Nontrivial gapless electronic states at the stacking faults of weak topological insulators

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Lattice defects such as stacking faults may obscure electronic topological features of real materials. In fact, defects are a source of disorder that can enhance the density of states and conductivity of the bulk of the system and they break crystal symmetries that can protect the topological states. On the other hand, in recent years it has been shown that lattice defects can act as a source of nontrivial topology. Motivated by recent experiments on three-dimensional (3D) topological systems such as Bi2TeI and Bi14Rh3I9, we examine the effect of stacking faults on the electronic properties of weak topological insulators (WTIs). Working with a simple model consisting of a 3D WTI formed by weakly-coupled two-dimensional (2D) topological layers separated by trivial spacers, we find that 2D stacking faults can carry their own, topologically gapless states. Depending on the WTI properties, as well as the way in which the stacking fault is realized, the latter can form a topologically protected 2D semimetal, but also a 2D topological insulator which is embedded in the higher-dimensional WTI bulk. This suggests the possibility of using stacking faults in real materials as a source of topologically nontrivial, symmetry-protected conducting states.

I. INTRODUCTION

Topological phases hosting the quantum Hall effect [1, 2] or the quantum spin-Hall effect (QSHE) [3–6] are characterized by an insulating bulk and gapless, topologically-protected boundary states. It is possible to classify different topological phases depending on the symmetries preserving these boundary states and the topology of the bulk Hamiltonian [7, 8]. In a so-called strong topological insulator, only local symmetries (chiral, time-reversal, or particle-hole symmetry) are required to protect the edge states [9]. In recent years, it was found out that some types of topological insulators, such as weak topological insulators (WTIs) [10–12], higher-order topological insulators [13, 14], and topological crystalline insulators [15–17] have boundary states protected by lattice symmetries [18], in addition to the local ones.

The presence of defects in topological insulators can be detrimental to the experimental detection of their topological properties [19]. Point-like impurities, for instance, can increase both the the density of states as well as the conductivity of the otherwise insulating bulk of the material. This can make it difficult to separate bulk and boundary contributions in transport. Furthermore, defects break lattice symmetries, which means they may be doubly detrimental for the experimental detection of the topological properties of materials which rely on these lattice symmetries to protect the conducting states.

In spite of this, lattice defects can carry topologically nontrivial states in and of themselves [20]. This has been shown to occur for a variety of so-called ‘topological defects’, such as full dislocations [20–24] and disclinations [25, 26], which break lattice symmetries only locally and for this reason are locally undetectable far from their core. The resulting classification of topological defects now includes those present in strong and weak topological insulators [27–30], as well as in topological crystalline phases [25, 26, 31, 32] and higher-order topological insulators [33, 34].

In contrast to topological defects, which are by now well understood, nontopological defects, which produce large-scale, visible distortions of the lattice, have only recently begun drawing attention. Partial dislocations, which lead to the formation of stacking faults, have been shown to host topologically protected modes in higher-order topological insulators [33, 35]. Even when the bulk of the material itself is trivial, it has been shown that stacking faults can form lower-dimensional topologically nontrivial subsystems. In such cases, the stacking faults present in the otherwise trivial bulk have been dubbed ‘embedded’ [36, 37] topological insulators and semimetals.

In this work, we focus instead on weak topological insulators, which have recently been experimentally realized in the Bi2TeI [38] and Bi14Rh3I9 [39] material families. In Bi14Rh3I9, the nontrivial layer is a (Bi4Rh3I12)22+ honeycomb lattice, which is topologically equivalent to graphene [40], and the spacer is formed by two coupled, one-dimensional (BiI4)− chains. There is one spacer fragment per honeycomb QSHE entity. In Bi2TeI, the QSHE is hosted by bismuth bilayers Bi2 [41, 42]. If interlayer
coupling is disregarded in this system, the QSHE bismuth bilayers are separated by two consecutive BiTeI spacer layers that are topologically trivial. In the real Bi$_2$TeI$_3$ material, a topological band inversion is realized via interlayer interaction between the Bi$_2$ and BiTeI layers that results in the formation of QSHE sandwiches BiTeI · Bi$_2$ · BiTeI [38]. This coupling is further strengthened when the Bi$_2$-to-BiTeI layer ratio reaches 1:1 in Bi$_2$TeI which shows an electronic spectrum of a topological metal [43]. Interestingly, the chemical composition (bromine for iodine substitution) and the number of spacer layers (one or two) per Bi$_2$ QSHE layer can be controlled by crystal growth parameters [43, 44]. Here, we model these systems as weakly-coupled stackings of QSHE layers and a varying number of trivial spacers.

Given the recent advances in the growth and control of tailor-made topological materials, we examine the phenomenology of stacking faults in WTIs from a theoretical point of view. These defects break the lattice translation symmetry, which is required to protect the WTI phase. Nevertheless, we show that stacking faults can carry their own topologically nontrivial modes, forming either two-dimensional (2D) semimetal phases or 2D QSHE phases which are embedded within the 3D bulk of the WTI. Which type of topology is realized depends on the properties of the stacking fault itself, namely on the fractional lattice translation used to realize it, and it also depends on the number of spacer layers in the parent WTI.

In what follows, we start by introducing a tight-binding model to describe the topological properties of the WTIs without a stacking fault (Sec. II). We explain then how to construct a stacking fault in our model and in which cases it is possible to have topological states at the stacking fault plane using a heuristic argument (Sec. III). In Sec. IV, we solve numerically the tight-binding model in the presence of a stacking fault in WTIs with different numbers of spacer layers, showing that the numerical results support our initial argument and that it is possible to obtain embedded 2D semimetals and topological insulators at the defect. We conclude and discuss directions for future work in Sec. V.

II. WEAK TOPOLOGICAL INSULATOR MODEL

Throughout this paper, we will work with a WTI tight-binding system consisting of a 3D stack of weakly-coupled, 2D QSHE layers separated by trivial spacers. Each of the 2D layers is a honeycomb lattice, with sublattices labeled $A$ and $B$, and the stacking is of the $A – A$ type. We choose the lattice constant such as to obtain Bravais vectors $\mathbf{a}_x = (1, 0, 0)$, $\mathbf{a}_y = (1/2, \sqrt{3}/2, 0)$, and $\mathbf{a}_z = (0, 0, 1)$, the latter vector corresponding to the stacking direction. Thus, unit cells are labeled by position vectors $\mathbf{r} = n_x \mathbf{a}_x + n_y \mathbf{a}_y + n_z \mathbf{a}_z$, $n_x, y, z \in \mathbb{Z}$.

To model the 2D topological layers, we use the well-known QSHE phase realized by the Kane-Mele model [3, 4]. The real-space Hamiltonian of each 2D layer (thus at constant $n_z$) reads

$$H_{KM} = \sum_{N.N.} t_{r,i,\alpha} c_{r,j,\beta}^\dagger s_i^{\alpha,\beta} + \sum_{N.N.N.} it_{2} \nu_{r,r',j,\alpha} c_{r,j,\beta}^\dagger s_i^{\alpha,\beta} \tag{1}$$

where $c_{r,i,\alpha}^\dagger$ creates an electron with spin $\alpha = \uparrow, \downarrow$ on the $j = A, B$ sublattice of the unit cell at position $\mathbf{r}$. The Pauli matrices $s_i^{\alpha,\beta}$ encode the spin degree of freedom, whereas N.N. and N.N.N. denote nearest and next-nearest neighbor sites in the plane of the honeycomb lattice. In the Kane-Mele model, the next-nearest-neighbor term $t_2$ opens a gap in the graphene-like spectrum of the model, leading to a topological insulating phase [3, 4]. The coefficient $\nu_{r,r',j,\alpha}$ is either +1 or −1 depending on whether hoppings go in the clockwise or counter-clockwise direction along a hexagonal plaquette of the honeycomb lattice.

To model the trivial spacers in the WTI, we use a honeycomb lattice with only nearest-neighbor hoppings and add an on-site term which has an opposite sign for sites belonging to different sublattices. Similar to the Kane-Mele term, this gaps out the Dirac cones at the corners of the Brillouin zone [3, 4, 45], but it produces a trivial system. The 2D Hamiltonian of each spacer, again at constant $n_z$, is

$$H_1 = \sum_{N.N.} t \, c_{r,i,\alpha}^\dagger c_{r,j,\beta} s_i^{\alpha,\beta} + \sum_{r,j,\alpha} m \lambda_{r,j,\alpha} \, c_{r,j,\alpha}^\dagger c_{r,j,\alpha} \tag{2}$$

where $m$ is a constant, $\lambda_{r,j,\alpha}$ is either 1 or −1 for the $A$ and $B$ sublattice, respectively, and the first sum runs, as before, over the nearest-neighbor sites within a given layer.

Using the two ingredients above, we can now construct a tight binding model for a WTI whose unit cell is composed of a single QSHE layer and a single spacer [46]. Thus, the in-plane Hamiltonian takes the form of Eq. (1) whenever $n_z$ is even, and the form of Eq. (2) whenever $n_z$ is odd. We add spin-dependent hoppings in the $\mathbf{a}_z$ direction between the layers, in such a way that the edge states moving in opposite directions in adjacent topological layers will couple with each other. The inter-layer coupling term takes the form

$$H_z = \sum_{r,j,\alpha,\beta} it_{2} c_{r,j,\alpha}^\dagger c_{r+n_z,\beta} s_i^{\alpha,\beta} + \sum_{r,j,\alpha} m \lambda_{r,j,\alpha} \, c_{r,j,\alpha}^\dagger c_{r,j,\alpha} \tag{3}$$

Thus, the nearest-neighbor hoppings in the $\mathbf{a}_z$ direction couple the QSHE layers to the neighboring spacers with a strength $t_z$. We choose a diagonal term in the spin degree of freedom, $s_i^z$, similar to the one in the Kane-Mele model. The second-nearest-neighbor hopping with strength $t_{2z} \leq t_z$, on the other hand, couples the QSHE edge modes in adjacent unit cells, leading to a larger dispersion of the
We set the nearest-neighbor hopping to $t_L$ and consider a WTI with a unit cell composed of one topological layer and two spacer layers. Their Hamiltonians are, as before, given by Eqs. (1) and (2), with the difference that we use the Kane-Mele model whenever $n_z$ is divisible by three and use the trivial Hamiltonian for all other values of $n_z$. To ensure a good coupling between the edge modes of QSHE layers in adjacent unit cells, we modify the inter-layer coupling Hamiltonian of Eq. (3) by adding a third-nearest-neighbor hopping. The latter has the same spin structure, $s_j^y$, and a magnitude $t_{z3} \leq t_{z2}$. The resulting, inter-layer coupling Hamiltonian is:

$$
\tilde{H}_z = \sum_{r,j,\alpha,\beta} it_z c^\dagger_{r,j,\alpha} c_{r+a_z,j,\beta} s^z_{\alpha,\beta} + it_{z2} c^\dagger_{r,j,\alpha} c_{r+2a_z,j,\beta} s^z_{\alpha,\beta} + it_{z3} c^\dagger_{r,j,\alpha} c_{r+3a_z,j,\beta} s^z_{\alpha,\beta} + \text{h.c.}
$$

For the two-spacer WTI, the band structure in a slab geometry (not shown) is qualitatively similar to that of Fig. 1. Two Dirac cones form on each surface, at the same points of the surface Brillouin zone, and are displaced in energy by the staggered on-site potentials of the spacers.

### III. STACKING FAULT CONSTRUCTION AND PHENOMENOLOGY

We create a stacking fault in the WTI by using the standard ‘cut and glue’ procedure [21]. First, we cut the WTI slab into two halves along its finite direction, $a_y$. One of these halves is then shifted along the stacking direction, $a_z$, by a fraction of the unit cell. For the single-spacer WTI, we shift by one layer, corresponding to half of the unit cell, whereas for the double-spacer WTI the shift is by either one or two layers, corresponding to $\frac{1}{3}$ or $\frac{2}{3}$ of a unit cell. Finally, the two halves are glued back together using the nearest-neighbor hoppings, $t$, that appear both in Eqs. (1) and (2). Notice that after the stacking fault has been created, each topological layer on one side of the resulting planar defect will have a trivial spacer layer at the same $n_z$ coordinate on the other side. This construction is shown schematically in Fig. 2(a, b) for the single- and double-spacer WTI, respectively.

If the two halves of the system are disconnected from each other (thus, before gluing), each will host two Dirac cones on its surface. When these gapless surfaces are reconnected, the two pairs of Dirac cones on either side will couple, producing either a gapped system, or a gapless one. We predict which of these scenarios will occur based on an analogy to the well-known Su-Schrieffer-Heeger (SSH) model [49], sketched in Fig. 2(c). To this end, we imagine the stacking fault to be made up of the edge modes of the individual QSHE layers on either side of the defect, which are coupled to each other with varying strength depending on the distance between them. The largest of these coupling terms, corresponding to the shortest distances, are labeled as $v$ and $w$ in Fig. 2.

Since the QSHE boundary modes are characterized by a $\mathbb{Z}_2$ topological classification, the states in one layer may be gapped out when coupled to those of a nearby layer. Similar to the sites of the SSH chain, the QSHE edge
modes on nearby layers may form dimers when the coupling between them is staggered in the a_z direction, as shown in Fig. 2(b). In this case, we expect the stacking fault to be gapped, but its boundaries should host topologically nontrivial states, depending on whether it is terminated with a weak or a strong coupling term (see Fig. 3). In contrast, when the boundaries of adjacent topological layers are equally spaced [v = w, as shown in Fig. 2(a)], then no dimers are formed, and we expect to observe a gapless stacking fault. This behavior is again similar to that of the SSH model, which is at a gapless topological phase transition point in the absence of dimerization.

IV. STACKING FAULT RESULTS

A. Single spacer

We begin by considering a stacking fault in the single-spacer WTI model, which is formed between the unit cells at n_x = 19 and n_y = 20 in a slab consisting of L_y = 40 unit cells in the a_y direction (in our convention the first unit cell is at n_y = 0). Before gluing the two halves of the system back together, the two adjacent surfaces host a total of four Dirac cones, as discussed above. After including the nearest neighbor hoppings to reconnect the two halves, we find that the two Dirac cones at (k_x, k_z) = (π, 0) are still present at the planar defect, while the other two Dirac cones at (k_x, k_z) = (π, π) gap out.

We show in Fig. 4 the band structure of the system (panel a) at k_z = 0, as well as the real-space probability distribution of the in-gap modes at the Dirac cone momenta (panel b). Each surface of the WTI slab still shows gapless modes, as expected, which are colored in red and blue, as in Fig. 1. In addition, however, there now appear gapless states localized at the stacking fault plane, colored green, which are 2D gapless states embedded in the middle of the gapped, 3D WTI bulk.

The analogy to the 1D SSH model proves to be correct in predicting the gapless nature of the stacking fault in the single-spacer WTI. This simple heuristic argument, however, fails to accurately describe the robustness of the stacking fault modes. In the SSH chain, while the absence of dimerization does produce a gapless phase, the latter is a fine-tuned point in the parameter space, one at which a topological phase transition takes place.

In contrast, as we show in the following, the 2D gapless phase of the planar defect is robust against perturbations, being protected by mirror symmetry.

Consider a WTI slab in the presence of a stacking fault, infinite along a_x, but having a finite width along a_y, as shown in Fig. 2(a). The unit cell of the full system (shown in red) is composed of four building blocks: the topological layers on the left and right of the defect, which we label H_{KM,l} and H_{KM,r}, and the trivial layers on the left and right, labeled H_{l,r}. We write the full slab...
Hamiltonian as $H_{\text{full}} = c^\dagger H_{\text{full}} c$, with $c$ a column vector formed by all annihilation operators of the system and

$$H_{\text{full}} = \begin{pmatrix}
H_{K,M,l/r} + T_v^2 \sin(k_z) & T_s & T_s & T_s
\end{pmatrix}
\begin{pmatrix}
T_s & H_{I,r} + T_v^2 \sin(k_z) & 0 & T_s
\end{pmatrix}
\begin{pmatrix}
T_s & T_s(1 - e^{-ik_z}) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
T_s \sin(k_z) & H_{K,M,r} + T_v^2 \sin(k_z)
\end{pmatrix}. \quad (5)

Here we have considered that the stacking fault is positioned in the middle of the slab, such that $H_{K,M,l/r}$ and $H_{I,l/r}$ are matrices of equal size, with thicker slabs leading to larger matrices. The term, $T_s$ is a Hamiltonian block containing all of the nearest-neighbor hopping terms across the stacking fault, the ones which serve to glue the two halves of the system back together. $H_{K,M,l/r}$, $H_{I,l/r}$, as well as $T_s$ depend on the momentum $k_z$ but are independent of $k_z$, since they only encode the on-site and hopping terms within the trivial and topological layers, as well as the in-plane hoppings between them.

All of the hoppings along the stacking direction, $a_z$, are included in the term $T_v^2(1 - e^{-ik_z})$ appearing in the off-diagonal blocks of Eq. (5) and in the diagonal term $H_{K,M,l/r}$. Part of the off-diagonal term includes the vertical hoppings connecting the trivial and topological layers within the same unit cell, and is thus momentum independent, whereas the second part contains the hoppings connecting layers across the unit cell boundary, hence the factor $e^{-ik_z}$. These vertical hoppings are given by the inter-layer coupling Hamiltonian of Eq. (3), meaning that $T_v = it_z s^z \otimes \mathbb{1}$, with $\mathbb{1}$ an identity matrix of the same size as the number of sites in $H_{K,M,l/r}$ and $H_{I,l/r}$. It follows that $T_v = -T_{v}^\dagger$, which explains the relative minus sign. The diagonal term $T_{v}^2 = 2t_v^2 s^y \otimes \mathbb{1}$ contains the next nearest neighbor hoppings connecting the TI (and the trivial) layers to each other. This term obeys $s^y T_{v}^2 s^y = -T_{v}^2$.

Using the form of Eq. (5) and the properties of $T_v$ and $T_{v}^2$, it becomes apparent that the slab Hamiltonian obeys a mirror symmetry. The mirror plane is perpendicular to the stacking direction, thus mapping $k_z \rightarrow -k_z$, such that

$$M^\dagger(k_z) H_{\text{full}}(k_z) M(k_z) = H_{\text{full}}(-k_z). \quad (6)$$

The mirror operator, written in the grading of Eq. (5), is a block-diagonal matrix composed of four blocks: $M = is^z \otimes \text{diag}(\mathbb{1}, \mathbb{1}, -Ie^{ik_z}, -Ie^{-ik_z})$. Note that $M$ is a spinful mirror symmetry which commutes with the time-reversal symmetry operator, $\mathcal{T} = is^y \otimes \mathbb{1}_{\text{full}} \mathcal{K}$, where $\mathbb{1}_{\text{full}}$ is an identity matrix acting on all of the sites of the slab’s unit cell and $\mathcal{K}$ is complex conjugation.

The mirror symmetry protects the Dirac cones at $k_z = 0$ but not the ones at $k_z = \pi$. On the mirror-invariant plane of the Brillouin zone, $k_z = 0$, the trivial and topological layers are effectively decoupled in the vertical direction. The states of $H_{K,M,l/r}$ and $H_{I,l/r}$ are associated to a mirror eigenvalue $+i$, whereas those of $H_{K,M,r}$, $H_{I,r}$ are orthogonal to them, having a mirror eigenvalue $-i$ (as shown in Fig. 1). Mirror symmetry thus prevents the topological modes of the QSHE layers on either side of the stacking fault from coupling to each other, guaranteeing the formation of gapless phase at the planar defect. The latter forms an embedded, 2D topological semimetal protected by mirror symmetry. In contrast, at $k_z = \pi$, the QSHE modes on the left and right of the stacking fault have the same mirror eigenvalues, and are therefore not protected.

### B. Double spacer

We now consider a stacking fault in the two-spacer WTI. In this case we shift half of the system in the $a_z$ direction by either $\frac{1}{4}$ or $\frac{3}{4}$ of the unit cell. When the system has an infinite number of layers, these two scenarios differ only in the choice of the unit cell. In both cases, the corresponding infinite SSH chain is in a insulting phase and the spectrum of the system should be gapped.

We have created, as before, two identical copies of the system shifted with respect to each other in the $a_z$ by $\frac{1}{4}$ of the unit cell. In this way every TI layer on one side of the stacking fault has a trivial layer on the opposite side. The two sides are glued together, as before, using the nearest-neighbor hopping $t$.

In this case, the helical modes at the stacking fault develop a gap at both the Dirac points at $k_z = 0, \pi$, and the system has no mirror symmetry. We obtain an embedded SSH-like system with $v \neq w$ at the defect, as we expected.

To distinguish between the topological and trivial phase of the SSH model, we study a system with a finite number of layers to detect the boundary states at the beginning and at the end of the stacking fault. In a system with a finite number of layers we will have more bands near the the Fermi level $E = 0$. In order to distinguish the bands associated with the topological states at the boundaries of the stacking fault, is convenient to make the gap larger. We achieve this by replacing the
Hamiltonian in Eq. (4) with:

\[
\hat{H}_z = \sum_{r,j,\alpha,\beta} it_z c_{r,j+1,\alpha}^\dagger c_{r,j,\beta} + it_z c_{r,j+2,\alpha}^\dagger c_{r,j,\beta} + it_z c_{r,j,\alpha}^\dagger c_{r,j+3,\beta} (s_{\alpha,\beta}^y + s_{\alpha,\beta}^z) + \sqrt{2} + \text{h.c.}
\]

where \( \lambda_j = \pm 1 \) for sites that belong to the A or B sublattice respectively. The new term, \( s_{\alpha,\beta}^z \lambda_j \), couples equal-spin components of the surface states. Components with the same spin of helical modes at opposite sides of the defect will have opposite velocity, so this new term will produce a bigger gap at the stacking fault, see Fig. 5.

**FIG. 5.** Band structure of the double-spacer WTI with a stacking fault, computed at \( k_z = 0 \). Bulk states are shown in dark green, states localized at the surfaces of the slab are shown in purple, whereas modes at the stacking fault are shown in green. In this case there is no mirror symmetry protecting the states at the stacking fault and they gap out. The states at the external surface are still present because they are protected by the time-reversal and translation symmetry along the stacking direction. We have used the inter-layer coupling Hamiltonian in Eq. (7) with \( t_2 = 1, m = 2, t_z = 0.8, t_{z2} = 0.2, t_{z3} = 0.2 \) in units of \( t \). The WTI slab consists of \( L_y = 40 \) unit cells, and the stacking fault is created between the unit cells at \( n_y = 19 \) and \( n_y = 20 \).

C. Topological edge states at the stacking fault

In order to check the presence of topological states at the defect in the tight-binding model, we build a system with a finite height, \( L_z \). To better visualize the stacking fault modes, we add periodic boundary conditions in the \( \mathbf{a}_y \) direction, such that the system contains two stacking faults. In this way we eliminate the surface modes and obtain only the spectrum of the two defects.

The SSH chain associated with the WTI with two trivial spacers will be in the trivial \( (v > w) \) or in the topological \( (v < w) \) phase depending on the boundary conditions at the stacking fault. If the defect begins and ends with a couple of dimerized TI layers, we do not find any gapless states (see Fig. 6), similar to the infinite case.

For a system with two weakly-coupled TI layers at the boundary of the defect, the corresponding SSH chain is in the topological phase. In this case, we find topological gapless bands located at the boundaries of the two stacking faults, see Fig. 6. We confirm their location by plotting the real-space probability distribution, \( P(x,z) = |\langle x,z | \Psi \rangle|^2 \), as a function of the position \((y,z)\) in the unit cell in Fig. 7.
V. CONCLUSIONS

In order to explore the effect of stacking faults in weak topological insulators like Bi$_2$TeI and Bi$_{14}$Rh$_3$I$_9$, we have used a tight binding model composed of stacked honeycomb QSHE layers. We have then used a heuristic argument to draw an analogy between the physics of the topological states at the defect and an SSH chain.

The stacking fault in a WTI with one trivial spacer per unit cell behaves like an SSH at the gap closing point. Thus, it forms an embedded topological semimetal, which we have shown to be protected by mirror symmetry. In contrast, a WTI with two insulating spacers per unit cell behaves like an SSH chain in the topological or trivial phase, depending whether or not it has weakly-coupled TI layers at the edges of the stacking fault. In the latter case, an embedded topological insulator is formed: the stacking fault hosts 1D gapless modes at its boundaries.

This means that it is in principle possible to obtain embedded topological insulators or semimetals inside the bulk of a WTI in the presence of a stacking fault. In this case the defect is not destroying the topological properties of the system, but is a source of new, robust states.

Our toy model is only qualitative in nature. A quantitative prediction of the real properties of stacking faults in Bi$_2$TeI and Bi$_{14}$Rh$_3$I$_9$ would require more accurate numerical methods, such as density functional theory.
Nevertheless, the analogy to the SSH model may still be used to guide this endeavor. For instance, while stacking faults obtained by a fractional translation have not been reported to date in these materials, it has been shown that Bi$_2$TeI forms inverted twin domains [43, 44]. Since its unit cell is composed of one QSHE layer and two spacers, we conjecture that these stacking faults will host protected gapless modes at their boundaries whenever they are terminated by weakly-coupled TI layers.

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[33] R. Queiroz, I. C. Fulga, N. Avraham, H. Beidenkopf, and J. Cano, Partial lattice defects in higher-order topological insulators, Phys. Rev. Lett. **123**, 266802 (2019).

[34] B. Roy and V. Jurcič, Dislocation as a bulk probe of higher-order topological insulators, Phys. Rev. Research **3**, 033107 (2021).

[35] S. S. Yamada, T. Li, M. Lin, C. W. Peterson, T. L. Hughes, and G. Bahl, Bound states at partial dislocation defects in multipole higher-order topological insulators, Nat. Commun. **13**, 2035 (2022).

[36] T. I. Tuegel, V. Chua, and T. L. Hughes, Embedded topological insulators, Phys. Rev. B **100**, 115126 (2019).

[37] S. Velury and T. L. Hughes, Embedded topological semimetals, arXiv.2108.07813 (2021).

[38] I. P. Rusinov, T. V. Menshchikova, A. Isaeva, S. V. Ereimeev, Y. M. Koroteev, M. G. Vergniory, P. M. Echenique, and E. V. Chulkov, Mirror-symmetry protected non-TRIM surface state in the weak topological insulator Bi$_2$TeI, Sci. Rep. **6**, 20734 (2016).

[39] B. Rasche, A. Isaeva, M. Ruck, S. Borisenko, V. Zabolotnyy, B. Büchner, K. Koepernik, C. Ortix, M. Richter, and J. van den Brink, Stacked topological insulator built from Bismuth-based graphene sheet analogues, Nat. Mater. **12**, 422 (2013).

[40] B. Rasche, A. Isaeva, M. Ruck, K. Koepernik, M. Richter, and J. van den Brink, Correlation between topological band character and chemical bonding in a Bi$_{14}$Rh$_3$I$_9$-based family of insulators, Sci. Rep. **6**, 20645 (2016).

[41] S. Murakami, Quantum spin Hall effect and enhanced magnetic response by spin-orbit coupling, Phys. Rev. Lett. **97**, 236805 (2006).

[42] N. Avraham, A. K. Nayak, A. Steinbok, A. Norris, H. Fu, Y. Sun, Y. Qi, L. Pan, A. Isaeva, A. Zeugner, C. Felser, B. Yan, and H. Beidenkopf, Visualizing coexisting surface states in the weak and crystalline topological insulator bi$_2$tei, Nat. Mater. **19**, 610 (2020).

[43] A. Zeugner, M. Kaiser, P. Schmidt, T. V. Menshchikova, I. P. Rusinov, A. V. Markelov, W. Van den Broek, E. V. Chulkov, T. Doert, M. Ruck, and A. Isaeva, Modular design with 2d topological-insulator building blocks: Optimized synthesis and crystal growth and crystal and electronic structures of Bi$_2$TeI ($x = 2, 3$), Chemistry of Materials **29**, 1321 (2017).

[44] A. Zeugner, J. Teichert, M. Kaiser, T. V. Menshchikova, I. P. Rusinov, A. V. Markelov, E. V. Chulkov, T. Doert, M. Ruck, and A. Isaeva, Synthesis, crystal and topological electronic structures of new bismuth tellurohalides Bi$_2$TeBr and Bi$_3$TeBr, Chemistry of Materials **30**, 5272 (2018).

[45] M. O. Goerbig, Electronic properties of graphene in a strong magnetic field, Rev. Mod. Phys. **83**, 1193 (2011).

[46] S. K. Das, B. Yan, J. van den Brink, and I. C. Fulga, Topological crystalline insulators from stacked graphene layers, Phys. Rev. B **99**, 165418 (2019).

[47] C. W. Groth, M. Wimmer, A. R. Akhmerov, and X. Waintal, Kwant: a software package for quantum transport, New J. Phys. **16**, 063065 (2014).

[48] G. Naselli, V. Könye, S. K. Das, G. G. N. Angilella, A. Isaeva, J. van den Brink, and C. Fulga, Nontrivial gapless electronic states at the stacking faults of weak topological insulators, Zenodo: https://doi.org/10.5281/zenodo.6638289 (2022).

[49] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Solitons in polyacetylene, Phys. Rev. Lett. **42**, 1698 (1979).