Abstract

We present a detailed study of the representations of the algebra of functions on the quantum group $GL_q(n)$. A q-analogue of the root system is constructed for this algebra which is then used to determine explicit matrix representations of the generators of this algebra. At the end a q-boson realization of the generators of $GL_q(n)$ is given.
1. Introduction

Although a lot of results exist concerning representations of the quantum algebras [1-5], besides some general theorems [6-7], very few explicit representations have been constructed [8-12] for the dual objects, that is, the quantum matrix algebras or more precisely the deformation of the algebra of functions on the group. In a recent letter [13] we outlined the general method for construction of the finite dimensional representations of the quantum matrix group $GL_q(n)$ (more precisely the quantization of the algebra of functions on $GL_q(n)$). It was proved in [13], that finite dimensional irreducible representations of this algebra exist only when $q$ is a root of unity ($q^p = 1$) and the dimensions of these representations can only be one of the following values: \( \frac{n^N}{2^k} \) where \( N = \frac{n(n-1)}{2} \) and \( k \in \{0, 1, 2, \ldots, N\} \). The topology of the space of states was also clarified (see also prop. 8 of the present article). The method developed in [13] was based on the introduction of a certain subalgebra of $GL_q(n)$ denoted by $\Sigma_n$ for which one could construct finite dimensional representations in a very straightforward way. This subalgebra is in fact nothing but a nice root decomposition of the original algebra. It was then shown that from each irreducible $\Sigma_n$ module one can construct an irreducible $GL_q(n)$ module. This strategy has already been carried out by the present author for the quantum groups $GL_{q,p}(2)$ [9] and $GL_q(3)$ [10-11]. What has remained to be done however is the explicit construction of the general $\Sigma_n$ modules in all its details. This is the subject of the present letter.

Two basic steps in this construction are:

i) A further redefinition of the generators of $\Sigma_n$ such that all the roots decouple into mutually commuting pairs (see eq. (24)).

ii) Introduction of a new identity concerning the quantum determinants (eq. (40)) which paves the way for the determination of the weights of representations.

2. The Root system of $GL_q(n)$

The quantum matrix algebra $GL_q(n)$ [14] is a Hopf algebra generated by unity and the elements $t_{ij}$ of an $n \times n$ matrix $T$, subject to the relations [15]:

\[
R T_1 T_2 = T_2 T_1 R
\] (1)
where $R$ is the solution of the Yang-Baxter equation corresponding to $SL_q(n)$ [16]:

$$
R = \sum_{i \neq j} e_{ii} \otimes e_{jj} + \sum_i q e_{ii} \otimes e_{ii} + (q - q^{-1}) \sum_{i < j} e_{ji} \otimes e_{ij}
$$

The commutation relations derived from (1) can be neatly expressed in the following way. For any for elements $a, b, c,$ and $d$ in the respective positions specified by rows and columns $(ij), (ik), (lj)$ and $(lk)$, the following relations hold:

$$
\begin{align*}
ab &= qba & cd &= qdc \\
ac &= qca & bd &= qdb \\
bc &= cb & ad - da &= (q - q^{-1})bc
\end{align*}
$$

For any matrix $T \in GL_q(n)$, a quantum determinant $D_q(T)$ is defined with the properties:

$$
[D_q(T), t_{ij}] = 0 \quad \forall t_{ij} \in T \\
\Delta D_q(T) = D_q(T) \otimes D_q(T)
$$

The quantum determinant of $T$ acquires a natural meaning as the $q$-analogue of the volume form when the quantum group is considered as the automorphism group on the quantum vector space associated to $GL_q(n)$ [17]. It has the following explicit expression:

$$
D_q(T) = \sum_{i=1}^{n} (-q)^{i-1} t_{1i} \Delta_{1i}
$$

where $\Delta_{1i}$ is the $q$-minor corresponding to $t_{1i}$ and is defined by a similar formula.

In eq. 2 $D_q(T)$ has been expanded in terms of the elements in the first row of $T$. Another useful expansion is in terms of the last column of $T$:

$$
D_q(T) = \sum_{i=1}^{n} (-q)^{n-i} \Delta_{in} t_{in}
$$

To proceed toward constructing the root system of $GL_q(n)$ let us label the elements of the matrix $T$ as follows:

$$
T = 
\begin{pmatrix}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .&n

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Consider the elements $H_i, X_i$ and $Y_i$ together with the q-minors (q-determinants of the submatrices)

\[
H_{ij} = \det_q \begin{pmatrix} \ldots & H_i \\ \ldots & \ldots \\ \ldots & \ldots \\ H_j & \ldots \end{pmatrix}
\]

\[
X_{ij} = \det_q \begin{pmatrix} \ldots & X_i \\ \ldots & \ldots \\ \ldots & \ldots \\ X_j & \ldots \end{pmatrix}
\]

\[
Y_{ij} = \det_q \begin{pmatrix} \ldots & Y_i \\ \ldots & \ldots \\ \ldots & \ldots \\ Y_j & \ldots \end{pmatrix}
\]

For convenience we sometimes denote $H_i, X_i$ and $Y_i$ by $H_{ii}, X_{ii}$ and $Y_{ii}$ respectively.

The subalgebra $\Sigma_n$ is equal to $\Sigma_0 \oplus \Sigma^+ \oplus \Sigma^-$ where the latter are generated respectively by the elements $H_{ij} i \leq j, X_{ij} i \leq j$ and $Y_{ij} i \leq j$

We call the elements $X_i$ and $Y_i$ simple roots and the elements $X_{ij} \quad i < j$ and $Y_{ij} \quad i < j$ non-simple roots. As will be shown below the generators $H_i$ will play the role of Cartan subalgebra elements and the elements $X_{ij} \quad i \leq j$ ( resp.$Y_{ij} \quad i \leq j$) will act as raising and lowering operators. We use the word root in a special sense, by which we mean that from representations of roots, representations of all the other elements of the quantum group can be constructed. For $GL_q(n)$ there are $N = \frac{n(n-1)}{2}$ pair of positive and negative roots.

The reason why constructing $\Sigma_n$ modules is easy is due to the very crucial fact that almost all the relations between generators of $\Sigma_n$ are multiplicative or of Heisenberg-Weyl type. By multiplicative relation between two element $x$ and $y$, we mean a relation of the form $xy = q^{\alpha}yx$, where $\alpha$ is an integer.

**Remark:** In the rest of this paper a multiplicative relation between $x$ and $y$ is indicated as $xy \approx yx$
The important properties of $\Sigma_n$ is encoded in the following propositions (see ref. [10] for their proof):

**proposition 1:** For all $i, j, k,$ and $l$:

\begin{align*}
[H_{ij}, H_{kl}] &= 0 \quad (4) \\
[X_{ij}, X_{kl}] &= 0 \quad (5) \\
[Y_{ij}, Y_{kl}] &= 0 \quad (6)
\end{align*}

Thus $\Sigma_n^0$ and $\Sigma_n^\pm$ are three commuting subalgebras of $GL_q(n)$. For the relations between the generators of $\Sigma^0$ and $\Sigma^\pm$ we have:

**proposition 2**

\begin{align*}
H_{i}X_{ij} &= qX_{ij}H_{i} \quad \forall j \geq i \quad (7) \\
H_{j+1}X_{ij} &= qX_{ij}H_{j+1} \quad \forall i \leq j \quad (8) \\
H_{k}X_{ij} &= X_{ij}H_{k} \quad k \neq i, j + 1 \quad (9) \\
H_{ij}X_{kl} &\approx X_{kl}H_{ij} \quad \forall i, j, k, l \quad (10)
\end{align*}

with $(q \rightarrow q^{-1}, X_{ij} \rightarrow Y_{ij})$

**Remark:** The exact coefficients in relation (10) can easily be determined (see Lemma 10 of ref. [10]). We need in particular the relations:

\begin{align*}
H_{ij}X_{k} &= X_{k}H_{ij} \quad i \leq k \leq j - 1 \quad (11) \\
H_{ij}X_{ij} &= qX_{ij}H_{ij} \quad (12) \\
H_{i+1,j+1}X_{ij} &= qX_{ij}H_{i+1,j+1} \quad (13) \\
[H_{i,j+1}, X_{ij}] &= [H_{i+1,j}, X_{ij}] = 0 \quad (14)
\end{align*}

The relations between elements of $\Sigma^+_n$ and $\Sigma^-_n$. 
proposition 3:

\[ Y_{kl}X_{ij} \approx X_{ij}Y_{kl} \quad (k, l) \neq (i, j) \] (15)

\[ Y_iX_i - X_iY_i = (q - q^{-1})H_iH_{i+1} \] (16)

\[ q^{-1}Y_{ij}X_{ij} - qX_{ij}Y_{ij} = (q^{-1} - q)H_{i,j+1}H_{i+1,j} \] (17)

proposition 4. For \( q^p = 1 \) the p-th power of all the elements of \( \Sigma_n \) are central.

Proof: For the multiplicative relations this is obvious. The only non-multiplicative relations are (16) and (17). From (16) we have:

\[ H_iH_{i+1}X_i = q^2X_iH_iH_{i+1} \] (18)

using this relation and (16) we find by induction:

\[ Y_iX_i^n = X_i^nY_i + (q - q^{-1})\left\{ \frac{q^{2n} - 1}{q^2 - 1} \right\}X_i^{n-1}H_iH_{i+1} \] (19)

which shows that for \( q^p = 1 \)

\[ Y_iX_i^p = X_i^pY_i \] (20)

A similar argument shows that \( Y_iX_i^p = X_i^pY_i \)

For the relation (17) we use the fact that \( H_{i,j+1}H_{i+1,j}X_{ij} = X_{ij}H_{i,j+1}H_{i+1,j} \). By induction from (17) we obtain:

\[ Y_{ij}X_{ij}^n = q^{2n}X_{ij}^nY_{ij} + (1 - q^{2n})X_{ij}^{n-1}H_{i,j+1}H_{i+1,j} \] (21)

which again shows that:

\[ [Y_{ij}, X_{ij}^p] = [Y_{ij}^p, X_{ij}] = 0 \] (22)

Let \( V \) be a \( \Sigma_n \) module. We call this module trivial if, the action of one or more of the elements of \( \Sigma_n \) on it, is identically zero. We are interested in nontrivial \( \Sigma_n \)-modules. (the trivial one’s are representations of reductions of \( \Sigma_n \)).
proposition 5. A $\Sigma_n$ module $V$ is nontrivial only if all the subspaces

$$K_{ij} \equiv \{|v> \in V| H_{ij}|v> = 0\}$$

are zero dimensional.

**Proof**: Suppose that for some $i$ and $j$ $\dim K_{ij} \neq 0$. We choose a basis like $\{|e_i>, i = 1, ... N\}$ for $K_{ij}$. Due to the multiplicative relation of $H_{ij}$ with all the elements of $\Sigma_n$ it is clear that for any $m \in \Sigma_n$ we have:

$$H_{ij} m |e_k> \approx m H_{ij} |e_k> = 0$$

Therefore $m e_k \in K_{ij}$ which means that the basis vectors $e_k$ transform among themselves under the action of $\Sigma_n$. Since $V$ is assumed to be irreducible we have $K_{ij} = V$ and

$$H_{ij} V = H_{ij} K_{ij} = 0$$

which shows that $V$ is a trivial $\Sigma_n$ module.

proposition 6:
i - Finite dimensional irreducible representations of $\Sigma_n$ exist only when $q$ is a root of unity.

ii- Any non-trivial $\Sigma_n$ module $V$ is also an $GL_q(n)$ module and vice versa.

**Proof**: The proof of this proposition is exactly parallel to the case of $GL_q(2)$ [9] and $GL_q(3)$[10]. One uses the expressions (2)(resp. 3) for the q-determinants $Y_{ij}$(resp. $X_{ij}$ (starting from $j = i + 1$, continuing to $j = i + 2, i + 3...$) and uses the fact that in the representation of $\Sigma_n$, all the elements $H_{ij}$ are invertible diagonal matrices. As an example, in the appendix we carry out this procedure explicitly for the quantum group $GL_q(4)$. Note that invertibility of $H_{ij}$’s (due to proposition. 5) is crutial here, otherwise one can not define the actions of the remaining elements of T or V.

3. Representations

To develop the full representation theory we rescale the roots as follows:

$$h_{ij} = H_{ij} \quad x_{ij} = \mu_{ij}^{-1} X_{ij} \quad y_{ij} = \mu_{ij}^{-1} Y_{ij}$$

(23)
where $\mu_{ij} = (H_{ij}H_{i+1,j+1})$

As the reader can verify, with this redefinition the root system is completely disentangled into mutually commuting pairs, while all the relations between $H_{ij}$ and $X_{ij}$ ($Y_{ij}$) remain intact. Instead of (15-17) one will have:

$$[x_{ij}, y_{kl}] = 0 \quad (k, l) \neq (i, j)$$  \hspace{1cm} (24)$$

$$q^{-1}x_{i}y_{i} - qy_{i}x_{i} = q^{-1} - q$$  \hspace{1cm} (25)$$

$$[x_{ij}, y_{ij}] = (q - q^{-1}) \frac{h_{i,j+1}h_{i+1,j}}{h_{ij}h_{i+1,j+1}}$$  \hspace{1cm} (26)$$

From these relations one can also obtain the more general relations:

$$y_{i}x_{i}^{l} = q^{-2l}x_{i}^{l}y_{i} + (1 - q^{-2l})x_{i}^{l-1}$$  \hspace{1cm} (27)$$

$$y_{ij}x_{ij}^{l} = x_{ij}^{l}y_{ij} + q(q^{-2l} - 1)x_{ij}^{l-1} \frac{h_{i,j+1}h_{i+1,j}}{h_{ij}h_{i+1,j+1}}$$  \hspace{1cm} (28)$$

With this redefinition the only structure constants of the algebra are the coefficients between the $h_{ij}$ and $x_{ij}$. Table 1 shows these structure constants for $GL_{q}(4)$.

Consider a common eigenvector of $h_{ij}$'s which we denote by $|0\rangle$ with eigenvalues $h_{ij}|0\rangle = \lambda_{ij}|0\rangle$ and construct an $N = \frac{n(n-1)}{2}$ dimensional hypercube of states

$$W = \{ |l\rangle = \prod_{i,j} (x_{ij})^{l_{ij}}|0\rangle \quad 0 \leq l_{ij} \leq p - 1 \}$$  \hspace{1cm} (29)$$

where $l$ is a vector $l = \sum_{i,j} l_{ij}e_{ij}$ in the lattice. From (10) all the states of $W$ are eigenstates of $h_{ij}$'s.

$$h_{ij}|l\rangle = q^{c_{ij}(l)}\lambda_{ij}|l\rangle$$  \hspace{1cm} (30)$$

The parameters $c_{ij}(l)$ can be easily calculated by using the structure constants.( see the appendix where the case of $GL_{q}(4)$ is considered as an example)
Each positive root generates one direction of this hypercube. Because of (5) we have:

\[ x_i | l > = | l + e_i > \]  \hspace{1cm} (31) 

\[ x_{ij} | l > = | l + e_{ij} > \]  \hspace{1cm} (32) 

Since \( x^p_{ij} \) is central we can set its value on \( W \) equal to a c-number \( \eta_{ij} \). Therefore we have:

\[ x_{ij} | (p - 1)e_{ij} > = \eta_{ij} | 0 > \]  \hspace{1cm} (33) 

The last relation says much more. We need some terminology. Denote by \( F^0_{ij} \) and \( F^1_{ij} \) the two faces which are perpendicular to the vector \( e_{ij} \) respectively passing through the origin and the point \((p - 1)e_{ij} \). Now if \( v \) is any vector in \( F^1_{ij} \) then by eq. (10) we have:

\[ x_{ij} | (p - 1)e_{ij} + v > = \eta_{ij} | v > \]  \hspace{1cm} (34) 

In this way when \( \eta_{ij} \) is nonzero the generator \( x_{ij} \) folds each face \( F^1_{ij} \) onto the face \( F^0_{ij} \). Define the action of \( y_{ij} \) on \( | 0 > \) by:

\[ y_{ij} | 0 > = \alpha_{ij} | (p - 1)e_{ij} > \]  \hspace{1cm} (35) 

By the same reasoning as in the case of \( x_{ij} \) one can show that when \( \alpha_{ij} \) is nonzero the generator \( y_{ij} \) folds the face \( F^0_{ij} \) onto the face \( F^1_{ij} \), i.e: for any vector \( u \) lying in \( F^0_{ij} \)

\[ y_{ij} | u > = \alpha_{ij} | (p - 1)e_{ij} + u > \]  \hspace{1cm} (36) 

We now calculate the action of the negative roots on the other states of \( W \). Thanks to the commutation relations (24) one can calculate the action of any root like \( y_k \) on any state as follows:

\[ y_k | l > = y_k(\prod_i x^{l_i})| 0 > = \prod_{i \neq k} x^{l_i} y_k x_k^{l_k} | 0 > \]  \hspace{1cm} (37) 

For simplicity of notation, in this equation we have represented any positive (resp. negative) root by the symbol \( x_k \) (resp. \( y_k \)) and have not distinguished between simple and nonsimple roots. One then uses eqs. (25-26) to complete the calculation. The result is:

\[ y_i | l > = (q^{-2l_i} \alpha_i \eta_i + (1 - q^{-2l_i})) | l - e_i > \]  \hspace{1cm} (38)
\[ y_{ij} | l > = (\alpha_{ij} \eta_{ij} + q(1 - q^{-2l_{ij}})s_{ij})| l - e_{ij} > \]  \hspace{1cm} (39)

where \( s_{ij} = \frac{\lambda_{i,j+1} \lambda_{i+1,j}}{\lambda_{ij} \lambda_{i+1,j+1}} \). This shows that each \( y_{ij} \) acts as a lowering operator in the direction \( e_{ij} \) of the hypercube.

It remains to determine the parameters \( \lambda_{ij} \). Clearly calculation of these parameters by direct expansion of \( h_{ij} \) is cumbersome. Instead we proceed as follows: Denote by \( E_{i,j} \) (resp. \( E_{ij,kl} \)) the q-minors obtained from a quantum matrix \( E \) by deleting the rows \( i \) (resp. \( i \) and \( j \)) and columns \( k \) (resp. \( k \) and \( l \)). Then we conjecture that the following identity is true:

\[ E_{jl} E_{ik} - q E_{jk} E_{il} = E_{ij,kl} \text{Det}_q E \]  \hspace{1cm} (40)

The classical limit of this identity is well known. In the quantum case it can be checked by direct computation for low dimensional \( GL_q(n) \) matrices. Later on we will give further justification for it using the conjugation properties of \( \Sigma_n \). We now use this relation to determine the parameters \( \lambda_{ij} \). Eq. (40) implies the following relation in \( \Sigma_n \):

\[ Y_{ij} X_{ij} = qH_{ij}H_{i+1,j+1} + H_{i,j+1}H_{i+1,j} \]  \hspace{1cm} (41)

Further justification is obtained by using the conjugation properties of \( T \) as follows. For \( q \) on the unit circle the elements of \( T \) allow the following conjugation:

\[ t_{ij}^\dagger = t_{ij} \]  \hspace{1cm} (42)

This results in the following conjugation properties in \( \Sigma_n \):

\[ X_{ij}^\dagger = X_{ij} \hspace{1cm} Y_{ij}^\dagger = Y_{ij} \hspace{1cm} H_{ij}^\dagger = H_{ij} \]  \hspace{1cm} (43)

One can then conjugate both sides of this equation to obtain:

\[ X_{ij} Y_{ij} = q^{-1}H_{ij}H_{i+1,j+1} + H_{i,j+1}H_{i+1,j} \]  \hspace{1cm} (44)

Combination of eqs. (41) and (44) then leads to eq. (17) which has already been proved in [10].

In terms of the rescaled generators relation (41) takes the following form:

\[ x_i y_i = 1 + q \frac{h_i h_{i+1}}{h_{i+1}} \]  \hspace{1cm} (45)
\[ x_{ij} y_{ij} = 1 + q \frac{h_{ij+1} h_{i+1,j}}{h_{ij} h_{i+1,j+1}} \] (46)

Now these relations help us to determine the parameters \( \lambda_{ij} \): Acting on the state \( |0> \) by both sides of (45,46) we obtain:

\[ \alpha_i \eta_i = 1 + q \frac{\lambda_{i,i+1}}{\lambda_i \lambda_{i+1}} \] (47)

\[ \alpha_{ij} \eta_{ij} = 1 + q \frac{\lambda_{i,j+1} \lambda_{i+1,j}}{\lambda_{ij} \lambda_{i+1,j+1}} \] (48)

or

\[ \lambda_{i,i+1} = q^{-1} \lambda_i \lambda_{i+1} (\alpha_i \eta_i - 1) \] (49)

\[ \lambda_{i,j+1} = q^{-1} \lambda_i \lambda_{i+1,j+1} (\alpha_{ij} \eta_{ij} - 1) \] (50)

Let us call \( \lambda_{ij} \) the weights of the representation and call each \( \lambda_{i,i+k} \) a weight at level \( k \). Eqs. (49-50) express the weights at each level in terms of the weights at the lower level. (see the appendix for the example of \( GL_q(4) \))

4. Types of Representations

We complete our analysis of representation of \( GL_q(n) \) by a discussion on the various types of representations. Each representation is defined by the \( n^2 \) parameters \( \alpha_{ij}, \eta_{ij} \) and \( \lambda_i \). The type of representation depends on the values of the parameters \( \alpha_{ij} \) and \( \eta_{ij} \). More precisely we have:

**Proposition 8**: The dimensions of the irreducible representations of \( GL_q(n) \) can only be one of the following values: \( \frac{N^N}{2^k} \) where \( N = \frac{n(n-1)}{2} \) and \( k \in \{0, 1, 2, ..., N\} \). For each \( k \) the topology of the space of states is \( (S^1)^{(N-k)} \times [0,1]^{(k)} \) (i.e. an \( N \) dimensional torus for \( k = 0 \) and an \( N \) dimensional cube for \( k = N \)).

**Proof**: Our style of proof is a generalization of the one given in [8] and [10] for the case of \( GL_{q,p}(2) \) and \( GL_q(3) \) respectively.
Let $V$ be an $GL_q(n)$ module with dimension $d$. Depending on the values of the parameters $\alpha_{ij}$ and $\eta_{ij}$ three cases can happen:

Case a: $\alpha_{ij} \neq 0 \neq \eta_{ij}$ \forall i, j

In this case $d$ can not be greater than $p^N$, otherwise the cube $W$ will span an invariant submodule which contradicts the irreducibility of $V$. The dimension of $V$ can not be less than $p^N$ either since this means that the length of one of the sides of the cube $W$ (say in the $i$-th direction) must be less than $p$. Therefore there must exist a positive integer $r < p$ such that $x_{ij}^r |0 >= 0$ which means that $\eta_i |0 >= x_{ij}^{p-r} x_{ij}^r |0 >= 0$ contradicting the original assumption. The topology of the space of states in this case is an $N$ dimensional torus ($S^{1 \times N}$)

Case b: For some $(ij) \alpha_{ij} \neq 0$, but $\eta_{ij} = 0$ or vice versa:

In this case the representation is semicyclic in the $ij$ direction.

Case c: for some $(ij) \alpha_{ij} = \eta_{ij} = 0$:

In this case the representation has a highest and a lowest weight in the $ij$-th direction.

If $d < P^N$ there must exist an integer like $r < p$ such that $x_{ij}^r |0 >= 0$ and $x_{ij}^l |0 \neq 0$ for $l < r$. Now denote $x_{ij}^r |0 >$ by $u_0$ and consider the string of states $y_{ij}^r u_0$. This string of states must terminate somewhere. Thats must exists an integer like $r'$ such that $y_{ij}^{r'} u_0 = 0$ and $y_{ij}^{r'-1} u_0 \neq 0$ Therefore

$$0 = x_{ij} y_{ij}^{r'} u_0 = (y_{ij}^{r'} x_{ij} + q(q^{-2r'} - 1) y_{ij}^{r'-1} h_{ij} h_{i+1,j+1} = q(q^{-2r'} - 1) s_{ij} u_0$$

which means that $q^{2r'} = 1$ or $r' = \frac{p}{2}$. $r'$ is in fact the length of the edge of the cube $W$ in the $ij$-th direction, the other edges being of length $p$. The dimension of $V$ is in this case $\frac{p^N}{2}$. The topology of the space of states is in this case $[0, 1] \times S^{1 \otimes N-1}$. By repeating this analysis for other pairs of the parameters the assertion is proved.
Q-Boson Realization

One can construct an infinite dimensional representation (q-analogue of Verma Module) by setting all $\alpha_i = 0$ and relaxing all the conditions of periodicity. It is then very easy to determine the q-boson realization of all the generators of $\Sigma_n$ and hence of $GL_q(n)$.

The q-boson algebra [18-20] $B_q$ is generated by three elements $a, a^\dagger$, and $N$ satisfying the relations:

\[
\begin{align*}
    aa^\dagger - q^{\pm 1} a^\dagger a &= q^{\mp N} \quad (51) \\
    q^{\pm N} a &= q^{\mp 1} a q^{\pm N} \quad (52)
\end{align*}
\]

A more useful form of the algebra is obtained if one replaces the above equations by the following pair of relations:

\[
\begin{align*}
    aa^\dagger &= \lfloor N + 1 \rfloor \\
    a^\dagger a &= \lfloor N \rfloor
\end{align*}
\]

where the symbol $\lfloor N \rfloor$ as usual stands for $\frac{q^N - q^{-N}}{q - q^{-1}}$ for $N$ being a number or an operator.

On the q-Fock space $F_q$ spanned by the states $|n > \equiv a^{\dagger n}|0 >$ the action of the generators are:

\[
\begin{align*}
    a^\dagger |n > &= |n + 1 > \quad (54) \\
    a |n > &= [n]_q |n - 1 > \quad (55) \\
    N |n > &= n |n > \quad (56)
\end{align*}
\]

Consider $N$ commuting q-bosons (i.e. $a_i, a_i^\dagger, N_i; i = 1...N$) and their representation on the q-Fock space $F^{\otimes N}_q$. Then if $\Psi$ is the natural isomorphism from $W$ to $F^{\otimes N}_q$, satisfying:

\[
\Psi : |l > \rightarrow \prod_{i=1}^{N} a_i^{l_i}|0 >
\]

the induced representation $\Psi^*$ is defined by [13] :

\[
\Psi^*(g) = \Psi \circ g \circ \Psi^{-1} \quad \forall g \in End \ W
\]

We will then have the following $n^2$ parameter family of q-boson realization of the quantum group $GL_q(n)$.

\[
\begin{align*}
    x_i &= a_i^\dagger \\
    x_{ij} &= a_{ij}^\dagger
\end{align*}
\]
\[ y_i = (q - q^{-1})a_i q^{-N_i} \quad \quad y_{ij} = q(q^{-1} - q)s_{ij}a_{ij}q^{-N_{ij}} \quad \quad (60) \]

\[ h_i = \lambda_i q^{C_i(N)} \quad \quad h_{ij} = q^{C_{ij(N)}}\lambda_{ij} \quad \quad (61) \]

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**Appendix**  An Example : The Case of \( GL_q(4) \)

The structure constants of \( GL_q(4) \) (see \[13\]) are indicated in table 1. Consequently we obtain the following actions:

\[ h_1|l> = q^{l_1 + l_{12} + l_{13}}\lambda_1|l> \]
\[ h_2|l> = q^{l_1 + l_{2} + l_{13}}\lambda_2|l> \]
\[ h_3|l> = q^{l_2 + l_3 + l_{12}}\lambda_3|l> \]
\[ h_4|l> = q^{l_3 + l_{13} + l_{23}}\lambda_4|l> \]
\[ h_{12}|l> = q^{l_{12} + l_{23} + l_{13}}\lambda_{12}|l> \]
\[ h_{23}|l> = q^{l_{1} + l_3 + l_{12} + l_{23}}\lambda_{23}|l> \]
\[ h_{34}|l> = q^{l_2 + l_{12} + l_{23} + l_{13}}\lambda_{34}|l> \]
\[ h_{13}|l> = q^{l_3 + l_{23} + l_{13}}\lambda_{13}|l> \]
\[ h_{24}|l> = q^{l_1 + l_{12} + l_{23}}\lambda_{24}|l> \]

The weights \( \lambda_{ij} \) are determined from (49-50) to be:

\[ \lambda_{12} = q^{-1}\lambda_1\lambda_2(\alpha_1\eta_1 - 1) \]
\[ \lambda_{23} = q^{-1}\lambda_2\lambda_3(\alpha_2\eta_2 - 1) \]
\[ \lambda_{34} = q^{-1}\lambda_3\lambda_4(\alpha_3\eta_3 - 1) \]
\[
\begin{align*}
\lambda_{13} &= q^{-1} \frac{\lambda_{12} \lambda_{23}(\alpha_{12} \eta_{12} - 1)}{\lambda_2} \\
\lambda_{24} &= q^{-1} \frac{\lambda_{23} \lambda_{34}(\alpha_{23} \eta_{23} - 1)}{\lambda_3} \\
\lambda_{14} &= q^{-1} \frac{\lambda_{13} \lambda_{24}(\alpha_{13} \eta_{13} - 1)}{\lambda_{23}}
\end{align*}
\]

In the following we carry out explicitly the process of reconstruction of \( GL_q(4) \) from \( \Sigma_4 \)

Let us label the elements of \( T \in GL_q(4) \) as follows:

\[
T = \begin{pmatrix}
p & l_1 & Y_1 & H_1 \\
l_2 & Y_2 & H_2 & X_1 \\
Y_3 & H_3 & X_2 & m_1 \\
H_4 & X_3 & m_2 & n
\end{pmatrix}
\]

Here we have:

\[
\begin{align*}
X_{12} &= H_2 m_1 - q X_1 X_2 \\
X_{23} &= H_3 m_2 - q X_2 X_3 \\
Y_{12} &= l_1 H_2 - q Y_1 Y_2 \\
Y_{23} &= l_2 H_3 - q Y_2 Y_3
\end{align*}
\]

From which we obtain:

\[
\begin{align*}
m_1 &= H_2^{-1} (X_{12} + q X_1 X_2) \\
m_2 &= H_3^{-2} (X_{23} + q X_2 X_3) \\
l_1 &= (Y_{12} + q Y_1 Y_2) H_2^{-1} \\
l_2 &= (Y_{23} + q Y_2 Y_3) H_3^{-1}
\end{align*}
\]

We also have:

\[
\begin{align*}
X_{13} &= H_{23} n - q (Y_2 m_2 - q H_2 X_3) m_1 + q^2 X_{23} X_1 \\
Y_{13} &= p H_{23} - q l_1 (l_2 X_2 - q H_2 Y_3) + q^2 Y_1 Y_{23}
\end{align*}
\]

From which we obtain:

\[
n = H_{23}^{-1} \{ X_{13} + q (Y_2 m_2 - q H_2 X_3) m_1 + q^2 X_{23} X_1 \} \]
\[ p = \{Y_{13} + ql_1(l_2 X_2 - qH_2 Y_3) + q^2 Y_1 Y_{23}\} H^{-1}_{23} \]

These equations show that once the action of \( \Sigma_4 \) is known on \( V \) the action of \( GL_q(4) \) can be determined uniquely.
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Table 1 - The structure constants of $GL_q(4)$

\[
\begin{pmatrix}
    x_1 & x_2 & x_3 & x_{12} & x_{23} & x_{13} \\
    h_1 & q & 1 & 1 & q & 1 & q \\
    h_2 & q & q & 1 & 1 & q & 1 \\
    h_3 & 1 & q & q & q & 1 & 1 \\
    h_4 & 1 & 1 & q & 1 & q & q \\
    h_{12} & 1 & q & 1 & q & q & q \\
    h_{23} & q & 1 & q & q & q & 1 \\
    h_{34} & 1 & q & 1 & q & q & q \\
    h_{13} & 1 & 1 & q & 1 & q & q \\
    h_{24} & q & 1 & 1 & q & 1 & q
\end{pmatrix}
\]

\(i.e.: \quad h_{12}x_{12} = qx_{12}h_{12}\)