A minimal model of an autonomous thermal motor

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Abstract – We consider a model of a Brownian motor composed of two coupled overdamped degrees of freedom moving in periodic potentials and driven by two heat reservoirs. This model exhibits a spontaneous breaking of symmetry and gives rise to directed transport in the case of a non-vanishing interparticle interaction strength. For strong coupling between the particles we derive an expression for the propagation velocity valid for arbitrary periodic potentials. In the limit of strong coupling the model is equivalent to the Büttiker-Landauer model for a single particle diffusing in an environment with position-dependent temperature. By using numerical calculations of the Fokker-Planck equation and simulations of the Langevin equations we study the model for arbitrary coupling, retrieving many features of the strong-coupling limit. In particular, directed transport emerges even for symmetric potentials. For distinct heat reservoirs the heat currents are well-defined quantities allowing a study of the motor efficiency. We show that the optimal working regime occurs for moderate coupling. Finally, we introduce a model with discrete phase space which captures the essential features of the continuous model, can be solved in the limit of weak coupling, and exhibits a larger efficiency than the continuous counterpart.

Introduction. – There is currently numerous scientific investigations aimed at characterizing the functioning of micro- and nano-motors. There has, for example, been a rapid development of various artificial nanomotors with the aim of mimicking the performance of biological machines [1–3].

From the point of view of man-made engineered micro- and nano-motors, ideally one would like to design autonomous machines which are able to cyclically extract energy from the resources available in the environment and convert it to useful work. Similarly to their macroscopic counterparts, such machines must be driven out-of-equilibrium by means of one or more thermodynamic forces.

In the present paper we focus in particular on a motor driven by temperature gradients. A Brownian motor has long been the paradigmatic model for a microscopic machine, working either in time-dependent or steady-state conditions. One well-known example is a Brownian particle moving in a periodic and asymmetric potential, a so-called ratchet potential. In such a spatially periodic system, the breaking of the spatial inversion symmetry and of thermal equilibrium, obtained by modulating the force acting on the particle, results in the emergence of directed transport [4–6]. Another typical example is represented by a Brownian particle driven by both a periodic temperature variation and an external parameter, periodically changing the system energy [7,8]. This model, which mimics the operation of a heat engine cyclically in contact with different heat reservoirs, has been implemented in a recent experiment [9]. In all these models there is an external agent that changes periodically some parameters, typically a thermodynamic force, according to the motor state in its phase space.

However, the optimal design for a thermal engine is achieved by an autonomous motor which can operate in steady-state conditions without any external time-dependent drive. A well-known example of an autonomous motor is the so-called Büttiker-Landauer model [10–12], consisting of a Brownian particle moving in a periodic potential and a periodic temperature profile. In this model the spatial symmetry is broken by a phase shift between the potential and the temperature profile [13], resulting in a direct particle current. However, for such a system the definition of efficiency presents an issue [14], e.g., the heat transfer cannot be evaluated
without ambiguity in the overdamped regime [11]. Still, the most remarkable example of autonomous design is the Feynman ratchet [15], where both spatial symmetry and thermal equilibrium are explicitly broken. In the context of Brownian motion, such a ratchet has been modelled, for example, with asymmetric objects moving in separate thermal baths [16–18]. Another class of autonomous machines is represented by Brownian gyros [19–21] where a Brownian particle in a parabolic asymmetric potential rotates around the potential minimum when connected to two different heat reservoirs. The particle mean rotation velocity in the phase space can be calculated [20], although the problem of how to extract useful work from such a setup has not been addressed. In [22] the authors introduced a Brownian motor consisting of two Brownian particles with linear and strong coupling maintained at different temperatures and moving in asymmetric ratchet potentials, so as to mimic the asymmetric features of the classical Feynman ratchet and pawl system.

In the present paper, inspired by the last model above, we present a minimal model of an autonomous thermal motor composed of two Brownian particles moving in two (possibly symmetric) periodic potentials, interacting with a general periodic potential, and maintained at different temperatures. We show that such a system does not require ratchet potentials (with, e.g., an asymmetric saw-tooth shape) in order to exhibit directed transport, the spatial symmetry being broken by the interaction between the particles. We solve the model analytically in the strong-coupling limit for general potentials and show that in this limit the model is equivalent to the Büttiker-Landauer model [4,10–12]. We study the model by numerically solving the Fokker-Planck equation and by numerical integration of the Langevin equation for arbitrary coupling strength, and investigate the dependence of the system velocity on the relevant set of parameters. We show that the particle current arises as soon as there is a non-vanishing coupling between the particles, and find that several features of the strong-coupling limit are also present in the weak- to moderate-coupling regime. We derive an expression for the heat current and, by applying an external force, also evaluate the motor thermodynamic efficiency. Our results indicate that the optimal regime, as far as the motor velocity and efficiency are concerned, occurs in the moderate coupling regime. We finally introduce and discuss a minimal discrete model, that can be solved exactly for any coupling, in particular we obtain the exact expression for the motor current and the heat currents. Such expressions corroborate our findings for the continuous model in the weak-coupling regime.

**Model.**—The model consists of two overdamped coupled degrees of freedom moving in periodic potentials and driven by two heat reservoirs maintained at different temperatures $T_1$ and $T_2$. Denoting the degrees of freedom by $x_1$ and $x_2$, the model is characterized by the potential

$$V(x_1, x_2) = V_1(x_1) + V_2(x_2) + k u (x_1 - x_2), \quad (1)$$

where $V_i$ are periodic potentials with period $L_i$, $i = 1, 2$, and $u(x_1 - x_2)$ a periodic interaction potential, with interaction strength $k$ and period $L_u$. We assume that the periods $L_i$ and $L_u$ are commensurable, such that $L = \max(L_1, L_2, L_u)$ is the total potential period, and $L = n L_1 = m L_2 = l L_u$, with $n, m, l$ integer numbers. Setting the friction constant $\Gamma = 1$ and denoting the forces by $F_i = -dV_i/dx_i$, the overdamped coupled Langevin equations have the form (with a dot denoting a time derivative)

$$\dot{x}_1 = F_1(x_1) - ku'(x_1 - x_2) + \eta_1(t), \quad (2)$$

$$\dot{x}_2 = F_2(x_2) - ku'(x_2 - x_1) + \eta_2(t); \quad (3)$$

where the white Gaussian noises $\eta_1$ and $\eta_2$, characterizing the heat reservoirs at temperatures $T_1$ and $T_2$, are correlated according to $\langle \eta_i(t) \eta_j(t') \rangle = 2 T_i \delta _{ij} \delta (t - t')$. In the non-equilibrium case for $T_1 \neq T_2$ a heat flux is established between the reservoirs. We show that if the following conditions are met, i) $k \neq 0$ and ii) $V_1 \neq V_2$, the system behaves as a motor and part of the integrated heat flux is used to sustain a non-vanishing velocity of the center of mass $\bar{v}$. In the following we will give a precise formulation of the condition $V_1 \neq V_2$.

According to the standard definition in stochastic thermodynamics [23], the rate of heat exchanged with each reservoir along a single stochastic trajectory is $Q_i = \dot{x}_i(t) \partial V(x_1, x_2)$. Using a standard approach [24–26] we then obtain the average heat rate

$$\langle Q_i \rangle = \langle T_i \partial^2 V(x_1, x_2) - (\partial_i V(x_1, x_2))^2 \rangle; \quad (4)$$

for the details of the calculation see the supplementary material [Supplementary material.pdf](SM).

In order to evaluate the thermodynamic efficiency of the motor, we apply a force $f_i$ to one of the particles and choose the sign of $f_i$ such that the force opposes the center-of-mass motion, whose direction we assume as the positive one. The Brownian motor will thus do work against the external force and the corresponding output power is $-f_i \dot{v}_i$. Consequently, the efficiency is given by

$$\eta = -f_i \dot{v}_i / \langle Q_i \rangle,$$

where the index $H$ labels the hot reservoir.

**Analysis for large $k$.**—The coupled Langevin equations (2) and (3) as well as the associated Fokker-Planck equation are difficult to analyze. However, in the adiabatic strong-coupling limit for large $k$ the model is amenable to analysis; details of the calculations are reported in the SM. Following [22] we note that the relative coordinate $y = (x_1 - x_2)/2$ is suppressed and its dynamics quenched, i.e., $y \sim 0$ and $\dot{y} \sim 0$. Moreover, introducing also the center-of-mass coordinate $x = (x_1 + x_2)/2$, setting $\bar{y} = 0$, and eliminating the fast variable $y$, we obtain a single Langevin equation for $x$,

$$\dot{x} = h(x) + g(x) \xi(t), \quad (6)$$
with \( \xi(t) \) a Gaussian white noise, \( \langle \xi(t) \xi(t') \rangle = 2 \delta(t-t') \). Here the drift term \( h \) is given by
\[
h(x) = F_1(x)s_1(x) + F_2(x)s_2(x),
\]
where the space-dependent diffusion coefficient \( g^2 \) depends on the reservoir temperatures and on the particle potentials. It has the form
\[
g^2(x) = T_1 s_1(x)^2 + T_2 s_2(x)^2,
\]
\[
s_{1,2}(x) = \frac{2k - F'_{1,2}(x)}{4k - (F'_{1}(x) + F'_{2}(x))}.
\]

From the definitions it follows that the drift and diffusion are periodic functions of \( x \) with period \( L \). For a constant \( g = \sqrt{T} \), \( T = (T_1 + T_2)/2 \), the Langevin equation (6) describes a Brownian particle subject to the force \( h(x) \). However, for a periodic “temperature” \( T(x) = g(x)^2 \) the Langevin equation exhibits the “blow torch” effect as in the Büttiker-Landauer model \([10,12]\) and thus give rise to a motor effect, as detailed below.

In order to determine the center-of-mass velocity \( \bar{v} = \langle \dot{x} \rangle \) we consider the non-linear Langevin equation (6) driven by multiplicative noise \( g(x) \xi(t) \) and derive the associated Fokker-Planck (FP) equation \([27]\). Adhering to the Stratonovich interpretation the FP equation has the form \( dP/dt = -dJ/dx \), where the probability current is given by \( J(x) = (h(x) - g(x)g(x)/P(x) - g^2(x)P'(x). \) The L-periodic stationary solution of the FP equation reads
\[
P(x) = \frac{J}{(1-e^{Lx})g(x)} \int_x^{x+L} e^{U(y)} g(y) dy,
\]
where we have introduced the effective potential
\[
U(x) = - \int_x^x dy h(y)/g^2(y).
\]
The normalization condition \( \int_0^L dx P(x) = 1 \) then yields the constant steady-state current
\[
\bar{J} = (1-e^{Lx}) \int_0^L dx \frac{e^{-U(x)} g(x)}{g(y)} \int_x^{x+L} dy e^{U(y)}/g(y)
\]
and, thus, the non-zero propagation velocity \( \bar{v} = L \bar{J} \). Here the quantity \( \bar{J} = [U(x+L) - U(x)]/L \) quantifies the breaking of the right-left symmetry. The expression (12) for the current is a central result. We infer that although \( h(x) \) and \( g(x) \) are periodic functions the average \( h(x)/g^2(x) \) over one period, as given by eq. (11), must be non-vanishing in order to ensure directed transport. The condition that \( \bar{J} \neq 0 \) in order for the present model to exhibit directed transport is the same as in the Büttiker-Landauer model for a single particle in a force field \( h(x) \) and a position-dependent profile \( T(x) = g^2(x) \) \([13]\).

Expressing the potentials in their Fourier representation \( V_i(x) = \sum_{q_i} v_{i,q_i} \exp(iq_i x) \), with \( v_{i,q_i} = v_{i,-q_i} \), and \( q_i = 2 \pi n_i/L_i \), and evaluating the ratio \( h(y)/g^2(y) \) in eq. (11) to leading order in \( 1/k \), we obtain
\[
V(x) = U_0(x) - x \bar{f},
\]
where \( U_0(x) \) is a \( L \)-periodic potential, that can be written in terms of the Fourier components of the two potentials \( V_i(x) \) and of the two temperatures \( T_{1,2} \), while for \( \bar{f} \) we obtain
\[
\bar{f} = -2 \frac{(T_1 - T_2)}{k(T_1 + T_2)^2} \sum_q q^2 \Im(v_{1,q} v_{2,q}^*)
\]
see the SM for the details. Inspection of eq. (13) suggests that the quantity \( \bar{f} \) plays the role of a constant tilting force for the periodic potential \( U_0(x) \), as found in models of isothermal molecular motors \([28,29]\), where a Brownian particle moves in a tilted periodic potential. By inspection of eq. (14) we observe that for general unequal periodic potentials the necessary conditions for \( \bar{f} \neq 0 \) are \( T_1 \neq T_2 \) and at least one common mode of the two potentials. Furthermore, if the potentials \( V_1 \) and \( V_2 \) are identical but shifted with respect to one another, \( V_2(x) = V_1(x + \phi) \), we find
\[
\bar{f} = 2 \frac{(T_1 - T_2)}{k(T_1 + T_2)^2} \sum_q q^2 |v_{1,q}|^2 \sin(q \phi),
\]
implying that the current and thus the steady-state velocity in this case is non-zero if, for at least one mode in the potential decomposition, \( \phi \neq \pi m \), with \( m \) integer.

**Arbitrary coupling strength.** – In the case of arbitrary coupling strength \( k \) and general periodic potentials in eq. (1) a numerical solution of the Fokker-Planck equation in the long-time limit yields the steady-state PDF \( P_{ss}(x_1,x_2) \). The steady-state velocity is then obtained from eqs. (2), (3) according to
\[
\bar{v} = \frac{1}{2} (\bar{x}_1 + \bar{x}_2) = \frac{1}{2} (F_1(x_1) + F_2(x_2)),
\]
where the last average is calculated with respect to \( P_{ss}(x_1,x_2) \). We have, moreover, corroborated our findings by means of direct numerical simulations of the Langevin equations (2), (3). In the following we choose the potential
\[
V(x_1,x_2) = a_1 \cos(n_1 x_1) + a_2 \cos(n_2 x_2 + \varphi) + ku(x_1 - x_2),
\]
with \( u(z) = -\cos(n_z z) \) if not otherwise stated, and with arbitrary coupling strength \( k \). We notice that while each single contribution on the r.h.s. of eq. (17) is a symmetric function, the total potential is not. We commence our analysis by considering the case in which the three terms in the potential (17) have the same period. The results are shown in fig. 1.

We find excellent agreement with the large-\( k \) result discussed above, while for fixed \( k \) the velocity increases with the potentials common frequency. As anticipated, the
can make a quadratic approximation for the periodic potential as a function of the phase shift $\varphi$ for the potential (17) with $n_1 = n_2 = n_u = 1$, $T_1 = 1$, $T_2 = 2.5$, $\varphi = \pi/2$, $a_1 = a_2 = 1$. The full lines correspond to the analytic solution $\bar{v} = L\bar{J}$ in the limit of large $k$ with $\bar{J}$ given by eq. (12). Inset: comparison with numerical simulations for $n = 1$. The error bar points are obtained by numerical integration of the Langevin equations (2), (3), with $10^5$ independent trajectories. Filled circles: periodic interaction potential $u(z) = -\cos(z)$. Open circles: quadratic interaction potential $u(z) = z^2/2$, the line is a guide to the eye. In the limit of large $k$ the two potentials give the same velocity, since the relative coordinate $x_1 - x_2$ is small, and one can make a quadratic approximation for the periodic potential $-\cos(x_1 - x_2) \simeq (-1 + (x_1 - x_2)^2/2)$.

optimal velocity is obtained in the moderate-coupling-strength regime. Next we consider the cases in which the coupling $k$ is fixed and we change a) the phase $\varphi$ between the two potentials $V_i(x_i)$ and b) the temperature difference, see fig. 2.

As in the case of a large coupling strength, we find that if the two potentials are identical with no phase shift, the center-of-mass velocity vanishes.

As anticipated the velocity vanishes for $T_1 = T_2$, independently of $k$. The optimal temperature bias $T_1 - T_2$ depends on the coupling strength $k$ and the largest value of the velocity is achieved in the moderate coupling regime. For large values of the temperature difference the thermal fluctuations become too large to favour a coordinated motion of the center of mass in a given direction.

Applying a force $f_1$ to particle 1 we evaluate the efficiency using eq. (5). The results for $n_1 = n_2 = n_u = 1$ are shown in fig. 3(a). We observe that the maximal efficiency one can achieve with this set of parameters is quite small, of the order of $4 \times 10^{-3}\%$. As long as the three potentials in eq. (1) have the same period $L$, changing $L$ corresponds to rescaling the single unit length, and, thus, the velocity $\bar{v}$ will decrease linearly with the potential period, while the heat rate (4) scales as $1/L^2$, see the SM. Thus, one cannot improve the motor maximal efficiency at constant $k$ just by changing the common period $L$. Inspection of eq. (14) suggests that, in the strong-coupling limit, the contribution of each harmonic to the linear tilt in the effective potential $U(x)$ scales as $L^3$ at constant $k$. This suggests a strategy to enhance the velocity and, thus, possibly the efficiency. In the following we will thus evaluate the efficiency $\eta$ by fixing the period of the interacting potential $u(z)$ and increase the period of the two potentials $V_1(x_1)$ and $V_2(x_2)$. The results for a given choice of parameters are shown in fig. 3, and we find indeed an increase in $\eta$ with a maximal value of the order of 0.1%, any further increase in $n_1 = n_2$ does not give rise to a higher maximal value of $\eta$ (data not shown).

Another possible strategy to increase the system efficiency is to increase the amplitudes $a_1$, $a_2$ of the potentials $V_1(x_1)$ and $V_2(x_2)$ so as to allow the motor to sustain a larger force before reaching the stall condition, while keeping the fluctuations of the relative coordinate small, in order to reduce the heat currents, and thus the denominator in eq. (5). While this approach does increase the efficiency, see fig. 3(c), the achieved values are still quite small.

In general our results show that the efficiency of the model motor is not very high. This is due to the lack of strong coupling between the input and the output energy currents: the heat currents flow between the two reservoirs even when the center of mass wanders about a given position, without advancing in the positive direction. We discuss this point in detail below, when comparing the efficiency of the continuous and the discrete models.

We want to stress that the requirement for the interaction potential $u(z)$ to be periodic is not necessary for the system to exhibit a non-zero velocity. A quadratic potential $ku(z) = k\bar{z}^2/2$ also results in a directed motion, see inset in fig. 1, since the total potential still breaks the left-right symmetry. This is the case, for example, in the strong-coupling regime, where the relative coordinate is small, and the cosine interaction potential can be
approximated by a quadratic polynomial. However, by choosing a periodic total potential one can solve the FP equation for \( P_s(x_1, x_2) \) by imposing periodic boundary conditions. On the contrary, when taking a non-periodic \( u(z) \) one is only left with the results of the numerical integration of the Langevin equations.

It is interesting to draw an analogy between our periodic model in its simplest form, eq. (17), and the Hamiltonian for the \( xy \)-model, describing the elastic free energy in ferromagnetic or liquid-crystal systems [30]. The \( xy \)-model Hamiltonian for a spin model on a plane reads \( H = -J \sum_{i,j} s_i \cdot s_j - \sum_i h_i \cdot s_i \), with \( s_i = (\cos \theta_i, \sin \theta_i) \). We now isolate the contribution from the spins \( i = 1 \) and \( j = 2 \), and take \( h_1 = (h_1, 0) \) and \( h_2 = (0, h_2) \). We obtain \( H_{1,2} = -J \cos(\theta_1 - \theta_2) - h_1 \cos(\theta_1) - h_2 \sin(\theta_2) \), which is equivalent to eq. (17), provided that one takes \( h_1 = h_2 = -1 \), \( n_1 = n_2 = n_n = 1 \), \( \varphi = \pi/2 \), and shifts the coordinate \( x_2 \to x_2 - \pi \).

**A discrete model**. – Here we introduce a model with a discrete phase space which captures the essential features of the model described above. We consider two particles \( n = 1, 2 \) that can occupy different positions \( a \cdot i_n \) on two regular, periodic lattices, where \( a \) is the common lattice step and \( i_n = \ldots -2, -1, 0, 1, 2, \ldots \). Each particle is in contact with a thermal reservoir at inverse temperature \( \beta_n \). The particles thus have right and left transition rates \( \omega_n^+ \) and \( \omega_n^- \), respectively. We also assume that the diffusion is unbiased when the particles are uncoupled, i.e., for \( \omega_n^+ = \omega_n^- = \omega_n^0 \), corresponding to a vanishing average velocity.

Next we introduce an interaction periodic potential depending on the distance of the particles on the lattice. Without loss of generality we assume that \( U \) is a \( 2\pi \)-periodic function of the particle distance \( (i_1 - i_2) a \), i.e.,

\[
U(i_1, i_2) = \frac{k}{2} \cos[(i_1 - i_2) a + \varphi].
\]

By taking \( k = 2\pi/N_s \), this corresponds to the energy of the clock model (discrete \( xy \)-model) [30] with lattice spin variables constrained to point in one of the \( N_s \) directions. If we choose the lattice step to be \( a = \pi \), the interaction energy can assume two different energy values, \( U = \pm k \cos(\varphi)/2 \). However, with this choice the interaction potential does not break the left-right symmetry: \( U(-i_1, -i_2) = U(i_1, i_2) \) for any value of the phase \( \varphi \). Consequently, our next choice is \( a = 2\pi/3 \), corresponding to three degenerate values of the energy \( U = k/2 \cos(\varphi), k/2 \cos(\varphi \pm 2\pi/3) \) as long as \( \varphi \neq 0, \pi \). With this choice the system potential breaks the left-right symmetry: \( U(-i_1, -i_2) \neq U(i_1, i_2) \) for \( \varphi \neq 0, \pi \) (see footnote 1).

To simplify the model we assume that only one particle at a time can jump to the left or right. We are thus left with the following choice of transition rates \( W(i_1, i_2 \to i_1', i_2) \) and \( W(i_1, i_2 \to i_1, i_2') \). These rates must be chosen such that when the two temperatures are equal \( \beta_1 = \beta_2 = \beta \), the system reaches the thermal equilibrium state \( P_{eq}(i_1, i_2) \propto \exp(-\beta U(i_1, i_2)) \), with \( i_n = 0, 1, 2 \). We thus impose that the transition rates of each particle obey a local detailed balance condition dictated by the particle’s reservoir of the form

\[
W(i_1, i_2 \to i_1', i_2) = e^{-\beta [U(i_1', i_2) - U(i_1, i_2)]},
\]

\[
W(i_1, i_2 \to i_1, i_2') = e^{-\beta [U(i_1, i_2') - U(i_1, i_2)]}.
\]

There are several choices that enforce this condition. In order to make the rates symmetric we choose

\[
W(i_1, i_2 \to i_1', i_2) = \omega_0 e^{-\beta_1 [U(i_1', i_2) - U(i_1, i_2)]^2},
\]

\[
W(i_1, i_2 \to i_1, i_2') = \omega_0 e^{-\beta_2 [U(i_1, i_2') - U(i_1, i_2)]^2},
\]

where we have assumed identical microscopic transition rates, i.e., \( \omega_0 = \omega_0^1 = \omega_0^2 \); in the following we also take \( \varphi = \pi/6 \). The system consists of 9 different states, and the steady state \( P_{ss}(i_1, i_2) \) can be solved for, together with the currents. We find that both currents read

\[
J_n = -\omega_0 \frac{(\gamma_n - 1)(\gamma_n - \gamma_0^2)(\gamma_2 - 1)}{3(1 + \gamma_1 \gamma_2 + \gamma_1^2 \gamma_2^2)},
\]

where

\[
\gamma_n = \exp(\beta_n k); \quad k = \cos(\pi/6)k/4.
\]
Expanding in Taylor series for small $k$, one obtains
\[ J_n = -\frac{\omega_0}{9} \beta_1 \beta_2 (\beta_1 - \beta_2) \left[ k^3 - \frac{1}{4} (\beta_1^2 + \beta_1 \beta_2 + \beta_2^2) k^5 \right] + O(k^7). \] 
(25)

The particle currents are odd functions of $k$ because of the particular choice of the potential (18); changing $k \to -k$ correspond to a shift of $\pi$ in one of the two coordinates $i_n a$. Changing the phase $\varphi$ changes the prefactors in the definition of $\bar{k}$ but not the current scaling behaviour.

In the limit of large $k$, eq. (23) becomes
\[ J_n \approx \frac{\omega_0}{3} \left[ \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right]. \] 
(26)

One might have chosen another expression for the transition rates, e.g., Glauber’s rates [31]. However, with this choice one does not obtain a compact expression for the current as in eq. (23), but the scaling behaviour for small $k$ is the same as in eq. (25) with the leading order being proportional to $k^3$. This is compatible with what we find numerically for the continuous model, where the velocity $\bar{v}$ appears to have a vanishing first derivative for $k = 0$, see fig. 1.

We can now calculate the entropy flow rate for each reservoir [32–34],
\[ \dot{S}_{p,1} = \sum_{i_1, i_1'} W(i_1, i_2 \to i_1', i_2) P_{ss}(i_1, i_2) \times \ln \frac{W(i_1', i_2 \to i_1, i_2)}{W(i_1, i_2 \to i_1', i_2)}, \]
\[ \dot{S}_{p,2} = \sum_{i_2, i_2'} W(i_1, i_2 \to i_1, i_2') P_{ss}(i_1, i_2) \times \ln \frac{W(i_1, i_2' \to i_1, i_2)}{W(i_1, i_2 \to i_1, i_2')}, \]

and given that the transition rates (21), (22) obey the detailed balance condition for each reservoir, we obtain
\[ T_1 \dot{S}_{p,1} = \langle \dot{Q}_1 \rangle = \sum_{i_1, i_1'} W(i_1, i_2 \to i_1', i_2) P_{ss}(i_1, i_2) \times \left[ U(i_1', i_2) - U(i_1, i_2) \right], \] 
(27)
\[ T_2 \dot{S}_{p,2} = \langle \dot{Q}_2 \rangle = \sum_{i_2, i_2'} W(i_1, i_2 \to i_1, i_2') P_{ss}(i_1, i_2) \times \left[ U(i_1, i_2') - U(i_1, i_2) \right], \] 
(28)

yielding
\[ \langle \dot{Q}_1 \rangle = 3\omega_0 k \left[ \frac{\gamma_1 - \gamma_2}{1 + \gamma_1 \gamma_2 + (\gamma_1 \gamma_2)^2} \right], \] 
(29)

with
\[ \langle \dot{Q}_2 \rangle = -\langle \dot{Q}_1 \rangle. \]

Expanding to leading order in $k$ we find
\[ \langle \dot{Q}_1 \rangle = 4\omega_0 k^2 (\beta_1 - \beta_2) + O(k^4). \] 
(30)

Fig. 4: (Colour online) Efficiency (top) and output power (bottom) for the discrete model as functions of the coupling strength $k$ and of the applied force $f_1$, with $T_1 = 2.5$, $T_2 = 1$, $N_s = 3$, $\varphi = \pi/6$. The time scale is set such that $\omega_0 = 1$.

One can evaluate the efficiency by following the same approach as for the continuous model. By applying a force to one of the two particles, say particle 1, the system potential becomes $U(i_1, i_2) - f_1 a \cdot i_1$, where $U(i_1, i_2)$ is given by eq. (18). The transition rates (19), (20) are modified accordingly, and one can evaluate the delivered power, as given by the product of the particle average velocity and the applied force $P_{out} = -\langle \dot{Q}_1 \rangle$, and the average heat rate from the hot reservoir, eqs. (27), (28). In the presence of the force, one does not obtain a compact expression for the particle and heat currents however, the master equation for the steady state $P(i_1, i_2)$ can be easily solved. The efficiency $\eta = P_{out}/\langle \dot{Q}_H \rangle$ and the output power are shown in fig. 4: we find efficiency values which are higher than those obtained for the continuous model, although the largest values of the efficiency ($\sim 50\%$) are obtained for a large coupling constant $k$ but close to the stall condition, where both $P_{out}$ and $\langle \dot{Q}_H \rangle$ vanish. In the region where the delivered power is maximum, the efficiency is $\sim 20\%$. Such a difference in the efficiency between the continuous and the discrete models can be understood if one notices that the continuous model does not exhibit a strong coupling between the input and the output energy currents. During its dynamic evolution the continuous model can exhibit time lapses where the center-of-mass coordinate does not advance, fluctuating back and forth. This is accompanied by simultaneous fluctuations of the relative coordinate $y$, leading to heat flowing between the two reservoirs. A typical example of such trajectories can occur close to the stall condition: if one applies an external force on the system which is large enough to stall the motor, the system mean velocity, and thus the extracted power will become small, but the heat current between the two reservoirs will not vanish. On the contrary the discrete model exhibits a stronger coupling between the input and the output cycles: when the particle on which the force is applied advances in the positive direction, there is a simultaneous contribution to the extracted work and to the heat current, and this results in higher
values for the efficiency and the efficiency at maximum power [35].

Conclusions. – In conclusion, we have shown that a periodic system, consisting of two Brownian particles, can exhibit direct transport, and behave as an autonomous heat engine when an external mechanical force is applied. In the large-coupling regime, the model is equivalent to a single Brownian particle in a position-dependent temperature profile. However, the heat rates are well-defined quantities, given that each degree of freedom is in contact with its heat reservoir, and thus the efficiency of the heat engine can be evaluated for any value of the interaction strength. We introduce a minimal discrete model that captures the essential features of the continuous Brownian motor, in particular the velocity scaling behaviour for small coupling, and that exhibits a larger efficiency than the continuous case, given the stronger coupling between the input and the output energy currents. Finally, we emphasize that the model engine we propose is feasible in the continuous case, given the stronger coupling between its heat reservoir, and thus the efficiency of the heat engine can be evaluated for any value of the interaction strength.

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