1}) + S(\psi^{AB_2}) \geq S(\psi^A) \tag{B26}
$$

which follows from subadditivity of von Neumann entropy.

It follows that we can apply similar arguments as in Case 1, depending on whether the rate $E_2^2$ is smaller or larger than $GS(\phi^{B_3})$. If $E_2^2 < GS(\phi^{B_3})$, there exists a convex combination of the rate triples (B24) and (B25) such that the second rate is equal to $GS(\phi^{B_3})$. The existence of such a convex combination is guaranteed by Eq. (B17b), and the triple has the same form as Eq. (B20). As discussed below Eq. (B20), this triple fulfills the desired inequality (B4), and the proof of the theorem is complete in this case. If $E_2^2 \geq GS(\phi^{B_3})$, the triple (B25) fulfills the desired inequality (B4) due to Eq. (B17a), and the proof of the theorem is complete for Case 3.

**Case 4.** In this case we consider convex combinations of the second and third line of Eqs. (B5), aiming for a rate triple with the third rate equal to $GS(\phi^{B_3})$. The existence of such a convex combination is guaranteed by Eq. (B13a), and the triple takes the same form as Eq. (B24),

$$
(S(\psi^A) - S(\psi^{AB_1}), GS(\phi^{B_2}), S(\psi^{AB_2}), GS(\phi^{B_3})) \tag{B27}
$$

We also consider convex combinations of the fifth and sixth line of Eqs. (B5), aiming for a rate triple with the third rate equal to $GS(\phi^{B_3})$. The existence of such a convex combination is guaranteed by Eq. (B13b). The corresponding rate triple has the same form as Eq. (B22), and we repeat it here for clarity, as

$$
(S(\psi^{AB_2}), S(\psi^A) - S(\psi^{AB_1}), GS(\phi^{B_2}), GS(\phi^{B_3})) \tag{B28}
$$

As in the previous cases, we denote the rates in Eq. (B27) by $(E_1^1, E_2^1, E_3^1)$, and the rates in Eq. (B28) will be denoted by $(E_1^2, E_2^2, E_3^2)$.

Also in this case Eqs. (B17) hold true. Eqs. (B17a) and (B17b) follow from the definition of $G$ in Eq. (B8). Eq. (B17c) follows from subadditivity of von Neumann entropy and Eq. (B13a), as we will discuss in the following. In particular, subadditivity of von Neumann entropy implies

$$
S(\psi^{AB_1}) + S(\psi^{AB_2}) \geq S(\psi^{AB_1}) \tag{B29}
$$

On the other hand, Eq. (B13a) implies the inequality

$$
GS(\phi^{B_3}) \geq E_3^3 = S(\psi^A) - S(\psi^{AB_1}) \tag{B30}
$$

where for $E_3^3$ we have used the third rate in the third line in Eqs. (B5). When combined with Eq. (B29), we arrive at the result

$$
S(\psi^{AB_1}) + S(\psi^{AB_2}) \geq S(\psi^A) - GS(\phi^{B_3}) \tag{B31}
$$

This is equivalent to $E_2^2 \geq GS(\phi^{B_3})$, due
to Eq. (B17b) there exists a convex combination of the triples (B27) and (B28) resulting in a rate triple of the form (B20). Since this rate triple fulfills the desired inequality (B4), the proof of the theorem is complete in this case. On the other hand, if $E_2^3 \geq GS(\rho^{B_2})$, the triple in Eq. (B28) fulfills the desired inequality (B4) due to Eq. (B17a). This completes the proof of the theorem for Case 4.

**Case 5.** In this case we consider the rate triples in the third and sixth line of Eqs. (B5),

\[
(S(\rho^{AB_1}) - S(\rho^{AB_2}), S(\rho^{AB_3}), S(\rho^A) - S(\rho^{AB_2})), \quad \text{(B32)}
\]

\[
(S(\rho^{AB_1}), S(\rho^{AB_2}) - S(\rho^{AB_3}), S(\rho^A) - S(\rho^{AB_2})), \quad \text{(B33)}
\]

Following the arguments from the previous cases, we will denote the rates in Eq. (B32) by $(E_1^3, E_3^3, E_1^1)$, and the rates in Eq. (B33) will be denoted by $(E_1^1, E_3^3, E_1^3)$. It is important to note that Eqs. (B17) also hold in this case. Eqs. (B17a) and (B17b) follow from the definition of $G$ in Eq. (B8) and Eq. (B17c) follows from the inequality (B29).

We can now use similar arguments as in the previous cases to complete the proof of the theorem. In particular, if $E_2^3 < GS(\rho^{B_2})$, then due to Eq. (B17b) there exists a convex combination of the triples (B32) and (B33) such that the second rate is equal to $GS(\rho^{B_2})$. The resulting rate triple has the form

\[
(S(\rho^{AB_1}) - GS(\rho^{B_2}), GS(\rho^{B_2}), S(\rho^A) - S(\rho^{AB_2})), \quad \text{(B34)}
\]

This rate triple fulfills the desired inequality (B4), as we will show in the following. In particular, the third entry of this triple fulfills $S(\rho^A) - S(\rho^{AB_2}) \geq GS(\rho^{B_2})$ due to Eq. (B14a). It remains to show that the first entry fulfills $S(\rho^{AB_1}) - GS(\rho^{B_2}) \geq GS(\rho^{B_2})$. This follows from the definition of $G$ in Eq. (B8). This completes the proof for the case $E_2^3 < GS(\rho^{B_2})$. If $E_2^3 \geq GS(\rho^{B_2})$, due to Eq. (B17a) the rate triple in Eq. (B33) fulfills the desired inequality (B4). This completes the proof of the theorem.

**Appendix C: Proof of Eq. (22)**

In this section, we will prove the inequality

\[
R(\rho \rightarrow \sigma) \leq \min_{P, q_{[\sigma]}} \frac{E_{\rho \rightarrow q_{[\sigma]}}}{E_{\rho \rightarrow q_{[\sigma]}}} \quad \text{(C1)}
\]

for multi-party mixed states $\rho$ and $\sigma$, where $P_{[\sigma]}$ denotes a bipartition of the total system in two parts $P$ and $[\sigma]$.

For this we will use the asymptotic continuity relation for the regularized relative entropy of entanglement [28, 29]

\[
|E_{\rho}^{XY}(\tau^{XY}) - E_{\rho}^{XY}(\nu^{XY})| \leq \epsilon \log_2 d_X + (1 + \epsilon)h\left(\frac{\epsilon}{1 + \epsilon}\right), \quad \text{(C2)}
\]

where $\tau^{XY}$ and $\nu^{XY}$ are any two states with

\[
||\tau^{XY} - \nu^{XY}||_1 \leq 2\epsilon,
\]

and $h : [0, 1] \rightarrow [0, 1]$ is the binary entropy defined as

\[
h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x). \quad \text{(C4)}
\]

Moreover, $d_X$ is the dimension of the subsystem $X$.

According to the definition of the transformation rate $R$, for any $\epsilon > 0$ there exists an LOCC protocol $\Lambda$ and an integer $n \geq 1$ such that

\[
||\Lambda[\rho^{[\sigma]}] - \sigma^{[\Lambda^\epsilon]}||_1 \leq 2\epsilon. \quad \text{(C5)}
\]

Without loss of generality, we can assume that the states $\Lambda[\rho^{[\sigma]}]$ and $\sigma^{[\Lambda^\epsilon]}$ have the same dimension. Together with the continuity relation (C2) this implies the inequality

\[
E_{\rho}^{\Lambda^\epsilon}(\sigma^{[\Lambda^\epsilon]}) - E_{\rho}^{\Lambda^\epsilon}(\Lambda[\rho^{[\sigma]}]) \leq \epsilon [R_n] \log_2 d_P + (1 + \epsilon)h\left(\frac{\epsilon}{1 + \epsilon}\right). \quad \text{(C6)}
\]

In combination with Eq. (C6) we obtain

\[
\frac{[R_n]}{n} E_{\rho}^{\Lambda^\epsilon}(\sigma) - n E_{\rho}^{\Lambda^\epsilon}(\Lambda[\rho^{\epsilon}]) \leq \epsilon \frac{[R_n]}{n} \log_2 d_P + (1 + \epsilon)h\left(\frac{\epsilon}{1 + \epsilon}\right). \quad \text{(C7)}
\]

In the last step of the proof, we rewrite this inequality as [36]

\[
\frac{[R_n]}{n} \leq \frac{E_{\rho}^{\Lambda^\epsilon}(\sigma)}{E_{\rho}^{\Lambda^\epsilon}(\tau)} + \frac{1}{E_{\rho}^{\Lambda^\epsilon}(\sigma)} \left\{ \epsilon \frac{[R_n]}{n} \log_2 d_P + (1 + \epsilon)h\left(\frac{\epsilon}{1 + \epsilon}\right) \right\}. \quad \text{(C10)}
\]

In particular, for any $\epsilon > 0$ the above inequality is fulfilled for some $n \geq 1$. The proof is complete by noting that $R \leq [R_n]/n$.

**Appendix D: Proof of Eq. (24)**

In the following proof we will denote the parties by Alice (A) and $n$ Bobs ($B_i$), i.e., the total number of parties is $N = n + 1$. We will now show that any $n + 1$-partite mixed state $\sigma = \sigma^{AB_1...B_n}$ can be obtained from the GHZ state vector $|\text{GHZ}\rangle = (|0^{|n+1}\rangle + |1^{|n+1}\rangle)/\sqrt{2}$ via asymptotic $n + 1$-partite LOCC at a rate bounded below as

\[
R(|\text{GHZ}\rangle, |\text{GHZ}\rangle \rightarrow \sigma) \geq \frac{1}{E_{\epsilon}^{B_1|AB_2...B_n}(\sigma)} + \sum_{j=1}^{n} S(\sigma^{B_j}), \quad \text{(D1)}
\]

where $E_{\epsilon}$ denotes the entanglement cost. By interchanging the role of the parties, this inequality is equivalent to Eq. (24).
For proving this statement, we first apply entanglement combing to the GHZ state, i.e., the asymptotic transformation

\[ \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n+1} + |1\rangle^{\otimes n+1}) \rightarrow \mu_1^{A_1B_1} \otimes \cdots \otimes \mu_n^{A_nB_n} \]  

(D2)

with \( n + 1 \) pure states \( \mu_i^{A_iB_i} \). A necessary and sufficient condition for this transformation is that

\[ \sum_i E(\mu_i^{A_iB_i}) \leq 1, \]  

(D3)

as can be seen by applying multi-partite assisted entanglement distillation [18, 20] and time-sharing. The combing is now performed in such a way that the following equalities hold for some parameter \( r \geq 0 \):

\[ E(\mu_1^{A_1B_1}) = r E_c^{B_1|AB_2...B_n}(\sigma^{AB_2...B_n}), \]  

(E4a)

\[ E(\mu_j^{A_jB_j}) = r S(\sigma^{B_j}) \text{ for } 2 \leq j \leq n. \]  

(E4b)

The parameter \( r \) will be determined below.

After combing, Alice and the first Bob use their state \( \mu_1^{A_1B_1} \) for creating the desired final state \( \sigma \) via bipartite LOCC. The optimal rate for this procedure is \( E(\mu_1^{A_1B_1})/E_c^{B_1|AB_2...B_n}(\sigma) \), which is equal to our parameter \( r \) due to Eqs. (D4). In the next step, Alice applies Schumacher compression to those subsystems of \( \sigma \) which are in her possession. The overall compression rate per copy of the initial state vector \( \langle \text{GHZ} | \text{GHZ} \rangle \) is given as \( rS(\sigma^X) \), where \( X \) is the corresponding subsystem. In a final step, Alice teleports compressed parts of the state \( \sigma \) to the other Bobs [24, 25]. Because of Eqs. (D4), the parties share exactly the right amount of entanglement for this procedure. The overall process achieves the transformation \( \langle \text{GHZ} | \text{GHZ} \rangle \rightarrow \sigma \) at rate \( r \).

Finally, by inserting Eqs. (D4) in Eq. (D3), we see that the parameter \( r \) can take any value compatible with the inequality

\[ r \leq \frac{1}{E_c^{B_1|AB_2...B_n}(\sigma) + \sum_{j=2}^n S(\sigma^{B_j})}, \]  

(D5)

which completes the proof of Eq. (D1).

Appendix E: Proof of Eqs. (B6)

The first inequality \( E_1^1 \geq E_3^1 \) follows from strong subadditivity of von Neumann entropy

\[ S(\psi^{AB_1B_2}) + S(\psi^{AB_1B_3}) \geq S(\psi^{AB_1}) + S(\psi^{AB_1B_2B_3}). \]  

(E1)

Since \( \psi^{AB_1B_2B_3} \) is a pure state, we arrive at

\[ S(\psi^{AB_1B_2}) \geq S(\psi^{AB_1}) - S(\psi^{AB_1B_3}), \]  

(E2)

which is the desired result. The second inequality \( E_3^2 \geq E_3^3 \) also follows from strong subadditivity as

\[ S(\psi^{AB_1}) + S(\psi^{AB_1B_3}) \geq S(\psi^{AB_1}) + S(\psi^{AB_1B_3}). \]  

(E3)

This inequality implies that

\[ S(\psi^{AB_1}) - S(\psi^{AB_1B_3}) \geq S(\psi^{AB_1}) - S(\psi^{AB_1}), \]  

(E4)

which is the desired result. The remaining inequalities \( E_3^4 \geq E_1^4 \) follow in the same way by swapping the roles of \( B_1 \) and \( B_2 \).