ON THE CONFIGURATION-SPACE TOPOLOGY
IN GENERAL RELATIVITY

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Abstract

The configuration-space topology in canonical General Relativity depends on the choice of the initial data 3-manifold. If the latter is represented as a connected sum of prime 3-manifolds, the topology receives contributions from all configuration spaces associated to each individual prime factor. There are by now strong results available concerning the diffeomorphism group of prime 3-manifolds which are exploited to examine the topology of the configuration spaces in terms of their homotopy groups. We explicitly show how to obtain these for the class of homogeneous spherical primes, and communicate the results for all other known primes except the non-sufficiently large ones of infinite fundamental group.

Section 1. Introduction

In recent years mathematicians have made progress in understanding the diffeomorphism group of 3-dimensional manifolds. The object of this paper is to show how this can be exploited to deepen our topological understanding of configuration spaces occurring in pure General Relativity. In particular, we shall investigate their homotopy groups and thus generalize already existing work on the fundamental group [Wi]. Besides for its intrinsic interest, a major motivation to study these topological structures steems from the canonical quantisation programme for General Relativity. Here, general arguments suggest a topological origin of certain interesting features of quantum gravity (e.g. degenerate vacuum structure, absence of anomalies, superselection sectors), resembling those already familiar from other

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(successfully quantized) theories. Certainly, the arguments given in the context of quantum gravity are primarily meant to be of heuristic value, that is, they are believed to really give insight into some aspects of quantum gravity by using methods which are not necessarily believed to survive an eventual rigorous formulation of it. Amongst others, there are two reasons that entertain this belief: Firstly, arguments identical in structure work in other field theories (e.g. Yang-Mills), where there is a quantum theory, secondly, the arguments (properly formulated) rest merely on general covariance and do not require more structural details about quantum gravity.

In any generally covariant theory the topological structure of configuration space receives characteristic imprints from the diffeomorphism group, which is used to mutually identify physically equivalent points on an auxiliary space that labels physical states in a redundant way. If this auxiliary space is topologically trivial, as it is the case in General Relativity, all the topological information in the homotopy groups of the quotient is determined by those of the diffeomorphism group. Generally, this holds whenever the configuration space is given as the base of a principal fiber-bundle with structure group the diffeomorphisms and contractible total space, as will be explained below. In this case the topology of the base is directly related to the topology of the fibres, and it is their topology which we are going to investigate. In theories where besides the diffeomorphisms there is an additional gauge group acting (which also occurs in the “connection” formulation of General Relativity [Ash]), additional topological structure is induced. In these cases our analysis can be used to provide the diffeomorphism contribution. In order to work within a fixed framework, we shall argue within the standard framework of General Relativity. But, as will become apparent, the investigation is really of a more general kind.

In the sequel of this introductory section and Section 2 we shall provide some basic material concerning the notion of configuration spaces in General Relativity, 3-manifolds and their diffeomorphism groups. In particular, the notion of a spinorial manifold is introduced. A more technical point is deferred to Appendix 1. Proofs of already existing results are only included when it seems appropriate. Their setting given here might differ from the one originally given. This sets the stage for the derivations of some new results in Section 3. In Section 4 all the results next to some other useful information is combined in a table, and some first observations
are made. This section should be accessible without going through the main body of the paper. Appendix 2 combines into five theorems some scattered results from the literature which we made essential use of.

**Configuration Spaces, 3-Manifolds and Diffeomorphisms**

The specification of initial data in General Relativity starts with the selection of a 3-manifold, $\Sigma$, on which initial data are constructed in form of a Riemannian 3-metric and the extrinsic curvature. Together they satisfy an elliptic system of four differential equations, the constraints, which are separate from the evolution equations. As configuration space we address the quotient-space obtained from the space of all 3-metrics on $\Sigma$, where those metrics which label the same physical state are mutually identified. This reduces three (the so-called momentum constraints, which are linear in momenta) of the four constraint equations, the remaining one being the so called Hamiltonian constraint (which is quadratic in the momenta). The identification is generically given by the action of some normal subgroup (possibly the whole group) of the diffeomorphism group, which we choose to call its gauge part, since it connects redundant labels for the same physical state. General covariance then implies that the quotient of the full diffeomorphism group with respect to the gauge part acts on the configuration space as proper symmetries, which now connect different physical states. We shall refer to those simply as symmetries. In case the gauge part exhausts all of the diffeomorphism group, there will simply be no symmetries.

Exactly how much of all diffeomorphisms are considered to be gauge part depends on the physical situation one likes to describe and cannot be answered a priori within the formalism. In General Relativity two major situations arise: Firstly, $\Sigma$ is closed and represents the whole universe, in which case there are no symmetries and the gauge part is the whole diffeomorphism group. This situation we shall refer to as the closed case. It is usually employed in classical- and quantum-cosmology. Secondly, $\Sigma$ represents an isolated part of the universe, so that $\Sigma$ is a manifold that outside some connected compact set is homeomorphic to the complement of a closed ball in $R^3$, i.e., to the cylinder $R \times S^2$. This cylinder can be thought of as the transition region between the system under study and the ambient universe relative to which the system is described. The gauge part is then given by those diffeomorphisms of $\Sigma$ that asymptotically die-off as one moves along the cylinder.
in an outward direction. Slightly more precise, we may compactify $\Sigma$ by a 2-sphere boundary at the outer end of the cylinder and take the gauge part as those diffeomorphisms that fix the boundary. They form a proper normal subgroup within the full diffeomorphism group of $\Sigma$. The action of symmetries is then interpreted as changing the relative positions of the system with respect to the ambient universe. This situation we shall refer to as the open case.

Mathematically the situations just described are surprisingly unique, in the sense that essentially (i.e. up to discrete groups) no other choice of a quotient symmetry group could have been made. This is due to two facts. Firstly, that the connected component of the diffeomorphism group of a closed manifold is simple, and, secondly, that the connected component of the group of boundary-fixing diffeomorphisms of a manifold with connected boundary is simple, and given by the unique non-trivial normal subgroup of the connected component of all diffeomorphisms [McD]. For more than one boundary component (i.e. a $\Sigma$ with more than one asymptotic region, a case which we do not consider here), there will be more normal subgroups according to those diffeomorphisms that fix only some of the boundary components [McD]. For a closed manifold, a minimal non-trivial normal subgroup of the diffeomorphism group is given by its connected component, whereas in the open case a minimal choice is given by the connected component of asymptotically trivial diffeomorphisms (larger choices would be the connected component of all diffeomorphisms, or all asymptotically trivial diffeomorphisms). In any case, the only way in which continuous symmetry groups can arise is via asymptotic regions or boundaries. Clearly, selecting minimal normal subgroups as gauge part corresponds to a maximal choice of symmetries.

As it stands, the open and closed case do not seem to be intimately related. We will argue, however, that, in a sense explained below, our topological investigations cover both situations at the same time. Recall that in the closed case the configuration space, as defined above, namely as the quotient of the space of metrics modulo all diffeomorphisms, has a non-trivial singularity structure (described in [Fi1]), due to the changing dimensions of isotropy groups at metrics with different isometries. Rather then working with the singular configuration space $\mathcal{S}(\Sigma)$ (for superspace), where e.g. the global dynamics is only defined by some regular dynamics on a singularity free resolution space (reflection conditions etc.), one may instead use the resolution space from the start. First arguments as to why canonical quantum
gravity should also be formulated on a resolution space of superspace where already given in [DeW]. It turns out that there is a canonical resolution space for $S(\Sigma)$ which we call $Q_R(\Sigma)$. Its construction is explained in lucid detail in [Fi2].

On the other hand, in the open case, an admissible and convenient way for our topological investigations is to consider the one-point compactification $\bar{\Sigma} := \Sigma \cup \{\infty\}$ of $\Sigma$ by a point called $\infty$, and then define the configuration space, $Q(\Sigma)$, by the space of all metrics on $\bar{\Sigma}$, $Riem(\bar{\Sigma})$, modulo the diffeomorphisms that fix the frames at $\infty$. To answer topological questions we neither need to specify fall-off conditions nor the precise functional space for the metric. To start the construction, we fix an oriented frame $u$ at $\infty$. A general linear transformation of the tangent space $T_{\infty}(\bar{\Sigma})$ is said to be $\in SO(3)$, if its matrix representative with respect to $u$ is $\in SO(3)$.

Clearly, all conclusions to follow are independent of the choice of $u$. Let us now define:

$$D(\bar{\Sigma}) := \{\text{orientation preserving } C^\infty - \text{diffeomorphisms on } \bar{\Sigma}\}$$
$$D_\infty(\bar{\Sigma}) := \{\phi \in D / \phi(\infty) = \infty, \ \phi_*|_\infty \in SO(3)\} \ \ (1.1)$$
$$D_F(\bar{\Sigma}) := \{\phi \in D_\infty / \phi_*|_\infty = \text{id.}\}$$

Here, $D_\infty(\bar{\Sigma})$ represents those diffeomorphisms of $\Sigma$ which induce “rigid” rotations on the 2-sphere at the end of the cylinder. But since the space of orientation-preserving diffeomorphisms of the 2-sphere is homotopy equivalent to its isometries $SO(3)$ [Sm], we may for our topological purposes represent $D(\Sigma)$ by $D_\infty(\bar{\Sigma})$. As an important example let us consider the particular class of open cases, where the system under consideration is represented by asymptotically flat metrics. Here, the restriction to $D_F$ as the gauge part is well motivated, since we want to include configurations with non-vanishing angular momentum at spatial infinity ($\infty$). These are included in $Riem(\bar{\Sigma})/D_F(\bar{\Sigma})$ but not in the smaller space $Riem(\bar{\Sigma})/D_\infty(\bar{\Sigma})$. Diffeomorphisms on $\Sigma$ must not disturb the fixed asymptotically euclidean structure, and are therefore faithfully represented by $D_\infty(\bar{\Sigma})$. The symmetry group is then given by $D_\infty(\bar{\Sigma})/D_F(\bar{\Sigma})$. However, motivated by its natural appearance in canonical quantum gravity, where only the connected component $D_F^0$ of $D_F$ appears as gauge part, we introduce the slightly larger symmetry group $G$ (note that by the discussion above this is the maximal symmetry group):

$$G(\Sigma) := D_\infty(\bar{\Sigma})/D_F^0(\bar{\Sigma}) \ \ (1.2)$$
It is now true that $Q(\Sigma)$ is the basis of a $D_F$-principal fibre bundle with total space $Riem(\Sigma)$ [Bou][Fi2]:

$$D_F(\Sigma) \rightarrow Riem(\Sigma) \quad \text{under} \quad \pi \quad \text{onto} \quad Q(\Sigma)$$ (1.3)

We can now compare this to the closed case by considering the space $Q_R(\bar{\Sigma})$ for the closed manifold $\bar{\Sigma}$. If we denote by $F(\bar{\Sigma})$ the bundle of oriented frames over $\bar{\Sigma}$, the resolved configuration space is defined as the base of the following principal fibre bundle [Fi2]:

$$D(\bar{\Sigma}) \rightarrow Riem(\bar{\Sigma}) \times F(\bar{\Sigma}) \quad \text{under} \quad \pi \quad \text{onto} \quad Q_R(\bar{\Sigma})$$ (1.4)

Here, $D(\bar{\Sigma})$ acts on $F(\bar{\Sigma})$ by its standard lift, and the action is free by the same argument as above. We now have the following result which is part of Theorem 6.1 in [Fi2]:

**Theorem 1.** The spaces $Q(\Sigma)$ and $Q_R(\bar{\Sigma})$ are diffeomorphic (as ILH-manifolds).

For us this implies that we can focus attention to $Q(\Sigma)$. All the statements we are going to make about the abstract topology of $Q(\Sigma)$ hold equally well for $Q_R(\bar{\Sigma})$. Keeping this in mind we shall never mention $Q_R(\bar{\Sigma})$ again.

$Riem(\bar{\Sigma})$ is a convex open cone in the topological vector space of smooth (0,2)-tensor fields. $D_F$ is topologized to make it a topological group (i.e. at least as fine as compact open) with topological action on $Riem(\bar{\Sigma})$. $Q$ is given the quotient topology which is the finest topology for which $\pi$ is continuous (for more information on topologies of mapping spaces see [Mi]). There are two important points to make.

1. The map $\pi$ is open and hence $Q$’s topology unique.

Proof: take an open set $O$ in $Riem(\bar{\Sigma})$. Since $D_F$ acts as a topological group, its orbit, given by $\pi^{-1}(\pi(O))$, is also open. Hence $\pi(O)$ is an open set and therefore $\pi$ an open map. Conversely, let $\pi$ be an open map and $U \subset Q(\bar{\Sigma})$ an arbitrary set such that $\pi^{-1}(U)$ is open. Then $U = \pi(\pi^{-1}(U))$ is open and therefore $Q(\Sigma)$’s topology equal or stronger than the quotient topology. But the
quotient topology is already the strongest one compatible with the continuity of \( \pi \). Hence it is unique. With respect to \( Q \) it therefore makes sense to talk of its topology.

2. Due to a theorem of Cerf’s [C], the diffeomorphism and homeomorphism groups of \( \Sigma \) are homotopy equivalent spaces. This allows us to be imprecise about the function spaces we work in. In particular, it allows us to use interchangeably \( D_F(\Sigma) \) and \( H_D(\Sigma) \), where the latter denotes the space of homeomorphisms fixing a disc containing \( \infty \) which is often employed in the literature (e.g. [Fr-Wi], [He - L]).

From the contractibility of \( \text{Riem}(\Sigma) \) and the homotopy exact sequence associated with the bundle (2), we immediately obtain for all \( n \geq 0 \):

\[
\pi_n(D_F(\Sigma)) \cong \pi_{n+1}(Q(\Sigma)).
\] (1.5)

The investigation of the homotopy groups of \( Q(\Sigma) \) is thus reduced to those of \( D_F(\Sigma) \). No reference to the space of metrics is made anymore. It has dropped out of the homotopy exact sequence due to its contractibility (\( Q(\Sigma) \) is a classifying space for the group \( D_F \)) and the only topological features are those of \( D_F \). This is why investigations of this type bear a high degree of generality. For example, in so-called higher derivative theories of gravity, \( \text{Riem}(\Sigma) \) is replaced by \( \text{Riem}(\Sigma) \times K \), where \( K \) is the linear (and hence contractible) space of sections in some tensor bundle. The \( D_F \) action is then still free and the total space still contractible. The corresponding configuration spaces therefore still satisfy (1.5).

Let us now decompose \( \Sigma \) into its prime factors \( \Sigma_i \), explicitly separating the irreducible primes \( \bar{P}_i \) from the non-irreducible handles \( S^2 \times S^1 \).

A prime \( \Sigma_i \) is irreducible \( \iff \) in \( \Sigma_i \) every 2-sphere bounds a disc. For closed, orientable primes, the only non-irreducible one is the handle.

\[
\Sigma = \biguplus_{i=1}^n \Sigma_i = \left( \biguplus_{i=1}^l \bar{P}_i \right) \sqcup \left( \biguplus_{i=l+1}^n S^2 \times S^1 \right) \quad \bar{P}_i \not\cong S^2 \times S^1,
\] (1.6)

where we used \( \sqcup \) to denote the operation of taking the connected sum. It is known (e.g. [McCu]) that there is a homotopy equivalence (\( \sim \))

\[
D_F(\Sigma) \sim \left( \bigtimes_{i=1}^l D_F(\bar{P}_i) \right) \times \left( \bigtimes_{i=l+1}^n \Omega SO(3) \right) \times \Omega \mathcal{C}(\Sigma),
\] (1.7)

where the symbol \( \Omega A \) stands for the loop-space of \( A \), and where \( \mathcal{C}(\Sigma) \) is a space that (vaguely speaking) labels and topologizes the relative configurations (sites) of
the prime-manifolds $\bar{\Sigma}_i$ in the connected sum $\bar{\Sigma}$. In general its determination seems to be difficult and we refer to [McCu][He-L] for more elaborate treatements. What interests us here is that consequently we have the direct-product structure for the homotopy groups $\pi_k$ ($k \geq 0$)

$$\pi_k (DF(\Sigma)) = \left( \prod_{i=1}^l \pi_k (DF(\bar{P}_i)) \right) \times \left( \prod_{i=l+1}^n \pi_{k+1}(SO(3)) \right) \times \pi_{k+1}(C). \quad (1.8)$$

This result rests on the restriction to the asymptotically trivial diffeomorphisms $DF$, for only in this case there is no topological intertwinemnt of the diffeomorphisms with support inside the prime factors $\bar{\Sigma}_i$ (called internal diffeomorphisms) with those of general support (called external diffeomorphisms). This would fail to hold if one considered $D$ instead of $DF$ as a simple counterexample shows (see remark 1 in [He-McCu]). As far as the homotopy groups are concerned, we now see how the determination of $Q(\Sigma)$’s topology is directly related to the determination of those for its prime factors $Q(\Sigma_i)$.

Section 2. Some preparatory material

At the end of this section and the whole of Section 3 we will perform explicit calculations for $\pi_0(DF(\Sigma))$ and $\pi_k(DF(\Sigma))$ ($k \geq 1$) respectively, where $\Sigma$ is taken from the subclass of homogeneous spherical primes. In order not to overload the actual proofs, we shall establish some preparatory material first.

A Closer Look at Diffeomorphisms

The strategy for our calculations is very simple: there are hard theorems available concerning the topological structure of $D(\Sigma)$ for $\Sigma$ prime. For convenience we collect some of them in Appendix 2. Most interesting for us is the question of validity of a conjecture made by Hatcher [Ha2] (abbreviated HC, in the literature also referred to as the generalized Smale conjecture). It asserts that for spherical primes the spaces of diffeomorphisms and isometries are homotopy equivalent. Restricted to the 3-sphere, this is known as the Smale conjecture which has been shown to hold in [Ha4]. Some of the proofs presented in this paper require the validity of (HC), the status of which has been indicated in the table presented at the end.
From now on we shall sometimes drop the explicit reference to the manifold \( \Sigma \) by just writing \( DF, D_\infty \), etc., without any argument. We obtain information about \( DF \) by relating it to \( D \) by some standard fibrations which we shall now describe. First note that orientable 3-manifolds are always parallelizable. Their bundle of oriented frames, \( F(\Sigma) \), is thus topologically the product \( \Sigma \times GL^+(3, R) \). The different diffeomorphism groups in (1.1) are topologically related by the following three principal fibre bundles (\( u \) still denotes the fixed frame at \( \infty \)):

\[
\begin{align*}
DF & \xrightarrow{i} D \\
\downarrow \quad \hat{p} & \quad \hat{p}(\phi) := T\phi(u) = (\phi(\infty), \phi_*|_{\infty}(u)) \\
F(\Sigma) & 
\end{align*}
\]

\[
\begin{align*}
D_\infty & \xrightarrow{i} D \\
\downarrow p & \quad p(\phi) := \phi(\infty) \\
\Sigma & 
\end{align*}
\]

\[
\begin{align*}
DF & \xrightarrow{\tilde{i}} D_\infty \\
\downarrow \tilde{p} & \quad \tilde{p}(\phi) := \phi_*|_{\infty} \\
SO(3) & 
\end{align*}
\]

As a first application we introduce the concept of spinoriality of a manifold \( \Sigma \). Associated with (2.3) is the fibration

\[
\begin{align*}
DF(\Sigma)/D_F^0(\Sigma) & \xrightarrow{i} \mathcal{G}(\Sigma) \\
\downarrow \tilde{p} & \\
SO(3) & 
\end{align*}
\]

where \( \mathcal{G}(\Sigma) \) is the symmetry group defined in (1.2). From the associated exact homotopy sequence we infer

\[
\begin{align*}
0 & \rightarrow \pi_1(\mathcal{G}) \xrightarrow{\tilde{p}_*} \mathbb{Z}_2 \rightarrow \pi_0(D_F) \rightarrow \pi_0(\mathcal{G}) \rightarrow 0 
\end{align*}
\]

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which, by injectivity of $\tilde{p}_*$, tells us that there are only two possibilities:

1. $\pi_1(\mathcal{G}) = 0$ hence $\pi_0(D_F)$ is a $\mathbb{Z}_2$ extension of $\pi_0(\mathcal{G})$. In this case (2.4) is non-trivial and $\mathcal{G}$ is given by $\{\pi_0(D_F) \times SU(2)\}/\mathbb{Z}_2$. Here the $\mathbb{Z}_2$ is generated by $(-1,-1)$ where the $-1$ in the left factor generates some $\mathbb{Z}_2$ in $\pi_0(D_F)$ and the $-1$ in the right factor generates the centre of $SU(2)$.

2. $\pi_1(\mathcal{G}) = \mathbb{Z}_2$ and $\pi_0(D_F) = \pi_0(\mathcal{G})$, in which case (2.3) is trivial and $\mathcal{G}(\Sigma)$ a direct product $\pi_0(D_F) \times SO(3)$.

In the first case, $\mathcal{G}(\Sigma)$ contains $SU(2)$ but not $SO(3)$ as a subgroup. Let a manifold, $\Sigma$, for which this is the case, be for obvious reasons called spinorial. Whether a manifold is spinorial is a purely topological question and has been decided for all known prime manifolds (see Theorem 2 below). In view of Corollary 1 (below) this provides sufficient information for the general case. From (2.3) one obtains

$$0 \to \pi_1(D_F) \to \pi_1(D_\infty) \xrightarrow{\tilde{p}_*} Z_2 \xrightarrow{\partial_*} \pi_0(D_F) \to \pi_0(D_\infty) \to 1 \quad (2.6)$$

$$\Sigma \text{ is spinorial } \iff \partial_* \text{ into } \iff \pi_1(D_F(\bar{\Sigma})) \cong \pi_1(D_\infty(\bar{\Sigma}))$$
$$\Sigma \text{ is not spinorial } \iff \tilde{p}_* \text{ onto } \iff \pi_0(D_F(\bar{\Sigma})) \cong \pi_0(D_\infty(\bar{\Sigma})) \quad (2.7)$$

For later application let us cite the following known results in form of two lemmas, a theorem and a corollary.

**Lemma 1.** If $\bar{\Sigma}$ is an irreducible prime manifold (i.e. a prime different from $S^1 \times S^2$), then the bundle projection $p$ of (2.2) has the property that $p_*$ maps $\pi_1(D)$ onto the centre $C$ of $\pi_1(\bar{\Sigma})$.

**Proof.** That $p_*(\pi_1(D(\bar{\Sigma}))) \subset C$ follows from Corollary 5.22 of ref. [McCa] for arbitrary $\bar{\Sigma}$ (his/her remark 5.24). Surjectivity onto $C$ was shown in section V of ref. [Wi] under the hypothesis that homotopy implies isotopy of diffeomorphisms. According to Theorem A1 in Appendix 2 this is now known to hold for all primes in our table.

**Lemma 2.** If $\bar{\Sigma}$ is not spinorial then the bundle projection $\hat{p}$ of (2.1) has the property that $\hat{p}_*(\pi_1(D))$ contains $(1,-1) \in \pi_1(F(\bar{\Sigma})) \cong \pi_1(\bar{\Sigma}) \times \mathbb{Z}_2$, where $1$ is the identity in $\pi_1(\bar{\Sigma})$ and $-1$ the generator of $\mathbb{Z}_2$. 

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Proof. This is proven in Lemma 2.1 of ref. [Fr-Wi]. The simple and instructive proof is worth a look at this point. Let $s \mapsto R_s \in D_0^0$ (the connected component of $D_\infty$) be the rotation of a 3-disc $D_1$ against a slightly larger, concentric 2-sphere $S_2$ (compare Appendix 1). Non-spinoriality implies the existence of a path $s \mapsto \phi_s \in D_0^0$ so that $\phi_0 = R_1$ and $\phi_1 = id$. The product path, $\gamma$, defined by

$$\gamma := \begin{cases} R_{2s} & \text{for } s \in [0, \frac{1}{2}] \\ \phi_{(2s-1)} & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

defines a loop at $id$ in $D_0^0$ satisfying $\tilde{p}_*([\gamma]) = -1$. Moreover, from the end of the exact sequence for (2.2) (shown in (3.3)), we have $p_*([\gamma]) = p_* \circ i_*([\gamma]) = 1$, so that $\tilde{p}_*([\gamma]) = (p_*([\gamma]), \tilde{p}_*([\gamma])) = (1, -1) \in \pi_1(\tilde{\Sigma}) \times \mathbb{Z}_2$.

Theorem 2. The only non spinorial prime manifolds are the lens spaces, $L(p, q)$, and the handle $S^1 \times S^2$.

Proof. Non-spinoriality for $L(p, q)$ and $S^1 \times S^2$ can actually be visualized. The demonstration of this fact is deferred to the Appendix 1. In ref. [He], chapter 4.3, Theorem 1 implies spinoriality for the following prime manifolds: Those with infinite fundamental group different from $S^1 \times S^2$ (the so-called $K(\pi, 1)$’s), and those with finite fundamental group which have a non-cyclic 2-Sylow subgroup. The remaining cases consist of some $S^3/G$ with non cyclic $G$ for which HC is known to hold true. Using this fact, they where shown to be spinorial in ref. [Fr-Wi], Theorem 2.2 and the remark following Corollary 2.2. For the latter one needs to add that in the meantime the validity of HC for the spaces $S^3/D_{4m}^* m \geq 2$ has been shown in ref. [McC-C-R].

Due to our strategy to be explicit in the case of homogeneous $S^3/G$’s, and anticipating some results from the next subsection, we want to give a simple proof of spinoriality for non-cyclic $G$ under the hypothesis of validity of HC. (More precisely, it only relies on one of its implications, namely that $\pi_1(D)$ and $\pi_1(\text{Isom})$ have the same number of generators. We are, however, not aware of a single case where this but not the full HC is known to hold.) By discreteness of $S(G)$ we learn from (2.12) that $\tilde{p}_*$ maps $Z_2 \cong \pi_1(\text{Isom}(G))$ injectively into $G = \pi_1(S^3/G)$. Since $\tilde{p}$ is just the restriction of $p$ in (2.2), which also appears as the first component of $\tilde{p}$ in (2.1), we learn that $\tilde{p}_*(\pi_1(D))$ contains the element $(-1, 1)$ or $(-1, -1)$ [by Lemma 1, $\tilde{p}_*$ maps $\pi_1(D)$ into $Z_2 \times Z_2 = \text{centre} \times \pi_1(\text{SO}(3))$]. By HC, $\pi_1(D) \cong \pi_1(\text{Isom}) = Z_2$ so that $\tilde{p}_*(\pi_1(D))$ cannot in addition contain $(1, -1)$ by mapping only one generator. Lemma 2 then implies spinoriality.

Corollary 1. A closed, oriented, connected 3-manifold $\Sigma$ is non-spinorial, if and only if its prime decomposition consists entirely of lens spaces and handles.

Proof. In view of Theorem 2 we need to prove that such a manifold is non-spinorial, if and only if none of its prime factors is spinorial. For this we assume that $\Sigma$ is
the connected sum of \( n \) prime factors which build up \( \bar{\Sigma} \) as follows (we follow ref. [McCu]): Take a 3-sphere and on it \( n+1 \) closed, disjoint 3-discs \( D_i, 0 \leq i \leq n \), with 2-sphere boundaries \( \partial D_i = S_i \). We take \( D_0 \) as a neighbourhood of \( \infty \), remove the interiors of \( B_i \) for \( i \geq 1 \), and construct \( \bar{\Sigma} \) by identifying the \( n \) 2-sphere boundaries \( S_i \ (1 \leq i \leq n) \) with the corresponding boundaries that one obtains by removing an open 3-disk on each prime factor respectively. In \( \Sigma \) we now connect \( S_i \) with \( S_{i+1} \) for each \( 1 \leq i \leq n - 1 \) by a thin cylindrical tube (topology \( I \times S^1 \)) and obtain a single new 2-sphere, \( S \) (the connected sum of all \( S_i \) for \( i \geq 1 \)), which is isotopic to \( S_0 \). A \( 2\pi \)-rotation parallel to \( S_0 \) (i.e. a rotation parallel to \( S_0 \) and an arbitrarily close concentric one, as explained in Appendix 1) is also isotopic to such a rotation parallel to \( S \). The latter one may be chosen to have the connecting tubes as axis (i.e. the rotation acts only on the \( S^1 \) part of \( I \times S^1 \)). Shrinking the tubes to zero diameter defines an isotopy to rotations parallel to each \( S_i \) for \( 1 \leq i \leq n \) (compare the remark after Theorem 1, chapter 4.3, of reference [He]). From (1.8) for \( k = 0 \) and Theorem 2 we infer that this diffeomorphism corresponds to the identity element of \( \pi_0(D_F(\bar{\Sigma})) \), if and only if the connected sum does not contain a single spinorial prime.

Corollary 1 suggests that non-spinorial manifolds are somewhat more special than spinorial ones. Note, however, that the non-spinorial primes still suffice to build up (in a non-unique fashion) manifolds of any given homology by taking connected sums. Non-spinorial manifolds are, therefore, not homologically special. Note also that the lens spaces \( L(p, q) \) are homogeneous, if and only if \( q = \pm 1 \mod p \). These are therefore the only non-spinorial primes in the class of homogeneous space forms to which we specialize in the following subsection. Finally, we note that the existence of spinorial manifolds has been already used in the literature to speculate about the existence about certain spinorial states in quantum gravity [Fr-So]. We will come back to this point in Observation 1 of Section 4.

**Homogeneous Spherical Primes**

In this subsection we consider a special class of elliptic spaces, that is, spaces of the form \( S^3/G \), where \( G \) is a finite subgroup of \( SO(4) \) with free action on \( S^3 \). We identify \( S^3 \) with the group manifold of \( SU(2) \) with its standard bi-invariant (round) metric and use the isomorphism \( SO(4) \cong (SU(2) \times SU(2))/Z_2 \), where the \( Z_2 \) is generated by \((-1, -1)\). Elements of \( SO(4) \) are then written as \( Z_2 \)-equivalence
classes \([g, h]\) with \(g, h \in SU(2)\), and elements of \(S^3/G\) as \(G\)-equivalence classes \([x]\) with \(x \in S^3\). We can now write down the (left) \(SO(4)\)-action on \(S^3\)

\[
SO(4) \times S^3 \rightarrow S^3, \quad [g, h] \times x \mapsto g \cdot x \cdot h^{-1}.
\]

(2.8)

\(SO(4)\) has a left- and right- \(SU(2)\) subgroup, given by the sets \(SU(2)_L := [SU(2), 1]\) and \(SU(2)_R := [1, SU(2)]\) respectively. It also has an obvious diagonal- \(SO(3)\) subgroup given by all elements of the form \([g, g]\). We call it \(SO(3)_D\). If \(N_{SO(4)}(G)\) denotes the normalizer of \(G\) in \(SO(4)\), it is easy to see that the residual isometry group acting on \(S^3/G\) is given by \(Isom(S^3/G) = N_{SO(4)}(G)/G:\)

\[
N_{SO(4)}(G) \times S^3/G \rightarrow S^3/G, \quad [g, h] \times [x] \mapsto [g \cdot x \cdot h^{-1}].
\]

(2.9)

It acts transitively on \(S^3/G\), if and only if \(G\) is a finite, freely acting subgroup of either \(SU(2)_L\) or \(SU(2)_R\). As subgroups of \(SU(2)\) these are given by \(Z_p, D_{4n}^*\) for \(n \geq 2\), \(T^*\), \(O^*\) and \(I^*\) which denote the cyclic group of order \(p\) and the \(SU(2)\)-double covers (denoted by \(*\)) of the symmetry groups of the \(n\)-prism, tetrahedron, octahedron and icosahedron of orders \(4n\), 24, 48 and 120 respectively. We shall now restrict to these cases and may choose \(G = G_R\), that is, we may choose \(G\) to sit inside \(SU(2)_R\), for the manifold obtained by using \(SU(2)_L\) would be certainly homeomorphic.

For our topological purposes we may indeed restrict ourselves to the right action for identifying \(S^3\) to \(S^3/G\). But as prime manifolds in the oriented category one needs to distinguish those obtained by right and left identifications. This follows from the fact that a homogeneous \(S^3/G\) does not allow for any orientation reversing diffeomorphism if \(G \neq Z_i\) for \(i \leq 2\) [Wi] and uniqueness of the prime decomposition in the oriented category. For those \(G\) this implies that \(S^3/G \sqcup S^3/G\) is not homeomorphic to (the minus sign indicates reversed orientation) \(S^3/G \sqcup (-S^3/G)\). On the other hand, the diffeomorphism \(\phi\) that relates the two manifolds obtained by right and left identifications is just given by \([p]_R \mapsto \phi([p]_R) := [p^{-1}]_L\), where \(p \in S^3 \cong SU(2)\) and \([·]_R, [·]_L\) denote the right and left cosets respectively. But this is an orientation reversing diffeomorphism.

For all but the cyclic groups \(Z_p\) of odd order, \(G\) contains the centre \(Z_2\) of \(SU(2)\). Standard properties of groups and their quotients imply the following homomorphism equivalences:

\[
Isom(S^3/G) \cong SO(3) \times \frac{NSU(2)(G)}{G} \cong SO(3) \times \frac{NSO(3)(G/Z_2)}{G/Z_2}.
\]

(2.10)

Let \(S(G)\) denote the (unique) conjugacy class of stabilizer subgroups for \(Isom(G)\)'s action on \(S^3/G\). We shall usually identify it with the stabilizer subgroup at \([e]\) \(\in S^3/G\), where \(e\) is the identity element of \(S^3 \cong SU(2)\). One easily
shows that \( S(G) \cong N_{SU(2)}(G)/Z_2 \subset SO(3)_D \), where \( Z_2 \) is the centre of \( SU(2) \). For \( G \neq Z_p \), where \( p \) odd, this can be written as \( S(G) \cong N_{SO(3)}(G/Z_2) \). As a closed subgroup of \( SO(3) \) it must be either discrete, or a finite disjoint union of circles, or the whole of \( SO(3) \). The last case is realized for \( G = Z_2 \) and the second case for \( G = Z_p \ (p > 2) \), where \( S(G) \) is given by two circles in two perpendicular planes (viewed as subset of unit quaternions) one of which is the unique 1-parameter subgroup containing \( Z_p \). In the first case we have explicitly (note: \( D_{4n} \) is the projection of \( D_{8n}^* \) into \( SO(3) \)):

\[
S(G) = \begin{cases} 
O & \text{for } G = D_4^*, T^* \text{ and } O^* \\
D_{4n} & \text{for } G = D_{4n}^* \\
I & \text{for } G = I^*
\end{cases}
\]  

(2.11)

By restricting the total space of the bundle (2.2) to the isometries, and, accordingly, the fibre \( D_\infty \) to \( S(G) \), we obtain the principal-bundle

\[
\begin{array}{ccc}
S(G) & \xrightarrow{i} & Isom(S^3/G) \\
\downarrow{\bar{p}} & & \downarrow{\bar{p}} \\
S^3/G & & [g \cdot x \cdot h^{-1}],
\end{array}
\]

(2.12)

where \([g, h] \in N_{SO(4)}(G)\) and \( x \in S^3 \) represents the point \( \infty = [x] \) on \( S^3/G \). The projection \( \bar{p} \) is just the restriction of \( p \) in (2.2) to \( Isom(S^3/G) \). Next we prove a lemma which we shall use for the calculation of \( \pi_k(D_F) \), \( k \geq 2 \), later on.

**Lemma 3.** If \( G \neq Z_2 \) is a non-trivial finite subgroup of \( SU(2) \) which acts freely on \( S^3 \) via the standard orthogonal action, and if HC holds for \( S^3/G \), then the projection map \( p \) from bundle (2.2) induces isomorphisms

\[
p_* : \pi_k(D(S^3/G)) \longrightarrow \pi_k(S^3/G) \quad \forall k \geq 2.
\]

**Proof.** The validity of HC implies that the inclusion \( I : Isom(S^3/G) \hookrightarrow D(S^3/G) \) induces isomorphisms \( I_* : \pi_k(Isom(S^3/G)) \rightarrow \pi_k(D(S^3/G)) \forall k \). Since for \( G \neq Z_2 \) we have \( \pi_k(S(G)) = 0 \forall k \geq 2 \), the homotopy exact sequence for (2.12) implies that \( \bar{p}_* : \pi_k(Isom(S^3/G)) \rightarrow \pi_k(S^3/G) \) is an isomorphism \( \forall k \geq 2 \). On the other hand, \( \bar{p} = p \circ I \), and hence \( \bar{p}_* = p_* \circ I_* \), which proves the claim \( \bullet \)
Calculation of $\pi_0(D_F(S^3/G))$.

For the spherical primes $S^3/G$, the zeroth homotopy group of $D_F$ has already been calculated in [Wi]. Here we give a separate calculation in the homogeneous subclass.

We start with the lens spaces $L(p,1)$. Lemma 1 applied to the exact homotopy sequence for (2.2) implies that its very last two group entries are isomorphic: $\pi_0(D_\infty) \cong \pi_0(D)$. Theorem 2 and (2.7) then imply $\pi_0(D_F) \cong \pi_1(D) \cong \pi_0(\text{Isom})$ (Theorem A2, Appendix 2). This is isomorphic to $Z_2$ for $p > 2$, and to the trivial group for $p \leq 2$.

For $G$ non-cyclic, we have that $Z_2 := \{1, -1\} \subset SU(2)$ is the centre and the centralizer of $G$ in $SU(2)$. This implies that the action of $S(G)$ on $S^3/G$, $[p] \mapsto [gpg^{-1}]$, is effective on the coset $[e] \subset S^3$ ($e=$identity in $SU(2) \cong S^3$) and therefore on the fundamental group $\pi_1(S^3/G, [e])$. Thus no nontrivial element in $S(G)$ lies in the identity component in $D_\infty$ (we take $\infty = [e]$), so that $\pi_0(D_\infty)$ contains the subgroup $S(G)$. We wish to show that it actually saturates all of $\pi_0(D_\infty)$. This can be done if $\pi_0(D) \cong \pi_0(\text{Isom})$ (see Theorem A2), for then we infer from Lemma 1 that the last part of the sequence for (2.2) just says that $\pi_0(D_\infty)$ is a $G/Z_2$ - extension of $\pi_0(D)$, and therefore ($|\cdot|$ denotes the order of the group $\cdot$):

$$|\pi_0(D_\infty)| = \frac{|G|}{2} |\pi_0(D)| = \frac{|G|}{2} \frac{|NSO(3)(G/Z_2)|}{|G/Z_2|} = |NSO(3)(G/Z_2)|. \quad (2.13)$$

Next, by spinoriality (Theorem 1), we know from (2.6) and (2.7) that $\pi_0(D_F)$ is a $Z_2$ - extension of $D_\infty$. We wish to show that it must be the $SU(2)$-double cover $NSU(2)(G)$. To do this in detail requires some notation.

Let $NSU(2)(G) = \{g_1, \cdots, g_N\}$, $G = \{g_1, \cdots, g_K\}$, $K \leq N$, where $g_1 = e$, and $\{\theta_1, \cdots, \theta_N\} \subset su(2)$ (the Lie-algebra of $SU(2)$) such that $\exp(\theta_i) = g_i$ (we take indices $i, \cdots$ to run from 1 to $N$, and $a, \cdots$ from 1 to $K$). There is a homomorphism $\sigma : NSU(2)(G) \to P_K$ into the permutation group of $K$ objects, $g_i \mapsto \sigma(g_i) =: \sigma_i$, defined by $g_\alpha g_i^{-1} = g_i^{-1} g_{\sigma_i(a)}$. Let $r : S^3 \to R$ be the distance function from $e$ (with respect to the bi-invariant metric) and $\rho$ a $C^\infty$ step-function, such that $\rho(r) = 0$ for $r \leq \epsilon$ and $\rho(r) = 1$ for $r \geq 2\epsilon$. We then define $\lambda := \rho \circ r$. Further, we let $B^2_\epsilon$ and $B^1_\epsilon$ be the closed $2\epsilon$- and $\epsilon$- balls around $e = g_1$ respectively. We right-translate them to $2\epsilon$- respectively $\epsilon$- balls centred at $g_a$, i.e., $B^2_a = B^2_\epsilon \cdot g_a$ and $B^1_a = B^1_\epsilon \cdot g_a$ for each $a$. We further take $\epsilon$ small enough for $B^2_a$ to lie within a regular neighbourhood
with respect to the covering $S^3 \to S^3/G$. In particular, they are all disjoint. Their projections into $S^3/G$ are called $B^2$ and $B^1$.

A diffeomorphic action of $N_{SU(2)}(G)$ on $S^3 - \bigcup_{a=2}^{K} B^2_a$ is then defined via

$$T_i : p \mapsto \exp(\lambda(p)\theta_i) \cdot p \cdot \exp(-\lambda(p)\theta_i).$$

It is easy to see that: (i) $T_i$ leaves $B^1_1$ pointwise fixed, (ii) $T_i$ leaves $B^2_i - B^1_1$ invariant, (iii) $T_i$ leaves $S^3 - \cup_{a=1}^{K} B^2_a$ invariant and (iv) $T_i$ maps $\partial B^2_a$ onto $\partial B^2_{\sigma_i(a)}$. In fact, any point in a small closed outer collar-neighbourhood of $\partial B^2_1$ can be written as $pg_a$, where $p$ is from such a neighbourhood of $\partial B^2_1$. Under $T_i$ it is mapped according to

$$pg_a \mapsto g_i pg_a g_i^{-1} = T_i(p) g_{\sigma_i(a)}.$$

We now use this formula to smoothly extend the map to all of $S^3$, that is, for any point $pg_a \in B^2_a$, where $p \in B^2_1$, we set

$$T_i(pg_a) = T_i(p) g_{\sigma_i(a)},$$

where $T_i(p)$ is defined above. By construction, this action projects onto $S^3/G$ and acts on the fundamental group of $S^3/G$ by simple conjugation: $p \mapsto g_i pg_i^{-1}$; for, as before, a path on $S^3$ from $e$ to the $g_a$ is mapped to a path from $e$ to $g_{\sigma_i(a)}$. Only the centre $Z_2$ of $N_{SU(2)}(G)$ acts trivially on the fundamental group. However, the action of this $Z_2$ just corresponds to a relative $2\pi$-rotation of the spheres $\partial B^2_1$ and $\partial B^1_1$, which, by spinoriality, is not isotopic to the identity keeping a frame at $[e]$ fixed. We have thus shown that $\pi_0(D_F(S^3/G))$ contains, and is hence equal to $N_{SU(2)}(G)$. 

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Section 3. Calculation of $\pi_k(D_F(S^3/G))$ for $k \geq 1$

Within the class of spaces considered here, $\Sigma = S^3 (G = Z_1)$ and $\Sigma = RP^3 (G = Z_2)$ receive a special status due to them being group manifolds. This allows us to make a somewhat more concise statement than in the other cases. We have the following

**Lemma 4.**

a): If $\Sigma$ is a topological group, (2.2) is trivial.

b): For $\Sigma \cong RP^3$ or $\cong S^3$, (2.3) is trivial.

**Proof.**

a): Define a global section $\sigma : \Sigma \rightarrow D$, $x \mapsto L_x$ (=left translations). We have $p \circ \sigma = id|_{\Sigma}$. A global trivialisation is given by

$$\phi^{-1} : \Sigma \times D_{\infty} \rightarrow D; \quad (x, h) \mapsto L_x \circ h$$

$$\phi : D \rightarrow \Sigma \times D_{\infty}; \quad g \mapsto \left(p(g), L^{-1}_{p(g)} \circ g\right)$$

b): Define a global section $\sigma : SO(3) \rightarrow D_{\infty}$, $\alpha \mapsto \text{Ad}(\alpha)$, where $\text{Ad}(\alpha)(p) = \alpha p \alpha^{-1} \forall p \in \Sigma$. We have that $\text{Ad}(\alpha)|_e = \alpha$, and hence $\tilde{p} \circ \sigma = id|_{SO(3)}$. A corresponding trivialisation is given by

$$\phi^{-1} : SO(3) \times D_F \rightarrow D_{\infty}; \quad (\alpha, h) \mapsto \text{Ad}(\alpha) \circ h$$

$$\phi : D_{\infty} \rightarrow SO(3) \times D_F; \quad g \mapsto \left(\tilde{p}(g), \text{Ad}^{-1}(\tilde{p} \circ g)\right) \bullet$$

By HC we know that $D(S^3)$ and $D(RP^3)$ have the homotopy type of $SU(2) \times SO(3)$ and $SO(3) \times SO(3)$ respectively. Hence we have the

**Theorem 2.** $Q(\Sigma)$ has altogether trivial homotopy groups for $\Sigma = S^3$ or $\Sigma = RP^3$.

**Proof:** By the previous lemma, the bundles (2.2) and (2.3) are product bundles. We thus have for all $k \geq 0$:

$$\pi_k(D) \cong \pi_k(\Sigma) \times \pi_k(D_{\infty}) \quad (3.1)$$

$$\pi_k(D_{\infty}) \cong \pi_k(D_F) \times \pi_k(RP^3) \quad (3.2)$$

Setting $\Sigma$ either equal to $S^3$ or $RP^3$, and inserting the space Isom for $D$ proves the claim $\bullet$
Calculation of $\pi_1(D_F(\Sigma))$

Since for $\Sigma = S^3/G$ we have $\pi_2(\Sigma) = 0$, the exact sequences for (2.2) and (2.1) end as follows:

\[ 0 \rightarrow \pi_1(D_\infty) \overset{i^*}{\rightarrow} \pi_1(D) \overset{p^*_e}{\rightarrow} G \rightarrow \pi_0(D_\infty) \rightarrow \pi_0(D) \rightarrow 1 \quad (3.3) \]

\[ 0 \rightarrow \pi_1(D_F) \overset{\hat{i}^*}{\rightarrow} \pi_1(D) \overset{\hat{p}^*_e}{\rightarrow} G \times Z_2 \rightarrow \pi_0(D_F) \rightarrow \pi_0(D) \rightarrow 1 \quad (3.4) \]

We shall first deal with $G$ non-cyclic. For non-cyclic $G$ we have $centreG = Z_2$ and by $HC$ $\pi_1(D) = \pi_1(\text{Isom}) = Z_2$, so that Lemma 1 tells us that $p_*$ in (3.3) is an isomorphism, and, therefore, $\pi_1(D_\infty) = 0$. Spinoriality together with formulae (2.7) then imply $\pi_1(D_F) = 0$.

For $G = Z_p$, $\Sigma \cong L(p, 1)$ is non-spinorial so that the Lemmas 1 and 2 imply ($G = centreG$) that $p_*$ in (3.4) is onto. HC then implies that (3.4) reduces to

\[ 0 \rightarrow \pi_1(D_F) \rightarrow Z \times Z_2 \overset{\hat{p}^*_e}{\rightarrow} Z_p \times Z_2 \rightarrow 0, \quad (3.5) \]

where $\pi_1(D_F) \cong ker \, \hat{p}_*$. For even $p$ it is immediate that $ker \, \hat{p}_*$ cannot contain the $Z_2$ (within $Z \times Z_2$), for, otherwise, $\hat{p}_*$ cannot be onto. For $p$ odd we argue as follows: $N_{SU(2)}(Z_p)$ contains the (unique) circle group in which $Z_p$ lies. We consider the loop in $SO(3)_D$ obtained by tracing through half that circle in the right- and left- $SU(2)$ at the same time. This loop is easily seen to generate $(p, -1) \in Z \times Z_2 \cong \pi_1(D)$, and, since it fixes $[e]$ and induces a $2\pi$-rotation in the tangent space, has an image $(a, -1) \in Z_p \times Z_2$ under $\hat{p}_*$ for some $a$. Hence, since $p$ is odd, $(1, -1) \in Z \times Z_2$ cannot be in the kernel of $\hat{p}_*$. We have thus shown that $Z_2 \notin ker \, \hat{p}_*$ so that $\pi_1(D_F) \cong Z$. 

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Calculation of $\pi_k(D_F)$ for $k \geq 2$

A typical piece of the exact sequence for (2.2) looks like:

$$\cdots \to \pi_{k+1}(D) \xrightarrow{p^{(k+1)}_*} \pi_{k+1}(\Sigma) \to \pi_k(D_\infty) \to \pi_k(D) \xrightarrow{p^{(k)}_*} \pi_k(\Sigma) \to \cdots$$  \hspace{1cm} (3.6)

By Lemma 3, $p^{(k)}_*$ is an isomorphism for $k \geq 2$, so that $\pi_k(D_\infty) = 0 \ \forall k \geq 2$. The exact sequence for (2.3) then implies $\pi_k(D_F) \cong \pi_{k+1}(SO(3)) \ \forall k \geq 2$.

Summary

According to (1.5) we can summarize the homotopy groups of the (connected) configuration space $Q(S^3/G)$ for $S^3/G$ homogeneous:

$$\pi_k(Q(RP^3)) \cong 0 \ \forall k \geq 1$$  \hspace{1cm} (3.7)

$$\pi_k(Q(L(p, 1))) \cong \begin{cases} Z_2 & \text{for } k = 1 \\ Z & \text{for } k = 2 \\ \pi_k(S^3) & \text{for } k \geq 3 \end{cases}$$  \hspace{1cm} (3.8)

$$\pi_k(Q(S^3/G)) \cong \pi_k(S^3/N_{SU(2)}(G)) \ \forall k \geq 1, \ \text{and } G \text{ non-cyclic}$$  \hspace{1cm} (3.9)

In particular the last equation is very suggestive. We stress, however, that we did not establish a homotopy equivalence $Q(S^3/G) \sim S^3/N_{SU(2)}(G)$.  

Section 4. Results and some Observations

The table summarizes the results for the homotopy groups of the configuration spaces $Q(\Sigma)$. The last column comprises all homotopy groups higher than the second, i.e., $k \geq 3$. The table is divided into five horizontal blocks of which the first three represent the spherical space forms, the fourth the single handle-manifold (wormhole), and the fifth the sufficiently large $K(\pi, 1)$ primes. The calculation for the last two blocks is almost trivial in view of the strong results obtained in [Ha3]

| Prime $\Sigma$ | HC | $\pi_1(Q(\Sigma))$ | $\pi_2(Q(\Sigma))$ | $\pi_k(Q(\Sigma))$ |
|----------------|----|---------------------|---------------------|---------------------|
| $S^3/D_8^*$ | *  | $O^*$               | 0                   | $\pi_k(S^3)$        |
| $S^3/D_{4n}$ | *  | $D_{8n}^*$           | 0                   | $\pi_k(S^3)$        |
| $S^3/T^*$  | ?  | $O^*$               | 0                   | $\pi_k(S^3)$        |
| $S^3/O^*$  | w  | $O^*$               | 0                   | $\pi_k(S^3)$        |
| $S^3/I^*$  | ?  | $I^*$               | 0                   | $\pi_k(S^3)$        |
| $S^3/D_8^* \times Z_p$ | *  | $Z_2 \times O^*$   | $Z$                 | $\pi_k(S^3 \times S^3)$ |
| $S^3/D_{4n}^* \times Z_p$ | *  | $Z_2 \times D_{8n}^*$ | $Z$                 | $\pi_k(S^3 \times S^3)$ |
| $S^3/T^* \times Z_p$ | ?  | $Z_2 \times O^*$   | $Z$                 | $\pi_k(S^3 \times S^3)$ |
| $S^3/O^* \times Z_p$ | w  | $Z_2 \times O^*$   | $Z$                 | $\pi_k(S^3 \times S^3)$ |
| $S^3/I^* \times Z_p$ | ?  | $Z_2 \times I^*$   | $Z$                 | $\pi_k(S^3 \times S^3)$ |
| $S^3/D_{2k(2n+1)}^* \times Z_p$ | *  | $Z_2 \times D_{8(2n+1)}^*$ | $Z$                 | $\pi_k(S^3 \times S^3)$ |
| $S^3/T^*_{8,3k} \times Z_p$ | ?  | $O^*$               | $Z$                 | $\pi_k(S^3 \times S^3)$ |
| $L(p, q_1)$ | w  | $Z_2$               | $Z$                 | $\pi_k(S^3)$        |
| $L(p, q_2)$ | w, * | $Z_2 \times Z_2$   | $Z \times Z$       | $\pi_k(S^3 \times S^3)$ |
| $L(p, q_3)$ | w  | $Z_4$               | $Z \times Z$       | $\pi_k(S^3 \times S^3)$ |
| $L(p, q_4)$ | w  | $Z_2$               | $Z \times Z$       | $\pi_k(S^3 \times S^3)$ |
| $RP^3$     | *  | 1                   | 0                   | 0                   |
| $S^3$       | *  | 1                   | 0                   | 0                   |
| $S^2 \times S^1$ |    | $Z_2 \times Z_2$   | $Z$                 | $\pi_k(S^3 \times S^2)$ |
| $K(\pi, 1)_{sl}$ |    | $Aut_{\pi}^{Z^2}(\pi_1)$ | 0                   | $\pi_k(S^3)$        |
and [Ha1] respectively. $\text{Aut}^Z_2$ in the last block denotes a $Z_2$ extension (according to (2.6) imposed by spinoriality) of the orientation preserving automorphisms of $\pi_1$.

The calculations for the spherical space forms depend on the validity of the Hatcher conjecture (HC), which is not known to hold in all cases. The calculations of $\pi_1(Q)$, however, depend only on a weak implication thereof, namely $\pi_0(D_F) \cong \pi_0(\text{Isom})$. Since there are cases where this weak form of HC but not HC itself is known to hold, we indicated their status of validity separately in the second column (compare Appendix 2, Theorems A1 and A2). Here, an asterisk, (*), denotes the validity of HC, a $w$ the validity of the weak form only, and the question mark, (?), that no such result is known to us. Assuming $(w)$, the groups $\pi_1(Q(S^3/G))$ were first calculated in [Wi]. Subsequently the table was completed to the present form in [Gi1].

The first block contains the homogeneous spherical primes with non-cyclic fundamental group, the second the non-homogeneous ones. Here, the order of the additional cyclic group, $p$, has to be coprime to the order of the group the $Z_p$ is multiplied with, and $\geq 2$ in the first five, and $\geq 1$ in the remaining two cases. The third block contains all the spherical primes with cyclic fundamental group $Z_p$, otherwise known as the lens spaces $L(p,q)$, where $q$ has to be coprime to $p \geq 2$. Here, $q_1$ stands for $q = \pm 1 \mod p$, $q_2$ for $q \neq \pm 1 \mod p$ and $q^2 = 1 \mod p$, $q_3$ for $q^2 = -1 \mod p$, and $q_4$ for the remaining cases. Amongst all $L(p,q_2)$ are those of the form $L(4n,2n-1), n \geq 2$. For those the (*) is valid in the second column and $(w)$ for all others. Finally, $RP^3$ and $S^3$ are listed separately. Together with $L(p,q_1)$ they comprise the homogeneous spaces in the third block, and, taken together with the first block, all the homogeneous spherical primes in the list.

Some Observations

In standard canonical quantization one regards the wave function as a section of a possibly non-trivial complex line bundle over some configuration space. For not simply connected configuration spaces, this induces an action of the fundamental group on sections via some 1-dimensional representations and thus also on the space of quantum states. In the canonical quantization approach to quantum gravity (as in other field theories) one just tries to repeat the same construction, using the classical configuration space as domain space for the state functional.
Since this space is infinite dimensional, the construction of a quantum state space (Hilbert space of sections) is technically much more difficult and overshadowed by the lack of appropriate measures. This points towards the necessity to use a kind of distributional-dual as domain space, on which such measures are available. This is not particular to theories of gravity and renders the availability and construction of the Schrödinger representation a non-trivial technical problem in any field theory. In our case, for example, it is easy to see that the group $D_F$ does not act freely on the distributional dual of the space of Riemannian metrics, thus rendering an analogous structure to (1.3) and hence the given derivation of (1.5) impossible. But we feel that a better technical definition and understanding of canonical quantisation in field theory is required first to attack these issues profitably. The presently given standard arguments are therefore necessarily heuristic in nature, as already stated in the Introduction.

As far as the fundamental group is concerned, it is sometimes alternatively argued directly using a presumed action of the zeroth homotopy group of the group of gauge transformations (diffeomorphisms) on the space of quantum states, without trying to use the auxiliary structure of the classical configuration space for its explicit construction. The (pretended) Schrödinger representation might then be useful in trying to visualize this action in terms of physical operations (e.g. exchanges, rotations etc.) on preferred states.

Applied to the case at hand, the possible fundamental groups are determined by (1.8) with those occurring in the table above. In this context one would say that a spinorial manifold $\Sigma$ admits abelian spinorial states, if $\pi_1(\mathcal{Q}(\Sigma))$ allows for one dimensional representations which represent the $Z_2$ generated by a $2\pi$-rotation parallel to an asymptotic sphere non-trivially. States transforming non-trivially under such a rotation where first considered in [Fr-So]. Their assertion was that if a single state existed that was not left invariant by this $Z_2$, its antisymmetric combination with the $Z_2$ transformed one would be a state that changed sign under that $Z_2$. They, however, did not address the question of whether such a state carried any one-dimensional representation of $\pi_1(\mathcal{Q}(\Sigma))$. Here, we make the following

**Observation 1.** No 3-manifold, $\bar{\Sigma}$, whose prime decomposition consists entirely of primes taken from the upper four horizontal blocks in our table allows for abelian spinorial states.

**Proof.** Clearly we assume the presence of at least one spinorial prime, since the statement is trivial otherwise. As shown in the proof for Corollary 1, the rotation parallel to the sphere at $\infty$ is isotopic to rotations parallel to each connecting sphere for the primes. This generates an extending $Z_2$ subgroup (see 2.6) in each factor in (1.8) that corresponds to a spinorial prime. $\bar{\Sigma}$ thus allows for abelian spinorial states, if and only if at least one of the spinorial $\bar{\Sigma}_i$ does. However, none of the groups $O^*, D_{8m}^m m \geq 1, I^*$ (see table) has a 1-dimensional representations that represent the extending $Z_2$ of their $SO(3)$ counterparts non-trivially.
The latter fact can be proven from the presentations

\[ D^*_n = \{ x, y \mid x^2 = (xy)^2 = y^{2n} \} \]
\[ O^* = \{ x, y \mid x^2 = (xy)^3 = y^4; \, x^4 = 1 \} \]
\[ I^* = \{ x, y \mid x^2 = (xy)^3 = y^5; \, x^4 = 1 \} , \]

from which their abelianizations are readily determined to be \( Z_2 \times Z_2 \) (generated by \( x \) and \( y \), taken as abelianized generators), \( Z_2 \) (generated by \( x \)) and 1 respectively. Since in each case it is \( x^2 \) that generates the extending \( Z_2 \), the result follows. Note that e.g. \( D^*_4(2^{n+1}) \) would have the desired representation since its abelianisation is \( Z_4 \), generated by \( x \). It just does not happen to occur as a \( \pi_1(Q) \). Note also that \( x^2 \) is always a central element. For this reason one can define an action of 2\( \pi \)-rotations on general wavefunctions not carrying one-dimensional representations (see e.g. [Gi2]).

We do not know whether the result can easily be extended to include general \( K(\pi, 1) \)-primes as well. We explicitly checked, however, that e.g. the three-torus as well does not allow for abelian spinorial states. But the situation clearly changes if one allows for higher dimensional representations. For example, the group \( O \) has five irreducible representations of dimensions 1, 1, 2, 3 and 3, whereas \( O^* \) has in addition three more of dimensions 2, 2 and 4, all of which represent the extending \( Z_2 \) non-trivially. If one allows for multiple component wavefunctions (sections in higher dimensional complex vector bundles), spinorial states can be incorporated, albeit in this case one can not incorporate the whole group as transformation group [An][Gi2]. Let us note at this point that higher dimensional representations automatically appear in carrying through the standard quantum-mechanical formalism in presence of a discrete gauge group. Excluding them means that one a priori excludes potentially interesting sectors [Gi2].

Pushing the original setting a bit further, we remark that the possible, inequivalent line bundles with connection over \( Q \) are classified by \( H_1(Q, Z) \oplus FH_2(Q, Z) \) (see e.g. [Wo]), where \( F \) denotes the free part. It is also convenient to split \( H_1 \) into a free-, and a torsion part (denoted by \( T \)). The free part of \( H_1 \) alone then accounts for the different flat connections with unchanged bundle topology, whereas \( TH_1(Q, Z) \oplus FH_2(Q, Z) \cong H^2(Q, Z) \) labels the topologically inequivalent bundles. Only the latter define different sectors for the quantum theory, and may e.g. show up as a non-trivial spectral flow for the Dirac operator [Al-Nel]. Whereas for
non-trivial $FH_2$ one cannot make any statements without explicitly analysing the dynamics as to whether it actually makes use of a non-trivial class, a trivial $FH_2$ clearly excludes such possibilities from the beginning. It is therefore interesting to see whether there are $\Sigma$'s whose associated $Q(\Sigma)$'s have trivial $FH_2(Q, Z)$. From the table and (1.8) we see that this is generally not the case. However, we have the

**Observation 2.** Let $\bar{\Sigma}$ be a homogeneous spherical prime of non-cyclic fundamental group and $Q = Q(\bar{\Sigma})$ its associated configuration space; then $FH_2(Q, Z) = 0$.

**Proof.** From the table we infer that for the $\bar{\Sigma}$ in question the associated $Q(\bar{\Sigma})$ have finite universal cover of which the first two homotopy groups are trivial. A standard spectral sequence argument, which we suppress at this point, then shows that $H_2(Q, Q)$ (rational coefficients) is trivial, which is equivalent to the statement made.

Taken together with Observation 1 this excludes a possibly conjectured connection between spinoriality and the possibility of non-trivial spectral flows.

A very special case is that of $\bar{\Sigma} = RP^3$. Here we have immediately

**Observation 3.** $RP^3$ is the unique, non-trivial prime 3-manifold on our table whose configuration space has altogether trivial homotopy (and hence homology) groups.

From (1.8) one thus infers that a multi-$RP^3$ manifold, $\pi$, therefore receives all the non-trivial topology of $Q(\pi)$ from the factor $\Omega C$. A more detailed study of $C$ shows [He-McCu] that in this case $\pi_*(Q(\pi))$ contains as subgroups those for the configuration space of $n$ identical objects in $R^3$ (if $\bar{\Sigma}$ is the connected sum of $n$ $RP^3$'s). Its fundamental group is $P_n$, the permutation group of $n$ objects, and the higher ones are given in ref. [Fa-Neu]. This particular example has been used to discuss, and rule out, the obvious version of a spin-statistics relation, as generally discussed in [So]. However, that such a correlation cannot exist is a
general consequence of the product formula (1.8), which explicitly uncorrelates the representation chosen for rotations and exchanges, irrespectively of the different definitions of the latter [An].
Appendix 1

In this appendix we show how to prove non-spinoriality for $\Sigma = L(p, q)$ and $\Sigma = S^1 \times S^2$. To this end we pick a curve $\alpha: [0, 1] \to SO(3)$ that generates $Z_2 = \pi_1(SO(3), id)$, find a covering curve in $D_{\infty}$, starting at the identity, and show that it ends in the identity component $D_{\infty}^0$ of $DF$ (we refer to (2.3)). We may choose $\alpha(s) = \exp(2\pi s \hat{z})$ and denote the corresponding linear map in $\mathbb{R}^3$ by $R_z[2\pi s]$, or $R_z[\varphi]$ for general angles $\varphi$.

Let now $\sigma: \mathbb{R}^3 \supset B_2 \to \bar{\Sigma}$ be an embedding of $B_2 = \{x \in \mathbb{R}^3/ \|x\| \leq 2\}$ into $\bar{\Sigma}$. We let $r$ denote the distance from the origin of $B_2$ and set:

\[
\begin{align*}
D_2 &= \text{Image } \sigma\big|_{r \leq 2}, \quad D_1 = \text{Image } \sigma\big|_{r \leq 1}, \quad T = \text{Image } \sigma\big|_{1 \leq r \leq 2} \\
S_2 &= \text{Image } \sigma\big|_{r = 2}, \quad S_1 = \text{Image } \sigma\big|_{r = 1}, \quad \infty = \text{Image } \sigma\big|_{r = 0}
\end{align*}
\]

(A1)

On $B_2$ we can define a path of homeomorphisms, $\rho_s$, by

\[
\rho_s = \begin{cases}
R_z[2\pi s(2 - r)] & \text{for } 2 \geq r \geq 1 \\
R_z[2\pi s] & \text{for } r \leq 1
\end{cases}
\]

(A2)

which then defines a path of homeomorphisms, $R_s$, of $\bar{\Sigma}$, by setting

\[
R_s = \begin{cases}
\sigma \circ \rho_s \circ \sigma^{-1} & \text{on } D_2 \\
id & \text{on } \bar{\Sigma} - D_2
\end{cases}
\]

(A3)

This map is not differentiable on $S_2$ and $S_1$, but it may easily be smoothed by modifying it in arbitrarily small collar-neighbourhoods of these two spheres. We imagine this being done but without giving the details here, since in order to calculate the projection map $\tilde{p}$ in (2.3) we only need differentiability in a neighbourhood of $\infty$. $R_s$ “rigidly” rotates $D_1$ by an angle $2\pi s$ about the $z$-axis by progressing “rigid” rotations of the spheres $r = \text{const.}$ within $T$. For $s = 1$ we have $R_s\big|_{D_1} = id$ so that we call $R_1$ a rotation parallel to $S_1$ and $S_2$ (see e.g. [He][L]). Each $R_s$ fixes $\infty$ and projects to $\alpha(s)$ via $\tilde{p}$. $R_1$ is in $DF$ since it fixes a disc ($D_1$) containing $\infty$. In order to show that $R_1$ is in $D_{\infty}^0$, we now explicitly construct a path, $K_s$, from $id$ to $R_1$ within $DF$. As above, it is sufficient to construct a path of homeomorphisms
rather than diffeomorphisms that fix $D_1$ and which we then imagine to be smoothed appropriately. The details are irrelevant for us. For the construction it is convenient to represent the spaces $L(p,q)$ and $S^1 \times S^2$ by the following fundamental domains:

$L(p,q)$: Take a solid ball $\|x\| \leq 3$ in $\mathbb{R}^3$, and identify the 2-dimensional sectors $s_k : \frac{2\pi}{p}(k-1) \leq \phi \leq \frac{2\pi}{p}k$ on the upper hemisphere with the sectors $s_{k+q}$ on the lower hemisphere, by first reflecting them on the equatorial plane ($z = 0$), followed by a rotation about the $z$-axis. We take $T = \{x \in \mathbb{R}^3 / 1 \leq \|x\| \leq 2\}$, so that $\infty$ corresponds to $x = 0$.

$S^1 \times S^2$: Take a solid sphericall shell $1 \leq |x| \leq 6$ in $\mathbb{R}^3$, and identify the inner and outer 2-sphere boundaries radially (i.e. points of equal polar angles are identified). We take $T = \{x \in \mathbb{R}^3 / 1 \leq |x - (0,0,3)| \leq 2\}$ so that $\infty$ corresponds to $(0,0,3)$.

The crucial observation is that the $SO(3)$- rotation, $R_z[\varphi]$, applied to these domains is compatible with the boundary identifications, and, therefore, defines a homeomorphism of the manifolds in question. But then it is obvious how to reach $R_1$ by a path of homeomorphisms that fix $D_1$: instead of rotating $D_1$ against $S_2$, we rotate $\Sigma - D_2$ and with it $S_2$ against $S_1$ by just the negative amount. That is ($r$ still denotes the distance from $\infty$ and $\sigma$ is the identity since we work within $\mathbb{R}^3$)

$$K_s = \begin{cases} id & \text{on } D_1 : (r \leq 1) \\ R_z[-2\pi s(r-1)] & \text{on } D_2 - D_1 : (1 \leq r \leq 2) \\ R_z[-2\pi s] & \text{on } \Sigma - D_2 : r \geq 2 \end{cases} \quad (A4)$$

so that

$$K_1 = \begin{cases} id & \text{on } D_1 \\ R_z[2\pi(1-r)] & \text{on } D_2 - D_1 \\ R_z[-2\pi] & \text{on } \Sigma - D_2 \end{cases} \quad (A5)$$

which, by $R_z[\varphi + n 2\pi] = R_z[\varphi]$, is equal to $R_1$.

Although we work with oriented manifolds only, let us note for completeness that from the fundamental domain we used to construct $S^1 \times S^2$, we can also construct $S^2 \# S^2$, the unique non-orientable 2-sphere bundle over $S^1$. For this we just identify the inner and outer 2-sphere boundaries via the antipodal map (in standard polar angles: $(\theta, \phi) \mapsto (\pi - \theta, \phi + \pi)$), rather than using the identity. The rotations $R_z[\varphi]$ are then still compatible with the boundary identifications, hence proving non-spinoriality for $S^1 \times S^2$.  

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Appendix 2

In this appendix we collect some of the results on the diffeomorphism group of 3-manifolds which were of relevance in our investigations.

**Theorem A1.** For spherical primes \((S^3/G)\), the handle \((S^1 \times S^2)\), sufficiently large \(K(\pi, 1)\)'s \((K(\pi, 1)_{sl})\) and most of the non-sufficiently large \(K(\pi, 1)\)'s which are Seifert, two diffeomorphisms are isotopic if and only if they are homotopic.

**Proof.** For the spherical primes these statements are proven in [Ho-R] (Lens spaces \(L(p, q)\)), [Bir-R] (octahedral spaces \(S^3/O^* \times Z_p\)), [Asa] or [R] (prism and generalized prism spaces \(S^3/D^*_m \times Z_p, S^3/D^*_2.(2n+1) \times Z_p\) and [Boi-O] (icosahedral, tetrahedral and generalized tetrahedral \(S^3/I^* \times Z_p, S^3/T^* \times Z_p, S^3/T'_{8.3k} \times Z_p\)). For the handle this is proven in [Gl], for the \(K(\pi, 1)_{sl}\)'s in [Wa] and for the Seifert non-sufficiently large \(K(\pi, 1)\)'s in [Sc].

**Theorem A2.** For all spherical primes but the icosahedral, tetrahedral and generalized tetrahedral spaces one has the isomorphism \(\pi_0(D) \cong \pi_0(\text{Isom})\), where \(\text{Isom}\) denotes the space of orientation preserving isometries.

**Proof.** \(\pi_0(D)\) has been calculated in the references just given, except [Boi-O]. The calculations for \(\pi_0(\text{Isom})\) may be found in [Wi].

For the 3-sphere it is proven in [Ha4] that there is a homotopy equivalence \(D(S^3) \sim Isom(S^3)\). As generalisation it has been conjectured in [Ha2] that there is a homotopy equivalence for all spherical primes (Hatcher conjecture (HC)): \(D(S^3/G) \sim Isom(S^3/G)\). The efforts to prove this are so far summarized in the following

**Theorem A3.** HC holds for the real projective space \(RP^3\), the lens spaces \(L(4k, 2k-1)\), where \(k > 1\), and the prism and generalized prism spaces \(S^3/D^*_4 \times Z_p\) and \(S^3/D^*_2.(2n+1) \times Z_p\) respectively.

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**Proof.** The assertion for $RP^3$ is made in [Ha2]. For $S^3/D_{4n}^* \times Z_p$, $p > 1$, and $S^3/D_{2k(2n+1)}^* \times Z_p$, $p \geq 1$, this was proven in [I1][I2] and for $S^3/D_{4n}^*$ and $L(4k,2k-1)$ $k > 1$ in [McCu-R].

Similar results are known for the handle and the $K(\pi,1)_{st}$’s:

**Theorem A4.** The group of diffeomorphisms of $S^2 \times S^1$ has the homotopy type of $O(2) \times O(3) \times \Omega O(3)$, where $\Omega(\cdot)$ denotes the loop space of $(\cdot)$.

**Proof.** This is proven in [Ha3]. If one considers orientation preserving diffeomorphisms only, one may write: $D(S^2 \times S^1) \cong Z_2 \times S^1 \times SO(3) \times \Omega SO(3)$.

**Theorem A5.** Let $\bar{\Sigma}$ be an orientable, sufficiently large $K(\pi,1)$ prime of fundamental group $G$. Its diffeomorphism group, $Diff$, has homotopy groups

$\pi_0(Diff) \cong Out(G)$, $\pi_1(Diff) \cong centre G$ and $\pi_k(Diff) = 0 \quad \forall k \geq 2$.

**Proof.** The proof is given in [Ha1] for the larger class of $P^2$-irreducible (irreducible which contain no two-sided $RP^2$), sufficiently large $K(\pi,1)$’s. If one restricts to orientation preserving diffeomorphisms, one has $\pi_0(D) \cong Out_+(G)$, the outer automorphisms which respect the orientation homomorphism (a $Z_2$-quotient of $Out(G)$ if $\bar{\Sigma}$ allows for orientation reversing diffeomorphisms).
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